Gravitational Waves Induced by non-Gaussian Scalar Perturbations

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We study gravitational waves (GWs) induced by non-Gaussian curvature perturbations. We calculate the density parameter per logarithmic frequency interval, $\Omega_{GW}(k)$, given that the power spectrum of the curvature perturbation $\mathcal{P}_R(k)$ has a narrow peak at some small scale $k_*$, with a local-type non-Gaussianity, and constrain the nonlinear parameter $f_{NL}$ with the future LISA sensitivity curve as well as with constraints from the abundance of the primordial black holes (PBHs). We find that the non-Gaussian contribution to $\Omega_{GW}$ increases as $k^3$, peaks at $k/k_*=4/\sqrt{3}$, and has a sharp cutoff at $k=4k_*$. The non-Gaussian part can exceed the Gaussian part if $\mathcal{P}_R(k)f_{NL}^2 \gtrsim 1$. If both a slope $\Omega_{GW}(k) \sim k^\beta$ with $\beta \sim 3$ and the multiple-peak structure around a cutoff are observed, it can be recognized as a smoking gun of the primordial non-Gaussianity. We also find that if PBHs with masses of $10^{20}$--$10^{22}$ g are identified as cold dark matter of the Universe, the corresponding GWs must be detectable by LISA-like detectors, irrespective of the value of $\mathcal{P}_R$ or $f_{NL}$.

Introduction. The detection of gravitational waves (GWs) from mergers of black holes (BHs) or neutron stars (NSs) by LIGO/VIRGO\textsuperscript{[1--6]} has marked the beginning of the era of gravitational wave astronomy. Besides these GWs from mergers, there are other sources of GWs, like BH/NS binaries\textsuperscript{[7,17]}, phase transitions during the evolution of the universe\textsuperscript{[18--26]}, reheating/heating/cooling after inflation\textsuperscript{[27--33]}, and primordial scalar and tensor perturbations from inflation. For reviews of GW physics, see\textsuperscript{[34,35]}. We have already observed that the curvature perturbation $\mathcal{R}$ which seeds the cosmic microwave background (CMB) anisotropies and the large scale structure inhomogeneities is nearly scale invariant and Gaussian, and has an amplitude of $10^{-5}$ on scales larger than about 1 Mpc\textsuperscript{[37]}. However, the curvature perturbation on small scales remains unknown, because of the lack of observational constraints.

The amplitude of the primordial tensor perturbation is much smaller than that of the scalar curvature perturbation on CMB scales. The current constraint on the tensor-to-scalar ratio $r$ is $r < 0.064$ at 95% level\textsuperscript{[37]}. However, as the scalar and tensor perturbations are coupled at nonlinear level, we do have an induced tensor perturbation of order $\mathcal{R}^2$. In most of inflation models the induced tensor perturbation is much smaller than the primordial one from the vacuum fluctuations. Nevertheless, there are models of inflation that predict large curvature perturbations on small scales\textsuperscript{[38,56]}, for which the induced tensor perturbation may dominate over the primordial one.

Early works on GWs induced by the scalar perturbation at second order can be found in\textsuperscript{[57--62]}. In\textsuperscript{[63,64]}, the evolution of the induced GWs in the radiation-dominated era was studied. It was found that a $\delta$-function-like peak in the power spectrum of the curvature perturbation, $\mathcal{P}_R \sim \delta(k-k_*)$, may induce a characteristic GW power spectrum, which has a zero point at $k/k_*=\sqrt{2/3}$, and a peak at $k/k_* = 2/\sqrt{3}$. This behavior was then confirmed numerically and analytically in\textsuperscript{[65,75]}. Current CMB data do not exclude the possibility that the scalar perturbation is large on small scales\textsuperscript{[76,77]}. Typically, if the power spectrum for the primordial curvature perturbation has a peak on some small scale, there may be some regions where the density perturbation exceeds a threshold value $\delta_{th} \sim 0.3$ at horizon re-entry, and the matter inside the Hubble horizon collapses to form a primordial black hole (PBH)\textsuperscript{[78--82]}. The mass of a PBH is of the same order of the total energy inside the Hubble radius at horizon re-entry, which is hence determined by the wavenumber of the peak. Various constraints on the abundance of PBHs have been discussed\textsuperscript{[7,15,38,40,83,96]}.

The relation between the induced GWs and PBH formation was first studied in\textsuperscript{[97]}. For a $\delta$-function peak of power spectrum, and then for broad plateaus by\textsuperscript{[98,100]}. However, in those previous studies, the scalar perturbation was assumed to be Gaussian, which seems to be a rather naive assumption. When there appears a sharp peak in the curvature perturbation spectrum, it is natural to expect that there also appears a non-negligible non-Gaussianity. As PBHs are produced at the large amplitude tail of the probability distribution of the cur-
vature perturbation, any non-negligible non-Gaussianity would completely alter the PBH formation rate. This also suggests that we may have very different predictions on the amplitude and shape of the induced GW spectrum. This is the issue we discuss in this paper.

**Induced Gravitational Waves.** The perturbed metric in the Newton gauge is

\[ ds^2 = a(\eta)^2 \left[ -(1 - 2 \Phi) \, d\eta^2 + \left( 1 + 2 \Phi + \frac{1}{2} h_{ij} \right) \, dx^i dx^j \right], \]

where \( \eta \) is the conformal time, \( \Phi \) is the curvature perturbation in the Newton gauge, and \( h_{ij} \) is the tensor perturbation, and we have neglected the anisotropic stress perturbation \( GW \). For each polarization mode \( + \) or \( \times \), the equation for the Fourier component of the tensor perturbation at second order reads \[ 63 \]

\[ h_{kk}^l + 2 \mathcal{H} h_{kk}^l + k^2 h_k^l = \mathcal{S}(k, \eta), \]  

(1)

where the prime denotes a derivative with respect to \( \eta \), and we have omitted the superscript for polarizations. The source term \( \mathcal{S} \) is, in the radiation-dominated universe,

\[ \mathcal{S}(k, \eta) = 36 \int \frac{d^3l}{(2\pi)^3/2} \frac{l^2}{\sqrt{2}} \sin^2 \theta \left( \cos 2\varphi \text{ sin } 2\varphi \right) \Phi_l \Phi_{k-1} \]

\[ \times \left[ j_0(ux)j_0(vx) - 2j_1(ux)j_0(vx) \right] \]

\[ -2j_0(ux)j_1(vx) + 3j_1(ux)j_1(vx) \right]. \]

(2)

In \[ 2 \], \( \cos 2\varphi \) and \( \sin 2\varphi \) are for the polarizations of \( + \), \( \times \), respectively. \( \Phi_l \) is the Fourier mode of the curvature perturbation. \( j_i \) is the spherical Bessel function of order \( i \). We have also defined new variables \( u = |k - l|/k, v = l/k \) and \( x = \eta \sqrt{3} \) for simplicity. Equation (1) can be solved by the Green function method. After solving \( h_k \), we can use its two-point correlation function to calculate the density parameter \( \Omega_{GW} \) defined as the energy density of the GW per unit logarithmic frequency normalized by the critical density,

\[ \Omega_{GW}(k) = \frac{1}{12} \left( \frac{k}{H_0} \right)^2 \frac{k^3}{\pi^2} \left( \overline{h_k(\eta)h_k(\eta)} \right), \]

(3)

where the overline means the time average. For convenience, we change the variable to the curvature perturbation in comoving slices, \( R \), which is related to \( \Phi \) by \( \Phi = (2/3)R \) on superhorizon scales in the radiation-dominated universe. Up to the second order, it is expressed in terms of the Gaussian part as \[ 103 \, 109 \] :

\[ R(x) = R_g(x) + F_{NL} \left[ R_g^2(x) - \langle R_g^2(x) \rangle \right], \]

(4)

where we have introduced the nonlinear parameter for \( R, F_{NL} \), which is related to the nonlinear parameter for \( \Phi, f_{NL} \), by \( F_{NL} = (3/5)f_{NL} \) for the modes on CMB scales, i.e., for those on superhorizon scales at the epoch of matter-radiation equality. Then for the two-point correlation function of \( \Phi_k \), we have

\[ \langle \Phi_k \Phi_l \rangle \sim \frac{4}{9} \left( P_R(k) + 2F_{NL}^2 \int d^3l \, P_R(|l|) P_R(l) \right), \]

(5)

where we omitted an overall factor \( (2\pi)^3 \delta^{(3)}(k + p) \), and the power spectrum is defined as \( \langle R_k R_l \rangle = (2\pi)^3 P_R(k) \delta^{(3)}(k + p) \).

To step forward, we should specify the \( k \)-dependence of \( P_R(k) \), which in general can be different from the nearly scale-invariant spectrum we observe on the CMB scales. Here we study the case of a primordial curvature perturbation with a narrow peak at some specific scale \( k \), with a width \( \sigma \ll k \),

\[ P_R(k) = \frac{A_R}{(2\pi)^3/2}\sqrt{k^2} \exp \left( -\frac{(k - k_\sigma)^2}{2\sigma^2} \right). \]

(6)

The coefficient is to normalize \( \int d^3k P_R(k) = A_R \) where \( A_R \) is the dimensionless amplitude. This power spectrum with a narrow peak can be produced in various models of inflation \[ 38 \, 109 \], and easy to be extended to more general cases. We neglect the scale invariant contribution extrapolated from the CMB scales, since we assume \( A_R \) is much larger than \( 10^{-5} \). Keeping in mind that \( \sigma \ll k \), we can calculate the convolution of the power spectra in [5],

\[ \int d^3l \, P_R(|l|) P_R(l) \approx \frac{A_R^2}{(2\pi)^2} \frac{\sigma}{kk^2} \operatorname{erf} \left( \frac{k}{2\sigma} \right), \]

(7)

where some terms suppressed by higher orders of \( \sigma/k \) are neglected. When \( k > 2k_\sigma \), there is an exponentially suppressed tail which we can safely neglect. Then we can calculate the power spectrum of the tensor perturbation, up to the epoch of radiation-matter equality. Using [5], we obtain

\[ \Omega_{GW} = 6A_R^2 \frac{k^2}{2\pi \sigma^2} \left( \frac{k}{k_\sigma} \right)^4 \int_0^\infty dv \int_{1-v}^{1+v} du \, uv \, T(u, v) \]

\[ \times \left[ e^{-\frac{(u-k-k_\sigma)^2}{2\sigma^2}} + 2A_R F_{NL}^2 \frac{\sigma}{vk} \sqrt{\pi} \frac{2}{\sqrt{2\pi} \sigma} \operatorname{erf} \left( \frac{v}{2\sigma} \right) \right], \]

\[ \times \left[ e^{-\frac{(u-k-k_\sigma)^2}{2\sigma^2}} + 2A_R F_{NL}^2 \frac{\sigma}{u} \sqrt{\pi} \frac{2}{\sqrt{2\pi} \sigma} \operatorname{erf} \left( \frac{u}{2\sigma} \right) \right]. \]

(8)

where the integral kernel, \( T(u, v) \), was derived by [75].

\[ T(u, v) = \frac{1}{4} \left( \frac{4v^2 - (1 + v^2 - u^2)^2}{4uv} \right)^2 \left( \frac{u^2 + v^2 - 3}{2uv} \right)^2 \]

\[ \times \left\{ -2 + \frac{u^2 + v^2 - 3}{2uv} \ln \frac{3 - (u + v)^2}{3 - (u - v)^2} \right\}^2 \]

\[ + \pi^2 \left( \frac{u^2 + v^2 - 3}{2uv} \right)^2 \Theta \left( u + v - \sqrt{3} \right). \]

(9)
If the non-Gaussian contribution is small, the leading order is given by the Gaussian integral. For a $\delta$-function-like peak of the curvature perturbation, the main contribution comes from the neighborhood of $u \sim v \sim k_*/k_*$, which gives

$$\Omega_{\text{GW}}^{(0)} \simeq 6A_{\text{R}}^2 \left( \frac{k}{k_*} \right)^2 \mathcal{T} \left( \frac{k_*}{k}, \frac{k_*/k}{k} \right) \Theta(2k_* - k). \quad (10)$$

When $k \ll k_*$, the leading term of $\mathcal{T}(k_* / k, k_* / k)$ is approximately a constant, so $\Omega_{\text{GW}}^{(0)} \propto k^2$, with a peak about $\Omega_{\text{GW,peak}}^{(0)} \simeq 21.0A_{\text{R}}^2$ at $k_p^{(0)} \sim (2/\sqrt{3})k_*$. When $k < \sigma \ll k_*$, $\Omega_{\text{GW}}^{(0)} \propto k^3$. Detailed studies of this Gaussian case can be found in [63, 68, 69, 72, 75, 97, 98].

If $A_{\text{R}}F_{\text{NL}}^2 \gtrsim O(1)$, the contribution from the non-Gaussianity dominates the tensor power spectrum. The contributions from the terms proportional to $F_{\text{NL}}^2$ and $F_{\text{NL}}^4$, respectively, have the form,

$$\Omega_{\text{GW}}^{(2)} = 6A_{\text{R}}^3 F_{\text{NL}}^2 \left( \frac{k}{k_*} \right)^3 \Theta(3k_* - k) \times \int_{|1-k_*|}^{\min(1+k_* / 2k_*/k)} du \mathcal{T} \left( u, \frac{k_*}{k} \right) + \int_{\max(0,k_* / k - 1)}^{\min(2k_* / k, 1+k_* / k)} dv \mathcal{T} \left( \frac{k_*}{v}, v \right). \quad (11)$$

$$\Omega_{\text{GW}}^{(4)} = 6A_{\text{R}}^4 F_{\text{NL}}^4 \left( \frac{k}{k_*} \right)^4 \int_0^\infty dv \int_{|1-v|}^{1+v} du \mathcal{T}(u, v) \Theta(2k_* - v) \Theta(2k_* - uk). \quad (12)$$

If $A_{\text{R}}F_{\text{NL}}^2 \gg 1$, the term proportional to $F_{\text{NL}}^2$, $\Omega_{\text{GW}}^{(2)}$, overwhelms the terms proportional to $F_{\text{NL}}^4$, $\Omega_{\text{GW}}^{(4)}$. It is nonzero only for $k < 4k_*$, which is twice of the Gaussian cutoff at $2k_*$. It also has a peak at twice the frequency of the Gaussian peak, i.e. $k_p^{(2)} \sim (4/\sqrt{3})k_*$. The scaling law when $k \ll k_*$ can be estimated by requiring $u \sim v \gg 1$ in [12],

$$\Omega_{\text{GW}}^{(4)} \simeq 89.6 (A_{\text{R}}F_{\text{NL}})^4 \left( \frac{k}{k_*} \right)^3 \left[ 1 + \cdots \right] \Theta(4k_* - k). \quad (13)$$

The dots represent terms proportional to $\ln(k/k_*)$ and $(\ln(k/k_*)^2$ which can be neglected in the LISA sensitivity band. We can see it increases as $k^3$ when $k$ is small, which is faster than $k^2$. The GWs induced by non-Gaussian scalar perturbations are easily distinguishable if they dominate, which depends on the ratio of the peak amplitudes,

$$\frac{\Omega_{\text{GW}}^{(4)}}{\Omega_{\text{GW}}^{(0)}} \sim 4.3A_{\text{R}}^2 F_{\text{NL}}^4. \quad (14)$$

If this is larger than unity, we will clearly see the effect of the non-Gaussianity. For the power spectrum of $\mathcal{R}$, $F_{\text{NL}} \gtrsim 10$ will be enough for a peak amplitude of $A_{\text{R}} \sim 10^{-2}$. We emphasize that there is no observational constraint on $F_{\text{NL}}$ on small scales. If $A_{\text{R}}F_{\text{NL}}^2 \gtrsim 1$, which implies that $\Omega_{\text{GW}}^{(4)}$ and $\Omega_{\text{GW}}^{(2)}$ are larger than $\Omega_{\text{GW}}^{(0)}$, we find a series of peaks from the resonances around $k_p^{(4)} \sim (2/\sqrt{3})k_*$, $k_p^{(2)} \sim \sqrt{3}k_*$, and $k_p^{(2)} \sim (4/\sqrt{3})k_*$, which can be recognized as a smoking gun of the primordial non-Gaussianity at scale $k_*$. Observational Implications. The GW density parameter calculated in the previous section is valid from the horizon re-entry to matter-radiation equality. The GW density parameter today is given by $\Omega_{\text{GW}} = 10^{-2}$.

$$\Omega_{\text{GW},0}h^2 = 4 \times 10^{-9} \frac{\Omega_{\text{R}}h^2}{4 \times 10^{-5}} \left( \frac{A_{\text{R}}}{10^{-2}} \right)^2 \frac{\Omega_{\text{GW,eq}}}{A_{\text{R}}^2},$$

where $\Omega_{\text{GW,eq}}$ is the result obtained from [8], and we have neglected detailed dependence on the thermal his-
tory of the universe studied in [101, 102, 110, 114] which may be easily incorporated if necessary. We see that the amplitude of the GW density parameter is determined by the peak value of the primordial scalar perturbation, which may generate PBHs whose masses are also determined by the frequency of the peak. This was first studied by Saito et al. in [97],

\[ f_{GW} \sim 3\text{Hz} \left( \frac{M_{PBH}}{10^{16}\text{g}} \right)^{-1/2}, \]  

(15)

where \( f_{GW} \) is related to \( k \) by \( f_{GW} = k/(2\pi a) \) where \( a \) is the scale factor. We know that PBHs lighter than \( 5 \times 10^{14}\text{g} \) have already evaporated by today through Hawking radiation, while PBHs lighter than \( 10^{16}\text{g} \) are approaching their doomsday by radiating high energy particles which are strongly constrained by the observation of \( \gamma \)-ray background [84]. This implies there is an upper bound for the frequency of GWs induced by scalar perturbations, \( f_{GW} \lesssim 3\text{Hz} \). Therefore, unfortunately, we cannot expect any induced GWs to be detected by LIGO/VIRGO/KAGRA/ET (10Hz to 10^{3}\text{Hz}) [116–118]. However, we may see them by the next-generation GW observatories like LISA (10^{-4}\text{Hz} to 0.1Hz) [119, 121], Taiji [122], Tianqin [123], BBO (0.1Hz to 1Hz) [124, 125] or DECIGO (10^{-2}\text{Hz} to 1Hz) [126, 127]. In Fig. 1 the results of numerical integration of (8) for different \( F_{NL} \) and \( A_R \) and the corresponding current density parameter \( \Omega_{GW,h^2} \) are shown, together with the LISA sensitivity curve. As we can see, for a fixed \( A_R \), smaller \( F_{NL} \) will leave some resonance peaks as tails beyond the 2\( f_* \) peak, which may be difficult to be detected. On the contrary, large \( F_{NL} \) can make the resonance peaks prominent, while the peak around \((2/\sqrt{3})f_* \) becomes barely visible.

We can also constrain \( F_{NL} \) on small scales by the abundance of PBHs. For the non-Gaussian curvature perturbation, [4], the tadpole term in our case is given by \( \langle R_{g}^2(x) \rangle = \frac{1}{4} \int d^3 k P_R = A_R \). Then we can express the Gaussian perturbation \( R_g \) in terms of \( R \) as in [128],

\[ R_g(\mathcal{R}) = \frac{1}{2} F_{NL}^{-1} \left( -1 + \sqrt{1 + 4 F_{NL} (F_{NL} A_R + \mathcal{R})} \right). \]

PBHs will form if the curvature perturbation exceeds some threshold value \( R_{th} \approx 1 \) [129, 132]. The PBH mass fraction at the formation is

\[ \beta \simeq \begin{cases} \frac{1}{2} \text{erfc} \left( \frac{R_{g+}(R_{th})}{\sqrt{2A_R}} \right) - \frac{1}{2} \text{erfc} \left( -\frac{R_{g+}(R_{th})}{\sqrt{2A_R}} \right) & ; \text{ } F_{NL} > 0, \\ \frac{1}{2} \text{erfc} \left( \frac{R_{g-}(R_{th})}{\sqrt{2A_R}} \right) - \frac{1}{2} \text{erfc} \left( -\frac{R_{g-}(R_{th})}{\sqrt{2A_R}} \right) & ; \text{ } F_{NL} < 0. \end{cases} \]

(16)

For definiteness, we assume that the curvature perturbation peaks at \( 3 \times 10^{-3}\text{Hz} \), which generates PBHs with a single mass of \( 10^{22}\text{g} \). There are basically no observational constraints on the PBH abundance for the mass range \( 10^{20} \sim 10^{22}\text{g} \) [133] and \( 10^{17} \sim 10^{19}\text{g} \) [134, 135], except for the constraint that the PBH density cannot exceed that of dark matter, i.e. \( 1 \geq \Omega_{PBH}/\Omega_{DM} \approx 2.6 \times 10^{-9}(M_{PBH}/10^{16}\text{g})^{-1/2} \). This relation together with (15) gives

\[ \beta \lesssim 8.6 \times 10^{-15} \left( \frac{3 \times 10^{-3}\text{Hz}}{f_{GW}} \right). \]  

FIG. 2: The primordial black hole mass fraction at formation \( \beta \) depicted as a function of \( F_{NL} \) and \( F_{NL}^2A_R \), for the positive \( F_{NL} \) (up) and the negative \( F_{NL} \) (down), respectively. The constant \( \beta \) contours are drawn, where the upper bound given by \( \beta < 8.6 \times 10^{-16}, 8.6 \times 10^{-15} \) for the PBHs corresponding to PBH masses at \( M_{PBH} = (10^{20}\text{g}, 10^{22}\text{g}) \) can be seen as the border of the white and colored areas. The dashed lines are for \( A_R = 10^{-2}, 10^{-3}, \text{ and } 10^{-4} \) from left to right, while the shaded area is unphysical since \( A_R > 1 \). The thick black curve is the absolute constraint that the GW energy density be smaller than the current density of radiation, while the red and blue curves are the sensitivity bound of LISA at \( f_{GW} = 3 \times 10^{-2}\text{Hz} \) and \( 3 \times 10^{-3}\text{Hz} \), respectively. They correspond to PBH masses \( M_{PBH} = 10^{20}\text{g} \) and \( 10^{22}\text{g} \).
Constraint (17) is drawn in Fig. 2 for both positive and negative $F_{NL}$, together with the sensitivity bound of LISA from $f_{GW} = 3 \times 10^{-2}$Hz to $3 \times 10^{-3}$Hz. The white area in both figures is the parameter space allowed. When $F_{NL} \lesssim -0.3$, it is impossible to generate enough PBHs to account for dark matter since there would be too much GWs, which means there is no constraint from PBHs when $F_{NL}$ is negative.

For $F_{NL} > 0$, the parameter space is narrower. From the small $F_{NL}$ limit, we see that to avoid PBH overproduction we need $A_R \lesssim 1.5 \times 10^{-2}$. Besides, for a given $A_R$, there is an upper bound for $F_{NL}$ from the PBH abundance constraint (17): $F_{NL} < 0.017/A_R$. This can be found from the intersections of the PBH constraint and the equal-$A_R$ lines in Figure. 2. Interestingly, all of the possible PBH abundances are above the LISA sensitivity curve, which means that if PBHs with masses from $10^{20}g$ to $10^{22}g$ are the dominant dark matter, we must observe the corresponding GW signals by LISA, no matter how small $A_R$ is.

**Conclusion** We studied the effect of a local-type non-Gaussianity in the curvature perturbation on the induced tensor perturbation at second order as well as on the PBH formation. The scalar perturbation was assumed to have a narrow peak on small scale $1/k_*$, with a typical non-Gaussianity. Our result shows that if $A_R F_{NL} \gtrsim 1$, the non-Gaussian contribution becomes prominent, and the main features of the GW density parameter $\Omega_{GW}$ will be a series of peaks with the highest at $(4/\sqrt{3})k_*$ just before the cutoff at $4k_*$, and the $k^3$-slope on smaller $k$ side of the peaks. The detection of these features will be clear evidence for the primordial non-Gaussianity of the curvature perturbation at around $k_*$.

In this paper we only considered a narrow peak in the scalar perturbation spectrum, although broad plateaus may be generated in some other models of inflation [31, 36]. Nevertheless, our criterion for the existence of non-Gaussianity remains universal. We can see from the integral [5] that the power $\beta$ of $\Omega_{GW} \sim k^3$ induced by the Gaussian scalar perturbations will be around 3 when $k \ll \sigma \ll k_*$, but decreases as the width $\sigma$ increases, while $\sigma \rightarrow \infty$ will induce a scale-invariant GW spectrum as expected, which is also shown numerically in [99]. So we can conclude that $\beta \lesssim 3$ is characteristic for GWs induced by scalar perturbations. The first order electroweak phase transition may also give rise to stochastic GWs with $\beta \sim 3$ on the low frequency side. However, almost all of the previous results indicate that the peak frequency is below the LISA band, thus we can probably only detect the high frequency tail where $\beta < 0$ by LISA [18, 20]. Another possible source of stochastic GW background is incoherent superpositions of GWs from compact binaries. But it will have $\beta \sim 2/3$ [83, 38]. This means that the detection of GWs with $\beta \sim 3$ can be recognized as of induced origin, where multiple peaks will be a smoking gun of primordial non-Gaussianity. Further detailed studies are left for future work.

We also derived constraints on the PBH abundances. Currently it is possible for PBHs to serve as all the dark matter if $M_{PBH}$ locates in the range $10^{17}g$ to $10^{19}g$ or $10^{20}g$ to $10^{22}g$. The former case corresponds to GWs with peak frequency from $0.1Hz$ to $1Hz$, which can be fully explored by DECIGO, while the low frequency tail can be seen by LISA. In this paper we focus on the latter case which corresponds to the GW frequencies $3 \times 10^{-3}Hz$ to $3 \times 10^{-2}Hz$, right in the sensitivity frequency band of LISA. We found that if these PBHs consist a substantial portion of the dark matter, the corresponding GW signal must be detectable by LISA. Conversely, if we are unable to detect any induced GW signal by LISA, it will be impossible for PBH to serve as all dark matter in the mass range $10^{20}g$ to $10^{22}g$. Depending on the integration time, the abundance of PBHs can be further constrained. Therefore the induced GWs can be used as a powerful tool of probing the abundances of small PBHs. This will also be left for our future work.

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[1] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], Phys. Rev. Lett. 116, no. 6, 061102 (2016) doi:10.1103/PhysRevLett.116.061102 arXiv:1602.03837 [gr-qc].
[2] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], Phys. Rev. Lett. 116, no. 24, 241103 (2016) doi:10.1103/PhysRevLett.116.241103 arXiv:1606.04855 [gr-qc].
[3] B. P. Abbott et al. [LIGO Scientific and VIRGO Collaborations], Phys. Rev. Lett. 118, no. 22, 221101 (2017) doi:10.1103/PhysRevLett.118.221101 arXiv:1706.01812 [gr-qc].
[4] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], Astrophys. J. 851, no. 2, L35 (2017) doi:10.3847/2041-8213/aa9f0c arXiv:1711.05578 [astro-ph.HE].
[5] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], Phys. Rev. Lett. 119, no. 14, 141101 (2017) doi:10.1103/PhysRevLett.119.141101 arXiv:1709.09660 [gr-qc].
[6] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], Phys. Rev. Lett. 119, no. 16,
