REGULARITY FOR A FRACTIONAL $p$-LAPLACE EQUATION

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Abstract. In this note we consider regularity theory for a fractional $p$-Laplace operator which arises in the complex interpolation of the Sobolev spaces, the $H^{s,p}$-Laplacian. We obtain the natural analogue to the classical $p$-Laplacian situation, namely $C^{s+\alpha}_{loc}$ regularity for the homogeneous equation.

1. Introduction and main result

In recent years equations involving what we will call the distributional $W^{s,p}$-Laplacian, defined for test functions $\varphi$ as

$$(-\Delta)^s_p u[\varphi] := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{d+sp}} dy \, dx,$$

have received a lot of attention, e.g. [3, 6, 7, 12, 14, 15, 19]. The $W^{s,p}$-Laplacian $(-\Delta)^s_p$ appears when one computes the first variation of certain energies involving the $W^{s,p}$ semi-norm

$$(1.1) \quad [u]_{W^{s,p}(\mathbb{R}^d)} := \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p}{|x-y|^{d+sp}} \, dy \, dx \right)^{\frac{1}{p}},$$

which was introduced by Gagliardo [10] and independently by Sobolev [25] to describe the trace spaces of Sobolev maps.

Regularity theory for equations involving this fractional $p$-Laplace operator is a very challenging open problem and only partial results are known: $C^{0,\alpha}_{loc}$-regularity for suitable right-hand-side data was obtained by Di Castro, Kuusi and Palatucci [6, 7]; A generalization of the Gehring lemma was proven by Kuusi, Mingione and Sire [14, 15];...
A stability theorem similar to the Iwaniec stability result for the \( p \)-Laplacian was established by the first-named author \[21\]. The current state-of-the-art with respect to regularity theory is higher Sobolev-regularity by Brasco and Lindgren \[3\].

Aside from their origins as trace spaces, the fractional Sobolev spaces

\[
W^{s,p}(\mathbb{R}^d) := \{ u \in L^p(\mathbb{R}^d) : [u]_{W^{s,p}(\mathbb{R}^d)} < +\infty \}
\]

also arise in the real interpolation of \( L^p \) and \( \dot{W}^{1,p} \). If one alternatively considers the complex interpolation method, one is naturally led to another kind of fractional Sobolev space \( H^{s,p}(\mathbb{R}^d) \), where taking the place of the differential energy (1.1) one can utilize the semi-norm

\[
[u]_{H^{s,p}(\mathbb{R}^d)} := \left( \int_{\mathbb{R}^d} |D^s u|^p \right)^{\frac{1}{p}}.
\]

Here \( D^s = (\frac{\partial^s}{\partial x_1^s}, \ldots, \frac{\partial^s}{\partial x_d^s}) \) is the fractional gradient for

\[
\frac{\partial^s u}{\partial x_i^s}(x) := c_{d,s} \int_{\mathbb{R}^d} \frac{u(x) - u(y) x_i - y_i}{|x - y|^{d+s}} \frac{1}{|x - y|} dy, \quad i = 1, \ldots, d.
\]

Composition formulae for the fractional gradient have been studied in the classical work \[11\], while more recently they have been considered by a number of authors \[1, 4, 5, 20, 22, 24\]. While it is common in the literature (for example in \[17\]) to see \( H^{s,p}(\mathbb{R}^d) \) equipped with the \( L^p \)-norm of the fractional Laplacian \((-\Delta)^{\frac{s}{2}}\) (see Section 2 for a definition), we here utilize (1.2) because it preserves the structural properties of the spaces for \( s \in (0,1) \) more appropriately. In particular, for \( s = 1 \) we have \( D^1 = D \) (the constant \( c_{d,s} \) tends to zero as \( s \) tends to one), while for \( s \in (0,1) \) the fractional Sobolev spaces defined this way support a fractional Sobolev inequality in the case \( p = 1 \) (see \[23\]). Let us also remark that for \( p = 2 \) these spaces are the same, \( W^{s,2} = H^{s,2} \), but for \( p \neq 2 \) this is not the case.

Returning to the question of a fractional \( p \)-Laplacian, in the context of \( H^{s,p}(\mathbb{R}^d) \) computing the first variation of energies involving the \( H^{s,p} \) semi-norms (1.2) yields an alternative fractional version of a \( p \)-Laplacian, we shall call it the \( H^{s,p} \)-Laplacian

\[
\text{div}_s(|D^s u|^{p-2} D^s u) = \sum_{i=1}^d \frac{\partial^s}{\partial x_i^s}(|D^s u|^{p-2} \frac{\partial^s u}{\partial x_i^s}).
\]
Somewhat surprisingly while the regularity theory for the homogeneous equation of the $W^{s,p}$-Laplacian

$$(-\Delta)^s_p u = 0$$

is far from being understood, the regularity for the $H^{s,p}$-Laplacian

$$(1.3) \quad \text{div}_s(|D^s u|^{p-2} D^s u) = 0$$

actually follows the classical theory, which is the main result we prove in this note:

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^d$ be open, $p \in (2 - \frac{1}{d}, \infty)$ and $s \in (0, 1]$. Suppose $u \in H^{s,p}(\mathbb{R}^d)$ is a distributional solution to (1.3), that is

$$(1.4) \quad \int_{\mathbb{R}^d} |D^s u|^{p-2} D^s u \cdot D^s \varphi = 0 \quad \forall \varphi \in C_0^\infty(\Omega).$$

Then $u \in C^{s+\alpha}_{\text{loc}}(\Omega)$ for some $\alpha > 0$ only depending on $p$.

The key observation for Theorem 1.1 is that $v := I_{1-s} u$, where $I_{1-s}$ denotes the Riesz potential, actually solves an inhomogeneous classical $p$-Laplacian equation with good right-hand side.

**Proposition 1.2.** Let $u$ be as in Theorem 1.1. Then $v := I_{1-s} u$ satisfies

$$- \text{div}(|Dv|^{p-2} Dv) \in L^\infty_{\text{loc}}(\Omega).$$

Therefore, Theorem 1.1 follows from the regularity theory of the classical $p$-Laplacian: By Proposition 1.1, $v$ is a distributional solution to

$$\text{div}(|Dv|^{p-2} Dv) = \mu$$

and $\mu$ is sufficiently integrable whence $v \in C^{1,\alpha}_{\text{loc}}(\Omega)$ [8, 9, 26] (see also the excellent survey paper by Mingione [18]). In particular, one can apply the potential estimates by Kuusi and Mingione [13, Theorem 1.4, Theorem 1.6] to deduce that $Dv \in C^{0,\alpha}_{\text{loc}}(\Omega)$, which implies that $u \in C^{s+\alpha}_{\text{loc}}(\Omega)$.

Let us also remark, that the reduction argument used for Proposition 1.2 extends the class of fractional partial differential equations introduced in [24], which will be treated in a forthcoming work.
2. Proof of Proposition 1.2

With \((-\Delta)^{\frac{s}{2}}\) we denote the fractional Laplacian
\[
(-\Delta)^{\frac{s}{2}} f(x) := \tilde{c}_{d,s} \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x - y|^{d+s}} dy,
\]
and with \(I_s\) its inverse, the Riesz potential. Let \(v := I_{1-s}u\) where \(u\) satisfies (1.4), so that
\[
\int_{\mathbb{R}^d} |Dv|^{p-2} Dv \cdot D^s \varphi = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega).
\]
Now let \(\Omega_1 \Subset \Omega\) be an arbitrary open set compactly contained in \(\Omega\), and let \(\phi\) be a test function supported in \(\Omega_1\). Pick an open set \(\Omega_2\) so that \(\Omega_1 \Subset \Omega_2 \Subset \Omega\) and a cutoff function \(\eta\), supported in \(\Omega\) and constantly one in \(\Omega_2\). Then in particular one can take
\[
\varphi := \eta(-\Delta)^{\frac{1-s}{2}} \phi
\]
as a test function in (2.1) to obtain
\[
\int_{\mathbb{R}^d} |Dv|^{p-2} Dv \cdot D^s (\eta(-\Delta)^{\frac{1-s}{2}} \phi) = 0.
\]
That is,
\[
\int_{\mathbb{R}^d} |Dv|^{p-2} Dv \cdot D\phi = \int_{\mathbb{R}^d} |Dv|^{p-2} Dv \cdot D^s (\eta^c(-\Delta)^{\frac{1-s}{2}} \phi).
\]
where \(\eta^c := (1 - \eta)\). We set
\[
T(\phi) := D^s (\eta^c(-\Delta)^{\frac{1-s}{2}} \phi)
\]
Now we show that by the disjoint support of \(\eta^c\) and \(\phi\) we have
\[(2.2) \quad \|T(\phi)\|_{L^p(\mathbb{R}^d)} \leq C_{\Omega_1,\Omega_2, d, s, p} \|\phi\|_{L^1(\mathbb{R}^d)}.
\]
Once we have this, the claim is proven as Hölder’s inequality and realizing the \(L^\infty\) norm via duality implies
\[- \text{div}(|Dv|^{p-2} Dv) \in L^\infty_{\text{loc}}(\Omega).
\]
To see (2.2), we use the disjoint support arguments as in [2, Lemma A.1] [16, Lemma 3.6.]: First we see that since \(\eta^c(x)\phi(x) \equiv 0\),
\[
T(\phi) = \tilde{c}_{d,1-s} D^s \int_{\mathbb{R}^d} \frac{\eta^c(x)\phi(y)}{|x - y|^{N+1-s}} dy.
\]
Now taking a cutoff-function \(\zeta\) supported in \(\Omega_2\), \(\zeta \equiv 1\) on \(\Omega_1\) we have
\[
T(\phi) = \tilde{c}_{d,1-s} D^s \int_{\mathbb{R}^d} \frac{\eta^c(x)\zeta(y)\phi(y)}{|x - y|^{N+1-s}} dy = \tilde{c}_{d,1-s} \int_{\mathbb{R}^d} k(x, y) \phi(y) dy,
\]
where
\[ k(x, y) := D_x^s \kappa(x, y) := D_x^s \frac{\eta^c(x) \zeta(y)}{|x - y|^{N+1-s}}. \]
The positive distance between the supports of \( \eta^c \) and \( \zeta \) implies that these kernels \( k, \kappa \) are a smooth, bounded, integrable (both, in \( x \) and in \( y \)), and thus by a Young-type convolution argument we obtain (2.2). One can also argue by interpolation,
\[ \left\| \int_{\mathbb{R}^d} \kappa(x, y) \phi(y) \right\|_{L^p(\mathbb{R}^d)} \leq \| \phi \|_{L^1(\mathbb{R}^d)}, \]
as well as
\[ \left\| \int_{\mathbb{R}^d} D_x^s \kappa(x, y) \phi(y) \, dy \right\|_{L^p(\mathbb{R}^d)} \leq \| \phi \|_{L^1(\mathbb{R}^d)}. \]
Interpolating this implies the desired result that
\[ \left\| \int_{\mathbb{R}^d} D_x^s \kappa(x, y) \phi(y) \, dy \right\|_{L^p(\mathbb{R}^d)} \leq \| \phi \|_{L^1(\mathbb{R}^d)}. \]
Thus (2.2) is established and the proof of Proposition 1.2 is finished. \( \square \)

References
1. P. Biler, C. Imbert, and G. Karch, The nonlocal porous medium equation: Barenblatt profiles and other weak solutions, Arch. Ration. Mech. Anal. 215 (2015), no. 2, 497–529. MR 3294409
2. S. Blatt, Ph. Reiter, and A. Schikorra, Harmonic analysis meets critical knots. Critical points of the M"obius energy are smooth, Trans. Amer. Math. Soc. 368 (2016), no. 9, 6391–6438. MR 3461038
3. L. Brasco and E. Lindgren, Higher sobolev regularity for the fractional p-laplace equation in the superquadratic case, Adv.Math. (2015).
4. L. Caffarelli, F. Soria, and J.-L. Vázquez, Regularity of solutions of the fractional porous medium flow, J. Eur. Math. Soc. (JEMS) 15 (2013), no. 5, 1701–1746. MR 3082241
5. L. Caffarelli and J. L. Vazquez, Nonlinear porous medium flow with fractional potential pressure, Arch. Ration. Mech. Anal. 202 (2011), no. 2, 537–565. MR 2847534
6. A. Di Castro, T. Kuusi, and G. Palatucci, Local behaviour of fractional p-minimizers, preprint (2014).
7. , Nonlocal harnack inequalities, J. Funct. Anal. 267 (2014), 1807–1836.
8. E. DiBenedetto, \( C^{1+\alpha} \) local regularity of weak solutions of degenerate elliptic equations, Nonlinear Anal. 7 (1983), no. 8, 827–850.
9. L. C. Evans, A new proof of local \( C^{1,\alpha} \) regularity for solutions of certain degenerate elliptic p.d.e, J. Differential Equations 45 (1982), no. 3, 356–373.
10. E. Gagliardo, *Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili*, Rend. Sem. Mat. Univ. Padova 27 (1957), 284–305. MR 0102739 (21 #1525)

11. J. Horváth, *On some composition formulas*, Proc. Amer. Math. Soc. 10 (1959), 433–437. MR 0107788

12. J. Korvenpää, T. Kuusi, and G. Palatucci, *The obstacle problem for nonlinear integro-differential operators*, Calc. Var. Partial Differential Equations 55 (2016), no. 3, Art. 63, 29. MR 3503212

13. T. Kuusi and G. Mingione, *Universal potential estimates*, J. Funct. Anal. 262 (2012), no. 10, 4205–4269.

14. T. Kuusi, G. Mingione, and Y. Sire, *A fractional Gehring lemma, with applications to nonlocal equations*, Rend. Lincei - Mat. Appl. 25 (2014), 345–358.

15. ____, *Nonlocal self-improving properties*, Analysis & PDE 8 (2015), 57–114.

16. L. Martinazzi, A. Maalaoui, and A. Schikorra, *Blow-up behaviour of a fractional adams-moser-trudinger type inequality in odd dimension*, Comm. PDE (accepted) (2015).

17. V. Maz’ya, *Sobolev spaces with applications to elliptic partial differential equations*, augmented ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 342, Springer, Heidelberg, 2011. MR 2777530

18. G. Mingione, *Recent advances in nonlinear potential theory*, pp. 277–292, Springer International Publishing, Cham, 2014.

19. A. Schikorra, *Integro-differential harmonic maps into spheres*, Comm. Partial Differential Equations 40 (2015), no. 3, 506–539. MR 3285243

20. ____, *Lp-gradient harmonic maps into spheres and SO(N)*, Differential Integral Equations 28 (2015), no. 3-4, 383–408. MR 3306569

21. ____, *Nonlinear commutators for the fractional p-laplacian and applications*, Mathematische Annalen (2015), 1–26.

22. ____, *ε-regularity for systems involving non-local, antisymmetric operators*, Calc. Var. Partial Differential Equations 54 (2015), no. 4, 3531–3570. MR 3426086

23. A. Schikorra, D. Spector, and J. Van Schaftingen, *An L1-type estimate for Riesz potentials*, Rev. Mat. Iberoamer. (accepted) (2014).

24. T.-T. Shieh and D. Spector, *On a new class of fractional partial differential equations*, Adv. Calc. Var. 8 (2015), no. 4, 321–336. MR 3403430

25. L. N. Slobodeckii, *S. L. Sobolev’s spaces of fractional order and their application to boundary problems for partial differential equations*, Dokl. Akad. Nauk SSSR (N.S.) 118 (1958), 243–246. MR 0106325

26. N. N. Ural’ceva, *Degenerate quasilinear elliptic systems*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 7 (1968), 184–222.

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