A full-discrete exponential Euler approximation of the invariant measure for parabolic stochastic partial differential equations

Ziheng Chen, Siqing Gan, Xiaojie Wang
School of Mathematics and Statistics, Central South University, Changsha 410083, Hunan, China
November 6, 2018

Abstract
We discrete the ergodic semilinear stochastic partial differential equations in space dimension \(d \leq 3\) with additive noise, spatially by a spectral Galerkin method and temporally by an exponential Euler scheme. It is shown that both the spatial semi-discretization and the spatio-temporal full discretization are ergodic. Further, convergence orders of the numerical invariant measures, depending on the regularity of noise, are recovered based on an easy time-independent weak error analysis without relying on Malliavin calculus. To be precise, the convergence order is \(1 - \epsilon\) in space and \(\frac{1}{2} - \epsilon\) in time for the space-time white noise case and \(2 - \epsilon\) in space and \(1 - \epsilon\) in time for the trace class noise case in space dimension \(d = 1\), with arbitrarily small \(\epsilon > 0\). Numerical results are finally reported to confirm these theoretical findings.

Key words: stochastic partial differential equations, invariant measure, ergodicity, weak approximation, exponential Euler scheme

AMS subject classifications: 60H15, 60H35, 37M25

1 Introduction

This work concerns the semilinear stochastic partial differential equations (SPDEs)
\[dX(t) = AX(t) \, dt + F(X(t)) \, dt + dW^Q(t), \quad t > 0, \quad X(0) = X_0,\]
where the dominant linear operator \(A: \mathcal{D}(A) \subset H \rightarrow H\) generates an analytic semigroup \(E(t) = e^{tA}, t \geq 0\) on a real separable Hilbert space \((H, \langle \cdot, \cdot \rangle, \| \cdot \|)\) and \(F: H \rightarrow H\) is a nonlinear deterministic mapping. Moreover, \(\{W^Q(t)\}_{t \geq 0}\) is an \(H\)-valued (possibly cylindrical) \(Q\)-Wiener process on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a normal filtration \(\{\mathcal{F}_t\}_{t \geq 0}\), with the covariance operator \(Q\) obeying
\[\|(-A)^{-\frac{1}{2}}Q^{\frac{1}{2}}\|_{L_2(H)} < \infty, \quad \text{for some } \beta \in (0, 1].\]

Under Assumptions 2.1, 2.2 specified later, a unique mild solution \(\{X(t)\}_{t \geq 0}\) of (1.1) exists, given by
\[X(t) = E(t)X_0 + \int_0^t E(t-s)F(X(s)) \, ds + \int_0^t E(t-s) \, dW^Q(s), \quad \text{a.s., } \quad t \geq 0.\]

Moreover, the mild solution \(\{X(t)\}_{t \geq 0}\) is shown to be ergodic (see Section 2 below for the definition of ergodicity), that is, it possesses a unique invariant probability measure \(\nu\) on \((H, \mathcal{B}(H))\) such that
\[\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E}[\Phi(X(t))] \, dt = \int_H \Phi(y) \, \nu(dy), \quad \forall \Phi \in C_c^2(H, \mathbb{R}).\]
The ergodicity characterizes the longtime behaviour of the mild solution and has significant impacts on quantum mechanics, fluid dynamics, financial mathematics and many other scientific fields [15]. Since the explicit expressions for the mild solutions of SPDEs are rarely available, numerical schemes inheriting the ergodic property of the continuous system turn out to be very important.

For finite dimensional stochastic differential equations (SDEs), much progress has been made in the design and analysis of approximations of invariant measures (see, e.g., [1, 26, 27, 28, 30, 31] and other references therein). By contrast, approximations of invariant measures for SPDEs are at an early stage and just a very limited number of literature [6, 9, 10, 11, 18] are devoted to this topic. In 2014, Bréhier first studied the temporal semi-discretization by the linear implicit Euler scheme in [6] for semilinear SPDEs of parabolic type driven by additive space-time white noise. To achieve higher order accuracy, Bréhier and Vilmart [10] further introduced a kind of implicit-explicit postprocessed method in the temporal semi-discretization. In the more recent publication [9], Bréhier and Kopec analyzed spatio-temporal full discretizations of invariant measures for SPDEs like (1.1). Besides, we mention [11, 18], where approximations of invariant measures were well studied for stochastic nonlinear Schrödinger equations.

In the present work, we attempt to study numerical invariant measures for a general class of parabolic SPDE (1.1), covering both space-time white noise in space dimension $d = 1$ and trace class noise in multiple space dimension $d \leq 3$. First we discrete (1.1) in space by a spectral Galerkin method to get

\[ dX^n(t) = A_n X^n(t) dt + P_n F(X^n(t)) dt + P_n dW^Q(t), \quad X^n(0) = P_n X_0, \]  

where $P_n$ is a projection operator from $H$ to the finite-dimensional space $H_n \subset H, n \in N$ and $A_n := AP_n$ is a bounded linear operator in $H_n$ (see Subsection 3.1 below for precise description). Observing that (1.5) is a finite-dimensional SDE in $H_n$ (or equivalently in $\mathbb{R}^n$), we apply a general ergodicity theory established in [13] to verify the ergodicity of $\{X^n(t)\}_{t \geq 0}$, which possesses a unique invariant measure $\nu^n$. Further, we carry out the time-independent weak error analysis, thanks to the uniform boundedness of the mean square moment of $\{X^n(t)\}_{t \geq 0}$ and the improved regularity for the associated Kolmogorov equation (Theorem 3.1 and Proposition 3.3 below). Then the ergodicity properties as well as the time-independent weak error help us to derive the error between the invariant measures $\nu$ and $\nu^n$, given by

\[ \left| \int_H \Phi(y) \nu(dy) - \int_{H_n} \Phi(y) \nu^n(dy) \right| \leq C \lambda_n^{-\beta+\epsilon}. \]  

(1.6)

Here $\epsilon > 0$ is arbitrarily small, $\beta \in (0, 1]$ comes from (1.2) and $\lambda_n$ serves as the $n$-th eigenvalue of the linear operator $-A$.

Regarding the temporal discretization of (1.5), we resort to the exponential Euler (EE) scheme, which, given $\tau > 0$ a uniform time stepsize, takes the form of

\[ Y^n_{m+1} = E_n(\tau)Y^n_m + \tau \Phi_n(\tau)P_n F(Y^n_m) + E_n(\tau)P_n \Delta W^Q_m, \quad Y^n_0 = X^n_0, \]  

(1.7)

where $Y^n_0$ is the numerical approximation of $X^n(t_m)$. As one of key ingredients to guarantee the ergodicity and nice regularity of $\{Y^n_m\}_{m \in \mathbb{N}}$, the semigroup operator $E_n(\tau) = e^{\tau A_n}$ exhibits an exponentially decreasing property in the sense $\|E_n(\tau)\|_{L(H_n)} \leq e^{-\lambda_1 \tau}$, $\lambda_1 > 0, \tau > 0$. More formally, we rely on the general ergodicity theory for Markov chains established in [26] to show the ergodicity of $\{Y^n_m\}_{m \in \mathbb{N}}$, with a unique invariant measure $\nu^n$ (Theorem 4.7). Now it remains to do the time-independent weak error analysis of the temporal discretization, which starts from a weak error representation formula of (1.7) in [33]. There the weak error analysis was done on a finite time interval $[0, T]$. However, weak error estimates here must be time-independent and hold over long time. Again, owing to the ergodicity and time-independent weak error of $\{X^n(t)\}_{t \geq 0}$ and $\{Y^n_m\}_{m \in \mathbb{N}}$, the error between $\nu^n$ and $\nu^n$ can be measured as

\[ \left| \int_{H_n} \Phi(y) \nu^n(dy) - \int_{H_n} \Phi(y) \nu^n(dy) \right| \leq C \tau^{\beta-\epsilon}. \]  

(1.8)

Combining this with (1.6) results in the space-time full approximations of invariant measures (Corollary 4.13). Specializing (1.6) and (1.8) into the case of space dimension $d = 1$, implies that the convergence
order is $1 - \epsilon$ in space and $\frac{1}{2} - \epsilon$ in time for the space-time white noise case and $2 - \epsilon$ in space and $1 - \epsilon$ in time for the trace class noise case, with arbitrarily small $\epsilon > 0$.

To conclude, convergence orders of the numerical invariant measures, depending on the regularity of noise, are recovered based on an easy time-independent weak error analysis without relying on Malliavin calculus, which is required in the analysis in [6, 9]. Instead of the linear implicit Euler scheme as done by [6, 9], we examine an exponential integrator scheme, whose strong and weak approximation errors over a finite time interval have been extensively investigated by many authors [4, 22, 25, 32, 33]. Numerical results in Table 4 of Section 5 indicate that, the exponential Euler scheme is always considerably more accurate than the linear implicit Euler scheme. We finally mention that one can consult, e.g., [2, 3, 7, 8, 12, 16, 20, 34] and references therein, for other relevant works on weak approximations over a finite time interval.

The rest of this paper is organized as follows. Some preliminaries and assumptions are collected in the next section. Sections 3 and 4 focus on the ergodicity of the numerical approximations for both spatial and temporal discretizations as well as the error estimates between invariant measures. Numerical experiments are finally performed to illustrate the theoretical results in Section 5.

## 2 Setting and preliminaries

Throughout this paper, the following notation is used. Let $\mathbb{N} = \{1, 2, \cdots \}$ be the set of positive integers and $\epsilon > 0$ be an arbitrarily small parameter. Let $(H, \langle \cdot, \cdot \rangle_H, \cdot \| H)$ and $(U, \langle \cdot, \cdot \rangle_U, \cdot \| U)$ be two real separable Hilbert spaces. By $\mathcal{L}_k^U(U, H)$ we denote the space of not necessarily bounded mappings from $U$ to $H$ that have continuous and bounded Fréchet derivatives up to order $k$ for $k = 1, 2$. Furthermore, by $\mathcal{L}(U, H)$ we denote the space of all bounded linear operators from $U$ to $H$ with the usual operator norm $\| \cdot \|_{\mathcal{L}(U, H)}$ and write $\mathcal{L}(U) := \mathcal{L}(U, U)$ for simplicity. Moreover, we need space of nuclear operators from $U$ to $H$ denoted by $\mathcal{L}_1(U, H)$ and space of Hilbert-Schmidt operators from $U$ to $H$ by $\mathcal{L}_2(U, H)$. To lighten the notation, we also write $\mathcal{L}_1(U) := \mathcal{L}_1(U, U)$ and $\mathcal{L}_2(U) := \mathcal{L}_2(U, U)$. As usual, $\mathcal{L}_1(U)$ and $\mathcal{L}_2(U, H)$ are endowed with the nuclear norm $\| \cdot \|_{\mathcal{L}_1(U)}$ and the Hilbert-Schmidt norm $\| \cdot \|_{\mathcal{L}_2(U, H)}$, respectively,

$$
\| \Gamma \|_{\mathcal{L}_1(U)} = \text{Tr} \hat{\Gamma} = \sum_{i=1}^{\infty} \langle \hat{\Gamma} \psi_i, \psi_i \rangle, \quad \| \Gamma \|_{\mathcal{L}_2(U, H)} = \left( \sum_{i=1}^{\infty} \| \Gamma \psi_i \|^2_H \right)^{\frac{1}{2}},
$$

(2.1)

where $\hat{\Gamma} = (\Gamma^* \Gamma)^{\frac{1}{2}}$ and $\Gamma^*$ denotes the adjoint operator of $\Gamma$. Additionally, the norms defined above do not depend on the particular choice of the orthonormal basis $\{ \psi_i \}_{i \in \mathbb{N}}$ of $U$ (see, e.g., [14, Appendix C]). For the convenience of the following analysis, we list some norm inequalities (see, e.g., [29, Appendix B]). If $\Gamma_1 \in \mathcal{L}_1(U)$ and $\Gamma_2 \in \mathcal{L}(U)$, then $\Gamma_1^* \in \mathcal{L}_1(U)$, $\Gamma_1 \Gamma_2 \in \mathcal{L}_1(U)$, $\Gamma_2 \Gamma_1 \in \mathcal{L}_1(U)$ and

$$
\| \text{Tr} \Gamma_1 \|_{\mathcal{L}_1(U)} \leq \| \Gamma_1 \|_{\mathcal{L}_1(U)}, \quad \text{Tr}(\Gamma_1^*) = \text{Tr}(\Gamma_1), \quad \text{Tr}(\Gamma_1 \Gamma_2) = \text{Tr}(\Gamma_2 \Gamma_1).
$$

(2.2)

When $\Gamma_1 \in \mathcal{L}_2(U, H)$ and $\Gamma_2 \in \mathcal{L}_2(H, U)$, it holds that $\Gamma_1^* \in \mathcal{L}_2(H, U)$, $\Gamma_1 \Gamma_2 \in \mathcal{L}_1(H)$ and

$$
\| \Gamma_1^* \|_{\mathcal{L}_2(H, U)} = \| \Gamma_1 \|_{\mathcal{L}_2(U, H)}, \quad \| \Gamma_1 \Gamma_2 \|_{\mathcal{L}_1(H)} \leq \| \Gamma_1 \|_{\mathcal{L}_2(U, H)} \| \Gamma_2 \|_{\mathcal{L}_2(H, U)}.
$$

(2.3)

For $\Gamma \in \mathcal{L}(U, H)$ and $\Gamma_j \in \mathcal{L}_j(U, j = 1, 2$, we have $\Gamma \Gamma_j \in \mathcal{L}_j(U, H)$ and

$$
\| \Gamma \Gamma_j \|_{\mathcal{L}_j(U, H)} \leq \| \Gamma \|_{\mathcal{L}(U, H)} \| \Gamma_j \|_{\mathcal{L}_j(U)}, \quad j = 1, 2.
$$

(2.4)

To proceed, we make the following assumptions.

**Assumption 2.1.** Let $A : \mathcal{D}(A) \subset H \rightarrow H$ be a densely defined, self-adjoint, negative definite linear operator, which is not necessarily bounded but with compact inverse.
In the above setting, the dominant linear operator $A$ generates an analytic semigroup of contractions $E(t) = e^{tA}$, $t \geq 0$ in $H$ and there exists an increasing sequence of real numbers $\{\lambda_i\}_{i \in \mathbb{N}}$ and an orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$ of $H$ such that

$$- Ae_i = \lambda_i e_i, \quad \forall i \in \mathbb{N} \quad \text{with} \quad 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n (\to \infty). \quad (2.5)$$

This allows us to define fractional powers of $-A$, i.e., $(-A)^\gamma$, $\gamma \in \mathbb{R}$, in a much simple way, see [24, Appendix B.2]. So we introduce the Hilbert space $H^\gamma = D((-A)^\gamma)$ for $\gamma \in \mathbb{R}$, equipped with the inner product $\langle \varphi, \psi \rangle_{H^\gamma} = \langle (-A)^\gamma \varphi, (-A)^\gamma \psi \rangle = \sum_{i=1}^n \lambda_i^\gamma \langle \varphi, e_i \rangle \langle \psi, e_i \rangle$ and the corresponding norm $\|\varphi\|_{\gamma} = \sqrt{\langle \varphi, \varphi \rangle_{H^\gamma}}$ for $\varphi, \psi \in H^\gamma$. It is known in [6,9] that the semigroup $\{E(t)\}_{t \geq 0}$ enjoys the following smoothing properties

$$\|(-A)^\gamma E(t)\|_{L(H)} \leq Ce^{-\frac{\lambda_1 t}{2}}, \quad t > 0, \gamma \geq 0,$$

$$\|(-A)^{-\rho}(E(t) - E(s))\|_{L(H)} \leq C(t-s)^\rho e^{-\frac{\lambda_1 s}{2}}, \quad 0 \leq s < t, \rho \in [0, 1]. \quad (2.6)$$

Here and below, $C$ is a generic constant that may vary from one place to another.

**Assumption 2.2.** Let $\{W^Q(t)\}_{t \geq 0}$ be a cylindrical $Q$-Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a normal filtration $\{\mathcal{F}_t\}_{t \geq 0}$ with $Q: H \to H$ being a self-adjoint, positive definite bounded linear operator. Furthermore, let $A$ and $Q$ be commutable and satisfy

$$\|(-A)^{\frac{\beta}{2}} Q^\frac{1}{2}\|_{L_2(H)} < \infty, \quad \text{for some} \quad \beta \in (0, 1]. \quad (2.7)$$

In addition, let the initial data $X_0 \in H^{\max(2\beta, 1)}$ be deterministic. Let the nonlinear mapping $F: H \to H$ satisfy a one-sided Lipschitz condition

$$\langle \varphi - \psi, F(\varphi) - F(\psi) \rangle \leq L_F \|\varphi - \psi\|^2, \quad \text{with} \quad L_F < \lambda_1, \quad \forall \varphi, \psi \in H, \quad (2.8)$$

where $\lambda_1$ is the smallest eigenvalue of $-A$. Finally, also let $F$ be twice differentiable and there exist some constants $\delta \in [1, 2]$ and $\eta \in [0, 1)$ such that

$$\|F'(\varphi)\varphi\| \leq L \|\varphi\|, \quad \varphi, \psi \in H, \quad (2.9)$$

$$\|(-A)^{-\frac{\delta}{2}} F'(\varphi)\| \leq L(1 + \|\varphi\|_1) \|\psi\|_{-1}, \quad \varphi \in H^1, \psi \in H, \quad (2.10)$$

$$\|(-A)^{-\eta} F'F'(\varphi)(\psi_1, \psi_2)\| \leq L \|\psi_1\| \|\psi_2\|, \quad \varphi, \psi_1, \psi_2 \in H. \quad (2.11)$$

It is well-known that $\{W^Q(t)\}_{t \geq 0}$ can be represented as

$$W^Q(t) := \sum_{i=1}^\infty \sqrt{\eta_i} \beta(t) e_i, \quad t \geq 0, \quad (2.12)$$

where $\{\beta(t)\}_{t \geq 0}$ for $i \in \{n \in \mathbb{N}; \; q_n > 0\}$ are independent real-valued Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$. A class of semi-linear stochastic heat equations satisfying the above setting can be found in [33, Example 3.2]. Moreover, under Assumptions 2.1, 2.2, (1.1) admits a unique mild solution, see [15, Theorem 5.3.1].

**Theorem 2.3 (Existence, uniqueness of mild solution).** Let Assumptions 2.1, 2.2 hold. Then (1.1) admits a unique mild solution $\{X(t)\}_{t \geq 0}$ given by (1.3).
where $X(t, x)$ is the mild solution of (1.1) with initial value $X(0) = x \in H$. Then it is easy to check that $\{P_t\}_{t \geq 0}$ is a Markov semigroup on $B_0(H)$, see [13, Definition 5.1] for the precise definition of Markov semigroup.

Let us give some properties of $\{P_t\}_{t \geq 0}$. $\{P_t\}_{t \geq 0}$ is said to be strong Feller if $P_t \Phi \in C_b(H)$ for any $\Phi \in B_b(H)$ and any $t > 0$. Also, $\{P_t\}_{t \geq 0}$ is said to be irreducible if $P_t 1_{B(x_0, r)}(x) > 0$ for any $x, x_0 \in H, r > 0$ and any $t \geq 0$, where $B(x_0, r)$ is the open ball in $H$ with center $x_0$ and radius $r > 0$. Moreover, a probability measure $\mu$ on $(H, B(H))$ is said to be invariant for $\{P_t\}_{t \geq 0}$ if

$$\int_H P_t \Phi \, d\mu = \int_H \Phi \, d\mu, \quad \forall \Phi \in B_0(H), t \geq 0.$$ 

According to the Von Neumann theorem [13, Theorem 5.12], the limit

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T P_t \Phi \, dt, \quad \forall \Phi \in L^2(H, \mu)$$

always exists in $L^2(H, \mu)$, where $L^2(H, \mu)$ is the space of all square integrable functions $\Phi: H \to \mathbb{R}$ with respect to $\mu$.

**Definition 2.4.** Let $\mu$ be an invariant probability measure for $\{P_t\}_{t \geq 0}$. We say that $\{P_t\}_{t \geq 0}$ is ergodic if

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T P_t \Phi \, dt = \int_H \Phi(y) \mu(dy) \quad \text{in} \quad L^2(H, \mu),$$

(2.13)

for all $\Phi \in L^2(H, \mu)$.

Additionally, we say the stochastic process $\{X(t, x)\}_{t \geq 0}$ is ergodic if the associated Markov semigroup $\{P_t\}_{t \geq 0}$ is ergodic. Observing that (2.8) yields

$$\langle A \varphi + F(\varphi), \varphi \rangle \leq -c\|\varphi\|^2 + C$$

(2.14)

for some constants $c, C \in (0, \infty)$ and for all $\varphi \in D(A)$, which is a sufficient condition to show $\{X(t, x)\}_{t \geq 0}$ is ergodic, see, e.g., [15, Section 8.6], [10].

**Theorem 2.5 (Ergodicity of mild solution).** Let Assumptions 2.1, 2.2 hold. Then $\{X(t)\}_{t \geq 0}$ given by (1.3) is ergodic with a unique invariant probability measure $\nu$.

We will end this section by giving a sufficient condition for stochastic process $\{X(t, x)\}_{t \geq 0}$ to be ergodic, see Proposition 7.10, Theorem 7.6 and Theorem 5.16 in [13], which will be used to show the ergodicity of the semi-discretization approximations process $\{X^n(t)\}_{t \geq 0}$ in Theorem 3.2 below.

**Theorem 2.6.** Let $V: H \to [0, \infty]$ be a Borel function whose level sets

$$K_a := \{x \in H : V(x) \leq a\}, \quad \forall a > 0,$$

(2.15)

are compact. Let $\{X(t, x)\}_{t \geq 0}$ be the solution of (3.1) with initial value $X(0) = x \in H$ and assume that there exist $x \in H$ and $C(x) > 0$ such that

$$\mathbb{E}[V(X(t, x))] \leq C(x), \quad \forall t > 0.$$ 

(2.16)

Then $\{X(t, x)\}_{t \geq 0}$ possesses at least one invariant probability measure. If in addition it happens that the corresponding Markov semigroup $\{P_t\}_{t \geq 0}$ is strong Feller and irreducible, then $\{X(t, x)\}_{t \geq 0}$ possesses a unique invariant probability measure and hence is ergodic.
3 Spatial discretization and its ergodicity

The aim of this section is to analyze the error of invariant measures in the spatial direction. To this end, we first obtain a numerical solution \( \{X^n(t)\}_{t \geq 0} \) in space by applying a spectral Galerkin method to (1.1) and introduce a sufficient condition for this approximation to be ergodic in Subsection 3.1. The main result in Subsection 3.2 shows that \( \{X^n(t)\}_{t \geq 0} \) is ergodic with a unique invariant measure \( \nu_n \). This ergodicity and the time-independent weak error established in Subsection 3.3 finally imply the convergence order of invariant measures \( \nu \) and \( \nu_n \) in Subsection 3.4.

3.1 Spectral Galerkin method

For \( n \in \mathbb{N} \), we define finite dimensional subspaces \( H_n \) of \( H \) by \( H_n := \text{span}\{e_1, e_2, \ldots, e_n\} \) and projection operators \( P_n: H \to H_n \) by \( P_n \varphi = \sum_{i=1}^{n} \langle e_i, \varphi \rangle e_i \) for all \( \varphi \in H \). Now we introduce a Galerkin approximation to (1.1) in the finite-dimensional space \( H_n \) as follows

\[
\begin{aligned}
\frac{dX^n(t)}{dt} &= A_nX^n(t) \, dt + P_n F(X^n(t)) \, dt + P_n \, dW^Q(t), \quad t > 0, \\
X^n(0) &= X_0^n := P_n X_0 \in H_n,
\end{aligned}
\] (3.1)

where \( A_n: H_n \to H_n \) is defined by \( A_n := AP_n \) and generates a strongly continuous semigroup \( E_n(t) = e^{tA_n}, t \geq 0 \) in \( H_n \). Similarly as above, we can define \( (-A_n)^\gamma: H_n \to H_n \) for all \( \gamma \in \mathbb{R} \) as \( (-A_n)^\gamma \varphi := \sum_{i=1}^{n} \lambda_i^\gamma \langle \varphi, e_i \rangle e_i \) for all \( \varphi \in H_n \). Note that \( (-A_n)^\gamma P_n \varphi = (-A)^\gamma P_n \varphi \) and \( E_n(t)P_n \varphi = E(t)P_n \varphi \) hold for all \( \varphi \in H \) and all \( \gamma \in \mathbb{R} \). Furthermore, variants of conditions in Assumptions 2.1, 2.2 and (2.6) remain true and are frequently used in the following estimates. For example, we have

\[
\begin{align*}
\|(-A_n)^{\beta-1} P_n Q^n\|_{L_2(H,H_n)} &< \infty, \quad \text{for some} \quad \beta \in (0,1], \\
\langle A_n \varphi + P_n F(\varphi), \varphi \rangle &\leq -c\|\varphi\|^2 + C, \quad \varphi \in D(A_n), \\
\|(-A_n)^{-\frac{\alpha}{2}} P_n F'(\varphi)\| &\leq L(1 + \|\varphi\|_1)\|\varphi\|_{-1}, \quad \varphi \in H^1, \psi \in H, \alpha \in [1,2), \\
\|(-A_n)^{-\eta} P_n F''(\varphi)(\psi_1,\psi_2)\| &\leq L\|\psi_1\|\|\psi_2\|, \quad \varphi, \psi_1, \psi_2 \in H, \eta \in [0,1), \\
\|(-A_n)^{-\rho} E_n(t)\|_{L(H_n)} &\leq C t^{-\gamma} e^{-\frac{t}{4\rho}}, \quad t > 0, \gamma \geq 0, \\
\|(-A_n)^{-\rho} (E_n(t) - E_n(s))\|_{L(H_n)} &\leq C(t-s)^{\rho} e^{-\frac{t-s}{4\rho}}, \quad 0 \leq s < t, \rho \in [0,1],
\end{align*}
\] (3.2)-(3.7)

where \( \beta, \delta, \eta, \) are the same with the parameters in (2.7), (2.10), (2.11), respectively and the constants \( c, C, L \) are independent of \( n \) and \( t \). Moreover, the above assumption ensures (3.1) has a well-defined solution with a uniform mean square moment bound.

**Theorem 3.1 (Existence, uniqueness and moment boundedness of spatial approximation).**

Let Assumptions 2.1, 2.2 hold. Then (3.1) admits a unique solution \( X^n: [0,\infty) \times \Omega \to H_n \) with continuous sample path given by

\[
X^n(t) = E_n(t)X^n_0 + \int_0^t E_n(t-s)P_n F(X^n(s)) \, ds + \int_0^t E_n(t-s)P_n \, dW^Q(s), \quad \text{a.s.,} \quad t \geq 0.
\] (3.8)

Moreover, there exists a constant \( C = C(X_0) > 0 \) independent of \( n, t \) such that

\[
\mathbb{E}[\|X^n(t)\|^2] \leq C.
\] (3.9)

**Proof.** It suffices to show (3.9) since the existence of the unique solution \( \{X^n(t)\}_{t \geq 0} \) can be found in [22, Theorem 4.5.3]. In fact, set \( O^n(t) = \int_0^t E_n(t-s)P_n \, dW^Q(s) \) and define \( \tilde{X}^n(t) = X^n(t) - O^n(t) \), one can easily verify that \( \tilde{X}^n(0) = X^n_0 \) and that \( \tilde{X}^n(t) \) satisfies the following partial differential equation

\[
\frac{d\tilde{X}^n(t)}{dt} = A_n \tilde{X}^n(t) + P_n F(\tilde{X}^n(t) + O^n(t)), \quad t \geq 0.
\] (3.10)
As a direct consequence of (3.10), we have
\[
\frac{de^{ct}\|X^n(t)\|^2}{dt} = 2e^{ct}\langle A_nX^n(t) + P_nF(X^n(t) + O^\alpha(t)), X^n(t) \rangle + ce^{ct}\|X^n(t)\|^2, \tag{3.11}
\]
where the constant \(c\) comes from (3.3). Employing (3.3), (2.9), the Cauchy-Schwarz inequality and the weighted Young inequality \(ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon}b^2\) for all \(a, b \in \mathbb{R}\) with \(\varepsilon = c > 0\) leads to
\[
e^{ct}\|X^n(t)\|^2 = \|X^n_0\|^2 + 2 \int_0^te^{cs}\langle A_nX^n(s) + P_nF(X^n(s)), X^n(s) \rangle ds + c \int_0^te^{cs}\|X^n(s)\|^2 ds \]
\[
+ 2 \int_0^te^{cs}\langle P_nF(X^n(s) + O^\alpha(s)) - P_nF(X^n(s)), X^n(s) \rangle ds
\]
\[
\leq \|X^n_0\|^2 - c \int_0^te^{cs}\|X^n(s)\|^2 ds + 2C\frac{e^{ct} - 1}{c} + 2L \int_0^te^{cs}\|O^n(s)\||\|X^n(s)\| ds
\]
\[
\leq \|X^n_0\|^2 + 2C\frac{e^{ct} - 1}{c} + \frac{L^2}{c} \int_0^te^{cs}\|O^n(s)\|^2 ds.
\tag{3.12}
\]
According to Itô’s isometry, (3.2) and (3.6), we obtain
\[
\mathbb{E}[\|O^n(s)\|^2] = \int_0^s \|E^n(s - r)P_nQ^n_1\|^2_{L^2(H_n)} dr
\]
\[
\leq \|(-A_n)^{\frac{\alpha-1}{2}}P_nQ^n_1\|^2_{L^2(H_n)} \int_0^s \|(-A_n)^{\frac{1-\alpha}{2}}E^n(s - r)\|^2_{L^2(H_n)} dr
\tag{3.13}
\]
\[
\leq C \int_0^s (s - r)^{\beta-1}e^{-\lambda_1(s-r)} dr \leq C,
\]
where in the last step we used the well-known fact of Gamma function as follows
\[
\int_0^\infty x^{\varphi-1}e^{-x} dx < \infty, \quad \forall \varphi > 0.
\tag{3.14}
\]
Using (3.13) after taking expectations on both sides of (3.12) yields
\[
e^{ct}\mathbb{E}[\|X^n(t)\|^2] \leq \mathbb{E}[\|X^n_0\|^2] + 2\frac{C e^{ct} - 1}{c} + \frac{CL^2}{c} \int_0^te^{cs} ds
\]
\[
\leq \mathbb{E}[\|X^n_0\|^2] + Ce^{ct},
\tag{3.15}
\]
which results in the required conclusion (3.9) by multiplying \(e^{-ct}\) on both sides of (3.15).

### 3.2 Ergodicity for the spatial discretization

Given the above preparations, we use Theorem 2.6 to give the following result.

**Theorem 3.2 (Ergodicity of \(\{X^n(t)\}_{t \geq 0}\)).** Let Assumptions 2.1, 2.2 hold. Then \(\{X^n(t)\}_{t \geq 0}\) given by (3.8) is ergodic with a unique invariant measure \(\nu^n\).

**Proof.** To prove the ergodicity of \(\{X^n(t)\}_{t \geq 0}\), let us first give an equivalent form of (3.1). Since \(\{X^n(t)\}_{t \geq 0}\) is an \(H_n\)-valued stochastic process, we have
\[
X^n(t) = \sum_{i=1}^n x_i(t)e_i, \quad x_i(t) = \langle X^n(t), e_i \rangle, \quad i = 1, 2, \cdots, n.
\tag{3.16}
\]
Inserting (3.1) with (2.12) into \(x_i(t) = \langle X^n(t), e_i \rangle\) yields
\[
dx_i(t) = (\lambda_i x_i(t) + \langle P_nF(X^n(t)), e_i \rangle) dt + \sqrt{q_i}d\beta_i(t), \quad t \geq 0, i = 1, 2, \cdots, n.
\tag{3.17}
\]
From now on, we use $B'$ to denote the transpose of a vector or matrix $B$. By denoting
\[
x(t) = (x_1(t), x_2(t), \cdots, x_n(t))' \in \mathbb{R}^n, \quad \beta(t) = (\beta_1(t), \beta_2(t), \cdots, \beta_n(t))' \in \mathbb{R}^n,
\]
and
\[
\Lambda = \text{diag}(-\lambda_1, \cdots, -\lambda_n) \in \mathbb{R}^{n \times n},
\]
\[
g(x(t)) = ((P_n F(X^n(t)), e_1), \cdots, (P_n F(X^n(t)), e_n))' \in \mathbb{R}^n,
\]
\[
\tilde{Q} = \text{diag}(\sqrt{q_1}, \cdots, \sqrt{q_n}) \in \mathbb{R}^{n \times n},
\]
we can rewrite (3.17) as a $\mathbb{R}^n$-valued SDE
\[
dx(t) = (\Lambda x(t) + g(x(t))) \, dt + \tilde{Q} \, d\beta(t), \quad t \geq 0, \tag{3.18}
\]
and thus it suffices to show that $\{x(t)\}_{t \geq 0}$ is ergodic. Indeed, the ergodicity of $\{x(t)\}_{t \geq 0}$ implies there is a random variable $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ such that $\lim_{t \to \infty} x(t) = \xi$, i.e., $\lim x_i(t) = \xi_i$, $i = 1, 2, \cdots, n$.

It follows that $\lim_{t \to \infty} X^n(t) = \sum_{i=1}^n \xi_i e_i$, which immediately ensures that $\{X^n(t)\}_{t \geq 0}$ is ergodic by the definition of ergodicity. By Theorem 2.6 the proof of the ergodicity of $\{x(t)\}_{t \geq 0}$ is equivalent to show that $\{x(t)\}_{t \geq 0}$ is strong Feller, irreducible and satisfies the Lyapunov condition. In what follows we will validate these properties one by one. Thanks to Rank($\tilde{Q}$) = $n$, the strong Feller property of $\{x(t)\}_{t \geq 0}$ follows immediately by [17, Remark 1.4].

To show $\{x(t)\}_{t \geq 0}$ is irreducible, we denote $G(x(t)) := \Lambda x(t) + g(x(t))$ in (3.18) to get
\[
dx(t) = G(x(t)) \, dt + \tilde{Q} \, d\beta(t), \quad t \geq 0. \tag{3.19}
\]
Let $y, y^+ \in \mathbb{R}^n$, $\delta, t > 0$ be arbitrary and denote the solution of (3.19) with initial value $x(0) = y$ by $x(t, y)$. By the definition of irreducibility (see [13, Definition 5.2]), it suffices to prove that
\[
\mathbb{P}(|x(t, y) - y^+| < \delta) > 0. \tag{3.20}
\]
Here and below, we denote $(\cdot, \cdot)$ to be the usual Euclidean inner product in $\mathbb{R}^n$ and $|\cdot|$ be the corresponding norm in $\mathbb{R}^n$, or the Frobenius matrix norm in $\mathbb{R}^{n \times n}$. To show (3.20), we follow the idea stemed from [26] and consider the associated control problem
\[
\frac{d\bar{x}(t)}{dt} = G(\bar{x}(t)) + \tilde{Q} \frac{dU(t)}{dt}. \tag{3.21}
\]
Then for every fixed $t > 0$, we can find the control function $U \in C^1([0, t], \mathbb{R}^n)$ with $U(0) = 0$ such that (3.21) is satisfied and $\bar{x}(0) = y, \bar{x}(t) = y^+$. This can be achieved by polynomial interpolation between the end points using a linear polynomial in time with vector coefficients in $\mathbb{R}^n$ and by the invertibility of matrix $\tilde{Q}$. The integral forms of (3.19) and (3.21) show that
\[
x(s, y) - \bar{x}(s) = \int_0^s G(x(r, y)) - G(\bar{x}(r)) \, dr + \tilde{Q}(\beta(s) - U(s)), \quad s \in [0, t]. \tag{3.22}
\]
Note that the event $\{\omega \in \Omega : \sup_{0 \leq s \leq t} |\beta(s)(\omega) - U(s)| \leq \varepsilon\}$ occurs with positive probability for any $\varepsilon > 0$, since the Wiener measure of any such tube is positive (see [26, Lemma 3.4]). Assume this event occurs and note that $G$ is Lipschitz continuous because of (2.9), one sees that
\[
|x(s, y) - u(s)| \leq L_G \int_0^s |x(r, y) - u(r)| \, dr + |\tilde{Q}| \varepsilon \leq |\tilde{Q}| \varepsilon e^{tL_G},
\]
By Gronwall’s inequality, we have $|x(s, y) - u(s)| \leq |\tilde{Q}| \varepsilon e^{tL_G}$. Choosing $s = t$ and $\varepsilon = \delta/|\tilde{Q}| e^{tL_G}$, (3.20) can be promised and hence the irreducibility follows.
Now we are in a position to verify that \( \{x(t)\}_{t \geq 0} \) satisfies Lyapunov condition. For this, we choose Lyapunov function \( V(x) = |x|^2, x \in \mathbb{R}^n \). Because of the continuity of norm and the Heine-Borel theorem in the finite-dimensional space \( \mathbb{R}^n \), it follows that the level sets \( K_a \) are compact for any \( a > 0 \). In addition, by (3.18) and Itô’s formula, we have

\[
\begin{align*}
\frac{d}{dt}|x(t)|^2 &= 2\langle x(t), A x(t) \rangle + 2\langle x(t), g(x(t)) \rangle dt + \sum_{i=1}^{n} q_i dt + 2\langle x(t), \tilde{Q}d\beta(t) \rangle \\
&\leq -2\lambda_1 |x(t)|^2 dt + 2\langle x(t), g(x(t)) \rangle dt + \sum_{i=1}^{n} q_i dt + 2\langle x(t), \tilde{Q}d\beta(t) \rangle.
\end{align*}
\]

(3.23)

Recall the notation \( x(t) \) and \( g(x(t)) \), we use the self-adjointness of \( P_n \) and (2.8) to get

\[
\begin{align*}
(x(t), g(x(t))) &= \sum_{i=1}^{n} \langle X^n(t), e_i \rangle \langle P_n F(X^n(t)), e_i \rangle = \langle X^n(t), P_n F(X^n(t)) \rangle \\
&= \langle X^n(t), F(X^n(t)) - F(0) \rangle + \langle X^n(t), F(0) \rangle \\
&\leq \frac{\lambda_1 + L_F}{2} ||X^n(t)||^2 + \frac{||F(0)||^2}{2(\lambda_1 - L_F)},
\end{align*}
\]

(3.24)

where we also used the weighted Young inequality \( ab \leq \varepsilon a^2 + \frac{1}{2}b^2 \) for all \( a, b \in \mathbb{R} \) with \( \varepsilon = \frac{\lambda_1 - L_F}{2} > 0 \). Observing the fact \( ||X^n(t)||^2 = |x(t)|^2 \) because of (3.16) and taking expectations on the both sides of (3.23) show that

\[
\frac{d\mathbb{E}[|x(t)|^2]}{dt} \leq -(\lambda_1 - L_F)\mathbb{E}[|x(t)|^2] + \left( \frac{||F(0)||^2}{\lambda_1 - L_F} + \sum_{i=1}^{n} q_i \right),
\]

which leads to

\[
\mathbb{E}[|x(t)|^2] \leq e^{-(\lambda_1 - L_F)t}\mathbb{E}[|x(0)|^2] + \frac{1 - e^{-(\lambda_1 - L_F)t}}{\lambda_1 - L_F} \left( \frac{||F(0)||^2}{\lambda_1 - L_F} + \sum_{i=1}^{n} q_i \right), \quad t \geq 0.
\]

This means that \( \{x(t)\}_{t \geq 0} \) satisfies the Lyapunov condition and thus finishes the proof. \( \square \)

### 3.3 Weak spatial approximation error over long time

An important ingredient to obtain the time-independent weak error is the improved estimates on the derivatives of the solution of the associated Kolmogorov equation. To show this, for \( n \in \mathbb{N}, T > 0 \) and \( \Phi \in C^2_b(H, \mathbb{R}) \) we introduce the function \( v^n : [0, \infty) \times H_n \rightarrow \mathbb{R} \) by

\[
v^n(t, y) = \mathbb{E}[\Phi(X^n(t, y))], \quad t \geq 0, y \in H_n,
\]

(3.25)

where \( X^n(t, y) \) is the unique solution of (3.1) with the initial value \( y = X^n_0 \in H_n \). Recall that \( v^n(t, y) \) is continuously differentiable with respect to \( t \) and continuously twice differentiable with respect to \( y \) and acts as the unique strict solution of the following Kolmogorov equation [14, Theorem 9.16]

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial v^n}{\partial t}(t, y) = \langle Dv^n(t, y), A_n y + P_n F(y) \rangle + \frac{1}{2} \text{Tr} \left\{ D^2 v^n(t, y)(P_n Q^2 y)(P_n Q^2 y)^* \right\}, \\
v^n(0, y) = \Phi(y),
\end{array} \right.
\end{align*}
\]

(3.26)

under Assumptions 2.1, 2.2. Here by a strict solution of (3.26) we mean a function \( v^n \in C^{1,2}_b([0, \infty) \times H_n, \mathbb{R}) \) such that (3.26) holds. Moreover, by the Riesz representation theorem, we can always identify the first derivative \( Dv^n(t, y) \) at \( y \in H_n \) with an element in \( H_n \) such that \( Dv^n(t, y) \cdot g = \langle Dv^n(t, y), g \rangle \) for all \( g \in H_n \) and the second derivative \( D^2 v^n(t, y) \) at \( y \in H_n \) with a bounded linear operator on \( H_n \) such that \( D^2 v^n(t, y) \cdot (g, k) = \langle D^2 v^n(t, y)g, k \rangle \) for all \( g, k \in H_n \).

Repeating each lines in the proof of Propositions 5.1 and 5.2 in [6] with slight changes and taking Assumptions 2.1, 2.2 into account, we have the following regularity results on the derivatives of \( v^n(t, y) \).
Proposition 3.3 (Regularity of $Dv^n(t, y)$ and $D^2v^n(t, y)$). Let Assumptions 2.1, 2.2 hold and let $v^n(t, y)$ be defined by (3.25) with $\Phi \in C^2_b(H, \mathbb{R})$. For any $\gamma \in (0, 1)$ and $\gamma_1, \gamma_2 \in (0, 1)$ with $\gamma_1 + \gamma_2 < 1$ there exist constants $C_\gamma, C_{\gamma_1, \gamma_2}, \tilde{c} > 0$ such that

$$
\|(−A_n)^\gamma Dv^n(t, y)\|_{H_n} \leq C_\gamma(1 + t^{-\gamma})e^{-\tilde{c}t},
$$

(3.27)

$$
\|(−A_n)^{\gamma_2}D^2v^n(t, y)(−A_n)^{\gamma_1}\|_{L(H_n)} \leq C_{\gamma_1, \gamma_2}(1 + t^{-\eta} + t^{-(\gamma_1 + \gamma_2)})e^{-\tilde{c}t},
$$

(3.28)

for all $t \geq 0$, $y \in H_n$, where the parameter $\eta$ comes from (2.11).

With the above preparations, we can prove the following time-independent weak error.

Theorem 3.4 (Spatial weak error). Let Assumptions 2.1, 2.2 hold and let $\{X(t)\}_{t \geq 0}$ and $\{X^n(t)\}_{t \geq 0}$ be given by (1.1) and (3.1), respectively. For any $T > 0$, $n \in \mathbb{N}$, $\Phi \in C^2_b(H, \mathbb{R})$ and arbitrarily small $\epsilon > 0$ there exists a constant $C > 0$ independent of $T, n$ such that

$$
\left| E[\Phi(X(T))] - E[\Phi(X^n(T))] \right| \leq C\lambda_n^{-\beta + \epsilon}.
$$

(3.29)

Proof. We set $k \in \mathbb{N} \cap [n, \infty)$ and decompose the spatial approximation error as follows

$$
\left| E[\Phi(X(T))] - E[\Phi(X^n(T))] \right| \leq \left| E[\Phi(X(T))] - E[\Phi(X^k(T))] \right| + \left| E[\Phi(X^k(T))] - E[\Phi(X^n(T))] \right|.
$$

(3.30)

Taking $k \to \infty$ in (3.30) and employing the fact that $X^k(T)$ converges to $X(T)$ in mean square sense (see, e.g., [34, Lemma A.1]) lead to

$$
\left| E[\Phi(X(T))] - E[\Phi(X^n(T))] \right| \leq \limsup_{k \to \infty} \left| E[\Phi(X^k(T))] - E[\Phi(X^n(T))] \right|.
$$

(3.31)

By (3.25) and (3.26), it follows that

$$
E[\Phi(X^k(T))] - E[\Phi(X^n(T))] = E[v^k(T, X^n_0)] - E[v^k(T, X^0_0)] + E[v^k(T, X^n_0)] - E[v^k(0, X^n(T))].
$$

(3.32)

Before we calculate the first term on the right hand side of (3.32), we note that

$$
\left| (P_k - P_n)v \right| \leq \left| P_k \right|_{L(H_k)} \left| (I - P_n)(-A)^{-\beta} \right|_{L(H_k)} \left| (-A)^{\beta} v \right| \leq \lambda_n^{-\beta} \| v \|_{2\beta}
$$

(3.33)

for all $v \in H^{2\beta}$ and $\beta \in (0, 1)$. We then use Taylor’s formula, (3.27), (3.33) and $X_0 \in H^{2\beta}$ to obtain

$$
\left| E[v^k(T, X^n_0)] - E[v^k(T, X^0_0)] \right| = \left| \int_0^1 E[\langle Dv^k(T, x^n_0 + r(x^n - x^0_0)), x^n_0 - x^0_0 \rangle] \, dr \right|
$$

$$
\leq \int_0^1 E[\| Dv^k(T, x^n_0 + r(x^n - x^0_0)) \| \| (P_k - P_n)X_0 \|] \, dr
$$

$$
\leq C\lambda_n^{-\beta} e^{-cT} \| X_0 \|_{2\beta} \leq C\lambda_n^{-\beta}.
$$

(3.34)

Now we consider the second term on the right hand side of (3.32). Applying Itô’s formula to $v^k(T - t, X^n(t)), t \in [0, T]$, one then sees that

$$
E[v^k(0, X^n(T))] - E[v^k(0, X^0_0)] = E\left[ \int_0^T - \frac{\partial v^k(T - t, X^n(t))}{\partial t} \, dt \right]
$$

$$
+ \int_0^T E[\langle Dv^k(T - t, X^n(t)), A_nX^n(t) + P_nF(X^n(t)) \rangle] \, dt
$$

$$
+ \frac{1}{2} \int_0^T E[\text{Tr} \left\{ D^2v^k(T - t, X^n(t))(P_nQ^\frac{1}{2})(P_nQ^\frac{1}{2})^* \right\}] \, dt.
$$

(3.35)
Substituting (3.26) into (3.35) and using $A_0X^n(t) - A_kX^n(t) = 0$ for $k \in \mathbb{N} \cap [n, \infty)$ enable us to get

$$\mathbb{E}[v^k(0, X^n(T))] - \mathbb{E}[v^k(T, X^n_0)] = \int_0^T \mathbb{E}[\langle D v^k(T-t, X^n(t)), (P_n - P_k)F(X^n(t)) \rangle] \, dt$$

$$+ \frac{1}{2} \int_0^T \mathbb{E}[\text{Tr} \{ D^2 v^k(T-t, X^n(t))(P_n - P_k)Q_{\frac{3}{2}}(P_nQ_{\frac{3}{2}})^* \}] \, dt$$

$$+ \frac{1}{2} \int_0^T \mathbb{E}[\text{Tr} \{ D^2 v^k(T-t, X^n(t))(P_nQ_{\frac{3}{2}})((P_n - P_k)Q_{\frac{3}{2}})^* \}] \, dt := I_1 + I_2 + I_3.$$  

(3.36)

In the sequel we will estimate $I_1, I_2, I_3$ separately. By (3.27), (2.9), (3.9) and (3.14), we have

$$|I_1| \leq \int_0^T \mathbb{E}[\|D v^k(T-t, X^n(t))(-A_k)^{\beta-\epsilon}\| \|(-A_k)^{-\beta+\epsilon}((P_n - P_k)F(X^n(t)))\|] \, dt$$

$$\leq C\lambda_n^{-\beta+\epsilon} \int_0^T (1 + (T-t)^{-\beta+\epsilon})e^{-\tilde{c}(T-t)}\mathbb{E}[\|F(X^n(t))\|] \, dt$$

$$\leq C\lambda_n^{-\beta+\epsilon}. \tag{3.37}$$

Concerning $I_2$, we can derive from (2.2) and (2.4) that

$$|I_2| = \frac{1}{2} \left| \int_0^T \mathbb{E}[\text{Tr} \{ (-A_k)^{\frac{1-\beta}{2}} D^2 v^k(T-t, X^n(t))(P_n - P_k)Q_{\frac{3}{2}}((-A_k)^{-\frac{1}{2}}P_nQ_{\frac{3}{2}})^* \}] \, dt \right|$$

$$\leq \frac{1}{2} \int_0^T \mathbb{E}[\|(-A_k)^{\frac{1-\beta}{2}} D^2 v^k(T-t, X^n(t))(-A_k)^{\frac{1+\beta}{2}-\epsilon}\|_{\mathcal{L}(H_k)}] \, dt$$

$$\cdot \|(-A_k)^{-\frac{1+\beta}{2}+\epsilon}(P_n - P_k)(-A_k)^{-\frac{1}{2}}\|_{\mathcal{L}(H_k)}$$

$$\cdot \|(-A_k)^{\frac{\beta+1}{2}}P_nQ_{\frac{3}{2}}((-A_k)^{\frac{\beta+1}{2}}P_nQ_{\frac{3}{2}})^*\|_{\mathcal{L}(H_k)}. \tag{3.38}$$

Noticing that $\|(-A_k)^{\frac{\beta+1}{2}}P_nQ_{\frac{3}{2}}\|_{\mathcal{L}_2(H,H_k)} = \|(-A_n)^{\frac{\beta+1}{2}}P_nQ_{\frac{3}{2}}\|_{\mathcal{L}_2(H,H_n)}$ and applying (3.28), (2.3), (3.2) and (3.14) bring about

$$|I_2| \leq C \int_0^T (1 + (T-t)^{-\gamma} + (T-t)^{-(1-\epsilon)})e^{-\tilde{c}(T-t)} \, dt \|(-A_k)^{-\beta+\epsilon}(P_n - P_k)\|_{\mathcal{L}(H_k)}$$

$$\cdot \|(-A_k)^{\frac{\beta+1}{2}}P_nQ_{\frac{3}{2}}\|_{\mathcal{L}_2(H,H_k)} \|(-A_n)^{\frac{\beta+1}{2}}P_nQ_{\frac{3}{2}}\|_{\mathcal{L}_2(H,H_n)}$$

$$\leq C\lambda_n^{-\beta+\epsilon}. \tag{3.39}$$

Similarly, we can arrive at

$$|I_3| \leq C\lambda_n^{-\beta+\epsilon}. \tag{3.40}$$

Inserting (3.37), (3.39) and (3.40) into (3.36) gives

$$\left| \mathbb{E}[v^k(0, X^n(T))] - \mathbb{E}[v^k(T, X^n_0)] \right| \leq C\lambda_n^{-\beta+\epsilon}. \tag{3.41}$$

This together with (3.34) and (3.32) verifies the desired result (3.29).
3.4 Error of invariant measures for spatial discretization

**Theorem 3.5.** Let Assumptions 2.1, 2.2 hold. Let \( \nu \) and \( \nu^n \) be the corresponding unique invariant measures of \( \{X(t)\}_{t \geq 0} \) and \( \{X^n(t)\}_{t \geq 0} \), respectively. For any \( T > 0, n \in \mathbb{N} \) and \( \Phi \in C_b^2(H, \mathbb{R}) \) there exists a constant \( C > 0 \) independent of \( T, n \) such that

\[
\left| \int_{H} \Phi(y) \nu(dy) - \int_{H_n} \Phi(y) \nu^n(dy) \right| \leq C \lambda_n^{-\beta + \epsilon}.
\]  

(3.42)

**Proof.** From Theorem 2.5 and Theorem 3.2, we know \( \{X(t)\}_{t \geq 0} \) and \( \{X^n(t)\}_{t \geq 0} \) are ergodic. This together with the definition of ergodicity implies (1.4) and

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E}[\Phi(X^n(t))] \, dt = \int_{H_n} \Phi(y) \nu^n(dy), \quad \forall \Phi \in C_b^2(H, \mathbb{R}),
\]

(3.43)

and hence

\[
\left| \int_{H} \Phi(y) \nu(dy) - \int_{H_n} \Phi(y) \nu^n(dy) \right| \leq \lim_{T \to \infty} \frac{1}{T} \int_0^T \left| \mathbb{E}[\Phi(X(t))] - \mathbb{E}[\Phi(X^n(t))] \right| \, dt \leq C \lambda_n^{-\beta + \epsilon},
\]  

(3.44)

as required, where (3.29) was used in the last step. \( \square \)

**Remark 3.6.** Note that two important classes of noise are included here. One is the space-time white noise in the case \( Q = I \) and the other is the trace class noise in the case \( \text{Tr}(Q) < \infty \). For the space-time white noise, it is well-known that (2.7) is fulfilled with \( \beta < \frac{1}{2} \) in space dimension \( d = 1 \) [23, Remark 3.2]. In this situation our result indicates that the convergence order between \( \nu \) and \( \nu^n \) is \( 1 - \epsilon \) for arbitrarily small \( \epsilon > 0 \). For the trace class noise, (2.7) is satisfied with \( \beta = 1 \) [23, Remark 3.2] and our result implies that the convergence order between \( \nu \) and \( \nu^n \) is \( 2 - \epsilon \) for arbitrarily small \( \epsilon > 0 \), in space dimension \( d = 1 \).

4 Spatio-temporal full discretization and its ergodicity

We will apply an exponential Euler scheme to (3.1) to obtain a spatio-temporal full discretization approximation \( \{Y_m^n\}_{m \in \mathbb{N}} \) and give some regularity estimates in Subsection 4.1. Subsection 4.2 shows that \( \{Y_m^n\}_{m \in \mathbb{N}} \) is ergodic with a unique invariant measure \( \nu^\natural \) via the theory of geometric ergodicity of Markov chains. Based on a weak error representation formula, the time-independent weak error is investigated in Subsection 4.3. Armed with the ergodicity and weak error estimate, we finally obtain the error between invariant measures \( \nu^n \) and \( \nu^\natural \) in Subsection 4.4.

Throughout this section, we need the following notation. Let \( \tau > 0 \) be the uniform time stepsize. Further let \( m, M \in \mathbb{N} \) and set \( t_m = m\tau \) and \( T = M\tau \). Moreover, the generic constant \( C \) must be independent of the spatial dimension \( n \) and the final time \( T = M\tau \) but may depend on \( X_0, \Phi, L_F, L \) and other parameters.

4.1 Exponential Euler time-stepping scheme

Now we approximate equation (3.1) in time by the exponential Euler scheme

\[
Y^n_m = E_n(\tau)Y^n_{m-1} + \tau E_n(\tau)P_nF(Y^n_{m-1}) + E_n(\tau)P_n \Delta W^Q_{m-1}, \quad Y^n_0 = X^n_0,
\]  

(4.1)

where \( Y^n_m \) is an approximation of \( X^n(t_m) \) and \( E_n(\tau)P_n \Delta W^Q_{m-1} := \int_{t_{m-1}}^{t_m} E_n(\tau)P_n \, dW^Q(s) \) is well defined since \( E_n(\tau)P_n Q^{1/2} : H \to H_n \) is a Hilbert-Schmidt operator.

The following lemma concerns the regularity of \( \{Y_m^n\}_{m \in \mathbb{N}} \) over long time.
Lemma 4.1. Let Assumptions 2.1, 2.2 hold, let \( \{Y^n_m\}_{m \in \mathbb{N}} \) be given by (4.1) and let \( \tau < \tau_0 \leq \frac{\lambda_1 - L_F}{4L^2} \). For any \( n \in \mathbb{N}, m \in \mathbb{N} \) and \( \gamma \in (0, \frac{\lambda_1}{2}) \), there is a constant \( C \) independent of \( n, m \) such that

\[
\mathbb{E}[\|(-A_n)^{\gamma}Y^n_m\|^2] \leq C. \tag{4.2}
\]

Proof. We first prove the following inequalities

\[
\mathbb{E}[\|Y^n_m\|^2] \leq C, \quad \mathbb{E}[\|F(Y^n_m)\|^2] \leq C. \tag{4.3}
\]

Indeed, it suffices to verify the first inequality of (4.3) since the second one is an immediate consequence of the first one and (2.9). Now we can easily rewrite (4.1) as

\[
Y^n_m = E^n_m(\tau)Y^n_0 + \tau \sum_{i=0}^{m-1} E^n_m-i(\tau)P_n F(Y^n_i) + \sum_{i=0}^{m-1} E^n_m-i(\tau)P_n \Delta W^Q_i. \tag{4.4}
\]

Set \( |s| = t_i \) for \( s \in [t_i, t_{i+1}) \), \( i = 0, 1, \ldots, m - 1 \) and denote

\[
O^n_m := \sum_{i=0}^{m-1} E^n_m-i(\tau)P_n \Delta W^Q_i = \int_0^{t_m} E_n(t_m - |s|)P_n dW^Q(s), \tag{4.5}
\]

then by Itô’s isometry, (3.6), (3.2) and (3.14) we have

\[
\mathbb{E}[\|O^n_m\|^2] = \int_0^{t_m} \|E_n(t_m - |s|)P_n Q^\frac{1}{2}\|_{L_2(H,H_n)}^2 ds \leq \int_0^{t_m} \|(-A_n)^{\frac{1}{2}}E_n(t_m - |s|)\|^2_{L_2(H,H_n)} \|(-A_n)^{\frac{1}{2}}P_n Q^\frac{1}{2}\|^2_{L_2(H,H_n)} ds \leq C \int_0^{t_m} (t_m - |s|)^{\beta-1} e^{-\lambda_1(t_m - |s|)} ds \leq C. \tag{4.6}
\]

This together with (2.9) indicates

\[
\mathbb{E}[\|P_n F(O^n_m)\|^2] \leq 2L^2 \mathbb{E}[\|O^n_m\|^2] + 2\|F(0)\|^2 \leq C. \tag{4.7}
\]

Set \( \bar{Y}^n_m := Y^n_m - O^n_m \), it is obvious that \( \bar{Y}^n_0 = Y^n_0 \) and

\[
\bar{Y}^n_m = E^n_m(\tau)\bar{Y}^n_0 + \tau \sum_{i=0}^{m-1} E^n_m-i(\tau)P_n F(\bar{Y}^n_i + O^n_i), \tag{4.8}
\]

which immediately gives

\[
\bar{Y}^n_m = E_n(\tau)\bar{Y}^n_{m-1} + \tau E_n(\tau)P_n F(\bar{Y}^n_{m-1} + O^n_{m-1}). \tag{4.9}
\]

According to \( E_n(\tau)\|_{L_2(H_n)} \leq e^{-\lambda_1 \tau} \) and (2.8)–(2.9), we have

\[
\|\bar{Y}^n_m\|^2 \leq \|E_n(\tau)\|^2_{L_2(H_n)} \left( \|\bar{Y}^n_{m-1}\|^2 + \tau^2 \|P_n F(\bar{Y}^n_{m-1} + O^n_{m-1})\|^2 + 2\tau \langle \bar{Y}^n_{m-1}, P_n F(\bar{Y}^n_{m-1} + O^n_{m-1}) \rangle \right) \leq e^{-2\lambda_1 \tau} \left( \|\bar{Y}^n_{m-1}\|^2 + 2\tau^2 \|P_n F(\bar{Y}^n_{m-1} + O^n_{m-1})\|^2 + 2\tau \langle \bar{Y}^n_{m-1}, P_n F(\bar{Y}^n_{m-1}) \rangle \right) \leq (1 + 2\tau L_F + 2\tau^2 L^2 + \frac{\lambda_1 - L_F}{2\lambda_1} \tau) e^{-2\lambda_1 \tau} \|\bar{Y}^n_{m-1}\|^2 + 2\tau^2 + \frac{\lambda_1 - L_F}{2\lambda_1} \tau e^{-2\lambda_1 \tau} \|P_n F(O^n_{m-1})\|^2, \tag{4.10}
\]

where we used the weighted Young inequality \( ab \leq e^a b + \frac{1}{4e} b^2 \) for all \( a, b \in \mathbb{R} \) with \( \varepsilon = \frac{\lambda_1 - L_F}{4L^2} > 0 \). Observing \( \tau < \tau_0 \leq \frac{\lambda_1 - L_F}{4L^2} \), we have \( 2\tau^2 L^2 \leq \frac{\lambda_1 - L_F}{2} \tau \). This together with

\[
1 + 2\tau L_F + 2\tau^2 L^2 + \frac{\lambda_1 - L_F}{2} \tau \leq 1 + (L_F + \lambda_1) \tau \leq e^{(\lambda_1 + L_F)\tau}, \quad e^{-(\lambda_1 + L_F)\tau} \leq 1, \quad \tau \in (0, \tau_0).
\]
and (4.7) results in
\[
\mathbb{E}[\|\tilde{Y}^n_m\|^2] \leq e^{-(\lambda_1 - L_F)\tau} \mathbb{E}[\|\tilde{Y}^n_m\|^2] + CE^{-\lambda_1/L_F}^\tau \\
\leq e^{-(\lambda_1 - L_F)\tau} \mathbb{E}[\|\tilde{Y}^n_0\|^2] + C e^{-(\lambda_1 - L_F)\tau \tau} \\
\leq \|X_0\|^2 + \frac{C}{\lambda_1 - L_F},
\]
which yields the first inequality of (4.3) because of (4.6) and \(Y^n_m = \tilde{Y}^n_m + O^n_m\). With regard to (4.2), we derive from (4.4) that
\[
Y^n_m = E_n(t_m)Y^n_0 + \int_0^{t_m} E_n(t_m - [s])P_nF(Y^n_{[s]}) \, ds + \int_0^{t_m} E_n(t_m - [s])P_n \, dW^Q(s).
\]
Using Itô’s isometry (3.6), (4.3), (2.7) and \(X_0 \in H^\beta\) leads to
\[
\|(A_n)\gamma Y^n_m\|_{L^2(\Omega, H_{2n})} \leq \|(A_n)\gamma E_n(t_m)Y^n_0\|_{L^2(\Omega, H_{2n})} + \int_0^{t_m} \|(A_n)\gamma E_n(t_m - [s])P_nF(Y^n_{[s]})\|_{L^2(\Omega, H_{2n})} \, ds \\
+ \|E_n(t_m)\|_{\mathcal{L}(H_{2n})} \|\gamma Y^n_0\| \\
\leq \|E_n(t_m)\|_{\mathcal{L}(H_{2n})} \|\gamma Y^n_0\| + \int_0^{t_m} \|(A_n)\gamma E_n(t_m - [s])\|_{\mathcal{L}(H_{2n})} \|P_nF(Y^n_{[s]})\|_{L^2(\Omega, H_{2n})} \, ds \\
+ \left( \int_0^{t_m} \|(A_n)\gamma \|_{\mathcal{L}(H_{2n})} \|E_n(t_m - [s])\|_{\mathcal{L}(H_{2n})} \|P_n\|_{L^2(H_{2n})} \|P_nQ^{1/2}\|_{L^2(H_{2n})} \, ds \right)^{1/2} \\
\leq C + C \left( \int_0^{t_m} (t_m - [s])^{-\frac{3}{2} + \lambda_1/(2\lambda_1)} ds \right) + \left( \int_0^{t_m} \|E_n(t_m - [s])\|_{\mathcal{L}(H_{2n})}^{-2\gamma + \beta - 1 - \lambda_1/(t_m - [s])} ds \right)^{1/2}. 
\]
Observeing \(1 - \frac{3}{2} > 0, -2\gamma + \beta > 0\), we finally use (3.14) to obtain (4.2) and complete the proof.

Furthermore, we can follow the following result.

**Lemma 4.2.** Let Assumptions 2.1, 2.2 hold, let \(\{Y^n_m\}_{m \in \mathbb{N}}\) be given by (4.1) and let \(\tau < \tau_0 \leq \frac{\lambda_1 - L_F}{4L^2}\). For any \(n \in \mathbb{N}\), \(m \in \mathbb{N}\) and arbitrarily small \(\epsilon > 0\) there is a constant \(C > 0\) independent of \(n, m\) such that
\[
\mathbb{E}\left[\|(A_n)^{1/2} Y^n_m\|^2\right] \leq C \tau^{\beta - \epsilon - 1}.
\]

**Proof.** Making use of (4.1), elementary inequality, Hölder’s inequality and Itô’s isometry gives
\[
\mathbb{E}\left[\|(A_n)^{1/2} Y^n_m\|^2\right] \leq 3 \mathbb{E}\left[\|(A_n)^{1/2} E_n(\tau) Y^n_{m-1}\|^2\right] + 3 \mathbb{E}\left[\|(A_n)^{1/2} E_n(\tau) P_n F(Y^n_{m-1})\|^2\right] \\
+ 3 \mathbb{E}\left[\|E_n(\tau) P_n \Delta W^Q_{m-1}\|^2\right] \\
\leq 3 \mathbb{E}\left[\|E_n(\tau)\|_{\mathcal{L}(H_{2n})}^2 \mathbb{E}\left[\|(A_n)^{\beta - \epsilon/2} Y^n_{m-1}\|^2\right]\right] \\
+ 3 \mathbb{E}\left[\|E_n(\tau)\|_{\mathcal{L}(H_{2n})}^2 \mathbb{E}\left[\|P_n F(Y^n_{m-1})\|^2\right]\right] \\
+ 3 \mathbb{E}\left[\|E_n(\tau)\|_{\mathcal{L}(H_{2n})}^2 \mathbb{E}\left[\|P_n Q^{1/2}\|_{L^2(H_{2n})}^2\right]\right] \\
\leq C \tau^{\beta - \epsilon - 1} + C \tau + C \tau^{\beta - 1},
\]
where we also applied (3.6), (4.2)–(4.3) and (3.2) in the last step. The fact that \(\tau \in (0, \tau_0), \epsilon > 0, -(\beta - \epsilon) + 2 > 0\) finally ends the proof.

\[\square\]
4.2 Ergodicity for the space-time full discretization

To prove the ergodicity of \( \{Y^m_n\}_{m \in \mathbb{N}} \), we introduce the following theory of geometric ergodicity of Markov chains, which was first introduced by Mattingly, Stuart and Higham in [26] to prove ergodicity of several discretizations based backward Euler method for SDEs. Then it was applied in [11] to test ergodicity of a modified implicit Euler method for an ergodic one-dimensional damped stochastic nonlinear Schrödinger equation.

**Theorem 4.6.** To prove the ergodicity of \( \{Y^m_n\}_{m \in \mathbb{N}} \), we can prove the following result.

Armed with the above theorem, we can prove the following result.

**Theorem 4.7 (Ergodicity of \( \{Y^m_n\}_{m \in \mathbb{N}} \)).** Let Assumptions 2.1, 2.2 hold and let \( \tau < \tau_0 \leq \frac{\lambda_1 - L_F}{4L^2} \). Then \( \{Y^m_n\}_{m \in \mathbb{N}} \) given by (4.1) is ergodic with a unique invariant measure \( \nu^\tau_t \).

**Proof.** In view of (2.12), we can rewrite (4.1) as

\[
Y^m_n = E_n(\tau)Y^m_{n-1} + \tau E_n(\tau)P_n F(Y^m_{n-1}) + \sum_{i=1}^n \sqrt{q_i} e^{-\lambda_i \tau} \Delta \beta_i^{m-1} e_i,
\]

with the Wiener increments \( \Delta \beta_i^{m-1} := \beta_i(t_m) - \beta_i(t_{m-1}) \). Owing to the independence of \( \{\Delta \beta_i^{m-1}\}_{i=1,...,n; m \in \mathbb{N}} \), it follows from [5, Page xix] that the random variables make an Markov chain. According to Theorem 4.6, it suffices to show \( \{Y^m_n\}_{m \in \mathbb{N}} \) satisfies the minorization condition and the Lyapunov condition.

Let us first show the minorization condition. For any \( a, b \in H_n \), \( \{\Delta \beta_i^{m-1}\}_{i=1,...,n} \) can be uniquely determined to ensure that \( Y^m_{n-1} = a \) and \( Y^m_m = b \) due to \( \{e_i\}_{i=1}^n \) is an orthonormal basis of \( H_n \). This leads to
the irreducibility of \(\{Y^n_m\}_{m\in\mathbb{N}}\). Further, it is obvious that the \(\{\mathcal{F}_m\}_{m\in\mathbb{N}}\)-measurable process \(\{Y^n_m\}_{m\in\mathbb{N}}\) is uniquely defined by a continuous function

\[
Y^n_m = \varphi \left( Y^n_{m-1}, \frac{E_n(\tau) P_n \Delta W^Q_{m-1}}{\sqrt{\tau}} \right).
\]

Since \(\Delta W^Q_{m-1}\) is a Gaussian random variable and thus admits \(C^\infty\) density function, then for some fixed compact set \(C \subset \mathcal{B}(H^n)\) the transition kernel \(P_1(x,A)\) with \(x \in C, A \in \mathcal{B}(H^n) \cap \mathcal{B}(C)\) possesses a density \(p_1(x,y)\), which is jointly continuous in \((x,y) \in C \times C\). Finally, the time-homogeneous property of Markov chain \(\{Y^n_m\}_{m\in\mathbb{N}}\) gives the required densities \(p_m(x,y)\) for all \(m \in \mathbb{N}\).

Now we are ready to prove the Lyapunov condition of \(\{Y^n_m\}_{m\in\mathbb{N}}\). Choosing the Lyapunov function \(V(x) = \|x\|^2 + 1, x \in H_n\), it is easy to verify that \(V\) is essentially quadratic. From (4.1) and the properties of conditional expectation, we have

\[
\mathbb{E}[V(Y^n_{m+1}) | \mathcal{F}_m] = \|E_n(\tau)(Y^n_m + \tau P_n F(Y^n_m))\|^2 + \mathbb{E}[\|E_n(\tau) P_n \Delta W^Q_m\|^2] + 1. \tag{4.15}
\]

Observing \(\|E_n(\tau)\|_{\mathcal{L}(H_n)} \leq e^{-\lambda_1 \tau}\), (2.8)–(2.9) and applying the weighted Young inequality \(ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2\) for all \(a, b \in \mathbb{R}\) with \(\varepsilon = 6 > 0\) enable us to show that

\[
\|E_n(\tau)(Y^n_m + \tau P_n F(Y^n_m))\|^2 \leq e^{-2\lambda_1 \tau} \left( \|Y^n_m\|^2 + \tau^2 \|F(Y^n_m)\|^2 + 2\tau \|Y^n_m, F(Y^n_m)\| \right)
\]
\[
\leq e^{-2\lambda_1 \tau} \left( \|Y^n_m\|^2 + 2\tau^2 \|F(Y^n_m) - F(0)\|^2 + 2\tau^2 \|F(0)\|^2 \right) + 2\tau \|Y^n_m, F(Y^n_m) - F(0)\| + 2\tau \|Y^n_m, F(0)\| \nonumber \tag{4.16}
\]
\[
\leq e^{-2\lambda_1 \tau} \left( \|Y^n_m\|^2 + 8L^2 \tau^2 \|Y^n_m\|^2 + 2\tau^2 \|F(0)\|^2 + 2\tau L_F \|Y^n_m\|^2 + \frac{\|F(0)\|^2}{6L^2} \right)
\]
\[
\leq e^{-2\lambda_1 \tau} \left( 1 + 2\lambda_1 \tau \right) \|Y^n_m\|^2 + e^{-2\lambda_1 \tau} \|F(0)\|^2 \frac{1 + 12\tau^2 L^2}{6L^2},
\]

where we also used the fact \(\tau < \tau_0 \leq \frac{\lambda_1 - LF}{4L^2}\) in the last step. Moreover, employing Itô’s isometry, (3.6) and (3.2) implies

\[
\mathbb{E}[\|E_n(\tau) P_n \Delta W^Q_{m-1}\|^2] = \tau \|E_n(\tau) P_n Q^\frac{1}{2}\|^2_{\mathcal{L}_2(H,H_n)} \leq \tau \|(A_n)^{\frac{1}{2}} E_n(\tau)\|^2_{\mathcal{L}(H_n)} \leq \tau \|(A_n)^{\frac{1}{2}}\|^2_{\mathcal{L}_2(H,H_n)} \leq C \tau^\beta e^{-\lambda_1 \tau}. \tag{4.17}
\]

Inserting (4.16) and (4.17) into (4.15), one can derive

\[
\mathbb{E}[V(Y^n_{m+1}) | \mathcal{F}_m] \leq \alpha_1 \|Y^n_m\|^2 + \alpha_2, \tag{4.18}
\]

where

\[
\alpha_1 := \left( 1 + 2\lambda_1 \tau \right) e^{-2\lambda_1 \tau} \in (0, 1), \tag{4.19}
\]
\[
\alpha_2 := e^{-2\lambda_1 \tau} \|F(0)\|^2 \frac{1 + 12\tau^2 L^2}{6L^2} + C \tau^\beta e^{-\lambda_1 \tau} + 1 \in [0, \infty). \tag{4.20}
\]

Thus we complete the proof. \(\Box\)

### 4.3 Weak temporal approximation error over long time

Armed with our assumption, one can easily check that all conditions of the weak error representation formula introduced in [33, Theorem 2.2] are fulfilled. Therefore, we can apply this formula to carry out an easy weak error analysis by some elementary arguments. To adapt our analysis, the formula is listed below with some non-essential changes.
Theorem 4.8 (Weak error representation formula). Let Assumptions 2.1, 2.2 hold. For $T = M\tau$, $\Phi \in C^2_b(H; \mathbb{R})$ the weak error of the exponential Euler scheme (4.1) for the problem (3.1) has the representation

\[
E[\Phi(X^n(T))] - E[\Phi(Y^n_m)] = \sum_{m=0}^{M-1} \left\{ \int_{t_m}^{t_{m+1}} E[\langle Dv^n(T - t, \tilde{Y}^n(t)), P_n F(\tilde{Y}^n(t)) - E_n(t-t_m)P_n F(Y^m_n) \rangle] \, dt \right. \\
+ \frac{1}{2} \int_{t_m}^{t_{m+1}} E[\text{Tr} \{ D^2v^n(T - t, \tilde{Y}^n(t))((P_n Q^\frac{1}{2}))^* ((P_n Q^\frac{1}{2}))^* \} \right. \\
- (E_n(t-t_m)P_n Q^\frac{1}{2})(E_n(t-t_m)P_n Q^\frac{1}{2})^* )] \, dt \right\}.
\] (4.21)

Here $X^n(T)$ and $Y^m_n$ are determined by (3.8) and (4.1), respectively, and $\tilde{Y}^n(t)$ for $t \in [t_m, t_{m+1}]$ is a continuous extension of $Y^m_n$, defined by

\[
\tilde{Y}^n(t) = E_n(t-t_m)(Y^m_n + P_n F(Y^m_n)(t-t_m)) + P_n (W^Q(t) - W^Q(t_m)),
\] (4.22)

where $E_n(t-t_m)P_n (W^Q(t) - W^Q(t_m)) := \int_{t_m}^{t} E_n(t-t_m)P_n \, dW^Q(s)$.

An approximation result between $\tilde{Y}^n(t)$ and $Y^m_n$ is given by the following lemma.

Lemma 4.9. Let Assumptions 2.1, 2.2 hold, let $\tau < \tau_0 \leq \frac{\lambda - LP}{2}\sigma^2$ and let $\{Y^m_n\}_{m \in \mathbb{N}}$ and $\tilde{Y}^n(t)$ be given by (4.1) and (4.22), respectively. For any $n \in \mathbb{N}$, $m \in \mathbb{N}$, there is a constant $C > 0$ independent of $n, m$ such that

\[
E[\|\tilde{Y}^n(t) - Y^m_n\|^2] \leq C\tau^{\beta - \epsilon}, \quad t \in [t_m, t_{m+1}].
\] (4.23)

Proof. One can easily derive from (4.22) that

\[
\tilde{Y}^n(t) - Y^m_n = (E_n(t-t_m) - I)Y^m_n + \int_{t_m}^{t} E_n(t-t_m)P_n F(Y^m_n) \, ds + \int_{t_m}^{t} E_n(t-t_m)P_n \, dW^Q(s).
\] (4.24)

By elementary inequalities, Hölder’s inequality and Itô’s isometry, it follows that

\[
E[\|\tilde{Y}^n(t) - Y^m_n\|^2] \leq 3E[\|E_n(t-t_m) - I\|Y^m_n\|^2] + 3E\left[\left\|\int_{t_m}^{t} E_n(t-t_m)P_n F(Y^m_n) \, ds\right\|^2\right] \\
+ 3E\left[\left\|\int_{t_m}^{t} E_n(t-t_m)P_n \, dW^Q(s)\right\|^2\right] \leq 3E[\|E_n(t-t_m) - I\|Y^m_n\|^2] + 3\tau^2 E[\|E_n(t-t_m)P_n F(Y^m_n)\|^2] \\
+ 3(t-t_m)\|E_n(t-t_m)P_n Q^\frac{1}{2}\|^2_{L_2(H,H_n)} \\
\leq 3\|E_n(t-t_m) - I\|(-A_n)^{\frac{\beta + \epsilon}{2}}\|^2_{L_2(H,H_n)}E[\|(A_n)^{\frac{\beta + \epsilon}{2}}Y^m_n\|^2] \\
+ 3\tau^2 E[\|E_n(t-t_m)\|^2_{L_2(H,H_n)}E[\|F(Y^m_n)\|^2]] \\
+ 3(t-t_m)\|E_n(t-t_m)(-A_n)^{\frac{1 - \beta}{2}}\|^2_{L_2(H,H_n)}E[\|(A_n)^{\frac{1 - \beta}{2}}P_n Q^\frac{1}{2}\|^2_{L_2(H,H_n)}].
\] (4.25)

One can employ (3.6)–(3.7), (4.2)–(4.3), (3.2) and the stability of the semigroup $\{E_n(t)\}_{t \geq 0}$ to get the desired result (4.23).

The next theorem gives a time-independent weak error.
Theorem 4.10 (Temporal weak error). Let Assumptions 2.1, 2.2 hold, let $\tau < \tau_0 \leq \frac{L}{4L^2}$ and let $\{X^n(t)\}_{t \geq 0}$ and $\{Y^n_m\}_{m \in \mathbb{N}}$ be given by (3.1) and (4.1), respectively. For any $T > 0$, $n \in \mathbb{N}$, $M \in \mathbb{N}$ and $\Phi \in C_b^{b} (\mathcal{H}, \mathbb{R})$ there exists a constant $C > 0$ independent of $T, n, M$ such that for any $T = M\tau$

$$|E[\Phi(X^n(T))] - E[\Phi(Y^n_M)]| \leq C_{\tau}^{\beta - \epsilon}.$$  

(4.26)

Proof. We first use (4.21) to decompose the weak error at time $T = M\tau$ with a telescoping sum

$$E[\Phi(X^n(T))] - E[\Phi(Y^n_M)] = \sum_{m=0}^{M-1} \int_{t_m}^{t_{m+1}} E[\langle Dv^n(T - t, \tilde{Y}^n(t)), P_n F(\tilde{Y}^n(t)) - P_n F(Y^n_m)\rangle] dt$$

$$+ \int_{t_m}^{t_{m+1}} E[\langle Dv^n(T - t, \tilde{Y}^n(t)), (I - E_n(t - t_m)) P_n F(Y^n_m)\rangle] dt$$

$$+ \frac{1}{2} \int_{t_m}^{t_{m+1}} E[\text{Tr} \{D^2 v^n(T - t, \tilde{Y}^n(t))(I - E_n(t - t_m))(P_n Q^{\frac{1}{2}}) (P_n Q^{\frac{1}{2}})^*\}] dt$$

$$+ \frac{1}{2} \int_{t_m}^{t_{m+1}} E[\text{Tr} \{D^2 v^n(T - t, \tilde{Y}^n(t)) E_n(t - t_m)(P_n Q^{\frac{1}{2}}) ((I - E_n(t - t_m))(P_n Q^{\frac{1}{2}})^*\}] dt$$

$$:= \sum_{m=0}^{M-1} J_{1m} + J_{2m} + J_{3m} + J_{4m}.$$ 

Below we will estimate these terms separately. For $J_{1m}$, further decomposition leads to

$$|J_{1m}| \leq \int_{t_m}^{t_{m+1}} E[\langle Dv^n(T - t, \tilde{Y}^n(t)) - Dv^n(T - t, Y^n_m), P_n F(\tilde{Y}^n(t)) - P_n F(Y^n_m)\rangle] dt$$

(4.28)

$$+ \int_{t_m}^{t_{m+1}} E[\langle Dv^n(T - t, Y^n_m), P_n F(\tilde{Y}^n(t)) - P_n F(Y^n_m)\rangle] dt := J_{11m} + J_{12m}.$$ 

Applying Taylor’s formula in Banach space, (3.28) with $\gamma_1 = 0, \gamma_2 = 0$, (2.9) and (4.23) to $J_{11m}$, we get

$$J_{11m} \leq \int_{t_m}^{t_{m+1}} \int_0^1 E[\langle D^2 v^n(T - t, \chi(r))(\tilde{Y}^n(t) - Y^n_m), P_n F(\tilde{Y}^n(t)) - P_n F(Y^n_m)\rangle] dr dt$$

$$\leq C \int_{t_m}^{t_{m+1}} (1 + (T - t)^{-\eta}) e^{-\zeta(T - t)} E[\langle ||\tilde{Y}^n(t) - Y^n_m|| P_n F(\tilde{Y}^n(t)) - P_n F(Y^n_m)\rangle] dt$$

(4.29)

$$\leq C \int_{t_m}^{t_{m+1}} (1 + (T - t)^{-\eta}) e^{-\zeta(T - t)} E[||\tilde{Y}^n(t) - Y^n_m||^2] dt$$

$$\leq C_{\tau}^{\beta - \epsilon} \int_{t_m}^{t_{m+1}} (1 + (T - t)^{-\eta}) e^{-\zeta(T - t)} dt,$$

where $\chi(r) := Y^n_m + r(\tilde{Y}^n(t) - Y^n_m)$ for $r \in [0, 1]$. Using Taylor’s formula in Banach space again further decomposes $J_{12m}$ as follows

$$J_{12m} \leq \int_{t_m}^{t_{m+1}} E[\langle Dv^n(T - t, Y^n_m), P_n F'(Y^n_m)(\tilde{Y}^n(t) - Y^n_m)\rangle] dt$$

$$+ \int_{t_m}^{t_{m+1}} E[\langle Dv^n(T - t, Y^n_m), \int_0^1 P_n F''(Y^n_m + r(\tilde{Y}^n(t) - Y^n_m))$$

$$\parallel \tilde{Y}^n(t) - Y^n_m, \tilde{Y}^n(t) - Y^n_m\parallel (1 - r) dr\rangle] dt := J_{12a}^m + J_{12b}^m.$$ 

(4.30)
By the Cauchy-Schwarz inequality, (3.27), (3.4) and (2.9), we can derive from (4.24) that

\[
J_{12}^{m a} \leq \int_{t_m}^{t_{m+1}} \mathbb{E} \left[ \langle Dv^n(T - t, Y^n_m), P_n F'(Y^n_m)(E_n(t - t_m) - I)Y^n_m \rangle \right] dt \\
+ \int_{t_m}^{t_{m+1}} \mathbb{E} \left[ \langle Dv^n(T - t, Y^n_m), P_n F'(Y^n_m)E_n(t - t_m)P_n F(Y^n_m(t - t_m)) \rangle \right] dt \\
\leq C \int_{t_m}^{t_{m+1}} (1 + (T - t)^{-\frac{\epsilon}{2}}) e^{-\tilde{c}(T-t)} \mathbb{E} \left[ \|(-A_n)^{-\frac{\beta-\epsilon}{2}} P_n F'(Y^n_m)(E_n(t - t_m) - I)Y^n_m\| \right] dt \\
+ C \tau \int_{t_m}^{t_{m+1}} e^{-\tilde{c}(T-t)} \mathbb{E} \left[ \|P_n F'(Y^n_m)E_n(t - t_m)P_n F(Y^n_m)\| \right] dt \\
\leq C \int_{t_m}^{t_{m+1}} (1 + (T - t)^{-\frac{\epsilon}{2}}) e^{-\tilde{c}(T-t)} \mathbb{E} \left[ (1 + \|Y^n_m\|_1) \|(E_n(t - t_m) - I)Y^n_m\|_{-1} \right] dt \\
+ C \tau \int_{t_m}^{t_{m+1}} e^{-\tilde{c}(T-t)} \mathbb{E} \left[ \|E_n(t - t_m)P_n F(Y^n_m)\| \right] dt.
\]

Applying Hölder’s inequality, elementary inequality, the stability of the semigroup \( \{E_n(t)\}_{t \geq 0} \) and (4.3), we can deduce that

\[
J_{12}^{m a} \leq C \int_{t_m}^{t_{m+1}} (1 + (T - t)^{-\frac{\epsilon}{2}}) e^{-\tilde{c}(T-t)} (1 + \mathbb{E} \left[ \|Y^n_m\|_1^2 \right] ) \mathbb{E} \left[ \|(-A_n)^{-\frac{\beta-\epsilon}{2}} Y^n_m\|_2 \right] dt \\
\leq C \tau \beta^{-\epsilon} \int_{t_m}^{t_{m+1}} (1 + (T - t)^{-\frac{\epsilon}{2}}) e^{-\tilde{c}(T-t)} dt \\
\leq C \tau \beta^{-\epsilon} \int_{t_m}^{t_{m+1}} (1 + (T - t)^{-\frac{\epsilon}{2}}) e^{-\tilde{c}(T-t)} dt,
\]

where (3.7), Lemma 4.1 and Lemma 4.2 were employed in the second step. Thanks to (3.27), (3.5) and (4.23), one gets

\[
J_{12}^{m b} \leq C \int_{t_m}^{t_{m+1}} \int_0^1 \mathbb{E} \left[ \|(-A)^{-\eta} P_n F''(Y^n_m + r(\tilde{Y}^n(t) - Y^n_m)) \right] \left( \tilde{Y}^n(t) - Y^n_m \right) (1 + (T - t)^{-\eta}) e^{-\tilde{c}(T-t)} (1 - r) dr dt \\
\leq C \int_{t_m}^{t_{m+1}} \mathbb{E} \left[ \|\tilde{Y}^n(t) - Y^n_m\|_2 \right] (1 + (T - t)^{-\eta}) e^{-\tilde{c}(T-t)} dt \\
\leq C \tau \beta^{-\epsilon} \int_{t_m}^{t_{m+1}} (1 + (T - t)^{-\eta}) e^{-\tilde{c}(T-t)} dt.
\]

Putting (4.32)–(4.33) into (4.30) implies

\[
J_{12}^{m} \leq C \tau \beta^{-\epsilon} \int_{t_m}^{t_{m+1}} (1 + (T - t)^{-\frac{\epsilon}{2}} + (T - t)^{-\eta}) e^{-\tilde{c}(T-t)} dt,
\]

which together with (4.28)–(4.29) leads to

\[
|J_{12}^{m}| \leq C \tau \beta^{-\epsilon} \int_{t_m}^{t_{m+1}} (1 + (T - t)^{-\frac{\epsilon}{2}} + (T - t)^{-\eta}) e^{-\tilde{c}(T-t)} dt. \tag{4.34}
\]
As to $J_2^m$, with the help of (3.27), (4.3) and (3.7), we can conclude that

$$
|J_2^m| \leq \int_{t_m}^{t_{m+1}} E \left[ \| (A_n)^{1-\epsilon} Dv^n(T - t, \tilde{Y}^n(t)) \| \cdot \| (A_n)^{-1} (I - E_n(t - t_m)) \|_{L_{H_n}} \cdot \| P_n F(Y_m^n) \| \right] dt
$$

$$
\leq C \tau^{1-\epsilon} \int_{t_m}^{t_{m+1}} (1 + (T - t)^{-1-\epsilon}) e^{-\bar{c}(T-t)} dt.
$$

Concerning $J_3^m$, we employ (2.2), (2.4) and the self-adjointness of $A_n$ to obtain

$$
|J_3^m| = \frac{1}{2} \int_{t_m}^{t_{m+1}} E \left[ \text{Tr} \left\{ (-A_n)^{1-\beta} D^2 v^n(T - t, \tilde{Y}^n(t))(I - E_n(t - t_m)) \cdot (P_n Q^n)^* \right\} \right] dt
$$

$$
\leq \frac{1}{2} \int_{t_m}^{t_{m+1}} E \left[ \| (A_n)^{1-\beta} D^2 v^n(T - t, \tilde{Y}^n(t)) (A_n)^{1+\beta - \epsilon} \|_{L_{(H_n)}} \cdot \| (A_n)^{-1} (I - E_n(t - t_m)) \| (P_n Q^n)^* \|_{L_{1(H_n)}} \right] dt
$$

$$
\leq \frac{1}{2} \int_{t_m}^{t_{m+1}} E \left[ \| (A_n)^{1-\beta} D^2 v^n(T - t, \tilde{Y}^n(t)) (A_n)^{1+\beta - \epsilon} \|_{L_{(H_n)}} \cdot \| (A_n)^{-1} (I - E_n(t - t_m)) \| (P_n Q^n)^* \|_{L_{1(H_n)}} \right] dt.
$$

By (2.3), (3.28), (3.7) and (3.2), it follows that

$$
|J_3^m| \leq \frac{1}{2} \int_{t_m}^{t_{m+1}} E \left[ \| (A_n)^{1-\beta} D^2 v^n(T - t, \tilde{Y}^n(t)) (A_n)^{1+\beta - \epsilon} \|_{L_{(H_n)}} \cdot \| (A_n)^{-1} (I - E_n(t - t_m)) \| \right] dt
$$

$$
\leq C \tau^{\beta - \epsilon} \int_{t_m}^{t_{m+1}} (1 + (T - t)^{-\eta} + (T - t)^{-1-\epsilon}) e^{-\bar{c}(T-t)} dt.
$$

With regard to $J_4^m$, similarly to $J_3^m$, we can get

$$
|J_4^m| \leq C \tau^{\beta - \epsilon} \int_{t_m}^{t_{m+1}} (1 + (T - t)^{-\eta} + (T - t)^{-1-\epsilon}) e^{-\bar{c}(T-t)} dt.
$$

Inserting (4.34)–(4.35), (4.37)–(4.38) into (4.27) and using (3.14) yields the required conclusion. \qed

### 4.4 Error of invariant measures for the temporal discretization

**Theorem 4.11.** Let Assumptions 2.1, 2.2 hold, let $n \in \mathbb{N}$, $\tau < \tau_0 \leq \frac{\lambda_1 - L_F}{4L^2}$, and let $\nu^n$ and $\nu^n_\tau$ be the corresponding unique invariant measure of $\{X^n(t)\}_{t \geq 0}$ and $\{Y_m^n\}_{m \in \mathbb{N}}$, respectively. For any $\Phi \in C_b^0(H, \mathbb{R})$ there exists a constant $C > 0$ independent of $n$, $\tau$ such that

$$
\left| \int_{H_n} \Phi(y) \nu^n(dy) - \int_{H_n} \Phi(y) \nu^n_\tau(dy) \right| \leq C \tau^{\beta - \epsilon}.
$$

**Proof.** Theorem 4.7 and the definition of ergodicity imply

$$
\lim_{M \to \infty} \frac{1}{M} \sum_{m=0}^{M-1} E[\Phi(Y_m^n)] = \int_{H_n} \Phi(\nu^n(dy), \forall \Phi \in C_b^0(H, \mathbb{R}), \quad (4.40)
$$
which in combination with (3.43) and (4.26) results in
\[
\left| \int_{H_n} \Phi(y) \nu^n(dy) - \int_{H_n} \Phi(y) \nu^\tau(dy) \right|
\]
\[
\leq \lim_{M \to \infty} \frac{1}{T} \sum_{m=0}^{M-1} \int_{t_m}^{t_{m+1}} \left| \mathbb{E}[\Phi(X^n(t))] - \mathbb{E}[\Phi(Y^n_m)] \right| dt
\]
\[
\leq \lim_{M \to \infty} \frac{1}{T} \sum_{m=0}^{M-1} \int_{t_m}^{t_{m+1}} \left| \mathbb{E}[\Phi(X^n(t))] - \mathbb{E}[\Phi(X^n(t_m))] \right| dt + C\tau^{\beta-\epsilon} := K_1 + C\tau^{\beta-\epsilon}.
\]

Now it remains to treat $K_1$. Using (3.25)–(3.26), we can show that for $t \in [t_m, t_{m+1}]$,
\[
\mathbb{E}[\Phi(X^n(t))] - \mathbb{E}[\Phi(X^n(t_m))] = v^n(t, X^n_0) - v^n(t_m, X^n_0) = \int_{t_m}^{t} \frac{\partial v^n(s, X^n_0)}{\partial s} ds
\]
\[
= \int_{t_m}^{t} \langle Dv^n(s, X^n_0), A_n X^n_0 + P_n F(X^n_0) \rangle + \frac{1}{2} \text{Tr} \{ D^2 v^n(s, X^n_0)(P_n Q^n)^2(P_n Q^n)^* \} ds
\]
\[
= -\int_{t_m}^{t} \langle (-A_n)^{-\frac{\beta}{2}} Dv^n(s, X^n_0), (-A_n)^{\frac{\beta}{2}} X^n_0 \rangle ds + \int_{t_m}^{t} \langle Dv^n(s, X^n_0), P_n F(X^n_0) \rangle ds
\]
\[
+ \frac{1}{2} \int_{t_m}^{t} \text{Tr} \{ (-A_n)^{1-\beta} D^2 v^n(s, X^n_0)((-A_n)^{\frac{\beta}{2}} P_n Q^n)^2((-A_n)^{\frac{\beta}{2}} P_n Q^n)^*) \} ds.
\]

By some elementary inequalities, (3.27)–(3.28), (2.9), (3.2) and $X_0 \in H^\beta$, we have
\[
|\mathbb{E}[\Phi(X^n(t))] - \mathbb{E}[\Phi(X^n(t_m))]| \leq \int_{t_m}^{t} \|(-A_n)^{-\frac{\beta}{2}} Dv^n(s, X^n_0)\| \|(-A_n)^{\frac{\beta}{2}} X^n_0\| ds + \int_{t_m}^{t} \|Dv^n(s, X^n_0)\| \|P_n F(X^n_0)\| ds
\]
\[
+ \frac{1}{2} \int_{t_m}^{t} \|(-A_n)^{1-\beta} D^2 v^n(s, X^n_0)\| \mathcal{L}(H_n) \|(-A_n)^{\frac{\beta}{2}} P_n Q^n\|^2 \|L_2(H, H_n)\| ds
\]
\[
\leq C \|(-A_n)^{\frac{\beta}{2}} X^n_0\| \int_{t_m}^{t} (1 + s^{\frac{\beta}{2}-1}) e^{-\tilde{c}s} ds + CL(1 + \|X^n_0\|) \int_{t_m}^{t} e^{-\tilde{c}s} ds
\]
\[
+ C \|(-A_n)^{\frac{\beta}{2}} P_n Q^n\|^2 \|L_2(H, H_n)\| \int_{t_m}^{t} (1 + s^{-\eta} + s^{\beta-1}) e^{-\tilde{c}s} ds
\]
\[
\leq C \int_{t_m}^{t_{m+1}} (1 + s^{-\eta} + s^{\beta-1}) e^{-\tilde{c}s} ds.
\]

With this and (3.14), we can easily get $K_1 = 0$ and hence complete the proof. \hfill \Box

**Remark 4.12.** Bearing Remark 3.6 in mind and specializing Theorem 4.11 to the space-time white noise case with $\beta < \frac{1}{2}$ yields that the convergence rate between $\nu^n$ and $\nu^\tau$ is of order $\frac{1}{2} - \epsilon$ for arbitrarily small $\epsilon > 0$, which coincides with that in [6] for the linear implicit Euler scheme. Further applying this theorem to the trace class noise case with $\beta = 1$ gives an order $1 - \epsilon$ with arbitrarily small $\epsilon > 0$ for the convergence rate between $\nu^n$ and $\nu^\tau$ in space dimension $d = 1$.

As a direct consequence of Theorem 3.5 and Theorem 4.11, we have

**Corollary 4.13.** Let Assumptions 2.1, 2.2 hold, let $\tau < \tau_0 \leq \frac{\lambda_n - L_F}{4L_2}$, $n \in \mathbb{N}$, and let $\nu$ and $\nu^\tau$ be the corresponding unique invariant measure of $\{X(t)\}_{t \geq 0}$ and $\{Y^\tau_m\}_{m \in \mathbb{N}}$, respectively. For any $\Phi \in C^2_b(H, \mathbb{R})$ there exists a constant $C > 0$ independent of $n, \tau$ such that
\[
\left| \int_{H} \Phi(y) \nu(dy) - \int_{H_n} \Phi(y) \nu^\tau(dy) \right| \leq C(\lambda_n^{-\beta + \epsilon} + \tau^{-\beta - \epsilon}).
\]
Table 1: The temporal averages for different initial values

\[
\frac{1}{M+1} \sum_{m=0}^{M} \mathbb{E}[\Phi(Y^n_m)], n = 100, M = 2^6, \tau = 2^{-6}, \Phi(y) = e^{-|y|^2}, y \in \mathbb{R}^n
\]

| \(T\) | \(u_0^0(x) = 0, u_0^2(x) = \sqrt{2} \sin(\pi x), u_0^3(x) = \sum_{i=1}^{\infty} \sin(i\pi x)/i\) | \(u_0^0\) | \(u_0^2\) | \(u_0^3\) | \(u_0^0\) | \(u_0^2\) | \(u_0^3\) |
|-----|-------------------------------------------------|------|------|------|------|------|------|
| 10  | 0.93451 0.93095 0.93187                         | 0.93828 | 0.93471 | 0.93563 | 0.93828 | 0.93471 | 0.93563 |
| 20  | 0.93553 0.93375 0.93421                         | 0.93932 | 0.93753 | 0.93799 | 0.93932 | 0.93753 | 0.93799 |
| 50  | 0.93495 0.93424 0.93442                         | 0.93875 | 0.93803 | 0.93821 | 0.93875 | 0.93803 | 0.93821 |
| 100 | 0.93506 0.93471 0.93480                         | 0.93885 | 0.93850 | 0.93859 | 0.93885 | 0.93850 | 0.93859 |
| 200 | 0.93482 0.93465 0.93469                         | 0.93862 | 0.93844 | 0.93848 | 0.93862 | 0.93844 | 0.93848 |
| 500 | 0.93523 0.93516 0.93518                         | 0.93902 | 0.93895 | 0.93897 | 0.93902 | 0.93895 | 0.93897 |

5 Numerical experiments

In this section, some numerical experiments are performed to illustrate the previous findings. We consider the example SPDE from [33, Example 3.2],

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + 1 + u + \sin(u) + \dot{W}^Q, \quad t > 0, \quad x \in (0, 1), \\
\frac{\partial u}{\partial x} &= \sqrt{2} \sin(\pi x), \quad x \in (0, 1), \\
u(0, x) &= u(0, 0) = u(t, 1) = 0, \quad t > 0.
\end{align*}
\]

In order to fulfill (2.7) and (2.12), we take \(q_i = 1, i \in \mathbb{N}, \beta < \frac{1}{2}\) for the space-time white noise case \((Q = I)\) and \(q_i = i^{-1.005}, i \in \mathbb{N}, \beta = 1\) for the trace class noise case \((\text{Tr}(Q) < \infty)\). Then one can easily show that all conditions in Assumptions 2.1, 2.2 are satisfied in this setting. We also remark all the expectations are approximated by computing averages over 100 samples.

By ergodicity, we know that the temporal averages \(\frac{1}{M+1} \sum_{m=0}^{M} \mathbb{E}[\Phi(Y^n_m)]\) should be a constant for all initial values in the whole space and may vary for different test functions \(\Phi \in C^2_b(H, \mathbb{R})\). These facts are numerically verified by Table 1 with three different initial values \(u_0^0, u_0^2, u_0^3\) and Table 2 with three different test functions \(\Phi_1, \Phi_2, \Phi_3\). Additionally, both the spatial and temporal weak errors listed in Table 3 show that these errors are independent of time \(T\).

Next we test the weak convergence orders with \(u_0(x) = \sqrt{2} \sin(\pi x), x \in (0, 1)\) being the initial value. Since no explicit expression of the exact solution to (5.1) is available, we take \(\tau = 2^{-20}, n = 2^{-t}, i = 1, 2, \ldots, 7\) and \(n = 2^{10}\) as reference for the spatial test, and take \(n = 100, \tau = 2^{-2j}, j = 5, 6, \ldots, 12\) and \(\tau = 2^{-15}\) as reference for the temporal test, respectively. We mention that we choose \(\Phi(y) = \exp(-|y|^2), y \in \mathbb{R}^n\) to be the test function and set the final time \(T = 20\), which is large enough to ensure that the equilibrium is reached based on Tables 1 and 2. From Figure 1, one can observe that, the slopes of the error lines and the reference lines match well, indicating that the convergence order is \(1 - \epsilon\) in space and \(\frac{1}{2} - \epsilon\) in time for the space-time white noise case and \(2 - \epsilon\) in space and \(1 - \epsilon\) in time for the trace class noise case with arbitrarily small \(\epsilon > 0\).

Finally we fix \(n = 100\) and also compare weak errors of the exponential Euler scheme with those of the existing linear implicit Euler scheme in \([6,9]\). From Table 4, we can see that the exponential Euler scheme is always considerably more accurate than the linear implicit Euler scheme.
Table 2: The temporal averages for different test functions

\[ \frac{1}{M+1} \sum_{m=0}^{M} \mathbb{E}[\Phi(Y^m_n)], n = 100, M = \frac{T}{\tau}, \tau = 2^{-6}, u_0(x) = \sqrt{2} \sin(\pi x) \]

\[ \Phi_1(y) = e^{-|y|^2}, \Phi_2(y) = \sin(|y|), \Phi_3(y) = \cos(|y|), y \in \mathbb{R}^n \]

| $T$ | $\Phi_1$ | $\Phi_2$ | $\Phi_3$ |
|-----|----------|----------|----------|
|     | STWN     | TCN      | STWN     | TCN      |
| 10  | 0.93059  | 0.22854  | 0.96248  | 0.93471  |
| 20  | 0.93375  | 0.22412  | 0.96426  | 0.93753  |
| 50  | 0.93424  | 0.22338  | 0.96463  | 0.93803  |
| 100 | 0.93471  | 0.22260  | 0.96494  | 0.93850  |
| 200 | 0.93465  | 0.22269  | 0.96492  | 0.93844  |
| 500 | 0.93516  | 0.22176  | 0.96521  | 0.93859  |

Table 3: The spatial weak errors and the temporal weak errors

\[ T = M \tau, u_0(x) = \sqrt{2} \sin(\pi x), \Phi(y) = e^{-|y|^2}, y \in \mathbb{R}^n \]

\[ \mathbb{E}[\Phi(X(T))] - \mathbb{E}[\Phi(X^n(T))] \]

| $T$ | $n = 50, n_{ref} = 100, \tau = 2^{-5}$ | $n = 100, \tau = 2^{-5}, \tau_{ref} = 2^{-8}$ |
|-----|--------------------------------------|--------------------------------------|
|     | STWN                                 | TCN                                 |
|     | STWN                                 | TCN                                 |
| 10  | 0.00000043250                        | 0.0000004352                        |
| 20  | 0.00000032528                        | 0.00000032601                      |
| 50  | 0.00000030435                        | 0.00000030462                      |
| 100 | 0.00000025270                        | 0.00000025277                      |
| 200 | 0.00000035050                        | 0.00000035119                      |
| 500 | 0.00000029597                        | 0.00000029627                      |

Figure 1: The spatial weak convergence orders (left) and the temporal weak convergence orders (right)
Table 4: The temporal weak errors for exponential Euler method (EEM) and linear implicit Euler method (LIEM) with $n = 100$, $\tau_{ref} = 2^{-15}$ and $T = 20$

$$
\mathbb{E}[\Phi(X^n(T))] - \mathbb{E}[\Phi(Y^n_M)], M = T/\tau, u_0(x) = \sqrt{2}\sin(\pi x), \Phi(y) = e^{-|y|^2}, y \in \mathbb{R}^n
$$

| $\tau$ | space-time white noise | trace class noise |
|-------|------------------------|-------------------|
|       | EEM | LIEM           | EEM | LIEM           |
| $2^{-5}$ | 0.0458270571 | 0.0723812355 | 0.0270082268 | 0.0559207587 |
| $2^{-6}$ | 0.0330721299 | 0.0623557903 | 0.0167672984 | 0.0466522564 |
| $2^{-7}$ | 0.0227228795 | 0.0482238113 | 0.0094952116 | 0.0334507770 |
| $2^{-8}$ | 0.0157090897 | 0.0364990864 | 0.0054211588 | 0.0230548773 |
| $2^{-9}$ | 0.0106864989 | 0.0260774425 | 0.0030290821 | 0.0145068919 |
| $2^{-10}$ | 0.0069976095 | 0.0180278780 | 0.0016330434 | 0.0088346321 |
| $2^{-11}$ | 0.0044973623 | 0.0113633291 | 0.0008598901 | 0.0046737117 |
| $2^{-12}$ | 0.0027247770 | 0.0069201586 | 0.0004274154 | 0.0025819014 |

References

[1] A. Abdulle, G. Vilmart, and K. C. Zygalakis. High order numerical approximation of the invariant measure of ergodic SDEs. *SIAM J. Numer. Anal.*, 52(4):1600–1622, 2014.

[2] A. Andersson, R. Kruse, and S. Larsson. Duality in refined Sobolev-Malliavin spaces and weak approximation of SPDE. *Stochastics and Partial Differential Equations Analysis and Computations*, 4(1): 113–149, 2016.

[3] A. Andersson, S. Larsson. Weak convergence for a spatial approximation of the nonlinear stochastic heat equation. *Math. Comp.*, 85(299): 1335–1358, 2016.

[4] R. Anton, D. Cohen, L. Quer-Sardanyons. A fully discrete approximation of the one-dimensional stochastic heat equation, *arXiv:1711.08340*.

[5] A. A. Borovkov. *Ergodicity and Stability of Stochastic Processes*. Wiley, 1998.

[6] C.-E. Bréhier. Approximation of the invariant measure with an Euler scheme for stochastic PDEs driven by space-time white noise. *Potential Anal.*, 40(1):1–40, 2014.

[7] C.-E. Bréhier, and A. Debussche. Kolmogorov equations and weak order analysis for SPDEs with nonlinear diffusion coefficient, *Journal de Mathmatiques Pures et Appliques*, 2018.

[8] C.-E. Bréhier, M. Hairer, and A. M. Stuart. Weak error estimates for trajectories of SPDEs for Spectral Galerkin discretization. *J. Comp. Math.*, 36: 159–182, 2018.

[9] C.-E. Bréhier and M. Kopec. Approximation of the invariant law of SPDEs: error analysis using a Poisson equation for a full-discretization scheme. *IMA J. Numer. Anal.*, 37:1375–1410, 2017.

[10] C.-E. Bréhier and G. Vilmart. High order integrator for sampling the invariant distribution of a class of parabolic stochastic PDEs with additive space-time noise. *SIAM J. Sci. Comput.*, 38(4):A2283–A2306, 2016.

[11] C. Chen, J. Hong, and X. Wang. Approximation of invariant measure for damped stochastic nonlinear Schrödinger equation via an ergodic numerical scheme. *Potential Anal.*, 46(2):323–367, 2017.

[12] D. Conus, A. Jentzen, and R. Kurniawan Weak convergence rates of spectral Galerkin approximations for SPDEs with nonlinear diffusion coefficients. *arXiv:1408.1108*, Accepted in Ann. Appl. Probab., 2014.
[13] G. Da Prato. *An Introduction to Infinite-dimensional Analysis*. Springer Science & Business Media, 2006.

[14] G. Da Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, UK, 1992.

[15] G. Da Prato and J. Zabczyk. *Ergodicity for Infinite Dimensional Systems*, volume 229. Cambridge University Press, 1996.

[16] A. Debussche. Weak approximation of stochastic partial differential equations: the nonlinear case. *Math. Comp.*, 80(273):89–117, 2011.

[17] Z. Dong, X. Peng, Y. Song, X. Zhang, et al. Strong Feller properties for degenerate SDEs with jumps. *Ann. Inst. H. Poincaré Probab. Statist.*, 52(2):888–897, 2016.

[18] J. Hong, X. Wang, and L. Zhang. Numerical analysis on ergodic limit of approximations for stochastic NLS equation via multi-symplectic scheme. *SIAM J. Numer. Anal.*, 55(1):305–327, 2017.

[19] A. Jentzen and P. E. Kloeden. The numerical approximation of stochastic partial differential equations. *Milan J. Math.*, 77(1):205–244, 2009.

[20] A. Jentzen and R. Kurniawan. Weak convergence rates for Euler-type approximations of semilinear stochastic evolution equations with nonlinear diffusion coefficients, arXiv:1501.03539, 2015.

[21] P. E. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations*. Springer, Berlin, 1992.

[22] P. E. Kloeden, G. J. Lord, A. Neuenkirch, and T. Shardlow. The exponential integrator scheme for stochastic partial differential equations: pathwise error bounds. *J. Comput. Appl. Math.*, 235(5), 1245–1260, 2011.

[23] M. Kovács, S. Larsson, and F. Saedpanah. Finite element approximation of the linear stochastic wave equation with additive noise. *SIAM J. Numer. Anal.*, 48(2):408–427, 2010.

[24] R. Kruse. *Strong and Weak Approximation of Semilinear Stochastic Evolution Equations*. Springer, 2014.

[25] G. J. Lord, and A. Tambue. *Stochastic exponential integrators for the finite element discretization of SPDEs for multiplicative and additive noise, IMA J. Numer. Anal.*, 33: 515-543, 2013.

[26] J. C. Mattingly, A. M. Stuart, and D. J. Higham. Ergodicity for SDEs and approximations: locally Lipschitz vector fields and degenerate noise. *Stochastic Process. Appl.*, 101(2):185–232, 2002.

[27] J. C. Mattingly, A. M. Stuart, and M. V. Tretyakov. Convergence of numerical time-averaging and stationary measures via Poisson equations. *SIAM J. Numer. Anal.*, 48(2):552–577, 2010.

[28] G. N. Milstein and M. V. Tretyakov. Computing ergodic limits for Langevin equations. *Phys. D*, 229(1):81–95, 2007.

[29] C. Prévôt and M. Röckner. *A Concise Course on Stochastic Partial Differential Equations*, volume 1905. Springer, 2007.

[30] D. Talay. Second-order discretization schemes of stochastic differential systems for the computation of the invariant law. *Stochastics*, 29(1):13–36, 1990.

[31] D. Talay. Stochastic Hamiltonian systems: exponential convergence to the invariant measure, and discretization by the implicit Euler scheme. *Markov Process. Related Fields*, 8(2):163–198, 2002.
[32] A. Tambue and J. Ngnotchouye. Weak convergence for a stochastic exponential integrator and finite element discretization of stochastic partial differential equation with multiplicative and additive noise. *Appl. Numer. Math.*, 108 (2016), 57-86.

[33] X. Wang. Weak error estimates of the exponential Euler scheme for semi-linear SPDEs without Malliavin calculus. *Discrete Contin. Dyn. Sys.-Ser. A*, 36(1):481–497, 2016.

[34] X. Wang and S. Gan. Weak convergence analysis of the linear implicit Euler method for semilinear stochastic partial differential equations with additive noise. *J. Math. Anal. Appl.*, 398(1):151–169, 2013.