Rotation number and lifts of a Fuchsian action of the modular group on the circle

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Abstract
We characterize the semi-conjugacy class of a Fuchsian action of the modular group on the circle in terms of rotation numbers of two standard generators and that of their product. We also show that among lifts of a Fuchsian action of the modular group, only 5-fold lift admits a similar characterization. These results indicate similarity and difference between rotation number and linear character.

Keywords: Rotation number, modular group, group actions on the circle

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1 Introduction
Rotation number of an orientation-preserving homeomorphism of the circle has similar properties to absolute value of the trace of an element in PSL(2, R). For example, they are invariant under conjugation and furthermore, Jørgensen’s criterion of discreteness for subgroups of PSL(2, R) [11, Theorem 2], which can be described in terms of absolute value of the trace, has an analogue for the group of real analytic diffeomorphisms of the circle (see [13, Theorem 1.2]). In this article, we give another similarity between rotation number and linear character from a viewpoint given by D. Calegari and A. Walker [5].

1.1 Rotation number
We denote by Homeo+(S^1) the group of orientation-preserving homeomorphisms of the circle. We regard the circle S^1 as the quotient R/Z and denote by p: R → S^1 the projection. Let Homeo+(S^1) be the group of lifts of orientation-preserving homeomorphisms to R, namely, homeomorphisms of R commuting with integral translations.

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For $\tilde{f} \in \widetilde{\text{Homeo}}_+(S^1)$, we define the translation number $\widetilde{\text{rot}}(\tilde{f}) \in \mathbb{R}$ of $\tilde{f}$ by

$$\widetilde{\text{rot}}(\tilde{f}) = \lim_{n \to \infty} \frac{(\tilde{f})^n(\tilde{x}) - \tilde{x}}{n},$$

where $\tilde{x} \in \mathbb{R}$. Note that the limit exists and does not depend on the choice of a point $\tilde{x} \in \mathbb{R}$. For $f \in \text{Homeo}_+(S^1)$, we define the rotation number $\text{rot}(f) \in \mathbb{R}/\mathbb{Z}$ of $f$ by

$$\text{rot}(f) = \widetilde{\text{rot}}(\tilde{f}) \mod \mathbb{Z},$$

where $\tilde{f} \in \widetilde{\text{Homeo}}_+(S^1)$ is a lift of $f$ to $\mathbb{R}$.

Among several properties of rotation number, we recall that $\text{rot}(f) = \frac{p}{q}$, where $\frac{p}{q}$ is a reduced fraction if and only if $f$ has a period point of period $q$. In particular, $\text{rot}(f) = 0$ if and only if $f$ has a fixed point (see for example [9] in detail and other properties of rotation number).

### 1.2 Lifts of a group action on the circle

For a group $\Gamma$, we denote by $R(\Gamma)$ the space of homomorphisms from $\Gamma$ to $\text{Homeo}_+(S^1)$. We equip $R(\Gamma)$ with the uniform convergence topology on generators if necessary.

We define a lift of a group action on the circle.

Let $k \geq 2$ be a positive integer and denote by $p_k: S^1 \to S^1$ the $k$-fold covering map. For a group $\Gamma$, a homomorphism $\phi \in R(\Gamma)$ is a $k$-fold lift of a homomorphism $\psi \in R(\Gamma)$ if $p_k \circ \phi(\gamma) = \psi(\gamma) \circ p_k$ for every $\gamma \in \Gamma$.

We remark that if $\phi \in R(\Gamma)$ is a $k$-fold lift of a homomorphism $\psi \in R(\Gamma)$, then we have $k \text{rot}(\phi(\gamma)) = \text{rot}(\psi(\gamma))$ for every $\gamma \in \Gamma$.

### 1.3 Semi-conjugacy class

Semi-conjugacy between two actions of a group on the circle has been defined in several ways (see [8], [9], [1]). In this paper, we follow the way presented in [3].

For $\phi_1, \phi_2 \in R(\Gamma)$, we say that $\phi_1$ is semi-conjugate to $\phi_2$ if there exists a continuous degree-one monotone map such that $h \circ \phi_1(\gamma) = \phi_2(\gamma) \circ h$ for every $\gamma \in \Gamma$. Here, a map $h: S^1 \to S^1$ is called a degree-one monotone map if it admits a lift $\tilde{h}: \mathbb{R} \to \mathbb{R}$ commuting with integral translations, and nondecreasing on $\mathbb{R}$.

Note that semi-conjugacy is not symmetric and is not an equivalence relation. We consider the equivalence relation generated by semi-conjugacy, which is called monotone equivalence in [3]. We call the monotone equivalence class of $\phi \in R(\Gamma)$ the semi-conjugacy class of $\phi$. Note that if two minimal homomorphisms belong to the same semi-conjugacy class, then they are topologically conjugate. We define the semi-conjugacy class of an orientation-preserving homeomorphism of the circle in a similar way.
A classical result due to H. Poincaré says that two homeomorphisms are in the same semi-conjugacy class if and only if their rotation numbers coincide, which is similar to the fact that two matrices in $\text{SL}(2, \mathbb{R}) \setminus \{\pm E\}$ are conjugate if and only if their traces coincide.

As for group actions, however, $\phi_1, \phi_2 \in \text{R}(\Gamma)$ do not belong to the same semi-conjugacy class if we only suppose that $\text{rot}(\phi_1(\gamma)) = \text{rot}(\phi_2(\gamma))$ for every $\gamma$. It can be seen by considering Fuchsian actions corresponding to hyperbolic structures on 2-orbifolds (see for example [6] about 2-orbifolds and hyperbolic structures on them).

### 1.4 Fuchsian actions

Let $\mathcal{O}$ be a compact, connected, oriented 2-orbifold with negative orbifold Euler characteristic $\chi_{\text{orb}}(\mathcal{O}) < 0$. For each hyperbolic structure on the interior of $\mathcal{O}$ compatible with the orientation of $\mathcal{O}$, we have a homomorphism from the orbifold fundamental group $\pi_{1,\text{orb}}(\mathcal{O})$ to $\text{PSL}(2, \mathbb{R})$ by identifying the universal cover $\tilde{\mathcal{O}}$ with the hyperbolic plane $\mathbb{H}^2$. By considering the action on the ideal boundary $\partial \mathbb{H}^2 \cong S^1$, we obtain a homomorphism $\phi_{\mathcal{O}} \in \text{R}(\pi_{1,\text{orb}}(\mathcal{O}))$. We call such a homomorphism a Fuchsian action associated to $\mathcal{O}$. Note that the semi-conjugacy class of a Fuchsian action associated to a fixed 2-orbifold $\mathcal{O}$ is independent of the choice of a hyperbolic structure and that a Fuchsian action corresponding to a hyperbolic structure with finite area is minimal.

In general, we cannot characterize the semi-conjugacy class of a Fuchsian action only by rotation numbers of all elements. In fact, for a Fuchsian action $\phi_S$ associated to a compact, connected, oriented surface $S$ with negative Euler characteristic, the homeomorphism $\phi_S(\gamma)$ has a fixed point for every $\gamma \in \Gamma$ but there is no global fixed point. This means that $\text{rot}(\phi_S(\gamma)) = 0$ for every $\gamma \in \Gamma$ but the Fuchsian action $\phi_S$ does not belong to the semi-conjugacy class of the trivial action.

Now we show, however, that we can characterize the semi-conjugacy classes of a Fuchsian action of a specific 2-orbifold and its certain lift by only rotation numbers of finite elements.

### 1.5 Main result

We focus on a special 2-orbifold. Let $\mathcal{O}_{2,3}$ be the 2-orbifold which is obtained from a 2-disk by making two cone-points of orders 2, 3. Note that the interior of $\mathcal{O}_{2,3}$ is homeomorphic to $\mathbb{H}^2/\text{PSL}(2; \mathbb{Z})$ and $\pi_{1,\text{orb}}^{\text{tr}}(\mathcal{O}_{2,3})$ is isomorphic to the modular group $\text{PSL}(2, \mathbb{Z})$. We fix a presentation

$$
\pi_{1,\text{orb}}^{\text{tr}}(\mathcal{O}_{2,3}) = \langle \alpha, \beta \mid \alpha^2 = \beta^3 = 1 \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_3,.
$$

where $\alpha = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\beta = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$. Let $\phi_{\mathcal{O}_{2,3}}$ be a Fuchsian action of $\mathcal{O}_{2,3}$ which is equal to the action by linear fractional transformations on
\( \mathbb{R} \cup \{ \infty \} \simeq S^1 \). It follows that
\[
\phi_{O_{2,3}}(\alpha)(0) = \infty, \quad \phi_{O_{2,3}}(\alpha)(\infty) = 0 \\
\phi_{O_{2,3}}(\beta)(0) = \infty, \quad \phi_{O_{2,3}}(\beta)(\infty) = -1, \quad \phi_{O_{2,3}}(\beta)(\pm 1) = 0 \\
\phi_{O_{2,3}}(\alpha \beta)(0) = 0.
\]
Hence we have
\[
(\text{rot}(\phi_{O_{2,3}}(\alpha)), \text{rot}(\phi_{O_{2,3}}(\beta)), \text{rot}(\phi_{O_{2,3}}(\alpha \beta))) = \left( \frac{1}{2}, \frac{1}{3}, 0 \right).
\]

It follows from the presentation of \( \pi_{1}^{orb}(O_{2,3}) \) that there exists a \( k \)-fold lift \( \phi_{O_{2,3}}^{(k)} \) of \( \phi_{O_{2,3}} \) if and only if \( k \equiv \pm 1 \mod 6 \) and that such a lift is unique if it exists. We also have
\[
(\text{rot}(\phi_{O_{2,3}}^{(k)}(\alpha)), \text{rot}(\phi_{O_{2,3}}^{(k)}(\beta)), \text{rot}(\phi_{O_{2,3}}^{(k)}(\alpha \beta)))
\]
\[
= \begin{cases}
(1, \frac{1}{3}, \frac{k-1}{k}) & (k \equiv 1 \mod 6) \\
(1, \frac{2}{3}, \frac{1}{k}) & (k \equiv -1 \mod 6)
\end{cases}
\]

Now we are ready to state the main result.

**Theorem 1.1.** Let \( \phi \in R(\pi_{1}^{orb}(O_{2,3})) \).

1. If \( (\text{rot}(\phi(\alpha)), \text{rot}(\phi(\beta)), \text{rot}(\phi(\alpha \beta))) = \left( \frac{1}{2}, \frac{1}{3}, 0 \right) \), then \( \phi \) belongs to the semi-conjugacy class of a Fuchsian action \( \phi_{O_{2,3}} \).

2. If \( (\text{rot}(\phi(\alpha)), \text{rot}(\phi(\beta)), \text{rot}(\phi(\alpha \beta))) = \left( \frac{1}{2}, \frac{2}{3}, \frac{1}{5} \right) \), then \( \phi \) belongs to the semi-conjugacy class of the 5-fold lift \( \phi_{O_{2,3}}^{(5)} \) of a Fuchsian action \( \phi_{O_{2,3}} \).

**Remark 1.2.** 1. Theorem 1.1 cannot be generalized to the other lifts of \( \phi_{O_{2,3}} \). Indeed for each positive integer \( k \geq 2 \) we denote by \( O_{2,3,k} \) a compact, connected, oriented 2-orbifold which is obtained from a 2-sphere by making three cone-points of orders \( 2, 3, k \). Now suppose that \( k \equiv \pm 1 \mod 6 \) and \( k \neq 5 \). Then we have \( \chi^{orb}(O_{2,3,k}) < 0 \). Let \( \phi_{O_{2,3,k}} \in R(\pi_{1}^{orb}(O_{2,3,k})) \) be a Fuchsian action of \( O_{2,3,k} \). For a suitable presentation
\[
\pi_{1}^{orb}(O_{2,3,k}) = \langle \alpha, \beta, \gamma \ | \ \alpha^2 = \beta^3 = \gamma^k = \alpha \beta \gamma = 1 \rangle,
\]
we have
\[
(\text{rot}(\phi_{O_{2,3,k}}(\alpha)), \text{rot}(\phi_{O_{2,3,k}}(\beta)), \text{rot}(\phi_{O_{2,3,k}}(\gamma)))
\]
\[
= \left( \frac{1}{2}, \frac{1}{3}, \frac{1}{k} \right) \}
and hence
\[
\begin{align*}
& \left( \rot(\phi_{\O_2,3,k}(\alpha)), \rot(\phi_{\O_2,3,k}(\beta)), \rot(\phi_{\O_2,3,k}(\alpha\beta)) \right) \\
= & \left( \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{k-1}{k} \right)
\end{align*}
\]
Let \( q \) be the homomorphism from \( \pi_{1}^{orb}(\O_2,3) \) onto \( \pi_{1}^{orb}(\O_{2,3,k}) \) such that
\( q(\alpha) = \alpha \) and \( q(\beta) = \beta \) and let \( \iota \) be the automorphism of \( \pi_{1}^{orb}(\O_2,3) \) such that
\( \iota(\alpha) = \alpha \) and \( \iota(\beta) = \beta^{-1} \). We define a homomorphism \( \hat{\phi}_{\O_{2,3,k}} \in R(\pi_{1}^{orb}(\O_{2,3})) \) by
\[
\hat{\phi}_{\O_{2,3,k}} = \begin{cases} \\
\phi_{\O_{2,3,k}} \circ q & (k \equiv 1 \mod 6) \\
\phi_{\O_{2,3,k}} \circ q \circ \iota & (k \equiv -1 \mod 6),
\end{cases}
\]
Since both \( \phi_{\O_{2,3,k}} \) and \( \phi_{\O_{2,3}} \) are minimal, it follows that both \( \hat{\phi}_{\O_{2,3,k}} \) are \( \phi^{(k)}_{\O_{2,3}} \) also minimal. It follows that
\[
\begin{align*}
& \left( \rot(\hat{\phi}_{\O_{2,3,k}}(\alpha)), \rot(\hat{\phi}_{\O_{2,3,k}}(\beta)), \rot(\hat{\phi}_{\O_{2,3,k}}(\alpha\beta)) \right) \\
= & \left( \rot(\phi_{\O_{2,3,k}}^{(k)}(\alpha)), \rot(\phi_{\O_{2,3,k}}^{(k)}(\beta)), \rot(\phi_{\O_{2,3,k}}^{(k)}(\alpha\beta)) \right).
\end{align*}
\]
Note that if \( k \equiv -1 \mod 6 \), then we have
\[
\begin{align*}
& \rot(\hat{\phi}_{\O_{2,3,k}}(\alpha\beta)) \\
= & \rot(\hat{\phi}_{\O_{2,3,k}}(\alpha\beta^{-1})) \\
= & \rot(\phi_{\O_{2,3,k}}(\beta)(\phi_{\O_{2,3,k}}(\alpha\beta))^{-1}(\phi_{\O_{2,3,k}}(\beta))^{-1}) \\
= & -\rot(\phi_{\O_{2,3,k}}(\alpha\beta)).
\end{align*}
\]
On the other hand \( \hat{\phi}_{\O_{2,3,k}} \) and \( \phi^{(k)}_{\O_{2,3}} \) do not belong to the same semi-conjugacy class. Indeed if they belonged the same conjugacy class, then they would be topologically conjugate by minimality. However this contradicts the fact that
\[
\hat{\phi}_{\O_{2,3,k}}((\alpha\beta)^k) = \id \neq \phi^{(k)}_{\O_{2,3}}((\alpha\beta)^k).
\]
2. We can prove Theorem [1,1] (1) by generalizing the notion of the bounded Euler number defined in [2] to actions of 2-orbifold groups. It will be indicated in a forthcoming paper together with generalizations of Theorem [1,1] to actions of other 2-orbifold groups.

3. Theorem [1,1] can be considered as a weak analogue of the following classical theorem about linear character [7], which we write in a specified form. Let \( F(\alpha, \beta) \) be a free group of rank two with a basis \( \alpha, \beta \).

**Theorem 1.3.** Let \( \phi, \psi: F(\alpha, \beta) \to \text{SL}(2, \R) \) be homomorphisms. If we have
\[
\begin{align*}
& (\tr(\phi(\alpha)), \tr(\phi(\beta)), \tr(\phi(\alpha\beta))) \\
= & (\tr(\psi(\alpha)), \tr(\psi(\beta)), \tr(\psi(\alpha\beta))) \\
= & (x, y, z)
\end{align*}
\]
with \(x^2 + y^2 + z^2 - xyz \neq 4\), then \(\phi\) and \(\psi\) are conjugate by an element of \(\text{PSL}(2, \mathbb{R})\).

4. When the author mentioned Theorem [14] in his talk given in the conference “Geometry and Foliations 2013”, E. Ghys informed us the following theorem about linear character.

**Theorem 1.4.** [10, Example 8.2] Let \(F_m\) be a free group of rank \(m \geq 2\). For every positive integer \(n\), there exist mutually non-conjugate elements \(w_1, \ldots, w_n\) of \(F_m\) such that for every homomorphism \(\phi: F_m \to \text{SL}(2, \mathbb{R})\), we have

\[
\text{tr}(\phi(w_1)) = \cdots = \text{tr}(\phi(w_n)).
\]

After that, he asked the following question.

**Question 1.5.** Does the following analogue of Theorem 1.4 hold for \(\text{Homeo}_+^+(\mathbb{S}^1)\)? Namely, for every positive integer \(m \geq 2\) and every positive integer \(n\), does there exist mutually non-conjugate elements \(w_1, \ldots, w_n\) of \(F_m\) such that for every homomorphism \(\phi \in \text{R}(F_m)\), we have

\[
\text{rot}(\phi(w_1)) = \cdots = \text{rot}(\phi(w_n))?
\]

Note that D. Calegari asked this question for the case where \(m = 2, n = 2\) and \(w_2\) is fixed as the identity element [4].

### 2 Proof of Theorem 1.1

For \(r_1, r_2, r_3 \in \mathbb{R}/\mathbb{Z}\), we put

\[
\text{R}(r_1, r_2, r_3) = \{\phi \in \text{R}(\pi_1^{orb}(O_{2,3})) | (\text{rot}(\phi(\alpha)), \text{rot}(\phi(\beta)), \text{rot}(\phi(\alpha \beta))) = (r_1, r_2, r_3)\}.
\]

#### 2.1 Proof of (1)

Let \(\phi \in \text{R} \left(\frac{1}{2}, \frac{1}{3}, 0\right)\). The following sufficient condition for belonging to the same semi-conjugacy class given in [12] is a corollary of a criterion in [14].

**Proposition 2.1.** [12, Corollary 7.5] Let \(\Gamma\) be a group and \(U \subset \text{R}(\Gamma)\) be connected. Suppose that \(\text{rot}(\phi_1(\gamma)) = \text{rot}(\phi_2(\gamma))\) for every \(\phi_1, \phi_2 \in U\) and every \(\gamma \in \Gamma\), then \(U\) is contained in a single semi-conjugacy class.

In view of Proposition 2.1 it suffices to show the following.

**Lemma 2.2.** \(\text{rot}(\phi(\gamma)) = \text{rot}(\phi_{O_{2,3}}(\gamma))\) for every \(\gamma \in \pi_1^{orb}(O_{2,3})\).
Lemma 2.3. The space \( R \left( \frac{1}{2}, \frac{1}{3}, 0 \right) \) is path-connected.

Proof of Lemma 2.2 We denote by \( \tilde{a} \) (resp. \( \tilde{b} \)) the lift of \( \phi(\alpha) \) (resp. \( \phi(\beta) \)) with \( \tilde{\text{rot}}(\tilde{a}) = \frac{1}{2} \) (resp. \( \tilde{\text{rot}}(\tilde{b}) = \frac{1}{3} \)). Since \( 0 < \tilde{\text{rot}}(\tilde{a}) < 1 \), we have

\[
\tilde{x} < \tilde{a}(\tilde{x}) < \tilde{x} + 1
\]

for every \( \tilde{x} \in \mathbb{R} \). Hence we have

\[
\tilde{b}(\tilde{x}) < (\tilde{a}\tilde{b})(\tilde{x}) < \tilde{b}(\tilde{x}) + 1
\]

for every \( \tilde{x} \in \mathbb{R} \). This implies that

\[
\frac{1}{3} = \tilde{\text{rot}}(\tilde{b}) \leq \tilde{\text{rot}}(\tilde{a}\tilde{b}) \leq \tilde{\text{rot}}(\tilde{b}) + 1 = \frac{4}{3}.
\]

Since \( \text{rot}(\phi(\alpha\beta)) = 0 \), we have \( \tilde{\text{rot}}(\tilde{a}\tilde{b}) = 1 \). Then there exists a point \( \tilde{x}_0 \in \mathbb{R} \) such that \( (\tilde{a}\tilde{b})(\tilde{x}_0) = \tilde{x}_0 + 1 \). Since both \( \tilde{a}^2 \) and \( \tilde{b}^3 \) are the translation by one, we have

\[
\tilde{x}_0 < \tilde{a}(\tilde{x}_0) = \tilde{b}(\tilde{x}_0) < \tilde{b}^2(\tilde{x}_0) < \tilde{x}_0 + 1.
\]

We put

\[
I = p([\tilde{x}_0, \tilde{b}(\tilde{x}_0)]) \quad \text{and} \quad J = p([\tilde{b}(\tilde{x}_0), \tilde{x}_0 + 1]).
\]

Then we have

\[
\phi(\alpha)(J) = I \quad \text{and} \quad \phi(\beta^{\pm 1})(I) \subset J.
\]

We claim that if \( \gamma \in \Gamma \) is not conjugate to a power of \( \alpha, \beta \), then there exists a closed interval \( K \subset S^1 \) such that \( \phi(\gamma)(K) \subset K \). Indeed by taking conjugates if necessary, we may assume that

\[
\gamma = \alpha^{e_1} \cdots \alpha^{e_n}, \quad \text{where} \quad e_i \in \{1, \ldots, 3\}.
\]

Then we have \( \phi(\gamma)(I) \subset I \).

This implies that if \( \gamma \) is not conjugate to a power of \( \alpha, \beta \), then \( \text{rot}(\phi(\gamma)) = 0 \). This finishes the proof of the lemma.

Proof of Lemma 2.3 Let \( \phi_0, \phi_1 \in R \left( \frac{1}{2}, \frac{1}{3}, 0 \right) \). We show that there exists a path in \( R \left( \frac{1}{2}, \frac{1}{3}, 0 \right) \) from \( \phi_0 \) to \( \phi_1 \). For \( t \in \{0, 1\} \), we denote by \( \tilde{a}_t \) (resp. \( \tilde{b}_t \)) the lift of \( \phi_t(\alpha) \) (resp. \( \phi_t(\beta) \)) with \( \tilde{\text{rot}}(\tilde{a}_t) = \frac{1}{2} \) (resp. \( \tilde{\text{rot}}(\tilde{b}_t) = \frac{1}{3} \)). By taking conjugates, we may assume that both \( \phi_0(b) \) and \( \phi_1(b) \) are the rotation by \( \frac{1}{3} \).
and that \((\tilde{a}_t \tilde{b}_t)(0) = 1\) for \(t \in \{0, 1\}\). We take a path \(\{\tilde{a}_t\}_{t \in [0,1]}\) in \(\overline{\text{Homeo}^+}(S^1)\) from \(\tilde{a}_0\) to \(\tilde{a}_1\) such that \((\tilde{a}_t) \left(\frac{1}{3}\right) = 1\) and \((\tilde{a}_t)^2\) is the translation by one. We denote by \(a_t \in \text{Homeo}^+(S^1)\) the projection of \(\tilde{a}_t\). Then the path \(\{\phi_t\}_{t \in [0,1]}\) in \(R\left(\frac{1}{2}, \frac{1}{3}, 0\right)\) defined by the condition that \(\phi_t(\alpha) = a_t\) and \(\phi_t(\beta)\) is the rotation by \(\frac{1}{3}\) is a desired one.

\section{Proof of (2)}

Let \(\phi \in R\left(\frac{1}{2}, \frac{2}{3}, \frac{1}{5}\right)\). Then \(\phi\) has no finite orbits. In fact if there were finite orbits, then the map \(\text{rot} \circ \phi\): \(\mathbb{Z}_2 \ast \mathbb{Z}_3 \to \mathbb{R}/\mathbb{Z}\) must be a homomorphism, which is impossible since \(\text{rot}(\phi(\alpha)) = \frac{1}{2}, \text{rot}(\phi(\beta)) = \frac{2}{3}\) and \(\text{rot}(\phi(\alpha \beta)) = \frac{1}{5}\). Therefore the action \(\phi\) admits a unique minimal set, either a Cantor set or the whole circle. Passing to a semi-conjugate action, we may assume the latter, that is, the action is minimal.

By Theorem \ref{thm2.1} (1), it suffices to show that \(\phi\) is the \(5\)-fold lift of some action, namely, there exists a homeomorphism \(\theta \in \text{Homeo}^+(S^1)\) which is \(\phi(\pi_1^{orb}(O_{2,3}))\)-equivariant and periodic of period \(5\).

We denote by \(\tilde{a}\) (resp. \(\tilde{b}\)) the lift of \(\phi(\alpha)\) (resp. \(\phi(\beta)\)) with \(\tilde{\text{rot}}(\tilde{a}) = \frac{1}{2}\) (resp. \(\tilde{\text{rot}}(\tilde{b}) = \frac{2}{3}\)). Since \(0 < \tilde{\text{rot}}(\tilde{a}) < 1\), we have

\[\tilde{x} < \tilde{a}(\tilde{x}) < \tilde{x} + 1\]

for every \(\tilde{x} \in \mathbb{R}\). Hence we have

\[\tilde{b}(\tilde{x}) < (\tilde{a} \tilde{b})(\tilde{x}) < \tilde{b}(\tilde{x}) + 1\]

for every \(\tilde{x} \in \mathbb{R}\). This implies that

\[\frac{2}{3} = \tilde{\text{rot}}(\tilde{b}) \leq \tilde{\text{rot}}(\tilde{a} \tilde{b}) \leq \tilde{\text{rot}}(\tilde{b}) + 1 = \frac{5}{3} .\]

Since \(\text{rot}(\phi(\alpha \beta)) = \frac{1}{5}\), we have \(\tilde{\text{rot}}(\tilde{a} \tilde{b}) = \frac{6}{5}\). We denote by \(\tilde{a} \tilde{b}\) the lift of \(\phi(\alpha \beta)\) with \(\tilde{\text{rot}}(\tilde{a} \tilde{b}) = \frac{1}{5}\). Then there exists a point \(\tilde{x}_0 \in \mathbb{R}\) such that \((\tilde{a} \tilde{b})^5(\tilde{x}_0) = \tilde{x}_0 + 1\).

Note that \(\tilde{a} \tilde{b}(\tilde{x}) = \tilde{a} \tilde{b}(\tilde{x}) + 1\) for every \(\tilde{x} \in \mathbb{R}\).

\begin{lemma}
We have the following.
\begin{enumerate}
  \item \(\tilde{a}(\tilde{x}) < \tilde{b}(\tilde{x})\) for every \(\tilde{x} \in \mathbb{R}\).
  \item \((\tilde{a} \tilde{b})^2 \tilde{a}(\tilde{x}) < \tilde{x} + 1\) for every \(\tilde{x} \in \mathbb{R}\).
  \item \((\tilde{a} \tilde{b})^l(\tilde{x}_0) < \tilde{b}(\tilde{a} \tilde{b})^{l+2}(\tilde{x}_0) - 1 < \tilde{b}^2(\tilde{a} \tilde{b})^{l+4}(\tilde{x}_0) - 2 < (\tilde{a} \tilde{b})^{l+1}(\tilde{x}_0)\) for every \(l \in \mathbb{Z}\).
\end{enumerate}
\end{lemma}
Proof. (1) Since \( \text{rot}(\tilde{a}b) = \frac{6}{5} > 1 \), we have
\[
\tilde{a}^2(\tilde{x}) = \tilde{x} + 1 < \tilde{a}\tilde{b}(\tilde{x})
\]
for every \( \tilde{x} \in \mathbb{R} \). This implies the desired inequality.

(2) It follows from (1) that for every \( \tilde{x} \in \mathbb{R} \) we have
\[
(\tilde{a}b)^2\tilde{a}(\tilde{x}) = (\tilde{a}b)^2\tilde{a}(\tilde{x}) - 2 < \tilde{a}b^2\tilde{a}(\tilde{x}) - 2 = \tilde{a}^2(\tilde{x}) = \tilde{x} + 1.
\]

(3) By substituting \( \tilde{b}(\tilde{a}b)^i(\tilde{x}_0) \) for \( \tilde{x} \) in inequality (2), it follows that
\[
(\tilde{a}b)^2\tilde{a}(\tilde{a}b)^{i+2}(\tilde{x}_0) < \tilde{b}(\tilde{a}b)^{i+2}(\tilde{x}_0) + 1.
\]
Since we have
\[
(\tilde{a}b)^2\tilde{a}(\tilde{a}b)^2((\tilde{a}b)^i(\tilde{x}_0)) = (\tilde{a}b)^5((\tilde{a}b)^i(\tilde{x}_0)) + 1 = (\tilde{a}b)^i(\tilde{x}_0) + 2,
\]
we obtain the first inequality. Since \( l \in \mathbb{Z} \) is an arbitrary integer, it follows that
\[
(\tilde{a}b)^{i+2}(\tilde{x}_0) < \tilde{b}(\tilde{a}b)^{i+4}(\tilde{x}_0) - 1.
\]
This implies the second inequality. Similarly we have
\[
(\tilde{a}b)^{i+4}(\tilde{x}_0) < \tilde{b}(\tilde{a}b)^{i+6}(\tilde{x}_0) - 1 = \tilde{b}(\tilde{a}b)^{i+1}(\tilde{x}_0).
\]
This implies the third inequality. \( \square \)

The following lemma follows from Lemma 2.4 (3) and the equality \( \tilde{a}(\tilde{a}b)^i(\tilde{x}_0) = \tilde{b}(\tilde{a}b)^{i+4}(\tilde{x}_0) - 1 \).

Lemma 2.5. For every integer \( l \in \mathbb{Z} \), we put
\[
\tilde{I}_l = [(\tilde{a}b)^i(\tilde{x}_0), (\tilde{b}(\tilde{a}b)^{i+2})(\tilde{x}_0) - 1] \quad \text{and} \quad \tilde{J}_l = [(\tilde{b}(\tilde{a}b)^{i+2})(\tilde{x}_0) - 1, (\tilde{a}b)^{i+1}(\tilde{x}_0)].
\]

Then we have the following.

1. \( \tilde{b}^{-1}((\tilde{a}b)^i(\tilde{x}_0)) \in \text{Int}(\tilde{J}_{l-4}) \) and \( (\tilde{b}a)((\tilde{a}b)^i(\tilde{x}_0)) \in \text{Int}(\tilde{J}_{l+5}) \).

2. \( \tilde{a}(\tilde{J}_l) = \tilde{I}_{l+3}, \quad \tilde{b}(\tilde{I}_l) \subset \tilde{J}_{l+3} \quad \text{and} \quad \tilde{b}^{-1}(\tilde{I}_l) \subset \tilde{J}_{l-4} \).

We denote by \( \phi(\pi^{orb}_1(\mathcal{O}_{2,3}) \cong \text{Homeo}_+(S^1) \) consisting of lifts of elements of \( \phi(\pi^{orb}_1(\mathcal{O}_{2,3})) \) to \( \mathbb{R} \). We define a map \( \tilde{\theta} \phi(\pi^{orb}_1(\mathcal{O}_{2,3}))(\tilde{x}_0) \) onto itself by
\[
\tilde{\theta}(\phi(\gamma))(\tilde{x}_0) = \phi(\gamma)(\tilde{a}(\tilde{x}_0)),
\]
where \( \gamma \in \pi^{orb}_1(\mathcal{O}_{2,3}) \) and \( \phi(\gamma) \) is a lift of \( \phi(\gamma) \) to \( \mathbb{R} \).
Lemma 2.6. The map $\tilde{\theta}$ is well-defined and strictly increasing.

Proof. First we prove that $\tilde{\theta}$ is well-defined. It suffices to show that for $\phi(\gamma) \in \phi(\pi^r_{\Omega}(\mathcal{O}_{2,3}))$ with $\phi(\gamma)(\hat{x}_0) = \hat{x}_0$, we have $\phi(\gamma)(ab(\hat{x}_0)) = ab(\hat{x}_0)$.

If $\gamma = \beta^{e_0}\alpha\beta^{e_1} \cdots \alpha\beta^{e_n}$, where $e_0 \in \{0, \pm 1\}$ and $e_i \in \{\pm 1\}$ for $i \in \{1, \ldots, n\}$, then we have $e_i \neq -1$ for $i \in \{0, 1, \ldots, n\}$. Indeed if $(e_i, e_{i+1}, \ldots, e_n) = (-1, 1, \ldots, 1)$ for some $i \in \{0, 1, \ldots, n\}$, then it would follow from Lemma 2.5 (1) that
\[
\tilde{b}^{e_i} \cdots \tilde{a}^{e_n}(\hat{x}_0) = \tilde{b}^{-1}(\tilde{a}\tilde{b})^{-n-i}(\hat{x}_0)) = \tilde{b}^{-1}((\tilde{a}\tilde{b})^{-n-i}(\hat{x}_0)) + (n - i) \in \text{Int}(\hat{J}_{6(n-i)-4})
\]
and hence
\[
\phi(\gamma)(\hat{x}_0) \in \text{Int}(\hat{J}_l) \cup \text{Int}(\hat{J}_i)
\]
for some $l \in \mathbb{Z}$ by Lemma 2.5 (2), which contradicts the assumption.

Therefore we have $\gamma = \beta^{e_0}(\alpha\beta)^n$, where $e_0 \in \{0, 1\}$ and it follows from Lemma 2.4 (3) we have $e_0 \neq 1$. Hence there exists an integer $m \in \mathbb{Z}$ such that
\[
\phi(\gamma)(\tilde{x}) = (\tilde{a} \tilde{b})^n(\tilde{x}) + m
\]
for every $\tilde{x} \in \mathbb{R}$. We have $n = -5m$ by the assumption and hence
\[
\phi(\gamma)(ab(\hat{x}_0)) = (ab)^{-5m+1}(\hat{x}_0) + m = ab(\hat{x}_0).
\]
If $\gamma = \beta^{e_0}\alpha\beta^{e_1} \cdots \alpha\beta^{e_n}\alpha$, where $e_0 \in \{0, \pm 1\}$ and $e_i \in \{\pm 1\}$ for $i \in \{1, \ldots, n\}$, then we have $e_i \neq -1$ for $i \in \{0, 1, \ldots, n\}$. Indeed if $(e_i, e_{i+1}, \ldots, e_n) = (1, -1, \ldots, -1)$ for some $i \in \{0, 1, \ldots, n\}$, then it would follow from Lemma 2.5 (1) that
\[
\tilde{b}^{e_i} \cdots \tilde{a}^{e_n}(\hat{x}_0) = (\tilde{b}\tilde{a})(\tilde{b}^{-1}\tilde{a})^{-n-i}(\hat{x}_0)) = (\tilde{b}\tilde{a})(\tilde{a}^{-1}(\tilde{a}\tilde{b})^{-n-i}(\hat{x}_0)) \in \text{Int}(\hat{J}_{-n-i}+5)
\]
and hence
\[
\phi(\gamma)(\hat{x}_0) \in \text{Int}(\hat{J}_l) \cup \text{Int}(\hat{J}_i)
\]
for some $l \in \mathbb{Z}$ by Lemma 2.5 (2), which contradicts the assumption.

Therefore we have $\gamma = \beta^{e_0}\alpha(\beta\alpha)^{-1}$, where $e_0 \in \{0, -1\}$ and it follows from Lemma 2.4 (3) that we have $e_0 \neq 0$. Hence there exists an integer $m \in \mathbb{Z}$ such that
\[
\phi(\gamma)(\tilde{x}) = (\tilde{a} \tilde{b})^{-(n+1)}(\tilde{x}) + m
\]
for every $\tilde{x} \in \mathbb{R}$. We have $n = 5m - 1$ by the assumption and hence
\[
\phi(\gamma)(ab(\hat{x}_0)) = (ab)^{-5m+1}(\hat{x}_0) + m = ab(\hat{x}_0).
\]
Next we prove that \( \hat{\theta} \) is strictly increasing. It suffices to show that for \( \phi(\gamma) \in \phi(\pi^{orb}_1(O_{2,3})) \) with \( \hat{x}_0 = \phi(\gamma)(\tilde{x}_0) \), we have \( \hat{\theta}(\hat{x}_0) < \hat{\theta}(\phi(\gamma)(\tilde{x}_0)) \).

If \( \gamma = \beta^{e_0}\alpha\beta^{e_1}\cdots\alpha\beta^{e_n} \), where \( e_0 \in \{0, \pm 1\} \) and \( e_i \in \{\pm 1\} \) for \( i \in \{1, \ldots, n\} \), then it follows from Lemma 2.5 (2) that

\[
\phi(\gamma)(\tilde{I}_0) \subset \tilde{I}_l \cup \tilde{J}_l
\]

for some non-negative integer \( l \in \mathbb{Z} \). This implies that

\[
\phi(\gamma)(\tilde{I}_1) \subset \tilde{I}_{l+1} \cup \tilde{J}_{l+1}
\]

and hence \( \hat{\theta}(\hat{x}_0) < \hat{\theta}(\phi(\gamma)(\tilde{x}_0)) \).

If \( \gamma = \beta^{e_0}\alpha\beta^{e_1}\cdots\alpha\beta^{e_n}\alpha \), where \( e_0 \in \{0, \pm 1\} \) and \( e_i \in \{\pm 1\} \) for \( i \in \{1, \ldots, n\} \), then it follows from Lemma 2.5 (2) that

\[
\phi(\gamma)(\tilde{J}_{-1}) \subset \tilde{I}_l \cup \tilde{J}_l
\]

for some non-negative integer \( l \in \mathbb{Z} \). This implies that

\[
\phi(\gamma)(\tilde{J}_0) \subset \tilde{I}_{l+1} \cup \tilde{J}_{l+1}
\]

and hence \( \hat{\theta}(\hat{x}_0) < \hat{\theta}(\phi(\gamma)(\tilde{x}_0)) \). \( \Box \)

The map \( \hat{\theta} = \phi(\pi^{orb}_1(O_{2,3})) \)-equivariant and we have \( \hat{\theta}^5(\phi(\gamma)(\tilde{x}_0)) = \hat{\theta}(\phi(\gamma)(\tilde{x}_0)) + 1 \) for every element \( \phi(\gamma) \) of \( \phi(\pi^{orb}_1(O_{2,3})) \). Since \( \phi \) is minimal, \( \phi(\pi^{orb}_1(O_{2,3}))(\tilde{x}_0) \) is dense in \( \mathbb{R} \) and hence \( \hat{\theta} \) can be extended to an element of \( \text{Homeo}_+(S^1) \), which we also denote by \( \hat{\theta} \). The homeomorphism \( \hat{\theta} \) is \( \phi(\pi^{orb}_1(O_{2,3})) \)-equivariant and we have \( \hat{\theta}^5(\hat{x}) = \hat{x} + 1 \) for every \( \hat{x} \in \mathbb{R} \). This gives the desired homeomorphism \( \theta \in \text{Homeo}_+(S^1) \).

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