Einstein Equation of State and Black Hole Membrane Paradigm in f(R) Gravity

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We study spacetime thermodynamics for non-equilibrium processes. For this purpose, we generalize the formulation of spacetime thermodynamics constructed independently by Jacobson and Padmanabhan, in which the Einstein equation is derived as the equation of state. Based on the generalized formulation, we reach the same picture and results as the black hole membrane paradigm, in which, to an external observer, a black hole appears to behave exactly like a dynamical fluid membrane. Using this generalized formulation, we construct the first law of spacetime thermodynamics for non-equilibrium processes in f(R) gravity. We find that the coefficients of the expansion and shear terms are equal to the viscosities of the membrane paradigm, and that a new term appears, which can be interpreted as another heat term.

§1. Introduction

Black hole solutions originally came from the Einstein equation. The four laws of the mechanics were analogous to those of thermodynamics.1 With the discovery of the quantum Hawking radiation,2 it became clear that the analogy is an identity, and that black holes are thermodynamic objects. Among those, the result of Gibbons and Hawking left a mystery.3 In the paper, black hole’s entropy $S = \frac{1}{4}A$ is derived from the free energy for the canonical system of a black hole, by using WKB approximation in Euclidean field theory. Then the finite statistical entropy results from a single classical black hole configuration. This may indicate that a solution of the Einstein equation corresponds to a thermodynamic state.

One might think that the above thermodynamic nature of spacetime cannot be restricted in the black hole spacetime. This speculation was investigated by Jacobson, and he concluded that, even in the non black hole spacetime, the spacetime has some thermodynamic property in the sense that the Einstein equation plays a role as “the equation of state”.4 He considered a part of any spacetime as a thermodynamic system by using the fact that a uniformly accelerating observer at any point in arbitrary spacetime has his own horizon (see the next section). He assumed the Uuruh effect,5 the entropy area law, local equilibrium, quasi-statistical process, and that the all energy is the heat ($\delta E = \delta'Q$). From the Raychaudhuri equation and the Clausius definition of entropy ($T \delta S = \delta Q$), he derived the Einstein equation. In this sense, the Einstein equation can be regarded as the equation of state.

However, Jacobson’s observer is strange. His observer is inside the horizon (inside the system) and measures energy flow into the system (see the next section). This is contrary to the spirit of thermodynamics because thermodynamic quanti-

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ties are conventionally measured by an external observer. Therefore it is difficult to apply Jacobson’s observer to a system of a black hole, but this fact is undesirable because Jacobson’s idea should be general enough to be applicable to black hole thermodynamics. Note that Padmanabhan generalized Jacobson’s idea to more general theories of gravity by using observers outside the horizon and Wald’s entropy, though the formulation is different from Jacobson’s and useful only to quasi-static processes.

Therefore we apply Padmanabhan’s observer to a dynamical spacetime, use the Raychaudhuri equation as Jacobson, and construct the first law of spacetime thermodynamics for non-equilibrium processes in f(R) gravity, which is the simplest theory in higher-curvature theories of gravity. Additionally, we show that Padmanabhan’s observer sees the timelike surface near his own causal horizon in the same way as an observer on the “stretched horizon” of the black hole membrane paradigm, which is the view that, to an external observer, a black hole appears to behave exactly like a dynamical fluid membrane. In our derived law, the coefficients of the expansion and shear terms are equal to the viscosities of the membrane paradigm and become those of Jacobson et al. in an infinitesimal time limit, and a new term appears, which can be interpreted as another heat term.

This paper is organized as follows. In section 2, Jacobson’s idea and Padmanabhan’s observer are introduced, where the difference between Padmanabhan’s observers and those of Jacobson becomes clear. In section 3, the black hole membrane paradigm is reviewed briefly, and it is found that Padmanabhan’s observer sees the timelike surface near his own causal horizon in the same way as an observer on the “stretched horizon” of the paradigm. In section 4, the idea of spacetime thermodynamics developed in the section 2 is applied to a dynamical spacetime. In Einstein’s gravity, the first law of spacetime thermodynamics is derived. In section 5, the first law in f(R) gravity is derived in almost the same way. In section 6, conclusions and discussions are given.

In this paper, we use the units \( (G = c = \hbar = k_B = 1) \) and a spacetime metric with the signature \( (-, +, +, +) \). Our sign conventions are those of MTW, with the exception of the relation between extrinsic curvature and expansion \( K_{\mu}^\nu = \theta \).

§2. Spacetime Thermodynamics: The Framework

In order to consider a part of spacetime as a thermodynamic system, we introduce some ingredients such as observer, system, and energy flow. The basic idea is based on Jacobson’s but we use Padmanabhan’s observer to measure the physics.

2.1. The Definition of System, External World and Heat

In general, heat is transfer of energy which cannot be identified and controlled by an external observer. Therefore, in spacetime thermodynamics, heat can be defined as energy flow through any causal horizon, and this can define the system and the external world. That is, the system is the region inside the horizon, and the external world is the region outside the horizon. A conventional observer is defined as an observer in the external world, who measures the thermodynamic quantities. He
cannot identify any energy flow passed through the causal horizon, so the energy flow is regarded as heat for him. A good example is a black hole event horizon. An observer outside the event horizon regards the inside as the system, the outside as the external world, and energy flow through the horizon as heat for him. However, the above definition is not limited to a black hole event horizon, but applicable to any causal horizon. A way to construct a causal horizon at any point in any spacetime is the use of a Rindler horizon.

Rindler horizon can be constructed as follows. Firstly, we take a point $P$ in any spacetime. Secondly, we invoke the equivalence principle to introduce a local inertial frame for an observer near the point. This is always allowed if the size of the region $l$ is restricted to $l \ll R|_{P}$, where $R|_{P}$ is the radius of curvature at $P$. The metric of this region is approximately Minkowski:

$$g_{\mu\nu} = \eta_{\mu\nu} + O(l^2).$$

Thirdly, the local patch is described by the Riemann normal coordinates $x^\mu$, such that $P$ stays at $x^\mu = 0$. Finally, we uniformly accelerate an observer near $P$ for the $X$ direction. The corresponding transformation is

$$T = x \sinh(\kappa t), \quad X = x \cosh(\kappa t).$$

Then, the local coordinate around $P$ is the local Rindler coordinate:

$$ds^2 = -\kappa^2 x^2 dt^2 + dx^2 + dy^2 + dz^2,$$

where $\kappa$ is an arbitrary scaling factor. The 4 vector of the observer at $x = \text{const}$ is given by $u = \frac{\partial}{\partial \tau} = \frac{1}{\kappa x} \frac{\partial}{\partial t}$, and the proper acceleration is given by $a = \frac{1}{\kappa}$.

In figure 1, the dashed line shows the horizon that hides the inside from the view of the observer. Therefore, his side is the external world for him, and the other side is the system for him. It is the observer that measures energy flow $\delta E$ into the system. Energy flow between his proper time $\tau_1$ and $\tau_2$ is given by

$$\delta E = \int_{\tau_1}^{\tau_2} d\tau \int_{S(\tau)} dA \ T_{\mu\nu} u^\mu n^\nu.$$
where $\bar{S}(\tau)$ is the 2 dimensional spacial area of the timelike surface near the horizon at proper time $\tau$, $T_{\mu\nu}$ is the energy-momentum tensor, $n^\nu$ is his normal vector inward the horizon, and $d\bar{A} = \sqrt{\bar{h}} d\bar{x}^2$ is the area element, where $\bar{h}$ is the determinant of the spacial metric $\bar{h}_{ab}$. In this paper, the symbol $\delta$ means variation in a thermodynamic quantity which occurs in the process. Moreover, we assume that all energy flow is heat:

$$\delta E = \delta' Q . \quad (2.5)$$

Here $\delta'$ means variation which depends upon the particular path taken through the space of thermodynamic parameters. Therefore, it is natural that, like heat in conventional thermodynamics, $\delta E$ given by (2.4) depends on the process, that is, $u^\mu$, $n^\mu$ and $\bar{S}(\tau)$.

We present here three comments.

(a) Our observer should be sufficiently close to the point P so that he can take the local Rindler coordinate and interpret the spacetime over his horizon as the thermodynamic system. Therefore, we should take the observer at $x \sim 0$, and then energy flow $\delta E$ given by (2.4) asymptotically becomes

$$\delta E = (\kappa x)^{-1} \int_{t_1}^{t_2} dt \int_{S(t)} dA T_{\mu\nu} k^\mu k^\nu , \quad (2.6)$$

where $u^\mu \simeq (\kappa x)^{-1} k^\mu$ and $n^\mu \simeq (\kappa x)^{-1} k^\mu$ for $x \sim 0$, $k = \frac{\partial}{\partial t}$ is horizon’s generator, $S(t)$ is the 2 dimensional spacial area of the null horizon at time $t$, and $dA = \sqrt{\bar{h}} d\bar{x}^2$ is the area element. In the limit where $x \to 0$, both (2.6) and $T_U$ in the next subsection diverge. However, in the first law which will be constructed in section 4 and 5, their ratio approaches a finite limit.\[ Note that (2.4) is integration on a timelike surface, and (2.6) is one on a null surface, though they are asymptotically equal in $x \sim 0$. Therefore, in the following discussion, we use $\bar{S}(\tau) \simeq S(\tau)$ and $d\bar{A} \simeq dA$ near the horizon.

(b) In general, a causal horizon is a virtual wavefront of light. Let us imagine the following situation. We take some spacial region. A virtual light emanates outward from the boundary. As figure 2, our observer is accelerating in front of the wavefront of the light. Then, he cannot notice things swallowed up by the light and can regard the energy flow as the heat. Here we should always arrange each observer to be the same distance $x$ from the wavefront and have equal proper time $\tau = \kappa x t$ and temperature $T_U$ in subsection 2.2, which means that the process is isothermal (see the section 4). Therefore, the construction of a thermodynamic system so far can also be applied to any wavefront of light. A good example is a black hole event horizon. A future event horizon is defined as the boundary of the closure of the causal past of the future null infinity. That is, a black hole is a region from where even light cannot escape eternally, and the event horizon is the waveform of the light which is the boundary. Note that a black hole is a spatially closed thermodynamic system, but Jacobson’s original system is an open system, which is not a natural thermodynamic system.

(c) Our observer is the same as Padmanabhan’s, not as Jacobson’s. We take Jacobson’s idea, but our energy flow is not his but Padmanabhan’s.\[ Jacobson
2.2. Temperature and Entropy

In this subsection, fundamental constants are introduced.

Our observer is uniformly accelerating at $x \sim 0$ in the local Rindler coordinate, so he feels the temperature of the Unruh effect\(^5\):

$$T_U = \frac{\hbar a}{2\pi c k_B} = \frac{\hbar c x^{-1}}{2\pi k_B}. \quad (2.7)$$

He is near the system and sees it contact with the external world at the temperature.

Next, the Rindler observer cannot get information about things which have gone into the horizon. This situation resembles Bekenstein’s gedankenexperiment, who thought black hole’s entropy as information defect for outside observers and derived

\(^5\) Another attempt to demonstrate differences between Jacobson’s and Padmanabhan’s formulation is discussed in.\(^\text{14}\)
Thus, we assume that variation in the thermodynamic system’s entropy, though observer-dependent, is Bekenstein’s entropy:

\[
\delta S = \delta \left( \frac{k_B}{4l_p^2} \int \gamma(x) dA \right),
\]

where \( l_p^2 = \frac{G \hbar}{c^3} \) is the Planck area, \( S(\tau) \) is the area of the wavefront of light at proper time \( \tau \), and \( \gamma(x) \) is the entropy density on it. For example, in the case of Einstein’s gravity, \( \gamma(x) = 1 \).

Instantaneous equilibrium condition is as follows:

\[
\frac{dS}{d\tau} \bigg|_{\tau=0} = 0.
\]

§3. Black Hole Membrane Paradigm and the Relation

We briefly review the black hole membrane paradigm\(^{13}\). Especially we summarize the physical framework and viewpoint and the results. Then, we find out that our observer sees the timelike surface near his own causal horizon in the same way as the FIDO on the “stretched horizon” of the paradigm.

3.1. Review of Black Hole Membrane Paradigm

This idea was originally designed to allow us to understand intuitively and compute quantitatively behaviors of a black hole in complex external environments. There are two main difficulties in black hole physics. The first is the boundary condition at the horizon. Even if things fall in a black hole, outside observers see them stick to it eternally because of the redshift. They leave complex relic structures near the horizon. The second is to imagine physics in 4-dimensional spacetime. Especially, a dynamical spacetime is hard to picture.

The notion of a “stretched horizon” resolves these difficulties. Things too near the horizon have almost no effect on outside time evolution because of the infinite red-shift delay. We can “stretch” the horizon to cover the irrelevant relic structures and ignore them. This manipulation introduces a timelike “stretched horizon”, which lives close enough to the true horizon that it can describe its physics consistently with general relativity.

More specifically, the idea is to perform a 2+1+1 split of spacetime. The foliating spacelike surfaces are surfaces \( \Sigma_t \) of constant time \( t \) according to a family of accelerated fiducial observers (FIDOs) with 4-velocity \( u^\mu \) defined such that \( u_\mu = \alpha dt \), where \( \alpha \) is the lapse function and defines the true horizon as the location such that \( \alpha = 0 \). Each FIDO measures physics at each own spatial point. In figure 4, their world lines are shown dashed. For example, in the case of a Schwarzschild black hole, \( \alpha = (1 - \frac{2M}{r})^{1/2} \) and \( \Sigma_t \) are surfaces of constant Schwarzschild time \( t \). The event horizon itself is characterized by the null generator \( k^\mu \) and can be foliated into spacelike 2-surfaces by surfaces of constant horizon time. In the Schwarzschild example, \( k^\mu = \frac{d}{dv} \) and \( v = \text{const} \), where \( v \) is the Killing time on the horizon because the event horizon of a stationary asymptotically flat spacetime is a Killing horizon. The
distance from the horizon is naturally parameterized by the affine parameter along
the ingoing null rays (for example just the radial coordinate $r$ in Schwarzschild), or
$\alpha$ equivalently by a change of variables.

\[ \alpha u^\mu \rightarrow k^\mu, \quad \alpha n^\mu \rightarrow k^\mu. \quad (3.1) \]

A stretched horizon in the limit where $\alpha \rightarrow 0$ will be used to approximate the true
event horizon. Note that a stretched horizon is a globally defined object just outside
the true horizon, which is globally defined.

The equation of motion for the stretched horizon is derived from the Einstein
equation and Israel’s junction condition\cite{12,16} and the appropriate boundary condi-
tion. The equation is a conservation law of the surface energy-momentum tensor on
the stretched horizon and the energy-momentum flow only from the outside. Espe-
sially, the momentum conservation is the form of the Navier-Stokes equation. The
bulk viscosity $\zeta$ and the shear viscosity $\eta$ are

\[ \zeta = -\frac{1}{16\pi}, \quad \eta = \frac{1}{16\pi}. \quad (3.2) \]

Therefore, the FIDOs on the stretched horizon see the stretched horizon behave
like 2 dimensional viscous fluid, whose fluid velocity is defined as $u^\mu$. Note that,
unlike ordinary fluids, the membrane has the negative bulk viscosity. From mechani-
cal viewpoint, this indicates an instability against generic perturbations triggering
expansion or contraction. This meaning is discussed in the next section.

Fig. 4. 2+1+1 split and FIDO’s world lines

The stretched horizon is defined as their generators equal to FIDO’s world lines
at $\alpha \ll 1$. The stretched horizon itself has unit spacelike normal $n_\mu$, which is inward
in our choice of section 2. Since this vector field can be extended throughout the
spacetime as the normal to all surfaces of constant $\alpha$, we have a 2+1+1 split defined
by $u_\mu$ and $n_\mu$. We always work in the limit of the true horizon $\alpha \rightarrow 0$, where

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unlike ordinary fluids, the membrane has the negative bulk viscosity. From mechani-
cal viewpoint, this indicates an instability against generic perturbations triggering
expansion or contraction. This meaning is discussed in the next section.
3.2. Our Observers and the FIDOs on the Stretched Horizon

Now, compare the FIDOs on the stretched horizon with our observers in section 2. Our observers are always just outside their own causal horizon and measure the energy flow given by (2.4), which is estimated on the timelike surface near the horizon. This timelike surface corresponds to the stretched horizon. The Rindler coordinate corresponds to the near-horizon coordinate, and, especially, the lapse function is \( \alpha = \kappa x \). Therefore, our observers see the timelike surface near their own causal horizon behave like 2 dimensional viscous fluid, though it is observer-dependent, as well as the FIDOs on the stretched horizon. Note that, in the case of a black hole, physics from our observers’ point of view is not observer-dependent in the sense of Jacobson’s, but rather physics from the blue-shifted point of view. For example, the Unruh temperature given by (2.7) becomes equal to the blue-shifted Hawking temperature near horizon, according to the Tolman law. This is natural in black hole physics.

§4. First Law of Spacetime Thermodynamics in Einstein’s gravity

Let’s derive the first law of spacetime thermodynamics for non-equilibrium processes in Einstein’s gravity. We use, like Jacobson, the Raychaudhuri equation, unlike him, Padmanabhan’s observers and non-affine parameter, that is, observers near the causal horizon and their proper time. We will derive not the Einstein equation but the first law, so we assume the Einstein equation in the following derivation. This approach is based on the conventional derivation of black hole’s first law for quasi-static processes.  

4.1. Derivation

Imagine the following physical situation. In the initial state, a spacetime thermodynamic system is equilibrium. Next, it interacts with its surroundings in many ways such as inward flow of matter, gravitational wave, and thermal radiation. In the final state, it becomes a new equilibrium state. What form does the first law of spacetime thermodynamics take in the process? In order to construct it, we use our observers near the horizon in section 2, the Raychaudhuri equation, the Einstein equation, the entropy area law (2.8), and the Unruh temperature \( T_U \).

The Raychaudhuri equation for null congruence in a non-affine parameter \( t \) is as follows:

\[
\frac{d\theta}{dt} = \kappa \theta - \frac{\theta^2}{2} - \sigma_{\mu\nu}\sigma^{\mu\nu} - R_{\mu\nu}k^\mu k^\nu, 
\]

(4.1)

where \( k = \frac{\partial}{\partial t} \) is null generator of the horizon, \( \theta \) is the expansion, \( \sigma_{\mu\nu} \) is the shear tensor, and \( \kappa \) is defined as \( k^\mu k^\nu = \kappa k^\mu \). Note that we use the Raychaudhuri equation not for timelike congruence but for null congruence, for we will formulate the first law of the spacetime thermodynamic system constructed by the null horizon. However, in this spacetime, unlike a static black hole spacetime, the global Killing vector does not exist, so we use the 4 vector \( u^\mu \) of the observer near the local horizon.
to estimate energy flow $\delta E$ as follows. Expansion $\theta$ can also be written as follows:

$$\theta = \frac{1}{\Delta A} \frac{d\Delta A}{dt}, \quad (4.2)$$

where $\Delta A = \sqrt{h}(\Delta x)^2$ is the area element of the wavefront, and $h$ is the determinant of the spacial metric on it. By this, (4.1) is expressed as follows:

$$\frac{d^2 \Delta A}{dt^2} = \left( \kappa \theta + \frac{\theta^2}{2} - \sigma_{\mu\nu} \bar{\sigma}^{\mu\nu} - R_{\mu\nu} \bar{k}^\mu \bar{k}^\nu \right) \Delta A. \quad (4.3)$$

Here we transform the non-affine parameter from $t$ into the observer’s proper time $\tau$:

$$\frac{d^2 \Delta A}{d\tau^2} = \left( x^{-1} \bar{\theta} + \frac{\bar{\theta}^2}{2} - \bar{\sigma}_{\mu\nu} \bar{\sigma}^{\mu\nu} - R_{\mu\nu} \bar{k}^\mu \bar{k}^\nu \right) \Delta A. \quad (4.4)$$

where $d\tau = \kappa x dt$, $\bar{\theta} = \frac{\partial \theta}{\partial \tau}$, $\sigma_{\mu\nu} = \frac{\partial}{\partial \tau} \bar{\sigma}_{\mu\nu} = \kappa x \bar{\sigma}_{\mu\nu}$, and $k^\mu = \kappa x \bar{k}^\mu$. Note that as discussed in the comment (b) of the subsection 2.1, we always arrange each observer to be the same distance $x$ from the horizon and have equal proper time $\tau$ and temperature $T_U$. Therefore, we can regard $x^{-1}$ as a constant. Then we perform area integral on the horizon at $\tau$:

$$\frac{d^2 A}{d\tau^2} = x^{-1} \frac{dA}{d\tau} + \int_{S(\tau)} \frac{dA}{d\tau} \left( \frac{\bar{\theta}^2}{2} - \bar{\sigma}_{\mu\nu} \bar{\sigma}^{\mu\nu} - R_{\mu\nu} \bar{k}^\mu \bar{k}^\nu \right) \Delta A. \quad (4.5)$$

where $A = \int_{S(\tau)} dA$. Then, we use the time-independence of $x^{-1}$ and perform time integral between $\tau_1$ and $\tau_2$:

$$\left. \frac{dA}{d\tau} \right|_{\tau_1}^{\tau_2} = x^{-1} \left. A \right|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} d\tau \int_{S(\tau)} \frac{dA}{d\tau} \left( \frac{\bar{\theta}^2}{2} - \bar{\sigma}_{\mu\nu} \bar{\sigma}^{\mu\nu} - R_{\mu\nu} \bar{k}^\mu \bar{k}^\nu \right). \quad (4.6)$$

The Einstein equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \quad (4.7)$$

and the null vector $k$ lead to

$$\frac{x^{-1}}{2\pi} \frac{1}{4} \delta A = \frac{1}{8\pi} \delta \left( \frac{dA}{d\tau} \right) + \frac{1}{8\pi} \int_{\tau_1}^{\tau_2} d\tau \int_{S(\tau)} \frac{dA}{d\tau} \left( \frac{\bar{\theta}^2}{2} - \bar{\sigma}_{\mu\nu} \bar{\sigma}^{\mu\nu} \right) \delta E \quad (4.8)$$

Finally, we use the asymptotic expression (2.6) of energy flow $\delta E$, the entropy formula (2.8) for $\gamma = 1$ and the Unruh temperature given by (2.7):

$$T_U \delta S = \frac{1}{2\pi} \delta \left( \frac{dS}{d\tau} \right) + \frac{1}{8\pi} \int_{\tau_1}^{\tau_2} d\tau \int_{S(\tau)} \frac{dA}{d\tau} \left( \frac{\bar{\theta}^2}{2} + \bar{\sigma}_{\mu\nu} \bar{\sigma}^{\mu\nu} \right) \delta E \quad (4.9)$$

By the assumption that the initial and final state are equilibrium and the equilibrium condition (2.9), we obtain the following formula:

$$T_U \delta S = \int_{\tau_1}^{\tau_2} d\tau \int_{S(\tau)} dA \left( \frac{1}{16\pi} \bar{\theta}^2 + \frac{1}{16\pi} 2\bar{\sigma}_{\mu\nu} \bar{\sigma}^{\mu\nu} \right) \delta E. \quad (4.10)$$
This is the first law of spacetime thermodynamics for non-equilibrium processes in Einstein’s gravity.

4.2. Interpretation

Before considering the meaning of (4.10), we summarize thermodynamics. The first law for any process between equilibrium states is as follows:

$$\delta U = \delta'Q + \delta'W,$$

(4.11)

where $\delta U$ is the energy change in the system, $\delta'Q$ is the heat from the external world to the system, and $\delta'W$ is the work done by the external world to the system.

The second law is, in Clausius’s form,

$$T^{(ex)}\delta S \geq \delta'Q,$$

(4.12)

where $\delta S$ is the variation in system’s entropy and $T^{(ex)}$ is the external temperature which is constant in the process. If the process is quasi-static, $T^{(ex)} = T$ and $T\delta S = \delta'Q$, where $T$ is the temperature of the system. The second law can also be written in terms of the internal heat production $\delta'D$, which leads to entropy production, such as friction, diffusion, and heat conduction:

$$T^{(ex)}\delta S = \delta'Q + \delta'D, \quad \delta'D \geq 0,$$

(4.13)

where, if the process is quasi-static process, $T^{(ex)} = T$ and $\delta'D = 0$. If energy is the only thermodynamic parameter in the system, all work is converted to internal heat production ($\delta'W = \delta'D \geq 0$). (For example, in an adiabatic system without a piston, all kinetic energy is converted irreversibly to internal heat production.) Therefore, combined as follows:

$$T^{(ex)}\delta S = \delta'D + \delta'Q = \delta U, \quad \delta'D \geq 0.$$

(4.14)

Let’s compare the formula (4.10) with the basic equation (4.14). Firstly, it is clear that $\delta E$ is considered as the heat because of the assumption (2.5) and $\delta S$ as the variation of th entropy of the system for the observer near the horizon. Secondly, we assume that our observer near the horizon feels the Unruh temperature $T_U$, which is constant as mentioned in the comment (b) of subsection 2.1. Thus we can regard the temperature as the external temperature $T^{(ex)}$ at which the system contacts with the external world, which means that the observer can see the process isothermal. Thirdly, the shear $\bar{\sigma}_{\mu\nu}$ comes mainly from Weyl tensor in not-too-dynamical processes which is pure gravitational degrees of freedom. So, the shear term $2\bar{\sigma}_{\mu\nu}\bar{\sigma}^{\mu\nu}$ would be a reversible mechanical term and would correspond to the work $\delta'W$. However, in the system whose thermodynamic parameter is energy, $\delta'W = \delta'D \geq 0$, as mentioned below (4.13). Indeed the term is always positive and increases the entropy for any dynamical process. Moreover, in the case of black hole, this term coincides with the Hartle-Hawking formula for the tidal heating of a classical black hole. Thus, the shear term corresponds to $\delta'D$. Finally, from (2.8) for $\gamma = 1$, the expansion $\dot{\theta}$ is the density of entropy increase per unit proper time, so the term can regarded as an entropy production term. Therefore, both of
them are dissipative terms in $\delta'D$. The expansion term’s coefficient $\zeta$ and the shear term’s one $\eta$ are respectively

$$\zeta = -\frac{1}{16\pi}, \quad \eta = \frac{1}{16\pi}.$$  

Indeed they are equal to viscosities of the membrane paradigm given by (3.2). In this sense, the equivalence in subsection 3.2 between our observer’s viewpoint and the black hole membrane paradigm holds in Einstein’s gravity. Note that each term of the RHS of (4.10) depends on the process like $\delta'D$ and $\delta'Q$, but the total does not, for the LHS is $T_U\delta S = T_U(S(\tau_2) - S(\tau_1))$.

There are some remarks.

(a) The derivation of (4.10) in the previous subsection is applicable to any space-time thermodynamic system in section 2. For the initial and final state, the equilibrium condition (2.9) is assumed there. Locally, this is equal to $\bar{\theta}|_{\tau_1,\tau_2} = 0$. If we take such a small “patch” system of a black hole event horizon that it can be interpreted as a local inertial frame, this condition is automatically satisfied, for the event horizon of a stationary black hole coincides the apparent horizon, which has $\bar{\theta} = 0$. Therefore, (4.10) for the above system has the most physical meaning. However, if the assumption (2.9) is made, (4.10) can be, in principle, applied to any spacetime thermodynamic system, though it may be an open system.

(b) We estimate all quantities in proper time $\tau$ of our observer near the horizon. Especially, $\delta E$ in (2.3) is estimated by 4-vector $u$ satisfying $u^2 = -1$. The normalization is unique, and so we can always interpret $\delta E$ as energy flow into the system, at least, for the local observer. Therefore, the physical meaning of the thermodynamic system is clear, though observer-dependent. This is the reason for using proper time $\tau$.

(c) In dynamical spacetime, matter energy and gravitational energy are indistinguishable from each other, and, in the case of asymptotically flat spacetime, the total energy is the ADM energy. Our formula is exact, so it can be applied to any dynamical processes and contains both matter and gravitational energy flow, which, if the system is a small patch system of a black hole event horizon in (a), contribute to the variation $\delta M_{BH}$ of the ADM mass of the black hole. Indeed, in quasi-static processes, (4.10) becomes

$$T_U\delta S = \delta E,$$

where, if the variation is so small that the Hawking temperature $T_{BH}$ can be regarded as constant in the process, $T_U = \sqrt{T_{(\text{initial})}^H_{(r\sim r_H)}}$. This is the blue-shifted conventional first law of black hole thermodynamics, in which only matter energy flow are taken into account through $T_{\mu\nu}$. Note that the Hartle-Hawking formula contains only the leading gravitational energy in not-too-dynamical processes.

(d) In the case of black hole thermodynamics, observers at infinity considers all energy flow as ADM energy. On the other hand, our observers near the horizon can distinguish between matter energy and gravitational energy in the sense that $T_{\mu\nu}$ corresponds to matter energy flow into/from the horizon and $\bar{\theta}^2$ and $\bar{\sigma}^2_{\mu\nu}$ correspond
to the horizon dynamics, which produces purely gravitational energy. Therefore, our observers near the horizon can distinguish between $\delta'Q$ and $\delta'D$.

(e) The expansion term’s coefficient $\zeta$ is negative. This is one of properties of the Raychaudhuri equation (4.1) for null congruence in a non-affine parameter. The negative coefficient would mean that the entropy of the system can decrease and be thermodynamically unstable. However, this does not happen at least classically. The first reason is that the equation is teleological in the sense that it is subject not to the initial condition that $\bar{\theta}(0) = 0$ but to the final condition that $\bar{\theta}(\infty) = 0$, which means that in the final state the system is locally equilibrium. In the case of a black hole, the teleological property is natural because the event horizon is globally defined, and so, determining the location at a time requires all the future information. Note that our basic time scale is $x \ll 1$, and so, if the black hole is instantaneously equilibrium at $\tau_1$ and $\tau_2$, the teleological effect can be neglected as the discussion by Carter. The second reason is that, from Hawking’s area theorem, the black hole area never decreases classically. In a stationary black hole the event horizon is the same as the apparent horizon, and so, in a spherical process ($\bar{\sigma}_{\mu\nu} = 0$), $\bar{\theta} \neq 0$ must be accompanied by $\delta E \neq 0$. Therefore, no matter how dynamical the spherical process is, $\delta E$ must be larger than the absolute value of the expansion term in (4.10) and the area must increase. Thus a black hole is thermodynamically stable in processes without the Hawking radiation. The effect of the negative coefficient in the radiation is discussed in section 6.

(f) An equation similar to (4.10) was derived in the black hole membrane paradigm. However, it is a hydrodynamic equation and its derivation is based on the membrane formulation of the stretched horizon. Our formula (4.10) is a thermodynamic equation and its derivation is based on the use of the Raychaudhuri equation. Note that by a different way, Carter also derived an area formula for dynamical processes, which is not the first law of black hole thermodynamics but a geometric formula because he used global time $t$ and did not identify energy flow for dynamical processes.

§5. First Law of Spacetime Thermodynamics in f(R) Gravity

Now we turn to the first law in f(R) gravity.

5.1. Derivation

The physical situation and the derivation are almost the same as the previous section. The main differences are “the equation of state” and the entropy density. f(R) gravity is defined as the following action:

$$ I = \frac{1}{16\pi} \int d^4x \sqrt{-g} f(R) , \quad (5.1) $$

where $f(R)$ is an arbitrary function of Ricci scalar $R$. Variation principle gives the equation of motion:

$$ f'(R)R_{\mu\nu} - \nabla_\mu \nabla_\nu f'(R) + \left( \nabla^2 f'(R) - \frac{1}{2} f'(R) \right) g_{\mu\nu} = 8\pi T_{\mu\nu} , \quad (5.2) $$
The variance is

\[ S = \frac{1}{4} \int_S dA \, f'(R) \] (5.3)

The variance is

\[ \delta S = \frac{1}{4} \int_{t_1}^{t_2} dt \left( \int_{S(t)} dA \, \frac{df'(R)}{dt} + \int_{S(t)} dA \, \frac{df'(R)}{dt} \right) \]

\[ = \frac{1}{4} \int_{t_1}^{t_2} dA \left( \theta f'(R) + \frac{df'(R)}{dt} \right) , \] (5.4)

which is (2.8) for \( \gamma = f'(R) \).

Let’s derive the first law. First, multiplying (4.4) by \( \bar{k}^2 \), we obtain

\[ f' \frac{d^2 \Delta A}{d\tau^2} = x^{-1} f' \bar{\theta} \Delta A + f' \left( \frac{\bar{\sigma}^2}{2} - \bar{\sigma}_{\mu \nu} \bar{\sigma}^{\mu \nu} \right) \Delta A - f' R_{\mu \nu} \bar{k}^\mu \bar{k}^\nu \Delta A . \] (5.5)

Using (5.2) and (5.4), we rewrite (5.5) as follows:

\[ f' \frac{d^2 \Delta A}{d\tau^2} = 4x^{-1} \frac{1}{4} \left( f' \bar{\theta} + \frac{df'}{d\tau} \right) \Delta A - x^{-1} \frac{df'}{d\tau} \Delta A + f' \left( \frac{\bar{\sigma}^2}{2} - \bar{\sigma}_{\mu \nu} \bar{\sigma}^{\mu \nu} \right) \Delta A \]

\[ - \bar{k}^\mu \bar{k}^\nu \nabla_\mu \nabla_\nu f' \Delta A - 8 \pi T_{\mu \nu} \bar{k}^\mu \bar{k}^\nu \Delta A , \] (5.6)

where we use \( \bar{k}^2 = 0 \). After the area integral, we perform the time integral, and thus obtain

\[ 4x^{-1} \delta S + \int_{\tau_1}^{\tau_2} d\tau \int_{S(\tau)} dA \, f' \left( \frac{\bar{\sigma}^2}{2} - \bar{\sigma}_{\mu \nu} \bar{\sigma}^{\mu \nu} \right) - 8 \pi \int_{\tau_1}^{\tau_2} d\tau \int_{S(\tau)} dA \, T_{\mu \nu} \bar{k}^\mu \bar{k}^\nu \]

\[ = \int_{\tau_1}^{\tau_2} d\tau \int_{S(\tau)} \frac{d^2 A}{d\tau^2} f' + \int_{\tau_1}^{\tau_2} d\tau \int_{S(\tau)} dA \left( x^{-1} \frac{df'}{d\tau} + \bar{k}^\mu \bar{k}^\nu \nabla_\mu \nabla_\nu f' \right) . \] (5.7)

Now we have

\[ \frac{df'(R)}{d\tau} = f''(R) R_{\mu \nu} u^\mu u^\nu \approx f''(R) R_{\mu \nu} \bar{k}^\mu \]

(5.8)

and \( \bar{k}^\mu \bar{k}^\nu = x^{-1} \bar{k}^\mu \), and thus, reach

\[ \frac{d^2 f'}{d\tau^2} = \bar{k}^\mu \bar{k}^\nu \nabla_\mu \nabla_\nu f' + x^{-1} \frac{df'}{d\tau} . \] (5.9)

From this and (5.4),

\[ \text{RHS of (5.7)} = \int_{\tau_1}^{\tau_2} d\tau \int_{S(\tau)} \frac{d^2 A}{d\tau^2} f' + \int_{\tau_1}^{\tau_2} d\tau \int_{S(\tau)} dA \frac{d^2 f'}{d\tau^2} \]

\[ = \int_{\tau_1}^{\tau_2} d\tau \frac{d}{d\tau} \int_{S(\tau)} \left( \frac{dA}{d\tau} f' + \frac{dA}{d\tau} \frac{df'}{d\tau} \right) - 2 \int_{\tau_1}^{\tau_2} d\tau \int_{S(\tau)} dA \frac{df'}{d\tau} \]

\[ = 4 \delta \left( \frac{dS}{d\tau} \right) - 2 \int_{\tau_1}^{\tau_2} d\tau \int_{S(\tau)} dA \bar{\theta} \frac{df'}{d\tau} . \] (5.10)
Using this, \( u \approx n \approx \bar{k} \), and (2.6), we rewrite (5.7) as follows:

\[
\frac{-1}{2\pi} \delta S = \frac{1}{2\pi} \delta \left( \frac{dS}{d\tau} \right) - \frac{1}{4\pi} \int_{\tau_1}^{\tau_2} \frac{dA}{d\tau} \frac{df'}{d\tau} d\tau \\
+ \frac{1}{16\pi} \int_{\tau_1}^{\tau_2} d\tau \int_{S(\tau)} dA \left[ \frac{-f''(R)}{16\pi} \bar{\theta}^2 + \frac{f''(R)}{16\pi} 2\bar{\sigma}_{\mu\nu} \bar{\sigma}^{\mu\nu} \right] + \delta E .
\]

Finally, by (2.7) and the assumption that the initial and final state are equilibrium, we arrive at

\[
T_U \delta S = \int_{\tau_1}^{\tau_2} d\tau \int_{S(\tau)} dA \left[ \frac{-f''(R)}{16\pi} \bar{\theta}^2 + \frac{f''(R)}{16\pi} 2\bar{\sigma}_{\mu\nu} \bar{\sigma}^{\mu\nu} \right] \\
- \frac{1}{4\pi} \int_{\tau_1}^{\tau_2} d\tau \int_{S(\tau)} dA \left( \frac{df}{d\tau} \right) \frac{df'}{d\tau} d\tau + \delta E .
\]

This is the first law generalized to f(R) gravity.

5.2. Interpretation

The meaning of (5.12) is essentially the same as the case of Einstein’s gravity. In (5.12), the expansion term’s coefficient \( \zeta \) and the shear term’s one \( \eta \) are found as follows:

\[
\zeta = -\frac{1}{16\pi} f'(R), \quad \eta = \frac{1}{16\pi} f'(R),
\]

which are equal to the viscosities of the membrane paradigm\[10\] in which the first law (5.12) was not derived and only a momentum conservation equation like the Navier-Stokes equation was obtained. In this sense, the equivalence in subsection 3.2 between our observer’s viewpoint and the black hole membrane paradigm holds in f(R) gravity, too.

There are some remarks.

(a) \( \zeta(x) \) and \( \eta(x) \) depend on the spacetime point, which comes from the entropy density \( \gamma(x) = f'(R(x)) \). The spacetime dependence of the entropy density and viscosities may reflect a microscopic structure of spacetime, for f(R) gravity includes higher-curvature terms and their coefficients are determined by renormalization of quantum field in the curved spacetime\[23\].

(b) The following term inevitably arises in (5.12):

\[
- \frac{1}{4\pi} \int_{\tau_1}^{\tau_2} d\tau \int_{S(\tau)} dA \left( \frac{df}{d\tau} \right) \frac{df'}{d\tau} .
\]

This term is second derivatives respective with proper time \( \tau \) of our observer, and thus, if quasi-static process, it vanishes as fast as the expansion term \( \bar{\theta}^2 \) and the shear term \( \bar{\sigma}_{\mu\nu}^2 \). In this sense, this term is effective only in non-equilibrium processes. Moreover, unlike the expansion term \( \bar{\theta}^2 \) and the shear term \( \bar{\sigma}_{\mu\nu}^2 \), the sign is not fixed. By choosing some appropriate process, the sign can be either positive or negative. Now, this spacetime thermodynamic system has a single thermodynamic parameter, that is, energy. Therefore, we can conjecture that the term corresponds to another
The heat term of $\delta'Q$ in (4.14) only for non-equilibrium processes. However, the meaning is not clear yet, which is discussed in the next section.

(c) Our bulk viscosity $\zeta$ in (5.13) is equal to that of the membrane paradigm, but not to that of Jacobson et al.\cite{11} which is $\zeta = \frac{3}{16\pi} f'(R)$. They used $\lambda_0$ such that $\frac{dS}{d\lambda}\big|_{\lambda_0} = 0$, and expanded the equation around it, where $\lambda$ was the affine parameter. Therefore, in our formula (5.12) we take $\tau_1 = \tau_0$, $\tau_2 = \tau_0 + \delta\tau$, $\delta\tau \ll 1$ and expand our formula. Note that $\delta\lambda \propto \delta\tau$, so this limit corresponds to the same situation.

Then we use

$$\bar{\theta} f' + \frac{df'}{d\tau} = 0 \quad \text{for} \quad \tau = \tau_0$$

and eliminate $\frac{df'}{d\tau}$ in (5.12). We arrive at

$$T_U \delta S = \int_{\tau_0}^{\tau_0 + \delta\tau} d\tau \int_{S(\tau)} dA \left( \frac{3f'(R)}{16\pi} \bar{\theta}^2 + \frac{f'(R)}{16\pi} 2\bar{\sigma}_{\mu\nu} \bar{\sigma}^{\mu\nu} \right) + \delta E .$$

Thus, we obtain

$$\zeta = \frac{3f'(R)}{16\pi} ,$$

which is the same as that of Jacobson et al. This meaning is discussed in the next section.

§6. Conclusions and Discussions

We have showed that Padmanabhan’s observer sees the timelike surface near his own causal horizon in the same way as the FIDO on the “stretched horizon” of the black hole membrane paradigm. By applying Padmanabhan’s observer to a dynamical spacetime and using the Raychaudhuri equation for null congruence in a non-affine parameter, we have constructed (4.10) and (5.12), that is, the first law of spacetime thermodynamics for non-equilibrium processes in Einstein’s gravity and f(R) gravity, respectively. In the laws, both matter and gravitational energy are taken into account, which, in the case of the “patch” system of a black hole, contribute to the variation $\delta M_{BH}$ of the ADM mass of the black hole. In quasi-static processes, the laws become the blue-shifted ordinary first law.\cite{1},\cite{16},\cite{22} In the derived laws, the coefficients of the expansion and shear terms are equal to the viscosities of the membrane paradigm\cite{9},\cite{10} and a new term given by (5.14) appears, which is conjectured as another heat term only for non-equilibrium processes. Moreover, in the infinitesimal time limit, our coefficients agree with those of Jacobson et al.\cite{11}

There remain some open questions.

(a) We should understand the expansion $\bar{\theta}$ in f(R) gravity more correctly. Though in Einstein’s gravity $\bar{\theta}$ corresponds to the density of entropy increase per unit proper time, in f(R) gravity this interpretation is not correct. Unlike the shear $\bar{\sigma}_{\mu\nu}$, which is interpreted as purely gravitational effect as mentioned in section 4, the expansion is sensitive to $T_{\mu\nu}$, $\bar{\theta}$ and $\bar{\sigma}_{\mu\nu}$, so the meaning is less clear. The remarks (b) and (c) in subsection 5.2 are intimately related to the above fact. $\zeta(x)$ in (5.13) is equal to the bulk viscosity in the membrane paradigm,\cite{10} in which the stretched horizon is
interpreted as a viscous fluid through the momentum conservation equation like the Navier-Stokes equation, and the new term in (5.14) inevitably arises in (5.12). Note that even if we rewrite the new term as
\[ 4 \int_{\tau_1}^{\tau_2} d\tau \int_{S(\tau)} dA \bar{\theta} \frac{d\zeta(x)}{d\tau}, \]
this is not correct because the spacetime-dependence of the bulk viscosity is already contained in (5.12), as ordinary fluid. In the infinitesimal time limit, the new term disappears and \( \zeta(x) \) becomes that of Jacobson et al. In these senses, the interpretations of the expansion term \( \bar{\theta}^2 \) and the new term may depend on the time scale of the process.

(b) In black hole thermodynamics, the generalized second law plays a fundamental role, which is an assumption made by Bekenstein\(^{15}\) that the sum of the black hole entropy \( S_{BH} = \frac{1}{4} A \) and the entropy \( S_{\text{matter}} \) of the usual matter and gravitational radiation outside a black hole never decreases. Though an explicit general proof of this law has not been given until now, the validity of the law for special cases have been verified, such as quasi-static processes without the back-reaction of quantum field energy taken into account.\(^{25}\) In a full proof, arbitrary dynamical processes and the back-reaction should be considered. Therefore, our formulae are useful to prove the generalized second law for dynamical processes because they can be applied to any dynamical processes with the Hawking radiation, though they are applicable only to a “patch” system of a black hole. Especially, when the effect of the evaporation is large in the process such as the evaporation process of a small black hole, the expansion term \( \bar{\theta}^2 \) in the laws will become more effective.

(c) What is the first law for more general theory of gravity? Our formula is restricted to f(R) gravity, which is the simplest model in higher-curvature theories of gravity. However, a complete proof of the generalized second law should need more general higher-curvature terms, such as the Gauss-Bonnet term, due to back reaction from quantum field renormalization.\(^{29}\) Thus, we are interested in the first law for non-equilibrium processes in the Lovelock gravity, which is the most general second-order gravity theory in higher dimensional spacetime. Note that Padmanabhan studied the first law for quasi-static processes through the Wald entropy.\(^{3}\)

(d) If the logic of our derivation are reversed, the Einstein equation can be derived as “the equation of state” from the first law of (4.10) for non-equilibrium processes in a spacetime thermodynamic system, as Jacobson\(^{3,11,13}\). However, as mentioned in subsection 4.2, it is difficult to interpret systems other than a black hole and the derivation physically. In order to understand this observer-dependent interpretation more physically, we should shed light on the microscopic meaning.

These issues require further study.

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Einstein Equation of State and Black Hole Membrane Paradigm

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