RELATIVE TWISTED HOMOLOGY AND COHOMOLOGY GROUPS ASSOCIATED WITH LAURICELLA’S $F_D$

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ABSTRACT. We introduce relative twisted homology and cohomology groups associated with Euler type integrals of solutions to Lauricella’s system $F_D(a, b, c)$ of hypergeometric differential equations. We define an intersection form between relative twisted homology groups and that between relative twisted cohomology groups, and show their compatibility. We prove that the relative twisted homology group is canonically isomorphic to the space of local solutions to $F_D(a, b, c)$ for any parameters $a, b, c$. Through this isomorphism, we study $F_D(a, b, c)$ by the relative twisted homology and cohomology groups and the intersection forms without any conditions on $a, b, c$.

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1. INTRODUCTION

There are many generalizations of the Gauss hypergeometric differential equation. It is known that Lauricella’s system $F_D(a, b, c)$ given in (2.2) is the simplest regular integrable system with multiple independent variables $x_1, \ldots, x_m$, where each of $(a, b, c) = (a_1, \ldots, b_m, c)$ is a complex parameter. This system is of rank

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m + 1, and local solutions to this system admit the path-integral representations of Euler type in (2.4). By separating

\[ u(t) = u(t, x) = t^{b_1 + \cdots + b_m - c}(t - x_1)^{-b_1} \cdots (t - x_m)^{-b_m}(t - 1)^{c-a} \]

and \( \varphi_0 = dt/(t-1) \) from the integrand in (2.4), we have twisted homology and cohomology groups as in [AK, §2]. We array the exponents of \( u(t) \) at \( 0 = x_0, x_1, \ldots, x_m, x_{m+1} = 1 \) and \( x_{m+2} = \infty \) as

\[ \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \alpha_{m+2}) = (\sum_{i=1}^{m} b_i - c, -b_1, \ldots, -b_m, c - a, a), \]

which satisfy

\[ \sum_{i=0}^{m+2} \alpha_i = 0. \]

Under a non-integral condition \( \alpha \in (\mathbb{C} - \mathbb{Z})^{m+3} \), several properties of \( \mathcal{F}_D(a, b, c) \) are studied by twisted homology and cohomology groups, for details refer to [AK], [CM], [KY], [M2], [OT], [Y2] and the references therein. Moreover, the monodromy representation of \( \mathcal{F}_D(a, b, c) \) is studied in [M3] under mild conditions. These studies are based on the key fact that the space \( \text{Sol}_c(a, b, c) \) of local solutions to \( \mathcal{F}_D(a, b, c) \) around \( x \) is canonically isomorphic to that of sections of the trivial vector bundle over a small neighborhood of \( x \) with fiber consisting of the twisted homology group, where \( x \) is a point in the complement \( X \) of the singular locus of \( \mathcal{F}_D(a, b, c) \). However, in case of \( \alpha \in \mathbb{Z}^{m+3} \), this key fact does not hold since the dimension of the twisted homology group is different from the rank of \( \mathcal{F}_D(a, b, c) \).

In this paper, we remove the non-integral condition \( \alpha \in (\mathbb{C} - \mathbb{Z})^{m+3} \) from the studies above by extending twisted homology and cohomology groups. As our extension of the twisted homology group, we introduce a relative twisted homology group \( H_1(T, D; \mathcal{L}) \), where the space \( T = T_x \) is a subset of the complex projective line \( \mathbb{P}^1 \) consisting of points at which \( u(t)\varphi_0 \) is a locally single-valued holomorphic 1-form, the relative set \( D = D_x \) is the intersection of \( T \) and \( \mathbb{P}^1 \), \( \mathcal{L} = \mathcal{L}_x \) is the local system associated with \( u(t) \), and they depend on the parameters and the variables of \( \mathcal{F}_D(a, b, c) \). We show that \( H_1(T, D; \mathcal{L}) \) is \( m + 1 \) dimensional for any parameters \( a, b, c \), in particular, this property holds in the case \( \alpha \in \mathbb{Z}^{m+3} \). By aligning points \( x_1, \ldots, x_m \) in convenient order, we give its basis \( (\gamma_1^u, \ldots, \gamma_{m+1}^u) \) in both two cases \( \alpha \notin \mathbb{Z}^{m+3} \) and \( \alpha \in \mathbb{Z}^{m+3} \).

We have a relative twisted cohomology group \( H^1(T, D; \mathcal{L}) \) as the dual of \( H_1(T, D; \mathcal{L}) \). We define \( H^1_{\text{alg}}(T, D; \mathcal{L}), H^1_{\mathcal{L}^\infty}(T, D; \mathcal{L}) \) and \( H^1_{C^\infty}(T, D; \mathcal{L}) \) as the first cohomology groups of relative twisted de Rham complexes consisting of rational \( k \)-forms, smooth ones and those with certain vanishing property, respectively. We show that they are canonically isomorphic to \( H^1(T, D; \mathcal{L}) \) through the pairing

\[ \langle \varphi, \gamma^u \rangle = \int_{\gamma} u(t) \varphi, \]

where \( \varphi \in H^1(T, D; \mathcal{L}) \) (\( * = \text{alg}, C^\infty, C^\infty \)) is represented by a 1-form, and \( \gamma^u \in H^1(T, D; \mathcal{L}) \) is represented by a 1-chain \( \gamma \) on which a branch of \( u(t) \) is assigned. We utilize the canonical isomorphisms among these relative twisted de Rham cohomology groups several times in this paper.
This pairing is extended to that between sections of the trivial vector bundles over a small neighborhood of \( x \) with fibers \( H^1(T, D; \mathcal{L}) \) and \( H_1(T, D; \mathcal{L}) \). For simplicity, the spaces of sections of these trivial vector bundles are denoted by the same symbols \( H^1(T, D; \mathcal{L}) \) and \( H_1(T, D; \mathcal{L}) \) as their fibers. We show that the map
\[
j_{\varphi_0} : H_1(T, D; \mathcal{L}) \ni \gamma^u \mapsto \langle \varphi_0, \gamma^u \rangle = \int_{\gamma} u(t)\varphi_0 \in \text{Sol}_x(a, b, c)
\]
is isomorphic for any parameters \( a, b, c \). This key fact enables us to study \( \mathcal{F}_D(a, b, c) \) by the relative twisted homology and cohomology groups \( H_1(T, D; \mathcal{L}) \) and \( H^1(T, D; \mathcal{L}) \).

In our proof of the key fact, we consider
\[
\partial_j \langle \varphi_0, \gamma^u \rangle = \frac{\partial}{\partial x_j} \langle \varphi_0, \gamma^u \rangle \quad (1 \leq j \leq m).
\]
For the pairing \( \langle \phi, \gamma^u \rangle \) between sections \( \phi \in H^1_{\mathcal{C}^\psi}(T, D; \mathcal{L}) \) and \( \gamma^u \in H_1(T, D; \mathcal{L}) \), we have
\[
\partial_j \langle \phi, \gamma^u \rangle = \int_{\gamma} (u(t, x)\partial_j(\phi) + \partial_j(u(t, x))\phi) = \int_{\gamma} \left( \partial_j \phi + \frac{\partial_j u(t, x)}{u(t, x)} \right) \phi = \langle \nabla_j \phi, \gamma^u \rangle,
\]
where \( \nabla_j = \partial_j - \frac{\partial_j u(t, x)}{u(t, x)} = \partial_j - \frac{\alpha_j}{t - x_j} \). We can regard \( \nabla_j \) as an action on \( H^1_{\mathcal{C}^\psi}(T, D; \mathcal{L}) \). However, it cannot act directly on \( H^1_{\text{alg}}(T, D; \mathcal{L}) \) and \( H^1_{\mathcal{C}^\psi}(T, D; \mathcal{L}) \) since their coboundary groups are not kept invariant under this action. It acts on \( H^1_{\text{alg}}(T, D; \mathcal{L}) \) and \( H^1_{\mathcal{C}^\psi}(T, D; \mathcal{L}) \) through the canonical isomorphisms \( H^1_{\text{alg}}(T, D; \mathcal{L}) \simeq H^1_{\mathcal{C}^\psi}(T, D; \mathcal{L}) \) and \( H^1_{\text{alg}}(T, D; \mathcal{L}) \simeq H^1_{\mathcal{C}^\psi}(T, D; \mathcal{L}) \), respectively.

Let \( D^\psi \) be the set of poles of the multivalued 1-form \( \varphi_0u(t) \), \( T^\psi \) be the complement of \( \overline{\mathbb{P}}^1 - D^\psi \) in \( \mathbb{P}^1 \) and \( \mathcal{L}^\psi \) be the local system associated with \( 1/u(t) \). We have the relative twisted homology and cohomology groups \( H_1(T^\psi, D^\psi; \mathcal{L}^\psi) \) and \( H^1(T^\psi, D^\psi; \mathcal{L}^\psi) \), where \( * \) is \( \text{alg}, \mathcal{C}^{\infty}, \mathcal{C}^\psi \) and the blank. We define the intersection form \( \mathcal{I}_h \) between \( H^1_{\mathcal{C}^\psi}(T^\psi, D^\psi; \mathcal{L}^\psi) \) and \( H_1(T, D; \mathcal{L}) \) by a similar way in [KY] §1.4. We also define the intersection form \( \mathcal{I}_c \) between \( H^1_{\mathcal{C}^\psi}(T^\psi, D^\psi; \mathcal{L}^\psi) \) and \( H^1_{\mathcal{C}^\psi}(T^\psi, D^\psi; \mathcal{L}^\psi) \) by
\[
\int \int_{T \cap T^\psi} \varphi \wedge \psi,
\]
where \( \varphi \in H^1_{\mathcal{C}^\psi}(T, D; \mathcal{L}) \) and \( \psi \in H^1_{\mathcal{C}^\psi}(T^\psi, D^\psi; \mathcal{L}^\psi) \). This double integral converges by the vanishing property of 1-forms \( \varphi \) and \( \psi \). We can extend it to the intersection form \( \mathcal{I}_c \) between \( H^1_{\mathcal{C}^\psi}(T, D; \mathcal{L}) \) and \( H^1_{\mathcal{C}^\psi}(T^\psi, D^\psi; \mathcal{L}^\psi) \) by utilizing the canonical isomorphisms between the relative twisted cohomology groups. We generalize the evaluation formula of \( \mathcal{I}_c \) in [CM] Theorem 1 so that it is valid without the condition \( \alpha \in (\mathbb{C} - \mathbb{Z})^{m+3} \). By using this formula, we give bases of \( H^1_{\mathcal{C}^\psi}(T, D; \mathcal{L}) \) and \( H^1_{\mathcal{C}^\psi}(T^\psi, D^\psi; \mathcal{L}^\psi) \), which are dual to each other.

We show the compatibility of the pairings between relative twisted homology and cohomology groups and the intersection forms \( \mathcal{I}_h \) and \( \mathcal{I}_c \). Our proof is elementary one based on Stokes’ theorem. This compatibility yields relations between period matrices and intersections matrices with respect to any bases of \( H_1(T, D; \mathcal{L}) \), \( H^1_{\mathcal{C}^\psi}(T, D; \mathcal{L}) \), \( H^1_{\mathcal{C}^\psi}(T^\psi, D^\psi; \mathcal{L}^\psi) \) and \( H_1(T^\psi, D^\psi; \mathcal{L}^\psi) \). These are regarded as generalizations of results in [CM] §3.

A Pfaffian system and the monodromy representation of \( \mathcal{F}_D(a, b, c) \) are studied by the intersection forms \( \mathcal{I}_c \) and \( \mathcal{I}_h \) under the condition \( \alpha \in (\mathbb{C} - \mathbb{Z})^{m+3} \) in
2. Lauricella’s system $F_D(a, b, c)$

In this section, we prepare facts on Lauricella’s hypergeometric system $F_D(a, b, c)$ by referring to [IKSY] §9.1 and [Y1] §6. Lauricella’s hypergeometric series $F_D(a, b, c; x)$ is defined by

$$F_D(a, b, c; x) = \sum_{n \in \mathbb{N}_0} \frac{(a, \sum_{i=1}^m n_i) \prod_{i=1}^m (b_i, n_i)}{(c, \sum_{i=1}^m n_i) \prod_{i=1}^m (1, n_i)} \prod_{i=1}^m x_i^{n_i},$$

where $x_1, \ldots, x_m$ are complex variables with $|x_i| < 1$ ($1 \leq i \leq m$), $a$, $b = (b_1, \ldots, b_m)$ and $c$ are complex parameters, $c \notin -\mathbb{N}_0 = \{0, -1, -2, \ldots\}$, and $(b_i, n_i) = b_i(b_i + 1) \cdots (b_i + n_i - 1)$. It admits an Euler type integral:

$$(2.1) \quad \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_1^\infty u(t, x) \varphi_0, \quad u(t, x) = t^{\sum_{i=1}^m b_i - c} (t-1)^{c-a} \prod_{i=1}^m (t-x_i)^{-b_i}, \quad \varphi_0 = \frac{dt}{t-1}$$

where the parameters $a$ and $c$ satisfy $0 < \Re(a) < \Re(c)$.

The differential operators

$$x_i(1-x_i) \partial_i^2 + (1-x_i) \sum_{1 \leq j \leq m} \frac{j \neq i}{x_j \partial_j} + [c - (a + b_i + 1)x_i] \partial_i - b_i \sum_{1 \leq j \leq m} x_j \partial_j - ab_i,$$

$$(2.2) \quad (1 \leq i \leq m)$$

annihilate the series $F_D(a, b, c; x)$, where $\partial_i = \frac{\partial}{\partial x_i}$ ($1 \leq i \leq m$). Lauricella’s system $F_D(a, b, c)$ is defined by the ideal generated by these operators in the ring of differential operators with rational function coefficients $\mathbb{C}(x_1, \ldots, x_m)\langle \partial_1, \ldots, \partial_m \rangle$. Though the series $F_D(a, b, c; x)$ is not defined when $c \in -\mathbb{N}_0$, the system $F_D(a, b, c)$ can be defined even in this case. It is a regular holonomic system of rank $m + 1$ with singular locus

$$(2.3) \quad S = \{ x \in \mathbb{C}^m \mid \prod_{i=1}^m [x_i(1-x_i)] \prod_{1 \leq i < j \leq m} (x_i - x_j) = 0 \} \cup \cup_{i=1}^\infty \{ x_i = \infty \} \subset (\mathbb{P}^1)^m.$$
We set
\[ X = (\mathbb{P}^1)^m - S = \{ (x_1, \ldots, x_m) \in \mathbb{C}^m \mid \prod_{0 \leq i < j \leq m+1} (x_j - x_i) \neq 0 \}, \]
where \( x_0 = 0 \) and \( x_{m+1} = 1 \). We introduce a notation
\[ \hat{x} = (x_0, x_1, \ldots, x_m, x_{m+1}, x_{m+2}) = (0, x_1, \ldots, x_m, 1, \infty) = (0, x, 1, \infty) \]
for \( x \in X \). Let \( \text{Sol}_x(a,b,c) \) be the vector space of solutions to \( \mathcal{F}_D(a,b,c) \) on a small simply connected neighborhood \( W(\subset X) \) of \( x \). It is called the local solution space to \( \mathcal{F}_D(a,b,c) \) around \( x \), and it is \( m+1 \) dimensional. If a 1-chain \( \gamma \) satisfies certain vanishing properties for its boundary then the integral
\[ \int_\gamma u(t,x) \varphi_0 \]
gives an element of \( \text{Sol}_x(a,b,c) \). We remark that it happens that this integral degenerates into the zero solution.

3. Relative twisted homology groups

Recall that
\[ \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \alpha_{m+2}) = (-c + \sum_{i=1}^{m+2} b_i, -b_1, \ldots, -b_m, c-a, a), \quad \sum_{i=0}^{m+2} \alpha_i = 0, \]
where \( a, b_1, \ldots, b_m \) and \( c \) are the parameters of Lauricella’s \( F_D \) belonging to \( \mathbb{C} \). We fix \( \alpha \) and \( x \in X \). We divide the index set \( I = \{0, 1, 2, \ldots, m, m+1, m+2\} \) of \( \alpha \) into two disjoint subsets
\[ I_Z = \{ i \in I \mid \alpha_i \in \mathbb{Z} \}, \quad I_{Z^c} = \{ i \in I \mid \alpha_i \notin \mathbb{Z} \}. \]
Moreover, we divide \( I_Z \) into two disjoint subsets
\[ (3.1) \quad I_{N_0}^0 = \{ i \in I_Z \mid \text{ord}_x(u(t)\varphi_0) \geq 0 \}, \quad I_{N}^0 = \{ i \in I_Z \mid \text{ord}_x(u(t)\varphi_0) < 0 \}, \]
where \( u(t) \) and \( \varphi_0 \) are in the integral \[ (2.1) \]
and \( \text{ord}_x \) denotes the order of zero of meromorphic functions or 1-forms at \( t = x_i \). We remark that though we have
\[ \{ \alpha_i \mid i \in I_Z \} = \{ \alpha_i \mid i \in I_Z \} = \{ \alpha_i \mid i \in \alpha \mid \alpha_i \notin \mathbb{Z} \}, \]
it happens that
\[ \{ \alpha_i \mid i \in I_{N_0}^0 \} \subsetneq \{ \alpha_i \mid i \in \alpha \mid \alpha_i \in \mathbb{N}_0 = \{0,1,2,\ldots\} \}, \]
\[ \{ \alpha_i \mid i \in I_{N}^0 \} \supsetneq \{ \alpha_i \mid i \in \alpha \mid \alpha_i \in \mathbb{N} = \{-1,-2,-3,\ldots\} \}, \]
since we count the order of zero by not the function \( u(t) \) but the 1-form \( u(t)\varphi_0 \).

We set \#I_{N_0}^0 = r, \#I_{N}^0 = s, \#I_Z = m+3-r-s, and
\[ I_{N_0}^0 = \{ i_1, \ldots, i_r \}, \quad I_{N}^0 = \{ i_{r+1}, \ldots, i_{r+s} \}, \quad I_Z = \{ i_0, i_{r+s+1}, \ldots, i_{m+2} \}. \]
If the set \( I_Z \) is empty, then neither \( I_{N_0}^0 \) nor \( I_{N}^0 \) is empty since the total sum of the orders of zeros of \( u(t)\varphi_0 \) is \(-2\); in this case we regard the set \( I_{N}^0 \) as
\[ \{ i_0, i_{r+1}, \ldots, i_{r+s-1} \} \quad (s = m+3-r, \ r > 0, \ s > 0). \]

We define a subspace \( T \) of \( \mathbb{P}^1 \) and a subset \( D \) in \( T \) by
\[ T = T_x = \mathbb{P}^1 - \{ x_i \mid i \in I_Z \cup I_{N_0}^0 \} = \mathbb{P}^1 - \{ x_{i_0}, x_{i_{r+s}}, x_{i_{r+s+1}}, \ldots, x_{i_{m+2}} \}, \]
\[ D = D_x = \{ x_i \mid i \in I_{N_0}^0 \} = \{ x_{i_1}, \ldots, x_{i_r} \}. \]
Note that the space $T$ consists points at which $u(t)\varphi_0$ is a locally single-valued holomorphic 1-form. We set $B = B_x = \{x_i, x_{i+1}, \ldots, x_{i+n}\} \subset T^c$ if $I_{2^n} \neq \emptyset$. Let $\mathcal{L} = \mathcal{L}_x$ be the locally constant sheaf on $T = T_x$ defined by $u(t) = u(t, x)$. We define $C_k(T; \mathcal{L})$ by the $\mathbb{C}$-vector space of twisted $k$-chains which are finite linear combinations of $k$-simplices in $T$ on which branches of $u(t)$ are assigned. Let $C_k(D; \mathcal{L})$ be the subspace defined by the restrictions of elements in $C_k(T; \mathcal{L})$ to $D$:

$$C_k(D; \mathcal{L}) = C_k(T; \mathcal{L})|_D.$$  

It is clear that

$$C_1(D; \mathcal{L}) = C_2(D; \mathcal{L}) = 0.$$

Since the space $C_0(D; \mathcal{L})$ is generated by $x_i \in D$ with the germ $u(t)|_{x_i}$ of a branch $u(t)$ at $t = x_i$, we have

$$\dim C_0(D; \mathcal{L}) = r.$$

Here note that the germ $u(t)|_{x_i}$ is non-zero even in the case $u(x_i) = 0$. The space of relative twisted $k$-chains is defined by the quotient

$$C_k(T, D; \mathcal{L}) = C_k(T; \mathcal{L})/C_k(D; \mathcal{L}).$$

We have the boundary operator $\partial^u : C_k(T; \mathcal{L}) \rightarrow C_{k-1}(T; \mathcal{L})$ by extending

$$\partial^u(\mu^{u(t)}|_\mu) = (\partial \mu)^{u(t)}|_{\partial \mu}$$

linearly, where $\mu^{u(t)}|_\mu$ is a twisted $k$-chain given by a $k$-simplex $\mu$ in $T$ and a branch $u(t)|_\mu$ of $u(t)$ on $\mu$. $\partial$ is the usual boundary operator, and $u(t)|_{\partial \mu}$ is the restriction of the branch $u(t)|_\mu$ to $\partial \mu$. We have an exact sequence of chain complexes

$$0 \rightarrow C_\bullet(D; \mathcal{L}) \rightarrow C_\bullet(T; \mathcal{L}) \rightarrow C_\bullet(T, D; \mathcal{L}) \rightarrow 0,$$

where the boundary operators of $C_\bullet(D; \mathcal{L})$ and $C_\bullet(T, D; \mathcal{L})$ are naturally induced from $\partial^u$ on $C_\bullet(T; \mathcal{L})$. We define $H_k(D; \mathcal{L})$, $H_k(T; \mathcal{L})$ and $H_k(T, D; \mathcal{L})$ by the $k$-th homology groups of the complexes $C_\bullet(D; \mathcal{L})$, $C_\bullet(T; \mathcal{L})$ and $C_\bullet(T, D; \mathcal{L})$, respectively. We call $H_k(T, D; \mathcal{L})$ the $k$-th relative twisted homology group. We have an exact sequence

$$0 \rightarrow H_2(D; \mathcal{L}) \rightarrow H_2(T; \mathcal{L}) \rightarrow H_2(T, D; \mathcal{L})$$

$$\begin{array}{ccc}
0 & \to & H_2(D; \mathcal{L}) \\
\partial^u & \to & H_1(D; \mathcal{L}) \\
\partial & \to & H_0(D; \mathcal{L})
\end{array} \quad \begin{array}{ccc}
H_2(T; \mathcal{L}) & \to & H_1(T; \mathcal{L}) \\
\partial^u & \to & H_0(T; \mathcal{L}) \\
\partial & \to & H_0(T, D; \mathcal{L})
\end{array} \quad 0.$$  

Here an element of $H_k(T, D; \mathcal{L})$ is represented by a $k$-chain $\ell^{u(t)}\in C_k(T; \mathcal{L})$ with its boundary in $C_{k-1}(D; \mathcal{L})$, and the connection map $\partial^u$ is naturally defined by the boundary operator as

$$H_k(T, D; \mathcal{L}) \ni \ell^{u(t)} \mapsto \partial \ell^{u(t)}|_{\partial \ell} \in C_{k-1}(D; \mathcal{L}) = \begin{cases} 0 & \text{if } k = 2, \\ H_0(D; \mathcal{L}) & \text{if } k = 1. \end{cases}$$

**Theorem 3.1.** For any parameters $\alpha$, we have

$$H_0(T, D; \mathcal{L}) = H_2(T, D; \mathcal{L}) = 0, \quad \dim H_1(T, D; \mathcal{L}) = m + 1.$$  

**Proof.** By the definition, it is easy to see that $H_2(T; \mathcal{L}) = 0$. Since $H_2(D; \mathcal{L}) = H_1(D; \mathcal{L}) = 0$, we have $H_2(T, D; \mathcal{L}) \simeq H_2(T; \mathcal{L}) = 0$ by the exact sequence (3.3). Since $T$ is connected, the map $H_0(D; \mathcal{L}) \rightarrow H_0(T; \mathcal{L})$ in the exact sequence (3.3) is surjective. Thus the kernel of the surjective map $H_0(T; \mathcal{L}) \rightarrow H_0(T, D; \mathcal{L})$ is the
Thus we have
\[ \dim H_1(T; \mathcal{L}) - \dim H_1(T, D; \mathcal{L}) + \dim H_0(D; \mathcal{L}) - \dim H_0(T; \mathcal{L}) = 0. \]

Note that \( \dim H_0(D; \mathcal{L}) = r \) and
\[ \dim H_1(T; \mathcal{L}) - \dim H_0(T; \mathcal{L}) = - \dim H_2(T; \mathcal{L}) + \dim H_1(T; \mathcal{L}) - \dim H_0(T; \mathcal{L}) = -\chi(T) = -(2 - (m + 3 - r)) = m + 1 - r, \]
where \( \chi(T) \) denotes the Euler number of \( T \). Hence we have \( \dim H_1(T, D; \mathcal{L}) = m+1 \).

Remark 3.2. The quotient space \( H_1(T, D; \mathcal{L})/H_1(T; \mathcal{L}) \) is isomorphic to the image of the map \( \partial^n \) in (3.4). It coincides with the kernel of the surjective map \( H_0(D; \mathcal{L}) \to H_0(T; \mathcal{L}) \) in (3.4). If \( \alpha \in \mathbb{Z}^{m+3} \) then \( H_0(T; \mathcal{L}) \) is one dimensional, otherwise \( H_0(T; \mathcal{L}) = 0 \). Thus we have
\[ \dim H_1(T, D; \mathcal{L})/H_1(T; \mathcal{L}) = \hat{r} = \begin{cases} r & \text{if } \alpha \notin \mathbb{Z}^{m+3}, \\ r - 1 & \text{if } \alpha \in \mathbb{Z}^{m+3}. \end{cases} \]

We call an element \( \gamma^u \in H_1(T, D; \mathcal{L}) \) satisfying \( 0 \neq \partial^n(\gamma^u) \in H_0(D; \mathcal{L}) \) a relative cycle, which represents a non-zero element of the quotient space \( H_1(T, D; \mathcal{L})/H_1(T; \mathcal{L}) \).

We give \( m + 1 \) elements of \( H_1(T, D; \mathcal{L}) \). We take a base point \( \hat{x} \in X \) so that
\[ (3.5) \]
\[ x_{i_0} < x_{i_1} < \cdots < x_{i_{m+2}} = x_{m+2} = \infty \quad \text{if } m + 2 \in I_{2^p}, \]
\[ -\infty = x_{m+2} = x_{i_1} < \cdots < x_{i_{m+2}} < x_{i_0} \quad \text{if } m + 2 \in I_{00}^{20}, \]
\[ x_{i_{r+s+1}} < \cdots < x_{i_{m+2}} < x_{i_0} < x_{i_1} < \cdots < x_{i_{r+s}} = x_{m+2} = \infty \quad \text{if } m + 2 \in I_{-m}^{00}. \]

We choose a base point \( \hat{i} \) in the upper half space \( \mathbb{H}_T \) of \( T \). Let \( \ell_{ij} \) \((0 \leq j \leq m + 2)\) be a path from \( \hat{i} \) to \( x_{ij} \) via \( \mathbb{H}_T \). Let \( \circ_{ij} \left(0 \leq j \leq m + 2\right)\) be a loop starting from \( \hat{i} \), approaching to \( x_{ij} \) in \( \mathbb{H}_T \), turning once around \( x_{ij} \) positively, and tracing back to \( \hat{i} \); see Figure 3.

![Figure 1. Chains and relative chains](image)

We fix a branch of \( u(t) \) on \( \mathbb{H}_T \) by the assignment \( 0 < \arg(t - x_i) < \pi \) \((0 \leq i \leq m + 1)\) for \( t \in \mathbb{H}_T \).
We consider two cases: (1) \( \alpha \notin \mathbb{Z}^{m+3} \); (2) \( \alpha \in \mathbb{Z}^{m+3} \).

(1) In this case, we have
\[
\gamma \in \mathbb{Z}, \ \alpha_{i_0}, \alpha_{i_{i+1}}, \ldots, \alpha_{i_{m+2}} \notin \mathbb{Z}, \ \alpha_{i_1}, \ldots, \alpha_{i_r}, \alpha_{i_{r+1}}, \ldots, \alpha_{i_{m+1}} \in \mathbb{Z}, \text{ and } x_{i_1}, \ldots, x_{i_r} \in D. \]
We set
\[
\gamma_j^u = \begin{cases} 
\ell_{ij} - \frac{\gamma_{ij}^u}{1 - \lambda_{i_0}} & \text{if } 1 \leq j \leq r, \\
\gamma_{ij}^u & \text{if } r + 1 \leq j \leq r + s, \\
\frac{1 - \lambda_i}{1 - \lambda_{i_0}} \gamma_{ij}^u & \text{if } r + s + 1 \leq j \leq m + 1.
\end{cases}
\]

(2) In this case, we have \( x_{i_1}, \ldots, x_{i_r} \in D, \) and \( r + s = m + 3 \). We set
\[
\gamma_j^u = \begin{cases} 
\ell_{ij} - \ell_{i_1} & \text{if } 1 \leq j \leq r - 1, \\
\gamma_{ij}^u & \text{if } r \leq j \leq r + s - 2 = m + 1.
\end{cases}
\]

**Theorem 3.3.** The elements \( \gamma_1^u, \ldots, \gamma_{m+1}^u \) form a basis of \( H_1(T, D; \mathcal{L}) \).

**Proof.** In the case (1), it is shown in [MM 3] that \( \gamma_1^u, \ldots, \gamma_{m+1}^u \) form a basis of \( H_1(T; \mathcal{L}) \). Since the image \( \gamma_j^u (1 \leq j \leq r) \) under the map \( \partial^u \) is a non-zero element of \( H_0(D; \mathcal{L}) \) given by the point \( t = x_{i_j} \) with the germ \( u(t) \) at \( x_{i_j} \), \( \gamma_1^u, \ldots, \gamma_r^u \) are linearly independent and they do not belong to \( H_1(T; \mathcal{L}) \). Hence \( \gamma_1^u, \ldots, \gamma_{m+1}^u \) form a basis of \( H_1(T, D; \mathcal{L}) \).

In the case (2), we see that \( \gamma_1^u, \ldots, \gamma_{m+1}^u \) form a basis of \( H_1(T; \mathcal{L}) \) similarly to the case (1). Recall that
\[
\dim H_0(D; \mathcal{L}) = r, \quad \dim H_0(T; \mathcal{L}) = 1, \quad \dim \partial^u (H_1(T, D; \mathcal{L})) = r - 1
\]
in this case. In the images \( \partial^u (\gamma_1^u), \ldots, \partial^u (\gamma_{r-1}^u) \in H_0(D; \mathcal{L}) \), the 0-chain \( x_{i_{j+1}} \) \( (1 \leq j \leq r - 1) \) appears only in \( \partial^u (\gamma_j^u), \gamma_1^u, \ldots, \gamma_{r-1}^u \) are linearly independent, and they do not belong to \( H_1(T; \mathcal{L}) \). Hence \( \gamma_1^u, \ldots, \gamma_{m+1}^u \) form a basis of \( H_1(T, D; \mathcal{L}) \). \( \square \)

4. Relative twisted cohomology groups

We define \( H^k(T, D; \mathcal{L}) \), \( H^k(T; \mathcal{L}) \) and \( H^k(D; \mathcal{L}) \) by the \( k \)-th cohomology groups of the cochain complexes \( \mathcal{C}^*(T, D; \mathcal{L}) \), \( \mathcal{C}^*(T; \mathcal{L}) \) and \( \mathcal{C}^*(D; \mathcal{L}) \), which are the dual complexes of chain complexes in [4.2]. We call \( H^k(T, D; \mathcal{L}) \) the \( k \)-th relative twisted cohomology group. Since cochain complexes satisfy
\[
0 \rightarrow \mathcal{C}^*(T, D; \mathcal{L}) \rightarrow \mathcal{C}^*(T; \mathcal{L}) \rightarrow \mathcal{C}^*(D; \mathcal{L}) \rightarrow 0,
\]
we have an exact sequence
\[
0 \rightarrow H^0(T; \mathcal{L}) \rightarrow H^0(D; \mathcal{L}) \rightarrow H^1(T, D; \mathcal{L}) \rightarrow H^1(T; \mathcal{L}) \rightarrow 0.
\]
By Theorem 3.1 we have the following corollary.

**Corollary 4.1.** For any parameters \( \alpha \), we have
\[
H^0(T, D; \mathcal{L}) = H^2(T, D; \mathcal{L}) = 0, \quad \dim H^1(T, D; \mathcal{L}) = m + 1.
\]

We give three kinds of relative twisted de Rham cohomology groups isomorphic to \( H^1(T, D; \mathcal{L}) \) in this section. We define a twisted exterior derivative \( \nabla_t \) by \( d + \omega \wedge \), where
\[
\omega = d \log u(t, x) = \sum_{i=0}^{m+1} \frac{\alpha_i dt}{t - x_i}.
\]
Let $\Omega^k(\tilde{x})$ ($k = 0, 1, 2$) be the vector space of rational differential $k$-forms with poles only on entries of $\tilde{x} = (0, x_1, \ldots, x_m, 1, \infty)$. We define subspaces of $\Omega^k(\tilde{x})$ by
\[
\Omega^0(T, D; \mathcal{L}) = \{f(t) \in \Omega^0(\tilde{x}) \mid \text{ord}_{x_i}(u(t) \cdot f(t)) \geq 1 \text{ for any } x_i \in D\},
\]
\[
\Omega^1(T, D; \mathcal{L}) = \{\varphi(t) \in \Omega^1(\tilde{x}) \mid \text{ord}_{x_i}(u(t) \cdot \varphi(t)) \geq 0 \text{ for any } x_i \in D\},
\]
\[
\Omega^2(T, D; \mathcal{L}) = 0.
\]

Note that $d(u(t) \cdot f(t)) = u(t) \cdot \nabla_t f(t)$ and that if $\nabla_t f(t)$ is not identically 0 then $\text{ord}_{x_i}(\nabla_t f(t)) = \text{ord}_{x_i}(f(t)) - 1$ for $0 \leq i \leq m + 2$. Thus we see that $\nabla_t f(t) \in \Omega^2(T, D; \mathcal{L})$ for any $f(t) \in \Omega^0(T, D; \mathcal{L})$. We define relative twisted algebraic de Rham cohomology groups by
\[
H^0_{\text{alg}}(T, D; \mathcal{L}) = \ker(\nabla_t : \Omega^0(\tilde{x}) \to \Omega^1(\tilde{x})) = \Omega^0_{\text{alg}}(T, D; \mathcal{L}),
\]
\[
H^1_{\text{alg}}(T, D; \mathcal{L}) = \Omega^1_{\text{alg}}(T, D; \mathcal{L}) = \Omega^1(T, D; \mathcal{L})/\nabla_t(\Omega^0(T, D; \mathcal{L})),
\]
\[
H^2_{\text{alg}}(T, D; \mathcal{L}) = 0.
\]

**Proposition 4.2.** We have
\[
H^0_{\text{alg}}(T, D; \mathcal{L}) = 0, \quad \dim H^1_{\text{alg}}(T, D; \mathcal{L}) = m + 1.
\]

**Proof.** Since
\[
\ker(\nabla_t : \Omega^0(\tilde{x}) \to \Omega^1(\tilde{x})) = \begin{cases} 0 & \text{if } \alpha \notin \mathbb{Z}^{m+3}, \\ (u(t)^{-1}) & \text{if } \alpha = \mathbb{Z}^{m+3}, \end{cases}
\]
and $u(t) \cdot u(t)^{-1} = 1$ does not vanish at $x_i \in D \neq \emptyset$ in the case $\alpha \in \mathbb{Z}^{m+3}$, we have $H^0_{\text{alg}}(T, D; \mathcal{L}) = 0$ for any $\alpha$.

We have a short exact sequence of complexes of sheaves
\[
\begin{array}{c}
0 \to \Omega^0(D; \mathcal{L}) \to \Omega^0(\tilde{x}) \to \sum_{x_i \in D} \mathbb{C} \cdot x_i \to 0 \\
\downarrow \quad \quad \downarrow \nabla_t \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\quad 0 \to \Omega^1(D; \mathcal{L}) \to \Omega^1(\tilde{x}) \to 0 \to 0,
\end{array}
\]
where $\Omega^k(D; \mathcal{L})$ and $\Omega^k(\tilde{x})$ are sheaves over $T$ satisfying
\[
H^0(\Omega^k(\tilde{x})) = \Omega^k(T, D; \mathcal{L}),
\]
\[
H^1(\Omega^k(\tilde{x})) = \Omega^k(T, D; \mathcal{L}) = \{\varphi \in \Omega^k(\tilde{x}) \mid \text{ord}_{x_i}(u(t) \cdot \varphi(t)) \geq 0 \text{ for any } x_i \in D\},
\]
and $\mathbb{C} \cdot x_i$ denotes the skyscraper sheaf at $x_i$. As in [EV Appendix], it induces a long exact sequence of hypercohomology groups
\[
\begin{array}{c}
0 \to \mathbb{H}^0(\Omega^k_{\star}(D; \mathcal{L})) \to \mathbb{H}^0(\Omega^k_{\star}(\tilde{x})) \to \mathbb{H}^0(\bigoplus_{x_i \in D} \mathbb{C} \cdot x_i) \to \mathbb{H}^1(\Omega^k_{\star}(D; \mathcal{L})) \to \mathbb{H}^1(\Omega^k_{\star}(\tilde{x})) \to 0.
\end{array}
\]
Since $T$ is an affine space, the cohomology groups $H^1(\Omega^k_{\mathbb{R}}(D; \mathcal{L}))$ and $H^1(\Omega^k_{\mathbb{C}}(\tilde{x}))$ vanish. As in [EV Appendix], hypercohomology groups reduce to
\[
\mathbb{H}^j(\Omega^k_{\star}(D; \mathcal{L})) = H^j(\mathbb{H}^0(\Omega^k_{\star}(D; \mathcal{L}))) = H^j_{\text{alg}}(T, D; \mathcal{L}),
\]
\[
\mathbb{H}^j(\Omega^k_{\star}(\tilde{x})) = H^j(\mathbb{H}^0(\Omega^k_{\star}(\tilde{x}))) = H^j_{\text{alg}}(T; \mathcal{L})
\]
\[
= \begin{cases} \ker(\nabla_t : \Omega^0(T; \mathcal{L}) \to \Omega^1(T; \mathcal{L})) & \text{if } j = 0, \\ \Omega^1(T; \mathcal{L})/\nabla_t(\Omega^0(T; \mathcal{L})) & \text{if } j = 1. \end{cases}
\]
Hence we have an exact sequence of cohomology groups

\[ 0 \rightarrow H^0_{\text{alg}}(T; \mathcal{L}) \rightarrow H^0_{\text{alg}}(D; \mathcal{L}) \xrightarrow{\nabla_{t}} H^1_{\text{alg}}(T; \mathcal{L}) \rightarrow H^1_{\text{alg}}(T; \mathcal{L}) \rightarrow 0, \]

where \( H^0_{\text{alg}}(D; \mathcal{L}) = H^0( \bigoplus_{x_1 \in D} \mathbb{C} \cdot x_1 ) \). Note that

\[ \dim H^0_{\text{alg}}(T; \mathcal{L}) - \dim H^0_{\text{alg}}(D; \mathcal{L}) + \dim H^1_{\text{alg}}(T; D; \mathcal{L}) - \dim H^1_{\text{alg}}(T; \mathcal{L}) = 0. \]

Since \( \dim H^0_{\text{alg}}(D; \mathcal{L}) = \#D = r \), \( \dim H^0_{\text{alg}}(T; \mathcal{L}) - \dim H^1_{\text{alg}}(T; \mathcal{L}) = \chi(T) = -m - 1 + r \), we have \( \dim H^1_{\text{alg}}(T, D; \mathcal{L}) = m + 1 \).

**Remark 4.3.** Though the rational 1-form \( \omega = \nabla_t(1) \), the constant function 1 does not belong to \( H^0_{\text{alg}}(T, D; \mathcal{L}) \) in general, \( \omega \) is not always the zero of \( H^1_{\text{alg}}(T, D; \mathcal{L}) \). Refer to Theorem 6.4 for details.

**Theorem 4.4.** The space \( H^1_{\text{alg}}(T, D; \mathcal{L}) \) is dual to \( H_1(T, D; \mathcal{L}) \) by the integral

\[ \langle \varphi, \gamma \rangle = \sum_i c_i \int_{\mu_i} u(t) \mid_{\mu_i} \varphi \in \mathbb{C}. \]

Here an element \( H^1_{\text{alg}}(T, D; \mathcal{L}) \) is represented by \( \varphi \in \Omega^1(T, D; \mathcal{L}) \), and an element \( \gamma \) of \( H_1(T, D; \mathcal{L}) \) is represented by \( \sum_i c_i \mu_i \in \mathbb{C}_1(T, D; \mathcal{L}) \), where \( c_i \in \mathbb{C} \) and \( \mu_i \) denotes a 1-simplex \( \mu_i \), on which a branch \( u(t) \mid_{\mu_i} \) of \( u(t) \) is assigned.

**Proof.** Since the integral \( (4.3) \) converges, there is a linear map from \( H_1(T, D; \mathcal{L}) \) to \( \mathbb{C} \) given by

\[ j_\varphi : H_1(T, D; \mathcal{L}) \ni \gamma \mapsto \langle \varphi, \gamma \rangle \in \mathbb{C}. \]

for any element \( \varphi \in H^1_{\text{alg}}(T, D; \mathcal{L}) \). We show the map

\[ j : H^1_{\text{alg}}(T, D; \mathcal{L}) \ni \varphi \mapsto j_\varphi \in H_1(T, D; \mathcal{L})^* \]

is bijective, where \( H_1(T, D; \mathcal{L})^* \) denotes the dual space of \( H_1(T, D; \mathcal{L}) \) isomorphic to \( H^1(T, D; \mathcal{L}) \). Though we can show it by applying the five lemma to the exact sequences \( (4.2) \) and \( (4.1) \), we give a direct proof. Since \( \dim H_1(T, D; \mathcal{L}) = \dim H^1_{\text{alg}}(T, D; \mathcal{L}) = m + 1 \) by Theorem 3.1 and Proposition 4.2, we have only to show that \( j \) is injective. Suppose that \( \varphi \) is an element of the kernel of \( j \). This means that

\[ \langle \varphi, \gamma_j \rangle = 0 \quad (1 \leq j \leq m + 1), \]

where \( (\gamma_1^u, \ldots, \gamma_{m+1}^u) \) is the basis of \( H_1(T, D; \mathcal{L}) \) given in \( (3.6) \) or \( (3.7) \). By the exact sequence \( (4.2) \), \( H^1_{\text{alg}}(T, D; \mathcal{L}) \) is regarded as the direct sum of \( H^0_{\text{alg}}(T; \mathcal{L}) \) and \( \nabla_t(H^0_{\text{alg}}(D; \mathcal{L})) \). Since the spaces \( H^1_{\text{alg}}(T; \mathcal{L}) \) and \( H_1(T; \mathcal{L}) \) are dual to each other and \( \langle \varphi, \gamma_j^u \rangle = 0 \) for \( \gamma_j^u \in H_1(T; \mathcal{L}) \), \( \varphi \) belongs to \( \nabla_t(H^0_{\text{alg}}(D; \mathcal{L})) \). Thus there exists \( f(t) \in \Omega^1(\bar{x}) \) such that \( \nabla_t(f(t)) = \varphi \). We consider two cases \( (1) \alpha \notin \mathbb{Z}^{m+3} \) and \( (2) \alpha \in \mathbb{Z}^{m+3} \).

(1) \( \alpha \notin \mathbb{Z}^{m+3} \). In this case, we have

\[ 0 = \langle \varphi, \gamma_j^u \rangle = \langle \nabla_t f(t), \gamma_j^u \rangle = \left[ u(t)f(t) \right]_{t=\partial(\gamma_j)} = u(x_{i_j})f(x_{i_j}), \]

where \( \gamma_j^u (1 \leq j \leq r) \) is a relative cycle with topological boundary \( x_{i_j} \in D \). This means that \( f \) belongs to \( \Omega^0(T, D; \mathcal{L}) \) and \( \varphi \) is the zero of \( H^1_{\text{alg}}(T, D; \mathcal{L}) \).
We define subspaces of $H^0(T; \mathcal{L})$ canonically isomorphic to $H^0(T; \mathcal{L})$ as the relative twisted (co)homology groups $H^1_{alg}(T; D; \mathcal{L}) \to H_1(T; D; \mathcal{L})^\ast$ is injective for any $\alpha$. □

By Theorem 4.4 together with Proposition 4.2 yields the following.

**Corollary 4.5.** The relative twisted algebraic de Rham cohomology group $H^k_{alg}(T; D; \mathcal{L})$ is canonically isomorphic to $H^k(T, T \cup D)$.

Let $\mathcal{E}^k(\tilde{x})$ be the vector space of $C^\infty$-differential $k$-forms on $T - D$. We define subspaces of $\mathcal{E}^k(\tilde{x})$ by

$$
\mathcal{E}^0(T, D; \mathcal{L}) = \{ f(t) \in \mathcal{E}^0(\tilde{x}) \mid u(t) \cdot f(t) is C^\infty on U_i, \lim_{t \to x_i} u(t) \cdot f(t) = 0 for any i \in I_{\tilde{U}} \},
$$

$$
\mathcal{E}^k(T, D; \mathcal{L}) = \{ \varphi(t) \in \mathcal{E}^k(\tilde{x}) \mid u(t) \cdot \varphi(t) is C^\infty on U_i for any i \in I_{\tilde{U}} \} \quad (k = 1, 2),
$$

$$
\mathcal{E}^k_T(T, D; \mathcal{L}) = \{ \varphi(t) \in \mathcal{E}^k(\tilde{x}) \mid u(t) \cdot \varphi(t) is identically 0 on V_i for any i \in I_{\tilde{U}} \}.
$$

where $U_i$ and $V_i$ are sufficiently small neighborhood of $x_i$ satisfying $V_i \subset U_i$. We define relative twisted $(C^\infty$ de Rham) $k$-th cohomology groups $H^k_{C^\infty}(T, D; \mathcal{L})$ and $H^k_{C^\infty}(T, D; \mathcal{L})$ as the $k$-th cohomology groups of the complexes

$$
\mathcal{E}^0(T, D; \mathcal{L}) \xrightarrow{\nabla} \mathcal{E}^1(T, D; \mathcal{L}) \xrightarrow{\nabla} \mathcal{E}^2(T, D; \mathcal{L}) \xrightarrow{\nabla} \mathcal{E}^3(T, D; \mathcal{L}) \xrightarrow{\nabla} 0,
$$

$$
\mathcal{E}^0_T(T, D; \mathcal{L}) \xrightarrow{\nabla} \mathcal{E}^1_T(T, D; \mathcal{L}) \xrightarrow{\nabla} \mathcal{E}^2_T(T, D; \mathcal{L}) \xrightarrow{\nabla} \mathcal{E}^3_T(T, D; \mathcal{L}) \xrightarrow{\nabla} 0,
$$

respectively, i.e.,

$$
H^k_{C^\infty}(T, D; \mathcal{L}) = \ker(\nabla) : \mathcal{E}^k(T, D; \mathcal{L}) \to \mathcal{E}^{k+1}(T, D; \mathcal{L})/\mathcal{E}^{k-1}(T, D; \mathcal{L}),
$$

$$
H^k_{C^\infty}(T, D; \mathcal{L}) = \ker(\nabla) : \mathcal{E}^k_T(T, D; \mathcal{L}) \to \mathcal{E}^{k+1}_T(T, D; \mathcal{L})/\mathcal{E}^{k-1}(T, D; \mathcal{L})).
$$

**Theorem 4.6.** The natural inclusions

$$
H^k_{alg}(T, D; \mathcal{L}) \to H^k_{C^\infty}(T, D; \mathcal{L}), \quad H^k_{alg}(T, D; \mathcal{L}) \to H^k_{C^\infty}(T, D; \mathcal{L}).
$$

are isomorphisms. The relative twisted cohomology groups $H^k_{alg}(T, D; \mathcal{L})$, $H^k_{C^\infty}(T, D; \mathcal{L})$ and $H^k_{C^\infty}(T, D; \mathcal{L})$ are canonically isomorphic to $H^k(T, T \cup D)$. In particular,

$$
H^0_{C^\infty}(T, D; \mathcal{L}) = H^0_{C^\infty}(T, D; \mathcal{L}) = 0, \quad H^2_{C^\infty}(T, D; \mathcal{L}) = H^2_{C^\infty}(T, D; \mathcal{L}) = 0,
$$

$$
\dim H^1_{C^\infty}(T, D; \mathcal{L}) = \dim H^1_{C^\infty}(T, D; \mathcal{L}) = m + 1.
$$

**Proof.** Let $\mathcal{E}^k_T(D; \mathcal{L})$ and $\mathcal{E}^k_{T^\ast}(D; \mathcal{L})$ be sheaves over $T$ satisfying

$$
H^0(\mathcal{E}^k_T(D; \mathcal{L})) = \mathcal{E}^k(T, D; \mathcal{L}), \quad H^0(\mathcal{E}^k_{T^\ast}(D; \mathcal{L})) = \mathcal{E}^k(T, D; \mathcal{L}),
$$

(2) $\alpha \in \mathbb{Z}^{m+3}$. In this case, we have

$$
0 = \langle \varphi, \gamma_j \rangle = \langle \nabla_t f(t), \gamma_j \rangle = \left[u(t) f(t) \right]_{(\gamma_j)} = u(x_{i+1}) f(x_{i+1}) - u(x_i) f(x_i),
$$

for $1 \leq j \leq r - 1$, where $\gamma_j$ is the relative cycles with topological boundary consisting of $x_{i+1}, x_i \in D$. In this case, $H^0_{alg}(T; \mathcal{L})$ is a 1-dimensional space spanned by $1/u(t)$, which satisfies $\nabla_t(1/u(t)) = 0$. Since the element $f(t) - f(x_i) u(x_i)/u(t)$ satisfies

$$
\nabla_t f(t) - f(x_i) u(x_i)/u(t) = \varphi, \quad \left[u(t) \cdot \left(f(t) - f(x_i) u(x_i)/u(t)\right)\right]_{t=x_i} = 0
$$

for any $x_i \in D$, it belongs to $H^0(T; \mathcal{L})$ and $\varphi$ is the zero of $H^1_{alg}(T, D; \mathcal{L})$. Therefore the map $j : H^1_{alg}(T, D; \mathcal{L}) \to H_1(T, D; \mathcal{L})$ is injective for any $\alpha$. □
respectively. The natural inclusions

\[ \Omega^k_T(D; \mathcal{L}) \hookrightarrow \mathcal{E}^k_T(D; \mathcal{L}), \quad \mathcal{E}^k_{\mathcal{L}_v}(D; \mathcal{L}) \hookrightarrow \mathcal{E}^k_T(D; \mathcal{L}) \]

induce quasi isomorphisms between complexes of sheaves

\[ \Omega^k_T(D; \mathcal{L}) \rightarrow \mathcal{E}^k_T(D; \mathcal{L}), \quad \mathcal{E}^k_{\mathcal{L}_v}(D; \mathcal{L}) \rightarrow \mathcal{E}^k_T(D; \mathcal{L}) \]

with differential \( \nabla_\lambda \). As in [EV, Appendix], we have isomorphisms of hypercohomology groups

\[ H^j(\Omega^k_T(D; \mathcal{L})) \simeq H^j(\mathcal{E}^k_T(D; \mathcal{L})) \simeq H^j(\mathcal{E}^k_{\mathcal{L}_v}(D; \mathcal{L})). \]

Since \( \mathcal{E}^k_T(D; \mathcal{L}) \) and \( \mathcal{E}^k_{\mathcal{L}_v}(D; \mathcal{L}) \) are fine sheaves, \( H^j(\mathcal{E}^k_T(D; \mathcal{L})) = H^j(\mathcal{E}^k_{\mathcal{L}_v}(D; \mathcal{L})) = 0 \) for \( j \geq 1 \). Thus we have

\[ H^j_{\text{alg}}(T, D; \mathcal{L}) = H^j(\Omega^k_T(D; \mathcal{L})) \simeq H^j(\mathcal{E}^k_T(D; \mathcal{L})) = H^j(H^0(\mathcal{E}^k_T(D; \mathcal{L}))) = H^j_{C^\infty}(T, D; \mathcal{L}), \]

\[ \simeq H^j(\mathcal{E}^k_{\mathcal{L}_v}(D; \mathcal{L})) = H^j(H^0(\mathcal{E}^k_{\mathcal{L}_v}(D; \mathcal{L}))) = H^j_{C^\infty}(T, D; \mathcal{L}). \]

The rest can be obtained from Proposition 4.2 and Corollary 4.5. \( \Box \)

Here we give an expression of the inverse

\[ i_D : H^1_{C^\infty}(T, D; \mathcal{L}) \rightarrow H^1_{C^\infty}(T, D; \mathcal{L}) \]

of the natural inclusion by following [MH, §4]. For any element \( \varphi \in \mathcal{E}^1(T, D; \mathcal{L}) \), there exists a \( C^\infty \) function \( f_i(t) \) around \( x_i \) (\( i \in \mathbb{I}_{I_0^0} \cup I_{\mathbb{Z}} \)) such that \( \nabla_\lambda(f_i) = \varphi \) and \( f_i(x_i) = 0 \) for \( x_i \in D \). It admits the expression

\[ f_i(t) = \begin{cases} 
\frac{1}{u(t)} \int_{x_i}^t u(t) \varphi & \text{if } x_i \in D, \\
\frac{1}{(\lambda_i - 1)u(t)} \int_{\wedge_i(t)}^t u(t) \varphi & \text{if } x_i \in B,
\end{cases} \]

where \( \wedge_i(t) \) is a positively oriented circle with center \( x_i \) and terminal \( t \). In case of \( \varphi \in \Omega^1(T, D; \mathcal{L}) \), \( f_i \) is a meromorphic function around \( x_i \) and admits the Laurent expansion at \( x_i \). Though \( f_i \) is defined locally, the function

\[ \sum_{i \in \mathbb{I}_{I_0^0} \cup I_{\mathbb{Z}}} h_i(t) \cdot f_i(t) \]

can be regarded as defined on \( T \), and it belongs to \( \mathcal{E}^0(T, D; \mathcal{L}) \), where \( h_i \) is a \( C^\infty \) function on \( T \) satisfying

\[ h_i(t) = \begin{cases} 
1 & \text{if } t \in V_i, \\
0 & \text{if } t \in U_i^c,
\end{cases} \]

for \( x_i \in V_i \subset U_i \). The element

\[ \varphi - \nabla_\lambda \left( \sum_{i \in \mathbb{I}_{I_0^0} \cup I_{\mathbb{Z}}} h_i(t) \cdot f_i(t) \right) \]

belongs to \( \mathcal{E}^1_T(T, D; \mathcal{L}) \) and represents \( i_D(\varphi) \in H^1_{C^\infty}(T, D; \mathcal{L}) \).

We will give a basis of \( H^1_{C^\infty}(T, D; \mathcal{L}) \) in [6].
5. Relative twisted dual homology groups

We set
\[ T^\vee = \mathbb{P}^1 - \{ x_i : i \in I_{\mathbb{Z}} \cup I_{\mathbb{N}}^{\phi_0} \} = \mathbb{P}^1 - \left\{ \begin{array}{ll}
\{ x_{i_0}, x_{i_1}, \ldots, x_{i_r}, x_{i_{r+1}}, \ldots, x_{i_{m+2}} \} & \text{if } I_{\mathbb{Z}} \neq \emptyset,
\{ x_{i_1}, \ldots, x_{i_r} \} & \text{if } I_{\mathbb{Z}} = \emptyset,
\end{array} \right. \]
\[ D^\vee = \{ x_i : i \in I_{\mathbb{N}}^{\phi_0} \} = \left\{ \begin{array}{ll}
\{ x_{i_{r+1}}, \ldots, x_{i_{r+s}} \} & \text{if } I_{\mathbb{Z}} \neq \emptyset,
\{ x_{i_0}, x_{i_{r+1}}, \ldots, x_{i_{r+s-1}} \} & \text{if } I_{\mathbb{Z}} = \emptyset.
\end{array} \right. \]

**Remark 5.1.** Note that \( T^\vee \) (resp. \( D^\vee \)) is different from the space \( T' \) (resp. \( D' \)) defined by the differential 1-form
\[
\frac{\varphi_0}{u(t,x)} = \frac{dt}{u(t,x)(t-1)}
\]
as in §3. For an example, in the case \( m = 1 \) and \( u(t) = t^0(t-x_1)^0(t-1)^0 = 1 \), we have
\[
T = \mathbb{P}^1 - \{ 1, \infty \}, \quad D = \{ 0, x_1 \}, \quad T^\vee = \mathbb{P}^1 - \{ 0, x_1 \}, \quad D^\vee = \{ 1, \infty \},
\]
since \( u(t)\varphi_0 = \frac{dt}{t-1} \). On the other hand, \( T' \) and \( D' \) defined by \( \varphi_0/u(t) \) as in §3 are
\[
T' = \mathbb{P}^1 - \{ 1, \infty \} = T, \quad D' = \{ 0, x_1 \} = D,
\]
since \( 1/u(t) = u(t) = 1 \).

Let \( \mathcal{L}^\vee \) be the locally constant sheaf defined by \( 1/u(t) \). We define \( \mathcal{C}_k(T^\vee; \mathcal{L}^\vee) \) by the vector space of finite linear combinations of \( k \)-simplices in \( T^\vee \) on which a branch of \( u(t)^{-1} \) is assigned. As in the previous section, we have an exact sequence of chain complexes
\[
0 \rightarrow \mathcal{C}_\bullet(D^\vee; \mathcal{L}^\vee) \rightarrow \mathcal{C}_\bullet(T^\vee; \mathcal{L}^\vee) \rightarrow \mathcal{C}_\bullet(T^\vee, D^\vee; \mathcal{L}^\vee) \rightarrow 0
\]
with the boundary operator
\[
\partial^{u^{-1}} : \mathcal{C}_k(T^\vee; \mathcal{L}^\vee) \rightarrow \mathcal{C}_{k-1}(T^\vee, D^\vee; \mathcal{L}^\vee),
\]
which induces an exact sequence of the twisted homology groups:
\[
(5.1) \quad 0 \rightarrow H_1(T^\vee; \mathcal{L}^\vee) \rightarrow H_1(T^\vee, D^\vee; \mathcal{L}^\vee) \rightarrow H_0(D^\vee; \mathcal{L}^\vee) \rightarrow 0.
\]
By Theorem 3.1, \( H_1(T^\vee, D^\vee; \mathcal{L}^\vee) \) is \( m+1 \) dimensional for any \( \alpha \).

We define the intersection form between \( H_1(X^\vee, D^\vee; \mathcal{L}^\vee) \) and \( H_1(X, D; \mathcal{L}) \) as follows.

**Definition 5.2** (The intersection form). Let elements \( \gamma^\alpha \in H_1(X, D; \mathcal{L}) \) and \( \delta^{u^{-1}} \in H_1(X^\vee, D^\vee; \mathcal{L}^\vee) \) be represented by
\[
\sum c_i \mu_i^\alpha \in \mathcal{C}_1(T, D; \mathcal{L}), \quad \sum d_j \nu_j^{u^{-1}} \in \mathcal{C}_1(T^\vee, D^\vee; \mathcal{L}^\vee),
\]
where \( c_i, d_j \in \mathbb{C} \), and \( \mu_i \) and \( \nu_j \) are 1-simplices in \( T \) and \( T^\vee \), respectively. We suppose that if \( \mu_i \cap \nu_j \neq \emptyset \) then \( \mu_i \) and \( \nu_j \) intersect transversely at a point \( p_{ij} \). The intersection form \( \mathcal{I}_h \) is defined by
\[
\mathcal{I}_h(\delta^{u^{-1}}, \gamma^\alpha) = - \sum_{p_{ij} \in \mu_i \cap \nu_j} (d_j \cdot c_i) \times [\nu_j \cdot \mu_i]_{p_{ij}} \times (u^{-1}|_{\nu_j}(p_{ij}) \cdot u|_{\mu_i}(p_{ij})),
\]
where \([\nu_j \cdot \mu_i]_{p_{ij}} (= \pm 1)\) is the topological intersection number of \(\mu_i\) and \(\nu_j\) at \(p_{ij}\), and \(u|_{\mu_i}(p_{ij})\) and \(u^{-1}|_{\nu_j}(p_{ij})\) are the values of \(u|_{\mu_i}(t)\) and \(u^{-1}|_{\nu_j}(t)\) at \(p_{ij}\).

**Remark 5.3.** If \(\alpha \in (\mathbb{C} - \mathbb{Z})^{m+3}\) then \(I_h(\delta_{i}^{-1}, \gamma_{i})\) is equal to the intersection number \(\gamma_{i} \cdot \delta_{i}^{-1}\) defined in [AK §2.3] and [Y2 §4.7]. Pay your attention to the layout of \(\gamma_{i}\) and \(\delta_{i}^{-1}\) in our intersection form \(I_h\) and to the construction of the intersection matrix \(H\) in Proposition 5.4. There is an advantage of our setting in the study of twisted period relations in [7].

We give \(m + 1\) elements \(\delta_{1}^{-1}, \ldots, \delta_{m+1}^{-1}\) of \(H_{1}(X^{\vee}, \mathcal{D}^{\vee}; \mathcal{L}^{\vee})\).

1. In the case \(\alpha \notin \mathbb{Z}^{m+3}\),

\[
\delta_{j}^{-1} = \begin{cases} 
- \gamma_{ij}^{-1} & \text{if } 1 \leq j \leq r, \\
\ell_{ij}^{-1} - \frac{\gamma_{ij}^{-1}}{1 - \lambda_{j}^{-1}} & \text{if } r + 1 \leq j \leq r + s, \\
\gamma_{ij}^{-1} - \frac{\lambda_{j}^{-1}}{1 - \lambda_{j}^{-1}} + \gamma_{ij}^{-1} & \text{if } r + s + 1 \leq j \leq m + 1.
\end{cases}
\]

2. In the case \(\alpha \in \mathbb{Z}^{m+3}\),

\[
\delta_{j}^{-1} = \begin{cases} 
- \gamma_{ij}^{-1} & \text{if } 1 \leq j \leq r - 1, \\
\ell_{ij}^{-1} - \ell_{ij}^{-1} & \text{if } r \leq j \leq r + s - 2 = m + 1.
\end{cases}
\]

**Proposition 5.4.** The intersection matrix \(H = (I_h(\delta_{i}^{-1}, \gamma_{j}))_{1 \leq i \leq m+1, \ 1 \leq j \leq m+1}\) is as follows.

1. In the case \(\alpha \notin \mathbb{Z}^{m+3}\), it is

\[
\begin{pmatrix}
E_{r} & O & O \\
O & E_{s} & H_{32} \\
O & O & H_{33}
\end{pmatrix}, \quad H_{32} = \begin{pmatrix}
\lambda_{i_{r}+1} - 1 & \cdots & \lambda_{i_{m}+1} - 1 \\
\vdots & \ddots & \vdots \\
\lambda_{i_{r}+1} - 1 & \cdots & \lambda_{i_{m}+1} - 1
\end{pmatrix},
\]

\[
H_{33} = \begin{pmatrix}
\lambda_{i_{r}+1} - 1 & (\lambda_{i_{r}+2} - 1)(1 - \lambda_{i_{r}+1}) & \cdots & (\lambda_{i_{m}+1} - 1)(1 - \lambda_{i_{r}+1}) \\
0 & \lambda_{i_{r}+2} - 1 & \cdots & (\lambda_{i_{m}+1} - 1)(1 - \lambda_{i_{r}+1}) \\
0 & 0 & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_{i_{m}+1} - 1
\end{pmatrix},
\]

where \(E_{r}\) is the unit matrix of size \(r\).

2. In the case \(\alpha \in \mathbb{Z}^{m+3}\), it is the unit matrix \(E_{m+1}\).

The elements \(\delta_{i}^{-1}, \ldots, \delta_{m+1}^{-1}\) form a basis of \(H_{1}(X^{\vee}, \mathcal{D}^{\vee}; \mathcal{L}^{\vee})\).

**Proof.** We can easily evaluate \(I_h(\delta_{i}^{-1}, \gamma_{j})\) with following variations of branches of \(u(t)\) and \(u(t)^{-1}\). The regularity of the intersection matrix shows that \(\delta_{i}^{-1}, \ldots, \delta_{m+1}^{-1}\) is a basis of \(H_{1}(X^{\vee}, \mathcal{D}^{\vee}; \mathcal{L}^{\vee})\). \(\square\)
6. Relative twisted dual cohomology groups

The relative twisted dual cohomology group \( H^k(X, D; \mathcal{L}) \) is defined as the dual space of \( H_k(X, D; \mathcal{L}) \). We define three complexes \( \Omega^\bullet(T^\vee, D^\vee; \mathcal{L}) \), \( \mathcal{E}^\bullet(T^\vee, D^\vee; \mathcal{L}) \), and \( \mathcal{E}^\bullet_\mathcal{L}(T^\vee, D^\vee; \mathcal{L}) \) by changing the roles in \( \Omega^\bullet(T, D; \mathcal{L}) \), \( \mathcal{E}^\bullet(T, D; \mathcal{L}) \) and \( \mathcal{E}^\bullet_\mathcal{L}(T, D; \mathcal{L}) \) as

\[
T \rightarrow T^\vee, \quad D \rightarrow D^\vee, \quad u(t) \rightarrow 1/u(t), \quad \nabla_i \rightarrow \nabla_i^\vee = d - \omega \wedge;
\]

that is,

\[
\begin{align*}
\Omega^0(T^\vee, D^\vee; \mathcal{L}) &= \{ f(t) \in \Omega^0(\tilde{\mathcal{E}}) \mid \ord_{x_i} f(t)/u(t) \geq 1 \text{ for any } x_i \in D^\vee \}, \\
\Omega^1(T^\vee, D^\vee; \mathcal{L}) &= \{ \varphi(t) \in \Omega^0(\tilde{\mathcal{E}}) \mid \ord_{x_i} \varphi(t)/u(t) \geq 0 \text{ for any } x_i \in D^\vee \}, \\
\mathcal{E}^0(T^\vee, D^\vee; \mathcal{L}) &= \{ \varphi(t) \in \mathcal{E}^0(\tilde{\mathcal{E}}) \mid \lim_{t \rightarrow \tilde{x}_i} f(t)/u(t) = 0 \text{ for any } i \in I^0_{\mathcal{L}} \}, \\
\mathcal{E}^k(T^\vee, D^\vee; \mathcal{L}) &= \{ \varphi(t) \in \mathcal{E}^k(\tilde{\mathcal{E}}) \mid \varphi(t)/u(t) \text{ is } C^\infty \text{ on } U_i \text{ for any } i \in I^0_{\mathcal{L}} \} \text{ (} k = 1, 2 \), \\
\mathcal{E}^k_\mathcal{L}(T^\vee, D^\vee; \mathcal{L}) &= \{ \varphi(t) \in \mathcal{E}^k(\tilde{\mathcal{E}}) \mid \varphi(t) \text{ is identically 0 on } V_i \} \text{ for any } i \in I^0_{\mathcal{L}} \cup I^\infty_{\mathcal{L}} \}.
\end{align*}
\]

We have relative twisted de Rham dual cohomology groups

\[
H^k_{\text{alg}}(T^\vee, D^\vee; \mathcal{L}), \quad H^k_{C^\infty}(T^\vee, D^\vee; \mathcal{L}), \quad H^k_{C^\infty_\mathcal{L}}(T^\vee, D^\vee; \mathcal{L}),
\]

as the \( k \)-th cohomology groups of the complexes \( \Omega^\bullet(T^\vee, D^\vee; \mathcal{L}) \), \( \mathcal{E}^\bullet(T^\vee, D^\vee; \mathcal{L}) \), and \( \mathcal{E}^\bullet_\mathcal{L}(T^\vee, D^\vee; \mathcal{L}) \), respectively. There is a natural pairing between \( H^1(X, D^\vee; \mathcal{L}) \) and \( H^1_{C^\infty}(T^\vee, D^\vee; \mathcal{L}) \) (resp. \( H^1_{\text{alg}}(T^\vee, D^\vee; \mathcal{L}) \) and \( H^1_{C^\infty_\mathcal{L}}(T^\vee, D^\vee; \mathcal{L}) \)) defined by

\[
\langle \delta^u, \varphi \rangle = \int_{\delta^u} \frac{\psi}{u(t)}
\]

for \( \delta^u \in H^1(X, D^\vee; \mathcal{L}) \) and \( \psi \in H^1_{C^\infty}(T^\vee, D^\vee; \mathcal{L}) \). Here note that the layout of \( \delta^u \) and \( \psi \) in this pairing is different from that in (4.3). As is shown in (4.1) \( H^k(X, D^\vee; \mathcal{L}) \) is canonically isomorphic to \( H^k_{\text{alg}}(T^\vee, D^\vee; \mathcal{L}) \), \( H^k_{C^\infty}(T^\vee, D^\vee; \mathcal{L}) \) and \( H^k_{C^\infty_\mathcal{L}}(T^\vee, D^\vee; \mathcal{L}) \).

**Definition 6.1** (Intersection form between \( H^1(T, D; \mathcal{L}) \) and \( H^1(T^\vee, D^\vee; \mathcal{L}) \)). The intersection form \( \mathcal{I}_\mathcal{L} \) between \( H^1(T, D; \mathcal{L}) \) and \( H^1(T^\vee, D^\vee; \mathcal{L}) \) is defined by

\[
\mathcal{I}_\mathcal{L}(\varphi, \psi) = \int_{T \cap T^\vee} \varphi \wedge \psi
\]

for \( \varphi \in H^1_{C^\infty}(T, D; \mathcal{L}) \) and \( \psi \in H^1_{C^\infty_\mathcal{L}}(T^\vee, D^\vee; \mathcal{L}) \).

Though the supports of \( \varphi \in H^1_{C^\infty}(T, D; \mathcal{L}) \) and \( \psi \in H^1_{C^\infty_\mathcal{L}}(T^\vee, D^\vee; \mathcal{L}) \) are not necessarily compact, \( \varphi \wedge \psi \) is a \( C^\infty \) 2-form with a compact support included in \( \mathbb{P}^1 - \bigcup_{i=0}^{n+2} V_i \). Thus the intersection form \( \mathcal{I}_\mathcal{L}(\varphi, \psi) \) is well-defined. Even in the case where

\[
\int_{T \cap T^\vee} \varphi \wedge \psi
\]

is not well-defined for elements \( \varphi \in H^1_{C^\infty}(T, D; \mathcal{L}) \) and \( \psi \in H^1_{C^\infty_\mathcal{L}}(T^\vee, D^\vee; \mathcal{L}) \), \( \mathcal{I}_\mathcal{L}(\varphi, \psi) \) is always defined by elements \( \varphi' \in \mathcal{E}^1_\mathcal{L}(T, D; \mathcal{L}) \) and \( \psi' \in \mathcal{E}^1_\mathcal{L}(T^\vee, D^\vee; \mathcal{L}) \) cohomological to \( \varphi \) and \( \psi \) as elements of \( H^1_{C^\infty}(T, D; \mathcal{L}) \) and \( H^1_{C^\infty_\mathcal{L}}(T^\vee, D^\vee; \mathcal{L}) \), respectively. In particular, we have the following.
Theorem 6.2. The isomorphisms \( i_D \) in (4.4) and
\[
i_D^*: H^1_{\text{alg}}(T^\vee, D^\vee; \mathcal{L}^\vee) \to H^1_\text{alg}(T^\vee, D^\vee; \mathcal{L}^\vee)
\]
induce the intersection form between \( H^1_{\text{alg}}(T, D; \mathcal{L}) \) and \( H^1_{\text{alg}}(T^\vee, D^\vee; \mathcal{L}^\vee) \), which is expressed as
\[
\mathcal{I}_i(\varphi, \psi) = \mathcal{I}_i(i_D(\varphi), i_D^*(\psi)) = \int_{T^\vee/T^\vee} i_D(\varphi) \wedge i_D^*(\psi)
\]
\[
= 2\pi \sqrt{-1} \left( \sum_{i \in I^0_{\text{alg}}} \text{Res}_x(f_i \cdot \psi) - \sum_{i \in I^0_{\text{alg}}} \text{Res}_x(g_i \cdot \varphi) + \frac{1}{2} \sum_{i \in I^0_{\text{alg}}} \text{Res}_x(f_i \cdot \psi - g_i \cdot \varphi) \right),
\]
where \( \varphi \in H^1_{\text{alg}}(T, D; \mathcal{L}), \psi \in H^1_{\text{alg}}(T^\vee, D^\vee; \mathcal{L}^\vee) \), \( \nabla_i f_i = \varphi \) around \( x_i \) for \( i \in I^0_{\text{alg}}, f_i(x_i) = 0 \) for \( i \in I^0_{\text{alg}} \), \( \nabla_i g_i = \psi \) around \( x_i \) for \( i \in I^0_{\text{alg}} \), \( g_i(x_i) = 0 \) for \( i \in I^0_{\text{alg}} \), and \( \text{Res}_x(\eta) \) denotes the residue of a meromorphic 1-form \( \eta \) at \( t = x_i \).

Proof. By the expression (4.4), we see that the support of \( i_D(\varphi) \wedge i_D^*(\psi) \) is included in the closure of \( \bigcup (U_i - V_i) \). The restriction of \( i_D(\varphi) \wedge i_D^*(\psi) \) to \( U_i - V_i \) becomes as follows:
\[
i \in I^0_{\text{alg}} \implies \varphi - \nabla_i(h_i f_i) \wedge \psi = -d(h_i f_i) \wedge \psi = -d(h_i f_i \psi),
\]
\[
i \in I^0_{\text{alg}} \implies \varphi \wedge (\psi - \nabla_i(h_i g_i)) = -\varphi \wedge d(h_i g_i) = d(h_i g_i \varphi),
\]
\[
i \in I^\vee_{\text{alg}} \implies (\varphi - \nabla_i(h_i f_i)) \wedge (\psi - \nabla_i(h_i g_i))
\]
\[
= d(-h_i f_i \psi + h_i g_i \varphi) - d(h_i f_i) \wedge h_i g_i + d(h_i g_i) \wedge d(h_i g_i)
\]
\[
= d(-h_i f_i \psi + h_i g_i \varphi) + d(h_i g_i) \wedge (f_i \psi + g_i \varphi) + (f_i \psi, g_i \varphi)
\]
\[
= d(h_i(\varphi - f_i \psi + g_i \varphi)) + \frac{1}{2} d(h_i^2(f_i \psi - g_i \varphi)).
\]

Thus we have
\[
\int_{U_i - V_i} i_D(\varphi) \wedge i_D^*(\psi)
\]
\[
= \begin{cases} 
\int_{U_i - V_i} -d(h_i f_i \psi) = \int_{\partial V_i} f_i \psi, & \text{if } i \in I^0_{\text{alg}}, \\
\int_{U_i - V_i} d(h_i g_i \varphi) = \int_{\partial V_i} g_i \varphi, & \text{if } i \in I^0_{\text{alg}}, \\
\int_{U_i - V_i} d\left(\frac{1}{2} h_i(\varphi - f_i \psi + g_i \varphi)\right) = \int_{\partial V_i} \frac{1}{2} (f_i \psi - g_i \varphi), & \text{if } i \in I^\vee_{\text{alg}},
\end{cases}
\]
by Stokes’ theorem, since \( h_i \) is identically 1 on \( \partial(V_i) \) and identically 0 on \( \partial(U_i) \).
Apply the residue theorem to the integrals along \( \partial V_i \).
Remark 6.3. Since $\text{Res}_{x_i}(f_i \cdot \psi) = \text{Res}_{x_i}(-g_i \cdot \varphi)$ for $i \in I_{Z^c}$, we have

$$I_c(\varphi, \psi) = 2\pi \sqrt{-1} \left( \sum_{i \in I_{Z^c}^0 \cup I_{Z^c}^c} \text{Res}_{x_i}(f_i \cdot \psi) - \sum_{i \in I_{Z^c}^0} \text{Res}_{x_i}(g_i \cdot \varphi) \right)$$

$$= 2\pi \sqrt{-1} \left( \sum_{i \in I_{Z^c}^0} \text{Res}_{x_i}(f_i \cdot \psi) - \sum_{i \in I_{Z^c}^0 \cup I_{Z^c}^c} \text{Res}_{x_i}(g_i \cdot \varphi) \right).$$

We give $(m + 2)$ elements $\varphi_{i,m+2}$ $(0 \leq i \leq m + 1)$ of $H^1_{C,\infty}(T, D; \mathcal{L})$ by

$$\varphi_{i,m+2} = \begin{cases} \frac{\alpha_i dt}{t - x_i} & \text{if } \alpha_i \neq 0, \\ -u(x_i)dh_i(t) & \text{if } \alpha_i = 0, \end{cases} \quad (0 \leq i \leq m),$$

(6.2)

$$\varphi_{m+1,m+2} = \varphi_0 = \frac{dt}{t - 1}.$$ We check that $\varphi_{i,m+2} \in \mathcal{E}^1(T, D; \mathcal{L})$ for $0 \leq i \leq m + 1$. If $x_i$ $(0 \leq i \leq m)$ belongs to $D$, then $\alpha_i = 0$ or $\alpha_i \in \mathbb{N}$, and we can see that $u(t)\varphi_{i,m+2}$ is smooth around $x_i$ in both cases. If $x_{m+1} = 1$ belongs to $D$, then $\alpha_{m+1} \in \mathbb{N}$ and $u(t)\varphi_{m+1,m+2}$ is holomorphic around $x_{m+1} = 1$. If $x_{m+2} = \infty$ belongs to $D$, then $\alpha_{m+2} \in \mathbb{N}$, and $u(t)\varphi_{i,m+2}$ $(0 \leq i \leq m + 1)$ is smooth around $x_{m+2} = \infty$, since $\varphi_{i,m+2}$ has a simple pole at $t = \infty$ or vanishes identically around $t = \infty$. Note that if $\alpha_i = 0$ then $u(t)$ is non-zero holomorphic around $t = x_i$. Thus $\varphi_{i,m+2}$ belongs to $\mathcal{E}^1(T, D; \mathcal{L})$ in any cases. It is clear that $\nabla \varphi_{m+1,m+2} = 0$ and $\nabla \varphi_{i,m+2} = 0$ for $\alpha_i \neq 0$. For $\alpha_i = 0$, we have

$$\nabla \varphi_{m+1,m+2} = -u(x_i)\nabla \left( \frac{1}{u(t)} \cdot dh_i(t) \right) = -u(x_i) \left( (\nabla \frac{1}{u(t)}) \wedge dh_i(t) + \frac{d(dh_i(t))}{u(t)} \right) = 0$$

since $\nabla \frac{1}{u(t)} = 0$. Thus $\varphi_{i,m+2}$’s represent elements of $H^1_{C,\infty}(T, D; \mathcal{L})$.

We also give $(m + 2)$ elements $\psi_{0,i}$ $(1 \leq i \leq m + 2)$ of $H^1_{C,\infty}(T^\vee, D^\vee; \mathcal{L}^\vee)$ by

$$\psi_{0,i} = \frac{dt}{t - x_i} - \frac{dt}{t} \quad (1 \leq i \leq m),$$

(6.3)

$$\psi_{0,m+1} = \begin{cases} \frac{\alpha_{m+1}}{u(t)dh_{m+1}(t)} & \text{if } \alpha_{m+1} \neq 0, \\ \frac{dt}{t - 1} & \text{if } \alpha_{m+1} = 0, \end{cases}$$

$$\psi_{0,m+2} = \begin{cases} -\alpha_{m+2} \frac{dt}{t} & \text{if } \alpha_{m+2} \neq 0, \\ u(t)dh_{m+2}(t) & \text{if } \alpha_{m+2} = 0. \end{cases}$$

As shown previously, we can check that $\psi_{i,m+2} \in H^1(T^\vee, D^\vee; \mathcal{L}^\vee)$ for $1 \leq i \leq m + 2$. Here we use the property $\nabla \psi_{i,m+2} = 0$.

Theorem 6.4. (1) For $1 \leq i, j \leq m + 1$, we have

$$I_c(\varphi_{i,m+2}, \psi_{0,j}) = 2\pi \sqrt{-1} \delta_{[i,j]}.$$
where \( \delta_{i,j} \) denotes Kronecker’s symbol. In particular, \( \varphi_{i,m+2} \)'s and \( \psi_{0,i} \)'s \((1 \leq i \leq m + 1)\) are bases of \( H^1_{C,\infty}(T, D; \mathcal{L}) \) and \( H^1_{C,\infty}(T^\vee, D^\vee; \mathcal{L}^\vee) \), respectively.

(2) We have

\[
I_c(\varphi_{0,m+2}, \psi_{0,j}) = -2\pi \sqrt{-1} 1 \quad (1 \leq j \leq m), \quad I_c(\varphi_{0,m+2}, \psi_{0,m+1}) = -2\pi \sqrt{-1} \alpha_{m+1}.
\]

In particular, there is a linear relation

\[
\varphi_{0,m+2} + \sum_{i=1}^{m} \varphi_{i,m+2} + a_{m+1} \varphi_{m+1,m+2} = \omega + \sum_{i \in I_{0,0}^0, \alpha_i = 0} \varphi_{i,m+2} = 0
\]

as elements of \( H^1_{C,\infty}(T, D; \mathcal{L}) \), and \( m + 1 \) elements

\[
\varphi_{0,m+2}, \ldots, \varphi_{i-1,m+2}, \varphi_{i+1,m+2}, \ldots, \varphi_{m+1,m+2} \quad (1 \leq i \leq m)
\]

are linearly independent.

(3) We have

\[
I_c(\varphi_{i,m+2}, \psi_{0,m+2}) = -2\pi \sqrt{-1} \alpha_i \quad (1 \leq i \leq m), \quad I_c(\varphi_{m+1,m+2}, \psi_{0,m+2}) = -2\pi \sqrt{-1}.
\]

In particular, there is a linear relation

\[
\sum_{i=1}^{m} \alpha_i \psi_{0,i} + \psi_{0,m+1} + \psi_{0,m+2} = \omega + \sum_{i \in I_{0,0}^0, \alpha_i = 0} \psi_{0,i} = 0
\]

as elements of \( H^1_{C,\infty}(T^\vee, D^\vee; \mathcal{L}^\vee) \).

**Proof.** (1) In case of \( \alpha_i \neq 0 \), we use Remark[6.3]. At least one of differential equations \( \nabla f(t) = \varphi_{i,m+2} \) or \( \nabla g(t) = \psi_{0,j} \) admits a meromorphic local solution \( f_k(t) \) or \( g_k(t) \) around \( x_k \) \((0 \leq k \leq m + 2)\). If \( f_k(t) \) (or \( g_k(t) \)) is a solution, then it satisfies

\[
\text{ord}_{x_k} f_k(t) = 1 + \text{ord}_{x_k} \varphi_{i,m+2}(t) \quad \text{(or \quad \text{ord}_{x_k} g_k(t) = 1 + \text{ord}_{x_k} \psi_{0,j}(t))}.
\]

Since \( \varphi_{i,m+2} \) and \( \psi_{0,j} \) admit simple poles only on \( t = x_i, \infty \) and on \( t = 0, x_j \), if \( i \neq j \) then \( \text{Res}_{x_k} f_k(t) \psi_{0,j} = 0 \) or \( \text{Res}_{x_k} g_k(t) \varphi_{i,m+2} = 0 \) holds. Thus we have \( I_c(\varphi_{i,m+2}, \psi_{0,j}) = 0 \) for \( i \neq j \). In case of \( i = j = k \), if \( f_i(t) \) (or \( g_i(t) \)) is a solution, then it takes a form

\[
f_i(t) = \begin{cases} 1 + O(t - x_i) & \text{if } 1 \leq i \leq m, \\ \frac{1}{\alpha_{m+1}} + O(t - 1) & \text{if } i = m + 1, \end{cases}
\]

or

\[
g_i(t) = \begin{cases} -\frac{1}{\alpha_i} + O(t - x_i) & \text{if } 1 \leq i \leq m, \\ -1 + O(t - 1) & \text{if } i = m + 1. \end{cases}
\]

where \( O \) denotes Landau’s symbol. The intersection number \( I_c(\varphi_{i,m+2}, \psi_{0,i}) \) is equal to \( 2\pi \sqrt{-1} \times \text{Res}_{x_k} f_k(t) \psi_{0,j} \) or \( -\text{Res}_{x_k} g_k(t) \varphi_{i,m+2} \); it becomes \( 2\pi \sqrt{-1} \) in both cases.

In case of \( \alpha_i = 0 \) \((1 \leq i \leq m)\), note that

\[
\varphi_{i,m+2} \land \psi_{0,j} = -u(x_i) dh_i(t) \land \frac{\psi_{0,j}}{u(t)} = -u(x_i) d\left( h_i(t) \frac{\psi_{0,j}}{u(t)} \right).
\]
and that its support is the closure of $U_i - V_i$. By Stokes’ theorem and the residue theorem, we have

$$\mathcal{I}_c(\varphi_{i,m+2}, \psi_{0,j}) = \iint_{U_i - V_i} \varphi_{i,m+2} \land \psi_{0,j} = -u(x_i) \int_{\partial(U_i - V_i)} h_i(t) \frac{\psi_{0,j}}{u(t)} = u(x_i) \int_{\partial V_i} \frac{\psi_{0,j}}{u(t)}$$

$$= 2\pi \sqrt{-1} \cdot u(x_i) \cdot \text{Res}_{x_i} \frac{\psi_{0,j}}{u(t)} = \begin{cases} 2\pi \sqrt{-1} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Here note that $u(t)$ is non-zero holomorphic around $x_i$ by $\alpha_i = 0$. In case of $\alpha_{m+1} = 0$, we can similarly show.

Since $H^1_{\mathcal{C} \mathcal{L}}(T, D; \mathcal{L})$ and $H^1_{\mathcal{C} \mathcal{L}}(T^{\gamma}, D^{\gamma}; \mathcal{L}^{\gamma})$ are $m + 1$ dimensional, $\varphi_{i,m+2}$’s and $\psi_{0,j}$’s are bases of these spaces.

(2) We can evaluate the intersection number similarly to (1). We can express $\varphi_{0,m+2}$ as a linear combination:

$$\varphi_{0,m+2} = c_1 \varphi_{1,m+2} + \cdots + c_m \varphi_{m,m+2} + c_{m+1} \varphi_{m+1,m+2}.$$

By comparing $\mathcal{I}_c(\varphi_{0,m+2}, \psi_{0,j})$ with

$$\mathcal{I}_c(\sum_{i=1}^{m+1} c_i \varphi_{i,m+1}, \psi_{0,j})$$

we have $c_1 = \cdots = c_m = -1$ and $c_{m+1} = -\alpha_{m+1}$. Thus we obtain the linear relation. This relation is also obtained by the property that

$$\nabla_i (1 - \sum_{i \in I^0_{\mathcal{L}_0}} u(x_i) \frac{h_i(t)}{u(t)})$$

is the zero of $H^1_{\mathcal{C} \mathcal{L}}(T, D; \mathcal{L})$. In fact, since

$$1 - \sum_{i \in I^0_{\mathcal{L}_0}} u(x_i) \frac{h_i(t)}{u(t)} \in \mathcal{E}^0(T, D; \mathcal{L}), \quad \nabla_i 1 = \omega,$$

$$\nabla_i \left( u(x_i) \frac{h_i(t)}{u(t)} \right) = u(x_i) h_i(t) \nabla_i \left( \frac{1}{u(t)} \right) + u(x_i) \frac{dh_i(t)}{u(t)} = u(x_i) \frac{dh_i(t)}{u(t)},$$

we have

$$0 = \nabla_i \left( 1 - \sum_{i \in I^0_{\mathcal{L}_0}} u(x_i) \frac{h_i(t)}{u(t)} \right) = \omega - \sum_{i \in I^0_{\mathcal{L}_0}} u(x_i) \frac{dh_i(t)}{u(t)} + \sum_{i \in I^0_{\mathcal{L}_0}, \alpha_i = 0} \varphi_{i,m+2}.$$

Here note that $u(x_i) = 0$ for $i \in I^0_{\mathcal{L}_0}$ with $\alpha_i \neq 0$.

For $1 \leq i \leq m$, $\varphi_{i,m+2}$ can be expressed as a linear combination of the others, we have the linear independence of the $m + 1$ elements.

(3) We can show the claims similarly to (2). $\square$

**Remark 6.5.** (1) By Theorem 6.4 (2), the $m+1$ elements $\varphi_{0,m+2}, \varphi_{1,m+2}, \ldots, \varphi_{m,m+2}$ are linearly dependent if $\alpha_{m+1} = 0$.

(2) By Theorem 6.4 (3), the $m+1$ elements $\psi_{0,1}, \ldots, \psi_{0,i-1}, \psi_{0,i+1}, \ldots, \psi_{0,m+1}$, $\psi_{0,m+2}$ are linearly dependent if $\alpha_i = 0$ for $1 \leq i \leq m$. 

Proposition 6.6. (1) If \( \alpha_i = 0 \) for \( 0 \leq i \leq m \) then \( \varphi_{i,m+2} \) is cohomologous to
\[
\nabla_t \left( \prod_{j \in I_0^{m+2}} \frac{t-x_j}{x_i-x_j} \right)^{1-\alpha_i} \in \Omega^1(T,D;L)
\]
as elements of \( H^1_{C(M)}(T,D;L) \).
(2) Suppose that one of \( \alpha_{m+1} \) and \( \alpha_{m+2} \) is 0. If \( \alpha_i = 0 \) (\( i = m+1, m+2 \)) then \( \psi_{0,i} \) is cohomologous to \(-\omega \in \Omega^1(T^\prime,D^\prime;L^\prime)\) as elements of \( H^1_{C(M)}(T^\prime,D^\prime;L^\prime) \).
(3) If \( \alpha_{m+1} = \alpha_{m+2} = 0 \) then \( \psi_{0,m+1}, \psi_{0,m+2} \) are cohomologous to
\[
\nabla_t \left( \frac{1-x_j}{t-x_j} \right), \nabla_t \left( \frac{t-1}{t-x_j} \right) \in \Omega^1(T^\prime,D^\prime;L^\prime)
\]
as elements of \( H^1_{C(M)}(T^\prime,D^\prime;L^\prime) \), respectively, where \( j \) is an element of \( I_0^{m+2} \).

Proof. (1) If \( \alpha_i = 0 \) then
\[
\langle \varphi_{i,m+2}, \gamma^u \rangle = \int_\gamma \frac{u(t) \, dh_i(t)}{u(t)} = \left[ u(x_i) \delta_i(t) \right]_{\partial(\gamma)}.
\]
Thus it vanishes for \( \gamma^u \in H^1(T,D;L) \) such that \( x_i \not\in \partial\gamma \), and becomes \( u(x_i) \) for \( \gamma^u \in H^1(T,D;L) \) with a path \( \gamma \) ending at \( x_i \). On the other hand, we have
\[
\langle \nabla_t \left( \prod_{j \in I_0^{m+2}} \frac{t-x_j}{x_i-x_j} \right)^{1-\alpha_j}, \gamma^u \rangle = \int_\gamma u(t) \nabla_t \left( \prod_{j \in I_0^{m+2}} \frac{t-x_j}{x_i-x_j} \right)^{1-\alpha_j} \frac{du(t)}{u(t)} = \left[ u(t) \prod_{j \in I_0^{m+2}} \frac{t-x_j}{x_i-x_j} \right]_{\partial(\gamma)}.
\]
The last term vanishes for \( \gamma^u \in H^1(T,D;L) \) such that \( x_i \not\in \partial\gamma \), and becomes \( u(x_i) \) for \( \gamma^u \in H^1(T,D;L) \) with a path \( \gamma \) ending at \( x_i \). Here we regard \( \frac{t-x_{m+2}}{x_i-x_{m+2}} \) as 1 when \( m+2 \in I_0^{m+2} \). Hence \( \varphi_{i,m+2} \) is cohomologous to this algebraic 1-form as elements of \( H^1_{C(M)}(T,D;L) \).
(2) The assertion is obvious from Theorem 6.4 (3).
(3) If \( \alpha_{m+1} = \alpha_{m+2} = 0 \) then we have \( \lim_{t \to \infty} u(t) = 1 \),
\[
\left[ \frac{t-x_j}{t-x_j} \right]_{t=1} = 1, \quad \left[ \frac{1-x_j}{t-x_j} \right]_{t=\infty} = 0, \quad \left[ \frac{t-1}{t-x_j} \right]_{t=1} = 0, \quad \left[ \frac{t-1}{t-x_j} \right]_{t=\infty} = 1,
\]
\[
\langle \gamma^{u-1}, \psi_{0,m+1} \rangle = \int_\gamma \frac{1}{u(t)} \frac{u(t) dh_{m+1}(t)}{u(1)} = \left[ \frac{h_{m+1}(t)}{u(1)} \right]_{\partial(\gamma)},
\]
\[
\langle \gamma^{u-1}, \nabla_t \left( \frac{1-x_j}{t-x_j} \right) \rangle = \int_\gamma \frac{1}{u(t)} \nabla_t \left( \frac{1-x_j}{t-x_j} \right) = \left[ \frac{1}{u(t)} \frac{1-x_j}{t-x_j} \right]_{\partial(\gamma)},
\]
\[
\langle \gamma^{u-1}, \psi_{0,m+2} \rangle = \int_\gamma \frac{1}{u(t)} \frac{u(t) dh_{m+1}(t)}{u(1)} = \left[ \frac{h_{m+2}(t)}{u(1)} \right]_{\partial(\gamma)},
\]
\[
\langle \gamma^{u-1}, \nabla_t \left( \frac{t-1}{t-x_j} \right) \rangle = \int_\gamma \frac{1}{u(t)} \nabla_t \left( \frac{t-1}{t-x_j} \right) = \left[ \frac{1}{u(t)} \frac{t-1}{t-x_j} \right]_{\partial(\gamma)}.
\]
Note that if \( x_k \in D^\vee \) (0 \( \leq k \leq m \)) then \( \alpha_k \in -\mathbb{N} \) and \( 1/u(x_k) = 0 \). Hence we have
\[
\langle \gamma^{-1}, \psi_{0,m+1} \rangle = \langle \gamma^{-1}, \nabla^\gamma_i \left( \frac{1-x_i}{t-x_j} \right) \rangle, \quad \langle \gamma^{-1}, \psi_{0,m+2} \rangle = \langle \gamma^{-1}, \nabla^\gamma_i \left( \frac{t-1}{t-x_j} \right) \rangle,
\]
which yield the assertion. \( \square \)

7. Twisted period relations

In this section, we show the compatibility of the pairings between relative twisted homology and cohomology groups and the intersection forms \( \mathcal{I}_c \) and \( \mathcal{I}_h \).

**Theorem 7.1.** The intersection form \( \mathcal{I}_c \) is compatible with \( \mathcal{I}_h \) through the isomorphisms \( H^1(T, D; \mathcal{L}) \cong H_1(T^\vee, D^\vee; \mathcal{L}^\vee) \) and \( H^1(T^\vee, D^\vee; \mathcal{L}^\vee) \cong H_1(T, D; \mathcal{L}) \).

**Proof.** By the perfectness of the pairing between \( H_1(T, D; \mathcal{L}) \) and \( H^1(T, D; \mathcal{L}) \), and that of \( \mathcal{I}_h \) between \( H_1(T, D; \mathcal{L}) \) and \( H_1(T^\vee, D^\vee; \mathcal{L}^\vee) \), there exists an isomorphism
\[
\kappa : H_1(T, D; \mathcal{L}) \rightarrow H^1(T^\vee, D^\vee; \mathcal{L}^\vee)
\]
such that
\[
\langle \varphi, \gamma_u \rangle = \mathcal{I}_c(\varphi, \kappa(\gamma_u))
\]
for any \( \varphi \in H^1(T, D; \mathcal{L}) \). We show that this isomorphism also satisfies
\[
\mathcal{I}_h(\delta^{-1}, \gamma_u) = \langle \delta^{-1}, \kappa(\gamma_u) \rangle
\]
for any \( \delta^{-1} \in H_1(T^\vee, D^\vee; \mathcal{L}^\vee) \).

For a twisted cycle \( \triangledown^u_i \) (\( i \in \mathbb{Z}_{\geq 0}^n \)) and any element \( \varphi \in H^1_{C^\omega}(T, D; \mathcal{L}) \), we have
\[
\langle \varphi, \triangledown^u_i \rangle = \int_{\partial V_i} u(t) \varphi = \int_{\int U_i-V_i} u(t) \varphi \wedge dh_i(t) = \int_{\int T \cap T^\vee} \varphi \wedge \zeta^\gamma_i = \mathcal{I}_c(\varphi, \zeta^\gamma_i),
\]
where \( \zeta^\gamma_i = \nabla^\gamma_i (u(t)h_i(t)) = u(t)dh_i(t) \in H^1_{C^\omega}(T, D; \mathcal{L}) \). Thus we have \( \kappa(\triangledown^u_i) = \zeta^\gamma_i \). For \( \delta^{-1}_j \in H_1(T^\vee, D^\vee; \mathcal{L}^\vee) \) (\( 1 \leq j \leq m + 1 \)) given in (5.2) or (5.3), we have
\[
\langle \delta_j^{-1}, \kappa(\triangledown^u_i) \rangle = \langle \delta_j^{-1}, \zeta^\gamma_i \rangle = \int_{\delta_j} \frac{\zeta^\gamma_i}{u(t)} = \int_{\delta_j} dh_i(t) = [h_i(t)]_{\delta(\delta_j)} = \mathcal{I}_h(\delta_j^{-1}, \zeta^\gamma_i).
\]

Let \( \gamma_u \in H_1(T, D; \mathcal{L}) \) be represented by a path \( \gamma \) connecting \( x_i \) and \( x_j \) (\( i, j \in \mathbb{Z}_x \cup I_{\mathbb{Z}_0} \)) with a branch of \( u(t) \) on it. Though we consider a small circle with center \( x_i \) for \( x_i \in B \) in our construction of a basis of \( H_1(T, D; \mathcal{L}) \), we may ignore it since \( \varphi \in H^1_{C^\omega}(T, D; \mathcal{L}) \) is identically 0 around \( x_i \). We define a \( C^\infty \) function \( h_\gamma(t) \) on \( \mathbb{P}^1 - \gamma \) satisfying
\[
h_\gamma(t) = \begin{cases} 1 & \text{if } V_\gamma, \\ 0 & \text{if } U_\gamma, \end{cases}
\]
where open sets \( V_\gamma \subset U_\gamma \) are in the right side of \( \gamma \) with respect to its orientation, see Figure 2. We define \( \zeta^\gamma_i \in E^1(T^\vee, D^\vee; \mathcal{L}^\vee) \) by
\[
\zeta^\gamma_i = \nabla^\gamma_i (u(t)h_\gamma(t)) = u(t)dh_\gamma(t) + h_\gamma(t)\nabla^\gamma_i (u(t)) = u(t)dh_\gamma(t).
\]
Here note that \( u(t) = u|_{\gamma}(t) \) is a (single valued) branch on \( U_{\gamma} \), \( u(t)h_\gamma(t) \) can be regarded as single valued on \( T^\vee - \gamma \), and \( \zeta^\vee_\gamma \) is extended to 0 on \( \gamma \) and it is \( C^\infty \) on \( T^\vee \). Then we have

\[
\int_{T^\vee - \gamma} \varphi \land \zeta^\vee_\gamma = \int_{U_{\gamma}} \varphi \land \zeta^\vee_\gamma = \int_{\partial U_{\gamma}} - (u(t)h_\gamma(t)) \cdot \varphi = \int_{\gamma} u(t) \varphi = \langle \varphi, \gamma^u \rangle,
\]

by Stokes’ theorem since

\[-d((u(t)h_\gamma(t)) \cdot \varphi) = -(d(u(t)h_\gamma(t)) + u(t)dh_\gamma(t)) \land \varphi = (u(t)h_\gamma(t))d\varphi\]

\[= \varphi \land u(t)dh_\gamma(t) - u(t)h_\gamma(t)(\omega \land \varphi + d\varphi) = \varphi \land \zeta^\vee_\gamma - u(t)h_\gamma(t)(\nabla_\varphi \varphi) = \varphi \land \zeta^\vee_\gamma.
\]

Thus we have \( \kappa(\gamma^u) = \zeta^\vee_\gamma \).

Let \( \delta_{\pm}^{-1} \) be elements of \( H_1(T^\vee, D^\vee; L^\vee) \) with paths \( \delta_+ \) and \( \delta_- \) intersecting \( \gamma \) with topological intersection number +1 and -1, respectively. We assume that the branches \( u|_{\delta_{\pm}^{-1}} \) satisfy \( u|_{\gamma}(p_{\pm}) \cdot u|_{\delta_{\pm}^{-1}}(p_{\pm}) = 1 \) at the intersection points \( p_{\pm} = \gamma \cap \delta_{\pm} \). We have

\[
\int_{\delta_{\pm}} u|_{\delta_{\pm}^{-1}}(t) \zeta^\vee_\gamma = \int_{\delta_{\pm} \cap U_{\gamma}} u|_{\delta_{\pm}^{-1}}(t)u(t)dh_\gamma(t) = [h_\gamma(t)]_{\partial(\delta_{\pm} \cap U_{\gamma})} = \pm 1,
\]

which means that \( \langle \delta_{\pm}^{-1}, \kappa(\gamma^u) \rangle = \mathcal{I}_h(\delta_{\pm}^{-1}, \gamma^u) \).

Let \( (\gamma^u_1, \ldots, \gamma^u_{m+1}) \), \( (\varphi_1, \ldots, \varphi_{m+1}) \), \( (\delta^u_1, \ldots, \delta^u_{m+1}) \) and \( (\psi_1, \ldots, \psi_{m+1}) \) be any bases of \( H_1(T, D; L) \), \( H^1_{L^\vee}(T, D; L) \), \( H_1(T^\vee, D^\vee; L^\vee) \), and \( H^1_{L^\vee}(T^\vee, D^\vee; L^\vee) \), respectively. We define four matrices by

\[
\Phi = \left( \langle \varphi_i, \gamma^u_j \rangle \right)_{1 \leq i \leq m+1 \atop 1 \leq j \leq m+1}, \quad \Psi = \left( \langle \delta^u_i, \psi_j \rangle \right)_{1 \leq i \leq m+1 \atop 1 \leq j \leq m+1},
\]

\[
H = \left( \mathcal{I}_h(\delta^u_i, \gamma^u_j) \right)_{1 \leq i \leq m+1 \atop 1 \leq j \leq m+1}, \quad C = \left( \mathcal{I}_c(\varphi_i, \psi_j) \right)_{1 \leq i \leq m+1 \atop 1 \leq j \leq m+1}.
\]

**Theorem 7.2.** The matrices \( \Phi, \Psi, H \) and \( C \) satisfy a twisted period relation

\[
(7.2) \quad H = \Psi C^{-1} \Phi \quad (\Leftrightarrow \quad C = \Phi H^{-1} \Psi).
\]

**Proof.** Let \( K \) be the representation matrix of \( \kappa \) in \( (7.1) \) with respect to the bases \( (\gamma^u_1, \ldots, \gamma^u_{m+1}) \) and \( (\psi_1, \ldots, \psi_{m+1}) \) of \( H_1(T, D; L) \) and \( H^1_{L^\vee}(T^\vee, D^\vee; L^\vee) \). Then the matrix \( K \) satisfies

\[
(\kappa(\gamma^u_1), \ldots, \kappa(\gamma^u_{m+1})) = (\psi_1, \ldots, \psi_{m+1})K.
\]
Since
\[ \mathcal{I}_c(\varphi_i, \kappa(\gamma^u_j)) = \langle \varphi_i, \gamma^u_j \rangle, \quad \langle \delta_t^{u-1}, \kappa(\gamma^u_j) \rangle = \mathcal{I}_h(\delta_t^{u-1}, \gamma^u_j), \]
we have
\[
\Phi = t(\varphi_1, \ldots, \varphi_{m+1}) \cdot (\gamma^u_1, \ldots, \gamma^u_{m+1}) = t(\varphi_1, \ldots, \varphi_{m+1}) \cdot (\kappa(\gamma^u_1), \ldots, \kappa(\gamma^u_{m+1})) \\
= t(\varphi_1, \ldots, \varphi_{m+1}) \cdot (\psi_1, \ldots, \psi_{m+1}) \cdot K = CK,
\]
\[
H = t(\delta_t^{u-1}, \ldots, \delta_t^{u-1}) \cdot (\gamma^u_1, \ldots, \gamma^u_{m+1}) = t(\delta_t^{u-1}, \ldots, \delta_t^{u-1}) \cdot (\kappa(\gamma^u_1), \ldots, \kappa(\gamma^u_{m+1})) \\
= t(\delta_t^{u-1}, \ldots, \delta_t^{u-1}) \cdot (\psi_1, \ldots, \psi_{m+1}) \cdot K = \Psi K,
\]
where \( \varphi_i \cdot \gamma^u_j, \varphi_i \cdot \psi_j, \delta_t^{u-1} \cdot \gamma^u_j \) and \( \delta_t^{u-1} \cdot \psi_j \) are regarded as \( \langle \varphi_i, \gamma^u_j \rangle, \mathcal{I}_c(\varphi_i, \psi_j), \mathcal{I}_h(\delta_t^{u-1}, \gamma^u_j) \) and \( \langle \delta_t^{u-1}, \psi_j \rangle \), respectively. By eliminating \( K \) from these, we obtain the twisted period relation. \( \square \)

**Example 7.3.** We give examples of twisted period relations for \( m = 2 \).

(1) \( \alpha_0 = \alpha_1 = \cdots = \alpha_4 = 0 \).

In this case, we have \( u(t) = 1, \omega = 0, \nabla_t = d, I = I_\mathbb{Z} = \{0, 1, \ldots, 4\}, I_{2^v} = \emptyset, I_{\mathbb{F}_0} = \{0, 1, 2\}, I_{\mathbb{F}_0} = \{3, 4\}, T = \mathbb{P}^1 - \{1, \infty\} = \mathbb{C} - \{1\}, D = \{0, x_1, x_2\}, T^\vee = \mathbb{P}^1 - \{0, x_1, x_2\} \) and \( D^\vee = \{1, \infty\} \). We array \( x_i \) (\( 0 \leq i \leq 4 \)) as
\[ 0 = x_0 < x_1 < x_2 < x_3 = 1 < x_4 = \infty. \]

We give bases of \( H_1(T, D; \mathcal{L}), H_{alg}^1(T, D; \mathcal{L}), H_1(T^\vee, D^\vee; \mathcal{L}^\vee) \) and \( H_{alg}^1(T^\vee, D^\vee; \mathcal{L}^\vee) \) as
\[
\langle \varphi_0, \psi_0 \rangle = \langle \nabla_t(t), \nabla_t(t^2) \rangle, \\
\begin{pmatrix}
\varphi_0 \\
\nabla_t(t) \\
\nabla_t(t^2)
\end{pmatrix}, \\
\begin{pmatrix}
\ell_0 \\
\ell_1^{u-1} - \ell_2^{u-1} \\
\ell_2^{u-1}
\end{pmatrix}, \\
\begin{pmatrix}
\ell_1^{u-1} - \ell_2^{u-1} \\
\ell_1^{u-1} \\
\ell_2^{u-1}
\end{pmatrix}.
\]

Then the period matrices \( \Phi, \Psi \) and the intersection matrices \( H, C \) become
\[
\Phi = \begin{pmatrix}
2\pi \sqrt{-1} & \log(1 - x_1) & \log(1 - x_2) \\
0 & x_1 & x_2 \\
0 & x_1^2 & x_2^2
\end{pmatrix}, \quad 
H = \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix},
\]
\[
\Psi = \begin{pmatrix}
1 - \log(1 - x_1) & -\log(1 - x_2) \\
2\pi \sqrt{-1} & 0 \\
0 & 2\pi \sqrt{-1}
\end{pmatrix}, \quad 
C = -2\pi \sqrt{-1} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & x_1 & x_2
\end{pmatrix},
\]
which satisfies (7.2).

(2) \( \alpha_0 = \alpha_1 = \alpha_2 = 0, \alpha_3 = 1, \alpha_4 = -1 \).

In this case, we have \( u(t) = t - 1, \omega = \frac{dt}{t - 1}, \nabla_t = d + \frac{dt}{t - 1}, \varphi_0 = \nabla_t(1), I = I_\mathbb{Z} = \{0, 1, \ldots, 4\}, I_{2^v} = \emptyset, I_{\mathbb{F}_0} = \{0, 1, 2, 3\}, I_{\mathbb{F}_0} = \{4\}, T = \mathbb{P}^1 - \{\infty\} = \mathbb{C}, D = \{0, x_1, x_2, x_3\}, T^\vee = \mathbb{P}^1 - \{0, x_1, x_2, 1\} \) and \( D^\vee = \{\infty\} \). Note that \( H_1(T; \mathcal{L}) = 0 \).
We give bases of $H_1(T, D; \mathcal{L})$, $H^1_{alg}(T, D; \mathcal{L})$, $H_1(T^\vee, D^\vee; \mathcal{L}^\vee)$ and $H^1_{alg}(T^\vee, D^\vee; \mathcal{L}^\vee)$ by
\[
(\ell^u_0 - \ell^u_3, \ell^u_1 - \ell^u_3, \ell^u_2 - \ell^u_3), \quad \begin{pmatrix} \varphi_0 \\ \nabla t(t - 1) \\ \nabla t(t - 1)^2 \end{pmatrix}, \quad \begin{pmatrix} \bigodot^1_0^{-1} \\ \bigodot^1_1^{-1} \\ \bigodot^2_1^{-1} \end{pmatrix}, \quad \begin{pmatrix} dt \\ dt \\ dt \end{pmatrix}.
\]
Then the period matrices $\Phi, \Psi$ and the intersection matrices $H, C$ become
\[
\Phi = \begin{pmatrix} -1 & x_1 - 1 & x_2 - 1 \\
1 & (x_1 - 1)^2 & (x_2 - 1)^2 \\
-1 & (x_1 - 1)^3 & (x_2 - 1)^3 \end{pmatrix}, \quad H = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix},
\]
\[
\Psi = 2\pi\sqrt{-1} \begin{pmatrix} -1 & 1 \\
\frac{1}{x_1 - 1} & \frac{1}{x_2 - 1} \end{pmatrix}, \quad C = -2\pi\sqrt{-1} \begin{pmatrix} 1 & 1 & 1 \\
-1 & (x_1 - 1)^2 & (x_2 - 1)^2 \\ 1 & (x_1 - 1)^2 & (x_2 - 1)^2 \end{pmatrix},
\]
which satisfy (7.2).

8. The isomorphism between $H_1(T, D; \mathcal{L})$ and $\text{Sol}_x(a, b, c)$

Let $x$ vary in a small simply connected domain $W$ in $X$. We have trivial vector bundles
\[
\prod_{x \in W} H_1(T_x, D_x; \mathcal{L}_x), \quad \prod_{x \in W} H^1_{alg}(T_x, D_x; \mathcal{L}_x),
\]
where $*$ is $alg$, $C^\infty$, $C^\infty_0$ and the blank. We can extend the pairing $\langle \varphi, \gamma^u \rangle$ to that between sections of these trivial vector bundles. Hereafter, we identify the spaces of local sections of these trivial vector bundles with the fibers $H^1(T, D; \mathcal{L})$ and $H_1(T, D; \mathcal{L})$ at $x \in W$ by the local triviality.

The partial differential operator $\partial_j = \frac{\partial}{\partial x_j}$ $(1 \leq j \leq m)$ acts on $\langle \varphi, \gamma^u \rangle$ for $\varphi \in H^1_{C^\infty_0}(T, D; \mathcal{L})$ and $\gamma^u \in H_1(T, D; \mathcal{L})$ as
\[
\partial_j(\varphi, \gamma^u) = \int_\gamma \partial_j(u(t)\varphi) = \int_\gamma u(t, x)(\partial_j \varphi + \frac{\partial_j u(t, x)}{u(t)}\varphi) = \langle \nabla_j \varphi, \gamma^u \rangle,
\]
where $\nabla_j$ denotes the operator
\[
\partial_j + \frac{\partial_j u(t, x)}{u(t)} = \partial_j + \frac{-\alpha_j}{t - x_j}
\]
in $\mathbb{C}(t, x_1, \ldots, x_m; \partial_1, \ldots, \partial_m)$. The operator $\partial_j$ induces a linear transformation $\nabla_j$ on $H^1_{C^\infty_0}(T, D; \mathcal{L})$.

Though the identity
\[
\partial_j(\varphi, \gamma^u) = \langle \nabla_j \varphi, \gamma^u \rangle
\]
holds for any element $\varphi \in E^1(T, D; \mathcal{L})$ and any element $\gamma^u$ in $H_1(T; \mathcal{L}) \subset H_1(T, D; \mathcal{L})$, the operator $\nabla_j$ cannot act directly on the spaces
\[
H^1_{alg}(T, D; \mathcal{L}), \quad H^1_{C^\infty}(T, D; \mathcal{L}),
\]
since $\Omega^0(T, D; L)$ and $E^0(T, D; L)$ are not kept invariant under the action of $\nabla_j$.
In fact, in case of $\alpha_j = 1$ and $\alpha_i \notin \mathbb{Z} \ (0 \leq i \leq m + 2, \ i \neq j)$, $\omega = \nabla_t(1)$ is the zero of $H^1_{C^\infty}(T, D; L)$ by $\lim_{t \to x_j} u(t) \cdot 1 = 0$ by $\alpha_j = 1$. However

$$\nabla_j(\omega) = \nabla_j \nabla_t(1) = \nabla_{\frac{-1}{t - x_j}} = \sum_{0 \leq i \leq m + 1} \frac{-\alpha_i}{(t - x_j)(t - x_i)} \in \mathcal{E}^1(T, D; L)$$

is not the zero of $H^1_{C^\infty}(T, D; L)$ since $\lim_{t \to x_j} u(t) \cdot \frac{-1}{t - x_j} \neq 0$.

By using the isomorphism $t_D : H^1_{C^\infty}(T, D; L) \to H^1_{C^\infty}(T, D; L) \to H^1_{C^\infty}(T, D; L)$
on $H^1_{C^\infty}(T, D; L)$. Similarly, we have an action on $H^1_{alg}(T, D; L)$ defined by the composition of $\nabla_j \circ t_D$ and the canonical isomorphism $H^1_{C^\infty}(T, D; L) \to H^1_{alg}(T, D; L)$. These actions are simply denoted by $\nabla_j$.

**Lemma 8.1.** Let $\phi_0$ be an element of $H^1_{C^\infty}(T, D; L)$ cohomologous to $\varphi_0 = \varphi_{m+1,m+2} = \frac{dt}{t - 1}$ as elements of $H^1_{C^\infty}(T, D; L)$. Then $\nabla_j \phi_0$ is cohomologous to

$$\frac{\alpha_j \varphi_{m+1,m+2} - \varphi_{j,m+2}}{x_j - 1} = \left\{ \begin{array}{ll}
-\alpha_j \frac{dt}{t - 1} = \alpha_j \left( \frac{dt}{t - 1} - \frac{dt}{t - x_j} \right) & \text{if } \alpha_j \neq 0, \\
\nabla_t \left( \frac{h_j(t)}{u(t)}, \frac{u(x_j)}{x_j - 1} \right) & \text{if } \alpha_j = 0,
\end{array} \right.$$

as elements of $H^1_{C^\infty}(T, D; L)$.

**Proof.** Note that

$$\phi_0 = \varphi_0 - \sum_{i \in I^0_{\bar{R}^0} \cup I_{\bar{F}^e}} \nabla_t h_i(t) f_i(t),$$

where $f_i(t)$ is a single valued meromorphic function on $U_i$ satisfying $\nabla_t(f_i(t)) = \varphi_0$ and $f_i(x_i) = 0$ for $i \in I^0_{\bar{R}^0}$. Since $\nabla_j \nabla_t = \nabla_t \nabla_j$ and $\partial_j h_i(t) = 0$ on $V_k$ for any $k \in I^0_{\bar{R}^0} \cup I_{\bar{F}^e}$, we have

$$\nabla_j \phi_0 = \frac{-\alpha_j dt}{(t - x_j)(t - 1)} - \sum_{i \in I^0_{\bar{R}^0} \cup I_{\bar{F}^e}} \nabla_t \nabla_j(h_i(t)f_i(t))$$

$$= \frac{-\alpha_j dt}{(t - x_j)(t - 1)} - \sum_{i \in I^0_{\bar{R}^0} \cup I_{\bar{F}^e}} \left[ \nabla_t \left( \partial_j h_i(t) \cdot f_i(t) \right) + \nabla_t \left( h_i(t) \nabla_j f_i(t) \right) \right]$$

$$= \frac{-\alpha_j dt}{(t - x_j)(t - 1)} - \sum_{i \in I^0_{\bar{R}^0} \cup I_{\bar{F}^e}} \nabla_t \left( h_i(t) \nabla_j f_i(t) \right).$$

Here note that $\partial_j h_i(t) \cdot f_i(t) \in E^0_{C^\infty}(T, D; L) \subset E^0_{C^\infty}(T, D; L)$. It is easy to see that $h_i(t) \nabla_j f_i(t)$ belongs to $E^0_{C^\infty}(T, D; L)$ for $t \in I_{\bar{F}^e}$. For $i \in I^0_{\bar{R}^0}$, if

$$\lim_{t \to x_i} u(t) \cdot \nabla_j f_i(t) = 0$$
then $h_i(t) \nabla_j f_i(t)$ belongs to $E^{0}_{C,\infty}(T, D; \mathcal{L})$. Since $\text{ord}_{x_i}(\nabla_j \varphi_0) = \text{ord}_{x_i}(\frac{-\alpha_j}{(t - x_j)(t - 1)}) \geq -1$ and $\nabla_j f_i(t)$ satisfies $\nabla_t(\nabla_j f_i(t)) = \nabla_j \varphi_0$, we have $\text{ord}_{x_i} \nabla_j f_i(t) \geq 0$. Note also that $\text{ord}_{x_i} u(t) \geq 0$ for any $i \in I^u_{\text{rel}}$. Thus if $\text{ord}_{x_i} u(t) > 0$ or $\text{ord}_{x_i} f_i(t) > 0$ then $\nabla_t\left(h_i(t)\nabla_j f_i(t)\right)$ is the zero of $H^{1}_{C,\infty}(T, D; \mathcal{L})$. If $I^u_{\text{rel}} \ni i \neq j, m + 1$ then $\text{ord}_{x_i} f_i(t) > 0$, if $I^u_{\text{rel}} \ni i = j, \alpha_j > 0$ then $\text{ord}_{x_j} u(t) > 0$, and if $I^u_{\text{rel}} \ni i = m + 1$ then $\text{ord}_{x_{m+1}} u(t) > 0$ by $\text{ord}_{x_{m+1}}(u(t)\varphi_0) \geq 0$. Hence if $\alpha_j \neq 0$ then

$$\sum_{i \in I^u_{\text{rel}} \cup \{\infty\}} \nabla_t\left(h_i(t)\nabla_j f_i(t)\right)$$

is the zero of $H^{1}_{C,\infty}(T, D; \mathcal{L})$ and $\nabla_t \varphi_0$ is cohomologous to $\frac{-\alpha_j dt}{(t - x_j)(t - 1)}$. If $\alpha_j = 0$ then

$$\sum_{i \in I^u_{\text{rel}} \cup \{\infty\}} \nabla_t\left(h_i(t)\nabla_j f_i(t)\right)$$

is cohomologous to $\nabla_t\left(h_j(t)\nabla_j f_j(t)\right)$. In this case, $f_j(t)$ admits an integral representation

$$\frac{1}{u(t)} \int_{x_j}^{t} u(t') dt' - 1.$$ 

Here note that an integral

$$\frac{1}{u(t)} \int_{p}^{t} \frac{u(t') dt'}{t' - 1}$$

is a solution to $\nabla_t f(t) = \varphi_0$ for any starting point $p$, and that $p$ should be $x_j$ by the vanishing property at $t = x_j$ for the condition $h_j(t)f_j(t) \in E^{0}(T, D; \mathcal{L})$. Hence we have

$$\nabla_j f_j(t) = \nabla_j \left(\frac{1}{u(t)}\right) \cdot \int_{x_j}^{t} \frac{u(t') dt'}{t' - 1} + \frac{1}{u(t)} \cdot \partial_j \left(\int_{x_j}^{t} \frac{u(t') dt'}{t' - 1}\right) = \frac{1}{u(t)} \frac{-u(x_j)}{x_j - 1},$$

since $u(t)$ is independent of $x_j$. Therefore, $\nabla_j(\varphi_0)$ is cohomologous to $\nabla_t\left(h_j(t)\frac{\varphi_0(t)}{u(t)}\right)$ as elements of $H^{1}_{C,\infty}(T, D; \mathcal{L})$ in the case $\alpha_j = 0$.

**Theorem 8.2.** The space of sections of the trivial vector bundle $H_{1}(T, D; \mathcal{L})$ around $x$ is isomorphic to the space $\text{Sol}_x(a, b, c)$ of local solutions to $\mathcal{F}_D(a, b, c)$ around $x \in X$ by the map

$$\mathcal{S} : H_{1}(T, D; \mathcal{L}) \ni \gamma^u \mapsto \langle \varphi_0, \gamma^u \rangle = \int_{\gamma} u(t) \varphi_0 \in \text{Sol}_x(a, b, c).$$

**Proof.** By similar way to [Y1, §6.4], we can show that $\mathcal{S}(\gamma^u) = \langle \varphi_0, \gamma^u \rangle$ is a local solution to $\mathcal{F}_D(a, b, c)$ around $x \in X$ for any $\gamma^u \in H_{1}(T, D; \mathcal{L})$. Since $H_{1}(T, D; \mathcal{L})$ and $\text{Sol}_x(a, b, c)$ are $m + 1$ dimensional, we show that the map $\mathcal{S}$ is surjective. Let $\gamma_i^u (1 \leq i \leq m + 1)$ be the basis of $H_{1}(T, D; \mathcal{L})$ given in (3.6) or (3.7). Since

$$\langle \varphi_0, \gamma_i \rangle = \langle \varphi_0, \gamma_i \rangle, \quad \partial_j \langle \varphi_0, \gamma_i \rangle = \langle \nabla_j \varphi_0, \gamma_i \rangle = \frac{1}{x_j - 1}(\alpha_j \varphi_0 - \varphi_j, m+1, \gamma_i),$$

we complete the proof.
for \(1 \leq i \leq m + 1\) and \(1 \leq j \leq m\), the Wronskian

\[
\left| \begin{array}{ccc}
\langle \phi_0, \gamma_1^u \rangle & \cdots & \langle \phi_0, \gamma_{m+1}^u \rangle \\
\partial_1 \langle \phi_0, \gamma_1^u \rangle & \cdots & \partial_1 \langle \phi_0, \gamma_{m+1}^u \rangle \\
\vdots & \ddots & \vdots \\
\partial_m \langle \phi_0, \gamma_1^u \rangle & \cdots & \partial_m \langle \phi_0, \gamma_{m+1}^u \rangle 
\end{array} \right| = \frac{(-1)^m}{(x_1 - 1) \cdots (x_m - 1)} \left| \begin{array}{ccc}
\langle \varphi_0, \gamma_1^u \rangle & \cdots & \langle \varphi_0, \gamma_{m+1}^u \rangle \\
\langle \varphi_1, \gamma_1^u \rangle & \cdots & \langle \varphi_{m+1}, \gamma_{m+1}^u \rangle \\
\vdots & \ddots & \vdots \\
\langle \varphi_{m+1}, \gamma_1^u \rangle & \cdots & \langle \varphi_{m+1}, \gamma_{m+1}^u \rangle 
\end{array} \right|
\]

does not vanish by the perfectness of the pairing between the relative twisted homology and cohomology groups and Theorem 6.4. Hence \(\langle \varphi_0, \gamma_i \rangle \quad (1 \leq i \leq m + 1)\) are linearly independent as functions in \(x_1, \ldots, x_m\), and the map \(J_{\varphi_0}\) is surjective.

9. INARIANT SUBSPACES OF \(H^1_{C^p}(T; D; \mathcal{L})\) UNDER PARTIAL DIFFERENTIALS

**Proposition 9.1.** The space

\[\nabla_t \mathcal{E}^0(T; \mathcal{L}) = \{ \varphi \in H^1_{C^p}(T; D; \mathcal{L}) \mid \exists f \in \mathcal{E}^0(T; \mathcal{L}) \text{ s.t. } \varphi = \nabla_t f \}\]

is invariant under the action \(\nabla_j \) (\(j = 1, \ldots, m\)), where

\[\mathcal{E}^0(T; \mathcal{L}) = \{ f(t) \in \mathcal{E}^0(\mathbb{R}) \mid u(t) \cdot f(t) \text{ is } C\infty \text{ on } U_i \text{ for any } i \in I_{0_0}^m \}.\]

This space is spanned by \(\nabla_t \left( \frac{h_i(t)}{u(t)} \right)\) for \(i \in I_{0_0}^m\). If \(\alpha \in \mathbb{Z}^{m+3}\) then they satisfy

\[
\sum_{i \in I_{0_0}^m} \nabla_t \left( \frac{h_i(t)}{u(t)} \right) = 0.
\]

The dimension of this space is

\[
\tilde{r} = \begin{cases} 
\frac{r}{r - 1} & \text{if } \alpha \notin \mathbb{Z}^{m+3}, \\
\alpha \in \mathbb{Z}^{m+3} & (r = \#D).
\end{cases}
\]

Each 1-dimensional span of \(\nabla_t \left( \frac{h_i(t)}{u(t)} \right)\) is invariant under the action \(\nabla_j \) (\(j = 1, \ldots, m\)).

**Proof.** Note that if \(D\) is empty then \(\nabla_t \mathcal{E}^0(T; \mathcal{L}) = 0\), since \(\nabla_t f\) is the zero of \(H^1_{C\infty}(T; D; \mathcal{L}) \cong H^1_{C^p}(T; D; \mathcal{L})\) in this case. Let \(\varphi\) be any element of \(\nabla_t \mathcal{E}^0(T; \mathcal{L})\). Then there exists \(f \in \mathcal{E}^0(T; \mathcal{L})\) such that \(\varphi = \nabla_t f\). Note that \(\nabla_j \varphi\) belongs to \(\mathcal{E}^1(T; D; \mathcal{L})\) and admits an expression

\[\nabla_j \varphi = \nabla_j (\nabla_t f) = \nabla_t (\nabla_j f).
\]

Since \(\varphi\) vanishes identically around \(x_i \in B \cup D\), \(f\) is identically 0 around \(x_i \in B\) and \(f\) takes a form \(g(x)/u(t)\) around \(x_i \in D\), where \(g(x)\) is a function independent
of $t$. This local property of $f$ is preserved under $\nabla_j$ by
\[
\nabla_j \left( \frac{g(x)}{u(t)} \right) = g(x) \nabla_j \frac{1}{u(t)} + \frac{\partial_j g(x)}{u(t)} = \frac{\partial_j g(x)}{u(t)}.
\]
$\nabla_t \mathcal{E}^0(T; L)$ is invariant under the action $\nabla_j$ ($j = 1, \ldots, m$). We also see that this space is spanned by
\[
\nabla_t \left( \frac{h_i(t)}{u(t)} \right) = \frac{dh_i(t)}{u(t)}, \quad i \in I^{\varphi}_0.
\]
We show that if $\alpha \in \mathbb{Z}^{m+3}$ then they satisfy the linear relation
\[
\sum_{i \in I^{\varphi}_0} \nabla_t \left( \frac{h_i(t)}{u(t)} \right) = 0.
\]
Under this condition, $1/u(t)$ becomes single valued on $T$ and satisfies $\nabla_t (1/u(t)) = 0$. Thus this linear combination is equal to
\[
-\nabla_t \frac{1}{u(t)} + \sum_{i \in I^{\varphi}_0} \nabla_t \left( \frac{h_i(t)}{u(t)} \right) = \nabla_t \left( -\frac{1}{u(t)} + \sum_{i \in I^{\varphi}_0} h_i(t) \right).
\]
Since
\[
\left( -\frac{1}{u(t)} + \sum_{i \in I^{\varphi}_0} h_i(t) \right) u(t) = -1 + \sum_{i \in I^{\varphi}_0} h_i(t)
\]
vanesishes identically around $x_i \in D$, the function $-\frac{1}{u(t)} + \sum_{i \in I^{\varphi}_0} h_i(t)$ belongs to $\mathcal{E}^0(T, D; L)$ and its $\nabla_t$-image is the zero of $H^1_{1,\varphi}(T, D; L)$. Hence we have the linear relation, and the claim on the dimension of this space.

Note that
\[
\nabla_j \frac{dh_i(t)}{u(t)} = \nabla_h \frac{h_i(t)}{u(t)} = \nabla_i \left( h_i(t) \nabla_j \frac{1}{u(t)} + \frac{\partial_j h_i(t)}{u(t)} \right) = \nabla_i \frac{\partial_j h_i(t)}{u(t)}
\]
is cohomologous to 0, since $\partial_j h_i(t)$ vanishes identically around $x_i \in B \cup D$ for $1 \leq j \leq m$. Thus we have
\[
\nabla_j \left( g(x) \frac{dh_i(t)}{u(t)} \right) = \frac{dh_i(t)}{u(t)} \partial_j g(x) + g(x) \nabla_j \frac{dh_i(t)}{u(t)} = \left( \partial_j g(x) \right) \cdot \frac{dh_i(t)}{u(t)},
\]
which means that each 1-dimensional span of $\nabla_t \left( \frac{h_i(t)}{u(t)} \right)$ is invariant under the action $\nabla_j$ ($1 \leq j \leq m$).

\begin{corollary}
The space $\nabla_t \mathcal{E}^0(T; L)$ coincides with $H_1(T; L)^+ = \{ \varphi \in H^1_{1,\varphi}(T, D; L) : \langle \varphi, \gamma^u \rangle = 0 \text{ for any } \gamma^u \in H_1(T; L) \subset H_1(T, D; L) \}$.\end{corollary}

\begin{proof}
For any elements $\varphi = \nabla_t f \in \nabla_t \mathcal{E}^0(T; L)$ and $\gamma^u \in H_1(T; L)$, we have
\[
\langle \varphi, \gamma^u \rangle = \langle \nabla_t f, \gamma^u \rangle = \langle f, \partial^u \gamma^u \rangle = 0,
\]
which yields that $\nabla_t \mathcal{E}^0(T; L) \subset H_1(T; L)^+$. Since they are of same dimension, they coincide.
\end{proof}
Remark 9.3. Since the spaces $H^1_{\text{alg}}(T, D; \mathcal{L})$, $H^1_{C^\infty}(T, D; \mathcal{L})$ and $H^1_{C^\infty}(T, D; \mathcal{L})$ are canonically isomorphic to $H^1(T, D; \mathcal{L})$, the subspaces

$\{\varphi \in H^1_{\text{alg}}(T, D; \mathcal{L}) \mid \langle \varphi, \gamma^u \rangle = 0 \text{ for any } \gamma^u \in H_1(T; \mathcal{L})\}$,

$\{\varphi \in H^1_{C^\infty}(T, D; \mathcal{L}) \mid \langle \varphi, \gamma^u \rangle = 0 \text{ for any } \gamma^u \in H_1(T; \mathcal{L})\}$,

$\{\varphi \in H^1_{C^\infty}(T, D; \mathcal{L}) \mid \langle \varphi, \gamma^u \rangle = 0 \text{ for any } \gamma^u \in H_1(T; \mathcal{L})\}$

are isomorphic to one another.

For the relative twisted dual homology and cohomology groups, we have trivial vector bundles

$$\prod_{x \in X} H_1(T^\vee, D^\vee; \mathcal{L}^\vee), \quad \prod_{x \in W} H^1_1(T^\vee, D^\vee; \mathcal{L}^\vee),$$

over a simply connected domain $W$ in $X$, where $*$ is alg, $C^\infty$, $C^\infty$ and the blank. We can regard the natural pairing

$$\langle \delta^u, \psi \rangle = \int_\delta \frac{\psi}{u(t)}$$

between $H_1(T^\vee, D^\vee; \mathcal{L}^\vee)$ and $H^1_{C^\infty}(T^\vee, D^\vee; \mathcal{L}^\vee)$ as that between the spaces of local sections of these trivial vector bundles. As mentioned previously, the partial differential operator $\partial_j$ induces a linear transformation

$$\nabla_j^\vee = \partial_j - \frac{\partial_j u(t, x)}{u(t)} = \partial_j + \frac{\alpha_j}{t - x_j}$$

on $H^1_{C^\infty}(T^\vee, D^\vee; \mathcal{L}^\vee)$, $H^1_{C^\infty}(T^\vee, D^\vee; \mathcal{L}^\vee)$ and $H^1_{\text{alg}}(T^\vee, D^\vee; \mathcal{L}^\vee)$. It also acts on $\mathcal{I}_c(\varphi, \psi)$ as

$$\partial_j \mathcal{I}_c(\varphi, \psi) = \int_{T \cap T^\vee} \left[ \partial_j (u(t) \varphi) \wedge \frac{\psi}{u(t)} + u(t) \varphi \wedge \frac{\partial_j \psi}{u(t)} \right]$$

$$= \int_{T \cap T^\vee} u(t) \nabla_j (\varphi) \wedge \frac{\psi}{u(t)} + \int_{T \cap T^\vee} u(t) \varphi \wedge \frac{\nabla_j^\vee (\psi)}{u(t)} + \mathcal{I}_c(\varphi, \psi)$$

(9.1)

where $\varphi \in H^1_{C^\infty}(T, D; \mathcal{L})$ and $\psi \in H^1_{C^\infty}(T^\vee, D^\vee; \mathcal{L}^\vee)$.

Corollary 9.4. Suppose that $k \in I^m_{\mathbb{Z}}$. Then the space

$$\{u(t) dh_k(t) \mid \mathcal{I}_c(\varphi, u(t) dh_k(t)) = 0\}$$

is invariant under the action $\nabla_j$ ($j = 1, \ldots, m$), and

$$\dim(u(t) dh_k(t)) = \begin{cases} m + 1 & \text{if } \alpha \in \mathbb{Z}^{m+3}, \#(I^m_{\mathbb{Z}}) = 1, \\ m & \text{otherwise.} \end{cases}$$

The space $(u(t) dh_k(t))$ coincides with

$$(\mathcal{C}^\vee_k)^+ = \{\varphi \in H^1_{C^\infty}(T, D; \mathcal{L}) \mid \langle \varphi, \mathcal{C}^\vee_k \rangle = 0\}.$$
for $1 \leq j \leq m$. Since
\[ \nabla^\gamma_j(u(t)dh_k(t)) = \nabla^\gamma_j\nabla^\gamma_i(u(t)h_k(t)) = \nabla^\gamma_i\nabla^\gamma_j(u(t)h_k(t)) \]
and $\partial_j h_k(t)$ vanishes identically around $x_i$ for $i \in I^{\infty}_T$, $\nabla^\gamma_j(u(t)dh_k(t))$ is the zero of $H^1_{\nabla^\gamma_\psi}(T^\vee, D^\vee; \mathcal{L}^\vee)$. Hence we have $\mathcal{I}_c(\nabla_j\varphi, u(t)dh_k(t)) = 0$ and $\nabla_j\varphi \in (u(t)dh_k(t))^\perp$.

By the perfectness of $\mathcal{I}_c$, if $u(t)dh_k(t)$ is not the zero of $H^1_{\nabla^\gamma_\psi}(T^\vee, D^\vee; \mathcal{L}^\vee)$ then $\nabla_j\varphi \in (u(t)dh_k(t))^\perp$ is $m$ dimensional. We show that $u(t)dh_k(t)$ degenerates only the case $\alpha \in \mathbb{Z}^{m+3}$ and $I^{\infty}_{\nabla} = \{k\}$. In this case, we have
\[ \nabla^\gamma_i(u(t)h_k(t)) = \nabla^\gamma_i(u(t)(h_k(t) - 1)) \]
by $\nabla^\gamma_i u(t) = 0$. Since $I^{\infty}_{\nabla} = \{k\}$ and $u(t)(h_k(t) - 1)$ vanishes identically around $x_k$, it belongs to $\mathcal{E}_1^{\nabla}(T^\vee, D^\vee, \mathcal{L}^\vee)$, which means that $\nabla^\gamma_i(u(t)(h_k(t) - 1))$ is the zero of $H^1_{\nabla^\gamma_\psi}(T^\vee, D^\vee; \mathcal{L}^\vee)$. Except in this case, we can make a relative cycle $\delta^{u-1}$ by $\ell_k^{u-1}$, it satisfies $\langle \delta^{u-1}, u(t)dh_k(t) \rangle = 1$. Thus $u(t)dh_k(t)$ is different from the zero of $H^1_{\nabla^\gamma_\psi}(T^\vee, D^\vee; \mathcal{L}^\vee)$.

We have shown in Proof of Theorem 7.1 that $\langle \varphi, \mathcal{C}_k \rangle = \mathcal{I}_c(\varphi, (u(t)dh_k(t)))$ which yields that $(u(t)dh_k(t))^\perp = (\mathcal{C}_k)^\perp$.

10. THE GAUSS-MANIN CONNECTION AND A PFAFFIAN SYSTEM OF $\mathcal{F}_D(a,b,c)$

Let \(\{W_n\}_{n \in \mathbb{N}}\) be an open covering of $X$, where $W_n$ are small simply connected domain in $X$. By patching the trivial vector bundles
\[ \prod_{x \in W_n} H^1_C(T_x, D_x; \mathcal{L}_x), \prod_{x \in W_n} H^1_C(T^\vee_x, D^\vee_x; \mathcal{L}^\vee_x), \]
we have local systems
\[ \mathcal{H}^1(\mathcal{L}) = \bigcup_{n \in \mathbb{N}} \prod_{x \in W_n} H^1_C(T_x, D_x; \mathcal{L}_x), \mathcal{H}^1(\mathcal{L}^\vee) = \bigcup_{n \in \mathbb{N}} \prod_{x \in W_n} H^1_C(T^\vee_x, D^\vee_x; \mathcal{L}^\vee_x) \]
over $X$.

**Lemma 10.1.** We can extend the local sections $\varphi_{i,m+2}$ in (6.3) and $\psi_{i,1}$ in (6.3) to global sections of $\mathcal{H}^1(\mathcal{L})$ and $\mathcal{H}^1(\mathcal{L}^\vee)$, respectively. The spaces $\mathcal{H}^1(\mathcal{L})$ and $\mathcal{H}^1(\mathcal{L}^\vee)$ admit the structure of a trivial vector bundle over $X$. The sections $\varphi_{i,m+2}$’s and $\psi_{0,i}$’s $(1 \leq i \leq m + 1)$ form a frame of $\mathcal{H}^1(\mathcal{L})$ and that of $\mathcal{H}^1(\mathcal{L}^\vee)$, respectively. They are dual to each other with respect to the intersection form $\mathcal{I}_c$.

**Proof.** It is obvious that $\varphi_{i,m+2}$ and $\psi_{0,i}$ are global sections for $\alpha_i \neq 0$. In case of $\alpha_i = 0$, we can regard $\varphi_{i,m+2} = u(x_i)dh_i/u(t)$ as a global section of $\mathcal{H}^1(\mathcal{L})$ since $u(x_i)/u(t)$ is single valued in a tubular neighborhood of $t = x_i$. By Theorem 6.4(1), we see that $\varphi_{i,m+2}$’s and $\psi_{0,i}$’s are dual frames of $\mathcal{H}^1(\mathcal{L})$ and $\mathcal{H}^1(\mathcal{L}^\vee)$. $\square$

**Remark 10.2.** We cannot regard local systems
\[ \bigcup_{n \in \mathbb{N}} \prod_{x \in W_n} H^1(T_x, D_x; \mathcal{L}_x), \bigcup_{n \in \mathbb{N}} \prod_{x \in W_n} H^1(T^\vee_x, D^\vee_x; \mathcal{L}^\vee_x) \]
as vector bundles over $X$, since their monodromy representations are not trivial in general.

Note that

$$d_x \langle \varphi, \gamma^u \rangle = \sum_{i=1}^{m} dx_i \wedge \partial_i \langle \varphi, \gamma^u \rangle = \sum_{i=1}^{m} dx_i \wedge \langle \nabla_i \varphi, \gamma^u \rangle$$

for local sections $\varphi \in H^1_{T,\infty}(T, D; L)$ and $\gamma^u \in H^1(T, D; L)$, where $d_x$ is the exterior derivative on the space $X$, and $\nabla_i$ means the operator given in (8.1). Thus the Gauss-Manin connection on the vector bundle $H^1(L)$ is expressed as

$$\nabla_x = \sum_{i=1}^{m} dx_i \wedge \nabla_i : \varphi \mapsto \sum_{i=1}^{m} dx_i \wedge \nabla_i(\varphi),$$

which is a map from the space of local sections of $H^1(L)$ to that of the tensor product of the holomorphic cotangent bundle over $X$ and $H^1(L)$.

We have also the dual connection

$$\nabla_x^\vee = \sum_{i=1}^{m} dx_i \wedge \nabla_i^\vee : \varphi \mapsto \sum_{i=1}^{m} dx_i \wedge \nabla_i^\vee(\varphi)$$

on the dual vector bundle $H^1(L^\vee)$ of $H^1(L)$ with respect to the intersection form $I_c$.

In this section, we express the connection $\nabla_x$ by the intersection form $I_c$, and represent its connection matrix with respect to a frame of $H^1(L)$, which can be regarded as that of a Pfaffian system of $F_D$.

Let $H^1_c(L)$ and $H^1_c(L^\vee)$ be the $\mathbb{C}$-spans of $\varphi_{i,m+2}$ and $\psi_{0,i}$ ($1 \leq i \leq m+1$), respectively. By Theorem 6.4 (2), (3), we have

$$\varphi_{0,m+2} \in H^1_c(L), \quad \psi_{0,m+2} \in H^1_c(L^\vee).$$

**Lemma 10.3.** For $1 \leq i \leq m$, we have

$$\nabla_i(\varphi_{m+1,m+2}) = -\frac{\varphi_{i,m+2} - \alpha_i \varphi_{m+1,m+2}}{x_i - 1}$$

and

$$\nabla_i(\varphi_{j,m+2}) = -\frac{\alpha_j \varphi_{i,m+2} - \alpha_i \varphi_{j,m+2}}{x_i - x_j} \quad (0 \leq j \leq m, \ j \neq i).$$

**Proof.** The first identity is essentially shown in Lemma 8.1. We show the second identity. In case of $\alpha_i \alpha_j \neq 0$ ($0 \leq j \leq m$), we have

$$\nabla_i(\varphi_{j,m+2}) = \frac{-\alpha_i\alpha_j}{t - x_i} \frac{dt}{t - x_j} = -\frac{\alpha_i \alpha_j}{x_i - x_j} \left( \frac{dt}{t - x_i} - \frac{dt}{t - x_j} \right) = -\frac{\alpha_j \varphi_{i,m+2} - \alpha_i \varphi_{j,m+2}}{x_i - x_j}.$$

In case of $\alpha_i \neq 0, \alpha_j = 0$, we have

$$\nabla_i(\varphi_{j,m+2}) = \nabla_i(\frac{-u(x_j)dh_j}{u(t)}) = -\nabla_i \nabla_i \frac{u(x_j)h_j}{u(t)} = -\nabla_i \nabla_i \frac{u(x_j)h_j}{u(t)} = -\nabla_i \left( h_j \frac{\partial_i (u(x_j)) + u(x_j) \partial_i (h_j)}{u(t)} \right).$$
Here note that $\nabla_i (1/u(t)) = 0$. Since $\partial_t (h_j)$ is identically zero around $x_k$ ($0 \leq k \leq m + 2$), $\nabla_i \frac{u(x_j) \partial_t (h_j)}{u(t)}$ is the zero of $H^1_{\mathcal{C}} (T; D; \mathcal{L})$. Thus $\nabla_i (\varphi_{j,m+2})$ is equal to

$$-\partial_t (u(x_j)) \nabla_i (\frac{h_j}{u(t)}) = \alpha_i \frac{u(x_j)}{x_j - x_i} \frac{dh_j}{x_j - x_i} = \frac{\alpha_i}{x_i - x_j} \varphi_{j,m+2}$$

as elements of $H^1_{\mathcal{C}} (T; D; \mathcal{L})$.

In case of $\alpha_i = 0, \alpha_j \neq 0$, By following Proof of Lemma 8.1 we have

$$\nabla_i \varphi_{j,m+2} = \frac{-\alpha_i dt}{(t - x_i)(t - x_j)} - \sum_{k \in I_{\mathcal{K}^0} \cup I_{\mathcal{E}}} \nabla_i (h_k(t) \nabla_i f_k(t)) = -\nabla_i (h_i(t) \nabla_i f_i(t))$$

$$= -\nabla_i \left( h_i(t) \nabla_i \left( \frac{1}{u(t)} \int_{x_i}^{t} \alpha_j u(t') dt' \right) \right) = \nabla_i \left( h_i(t) \frac{\alpha_j u(x_i)}{u(t)} \right) = \frac{\alpha_j}{x_i - x_j} \varphi_{i,m+2},$$

where $f_k(t)$ is a holomorphic solution to $\nabla_i (f(t)) = \varphi_{j,m+2}$ around $x_k$ ($k \in I_{\mathcal{L}^0} \cup I_{\mathcal{E}}$).

In case of $\alpha_i = \alpha_j = 0$, we have

$$\nabla_i (\varphi_{j,m+2}) = -\nabla_i \left( \nabla_i \left( \frac{u(x_j) h_j(t)}{u(t)} \right) \right) = -\nabla_i \left( \nabla_i \left( \frac{u(x_j) h_j(t)}{u(t)} \right) \right) = -\nabla_i \left( \partial_t \left( \frac{u(x_j) h_j(t)}{u(t)} \right) \right).$$

Since $\partial_t u(x_j) = 0$ by $\alpha_i = 0$, the numerator of the last term reduces to

$$\partial_t (u(x_j) h_j(t)) = h_j(t) \partial_t u(x_j) + u(x_j) \partial_t h_j(t) = u(x_j) \partial_t h_j(t),$$

which vanishes identically around $x_k$ ($0 \leq k \leq m + 2$). Hence $\nabla_i (\varphi_{j,m+2})$ is the zero of $H^1_{\mathcal{C}} (T; D; \mathcal{L})$.

By Lemma 10.3 together with Theorem 6.4 (2), we can define linear transformations

(10.1) \hspace{1cm} \mathcal{R}_{i,j} : \mathcal{H}^1_{\mathcal{C}} (\mathcal{L}) \ni \varphi \mapsto \lim_{x_i \to x_j} (x_i - x_j) \nabla_i (\varphi) \in \mathcal{H}^1_{\mathcal{C}} (\mathcal{L})

for $1 \leq i \leq m$ and $j = 0, \ldots, i - 1, i + 1, \ldots, m + 1$, and decompose the operator $\nabla_i$ into

(10.2) \hspace{1cm} \nabla_i = \sum_{0 \leq j \leq m+1} \frac{\mathcal{R}_{i,j}}{x_i - x_j}.$

**Lemma 10.4.** The eigenvalues of $\mathcal{R}_{i,j}$ are 0 and $\alpha_i + \alpha_j$. If $\alpha_i = \alpha_j = 0$ then $\mathcal{R}_{i,j}$ is the zero map. Otherwise, the 0-eigenspace of $\mathcal{R}_{i,j}$ is $m$ dimensional and an $(\alpha_i + \alpha_j)$-eigenvector of $\mathcal{R}_{i,j}$ is given by
where $\varphi_{i,m}$ are given in (6.3).

**Remark 10.5.**

(1) Note that $\varphi_{i,j} = -\varphi_{i,j}$ for $1 \leq i, j \leq m$, $i \neq j$. We set

$$\varphi_{0,i} = -\varphi_{i,0}, \quad \varphi_{m+1,i} = -\varphi_{i,m+1}.$$ 

(2) If $\alpha_i + \alpha_j = 0$ and $\alpha_i \alpha_j \neq 0$ then the 0-eigenspace of $R_{i,j}$ is $m$ dimensional and this space includes $\varphi_{i,j}$. In this case, $R_{i,j}$ is not diagonalizable.

**Proof.** We fix $i$ and $j$ satisfying $1 \leq i \leq m$, $0 \leq j \leq m + 1$ and $j \neq i$. For $0 \leq k \leq m + 1$, $k \neq i, j$, we have

$$\nabla_i (\varphi_{k,m+2}) = \left\{ \begin{array}{ll}
-\frac{\alpha_i \alpha_j}{t - x_i} - \frac{dt}{t - x_j} & \text{if } j \leq m, \alpha_i \neq 0, \alpha_j \neq 0, \\
\frac{\alpha_i}{u(x_j)\alpha_j} & \text{if } j \leq m, \alpha_i \neq 0, \alpha_j = 0, \\
\frac{u(t)}{\alpha_j} - \frac{dt}{u(x_i)\alpha_j} & \text{if } j \leq m, \alpha_i = 0, \alpha_j \neq 0, \\
0 & \text{if } j \leq m, \alpha_i = 0, \alpha_j = 0,
\end{array} \right.$$ 

by Lemma [10.3]. Thus we have

$$\lim_{x_i \to x_j} (x_i - x_j) \nabla_i (\varphi_{k,m+2}) = 0,$$

which means that $\varphi_{k,m+2}$ is a 0-eigenvector of $R_{i,j}$. By Theorem [6.4] (2), the dimension of the 0-eigenspace of $R_{i,j}$ is greater than or equal to $m$. Since we have

$$\varphi_{i,m+2} = - \sum_{0 \leq k \leq m} \varphi_{k,m+2} - \alpha_{m+1} \varphi_{m+1,m+2}$$

by Theorem [6.4] (2), $\varphi_{i,j}$ is expressed as

$$\varphi_{i,j} = \left\{ \begin{array}{ll}
-(\alpha_i + \alpha_j) \varphi_{j,m+2} - \sum_{0 \leq k \leq m} \alpha_j \varphi_{k,m+2} - \alpha_j \alpha_{m+1} \varphi_{m+1,m+2} & \text{if } j \leq m + 1, \\
-(\alpha_i + \alpha_{m+1}) \varphi_{m+1,m+2} - \sum_{0 \leq k \leq m} \varphi_{k,m+2} & \text{if } j = m + 1.
\end{array} \right.$$ 

By Lemma [10.3] we have

$$\lim_{x_i \to x_j} (x_i - x_j) \nabla_i (\varphi_{i,j}) = -(\alpha_i + \alpha_j)(-\varphi_{i,m+2} + \alpha_i \varphi_{j,m+2}) = (\alpha_i + \alpha_j) \varphi_{i,j},$$

$$\lim_{x_i \to 1} (x_i - 1) \nabla_i (\varphi_{i,m+1}) = -(\alpha_i + \alpha_{m+1})(-\varphi_{i,m+2} + \alpha_i \varphi_{m+1,m+2}) = (\alpha_i + \alpha_{m+1}) \varphi_{i,m+1}.$$ 

Thus $\varphi_{i,j}$ is an $(\alpha_i + \alpha_j)$-eigenvector of $R_{i,j}$.
If \( \alpha_i + \alpha_j = 0 \) and \( \alpha_i \alpha_j \neq 0 \) then \( \varphi_{i,j} \) is different from the zero vector and it belongs to the 0-eigenspace of \( R_{i,j} \). By Lemma [10.3]

\[
R_{i,j}(\varphi_{j,m+2}) = \begin{cases} 
-\alpha_j \varphi_{i,m+2} + \alpha_i \varphi_{j,m+2} = -\varphi_{i,j} & \text{if } 0 \leq j \leq m, j \neq i, \\
-\varphi_{i,m+2} + \alpha_i \varphi_{m+1,m+2} = -\varphi_{i,m+1} & \text{if } j = m + 1,
\end{cases}
\]

\( R_{i,j} \) is not the zero map. Since the 0-eigenspace of \( R_{i,j} \) is \( m \)-dimensional, any element \( \varphi \in \mathcal{H}_C^1(\mathcal{L}) \) is expressed as a linear combination of \( \varphi_{j,m+2} \) and elements of the 0-eigenspace of \( R_{i,j} \). Thus \( R_{i,j}(\varphi) \) is a scalar multiple of \( \varphi_{i,j} \), which belongs to the 0-eigenspace of \( R_{i,j} \). Hence \( R_{i,j}^2 \) is the zero map and the set of eigenvalues of \( R_{i,j} \) consists of 0.

If \( \alpha_i = \alpha_{m+1} = 0 \) then \( \varphi_{i,m+1} \) is different from the zero vector. Thus the 0-eigenspace of \( R_{i,m+1} \) is \( m + 1 \)-dimensional, which means that \( R_{i,m+1} \) is the zero map. If \( \alpha_i = \alpha_j = 0, 0 \leq j \leq m, j \neq i \) then \( \varphi_{i,j} \) degenerates to the zero vector. In this case, \( \varphi_{j,m+2} \) satisfies

\[
R_{i,j}(\varphi_{j,m+2}) = -\varphi_{i,j} = 0
\]

by \( \alpha_i = \alpha_j = 0 \), it is a 0-eigenvector of \( R_{i,j} \). Hence the 0-eigenspace of \( R_{i,j} \) is \( m + 1 \)-dimensional, and \( R_{i,j} \) is the zero map. \( \square \)

**Theorem 10.6.** The linear transformation \( R_{i,j} \) in (10.1) is expressed by the intersection form \( \mathcal{I}_c \) as

\[
R_{i,j} : \mathcal{H}_C^1(\mathcal{L}) \ni \varphi \mapsto -\frac{1}{2\pi \sqrt{-1}} \mathcal{I}_c(\varphi, \psi_{i,j}) \varphi_{i,j} \in \mathcal{H}_C^1(\mathcal{L}),
\]

where \( \varphi_{i,j} \) are given in (10.3) and

\[
\psi_{i,j} = \begin{cases} 
-\psi_{0,i} = \frac{dt}{t} - \frac{dt}{t - x_i} & \text{if } j = 0, \\
\psi_{0,j} - \psi_{0,i} = \frac{dt}{t - x_j} - \frac{dt}{t - x_i} & \text{if } 1 \leq j \leq m, j \neq i,
\end{cases}
\]

(10.4)

\[
\psi_{i,m+1} = \psi_{0,m+1} - \alpha_{m+1} \psi_{0,i} = \begin{cases} 
\alpha_{m+1} \frac{dt}{t - 1} - \frac{dt}{t - x_i} & \text{if } \alpha_{m+1} \neq 0, \\
\frac{u(t) dh_{m+1}(t)}{u(1)} & \text{if } \alpha_{m+1} = 0.
\end{cases}
\]

Here \( \psi_{0,i} \) are given in (6.3). The Gauss-Manin connection \( \nabla_x \) restricted to \( \mathcal{H}_C^1(\mathcal{L}) \) is expressed as

\[
\nabla_x(\varphi) = \sum_{i=1}^{m} \sum_{0 \leq j \leq m+1} \frac{dx_i}{x_i - x_j} \wedge R_{i,j}(\varphi) = \sum_{0 \leq i < j \leq m+1} \frac{dx_i - dx_j}{x_i - x_j} \wedge R_{i,j}(\varphi)
\]

\[
= \frac{-1}{2\pi \sqrt{-1}} \sum_{0 \leq i < j \leq m+1} \frac{dx_i - dx_j}{x_i - x_j} \wedge \mathcal{I}_c(\varphi, \psi_{i,j}) \varphi_{i,j},
\]

where \( \varphi \in \mathcal{H}_C^1(\mathcal{L}) \) and \( dx_0 = dx_{m+1} = 0 \).

**Remark 10.7.** (1) The kernel of \( R_{i,j} \) is

\[
(\psi_{i,j})^\perp = \{ \varphi \in \mathcal{H}_C^1(\mathcal{L}) \mid \mathcal{I}_c(\varphi, \psi_{i,j}) = 0 \}.
\]
(2) If $\alpha_i + \alpha_j \neq 0$ then $R_{i,j}$ admits the expression

$$R_{i,j} : \varphi \mapsto (\alpha_i + \alpha_j) I_c(\varphi, \psi_{i,j})$$

which is invariant under non-zero scalar multiples of $\varphi_{i,j}$ and $\psi_{i,j}$. We can see that $\varphi_{i,j}$ is an $(\alpha_i + \alpha_j)$-eigenvector of $R_{i,j}$ by this expression.

(3) Any section $\varphi(x)$ of $H^1(\mathcal{L})$ can be expressed as

$$\sum_{i=1}^{m} c_i(x) \varphi_{i,m+2},$$

where $c_i(x)$ are holomorphic functions on $X$. Its image under $\nabla_x$ is

$$\nabla_x \varphi(x) = \sum_{i=1}^{m} [c_i(x) \nabla_x \varphi_{i,m+2} + (dx_i(x)) \varphi_{i,m+2}].$$

**Proof.** We set

$$R'_{i,j} : \varphi \mapsto -\frac{1}{2\pi i} I_c(\varphi, \psi_{i,j}) \varphi_{i,j}$$

and study its eigenspaces. By Lemma 10.1 together with (10.3) and (10.4), we have

$$I_c(\varphi_{k,m+2}, \psi_{i,j}) = 0$$

for $0 \leq k \leq m + 1, k \neq i,j$ and

$$I_c(\varphi_{i,j}, \psi_{i,j}) = -2\pi i (\alpha_i + \alpha_j).$$

Thus $\varphi_{k,m+2}$'s belong to the 0-eigenspace of $R'_{i,j}$ and $\varphi_{i,j}$ is an $(\alpha_i + \alpha_j)$-eigenvector of $R'_{i,j}$ unless $\alpha_i = \alpha_j = 0$. Hence if $\alpha_i + \alpha_j \neq 0$ then the eigenspaces of $R_{i,j}$ coincide with those of $R'_{i,j}$, and this means that $R_{i,j} = R'_{i,j}$. If $\alpha_i + \alpha_j = 0, \alpha_i \alpha_j \neq 0$, then we have seen that $R_{i,j}(\varphi_{j,m+2}) = -\varphi_{i,j}$ in Proof of Lemma 10.4. In this case, we have

$$R'_{i,j}(\varphi_{j,m+2}) = -\varphi_{i,j}$$

by

$$I_c(\varphi_{j,m+2}, \psi_{i,j}) = 2\pi i;$$

hence $R_{i,j} = R'_{i,j}$ holds. If $\alpha_i = \alpha_j = 0$ then $R_{i,j}$ is the zero map. In this case, if $j \leq m$ then $\varphi_{i,j}$ is the zero vector and $R'_{i,j}$ is the zero map; otherwise, $\varphi_{i,m+1}$ is an eigenvector of $R'_{i,m+1}$ of eigenvalue $\alpha_i + \alpha_j = 0$, and $R'_{i,m+1}$ is the zero map. Therefore, $R_{i,j} = R'_{i,j}$ holds for any case.

By the expression of $R_{i,j}$ and $\varphi_{j,i} = -\varphi_{i,j}, \psi_{j,i} = -\psi_{i,j}$ for $i \neq j$, $R_{j,i}$ coincides with $R_{i,j}$. We can unite $dR_{i,j} = dx_i - x_j R_{i,j}$ and $dR_{j,i} = dx_j - x_i R_{j,i}$ to $dR_{i,j} = dx_i - dx_j R_{i,j}$. □

**Corollary 10.8.** Let $R_{i,j}$ be the representation matrix of the linear transformation $R_{i,j}$ with respect to the frame $\{\varphi_{1,m+2}, \ldots, \varphi_{m,m+2}, \varphi_{m+1,m+2}\}$ of $H^1(\mathcal{L})$. Then it admits an expression

$$R_{i,j} = -w_{i,j} v_{i,j},$$
where
\begin{align*}
v_{i,j} &= \begin{cases} 
\alpha_0 e_i - \alpha_i e_0 = (\alpha_1, \ldots, \alpha_i, \ldots, \alpha_{m+1}) & \text{if } j = 0, \\
\alpha_j e_i - \alpha_i e_j = (0, \ldots, \alpha_j, \ldots, -\alpha_i, \ldots, 0) & \text{if } 1 \leq i < j \leq m, \\
e_i - \alpha_i e_{m+1} = (0, 1, \ldots, 0, -\alpha_i) & \text{if } j = m + 1,
\end{cases}
\quad w_{i,j} = \begin{cases} 
-t e_i = t(0, \ldots, 1, 0, \ldots, 0) & \text{if } j = 0, \\
t e_j - t e_i = t(0, \ldots, -1, \ldots, 1, \ldots, 0) & \text{if } 1 \leq i < j \leq m, \\
t e_{m+1} - \alpha_{m+1} e_i = t(0, \ldots, -\alpha_{m+1}, \ldots, 0, 1) & \text{if } j = m + 1,
\end{cases}
\end{align*}
and \( v_{j,i} = -v_{i,j} \), \( w_{j,i} = -w_{i,j} \). Here \( e_k \) \((1 \leq k \leq m + 1)\) is the \( k \)-th unit row vector of size \( m + 1 \), and \( e_0 = (-1, \ldots, -1, -\alpha_{m+1}) \). The Gauss-Manin connection is represented as
\[ \nabla^\vee x \ t(\varphi_{1,m+2}, \ldots, \varphi_{m,m+2}, \varphi_{m+1,m+2}) = R(x) \ t(\varphi_{1,m+2}, \ldots, \varphi_{m,m+2}, \varphi_{m+1,m+2}), \]
where
\[ R(x) = \sum_{0 \leq i < j \leq m+1} \frac{dx_i - dx_j}{x_i - x_j} R_{i,j}. \]

**Proof.** We identify elements
\[ \varphi = \sum_{k=1}^{m+1} v_k \varphi_{k,m+2} \in \mathcal{H}^1(\mathcal{L}), \quad \psi = \sum_{k=1}^{m+1} w_k \psi_{0,k} \in \mathcal{H}^1(\mathcal{L}^\vee), \]
with a row vector \( v = (v_1, \ldots, v_m, v_{m+1}) \) and a column vector \( w = t(v_1, \ldots, v_m, v_{m+1}) \), respectively. Since
\[ \mathcal{I}_c(\varphi, \psi_{i,j}) = 2\pi \sqrt{-1} \cdot v \cdot w_{i,j}, \quad \varphi_{i,j} = v_{i,j} \cdot t(\varphi_{1,m+2}, \ldots, \varphi_{m,m+2}, \varphi_{m+1,m+2}), \]
we have
\[ R_{i,j}(\varphi) = v \cdot (-w_{i,j} v_{i,j}) \cdot t(\varphi_{1,m+2}, \ldots, \varphi_{m,m+2}, \varphi_{m+1,m+2}), \]
which means \( R_{i,j} = -w_{i,j} v_{i,j} \). \( \square \)

**Corollary 10.9.**
1. The dual connection \( \nabla^\vee x \) on \( \mathcal{H}^1(\mathcal{L}^\vee) \) is expressed as
\[ \nabla^\vee x(\psi) = \frac{1}{2\pi \sqrt{-1}} \sum_{0 \leq i < j \leq m+1} \frac{dx_i - dx_j}{x_i - x_j} \wedge \mathcal{I}_c(\varphi_{i,j}, \psi) \psi_{i,j}, \]
where \( \psi \in \mathcal{H}^1(\mathcal{L}^\vee) \), \( dx_0 = dx_{m+1} = 0 \) and \( \varphi_{i,j} \) and \( \psi_{i,j} \) are given in \( \text{[10.3]} \) and \( \text{[10.4]} \), respectively.
2. Let \( R_{i,j}^\vee \) be the representation matrix of the linear transformation
\[ R_{i,j}^\vee(\psi) = \frac{\mathcal{I}_c(\varphi_{i,j}, \psi) \psi_{i,j}}{2\pi \sqrt{-1}} \]
with respect to the frame \((\psi_{0,1}, \ldots, \psi_{0,m}, \psi_{0,m+1})\) of \( \mathcal{H}^1(\mathcal{L}^\vee) \). Then it coincides with \(-R_{i,j}\) in Corollary \( \text{[10.8]} \).

**Remark 10.10.**
1. Since \( T \neq T^\vee \) in general, we cannot regard the dual connection \( \nabla^\vee x \) as the Gauss-Manin connection for the parameter \(-\alpha\).
(2) The matrix $R_{i,j}$ in Theorem 10.6 acts on the frame $\{\varphi_{1,m+2}, \ldots, \varphi_{m,m+2}, \varphi_{m+1,m+2}\}$ of $H^1(\mathcal{L})$ from the left, on the other hand, the matrix $-R_{i,j}$ in Corollary 10.9 acts on the frame $(\psi_{0,1}, \ldots, \psi_{0,m}, \psi_{0,m+1})$ of $H^1(\mathcal{L}')$ from the right.

PROOF. Since $\mathcal{I}_c(\varphi, \psi)$ is independent of $x_1, \ldots, x_m$ for $\varphi \in H^1_c(\mathcal{L})$ and $\psi \in H^1_c(\mathcal{L}')$, we have

$$0 = d_x \mathcal{I}_c(\varphi, \psi) = \mathcal{I}_c(\nabla_x \varphi, \psi) + \mathcal{I}_c(\varphi, \nabla_x' \psi)$$

by (9.1). Thus $\nabla'_x$ admits a decomposition

$$\nabla'_x = \sum_{0 \leq i < j \leq m+1} \frac{dx_i - dx_j}{x_i - x_j} \wedge \mathcal{R}_{i,j}^\vee,$$

and each $\mathcal{R}_{i,j}^\vee$ satisfies

$$\mathcal{I}_c(\varphi, \mathcal{R}_{i,j}^\vee(\psi)) = -\mathcal{I}_c(\mathcal{R}_{i,j}(\varphi), \psi).$$

Array the identities for $\varphi_{k,m+2}$'s and $\psi_{0,l}$'s for $1 \leq k, l \leq m + 1$ as

$$\left(\mathcal{I}_c(\varphi_{k,m+2}, \mathcal{R}_{i,j}^\vee(\psi_{0,l}))\right)_{1 \leq k \leq m+1} = -\left(\mathcal{I}_c(\mathcal{R}_{i,j}(\varphi_{k,m+2}), \psi_{0,l})\right)_{1 \leq k \leq m+1}.$$

Since

$$(\ldots, \mathcal{R}_{i,j}^\vee(\psi_{0,l}), \ldots) = (\ldots, \psi_{0,t}, \ldots) R_{i,j}^\vee, \quad \left(\begin{array}{c} \vdots \cr \mathcal{R}_{i,j}(\varphi_{k,m+2}) \cr \vdots \end{array}\right) = R_{i,j} \left(\begin{array}{c} \vdots \cr \varphi_{k,m+2} \cr \vdots \end{array}\right),$$

we have an identity

$$\left(\mathcal{I}_c(\varphi_{k,m+2}, \psi_{0,l})\right)_{1 \leq k \leq m+1} R_{i,j}^\vee = -R_{i,j} \left(\mathcal{I}_c(\varphi_{k,m+2}, \psi_{0,l})\right)_{1 \leq k \leq m+1}.$$ 

By the duality $\mathcal{I}_c(\varphi_{k,m+2}, \psi_{0,l}) = 2\pi \sqrt{-1} \delta_{[k,l]}$, $R_{i,j}^\vee = -R_{i,j}$ is obtained.

Since the set of eigenvalues of $R_{i,j}$ consists of 0 and $\alpha_i + \alpha_j$, that of $R_{i,j}^\vee$ consists of 0 and $-(\alpha_i + \alpha_j)$. The identity

$$R_{i,j}^\vee w_{i,j} = (w_{i,j} v_{i,j}) w_{i,j} = w_{i,j} (v_{i,j} w_{i,j}) = w_{i,j} \mathcal{I}_c(\varphi_{i,j}, \psi_{i,j}) = -(\alpha_i + \alpha_j) w_{i,j},$$

means that $w_{i,j}$ is a $-(\alpha_i + \alpha_j)$-eigenvector of $R_{i,j}^\vee$, i.e., $v_{i,j}$ is a $-(\alpha_i + \alpha_j)$-eigenvector of $\mathcal{R}_{i,j}^\vee$. Since $v_{i,j} w = \mathcal{I}_c(\varphi_{i,j}, \psi)$ for $\psi = \psi_{0,1} + \cdots + \psi_{m+1, \psi_{0,m+1}}$, if $\mathcal{I}_c(\varphi_{i,j}, \psi) = 0$ then $R_{i,j}^\vee w = (w_{i,j} v_{i,j}) w = 0$ i.e., $\mathcal{R}_{i,j}^\vee(\psi) = 0$. Hence if $\alpha_i + \alpha_j \neq 0$ then the $-(\alpha_i + \alpha_j)$-eigenspace of $R_{i,j}^\vee$ is spanned by $\psi_{i,j}$ and the 0-eigenspace of $\mathcal{R}_{i,j}^\vee$ is

$$\{\psi \in H^1_c \mid \mathcal{I}_c(\varphi_{i,j}, \psi) = 0\}.$$

Therefore, $\mathcal{R}_{i,j}^\vee$ admits the expression

$$\mathcal{R}_{i,j}^\vee(\psi) = -(\alpha_i + \alpha_j) \frac{\mathcal{I}_c(\varphi_{i,j}, \psi)}{\mathcal{I}_c(\varphi_{i,j}, \psi_{i,j})} \psi_{i,j} = \frac{1}{2\pi \sqrt{-1}} \mathcal{I}_c(\varphi_{i,j}, \psi) \psi_{i,j}.$$ 

If $\alpha_i + \alpha_j = 0$, then we can get this expression by case study in Proof of Theorem 10.6. □
A Pfaffian system of $\mathcal{F}_D(a, b, c)$ is a first order differential equation

$$d_x \mathbf{F}(x) = \Xi(x) \mathbf{F}(x)$$

(10.5)
of a vector-valued unknown function $\mathbf{F}(x) = \{f_0(x), f_1(x), \ldots, f_m(x)\}$ equivalent to the system $\mathcal{F}_D(a, b, c)$. Here, $f_0(x)$ is supposed to be a local solution to $\mathcal{F}_D(a, b, c)$ and each $f_i(x)$ ($1 \leq i \leq m$) is given by an action of $\mathcal{O}(X)(\partial_1, \ldots, \partial_m)$ on $f_0(x)$, where $\mathcal{O}(X)$ is the ring of holomorphic function on $X$. Each entry of the connection matrix $\Xi(x)$ in (10.5) belongs to the space $\Omega^1(X)$ of holomorphic 1-forms on $X$, and the integrability condition

$$d_x \Xi(x) = \Xi(x) \wedge \Xi(x)$$

holds.

Since $\langle \varphi_{m+1,m+2}, \gamma^u \rangle$ is a local solution to $\mathcal{F}_D(a, b, c)$ for any local section $\gamma^u \in H_1(T, D; \mathcal{L})$ and $\partial_i \langle \varphi_{m+1,m+2}, \gamma^u \rangle = \langle \nabla_i \varphi_{m+1,m+2}, \gamma^u \rangle$, we obtain a Pfaffian system of $\mathcal{F}_D(a, b, c)$ from the Gauss-Manin connection by relating $\mathbf{F}(x)$ to our frame of $\mathcal{H}^1(\mathcal{L})$ (in other words, by determining a Gauss-Manin vector introduced in [GM]).

**Theorem 10.11.** Let $f_0(x)$ be a local solution to $\mathcal{F}_D(a, b, c)$. We define a vector valued function $\mathbf{F}(x) = \{f_0(x), f_1(x), \ldots, f_m(x)\}$ by

$$f_i(x) = (x_i - 1)\partial_i f_0(x) \quad (1 \leq i \leq m).$$

Then $\mathbf{F}(x)$ satisfies a Pfaffian system

$$d_x \mathbf{F}(x) = \Xi(x) \mathbf{F}(x),$$

where

$$\Xi(x) = PR(x)P^{-1}, \quad P = \begin{pmatrix} 0_m & 1 \\ \alpha_1 & \vdots \\ \alpha_m & \end{pmatrix} \in GL_{m+1}(\mathbb{C}).$$

**Proof.** There exists $\gamma^u \in H_1(T, D; \mathcal{L})$ such that

$$f_0(x) = \langle \frac{dt}{t-1}, \gamma^u \rangle.$$

By Lemma 8.1 together with (6.2), we have

$$\partial_i f_0(x) = \langle \nabla_i \frac{dt}{t-1}, \gamma^u \rangle = \frac{1}{x_i - 1} \langle \alpha_i \varphi_{m+1,m+2} - \varphi_{i,m+2}, \gamma^u \rangle.$$

Thus we can identify $\mathbf{F}(x)$ with

$$P \langle \varphi_{1,m+2}, \varphi_{m,m+2}, \varphi_{m+1,m+2} \rangle.$$

Since the matrix $P$ is independent of $x_1, \ldots, x_m$, the connection matrix $\Xi(x)$ is obtained by $PR(x)P^{-1}$. \hfill \square

**Corollary 10.12.** Let $f_0(x)$ be a local solution to $\mathcal{F}_D(a, b, c)$. A vector valued function $\mathbf{f}(x) = \{f_0(x), \partial_1 f_0(x), \ldots, \partial_m f_0(x)\}$ satisfies a Pfaffian system

$$d_x \mathbf{f}(x) = \Theta(x) \mathbf{f}(x),$$

where

$$\Theta(x) = Q(x)\Xi(x)Q(x)^{-1} + [d_x Q(x)]Q(x)^{-1}, \quad Q(x) = \text{diag}(\frac{1}{x_1 - 1}, \ldots, \frac{1}{x_m - 1}).$$
PROOF. Since \( f(x) = Q(x)\mathbf{F}(x) \), it satisfies
\[
d_x f(x) = [d_x Q(x)]\mathbf{F}(x) + Q(x)d_x\mathbf{F}(x)
\]
\[
= [d_x Q(x)Q(x)^{-1}]Q(x)\mathbf{F}(x) + Q(x)\Xi(x)Q(x)^{-1}Q(x)\mathbf{F}(x)
\]
\[
= [d_x Q(x)Q(x)^{-1}]f(x) + Q(x)\Xi(x)Q(x)^{-1}f(x) = \Theta(x)f(x);
\]
we have this corollary. \hfill \Box

11. THE MONODROMY REPRESENTATION OF \( \mathcal{F}_D(a, b, c) \)

By patching the trivial vector bundles
\[
\prod_{x \in W_n} H_1(T_x, D_x; \mathcal{L}_x)
\]
for an open covering \( \{W_n\}_{n \in N} \) of \( X \), we have a local system
\[
\mathcal{H}_1(\mathcal{L}) = \bigcup_{n \in N} \prod_{x \in W_n} H_1(T_x, D_x; \mathcal{L}_x)
\]
of rank \( m + 1 \) over \( X \). We take a base point \( \hat{x} \in W_0 \in \{W_n\}_{n \in N} \) so that
\[
(\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_m, \hat{x}_{m+1}, \hat{x}_{m+2}) = (0, \hat{x}_1, \ldots, \hat{x}_m, 1, \infty)
\]
are aligned as in (3.5) for a fixed parameter \( \alpha \). By the continuation of any section of the trivial vector bundle
\[
H_1(T_x, D_x; \mathcal{L}_x) = \prod_{x \in W_0} H_1(T_x, D_x; \mathcal{L}_x)
\]
along a path in \( X \) from \( \hat{x} \) to any point \( x' \in X \), we have a linear isomorphism from \( H_1(T_{\hat{x}}, D_{\hat{x}}; \mathcal{L}_{\hat{x}}) \) to \( H_1(T_{x'}, D_{x'}; \mathcal{L}_{x'}) \). In particular, for a loop \( \rho \) in \( X \) with terminal \( \hat{x} \), we can make the continuation \( \rho_*(\gamma^u) \) of any section \( \gamma^u \in H_1(T, D; \mathcal{L}) \). The linear transformation
\[
\mathcal{M}_\rho : H_1(T, D; \mathcal{L}) \ni \gamma^u \mapsto \rho_*(\gamma^u) \in H_1(T, D; \mathcal{L})
\]
is called the circuit transformation of \( H_1(T, D; \mathcal{L}) \) along \( \rho \), and the homomorphism
\[
\mathcal{M} : \pi_1(X, \hat{x}) \ni \rho \mapsto \mathcal{M}_\rho \in GL(H_1(T, D; \mathcal{L}))
\]
is called the monodromy representation of \( \mathcal{H}_1(\mathcal{L}) \). Thanks to Theorem 8.2, we can study the monodromy representation of \( \mathcal{F}_D(a, b, c) \) as that of the local system \( \mathcal{H}_1(\mathcal{L}) \).

We give generators of \( \pi_1(X, \hat{x}) \). We set
\[
\mathbb{C}_p(\hat{x}) = \{(x_1, \ldots, x_{p-1}, x_p, x_{p+1}, \ldots, x_m) \mid x_p \in \mathbb{C}\} \quad (1 \leq p \leq m),
\]
which are lines in \( \mathbb{C}^m \) passing through \( \hat{x} \). For distinct indices \( 1 \leq p \leq m \) and
\( 0 \leq q \leq m + 1 \), let \( \rho_{p,q} \) be a loop in \( X \cap \mathbb{C}_p(\hat{x}) \) starting from \( x_p = \hat{x}_p \), approaching to \( \hat{x}_q \) via the upper half space in \( \mathbb{C}_p(\hat{x}) \), turning \( \hat{x}_q \) once positively, and tracing back to \( \hat{x}_p \). It is known that \( \pi_1(X, \hat{x}) \) is generated by the loops \( \rho_{p,q} \), where \( 0 \leq p < q \leq m + 1 \), \( (p, q) \neq (0, m + 1) \), and \( \rho_{0,p} \) is regarded as the loop \( \rho_{p,0} \) in \( X \cap \mathbb{C}_p(\hat{x}) \).

**Theorem 11.1.** The circuit transformation \( \mathcal{M}_{p,q} = \mathcal{M}(\rho_{p,q}) \) is expressed as
\[
\gamma^u \mapsto \gamma^u - \mathcal{L}_h(\delta_{p,q}^{u-1}, \gamma^u)\gamma^u_{p,q},
\]
where \( \gamma^u_{p,q} \) and \( \delta_{p,q}^{u-1} \) are given in Table 4.
with the regard in (2). For examples, in case of $x$ in which $\lambda$ should be multiplied by $-1$.

Our proof of this theorem is based on [M3, Theorem 5.4], can be regarded as $I_{\mathbb{Z}}^0$. For examples, $\alpha$ are given by the $\lambda$ convergence.

Table 1. $\gamma_{p,q}^{u}$ and $\delta_{p,q}^{u^{-1}}$

Remark 11.2. (1) Our proof of this theorem is based on [M3, Theorem 5.4], in which

$$\gamma_{i(p),(q)} \alpha = (\lambda_{p}^{-1} - 1) \gamma_{i(q)} \alpha - (\lambda_{q}^{-1} - 1) \gamma_{i(p)} \alpha$$

should be multiplied by $(-1)$.

(2) For $\alpha_p \notin \mathbb{Z}$, we have

$$\lim_{t \to t_p} \int_{t}^{u(t) \phi_0} \frac{1}{1 - \lambda_p} \int_{C_p} u(t) \varphi_0$$

where $\phi_0 = \iota_D(\varphi_0) \in H^1_{C_p}(T, D; \mathcal{L})$, and $\overrightarrow{tt}$ denotes the segment from $i$ to $t$. Thus we can regard $\frac{1}{1 - \lambda_p} \mathcal{C}_p$ as $\frac{1}{1 - \lambda_p} \mathcal{C}_p$. Under the limit as $\alpha_p$ converges to an integer such that $x_p \in D$, we see that this regard is valid. Similarly, $\frac{1}{1 - \lambda_p} \mathcal{C}_p - \frac{1}{1 - \lambda_p} \mathcal{C}_p$ can be regarded as $\ell_{p^{-1}}$ if $\alpha_p$ is a non-integer or an integer such that $x_p \in D$. For examples, $\frac{1}{1 - \lambda_p} \mathcal{C}_p - \frac{1}{1 - \lambda_p} \mathcal{C}_p$ can be regarded as $\ell_{p^{-1}} - \ell_{p}$ for $p, q \in I_{\mathbb{Z}}^0 \cup I_{\mathbb{Z}}$, and $-\lambda_q(1 - \lambda_p) \mathcal{C}_q + \lambda_p(1 - \lambda_q) \mathcal{C}_p$ can be regarded as $(1 - \lambda_p)(1 - \lambda_q)(\ell_{p^{-1}} - \ell_{p^{-1}})$ for $p, q \in I_{\mathbb{Z}}^0 \cup I_{\mathbb{Z}}$.

(3) In case of $p \in I_{\mathbb{Z}}^0$ or $q \in I_{\mathbb{Z}}^0$, the cycles $\gamma_{p,q}^{u}$ and $\delta_{p,q}^{u^{-1}}$ are given by the multiplications

$$\prod_{i \in (p,q) \cap t_{\mathbb{Z}}^0} (1 - \lambda_i) \cdot \left( \frac{1}{1 - \lambda_q} \mathcal{C}_p^{u^{-1}} - \frac{1}{1 - \lambda_p} \mathcal{C}_p^{u} \right),$$

$$\prod_{i \in (p,q) \cap t_{\mathbb{Z}}^0} \left( 1 - \lambda_q(1 - \lambda_p) \mathcal{C}_q^{u^{-1}} + \lambda_p(1 - \lambda_q) \mathcal{C}_p^{u^{-1}} \right)$$

with the regard in (2). For examples, in case of $p \in I_{\mathbb{Z}}$ and $q \in I_{\mathbb{Z}}^0$, they are

$$\mathcal{C}_q^{u} - \frac{1}{1 - \lambda_p} \mathcal{C}_p^{u} = -\lambda_q(1 - \lambda_p) \mathcal{C}_q^{u^{-1}} + \lambda_p(1 - \lambda_q) \mathcal{C}_p^{u^{-1}}$$
regarded as $\gamma_q^u$ and $(1 - \lambda_p)\ell_q^u - (1 - \lambda_q)\ell_p^u + \lambda_p \gamma_q^{u^{-1}}$, respectively; in case of $p, q \in I_{\alpha N}^\omega$, they are

$$(1 - \lambda_p)\gamma_q^u -(1 - \lambda_q)\gamma_p^u, \quad \frac{1}{1 - \lambda_q} \gamma_q^{u^{-1}} - \frac{1}{1 - \lambda_p} \gamma_p^{u^{-1}},$$

regarded as 0 and $\ell_q^{u^{-1}} - \ell_p^{u^{-1}}$, respectively.

(4) Note that

$$I_h(\delta_{u,q}^{-1}, \gamma_{p,q}) = 1 - \lambda_p \lambda_q$$

for any $\alpha_p, \alpha_q$. If it does not vanish, then $M_{p,q}$ is the complex reflection

$$\gamma^u \mapsto \gamma^u - (1 - \lambda_p \lambda_q)I_h(\delta_{u,q}^{-1}, \gamma^u)\gamma_{p,q}^u$$

with respect to $I_h$.

(5) Suppose that one of $p$ and $q$ is in $I_{\alpha N}^\omega$ and the other is in $I_{\alpha N}^\omega$, and consider the limit $x_p \to x_q$ in the upper half space $\mathbb{H}_T$ of $T$. Then either $\gamma_{p,q}^u$ is a vanishing cycle of $H_1(T, D; \mathcal{L})$ or $\delta_{p,q}^{-1}$ is that of $H_1(T', D'; \mathcal{L}')$, but not both occur at the same time. Though the circuit transformation $M_{p,q}$ is characterized by vanishing cycles as $x_p \to x_q$ in the other cases of parameters, it is not true in this case.

**Proof.** The circuit transformation $M_{p,q}$ is studied in [M3 Theorem 5.4] except the case one of $p$ and $q$ is in $I_{\alpha N}^\omega$ and the other is in $I_{\alpha N}^\omega$. In this case, we may assume that $p \in I_{\alpha N}^\omega$ and $q \in I_{\alpha N}^\omega$. We see that the 1-eigenspace of $M_{p,q}$ is

$$\left(\gamma_p^{u^{-1}}\right)^\perp = \{ \gamma^u \in H_1(T, D; \mathcal{L}) \mid I_h(\gamma_p^{u^{-1}}, \gamma^u) = 0 \},$$

since we can select 1-chains representing any element $\gamma^u \in \left(\gamma_p^{u^{-1}}\right)^\perp$ so that they are not involved in the movement of $x_p$ and $x_q$ caused by $\rho_{p,q}$. Thus the linear transformation $M_{p,q}$ is characterized by its image of $\gamma^u \in H_1(T, D; \mathcal{L})$ satisfying $I_h(\gamma_p^{u^{-1}}, \gamma^u) = -1$ and its 1-eigenspace $\left(\gamma_p^{u^{-1}}\right)^\perp$. By tracing the deformation of $\gamma^u$ along $\rho_{p,q}$, we can see that

$$M_{p,q}(\gamma^u) = \gamma^u + \gamma_q^u.$$

On the other hand, we have

$$\gamma^u - I_h(\delta_{p,q}^{-1}, \gamma^u)\gamma_{p,q}^u = \gamma^u$$

for any $\gamma^u \in \left(\gamma_p^{u^{-1}}\right)^\perp$, and

$$\gamma^u - I_h(\delta_{p,q}^{-1}, \gamma^u)\gamma_{p,q}^u = \gamma^u + \gamma_q^u$$

since $\gamma_{p,q}^u = \gamma_q^u$, $\delta_{p,q}^{-1} = \gamma_p^{u^{-1}}$, $I_h(\gamma_p^{u^{-1}}, \gamma^u) = 0$ and $I_h(\gamma_p^{u^{-1}}, \gamma^u) = -1$. Hence $M_{p,q}(\gamma^u)$ coincides with $\gamma^u - I_h(\delta_{p,q}^{-1}, \gamma^u)\gamma_{p,q}^u$ for any $\gamma^u \in H_1(T, D; \mathcal{L})$. \(\square\)

**Corollary 11.3.** With respect to the basis $(\gamma_1^u, \ldots, \gamma_{m+1}^u)$ of $H^1(T, D; \mathcal{L})$ given in [3.6] or [3.7], the representation matrix $M_{p,q}$ of $M_{p,q}$ is expressed as

$$M_{p,q} = E_{m+1} - H y_{p,q} \gamma_{p,q}.$$
where the basis $\{\delta_{m+1}^{-1}, \ldots, \delta_{m+1}^{-1}\}$ of $H^1(T'; D'; L')$ is given in (5.2) or (5.3), the intersection matrix $H$ with respect to $\{\delta_{m+1}^{-1}, \ldots, \delta_{m+1}^{-1}\}$ and $(y_1^u, \ldots, y_{m+1}^u)$ is given in Proposition 5.4 and $y_{p,q}$ and $z_{p,q}$ are column and row vectors satisfying

$\gamma_{p,q}^u = (\gamma_1^u, \ldots, \gamma_{m+1}^u)y_{p,q}, \quad \delta_{p,q}^{u-1} = z_{p,q}^u(\delta_{m+1}^{u-1}, \ldots, \delta_{m+1}^{u-1}).$

**Proof.** Let $y$ be a column vector satisfying

$\gamma^u = (\gamma_1^u, \ldots, \gamma_{m+1}^u)y$

for any $\gamma^u \in H^1(T, D; L).$ Since $I_h(\delta_{p,q}^{u-1}, \gamma^u) = z_{p,q}Hy,$ we have

$M_{p,q}(\gamma^u) = \gamma^u - I_h(\delta_{p,q}^{u-1}, \gamma^u)\gamma_{p,q}^u = (\gamma_1^u, \ldots, \gamma_{m+1}^u)(y - (z_{p,q}Hy)y_{p,q})$

$= (\gamma_1^u, \ldots, \gamma_{m+1}^u)(E_{m+1} - y_{p,q}z_{p,q}Hy).$

Hence the representation matrix $M_{p,q}$ is obtained. $\square$

**Example 11.4.** We give examples of circuit matrices $M_{p,q}$ for $m = 3.$

(1) $a_0, a_2 \notin \mathbb{Z}, a_1, a_0, a_3, a_4 \in -\mathbb{N}.$

In this case, we have $T = \mathbb{P}^1 - \{0, x_3, 1, \infty\}, D = \{x_1, x_2\}, \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1,$ $\lambda_5 = \lambda_0^{-1}(\neq 1).$ We have $H = E_4,$ and list $y_{p,q}, z_{p,q}$ and $M_{p,q}$ in Table 2.

(2) $a_0, a_3, a_4 \in \mathbb{Z}, a_3 \in \mathbb{N}, a_2 \in -\mathbb{N}.$

In this case, we have $T = \mathbb{P}^1 - \{0, x_2, x_3, 1, \infty\}, D = \{x_1\}, \lambda_1 = \lambda_2 = 1,$ $\lambda_3 = \lambda_0^{-1}\lambda_5^{-1}\lambda_4^{-1}(\neq 1).$ The intersection matrix becomes

$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \lambda_3 - 1 & \lambda_4 - 1 \\ 0 & 0 & \lambda_3 - 1 & (1 - \lambda_3)(\lambda_4 - 1) \\ 0 & 0 & 0 & \lambda_4 - 1 \end{pmatrix}.

It is easy to express $\gamma_{p,q}^u$ as a linear combination of $\gamma_1^u, \ldots, \gamma_4^u.$ To express $\delta_{p,q}^{u-1}$ as a linear combination of $\delta_1^{u-1}, \ldots, \delta_4^{u-1},$ we express $\frac{1}{1 - \lambda_5} \gamma_{p,q}^u - \frac{1}{1 - \lambda_0} \gamma_{p,q}^u$ (regarded as $\ell_{0,5}^{u-1}$) in terms of them. Let $\ell_{0,5}^{u-1}$ be expressed as $(z_1, \ldots, z_4)\ell(\delta_{1}^{u-1}, \ldots, \delta_{4}^{u-1}).$ Then we have

$(z_1, \ldots, z_4)H = (I_h(\ell_{0,5}^{u-1}, \gamma_1^u), \ldots, I_h(\ell_{0,5}^{u-1}, \gamma_4^u)) = \left(\frac{\lambda_0 - 1}{\lambda_0 - 1 - 1}, 0, (1 - \lambda_3)\lambda_0^{-1}, (1 - \lambda_4)\lambda_0^{-1}\right),$

and

$(z_1, \ldots, z_4) = \left(\frac{\lambda_0 - 1}{\lambda_0 - 1 - 1}, 0, (1 - \lambda_3)\lambda_0^{-1}, (1 - \lambda_4)\lambda_0^{-1}\right)H^{-1},$

and

$\ell_{0,5}^{u-1} = \left(\frac{1}{1 - \lambda_0}0, \frac{1}{1 - \lambda_0} - 1\right)(\delta_1^{u-1}, \ldots, \delta_4^{u-1}).$

We list $y_{p,q}, z_{p,q}$ and $M_{p,q}$ in Table 3.

**Theorem 11.5.** If there exists an integral parameter $a_i,$ then the monodromy representation of $\mathcal{F}_D(a, b, c)$ is reducible. In particular, if $a \in \mathbb{Z}^{m+3}$ and $\#(I_{0}^{u-1}) = 1$ or $\#(I_{0}^{u}) = m + 2$ then the monodromy representation of $\mathcal{F}_D(a, b, c)$ becomes trivial.
Table 2. List of $y_{p,q}$, $z_{p,q}$ and $M_{p,q}$ for $\alpha_0, \alpha_5 \notin \mathbb{Z}$, $\alpha_1, \alpha_2 \in \mathbb{N}_0$, $\alpha_3, \alpha_4 \in -\mathbb{N}$.

| $p,q$ | $y_{p,q}$ | $z_{p,q}$          | $M_{p,q}$                              |
|-------|----------|-------------------|----------------------------------------|
| 0,1   | $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ | $(1 - \lambda_0, 0, 0, 0)$ | $\text{diag}(\lambda_0, 1, 1, 1)$ |
| 0,2   | $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ | $(0, 1 - \lambda_0, 0, 0)$ | $\text{diag}(1, \lambda_0, 1, 1)$ |
| 0,3   | $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ | $(1, 1, 1 - \lambda_0, 0)$ | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & \lambda_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ |
| 1,2   | $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ | $(0, 0, 0, 0)$ | $E_4$ |
| 1,3   | $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ | $(-1, 0, 0, 0)$ | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ |
| 1,4   | $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ | $(-1, 0, 0, 0)$ | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ |
| 2,3   | $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ | $(0, -1, 0, 0)$ | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ |
| 2,4   | $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ | $(0, -1, 0, 0)$ | $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ |
| 3,4   | $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ | $(0, 0, -1, 1)$ | $E_4$ |

Proof. (1) $\alpha \notin \mathbb{Z}^{m+3}$. Suppose that $\alpha_i \in \mathbb{Z}$. If $i \in I_{H_0}^{\mathbb{N}}$ then the 1-dimensional span $\langle \mathcal{L}^v_i \rangle \subset H_1(T; \mathcal{L})$ is invariant under any circuit transformation. If $i \in I_{H_0}^{\mathbb{N}}$ then the space $H_1(T; \mathcal{L})$ is a non-zero proper subspace of $H_1(T, D; \mathcal{L})$ and it is invariant under any circuit transformation.
If $\{\alpha\}$ then the space $T_{1,3}$ is simply connected. Hence the monodromy is single valued. If $\{\alpha\}$ are element-wise invariant under any circuit transformation as studied in (1) if $\{\alpha\}$ is generated by $m+1$ twisted cycles $C_{t_2}^p, \ldots, C_{t_{m+2}}^u$, which are element-wise invariant under any circuit transformation since $u(t)$ is single valued. If $\{\alpha\} = m+2$ then the space $T$ is $\mathbb{P}^1 - \{x_{i_0}\}$, which is simply connected. Hence the monodromy

| $p, q$ | $y_{p,q}$ | $z_{p,q}$ | $M_{p,q}$ |
|-------|-----------|-----------|-----------|
| 0, 1  | 1         | $(1 - \lambda_0, 0, 0, 0)$ | $\text{diag}(\lambda_0, 1, 1, 1)$ |
| 0, 2  | 1         | $(1, -1, -1, -\lambda_3^{-1})$ | |
| 0, 3  | 1         | $(1 - \lambda_3, 0, \lambda_0 \lambda_3 - 1, 1 - \lambda_3^{-1})$ | |
| 1, 2  | 1         | $(-1, 0, 0, 0)$ | |
| 1, 3  | 1         | $(-1 - \lambda_3, 0, 0, 0)$ | |
| 1, 4  | 1         | $(-1 - \lambda_4, 0, 0, 0)$ | |
| 2, 3  | 1         | $(0, -1 - \lambda_3, -\lambda_3, 0)$ | |
| 2, 4  | 1         | $(0, -1 - \lambda_4, 0, -\lambda_4)$ | |
| 3, 4  | 1         | $(0, 0, \lambda_3(1 - \lambda_3), -\lambda_4(1 - \lambda_3))$ | |

Table 3. List of $y_{p,q}$, $z_{p,q}$ and $M_{p,q}$ for $\alpha_0, \alpha_3, \alpha_4, \alpha_5 \notin \mathbb{Z}$, $\alpha_1 \in \mathbb{N}_0$, $\alpha_2 \in -\mathbb{N}$. 

(2) $\alpha \in \mathbb{Z}^{m+3}$. 

If $\#(I_{\alpha_0}^{y_0}) \neq 1$ and $\#(I_{\alpha_0}^{y_0}) \neq m+2$ then there is an invariant subspace under any circuit transformation as studied in (1).
representation of $F_D(a, b, c)$ becomes trivial.

\[\square\]

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