Stabilization of an inverted pendula system in the presence of elastic bonds

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Abstract. We developed a mathematical model of an unstable system of set of $N$ connected inverted pendula. We described principles of stabilization of such a system at the neighborhood of the unstable upright position. Also, we studied dynamics of this system and obtained the limiting conditions ensuring the stability of this system. Simulation results demonstrate several important features of the system’s dynamics.

1. Introduction
The problem of inverted pendulum is known to play a central role in the control theory [1–7]. This problem, as a test model, illustrates many challenging problems in control design. Due of their nonlinear nature, pendulums are widely used to illustrate many of the ideas emerging in the field of nonlinear control [8, 9]. Typical examples are the feedback stabilization, variable structure control, passivity-based control, back-stepping and forwarding, nonlinear observers, friction compensation and nonlinear model reduction. The challenges of control made the inverted pendulum systems a classical tool in control laboratories. Note that the problem of stabilization of such a system is a classical problem of the dynamics and control theory. Moreover, the model of inverted pendulum is widely used as a standard model for testing of control algorithms (including PID controller, neural networks, fuzzy control etc.). Such a mechanical system can be found in various fields of technical science, from robotics to space technologies. For example, the stabilization of inverted pendulum is considered in the problem of missile pointing because the engine of missile is placed lower than the center of mass and this leads to aerodynamic instability. It can be used as an ethalon system for development the appropriate algorithm ensuring the stability of missel at starting moment of motion (in other words on the initial part of trajectory). A similar problem is solved in the self-balancing transport device (the so-called segway or self-balancing scooter). Thus, the inverted pendulum can be stated as a convenient model in the problem of stabilization of unstable robotic devices.
Moreover, the model of inverted pendulum (especially, under various kinds of control of the suspension point motion) is widely used in physics, applied mathematics, engineering science, neuroscience, economics, and other fields [10].

The first theoretical description of the inverted pendulum was carried out by Stephenson [11], and the first experimental investigation of the stabilization process for such a system (using oscillations of the suspension point) was considered by Kapitza [12]. In fact, the problem of inverted pendulum is the one which is more than one hundred years old, but it is still relevant. Note that recently the systems of inverted pendula, namely the double and triple pendula, attracted special interest, especially because in such systems the deterministic chaos can be obtained. The problem of stabilization of such an otherwise unstable, autonomous, and mechanical system is a fascinating task, both from theoretical (various methods of nonlinear analysis) and applied (modeling of real-life mechanical systems) points of view [13–15].

In this paper we present the results of generalization of the mathematical model of two coupled inverted pendula [16] and investigation of the dynamics of a mechanical system consisting of an arbitrary number of inverted pendula hinged on the moving platform and connected by a spring (we would like to note that this paper is a continuation of the cited paper [16]). The force applied to the platform (and caused its horizontal motion) is treated as a control action. The purpose of this investigation is to solve the problem of stabilization of the pendula system in the vertical position using a horizontal motion of the platform when the angles of deviation are known. To solve this problem an algorithm ensuring stabilization of the system near the vertical position has been developed (this algorithm is based on the well-known feedback principles and realized using the standard methods of numerical simulations). The stability zones for the described system as well as the dependence of the form of stability zones on the spring stiffness have also been obtained. More specific, these are the main aims of our research: 1) to investigate the dynamics of the mechanical system described above; 2) to obtain explicit expressions for parameters ensuring the stabilization of the system in the vicinity of the upright position; 3) to identify the stability zones depending on the system’s parameters.

2. Physical model
Let us consider a system of $N$ inverted pendula with mass $m$ that are rigidly fixed on a movable cart and connected by a spring with stiffness $k$ (when the pendulums are in vertical position the spring is non-stretched). We assume that the cart has no mass and moves without friction, and the control action applied to the cart determines acceleration $u$ (see Fig. 1).

![Physical model of coupled pendulums.](image)

To describe the dynamics of such a system we write the equations for the moments (neglecting
damping) in the form:

\[
\begin{align*}
I_1 \frac{d\omega_1}{dt} &= M_1^{(g)} + M_1^{(kr)} + M_1^{(u)}, \\
I_i \frac{d\omega_i}{dt} &= M_i^{(g)} + M_i^{(kr)} + M_i^{(kl)} + M_i^{(u)}, \\
I_n \frac{d\omega_n}{dt} &= M_n^{(g)} + M_n^{(kl)} + M_n^{(u)},
\end{align*}
\]

(1)

where \(I_i\) are the moments of inertia of the pendulums, \(\omega_i\) are the angular velocities, \(M_i^{(g)}\) are the returning moments of gravitational force, \(M_i^{(kl)}\) and \(M_i^{(kr)}\) are the returning moments of elastic forces from left and right sides relative to the \(i\)-th pendulum, \(M_i^{(u)}\) is the returning moment of control.

Since the motion is flat, all these vectors are orthogonal to the plane of the figure. This allows us to transform vector equations to a scalar form. Positions of the pendula are characterized by the deviation angles (relative to the vertical axis) \(\varphi_i\). We consider small (linear) oscillations of pendula (it is assumed that \(\varphi_i \ll 1\)). For the elongation of the spring \(\Delta x_i\) we can write the following approximate expression:

\[
\Delta x_i \approx l (\varphi_{i+1} - \varphi_i).
\]

(2)

Also, we introduce the eigen-frequency for a single pendulum:

\[
\omega = \sqrt{\frac{mgl}{I}}.
\]

(3)

When \(I = ml^2\), \(\omega = \sqrt{\frac{g}{l}}\) and we have the case of mathematical-like pendulum. From System (1), taking into account (2) and (3), we obtain the following equations:

\[
\begin{align*}
\ddot{\varphi}_1 &= \omega^2 \varphi_1 + a (\varphi_2 - \varphi_1) - c, \\
\ddot{\varphi}_i &= \omega^2 \varphi_i + a (\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}) - c, \\
\ddot{\varphi}_n &= \omega^2 \varphi_n - a (\varphi_n - \varphi_{n-1}) - c,
\end{align*}
\]

(4)

where \(a_i = \frac{k}{m}\) and \(c = \frac{c}{l}\).

At the next stage, we consider the control based on the feedback principle [10, 13, 16], namely, the control action which can be described by the following expression:

\[
u = A \cdot \text{sign} (Bs + \dot{s}),
\]

(5)

where \(s = \sum_{i=1}^{n} \varphi_i\), \(A = \text{const}\) and \(B = \text{const}\).

If \(\varphi_i = 0\) (when the pendulums are at the opposite sides relative to the vertical axis) the control \(u\) does not act, and the spring stiffness \(k\) defines the positions of pendulums.

3. Solution to the stabilization problem

To achieve the stabilization of the system under consideration we use the standard principles of feedback control. The first step in the solution to the stabilization problem is to investigate the dynamics of the mechanical system, described earlier. To do this, we need to sum up all equations in (4). In this case we obtain the following equations:

\[
\ddot{s} = \omega^2 s - nc, \quad s = \sum_{i=1}^{n} \varphi_i.
\]

(6)
The sum of deviation angles $s$ is proportional to the average position of pendula in space. It is known (see, e.g., [16]) that the control $u$ affects the average positions of pendula, but does not affect their relative positions. At the same time, the stiffness of the spring $k$ affects the relative positions, but does not affect the average positions. Thus, the control $u$ is suitable for stabilization of the pendula in the case when the stiffness of the spring $k$ is sufficient to hold System (4).

The solution to Equations (6) has the form:

$$s(t) = C_1 e^{\omega t} + C_2 e^{-\omega t} + \frac{nu}{\omega^2 l}. \quad (7)$$

Let the initial values of the deviation and velocity are $s(0) = s_0$ and $\dot{s}(0) = \dot{s}_0$, respectively. In this case, the solution to the Cauchy’s problem for the first equation has the following form:

$$s(t) = \frac{1}{2} \left( s_0 + \frac{\dot{s}_0}{\omega} - \frac{nu}{\omega^2 l} \right) e^{\omega t} + \frac{1}{2} \left( s_0 - \frac{\dot{s}_0}{\omega} - \frac{nu}{\omega^2 l} \right) e^{-\omega t} + \frac{nu}{\omega^2 l}. \quad (8)$$

We investigate the phase trajectory of the system (see the Fig. 2) based on the solution to (8) following [16]. When $u = 0$ (as can be seen from $\omega s + \dot{s} = 0$), the phase coordinates tend to zero equilibrium point. Thus, if the control $u$ is able to bring the pendulum’s phase point to a straight line (as is shown in the figure 2), the system becomes stable [10]. In this case, we can write:

$$s_0 + \frac{\dot{s}_0}{\omega} - \frac{nu}{\omega^2 l} = 0, \quad (9)$$

where

$$u = \frac{\omega l}{n} (\omega s_0 + \dot{s}_0). \quad (10)$$

Equation (10) can be rewritten in the following form:

$$u = \frac{\omega l}{n} [\omega s_0 + \dot{s}_0] \text{sign} (\omega s_0 + \dot{s}_0). \quad (11)$$

Now we compare Equations (5) and (11). Control coefficients $A$ and $B$ ensuring the stabilization of the pendula can be obtained in the following form:

$$\begin{cases} A = \frac{\omega l}{n} |\omega s_0 + \dot{s}_0|, \\ B = \omega. \end{cases} \quad (12)$$
Note that this stable position is not vertical (in Expression (9) the sum of the angles is \( s = \frac{nu}{\omega l} \)). However, as follows from the solution to (8):

\[
s_0 + \frac{\dot{s}_0}{\omega} - \frac{nu}{\omega^2 l} < 0,
\]

(13)

the control feedback will keep the pendulum in the vicinity of the vertical position. At the time instants when the pendula cross the vertical points, the control’s sign reverses and the process repeats. Thus, Inequality (13) is the criterion for the stabilization of the pendula in the vicinity of the vertical position. Based on this fact, we can rewrite Inequality (12) in the following way:

\[
\begin{align*}
A & \geq \frac{\omega l}{n} |\omega s_0 + \dot{s}_0|, \\
B &= \omega.
\end{align*}
\]

(14)

The resulting inequality is the criterion for the stabilization of the pendulums.

Now we investigate the phase trajectory of the system based on Criterions (14). As follows from this inequality:

\[
\begin{align*}
A & \geq \frac{\omega l}{n} (\omega s_0 + \dot{s}_0), & \text{if } s_0 > 0, \quad \dot{s}_0 > 0, \\
-A & \leq \frac{\omega l}{n} (\omega s_0 + \dot{s}_0), & \text{if } s_0 < 0, \quad \dot{s}_0 < 0.
\end{align*}
\]

(15)

Inequalities (15) can be presented on the phase plane as the area of all possible initial states for which the system with fixed control can be stabilized (see Fig. 3).

\[\text{Figure 3. Stability zone of the system described by Inequality (15).}\]

Thus, areas \( C_1 \) and \( C_3 \) in Fig. 3 show all possible initial positions of the system when the pendula’s stable vertical positions are not possible, and area \( C_2 \) show the stability zone of the system.

Note that Condition (14) is a necessary but not sufficient for stabilization of the system. To stabilize the system it is also necessary to know the range of the values of the coefficient of spring stiffness \( k \) which ensures the retention of the pendulums. To identify this range of values we introduce a new variable \( d_i \), which is defined as \( d_i = \varphi_i - \varphi_{i+1} \), where \( m = n - 1 \). In this case,
we obtain the following expressions (the matrix notations can be used for simplicity together with the corresponding notations for matrix product):

\[
\begin{align*}
\ddot{d}_1 &= \omega^2 d_1 + \frac{k}{m} (d_2 - 2d_1), \\
\ddot{d}_i &= \omega^2 d_i + \frac{k}{m} (d_{i-1} - 2d_i + d_{i+1}), \\
\ddot{d}_m &= \omega^2 d_m - \frac{k}{m} (d_{m-1} - 2d_m),
\end{align*}
\]

\[\Rightarrow \ddot{D} = C \times D, \quad (16)\]

where

\[
C = \begin{cases} 
\begin{align*}
a_{i,i} &= \omega^2 - \frac{2k}{m}, & i = 1, m, \\
a_{i,i+1} &= \frac{k}{m}, & i = 1, m-1, \\
a_{i,j} &= 0, & i = 1, m-2, j = i+2, m, \\
a_{j,i} &= 0, & i = 1, m-2, j = i+2, m.
\end{align*}
\end{cases} \quad (17)\]

Parameters of the system required for stability can be found using the following expression:

\[\text{Re} \lambda_1 < 0 \land \text{Re} \lambda_2 < 0 \ldots \land \text{Re} \lambda_m < 0, \quad (18)\]

where \(\lambda_i\) are the eigenvalues of matrix \(C\) (17).

Inequalities (18) allow us to find the range of coefficient \(k\), required for stabilization of the pendula relative to each other. For example, for 6 pendulums we can obtain the following range:

\[k > \frac{\omega^2 m}{2 - \sqrt{3}}, \quad (19)\]

Note that the solution to System (18) becomes rather complex when \(n\) increases.

4. Numerical simulation

The numerical simulation of the dynamics of the system under consideration (we use the standard 4th order Runge-Kutta method realized in MATLAB) has been performed for the following system parameters (these parameters correspond to physically realized system of coupled inverted pendula):

\[k = 200, \quad m = 0.1, \quad l = 1, \quad n = 40. \quad (20)\]

The following initial values of the deviation and velocity have been used:

\[s_1 (0) = 0, \quad \dot{s}_1 (0) = 0.1, \quad s_i (0) = \dot{s}_i (0) = 0, \quad i = 2, n. \quad (21)\]

Figure 4 shows the results for the sum of angles of deviations when Condition (14) is satisfied and not satisfied. As it can be seen from this figure, the system can be stabilized near the vertical position using control in the form (5) (the criterion of stabilization (14) was taken into account). Note that Condition (14) imposes the restriction on the low values of the coefficient \(A\); even fairly large values of this coefficient lead the system to a stable state (in this case, large amplitude oscillations of the pendulums relative to the vertical axis are expected).

5. Elastic waves

Excitation of the chain of coupled oscillators (taking into account elastic bonds between them) is known to lead to wave motion. In this case the wave phase front moves with a finite velocity and carries the energy of oscillations (here we would like to note that the presented in the previous section initial conditions determine a single wave traveling from one edge to another one and back). Note also that the velocity of this front is constant only for homogeneous medium (here we
consider just a set of pendula, i.e. the discrete system). This phenomenon is known as traveling wave. Examples of traveling waves include elastic waves in a metal rod, a column of gas or liquid, and electromagnetic waves. In the mechanical system under consideration elastic waves arise in the process of stabilization (see Fig. 5). This occurs under certain initial conditions.

The left panel of Figure 5 shows a single wave while the right one shows two opposing waves (determined by the corresponding initial conditions, namely \( \dot{\varphi}_1 (0) = 0.1, \dot{\varphi}_n (0) = -0.1, \varphi_1 (0) = \varphi_n (0) = 0, \varphi_i (0) = \dot{\varphi}_i (0) = 0, i = 2, n - 1 \)). As can be seen from this figure, the elastic waves produced in our mechanical system form the interference-like pattern.

6. Conclusions
The dynamics of coupled inverted pendula is investigated using numerical simulations. The stability zones for such a system in the phase space, depending on the system’s parameters, are predicted. The analysis of the dynamics of the model has shown that stabilization of such a
complex unstable system is possible using a relatively simple control action in the form of an alternating force with a constant amplitude. The elastic waves and interference-like patterns for these waves are predicted during the stabilization process.

7. References
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