Flow with $A_\infty(\mathbb{R})$ density and transport equation in $\text{BMO}(\mathbb{R})$

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Abstract. We show that, if $b \in L^1(0, T; L^1_{\text{loc}}(\mathbb{R}))$ has spatial derivative in the John-Nirenberg space $\text{BMO}(\mathbb{R})$, then it generalizes a unique flow $\phi(t, \cdot)$ which has an $A_\infty(\mathbb{R})$ density for each time $t \in [0, T]$. Our condition on the map $b$ is optimal and we also get a sharp quantitative estimate for the density. As a natural application we establish a well-posedness for the Cauchy problem of the transport equation in $\text{BMO}(\mathbb{R})$.

1 Statement of main results

Given an integer $n \geq 1$, a real $T \geq t > 0$ and an evolutionary self-map $b(t, \cdot)$ of $\mathbb{R}^n$ with

$$b \in L^1(0, T; L^1_{\text{loc}}(\mathbb{R}^n)),$$

consider the flow

$$\phi(t, x) = x + \int_0^t b(r, \phi(r, x)) \, dr.$$

We are motivated by the composition and transportation problems in $\text{BMO}$ space to answer the question:

What condition is needed on a vector field such that it generalizes a flow $\phi$ with $A_\infty$ density?

On $\mathbb{R}^n$, $n \geq 2$, the question has a satisfactory solution by Reimann [27] via the following $(Q)$-condition

$$(Q) \quad \sup_{(x,y,z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \ |z|=|y| > 0} \frac{|\langle y, b(x+y) - b(x) \rangle - \langle z, b(x+z) - b(x) \rangle|}{|y|^2 - |z|^2} < \infty$$

which is equivalent to the anti-conformal part

$$S_A b = \frac{1}{2} (Db + Db^T) - \frac{\text{div} b}{n} I_{n \times n}$$

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is bounded - moreover (cf. [27]) -

\[ S_A b \in L_+^\infty(\mathbb{R}^n) \Rightarrow Db \in \text{BMO}(\mathbb{R}^n). \]

More precisely, [27] shows if \( b \) satisfies \((Q)\) then it generalizes a unique flow \( \phi(t, x) \), which at each time \( t \) is a quasi-conformal mapping and so the Jacobian \( J_\phi \) of \( \phi \) is of \( A_\infty(\mathbb{R}^n) \) (cf. [5]) where

\[ 0 \leq w \in A_\infty(\mathbb{R}^n) \iff [w]_{A_\infty(\mathbb{R}^n)} = \sup_{I \subset \mathbb{R}^n} \left( \frac{1}{|I|} \int_I w \, dx \right) \exp \left( -\frac{1}{|I|} \int_I \log w \, dx \right) < \infty. \]

However, less known is the situation on \( \mathbb{R} \). Note that the 1-dimensional \((Q)\)-condition coincides with the Zygmund condition for a constant \( C > 0 \):

\[ (Z) \quad |b(x + y) + b(x - y) - 2b(x)| \leq C|y| \quad \forall \ (x, y) \in \mathbb{R} \times \mathbb{R}. \]

Reimann [27] showed that for functions satisfying \((Q)\) the induced flows are quasi-symmetric mappings - unfortunately - quasi-symmetric mappings are not necessarily absolutely continuous in \( \mathbb{R} \) and a function satisfying \((Z)\) needs not be absolutely continuous (cf. [3, 27] and [14]). In view of this, some more restrictions on \( b \) seem to be necessary for the generalized flow to have an \( A_\infty \) density. Observe that the notion \( S_A b = 0 \) in \( \mathbb{R} \) does not carry any information.

In this paper, we show that if \( b' = \text{BMO}(\mathbb{R}) \) then \( b \) generalizes a (unique) flow with \( A_\infty(\mathbb{R}) \) densities. To see this clearly, recall that

\[ f \in \text{BMO}(\mathbb{R}^n) \iff \|f\|_{\text{BMO}(\mathbb{R}^n)} = \sup_{\text{cubes } I \subset \mathbb{R}^n} |I|^{-1} \int_I |f - f_I| \, dx < \infty, \]

where

\[ f_I = |I|^{-1} \int_I f(x) \, dx \]

denotes the integral average of \( f \) over \( I \) whose Lebesgue measure is written as \( |I| \). Since all constant functions have zero \( \text{BMO}(\mathbb{R}^n) \)-norm, and any constant does effect the flow, we make a modification on \( \text{BMO}(\mathbb{R}^n) \) functions \( f \) as

\[ \|f\|_* = \|f\|_{\text{BMO}(\mathbb{R}^n)} + \int_{B(0, 1)} |f| \, dx, \]

where \( B(0, 1) \) is the unit ball of \( \mathbb{R}^n \). Obviously,

\[ f \in \text{BMO}(\mathbb{R}^n) \iff \|f\|_* < \infty, \]

however, \( \|f\|_* \) is not comparable to \( \|f\|_{\text{BMO}(\mathbb{R}^n)} \). In what follows,

\[ \frac{\partial}{\partial x} b(t, x) \in L^1(0, T; \text{BMO}(\mathbb{R})) \]

stands for

\[ \int_0^T \left\| \frac{\partial}{\partial x} b(t, x) \right\|_* \, dt < \infty. \]

Our first main result reads as follows.
Theorem 1.1. Let

\[ b(t, x) : [0, T] \times \mathbb{R} \mapsto \mathbb{R} \text{ be in } L^1(0, T; L^1_{\text{loc}}(\mathbb{R})) \text{ with } \frac{\partial b(t, x)}{\partial x} \in L^1(0, T; \text{BMO}(\mathbb{R})). \]

Then there exists a unique flow \( \phi(t, x) \) satisfying

\[
\begin{cases}
\frac{\partial}{\partial t} \phi(t, x) = b(t, \phi(t, x)) & \forall (t, x) \in [0, T] \times \mathbb{R}; \\
\phi(0, x) = x & \forall x \in \mathbb{R}.
\end{cases}
\]

Moreover, for each \( t \in [0, T] \),

\[
\frac{\partial}{\partial x} \phi(t, \cdot)
\]

is an \( A_\infty(\mathbb{R}) \)-weight, and there exist constants \( C_1, c > 0 \) such that

\[
(1.2) \quad \left\| \log \left| \frac{\partial}{\partial x} \phi(t, \cdot) \right| \right\|_{\text{BMO}(\mathbb{R})} \leq \frac{\int_0^t C_1 \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{\text{BMO}(\mathbb{R})} ds}{\exp \left(-c \int_0^t \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{\text{BMO}(\mathbb{R})} ds \right)}.
\]

Some remarks are in order. First, from the well-known fact that the logarithm of an \( A_\infty \) weight is a BMO function (see Lemma 2.4), and the formula

\[
\log \left| \frac{\partial}{\partial x} \phi(t, x) \right| = \int_0^t \frac{\partial}{\partial x} b(s, \phi(s, x)) \, ds \in \text{BMO}(\mathbb{R}),
\]

we see that our condition (1.1) is critical, i.e., for each \( t \),

\[ x \mapsto \frac{\partial}{\partial x} b(t, x) \]

is necessarily a BMO(\( \mathbb{R} \))-function. Second, taking

\[ b(x) = x \log |x| \]

for example, indicates that \( b \) generalizes a flow \( \phi(t, x) \) with

\[
\begin{cases}
\phi(t, x) = \text{sign} x |x|^{e'} \\
\frac{\partial}{\partial x} \phi(t, x) = e'|x|^{e'-1} \in A_\infty(\mathbb{R}) \\
\left\| \log \left| \frac{\partial}{\partial x} \phi(t, x) \right| \right\|_{\text{BMO}(\mathbb{R})} \leq (e' - 1) \| |x| \|_{\text{BMO}(\mathbb{R})} \leq Cte'.
\end{cases}
\]

This implies that our estimate (1.2) is sharp.

For the proof, we shall first provide a version of the result in smooth setting, namely,

\[
(1.3) \quad b \in L^1(0, T; C^1(\mathbb{R})) \text{ with } \frac{\partial b(t, x)}{\partial x} \in L^1(0, T; \text{BMO}(\mathbb{R})),
\]
and then use the compactness argument based on development of non-smooth flows from [2, 9, 12, 13]. Note that since the Zygmund condition is satisfied for $b$, existence and uniqueness follow already from Reimann [27]. The key of the proof is to establish (1.2), which even in the smooth setting seems non-trivial. By the composition result of Jones [21], a homeomorphism $\phi$ preserves $\text{BMO}(\mathbb{R})$ if and only if $\phi'$ is an $A_{\infty}(\mathbb{R})$ weight. However, even we assume that $b$ is smooth on $\mathbb{R}$, it seems mysteries to us whether one can prove the generalized flow carries $A_{\infty}(\mathbb{R})$ density directly from (1.1).

In order to overcome the difficulties, we further consider the simpler case

$$
(1.4) \quad b \in L^1(0, T; C^1(\mathbb{R})) \quad \text{with} \quad \frac{\partial b(t, x)}{\partial x} \in L^1(0, T; L^\infty(\mathbb{R})),
$$

where the generalized flow carries $A_{\infty}(\mathbb{R})$-density following from the Cauchy-Lipschitz theory. Then we observe that for a function $v$ with small $\text{BMO}(\mathbb{R})$-norm, $e^v$ lies in the $A_{\infty}(\mathbb{R})$ class with its norm controlled by the $\text{BMO}(\mathbb{R})$-norm of $v$ linearly. Then by using the flow with $A_{\infty}(\mathbb{R})$-density in the smooth setting, a quantitative estimate of the norm of composition in $\text{BMO}(\mathbb{R})$, and a bootstrap argument, we succeed in showing (1.2) in the Lipschitz case (1.4). Finally a truncation argument involving the Arzelà-Ascoli theorem allows us to pass to the case (1.3); see Section 3.

One may wonder if a quantitative estimate of the $A_{\infty}(\mathbb{R})$-norm of

$$
\left| \frac{\partial}{\partial x} \phi(t, \cdot) \right|
$$

can be established. Although we do not know a positive answer, we doubt it since a quantitative bound for an $A_{\infty}(\mathbb{R})$-weight $e^v$ holds only for $v$ with small $\text{BMO}(\mathbb{R})$-norm; see Lemma 2.3 and Lemma 2.4 below. However, there is a nice result regarding homeomorphisms preserving $A_p(\mathbb{R})$-weights by [20].

We next apply the result on flow to study the transportation problem in $\text{BMO}$ space. Besides its own interest, this problem and its dual equation also arise naturally from the study of conservation laws (see [6] for instance). In [10] (somewhat related to [25]), a well-posedness of the Cauchy problem of the transport equation in $\text{BMO}(\mathbb{R}^n)$ has been established for $n \geq 2$ and then pushed to the case $n = 1$ in [29]. The main step over there is to use the hypothesis that

$$(t, x) \mapsto \begin{cases} 
S_a b(t, x) & \forall \ n \geq 2 \\
\frac{\partial}{\partial t} b(t, x) & \forall \ n = 1
\end{cases}
$$

belongs to $L^1(0, T; L^\infty(\mathbb{R}^n))$ with a suitably small norm, the quasi-conformal flows of [27] and the composition results obtained in [23, 26] for $n \geq 2$ (cf. [22] [28] [30]) and in [21] for $n = 1$. But nevertheless, as our second main result we utilize Theorem (1.1) and [21, Theorem 1] to discover the following stronger well-posedness of the transport equation in $\text{BMO}(\mathbb{R})$.

**Theorem 1.2.** Let $b(t, x) : [0, T] \times \mathbb{R} \mapsto \mathbb{R}$ be in $L^1(0, T; L^1_{\text{loc}}(\mathbb{R}))$ and satisfy

$$
\frac{\partial b(t, x)}{\partial x} \in L^1(0, T; \text{BMO}(\mathbb{R})).
$$
Then for \( u_0 \in \text{BMO}(\mathbb{R}) \) there exists a unique solution \( u \in L^\infty(0, T; \text{BMO}(\mathbb{R})) \) to the Cauchy problem of the transport equation

\[
\begin{cases}
\left( \frac{\partial u}{\partial t} - b \cdot \frac{\partial u}{\partial x} \right)(t, x) = 0 & \forall (t, x) \in (0, T) \times \mathbb{R}; \\
u(0, x) = u_0(x) & \forall x \in \mathbb{R}.
\end{cases}
\]

Moreover, for each \( t \in [0, T] \), it holds that

\[
\begin{cases}
u(t, x) = u_0(\phi(t, x)); \\
\frac{\partial}{\partial t}\phi(t, x) = b(t, \phi(t, x)),
\end{cases}
\]

and there exist \( C_2, c > 0 \) such that

\[
(1.5) \quad \|u\|_{\text{BMO}(\mathbb{R})} \leq C_2\|u_0\|_{\text{BMO}(\mathbb{R})} \exp \left( c \int_0^t \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \, ds \right).
\]

Based on the duality of Hardy space \( H^1 \) and BMO by Fefferman and Stein [16], the above theorem provides the existence of solutions in Hardy space \( H^1 \) to the continuity equation

\[
\begin{cases}
\left( \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} (bu) \right)(t, x) = 0 & \forall (t, x) \in (0, T) \times \mathbb{R}; \\
u(0, x) = u_0(x) & \forall x \in \mathbb{R};
\end{cases}
\]

see [11] for a study of the equation in higher dimensions and a proof of uniqueness (cf. [11, Theorem 3]).

The paper is organized as follows. In Section 2, we recall and establish some results concerning Muckenhoupt weights, \( \text{BMO}(\mathbb{R}) \), and continuity estimates. In Section 3, we present the key a priori estimation for the flow, i.e., the version of Theorem 1.1 in the smooth setting. In Section 4, we verify the above main results.

**Notation.** In the above and below, \( C, C_1, C_2, \ldots \) and \( c, c_1, c_2, \ldots \) stand for positive constants.

## 2 Weights and bounded mean oscillation

For a locally integrable function \( f \) and an open interval \( I \subset \mathbb{R} \), we denote by \( f_I \) the integral average of \( f \) on \( I \). We say that a locally integrable nonnegative function \( w \) belongs to the Muckenhoupt \( A_p(\mathbb{R}) \) class, \( 1 < p < \infty \), if

\[
[w]_{A_p(\mathbb{R})} = \sup_{\text{intervals } I \subset \mathbb{R}} \left( \frac{1}{|I|} \int_I w \, dx \right) \left( \frac{1}{|I|} \int_I w^{-\frac{1}{p-1}} \, dx \right)^{p-1} < \infty,
\]

and that \( w \in A_\infty(\mathbb{R}) \), if

\[
[w]_{A_\infty(\mathbb{R})} = \sup_{\text{intervals } I \subset \mathbb{R}} \left( \frac{1}{|I|} \int_I w \, dx \right) \exp \left( - \frac{1}{|I|} \int_I (\log w) \, dx \right) < \infty.
\]
Note that, if \( w > 0 \) a.e., then \([w]_{A_\infty(\mathbb{R})} \geq 1\) follows from the Jensen inequality that
\[
[w]_{A_\infty(\mathbb{R})} \geq w_I \exp\left((- \log w_I)\right) \geq \exp\left((\log w_I)\right) \exp\left((- \log w_I)\right) = 1,
\]
and similarly
\[\[w\]_{A_p(\mathbb{R})} \geq [w]_{A_\infty(\mathbb{R})} \quad \forall \ p \in (1, \infty).\]

We need the following quantitative version of reverse Hölder inequality for \(A_\infty(\mathbb{R})\)-weight from [19]; see also [24].

**Lemma 2.1.** Let \( w \in A_\infty(\mathbb{R}) \) and \( I \subset \mathbb{R} \) be an arbitrary interval. Then there exits
\[
\begin{align*}
\tau > 0; \\
r_w = 1 + \left(\tau[w]_{A_\infty(\mathbb{R})}\right)^{-1}; \\
e_w = \left(1 + \tau[w]_{A_\infty(\mathbb{R})}\right)^{-1},
\end{align*}
\]
such that
\[
\begin{align*}
\left(\frac{|I|^{-1}}{w(I)}\right)^{1/r_w} \leq 2|I|^{-1} \int_{I} w dx; \\
\frac{w(E)}{w(I)} = \frac{\int_E w(x) dx}{\int_{I} w(x) dx} \leq 2 \left(\frac{|E|}{|I|}\right)^{e_w} \text{ for any measurable set } E \subset I.
\end{align*}
\]

By [21, Theorem], we know that an increasing homeomorphism \( \varphi \) of \( \mathbb{R} \) preserves BMO if and only if \( \varphi' \) belongs to \( A_\infty(\mathbb{R}) \). By using the previous lemma we deduce the following quantitative version; see [1] for an explicit bound in terms of reverse Hölder index and [4, 15, 17] for related results.

**Lemma 2.2.** Let \( \varphi \) be an increasing homeomorphism on \( \mathbb{R} \) with \( \varphi' \in A_\infty(\mathbb{R}) \). Then there is \( C_3 > 0 \) such that
\[\|f \circ \varphi^{-1}\|_{\text{BMO}(\mathbb{R})} \leq C_3[\varphi']_{A_\infty(\mathbb{R})}\|f\|_{\text{BMO}(\mathbb{R})}.\]

**Proof.** Recall that for a BMO(\( \mathbb{R} \))-function \( f \), the John-Nirenberg inequality states that, for all \( I \subset \mathbb{R} \), there exists \( c_1, c_2 > 0 \) such that
\[
|\{x \in I : |f(x) - f_I| > \lambda\}| \leq c_1|I| \exp\left(-\frac{c_2 \lambda}{\|f\|_{\text{BMO}(\mathbb{R})}}\right) \quad \forall \ \lambda > 0;
\]
see [18] for instance.

Suppose that \( \varphi \) is an increasing homeomorphism of \( \mathbb{R} \) with \( \varphi' \in A_\infty(\mathbb{R}) \). By [21, Theorem], we have
\[f \circ \varphi^{-1} \in \text{BMO}.\]

For every interval
\[I = (a, b) \subset \mathbb{R},\]
set
\[E_\lambda = \{x \in I : |f \circ \varphi^{-1}(x) - f_{\varphi^{-1}(I)}| > \lambda\}.\]
Then 
\[ \varphi^{-1}(E_{\lambda}) = \{ y \in \varphi^{-1}(I) : |f(y) - f_{\varphi^{-1}(I)}| > \lambda \}, \]
and hence, by Lemma 2.1 and the John-Nirenberg inequality, we get
\[ \frac{|E_{\lambda}|}{|I|} \leq 2 \left( \frac{|\varphi^{-1}(E_{\lambda})|}{|\varphi^{-1}(I)|} \right)^{\epsilon_w} \leq 2c_2 \epsilon_w \lambda \frac{1}{\|f\|_{\text{BMO}(\mathbb{R})}} \]
where \( \epsilon_w = (1 + \tau[\varphi']_{A_\infty(\mathbb{R})})^{-1} \), thereby finding
\[ \|f \circ \varphi^{-1}\|_{\text{BMO}(\mathbb{R})} \leq C(1 + \tau[\varphi']_{A_\infty(\mathbb{R})})\|f\|_{\text{BMO}(\mathbb{R})} \leq C_3 \|f\|_{\text{BMO}(\mathbb{R})}, \]
where we have used the fact that \( \varphi \) is an increasing homeomorphism on \( \mathbb{R} \) with \( [\varphi']_{A_\infty(\mathbb{R})} \geq 1 \).

\[ \square \]

The following result is well-known; see [7, 18] for instance.

**Lemma 2.3.** There exists \( \alpha < 1 < \beta \) such that for
\[
\begin{cases}
    f \in \text{BMO}(\mathbb{R}); \\
    s \in \mathbb{R}; \\
    |s| \leq \alpha \|f\|_{\text{BMO}(\mathbb{R})}^{-1},
\end{cases}
\]
it holds that
\[ e^{sf} \in A_2(\mathbb{R}) \text{ with } [e^{sf}]_{A_2(\mathbb{R})} \leq \beta^2. \]

Here it is perhaps appropriate to mention that the requirement
\[ |s| \leq \alpha \|f\|_{\text{BMO}(\mathbb{R})}^{-1} \]
is critical since
\[ x \mapsto f(x) = \log |x| \]
is in \( \text{BMO}(\mathbb{R}) \) but
\[ x \mapsto e^{-f(x)} = |x|^{-1} \]
is not a Muckenhoupt weight.

**Lemma 2.4.** If
\[ 0 \leq w \in A_\infty(\mathbb{R}) \]
then
\[ \| \log w \|_{\text{BMO}(\mathbb{R})} \leq 2 \log ([w]_{A_\infty(\mathbb{R})} + 1). \]
Conversely, if
\[ \nu \in \text{BMO}(\mathbb{R}) \text{ & } \nu \geq 0 \text{ a.e. on } \mathbb{R} \]
then there exists a sufficiently small \( \epsilon_0 \in (0, 1] \) such that
\[ \|\nu\|_{\text{BMO}(\mathbb{R})} < \epsilon_0 \Rightarrow e^\nu \in A_\infty(\mathbb{R}) \text{ with } [e^\nu]_{A_\infty(\mathbb{R})} \leq 1 + C_4 \|\nu\|_{\text{BMO}(\mathbb{R})}. \]
Proof. On the one hand, for any $0 \leq w \in A_\infty(\mathbb{R})$ we have

$$\int_I \log w - (\log w)_I \, dx = \int_I [\log w - (\log w)_I]_+ \, dx + \int_I [\log w - (\log w)_I]_- \, dx$$

$$= 2 \int_I [\log w - (\log w)_I]_+ \, dx,$$

where $[f]_+$ and $[f]_-$ denote the positive and negative parts of $f$ respectively. In virtue of Jensen’s inequality we obtain

$$\left| I \right| - 1 \int_I \log w - (\log w)_I \, dx = 2 \left| I \right| - 1 \int_I [\log w - (\log w)_I]_+ \, dx$$

$$\leq 2 \log \left( \left| I \right| - 1 \int_I \exp [\log w - (\log w)_I]_+ \, dx \right)$$

$$\leq 2 \log \left( \left| I \right| - 1 \int_I \exp [\log w - (\log w)_I] \, dx + 1 \right)$$

$$\leq 2 \log \left( \left| w \right|_{A_\infty(\mathbb{R})} + 1 \right),$$

whence

$$\| \log w \|_{BMO(\mathbb{R})} \leq 2 \log \left( \left| w \right|_{A_\infty(\mathbb{R})} + 1 \right).$$

On the other hand, note that

(2.1) $\left[ e^v \right]_{A_\infty(\mathbb{R})} = \left( \sup_{I = (a,b) \subset \mathbb{R}} \left| I \right| - 1 \int_I e^{v(x)} \, dx \right) \exp \left( \left| I \right|_I \right) = \sup_{I = (a,b) \subset \mathbb{R}} \left| I \right| - 1 \int_I e^{v(x)-v_I} \, dx.$

So, if $v \in BMO(\mathbb{R})$, then the John-Nirenberg inequality gives

$$\left| \left| x \in I : |v(x) - v_I| > \lambda \right| \right| \leq c_1 |I| \exp \left( -\frac{c_2 \lambda}{\| v \|_{BMO(\mathbb{R})}} \right).$$

Inserting this into (2.1), we find that if

$$\| v \|_{BMO(\mathbb{R})} < c_2$$

then

$$\left| I \right| - 1 \int_I e^{v(x)-v_I} \, dx = \frac{1}{|I|} \int_{x \in I : v(x) - v_I < 0} e^{v(x)-v_I} \, dx + \frac{1}{|I|} \int_{x \in I : v(x) - v_I \geq 0} e^{v(x)-v_I} \, dx$$

$$\leq 1 + c_1 \int_0^\infty \exp \left( \lambda - \frac{c_2 \lambda}{\| v \|_{BMO(\mathbb{R})}} \right) d\lambda$$

$$\leq 1 + \frac{c_1 \| v \|_{BMO(\mathbb{R})}}{c_2 - \| v \|_{BMO(\mathbb{R})}}.$$

Accordingly,

$$\| v \|_{BMO(\mathbb{R})} < 2^{-1} c_2 \Rightarrow \left[ e^v \right]_{A_\infty(\mathbb{R})} \leq 1 + 2 c_1 c_2^{-1} \| v \|_{BMO(\mathbb{R})}.$$
Letting 
\[ \epsilon_0 = \min\{1, 2^{-1}c_2\} \]
yields the assertion. □

**Proposition 2.5.** Suppose that \( b \in L^1_{\text{loc}}(\mathbb{R}) \) has its derivative \( b' \in \text{BMO}(\mathbb{R}) \). Then \( b \) satisfies the Zygmund condition with

\[ |b(x + y) + b(x - y) - 2b(x)| \leq 2|y|\|b'||_{\text{BMO}(\mathbb{R})} \quad \forall (x, y) \in \mathbb{R} \times \mathbb{R}. \]

**Proof.** This follows from

\[
|b(x + y) + b(x - y) - 2b(x)| \\
= \left| \int_x^{x+y} b'(z) \, dz - \int_{x-y}^x b'(z) \, dz \right| \\
\leq \left| \int_x^{x+y} b'(z) \, dz - \frac{1}{2} \int_{x-y}^{x+y} b'(z) \, dz \right| + \left| \frac{1}{2} \int_{x-y}^{x+y} b'(z) \, dz - \int_{x-y}^x b'(z) \, dz \right| \\
\leq \int_{x-y}^{x+y} |b'(z) - b'_{[x-y,x+y]}| \, dz + \int_{x-y}^x |b'(z) - b'_{[x-y,x+y]}| \, dz \\
\leq 2|y|\|b'||_{\text{BMO}(\mathbb{R})}. 
\]

Recall that for a \( \text{BMO}(\mathbb{R}) \) function \( f \) we have

\[ \|f\|_* = \|f\|_{\text{BMO}(\mathbb{R})} + \int_{[-1,1]} |f| \, dx < \infty. \]

In what follows, for a positive constant \( C \), denote by

\[ \log^+ C = \max\{1, \log C\}. \]

**Proposition 2.6.** Suppose that \( b \in L^1_{\text{loc}}(\mathbb{R}) \) has its derivative \( b' \in \text{BMO}(\mathbb{R}) \). Then \( b \) satisfies

\[ |b(x) - b(0)| \leq C_5\|b'||_\ast |x|(1 + |\log |x||) \quad \forall \ x \in \mathbb{R} \]

and

\[ |b(x + h) - b(x)| \leq C_5\|b'||_\ast (\log^+ |x|) (|h|(1 + |\log |h||)) \quad \forall (x, h) \in \mathbb{R} \times \mathbb{R}. \]

**Proof.** From [27, Proposition 5] and Proposition 2.5 it follows that if

\[ y \neq 0; \ z \neq 0; \ x \in \mathbb{R}, \]
then
\[
\left| \frac{(y, b(x+y) - b(x)) - (z, b(x+z) - b(x))}{|y|^2} \right| \leq 5\|b'\|_{\text{BMO}(\mathbb{R})} + \frac{\|b'\|_{\text{BMO}(\mathbb{R})}}{\log 2} \left| \log \frac{|y|}{|z|} \right|.
\]

Letting \( x = 0 \) and \( z = 1 \) in (2.2) gives the first inequality in Proposition 2.6 via
\[
|b(y) - b(0)| \leq |y| \left( |b(1) - b(0)| + 5\|b'\|_{\text{BMO}(\mathbb{R})} + \frac{\|b'\|_{\text{BMO}(\mathbb{R})}}{\log 2} \left| \log |y| \right| \right)
\leq C_5\|b'\| |y|(1 + |\log |y|).
\]

Also, by using structure of \( \text{BMO}(\mathbb{R}) \) (cf. [18]) we see that if \( x \in \mathbb{R} \) then
\[
|b(x + 1) - b(x)| = \left| \int_x^{x+1} b' dy - \int_0^1 b' dy \right| + \left| \int_0^1 b' dy \right|
\leq 2(\log^+ |x|)\|b'\|_{\text{BMO}(\mathbb{R})} + \left| \int_0^1 b' dy \right|
\leq 2(\log^+ |x|)\|b'\|_\ast.
\]

This, along with (2.2), derives the second inequality in Proposition 2.6 via
\[
|b(x + h) - b(x)| \leq |h| \left( |b(x + 1) - b(x)| + 5\|b'\|_{\text{BMO}(\mathbb{R})} + \frac{\|b'\|_{\text{BMO}(\mathbb{R})}}{\log 2} \left| \log |h| \right| \right)
\leq C_5\|b'\|_\ast(\log^+ |x|)|h|(1 + |\log |h|).
\]

\( \square \)

3 Key a priori estimates for the flow

We say that \( \phi \) is a forward flow associated to \( b \) if for each \( s \in [0, T] \) and almost every \( x \in \mathbb{R}^n \) the map
\[
t \mapsto |b(t, \phi_s(t, x))| \text{ belongs to } L^1(s, T)
\]
and
\[
\phi_s(t, x) = x + \int_s^t b(r, \phi_s(r, x)) \, dr.
\]

If the flow starts at \( s = 0 \), then we simply denote \( \phi_0(t, x) \) by \( \phi(t, x) \).

Meanwhile, we say that \( \tilde{\phi} \) is a backward flow associated to \( b \) if for each \( t \in [0, T] \) and almost every \( x \in \mathbb{R}^n \) the map
\[
s \mapsto |b(s, \tilde{\phi}_s(s, x))| \text{ belongs to } L^1(0, t)
\]
and
\[
\tilde{\phi}_s(s, x) = x - \int_s^t b(r, \tilde{\phi}_s(r, x)) \, dr.
\]
Theorem 3.1. Let 
\[ b(t, x) : [0, T] \times \mathbb{R} \mapsto \mathbb{R} \text{ be in } L^1(0, T; C^1(\mathbb{R})) \text{ with } \int_0^T \left\| \frac{\partial b(t, \cdot)}{\partial x} \right\|_{L^\infty(\mathbb{R})} \, dt < \infty. \]

Then there exists a unique flow \( \phi(t, x) \) satisfying
\[
\begin{cases}
    \frac{\partial}{\partial t} \phi(t, x) = b(t, \phi(t, x)) & \forall (t, x) \in [0, T] \times \mathbb{R}; \\
    \phi_0(x) = x & \forall x \in \mathbb{R}.
\end{cases}
\]

Moreover, for each \( t \in [0, T] \), it holds that
\[
\left\| \log \left( \frac{\partial}{\partial x} \phi(t, x) \right) \right\|_{\text{BMO}(\mathbb{R})} \leq \frac{\int_0^t C_6 \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{\text{BMO}(\mathbb{R})} ds}{\exp \left( -C_7 \int_0^t \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{\text{BMO}(\mathbb{R})} ds \right)}.
\]

Proof. The argument is divided into four steps.

Step 1 - initialing argument. Since
\[ b(t, x) : [0, T] \times \mathbb{R} \mapsto \mathbb{R} \]
satisfies
\[ b \in L^1(0, T; C^1(\mathbb{R})) \text{ with } \frac{\partial b(t, x)}{\partial x} \in L^1(0, T; L^\infty(\mathbb{R})), \]
the classical Cauchy-Lipschitz theory produces a unique flow \( \phi_s(t, x) \) with
\[
\begin{cases}
    \frac{\partial}{\partial t} \phi_s(t, x) = b(t, \phi_s(t, x)) & \forall (t, x) \in [s, T] \times \mathbb{R}; \\
    \phi_s(s, x) = x & \forall x \in \mathbb{R}.
\end{cases}
\]

Moreover, for each \( t \in [s, T] \), \( \phi_s(t, \cdot) \) is a bi-Lipschitz map on \( \mathbb{R} \). Differentiating the equation with respect to the spatial direction, we have
\[
\begin{align*}
    \frac{\partial}{\partial x} \left( \frac{\partial}{\partial t} \phi_s(t, x) \right) &= \left( \frac{\partial}{\partial x} b(t, \phi_s(t, x)) \right) \frac{\partial}{\partial x} \phi_s(t, x); \\
    \frac{\partial}{\partial t} \log \left| \frac{\partial}{\partial x} \phi_s(t, x) \right| &= \frac{\partial}{\partial x} b(t, \phi_s(t, x)).
\end{align*}
\]

As \( \phi_s(t, \cdot) \) is a bi-Lipschitz map on \( \mathbb{R} \) for each \( t \in [s, T] \), its \( x \)-derivative has lower and upper bounds, i.e.,
\[ e^{-\int_s^t A(r) \, dr} \leq \left| \frac{\partial}{\partial x} \phi_s(t, x) \right| \leq e^{\int_s^t A(r) \, dr}, \]
where
\[ A(r) = \left\| \frac{\partial}{\partial x} b(r, \cdot) \right\|_{L^\infty(\mathbb{R})}. \]
In particular, this implies that for each \( t \), the function
\[
\left| \frac{\partial}{\partial x} \phi_s(t, \cdot) \right|
\]
is an \( A_{\infty}(\mathbb{R}) \)-weight with
\[
\left[ \frac{\partial}{\partial x} \phi_s(t, \cdot) \right]_{A_{\infty}(\mathbb{R})} \leq e^{2 \int_r^s A(r) \, dr}.
\]
Note that the same estimate holds for the backward flow \( \tilde{\phi}_t(s, x) \), which is the inverse of \( \phi_s(t, x) \).

Upon applying Lemma 2.2, we achieve
\[
\left\| \log \left| \frac{\partial}{\partial x} \phi_s(t, \cdot) \right| \right\|_{BMO(\mathbb{R})}
= \left\| \int_s^r \frac{\partial}{\partial x} b(r, \phi_s(r, \cdot)) \, dr \right\|_{BMO(\mathbb{R})}
\leq \int_s^r \left\| \frac{\partial}{\partial x} b(r, \phi_s(r, \cdot)) \right\|_{BMO(\mathbb{R})} \, dr
\leq \int_s^r C_3 \left\| \frac{\partial}{\partial x} b(r, \cdot) \right\|_{BMO(\mathbb{R})} \left[ \frac{\partial}{\partial x} \tilde{\phi}_s(s, \cdot) \right]_{A_{\infty}(\mathbb{R})} \, dr
\leq \int_s^r C_3 \left\| \frac{\partial}{\partial x} b(r, \cdot) \right\|_{BMO(\mathbb{R})} e^{2 \int_r^s A(z) \, dz} \, dr.
\]

**Step 2 - starting from short time.** By letting \( T_0 > s \geq 0 \) be small enough with
\[
\int_s^{T_0} C_3 \left\| \frac{\partial}{\partial x} b(r, \cdot) \right\|_{BMO(\mathbb{R})} e^{2 \int_r^s A(z) \, dz} \, dr < \epsilon_0,
\]
where \( \epsilon_0 \) is as in Lemma 2.4, we utilize (3.1) to get
\[
\sup_{s \leq t \leq T_0} \left\{ \left\| \log \left| \frac{\partial}{\partial x} \phi_s(t, \cdot) \right| \right\|_{BMO(\mathbb{R})}, \left\| \log \left| \frac{\partial}{\partial x} \tilde{\phi}_s(s, \cdot) \right| \right\|_{BMO(\mathbb{R})} \right\} < \epsilon_0.
\]

Hence, by applying Lemma 2.4, we see
\[
\left[ \left| \frac{\partial}{\partial x} \phi_s(t, \cdot) \right| \right]_{A_{\infty}(\mathbb{R})} < 1 + C_4 \left\| \log \left| \frac{\partial}{\partial x} \phi_s(t, \cdot) \right| \right\|_{BMO(\mathbb{R})}.
\]

Inserting this estimate into (3.1), we conclude
\[
\left\| \log \left| \frac{\partial}{\partial x} \phi_s(t, \cdot) \right| \right\|_{BMO(\mathbb{R})} \leq \int_s^r C_3 \left\| \frac{\partial}{\partial x} b(r, \cdot) \right\|_{BMO(\mathbb{R})} \left[ \frac{\partial}{\partial x} \tilde{\phi}_s(s, \cdot) \right]_{A_{\infty}(\mathbb{R})} \, ds
\leq \int_s^r C_3 \left\| \frac{\partial}{\partial x} b(r, \cdot) \right\|_{BMO(\mathbb{R})} \left( 1 + C_4 \left\| \log \left| \frac{\partial}{\partial x} \tilde{\phi}_s(s, \cdot) \right| \right\|_{BMO(\mathbb{R})} \right) ds.
\]
Set
\[ I_s(t) = \sup_{s \leq s_0 \leq t} \left\{ \left\| \log \left| \frac{\partial}{\partial x} \phi_s(r, \cdot) \right| \right\|_{\text{BMO}(\mathbb{R})}, \left\| \log \left| \frac{\partial}{\partial x} \phi_r(s, \cdot) \right| \right\|_{\text{BMO}(\mathbb{R})} \right\}. \]

The above estimates yield
\[ I_s(t) \leq \int_s^t C_3 \left\| \frac{\partial}{\partial x} b(r, \cdot) \right\|_{\text{BMO}(\mathbb{R})} (1 + C_4 I_s(r)) \, dr \quad \forall \ t \in [s, T_0]. \]

The Gronwall inequality then implies
\[ I_s(t) \leq \exp \left( - \int_s^t C_3 C_4 \left\| \frac{\partial}{\partial x} b(r, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \, dr \right) \quad \forall \ t \in [s, T_0]. \]

**Step 3 - removing the dependence of Lipschitz constant.** Let \( T_1 \in (s, T) \) obey

\[ \int_s^{T_1} C_3 \left\| \frac{\partial}{\partial x} b(r, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \, dr \exp \left( C_3 C_4 \int_s^{T_1} \left\| \frac{\partial}{\partial x} b(r, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \, dr \right) \leq 2^{-1} \epsilon_0, \]

We claim that (3.2) holds for all \( t \in (s, T_1] \).

If \( T_1 \leq T_0 \), then the claim follows from (3.2).

Suppose now \( T_0 < T_1 \). Assume that for some \( t_0 \in [T_0, T_1) \), (3.2) holds for all \( t \in (s, t_0] \). Then

\[ I_s(t_0) \leq \frac{\int_s^{t_0} C_3 \left\| \frac{\partial}{\partial x} b(r, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \, dr \exp \left( - \int_s^{t_0} C_3 C_4 \left\| \frac{\partial}{\partial x} b(r, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \, dr \right)}{2^{-1} \epsilon_0}. \]

Since
\[ \frac{\partial b(t, \cdot)}{\partial x} \in L^1(0, T; L^\infty(\mathbb{R})), \]

We can choose \( t_1 \in (t_0, T_1] \) such that

\[ \int_{t_1}^{t_0} C_3 \left\| \frac{\partial}{\partial x} b(r, \cdot) \right\|_{\text{BMO}(\mathbb{R})} e^{2 \int_0^{t_1} \lambda(s) ds} \, dr < \epsilon_0 \]

and

\[ \int_{t_1}^{t_0} C_3 C_4 \left\| \frac{\partial}{\partial x} b(r, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \, dr \exp \left( - \int_{t_1}^{t_0} C_3 C_4 \left\| \frac{\partial}{\partial x} b(r, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \, dr \right) < \frac{\epsilon_0}{2C_3(1 + C_4 2^{-1} \epsilon_0)}. \]
The same argument as in proving (3.2) then implies that for \( t_0 < t \leq t_1 \) it holds

\[
I(t_0) \leq \frac{\int_{t_0}^t C_3 \left\| \frac{\partial}{\partial x} b(r, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \, dr}{\exp \left( -\int_{t_0}^t C_3 C_4 \left\| \frac{\partial}{\partial x} b(r, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \, dr \right)} < \frac{\epsilon_0}{2C_3(1 + C_4 2^{-1} \epsilon_0)}.
\]

For any \( t \in (t_0, t_1] \), we have via the semigroup property of the flow that

\[
\phi_s(t, x) = \phi_{t_0}(t, \phi_s(t_0, x)).
\]

By applying Lemma 2.2, Lemma 2.4 and (3.6), we find

\[
\left\| \log \left| \frac{\partial}{\partial x} \phi_s(t, \cdot) \right| \right\|_{\text{BMO}(\mathbb{R})} = \left\| \log \left| \frac{\partial}{\partial x} \phi_{t_0}(t, \phi_s(t_0, \cdot)) \right| \right\|_{\text{BMO}(\mathbb{R})} \\
\leq \left\| \log \left| \frac{\partial}{\partial x} \phi_{t_0}(t, \cdot) \right| \right\|_{\text{BMO}(\mathbb{R})} + \left\| \log \left| \frac{\partial}{\partial x} \phi_{t_0}(t_0, \cdot) \right| \right\|_{\text{BMO}(\mathbb{R})} \\
\leq C_3 \left\| \log \left| \frac{\partial}{\partial x} \phi_{t_0}(t, \cdot) \right| \right\|_{\text{BMO}(\mathbb{R})} + \left\| \log \left| \frac{\partial}{\partial x} \phi_{t_0}(t_0, \cdot) \right| \right\|_{\text{BMO}(\mathbb{R})} \\
< \frac{\epsilon_0 C_3(1 + C_4 2^{-1} \epsilon_0)}{2C_3(1 + C_4 2^{-1} \epsilon_0)} + \frac{\epsilon_0}{2} \\
= \epsilon_0.
\]

This derives

\[
\sup_{s \leq s \leq t_1} \left\{ \left\| \log \left| \frac{\partial}{\partial x} \phi_s(t, \cdot) \right| \right\|_{\text{BMO}(\mathbb{R})}, \left\| \log \left| \frac{\partial}{\partial x} \phi_t(s, \cdot) \right| \right\|_{\text{BMO}(\mathbb{R})} \right\} < \epsilon_0.
\]

Using this estimate in Step 2, we further have the following estimate

\[
\sup_{s \leq s \leq t_1} \left\{ \left\| \log \left| \frac{\partial}{\partial x} \phi_s(t, \cdot) \right| \right\|_{\text{BMO}(\mathbb{R})}, \left\| \log \left| \frac{\partial}{\partial x} \phi_t(s, \cdot) \right| \right\|_{\text{BMO}(\mathbb{R})} \right\} \\
\leq \int_s^{t_1} C_3 \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \exp \left( C_3 C_4 \int_s^{t_1} \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \, ds \right) \, ds \\
< 2^{-1} \epsilon_0,
\]

which implies that (3.2) holds for all \( t \in (s, t_1] \).

Since in (3.4) and (3.5) the extension of time only depends on \( b \) itself, we may iterate this argument finite times and conclude that (3.2) holds for all \( t \in (s, T_1] \).

**Step 4 - completing argument.** Since \( b \) satisfies

\[
\frac{\partial b(t, \cdot)}{\partial x} \in L^1(0, T; L^\infty(\mathbb{R})),
\]

we may choose a sequence of increasing numbers \( \{T_i\}_{i=1,...,k_0} \) such that \( T_1 = 0, T_{k_0} = T \) and

\[
\frac{\int_{T_i}^{T_{i+1}} C_3 \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{BMO(\mathbb{R})} ds}{\exp \left( -\int_{T_i}^{T_{i+1}} C_3 C_4 \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{BMO(\mathbb{R})} ds \right)} = 2^{-1} \epsilon_0 \quad \forall \ i \in \{1, ..., k_0 - 2\},
\]

and

\[
\frac{\int_{T_{k_0-1}}^{T_{k_0}} C_3 \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{BMO(\mathbb{R})} ds}{\exp \left( -\int_{T_{k_0-1}}^{T_{k_0}} C_3 C_4 \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{BMO(\mathbb{R})} ds \right)} \leq 2^{-1} \epsilon_0
\]

If \( t \in (T_1, T_2] \), then \textbf{Step 3} gives

\[
(3.7) \quad \left\| \log \left\| \frac{\partial}{\partial x} \phi(t, \cdot) \right\|_{BMO(\mathbb{R})} \right\|_{BMO(\mathbb{R})} \leq \int_0^t C_3 \left\| \frac{\partial}{\partial x} b(r, \cdot) \right\|_{BMO(\mathbb{R})} \exp \left( -C_3 C_4 \int_0^r \left\| \frac{\partial}{\partial x} b(r, \cdot) \right\|_{BMO(\mathbb{R})} dr \right) ds.
\]

Suppose that \( t \) belongs to some \((T_i, T_{i+1}]\) with \( 2 \leq i \leq k_0 - 1 \).

By using the semigroup property of the flow \( \phi \), we have

\[
\phi(t, x) = \phi_{T_i}(t, \cdot) \circ \phi_{T_{i-1}}(T_i, \cdot) \circ \cdots \circ \phi_{T_2}(T_2, x).
\]

By using Lemma 2.2, Lemma 2.4 and \textbf{Step 3}, we conclude

\[
\left\| \log \left\| \frac{\partial}{\partial x} \phi(t, \cdot) \right\|_{BMO(\mathbb{R})} \right\|_{BMO(\mathbb{R})} = \left\| \log \left\| \frac{\partial}{\partial x} \phi_{T_2}(t, \phi_{T_1}(T_2, \cdot)) \right\|_{BMO(\mathbb{R})} \right\|_{BMO(\mathbb{R})}
\]

\[
\leq \left\| \log \left\| \frac{\partial}{\partial x} \phi_{T_2}(t, z) \right\|_{z = \phi_{T_1}(T_2, \cdot)} \right\|_{BMO(\mathbb{R})} + \left\| \log \left\| \frac{\partial}{\partial x} \phi_{T_1}(T_2, \cdot) \right\|_{BMO(\mathbb{R})} \right\|_{BMO(\mathbb{R})}
\]

\[
\leq C_3 \left\| \log \left\| \frac{\partial}{\partial x} \phi_{T_2}(t, \cdot) \right\|_{BMO(\mathbb{R})} \right\|_{BMO(\mathbb{R})} + \left\| \log \left\| \frac{\partial}{\partial x} \phi_{T_1}(T_2, \cdot) \right\|_{BMO(\mathbb{R})} \right\|_{BMO(\mathbb{R})}
\]

\[
\leq C_3 (1 + C_4 2^{-1} \epsilon_0) \left\| \log \left\| \frac{\partial}{\partial x} \phi_{T_2}(t, \cdot) \right\|_{BMO(\mathbb{R})} \right\|_{BMO(\mathbb{R})} + 2^{-1} \epsilon_0
\]

\[
\leq C_3 (1 + C_4) \left\| \log \left\| \frac{\partial}{\partial x} \phi_{T_2}(t, \cdot) \right\|_{BMO(\mathbb{R})} \right\|_{BMO(\mathbb{R})} + 1
\]
\[ \leq (C_3(1 + C_4))^2 \left\| \log \left| \frac{\partial}{\partial x} \phi_{T_3(t, \cdot)} \right| \right\|_{BMO(\mathbb{R})} + C_3(1 + C_4) + 1 \]
\[ \leq \cdots \]
\[ \leq (C_3(1 + C_4))^{i-1} \left\| \log \left| \frac{\partial}{\partial x} \phi_{T_i(t, \cdot)} \right| \right\|_{BMO(\mathbb{R})} + \sum_{j=0}^{i-2} (C_3(1 + C_4))^j \]
\[ \leq (C_3(1 + C_4) + 1)^i. \]

Let \( \delta_0 > 0 \) obey
\[ C_3 \delta_0 e^{C_4 \delta_0} = 2^{-1} \epsilon_0. \]

As
\[
\begin{cases}
\epsilon_0 \leq 1; \\
\delta_0 < 1; \\
t \in (T_i, T_{i+1}],
\end{cases}
\]

by our choice of \( \{T_i\} \) we find
\[ (i - 1) \delta_0 < \int_0^t \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{BMO(\mathbb{R})} ds \leq i \delta_0, \]

whence
\[
\left\| \log \left| \frac{\partial}{\partial x} \phi(t, \cdot) \right| \right\|_{BMO(\mathbb{R})} \leq \frac{\frac{1}{\delta_0} \int_0^t \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{BMO(\mathbb{R})} ds}{\exp \left( -C \int_0^t \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{BMO(\mathbb{R})} ds \right)}.
\]

This, together with (3.7), implies
\[
\left\| \log \left| \frac{\partial}{\partial x} \phi(t, \cdot) \right| \right\|_{BMO(\mathbb{R})} \leq \frac{\int_0^t C_3 \left\| \frac{\partial}{\partial x} b(r, \cdot) \right\|_{BMO(\mathbb{R})} dr \right\|_{BMO(\mathbb{R})} ds}{\exp \left( -C \int_0^t \left\| \frac{\partial}{\partial x} b(r, \cdot) \right\|_{BMO(\mathbb{R})} dr \right) ds},
\]
as desired.

\[ \square \]

Rather surprisingly, the hypothesis
\[ \int_0^T \left\| \frac{\partial b(t, \cdot)}{\partial x} \right\|_{L^\infty(\mathbb{R})} dt < \infty \]

in Theorem 3.1 can be replaced by a weaker one
\[ \int_0^T \left\| \frac{\partial b(t, \cdot)}{\partial x} \right\| dt < \infty \]
in the following assertion.
Theorem 3.2. Let

\[ b(t, x) : [0, T] \times \mathbb{R} \mapsto \mathbb{R} \text{ be in } L^1(0, T; C^1(\mathbb{R})) \text{ with } \int_0^T \left\| \frac{\partial b(t, \cdot)}{\partial x} \right\|_* \, dt < \infty. \]

Then there exists a unique flow \( \phi(t, x) \) satisfying

\[
\begin{align*}
\frac{\partial}{\partial t} \phi(t, x) &= b(t, \phi(t, x)) \quad \forall \ (t, x) \in [0, T] \times \mathbb{R}; \\
\phi_0(x) &= x \quad \forall \ x \in \mathbb{R}.
\end{align*}
\]

Moreover

\[
\left\| \log \left| \frac{\partial}{\partial x} \phi(t, x) \right| \right\|_{\text{BMO}(\mathbb{R})} \leq \frac{\int_0^t 2C_6 \left| \frac{\partial}{\partial x} b(s, \cdot) \right|_{\text{BMO}(\mathbb{R})} \, ds}{\exp \left( -2C_7 \int_0^t \left| \frac{\partial}{\partial x} b(s, \cdot) \right|_{\text{BMO}(\mathbb{R})} \, ds \right)} \quad \forall \ t \in [0, T].
\]

Proof. The existence and uniqueness has essentially been established in [27]. So it remains to verify the last \( \text{BMO}(\mathbb{R}) \)-size estimate.

For each \((k, t) \in \mathbb{N} \times [0, T]\) set

\[
\begin{align*}
v_k(t, x) &= \min \{ \max \{ -k, \partial_x b(t, x) \}, k \}; \\
b_k(t, x) &= b(t, 0) + \int_0^t v_k(t, y) \, dy.
\end{align*}
\]

Then

\[
\begin{align*}
\partial_x b_k(t, \cdot) &\in L^1(0, T; L^\infty(\mathbb{R})); \\
\left\| v_k(t, \cdot) \right\|_{\text{BMO}(\mathbb{R})} &\leq 2 \left\| \partial_x b(t, \cdot) \right\|_{\text{BMO}(\mathbb{R})}; \\
\left\| v_k(t, \cdot) \right\|_* &\leq 2 \left\| \partial_x b(t, \cdot) \right\|_*.
\end{align*}
\]

In accordance with Propositions 2.5, 2.6, we see that \{\(b_k\) and \(b\) satisfy the Zygmund condition with a uniform constant.

Let \{\(\phi_k, \phi\) be the unique flow pair generalized by \{\(b_k(t, x), b(t, x)\). Then by [27] Proposition 4, we see that \(\phi(t, \cdot) \) and \(\phi_k(t, \cdot) \) are locally H"{o}lder continuous on \(\mathbb{R}\) for each \(t \in [0, T]\). Moreover for each compact set \(K \subset \mathbb{R}\), both \(\phi(t, \cdot) \) and \(\phi_k(t, \cdot) \) are H"{o}lder continuous on \(K\) for each \(t \in [0, T]\) with the H"{o}lder exponent and constant depending only on

\[
\int_0^t \left\| \partial_x b(s, \cdot) \right\|_* \, ds.
\]

On the other hand, by the construction of \(b_k\) and Proposition 2.6 we have

\[
|b_k(t, x) - b(t, 0)| \leq C_5 \left\| v_k(t, \cdot) \right\|_* |x| (1 + |\log |x||) \leq 2C_5 \left\| \partial_x b(t, \cdot) \right\|_* |x| (1 + |\log |x||),
\]

Moreover

\[
\left\| \frac{\partial}{\partial x} \phi(t, x) \right\|_{\text{BMO}(\mathbb{R})} \leq \frac{\int_0^t 2C_6 \left| \frac{\partial}{\partial x} b(s, \cdot) \right|_{\text{BMO}(\mathbb{R})} \, ds}{\exp \left( -2C_7 \int_0^t \left| \frac{\partial}{\partial x} b(s, \cdot) \right|_{\text{BMO}(\mathbb{R})} \, ds \right)} \quad \forall \ t \in [0, T].
\]
thereby getting that

$$\{ |\phi_k(t, x)| : (t, x) \in [0, T] \times K \}$$

is uniformly bounded. Denote by

$$C_8(K) := \sup \{ |\phi_k(t, x)| + |\phi(t, x)| : (t, x, k) \in [0, T] \times K \times \mathbb{N} \}.$$

Then it holds for each $x \in K$ and all $0 \leq s < t \leq T$ that

$$|\phi_k(t, x) - \phi_k(s, x)| \leq \int_s^t |b_k(r, \phi_k(r, x))| \, dr$$

$$\leq \int_s^t (|b(r, 0)| + 2C_5||\partial_s b(r, \cdot)||C_8(K)(1 + |\log |C_8(K)||)) \, dr.$$

This, together with the previous discussion on the Hölder continuity in the spatial direction, implies that $\{\phi_k\}_k$ are equicontinuous on $[0, T] \times K$. Applying the Arzelá-Ascoli theorem, we conclude that there is a subsequence of $\{\phi_k\}_k$, denoted by $\{\phi_{K, k}\}_k$, such that $\phi_{K, k}$ converges uniformly on $[0, T] \times K$.

By construction we have

$$b_k(t, x) \to b(t, x) \text{ as } k \to \infty,$$

thereby concluding that if $(t, x) \in [0, T] \times K$ then

$$\lim_{k \to \infty} \phi_{K, k}(t, x) = x + \lim_{k \to \infty} \int_0^t b_{K, k}(s, \phi_{K, k}(s, x)) \, ds$$

$$= x + \lim_{k \to \infty} \int_0^t \int_0^{\phi_{K, k}(s, x)} [v_{K, k}(s, y) - \partial_s b(s, y)] \, dy \, ds + \lim_{k \to \infty} \int_0^t b(s, \phi_{K, k}(s, x)) \, ds.$$

Since

$$|\phi_k(s, x)| \leq C_8(K),$$

one has

$$\left| \int_0^t \int_0^{\phi_{K, k}(s, x)} [v_{K, k}(s, y) - \partial_s b(s, y)] \, dy \, ds \right| \leq \int_0^t \int_{-C_8(K)} |\partial_s b(s, y)| \, dy \, ds < \infty,$$

and hence the dominated convergence theorem and continuity of $b(t, \cdot)$ guarantee

$$\lim_{k \to \infty} \phi_{K, k}(t, x) = x + \int_0^t b(s, \lim_{k \to \infty} \phi_{K, k}(s, x)) \, ds.$$

By choosing a sequence of increasing compacts $K_j$ such that $\mathbb{R} = \bigcup_j K_j$ and passing to further subsequences, we see that there is a subsequence of $\{\phi_k\}$, still denoted by $\{\phi_{K, k}\}$, such that $\phi_{K, k}(t, x)$ converges on $[0, T] \times \mathbb{R}$, and uniformly on any compact subset $[0, T] \times \tilde{K}$, and consequently,

$$\lim_{k \to \infty} \phi_{K, k}(t, x) = x + \int_0^t b(s, \lim_{k \to \infty} \phi_{K, k}(s, x)) \, ds, \forall (t, x) \in [0, T] \times \mathbb{R}.$$
By the uniqueness, we see that
\[ \phi(t, x) = \lim_{k \to \infty} \phi_{K,k}(t, x), \forall (t, x) \in [0, T] \times \mathbb{R}, \]
and the convergence is uniform on any compact set.

Since
\[ b(t, x) \in L^1(0, T; C^1(\mathbb{R})), \]
and so is any \( b_k(t, x) \). Accordingly, the proof of Theorem 3.1 yields that if \( (t, x) \in [0, T] \times \mathbb{R} \) then
\[ \log \left| \frac{\partial}{\partial x} \phi(t, x) \right| = \int_0^t \frac{\partial}{\partial x} b(s, \phi(s, x)) \, ds \]
\[ = \int_0^t \lim_{k \to \infty} v_k(s, \phi_k(s, x)) \, ds \]
\[ = \lim_{k \to \infty} \log \left| \frac{\partial}{\partial x} \phi_k(t, x) \right|. \]

By (3.8) and Theorem 3.1, we see that for each \( k \in \mathbb{N} \), it holds
\[ \left\| \log \left| \frac{\partial}{\partial x} \phi_k(t, x) \right| \right\|_{BMO(\mathbb{R})} \leq \frac{\int_0^t 2C_6 \left\| \frac{\partial}{\partial x} b(s, \phi(s, x)) \right\|_{BMO(\mathbb{R})} \, ds}{\exp \left( -2C_7 \int_0^t \left\| \frac{\partial}{\partial x} b(s, \phi(s, x)) \right\|_{BMO(\mathbb{R})} \, ds \right)}. \]

By this, the weak-* compactness in \( BMO(\mathbb{R}) \), and the pointwise convergence of \( \frac{\partial}{\partial x} \phi_k(t, x) \),
we conclude that the last estimation holds also for
\[ \log \left| \frac{\partial}{\partial x} \phi(t, x) \right|, \]
thereby completing the proof. \( \Box \)

4 Proof of main results

Proof of Theorem 1.1 The argument consists of three steps.

Step 1 - an Orlicz space estimate. Let \( \mu \) denote the Gaussian measure on \( \mathbb{R} \), i.e.,
\[ \mu(x) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{|x|^2}{2} \right), \]
and \( \text{div}_\mu b \) denotes the distributional divergence of \( b \) with respect to \( \mu \). We say that a measurable function
\[ f \in \text{Exp}_\mu \left( \frac{L}{\log L} \right), \]
provided
\[ \|f\|_{\text{Exp}, \left(\exp_{\frac{t}{x^2}}\right)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}} \left[ \exp \left( \frac{|f(x)|/\lambda}{1 + \log^+ (|f(x)|/\lambda)} \right) - 1 \right] d\mu \leq 1 \right\}. \]

Let \( b(t, x) \) obey (1.1). Then
\[ (4.1) \]
\[
\begin{cases}
\frac{b(t, x)}{1 + |x| \log^+ |x|} \in L^1(0, T; L^\infty(\mathbb{R})); \\
\text{div}_t b(t, x) \in L^1(0, T; \text{Exp}_x (\frac{t}{\log L})).
\end{cases}
\]

As a matter of fact, the first estimate of (4.1) follows from Proposition 2.5 as
\[ \frac{|b(t, x)|}{1 + |x| \log^+ |x|} \leq \frac{|b(t, x) - b(t, 0) + b(t, 0)|}{1 + |x| \log^+ |x|} \leq |b(t, 0)| + C \left\| \frac{\partial}{\partial x} b(t, \cdot) \right\|_{s}. \]

To verify the second relation in (4.1), set
\[ \beta(t) = |b(t, 0)| + C \left\| \frac{\partial}{\partial x} b(t, \cdot) \right\|_{\text{BMO}(\mathbb{R})}. \]

Noting that
\[ \int_{\mathbb{R}} \exp \left( \frac{c|x| b(t, x)}{1 + \log^+ (c|x| b(t, x))} \right) d\mu(x) \leq \int_{\mathbb{R}} \exp \left( \frac{c|x| (1 + |x| \log^+ |x|) \beta(t)}{1 + \log^+ (c|x| (1 + |x| \log^+ |x|) \beta(t))} \right) d\mu(x), \]
we obtain
\[ \|xb(t, x)\|_{\text{Exp}, \left(\exp_{\frac{t}{x^2}}\right)} \leq C\beta(t). \]

On the other hand, for a BMO \((\mathbb{R})\)-function \( f \), we utilize the John-Nirenberg inequality:
\[ ||x I : |f(x) - f_I| > \lambda|| \leq c_1 |I| \exp \left( \frac{-c_2 \lambda}{\|f\|_{\text{BMO}(\mathbb{R})}} \right) \forall \text{ interval } I \subset \mathbb{R} \]
to obtain that if
\[ I = [x - r, x + 1]; \]
\[ (x, r) \in \mathbb{R} \times [1, \infty); \]
\[ \gamma(t) = \left\| \frac{\partial}{\partial x} b(t, \cdot) \right\|_{s}; \]
\[ \alpha = c_2 (2\gamma(t))^{-1}, \]
then
\[ |f_I| \leq |f_{I - [1, 1]}| + |f_{[-1, 1]}| \leq C(1 + \log^+ |x|)\|f\|_s, \]
and hence
\[ \int_{\mathbb{R}} \exp \left( \alpha \left| \frac{\partial}{\partial x} b(t, x) \right| \right) d\mu(x) \]
\[ \leq \int_{[-1, 1]} \exp \left( \alpha \left| \frac{\partial}{\partial x} b(t, x) \right| \right) d\mu(x) + \sum_{k=1}^{\infty} \left( \int_{[2^k-1, 2^k]} + \int_{[-2^k, -2^{k-1}]} \right) \exp \left( \alpha \left| \frac{\partial}{\partial x} b(t, x) \right| \right) d\mu(x) \]
\[ \leq e^{2\gamma(t)} \sum_{k=0}^{\infty} a 2^k e^{-2^k + 1 + k} \left( \frac{\alpha \gamma(t)}{c_2 - \alpha \gamma(t)} \right) \leq C. \]

Consequently we achieve the desired inequality

\[ \left\| \frac{\partial}{\partial x} b(t, \cdot) \right\|_{\text{Exp}_p(\mathbb{R}^d)} \leq \left\| \frac{\partial}{\partial x} b(t, \cdot) \right\|_{\text{Exp}_p(L)} \leq C \left\| \frac{\partial}{\partial x} b(t, \cdot) \right\|_{\text{Exp}_p(\mathbb{R}^d)}. \]

**Step 2 - existence-uniqueness-size of flow.** Under (1.1) we conclude via Proposition 2.5 for a.e. \( q \) for any \( t \), that \( b \) is in the Zygmund class, which implies that the flow exists and is unique; see [27] for instance.

Moreover, from **Step 1** above it follows that \( b \) satisfies requirements from [9, Main Theorem] and so that \( \phi(t, x) \) is absolutely continuous and differentiable. Indeed, by using [9, Theorem 1.2] and that \( b(t, \cdot) \) is in the Zygmund class, one can deduce that

\[ \left| \frac{\partial}{\partial x} \phi(t, \cdot) \left( 1 + \log \left| \frac{\partial}{\partial x} \phi(t, \cdot) \right| \right) \right|^{q} \in L^{1}_{\text{loc}}(\mathbb{R}) \]

for any \( q \in [1, \infty) \). As \( \partial_t b(t, x) \in \text{BMO}(\mathbb{R}) \) is locally exponentially integrable, we deduce that

\[ \frac{\partial}{\partial t} \left( \frac{\partial}{\partial x} \phi(t, x) \right) = \left( \frac{\partial}{\partial z} b(t, z)_{|z=\phi(t, x)} \right) \frac{\partial}{\partial x} \phi(t, x) \]

and

\[ \log \left| \frac{\partial}{\partial x} \phi(t, x) \right| = \int_{0}^{t} \frac{\partial}{\partial x} b(s, \phi(s, x)) \, ds. \]

For \( \epsilon > 0 \) and \( x \in \mathbb{R} \) set

\[ \begin{cases} 0 \leq \rho \in C^\infty_c(\mathbb{R}); \\
\text{supp} \rho \subset (-1, 1); \\
\int_{\mathbb{R}} \rho(x) \, dx = 1; \\
\rho_\epsilon(x) = \frac{1}{\epsilon} \rho \left( \frac{x}{\epsilon} \right); \\
b_\epsilon(t, x) = b(t, \cdot) * \rho_\epsilon(x). \end{cases} \]

Note that

\[ \frac{\partial}{\partial x} b(t, x) \in L^1(0, T; \text{BMO}(\mathbb{R})) \Rightarrow \frac{\partial}{\partial x} b_\epsilon(t, x) \in L^1(0, T; \text{BMO}(\mathbb{R})) \cap L^1(0, T; C^\infty(\mathbb{R})). \]

Thus we have

\[ \int_{0}^{t} \left\| \frac{\partial}{\partial x} b_\epsilon(s, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \, ds \leq \int_{0}^{t} \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \, ds \quad \forall \ t \in (0, T] \]
and so for any $\epsilon \in (0, 1)$
\[
\left\| \frac{\partial}{\partial x} b_\epsilon(t, \cdot) \right\|_{BMO(\mathbb{R})} \leq 2 \left\| \frac{\partial}{\partial x} b(t, \cdot) \right\|_{BMO(\mathbb{R})} \text{ for a.e. } t \in (0, T].
\]

Let $\phi_\epsilon(t, x)$ be the flow generated by $b_\epsilon$, i.e.,
\[
\frac{\partial}{\partial t} \phi_\epsilon(t, x) = b_\epsilon(t, \phi_\epsilon(t, x)).
\]

Then Theorem 3.2 is utilized to imply
\[
\left\| \log \left| \frac{\partial}{\partial x} \phi_\epsilon(t, \cdot) \right| \right\|_{BMO(\mathbb{R})} \leq \frac{\int_0^t 2C_6 \left\| \frac{\partial}{\partial x} b_\epsilon(s, \cdot) \right\|_{BMO(\mathbb{R})} ds}{\exp\left(-2C_7 \int_0^t \left\| \frac{\partial}{\partial x} b_\epsilon(s, \cdot) \right\|_{BMO(\mathbb{R})} ds\right)} \forall \epsilon > 0.
\]

The proof of [9 Main Theorem] infers that, up to a subsequence $\{\epsilon_k\}_{k \in \mathbb{N}}$,
\[
\lim_{k \to \infty} \phi_{\epsilon_k}(t, x) = \phi(t, x) \forall t \in (0, T].
\]

From this, (4.2) and the weak-* compactness in $BMO(\mathbb{R})$, we conclude that $\frac{\partial}{\partial x} \phi$ is the weak-* limit of $\frac{\partial}{\partial x} \phi_{\epsilon_k}$ for each $t \in (0, T]$. This implies
\[
\left\| \log \left| \frac{\partial}{\partial x} \phi(t, \cdot) \right| \right\|_{BMO(\mathbb{R})} \leq \frac{\int_0^t 2C_6 \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{BMO(\mathbb{R})} ds}{\exp\left(-2C_7 \int_0^t \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{BMO(\mathbb{R})} ds\right)},
\]

namely, the size estimate (1.2) holds.

**Step 3 - $A_{\infty}(\mathbb{R})$ density of flow.** It remains to show that for each $t \in [0, T]$,
\[
\left| \frac{\partial}{\partial x} \phi(t, \cdot) \right|
\]
is an $A_{\infty}(\mathbb{R})$-weight. But, from Theorem 1.2 (to be proved later on), we see that
\[
u_0 \in BMO(\mathbb{R}) \Rightarrow u_0 \circ \phi(t, \cdot) \in BMO(\mathbb{R}) \forall t \in (0, T].
\]

Then we apply [21 Theorem] to conclude that for each $t \in [0, T]$,
\[
\left| \frac{\partial}{\partial x} \phi(t, x) \right|
\]
is an $A_{\infty}(\mathbb{R})$-weight.
Proof of Theorem 1.2. The argument consists of three steps.

**Step 1 - existence of solution.** Let $\phi$ be the flow generated by $b$, i.e.,

$$
\begin{cases}
\frac{\partial}{\partial t} \phi(t, x) = b(t, \phi(t, x)) & \forall (t, x) \in (0, T] \times \mathbb{R}; \\
\phi_0(x) = x & \forall x \in \mathbb{R}.
\end{cases}
$$

Then the same proof of [10, Theorem 1] derives that $u_0 \circ \phi$ is a solution to the transport equation.

**Step 2 - size of solution.** Let $\epsilon_0$ be the same as in Lemma 2.4 and

$$
\delta_0 > 0 \text{ and } 2C_6\delta_0 e^{2C_7\delta_0} = 2^{-1}\epsilon_0.
$$

We choose a sequence of increasing numbers

$$
0 = T_0 < T_1 < \cdots < T_{k_0} = T
$$

such that

$$
\frac{\int_{T_{i-1}}^{T_i} 2C_6 \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \, ds}{\exp \left( - \int_{T_{i-1}}^{T_i} 2C_7 \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \, ds \right)} = 2^{-1}\epsilon_0 \quad \forall \ i \in \{1, \ldots, k_0 - 1\},
$$

and

$$
\frac{\int_{T_{k_0-1}}^{T_{k_0}} 2C_6 \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \, ds}{\exp \left( - \int_{T_{k_0-1}}^{T_{k_0}} 2C_7 \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \, ds \right)} \leq 2^{-1}\epsilon_0.
$$

Suppose that $t$ belongs to

some $(T_i, T_{i+1}]$ where $i = 0, \ldots, k_0 - 1$.

If $i = 0$, then by Lemma 2.2 and Lemma 2.4 we obtain

$$
\|u(t, \cdot)\|_{\text{BMO}(\mathbb{R})} \leq C_3 \|u_0\|_{\text{BMO}(\mathbb{R})} \left( 1 + C_4 \left\| \log \left\| \frac{\partial}{\partial x} \tilde{\phi}_t(0, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \right) 
$$

$$
(4.3)
$$

$$
\leq C_3 \|u_0\|_{\text{BMO}(\mathbb{R})} \left( 1 + \exp \left( - \int_0^t 2C_7 \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \, ds \right) \right) 
$$

$$
\leq C_3 \|u_0\|_{\text{BMO}(\mathbb{R})} \exp \left( \int_0^t C \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \, ds \right).
$$

Suppose next $i \geq 1$. By the semigroup property of the flow, we may write

$$
u(t, x) = u_0 \circ \phi_{T_i}(t, \cdot) \circ \cdots \circ \phi_{T_0}(T_1, x).$$
By Theorem 1.1 we have

\[(4.4) \quad \left\| \log \left| \frac{\partial}{\partial x} \tilde{\phi}_t(T_i, \cdot) \right| \right\|_{\text{BMO}(\mathbb{R})} \leq \frac{\int_{T_i}^T 2C_6 \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{\text{BMO}(\mathbb{R})} ds}{\exp\left(-2C_7 \int_{T_i}^T \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{\text{BMO}(\mathbb{R})} ds \right)} \leq 2^{-1} \varepsilon_0 \ \forall \ t \in (T_i, T_{i+1}].\]

A combination of (4.4) and Lemma 2.4 derives

\[\left\{ \begin{array}{l}
\left| \frac{\partial}{\partial x} \tilde{\phi}_t(T_i, \cdot) \right| \in A_\infty(\mathbb{R}); \\
\left| \frac{\partial}{\partial x} \tilde{\phi}_t(T_i, \cdot) \right| \leq 1 + C_4 \left\| \log \left| \frac{\partial}{\partial x} \tilde{\phi}_t(T_i, \cdot) \right| \right\|_{\text{BMO}(\mathbb{R})}.
\end{array} \right.

Then Lemma 2.2 implies

\[\left\| v \circ \phi_{T_i}(t, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \leq C_3 \left\| v \right\|_{\text{BMO}(\mathbb{R})} \left( 1 + C_4 \left\| \log \left| \frac{\partial}{\partial x} \tilde{\phi}_{T_i}(t, \cdot) \right| \right\|_{\text{BMO}(\mathbb{R})} \right) \ \forall \ v \in \text{BMO}(\mathbb{R}).\]

Upon repeating this argument for \(i\) times more, we gain

\[\left\| u(t, \cdot) \right\|_{\text{BMO}(\mathbb{R})} = \left\| u_0 \circ \phi_{T_i}(t, \cdot) \circ \cdots \circ \phi_{T_0}(T_1, \cdot) \right\|_{\text{BMO}(\mathbb{R})} \leq C_3^{i+1} \left\| u_0 \right\|_{\text{BMO}(\mathbb{R})} \left( \prod_{j=1}^{i} \left( 1 + C_4 \left\| \log \left| \frac{\partial}{\partial x} \tilde{\phi}_{T_j}(T_{j-1}, \cdot) \right| \right\|_{\text{BMO}(\mathbb{R})} \right)^{-1} \right) \leq C_3^{i+1} \left( 1 + C_4 2^{-i} \varepsilon_0 \right)^{i+1} \left\| u_0 \right\|_{\text{BMO}(\mathbb{R})} \leq \left\| u_0 \right\|_{\text{BMO}(\mathbb{R})} \exp\left(C \int_0^t \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{\text{BMO}(\mathbb{R})} ds \right),\]

where in the last inequality we have used

\[i \varepsilon_0 < \int_0^t \left\| \frac{\partial}{\partial x} b(s, \cdot) \right\|_{\text{BMO}(\mathbb{R})} ds \leq (i + 1) \varepsilon_0.\]

This, together with (4.3), gives the desired size estimate.

**Step 3 - uniqueness of solution.** This follows easily as an application of the renormalized property of solutions established by DiPerna-Lions [13] and the well-posedness of solutions in \(L^\infty(0, T; L^\infty(\mathbb{R}))\) established in [8]; see the proof of [10, Theorem 1] for instance.

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