Renormalon-inspired resummations for vector and scalar correlators—estimating the uncertainty in $\alpha_s(M_T^2)$ and $\alpha(M_Z^2)$

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Abstract

We perform an all-orders resummation of the QCD Adler $D$-function for the vector correlator, in which the portion of perturbative coefficients containing the leading power of $b$, the first beta-function coefficient, is resummed. To avoid a renormalization scale dependence when we match the resummation to the exactly known next-to-leading order (NLO), and next-NLO (NNLO) results, we employ the Complete Renormalization Group Improvement (CORG) approach in which all RG-predictable ultra-violet logarithms are resummed to all-orders, removing all dependence on the renormalization scale. We can also obtain fixed-order CORGI results. Including suitable weight-functions we can numerically integrate these results for the $D$-function in the complex energy plane to obtain so-called “contour-improved” results for the ratio $R$ and its tau decay analogue $R_\tau$. We use the difference
between the all-orders and fixed-order (NNLO) results to estimate the uncertainty in $\alpha_s(M_Z^2)$ extracted from $R_\tau$ measurements, and find $\alpha_s(M_Z^2) = 0.120 \pm 0.002$. We also estimate the corresponding uncertainty in $\alpha(M_Z^2)$ arising from hadronic corrections by considering the uncertainty in $R(s)$, in the low-energy region, and compare with other estimates. Analogous resummations are also given for the scalar correlator. As an adjunct to these studies we show how fixed-order contour-improved results can be obtained analytically in closed form at the two-loop level in terms of the Lambert $W$-function and hypergeometric functions.

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1 Introduction

The correlator of two vector currents in the Euclidean region is a fundamental ingredient in constructing a number of inclusive hadronic QCD observables of great importance in testing the theory. By taking a logarithmic energy derivative one can define the so-called Adler $D$-function, $D(s)$. By analytical continuation to the Minkowski region this quantity can then be directly related to the ratio $R(s)$, of the total $e^+e^-$ hadronic cross section to the point leptonic cross-section, and also to the analogous ratio $R_\tau$ of the total hadronic decay width of the $\tau$ lepton normalized to the leptonic decay width. The analytical continuation can be elegantly formulated as a contour integration of $D(s)$ together with a weight function around a circle in the complex energy $s$-plane [1, 2]. Performing this integration numerically with $D(s)$ approximated at some fixed order of perturbation theory then automatically resums to all-orders an infinite subset of potentially large analytical continuation terms involving powers of $\pi^2$ and beta-function coefficients, which arise in the running of the coupling around the integration contour. These terms are usually truncated in the direct fixed-order perturbative expansion of the Minkowskian quantity. Such approximations are referred to as “contour-improved”.

The remaining uncertainty in these “contour-improved” predictions for $R$ and $R_\tau$ comes from the uncalculated higher order terms in the perturbation series for $D(s)$, which has presently only been computed exactly to $O(\alpha_s^3)$ in the limit of massless quarks [3], and with some approximations including the contribution of top and bottom quarks [4]. We shall assume massless quarks in these investigations. There are also effects due to non-perturbative power corrections [2, 5, 6]. We shall focus in this work on the former perturbative uncertainties. There are two interrelated aspects to these. Fixed order perturbation theory predictions have a dependence on the renormalization scheme (RS) chosen to define the coupling. In particular they depend on a dimensionful renormalization scale $\mu$. Further, given a choice of RS there is an uncertainty due to the unknown $O(\alpha_s^4)$ and higher uncalculated perturbative terms. To estimate this one needs to perform a necessarily approximate all-orders resummation of these terms. A well-motivated framework to accomplish this is provided by the so-called “leading-$b$” approximation [7, 8] (sometimes also referred to as “naive nonabelianization” [9-11]), which
amounts to resumming to all-orders the portion of perturbative coefficients containing the highest power of $b = \frac{1}{6}(11N - 2N_f)$, the first beta-function coefficient for SU($N$) QCD with $N_f$ active massless quark flavours. This can be accomplished since in the large-$N_f$ limit one has an exact all-orders result for the Adler $D$-function [12-14]. The leading-$b$ resummation is then performed by replacing $N_f$ by $(\frac{11}{2}N - 3b)$. Whilst the leading-$b$ approach is motivated by the structure of renormalon singularities in the Borel plane [7, 8, 10, 11], and also by a QCD skeleton expansion [15], it is effectively just the first “one chain” term in the skeleton expansion, and does not include the multiple exchanges of renormalon chains needed to build the full asymptotic behaviour of the perturbative coefficients, and there are no firm guarantees as to its accuracy. The strongest statement that can be made follows from an analysis of the operators which build the leading ultraviolet renormalon singularity [16, 17]. One can prove that in the case of the vector Adler function the re-expansion of the leading-$b$ result in powers of $N_f$ correctly gives the asymptotics of the portion of perturbative coefficients proportional to $N_f^{n-r}N^r$ in $n^{\text{th}}$-order perturbation theory, with accuracy $O(1/n)$ [17]. In practice for the exact NLO and NNLO coefficients of the Adler $D$-function the level of agreement of these individual coefficients is at the ten percent level, much better than would be expected from the above weak asymptotic result, remarkably in the $N_f = 0$ or large-$N$ limit agreement is at the few percent level [8]. We may therefore hope that the leading-$b$ approximation is indicative of the size of uncalculated higher-order corrections. A remaining difficulty, first emphasised in [8], is the scale dependence of the leading-$b$ resummations, if one tries to match them to the known exact NLO and NNLO perturbative coefficients. This matching ambiguity means that any result may be obtained by varying the scale. It was pointed out that claims [5, 11] that comparison of fixed-order and “leading-$b$” resummed results indicated rather large uncalculated perturbative corrections for $R_\tau$ were undermined by this matching problem. In Ref. [18, 19] the difficulty was resolved by performing a leading-$b$ resummation for the Effective Charge beta-function corresponding to $D(s)$. This scheme-invariant construct then unambiguously determines $R_\tau$. This approach revealed a rather small uncertainty due to uncalculated higher-order corrections.

In this paper we wish to formulate the resummations in a very closely related, but technically much more straightforward way. The renormalization
scale, $\mu$, dependence of fixed-order QCD perturbation theory is an artefact of the way renormalization group (RG) improvement is customarily performed. The two crucial features are the use of a scale $\mu$ proportional to the physical energy scale, $Q$, of the process, and the truncation of the perturbation series at fixed-order. As argued recently \[21\] one should rather keep $\mu$ strictly independent of $Q$. Fixed-order perturbation theory with $\mu$ constant does not then satisfy asymptotic freedom, and one is forced to sum to all-orders the RG-predictable unphysical logarithms of $\mu$ and physical ultraviolet (UV) logarithms of $Q$ from which the perturbative coefficients are built. This so-called “Complete RG-improvement” (CORGI) \[21\] serves to cancel all $\mu$-dependence between the unphysical renormalised coupling $\alpha_s(\mu)$, and the unphysical logarithms of $\mu$ in the coefficients, and one directly trades unphysical $\mu$ dependence for the physical $Q$-dependence. The idea can also be generalized to processes, such as structure function moments, which involve a factorization scale as well as a renormalization scale \[21\]. The CORGI approach as formulated in Ref.\[21\] is exactly equivalent at NLO to the Effective Charge approach of Grunberg \[20\] in which the UV logarithms are also completely resummed in exactly the same way, whilst the remaining RG-predictable effects are parametrized in a different, but \textit{a priori} equally reasonable way. Our plan is to perform a leading-$b$ resummation for the Adler-$D$ function in the CORGI approach. As we shall see this is extremely straightforward to implement and the resulting resummed result can be written as a sum over exponential integral functions, representing the contributions of the ultraviolet and infra-red renormalons in the Borel plane. In contrast the resummation of the Effective Charge beta-function in Refs.\[18, 19\] involved a complicated numerical inversion of a function. We anticipate that the two approaches should yield very similar results. We shall obtain leading-$b$ resummed and contour-improved CORGI results for the quantities $R(s)$, and $R_\tau$. As in Refs.\[18, 19\] we shall use the difference in fixed-order and resummed results to estimate the uncertainty in $\alpha_s(M_Z^2)$ obtained from measurements of $R_\tau$, using more recent experimental data \[22\]. We shall also fit the leading-$b$ resummation to the spectral distribution for hadronic $\tau$ decay \[22, 23\]. We shall attempt to estimate the uncertainty in the hadronic corrections to the value of the QED coupling at the $Z$ pole, $\alpha(M_Z^2)$, using the resummed and fixed-order results for $R(s)$ in the energy ranges $5 < \sqrt{s} < \infty$ GeV, and $2.8 < \sqrt{s} < 3.74$ GeV, and using inclusive data in the remaining ranges, as in Ref.\[24\]. We shall compare our result for $\alpha(M_Z^2)$ with that of Ref.\[24\].
which uses standard fixed-order perturbation theory. Using recent large-\(N_f\) results on the scalar correlator \cite{25} we shall also perform a contour-improved leading-\(b\) resummation for the Higgs decay width. Finally, we show how the analytical continuation to the Minkowski region can be performed in closed analytical form at the two-loop level in terms of the Lambert W-function and hypergeometric functions.

The organization of the paper is as follows. In Section 2 we shall review the contour integral representation of \(R(s)\) and \(R_\tau\) in terms of \(D(s)\), and describe a simple numerical algorithm for evaluating it. In Section 3 we shall review the fixed-order perturbative, and all-orders large-\(N_f\) results for \(D(s)\), and show how the leading-\(b\) all-orders resummation in the CORGI approach can be written in closed form as a sum of exponential integral functions. In Section 4 we estimate the uncertainty in \(\alpha_s(M_Z^2)\) obtained from \(R_\tau\) measurements, and also fit the leading-\(b\) resummed results to the spectral function. In Section 5 we shall estimate the uncertainty in the hadronic corrections to \(\alpha(M_Z^2)\) as discussed above. In Section 6 we perform a contour-improved leading-\(b\) resummation for the Higgs decay width, and in Section 7 discuss how the analytical continuation to the Minkowski region can be performed analytically at the two-loop level in terms of the Lambert W-function and hypergeometric functions. Section 8 contains a discussion and our conclusions.

2 Contour integral representation of Minkowski observables

We shall mainly be concerned in this work with two inclusive QCD observables. The first is the \(e^+e^-\) \(R\)-ratio, defined by

\[
R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}. \tag{1}
\]

In SU(\(N\)) QCD perturbation theory

\[
R(s) = N \sum_f Q_f^2 \left(1 + \frac{3}{4}C_F \bar{R}(s)\right) + \left(\sum_f Q_f\right)^2 \bar{R}(s), \tag{2}
\]
with $Q_f$ denoting the quark charges, summed over the flavours accessible at a given energy. $C_F$ is the SU($N$) Casimir $C_F = (N^2 - 1)/2N$, and $s$ denotes the timelike Minkowski squared momentum transfer. $\tilde{R}$ denotes the perturbative corrections to the parton model result. It has the perturbative expansion

$$\tilde{R}(s) = a(1 + \sum_{n>0} r_n a^n),$$  

(3)

where $a \equiv \alpha_s(\mu^2)/\pi$ denotes the RG-improved coupling. $\tilde{R}$ denotes so-called “light-by-light” contributions which enter at $O(a^3)$. We shall ignore this term in our all-orders resummations. The ratio $R_\tau$ is defined analogously as a ratio of the total $\tau$ hadronic decay width to its leptonic decay width,

$$R_\tau = \frac{\Gamma(\tau \to \nu_\tau + \text{hadrons})}{\Gamma(\tau \to \nu_\tau e^- \bar{\nu_e})}. \quad (4)$$

Its perturbative expansion has the form

$$R_\tau = N(|V_{ud}|^2 + |V_{us}|^2)S_{EW} \left[ 1 + \frac{5}{12} \frac{\alpha(m_\tau^2)}{\pi} + \tilde{R}_\tau + \delta_{PC} \right], \quad (5)$$

with $V_{ud}$ and $V_{us}$ CKM mixing matrix elements. Since the energy scale $s = m_\tau^2$ lies below the threshold for charmed hadron production only three flavours $u, d, s$ are active. The $\alpha(m_\tau^2)$ term denotes the leading QED electromagnetic corrections, and $S_{EW} \approx 1.0194$ represents further electroweak corrections. $\delta_{PC}$ denotes possible power corrections. $\tilde{R}_\tau$ has a perturbative expansion

$$\tilde{R}_\tau = a(1 + \sum_{n>0} r_n^\tau a^n). \quad (6)$$

In this case there are no ”light-by-light” corrections because summing over $u, d$ and $s$ quarks $(\sum Q_f)^2 = 0$. Both $R$ and $R_\tau$ can be directly expressed in terms of the transverse part of the correlator of two vector currents in the Euclidean region,

$$(g_\mu g_\nu - g_{\mu\nu}q^2)\Pi(s) = 4\pi^2 i \int d^4xe^{iq.x} < 0|T[j_\mu(x)j_\nu(x)]|0 > , \quad (7)$$

where $s = -q^2 > 0$. In fact it is convenient to take a logarithmic derivative with respect to $s$ and define the Adler $D$-function,

$$D(s) = -s \frac{d}{ds} \Pi(s). \quad (8)$$
This can be represented by an expression analogous to Eq.(2) involving $\tilde D$ and $\bar D$, where $\tilde D(s)$ has the perturbative expansion

$$\tilde D(s) = a(1 + \sum_{n>0} d_n a^n) .$$

(9)

A generic Minkowskian observable $\hat R(s_0)$ can then be related to $\tilde D(-s)$ by analytical continuation from Euclidean to Minkowski. This can be elegantly formulated as an integration around a circular contour in the complex energy squared $s$-plane [19],

$$\hat R(s_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(\theta) \tilde D(s_0 e^{i\theta}) d\theta ,$$

(10)

where $W(\theta)$ is a weight function which depends on the observable $\hat R$. For $W(\theta) = 1$ one has $\hat R(s_0) = \check R(s_0)$, and for $W(\theta) = (1 + 2e^{i\theta} - 2e^{3i\theta} - e^{4i\theta})$ one has $\hat R(m^2) = \check R_\tau$. If one expands $\tilde D(s_0 e^{i\theta})$ as a perturbation series in $\bar a \equiv a(s_0 e^{i\theta})$ and numerically performs the $\theta$ integration term-by-term one obtains “contour-improved” perturbative results in which at each order an infinite subset of analytical continuation terms present in the conventional perturbation series of Eqs.(3),(6) are resummed. These terms are potentially large and involve powers of $\pi^2$ and beta-function coefficients, as is easily seen by expanding $\bar a$ in powers of $a(s_0)$ and integrating. In this paper we shall focus on this “contour-improved” version of perturbation theory. In Ref.[19] detailed comparisons of the performance of the two versions were made, and the importance of resumming the analytical continuation terms was emphasised.

An obvious numerical algorithm for evaluating the integral in Eq.(10) is to split the range from $\theta=0,\pi$ into $K$ steps of size $\Delta\theta = \pi/K$ and perform a sum over the integrand evaluated at $\theta_n=n\Delta\theta$, $n=0,1,\ldots,K$. So that

$$\hat R(s_0) \simeq \frac{\Delta\theta}{2\pi} [W(0) \tilde D(s_0) + 2Re \sum_{n=1}^{K} W(\theta_n) \tilde D(s_n)]$$

(11)

where $s_n \equiv s_0 e^{in\Delta\theta}$. In practice we perform a Simpson’s Rule evaluation. Writing the perturbation expansion for $\tilde D(s_n)$ we have

$$\tilde D(s_n) = \bar a_n + d_1 \bar a_n^2 + d_2 \bar a_n^3 + \ldots ,$$

(12)
where we have defined \( \bar{a}_n \equiv a(s_n) \). An efficient strategy \([27]\) is to start with \( \bar{a}_0 = a(s_0) \) and use Taylor’s theorem step-by-step to evolve \( \bar{a}_n \) to \( \bar{a}_{n+1} \), using

\[
\bar{a}_{n+1} = \bar{a}_n - i \frac{\Delta \theta}{2} b B(\bar{a}_n) - \frac{\Delta \theta^2}{8} b^2 B(\bar{a}_n) B'(\bar{a}_n) + i \frac{\Delta \theta^3}{48} b^3 [B(\bar{a}_n) B'(\bar{a}_n)]^2
+ B(\bar{a}_n)^2 B''(\bar{a}_n)] + O(\Delta \theta^4) + ... \tag{13}
\]

where \( B(x) \) is the truncated beta-function

\[
B(x) = x^2 + cx^3 + c_2 x^4 + ... \tag{14}
\]

so that \( \bar{a} \) satisfies

\[
\frac{\partial \bar{a}}{\partial \ln s} = - \frac{b}{2} (\bar{a}^2 + c \bar{a}^3 + c_2 \bar{a}^4 + ...) = - \frac{b}{2} B(\bar{a}) . \tag{15}
\]

Here \( b = (33 - 2N_f)/6 \), and \( c = (153 - 19N_f)/12b \) are the first two universal beta-function coefficients for SU(3) QCD with \( N_f \) active massless quark flavours. The higher coefficients \( c_i, i > 1 \) are scheme-dependent. The above use of Taylor’s theorem is much faster to implement than the standard approach \([5]\) of solving the integrated beta-function equation with complex renormalization scale \( s_n \) to find \( \bar{a}_n \) at each step.

### 3 Fixed-order and resummed expressions for \( D(s) \) in the CORGI approach

In the CORGI approach one avoids renormalization scale \( \mu \)-dependence by performing a complete resummation of the ultraviolet logarithms which build the dependence of the observable on the physical energy scale \([21]\). This is equivalent to directly relating the observable to the dimensional transmutation parameter of the theory \([28]\), \( \Lambda_{\overline{MS}} \) say. In this way one can write the CORGI series for \( \bar{D}(s) \),

\[
\bar{D}(s) = a_0(s) + X_2 a_0^3(s) + X_3 a_0^4 + ... + X_n a_0^{n+1} + ... \tag{16}
\]

Here \( a_0(s) \) is the CORGI coupling which may be written in terms of the Lambert \( W \)-function defined implicitly by \( W(z) \exp(W(z)) = z \) \([23]\) as,

\[
a_0(s) = - \frac{1}{c[1 + W(z(s))]} \]

9
\[ z(s) \equiv -\frac{1}{e} \left( \frac{\sqrt{s}}{\Lambda_D} \right)^{b/c}, \]  

(17)

where \( \Lambda_D \equiv e^{d/b}(2c/b)^{-c/b} \Lambda_{\overline{MS}} \), with \( d \) the NLO perturbative coefficient \( d_1 \) for \( \tilde{D}(s) \) in Eq.(9), in the \( \overline{MS} \) scheme with \( \mu^2 = s \). \( a_0(s) \) is the coupling in the scheme with \( \mu^2 = e^{-2d/b}s \), and the non-universal beta-function coefficients, \( c_i, (i > 1) \) all zero. In this scheme \( d_1 = 0 \), and it is exactly equivalent at NLO to the Effective Charge approach of Grunberg [20]. Standard RG-improvement in this scheme completely resums all ultraviolet logarithms, and is equivalent to the CORGI approach which can be formulated in any scheme [21]. \( X_2 \) is the NNLO scheme-invariant combination

\[ X_2 = c_2 + d_2 - cd_1 - d_1^2, \]  

(18)

built from the perturbative coefficients \( d_1 \) and \( d_2 \) and beta-function coefficients. The NLO and NNLO coefficients \( d_1 \) and \( d_2 \) are known exactly [3] and so NNLO contour-improved CORGI predictions can be straightforwardly obtained for Minkowski observables \( \hat{R}(s_0) \), using the numerical integration described in Section 2. Since \( a_0(s) \) is known in closed form in terms of the Lambert \( W \)-function, which has a well-defined branch structure in the complex plane, one can evaluate it directly, avoiding the Taylor’s theorem trick in Eq.(13). In fact one needs to use the \( W_{-1} \) branch of the function (in the nomenclature of Ref.[29]) on the range of integration \([0, \pi]\), and the \( W_1 \) branch on the range \([-\pi, 0]\). As we shall discuss in Section 7 one can, in fact, avoid using the numerical Simpson’s Rule integration all together for the case of the \( e^+e^-R \)-ratio where \( W(\theta) = 1 \), and perform the integral in closed form in terms of logarithms of the \( W \)-function.

In order to assess the likely accuracy of the fixed-order perturbative approximation we can attempt to approximate at the present uncalculated coefficients \( d_i, (i > 2) \) in \( \tilde{D}(s) \) using the so-called “leading-\( b^\prime \)” approximation. A given coefficient \( d_n \) can be written as an expansion in powers of \( N_f \), so that we have

\[ d_n = d_n^{[n]} N_f^n + d_n^{[n-1]} N_f^{n-1} + \ldots + d_n^{[0]} . \]  

(19)

The large-\( N_f \) coefficient \( d_n^{[n]} \) can be computed exactly to all-orders since it derives from a restricted set of diagrams in which a chain of \( n \) fermion bubbles is inserted in the initiating quark loop [12, 13]. Motivated by the structure
of renormalon singularities in the Borel plane one can convert this expansion into the so-called leading-$b$ expansion \[7, 8\], by substituting $N_f = (33/2 - 3b)$, to obtain

$$d_n = d_n^{(0)} + d_n^{(1)} b^{n-1} + \ldots + d_n^{(0)}.$$  \tag{20}

The leading-$b$ term $d_n^{(L)} \equiv d_n^{(0)} b^n$ is then used to approximate $d_n$. Since $d_n^{(L)} = (\frac{-3}{2} - 3b)$ it is known to all-orders. Using the exact large-$N_f$ result one finds that the explicit all-orders result for $d_n^{(L)}$ in the so-called $V$-scheme, i.e. $\overline{MS}$ with scale $\mu^2 = e^{-5/3} s$, is given by \[13\]

$$d_n^{(L)}(V) = \frac{-2}{3} n! (n + 1) \frac{2^n}{2^{n+2}} [-2n - n + 6] b^n + \frac{16}{n + 1} \sum_{n + 1 > s > 0} s(1 - 2^{-2s})(1 - 2^{2s - n - 2 - 2\zeta_{2s + 1}}) b^n. \tag{21}$$

The resulting leading-$b$ resummation

$$\tilde{D}^{(L)} = a(1 + \sum_{k=0}^{\infty} d_k^{(L)} a^k), \tag{22}$$

may then be defined as a principal value (PV) regulated Borel Sum,

$$\tilde{D}^{(L)}(1/a) = PV \int_0^\infty dz e^{-z/a} B[\tilde{D}^{(L)}](z). \tag{23}$$

Here $B[\tilde{D}^{(L)}](z)$, denotes the Borel transform which contains an infinite set of single and double poles at $z = z_l = \frac{2l}{\bar{\alpha}_s}$ corresponding to infra-red renormalons, $IR_l$, and an infinite set of ultra-violet renormalons, $UV_l$, at $z = -z_l$. The structure is

$$B[\tilde{D}^{(L)}](z) = \sum_{j=1}^{\infty} \frac{A_0(j) + A_1(j) z}{(1 + \frac{z}{z_j})^2} + \frac{B_0(2)}{(1 - \frac{z}{z_j})^2} + \sum_{j=3}^{\infty} \frac{B_0(j) + B_1(j) z}{(1 - \frac{z}{z_j})^2}. \tag{24}$$

The residues at these poles can be computed from the exact all-orders large-$N_f$ result. The $UV$ and $IR$ renormalon contributions can then be easily expressed in terms of the exponential integral function,

$$Ei(x) = -\int_{-x}^{\infty} dt \frac{e^{-t}}{t}, \tag{25}$$
where for IR renormalons $x > 0$ and one defines $Ei(x)$ by taking the Cauchy principal value of the integral. The arbitrariness in regulating the IR renormalon contributions reflects the fact that the perturbation series needs to be combined with the power corrections of the operator product expansion (OPE) to obtain a well-defined result \cite{30}. The absence of a relevant operator of dimension two in the OPE for the vector correlator is in accord with the fact that the singularity $IR_1$ is not present, and the nearest singularity to the origin in the Borel plane is in fact $UV_1$, which generates the leading asymptotic behaviour \cite{8},

$$d_n^{(L)}(V) \approx \frac{(12n + 22)}{27} n! \left(-\frac{1}{2}\right)^n b^n. \quad (26)$$

One can then write the $UV$ renormalon and IR renormalon contributions as infinite sums over the $Ei$ functions,

$$\tilde{D}^L(F)|_{UV} = \sum_{j=1}^{\infty} z_j \{e^{F(a)z_j} Ei(-Fz_j)[Fz_j(A_0(j) - z_jA_1(j)) - z_jA_1(j)] + (A_0(j) - z_jA_1(j))\} , \quad (27)$$

and

$$\tilde{D}^L(F)|_{IR} = e^{-Fz_2} z_2 B_0(2) Ei(Fz_2) \sum_{j=3}^{\infty} z_j \{e^{-Fz_j} Ei(Fz_j)[Fz_j(B_0(j) + z_jB_1(j)) - z_jB_1(j)] - (B_0(j) + z_jB_1(j))\} . \quad (28)$$

Here we have defined $F \equiv 1/a_V$, where $a_V$ is the coupling in the V-scheme. The $A_0(j), A_1(j)$ are related to the residues of the $UV_j$ poles, with \cite{8}

$$A_0(j) = \frac{8}{3} \frac{(-1)^{j+1}(3j^2 + 6j + 2)}{j^2(j + 1)^2(j + 2)^2}, \quad A_1(j) = \frac{4}{3} \frac{b(-1)^{j+1}(2j + 3)}{j^2(j + 1)^2(j + 2)^2}. \quad (29)$$

Because of the conformal symmetry \cite{31} of the vector correlator the $UV$ residues are directly related to the $IR$ residues with $B_0(j) = -A_0(-j)$ and $B_1(j) = -A_1(-j)$ for $j > 2$, and $B_0(1) = B_1(1) = B_1(2) = 0$, and $B_0(2) = 1$ \cite{8}. To evaluate the contour integral in the complex s-plane using this $D^{(L)}(F)$ result
one needs to modify the definition of the $Ei$ functions to cope with the fact that their argument involves $1/a_V(s_0e^{i\theta})$ which is complex for nonzero $\theta$. The appropriate generalization uses the function $Ei(n, z)$ defined by

$$Ei(n, z) = \int_1^\infty dt \frac{e^{-tz}}{tn}.$$  

This function is analytic everywhere in the cut complex $z$-plane, but has a branch cut along the negative real axis. One needs to replace $Ei(-Fz_j)$ in the UV contribution by $-Ei(1, Fz_j)$, and $Ei(Fz_j)$ in the IR contribution by $-Ei(1, -Fz_j) + i\pi \text{sign}(\text{Im}(Fz_j))$, where the discontinuity across the branch cut is removed by the final $i\pi$ contribution. The final result for $\tilde{D}(F)$ is simply the sum of the UV and IR contributions. The sums in Eqs.(27),(28) are rapidly convergent since the $A(j)$ and $B(j)$ coefficients have a $j^{-4}$ dependence for large $j$. For the numerical results to be reported in Section 4 we used $N_{UV} = 15$ and $N_{IR} = 17$ terms respectively in the two sums. It is sensible to arrange that $N_{IR} = N_{UV} + 2$, since the symmetry properties above mean that $A_0(j) = -B_0(j + 2)$, this ensures that the O($a$) term in the perturbation series of Eq.(9) has the correct unit coefficient $B_0(2) = 1$.

The final step is to use the above results to perform an all-orders resummation in the CORGI approach. We would like to formally perform the resummation

$$\tilde{D}_{\text{CORGI}} = a_0 + X_2 a_0^3 + \sum_{n>2} X_n^{(L)} a_0^{n+1},$$

so that the exactly known NNLO $X_2$ coefficient is included, with the remaining unknown coefficients approximated by $X_3^{(L)}, X_4^{(L)}, \ldots$, the leading-$b$ approximations. Note that $a_0$ is the full CORGI coupling defined in Eq.(17), so that all the RG-predictable ultraviolet logarithms involving the exact NLO coefficient $d_1$ are completely resummed. This resummation is most easily achieved by noting that the combination

$$\rho_0 = b\ln \left( \frac{\mu}{\Lambda} \right) - d_1(\mu),$$

is scheme-independent. At the leading-$b$ level the coupling $a^{(L)}(s)$ is defined by the one-loop formula

$$a^{(L)}(s) = \frac{1}{b\ln(\sqrt{s}/\Lambda)}. $$

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In the CORGI scheme in leading-\( b \) approximation \( d_1^{(L)} = 0 \), and so by evaluating the invariant combination \( \rho_0 \) in Eq.(32) in the \( V \) scheme and the CORGI scheme one can relate the couplings in the two schemes,

\[
\frac{1}{a_V^{(L)}} = \frac{1}{a_0^{(L)}} + d_1^{L}(V). \tag{34}
\]

It then follows straightforwardly that the formal resummation in Eq.(31) is given by

\[
\tilde{D}_{CORGI} = \tilde{D}^{(L)} \left( \frac{1}{a_0} + d_1^{(L)}(V) \right) + (X_2 - X_2^{(L)})a_0^3, \tag{35}
\]

in which the \( \tilde{D}^{(L)} \) term is the all-orders sum with the exact \( X_2 \) replaced by \( X_2^{(L)} \), and the second term corrects for this. One can obtain approximate N\( ^3 \)LO and higher CORGI results by truncating the sum in Eq.(31). The \( X_n^{(L)} \) can be readily calculated by using the leading-\( b \) relation between the \( V \)-scheme and CORGI couplings in (34). One easily finds

\[
X_n^{(L)} = C_{n+1} \left[ \sum_{k=1}^{\infty} d_n^{(L)}(V) \left( \frac{a}{1 + ad_1^{(L)}(V)} \right)^{k+1} \right], \tag{36}
\]

where the symbol \( C_n[f(a)] \) denotes the coefficient of \( a^n \) in the power series expansion of \( f(a) \). The \( d_n^{(L)}(V) \) can be directly generated using the explicit result in Eq.(21).

Using the above results we can now straightforwardly generate all-orders resummed and fixed-order contour-improved CORGI results for the Minkowski observables \( R_\tau \) and \( R \). We shall perform some phenomenological studies in the next two sections.

### 4 Resummed versus fixed-order predictions for \( R_\tau \)

The ratio \( R_\tau \) defined by Eq.(4) has been the subject of a wide-ranging experimental study by the ALEPH collaboration [22]. If events involving strange quarks are removed from the data, they find \( R_\tau = 3.492 \pm 0.016 \). Setting \( V_{us} = 0 \), and \( V_{ud} = 0.9754 \pm 0.0007 \), and estimating the power correction
contribution to be $\delta_{PC} = -0.003 \pm 0.004$. One finds from Eq.(5) the experimental value $\hat{R}_\tau = 0.2032^{+0.0160}_{-0.0159}$. The QED contribution has been neglected. One can then obtain all-orders leading-$b$ resummed and fixed-order contour-improved CORGI results as described in Sections 2 and 3. We use $N_f = 3$ and fix $\Lambda^{(3)}_{MS}$ so that the all-orders result reproduces the measured central value $\hat{R}_\tau = 0.203$. The results are shown in Figure 1. The solid line is the all-orders resummed result fixed to the data, and the starred points show the $N^n$LO fixed-order CORGI results. We see that the NNLO ($n=2$) fixed-order result, which is the highest order exactly known, is in rather good agreement with the all-orders resummation. The leading-$b$ approximated $N^n$LO results show an oscillatory trend which becomes explosive for $n > 7$, where fixed-order perturbation theory breaks down. The oscillatory behaviour is exactly what one would anticipate from the alternating-sign factorial growth of the contribution of the leading $UV_1$ renormalon, given by Eq.(26). To attempt
to estimate the uncertainty in $\alpha_s(m_T^2)$ extracted from $R_\tau$ measurements we can use the difference between the resummed and exact NNLO fixed-order CORGI results to estimate the possible effects of uncalculated higher order terms. In Figure 2 we have plotted $\tilde{R}_\tau$ versus $\alpha_s(m_T^2)$. The upper solid curve is the all-orders CORGI result, whilst the lower dashed curve is the NNLO fixed-order CORGI result. We note that the separation of the curves increases rapidly with increasing $\tilde{R}_\tau$, so we are fortunate that for the experimentally measured $\tilde{R}_\tau \approx 0.2$ the separation of the curves is reasonably small. Using the ALEPH data we find $\alpha_s(m_T^2) = 0.330^{+0.014}_{-0.013}$ from the all-orders CORGI result, and $\alpha_s(m_T^2) = 0.355^{+0.022}_{-0.022}$ from NNLO fixed order CORGI. The corresponding results which would have been obtained by integrating up the Effective Charge (EC) beta-function for $\tilde{D}$ as in Ref. [19] are $\alpha_s(m_T^2) = 0.337^{+0.015}_{-0.016}$ and $\alpha_s(m_T^2) = 0.347^{+0.021}_{-0.022}$ for the resummed and NNLO EC results. So, as expected, the two approaches yield similar results. If we evolve these $\alpha_s(m_T^2)$ results through flavour thresholds up to $\mu = M_Z$ using

Figure 2: $\tilde{R}_\tau$ versus $\alpha_s(m_T^2)$. The dotted curve is the exact NNLO CORGI fixed-order result, and the upper solid curve is the approximate all-orders CORGI resummation.
the three-loop matching conditions \[33, 34\], we find $\alpha_s(M_Z^2) = 0.120^{+0.002}_{-0.022}$ from the resummed CORGI result, and $\alpha_s(M_Z^2) = 0.123^{+0.002}_{-0.022}$ from the NNLO CORGI result. Thus, we conservatively estimate an uncertainty $\delta\alpha_s(M_Z^2) \approx 0.003$. A direct plot of the resummed and NNLO results for $R_\tau$ versus $\alpha_s(M_Z^2)$ is given in Figure 3. The invariant mass distribution of the produced hadrons in $\tau$ decay is well-measured experimentally \[22, 23\]. We define the quantity $R_\tau(s_0)$ as

$$R_\tau(s_0) = \frac{\Gamma(\tau \rightarrow \nu_\tau + \text{hadrons}; s_{\text{had}} > s_0)}{\Gamma(\tau \rightarrow \nu_\tau e \bar{\nu}_e)} = \int_0^{s_0} ds \frac{dR_\tau(s)}{ds},$$

where $\frac{dR_\tau}{ds}$ denotes the measured inclusive hadronic spectrum.

$$R_\tau(s_0) = N(|V_{ud}|^2)S_{\text{EW}}[(2x - 2x^3 + x^4) + \frac{3}{4}c_F R_\tau(s_0) + \delta_{PC}],$$

with $x \equiv s_0/m_\tau^2$. The perturbative part $\tilde{R}_\tau(s_0)$ can be computed from Eq.(10) with the choice of weight function

$$W(\theta) = 2x(1 + e^{i\theta}) - 2x^3(1 + e^{3i\theta}) + x^4(1 - e^{4i\theta}).$$
It is then straightforward to obtain contour-improved fixed-order and resummed CORGI results for $\tilde{R}(s_0)$. In Figure 4 we show the fit of the all-orders leading-$b$ CORGI resummation (solid line) to the ALEPH data for $R_\tau(s)$ (open circles) \[22\]. The resummation is fitted to the data at $s = m_\tau^2$, where $R_\tau(m_\tau^2) = R_\tau$. The CORGI coupling has a Landau pole at $\sqrt{s} = \Lambda_D$, as is apparent from Eq.\(17\). Fitting to the experimental value of $R_\tau$ determines $\Lambda_D = 0.725$ GeV, and so the resummed prediction is only defined for $s > 0.525$ GeV$^2$. There is excellent agreement with the data. On this scale the fixed-order NNLO CORGI result would not be distinguishable from the all-orders result, and so we have not included it on the plot.

5 Estimating the uncertainty in hadronic corrections to $\alpha(M_Z^2)$
In this section we wish to make use of the difference between the NNLO fixed-order and resummed CORGI results for $\tilde{R}(s)$ in $e^+e^-$ annihilation to estimate the uncertainty in $\alpha(M_Z^2)$, the QED coupling at the Z pole, which plays a crucial role in constraining the Standard Model Higgs mass from precision electroweak fits to radiative corrections [35]. We begin, however, by plotting some figures, analogous to Figure 1, to indicate the performance of fixed-order perturbation theory versus the resummed results at various energies. In Figure 5 we show the all-orders CORGI leading-$b$ resummation (solid line) and fixed order results (starred points) for $\tilde{R}(s)$ at $\sqrt{s} = 1.777$ GeV, corresponding to $m_\tau$, so the performance can be directly compared to Figure 1. The only difference in the two calculations is the choice of weight function, $W(\theta)$ in Eq.(10). The oscillatory trend due to the leading ultraviolet renormalon is again evident, with wild oscillations setting in at $n > 9$ where fixed-order perturbation theory breaks down.

In Figure 6 we present a corresponding plot at LEP1 energy $\sqrt{s} = M_Z$. 

Figure 5: Fixed order results (starred points) for $\tilde{R}$ versus different orders of perturbation theory at $\sqrt{s} = M_\tau =1.777$ GeV. The solid line shows $\tilde{R}$ for the all-orders contour-improved resummation.
Figure 6: As Fig 5, but at $\sqrt{s} = M_Z$

Clearly at the higher energy the agreement is much improved. With the fixed-order results exactly tracking the all-orders result for $n > 4$. Wild oscillations only set in for $n > 30$ at this higher energy.

Finally, in Figure 7 we show a plot of $\tilde{R}(s)$ versus $\ln(\sqrt{s}/\text{GeV})$, in the range $1 < \sqrt{s} < 91$ GeV. The solid line corresponds to the all-orders resummed result and the dashed line to the NNLO fixed-order CORGI result. We assume $\alpha_s(M_Z^2) = 0.119$, and evolve through flavour thresholds using the three-loop matching condition [33, 34].

The QED fine structure constant is extremely well-measured with

$$\alpha^{-1} = \alpha(0)^{-1} = 137.03599976(50).$$  (40)

If one wishes to evolve $\alpha$ away from $s = 0$ to obtain $\alpha(s)$, however, then one needs to know leptonic and hadronic corrections,

$$\alpha(s)^{-1} = (1 - \Delta\alpha_{\text{lep}}(s) - \Delta\alpha_{\text{had}}(s) - \Delta\alpha_{\text{top}}(s))\alpha^{-1}.$$  (41)

Whilst the leptonic corrections are known at three loops and are well-determined [30], the hadronic corrections for the contribution of the five lightest
flavours, which we have denoted $\Delta \alpha_{\text{had}}(s)$, is rather poorly determined and has to be reconstructed from the $s$-dependence of $\tilde{R}(s)$ using a dispersion relation. The contribution of the heaviest flavour $\Delta \alpha_{\text{top}}(s)$ is rather well-determined and can be included separately. The value of $\alpha(M_Z^2)$ is of particular relevance since it limits the precision with which the unknown Higgs mass $M_H$ of the Standard Model can be predicted from precision electroweak corrections \[35\]. Taking $s = M_Z^2$ we have \[36\] $\Delta \alpha_{\text{lep}}(M_Z^2) = 314.98 \times 10^{-4}$ and $\Delta \alpha_{\text{top}}(M_Z^2) = -0.76 \times 10^{-4}$. For the hadronic contribution we can use the dispersion relation,

$$\Delta \alpha_{\text{had}}(M_Z^2) = -\frac{\alpha M_Z^2}{3\pi} PV \int_{4m_t^2}^{\infty} ds \frac{R(s)}{s(s - M_Z^2)} .$$ \hspace{1cm} (42)

In Ref.\[24\] new exclusive data from BES-II \[37\] and Novosibirsk \[38\] have been used to extract $R(s)$ in the low energy region, with NNLO fixed order perturbative QCD used to evaluate it in the ranges $2.8 < \sqrt{s} < 3.74$ and $5 < \sqrt{s} < \infty$. We plan to approximate $R(s)$ in these latter ranges
by the all-orders and NNLO fixed-order results for $R(s)$, as plotted in Figure 7. We shall use the exclusive data results as in Ref.\[24\], in the remaining energy ranges. Taking $\alpha_s(M_Z^2) = 0.119$ we shall then determine $\alpha(M_Z^2)$ from the fixed-order CORGI results, and the all-orders leading-$b$ resummed results. Since these results are contour-improved they include a resummation of analytical continuation terms not included in the fixed-order perturbative results used in \[24\]. We are interested in establishing if these terms and the uncalculated higher-order corrections, as estimated by the leading-$b$ approximation, cause a significant shift in $\alpha(M_Z^2)$, and whether this has any ramifications for the constraints on $M_H$. In the region $2.8 < \sqrt{s} < 3.74$ we obtain $\Delta\alpha_{\text{had}}(M_Z^2) = (9.5424 \times 10^{-4}, 9.7075 \times 10^{-4})$ for the (fixed-order, all-orders) CORGI results, and in the region $5 < \sqrt{s} < \infty$ we find $\Delta\alpha_{\text{had}}(M_Z^2) = (170.788 \times 10^{-4}, 170.635 \times 10^{-4})$. We find correspondingly using Eq.(41), $\alpha(M_Z^2)^{-1} = (128.967, 128.971)$, to be compared with $\alpha(M_Z^2)^{-1} = 128.978 \pm 0.027$ quoted in Ref \[24\]. We conclude that the analytical continuation terms and uncalculated higher order perturbative corrections do not cause a significant change in $\alpha(M_Z^2)$, and their inclusion does nothing modify the conclusions of Ref \[24\].

6 All-orders CORGI resummations for the scalar correlator

The Higgs decay width to a quark anti-quark pair will be of fundamental phenomenological importance. In practice the decay to a $b\bar{b}$ will be the dominant contribution

$$\Gamma(H \to b\bar{b}) = \frac{3G_F}{4\sqrt{2}\pi} M_H m_b^2 \frac{M_H^2}{M_H^2} R(M_H^2).$$

(43)

Here $M_H$ is the Higgs mass and $m_b(M_H^2)$ is the running $b$-quark mass. $R(M_H^2)$ is a coefficient function with a perturbative expansion

$$R(M_H^2) = 1 + \sum_{n>0} r_n a^n,$$

(44)

where the coefficients $r_1, r_2, r_3$ have been exactly computed \[39\]. $R$ can be straightforwardly related to the scalar correlator $\Pi_s(s)$. One can define an
analogue of the vector Adler $D$-function so that (cf. Eq. (8))

$$D(s) = s \frac{d}{ds} \left[ \frac{\Pi_b(s)}{s} \right],$$  \hspace{1cm} (45)

This may be written in terms of the coefficient function $D(s)$ where

$$D(s) = 3 \frac{3}{8 \pi^2} (m_b(s))^2 D(s),$$  \hspace{1cm} (46)

and $D$ has the perturbative expansion,

$$D(s) = 1 + \sum_{n>0} d_n a_n.$$  \hspace{1cm} (47)

$m_b^2(s) R(s)$ can be related to $m_b^2(-s) D(-s)$ by analytical continuation from Euclidean to Minkowski, and one can write a representation of the same form as Eq. (10) with

$$m_b^2(M_H^2) R(M_H^2) = \frac{1}{2 \pi} \int_{-\pi}^{\pi} d\theta \ m_b^2(e^{i\theta} M_H^2) D(e^{i\theta} M_H^2).$$  \hspace{1cm} (48)

To proceed further we can express the running mass in terms of an RG-invariant mass $\hat{m}_b$ and the mass anomalous dimension $\gamma_m(a)$, defined by

$$-\frac{d \ln(m(s))}{d \ln(s)} = \gamma_m(a) = \sum_{i>0} \gamma_i a^{i+1}.$$  \hspace{1cm} (49)

We can then write \cite{21}

$$m_b^2(s) = \hat{m}_b^2 b^{\frac{4 m}{b}} \left( \frac{a}{1 + ca} \right)^{\frac{4 m}{b}} \exp \left[ 4 \int_0^a dx \frac{\gamma_1 + (\gamma_1 c + \gamma_2 - \gamma_0 c_2) x + \ldots}{b(1 + cx)(1 + cx + c_2 x^2 + \ldots)} \right].$$  \hspace{1cm} (50)

The $b^{\frac{4 m}{b}}$ is the standard normalization of the definition of the RG-invariant mass $\hat{m}_b$. One can then write a CORGI series for $m_b^2(s) D(s)$ exactly equivalent to that for the moments of structure functions in Ref. \cite{21}.

$$m_b^2(s) D(s) = \hat{m}_b^2 b^{\frac{4 m}{b}} \left( \frac{a_0(s)}{1 + ca_0(s)} \right)^{\frac{4 m}{b}} (1 + X_2 a_0^2(s) + X_3 a_0^3(s) + \ldots + X_n a_0^n(s) + \ldots)$$  \hspace{1cm} (51)
where $a_0(s)$ denotes the CORGI coupling which is again defined in terms of the Lambert W-function as in Eq.(17), and with the anomalous dimension present one now has $\Lambda_D = \exp[(\frac{2\pi}{\gamma_1}) + (\frac{d}{4\gamma_0})][\frac{2\pi}{\gamma_0}]^{-\frac{2}{3}} \Lambda_{\overline{MS}}$, with $d$ the coefficient $d_1$ in the $\overline{MS}$ factorization and renormalization scheme with $M^2 = \mu^2 = s$ ($M$ denoting the factorization scale) [21]. The exactly known CORGI invariants $X_2$ and $X_3$ follow from Eqs.(18) of Ref.[21], and allowing for the different definition of the anomalous dimension one needs to replace $d_i$ in Ref.[21] by $4\gamma_i$. Lumping various inessential prefactors together we can define

$$\Gamma(H\rightarrow b\bar{b}) = \frac{3G_F}{4\sqrt{2\pi}} \hat{m}_b^2 b^{4\gamma_0/b} \Gamma,$$

where $\Gamma$ has the contour-improved CORGI representation,

$$\Gamma = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \left( \frac{\tilde{a}_0}{1 + c\tilde{a}_0} \right)^{4\gamma_0/b} \left( 1 + X_2 \tilde{a}_0^2 + X_3 \tilde{a}_0^3 + \sum_{n>3} X_{n}^{(L)} \tilde{a}_0^n \right),$$

with $\tilde{a}_0 = a_0(e^{i\theta} M_R^2)$. In the scalar case one will have coefficients with the structure

$$d_n = d_n^{(n-1)} N_f^{n-1} + d_n^{(n-1)} N_f^{n-2} + \ldots + d_n^{(0)} ,$$

and after replacing $N_f = (\frac{33}{2} - 3b)$, as before, one arrives at a leading-$b$ term with the structure $d_n^{(L)} = (-3)^{n-1} d_n^{(n-1)} b^{n-1}$, with one less power of $b$. The anomalous dimension coefficients will have the structure $\gamma_n^{(L)} = \gamma_n^{(n)} b^n$, but since the anomalous dimension $\gamma_m(a)$ does not contain renormalons there is no motivation for making this approximation, and it is poor in practice, as noted in Ref. [25]. Whilst an all-orders result for $\gamma_n^{(L)}$ does exist [25], we shall follow Ref. [25] and set $\gamma_n^{(L)} = 0$ for $n > 0$, retaining only $\gamma_0$. The all-orders result for $X_n^{(L)}$ follow straightforwardly from $d_n^{(L)}$. From the large-$N_f$ results of Ref.[23] for the scalar correlator one can obtain an explicit all-orders expression for $d_n^{(L)}(V)$ (in the $V$-scheme) analogous to Eq.(21) in the vector case. For $n$ even one has,

$$d_n^{(L)}(V) = -\frac{32}{3} \frac{1}{2^{n+1}} \left( 1 - \frac{1}{2^n} \right) \zeta(n+1) n! b^{n-1} + \left( \frac{4}{n+1} \right) \left( \frac{1}{2} \right)^{n-1} n! b^{n-1}$$

$$- \left( \frac{1}{n} + \frac{1}{3} \right) \left( \frac{1}{4} \right)^{n-1} n! b^{n-1} ,$$

(55)
whilst for odd \( n \) one has,
\[
d_{n}^{(L)}(V) = \left( \frac{4}{n} + \frac{4}{3} \right) \left( \frac{1}{2} \right)^{n-1} n! b^{n-1} - \left( \frac{1}{n} + \frac{1}{3} \right) \left( \frac{1}{4} \right)^{n-1} n! b^{n-1} .
\] (56)

As in the vector case one can define a leading-\( b \) resummation
\[
\mathcal{D}^{(L)} = 1 + \sum_{k=1}^{\infty} d_{k}^{(L)}(V) a^{k} ,
\] (57)

analogous to Eq.(22), which may be defined as a regulated Borel sum
\[
\mathcal{D}^{(L)}(1/a) = 1 + PV \int_{0}^{\infty} dz \, e^{-z/a} [G_{-}(z) + G_{+}(z)] .
\] (58)

Here \( G_{-}(z) \) and \( G_{+}(z) \) are the contributions to the Borel transform from UV and IR renormalons, respectively. One has (in the V-scheme) [25]
\[
G_{-}(z) = -4 \sum_{k=1}^{\infty} \frac{(-1)^{k}}{k^{2} (1 + bz/2k)^{2}}
\]
\[
G_{+}(z) = \frac{4}{1 - bz/2} - \frac{1}{1 - bz/4} + 4 \sum_{k=3}^{\infty} \frac{(-1)^{k}}{3 k^{2} (1 - bz/2k)^{2}}
\] (59)

From these expressions one can read off the residues \( A_{0}, B_{0} \) (cf. Eq.(24)), and one can then calculate \( \mathcal{D}^{(L)}(F)\rvert_{UV} \) and \( \mathcal{D}^{(L)}(F)\rvert_{IR} \) using Eqs.(27),(28).

Finally
\[
\mathcal{D}^{(L)}(F) = \mathcal{D}^{(L)}(F)\rvert_{UV} + \mathcal{D}^{(L)}(F)\rvert_{IR} ,
\] (60)

with \( F \equiv 1/a_{V} \). To perform the leading-\( b \) CORGI resummation in Eq.(53) we simply need to relate \( a_{V}^{(L)} \) and \( a_{0}^{(L)} \) as we did in Section 3. In the presence of the anomalous dimension the RS-invariant combination \( \rho_{0} \) in Eq.(32) is replaced by the factorization scheme and RS (FRS) invariant combination \( X_{1} \) introduced in Ref.[21],
\[
X_{1} = 4 \gamma_{0} \ln \left( \frac{M}{\Lambda} \right) - \frac{4 \gamma_{1}}{b} - d_{1}(M) ,
\] (61)

where \( M \) is the factorization scale. Recalling that we have decided to set \( \gamma_{i}^{(L)} = 0 \) for \( i > 0 \) in our leading-\( b \) resummations , we can use Eq.(61) to relate \( a_{0}^{(L)} \) and \( a_{V}^{(L)} \),
\[
\frac{1}{a_{V}^{(L)}} = \frac{1}{a_{0}^{(L)}} + \frac{b}{4 \gamma_{0} d_{1}^{(L)}(V)} .
\] (62)
It then follows that the all-orders formal resummation in Eq.(53) is given by

\[ \Gamma = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \left( \frac{\bar{a}_0}{1 + c\bar{a}_0} \right)^{4\gamma_0/b} \left[ 1 - \frac{4\gamma_0}{b} \ln \left( 1 + \frac{4\gamma_0}{b} d_1^{(L)}(V)\bar{a}_0 \right) \right] \]

\[ + \mathcal{D}^{(L)} \left( \frac{1}{\bar{a}_0} + \frac{b}{4\gamma_0} d_1^{(L)}(V) \right) + (X_2 - X_2^{(L)})\bar{a}_0^2 + (X_3 - X_3^{(L)})\bar{a}_0^3 \right] . \quad (63) \]

The logarithm term arises because of the fractional power $a^{4\gamma_0/b}$. Relating the $V$-scheme and CORGI couplings at the leading-$b$ level one has,

\[ a_V^{(L)4\gamma_0/b} = a_0^{(L)4\gamma_0/b} \left( 1 + \frac{4\gamma_0}{b} d_1^{(L)}(V)a_0^{(L)} \right)^{-4\gamma_0/b} . \quad (64) \]

On expanding using the binomial theorem only the terms linear in $\gamma_0$ are leading in $b$, the remainder should be discarded. Writing the binomial expansion as $\exp[(-4\gamma_0/b)\ln S] = 1 - (4\gamma_0/b)\ln S + O(\gamma_0^2)$, the result follows.
The same subtlety enters in deriving an analogue of Eq.(36) to generate the $X_n^{(L)}$ in terms of $d_n^{(L)}(V)$ explicitly given by Eqs.(55),(56). One finds

$$X_n^{(L)} = C_n \left[ \sum_{k=1}^{\infty} d_n^{(L)}(V) \left( \frac{a}{1 + (4\gamma_0/b)d_1^{(L)}(V)a} \right)^k - \frac{4\gamma_0}{b} \ln \left( 1 + \frac{4\gamma_0}{b} d_1^{(L)}(V)a \right) \right].$$

(65)

These results can then be used to calculate all-orders and fixed-order CORGI predictions. We give in Figure 8 the analogue of Figs.1,5,6, for the Higgs decay width (with pre-factor set to unity) $\Gamma$, we set $M_H = 115$ GeV and $\alpha_s(M^2_Z) = 0.119$. The starred points show the fixed-order CORGI results, and the solid line the all-orders resummation. As before the agreement of the highest exactly calculated $n = 3$ fixed-order result with the all-orders result is good. The fixed-order results track the resummed result up to $n = 12$, beyond which an oscillatory trend is noticeable. From Eq.(59) one can see that $UV_1$ and $IR_1$ renormalon singularities are present, and so the leading asymptotics are not dominated by $UV_1$ as in the vector case. The process of analytical continuation, however, serves to remove $IR_1$ and so the leading asymptotics of $\Gamma$ is expected to be dominated by the leading $UV_1$ renormalon, with resulting alternating-sign factorial behaviour. As discussed in Ref.[25] the presence of the leading $IR_1$ renormalon in $D$ suggests that the obvious generalization of the Adler function in Eq.(45) may not be optimal, and an alternative is suggested. For our purposes here we are simply using $D$ as a tool to compute the physical quantity $\Gamma$, and so this is not a problem.

7 Analytic expressions for the CORGI contour improvement

In this section we wish to point out that we can obtain explicit analytic expressions for the CORGI fixed-order contour improved results for $\bar{R}(s)$ in terms of the Lambert $W$-function, eliminating the need for numerical Simpson’s rule evaluation. From Eq.(17) we see that we can write the CORGI coupling $\bar{a}_0 = a_0(e^{i\theta}s)$ in terms of the Lambert $W$-function as,

$$\bar{a}_0 = \frac{-1}{c[1 + W(Ae^{iK\theta})]},$$

(66)
where

\[ A(s) = \frac{-1}{e} \left( \frac{\sqrt{s}}{\Lambda_D} \right)^{-b/c}, \quad K = \frac{-b}{2c}. \]  

(67)

Thus after the contour integration the \( X_{n-1} \) coefficient multiplies

\[
A_n(s) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \bar{a}_0^n = \frac{1}{2\pi} \int_{-\pi}^{0} d\theta \frac{(-1)^n}{c^n} [1 + W_1(A(s)e^{iK\theta})]^n
\]

\[ + \frac{1}{2\pi} \int_{0}^{\pi} d\theta \frac{(-1)^n}{c^n} [1 + W_{-1}(A(s)e^{iK\theta})]^n, \]

(68)

where the appropriate branches of the \( W \)-function are to be used in the two regions of integration. By making the change of variable \( w = W(A(s)e^{iK\theta}) \) we can then obtain the above integrals in the form,

\[
\frac{(-1)^n}{2iKe^n\pi} \int \frac{dw}{w(1+w)^{n-1}}.
\]

(69)

The \( w \)-integral is elementary, and including the limits of integration, and noting that \( W_1(A(s)e^{-iK\pi}) = [W_{-1}(A(s)e^{iK\pi})]^* \), we obtain the explicit result,

\[
A_n(s) = \frac{(-1)^n}{e^{nK\pi}I_m} \left[ \ln \left( \frac{W_{-1}(A(s)e^{iK\pi})}{1 + W_{-1}(A(s)e^{iK\pi})} \right) + \sum_{k=1}^{n-2} \frac{1}{k(1 + W_{-1}(A(s)e^{iK\pi})^k) \ln W_{-1}(A(s)e^{iK\pi})} \right],
\]

(70)

for \( n > 2 \). For \( n=1 \) we have \( A_1(s) = (-1/(\pi Kc))Im[\ln W_{-1}(A(s)e^{iK\pi})] \). We finally obtain the CORGI contour-improved fixed-order results in the form,

\[
\tilde{R}(s) = A_1(s) + \sum_{k=2}^{\infty} X_k A_{k+1}(s).
\]

(71)

In the scalar correlator case analytic results can also be obtained. The \( X_n \) coefficient in the CORGI series for \( \Gamma \) in Eq.(53) will multiply

\[
A_n(M_H^2) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \left( \frac{\bar{a}_0}{1 + c\bar{a}_0} \right)^{4\gamma_0/b} \bar{a}_0^n
\]

\[ = \frac{1}{2\pi} \int_{-\pi}^{0} d\theta \frac{(-1)^n}{c^n} \frac{c^{-n-4\gamma_0/b}}{[-W_1(Ae^{iK\theta})]^{4\gamma_0/b} [1 + W_1(Ae^{iK\theta})]^n}
\]

\[ + \frac{1}{2\pi} \int_{0}^{\pi} d\theta \frac{(-1)^n}{c^n} \frac{c^{-n-4\gamma_0/b}}{[-W_{-1}(Ae^{iK\theta})]^{4\gamma_0/b} [1 + W_{-1}(Ae^{iK\theta})]^n}. \]

(72)
Here $A = A(M_H^2)$. Making the change of variable $\omega = -W(Ae^{iK\theta})$ one can then obtain the above integrals in the form,

$$\frac{(-1)^n}{2\pi i K e^n + 4\gamma_0 / b} \int d\omega \frac{\omega^{-4\gamma_0/b}}{(1 - \omega)^{n-1}} .$$

These integrals may be evaluated in terms of the Hypergeometric function $F(a, b; c; z)$ [10]. Inserting the limits of integration we obtain the explicit result for $n > 0$,

$$A_n(M_H^2) = \frac{-(-1)^n b}{4\pi K e^n + 4\gamma_0 / b \gamma_0} Im \left\{ \left[ W_{-1}(Ae^{iK\pi}) \right]^{-4\gamma_0/b} \right. \right. \left. \left. F\left(n - 1, -\frac{4\gamma_0}{b}, 1 - \frac{4\gamma_0}{b}, W_{-1}(Ae^{iK\pi}) \right) \right\} .$$

We can finally write the CORGI contour-improved result in the form

$$\Gamma = A_0(M_H^2) + \sum_{k=2}^{\infty} X_k A_k(M_H^2) ,$$

with

$$A_0(M_H^2) = \frac{1}{\pi K e^{4\gamma_0 / b}} Im \left[ \frac{b}{4\gamma_0} (W_{-1}(Ae^{iK\pi})^{-4\gamma_0/b} + \frac{(W_{-1}(Ae^{iK\pi})^{-4\gamma_0/b}}{(1 - \frac{4\gamma_0}{b})} \right] .$$

8 Conclusions

In this paper we have focussed on obtaining exact fixed-order, and leading-$b$ estimated all-orders results for various inclusive QCD Minkowski observables, related to the vector correlator. These could be expressed as a contour integral of the suitably weighted Euclidean Adler $D(-s)$-function in the complex energy squared plane. $D(s)$ is truncated at some fixed-order and the integral performed numerically as described in Section 2. In this way contour-improved predictions are obtained, in which an infinite subset of known and potentially large analytical continuation terms are resummed to all-orders. By employing the CORGI approach, as discussed in Section 3, we could
further resum to all-orders the complete set of ultraviolet logarithms involving $s$, which build the $s$-dependence of $D(s)$, avoiding any dependence on an arbitrary renormalization scale $\mu$. The remaining approximation is the missing higher-order CORGI invariants $X_i$ ($i > 2$), which remain unknown since the perturbative coefficients have only been calculated to NNLO so far. We used the so-called leading-$b$ approximation to estimate these. Using the exact large-$N_f$ all-orders results for $D(s)$, we were able to sum the CORGI series to all-orders in terms of a sum of exponential integral functions, corresponding to the contributions of the ultraviolet and infrared renormalons in the Borel plane. This was technically far more straightforward then previous analogous resummations of the Effective Charge beta-function for $D(s)$. By comparing the NNLO fixed-order CORGI results to the all-orders resummations we estimated the uncertainty in $\alpha_s(M_Z^2)$, extracted from experimental measurements of $R_\tau$, to be $\delta \alpha_s(M_Z^2) \approx 0.003$. We also showed that using all-orders and fixed-order contour-improved CORGI results for $R(s)$ gave results for the hadronic corrections to the QED coupling $\alpha(M_Z^2)$ which did not differ significantly from those obtained using conventional fixed-order perturbation theory in Ref. Finally we showed how recent all-orders large-$N_f$ results for the scalar correlator could be used to perform analogous resummations for the Higgs decay width to a heavy quark pair. We finally noted that the CORGI contour-improvement for the $R$-ratio can be written in analytic form in terms of the Lambert $W$-function, and for the Higgs width in terms of hypergeometric functions and the Lambert $W$-function, thus avoiding the need to use a numerical Simpson’s Rule evaluation.

There are many possible investigations to be pursued using these methods. In particular it would be interesting to investigate hadronic corrections to $\alpha(M_Z^2)$ using the resummed results for $R(s)$ in lower energy ranges where conventionally inclusive or exclusive data has been used. The scalar correlator results could also be used to investigate more carefully the uncertainties in estimates of the bottom quark mass using Sum Rule techniques.
Acknowledgements

We would like to thank Andrei Kataev for many entertaining and informative discussions about contour-improvement and all-orders resummations. A.M. acknowledges the financial support of the Iranian government.

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