Combinatorics of poly-Bernoulli numbers

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What is combinatorics?

There are several “types” of combinatorics.

**Extremal/Hungarian combinatorics**

Given a set of discrete structures: \( S \) and a parameter \( p \).

Determine \( \max \{ p(S) : S \in S_n \} \).

**Enumerative/algebraic combinatorics**

Given a set of finite sets \( \{ S_n \} \).

Determine/bound \( |S_n| \).
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Given a set of finite set $\{S_n\}$. 
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Determine

\[
\max\{p(S) : S \in S_n\}.
\]

**Enumerative/algebraic combinatorics**

Given a set of finite set \( \{S_n\} \). Determine/bound

\[
|S_n|.
\]
An example for an extremal question

**Question**

What is the maximum number of 1’s in a 0-1 matrix of size $n \times k$ without the configuration

$\begin{pmatrix} 1 & 1 \\ 1 & \ast \end{pmatrix}$?
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\[
\begin{pmatrix}
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**The answer**

$n + k - 1$. 
An example for an enumerative question

Question
How many permutation matrices $P$ are there of size $n \times n$ such that $P$ does not contain a submatrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

The answer is $C_n = \frac{1}{n+1} \binom{2n}{n}$, the $n$-th Catalan number.
An example for an enumerative question

**Question**
How many permutation matrices $P$ are there of size $n \times n$ such that $P$ does not contain a submatrix

$$
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\end{pmatrix}
$$

**The answer**

$$
C_n = \frac{1}{n+1} \binom{2n}{n},
$$

the $n^{th}$ Catalan number.
Further examples

Füredi-Hajnal conjecture

Let $\pi$ be a forbidden configuration where the 1’s form a permutation matrix. Then the maximum number of 1’s in a matrix of size $n \times n$ without $\pi$ is $\mathcal{O}(n)$. 
### Füredi-Hajnal conjecture

Let $\pi$ be a forbidden configuration where the 1’s form a permutation matrix. Then the maximum number of 1’s in a matrix of size $n \times n$ without $\pi$ is $O(n)$.

### Stanley-Wilf conjecture

Let $\pi$ be any permutation matrix. The number of permutation matrices of size $n \times n$ without the submatrix $\pi$ is $2^{O(n)}$. 
A connection

**Klazar thereom**

Füredi-Hajnal conjecture implies Stanley-Wilf conjecture.
A connection

**Klazar theorem**
Füredi-Hajnal conjecture implies Stanley-Wilf conjecture.

**Marcus - Tardos theorem**
The Füredi-Hajnal conjecture is true.
A connection

**Klazar theorem**

Füredi-Hajnal conjecture implies Stanley-Wilf conjecture.

**Marcus - Tardos theorem**

The Füredi-Hajnal conjecture is true. Hence the Stanley-Wilf conjecture is true too.
How many $0$-$1$ matrices $M$ are there of size $n \times k$ such that $M$ does not contain the configuration $\begin{pmatrix} 1 & 1 \\ \ast & \end{pmatrix}$?

Observation

The answer should be $B_{-k}^n$, poly-Bernoulli numbers.
A question

How many 0-1 matrices $M$ are there of size $n \times k$ such that $M$ does not contain the configuration

\[
\begin{pmatrix}
1 & 1 \\
1 & *
\end{pmatrix}
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Observation

The answer should be $B^{-k} n^{\text{poly-Bernoulli numbers}}$. 

Szeged–Novi Sad Workshop on Combinatorics

Combinatorics of poly-Bernoulli numbers
A question

How many 0-1 matrices $M$ are there of size $n \times k$ such that $M$ does not contain the configuration

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\begin{pmatrix}
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1 & *
\end{pmatrix}
$$

Observation

The answer should be $B_n^{-k}$, poly-Bernoulli numbers.
What are the poly-Bernoulli numbers?

\[
\sum_{n=0}^{\infty} B^{(k)}_n x^n = \text{Li}_k(1 - e^{-x}),
\]
for all \(k \in \mathbb{Z}\),

where \(\text{Li}_k(x) = \sum_{i=1}^{\infty} x^i / i^k\).
What are the poly-Bernoulli numbers?

(Kaneko 1997)

\[
\sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!} = \frac{\text{Li}_k(1 - e^{-x})}{1 - e^{-x}}, \quad \text{for all } k \in \mathbb{Z}
\]

where

\[
\text{Li}_k(x) = \sum_{i=1}^{\infty} \frac{x^i}{i^k}.
\]
Let us see the $B_n^{(k)}$ numbers!

| $k$ | $n$ = 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---------|---|---|---|---|---|---|---|---|
| $-5$ | $1$ | $16$ | $146$ | $1066$ | $6902$ | $41506$ | $237686$ | $1315666$ | $-3$ |
| $-4$ | $1$ | $8$ | $46$ | $230$ | $1066$ | $4718$ | $20266$ | $85310$ | $-2$ |
| $-3$ | $1$ | $4$ | $14$ | $46$ | $146$ | $454$ | $1394$ | $4246$ | $-1$ |
| $-2$ | $1$ | $2$ | $8$ | $16$ | $32$ | $64$ | $128$ | $256$ | $0$ |
| $-1$ | $1$ | $2$ | $4$ | $8$ | $16$ | $32$ | $64$ | $128$ | $0$ |
| $0$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $1$ | $1$ | $1$ | $2$ | $4$ | $8$ | $16$ | $32$ | $64$ | $0$ |
| $2$ | $1$ | $1$ | $3$ | $6$ | $10$ | $15$ | $21$ | $28$ | $0$ |
| $3$ | $1$ | $1$ | $4$ | $10$ | $20$ | $35$ | $56$ | $84$ | $0$ |
| $4$ | $1$ | $1$ | $5$ | $15$ | $35$ | $70$ | $126$ | $210$ | $0$ |
Let us see the $B_n^{(k)}$ numbers!

| $n$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7    |
|-----|----|----|----|----|----|----|----|------|
| $k = -5$ | 1  | 32 | 454| 4718| 41506| 329462| 2441314| 17234438 |
| $-4$  | 1  | 16 | 146| 1066| 6902 | 41506 | 237686 | 1315666 |
| $-3$  | 1  | 8  | 46 | 230 | 1066 | 4718  | 20266  | 85310   |
| $-2$  | 1  | 4  | 14 | 46  | 146  | 454   | 1394   | 4246    |
| $-1$  | 1  | 2  | 4  | 8   | 16   | 32    | 64     | 128     |
| 0    | 1  | 1  | 1  | 1   | 1    | 1     | 1      | 1       |
| 1    | 1  | $\frac{1}{2}$ | $\frac{1}{6}$ | 0 | $-\frac{1}{30}$ | 0 | $\frac{1}{42}$ | 0 |
| 2    | 1  | $\frac{1}{4}$ | $-\frac{1}{36}$ | $-\frac{1}{24}$ | $\frac{7}{450}$ | $\frac{1}{40}$ | $-\frac{38}{2205}$ | $-\frac{5}{168}$ |
| 3    | 1  | $\frac{1}{8}$ | $-\frac{11}{216}$ | $-\frac{1}{288}$ | $\frac{1243}{54000}$ | $-\frac{49}{7200}$ | $-\frac{75613}{3704400}$ | $\frac{599}{35280}$ |
| 4    | 1  | $\frac{1}{16}$ | $-\frac{49}{1296}$ | $\frac{41}{3456}$ | $\frac{26291}{3240000}$ | $-\frac{1921}{144000}$ | $\frac{845233}{155848000}$ | $\frac{1048349}{59270400}$ |
What are the poly-Bernoulli numbers of negative upper index?

\[(\text{Arakawa-Kaneko 1999}) \quad k \in \mathbb{N}\]

\[B_n^{(-k)} = \sum_{m=0}^{\min\{n,k\}} m! \binom{n+1}{m+1} m! \binom{k+1}{m+1}.\]
The combinatorial interpretation of Arakawa-Kaneko’s formula

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Let \( N \) be a set of \( n \) elements and \( K \) a set of \( k \) elements. One can think as \( N = \{1, 2, \ldots, n\} =: [n] \) and \( K = [k] \).
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Take \( \mathcal{P}_{\hat{N}} \) a partition of \( \hat{N} \) and \( \mathcal{P}_{\hat{K}} \) a partition of \( \hat{K} \) with the same number of classes as \( \mathcal{P}_{\hat{N}} \).
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The easy combinatorial definition

\[ B(k, n) = |A(k, n)| \]
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\[ B_n^{(k)} := |A_n^{(k)}| \]
Equivalent combinatorial definitions

**Brewbaker**

Let $\mathcal{L}_n^{(k)}$ be the set of 0-1 matrices that can be reconstructed from their row and column sums.
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**Callan**

Let $C_n^{(k)}$ be the set of permutations of $1, 2, 3, \ldots, n, 1, 2, 3, \ldots, k$, such that each monochromatic segment is increasing.
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**Vesztergombi**

Let $\mathcal{C}_n^{(k)}$ be the set of permutations of $[n + k]$ such that

$$-n \leq \pi(i) - i \leq k,$$

for each $i$. 
With Celia Glass and Robert Schumacher, I recently found a combinatorial interpretation of the poly-Bernoulli numbers of negative order ...

Let $\mathcal{O}(k)_n$ be the set of acyclic orientations of $K_k$, $k$. 

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Let $\mathcal{O}_n^{(k)}$ be the set of acyclic orientations of $K_{n,k}$. 
If a formula is simple and combinatorial, then there must be a simple and combinatorial explanation for that.
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See

Stanley, Bijective proof problems,
http://www-math.mit.edu/~rstan/bij.pdf
Theorem

There is a bijection between the set of 0-1 matrices of size $n \times k$ without the configuration

$$
\begin{pmatrix}
1 & 1 \\
1 & *
\end{pmatrix}
$$

and

$A_n^{(k)}$. 

The proof: The first steps

$N$ is the set of rows, $K$ is the set of columns.
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$N$ is the set of rows, $K$ is the set of columns.

We add an additional all-0 row and an additional all-0 column. $\hat{N}$ is the set of rows, $\hat{K}$ is the set of columns.

Two columns are equivalent iff their top 1's are in the same row. That gives us a partition of $\hat{K}$. The special class is the set of all-0 columns. By knowing this partition of columns we know a lot about our matrix, except elements at the last columns of the ordinary classes.
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In each not all-0 row we define an important 1:
- it is a top 1, if it contains a top 1,
- it is the first 1, if it does NOT contain a top 1.
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- it is a top 1, if it contains a top 1,
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Two not all-0 rows are equivalent iff their important 1’s are in the same columns.

There is a natural bijection between the classes of the two partitions.
Corollaries

\[ B_n^{(-k)} = B_k^{(-n)}. \]
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\[ B_n^{(-k)} = B_n^{(-(k-1))} + \sum_{i=1}^{n} \binom{n}{i} B_{n-(i-1)}^{(-(k-1))}. \]
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\[ B_n^{(-k)} = B_k^{(-n)}. \]

\[ B_n^{(-k)} = B_n^{(-(k-1))} + \sum_{i=1}^{\frac{n}{2}} \binom{n}{i} B_{n-(i-1)}^{(-(k-1))}. \]

\[ \sum_{i,j \in \mathbb{N} : i+j=N \text{ and } i \text{ even}} B_i^{(-j)} = \sum_{i,j \in \mathbb{N} : i+j=N \text{ and } i \text{ odd}} B_i^{(-j)}. \]
Thank you for your attention