The Freedman group: a physical interpretation for the $SU(3)$-subgroup $D(18, 1, 1; 2, 1, 1)$ of order 648

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Abstract

We study a subgroup $Fr(162 \times 4)$ of $SU(3)$ of order 648 which is an extension of $D(9, 1, 1; 2, 1, 1)$ and whose generators arise from anyonic systems. We show that this group is isomorphic to a semi-direct product $(\mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}) \rtimes S_3$ with respect to conjugation and we give a presentation of the group. We show that the group $D(18, 1, 1; 2, 1, 1)$ from the series $(D)$ in the existing classification for finite $SU(3)$-subgroups is also isomorphic to a semi-direct product $(\mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}) \rtimes S_3$, with respect to conjugation. We next exhibit the isomorphism between both groups. We prove that $Fr(162 \times 4)$ is not isomorphic to the exceptional $SU(3)$ subgroup $\Sigma(216 \times 3)$ of the same order 648. We further prove that the only $SU(3)$ finite subgroups from the 1916 classification by Blichfeldt or its extended version, in which $Fr(162 \times 4)$ may be isomorphic, belong to the $(D)$-series. Finally, we show that $Fr(162 \times 4)$ and $D(18, 1, 1; 2, 1, 1)$ are both conjugate under the orthogonal matrix which we provide.

Keywords: $SU(3)$ finite subgroups, anyonic systems, quantum gates, $SU(2)$ Chern–Simons theory at level 4, particle physics, universal quantum computation

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1. Introduction

1.1. Motivation of the work

The group studied in the present paper arises in topological quantum computation and the physics governing anyons. The mathematical framework is the Kauffman–Jones version of
**1.2. Introduction and definition of the group**

Finite subgroups of $SU(3)$ are used in particle physics and topological quantum computation among other fields. The first tentative classification of the finite subgroups of $SU(3)$ appears in the works of GA Miller, HF Blichfeldt and LE Dickson [15], dating from 1916. The part written by Blichfeldt describes in terms of generators all the finite $SU(3)$ subgroups currently known, excluding two more recent ones, namely direct products with a cyclic group of order 3 of the two smallest non-abelian simple groups, that is $A_3 \times \mathbb{Z}/3\mathbb{Z}$ and $PSL(2, 7) \times \mathbb{Z}/3\mathbb{Z}$. Since then, many efforts have been pursued and progress is still made. For a clear historical summary and table of the existing classification, see [13]. The group we study in this paper is not one of the 6 exceptional subgroups of $SU(3)$, although it has the same order as $\Sigma(216 \times 3)$, that is 648, and is like $\Sigma(216 \times 3)$ a group extension of $D(9, 1, 1; 2, 1, 1)$ by $[14]$. It is rather the group $D(18, 1, 1; 2, 1, 1)$ from the series $(D)$. Our group arises up to phase as the result of unitary quantum gates obtained by braiding four anyons of topological charge 2 in the Jones–Kauffman version of $SU(2)$ Chern–Simons theory at level 4 on the one hand.
and by fusing a pair of 4’s out of the vacuum on the other hand, like shown on the figures below.

\[ \text{The Freedman fusion operation} \]

The latter operation is due to Mike Freedman\(^1\) and has the effect of exchanging particles of respective topological charges 0 and 4 on the quantum trit, thus allowing to get an extra quantum gate, namely a permutation matrix up to phase, see appendix. This extra quantum gate discovered by Mike Freedman provides the third generator of our group, which we name for that reason \( Fr(162 \times 4) \). In [7], P. O. Ludl studies the principal series of some exceptional groups, thus establishing their structure. His work shows in particular that \( \Sigma(216 \times 3) \) is isomorphic to

\[ \Delta(54) \rtimes A_4 = ((\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \rtimes S_4) \times (V_4 \times \mathbb{Z}/3\mathbb{Z}). \]

The Freedman group \( Fr(162 \times 4) \) is rather isomorphic to

\[ (\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times V_4) \rtimes S_3. \]

Let us now proceed to the definition of \( Fr(162 \times 4) \) by providing its generators. Our generators are the following: the two braid matrices from [1] which we recall below and the fusion matrix, which we will denote by FUM.

\[
G_1 = \begin{pmatrix}
e^{\frac{3\pi}{4}} & 0 & 0 \\
0 & -e^{\frac{3\pi}{4}} & 0 \\
0 & 0 & -e^{\frac{3\pi}{4}}
\end{pmatrix},
G_2 = \begin{pmatrix}
e^{\frac{3\pi}{4}} & e^{\frac{3\pi}{4}} & e^{\frac{3\pi}{4}} \\
e^{\frac{3\pi}{4}} & 0 & e^{\frac{3\pi}{4}} \\
e^{\frac{3\pi}{4}} & e^{\frac{3\pi}{4}} & 0
\end{pmatrix},
FUM = \begin{pmatrix}
0 & 0 & -e^{\frac{3\pi}{4}} \\
0 & -e^{\frac{3\pi}{4}} & 0 \\
-e^{\frac{3\pi}{4}} & 0 & 0
\end{pmatrix}.
\]

\(^1\) The Freedman fusion operation was communicated by Mike Freedman to the author during a Skype conversation between Station Q, Santa Barbara and HRI, Allahabad, where the author was visiting. She thanks Harish Chandra Research Institute for being a family.
The subgroup of SU(3) generated by these three matrices acts on the complex vector space spanned by the labels 0, 2, 4 on the lowest edge, see [1, 10, 18]. Our result is the following. For clarity, we denote the group $G$ of [1] by $Fr(162)$.

**Theorem 1.** The subgroup $Fr(162 \times 4)$ of SU(3) generated by the three matrices above is a finite group extension of $D(9, 1; 2, 1, 1)$ of order 648 and is isomorphic to $((\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}) \rtimes V_4) \rtimes S_3$, where the semi-direct product is with respect to conjugation (the detailed action is provided in the presentation below). Any element of $Fr(162 \times 4)$ can be written uniquely as a product $A'B'TH$ with $A$ and $B$ defined like further below, $0 \leq i \leq 8$, $0 \leq j \leq 2$, $H$ the identity matrix or one of the matrices $H_i$ with $i \in \{0, 1, 2, 3, 4\}$ whose definitions also appear further below and $T$ an element of the Klein group

$$
\mathcal{V} = \begin{cases}
I_3, (\text{FUM})^3 = \begin{pmatrix}
-1 & -1 \\
-1 & 1
\end{pmatrix}, G_1(\text{FUM})^3G_1^{-1} = \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}
\end{cases}
$$

Set $C_{18} = A(\text{FUM})^3$ and $C_6 = B G_1(\text{FUM})^3 G_1^{-1}$.

A presentation of $Fr(162 \times 4)$ is as follows.

$$
\langle C_6, C_{18}, H_1, H_3 \mid C_{18} = C_6, C_{18} = H_1^2 = H_3^2 = (H_1H_3)^2 = I
\rangle
$$

$Fr(162 \times 4)$ contains exactly four 3-Sylow subgroups, one of which is the unique 3-Sylow subgroup of $Fr(162)$ given by

$$
S_3(F) = N \bigcup N H_3 \bigcup N H_3^2
$$

where $N = < A, B >$. The other three are the $\mathcal{V}$-conjugates of $S_3(F)$.

The group $Fr(162 \times 4)$ is conjugate to the SU(3) finite subgroup $D(18, 1, 1; 2, 1, 1)$, under an orthogonal matrix as follows

$$
\begin{pmatrix}
1/\sqrt{2} & 0 & 1/\sqrt{2} \\
0 & 1 & 0 \\
-1/\sqrt{2} & 0 & 1/\sqrt{2}
\end{pmatrix}
Fr(162 \times 4)
\begin{pmatrix}
1/\sqrt{2} & 0 & -1/\sqrt{2} \\
0 & 1 & 0 \\
1/\sqrt{2} & 0 & 1/\sqrt{2}
\end{pmatrix}
= D(18, 1, 1; 2, 1, 1).
$$

The paper is organized as follows. In the next part of the paper, we identify the structure of the Freedman group, based on the knowledge we already have of the structure of its subgroup $D(9, 1; 2, 1, 1)$, generated by the two braid matrices, see [1]. We then study the structure of another SU(3) finite subgroup, namely $D(18, 1, 1; 2, 1, 1)$. Along the way, we prove in particular that a semi-direct product $(\mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}) \rtimes S_3$ arises as a finite subgroup of SU(3). We next exhibit an isomorphism between $Fr(162 \times 4)$ and $D(18, 1, 1; 2, 1, 1)$. We further show that the group $Fr(162 \times 4)$ is isomorphic to at least four other $D$-groups and that the five $D$-groups are identical. Only to finite SU(3)-subgroups from the series $D$ may the Freedman group be isomorphic to. Along the way we get partial informations about the structure of $D$-groups based on what is known from [13] on the complete structure of $C$-groups. Finally, we show that the two groups $Fr(162 \times 4)$ and $D(18, 1, 1; 2, 1, 1)$ are conjugate by using the local structure of the group as studied in section 3. We later recover this result by different means in [12]. Toward the end of the paper, we explain how we could verify some of our results by using the computer program GAP.
2. Structure of the group

First, we recall some facts from [1] about the structure of

\[ D(9, 1; 2, 1, 1) = \langle G_1, G_2 \rangle. \]

In [1] we show that this group is isomorphic to the semidirect product \( \mathbb{Z}_9 \times \mathbb{Z}_3 \rtimes S_3 \) with respect to conjugation. The elements of the abelian group \( \mathbb{Z}_9 \times \mathbb{Z}_3 \) are the matrices \( A_i B_j \) with \( 0 \leq i \leq 8 \) and \( 0 \leq j \leq 2 \). The matrices \( A \) and \( B \) commute and are provided below:

\[
A = \begin{pmatrix}
\frac{1}{2} e^{\frac{2\pi i}{9}} (-1 + e^{\frac{2\pi i}{3}}) & \frac{1}{2} e^{\frac{2\pi i}{9}} (1 + e^{\frac{2\pi i}{3}}) \\
0 & -e^{\frac{2\pi i}{3}} & 0 \\
\frac{1}{2} (1 + e^{\frac{2\pi i}{3}}) & 0 & \frac{1}{2} (-1 + e^{\frac{2\pi i}{3}})
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
\frac{1}{2} (1 + e^{\frac{2\pi i}{3}}) & 0 & \frac{1}{2} (-1 + e^{\frac{2\pi i}{3}}) \\
0 & -e^{\frac{2\pi i}{3}} & 0 \\
\frac{1}{2} (-1 + e^{\frac{2\pi i}{3}}) & 0 & \frac{1}{2} (1 + e^{\frac{2\pi i}{3}})
\end{pmatrix}
\]

In terms of generators, these matrices are

\[
A = G_1 G_2 G_1^{-1} \tag{1}
\]

\[
B = G_1 G_2^2 G_1. \tag{2}
\]

It is shown in [1] that the subgroup of \( \langle G_1, G_2 \rangle \) generated by the matrices \( A \) and \( B \) is a normal subgroup \( \mathcal{N} \) of \( D(9, 1; 2, 1, 1) \). Moreover, there is in \( D(9, 1; 2, 1, 1) \) a subgroup \( \mathcal{H} \) isomorphic to the symmetric group \( S_3 \), whose six matrices are given by the identity matrix, the matrices \( H_0, H_1, H_2, H_3, H_4 \), and the non-diagonal matrices \( H_i, 1 \leq i \leq 4 \) with

\[
H_0 = \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

and the non-diagonal matrices

\[
H_1 = \begin{pmatrix}
-\frac{1}{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{2} \\
-\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
-\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2}
\end{pmatrix}, \quad
H_2 = \begin{pmatrix}
-\frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\
-\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
\frac{1}{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{2}
\end{pmatrix},
\]

\[
H_3 = \begin{pmatrix}
\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{2}
\end{pmatrix}, \quad
H_4 = \begin{pmatrix}
\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
-\frac{1}{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{2}
\end{pmatrix}.
\]

The matrices \( H_0, H_1 \) and \( H_2 \) are the three elements of order 2 in the group and the matrices \( H_3 \) and \( H_4 \) the two elements of order 3.

We recall from [1] that the intersection \( \mathcal{N} \cap \mathcal{H} \) is trivial, and that

\[ \langle G_1, G_2 \rangle = \mathcal{N} \cdot \mathcal{H}. \]
And so any element of $D(9, 1, 1; 2, 1, 1)$ can be uniquely written as the product of an element of $\mathcal{N}$ and an element of $\mathcal{H}$.

We now use our extra generator FUM to identify the Klein group

$$\mathcal{V} = \langle I_3, (\text{FUM})^3, G_1(\text{FUM})^3G_1^{-1}, (\text{FUM})^3G_2(\text{FUM})^3G_2^{-1} \rangle \cong V_4$$

inside the group $\langle G_1, G_2, \text{FUM} \rangle$. We refer the reader to [11] where it is shown that

$$Fr(162 \times 4) = (\mathcal{N} \times \mathcal{V}) \rtimes \mathcal{H} \cong (\mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}) \rtimes S_3.$$ 

3. Analysis

In this part, we show that the Freedman group $Fr(162 \times 4)$ is isomorphic to the group $D(18, 1; 1; 2, 1, 1)$. Along the way, we prove the fact that $\mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \rtimes S_3$ is one of the $SU(3)$ finite subgroups from the series $(D)$.

In [13], the author studies the structure of the series $(D)$. To that aim, he introduces a new set of generators different from the original set of generators which enlightens the structure of the group. In the case of $D(18, 1; 1; 2, 1, 1)$ however, this new set of generators coincides with the old one as two of the generators from the new set are simply the identity matrices. We recall some material from [8] on the series $(D)$, applied to $D(18, 1; 1; 2, 1, 1)$. According to [13], the group $D(18, 1; 1; 2, 1, 1)$ is generated by the following matrices

$$F(18, 1; 1) = \begin{pmatrix} e^{\frac{3\pi i}{18}} & e^{\frac{\pi i}{18}} & e^{-\frac{3\pi i}{18}} \\ 0 & e^{\frac{3\pi i}{18}} & 0 \\ 0 & 0 & e^{\frac{3\pi i}{18}} \end{pmatrix},$$

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

where the matrix $E$ acts by conjugation is by doing the row and column cycle (132). The way the matrix $B$ acts by conjugation is by swapping the second row and third row and swapping the second column and third column. Consider the subgroup $N(18, 1; 1; 2, 1, 1)$ generated by the three diagonal matrices $F = F(18, 1; 1)$, its (identical) $E$- or $B$-conjugate $F' = F'(18, 1; 1)$ and the $E$-conjugate $F'' = F''(18, 1; 1)$ of $F'$, which we write below for clarity

$$F'(18, 1; 1) = \begin{pmatrix} e^{\frac{3\pi i}{18}} & e^{-\frac{3\pi i}{18}} & e^{-\frac{3\pi i}{18}} \\ 0 & e^{\frac{3\pi i}{18}} & 0 \\ 0 & 0 & e^{\frac{3\pi i}{18}} \end{pmatrix}, \quad F''(18, 1; 1) = \begin{pmatrix} e^{\frac{3\pi i}{18}} & e^{-\frac{3\pi i}{18}} & e^{-\frac{3\pi i}{18}} \\ 0 & e^{\frac{3\pi i}{18}} & 0 \\ 0 & 0 & e^{\frac{3\pi i}{18}} \end{pmatrix}.$$ 

We can show (see the arXiv version) that

$$N(18, 1; 1; 2, 1, 1) = \langle F, F', F'' \rangle \cong \mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}.$$ 

Moreover, we have

$$N(18, 1; 1; 2, 1, 1) \rtimes D(18, 1; 1; 2, 1, 1) \cong S_3,$$

$$\langle E, B \rangle \simeq S_3,$$

$$\langle E, B \rangle \cap N(18, 1; 1; 2, 1, 1) = \{I_3\}.$$
Then \( D(18, 1, 1; 2, 1, 1) = N(18, 1, 1; 2, 1, 1) \times S_3 \), where the action by \( S_3 \) on the group \( N(18, 1, 1; 2, 1, 1) \) is given by

\[
\begin{align*}
BFB^{-1} &= EFE^{-1} = F' \\
BF'B^{-1} &= F \\
EFE^{-1} &= F'' \\
BF''B^{-1} &= F''
\end{align*}
\]

We must still investigate whether there could be an isomorphism between the two semi-direct products

\[
\mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \rtimes S_3 \quad \text{and} \quad \mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \rtimes S_3
\]

where

\[
\begin{align*}
\varphi_D : S_3 &\longrightarrow \text{Aut}(\mathbb{Z}_{18} \times \mathbb{Z}_6) \\
\varphi_F : S_3 &\longrightarrow \text{Aut}(\mathbb{Z}_{18} \times \mathbb{Z}_6)
\end{align*}
\]

are the two homomorphisms provided in this paper for the respective groups \( D(18, 1, 1; 2, 1, 1) \) and \( Fr(162 \times 4) \). First, we clearly state that the groups \( Fr(162) \) and \( D(9, 1, 1; 2, 1, 1) \) are isomorphic groups like claimed in [1] and mentioned several times along this paper. We provide an explicit isomorphism between both groups.

**Theorem 2.** The groups \( Fr(162) \) and \( D(9, 1, 1; 2, 1, 1) \) are isomorphic groups.

First we establish the following result.

**Theorem 3.** The unique 3-Sylow subgroup of \( Fr(162) \) is

\[
S_3(F) = N \bigcup N H_3 \bigcup N H_3^2.
\]

The fact that the unions are disjoint comes from the following facts below. A proof of equation (3) is then given.

**Definition 1.** Call 'cross matrix' a \( 3 \times 3 \) matrix having zeroes in positions \((i, j)\) where exactly one of \(i\) or \(j\) is even.

**Lemma 1.** The matrices \( A \) and \( B \) are cross matrices and the set of special unitary cross matrices forms a subgroup of \( SU(3) \).

The proof of the second point of lemma 1 is straightforward since the inverse of a unitary matrix is its conjugate transpose.

**Corollary 1.** All the matrices \( A'B' \) with \( 0 \leq i \leq 8 \) and \( 0 \leq j \leq 3 \) are cross matrices.

Since \( H_3 \) and \( H_3^2 \) are not cross matrices, by the corollary, the unions above are indeed disjoint. This provides \( 27 + 27 + 27 = 3^4 \) elements. We now finish proving that equation (3) holds.

**Lemma 2.** Let \( S_3 \) denote the unique 3-Sylow subgroup of \( Fr(162) \). If \( x \) does not belong to \( S_3 \), then \( x^3 \) belongs to \( S_3 \).

**Proof.** Since \( 162 = 3^4 \times 2 \), the number \( n_3 \) of 3-Sylow subgroups of \( Fr(162) \) satisfies to

\[
\begin{align*}
n_3 &\equiv 1 \quad (mod 3).
\end{align*}
\]

Both conditions imply that \( n_3 = 1 \). Hence \( Fr(162) \) contains a unique 3-Sylow subgroup, say \( S_3 \), which is normal in \( Fr(162) \). It now suffices to consider the quotient group \( Fr(162)/S_3 \). This is a group of order 2. If an element \( x \) of \( Fr(162) \) does not belong to the 3-Sylow, then its image in the quotient group has order 2. That is \( x^2 \) belongs to the 3-Sylow. \( \square \)
Corollary 2. Both $H_3$ and $H_3^2$ belong to the 3-Sylow.

Proof. By contradiction, if $H_3 \not\in S_3$, then by the lemma above, we must have $H_3^2 \in S_3$. Then both elements are in fact in the 3-Sylow since $H_3^2$ has order 3. \hfill $\square$

End of the proof of theorem 3. If we can show that $S_3$ contains $N$, we are done. But the fact that $B$ belongs to $S_3$ follows from lemma 2 by the same argument as before. And the fact that $A$ belongs to $S_3$ is also a consequence of lemma 2. Indeed, by contradiction, if $A$ does not belong to $S_3$, then $A^2$ belongs to $S_3$. But because the integer 2 is prime to 9, the element $A^2$ also generates the cyclic group $(A)$ of order 9.

The fact that $N$ and $\langle H_3 \rangle = \{I, H_3, H_3 A^2\}$ are both contained in $S_3$ and the count from above suffice to imply that the disjoint union of the right-hand side of equation (3) is a group which is the unique 3-Sylow of $Fr(162)$. \hfill $\square$

We proceed by now determining the unique 3-Sylow $S_3(D)$ of $D(9, 1, 1; 2, 1, 1)$. The group $D(9, 1, 1; 2, 1, 1)$ is the group generated by the three matrices $E$, $B$ and $F^2$, where we used the same notations as before. The abelian group of diagonal matrices contains a subgroup, say $\mathcal{F}$ with

$$\mathcal{F} = \langle F^3 \rangle \times \langle [F(F^2)^{-1}] \rangle.$$

This group is isomorphic to $\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. It is also a subgroup of $D(9, 1, 1; 2, 1, 1)$.

Lemma 3. The unique 3-Sylow subgroup $S_3(D)$ of $D(9, 1, 1; 2, 1, 1)$ must contain all the elements of odd order.

Indeed, suppose $x$ is an element of odd order of $D(9, 1, 1; 2, 1, 1)$ which is not in $S_3(D)$. Then, by considering the quotient, we must have $x^2$ belongs to $S_3(D)$. But since $x$ has odd order, the element $x^2$ is also a generator of the cyclic group $\langle x \rangle$. This represents a contradiction. As a corollary to the lemma, the direct product $\mathcal{F}$ is contained in the 3-Sylow $S_3(D)$. And the two matrices $E$ and $F^2$ of order 3 also belong to $S_3(D)$. Moreover, these two matrices are not diagonal. Thus, we conclude

Theorem 4. The unique 3-Sylow subgroup of $D(9, 1, 1; 2, 1, 1)$ is

$$S_3(D) = \mathcal{F} \bigcup \mathcal{F}E \bigcup \mathcal{F}E^2.$$  \hfill (4)

Moreover, this group is $C(9, 1, 1)$.

From the previous study, we derive some facts about the structure of the bigger groups $D(18, 1, 1; 2, 1, 1)$ and $Fr(162 \times 4)$.

Theorem 5. Both $D(18, 1, 1; 2, 1, 1)$ and $Fr(162 \times 4)$ contain exactly four 3-Sylows.

Proof. For each group, the number of 3-Sylows is either 1 or 4. Suppose for a contradiction that this number is one. The unique 3-Sylow subgroups would have to be $S_3(D)$ and $S_3(F)$ respectively. Also, they would have to contain all the elements of odd order of their respective groups. Indeed if an element $x$ does not belong to the unique 3-Sylows, then by arguments similar as before, $x^2$ or $x^4$ or $x^8$ belongs to the 3-Sylow. When $x$ has an odd order, any of these elements is also a generator for the cyclic group $\langle x \rangle$, hence a contradiction. As far as $D(9, 1, 1; 2, 1, 1)$, simply notice the product matrix $F'E$ has order 3 and does not belong to $S_3(D)$ since

$$F' = F^{-2}F(F'^{-1})^{-1} \notin \mathcal{F}.$$ 

And in the case of $Fr(162 \times 4)$, the element $(FUM)H_3$ has order 3 and does not belong to $S_3(F)$, as neither of $(FUM)H_3$ nor $(FUM)H_3^{-1}$ is a cross matrix. \hfill $\square$
In light of theorem 3 and 4, it is now easy to conclude that $D(9, 1; 2, 1, 1)$ and $Fr(162)$ are isomorphic, like stated in theorem 2. An isomorphism between $Fr(162)$ and $D(9, 1; 2, 1, 1)$ must map $S_3(F)$ onto $S_3(D)$.

**Lemma 4.**

(i) The only elements of order 3 which belong to the center of the unique 3-Sylow $S_3(F)$ of $Fr(162)$ are $A^3$ and $A^6$.

(ii) The only elements of order 3 which belong to the center of the unique 3-Sylow $S_3(D)$ of $D(9, 1; 2, 1, 1)$ are $F^6$ and $F^{12}$.

**Proof.**

(i) The matrices $H_3$ and $B$ do not commute. Likewise, the matrices $H_3^2$ and $B$ do not commute. Therefore, an element of order 3 belonging to $Z(S_3(F))$ would have to belong to $N$. The elements of order 3 of $N$ are

$$A^3, A^6, B, B^3, A^3B, A^3B^2, A^6B, A^6B^2.$$  

Among these, only $A^3$ and $A^6$ do commute to $H_3$.

(ii) The matrices $E$ (resp $E^2$) and $F^2$ do not commute. Hence an element of order 3 belonging to $Z(S_3(D))$ must belong to $F$. The fact that $F^6$ commutes to $E$ while $[F(F''^{-1})^2]$ does not commute to $E$ implies the result.

A corollary to this lemma is that if $f$ is an isomorphism between $Fr(162)$ and $D(9, 1; 2, 1, 1)$, then

$$\begin{align*}
f(A^3) &= F^6 \quad \text{and} \quad f(A^6) = F^{12} \\
or \quad f(A^3) &= F^{12} \quad \text{and} \quad f(A^6) = F^6
\end{align*}$$

It gives some ingredients for the following theorem whose proof appears in [11].

**Theorem 6.** The map

$$\begin{align*}
Fr(162) &\xrightarrow{\sim} D(9, 1; 2, 1, 1) \\
\langle A, B \rangle \rtimes \langle H_3, H_5 \rangle &\sim \langle F^2, [F(F''^{-1})^2] \rtimes \langle B, E \rangle \\
A &\mapsto F^2 \\
B &\mapsto F^{12}[F(F''^{-1})^2] \\
H_3 &\mapsto BE \\
H_5 &\mapsto E
\end{align*}$$

defines an isomorphism of groups between $Fr(162)$ and $D(9, 1; 2, 1, 1)$.

We now continue our study of the group extensions. It will be useful to explicit the four 3-Sylow subgroups. A first result is as follows.

**Lemma 5.** The four 3-Sylows of $Fr(162 \times 4)$ are the following.

(i) $S_3(F)$

(ii) $V_2 S_3(F) V_2^{-1}$

(iii) $V_3 S_3(F) V_3^{-1}$

(iv) $V_4 S_3(F) V_4^{-1}$

where the elements of the Klein group $V$ have been renamed $V_2$, $V_3$ and $V_4$ for convenience. For instance, $V_2 = (FUM)^3$. 

9
We have the following corollary.

**Corollary 3.** Denote by $3\text{Syl}_F$ the subgroup of $Fr(162 \times 4)$ generated by the four 3-Sylows.

We have

$$3\text{Syl}_F = \langle \mathcal{V}, S_3(F) \rangle \simeq \mathcal{V} \times S_3(F).$$

Consequently, $Fr(162 \times 4)$ is not generated by its 3-Sylow subgroups. In other words, $3\text{Syl}_F$ is a proper normal subgroup of $Fr(162 \times 4)$. We have the series

$$\mathcal{N}\mathcal{V} \lhd 3\text{Syl}_F \lhd Fr(162 \times 4),$$

where the quotients are simple.

**Proof.** By theorem 3, we know that

$$S_3(F) = \mathcal{N} \bigcup \mathcal{N}H_3 \bigcup \mathcal{N}H_4$$

is one of the four 3-Sylows of $D$. Another distinct one is

$$(FUM)S_3(F)(FUM)^{-1} = \mathcal{N} \bigcup \mathcal{N}[(FUM)H_3(FUM)^{-1}] \bigcup \mathcal{N}[(FUM)H_4(FUM)^{-1}]$$

since both $A$ and $B$ are self-conjugate under $FUM$. We have

$$H_4(FUM)H_3(FUM)^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in 3\text{Syl}_F$$

and

$$(FUM)H_3(FUM)^{-1}H_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in 3\text{Syl}_F.$$ 

The product of these two matrices is

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$ 

So, $(FUM)^3$ belongs to $3\text{Syl}_F$ (and so does $FUM$ by equation (13)). Then, by the way it is defined, the whole Klein group $\mathcal{V}$ is contained in $3\text{Syl}_F$. This implies

$$\langle \mathcal{V}, S_3(F) \rangle \subseteq 3\text{Syl}_F.$$ 

(5)

The subgroups of lemma 5 are all 3-Sylows. It remains to show that they are all distinct. Notice the products

$$V_jH_jV_j^{-1}H_k$$

with $j \in \{2, 3, 4\}$ and $k \in \{3, 4\}$, are either not cross matrices or have order two or are $(FUM)^3$ or $V_3$. In any case, they do not belong to $\mathcal{N}$. This shows the result. Lemma 5 implies that

$$3\text{Syl}_F \subseteq \langle \mathcal{V}, S_3(F) \rangle.$$ 

(6)

Gathering the two inclusions (5) and (6), we obtain

$$3\text{Syl}_F = \langle \mathcal{V}, S_3(F) \rangle,$$

(7)

as stated in corollary 3. Recall that $\mathcal{V}$ is a normal subgroup of $Fr(162 \times 4)$. From (46), we then derive

$$3\text{Syl}_F/\mathcal{V} \simeq S_3(F).$$ 

(8)
Lemma 7. Of $D$ are conjugate or follows from the fact that $3$ is not diagonal. And the groups of $N$ with $k \in \{3, 4\}$ and $V_i = I_3$ and $i \in \{1, 2, 3, 4\}$, is an isomorphism of groups which allows to identify the subgroup $3SylF$ of $Fr(162 \times 4)$ generated by all the $3$-Sylows of $Fr(162 \times 4)$ with the semi-direct product $V \rtimes S_3(F)$ with respect to conjugation, where $S_3(F)$ is the unique $3$-Sylow subgroup of $Fr(162)$ and $V$ is the Klein group formerly defined.

Lemma 6. The four $3$-Sylows of $D(18, 1, 1; 2, 1, 1)$ are

(i) $S_3(D)$
(ii) $F S_3(D) F^{-1}$
(iii) $E F S_3(D) F^{-1} E^{-1}$
(iv) $E^2 F S_3(D) F^{-1} E^{-2}$.

First, we show that

$$FS_3(D)F^{-1} = \mathcal{F} \bigcup \mathcal{F}(FEF^{-1}) \bigcup \mathcal{F}(FE^2F^{-1})$$

$$FS_3(D)F^{-1} \neq S_3(D).$$

This follows from three simple facts:

(1) $FEF^{-1}$ is not a diagonal matrix, hence the unions above are disjoint.
(2) $FEF^{-1} E^{-1}$ is diagonal but has order 6, hence does not belong to the group $\mathcal{F}$ of odd order $27$.
(3) $FEF^{-1} E^{-2}$ is not a diagonal matrix.

Next, the groups of (i) and (iii) are distinct since $F^{-1} E F^{-1}$ has order 6 and $F^{-1} E F^{-2}$ is not diagonal. And the groups of (i) and (iv) are distinct since $E^{-1} F^{-1} E^2 F E^{-2}$ has order 6 and $E^{-1} F^{-1} E^2 F E^{-2}$ is not diagonal. It remains to show that the two groups in (ii) and (iii) are distinct and so are those of (ii) and (iv). This follows from similar arguments.

Lemma 7. Denote by $3SylD$ the subgroup of $D(18, 1, 1; 2, 1, 1)$ generated by all the $3$-Sylows of $D(18, 1, 1; 2, 1, 1)$. The group $3SylD$ contains the Klein group, say $V_D$.

$$I_3, \ W_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \ W_3 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \ W_4 = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}.$$”

Moreover, $V_D$ is normal in $D(18, 1, 1; 2, 1, 1)$.

Proof. Notice the matrices $FEF^{-1} E^2$ and $E^2 F E F^{-1}$ both have order 6. The cube of the first one (resp second one) is the second (resp third) matrix from the lemma. The product of their respective cubes is the last matrix of the lemma.
Lemma 8. The four 3-Sylows of $D(18, 1, 1; 2, 1, 1)$ are also given by $S_3(D)$ and its $W_i$-conjugates.

Proof. It is easy to check by similar arguments as already used before that they are all distinct. Hence the immediate corollary. □

Corollary 4. $3\text{Syl} D = < V_D, S_3(D) > \simeq V_D \rtimes S_3(D)$. Moreover, we have

$$N(18, 1, 1; 2, 1, 1) = V_D \rtimes \mathcal{F} \simeq V_D \rtimes \mathcal{F}$$

and the series

$$V_D \rtimes \mathcal{F} \triangleleft 3\text{Syl} D \triangleleft D(18, 1, 1; 2, 1, 1),$$

where the quotients are simple.

In light of corollary 3 and 4, we are now ready to show the following theorem.

Theorem 7. The Freedman group is isomorphic to the finite $SU(3)$-subgroup $D(18, 1, 1; 2, 1, 1)$ from the series $D$.

Proof. Suppose that there exists an isomorphism of groups, say $g$ between $Fr(162 \times 4)$ and $D(18, 1, 1; 2, 1, 1)$. Then, $g$ maps

$$3\text{Syl} F = V \times S_3(F)$$

onto

$$3\text{Syl} D = V_D \rtimes S_3(D).$$

Further, we claim that such an isomorphism $g$ maps $V$ onto $V_D$. Indeed, an element of order 2 belonging to $V$ must be mapped to an element of order 2 of $3\text{Syl} D$. The elements of $3\text{Syl} D$ can be uniquely written as a product of an element of $V_D$ times an element of $S_3(D)$. The matrices of $V_D$ are all diagonal matrices, $W_i$’s as we called them earlier. An element of $S_3(D)$ is a diagonal matrix of $F$, say $\Delta$, times $I_2$ or $E$ or $E^2$. It is now easy to see that the only elements of order two of $3\text{Syl} D$ are those belonging to the Klein group $V_D$. Indeed, we have

$$(W_i \Delta E) (W_i \Delta E) = (W_i \Delta) (E W_i \Delta E) \quad (10)$$

$$(W_i \Delta E^2) (W_i \Delta E^2) = (W_i \Delta) (E^2 W_i \Delta E^2). \quad (11)$$

The effect of multiplying a matrix to the right and to the left by the permutation matrix $E = P_6$ is to do the cycle $\sigma = (132)$ on the rows, and the inverse cycle $\sigma^{-1} = (123)$ on the columns. When doing this operation on a diagonal matrix, we obtain a non-diagonal matrix. In particular, the respective right-hand sides of equations (10) and (11) cannot be the identity matrix. Moreover, since $W_i$ has order 2, $\Delta$ has odd order and $W_i$ and $\Delta$ commute, their product cannot have order 2. We conclude like announced. Then, we have

$$g(FUM)^3 \in \{(E^2FEF^{-1})^3, (FEF^{-1}E^2)^3, (E^2FE^2FE^{-1}E^2)^3\}.$$

Further, the 3-Sylow $S_3(F)$ is mapped to some $W_i$ conjugate of $S_3(D)$. Now, an element of order 3 of the center $Z(W_i S_3(F) W_i^{-1})$ is a $W_i$ conjugate of an element of order 3 belonging to $Z(S_3(D))$, hence is $F^6$ or $F^{12}$ as shown earlier. Using some facts from before, it follows that

$$\begin{cases} g(A^3) = F^6 \quad \text{and} \quad g(A^6) = F^{12} \\ g(A^3) = F^{12} \quad \text{or} \quad g(A^6) = F^6 \end{cases}$$

□
**Claim 1.** We have \( F^6 = \omega I_3 \) and \( F^{12} = \omega^2 I_3 \).

Hence it is not straightforward to conclude. We set for instance \( g(A) = F^2 \) like above. Furthermore, we note that the image \( g(H_3) \) cannot be in \( F \) since \( F \) commutes to \( F^2 \), but \( H_3 \) does not commute to \( A \). Then it must be of the form

\[
g(H_3) = W_i \Delta E_i W_i^{-1}
\]

with \( \Delta \) some diagonal matrix of \( F \), \( k \in \{1, 2\} \), \( i \in \{1, 2, 3, 4\} \) and \( W_i = I_3 \).

We have the expression

\[
g(C_6) = g(H_3) g(C_{18}) g(H_3) g(C_{18})^8
\]

read out of the presentation given in theorem 1. Since \( C_{18} = A (FUM)^3 \), the image \( g(C_{18}) \) is a diagonal matrix. The \( E^k \) conjugate of a diagonal matrix is again a diagonal matrix. Therefore,

\[
g(H_3) g(C_{18}) g(H_3)^{-1} = W_i \Delta E_i W_i^{-1} g(C_{18}) W_i E^{-k} \Delta^{-1} W_i^{-1}
\]

and so

\[
g(C_6) = E^k g(C_{18}) E^{-k}
\]

and so by similar arguments as before,

\[
g(C_6) = E^k g(C_6) E^{-k} g(C_{18})^3.
\]

By gathering equations (14) and (16), we now get

\[
g(C_6) = E^k g(C_6) E^{-k} g(C_{18})^3 E^{-k} E^k g(C_{18})^3
\]

where \( \tilde{k} \) is the ‘conjugate’ of \( k \), namely \( \tilde{k} = 1 \) if \( k = 2 \) and \( \tilde{k} = 2 \) if \( k = 1 \).

With

\[
\begin{align*}
| & k = 2 \quad g((FUM)^3) = (E^2 FEF^{-1})^3, \\
& g(H_1) = \sim BE
\end{align*}
\]

all the relations of the presentation given in theorem 1 are verified on the images. Hence the theorem.

**Theorem 8.** The map

\[
Fr(162 \times 4) \longrightarrow D(18, 1, 1; 2, 1, 1)
\]

| \( H_3 \) | \( H_1 \) | \( C_{18} \) | \( C_6 \) |
|---|---|---|---|
| \( \longrightarrow E^2 \) | \( \longrightarrow \sim BE \) |
| \( \begin{pmatrix} e^{-i7\pi/9} & e^{i2\pi/9} \\ e^{i5\pi/9} & e^{i\pi/9} \end{pmatrix} \) |
| \( \begin{pmatrix} e^{i\pi/3} \\ e^{i2\pi/3} \end{pmatrix} \) |
where
\[ g(C_{18}) = F^2(E^2FEF^{-1})^3 \]
\[ g(C_6) = Eg(C_{18})E^{-2}g(C_{18})^8E^{-2}g(C_{18})^3 \]
defines an isomorphism of groups between \( Fr(162 \times 4) \) and \( D(18, 1, 1; 2, 1, 1) \).

4. Epilogue

We have just shown that the groups \( D(18, 1, 1; 2, 1, 1) \) and \( Fr(162 \times 4) \) are isomorphic groups and both groups are isomorphic to a semi-direct product \( (\mathbb{Z}_6 \times \mathbb{Z}_{18}) \rtimes S_3 \). In this part, we show that the only \( SU(3) \)-subgroups from the extended version of the 1916 classification the Freedman group may be isomorphic to are of \( D \)-type and we provide some instances of \( D \)-groups the Freedman group is isomorphic to. At first sight, we can already rule out the series \( /\Delta_1(3n^2) \) and \( /\Delta_1(6n^2) \) because
\[ 648 = 6 \times 108 = 3 \times 216 \]
but neither of 108 or 216 is a square. Further, for orders considerations, the only candidate in the series \( T_n \) would be \( T_216 \) but the series exists only for special values of the integer \( n \) (see [5]) and 216 being even is not a product of primes of the form \( 6k + 1 \) with \( k \geq 1 \) an integer.
The only candidate among the exceptional groups or direct products of \( \mathbb{Z}_3 \) by an exceptional group would in turn be \( /\Sigma_1(216 \times 3) \), still for orders purposes. It is a result from PO Ludl thesis that this group is generated by only two matrices which we recall below.
\[ /\Sigma_1(216 \times 3) = \langle D = \begin{pmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \omega \end{pmatrix}, \ V = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \rangle \]
where
\[ \varepsilon = e^{\frac{2\pi}{3}} \text{ and } \omega = e^{\frac{2\pi}{9}}. \]
However, we have the following result.

**Theorem 9.** The Freedman group \( Fr(162 \times 4) \) is not isomorphic to the exceptional group \( /\Sigma(216 \times 3) \).

**Proof.** We have the following principal series, where the numbers at the top denote the degrees of the group extensions.
\[
\{e\} < \mathbb{Z}_1 < \mathbb{Z}_1 \times \mathbb{Z}_3 < \mathbb{Z}_1 \times \mathbb{Z}_3 \times \Delta(27) < \Delta(54) < \Sigma(36 \times 3) < \Sigma(72 \times 3) < \Sigma(216 \times 3) < Fr(162 \times 4)
\]
\[
\{e\} < A^3 < \Delta^2 < \Delta(27) < \Delta(54) < \Sigma(36 \times 3) < \Sigma(72 \times 3) < \Sigma(216 \times 3) < 3SylF < Fr(162 \times 4)
\]
The first one is a result of [7] where the principal series of the exceptional \( SU(3) \) finite subgroups are determined. The second one is derived from the present paper. It is also shown in [7] that \( \Sigma(72 \times 3)/\Delta(27) \) is isomorphic to the quaternion group \( Q_8 \).

**Lemma 9.** Suppose \( Fr(162 \times 4) \) contains a proper subgroup \( H \) such that \( H \) has a quotient \( H/K \) with \( H/K \cong Q_8 \). Then, all the 2-Sylow subgroups of \( Fr(162 \times 4) \) are isomorphic to \( Q_8 \).
Proof. If such subgroups $H$ and $K$ exist, then $2^3$ divides $|H|$, and so $|H| = 2^3 \cdot 3^k$ for some integer $k \in \{0, 1, 2, 3\}$ and $|K| = 3^k$. First, if $k = 0$, the result clearly holds. Assume now $k \in \{1, 2, 3\}$. Since $K$ is normal in $H$, then $K$ must be the unique 3-Sylow subgroup $S_3$ of $H$. Let $S_2$ be a 2-Sylow subgroup of $H$. Since $S_2 \cap S_3 = \{e\}$, we then have $S_3 S_2 = H \simeq S_3 \rtimes S_2$.

It follows that $S_2 \simeq H/S_3 \simeq Q_8$.

Thus, $S_2$ is isomorphic to $Q_8$. Further, $S_2$ is also a 2-Sylow subgroup of $Fr(162 \times 4)$ and all the 2-Sylows of $Fr(162 \times 4)$ are isomorphic since they are all conjugate. We deduce that any 2-Sylow subgroup of $Fr(162 \times 4)$ must then be isomorphic to the quaternion group $Q_8$. □

Lemma 10. $V \rtimes \langle H_1 \rangle$ is a 2-Sylow subgroup of $Fr(162 \times 4)$ which is not isomorphic to $Q_8$.

Proof. Recall $V$ is a normal subgroup of $Fr(162 \times 4)$. The semi-direct product from the statement of the lemma has the right cardinality, hence is a 2-Sylow subgroup of $Fr(162 \times 4)$. One of the characteristics of $Q_8$ is that all its subgroups are normal. However, 

$$V_2 H_1 V_2^{-1} \notin \langle H_1 \rangle,$$

so that $\langle H_1 \rangle$ is a subgroup of $V \rtimes \langle H_1 \rangle$ which is not normal. This shows the lemma.

Now, lemma 9 and lemma 10 imply theorem 9. □

We now study the $SU(3)$-subgroups from the series $(D)$. It is known only since recently (2011) in the work of [13] that an $SU(3)$ finite subgroup from the series $(D)$ has the general structure $A \times S_3 \simeq \mathbb{Z}_m \times \mathbb{Z}_p \rtimes S_3$ with $A$ the normal subgroup of all the diagonal matrices of the given $D$-group. Moreover, $m$ divides $p$ by theorem 2.1 of [13] which states that any finite abelian subgroup of $SU(3)$ is of the form $\mathbb{Z}_r \times \mathbb{Z}_s$ with $r$ divides $s$.

Furthermore, we prove the following lemma.

Lemma 11. The Freedman group $Fr(162 \times 4)$ does not contain any element of order 24, 27, 54, 72 or 108.

Proof. Suppose it does contain an element of order $k$ with $k \in \{24, 27, 54, 72, 108\}$.

First, there does not exist any element of such order inside the direct product $\mathbb{Z}_3 \times \mathbb{Z}_9 \times V_4$. Indeed, if $d_1 | 3$, $d_2 | 9$ and $d_3 | 2$, then none of $24, 27, 54, 72, 108$ does divide $lcm(d_1, d_2, d_3)$.

So an element of order $k$ must be a product $NH_i$. 

15
with \(N\) in the direct product and \(H_i\) in the symmetric group. Moreover, by the way the semi-direct product is defined, we must have

\[
H_i^k = I_3
\]

This implies \(i \in \{0, 1, 2\}\) if \(k\) is even and \(i \in \{3, 4\}\) if \(k\) is odd.

A generic form for \(N\) is

\[
N = A'B'B'V_r.
\]

Let us first deal with the case \(k = 27\). We found out using Mathematica that all the products \(NH_3\) or \(NH_4\) have order 3. So there is no element of order 27 in the Freedman group.

Since \(54 = 27 \times 2\) and \(108 = 54 \times 2\), there are no elements of order 54 or 108 either since their respective squares would then have order 27 and 54.

Finally, we see with Mathematica that the possible orders for the products \(NHi\) with \(N \in \mathcal{N}\mathcal{V}\) and \(i \in \{0, 1, 2\}\) are not among 24 or 72. \(\square\)

Since \(108 = 2 \times 54 = 3 \times 36 = 4 \times 27 = 6 \times 18 = 9 \times 12\) we deduce that the only candidates from the series \((D)\) of order 648 are of the form

\[
\mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \rtimes S_3
\]

\[
\mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_4 \rtimes S_3.
\]

We now prove the following theorem.

**Theorem 10.**

(i) Assume without loss of generality that \(\gcd(n, a, b) = 1\). A necessary condition for the Freedman group to be isomorphic to the \(D\)-group \(D(n, a, b; d, r, s)\) is that

\[
n \in \{2, 3, 6, 9, 18\}.
\]

(ii) The Freedman group is isomorphic to many \(SU(3)\) subgroups from the series \((D)\). For instance, it is isomorphic to all the identical groups

\[
D(9, 1, 1; 2, 0, 1), \quad D(9, 1, 1; 2, 0, 0),
\]

\[
D(18, 1, 1; 2, 1, 1), \quad D(18, 1, 1; 2, 0, 0), \quad D(18, 1, 1; 2, 0, 1).
\]

**Proof.** Notice that

\[
C(n, a, b) \subseteq D(n, a, b; d, r, s).
\]

As shown in [7], the structure of an \(SU(3)\) subgroup from the series \((C)\) is

\[
\mathbb{A} \times \mathbb{Z}_3 \simeq (\mathbb{Z}_m \times \mathbb{Z}_p) \times \mathbb{Z}_3
\]

with \(\mathbb{A}\) the normal subgroup of all the diagonal matrices of \(C(n, a, b)\), the integer \(p\) the maximal order of the diagonal matrices and \(m\) the divisor of \(p\) provided in [13].

In particular, in a \(C\)-group, all the elements of order 2 commute. Thus, the inclusion above is strict. If \(|C(n, a, b)|\) divides strictly \(3^4 \cdot 2^3\), this implies that

\[
|C(n, a, b)| \leq 324.
\]
Further, let $g = \gcd(n, a, b)$. Notice that
\[
C\left(\frac{n}{g}, \frac{a}{g}, \frac{b}{g}\right) = C(n, a, b).
\]
Thus, without loss of generality, we may assume that $\gcd(n, a, b) = 1$ and so the order of $F(n, a, b)$ is $n$. From before, a diagonal matrix of a $D$-group isomorphic to the Freedman group has order among
\[
2, 3, 4, 6, 9, 12, 18, 36.
\]
And so,
\[
n \in \{2, 3, 6, 9, 18\} \quad (\ast)
\]
onlyear{or}
\[
n \in \{4, 12, 36\} \quad (\ast \ast).
\]

We claim that the second row of values are to exclude. Our proof is based on the following set of propositions.

**Proposition 1.** We have
\[
C(k, \tilde{a}, \tilde{b}) \subseteq C(2k, a, b)
\]
with
\[
\tilde{a} = \begin{cases} 
  a & \text{if } 0 \leq a \leq k - 1 \\
  a - k & \text{if } k \leq a \leq 2k - 1.
\end{cases}
\]
We also have
\[
C(k, \hat{a}, \hat{b}) \subseteq C(3k, a, b)
\]
with
\[
\hat{a} = \begin{cases} 
  a & \text{if } 0 \leq a \leq k - 1 \\
  a - k & \text{if } k \leq a \leq 2k - 1 \\
  a - 2k & \text{if } 2k \leq a \leq 3k - 1.
\end{cases}
\]

**Proof.** Straightforward. \hfill \Box

**Proposition 2.**
\[
C(2, 0, 1) = C(2, 1, 0) = C(2, 1, 1).
\]
Moreover, these groups contain a subgroup which is isomorphic to the Klein group $V_4$.

**Proof.** It suffices to notice that the matrices $F(2, 0, 1)$, $F(2, 1, 0)$ and $F(2, 1, 1)$ are all conjugate under $E$ or $E^2$. \hfill \Box

**Proposition 3.** Suppose $\gcd(4, a, b) = 1$. A $D$-group isomorphic to the Freedman group cannot contain a $C$-group of type $C(4, a, b)$.

**Proof.** Suppose it does. Then it also contains a $C$-group of type $C(2, \tilde{a}, \tilde{b})$ by proposition 1. Then, by proposition 2, the group $C(4, a, b)$ contains a Klein group formed of diagonal matrices. Since $\gcd(4, a, b) = 1$, the group $C(4, a, b)$ must also contain a diagonal matrix of order 4. Then, the $D$-group containing $C(4, a, b)$ would contain a direct product $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ of diagonal matrices. However, we know from before that the maximal power of two dividing the order of the subgroup of diagonal matrices is 4, thus a contradiction. \hfill \Box
Proposition 4. Suppose gcd(12, a, b) = 1. A D-group isomorphic to the Freedman group cannot contain a C-group of type C(12, a, b).

Proof. Follows from propositions 1 and 3. □

Proposition 5. Suppose gcd(36, a, b) = 1. A D-group isomorphic to the Freedman group cannot contain a C-group of type C(36, a, b).

Proof. Follows from propositions 1 and 4.

This finishes the proof of point (i) in the theorem. Our study of which D-groups arise as the Freedman group is far from being complete at this point. However, point (i) and its proof provide in fact many more informations than those already disclosed about the integers n, a and b. Let us give an example. □

Fact 1. Suppose s is an integer such that gcd(3, s) = 1. The group C(9, 3, s) never arises as a subgroup of the Freedman group. Consequently also, nor does the group C(18, 3, s) arise as a subgroup of the Freedman group.

Proof. The second point follows from the first one by applying proposition 1. We will show that when gcd(3, s) = 1, the structure of C(9, 3, s) is

\[ Z_9 \times Z_9 \cong Z_3. \]

Then we have 3^3 divides the order of C(9, 3, s), which prevents this group from being isomorphic to a subgroup of the Freedman group.

Since gcd(9, 3, s) = 1, the maximal order of a diagonal matrix of C(9, 3, s) is 9. Hence

\[ C(9, 3, s) \cong Z_9 \times Z_3, \]

with \( p \in \{3, 9\} \). Following [13], the integer \( p \) must satisfy to

\[ 9 \mid p(s - 3t) \quad 1 \leq t \leq \frac{9}{p} - 1, \quad \text{and } p \text{ the smallest} \]

\[ 9 \mid p(3 + s(t + 1)). \]

If \( p = 3 \), then we must have 3|s−3t which implies 3|s, impossible. Thus, we rather have \( p = 9 \). □

We provide below more inclusions of C-groups inside D-groups which impose more restrictions on the choice for the integers \( d \) and \( r \). First, we deal with the case when \( d \) is even.

Lemma 12. Assume \( d \) is even. Then, the following inclusions hold.

\[
\begin{align*}
C \left( d, 2r, \frac{d - 2r}{2} \right) & \subset D(n, a, b; d, r, s) \quad \text{if } r \leq \frac{d}{2} - 1 \\
C \left( d, 2r - d, \frac{3d - 2r}{2} \right) & \subset D(n, a, b; d, r, s) \quad \text{if } r \geq \frac{d}{2} + 1 \\
C(d, 0, 0) & \subset D \left( n, a, b; d, \frac{d}{2}, s \right) \quad \text{if } r = \frac{d}{2}. 
\end{align*}
\]
Proof. Read from
\[ \tilde{G}(d, r, s)^2 = \begin{pmatrix} e^{2\pi i/d} & 0 & 0 \\ 0 & -e^{-2\pi i/d} & 0 \\ 0 & 0 & -e^{-2\pi i/d} \end{pmatrix}. \]

The case \( d \) is odd requires a bit more effort. Instead, we must consider the fourth power of the generator \( \tilde{G}(d, r, s) \). We have
\[ \tilde{G}(d, r, s)^4 = \begin{pmatrix} e^{4\pi i/2r} & e^{2\pi i (-2r)} \\ e^{2\pi i (-2r)} & e^{4\pi i/2r} \end{pmatrix}. \]

From there, we deduce the following lemma.

Lemma 13. Suppose \( d \) is odd. The following inclusions hold.
\[
\begin{align*}
C(d, 4r - 3d, 2d - 2r) &\subset D(n, a, b; d, r, s) \quad \text{if } d \geq 5 \& r \geq \lfloor \frac{3d}{4} \rfloor + 1 \\
C(d, d - 3, \frac{d+3}{2}) &\subset D(n, a, b; d, \lfloor \frac{d+3}{2} \rfloor, s) \quad \text{if } d \geq 5 \& r = \lfloor \frac{d}{2} \rfloor \& d \equiv 1 \text{ mod } 4 \\
C(d, d - 1, \frac{d+1}{2}) &\subset D(n, a, b; d, \lfloor \frac{d+1}{2} \rfloor, s) \quad \text{if } d \geq 3 \& r = \lfloor \frac{d}{2} \rfloor \& d \equiv 3 \text{ mod } 4 \\
C(d, 4r - 2d, 2d - 2r) &\subset D(n, a, b; d, r, s) \quad \text{if } d \geq 7 \& \lfloor \frac{d}{2} \rfloor + 1 \leq r \leq \lfloor \frac{3d}{4} \rfloor - 1 \\
C(d, d - 2, 1) &\subset D(n, a, b; d, \lfloor \frac{d}{2} \rfloor, s) \quad \text{if } d \geq 3 \& r = \lfloor \frac{d}{2} \rfloor \\
C(d, 4r - d, d - 2r) &\subset D(n, a, b; d, r, s) \quad \text{if } d \geq 7 \& \lfloor \frac{d}{2} \rfloor + 1 \leq r \leq \lfloor \frac{d}{2} \rfloor - 1 \\
C(d, d - 1, \frac{d+1}{2}) &\subset D(n, a, b; d, \lfloor \frac{d+1}{2} \rfloor, s) \quad \text{if } d \geq 3 \& r = \lfloor \frac{d}{2} \rfloor \& d \equiv 1 \text{ mod } 4 \\
C(d, d - 3, \frac{d+3}{2}) &\subset D(n, a, b; d, \lfloor \frac{d+3}{2} \rfloor, s) \quad \text{if } d \geq 5 \& r = \lfloor \frac{d}{2} \rfloor \& d \equiv 3 \text{ mod } 4 \\
C(d, 4r, d - 2r) &\subset D(n, a, b; d, r, s) \quad \text{if } d \geq 13 \& 1 \leq r \leq \lfloor \frac{d}{2} \rfloor - 1 \\
C(d, 4, d - 2) &\subset D(n, a, b; d, 1, s) \quad \text{if } d \geq 5 \& r = 1 \\
C(3, 1, 1) &\subset D(n, a, b; 3, 1, s) \quad \text{if } d = 3 \& r = 1 \\
C(d, 0, 0) &\subset D(n, a, b; d, 0, s) \quad \text{if } r = 0
\end{align*}
\]

where \( \lfloor x \rfloor \) denotes the integer part of \( x \).

Knowing from Ludl’s work in [13] the structure of the \( C \)-groups, we obtain this way many informations on the structure of the \( D \)-groups.

We next focus our attention on some \( D \)-groups \( D(n, a, b; d, r, s) \) with \( n \in \{9, 18\} \), \((a, b) = (1, 1)\) and \( d = 2 \). We have
\[ \tilde{G}(2, 0, s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & (-1)^r \\ 0 & (-1)^{s+1} & 0 \end{pmatrix}. \]

We thus see that the two groups \( D(18, 1, 1; 2, 0, 0) \) and \( D(18, 1, 1; 2, 0, 1) \) are identical since
\[ \tilde{G}(2, 0, 0) = \tilde{G}(2, 0, 1)^{-1}. \]
Moreover, we notice that
\[ \tilde{B}W_4 = G(2, 0, 1). \]
Thus we have
\[ D(18, 1; 2, 0, 1) \subseteq D(18, 1; 2, 1, 1). \]
The latter inclusion is in fact an equality since we have
\[ F(18, 1, 1)^9 = W_4. \]
So the three groups \( D(18, 1; 2, 1, 1), D(18, 1; 2, 0, 1) \) and \( D(18, 1; 2, 0, 0) \) are identical.

**Fact 2.** For the smaller groups, we have
\[ D(9, 1, 1; 2, 1, 1) \subseteq D(9, 1, 1; 2, 0, 1) = D(9, 1, 1; 2, 0, 0) = D(18, 1, 1; 2, 1, 1). \]

**Proof.** We already know that the first equality holds. Next, notice that
\[ (G(2, 0, 1)E)^2 = W_4. \]
Hence
\[ W_4 \in D(9, 1, 1; 2, 0, 1). \]
Then \( B \in D(9, 1, 1; 2, 0, 1) \) and so
\[ D(9, 1, 1; 2, 1, 1) \subseteq D(9, 1, 1; 2, 0, 1) \subseteq D(18, 1, 1; 2, 1, 1). \]
Because in \( D(9, 1, 1; 2, 1, 1) \), there is no diagonal matrix of even order, we know that \( W_4 \)
cannot belong to \( D(9, 1, 1; 2, 1, 1) \). Hence the first inclusion above is strict.

Then, either \( |D(9, 1, 1; 2, 0, 1)| = 324 \) or \( D(9, 1, 1; 2, 0, 1) = D(18, 1, 1; 2, 1, 1) \). To decide, it suffices to notice that \( W_4 \) belongs to \( D(9, 1, 1; 2, 0, 1) \) implies that the whole Klein group \( V_9 \) is contained in \( D(9, 1, 1; 2, 0, 1) \). Indeed, the matrices \( W_2 \) and \( W_3 \) are the respective \( E \) and \( E^2 \) conjugates of the matrix \( W_4 \).

By [13], the D-group \( D(9, 1, 1; 2, 0, 1) \) has the structure
\[ A \times S_3 \]
with \( A \) the normal subgroup of all the diagonal matrices of \( D(9, 1, 1; 2, 0, 1) \).

So,
\[ 4 \big| |A|. \]
Then,
\[ 24 \big| |D(9, 1, 1; 2, 0, 1)|. \]
This prevents to have
\[ |D(9, 1, 1; 2, 0, 1)| = 324. \]
And so,
\[ D(9, 1, 1; 2, 0, 1) = D(18, 1, 1; 2, 1, 1). \]
This ends the proof of theorem 10. \(\square\)
Remarks.

(i) The order of the direct product $\mathbb{Z}_6 \times \mathbb{Z}_{18}$ is the number of distinct semi-direct products $\mathbb{Z}_6 \times \mathbb{Z}_{18} \rtimes S_3$ that are all isomorphic to the Freedman group.

(ii) The order of the automorphism group $\text{Aut}(\mathbb{Z}_6 \times \mathbb{Z}_{18})$ is the order of the Freedman group.

Point (i) was obtained by writing a program in GAP which lists all the semi-direct products $\mathbb{Z}_6 \times \mathbb{Z}_{18} \rtimes S_3$ from the list of all the homomorphisms from $S_3$ to $\text{Aut}(\mathbb{Z}_6 \times \mathbb{Z}_{18})$ and then returns the list of their GAP ID. There are 36 distinct GAP ID’s in this list, that is there are 36 non-isomorphic semi-direct products $\mathbb{Z}_6 \times \mathbb{Z}_{18} \rtimes S_3$. We counted the number of occurrences in this list of the GAP ID [468, 259] (see forthcoming (24)) and found the number 108.

We now investigate the series (C). We have the following statement whose proof is straightforward.

**Theorem 11.** The Freedman group is not isomorphic to any $\text{SU}(3)$ finite subgroup from the series (C).

**Proof.** As already mentioned before, a result of [13] states that a (C) group is of the form $\mathbb{Z}_n \times \mathbb{Z}_m \rtimes \mathbb{Z}_3$. Hence, in such a group all the elements of even order commute. In a (D)-group however, the elements of order two of the symmetric group do not commute. □

Finally, the Freedman group is not isomorphic to a group of matrices of the shape

$$
\begin{pmatrix}
\text{det}(A^\dagger) & 0_{1,2} \\
0_{2,1} & A
\end{pmatrix},
$$

with $A$ a unitary matrix of size 2. So it does not belong to the ‘B type’ as it is referred to in the classification of [13]. Indeed, we have the following statement.

**Theorem 12.** The Freedman group is not isomorphic to any $\text{SU}(3)$ finite subgroup of the type B. In other words, the Freedman group is not isomorphic to a finite subgroup of $\text{U}(2)$.

However, the Freedman group contains subgroups of type (B) of various orders. It is worth mentioning one which has order 54.

**Theorem 13.** The Freedman group contains a (B)-subgroup which is isomorphic to $\mathbb{Z}_9 \times S_3$.

**Proof.** We first begin with the latter theorem. Our proof here relies on a table which can be found by clicking on a link created by the same authors in the 2011 paper [16] by Parattu and Wingerter. This table shows in particular whether groups of GAP ID [54, n] are subgroups of $\text{SU}(3)$ or not and whether they are subgroups of $\text{U}(2)$ or not. We summarize below the status with respect to $\text{U}(2)$ for those groups belonging to $\text{SU}(3)$. The informations below have been copied out of Parattu and Wingerter’s table. We added an extra row and extra column to the table for the sake of $(\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, which drew our attention while writing this proof.

| GAP ID | Group | SU(3) | U(2) |
|--------|-------|-------|------|
| [54,1] | $D_{27}$ | YES | YES |
| [54,3] | $\mathbb{Z}_3 \times D_9$ | YES | YES |
| [54,4] | $\mathbb{Z}_9 \times S_3$ | YES | YES |
| [54,7] | $(\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ | NO | NO |
| [54,8] | $(\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ | YES | NO |
Our study of the Freedman group shows that it contains a subgroup
\[ S = \mathcal{N} \rtimes (H_i) \]
with \( i \in \{0, 1, 2\} \), which is isomorphic to a semi-direct product
\[ (\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2. \]
We notice the semi-direct product \((\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2\) which is recorded in Parattu and Wingerter’s table does not arise as a subgroup of \( SU(3) \). This means the two semi-direct products are not isomorphic. By lemma 11, the Freedman group does not contain any element of order 27. Thus, \( S \) cannot be isomorphic to \( D_{27} \) either. By [13], the group of GAP ID [54, 8] is \( \Delta(54) \). In order to rule it out, it suffices to notice that a group of order 54 has a unique 3-Sylow subgroup. In the case of \( S \), this unique 3-Sylow subgroup is the group \( \mathcal{N} \) of order 3³. Recall that the groups of order 3³ are up to isomorphism
\[ \text{The two non-abelian groups} \quad \text{The three abelian groups} \]
\[ (\mathbb{Z}_3 \times \mathbb{Z}_3) \times \mathbb{Z}_3 \quad \mathbb{Z}_9 \times \mathbb{Z}_3 \quad \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \quad \mathbb{Z}_9 \times \mathbb{Z}_3 \quad \mathbb{Z}_27. \]
Now, the group \( S \) cannot be isomorphic to \(((\mathbb{Z}_3 \times \mathbb{Z}_3) \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2\), else it would contain a non-abelian 3-Sylow subgroup. However, its unique 3-Sylow subgroup is \( \mathcal{N} \) which is abelian. Note that we also recover the fact that \( S \) is not isomorphic to \( D_{27} = \mathbb{Z}_{27} \rtimes \mathbb{Z}_2 \), by a similar argument. By combining all the preceding results and the content of the table above, the group \( S \) must then be isomorphic to one of the two groups
\[ \mathbb{Z}_9 \times D_9 \]
\[ \mathbb{Z}_9 \times S_3. \]
In what follows, decide to set
\[ S = \langle A \rangle \times \langle B \rangle \rtimes \langle H_2 \rangle. \]
By the same arguments as in the proof of lemma 3, all the elements of odd order of \( S \) belong to \( \mathcal{N} \). To proceed, we will need another fact, this time about the elements of order 2.

**Lemma 14.** There are exactly three elements of order 2 in \( S \), namely \( H_2, BH_2 \) and \( B^3H_2 \).

**Corollary 5.** The group \( S \) is isomorphic to the direct product \( \mathbb{Z}_9 \times S_3 \).

**Proof.** (Corollary) In the dihedral group \( D_9 \), there are exactly 9 elements of order 2. Thus, the number of elements of order 2 in \( S \) is not sufficient for the group to be isomorphic to \( \mathbb{Z}_9 \times D_9 \).

**Proof.** (Lemma) We know from studying the structure of the Freedman group that the elements of \( S \) can be uniquely written as products \( A^iB^jH_2 \) with \( i \in \{0, 8\} \) and \( j \in \{0, 1, 2\} \). So, an element has order two means
\[ H_2 A^i B^j H_2 = A^{-i} B^{-j}. \]  
(18)
From there, since
\[ H_2 A H_2^{-1} = AB \]  
(19)
\[ H_2 B H_2^{-1} = B^3 \]  
(20)
we claim that the values \( i = 1, 2, 4, 5, 7, 8 \) are to exclude to allow equation (18) to be verified. Indeed, we have the following lemma.
Lemma 15. Out of the 18 elements of order 9

\[ A, A^2, A^4, A^5, A^7, A^8, AB, A^2B, A^4B, A^5B, A^7B, A^8B, A^3B^2, A^3B^3, A^3B^4, A^4B^2, A^4B^3, A^4B^4, A^5B^2, A^5B^3, A^5B^4, A^7B^2, A^7B^3, A^7B^4. \]

of \( S \), the ones that are self-conjugate under \( H_2 \) are exactly those below.

\[ AB^2, A^2B^2, A^4B^2, A^5B^2, A^7B^2, A^8B^2. \]

These are the only elements of \( S \) which have order 9 and belong to \( Z(S) \).

This lemma follows from the key relation

\[ H_2 A^iB^j H_2 = A^{i+j} \tag{21} \]

which holds for any \( i \) and \( j \) with \( i \in [0, 8] \) and \( j \in \{0, 1, 2\} \). In this equality, \( \bar{j} \) denotes the ‘conjugate’ of \( j \), that is \( \bar{j} = 2 \) if \( j = 1 \) and \( \bar{j} = 1 \) if \( j = 2 \).

Indeed, it suffices to notice that

\[ \forall (i, j) \in \{(1, 2), (2, 1), (4, 2), (5, 1), (7, 2), (8, 1)\}, \ i + \bar{j} \equiv j \pmod{3}. \tag{22} \]

We now return to the proof of lemma 14.

By lemma 15, for the values of \( i \) and \( j \) above, the product matrix \( A^iB^j \) is self-conjugate under \( H_2 \). Moreover, it cannot equal its inverse since it has order 9. Thus, equation (18) is not satisfied for such \( i \)’s and \( j \)’s. And still for the same values of \( i \), equation (18) is still not satisfied when \( j = 0 \). This follows from equation (19) and the fact that for these given values of \( i \), we have \( B^j \in \{B, B^2\} \). Now the contradiction comes from

\[ \langle A \rangle \cap \langle B \rangle = \{I_3\}. \tag{23} \]

Also, for these values of \( i \) and the respective conjugate values for \( j \), the \( H_2 \) conjugate of \( A^iB^j \) is \( A^i \). Again, equation (22) prevents equation (18) from happening. Thus, it remains to check the \( H_2 \)-conjugates of the elements of \( N \) of order 3. From equation (21), these are ruled by the relation

\[ H_2 A^iB^j H_2 = A^{i+j} \tag{23} \]

as now \( i \in \{0, 3, 6\} \). By (22), both equations (18) and (23) imply that \( i \) must then be zero. And since \( j + \bar{j} = 3 \), equation (18) gets satisfied when \( i = 0 \) and \( j \in \{1, 2\} \), thus ending the proof of lemma 14.

Proposition 6.

\[ S = \langle AB^2 \rangle \times \langle \langle B \rangle \rtimes \langle H_2 \rangle \rangle \simeq \mathbb{Z}_9 \times S_3. \]

Corollary 6. There exists semi-direct products so that

\[ (\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2 \simeq \mathbb{Z}_9 \times (\mathbb{Z}_3 \times \mathbb{Z}_2). \]

Proof. By lemma 14, the group \( S \) contains a subgroup \( \langle B \rangle \rtimes \langle H_2 \rangle \) which is isomorphic to \( S_3 \). Further, by lemma 15, the product \( AB^2 \) is self-conjugate under \( H_2 \) and thus commutes to all the other elements in the group. Moreover, we have

\[ S = \langle AB^2 \rangle \langle B \rangle \langle H_2 \rangle. \]

Then \( S \) is the direct product announced in the statement of proposition 6.

Corollary 6 follows.
Proof. We now deal with the proof of theorem 13. Again, it relies on the table from the link [17] in [16]. If $Fr(162 \times 4)$ were isomorphic to a finite subgroup of $U(2)$, then the four isomorphic 3-Sylow subgroups of $Fr(162 \times 4)$ of order 81 would also be isomorphic to a finite subgroup of $U(2)$. However, when looking at the short list of non-abelian groups of order 81 in the table, it appears that the only non-abelian $SU(3)$ finite subgroup of that order is

$$C(9, 1, 1) = (\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3,$$

which, as read from the table, cannot be identified with a finite subgroup of $U(2)$. □

5. Conclusion

5.1. Computer verification with GAP

Around the 2000 millennium time, Hans Besche, Bettina Eick and Eamonn O’Brien announced the construction up to isomorphism of the 49 910 529 484 groups of order at most 2000 [6]. Their work is of course amazing and from their table 1 of [3], the most difficult orders (in terms of the largest numbers of groups) appear as some products of a power of 2 by a power of 3. We know from their work [3] that there are 757 non-isomorphic groups of order 648. We provide below a small table which contains a single row of their big table.

| Order | 640 | 641 | 642 | 643 | 644 | 645 | 646 | 647 | 648 | 649 |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Number of groups | 21 541 | 1 | 4 | 1 | 9 | 2 | 4 | 1 | 757 | 1 |

We entered the presentation in GAP and found out that $Fr(162 \times 4)$ is the group of GAP ID [648, 259].

As for the structure returned by GAP for this group ID, it shows

$$((\mathbb{Z}_{18} \times \mathbb{Z}_6) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2.$$  \hspace{1cm} (24)

We then entered the presentation determined in the present paper for the group $D(18, 1, 1; 2, 1, 1)$ with the reassuring outcome that both groups have the same ID, thus confirming that they are isomorphic. We also played the same game with the smaller groups $D(9, 1, 1; 2, 1, 1)$ and $Fr(162)$. A presentation for the first group is part of the material exposed in the present paper, and a presentation for the second group can be found at the end of [1]. Note that one of the relations in the presentation of [1] is redundant and the commutator relation was unwillingly forgotten by the authors. Both groups have the same GAP ID, that is [162, 14].

For this ID, GAP provides the structure

$$((\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2.$$  \hspace{1cm} (26)

Remark 1. Combining the results from GAP and our own study, we see that we obtain some isomorphisms:

- $((\mathbb{Z}_{18} \times \mathbb{Z}_6) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2 \cong (\mathbb{Z}_{18} \times \mathbb{Z}_6) \rtimes (\mathbb{Z}_3 \rtimes \mathbb{Z}_2)$ by equation (25) and section 2
- $((\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2 \cong (\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes (\mathbb{Z}_3 \rtimes \mathbb{Z}_2)$ by equation (27), [1] and section 3
Associativity of the semi-direct product up to isomorphism is not automatic. But holds for instance when $G_1$, $G_2$, $G_3$ are subgroups of a same group $G$ and the group $G_2 \rtimes G_3$ acts on $G_1$ by first conjugating by the element of $G_3$ and then conjugating by the element of $G_2$.

More generally, if

$$\begin{align*}
\varphi : & \ G_3 \to Aut(G_1 \rtimes G_2), \quad \psi : \ G_2 \to Aut(G_1), \\
\alpha : & \ G_3 \to Aut(G_2), \quad \beta : \ G_2 \rtimes G_3 \to Aut(G_1),
\end{align*}$$

are homomorphisms satisfying to

$$\forall (g_2, g_3) \in G_2 \times G_3, \quad \beta(g_2, g_3) \circ p_1 = \psi(g_2) \circ p_1 \circ \varphi(g_3)$$

$$\forall (g_2, g_3) \in G_2 \times G_3, \quad \alpha(g_3) \circ p_2 = p_2 \circ \varphi(g_3)$$

where $p_1$ and $p_2$ denote the respective projections with respect to the first and second coordinates of the Cartesian product, then we have

$$(G_1 \rtimes G_2) \rtimes G_3 \simeq G_1 \rtimes (G_2 \rtimes G_3).$$

5.2. Concluding remarks

One of the weaknesses of the classification of [15] or its extended version is that this classification is not a classification up to isomorphism as well pointed out during our mini studies involving the $SU(3)$ finite subgroups from the series $(D)$. Indeed, we showed that several $D$-groups are identical within the series $D$. Also, a systematic analysis of the structure of some of the groups is not yet complete. The classification of all the finite subgroups of $SU(3)$ up to isomorphism remains an open problem. However, a stronger classification, this time up to conjugacy, makes more sense when dealing with groups of matrices. Of course a necessary condition for conjugacy is isomorphism. We investigated whether the Freedman group $Fr(162 \times 4)$ is conjugate to $D(18, 1; 2, 1, 1)$ and found out this answer is yes. The conclusions we drew are summarized below.

**Theorem 14.**

(i) The Freedman group $Fr(162 \times 4)$ is conjugate to $D(18, 1; 2, 1, 1)$.

(ii) There exists an orthogonal matrix $O$ such that

$$OFr(162 \times 4)O^T = D(18, 1; 2, 1, 1)$$

with

$$O = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}.$$  

(iii) The same conjugation relation holds for the respective subgroups:

$$OFr(162)O^T = D(9, 1; 2, 1, 1).$$

**Proof.** Suppose that there exists an invertible matrix $P \in GL_3(\mathbb{C})$ such that

$$P[Fr(162 \times 4)]P^{-1} = D(18, 1; 2, 1, 1).$$

□

**Lemma 16.** The invertible matrix $P$ must be a transition matrix from a common basis of diagonalization for $N$ to the canonical basis of $\mathbb{C}^3$.  

25
Proof. Under conjugation, a 3-Sylow subgroup of $Fr(162 \times 4)$ must be mapped to a 3-Sylow subgroup of $D(18, 1; 2, 1, 1)$. In particular, we must have

$$PS_3(F)P^{-1} = W_iS_3(D)W_i^{-1} \text{ some } i \in \{1, 2, 3, 4\} \text{ with } W_i = I_3.$$  \hspace{1cm} (28)

So, we have

$$PS_3(F)P^{-1} = F \sqcup FW_iE^2W_i^{-1}.$$  \hspace{1cm} (29)

The key idea now is to notice that the traces of elements of $FW_iE^2W_i^{-1}$ are all zero, this for all $i \in \{1, 2, 3, 4\}$. Since the trace of $A$ is not zero, we must then have

$$PA^{-1}P^{-1} = I_3.$$  \hspace{1cm} (30)

The same trick does not work with $B$ because $B$ has trace zero. However, the trace of the product $AB$ is non-zero. Therefore,

$$P(AB)P^{-1} = (PAP^{-1})(PB^{-1}) \in F.$$  \hspace{1cm} (31)

Now (30) and (31) imply that $PB^{-1}$ is also in $F$. Hence the immediate corollary. \hspace{1cm} $\Box$

**Corollary 7.**

$$PN^{-1} = F.$$  \hspace{1cm} (32)

This finishes the proof of lemma 16.

To move further, we will need to use some facts from before. We have seen in the proof of theorem 7 that any isomorphism of groups between $Fr(162 \times 4)$ and $D(18, 1; 2, 1, 1)$ must map the Klein group $V$ onto the Klein group $V_D$. We apply this fact to our isomorphism by conjugation and we get

$$PV^{-1} = V_D.$$  \hspace{1cm} (33)

It now suffices to recall that

$$(N \times V) \cap S_3(F) = [I_3]$$

$$(F \times V_D) \cap S_3(D) = [I_3].$$

Then, we also have

$$PS_3(F)P^{-1} = S_3(D).$$  \hspace{1cm} (34)

In order to conclude, we cannot bypass to investigate the shape of the transition matrix $P$. The matrix $A$ has three distinct eigenvalues, hence each eigenspace has dimension 1 over $C$. Explicitly, the eigenspaces are

$$C \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, C \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, C \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$  

Then, the most general form for $P$ is as follows

$$P = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} P_\sigma^{-1},$$  \hspace{1cm} (35)

where $d_1, d_2, d_3$ are non-zero complex numbers and $P_\sigma$ is the permutation matrix associated with a given permutation $\sigma$ of $Sym(3)$.

Equality (34) imposes some restrictions on $d_1, d_2, d_3$ and $\sigma$. First, we computed the product

$$\begin{pmatrix} 1/d_1 \\ 1/d_2 \\ 1/d_3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1} H_3 \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$  \hspace{1cm} (36)
and found out this product is:
\[
\begin{pmatrix}
0 & -\frac{d_2}{\sqrt{2}d_1} & 0 \\
0 & 0 & \frac{\sqrt{2}d_3}{d_2} \\
-\frac{d_1}{d_3} & 0 & 0
\end{pmatrix}.
\]
This matrix must also equal one of $E$ or $BEB$. By looking at the shape of all these matrices and also allowing conjugating equation (36) by a permutation matrix, we see that the following conditions must hold anyway.
\[
\begin{aligned}
d_3 &= -d_1 \\
d_2 &= -\sqrt{2}d_1.
\end{aligned}
\]
That is all the non-zero coefficients must be 1’s. Conversely,
\[
\begin{align*}
&\begin{aligned}
&d_1 = -\frac{1}{\sqrt{2}} \\
&d_2 = 1 \\
&d_3 = \frac{1}{\sqrt{2}} \\
&\sigma = id.
\end{aligned}
\end{align*}
\]
Then, we have
\[
\begin{align*}
PH_3P^{-1} &= E \\
PH_1P^{-1} &= BE \\
P(FUM)^3P^{-1} &= W_3 \\
PG_1(FUM)^3G_1^{-1}P^{-1} &= W_4 \\
P\mathcal{N}P^{-1} &= F
\end{align*}
\]
so that the first two points of theorem 14 hold. As for point (iii), it is part of our latter work since for this $O$, we have just seen that
\[
\begin{pmatrix}
O & \mathcal{N} & O^2 = F \\
O & S_3(F) & O^2 = S_3(D)
\end{pmatrix}.
\]

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Appendix

This part presents a brief explanation for the non-expert for how to obtain the permutation matrix from the Freedman fusion operation. We use the unitary normalization of Kauffman–
Lins theory at level 4 and refer the unfamiliar reader to [10] and [18]. Here is how we evaluate the diagram

First, we do an $F$-move on the edge labeled ‘0’. From the fusion rules, set

\begin{align*}
  i &= 4 \quad \text{when} \quad k = 0 \\
  i &= 2 \quad \text{when} \quad k = 2 \\
  i &= 0 \quad \text{when} \quad k = 4.
\end{align*}

We obtain

\[
\begin{bmatrix} 4 & 4 & i \\ k & k & 0 \end{bmatrix}_u
\]

where the brackets are used to denote unitary $6j$-symbols. Note the formula for the unitary $6j$-symbol is provided at the beginning of the forthcoming page. Do two more $F$-moves on the ‘internal’ edges labeled ‘2’. For the sake of not over-loading the next figure, the unitary $6j$-symbols

\[
\begin{bmatrix} 2 & 4 & i \\ k & 2 & 2 \end{bmatrix}_u \quad \text{and} \quad \begin{bmatrix} 4 & 2 & i \\ 2 & k & 2 \end{bmatrix}_u
\]

have been omitted since they are equal by elementary symmetries of the $6j$-symbols and they happen to be both equal to 1 or both equal to $-1$ depending on the admissible couple $(i, k)$. Get

\[
\begin{bmatrix} 4 & 4 & i \\ k & k & 0 \end{bmatrix}_u
\]

Next, undo the two loops by multiplying by adequate unitary theta symbols divided by the quantum dimension $\Delta_i$ of the particle of topological charge $i$. Get

\[
\begin{bmatrix} 4 & 4 & i \\ k & k & 0 \end{bmatrix}_u \left( \left( \begin{array}{c} 4, k, i \\ 2 \end{array} \right) \right) / \Delta_i
\]

We see that the particles of respective topological charge 0 and 4 have been interchanged as a result of the Freedman fusion operation.
Recall from [18] that
\[ \Theta^a(a, b, c) = \sqrt{\Delta_a} \sqrt{\Delta_b} \sqrt{\Delta_c}. \]
Also, the unitary version of the 6j-symbol is the following.
\[
\begin{bmatrix} G & B & E \\ C & D & F \end{bmatrix}^u = \frac{Tet \begin{bmatrix} G & B & E \\ C & D & F \end{bmatrix}}{\sqrt{\Theta(G, D, E) \sqrt{\Theta(C, D, F) \sqrt{\Theta(C, B, E) \sqrt{\Theta(G, B, F)}}}}}
\]
where Tet is the tetrahedron of [10] and \( \Theta \) denotes the non-unitary theta symbol.

We evaluated the coefficient in front of the final diagram and found the value 1 for all the couples \((k, i) = (0, 4), (k, i) = (2, 2)\) and \((k, i) = (4, 0)\). The 'k' corresponds to qutrit of the column of the matrix and the 'i' to the qutrit of the row of the matrix since the action is written with respect to the basis \((0, 2, 4)\) where the integers are the labels of the lowest edge. Thus, the resulting unitary matrix of the Freedman fusion operation is
\[
\begin{pmatrix} 0 & 2 & 4 \\ 0 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
\]
As a matrix of \( SU(3) \) we obtain the fusion matrix which we named FUM.

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