Stochastic Soliton Lattices

by

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Abstract

We introduce a new concept, Stochastic Soliton Lattice, as a random process generated by a finite-gap potential of the Schrödinger operator. We study the basic properties of this stochastic process and consider its KdV evolution.

1 Introduction

The finite-gap solutions of completely integrable systems are almost periodic functions [N], [L], and, therefore, possess their natural stochastic structure determined by intrinsic shift group (compact group with the Haar measure) [PF]. Ergodic properties of finite-gap potentials were first used by Flaschka, Forest, and McLaughlin [FFM] in constructing the theory of multiphase averaging.

The aim of the present work is the consistent introduction of the basic notions of the finite-gap stochastics, namely, we introduce a concept of a Stochastic Soliton Lattice (SSL) and study its basic properties. The stochastic structure of $g$-gap soliton lattice (SL) (this term has been introduced by Dubrovin and Novikov in [DN] for finite-gap potentials) is the uniform distribution on $g$-dimensional phase torus (the real section of the correspondent Jacobian). We also generalize the important notion of the slowly modulated SL to the stochastic case. The slow modulations of SSL are described by the hydrodynamic-type Whitham equations [W], [FFM], [LL], [DN] as well as in the case of usual SL.

Following Johnson and Moser [JM] we define the rotation number for the SSL. It is well known that the rotation number coincides with the ‘density of states’ per unit length $\nu(\lambda)$ which is one of the most important quantities in physics of disordered systems [LGP]. For the SL the rotation number is equal to the quasimomentum [D1] that leads to a considerable effectivization of some general results of the theory of almost periodic functions.

Passage to the stochastic description is especially natural when one considers the wave systems with a big number of degrees of freedom.

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The case of the SSL with \( g >> 1 \) and \( \nu << 1 \) (weakly interacting rare solitons Poisson-
ically distributed on the line) corresponds to the ‘soliton turbulence’, studied by Zakharov in [Z] invoking the kinetic description. The similar object is studied by a different method in the paper of Gurevich, Zybin, and El [GZE] presented in these Proceedings.

We note also that the introduced uniform phase distribution agrees with the Whitham approach to the description of nonlinear dispersive waves [W], [DN] where the principal term of the WKB - decomposition (\( g \)-gap SL) is determined up to an arbitrary phase shift.

Calculation of various averaged values on the realization space of SSL is an important applied aspect of the proposed theory. We present explicit formulas for the one-point probability function and effective expressions for two first moments in the \( g \)-gap SSL. The challenge is to find the correlation function (the second mixed moment).

2 Basic Preliminaries

The soliton lattice (SL) is a function (\( g \)-gap potential of the Schroedinger equation)

\[
 u_g(x; r) = C(r) - 2\partial^2_{xx} \ln \Theta[z(x)|B(r)],
\]

\( x \in \mathbb{R} \), \( r = \{r_1, \ldots, r_{2g+1}\} \), \( r_1 < r_2 < \ldots < r_{2g} < r_{2g+1} \).

Here

\[
 \Theta[z|B(r)] = \sum_m \exp\{\pi i [2(m, z) + (m, Bm)]\},
\]

\( m = \{m_1, \ldots, m_g\} \in \mathbb{Z}^g \), \( z \in \mathbb{C}^m \)
is the Jacobi theta-function of the hyperelliptic Riemann surface of genus \( g \)

\[
 \mu^2 = \prod_{j=1}^{2g+1} (\lambda - r_j) \equiv R_{2g+1}(r, \lambda).
\]

with cuts along the bands.

The Riemann matrix \( B(r) \) and the constant \( C(r) \) are expressed in terms of the basis holomorphic differentials \( \psi_j \) :

\[
 B_{ij} = \oint_{\beta_j} \psi_i, \quad C(r) = \sum_{j=1}^g r_j - 2 \sum_{j=1}^g \oint_{\alpha_j} \lambda \psi_j,
\]

Here

\[
 \psi_j = \sum_{k=0}^{g-1} a_{jk} \frac{\lambda^k}{\sqrt{R_{2g+1}(r, \lambda)}} d\lambda,
\]

and dependence \( a_{jk}(r) \) is given by the normalization

\[
 \oint_{\beta_k} \psi_j = \delta_{jk},
\]
where the $\alpha$-cycles surround the bands clockwise, and the $\beta$-cycles are canonically conjugated to $\alpha$’s.

The imaginary phases $z(x)$ are given by the formula

$$
    z = -2ia_{g-1}x + d,
$$

where $d$ is the initial imaginary phase vector.

Using the substitution

$$
    z = \frac{1}{2\pi}By
$$

we rewrite (1) in the form with real phases $y$ [FFM]

$$
    u_g(x|r) = u_g(y_1(x), \ldots, y_g(x)|r),
$$

$$
    y_j(x) = k_jx + f_j, \quad f_j(\text{mod} 2\pi),
$$

$$
    k = -2\pi iB^{-1}a_{g-1}.
$$

Here $k = k(r) = \{k_1, \ldots, k_g\}$ is the wave number vector, and $f = \{f_1, \ldots, f_g\}$ is the initial (angle) phase vector, $-\pi < f_j \leq \pi$.

Note that in (2)

$$
    u_g(y_1, \ldots, y_j + 2\pi, \ldots, y_g|r) = u_g(y_1, \ldots, y_j, \ldots, y_g|r),
$$

that is $u_g(x|r)$ is $g$-quasiperiodic on $x$ (from here and on we consider only incommensurate $k_j, j = 1, \ldots, g$. As for any almost periodic function we have for $u_g(x|r)$ the Fourier representation

$$
    u_g(x|r) = \sum_j c_je^{i(l_jx+h_j)},
$$

where $c_j, l_j, h_j$ are real, $-\pi < h_j \leq \pi$. (Really, $l_j \in M$, where $M$ is a frequency modulo [JM] and $h_j$ are the integer linear combinations of $f_j(\text{mod} 2\pi)$).

The KdV evolution of (7) is isospectral and is described by the linear motion of the phases on the Jacobian:

$$
    u_g(x, t|r) = u_g(y_1(x, t), \ldots, y_g(x, t)|r),
$$

$$
    y_j(x, t) = k_jx + \omega_jt + f_j,
$$

where

$$
    \omega = \omega(r) = \{\omega_1, \ldots, \omega_g\} = -4\pi iB^{-1}(a_{g-1} \sum_{j=1}^{2g+1} r_j + 2a_{g-2})
$$

is the frequency vector.
We call \( u_g(x|r) \) in (7), (8) the modulated SL if \( r = r(\epsilon x), \epsilon \ll 1 \) (the possibility of \( g = g(\epsilon x) \) is included). The initial value problem for the KdV equation

\[
    u_t - 6uu_x + u_{xxx} = 0, \quad (15)
\]

\[
    u(x,0) = u_g(x|r(\epsilon x)) \quad (16)
\]

has the solution

\[
    u(x,t) = u_g(x,t|\epsilon x, \epsilon t) \quad (17)
\]

where \( u_g(x,t|\epsilon x, \epsilon t) \) is given by (12), (13) and the evolution of modulation parameters \( r(X,T) \) where \((X,T) = (\epsilon x, \epsilon t)\) is given by the Whitham equations [W], [FFM], [LL], [DN]

\[
    \partial_T r_j + V_j(r) \partial_X r_j = 0, \quad j = 1, \ldots, 2g + 1, \quad (18)
\]

\[
    r_j(X,0) = r_j(X) \quad (19)
\]

Here \( g = g(X,T), g(X,0) = g(X) \) and the Whitham equations [W], [FFM], [LL], [DN] must be supplemented with the additional information about evolution of \( g \) [LL], [V1], [DVZ], [D1].

### 3 Stochastic Soliton Lattices: Definitions and Basic Properties

It is well known that any almost periodic function generates the stochastic stationary process (see [JM], [PF]).

**Definition 1.** The stochastic process generated by SL \( u_g(x|r) \) we call *Stochastic Soliton Lattice* (SSL) and denote as \( \nu_g(x|r) \).

The general construction of \( \nu_g(x|r) \) admits a very simple and clear description. The realization set \( \Omega \) of it consists of functions (7), (8) where \( f \in T^g \); \( T^g \) is \( g \)-dimensional torus (\( -\pi, \pi \]^g). The probability measure \( d\mu \) is the uniform (Lebesque) measure on the torus (on Borel sets on it). It corresponds to the description of \( \nu_g(x|r) \) following from (7), (8):

\[
    \nu_g(x|r) = u_g(\ldots k_j x + \phi_j \ldots |r) \quad (20)
\]

where \( \phi_1, \ldots, \phi_g \) are independent random values uniformly distributed on \((-\pi, \pi]\), that is \( \phi = \{\phi_1, \ldots, \phi_g\} \) is uniformly distributed on \( T^g \). As \( k_j \) are incommensurate then \( \nu_g(x|r) \) is an ergodic process [CSF].

As \( \nu_g(x|r) \) is the stationary process then it has the Stone-Kolmogorov spectral decomposition [I]; due to the ergodicity this decomposition has the form (cf. [II])

\[
    \nu_g(x|r) = \sum_j c_j e^{i(l_j x + \theta_j)}, \quad (21)
\]

where \( \theta_j \) are the uniformly distributed on \((-\pi, \pi]\) *noncorrelated* random values [PR].
The well known formula (Bochner - Khintchin) \([I]\) gives us the covariance function \(K(h)\) of the stationary process \(\nu_g(x|r)\):

\[
K(h) \equiv \langle \hat{\nu}_g(x|r) \cdot \hat{\nu}_g(x|r) \rangle = \sum_j |c_j|^2 e^{i\lambda_j x}.
\] (22)

Here \(\hat{\xi}(x) \equiv \xi(x) - \langle \xi \rangle\) is the centered process.

**Theorem 1.** Consider the KdV equation (15) as the equation describing the evolution in the phase space of stationary processes. Let the initial data has the form of the SSL:

\[
u_g(x|r) = u(x,0).
\] (23)

Then the solution of the KdV is

\[
u_g(x|r) = u(x,0),
\] (24)

that is \(u(x,0) = \nu_g(x|r)\) is a stationary point.

**Proof.** The evolution of realizations (7), (8) is described by (12), (13). For any moment \(t\) one can introduce the new ‘initial phase’ \(f_j^* = \omega_j t + f_j\) which is also uniformly distributed on the torus \(T^9\). Therefore, the KdV evolution changes neither realization set \(\Omega\) nor the probability measure.

**Remark.** In the connection with the Theorem 1 what can one say about the closure (in some topology) of the set \(\{\nu_g(x|r), r \in R^g, g \in Z\}\)?

**Definition 2.** We call \(\nu_g(x|r)\) from (20) a modulated SSL if \(r = r(\epsilon x)\).

The modulated SSL (in opposite to the nonmodulated one) does evolve according to the KdV.

**Theorem 2.** Consider the KdV initial value problem in the sense of the Theorem 1 with the same form of the initial data (23) but with \(r = r(\epsilon x)\). Then the solution of it is

\[
u_g(x|r(\epsilon x, \epsilon t))\),
\] (25)

where \(r(X,T)\) is described by (18),(19).

**Remark.** In a more general case the modulation function \(r(X)\) can be random itself (the simplest example is almost periodic \(r(X)\)).

4 Some Mean Values (Moments) for SSL

If \(Q(f)\) is an arbitrary smooth function then the mean value \(Q(\xi)\) of the stochastic process \(\xi\) is

\[
\langle Q(\xi) \rangle = \int_\Omega d\mu Q(f), \quad f \in \Omega.
\]

(We recall that \(\xi(x) \equiv \{\Omega = \{f(x)\}, B, \mu\},\) and \(B\) is some \(\sigma\)-algebra of measurable Borel sets of \(\Omega\).)
For $\nu_g(x|r)$ we have [FFM]

$$
\langle Q(\nu_g(x|r)) \rangle = \frac{1}{(2\pi)^g} \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} d\phi_1 \ldots d\phi_g Q(\nu_g(x|r)) = \lim_{L \to \infty} \frac{1}{L} \int_0^L Q(\nu_g(x|r)).
$$

(26)

Direct calculation using (1), (2) gives surprisingly simple formulas for the mean value [V3] and the mean square of

$$
\tilde{\nu}_g(x|r) = \nu_g(x|r) - C(r).
$$

(27)

Namely,

$$
\langle \tilde{\nu}_g(x|r) \rangle = \frac{1}{4\pi i} (k, Bk), \quad \langle \tilde{\nu}_g^2(x|r) \rangle = -\frac{1}{4\pi i} (\omega, Bk),
$$

(28)

and $k(r), \omega(r)$ are given by [14], [14]. It should be noted that expressions (28) are obtained for the particular choice of the canonical basis of cycles (see [4]). However, namely this normalizatiton is preferrable for many considerations of finite-gap solutions (see for example [FFM], [V2]).

We still do not know the expression for more important value: the second mixed moment $\langle \tilde{\nu}_g(x|r)\tilde{\nu}_g(x+h|r) \rangle$.

5 One - Point Generation Function for the Moments

The function

$$
f(\lambda) = \langle e^{i\lambda u} \rangle
$$

(29)

gives the moments $\langle u \rangle, \langle u^2 \rangle, \ldots$ as the coefficients in expansion of $f(\lambda)$ in powers of $\lambda$. Obviously, the one-point probability function is connected with $f(\lambda)$ by a Fourier transform:

$$
w(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\lambda)e^{-i\lambda u} d\lambda
$$

(30)

To calculate $f^g(\lambda)$ for the SSL we make use of the trace formula for the SL $u = u_g(y)$ [7] [Lev], [FFM]

$$
u_g = r_{2g+1} - 2 \sum_{j=1}^g (\mu_j + \eta_j^2),
$$

(31)

where

$$
\eta_j^2 = -\frac{r_{2j-1} + r_{2j}}{2}
$$

(32)

is the central point of the band and the functions $\mu_j$ obey the Dubrovin’s ODE’s [D2] and each $\mu_j$ lives inside the $j$ -th gap. Then the ensemble integral (26) can be represented as

$$
\langle e^{iu}\lambda \rangle = \frac{1}{(2\pi)^g} \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} e^{i\lambda u_g(y)} dy_1 \ldots dy_g =
$$
The Jacobian $\frac{\partial y}{\partial \mu}$ appears in [FFM] and has the form

$$\frac{1}{\pi^g} \frac{\partial y}{\partial \mu} = \frac{\det \frac{\mu_k^{j-1}}{\sqrt{R(\mu)}}}{\det \left[ \int_{r_2}^{r_2+1} \frac{\mu^{k-1}}{\sqrt{R(\mu)}} d\mu \right]}.$$  \hspace{1cm} (34)

Substitution of (34) into (33) allows to represent the multiple integral in the form of the product

$$f_g(\lambda) = e^{i\lambda_{r_2+1}} \frac{\det \left[ \int_{r_2}^{r_2+1} e^{-2i\lambda(\mu+\eta^2)} \mu^{j-1} \sqrt{R(\mu)} d\mu \right]}{\det \left[ \int_{r_2}^{r_2+1} \mu^{k-1} \sqrt{R(\mu)} d\mu \right]}.$$  \hspace{1cm} (35)

Taking the Fourier transform of (35) we get the probability density $w_g(u)$ [30]. (As a matter of fact, $\int_{-\infty}^{\infty} w_g(u) du = 1$ as $f_g(0) = 1$).

The next step is to calculate the two-point generating function $f_g(\lambda_1, \lambda_2; h)$

$$f_g(\lambda_1, \lambda_2; h) = \langle e^{-i\lambda_1 u(x)} e^{-i\lambda_2 u(x+h)} \rangle,$$  \hspace{1cm} (36)

which gives the covariance function [24] as the term before $\lambda_1 \lambda_2$ in the decomposition of (36).

6 Density of States (Rotation Number) for SSL

Consider the Schrödinger equation with almost periodic potential $q(x)$

$$L\phi = (-\partial_{xx}^2 + q(x))\phi = \lambda \phi, \quad x \in \mathbb{R}.$$  \hspace{1cm} (37)

The potential $q(x)$ has a very important characteristics, the rotation number, which is defined for real $\lambda$ as [JM]

$$\alpha(\lambda) = \lim_{x \to \infty} \frac{1}{x} \arg(\phi(x, \lambda) - i\phi(x, \lambda)).$$  \hspace{1cm} (38)

We will use also the density of states $\rho(\lambda)$ which is connected with the rotation number by simple relation

$$\rho(\lambda) = \frac{1}{\pi} \alpha(\lambda).$$  \hspace{1cm} (39)
and the integral density of states \[LG\], \[PF\],

\[ N(\lambda) \equiv \int_{-\infty}^{\infty} \rho(\lambda)d\lambda. \]  

(40)

If \(q(x)\) is a finite-gap potential, \(q(x) = u_g(x|\mathbf{r})\), then \[DF\]

\[ \alpha(\lambda) = \frac{dp(\lambda)}{d\lambda}, \]  

(41)

where \(dp(\lambda)\) is the quasimomentum \[CR\], \[DN\] normalized by

\[ \oint_{\beta_j} dp(\lambda) = 0, \quad j = 1, \ldots, g. \]  

(42)

As the value of \(\alpha(\lambda)\) is the same for any realization from the set \(\Omega\) from Def.1 then it is the spectral characteristics of the whole SSL \(\nu(x|\mathbf{r})\). Due to the Theorem 1 the rotation number does not change under the KdV evolution.

We introduce also the full integral density of states in the SSL \(\nu\) as

\[ \nu \equiv N(\infty) = \frac{1}{\pi} \int_{-\infty}^{\infty} dp(\lambda) = \frac{1}{2\pi} \sum_{j=1}^{g} \oint_{\alpha_j} dp(\lambda) = \frac{1}{2\pi} \sum_{j=1}^{g} k_j. \]  

(43)

where \(k_j\) are the components of the wave number vector \(\mathbf{K}\). Here we have used the known relationship \[CR\],\[DN\] (recall that the \(\alpha\)-cycles surround the bands in our normalization (see Sec.2))

\[ \oint_{\alpha_j} dp(\lambda) = k_j. \]  

(44)

One can see that the full integral density of states has here the natural meaning of the mean number of waves per unit length.

Remark.

As the SL \(\mathbf{Z}\), \(\mathbf{Z}\) is the quasiperiodic function both in \(x\) and \(t\) we can deduce that an analogous result is valid for the full temporal density of states as well, namely

\[ \nu' = \frac{1}{2\pi} \sum_{j=1}^{g} \omega_j, \]  

(45)

where \(\omega_j\) are the frequencies \(\mathbf{F}\), that implies that the temporal rotation number in the SSL is

\[ \alpha'(\lambda) = \frac{dq(\lambda)}{d\lambda}, \]  

(46)

where \(dq(\lambda)\) is the quasienergy \[CR\],\[DN\] (\( \oint_{\alpha_j} dq(\lambda) = \omega_j \)).
7 Distribution of Random Initial Phases in SSL.

Poisson and Normal Limits

We introduce the linear phases

\[ l_j \equiv \frac{\phi_j}{k_j}, \quad (47) \]

where \( l_j \) \((j = 1, \ldots, g)\) are independent random values uniformly distributed on \((-\frac{\pi}{k_j}, \frac{\pi}{k_j}]\) each.

Introduce also the random value \( \xi_j = \chi_{(0,1)}(l_j) \) which is the number of hitting of \( l_j \) into the fixed interval \((0, 1) \subset \mathbb{R}\) (we suppose that \( \pi/k_j \geq 1 \)). The variable \( \xi_j \) takes two values: 1 and 0 with the probabilities \( p_j(1) = 1/2\pi \) and \( p_j(0) = q_j = 1 - p_j = 1 - k_j/2\pi \). The generating function \( \varphi_j(z) \) of \( \xi_j \) is

\[ \varphi_j(z) = (1 - p_j) + zp_j \quad (48) \]

The sum \( \xi^{(g)} \equiv \sum_{j=1}^{g} \xi_j \) is the number of hitting of all linear phases into \((0, 1)\). As \( \xi_j \) are independent, the generating function for \( \xi^{(g)} \) has the form

\[ \varphi^{(g)}(z) = \prod_{j=1}^{g} \varphi_j(z) = \prod_{j=1}^{g} (1 + (z - 1)p_j) = \prod_{j=1}^{g} (1 + \frac{(z - 1)k_j}{2\pi}). \quad (49) \]

Consider now the important particular case \( g \gg 1 \) There are two subcases to consider.

7.1 Finite full integral density of states

Suppose the following scaling

\[ k_j = O(g^{-1}). \quad (50) \]

Then taking the logarithm of (49) we obtain

\[ \ln \varphi^{(g)}(z) = (z - 1)\frac{1}{2\pi} \sum_{j=1}^{g} k_j + O(g^{-1}). \]

Therefore

\[ \varphi^{(\infty)}(z) = \exp\{(z - 1)\nu\}. \quad (51) \]

It is well known that the right-hand part of (51) is the generating function for the Poisson distribution with the mean value \( \nu \). That means that \( \xi^{(\infty)} \) is a Poissonic random value with the full integral density of states as a mean value.

Remark 1.

The scaling (50) arises in particular in the semiclassical asymptotics for periodic KdV potentials. potentials [V1][WK].

Remark 2.

If \( \nu \ll 1 \) then we arrive at the case of the separate oscillations asymptotically close to solitons Poissonically distributed on the line. The linear phases \( l_j \) in this case give the asymptotic positions of the "solitons". This case corresponds to Zakharov’s ‘soliton turbulence’ [Z].
7.2 Large full integral density of states

Suppose now that the following conditions fulfil
1. \( \sum_{j=1}^{g} k_j = \nu \gg 1, \)
2. \( k_j = o(1). \)
Then the value \( \xi^{(g)} \) is distributed asymptotically as \( g \to \infty \) by a normal distribution. This follows from the Lyapunov Central Limit Theorem in Berry - Essen formulation [F]. We present here the corresponding estimates for the distribution function \( F_g(x) \) for the normed sums

\[
\frac{\sum_{k=1}^{g} \xi_k - \sum_{k=1}^{g} p_k}{\left( \sum_{k=1}^{g} p_k q_k \right)^{1/2}}.
\]

The estimate is

\[
|F_g(x) - \Phi(x)| = O(g^{-1/2}), \tag{52}
\]
where

\[
\Phi(x) = \frac{1}{2\pi} \int_{-\infty}^{x} e^{-y^2/2} dy.
\]

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REFERENCES

[CSF] I.P.Cornfeld, Ya.G.Sinai, S.V.Fomin, Ergodic theory (1980) Nauka, Moscow.
[DVZ] P.Deift, S.Venakides, and X.Zhou, International Math. Research. Journ., No 4 (1997) 285.
[D1] B.A.Dubrovin, Amer.Math.Soc.Transl (2) 179 (1997) 35.
[D2] B.A.Dubrovin, Russian Math. Surveys 36 (1981) 215.
[DN] B.A.Dubrovin and S.P.Novikov, Russian Math. Surveys 44 (1989) 35.
[F] W.Feller Introduction to the Probability Theory
[FFM] H. Flaschka, G. Forest, D.W. McLaughlin, Comm. Pure Appl. Math. 33 (1979) 739.

[GZE] A.V. Gurevich, K.P. Zybin, and G.A. El, Development of stochastic oscillations in one-dimensional dynamical system described by the Korteweg – de Vries equation, these Proceedings.

[I] Ito, Stochastic Processes, (1970) Mir, Moscow.

[JM] R. Johnson and J. Moser, Comm. Math. Phys., 84 (1982) 403.

[L] P.D. Lax, Comm. Pure Appl. Math. 26 (1975) 141.

[Lev] B.M. Levitan, Inverse Sturm-Liouville problems, (1984), Nauka, Moscow.

[LL] P.D. Lax and C.D. Levermore, Comm. Pure Appl. Math. 36 (1983) 253, 571, 809.

[LGP] I.M. Lifshitz, S.A. Gredeskul, A.L. Pastur, Introduction to the disordered systems theory, (1982) Nauka, Moscow.

[N] S.P. Novikov, Func. Anal. Pril., 8 (1974) 54.

[PF] L.A. Pastur, A.L. Figotin, Random and almost periodic self-adjoint operators, (1991) Nauka, Moscow.

[PR] Yu.V. Prokhorov, Yu.A. Rozanov, Probability Theory, Nauka, Moscow, 1970.

[V1] S. Venakides, AMS Transaction 301 (1987) 189.

[V2] S. Venakides, Comm. Pure Appl. Math. 42 (1989) 711.

[V3] S. Venakides, private communication.

[Wk] M.I. Weinstein, J.B. Keller, SIAM Journal Appl. Math. 47, 941.

[W] G.B. Whitham, Proc. Roy. Soc. A283 (1965) 238.

[Z] V.E. Zakharov, Sov. Phys. JETP 60 (1971) 1012 [In Russian].