Abstract—Classical discrete-time adaptive controllers provide asymptotic stabilization and tracking; neither exponential stabilization nor a bounded noise gain is typically proven. In recent work it has been shown, in both the pole placement stability setting and the first-order one-step-ahead tracking setting, that if the original, ideal, Projection Algorithm is used (subject to the common assumption that the plant parameters lie in a convex, compact set and that the parameter estimates are restricted to that set) as part of the adaptive controller, then a linear-like convolution bound on the closed loop behaviour can be proven; this immediately confers exponential stability and a bounded noise gain, and it can be leveraged to provide tolerance to unmodelled dynamics and plant parameter variation. In this paper we extend the approach to the $d$-step-ahead adaptive controller setting and prove comparable properties.

I. INTRODUCTION

Adaptive control is an approach used to deal with systems with uncertain and/or time-varying parameters. In the classical approach to adaptive control, one combines a linear time-invariant (LTI) compensator together with a tuning mechanism to adjust the compensator parameters to match the plant. The first general proofs that parameter adaptive controllers could work came around 1980, e.g. see [2], [15], [3], [18], and [17]. However, such controllers are typically not robust to unmodelled dynamics, do not tolerate time-variations well, have poor transient behaviour, and do not handle noise (or disturbances) well, e.g. see [19]. During the following two decades a good deal of research was carried out to address these shortcomings. The most common approach was to make small controller design changes, such as the use of signal normalization, deadzones, and $\sigma$-modification, e.g. see [10], [9], [20], [8], [5]. It turns out that simply using projection (onto a convex set of admissible parameters) has proved quite powerful, and the resulting controllers typically provide a bounded-noise bounded-state property, as well as tolerance of some degree of unmodelled dynamics and/or time-variations, e.g. see [23], [24], [16], [22], [21] and [6]. However, in general these controllers provide only asymptotic stability and not exponential stability, with no bounded gain on the noise. Our goal is to investigate the redesign of adaptive controllers so that they have more desirable properties.

Here we return to a common approach in classical adaptive control - the use of a Projection Algorithm based estimator together with a tuneable compensator whose parameters are chosen via the Certainty Equivalence Principle. In the literature it is the norm to use a modified version of the ideal Projection Algorithm in order to avoid division by zero; it turns out that an unexpected consequence of this minor adjustment is that some inherent properties of the scheme are destroyed. In earlier work by the first co-author on the first order setting [11] and in the pole placement setting of [12] and [14], linear-like convolution bounds on the closed-loop behaviour are proven; such bounds are highly desirable and have never before been proven in the adaptive setting. They confer exponential stability and a bounded gain on the noise, and allows a modular approach to analyse robustness and tolerance to time-varying parameters. The objective of the present paper is to use this approach to analyse the $d$-step-ahead adaptive control problem. While we initially expected it to follow in a straight-forward manner from the pole placement setting of [12] and [14], this has not proven to be the case; the difficulty stems from the fact the importance of the system delay in this setting creates significant additional complexity, as does the fact that in this problem there is a tracking objective which is not present in the pole placement problem. We have adopted ideas from [12] and [14] as a starting point, and we have proven the same highly desirable linear-like properties enjoyed in the adaptive pole placement setting.

Before proceeding we present some mathematical preliminaries. Let $\mathbf{Z}$ denote the set of integers, $\mathbf{Z}^+$ the set of non-negative integers, $\mathbf{N}$ the set of natural numbers, $\mathbf{R}$ the set of real numbers, and $\mathbf{R}^+$ the set of non-negative real numbers. We use the Euclidean 2-norm for vectors and the corresponding induced norm for matrices, and denote the norm of a vector or matrix by $\| \cdot \|$. We let $l_\infty(\mathbf{R}^p)$ denote the set of $\mathbf{R}^p$-valued bounded sequences.

If $\mathcal{S} \subset \mathbf{R}^p$ is a convex and compact set, we define $\| \mathcal{S} \| := \max_{x \in \mathcal{S}} \| x \|$ and the function $\pi_{\mathcal{S}} : \mathbf{R}^p \rightarrow \mathcal{S}$ denotes the projection onto $\mathcal{S}$; it is well-known that $\pi_{\mathcal{S}}$ is well-defined.

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2An exception is the work of Ydstie [23], [24], who considers the ideal Projection Algorithm as a special case; however, a crisp bound on the effect of the initial condition and a convolution bound on the effect of the exogenous inputs are not proven. Another notable exception is the work of Akhtar and Bernstein [1], where they are able to prove Lyapunov stability; however, they do not prove a convolution bound on the effect of the exogenous inputs either, and they assume that the high frequency gain is known.
II. THE SETUP

In this paper we start with a linear time-invariant discrete-time plant described by

\[ \sum_{i=0}^{n} a_i y(t-i) = \sum_{i=0}^{m} b_i u(t-d-i) + w(t), t \in \mathbb{Z}, \]  

(1)

with

- \( y(t) \in \mathbb{R} \) the measured output,
- \( u(t) \in \mathbb{R} \) the control input,
- \( w(t) \in \mathbb{R} \) the disturbance (or noise) input;
- the parameters regularized so that \( a_0 = 1 \), and
- the system delay is exactly \( d \), i.e. \( b_0 \neq 0 \).

Associated with this plant model are the polynomials \( A(z^{-1}) := \sum_{i=0}^{n} a_i z^{-i} \) and \( B(z^{-1}) := \sum_{i=0}^{m} b_i z^{-i} \), as well as the transfer function \( z^{-d} \frac{B(z^{-1})}{A(z^{-1})} \) and the list of plant parameters:

\[ \theta_{ab}^* := [a_1 \cdots a_n \ b_0 \cdots b_m]^T. \]

It is assumed that \( \theta_{ab}^* \) lies in a known set \( S_{ab} \subset \mathbb{R}^{n+m+1} \).

Remark 1: It is straightforward to verify that if the system has a disturbance at both the input and output, then it can be converted to a system of the above form.

The goal is closed-loop stability and asymptotic tracking of an exogenous reference input \( y^*(t) \). We impose several assumptions on the set of admissible parameters.

Assumption 1: The parameter set \( S_{ab} \) is compact, and for each \( \theta \in S_{ab} \), the corresponding polynomial \( B(z^{-1}) \)

- has all of its zeros in the open unit disk, and
- the sign of \( b_0 \) is always the same.

Remark 2: We have implicitly assumed knowledge of the system delay \( d \) as well as upper bounds on the order of \( A(z^{-1}) \) and \( B(z^{-1}) \).

The boundedness requirement on \( S_{ab} \) is quite reasonable in practical situations; it is used here to prove uniform bounds and decay rates on the closed-loop behaviour. The constraint on the zeros of \( B(z^{-1}) \) is a requirement that the plant be minimum phase; this is necessary to ensure tracking of an arbitrary bounded reference signal [13]. Knowledge of the sign of \( b_0 \) is a common one in adaptive control [4].

To proceed we use a parameter estimator together with an adaptive \( d \)-step-ahead control law. To design the estimator it is convenient to put the plant into the so-called predictor form. To this end, following [4], we carry out long division by dividing \( A(z^{-1}) \) into one, and define \( F(z^{-1}) = \sum_{i=0}^{d-1} f_i z^{-i} \) and \( G(z^{-1}) = \sum_{i=0}^{n-1} g_i z^{-i} \) satisfying

\[ \frac{1}{A(z^{-1})} = F(z^{-1}) + \frac{z^{-d} G(z^{-1})}{A(z^{-1})}. \]

Hence, if we define

\[ \beta(z^{-1}) = \sum_{i=0}^{m+d-1} \beta_i z^{-i} := F(z^{-1})B(z^{-1}), \]

\[ \alpha(z^{-1}) = \sum_{i=0}^{n-1} \alpha_i z^{-i} := G(z^{-1}), \]

then we can rewrite the plant model as

\[ y(t+d) = \sum_{i=0}^{n-1} \alpha_i y(t-i) + \sum_{i=0}^{m+d-1} \beta_i u(t-i) + \bar{w}(t), t \in \mathbb{Z}. \]

(2)

Let \( S_{\alpha \beta} \) denote the set of admissible \( \theta^* \) which arise from the original plant parameters which lie in \( S_{ab} \); since the associated mapping is continuous, it is clear that the compactness of \( S_{ab} \) means that \( S_{\alpha \beta} \) is compact as well. Furthermore, it is easy to see that \( f_0 = 1 \), so \( \beta_0 = b_0 \), which means that the sign of \( b_0 \) is always the same. It is convenient that the set of admissible parameters in the new parameter space be convex and closed; so at this point let \( S \subset \mathbb{R}^{n+m+d} \) be any compact and convex set containing \( S_{\alpha \beta} \) for which the \( n+1 \)th element (the one which corresponds to \( \beta_0 \)) is never zero, e.g. the convex hull of \( S_{\alpha \beta} \) would do.

The \( d \)-step-ahead control law is the one given by

\[ y^*(t+d) = \sum_{i=0}^{n-1} \alpha_i y(t-i) + \sum_{i=0}^{m+d-1} \beta_i u(t-i); \]

in the absence of a disturbance, and assuming that this controller is applied for all \( t \in \mathbb{Z} \), we have \( y(t) = y^*(t) \) for all \( t \in \mathbb{Z} \). Of course, if the plant parameters are unknown, we need to use estimates; also, the adaptive version of the \( d \)-step-ahead control law is only applied after some initial time, i.e. for \( t \geq t_0 \).

A. Initialization

In most adaptive controllers the goal is to prove asymptotic results, so the details of the initial condition is unimportant. Here, however, we wish to get a bound on the transient behaviour so we must proceed carefully. If we wish to solve (2) for \( y(t) \) starting at time \( t_0 \), it is clear that we need an initial condition of

\[ x_0 := [y(t_0-1) \cdots y(t_0-n-d+1) \ u(t_0-1) \cdots u(t_0-m-2d+1)]^T. \]

B. Parameter Estimation

We can rewrite the plant (2) as

\[ y(t+1) = \phi(t-d+1)^T \theta^* + \bar{w}(t-d+1), t \geq t_0 - 1. \]

(3)

Given an estimate \( \hat{\theta}(t) \) of \( \theta^* \) at time \( t \), we define the prediction error by

\[ e(t+1) := y(t+1) - \phi(t-d+1)^T \hat{\theta}(t); \]
this is a measure of the error in \( \hat{\theta}(t) \). A common way to obtain a new estimate is from the solution of the optimization problem
\[
\arg\min_{\theta} \{ \| \theta - \hat{\theta}(t) \| : y(t + 1) = \phi(t - d + 1)^T \theta \},
\]
yielding the ideal (projection) algorithm
\[
\hat{\theta}(t + 1) = \begin{cases} 
\hat{\theta}(t) + \frac{\phi(t-d+1)}{\|\phi(t-d+1)\|^2} e(t + 1) & \text{if } \phi(t-d+1) = 0 \\
\hat{\theta}(t) & \text{otherwise;}
\end{cases}
\]
(4)
at this point, we can also restrain it to \( S \) by projection. Of course, if \( \phi(t-d+1) \) is close to zero, numerical problems can occur, so it is the norm in the literature (e.g. [3] and [4]) to add a constant to the denominator, but as pointed out in [11], [12], and [14], this can lead to the loss of exponential stability and a loss of a bounded gain on the noise. We propose a middle ground: as proposed in [12] and [14], we turn off the estimation if it is clear that the disturbance signal \( \bar{w}(t) \) is swamping the estimation error. To this end, with \( \delta \in (0, \infty) \), we turn off the estimator if the update is larger than \( 2\|S\| + \delta \) in magnitude; so define \( \rho_\delta : \mathbb{R}^{n+m+d} \times \mathbb{R} \rightarrow \{0, 1\} \) by
\[
\rho_\delta(\phi(t-d+1), e(t + 1)) :=
\begin{cases} 
1 & \text{if } |e(t + 1)| < (2\|S\| + \delta)\|\phi(t-d+1)\| \\
0 & \text{otherwise;}
\end{cases}
\]
given \( \hat{\theta}(t_0 - 1) = \theta_0 \), for \( t \geq t_0 - 1 \) we define
\[
\hat{\theta}(t + 1) = \hat{\theta}(t) + \rho_\delta(\phi(t-d+1), e(t + 1)) \times 
\frac{\phi(t-d+1)}{\|\phi(t-d+1)\|^2} e(t + 1), \tag{5}
\]
which we then project onto \( S \):
\[
\hat{\theta}(t + 1) := \pi_\delta(\hat{\theta}(t + 1)). \tag{6}
\]

C. Properties of the Estimation Algorithm

Analysing the closed-loop system will require a careful analysis of the estimation algorithm. We define the parameter estimation error by \( \varepsilon(t) := \hat{\theta}(t) - \theta^* \) and the corresponding Lyapunov function associated with \( \hat{\theta}(t) \), namely \( V(t) := \hat{\theta}(t)^T \hat{\theta}(t) \). In the following result we list a property of \( V(t) \); it is a straightforward generalization of what holds in the pole placement setup of [12] and [14].

\footnote{If \( \delta = \infty \), then we adopt the understanding that \( \infty \times 0 = 0 \), in which case this formula collapses into the original one.}

**Proposition 1:** For every \( t_0 \in \mathbb{Z} \), \( \phi_0 \in \mathbb{R}^{n+m+d} \), \( \theta_0 \in S \), \( \theta_{ab}^* \in S_{ab} \), \( y^* \in L_\infty \), and \( \delta \in (0, \infty) \), when the estimator (5) and (6) is applied to the plant (1), the following holds:
\[
\| \hat{\theta}(t + 1) - \hat{\theta}(t) \| \leq \rho_\delta(\phi(t-d+1), e(t + 1)) \times 
\frac{|e(t + 1)|}{\|\phi(t-d+1)\|^2}, \quad t \geq t_0 - 1, \tag{7}
\]
\[
V(t) \leq V(t_0 - 1) + \sum_{j=t_0-1}^{t-1} \rho_\delta(\phi(j-d+1), e(j + 1)) \times
\left[ \frac{1}{2} \frac{|e(j + 1)|^2}{\|\phi(j-d+1)\|^2} + \frac{1}{2} \frac{|\bar{w}(j-d+1)|^2}{\|\phi(j-d+1)\|^2} \right], \quad t \geq t_0 - 1.
\]

D. The Control Law

The elements of \( \hat{\theta}(t) \) are partitioned in a natural way as
\[
[ \hat{\alpha}_0(t) \cdots \hat{\alpha}_{n-1}(t) \quad \hat{\beta}_0(t) \cdots \hat{\beta}_{m+d-1}(t) ]^T.
\]
The one-step-ahead adaptive control law is that of
\[
y^*(t + d) = \hat{\theta}(t)^T \phi(t), \quad t \geq t_0,
\]
or equivalently
\[
\sum_{i=0}^{m+d-1} \hat{\beta}_i(t) u(t-i) = y^*(t + d) - \sum_{i=0}^{n-1} \hat{\alpha}_i(t) y(t-i). \tag{8}
\]
Hence, as is common in this setup, we assume that the controller has access to the reference signal \( y^*(t) \) exactly \( d \) time units in advance.

**Remark 3:** With this choice of control law, it is easy to prove that the prediction error \( e(t) \) and the tracking error \( \varepsilon(t) := y^*(t) - y(t) \) are different if \( d \neq 1 \). Indeed, it is easy to see that
\[
\varepsilon(t) = -\phi(t-d)^T \hat{\beta}(t-d) + \bar{w}(t-d), \quad t \geq t_0 + d, \tag{9}
\]
\[
epsilon(t) = -\phi(t-d)^T \hat{\beta}(t-1) + \bar{w}(t-d), \quad t \geq t_0. \tag{10}
\]

The goal of this paper is to prove that the adaptive controller consisting of the estimator (5-6) together with the control equation (8) yields highly desirable linear-like convolution bounds on the closed-loop behaviour. While the approach is similar to that in our earlier work [12] and [14], it requires a much more nuanced analysis. In the next section we develop several models used in the development, after which we state and prove the main result.

III. PRELIMINARY ANALYSIS

A. A Good Closed-Loop Model

In our pole-placement adaptive control setup [12], [14], a key closed-loop model consists of an update equation for \( \phi(t) \), with the state matrix consisting of controller and plant estimates; this was effective - the characteristic polynomial of this matrix is time-invariant and has all roots in the open unit disk. If we were to mimic this in the one-step-ahead setup, the characteristic polynomial would have roots which are time-varying, with some at zero and the rest at
the zeros of \( \hat{\beta}(t, z^{-1}) \), which is time-varying and may not have roots in the open unit disk. Hence, at this point we make an important deviation from the approach of [12] and [14] and construct the following update equation for \( \phi(t) \) which avoids the use of plant parameter estimates, but is driven by the tracking error. Only two elements of \( \phi \) have a complicated description:

\[
\phi_1(t+1) = y(t+1) = \varepsilon(t+1) + y^*(t+1),
\]

and the \( u(t+1) \) term, for which we use the original plant model to write:

\[
\phi_{n+1}(t+1) = u(t+1)
\]

\[
= \frac{1}{b_0} \left( \sum_{i=0}^{d} a_t \varepsilon(t+d+1-i) + y^*(t+d+1-i) \right) + \sum_{i=d+1}^{n} a_i y(t+d+1-i) - \sum_{i=0}^{d} b_{t+1} u(t-i) - w(t+d+1)].
\]

With \( e_i \in \mathbb{R}^{n+m} \) the \( i \)th normal vector, if we now define

\[
B_1 := e_1, \quad B_2 := e_{n+1},
\]

then it is easy to see that there exists a matrix \( A_g \in \mathbb{R}^{(n+m+d) \times (n+m+d)} \) so that the following equation holds:

\[
\phi(t+1) = A_g \phi(t) + B_1 \varepsilon(t+1) + B_2 \sum_{j=0}^{d} \frac{a_{d-j}}{b_0} \varepsilon(t+1+j) + \frac{a_{d-j}}{b_0} y^*(t+1+j) \]

\[
= B_1 y^*(t+1) - \frac{1}{b_0} B_2 w(t+d+1), \quad t \geq t_0 - 1.
\]

The characteristic equation of \( A_g \) equals \( \frac{1}{b_0} z^{n+m+d} B(z^{-1}) \), so all of its roots are in the open unit disk.

**B. A Crude Closed-Loop Model**

At times we will need to use a crude model to bound the size of the growth of \( \phi(t) \) in terms of the exogenous inputs. Once again, only two elements of \( \phi(t) \) have a complicated description: to describe \( y(t+1) \) we use the plant model:

\[
\phi_1(t+1) = y(t+1)
\]

\[
= - \sum_{i=1}^{n} a_i y(t+1-i) + \sum_{i=0}^{m} b_i u(t+1-d-i) + w(t+1)
\]

\[
= \tilde{\theta}_{ab}^* \phi(t) + w(t+1),
\]

and to describe \( u(t+1) \) we use the control law:

\[
y^*(t+d) = \hat{\theta}(t)^T \phi(t)
\]

\[
y^*(t+d+1) = \hat{\theta}(t+1)^T \phi(t+1), \quad t \geq t_0 - 1;
\]

it is easy to define \( \tilde{\theta}_{\alpha \beta}(t) \) in terms of the elements of \( \hat{\theta}(t+1) \) so that

\[
y^*(t+d+1) = \tilde{\theta}_{\alpha \beta}(t)^T \phi(t) + \hat{\theta}_{\alpha \beta}(t) y^*(t+d+1)
\]

\[
\hat{\theta}_{\alpha \beta}(t+1) y^*(t+d+1) + \tilde{\theta}_{\alpha \beta}(t+1) u(t+1), \quad t \geq t_0 - 1.
\]

If we combine this with the formula for \( y(t+1) \) above, we end up with

\[
\frac{1}{\beta_0(t+1)} y^*(t+d+1) - \frac{\tilde{\theta}_{\alpha \beta}(t+1)}{\beta_0(t+1)} u(t+1), \quad t \geq t_0 - 1.
\]

Hence, we can define matrices \( A_0(t), B_3(t) \) and \( B_4(t) \) so that

\[
\phi(t+1) = A_0(t) \phi(t) + B_3(t) y^*(t+d+1) + B_4(t) u(t+1), \quad t \geq t_0 - 1;
\]

\[
\text{due to the compactness of } S_{ab}, \ S_{\alpha \beta}, \text{and } S, \text{the following is immediate:}
\]

**Proposition 2:** There exists a constant \( c_1 \) so that for every \( t_0 \in \mathbb{Z}, \phi_0 \in \mathbb{R}^{n+m+d}, \theta_0 \in S, \theta_{ab}^* \in S_{ab}, \ y^*, \ w \in l_{\infty}, \text{and } \delta \in (0, \infty), \text{when the adaptive controller (5), (8) and (8) is applied to the plant (1), the following holds:}

\[
\|A_0(t)\| \leq c_1, \quad \|B_3(t)\| \leq c_1, \quad \|B_4(t)\| \leq c_1, \quad t \geq t_0 - 1.
\]

**C. A Better Closed-Loop Model**

The good closed-loop model (12) is driven by future tracking error signals. We can now combine this with the crude closed-loop model (13) to create a new model which is driven by perturbed versions of the present and past values of \( \phi \), with the weights associated with the parameter update law. To this end, first define

\[
\nu(t-1) := \rho_\delta(\phi(t-1)), \quad \phi(t) \times \frac{\phi(t-d)}{[\phi(t-d)]^2} \varepsilon(t), \quad t \geq t_0.
\]

The following result plays a pivotal role in the analysis of the closed-loop system.

**Proposition 3:** There exists a constant \( c_2 \) so that for every \( t_0 \in \mathbb{Z}, \phi_0 \in \mathbb{R}^{n+m+d}, \theta_0 \in S, \theta_{ab}^* \in S_{ab}, \ y^*, \ w \in l_{\infty}, \text{and } \delta \in (0, \infty), \text{when the adaptive controller (5), (8) and (8) is applied to the plant (1), the following holds:}

\[
\phi(t+1) = A_g \phi(t) + \sum_{j=0}^{d-1} \Delta_j(t) \phi(t-j) + \eta(t), \quad t \geq t_0 + d-1
\]

with

\[
\|\eta(t)\| \leq c_2(1 + \nu(t+2) + \cdots + \nu(t+d+1)) \times \left( \sum_{j=1}^{d-1} |y^*(t+j)| + \sum_{j=1}^{d} (|w(t+j)| + |\hat{w}(t+j)|) \right)
\]

and

\[
\|\Delta_j(t)\| \leq c_2(\nu(t - d + 2) + \cdots + \nu(t + d + 1)), \quad j = 0, ..., d - 1.
\]

**Proof:** See the Appendix. □
To make the model of Proposition 3 amenable to analysis, we define a new extended state variable and associated matrices:

$$
\tilde{\phi}(t) := \begin{bmatrix} 
\phi(t) \\
\phi(t-1) \\
\vdots \\
\phi(t-d+1) 
\end{bmatrix}, \quad A_{nom} = \begin{bmatrix} 
A_g & I \\
& & \ddots \\
& & & I 
\end{bmatrix}
$$

(14)

and

$$
B_1 := \begin{bmatrix} 
I \\
0 \\
\vdots \\
0 
\end{bmatrix}, \quad \Delta(t) = \begin{bmatrix} 
\Delta_0(t) & \Delta_1(t) & \cdots & \Delta_{d-1}(t) \\
0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 
\end{bmatrix}
$$

(15)

which gives rise to a state-space model which will play a key role in our analysis:

$$
\tilde{\phi}(t+1) = [A_{nom} + \Delta(t)]\tilde{\phi}(t) + B_1 \eta(t), \quad t \geq t_0 + d - 1.
$$

(16)

Now $A_g$ arises from $\theta_{ab}^* \in S_{ab}$, and lies in a corresponding compact set $A$; furthermore, its eigenvalues are at the zeros of $B(z^{-1})$ which has all of its roots in the open unit disk, so we can use classical arguments to prove that there exists $\gamma_1$ and $\sigma \in (0, 1)$ so that

$$
\|A_{nom}\| \leq \gamma_1 \sigma^i, \quad i \geq 0.
$$

Indeed, we can choose any $\sigma$ larger than

$$
\Delta := \max_{\theta_{ab}^* \in S_{ab}} \{ |\lambda| : B(\lambda^{-1}) = 0 \}.
$$

Equations of the form given in (16) arise in classical adaptive control approaches; the following proposition follows easily from the lemma of Kreisselmeier [7].

**Proposition 4:** Consider the discrete-time system with $\Phi(t, \tau)$ denoting the state transition matrix corresponding to $A_{nom} + \Delta(t)$. Suppose that there exist constants $\beta_1 \geq 0$ so that for all $t > \tau \geq t_0 + d - 1$ we have

$$
\sum_{i=\tau}^{t-1} \|\Delta(i)\| \leq \beta_0 + \beta_1 (t-\tau)^{1/2} + \beta_2 (t - \tau),
$$

and there exists a $\mu \in (\sigma, 1)$ and $N \in \mathbb{N}$ satisfying

$$
\beta_2 \leq \frac{1}{\gamma_1} \left(\frac{\mu}{\gamma_1 - \sigma}\right)^N.
$$

Then there exists a constant $\gamma_2$ so that the transition matrix satisfies

$$
\|\Phi(t, \tau)\| \leq \gamma_2 \mu^{t-\tau}, \quad t \geq \tau.
$$
Proposition 1 simplifies to
\[
\left( \sum_{j=1}^{d+1} |y^* (t+j)| + \sum_{j=-2d+1}^{d+1} (|w(t+j)| + |\tilde{w}(t)|) \right) =: \tilde{w}(t)
\]
and
\[
\| \Delta(t) \| \leq c_2 \nu(t-d+2) + \cdots + \nu(t+d+1),
\]
\[
t \geq t_0 + d - 1.
\]
Before proceeding, we provide a useful preliminary result; it follows immediately from Proposition 2.

**Claim 1:** There exists a constant \( \gamma_3 \) so that
\[
\| \phi(t+i) \| \leq \gamma_3 \| \phi(t) \| + \gamma_3 \sum_{j=1}^{t-1} |y^* (t+j)| + |w(t+j)|
\]
for \( t \geq t_0 - 1 \) and \( i = 1, \ldots, 2d \).

In order to apply Proposition 4, we need to compute a bound on a sum; the following result provides an avenue.

**Claim 2:** There exists a constant \( \gamma_4 \) so that for every \( t_2 > t_1 \geq t_0 + d - 1 \),
\[
\sum_{j=t_1}^{t_2} \| \Delta(j) \| \leq \gamma_4 \sum_{j=t_1-d+2}^{t_2+d} \nu(j)^2/2 (t_2 - t_1)^{1/2}.
\]

**Proof:** It follows from \((19)\) that
\[
\sum_{j=t_1}^{t_2} \| \Delta(j) \| \leq 2c_2 d \sum_{j=t_1-d+2}^{t_2+d} \nu(j).
\]
If we apply the Cauchy-Schwarz inequality and observe that \( (t_2 - t_1)^{1/2} \leq (2d)^{1/2} (t_2 - t_1 + 2d - 2)^{1/2} \), then the result follows.

At this point we consider two cases: the easier case in which there is no noise, and the harder case in which there is noise.

**Case 1:** \( w(t) = 0, t \geq t_0 - d \).

Using the definition of \( \nu(j) \), the bound on \( V(t) \) given by Proposition 1 simplifies to
\[
V(t) \leq V(t_0 - 1) - \frac{1}{2} \sum_{j=t_0-1}^{t-1} \nu(j)^2, t \geq t_0.
\]

Since \( V(\cdot) \geq 0 \) and \( V(t_0 - 1) = \| \theta_0 - \theta^* \|^2 \leq 4 \| S \|^2 \), this means that
\[
\sum_{j=t_0-1}^{t-1} \nu(j)^2 \leq 8 \| S \|^2, t \geq t_0.
\]

From Claim 2 we conclude that
\[
\sum_{j=t_1}^{t_2} \| \Delta(j) \| \leq 8^{1/2} \gamma_4 \| S \| (t_2 - t_1)^{1/2}, t_2 > t_1 \geq t_0 + d.
\]

Now we apply Proposition 4: we set
\[
\beta_0 = \beta_2 = 0, \ \beta_1 = 8^{1/2} \gamma_4 \| S \|, \ \mu = \lambda.
\]
It is now trivial to choose \( N \in \mathbb{N} \) so that \( \frac{1}{\gamma_4} - \lambda_1 > 0 \), namely \( N = \min \{ \ln(\gamma_4)/\ln(\lambda_1) \} + 1 \), which means that \( \beta_2 < \gamma_4 (\frac{1}{\gamma_4} - \lambda_1) \). From Proposition 4 we see that there exists a constant \( \gamma_2 \) so that the state transition matrix \( \Phi(t, \tau) \) corresponding to \( \lambda_{nom} + \lambda(t) \) satisfies
\[
\| \Phi(t, \tau) \| \leq \gamma_2 \lambda_{nom} - \tau, t \geq \tau \geq t_0 + d.
\]
Also, we see from \((18), (20)\) and Proposition 3 that
\[
\| \tilde{w}(t) \| \leq c_2 (1 + 8^{1/2} \| S \|) \tilde{w}(t).
\]
If we now apply this to \((16)\), we see that there exists a constant \( \gamma_5 \) so that
\[
\| \tilde{w}(k) \| \leq \gamma_5 \lambda_{nom} - k \| \tilde{w}(t_0 + d) \| + \sum_{j=t_0+d}^{k-1} \gamma_5 \lambda_{nom} - k \| \tilde{w}(j) \|, k \geq t_0 + d.
\]
At this point we can use Claim 1 to find a bound on \( \| \tilde{w}(t_0 + d) \| \) in terms of \( x_0, y^* \) and \( w \); if we convert the bounds on \( \tilde{w}(j) \) to bounds on \( |w(t)| \) and \( |y^*(t)| \), then \((12)\) holds for this case.

**Case 2:** \( w(t) \neq 0 \) for some \( t \geq t_0 - d \).

This case is much more involved since noise can radically affect parameter estimation. Indeed, even if the parameter estimate is quite accurate at a point in time, the introduction of a large noise signal (large relative to the size of \( \phi(t) \)) can create a highly inaccurate parameter estimate. Following \([12, 14]\), we partition the timeline into two parts: one in which the noise is small versus \( \phi \) and one where it is not. To this end, with \( \xi > 0 \) to be chosen shortly, partition \( \{ j \in \mathbb{Z} : j \geq t_0 \} \) into \( S_{good} \) and \( S_{bad} \), respectively:
\[
\{ j \geq t_0 : \phi(j-d+1) \neq 0 \} \quad \text{and} \quad \| \tilde{w}(j-d+1) \|^2 < \xi, \quad \| \phi(j-d+1) \|^2 \leq 4 \| S \|^2.
\]
\[
\{ j \geq t_0 : \phi(j-d+1) = 0 \} \quad \text{or} \quad \| \tilde{w}(j-d+1) \|^2 \geq \xi; \quad \| \phi(j-d+1) \|^2 \geq 4 \| S \|^2.
\]
clearly \( \{ j \in \mathbb{Z} : j \geq t_0 \} = S_{good} \cup S_{bad} \). Observe that this partition clearly depends on \( \theta_0, \theta^* \), etc. We will apply Proposition 4 to analyse the closed-loop system behaviour on \( S_{good} \); on the other hand, we will easily obtain bounds on the system behaviour on \( S_{bad} \). Before doing so, following \([12, 14]\), we partition the time index \( \{ j \in \mathbb{Z} : j \geq t_0 \} \) into intervals which oscillate between \( S_{good} \) and \( S_{bad} \). To this end, it is easy to see that we can define a (possibly infinite) sequence of intervals of the form \( \{ k_j, k_{j+1} \} \) satisfying: (i) \( k_1 = t_0 \); (ii) \( \{ k_j, k_{j+1} \} \) either belongs to \( S_{good} \) or \( S_{bad} \); and (iii) if \( k_{j+1} \neq \infty \) and \( \{ k_j, k_{j+1} \} \) belongs to \( S_{good} \) (respectively, \( S_{bad} \)), then the interval \( [k_{j+1}, k_{j+2}] \) must belong to \( S_{bad} \) (respectively, \( S_{good} \)).

Now we analyse the behaviour during each interval.

**Sub-Case 2.1:** \( \{ k_j, k_{j+1} \} \) lies in \( S_{bad} \).
Let $j \in [k_i, k_{i+1})$ be arbitrary. In this case either $\phi(j - d + 1) = 0$ or $\frac{|\tilde{w}(j - d + 1)|^2}{\|\phi(j - d + 1)\|^2} \geq \xi$ holds. In either case we have
\[\|\phi(j - d + 1)\| \leq \frac{1}{\xi^{1/2}}|\tilde{w}(j - d + 1)|, \ j \in [k_i, k_{i+1}). \quad (21)\]
From (13) and Proposition 2 we have
\[\|\phi(j - d + 2)\| \leq \frac{c_1}{\xi^{1/2}}|\tilde{w}(j - d + 1)| + c_1|y^*(j + 1)| + c_1|w(j + 1)|, \ j \in [k_i, k_{i+1}). \quad (22)\]
If we combine this with (21) we end up with
\[\|\phi(j)\| \leq \begin{cases} \frac{1}{\xi^{1/2}}|\tilde{w}(j)| & \text{if } j = k_i \\ c_1(1 + \frac{1}{\xi^{1/2}})|\tilde{w}(j - 1)| & \text{if } j = k_i + 1, \ldots, k_{i+1}. \end{cases} \quad (23)\]

**Sub-Case 2.2:** $[k_i, k_{i+1})$ lies in $S_{good}$.

This case is much more involved than in the proof of [12] and [14] since the bound on $\|\Delta(t)\|$ provided by Claim 2 extends both forward and backward in time, occasionally outside $S_{good}$. Hence, we need to handle the first $d$ and last $d$ time units separately.

To this end, first suppose that $k_{i+1} \leq k_i + 2d$. Then by Claim 1 we see that there exists a constant $\gamma_9$ so that
\[\|\phi(k)\| \leq \gamma_9\lambda^{k-k_i}\|\phi(k_i)\| + \sum_{j=k_i}^{k_{i+1}} \gamma_9\lambda^{j-k_i}|\tilde{w}(j)|, \ k_i \leq k \leq k_{i+1}. \quad (24)\]
Now suppose that $k_{i+1} > k_i + 2d$. Define $\bar{k}_i = k_i + d$ and $\bar{k}_{i+1} = k_{i+1} - d$. Let $j \in [k_i, k_{i+1})$ be arbitrary; then
\[\rho_8(\phi(j - d + 1), e_{\tau}(j + 1))^2 \|\tilde{w}(j - d + 1)\|^2 < \xi. \quad (25)\]
Combining this with Proposition 1 we have that
\[\sum_{j=\bar{k}}^{\bar{k}_{i+1}} \nu(j)^2 \leq 8\|S\|^2 + 4\xi(\bar{k} - \bar{k}_i), \ \bar{k}_i \leq \bar{k} < \bar{k}_i \leq \bar{k}_{i+1}. \quad (26)\]
From Claim 2 there exists a constant $\gamma_6$ so that
\[\sum_{j=\bar{k}}^{\bar{k}_{i+1}} \|\Delta(j)\| \leq \gamma_6(\bar{k} - \bar{k}_i)^{1/2} + 2\gamma_6\xi^{1/2}(\bar{k} - \bar{k}_i), \quad \bar{k}_i \leq \bar{k} < \bar{k}_i \leq \bar{k}_{i+1}. \quad (27)\]
Now we will apply Proposition 4: we set
\[\beta_0 = 0, \ \beta_1 = \gamma_6, \ \beta_2 = \gamma_6\xi^{1/2}, \ \lambda = \lambda. \]
With $N$ chosen as in Case 1, we have that $\hat{\lambda} := \frac{1}{\gamma_6}\lambda_1 > 0$; we need $\beta_2 < \frac{1}{\hat{\lambda}^2}\gamma_6$, which will certainly be the case if we set $\xi := \frac{\delta}{2\gamma_6^2}$. From Proposition 4 we see that there exists a constant $\gamma_7$ so that the state transition matrix $\Phi(t, \tau)$ corresponding to $A_{nom} + \Delta(t)$ satisfies
\[\|\Phi(t, \tau)\| \leq \gamma_7\lambda^{\tau - t}, \ \bar{k}_i \leq \tau \leq t \leq \bar{k}_{i+1}. \quad (28)\]
Hence, we see from (13) and (26) that
\[|\eta(t)| \leq c_2(1 + 8^{1/2}\|S\|^2 + 4\xi d)|\tilde{w}(t)|, \ \bar{k}_i \leq t \leq \bar{k}_{i+1}; \quad (29)\]
if we now apply this to (10) then we see that there exists a constant $\gamma_8$ so that
\[|\tilde{\phi}(k)\| \leq \gamma_8\lambda^{k-k_i}|\tilde{\phi}(\bar{k}_i)\| + \sum_{j=\bar{k}_i}^{k_{i+1}} \gamma_8\lambda^{k-j-1}|\tilde{w}(j)| \quad (30)\]
for $\bar{k}_i \leq k \leq \bar{k}_{i+1}$. We can use Claim 1 to extend the bound to the rest of $[k_i, k_{i+1})$: there exists a constant $\gamma_9$ so that
\[|\tilde{\phi}(k)\| \leq \gamma_9\lambda^{k-k_i}|\tilde{\phi}(\bar{k}_i)\| + \sum_{j=\bar{k}_i}^{k_{i+1}} \gamma_9\lambda^{k-j-1}|\tilde{w}(j)| \quad (31)\]
for $\bar{k}_i \leq k \leq \bar{k}_{i+1}$. This completes Sub-Case 2.2.

Using an argument virtually identical to that used in the last part of the proof of Theorem 1 of [12] and [14], we can glue the bounds from Sub-Case 1 and Sub-Case 2 together: using Claim 1, we conclude that there exists a constant $\gamma_{10}$ so that
\[|\phi(k)| \leq \gamma_{10}\lambda^{k-t_0}|x_{t_0}| + \sum_{j=t_0}^{k_{i+1}} \gamma_{10}\lambda^{k-j}(|y^*(j)| + |w(j)|) \quad (32)\]
for $k \geq t_0$. This completes Case 2.

Now suppose that $w = 0$. From (9) and (10) we have
\[e(t) = e(t) + \phi(t - d)^T\left[\hat{\theta}(t - 1) - \hat{\theta}(t - d)\right], \ t \geq t_0 + d \quad (33)\]
so if $|\phi(t - d)| \neq 0$, we have
\[\frac{|e(t)|}{\|\phi(t - d)\|} \leq \sum_{j=0,\phi(t-d-j)\neq0}^{d-1} \frac{|e(t-j)|}{\|\phi(t-d-j)\|}. \quad (34)\]
From the first estimator property of Proposition 1 we obtain
\[\frac{|e(t)|^2}{\|\phi(t - d)\|^2} \leq \sum_{j=0,\phi(t-d-j)\neq0}^{d-1} \frac{|e(t-j)|^2}{\|\phi(t-d-j)\|^2}. \quad (35)\]
By Cauchy-Schwartz we obtain
\[\sum_{t=t_0+2d-1,\phi(t-d)\neq0}^{T} \frac{|e(t)|^2}{\|\phi(t - d)\|^2} \leq \sum_{j=0,\phi(t-d-j)\neq0}^{d-1} \frac{|e(t-j)|^2}{\|\phi(t-d-j)\|^2}. \quad (36)\]
Hence, for $T > t_0 + 2d - 1$:
\[\sum_{t=t_0+2d-1,\phi(t-d)\neq0}^{T} \frac{|e(t)|^2}{\|\phi(t - d)\|^2} \leq d \sum_{j=0,\phi(t-d-j)\neq0}^{d-1} \frac{|e(t-j)|^2}{\|\phi(t-d-j)\|^2} \leq 8d^2\|S\|^2 \quad (37) \] (by Proposition 1).
Since \( \varepsilon(t) = 0 \) if \( \phi(t - d) = 0 \), we now apply the bound on \( \phi(t) \) proven above, we conclude that
\[
\sum_{t=t_0+2d-1}^{\infty} \varepsilon(t)^2 \leq 8d^2\|\mathbf{S}\|^2 \times \sup_{j \geq t_0} \|\phi(j)\|^2
\]
\[
\leq 8d^2\|\mathbf{S}\|^2 c^2 \|[x_0]\|^2 + \left(\frac{1}{1-\lambda}\right)^2 \sup_{j \geq t_0} |y^*(j)|^2,
\]
which yields the desired result. \( \square \)

**Remark 7:** The linear-like bound proven in Theorem 1 can be leveraged to prove that parametric time-variations can be tolerated. So suppose that the actual plant model is
\[
y(t + d) = \phi(t)^T \theta^*(t) + \tilde{w}(t), \quad \phi(0) = \phi_0,
\]
with \( \theta^*(t) \in \mathcal{S} \) for all \( t \in \mathbb{Z} \). We adopt a common model of acceptable time-variations used in adaptive control: with \( \mathcal{S} \) the set of all \( n \times m \) matrices with full column rank, \( \mathcal{S} \) denotes the subset of all matrices \( \mathcal{S} \) whose elements \( \theta^* \) satisfy \( \theta^*(t) \in \mathcal{S} \) for every \( t \in \mathbb{Z} \) as well as
\[
\sum_{t=t_1}^{t_2-1} \|\theta^*(t + 1) - \theta^*(t)\| \leq c_0 + \epsilon(t_2 - t_1), \quad t_2 > t_1
\]
for every \( t_1 \in \mathbb{Z} \). If we argue as in [12] and [14], we can show that for every \( \mathcal{S} \) with \( c_0 > 0 \), if \( \epsilon \) is small enough then the proposed controller will still provide linear-like bounds on \( \phi(t) \) for all \( \theta^* \in s(\mathcal{S}, c_0, \epsilon) \).

**Remark 8:** The linear-like bounds proven in Theorem 1 can be used in conjunction with the Small Gain Theorem to prove that the closed-loop system tolerates a degree of unmodelled dynamics.

**V. A Simulation Example**

Here we provide a simulation example to illustrate the results of this paper. Consider the time-varying plant
\[
y(t + 1) = -a_1(t)y(t) - a_2(t)y(t - 1) + b_0(t)u(t) + b_1(t)u(t - 1) + w(t)
\]
with \( a_1(t) \in [-2, 2] \), \( a_2(t) \in [-2, 2] \), \( b_0(t) \in [1.5, 5] \) and \( b_1(t) \in [-1, 1] \). We apply the proposed adaptive controller (with \( \delta = \infty \)) to this plant when
\[
a_1(t) = 2 \cos(0.1t), \quad a_2(t) = -2 \sin(0.07t), \quad b_0(t) = 3.25 - 1.75 \cos(0.08t), \quad b_1(t) = -\cos(0.02t),
\]
\[
y^*(t) = \cos(t), \quad w(t) = \begin{cases} 0.1 \cos(10t) & 0 < t \leq 500 \\ 0 & \text{otherwise}; \end{cases}
\]
we set \( y(-1) = y(0) = -1, u(-1) = 0 \), and the initial parameter estimates to the midpoint of the respective intervals. Figure 1 shows the results. As expected, the controller does a good job of tracking when there is no disturbance; while the tracking degrades when the disturbance is turned on at \( t = 200 \), it quickly improves when the disturbance returns to zero at \( t = 500 \). Furthermore, the estimator tracks the time-varying parameters fairly well.

**VI. SUMMARY AND CONCLUSIONS**

Under suitable assumptions, here we show that if the original, ideal, projection algorithm is used in the estimation process, then the corresponding \( d \)-step-ahead adaptive controller guarantees linear-like convolution bounds on the closed-loop behaviour; this offers exponential stability and a bounded noise gain, unlike almost all other parameter adaptive controllers. This can be leveraged in a modular way to prove tolerance to unmodelled dynamics and plant parameter variation.

In the case of a zero disturbance, it is proven that asymptotic tracking is achieved; we are presently working on obtaining a bound on the tracking quality in terms of the size of the disturbance. In this approach we assumed that the sign of the high frequency gain is known; we are presently trying to use a multi-estimator approach to remove this assumption.

**VII. APPENDIX**

**Proof of Proposition 3:**

To proceed, we analyse the good closed-loop model of Section III.A. From \[13\], it is clear that we need to obtain a bound on the terms \( B_1 \varepsilon(t) \) and \( B_2 \varepsilon(t + j) \) for \( j = 1, \ldots, d + 1 \). It will be convenient to define an intermediate quantity \[15\]:
\[
\tilde{v}(t - 1) := \rho_3(\phi(t - d), \varepsilon(t)) \times \frac{\phi(t - d)}{\|\phi(t - d)\|^2}^T \varepsilon(t)
\]

**Step 1:** Obtain a desirable bound on \( B_1 \varepsilon(t) \) in terms of \( \tilde{v}(t - 1), \phi(t - d) \) and \( \tilde{w}(t - d) \).

First of all, for \( i = 1, 2 \) define
\[
\tilde{A}_i(t) := \rho_3(\phi(t - d), \varepsilon(t)) \frac{\varepsilon(t)}{\|\phi(t - d)\|^2} B_i \phi(t - d)^T.
\]
It is easy to see that
\[
\tilde{A}_i(t) \phi(t - d) = \rho_3(\phi(t - d), \varepsilon(t)) B_i \varepsilon(t).
\]

\[4\] It is similar to \( v(t - 1) \) except for the \( \varepsilon(t) \) rather than \( \varepsilon(t) \) at the end.
So

\[ B_i \varepsilon(t) = \rho_b(\phi(t-d), e(t)) B_i \varepsilon(t) + [1 - \rho_b(\phi(t-d), e(t))] \varepsilon(t) B_i =: \eta_0(t) \]
\[ = \bar{\Delta}_i(t) \phi(t-d) + B_i \eta_0(t). \]  
(29)

Using (9) and (10) and the definition of \( \rho_b(\phi(t-d), e(t)) \) it is easy to show that

\[ |\eta_0(t)| \leq 4||S|| + \frac{\delta}{\delta} |\bar{w}(t-d)|; \]  
(30)

furthermore, it is clear that

\[ \| \bar{\Delta}_i(t) \| = \rho_b(\phi(t-d), e(t)) \| \varepsilon(t) \| \| \phi(t-d) \| = |\nu(t-1)|. \]  
(31)

**Step 2: Bound \( \bar{\nu}(t-1) \) in terms of \( \nu(t-1), \ldots, \nu(t-d) \).**

It follows from the formulas for \( \varepsilon(t) \) and \( e(t) \) given in (9) and (10) that

\[ \varepsilon(t) = e(t) + \phi(t-d)^T [\hat{\theta}(t) - \hat{\theta}(t-d)]. \]

Using the definition of \( \bar{\nu}(t-1) \) we have

\[ |\bar{\nu}(t-1)| \leq \rho_b(\phi(t-d), e(t)) \| e(t) \| \| \phi(t-d) \| + \| \hat{\theta}(t-1) - \hat{\theta}(t-d) \| \]
\[ = |\nu(t-1)| + \| \nu(t-1) \| - \| \hat{\theta}(t-1) - \hat{\theta}(t-d) \|. \]

Now it follows from the estimator update law that

\[ |\| \hat{\theta}(t-1) - \hat{\theta}(t-d) \| \| \leq |\| \hat{\theta}(t-1) - \hat{\theta}(t-2) \| + \cdots + \| \| \hat{\theta}(t-d+1) - \hat{\theta}(t-d) \| \]
\[ \leq \rho_b(\phi(t-d), e(t-d)) \| e(t) \| \| \phi(t-d) \| + \cdots + \rho_b(\phi(t-2d+1), e(t-2d+1)) \| e(t-2d+1) \| \| \phi(t-2d+1) \|
\[ \leq \sum_{j=2}^{d} |\nu(t-j)|. \]

We conclude that

\[ |\bar{\nu}(t-1)| \leq \sum_{j=1}^{d} |\nu(t-j)|. \]  
(32)

**Step 3: Obtain a bound on \( B_i \varepsilon(t) \) in terms of \( \nu(t), \ldots, \nu(t-d), \phi(t-d) \) and \( \bar{w}(t-d) \).**

If we combine (29), (30), (31) and (32), we see that

\[ B_i \varepsilon(t) = \bar{\Delta}_i(t) \phi(t-d) + B_i \eta_0(t) \]

with

\[ \| \bar{\Delta}_i(t) \| \leq \sum_{j=0}^{d-1} |\nu(t-j)| \]

and

\[ |\eta_0(t)| \leq 4||S|| + \frac{\delta}{\delta} |\bar{w}(t-d)|. \]

**Step 4: Apply the result of Step 3 to (12).**

We can apply the above result to each of the terms on the RHS of (12) containing \( \varepsilon(\cdot) \). So from Step 3 we have

\[ B_1 \varepsilon(t+1) = \bar{\Delta}_1(t+1) \phi(t+1 - d) + B_1 \eta_0(t+1) \]
\[ = \frac{a_{d-j}}{b_0} B_2 \varepsilon(t+1+j) + \frac{a_{d-j}}{b_0} \bar{\Delta}_2(t+1+j) \phi(t+1 - d + j) + \frac{a_{d-j}}{b_0} B_2 \eta_0(t+1-j), \quad j = 0, 1, \ldots, d. \]

(34)

Each term except one is of the desired form: the case of \( j = d \) is problematic since it contains a \( \phi(t+1) \) term. However, we can use the crude model given in (13) to see that

\[ \frac{a_0}{b_0} \bar{\Delta}_2(t+d+1) \phi(t+1) = \frac{a_0}{b_0} \bar{\Delta}_2(t+d+1) \times [A_0(t) \phi(t) + B_3(t)y^*(t+d+1) + B_4 w(t+1)]. \]

(35)

If we now combine (33), (34) and (35), we see that we should define

\[ \Delta_0(t) = \frac{a_0}{b_0} \bar{\Delta}_2(t+d+1) A_0(t) + \frac{a_1}{b_0} \bar{\Delta}_2(t+d), \]
\[ \Delta_j(t) = \frac{a_{j+1}}{b_0} \bar{\Delta}_2(t+d - j), \quad j = 1, \ldots, d-2, \]

and

\[ \Delta_{d-1}(t) = \bar{\Delta}_1(t+1) + \frac{a_d}{b_0} \bar{\Delta}_2(t+1). \]

It is clear from Step 3 that this choice of \( \bar{\Delta}_i \) has the desired property. Last of all, we group all of the remaining terms into \( \eta(t) \):

\[ \eta(t) := B_1 y^*(t+1) + B_2 \frac{1}{b_0} \sum_{j=0}^{d} a_{d-j} y^*(t+1+j) - \]
\[ \frac{1}{b_0} B_2 w(t+d+1) + B_1 \eta_0(t+1) + \]
\[ \frac{a_0}{b_0} \bar{\Delta}_2(t+d+1) [B_3(t) y^*(t+d+1) + \]
\[ B_4 w(t+1)] + \frac{1}{b_0} \sum_{j=0}^{d-1} a_{d-j} \eta_0(t+1-j). \]

If we apply Proposition 2 and use the bound on \( \eta_0(t) \) given in Step 3, we see that \( \eta(t) \) has the desired property. \( \square \)

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