Colouring \((sP_1 + P_5)\)-Free Graphs: a Mim-Width Perspective

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Abstract
We prove that the class of \((K_t, sP_1 + P_5)\)-free graphs has bounded mim-width for every \(s \geq 0\) and \(t \geq 1\), and that there is a polynomial-time algorithm that, given a graph in the class, computes a branch decomposition of constant mim-width. A large number of \#P-complete graph problems become polynomial-time solvable on graph classes with bounded mim-width and for which a branch decomposition is quickly computable. The \(\chi\)-Colouring problem is an example of such a problem. For this problem, we may assume that the input graph is \(K_{k+1}\)-free. Then, as a consequence of our result, we obtain a new proof for the known result that for every fixed \(k \geq 1\) and \(s \geq 0\), \(\chi\)-Colouring is polynomial-time solvable for \((sP_1 + P_5)\)-free graphs. In fact, our findings show that the underlying reason for this polynomial-time algorithm is that the class has bounded mim-width.

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1 Introduction

A graph class \(G\) has bounded “width” if there exists a constant \(c\) such that every graph in \(G\) has width at most \(c\). We continue a recent study \cite{Vatshelle2012} on the boundedness of a particular width parameter, namely mim-width, for hereditary graph classes. Mim-width was introduced by Vatshelle \cite{Vatshelle2012}, who proved that it is more powerful than clique-width, and several well-known width parameters of equivalent strength, including boolean width, module-width, NLC-width and rank-width. That is, every graph class of bounded clique-width has bounded mim-width, but there exist graph classes of bounded mim-width that have unbounded clique-width. Hence, proving that a problem is polynomial-time solvable for graphs of bounded mim-width yields larger “islands of tractability” than doing this for clique-width (or any parameter equivalent to clique-width) in terms of the graphs that admit a polynomial-time algorithm. However, fewer problems have such an algorithm; see \cite{Horsfield2012,Horsfield2013,Horsfield2014,Horsfield2015,Horsfield2016,Horsfield2017,Horsfield2018} for some examples.

To define mim-width, we need the notion of a branch decomposition for a graph \(G\), which is a pair \((T, \delta)\), where \(T\) is a subcubic tree and \(\delta\) is a bijection from \(V(G)\) to the leaves of \(T\). We note that each edge \(e \in E(T)\) partitions the leaves of \(T\) into two classes, \(L_e\) and \(\overline{L}_e\), depending on which component of \(T - e\) they belong to. In this way, each edge \(e\) induces a partition \((A_e, \overline{A}_e)\) of \(V(G)\), where \(\delta(A_e) = L_e\) and \(\delta(\overline{A}_e) = \overline{L}_e\). Let \(G[A_e, \overline{A}_e]\) denote the bipartite subgraph of \(G\) induced by the edges with one end-vertex in \(A_e\) and the other in \(\overline{A}_e\). A matching \(F \subseteq E(G)\) of \(G\) is induced if there is no edge in \(G\) between vertices of different edges of \(F\). We let \(\text{cutmim}_G(A_e, \overline{A}_e)\) denote the size of a maximum induced matching in \(G[A_e, \overline{A}_e]\). The mim-width \(\text{mimw}_G(T, \delta)\) of \((T, \delta)\) is the maximum value of \(\text{cutmim}_G(A_e, \overline{A}_e)\) over all edges \(e \in E(T)\). The mim-width \(\text{mimw}(G)\) of \(G\) is the minimum value of \(\text{mimw}_G(T, \delta)\) over all branch decompositions \((T, \delta)\) for \(G\).
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Sæther and Vatshelle \cite{23} proved that computing mim-width is \textup{NP}-hard. In the same paper they showed that deciding if the mim-width is at most \(k\) is \textup{W[1]}-hard when parameterized by \(k\), and that there is no polynomial-time algorithm for approximating the mim-width of a graph to within a constant factor of the optimal, unless \textup{NP} = \textup{ZPP}. However, there are many graph classes \cite{5,7,17} for which mim-width is bounded and quickly computable. By the latter, we mean that the class admits a polynomial-time algorithm for computing a branch decomposition whose mim-width is bounded by a constant.

A class of graphs is \textit{hereditary} if it is closed under vertex deletion, or equivalently, can be characterized by a set of forbidden induced subgraphs. In particular, a graph \(G\) is \textit{\(H\)-free} for some graph \(H\) if \(G\) does not contain \(H\) as an induced subgraph, and \(G\) is \((H_1, H_2)\)-free for some graphs \(H_1\) and \(H_2\) if \(G\) is both \(H_1\)-free and \(H_2\)-free. It is not difficult to prove that a class of \(H\)-free graphs has bounded mim-width if and only if \(H\) is an induced subgraph of the 4-vertex path \(P_4\); see \cite{5}, in which a study into the boundedness of mim-width of \((H_1, H_2)\)-free graphs was initialized. We also refer to \cite{3} for an up-to-date summary and list of open cases. In this brief note, we focus on a specific family of \((H_1, H_2)\)-free graphs. The \textit{disjoint union} \(G + H\) of graphs \(G\) and \(H\) is the graph with vertex set \(V(G) \cup V(H)\) and edge set \(E(G) \cup E(H)\). The graph \(kG\) denotes the disjoint union of \(k\) copies of \(G\). The path and complete graph on \(n\) vertices are denoted by \(P_n\) and \(K_n\), respectively. We can now state our main result.

\noindent\textbf{Theorem 1.} For every \(s \geq 0\) and \(t \geq 1\), the mim-width of the class of \((K_1, sP_1 + P_5)\)-free graphs is bounded and quickly computable.

Apart from solving an infinite family of open cases, we believe this result is of interest as it sheds some light on a known result on graph colouring. A \textit{colouring} of a graph \(G = (V, E)\) is a mapping \(c : V \to \{1, 2, \ldots\}\) that assigns each vertex \(u \in V\) a \textit{colour} \(c(u)\) in such a way that \(c(u) \neq c(v)\) whenever \(u\) and \(v\) are adjacent. If \(1 \leq c(u) \leq k\) for each \(u \in V\), then \(c\) is also called a \(k\)-\textit{colouring} of \(G\). The \textit{Colouring} problem is to decide, on input a graph \(G\) and an integer \(k\), whether \(G\) has a \(k\)-colouring. For fixed \(k\) (that is, \(k\) is not part of the input), the problem is known as \(k\)-\textit{Colouring}.

It is well known that 3-\textit{Colouring} is \textup{NP}-complete \cite{22}, but the problem becomes polynomial-time solvable under certain input restrictions. This led to a large number of complexity results for \textit{Colouring} and \(k\)-\textit{Colouring} on special graph classes. The complexity of Colouring for \(H\)-free graphs has been settled \cite{20}, but there are still a number of open cases for \(k\)-\textit{Colouring} restricted to \(H\)-free graphs when \(H\) is a \textit{linear forest}, that is, a disjoint union of paths; see \cite{12} for a survey and \cite{8,18} for some later summaries. In particular, Hoàng et al. \cite{13} proved that for every integer \(k \geq 1\), \(k\)-\textit{Colouring} is polynomial-time solvable for \(P_5\)-free graphs. This result was generalized by Couturier et al. \cite{9}.

\noindent\textbf{Theorem 2 (\cite{9}).} For every \(k \geq 1\) and \(s \geq 0\), \(k\)-\textit{Colouring} is polynomial-time solvable for \((sP_1 + P_5)\)-free graphs.

We note that Theorem \cite{1} implies Theorem \cite{2}, as \(k\)-\textit{Colouring}, for every fixed integer \(k \geq 1\), is polynomial-time solvable for a graph class where mim-width is bounded and quickly computable, and we may assume that an instance of \(k\)-\textit{Colouring} is \(K_{k+1}\)-free, for otherwise it is clearly not \(k\)-colourable.

For an integer \(k \geq 1\), a \textit{k-list assignment} of a graph \(G = (V, E)\) is a function \(L\) that assigns each vertex \(u \in V\) a \textit{list} \(L(u) \subseteq \{1, 2, \ldots, k\}\) of \textit{admissible} colours for \(u\). A colouring \(c\) of \(G\) \textit{respects} \(L\) if \(c(u) \in L(u)\) for every \(u \in V\). For fixed \(k \geq 1\), the \textit{List} \(k\)-\textit{Colouring} problem is to decide if a graph \(G\) has a colouring that respects a \(k\)-list assignment \(L\). Theorem \cite{2} holds even for \textit{List} \(k\)-\textit{Colouring}. It is known that \textit{List} \(k\)-\textit{Colouring} is polynomial-time solvable for graph classes of bounded clique-width \cite{19}. Kwon \cite{21} observed the following. Given an instance \((G, L)\) of \textit{List} \(k\)-\textit{Colouring}, one can construct an equivalent instance \(G'\) of \(k\)-\textit{Colouring} by adding a clique on new vertices \(u_1, \ldots, u_k\) to \(G\) and adding an edge.

\footnote{This is not true for the \textit{Colouring} problem, where \(k\) is part of the input; indeed \textit{Colouring} is \textup{NP}-complete for circular-arc graphs \cite{11}, despite the fact that this is a class for which mim-width is bounded and quickly computable \cite{2}.}
between $u_i$ and $v \in V(G)$ if and only if $i \notin L(u)$. As $\text{minw}(G') \leq \text{minw}(G) + k$, this means that $\text{List}$ $k$-$\text{Colouring}$ is polynomial-time solvable even for graph classes of bounded mim-width. Hence, by the same arguments as before, Theorem 1 also serves as an alternative proof for the aforementioned $\text{List}$ $k$-$\text{Colouring}$ generalization of Theorem 2.

Let $\omega(G)$ denote the size of a largest clique in a graph $G$. Note that $G$ is $Kr$-free if and only if $\omega(G) \leq t - 1$. Very recently, Chudnovsky et al. independently gave for the class of $P_3$-free graphs, an $n^{O(\omega(G))}$-time algorithm for Max Partial $H$-$\text{Colouring}$, which is equivalent to Independent Set if $H = P_1$ and to Odd Cycle Transversal if $H = P_2$. They noted that Max Partial $H$-$\text{Colouring}$ is polynomial-time solvable for graph classes whose mim-width is bounded and quickly computable. Hence, Theorem 1 also provides a polynomial-time algorithm for Max Partial $H$-$\text{Colouring}$ restricted to $(K_t, sP_1 + P_5)$-free graphs for every $s \geq 0$ and $t \geq 1$, or equivalently, $(sP_1 + P_5)$-free graphs $G$ with $\omega(G) \leq t - 1$. However, the running time of this algorithm is worse than $n^{O(\omega(G))}$ (see [6] for details).

The rest of the paper is devoted to the proof of Theorem 1.

2 The Proof of Theorem 1

For a graph $G = (V, E)$, a set $D \subseteq V$ is dominating if every vertex of $G$ belongs to $D$ or is adjacent to a vertex of $D$: we also say that $G[D]$ is dominating. We need the following two known results on $(sP_1 + P_5)$-free graphs, which distinguish the cases $s = 0$ and $s > 0$.

- **Lemma 3 ([1]).** Every connected $P_5$-free graph $G$ has a dominating $P_3$ or a dominating complete graph.
- **Lemma 4 ([9]).** For an integer $s \geq 1$, let $G$ be an $(sP_1 + P_5)$-free graph. If $G$ contains an induced $P_5$, then $G$ contains a dominating induced $rP_1 + P_5$ for some $r < s$.

Let $G_1$ and $G_2$ be two vertex-disjoint graphs. We identify two vertices $u \in V(G_1)$ and $v \in V(G_2)$ by adding an edge between them, which we then contract.

We will need a new result on mim-width, which might be of independent interest. It shows that, given a partition of the vertex set of a graph $G$, we can bound the mim-width of $G$ in terms of the mim-width of the graphs induced by each part and the mim-width between any two of the parts.

- **Lemma 5.** Let $G$ be a graph, and let $(X_1, \ldots, X_p)$ be a partition of $V(G)$ such that $\text{cutmim}_G(X_i, X_j) \leq c$ for all distinct $i, j \in \{1, \ldots, p\}$, and $p \geq 2$. Then

\[
\text{minw}(G) \leq \max \left\{\left. c \left(\frac{p}{2}\right)^2, \max_{i \in \{1, \ldots, p\}} \{\text{minw}(G[X_i])\} + c(p - 1) \right\} \right. .
\]

Moreover, if $(T_i, \delta_i)$ is a branch decomposition of $G[X_i]$ for each $i$, then we can construct, in $O(1)$ time, a branch decomposition $(T, \delta)$ of $G$ with $\text{minw}(T, \delta) \leq \max\{c[\left(\frac{p}{2}\right)^2], \max_{i \in \{1, \ldots, p\}} \{\text{minw}(T_i, \delta_i)\} + c(p - 1)\}$.

**Proof.** We construct a branch decomposition $(T, \delta)$ of $G$ with the desired mim-width, as follows. Let $T_0$ be an arbitrary subcubic tree having $p$ leaves $\ell_1, \ldots, \ell_p$. For each $i \in \{1, \ldots, p\}$, we choose an arbitrary leaf vertex $v_i$ of $T_i$, we identify $v_i$ with $\ell_i$ calling the resulting vertex $\ell_i$, and we create a new pendant edge incident to $\ell_i$, where the new leaf vertex adjacent to $\ell_i$ is called $v_i$. Then $T$ is a subcubic tree whose set of leaves is the disjoint union of the leaves of $T_i$ for each $i \in \{1, \ldots, p\}$. See Figure 1 for example. For a leaf $v$ of $T$, we set $\delta(v) = \delta_i(v)$ where $v$ is a leaf of $T_i$. Now $(T, \delta)$ is a branch decomposition of $G$, and clearly this branch decomposition can be constructed in $O(1)$ time. It remains to prove the upper bound for $\text{minw}(T, \delta)$.

Consider $e \in E(T)$ and the partition $(A_e, \overline{A_e})$ of $V(G)$. If $e \in E(T_0)$, then $A_e = \bigcup_{j \in J} X_j$ for some $J \subseteq \{1, \ldots, p\}$. If $e \in E(T_i)$ for some $i \in \{1, \ldots, p\}$, then either $A_e$ or $\overline{A_e}$ is properly contained in $X_i$. The only other possibility is that $e$ is one of the newly created pendant edges, in which case either $A_e$ or $\overline{A_e}$ has size 1.
First suppose \( e \in E(T_b) \), so \( A_e = \bigcup_{j \in J} X_j \) for some \( J \subseteq \{1, \ldots, p\} \). We claim that cutmim\(_G(A_e, \overline{A_e}) \leq \left\lceil \left( \frac{e}{4} \right)^2 \right\rceil \). Let \( M \) be a maximum-sized induced matching in \( G[A_e, \overline{A_e}] \). Let \( K = \{1, \ldots, p\} \setminus J \). For each \( j \in J \) and \( k \in K \), there are at most \( e \) edges of \( M \) with one end in \( X_j \) and the other end in \( X_k \), since cutmim\(_G(X_j, X_k) \leq e \). Thus cutmim\(_G(A_e, \overline{A_e}) \leq \left\lceil \frac{|J||K|}{5} \right\rceil = e \left\lceil \left( \frac{e}{4} \right)^2 \right\rceil \), the claim follows.

Now suppose \( e \in E(T_i) \), for some \( i \in \{1, \ldots, p\} \), so, without loss of generality, \( A_e \) is properly contained in \( X_i \). We claim that cutmim\(_G(A_e, \overline{A_e}) \leq \text{mmw}(G[X_i]) + c(p - 1) \). Consider a maximum-sized induced matching \( M \) in \( G[A_e, \overline{A_e}] \). As \( A_e \subseteq X_i \), all the edges of \( M \) have one end in \( X_i \). For each \( j \in \{1, \ldots, p\} \) with \( j \neq i \), there are at most \( e \) edges of \( M \) with one end in \( X_j \), since cutmim\(_G(X_i, X_j) \leq e \). Since there are at most mmw\(_G(X_i)\) edges of \( M \) with both ends in \( X_i \), we deduce that cutmim\(_G(A_e, \overline{A_e}) \leq \text{mmw}(G[X_i]) + c(p - 1) \), as claimed. The lemma follows.

We are now ready to prove Theorem \( 1 \) which we restate below.

**Theorem 1 (restated).** For every \( s \geq 0 \) and \( t \geq 1 \), the mim-width of the class of \((K_t, sP_1 + P_3)\)-free graphs is bounded and quickly computable.

**Proof.** Let \( G = (V, E) \) be a \((K_t, sP_1 + P_3)\)-free graph for some \( s \geq 0 \) and \( t \geq 1 \). We may assume without loss of generality that \( G \) is connected. We use induction on \( t \). If \( t = 1 \), then the statement of the theorem holds trivially.

Suppose that \( t \geq 2 \). First suppose that \( G \) is \( P_5 \)-free. Then, by Lemma 3 \( G \) has a dominating set of size at most \( \max\{3, t - 1\} \). Now suppose that \( G \) is not \( P_5 \)-free. Then, by Lemma 4 \( G \) has a dominating set \( D \) of size at most \( s + 4 \). Hence, in both cases, \( G \) has a dominating set \( D \) of constant size, as \( s \) and \( t \) are fixed; we let \( p = |D| \). Moreover, we can find \( D \) in polynomial time by brute force.

We will partition the vertex set of \( G \) with respect to \( D \) in the same way as in \( 9 \) \( 13 \). That is, we first assign an arbitrary ordering \( d_1, \ldots, d_p \) on the vertices of \( D \). We then let \( X_1 \) be the set of vertices in \( V \setminus D \) adjacent to \( d_1 \), and for \( i = 2, \ldots, p \), let \( X_i \) be the set of vertices in \( V \setminus D \) adjacent to \( d_i \), but non-adjacent to any \( d_h \) with \( h \leq i - 1 \). Note that \( D \) and the sets \( X_1, \ldots, X_p \) partition \( V \) (but some of the sets \( X_i \) might be empty).

As \( \{d_i\} \) dominates \( X_i \) for every \( i \in \{1, \ldots, p\} \), each \( X_i \) induces a \((K_{t-1}, sP_1 + P_3)\)-free subgraph of \( G \), which we denote by \( G_i \). By the induction hypothesis, the mim-width of \( G_i \) is bounded and quickly computable for every \( i \in \{1, \ldots, p\} \).

Consider two sets \( X_i \) and \( X_j \) with \( i < j \). By Ramsey’s Theorem, for every two positive integers \( p \) and \( q \), there exists an integer \( R(p, q) \) such that if \( G \) is a graph on at least \( R(p, q) \) vertices, then \( G \) has a clique of size \( p \) or an independent set of size \( q \). We claim that cutmim\(_G(X_i, X_j) < c = R(t - 1, R(t - 1, s + 2)) \). Towards a contradiction, suppose that cutmim\(_G(X_i, X_j) \geq c \). Let \( A = \{a_1, a_2, \ldots, a_c\} \subseteq X_i \) and \( B = \{b_1, b_2, \ldots, b_c\} \subseteq X_j \) such that \( a_ib_j \) is a distinct edge of a corresponding induced matching. Then, as
Assume without loss of generality that one leaf of $G[N]$ contains an independent set $A'$ of size $c' = R(t−1, s+2)$. Let $B' = \{b_1, \ldots, b_s\}$. As $G[X_j]$ is $K_{t−1}$-free, the subgraph of $G$ induced by $B'$ contains an independent set $B''$ of size $s + 2$. Assume without loss of generality that $B'' = \{b_1, \ldots, b_{s+2}\}$. Then $\{b_1, a_1, d, a_2, b_2\} \cup \{b_3, \ldots, b_{s+2}\}$ induces an $sP_1 + P_3$, a contradiction. We deduce that $cutmim_G(X_i, X_j) < c$.

We now apply Lemma 5 to find that the mim-width of $G − D$ is bounded and quickly computable. Let $(T, δ)$ be a branch decomposition of $G − D$ having mim-width $k$. As $|D| = p$, we can readily extend $(T, δ)$ to a branch decomposition $(T^*, δ^*)$ of mim-width at most $k + p$, where $T^*$ is obtained by identifying one leaf of $T$ with a leaf of an arbitrary subcubic tree having $|D| + 2$ leaves.

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