The number of paperfolding curves in a covering of the plane
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ABSTRACT. Let $C$ be a covering of $\mathbb{R}^2$ by a set of complete folding curves which satisfies the local isomorphism property. We show that $C$ is locally isomorphic to an essentially unique covering generated by a curve associated to an $\infty$-folding sequence indexed by $\mathbb{N}$. We also prove that $C$ consists of 1, 2, 3, 4 or 6 curves. We give examples for each case; the last one is realized if and only if $C$ is generated by a curve associated to the alternating folding sequence or to one of its primitives.

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In the present paper, we use the notations and the results of [2]. A curve consists of segments whose supports are intervals $[(x, y), (x + 1, y)]$ or $[(x, y), (x, y + 1)]$ with $x, y \in \mathbb{Z}$. The length of a bounded curve is the number of its segments.

Throughout the paper, we consider a covering $C$ of $\mathbb{R}^2$ by a set of complete folding curves which satisfies the local isomorphism property. We do not mention explicitly the orientation of the curves since it is not used in our results.

We prove that $C$ necessarily consists of 1, 2, 3, 4 or 6 curves, and that all the possibilities are realized. We also consider the following question: For which integers $n$ does there exist a covering $D$ of $\mathbb{R}^2$ by $n$ complete folding curves which is locally isomorphic to $C$?

The theorem and the lemma below will be used in the proofs of the other results:

**Theorem 1.** $C$ is locally isomorphic to a covering generated by a curve associated to an $\infty$-folding sequence.

**Proof.** For $n \in \mathbb{N}$ and $(u, v) \in E_{n}(C)$, we denote by $C_{n}(u, v)$ the set of curves obtained from $C$ by keeping only the segments with supports in $[u - 2^n, u + 2^n] \times [v - 2^n, v + 2^n]$. For $m < n$, $C_{m}(u, v)$ is the restriction of $C_{n}(u, v)$ to $[u - 2^m, u + 2^m] \times [v - 2^m, v + 2^m]$.

By König’s Lemma, there exists a sequence $(X_{n})_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} E_{n}(C)$ such that, for $m < n$, the translation $X_{m} \to X_{n}$ induces an embedding of $C_{m}(X_{m})$
in $C_n(X_n)$. As $C$ satisfies the local isomorphism property, the inductive limit $D$ of the coverings $C_n(X_n)$ relative to these embeddings is a covering of $\mathbb{R}^2$ by a set of complete folding curves, $D$ satisfies the local isomorphism property, $D$ is locally isomorphic to $C$ and the image $X$ of the elements $X_n$ in $D$ belongs to $E_\infty(D)$. Each of the two halves of curves of $D$ which start at $X$ is associated to an $\infty$-folding sequence. □

**Remark.** According to [2, Theorem 3.10], the covering $D$ given by Theorem 1 is essentially unique: two such coverings only differ by a translation or/and a change in the connections at the $E_\infty$ point.

We denote by $\mathbb{R}^2 - C$ the exterior of the union of the curves of $C$. For each $A \in \mathbb{Z}^2$, the square $(A, A + (1, 0), A + (1, 1), A + (0, 1))$ is essentially contained in one of the connected components of $\mathbb{R}^2 - C$. On the other hand, each of its 4 vertices can belong to that component or to another one.

Now let $S, T$ be two such squares with exactly 1 common vertex $X$. We say that $S$ and $T$ are connected if $X$ and their centers belong to the same connected component of $\mathbb{R}^2 - C$.

For each $X \in \mathbb{Z}^2$, we write $P(X)$ if $X \in E_2(C)$ and if each square with vertex $X$ is connected to 2 squares without vertex $X$.

**Lemma 2.** There exists $A \in \mathbb{Z}^2$ such that \{ $X \in \mathbb{Z}^2$ $|$ $P(X)$ $\} = A + \mathbb{Z}(2,-2) + \mathbb{Z}(2,2)$.

**Proof.** For each $X \in E_2(C)$, the 4 nonoriented subcurves of length 4 with endpoint $X$ of the curves of $C$ are all obtained from one of them by successive rotations of center $X$ and angle $\pi/2$. Consequently, for each of them, $X$ satisfies $P$ if and only if the second and the third segment starting from $X$ are obtained from the first one by turning left then right, or right then left.

It follows that each $X \in E_2(C)$ satisfies $P$ if and only if $X + (2,0)$ (resp. $X + (0,2)$) does not satisfy $P$. □

**Notation.** We denote by $O$ the point $(0,0) \in \mathbb{R}^2$.

**Theorem 3.** One of the two following properties is true:
1) $C$ consists of 1, 2, 3 or 4 curves;
2) $C$ consists of 6 curves and $C$ is generated by a curve associated to the alternating folding sequence or to one of its primitives.

**Proof.** By [2, Th. 3.15], if $E_\infty(C) \neq \emptyset$, then $C$ consists of 2 curves or the property 2) above is true. It remains to be proved that, if $E_\infty(C) = \emptyset$, then $C$ consists of at most 4 curves.

For each $X \in \mathbb{R}^2$ and each curve $D$, we denote by $\delta(X,D)$ the minimum distance between $X$ and a vertex of $D$. In the proof of [2, Th. 3.12], we saw
that there exist $k \in \mathbb{N}$ and $X \in \mathbb{R}^2$ such that $\delta(X, D) < 1.16$ for each $D$ in the $k$-th derivative $C^{(k)}$ of $C$. Moreover, $C$ and $C^{(k)}$ have the same number of curves and $E_{\infty}(C) = \emptyset$ implies $E_{\infty}(C^{(k)}) = \emptyset$. Consequently, we can replace $C$ with $C^{(k)}$, and therefore suppose for the remainder of the proof that there exists $(x, y) \in \mathbb{R}^2$ such that $\delta((x, y), C) < 1.16$ for each $C \in \mathcal{C}$.

By Lemma 2, for $C$, there exists $A \in \mathbb{Z}^2$ such that $\{B \in \mathbb{Z}^2 \mid P(B)\} = A + \mathbb{Z}(2, -2) + \mathbb{Z}(2, 2)$, and therefore $(c, d) \in \mathbb{Z}^2$ such that $P(c, d)$ and $|x - c| + |y - d| \leq 2$.

For each $n \in \mathbb{N}$, we consider the images $(c_n, d_n)$ and $(x_n, y_n)$ of $(c, d)$ and $(x, y)$ in $C^{(n)}$. We have $|x_n - c_n| + |y_n - d_n| \leq 2$. In the proof of [2, Th. 3.12], we saw that $\delta((x, y), C) < 1.16$ for each $C \in \mathcal{C}$ implies $\delta((x_n, y_n), D) < 1.16$ for each $D \in C^{(n)}$.

As $E_{\infty}(C) = \emptyset$, there exists a maximal integer $k$ such that $(c_k, d_k)$ satisfies $P$ for $C^{(k)}$. We can assume $(c_k, d_k) = (c_{k+1}, d_{k+1}) = 0$. Replacing $C$ with $C^{(k)}$ if necessary, we can also assume $k = 0$. Then we have two cases:

1) $O \in F_2(C)$;
2) $O \in E_3(C)$ and, in the derived covering $C'$, each square with vertex $O$ is connected to exactly one square without vertex $O$.

Figures 1A and 1B represent these two cases. Whatever the case, there are 2 possible dispositions for the subcurves of length 4 with endpoint $O$ of the curves of $C$. We only consider one of them since the other one is equivalent modulo a symmetry. Similarly, we only consider one of the 2 possible choices for the connections in $O$. We note that the ball $B((x, y), 1.16)$ is contained in the interior of the square $(Z_1, Z_2, Z_3, Z_4)$ because $(x, y)$ belongs to the square $(X_1, X_2, X_3, X_4)$.

In Figure 1A, the connexions in $Y_1, Y_3$ are imposed by the connexions in $O$, since the property $O \in F_2(C)$ implies $X_1, X_2, X_3, X_4 \in E_3(C)$. Because of the existence of connexions in $X_1, X_2, X_3, X_4$, all the subcurves represented are contained in at most 4 curves. As no other curve of $C$ can reach the vertices in $B((x, y), 1.16)$, $C$ contains at most 4 curves.

In Figure 1B, the connexions in $X_1, X_2, X_3, X_4$ are imposed since, in $C'$, each square with vertex $O$ is connected to only one square without vertex $O$. Consequently, $C$ contains at most 2 curves with segments in the interior of the square $(Y_1, Y_2, Y_3, Y_4)$. As at most 2 other curves of $C$ can reach the vertices in $B((x, y), 1.16)$, it follows that $C$ contains at most 4 curves. ■
By [2, Th. 3.2, Cor. 3.6 and Th. 3.7], \( \mathcal{C} \) is locally isomorphic to a covering of \( \mathbb{R}^2 \) by 1 curve. We also have:

**Proposition 4.** \( \mathcal{C} \) is locally isomorphic to a covering of \( \mathbb{R}^2 \) by 2 curves.

**Proof.** According to Theorem 1, it suffices to prove Proposition 4 when \( \mathcal{C} \) is generated by a curve associated to an \( \infty \)-folding sequence \( S \). In that case, by [2, Th. 3.15], \( \mathcal{C} \) itself consists of 2 curves except if \( S \) is the alternating folding sequence or one of its primitives. So we can suppose for the remainder of the proof that there exists \( k \in \mathbb{N} \) such that \( S(k) \) is the alternating folding sequence \( T \). Then \( \mathcal{C}(k) \) is generated by a curve associated to \( T \).

If there exists a covering \( \mathcal{D} \) of \( \mathbb{R}^2 \) by 2 curves which is locally isomorphic to \( \mathcal{C}(k) \), then there exists a \( k \)-th primitive \( \mathcal{E} \) of \( \mathcal{D} \) which is locally isomorphic to \( \mathcal{C} \), and \( \mathcal{E} \) also consists of 2 curves. So we can also suppose \( k = 0 \). Then \( \mathcal{C} \) is the covering of \( \mathbb{R}^2 \) shown in [2, Fig. 8].

For each \( n \in \mathbb{N}^* \) and each \( r \in \mathbb{Z} \), two bounded subcurves of distinct curves of \( \mathcal{C} \) form a covering \( \mathcal{C}_{n,r} \) of the triangle

\[
T_{n,r} = ((0, (2r)2^n), (2^n, (2r + 1)2^n), (-2^n, (2r + 1)2^n)),
\]

in the sense given in the proof of [2, Example 3.8]. For each \( s \in \mathbb{Z} \), the translation \( X \to X + (0, s.2^{n+1}) \) induces an isomorphism from \( \mathcal{C}_{n,r} \) to \( \mathcal{C}_{n,r+s} \).

Now, for each \( n \in \mathbb{N}^* \), we consider the embedding \( \pi_n : \mathcal{C}_{2n,0} \to \mathcal{C}_{2n,1} \subseteq \mathcal{C}_{2n+2,0} \) induced by the translation \( \tau_n : X \to X + (0, 2^{n+1}) \). We observe that

\[
\bigcup_{n \in \mathbb{N}} \tau_n^{-1}\big((\cdots(\tau_1^{-1}(T_{2n+2,0}))\cdots) = \mathbb{R}^2.
\]

It follows that the inductive limit of \( (\mathcal{C}_{2n,0})_{n \in \mathbb{N}} \) relative to the embeddings \( \pi_n \) is a covering of \( \mathbb{R}^2 \) by 2 complete folding curves which is locally isomorphic to \( \mathcal{C} \). ■
We do not know presently for which coverings $C$ there exists a covering $D$ consisting of 3 or 4 curves which is locally isomorphic to $C$. However, we shall give examples of the two situations. For that purpose, we prove the following result, which gives a general method to construct $D$ from $C$. It is similar to the deflation process used by R. Penrose to construct aperiodic tilings (see for instance [1, p. 115]).

**Proposition 5.** Let $n \in \mathbb{N}^*$ be an integer. Suppose that $C$ is generated by a curve associated to an $\infty$-folding sequence $(a_k)_{k \in \mathbb{N}^*} \subset \{-1, +1\}$ with $a_{2r+n} = a_{2r}$ for each $r \in \mathbb{N}$. Identify $C^{(n)}$ with $C$ via the unique isometry $\sigma$ such that $\sigma(C^{(n)}) = C$ for each $C \in C$. Denote by $\Delta$ the $n$-th derivation defined on $C$. Let $D$ be a set of disjoint bounded curves, each of them contained in a curve of $C$. Let $\pi$ be an embedding of $D$ in $\Delta^{-1}(D)$, defined by a translation, such that no two curves of $D$ are embedded in the same curve of $\Delta^{-1}(D)$. Suppose that, for each endpoint $X$ of a segment of $\pi(D)$, the 4 intervals $[X - (1,0), X]$, $[X, X + (1,0)]$, $[X - (0,1), X]$, $[X, X + (0,1)]$ are the supports of two pairs of consecutive segments in $\Delta^{-1}(D)$. Then the pair $(D, \pi)$ induces a covering of $\mathbb{R}^2$, obtained as an inductive limit, locally isomorphic to $C$ and having the same number of curves as $D$.

**Proof.** For each $k \in \mathbb{N}$, $\pi$ induces an embedding $\pi_k : \Delta^{-k}(D) \to \Delta^{-(k+1)}(D)$ defined by a translation. For each endpoint $X$ of a segment of $\pi_k(\Delta^{-k}(D))$, the 4 intervals $[X - (1,0), X]$, $[X, X + (1,0)]$, $[X - (0,1), X]$, $[X, X + (0,1)]$ are the supports of two pairs of consecutive segments in $\Delta^{-(k+1)}(D)$.

Consequently, the inductive limit of the sets of curves $\Delta^{-k}(D)$ relative to the embeddings $\pi_k$ is a covering of $\mathbb{R}^2$. It is locally isomorphic to $C$ with the same number of curves as $D$. ■

**Example 6.** Let $C$ be the covering of $\mathbb{R}^2$ generated by the dragon curve associated to the $\infty$-folding sequence $(a_k)_{k \in \mathbb{N}^*}$ with $a_{2^n} = +1$ for each $n \in \mathbb{N}$. Then $C$ is locally isomorphic to two coverings of $\mathbb{R}^2$ by 3 curves. In one of them, each pair of curves has common vertices. In the other one, two curves are separated by the third one.

**Proof.** For the first covering, we apply Proposition 5 to the set of curves $D$ shown in Figure 2A with horizontal and vertical segments. It is embedded in $C$ since it appears in [2, Fig. 7] between the points $(2,0)$ and $(2,2)$. We consider the first primitive of $D$, shown in Figures 2A and 2B with diagonal segments, and the second primitive, shown in Figure 2B with horizontal and vertical segments. They are also embedded in $C$. The second primitive contains in its interior the image of $D$ under a rotation by $-\pi/2$. Repeting this process 3 more times, we obtain a set of bounded curves embedded in $C$, which contains in its interior a copy of $D$. 
For the second covering, we apply Proposition 5 to the set of curves \( \mathcal{E} \) shown in Figure 3A with horizontal and vertical segments. It is embedded in \( \mathcal{C} \) since it appears in [2, Fig. 7] between \( O \) and \((0, 3)\). We consider the first primitive of \( \mathcal{E} \), shown in Figures 3A and 3B, and the second primitive, shown in Figure 3B. They are also embedded in \( \mathcal{C} \). The second primitive contains in its interior the image of \( \mathcal{E} \) under a rotation by \( \pi \). Repeting this process 1 more time, we obtain a set of bounded curves embedded in \( \mathcal{C} \), which contains in its interior a copy of \( \mathcal{E} \). ■

Example 7. Let \( \mathcal{C} \) be the covering of \( \mathbb{R}^2 \) generated by a curve associated to the \( \infty \)-folding sequence \( (a_k)_{k \in \mathbb{N}^*} \), with \( a_{2n} = a_{2n+1} = -1 \) and \( a_{2n+2} = a_{2n+3} = +1 \) for each \( n \in \mathbb{N} \). Then \( \mathcal{C} \) is locally isomorphic to a covering of \( \mathbb{R}^2 \) by 4 curves.
Proof. $\mathcal{C}$ is represented in Figure 4. We apply Proposition 5 to the set of curves $\mathcal{D}$ shown in Figure 5A with horizontal and vertical segments. It is embedded in $\mathcal{C}$ since it appears in Figure 4 between $(-3, 0)$ and $O$.

We consider the first primitive of $\mathcal{D}$, shown in Figures 5A and 5B, and the second primitive, shown in Figure 5B. The second primitive is embedded in the covering of $\mathbb{R}^2$ generated by a curve associated to the $\infty$-folding sequence $(a_k)_{k \in \mathbb{N}^*}$ with $a_{2^{4n}} = a_{2^{4n}+1} = +1$ and $a_{2^{4n}+2} = a_{2^{4n}+3} = -1$ for each $n \in \mathbb{N}$. It contains in its interior a copy of the image of $\mathcal{D}$ under a reflection about the $y$-axis.

For the definition of the third and the fourth primitives of $\mathcal{D}$, we apply a process which is the image of the previous one under a reflection about the $y$-axis. The fourth primitive is a set of bounded curves embedded in $\mathcal{C}$, which contains in its interior a copy of $\mathcal{D}$. ■
References

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