The Intrinsic Robustness of Stochastic Bandits to Strategic Manipulation

Zhe Feng  
Harvard University  
Cambridge, MA 02138  
zhe_feng@g.harvard.edu

David C. Parkes  
Harvard University  
Cambridge, MA 02138  
parkes@eecs.harvard.edu

Haifeng Xu  
Harvard University  
Cambridge, MA 02138  
hxu@seas.harvard.edu

Abstract

We study the behavior of stochastic bandits algorithms under strategic behavior conducted by rational actors, i.e., the arms. Each arm is a strategic player who can modify its own reward whenever pulled, subject to a cross-period budget constraint. Each arm is self-interested and seeks to maximize its own expected number of times of being pulled over a decision horizon. Strategic manipulations naturally arise in various economic applications, e.g., recommendation systems such as Yelp and Amazon. We analyze the robustness of three popular bandit algorithms: UCB, $\varepsilon$-Greedy, and Thompson Sampling. We prove that all three algorithms achieve a regret upper bound $O(\max\{B, \ln T\})$ under any (possibly adaptive) strategy of the strategic arms, where $B$ is the total budget across arms. Moreover, we prove that our regret upper bound is tight. Our results illustrate the intrinsic robustness of bandits algorithms against strategic manipulation so long as $B = o(T)$. This is in sharp contrast to the more pessimistic model of adversarial attacks where an attack budget of $O(\ln T)$ can trick UCB and $\varepsilon$-Greedy to pull the optimal arm only $o(T)$ number of times. Our results hold for both bounded and unbounded rewards.

1 Introduction

Nowadays multi-armed bandits (MAB) play a significant role across the digital economy, for online/display advertising [9, 13], search engines [17] and online recommendation systems [18], to name a few. Classical stochastic multi-armed bandit models assume that the feedback (reward or loss) of each arm is drawn from a fixed distribution. However, in many economic applications, an arm may be strategic and able to modulate its own reward feedback to the learner in order to increase its own number of times of being selected. For example, restaurants are known to offer discounts or free dishes in order to entice customers to return; online stores on Amazon may offer discounts or coupons to potential buyers in order to receive higher ratings and thus increase its ranking.

To facilitate our description, we will distinguish the two different parties in our strategic setting: the principal agent (or principal for short) and arms. The principal represents a multi-armed bandit algorithm (e.g., UCB or Thompson Sampling) run by some system (e.g., the Amazon platform) whereas the arms represent all the parties who generate reward feedbacks to the principal (e.g. online stores on Amazon). As is typical in stochastic bandits, the true reward of each arm is drawn from an underlying distribution. However, we assume that each arm $i$ is a strategic agent and can increase its
own reward revealed to the principal, but with a cross-period total budget $B_i$ for the manipulation. All arms are self-interested, equipped with a natural objective of maximizing its expected number of times being pulled. Each arm can only modify its own reward at the time it is pulled, but has no control over the rewards of other arms. The strategy of each arm can be adaptive — i.e., the amount by which it changes the current reward can depend on the history. Since arms’ strategies affect each other, this dynamic interaction forms a strategic game among arms, more precisely, a stochastic game. Our study is motivated by various economic applications of MAB, where we believe strategic manipulations appear more realistic than the pessimistic consideration of adversarial attacks \cite{Jun2016,Lykouris2018} (see more illustrations in Section 2). The central question we study in this paper is the following:

Are existing stochastic bandit algorithms robust to arms’ strategic manipulations? Quantitatively, can we characterize their regret bounds?

Our results and contributions. Our main results illustrate that the three popular stochastic bandits algorithms — i.e., Upper Confidence Bound (UCB), $\epsilon$-Greedy and Thompson Sampling — are essentially all robust to strategic manipulations.

Concretely, we prove that under any (possibly adaptive) strategic manipulations of the arms, the regret of all three algorithms is upper bounded by $O\left(\sum_{i \neq i^*} \max\{B_i, \ln T\}\right)$, where $i^*$ indexes the optimal arm. Moreover, we prove that this regret upper bound is tight by exhibiting a simple and natural manipulation strategy which results in a regret of the same order in all three algorithms. The regrets here are with respect to the underlying true rewards, and they hold for both bounded and unbounded rewards. This shows that performances of all these algorithms deteriorate linearly in the summed budget $B = \sum_{i \neq i^*} B_i$. As long as $B = o(T)$, the optimal arm will be pulled for $T - o(T)$ times. Our simulation results also validate such linear dependence on $B$. This is in sharp contrast to the situation of adversarial attacks where an attack budget of $O(\ln T)$ can trick UCB and $\epsilon$-Greedy to pull the optimal arm only $o(T)$ number of times, resulting in regret $\Omega(T)$ \cite{Jun2016}. We remark that such contrasting performances in adversarial vs strategic settings has been observed in other fields such as Cryptography \cite{Coppersmith1998}. However, to the best of our knowledge, this appears to be the first such example in stochastic bandits. From the technical viewpoint, our setting strictly generalizes the standard stochastic bandits (corresponding to $B_i = 0$ for all arm $i$’s), therefore the proofs also have to generalize previous techniques for proving regret bounds. To do so, we devise new ways to bound the expected number of times that an arm is pulled, which may be of independent interest.

Additional related work. Most relevant to ours are the two recent works on adversarial attacks to stochastic bandits \cite{Jun2016,Lykouris2018}. Jun et al. \cite{Jun2016} design attack strategies that only use $O(\ln T)$ attack budget and can trick an UCB and $\epsilon$-Greedy principal to pull the optimal arm only $o(T)$ number of times. The key difference between our model and the adversarial attack in \cite{Jun2016} (as well as \cite{Lyu2020}) is that the adversary there can modify any arm’s rewards (crucially, can decrease the optimal arm’s rewards) whereas in our setting each arm can only increase its own rewards due to our strategic behavior assumption. This is the fundamental reason of the contrasting results in these two models. Moreover, \cite{Jun2016} only studies UCB and $\epsilon$-Greedy with unbounded rewards. Efficient attacks for bounded rewards and for Thompson Sampling are left as open problems in \cite{Jun2016}. However, our regret bound holds also for Thompson sampling and bounded rewards. Lykouris et al. \cite{Lyu2020} consider a complementary question: can we design a stochastic bandit algorithm that is robust to adversarial corruptions? They answer the question in the affirmative by designing a new algorithm whose performance deteriorates also linearly in the amount of adversarial corruptions but with an additional $\ln T$ multiplicative factor. In contrast, our goal is to prove the robustness of existing stochastic bandits algorithms to strategic manipulations. Moreover, our regret bound is linear in $B$ without the additional $\ln T$ factor. This is possible due to again the aforementioned model difference.

\footnote{Here to state regret bounds concisely, we treated $K$ and other bandit parameters (e.g. $\Delta_i$ and $\sigma$) as constants.}

\footnote{Indeed, since our regret bound happens to be linear in $B = \sum_{i \neq i^*} B_i$, if all arms’ budgets are not endogenously given and instead controlled by a meta adversary who can only use this budget to modify non-optimal arms’ rewards, our regret bounds still hold. This would not have been true if the regret bound was nonlinear in $B_i$'s.}
We will distinguish the two different parties in our strategic setting: the principal agent (or principal for short) and arms. In particular, the principal represents a bandit algorithm (e.g., UCB or e-greedy or Thompson Sampling). At each time $t = 1, \cdots, T$, the principal pulls arm $I_t$ which generates a reward $r_{t}$. Here $T$ is some fixed time horizon. Let $n_i(t) = \sum_{r=1}^{t} I(I_r = i)$ denote the number of times that arm $i$ has been pulled up to (including) time $t$ and $ar{\mu}_i(t) = \frac{1}{n_i(t)} \sum_{\tau=1}^{t} r_\tau \cdot I(I_\tau = i)$ denote the average rewards obtained from pulling arm $i$ up to (including) time $t$.

### Strategic arms

Different from classic stochastic bandits, each arm $i \in [K]$ in our setting is a strategic actor, equipped with the natural objective of maximizing $\mathbb{E}[n_i(T)]$, i.e., the expected total number of times arm $i$ is pulled. The actions available to arm $i$ is to modify its own rewards when it is pulled under a total budget $B_i$. Concretely, when $I_t = i$, arm $i$ can add an additional reward amount $\alpha_i(t)$ to the realized reward $r_t$ — subject to the budget constraint $\sum_{t=1}^{T} |\alpha_i(t)| \leq B_i$ — so that the revealed reward to the principal is $\bar{r}_t = r_t + \alpha_i(t)$. Note that arm $i$ has no control over other arms’ rewards, i.e., $\alpha_i(t)$ must equal 0 when $I_t \neq i$. To distinguish, we call $r_t$ the true reward and $\bar{r}_t$ the manipulated reward. The principal only knows $\bar{r}_t$ but not $r_t$, however her goal is to minimize regret with respect to the true reward $r_t$ (see motivations later). Without loss of generality, we assume $\alpha_i(t) \geq 0$ since in order to maximize $\mathbb{E}[n_i(T)]$ arm $i$ should never use a negative $\alpha_i(t)$. We thus call $\bar{\alpha}(i) = (\alpha_i(1), \cdots, \alpha_i(T))$ an manipulation strategy of arm $i$. Note that, $\bar{\alpha}(i)$ can be adaptive, i.e., $\alpha_i(t)$ can depend on what happens at round $\tau = 1, \cdots, t - 1$. As a more convenient notation, let $\beta_i(t) = \sum_{\tau \leq t} \alpha_i(\tau)$ denote the total manipulations until time $t$, which satisfies $\beta_i(t) \leq B_i$. Note that $\bar{\beta}(i) = (\beta_i(1), \cdots, \beta_i(T))$ is just an equivalent representation of $\bar{\alpha}(i)$. We call $\bar{\beta}(i) = (\beta_i(1), \cdots, \beta_i(K))$ a strategy profile. The objective of any arm $i$ is to find an manipulation strategy $\bar{\beta}(i)$ to maximize $\mathbb{E}[n_i(T)] = \mathbb{E}[\sum_{\tau=1}^{T} I(I_\tau = i)]$. For convenience, we sometimes omit the superscript $\beta$ when it is clear from the context.

Our main goal is to upper bound the principal’s regret under arms’ strategic manipulations. Note that all the regrets are with respect to the true reward sequence $\{r_t\}$, while not $\{\bar{r}_t\}$. For convenience, we assume that $B_i = 0$ and thus $\bar{\beta}(i)$ is the trivial vector of all zeros. This is without loss of generality since $B_i > 0$ would only reduce the regret for any manipulation strategy of the optimal arm $i^*$. 

### Motivations of the strategic model

Here we give one practical motivation of our model but it shall be easy to see that similar situations happen in many other applications. Consider the recon-
We consider a standard recommendation system Yelp running a stochastic bandit algorithm. The bandits correspond to restaurants to be recommended and each user access corresponds to a pull of the arms. The true service quality of each restaurant follows some underlying distribution. However, restaurants are strategic agents, and one natural objective for them to be on Yelp is to maximize the expected number of times it gets recommended to users, i.e., $\mathbb{E}[n_i(T)]$ in our notation. To do so, a widely observed approach in practice is to provide discounts or coupons to some users (i.e., additional rewards to users), subject to budget constraints because the restaurants cannot provide arbitrarily many coupons. This phenomenon is captured precisely by our modeling of arms’ strategic manipulations. The Yelp platform, on the other hand, would like to recommend the restaurants with truly good service and thus want to minimize the regret with respect to the true reward. In this context, our goal is to understand how these restaurants’ strategic behaviors affect the platform’s regret.

**Solution concept.** Since the strategies of these self-interested arms affect each others, their interaction induces a strategic game, more precisely, a stochastic game. The standard approach then is to analyze the principal’s regret at some Nash equilibrium among the strategic arms. However, the equilibria of stochastic games are extremely difficult to characterize and computationally intractable (see, e.g., [4][10] and references therein). We instead consider a stronger benchmark and prove the robustness of the regret upper bound against arbitrary strategy profiles — regardless whether it forms a Nash Equilibrium or not. This is the strongest robustness guarantee in this strategic model.

### 3 UCB and $\varepsilon$-Greedy are Robust to Strategic Manipulations

In this section, we analyze the regret guarantee when the principal agent uses either the Upper Confidence Bound (UCB) [3] or the $\varepsilon$-Greedy algorithm. It turns out that they admit the same regret upper bound under strategic manipulations, up to constant factors. Note that our proofs do not depend on whether the rewards are bounded or unbounded, so the upper bounds hold for both cases. The full proofs for UCB and $\varepsilon$-Greedy can be found in Appendix [B.1] and [B.2] respectively.

#### 3.1 UCB Principal

We consider a standard $(\alpha, \psi)$-UCB with $\alpha = 4.5$ and $\psi : x \to \frac{x^2 \lambda^2}{2}$ (thus $(\psi^*)^{-1}(\epsilon) = \sqrt{2 \sigma^2 \epsilon}$) mentioned in [7] and used in [16] as well. Concretely, the algorithm selects each arm once in the first $K$ rounds, i.e. $I_i = t$, $\forall t < K$. For $t \geq K$,

$$I_i = \arg \max_i \left\{ \hat{\mu}_i(t-1) + 3\sigma \sqrt{\frac{\ln t}{n_i(t-1)}} + I(i \neq i^*) \frac{\beta_{i-1}(t)}{n_i(t-1)} \right\},$$

Recall that the term $\hat{\mu}_i(t-1) + 3\sigma \sqrt{\frac{\ln t}{n_i(t-1)}}$ is the standard UCB term for any arm $i \in [K]$ at time $t$, which we refer to as $\text{UCB}_i(t)$. $\hat{\text{UCB}}_i(t) = \text{UCB}_i(t) + \beta_{i-1}(t) / n_i(t-1)$ represents the modified UCB term for the strategic arm $i (i \neq i^*)$ with manipulation strategy $\bar{\beta}^{(i)}$.

We now prove the upper bound of regret $\mathbb{E}[R(T)]$ under an arbitrary manipulation strategy $\bar{\beta} = (\bar{\beta}^{(1)}, \ldots, \bar{\beta}^{(K)})$. Our proof technique strictly generalizes the regret analysis for the standard UCB, which corresponds to the case with no budget for each arm to manipulate.

Fix any manipulation strategy $\bar{\beta}$. The regret can be rewritten in the following way:

$$\mathbb{E}[R(T)] = \sum_{i \neq i^*} \Delta_i \cdot \mathbb{E}[n_i^{\bar{\beta}}(T)].$$

What remains is to bound $\mathbb{E}[n_i^{\bar{\beta}}(T)]$ for each arm $i$. For notational convenience, we omit the superscript $\bar{\beta}$ henceforth when it is clear from the context. Lemma [3.1] proves the upper bound of $\mathbb{E}[n_i(T)]$ for each arm $i$. The main result of this section is summarized in Theorem [3.2].

**Lemma 3.1.** For any manipulation strategy $\bar{\beta}$ of strategic arms, the expected number of times that arm $i (i \neq i^*)$ being pulled up to time $T$ if the principal uses UCB, can be bounded as follows,

$$\mathbb{E}[n_i(T)] \leq \max \left\{ \frac{3B_i}{\Delta_i} \frac{81\sigma^2 \ln T}{\Delta_i^2} \right\} + 1 + \frac{\pi^2}{3}.$$
Proof Sketch. The key idea of this proof is to carefully choose a threshold \( C_i(T) \) for \( n_i(t-1) \) so that we can have the best trade-off between the two terms in the following decomposition of \( \mathbb{E}[n_i(T)] \):

\[
\mathbb{E}[n_i(T)] \leq 1 + \mathbb{E} \left[ \sum_{t=K+1}^{T} \mathbb{I}\{I_t = i, n_i(t-1) \leq C_i(T)\} \right] + \mathbb{E} \left[ \sum_{t=K+1}^{T} \mathbb{I}\{I_t = i, n_i(t-1) \geq C_i(T)\} \right].
\]

It turns out that setting \( C_i(T) = \max \left\{ \frac{81 \sigma^2 \ln T}{\Delta_i}, \frac{3B_i}{\Delta_i} \right\} \) gives the best regret bound after bounding the first term directly by \( C_i(T) \) and bounding the second term via the Chernoff-Hoeffding inequality.

Combining Lemma 3.1 and Equation (1) yields a proof of our main theorem in this section.

**Theorem 3.2.** For any manipulation strategy \( \hat{\beta} \) of strategic arms, the regret of the UCB principal is bounded by

\[
\mathbb{E}[R(T)] \leq \sum_{i \neq i^*} \left[ \max \left\{ 3B_i, \frac{81 \sigma^2 \ln T}{\Delta_i} \right\} + (1 + \frac{\pi^2}{3}) \Delta_i \right].
\]

Theorem 3.2 reveals that the standard UCB algorithm is robust to the strategic manipulations of arms. If the budget of each arm is bounded by \( \mathcal{O}(\ln T) \), the regret of the principal agent is still bounded by \( \mathcal{O}(\ln T) \). If \( B_i = \Omega(\ln T) \) for some arm \( i \)'s, the regret is upper bounded by \( \mathcal{O}(\sum_{i \neq i^*} B_i) \). This is sublinear in \( T \) so long as \( B = \sum_{i \neq i^*} B_i = o(T) \), which we believe is typically the case in practice.

### 3.2 \( \varepsilon \)-Greedy Principal

Different from UCB, the \( \varepsilon \)-Greedy algorithm involves a random exploration phase, which makes the analysis more intricate. Like in UCB, we similarly assume that the algorithm pulls arm \( t \) when \( t \leq K \), i.e., first exploring each arm once. At round \( t > K \), the algorithm selects an arm as follows:

\[
I_t = \begin{cases} 
\arg \max_i \left\{ \hat{\mu}_i(t-1) + \mathbb{I}(i \neq i^*) \frac{\beta^{(i)}_{t-1}}{n_i(t-1)} \right\}, & \text{w.p. } \varepsilon_t \text{ (exploration)} \\
, & \text{otherwise } \text{(exploitation)}
\end{cases}
\]

We usually decay \( \varepsilon_t \) over time in order to achieve better performance (see, e.g., the guidance in [3]). Similar to the analysis of UCB, we derive the following regret guarantee for \( \varepsilon \)-Greedy.

**Theorem 3.3.** Let \( \varepsilon_t = \min\{1, \frac{c}{t}\} \), where constant \( c = \max\{20, \frac{36\sigma^2}{\Delta} \} \), for any manipulation strategy \( \hat{\beta} \) of strategic arms, the regret of the \( \varepsilon \)-Greedy principal is bounded by

\[
\mathbb{E}[R(T)] \leq \sum_{i \neq i^*} \left[ 3B_i + \mathcal{O} \left( \frac{\ln T}{\Delta_i} \right) \right].
\]

### 4 The Robustness of Thompson Sampling

In this section, we study the performance of Thompson Sampling (TS) [21], another well-known class of MAB algorithms. TS is widely known to be challenging to analyze — indeed, its regret bound was proved only recently in [11, 12]. This is because the algorithm does not directly depend on the empirical mean of each arm, but relies on random samples from the prior distribution centered at the empirical mean. This sampling process further complicates the analysis of the stochasticity in the algorithm. Moreover, it is also not clear that whether there exists an effective adversarial attack to TS, which was left as an open problem in [16]. Nevertheless, we prove that for strategic manipulations, TS admits the same regret upper bound as UCB and \( \varepsilon \)-Greedy, up to constant factors. These results serve as an evidence of the intrinsic robustness of stochastic bandits to strategic manipulations, regardless which no regret learning algorithm is used. All proofs from this section can be found in Appendix C.

TS employs the Bayesian updating during arm selection [8, 21]. In this paper, we use Gaussian priors and likelihood to handle the general rewards setting (Beta priors are usually used for binary reward feedback). Like UCB and \( \varepsilon \)-Greedy, we also assume that the algorithm pulls each arm once in the first \( K \) rounds. For \( t > K \), the algorithm selects an arm according to the following procedure:
To begin with, we adopt some notations from [2]. For each arm \( k \in [K] \), we denote two thresholds \( x_k \) and \( y_k \) such that \( \mu_k \leq x_k \leq y_k \leq \mu_{i^*} \), let \( E_k^\mu(t) \) be the event \( \mu_k(t-1) \leq x_k \) and \( \mathbb{E}_k^\theta(t) \) be the event \( \theta_k(t) \leq y_k \). We also denote \( \mathcal{F}_t \) as the history of plays until time \( t \). Let \( \tau_k, i \) be the time step at which arm \( k \) is played for the \( i \)th time and \( p_{k,t} \) be the probability that \( p_{k,t} = \mathbb{P}(\theta_i(t) \geq y_k | \mathcal{F}_{t-1}) \).

Lemma 4.1 shows the upper bound of \( n_i(T) \) under any manipulation strategy \( \beta \) when the principal runs Thompson Sampling. Theorem 4.2 follows from this lemma. Unsurprisingly, the proof of Lemma 4.1 turns out to be much more involved than that of Lemma 4.1 due to the aforementioned challenges. Our proof strictly generalizes the analysis in [2] to incorporate each arm’s manipulation.

**Lemma 4.1.** For any manipulation strategy \( \beta \), the expected number of times of arm \( i \) being pulled up to time \( T \) can be bounded as follows:

\[
\mathbb{E}[n_i(T)] \leq \max \left\{ \frac{6B_i}{\Delta_i}, \frac{72\sigma^2 \ln T}{\Delta_i^2} \right\} + O\left( \frac{\ln T}{\Delta_i} \right).
\]

**Proof Sketch.** First we can decompose \( \mathbb{E}[n_{i,T}] \) as follows,

\[
\mathbb{E}[n_{i,T}] \leq 1 + \mathbb{E}\left[ \sum_{t=K+1}^{T} \mathbb{I}\{ I_t = i, E_i^\mu(t), E_i^\theta(t) \} \right] + \mathbb{E}\left[ \sum_{t=K+1}^{T} \mathbb{P}\{ I_t = i, E_i^\mu(t), E_i^\theta(t) \} \right] + \mathbb{E}\left[ \sum_{t=K+1}^{T} \mathbb{I}\{ I_t = i, E_i^\theta(t) \} \right]
\]

The proof then proceeds by bounding each of the above terms separately. We set \( x_i = \mu_i + \frac{\Delta_i}{4} \), \( y_i = \mu_{i^*} - \frac{\Delta_i}{4} \). The first term can be bounded by \( \left( \frac{\ln T}{\Delta_i^2} + 1 \right) \) using a result of Agrawal and Goyal [2].

The second term can be bounded by \( \sum_{t=K+1}^{T-1} \mathbb{E}\left[ \frac{1}{p_{i,\tau_{i^c},s^c+1}} - 1 \right] \). We then bound each summand by the following bounds (Lemma C.3 in the Appendix):

\[
\mathbb{E}\left[ \frac{1}{p_{i,\tau_{i^c},s^c+1}} - 1 \right] \leq \left\{ \begin{array}{ll}
eq e^{11/4\sigma^2} + \frac{\pi^2}{3} \frac{1}{\Delta_i^2} & \text{if } s \geq \frac{72\ln(T\Delta_i^2)\max\{1,\sigma^2\}}{\Delta_i^2} \\ \end{array} \right.
\]

Finally, utilizing a similar technique as in Lemma 3.1 we can bound the third term by \( \max \left\{ \frac{6B_i}{\Delta_i}, \frac{144\sigma^2 \ln T}{\Delta_i^2} \right\} + 1 \) (Lemma C.4 in the Appendix).

**Theorem 4.2.** For any manipulation strategy \( \beta \) of strategic arms, the regret of the Thompson Sampling principal can be bounded as

\[
\mathbb{E}[R(T)] \leq \sum_{i \neq i^*} \max \left\{ 6B_i, \frac{72\sigma^2 \ln T}{\Delta_i} \right\} + O\left( \frac{\ln T}{\Delta_i} \right).
\]

## 5 Tightness of the Regret Bounds under a Natural Manipulation Strategy

In this section, we prove that the previous regret upper bounds for UCB, \( \epsilon \)-Greedy and TS are all tight. Specifically, we consider a specific manipulation strategy and prove that it results in a regret of at least \( \Omega(\sum_{i \neq i^*} |B_i - \ln T|) \) for the principal in all three algorithms. In this manipulation strategy, each arm simply spends all of its budget at the first time when it is pulled. We thus coin the name Lump Sum Investing (LSI). Note that LSI is a non-adaptive manipulation strategy, though our regret upper bound holds even for adaptive strategies.

We remark that our lower bounds hold for arbitrary \( \sigma \)-Gaussian distributions. This differs from classical regret lower bounds, which are typically distribution-dependent and proved by constructing a particular class of distributions, i.e., Bernoulli [7]. Our proof also employs different techniques
which are tailored particularly to achieve tight bounds w.r.t. to manipulation budget $B_i$'s. We start with a lower bound for the regret $\mathbb{E}[R(T)]$ by utilizing Equation (1). That is,

$$\mathbb{E}[R(T)] = \sum_{i \neq i^*} \Delta_i \mathbb{E}[n_i(T)] \geq \Delta \cdot \sum_{i \neq i^*} \mathbb{E}[n_i(T)].$$

(5)

Next, we focus on lower bounding $\sum_{i \neq i^*} \mathbb{E}[n_i(LSI)(T)]$ under the manipulation strategy LSI. For convenience, we omit the superscript LSI since it is clear that this will be the manipulation strategy of our focus. The proof proceeds by proving an upper bound for $\mathbb{E}[n_i(T)]$, which then is translated to a lower bound for $\sum_{i \neq i^*} \mathbb{E}[n_i(T)]$. The following theorems state the regret lower bound of UCB, $\epsilon$-Greedy and Thompson Sampling, respectively. The proofs are in Appendix D.

**Theorem 5.1.** Suppose the principal uses UCB and each strategic arm uses LSI. For any $\sigma$-Gaussian reward distributions, the regret of the principal satisfies

$$\mathbb{E}[R(T)] \geq \Delta \sum_{i \neq i^*} \frac{B_i}{2\Delta_i} - O\left(\frac{\ln T}{\Delta}\right).$$

**Theorem 5.2.** Suppose the principal uses $\epsilon$-Greedy with $\epsilon_t = \min\{1, \frac{cK_t}{t}\}$, $\forall t > K$ where $c = \max\{20, \frac{\ln^2 \sigma}{\Delta}\}$, and each strategic arm uses LSI. For any $\sigma$-Gaussian reward distributions, the regret of the principal satisfies

$$\mathbb{E}[R(T)] \geq \Delta \sum_{i \neq i^*} \frac{B_i}{2\Delta_i} - O\left(\frac{\ln T}{\Delta}\right).$$

**Theorem 5.3.** Suppose the principal uses Thompson Sampling and each strategic arm uses LSI. For any $\sigma$-Gaussian reward distributions, the regret of the principal satisfies,

$$\mathbb{E}[R(T)] \geq \Delta \sum_{i \neq i^*} \frac{B_i}{2\Delta_i} - O\left(\frac{\ln T}{\Delta}\right).$$

**Remarks:** The above lower bounds hold for arbitrary $\sigma$-Gaussian distributions, however they can be easily converted to a distribution-dependent lower bound $\max\{\Delta \sum_{i \neq i^*} \frac{B_i}{2\Delta_i} - O\left(\frac{\ln T}{\Delta}\right), \Omega(\ln T)\}$ because there exist distributions such that any no-regret learning algorithm will suffer regret $\Omega(\ln T)$ and the manipulation strategy LSI of non-optimal arms will not increase this regret. Therefore, the distribution-dependent lower bound for all three algorithms can be written as $\Omega\left(\max\{B - \ln T, \ln T\}\right)$, which precisely matches the order of the upper bound $O\left(\max\{B, \ln T\}\right)$.

### 5.1 Generalization to Bounded Rewards

In many applications, such as customers’ rating in Yelp/Amazon, the reward signal is bounded and the bounds are common knowledge (e.g. $0 \sim 5$ stars rating). Without loss of generality, we assume that the reward is bounded within $[0, 1]$. In such settings, the LSI manipulation strategy may be infeasible since the strategic arm can only manipulate the reward to the upper bound at each time. We thus use the following natural variant of LSI tailored for bounded rewards: each strategic arm $i$ spends the budget to promote the observed reward to 1 when it is pulled, until it runs out of all the budget $B_i$, and term this strategy LSI for Bounded Rewards or LSIBR for short.

To prove lower bounds for bounded reward settings, we provide a unified reduction from any lower bound under LSI to a lower bound under LSIBR, with an additional $\Theta(\ln T)$ loss in the lower bound. Our reduction works for any no-regret learning algorithm, and thus yields the following theorem.

**Theorem 5.4.** Suppose rewards are bounded within $[0, 1]$ and all strategic arms use LSIBR. For a principal running UCB, $\epsilon$-Greedy or Thompson Sampling, the regret satisfies

$$\mathbb{E}[R(T)] \geq \Delta \sum_{i \neq i^*} \frac{B_i}{2\Delta_i} - O\left(\frac{\ln T}{\Delta} + \frac{\Delta \ln T}{(1 - \mu_{i^*})^2}\right).$$
Therefore, it's budget will be "diluted" significantly in later rounds and thus does not affect the regret achieved by each stochastic bandit algorithm with strategic manipulations is linear in the total budget of the strategic arms. We consider three settings: (1) $B_1 = B_2 = B_2/3$, $B_3 = 0$, (2) $B_1 = B_2 = B_2 = B_3 = 0$, and (3) $B_1 = B_2 = B_3 = B_3 = B_2/3$. For setting (1), we uniformly partition the budget to arm 1 and arm 2. For setting (2), we put all the budget to arm 1. For setting (3), the optimal arm also has budget to manipulate its reward, and we assume arm 3 also uses strategy LSI. Figure 2 shows the regret at the end of $10^4$ rounds achieved by three algorithms as the total budget $B = B_1 + B_2$ varies. We find that the regret is indeed linear with total budget in general, which validates our theoretical findings. Interestingly, even if the optimal arm also has budget to change its reward, the regret still becomes worse as the budget for arm 1 and 2 increase. In fact, the green line always stays close to the other two lines where arm 3 does not have any budget. This is because the optimal arm will be pulled for many times. Therefore, its budget will be "diluted" significantly in later rounds and thus does not affect the regret much.

6 Simulations

In this section, we run simulations to validate our theoretical results. For space limit, we only report simulation results for the unbounded reward setting. The results for bounded reward setting are similar, and can be found in Appendix E.

**Setup.** There are 3 different arms, the reward distribution of arm 1, arm 2 and arm 3 are $\mathcal{N}(\mu_1, \sigma^2)$, $\mathcal{N}(\mu_2, \sigma^2)$ and $\mathcal{N}(\mu_3, \sigma^2)$. We assume that $\mu_1 < \mu_2 < \mu_3$, i.e. arm 3 is the optimal arm. In $\epsilon$-Greedy algorithm, we set $\epsilon_t = \min\{1, \frac{1}{t}\}$. Throughout the simulations, we fix $\mu_1 = 5$, $\mu_2 = 8$, $\mu_3 = 10$ and $\sigma = 1$. All the arms use LSI strategy. We run bandit algorithms for $10^4$ rounds, which forms one trial. We repeat 100 trials and only report the average results over 100 trials.

**Regret of principal with different budgets.** We consider the regret for three stochastic bandit algorithms (UCB, $\epsilon$-Greedy and Thompson Sampling) with different budgets among strategic arms. For each algorithm, arm 1 and arm 2 have the same budget $B_i$, chosen from $\{0, 10, 100\}$. We show the regret as a function of $\ln t$ in Figure 2. We observe that for small budgets (i.e., $B_i = 0$, 10), the $\Theta(\ln t)$ term dominates the regret whereas for large budget, the budget term $B_i$ gradually dominates the regret as $t$ becomes large. This is why we see a turning point in the green lines, at which the plot transitions to a relatively flat curve (since $B_i = 100$ is fixed). Interestingly, we find that Thompson sampling performs better than both UCB and $\epsilon$-Greedy in this strategic manipulation scenario.

**Regret is linear with total budget.** We validate that the regret achieved by each stochastic bandit algorithm with strategic manipulations is linear in the total budget of the strategic arms. We consider three settings: (1) $B_1 = B_2 = B_2/3$, $B_3 = 0$, (2) $B_1 = B_2 = B_2 = B_3 = 0$, and (3) $B_1 = B_2 = B_3 = B_3 = B_2/3$. For setting (1), we uniformly partition the budget to arm 1 and arm 2. For setting (2), we put all the budget to arm 1. For setting (3), the optimal arm also has budget to manipulate its reward, and we assume arm 3 also uses strategy LSI. Figure 2 shows the regret at the end of $10^4$ round achieved by three algorithms as the total budget $B = B_1 + B_2$ varies. We find that the regret is indeed linear with total budget in general, which validates our theoretical findings. Interestingly, even if the optimal arm also has budget to change its reward, the regret still becomes worse as the budget for arm 1 and 2 increase. In fact, the green line always stays close to the other two lines where arm 3 does not have any budget. This is because the optimal arm will be pulled for many times. Therefore, its budget will be "diluted" significantly in later rounds and thus does not affect the regret much.
7 Conclusion and Further Work

In this paper, we study the effects of strategic manipulations on three popular stochastic MAB algorithms: UCB, $\varepsilon$-Greedy and Thompson sampling. We prove that the regrets they suffer under any strategic manipulations of arms in our model are all upper bounded by $O(\max\{B, \ln T\})$ where $B$ is the summed budgets across arms. Moreover, this bound is tight under a simple manipulation strategy. For future work, one direction is to understand the equilibrium among arms and see whether it is possible to achieve sublinear (in summed budget $B$) regret at equilibrium (recall that our regret bounds hold for the set of arbitrary strategy profiles). Another interesting direction is to design new stochastic bandits algorithms that can achieve sublinear (in $B$) regret bounds in our strategic model. Finally, as machine learning algorithms are used more and more in the digital economy, it becomes crucial to understand the performance of learning algorithms in presence of strategic behaviors, and we hope our study can encourage more related works in this space.

References

[1] Shipra Agrawal and Navin Goyal. Analysis of Thompson sampling for the multi-armed bandit problem. In Conference on Learning Theory, pages 39–1, 2012.

[2] Shipra Agrawal and Navin Goyal. Near-optimal regret bounds for Thompson sampling. J. ACM, 64(5):30:1–30:24, September 2017.

[3] Peter Auer, Nicolò Cesa-Bianchi, and Paul Fischer. Finite-time analysis of the multi-armed bandit problem. Mach. Learn., 47(2-3):235–256, May 2002.

[4] Elchanan Ben-Porath. The complexity of computing a best response automaton in repeated games with mixed strategies. Games and Economic Behavior, 2(1):1–12, 1990.

[5] Avrim Blum, Vijay Kumar, Atri Rudra, and Felix Wu. Online learning in online auctions. Theoretical Computer Science, 324(2-3):137–146, 2004.

[6] Mark Braverman, Jiemeing Mao, Jon Schneider, and S. Matthew Weinberg. Multi-armed bandit problems with strategic arms. In Conference on Learning Theory, 2019.

[7] Sébastien Bubeck and Nicolò Cesa-Bianchi. Regret analysis of stochastic and nonstochastic multi-armed bandit problems. Foundations and Trends® in Machine Learning, 5(1):1–122, 2012.

[8] Olivier Chapelle and Lihong Li. An empirical evaluation of Thompson sampling. In Advances in Neural Information Processing Systems 24, pages 2249–2257. Curran Associates, Inc., 2011.

[9] Olivier Chapelle, Eren Manavoglu, and Romer Rosales. Simple and scalable response prediction for display advertising. ACM Trans. Intell. Syst. Technol., 5(4):61:1–61:34, December 2014.

[10] Vincent Conitzer and Tuomas Sandholm. Complexity results about nash equilibria. In Proceedings of the 18th international joint conference on Artificial intelligence, pages 765–771. Morgan Kaufmann Publishers Inc., 2003.

[11] Michal Feldman, Tomer Koren, Roi Livni, Yishay Mansour, and Aviv Zohar. Online pricing with strategic and patient buyers. In Advances in Neural Information Processing Systems 29, pages 3864–3872. Curran Associates, Inc., 2016.

[12] Zhe Feng, Chara Podimata, and Vasilis Syrgkanis. Learning to bid without knowing your value. In Proceedings of the 2018 ACM Conference on Economics and Computation, EC ‘18, pages 505–522, New York, NY, USA, 2018. ACM.

[13] Zhe Feng, Okke Schrijvers, and Eric Sodomka. Online learning for measuring incentive compatibility in ad auctions. In Proceedings of the Web Conference, WWW ’19, 2019.

[14] S Dov Gordon and Jonathan Katz. Rational secret sharing, revisited. In International Conference on Security and Cryptography for Networks, pages 229–241. Springer, 2006.
[15] Nicole Immorlica, Jieming Mao, Aleksandrs Slivkins, and Zhiwei Steven Wu. Bayesian exploration with heterogeneous agents. In Proceedings of the 2019 World Wide Web Conference on World Wide Web. International World Wide Web Conferences Steering Committee, 2019.

[16] Kwang-Sung Jun, Lihong Li, Yuzhe Ma, and Jerry Zhu. Adversarial attacks on stochastic bandits. In Advances in Neural Information Processing Systems 31, pages 3644–3653. Curran Associates, Inc., 2018.

[17] Branislav Kveton, Csaba Szepesvári, Zheng Wen, and Azin Ashkan. Cascading bandits: Learning to rank in the cascade model. In Proceedings of the 32Nd International Conference on International Conference on Machine Learning - Volume 37, ICML ’15, pages 767–776. JMLR.org, 2015.

[18] Lihong Li, Wei Chu, John Langford, and Robert E. Schapire. A contextual-bandit approach to personalized news article recommendation. In Proceedings of the 19th International Conference on World Wide Web, WWW ’10, pages 661–670, New York, NY, USA, 2010. ACM.

[19] Thodoris Lykouris, Vahab Mirrokni, and Renato Paes Leme. Stochastic bandits robust to adversarial corruptions. In Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2018, pages 114–122, New York, NY, USA, 2018. ACM.

[20] Yishay Mansour, Aleksandrs Slivkins, and Vasilis Syrgkanis. Bayesian incentive-compatible bandit exploration. In Proceedings of the Sixteenth ACM Conference on Economics and Computation, pages 565–582. ACM, 2015.

[21] William R. Thompson. On the likelihood that one unknown probability exceeds another in view of the evidence of two samples. Biometrika, 25(3/4):285–294, 1933.

[22] Jonathan Weed, Vianney Perchet, and Philippe Rigollet. Online learning in repeated auctions. In Conference on Learning Theory, pages 1562–1583, 2016.
The Intrinsic Robustness of Stochastic Bandits to Strategic Manipulation

Appendix

A Useful Definitions and Inequalities

**Definition A.1** (σ-sub-Gaussian). A random variable $X \in \mathbb{R}$ is said to be sub-Gaussian with variance proxy $\sigma^2$ if $\mathbb{E} [X] = \mu$ and satisfies,

$$\mathbb{E} [\exp(s(X - \mu))] \leq \exp \left( \frac{\sigma^2 s^2}{2} \right), \forall s \in \mathbb{R}$$

Note the distribution defined on $[0, 1]$ is a special case of 1/2-sub-Gaussian.

**Fact A.2.** Let $X_1, X_2, \ldots, X_n$ i.i.d drawn from a σ-sub-Gaussian, $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ and $\mathbb{E}[X]$ be the mean, then

$$\mathbb{P} (\bar{X} - \mathbb{E}[X] \geq a) \leq e^{-na^2/2\sigma^2} \quad \text{and} \quad \mathbb{P} (\bar{X} - \mathbb{E}[X] \leq -a) \leq e^{-na^2/2\sigma^2}$$

**Fact A.3** (Harmonic Sequence Bound). For $t_2 > t_1 \geq 2$, we have

$$\ln \frac{t_2}{t_1} \leq \sum_{t=t_1}^{t_2} \frac{1}{t} \leq \ln \left( \frac{t_2}{t_1 - 1} \right)$$

**Fact A.4.** For a Gaussian distributed random variable $Z$ with mean $\mu$ and variance $\sigma^2$, for any $z$,

$$\mathbb{P} (|Z - \mu| > z\sigma) \leq \frac{1}{2} e^{-z^2/2}$$

**Lemma A.5** (Theorem 3 in [3]). In $\varepsilon$-Greedy, for any arm $k \in [K]$, $t > K$, $n \in \mathbb{N}_+$, we have

$$\mathbb{P} \left( \tilde{\mu}_k(t) - \frac{\Delta_k}{n} \leq m_k(t - 1) \leq \frac{\Delta_k}{n} \right) \leq x_t \cdot e^{-x_t/5} + \frac{2\sigma^2 n^2}{\Delta^2_k} e^{-\Delta^2_k |x_t|/2 \sigma^2 n^2}, \quad \text{and}$$

$$\mathbb{P} \left( \tilde{\mu}_i (t - 1) \geq \mu_i + \frac{\Delta_i}{n} \right) \leq x_t \cdot e^{-x_t/5} + \frac{2\sigma^2 n^2}{\Delta^2_i} e^{-\Delta^2_i |x_t|/2 \sigma^2 n^2},$$

where $x_t = \frac{1}{2K} \sum_{s=K+1}^{t} \varepsilon_s$.

B Omitted Proofs in Section 3

**B.1 Proof of Lemma 3.1**

*Proof.* Let $C_1(T) = \max \left\{ \frac{81\sigma^2 \ln T}{\Delta^2}, \frac{3B_1}{\Delta^2} \right\}$. By Fact A.2 we have for any $s \geq 1$ and $\ell \geq C_1(T)$

$$\forall k, \quad \mathbb{P} \left( \mu_k - \tilde{\mu}_k(t - 1) \geq 3\sigma \sqrt{\frac{\ln t}{n_k(t - 1)}} | n_k(t - 1) = s \right) \leq \frac{1}{t^{3/2}}$$

$$\mathbb{P} \left( \tilde{\mu}(t - 1) - \mu_i \geq \frac{\Delta_i}{3} | n_i(t - 1) = \ell \right) \leq \frac{1}{t^{3/2}} \quad (6)$$

We first decompose $\mathbb{E}[n_i(T)]$ as follows,

$$\mathbb{E} [n_i(T)] \leq 1 + \mathbb{E} \left[ \sum_{t=K+1}^{T} \mathbb{I} \{ I_t = i, n_i(t - 1) \leq C_i(T) \} \right] + \mathbb{E} \left[ \sum_{t=K+1}^{T} \mathbb{I} \{ I_t = i, n_i(t - 1) \geq C_i(T) \} \right]$$

$$\leq 1 + C_i(T) + \mathbb{E} \left[ \sum_{t=K+1}^{T} \mathbb{I} \{ I_t = i, n_i(t - 1) \geq C_i(T) \} \right]$$

$$\leq 1 + C_i(T) + \sum_{t=K+1}^{T} \mathbb{P} \left( \text{UCB}_i(t) + \frac{\beta_i^{(i)}(t - 1)}{n_i(t - 1)} \geq \text{UCB}_i(t), n_i(t - 1) \geq C_i(T) \right) \quad (7)$$

11
We then bound the probability \( \mathbb{P} \left( \text{UCB}_i(t) + \frac{\beta_{i-1}^{(i)}}{n_i(t-1)} \geq \text{UCB}_{i^*}(t), n_i(t-1) \geq C_i(T) \right) \) by union bound, and decompose this probability term as follows,

\[
\begin{align*}
\mathbb{P} \left( \text{UCB}_i(t) + \frac{\beta_{i-1}^{(i)}}{n_i(t-1)} \geq \text{UCB}_{i^*}(t), n_i(t-1) \geq C_i(T) \right) \\
\leq \sum_{s=1}^{t-1} \sum_{t \geq C_i(T)} \mathbb{P} \left( \text{UCB}_i(t) + \frac{\beta_{i-1}^{(i)}}{n_i(t-1)} \geq \text{UCB}_{i^*}(t) \mid n_i(t-1) = \ell, n_{i^*}(t-1) = s \right).
\end{align*}
\]

(8)

What remains is to upper bound the summand in the above term. Consider for \( 1 \leq s \leq t-1 \) and \( C_i(T) \leq \ell \leq t-1 \), we have

\[
\begin{align*}
\mathbb{P} \left( \text{UCB}_i(t) + \frac{\beta_{i-1}^{(i)}}{n_i(t-1)} \geq \text{UCB}_{i^*}(t) \mid n_i(t-1) = \ell, n_{i^*}(t-1) = s \right) \\
\leq \mathbb{P} \left( \hat{\mu}_i(t-1) + 3\sigma \sqrt{\frac{\ln t}{n_i(t-1)}} + \frac{\Delta_i}{3} \geq \tilde{\mu}_{i^*}(t-1) + 3\sigma \sqrt{\frac{\ln t}{n_{i^*}(t-1)}} \mid n_i(t-1) = \ell, n_{i^*}(t-1) = s \right) \\
\leq \mathbb{P} \left( \hat{\mu}_i(t-1) + \frac{\Delta_i}{3} \geq \tilde{\mu}_{i^*}(t-1) + 3\sigma \sqrt{\frac{\ln t}{n_{i^*}(t-1)}} \mid n_i(t-1) = \ell, n_{i^*}(t-1) = s \right)
\end{align*}
\]

The first inequality relies on the fact that \( \ell \geq C_i(T) \geq \frac{3B_i}{\Delta_i} \geq \beta_{i-1}^{(i)} \) and second inequality holds because \( \ell \geq C_i(T) \geq \frac{81\sigma^2 \ln T}{\Delta_i^2} \). By union bound and Equation (6), we can further upper bound the last term in the above inequality by

\[
\begin{align*}
\mathbb{P} \left( \hat{\mu}_i(t-1) - \mu_i \geq \frac{\Delta_i}{3} \mid n_i(t-1) = \ell \right) + \mathbb{P} \left( \mu_{i^*} - \hat{\mu}_{i^*}(t-1) \geq 3\sigma \sqrt{\frac{\ln t}{n_{i^*}(t-1)}} \mid n_{i^*}(t-1) = s \right) \\
\leq \frac{1}{T^{9/2}} + \frac{1}{t^{9/2}} \leq \frac{2}{t^{9/2}}
\end{align*}
\]

Combining equations (8) and the fact that

\[
\sum_{t=K+1}^{T} \sum_{s=1}^{t-1} \sum_{t \geq C_i(T)} \frac{2}{t^{9/2}} \leq \sum_{t=K+1}^{T} \frac{2}{t^2} \leq \frac{\pi^2}{3},
\]

we complete the proof. \( \square \)

### B.2 Proof of Theorem 3.3

To prove this theorem, we instead prove the following Lemma B.1 to bound \( \mathbb{E}[n_i(T)] \) for each arm \( i \neq i^* \). Given this Lemma, it is then easy to show Theorem 3.3.

**Lemma B.1.** Suppose the principal runs the \( \varepsilon \)-Greedy algorithm with \( \varepsilon_t = \min \{ 1, \frac{cK}{t} \} \) when \( t > K \), where the constant \( c = \max \{ 20, \frac{36\sigma^2}{\Delta_i^2} \} \). Then for any strategic manipulation strategy \( \hat{\beta} \), the expected number of times of arm \( i \) being pulled up to time \( T \) can be bounded by

\[
\mathbb{E}[n_i(T)] \leq \frac{3B_i}{\Delta_i} + \mathcal{O} \left( \frac{\ln T}{\Delta_i^2} \right).
\]

**Proof.** Let \( C_i = \frac{3B_i}{\Delta_i} \), \( x_t = \frac{1}{2K} \sum_{s=K+1}^{t} \varepsilon_s \) and for \( t \geq \lfloor cK \rfloor + 1 \), Given Fact A.3 we have

\[
x_t \geq \sum_{s=K+1}^{\lfloor cK \rfloor + 1} \frac{\varepsilon_s}{2K} + \sum_{t=\lfloor cK \rfloor + 1}^{t} \frac{\varepsilon_s}{2K} \geq [cK] - K + \frac{c}{2} \sum_{s=\lfloor cK \rfloor + 1}^{t} \frac{1}{s} \geq [cK] - K + \frac{c}{2} \ln \frac{t}{[cK]} + 1 \tag{9}
\]

12
We observe the fact that we do the decomposition for $\sum_{t=K+1}^T \mathbb{I}\{I_t = i, n_i(t-1) \leq C_i\} + \mathbb{E} \left[ \sum_{t=K+1}^T \mathbb{I}\{I_t = i, n_i(t-1) \geq C_i\} \right]

\leq 1 + C_i + \sum_{t=K+1}^T \frac{\epsilon_t}{K} + \mathbb{E} \left[ \sum_{t=K+1}^T (1 - \epsilon_t) \cdot \mathbb{I}\{\bar{\mu}_i(t-1) \geq \bar{\mu}_{i,t-1}, n_i(t-1) \geq C_i\} \right]

\leq 1 + C_i + \sum_{t=K+1}^T \frac{\epsilon_t}{K} + \sum_{t=K+1}^T \mathbb{P}(\bar{\mu}_i(t-1) + \frac{\beta_{t-1}}{n_i(t-1)} \geq \bar{\mu}_{i,t-1}, n_i(t-1) \geq C_i)

\leq 1 + C_i + \sum_{t=K+1}^T \frac{\epsilon_t}{K} + \sum_{t=K+1}^T \mathbb{P}(\bar{\mu}_i(t-1) + \frac{\beta_{t-1}}{n_i(t-1)} \geq \bar{\mu}_{i,t-1}, n_i(t-1) \geq C_i)

The last inequality holds because $\epsilon_t = 1$ when $t \leq |cK|$ and $1 - \epsilon_t \leq 1, \forall t$. What remains is to bound the last term above. Since $n_i(t-1) \geq C_i, \beta_{t-1} \leq B_t, \forall t \leq T$, this term is always upper bounded by

$$\mathbb{P}(\bar{\mu}_i(t-1) + \frac{\beta_{t-1}}{n_i(t-1)} \geq \bar{\mu}_{i,t-1}, n_i(t-1) \geq C_i) \leq \mathbb{P}(\bar{\mu}_i(t-1) + \frac{B_t}{C_i} \geq \bar{\mu}_{i,t-1}) \leq \mathbb{P}(\bar{\mu}_i(t-1) \geq \mu_i + \frac{\Delta_i}{3}) + \mathbb{P}(\bar{\mu}_i(t-1) \leq \mu_i - \frac{\Delta_i}{3})$$

By union bound, we have $\mathbb{P}(\bar{\mu}_i(t-1) + \frac{\Delta_i}{3} \geq \bar{\mu}_{i,t-1}) \leq \mathbb{P}(\bar{\mu}_i(t-1) \geq \mu_i + \frac{\Delta_i}{3})$. Based on Lemma A.5 we have

$$\mathbb{P}(\bar{\mu}_i(t-1) + \frac{\Delta_i}{3} \geq \bar{\mu}_{i,t-1}) \leq 2x_t \cdot e^{-x_t/5} + \frac{18\sigma^2}{\Delta^2} e^{-\Delta^2 |x_t|/18\sigma^2}$$

We observe the fact that $x_t \geq |cK| - K + \frac{\sigma}{2} \ln \frac{t}{|cK|+1} > 5$. Given $xe^{-x} \leq ye^{-y}, \forall x \geq y \geq 5$ and $e^{-x} \leq e^{-y}, \forall x \geq y$, we have

$$xe^{-x_t/5} \leq \left(|cK| - K + \frac{\sigma}{2} \ln \frac{t}{|cK|+1}\right) e^{-\frac{\sigma}{2} \ln \frac{t}{|cK|+1}} = \left(|cK| - K + \frac{\sigma}{2} \ln \frac{t}{|cK|+1}\right) e^{-\frac{\sigma}{2} \ln \frac{t}{|cK|+1}} e^{\sigma \Delta_t^2 |x_t| \Delta_t^2/36\sigma^2} = e^{\frac{\sigma^2 \Delta_t^2}{2} e^{\Delta_t^2 |x_t|} |x_t| \Delta_t^2/36\sigma^2}$$

Combining the above inequalities and Fact A.3, we can bound

$$\sum_{t=|cK|+1}^T 2x_t \cdot e^{-x_t/5} + \frac{18\sigma^2}{\Delta^2} e^{-\Delta^2 |x_t|/18\sigma^2} \leq \sum_{t=|cK|+1}^T \left(2|cK| - 2K + c \ln \frac{t}{|cK|+1}\right) + \left|\frac{|cK|+1}{t}\right|^2 + \frac{18\sigma^2}{\Delta^2} \frac{|cK|+1}{t} \frac{1}{t} \leq \left(|cK| - K\right) \cdot \frac{2(|cK|+1)^2 \pi^2}{3} + \left(c + \frac{18\sigma^2}{\Delta^2}\right) \sum_{t=|cK|+1}^T \frac{|cK|+1}{t} \frac{1}{t} \leq \left(|cK| - K\right) \cdot \frac{2(|cK|+1)^2 \pi^2}{3} + \left(c + \frac{18\sigma^2}{\Delta^2}\right) \ln \frac{T}{|cK|}$$

The first inequality in the above holds because $c \geq \max\{20, \frac{36\sigma^2}{\Delta^2}\}$, and the second inequality is based on the fact that $\ln x < x, \forall x > 1$ and $\sum_{t=1}^T \frac{1}{t} \leq \frac{\pi^2}{6}$. The last inequality is the implication of Fact A.3. Moreover, utilizing Fact A.3 we bound $\sum_{t=K+1}^T \frac{\epsilon_t}{K}$ in the following way,

$$\sum_{t=K+1}^T \frac{\epsilon_t}{K} = \sum_{t=K+1}^T \frac{1}{K} + \sum_{t=|cK|+1}^T \frac{\epsilon_t}{K} \leq \frac{|cK| - K}{K} + c \ln \frac{T}{|cK|},$$

Combining Equations (10), (11), (12) and (14), we complete the proof.
C Omitted Proofs in Section 4

C.1 Proof of Lemma 4.1

We bound the terms in the decomposition of $E[n_i(T)]$ in Eq. (3) using Lemma C.1 – Lemma C.4

**Lemma C.1** (Lemma 2.16 in [2]). Let $x_i = \mu_i + \frac{\Delta_i}{\sqrt{s}}$ and $y_i = \mu_{i^*} - \frac{\Delta_i}{\sqrt{s}}$.

$$E\left[ \sum_{i=K+1}^{T} \mathbb{I}\{I_i = i, E_i^p(t), \mathbb{E}_i^q(t)\} \right] \leq \frac{18 \ln T}{\Delta^2} + 1$$

**Lemma C.2** (Eq. (4) in [2]). $\sum_{t=K+1}^{T} \mathbb{P}\{I_t = i, E_i^p(t), \mathbb{E}_i^q(t)\} \leq \sum_{s=K+1}^{T-1} E\left[ \frac{1}{p_i(r_i, s+1)} - 1 \right]$.

**Lemma C.3** (Extension of Lemma 2.13 in [2]). Let $y_i = \mu_{i^*} - \frac{\Delta_i}{\sqrt{s}}$.

$$E\left[ \frac{1}{p_i(r_i, s+1)} - 1 \right] \leq \left\{ \begin{array}{ll}
\frac{e^{11/4\sigma^2} + \frac{\pi^2}{3}}{4T\Delta^2} & \text{if } s \geq \frac{72 \ln(T \Delta^2) \max(1, \sigma^2)}{\Delta^2}
\end{array} \right.$$  

**Proof.** This lemma extends Lemma 2.13 in [2] to our setting, and we mainly emphasize the required changes to the proof. Using the same notation as in [2], let $\Theta_j$ denote the Gaussian random variable follows $N(\mu_j, (\tau_j + 1, \frac{1}{\gamma})$, given $F_j$. Let $G_j$ be the geometric random variable denoting the number of consecutive independent trials until a sample of $\Theta_j$ becomes greater than $y_i$. Let $\gamma \geq 1$ be an integer and $z = 2\sigma \sqrt{\ln n}$ Then we have $E\left[ \frac{1}{p_i(r_i, s+1)} - 1 \right] = E[G_j]$. Following the same argument proposed in [2], we have for any $\gamma > e^{11/4\sigma^2}$,

$$\mathbb{P}(G_j < \gamma) \geq \left( 1 - \frac{1}{\gamma^2} \right) \mathbb{P}\left( \mu_{i^*} + \frac{z}{\sqrt{\gamma}} \geq y_i \right)$$

For $n_i^*(t - 1) = j, F_j$, we have

$$\mathbb{P}\left( \mu_{i^*} + \frac{z}{\sqrt{\gamma}} \geq y_i \right) \geq 1 - e^{-\frac{z^2}{2\sigma^2}}$$

Then $\mathbb{P}(G_j < \gamma) \geq 1 - \frac{1}{\gamma} - \frac{1}{\gamma} = 1 - \frac{2}{\gamma}$. Therefore,

$$E[G_j] = \sum_{\gamma=0}^{\infty} \mathbb{P}(G_j \geq \gamma) \leq e^{11/4\sigma^2} + \sum_{\gamma=1}^{\infty} \frac{2}{\gamma^2} \leq e^{11/4\sigma^2} + \frac{\pi^2}{3}$$

By the proof of Lemma 2.13 in [2], we have for any $D_i(t) \geq 0$,

$$E\left[ \frac{1}{p_i(r_i, s+1)} - 1 \right] \leq \left( \frac{1}{1 - \frac{1}{2} e^{-D_i(t)} \Delta^2 / 72} \right) \left( 1 - e^{-D_i(t) \Delta^2 / 72\sigma^2} \right)$$

Since $D_i(t) = \frac{72 \ln(T \Delta^2) \max(1, \sigma^2)}{\Delta^2}$, we have both $1 - \frac{1}{2} e^{-D_i(t) \Delta^2 / 72}$ and $1 - e^{-D_i(t) \Delta^2 / 72\sigma^2}$ are larger than or equal to $1 - \frac{1}{T\Delta^2}$. Thus, $E\left[ \frac{1}{p_i(r_i, s+1)} - 1 \right]$ can be bounded by $\frac{1}{T\Delta^2}$ when $j \geq D_i(t)$. 

**Lemma C.4.**

$$E\left[ \sum_{t=K+1}^{T} \mathbb{I}\{I_t = i, E_i^p(t)\} \right] \leq \max\left\{ \frac{6B_i}{\Delta_i}, \frac{144\sigma^2 \ln T}{\Delta_i^2} \right\} + 1 \quad (15)$$
Proof. Let $C_i(T) = \max \left\{ \frac{6\hat{B}_i}{\Delta_i}, \frac{144\sigma^2 \ln T}{\Delta_i} \right\}$. We first decompose the left hand side in Equation (15) as below,

$$
\mathbb{E} \left[ \sum_{t=K+1}^{T} \mathbb{I} \{ I_t = i, \hat{E}^p_i(t) \} \right] \leq \mathbb{E} \left[ \sum_{t=K+1}^{T} \mathbb{I} \{ I_t = i, \hat{E}^p_i(t), n_i(t-1) \leq C_i(T) \} \right] + \mathbb{E} \left[ \sum_{t=K+1}^{T} \mathbb{I} \{ I_t = i, \hat{E}^p_i(t), n_i(t-1) \geq C_i(T) \} \right]
$$

(16)

The first term in the above decomposition is trivially bounded by $c_i(T)$. What remains is to bound the second term

$$
\mathbb{E} \left[ \sum_{t=K+1}^{T} \mathbb{I} \{ I_t = i, \hat{E}^p_i(t), n_i(t-1) \geq C_i(T) \} \right]
$$

$$
\leq \sum_{t=K+1}^{T} \mathbb{P} \left( \hat{E}^p_i(t), n_i(t-1) \geq C_i(T) \right)
$$

$$
\leq \sum_{t=K+1}^{T} \mathbb{P} \left( \hat{\mu}_{i,t-1} + \frac{\beta_{t-1}}{n_i(t-1)} \geq x_i \ \big| \ n_i(t-1) \geq C_i(T) \right)
$$

By union bound, we have

$$
\mathbb{P} \left( \hat{\mu}_{i,t-1} + \frac{\beta_{t-1}}{n_i(t-1)} \geq x_i \ \big| \ n_i(t-1) \geq C_i(T) \right)
$$

$$
\leq \sum_{s=C_i(T)}^{t-1} \mathbb{P} \left( \hat{\mu}_{i,t-1} + \frac{B_i}{n_i(t-1)} \geq x_i \ \big| \ n_i(t-1) = s \right)
$$

$$
\leq \sum_{s=C_i(T)}^{t-1} \exp \left( -\frac{(x_i - \hat{x}_i) \Delta_i}{2 \hat{B}_i} \right) \leq \sum_{s=1}^{t-1} \frac{1}{T^2}
$$

The last inequality above uses Fact (A.2) and the fact $s \geq c_i(T) \geq \frac{6\hat{B}_i}{\Delta_i}$ and $s \geq \frac{144\sigma^2 \ln T}{\Delta_i}$. Then the second term of the right hand side in Equations (16) can be bounded by $\sum_{t=K+1}^{T} \sum_{s=1}^{t-1} \frac{1}{T^2} \leq 1$.  

D Omitted Proofs in Section 5

D.1 Proof of Theorem 5.1

As mentioned in Section 5, we show the lower bound of the regret by deriving the upper bound of the expected number of times that arm $i^*$ being pulled, which is summarized in Lemma D.1. Given Lemma D.1 and Eq. 5, it is straightforward to conclude Theorem 5.1.

Lemma D.1. Suppose each strategic arm $i (i \neq i^*$) uses LSI and $\Delta_i = \min_{i \neq i^*} \Delta_i$, the expected number of times that optimal arm $i^*$ being pulled up to time $T$ is bounded by,

$$
\mathbb{E} \left[ n_{i^*}(T) \right] \leq T - \sum_{i \neq i^*} \frac{B_i}{2 \Delta_i} + \mathcal{O} \left( \frac{\ln T}{\Delta_i^2} \right)
$$
Proof. Let $\Delta_i = \min_{i' \neq i'} \Delta_i$, $C(T) = \frac{3\sigma^2 \ln T}{\Delta_i}$, $D_i = \frac{B_i}{\Delta_i}$. First, by Fact A.2 we have for any $\ell \geq C(T)$, $s \geq 1$ and any $i$,

$$
P \left( \mu_i - \tilde{\mu}_i(t-1) \geq 3\sigma_i \sqrt{\frac{\ln t}{n_i(t-1)}} | n_i(t-1) = s \right) \leq \frac{1}{t^{\alpha/2}}
$$

\[
P \left( \tilde{\mu}_i(t-1) - \mu_i \geq \frac{\Delta_i}{2} | n_i(t-1) = \ell \right) \leq \exp \left( -\frac{\ell \Delta_i^2}{8\sigma_i^2} \right) \leq \frac{1}{t^{\alpha/2}}.
\]  

(17)

First, we decompose $E[n_i^*(T)]$ as follows,

$$
E[n_i^*,T] \leq 1 + E \left[ \sum_{t=K+1}^{T} \mathbb{I}\{I_t = i^*, n_i^*(t-1) \leq C(T)\} \right] + E \left[ \sum_{t=K+1}^{T} \mathbb{I}\{I_t = i^*, n_i^*(t-1) \geq C(T)\} \right]
$$

$$
\leq 1 + E \left[ \sum_{t=K+1}^{T} \mathbb{I}\{I_t = i^*, n_i^*(t-1) \leq C(T)\} \right]
$$

$$
+ E \left[ \sum_{t=K+1}^{T} \mathbb{I}\{I_t = i^*, n_i^*(t-1) \geq C(T), \forall i \neq i^*, n_i(t-1) \geq D_i\} \right]
$$

$$
+ E \left[ \sum_{t=K+1}^{T} \mathbb{I}\{I_t = i^*, n_i^*(t-1) \geq C(T), \exists i \neq i^*, n_i(t-1) \leq D_i\} \right]
$$

(18)

For the first term in the above decomposition, it can be trivially bounded by $C(T)$. For the second term, since $n_i^*(t) \leq T - \sum_{i \neq i^*} n_i(t), \forall t$, we have

$$
E \left[ \sum_{t=K+1}^{T} \mathbb{I}\{I_t = i^*, n_i^*(t-1) \geq C(T), \forall i \neq i^*, n_i(t-1) \geq D_i\} \right]
$$

$$
\leq E \left[ \sum_{t=K+1}^{T} \mathbb{I}\{I_t = i^*, n_i^*(t-1) \leq T - \sum_{i \neq i^*} D_i\} \right] \leq T - \sum_{i \neq i^*} B_i
$$

(19)

What remains is to bound the third term in Equations (18). By union bound, we have

$$
E \left[ \sum_{t=K+1}^{T} \mathbb{I}\{I_t = i^*, n_i^*(t-1) \geq C(T), \exists i \neq i^*, n_i(t-1) \leq D_i\} \right]
$$

$$
= \sum_{i \neq i^*} \sum_{t=K+1}^{T} P(I_t = i^*, n_i^*(t-1) \geq C(T), n_i(t-1) \leq D_i)
$$

Note $I_t = i^*$ implies UCB$_i^*$ $(t) \geq \hat{\text{UCB}}_i(t)$, combining the facts that $3\sigma \sqrt{\frac{\ln t}{n_i(t-1)}} \leq \Delta_i/2$ and $\frac{B_i}{n_i(t-1)} \geq 2\Delta_i$, and standard union bound, we have

$$
P(I_t = i^*, n_i^*(t-1) \geq C(T), n_i(t-1) \leq D_i)
$$

$$
\leq \sum_{s=1}^{D_i} \sum_{t=K+1}^{T} P(\hat{\mu}_i(t-1) + 3\sigma_i \sqrt{\frac{\ln t}{n_i(t-1)}} \geq \text{UCB}_i(t) + \frac{B_i}{n_i(t-1)} | \mu_i(t-1) = s)
$$

$$
\leq \sum_{s=1}^{D_i} \sum_{t=K+1}^{T} P(\hat{\mu}_i(t-1) + \frac{\Delta_i}{2} \geq \tilde{\mu}_i(t-1) + 3\sigma_i \sqrt{\frac{\ln t}{n_i(t-1)}} + 2\Delta_i | n_i^*(t-1) = \ell, n_i(t-1) = s)
$$

$$
\leq \sum_{s=1}^{D_i} \sum_{t=K+1}^{T} P(\hat{\mu}_i(t-1) - \mu_i \geq \frac{\Delta_i}{2} | n_i^*(t-1) = \ell) + P(\mu_i - \tilde{\mu}_i(t-1) \geq 3\sigma_i \sqrt{\frac{\ln t}{n_i(t-1)}} | n_i(t-1) = s)
$$

(19)
The last inequality is based on union bound, if both $\tilde{\mu}_i^*(t - 1) - \mu_i < \Delta_i/2$ and $\mu_i - \tilde{\mu}_i(t - 1) < 3\sigma \sqrt{\frac{\ln t}{n_i(t - 1)}}$ hold when $n_i^*(t - 1) = \ell, n_i(t - 1) = s$, then

$$\tilde{\mu}_i^*(t - 1) + \frac{\Delta_i}{2} < \mu_i^* + \frac{\Delta_i}{2} + \frac{\Delta_i}{2} \leq \mu_i + \Delta_i + \Delta_i$$

$$< \tilde{\mu}_i(t - 1) + 3\sigma \sqrt{\frac{\ln t}{n_i(t - 1)}} + 2\Delta_i$$

Given Equation (17), we have

$$\mathbb{P}(I_t = i^*, n_i^*(t - 1) \geq C(T), n_i(t - 1) \leq D_i) \leq \sum_{s=1}^{t-1} \sum_{t=1}^{t-1} \frac{2}{t^2} = \frac{2}{t}.$$

Combining Equation (18), we get

$$\mathbb{E}[n_i^*(T)] \leq 1 + C(T) + T - \sum_{i \neq i^*} \frac{B_i}{2\Delta_i} + \sum_{i \neq i^*} \sum_{t=K+1}^{T} \frac{2}{t^2} \leq T + \frac{36\sigma^2 \ln T}{\Delta^2} - \sum_{i \neq i^*} \frac{B_i}{2\Delta_i} + 1 + \frac{(K - 1)\pi^2}{3}$$

D.2 Proof of Theorem 5.2

Analogous to Theorem 5.1, we derive the upper bound of $\mathbb{E}[n_i^*(T)]$ when all strategic arms use LSI manipulation strategy in Lemma D.2.

**Lemma D.2.** \(\forall t > K, \text{let } \epsilon_t = \min\{1, \frac{cK}{t}\}, \text{where a constant } c = \max\{20, \frac{16\sigma^2}{\Delta^2}, \forall k \in [K]\}\). \(B_i\) be the total budget for strategic arm. The expected number of plays of arm \(i^*\) up to time \(T\), if all strategic arms use LSI, is bounded by

$$\mathbb{E}[n_i^*(T)] \leq T - \sum_{i \neq i^*} \frac{B_i}{2\Delta_i} + O\left(\frac{\ln T}{\Delta^2}\right)$$

**Proof.** Let \(C_i = \frac{B_i}{2\Delta_i}, x_t = \frac{1}{2K} \sum_{s=K+1}^{T} \epsilon_s\) and for \(t \geq |cK| + 1\), by Equation (9) \(x_t \geq |cK| - K + \frac{c}{\ln \left(\frac{t}{|cK| + 1}\right)}\).

We first bound the probability of $\mathbb{P}\left(\tilde{\mu}_i^*(t - 1) \geq \tilde{\mu}_i(t - 1), n_i(t - 1) \leq C_i\right)$ for $t \geq K + 1$,

$$\mathbb{P}\left(\tilde{\mu}_i^*(t - 1) \geq \tilde{\mu}_i(t - 1, n_i(t - 1) \leq C_i\right)$$

$$= \mathbb{P}\left(\tilde{\mu}_i^*(t - 1) \geq \tilde{\mu}_i(t - 1) + \frac{B_i}{n_i(t - 1)}, n_i(t - 1) \leq C_i\right)$$

$$\leq \mathbb{P}\left(\tilde{\mu}_i^*(t - 1) \geq \tilde{\mu}_i(t - 1) + 2\Delta_i\right)$$

$$\leq \mathbb{P}\left(\tilde{\mu}_i^*(t - 1) \geq \mu_i^* + \frac{\Delta_i}{2}\right) + \mathbb{P}\left(\tilde{\mu}_i(t - 1) \leq \mu_i - \frac{\Delta_i}{2}\right)$$

$$\leq 2x_t \cdot e^{-x_t/5} + \frac{8\sigma^2}{\Delta^2} e^{-\Delta^2 |x_t|/8\sigma^2} \text{(By Lemma A.3)}$$

We can decompose the expected number of plays of the optimal arm \(i, \mathbb{E}[n_i^*, T]\), as follows,

$$\mathbb{E}[n_i^*(T)] = 1 + \mathbb{E} \left[ \sum_{t=K+1}^{T} \mathbb{I}\{I_t = i^*, \forall i \neq i^* n_i(t - 1) \geq C_i\} \right]$$

$$+ \mathbb{E} \left[ \sum_{t=K+1}^{T} \mathbb{I}\{I_t = i^*, \exists i \neq i^* n_i(t - 1) \leq C_i\} \right]$$

17
The first term in the above decomposition can be bounded by \( T - \sum_{i \neq i^*} C_i \). This is because
\[
E \left[ \sum_{t=K+1}^{T} \mathbb{I}\{I_t = i^*, \forall i \neq i^*, n_i(t-1) \geq C_i\} \right] \\
\leq E \left[ \sum_{t=K+1}^{T} \mathbb{I}\{I_t = i^*, n_{i^*}(t-1) \leq T - \sum_{i \neq i^*} C_i\} \right] \leq T - \sum_{i \neq i^*} C_i.
\]

By union bound, the second term is bounded by \( \sum_{i \neq i^*} E \left[ \sum_{t=K+1}^{T} \mathbb{I}\{I_t = i^*, n_i(t-1) \leq C_i\} \right] \). Then, we bound the above summand using Equations (20) and the fact that \( 1 - \epsilon_t = 0 \) when \( t \leq [cK] \),
\[
E \left[ \sum_{t=K+1}^{T} \mathbb{I}\{I_t = i^*, n_i(t-1) \leq C_i\} \right] \\
\leq \sum_{t=K+1}^{T} \frac{\epsilon_t}{K} + \sum_{t=K+1}^{T} (1 - \epsilon_t) \cdot \mathbb{P}(\tilde{\mu}_{i^*}(t-1) \geq \tilde{\mu}_i(t-1), n_i(t-1) \leq C_i) \tag{22}
\leq \sum_{t=K+1}^{T} \frac{\epsilon_t}{K} + \sum_{t=[cK]+1}^{T} 2x_t \cdot e^{-x_t/5} + \frac{8\sigma^2}{\Delta_i^2} e^{-\Delta_i^2[x_t]/8\sigma^2}
\]

What remains is to bound the last term in the above equations. Following the same arguments and proof procedure in Equations (13), we can bound
\[
\sum_{t=[cK]+1}^{T} 2x_t \cdot e^{-x_t/5} + \frac{8\sigma^2}{\Delta_i^2} e^{-\Delta_i^2[x_t]/8\sigma^2}
\leq ([cK] - K) \cdot \frac{2([cK] + 1)^2\pi^2}{3} + ([cK] + 1) \left( c + \frac{8\sigma^2}{\Delta_i} \right) \ln \frac{T}{[cK]} \tag{23}
\]

By Eq. (14), we have
\[
E[n_{i^*}(T)] \leq T - \sum_{i \neq i^*} \frac{B_i}{2\Delta_i} + \frac{[cK]}{K} + c \ln \frac{T}{[cK]}
+ \sum_{i \neq i^*} \left( ([cK] - K) \cdot \frac{2([cK] + 1)^2\pi^2}{3} + ([cK] + 1) \left( c + \frac{8\sigma^2}{\Delta_i} \right) \ln \frac{T}{[cK]} \right)
\leq T - \sum_{i \neq i^*} \frac{B_i}{2\Delta_i} + O \left( \frac{\ln T}{\Delta_i^2} \right)
\]

D.3 Proof of Theorem 5.3

To prove Theorem 5.3, we follow the same approach as for UCB/ε−Greedy algorithms. In particular, we instead prove the upper bound of \( E[n_{i^*}(T)] \), shown in Theorem D.3. Given this Theorem, our main theorem in this section (Theorem 5.3) is straightforward. Here we slightly abuse notations, and use \( E_{i^*}^0(t) \) to denote the event that \( \tilde{\mu}_{i^*}(t-1) \leq v_i \) whereas \( E_{i^*}^0(t) \) to denote the event that \( \theta_{i^*}(t) \leq w_i \), where \( \mu_{i^*} < v_i < w_i \).

Theorem D.3.
\[
E[n_{i^*}(T)] \leq T - \sum_{i \neq i^*} \frac{B_i}{2\Delta_i} + O \left( \frac{\ln T}{\Delta_i^2} \right)
\]
Proof. We decompose the expected number of plays of the optimal arm $i^*$ as follows,

$$E[n_{i^*}(T)] \leq 1 + \sum_{t=K+1}^{T} \mathbb{P}(I_t = i^*, E_{i^*}^*(t)) + \sum_{t=K+1}^{T} \mathbb{P}(I_t = i^*, E_{i^*}^0(t), E_{i^*}^1(t))$$

$$+ \sum_{t=K+1}^{T} \mathbb{P}(I_t = i^*, E_{i^*}^0(t), E_{i^*}^1(t))$$

Then we bound each of the above terms. Lemma D.4, D.5, and D.8 show the upper bound of each term and complete the proof.

Lemma D.4. Let $v_i = \mu_{i^*} + \frac{2\Delta}{\sigma}$.

$$\sum_{t=K+1}^{T} \mathbb{P}(I_t = i^*, E_{i^*}^*(t)) \leq \frac{18\sigma^2}{\Delta^2}$$

Proof. Following the proof of Lemma 2.11 in [2], we have

$$\sum_{t=K+1}^{T} \mathbb{P}(I_t = i^*, E_{i^*}^*(t)) \leq \sum_{s=1}^{T-1} \mathbb{P}(E_{i^*}^*(\tau_{i^*,s+1})) = \sum_{s=1}^{T-1} \mathbb{P}(\hat{\mu}_{i^*}(\tau_{i^*,s+1}) > v_i)$$

$$\leq \sum_{s=1}^{T-1} \exp\left(-\frac{s(v_i - \mu_{i^*})^2}{2\sigma^2}\right) \leq \frac{2\sigma^2}{(v_i - \mu_{i^*})^2}$$

The first inequality holds because each summand on the right hand side in this inequality is a fixed number since the distribution of $\hat{\mu}_{i^*}(\tau_{i^*,s+1})$ only depends on $s$. The second inequality is based on Fact A.4 and the third inequality goes through because $\sum_{k=1}^{\infty} e^{-kx} \leq \frac{1}{x}$, $\forall x > 0$.

Notice that Lemma C.2 holds independently with the identity of the arm. Then the following Lemma can be directly implied.

Lemma D.5. Let $v_i = \mu_{i^*} + \frac{2\Delta}{\sigma}$ and $w_i = \mu_{i^*} + \frac{2\Delta}{\sigma}$.

$$\sum_{t=K+1}^{T} \mathbb{P}(I_t = i^*, E_{i^*}^0(t), E_{i^*}^1(t)) \leq \frac{18\ln T}{\Delta^2} + 1$$

Proof. The proof of Lemma 2.16 in [2] can be directly applied here by regarding arm $i^*$ as a standard sub-optimal arm $i$.

What remains is to bound $\sum_{t=K+1}^{T} \mathbb{P}(I_t = i^*, E_{i^*}^0(t), E_{i^*}^1(t))$. To this end, we show some auxiliary lemmas in the following. Lemma D.6 mimics Lemma 2.8 in [2], which bridges the probability that arm $i^*$ will be pulled and the probability that arm $i$ will be pulled at time $t$. Lemma D.7 bounds the term $E\left[\frac{1}{q_{i,t+1}} - 1\right]$ by a reduction to the case shown in Lemma C.2.

Lemma D.6. For any instantiation $F_{t-1}$ of $F_{t-1}$, let $q_{i,t} := \mathbb{P}\left(\theta_i(t) > w_i \mid F_{t-1}\right)$, we have

$$\mathbb{P}\left(I_t = i^*, E_{i^*}^0(t), E_{i^*}^1(t) \mid F_{t-1}\right) \leq \frac{1 - q_{i,t}}{q_{i,t}} \mathbb{P}\left(I_t = i, E_{i^*}^0(t), E_{i^*}^1(t) \mid F_{t-1}\right)$$

Proof. Since $E_{i^*}^0(t)$ is only determined by the instantiation $F_{t-1}$ of $F_{t-1}$, we can assume event $E_{i^*}^0(t)$ is true without loss of generality. Then, it is sufficient to show that for any $F_{t-1}$ we have

$$\mathbb{P}\left(I_t = i^* \mid E_{i^*}^0(t), F_{t-1}\right) \leq \frac{1 - q_{i,t}}{q_{i,t}} \mathbb{P}\left(I_t = i, E_{i^*}^0(t), F_{t-1}\right)$$
Note, given $E^0_i(t), I_t = i^*$ implies $\theta_j(t) \leq w_i, \forall j$, meanwhile, $\theta_i(t)$ is independent with $\theta_j(t), j \neq i$, given $F_{t - 1} = F_{t - 1}$. Therefore, we have

$$
\mathbb{P}\left(I_t = i^* \mid E^0_i(t), F_{t - 1}\right) \leq \mathbb{P}\left(\theta_j(t) \leq w_i, \forall j \mid E^0_i(t), F_{t - 1}\right) = \mathbb{P}\left(\theta_i(t) \leq w_i \mid F_{t - 1}\right) \cdot \mathbb{P}\left(\theta_j(t) \leq w_i, \forall j \neq i \mid E^0_i(t), F_{t - 1}\right)
$$

On the other side,

$$
\mathbb{P}\left(I_t = i \mid E^0_i(t), F_{t - 1}\right) \geq \mathbb{P}\left(\theta_i(t) > w_i \geq \theta_j(t), \forall j \neq i \mid E^0_i(t), F_{t - 1}\right) = \mathbb{P}\left(\theta_i(t) > w_i \mid F_{t - 1}\right) \cdot \mathbb{P}\left(\theta_j(t) \leq w_i, \forall j \neq i \mid E^0_i(t), F_{t - 1}\right)
$$

Thus, the above two inequalities implies the correctness of the Lemma.

Lemma D.7. Let $w_i = \mu_i + \frac{2\Delta_i}{T}$. For any $s \geq 1$, given $n_i(\tau_{i,s}) \leq \frac{B_i}{\Delta_i}$, we have

$$
\mathbb{E}\left[\frac{1}{q_{i,\tau_{i,s}+1}} - 1 \mid n_i(\tau_{i,s}) \leq \frac{B_i}{2\Delta_i}\right] \leq \begin{cases} 0 & \forall s \text{ if } s \geq L_i(T) \\ e^{11/4\sigma^2} + \frac{\pi^2}{3} & \end{cases}
$$

where $L_i(T) = \frac{72\ln(T\Delta_i^2)\max(1,\sigma^2)}{\Delta_i}$.

Proof. We prove this Lemma by a reduction to Lemma C.3. First, we observe $\theta_i(\tau_{i,s} + 1) \sim \mathcal{N}\left(\tilde{\mu}_i(\tau_{i,s}), \frac{1}{n_i(\tau_{i,s})}\right)$, where $\tilde{\mu}_i(\tau_{i,s}) = \tilde{\mu}_i(\tau_{i,s}) + \frac{B_i}{n_i(\tau_{i,s})}$. Given $n_i(\tau_{i,s}) \leq \frac{B_i}{\Delta_i}$, we have $\tilde{\mu}_i(\tau_{i,s}) \geq \mu_i(\tau_{i,s}) + 2\Delta_i$. Let $\xi_i(\tau_{i,s} + 1)$ denote the random variable of Gaussian distribution $\mathcal{N}\left(\tilde{\mu}_i(\tau_{i,s}), \frac{1}{n_i(\tau_{i,s})}\right)$. By the fact that a Gaussian random variable $a \sim \mathcal{N}(m, \sigma^2)$ is stochastically dominated by any $b \sim \mathcal{N}(m', \sigma^2)$ when $m < m'$, we have for any $F_{t - 1}$ of $F_{t - 1}$

$$
q_{i,\tau_{i,s}+1} = \mathbb{P}\left(\theta_i(\tau_{i,s} + 1) > w_i \mid F_{t - 1}\right) \geq \mathbb{P}\left(\xi_i(\tau_{i,s} + 1) + 2\Delta_i > w_i \mid F_{t - 1}\right) = \mathbb{P}\left(\xi_i(\tau_{i,s} + 1) > \mu_i - \frac{\Delta_i}{3} \mid F_{t - 1}\right) := \eta_i,\tau_{i,s},+1
$$

Therefore, $\mathbb{E}\left[\frac{1}{q_{i,\tau_{i,s}+1}} - 1\right] \leq \mathbb{E}\left[\frac{1}{\eta_i,\tau_{i,s},+1} - 1\right]$. Denote $u_i := \mu_i - \frac{\Delta_i}{3}$. Recall

$$
p_{i,\tau_{i,s}+1} = \mathbb{P}\left(\theta_i(\tau_{i,s} + 1) > \mu_i - \frac{\Delta_i}{3} \mid F_{t - 1}\right),
$$

we observe $\eta_i,\tau_{i,s},+1$ is analogous to $p_{i,\tau_{i,s}+1}$ in formula, when we replace $\mu_i$ and $\tilde{\mu}_i(\tau_{i,s} + 1)$ by $\mu_{i^*}$ and $\tilde{\mu}_{i^*}(\tau_{i^*,s} + 1)$ respectively (i.e. change arm $i$ by $i^*$). Recall the proof in Lemma C.2, it only depends on the relationship between $y_i = \mu_i - \frac{\Delta_i}{3}$ and $\mu_{i^*}$, which is the same as $u_i$ and $\mu_i$ in $\eta_i,\tau_{i,s},+1$. Thus, the proof of Lemma C.2 can be directly applied here to bound $\mathbb{E}\left[\frac{1}{\eta_i,\tau_{i,s},+1} - 1\right]$.

Lemma D.8.

$$
\sum_{t = K + 1}^{T} \mathbb{P}\left(I_t = i^*, E^0_i(t), E^0_i(t)\right) \leq T - \sum_{i \neq i^*} \frac{B_i}{2\Delta_i} + \sum_{i \neq i^*} \left(\frac{e^{11/4\sigma^2} + \frac{\pi^2}{3}}{3}\right) \cdot \frac{72\ln(T\Delta_i^2)\max(1,\sigma^2)}{\Delta_i^2} + \frac{4}{\Delta_i^2}
$$

20
Proof. We first decompose the target term by thresholding \( n_i(t-1) \) as follows,

\[
\sum_{t=K+1}^{T} \mathbb{P} \left( I_t = i^*, E_i^0(t), E_i^n(t) \right) \\
\leq \mathbb{E} \left[ \sum_{t=K+1}^{T} \mathbb{I} \left\{ I_t = i^*, E_i^0(t), E_i^n(t), \forall \hat{i} \neq i^*, n_{\hat{i}}(t-1) \geq \frac{B_{\hat{i}}}{2\Delta_i} \right\} \right] + \mathbb{P} \left( I_t = i^*, E_i^0(t), E_i^n(t), \exists \hat{i} \neq i^*, n_{\hat{i}}(t-1) \leq \frac{B_{\hat{i}}}{2\Delta_i} \right) \\
\leq \sum_{i \neq i^*} \sum_{t=K+1}^{T} \mathbb{P} \left( I_t = i^*, E_i^0(t), E_i^n(t), \forall \hat{i} \neq i^*, n_{\hat{i}}(t-1) \leq \frac{B_{\hat{i}}}{2\Delta_i} \right) \\
= \sum_{i \neq i^*} \sum_{t=K+1}^{T} \mathbb{E} \left[ \mathbb{P} \left( I_t = i^*, E_i^0(t), E_i^n(t), n_{\hat{i}}(t-1) \leq \frac{B_{\hat{i}}}{2\Delta_i}, F_{t-1} \right) \right] \\
= \sum_{i \neq i^*} \sum_{t=K+1}^{T} \mathbb{E} \left[ \frac{1-q_{i,t}}{q_{i,t}} \cdot \mathbb{P} \left( I_t = i, E_i^0(t), E_i^n(t), n_i(t-1) \leq \frac{B_i}{2\Delta_i}, F_{t-1} \right) \right] \\
= \sum_{i \neq i^*} \sum_{t=K+1}^{T} \mathbb{E} \left[ \frac{1-q_{i,t}}{q_{i,t}} \cdot \mathbb{I} \left\{ I_t = i, E_i^0(t), E_i^n(t) \right\} n_i(t-1) \leq \frac{B_i}{2\Delta_i} \right] \\
\leq \sum_{i \neq i^*} \sum_{t=K+1}^{T} \mathbb{E} \left[ \frac{1-q_{i,t}}{q_{i,t}} \cdot \mathbb{I} \left\{ I_t = i, E_i^0(t), E_i^n(t) \right\} n_i(t-1) \leq \frac{B_i}{2\Delta_i} \right]
\]

Observe that \( q_{i,t} = \mathbb{P} \left( \theta_i(t) > w_i, F_{t-1} \right) \) changes only at the time step after each pull of arm \( i \). Therefore we can bound the above term by,

\[
\sum_{s=1}^{T-1} \mathbb{E} \left[ \frac{1-q_{i,\tau_{i,s}+1}}{q_{i,\tau_{i,s}+1}} \cdot \sum_{t=\tau_{i,s}+1}^{\tau_{i,s+1}} \mathbb{I} \left\{ I_t = i, E_i^0(t), E_i^n(t) \right\} n_i(\tau_{i,s}) \leq \frac{B_i}{2\Delta_i} \right] \\
\leq \sum_{s=1}^{T-1} \mathbb{E} \left[ \frac{1-q_{i,\tau_{i,s}+1}}{q_{i,\tau_{i,s}+1}} \cdot n_i(\tau_{i,s}) \leq \frac{B_i}{2\Delta_i} \right]
\]

Combining Lemma [D.7] and Equation (24), we complete the proof. \( \square \)

D.4 Proof of Theorem 5.4

To prove Theorem 5.4, we first show the following Lemma.

Lemma D.9. Suppose all the strategic arms use LSIBR, and let time step \( n \) be the last time that a strategic arm spend budget for some \( n \leq T \). Then if principal agent runs UCB or \( \epsilon \)-Greedy or Thompson Sampling, the expected number of plays of the optimal arm \( i^* \) from time \( n+1 \) to \( T \) is bounded by,

\[
\mathbb{E} \left[ \sum_{t=n+1}^{T} \mathbb{I} \left\{ I_t = i^* \right\} \right] \leq T - \sum_{i \neq i^*} \frac{B_i}{2\Delta_i} \cdot \mathcal{O} \left( \frac{\ln T}{\Delta_i^2} \right).
\]
Proof. The proof follows a simple reduction to the setting with arms using LSI. By using LSIBR, any strategic arm \( i \) has no budget to manipulate after (includes) time step \( n + 1 \), which is analogous to the case that arm \( i \) has no budget to manipulate after time \( K + 1 \) using LSI in unbounded reward setting. Then after time \( n + 1 \), the \( \hat{\mu}_i(t - 1) = \hat{\mu}_i(t - 1) + \frac{B_i}{n_i(t - 1)}, \forall \in [K] \), which shares the same formula with it in LSI setting. Finally, we notice that the proofs of the upper bounds of \( E \left[ \sum_{t=K+1}^{T} 1\{I_t = i^*\} \right] \) in LSI settings (Lemma D.1, D.2 and Theorem D.3) don’t depend on the starting time step in the summand. Therefore, the proofs in these previous results can be directly applied here. \( \square \)

Next, we prove Theorem 5.4 using the above Lemma.

Proof of Theorem 5.4 Let \( n \) be the last time step that any arm can spend the budget. First we show the upper bound of \( E [n_2^{LSIBR}(T)] \). Note, from time 1 to \( n - 1 \), any strategic arm \( i \) always promote its reward to 1, which makes arm \( i \) the “optimal arm” from time 1 to \( n \) (the arm selection at time \( n \) only depends on previous feedback). Then following the standard analysis in stochastic MAB algorithms (UCB, \( \varepsilon \)-Greedy and Thompson Sampling), \( E [n_2^{LSIBR}(n)] \leq O \left( \frac{\ln n}{(1 - \mu_i)^2} \right) \). Thus, \( E [n_2^{LSIBR}(T)] \) can be bounded by,

\[
E [n_2^{LSIBR}(T)] \leq T - \sum_{i \neq i^*} \frac{B_i}{2 \Delta_i} + O \left( \frac{\ln T}{\Delta^2} + \frac{\ln n}{(1 - \mu_i)^2} \right)
\]

Consequently, we can show the lower bound of regret when all strategic arms use LSIBR, as follows

\[
E [R(T)] \geq \Delta \sum_{i \neq i^*} \frac{B_i}{2 \Delta_i} - O \left( \frac{\ln T}{\Delta} + \frac{\Delta \ln T}{(1 - \mu_i)^2} \right)
\]

\( \square \)

E Additional Simulations

We report our simulation results for bounded rewards in this section. Similarly, we also consider a stochastic bandit setting with three arms. The reward of each arm lies within the interval \([0, 1]\). The distributions of rewards of each arm are Beta(1, 1), Beta(2, 1) and Beta(3, 1) respectively. In \( \varepsilon \)-Greedy algorithm, we use a different \( \varepsilon_t \) parameter, i.e. \( \varepsilon_t = \min \{1, \frac{\Delta}{T} \} \). We run simulations for the same settings as those in Section 6 and report the results in Figure 3 and 4. These figures illustrate similar performances for bounded rewards as for unbounded rewards.

![Figure 3: Bounded rewards](image)

Figure 3: \([0, 1]\) bounded rewards: plots of regret with \( \ln t \) for UCB principal (left), \( \varepsilon \)-Greedy principal (middle), and Thompson Sampling principal (right), as \( B_1 \) and \( B_2 \) vary. We set \( B_3 = 0 \) for the three algorithms.
Figure 4: $[0, 1]$ bounded rewards: plots of regret with total budget $B$ of strategic arms (arm 1 and 2) for UCB principal (left), $\varepsilon$-Greedy principal (middle), and Thompson Sampling principal (right), as $B$ varies.