Nucleon-nucleon interaction

- Shell structure in nuclei and lots more to be explained on the basis of how nucleons interact with each other in free space
- QCD
- Lattice calculations
- Effective field theory
- Exchange of lowest bosonic states
- Phenomenology

- Realistic NN interactions: describe NN scattering data up to pion production threshold plus deuteron properties
- Note: extra energy scale from confinement of nucleons
(Effective) central force

N. Ishii, S. Aoki, T. Hatsuda, Phys. Rev. Lett. 99, 022001 ('07).

The following diagram is contained.

This leads to the one pion exchange at the large spatial separation.

Ishii talk at 2009 Oak Ridge workshop

Both of these are smaller than we expect. This is answered by the quark mass dependence.

repulsive core: 500 - 600 MeV

attractive pocket: about 30 MeV
Quark mass dependence of the central force:

(1) $m_\pi = 380 \text{MeV}$: $N_{\text{conf}} = 2034$
   
   [28 exceptional configurations have been removed]

(2) $m_\pi = 529 \text{MeV}$: $N_{\text{conf}} = 2000$

(3) $m_\pi = 731 \text{MeV}$: $N_{\text{conf}} = 1000$

Strong quark mass dependence is found.

In the light quark mass region,

✓ the repulsive core grows rapidly.

✓ attractive pocket is enhanced mildly.

The lattice QCD calculation at light quark mass region is quite important.

Ishii talk at 2009 Oak Ridge workshop
Two-body interactions and matrix elements

- To determine \( \hat{V} = \frac{1}{2} \sum_{\alpha\beta\gamma\delta} (\alpha\beta|V|\gamma\delta) a_\alpha^+ a_\beta^+ a_\delta a_\gamma \)
- we need a basis and calculate \( (\alpha\beta|V|\gamma\delta) \) for given interaction
- Simplest type: spin-independent & local (also for spinless bosons)

\[
(r_1 r_2|V|r_3 r_4) = (Rr|V|R'r') = \delta(R - R') \langle r|V|r' \rangle = \delta(R - R') \delta(r - r') V(r)
\]

- with

\[
R = \frac{1}{2} (r_1 + r_2)
\]

\[
r = r_1 - r_2
\]
Nucleon-nucleon interaction

• Yukawa 1935

• short-range interaction requires exchange of massive particle

\[ V_Y(r) = V_0 \frac{e^{-\mu r}}{\mu r} \]

• mass of particle \( \mu \hbar c = mc^2 \)

• mesons are the bosonic excitations of the QCD vacuum

• many quantum numbers; most important: pion \( T=1, 0^- \) lowest mass!

• So one encounters also spin and isospin dependence

\[ V_{\text{spin}} = V_\sigma(r) \sigma_1 \cdot \sigma_2 \]
\[ V_{\text{isospin}} = V_\tau(r) \tau_1 \cdot \tau_2 \]
\[ V_{s-i} = V_{\sigma\tau}(r) \sigma_1 \cdot \sigma_2 \tau_1 \cdot \tau_2 \]
Spin and isospin matrix elements

• Pauli spin matrices $\sigma_1 \cdot \sigma_2$

• represent
  $$\frac{4}{\hbar^2} s_1 \cdot s_2$$

• Use
  $$S = s_1 + s_2$$

• Then
  $$s_1 \cdot s_2 = \frac{1}{2} (S^2 - s_1^2 - s_2^2)$$

• So coupled states are required
  $$\langle S' M_S' | \sigma_1 \cdot \sigma_2 | S M_S \rangle = (2S (S + 1) - 3) \delta_{S,S'} \delta_{M_S,M_S'}$$

• Same for isospin
  $$\langle T' M_T' | \tau_1 \cdot \tau_2 | T M_T \rangle = (2T (T + 1) - 3) \delta_{T,T'} \delta_{M_T,M_T'}$$
Realistic NN interaction

- Required for NN scattering

\[
\begin{array}{cccc}
1 & \tau_1 \cdot \tau_2 & \sigma_1 \cdot \sigma_2 & \sigma_1 \cdot \sigma_2 \tau_1 \cdot \tau_2 \\
S_{12} & S_{12} \tau_1 \cdot \tau_2 & L \cdot S & L \cdot S \tau_1 \cdot \tau_2 \\
L^2 & L^2 \tau_1 \cdot \tau_2 & L^2 \sigma_1 \cdot \sigma_2 & L^2 \sigma_1 \cdot \sigma_2 \tau_1 \cdot \tau_2 \\
(L \cdot S)^2 & (L \cdot S)^2 \tau_1 \cdot \tau_2 \\
\end{array}
\]

- plus radial dependence

- Tensor force \( S_{12}(\hat{r}) = 3 (\sigma_1 \cdot \hat{r}) (\sigma_2 \cdot \hat{r}) - \sigma_1 \cdot \sigma_2 \)

- Short-range interaction suggests use of angular momentum basis
- Angular momentum algebra
- Spherical tensor algebra
- Often calculations are done in momentum space
Operator content pion exchange

- Decomposition of static pion exchange

\[
V^\pi(Q, 0) = -\frac{1}{3} \frac{f^2_{\pi NN}}{\mu^2_\pi} \left[ \frac{3\sigma_1 \cdot Q c \sigma_2 \cdot Q c - \sigma_1 \cdot \sigma_2 Q^2 c^2}{\hbar^2 c^2 \mu^2_\pi + Q^2 c^2} \right] \tau_1 \cdot \tau_2 \\
+ \frac{1}{3} \frac{f^2_{\pi NN}}{\mu^2_\pi} \left[ \frac{\sigma_1 \cdot \sigma_2}{\hbar^2 c^2 \mu^2_\pi + Q^2 c^2} \right] \tau_1 \cdot \tau_2 \\
- \frac{1}{3} \frac{f^2_{\pi NN}}{\mu^2_\pi} \sigma_1 \cdot \sigma_2 \tau_1 \cdot \tau_2
\]

- First term: tensor force

- Rewrite \( S_{12}(\hat{Q}) = 3 \sigma_1 \cdot \hat{Q} \sigma_2 \cdot \hat{Q} - \sigma_1 \cdot \sigma_2 = \sqrt{24\pi} \left[ [\sigma_1 \otimes \sigma_2]^2 \otimes Y_2 \right]_0 \)

\[
= \sqrt{24\pi} \sum_\mu (2 \mu 2 - \mu | 0 0) [\sigma_1 \otimes \sigma_2]_\mu^2 Y_{2,-\mu}(\hat{Q})
\]

- Can couple states with different orbital angular momentum but total spin must be 1, also \([\sigma_1 \otimes \sigma_2]_\mu^2 = \sum_{m_1 m_2} (1 m_1 1 m_2 | 2 \mu) (\sigma_1)_{m_1}^1 (\sigma_2)_{m_2}^1\)

- Responsible for quadrupole moment of the deuteron

- Second term: Yukawa

Remaining: “delta-function”
Momentum space

- Transform to total and relative momentum basis

\[(p_1 p_2 | V | p_3 p_4) = (Pp | V | P'p') = \delta_{P,P'} \langle p | V | p' \rangle\]

- or wave vectors

\[\langle k | V | k' \rangle = \frac{1}{V} \int d^3r \exp\{i(k' - k) \cdot r\}V(r)\]

- Use

\[\exp\{iq \cdot r\} = 4\pi \sum_{\ell m} i^{\ell}Y^{*}_{\ell m}(\hat{r})Y_{\ell m}(\hat{q}) j_{\ell}(qr)\]

- to find

\[\langle k | V | k' \rangle = \frac{4\pi}{V} \int dr\ r^2 \ j_0(qr)V(r) \quad \text{with} \quad q = |k - k'|\]

- Yukawa

\[\langle k | V_Y | k' \rangle = \frac{4\pi}{V} \frac{V_0}{\mu} \ \frac{1}{\mu^2 + (k' - k)^2}\]

- Helps for Coulomb

\[\langle k | V_C | k' \rangle = \frac{4\pi}{V} \ \frac{q_1 q_2 e^2}{(k' - k)^2} \quad \text{when} \ k \neq k'\]
Partial wave basis

- Requires matrix elements of the form

\[
\langle kLM_L | V | k' L' M'_L \rangle = \int d\hat{k} \langle LM_L | \hat{k} \rangle \int d\hat{k}' \langle \hat{k}' | L' M'_L \rangle \langle k | V(r) | k' \rangle
\]

- For Yukawa write

\[
\langle k | V_Y(r) | k' \rangle = \frac{4\pi}{V} \frac{V_0}{\mu} \frac{1}{2k k'} \frac{1}{\frac{\mu^2 + k^2 + k'^2}{2k k'}} - \cos \theta_{kk'}
\]

- and use

\[
\frac{1}{\frac{\mu^2 + k^2 + k'^2}{2k k'}} - \cos \theta_{kk'} = \sum_{\ell=0}^{\infty} (2\ell + 1) Q_\ell \left( \frac{\mu^2 + k^2 + k'^2}{2k k'} \right) P_\ell(\cos \theta_{kk'})
\]

\[
= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} 4\pi Q_\ell \left( \frac{\mu^2 + k^2 + k'^2}{2k k'} \right) Y^*_\ell m(\hat{k}) Y_\ell m(\hat{k}')
\]

- with Legendre functions

\[
Q_0(z) = \frac{1}{2} \ln \left( \frac{z + 1}{z - 1} \right)
\]
\[
Q_1(z) = \frac{z}{2} \ln \left( \frac{z + 1}{z - 1} \right) - 1
\]
\[
Q_2(z) = \frac{3z^2 - 1}{4} \ln \left( \frac{z + 1}{z - 1} \right) - \frac{3}{2} z
\]

- yields

\[
\langle kLM_L | V | k' L' M'_L \rangle = \delta_{L,L'} \delta_{M_L,M'_L} \frac{(4\pi)^2 V_0}{V \mu 2k k'} Q_L \left( \frac{\mu^2 + k^2 + k'^2}{2k k'} \right)
\]
Example

- Reid soft-core interaction (1968)
  - solid $^1S_0$
  - no bound state
  - dashed $^3S_1$
  - deuteron
  - ??

Note similarity to atom-atom interaction
Two-particle states and interactions

- Pauli principle has important effect on possible states
- Free particles $\Rightarrow$ plane waves
- Eigenstates of $T = \frac{p^2}{2m}$ notation (isospin)
- Use box normalization
- Nucleons $|p_1 s = \frac{1}{2} m_1 s_1 t = \frac{1}{2} m_1 t_1 \rangle \equiv |p m_s m_t \rangle$

- Use successive basis transformations for two-nucleon states to survey angular momentum restrictions
- Total spin & isospin; CM and relative momentum; orbital angular momentum relative motion; total angular momentum
Antisymmetric two-nucleon states

- Start with
  \[
  |p_1 m_{s_1} m_{t_1}; p_2 m_{s_2} m_{t_2}\rangle = \frac{1}{\sqrt{2}} \left\{ |p_1 m_{s_1} m_{t_1}\rangle |p_2 m_{s_2} m_{t_2}\rangle - |p_2 m_{s_2} m_{t_2}\rangle |p_1 m_{s_1} m_{t_1}\rangle \right\}
  \]

  \[
  = \frac{1}{\sqrt{2}} \sum_{SM_S T M_T} \left\{ (\frac{1}{2} m_{s_1} \frac{1}{2} m_{s_2} |S M_S\rangle (\frac{1}{2} m_{t_1} \frac{1}{2} m_{t_2} |T M_T\rangle |p_1 p_2 S M_S T M_T\rangle
  
  - (\frac{1}{2} m_{s_2} \frac{1}{2} m_{s_1} |S M_S\rangle (\frac{1}{2} m_{t_2} \frac{1}{2} m_{t_1} |T M_T\rangle |p_2 p_1 S M_S T M_T\rangle \right\}
  \]

- then
  \[
  P = p_1 + p_2
  \]

  \[
  p = \frac{1}{2} (p_1 - p_2)
  \]

- and use
  \[
  |p\rangle = \sum_{LM_L} |pLM_L\rangle \langle LM_L|\hat{p}\rangle = \sum_{LM_L} |pLM_L\rangle Y^*_{LM_L}(\hat{p})
  \]

  \[
  |-p\rangle = \sum_{LM_L} |pLM_L\rangle \langle LM_L|\hat{p}\rangle = \sum_{LM_L} |pLM_L\rangle (-1)^L Y^*_{LM_L}(\hat{p})
  \]

  \[
  Y^*_{LM_L}(-\hat{p}) = Y^*_{LM_L}(\pi - \theta_p, \phi_p + \pi) = (-1)^L Y^*_{LM_L}(\hat{p})
  \]

- as well as
  \[
  (\frac{1}{2} m_{s_2} \frac{1}{2} m_{s_1} |S M_S\rangle = (-1)^{\frac{1}{2}+\frac{1}{2}} - S (\frac{1}{2} m_{s_1} \frac{1}{2} m_{s_2} |S M_S\rangle
  \]

  \[
  (\frac{1}{2} m_{t_2} \frac{1}{2} m_{t_1} |T M_T\rangle = (-1)^{\frac{1}{2}+\frac{1}{2}} - T (\frac{1}{2} m_{t_1} \frac{1}{2} m_{t_2} |T M_T\rangle
  \]
Antisymmetry constraints for two nucleons

- Summarize

\[ |p_1 m_{s_1} m_{t_1}; p_2 m_{s_2} m_{t_2} \rangle = \]
\[ \frac{1}{\sqrt{2}} \sum_{S M S T M T L M L} (\frac{1}{2} m_{s_1} \frac{1}{2} m_{s_2} | S M S \rangle (\frac{1}{2} m_{t_1} \frac{1}{2} m_{t_2} | T M T \rangle Y_{L L}^{*} (\hat{p}) \]
\times \left[ 1 - (-1)^{L+S+T} \right] |P p L M L S M S T M T \rangle \]
\[ = \frac{1}{\sqrt{2}} \sum_{S M S T M T L M L J M J} (\frac{1}{2} m_{s_1} \frac{1}{2} m_{s_2} | S M S \rangle (\frac{1}{2} m_{t_1} \frac{1}{2} m_{t_2} | T M T \rangle Y_{L L}^{*} (\hat{p}) \]
\times (L M L S M L | J M J \rangle \left[ 1 - (-1)^{L+S+T} \right] |P p (L S) J M J T M T \rangle \]

- L + S + T must be odd!

- Notation

| L=0 | L=1 |
|---|---|
| $^3 S_1 - ^3 D_1$ | $^1 S_0$ |
| $^1 P_1$ | $^3 P_0$ |
| $^3 D_2$ | $^3 P_1$ |
| ... | $^3 P_2 - ^3 F_2$ |
| | $^1 D_2$ |
Phase shifts 1968...

Dynamic
Static

Nucleon correlations
Phase shifts 1968...

Nucleon correlations