Prolate stars due to meridional flows

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ABSTRACT

We have shown analytically that shapes of incompressible stars could be prolate if appropriate meridional flows exist. Although this result is strictly valid only if either the meridional flow or the rotation is absent and the vorticity is associated uniformly with meridional flow, this implies that perpendicular forces against centrifugal and/or magnetic forces might play important roles within stars. A consequence of the presence of meridional flows might be to decrease stellar oblateness due to centrifugal and/or magnetic fields.

Key words: stars: rotation.

1 INTRODUCTION

Stellar shapes have long been considered to be oblate (including a spherical shape) due to the effects of centrifugal and/or magnetic forces. However, recently Kuhn et al. (2012) have revealed that the shape of our Sun is ‘perfectly round’, against the common expectation of an oblate shape due to its rotation. At present, no clues have been proposed to solve this ‘strange’ problem (see e.g. Gough 2012).

Concerning stellar deformation, the effect of magnetic fields has been widely investigated (see e.g. Chandrasekhar & Fermi 1953; Ferraro 1954; Tomimura & Eriguchi 2005; Yoshida & Eriguchi 2006; Yoshida, Yoshida & Eriguchi 2006; Haskell et al. 2008; Lander & Jones 2009; Fujisawa, Yoshida & Eriguchi 2012).

The results found thus far are that purely poloidal magnetic fields make stars oblate, while purely toroidal magnetic fields lead stars to become prolate. Very recently, Ciolfi & Rezzolla (2013) succeeded in obtaining equilibrium states of magnetized stars with mixed poloidal–toroidal magnetic fields, even for configurations with very large toroidal magnetic fields. They showed that configurations with strong toroidal magnetic fields could be prolate. Thus, poloidal and toroidal magnetic fields act as increasing and decreasing mechanisms for stellar oblateness, respectively.

However, we should point out that flows within stars might work as one deforming mechanism of stellar configurations. In particular, the effect of meridional flows might make stellar shapes prolate, as shown in Eriguchi, Mueller & Hachisu (1986) and Birkl, Stergioulas & Muller (2011), although they did not describe their results quantitatively from the point of deformation due to meridional flows.

In this Letter, we deal analytically with meridional circulations of incompressible stars with slow flow velocities in order to show the deformation due to meridional flows quantitatively and clearly.

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We need to note that our calculations should be used either for purely rotating stars or for stars with purely circulating flows within meridional planes. The reason for this is that we have not taken into account the \( \psi \) component of the equation of motion, i.e.
\[
\mathbf{v} \cdot \nabla (R\Omega) = 0. \tag{8}
\]
As is explained in Eriguchi et al. (1986), for non-singular angular velocity distributions this condition results in two situations: (i) purely rotating stars, i.e. \( \nu_r = 0 \) and \( \nu_\theta = 0 \), or (ii) configurations with constant angular momentum throughout the whole star, i.e. \( \Omega \) needs to vanish to avoid singular behaviour of the angular velocity on the rotation axis. Thus our solutions in this Letter need to be used for either rotating stars without meridional flows or non-rotating stars with meridional flows.

The streamfunction must satisfy the following equation:
\[
\frac{\partial^2 \psi}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta} \right) = -r \sin \theta \rho_0 \omega_0. \tag{9}
\]

The distributions of the density, pressure and streamfunction can be obtained from the above stationary condition and the equation for the streamfunction, once the forms of arbitrary functions \( \nu(\psi) \) and \( \Omega(R) \) and the barotropic relation between the density and the pressure are specified.

The boundary conditions for the gravitational potential and the streamfunction are as follows: (i) the gravitational potential behaves as \( 1/r \) at infinity and the streamfunction is constant along the (unknown) stellar surface. Considering these boundary conditions, the gravitational potential and the streamfunction can be expressed by integral forms as
\[
\begin{align*}
\phi(r, \theta) &= -4\pi G \sum_{n=0}^{\infty} P_{2n}(\cos \theta) \int_0^{\pi/2} d\theta' \sin \theta P_{2n}(\cos \theta') \\
&\quad \times \int_0^{\alpha_{2n}^{(0)}} dr' r'^2 f_n(r, r') \rho_0, \tag{10}
\end{align*}
\]
\[
\begin{align*}
\psi(r, \theta) &= r \sin \theta \sum_{n=0}^{\infty} P_{2n-1}(\cos \theta) \int_0^{\pi/2} d\theta' \sin \theta' \\
&\quad \times P_{2n-1}^{(1)}(\cos \theta') \int_0^{\alpha_{2n-1}^{(0)}} dr' r'^2 f_{2n-1}(r, r') \rho_0 \omega_0(r', \theta') \\
&\quad + r \sin \theta \sum_{n=1}^{\infty} \alpha_{2n-1} P_{2n-1}^{(1)}(\cos \theta) r^{2n-1}, \tag{11}
\end{align*}
\]
where \( P_n \) is the Legendre function and \( r_{\alpha}^{(0)} \) expresses the shape of the deformed surface of the star,
\[
f_n(r, r') = \begin{cases} 
\frac{r_n^{(n)}/r^{n+1}}{r^{n+1}} & (r \geq r'), \\
\frac{r_n^{(n)}/r^{n+1}}{r^{n+1}} & (r \leq r'), 
\end{cases} \tag{12}
\]
and the \( \alpha_n \) are coefficients. Since we have assumed that there is no external wind, the \( r \) component of the velocity \( (\nu_r) \) must vanish at the surface. Therefore, the boundary condition for the streamfunction on the surface is as follows:
\[
\psi(r_s, \theta) = 0. \tag{13}
\]

We fix the coefficients to fulfil this boundary condition (Eriguchi et al. 1986; Fujisawa & Eriguchi 2013).

### 2.2 Stationary configurations of incompressible fluids with very slow flow velocities

In order to find analytical solutions to the basic equations described above, we further assume the following situation. (i) The form of the arbitrary function \( \nu(\psi) \) is specified as follows:
\[
\nu(\psi) = \varepsilon \nu_0 (\text{constant}). \tag{14}
\]
(ii) The rotational velocity is a constant, i.e.
\[
\Omega = \varepsilon \Omega_0 (\text{constant}), \tag{15}
\]
where \( \varepsilon \) is a small constant that expresses the slow fluid velocities in both the meridional plane and the \( \varphi \) direction.

For an incompressible body, we need only to solve for the surface shape by setting \( p = 0 \) on the surface, instead of solving for the density distribution. Thus, the solutions for our problem can be studied by expanding the quantities with respect to the small quantity \( \varepsilon \), as
\[
\begin{align*}
\rho(r, \theta, \varphi) &= \sum_{n=0}^{\infty} \varepsilon^n \rho^{(n)}(r, \theta), \\
p(r, \theta, \varphi) &= \sum_{n=0}^{\infty} \varepsilon^n p^{(n)}(r, \theta), \\
\end{align*}
\]
where \( R^{(0)} \) and \( \psi^{(0)} \) are the corresponding quantities of the surface shape and the streamfunction, respectively. Other physical quantities are also expanded, as
\[
\begin{align*}
F(r, \theta) &= \sum_{n=0}^{\infty} \varepsilon^n F^{(n)}(r, \theta), \tag{18}
\end{align*}
\]
where \( F(r, \theta) \) expresses a certain physical quantity.

For simplicity, we choose spherical configurations without flows as the \( n = 0 \) terms, i.e.
\[
\begin{align*}
\rho_0^{(0)}(r) &= r_0^{(0)} (\text{constant}), \tag{19}
\psi^{(0)}(r, \theta) &= 0, \tag{20}
\frac{d\rho_0^{(0)}(r)}{dr} &= -\rho_0 \frac{d\rho_0^{(0)}}{dr}. \tag{21}
\end{align*}
\]

The stationary equations are written to the second lowest order with respect to the small quantity \( \varepsilon \) as follows:
\[
\begin{align*}
\frac{\rho_0^{(0)}}{\rho_0^{(0)}} + \varepsilon^2 \frac{p_n^{(2)}}{p_0^{(0)}} &= -\phi^{(0)}(r) + C^{(0)} \\
&\quad + \varepsilon^2 \left[ -\phi^{(0)}(r, \theta) - \frac{1}{2} \left( \nu^{(1)}(r, \theta) + \nu^{(2)}(r, \theta) \right) \right] \\
&\quad + \nu_0 \psi^{(1)}(r, \theta) + \frac{1}{2} \varepsilon^2 \sin^2 \theta \Omega_0^2 + C^{(2)}, \tag{22}
\end{align*}
\]
where
\[
\begin{align*}
\varepsilon^2 \psi^{(1)}(r, \theta) &= -\frac{1}{30} \varepsilon \nu_0 \omega_0 r^2 (5r^2 - 3r^2) \sin^2 \theta \\
&\quad + r_0 \sin \theta \varepsilon \sum_{n=1}^{\infty} \alpha_{2n-1} P_{2n-1}^{(1)}(\cos \theta) r^{2n-1}. \tag{23}
\end{align*}
\]
When we apply the boundary condition to the above stationary equation on a deformed surface, i.e.,
\[
\begin{align*}
\rho^{(0)}(r_0) + \varepsilon^2 \frac{d\rho^{(0)}(r)}{dr} \big|_{r=r_0} &= r_0^{(2)}(\theta) + \varepsilon^2 p^{(2)}(r_0, \theta) = 0, \tag{24}
\end{align*}
\]
we obtain
\[0 = -\phi^{(0)}(r_0) - \varepsilon^2 \frac{d\phi^{(0)}(r)}{dr}_r + \frac{\nu}{2} \left( v^{(1)}(r_0, \theta) + v^{(1)}_0(r_0, \theta) \right) + \nu \psi^{(1)}(r_0, \theta) + \frac{1}{2} v^2 \sin^2 \theta \Omega_0^2 + C^{(2)} \].
\[(25)\]

Here,
\[\phi^{(0)}(r) = \frac{2\pi G}{3} \rho \varepsilon^2 r^2 - 2\pi G \rho_0 r^2,\]
\[(26)\]

\[\psi^{(0)}(r, \theta) = r \sin \theta \sum_{n=1}^\infty \alpha^{(0)}_{2n-1} P_{2n-1}(\cos \theta) r^{2n-1} = 0,\]
\[(27)\]

\[\psi^{(1)}(r, \theta) = -\frac{1}{30} \rho_0^2 v_0 r^2 \left( -3 r^2 + 5 r_0^2 \right) \sin^2 \theta + r \sin \theta \sum_{n=1}^\infty \alpha^{(1)}_{2n-1} P_{2n-1}(\cos \theta) r^{2n-1}.\]
\[(28)\]

Since this quantity \(\psi^{(1)}\) is the first-order term with respect to \(\varepsilon\), we only need to consider the boundary condition for the streamfunction on the undeformed surface. Thus,
\[\psi^{(1)}(r, \theta) = \frac{1}{10} \rho_0^2 v_0 \sin^2 \theta \varepsilon r^2 (r^2 - r_0^2),\]
\[(29)\]

\[v_r(r, \theta) = \frac{1}{5} \rho_0 v_0 (r^2 - r_0^2) \cos \theta,\]
\[(30)\]

\[v_\theta(r, \theta) = -\frac{1}{5} \rho_0 v_0 (2r^2 - r_0^2) \sin \theta,\]
\[(31)\]

\[\alpha^{(0)}_{2n-1} = 0,\]
\[\alpha^{(1)}_1 = \frac{1}{15} \rho_0^2 v_0 r_0^2,\]
\[\alpha^{(1)}_2 = 0,\]
\[(32)\]

where \(n = 1, 2, \ldots\).

From the second-order terms with respect to \(\varepsilon\), the following condition can be derived:
\[0 = \frac{4\pi G}{3} \rho \varepsilon^2 r^2 (\theta) + 4\pi G \rho_0 r^2 \sum_{n=0}^\infty P_{2n}(\cos \theta) \times \int_0^{\pi/2} d\theta' \sin \theta' P_{2n}(\cos \theta')(r^{(2)}(\theta')) - \frac{1}{2} \nu \psi^{(1)}(r_0, \theta) + \frac{1}{2} r_0^2 \sin^2 \theta \Omega_0^2 + C^{(2)}\]
\[(33)\]

By expanding the quantity \(r^{(2)}(\theta)\) as follows:
\[r^{(2)}(\theta) = \sum_{n=0}^\infty \beta^{(2)}_{2n} P_{2n}(\cos \theta),\]
\[(34)\]

where \(\beta^{(2)}_{2n}\) are certain constants, we obtain the following results for unknown quantities:
\[C^{(0)} = -\frac{4\pi G}{3} \rho_0^2,\]
\[(35)\]
\[C^{(2)} = -\frac{8\pi G}{3} \rho_0 \beta^{(2)}_2 + \frac{1}{75} \nu \rho_0^3 \Omega_0^2 - \frac{1}{3} \rho_0^3 \Omega_0^2,\]
\[(36)\]
\[\beta^{(2)}_2 = \frac{1}{8\pi G} \left[ \frac{1}{5} \rho_0 v_0^2 (\theta_0 - \frac{5}{\rho_0} r_0^2 \Omega_0^2) \right],\]
\[(37)\]
\[\beta^{(2)}_2 = 0 \quad (n = 2, 3, \ldots).\]
\[(38)\]

Here, if we require some condition for the scale of the star, we can calculate the value of \(\beta^{(2)}_2\) and complete solutions for our problem to second order in \(\varepsilon\) can be obtained.

### 2.3 Rotation versus circulation

From the analysis in the previous subsection, the change of the surface due to circulation and/or rotation can be expressed to second order of a certain small quantity \(\varepsilon\) as follows:
\[r_1(\theta) = r_0 + \varepsilon \left[ \beta_0 + \frac{1}{8\pi G} \left( \frac{1}{5} \rho_0 v_0^2 (r_0 - \frac{5}{\rho_0} \Omega_0^2) \right) P_2(\cos \theta) \right].\]
\[(39)\]

From this equation,
\[\frac{r_{\text{equator}} - r_{\text{pole}}}{r_0} = -\frac{3}{16\pi G} \left( \frac{1}{5} \rho_0 v_0^2 r_0^2 - \frac{5}{5} \Omega_0^2 \right),\]
\[(40)\]

where \(r_{\text{pole}}\) and \(r_{\text{equator}}\) are the polar and equatorial radii, respectively. As is easily seen, uniformly rotating configurations without circulation (\(\Omega_0 \neq 0, v_0 = 0\)) become oblate, while non-rotating configurations with circulation (\(\Omega_0 = 0, v_0 \neq 0\)) are prolate. In other words, the presence of meridional flows acts as an effective force perpendicular to the equatorial plane.

This can be clearly seen by defining the effective force due to meridional flows with \(\Omega_0 = 0\) as
\[F = \frac{\beta_0}{2} \left[ v_0^2 + \frac{v_0^2}{5} \right] + \rho_0 (v \times \omega).\]
\[(41)\]

The \(r\) and \(\theta\) components of this force for our incompressible configurations with \(\Omega_0 = 0\) are
\[F_r = \frac{v_0^2 \rho_0}{50} \left( -4r^3 + 4r_0^2 r \right) \left( 8r^3 - 6r_0^2 r \right) \sin^2 \theta.\]
\[(42)\]

\[F_\theta = \frac{v_0^2 \rho_0}{50} \left( 4r^3 - 6r_0^2 r \right) \sin \theta \cos \theta.\]
\[(43)\]

They are shown in the central panel of Fig. 1. In this figure, the centrifugal force due to uniformly rotating configurations, i.e. \(\Omega_0^2\), is also shown in the right panel.

In order to estimate deformation by meridional flows quantitively, we define the deformation ratio \(\Lambda\) as
\[\Lambda = \frac{r_{\text{equator}} - r_{\text{pole}}}{r_{\text{equator}} - r_{\text{pole}}},\]
\[(44)\]

where subscripts ‘c’ and ‘t’ denote configurations with meridional circulation and with rotation, respectively. For incompressible fluids
with very slow flow velocities,\[\]
\[
\Lambda = \frac{1}{25} \frac{r_o^2 \nu_p^2 \rho_0^2}{\Omega_0^2}
\]
\[
= \frac{v_p^2(r_o \pi /2)}{r_o^2 \Omega_0^2},
\]
\[
(45)
\]
where \(v_p\) is the poloidal velocity, defined as \(v_p^2 = v_r^2 + v_\theta^2\). Therefore, the stellar deformation depends on the ratio of the meridional velocity to the rotational velocity at the equatorial surface. When we use the solar radius and rotational velocity, for example,
\[
\Lambda \sim 1.0 \left( \frac{v_p}{1.81 \times 10^3 \text{ cm s}^{-1}} \right)^2
\]
\[
\times \left( \frac{r_o}{6.96 \times 10^{10} \text{ cm}} \right)^{-2} \left( \frac{\Omega_0}{2 \pi /28.0 \text{ day}} \right)^{-2}.
\]
\[
(46)
\]
The commonly believed solar convective flow or meridional flow velocities (e.g. Miesch 2005; Nordlund, Stein & Asplund 2009) are a few tens or hundred times smaller than the meridional flow velocity chosen here, but those values are not too small to be neglected. Thus, this simple model shows that meridional flows have small but non-zero influence on stellar deformation compared with stellar rotation, which is considered as the most powerful deformation mechanism within stars.

### 3 DISCUSSION AND CONCLUSION

In this Letter, we have obtained the expression for stellar deformation due to meridional flows analytically. In order to treat the problem analytically, we have assumed stationary incompressible stars and imposed a \(\psi = 0\) boundary condition on stellar surfaces.

We have shown that meridional flows make stars prolate under the conditions of our model. This might imply that meridional flows within stars work to decrease the oblateness of rotating stars. Although we have assumed incompressible fluids, the role of meridional flows would not disappear for compressible stars. As explained in the Introduction, according to the recent very accurate observation by Kuhn et al. (2012), the solar oblateness is unexpectedly smaller than the theoretical value, which is derived by considering only rotation (Armstrong & Kuhn 1999). Gough (2012) argued that magnetic fields and/or stresses due to turbulence could be possible mechanisms causing the small oblateness. However, they did not consider the influence of meridional flows, which could be one of the possible mechanisms, as we have shown in this Letter.

One might argue that the velocities of solar meridional flows are believed to be much smaller than the values required to reduce the rotational effect, as seen from equation (46). This belief has been based on theoretical analysis of many observational data on surface phenomena (Zhao et al. 2012). On the other hand, we can rely on the theories that are used to ‘estimate’ physical quantities within the solar interior. One approach for us is to take even the ‘curious’ observational data seriously and to consider possible mechanisms thoroughly, even though those mechanism might seem to be far from the widely believed theories and/or values. The ‘perfectly round’ Sun might be offering us an important hint about its interior.

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