Characterization of Complex Networks: A Survey of measurements

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September 19, 2005

Abstract

Each complex network (or class of networks) presents specific topological features which characterize its connectivity and highly influence the dynamics and function of processes executed on the network. The analysis, discrimination, and synthesis of complex networks therefore rely on the use of measurements capable of expressing the most relevant topological features. This article presents a survey of such measurements. It includes general considerations about complex network characterization, a brief review of the principal models, and the presentation of the main existing measurements organized into classes. Special attention is given to relating complex network analysis with the areas of pattern recognition and feature selection, as well as on surveying some concepts and measurements from traditional graph theory which are potentially useful for complex network research. Depending on the network and the analysis task one has in mind, a specific set of features may be chosen. It is hoped that the present survey will help the identification of suitable measurements.
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1 Introduction

Complex networks research can be thought of as lying at the intersection between graph theory and statistical mechanics, which confers a truly multidisciplinary nature to this area. While its origin can be traced back to the pioneering works on percolation and random graphs by Flory [1], Rapoport [2, 3, 4], and Erdős and Rényi [5, 6, 7], complex networks only more recently became a focus of attention. The main reasons for this was the discovery that real networks have characteristics which are not uniformly random. Instead, networks derived from real data may involve community structure, power law degree distributions and hubs, among other features. Two particular developments motivated a good deal of ongoing related research: Watts and Strogatz’s investigation of small-world networks [8] and Barabási and Albert’s characterization of scale-free models [9].

Although graph theory is a well-established and developed area in mathematics and theoretical computer science, many of the recent developments in complex networks have taken place in areas such as sociology and physics. Current interest has focused not only on applying the developed concepts to many real data and situations, but also on studying the dynamical evolution of network topology. Supported by the availability of high performance computers and large data collections on real world networks, results like the discovery of the scale-free structure of the Internet [10] and the WWW [11, 12] were of major importance for the increased interest on the new area of complex networks, whose growing importance is substantiated by the large number of recent related publications. A good review of such developments can be obtained from three excellent surveys [13, 14, 15], introductory papers [16, 17, 18, 19] and books [20, 21, 22]. For additional information about the related areas of percolation, disordered systems and fractals see [23, 24, 25].

One of the main reasons why complex networks have become so popular is their flexibility and generality for representing virtually any natural structure, including those undergoing dynamical changes of topology. As a matter of fact, every discrete structure such as lists, trees, or even lattices, can be suitably represented as graphs. It is thus little surprising that several investigations in complex network involve the representation of the structure of interest as a complex network, followed by an analysis of the topological features of the obtained representation performed in terms of a set of informative measurements. Another interesting problem consists of measuring the structural properties of evolving networks in order to characterize how the connectivity of the investigated structures change along the process. Both such activities can be understood as being aimed at the topological characterization of the studied structures. Another related application is to use the obtained measurements in order to discriminate between different classes of structures, which is a problem in the area of pattern recognition [26, 27]. Even when modeling networks, it is often necessary to compare the realizations of the model with real networks, which can be done by comparing respective measurements. Provided the measurements are comprehensive (ideally the representation by the measurements should be one-to-one or invertible), the fact that the simulated net-
works yield measurements similar to those of the real counterparts supports the validity of the model.

Network measurements are therefore essential as a direct or subsidiary resource in many network investigations, including representation, characterization, classification and modeling. Figure 1 shows the mapping of a generic complex network into the feature vector $\vec{\mu}$, i.e. a vector of related measurements such as average vertex degree, average clustering coefficient, the network diameter, and so on. Mappings quantifying important topological aspects are used to obtain the characterization and analysis of complex networks. In case the mapping is invertible, in the sense that the network can be recovered from the feature vector, the mapping is said to provide a representation of the network. An example of invertible mapping is the adjacency matrix. Observe that the characterization and classification of networks does not necessarily require invertible measurements. An interesting strategy which can be used in order to obtain additional information about the structure of complex networks involves transforming the original network and to obtain the measurements from the resulting network, as illustrated in Figure 2. In this figure, a transformation $T$ (e.g. remove the extremity vertices) is applied over the original network in order to obtain a transformed structure from which new measurements $\vec{\mu}_T$ are extracted. In case the feature vectors $\vec{\mu}$ and $\vec{\mu}_T$ correspond to the same set of measurements, it is also possible to consider the difference between these two vectors in order to obtain additional information about the network under analysis.

Perturbations of the network under analysis, which can be understood as a special case of the transformation framework outlined above, can be used to investigate the sensitivity of the measurements. Informally speaking, if the measurements considered in the feature vector are such that small changes of the network topology (e.g. add/remove a few edges or nodes) imply large values of $||\Delta \vec{\mu}||$, those measurements can be considered as being highly sensitive or unstable. One examples of such an unstable measurement is the average shortest path between two nodes. More formal approaches to the analysis of stability of measurements may involve the characterization of the divergent and Lyapunov coefficients in the feature space.

Another possibility to obtain a richer set of measurements involves the consideration of several instances along its development. A feature vector $\vec{\mu}(t)$ is obtained at each time instant $t$. Figure 3 shows four instances of an evolving network and the respective trajectory defined in one of the possible feature spaces involving two generic measurements $\mu_1$ and $\mu_2$. In such a way, the evolution of a network can now be investigated in terms of a trajectory in a phase space.

Both the characterization and classification of natural and human-made structures using complex networks imply the same important question of how to choose the most appropriate measurements. While such an optimal set of measurements should reflect the specific interests and application, it is unfortunate that there is no mathematical procedure for identifying the best measurements. To begin with, there is an unlimited set of topological measurements. Next, we have the fact that measurements are often correlated, implying redundancy. Still, while statistical approaches to decorrelation (e.g. principal component analysis) can help select and
Figure 1: The mapping from a complex network to a feature vector. Generic mappings can be used in order to obtain the characterization of the network in terms of a suitable set of measurements. In case the mapping is invertible, we have a complete representation of the original structure.

Figure 2: Additional measurements of an complex network can be obtained applying a transforming $T$ on it and obtaining a new feature vector $\vec{\mu}_T$ from the transformed network. The difference $\Delta\vec{\mu}$ between the original and transformed features vectors can also be considered in order to obtain additional insights about the properties of the original network.
Figure 3: Given a network undergoing some dynamical evolution (a) and a set of measurements (e.g. $\mu_1$ and $\mu_2$), trajectories can be defined in the feature space (b).
enhance measurements, they are not guaranteed to produce optimal results [26]. Ultimately, one has to rely on her/his knowledge of the problem and available measurements in order to select a suitable set of features to be made. For such reasons, it is of paramount importance to have a good knowledge not only of the most representative measurements, but also of their respective properties and interpretation. Although a small number of topological measurements, namely the average vertex degree, clustering coefficient and average shortest path, were typically considered for complex network characterization during the initial stages of this area, a series of new and more sophisticated features have been proposed and used in the literature along the last years. Actually, the fast pace of developments and new results reported in this very dynamic area makes it particularly difficult to follow and to organize the existing measurements. The purpose of the current survey consists precisely in providing an integrated and comprehensive guide to the main existing topological measurements, as well as their main interpretation, which can be applied to the characterization of complex networks. This review starts by presenting the basic concepts and notation in complex networks and follows by presenting several topological measurements. Illustrations of some of these measurements respectively to Erdős-Rényi, Watts-Strogatz and Barabási-Albert models are also included.

A WWW homepage has been designed to complement this survey, including resources on related books, groups, databases, software and general edges. The homepage can be accessed at:

http://cyvision.if.sc.usp.br/~francisco/networks/

2 Basic Concepts

Figure 4 shows the four main types of complex networks, which include weighted digraphs (directed graphs), unweighted digraphs, weighted graphs and unweighted graphs. The operation of symmetry can be used to transform a digraph into a graph, and the operation of thresholding can be applied to transform a weighted graph into its unweighted counterpart. These types of graphs and operations are defined more formally in the following, starting from the concept of weighted digraph, from which all the other three types can be obtained.

A weighted directed graph, \( G \), is defined by a set \( N(G) \) of \( N \) vertices (or nodes), a set \( E(G) \) of \( n \) edges (or links), and a mapping \( \omega : E(G) \rightarrow \mathbb{R} \). Each vertex can be identified by an integer value \( i = 1, 2, \ldots, N \); the edges are identified by a pair \( (i, j) \) that represents a connection going from vertex \( i \) to vertex \( j \) to which a weight \( \omega(i, j) \) is associated. In the complex network literature, it is often assumed that no self-connections or duplicate connections exist; that is, there are no edges of the form \( (i, i) \) and for each pair of edges \( (i_1, j_1) \) and \( (i_2, j_2) \) it holds that \( i_1 \neq i_2 \) or \( j_1 \neq j_2 \). Graphs with self- or duplicate connections are sometimes
called **multigraphs**, or **degenerate** graphs. In the following only non-degenerate graphs are considered. In an **unweighted digraph**, the edges have no weight, and the mapping $\omega$ is not needed. For **undirected graphs** (weighted or unweighted), the edges have no directions; the presence of a edge $(i, j)$ in $E(G)$ thus means that a connection exist from $i$ to $j$ and from $j$ to $i$.

A weighted digraph can be completely represented in terms of its **weight matrix** $W$, whose elements $w_{ij} = \omega(i, j)$ express the weight of the connection from vertex $i$ to vertex $j$. The operation of **thresholding** can be applied to a weighted digraph to produce an unweighted counterpart. This operation, henceforth represented as $A = \delta_T(W)$, is applied to each element of the matrix $W$, yielding the matrix $A$. In case $|w_{ij}| > T$ we have $a_{ij} = 1$, otherwise $a_{ij} = 0$. The resulting matrix $A$ can be understood as the **adjacency matrix** of the unweighted digraph obtained as a result of the thresholding operation. Any weighted digraph can be transformed into a graph by using the **symmetry** operation $\sigma(W) = W + W^T$, where $W^T$ is the transpose of $W$. For undirected graphs, two vertices $i$ and $j$ are said to be **adjacent** or **neighbors** if $a_{ij} \neq 0$. For directed graphs, the corresponding concepts are **predecessor** and **successor**: if $a_{ij} \neq 0$ then $i$ is a predecessor of $j$ and $j$ is a successor of $i$. The concept of adjacency can also be used in digraphs by considering predecessors and successors as adjacent vertices. The **neighborhood** of a vertex $i$, henceforth represented as $\nu(i)$, corresponds to the set of vertices adjacent to $i$. A **loop** or **cycle** is defined as a sequence of edges starting and terminating in the same vertex $i$ and passing only once through each vertex.

The **degree of a vertex** $i$, hence $k_i$, is the number of edges connected to that vertex, i.e. the cardinality of the set $\nu(i)$ (in the physics literature, this quantity is often called "connectivity" [14]). In the case of directed networks, there are two kinds of degrees: the **out-degree**, $k_i^{\text{out}}$, equal to the number of outgoing edges, and the **in-degree**, $k_i^{\text{in}}$, corresponding to the number of incoming edges. Note that $k_i = k_i^{\text{in}} + k_i^{\text{out}}$. For weighted networks, the definitions of degree given above can
Table 1: List of basic symbols used in the text

| Symbol | Concept |
|--------|---------|
| \( \mathcal{N}(G) \) | Set of vertices of graph \( G \) |
| \( \mathcal{E}(G) \) | Set of edges of graph \( G \) |
| \( N \) | Number of vertices |
| \( e_i \) | \( i^{th} \) edge |
| \( W \) | Weight matrix |
| \( w_{ij} \) | Element of weight matrix |
| \( A \) | Adjacency matrix |
| \( a_{ij} \) | Element of adjacency matrix |
| \( k_i \) | Degree of a vertex \( i \) |
| \( k_{\text{out}}^i \) | Out-degree of a vertex \( i \) |
| \( k_{\text{in}}^i \) | In-degree of a vertex \( i \) |
| \( s_i \) | Strength of a vertex \( i \) |
| \( \nu(i) \) | Set of neighbors of vertex \( i \) |
| \( |S| \) | Cardinality of set \( S \) |
| \( E \) | Number of different edge types |

be used, but a quantity called strength of \( i \), \( s_i \), defined as the sum of the weights of the corresponding edges, is generally used [23]:

\[
s_{\text{out}}^i = \sum_j w_{ij} \tag{1}
\]

\[
s_{\text{in}}^i = \sum_j w_{ji} \tag{2}
\]

Table 1 lists the basic symbols used in the paper.

3 Network Models

Modeling of complex networks is an important tool to improve the understanding of real networks. The three most important network models are discussed below.

3.1 The Random Graph of Erdős and Rényi

The random graph, developed by Rapoport [2, 3, 4] and independently by Erdős and Rényi [5, 6, 7], can be considered the most basic model of complex networks. This model, known as Erdős-Rényi (ER) graph, is defined by the number of vertices \( N \) and the probability \( p \) that a edge between two given vertices exists. The expected degree of a vertex in the network is

\[
\langle k \rangle = p(N - 1). \tag{3}
\]
When dealing with the large network size limit \( N \to \infty \), \( \langle k \rangle \) diverges if \( p \) is fixed. Instead, \( p \) is chosen as a function of \( N \) to keep \( \langle k \rangle \) fixed: \( p = \langle k \rangle / (N - 1) \).

The probability of a randomly chosen vertex having degree \( k \) (the degree distribution of the network, see Section 6), is binomial:

\[
P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}.
\]

(4)

For large \( N \) and \( \langle k \rangle \) fixed, this distribution approaches Poisson distribution with mean value \( \langle k \rangle \):

\[
P(k) \cong \frac{\langle k \rangle^k e^{-\langle k \rangle}}{k!},
\]

(5)

which is sharply peaked at \( \langle k \rangle \), as seen in Figure 5(a). Random graphs are studied in depth in the book of Bollobás [20].

### 3.2 The Small-World Model of Watts and Strogatz

Many real world networks present what is called the “small world” property, i.e., all vertices can be reached from the others through a small number of edges. For example, in the social context, everyone in the world can be reached through a short chain of social acquaintances. This concept was born from the famous experiment made by Milgram in 1967 [29], who found that two US citizens chosen at random were connected by an average of six acquaintances. Other property found in many networks is the presence of a large number of loops of size three, i.e., if vertex \( i \) is connected to vertices \( j \) and \( k \), there is a high probability of vertices \( j \) and \( k \) being connected (the clustering coefficient, Section 5 is high); for example, if \( B \) and \( C \) are friends of \( A \), there is a high probability that \( B \) and \( C \) are friends. ER networks have the first property but not the second; regular network with the second property are easy to construct. The most popular model of random networks with small
world characteristics and an abundance of short loops was developed by Watts and Strogatz \cite{8} and called the Watts-Strogatz (WS) small-world model. They showed that small-world networks are common in a variety of different realms ranging from the \textit{C. elegans} neuronal system to power grids. The model is situated between an ordered finite lattice and a random graph.

To construct a small-word network, one starts with a regular lattice of \( N \) vertices (Figure 6) in which each vertex is connected to \( \kappa \) neighbors in each direction, where \( N \gg \kappa \gg \ln(N) \gg 1 \). Each edge is then randomly rewired with probability \( p \). When \( p = 0 \) we have an ordered lattice with high number of loops but large distances and when \( p \to 1 \), the network becomes a random graph with short distances but no loops. Watts and Strogatz have shown that, in an intermediate regime, both short distances and a large number of loops are present.

The degree distribution for small-world networks is similar to random networks, with a peak at \( \langle k \rangle = 2\kappa \).

### 3.3 Scale-free networks of Barabási and Albert

After the creation of the model of Watts and Strogatz, Barabási and Albert \cite{9} showed that the degree distribution of many real systems are characterized by an uneven distribution of connectedness. Instead of the vertices of these networks having a random pattern of connections, some vertices are highly connected while others have few connections. The degree distribution follows a power law for large \( k \),

\[
P(k) \sim k^{-\gamma}. \tag{6}
\]

Because of this relation, these networks are called \textit{scale-free} networks.

A characteristic of this kind of network is the existence of hubs, i.e., vertices that collect a significant fraction of the total number of edges of the network. In Figure 7, hubs are represented by black vertices.

The Barabási-Albert (BA) network model is created by starting with a set of \( m_0 \) vertices; afterwards, at each step of the construction the network grows with the addition of new vertices. For each vertex, \( m \) new edges are inserted with one end on
the new vertex and the other end on a previous vertex. The vertices which receive
the new edges are chosen following a linear preferential attachment rule, where the
most connected vertices have a greater probability to receive new vertices. This
is known as “the rich get richer” paradigm. The procedure is repeated until the
desired number of vertices is reached.

4 Distance

In the general case, two vertices of a complex network are not adjacent. In fact,
most of the networks of interest are sparse, in the sense that only a small fraction of
all possible edges are present. Nevertheless, two non-adjacent vertices $i$ and $j$ can
be connected through a sequence of $m$ edges $(i, k_1), (k_1, k_2), \ldots, (k_{m-1}, j)$; such
set of edges is called a path between $i$ and $j$, and $m$ is the length of the path. We
say that two vertices are connected if there is at least one path connecting them.
Many measurements are based on the length of these connecting paths.

For undirected, unweighted graphs, the number of edges in a path connecting
vertices $i$ and $j$ is called the length of the path. A geodesic path (or shortest path)
between vertices $i$ and $j$, is one of the paths connecting these vertices with mini-
num length (many geodesic paths may exist between two vertices); the length of
the geodesic paths is the geodesic distance $d_{ij}$ between vertices $i$ and $j$. If the
graph is weighted, the same definition can be used, but generally one is interested
Table 2: Values of some measurements for the Erdős-Rényi, Watts-Strogatz and Barabási-Albert network models.

| Measure                  | Erdős-Rényi                                | Watts-Strogatz             | Barabási-Albert       |
|--------------------------|--------------------------------------------|----------------------------|-----------------------|
| Degree distribution      | $P(k) = \frac{e^{-\langle k \rangle} k^k}{k!}$ | —                          | $P(k) \sim k^{-3}$    |
| Average vertex degree    | $\langle k \rangle = p(N - 1)$              | $\langle k \rangle = K*$   | —                     |
| Clustering coefficient   | $C = p$                                     | $C = \frac{3(k-1)}{2(2k-1)+4kp+2}$ | —                     |
| Average path length      | $l \sim \frac{\ln(N)}{\ln(k)}$            | $l(N, p) \sim \frac{N}{K} f(pKN^d)^*$ | —                     |

* In WS networks, the value $K$ represents the number of neighbors of each vertex in the initial regular network (in Figure 6, $K = 4$).
* The function $f(u) = \text{constant if } u \ll 1 \text{ or } f(u) = \ln(u)/u \text{ if } u \gg 1$.

in taking into account the edge weights. Two main possibilities come out: first, the edge weights may be proportionally related to some physical distance, for example if the vertices correspond to cities and the weights to distances between these cities through given highways. In this case, one can compute the distance along a path as the sum of the weights of the edges in the path. Second, the edge weights may reflect the strength of connection between the vertices, for example if the vertices are Internet routers and the weights are the bandwidth of the edges, the distance corresponding to each edge can be taken as the reciprocal of the edge weight, and the path length is the sum of the reciprocal of the weight of the edges along the path. If there are no paths from vertex $i$ to vertex $j$, then $d_{ij} = \infty$. For digraphs, the same definitions can be used, but in general $d_{ij} \neq d_{ji}$, as the paths from vertex $i$ to vertex $j$ are different from the paths from $j$ to $i$.

We can define a network measurement by computing the mean value of $d_{ij}$, known as mean geodesic distance:

$$
\ell = \frac{1}{N(N-1)} \sum_{i \neq j} d_{ij}.
$$

(7)

A problem with this definition is that it diverges if there are unconnected vertices in the network. To circumvent this problem, only connected pairs of vertices are included in the sum. This avoids the divergence, but introduces a distortion for networks with many unconnected pairs of vertices, which will show a small value of mean distance, similarly with networks with very high number of connections. Latora and Marchiori [30] proposed a closely related measurement that they called
global efficiency:

\[ E = \frac{1}{N(N - 1)} \sum_{i \neq j} \frac{1}{d_{ij}}, \quad (8) \]

where the sum takes all pairs of vertices into account. This measurement quantifies the efficiency of the network in sending information between vertices, assuming that the efficiency for sending information between two vertices \(i\) and \(j\) is proportional to the reciprocal of their distance. The reciprocal of the global efficiency is the harmonic mean of the geodesic distances:

\[ h = \frac{1}{E}. \quad (9) \]

As Eq. (9) does not present the divergence problem of Eq. (7), it is therefore a more appropriate measurement for disjoint graphs.

The determination of shortest distances in a graph is only possible with global information on the structure of the graph. This information is not always available. Therefore, navigation in a graph must happen using limited, local information and a specific algorithm. The effective distance between two nodes is thus generally larger than the shortest distance, and dependent on search algorithm as well as network structure [31].

5 Clustering Coefficient

A characteristic of the Erdős-Rényi model is that the local structure of the network near a vertex is a tree. More precisely, the probability of loops involving a small number of vertices tends to 0 in the large network size limit. This is in marked contrast with the profusion of short loops which shows up in many real-world networks. One way to characterize the presence of such loops is through the clustering coefficient.

Two different clustering coefficients are frequently used. Barrat and Weigt [32] proposed the following definition for undirected unweighted networks:

\[ C = \frac{3N_\Delta}{N_3}, \quad (10) \]

where \(N_\Delta\) is the number of triangles in the network and \(N_3\) is the number of connected triples. The factor three accounts for the fact that each triangle can be seen as consisting of three different connected triples, one with each of the vertices as central vertex. A triangle is a set of three vertices with edges between each pair of vertices; a connected triple is a set of three vertices where each vertex can be reached from each other (directly or indirectly), that is, two vertices must be adjacent to another vertex (the central vertex):

\[ N_\Delta = \frac{1}{3} \sum_{(i,j,k)} a_{ij}a_{ik}a_{jk}, \quad (11) \]

15
\[ N_3 = \sum_{(i,j,k)} a_{ij}a_{jk}, \] (12)

where the sum should be taken over all ordered triples \((i, j, k)\) of distinct vertices \(i, j,\) and \(k\).

It is also possible to define the clustering coefficient of a given vertex \(i\) as:
\[ C_i = \frac{N_\Delta(i)}{N_3(i)}, \] (13)

where \(N_\Delta(i)\) is the number of triangles involving vertex \(i\) and \(N_3(i)\) is the number of connected triples with \(i\) as the central vertex:
\[ N_\Delta(i) = \sum_{j>k} a_{ij}a_{ik}a_{jk}, \] (14)
\[ N_3(i) = \sum_{j>k} a_{ij}a_{ik}. \] (15)

If \(k_i\) is the number of neighbors of vertex \(i\), then \(N_3(i) = k_i(k_i - 1)\); also \(N_\Delta(i)\) counts the number of edges between neighbors of \(i\). Writing \(l_i\) for the number of edges between neighbors of \(i\), Eq. (13) can be written as:
\[ C_i = \frac{2l_i}{k_i(k_i - 1)}. \] (16)

Using \(C_i\), an alternative definition of the network clustering coefficient (different from that in Eq. (10)) is
\[ \bar{C} = \frac{1}{N} \sum_i C_i. \] (17)

The difference between the two definitions is that Eq. (10) gives the same weight to each triangle in the network, while Eq. (17) gives the same weight to each vertex, resulting in different values because vertices of higher degree are possibly involved in a larger number of triangles than vertices of smaller degree.

For weighted graphs, Barthélemy [28] introduced the concept of weighted clustering coefficient:
\[ C_i^* = \frac{1}{s_i(k_i - 1)} \sum_{j>k} \frac{w_{ij} + w_{ik}}{2} a_{ij}a_{ik}a_{jk}, \] (18)

where the normalizing factor \(s_i(k_i - 1)\) \((s_i\) is the strength of the vertex, see Section 2\) assures that \(0 \leq C_i^* \leq 1\). From this equation, a clustering coefficient weighted network can be defined as
\[ C^* = \frac{1}{N} \sum_i C_i^*. \] (19)
Another definition for clustering in weighted networks is based on the intensity of the triangle subgraphs (Section 14.2):

\[ \tilde{C}_i^* = \frac{2}{k_i(k_i - 1)} \sum_{j > k} (\tilde{w}_{ij}\tilde{w}_{jk}\tilde{w}_{ki})^{1/3}, \]  

(20)

where \( \tilde{w}_{ij} = \frac{w_{ij}}{\max w_{ij}} \).

Given the clustering coefficients of the vertices we can compute the clustering coefficient as a function of the degree of the vertices:

\[ C(k) = \frac{\sum_i C_i \delta_{k_i k}}{\sum_i \delta_{k_i k}}. \]  

(21)

For some networks, this function has the form \( C(k) \sim k^{-\alpha} \). This behavior was associated with a hierarchical structure in the network and the exponent \( \alpha \) is called its hierarchical exponent. Soffer and Vázquez found that this dependence of the clustering coefficient with \( k \) is to some extent due to the degree correlations of the networks, with vertices of high degree connecting with vertices of low degree. They suggested a new definition of clustering coefficient without degree correlation bias:

\[ \hat{C}_i = \frac{l_i}{\omega_i}, \]  

(22)

where \( \omega_i \) is the maximum number of edges possible between the neighbors of vertex \( i \), considering their vertex degrees and the fact that one of their edges is to vertex \( i \).

6 Degree related measurements

The degree is an important characteristic of a vertex. Based on the degree of the vertices, it is possible to construct measurements for the network. One of the simplest is the maximum degree:

\[ k_{\text{max}} = \max_i k_i. \]  

(23)

More detail is provided by the degree distribution, \( P(k) \), which expresses the fraction of vertices in a network with degree \( k \). For directed networks there are an out-degree distribution \( P^{\text{out}}(k^{\text{out}}) \), an in-degree distribution \( P^{\text{in}}(k^{\text{in}}) \), and the joint in-degree and out-degree distribution \( P^{\text{io}}(k^{\text{in}}, k^{\text{out}}) \). For weighted networks, similar definitions use the strength of the vertices.

One may be interested in finding out if there is a correlation between the degrees of different vertices. Such correlations were found to have an important role in many network structural and dynamical properties. The most common choice is to find correlations between two vertices connected by an edge. This correlation can be expressed by the joint degree distribution \( P(k, k') \), i.e., as the
probability of a edge connecting two vertices of degree \( k \) and \( k' \). Another way to express this is by giving the conditional probability that an arbitrary neighbor of a vertex of degree \( k' \) has degree \( k \),

\[
P(k|k') = \frac{\langle k \rangle P(k', k)}{k' P(k')}. \tag{24}
\]

For undirected networks, \( P(k, k') = P(k', k) \); for directed networks, in general \( P(k, k') \neq P(k, k') \), \( k \) is the degree at the tail of the edge and \( k' \) is the degree at the head, and both \( k \) and \( k' \) may be in-, out-, or total degrees. For weighted networks the strength \( s \) can be used instead of \( k \). This distribution gives a very detailed description of vertex degree correlations but, for fat tailed degree distributions as in scale-free networks, it is difficult to evaluate experimentally, because of the poor statistics. A measure with better statistics is to computing the mean degree of the neighbors of vertices with a given degree \( k \), given by

\[
k_{nn}(k) = \sum_{k'} k' P(k'|k). \tag{25}
\]

A related scalar measurement \( r \) is the Pearson correlation coefficient of the degrees at both ends of the edges:

\[
r = \frac{\frac{1}{n} \sum_{i \to j} k_i k_j - \left[ \frac{1}{n} \sum_{i \to j} \frac{1}{2} (k_i + k_j) \right]^2}{\frac{1}{n} \sum_{i \to j} \frac{1}{2} (k_i^2 + k_j^2) - \left[ \frac{1}{n} \sum_{i \to j} \frac{1}{2} (k_i + k_j) \right]^2}, \tag{26}
\]

where the notation \( \sum_{i \to j} \) means that the sum must be carried out for all edges of the network and the vertices connected by the edge are named \( i \) and \( j \); \( n \) is the total number of edges.

## 7 Assortativity

Some networks consist of vertices of different types. For example, in a sociological network where the vertices are people and the edges are a social relation between two persons, e.g. friendship, one may be interested in answering questions like: how probable is it for a friendship to exist between two persons of different economic classes? The vertices are in this case not homogeneous, but classified in different types.

For networks with different types of vertices, a type mixing matrix \( M \) can be defined, with elements \( m_{st} \) such that \( m_{st} \) is the number of edges connecting vertices of type \( s \) to vertices of type \( t \) (or the total strength of the edges connecting the two vertices of the given types, for weighted networks). It can be normalized as

\[
\hat{M} = \frac{M}{||M||}. \tag{27}
\]
where \( \|X\| \) represents the sum of all elements of matrix \( X \).

The probability of a vertex of type \( s \) having a neighbor of type \( t \) is then

\[
P^{\text{(type)}}(t|s) = \frac{\hat{m}_{st}}{\sum_u \hat{m}_{su}}. \tag{28}
\]

Note that \( \sum_t P^{\text{(type)}}(t|s) = 1 \).

\( P^{\text{(type)}}(s,t) \) and \( \hat{M} \) can be used to quantify the tendency in the network of vertices of some type to connect to vertices of the same type, called assortativity.

We can define an assortativity coefficient \([38]\) as:

\[
Q = \frac{\sum_s P^{\text{(type)}}(s|s) - 1}{T - 1}, \tag{29}
\]

where \( T \) is the number of different vertex types in the network. It is clear that \( 0 \leq Q \leq 1 \), where \( Q = 1 \) for a perfectly assortative network (only edges between vertices of the same type) and \( Q = 0 \) for random mixing. But each vertex type has the same weight in \( Q \), regardless of the number of vertices of that type. An alternative definition that avoids this problem \([39]\) is:

\[
R = \frac{\text{Tr } \hat{M} - \|\hat{M}^2\|}{1 - \|\hat{M}^2\|}. \tag{30}
\]

It is interesting to associate the type of the vertex to its degree. The Pearson correlation coefficient of vertex degrees, Eq. \([26]\), can be seen as an assortativity coefficient for this case. With this interpretation, if \( r > 0 \) vertices of high degree connect mainly to vertices of high degree, we call the network assortatively mixed; if \( r < 0 \), vertices of high degree connect mainly with vertices of low degree, and the network is dissortatively mixed. For \( r = 0 \), the degrees are uncorrelated.

8 Vulnerability

In infrastructure networks (like WWW, the Internet, energy supply, etc), it is important to know which components (vertices or edges) are crucial to their best functioning. Intuitively, the critical vertices of a network are their hubs (vertices with higher degree), however there are situations in which they are not necessarily most vital for the system performance. For instance, all vertices of a network in the form of a binary tree have equal degree, therefore there is no hub, but the vertices closer to the root and the root itself are much more important than those near the leaves. This suggests that networks have a hierarchical property, which means that the most crucial components are those in higher positions in the hierarchy.

A way to find critical components of a network is by looking for the most vulnerable vertices. The vulnerability of a network is defined as the drop in performance when a vertex and all its edges are removed from the network. The vulnerability \( V_i \) of a network associated with a vertex \( i \) is expressed as \([40]\):

\[
V_i = \frac{E - E_i}{E} \tag{31}
\]
where $E$ is the global efficiency Eq. (8) and $E_i$ is the global efficiency after the removal of the vertex $i$ and all its edges. As suggested by Gol’dshein et al. [40], the ordered distribution of vertices with respect to their vulnerability $V_i$ is related to the network hierarchy, thus the most vulnerable (critical) vertex occupies the highest position in the network hierarchy.

A measurement of network vulnerability [41] is the maximum vulnerability for all of its vertices:

$$V = \max_i V_i. \quad (32)$$

### 9 Entropy

The structure of the network is related to its reliability and the speed of information propagation. The difficulty of searching information in the network can be quantified [42, 43] through the information entropy of the network. If a random walk starts on vertex $i$ going to vertex $j$, the probability that it goes through a given shortest path $\pi(i, j)$ between these vertices is:

$$P\{\pi(i, j)\} = \frac{1}{k_i} \prod_{b \in N(\pi(i, j))} \frac{1}{k_b - 1}. \quad (33)$$

where $N(\pi(i, j))$ is the set of vertices in the path $\pi(i, j)$ excluding $i$ and $j$.

The search information is the total information needed to identify one of all the shortest paths between $i$ and $j$ and is given by:

$$S(i, j) = -\log_2 \sum_{\pi(i, j)} P\{\pi(i, j)\}, \quad (34)$$

where the sum is over all shortest paths between $i$ and $j$. The search information of the network is given by the average over all pairs $(i, j)$:

$$S = \frac{1}{N(N-1)} \sum_{i \neq j} S(i, j). \quad (35)$$

The access ($A_i$) and hidden ($H_i$) information of vertex $i$ can be defined as:

$$A_i = \frac{1}{N-1} \sum_{j \neq i} S(i, j), \quad (36)$$

$$H_i = \frac{1}{N-1} \sum_{j \neq i} S(j, i). \quad (37)$$

The former quantifies how difficult it is to find other vertices starting from vertex $i$; the latter quantifies how difficult it is to find the vertex starting from the other vertices in the network. Note that the average value of $A_i$ and $H_i$ for a network is $S$: $\sum_i A_i = \sum_i H_i = SN$. 20
Considering the exchange of messages in the network, it is possible to define entropies to quantify the predictability of the message flow. Assuming that messages always flow through shortest paths and all pairs of vertices exchange the same number of messages at the same rate, the following entropies can be defined \[43\]:

\[
T_i = -\sum_{j \in \text{pred}(i)} c_{ij} \log_2 c_{ij},
\]

\[
R_i = -\sum_{j \in \text{pred}(i)} b_{ij} \log_2 b_{ij},
\]

where \(\text{pred}(i)\) is the set of predecessors of vertex \(i\) (Section 2), \(c_{ij}\) is the fraction of messages targeted at vertex \(i\) that comes through vertex \(j\), and \(b_{ij}\) is the fraction of messages that goes through vertex \(i\) coming from vertex \(j\). \(T_i\) is the target entropy of vertex \(i\); \(R_i\) is the road entropy of vertex \(i\). Low values of these entropies mean that the vertex from where the next message (coming to vertex \(i\) or passing through vertex \(i\), respectively) will come is easily predictable.

For the network, we can define target and road entropies as averages for all vertices:

\[
T = \frac{1}{N} \sum_i T_i,
\]

\[
R = \frac{1}{N} \sum_i R_i.
\]

These quantities are related with the organization of the network, as shown in \[43\]: a network with a low value of \(T\) has a star structure and a low value of \(R\) means that the network is composed by hubs connected in a string.

### 10 Size of Clusters

In undirected graphs, if vertices \(i\) and \(j\) are connected (Section 4) and vertices \(j\) and \(k\) are connected, then \(i\) and \(k\) are also connected. This property can be used to partition the vertices of a graph in non-overlapping subsets of connected vertices. These subsets are called connected components or clusters.

If a network has too few edges, that is, the average connectivity of its vertices \(\langle k \rangle\) is too small, there will be many isolated vertices and clusters with a small number of vertices. As more edges are added to the network, the small clusters are connected to larger clusters; after some critical value of the connectivity, most of the vertices are connected into a giant cluster, characterizing the percolation \[24\] of the network. For the Erdős-Rényi graph in the limit \(N \to \infty\) this happens at \(\langle k \rangle \approx 1\). Of special interest is the distribution of sizes of the clusters in the percolation point and the fraction of vertices in the giant cluster. The critical density of edges...
(as well as average and standard deviation) needed to achieve percolation can be used to characterize network models or experimental phenomena.

It is also possible to consider the $L$-percolations [44] of the investigated networks. Given a network, its $L$-expansion is constructed by adding a edge between each pair of vertices that are connected by a self-avoiding loop of length $L$. The intersection between the original network and its $l$-expansion is its $L$-conditional expansion. An $L$-percolation is the percolation of the $L$-conditional expansion of the network.

11 Centrality Measurements

Looking at a network as a description of the interaction between agents represented by the vertices, and considering that not all pairs of vertices are adjacent, the presence of intermediate vertices and edges forming a path that connect these non-adjacent vertices is of high importance. In that sense, the greater the number of paths in which a vertex or edge takes part, the greater the importance of this vertex or edge for the network. Assuming that the interactions follow the shortest paths between two vertices, it is possible to quantify the importance of a vertex or an edge in this sense by its *betweenness centrality* [45]:

$$B_i = \sum_{jk} \frac{\sigma(j, i, k)}{\sigma(j, k)}, \quad (42)$$

where $\sigma(j, i, k)$ is the number of shortest paths between vertices $j$ and $k$ that pass through vertex or edge $i$, $\sigma(j, k)$ is the total number of shortest paths between $j$ and $k$ and the sum is over all pairs $(j, k)$ of distinct vertices. When one takes into account the fact that the shortest paths might not be known and instead a search algorithm is used for navigation (see Section 4), the betweenness of a vertex or edge must be defined in terms of the probability of it being visited by the search algorithm; this generalization was introduced by Arenas *et al.* [46], and subsumes the betweenness centrality measure using random walks proposed by Newmann [47].

Other centrality measures include (see [48]) *closeness centrality*:

$$CC_i = \frac{1}{\sum_j d_{ij}}; \quad (43)$$

*graph centrality*:

$$GC_i = \frac{1}{\max_j d_{ij}}; \quad (44)$$

and *stress centrality*:

$$TC_i = \sum_{jk} \sigma(j, i, k). \quad (45)$$
12 Spectral Measurements

Taking the adjacency matrix $A$ of a graph, we can compute its eigenvalues and eigenvectors. If $\lambda_i$ are the eigenvalues of $A$, the spectral density of the graph is defined as:

$$\rho(\lambda) = \frac{1}{N} \sum_i \delta(\lambda - \lambda_i),$$  \hspace{1cm} (46)

where $\delta(x)$ is the delta function. $\rho$ approaches a continuous function as $N \to \infty$. For Erdős-Rényi networks, if $p$ is constant as $N \to \infty$, $\rho(\lambda)$ converges to a semicircle; if $p$ decreases with $N$ so that $pN$ is constant, the form of $\rho(\lambda)$ is different, an important characteristic of real-world networks.

The eigenvalues can be used to compute the $k$th-moments,

$$M_k = \frac{1}{N} \sum_i (\lambda_i)^k = \frac{1}{N} \sum_{i_1, i_2, \ldots, i_k} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}.$$  \hspace{1cm} (47)

The quantity $D_k = NM_k$ is the number of paths returning to the same vertex in the graph passing through $k$ edges, where these paths can contain vertices that were already visited. Because in a tree-like graph a return path is only possible going back through the already visited edges, the presence of odd moments is a sure sign of cycles in the graph; specially, as a path can go through three edges and return to its starting vertex only by following three different edges (if self-connections are not allowed), $D_3$ is related with the number of triangles in the network.

13 Community structure

Networks can have a non homogeneous structure formed by a group of vertices strongly connected but with few edges connecting them. These networks have a modular (or community) structure, as the communities of social networks or the World Wide Web, where the pages can be grouped by topics. Figure 8 presents an example of network with community structure.

There are many methods to identify communities in networks as described in. A given community division of a network can be evaluated by computing its modularity, a measure introduced by Newman and Girvan.

Modularity is based on the concept of assortative mixing (see Section 7). The type of a vertex is the community to which it is associated. Then a coefficient like that of Eq. (30), but without the normalization with respect to perfect mixing can be used:

$$R_c = \text{Tr} \hat{M} - \|\hat{M}\|^2,$$  \hspace{1cm} (48)

where the $\hat{m}_{st}$ of $\hat{M}$ is the fraction of edges connecting communities $s$ and $t$. Values of $R_c$ near 0 indicate a bad division of the network in communities while
Figure 8: A network with community structure represented by the dashed lines. The communities are the groups of more intensely connected vertices.

$R_c$ near 1 indicates a good division (edges are mostly between vertices in the same community).

After community identification, it is possible to determine the role of vertices [61] using the z-score of the within-module degree, $z_i$, and the participation coefficient, $P_i$. The z-score measures how “well-connected” vertex $i$ is to the other vertices in the community,

$$z_i = \frac{q_i - \bar{q}_{s_i}}{\sigma_{q_{s_i}}}$$  \hspace{1cm} (49)

where $q_i$ is the number of edges of vertex $i$ to other vertices in its own community $s_i$, $\bar{q}_{s_i}$ is the average of $q$ over all vertex in $s_i$, and $\sigma_{q_{s_i}}$ is the standard deviation of $q$ in $s_i$.

The participation coefficient measures how “well-distributed” the edges of vertex $i$ are among different communities,

$$P_i = 1 - \sum_{s=1}^{N_M} \left( \frac{q_{is}}{k_i} \right)^2$$ \hspace{1cm} (50)

where $q_{is}$ is the number of edges from vertex $i$ to community $s$ and $k_i$ is the degree of vertex $i$. This value is zero if all edges are within its own community and it is close to one if its edges are uniformly distributed among all communities. Based on these two index, a $zP$ parameter-space is constructed, allowing the classification of vertices in seven different roles [61].
14 Subgraphs

A graph \( g \) is a subgraph of the graph \( G \) if \( \mathcal{N}(g) \subseteq \mathcal{N}(G) \) and \( \mathcal{E}(g) \subseteq \mathcal{E}(G) \), with the edges in \( \mathcal{E}(g) \) extending over vertices in \( \mathcal{N}(g) \). If \( g \) contains all edges of \( G \) that connect vertices in \( \mathcal{N}(g) \), the subgraph \( g \) is said to be implied by \( \mathcal{N}(g) \). Important subgraphs are loops, trees (connected graphs without loops) and complete subnetworks (graphs with edges between each pair of vertices). Figure 9 shows a network and some subnetworks. The probability distribution of subgraphs in random graphs has been studied for some time \([20]\), but interest has increased recently due to the discovery of network motifs discussed below.

14.1 Network motifs

Network motifs are subgraphs that appear more often in real networks than in randomly generated networks \([62, 63]\); and they are illustrated in Figure 10. Given a real network, a large number of randomized networks with the same degrees for the vertices are generated. If the probability \( P \) of a subgraph appear a larger or the same number of times in the randomized networks is smaller than a given threshold (usually \( P = 0.01 \)), the subgraph is considered a motif of the network. Figure 11 shows some possible motifs of directed networks and their conventional names.

To quantify the significance of a given motif, its \( Z \)-score can be computed. If \( \langle N_i^{\text{real}} \rangle \) is the number of times that a motif \( i \) appears in the real network, \( \langle N_i^{\text{rand}} \rangle \)
Figure 11: Some types of motifs: (a) three-vertex feedback, (b) three chain, (c) feed-forward loop, (d) bi-parallel, (e) four-vertex feedback, (f) bi-fan, (g) feedback with two mutual dyads, (h) fully connected triad and (i) uplinked mutual dyad.

the average number of times it appears in the randomized networks, and $\sigma_i^{(\text{rand})}$ the standard deviation of the number of occurrences in the randomized networks, then:

$$Z = \frac{N_i^{(\text{real})} - \langle N_i^{(\text{rand})} \rangle}{\sigma_i^{(\text{rand})}}.$$  \hfill (51)

It is also possible to categorize different networks by the $Z$-scores of their motifs: networks that show emphasis on the same motifs can be considered as part of the same family [64]. To this purpose, the significance profile of the network is computed. The significance profile is a vector that, for each motif of interest $i$, computes the importance of this motif with respect to the other on the network:

$$SP_i = \frac{Z_i}{\sum_j Z_j^2}. \hfill (52)$$

14.2 Subgraphs in Weighted Networks

In weighted networks, a subgraph may be present with different values for the weights of the edges. Onnela et al. [65] suggested a definition for the intensity of a subgraph based on the geometric mean of its weights on the network. Given a
subgraph \( g \), its intensity is defined as:

\[
I(g) = \left( \prod_{(i,j) \in E(g)} w_{ij} \right)^{1/n_g},
\]

where \( n_g = |E(g)| \) is the number of edges of subgraph \( g \).

To distinguish if the intensity of a subgraph is small because all its edges have small weight values or if just one of the weights is too small, the coherence of the subgraph, defined as the ratio between geometric and arithmetic mean of its weights, can be used:

\[
Q(g) = \frac{I(g)n_g}{\sum_{(i,j) \in E(g)} w_{ij}}.
\]

15 Hierarchical Measurements

Using concepts of mathematical morphology [66, 67, 68, 69], it is possible to extend some of the traditional measures and develop new ones [70, 71]. Two fundamental operations of mathematical morphology are dilation and erosion (see Figure 12). Given a subgraph \( g \) of a graph \( G \), the complement of \( g \), denoted \( \overline{g} \) is the subgraph implied by the set of vertices

\[
N(\overline{g}) = N(G) \setminus N(g).
\]

The dilation of \( g \) is the subgraph \( \delta(g) \) implied by the vertices in \( g \) plus the vertices directly connected to a vertex in \( g \). The erosion of \( g \), denoted \( \varepsilon(g) \), is defined as the complement of the dilation of the complement of \( g \):

\[
\varepsilon(g) = \overline{\delta(\overline{g})}.
\]

These operations can be applied repeatedly to generate the \( d \)-dilations and \( d \)-erosions:

\[
\delta_d(g) = \delta(\delta(...(g)...))_{d}
\]

\[
\varepsilon_d(g) = \varepsilon(\varepsilon(...(g)...))_{d},
\]

The first operation converges to the network \( G \) and the second converges to an empty network.

The \( d \)-ring of subgraph \( g \), denoted \( R_d(g) \), is the subgraph implied by the set of vertices

\[
N(\delta_d(g)) \setminus N(\delta_{d-1}(g));
\]

the \( rs \)-ring of \( g \), denoted \( R_{rs}(g) \), is the subgraph implied by

\[
N(\delta_s(g)) \setminus N(\delta_{s-1}(g)).
\]
Figure 12: Example of morphological operations: (a) Dilation: the dilation of the initial subnetwork (black vertices) corresponds to the black and gray vertices; (b) Erosion: the erosion of the original network, given by the black vertices in (a), results in the subnetwork represented by the black vertices in (b).

Note that \( R_d(g) = R_{dd}(g) \). The same definitions can be extended to a single vertex by considering the subgraph implied by that vertex and to an edge by considering the subgraph formed by the edge and the two vertices connected by the edge. In the case of a single vertex \( i \) the simplified notations \( R_d(i) \) and \( R_{rs}(i) \) are used. For example, in Figure 13, \( R_1(15) \) includes the vertices \{8, 16, 14, 17\}; \( R_2(15) \) includes \{1, 13, 18, 19\}; for the graph \( g \) implied by the vertices \{1, 15, 22\} (in black), \( R_1(g) \) includes the vertices in white: \{2, 3, 4, 5, 6, 7, 8, 9, 14, 16, 17, 26, 27\}.

The hierarchical degree of a subgraph \( g \) at distance \( d \), henceforth represented as \( k_d(g) \), can be defined as the number of edges connecting rings \( R_d(g) \) to \( R_{d+1}(g) \).

Another measurement which can be extended hierarchically is the clustering coefficient. Thus, the \( rs \)-clustering coefficient of \( g \), \( C_{rs}(g) \), can be defined as the number of edges in the respective \( rs \)-ring \( n_{rs} \), divided by the total of possible edges between the vertices in that ring, i.e.,

\[
C_{rs}(g) = \frac{2n_{rs}(g)}{|\mathcal{N}(R_{rs}(g))|(|\mathcal{N}(R_{rs}(g))| - 1)}, \tag{57}
\]

where \(|S|\) denotes the cardinality of the set \( S \).

Other possible hierarchical measurements are briefly described in the following. The convergence ratio at distance \( d \), \( C_d(i) \), corresponds to the ratio between the hierarchical vertex degree of a vertex \( i \) at distance \( d - 1 \) and the number of vertices in the ring at distance \( d \); it can be understood as the average number of
Figure 13: The subgraph of interest is defined by black vertices, \( g = \{1, 15, 22\} \). The first hierarchical level of \( g \) is given by the first dilation around \( g \), represented by the white vertices, resulting as first hierarchical degree \( k_1(g) = 12 \). The second hierarchical level is obtained dilating the subnetwork again, represented by the gray vertices, yielding \( k_2(g) = 12 \).

edges received by each vertex in the hierarchical level \( d \) from the previous level,

\[
C_d(i) = \frac{k_{d-1}(i)}{|N(R_d(i))|},
\]

(58)

The *intra-ring degree*, \( A_d(i) \), is obtained by taking the average among the degrees of vertices in the subnetwork \( R_d(i) \); note that only internal ring edges are considered. On the other hand, the *inter-ring degree*, \( E_d(i) \), is defined by the average number of connections between vertices in ring \( R_d(i) \) and those in \( R_{d+1}(i) \) (that is \( E_d(i) = k_d(i)/|N(R_d(i))| \)). Finally, the *hierarchical common degree*, \( H_d(i) \), is the average vertex degree among the vertices in \( R_d(i) \), considering all edges in the original network.

### 16 Subgraph Centrality

Recently, a way to quantify the centrality of a vertex based on the number of subgraphs in which the vertex takes part was proposed \([72]\).

The measure, called *subgraph centrality*, is related with the moments of the adjacency matrix, Eq. (47):

\[
SC_i = \sum_{k=0}^{\infty} \frac{(A^k)_{ii}}{k!},
\]

(59)

where \((A^k)_{ii}\) is the \( i \)th diagonal element of the \( k \)th power of the adjacency matrix \( A \), and the factor \( k! \) assures that the sum converges and that smaller subgraphs have more weight in the sum. In \([72]\) it is shown that the subgraph centrality can
be easily computed from the spectral decomposition of the adjacency matrix:

\[
SC_i = \sum_{j=1}^{N} v_j(i)^2 e^{\lambda_j},
\]

where \(\lambda_j\) is the \(j\)th eigenvalue and \(v_j(i)\) is the \(i\)th element of the \(j\)th eigenvector. This set of eigenvectors should be orthogonalized. The subgraph centrality of a graph is given by [73]:

\[
SC = \frac{1}{N} \sum_{i=1}^{N} SC_i = \frac{1}{N} \sum_{i=1}^{N} e^{\lambda_i}.
\]

17 Complex networks with multiple edge types

Many real world systems and phenomena involve several types of relationships between the same entities. For instance, given a society with fixed individuals, a large number of distinct relationships can be defined, including but not limited to: friendship, disease transmission, religion, family ties and relations at the workplace. A natural approach to represent and model such systems in terms of complex networks involves mapping each individual or entity into respective vertices and having different types of edges to interconnect such vertices. Such a kind of structures are henceforth called complex networks with multiple edge types. Interestingly, relatively little attention has been drawn to such networks.

At least three types of measurements of networks with multiple edge types are possible: (a) derive subgraphs and apply traditional measurements; (b) obtain measurements by combining those obtained in (a); and (c) look for statistics of specific types of paths. These three possibilities are covered in more detail in the following.

Given a complex network with \(N\) vertices and \(E\) edge types, it is possible to obtain \(E\) independent subnetworks, each corresponding to the subgraph defined by the original vertices and the edges of the respective type. Once such subnetworks are obtained, any of the measurements defined for networks with a simple type of edge can be immediately applied and used to characterize the structure under analysis. Such an approach has the immediate advantage of allowing straightforward use of a large number of available measurements. However, it does not take into account topological properties arising from the co-existence of the different types of edges.

A more comprehensive characterization of networks with multiple types of edges can be obtained by combining the measurements obtained for each derived subnetwork. Several approaches are possible. For instance, one may be interested in the relative vertex degree of a vertex with respect to a given type of edge, which can be calculated by dividing the vertex degree of the respectively derived subgraph and the vertex degree considering all types of edges. Relative measurements can also be obtained with respect to two different types of edges.
The third possibility to obtain additional information about the topological properties of a complex network with multiple edges involves specifying types of paths and calculating respective statistics. Examples of specific paths, considering $E = 3$, are:

\[ e_1 e_1 e_2 e_3 e_3 e_3 e_1 e_1 e_2 e_1 e_1 e_3 \]

where $e_i$ stands for an edge of type $i \in \{1, 2, 3\}$.

A particularly effective way to specify such paths is by using formal languages (e.g. [74]). For instance,

\[
\begin{align*}
S & \rightarrow AB \\
A & \rightarrow e_1 A \\
B & \rightarrow e_2 B \\
A & \rightarrow e_1 \\
B & \rightarrow e_2
\end{align*}
\]

It can be easily verified that such a grammar produces sentences $S$ composed by at least one edge $e_1$ followed by at least one edge $e_2$.

Once a reference path (or class of paths) is specified, it becomes interesting to find the number of such paths in the complex network with multiples types of edges of interest. A similar approach can be defined with respect to cycles.

18 Bipartivity Degree

A network is called bipartite if its vertices can be separated in two sets such that a edge exists only between vertices of different sets. It is a known fact that a network is bipartite if and only if it has no loops of odd length. Although some networks are bipartite by construction, like a network with actors and films where they participate, some networks are only approximately bipartite, like a network of sexual contacts. A way to quantify how much a network is bipartite is therefore needed. A possible measure is based on the number of edges between vertices of the same subset in the best possible division [75]:

\[
b = 1 - \frac{n_{\text{same}}}{n},
\]

where $n$ is the total number of edges and $n_{\text{same}}$ is the number of edges between vertices of the same subset. The smallest value of $b$ for all possible divisions is the bipartivity of the network. The problem with this measure is that its computation is NP-complete, due to the necessity of evaluating $b$ for the best possible division. A measure that approximates $b$ but is computationally easier was proposed in [75].
based on a process of marking the minimum possible number of edges as responsible for the creation of loops of odd size.

Another approach is based on the subgraph centrality (Section 16). The subgraph centrality of the network, Eq. (61), is divided in a part due to even closed walks and a part due to odd closed walks (a closed walk is a path, possibly with repetition of vertices, ending on the starting vertex). As odd closed walks are not possible in bipartite networks, the fraction of the subgraph centrality of the network due to even closed walks can be used as the bipartivity degree:

\[ \beta = \frac{SC_{\text{even}}}{SC} = \frac{\sum_{j=1}^{N} \cosh \lambda_j}{\sum_{j=1}^{N} e^{\lambda_j}}. \]  

(63)

19 Edge Reciprocity

For directed networks, it is of interest to know how much reciprocity is shown by the edges, i.e., if vertex \( i \) edges to vertex \( j \), does vertex \( j \) also edge to vertex \( i \)? This helps in a better characterization of the network, can be used to test network models against real networks and gives indication of how much information is lost when the direction of the edges must be discarded (e.g., for the computation of the clustering coefficient).

A standard way to do this is to compute the relation between the number of edges with reciprocal edges, \( n^{\leftrightarrow} \) and the total number of edges \( n \):

\[ \varrho = \frac{n^{\leftrightarrow}}{n}. \] 

(64)

The problem with this relation is that its value is only significative with respect to a random version of the network, as it depends on the average degree.

Garlaschelli and Loffredo proposed the use of the correlation coefficient of the adjacency matrix:

\[ \rho = \frac{\sum_{i \neq j} (a_{ij} - \bar{a})(a_{ji} - \bar{a})}{\sum_{i \neq j} (a_{ij} - \bar{a})^2}, \] 

(65)

where \( \bar{a} \) is the mean value of the elements of the adjacency matrix. This expression reduces to:

\[ \rho = \frac{\rho - \bar{a}}{1 - \bar{a}}. \] 

(66)

This value is an absolute quantity, in the sense that values of \( \rho \) greater than zero imply larger reciprocity than the random version (reciprocal networks), while values below zero imply smaller reciprocity than a random network (antireciprocal networks). Another advantage is that the concept can be easily extended to networks with weight substituting \( w_{ij} \) for \( a_{ij} \) in the above expressions.
20 Cyclic Coefficient

The cyclic structure has attracted much attention in recent studies, since the presence of loops (cycles) can be related to propagation processes in complex networks. The presence of cycles increases the number of paths available for spreading information through the network.

The clustering coefficient accounts for cycles of order three. Kim and Kim \[77\] defined a cyclic coefficient for the measurement of how cyclic a network is. The local cyclic coefficient of a vertex \(i\) is defined as the average of the inverse of the sizes of the smallest cycles formed by vertex \(i\) and its neighbors,

\[
\Theta_i = \frac{2}{k_i(k_i - 1)} \sum_{j,k \in \mathcal{V}(i) \atop j \neq k} \frac{1}{S_{jk}^i},
\]

(67)

where \(S_{jk}^i\) is the size of the smallest cycle which passes through vertices \(i\), \(j\) and \(k\). Note that if vertices \(j\) and \(k\) are connected, the smallest cycle is a triangle and \(S_{jk}^i = 3\). If there is no loop passing through \(i\), \(j\) and \(k\), then there is a tree-like structure among these vertices and \(S_{jk}^i = \infty\). The cyclic coefficient of a network is the average of the cyclic coefficient of all its vertices:

\[
\Theta = \frac{1}{N} \sum_i \Theta_i.
\]

(68)

21 Fractal Dimensions and Network Complexity

Fractals are objects or quantities that display self-similarity in all scales. For complex networks, the concept of self-similarity under a length-scale transformation was not expected because of the small-world property, which shows that the average length of a network increases in logarithmically with the number of vertices.

However, Song et al. \[78\] analyzed complex networks using fractal methodologies and verified that complex networks consist of self-repeating patterns on all length scales.

To measure the fractal dimension of complex networks, a box counting method and a cluster growing method were considered \[78\]. For the first method, the network is covered with \(N_B\) boxes, where all vertices in a box are connected by a minimum distance smaller than \(l_B\). \(N_B\) and \(l_B\) are found to be related by

\[
N_B \sim l_B^{-d_B},
\]

(69)

where \(d_B\) is the fractal box dimension of the network.

For the cluster growing method, a seed vertex is chosen at random and a cluster is formed by vertices distant at most \(l\) from the seed. This process is repeated many times and the average mass of resulting clusters is calculated as a function of \(l\), resulting in the relation

\[
\langle M_c \rangle \sim l^{d_f},
\]

(70)
where the average mass $\langle M_c \rangle$ is defined as the number of vertices in the cluster and $d_f$ is the fractal cluster dimension.

For a network in which the vertices have a typical number of connections, the two exponents are the same. But for scale-free networks this is not the case.

Another scaling relation is found with a renormalization procedure base on the box counting method. A renormalized network is created with each box of the original network transformed in a vertex and a edge between two vertices if at least a edge exists between vertices of the two boxes in the original network. Plotting the degree $k'$ of each vertex of the renormalized network versus the maximum degree $k$ in each box of original network yields:

$$k' \approx l_B^{-d_k} k,$$

(71)

The exponents $\gamma$, $d_B$ and $d_k$ are related by [78]:

$$\gamma = 1 + d_B / d_k.$$  

(72)

Thus, scale-free networks, characterized by the exponent $\gamma$, can also be described by the two length invariant exponents $d_B$ and $d_k$.

Machta and Machta [79] proposed the use of the computational complexity of a parallel algorithm [80] for the generation of a network as a complexity measure for the network model. If a parallel algorithm for the generation of order $O(f(N))$, with $f(x)$ a given function, is known, then the complexity of the network model is also $O(f(N))$. For example, Barabási-Albert networks can be generated in $O(\log \log N)$ parallel steps. Meyer-Ortmanns [81] associates the complexity of the network with the number of topologically non-equivalent graphs generated by splitting vertices and partitioning the edges of the original vertex among the new vertices.

### 22 Planar Networks

An important class of graphs, namely planar graphs, are characterized by the property that they can be drawn on a two-dimensional surface while completely avoiding edge crossing, therefore defining the planar embedding of the graph. Such embeddings define the faces of the graph, namely the empty maximal regions of the plane. A planar graph will always involve one unbounded face. Planar graphs are known to obey Euler’s formula:

$$n - e + f = 2$$

(73)

where $n$ is the number of vertices, $e$ is the number of edges, and $f$ is the number of faces in the graph. Figure 14 illustrates a simple planar graph (a) and one of its possible planar embeddings (b). Since $n = 6$, $e = 11$, and $f = 7$ we have that $n - e + f = 2$. 
Figure 14: A simple planar graph (a), one of its possible planar embeddings (b) and its respective dual (c).

Given a planar embedding of a graph, it is possible to transform it into its dual, which can be obtained by assigning a vertex to each face of the embedding and an edge to correspond to each interface between the faces. An interesting feature of a face in a planar graph regards its length, corresponding to the number of edges along the respective closed walk around the face. Several properties and features of planar graphs are available, including the interesting Kuratowski’s theorem and convex embeddings (e.g. [82]), which can be eventually used to characterize complex networks.

23 Coloring

The main problem of graph coloring is the determination of the number of colors necessary to label vertices or edges of a graph in such a way that adjacent vertices or edges receive different colors. Given a graph $G$ and a set $S$ (whose elements are called colors), a mapping

$$c_v : \mathcal{N}(G) \mapsto S$$

is a vertex coloring of $G$ if it assigns different colors to adjacent vertices, that is, if

$$(i, j) \in \mathcal{E}(G) \Rightarrow c_v(i) \neq c_v(j).$$

Two edges $e_1 = (i_1, j_1), e_2 = (i_2, j_2)$, with $e_1, e_2 \in \mathcal{E}(G)$, are called adjacent, denoted $\text{adj}(e_1, e_2)$, if they have at least a vertex in common

$$\text{adj}(e_1, e_2) \equiv (i_1 = i_2) \vee (i_1 = j_2) \vee (j_1 = i_2) \vee (j_1 = j_2).$$

A mapping

$$c_e : \mathcal{E}(G) \mapsto S$$

is an edge coloring if two adjacent edges have different colors

$$\forall e_1, e_2 \in \mathcal{E}(G), \quad \text{adj}(e_1, e_2) \Rightarrow c_e(e_1) \neq c_e(e_2).$$

In general, techniques for graph coloring concentrate on finding the least number of colors needed to color the vertices or edges of a graph, called respectively
Figure 15: A vertex coloring $V \rightarrow \{1, 2, 3, 4\}$

its **vertex chromatic number** $\chi(G)$ or **edge chromatic number** $\chi'(G)$. An example of graph coloring is shown at Figure 15. When $k = \chi(G)$, the graph is called $k$-**chromatic** and when $k \geq \chi(G)$, it is called $k$-**colorable**.

Some coloring-related properties of interest for the characterization of complex networks involve the relations between colorability and (i) planarity (every planar graph if 4-colorable); (ii) vertex degrees ($\chi(G) \leq k_{\text{max}} + 1$, $k_{\text{max}} \leq \chi'(G) \leq k_{\text{max}} + 1$); and (iii) cycles ($\chi(G) \geq 3$, if only if $G$ has an odd cycle).

Further results and properties related with graph coloring can be seen in graph theory books, e.g. [83, 84, 82].

### 24 Flows

Considering a graph $G$ as a representation of a transportation system, the edges can be labeled with the amount of flow between the two linked vertices. This flow can be characterized by a function $f(i, j)$, with $(i, j) \in E(G)$. If vertex $i$ is sending flow to vertex $j$, then vertex $j$ receives flow from vertex $i$, and therefore:

$$f(i, j) = -f(j, i). \quad (74)$$

Also, for vertices that do not create or consume flow:

$$\sum_{j \in \nu(i)} f(i, j) = 0. \quad (75)$$

A mapping that satisfies Eqs. (74) and (75) is called a **flow**.

Starting with a graph $G$, associate to each of its edges a capacity $c(i, j) \geq 0$\footnote{The case where each direction of an edge can have a different capacity is not considered here.} that fixes the maximum flow possible on the edge:

$$|f(i, j)| \leq c(i, j). \quad (76)$$

Choose now two vertices $s, t$, called source and target, for which Eq. (76) does not need to hold. The total value of a flow, represented as $|f|$, is given by the absolute value:

$$|f| = \sum_{(i, j) \in E(G)} |f(i, j)|.$$
value of the flow from source or target:

\[ |f| = \left| \sum_{j \in \nu(s)} f(s, j) \right| = \left| \sum_{j \in \nu(t)} f(t, j) \right|. \quad (77) \]

The problem of maximum flow is to determine the maximum possible value of \( |f| \) satisfying Eqs. (74), (75) for vertices other than \( s \) and \( t \), and (76). A well known theorem states that the maximum total value of a flow is the same as the minimum capacity of a cut. The capacity of a cut is defined as follows. Consider a partition of the set of vertices of the graph in two subsets \( S \) and \( T \), such that \( s \in S \) and \( t \in T \). The cut of this partition is the set of edges that join vertices in \( S \) to vertices in \( T \). The capacity of the cut is the sum of the capacities of its the edges. The theorem therefore states that the maximum possible flow is limited by the minimum value of the capacity of the cuts determined by all possible partitions of the vertices of \( G \). This shows that the minimum cut capacity, a property of the network and the source and target vertices if the capacity is associated with vertex weights, can be used as a measure of the network ability to transmit information between source and target.

Another property of the network related with flows is its flow number. A \( k \)-flow is a flow \( f \) with values that, additionaly to the previous restrictions Eqs. (74) and (75), the second now valid for all vertices, satisfy

\[ 0 < |f(i, j)| < k, \quad (i, j) \in E(G). \quad (78) \]

Figure 16 shows an example of a flow with \( k = 3 \). The flow number of \( G \), denoted \( \phi(G) \), is the least value of \( k \) for which such a flow is possible. Various results relate the flow number of a graph with its topological properties, specially restriction on the degrees of the vertices (see [83]); unfortunately, these results are in general not useful for complex networks, where the degrees of the edges show no regularity. An interesting result is the equivalence between the chromatic number of a planar graph and the flow number of its dual graph.

Figure 16: A flow obeying Eq. (78) for \( k = 3 \). Numbers on the edges indicate the value of the flow; arrows indicate the direction of positive flow.
25 Perspectives for Future Works

The characterization of the properties of complex networks in terms of an ever growing number of measurements has involved a good deal of efforts from researchers and produced several interesting results. Yet, there is a number of pending important issues related to the characterization of complex networks. Some of the most representative perspectives for further related investigations are discussed in the following.

25.1 Feature Selection

One of the key issues related to the use of measurements to characterize networks is the fact that the adopted sets of measurements are almost invariably degenerated, indicating that the original network can not be completely recovered from the measurements. For instance, there is a virtually infinite number of networks mapping to a same vertex degree distribution. It would be interesting to obtain a comprehensive identification of the combined sets of measurements yielding inversible representation of networks.

Another important related point is the selection of features to be used to best discriminate between a number of given classes of networks. Note that such measurements no longer need to be complete, as it is no longer necessary to recover the networks, but just express the features which are most characteristic and discriminative of each network class. For instance, what is the combination of measurements allowing the best separation between scale-free and small-world models? Such a question is particularly difficult to answer because it involves several preliminary choices and parameters. Insights on such a problem can be obtained from the well-developed (and challenging) area of Pattern Recognition (e.g. [27, 26]).

Figure 17 illustrates the two basic steps involved in a pattern recognition problem. It starts by obtaining a set of measurements or features from the network to be classified. Such features are then fed to a classifier, which produces as output the most likely class.

There are two main types of classifiers: supervised and unsupervised. The former is characterized as having a preliminary training stage where examples of each of the classes of patterns are provided to the system. Unsupervised classification avoids such a stage and tries to organized the presented patterns into coherent clusters. Needless to say, unsupervised classification is normally more challenging than the supervised counterpart. Going back to the question of finding the features yielding the best classification for a given data set, the first important issue regards the original data itself. For instance, it may be enough to consider the straightness of the loglog dispersion of vertex degrees in order to distinguish a scale-free from a random network. However, one will need to seek additional features to distinguish between two alternative scale free classes of networks. A second important point affecting the selection of measurements is the definition of a performance index. Among the several possibilities, one may consider the number of correct classifica-
Figure 17: The classification (or identification) of one or more given networks involve the extraction of some meaningful measurements, their organization into the feature vector, and its subsequent consideration by a classifier in order to produce the respectively inferred classes $C_i$. 

$$F = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_N \end{bmatrix}$$

{\{C_1, C_2, \ldots\}}
tions or the distance between the obtained clusters as opposed to the dispersion of the respective measurements in the feature space. The same set of measurements may often lead to different performance as quantified by such different methods. The third issue affecting the choice of features is the method chosen for classification, because different performances can be obtained by using the same set of features and same performance criterion, but different classifiers. All in all, the identification of the best set of measurements depends on: (i) the properties of the data itself, (ii) the adopted performance index; (iii) the method chosen for classification. One of the main difficulties in pattern recognition stems for the non-trivial dependencies between such elements.

Because of such challenges and lack of definite mathematical guidelines, the systematic investigation of the best set of features has implied the creation of a new research area, namely feature selection (e.g. [85]). Several heuristics and semi-optimal methods have been proposed in order to try to identify the best set of measurements given specific data sets, performance indices, and classifiers. Two particularly interesting classes of methods include the wrapper and filter approaches (e.g. [85]). The former family of approaches consider some measurement of the separability of the classes (e.g. the distance between the cluster averages normalized by their dispersion) as a performance criterion. Wrapper methods take into account the number of correct and incorrect classifications as provided by a specific classifier. Therefore, the main difference between these two families of feature selection methods is the fact that the wrapper approach includes the classifier while the filter does not. Such methods are applied over many sets of measurements, trying to identify that set leading to the best classification.

Generally, it would be particularly useful to obtain the distribution of several measurements for the main network models and data sets. Given the large number of measurements and the algorithmic complexity characterizing several of them, an interesting possibility would be to establish an internet resource containing data in some standard format for the main network models (at several evolution stages) and experimental data, with the possibility of incorporating statistics of respective measurements.

25.2 Dynamical Measurements

One of the distinctive properties of a given complex network is that it often corresponds to one among the several configurations along some network evolution dynamics [86, 87, 88]. For instance, a Barabási-Albert network starts with \( m_0 \) vertices and evolves continuously through the addition of new vertices with \( m \) connections. The immediate consequence is that the measurements of complex networks are a function of time, in the sense that two networks obtained at two distinct instants from the same underlying dynamics will typically be characterized by different features. Although we often only have access to a given configuration obtained from an unknown dynamical evolution, sometimes a series of samples along such an evolution is available. This is typical of systems which can be mon-
itored along time, such as the evolution of citation bases or the internet. In such cases, it becomes particularly interesting to consider measurement trajectories, and not only isolated points, in the feature space as a means for analyzing and classifying networks, because such time sequences of measurements provide a more comprehensive characterization of the networks under analysis.

In addition to considering measurements which are a function of time, it would also be interesting to try to characterize classes of complex networks by considering the dynamics of the respectively defined feature space. More specifically, once the measurements are chosen, the respective feature space can be considered as a phase space where trajectories can be obtained by following the system speed while starting from a given initial point. The dynamics of the trajectories defined in such spaces as a consequence of the evolution of the networks may define attractors and other dynamics features which can be explored as means for better understanding the respective network classes.

25.3 Integration of Multiple Networks

The entities represented as vertices in complex networks often have several properties and relationships of different natures. For instance, people have relationships related to politics, club participation, work relations, nationality, among others. A different network can be obtained for each relationship, yielding multiple sets of complex nets, which are related to the concept of Solomon networks.\[89\]

26 Concluding Remarks

Measurements of the connectivity and topology of complex networks are essential for the characterization, analysis, classification, modeling and validation of complex networks. Although initially limited to simple features such as vertex degree, shortest path and clustering coefficient, several novel and powerful measurements have been proposed in the related literature. We hope it has become clear that the several available measurements, as well as the many new alternatives to come, often provide complementary characterization of distinct connectivity properties of the structures under analysis. It is only by becoming familiar with such measurements that one can expect to identify proper sets of features to be used for the characterization of complex networks of interest. At the same time, it should also have become clear that the topological properties quantified by different measurements have diverse implications and relationship with the dynamics of the systems running on complex networks. For instance, the dynamical evolution of such systems is closely related by the adjacency matrix spectra. All in all, the study of complex networks measurements emphasizes and underlies the shape/function paradigm.\[90\]

Finally, though we have done our best efforts to achieve a comprehensive and practical discussion of measurements, the uncompleteness of this review is guar-
anteed and tends to increase with time. We plan to keep this survey updated, so that suggestions for inclusion will be very welcomed.

Acknowledgments

We are grateful to Ernesto Estrada, Xiang Lee, Adilson E. Motter, Andrea Scharnhorst, Albert Diaz-Guilera, Shalev Itzkovitz and Jon Machta for suggestions. Luciano da F. Costa is grateful to FAPESP (proc. 99/12765-2), CNPq (proc. 308231/03-1) and the Human Frontier Science Program (RGP39/2002) for financial support. Francisco A. Rodrigues is grateful to FAPESP (proc. 04/00492-1).

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### Table 3: Summary of discussed measurements.

| Measurement                                      | Symbol | Equation |
|--------------------------------------------------|--------|----------|
| Mean geodesic distance                           | $\ell$ | (7)      |
| Global efficiency                                | $E$    | (8)      |
| Harmonic mean distance                           | $h$    | (9)      |
| Network clustering coefficient                    | $C$ and $\tilde{C}$ | (10) and (17) |
| Weighted clustering coefficient                   | $C^*$  | (19)     |
| Maximum degree                                    | $k_{\text{max}}$ | (23)     |
| Mean degree of the neighbors                      | $k_{\text{nn}}$ | (25)     |
| Degree-degree correlation coefficient             | $r$    | (26)     |
| Assortativity coefficient                         | $Q$, $R$ | (29) and (30) |
| Vulnerability                                     | $V$    | (32)     |
| Average search information                        | $S$    | (35)     |
| Access information                                | $A_i$  | (50)     |
| Hide information                                  | $H_i$  | (37)     |
| Target entropy                                    | $T$    | (40)     |
| Road entropy                                      | $R$    | (41)     |
| Betweenness centrality                            | $B_i$  | (42)     |
| Closeness centrality                              | $CC$   | (43)     |
| Graph centrality                                  | $GC$   | (44)     |
| Stress centrality                                 | $TC$   | (45)     |
| Hierarchical clustering coefficient               | $C_{rs}$ | (57)     |
| Convergence ratio                                 | $C_d(i)$ | (58)   |
| Subgraph centrality                               | $SC$   | (61)     |
| $k$th moment                                      | $M_k$  | (47)     |
| Modularity                                        | $Q$    | (48)     |
| $z$-score                                         | $Z_i$  | (51)     |
| Participation coefficient                         | $P_i$  | (50)     |
| Significance profile                              | $SP_i$ | (52)     |
| Bipartivity degree                                | $b$ and $\beta$ | (62) and (63) |
| Edge reciprocity                                  | $g$ and $\rho$ | (64) and (65) |
| Cyclic coefficient                                | $\Theta$ | (68)     |
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