A FOOTNOTE TO A THEOREM OF KAWAMATA

MARGARIDA MENDES LOPES, RITA PARDINI, AND SOFIA TIRABASSI

Abstract. Kawamata has shown that the quasi-Albanese map of a quasi-projective variety with log-irregularity equal to the dimension and log-Kodaira dimension 0 is birational. In this note we show that under these hypotheses the quasi-Albanese map is proper in codimension 1 as conjectured by Iitaka.

Introduction

Given a smooth complex quasi-projective variety $V$, by Hironaka’s theorem on the resolution of singularities we can embed $V$ into a smooth projective variety $X$ such that the complement $D := X \setminus V$ is a reduced divisor on $X$ with simple normal crossings.

One can use the compactification $X$ to define the logarithmic invariants of $V$. In particular, if $K_X$ denotes the canonical divisor of $X$, then:

- the log-Kodaira dimension of $V$ is $\kappa(V) := \kappa(X, K_X + D)$;
- the log-irregularity of $V$ is $\overline{q}(V) := h^0(X, \Omega_X^1(\log D))$.

It easy to show that these invariants do not depend on the choice of the compactification $X$.

In a similar way to projective varieties, we can associate to $V$ a quasi-abelian variety (i.e., an algebraic group which does not contain $G_a$) $A(V)$, called the quasi-Albanese variety of $V$ and a morphism $a_V : V \to A(V)$ which is called the quasi-Albanese morphism.

Kawamata, in his celebrated work [6], provides a numerical criterion for the birationality of the quasi-Albanese map:

Theorem (Kawamata, Theorem 28 of [6]). Let $V$ be a smooth complex algebraic variety of dimension $n$. If $\kappa(V) = 0$ and $\overline{q}(V) = n$ then the quasi-Albanese morphism $a_V : V \to A(V)$ is birational.

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In this note, we strengthen Kawamata’s Theorem in a way that was conjectured by Iitaka in [4, p. 501]. Our main result is the following:

**Theorem A.** Let \( V \) be a smooth complex algebraic variety of dimension \( n \) with \( \kappa(V) = 0 \) and \( q(V) = n \). Denote by \( a_V : V \to A(V) \) the quasi-Albanese morphism (which is birational).

Then there is a closed subset \( Z \subset A(V) \) of codimension > 1 such that, setting \( V^0 := V \setminus a_V^{-1}(Z) \), the restriction \( a_V|_{V^0} : V^0 \to A(V) \setminus Z \) is proper.

In [4] Iitaka defined the quite involved notion of WWPB (“weakly weak proper birational”) equivalence. With this language, we can rewrite our statement in the following way:

**Theorem A*.** Let \( V \) be a smooth complex algebraic variety of dimension \( n \). Then \( V \) is WWPB equivalent to a quasi-abelian variety if and only if \( \kappa(V) = 0 \) and \( q(V) = n \).

While the usual Kodaira dimension and irregularity are birational invariants, this is not the case for logarithmic invariants. As an easy example of this phenomenon, one can think of \( \mathbb{P}^1 \) and \( \mathbb{P}^1 \setminus \{p_0 + p_1\} \): they are birationally equivalent but their invariants are completely different. However, WWPB-equivalent varieties share the same logarithmic invariants. Therefore this seems to be the right notion of equivalence to consider when studying the birational geometry of open varieties.

We conclude this section by remarking that the condition of being WWPB-equivalent is a really strong one: WWPB maps between normal affine varieties are indeed isomorphisms. Thus we get the following corollary of Theorem A:

**Corollary B.** A smooth complex affine variety \( V \) of dimension \( n \) is isomorphic to \( G_m^n \) if and only if \( \kappa(V) = 0 \) and \( q(V) = n \).

**Notice.** Our original proof contained an unsubstantiated claim that was pointed to us by Prof. Osamu Fujino. In the attached addendum/correction (in collaboration with Prof. Fujino) we provide a new proof and we also explain how to avoid the gap in the original argument.

**Notation.** We work over the complex numbers. If \( X \) is a smooth projective variety, we denote by \( K_X \) the canonical class, and by \( q(X) := h^0(X, \Omega^1_X) = h^1(X, \mathcal{O}_X) \) its irregularity.

We identify invertible sheaves and Cartier divisors and we use the additive and multiplicative notation interchangeably. Linear equivalence is denoted by \( \sim \). Given two divisors \( D_1 \) and \( D_2 \) on \( X \), we write \( D_1 \geq D_2 \) (respectively \( D_1 > D_2 \)) if the divisor \( D_1 - D_2 \) is (strictly) effective.

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1. Preliminaries

In this section, we provide the background material and preliminary lemmas which are essential for our main result. In particular, we review the logarithmic ramification formula in 1.1, and give a quick introduction to quasi-abelian varieties and quasi-Albanese morphism in 1.2. For the sake of brevity, we do not cover here the general theory of log-varieties and WWPB maps. An exhaustive introduction to the theory of logarithmic forms can be found in [5]. For a quick overview, with also an introduction to WWPB equivalence, we refer to our own [7].

1.1. Logarithmic ramification formula. Let $V, W$ be smooth varieties of dimension $n$ and let $h: V \rightarrow W$ be a dominant morphism. Let $g: X \rightarrow Y$ be a morphism extending $h$, where $X, Y$ are smooth compactifications of $V, W$, such that $D := X \setminus V$ and $\Delta := Y \setminus W$ are simple normal crossings (snc for short) divisors. Then the pull back of a logarithmic $n$-form on $Y$ is a logarithmic $n$-form on $X$, and a local computation shows that there is an effective divisor $R_g$ of $X$ - the logarithmic ramification divisor - such that the following linear equivalence holds:

\begin{equation}
K_X + D \cong g^* (K_Y + \Delta) + R_g.
\end{equation}

Equation (1) is called the logarithmic ramification formula (cf. [5, §11.4]).

In [7, Lemma 1.8] we observed the following useful fact:

**Lemma 1.1.** In the above set-up, denote by $R_g$ the (usual) ramification divisor of $g$. Let $\Gamma$ be an irreducible divisor such that $g(\Gamma) \not\subseteq \Delta$. Then $\Gamma$ is a component of $R_g$ if and only if $\Gamma$ is a component of $D + R_g$.

**Proof.** Let $p \in E$ general and consider $q := g(p) \in g(E)$.

If $g(E)$ is not contained in $D_Y$, then $q$ is not in $D_Y$ and so, by Lemma 1.1, $E \not\subseteq R_g$. Next we characterize the crepant exceptional divisors, namely those that do not contribute to $R_g$:

**Lemma 1.2.** Let $g: X \rightarrow Y$ be a birational morphism of smooth projective varieties, let $D_X \subseteq X$ and $D_Y \subseteq Y$ be snc divisors such that $g(X \setminus D_X) \subseteq Y \setminus D_Y$, and consider $E$ an irreducible $g$-exceptional divisor. If $E$ does not appear in $R_g$, then $g(E)$ is an irreducible component of the intersection of some components of $D_Y$.

**Proof.** Let $p \in E$ general and consider $q := g(p) \in g(E)$.
Thus, we can assume that \( g(E) \subseteq D_Y \) and, therefore, \( E \leq D_X \). Denote by \( D_1, \ldots, D_k \) the components of \( D_Y \) containing \( g(E) \) and let \( x_i \) be local equations for \( D_i \) near \( q \).

Assume by contradiction that \( g(E) \) is a proper subset of \( D_1 \cap \ldots \cap D_k \). Since \( q \) is general, \( g(E) \) is smooth near \( q \), and hence locally a complete intersection. Since \( D_Y \) is snc, we can complete \( x_1, \ldots, x_k \) to a system of local coordinates \( x_1, \ldots, x_n \) such that around \( q \) the set \( g(E) \) is locally defined by the vanishing of \( x_1, \ldots, x_{k+s} \) for some \( s \geq 1 \). In this setting, locally around \( q \) the line bundle \( O_Y(K_Y + D_Y) \) is generated by

\[
\sigma := \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_k}{x_k} \wedge dx_{k+1} \wedge \cdots \wedge dx_n \\
= (x_{k+1} \cdots x_{k+s}) \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_{k+s}}{x_{k+s}} \wedge dx_{k+s+1} \wedge \cdots \wedge dx_n.
\]

Choose a local equation \( t \) for \( E \) near \( p \). Then we can write

\[
g^* x_i = t^{\alpha_i} h_i,
\]

with \( h_i \) regular and not divisible by \( t \). Observe that \( \alpha_i > 0 \) if \( i \leq k + s \), while \( \alpha_i = 0 \) for \( i > k + s \). Thus we get

\[
g^* \sigma = h t^{\alpha_{k+1} + \cdots + \alpha_{k+s}} \cdot \frac{dt}{t} \wedge \tau,
\]

where \( h \) is a regular function and \( \tau \) is a \((n - 1)\)-form without poles along \( E \). We conclude that \( E \) appears in \( \overline{R_g} \) with multiplicity at least \( \alpha_{k+1} + \cdots + \alpha_{k+s} > 0 \). This is a contradiction because we are assuming that \( E \) does not appear in \( \overline{R_g} \). \( \square \)

We close the section by proving a partial converse of the previous lemma:

**Lemma 1.3.** Let \( W \) be a quasi-projective variety with compactification \( Y \) and snc boundary \( \Delta \). Let \( C \subseteq \Delta \) be an irreducible component of the intersection of some components of \( \Delta \), and denote by \( \epsilon_C : Y_C \to Y \) the blow-up of \( C \). If \( \Delta_C := \epsilon_C^{-1}\Delta \) is the set-theoretic pre-image of \( \Delta \), then \( Y_C \) is also a compactification of \( W \) with snc boundary \( \Delta_C \) and the logarithmic ramification divisor \( \overline{R_{\epsilon_C}} \) is trivial.

**Proof.** It is clear that \( W \approx Y_C \setminus \Delta_C \) and that \( \Delta_C \) has snc support. Thus we need just to verify the statement about the logarithmic ramification divisor. Let \( k \) be the codimension of \( C \) in \( Y \), and denote by \( E \) the exceptional divisor of the blow-up. Then

\[
K_{Y_C} + \Delta_C \simeq f^*(K_Y) + (k-1)E + f^*(\Delta) - (k-1)E = f^*(K_Y + \Delta).
\]

We conclude directly by the logarithmic ramification formula (1) that \( \overline{R_{\epsilon_C}} = 0 \). \( \square \)
1.2. **Quasi-abelian varieties and their compactifications.** A *quasi-abelian variety* — in some sources also called a *semiabelian variety* — is a connected algebraic group $G$ that is an extension of an abelian variety $A$ by an algebraic torus. More precisely, $G$ sits in the middle of an exact sequence of the form

\[(1) \quad 1 \to G^\text{r}_m \to G \to A \to 0.\]

We call $A$ the *compact part* and $G^\text{r}_m$ the *linear part* of $G$.

We recall the following from [3, §10]:

**Proposition 1.4.** Let $G$ be a quasi-abelian variety, let $A$ be its compact part and let $r := \dim G - \dim A$. Then there exists a compactification $G \subset Y$ such that:

(a) $Y$ is a $\mathbb{P}^r$-bundle over $A$;
(b) $\Delta := Y \setminus G$ is a simple normal crossing divisor and $K_Y + \Delta = 0$;
(c) the natural $G^\text{r}_m$-action on $G$ extends to $Y$.

**Proof.** As explained in [3, §10], the variety $G$ is a principal $G^\text{r}_m$-bundle over $A$. We let $\rho: G^\text{r}_m \to \text{Aut}(\mathbb{P}^r)$ be the inclusion that maps $(\lambda_1, \ldots, \lambda_r)$ to the automorphism represented by $\text{Diag}(1, \lambda_1, \ldots, \lambda_r)$ and take $Y := G \times_{\rho} \mathbb{P}^r$. By construction, $Y = \text{Proj}_A(\mathcal{O}_A \oplus L_1 \oplus \cdots \oplus L_r)$, for some $L_i \in \text{Pic}(A)$, the boundary $\Delta$ is the sum of the relative hyperplanes $\Delta_0, \Delta_1, \ldots, \Delta_r$ given by the inclusions of $\mathcal{O}_Y, L_1, \ldots, L_r$ into $\mathcal{O}_A \oplus L_1 \oplus \cdots \oplus L_r$ and the $G^\text{r}_m$-action clearly extends to $Y$. Since $K_A$ is trivial, the formula for the canonical class of a projective bundle reads $K_Y \simeq -(\Delta_0 + \cdots + \Delta_r)$. □

In the sequel, we will denote by $q: Y \to A$ the structure map giving the $\mathbb{P}^r$ bundle structure.

1.2.1. **The quasi-Albanese map.** The classical construction of the Albanese variety of a projective variety can be extended to the non projective case, by replacing regular 1-forms by logarithmic ones and abelian varieties by quasi-abelian ones. The key fact is that by Deligne [1] logarithmic 1-forms are closed (for the details of the construction see [3, §2, Section 3]). As a consequence, for every $V$ smooth quasi projective variety, there exists a quasi-abelian variety $A(V)$ and a morphism $a_V: V \to A(V)$ such that any morphism $h: V \to G$ to a quasi-abelian variety factors through $a_V$ in a unique way up to translation.

## 2. Proof of theorem A

Let $V$ be a smooth complex quasi-projective variety of dimension $n \geq 2$ satisfying $\overline{κ}(V) = 0$ and $\overline{q}(V) = n$. Let $A(V)$ be the quasi-Albanese variety of $V$, and $a_V: V \to A(V)$ the quasi-Albanese morphism.
Pick a compactification \((Y, \Delta)\) of \(A(V)\) as in Proposition 1.4. Choose then a compactification \((X, D)\) of \(V\) such that the quasi-Albanese morphism \(a_V: V \to A(V)\) extends to a morphism \(f: X \to Y\). Recall that \(f\) is birational by Kawamata’s theorem ([6, Thm. 28]).

We observe that proving Theorem A is equivalent to showing that every irreducible component of \(D\) not contracted by \(f\) is mapped to \(\Delta\). In fact, let \(D'\) be the divisor formed by the irreducible components of \(D\) which are not mapped to \(\Delta\), assume that \(D'\) is contracted by \(f\) and set \(Z' := f(D')\). This is a closed subset of codimension at least 2 in \(Y\) and thus \(Z := Z' \cap A(V)\) is a closed subset of codimension at least 2. Denoting by \(V^0\) the open set \(V \setminus a_V^{-1}(Z)\), we see that the restriction
\[
a_V|_{V^0}: V^0 \to A(V) \setminus Z
\]
is proper.

So we argue by contradiction supposing that there is an irreducible component \(H\) of \(D\) such that \(f(H) =: \overline{H}\) is a divisor not contained in \(\Delta\). Then
\[
H = f'(\overline{H}) - \sum m_i E_i,
\]
where the divisors \(E_i\) are \(f\)-exceptional divisors. By the logarithmic ramification formula we have
\[
K_X + D \simeq f^*(K_Y + \Delta) + R_f \simeq \overline{R}_f,
\]
where the last equality holds because \(K_Y + \Delta \simeq 0\) (see Proposition 1.4).

We split the proof in two main steps: first, we give an argument in the special case in which all the \(E_i\) appearing in (3) belong to \(\overline{R}_f\); afterwards, we show that the general situation can always be reduced to the special case.

2.1. Special case: all the \(E_i\)'s are components of \(\overline{R}_f\). Since \(H\) is also a component of \(\overline{R}_f\) by Lemma 1.1, we have that
\[
K_X + D \simeq \overline{R}_f \geq H + \sum E_i = f^*\overline{H} - \sum (m_i - 1)E_i.
\]
Choose \(\alpha\) a rational number, \(\alpha \in (0, 1)\) such that \(\alpha \geq \frac{m_i - 1}{m_i}\) for every \(i\). Then we can write
\[
f^*\overline{H} - \sum (m_i - 1)E_i = \alpha f^*\overline{H} - \sum (m_i - 1)E_i + (1 - \alpha)f^*\overline{H}
\geq \alpha \left( f^*\overline{H} - \sum m_i E_i \right) + (1 - \alpha)f^*\overline{H}
= \alpha H + (1 - \alpha)f^*\overline{H}.
\]
It follows that \(\kappa(Y, \overline{H}) \leq \overline{\kappa}(V)\). We are going to show that \(\kappa(Y, \overline{H}) > 0\), against the hypothesis \(\overline{\kappa}(V) = 0\).
If $\overline{H}$ is the pullback of an effective divisor $\Gamma$ via the map $q: Y \to A$, then $\kappa(Y, \overline{H}) \geq \kappa(A, \Gamma) > 0$, because an effective divisor on an abelian variety is the pullback of an ample divisor on a quotient abelian variety.

Thus we may assume that this is not the case, namely that $\overline{H}$ restricts to a divisor on the general fiber $F$ of the projective bundle $q: Y \to A$. Since $\overline{H}$ is not contained in $\Delta$, its restriction to the general fiber of $q$ is not contained in $\Delta | F$. In particular $\overline{H} | F$ is not invariant with respect to the $G_m$-action on $F$ (see Proposition 1.4). This means that we can find a $g \in G_m$ such that $g^* \overline{H} | F \neq \overline{H} | F$. Since $g^* \overline{H}$ and $\overline{H}$ are linearly equivalent, we obtain $h^0(Y, \overline{H}) \geq 2$ and we are done.

2.2. Conclusion of the proof. Up to reordering, we can assume that $E_1 \not\subseteq \overline{R_f}$. By Lemma 1.2 we have that $f(E_1)$ is a component of a complete intersection of components of $\Delta$. Let $\epsilon: Y_1 \to Y$ be the blow up of $f(E_1)$. Observe that $Y_1$ is smooth, since $f(E_1)$ is so. By Lemma 1.3, $(Y_1, \Delta_1)$ is a compactification of $A(V)$ with $K_{Y_1} + \Delta_1 \simeq 0$. In addition the $G_m$-action on $A(V)$ extends to $Y_1$ and preserves the components of $\Delta_1$.

By the universal property of the blow-up we can factor $f$ as follows:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
Y_1 & \xrightarrow{\epsilon_1} & Y
\end{array}
$$

Let $\overline{H}_1 := f_1(H)$. This is a divisor not contained in $\Delta_1$, otherwise $f(H)$ would be contained in $\Delta$. Then

$$
f_1 \overline{H}_1 = H + \sum_{i \geq 2} m_i E'_i,
$$

where the $E'_i$ are $f_1$-exceptional divisors. If all the $E'_i$ appear in the logarithmic ramification divisor $\overline{R}_{f_1}$, we can argue as in the special case and we are done. Otherwise, we can repeat the above construction. As at every stage we provide a blow-up $Y_n$ of $Y_{n-1}$ and a factorization $f = \epsilon_1 \circ \cdots \circ \epsilon_n \circ f_n$, after finitely many iterations this process must stop, and again we can argue as in the special case.

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CAMGSD/Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

Email address: mmendeslopes@tecnico.ulisboa.pt

Dipartimento di Matematica, Università degli studi di Pisa, Largo Pontecorvo 5, 56127 Pisa (PI), Italy

Email address: rita.pardini@unipi.it

Department of Mathematics, Stockholm University, Albano Campus, Stockholm, Sweden

Email address: tirabassi@math.su.se
ADDENDUM TO “A FOOTNOTE TO A THEOREM OF KAWAMATA”

OSAMU FUJINO, MARGARIDA MENDES LOPE, RITA PARDINI, AND SOFIA TIRABASSI

Abstract. We give an alternative proof of Theorem A in the paper: M. Mendes Lopes, R. Pardini, S. Tirabassi, A footnote to a theorem of Kawamata. We also explain how to fill a gap in the original proof.

1. Introduction

In this paper, we give an alternative proof of the following theorem, which is the main result of [MPT]:

Theorem 1.1 (see [MPT, Theorem A]). Let $X$ be a smooth variety defined over $\mathbb{C}$ with logarithmic Kodaira dimension $\kappa(X) = 0$ and logarithmic irregularity $\tau(X) = \dim X$. Then the quasi-Albanese map $\alpha : X \to A$ is birational and there exists a closed subset $Z$ of $A$ with $\text{codim}_A Z \geq 2$ such that $\alpha : X \setminus \alpha^{-1}(Z) \to A \setminus Z$ is proper.

The proof given in [MPT] contains a gap, noticed by the first named author of this paper; in §3 we explain this gap and how to avoid it.

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Conventions: We work over the field $\mathbb{C}$ of complex numbers and we use freely Iitaka’s theory of quasi-Albanese maps and logarithmic Kodaira dimension developed in [I1] and [I2] (see also [F1]).

2. Proof of Theorem 1.1

In this section we prove Theorem 1.1. The birationality of $\alpha : X \to A$ is a well-known theorem by Kawamata (see [K2]), hence we are going to prove the existence of the desired closed subset $Z$. The proof given here uses Kawamata’s subadditivity formula in [K1]. Before we start the proof of Theorem 1.1, we note the following fact (see also [MPT, Lemma 2.2]):

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Remark 2.1 (Log canonical centers). Let $X$ be a smooth variety and let $\Delta_X$ be a simple normal crossings divisor on $X$, so that $(X, \Delta_X)$ is log canonical. Let $\Delta_X = \sum_{i \in I} \Delta_i$ be the irreducible decomposition of $\Delta_X$; then a closed subset $W$ of $X$ is a log canonical center of $(X, \Delta_X)$ if and only if $W$ is an irreducible component of $\Delta_i \cap \ldots \cap \Delta_{i_k}$ for some $\{i_1, \ldots, i_k\} \subset I$. When $\Delta_i \cap \ldots \cap \Delta_{i_k}$ is connected for every $\{i_1, \ldots, i_k\} \subset I$, we say that $\Delta_X$ is a strong normal crossings divisor in the terminology of [MS].

We now turn to the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\alpha: X \to A$ be the quasi-Albanese map. By Kawamata’s theorem (see [K2, Corollary 29] and [F1, Corollary 10.2]), we see that $\alpha$ is birational.

Step 1. Let

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{G}_m^d \\
\alpha & \longrightarrow & A \\
\pi & \longrightarrow & B \\
\end{array}
$$

be the Chevalley decomposition. Then $A$ is a principal $\mathbb{G}_m^d$-bundle over an abelian variety $B$ in the Zariski topology (see, for example, [BCM, Theorems 4.4.1 and 4.4.2]) and there is a natural completion $\pi: \overline{A} \to B$ of $\pi: A \to B$ where $\overline{A}$ is a $\mathbb{P}^d$-bundle. We set $\Delta_{\overline{A}} := \overline{A} \setminus A$; then $\Delta_{\overline{A}}$ is a simple normal crossings divisor on $\overline{A}$, and $(\overline{A}, \Delta_{\overline{A}})$ is a log canonical pair.

Let $\pi: X \to \overline{A}$ be a compactification of $\alpha: X \to A$, that is, $X$ is a smooth complete algebraic variety containing $X$, $\Delta_X := X \setminus X$ is a simple normal crossing divisor on $X$ and $\overline{A}$ is a morphism extending $\alpha$.

Claim. Let $D$ be an irreducible component of $\Delta_X$ such that $\pi(D)$ is a divisor. Then $\pi: \pi(D) \to B$ is dominant.

Proof of Claim. We set $D_1 := \pi(D)$. If $\pi: D_1 \to B$ is not dominant, then we can write $D_1 = \pi(D_2)$ for some prime divisor $D_2$ on $B$. By Remark 2.1 every log canonical center of $(\overline{A}, \Delta_{\overline{A}})$ dominates $B$, therefore $D_1$ does not contain any log canonical center (so in particular it is not a component of $\Delta_{\overline{A}}$). Hence

$$K_{\overline{A}} + \Delta_{\overline{A}} - \pi^*(K_A + \Delta_A) \geq \varepsilon \pi^* D_1$$

for some $0 < \varepsilon \ll 1$ since the support of $D_1$ does not contain any log canonical center of $(\overline{A}, \Delta_{\overline{A}})$. By construction, we have $K_{\overline{A}} + \Delta_{\overline{A}} \sim 0$. Thus we obtain

$$0 = \pi(X) = K_{\overline{A}} + \Delta_{\overline{A}} \geq \kappa(X, \pi^* D_1) = \kappa(\overline{A}, D_1) = \kappa(B, D_2) > 0,$$

where the last inequality follows from the fact that $D$ is a nonzero effective divisor on the abelian variety $B$. This contradiction proves the claim.

Step 2. We assume that there exists an irreducible component $D$ of $\Delta_X$ such that $\pi(D)$ is a divisor with $\pi(D) \not\subseteq \overline{A} \setminus A$ and we set $D' := \pi(D) \cap A$. By the Claim in Step 1, $D'$ dominates $B$, therefore we can find a subgroup $\mathbb{G}_m$ of $A$ such that $\varphi|_{D'}: D' \to A_1$ is dominant, where

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{G}_m \\
\alpha & \longrightarrow & A \\
\varphi & \longrightarrow & A_1 \\
\end{array}
$$

Note that $A$ is a principal $\mathbb{G}_m^d$-bundle over $A_1$ in the Zariski topology (see, for example, [BCM, Theorems 4.4.1 and 4.4.2]). We take a compactification

$$
\begin{array}{ccc}
f^\dagger: X^\dagger & \longrightarrow & A^\dagger \\
\alpha^\dagger & \longrightarrow & A_1^\dagger \\
\end{array}
$$

of

$$
\begin{array}{ccc}
f: X & \longrightarrow & A \\
\alpha & \longrightarrow & A_1, \\
\end{array}
$$
where $X^†, A^†$, and $A^↓i$ are smooth complete algebraic varieties such that $X^† \setminus X$, $A^† \setminus A$, and $A^↓i \setminus A_↓1$ are simple normal crossing divisors. Let $F$ be a general fiber of $f$. Then $\pi(F) = 1$ since $\varphi|_{D^′} : D^′ \to A_↓1$ is dominant. Note that $A_↓1$ is a quasi-abelian variety, hence we have $\overline{\pi}(A_↓1) = 0$. By Kawamata’s theorem (see [K1, Theorem 1] and [F2, Chapter 8]), we obtain
\[
0 = \pi(X) \geq \pi(F) + \pi(A_↓1) = 1.
\]
This is a contradiction, showing that every irreducible component of $\Delta$ which is not contracted by $\overline{\pi}$ is mapped to $\Delta$. Let $\Delta^′$ be the sum of the irreducible components of $\Delta$, which are not mapped to $\Delta$. It is obvious that $\text{codim}_A \overline{\pi}(\Delta^′) \geq 2$ holds since $\Delta^′$ is $\overline{\pi}$-exceptional. We put $Z := \overline{\pi}(\Delta^↓) \cap A$. Then $Z$ is a closed subset of $A$ with $\text{codim}_A Z \geq 2$. It is easy to see that
\[
\alpha : X \setminus \alpha^{-1}(Z) \to A \setminus Z
\]
is proper.

This finishes the proof of Theorem 1.1. □

3. ON THE ORIGINAL PROOF IN [MPT]

The final part (§3.2) of the original proof of Theorem 1.1 given in [MPT] contains the unsubstantiated claim that, if $X$ contains a divisor contracted to a log canonical center $W$ of $(\overline{A}, \Delta)$, then $\overline{\pi}$ factors though the blow-up of $\overline{A}$ along $W$. This claim is then used to reduce the proof to the special case treated in §3.1 of [MPT].

We explain here how to reduce the proof to the special case by a different argument.

Let $T$ be a smooth variety and let $\Delta_T$ be a simple normal crossing divisor on $T$. Let $f : T^′ \to T$ be the blow-up of a log canonical center of $(T, \Delta_T)$. We set
\[
K_{T^′} + \Delta_{T^′} := f^*(K_T + \Delta_T).
\]
Then it is easy to see that $T^′$ is smooth and $\Delta_{T^′}$ is a simple normal crossing divisor on $T^′$. We call $f : (T^′, \Delta_{T^′}) \to (T, \Delta_T)$ a crepant pull-back of $(T, \Delta_T)$.

Now, with the same notation and assumptions of §2, let $D$ be a component of $\Delta$ such that $\overline{\pi}(D)$ is a divisor not contained in $\Delta$. In the terminology of [MS] the divisor $\Delta$ is a strong normal crossings divisor (see Remark 2.1). Then we are precisely in the situation of §5.1 of [MS] and by Corollary 2, ibid., there is a finite sequence of crepant pull-backs
\[
(\overline{A}, \Delta) =: (T_0, \Delta_{T_0}) \xleftarrow{f_1} (T_1, \Delta_{T_1}) \xleftarrow{f_2} \cdots \xleftarrow{f_s} (T_k, \Delta_{T_k})
\]
such that the strict transform of $\overline{\pi}(D)$ on $T_k$ does not contain any log canonical center of $(T_k, \Delta_{T_k})$. In addition, the components of $\Delta$ are preserved by the $G_m^d$-action and therefore it is easy to check that $G_m^d$ acts also on $T_1$ and the action preserves the components of $\Delta_{T_1}$. An inductive argument then shows that $G_m^d$ acts on $T_k$ and preserves the components of $\Delta_{T_k}$.

So, up to replacing $(\overline{A}, \Delta)$ by $(T_k, \Delta_{T_k})$ and modifying $X$ accordingly, we may assume that $\overline{\pi}(D)$ does not contain any log canonical center and that the $G_m^d$-action extends to $\overline{A}$. We conclude by observing that the argument in [MPT, §3.1] (the “special case”) works in this situation.
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Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan

Email address: fujino@math.kyoto-u.ac.jp

CAMGSD/Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

Email address: mmendeslopes@tecnico.ulisboa.pt

Dipartimento di Matematica, Università degli studi di Pisa, Largo Pontecorvo 5, 56127 Pisa (PI), Italy

Email address: rita.pardini@unipi.it

Department of Mathematics, Stockholm University, Albano Campus, Stockholm, Sweden

Email address: tirabassi@math.su.se