ON THE MELLIN TRANSFORM OF A $\mathcal{D}$–MODULE.

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Abstract

Let $k$ be an algebraically closed field of characteristic zero. Given a holonomic $k[z, z^{-1}][\partial_z]$-module $M$, following [21] one can consider its Mellin transform, which is a difference system on the affine line over $k$. In this note we prove a stationary phase formula, which shows that its formal behavior at infinity is determined by the local germs defined by $M$ at its singular points.

1. Introduction.

The Fourier transform for $\mathcal{D}$-modules has been extensively studied, the most precise results available are in dimension one, that is, for holonomic modules over $\mathbb{C}[x][\partial_x]$, see [22]. In the analogous situation for $\ell$-adic sheaves, G. Laumon defined in [16] so-called local Fourier transformations, which are related to the global $\ell$-adic Fourier-Deligne transform via a stationary phase formula. These local transformations allowed him to give a product formula for local constants, a construction of the Artin representation in equal characteristic and a simplification of Deligne’s proof of the Weil conjecture.

Having Laumon’s work as a guideline, local Fourier transforms have been defined in the $\mathcal{D}$-module setting ([6, 12, 23, 3]), where they also satisfy a stationary phase formula (see [12]), albeit only at the formal level. Beyond this, Stokes structures have to be considered and the study becomes much more complicated, see [14, 7, 24, 8].

In [21], F. Loeser and C. Sabbah defined the Mellin transform of a $\mathcal{D}$-module on an algebraic torus (see also [17]), and they used it to prove a product formula for the determinant of the Aomoto complex ([21, Théorème 2.3.1]).

In [13], A. Graham-Squire defined local Mellin transforms for formal germs of meromorphic connections in one variable, and computed them explicitly.

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They might be regarded as local analogues of the global Mellin transform of Loeser and Sabbah. In this note we prove that a stationary phase formula holds also in this case, in particular we show that the local (formal) behavior at infinity of the global Mellin transform of a holonomic $k[z, z^{-1}][\partial_z]$-module $\mathcal{M}$ is determined by the (formal) connection germs defined by $\mathcal{M}$ at its singular points, and no global information is required. Our definition of the local Mellin transforms is microlocal, in the spirit of [12], and differs from that in [13]. This allows to remove the assumptions made in loc. cit. about non-existence of horizontal sections.

Through this note, $k$ will denote an algebraically closed field of characteristic zero. If $x$ is a coordinate, we denote $K_x = k[[x]][x^{-1}]$ the ring of Laurent series with coefficients in $k$.

2. Differential and difference modules.

We recall a few well-known notions and results from the local theory of differential and difference modules, we refer to [22] and [26] for more details and proofs:

**Definition 1.** A differential module $(V, \nabla)$ is a finite-dimensional vector space $V$ over $K_x$ endowed with a $k$-linear operator $\nabla : V \rightarrow V$ such that, for all $f \in K_x$ and $v \in V$ one has

$$\nabla(f \cdot v) = \frac{df}{dx} \cdot v + f \cdot \nabla(v).$$

Using the cyclic vector lemma, one attaches to a differential module $V$ its Newton polygon, the slopes of its non-vertical sides are called the formal slopes of $V$. One has a canonical, functorial decomposition

$$V = \bigoplus_{\lambda} V^\lambda,$$

where $\lambda$ runs over the set of slopes of $V$ and $V^\lambda$ is a differential module which has only slope $\lambda$ (see for example [22, Chapter III]).

If $V$ is a differential module, we denote by $irr(V)$ its irregularity, defined as the height of its Newton polygon ([22 Chapitre IV, (4.4)]) and by $\mu(V)$

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3But then, while in [13] the local Mellin transforms were equivalences of categories, here they are not.
the dimension of its space of vanishing cycles ([loc. cit., §4]). By [loc. cit., Chapitre IV, Corollaire 4.10], we have \( \mu(V) = \dim(V) + \text{irr}(V) \).

**Definition 2.** Let \( \theta \) be a coordinate, denote \( \phi : K_\theta \to K_\theta \) the automorphism given by \( \phi(a(\theta)) = a(\theta/1 + \theta) \). A difference module \((V, \Phi)\) is a finite-dimensional vector space \( V \) over \( K_\theta \) endowed with a \( k \)-linear invertible operator \( \Phi : V \to V \) such that for all \( f \in K_\theta \) and \( v \in V \) one has

\[
\Phi(f(\theta) \cdot v) = f(\phi(\theta)) \cdot \Phi(v).
\]

Taking as morphisms those \( k \)-linear maps which commute with the difference operators, difference modules over \( K_\theta \) form an abelian category.

We briefly recall the construction of the Newton polygon attached to a difference operator (see e.g. [26]): Let \( K_\theta(\Phi) \) denote the skew-polynomial ring determined by the relations \( \Phi \cdot f = \phi(f) \cdot \Phi \) for \( f \in K_\theta \). With respect to the degree function, \( K_\theta(\Phi) \) is an euclidean ring and every finitely generated (left or right) \( K_\theta(\Phi) \)-module is a direct sum of cyclic modules. The datum of a difference module is equivalent to that of a \( K_\theta(\Phi) \)-module, of finite dimension as a \( K_\theta \)-vector space and such that the action of \( \Phi \) is invertible.

Given \( P = \sum_{i=0}^{m} a_i \Phi^i \in K_\theta(\Phi) \), the Newton polygon \( \mathcal{N}(P) \) of \( P \) is the convex envelope in \( \mathbb{R}^2 \) of the union of the half–lines \( \{(x, y) \in \mathbb{R}^2 \mid x = i, \ y \geq v(a_i)\} \), where \( v : K_\theta \to \mathbb{Z} \cup \{\infty\} \) is the valuation defined by \( v(\sum_i a_i \theta^i) = \min\{j \mid a_j \neq 0\} \), \( v(0) = \infty \).

It is proved in [26, pg. 257, Remark 3] that, up to a vertical translation corresponding to multiplication by a power of \( \theta \), \( \mathcal{N}(P) \) depends only on the difference module \( D_P = K_\theta(\Phi) / K_\theta(\Phi) \cdot P \). In particular, it follows easily from the definitions that the width of \( \mathcal{N}(P) \) coincides with the dimension of \( D_P \) as a \( K_\theta \)-vector space.

\[\text{In [26, the condition defining the half-lines is } y \leq v(a_i) \text{ which, in view of the claimed properties of slopes, seems to be a misprint. Notice also that Praagman’s polygon is not identical to the one considered in [9]. They differ by a reflexion.}\]
In the sequel, the polygon $\mathcal{N}(P)$ will be always considered up to a vertical translation, the slopes of its non-vertical sides will be called the slopes of $D_P$. 

In fact, if $V$ is a difference module over $K_\theta$, a version of the cyclic vector lemma allows to attach to $V$ an operator $P \in K_\theta \langle \Phi \rangle$ such that $V \cong D_P$ as difference modules. The operator $P$ is not unique, but $\mathcal{N}(P)$ is independent of $P$, see [4, section 2].

Given $q \geq 1$, set $L_q = k((\theta^{1/q}))$. The automorphism $\phi(\theta) = \frac{\theta}{1+\theta}$ of $K_\theta$ has a unique extension $\phi_q$ to $k((\theta^{1/q}))$ ([26, §1]), then one can define difference modules over $L_q$ as done for $K_\theta$, the definition of the Newton polygon of an operator extends as well. Given $g = \sum_{i \geq r} b_i \theta^{i/q} \in L_q$ with $b_r \neq 0$, denote $D_{g,q}$ the $L_q$-difference module $(L_q, g \phi_q)$.

Given $m \geq 0$, put $T_m = (K_\theta^m, (Id + \theta N_m)\phi)$, where $N_m$ is the nilpotent Jordan block of size $m$. These are unipotent objects in the category of difference modules over $K_\theta$. The classification theorem for formal difference modules is the following (see [26, Theorem 8], [9, Theorem 3.3])

**Theorem 1.** Let $V$ be a difference module over $K_\theta$ of rank $m$. Then, there is a finite cyclic extension $K_\theta \subset L_q$ and an isomorphism of $L_q$-difference modules

$$V \otimes L_q \cong \bigoplus_{i \in I} (D_{g_i,q_i} \otimes_{K_\theta} T_{m_i}) \otimes_{L_{q_i}} L_q,$$

where $I$ is a finite set, $m_i, q_i > 0$ are positive integers, $q_i | q$, the $D_{g_i,q_i}$ are simple difference modules and $\sum m_i = m$. Also, $g_i \in k((\theta^{1/q}))$ are of the form $g_i = \sum_{h=0}^{q_i} a_{i,h} \theta^{\lambda_i + \frac{h}{q_i}}$, where $\lambda_i \in (1/q_i)\mathbb{Z}$ is the only slope of $D_{g_i,q_i}$.

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5 In [13] and [26] the terminology differs. The orders considered by Graham-Squire are minus the slopes in Praagman’s article.

6 In loci cit. it is assumed that $k$ is the field of complex numbers, but no transcendental arguments are needed to prove this theorem.
and \( a_{i,0} \in k - \{0\} \). In this decomposition, the rational numbers \( \{\lambda_i\}_{i \in I} \) are the formal slopes of \( V \), the integers \( m_i, q_i \) are uniquely determined, and the \( g_i \in k((\theta^{1/q})) \) are uniquely determined up to addition of an integer multiple of \( a_{i,0}/q_i \) to \( a_{i,q_i} \in k \).

It follows from the theorem that we have:

**Corollary 1.** If \( V, W \) are difference modules with no common slope, every morphism of difference modules \( V \rightarrow W \) is zero.

To prove a stationary phase formula for the Mellin transform, we will need some more information on the formal structure of a difference module than the one provided by formal slopes\(^7\). If \( \sigma \) is a non-vertical side of \( \mathcal{N}(P) \), consider the polynomial \( p_\sigma(t) = \sum_i a_i t^i \in k[t] \), where the sum is over those indexes \( i \) corresponding to monomials in \( P \) giving a point on the side \( \sigma \). Then, the roots of \( p_\sigma \) are formal invariants of \( D_P \), see e.g. [4, section 2.3], and we can make the following definition:

**Definition 3.** Let \( V \) be a difference module over \( K_\theta \), choose \( P \in K_\Phi(\Phi) \) with \( V \cong D_P \) as difference modules. We define a finite set \( \text{Horz}(V) \) of horizontal zeros as follows: If the Newton polygon \( \mathcal{N}(P) \) has no horizontal side, we put \( \text{Horz}(V) = \emptyset \). If it has a horizontal side \( \sigma_{\text{hor}} \), we put \( \text{Horz}(V) = \{ \text{zeros of } p_{\sigma_{\text{hor}}} \} \) (regardless of multiplicities). Notice that for an extension \( K_\theta \subset L_q = k((\theta^{1/q})) \), we have \( \text{Horz}(V) = \text{Horz}(V \otimes L_q) \).

**Lemma 1.** Let \( V, W \) be difference module over \( K_\theta \), both of them purely of slope zero, with no common horizontal zero. Then every morphism \( V \rightarrow W \) is the zero map.

**Proof.** Taking an extension \( K_\theta \subset L_q \) we can assume both \( V \) and \( W \) decompose as in Theorem 1. Given a summand \( D_{g,q} \otimes T_m \), a computation as in [26] shows that

\[
D_{g,q} \otimes T_m \cong \frac{L_q(\Phi)}{(\theta^{-1} \Phi - g)^m}
\]

\(^7\)This is contrast with the situation for the stationary phase formula for the Fourier transform. In that case, formal slopes are enough, this difference is ultimately due to the different behavior of slopes with respect to tensor product in the differential and in the difference case.
where \( g = a_0 + a_1 \theta^{1/q} + \cdots + a_q \theta \) and \( a_i \in k \) for \( 1 \leq i \leq q \). A direct calculation shows that the only horizontal zero of this difference module is \( a_0 \). Then, by the classification theorem the lemma follows. □

For later use, we recall a few definitions and results concerning differential and difference modules over tori and affine lines. We denote by \( k[z, z^{-1}]\langle \partial_z \rangle \) the localized Weyl algebra (where \([\partial_z, z] = 1\)) and by \( k[\eta] \langle \Phi, \Phi^{-1} \rangle \) the algebra of invertible difference operators on the affine \( k \)-line (where \([\Phi, \eta] = \Phi\)).

i) If \( \mathcal{M} \) is a holonomic \( k[z, z^{-1}]\langle \partial_z \rangle \)-module and \( s \in k \), we set \( z_s = z - s \) and we denote \( \mathcal{M}_s \) the \( K_{z_s} \)-differential module \( \mathcal{M} \otimes_k K_{z_s} \), where the connection is given by \( \nabla(m \otimes 1) = \partial_z m \otimes 1 \). We denote \( \mathcal{M}_\infty \) the \( K_y \)-differential module \( \mathcal{M} \otimes_k K_y \) where \( z \mapsto y \) and the connection is given by \( \nabla(m \otimes 1) = -z^2 \partial_z m \otimes 1 \).

ii) The finite set of points \( s \in k \setminus \{0\} \) such that \( \mathcal{M} \otimes_k k[[z_s]] \) is not free of finite type over \( k[[z_s]] \) will be denoted \( S(\mathcal{M}) \) (the singular set of \( \mathcal{M} \), see e.g. [27, III. Proposition 1.1.5]).

iii) The global Newton polygon attached by J. P. Ramis and B. Malgrange to an operator \( P \in k[z, z^{-1}]\langle z\partial_z \rangle \) is defined as follows:

Write \( P \) as a finite sum \( P = \sum \alpha_r (z \partial_z) z^r \) where \( \alpha_r \in k[X] \) for all \( r \in \mathbb{Z} \) and, for each \( \alpha_r \neq 0 \), consider the half-line \( \{(u, v) \in \mathbb{R}^2 \mid u \leq \deg \alpha_r, v = r\} \). The Newton polygon \( \mathcal{N}(P) \) of \( P \) is the convex envelope of these half-lines. It depends only on the quotient module \( \mathcal{M} = k[z, z^{-1}]\langle z\partial_z \rangle / k[z, z^{-1}]\langle z\partial_z \rangle P \) and, in particular, it follows from the definitions (see [22, V.1]) that the height \( h(P) \) of \( \mathcal{N}(P) \) equals

\[
\text{irr}(\mathcal{M}_0) + \text{irr}(\mathcal{M}_\infty) + \sum_{s \in S(\mathcal{M})} \mu(\mathcal{M}_s).
\]

The definition in [22, V.1] looks slightly different to the one given here, but both give the same polygon. In fact, in loc. cit. the case considered is that of a module over \( k[z]\langle \partial_z \rangle \). Since we have inverted \( z \), we consider the global Newton polygon only up to horizontal translation.
As shown in loc. cit., in fact a Newton polygon can be attached to any holonomic \(k[z, z^{-1}]⟨z∂_z⟩\)-module.

iv) If \(\mathfrak{M}\) is a \(k[η]\langle Φ, Φ^{-1}\rangle\)-module, we define its germ at infinity as the \(K_0\)-difference module \(\mathfrak{M}\_∞ = \mathfrak{M} \otimes_{k[η]} K_0\), where \(η \mapsto θ^{-1}\) and where the difference operator is given by

\[
n \otimes a(θ) \longmapsto Φ \cdot (n \otimes a\left(\frac{θ}{1+θ}\right)) .
\]

Global Mellin transform: Denote \(\mathfrak{M} : k[z, z^{-1}]⟨z∂_z⟩ \longrightarrow k[η]\langle Φ, Φ^{-1}\rangle\) the morphism of \(k\)-algebras defined by \(\mathfrak{M}(z∂_z) = -η\), \(\mathfrak{M}(z) = Φ\). Following \([21]\), if \(\mathfrak{M}\) is a \(k[z, z^{-1}]⟨z∂_z⟩\)-module, its Mellin transform is defined as the \(k[η]\langle Φ, Φ^{-1}\rangle\)-module

\[
\mathfrak{M}(\mathfrak{M}) = \mathfrak{M} \otimes_{k[z, z^{-1}]⟨z∂_z⟩} k[η]\langle Φ, Φ^{-1}\rangle .
\]

Remark. Consider the example \(\mathfrak{M} = k[z, z^{-1}]⟨z∂_z⟩/(z - s)^m\) where \(s ∈ k\) and \(m ≥ 1\). We claim that \(\mathfrak{M}(\mathfrak{M})\_∞ = 0\). By induction on \(m\) and exactness of the Mellin transform functor, we can assume \(m = 1\). Then, the Mellin transform of \(\mathfrak{M}\) is

\[
\frac{k[η]\langle Φ, Φ^{-1}\rangle}{k[η]\langle Φ, Φ^{-1}\rangle(Φ - s)} .
\]

We have the relation \(Φη - ηΦ = Φ\) and, since in the quotient we have \(η\bar{Φ} - sη = 0\), it follows that \((\bar{Φ} - s)η = \bar{Φ}\). Inverting \(η\) we get \(\bar{Φ} η^{-1} = 0\), thus \(\mathfrak{M}(\mathfrak{M})\_∞ = 0\). Therefore, if we want to analyze the germ at infinity of the Mellin transform of a \(D\)-module on a torus, modules with punctual support are not relevant. More precisely, given a holonomic \(k[z, z^{-1}]⟨z∂_z⟩\)-module \(\mathfrak{M}\), there is an exact sequence

\[
0 \longrightarrow P \longrightarrow \mathfrak{M} \longrightarrow k[z, z^{-1}]⟨z∂_z⟩ \longrightarrow 0
\]
where $P$ is a $k[z,z^{-1}]\langle z\partial_z \rangle$-module with punctual support (see e.g. [27 III. Proposition 1.1.4]) and $Q \in k[z,z^{-1}]\langle z\partial_z \rangle$. Then we have
\[ \mathcal{M}(M) = \mathcal{M}(k[z,z^{-1}]\langle z\partial_z \rangle) / k[z,z^{-1}]\langle z\partial_z \rangle \cdot Q. \]

3. Microdifference operators and local Mellin transforms.

We consider the set of formal sums
\[ \mathcal{M} = \left\{ \sum_{i \geq r} a_i(u)\eta^{-i} \mid r \in \mathbb{Z} \text{ and } a_i(u) \in k((u)) \right\}. \]

The degree of an operator $P \in \mathcal{M}$ is $\deg(P) = \max\{i \mid a_i \neq 0\}$. If $\delta$ is a $k$-derivation of $k((u))$, we consider in $\mathcal{M}$ the multiplication
\[ P \circ_\delta Q = \sum_{\alpha \geq 0} \frac{1}{\alpha!} \partial^\alpha_P \cdot \delta^\alpha Q \]
where, on the right-hand side, it is understood that the product is first performed in $k((u))((\eta^{-1}))$, and then the result is reordered so as to obtain an element of $\mathcal{M}$. It is easy to see that the set $\mathcal{M}$, endowed with the obvious addition and the multiplication $\circ_\delta$, is a unitary ring. We will only consider the derivations $\delta_s = -(u+s)\partial_u$, where $s \in k$, and $\delta_\infty = u\partial_u$. In case $s = 0$, the ring obtained is exactly the one defined by A. Duval in [10].

**Definition 4.** We denote $\mathcal{M}^{(s,\infty)}$ the ring which is $\mathcal{M}$ as a set, and where the product is $\circ_{\delta_s}$, and we let $\xi_s : k((\theta)) \to \mathcal{M}^{(s,\infty)}$ be the ring homomorphism defined by $\theta \mapsto -\eta^{-1}$. Also, we denote $\mathcal{M}^{(\infty,\infty)}$ the set $\mathcal{M}$ endowed with the product $\circ_{\delta_\infty}$ and $\xi_\infty : k((\theta)) \to \mathcal{M}^{(\infty,\infty)}$ the ring homomorphism defined by $\theta \mapsto \eta^{-1}$.

The proof of the following theorem is completely analogous to the one for the usual formal microdifferential operators (see [5] or [10, Théorème 1.c.2]).

**Theorem 2.** (division) For $* \in k \cup \{\infty\}$, let $P = \sum_{i \leq d} a_i(u)\eta^i \in \mathcal{M}^{(*,\infty)}$ be an operator of degree $d$. Assume $a_d(u)$ has $u$-adic valuation $v \in \mathbb{Z}$. Then, for all $S \in \mathcal{M}^{(*,\infty)}$ there are unique $Q, R \in \mathcal{M}^{(*,\infty)}$ such that
\[ S = Q \circ_{\delta_s} P + R \quad \text{with} \quad \deg(R) \leq \deg(S), \quad \deg(Q) \leq \deg(S) - \deg(P) \]
and

\[ R = \sum_{i=0}^{n-1} w^i R_j , \quad R_j \in \mathcal{M}(s,\infty) \cap k[[\eta^{-1}]]. \]

**Definition 5.** Let \( \mathcal{M} = \frac{k((x)) \langle x \partial_x \rangle}{k((x)) \langle x \partial_x \rangle \cdot P(x, x \partial_x)} \) be a holonomic module with \( \mathcal{M} = \mathcal{M}[x^{-1}] \) (i.e., a germ of formal meromorphic connection). For \( s \in k \), put

\[ \mathcal{M}^{(s,\infty)}(\mathcal{M}) = \mathcal{M} \otimes_{k((x)) \langle x \partial_x \rangle} \mathcal{M}^{(s,\infty)}, \]

where the tensor product is given by the ring homomorphism \( \varphi_s : k((x)) \langle x \partial_x \rangle \rightarrow \mathcal{M}^{(s,\infty)} \) defined by \( x \mapsto u \) and \( (x + s) \partial_x \mapsto -\eta \). If we put \( \theta = \eta^{-1} \), then \( \mathcal{M}^{(s,\infty)}(\mathcal{M}) \) is a \( K_{\theta} \)-vector space where scalar multiplication is given by \( f(\theta) \cdot (m \otimes P(u, \eta)) = m \otimes (f(\theta) \circ_{\partial_x} P(u, \eta)) \). It is easy to check that the automorphism \( \mathcal{M}^{(s,\infty)}(\mathcal{M}) \rightarrow \mathcal{M}^{(s,\infty)}(\mathcal{M}) \) given by \( m \otimes P(u, \eta) \mapsto m \otimes (u + s) \cdot P(u, \eta) \) endows \( \mathcal{M}^{(s,\infty)}(\mathcal{M}) \) with a structure of \( K_{\theta} \)-difference module.

Put also \( \mathcal{M}^{(\infty,\infty)}(\mathcal{M}) = \mathcal{M} \otimes_{k((x)) \langle x \partial_x \rangle} \mathcal{M}^{(\infty,\infty)} \), where the tensor product is given by the ring homomorphism \( \varphi_{\infty} : k((x)) \langle x \partial_x \rangle \rightarrow \mathcal{M}^{(\infty,\infty)} \) defined by \( x \mapsto u \) and \( x \partial_x \mapsto \eta \). Then, as in the previous case, \( \mathcal{M}^{(\infty,\infty)}(\mathcal{M}) \) is a \( K_{\theta} \)-vector space, endowed with the difference operator given by left multiplication by \( u^{-1} \).

**Remark.** If the action of \( \partial_x \) is invertible in \( \mathcal{M} \), one says that \( \mathcal{M} \) is of microlocal type, then it is isomorphic to its microlocalization (in the sense of [25, 11.8]). From this fact one can derive that, in this case, \( \mathcal{M}^{(\infty,\infty)}(\mathcal{M}) \) is isomorphic, as a difference module, to the local Mellin transform defined by Graham-Squire in [13]. For the other microlocalizations, analogous statements hold as well.

To relate global and local Mellin transforms, we will need the following theorem, proved in [15, Theorem 2.4.10]:

**Theorem 3.** *(Katz’s extensions):* Let \((V, \nabla)\) be a differential module over \( K_z \). Then, there exists a holonomic \( k[z] \langle \partial_z \rangle \)-module \( \mathcal{K}(V) \) with no singularity on \( A^1_k - \{0\} \), regular at infinity and such that \( \mathcal{K}(V)_0 \cong (V, \nabla) \).

If \( s \in k \), \( \tau : A^1_k \rightarrow A^1_k \) is the translation \( t \mapsto t-s \) and \( i : G^1_k = A^1_k - \{0\} \hookrightarrow A^1_k \) is the inclusion map, we put \( \mathcal{K}_s(V) = i^* \tau^s(\mathcal{K}(V)) \). If \( j : G^1_k \rightarrow G^1_k \) is the inversion \( t \mapsto t^{-1} \), we put \( \mathcal{K}_\infty(V) = j^*(\mathcal{K}_0(V)) \).

**Lemma 2.**

i) The slopes of \( \mathcal{M}(\mathcal{K}_0(V))_\infty \) are strictly negative.
ii) For $s \in k - \{0\}$, $\mathcal{M}(K_s(V))_\infty$ has only slope 0 and $\text{Horz}(\mathcal{M}(K_s(V))_\infty) \subset \{s\}$.

iii) The slopes of $\mathcal{M}(K_\infty(V))_\infty$ are strictly positive.

Proof. It follows from the definitions that, for any holonomic $k[z, z^{-1}](z\partial_z)$-module $\mathbb{M} = k[z, z^{-1}](z\partial_z)/k[z, z^{-1}](z\partial_z) \cdot P$, the Newton polygon of $\mathcal{M}(\mathbb{M})_\infty$ equals the polygon obtained from the Newton polygon of $\mathbb{M}$ by applying a rotation of ninety degrees in the clockwise direction. The Newton polygon of $K_0(V)$ has only sides of positive slope (corresponding to the slopes of $V$) and no vertical side (since it has no singular point in $A^1_k - \{0\}$), and so i) follows. The claims about slopes in ii) and iii) follow in the same way. In case ii), since the only singular point of $K_s(V)$ is $s \in k$ and $K_s(V)$ is of exponential type in the sense of [22, Chapitre XII], up to modules with punctual support we have $K_s(V) = k[z, z^{-1}](z\partial_z)/k[z, z^{-1}](z\partial_z) \cdot P$, where

$$P = (z - s)^{m} \partial_z^{n} + \ldots, \quad m = \deg_z(P) \text{ and } n = \deg_{\partial_z}(P).$$

Applying the global Mellin transform, localizing at infinity and computing the Newton polygon, one obtains $\text{Horz}(\mathcal{M}(K_s(V))_\infty) \subset \{s\}$. □

Lemma 3. Let $\mathbb{M}$ be a holonomic $k[z, z^{-1}](z\partial_z)$-module. For $\star \in k \cup \{\infty\}$, there is a functorial epimorphism of $K_\theta$-difference modules

$$\Xi_\star : \mathcal{M}(\mathbb{M})_\infty \longrightarrow \mathcal{M}^{(\star, \infty)}(\mathbb{M}_\star).$$

Proof. We consider the case $\star = 0$, the other cases are similar. We can assume that $\mathbb{M}$ is given by a single operator, i.e. that we have a presentation

$$\mathbb{M} = \frac{k[x, x^{-1}](x\partial_x)}{k[x, x^{-1}](x\partial_x) \cdot P(x, x\partial_x)}.$$ 

Using the definitions, we have

$$\mathcal{M}(\mathbb{M})_\infty = \frac{k[\Phi, \Phi^{-1}](\eta)}{k[\Phi, \Phi^{-1}](\eta) \cdot P(\Phi, -\eta)} \otimes_{k[\eta]} k((\theta)),$$

$$\mathcal{M}^{(0, \infty)}(\mathbb{M}_0) = \frac{k((x))(x\partial_x)}{k((x))(x\partial_x) \cdot P(x, x\partial_x)} \otimes_{x\neq 0} \mathcal{M}^{(0, \infty)} = \frac{\mathcal{M}^{(0, \infty)}}{\mathcal{M}^{(0, \infty)} \cdot P(u, -\eta)},$$

and the desired map $\Xi_0$ is given by $[q(\eta, \Phi)] \otimes a(\theta) \mapsto [q(\eta, u) \circ_{\delta_0} a(\eta^{-1})]$. Surjectivity follows from the division theorem. □
Lemma 4. Let \((V, \nabla)\) be a differential module over \(K_x\).

i) All slopes of \(M^{(0,\infty)}(V)\) are strictly negative and \(\dim_{K_\theta} M^{(0,\infty)}(V) = \text{irr}(V)\).

ii) If \(s \in k - \{0\}\), all slopes of \(M^{(s,\infty)}(V)\) are equal to zero, \(\dim_{K_\theta} M^{(s,\infty)}(V) = \mu(V)\), and \(\text{Horz}(M^{(s,\infty)}(V)) \subset \{s\}\).

iii) All slopes of \(M^{(\infty,\infty)}(V)\) are strictly positive and \(\dim_{K_\theta} M^{(\infty,\infty)}(V) = \text{irr}(V)\).

Proof. In all three cases, the assertion about slopes follows from Lemma 2 and Lemma 3 (the last applied to \(M = K_x(V)\)). We prove the remaining claim in i): Put

\[
V = \frac{k((x))\langle x \partial_x \rangle}{k((x))\langle x \partial_x \rangle \cdot P(x, x \partial_x)},
\]

where

\[P(x, x \partial_x) = a_d(x)(x \partial_x)^d + \cdots + a_0(x) \quad \text{and} \quad a_d(x) \neq 0.\]

Then, the irregularity of \(V\) is \(\text{ord}(a_d(x))\) and we have

\[M^{(0,\infty)}(V) = \frac{M^{(0,\infty)}}{M^{(0,\infty)} \cdot P(u, -\eta)}.
\]

It follows from the division theorem that \(\text{ord}(a_d(x)) = \dim_{K_\theta}(M^{(0,\infty)}(V))\), as claimed.

If \(s \neq 0\), then

\[M^{(s,\infty)}(V) = \frac{M^{(s,\infty)}}{M^{(s,\infty)} \cdot P\left(u, \left(\sum_{i \geq 0} \left(\frac{-u}{s} \right)^{i+1}\right) \eta\right)},\]

and we have \(P(u, \left(\sum_{i \geq 0} \left(\frac{-u}{s} \right)^{i+1}\right) \eta) = b_d(u)\eta^d + b_{d-1}\eta^{d-1} + \ldots\) with \(\text{ord}(b_d(u)) = d + \text{ord}(a_d(u))\). Again by the division theorem, we have \(\dim_{K_\theta} M^{(s,\infty)}(V) = \mu(V)\), as desired. The assertion about the horizontal zeros follows from the classification theorem together with Lemmas 1, 2 and 3. The case \(s = \infty\) is analogous. \(\square\)
4. Formal stationary phase for the Mellin transform.

The main result of this note is:

**Theorem 4.** Let $\mathcal{M}$ be a holonomic $k[z, z^{-1}](z\partial_z)$-module. The map

$$\Xi = \bigoplus_s \Xi_s : \mathcal{M}(\mathcal{M})_{\infty} \longrightarrow \bigoplus_{s \in S(\mathcal{M}) \cup \{\infty\}} \mathcal{M}^{(s, \infty)}(\mathcal{M}_s)$$

is a functorial isomorphism of difference modules over $K_\theta = k((\theta))$.

**Proof.** We show first that the map $\Xi$ is onto: We can decompose $\mathcal{M}(\mathcal{M})_{\infty}$ according to slopes

$$\mathcal{M}(\mathcal{M})_{\infty} \cong \mathcal{M}(\mathcal{M})^{>0}_{\infty} \oplus \mathcal{M}(\mathcal{M})^{=0}_{\infty} \oplus \mathcal{M}(\mathcal{M})^{<0}_{\infty}$$

and then, by Corollary 1 and Lemmas 3 and 4 above, it follows that it suffices to prove that the map

$$\bigoplus_s \Xi_s : \mathcal{M}(\mathcal{M})^{=0}_{\infty} \longrightarrow \bigoplus_{s \in S(\mathcal{M})} \mathcal{M}^{(s, \infty)}(\mathcal{M}_s)$$

is onto. By the classification theorem and (the proof of) Lemma 1, after a cyclic extension of $K_\theta$ we can assume that $\mathcal{M}(\mathcal{M})^{=0}_{\infty}$ can be decomposed according to its horizontal zeros. But from Lemma 4 we have $Horz(\mathcal{M}^{(s, \infty)}(\mathcal{M}_s)) \subset \{s\}$ for $s \in S(\mathcal{M})$ and then by Lemma 1 the surjectivity of $\bigoplus_s \Xi_s$ follows.

It is therefore enough to show that the dimensions over $K_\theta$ of the source and the target of $\Xi$ are equal: As remarked in the proof of Lemma 2, the Newton polygon of $\mathcal{M}(\mathcal{M})_{\infty}$ equals the polygon obtained from the Newton polygon of $\mathcal{M}$ by applying a rotation of ninety degrees in the clockwise direction. Comparing the width of the former with the height of the latter, it follows that

$$\dim_{K_\theta}(\mathcal{M}(\mathcal{M})_{\infty}) = \text{irr}(\mathcal{M}_0) + \text{irr}(\mathcal{M}_{\infty}) + \sum_{s \in S(\mathcal{M})} \mu(\mathcal{M}_s)^p$$

By Lemma 4, we are done. □

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9To my knowledge, this formula was first proved by C. Sabbah, using a different method (unpublished, but see [20] for a special case).
Remark. If the base field is the field $\mathbb{C}$ of complex numbers, then local Fourier
transforms can be defined at the analytic level (this is well-known, take first
Katz’s extension, apply the global Fourier transform and localize at infinity),
denote $\Phi^{(*,\infty)}$ the functors so defined ($* \in \mathbb{C} \cup \{\infty\}$). For differential modules
of rank one in one variable, the formal and the analytic classification coincide,
so the formal stationary phase isomorphism for the Fourier transform \cite[section 1]{12}
implies that if $M$ is a holonomic module over the affine line, then
there is an analytic isomorphism\footnote{See \cite[(5.11)]{24} for a description in terms of local data of
the determinant of the Fourier transform (and not just of its germ at infinity).}
$$\det \mathfrak{F}our(M)_\infty \cong \bigotimes_{* \in \mathbb{C} \cup \{\infty\}} \det \Phi^{(*,\infty)}(M_*)$$

Following the analogous procedure, local Mellin transforms can also be de-
defined at the analytic level. However, for difference modules of rank one, the
analytic classification is much finer than the formal one (see \cite[10.2]{28}), and
therefore Theorem 4 above does not allow to derive a similar conclusion as
in the Fourier case. If $M$ is a module with regular singularities, then the
analytic type of the determinant of its Mellin transform was determined in
\cite{21} (as explained in loc. cit., in fact only regularity at zero and at infinity
is needed).

One could also ask about possible $\ell$–adic analogues of the local Mellin trans-
forms. The global Mellin transform does have a $\ell$-adic analogue, see \cite{11},
and its determinant was computed in \cite{18}. Also, since in our approach the
local Mellin transforms are a kind of modified microlocalizations and since
there is a good theory of $p$-adic microdifferential operators and local Fourier
transforms (see \cite{1, 2}), one could also hope for a $p$-adic theory of local Mellin
transforms, which might be related to the results in \cite{19}.

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