Multi-body correlations in SU(3) Fermi gases

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1 Introduction

Ultracold atomic gases give us ideal testing grounds for the study of various strongly correlated quantum systems [1,2]. The controllability of physical parameters such as interatomic interactions enables us to use these atomic systems as quantum simulators for other systems, ranging from high-$T_c$ superconductors [3,4,5] to neutron star matter [6,7,8,9,10]. In particular, a three-component Fermi gas is expected to be analogous to quantum chromodynamics (QCD) [11] where quarks with three colors strongly interact with each other. The crossover from a trimer phase to a color superfluid phase [12,13,14] has been theoretically proposed in this system [15,16,17] in analogy with

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the hadron phase and color superconducting phase of QCD. Conventional superfluids have already been realized in two-component Fermi gases of \(^{40}\text{K}\) \cite{18} and \(^{6}\text{Li}\) \cite{19} atoms and have been extensively discussed \cite{12}. In the case of three components, Fermi degeneracy has been achieved experimentally \cite{20,21} and the existence of three-body bound states called the Efimov trimers \cite{22,23,24,25} has been experimentally confirmed \cite{26,27,28}. However, neither the color superfluidity nor Efimov trimer phase have been realized in current experiments yet. The study of these many-body states constitutes a great challenge for understanding strong-coupling effects in both cold atom systems and dense QCD matter.

In this paper, we theoretically investigate two-body and three-body correlations in a symmetric three-component Fermi gas. By using the many-body \(T\)-matrix approximation (TMA) \cite{29,30,31}, which successfully describes the crossover from weak-coupling Bardeen-Cooper-Schrieffer (BCS) Fermi superfluidity to the Bose-Einstein condensation (BEC) of molecules in two-component Fermi gases, we first incorporate effects of superfluid fluctuations associated with two-body correlations. Specifically, we consider a two-channel model that physically describes a narrow resonance with finite negative effective range \cite{16}. We calculate the superfluid phase transition temperature \(T_c\) and critical chemical potential \(\mu_c\) as a function of the effective range \(r_e\) where the scattering length \(a\) diverges. We then investigate effects of the medium on the trimer binding energy \(E_3\) by means of the Skorniakov-Ter-Martirosian (STM) equation \cite{32} with medium corrections, where the STM equation is known as an exact equation to depict Efimov physics in the three-body problem \cite{25}. In the following, we use \(\hbar = k_B = 1\) and the system volume is taken to be unity, for simplicity.

2 Formulation

We start from the two-channel Hamiltonian for three-component symmetric fermions given by

\[
H = \sum_{i=1,2,3} \sum_{\mathbf{p}} \xi_{\mathbf{p},i} \hat{c}_{\mathbf{p},i}^{\dagger} \hat{c}_{\mathbf{p},i} + \sum_{i<j} \sum_{\mathbf{q}} \left( \varepsilon_{\mathbf{q},ij} + \nu - 2\mu \right) \hat{b}_{\mathbf{q},ij}^{\dagger} \hat{b}_{\mathbf{q},ij} \\
+ g \sum_{i<j} \sum_{\mathbf{p},\mathbf{q}} \left( \hat{b}_{\mathbf{q},ij}^{\dagger} \hat{c}_{\mathbf{p},i} \hat{c}_{\mathbf{q},j} + H.c. \right),
\]  

(1)

where \(\xi_{\mathbf{p}} = \mathbf{p}^2/2m - \mu\) and \(\varepsilon_{\mathbf{q},ij} = \mathbf{q}^2/4m\) are the kinetic energies of a Fermi atom with mass \(m\) measured from the chemical potential \(\mu\) and a diatomic molecules, respectively (\(\mathbf{p}\) and \(\mathbf{q}\) are the momenta). \(\hat{c}_{\mathbf{p},i}\) (\(i = 1, 2, 3\)) and \(\hat{b}_{\mathbf{q},ij}\) (\(i < j\)) are the annihilation operators of a Fermi atom with the hyperfine state \(i\) and a diatomic molecule of \(i-j\) pair, respectively. In our model, the threshold energy of a diatomic molecule \(\nu\) and the Feshbach coupling \(g\) can be written in terms of the scattering length \(a\) and effective range \(r_e\) as follows,

\[
\frac{1}{a} = \frac{1}{2} r_e \nu + \frac{2}{\pi} A, \quad r_e = \frac{8\pi}{m^2 g^2}.
\]  

(2)
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\[ \Phi_{ij} = \sum_i G_i^0 \]

\[ \sum_i = g \]

\[ D_{ij} \]

\[ \sum_i = + \]

\[ G_0 \]

\[ G_0 k \]

\[ D_{ik} \]

\[ \Phi_{ij} = \frac{1}{i\nu_n' - \epsilon_B} \]

\[ \xi_p + 2\mu - \Phi_{ij}(q, i\nu_n') \]

\[ \frac{1}{i\nu_n' - \epsilon_B - \Sigma_i(p, i\omega_n)} \]

\[ G_i(p, i\omega_n) = \frac{1}{i\omega_n - \xi_p - \Sigma_i(p, i\omega_n)} \]

\[ \Sigma_i(p, i\omega_n) = g^2 T \sum_{q, i\nu_n'} \left[ D_{ij}(q, i\nu_n') G_j^0(q - p, i\omega_n - i\nu_n') \right. \]

\[ + D_{ik}(q, i\nu_n') G_k^0(q - p, -i\omega_n) \left. \right] (i \neq j) \]

\[ \Phi_{ij}(q, i\nu_n') = -g^2 T \sum_{p, i\omega_n} G_i^0(p + q, i\omega_n + i\nu_n') G_j^0(-p, -i\omega_n) \]

\[ N = T \sum_{i=1,2,3} \sum_{p, i\omega_n} G_i(p, i\omega_n) - 2T \sum_{i<j} \sum_{q, i\nu_n'} D_{ij}(q, i\nu_n') \]

\( \Lambda \) is the ultraviolet momentum cutoff. In this paper, we focus on \( 1/a = 0 \).

We calculate the superfluid phase transition temperature \( T_c \) and chemical potential \( \mu_c \) within the framework of the many-body \( T \)-matrix approximation (TMA). The atomic thermal Green’s function \( G_i(p, i\omega_n) \) with the fermionic Matsubara frequency \( \omega_n = (2n + 1)\pi T \) is given by

\[ G_i(p, i\omega_n) = \frac{1}{i\omega_n - \xi_p - \Sigma_i(p, i\omega_n)} \]

\( \Phi_{ij} \) is the thermal Green’s function of a \( i-j \) diatomic pair (\( \nu_n' = 2n'\pi T \) is the bosonic Matsubara frequency), which involves the self-energy \( \Phi_{ij}(q, i\nu_n') \) (see Fig. 1) given by

\[ \Phi_{ij}(q, i\nu_n') = -g^2 T \sum_{p, i\omega_n} G_i^0(p + q, i\omega_n + i\nu_n') G_j^0(-p, -i\omega_n) \]

We note that \( G_i^0(p, i\omega_n) = 1/(i\omega_n - \xi_p) \) in Eqs. (1) and (3) is the bare atomic Green’s function. \( T_c \) and \( \mu_c \) are determined by solving the particle number equation

\[ N = T \sum_{i=1,2,3} \sum_{p, i\omega_n} G_i(p, i\omega_n) - 2T \sum_{i<j} \sum_{q, i\nu_n'} D_{ij}(q, i\nu_n') \]
where \( N \) is the total atomic number and the Thouless criterion \[31],

\[
|D_{ij}(q = 0, \nu n' = 0)|^{-1} = 0.
\] (8)

After obtaining \( T_c \) and \( \mu_c \), we determine the trimer binding energy \( E_3 \) by solving the Skorniakov-Ter-Martirosian (STM) equation \[32\] in the presence of medium corrections. In our model, the STM equation is given by

\[
\left[ \frac{r_c \kappa(q)}{2} + 4\pi \sum_p \left\{ \frac{F(p, q)}{p^2 + \kappa(q)^2} - \frac{1}{p^2} \right\} \right] \chi(q) \\
= -8\pi \sum_{p'} F(p', q) \chi(p' + q/2) \frac{\chi(p' + q/2)}{p'^2 + \kappa(q)^2},
\] (9)

where \( \kappa(q)^2 = \frac{2}{3} q^2 - E_3 \). The medium corrections are included in the statistical factor \( F(p, q) \). Considering the Pauli-blocking effect on Fermi atoms, we introduce

\[
F(p, q) = [1 - f(\xi_{p+q/2})][1 - f(\xi_{p-q/2})],
\] (10)

where \( f(\xi_p) = 1/ \left[ \exp(\xi^2/2m - \mu_c)/T_c + 1 \right] \) is the Fermi-Dirac distribution function. Eq. (10) is a generalization of Ref. \[33\] to the finite temperature where the step functions are replaced by \( f(\xi_p) \). We note that it can be regarded as a particle-particle (pp) pair contribution above the Fermi sea. In addition, we also calculate \( E_3 \) by using \( F(p, q) \) including the hole-hole (hh) pair contribution below the Fermi sea, given by

\[
F(p, q) = [1 - f(\xi_{p+q/2})][1 - f(\xi_{p-q/2})] - f(\xi_{p+q/2})f(\xi_{p-q/2}).
\] (11)

We note that both factors go to unity in the vacuum limit \( \mu \rightarrow -\infty \) and Eq. (9) reduces to the ordinary STM equation for a three-body system in this limit.

3 Results

Figure 2 shows the effective-range dependence of the superfluid phase transition temperature \( T_c/T_F \) and the critical chemical potential \( \mu_c/\varepsilon_F \) at \( 1/a = 0 \), where \( T_F \) and \( \varepsilon_F \) are the Fermi temperature and Fermi energy, respectively. Both quantities gradually decrease with increasing the absolute value of the effective range. A similar behavior can be seen in a strongly interacting two-component Fermi gas with finite negative effective range \[31\]. In the narrow resonance limit \( (g \rightarrow 0, r_e \rightarrow -\infty) \), since the self-energy corrections disappears in Eqs. (5) and (8), \( \mu_c \) goes to \( \nu/2 \) (= 0). Therefore, in the large-negative-effective-range region, \( T_c \) approaches \( T_c^{\text{NRL}} = 0.133T_F \), which is obtained by solving

\[
N = 3 \sum_p f(\varepsilon_p) + 6 \sum_q b(\varepsilon_q),
\] (12)
where $\varepsilon_p = p^2/2m$ and $b(\varepsilon_q^B) = 1/(e^{\varepsilon_q^B/T} - 1)$ is the Bose-Einstein distribution function. Eq. (12) is obtained from Eq. (7) by taking limits of $\mu \rightarrow 0$ and $g \rightarrow 0$. $T_{c}^{\text{NRL}}$ is close to the BEC temperature of molecules in the strong-coupling limit at zero effective range given by $T_{\text{BEC}} \simeq 0.137T_F$ since the system is dominated by diatomic molecules. We note that the small difference between $T_{c}^{\text{NRL}}$ and $T_{\text{BEC}}$ originates from the first term of Eq. (12) corresponding to the contribution of thermal-excited atoms.

Figure 3 shows the effective-range dependence of the binding energy of an Efimov trimer in medium $E_M^3$ calculated by solving Eq. (9) with $T_c$ and $\mu_c$ shown in Fig. 2. The dashed and solid curves are obtained by using Eqs. (10) and (11) for $F(p,q)$, respectively. In the zero effective-range limit, both curves coincide with the Efimov trimer binding energy in vacuum given by $E_V^3 = -0.0138542/(4mr_e^2)$ since the contribution of the high-energy region in the integral of Eq. (9) is rather important there. If one regards the horizontal axis $-r_e k_F$ as a measure of the particle density $N = k_F^2/(2\pi^2)$ with fixed $r_e$, the limit ($r_e k_F \rightarrow 0$) corresponds to the low-density limit. In this sense, this cold atomic system has a phase structure resembling dense QCD matter where all quarks are confined in hadrons in the low-density regime. With an increasing negative effective range, medium effects suppress the binding of Efimov trimers and finally $E_M^3$ disappears around $r_e k_F \simeq -0.17$. However, this behavior does not necessarily mean the disappearance of the trimer states at this point. There may still be trimer state solutions of the STM equation at a positive energy ($E_3 > 0$), called Cooper triple states [33]. These states can be understood as a generalization of the Cooper problem, where two electrons can form a so-called Cooper pair in the presence of a Fermi surface and an
infinitesimally attractive interaction [34]. In the case of Cooper triples, one also has to consider the Pauli-blocking effect on fermionic trimers, in contrast to Cooper pairs which are bosonic. To understand such a many-body state, a self-consistent treatment of two-body and three-body correlations is necessary, which is left as an interesting future work.

The difference between two curves of $E_3^M$ in Fig. 3 comes from the hole-hole (hh) pair contribution described by the second term of Eq. (11). The appearance of hole-hole pair excitations at the same time as the particle-particle excitations would be natural in the presence of Fermi seas. As shown in Fig. 3 this effect becomes slightly larger with increasing the negative effective range. One can find that the qualitative behavior of negative $E_3^M$ can be captured by considering only the particle-particle (pp) pair contribution.

4 Summary

To summarize, we have theoretically investigated the effects of two-body and three-body correlations in a three-component Fermi gas. By using the many-body $T$-matrix approximation to incorporate effects of two-body pairing fluctuations, we have numerically calculated the superfluid phase transition temperature and critical chemical potential as functions of the negative effective range. Furthermore, we have solved the Skorniaakov-Ter-Martirosian equation in the medium background. The Efimov trimer binding is suppressed with increasing the density or negative effective range by medium corrections associated with the Pauli-blocking effects on Fermi atoms in the intermediate state. This behavior is quite similar to the quark deconfinement, where the
finite density breaks up a hadron into three quarks. We hope that our study contributes to the understanding of this phenomenon in both condensed matter and high-energy physics.

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