REGULARITY CRITERIA IN WEAK SPACES FOR NAVIER-STOKES EQUATIONS IN $\mathbb{R}^3$

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Abstract. In this paper we establish a Serrin type regularity criterion on the gradient of pressure in weak spaces for the Leray-Hopf weak solutions of the Navier-Stocks equations in $\mathbb{R}^3$. It partly extends the results of Zhou[2] to $L^p_w$ spaces instead of $L^p$ spaces.

1. Introduction

In this paper, we consider the following Cauchy problem for the incompressible Navier-Stokes equations:

$$\begin{cases}
\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = \Delta u, \\
div u = 0, \\
u(x, 0) = u_0(x),
\end{cases}$$

(1.1)
in $\mathbb{R}^3 \times (0, T)$. Here $u = u(x, t) \in \mathbb{R}^3$ is the velocity field, $p(x, t)$ is a scalar pressure field of an incompressible fluid at the point $(x, t)$, and $u_0(x)$ with $\text{div } u_0 = 0$ in the sense of distribution is the initial velocity field.

The global existence of weak solutions in time was proved by Leray[9] and Hopf[9]. However, the answer to the problem of global regularity for the three dimensional incompressible Navier-Stokes equations is not known. By weak solutions of the Navier-Stokes equations, we mean the usual Leray-Hopf solutions:

Definition 1.1. A vector field $u = u(x, t)$ on $\mathbb{R}^3 \times (0, T)$ is called a weak solution of (1.1) in $\mathbb{R}^3 \times (0, T)$, provided that

(a) $u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^{1,2}(\mathbb{R}^3))$,

(b) $\text{div } u = 0$ in $\mathbb{R}^3 \times (0, T)$

(c) $\int_0^T \int_{\mathbb{R}^3} \{-u \cdot \Phi_t + \nabla u \cdot \nabla \Phi + (u \cdot \nabla u) \cdot \Phi\} dx dt = 0$

for all $\Phi \in C^\infty_c(\mathbb{R}^3 \times (0, T))$ with $\text{div } \Phi = 0$ in $\mathbb{R}^3 \times (0, T)$.

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Serrin\cite{6} proved that if \( u \in L^\alpha(0, T; L^\gamma(\mathbb{R}^3)) \) is a Leray-Hopf weak solution with \( 2/\alpha + 3/\gamma < 1, 3 > \gamma < \infty \), then the solution \( u(x, t) \in C^\infty(\mathbb{R}^3 \times (0, T]) \). And then Sohr\cite{10} solved the limit case \( 2/\alpha + 3/\gamma = 1 \).

There are some regularity criteria in terms of \( p \) or \( \nabla p \) for the whole space. In 2004, Zhou\cite{2} established a final regularity criterion in terms of \( \nabla p \) :

\[
\nabla p \in L^\alpha(0, T; L^\gamma(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{\alpha} + \frac{3}{\gamma} < 3, \quad \frac{2}{3} < \alpha < \infty, \quad 1 < \gamma < \infty
\]
or \( \nabla p \in L^{2/3}(0, T; L^\infty(\mathbb{R}^3)) \), or else \( \|\nabla p\|_{L^\infty(0, T; L^\infty(\mathbb{R}^3))} \) is sufficiently small.

Also in 2004, Kim and Kozono\cite{11} obtained an interior regularity criteria in weak spaces in \( \Omega \times (0, T) \) under the assumption that \( \|u\|_{L^2_{w, \infty}(0, T; L^\infty(\Omega))} \) is sufficiently small for some \( (r, s) \) with \( \frac{2}{s} + \frac{3}{r} = 1 \) and \( 3 \leq r < \infty \). They extended the criteria of Serrin\cite{6} to the weak space-time spaces.

The main purpose of this paper is to establish a regularity criterion in weak spaces instead of \( L^\gamma \) in Zhou\cite{2}. Our main result is

**Theorem 1.2.** Let \( u_0(x) \in L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3) \), for \( q \geq 4 \), and let \( \text{div} \, u_0 = 0 \) in the sense of distribution. Suppose that \( u(x, t) \) is Leray-Hopf weak solution of (1.1). If

\[
\nabla p \in L^\alpha(0, T; L^\gamma_{w}(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{\alpha} + \frac{3}{\gamma} < 3, \quad \frac{2}{3} < \alpha < \infty, \quad 1 < \gamma < \infty,
\]
or \( \nabla p \in L^\alpha(0, T; L^\gamma_{w}(\mathbb{R}^3)) \) is sufficiently small when \( \frac{2}{\alpha} + \frac{3}{\gamma} = 3, \quad \frac{2}{3} < \alpha < \infty, \quad 1 < \gamma < \infty \), or else \( \|\nabla p\|_{L^\infty(0, T; L^\infty(\mathbb{R}^3))} \) is sufficiently small, then \( u(x, t) \) is a regular solution on \([0, T] \).

Here \( L^\gamma_{w}(\mathbb{R}^3) \) denotes the weak \( L^\gamma(\mathbb{R}^3) \)-space

\[
L^\gamma_{w}(\mathbb{R}^3) = \{ v \in L^1_{loc}(\mathbb{R}^3) : \|v\|_{L^\gamma_{w}(\mathbb{R}^3)} = \sup_{\sigma > 0} \sigma |\{ x \in \mathbb{R}^3 : |v(x)| > \sigma \}|^{1/\gamma} < \infty \}
\]

The Lorentz space \( L^{p, r} \) is defined as follows. We have \( f \in L^{p, r}, 1 \leq p \leq \infty \), if and only if

\[
\|f\|_{L^{p, r}} = \left( \int_0^\infty (t^{1/p} f^*(t))^{r} dt / t \right)^{1/r} < \infty \quad \text{when} \quad 1 \leq r < \infty
\]

\[
\|f\|_{L^{p, \infty}} = \sup_t t^{1/p} f^*(t) < \infty \quad \text{when} \quad r = \infty
\]

where

\[
f^*(t) = \inf \{ \sigma : m(\sigma, f) \leq t \}, \quad m(\sigma, f) = \mu(\{ x : |f(x)| > \sigma \}) \]

We have the following properties which are useful in this paper, with equality of norms,

\[
L^{p, p} = L^p, \quad L^{p, \infty} = L^p_{w}, \quad \text{when} \quad 1 \leq p \leq \infty.
\]

and

\[
L^{p, r_1} \subset L^{p, r_2}, \quad \text{if} \quad r_1 \leq r_2 \]
Regularity criteria in weak spaces

In particular,

\[ L^p \subset L^p_w. \]

2. Proof of theorem 1.2

Taking \( \nabla \text{div} \) on both side of (1.1) for smooth \((u, p)\), one can obtain

\[ -\Delta (\nabla p) = \sum_{i,j=1}^3 \partial_i \partial_j (\nabla (u_i u_j)) \]

Therefore the Calderon-Zygmund inequality

\[ \| \nabla p \|_{L^q} \leq C_1 \| |u| \| \nabla u \|_{L^q} \]

holds for any \( 1 < q < \infty \).

Multiply both side of equation (1.1) by \( 4|u|^2 \), and integrate over \( \mathbb{R}^3 \):

\[ \frac{d}{dt} \| u \|_{L^4}^4 + 4 \| \nabla u \|_{L^2}^2 + 2 \| \nabla u \|_{L^2}^2 \leq 4 \int_{\mathbb{R}^3} |\nabla p| |u|^3 dx \]  

\[ \int_{\mathbb{R}^3} |\nabla p| |u|^3 dx = \int_{\mathbb{R}^3} |\nabla p|^{1/2} |\nabla p|^{1/2} |u|^3 dx \]  

(use Hölder’s inequality with \( p = 4, q = \frac{4}{3} \))

\[ \leq \| \nabla |p|^{1/2} \|_{L^4} \| |\nabla |p|^{1/2} \|_{L^4} \]

\[ = \left( \int_{\mathbb{R}^3} |\nabla |p|^2 \right)^{\frac{1}{4}} \left( \int_{\mathbb{R}^3} |\nabla p|^{\frac{2}{3}} |u|^4 \right)^{\frac{3}{4}} = \| \nabla |p|^{\frac{1}{2}} \|_{L^2} \left( \int_{\mathbb{R}^3} |\nabla p|^{\frac{2}{3}} |u|^4 \right)^{\frac{3}{4}} \]  

(use inequality \( \int |fg| \leq \| f \|_{L^{p,\infty}} \| g \|_{L^{p',1}}, \) \( 1 \leq p \leq \infty \))

\[ \leq \| \nabla p \|_{L^2} \left( \| \nabla |p|^{\frac{2}{3}} \|_{L^{p,\infty}} \| |u|^4 \|_{L^{p',1}} \right)^{\frac{3}{4}} \]  

\( (p = \frac{3}{2} \gamma, \ p' = \frac{3\gamma}{3\gamma - 2}) \)

By definition

\[ \| f \|_{L^{p,\infty}} = \sup_t t^{1/p} f^*(t), \ |f|_{L^{p,1}} = \int_0^\infty t^{1/p} f^*(t) \frac{dt}{t} \]

\[ f^*(t) = \inf \{ \sigma : m(\sigma, f) \leq t \}, \ m(\sigma, f) = \mu(\{ x : |f(x)| > \sigma \}) \]

Let \( k > 0 \), then

\[ m(\sigma, f^k) = \mu(\{ x : |f^k(x)| > \sigma \}) = \mu(\{ x : |f(x)| > \sigma^{1/k} \}) = m(\sigma^{1/k}, f) \]

\[ \Rightarrow (f^k)^*(t) = \inf \{ \sigma : m(\sigma, f^k) \leq t \} = \inf \{ \sigma : m(\sigma^{1/k}, f) \leq t \} = (f^*(t))^k \]
Then
\[
\left\| \nabla p \right\|_{L^{2, \infty}}^2 = \sup_t t^{1/p} \left( \left\| \nabla p \right\|_{L^2}^p \right)^{1/p} \\
= \sup_t t^{1/p} \left( \left\| \nabla p \right\|_{L^2}^p \right)^{1/p} \\
= (\sup_t t^{\frac{2}{3\gamma}} \left\| \nabla p \right\|_{L^2}^p)^{\frac{2}{3\gamma}} \\
= \left\| \nabla p \right\|_{L^{2, \infty}}^{\frac{2}{3\gamma}}
\]

We obtain
\[
4 \int_{\mathbb{R}^3} \left| \nabla p \right| u^3 dx \leq 4 \left\| \nabla p \right\|_{L^2}^\frac{1}{2} \left\| \nabla p \right\|_{L^{2, \infty}}^{\frac{3}{2}} \left\| u \right\|_{L^{p,1}}^{\frac{3}{2}}
\]

(\text{use Cauchy's inequality with } \epsilon)

\[
(2.4)
\]
where \( 4 < 4p' = \frac{12\gamma}{3\gamma - 2} < 12 \) as \( 1 < \gamma < \infty \).

Next, we want to estimate \( \left\| u \right\|_{L^{p,1}}^{4} \).

**Claim 2.1.**
\[
\left\| u \right\|_{L^{p,1}}^{4} \leq \left\| u \right\|_{L^{4,1}}^{4(1-\frac{1}{p})} \left\| u \right\|_{L^{12}}^{\frac{4}{3\gamma - 2}}, \quad 1 < \gamma < \infty.
\]

In fact, \( \frac{4}{3\gamma - 2} = u \frac{12\gamma}{3\gamma - 1} u \frac{12\gamma}{3\gamma - 2} \), using Hölder inequality with \( p = \frac{3\gamma - 2}{3(\gamma - 1)}, \quad q = 3\gamma - 2 \), one can easily obtain this inequality.

**Claim 2.2.** \( \forall f \in L^{p,1}(\mathbb{R}^3), \forall 0 < p_1 < p < p_2, \)
\[
\left\| f \right\|_{L^{p,1}} \leq C(p, p_1) \left\| f \right\|_{L^{p_1}} + C(p, p_2) \left\| f \right\|_{L^{p_2}}
\]

Proof.
\[
\left\| f \right\|_{L^{p,1}} = \int_0^\infty t^{1/p} f^*(t) \frac{dt}{t} = \int_0^1 t^{1/p} f^*(t) \frac{dt}{t} + \int_1^\infty t^{1/p} f^*(t) \frac{dt}{t}
\]
\[
\leq \sup_t (t^{1/p} f^*(t)) \int_0^1 t^{\frac{1}{p} - \frac{1}{p_2}} dt + \sup_t (t^{1/p} f^*(t)) \int_1^\infty t^{\frac{1}{p} - \frac{1}{p_1}} dt
\]
\[
= \frac{pp_1}{p - p_1} \left\| f \right\|_{L^{p_2,1}} + \frac{pp_2}{p_2 - p} \left\| f \right\|_{L^{p_2,1}}
\]

We note that \( L_p \subset L^s_p \) this completes the proof.

Let \( q(\gamma) = \frac{12\gamma}{3\gamma - 2}, \quad 0 < \gamma_1 < \gamma < \gamma_2 \). As \( q(\gamma) \) is a decreasing function, \( q(\gamma_1) > q(\gamma) > q(\gamma_2) \). Due to Claim 2.2
\[
\left\| u \right\|_{L^{p,1}}^{4} \leq C(\gamma, \gamma_1) \left\| u \right\|_{L^{3\gamma_1 - 2}}^{3\gamma_1 - 2} + C(\gamma, \gamma_2) \left\| u \right\|_{L^{3\gamma_2 - 2}}^{3\gamma_2 - 2}
\]

(2.5)

\[
= C(\gamma, \gamma_1) \left\| u \right\|_{L^{3\gamma_1 - 2}}^{3\gamma_1 - 2} + C(\gamma, \gamma_2) \left\| u \right\|_{L^{3\gamma_2 - 2}}^{3\gamma_2 - 2}
\]

**Case 1:** \( \frac{2}{\alpha} + \frac{4}{\gamma} = 3(\nabla p \text{ is sufficiently small in } L^0(0, T; L^0_0(\mathbb{R}^3))) \).
Regularly criteria in weak spaces

Let $\gamma_1 = 1$, then $q(\gamma_1) = 12$. Use Claim 2.1 We obtain

$$\|u\|_{L^{2'}} = C(\gamma, 1)\|u\|_{L^{12/9}} + C(\gamma, \gamma_2)\|u\|_{L^{4}} + C(\gamma, \gamma_2)\|u\|_{L^{4}}^{4(1 - \frac{1}{12})} \|u\|_{L^{12}}^{\frac{1}{12}}$$

Since

$$\|u\|_{L^{12}} = \|u\|_{L^6}^2 \leq C\|\nabla u\|_{L^2}^2$$

Apply (2.1) and Claim 2.1 to (2.4), we have

$$\frac{d}{dt}\|u\|_{L^4}^4 + 4\|\nabla u\|_{L^2}^2 + 2\|\nabla |u|\|_{L^2}^2 \leq 4 \int R^3 |\nabla p| |u|^3 dx$$

$$\leq \epsilon C\|\nabla u\|_{L^2}^2 + C(\epsilon)\|\nabla p\|_{L^{\gamma, \infty}}^2 \|C(\gamma, 1)\|_{L^{4}}^{4(1 - \frac{1}{12})} \|\nabla u\|_{L^{12}}$$

$$\leq \epsilon C\|\nabla u\|_{L^2}^2 + C_1(\epsilon, \gamma)\|\nabla p\|_{L^{\gamma, \infty}}^2 \|\nabla u\|_{L^2}^2 + C_2(\epsilon, \gamma)\|\nabla p\|_{L^{\gamma, \infty}}^2 \|\nabla u\|_{L^2}^2 + \frac{2}{\gamma - 1} \|\nabla u\|_{L^4}^4 \|\nabla u\|_{L^4}^4 + \delta \|\nabla u\|_{L^{12}}$$

After choosing suitable $\epsilon$ and $\delta$, we have

$$\frac{d}{dt}\|u\|_{L^4}^4 \leq C\|\nabla p\|_{L^{\gamma, \infty}}^{\frac{2}{\gamma - 1}} \|u\|_{L^4}^4$$

note that $\frac{2}{\gamma - 1} < \frac{2}{\alpha} < \frac{2}{\gamma}$ due to the integrability of $\nabla p$, it follows that

$$\sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{L^4}^4 \leq C(T)\|u_0\|_{L^4}^4.$$

Case 2: $\frac{\alpha}{\gamma} + \frac{3}{7} < 3$ ($\nabla p$ is bounded in $L^\alpha(0, T; L^\gamma_u(R^3))$).

Let $\gamma_1 = \frac{3\alpha}{3\alpha - 2}$, then $\frac{\alpha}{\gamma_1} + \frac{3}{7} = 3$. It follows from (2.5) that

$$\|u\|_{L^{2'}} = C(\gamma, \gamma_1)\|u\|_{L^{12/9}}^{\frac{12\gamma_1}{12 - \gamma_1}} + C(\gamma, \gamma_2)\|u\|_{L^{4}}^{4(1 - \frac{1}{12})} \|u\|_{L^{12}}^{\frac{1}{12}}$$

The same as Case 1, we can get

$$\frac{d}{dt}\|u\|_{L^4}^4 + 4\|\nabla u\|_{L^2}^2 + 2\|\nabla |u|\|_{L^2}^2 \leq 4 \int R^3 |\nabla p| |u|^3 dx$$

$$\leq \epsilon C\|\nabla u\|_{L^2}^2 + C(\epsilon)\|\nabla p\|_{L^{\gamma, \infty}}^2 \|C(\gamma, \gamma_1)\|_{L^{4}}^{4(1 - \frac{1}{12})} \|\nabla u\|_{L^{12}}$$

$$\leq \epsilon C\|\nabla u\|_{L^2}^2 + (C_2(\epsilon, \gamma, \delta_1)\|\nabla p\|_{L^{\gamma, \infty}}^{\frac{2}{\gamma - 1}} + C_2(\epsilon, \gamma, \delta_2)\|\nabla p\|_{L^{\gamma, \infty}}^{\frac{2}{\gamma - 1}}) \|u\|_{L^4}^4 + (\delta_1 + \delta_2) \|u\|_{L^{12}}$$

Then, with suitable $\epsilon$, $\delta_1$ and $\delta_2$, we have

$$\frac{d}{dt}\|u\|_{L^4}^4 \leq (C_1\|\nabla p\|_{L^{\gamma, \infty}} + C_2\|\nabla p\|_{L^{\gamma, \infty}}^{\frac{2}{\gamma - 1}}) \|u\|_{L^4}^4$$

As above, we can get (2.7).
Zhihui Cai, Jian Zhai

Case 3: \((\alpha, \gamma) = \left(\frac{2}{3}, \infty\right)\) \((\|\nabla p\|_{L^2(0,T;L^3_{\infty}(\mathbb{R}^3))} \) is sufficiently small).

Taking the limit case in (2.4), we have

\[4 \int_{\mathbb{R}^3} |\nabla p|^3 \, dx \leq 4 \|\nabla p\|_{L^2}^{\frac{1}{2}} \|\nabla p\|_{L^\infty}^{\frac{1}{2}} \|u^4\|_{L^{1,1}}\]

\[\leq \epsilon \|\nabla p\|_{L^2}^2 + C(\epsilon) \|\nabla p\|_{L^\infty}^2 \|u^4\|_{L^1}\]

\[\leq \epsilon C \|\nabla u\|_{L^2}^2 + C(\epsilon) \|\nabla p\|_{L^\infty}^2 \|u^4\|_{L^4}^4\]

The same as Case 1, after choosing suitable \(\epsilon\), then use Gronwall inequality, we can get (2.7).

This apriori estimate (2.7) is what we want. Then we use a result of Giga [5]:

Theorem 2.3. Suppose \(u_0 \in L^s(\mathbb{R}^3), s \geq 3\). Then there exists \(T_0\) and a unique classical solution \(u \in BC([0,T_0);L^s(\mathbb{R}^3))\). Moreover, let \((0,T_\ast)\) be the maximal interval such that \(u\) solves (1.1) in \(C((0,T_\ast);L^s(\mathbb{R}^3))\), \(s > 3\). Then

\[(2.9) \quad \|u(\cdot,\tau)\|_{L^s} \geq \frac{C}{(T_\ast - \tau)^{(s-3)/2s}}\]

with constant \(C\) independent of \(T_\ast\) and \(s\).

Proof of Theorem 1.2 We follow the method of Zhou [2]; Since \(u_0(x) \in L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)\) for \(q \geq 4\), due to Theorem 2.3 \((s = 4)\), there exists a unique solution \(\tilde{u}(x,t) \in BC([0,T_\ast);L^4(\mathbb{R}^3))\). Since \(u\) is a Leray-Hopf weak solution, we have by the uniqueness criterion of Serrin-Masuda [6] [7]

\[u \equiv \tilde{u} \quad \text{on} \quad [0,T_\ast).\]

By the apriori estimate (2.7), and the standard continuation argument, we can continue our local smooth solution corresponding to \(u_0 \in L^4(\mathbb{R}^3)\) to obtain \(u \in BC([0,T];L^4(\mathbb{R}^3)) \cap C(\mathbb{R}^3 \times (0,T])\). This completes the proof of Theorem 1.2.

Remark The limit case \((\alpha, \gamma) = (\infty, 1)\) is not solved in this paper.

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Regularity criteria in weak spaces

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