A WEIGHTED $L_p$-REGULARITY THEORY FOR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH TIME-MEASURABLE PSEUDO-DIFFERENTIAL OPERATORS

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ABSTRACT. We obtain the existence, uniqueness, and regularity estimates of the following Cauchy problem
\begin{align*}
\partial_t u(t, x) &= \psi(t, -i\nabla)u(t, x) + f(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^d, \\
u(0, x) &= 0, \quad x \in \mathbb{R}^d,
\end{align*}
in (Muckenhoupt) weighted $L_p$-spaces with time-measurable pseudo-differential operators
\[\psi(t, -i\nabla)u(t, x) := \mathcal{F}^{-1}[\psi(t, \cdot)\mathcal{F}[u(t, \cdot)](x)].\]
More precisely, we find sufficient conditions of the symbol $\psi(t, \xi)$ (especially depending on the smoothness of the symbol with respect to $\xi$) to guarantee that equation (0.1) is well-posed in (Muckenhoupt) weighted $L_p$-spaces. Here the symbol $\psi(t, \xi)$ is merely measurable with respect to $t$, and the sufficient smoothness of $\psi(t, \xi)$ with respect to $\xi$ is characterized by a property of each weight. In particular, we prove the existence of a positive constant $N$ such that for any solution $u$ to equation (0.2)
\begin{align*}
\int_0^T \int_{\mathbb{R}^d} |(-\Delta)^{\gamma/2} u(t, x)|^p (t^2 + |x|^2)^{\alpha/2} \,dx \,dt &\leq N \int_0^T \int_{\mathbb{R}^d} |f(t, x)|^p (t^2 + |x|^2)^{\alpha/2} \,dx \,dt,
\end{align*}
and
\begin{align*}
\int_0^T \left( \int_{\mathbb{R}^d} |(-\Delta)^{\gamma/2} u(t, x)|^p |x|^{\alpha_2} \,dx \right)^{\gamma/p} t^{\alpha_1} \,dt &\leq N \int_0^T \left( \int_{\mathbb{R}^d} |f(t, x)|^p |x|^{\alpha_2} \,dx \right)^{\gamma/p} t^{\alpha_1} \,dt,
\end{align*}
where $p, q \in (1, \infty)$, $-d - 1 < \alpha < (d + 1)(p - 1)$, $-1 < \alpha_1 < q - 1$, $-d < \alpha_2 < d(p - 1)$, and $\gamma$ is the order of the operator $\psi(t, -i\nabla)$.

1. Introduction

Pseudo-differential operators have been interesting operators in mathematics for a long time. They include not only classical differential operators with a natural number order but also fractional differential operators. Moreover, many interesting non-local operators are contained in the class of pseudo-differential operators and it naturally connects them to generators of Markov processes (cf. [13, 15, 16, 45]).

$L_p$-theories also have a long history in mathematics. There are tremendous results showing $L_p$-boundedness of many interesting operators in Fourier analysis (cf. [10, 44]). They have played important roles in theories of partial differential equations (PDEs) to show the well-posedness of solutions to PDEs in the Sobolev space. Especially, they commonly appear singularities to obtain optimal regularity estimates of solutions to PDEs and they are overcome by collaboration of many deep theories in analysis and PDEs. Classically, $L_p$-theories in PDEs mostly targeted second-order equations (cf. [28, 31]). However, these days beyond second-order operators, there occur numerous interesting kinds of research handling high-order operators, non-local operators, and pseudo-differential operators (cf. [2, 3, 4, 9, 17, 19, 20, 22, 23, 25, 33, 34, 35, 36, 40, 41, 47]). Even for stochastic PDEs, there have been improvements in $L_p$-theories for various operators (cf. [1, 13, 18, 21, 25, 27]). Particularly, we need to mention that there is progress for these theories in weighted $L_p$-spaces recently (cf. [6, 7, 9, 12, 22, 35, 39, 40, 41, 42]).

As mentioned above, there have been tons of researches studying the properties of pseudo-differential operators and the well-posedness of PDEs with them. However, these theories are built up in an elliptic setting mostly. In other words, there are not many results handling PDEs with pseudo-differential operators in a parabolic setting.
even though generalizations from elliptic theories to parabolic theories are considered difficult and important in theories of mathematics (especially in PDEs). In particular, if an additional (time) variable does not satisfy any regularity condition, then this parabolic generalization becomes complicated and non-trivial even though there are many well-constructed theories in an elliptic setting. Our plan of this paper is to construct weighted $L_p$-theories to equation (1.1) with time-measurable differential operators defined in (1.2). To the best of our knowledge, there are not many results handling general time-measurable differential operators in $L_p$-spaces even without weights. For previous results with these operators, we refer readers to [19] [22] [23] in $L_p$, $L_q(L_p)$, and $L_p(C^\alpha)$ spaces without weights.

Next, we explain a connection between our results and classical important theories in Fourier analysis. Recall classical Mikhlin’s multiplier theorem (cf. [10] Section 6.2). Let $\psi(\xi)$ be a complex-valued function defined on $\mathbb{R}^d$ and assume that there exists a positive constant $M$ such that

$$|D^\alpha \psi(\xi)| \leq M|\xi|^{-|\alpha|}, \quad \forall \xi \in \mathbb{R}^d$$

for any multi-index $\alpha$ with $|\alpha| \leq [d/2] + 1$, (1.1)

where $[d/2]$ denotes the integer which is less than or equal to $d/2$. Then the operator

$$T_\psi f(x) := \mathcal{F}^{-1}[\psi \mathcal{F}[f]](x)$$

becomes $L_p$-bounded, i.e., for any $p \in (1, \infty)$, there exists a positive constant $N(d, p, M)$ such that

$$\int_{\mathbb{R}^d} |T_\psi f(x)|^p dx \leq N \int_{\mathbb{R}^d} |f(x)|^p dx \quad \forall f \in L_p(\mathbb{R}^d).$$

However, if a weight $w$ is given, then the smoothness on $\psi$ in (1.1) are not sufficient to guarantee

$$\int_{\mathbb{R}^d} |T_\psi f(x)|^p w(x) dx \leq N \int_{\mathbb{R}^d} |f(x)|^p w(x) dx \quad \forall f \in L_p(\mathbb{R}^d).$$

In other words, we expect an extra condition on a symbol $\psi$ if a weight $w$ is additionally given in $L_p$-estimates. Particularly, it is well-known that (1.2) holds if $\psi$ satisfies stronger smooth conditions than (1.1) depending on the weight $w$. For instance, let $w$ be a weight in Muckenhoupt’s class $A_{p/d}(\mathbb{R}^d)$ (see Definition [41]) with $s \in (1, 2]$, $d/s < l \leq d$, and $d/l < p$. Then, if

$$|D^\alpha \psi(\xi)| \leq N|\xi|^{-|\alpha|}, \quad \forall \xi \in \mathbb{R}^d$$

we have (1.2) (cf. [40] Theorem 1). Based on these theories in Fourier analysis, we find appropriate conditions on $\psi(t, x)$ to enable equation (0.1) to be well-posed in weighted $L_p$-spaces. Formally, the solution $u$ to (0.1) is given by

$$u(t, x) = \int_0^t \int b \mathbb{F}(r, -i\nabla)(x - y) f(s, y) dy ds := \int_0^t \int R^d p(t, s, x - y) f(s, y) dy ds,$$

where

$$p(t, s, x) := 1_{0 < s < t} \cdot \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp \left(\int_s^t \psi(r, \xi) dr\right) e^{ix\cdot \xi} d\xi.$$ 

Moreover, if $\psi(r, \xi) \approx |\xi|^\gamma$, then

$$f \mapsto (-\Delta)^{\gamma/2} u$$

becomes a (parabolic) singular integral operator (see Section 5), where

$$(-\Delta)^{\gamma/2} u(t, x) = \int_0^t \int_{\mathbb{R}^d} \left((-\Delta)^{\gamma/2} p\right)(t, s, x - y) f(s, y) dy ds$$

and

$$\left((-\Delta)^{\gamma/2} p\right)(t, s, x) := 1_{0 < s < t} \cdot \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} |\xi|^\gamma \exp \left(\int_s^t \psi(r, \xi) dr\right) e^{ix\cdot \xi} d\xi.$$ 

Similarly to elliptic cases, if

$$|D^\alpha \psi(t, \xi)| \leq N|\xi|^{\gamma - |\alpha|}, \quad \forall (t, \xi) \in \mathbb{R} \times (\mathbb{R}^d \setminus \{0\})$$

for any multi-index $\alpha$ with $|\alpha| \leq [d/2] + 1$, (1.4)
then $L_p$ and $L_q(L_p)$ norms of $(-\Delta)^{\gamma/2}u(t,x)$ can be controlled by $f$’s (see [19] [22]). However, if a weight $w$ is given, the upper bound condition of $\psi(t,\xi)$ in (1.4) should be enhanced. That is, it needs to hold for any multi-index with $|\alpha|$ which is greater than $|d/2|$ depending on the weight $w$. We characterize this as requiring smoothness based on a constant related to each weight. We call this constant a regularity constant and denote it by $R_{p,d}$ (see (2.2)). In particular, one can easily check that

$$\left| \frac{d}{\gamma} \right| \leq \frac{d}{R_{p,d}} \leq d + 1.$$  

In our main theorems (Theorem 2.14 and Theorem 2.15), we characterize numbers of smoothness of symbols $\psi(t,\xi)$ satisfying (1.4) depending on $A_p$-weights $w_1$, and $w_2$ with a help of regularity constants for us to find a positive constant $N$ which is independent of $f$ such that

$$\int_0^T \int_{\mathbb{R}^d} |(-\Delta)^{\gamma/2}u(t,x)|^p w(t,x)dxdt \leq N \int_0^T \int_{\mathbb{R}^d} |f(t,x)|^p w(t,x)dxdt$$

and

$$\int_0^T \left( \int_{\mathbb{R}^d} |(-\Delta)^{\gamma/2}u(t,x)|^p w_2(x)dx \right)^{q/p} w_1(t)dt \leq N \int_0^T \left( \int_{\mathbb{R}^d} |f(t,x)|^p w_2(x)dx \right)^{q/p} w_1(t)dt.$$ 

Estimates (1.3) and (1.3) are given as particular cases of our theorems. We need to mention that if a weight is given by a constant or depends only on time, then our regularity constants are not optimal (Remark 2.16) and this initiates for us to prepare the next papers handling only time-dependent weights.

Due to (1.3), the most important part of our theory is to show $L_p$-boundedness of parabolic singular integral operators. For future applications, we consider general parabolic singular integral operators in the form of

$$\mathcal{T}_\varepsilon f(t,x) = \int_{-\infty}^t \int_{\mathbb{R}^d} K_\varepsilon(t,s,x-y)f(s,y)dyds,$$

where $K_\varepsilon$ is a kernel satisfying

$$\sup_{x \in \mathbb{R}^d} |x|^m |\partial_\alpha D_\xi^\varepsilon K_\varepsilon(t,s,x)| \leq N_1 |t-s|^{-(m+\varepsilon)-(d+|\alpha|-n)/\gamma} \tag{1.5}$$

and

$$\left( \int_{\mathbb{R}^d} |x|^{2m} |\partial_\alpha D_\xi^\varepsilon K_\varepsilon(t,s,x)|^2 dx \right)^{1/2} \leq N_2 |t-s|^{-(m+\varepsilon)-(d+|\alpha|-n)/\gamma} + \frac{M}{\varepsilon}, \tag{1.6}$$

where $\alpha$ is a multi-index. For weights $w \in A_p(\mathbb{R}^{d+1})$, $w_2 \in A_p(\mathbb{R}^d)$, and $w_1 \in A_p(\mathbb{R})$, we show (Theorem 5.2)

$$\int_0^T \int_{\mathbb{R}^d} |\mathcal{T}_\varepsilon f(t,x)|^p w(t,x)dxdt \leq NT^{1-\varepsilon} \int_0^T \int_{\mathbb{R}^d} |f(t,x)|^p w(t,x)dxdt \tag{1.7}$$

and

$$\int_0^T \left( \int_{\mathbb{R}^d} |\mathcal{T}_\varepsilon f(t,x)|^p w_2(x)dx \right)^{q/p} w_1(t)dt \leq NT^{1-\varepsilon} \int_0^T \left( \int_{\mathbb{R}^d} |f(t,x)|^p w_2(x)dx \right)^{q/p} w_1(t)dt \tag{1.8}$$

if $K_\varepsilon$ satisfies (1.3) and (1.3) for any multi-index $\alpha$ which is less than or equal to a positive integer related to regularity constants $R_{p,d+1}$, $R_{p,d}$, and $R_{q,1}$. Our main tool is Hardy-Littlewood’s maximal function and Fefferman-Stein’s sharp function. Based on $L_2$-boundedness of $\mathcal{T}_\varepsilon$ and kernel estimates given in (1.5) and (1.6), we prove

$$|\mathcal{T}_\varepsilon f|^2(t,x) \leq NT^{1-\varepsilon} (M|f|^{p_0}(t,x))^{1/p_0}, \tag{1.9}$$

for $p_0 \in (1,2]$ depending on a given weight, where $|\mathcal{T}_\varepsilon f|^2$ denotes the sharp function of $\mathcal{T}_\varepsilon f$ and $M|f|$ is the standard maximal function of $f$ (see Section 2.2 for the explicit definitions). Due to the equivalence of the sharp and maximal operators in weighted $L_p$-spaces, (1.9) leads us to obtain (1.7) and (1.8).

Our main results are given in Section 2 and the existence and uniqueness of solutions to (1.1) for a smooth data $f$ is shown in Section 3. In Section 4, we obtain an $L_2$-estimate and auxiliary estimates of a fundamental solution to (0.1). The Boundedness of general parabolic singular integral operators is handled in Section 5 and 6. Many
interesting properties of weighted $L_p$-spaces are self-contained in Appendix A, since most well-known properties of Sobolev spaces become unclear due to the effects of unbounded weights.

We finish this section with the notations used in the article.

- Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ denote the natural number system, the integer number system, the real number system, and the complex number system, respectively. For $d \in \mathbb{N}$, $\mathbb{R}^d$ denotes the $d$-dimensional Euclidean space.
- For $i = 1, \ldots, d$, a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $\alpha_i \in \{0, 1, 2, \ldots\}$, and function $g$, we set
  \[
  \frac{\partial g}{\partial x^i} = D_{x^i}g, \quad D^\alpha g = D_{x_1}^{\alpha_1} \cdots D_{x_d}^{\alpha_d}g, \quad |\alpha| := \sum_{i=1}^d \alpha_i.
  \]
  For $\alpha_i = 0$, we define $D_{x^i}^0 f = f$. We denote the gradient of a function $g : \mathbb{R}^d \to \mathbb{R}$ by
  \[
  \nabla g = (D_{x^1}g, D_{x^2}g, \ldots, D_{x^d}g).
  \]
  If $g = g(t, x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, then we denote
  \[
  D^2 g = D_{x^1}^{\alpha_1} \cdots D_{x^d}^{\alpha_d}g,
  \]
  where $\alpha = (\alpha_1, \ldots, \alpha_d)$.
- Let $C_c^\infty(\mathbb{R}^d)$ denote the space of infinitely differentiable functions with compact support, $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space on $\mathbb{R}^d$, and $\mathcal{S}'(\mathbb{R}^d)$ be the space of tempered distributions on $\mathbb{R}^d$. The convergence $f_n \to f$ in $\mathcal{S}(\mathbb{R}^d)$ as $n \to \infty$ implies
  \[
  \sup_{x \in \mathbb{R}^d} |(x^1)^{\alpha_1} \cdots (x^d)^{\alpha_d}(D^\beta(f_n - f))(x)| \to 0 \quad \text{as } n \to \infty \quad \forall \alpha_1, \ldots, \alpha_d, \beta.
  \]
- Let $F$ be a normed space and $(X, \mathcal{M}, \mu)$ be a measure space.
  - $\mathcal{M}^\mu$ denotes the completion of $\mathcal{M}$ with respect to the measure $\mu$.
  - For $p \in [1, \infty)$, the space of all $\mathcal{M}^\mu$-measurable functions $f : X \to F$ with the norm
    \[
    \|f\|_{L_p(X, \mathcal{M}, \mu; F)} := \left(\int_X \|f(x)\|_F^p \, \mu(dx)\right)^{1/p} < \infty
    \]
    is denoted by $L_p(X, \mathcal{M}, \mu; F)$. We also denote by $L_\infty(X, \mathcal{M}, \mu; F)$ the space of all $\mathcal{M}^\mu$-measurable functions $f : X \to F$ with the norm
    \[
    \|f\|_{L_\infty(X, \mathcal{M}, \mu; F)} := \inf \{r \geq 0 : \mu(\{x \in X : \|f(x)\|_F \geq r\}) = 0\} < \infty.
    \]
    We usually omit the given measure and $\sigma$-algebra if there is no confusion (e.g., Lebesgue (or Borel) measure and $\sigma$-algebra).
  - We denote by $C(X; F)$ the space of all $F$-valued continuous functions $f : X \to F$ with the norm
    \[
    |f|_{C(X; F)} := \sup_{x \in X} |f(x)|_F < \infty.
    \]
    We omit $F$ if $F = \mathbb{R}$ or $F = \mathbb{C}$.
- For $r > 0$, \[B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}, \quad B_r(x) := \{y \in \mathbb{R}^d : |x - y| \leq r\}\]
- For $\mathcal{O} \subseteq \mathbb{R}^d$, the set of all Borel sets contained in $\mathcal{O}$ is denoted by $\mathcal{B}(\mathcal{O})$. We denote by $|\mathcal{O}|$ the Lebesgue measure of a measurable set $\mathcal{O} \subset \mathbb{R}^d$. For locally integrable function $f$ on $\mathbb{R}^d$ and bounded measurable set $A \subseteq \mathbb{R}^d$ satisfying $|A| > 0$,
  \[
  \int_A f(x) \, dx := \frac{1}{|A|} \int_A f(x) \, dx
  \]
- For integrable function $f$ on $\mathbb{R}^d$, we denote the $d$-dimensional Fourier transform of $f$ by
  \[
  \mathcal{F}[f](\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) \, dx
  \]
and the $d$-dimensional inverse Fourier transform of $f$ by
\[
\mathcal{F}^{-1}[f](x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot \xi} f(\xi) d\xi.
\]

We also use the same notations $\mathcal{F}$ and $\mathcal{F}^{-1}$ as Fourier transform operator on tempered distributions or $L_2(\mathbb{R}^d)$ functions.

- We write $N = N(a, b, \cdots)$, if the constant $N$ depends only on $a, b, \cdots$.
- For $a, b \in \mathbb{R}$,
  \[ a \wedge b := \min\{a, b\}, \quad a \vee b := \max\{a, b\}, \quad [a] := \max\{n \in \mathbb{Z} : n \leq a\}. \]

For $z \in \mathbb{C}$, $\Re[z]$ denotes the real part of $z$, $\Im[z]$ is the imaginary part of $z$ and $\bar{z}$ is the complex conjugate of $z$.

## 2. Main results

Throughout the paper, we fix $T \in (0, \infty)$ and $d \in \mathbb{N}$. For a suitable complex-valued locally integrable function $\psi(t, \xi)$ on $[0, T] \times \mathbb{R}^d$ and almost every $t \in [0, T]$, we can consider a pseudo-differential operator $\psi(t, i\nabla)$ given by
\[
\psi(t, -i\nabla)u(x) := \mathcal{F}^{-1} [\psi(t, \cdot) \mathcal{F}[u]](x), \quad u \in C^\infty_c(\mathbb{R}^d).
\]

More generally, for a complex-valued nice function $u(t, x)$ on $[0, T] \times \mathbb{R}^d$
\[
\psi(t, -i\nabla)u(t, x) := \mathcal{F}^{-1} [\psi(t, \cdot) \mathcal{F}[u](t, \cdot)](x).
\]

This function $\psi(t, \xi)$ is usually called the symbol of the pseudo-differential operator. The operator $\psi(t, -i\nabla)$ is called a time-measurable pseudo-differential operator if there is no regularity condition on the symbol $\psi$ with respect to $t$. By considering natural constant extensions at $t = 0$ and $t = T$, we may assume the symbol $\psi(t, \xi)$ is defined on $\mathbb{R} \times \mathbb{R}^d$. In this paper, we study the following Cauchy problem with a time-measurable pseudo-differential operator
\[
\begin{aligned}
\partial_t u(t, x) &= \psi(t, -i\nabla)u(t, x) + f(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^d, \\
u(0, x) &= 0, \quad x \in \mathbb{R}^d
\end{aligned}
\tag{2.1}
\]
and find appropriate conditions on the symbol for us to establish a well-posedness theory to (2.1) in weighted $L_p$-spaces. First, we introduce related functions spaces handling solutions, and inhomogeneous data.

**Definition 2.1** (Muckenhoupt weight). For $p \in (1, \infty)$, let $A_p(\mathbb{R}^d)$ be the class of all nonnegative and locally integrable functions $w$ satisfying
\[
[w]_{A_p(\mathbb{R}^d)} := \sup_{x_0 \in \mathbb{R}^d, r > 0} \left( \int_{B_r(x_0)} w(x) \, dx \right) \left( \int_{B_r(x_0)} w(x)^{-1/(p-1)} \, dx \right)^{p-1} < \infty.
\]

We need to relate each $A_p$-weight to a nonnegative integer to mention the appropriate smoothness of symbols guaranteeing our $L_p$-theory.

**Remark 2.2.** The class $A_p(\mathbb{R}^d)$ is increasing as $p$ increases, and it holds that
\[
A_p(\mathbb{R}^d) = \bigcup_{q \in (1, p)} A_q(\mathbb{R}^d).
\]

More precisely, by [10] Corollary 7.2.6, for any $w \in A_p(\mathbb{R}^d)$, there exist $c_w = c_w(d, p, [w]_{A_p(\mathbb{R}^d)}) > 0$ and $C_w = C_w(d, p, [w]_{A_p(\mathbb{R}^d)}) > 0$ such that
\[
1 < q := \frac{p + c_w}{1 + c_w} < p, \quad \text{and} \quad [w]_{A_q(\mathbb{R}^d)} \leq C_w [w]_{A_p(\mathbb{R}^d)}.
\]

Note that the constants $C_w$ and $c_w^{-1}$ increase according to $[w]_{A_p(\mathbb{R}^d)}$. 

Definition 2.3 (Regularity constant of a weight in $A_p$). Let $w \in A_p(\mathbb{R}^d)$ and define

$$R_{p,d}^w := \sup \{ p_0 \in (1,2) : w \in A_{p/p_0}(\mathbb{R}^d) \}. \quad (2.2)$$

We say that $R_{p,d}^w$ is the regularity constant of the weight $w \in A_p(\mathbb{R}^d)$ since this constant plays an important role to characterize the differentiability of a symbol to make $L_p$-theories possible.

Remark 2.4. In this remark, we assume $w \in A_p(\mathbb{R}^d)$, and we denote

$$p_w := \frac{p(1 + c_w)}{p + c_w} \wedge 2 \in (1,2].$$

As pointed out in Remark 2.2 we have $w \in A_{p/p_0}(\mathbb{R}^d)$. Therefore, the set $I_{w,p} := \{ r \in [p/2, \infty) : w \in A_r(\mathbb{R}^d) \}$ is either $[p/2, \infty)$ or $(q, \infty)$ with $p/2 \leq q < p$. This implies that the constant $R_{p,q}^w$ is well-defined (finite).

We do not know if $w \in A_{p/R_{p,d}^w}(\mathbb{R}^d)$ for each $w \in A_p(\mathbb{R}^d)$ in general since there is no guarantee that $p/R_{p,d}^w \in I_{w,p}$. However, there exists a constant $p_0 \in (1,2)$ such that $w \in A_{p/p_0}(\mathbb{R}^d)$, $p_0 \leq R_{p,d}^w \wedge p$, and

$$\left| \frac{d}{p_0} \right| = \left\lfloor \frac{d}{R_{p,d}^w} \right\rfloor.$$

To prove this claim, note that $|d/p|$ is left-continuous and piecewise-constant with respect to $p$. Thus, there exists a $p_0 \in \left(1, R_{p,d}^w\right)$ such that $|d/p_0| = \left\lfloor d/R_{p,d}^w \right\rfloor$ and $w \in A_{p/p_0}(\mathbb{R}^d)$. It remains to show that $p_0 \leq R_{p,d}^w \wedge p$.

If $p \geq 2$, then it is obvious that $p_0 < 2$, so we only need to consider the case $p \in (1,2)$. However, it should be noted that $A_p(\mathbb{R}^d)$ is only defined for $p > 1$, and $A_1(\mathbb{R}^d)$-class is not introduced in this paper (see Definition 2.1). Therefore, for any $p_0 \in (1,2)$ such that $w \in A_{p/p_0}(\mathbb{R}^d)$, it is clear that $p_0$ is less than $p$.

Example 2.5. The easiest examples of a function in $A_p(\mathbb{R}^d)$ are polynomials, $w_\alpha(x) := |x|^\alpha$. It is well-known that $w_\alpha \in A_p(\mathbb{R}^d)$ if and only if $-d < \alpha < d(p - 1)$ (cf. Remark 7.1.7). This implies that for $p_0 > 1$,

$$p_0 < \frac{p}{\alpha + d} \quad \Longleftrightarrow \quad w_\alpha \in A_{p/p_0}(\mathbb{R}^d).$$

Therefore, for $w_\alpha(x) := |x|^\alpha$ with $-d < \alpha < d(p - 1)$, one can easily check that the exact value of $R_{p,d}^{w_\alpha}$ is given by $\frac{pd}{\alpha + d}$.

Next, we introduce two types of weighted $L_p$-spaces.

Definition 2.6. Let $p, q \in (1, \infty)$.

(i) For $w_1 \in A_q(\mathbb{R})$ and $w_2 \in A_{p}(\mathbb{R}^d)$, we write $f \in L_q(\mathbb{R}, w_1; L_p(\mathbb{R}^d, w_2))$ if

$$\| f \|_{L_q(\mathbb{R}, w_1; L_p(\mathbb{R}^d, w_2))} := \left( \int_{-\infty}^{\infty} \left( \int_{\mathbb{R}^d} |f(t, x)|^{p} w_2(x) dx \right)^{q/p} w_1(t) dt \right)^{1/q} < \infty.$$

(ii) For $w \in A_p(\mathbb{R}^{d+1})$, we write $f \in L_p(\mathbb{R}^{d+1}, w)$ if

$$\| f \|_{L_p(\mathbb{R}^{d+1}, w)} := \left( \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} |f(t, x)|^{p} w(t,x) dx dt \right)^{1/p} < \infty.$$

Note that even if $p = q \in (1, \infty)$, there is no inclusion between $L_q(\mathbb{R}, w_1; L_p(\mathbb{R}^d, w_2))$ and $L_p(\mathbb{R}^{d+1}, w)$.

Next, we introduce Sobolev spaces (Bessel potentials) which fit our weighted spaces. Recall that $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ denote the Schwartz space and the space of tempered distributions on $\mathbb{R}^d$, respectively.

Definition 2.7. Let $p \in (1, \infty)$.

(i) For $\nu \in \mathbb{R}$ and $w \in A_p(\mathbb{R}^d)$, $H_p^\nu(\mathbb{R}^d, w)$ denotes the set of all $f \in \mathcal{S}'(\mathbb{R}^d)$ satisfying

$$\| f \|_{H_p^\nu(\mathbb{R}^d, w)} := \| (1 - \Delta)^{\nu/2} f \|_{L_p(\mathbb{R}^d, w)} < \infty,$$
where 
\[(1 - \Delta)^{\nu/2} f(x) := F^{-1} \left[ (1 + |\cdot|^2)^{\nu/2} F[f] \right](x).\]

If \(w \equiv 1\), then we omit \(w\).

(ii) For \(\nu \in [0, \infty)\) and \(w \in A_p(\mathbb{R}^{d+1})\), \(\mathcal{H}_p^\nu((0, T) \times \mathbb{R}^d, w)\) denotes the set of all \(\mathcal{S}'(\mathbb{R}^d)\)-valued measurable functions \(f\) on \((0, T)\) satisfying
\[
\|f\|_{\mathcal{H}_p^\nu((0, T) \times \mathbb{R}^d, w)} := \|f\|_{L_p((0, T) \times \mathbb{R}^d, w)} + \|(-\Delta)^{\nu/2} f\|_{L_p((0, T) \times \mathbb{R}^d, w)} < \infty,
\]
where
\[
(-\Delta)^{\nu/2} f(t, x) := F^{-1} \left[ |\cdot|^\nu F[f(t, \cdot)] \right](x).
\]

If \(\nu < 0\), then \(\mathcal{H}_p^\nu((0, T) \times \mathbb{R}^d, w)\) denotes the set of all \(\mathcal{S}'(\mathbb{R}^d)\)-valued measurable functions \(f\) on \((0, T)\) satisfying
\[
\|f\|_{\mathcal{H}_p^\nu((0, T) \times \mathbb{R}^d, w)} := \|(1 - \Delta)^{\nu/2} f\|_{L_p((0, T) \times \mathbb{R}^d, w)} < \infty.
\]

**Remark 2.8.** Generally, there is no guarantee that two norms
\[
\|f\|_{L_p((0, T) \times \mathbb{R}^d, w)} + \|(-\Delta)^{\nu/2} f\|_{L_p((0, T) \times \mathbb{R}^d, w)} \quad \text{and} \quad \|(1 - \Delta)^{\nu/2} f\|_{L_p((0, T) \times \mathbb{R}^d, w)}
\]
are equivalent due to the effect of weights. However, if our weight \(w\) additionally satisfies
\[
\text{ess sup}_{t \in [0, T]}|w(t, \cdot)|_{A_p(\mathbb{R}^d)} =: N_0 < \infty, \tag{2.3}
\]
then two norms are equivalent, which will be shown in Appendix A. In particular, we show that the polynomial \(w(t, x) := (t^2 + |x|^2)^{\alpha/2}\) with \(-d < \alpha < d(p - 1)\) satisfies (2.3) (see Lemma A.8).

**Definition 2.9.** Let \(p \in (1, \infty)\).

(i) \(C_p^\infty([0, T] \times \mathbb{R}^d)\) denotes the set of all \(B([0, T] \times \mathbb{R}^d)\)-measurable functions \(f\) on \([0, T] \times \mathbb{R}^d\) such that for any multi-index \(\alpha\) with respect to the space variable, their derivatives
\[
D^\alpha f \in L_\infty([0, T]; L_2(\mathbb{R}^d) \cap L_p(\mathbb{R}^d)).
\]

(ii) \(C_p^{1, \infty}([0, T] \times \mathbb{R}^d)\) denotes the set of all \(f \in C_p^\infty([0, T] \times \mathbb{R}^d)\) such that for any multi-index \(\alpha\), their derivatives
\[
D^\alpha f \in C([0, T] \times \mathbb{R}^d) \quad \text{and} \quad \partial_t f \in C_p^\infty([0, T] \times \mathbb{R}^d),
\]
where \(\partial_t f(t)\) denotes the right derivative and the left derivative at \(t = 0\) and \(t = T\), respectively.

**Remark 2.10.** (i) In our previous work [3], we introduced similar smooth function spaces. The operators discussed in [3] have integral representations and are \(L_1\)-bounded. However, in the present article, we address general pseudodifferential operators that do not possess \(L_1\)-integrability and lack integral representations. Therefore, it becomes necessary to consider a larger class of functions than those presented in [3], in order to approximate solutions.

(ii) Let \(f \in C_p^\infty([0, T] \times \mathbb{R}^d)\). Note that for any \((d\text{-dimensional})\) multi-index \(\alpha\) and almost all \(t \in [0, T]\),
\[
D^\alpha f(t, \cdot) \in C(\mathbb{R}^d)
\]
due to the Sobolev Embedding theorem (e.g. [28, Theorem 13.1.8]).

**Definition 2.11** (Solution). Let \(\nu \in \mathbb{R}, p, q \in (1, \infty), w \in A_p(\mathbb{R}^{d+1}), w_1 \in A_q(\mathbb{R})\), and \(w_2 \in A_p(\mathbb{R}^d)\). For a given
\[
f \in \mathcal{H}_p^\nu((0, T) \times \mathbb{R}^d, w) \quad \text{resp.} \quad f \in L_q((0, T), w_1; H_p^\nu(\mathbb{R}^d, w_2)),
\]
we say that \(u \in \mathcal{H}_p^\nu(\mathbb{R}^{d+1})\) is a solution to the Cauchy problem
\[
\begin{aligned}
\partial_t u(t, x) &= \psi(t, -i\nabla)u(t, x) + f(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^d, \\
u(0, x) &= 0, \quad x \in \mathbb{R}^d,
\end{aligned}
\]
if there exists a sequence of functions \(u_n \in C_p^{1, \infty}([0, T] \times \mathbb{R}^d)\) such that \(u_n(0, \cdot) = 0\),
\[
\partial_t u_n - \psi(t, -i\nabla)u_n \to f \quad \text{in} \quad \mathcal{H}_p^\nu((0, T) \times \mathbb{R}^d, w), \quad \text{resp.} \quad L_q((0, T), w_1; H_p^\nu(\mathbb{R}^d, w_2)),
\]
and
\[
u_n \to u \quad \text{in} \quad \mathcal{H}_p^\nu(\mathbb{R}^{d+1}), \quad \text{resp.} \quad L_q((0, T), w_1; H_p^{\nu+\gamma}(\mathbb{R}^d, w_2))
\]
as \( n \to \infty \).

**Definition 2.12 (Ellipticity condition of a symbol).** We say that a symbol \( \psi(t, \xi) \) satisfies the ellipticity condition (with \( (\gamma, \kappa) \)) if there exists a \( \gamma \in (0, \infty) \) and \( \kappa \in (0, 1] \) such that

\[
\Re[-\psi(t, \xi)] \geq \kappa|\xi|^\gamma, \quad \forall (t, \xi) \in \mathbb{R} \times \mathbb{R}^d,
\]

where \( \Re[z] \) denotes the real part of the complex number \( z \).

**Definition 2.13 (regular upper bounds of a symbol).** Let \( n \in \mathbb{N} \). We say that a symbol \( \psi \) has a \( n \)-times regular upper bound (with \( (\gamma, M) \)) if there exist positive constants \( \gamma \) and \( M \) such that

\[
|D_\xi^{\alpha}\psi(t, \xi)| \leq M|\xi|^{|\alpha|-1}, \quad \forall (t, \xi) \in \mathbb{R} \times (\mathbb{R}^d \setminus \{0\}),
\]

for any \( (d\text{-dimensional})multi-index \( \alpha \) with \( |\alpha| \leq n \).

Recall the regularity constant for a weight \( w \in A_p(\mathbb{R}^d) \), i.e.,

\[
R_{p,d}^w := \sup\{p_0 \in (1, 2) : w \in A_{p_0}/p_0(\mathbb{R}^d)\}.
\]

Here are the main results of this article and we prove them in Section 7.

**Theorem 2.14.** Let \( p \in (1, \infty) \) and \( w \in A_p(\mathbb{R}^{d+1}) \). Suppose that \( \psi(t, \xi) \) is a symbol satisfying the ellipticity condition with \( (\gamma, \kappa) \) (Definition 2.12) and having a \((d/R_{p,d+1}^w plus 2\)-times regular upper bound with \( (\gamma, M) \) (Definition 2.13). Then for any \( f \in L_p((0, T) \times \mathbb{R}^d, w) \), the Cauchy problem (2.1) has a unique solution \( u \in \mathbb{H}^\gamma_p((0, T) \times \mathbb{R}^d, w) \) with estimates

\[
\|\partial_t u\|_{L_p((0, T) \times \mathbb{R}^d, w)} + \|(-\Delta)^{\gamma/2} u\|_{L_p((0, T) \times \mathbb{R}^d, w)} \leq N\|f\|_{L_p((0, T) \times \mathbb{R}^d, w)},
\]

where \( N = N(d, p, \gamma, \kappa, M, [w]_{A_p(\mathbb{R}^{d+1})}) \). Additionally assume that \( w(t, \cdot) \in A_p(\mathbb{R}^d) \) for almost all \( t \in [0, T] \) and \( \text{ess sup}_{t \in [0, T]}[w(t, \cdot), A_p(\mathbb{R}^d) =: N_0 < \infty \).

Then, for all \( \nu \in \mathbb{R} \) and \( f \in \mathbb{H}^\gamma_p((0, T) \times \mathbb{R}^d, w) \), the Cauchy problem (2.1) has a unique solution \( u \in \mathbb{H}^{\nu+\gamma}_p((0, T) \times \mathbb{R}^d, w) \) with estimates

\[
\|\partial_t u\|_{\mathbb{H}^\nu_p((0, T) \times \mathbb{R}^d, w)} + \|(-\Delta)^{\gamma/2} u\|_{\mathbb{H}^\nu_p((0, T) \times \mathbb{R}^d, w)} \leq N\|f\|_{\mathbb{H}^\gamma_p((0, T) \times \mathbb{R}^d, w)},
\]

where \( N = N(d, p, \gamma, \kappa, M, [w]_{A_p(\mathbb{R}^{d+1})}, \nu, N_0) \).

**Theorem 2.15.** Let \( p, q \in (1, \infty) \), \( w_1 \in A_q(\mathbb{R}) \), and \( w_2 \in A_p(\mathbb{R}^d) \). Suppose that \( \psi(t, \xi) \) is a symbol satisfying the ellipticity condition with \( (\gamma, \kappa) \) (Definition 2.12) and having a \((d/R_{p,d+1}^w plus 2\)-times regular upper bound with \( (\gamma, M) \) (Definition 2.13). Then, for any \( f \in L_q((0, T), w_1; H^\nu_p(\mathbb{R}^d, w_2)) \), the Cauchy problem (2.1) has a unique solution \( u \in L_q((0, T), w_1; H^{\nu+\gamma}_p(\mathbb{R}^d, w_2)) \) with estimates

\[
\|\partial_t u\|_{L_q((0, T), w_1; H^\nu_p(\mathbb{R}^d, w_2))} + \|(-\Delta)^{\gamma/2} u\|_{L_q((0, T), w_1; H^\nu_p(\mathbb{R}^d, w_2))} \leq N\|f\|_{L_q((0, T), w_1; H^\gamma_p(\mathbb{R}^d, w_2))},
\]

where \( N = N(d, p, \nu, \gamma, \kappa, M, [w_1]_{A_q(\mathbb{R})}, [w_2]_{A_p(\mathbb{R}^d)}) \).

**Remark 2.16.** If our weight is a constant function (i.e., without weight), the differentiability \((d/R_{p,d+1}^w plus 2\) for the symbol \( \psi(t, \xi) \) in Theorem 2.14 is not optimal. Indeed, if \( w \) is a constant, then

\[
\left|\frac{d}{R_{p,d+1}^w} + 2\right| = \left|\frac{d}{2}\right| + 2.
\]

However, if there is no weight term in the estimate, \( 4/2 + 1 \) is enough differentiability for the symbol \( \psi(t, \xi) \) to obtain (2.4) as shown in [19] (or see Theorem 6.9). Moreover, if one considers a weight depending only on the time variable \( t \), then the differentiability \( |d/R_{w,p}| + 2 \) can be relaxed to \( |d/R_{w,p}| + 1 \). We will show this result in our
satisfying the ellipticity condition with \(w\) depending on both the time variable \(t\) and space variable \(x\), we do not know yet whether the regular constant \(\frac{d}{R_{p,d+1}} + 2\) for the symbol is optimal to guarantee \(2.1\), which seems to be an interesting open problem.

For applications, we believe that the most important example of weights is a polynomial type. We give explicit statements for the reader’s convenience when weights are given by a polynomial type.

**Theorem 2.17.** Let \(p \in (1, \infty)\) and \(\alpha \in (-d-1, (d+1)(p-1))\). Suppose that \(\psi(t,\xi)\) is a symbol satisfying the ellipticity condition with \((\gamma, \kappa)\) and having \(a \left(\frac{d}{p} + 1\right)\) times regular upper bound with \((\gamma, M)\) (Definition 2.13).

Then, for any \(f \in L_p((0, T) \times \mathbb{R}^d, (t^2 + |x|^2)^{\alpha/2})\), the Cauchy problem (2.1) has a unique solution \(u \in H_p^\gamma((0, T) \times \mathbb{R}^d, (t^2 + |x|^2)^{\alpha/2})\) with estimates
\[
\int_0^T \int_{\mathbb{R}^d} |(-\Delta)^{\gamma/2} u(t, x)|^2 (t^2 + |x|^2)^{\alpha/2} \, dx \, dt \leq N \int_0^T \int_{\mathbb{R}^d} |f(t, x)|^p (t^2 + |x|^2)^{\alpha/2} \, dx \, dt
\]
and
\[
\int_0^T \int_{\mathbb{R}^d} |u(t, x)|^2 (t^2 + |x|^2)^{\alpha/2} \, dx \, dt \leq NT \int_0^T \int_{\mathbb{R}^d} |f(t, x)|^p (t^2 + |x|^2)^{\alpha/2} \, dx \, dt,
\]
where \(N = N(d, p, \gamma, \kappa, M, \alpha)\).

**Theorem 2.18.** Let \(p, q \in (1, \infty)\), \(\alpha_1 \in (-1, q-1)\), and \(\alpha_2 \in (-d, d(p-1))\). Suppose that \(\psi(t,\xi)\) is a symbol satisfying the ellipticity condition with \((\gamma, \kappa)\) and having \(a \left(\frac{d(\alpha_1+1)}{q} \vee \frac{\alpha_2 + d}{p} + 2\right)\) times regular upper bound with \((\gamma, M)\) (Definition 2.13). Then, for any \(f \in L_q((0, T), t^{\alpha_1}; L_p(\mathbb{R}^d, (t^2 + |x|^2)^{\alpha_2}))\), the Cauchy problem (2.1) has a unique solution \(u \in L_q((0, T), t^{\alpha_1}; H^\gamma_p(\mathbb{R}^d, (t^2 + |x|^2)^{\alpha_2}))\) with estimates
\[
\int_0^T \left( \int_{\mathbb{R}^d} |(-\Delta)^{\gamma/2} u(t, x)|^p (t^2 + |x|^2)^{\alpha_2} \, dx \right)^{q/p} \, t^{\alpha_1} \, dt \leq N \int_0^T \left( \int_{\mathbb{R}^d} |f(t, x)|^p (t^2 + |x|^2)^{\alpha_2} \, dx \right)^{q/p} \, t^{\alpha_1} \, dt
\]
and
\[
\int_0^T \left( \int_{\mathbb{R}^d} |u(t, x)|^2 (t^2 + |x|^2)^{\alpha_2} \, dx \right)^{q/p} \, t^{\alpha_1} \, dt \leq NT \int_0^T \left( \int_{\mathbb{R}^d} |f(t, x)|^p (t^2 + |x|^2)^{\alpha_2} \, dx \right)^{q/p} \, t^{\alpha_1} \, dt,
\]
where \(N = N(d, p, \gamma, \kappa, M, \alpha_1, \alpha_2)\).

### 3. The Cauchy problems with smooth functions

Recall the target equation
\[
\begin{align*}
\partial_t u(t, x) &= \psi(t, -i\nabla) u(t, x) + f(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^d, \\
u(0, x) &= 0, \quad x \in \mathbb{R}^d,
\end{align*}
\]
(3.1)

where
\[
\psi(t, -i\nabla) u(t, x) := \mathcal{F}^{-1} \left[ \psi(t, \cdot) \mathcal{F}[u](t, \cdot) \right](x).
\]

Then the fundamental solution to (3.1) is given by
\[
p(t, s, x) := 1_{0 < s < t} \cdot \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp \left( \int_s^t \psi(r, \xi) \, dr \right) e^{ix\xi} \, d\xi,
\]
that is, the integral operator
\[
K f(t, x) := \int_{-\infty}^t \int_{\mathbb{R}^d} p(t, s, x - y) 1_{0 < s < T} f(s, y) \, dy \, ds
\]
(3.2)
becomes a solution to (3.1) if \(f\) is a sufficiently smooth function. In this section, we prove that the function \(p(t, s, x)\) becomes the fundamental solution to (3.1) indeed. We prove some preliminaries first.
Lemma 3.1. Let $p \in (1, \infty)$.

(i) For any $f \in C_p^\infty([0, T] \times \mathbb{R}^d)$,
\[ \int_0^t f(s, x)ds \in C_p^{1, \infty}([0, T] \times \mathbb{R}^d). \]

(ii) For all $\nu \in \mathbb{R}$, $(1 - \Delta)^{\nu/2}$ is a bijective linear operator on $C_p^\infty([0, T] \times \mathbb{R}^d)$.

(iii) Suppose that $\psi(t, \xi)$ has a $(\lfloor \frac{d}{2} \rfloor + 1)$-times regular upper bound with $(\gamma, M)$. Then $\psi(t, -i\nabla)$ is a bounded linear operator on $C_p^\infty([0, T] \times \mathbb{R}^d)$.

Proof. (i) Put
\[ g(t, x) := \int_0^t f(s, x)ds. \]
By the dominated convergence theorem and Minkowski’s inequality,
\[ \|D_x^\alpha g(t_2, \cdot) - D_x^\alpha g(t_1, \cdot)\|_{L_p([0, T] \times \mathbb{R}^d)} \leq |t_2 - t_1| \|D_x^\alpha f\|_{L_p([0, T] \times \mathbb{R}^d)} \tag{3.3} \]
for all positive numbers $t_1$ and $t_2$. This implies $g \in C_p^\infty([0, T] \times \mathbb{R}^d)$ and by Remark 2.10 (ii) and (3.3), $g$ is continuous on $[0, T] \times \mathbb{R}^d$. The fundamental theorem of calculus yields for almost all $t \in [0, T]$,
\[ \partial_t g(t, x) = f(t, x), \quad \forall x \in \mathbb{R}^d. \]
Therefore, $g \in C_p^{1, \infty}([0, T] \times \mathbb{R}^d)$.

(ii) Let $f \in C_p^\infty([0, T] \times \mathbb{R}^d)$ and $\nu \in \mathbb{R}$. It is well-known that $(1 - \Delta)^{\nu/2}$ is a bijective mapping on the Schwartz class (cf. [28, Chapter 13]). Thus it suffices to show
\[ (1 - \Delta)^{\nu/2} f \in C_p^\infty([0, T] \times \mathbb{R}^d) \]
since $\nu$ is an arbitrary real number, Let $\alpha$ be a $(d$-dimensional) multi-index $\alpha$ (with respect to space variable). By a well-known property of the operator $(1 - \Delta)^{\nu/2}$ in $L_p$-spaces,
\[ \|D_x^\alpha (1 - \Delta)^{\nu/2} f\|_{L_p([0, T] \times \mathbb{R}^d)} \leq \|D_x^\alpha f\|_{L_p([0, T] \times \mathbb{R}^d)} \]
where $N$ is independent of $f$. This certainly implies $(1 - \Delta)^{\nu/2} f \in C_p^\infty([0, T] \times \mathbb{R}^d)$.

(iii) Let $m(t, \xi) := |\xi|^{-\gamma} \psi(t, \xi)$. Since $\psi$ has a $(\lfloor \frac{d}{2} \rfloor + 1)$-times regular upper bound with $\gamma$ and $M$, we have
\[ |D_{\xi}^\alpha m(t, \xi)| \leq N(\alpha, \gamma, M) |\xi|^{-\lfloor \frac{d}{2} \rfloor}, \quad \forall |\alpha| \leq \left\lfloor \frac{d}{2} \right\rfloor + 1, \quad \forall (t, \xi) \in [0, T] \times (\mathbb{R}^d \setminus \{0\}) \]
by a simple multiplicative law of derivatives. Thus, using Mikhlin’s multiplier theorem (e.g. [10, Theorem 6.2.7]), for all $f \in C_p^\infty([0, T] \times \mathbb{R}^d)$, $(d$-dimensional) multi-index $\alpha$, and $s \in [0, T]$, we have
\[ \|D_s^\alpha \psi(t, -i\nabla) f(s, \cdot)\|_{L_p(\mathbb{R}^d)} \leq N \|(1 - \Delta)^{\gamma/2} D_s^\alpha f(s, \cdot)\|_{L_p(\mathbb{R}^d)} \]
where $N = N(d, p, \gamma, M)$. Moreover, since $p \in (1, \infty)$, by a well-known embedding theorem which can be easily obtained from Mikhlin’s multiplier theorem, we have
\[ \|(1 - \Delta)^{\gamma/2} D_s^\alpha f(s, \cdot)\|_{L_p(\mathbb{R}^d)} \leq N \sum_{|\beta| \leq |\alpha| + |\gamma| + 1} \|D_s^\beta f(s, \cdot)\|_{L_p(\mathbb{R}^d)} \]
where $N$ is independent of $f$. The lemma is proved. \qed
Theorem 3.2. Let \( p \in (1, \infty) \) and \( f \in C_p^\infty([0, T] \times \mathbb{R}^d) \). Assume that \( \psi(t, \xi) \) satisfies the ellipticity condition with \( (\gamma, \kappa) \) and has a \( (\lfloor \frac{d}{2} \rfloor + 1) \)-times regular upper bound with \( (\gamma, M) \). Then, there exists a unique solution \( u \in C_p^\infty([0, T] \times \mathbb{R}^d) \) to the Cauchy problem

\[
\begin{align*}
\partial_t u(t, x) &= \psi(t, -i\nabla)u(t, x) + f(t, x), & (t, x) \in (0, T) \times \mathbb{R}^d, \\
 u(0, x) &= 0, & x \in \mathbb{R}^d,
\end{align*}
\]

Moreover, the solution \( u \) has the following representation :

\[ u(t, x) = Kf(t, x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d, \]

where \( K \) is defined in \([3.2]\).\]

Proof. (Uniqueness) Suppose that \( u \in C_p^1([0, T] \times \mathbb{R}^d) \) is a solution of the Cauchy problem

\[
\begin{align*}
\partial_t u(t, x) &= \psi(t, -i\nabla)u(t, x), & (t, x) \in (0, T) \times \mathbb{R}^d, \\
 u(0, x) &= 0, & x \in \mathbb{R}^d.
\end{align*}
\]

Taking the Fourier transform to \((3.5)\) and using the fundamental theorem of calculus, we have

\[ |\mathcal{F}[u(t, \cdot)](\xi)| \leq M|\xi|^\gamma \int_0^t |\mathcal{F}[u(s, \cdot)](\xi)|ds. \]

By Gronwall’s inequality, we conclude that \( u = 0. \) From the linearity of equation \((3.4)\), we obtain the uniqueness.

(Existence) Define

\[ u(t, x) := \mathcal{K}f(t, x). \]

First, we claim that \( u \in C_p^\infty([0, T] \times \mathbb{R}^d) \). In virtue of the dominated convergence theorem, for a \((d\)-dimensional\) multi-index \( \alpha \) and \((t, x) \in [0, T] \times \mathbb{R}^d, \)

\[ D_\alpha^x u(t, x) = D_\alpha^x \mathcal{K}f(t, x) = \mathcal{K}(D_\alpha^x f)(t, x). \]

Note that by \([23, Corollary 3.6]\),

\[ \sup_{s < t} \int_{\mathbb{R}^d} |p(t, s, x)|dx \leq N, \]

where \( N = N(d, \gamma, \kappa, M) \). Using Minkowski’s inequality and \([3.9]\), for all positive numbers \( t_1 \) and \( t_2 \), we have

\[ \left| \mathcal{K}(D_\alpha^x f)(t_2, \cdot) - \mathcal{K}(D_\alpha^x f)(t_1, \cdot) \right|_{L_{\mu}(\mathbb{R}^d)} \leq \left| \int_{t_1}^{t_2} \|p(t, s, \cdot)\|_{L_1(\mathbb{R}^d)}\|D_\alpha^x f(s, \cdot)\|_{L_{\mu}(\mathbb{R}^d)}ds \right| \leq N|t_2 - t_1|\|D_\alpha^x f\|_{L_{\infty}([0, T]; L_{\mu}(\mathbb{R}^d))}, \]

where \( N = N(d, \gamma, \kappa, M) \). Thus, since \( f \in C_p^\infty([0, T] \times \mathbb{R}^d) \), we conclude that \( u \in C_p^\infty([0, T] \times \mathbb{R}^d) \). Next, define

\[ v(t, x) := \int_0^t (\psi(t, -i\nabla)u(s, x) + f(s, x))ds. \]

By Lemma \([3.11] (i)\), \( v \in C_p^{1, \infty}([0, T] \times \mathbb{R}^d) \). We claim that

\[ u(t, x) = v(t, x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d, \]

which obviously completes the proof. For \((t, x) \in [0, T] \times \mathbb{R}^d\), applying Fubini’s theorem, we have

\[ v(t, x) - \int_0^t f(s, x)ds = \int_0^t \mathcal{F}^{-1} \left[ \psi(s, \cdot) \int_0^s \mathcal{F} \left[ f(l, \cdot) |e_t^l \psi(r, \cdot)dr \right] \right](x)ds \]

\[ = \int_0^t \mathcal{F}^{-1} \left[ \mathcal{F} \left[ f(l, \cdot) |e_t^l \psi(r, \cdot)dr \right] - 1 \right](x)dl = u(t, x) - \int_0^t f(s, x)ds. \]

The theorem is proved. \(\square\)
4. AN $L_2$-BOUNDEDNESS AND ESTIMATES OF THE FUNDAMENTAL SOLUTION

In this section, we obtain $L_2$-estimates of the Cauchy problem and prove some properties of the fundamental solution. Recall

$$ p(t, s, x) := 1_{0 < s < t} \cdot \frac{1}{(2\pi)^d/2} \int_{\mathbb{R}^d} \exp \left( \int_s^t \psi(r, \xi)dr \right) e^{ix \cdot \xi} d\xi. $$

Let $f \in C^\infty_p([0, T] \times \mathbb{R}^d)$. Then by Theorem 3.2

$$ u(t, x) = K f (t, x) := \int_{-\infty}^t \int_{\mathbb{R}^d} p(t, s, x - y) 1_{0 < s < T} f(s, y) dy ds $$

becomes the solution to (3.4). Moreover, one can easily check that for any $\varepsilon \in [0, 1]$,

$$ (-\Delta)^{\frac{\varepsilon}{2}} u(t, x) = (-\Delta)^{\frac{\varepsilon}{2}} K f (t, x) $$

(4.1)

Thus to show the boundedness of $(-\Delta)^{\frac{\varepsilon}{2}} u$ in an $L_p$-space, it is required to estimate the kernel $P_\varepsilon$.

**Lemma 4.1.** Let $n \in \mathbb{N}$ and assume that $\psi(t, \xi)$ satisfies the ellipticity condition with $(\gamma, \kappa)$ and has a $n$-times regular upper bound with $(\gamma, M)$. Then there exists a positive constant $N = N(n, M)$ such that for all $t > s$ and $\xi \in \mathbb{R}^d$,

$$ \left| D^n_\xi \left( \exp \left( \int_s^t \psi(r, \xi)dr \right) \right) \right| \leq N |\xi|^{-n} \exp(-\kappa(t-s)|\xi|^\gamma) \sum_{k=1}^n |t-s|^k |\xi|^{k\gamma}, \quad (4.2) $$

where $D^n_\xi$ denotes a (d-dimensional) partial derivative with respect to the variable $\xi$ whose order is $n$.

**Proof.** We use mathematical induction. The case $n = 1$ is clear since $|D_\xi \psi(t, \xi)|$ has a upper bound $N|\xi|^{\gamma-1}$ and $|\psi(t, \xi)|$ has both lower and upper bounds which are given by $|\xi|^{\gamma}$ multiplied by positive constants. Next, suppose that (4.2) holds for all $1, 2, \cdots, k$. Then, by Leibniz’s product rule and the hypothesis of the induction,

$$ \left| D^{k+1}_\xi \left( \exp \left( \int_s^t \psi(r, \xi)dr \right) \right) \right| = \left| D^k_\xi \left( \exp \left( \int_s^t \psi(r, \xi)dr \right) \right) \int_s^t D^1_\xi \psi(r, \xi)dr \right| $$

$$ \leq N \sum_{j=0}^k \left| D^j_\xi \left( \exp \left( \int_s^t \psi(r, \xi)dr \right) \right) \right| \left| \int_s^t D^{k-j+1}_\xi \psi(r, \xi)dr \right| $$

$$ \leq N \sum_{j=0}^k \left( |\xi|^{-j} \exp(-\kappa(t-s)|\xi|^\gamma) \sum_{l=1}^j |t-s|^l |\xi|^{l\gamma} \right) $$

$$ = N |\xi|^{-k-1} \exp(-\kappa(t-s)|\xi|^\gamma) \sum_{j=0}^k \sum_{l=1}^j |t-s|^{l+1} |\xi|^{(l+1)\gamma} $$

$$ \leq N |\xi|^{-k-1} \exp(-\kappa(t-s)|\xi|^\gamma) \sum_{l=1}^{k+1} |t-s|^l |\xi|^{l\gamma}, $$

where $N = N(k, M)$. The lemma is proved. \hfill $\Box$

**Theorem 4.2.** Let $n \in \mathbb{N}$, $\alpha$ be a (d-dimensional) multi-index, and $m \in \{0, 1\}$. Assume that $\psi(t, \xi)$ satisfies the ellipticity condition with $(\gamma, \kappa)$ and has a $n$-times regular upper bound with $(\gamma, M)$. Then there exists a positive
constant \( N = N(n, \kappa, \gamma, M, \varepsilon, m, |\alpha|) \) such that for all \( s < t \),
\[
\sup_{x \in \mathbb{R}^d} |x|^n (|\partial_t^m D_x^\alpha \psi(t, s, x)|) \leq N |t - s|^{-(m+\varepsilon) - \frac{(d+|\alpha|)-n}{2}},
\]
\[
\left( \int_{\mathbb{R}^d} |x|^{2n} |\partial_t^m D_x^\alpha \psi(t, s, x)|^2 \, dx \right)^{1/2} \leq N |t - s|^{-(m+\varepsilon) - \frac{(d+|\alpha|)-n}{2} + \frac{d}{2}},
\]

**Proof.** It is well-known that
\[
|x| \simeq |x^1| + \cdots + |x^d| \quad \forall x \in \mathbb{R}^d.
\]
In particular, we have,
\[
|x|^n \leq N(n)(|x^1|^n + \cdots + |x^d|^n), \quad n \in \mathbb{N}.
\]
By (4.3) and elementary properties of the Fourier transform,
\[
|x|^n |\partial_t^m D_x^\alpha \psi(t, s, x)| \leq N \sum_{i=1}^d |x|^i |\partial_t^m D_x^\alpha \psi(t, s, x)|
\]
\[
\leq N \sum_{|\beta|=n} \int_{\mathbb{R}^d} D_\xi^\beta \left( \xi^\alpha |\xi|^\gamma \partial_t^m \exp \left( \int_s^t \psi(r, \xi) \, dr \right) \right) \, d\xi.
\]
By Leibniz's product rule,
\[
D_\xi^\beta \left( \xi^\alpha |\xi|^\gamma \partial_t^m \exp \left( \int_s^t \psi(r, \xi) \, dr \right) \right)
\]
\[
= D_\xi^\beta \left( \psi(t, \xi)^m \xi^\alpha |\xi|^\gamma \exp \left( \int_s^t \psi(r, \xi) \, dr \right) \right)
\]
\[
= \sum_{\beta_0+\beta_1=\beta} c_{\beta_0,\beta_1} D_\xi^{\beta_0} \psi(t, \xi)^m \xi^\alpha |\xi|^\gamma D_\xi^{\beta_1} \exp \left( \int_s^t \psi(r, \xi) \, dr \right).
\]
By Definition 2.12 and Lemma 4.1,
\[
\left| D_\xi^\beta \left( \psi(t, \xi)^m \xi^\alpha |\xi|^\gamma \exp \left( \int_s^t \psi(r, \xi) \, dr \right) \right) \right|
\]
\[
\leq N |\xi|^{(m+\varepsilon)\gamma + |\alpha| - n} \exp(-\kappa(t - s)|\xi|^\gamma) \sum_{k=1}^n (t-s)^k |\xi|^k \gamma.
\]
Therefore,
\[
\sum_{|\beta|=n} \int_{\mathbb{R}^d} D_\xi^\beta \left( \xi^\alpha |\xi|^\gamma \partial_t^m \exp \left( \int_s^t \psi(r, \xi) \, dr \right) \right) \, d\xi \leq N(t - s)^{-(m+\varepsilon) - \frac{(d+|\alpha|)-n}{2}},
\]
where \( N = N(n, \kappa, \gamma, \varepsilon, m, |\alpha|) \). Similarly, by (4.3), (4.4), and Plancherel's theorem,
\[
\int_{\mathbb{R}^d} |x|^{2n} |\partial_t^m D_x^\alpha \psi(t, s, x)|^2 \, dx \leq N \int_{\mathbb{R}^d} |x|^{2n} |\psi(t, -i\nabla)^m D_x^\alpha \psi(t, s, x)|^2 \, dx
\]
\[
\leq N \sum_{|\beta|=n} \int_{\mathbb{R}^d} D_\xi^\beta \left( \psi(t, \xi)^m \xi^\alpha |\xi|^\gamma \exp \left( \int_s^t \psi(r, \xi) \, dr \right) \right) \, d\xi
\]
\[
\leq N |t - s|^{-2(m+\varepsilon) - \frac{(d+2|\alpha|-2n)}{2}},
\]
where \( N = N(n, \kappa, \gamma, \varepsilon, m, |\alpha|) \). The theorem is proved.

Recalling (4.4) and handling estimates of various regularities of the solution simultaneously, we introduce the following family of iterated integral operators
\[
K_{\varepsilon,T} f(t, x) := \int_{-\infty}^t \int_{\mathbb{R}^d} \left[ (1_{\varepsilon[0,1]}(1_{t-s}<\varepsilon) + 1_{\varepsilon=1}) P_\varepsilon(t, s, x-y) f(s, y) \right] \, dy \, ds,
\]
where \( f \in C_c^\infty(\mathbb{R}^{d+1}) \). It is remarkable that \( P_\varepsilon(t, \cdot, \cdot) \) is integrable on \((0, t) \times \mathbb{R}^d\) for all \( t > 0 \) and \( \varepsilon \in [0, 1) \). Thus the integral operator \( K_{\varepsilon,T}f \) is well-defined in a point-wise sense. However, \( P_1(t, \cdot, \cdot) \) is not integrable on \((0, t) \times \mathbb{R}^d\) for any \( t > 0 \). In this sense, we say that the operator \( K_{1,T}f \) is (parabolic) singular. However, one can understand the function \( K_{1,T}f(t, x) \) in a point-wise sense if one regards the integral as an iterated integral. Indeed,

\[
K_{1,T}f(t, x) := \int_{-\infty}^t \left[ \int_{\mathbb{R}^d} P_1(t, s, x - y)f(s, y)dy \right] ds
\]

\[
= \int_{-\infty}^t \left[ \int_{\mathbb{R}^d} p(t, s, x - y) \left( (-\Delta)^{\frac{2}{d}}_y f \right)(s, y)dy \right] ds
\]

\[
= \int_{-\infty}^t \int_{\mathbb{R}^d} p(t, s, x - y) \left( (-\Delta)^{\frac{2}{d}}_y f \right)(s, y)dy ds.
\]

Moreover, since the mapping \( f \mapsto K_{\varepsilon,T}f \) is a pseudo-differential operator as well, the \( L_2 \)-boundedness can be easily obtained based on Plancherel’s theorem without any regularity condition on the symbol.

**Theorem 4.3.** Assume that the symbol \( \psi \) satisfies the ellipticity condition with \((\gamma, \kappa)\). Then there exists a constant 
\( N = N(\kappa) \) such that

\[
\|K_{\varepsilon,T}f\|_{L_2(\mathbb{R}^{d+1})} \leq NT^{1-\varepsilon}\|f\|_{L_2(\mathbb{R}^{d+1})} \quad \forall f \in C_c^\infty(\mathbb{R}^{d+1}).
\]

**Proof.** By Plancherel’s theorem, Fubini’s theorem, and Minkowski’s inequality,

\[
\int_{-\infty}^\infty \int_{\mathbb{R}^d} |K_{\varepsilon,T}f(t, x)|^2dx dt
\]

\[
= \int_{-\infty}^\infty \int_{\mathbb{R}^d} \left| |\xi|^{\gamma} \exp \left( \int_{t}^{\infty} \psi(r, \xi)dr \right) \mathcal{F}[f(s, \cdot)](\xi)h(\varepsilon, t - s, T)ds \right|^2 d\xi dt
\]

\[
\leq \int_{-\infty}^\infty \int_{\mathbb{R}^d} \left( \int_{t}^{\infty} |\xi|^{\gamma} \exp \left( -\kappa(t - s)|\xi|^{\gamma} \right) |\mathcal{F}[f(s, \cdot)](\xi)|h(\varepsilon, t - s, T)ds \right)^2 d\xi dt
\]

\[
= \int_{-\infty}^\infty \int_{\mathbb{R}^d} \left( \int_{t}^{\infty} |\xi|^{\gamma} \exp \left( -\kappa s|\xi|^{\gamma} \right) |\mathcal{F}[f(t - s, \cdot)](\xi)|h(\varepsilon, s, T)ds \right)^2 d\xi dt
\]

\[
\leq \int_{\mathbb{R}^d} \left( \int_{0}^{\infty} |\xi|^{\gamma} \exp \left( -2\kappa s|\xi|^{\gamma} \right) h(\varepsilon, s, T) \left( \int_{s}^{\infty} |\mathcal{F}[f(t - s, \cdot)](\xi)|^2 dt \right)^{1/2} ds \right)^2 d\xi
\]

\[
\leq \int_{\mathbb{R}^d} \int_{-\infty}^{\infty} |\mathcal{F}[f(t, \cdot)](\xi)|^2 dt \left( \int_{0}^{\infty} |\xi|^{\gamma} \exp \left( -2\kappa s|\xi|^{\gamma} \right) h(\varepsilon, s, T)ds \right)^2 d\xi,
\]

where

\[
h(\varepsilon, s, T) := 1_{\varepsilon \in [0, 1)}1_{0 < s < T} + 1_{\varepsilon = 1}.
\]

By changing variables,

\[
\int_{0}^{\infty} |\xi|^{\gamma} \exp \left( -2\kappa s|\xi|^{\gamma} \right) h(\varepsilon, s, T)ds = N|\xi|^{(\varepsilon-1)\gamma} \int_{0}^{\infty} e^{-s}h(\varepsilon, s, T|\xi|^{\gamma})ds,
\]

where \( N = N(\kappa) \). If \( \varepsilon = 1 \), then

\[
|\xi|^{(\varepsilon-1)\gamma} \int_{0}^{\infty} e^{-s}h(\varepsilon, s, T|\xi|^{\gamma})ds = \int_{0}^{\infty} e^{-s}ds = 1.
\]

If \( \varepsilon \in [0, 1) \), then we split the estimate into two cases, \( T|\xi|^{\gamma} \geq 1 \) and \( 0 < T|\xi|^{\gamma} < 1 \). If \( T|\xi|^{\gamma} \geq 1 \), then

\[
|\xi|^{(\varepsilon-1)\gamma} \int_{0}^{\infty} e^{-s}h(\varepsilon, s, T|\xi|^{\gamma})ds = |\xi|^{(\varepsilon-1)\gamma} \int_{0}^{T|\xi|^{\gamma}} e^{-s}ds \leq |\xi|^{(\varepsilon-1)\gamma} \leq T^{1-\varepsilon}.
\]

If \( 0 < T|\xi|^{\gamma} < 1 \), then

\[
|\xi|^{(\varepsilon-1)\gamma} \int_{0}^{T|\xi|^{\gamma}} e^{-s}ds \leq T|\xi|^{\gamma} \leq T^{1-\varepsilon}.
\]
Therefore, by Plancherel’s theorem,
\[
\int_{-\infty}^{\infty} \int_{\mathbb{R}^d} |K_{t,T} f(t,x)|^2 \, dx \, dt \leq NT^{2(1-\varepsilon)} \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} |\mathcal{F}[f(t,\cdot)](\xi)|^2 \, d\xi \, dt = NT^{2(1-\varepsilon)} \|f\|^2_{L^2(\mathbb{R}^{d+1})},
\]
where \(N = N(\kappa)\). The theorem is proved.

5. **Weighted boundedness for a family of parabolic singular integral operators**

Let \(u\) be a solution to (3.4) with a smooth \(f\). Due to the solution representation given in Theorem 3.2 we have
\[
u(t,x) = K_t(t,x) = \int_{-\infty}^{t} \int_{\mathbb{R}^d} p(t,s,x-y)1_{0<s<T} f(s,y) \, dy \, ds
\]
and
\[
(-\Delta)^{\frac{\varepsilon}{2}} u(t,x) = \int_{-\infty}^{t} \left[ \int_{\mathbb{R}^d} P_{\varepsilon}(t,s,x-y)1_{0<s<T} f(s,y) \, dy \right] \, ds
\]
\[
:= \int_{-\infty}^{t} \left[ \int_{\mathbb{R}^d} (-\Delta)^{\frac{\varepsilon}{2}} p(t,s,x-y)1_{0<s<T} f(s,y) \, dy \right] \, ds.
\]
Therefore, to obtain a priori estimates (2.4) and (2.5), we need to show the boundedness of the above integral operator related to \(K_{\varepsilon}(t,s,x,y)\) is not singular and the given weight is a constant, then the boundedness of the integral operator related to \(K_{\varepsilon}(t,s,x,y)\) is singular. A typical example of these kernels are classical heat kernels, i.e.,
\[
K_{\varepsilon}(t,x,y) = \Delta^\varepsilon \left( (2\pi t)^{d/2} e^{-|x|^2/(2t)} \right).
\]
In particular, the kernel \(K_1(t,x,y) = \Delta \left( (2\pi t)^{d/2} e^{-|x|^2/(2t)} \right)\) is singular.

Note that if the kernel \(K_{\varepsilon}(t,s,x)\) is not singular and the given weight is a constant, then the boundedness of the operator (5.1) is easily obtained on the basis of the generalized Minkowski inequality. However, if a general weight is given, then the boundedness becomes non-trivial even though the kernel is not singular. Therefore, we have to show the boundedness of the operators for all \(\varepsilon \in (0,1]\) based on weighted estimates, and the cut-off function \(1_{|t-s|<T}\) is necessary to overcome the lack of the integrability of the kernels at large \(t\) when they are not singular. For a simplification of the notation, we denote
\[
h(\varepsilon,s,T) := 1_{\varepsilon \in (0,1]}1_{0<s<T} + 1_{\varepsilon=1}
\]
and
\[
T_{\varepsilon,T} f(t,x) := \int_{-\infty}^{t} \int_{\mathbb{R}^d} h(\varepsilon,t-s,T) K_{\varepsilon}(t,s,x-y) f(s,y) \, dy \, ds.
\]
Since we can obtain a more exact upper bound of the integral operator related to \(T\) with the help of the cut-off function \(h(\varepsilon,t-s,T)\), we separated it from the kernel \(K_{\varepsilon}(t,s,x,y)\).

**Definition 5.1** (Regularity condition on \(K_{\varepsilon}\)). For \(k \in \mathbb{N}\), we say that a kernel \(K_{\varepsilon}(t,s,x)\) (or the operator \(T_{\varepsilon,T}\)) satisfies the \(k\)-times regular condition with \((\gamma,N_1,N_2)\) if there exist positive constants \(\gamma\), \(N_1\), and \(N_2\) such that for all \((n,m,|\alpha|) \in \{0,1,2,\cdots,k\} \times \{0,1\} \times \{0,1,2\}\) and \(s < t\),
\[
\sup_{x \in \mathbb{R}^d} |x|^n |\partial_t^m D_x^\alpha K_{\varepsilon}(t,s,x)| \leq N_1 |t-s|^{-(m+\varepsilon)-\frac{(d+|\alpha|-n)}{\gamma}}
\]
and
\[ \left( \int_{\mathbb{R}^d} |x|^{2n} |\partial^m D_x^2 K_\varepsilon(t, s, x)|^2 dx \right)^{1/2} \leq N_2 |t-s|^{-(m+n)+\frac{d+|\alpha|-n}{2}}. \]

Recall that for any Muckenhoupt's weight w, the constant \( R_{p,d}^w \) denotes the regularity constant of the weight w (Definition 2.3).

**Theorem 5.2** (A weighted extrapolation theorem). Assume that there exists a constant \( N_3 > 0 \) such that
\[ \|T_{\varepsilon,T} f\|_{L_2(\mathbb{R}^{d+1})} \leq N_3 T^{1-\varepsilon} \|f\|_{L_2(\mathbb{R}^{d+1})} \quad \forall f \in C_c^\infty(\mathbb{R}^{d+1}). \]

(i) Let \( p \in (1, \infty) \) and \( w \in A_p(\mathbb{R}^{d+1}) \). Suppose that \( T_{\varepsilon,T} \) satisfies the \((d/R_{p,d+1}^w) + 2\)-times regular condition. Then, there exists a constant \( N = N(d, p, [w]_{A_p(\mathbb{R}^{d+1})}, \varepsilon, N_1, N_2, N_3) \) such that
\[ \|T_{\varepsilon,T} f\|_{L_p(\mathbb{R}^{d+1}, w)} \leq N T^{1-\varepsilon} \|f\|_{L_p(\mathbb{R}^{d+1}, w)} \quad \forall f \in C_c^\infty(\mathbb{R}^{d+1}). \]

(ii) Let \( p, q \in (1, \infty) \), \( w_1 \in A_q(\mathbb{R}) \), and \( w_2 \in A_p(\mathbb{R}^{d+1}) \). Suppose that \( T_{\varepsilon,T} \) satisfies the \((d/K_{p,d}^w) \vee (d/R_{q,d}^w) + 2\) times regular condition. Then, there exists a constant \( N = N(d, p, q, [w_1]_{A_q(\mathbb{R})}, [w_2]_{A_p(\mathbb{R}^{d+1})}, \varepsilon, N_1, N_2) \) such that
\[ \|T_{\varepsilon,T} f\|_{L_q(\mathbb{R}^{d+1}, w_1; L_p(\mathbb{R}^{d+1}, w_2))} \leq N T^{1-\varepsilon} \|f\|_{L_q(\mathbb{R}^{d+1}, w_1; L_p(\mathbb{R}^{d+1}, w_2))}. \]

**Proof of Theorem 5.2**

Recall that for each \( \varepsilon \in [0, 1] \), \( K_\varepsilon(t, s, x) \) is a complex-valued function defined on \( \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \). We fix a complex-valued function \( K_\varepsilon(t, s, x) \) defined on \( \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \) with some \( \varepsilon \in [0, 1] \) throughout the section. We start the proof with kernel estimates. In this section, \( \alpha = (\alpha_1, \ldots, \alpha_d) \) denotes a \((d\text{-dimensional})\) multi-index and \(|\alpha| = \alpha_1 + \cdots + \alpha_d\).

**6. Estimates of a family of parabolic singular kernels.**

**Theorem 6.1.** Let \( k \) be an integer such that \( k > [d/2] \) and suppose that \( K_\varepsilon \) satisfies the \( k \)-times regular condition with \((\gamma, N_1, N_2)\) (Definition 5.1).

(i) Let \( p \in [2, \infty) \), \((n, m, |\alpha|) \in \{0, 1, 2, \cdots, k\} \times \{0, 1\} \times \{0, 1, 2\} \), and \( \delta \in [0, 1) \) be a constant satisfying \( n + \delta \leq k \). Then, there exists a positive constant \( N = N(p, N_1, N_2) \) such that
\[ \|\cdot\|^{n+\delta}_{L_p(\mathbb{R}^d)} \|K_\varepsilon(t, s, \cdot)| \|_{L_p(\mathbb{R}^d)} \leq N |t-s|^{-(m+n)+\frac{d+|\alpha|-n-\delta}{2} + \frac{n}{p}}. \]  

(ii) Let \( p \in [1, 2) \) and \((m, |\alpha|) \in \{0, 1\} \times \{0, 1, 2\} \). Then there exists a positive constant \( N = N(p, N_1, N_2) \) such that
\[ \|\partial^m D_x^\alpha K_\varepsilon(t, s, \cdot)\|_{L_p(\mathbb{R}^d)} \leq N |t-s|^{-(m+n)+\frac{d+|\alpha|-n}{2} + \frac{d}{p}}. \]

**Proof.** (i) We divide the proof into two cases.

**Case 1.** \( \delta = 0. \)

If \( p = 2 \) or \( p = \infty \), then obviously (6.1) holds since \( K_\varepsilon \) satisfies the \( k \)-times regular condition with \((\gamma, N_1, N_2)\). If \( p \in (2, \infty) \), then
\[
\int_{\mathbb{R}^d} |x|^{2n} |\partial^m D_x^\alpha K_\varepsilon(t, s, x)|^p dx
\]
\[
= \int_{\mathbb{R}^d} |x|^{2n} |\partial^m D_x^\alpha K_\varepsilon(t, s, x)|^2 dx |(p-2)n| |\partial^m D_x^\alpha K_\varepsilon(t, s, x)|^{p-2} dx
\]
\[
\leq N(p, N_2) |t-s|^{(p-2)(-(m+n)+\frac{(d+|\alpha|-n)}{2})} \int_{\mathbb{R}^d} |x|^{2n} |\partial^m D_x^\alpha K_\varepsilon(t, s, x)|^2 dx
\]
\[
\leq N(p, N_1, N_2) |t-s|^{(p-2)(-(m+n)+\frac{(d+|\alpha|-n)}{2})} |t-s|^{-2(m+n)+\frac{(d+2|\alpha|-2n)}{2}}
\]
\[
= N(p, N_1, N_2) |t-s|^{-(p(m+n)+\frac{(d+|\alpha|-n)}{2}) + \frac{d}{2}}.
\]
Case 2. $\delta \in (0, 1)$.

We use the result from Case 1 repeatedly, i.e., we apply (6.1) with $\delta = 0$ several times for the proof of this case. Observe that for any $q \in [1, \infty)$,
\[
|x|^{q(n+\delta)}|\partial^m_x D^\alpha_t K_\varepsilon(t, s, x)|^q = \left(\frac{|x|^{q(n+1)}|\partial^m_x D^\alpha_t K_\varepsilon(t, s, x)|^q}{|x|^{q(n+1)}|\partial^m_x D^\alpha_t K_\varepsilon(t, s, x)|^q}\right)^{\delta}.
\]
(6.3)
Using (6.3) with $q = 1$, (6.1) holds for $p = \infty$. Thus it only remains to consider the case $p \in [2, \infty)$. Taking integrals and applying Hölder’s inequality to (6.3) with $q = p$, we have
\[
\left( \int_{\mathbb{R}^d} |x|^{p(n+\delta)}|\partial^m_x D^\alpha_t K_\varepsilon(t, s, x)|^p dx \right)^{1/p} \leq \left( \int_{\mathbb{R}^d} |x|^{p(n+1)}|\partial^m_x D^\alpha_t K_\varepsilon(t, s, x)|^p dx \right)^{\delta/p} \times \left( \int_{\mathbb{R}^d} |x|^{pn} |\partial^m_x D^\alpha_t K_\varepsilon(t, s, x)|^p dx \right)^{(1-\delta)/p} \leq N(p, N_1, N_2)|t-s|^{-(m+\delta)-\frac{(d+|\alpha|+\frac{1}{\gamma})}{\delta}.}
\]
(ii) The case $p = 2$ is obvious due to the assumption. Moreover, we claim that it is sufficient to show that (6.2) holds for $p = 1$. Indeed, for $p \in (1, 2)$ there exists a $\lambda \in (0, 1)$ such that $p = \lambda + 2(1 - \lambda)$. Thus taking integrals and applying Hölder’s inequality to
\[
|\partial^m_x D^\alpha_t K_\varepsilon(t, s, x)|^p = |\partial^m_x D^\alpha_t K_\varepsilon(t, s, x)|^\lambda \times |\partial^m_x D^\alpha_t K_\varepsilon(t, s, x)|^{2(1-\lambda)},
\]
we obtain (6.2). Thus, we focus on showing (6.2) with $p = 1$. Observe that
\[
\int_{\mathbb{R}^d} |\partial^m_x D^\alpha_t K_\varepsilon(t, s, x)| dx = \int_{|x| \leq |t-s|^{1/\gamma}} |\partial^m_x D^\alpha_t K_\varepsilon(t, s, x)| dx + \int_{|x| > |t-s|^{1/\gamma}} |\partial^m_x D^\alpha_t K_\varepsilon(t, s, x)| dx.
\]
By (i) with $p = \infty$,
\[
\int_{|x| \leq |t-s|^{1/\gamma}} |\partial^m_x D^\alpha_t K_\varepsilon(t, s, x)| dx \leq N(d, N_1)|t-s|^{-(m+\epsilon)-\frac{(d+|\alpha|+\frac{1}{\gamma})}{\delta}}.
\]
Put $d_2 := \lfloor d/2 \rfloor + 1$ and note that $d_2 \leq k$. Then by Hölder’s inequality and (i),
\[
\int_{|x| > |t-s|^{1/\gamma}} |\partial^m_x D^\alpha_t K_\varepsilon(t, s, x)| dx \leq \left( \int_{|x| > |t-s|^{1/\gamma}} |x|^{-2d_2} dx \right)^{1/2} \left( \int_{|x| > |t-s|^{1/\gamma}} |x|^{2d_2} |\partial^m_x D^\alpha_t K_\varepsilon(t, s, x)|^2 dx \right)^{1/2} \leq N(d, N_2)|t-s|^{-d_2 + \frac{d_2}{2}} \times |t-s|^{-(m+\epsilon)-\frac{(d+|\alpha|+\frac{1}{\gamma})}{\delta}} = N(d, N_2)|t-s|^{-(m+\epsilon)-\frac{(d+|\alpha|+\frac{1}{\gamma})}{\delta}}.
\]
The theorem is proved.

6.2. Auxiliary estimates for integral operators. In this subsection, we introduce an equivalence of weighted $L_p$-norms of sharp and maximal functions with filtration of partitions (cf. [3]) and obtain auxiliary estimates in an appropriate partition that fits our integral operators $T_{\varepsilon,T}$.

Definition 6.2. Let $B_0 = B_0(\mathbb{R}^{d+1})$ be a collection of all Borel sets $A \subseteq \mathbb{R}^{d+1}$ such that $|A| < \infty$. We say that a collection $P \subseteq B_0$ is a partition of $\mathbb{R}^{d+1}$ if and only if elements of $P$ are countable, pairwise disjoint, and
\[
\bigcup_{A \in P} A = \mathbb{R}^{d+1}.
\]

Definition 6.3. Let $\{P_n\}_{n \in \mathbb{Z}}$ be a sequence of partitions of $\mathbb{R}^{d+1}$. We say that $\{P_n\}_{n \in \mathbb{Z}}$ is a filtration of partitions on $\mathbb{R}^{d+1}$ if the followings are satisfied:
(i) The partitions are finer as $n \to \infty$. That is,
\[
\inf_{A \in \mathcal{P}_n} |A| \to \infty \quad \text{as} \quad n \to -\infty
\]
and for any $f \in L_1(\mathbb{R}^{d+1})$
\[
\lim_{n \to \infty} f_{|n}(t, x) := \lim_{n \to \infty} \frac{1}{A_n(t, x)} \int_{A_n(t, x)} f(s, y) \, dy \, ds = f(t, x) \quad \text{(a.e.)},
\]
where $A_n(t, x)$ is the elements of $\mathcal{P}_n$ containing $(t, x)$.

(ii) For each $n \in \mathbb{Z}$ and $A \in \mathcal{P}_n$, there is a (unique) $A' \in \mathcal{P}_{n-1}$ such that $A \subseteq A'$ and
\[
|A'| \leq N_0 |A|,
\]
where $N_0$ is a constant independent of $n, A$ and $A'$.

An example of filtration of partitions that will be used in the proof of Theorem 6.1 is given in the following lemma. Recall that a nonnegative function $d$ defined on $S \times S$ with a set $S$ is called a quasi-metric on $S$ if
\[
d(s_1, s_2) = 0 \quad \text{iff} \quad s_1 = s_2,
\]
and there exists a positive constant $N_d$ such that
\[
d(s_1, s_2) \leq N_d (d(s_1, s_3) + d(s_3, s_2)) \quad \forall s_1, s_2, s_3 \in S.
\]

**Lemma 6.4.** Let $\gamma \in (0, \infty)$ and $n \in \mathbb{Z}$. Define
\[
D_n(i_1, \ldots, i_d) := [i_12^{-n}, (i_1 + 1)2^{-n}) \times \cdots \times [i_d2^{-n}, (i_d + 1)2^{-n}),
\]
\[
\mathcal{P}_n^\gamma := \{[i_02^{-n\gamma}, (i_0 + 1)2^{-n\gamma}) \times D_n(i_1, \ldots, i_d) : i_0, \ldots, i_d \in \mathbb{Z}\},
\]
and
\[
d_\gamma((t, x), (s, y)) := |t - s|^{\frac{1}{\gamma}} + |x - y|, \quad B_n^\gamma(t, x) := \{(s, y) : d_\gamma((t, x), (s, y)) < b\}.
\]

Then
(i) $\{\mathcal{P}_n^\gamma\}_{n \in \mathbb{Z}}$ is a filtration of partitions.
(ii) $d_\gamma$ is a quasi-metric on $\mathbb{R}^{d+1}$.
(iii) $B_n^\gamma$ satisfies the doubling property. More precisely,
\[
|B_n^\gamma(t, x)| = 2^{\gamma+d}|B_0^\gamma(t, x)|.
\]
(iv) for each $A \in \mathcal{P}_n^\gamma$, there exist constants $\delta = \delta(\gamma) \in (1/2, 1)$ and $N = N(d)$ such that $A \subseteq B_n^{\gamma N\delta}(t, x)$ for all $(t, x) \in A$.

**Proof.** (i) It can be easily checked that $\{\mathcal{P}_n^\gamma\}_{n \in \mathbb{Z}}$ satisfies Definition 6.3

(ii) For $\gamma > 0$, there exists a constant $N = N(\gamma) \geq 1$ such that
\[
|t - r|^{\frac{1}{\gamma}} \leq N(|t - s|^{\frac{1}{\gamma}} + |s - r|^{\frac{1}{\gamma}}), \quad \forall t, s, r \in \mathbb{R},
\]
and this certainly implies $d_\gamma$ is a quasi-metric on $\mathbb{R}^{d+1}$.

(iii) Using the change of variable formula,
\[
|B_n^\gamma(t, x)| = \int_{\mathbb{R}^{d+1}} 1_{B_n^\gamma(t, x)}(s, y) \, dy \, ds = \int_{\mathbb{R}^{d+1}} 1_{B_0^\gamma(t/2^\gamma, x/2)}(s/2^\gamma, y/2) \, dy \, ds
\]
\[
= 2^{\gamma+d} \int_{\mathbb{R}^{d+1}} 1_{B_0^\gamma(t/2^\gamma, x/2)}(s, y) \, dy \, ds = 2^{\gamma+d}|B_0^\gamma(t/2^\gamma, x/2)| = 2^{\gamma+d}|B_0^\gamma(t, x)|.
\]

(iv) Let $\delta := (2^{-\gamma} \vee 2^{-1}) \in (1/2, 1)$. Note that for $A \in \mathcal{P}_n^\gamma$,
\[
diam(A) = \sqrt{4^{-n\gamma} + 4^{-n}} < N(d)\delta^n,
\]
and this certainly implies the result. The lemma is proved. \qed
For \((t, x) \in \mathbb{R} \times \mathbb{R}^d\) and \(b, \gamma > 0\), denote 
\[Q_b^\gamma(t, x) := (t - b^\gamma, t + b^\gamma) \times \{y \in \mathbb{R}^d : |x - y| < b\} \]

**Theorem 6.5.** Let \(p, q \in (1, \infty), \gamma \in (0, \infty), w_1 \in A_p(\mathbb{R}), w_2 \in A_p(\mathbb{R}^d),\) and \(w \in A_p(\mathbb{R}^{d+1})\). Suppose that 
\[|w_1|_{A_p(\mathbb{R})} \leq K_0, \quad |w_2|_{A_p(\mathbb{R}^d)} \leq K_1, \quad |w|_{A_p(\mathbb{R}^{d+1})} \leq K_2.\]

Then

(i) for any \(f \in L_q(\mathbb{R}, w_1; L_p(\mathbb{R}^d, w_2))\) (resp. \(f \in L_p(\mathbb{R}^{d+1}, w)\)),
\[
\|f\|_{L_q(\mathbb{R}, w_1; L_p(\mathbb{R}^d, w_2))} \leq N\|f\|_{L_q(\mathbb{R}, w_1; L_p(\mathbb{R}^{d+1}, w))},
\]
where \(N = N(d, \gamma, p, q, K_0, K_1)\) (resp. \(N = N(d, \gamma, p, K_2)\)) and
\[
f(t, x) := \sup_{(t,x) \in Q_b^\gamma} \frac{1}{|Q_b^\gamma|^2} \int_{Q_b^\gamma} \int_{Q_b^\gamma} |f(s_1, y_1) - f(s_0, y_0)| dy_1 ds_1 dy_0 ds_0.
\]

(ii) for any \(f \in L_q(\mathbb{R}, w_1; L_p(\mathbb{R}^d, w_2))\) (resp. \(f \in L_p(\mathbb{R}^{d+1}, w)\)),
\[
\|\mathcal{M}f\|_{L_q(\mathbb{R}, w_1; L_p(\mathbb{R}^d, w_2))} \leq N\|f\|_{L_q(\mathbb{R}, w_1; L_p(\mathbb{R}^{d+1}, w))},
\]
where \(N = N(d, \gamma, p, q, K_0, K_1)\) (resp. \(N = N(d, \gamma, p, K_2)\)) and
\[
\mathcal{M}(t, x) := \sup_{(t,x) \in Q_b^\gamma} \frac{1}{|Q_b^\gamma|^2} \int_{Q_b^\gamma} |f(s, y)| dy ds.
\]

**Proof.** By Lemma 6.4, the filtration \(\{\mathcal{P}_n\}_{n \in \mathbb{Z}}\) satisfies Theorem 2.1. Therefore, by Section 2, we obtain the results if we prove that there exists a constant \(N\) such that
\[
f_{\#p}^\gamma \leq N f_{\#p}^\gamma, \quad \mathcal{M}f \leq \mathcal{N} \mathcal{M}f,
\]
where
\[
f_{\#p}^\gamma(t, x) := \sup_{n \in \mathbb{Z}} \int_{A_n(t, x)} |f(s, y) - f_n(t, x)| dy ds, \quad \mathcal{M}f(t, x) := \sup_{(t,x) \in B_0^\gamma} \int_{B_0^\gamma} |f(s, y)| dy ds.
\]

Since \((t, x) \in A_n(t, x),\) by Lemma 6.2 (iv) and the definition of \(Q_b^\gamma\),
\[A_n(t, x) \subseteq B_{N\delta^n}(t, x) \subseteq Q_{N\delta^n}(t, x).
\]

One can observe that
\[|A_n(t, x)| = 2^{-n(\gamma + d)}, \quad |Q_{N\delta^n}(t, x)| = N_1(d, \gamma)\delta^{n(\gamma + d)}.
\]
Since \(\delta = (2^{-\gamma} \vee 2^{-1})\), we have \(1 \leq 2\delta \leq 1 + 2^{1-\gamma}\). Thus, \(|Q_{N\delta^n}(t, x)| \leq N_2(d, \gamma)|A_n(t, x)|\), and this implies
\[f_{\#p}^\gamma(t, x) \leq N(d, \gamma) f_{\#p}^\gamma(t, x).
\]

Note that \(Q_{b/2}^\gamma \subseteq B_0^\gamma \subseteq Q_b^\gamma\), and this implies
\[\mathcal{M}f(t, x) \leq N(d, \gamma) \mathcal{M}f(t, x).
\]

The theorem is proved. \(\square\)

**Lemma 6.6.** Let \(f \in C_c(\mathbb{R}^d)\) and \(g\) be a complex-valued continuously differentiable function on \(\mathbb{R}^d\) such that 
\[\lim_{|z| \to \infty} |g(z)| = 0.
\]
Suppose that \(x, y \in \mathbb{R}^d, |x - y| \leq R_3\) and \(f(y - z) = 0\) for \(|z| \leq R_2\). Then for any \(p_0 \in (1, \infty)\) and its Hölder conjugate \(p_0', i.e., 1/p_0 + 1/p_0' = 1,\)
\[|f \ast g(y)| \leq N \int_{R_2} \left( \int_{\mathbb{S}^{d-1}} |\nabla g(r\omega) \cdot \omega|^{p_0'} \sigma(d\omega) \right)^{1/p_0'} \left( \int_{B_{R_1+r}(x)} |f(z)|^{p_0} dz \right)^{1/p_0} dr,
\]
where \(N = N(d, p_0)\) and \(\mathbb{S}^{d-1}\) denotes the \(d - 1\)-dimensional unit sphere in \(\mathbb{R}^d\).
Lemma 6.7. □

The lemma is proved.

By Fubini’s theorem and Hölder’s inequality,

\[
\text{Obviously,}
\]

Due to the integration by parts,

\[
\text{Proof.}
\]

\[
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\]

\[
\text{(ii) for any } a \in (0, 1),
\]

\[
\text{(i) for all } a, a_1 \in (0, \infty),
\]

\[
\text{Lemma 6.7. Let } p_0 \in (1, 2) \text{ and } f \in C_c(\mathbb{R}^d). \text{ Suppose that } K_z \text{ satisfies the } \left( \left\lceil \frac{d}{p_0} \right\rceil + 2 \right) \text{-times regular condition with } (\gamma, N_1, N_2) \text{ (Definition 6.7). Then for any } (m, |\alpha|) \in \{0, 1\} \times \{0, 1, 2\}, \text{ there exists a positive constant } N = N(d, p_0, \varepsilon, N_1, N_2) \text{ such that}
\]

\[
(i) \text{ for all } a, a_1 \in (0, \infty),
\]

\[
(ii) \text{ for any } a_0 \in (0, \infty),
\]

\[
(6.4)
\]
(iii) For all $|t-s| \geq a^\gamma > 0$ and $c_0 > 0$,

$$
\int_0^\infty (c_0 a + \lambda)^d |t-s|^{-\frac{d}{\gamma}} \left( |t-s|^{-\frac{d}{\gamma}} 1_{\lambda \leq |t-s|^{-\frac{1}{\gamma}}} + |t-s|^{\frac{p_0 d(p_0)}{\gamma}} \lambda^{-p_0 d(p_0) - 1} 1_{|t-s|^{-\frac{1}{\gamma}} \leq \lambda} \right) d\lambda \leq N,
$$

(6.5)

where $N = N(d, p_0, c_0)$ and

$$
d(p_0) := \left\lfloor \frac{d}{p_0} \right\rfloor + 1
$$

and $p'_0$ is the Hölder conjugate of $p_0$, i.e., $1/p_0 + 1/p'_0 = 1$.

**Proof.** For simpler notations, we denote

$$
\mu := d(p_0) + \frac{1}{p_0} = \left\lfloor \frac{d}{p_0} \right\rfloor + 1 + \frac{1}{p_0} > \frac{d+1}{p_0},
$$

$$
H(r, \lambda) := r^{-\frac{d}{\gamma}} \left( r^{-\frac{d}{\gamma}} 1_{\lambda \leq r^{-\frac{1}{\gamma}}} + r^{\frac{p_0 d(p_0)}{\gamma}} \lambda^{-p_0 d(p_0) - 1} 1_{r^{-\frac{1}{\gamma}} \leq \lambda} \right),
$$

and

$$
K^{p'_0}_{\varepsilon, m, \alpha}(t, s, \lambda) := \int_{\mathbb{R}^{d-1}} |\partial_t^m D^s K_{\varepsilon}(t, s, \lambda w)|^{p'_0} \sigma(\omega).
$$

(i) By Hölder’s inequality and Lemma 6.1,

$$
\int_{a_1}^\infty \left( \int_{B_{a_0 + \lambda}(x)} |f(z)|^{p_0} \, dz \right)^{1/p_0} (\lambda^d K^{p'_0}_{\varepsilon, m, \alpha}(t, s, \lambda))^{1/p'_0} \, d\lambda
\leq \left( \int_{a_1}^\infty \lambda^{-p_0 \mu} \int_{B_{a_0 + \lambda}(x)} |f(z)|^{p_0} \, dz \, d\lambda \right)^{1/p_0} \left( \int_{a_1}^\infty \lambda^{d+p'_0 \mu} K^{p'_0}_{\varepsilon, m, \alpha}(t, s, \lambda) \, d\lambda \right)^{1/p'_0}
\leq \left( \int_{a_1}^\infty \lambda^{-p_0 \mu} \int_{B_{a_0 + \lambda}(x)} |f(z)|^{p_0} \, dz \, d\lambda \right)^{1/p_0} \left( \int_{\mathbb{R}^{d}} |z|^{1+p'_0 \mu} |\partial_t^m D^s K_{\varepsilon}(t, s, z)|^{p'_0} \, dz \right)^{1/p'_0}
\leq N \left( \int_{a_1}^\infty \lambda^{-p_0 d(p_0) - 1} \int_{B_{a_0 + \lambda}(x)} |f(z)|^{p_0} \, dz \, d\lambda \right)^{1/p_0} |t-s|^{-\left( m+\frac{d+p_0 d(p_0) - 1}{\gamma} + \frac{d}{p_0 \gamma} \right)},
$$

where $N = N(d, p_0, N_1, N_2)$.

(ii) Decompose the integrals in (6.4) into two parts,

$$
\int_0^\infty \left( \int_{B_{a_0 + \lambda}(x)} |f(z)|^{p_0} \, dz \right)^{1/p_0} (\lambda^d K^{p'_0}_{\varepsilon, m, \alpha}(t, s, \lambda))^{1/p'_0} \, d\lambda
= \int_0^{\infty} \cdots \, d\lambda + \int_{|t-s|^{-\frac{1}{\gamma}}}^\infty \cdots \, d\lambda := I_1(t, s) + I_2(t, s).
$$
By Hölder’s inequality and Lemma 6.1

\[ I_1(t, s) \leq \left( \int_0^{\left| t-s \right|^{1/\gamma}} \int_{B_{\lambda_0+r}(x)} |f(z)|^{p_0} d\lambda d\lambda \right)^{1/p_0} \left( \int_0^{\left| t-s \right|^{1/\gamma}} \lambda^d K_{\lambda,m,a}(t, s, \lambda) d\lambda \right)^{1/p_0} \]

\[ \leq \left( \int_0^{\left| t-s \right|^{1/\gamma}} \int_{B_{\lambda_0+r}(x)} |f(z)|^{p_0} d\lambda d\lambda \right)^{1/p_0} \left( \int_{\mathbb{R}^d} |z||\partial_{\lambda}^m D_{\lambda}^{p_0} K_{\lambda}(t, s, z)| d\lambda \right)^{1/p_0} \]

\[ \leq N \left( \int_0^{\left| t-s \right|^{1/\gamma}} \int_{B_{\lambda_0+r}(x)} |f(z)|^{p_0} d\lambda d\lambda \right)^{1/p_0} \left| t-s \right|^{-\frac{(m+\epsilon)-(d+|\alpha|+1)/\gamma}{\gamma}} + \frac{d}{p_0} \]

where \( N = N(d, p_0, N_1, N_2) \). For \( I_2 \), by (i) with \( a_1 = \left| t-s \right|^{1/\gamma} \),

\[ I_2(t, s) \leq N \left( \int_0^{\left| t-s \right|^{1/\gamma}} \int_{B_{\lambda_0+r}(x)} |f(z)|^{p_0} d\lambda d\lambda \right)^{1/p_0} \left| t-s \right|^{-\frac{(m+\epsilon)-(d+|\alpha|+1)/\gamma}{\gamma}}, \]

where \( N = N(d, p_0, c_0, N_1, N_2) \).

(iii) Decompose the integral in (6.5) into two parts,

\[ \int_0^\infty (c_0^a + \lambda)^d H\left( \left| t-s \right|, \lambda \right) d\lambda = \int_0^{\left| t-s \right|^{1/\gamma}} \cdots d\lambda + \int_0^{\left| t-s \right|^{1/\gamma}} \cdots d\lambda \equiv: I_3(t, s) + I_4(t, s). \]

By the definition of \( H \),

\[ I_3(t, s) = \int_0^{\left| t-s \right|^{1/\gamma}} (c_0^a + \lambda)^d \left| t-s \right|^{-\frac{(d+1)}{\gamma}} d\lambda \leq N(d, c_0) \int_0^{\left| t-s \right|^{1/\gamma}} \left| t-s \right|^{-\frac{1}{\gamma}} d\lambda = N(d, c_0), \]

and

\[ I_4(t, s) = \left| t-s \right|^{-\frac{(d+|\alpha|+d-p_0)\gamma}{\gamma}} \int_0^{\left| t-s \right|^{1/\gamma}} (c_0^a + \lambda)^d \lambda^{-p_0 d(p_0)-1} d\lambda \]

\[ \leq N(d, c_0) \left| t-s \right|^{-\frac{(d+|\alpha|+d-p_0)\gamma}{\gamma}} \int_0^{\left| t-s \right|^{1/\gamma}} \lambda^{-p_0 d(p_0)-1} d\lambda = N(d, p_0, c_0). \]

The lemma is proved. \( \square \)

**Lemma 6.8.** Let \( p_0 \in (1, 2) \), \( t_0 \in \mathbb{R} \), \( b > 0 \), \( s \in (t_0-b^7, t_0+b^7) \), and \( f \in C_c^\infty(\mathbb{R}^{d+1}) \). Then for any \( (t, x) \in Q_0^2(t_0, 0) \),

\[ \int_{-\infty}^{t_0-2b^7} \left( \int_0^\infty H\left( |s-r|, \lambda \right) \int_{B_{2b+\lambda}(x)} |f(r, z)|^{p_0} d\lambda \right)^{1/p_0} h_0(m, |\alpha|, \epsilon, s-r, T) dr \]

\[ \leq N(T^{1-\epsilon} h_1(0, 1, \epsilon) + T^{1-\epsilon} b^{-1} h_1(0, 2, \epsilon) + b^{-\gamma} h_1(1, 1, 1))(\mathcal{M}|f|^{p_0}(t, x))^{1/p_0}, \]

where \( N = N(d, p_0, \epsilon, N_1, N_2) \) and

\[ h_0(m, |\alpha|, \epsilon, s-r, T) := |s-r|^{-\frac{(m+\epsilon)-(1+|\alpha|)}{\gamma}} h_1(m, |\alpha|, \epsilon)(1_{\epsilon \in [0, 1]} 1_{s-r \leq T} + 1_{\epsilon = 1}), \]

\[ h_1(m, |\alpha|, \epsilon) := 1_{(m, |\alpha|) = (0, 1)} 1_{\epsilon \in [0, 1]} + 1_{(m, |\alpha|) = (0, 2)} + 1_{(m, |\alpha|) = (1, 1)} 1_{\epsilon = 1}, \]

\[ H(r, \lambda) := r^{-\frac{1}{\gamma}} \left( r^{-\frac{1}{\gamma}} \lambda \right)^{-\frac{\gamma}{\gamma}} \left( r^{-\frac{1}{\gamma}} \lambda \right)^{-\frac{\gamma}{\gamma}} \lambda^{-p_0 d(p_0)-1} \left( \frac{1}{r^{\gamma}} \right). \]
Proof. Let \((t, x) \in Q^3_1(t_0, 0)\). By Hölder’s inequality,
\[
\int_{-\infty}^{t_0 - 2b^2} \left( \int_{-\infty}^{\infty} H(|s - r|, \lambda) \int_{B_{2b + \lambda}(x)} |f(r, z)|^{p_0} \, dz \, d\lambda \right)^{1/p_0} h_0(m, |z|, \varepsilon, s - r, T) \, dr
\leq \left( \int_{-\infty}^{t_0 - 2b^2} \int_{-\infty}^{\infty} H(|s - r|, \lambda) h_0(m, |z|, \varepsilon, s - r, T) \int_{B_{2b + \lambda}(x)} |f(r, z)|^{p_0} \, dz \, d\lambda \, dr \right)^{1/p_0}.
\]
Due to the integration by parts formula,
\[
\int_{-\infty}^{t_0 - 2b^2} H(|s - r|, \lambda) h_0(m, |z|, \varepsilon, s - r, T) \int_{B_{2b + \lambda}(x)} |f(r, z)|^{p_0} \, dz \, dr
\leq \int_{-\infty}^{t_0 - 2b^2} \frac{\partial}{\partial r} (H(|s - r|, \lambda) h_0(m, |z|, \varepsilon, s - r, T)) \left( \int_{r}^{t_0 - 2b^2} \int_{B_{2b + \lambda}(x)} |f(\varphi, z)|^{p_0} \, dz \, d\varphi \, dr \right)
\leq M|f|^{p_0}(t, x)(2b + \lambda)^d \times \left( \int_{-\infty}^{t_0 - 2b^2} \frac{\partial}{\partial r} (H(|s - r|, \lambda) h_0(m, |z|, \varepsilon, s - r, T)) |t_0 + b^2 - r| \, dr \right).
\]
Note that if \(r \in (-\infty, t_0 - 2b^2)\) and \(s \in (t_0 - b^2, t_0 + b^2)\), then
\[
1 < \frac{t_0 + b^2 - r}{s - r} < 3.
\]
Also, observe that
\[
\frac{\partial}{\partial r} (H(|s - r|, \lambda) h_0(m, |z|, \varepsilon, s - r, T)) = N|s - r|^{-1} H(|s - r|, \lambda) h_0(m, |z|, \varepsilon, s - r, T),
\]
where \(N = N(d, p_0, \gamma, \varepsilon)\). Thus from (6.7),
\[
\int_{-\infty}^{t_0 - 2b^2} \int_{-\infty}^{\infty} H(|s - r|, \lambda) h_0(m, |z|, \varepsilon, s - r, T) \int_{B_{2b + \lambda}(x)} |f(r, z)|^{p_0} \, dz \, d\lambda \, dr
\leq N \|f\|^{p_0}(t, x) \int_{-\infty}^{t_0 - 2b^2} h_0(m, |z|, \varepsilon, s - r, T) \left( \int_{0}^{\infty} (2b + \lambda)^d H(|s - r|, \lambda) \, d\lambda \right) \, dr
\leq N \|f\|^{p_0}(t, x) \int_{-\infty}^{t_0 - 2b^2} h_0(m, |z|, \varepsilon, s - r, T) \, dr,
\]
where \(N = N(d, p_0, \gamma, \varepsilon, N_1, N_2)\). Therefore, combining (6.6) and (6.8), we have
\[
\int_{-\infty}^{t_0 - 2b^2} \left( \int_{-\infty}^{\infty} H(|s - r|, \lambda) \int_{B_{2b + \lambda}(x)} |f(r, z)|^{p_0} \, dz \, d\lambda \right)^{1/p_0} h_0(m, |z|, \varepsilon, s - r, T) \, dr
\leq N(\|f\|^{p_0}(t, x))^{1/p_0} \int_{-\infty}^{t_0 - 2b^2} h_0(m, |z|, \varepsilon, s - r, T) \, dr,
\]
where \(N = N(d, p_0, \gamma, \varepsilon, N_1, N_2)\). It only remains to observe
\[
\int_{-\infty}^{t_0 - 2b^2} h_0(m, |z|, \varepsilon, s - r, T) \, dr \leq N(T^{1-\varepsilon} h_1(0, 1, \varepsilon) + T^{1-\varepsilon} b^{-1} h_1(0, 2, \varepsilon) + b^{-\gamma} h_1(1, 1, 1))
\]
where \(N = N(d, \gamma, \varepsilon)\). The lemma is proved. \(\Box\)
6.3. $L_p$ estimates without weights for $p \in (1, 2)$. Recall
\[ h(\varepsilon, t - s, T) := 1_{\varepsilon \in [0, 1]} \cdot |t - s| < T + 1 \varepsilon = 1 \]
and
\[ T_{\varepsilon,T} f(t, x) := \int_{-\infty}^{t} \int_{\mathbb{R}^d} h(\varepsilon, t - s, T) K_\varepsilon(t, s, x - y) f(s, y) dy ds. \]

In this subsection, we prove the $L_p$-boundedness of $T_{\varepsilon,T}$ without weights. More precisely, we show the following theorem:

**Theorem 6.9.** Suppose that $K_\varepsilon$ satisfies the $([d_2] + 1)$-times regular condition with $(\gamma, N_1, N_2)$ (Definition 5.1) and there exists a constant $N_3 > 0$ such that
\[ \|T_{\varepsilon,T} f\|_{L_2(\mathbb{R}^{d+1})} \leq N_3 T^{1-\varepsilon} \|f\|_{L_2(\mathbb{R}^{d+1})}, \quad \forall f \in C_c^\infty(\mathbb{R}^{d+1}). \]
Then for $p \in (1, 2)$, there exists a constant $N = N(d, p, \gamma, \varepsilon, N_1, N_2, N_3)$ such that
\[ \|T_{\varepsilon,T} f\|_{L_p(\mathbb{R}^{d+1})} \leq NT^{1-\varepsilon} \|f\|_{L_p(\mathbb{R}^{d+1})}, \quad \forall f \in C_c^\infty(\mathbb{R}^{d+1}). \]

The proof of the theorem will be given in the last of this section.

We first show that our kernel satisfies the important condition so-called Hörmander’s condition to guarantee the $L_p$-boundedness. Recall
\[ P_n^\gamma := \{(i_0 2^{-n\gamma}, (i_0 + 1) 2^{-n\gamma}) \times D_n(i_1, \cdots, i_d) : i_0, \cdots, i_d \in \mathbb{Z}\}, \]
and \{P_n^\gamma\}_{n \in \mathbb{Z}} is a filtration of partitions by Lemma 6.4.

**Lemma 6.10 (Hörmander’s condition for the kernel $K_\varepsilon$).** Suppose that $K_\varepsilon$ satisfies the $([d_2] + 1)$-times regular condition with $(\gamma, N_1, N_2)$ (Definition 5.1). Then for each $A \in \cup_{n \in \mathbb{Z}} P_n^\gamma$, there exist constant $N = N(d, \gamma, \varepsilon, N_1, N_2)$ and closed set $A^*$ such that $\overline{A} \subset A^*$, $|A| \leq N|A^*|$ and
\[ \int_{\mathbb{R}^{d+1} \setminus A^*} |K_\varepsilon(t, s, x - y_0) h(\varepsilon, t - s, T) - K_\varepsilon(t, s, x - y_1) h(\varepsilon, t - s, T)| dx dt \leq NT^{1-\varepsilon} \quad (6.9) \]
whenever $(s_0, y_0), (s_1, y_1) \in A$.

**Proof.** Let
\[ A = [i_0 2^{-n\gamma}, (i_0 + 1) 2^{-n\gamma}] \times [0, 2^{-n}) d \in P_n^\gamma \]
and put
\[ A^* = [i_0 2^{-n\gamma} - 2^{-n\gamma}, i_0 2^{-n\gamma} + 2^{-n\gamma+1}] \times \overline{B_{2^{-(n-1)\sqrt{d}}}(0)}. \]

We show that (6.9) holds with a positive constant $N$. Denote
\[ I(s_0, y_0, s_1, y_1) := \int_{\mathbb{R}^{d+1} \setminus A^*} |K_\varepsilon(t, s, x - y_0) h(\varepsilon, t - s, T) - K_\varepsilon(t, s, x - y_1) h(\varepsilon, t - s, T)| dx dt, \]
\[ A_1 := [i_0 2^{-n\gamma} - 2^{-n\gamma}, i_0 2^{-n\gamma} + 2^{-n\gamma+1}] \times (\mathbb{R}^d \setminus \overline{B_{2^{-(n-1)\sqrt{d}}}(0)}), \]
\[ A_2 := (-\infty, i_0 2^{-n\gamma} - 2^{-n\gamma}) \times \mathbb{R}^d, \quad A_3 := (i_0 2^{-n\gamma} + 2^{-n\gamma+1}, \infty) \times \mathbb{R}^d. \]

Set
\[ I(s_0, y_0, s_1, y_1) = \left( \int_{A_1} \cdots dx dt \right) + \left( \int_{A_2} \cdots dx dt \right) + \left( \int_{A_3} \cdots dx dt \right) =: I_1(s_0, y_0, s_1, y_1) + I_2(s_0, y_0, s_1, y_1) + I_3(s_0, y_0, s_1, y_1). \]
and with the triangle inequality \((i = 1, 2, 3)\),

\[
I_i(s_0, y_0, s_1, y_1) \leq \int_{A_i} |K_\varepsilon(t, s_0, x - y_0)h(\varepsilon, t - s_0, T) - K_\varepsilon(t, s_0, x - y_1)h(\varepsilon, t - s_0, T)|dxdt \\
+ \int_{A_i} |K_\varepsilon(t, s_1, x - y_1)h(\varepsilon, t - s_0, T) - K_\varepsilon(t, s_1, x - y_1)h(\varepsilon, t - s_1, T)|dxdt \\
=: I_{i,1}(s_0, y_0, y_1) + I_{i,2}(s_1, s_1, y_1).
\]

Choose a sequence of nonnegative functions \(\eta_m \in C_0^\infty(\mathbb{R}^d)\) so that \(\eta_m \uparrow 1\) as \(m \to \infty\) and \(\eta_m(-x) = \eta_m(x)\) for all \(x \in \mathbb{R}^d\). First, we prove that \(I_1 \leq NT^{1-\varepsilon}\). If \(x \in \mathbb{R}^d \setminus B_{2^{-n}(n-1)\sqrt{d}}(0)\) and \(y \in [0, 2^{-n})^d\), then \(|x - y| > 2^{-n}\sqrt{d}\). By Hölder’s inequality and Definition [5.1]

\[
\int_{|x| \geq 2^{-(n-1)\sqrt{d}}} |K_\varepsilon(t, s, x - y)|dx \leq \int_{|x| \geq 2^{-n}\sqrt{d}} |K_\varepsilon(t, s, x)|dx \\
\leq \left( \int_{|x| \geq 2^{-n}\sqrt{d}} |x|^{-2d(2)}dx \right)^{1/2} \left( \int_{|x| \geq 2^{-n}\sqrt{d}} |x|^{2d(2)}|K_\varepsilon(t, s, x)|^2dx \right)^{1/2} \\
\leq N(2^{-n}\sqrt{d})^{-d(2)+\varepsilon}2^{-d(2)\varepsilon}\leq t-s|^{-\varepsilon-\frac{d(2)}{2d}+\frac{\varepsilon}{d}},
\]

where \(N = N(d, N_2)\). Thus, since \(s \in [i_02^{-n\gamma}, (i_0 + 1)2^{-n\gamma})\),

\[
I_1(s_0, y_0, s_1, y_1) \leq N2^{d(2)-\frac{\varepsilon}{d}}\int_{i_02^{-n\gamma}}^{i_02^{-n\gamma+2^{-n\gamma+1}}} |t-s|^{-\frac{d(2)}{2d}+\frac{\varepsilon}{d}}(10<|t-s|<T1_{\varepsilon \in [0,1]} + 1_{\varepsilon=1})dt \\
\leq NT^{1-\varepsilon}2^{d(2)-\frac{\varepsilon}{d}}\int_{i_02^{-n\gamma}}^{i_02^{-n\gamma+2^{-n\gamma+1}}} |t-s|^{-\frac{d(2)}{2d}+\frac{\varepsilon}{d}}dt \\
\leq NT^{1-\varepsilon}2^{d(2)-\frac{\varepsilon}{d}}2^{n(d-d(2))-\frac{\varepsilon}{d}} = NT^{1-\varepsilon},
\]

where \(N = N(d, \gamma, N_2)\).

Next we show that \(I_{2,1} + I_{3,1} \leq NT^{1-\varepsilon}\). By similarity, we only estimate \(I_{3,1}\). By the fundamental theorem of calculus, Fubini’s theorem and Theorem [6.1]

\[
\int_{\mathbb{R}^d} |K_\varepsilon(t, s_0, x - y_0) - K_\varepsilon(t, s_0, x - y_1)|dx \leq |y_1 - y_0| \int_{\mathbb{R}^d} \int_0^1 |\nabla_x K_\varepsilon(t, s_0, x - y_0)|d\theta dx \\
\leq 2^{-n}\sqrt{d} \int_{\mathbb{R}^d} |\nabla_x K_\varepsilon(t, s_0, x)|dx \leq N2^{-n}|t - s_0|^{-\varepsilon-\frac{1}{2}},
\]

where \(N = N(d, N_1, N_2)\) and \(y_0 := \theta y_0 + (1 - \theta)y_1\). Therefore, since \(s_0 \in [i_02^{-n\gamma}, (i_0 + 1)2^{-n\gamma})\),

\[
I_{3,1}(s_0, y_0, y_1) \leq N2^{-n}\int_{i_02^{-n\gamma}}^{i_02^{-n\gamma+2^{-n\gamma+1}}} |t - s_0|^{-\varepsilon-\frac{1}{2}}(10<|t-s_0|<T1_{\varepsilon \in [0,1]} + 1_{\varepsilon=1})dt \\
\leq NT^{1-\varepsilon}2^{-n}\int_{i_02^{-n\gamma}}^{i_02^{-n\gamma+2^{-n\gamma+1}}} |t - s_0|^{-1-\frac{1}{2}}dt = NT^{1-\varepsilon},
\]

where \(N = N(d, \gamma, \varepsilon, N_1, N_2)\). Finally we claim that \(I_{2,2} + I_{3,2} \leq NT^{1-\varepsilon}\). Due to similarity, we only estimate \(I_{3,2}\). For estimating \(I_{3,2}\), we consider the two cases: \(\varepsilon \in [0, 1)\) and \(\varepsilon = 1\).

**Case 1.** \(\varepsilon \in [0, 1)\).
By Fubini’s theorem and Theorem 6.1
\[
I_{3,2}(s_0, s_1, y_1) \leq \sum_{i=1}^{2} \int_0^\infty \int_{|y_1 - x| < t} |K_\varepsilon(t, s_i, x)|dx \cdot h(\varepsilon, t - s_i, T)dt
\]
\[
\leq N \sum_{i=1}^{2} \int_0^\infty \int_{|y_1 - x| < t} |t - s_i|^{-\varepsilon} 1_{0 < t < s_i + T}dt
\]
\[
\leq N \sum_{i=1}^{2} \int_{s_i}^{s_i + T} |t - s_i|^{-\varepsilon} dt = NT^{1-\varepsilon},
\]
where \( N = N(d, \varepsilon, N_1, N_2) \).

**Case 2.** \( \varepsilon = 1 \).

By the fundamental theorem of calculus, Fubini’s theorem, and Theorem 6.1
\[
\int_{\mathbb{R}^d} |K_1(t, s_0, x - y_1) - K_1(t, s_1, x - y_1)|dx \leq |s_1 - s_0| \int_{\mathbb{R}^d} \int_{0}^{1} |\partial_t K_1(t, s_\theta, x - y_1)|d\theta dx
\]
\[
\leq 2^{-n\gamma} \int_{\mathbb{R}^d} \int_{0}^{1} |\partial_t K_1(t, s_\theta, x)|dxd\theta
\]
\[
\leq N2^{-n\gamma} \int_{0}^{1} |t - s_\theta|^{-2} d\theta
\]
where \( N = N(d, N_1, N_2) \) and \( s_\theta := (1 - \theta)s_0 + \theta s_1 \). One can observe that for \( s \in [i_0 2^{-n\gamma}, (i_0 + 1)2^{-n\gamma}) \),
\[
\frac{1}{2} \leq \frac{|t - s|}{|t - s_\theta|} \leq 2.
\]
Therefore,
\[
I_{3,2}(s_0, s_1, y_1) \leq N2^{-n\gamma} \int_{i_0 2^{-n\gamma} + 2^{-n\gamma+1}}^\infty |t - s|^{-2} dt \leq N,
\]
where \( N = N(d, \gamma, N_1, N_2) \). The theorem is proved.

**Proof of Theorem 6.9**

By Lemma 6.10, \( K_\varepsilon \) is a Calderón-Zygmund kernel relative to \( \mathcal{P}_N^\varepsilon \) (See [29, Definition 3.1]). Therefore, by [29, Theorem 4.1], we obtain the desired result. The theorem is proved.

6.4. Sharp-Maximal function estimates and proof of Theorem 5.2. Throughout this subsection, we fix a \( p_0 \in (1, 2] \) and an operator \( T_{c,T} \) satisfying the \( \left( \left[ \frac{d}{\rho_0} \right] + 2 \right) \)-times regular condition with \( (\gamma, N_1, N_2) \). Recall
\[
Q_b^\gamma(t, x) = (t - b^\gamma, t + b^\gamma) \times \{ y \in \mathbb{R}^d : |x - y| < b \}.
\]

**Lemma 6.11.** Let \( t_0 \in \mathbb{R} \) and \( b > 0 \). Assume that \( f \in C_c^\infty(\mathbb{R}^{d+1}) \) has a support in \( (t_0 - 3b^\gamma, t_0 + 3b^\gamma) \times B_{3b}(0) \). Then for any \( (t, x) \in Q_b^\gamma(t_0, 0) \),
\[
\int_{Q_b^\gamma(t_0, 0)} |T_{c,T}f(s,y)|^{p_0} dy ds \leq NT^{p_0(1-\varepsilon)}M|f|^{p_0}(t, x),
\]
where \( N = N(d, p_0, \varepsilon, N_1, N_2, N_3) \).
Proof. By Theorem 6.9
\[
\int_{Q_L^0(t_0,0)} |\mathcal{T}_{\varepsilon,T} f(s,y)|^{p_0} dy ds \leq N T^{p_0(1-\varepsilon)} |Q_L^0(t_0,0)|^{-1} \int_{\mathbb{R}^{d+1}} |f(s,y)|^{p_0} dy ds = N T^{p_0(1-\varepsilon)} |Q_L^0(t_0,0)|^{-1} \int_{t_0-2b^\gamma}^{t_0} \int_{B_{3b}(0)} |f(s,y)|^{p_0} dy ds \\
\leq N T^{p_0(1-\varepsilon)} M |f|^{p_0}(t,x),
\]
where \(N = N(d, p_0, \gamma, \varepsilon, N_1, N_2, N_3)\). The lemma is proved. \(\square\)

Lemma 6.12. Let \(t_0 \in \mathbb{R}\) and \(b > 0\). Assume that \(f \in C_\varepsilon^\infty(\mathbb{R}^{d+1})\) has a support in \((t_0 - 3b^\gamma, \infty) \times \mathbb{R}^d\). Then for any \((t, x) \in Q_L^0(t_0, 0)\),
\[
\int_{Q_L^0(t_0,0)} |\mathcal{T}_{\varepsilon,T} f(s,y)|^{p_0} dy ds \leq N T^{p_0(1-\varepsilon)} M |f|^{p_0}(t,x),
\]
where \(N = N(d, p_0, \gamma, \varepsilon, N_1, N_2, N_3)\).

Proof. Choose a \(\eta \in C_\varepsilon^\infty(\mathbb{R})\) satisfying

- \(\eta(s) \in [0, 1]\) for all \(s \in \mathbb{R}\)
- \(\eta(s) = 1\) for all \(s \leq t_0 + 2b^\gamma\)
- \(\eta(s) = 0\) for all \(s \geq t_0 + 3b^\gamma\).

Observe that if \((t, x) \in Q_L^0(t_0, 0)\), then
\[
\mathcal{T}_{\varepsilon,T} f(t,x) = \mathcal{T}_{\varepsilon,T} f(t,x) \quad \text{and} \quad M |f\eta|^{p_0}(t,x) \leq M |f|^{p_0}(t,x).
\]

Thus we may assume that \(f(s,y) = 0\) if \(|s - t_0| \geq 3b^\gamma\). Next choose a \(\eta \in C_\varepsilon^\infty(\mathbb{R}^d)\) satisfying

- \(\eta(y) \in [0, 1]\) for all \(y \in \mathbb{R}^d\)
- \(\eta(y) = 1\) for all \(y \in B_{2b}(0)\)
- \(\eta(y) = 0\) for all \(y \in \mathbb{R}^d \setminus B_{3b/2}(0)\).

Note that \(\mathcal{T}_{\varepsilon,T} f = \mathcal{T}_{\varepsilon,T} (f \zeta) + \mathcal{T}_{\varepsilon,T}(f(1 - \zeta))\) and \(\mathcal{T}(f \zeta)\) can be estimated by Lemma 6.11. Thus it suffices to estimate \(\mathcal{T}_{\varepsilon,T}(f(1 - \zeta))\) and we may assume that \(f(s,y) = 0\) if \(|y| < 2b\). In total, we assume that \(f(s,y) = 0\) if \(|s - t_0| \geq 3b^\gamma\) or \(|y| < 2b\) without loss of generality. Hence if \((s,y) \in Q_L^0(t_0, 0)\) and \(|z| < b\), then \(|y - z| \leq 2b\) and \(f(r,y - z) = 0\). By Lemma 6.10 and Hölder’s inequality,
\[
\left| \int_{\mathbb{R}^d} K_\varepsilon(s,r,y-z) f(r,z) dz \right| = \left| \int_{|z| \geq b} K_\varepsilon(s,r,z) f(r,y-z) dz \right| \\
\leq N \int_b^\infty (\lambda^d K_{\varepsilon,0,0}^p(s,r,\lambda))^{1/p_0} \left( \int_{B_{2b+\lambda}(x)} |f(r,z)|^{p_0} dz \right)^{1/p_0} d\lambda,
\]
where \(N = N(d, p_0), |\alpha| = 1,\) and
\[
K_{\varepsilon,0,0}^p(s,r,\lambda) := \int_{\mathbb{R}^{d-1}} |D^\alpha_x K_\varepsilon(s,r,\lambda w)|^{p_0} \sigma(dw).
\]

Recalling
\[
h(\varepsilon, t-s, T) = 1_{\varepsilon \in [0,1]} 1_{|t-s| < T + 1_{\varepsilon = 1}},
\]
we also have
\[
|s-r|^{-\varepsilon - \frac{(d-d(p_0))}{p_0} + \frac{d}{p_0}} h(\varepsilon, s-r, T) \leq T^{1-\varepsilon} |s-r|^{-1 - \frac{(d-d(p_0))}{p_0} + \frac{d}{p_0}}.
\]
By Lemma 6.7 and Hölder's inequality,

\[ |T_{\varepsilon,T}(s,y)| \leq T^{1-\varepsilon} \int_{t_0}^{t_0+b^\gamma} \left( \int_b^\infty \lambda^{-p_0(d(p_0))-1} \left( \int_{B_{2b+\lambda}(x)} |f(r,z)|^{p_0} \, dz \, d\lambda \right)^{1/p_0} |s-r|^{-1-\frac{(d-d(p_0))}{p_0}} \frac{d}{p_0^\gamma} \, dr \right)^{1/p_0} \]

\[ \leq T^{1-\varepsilon} \left( \int_b^\infty \lambda^{-p_0(d(p_0))-1} \left( \int_{t_0}^{t_0+b^\gamma} \left( \int_{B_{2b+\lambda}(x)} |f(r,z)|^{p_0} \, dz \, d\lambda \right)^{1/p_0} |s-r|^{-1-\frac{(d-d(p_0))}{p_0}} \frac{d}{p_0^\gamma} \, dr \right) \right)^{1/p_0} \]

\[ \leq NT^{1-\varepsilon} b^{-\frac{(d-d(p_0))}{p_0^\gamma} + \frac{d}{p_0^\gamma}} \left( \int_b^\infty \lambda^{-p_0(d(p_0))-1} \left( \int_{t_0}^{t_0+b^\gamma} \left( \int_{B_{2b+\lambda}(x)} |f(r,z)|^{p_0} \, dz \, d\lambda \right)^{1/p_0} |s-r|^{-1-\frac{(d-d(p_0))}{p_0}} \frac{d}{p_0^\gamma} \, dr \right) \right)^{1/p_0} \]

By virtue of Fubini’s theorem,

\[ \int_{t_0-b^\gamma}^{t_0+b^\gamma} |T_{\varepsilon,T}(s,y)|^{p_0} \, ds \]

\[ \leq NT^{p_0(1-\varepsilon)} b^{-\frac{(p_0-1)(d-d(p_0))}{p_0}} + \frac{2p_0-1}{p_0} \]

\[ \leq NT^{p_0(1-\varepsilon)} b^{-\frac{(d-d(p_0))}{p_0^\gamma} + \frac{d}{p_0^\gamma}} \left( \int_b^\infty \lambda^{-p_0(d(p_0))-1} \left( \int_{t_0}^{t_0+b^\gamma} \left( \int_{B_{2b+\lambda}(x)} |f(r,z)|^{p_0} \, dz \, d\lambda \right)^{1/p_0} |s-r|^{-1-\frac{(d-d(p_0))}{p_0}} \frac{d}{p_0^\gamma} \, dr \right) \right)^{1/p_0} \]

\[ \leq NT^{p_0(1-\varepsilon)} b^{-\frac{(d-d(p_0))}{p_0^\gamma} + \frac{d}{p_0^\gamma}} + \gamma^{-p_0(d(p_0))} + d M\|f\|_{p_0}(t,x) \leq NT^{p_0(1-\varepsilon)} b^\gamma M\|f\|_{p_0}(t,x), \]

Therefore,

\[ \int_{Q_{b_0}^+} |T f(s,y)|^{p_0} \, dy \, ds \leq NT^{p_0(1-\varepsilon)} M\|f\|_{p_0}(t,x), \]

where \( N = N(d, p_0, \gamma, \varepsilon, N_1, N_2, N_3). \) The lemma is proved. \( \square \)

**Lemma 6.13.** Let \( t_0 \in \mathbb{R} \) and \( b > 0. \) Assume that \( f \in C_c^\infty(\mathbb{R}^{d+1}) \) has a support in \((\varepsilon, t_0 - 2b^\gamma) \times \mathbb{R}^d. \) Then for any \((t,x) \in Q_b^+(t_0,0),\)

\[ \sup_{(s_1,y_1), (s_2,y_2) \in Q_b^+(t_0,0),} |T_{\varepsilon,T} f(s_1,y_1) - T_{\varepsilon,T} f(s_2,y_2)|_{p_0} \leq NT^{p_0(1-\varepsilon)} M\|f\|_{p_0}(t,x), \]

where \( N = N(d, p_0, \gamma, \varepsilon, N_1, N_2, N_3). \)

**Proof.** Recall that

\[ h(\varepsilon, s-r, T) := 1_{\varepsilon \in [0,1)} 1_{|s-r| \leq T} + 1_{\varepsilon=1}, \]

\[ H(r, \lambda) := r^{-\frac{d}{p_0}} \left( r^{-\frac{1}{p_0^\gamma}} 1_{\lambda \leq r} + r^{\frac{p_0d(p_0)}{p_0^\gamma}} \lambda^{-p_0(d(p_0))-1} 1_{\lambda \leq r} \right), \]

\[ K_{p_0}^{\varepsilon,m,\alpha}(s,r,\lambda) := \int_{\mathbb{R}^{d-1}} |\partial_r^m D_2^\alpha K_{\varepsilon}(s,r,\lambda w)|^{p_0^\gamma} \sigma(dw). \]

By the triangle inequality,

\[ |T_{\varepsilon,T} f(s_1,y_1) - T_{\varepsilon,T} f(s_2,y_2)| \leq |T_{\varepsilon,T} f(s_1,y_1) - T_{\varepsilon,T} f(s_1,y_2)| + |T_{\varepsilon,T} f(s_1,y_2) - T_{\varepsilon,T} f(s_2,y_2)| =: I + II. \]
We claim that

$$I + II \leq NT^{1-\varepsilon}(M\|f|^{p_0}(t, x))^{1/p_0}.$$  

First, we estimate $I$. By the fundamental theorem of calculus, Lemma 6.6 with $R_1 = 2b$ and $R_2 = 0$ and Lemma 6.7, for $(s, y) \in Q_{b}^{0}(t, 0)$,

$$I = |T_{\varepsilon, T}f(s, y_{1}) - T_{\varepsilon, T}f(s, y_{2})|$$

$$= \left| \int_{-\infty}^{t_{0} - 2b^{2}} \int_{\mathbb{R}^{d}} K_{\varepsilon}(s, r, y_{1} - y) - K_{\varepsilon}(s, r, y_{2} - y) f(r, y) h(\varepsilon, s_{1} - r, T) dy dr \right|$$

$$\leq 2b \left| \int_{0}^{1} \int_{-\infty}^{t_{0} - 2b^{2}} \int_{\mathbb{R}^{d}} \nabla_{x} K_{\varepsilon}(s, r, y_{1} - y) f(r, y) h(\varepsilon, s_{1} - r, T) dy dr d\theta \right|$$

$$\leq N b \int_{-\infty}^{t_{0} - 2b^{2}} \left( \int_{0}^{\infty} \left| f(r, z) \right|^{p_0} \left( \lambda^{d} K_{\varepsilon, 0, 0}^{p_0}(s_1, r, \lambda) \right)^{1/p_0} \lambda^{d} d\lambda h(\varepsilon, s_{1} - r, T) dr \right)$$

$$\leq N \int_{-\infty}^{t_{0} - 2b^{2}} \left( \int_{0}^{\infty} H(|s_{1} - r|, \lambda) \left| f(r, z) \right|^{p_0} d\lambda \right)^{1/p_0} |s_{1} - r|^{-\varepsilon - 1/2} h(\varepsilon, s_{1} - r, T) dr,$$

where $N = N(d, p_0, \gamma, \varepsilon, N_1, N_2)$, $y_{0} := (1 - \theta) y_{1} + \theta y_{2}$ and $|\alpha| = 2$. By Lemma 6.8 with $(m, |\alpha|) = (0, 2)$,

$$I = |T_{\varepsilon, T}f(s, y_{1}) - T_{\varepsilon, T}f(s, y_{2})| \leq NT^{1-\varepsilon}(M\|f|^{p_0}(t, x))^{1/p_0}.$$  

For $II$, we consider the two cases: $\varepsilon \in [0, 1)$ and $\varepsilon = 1$.

Case 1. $\varepsilon \in [0, 1)$.

By Lemma 6.6 with $R_1 = 2b$ and $R_2 = 0$ and Lemma 6.7, for $(s, y) \in Q_{b}^{0}(t, 0)$,

$$|T_{\varepsilon, T}f(s, y)| = \left| \int_{-\infty}^{t_{0} - 2b^{2}} \int_{\mathbb{R}^{d}} K_{\varepsilon}(s, r, y - z) f(r, z) 1_{|s - r| < T} dz dr \right|$$

$$\leq N \int_{-\infty}^{t_{0} - 2b^{2}} \int_{0}^{\infty} \left( \lambda^{d} K_{\varepsilon, 0, 0}^{p_0}(s_1, r, \lambda) \right)^{1/p_0} \left( \int_{B_{2b, \lambda}(x)} \left| f(r, z) \right|^{p_0} dz \right)^{1/p_0} \lambda^{d} |s - r|^{-\varepsilon} 1_{|s - r| < T} dz dr$$

$$\leq N \int_{-\infty}^{t_{0} - 2b^{2}} \left( \int_{0}^{\infty} H(|s - r|, \lambda) \left| f(r, z) \right|^{p_0} d\lambda \right)^{1/p_0} |s - r|^{-\varepsilon} 1_{|s - r| < T} dz dr,$$

where $N = N(d, p_0, \varepsilon, N_1, N_2)$ and $|\alpha| = 1$. By Lemma 6.8 with $(m, |\alpha|) = (0, 1)$,

$$II \leq |T_{\varepsilon, T}f(s_1, y_2)| + |T_{\varepsilon, T}f(s_2, y_2)| \leq NT^{1-\varepsilon}(M\|f|^{p_0}(t, x))^{1/p_0}.$$  

Case 2. $\varepsilon = 1$. 

By the fundamental theorem of calculus, Lemma 6.6 with \( R_1 = 2b \) and \( R_2 = 0 \), and Lemma 6.7,
\[
T_{\epsilon,T} f(s_1, y_2) - T_{\epsilon,T} f(s_2, y_2)
\]
\[
= \left| \int_{-\infty}^{t_0 - 2b^{\gamma}} \int_{\mathbb{R}^d} K_1(s_1, r, y_2 - y) - K_1(s_2, r, y_2 - y) f(r, y) dy dr \right|
\]
\[
= |s_2 - s_1| \left| \int_{-\infty}^{t_0 - 2b^{\gamma}} \int_{\mathbb{R}^d} \partial_t K_1(s_\theta, r, y_2 - y) f(r, y) dy dr \right|
\]
\[
\leq 2b^{\gamma} \left| \int_0^1 \int_{-\infty}^{t_0 - 2b^{\gamma}} \int_{\mathbb{R}^d} \partial_t K_1(s_\theta, r, y) f(r, y) dy dr d\theta \right|
\]
\[
\leq N b^{\gamma} \int_0^1 \int_{-\infty}^{t_0 - 2b^{\gamma}} \int_0^\infty \left( \int_{B_{2b^{\lambda}(x)}} |f(r, z)|^{p_0} \right)^{1/p_0} \left( \lambda^{d} K_{1,1,0}^{p_0}(s_\theta, r, \lambda)^{1/p_0} d\lambda dr d\theta \right)
\]
\[
\leq N b^{\gamma} \int_0^1 \int_{-\infty}^{t_0 - 2b^{\gamma}} \int_0^\infty \left( \int_{H(\mathbb{R}^d)} |f(r, z)|^{p_0} d\lambda dr d\theta \right)
\]
where \( N = N(d, p_0, \gamma, \epsilon, N_1, N_3) \), \( s_0 := (1 - \theta)s_1 + \theta s_2 \) and \( |\alpha| = 1 \). By Lemma 6.8 with \( (m, |\alpha|) = (1, 1) \),
\[
|T_{\epsilon,T} f(s_1, y_2) - T_{\epsilon,T} f(s_2, y_2)| \leq N(M)|f|^{p_0}(t, x)^{1/p_0}
\]
The lemma is proved. \( \square \)

**Theorem 6.14.** There exists a constant \( N = N(d, p_0, \gamma, \epsilon, N_1, N_3) \) such that for any \( f \in C_0^\infty(\mathbb{R}^{d+1}) \),
\[
T_{\epsilon,T} f(t, x) \leq NT^{1-\epsilon}(M)|f|^{p_0}(t, x)^{1/p_0}
\]

**Proof.** Let \( b > 0 \), \( t_0 \in \mathbb{R} \), and \((t, x) \in Q_b(t_0, 0)\). Choose a \( \eta \in C^\infty(\mathbb{R}) \) satisfying
- \( \eta(t) \in [0, 1] \) for all \( t \in \mathbb{R} \).
- \( \eta(t) = 1 \) for all \( t < t_0 - 3b^{\gamma} / 3 \).
- \( \eta(t) = 0 \) for all \( t \geq t_0 - 7b^{\gamma} / 3 \).

Put \( \eta^* := 1 - \eta \). Then
\[
|T_{\epsilon,T} f(s_1, y_1) - T_{\epsilon,T} f(s_0, y_0)| \leq |T_{\epsilon,T} (f \eta)(s_1, y_1) - T_{\epsilon,T} (f \eta)(s_0, y_0)|
\]
\[
+ |T_{\epsilon,T} (f \eta^*)(s_1, y_1) - T_{\epsilon,T} (f \eta^*)(s_0, y_0)|.
\]
By Lemma 6.13,
\[
\int_{Q_b(t_0, 0)} \int_{Q_b(t_0, 0)} |T_{\epsilon,T} (f \eta)(s_1, y_1) - T_{\epsilon,T} (f \eta)(s_0, y_0)|^{p_0} dy_1 ds_1 dy_0 ds_0 \leq NT^{p_0(1-\epsilon)}M|f|^{p_0}(t, x)
\]
and by Lemma 6.12,
\[
\int_{Q_b(t_0, 0)} \int_{Q_b(t_0, 0)} |T_{\epsilon,T} (f \eta^*)(s_1, y_1) - T_{\epsilon,T} (f \eta^*)(s_0, y_0)|^{p_0} dy_1 ds_1 dy_0 ds_0
\]
\[
\leq 2 \int_{Q_b(t_0, 0)} |T_{\epsilon,T} (f \eta^*)(s, y)|^{p_0} dy ds \leq NT^{p_0(1-\epsilon)}M|f|^{p_0}(t, x).
\]
Hence,
\[
\int_{Q_b(t_0, 0)} \int_{Q_b(t_0, 0)} |T_{\epsilon,T} f(s_1, y_1) - T_{\epsilon,T} f(s_0, y_0)|^{p_0} dy_1 ds_1 dy_0 ds_0 \leq NT^{p_0(1-\epsilon)}M|f|^{p_0}(t, x),
\]
where \( N = N(d, p_0, \gamma, \epsilon, N_1, N_2, N_3) \). For \( x_0 \in \mathbb{R}^d \), denote
\[
\tau_{x_0} f(t, x) := f(t, x_0 + x).\]
Since $\mathcal{T}_{\tau_x}$ and $\tau_{x_0}$ are commutative,
\[
\int_{Q^+_\epsilon(t_0,x_0)} \int_{Q^+_\epsilon(t_0,x_0)} |\mathcal{T}_{\tau_x} f(s_1, y_1) - \mathcal{T}_{\tau_x} f(s_0, y_0)|^p d\nu_{1} ds_1 dy_0 ds_0
\]
\[
= \int_{Q^+_\epsilon(t_0,0)} \int_{Q^+_\epsilon(t_0,0)} |\mathcal{T}_{\tau_{x_0}} f(t, x) - \mathcal{T}_{\tau_{x_0}} f(s, y)|^p dx dt dy ds
\]
\[
\leq N T^{p_0(1-\epsilon)} M |\tau_{x_0} f|^p (t, x) = N T^{p_0(1-\epsilon)} M |f|^p (t, x_0 + x).
\]
Therefore, by Jensen’s inequality, for $(t, x) \in Q^+_\epsilon(t_0, 0)$ and $x_0 \in \mathbb{R}^d$
\[
\left(\int_{Q^+_\epsilon(t_0,x_0)} \int_{Q^+_\epsilon(t_0,x_0)} |\mathcal{T}_{\tau_x} f(s_1, y_1) - \mathcal{T}_{\tau_x} f(s_0, y_0)|^p d\nu_{1} ds_1 dy_0 ds_0\right)^{p_0}
\]
\[
\leq \int_{Q^+_\epsilon(t_0,x_0)} \int_{Q^+_\epsilon(t_0,x_0)} |\mathcal{T}_{\tau_x} f(s_1, y_1) - \mathcal{T}_{\tau_x} f(s_0, y_0)|^p d\nu_{1} ds_1 dy_0 ds_0
\]
\[
\leq N T^{p_0(1-\epsilon)} M |f|^p (t, x_0 + x),
\]
where $N = N(d, p_0, \gamma, \epsilon, N_1, N_2, N_3)$. Taking the supremum on both sides with respect to all $Q^+_\epsilon$ containing $(t, x_0 + x)$, we obtain the desired result. The theorem is proved. □

Proof of Theorem 5.2

Because of the similarity, we only prove (i). Let $w \in A_p(\mathbb{R}^{d+1})$. Choose a $p_0 \in (1, R^p_{p,d+1}] \subseteq (1, 2]$ so that $w \in A_{p/p_0}(\mathbb{R}^{d+1})$ and $[d/R^p_{p,d+1}] = [d/p_0]$. Then, it is obvious that $\mathcal{T}_{\tau_x}$ satisfies satisfies $[d/p_0] + 2$ times regular condition with $(\gamma, N_1, N_2)$. Thus, by Theorem 6.14
\[
\|(\mathcal{T}_{\tau_x} f)^\delta\|_{L_p(\mathbb{R}^{d+1}, w)} \leq N T^{1-\epsilon} \|(M |f|^p_0)^{1/p_0}\|_{L_p(\mathbb{R}^{d+1}, w)},
\]
where $N = N(d, p, \gamma, \epsilon, N_1, N_2, N_3, [w]_{A_p(\mathbb{R}^{d+1})})$. Moreover, recalling $w \in A_{p/p_0}(\mathbb{R}^{d+1})$ and applying Theorem 6.3 ii) and Remark 2.2 we have
\[
\|(M |f|^p_0)^{1/p_0}\|_{L_p(\mathbb{R}^{d+1}, w)} = \|M |f|^p_0\|_{L_p(\mathbb{R}^{d+1}, w)} \leq N \|f|^p_0\|_{L_p(\mathbb{R}^{d+1}, w)} = N \|f\|_{L_p(\mathbb{R}^{d+1}, w)},
\]
where $N = N(d, p, \gamma, [w]_{A_p(\mathbb{R}^{d+1})})$. Finally, using Theorem 6.5(i), (6.10), and (6.11), we obtain
\[
\|\mathcal{T}_{\tau_x} f\|_{L_p(\mathbb{R}^{d+1}, w)} \leq N \|\mathcal{T}_{\tau_x} f\|^\delta_{L_p(\mathbb{R}^{d+1}, w)} \leq N T^{1-\epsilon} \|(M |f|^p_0)^{1/p_0}\|_{L_p(\mathbb{R}^{d+1}, w)} \leq N T^{1-\epsilon} \|f\|_{L_p(\mathbb{R}^{d+1}, w)},
\]
where $N = N(d, p, \gamma, \epsilon, N_1, N_2, N_3, [w]_{A_p(\mathbb{R}^{d+1})})$. The theorem is proved.

7. Proof of Theorems 2.14 and 2.15

Recall the kernel
\[
p(t, s, x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp \left( \int_s^t \psi(r, \xi) dr \right) e^{ix \cdot \xi} d\xi \cdot 1_{t > s \geq 0},
\]
and the operator
\[
\mathcal{K}_{\tau_x} f(t, x) := \int_{-\infty}^t \int_{\mathbb{R}^d} \left( 1_{\epsilon \in [0, 1]} [1 - s] \times T + 1_{\epsilon = 1} \right) P_{\tau_x}(t, s, x - y) f(s, y) dy ds.
\]
Throughout the section, we fix $\epsilon \in [0, 1]$, $\nu \in \mathbb{R}$, $p, q \in (1, \infty)$, $w \in A_p(\mathbb{R}^{d+1})$, $w_1 \in A_q(\mathbb{R})$, and $w_2 \in A_p(\mathbb{R}^d)$. If $\nu \neq 0$, then we additionally assume that $w(t, \cdot) \in A_p(\mathbb{R}^d)$ for almost all $t \in [0, T]$ and
\[
\text{ess sup}_{t \in [0, T]} [w(t, \cdot)]_{A_p(\mathbb{R}^d)} =: N_0 < \infty.
\]
Below, to treat two types of weighted spaces (time-space mixed or not) simultaneously, we introduce a unified notation. We use the notation $\mathbb{B}_0(T)$ and $\mathbb{B}_1(T)$ to denote either
\[ (\mathbb{B}_0(T), \mathbb{B}_1(T)) = (L^p((0, T) \times \mathbb{R}^d, u), H^{\omega + \gamma}_{p,d}((0, T) \times \mathbb{R}^d, w)) \]
or
\[ (\mathbb{B}_0(T), \mathbb{B}_1(T)) = (L_q((0, T), u; H^p_{d}((\mathbb{R}^d, w_2)), u \in L_q((0, T), w_1; H^{\omega + \gamma}_{p,d}((\mathbb{R}^d, w_2)))). \]

**Theorem 7.1** (A priori estimate). Suppose that $\psi(t, \xi)$ satisfies the ellipticity condition with $(\gamma, \kappa)$ and has a $([d/R^w_{p,d+1}] + 2)$-times (resp. $([d/R^w_{p,d}] + 2)$-times) regular upper bound with $(\gamma, M)$. Then, for $f \in \mathbb{B}_0(T)$,
\[ ||\mathcal{K}_t, T(1_{[0,T]})||_{\mathbb{B}_1(T)} \leq N T^{1-\epsilon} ||f||_{\mathbb{B}_0(T)}, \]
\[ ||\psi(t, -i\nabla)\mathcal{K}_0, T(1_{[0,T]})||_{\mathbb{B}_1(T)} \leq N ||f||_{\mathbb{B}_0(T)}, \]
where $N = N(d, p, \gamma, \kappa, M, [w]_{\mathcal{A}_d}, \nu, N_0)$ (resp. $N = N(d, p, \nu, \gamma, \kappa, M, [w_1]_{\mathcal{A}_d}, [w_2]_{\mathcal{A}_d})$).

**Proof.** Due to Theorem 4.2 and 4.3, the family of the operators $\mathcal{K}_t, T$ is a particular case of $\mathcal{T}_{1: T}$ in Theorem 5.2. Therefore, these estimates can be easily derived using Theorem 5.2. The theorem is proved. \(\square\)

**Corollary 7.2** (Uniqueness of a solution). Let $f \in \mathbb{B}_0(T)$. Suppose that $\psi(t, \xi)$ satisfies the ellipticity condition with $(\gamma, \kappa)$ and has a $([d/R^w_{p,d+1}] + 2)$-times (or $([d/R^w_{p,d}] + 2)$-times) regular upper bound with $(\gamma, M)$. Then, the Cauchy problem
\[ \begin{cases} \partial_t u(t, x) = \psi(t, -i\nabla)u(t, x) + f(t, x), & (t, x) \in (0, T) \times \mathbb{R}^d, \\
u(0, x) = 0, & x \in \mathbb{R}^d, \end{cases} \]
has at most one solution in the sense of Definition 2.11.

**Proof.** Let $u, v \in \mathbb{B}_1(T)$ be solutions to Cauchy problem (7.1). Then by the definition of solution, there exist $u_n, v_n \in C_{p,d}^\infty([0, T] \times \mathbb{R}^d)$ such that $u_n(0, \cdot), v_n(0, \cdot) = 0$,
\[ \partial_t u_n - \psi(t, -i\nabla)u_n \rightarrow f, \quad \partial_t v_n - \psi(t, -i\nabla)v_n \rightarrow f \quad \text{in} \quad \mathbb{B}_0(T) \]
and
\[ u_n \rightarrow u \quad \text{and} \quad v_n \rightarrow v \quad \text{in} \quad \mathbb{B}_1(T). \]

Denote
\[ f_n := \partial_t u_n - \psi(t, -i\nabla)u_n \in C^\infty_p([0, T] \times \mathbb{R}^d), \]
\[ g_n := \partial_t v_n - \psi(t, -i\nabla)v_n \in C^\infty_p([0, T] \times \mathbb{R}^d). \]
Then by Theorem 5.2 we have
\[ u_n(t, x) = \mathcal{K}_{0,T} f_n(t, x), \quad v_n(t, x) = \mathcal{K}_{0,T} g_n(t, x). \]
Recall that the operator $\mathcal{K}_{0,T}$ is continuous from $\mathbb{B}_0(T)$ to $\mathbb{B}_1(T)$ due to Theorem 7.1. Therefore, taking limits in (7.3), we conclude that $u = \mathcal{K}_{0,T} f = v$ in $\mathbb{B}_1(T)$ since both $f_n$ and $g_n$ converge to $f$ in $\mathbb{B}_0(T)$. The corollary is proved. \(\square\)

**Corollary 7.3** (Existence of a solution). Let $f \in \mathbb{B}_0(T)$. Suppose that $\psi(t, \xi)$ satisfies the ellipticity condition with $(\gamma, \kappa)$ and has a $([d/R^w_{p,d+1}] + 2)$-times (or $([d/R^w_{p,d}] + 2)$-times) regular upper bound with $(\gamma, M)$. Then, the Cauchy problem
\[ \begin{cases} \partial_t u(t, x) = \psi(t, -i\nabla)u(t, x) + f(t, x), & (t, x) \in (0, T) \times \mathbb{R}^d, \\
u(0, x) = 0, & x \in \mathbb{R}^d, \end{cases} \]
has at least one solution in the sense of Definition 2.11.
has a solution \( u \in B_1(T) \) and \( u \) has the integral representation

\[
u(t, x) := \mathcal{K}_0,T f(t, x) = \int_0^1 \int_{\mathbb{R}^d} p(t, s, x - y) f(s, y) \, dy \, ds.
\]

**Proof.** By Theorem A.1, there exists a sequence \( \{f_n\}_{n=1}^\infty \subseteq C_c^{\infty}((0, T) \times \mathbb{R}^d) \) such that \( f_n \to f \) in \( B_0(T) \). For each \( f_n \), due to Theorem \( \ref{thm:approximation} \), there exists a unique solution \( u_n \) to (A3) such that

\[
u_n(t, x) = \mathcal{K}_0,T f_n(t, x) \in C^{1,\infty}_p([0, T] \times \mathbb{R}^d).
\]

By Theorem \( \ref{thm:embedding} \) and well-known embedding property of the Sobolev space (cf. Theorem A.1(iii) and (vii)),

\[
\nu \cdot (\Delta)^{\frac{\nu}{2}} u_n^\nu \|_{B_0(T)} = \| \mathcal{K}_\varepsilon,T f_n \|_{B_0(T)} \leq N T^{1-\varepsilon} \| f_n \|_{B_0(T)} \quad \forall \varepsilon \in [0, 1],
\]

\[
\| \partial_{t} u_n \|_{B_0(T)} \leq \| \psi(-i\nabla) \mathcal{K}_\varepsilon,T f_n \|_{B_0(T)} + \| f_n \|_{B_0(T)} \leq N \| f_n \|_{B_0(T)},
\]

where \( N \) is independent of \( n, \varepsilon \) and \( T \). Thus, due to the linearity of the equation and the operator, we have

\[
\| \partial_t u_n - \partial_t u_m \|_{B_0(T)} + \| \psi(-i\nabla) u_n - \psi(-i\nabla) u_m \|_{B_0(T)} + \| u_n - u_m \|_{B_1(T)} \leq N(1 + T) \| f_n - f_m \|_{B_0(T)}
\]

for all natural numbers \( n \) and \( m \). Finally, since \( B_0(T) \) and \( B_1(T) \) are Banach spaces (see Theorem A.1(i) and A.2(i)), there exists a \( u \in B_1(T) \) such that \( u_n(0, \cdot) = 0 \), \( u_n \to u \) in \( B_1(T) \) and

\[
\partial_t u_n - \psi(-i\nabla) u_n \to f \quad \text{in} \quad B_0(T).
\]

The corollary is proved. \( \square \)

**Appendix A. Properties of function spaces**

We prove some properties of Sobolev spaces with \( A_p \)-weights for the completeness of our paper. Most properties can be easily deduced from weighted theories in [3, 10, 11, 37] and classical Sobolev space theories. Note that most approximations based on Sobolev mollifiers are non-trivial in weighted spaces since our weights are neither bounded above nor bounded below in general.

**Theorem A.1.** Let \( p \in (1, \infty) \), \( w \in A_p(\mathbb{R}^d) \), and \( \nu \in \mathbb{R} \).

(i) \( H^\nu_p(\mathbb{R}^d, w) \) is a Banach space equipped with the norm

\[
\| f \|_{H^\nu_p(\mathbb{R}^d, w)} := \| (1 - \Delta)^{\nu/2} f \|_{L_p(\mathbb{R}^d, w)}
\]

(ii) For \( \nu_0 \in \mathbb{R} \), \( (1 - \Delta)^{\nu_0/2} \) is a bijective isometry from \( H^0_p(\mathbb{R}^d, w) \) to \( H^{\nu + \nu_0}_p(\mathbb{R}^d, w) \).

(iii) If \( \nu_1 \geq \nu_0 \), then \( H^{\nu_1}_p(\mathbb{R}^d, w) \) is continuously embedded into \( H^{\nu_0}_p(\mathbb{R}^d, w) \). More precisely, there exists a positive constant \( N \) such that

\[
\| f \|_{H^{\nu_1}_p(\mathbb{R}^d, w)} \leq N \| f \|_{H^{\nu_0}_p(\mathbb{R}^d, w)}, \quad \forall f \in \mathcal{S}(\mathbb{R}^d).
\]

(iv) If \( \nu \) is a nonnegative integer, then

\[
\| f \|_{H^\nu_p(\mathbb{R}^d, w)} \simeq \sum_{|\alpha| \leq \nu} \| D^\alpha f \|_{L_p(\mathbb{R}^d, w)}.
\]

(v) \( C_c^{\infty}(\mathbb{R}^d) \) is dense in \( H^\nu_p(\mathbb{R}^d, w) \).

(vi) The topological dual space of \( H^\nu_p(\mathbb{R}^d, w) \) is \( H^{-\nu}_p(\mathbb{R}^d, w) \), where

\[
\frac{1}{p} + \frac{1}{p'} = 1, \quad \bar{w} := w^{-\frac{1}{p'}} \in A_{p'}(\mathbb{R}^d).
\]

Moreover, for \( f \in H^\nu_p(\mathbb{R}^d, w) \),

\[
\sup_{\| \phi \|_{H^{-\nu}_p(\mathbb{R}^d, w)} \leq 1} \| (f, \phi) \| = \| f \|_{H^\nu_p(\mathbb{R}^d, w)}
\]

(vii) If \( \nu \geq 0 \), then the norm \( \| \cdot \|_{H^\nu_p(\mathbb{R}^d, w)} \) is equivalent to \( \| \cdot \|_{L_p(\mathbb{R}^d, w)} + \| \cdot \|_{H^\nu_p(\mathbb{R}^d, w)} \), where

\[
\| f \|_{H^\nu_p(\mathbb{R}^d, w)} := \| (1 - \Delta)^{\nu/2} f \|_{L_p(\mathbb{R}^d, w)}.
\]
Proof. For (i) \sim (iii), see [37, Section 3]. For (iv), see [37, Theorem 3.3].

(v) Note that $S(\mathbb{R}^d)$ is dense in $H^p_\nu(\mathbb{R}^d, w)$ (see [37, Section 3]). In particular,

$$S(\mathbb{R}^d) \subset H^p_\nu(\mathbb{R}^d, w) \quad \forall \nu \in \mathbb{R}.$$ 

Thus it is obviously sufficient to prove that $C_c^\infty(\mathbb{R}^d)$ is dense in $S(\mathbb{R}^d)$. Let $f \in S(\mathbb{R}^d)$ and $\eta \in C_c^\infty(\mathbb{R}^d)$ be a nonnegative function satisfying $\eta(0) = 1$ and $\eta_n(x) := \eta(x/n)$. Clearly, $f \eta_n \in C_c^\infty(\mathbb{R}^d)$. By Leibniz’s product rule and the dominated convergence theorem, for any multi-index $\alpha$,

$$\|D^\alpha f\eta_n - D^\alpha f\|_{L_p(\mathbb{R}^d, w)} \to 0.$$ 

Thus by (iv),

$$f \eta_n \to f \text{ in } H^p_\nu(\mathbb{R}^d, w) \quad \text{(A.1)}$$ 

if $\nu$ is a positive integer. Moreover, for general $\nu$, one can find a positive integer $\nu_1$ such that $\nu_1 \geq \nu$. Therefore, due to (iii), \text{(A.1)} holds for all $\nu \in \mathbb{R}$.

(vi) Since $(1 - \Delta)^{\nu/2}(1 - \Delta)^{-\nu/2} = (1 - \Delta)^{-\nu/2}(1 - \Delta)^{\nu/2}$ is an identity operator on $H^q_\nu(\mathbb{R}^d, w)$ for all $q \in (1, \infty)$ and $\nu' \in \mathbb{R}$, it suffices to show that the topological dual space of $L_p(\mathbb{R}^d, w)$ is $L_{p'}(\mathbb{R}^d, \nu)$ and

$$\sup_{\|\phi\|_{L_{p'}(\mathbb{R}^d, \nu)} \leq 1} \left| \int_{\mathbb{R}^d} f(x) \phi(x) dx \right| = \|f\|_{L_p(\mathbb{R}^d, w)}.$$ 

Recall Riesz’s theorem, first. That is, any bounded linear function on $L_p(\mathbb{R}^d)$ is given by a function in $\in L_{p'}(\mathbb{R}^d)$. We prove the statement by fitting our weighted classes in the setting of classical Riesz’s theorem as follows.

- Let $f$ be a bounded linear functional on $L_p(\mathbb{R}^d, w)$. Then it is easy to check that $f_{w^{-1/p}}$ is a bounded linear functional on $L_q(\mathbb{R}^d)$, thus, $f_{w^{-1/p}} \in L_{p'}(\mathbb{R}^d)$. This certainly implies that $f \in L_{p'}(\mathbb{R}^d, \tilde{w})$.
- By Hölder’s inequality,

$$\sup_{\|\phi\|_{L_{p'}(\mathbb{R}^d, \tilde{w})} \leq 1} \left| \int_{\mathbb{R}^d} f(x) \phi(x) dx \right| = \sup_{\|\phi\|_{L_{p'}(\mathbb{R}^d, \tilde{w})} \leq 1} \left| \int_{\mathbb{R}^d} f(x) w^{1/p}(x) \phi(x) w^{-1/p}(x) dx \right| \leq \|f\|_{L_p(\mathbb{R}^d, w)}.$$ 

For $f \in L_p(\mathbb{R}^d, w)$, $f_{w^{1/p}} \in L_{p'}(\mathbb{R}^d)$. Thus,

$$\|f\|_{L_p(\mathbb{R}^d, w)} = \|f_{w^{1/p}}\|_{L_{p'}(\mathbb{R}^d)} = \sup_{\|\phi\|_{L_{p'}(\mathbb{R}^d, \tilde{w})} \leq 1} \left| \int_{\mathbb{R}^d} f(x) (w(x))^{1/p} \phi(x) dx \right|.$$ 

Since $\|\phi\|_{L_{p'}(\mathbb{R}^d, \tilde{w})} \leq 1$,

$$\|\phi w^{1/p}\|_{L_{p'}(\mathbb{R}^d, \tilde{w})} = \int_{\mathbb{R}^d} |\phi(x)|^{p'} (w(x))^{1/p} (w(x))^{-1/p} dx = \|\phi\|_{L_{p'}(\mathbb{R}^d)} \leq 1.$$ 

Therefore,

$$\left| \int_{\mathbb{R}^d} f(x) (w(x))^{1/p} \phi(x) dx \right| \leq \sup_{\|\phi\|_{L_{p'}(\mathbb{R}^d, \tilde{w})} \leq 1} \left| \int_{\mathbb{R}^d} f(x) \phi(x) dx \right|.$$ 

(vii) Let

$$m_1(\xi) := \frac{\left|\xi\right|^\nu}{(1 + |\xi|^2)^{\nu/2}}, \quad m_2(\xi) := \frac{(1 + |\xi|^2)^{\nu/2}}{1 + |\xi|^{\nu}}.$$ 

One can easily check that (e.g. [10, Example 6.2.9]), for any multi-index $\alpha$,

$$|D^\alpha \xi m_1(\xi)| + |D^\alpha \xi m_2(\xi)| \leq N(\nu, |\alpha|) |\xi|^{-|\alpha|}.$$ 

Thus, for $s \in (1, \infty)$ and multi-index $\alpha$,

$$\sum_{i=1}^{2} \sup_{R > 0} \left( R^{s|\alpha|} \int_{R < |\xi| < 2R} \left| D^\alpha _{\xi} m_i(\xi) \right|^s d\xi \right)^{1/s} \leq N(d, \nu, |\alpha|).$$
By the weighted multiplier theorem ((30) Theorem 2), we have the following equivalence of the two norms
\[ \|f\|_{L_p(\mathbb{R}^d, w)} + \|(-\Delta)^{\nu/2} f\|_{L_p(\mathbb{R}^d, w)} \simeq \|f\|_{H^\nu_p(\mathbb{R}^d, w)}. \]

The theorem is proved. \( \square \)

**Theorem A.2.** Let \( p \in (1, \infty) \), \( w \in A_p(\mathbb{R}^{d+1}) \), and \( \nu \in \mathbb{R} \). Then,
(i) \( \mathbb{H}_p^\nu((0, T) \times \mathbb{R}^d, w) \) is a Banach space equipped with the norm
\[ \|f\|_{\mathbb{H}_p^\nu((0, T) \times \mathbb{R}^d, w)} := \|f\|_{L_p((0, T) \times \mathbb{R}^d, w)} + \|(-\Delta)^{\nu/2} f\|_{L_p((0, T) \times \mathbb{R}^d, w)}. \]
(ii) The topological dual space of \( L_p((0, T) \times \mathbb{R}^d, w) \) is \( L_{p'}((0, T) \times \mathbb{R}^d, \tilde{w}) \), where
\[ \frac{1}{p} + \frac{1}{p'} = 1, \quad \tilde{w} := w^{-\frac{1}{p'}} \in A_{p'}(\mathbb{R}^{d+1}). \]

**Proof.** (i) Note that \( \|f\|_{\mathbb{H}_p^\nu((0, T) \times \mathbb{R}^d, w)} \) is a norm due to some properties of \( L_p \)-space. Thus it suffices to prove the completeness. Suppose that \( \{f_n\}_{n=1}^\infty \subseteq \mathbb{H}_p^\nu((0, T) \times \mathbb{R}^d, w) \) is a Cauchy sequence.

**Case 1.** \( \nu \geq 0. \)

By the completeness of \( L_p \)-spaces, there exist \( f \) and \( g \) in \( L_p((0, T) \times \mathbb{R}^d, w) \) such that
\[ \lim_{n \to \infty} \left( \|f_n - f\|_{L_p((0, T) \times \mathbb{R}^d, w)} + \|(-\Delta)^{\nu/2} f_n - g\|_{L_p((0, T) \times \mathbb{R}^d, w)} \right) = 0. \]

For \( \phi \in C_0^\infty(\mathbb{R}^{d+1}) \),
\[ \int_0^T ((-\Delta)^{\nu/2} f(t, \cdot), \phi(t, \cdot)) dt = \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} f(t, x) 1_{0 < t < T} (-\Delta)^{\nu/2} \phi(t, x) dx dt \]
\[ = \lim_{n \to \infty} \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} f_n(t, x) 1_{0 < t < T} (-\Delta)^{\nu/2} \phi(t, x) dx dt \]
\[ = \lim_{n \to \infty} \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} (-\Delta)^{\nu/2} f_n(t, x) 1_{0 < t < T} \phi(t, x) dx dt = \int_0^T \int_{\mathbb{R}^d} g(t, x) \phi(t, x) dx dt. \]

By Theorem A.1 (v), \( (-\Delta)^{\nu/2} f1_{(0, T)} \) is a bounded linear functional on \( L_{p'}(\mathbb{R}^{d+1}, \tilde{w}) \). By virtue of Theorem A.1 (vi), \( (-\Delta)^{\nu/2} f1_{(0, T)} \in L_p(\mathbb{R}^{d+1}, w) \). Therefore,
\[ (-\Delta)^{\nu/2} f1_{(0, T)} = g, \]
which certainly implies that \( f_n \to f \) in \( \mathbb{H}_p^\nu((0, T) \times \mathbb{R}^d, w) \).

**Case 2.** \( \nu < 0. \)

There exists \( g \in L_p((0, T) \times \mathbb{R}^d, w) \) such that
\[ \lim_{n \to \infty} \|(1 - \Delta)^{-\nu/2} f_n - g\|_{L_p((0, T) \times \mathbb{R}^d, w)} = 0. \]

Denote
\[ f := (1 - \Delta)^{-\nu/2} g \in \mathbb{H}_p^\nu((0, T) \times \mathbb{R}^d, w), \]
then \( f_n \to f \) in \( \mathbb{H}_p^\nu((0, T) \times \mathbb{R}^d, w) \).

(ii) Let \( f \) be a bounded linear functional on \( L_p((0, T) \times \mathbb{R}^d, w) \). Then, it is easy to check that \( fw^{-1/p} \) is a bounded linear functional on \( L_p((0, T) \times \mathbb{R}^d) \), thus, \( fw^{-1/p} \in L_{p'}((0, T) \times \mathbb{R}^d) \). This certainly implies that \( f \in L_{p'}((0, T) \times \mathbb{R}^d, \tilde{w}) \). \( \square \)

**Assumption A.3.** For almost all \( t \in [0, T] \), \( w(t, \cdot) \in A_p(\mathbb{R}^d) \) and
\[ \text{ess sup}_{t \in [0, T]} \|w(t, \cdot)\|_{A_p(\mathbb{R}^d)} =: N_0 < \infty. \]
Lemma A.4. Let $p \in (1, \infty)$, $w \in A_p(\mathbb{R}^{d+1})$, and $f \in \mathcal{S}(\mathbb{R}^{d+1})$. Suppose that Assumption [A.3] holds. Additionally, assume that

$$
\sup_{R>0} \left( R^{2|\alpha|-d} \int_{R<|\xi|<2R} |D_\xi^\alpha m(\xi)|^2 d\xi \right)^{1/2} \leq N^*, \ \forall |\alpha| \leq d+1,
$$

(A.2)

where $m(\xi)$ is a function defined on $\mathbb{R}^d$. Then, there exists a constant $N = N(d, p, N^*, N_0)$ such that

$$
\|T_m f(t, \cdot)\|_{L_p(\mathbb{R}^d, w(t, \cdot))} \leq N \|f(t, \cdot)\|_{L_p(\mathbb{R}^d, \cdot)} \quad (t \text{ - a.e.}),
$$

where

$$
T_m f(t, x) := F^{-1}[m \mathcal{F}[f(t, \cdot)]](x).
$$

Proof. Consider the operator

$$
g \mapsto T_m g(x) := F^{-1}[m \mathcal{F}[g(\cdot)]](x).
$$

Then, by Theorem [10] Theorem 6.2.7, the operator $T_m : L_1(\mathbb{R}^d) \to L_{1, \infty}(\mathbb{R}^d)$ is bounded. Moreover, recall that $w(t, \cdot) \in A_p(\mathbb{R}^d)$ (t - a.e.) Thus, applying [8] Corollaries 6.10, 6.11, and Remark 6.14, we have, for almost all $t \in [0, T]$,

$$
\|T_m f(t, \cdot)\|_{L_p(\mathbb{R}^d, w(t, \cdot))} \leq N \|w(t, \cdot)^{1/(p-1)} f(t, \cdot)\|_{A'_p(\mathbb{R}^d)} \|f(t, \cdot)\|_{L_p(\mathbb{R}^d, \cdot)} \leq N \|f(t, \cdot)\|_{L_p(\mathbb{R}^d, \cdot)},
$$

where $N = N(d, p, N^*, N_0)$. The lemma is proved. □

Theorem A.5. Let $p \in (1, \infty)$, $w \in A_p(\mathbb{R}^{d+1})$, and $\nu \in \mathbb{R}$. Suppose that Assumption [A.3] holds.

(i) If $\nu_0 \in \mathbb{R}$, $(1 - \Delta)^{\nu_0/2}$ is a bijective mapping from $H^\nu_p((0, T) \times \mathbb{R}^d, w)$ to $H^{\nu_0+\nu_0}((0, T) \times \mathbb{R}^d, w)$.

(ii) If $\nu_1 \geq \nu_0$, then $H^\nu_p((0, T) \times \mathbb{R}^d, w)$ is continuously embedded into $H^{\nu_0}_p((0, T) \times \mathbb{R}^d, w)$. More precisely, there exists a positive constant $N$ such that

$$
\|f\|_{H^\nu_p((0, T) \times \mathbb{R}^d, w)} \leq N \|f\|_{H^{\nu_0}_p((0, T) \times \mathbb{R}^d, w)}, \quad \forall f \in \mathcal{S}(\mathbb{R}^{d+1})..
$$

(iii) If $\nu$ is a nonnegative integer, then

$$
\|f\|_{H^\nu_p((0, T) \times \mathbb{R}^d, w)} \simeq \sum_{|\alpha| \leq \nu} \|D_\xi^\alpha f\|_{L_p((0, T) \times \mathbb{R}^d, w)}.
$$

(iv) The topological dual space of $H^\nu_p((0, T) \times \mathbb{R}^d, w)$ is $H^{-\nu}((0, T) \times \mathbb{R}^d, \bar{w})$, where

$$
\frac{1}{p} + \frac{1}{p'} = 1, \quad \bar{w} := w^{-\frac{1}{p'}} \in A_p(\mathbb{R}^{d+1}).
$$

Proof. (i) Recall the definition of $H^\nu_p((0, T) \times \mathbb{R}^d, w)$ (Definition [A.7] (i)). Then it suffices to show that for all $\nu \geq 0$,

$$
\|f\|_{L_p((0, T) \times \mathbb{R}^d, w)} + \|(-\Delta)^{\nu/2} f\|_{L_p((0, T) \times \mathbb{R}^d, w)} \simeq \|f\|_{H^\nu_p((0, T) \times \mathbb{R}^d, w)}.
$$

(A.3)

We prove this equivalence of the two norms above by using Lemma [A.4]. Let $\nu \geq 0$ and put

$$
m_1(\xi) := \frac{1}{(1 + |\xi|^2)^{\nu/2}}, \quad m_2(\xi) := \frac{|\xi|^{2\nu}}{(1 + |\xi|^2)^{\nu/2}}, \quad m_3(\xi) := \frac{(1 + |\xi|^2)^{\nu/2}}{1 + |\xi|^{2\nu}}
$$

It is easy to check that $m_1$, $m_2$, and $m_3$ satisfy (A.2). Thus, by Lemma [A.4]

$$
\|f\|_{L_p((0, T) \times \mathbb{R}^d, w)} = \|T_m(1 - \Delta)^{\nu/2} f\|_{L_p((0, T) \times \mathbb{R}^d, w)} \leq N \|(1 - \Delta)^{\nu/2} f\|_{L_p((0, T) \times \mathbb{R}^d, w)},
$$

$$
\|(-\Delta)^{\nu/2} f\|_{L_p((0, T) \times \mathbb{R}^d, w)} = \|T_m(1 - \Delta)^{\nu/2} f\|_{L_p((0, T) \times \mathbb{R}^d, w)} \leq N \|(1 - \Delta)^{\nu/2} f\|_{L_p((0, T) \times \mathbb{R}^d, w)},
$$

$$
\|(1 - \Delta)^{\nu/2} f\|_{L_p((0, T) \times \mathbb{R}^d, w)} = \|T_m(1 - (-\Delta)^{\nu/2}) f\|_{L_p((0, T) \times \mathbb{R}^d, w)} \leq N \|f\|_{H^\nu_p((0, T) \times \mathbb{R}^d, w)}.
$$
(ii) The result is an easy consequence of (i). Indeed, define $m_4(\xi) := (1 + |\xi|^2)^{\nu_1/2}(1 + |\xi|^2)^{-\nu_1/2}$. Then $m_4$ satisfies (A.2). Thus by (ii) and Lemma A.4
\[ \|f\|_{L^p_0((0,T) \times \mathbb{R}^d, w)} \approx \|(1 - \Delta)^{\nu_1/2} f\|_{L^p((0,T) \times \mathbb{R}^d, w)} = \|T_{m_4}(1 - \Delta)^{\nu_1/2} f\|_{L^p((0,T) \times \mathbb{R}^d, w)} \leq N\|(1 - \Delta)^{\nu_1/2} f\|_{L^p((0,T) \times \mathbb{R}^d, w)} \approx \|f\|_{\overline{H}^{\nu_1}((0,T) \times \mathbb{R}^d, w)}. \]

(iii) For a multi-index $\alpha$ satisfying $|\alpha| \leq \nu$, we denote
\[ m_5(\xi) := \frac{(i\xi)^{\alpha}}{(1 + |\xi|^2)^{\nu/2}}. \]
It is easy to show that (A.2) holds with $m = m_5$. Thus by (ii) and Lemma A.4
\[ \|D_x \alpha f\|_{L^p((0,T) \times \mathbb{R}^d, w)} = \|T_{m_5}(1 - \Delta)^{\nu/2} f\|_{L^p((0,T) \times \mathbb{R}^d, w)} \leq N\|f\|_{\overline{H}^{\nu}((0,T) \times \mathbb{R}^d, w)}. \]
To prove the other direction, it is sufficient to show
\[ \|(1 - \Delta)^{1/2} f\|_{L^p((0,T) \times \mathbb{R}^d, w)} \leq N\|\|f\|_{L^p((0,T) \times \mathbb{R}^d, w)} + N \sum_{j=1}^d \|D_x^j f\|_{L^p((0,T) \times \mathbb{R}^d, w)} \tag{A.4} \]
due to the mathematical induction. By (A.3) and (ii),
\[ \|(1 - \Delta)^{1/2} f\|_{L^p((0,T) \times \mathbb{R}^d, w)} \leq N\|g\|_{L^p((0,T) \times \mathbb{R}^d, w)} + N\|\Delta g\|_{L^p((0,T) \times \mathbb{R}^d, w)} \leq N\|f\|_{L^p((0,T) \times \mathbb{R}^d, w)} + N \sum_{i=1}^d \|T_{m_i} D_x^i f\|_{L^p((0,T) \times \mathbb{R}^d, w)}, \]
where $g := (1 - \Delta)^{-1/2} f$ and
\[ m_i(\xi) := \frac{i\xi^i}{(1 + |\xi|^2)^{1/2}}, \quad i = 1, 2, \ldots, d. \]
One can check that $m_i$ is a form of $m_5$. Thus (A.2) holds with $m = m_i$. Finally, applying Lemma A.4 again, we obtain (A.3).

(iv) Due to (ii), $(1 - \Delta)^{\nu/2}(1 - \Delta)^{-\nu/2} = (1 - \Delta)^{-\nu/2}(1 - \Delta)^{\nu/2}$ is an identity operator on $\overline{H}^{\nu'}((0,T) \times \mathbb{R}^d)$ for all $q \in (1, \infty)$ and $\nu' \in \mathbb{R}$. Thus the statement can be proved by following the proof of Theorem A.1 (vi). The theorem is proved.

Definition A.6. Let $\overline{H}^{\infty}_c((0,T) \times \mathbb{R}^d)$ (or $H^{\infty}((0,T) \times \mathbb{R}^d)$, resp.) be a set of all functions $f$ on $(0,T) \times \mathbb{R}^d$ defined by
\[ f(t,x) := \sum_{i=1}^n f_i(t)g_i(x), \]
where $f_i \in C_c^{\infty}((0,T))$ and $g_i \in C_c^{\infty}(\mathbb{R}^d) \ (g_i \in S(\mathbb{R}^d))$, resp.) for all $i$.

Theorem A.7. (i) $\overline{H}^{\infty}_c((0,T) \times \mathbb{R}^d)$ is dense in $L^p_p((0,T) \times \mathbb{R}^d, w)$ for all $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^{d+1})$. If additionally Assumption A.3 holds, then $\overline{H}^{\infty}_p((0,T) \times \mathbb{R}^d)$ is dense in $\overline{H}^{\nu'}_p((0,T) \times \mathbb{R}^d, w)$ for all $\nu \in \mathbb{R}, p \in (1, \infty)$, and $w \in A_p(\mathbb{R}^{d+1})$.
(ii) $\overline{H}^{\infty}_c((0,T) \times \mathbb{R}^d)$ is dense in $L^p_q((0,T), w_1; H^{\nu'}_p(\mathbb{R}^d, w_2))$ for all $\nu \in \mathbb{R}, p,q \in (1, \infty), w_1 \in A_q(\mathbb{R})$ and $w_2 \in A_p(\mathbb{R}^d)$.

Proof. Because of the similarity of the proof, we only prove (i). Let $\nu \in \mathbb{R}, p \in (1, \infty)$, and $w \in A_p(\mathbb{R}^{d+1})$. We divide the proof depending on the value of $\nu$.

Case 1. Assume $\nu = 0$. Suppose that the statement is not true. Then by Hahn-Banach theorem (e.g. [43, Theorem 5.19]) and Theorem A.2 (ii), there is a nonzero $h \in L^p_p((0,T) \times \mathbb{R}^d, w)$ such that
\[ \int_0^T \int_{\mathbb{R}^d} h(t,x)f(t,x)dxdt = 0, \quad \forall f \in \overline{H}^{\infty}_c((0,T) \times \mathbb{R}^d). \]
In particular,  
\[ \int_{0}^{t_1} \int_{\mathbb{R}^d} h(t, x)g(x)dxdt = 0, \quad \forall g \in C_c^\infty(\mathbb{R}^d), \quad 0 < t_0 < t_1 < T. \]

By Fubini’s theorem, \( \int_{\mathbb{R}^d} h(t, x)g(x)dx \) is a measurable function with respect to \( t \) and moreover by a basic property of the Lebesgue integral, we have
\[ \int_{\mathbb{R}^d} h(t, x)g(x)dx = 0, \quad \text{(A.5)} \]
for almost all \( t \in [0, T] \). Finally, since (A.5) holds for all \( g \in C_c^\infty(\mathbb{R}^d) \), we conclude that \( h(t, x) = 0 \) for almost all \( (t, x) \in (0, T) \times \mathbb{R}^d \). It is a contradiction since \( h \) is not zero.

**Case 2.** Assume that \( \nu \neq 0 \). Note that for any \( f \in \mathcal{H}_c^\infty((0, T) \times \mathbb{R}^d) \),
\[ (1 - \Delta)^{\nu/2} f \in \mathcal{H}_c^\infty((0, T) \times \mathbb{R}^d). \]

Moreover, it is obvious that
\[ \mathcal{H}_c^\infty((0, T) \times \mathbb{R}^d) \subset \mathcal{H}_c^\infty((0, T) \times \mathbb{R}^d). \]
Thus, by Case 1, \( \mathcal{H}_c^\infty((0, T) \times \mathbb{R}^d) \) is dense in \( L_p((0, T) \times \mathbb{R}^d, w) \). Using Theorem A.5(i), we obtain the result. The theorem is proved. \( \square \)

We finish the appendix by proving the fact in Remark 2.8.

**Lemma A.8.** Let \( -d < \alpha < d(p - 1) \) and denote \( w_{\alpha}(t, x) := (t^2 + |x|^2)^{\alpha/2} \). Then there exists a positive constant \( N(d, \alpha) \) such that for any balls \( B_R(x_0) \subset \mathbb{R}^d \) and almost all \( t \in [0, T] \),
\[ \left( \int_{B_R(x_0)} w_{\alpha}(t, x)dx \right)^{p-1} \left( \int_{B_R(x_0)} w_{\alpha}(t, x)^{-1/p}dx \right)^{p-1} \leq N. \quad \text{(A.6)} \]

**Proof.** **Case 1.** Suppose that \( B_R(x_0) \) satisfies \( |x_0| \geq 3R/2 \). Then,
- for \( x \in B_R(x_0), |x_0| - R \leq |x| \leq |x_0| + R \).
- \( |x_0|/3 \leq |x_0| - R \leq |x_0| + R \leq 5|x_0|/3 \leq 4(|x_0| - R) \).
These imply that for \( t \in \mathbb{R}, \)
\[ \int_{B_R(x_0)} w_{\alpha}(t, x)dx \asymp (t^2 + |x_0|^2)^{\alpha/2}, \quad \forall \alpha \in \mathbb{R}, \]
where the comparability constant depends only on \( d \) and \( \alpha \). Therefore, (A.6) holds.

**Case 2.** Next, we assume that \( B_R(x_0) \) satisfies \( |x_0| < 3R/2 \) and \( t \in \mathbb{R} \setminus \{0\} \). Since \( B_R(x_0) \subset B_{3R}(0) \),
\[ \int_{B_R(x_0)} w_{\alpha}(t, x)dx \leq N(d) \int_{B_{3R}(0)} w_{\alpha}(t, x)dx, \]
and by changing variables \( x = |t|y, \)
\[ \int_{B_{3R}(0)} w_{\alpha}(t, x)dx = N(d)t^\alpha \int_{B_{3R/|t|}(0)} w_{\alpha}(1, y)dy \]
Due to the Lebesgue differentiation theorem, there exists \( R_1 > 0 \) such that if \( 0 < R < R_1 \), then
\[ \int_{B_R(0)} |w_{\alpha}(1, y) - 1|dy < 1. \quad \text{(A.7)} \]

**Case 2-1.** If \( 0 < 3R/|t| < R_1 \), then by (A.7),
\[ \int_{B_{3R/|t|}(0)} w_{\alpha}(1, y)dy < 2, \quad \forall \alpha \in \mathbb{R}. \]
Therefore, (A.6) holds.
Case 2-2. Now, suppose that $3R/|t| \geq R_1$. Note that

$$
\int_{B_{3R/|t|}(0)} w_\alpha(1,y)dy = \frac{N(d)|t|^d}{(3R)^d} \int_0^{3R/|t|} (1 + r^2)^{\alpha/2} r^{d-1}dr.
$$

If $-d < \alpha < 0$, then

$$
\frac{N(d)|t|^d}{(3R)^d} \int_0^{3R/|t|} (1 + r^2)^{\alpha/2} r^{d-1}dr \leq \frac{N(d)|t|^d}{(3R)^d} \min \left( \int_0^{3R/|t|} r^{d-1}dr, \int_0^{3R/|t|} r^{\alpha} r^{d-1}dr \right)
$$

$$
= N(d, \alpha) \min(1, R^\alpha/|t|^\alpha)
$$

(A.8)

If $\alpha \geq 0$, then

$$
\int_{B_{3R/|t|}(0)} w_\alpha(1,y)dy = \frac{N(d)|t|^d}{(3R)^d} \int_0^{3R/|t|} (1 + r^2)^{\alpha/2} r^{d-1}dr
$$

$$
\leq N(d, R_1)(1 + 9R^2/t^2)^{\alpha/2} \leq N(d, \alpha, R_1) \max(1, R^\alpha/|t|^\alpha).
$$

(A.9)

Combining (A.8) and (A.9),

$$
\left( \int_{B_R(x_0)} w_\alpha(t,x)dx \right) \left( \int_{B_R(x_0)} w_{-\alpha/(p-1)}(t,x)dx \right)^{p-1}
$$

$$
\leq N \left( \max(1, R^\alpha/|t|^\alpha) \min(1, R^{-\alpha}/|t|^{-\alpha}) 1_{\alpha \geq 0} + \min(1, R^\alpha/|t|^\alpha) \max(1, R^{-\alpha}/|t|^{-\alpha}) 1_{-d < \alpha < 0} \right)
$$

$$
\leq N(d, p, \alpha, R_1).
$$

The lemma is proved.

Declarations of interest

Declarations of interest: none

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