Continuum Determination of Light Quark Masses from Quenched Lattice QCD

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Abstract

We compute the strange and the average up/down quark masses in the quenched approximation of lattice QCD, by using the $O(a)$-improved Wilson action and operators and implementing the non-perturbative renormalization. Our computation is performed at four values of the lattice spacing, from which we could extrapolate to the continuum limit. Our final result for the strange quark mass is $m_{\bar{s}}^{\overline{\text{MS}}}(2\text{ GeV}) = (106 \pm 2 \pm 8)\text{ MeV}$. For the average up/down quark mass we obtain $m_{\ell}^{\overline{\text{MS}}}(2\text{ GeV}) = (4.4 \pm 0.1 \pm 0.4)\text{ MeV}$ and the ratio $m_{s}/m_{\ell} = (24.3 \pm 0.2 \pm 0.6)$.

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1 Introduction

In recent years, the determination of quark masses has become one of the main research topics of lattice QCD simulations. An accurate determination of these masses is in fact of great importance for both phenomenological and theoretical studies. The masses of the charm and bottom quarks, for instance, enter the theoretical predictions of beauty hadron decay rates which, in turn, are relevant for the phenomenological analysis of the CKM unitarity triangle and thus of CP violation in the Standard Model. On the more theoretical side, a precise knowledge of quark masses may give insight on the physics of flavour, by revealing relations between masses and mixing angles or specific textures of the quark mass matrices, eventually due to still uncovered flavour symmetries.

The values of quark masses cannot be directly measured in the experiments, since quarks are confined inside the hadrons. On the other hand, being fundamental parameters of the theory, quark masses cannot even be computed on the basis of purely theoretical considerations. Their evaluation is based on the comparison of the theoretical estimate of a physical quantity, which depends on quark masses, and its experimental value. This is typically realized on the lattice by using, as experimental input, the values of pseudoscalar or vector meson masses.

In this paper we present the results of an extensive lattice calculation of the strange and the average up/down quark masses in the quenched approximation. Particular attention has been dedicated to the reduction and control of the systematic uncertainties, particularly in the case of the strange quark mass. Leading $O(a)$ discretization effects (where $a$ is the lattice spacing) have been removed by using the non-perturbatively $O(a)$-improved Wilson action and operators [1]. The systematic uncertainty related to the evaluation of the quark mass renormalization constant in perturbation theory has been significantly reduced, by implementing the non-perturbative renormalization technique of ref. [2] in the RI/MOM scheme. Conversion of the quark masses from the RI/MOM scheme to the most popular $\overline{\text{MS}}$ scheme has been performed by using continuum perturbation theory at the $N^3\text{LO}$ [3]. Finite volume effects have also been studied. In order to estimate residual systematic uncertainties, we have compared the results obtained by using two alternative definitions of the lattice bare quark mass, related to the vector and axial-vector Ward identities respectively. Finally, with respect to previous determinations of light quark masses by our collaboration [4, 5], in this study we have performed a calculation at four different values of the lattice scale, corresponding to an inverse lattice spacing in the range between approximately 2 and 4 GeV. In this way, we have been able to extrapolate our results to the continuum limit. The main relevant source of uncertainty, which is left in our calculation of the strange quark mass, is therefore the quenched approximation.

An additional uncertainty is present in the determination of the average up/down quark mass. Typical values of the lightest quark masses, used in the present and most of current lattice calculations, are approximately of the order of $m_s/2$, where $m_s$ is the strange quark mass. Therefore, a large chiral extrapolation is required to reach the physical values of the up and down quark masses. Chiral perturbation theory may be used as a guidance in this extrapolation, but the inclusion of higher order terms in the chiral
expansion, necessary to increase the accuracy of this determination, requires simulations
with many more (and preferably lighter) quark masses. In the region of masses considered
in this paper, the pseudoscalar meson mass squared shows a good linear dependence on
quark masses, and a linear or quadratic fit has been considered in performing the chiral
extrapolation. The systematic uncertainty introduced by this extrapolation is difficult
to be reliably quantified. It will be assessed only when simulations on larger lattice vol-
umes and smaller values of quark masses become feasible. We stress, however, that this
uncertainty does not affect the determination of the strange quark mass.

We conclude this section by summarizing the main results of this paper. For the
strange quark mass and the average value of the up/down quark masses,

\[ m_{\ell} = \left( m_u + m_d \right)/2, \]

quoted in the \( \overline{\text{MS}} \) scheme at the renormalization scale \( \mu = 2 \text{ GeV} \), we obtain

\[ m_{s}^{\overline{\text{MS}}}(2 \text{ GeV}) = (106 \pm 2 \pm 8) \text{ MeV} \]

and

\[ m_{\ell}^{\overline{\text{MS}}}(2 \text{ GeV}) = (4.4 \pm 0.1 \pm 0.4) \text{ MeV}, \]

in good agreement with the current lattice world averages [6, 7]. For the ratio of the
strange to the average light quark mass we find

\[ \frac{m_s}{m_{\ell}} = 24.3 \pm 0.2 \pm 0.6. \]

This result is in good agreement with the prediction \( m_s/m_{\ell} = 24.4 \pm 1.5 \) based on NLO
chiral perturbation theory [8].

2 Details of the lattice calculation

In determining the values of quark masses we used the standard procedures based on the
vector and axial-vector Ward identities [9].

Vector Ward Identity (VWI): in the renormalized continuum theory the VWI reads

\[ \langle \partial_\mu V_\mu(x)O^\dagger(0) \rangle = (m_1(\mu) - m_2(\mu))\langle S(\mu;x)O^\dagger(0) \rangle, \]

where \( m_{1,2} \) are the quark masses, \( V_\mu = \bar{q}_1 \gamma_\mu q_2 \) is the vector current, and \( S = \bar{q}_1 q_2 \)
is the scalar density. The renormalized quark mass, \( m_q(\mu) = Z_m(\mu a) m_q(a) \), is ob-
tained from the bare mass \( m_q(a) \) which, on the lattice with Wilson fermions, is equal
to \( m_q(a) = (1/2 a)(1/\kappa_q - 1/\kappa_{cr}) \). \( \kappa_q \) is the Wilson hopping parameter and \( \kappa_{cr} \) is its
critical value, corresponding to the chiral limit. The VWI relates the quark mass renor-
malization constant to the one of the scalar density, i.e. \( Z_m = Z_s^{-1} \). The computation of
the quark mass, using the VWI, can be summarized by the following formula:

\[ m_q^{(\text{VWI})}(\mu) = Z_m(\mu a)m_q^{(\text{VWI})}(a) = Z_s^{-1}(\mu a)(1 + b m a m_q)m_q(a). \]
Notice that the parameter \( b_m = -b_S/2 \) provides the elimination of \( \mathcal{O}(am_q) \) effects [1].

**Axial-Vector Ward Identity (AWI):** the renormalized continuum AWI reads

\[
\langle \partial_\mu A_\mu (x) \mathcal{O}^\dagger(0) \rangle = 2m(\mu)\langle P(\mu; x)\mathcal{O}^\dagger(0) \rangle
\]

where \( A_\mu = \bar{q}\gamma_\mu\gamma_5 q \) is the axial current, \( P = \bar{q}\gamma_5 q \) is the pseudoscalar density, \( \mathcal{O} \) is a generic operator and we have considered quark fields with degenerate masses. The quark mass renormalization constant, in this case, is related by the AWI to the renormalization constant of the axial and of the pseudoscalar operators, i.e. \( \bar{Z}_m = Z_A/Z_P \). By choosing \( \mathcal{O} = P \) we have:

\[
m_q^{(AWI)}(\mu a) = \bar{Z}_m(\mu a)m_q^{(AWI)}(a) = \frac{Z_A}{Z_P(\mu a)}\left(1 + (b_A - b_P)am_q\right)\frac{\langle \sum_x \partial_0 A'_0(x)P^4(0) \rangle}{2 \langle \sum_x P(x)P^4(0) \rangle},
\]

where the bare axial current is improved at \( \mathcal{O}(a) \) as \( A'_\mu(x) = A_\mu(x) + a c_A \partial_\mu P(x) \). The coefficient \( (b_A - b_P) \) cancels the terms of \( \mathcal{O}(am_q) \). For the time derivative, we consider the symmetric \( \mathcal{O}(a) \)-improved form, i.e. \( \partial_0 f(t) = (f(t+a) - f(t-a))/2a \). The improvement coefficients \( b_m, b_A - b_P \) and \( c_A \), in eqs. (5) and (7), are only functions of the bare lattice coupling \( g_0^2 \).

Complete information about the lattice calculation performed in this study is provided in table 1. We generated \( \mathcal{O}(1000) \) gauge field configurations in the quenched approximation at four values of the gauge coupling constant, corresponding to an inverse lattice spacing in the range between approximately 2 and 4 GeV. For each value of the lattice spacing, quark propagators have been computed at four light values of the bare quark mass, by using the non-perturbatively \( \mathcal{O}(a) \)-improved Wilson action [1, 10].

In view of the final extrapolation of the lattice results to the continuum limit, an important requirement in this calculation is a precise determination of the lattice spacing which corresponds to the different values of the coupling used in this study. While the absolute value of the physical scale is affected by a rather large uncertainty in the quenched approximation (of the order of 10%), the ratio between two scales can be determined with better accuracy. To this purpose, we use the precise determination based on the study of the static quark anti-quark potential [14], which in the range 5.7 \( \leq \beta \leq 6.92 \) can be expressed in the form

\[
\ln \left( \frac{a^{-1}(\beta)}{a^{-1}(\beta = 6)} \right) = 1.7331 (\beta - 6) - 0.7849 (\beta - 6)^2 + 0.4428 (\beta - 6)^3.
\]

By using this formula, and varying the inverse lattice spacing at the reference point \( \beta = 6.0 \) in the conservative range between 1.9 and 2.1 GeV, we obtain the estimates of the scale given in table 1.
| \( \beta = 6/g_0^2 \) | 6.0 | 6.2 | 6.4 | 6.45 |
|-----------------|-----|-----|-----|-----|
| \( c_{SW} \) \(^{[10]} \) | 1.769 | 1.614 | 1.526 | 1.509 |
| \( L^3 \times T \) | \( 16^3 \times 52 \) | \( 24^3 \times 64 \) | \( 32^3 \times 70 \) | \( 32^3 \times 70 \) |
| \# conf. | 500 | 200 | 150 | 100 |
| \( a^{-1}(\text{GeV}) \) | 2.00(10) | 2.75(14) | 3.63(18) | 3.87(19) |
| \( \kappa_1 \) | 0.1335 | 0.1339 | 0.1347 | 0.1349 |
| \( \kappa_2 \) | 0.1338 | 0.1344 | 0.1349 | 0.1351 |
| \( \kappa_3 \) | 0.1340 | 0.1349 | 0.1351 | 0.1352 |
| \( \kappa_4 \) | 0.1342 | 0.1352 | 0.1353 | 0.1353 |
| \( \kappa_{cr} \) | 0.135175(4) | 0.135785(2) | 0.135734(2) | 0.135680(2) |
| \( Z_A \) \(^{[11]} \) | 0.804(2) | 0.809(2) | 0.824(2) | 0.825(4) |
| \( Z_S^{RI}(\mu = 3 \text{ GeV}) \) \(^{[11]} \) | 0.745(3) | 0.692(3) | 0.668(4) | 0.668(7) |
| \( Z_F^{RI}(\mu = 3 \text{ GeV}) \) \(^{[11]} \) | 0.598(3) | 0.575(4) | 0.576(4) | 0.579(8) |
| \( c_A \) \(^{[11], [12]} \) | -0.038 | -0.038 | -0.025 | -0.023 |
| \( b_m \) \(^{[13]} \) | -0.709 | -0.691 | -0.676 | -0.673 |
| \( (b_A - b_P) \) \(^{[13]} \) | 0.171 | 0.039 | 0.013 | 0.010 |

| \( am_P \): fit for \( t \in \) | \([11 \div 25]\) | \([12 \div 31]\) | \([17 \div 34]\) | \([17 \div 34]\) |
| \( am_V \): fit for \( t \in \) | \([11 \div 23]\) | \([12 \div 28]\) | \([17 \div 28]\) | \([17 \div 30]\) |
| \( (am_q)^{AWI} \): fit for \( t \in \) | \([11 \div 24]\) | \([13 \div 29]\) | \([17 \div 32]\) | \([17 \div 31]\) |

Table 1: Summary of the lattice details and parameters used in this work. We also give the values of the inverse lattice spacing, of the critical hopping parameter and of the renormalization constants and improvement coefficients (with corresponding references). In addition we supply the reader with the fit intervals that have been used for all the correlation functions considered in this work. Note that our time counting is 0, . . . , \((T - 1)\).
In the same table, we also give the results for the relevant renormalization constants, $Z_A$, $Z_S$ and $Z_P$, in the chiral limit, which have been determined by using the non-perturbative renormalization method of ref. [2], in the RI/MOM scheme. The scale dependent constants, $Z_S$ and $Z_P$, have been computed at the scale $\mu = 3$ GeV, which lies in the allowed range $\Lambda_{QCD} < \mu < \pi/a$ for all the values of the coupling considered in this study. For this reason, our non-perturbative results for the quark masses in the RI/MOM scheme will be given at the reference scale $\mu = 3$ GeV. Details of the non-perturbative calculation of the renormalization constants have been presented at the “Lattice 2002” conference and will be discussed in a separate publication [11].

Concerning the values of the improvement coefficients, $c_A$, $b_m$ and $(b_A - b_P)$, we use the non-perturbative determinations of refs. [10, 12, 13], whose results are collected in table 1. Notice that at $\beta = 6.0$ we opted for the value of $c_A$ obtained in [12] (and recently confirmed in [15]), whose absolute value is significantly smaller than the one obtained in ref. [10]. Had we used the value of $c_A$ obtained in ref. [10], we would have found values of the AWI quark masses, at $\beta = 6.0$, smaller by approximately 7%.

In table 2, we show the results for the pseudoscalar and vector meson masses, in lattice units, obtained by fitting the corresponding correlation functions at zero spatial momentum in the time intervals indicated in table 1. For each value of the hopping parameter, we also present in table 2 the corresponding values of the VWI and AWI quark masses, defined in eqs. (5) and (7), renormalized in the RI/MOM scheme at the scale $\mu = 3$ GeV.

In order to get the physical values of quark masses, we follow the usual procedure [4] and fit the pseudoscalar meson masses to the following form

\[
(\text{am}_P)^2 = Q_1 \left( \text{am}_1^{\text{VWI}} + \text{am}_2^{\text{VWI}} \right) + Q_2 \left( \text{am}_1^{\text{VWI}} + \text{am}_2^{\text{VWI}} \right)^2,
\]

and similarly for the AWI quark masses. The subscripts 1 and 2 denote the flavour of the two valence quarks in the meson. Since in this study we only considered mesons consisting of two degenerate quarks, we always have $m_1 = m_2$ in the fits. For the same reason, we did not include a term proportional to $(m_1 - m_2)^2$ in eq. (8). The physical values of the average up/down and of the strange quark masses are then obtained by substituting the experimental pion and kaon masses on the l.h.s. of eq. (9) and the values of the lattice spacing listed in table 1. Notice that we do not distinguish the up from the down quark mass and, by using eq. (9), we can only determine the average of the two, i.e. $m_\ell = (m_u + m_d)/2$. The fit of the pseudoscalar meson masses to eq. (9), at $\beta = 6.2$, and the resulting extrapolation to the physical values, is shown in figure 1, for both the VWI and AWI quark masses. As can be seen from the figure, the effect of including a quadratic term in the chiral extrapolation of quark masses is rather negligible. This is true for all values of the lattice spacing considered in this study.

The results for the strange and the average up/down quark masses, in the RI/MOM scheme, at the renormalization scale $\mu = 3$ GeV, are collected in table 3. Also shown in the table are the values of the ratio $m_s/m_\ell$ (which is a both scheme and scale independent quantity). Note that this quantity can be determined with much better accuracy than
| $\beta$ | $\kappa$ | $am_P$  | $am_V$  | $am_q^{(VWI)}$ | $am_q^{(AWI)}$ |
|-------|---------|--------|--------|---------------|---------------|
| 6.0   | 0.1335  | 0.391(1) | 0.524(4) | 0.0602(3) | 0.0672(4) |
|       | 0.1338  | 0.356(1) | 0.498(6) | 0.0496(2) | 0.0554(3) |
|       | 0.1340  | 0.331(1) | 0.480(7) | 0.0425(2) | 0.0475(3) |
|       | 0.1342  | 0.304(1) | 0.462(9) | 0.0353(2) | 0.0396(2) |
| 6.2   | 0.1339  | 0.357(1) | 0.443(3) | 0.0722(3) | 0.0781(4) |
|       | 0.1344  | 0.303(1) | 0.405(4) | 0.0534(2) | 0.0575(3) |
|       | 0.1349  | 0.243(1) | 0.370(7) | 0.0343(1) | 0.0370(2) |
|       | 0.1352  | 0.200(1) | 0.351(11)| 0.0228(1) | 0.0247(1) |
| 6.4   | 0.1347  | 0.228(1) | 0.306(2) | 0.0416(3) | 0.0440(3) |
|       | 0.1349  | 0.204(2) | 0.291(2) | 0.0336(2) | 0.0355(2) |
|       | 0.1351  | 0.178(2) | 0.277(3) | 0.0256(2) | 0.0271(2) |
|       | 0.1353  | 0.148(2) | 0.266(4) | 0.0176(1) | 0.0186(1) |
| 6.45  | 0.1349  | 0.195(2) | 0.272(4) | 0.0314(3) | 0.0332(4) |
|       | 0.1351  | 0.167(2) | 0.255(5) | 0.0234(2) | 0.0247(3) |
|       | 0.1352  | 0.152(2) | 0.247(6) | 0.0194(2) | 0.0205(2) |
|       | 0.1353  | 0.134(3) | 0.240(8) | 0.0154(1) | 0.0162(2) |

Table 2: Pseudoscalar and vector meson masses together with the corresponding VWI and AWI quark masses, in lattice units. Quark masses are renormalized in the RI/MOM scheme at the scale $\mu = 3$ GeV. Information about the fit intervals, the values of the renormalization constants and of the improvement coefficients can be found in table 3.
Figure 1: Quadratic fits of the pseudoscalar meson mass squared as a function of the renormalized VWI (left) and AWI (right) quark masses, at $\beta = 6.2$. Empty circles represent the lattice data, full diamonds show the physical values of the pion and kaon masses.

the mass $m_\ell$ itself, since statistical and systematic errors largely cancel in the ratio. We emphasize that the results for quark masses presented in table 3 are obtained in a completely non-perturbative way.

3 Conversion to the $\overline{\text{MS}}$ scheme and extrapolation to the continuum limit

We now convert the RI/MOM quark masses, obtained in the previous section, to the popular $\overline{\text{MS}}$ scheme, in which the light quark masses are conventionally expressed at the scale $\mu = 2$ GeV. That allows to compare our results to the results obtained by other lattice groups and to the ones obtained by using QCD sum rules. It is only at this stage of the calculation that we are forced to introduce perturbation theory. This is because the $\overline{\text{MS}}$ scheme, being related to dimensional regularization, is defined in perturbation theory only.

The conversion factor from the RI/MOM to the $\overline{\text{MS}}$ scheme is conveniently calculated by introducing the renormalization group invariant mass, $m_{\text{RGI}}$, defined by dividing out from the renormalized quark masses the perturbative scale dependence

$$m_{\text{RGI}} = \frac{m_{\text{RI}}(\mu)}{c_{\text{RI}}(\mu)} = \frac{m_{\overline{\text{MS}}}(\mu)}{c_{\overline{\text{MS}}}(\mu)}.$$  \hfill (10)
Table 3: Values of the strange and the average up/down quark masses in the RI/MOM scheme at the scale $\mu = 3$ GeV, as obtained from the VWI and the AWI methods. We also present the values of the scheme and scale independent ratio $m_s/m_\ell$.

The mass $m^{\text{RGI}}_q$ is, by definition, both renormalization scale and renormalization scheme independent. The beta function of QCD and the quark mass anomalous dimension, entering the functions $c(\mu)$ in eq. (10), are known to 4-loop accuracy, in both the RI/MOM [3] and the $\overline{\text{MS}}$ schemes [16]-[18]. From these papers, we extract

$$c^{\text{RI}}(\mu) = \left( \frac{\alpha_s(\mu)}{\pi} \right)^{\frac{12}{25}} \left[ 1 + 2.34747 \frac{\alpha_s(\mu)}{\pi} + 12.0599 \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 + 84.4076 \left( \frac{\alpha_s(\mu)}{\pi} \right)^3 \right],$$

and

$$c^{\overline{\text{MS}}}(\mu) = \left( \frac{\alpha_s(\mu)}{\pi} \right)^{\frac{12}{25}} \left[ 1 + 1.01413 \frac{\alpha_s(\mu)}{\pi} + 1.38921 \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 + 1.09054 \left( \frac{\alpha_s(\mu)}{\pi} \right)^3 \right],$$

by using $n_F = 4$ as the number of active flavours in the range of scales between 2 and 3 GeV. Then, by using $\alpha_s(m_Z) = 0.118$, we obtain the conversion factor

$$R^{(4)} = \frac{m^{\overline{\text{MS}}}_q(2 \text{ GeV})}{m^{\text{RGI}}_q(3 \text{ GeV})} = \frac{c^{\overline{\text{MS}}}(2 \text{ GeV})}{c^{\text{RI}}(3 \text{ GeV})} = 0.918.$$  

This result has a N$^3$LO accuracy in continuum perturbation theory. Therefore, we expect the perturbative error in our determination of the $\overline{\text{MS}}$ quark masses to be completely negligible.

We use eq. (13) to convert the results for the RI/MOM quark masses, presented in table 3, to the $\overline{\text{MS}}$ scheme. The resulting values of $m^{\overline{\text{MS}}}_\ell(2 \text{ GeV})$ and $m^{\overline{\text{MS}}}_s(2 \text{ GeV})$, as
Table 4: Values of the strange and the average up/down quark masses in the \( \overline{\text{MS}} \) scheme at the scale \( \mu = 2 \text{ GeV} \).

| \( \beta \) | \( m_s^{\text{VWI}} \) [MeV] | \( m_s^{\text{AWI}} \) [MeV] | \( m_\ell^{\text{VWI}} \) [MeV] | \( m_\ell^{\text{AWI}} \) [MeV] |
|---|---|---|---|---|
| 6.0 | 82(1) | 92(1) | 3.22(5) | 3.61(6) |
| 6.2 | 92(1) | 99(1) | 3.69(5) | 3.98(6) |
| 6.4 | 96(2) | 102(2) | 3.87(12) | 4.09(13) |
| 6.45 | 96(3) | 101(3) | 3.89(21) | 4.11(22) |

The last step of our calculation is the extrapolation to the continuum limit. As discussed in the previous section, we find it convenient for that purpose to fix the relative values of the lattice spacing by using eq. (8), and taking as input the central value \( a^{-1}(\beta = 6) = 2.0 \text{ GeV} \). Since our results for quark masses are free of leading \( O(a) \)-effects, the first term in the extrapolation is of \( O(a^2) \). For this reason, we extrapolate our data to the continuum limit linearly in \( a^2 \), and obtain the results

\[
m_s^{\text{VWI}} \text{[MeV]} = (102 \pm 2) - (2.01 \pm 0.25) a^2,
\]

\[
m_\ell^{\text{VWI}} \text{[MeV]} = (4.20 \pm 0.10) - (0.100 \pm 0.013) a^2,
\]

\[
(m_s/m_\ell)^{\text{VWI}} = (24.25 \pm 0.17) + (140 \pm 25) a^2,
\]

from the VWI, and

\[
m_s^{\text{AWI}} \text{[MeV]} = (106 \pm 2) - (1.39 \pm 0.28) a^2,
\]

\[
m_\ell^{\text{AWI}} \text{[MeV]} = (4.35 \pm 0.11) - (0.076 \pm 0.014) a^2,
\]

\[
(m_s/m_\ell)^{\text{AWI}} = (24.32 \pm 0.17) + (127 \pm 25) a^2,
\]

from the AWI, where the lattice spacing must be expressed in units of fm. The illustration of the extrapolation of the \( \overline{\text{MS}} \) strange quark mass to the continuum limit is shown in fig. 2.
A pleasant feature of our results is that the quark masses extrapolated to the continuum as obtained by using either the AWI or the VWI lead to fully consistent determinations. The $\mathcal{O}(a)$-improved quark masses, with the lattice spacings used in our simulations, are very close to the continuum limit. Although the central values for the quark masses obtained at $\beta = 6.45$ are slightly smaller than the ones obtained at $\beta = 6.4$, they are completely consistent within the statistical errors.

4 Systematic uncertainties and final results

The errors quoted with the continuum determination of quark masses, in eqs. (14) and (15), are statistical only. In this section, we discuss the systematic uncertainties and present our final results.

The main sources of systematic errors, present in our calculation, are discussed below.

- **Additive renormalization of the VWI quark mass:** when using the VWI method, the renormalized quark mass is obtained from the bare one by implementing both a multiplicative and an additive renormalization. The latter is defined by the critical value of the Wilson hopping parameter, $\kappa_{cr}$, which, in this study, has been determined from the
vanishing of the two-point correlation function of the divergence of the axial current. An equivalent possibility to fix $\kappa_{cr}$ is to require the pseudoscalar meson mass squared to vanish in the chiral limit. The values of light quark masses are rather sensitive to the precise choice of $\kappa_{cr}$, and, when using this alternative determination, we obtain the results

$$m_s^{\text{VWI}}[\text{MeV}] = (110 \pm 2) - (1.17 \pm 0.28) a^2,$$

$$m_\ell^{\text{VWI}}[\text{MeV}] = (4.57 \pm 0.11) - (0.052 \pm 0.016) a^2,$$

$$(m_s/m_\ell)^{\text{VWI}} = (23.94 \pm 0.18) + (20 \pm 27) a^2,$$

(16)

to be compared with those given in eq. (14). By combining the two sets of determinations in eqs. (14) and (16), we get the following estimates of the VWI quark masses

$$m_s^{\text{VWI}} = (106 \pm 2 \pm 4) \text{ MeV},$$

$$m_\ell^{\text{VWI}} = (4.38 \pm 0.10 \pm 0.18) \text{ MeV},$$

$$(m_s/m_\ell)^{\text{VWI}} = (24.10 \pm 0.17 \pm 0.15),$$

(17)

where the first error is statistical and the second represents the systematic uncertainty due to the spread of the two determinations of $\kappa_{cr}$.

The VWI results in eq. (17) are in perfect agreement with those obtained in eq. (15) by using the AWI method, the differences being smaller than 1%. The systematic uncertainty, however, is larger in the VWI case. For this reason, we will quote as our final central values of the results obtained from the AWI method, whereas the difference between the two methods will be included in the systematic error.

- **Determination of the lattice spacing**: our estimate of the lattice spacing has been performed by using eq. (8), in which the main source of uncertainty comes from the input value of the lattice scale at the reference point $\beta = 6$. Our choice $a^{-1}(\beta = 6) = 2.0(1)$ GeV covers, in a rather conservative way, determinations of the scale based on different physical quantities, like $f_\pi$, $m_\rho$, $m_{K^*}$, $r_0$, etc., which are not expected to produce the same estimate of the scale in the quenched approximation. In order to evaluate the effect of this uncertainty on the determination of the light quark masses, we have repeated the analysis by using for $a^{-1}(\beta = 6)$ the values 1.9 and 2.1 GeV respectively, and compared the results with those given in eqs. (14) and (15) obtained by using the central value $a^{-1}(\beta = 6) = 2.0$ GeV. In this way, we find that the quark masses vary by approximately 5% (the mass increases as the lattice spacing increases).

- **Renormalization constants**: a reasonable estimate of the systematic uncertainty involved in the non-perturbative RI/MOM calculation of the renormalization constants, $Z_A$, $Z_S$ and $Z_P$, can be obtained by comparing the results with those obtained by using the chiral
Ward identity method \cite{11, 12, 19}. In the latter case, only scale independent quantities can be computed. In particular, the values of the renormalization constant $Z_A$ obtained from the two methods are in perfect agreement within the statistical errors, while a systematic difference of the order of 5\% is observed in the case of the ratio $Z_S/Z_P$. We include this difference in the systematic uncertainty.

\begin{itemize}
\item **Finite volume effects:** the spatial extension of the lattices considered in this study is of the order of 1.6-1.7 fm, which is expected to be large enough for finite volume effects to be well under control. In order to verify this statement, we have performed an independent simulation, at $\beta = 6.0$, on the volume $24^3 \times 64$, which corresponds (in physical units) to a spatial extension of 2.3 fm. The results for the VWI and AWI strange quark masses, in the $\overline{\text{MS}}$ scheme at the scale $\mu = 2$ GeV, which are obtained from the simulation on the larger lattice, are $m_s^{\text{VWI}} = 85(1)$ MeV and $m_s^{\text{AWI}} = 93(3)$ MeV respectively. These results should be compared with the values $m_s^{\text{VWI}} = 82(1)$ MeV and $m_s^{\text{AWI}} = 92(1)$ MeV quoted in table 4. From this comparison, we get an estimate of finite volume effects which is of the order of 2\%, which we also account for in our final systematic error.

\item **Continuum extrapolation:** the extrapolation of quark masses to the continuum limit has been performed by considering only the effect of a linear term in $a^2$. One may wonder, however, whether higher order discretization effects are indeed negligible, particularly for the results obtained on the coarsest lattice. For this reason, we have also performed the continuum extrapolation without including the point at $\beta = 6.0$ in the fit. In this way, we find that the results for the strange and the average up/down quark masses decrease by 1\% and 2\% respectively.

\item **Perturbative matching:** the conversion factor $R$ of eq. (13), which translates the RI/MOM quark masses to the masses in the $\overline{\text{MS}}$ scheme, has been computed by using $\alpha_s(m_Z) = 0.118$ and the number $n_F = 4$ of active flavours in the range of scales between 2 and 3 GeV. By working in the quenched approximation, however, we could have equally computed this factor by using $n_F = 0$ and $\alpha_s$ from $\Lambda_{\text{QCD}}^{n_F=0} \approx 0.250$ GeV. By proceeding in this way, we would have obtained $R^{(0)} = 0.944$, instead of the result $R^{(4)} = 0.918$ given in eq. (13). The difference between the two determinations, which is of the order of 3\%, represents an intrinsic ambiguity in the quenched approximation. We include this difference in the final evaluation of the systematic uncertainty.

From the continuum results shown in eq. (14), and by adding in quadrature the systematic uncertainties discussed above, we finally obtain our best estimates of light quark masses,

\begin{equation}
\begin{align*}
m_s^{\overline{\text{MS}}}(2 \text{ GeV}) &= (106 \pm 2 \pm 8) \text{ MeV} , \\
m_U^{\overline{\text{MS}}}(2 \text{ GeV}) &= (4.4 \pm 0.1 \pm 0.4) \text{ MeV}
\end{align*}
\end{equation}

and

\begin{equation}
m_s/m_U = 24.3 \pm 0.2 \pm 0.6 ,
\end{equation}

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which have been also quoted in the abstract of this paper. Notice that most of statistical
and systematic uncertainties cancel in the ratio $m_s/m_ℓ$.

Our results in eqs. (18) and (19) are in agreement with the recent extensive lattice
QCD calculations of the quark masses performed by using Wilson fermions in the quenched
approximation [20]-[22]. They are also in good agreement with the current lattice world
averages, presented in the reviews [6, 7] and in the 2002 Review of Particle Physics [23].
The main feature of the present study is, in our opinion, the special attention dedicated
to the reduction and control of the systematic uncertainties within the quenched approx-
imation.

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