QUANTUM GEOMETRY AND QUIVER GAUGE THEORIES

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Abstract. We study macroscopically two dimensional $\mathcal{N} = (2, 2)$ supersymmetric gauge theories constructed by compactifying the quiver gauge theories with eight supercharges on a product $\mathbb{T}^d \times \mathbb{R}^2$ of a $d$-dimensional torus and a two dimensional cigar with $\Omega$-deformation. We compute the universal part of the effective twisted superpotential. In doing so we establish the correspondence between the gauge theories, quantization of the moduli spaces of instantons on $\mathbb{R}^{2-d} \times \mathbb{T}^{2+d}$ and singular monopoles on $\mathbb{R}^{2-d} \times \mathbb{T}^{1+d}$, for $d = 0, 1, 2$, and the Yangian $\mathcal{Y}_\epsilon(g_\Gamma)$, quantum affine algebra $\mathcal{U}_{\text{aff}}^q(g_\Gamma)$, or the quantum elliptic algebra $\mathcal{U}_{q,p}^\text{ell}(g_\Gamma)$ associated to Kac-Moody algebra $g_\Gamma$ for quiver $\Gamma$.

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### 1. Introduction

The Bethe/gauge correspondence between the supersymmetric gauge theories and quantum integrable systems is a subject of research spanning over a decade \([1–9]\) and even longer in the context of topological gauge theories \([10–13]\). It can also be viewed as a correspondence between the supersymmetric gauge theories and representation theory of infinite dimensional algebras typical for the two dimensional conformal field theories or integrable deformations thereof. This relation, dubbed the BPS/CFT correspondence in \([14]\), following the prior work in \([15–21]\), became a subject of intense development after the seminal work \([22]\) where the instanton partition functions of the \(S\)-class \(\mathcal{N} = 2\) gauge theories in the \(\Omega\)-background \([19]\) were conjectured to be the Liouville (and, more generally, the \(ADE\) Toda) conformal blocks.

#### 1.1. Gauge theories

Recently \([23]\) the Seiberg-Witten geometry, the curves \([24, 25]\) and the integrable systems associated to all \(\mathcal{N} = 2\) (affine) \(ADE\) quiver gauge theories, whose gauge group \(G_v = \prod SU(v_i)\) is a product of unitary groups, were systematically found. The direct application of quantum field theory techniques leads to the equivariant integration over the instanton moduli on \(\mathbb{R}^4\) \([1, 19, 21, 26, 27]\). The above mentioned geometry emerges in the flat space limit \(\epsilon_1, \epsilon_2 \to 0\) of the \(\Omega\)-deformed gauge theory \([19]\).

#### 1.1.1. Prepotentials and special geometry

The prepotential \(\mathcal{F}(a, m; q)\) of the effective low-energy theory relates to the gauge theory partition \(\mathcal{Z}\)-function \([19]\) in the flat space limit as follows:

\[
\mathcal{F}(a, m; q) = - \lim_{\epsilon_1, \epsilon_2 \to 0} \epsilon_1 \epsilon_2 \log \mathcal{Z}(a, m; q; \epsilon_1, \epsilon_2)
\]

where \(q\) here denotes the set of gauge coupling constants of the theory, \(m\) the set of masses of the hypermultiplets fields, and \(a\) is the set of the flat special coordinates.
on the moduli space of vacua $\mathcal{M}$ on the Coulomb branch of the theory. The latter is identified with the asymptotics of the scalar fields of $N = 2$ vector multiplets at infinity of the Euclidean space-time. The geometry of $\mathcal{M}$ and $\mathcal{F}(a, m; q)$ are captured by a complex analogue of a classical integrable system, an algebraic integrable system. The phase space $\mathfrak{P}$ of that system admits a partial compactification which is a complex symplectic manifold with the holomorphic symplectic form and projection to $\mathcal{M}$ with Lagrangian fibers, which are, generically, abelian varieties.

The gauge theories studied in [23] and in this paper are labeled by the classes of representations $\Gamma(v, w, \tilde{w})$ of quivers which correspond to the Dynkin diagrams $\Gamma$ of simply laced finite-dimensional or affine Kac-Moody algebras $\mathfrak{g}_\Gamma$. The dimension vectors $v = (v_i), w = (w_i), \tilde{w} = (\tilde{w}_i)$ encode the number of colors, the number of fundamental flavors and the number of anti-fundamental flavors charged with respect to the gauge group factor $U(v_i)$, respectively.

For affine Kac-Moody $\mathfrak{g}_\Gamma$, let $\mathfrak{g}_{\Gamma'}$ be the underlying finite-dimensional simple Lie algebra. The corresponding phase spaces $\mathfrak{P}$ turn out to be the moduli spaces of solutions to the BPS equations, $G_{\Gamma}$-monopoles (for finite quivers) on $\mathbb{R}^d \times \mathbb{T}^{d-d}$, or $G_{\Gamma'}$-instantons (for affine quivers) on $\mathbb{R}^d \times \mathbb{T}^{d-d}$. The gauge group $G_{\Gamma}$ ($G_{\Gamma'}$) for these BPS equations is not the original gauge group $G_v$ of the quiver theory. Yet the solution of the quantum $G_v$-gauge theory is captured by the moduli space of particular classical solutions to the $G_{\Gamma'}$-gauge theory.

1.1.2. Departure from the Seiberg-Witten theory. In this paper, we extend the results of [23] in two ways.

First of all, we study the effective low-energy theory of the two dimensional $\mathcal{N} = (2, 2)$ supersymmetric theory which is obtained from the original four dimensional $\mathcal{N} = 2$ theory by $\Omega$-deformation, affecting two out of four space-time dimensions. Secondly, we focus more on the theories in five dimensions, compactified on a circle. The four dimensional case is obtained by sending the radius of the circle to zero.

1.1.3. Effective twisted superpotential. In terms of the general $\Omega$-backgrounds [19], we send to zero only one equivariant parameter, say $\epsilon_2 = 0$, while keeping the other parameter $\epsilon_1 = \epsilon$ finite. Thus we work in the limit discussed in [5] and study the universal part of the effective twisted superpotential

$$W(a, m; q, \epsilon) = - \lim_{\epsilon_2 \to 0} \epsilon_2 \log Z(a, m; q; \epsilon_1 = \epsilon, \epsilon_2),$$

identified in the Refs. [4], [5], [6], [7].

Recall that the twisted superpotential is a specific $F$-term in the Lagrangian of a $\mathcal{N} = (2, 2)$ supersymmetric theory in two dimensions. A four dimensional gauge theory can be viewed as a two dimensional theory with an infinite number of degrees of freedom. The two dimensional $\Omega$-deformation of the type we study in this paper breaks the four dimensional $\mathcal{N} = 2$ supersymmetry down to two dimensional $\mathcal{N} = 2$ supersymmetry. However, in order to view the resulting theory as the two dimensional theory with the well-defined effective Lagrangian we need to specify the boundary conditions at infinity. One may view the choice of the boundary conditions at infinity as a choice of a three dimensional supersymmetric theory compactified on a circle,
coupled to the original four dimensional theory on the product of a cigar-like two-dimensional geometry and the two dimensional Minkowski world-sheet of the resulting theory. The three dimensional theory at infinity is compactified on a circle because of the asymptotics of the cigar-like geometry, which looks like $\mathbb{R}^1 \times S^1$ at infinity. The contribution $\mathcal{W}^\infty(a, m; \epsilon)$ of the three dimensional theory at infinity of the cigar is purely perturbative and is given by a sum of dilogarithm functions

$$\mathcal{W}^\infty(a, m; \epsilon) \sim \epsilon \sum \text{Li}_2 \left( e^{\frac{\text{linear}(a, m)}{\epsilon}} \right)$$

According to the Refs. [8, 9], the twisted superpotential $\mathcal{W}^{\text{eff}}(a, m; q, \epsilon)$ of the effective two-dimensional theory is the sum:

$$\mathcal{W}^{\text{eff}}(a, m; q, \epsilon) = \mathcal{W}(a, m; q, \epsilon) + \mathcal{W}^\infty(a, m; \epsilon) \quad (1.3)$$

which can be identified with the Yang-Yang function of some quantum integrable system [4], [5], [6], [7]. The $\Omega$-deformation parameter $\epsilon$ plays the role of a Planck constant. It is easy to see by comparing the Eqs. (1.2) and (1.1) that in the limit $\epsilon \to 0$, the superpotential $\mathcal{W}(a, m; q, \epsilon)$ behaves as

$$\frac{1}{\epsilon} \mathcal{F}(a, m; q)$$

and the quantum integrable system approaches the classical one with the phase space $\mathfrak{P}$, and the prepotential $\mathcal{F}(a, m; q)$ describing the special geometry of its base $\mathfrak{M}$.

1.1.4. Bethe equations and vacua. More precisely, the Bethe equations of that quantum integrable system read as follows:

$$\exp \left[ \frac{\partial \mathcal{W}^{\text{eff}}(a, m; q, \epsilon)}{\partial a_i} \right] = 1, \quad i = 1, \ldots, \dim \mathfrak{M} \quad (1.4)$$

The Eq. (1.4) describes in some specific Darboux coordinates, e.g. [3], the intersection of two Lagrangian subvarieties, cf. [8], one with the generating function $\mathcal{W}(a)$ and another, with the generating function $-\mathcal{W}^\infty(a)$.

1.1.5. Focus of the paper. In this paper we explore a particular formalism for the systematic computation of the universal part of the superpotential, $\mathcal{W}(a, m; q, \epsilon)$. We shall not discuss the choices of boundary conditions and representations of the non-commutative deformations of the algebra of functions on $\mathfrak{M}$.

Another aspect in which this paper develops the Ref. [23] is its focus on the five dimensional version of the theory. The $\mathcal{N} = 2$ gauge theory in four dimensions can be canonically lifted to the $\mathcal{N} = 1$ theory in five dimensions. The five dimensional $\mathcal{N} = 1$ supersymmetric gauge theory compactified on a circle $S^1_\ell$ of circumference $\ell$ can be viewed [28] as a particular deformation of the four dimensional $\mathcal{N} = 2$ theory. One studies the theory with twisted boundary conditions, such that in going around the circle the remaining four dimensional Euclidean space-time is rotated in the two orthogonal two-planes, by the angles $\ell \epsilon_1$ and $\ell \epsilon_2$, respectively. For

$$q_1 = e^{i \ell \epsilon_1}, \quad q_2 = e^{i \ell \epsilon_2}, \quad q = q_1 q_2 \quad (1.5)$$
we are interested in the limit
\[ q_2 \to 1, \quad q = q_1 \] (1.6)
and for convergence issues we shall assume in this paper
\[ |q| < 1 \] (1.7)
For the quiver which is a Dynkin graph of (affine) ADE Lie algebra \( \mathfrak{g}_\Gamma \), we find that the \( \epsilon \)-deformed limit shape partition profile equations of [21] can be mapped to the Bethe equations of a formal trigonometric XXZ \( \mathfrak{g}_\Gamma \)-spin chain. Despite the similarity to the problems (for finite \( A_r \) quivers in four dimensions) studied in [29, 31] and also in [2, 3, 5, 32, 46], there are important conceptual differences which we shall explain below. In fact, the universal part of the gauge theory twisted superpotential does not correspond to any spin chain. It corresponds to a commutative subalgebra of noncommutative associative algebra, which is the quantum affine algebra \( U_q^{\text{aff}}(\mathfrak{g}_\Gamma) \) associated to \( \mathfrak{g}_\Gamma \). One can then study the spectrum of this subalgebra in any representation of \( U_q^{\text{aff}}(\mathfrak{g}_\Gamma) \), which would lead to the ordinary Bethe equations.

1.1.6. Chiral ring generating functions. In gauge theory, the functions \( Y_i(x) \), where \( i \in I_\Gamma \) runs through the set \( I_\Gamma \) of nodes of \( \Gamma \), encode the expectation values of chiral ring observables. For a point \( u \in \mathcal{M} \) on the Coulomb branch moduli space \( \mathcal{M} \) and a simple gauge group factor \( U(v_i) \) we define the generating function, in the four and five dimensional versions given by
\[ 4d : \quad Y_i(x - \frac{x}{\ell}; u) = \exp(\langle \log \det(x - \phi_i) \rangle_u) \]
\[ 5d : \quad Y_i^+(\xi q^{-\frac{x}{\ell}}; u) = \exp(\langle \log \det(1 - e^{i\epsilon(\phi_i-x)}) \rangle_u), \quad \xi = e^{i\epsilon x} \] (1.8)
respectively, with \( \phi_i \) the complex adjoint scalar in the \( SU(v_i) \) gauge vector multiplet, \( \langle \rangle_u \) denotes the expectation value at the point \( u \in \mathcal{M} \) of the moduli space of vacua, and \( \ell \) is the circumference of the circle \( S_\ell^1 \). We give the geometric definition below, cf. Eqs. (2.36), (2.43).

1.2. Quantum groups. In this note we will be dealing with quantum algebras
\[ U_\epsilon(\mathfrak{g}_\Gamma(C_x)) : \quad Y_\epsilon(\mathfrak{g}_\Gamma), \quad U_q^{\text{aff}}(\mathfrak{g}_\Gamma), \quad U_q^{\text{aff}}(\mathfrak{g}_\Gamma) \] (1.9)
associated to the quiver theories on \( \mathbb{R}^4 \), on \( \mathbb{R}^4 \times S_\ell^1 \) and on \( \mathbb{R}^4 \times T^2_\ell - \ell/\tau_p \), respectively. Here \( \ell \) is the circumference of the circle \( S_\ell^1 \) on which we compactify the 5d gauge theory, and \( (\ell, -\ell/\tau_p) \) are the periods of the torus \( T^2_\ell - \ell/\tau_p \) on which we compactify the six dimensional gauge theory. The parameters \( \epsilon \), or \( q \)
\[ q = e^{i\epsilon}, \] (1.10)
are the quantization parameters (Planck constant) of quantum algebras (1.9).

The algebras (1.9) are defined using second Drinfeld realization [47], i.e. quantum \( \mathfrak{g}_\Gamma \) currents \( (\psi^\pm_1(x), e^\pm(x))_{i \in I_\Gamma} \), see [47, 2.1]. The domain \( C_x \) of the additive spectral parameter \( x \in C_x \) in quantum \( \mathfrak{g}_\Gamma \)-currents, for the four, five and the six dimensional case, is
- a plane \( C_x = \mathbb{R}^2 \cong \mathbb{C} \) for \( \mathfrak{g}_\Gamma \)-Yangian \( Y_\epsilon(\mathfrak{g}_\Gamma) \),
- a cylinder \( C_x = \mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z}) \cong \mathbb{R}^1 \times S^1 \cong \mathbb{C}^\times \) for \( \mathfrak{g}_\Gamma \)-quantum affinization \( U_q^{\text{aff}}(\mathfrak{g}_\Gamma) \),
• a torus $C_x = \mathbb{T}^2 \cong \mathbb{R}^2 / (\frac{2\pi}{\tau}) (\mathbb{Z} + \tau_p \mathbb{Z}) \cong \mathbb{C}^\times / p^\mathbb{Z} = \mathcal{E}_p$ for $\mathfrak{g}_\Gamma$-quantum elliptic algebra $U_{q,p}^{\text{ell}}(\mathfrak{g}_\Gamma)$; where the parameter $p = e^{2\pi i \tau}$ is the multiplicative elliptic modulus

In the classical limit $\epsilon = 0$, quantum algebras (1.9) reduce to the (universal enveloping of) classical current algebra $U_{\mathfrak{g}_\Gamma}(C_x)$ on the spectral domain $C_x$, and the results of this note reduce to the results of [23]. Namely, in [23], to each point on the Coulomb moduli space $\mathfrak{M}$ of the gauge theory, one associated a current $h(x) \in U_{\mathfrak{g}_\Gamma}(C_x)$, defined in terms of the functions $Y_i(x)$ and gauge theory data. It is shown in [23] that $h(x)$ satisfies the equation

$$ \chi_i(h(x)) = T_i(x), \quad i \in I_{\Gamma} \tag{1.11} $$

where $\chi_i$ is a (twisted) character of the $i$-th fundamental $\mathfrak{g}_\Gamma$-module, and $T_i(x)$ is a polynomial of degree $v_i$.

1.3. **Main result.** In this paper we show, the result (1.11) of [23] has natural quantum generalization 4: for $\epsilon \neq 0$, the algebra $U_{\mathfrak{g}_\Gamma}(C_x)$ is replaced by its quantum version $U_{q,\epsilon} \mathfrak{g}_\Gamma(C_x)$. We construct a quantum current $h(x) \in U_{q,\epsilon} \mathfrak{g}_\Gamma(C_x)$, defined in terms of the gauge theory data and the generating functions $Y_i(x)$ such that the character equation (1.11) still holds, but $\chi_i$ are replaced by the characters for quantum algebras known as $q$-characters [48, 49].

1.4. **Conventions.**

1.4.1. **Quantum parameter.** In this paper, the $q$-numbers are defined by

$$ [n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \tag{1.12} $$

Our conventions for the quantum parameter $q = e^{\epsilon \ell}$ of the quantum groups differ from [49] or [50], so that our $q$ is the $q^2$ of [49] or [50]. Our conventions for $q$ agree with e.g. [51, 54] so that the $SU(n)$ three dimensional Chern-Simons theory at the level $k$ relates to the quantum group $U_q(sl_n)$ with parameter

$$ q = \exp \left( \frac{2\pi i}{h^\vee + k} \right), \quad h^\vee_{sl_n} = n \tag{1.13} $$

and the invariant of the unknot $\bigcirc$ in fundamental representation $n$ is $\langle W_n(\bigcirc) \rangle = [n]_q$ in convention (1.12). The three dimensional Chern-Simons theory has been related to $\mathcal{N} = 4$ SYM on a four-manifold with a boundary in [55], and the parameter $q$ in the normalization (1.13) naturally counts instantons in the $\mathcal{N} = 4$ SYM and enjoys Langlands duality, or modular transformation: for a simply-laced gauge group for $q = e^{2\pi \epsilon} = e^{2\pi i \tau}$, the Langlands dual $\tau^L = -1/\tau$. It would be interesting to explore the modular properties of the $\epsilon$-'modular' parameter

$$ \tau^L = \frac{\epsilon}{2\pi} \tag{1.14} $$

in the context of this paper, dealing with quantum algebras (1.9), in particular in view of the applications to S-duality [56, 57]. We leave this task for the future.
1.4.2. Quantum algebras. There is a variety of conventions for naming the quantum algebras $U_\varepsilon(C_x)$ (1.9) in the literature. In this note we follow the naming scheme in which the quiver $\Gamma = A_1 = \bullet$ that has one node and no edges, associated to the Lie algebra $g_\Gamma = sl_2$, corresponds to

- 4d: the $sl_2$ Yangian $Y_\varepsilon(sl_2)$; XXX $sl_2$-spinchain for 4d
- 5d: the $sl_2$ quantum affine algebra $U_q^{aff}(sl_2) \simeq U_q(sl_2)$; XXZ $sl_2$-spinchain
- 6d: the $sl_2$ quantum elliptic algebra $U_{q,p}^{ell}(sl_2) \simeq E^{\varepsilon,\tau}(sl_2)$; XYZ $sl_2$-spinchain

It is known that the Yangian $Y_\varepsilon(g_\Gamma)$ or the quantum affinization $U_q^{aff}(g_\Gamma)$ can be defined using second Drinfeld current realization [47], see A.2.1, for any generalized symmetrizable affine Kac-Moody algebra $g_\Gamma$. In particular, if $g_\Gamma$ is an affine Kac-Moody Lie algebra $g_\Gamma = \widehat{g}$ where $g$ is a finite dimensional simple Lie algebra, then $U_q^{aff}(g_\Gamma)$

$$U_q^{aff}(g_\Gamma) = U_q^{aff}(\widehat{g}) = U_q(\widehat{\widehat{g}})$$ (1.15)

is often called in the literature $g$ quantum toroidal algebra because of double loop $\widehat{\widehat{g}}$ and $Y_\varepsilon(g_\Gamma) = Y_\varepsilon(\widehat{g})$ is called affine Yangian.

In this note we will be using consistent naming convention

- $g_\Gamma$-Yangian for $Y_\varepsilon(g_\Gamma)$,  
- $g_\Gamma$-quantum affine algebra for $U_q^{aff}(g_\Gamma)$,  
- $g_\Gamma$-quantum elliptic algebra for $U_{q,p}^{ell}(g_\Gamma)$.

For example, the necklace quiver with $r + 1$ nodes, $g_\Gamma = \widehat{A}_r$, is associated in our conventions with $gl_{r+1}$ quantum affine algebra, or, equivalently, with $gl_{r+1}$ quantum toroidal algebra, see [58].

In terms of Schur-Weyl duality, quantum deformations of $g$-Weyl groups are called $g$-Hecke algebras. If $g$ is a finite-dimensional simple Lie group, then

- $U_q(g)$ is Schur-Weyl dual to $g$-Hecke algebra

Meanwhile, for the $C_x = C^x$ versions of quantum algebras (1.9), considered in this note, associated to $g_\Gamma = g$ or $g_\Gamma = \widehat{g}$ where $g$ is finite dimensional simple Lie algebra

- $U_q(g)$ is Schur-Weyl dual to $g$-affine Hecke algebra
- $U_q^{aff}(g)$ is Schur-Weyl dual to $g$-double affine Hecke algebra (DAHA), see review [59].

See section 5 for more details and literature review on representation theory of quantum algebras.

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2. The partition $Z$-function of quiver gauge theories

We start with the class of supersymmetric quiver gauge theories studied in \cite{[23]}, that is the $\mathcal{N} = 2$ supersymmetric four-dimensional gauge theories with a Lagrangian microscopic description, with a gauge group $G_v = \times_i SU(v_i)$ and matter hypermultiplets in the fundamental, the bifundamental or the adjoint representations of $G_v$, such that the theory is asymptotically free or conformal in the ultraviolet.

2.1. The Lagrangian data. The theory is encoded in the following equipped quiver $\Gamma(v, w, \tilde{w})$, cf. \cite{[23]}:

- An oriented graph $\Gamma$ whose vertices label gauge groups while edges label bifundamental multiplets.
- The set of vertices of $\Gamma$ is denoted by $I_\Gamma$. For our purposes it is sufficient to consider non-multiple edges, so the set of edges comes as $E_\Gamma \subset I_\Gamma \times I_\Gamma$ with the two natural projection maps $s$ and $t$ on the first and second factor, respectively, with $s(e)$ called the source of the edge $e$ and $t(e)$ called the target of the edge $e$.
- There are three $\mathbb{Z}$-valued functions on $I_\Gamma$, $v$, $w$ and $\tilde{w}$. For the vertex $i$ we denote $v_i = v(i) \in \mathbb{Z}_{>0}$, $w_i = w(i) \in \mathbb{Z}_{\geq 0}$ and $\tilde{w}_i = \tilde{w}(i) \in \mathbb{Z}_{\geq 0}$.
  
  Then $v_i$ is the number of colors in the factor $SU(n_i)$ of the gauge group associated to the $i$-th vertex, $w_i$ and $\tilde{w}_i$ is the number of fundamental and anti-fundamental multiplets charged under the gauge group factor $SU(v_i)$, respectively. Note that in the gauge theory on flat space-time there is no distinction between the fundamental and anti-fundamental hypermultiplets. The distinction comes upon the coupling to the $\Omega$-background.
- Each edge $e \in E_\Gamma$ describes a bifundamental hypermultiplet transforming in $(\overline{v}_{s(e)}, v_{t(e)})$ (or adjoint hypermultiplet if the edge is a loop with $s(e) = t(e)$). Let $I_{ij}$ be the incidence matrix of the graph $\gamma$ with

$$I_{ij} = \#(s^{-1}(i) \cap t^{-1}(j)) + \#(s^{-1}(j) \cap t^{-1}(i))$$

equal to the number of oriented edges connecting the vertices $i$ and $j$ with either orientation. The Cartan matrix of $\Gamma$ is $a_{ij} = 2\delta_{ij} - I_{ij}$. The integers $(v_i, w_i, \tilde{w}_i)$
should satisfy the inequality
\[ \sum_{j \in I_\Gamma} a_{ij} v_j \geq w_i + \tilde{w}_i \] (2.1)

which implies that the four dimensional theory is sensibly defined in the ultraviolet as a quantum field theory. This implies that \( a_{ij} \) is the finite or affine ADE Cartan matrix and \( \Gamma \) is the finite or affine ADE Dynkin diagram. If the inequality (2.1) is saturated, we say that \( R_{\Gamma(v,w,\tilde{w})} \) is in \textit{conformal class}
\[ \sum_{j \in I_\Gamma} a_{ij} v_j = w_i + \tilde{w}_i \leftrightarrow R_{\Gamma(v,w,\tilde{w})} \text{ is in conformal class} \] (2.2)

All affine type ADE quivers \( \Gamma \) saturate the inequality (2.1) at \( w_i = \tilde{w}_i = 0 \) and hence belong to conformal class automatically. The assigned dimensions \( v_i \) for affine ADE are uniquely determined by a single integer \( N \) and are given by \( v_i = N a_i \) where \( a_i \) are Dynkin marks on the vertices of \( \Gamma \); so that the imaginary root \( \delta = \sum_i a_i \alpha_i \) where \( \alpha_i \) are simple roots.

The set \( \kappa = (\kappa_i)_{i \in I_\Gamma} \), where \( \kappa_i \) is the level of the five-dimensional Chern-Simons term for the \( i \)-th gauge group factor \( SU(v_i) \)
\[ \kappa_i \int \text{tr}_{v_i} \left( \frac{1}{3} A \wedge dA \wedge dA + \frac{1}{4} A^3 \wedge dA + \frac{1}{5} A^5 \right) \] (2.3)

Each level \( \kappa_i \) is constrained by the inequality
\[ \kappa^-_i \leq \kappa_i \leq \kappa^+_i \] (2.4)
where
\[ \kappa^-_i = -v_i + \tilde{w}_i + \sum_{j \in (s^{-1}(i))} v_j \]
\[ \kappa^+_i = v_i - w_i - \sum_{j \in (t^{-1}(i))} v_j \] (2.5)

The range of allowed Chern-Simons couplings (2.4) is non-empty under the conditions (2.1) when the four dimensional theory with the same quiver is in the asymptotically free or conformal class.

- \( q_i = \exp(2\pi i \tau) \), \( 0 < |q_i| < 1 \) is the exponentiated complexified coupling constant of the \( SU(v_i) \) gauge factor for the vertex \( i \in I_\Gamma \). The coupling constants \( q \) encode the Yang-Mills coupling \( g_{YM} \) and the \( \theta \)-parameter for each gauge group factor:
\[ q_i = \exp(2\pi i \tau), \quad \tau = \frac{4\pi \tau}{g_{YM}^2} + \frac{\theta}{2\pi} \] (2.6)

- \( a_{i,a}, i \in I_\Gamma, a = 1, \ldots, v_i \) are the eigenvalues of the scalars in the vector multiplet serving as the special coordinates on the classical Coulomb moduli space
- \( m^f_{i,f} (\tilde{m}^f_{i,f}) \) are the masses of the fundamental (anti-fundamental) matter multiplets for the vertex \( i \) with \( f = 1, \ldots, v_i \) (\( \tilde{f} = 1, \ldots, \tilde{w}_i \))
- \( m^{bf}_e \) is the mass of the bifundamental matter multiplet at edge \( e \in E_\Gamma \)
We will also use exponentiated notations for the order parameters of the theory

\[ w_{1,a} = e^{i\ell a_{1,a}} \]
\[ q_1 = e^{i\ell_1}, \quad q_2 = e^{i\ell_2} \]
\[ \mu_e = e^{i\ell \rm_{m_e}}, \]
\[ \mu^f_{1,b} = e^{i\ell \rm_{m^f_{1,b}}}, \quad \tilde{\mu}^f_{1,j} = e^{i\ell \rm_{\tilde{m}^f_{1,j}}} \] (2.7)

Let \( R_{\Gamma(v,w,\tilde{w})} \) be the representation of quiver \( \Gamma(v,w,\tilde{w}) \), that is the representation of the gauge group \( G_v = \times_i SU(v_i) \) (2.8)

for the matter hypermultiplets in the theory

\[ R_{\Gamma(v,w,\tilde{w})} = \bigoplus_{e \in E_{\Gamma}} \text{Hom}(v_{s(e)}, v_{t(e)}) \oplus \bigoplus_{i \in I_{\Gamma}} \text{Hom}(\tilde{w}_i, v_i) \oplus \bigoplus_{i \in I_{\Gamma}} \text{Hom}(v_i, \tilde{w}_i), \] (2.9)

where slightly abusing notations by \( v_i \) we denote complex vector space of dimension \( v_i \), etc. It may be useful to think about \( w_i \) fundamental matter multiplets for \( SU(v_i) \) as a bifundamental hypermultiplet with an auxiliary frozen gauge group \( SU(w_i) \), and the role of masses \( m^f_{1,i} \) is played by the frozen scalars of the \( SU(w_i) \) vector multiplet.

Equivalently,

\[ R_{\Gamma(v,w,\tilde{w})} = \bigoplus_{e \in E_{\Gamma}} \text{Hom}(v_{s(e)}, v_{t(e)}) \oplus \bigoplus_{i \in I_{\Gamma}} \text{Hom}(w_i, v_i) \oplus \bigoplus_{i \in I_{\Gamma}} \text{Hom}(v_i, \tilde{w}_i) \] (2.10)

The hypermultiplet space \( R_{\Gamma(v,w,\tilde{w})} \) is naturally acted by flavor symmetry group for the fundamental fields \( G_{w,\tilde{w}} = \times_i U(w_i) \times U(\tilde{w}_i) \) (2.11)

and bifundamentals \( G_{\text{edge}} = \times_{e \in E_{\Gamma}} U(1) \) (2.12)

The total flavor symmetry group is \( G_f = G_{\text{edge}} \times G_{w,\tilde{w}} \) (2.13)

The theories corresponding to two equipped quivers differing by the orientation of some arrows (or the choice of the fundamental/anti-fundamental multiplets) are equivalent after certain adjustments of the couplings \( \kappa_i \) and \( q_i \).

The rationale for the equation (2.11) is to require the non-positive beta-function

\[ \beta_i \leq 0 \]

for the four dimensional running of the i-th gauge coupling constant:

\[ \Lambda_{uv} \frac{d}{d\Lambda_{uv}} \tau_i = \beta_i = w_i + \tilde{w}_i - \sum_j a_{ij} v_j. \]

The actual reason for this requirement is not so much that the gauge theory is well-defined in the ultraviolet, since we are going to study the five dimensional theory whose definition requires some completion at high energies. The class of theories which can be sensibly completed at high energy by embedding in some brane configuration in string theory or by geometrical engineering in M-theory or by compactification of the
(0, 2)-theory from six dimensions seems to be much larger. However, there are reasons to suspect that once we go beyond the realm of the theories which are sensible quantum field theories in four dimensions, the microscopic physics of M-theory will not decouple. For example, the instanton partition function defined entirely in terms of the gauge theory degrees of freedom will diverge, or will cease to be gauge invariant.

2.1.1. Six dimensions and anomalies. Formally one can define and compute the partition $Z$-function even for the six dimensional theory compactified on the two-torus $\mathbb{T}_{\ell/\tau}^2$, as the equivariant elliptic genus of the four dimensional $G_{v}$ instanton moduli spaces [68], [69]. The Coulomb parameters $a_{i}$ of the six dimensional theory compactified on $\mathbb{T}_{\ell/\tau}^2$ parametrize holomorphic $G_{v,\mathbb{C}}$-bundles on $\mathbb{T}_{\ell/\tau}^2$. The latter is a quotient of the abelian variety by the action of finite group (the Weyl group of $G_{v}$). One has to check that the partition function $Z$ is invariant with respect to large gauge transformations on $\mathbb{T}_{\ell/\tau}^2$. Indeed, we find that if $\beta_{i} = 0$ then $Z$ is invariant, provided the coupling constants $q_{i}$ of $SU(v_{i})$ transform in a suitable way under such large gauge transformations. This has the following explanation. The six dimensional $\mathcal{N} = (1, 0)$ theory for affine ADE quiver $\Gamma$ with gauge group $G_{v} = \times SU(v_{i})$ where $v_{i} = N q_{i}$ and $a_{i}$ are Dynkin marks can be realized on the stack of $N$ D5 branes in IIB on $\mathbb{C}^2/\Gamma$, where $\Gamma \subset SU(2)$ is the finite subgroup associated to affine ADE quiver $\Gamma$ by McKay correspondence. In addition to the vector multiplets of the gauge group $G_{v} = \times SU(v_{i})$ one has $r$ tensor multiplets coming from the reduction of the IIB potentials on the $r$ cycles generating $H_{2}(\mathbb{C}^2/\Gamma)$. Now, the theory is not anomalous (see p.11 of [70] in the $A_{1}$ case and p.8 of [71] for the more general case) if the scalar fields of the tensor multiplets are coupled suitably to the gauge fields. Because of this coupling, indeed, we need to compensate the large gauge transformation, acting by the shift of $a$ by the change of $q_{i}$ when we compute the $Z$-function of the six dimensional theory on $\mathbb{T}_{\ell/\tau}^2$.

We recall the computation from [72] adopting to our notations. Consider $\mathcal{N} = 1$ theory in six dimensions with the gauge group $G_{v}$ and the hypermultiplet in the quiver representation $R_{\Gamma(v,w,\bar{w})}$. The quartic anomaly cancellation condition from [73] requires

$$\text{tr}_{adj} F^4 - \text{tr}_R F^4 = \sum_{i=1}^{\tilde{r}} (\alpha_{i}^i \text{tr} F_j)^2 \quad (2.14)$$

where $F$ is $G_{v}$ curvature, index $j$ labels the simple factors in the gauge group $G_{v}$ and $\tilde{r}$ is equal to the number of tensor multiplet that we need to add to cancel the anomaly in the vector and hypermultiplet. A useful identity for $SU(v_{i})$ gauge groups is

$$\text{tr}_{adj} F_{i}^4 = 2v_{i} \text{tr} F_{i}^4 + 6(\text{tr} F_{i}^2)^2 \quad (2.15)$$

Let $x_{i,\alpha}$ be Chern roots of $F_{i}$. Each $(i,j)$ bifundamental hypermultiplet contributes

$$- \sum_{\alpha,\beta} (x_{i,\alpha} - x_{j,\beta})^4 = -(v_{j} \text{tr} F_{i}^4 + 6 \text{tr} F_{i}^2 \text{tr} F_{j}^2 + v_{i} \text{tr} F_{j}^4) \quad (2.16)$$
For a quiver $\Gamma$ with Cartan matrix $a_{ij}$ we find that vector multiplet and all bifundamental together contribute
\begin{equation}
\sum_{ij} a_{ij} (v_i \text{tr} F_j^4 + 3 \text{tr} F_i^2 \text{tr} F_j^2) \tag{2.17}
\end{equation}
Adding up the contributions from all the fundamental and the anti-fundamental multiplets we find, for the total anomaly
\begin{equation}
\sum_{i,j} (v_i a_{ij} - w_j - \tilde{w}_j) \text{tr} F_j^4 + 3 \sum_{i,j} a_{ij} \text{tr} F_i^2 \text{tr} F_j^2 \tag{2.18}
\end{equation}
For $R_{\Gamma(v,w,\tilde{w})}$ in the conformal class (2.2) the dangerous $\text{tr} F^4$ term vanishes. To cancel the remaining anomaly by the $\tilde{r}$ tensor multiplets we want to present it as a sum of $\tilde{r}$ squares
\begin{equation}
\sum_{i=1}^{\tilde{r}} (\alpha_{i}^j \text{tr} F_j^2)^2
\end{equation}
with some real coefficients $\alpha_{i}^j$, so that the two-forms $B_i$ of the tensor multiplets interact with the gauge fields by means of the coupling
\begin{equation}
\alpha_{i}^j B_i \wedge \text{tr} F_j^2 \tag{2.19}
\end{equation}
shifting the gauge coupling
\begin{equation}
\tau_j \rightarrow \tau_j + \sum_i \alpha_{i}^j \int_{T^2} B_i \tag{2.20}
\end{equation}
In other words, we need to find $\tilde{r}$ one-forms $\alpha_{i}^j$, $i = 1, \ldots, \tilde{r}$ such that
\begin{equation}
a_{ij} = \sum_{i=1}^{\tilde{r}} \alpha_{i}^i \alpha_{i}^j \tag{2.21}
\end{equation}
The equation exactly coincides with the definition of the Cartan matrix $a_{ij}$ of quiver $\Gamma$ in terms of simple roots $\alpha_j = \sum_i \alpha_{i}^j e_i$ expressed in some orthonormal basis $e_i$. Therefore, $\tilde{r} = r$ where $r$ denotes the rank of Cartan matrix $a_{ij}$. Recall that the $\Gamma$ of finite type has $r = \#I_\Gamma$ nodes while the $\Gamma$ of affine type has $r = \#I_\Gamma - 1$ nodes. If $\Gamma$ is of affine type, then $\alpha_{i}^j$ for $i = 1, \ldots, r$ are the coefficients of simple roots $\alpha_{i}^j$ in the expansion in the basis $(e_i)_{i=0,\ldots,r}$, where $e_0 = \delta$ is the imaginary root of the affine system \cite{74}, and $(e_i)_{i=1,\ldots,r}$ is the orthonormal basis for the root system for the underlying quiver $\Gamma'$ of finite type obtained by removing the affine node '0' from $\Gamma$.

We conclude that the lift to the six dimensional $N^\prime = (1,0)$ theories of all of the quiver theories $R_{\Gamma(v,w,\tilde{w})}$ of the conformal class (2.2) is not prohibited by the quartic anomaly \cite{73}.

2.2. The $Z$-function. To the quiver representation $R_{\Gamma(v,w,\tilde{w})}$, the parameters $q, m, a$ and two additional parameters $\epsilon = (\epsilon_1, \epsilon_2)$ introduced in \cite{1, 20, 27} we associate the partition function $Z$ as the formal twisted Witten index of the five dimensional gauge theory \cite{19, 21}:
\begin{equation}
Z(t) = \text{Tr}_H (-1)^F e^{iH} t, \quad t \in \tilde{T} \tag{2.22}
\end{equation}
It is the partition function of the five dimensional theory on the five-dimensional manifold which is the twisted bundle $\mathbb{R}^4 \times S^1_{(\epsilon_1, \epsilon_2, \ell)}$ mentioned earlier, over the circle $S^1_{(\ell)}$ of
circumference $\ell$, with the fibers being $\mathbb{R}^4$. The boundary conditions involve the twists by flavor symmetry, the $R$-symmetry, the asymptotic gauge transformation, and the rotation of $\mathbb{R}^4$. The couplings $q_i$ are actually the expectation values of the complexified Wilson loop of the five dimensional vector multiplet which couples to the conserved topological current

$$J_i = \star \text{tr}_{\mathbf{v}_i} F \wedge F$$

Therefore the couplings $q_i$ and the couplings $\mu_e, \mu_{if}$ are of somewhat similar nature. In fact, in some theories (and most likely in all theories studied in our paper) the global symmetry of the theory extends to include the transformations generated by the topological charges. The torus $\tilde{T}$ is the maximal torus of the group $G_{eq} \times G_{top} = G_v \times G_f \times G_L \times G_{top}$, defined below. The group $G_{top}$ is generated by the topological charges. For the theories in this paper the it is $U(1)^I_r$. The factor $G_v$ is the group of global constant gauge transformations (or changing of the framing at infinity of the framed gauge bundle used to describe instantons on $\mathbb{R}^4$), the $G_f$ is the flavor symmetry group acting on hypermultiplets, the $G_L = SO(4)$ is the $\mathbb{R}^4$ rotation group. The four-dimensional instantons are lifted to particles in five dimensions. The partition function $Z$ factorizes as the product of the tree level, the perturbative and the instanton factors:

$$Z = Z^{\text{tree}} Z^{\text{pert}} Z^{\text{inst}}. \quad (2.23)$$

The perturbative part is ambiguous, as we mentioned above, it depends on the choice of boundary conditions. The instanton part is unambiguous and can be defined mathematically precisely as the $G_{eq}$-character of the graded vector space $\mathcal{H}_k^{\text{inst}}$

$$Z^{\text{inst}}(q; t) = \sum_k q^k \text{str}_{\mathcal{H}_k^{\text{inst}}}(t) \quad (2.24)$$

evaluated on the group element $t$ of the maximal torus $T$ of the equivariant group $G_{eq} = G_v \times G_f \times G_L$. The group element $t \in T$ is parametrized by $(a, m, \epsilon)$. The Coulomb parameters $a = (a_i)_{\Gamma}$ are coordinates in the Cartan algebra of global constant gauge transformations acting at the framing of the gauge bundle at infinity

$$G_v = \prod_{i \in I_r} SU(v_i) \quad (2.25)$$

the mass parameters $m = (m_{e}, m_{i,f}, \bar{m}_{i,f})$ are the coordinates on the Cartan subalgebra of the flavor symmetry group $G_f$ (2.13)

$$G_f = \times_{e \in E_r} U(1) \times_{i \in I_r} U(w_i) \times_{\bar{i} \in I_r} U(\bar{w}_{\bar{i}}) \quad (2.26)$$

and the epsilon parameters $\epsilon = (\epsilon_1, \epsilon_2)$ are the coordinates on the Cartan subalgebra of the four dimensional rotation group

$$G_L = SO(4) \quad (2.27)$$

acting on $\mathbb{R}^4$ by rotations about a particular point $0$. The set of integers $k \in \mathbb{Z}_{+}^{I_r} = \{k_i \in \mathbb{Z}_{\geq 0} | i \in I_r\} = \{k_i \in \mathbb{Z}_{\geq 0} | i \in I_r\}$ encodes the four dimensional instanton charges (the second Chern classes $c_2$) for the gauge group factors $SU(v_i)$ for $i \in I_r$. More precisely,

$$\mathcal{H}_k^{\text{inst}} = H^\bullet(\mathcal{M}_k^{\text{inst}}, \mathcal{E}^R) \quad (2.28)$$
is the sheaf cohomology of the virtual matter bundle \( \tilde{E}^{R_{\Gamma(v,w,\tilde{w})}} \to M_{k}^{\text{inst}} \) defined as follows. The base space \( M_{k}^{\text{inst}} \) is the framed \( G_{v} \)-instanton moduli space on \( \mathbb{R}^{4} \). Let \( \mathcal{A}_{G_{v}} \) be the space of connections on a principal \( G_{v} \)-bundle over \( \mathbb{R}^{4}_{\hat{e}_{1},\hat{e}_{2}} \) with fixed framing at infinity and fixed second Chern class \( k \). Then

\[
M_{k}^{\text{inst}} = \{ A \in \mathcal{A}_{G_{v}} \mid F_{A}^{+} = 0 \} / G_{\infty}
\]

where \( G_{\infty} \) denotes the group of framed gauge transformations, i.e. trivial at \( \infty = S^{4} \backslash \mathbb{R}^{4}_{\hat{e}_{1},\hat{e}_{2}} \). The fiber of the virtual matter bundle \( \mathcal{E}^{R}_{A} \) at the point \( A \in M_{k}^{\text{inst}} \) is the virtual space of the zero modes of Dirac operator in representation \( R_{\Gamma(v,w,\tilde{w})} \) in the background of the instanton connection \( A \) in the gauge bundle on \( S^{4} \), that is

\[
\mathcal{E}^{R}_{A} = \ker \mathcal{D}^{R}_{A} - \text{coker} \mathcal{D}^{R}_{A}
\]

The equivariant partition function \( Z(q,a,m;\epsilon) \) is analytic in the equivariant parameters \( (a,m;\epsilon) \) which henceforth are treated as complex variables.

2.3. Index computation. In this section we give only the brief review of the gauge theory partition \( Z \)-function, mostly in order to set the notations. We shall skip the arguments of the partition function \( Z \).

For more details one can consult \([1, 19, 21, 26, 27, 75–79]\). The \( T \)-equivariant path integral of the supersymmetric \( \Omega \)-deformed gauge theory on \( \mathbb{R}^{4} \) localizes to the \( T \)-equivariant integral over the instanton moduli space \( \bigcup_{k} M_{k}^{\text{inst}} \), where \( k = (k_{i})_{i \in I_{\Gamma}} \in \mathbb{Z}_{+}^{I_{\Gamma}} \) is the second Chern class of the anti-self-dual \( G_{v} \)-connection on \( \mathbb{R}^{4} \) and

\[
M_{k}^{\text{inst}} = \prod_{i \in I_{\Gamma}} M_{v_{i},k_{i}}
\]

Then, using the Atiyah-Bott theorem one reduces the integral over \( M_{k}^{\text{inst}} \) to the sum over the set of fixed points of the locally defined expressions. The fixed point formula is similar to the calculation of an integral of a meromorphic form via residues. To apply Atiyah-Bott localization we need to partly compactify the moduli space of instantons to the Gieseker-Nakajima moduli space \( M_{k}^{\text{inst}} \) in which vector bundles are replaced by the torsion free sheaves, or, in differential-geometric language, by the noncommutative instantons \([80]\). Thus, in particular, \( M_{v_{i},k_{i}} \) is the moduli space of framed torsion free rank \( v_{i} \) sheaves \( E_{i} \) on \( \mathbb{C}P^{2} \), trivialized over a complex line \( \mathbb{C}P^{1}_{\infty} \) at infinity, with the second Chern characters \( \text{ch}_{2}(E_{i}) = k_{i} \).

Let \( \mathcal{E}^{i} \) denote the universal torsion free sheaf over \( M_{v,k} \times \mathbb{C}P^{2} \) in the \( i \)-th factor, and let \( \pi : M_{v,k} \times \mathbb{C}P^{2} \to M_{v,k} \) be the projection. For a quiver representation \( R_{\Gamma(v,w,\tilde{w})} \) the matter sheaf cf. \( (2.19) \) over \( M_{v,k} \times \mathbb{C}P^{2} \) is

\[
\mathcal{E}^{R_{\Gamma(v,w,\tilde{w})}} = \bigoplus_{e \in E_{\Gamma}} \text{Hom}(\mathcal{E}_{s(e)}, \mathcal{E}_{t(e)}) \oplus \bigoplus_{i \in I_{\Gamma}} \text{Hom}(w_{i}, \mathcal{E}_{i}) \oplus \bigoplus_{i \in I_{\Gamma}} \text{Hom}(\mathcal{E}_{i}, \tilde{w}_{i})
\]

Let

\[
\tilde{E}^{R_{\Gamma(v,w,\tilde{w})}} = R\pi_{*}E^{R_{\Gamma(v,w,\tilde{w})}}
\]

be the derived pushforward of the matter sheaf to \( M_{v,k} \), which means that the virtual fiber \( \tilde{E}^{R_{\Gamma(v,w,\tilde{w})}}|_{m} \) at a point \( m \in M_{v,k} \) is the graded space of sheaf cohomology \( H^{*}(\mathbb{C}P^{2}, \mathcal{E}^{R_{\Gamma(v,w,\tilde{w})}}|_{m}) \).
2.4. Four dimensions. The instanton part of the four dimensional gauge theory partition function is

\[ Z_{\text{inst}}^{4d} = \sum_{k=0}^{\infty} q^k \int_{\mathcal{M}_{V,k}} e_T(\mathcal{E}^{R_T(V,W,W)}) \]  

(2.33)

where equivariant integration \( \int_{\mathcal{M}_{V,k}} \) denotes the pushforward of the \( T \)-cohomologies of \( \mathcal{M}_{V,k} \) by projection map from \( \mathcal{M}_{V,k} \) to a point, \( e_T \) is the \( T \)-equivariant Euler class, and

\[ q^k = \prod_{i \in \Gamma} q_i^{k_i} \]  

(2.34)

2.4.1. The \( Y \)-operators in four dimensions. The geometric definition of generating functions \( Y_i(x) \) (1.8) is conveniently given in terms of the spectral Chern class \( c(x,E) \). For \( x_I \) the virtual Chern roots of a vector bundle \( E \), define the rational \( x \)-spectral Chern class

\[ c(x,E) = \prod (x - x_I) \]  

(2.35)

so that \( c(1,E) = c(E^\vee) \) is the ordinary total Chern class. Then

\[ 4d : \quad Y_i(x - \frac{q}{2}) = \frac{\sum_{k=0}^{\infty} q^k \int_{\mathcal{M}_{V,k}} c_T(x,\hat{E}_I)e_T(\mathcal{E}^{R_T(V,W,W)})}{\sum_{k=0}^{\infty} q^k \int_{\mathcal{M}_{V,k}} e_T(\mathcal{E}^{R_T(V,W,W)})} \]  

(2.36)

and \( \hat{E}_I \) is the restriction of the sheaf \( E_I \) at the \( T \)-fixed point \( 0 \in \mathbb{C}^2 \). The restriction is defined for sheaves using the Koszul resolution complex and derived pushforward map

\[ \hat{E}_I = R\pi_*E_I \otimes (\Lambda^2 T_{\mathbb{C}^2}^* \rightarrow T_{\mathbb{C}^2}^* \rightarrow O) \]  

(2.37)

2.5. Five dimensions. The five dimensional, K-theoretic, version of the partition function is

\[ Z_{\text{inst}}^{5d} = \sum_{k} q^k \text{ch}_T H^*_T(\mathcal{M}_{V,k},\mathbb{I}(\mathcal{E}^{R_T(V,W,W)})) \]  

(2.38)

where \( \mathbb{I} \) is the index functor defined on sheaves by

\[ \mathbb{I} : \mathcal{E} \rightarrow \sum_{i=0}^{\infty} (-1)^i \Lambda^i \mathcal{E} \]  

(2.39)

which satisfies

\[ \mathbb{I}[\mathcal{E}_1 \oplus \mathcal{E}_2] = \mathbb{I}[\mathcal{E}_1] \mathbb{I}[\mathcal{E}_2] \]  

(2.40)

and

\[ \mathbb{I}[\mathcal{E}^\vee] = \mathbb{I}[\mathcal{E}]^\vee = (-1)^{rkE} (\Lambda^{rkE} \mathcal{E})^{-1} \otimes \mathbb{I}[\mathcal{E}] \]  

(2.41)

At the level of Chern characters, with \( x_I \) the virtual Chern roots,

\[ \mathbb{I} \left[ \sum_I e^{x_I} \right] = \left[ \prod_I (1 - e^{x_I}) \right] \]  

(2.42)

Remember, that in our theories we have the mass parameters, i.e. the equivariant parameters for the symmetries, which rescale the fibers of various summands in the matter sheaf (2.32). Thus the sum in (2.42) is more refined then it may appear.
2.5.1. The $y$-operators in five dimensions. The five-dimensional, K-theoretic geometric definition of the $y_i(\xi)$ function is

$$y_i(\xi q^{-\frac{1}{2}}) = \frac{\sum_{k=0}^{\infty} q^k \int_{M_{v,k}} \operatorname{td}(TM_{v,k}) \hat{c}_T(\xi, E_i) \hat{c}_T(1, \hat{R}^{T(v,w,\psi)})}{\sum_{k=0}^{\infty} q^k \int_{M_{v,k}} \operatorname{td}(TM_{v,k}) \hat{c}_T(1, \hat{R}^{T(v,w,\psi)})}$$  (2.43)

where the $K$-theoretic trigonometric $\xi$-spectral characteristic class $\hat{c}_T(\xi, E)$ is defined in terms of the $T$-equivariant Chern roots $x_I$ of the sheaf $E$ by

$$\hat{c}_T(\xi, E) = \prod_I \left(1 - \frac{\xi_I}{\xi} \right)$$  (2.44)

where $\ell$ is the circumference of the 5d compactification circle $S^1_{\ell}$. The class $\hat{c}_T(\xi, E)$ has the following transformation property:

$$\hat{c}_T(\xi, E^\vee) = e^{-c_1(E)}(-\xi)^{-\text{rk} E} \hat{c}_T(\xi^{-1}, E)$$  (2.45)

2.5.2. Asymptotics of the $y$-operators. Using the definitions above, and the fact that $\text{rk} E_i = v_i$, we get:

$$4d: \quad y_i(x) \to x^{v_i}, \quad x \to \infty$$

$$5d: \quad \begin{array}{l}
y_i(\xi) \to 1, \quad \xi \to \infty \\
y_i(\xi) \to \xi^{-v_i} \prod_{a=1}^{v_i} (-w_{i,a} q^{-\frac{1}{2}}), \quad \xi \to 0
\end{array}$$  (2.46)

In the limit $q = 0$ the functions $y_i(x), y_i(\xi)$ are (Laurent) polynomials

$$4d: \quad \lim_{q \to 0} y_i(x - \xi) = \prod_{a=1}^{v_i} (x - a_{i,a})$$

$$5d: \quad \lim_{q \to 0} y_i(\xi q^{-\frac{1}{2}}) = \prod_{a=1}^{v_i} (1 - w_{i,a}/\xi)$$  (2.47)

The shifts $x \mapsto x - \frac{\xi}{2}, \xi \mapsto \xi q^{-\frac{1}{2}}$ are introduced here for a later convenience of relating $y_i$ to the representation theory of quantum groups.

2.6. Partition function is a sum over partitions. The equivariant integration over $M_{v,k}$ reduces to the sum over the space $M_{v,k}^T$ of $T$-fixed points by Atiyah-Bott localization

$$Z_{\text{inst}} = \sum_k \sum_{\lambda \in M_{v,k}^T} Z_\lambda$$  (2.48)

Note that $Z_\lambda$ in (2.48) contains the factor

$$q^k = \prod_{i \in I} q_i^{k_i}$$

which helps to make the instanton sum a convergent series, at least for $|q_i| \ll 1$. The $T$-fixed points $\lambda$ are the sheaves which split as the sum of the $T$-equivariant rank one torsion free sheaves. The latter are the finite codimension monomial ideals $I$ in the ring of polynomials in two variables $\mathbb{C}[z_1, z_2]$. The latter are in one-to-one correspondence with partitions of the size codim $I$. The use of torsion free sheaves in lieu of the honest
bundles reduces the rotation symmetry group from $SO(4)$ to $U(2)$, since it requires a choice of complex structure on $\mathbb{R}^4 \approx \mathbb{C}^2$. Thus the symmetry group for the $\mathcal{M}_{v_i,k_i}$ moduli space is $U(v_i) \times U(2)$. The maximal torus $\mathbb{T}$ is unchanged by this reduction.

The set of fixed points $\mathcal{M}_{v,k}^T$ is a two indexed set of partitions labeled by the quiver node $i \in I_G$ and by the color $a = 1, \ldots, v_i$ of the $SU(v_i)$-factor in $G_v$:

$$\mathcal{M}_{v,k}^T = \{ \lambda_{i,a} \mid i \in I_G, a = 1, \ldots, v_i \} \quad \text{(2.49)}$$

where each partition $\lambda_{i,a}$ is a non-increasing sequence of non-negative integers

$$\lambda_{i,a} = (\lambda_{i,a,k})_{k \in \mathbb{N}} = (\lambda_{i,a,1} \geq \lambda_{i,a,2} \geq \ldots \geq 0 = 0 = \ldots) \quad \text{(2.50)}$$

whose size is defined as:

$$|\lambda_{i,a}| = \sum_{k=1}^{\infty} \lambda_{i,a,k}$$

Let us denote by $\lambda_i = (\lambda_{i,a})_{a=1}^{v_i}$ the colored partition corresponding to $U(v_i)$ (cf. [21]), and

$$|\lambda_i| = \sum_{a=1}^{v_i} |\lambda_{i,a}|$$

The fixed point $\lambda \in \mathcal{M}_{v,k}$ corresponding to $\lambda = (\lambda_i)_{i \in I_G}$ (we thus use the same letter $\lambda$ to denote both the fixed point in the moduli space of sheaves and the quiver-colored partition it corresponds to) is the collection $(E_{i,\lambda_i})_{i \in I_G}$ of the sheaves which are the direct sums of the monomial ideals corresponding to $\lambda_{i,a}$. The topology of the sheaf determines the sizes of the colored partitions:

$$|\lambda_i| = k_i \quad \text{(2.51)}$$

Let $\mathcal{E}_{i,\lambda_i}$ be the $U(v_i) \times U(2)$ equivariant Chern character of $E_{i,\lambda_i}$ restricted at $0 \in \mathbb{C}^2$.

Let $(a_{i,a}, \epsilon) = (\epsilon_1, \epsilon_2)$ be coordinates in the Cartan algebra of $SU(v_i) \times SO(4)$. From the ADHM [81] construction one derives (cf. [73])

$$\mathcal{E}_{i,\lambda_i} = W_i - (1 - q_1)(1 - q_2)V_{i,\lambda_i} \quad \text{(2.52)}$$

where $q_1, q_2$ are defined in [2.7],

$$W_i = \sum_{a=1}^{v_i} e^{i\ell a_{i,a}}$$

$$V_{i,\lambda_i} = \sum_{a=1}^{v_i} \sum_{s \in \lambda_{i,a}} e^{i\ell c_{i,a,s}} \quad \text{(2.53)}$$

where $s \in \lambda_{i,a}$ labels boxes in the partition $\lambda_{i,a}$ so that $s = (s_1, s_2)$ is a pair of integers with $s_1 = 1, 2, \ldots$, and $s_2 = 1, \ldots, \lambda_{i,a,s_1}$, and $c_{i,a,s}$ is the $a_{i,a}$-shifted content [21] of the box $s$

$$c_{i,a,s} = a_{i,a} + (s_1 - 1)\epsilon_1 + (s_2 - 1)\epsilon_2 \quad \text{(2.54)}$$
The contribution of the fixed point \( \lambda \) to the partition function is a product of factors labeled by the vector and hypermultiplets

\[
Z_\lambda = \left( \prod_{i \in I} Z_{i,\lambda}^{cs} Z_{i,\lambda}^{top} Z_{i,\lambda}^{vec} Z_{i,\lambda}^{f} Z_{i,\lambda}^{af} \right) \left( \prod_{e \in E} Z_{e,\lambda}^{bf} \right)
\]  

(2.55)

2.6.1. Plethystic exponents and the asymptotics \( \epsilon_2 \to 0 \). The contributions to the five dimensional index \( Z_\lambda \) are found by applying the plethystic exponent (index function) \( I \) to the corresponding characters \( \chi_\lambda \). Applying \( I \) to \( \chi \) is equivalent to using Atiyah-Singer formula to compute the \( T \)-equivariant index of the Dolbeault operator \( \bar{\partial} \) for the complex of \((0, \bullet)\)-forms on (virtual) vector space which is \( T \)-module with Chern character \( \chi \).

First, we introduce the \( q \)-analogue of the \( \log \Gamma \) function

\[
\gamma_q(\xi) = \log \prod_{n \geq 0} (1 - \xi q^n) = \log \prod_{n \geq 0} (1 - \xi q^n)
\]  

(2.56)

To compute \( Z \) in the limit \( \epsilon_2 \to 0 \) we will need the limit of \( \gamma_{q_2}(x) \) function for \( q_2 = e^{i\ell\epsilon_2} \to 1 \). We find

\[
\gamma_{q_2}(\xi) = \sum_{m=1}^{\infty} \frac{1}{m} \frac{\xi^m}{1 - q_2^m} \bigg|_{\epsilon_2 \to 0} \approx \exp \left( -\frac{1}{i\ell \epsilon_2} \text{Li}_2(\xi) \right)
\]  

(2.57)

We now proceed with the definitions of each factor in (2.55).

2.6.2. The bi-fundamental contribution. First, we describe explicitly the bi-fundamental factor \( Z_{e,\lambda}^{bf} \). Let

\[
\chi_{e,\lambda} = \text{ind} D_{e,\lambda}
\]  

(2.58)

be the index of the Dirac operator on \( \mathbb{R}^4 \) in the bifundamental representation \( e = (v_{t(e)}, \bar{v}_{s(e)}) \) evaluated in the background of the connection associated to the fixed point \( \lambda \in \mathcal{M}_{k}^{\text{inst}} \). From the equivariant Atiyah-Singer theorem we find the character

\[
\chi_{e,\lambda} = -\mu^{-1}_e \left( \frac{q_1 q_2}{(1 - q_1)(1 - q_2)} \right)^{\frac{1}{2}} \varepsilon_{t(e),\lambda_{t(e)}} \varepsilon_{s(e),\lambda_{a(e)}}
\]  

(2.59)

which contains the overall factor \( \mu^{-1}_e \) (2.7) accounting for the bi-fundamental mass. It is the \( U(1)_e \) flavor symmetry group element of the bi-fundamental hypermultiplet corresponding to the edge \( e \). By \( \varepsilon^\vee \) we denote the equivariant Chern character of the dual bundle \( \mathcal{E}^\vee \) in which all weights are the negatives of the weights in \( \mathcal{E} \).

The character \( \varepsilon_{i,\lambda} \) of the universal bundle \( \mathcal{E}_{i,\lambda} \) can be represented by the infinite sum

\[
\varepsilon_{i,\lambda} = \sum_{a=1}^{\infty} \sum_{k=1}^{\infty} (1 - q_1) e^{i\ell x_{i,a,k}}
\]  

(2.60)

where \( x_{i,a,k} \)

\[
x_{i,a,k} = a_i + \epsilon_1(k - 1) + \epsilon_2 \lambda_{i,a,k}
\]  

(2.61)
denotes the $a_{i,a}$-shifted content of the box $s = (s_1, s_2) = (k, \lambda_{i,a} + 1)$ (which is right outside the Young diagram of the partition $\lambda_{i,a}$). The profile of the partition $\lambda_{i,a}$ can be parametrized by the infinite sequence $\xi_{i,a} = (\xi_{i,a,1}, \xi_{i,a,2}, \ldots)$

$$\xi_{i,a,k} = e^{ix_{i,a,k}}$$ (2.62)

Let

$$\Xi_i = (\xi_{i,a,k})_{a=1,...,v_i;k=1,2,...}$$ (2.63)

denote the set of all $\xi_{i,a,k}$ with fixed $i \in I_T$ and $1 \leq a \leq v_i$, $k \in \mathbb{N}$. Then

$$\chi_{e,\lambda} = \frac{q_1^{1/2} - q_1^{1/2}}{q_2^{1/2} - q_2^{1/2}} \sum_{(\xi_t, \xi_s) \in \Xi_t(e) \times \Xi_s(e)} \frac{\xi_t(e)}{\mu_e \xi_s(e)}$$ (2.64)

The contribution $Z_{e,\lambda}^{bf}$ of the bi-fundamental hypermultiplets, computed from the Chern character $\chi_{e,\lambda}$ (2.64), asymptotes to

$$Z_{e,\lambda}^{bf} \sim \exp \left( -\frac{1}{2v_2} \sum_{(\xi_t, \xi_s) \in \Xi_t(e) \times \Xi_s(e)} L \left( \frac{\xi_t}{\mu_e \xi_s} \right) \right)$$ (2.65)

where

$$L(\xi) = \text{Li}_2(q_1^{1/2} \xi) - \text{Li}_2(q_1^{-1/2} \xi)$$ (2.66)

Actually, both (2.64) and (2.65) are formal expressions at this stage, as the convergence of the series for $\chi_{e,\lambda}$ requires $|q_1| < 1$. Of course, the original formula (2.59) makes sense, as it operates with finite expressions, so that the correct formula is

$$\chi_{e,\lambda} = -\mu_e^{-1} \frac{(q_1 q_2)^{1/2}}{(1 - q_1)(1 - q_2)} \left( W_{t(e),\lambda_t(e)} W_s^{\vee}(e),\lambda_s(e) \right) +$$

$$\mu_e^{-1} \left( (q_1 q_2)^{1/2} W_{t(e),\lambda_t(e)} V_s^{\vee}(e),\lambda_s(e) + (q_1 q_2)^{-1/2} V_{t(e),\lambda_t(e)} W_s^{\vee}(e),\lambda_s(e) \right) -$$

$$\mu_e^{-1} (q_1^{1/2} - q_1^{-1/2})(q_2^{1/2} - q_2^{-1/2}) V_{t(e),\lambda_t(e)} V_s^{\vee}(e),\lambda_s(e)$$ (2.67)

One can actually make sense out of (2.64) and (2.65) using a prescription where one sums over $\xi_s$ first, viewing it as a telescopic sum. The telescopic sum is actually finite. The remaining sum over $\xi_t$ is convergent. One should use a similar prescription for the evaluation of $Z_{e,\lambda}^{vec}$ below.

One can actually avoid using the formal expressions whatsoever, by working with the finite formulae like (2.67). Below we write the formal expressions for the sake of brevity.

2.6.3. The fundamental and the anti-fundamental hypermultiplet contributions. The contribution of the fundamentals $Z_{e,\lambda}^{f}$ is obtained by applying the $I$-functor to the simpler version of the character (2.59)

$$\chi_i^f = -\sum_{i=1}^{w_1} \frac{(q_1 q_2)^{1/2}}{(1 - q_1)(1 - q_2)} \xi_i \mu_i^{-1}$$ (2.68)
Thus, in the $\epsilon_2 \to 0$ limit,

$$Z_{i,\lambda}^f \sim \exp \left( -\frac{1}{i\ell\epsilon_2} \sum_{(\xi_t,\xi_s) \in \Xi_i \times \{\mu_i,\bar{\mu}_i\}} L(\xi_t/\xi_s) \right)$$  \hspace{1cm} (2.69)

and, similarly, the anti-fundamental contribution is

$$Z_{i,\lambda}^{af} = \exp \left( -\frac{1}{i\ell\epsilon_2} \sum_{(\xi_t,\xi_s) \in \{\tilde{\mu}_i,\bar{\mu}_i\} \times \Xi_i \times \{\mu_i,\bar{\mu}_i\}} L(\xi_t/\xi_s) \right)$$  \hspace{1cm} (2.70)

### 2.6.4. The vector multiplet contribution.

To compute the vector multiplet contribution $Z_{i,\lambda}^{vec}$ we need to find the $G_{eq}$-character of the tangent space $T_{\lambda \mathcal{M}_k^{\text{inst}}}$ to the moduli space at the fixed point $\lambda \in \mathcal{M}_k^{\text{inst}}$. From the deformation theory, the character of the tangent space is dual to the index of Dirac operator in the adjoint representation twisted by the square root of the canonical bundle. Thus,

$$\chi_{i}^{vec} = q_1 q_2 (1 - q_1)(1 - q_2) \chi_{i}^{\nu}$$  \hspace{1cm} (2.71)

Therefore, in the $\epsilon_2 \to 0$ limit,

$$Z_{i,\lambda}^{vec} = \exp \left( \frac{1}{i\ell\epsilon_2} \sum_{(\xi_t,\xi_s) \in \Xi_i \times \Xi_i} L(q_1^k \xi_t/\xi_s) \right)$$  \hspace{1cm} (2.72)

### 2.6.5. The topological contributions.

The factor $Z_{i,\lambda}^{top}$ accounts for the topological instanton charge given by the second Chern class $k_i$ for the $i$-th gauge group factor $U(v_i)$

$$Z_{i,\lambda}^{top} = q_i^{k_i} = \exp \left( \frac{2\pi i v_i}{i\ell\epsilon_2} \sum_{k=1}^{v_i} \sum_{a=1}^{\infty} \log \frac{\xi_{i,a,k}}{\xi_{i,a,k}} \right)$$  \hspace{1cm} (2.73)

where

$$\Xi_i = \left\{ \xi_{i,a,k} = w_{i,a}^{k-1} \right\}$$  \hspace{1cm} (2.74)

corresponds to the empty partition $(\lambda_{i,a,k}) = (0,0,\ldots,)$. The second Chern class of $\mathcal{E}_{i,\lambda_i}$ is the size $|\lambda_i|$ of the colored partition

$$k_i = \sum_{a=1}^{v_i} \sum_{k=1}^{\infty} \lambda_{i,a,k}$$  \hspace{1cm} (2.75)

Finally, the factor $Z_{i,\lambda}^{cs}$ accounts for the contribution from the Chern-Simons coupling. It is the descendant of the cubic term in the tree level prepotential

$$\mathcal{L}_{cs,i} = -\kappa_i \int d^5x d^4\vartheta \ tr \Phi_1^3 \sim \frac{i\ell k_i}{3!} \epsilon_1 \epsilon_2 \ tr \Phi_1^3$$  \hspace{1cm} (2.76)

The fixed point contribution of the cubic term $\frac{1}{6} tr \Phi_1^3$ is equal to the third component of the Chern character of the universal bundle, restricted at the fixed point $\lambda \times 0 \in \mathcal{M}_k^{\text{inst}}$. 


\[ \mathcal{M}_k^{\text{inst}} \times \mathbb{C}^2: \]
\[ \frac{1}{3!} \text{tr} \Phi_3^{(2)} \bigg|_{\lambda \times \{0\}} = \sum_a \left( \frac{a_3^a}{6} - \frac{\epsilon_1 \epsilon_2 (\epsilon_1 + \epsilon_2)}{2} |\lambda_{1,a}| - \epsilon_1 \epsilon_2 \sum_{s \in \lambda_{1,a}} c_s \right) \]
\[ (2.77) \]

which gives, in terms of the \( \xi_{1,a,k} \) variables
\[ Z_{i,\lambda}^{cs} = \exp \frac{\kappa_1}{2 \epsilon_2} \sum_{\xi \in \Xi} \log(\xi/\bar{\xi}) \left( \frac{\log(\xi/\bar{\xi})}{i \ell} + \epsilon_1 + \epsilon_2 \right) \]
\[ (2.78) \]

2.6.6. Six dimensional case. For a six dimensional theory compactified on the torus
\[ T_p = \mathbb{C}/(\ell \mathbb{Z} \oplus -\ell/\tau_p \mathbb{Z}) \]
\[ (2.79) \]

with the metric
\[ ds_p^2 = \frac{\ell^2}{|\tau_p|^2} (\text{Im}^2 \tau_p dz d\bar{z}), \quad z \sim z + \frac{\ell}{\tau_p} (n_1 + n_2 \tau_p) \]
\[ (2.80) \]

and the nome \( p = e^{2 \pi i \tau_p} \), the elliptic analogue of the index functor \( (2.42) \) is
\[ \mathbb{I}_p \left[ \sum_t e^{xt} \right] = \prod_t \theta_1(e^{xt}; p) \]
\[ (2.81) \]

where
\[ \theta_1(\xi; p) = -\sum_{r \in \mathbb{Z} + \frac{1}{2}} (-1)^r \xi^r p^\frac{r^2}{2} = ip^{\frac{1}{2}} \xi^{-\frac{1}{2}} \theta(\xi; p) \]
\[ \theta(\xi; p) = \sum_{n \in \mathbb{Z}} (-1)^n \xi^n p^{\frac{1}{2} n(n-1)} = (\xi, p)_\infty (p, p)_\infty (p^\xi, p)_\infty \]
\[ (2.82) \]

In terms of the additive arguments
\[ \xi = e^{ix}, \quad p = e^{2 \pi i \tau_p} \]
\[ (2.83) \]

Under the shifts
\[ x \mapsto \tilde{x} = x + \frac{2 \pi}{\ell} (n_1 + n_2 \tau_p), \quad n_1, n_2 \in \mathbb{Z} \]
\[ (2.85) \]

the function \( \vartheta_1(x; \tau_p) \) transforms as
\[ \vartheta_1(x; \tau_p) \mapsto e^{-(\pi \tau_p n_2^2 + \pi i (n_1 + n_2 + x n_2))} \vartheta_1(x; \tau_p) \]
\[ (2.86) \]

For future use, let us rewrite the multiplier for the \( \vartheta_1 \) transformation under the shifts by \( (n_1, n_2) = (n, 0) \) in the form:
\[ T_1^m : e^{-i \pi n} = e^{-\frac{\pi i}{2} (\tilde{x} - x)} \]
\[ (2.87) \]

under the shifts by \( (n_1, n_2) = (0, n) \) in the form:
\[ T_2^m : e^{-(\pi \tau_p n_2^2 + i n x + i \pi n)} = e^{-\frac{\pi i}{4 \tau_p} (\tilde{x}^2 - x^2) - \frac{\pi i}{2 \tau_p} (\tilde{x} - x)} \]
\[ (2.88) \]
Under the modular transformations of the parameter $\tau_p$:

$$
\vartheta_1 \left( \frac{x}{\tau_p}; -\frac{1}{\tau_p} \right) = e^{-\frac{3}{4} \pi \tau_p^2} e^{\frac{\mu^2 x^2}{4 \pi \tau_p}} \vartheta_1(x; \tau_p) \quad \vartheta_1(x, \tau_p + 1) = e^{\frac{3}{4} \pi \tau_p} \vartheta_1(x, \tau_p)
$$

(2.89)

2.6.7. The gauge and modular anomalies. Define

$$
\left[ \sum_I e^{i\ell w_I} \right]_d := \sum_I w_I^d
$$

The transformation of $I_p[\chi]$ under the large gauge transformations, which act on the Coulomb moduli $a_{i,a}$ by the shifts:

$$
a_{i,a} \mapsto \tilde{a}_{i,a} = a_{i,a} + \frac{2\pi}{\ell} (n_{i,a} + m_{i,a} \tau_p), \quad n_{i,a}, m_{i,a} \in \mathbb{Z}
$$

(2.90)

is given by $I_p[\chi] \mapsto I_p[\tilde{\chi}]$, cf. Eqs. (2.87), (2.88)

$$
T_1^n : I_p[\tilde{\chi}] - I_p[\chi] = -\frac{i\ell}{2} ([\tilde{\chi}]_1 - [\chi]_1)
$$

$$
T_2^n : I_p[\tilde{\chi}] - I_p[\chi] = -\frac{i\ell^2}{4\pi \tau_p} ([\tilde{\chi}]_2 - [\chi]_2) - \frac{i\ell}{2\tau_p} ([\tilde{\chi}]_1 - [\chi]_1)
$$

(2.91)

**Proposition.** The six dimensional theory compactified on a torus is gauge invariant, if the right hand side of (2.91) vanishes for all $(n_{i,a}, m_{i,a})$, or if the right hand side of (2.91) can be compensated by some consistent transformation of the coupling constants, i.e. $q_i$'s. For that to be the case, the right hand side of (2.91) must be proportional to $k_i$ and should not depend on the individual colored partitions $\lambda_i$.

The transformation of $I_p[\chi]$ under $\tau_p \to -1/\tau_p$, i.e. the modular anomaly, is proportional to the exponent of $[\chi]_2$.

**Proposition.** For the theory compactified on the torus to make sense on its own, the partition function must be modular invariant, perhaps at the expense of making the gauge couplings $q_i$ transform under the modular group. The latter means that $[\chi]_2$ can be linear in $k_i$, but should not depend on any moduli such as $a_{i,a}$.

2.6.8. Example: the $\hat{A}_0$ theory in six dimensions. For the $\hat{A}_0 = (\mathcal{N} = 2^*)$ theory the set of vertices consists of one element, so we skip the index $i$ in what follows. We have

$$
\chi^{\text{vec}} = \frac{q_1 q_2}{(1 - q_1)(1 - q_2)} \mathcal{E} \mathcal{E}^\vee
$$

(2.92)

and

$$
\chi^{\text{adj}} = -\frac{(q_1 q_2)^{\frac{1}{2}} \mu^{-1}}{(1 - q_1)(1 - q_2)} \mathcal{E} \mathcal{E}^\vee
$$

(2.93)

Then

$$
\chi^{\text{vec,inst}} = -W V^\vee - q_1 q_2 W^\vee V + (1 - q_1)(1 - q_2) V V^\vee
$$

(2.94)
For the $\hat{A}_0$ theory we find, in the $k$-instanton sector
\[
\left[ \chi_\lambda^{\text{vec}} + \chi_\lambda^{\text{adj}} \right]^2 = - \sum_{a,b,s \in \lambda_b} ((a_a - c_{b,s})^2 + (2\epsilon_+ + c_{b,s} - a_a)^2) + \\
+ \sum_{a,b,s \in \lambda_b} ((a_a - c_{b,s} - \epsilon_+ - m)^2 + (\epsilon_+ + c_{b,s} - a_a - m)^2) = \\
= 2k(v(m + \epsilon_+)(m - \epsilon_+)) 
\tag{2.95}
\]
Since this is $\alpha$-independent, there is no gauge anomaly. In addition, using (2.95) we see that under $\tau_p \rightarrow -1/\tau_p$
\[
Z_\lambda \mapsto \exp \left( |\lambda| \frac{IL^2v(m^2 - \epsilon_+^2)}{2\pi \tau_p} \right) Z_\lambda \tag{2.96}
\]
The instanton partition function of the $\hat{A}_0$ theory in six dimensions can be rendered modular invariant by requiring the gauge coupling to transform
\[
q \mapsto \exp \left( - \frac{IL^2v(m^2 - \epsilon_+^2)}{2\pi \tau_p} \right) q \tag{2.97}
\]
under the $\tau_p \rightarrow -1/\tau_p$ transformation.

The factor $m^2 - \epsilon_+^2$ should be related to the anomaly cancellation condition
\[
dH = F^2 - R^2 \propto m^2 - \epsilon_+^2 \tag{2.98}
\]
where $F$ and $R$ are the equivariant curvatures of the R-symmetry connection and spin-connection, respectively, on the compactification torus $\mathbb{T}_{\ell,-\ell/\tau_p}$.

2.6.9. Example: the $A_1$ theory in six dimensions. Here is an example of potentially sick theory. Again, we skip the $i$ index below. The $SU(v)$ theory with $v$ fundamental hypermultiplets and $v$ anti-fundamental hypermultiplets has, in the instanton sector:
\[
\left[ \chi_\lambda^{\text{vec}} + \chi_\lambda^{f} + \chi_\lambda^{af} \right]^2 = 2\epsilon_1 \epsilon_2 k^2 - \sum_{a,b,s \in \lambda_b} ((a_a - c_{b,s})^2 + (2\epsilon_+ + c_{b,s} - a_a)^2) + \\
+ \sum_{b,f,s \in \lambda_b} ((c_{b,s} + \epsilon_+ - m_f)^2 + (\epsilon_+ + \tilde{m}_f - c_{b,s})^2) = \\
= 2\epsilon_1 \epsilon_2 k^2 + 2 \left( \sum_{b,s \in \lambda_b} c_{b,s} \right) \left( \sum_{a=1}^{v} (2a_a - 2\epsilon_+ - m_a - \tilde{m}_a) \right) \\
+ k \sum_{a=1}^{v} (-2(a_a - \epsilon_+)^2 + m_a^2 + \tilde{m}_a^2 + 2\epsilon_+(\tilde{m}_a - m_a)) 
\tag{2.99}
\]
To make the right hand side gauge invariant we need to insist on the following traceless constraint:
\[
\sum_{a=1}^{v} (2a_a - 2\epsilon_+ - m_a - \tilde{m}_a) = 0 
\tag{2.100}
which means that only the $SU(v)$ part of the gauge group is possibly anomaly free, hardly a surprise. In addition, we must make the gauge coupling $q$ transform under the gauge transformations, so that the combination

$$q \exp - \frac{i l^2}{4 \pi \tau_p} \sum_{a=1}^{v} \left(-2(a_a - \epsilon_+)^2 + m_a^2 + \tilde{m}_a^2 + 2\epsilon_+(\tilde{m}_a - m_a)\right)$$

(2.101)

is invariant. However the modular invariance cannot be fixed because of the first term $\propto k^2$.

3. The twisted superpotential $W$ of quiver theories

We now wish to evaluate the $Z$-partition function in the limit $\epsilon_2 \to 0$ with $\epsilon_1 = \epsilon$ fixed. On the physical grounds (the extensivity of the partition function of the instanton gas) we expect the following behavior:

$$Z(a, m; q, \epsilon_1 = \epsilon, \epsilon_2 \to 0) \sim \exp \left(-\frac{W(a, m; q, \epsilon)}{\epsilon_2}\right)$$

(3.1)

Our task is to demonstrate the validity of the asymptotics (3.1) and to evaluate $W$.

3.1. The limit shape. We now investigate the limit $\epsilon_2 \to 0$. Observe that each contribution to the full partition function (2.55), for small $\epsilon_2$, if expressed through the quantities $x_{i,a,k}$, contains the factor $\frac{1}{\epsilon_2}$ in the exponential. This suggests to look for a dominant contribution to (2.55) among the configurations with finite $(x_{i,a,k})$, for all $i \in \Gamma, a = 1, \ldots, v, k = 1, 2, \ldots$. Recall that for finite sums

$$\lim_{\epsilon_2 \to 0} \epsilon_2 \log \left(\sum_{\lambda \in \Lambda} \exp \frac{S_{\lambda}}{\epsilon_2}\right) = S_{\lambda_*}$$

(3.2)

where $\lambda_{*}$ is the element of the set $\Lambda$ on which

$$\text{Re} \left(\frac{S_{\lambda_*}}{\epsilon_2}\right)$$

is maximal

(3.3)

under the assumption that the phases $\text{Im} \left(\frac{S_{\lambda}}{\epsilon_2}\right)$ of the contributions of the configurations $\lambda$ with $\text{Re} \left(\frac{S_{\lambda}}{\epsilon_2}\right)$ close the maximal one, are also close to each other. A typical example is the evaluation of the series

$$\sum_{n=0}^{\infty} \frac{(q/\epsilon_2)^n}{n!}$$

for $\text{Re}(q/\epsilon_2) > 0$. The dominant contribution comes from $n \sim q/\epsilon_2$, i.e. from $x = \epsilon_2 n \sim q$. To make the similar evaluation for all values of $q/\epsilon_2$ we use the analyticity of the series (assuming it converges, which in the present example is easy to establish), and then evaluate it with an arbitrary accuracy for $q/\epsilon_2 \in \mathbb{R}_+$.

In the case at hand we need to show that there is a domain in the space of parameters $a, m, \epsilon_1, \epsilon_2$ such that the phases of all contributions to the partition function in the vicinity of the dominant one are aligned. We shall make the argument in the four
dimensional case, with \( \tilde{w}_1 = 0 \), leaving the general five dimensional case to a curious student.

It is convenient, for this purpose, to choose all the parameters in the problem: \( a_{i,a} \), \( \epsilon_{1,2} \), \( m_i \), \( m^f_i \), \( q_i \) etc., to be real. We shall also assume that \( \epsilon_1 = \epsilon \gg -\epsilon_2 > 0 \), and that \( a_{i,a} - a_{i,b} \gg \epsilon \) for \( a \neq b \) for all \( i \). We shall also assume that all masses are real, and much larger then \( |a_{i,a}| \) for all \( (i,a) \).

The contribution \( 2.55 \) \( Z_{i\alpha} \) of a set \( \lambda = (\lambda_{i,a}) \) of quiver colored partitions to the partition function \( Z \) is then also real. We only need to make sure it is positive for all \( \lambda \)'s close to the dominant one.

Let us now compare the contributions of two close configurations, namely \( \lambda \) and \( \lambda' \) obtained from \( \lambda \) by adding one square to the \( \lambda_{i,a} \) partition. The ratio can be easily evaluated to be (cf. the Eq. \( 3.33 \) below)

\[
Z_{\lambda'}/Z_{\lambda} = (-1)^{v_1-1 + \sum_{e \in s^{-1}(i)} v_{t(e)} - 1} \times \\
\frac{q_i P_i(x) (\epsilon_1 + \epsilon_2)}{\epsilon_1 \epsilon_2} \times \\
\prod_{e \in t^{-1}(i)} V_{s(e)}(x + m_e + \epsilon_1 + \epsilon_2) \prod_{e \in s^{-1}(i)} V_{t(e)}(x - m_e) \\
\times \frac{Y'_i(x) y'_i(x + \epsilon_1 + \epsilon_2)}{Y_i(x) y_i(x + \epsilon_1 + \epsilon_2)} \times \prod_{e \in s^{-1}(i) \cap \Gamma^{-1}(i)} \frac{(m_e + \epsilon_1)(m_e + \epsilon_2)}{m_e(m_e + \epsilon_1 + \epsilon_2)}
\]  

(3.4)

where \( x \) is one of the zeroes of \( Y_i \). It determines the position of the box one can add to \( \lambda_{i,a} \); in fact, \( x \) is equal to the \( a_{i,a} \)-shifted content of that box. By taking \( |m_e| \gg \epsilon \) we can neglect the last line in (3.4). Also, by taking the masses of the fundamentals to be much larger then the Coulomb moduli, we can drop the fundamental polynomial \( P_i(x) \), replacing it by \(-1)^{w_i} \).

Using the fact that the zeroes and the poles of \( Y_i \) interlace for \( \epsilon_1/\epsilon_2 < 0 \), and assuming \( |a_{i,a} - a_{i,b}| \gg \epsilon \), the sign of the denominator of (3.4) is the same as the sign of \( \epsilon_1 + \epsilon_2 \). Now, the final adjustment is the choice of the chamber where for all \( e \in E_\Gamma \), \( a = 1, \ldots, v_{s(e)} \), \( b = 1, \ldots, v_{t(e)} \):

\[
a_{t(e),b} - a_{s(e),a} + m_e \gg 0
\]  

(3.5)

It implies that the period of the 1-cochain \( m_e \) on any closed 1-cycle in the quiver is non-trivial. The sign of the numerator in the third line of (3.4) is then equal to

\[
\prod_{e \in t^{-1}(i)} (-1)^{v_{s(e)}}
\]

Collecting all the signs and using the equality in (2.2), we obtain:

\[
\text{sign} \frac{Z_{\lambda'}}{Z_{\lambda}} = \text{sign}(q_i) (-1)^{v_1 - 1}
\]  

(3.6)

Thus, we shall assume \( q_i \in (-1)^{v_1-1} \cdot \mathbb{R}_+ \).

Therefore, in the limit \( \epsilon_2 \to 0 \) the sum over the set of all fixed points can be computed semi-classically by finding the dominant configuration, as was originally done in [21].
The important difference is that now, for finite $\epsilon_1$, the profile of the dominant partition cannot be assumed to be a smooth function. Instead, the profile of partition $\lambda_{i,a}$ shall be described by an infinite series of continuous variables $(\xi_{i,a,k})$. This approach was suggested in [29, 31]. An alternative approach is based on the Mayer expansion [20] and the $\epsilon_2 \to 0$ analysis of the asymptotics of the gauge partition function expressed via the contour integral form of the ADHM integrals [1], leading to the TBA-like integral equations [5, 82]. It will be discussed elsewhere.

3.2. Entropy estimates. Let us now explain why finding the dominant configuration suffices for the evaluation of $\mathcal{W}(a, m; q, \epsilon)$.

Indeed, the argument similar to (3.2) may fail if the number $N(\epsilon_2)$ of configurations $\lambda$ whose actions $S_\lambda$ fall within an order $\epsilon_2$ error in the vicinity of $S_{\lambda_*}$, grows as $e^{C/\epsilon_2}$ for some constant $C$, for $\epsilon_2 \to 0$. This is how, for example, one may understand the phase transition in the Ising model, starting with the high temperature expansion, leading to the counting of random walks on the lattice [83].

Let us now explain why in our problem the entropy factor $N(\epsilon_2)$ grows at most in a power-like fashion with $1/\epsilon_2$.

Recall that the configurations $\lambda$ in our problem are the multi-sets $\lambda_{i,a}$ of partitions of integers, indexed by the vertices $i \in I_\Gamma$ of the quiver, and the colors $a = 1, \ldots, v_i$ of the corresponding gauge group factor.

Let us estimate the number of partitions, close to the given set $\lambda_{i,a}$ of partitions $\lambda_{i,a}$, such that the action $S_\lambda$ is close to $S_{\lambda_*}$. This number is equal to the number of boxes one can add, or remove from the Young diagrams of the all these partitions $\lambda_{i,a}$. For each row $j = 1, \ldots, \lambda_{i_1}$ we can add one box, $\lambda_j \mapsto \lambda_j + 1$ if $\lambda_{j-1} > \lambda_j$. Analogously we can remove one box $\lambda_j \mapsto \lambda_j - 1$ if $\lambda_j > \lambda_{j+1}$. At any rate, the number of such modifications of a single partition is bounded above by $2^{\lambda_{i_1}}$, and for the whole configuration $\lambda$ it is bounded above by

$$e^{\mathcal{E}_{\text{entropy}}} \leq \exp K' \sum_{i,a} (\lambda_{i,a})_{i_1}^t$$

for some constant $K'$. Now let us estimate $(\lambda_{i,a})_{i_1}^t$. We shall show below, using the analysis of the limit shape equations in the form of the Eq. (4.10), that

$$\epsilon_2 \lambda_{i,a,k} \sim C_{i,a,k} q_i^k, \quad \text{for} \quad k \to \infty$$

where

$$C_{i,a,k} = \frac{A_i(a_{i,a} + \epsilon k)}{A'_i(a_{i,a})} \times \prod_{j=1}^{k} \left( \frac{P_i(a_{i,a} + \epsilon(j-1))}{A_i(a_{i,a} + \epsilon j)} \prod_{e \in t^{-1}(i)} A_i(e)(a_{i,a} + \epsilon j - m_e) \prod_{e \in s^{-1}(i)} A_i(e)(a_{i,a} + \epsilon j + m_e) \right)$$

(3.9)

where

$$A_i(x) = \prod_{a=1}^{v_i} (x - a_{i,a})$$
Since for large \( k \), \( C_{i,a,k} \propto k^{K''} \) for some constant \( K'' \) which depends only on \((a, m, \epsilon)\), we can conclude that \((\lambda_{i,a})_1^1 \sim \frac{\log \epsilon}{\log q_i} \) so that

\[
e^{\text{entropy}} \lambda \leq \epsilon_2^{-K}
\]

for some uniformly bounded constant \( K \). Thus, the entropy factor is subleading compared to the exponential of the critical action \( \exp(S_\lambda / \epsilon_2) \).

**Remark.** Note that the generating function of the leading asymptotic s (3.9)

\[
J_{i,a}(q) = \sum_{k=1}^{\infty} C_{i,a,k} q^k
\]

is a generalized hypergeometric function, and can be identified with a vortex partition function \([84]\) of some gauged linear sigma model (and the \( J \)-function of \([85]\)). It would be interesting to understand the physics of this coincidence.

### 3.3. Analysis of the limit shape equations.

In the limit \( q_2 = 1 \) the functions \( y_i(\xi) \) (2.43) can be represented as the infinite product over the set \( \Xi_i \) (2.63) as

\[
y_i(\xi) = y_i^+(\xi) = \frac{Q_i^+(q^{\frac{1}{2}}\xi)}{Q_i^+(q^{-\frac{1}{2}}\xi)}
\]

where the Baxter function

\[
Q_i^+(\xi) = \prod_{\xi' \in \Xi_i} (1 - \xi'/\xi).
\]

The product (3.13) converges because of the set \( \Xi_i \) (2.63) is asymptotic to \( \Xi_i^0 \):

\[
\xi_{i,a,k} \rightarrow \xi_{i,a,k}^0; \quad k \rightarrow \infty
\]

where \( \xi_{i,a,k} = w_{i,a} q^{-1} \) by (2.74).

Explicitly

\[
y_i^+(\xi) := \prod_{a=1}^{v_i} \left( (1 - q^{-\frac{1}{2}}\xi_{i,a,1}/\xi) \prod_{k=1}^{\infty} \frac{1 - q^{-\frac{1}{2}}\xi_{i,a,k+1}/\xi}{1 - q^{\frac{1}{2}}\xi_{i,a,k}/\xi} \right)
\]

It is convenient to introduce the notation \( y_i^-(\xi) \) for

\[
y_i^-(\xi) := y_i^+(\xi) \prod_{a=1}^{v_i} \left( -\frac{q^{\frac{1}{2}}\xi}{w_{i,a}} \right)
\]

and from the asymptotics (3.14), (2.74) we see\footnote{This product formula follows from}

\[
y_i^+(\xi) = y_i^-(\xi) \left( \prod_{a=1}^{v_i} \frac{1 - q^{-\frac{1}{2}}\xi_{i,a,1}/\xi}{1 - q^{\frac{1}{2}}\xi/\xi_{i,a,1}} \prod_{k=1}^{\infty} \frac{\xi_{i,a,k+1}}{q^{\xi_{i,a,k}}} \right)
\]
The asymptotics of $y_i^\pm(\xi)$ functions are (c.f. (2.46))

$$\xi \to 0 : \quad y_i^+(\xi) \to \xi^{-v_i} \prod_{a=1}^{v_i} \left(-w_{i,a} q^{-\frac{2}{p}}\right), \quad y_i^-(\xi) \to 1$$

$$\xi \to \infty : \quad y_i^+(\xi) \to 1, \quad y_i^-(\xi) \to \xi^{v_i} \prod_{a=1}^{v_i} \left(-w_{i,a} q^{\frac{2}{p}}\right)$$

(3.19)

Let

$$P_i^-(\xi) = \prod_{f=1}^{\hat{m}_i} (1 - \xi / \mu_{i,f}), \quad \tilde{P}_i^+(\xi) = \prod_{f=1}^{\hat{m}_i} (1 - \tilde{\mu}_{i,f} / \xi)$$

(3.20)

be the fundamental (anti-fundamental) matter polynomials.

The critical point equations are obtained by a simple variation of the exponents in the constituent factors of (2.55). Using

$$\partial_{\log \xi} \text{Li}_2(\xi) = \text{Li}_1(\xi) = -\log(1 - \xi)$$

(3.21)

we find the critical point equations on the dominant fixed point $\lambda_s$ encoded by the sets $\Xi_1$ to be

$$\exp(i \ell e_2 \partial_{\log \xi} \text{Li}_{\Xi_{1,a,k}} \log Z_{\lambda_s}) = 1$$

(3.22)

for each $\Xi_{1,a,k}$ denoted below as $\xi$. Explicitly, we find

$$\exp(i \ell e_2 \partial_{\log \xi} \log Z_{\lambda}) =$$

$$- q_i (q^\frac{2}{p} \xi)^{\kappa_i} \frac{P_i^-(\xi) \tilde{P}_i^+(\xi)}{y_i^-(q^\frac{2}{p} \xi) y_i^+(q^{-\frac{2}{p}} \xi)} \prod_{e \in e^{-1}(i)} y_{s(e)}(\mu e^{-1}) \prod_{e \in e^{+1}(i)} y_{l(e)}(\mu e), \quad \xi \in \Xi_1$$

(3.23)

Therefore, the critical profiles (2.48) are the solutions to the following infinite system of difference equations in the variables $\xi_{1,a,k}$: For each $i \in I_1$ and for each $\xi \in \Xi_1$,

$$y_i^-(q^\frac{2}{p} \xi) y_i^+(q^{-\frac{2}{p}} \xi) =$$

$$- q_i (q^\frac{2}{p} \xi)^{\kappa_i} P_i^-(\xi) \tilde{P}_i^+(\xi) \prod_{e \in e^{-1}(i)} y_{s(e)}(\mu e^{-1}) \prod_{e \in e^{+1}(i)} y_{l(e)}(\mu e), \quad \xi \in \Xi_1$$

(3.24)

It would appear that the left hand side of the equations (3.24) cannot be evaluated at $\xi = \xi_{1,a,k}$ because of the first order pole in $y_i^-(q^{1/2} \xi)$ at $\xi \to \xi_{1,a,k}$. However, $y_i^+(q^{-\frac{2}{p}} \xi)$ has a first order zero at $\xi \to \xi_{1,a,k}$. Therefore the product $y_i^-(q^\frac{2}{p} \xi) y_i^+(q^{-\frac{2}{p}} \xi)$ is non-singular in the limit $\xi \to \xi_{1,a,k}$. The equations (3.24) are formally equivalent to the algebraic trigonometric Bethe ansatz equations for the $\mathfrak{g}_1$-spin chain, with an infinite number of Bethe roots $\xi_{1,a,k}$, see e.g. (86,89). These Bethe roots come in the Regge-like trajectories, or Bethe strings. The strings are labeled by the quiver nodes $i$ and the individual colors $a = 1, \ldots, v_i$. Each string contains an infinite number of Bethe roots $\xi_{1,a,k}$ labeled by $k = 1, \ldots, \infty$. Recall, that the sequences $(\xi_{1,a,k})$ parametrize the asymptotic profiles of the colored partitions $(\lambda_{1,a,k})$. 

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We summarize: the critical profile equations of \cite{21,23} for the five dimensional $\mathfrak{g}_\Gamma$-quiver theory with $\epsilon_1 = \epsilon$ finite and $\epsilon_2 = 0$ are the Bethe equations for a formal $\mathfrak{g}_\Gamma$ XXZ spin chain. The number of Bethe roots is infinite with prescribed regular asymptotics at infinity: they have the slope given by $\epsilon$ and the intercepts equal to the Coulomb moduli $a$ of the theory.

3.3.1. On-shell versus off-shell. Note that these Bethe equations should not be confused with those in \cite{14} where one extremizes with respect to the Coulomb moduli $a$. Here we do no such minimization, so we are off-shell in that sense. In other words, in order to construct the off-shell superpotential, more precisely the universal twisted superpotential $\mathcal{W}(a, m, q; \epsilon)$ we solve the Bethe equation for the $U_q(\mathfrak{g}_\Gamma)$ XXZ spin chain; in order to find the states in a proper Hilbert space, i.e. to go on-shell we need to solve another Bethe equation which follows from the extremization of $\mathcal{W}^{\text{eff}}$ with expect to the Coulomb moduli $a$.

For $A_1$ theories the proposition was formulated with some restrictions in \cite{29,31,90}. The general ADE case was presented in \cite{60,61}, the formal construction in the four dimensional case can also be found in \cite{67}.

Remark. Since $Y_i^+(\xi)$ and $Y_i^-(\xi)$ as well as $P_i^-(\xi)$ and $P_i^+(\xi)$ differ only by a power of $\xi$ and a constant factor, one can change an orientation of an edge in the quiver, or replace the fundamental by anti-fundamental matter and adjust the Chern-Simons level $\kappa_i$ so that it compensates the change of the orientation in the equations (3.23); in this way such quiver theories are isomorphic.

Using (3.16) we can write the equations (3.23) only in terms of $Y_i^+(\xi)$ as

$$Y_i^-(q^{\frac{\sqrt{2}}{2}}\xi)Y_i^-(q^{-\frac{\sqrt{2}}{2}}\xi) = -P_i^-(\xi) \prod_{e \in t^{-1}(i)} Y_{s(e)}^-(\mu_e^{-1}\xi) \prod_{e \in s^{-1}(i)} Y_{t(e)}^-(\mu_e\xi), \quad \xi \in \Xi_i$$  \hspace{1cm} (3.25)

where

$$P_i^-(\xi) = -q_1(q^{\frac{\sqrt{2}}{2}}\xi)^{\kappa_i} \prod_{a=1}^{v_i} \left( -\frac{\xi}{w_{i,a}} \right) P_i^-(\xi) \tilde{P}_i^+(\xi) \left( \prod_{e \in s^{-1}(i)} \prod_{a=1}^{v_{i(e)}} \left( -\frac{w_{t(e),a}}{q^{\frac{\sqrt{2}}{2}}\mu_e\xi} \right) \right)$$  \hspace{1cm} (3.26)

or, equivalently, in term of $Y_i^+(\xi)$ as

$$Y_i^+(q^{\frac{\sqrt{2}}{2}}\xi)Y_i^+(q^{-\frac{\sqrt{2}}{2}}\xi) = -P_i^+(\xi) \prod_{e \in t^{-1}(i)} Y_{s(e)}^+(\mu_e^{-1}\xi) \prod_{e \in s^{-1}(i)} Y_{t(e)}^+(\mu_e\xi), \quad \xi \in \Xi_i$$  \hspace{1cm} (3.27)

where

$$P_i^+(\xi) = -q_1(q^{\frac{\sqrt{2}}{2}}\xi)^{\kappa_i} \prod_{a=1}^{v_i} \left( -\frac{w_{i,a}}{q^{\frac{\sqrt{2}}{2}}\xi} \right) P_i^-(\xi) \tilde{P}_i^+(\xi) \left( \prod_{e \in t^{-1}(i)} \prod_{a=1}^{v_{i(e)}} \left( -\frac{\xi}{q^{\frac{\sqrt{2}}{2}}\mu_e w_{t(e),a}} \right) \right)$$  \hspace{1cm} (3.28)

The matter polynomials $P_i^-(\xi) \in \mathbb{C}[\xi, \xi^{-1}]$ are Laurent polynomials in $\xi$, with the following asymptotics at $\xi \to 0, \infty$

$$\xi \to 0 : \quad P_i^-(\xi) \sim \xi^{-\tilde{w}_i + \kappa_i + v_i - \sum_{j \in \ell(i)} v_j}$$

$$\xi \to \infty : \quad P_i^-(\xi) \sim \xi^{w_i + \kappa_i + v_i - \sum_{j \in \ell(i)} v_j}$$  \hspace{1cm} (3.29)
Now pick the CS levels $\kappa_i$ in such a way to set asymptotics $\mathcal{P}^-_i(\xi) \sim \xi^0$ at $\xi \to 0$

$$\kappa^-_i = -v_i + \tilde{w}_i + \sum_{j \in \ell(i)} v_j$$

(3.30)

Then

$$\kappa_i = \kappa^-_i : \begin{cases} \xi \to 0 : & \mathcal{P}^-_i(\xi) \sim \xi^0 \\ \xi \to \infty : & \mathcal{P}^-_i(\xi) \sim \xi^{w_i + \tilde{w}_i} \end{cases}$$

(3.31)

Similarly, the matter polynomials $\mathcal{P}^+_i(\xi) \in \mathbb{C}[\xi, \xi^{-1}]$ have the following asymptotics at $\xi \to 0, \infty$

$$\xi \to 0 : \quad \mathcal{P}^+_i(\xi) \sim \xi^{-\tilde{w}_i + \kappa_i - \sum_j s_j v_j}$$

$$\xi \to \infty : \quad \mathcal{P}^+_i(\xi) \sim \xi^{w_i + \sum_j s_j v_j}$$

(3.32)

and we can pick the CS levels $\kappa_i^+$ in such a way to set asymptotics $\mathcal{P}^+_i(\xi) \sim \xi^0$ at $\xi \to \infty$

$$\kappa^+_i = v_i - w_i - \sum_{j \in \ell(s(i))} v_j$$

(3.33)

then

$$\kappa_i = \kappa^+_i : \begin{cases} \xi \to 0 : & \mathcal{P}^+_i(\xi) \sim \xi^{-\tilde{w}_i - w_i} \\ \xi \to \infty : & \mathcal{P}^+_i(\xi) \sim \xi^0 \end{cases}$$

(3.34)

Notice that $\kappa_i^\pm$ are precisely the boundary of the allowed range of CS couplings (2.24), and $\kappa^-_i = \kappa^+_i$ for the quiver gauge theories in the conformal class (2.22).

If the set of bifundamental masses $\mu_e$ is a trivial cocycle, i.e. if for any 1-cycle

$$c = \sum_e c_e [e] \in C_1(\gamma), \quad \partial c = \sum_e c_e ([t(e)] - [s(e)]) = 0 \in C_0(\gamma)$$

$$\prod_{e \in \text{cycle in } \gamma} \mu_e^{c_e} = 1,$$

(3.35)

then we can find the set $(\tilde{\mu}_i)_{i \in I}$ of compensators $\tilde{\mu}_i \in \mathbb{C}^\times$, such that

$$\mu_e = \tilde{\mu}_{s(e)}/\tilde{\mu}_{t(e)}$$

(3.36)

This is possible for all acyclic quivers, that is for all cases except affine $A$-quiver. Then, if we set

$$\tilde{\mathcal{P}}^\pm_i(\xi) = \mathcal{P}^\pm_i(\xi/\tilde{\mu}_i), \quad \tilde{\mathcal{P}}^\pm_i(\xi) = \mathcal{P}^\pm_i(\xi/\tilde{\mu}_i)$$

(3.37)

the equations (3.25) (3.27) convert to

$$\tilde{Y}_i(q^{\frac{1}{2}}\xi)\tilde{Y}_i(q^{-\frac{1}{2}}\xi) = -\tilde{\mathcal{P}}_i(\xi) \prod_{j \in \ell(s^{-1}(i)) \cup \ell(t^{-1}(i))} \tilde{y}_j(\xi), \quad \xi \in \Xi_i,$$

(3.38)

where $\tilde{Y}_i(\xi), \tilde{\mathcal{P}}(\xi)$ denotes $+$ or $-$ form of $\tilde{Y}_i^\pm(\xi), \tilde{\mathcal{P}}_i^\pm(\xi)$.

The equations (3.38) are the $q$-version of the cross-cut equations of [23].

For $A_r$ quiver we consider its universal covering, the $A_{\infty}$ quiver, and the infinite set of functions $Y_i(\xi)$ for $i \in \mathbb{Z}$ with twisted periodicity mod $r + 1$. 
Let $\Gamma$ be $\widehat{A}_r$ quiver with $r + 1$ nodes on a cycle, labeled $0, \ldots, r$, and $(\mu_e)_{e \in E_{\Gamma}}$ representing a non-trivial class in $H^1(\gamma, \mathbb{C})$. Let us define $\mu \in \mathbb{C}^\times$ by

$$\prod_e \mu_e = \mu^{r+1} \quad (3.39)$$

Then $(\mu_e/\mu)_e$ represents a trivial cocycle, for which one can find the compensators, and define $Y'_i$ as above. Now the equations $3.25$ $3.27$ convert to

$$\tilde{y}_i(q^{\frac{r}{2}}\xi)\tilde{y}_i(q^{-\frac{r}{2}}\xi) = -\tilde{P}_i\tilde{y}_{i-1}(\mu^{-1}\xi)\tilde{y}_{i+1}(\mu\xi), \quad i \in \mathbb{Z}/(r+1)\mathbb{Z}, \quad \xi \in \Xi'_i \quad (3.40)$$

Let us now define functions $\tilde{y}_i(\xi)$ labeled by the vertices $i \in \mathbb{Z}$ on the universal cover $\tilde{\Gamma}$ of $\gamma$

$$\tilde{y}_i(\xi) = \tilde{y}_{i_{\text{mod} r+1}}(\mu^i\xi) \quad i \in \mathbb{Z} \quad (3.41)$$

In terms of functions $\tilde{y}_i(\xi)$ the equations $3.40$ take the canonical form of equations $3.38$ for $A_{\infty}$ quiver

$$\tilde{y}_i(q^{\frac{1}{2}}\xi)\tilde{y}_i(q^{-\frac{1}{2}}\xi) = -\tilde{P}_i\tilde{y}_{i-1}(\xi)\tilde{y}_{i+1}(\xi), \quad i \in \mathbb{Z}, \quad \xi \in \Xi_i \quad (3.42)$$

In any case, the limit shape equations are $3.38$ associated to the universal cover $\tilde{\Gamma}$ of the original quiver $\Gamma$, with $\Gamma = \Gamma$ if $\Gamma$ is of the finite $ADE$ or affine $\widehat{D}\widehat{E}$ type, and $\Gamma = A_{\infty}$ if $\Gamma = \widehat{A}_r$. In the $\widehat{A}_r$ case the functions $y_i(\xi)$ are lifted to the cover with the twisted $r + 1$-periodicity. In the language of quantum groups, such lifting corresponds to the relation between $U_q^{\text{aff}}(\mu \widehat{g}_r)$ and $U_q(\mathfrak{gl}_\infty)$ see $[91, 92]$.

In the subsequent sections we drop the symbol tilde on functions $\tilde{y}_i(\xi)$ in the canonical form $3.38$, keeping in mind the change of variables $3.37$.

4. Quantum geometry

The set of algebraic equations on the set of Bethe roots $(\xi_{i,a,k})$ in the theory of $\mathfrak{g}_r$ spin chains (known as the nested algebraic Bethe Ansatz equations in the case $\mathfrak{g}_r = \mathfrak{sl}_r$), one equation per each Bethe root, can be converted to a set of functional $q$-difference equations on functions $y_1(\xi)$ (for elementary introduction to algebraic Bethe Ansatz see $[93]$), for most generic case see $[49, 94]$, and for solvable lattice models of statistical mechanics see $[95]$). The zeroes and poles of the functions $y_i(\xi)$ encode information about Bethe roots. In the limit $q \to 1$ the functional $q$-difference equations on $y_i(\xi)$ reduces to the ordinary character equations determining the Seiberg-Witten curve in $[23]$. The functions $y_1(x)$ of $[23]$ provide the classical limit of their more complicated $q$-relatives $y_1(x)$ of the present paper.

For illustration we first consider the quiver $\mathfrak{g}_r = A_1 = \mathfrak{sl}_2$. The Bethe Ansatz equations are simply (we can use here either plus or minus form of functions, $y_1^\pm(\xi) , P_1^\pm(\xi)$)

$$y_1(q^{\frac{1}{2}}\xi)y_1(q^{-\frac{1}{2}}\xi) = -P_1(\xi), \quad \xi \in \Xi_1 \quad (4.1)$$
with $\Xi_1$ being the set of poles of $Y_1(q^{1/2}\xi)$, and we remind that $\xi \in \Xi_1$ should be understood in the limit sense (see remark after (3.24)). Now consider the functional

$$\chi_1[Y_1(\xi)] = Y_1(\xi) + \frac{P_1(q^{-1/2}\xi)}{Y_1(q^{-1}\xi)}$$

(4.2)

The function $\chi_1(\xi)$ is regular function everywhere on the multiplicative plane $\mathbb{C}_*^{\mathbb{Z}}$ provided the equations (4.1). Indeed, the poles which appear in the first term, $Y_1(\xi)$, are precisely canceled by the second term due (4.1). Therefore, $\chi_1(\xi)$ is Laurent polynomial in $\mathbb{C}[\xi, \xi^{-1}]$, i.e. a holomorphic function on the cylinder $\mathbb{C}_*^{\mathbb{Z}}$. Now consider the asymptotics of $\chi_1(\xi)$ at $\xi \to 0$ and $\xi \to \infty$.

We find that the degree in $\xi$ of the first term and second term in $\chi_1(\xi)$ of (4.2) is respectively (using $Y_1(\xi), P_1(\xi)$ form)

$$\xi \to 0 : \ [\xi^0], \ [\xi^0]$$

$$\xi \to \infty : \ [\xi^v], \ [\xi^{w_1+w_1-v_1}]$$

(4.3)

Therefore, at $\xi \to \infty$ the first term dominates the second term by the positivity condition (2.1). We conclude that

$$\chi_1[Y_1(\xi), P_1(\xi)] = T_1(\xi)$$

(4.4)

where $T_1(\xi)$ is polynomial in $\xi$ of degree $v_1$, in which the highest and lowest coefficients in $\xi$ are fixed from the known asymptotics of $Y_1(\xi)$ and $P_1(\xi)$, i.e. in terms of masses and coupling constants. The functional $q$-difference equation (4.4) is quantum version of Seiberg-Witten for the $A_1$ quiver theory, that is $SU(v_1)$ gauge theory with $w_1$ and $\bar{w}_1$ fundamentals and anti-fundamental. The Laurent polynomial $\chi_1$ in $\mathbb{C}[Y(q^{1/2})^\pm]$ given by (4.2) is the Frenkel-Reshetikhin $q$-character for $U_q^{\text{aff}}(A_1)$ (49). The $v_1 - 1$ middle term coefficients in gauge polynomial $T_1(\xi)$ play the role of the Coulomb moduli of the theory and are implicitly defined in terms of the $SU(v_1)$ Coulomb parameters $a_{1,\alpha}$.

Note that in the four dimensional $\ell \to 0$ limit the Eq. (4.4) formally becomes identical to the Baxter equation for the $A_1$ XXX spin chain. It is not yet a vacuum equation of the four dimensional $\mathcal{N} = 2$ theory in the $\Omega$-background. As we discussed in the Introduction, the latter requires the second minimization with respect to the Coulomb moduli $a$ and this corresponds to finding the proper solution of Baxter equation. We know the definition of proper from the integrable model side only in a few cases — in particular in the cases considered in [3]. Similar interpretation holds for the five dimensional $A_1$ case (and XXZ spin chain), as well as for the arbitrary $ADE$ or affine $\tilde{A\tilde{D}\tilde{E}}$ quivers both in five and four dimensions.

The cancellation of poles between the two terms in (4.2) is the $q$-analogue of the invariance of the classical character used in (23) under crossing the cuts supporting the charge densities used to define the classical potentials $Y_i(\xi)$. In the classical limit $q \to 1$ the strings of Bethe roots condense on a certain finite intervals $I_{i,\alpha}$, which are interpreted as cuts of the analytic functions $Y_i(\xi)$. The remaining Bethe roots form the semi-infinite strings $(\xi_{i,\alpha,k} | k > k_{i,\alpha}^*)$ for certain $k_{i,\alpha}^*$, and the classical limit their slope approaches a constant so that their contribution to $Y_i(\xi)$ defined by the product formula (3.12) cancels.
Below we will use short-hand notations
\[ Y_{i,b} = Y_i(q^{-b}\xi) \]
\[ P_{i,b} = P_i(q^{-b}\xi) \] (4.5)

For any (affine) simply-laced \( \mathfrak{g}_r \) one can find recursively the \( q \)-characters \( \chi_i \) for \( i \)-th fundamental evaluation representation of \( U_q^{aff}(\mathfrak{g}_r) \) as follows. Start from the highest weight \( \bar{\chi}_i = Y_{i,0} + \ldots \) and add a term to cancel the poles in the highest weight term
\[ \bar{\chi}_i = Y_{i,0} + \ldots \] (4.6)

This term introduces new poles from the functions \( Y_j(\xi) \) of the nodes \( j \) linked with \( i \). To cancel these poles, we add new monomials \( Y_{j,b}(\xi) \) to (4.7) until all poles are canceled. The precise algorithm to compute the \( q \)-characters (4.4) for \( U_q^{aff}(\mathfrak{g}_r) \) is given by Frenkel-Mukhin [96] for all finite dimensional Lie algebras \( \mathfrak{g}_r \). It is generalized for the general symmetrizable Kac-Moody Lie algebra \( \mathfrak{g}_r \) in [97].

The symbols \( Y_{i,b}(\xi) \) in the \( q \)-character are the current-weights encoding the (generalized) eigenvalues of an element of the maximal commuting subalgebra of \( U_q^{aff}(\mathfrak{g}_r) \) on a (generalized) eigenspace in a representation of \( U_q^{aff}(\mathfrak{g}_r) \), see [49, 94] and appendix for details. We give explicit formula for the current \( h(x) \in U_q^{aff}(\mathfrak{g}_r) \) in (5.30), (5.12).

The \( q \)-analogue of the \( i \)Weyl group \( W_{\mathfrak{g}_r} \) of [23] is the braid group \( B(W_{\mathfrak{g}_r}) \), see e.g. [98]. We remind that as \( \xi \) crosses a classical cut of \( Y_i(\xi) \) the \( Y_i(\xi) \) is transformed by the reflection in \( W_{\mathfrak{g}_r} \) generated by \( i \)-th simple root [23]. The braid group \( B(W_{\mathfrak{g}_r}) \) is freely build on generators \( T_i, i \in \mathcal{I}_r \) with the relations (for simply laced \( \mathfrak{g}_r \))
\[ T_i T_j = T_j T_i, \quad a_{ij} = 0 \]
\[ T_i T_j T_i = T_j T_i T_j, \quad a_{ij} = -1. \] (4.8)

The braid group \( B(W_{\mathfrak{g}_r}) \) action on the weight space of \( U_q^{aff}(\mathfrak{g}_r) \) generated by fundamental weights \( Y_i \) is the natural analogue of the Weyl reflection
\[ T_i : Y_{i,0} \mapsto \frac{P_{i,b}}{Y_{i,1}} \prod_{j \in \{i,s^{-1}(i)\} U(s^{-1}(i))} Y_j^{\frac{1}{2}}, \quad Y_j \mapsto Y_j, \quad j \neq i. \] (4.9)

The central result of the present work is the following statement, which could be seen as the \( q \)-version of the cameral Seiberg-Witten curve of [23] for \( \mathfrak{g}_r \)-quiver gauge theories.

**Proposition.** The \( \mathcal{N} = 2 \) gauge theory functions \( \{Y_i(\xi)\}_{i \in \mathcal{I}_r} \) of \( \mathfrak{g}_r \)-quiver gauge theory on \( \mathbb{R}^4_x \times S^1_\ell \) (the \( \Omega \)-background of [5]) defined by (2.43) satisfy a system of functional \( q = e^{\ell t} \) difference equations
\[ \{\tilde{\chi}_i \left[ \{Y_j(q^{\frac{1}{2}}\xi), P_j(q^{\frac{1}{2}}\xi)\}_{i \in \mathcal{I}_r} \right] = T_i(\xi) \mid i \in \mathcal{I}_r \} \] (4.10)

where \( \tilde{\chi}_i \) is the twisted \( q \)-characters for the \( i \)-th fundamental module of \( U_q^{aff}(\mathfrak{g}_r) \) (see [49, 97] and Appendix), and \( T_i(\xi) \) is a polynomial of degree \( v_i \) with the highest and
lowest coefficient fixed by the masses and coupling constants of the theory, and the middle \(v_i - 1\) coefficients parametrizing the Coulomb branch.

4.1. The slope estimates. It follows from (4.10) that \(\xi_{i,a,k}\) approach \(\xi_{i,a,k}^0\) as \(k \to \infty\) exponentially fast (for \(|q_i| < 1\)):

\[
\xi_{i,a,k} = \xi_{i,a,k}^0 + \mathcal{C}_{i,a,k} q_i^k, \quad k \to \infty
\]

(4.11)

where \(\mathcal{C}_{i,a,k}\) all have a finite limit as \(q \to 0\), and behave at most power-like with \(k\) for large \(k\). The idea is to study the small \(q_i\) limit of (4.10) evaluated at \(\xi = \xi_{i,a,k}^0 = w_{i,a} q_i^{-k-1}\) in five dimensions, or, at \(x = a_i + \epsilon (k-1)\) in four dimensions. Let us stick with four dimensions. It is easy to see that

\[
\begin{align*}
\gamma_i(a_i + \epsilon (k-1)) &\sim q_i A_i(a_i + \epsilon (k-1)) \frac{\mathcal{C}_{i,a,k}}{\mathcal{C}_{i,a,k-1}}, \quad k > 1 \\
\gamma_i(a_i) &\sim q_i A_i(a_i) \frac{\mathcal{C}_{i,a,1}}{\mathcal{C}_{i,a,1}}, \quad k > 1 \\
\gamma_j(a_i + \epsilon (k-1)) &\sim A_j(a_i + \epsilon (k-1)), \quad j \in I_r, \quad j \neq i
\end{align*}
\]

(4.12)

so that in the \(q\)-character \(\chi_i\) it is the term proportional to \(q_i\) will survive the \(q \to 0\) limit, all other terms being suppressed either by the explicit powers of \(q_i\)'s, or by the asymptotics of \(\gamma_i\) itself, cf. (4.12). From this the Eq. (4.9) follows.

4.2. The convergence of \(q\)-characters. Because of the constraints (2.4) on the Chern-Simons couplings the highest term in (4.7) dominates the other terms in the limit \(\xi \to \infty\) (using \(-\) version of \(\mathcal{P}_i(\xi), \gamma_i(\xi)\)) and \(\xi \to 0\) (using \(+\) version of \(\mathcal{P}_i(\xi), \gamma_i(\xi)\)). For affine \(g_r\)-quiver theories in the conformal class (2.2) all terms have the same degree in \(\xi\). For affine \(g_r\) quiver, the \(q\)-character \(\chi_i\) is infinite convergent series in coupling constants \((q_i)_{i \in I_r}\) under the assumption \((|q_i| < 1)_{i \in I_r}\).

4.3. The observables. In this section we record the formulae for the observables of the gauge theory which are encoded in the functions \(\gamma_i(\xi)\) obeying (4.10). For the practical purposes, the expansion of functions \(\gamma_i(\xi)\) in the instanton parameters \(q_i\) can be conveniently obtained from (4.10) by representing the solution \(\gamma_i(\xi)\) in the continuous fraction form in terms of the polynomials \(T_i(\xi)\) and \(\mathcal{P}_i(\xi)\) at \(\xi \to \infty\) starting from the zeroth order approximation (3.19)

\[
\gamma_i(\xi) = \xi^{v_i} \prod_{a=1}^{v_i} (-w_{i,a}^{-1} q_i^{2}) + \ldots \quad \xi \to \infty
\]

(4.13)

The Chern character of the universal bundle \(\mathcal{E}_i\) over the instanton moduli space corresponds to the observable

\[
\mathcal{O}_{i,n} = \text{tr}_v \psi_i e^{i \psi_i \phi_i}
\]

(4.14)

in the gauge theory where \(\phi_i\) is the adjoint complex scalar of the \(N = 2\) supersymmetric \(SU(v_i)\) gauge vector multiplet. The relation between (1.8) and (2.43) follows from the
expansion of $Y_i(q^{-\frac{1}{2}}\xi)$ at $\xi = \infty$

$$\text{tr}_{\mathfrak{v}_i} e^{i\ell_{\phi_i}} = \int_{\xi=\infty} \xi^n d\log Y_i^{-}(q^{-\frac{1}{2}}\xi)$$

(4.15)

where the contour runs around $\xi = \infty$ and encloses all zeroes and poles of $Y_i$. Therefore, the gauge theoretic definition of the functions $Y_i^{\pm}(\xi)$ is:

$$Y_i^{-}(q^{-\frac{1}{2}}\xi) = \xi v_i^{\pm} \prod_{a=1}^{v_i} (-w_{i,a}^{-1})$$

$$Y_i^{+}(q^{-\frac{1}{2}}\xi) = \sum_{i=1}^{\infty} \frac{\xi^{-n}}{n} \left( \text{tr}_{\mathfrak{v}_i} e^{i\ell_{\phi_i}} \right)$$

$$= \prod_{a=1}^{v_i} (-w_{i,a}^{-1}) \exp \left( \log \det_{\mathfrak{v}_i} \left( \xi - e^{i\ell_{\phi_i}} \right) \right)$$

(4.16)

and

$$Y_i^{+}(q^{-\frac{1}{2}}\xi) = \exp(\langle \det_{\mathfrak{v}_i} (1 - e^{i\ell_{\phi_i}}/\xi) \rangle)$$

(4.17)

4.4. **Superpotential.** Here we compute the $q$-derivatives of the twisted superpotential

$$W(a, m; q, \epsilon) = -\lim_{\epsilon_2 \to 0} \epsilon_2 \log Z(a, m; q, \epsilon_1 = \epsilon, \epsilon_2)$$

(4.18)

From (3.1), (2.73) we find

$$W_i := \log_{q_i} W := -\sum_{\xi \in \Xi_i} \log \frac{\xi}{\xi_{\text{crit}}}$$

(4.19)

Then, using the Baxter function $Q_i^{\pm}(\xi; \Xi)$ (3.13) we find

$$W_i = -\frac{1}{2\pi i} \int_{C_i} \log \xi d\log \frac{Q_i^{\pm}(\xi; \Xi)}{Q_i^{\pm}(\xi; \Xi_i)}$$

(4.20)

in terms of the sets of Bethe roots $\Xi_i = \{\xi_{i,a,k}\}$ (2.63) and $\hat{\Xi}_i = \{\xi_{i,a,k}\}$ (2.74), where the contour $C_i$ encloses all points in $\Xi_i$ and $\hat{\Xi}_i$. In terms of the functions $Y_i^{\pm}(\xi)$ we find

$$\frac{Q_i^{\pm}(\xi)}{Q_i^{\pm}(\xi_i)} = \prod_{j=0}^{\infty} \frac{Y_i^{-}(q^{-j}\xi)}{Y_i^{+}(q^{-j}\xi)}$$

(4.21)

For practical purposes of computation $W_i$ in the $q$-expansion it is convenient to integrate by parts and represent the contour $C_i$ as the difference of contours around $\xi = \infty$ and $\xi = 0$. Then we get

$$W_i = \frac{1}{2\pi i} \int_{C_{\infty} - C_0} \frac{d\xi}{\xi} \log \prod_{j=0}^{\infty} \frac{Y_i^{-}(q^{-j}\xi)}{Y_i^{+}(q^{-j}\xi)}$$

(4.22)

Using the continuous fraction expression of $Y_i^{\pm}(\xi)$ from the system of $q$-characters the integrand can be expanded in the series in $q$ with coefficients in the rational functions of $\xi$ after which the contour integral is computed by taking the difference of coefficients at $\xi^{-1}$ in the series expansion at $\xi = 0$ and at $\xi = \infty$. 




4.5. Dual periods. Finally, we give the formula for the partial derivatives
\[ a_{i,a}^D = \partial_{ai,a} \mathcal{W} \] (4.23)
which can be obtained from the variational limit shape problem

\[ \exp \left( a_{i,a}^D \right) = \prod_{k=1}^{\infty} \frac{\prod_{c \in E} \gamma_{s(c)}(\mu^{-1}_c \xi_{i,a,k}) \prod_{b \in E} \gamma_{t(c)}(\mu_c \xi_{i,a,k})}{\prod_{b \neq a} \gamma_{i,b}(q^\pm \xi_{i,a,k}) \gamma_{i,b}(q^{-\pm} \xi_{i,a,k})} \] (4.24)

where we using the colored functions \( \gamma_{i,a}(\xi) \) defined as factors in the product over \( a \) in (3.15) and

\[ \gamma_{i,a}(\xi) := (1 - q^{\pm \xi} / \xi_{i,a,1}) \prod_{k=1}^{\infty} \frac{1 - q^{\pm \xi} / \xi_{i,a,k+1}}{1 - q^{-\pm \xi} / \xi_{i,a,k}} \] (4.25)

so that (c.f. 3.12)

\[ \gamma_{i,a}(\xi) = (-q^{\pm \xi} / w_{i,a}) \gamma_{i,a}(\xi) \] (4.26)

It would be interesting to compare the above computation of \( a_{i,a}^D \) with the approach of [99, 44].

5. Quantum Groups

5.1. Introduction. In [100, 101] Drinfeld and Jimbo introduced the notion of Quantum Groups generalizing the algebraic structures underlying quantum integrable systems known at that time after the work of Sklyanin, Takhtajan, Faddeev, Kulish, Reshetikhin and others [102–108]. Namely, for any symmetrizable Kac-Moody Lie algebra \( g \) and a complex parameter \( q \in \mathbb{C}^\times \) the works [100, 101] associated the Hopf algebra \( U_q(g) \), also often called the quantum group, by \( q \)-deforming the Serre relations of \( g \), this construction is often referred as first Drinfeld realization [100, 101].

Later Drinfeld found [47] that for an affine Kac-Moody Lie algebra \( \hat{g} \), the algebra \( U_q(\hat{g}) \) can be obtained by a certain canonical quantum affinization process applied to finite-dimensional simple Lie algebra \( g \), with the result being quantum affine algebra \( U^\text{aff}_q(g) \cong U_q(\hat{g}) \) [50, 109, 110]. The construction [47] is known as second Drinfeld realization of quantum affine algebras, or Drinfeld loop realization, or Drinfeld currents realization.

In fact, Drinfeld current realization of \( U^\text{aff}_q(\mathfrak{g}_\Gamma) \) is defined for any symmetrizable Kac-Moody algebra \( \mathfrak{g}_\Gamma \) [111, 112]. If \( \mathfrak{g}_\Gamma = \hat{g} \) is an affine Kac-Moody Lie algebra, then the \( \mathfrak{g}_\Gamma \)-quantum affine algebra \( U^\text{aff}_q(\mathfrak{g}_\Gamma) = U^\text{aff}_q(\hat{g}) \) is often called the \( g \)-quantum toroidal algebra [58, 113], because of the presence of two loops inside. The \( \mathfrak{g}_\Gamma \)-quantum affine algebra \( U^\text{aff}_q(\mathfrak{g}_\Gamma) \) prominently appears in Nakajima’s work [112, 114, 115] on equivariant \( K \)-theory of quiver varieties associated to \( \Gamma \).

5.1.1. Five dimensional version. In our work, \( \mathfrak{g}_\Gamma \)-quantum affine algebra \( U^\text{aff}_q(\mathfrak{g}_\Gamma) \), where \( \mathfrak{g}_\Gamma \) is ADE or affine ADE, appears in the study of the 5d \( \mathfrak{g}_\Gamma \)-quiver gauge theory on \( S^1 \times \mathbb{R}^4_{\epsilon_1,\epsilon_2} \) in the limit \( \epsilon_1 = \epsilon, \epsilon_2 = 0 \) [5] with quantization parameter

\[ q = e^{i \epsilon_1} \] (5.1)
5.1.2. *Four dimensional version.* In the *four-dimensional limit* $\mathfrak{g}_f$-quantum affine algebra $U_q^\text{aff}(\mathfrak{g}_f)$ contracts to $\mathfrak{g}_f$-Yangian $Y_\epsilon(\mathfrak{g}_f)$ \cite{116, 118}. The formulae in the present note are presented in terms of the multiplicative spectral parameter

$$\xi = e^{i\ell x}$$

(5.2)
on $C_x \cong \mathbb{C}^\times$ for the 5d case of $U_q^\text{aff}(\mathfrak{g}_f)$ but are easily adapted to the 4d case of $Y_\epsilon(\mathfrak{g}_f)$. It is sufficient to take the limit $\ell \to 0$, with the multiplicative variables such as $\xi = e^{i\ell x}$ and $q = e^{it\ell}$ sent to 1, but keeping additive variables $x$ and $\ell$ finite. At the level of the quantum integrable systems, 4d limit of the 5d theory is the isotropic limit in which the trigonometric $\mathfrak{g}_f$ XXX spin chain associated to the 4d theory; for classical Seiberg-Witten for $\mathfrak{g}_f = sl_f$, see \cite{36}.

5.1.3. *Six dimensional version.* The lift of the gauge theory to *six-dimensional space-time* $\mathbb{R}^4 \times T^2_{\ell, -\ell/\tau_p}$, where $(\ell, -\ell/\tau_p)$ denote the periods of the compactification torus $T^2_{\ell, -\ell/\tau_p}$, promotes $U_q^\text{aff}(\mathfrak{g}_f)$ to the $\mathfrak{g}_f$-quantum elliptic algebra $U_{q,p}^\text{ell}(\mathfrak{g}_f)$ \cite{119, 120} with $p$ denoting the multiplicative modulus of the compactification torus $T^2_{\ell, -\ell/\tau_p}$

$$p = e^{2\pi i \tau_p}$$

(5.3)

This is done perturbatively by summing over Kaluza-Klein modes \cite{28, 122} and non-perturbatively by the study of elliptic genera of instanton moduli spaces \cite{26, 68, 69, 123} replacing the quantum affine group $U_q^\text{aff}(\mathfrak{g}_f)$ by the $\mathfrak{g}_f$-quantum elliptic algebra $U_{q,p}^\text{ell}(\mathfrak{g}_f)$ \cite{121, 124}. The spectral additive parameter $x$ lives on the elliptic curve

$$C_x = \mathbb{C}/(\mathbb{Z} + \tau_p \mathbb{Z}) \cong \mathbb{C}^\times/p\mathbb{Z} = E_p$$

(5.4)
dual to the torus $T^2_{\ell, -\ell/\tau_p}$ and hence gets two periods

$$x \to x + \frac{2\pi}{\ell}, \quad x \to x + \frac{2\pi}{\ell} \tau_p$$

(5.5)

see \cite{214} for gauge theory discussion. The quantum affine algebra $U_q^\text{aff}(\mathfrak{g}_f)$ is promoted to quantum elliptic algebra $U_{q,p}^\text{ell}(\mathfrak{g}_f)$ \cite{121, 124}. (See \cite{119, 120} on earlier definitions, \cite{123, 126} for elliptic KZ equation and \cite{127} on the defining elliptic gamma function, \cite{128} for connection to elliptic $\mathcal{W}_{q,p}(\mathfrak{g}_f)$ algebra.) For generalized Kac-Moody algebra $\mathfrak{g}_f$ with symmetric Cartan matrix the elliptic algebra $U_{q,p}^\text{ell}(\mathfrak{g}_f)$ is defined using an *elliptic version of Drinfeld currents realization* in \cite{121}. For $\mathfrak{g}_f$ of finite-dimensional ADE type see definition in \cite{121}, appendix A; keeping in mind that our quantization parameter $q$ corresponds to $q^2$ in \cite{121}. Elliptic version, as usual, is achieved by promoting factors

$$\xi^{\frac{1}{2}} - \xi^{-\frac{1}{2}} \to \theta_1(\xi; p)$$

(5.6)

where $\theta_1(\xi; p)$ is \cite{2282}.

In the world of quantum integrable systems the lift to the six dimensional theory corresponds to the deformation of the $\mathfrak{g}_f$ XXX spin chain to the anisotropic $\mathfrak{g}_f$ XYZ spin chain with elliptic $R$-matrix. The classical limit $\epsilon = 0$ of the associated integrable system \cite{23} for $\mathfrak{g}_f$ of finite type is the $\mathfrak{g}_f$-monopoles on $S^1 \times \mathbb{R}_x = \mathbb{T}^3$. The classical SW theory for $\mathfrak{g}_f = sl_f$, quiver, was studied in \cite{37}. The classical limit $\epsilon = 0$ of
the associated integrable system \([23]\) for affine \(\mathfrak{g}_r = \widehat{\mathfrak{g}_r}\), where \(\mathfrak{g}_r\) is finite-dimensional simple Lie algebra, is the moduli space of \(G_r\) instantons on \(T^4 = \mathcal{E}_p \times \mathcal{E}_q\) for \(q = \prod_i q_i^a_i\), where \(a_i\) are Dynkin marks on \(\Gamma\).

5.1.4. Dimension uniform description. To say uniformly, we deal with a quantum current algebra \(U_\epsilon \mathfrak{g}_r(C_\chi)\), where complex one-dimensional curve \(C_\chi\) is the domain of the additive spectral parameter \(x \in C_\chi\), or loop variable in the definition of \(U_\epsilon \mathfrak{g}_r(C_\chi)\) by Drinfeld currents \((\psi_i^\pm(x), e_i^\pm(x))|_{i \in \Gamma}\), see \([A.2.1]\) so that

\[
U_\epsilon \mathfrak{g}_r(C_\chi) = \begin{cases} 
Y_\epsilon(\mathfrak{g}_r), & C_\chi = \mathbb{C} \\
U^\text{aff}_q(\mathfrak{g}_r), & C_\chi = \mathbb{C}/\mathbb{Z} \cong \mathbb{C}^\times \\
U^\text{aff}_{q,p}(\mathfrak{g}_r), & C_\chi = \mathbb{C}/(\mathbb{Z} + \tau_p \mathbb{Z}) \cong \mathbb{C}^\times/p^\mathbb{Z}
\end{cases}
\]  

The representation theory of quantum algebras \(U_\epsilon \mathfrak{g}_r(C_\chi)\) is the topic of active research with a vast literature. We shall mention only some facts of direct relevance to what follows. The basic result of Chari and Presley \([50, 110]\) is that the irreducible representations of \(\mathfrak{g}_r\) are classified by the \(r\)-tuples of polynomials, called Drinfeld polynomials. These \(U^\text{aff}_q(\mathfrak{g}_r)\)-modules can be described by the \(q\)-characters of Frenkel and Reshetikhin \([49]\), such that the tensor product of representations corresponds to the product of \(q\)-characters. In the rational limit the \(q\)-characters reduce to the characters of the Yangian \(Y_q(\mathfrak{g}_r)\) found in \([48]\), see also \([129]\). Frenkel and Mukhin \([96]\) gave combinatorial algorithm for the \(q\)-characters of \(U^\text{aff}_q(\mathfrak{g}_r)\). Nakajima \([112]\) realized geometrically representation of \(U^\text{aff}_q(\mathfrak{g}_r)\) on equivariant \(K\)-homology group of the quiver variety and in \([130, 131]\) he defined non-commutative \(t\)-deformation of the \(q\)-character building on the geometrical methods of quiver varieties \([15, 16, 132]\) and computed explicitly \((q, t)\)-characters of \(U^\text{aff}_q(\mathfrak{g})\) for the \(\mathfrak{g} = A_r, D_r\) series in \([77]\) and for \(\mathfrak{g} = E_6, E_7, E_8\) in \([133]\). Hernandez \([97]\) constructed \((q, t)\)-characters for \(\mathfrak{g}_r\)-quantum affine algebra \(U^\text{aff}_q(\mathfrak{g}_r)\) of generic symmetrizable Kac-Moody Lie algebra \(\mathfrak{g}_r\), developed representation theory in \([134]\) and constructed quantum fusion tensor category in \([135]\); Chari et al. \([136, 137]\) constructed fundamental \(U^\text{aff}_q(\mathfrak{g})\) \(q\)-characters for classical series \(\mathfrak{g} = A, B, C, D\) using the action of the braid group \(\mathcal{B}(W_\mathfrak{g})\) corresponding to the Weyl group \(W_\mathfrak{g}\).

More general category of infinite-dimensional representations for the Borel subalgebra \(U^\text{aff}_q(\mathfrak{g}_r)\) was studied by Hernandez and Jimbo \([138]\) following the work \([38, 139]\) on the case \(\mathfrak{g}_r = \mathfrak{sl}_2\). The recent paper \([94]\) interprets the Baxter functions for certain infinite-dimensional representations.

5.1.5. The twist mass for the \(A\)-type affine quivers. Recall that if \(\mathfrak{g}_r = \hat{\mathfrak{A}_r}\), the \(\mathfrak{g}_r\)-theory has mass parameter \(m\) associated to the \(H^1(\Gamma)\) that cannot be removed by the shifts of scalars in the \(U(1)\) vector multiplets \([23]\). The additional equivariant parameter \(\mu = e^{itm}\) appears in H. Nakajima’s work \([115]\) on \(K\)-theory of the quiver variety \(\mathcal{M}_r\) for \(\Gamma\) of type \(\hat{\mathfrak{A}_r}\). In his picture the multiplicative group \(\mathbb{C}^\times_q\), with defining character \(q\), acts in the \(K\)-theory of moduli space of sheaves on \(\mathbb{C}^2_{(z_1, z_2)}/\Gamma\) by \([3]\) via the scaling symmetry of

\[q^{1/2}_{\text{present}} = q\]  

\(^2\)With the identification \(q^{1/2}_{\text{present}} = q\) of \([115]\).  


the base space: \((z_3, z_4) \mapsto (z_3 q^{1/2}, z_4 q^{1/2})\). If \(\Gamma \subset SU(2)\) is cyclic \(\Gamma = \mathbb{Z}_{r+1}\), then there is another \(\mathbb{C}^x\) action on \(\mathbb{C}^2\) commuting with \(\Gamma\), namely \((z_3, z_4) \mapsto (\mu z_3, \mu^{-1} z_4)\). Hence, for cyclic \(\Gamma\) the equivariant K-theory of moduli space of sheaves on \(\mathbb{C}^2/\Gamma\) depends on two parameters \((q, \mu)\) for \(\mathbb{C}^x \times \mathbb{C}^x\) action on \(\mathbb{C}^2\). Naturally, we expect that there exists a \(\mu\)-twisted version of \(U_{q}^{\text{aff}}(gl_{r+1})\). Indeed, such \(\mu\)-twisted quantum toroidal algebra \(U_{q}^{\text{aff}}(\mu \hat{gl}_{r+1})\) was constructed by Varagnolo and Vasserot [140] and it has two parameters \(q\) and \(\mu\). Our conventions correspond to the conventions of [140] as follows: \(\mu\)-twisted quantum toroidal algebra \(U_{q}^{\text{aff}}(\mu \hat{gl}_{r+1})\) is \(U_{q,t}(\mathbb{Z}/n\mathbb{Z})\) of [140] for \(n=1\), \(q_{140} = q^2\mu\) and \(t_{140} = q^2\mu^{-1}\).

The affine Yangian \(Y_{\epsilon}(\mu \hat{gl}_{r+1})\) associated to \(\hat{A}_r\) quiver is a rational limit for \(\mu = e^{it\mu}\) of quantum toroidal algebra \(U_{q}^{\text{aff}}(\mu \hat{gl}_{r+1})\), which admits additional deformation parameter that we identify with the twist mass \(m\) coupled to the cycle in \(H_1(\Gamma)\), see more on affine Yangian in [141–144].

The twist mass parameter \(m\) in \(H^1(\Gamma)\) for \(\Gamma = \hat{A}_r\) corresponds to the parameter of the non-commutative deformation of \(\mathbb{C}^x \times E_q\). Recall that the phase space of the associated integrable system is the moduli space of \(U(r+1)\)-instantons on the non-commutative \(\mathbb{C}^x \times E_q\) [23]. The six dimensional case of \(gl_\infty = \hat{A}_r^\infty\) theory (i.e. \(N = 2^r\) theory), the phase space of classical integrable system is the moduli space of non-commutative \(U(1)\) instantons on \(T^4\) was studied in [123–145, 146]. Our work implies that the quantization of the integrable system for the six dimensional theory is provided by the elliptic version of the \(gl_1\)-toroidal algebra, that we call \(U_{q,p}^{\text{ell}}(\mu \hat{gl}_1)\).

5.1.6. Quantum toroidal algebras. Quantum \(g\)-toroidal algebra was introduced in [113] for all simple Lie algebras \(g\) and in [147] it was shown that quantum \(gl_{r+1}\)-toroidal algebra is Schur-Weyl dual to double affine Hecke algebra [59, 148]. The vertex representation at level 1 for \(sl_{r+1}\)-toroidal was constructed in [149]. Representation theory of quantum toroidal algebras is based on non-symmetric Macdonald polynomials [150].

In [113] for \(g = ADE\) and [140] for \(g = A\) was shown that equivariant \(K\)-theory of Nakajima’s quiver variety [16] realizes representation of quantum \(g\) toroidal algebra. Another construction of representations of quantum toroidal algebra was given using Schur-Weyl duality to Dunkl-Cherednik representation [151] of DAHA [151]. In [152, 153] was constructed representation of quantum \(gl_{r+1}\)-toroidal algebra in the q-Fock space (see [154] for quantum \(gl_{1}\)-affine version). The isomorphism between the two constructions has been shown in [155]. The braid group and automorphisms of quantum \(gl_{r+1}\) toroidal algebra was studied in [156].

Consider the algebra \((\mathbb{C}^x)^d\), called the non-commutative multiplicative d-torus, generated by the variables \(x_1^\pm, \ldots, x_d^\pm\) subject to the relations

\[x_i x_j = \mu_{ij} x_j x_i\]  \hspace{1cm} (5.8)

where \(\mu_{ij} = \mu_{ji}^{-1}\) is the matrix of the non-commutativity \(c\)-valued parameters. For example, for \(d = 2\), the algebra \((\mathbb{C}^x)^2\) is the non-commutative 2-torus, also known as algebra of the \(\mu\)-difference operators on \(\mathbb{C}^x\), generated by the two variables \(x_1^\pm, x_2^\pm\) modulo relations \(x_1 x_2 = \mu x_2 x_1\). Let \(g\) be a finite-dimensional simple Lie algebra. The
paper \cite{157} defined possibly $\mu$-non-commutative $(\mathbb{C}^*)^d$-toroidal $\mathfrak{g}$-algebras $\mathfrak{g}((\mathbb{C}^*)^d)_\mu$ and their central extensions. In particular, it was shown in \cite{157} that only the $A$-type $d$-toroidal $\mathfrak{g}$-algebras allow non-trivial non-commutative deformation by parameter $\mu$.

The toroidal algebra $U_q^{\text{aff}}(\hat{\mathfrak{g}})$ associated to the affine quiver gauge theory $\mathfrak{g}_\Gamma = \hat{\mathfrak{g}}$ is isomorphic to $U_q\mathfrak{g}((\mathbb{C}^*)^2)_\mu$, that is to the $q$-deformation of the universal enveloping algebra of the $d = 2$ $\mu$-noncommutative toroidal algebra $\mathfrak{g}((\mathbb{C}^*)^2)_\mu$ of \cite{157}; moreover, the non-commutativity parameter $\mu$ in $H^1(\Gamma)$ is possibly non-trivial only for the affine $A$-series, in agreement with our gauge theory considerations and the constructions of H. Nakajima.

In this way, the simplest, $\mathfrak{g}_1$ $d = 2$ toroidal algebra, $U_q^{\text{aff}}(\hat{\mathfrak{g}}_1)$, with quantization parameter $q$ and non-commutativity parameter $\mu$ was constructed and studied by Miki in \cite{158} (this $\mathfrak{g}_1$-toroidal algebra is called there $U_{q,\gamma}$ with the parameter $\gamma$ related to our $\mu$, cf. Eqs. (3.18) - (3.24) of \cite{158}).

Moreover, in \cite{158} Miki has shown that the tensor product of $n$ level 1 modules of $\mathfrak{g}_1$-toroidal algebra relates to the $q$-deformed $\mathcal{W}$-algebra $\mathcal{W}_q(\mathfrak{sl}_n)$ of \cite{159,160}. On the other hand, from Nakajima \cite{16,115,113,161} and related works \cite{113,140,141,147,152,162,167}, $\mathfrak{g}_1$-toroidal algebra $U_q^{\text{aff}}(\hat{\mathfrak{g}}_1)$ at level $n$ acts in the equivariant $K$-theory of the moduli space of framed torsion free rank $n$ sheaves on $\mathbb{C}^2_{q_1,q_2}$ with two equivariant parameters $q_3 = q^\frac{1}{2}\mu$ and $q_4 = q^\frac{1}{2}\mu^{-1}$. Hence, in \cite{158}, by relating the tensor product of $n$ level 1 modules of $\mathfrak{g}_1$-toroidal algebra to the $q$-deformed $\mathfrak{sl}_n$ quantum $\mathcal{W}$-algebra \cite{128,159,160} Miki has shown (see also remark A.7 in \cite{168}) a version of correspondence between equivariant rank $n$ gauge theory on $\mathbb{C}^2_{q_1,q_2}$ and $q$-deformed $\mathfrak{sl}_n$-Toda algebra that appeared later as an AGT conjecture \cite{22,169,171}.

In the 4d, rational limit, the non-commutative 2-torus $(\mathbb{C}^*)^2$, or the algebra of $\mu$-difference operators, contracts to the Lie algebra of differential operators of one-variable, called $\mathfrak{g}$. Its central extension $\hat{\mathfrak{g}}$ is also known as $\mathcal{W}_{1+\infty}$ algebra \cite{158}; and its quantization would produce the rational limit of $\mathfrak{g}_1$-toroidal algebra, which is $\mathfrak{g}_1$ affine Yangian $Y_\epsilon^{(m)}(\hat{\mathfrak{g}}_1)$. For explicit $R$-matrix for $Y_\epsilon^{(m)}(\hat{\mathfrak{g}}_1)$ see \cite{172}. The tensor product of $n$ level 1 modules of $\mathfrak{g}_1$ affine Yangian relates to the ordinary $\mathcal{W}(\mathfrak{sl}_n)$-algebra, the symmetry of the 2d conformal $\mathfrak{sl}_n$-Toda field theory; and since in the rational limit equivariant $K$-theory contracts to the equivariant cohomology, the ordinary $\mathcal{W}(\mathfrak{sl}_n)$ algebra relates to the equivariant rank $n$ gauge theory on $\mathbb{C}^2_{q_1,q_2}$ (AGT conjecture \cite{22,169}) that has been proved in this form in \cite{173,175} after \cite{171,176}.

The $\mathfrak{g}_1$ toroidal algebra $U_q^{\text{aff}}(\hat{\mathfrak{g}}_1)$ appeared under different names in the literature. It was called \textit{quantum continuous} $\mathfrak{g}_1$, or deformation of universal enveloping of $q$-difference operators in one-variable in \cite{177,178}, \textit{spherical elliptic Hall algebra} or $\mathfrak{g}_1$ DAHA in \cite{179} and \cite{165,180}. In \cite{181} the same algebra without Serre relations is called \textit{Ding-Iohara algebra} \cite{182} or trigonometric limit of Feigin-Odesskii \cite{183} \textit{elliptic shuffle algebra} (see also \cite{184}). The elliptic deformation $U_{q,p}^{\text{ell}}(\hat{\mathfrak{g}}_1)$ of the $\mathfrak{g}_1$ toroidal algebra $U_q^{\text{aff}}(\hat{\mathfrak{g}}_1)$ is called $\mathcal{A}(q_1,q_2,q_3,p)$ in \cite{168} with $q_1 = q^{\frac{1}{2}}\mu$, $q_2 = q^{\frac{1}{2}}\mu^{-1}$, $q_3 = q^{-1}$ and is equivalent without Serre relations to the Feigin-Odesskii elliptic shuffle algebra \cite{183}. For details in equivalence of $\mathfrak{g}_1$ toroidal algebra and shuffle algebra see \cite{185}.

The elliptic deformation of $\mathfrak{g}_1$ toroidal algebra in a certain limit has been related to
the quantum cohomology of Hilbert scheme of points on $\mathbb{C}^2$ \cite{186} in \cite{168}, and the relation to elliptic Macdonald operator was further studied in \cite{187, 188}.

The characters of quantum $\mathfrak{gl}_1$ toroidal algebra are computed as a sum over plane partitions \cite{189, 190}; and representation theory of quantum $\mathfrak{gl}_r$ toroidal algebra is discussed in \cite{191, 192}.

5.1.7. Other topics. The representation theory of quantum affine and toroidal algebras $U_q^{\text{aff}}(\mathfrak{g})$ is connected to many exciting topics in geometry, algebra, and mathematical physics, e.g. the cluster algebras, the $Y$- and the $T$-systems, affine and double affine Hecke algebras \cite{59}, knots and three dimensional theories \cite{193}, quantum Knizhnik-Zamolodchikov-Smirnov equation \cite{194, 196}, affine Toda integrable field theories and many other integrable systems, $W$-algebras \cite{128, 197}, the putative five dimensional version of the AGT \cite{22} correspondence, Stokes multipliers for ordinary differential equations and CFT correlation functions \cite{38, 39, 139, 198}, holomorphic anomaly equation in the limit of \cite{3} (see, e.g. \cite{199}), geometric Langlands correspondence \cite{200}, spectrum of anomalous dimensions in $\mathcal{N} = 4$ super Yang-Mills. We do not consider in the present note the above topics and the web of their interrelation leaving the discussion to the future.

5.2. Quantum group interpretation of the gauge theory results. There are two ways to think about the Coulomb moduli space $\mathfrak{M}$.

In the first interpretation the expectation values of the observables \cite{144} parametrize an element $h(\xi) \in U_q^{\text{aff}}(\mathfrak{g}_r)$ (in the maximal commutative subalgebra) which morally speaking is similar to picking a conjugacy class in a group (if $U_q^{\text{aff}}(\mathfrak{g}_r)$ were a group). For gauge groups of finite rank $\mathfrak{v}_1$, this parametrization is clearly redundant because of the relations between $\mathfrak{O}_{i,n}$ for high enough $n$. Conceptually, the elements $h(\xi)$ in the maximal commutative subalgebra of $U_q^{\text{aff}}(\mathfrak{g}_r)$ parametrize the union of Coulomb moduli spaces over all ranks $\mathfrak{v}_1$ of the gauge groups and couplings $q_i$. This interpretation in the classical limit $q \to 1$ reduces to the construction in \cite{23}.

In the second interpretation the polynomial $T_1(\xi)$ is an eigenvalue of the transfer matrix operator $t_{V_1} \in U_q(\hat{\mathfrak{g}})$ (where $V_1$ is a fundamental $U_q^{\text{aff}}(\mathfrak{g})$ module usually called an auxiliary space in the algebraic Bethe Ansatz) on an eigenstate $|w\rangle \in W$ where another $U_q^{\text{aff}}(\mathfrak{g}_r)$-module $W$ is the physical space of states of $\mathfrak{g}_r$-spinchain. This $U_q^{\text{aff}}(\mathfrak{g}_r)$-module $W$, defined by masses of the gauge theory, is usually not finite-dimensional, but the spectral problem in the $U_q^{\text{aff}}(\mathfrak{g}_r)$-module $W$ can be converted to the spectral problem in a finite-dimensional $U_q^{\text{aff}}(\mathfrak{g}_r)$-module $\tilde{W}$ for special discrete choices of Coulomb parameter $a_{i,a}$ as for example in \cite{30, 31} (special electric $a_{i,a}$) or \cite{5} (special magnetic $a_{i,a}$). For example, for $\mathfrak{g}_r = \mathfrak{sl}_2$ with $w_1 = 2\mathbf{v}_1$, as was advocated by Dorey et al \cite{30, 31}, one can split the set of masses $M = \{\mu_{1,a}\}$ into two disjoint sets $M^- = \{\mu_{1,a}\}$ and $M^+ = \{\mu_{1,a}\}$ with $a = 1, \ldots, v$ and then choose Coulomb parameters $w_1,a = q^{n_a}\mu_{1,a}$ for some integers $n_a \in \mathbb{Z}_{\geq 0}$. Then $U_q^{\text{aff}}(\mathfrak{sl}_2)$-module $\tilde{W}$ is the tensor product $\tilde{W} = W_{s_1,\zeta_1} \otimes W_{s_2,\zeta_2} \otimes \ldots \otimes W_{s_n,\zeta_n}$ of the $U_q^{\text{aff}}(\mathfrak{sl}_2)$ evaluation Verma modules $W_{s_n,\zeta_n}$ at spectral parameter $\zeta_n = \sqrt{\mu_{1,a}^+}$ and highest weight $s_n = \log_\mu(\mu_{1,a}^+/\mu_{1,a}^-)$. 


5.2.1. First interpretation of the gauge theory $q$-character equations. The conventions for the quantum affine algebras are summarized in the appendix A.

First, we find an element $\psi(\xi|\Xi) \in U_{q}^{\text{aff}}(g_{\Gamma})$ in the maximal commutative subalgebra of $U_{q}^{\text{aff}}(g_{\Gamma})$ such that its evaluation in a finite-dimensional $U_{q}^{\text{aff}}(g_{\Gamma})$-module $V$ has the generalized eigenvalues (the symbols $Y_{i,\zeta}$) equal to the gauge theory function $Y_{i}(\xi/\zeta)$

$$\text{ev}_{\psi(\xi|\Xi)} Y_{i,\zeta} = Y_{i}^{+}(\xi/\zeta)$$

Explicitly, comparing the evaluation definition (A.53) and the definition of the $Y$-functions in the gauge theory (3.12) we conclude that such element $\psi(\xi|\Xi) \in U_{q}^{\text{aff}}(g_{\Gamma})$ is given by

$$\psi(\xi|\Xi) = \prod_{i \in I_{\Gamma}} \prod_{\xi' \in \Xi_{i}} \psi_{i}^{+}(\xi/\xi')$$

where $\psi_{i}^{+}(\xi/\xi')$ is defined by (A.23).

For example, in the case $g_{\Gamma} = \mathfrak{sl}_{2}$, the $q$-character of the fundamental module $V_{1,q^{-\frac{1}{2}}}$ evaluated by the element (5.10) gives us

$$\text{ev}_{\psi(\xi|\Xi)} \chi_{q}(V_{1,q^{-\frac{1}{2}}}) = Y_{1}^{+}(q^{\frac{1}{2}}\xi) + \frac{1}{Y_{1}^{+}(q^{-\frac{1}{2}}\xi)}$$

Notice by comparing (4.16) with (5.10) that can represent the operator $\psi(\xi|\Xi) \in U_{q}^{\text{aff}}(g_{\Gamma})$ as follows

$$\psi(\xi|\Xi) = \exp \left( - \sum_{i \in I_{\Gamma}} \sum_{n=1}^{\infty} \frac{\xi^{-n}q^{-\frac{n}{2}}}{[n]_{q}} \left\langle \text{tr}_{V_{i}} e^{2\pi i n h_{i}} \right\rangle h_{1,n} \right)$$

For the gauge theory with fundamental matter we need to incorporate the extra factors of matter polynomials as in (4.2). We can do this by multiplying (5.10) by another element in $U_{q}^{\text{aff}}(g_{\Gamma})$, which depends only on the masses and coupling constants. To find explicit expression for $g_{\Gamma} = \mathfrak{sl}_{2}$, it is useful to notice that the equation

$$s(q^{-\frac{1}{2}}\xi)s(q^{\frac{1}{2}}\xi) = 1 - \xi^{-1}$$

is solved by the function (defined as the expansion near $\xi = \infty$)

$$s(\xi) = \exp \left( - \sum_{n=1}^{\infty} \frac{\xi^{-n}}{n q^{-\frac{n}{2}} + q^{\frac{n}{2}}} \right)$$

Define an operator $\psi_{1}^{\vee}(\xi) \in U_{q}^{\text{aff}}(\mathfrak{sl}_{2})$ as follows

$$\psi_{1}^{\vee}(\xi) := \exp \left( - \sum_{n=1}^{\infty} \frac{\xi^{-n}}{[n]_{q} q^{\frac{n}{2}} + q^{-\frac{n}{2}} h_{1,n}} \right)$$

The eigenvalues of the operators $p_{1}(\xi), \psi_{1}(x), \psi_{1}^{\vee}(x)$ on a fundamental eigenvector with generalized weight encoded by the Drinfeld polynomial $P(\xi) = 1 - \xi^{-1}$ are given
by

\[ p_1(\xi)|v\rangle = (1 - \xi^{-1}) \]

\[ \psi_1^\vee(\xi)|v\rangle = s(\xi) \] \hspace{1cm} (5.16)

\[ \psi_1(\xi)|v\rangle = \frac{1 - q^{-\frac{1}{2}}\xi^{-1}}{1 - q^{\frac{1}{2}}\xi^{-1}} \]

Given a set

\[ M_1 = \{\mu_i\} \]

of the masses for fundamental multiplets define the operator

\[ \psi_1^\vee(\xi|M_1) \in U^{\text{aff}}(\mathfrak{sl}_2) \]

\[ \psi_1^\vee(\xi|M_1) = \prod_{\mu_i \in M_1} \psi_1^\vee(\xi/\mu_i) \] \hspace{1cm} (5.17)

with eigenvalue given by the function

\[ s_{11}(\xi|M_1) = \prod_{\mu_i \in M_1} s(\xi/\mu_i) \] \hspace{1cm} (5.18)

that satisfies

\[ s_{11}(\xi q^{\frac{1}{2}}|M)s_{11}(\xi q^{-\frac{1}{2}}|M) = \hat{P}_1(\xi) \equiv \prod_{f} (1 - \mu_f/\xi) \] \hspace{1cm} (5.19)

Then \( \chi_q(V_{1,q^{-\frac{1}{2}}}) \) character evaluated on the element \( h(\xi) \in U^{\text{aff}}(\mathfrak{sl}_2) \)

\[ h(\xi) = q^{-\frac{1}{2}h_1,0} \psi_1(\xi|\Xi)/\psi_1^\vee(\xi|M) \] \hspace{1cm} (5.20)

equals

\[ \text{ev}_{h(\xi)} \chi_q(V_{1,q^{-\frac{1}{2}}}) = q^{-\frac{1}{2}} \frac{Y_1(\xi q^{\frac{1}{2}})}{s_{11}(\xi q^{\frac{1}{2}}|M)} + q^{\frac{1}{2}} s_{11}(\xi q^{-\frac{1}{2}}|M) \] \hspace{1cm} (5.21)

Finally, define the twisted \( q \)-character \( \tilde{\chi}_{1,\zeta} \) multiplying the \( q \)-character by a suitable scalar factor such the term corresponding to the highest weight vector in the fundamental module \( V_{1,\zeta} \) equals to \( \frac{y_1(\xi/\zeta)}{s_{11}(\xi|M)} \)

\[ \tilde{\chi}_{1,\zeta} = c_1(\xi/\zeta) \chi_q(V_{1,\zeta}) \] \hspace{1cm} (5.22)

where

\[ c_1(\xi) = q^{\frac{1}{2}} s_{11}(\xi|M) \] \hspace{1cm} (5.23)

Hence, we obtain the first interpretation of the limit-shape equations (4.10)

For \( \mathfrak{g}_\Gamma = \mathfrak{sl}_2 \), the twisted character \( \tilde{\chi}_{1,\zeta} \) evaluated on the element \( h(\xi) \in U^{\text{aff}}(\mathfrak{g}_\Gamma) \) is a polynomial of degree \( v_1 \) that we denote \( T_1(\xi) \)

\[ \text{ev}_{h(\xi)} \tilde{\chi}_{1,q^{-1/2}} = T_1(\xi) \hspace{1cm} (\Rightarrow) \hspace{1cm} \frac{Y_1(\xi q^{\frac{1}{2}})}{Y_1(\xi q^{-\frac{1}{2}})} = T_1(\xi) \] \hspace{1cm} (5.24)

This interpretation is completely in parallel with the construction of \( \mathbf{G}_\Gamma(C_\chi) \)-group element in [23] and becomes the one in the classical limit \( q \to 1 \).
The equation (5.13) is the multiplicative $q$-version of the equation $a \cdot \Lambda^\vee = \Lambda^\vee$ relating the fundamental coweights and the simple coroots. The higher rank $\mathfrak{g}$ requires a generalization of (5.13). The multiplicative $q$-version of the equation

$$\sum_j a_{ij} \Lambda_j^\vee = \alpha_i^\vee$$

for the inverse Cartan matrix is

$$s_{ik}(q^{\frac{1}{2}} \xi)s_{ik}(q^{-\frac{1}{2}} \xi) \prod_{j:(ij)=1} s_{jk}(\xi) = \begin{cases} 1 - \frac{1}{\xi}, & i = k \\ 1, & i \neq k \end{cases}$$

Then $s_{ik}(\xi)$ is the $q$-multiplicative decomposition of the fundamental coweight $\Lambda_k^\vee$ over the basis of simple coroots $\alpha_i^\vee$.

Let $a(q)$ be the $q$-Cartan matrix defined by replacing each entry of the Cartan matrix by its $q$-number, and $\tilde{a}(q)$ its inverse:

$$a(q)_{ij} = [a_{ij}]_q, \quad \sum_j a(q)_{ij} \tilde{a}(q)_{jk} = \delta_{ik}.$$ \hspace{1cm} (5.26)

Then, expanding the defining equation (5.25) in power series at $\xi = \infty$ we find

$$s_{ij}(\xi) = \exp\left(-\sum_{n=1}^{\infty} \frac{\xi^{-n}}{n} \tilde{a}_{ij}(q^n)\right)$$

and for a set $M_j = \{\mu_{j,i}\}$ define

$$s_{ij}(\xi|M_j) = \prod_{\mu_{j,i} \in M_j} s_{ij}(\xi/\mu_{j,i})$$

Consequently, the generalization of (5.15) to the $q$-coweight operator $\tilde{\psi}^\vee_i(\xi) \in \mathcal{U}_q^{aff}(\mathfrak{g})$ is

$$\tilde{\psi}^\vee_i(\xi) = \exp\left(-\sum_{n=1}^{\infty} \frac{\xi^{-n}}{n} \tilde{a}_{ij}(q^n) h_{j,n}\right)$$

The generalization of the formula (5.20) to the element $h(\xi) \in \mathcal{U}_q^{aff}(\mathfrak{g})$ is

$$h(\xi) = \prod_{i \in I} q^{\tilde{a}_{ij} h_{j,0}} \tilde{\psi}_i(\xi|\Xi_i)/\psi_i^\vee(\xi|M_i)$$

where $\Xi_i$ is the set of zeroes of $Q_i^+(\xi)$ and $M_i$ is the set of zeroes of $\mathcal{P}_i^+(\xi)$.

The gauge theory $q$-character equations are written using the twisted $q$-characters $\tilde{\chi}_{i,\xi}$ for the fundamental $\mathcal{U}_q^{aff}(\mathfrak{g})$ modules $V_{i,\xi}$

$$\tilde{\chi}_{i,\xi} = c_i(\xi/\xi) \chi_q(V_{i,\xi})$$

where the scalar factor is

$$c_i(\xi) = \prod_j q^{\tilde{a}_{ij} s_{ij}(\xi|M_j)}$$

We summarize the first interpretation by the proposition
Proposition. The element \( h(\xi) \in U^\text{aff}_q(\mathfrak{g}_\Gamma) \) that encodes by (5.30 (5.12)) the gauge theory chiral ring generating functions \( Y^+ (\xi) \) satisfies the equations

\[
\text{ev}_{h(\xi)} \tilde{\chi}_{i,\zeta} = T_i(\xi), \quad i \in I_{\Gamma}
\]

where \( \tilde{\chi}_{i,\zeta} \) is the twisted \( U^\text{aff}_q(\mathfrak{g}_\Gamma) \) \( q \)-character of the fundamental \( U^\text{aff}_q(\mathfrak{g}_\Gamma) \)-module \( V_{i,\zeta} \) defined by (5.31 (5.32)) and (A.54), and \( T_i(\xi) \in \mathbb{C}[\xi^{-1}] \) is a polynomial in \( \xi^{-1} \) of degree \( v_i \).

6. Examples and discussions

6.1. Finite \( A_r \) quiver. For \( \mathfrak{g}_\Gamma = A_r \) the system of \( r \) \( q \)-difference equations (4.10) can be reduced to a single \( q \)-difference equation of order \( r + 1 \). This reduction is the \( q \)-analogue of the expansion of the characteristic polynomial of an \( sl(r+1) \) matrix in the fundamental characters. Explicitly, we obtain

\[
\sum_{k=0}^{r+1} (-1)^k y_{i,k}^{r+1-k} T_{k,(k-1)/2} \prod_{j=1}^{k-1} \mathcal{P}_{j,j/2}^{[k-j]} = 0
\]

where

\[
f_{i,j}^{[k]} = \prod_{j=1}^{k} f_{i,j+k-1}, \quad f = y, \mathcal{P}
\]

Now, in terms of Baxter functions (3.13)

\[
y_{i,j} = \frac{Q_{i,j-\frac{1}{2}}}{Q_{i,j+\frac{1}{2}}}
\]

the \( q \)-determinant equation (6.1) reduces to the linear degree \( r + 1 \) \( q \)-difference equation on \( Q_1(\xi) \).

\[
\sum_{k=0}^{r+1} (-1)^k Q_{1,k-\frac{1}{2}} T_{k,(k-1)/2} \prod_{j=1}^{k-1} \mathcal{P}_{j,j/2}^{[k-j]} = 0
\]

The above is known as the first equation in the hierarchy of QT-system for the integrable \( A_r \) spin chain with inhomogeneous parameters encoded in the roots of \( \mathcal{P}_1(\xi) \).

6.2. \( A_\infty \) quiver. For \( \mathfrak{g}_\Gamma = A_\infty \), serving as the universal cover of \( \hat{A}_{r-1} \), we construct the \( q \)-character of quantum affine algebra \( U^\text{aff}_q(A_\infty) \) using the same requirement (cancellation of poles) and from (3.42) arrive to the formula

\[
\chi_i = \sum_{\lambda} \prod_{j=1}^{\ell_\lambda} \left( \mathcal{P}_{[\lambda_j]}^{[\lambda_i]} \frac{y_{i-j+1,\frac{1}{2}}^{i+j-1,\frac{1}{2}}}{y_{i+\lambda-j-j,\frac{1}{2}}^{i+j-1,\frac{1}{2}}} \right)^{y_{i-j+1,\frac{1}{2}}^{i+j-1,\frac{1}{2}}} y_{i-\ell_\lambda,\frac{1}{2}}^{\lambda_\lambda}
\]

where the sum is over all partitions \( \lambda \). In (6.5) we omitted symbol \( \tilde{\lambda} \) compared to (3.42), and where

\[
\mathcal{P}_{i,0}^{[\lambda]} \equiv \prod_{k=i}^{k=i+\lambda_j-1} \mathcal{P}_{k,\frac{1}{2}(k-i)}
\]
By restricting the range of indices in (6.5) one naturally recovers the characters of all fundamental modules for $U_q^\text{aff}(\mathfrak{g}_\Gamma)$ for $\mathfrak{g}_\Gamma = A_r$ with a finite $r$.

It is useful to express the formula (6.5) in terms of the variables $t_i(\xi)$ defined by

$$t_{i,0} := \frac{y_{i,0}}{y_{i-1,\frac{1}{2}}}$$

which are the $q$-versions of the $GL_\infty$ eigenvalues. We find

$$\chi_i = y_{i-\ell,\frac{1}{2},\ell} \sum_\lambda \prod_{j=1}^{\ell_\lambda} \left( b^{[\lambda_j]}_{i-j+1,\frac{1}{2}} t_{i+j+\lambda_j-(j-1),\frac{1}{2}}(\lambda_j+j-1) \right)$$

Now we define a non-commutative equivalent of the determinant generating function for all fundamental characters of $A$-type quivers. For $j \in \mathbb{Z}_{\geq 0}$ define

$$P_j^\rightarrow[i,0] = \prod_{j \in 1/2 + \mathbb{Z}_{\geq 0}} (1 - D^{j-1/2} P_{i,j-1,\frac{1}{2}} Y_{i-j+\frac{1}{2},\frac{1}{2}} D^{-j+1})$$

where the polynomial $P_{i,0}$ has been factorized into a product of two polynomials

$$P_{i,0} = P_{i,0}^\rightarrow P_{i,0}^\leftarrow$$

To get the canonical (minimal degree equation) for $A_r$ theory we set $i$ to be the vertex with the maximal $v_i$ and split arbitrarily $P_{i,0}$ into the polynomials $P_{i,0}^\rightarrow$ of degree $v_i-v_{i-1}$ and $P_{i,0}^\leftarrow$ of degree $v_i - v_{i+1}$ (c.f. the degree profile in section 7.1.1 of [23]).

Then we consider the operator

$$\Theta_i = (\Theta_i^\rightarrow, Y_{i,0}, \Theta_i^\leftarrow)$$

where

$$\Theta_i^\rightarrow = \prod_{j \in 1/2 + \mathbb{Z}_{\geq 0}} \left( 1 - D^{-j-1/2} P_{i,j-1,\frac{1}{2}} Y_{i-j-1/2,\frac{1}{2}} D^{-j+1/2} \right)$$

$$\Theta_i^\leftarrow = \prod_{j \in 1/2 + \mathbb{Z}_{\geq 0}} \left( 1 - D^{j+1/2} P_{i,j+1,\frac{1}{2}} Y_{i+j+1/2,\frac{1}{2}} D^{j-1/2} \right)$$

and $D$ is the shift operator

$$D = q^{E_0}$$
so that $Df_{i,j}D^{-1} = f_{i,j-1}$ where $f = Y,P$. The notation for the non-commutative products over an ordered index set $J$ is
\[
\prod_{j \in J} f_j = \ldots f_{*} f_{*+1} \ldots
\]
\[
\prod_{j \in J} f_j = \ldots f_{+1} f_{*} \ldots
\]

The operator $\Theta$, like the $A_\infty$ determinant or lattice theta-function, is the generating function in the operator variable $D$ of all fundamental $q$-characters:
\[
\Theta_i = \sum_{k<0} (-1)^k \left( \prod_{j=1}^{-k} P_{1,\frac{j}{2}-k-j} \right) \chi_{i+k,-\frac{k}{2}} D^k + \chi_{i,0} + \sum_{k>0} (-1)^k \left( \prod_{j=1}^{k} P_{1,\frac{j}{2}} \right) \chi_{i+k,-\frac{k}{2}} D^k
\]  (6.15)

Our gauge theory equations state that the characters $\chi_i(\xi)$ are the (Laurent) polynomials $T_i(\xi)$ in $\xi$. We call $T_i(\xi)$ the Coulomb polynomials. After substitution $\chi_i(\xi) = T_i(\xi)$ the operator version of the spectral curve is
\[
\Theta_i |Q_i(\xi)\rangle = 0
\]  (6.16)

Remark. If $A_\infty q$-characters is specialized to $\hat{A}_r^*$ quiver using periodicity modulo $r + 1$ than the polynomials $P_i(\xi)$ are of degree zero, i.e. $P_i(\xi) = q_i''$ where $q_i$ is a certain complex number made from coupling constants $q_i$, masses $\mu_e$ and $q$ using (3.26) (3.37).

The $q$-characters $\chi_i$ are defined for $i \in \mathbb{Z}$, but there are only $r + 1$ functionally independent ones.

6.2.1. Details on the $A_1$ quiver. The operator $\Theta_1$ is
\[
\Theta_1 = \left( 1 - D^{-\frac{1}{2}} P_{1,\frac{1}{2},\frac{1}{2}} D^{-\frac{1}{2}} \right) Y_{1,0} \left( 1 - D^{\frac{1}{2}} P_{1,0,\frac{1}{2}} D^{\frac{1}{2}} \right)
\]  (6.17)

and its expansion is
\[
\Theta_1 = P_{1,\frac{1}{2}} D^{-1} + \chi_{1,0} - P_{1,-\frac{1}{2}} D
\]  (6.18)

In the equation
\[
\Theta_1 |Q_1\rangle = 0
\]  (6.19)

we recognize the Baxter equation
\[
- P_{1,\frac{1}{2}} Q_{1,1} + T_{1,0} Q_{1,0} - P_{1,-\frac{1}{2}} Q_{1,-1} = 0
\]  (6.20)

where the polynomials $P_{1,\frac{1}{2}}, P_{1,-\frac{1}{2}}$ and $T_{1,0}$ would be conventionally called $A, D$ and $T$ Baxter polynomials. The equation (6.20) is the second order linear difference equation
with a basis \( \{ Q^{(0)}_1, Q^{(1)}_1 \} \) of two independent solutions. We choose one solution that it is annihilated the last factor of (6.17). Then we find

\[
\frac{Q_{1,-\frac{1}{2}}}{Q_{1,\frac{1}{2}}} = \frac{y_{i+\frac{1}{2}}}{P_{i,0}^{\frac{1}{2}}}
\]

so that \( Q_1 \) is \( Q_1 \) of (3.13) up to a factor of \( \gamma_{q,P_{i,0}^{\frac{1}{2}}} \).

Here the function \( \gamma_{q,P}(\xi) \) is a generalization of \( q \)-Gamma function, it is determined by the polynomial \( P(\xi) \) as the solution of the \( q \)-difference equation

\[
\frac{\gamma_{q,P}(q^{\frac{1}{2}}\xi)}{\gamma_{q,P}(q^{-\frac{1}{2}}\xi)} = \frac{1}{P(\xi)}
\]

6.2.2. Details on the \( A_3 = D_3 \) and \( A_r \) quivers. The new feature of the \( A_r \) quivers with \( r \geq 3 \) is the presence of the interior nodes. To be specific, consider the \( A_3 \) quiver with the maximal rank at the middle node \( i = 2 \). The \( q \)-character \( \chi_{i,0} \) can be displayed by the diagram

where each node denotes a term in the character and each arrow with a label \( j \) denotes the \( q \)-Weyl reflection in the quiver node \( j \). The character \( \chi_2 \) for \( A_3 \) theory is also the fundamental character of the vector representation for the \( D_3 \) theory.

The \( q \) characters \( \chi_{i-1,\frac{1}{2}} \) and \( \chi_{i+1,-\frac{1}{2}} \) are
satisfies equation

Let \( Q \) theory, and let \( A \) as we can see from the last factor of (6.23). For a generic above equation and a simple shift, the function \( Q \) satisfies equation (6.16) for \( A_3 \) theory, and let \( Q_i^{(3)} \) be solution annihilated by the last factor in the (6.23). Then \( Q_i^{(3)} \) satisfies equation

\[
Q_i^{(3)} - \mathcal{P}_{i,0}^{\frac{1}{2}} \frac{1}{y_{i+1,1}} Q_i^{(3)} = 0
\]

as we can see from the last factor of (6.23). For a generic \( A_r \) quiver the corresponding equation on \( Q_r \) is

\[
Q_r^{\frac{1}{2}} - \mathcal{P}_{i,0}^{\frac{1}{2}} \frac{1}{y_{r,1}^{1+r-i}} Q_r^{\frac{1}{2}} = 0
\]

or, equivalently

\[
\frac{Q_r^{-\frac{1}{2}}}{Q_r^{\frac{1}{2}}} = \frac{y_{r,2}^{1+r-i}}{\mathcal{P}_{i,0}^{\frac{1}{2}+r-i}}
\]

We conclude, that up to a product of \( \gamma_q \) factors coming from the \( \mathcal{P}_{i,0}^{\frac{1}{2}+r-i} \) in the above equation and a simple shift, the function \( Q_r(\xi) \) is the function \( Q_r(\xi) \) (see (3.13))
associated to the right terminal node \( r \) of the linear quiver

\[
Q_r = \gamma_q^r Q_{r, \frac{1}{2}(1+r-i)}
\]  

(6.29)

6.3. From Baxter equation to \( Y \)-functions from \( D \)-Wronskian. Here we will recover relation to the \( Y \) and \( Q \) functions of (3.13) from the solutions to the \( D \)-spectral equation (6.16), see [38, 89] for details.

Define operator \( L_k \) to be the product of \( k \) right factors in the \( \Theta_i \).

For \( k \leq r - i + 1 \) we find

\[
L_k = \prod_{j=r-i+\frac{1}{2}-(k-1)}^{r-i+\frac{1}{2}} \left( 1 - D_{i,j}^{\frac{1}{2}} Y_{i+j+\frac{1}{2}, -\frac{1}{2} + \frac{1}{2} j} D_{i,j}^{\frac{1}{2}} \right)
\]  

(6.30)

Let \( \langle Q_i^{(r-j;k)} \rangle_{j \in \{0,1,...,k-1\}} \) be the basis in the kernel of \( L_k \)

\[
\ker L_k = \bigoplus_{j=0}^{k-1} C Q_i^{(r-j;k)}
\]  

(6.31)

Let \( W_k \) be the \( D \)-Wronskian built on functions \( \langle Q_i^{(r-j;k)} \rangle_{j \in \{0,1,...,k-1\}} \)

\[
W_k = \det \left( Q_i^{(r-j;k)} \right)_{j \in \{0,1,...,k-1\}, j' \in \{0,1,...,k-1\}}
\]  

(6.32)

Expand \( L_k \) in the powers of \( D \)

\[
L_k = 1 + \sum_{j=1}^{k-1} c_j D^j + \frac{(-1)^k}{Y_{r-k+1, \frac{1}{2}(r-i-k+1)}} \left( \prod_{j=r-i+\frac{1}{2}-k}^{r-i+\frac{1}{2}} \mathcal{P}_{i,j} \right) D^k
\]  

(6.33)

where \( c_j \) are certain polynomials in \( Y \) and \( \mathcal{P} \). The equations \( (L_k Q_i^{(r-j;k)} = 0)_{j \in \{0,1,...,k-1\}} \) together with determinant definition (6.32) implies

\[
\left( 1 - \prod_{j=r-i+\frac{1}{2}-k}^{r-i+\frac{1}{2}} \mathcal{P}_{i,j} \right) D W_{r-k+1} = 0
\]  

(6.34)

so that finally

\[
\frac{W_{r-(k-1),-1}}{W_{r-(k-1),0}} = \frac{Y_{r-(k-1), \frac{1}{2}(r-i-(k-1))}}{\prod_{j=r-i+\frac{1}{2}-k}^{r-i+\frac{1}{2}} \mathcal{P}_{i,j}}
\]  

(6.35)
For \( k > r - i + 1 \) we find

\[
L_k = (-1)^{r-i-k+1} y_{r-k+1, \frac{r-i-k+1}{2}} \left( \prod_{j=1}^{k-1-r+i} \Phi_{i, k-r+i-j+\frac{1}{2}}^{j} \right) D^{r-i-k+1} + \sum_{j=r-i-k+2}^{r-i} c_j D^j + \]

\[
(-1)^{r-i+1} \left( \prod_{j=\frac{1}{2}}^{r-i-j+1} \Phi_{i-r-i+j+\frac{1}{2}}^j \right) D^{r-i+1} \quad (6.36)
\]

The equations \((L_k Q_i^{(r-j;k)} = 0)_{j \in \{0, 1, \ldots, k-1\}}\) and the determinant definition \((6.32)\) imply

\[
\frac{W_{r-k+1, k-r+i-2}}{W_{r-k+1, k-r+i-1}} = \frac{y_{r-k+1, \frac{r-i-k+1}{2}} \left( \prod_{j=1}^{k-i-r+1} \Phi_{i, k-r+i-j+\frac{1}{2}}^j \right)}{\left( \prod_{j=\frac{1}{2}}^{r-i-j+1} \Phi_{i-r-i+j+\frac{1}{2}}^j \right)}
\]

or, equivalently,

\[
\frac{W_{r-k+1, 1}}{W_{r-k+1, 0}} = \frac{y_{r-k+1, \frac{1}{2}} \left( \prod_{j=1}^{k-r-1+i} \Phi_{i-j+r-i-k+1+\frac{1}{2}}^j \right)}{\left( \prod_{j=\frac{1}{2}}^{r-i-j+\frac{1}{2}} \Phi_{i-r-i+j+\frac{1}{2}}^j \right)}
\]

This formula \((6.35)\) and \((6.38)\) together with the \( D\)-Wronskian definition \((6.32)\) allows us to compute all functions \( y_j \) from the complete basis in the ker \( \Theta_i \).

6.4. \( \tilde{A}_0 \) quiver. Consider the type II* theory corresponding to the \( \tilde{A}_0 \) quiver, i.e. the \( \mathcal{N} = 2^* \) theory with the gauge group \( SU(N) \), where \( N = \nu_0 \). There is one fundamental \( q \)-character in this case:

\[
\chi(\xi) = y(\xi) \sum_{\lambda} q^{\left| \lambda \right|} \prod_{\square \in \lambda} \frac{y(\sigma_{\square} \mu \xi) y(\sigma_{\square} \mu^{-1} \xi)}{y(\sigma_{\square} q^{\frac{1}{2} \xi}) y(\sigma_{\square} q^{-\frac{1}{2} \xi})} = \sum_{\lambda} q^{\left| \lambda \right|} \frac{\prod_{\square \in \partial_+ \lambda} y(\sigma_{\square} q^{\frac{1}{2} \xi})}{\prod_{\square \in \partial_- \lambda} y(\sigma_{\square} q^{-\frac{1}{2} \xi})},
\]

\((6.39)\)
Where for $\square = (i,j) \in \lambda$,
\[
\sigma_{\square} = \mu^{j-i} q^\frac{1-i+j}{2}, \quad \text{and} \quad 1 - \sum_{\square \in \lambda} \sigma_{\square} \left( q^{\frac{1}{2}} + q^{-\frac{1}{2}} - \mu - \mu^{-1} \right) = \\
\sum_{\square \in \partial_+ \lambda} \sigma_{\square} q^{\frac{1}{2}} - \sum_{\square \in \partial_- \lambda} \sigma_{\square} q^{-\frac{1}{2}}
\] (6.40)

The outer and inner boundaries $\partial_\pm \lambda$ correspond to the generators and relations of the monomial ideal $I_\lambda$ corresponding to the partition $\lambda$.

Let us now present the quantization of the curve $R(t, x) = 0$ found in [23] for this theory.

Introduce the notation (not to be confused with (4.5)):
\[
y_{i,j} = Y(\xi q^{-\frac{1}{2}} \mu^{j-i}), \quad i, j \in \mathbb{Z}
\] (6.41)

Then (6.39) can be rewritten in the form:
\[
\chi = \sum_{\lambda} q^{[\lambda]} y_{\lambda,0} \prod_{j=1}^{\ell_\lambda} \frac{y_{j-1,\lambda_j}}{y_{j,\lambda_j}}
\] (6.42)

Now let us introduce the operator
\[
U = q^{-\xi \partial_\xi}
\] (6.43)
such that
\[
U y_{i,j} U^{-1} = y_{i+1,j+1}
\] (6.44)

Introduce also the notations
\[
g_j = \frac{y_{j-1,j}}{y_{0,j}}, \quad \tilde{g}_j = \frac{y_{j-1,j}}{y_{j,j}}, \quad j \in \mathbb{Z}
\] (6.45)

Let us also introduce the shift operators
\[
U_\pm = \left( \mu q^{\pm \frac{1}{2}} \right)^{-\xi \partial_\xi}, \quad U_+ = U U_-
\] (6.46)

which commute with $U$. The quantities $g_j, \tilde{g}_j$ can be written also in the following form:
\[
g_j = U_{-j} g_0 U_{-j}, \quad \tilde{g}_j = U_{+j} \tilde{g}_0 U_{+j}
\] (6.47)

Using $U$, $g$, $\tilde{g}$ we can present $\chi$ in a more suggestive form:
\[
\chi = \sum_{\lambda} q^{[\lambda]} y_{\lambda,0} \prod_{i=1}^{\ell_\lambda} U^{\xi_\lambda} g_{\lambda,-i} U^{-i}
\]
\[
= \sum_{\lambda} q^{[\lambda]} y_{\lambda,0} U g_{\lambda_1 -1} U g_{\lambda_2 -2} \ldots U g_{\lambda_{\ell_\lambda} - \ell_\lambda} U^{-\ell_\lambda}
\]
\[
= \sum_{\lambda} q^{[\lambda]} \prod_{i=1}^{\ell_\lambda} q^{\lambda_{i-1} + \frac{1}{2}} U^{g_{\lambda_{i-1}} - i} y_{0,-\ell_\lambda} U^{-\ell_\lambda}
\]
\[
= \text{Coeff}_{U^0} \Theta(\xi, q; U)
\] (6.48)
with the *q-vertex operator* $\Theta$,

$$\Theta(\xi, q; U) = \Theta_+ y(\xi) \Theta_-$$  \hspace{1cm} (6.49)

whose factors are given by the ordered infinite products:

$$\Theta_- = \prod_{r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} (1 + q^r U g_{r-\frac{1}{2}})$$  \hspace{1cm} (6.50)

$$\Theta_+ = \prod_{r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} (1 + q^r \tilde{g}_{r-\frac{1}{2}} U^{-1})$$

The notations

$$\prod, \prod$$

for the noncommutative products read as follows:

$$\prod_{r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} a_r = \ldots a_{\frac{1}{2}} a_{\frac{1}{2}} a_{\frac{1}{2}}, \quad \prod_{r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} \tilde{a}_r = \tilde{a}_{\frac{1}{2}} \tilde{a}_{\frac{1}{2}} \tilde{a}_{\frac{1}{2}} \ldots$$  \hspace{1cm} (6.51)

The vertex operator $\Theta$ is the noncommutative analogue of the theta-product:

$$\vartheta(t; q) \equiv \sum_{n \in \mathbb{Z}} q^{n^2} t^n = \phi(q) \prod_{r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} (1 + t q^r) \left(1 + t^{-1} q^{-r}\right)$$  \hspace{1cm} (6.52)

Just like the theta-function $\vartheta(t; q)$, the operator $\Theta$ can be expanded as a $U$-series:

$$\Theta(\xi, q; U) = \sum_{n \in \mathbb{Z}} q^{n^2} \chi_n(\xi) U^n$$  \hspace{1cm} (6.53)

where

$$\chi_n(\xi) = \chi(\xi \mu_n), \quad \mu_\pm = \mu q^{\pm \frac{1}{2}}$$  \hspace{1cm} (6.54)

Now let us define the function $\tilde{Q}(\xi)$ by the property:

$$\left(1 + q^{\frac{1}{2}} U g_0\right) y_{0,0} \tilde{Q} = 0 \iff q^{\frac{1}{2}} \tilde{Q}(\xi q^{-1}) y(\xi \mu_+) = -y(\xi) \tilde{Q}(\xi)$$  \hspace{1cm} (6.55)

It is related to the $Q$-function

$$y(\xi) = \frac{Q(\xi)}{Q(\xi q^{-1})}$$  \hspace{1cm} (6.56)

in an obvious manner. Define

$$Q(\xi) = \left(\Theta_+^{-1} \tilde{Q}\right)(\xi)$$  \hspace{1cm} (6.57)

Then $\Theta(\xi, q; U) Q(\xi) = 0$, which is an infinite order linear difference equation:

$$\sum_{n \in \mathbb{Z}} q^{n^2} \chi(\xi \mu_-^n) Q(\xi q^{-n}) = 0$$  \hspace{1cm} (6.58)

which is an analogue of the Hirota difference equation \[201\]. We discuss the four-dimensional limit of the equation (6.58) in the section below.
7. Four dimensional limit and the opers.

7.1. The $A_1$ case. The $q$-character equation (4.2), up to non-essential $\epsilon$ shifts in the variable $x$, assumes the following form in the four dimensional limit:

$$y\left(x + \frac{\epsilon}{2}\right) + qy\left(x - \frac{\epsilon}{2}\right) = (1 + q)T(x)$$

(7.1)

with the degree $N_f = 2N$ monic polynomial $\tilde{P}(x)$ and degree $N$ monic polynomial $T(x)$. The roots of polynomial $\tilde{P}(x)$ are the masses of the fundamental hypermultiplets, while the polynomial $T(x)$ determines the vacuum of the theory. Now let us factorize the fundamental polynomial

$$\tilde{P}(x) = A(x - \epsilon)D(x),$$

(7.3)

where $A(x), D(x)$ are degree $N$ monic polynomials. Obviously, there are many representations like (7.3). Define Baxter function $Q(x)$ to be an entire function of $x$, such that (c.f. with (6.21) where the partial matter polynomial $A(x)$ corresponds to $P_{1,0}(\xi)$)

$$y(x) = A\left(x - \frac{\epsilon}{2}\right) Q\left(x + \frac{\epsilon}{2}\right) Q\left(x - \frac{\epsilon}{2}\right)$$

(7.4)

Comparing with (3.12) we find

$$A(x) Q\left(x + \epsilon\right) = \frac{Q(x + \epsilon)}{Q(x)}$$

(7.5)

therefore

$$Q(x) = \frac{Q(x)}{\Gamma_{A(x)}(x)}$$

(7.6)

where for a monic polynomial $A(x)$ the $\Gamma_{A(x)}(x)$ function satisfies

$$\Gamma_{A(x)}(x + \epsilon) = A(x)\Gamma_{A(x)}(x)$$

(7.7)

Up to an $\epsilon$-periodic function the $\Gamma_{A(x)}(x)$ function is a product of ordinary $\Gamma_{\epsilon}$-functions

$$\prod_{f=1}^{N} \Gamma_{\epsilon}(x - m_f)$$

(7.8)

its inverse is an entire function with zeroes at $m_f - \epsilon\mathbb{Z}_{\geq 0}$. The zeroes $\xi_{a,i}$ of $Q(x)$ behave as

$$\xi_{a,i} = a_a + \epsilon(i - 1) + x_{a,i}, \quad x_{a,i} \propto q^i \to 0, i \to \infty$$

(7.9)

The factor $\Gamma_{A(x)}(x)$ adds a string of zeroes going in the opposite direction

$$m_f - k\epsilon, \quad k \geq 0$$

(7.10)

If $m_a - a_a \in \epsilon\mathbb{Z}_{\geq 0}$, then an infinite string of zeroes $m_a + \epsilon\mathbb{Z}$ going in both directions can be removed from $Q(x)$ by factoring out the periodic function $\sin \frac{\pi}{\epsilon}(x - m_a)$, this is the case studied in [30, 31], in which $Q(x)$ becomes polynomial.
The equation (7.11) becomes the celebrated $TQ$-relation (c.f. eq. (6.20)):

$$A(x)Q(x + \epsilon) + qD(x)Q(x - \epsilon) = (1 + q)T(x)Q(x)$$  \hspace{1cm} (7.11)

Now define:

$$\Psi(t) = \sum_{x \in \Gamma} Q(x)t^{-x/\epsilon}$$  \hspace{1cm} (7.12)

where $\Gamma \subset \mathbb{C}$ is some lattice, invariant under the shifts by $\epsilon$: $\Gamma + \epsilon = \Gamma$. Then (7.11) implies:

$$\{ A(-\epsilon t \partial_t) + qD(-\epsilon t \partial_t)t^{-1} - (1 + q)T(-\epsilon t \partial_t) \} \Psi(t) = 0$$  \hspace{1cm} (7.13)

i.e. $\Psi$ is a solution of the $N$-order differential equation with $4$ regular singularities $t = 0, q, 1, \infty$. The monodromy of (7.13) around $t = 0$ and $t = \infty$ has $N$ distinct eigenvalues $e^{2\pi i m^\pm/\epsilon}$, $i = 1, \ldots, N$, where $m^\pm$ are the roots of $D(x)$ and $A(x)$, respectively. The monodromy of (7.13) around $t = q, 1$ has $N - 1$ eigenvalues equal to $1$, and one non-trivial eigenvalue $e^{2\pi i \mu^\pm/\epsilon}$, respectively, where $\mu^\pm$ can be computed from the $U(1)$ Coulomb moduli $u_1$ (7.2) and $m_i^\pm$.

7.2. Other examples.

7.2.1. $q$-characters for the $A_2$ theory. We have two $\Psi$-functions and two matter polynomials $P_{1,2}(x)$,

$$P_i(x) = (x - m_i^+)(x - m_i^-).$$  \hspace{1cm} (7.14)

The equations determining the vacua are:

$$y_1 \left( x - \frac{\epsilon}{2} \right) + q_1 \frac{P_1(x - \epsilon)y_2(x - \epsilon)}{y_1(x - \frac{3\epsilon}{2})} + q_1q_2 \frac{P_1(x - \epsilon)P_2(x - \frac{3\epsilon}{2})}{y_2(x - 2\epsilon)} =$$

$$= (1 + q_1 + q_1q_2)T_1 \left( x - \frac{\epsilon}{2} \right)$$

$$y_2 \left( x + \frac{\epsilon}{2} \right) + q_2 \frac{P_2(x)y_1(x)}{y_2(x - \frac{\epsilon}{2})} + q_1q_2 \frac{P_1(x - \epsilon)P_2(x)}{y_1(x - \epsilon)} =$$

$$= (1 + q_2 + q_1q_2)T_2 \left( x + \frac{\epsilon}{2} \right)$$  \hspace{1cm} (7.15)

where $T_{1,2}(x)$ are the monic degree $2$ polynomials:

$$T_i(x) = x^2 - u_{i,1}x - u_{i,2}, \hspace{1cm} i = 1, 2$$  \hspace{1cm} (7.16)

The parameters $u_{i,1}$ are the $U(1)$ Coulomb moduli, they are determined by the masses of the fundamental hypermultiplets and the mass of the bi-fundamental hypermultiplet.

Shift the argument of the first equation in (7.15) as $x \mapsto x + \epsilon/2$, multiply the result by $q_2P_2(x)/y_2 \left( x - \frac{\epsilon}{2} \right)$ and subtract the result from the second equation in (7.15) to obtain

$$y_2 \left( x + \frac{\epsilon}{2} \right) + (1 + q_1 + q_1q_2)q_2 \frac{P_2(x)T_1(x)}{y_2(x - \frac{\epsilon}{2})} -$$

$$- q_1q_2 \frac{P_1(x - \frac{\epsilon}{2})P_2(x - \epsilon)P_2(x)}{y_2(x - \frac{\epsilon}{2})y_2(x - \frac{3\epsilon}{2})} = (1 + q_2 + q_1q_2)T_2 \left( x + \frac{\epsilon}{2} \right)$$  \hspace{1cm} (7.17)
Now, write (cf. (6.28))
\[ Y^2(x) = P^2(x + \frac{\epsilon}{2}) \frac{Q(x + \frac{\epsilon}{2})}{Q(x - \frac{\epsilon}{2})} \]  
(7.18)
to reduce (7.17) to the linear equation (cf. (6.15), (6.16)):
\[
P^2(x + \epsilon)Q(x + \epsilon) - (1 + q_2 + q_1q_2)T_2(x + \frac{\epsilon}{2})Q(x) \\
+ q_2(1 + q_1 + q_1q_2)T_1(x)Q(x - \epsilon) - q_1q_2^2P_1(x - \frac{\epsilon}{2})Q(x - 2\epsilon) = 0 
\]  
(7.19)
Now let us make a Fourier transform:
\[ \Psi(t) = \sum_x Q(x)t^{-x/\epsilon} \]  
(7.20)
where the sum goes over some lattice \( x \in a + Z\epsilon \). Now the substitution:
\[ x \mapsto -\epsilon t \partial_t, \quad e^{t\partial_x} \mapsto t \]  
(7.21)
maps the Eq. (7.19) to the second order differential equation, obeyed by the function \( \Psi(t) \)
\[
[t^3P_2(-\epsilon t\partial_t) - (1 + q_2 + q_1q_2)t^2T_2(-\epsilon t\partial_t - \frac{\epsilon}{2}) + \\
+ tq_2(1 + q_1 + q_1q_2)T_1(-\epsilon t\partial_t - \epsilon) - q_1q_2^2P_1(-\epsilon t\partial_t - \frac{3}{2}\epsilon)] \Psi(t) = 0 
\]  
(7.22)
The symbol of the left hand side vanishes at the zeros of
\[ t^3 - (1 + q_2 + q_1q_2)t^2 + q_2(1 + q_1 + q_1q_2)t - q_1q_2^2 = (t - 1)(t - q_2)(t - q_1q_2) \]  
(7.23)
There are, then, five regular singularities, at \( t = 0, q_1q_2, q_2, 1, \infty \). Near \( t = 0 \) the dominant term is \( q_1q_2^2P_1(-\epsilon t\partial_t - \frac{3}{2}\epsilon) \), the eigenfunctions of the monodromy of the Eq. (7.22) around \( t = 0 \) are
\[
\psi^{\pm}_{(0)} \sim t^{-m_{\pm}/\epsilon - \frac{3}{2}}(1 + O(t)) 
\]  
(7.24)
and the corresponding eigenvalues are
\[ e^{-2\pi i(m_{\pm}/\epsilon + \frac{3}{4})} \]  
(7.25)
Analogously, near \( t = \infty \) the dominant term is \( t^3P_2(-\epsilon t\partial_t) \), and the eigenfunctions are
\[
\psi^{\pm}_{(\infty)} \sim t^{-m_{\pm}/\epsilon}(1 + O(t^{-1})) 
\]  
(7.26)
with the corresponding eigenvalues
\[ e^{-2\pi i m_{\pm}/\epsilon} \]  
(7.27)
Near the points \( t = z_i, z_1 = q_1q_2, z_2 = q_2, z_3 = 1 \) the equation (7.22) has one singular solution
\[ (t - z_i)^{-\mu_i/\epsilon} \]  
(7.28)
and one non-singular solution. The monodromy around these points has, therefore, one trivial eigenvalue 1 and one non-trivial eigenvalue
\[ e^{-2\pi i \mu_i/\epsilon}, \quad i = 1, 2, 3 \]  
(7.29)
The exponents $\mu_i$ are determined by the $U(1)$ Coulomb moduli and the masses, and in fact can be used as convenient parameters instead of $u_{1,1}, u_{2,1}$, subject to the obvious relation

$$
\sum_{i=1}^{3} \mu_i = m^+_2 + m^-_2 - m^+_1 - m^-_1 - 3\epsilon
$$

(7.30)

The redefinition

$$
\Psi(t) = t^{-\frac{m^+_1 - m^-_1 - 3\epsilon}{2\epsilon}} \prod_{i=1}^{3} (t - z_i)^{\frac{\mu_i}{2\epsilon}} \chi(t)
$$

(7.31)

maps (7.22) to the $sl_2$ oper (c.f. [202–208])

$$
\left[\epsilon^2 \partial^2_t + T(t)\right] \chi = 0
$$

(7.32)

where

$$
T = \sum_{i=0}^{3} \frac{\Delta_i}{(t - z_i)^2} + \frac{c_i}{t - z_i}
$$

$$
z_0 = 0, z_1 = q_1 q_2, z_2 = q_2, z_3 = 1, z_4 = \infty
$$

(7.33)

**Remark.** Observe that the equation obeyed by $\Psi(t)$ is the $\epsilon_2 \to 0$ limit of the equation obeyed by the degenerate conformal block of $A_{N-1}$ Toda conformal field theory, with $b^2 = \epsilon_2/\epsilon_1$, corresponding to the 5-punctured sphere. According to the AGT correspondence this is a $\mathbb{Z}$-partition function of the linear quiver theory of type $A_2$. It should be interesting to understand the relation between $\Psi(t)$ and $Z(a, m; q, t, \epsilon)$ in more general context. See [209] for some developments in this direction.

7.2.2. **Type II* $\hat{A}_0$ case.** We claim that in the four dimensional limit the equation (6.58) can be transformed to an interesting differential equation. Recall that our master equations state that $\chi(\xi) = T(\xi)$ where $T(\xi)$ is a (Laurent) polynomial of degree $N$ for the $v_0 = N$ theory. In the four dimensional limit, $T$ is a degree $N$ polynomial in $x$ (recall that $\xi = e^{i\ell x}$):

$$
T(x) = \frac{1}{\phi(q)} \left(x^N + u_1 x^{N-1} + \ldots + u_N\right)
$$

(7.34)

where we used the standard notation

$$
\phi(q) = \prod_{n=1}^{\infty} (1 - q^n), \quad \frac{1}{\phi(q)} = \sum_{\lambda} q^{\lambda} |\lambda|
$$

The equation (6.58) adapts to, again in the four dimensional limit,

$$
\sum_{n \in \mathbb{Z}} T(x + nm_-) Q(x - n\epsilon) q^{\epsilon^2} = 0,
$$

(7.35)

where

$$
m_{\pm} = m \pm \frac{\epsilon}{2}
$$
Now let us perform the Fourier transform:

$$\Psi(t) = \vartheta(t; q) \frac{m_+}{m} \sum_{x \in a + \mathbb{Z}} t^x Q(x)$$  \hspace{1cm} \text{(7.36)}$$

assuming this sum converges. Then \textbf{(7.35)} becomes the differential equation:

$$\mathcal{D} \Psi(t) = 0$$

where

$$\mathcal{D} = \frac{1}{\vartheta(t; q)} \sum_{n \in \mathbb{Z}} q^{\frac{n}{2} T} \left( e t \partial_t - \frac{u_1}{N} + (n - t \partial_t \log \vartheta(t; q)) m_+ \right)$$  \hspace{1cm} \text{(7.37)}$$

is the $N$-order differential operator on the torus $E_q = \mathbb{C}^n / q \mathbb{Z}$ with one puncture, which is at $t = -q^{\frac{1}{2}}$ in our normalization. The coefficients $v_i(t)$ are $q$-periodic, $v_i(qt) = v_i(t)$, and have the $i$'th order pole at $t = -q^{\frac{1}{2}}$. In line with \textbf{[8, 9]} we should view $\mathcal{D}$ as the $\mathfrak{pgl}_N$-oper.

\textbf{Example.} Let us now consider the small $N$ cases. For $N = 1$ the Eq. \textbf{(7.37)} reads:

$$e t \partial_t \Psi = 0 \implies \Psi(t) = \text{const}$$  \hspace{1cm} \text{(7.38)}$$

For $N = 2$ the Eq. \textbf{(7.37)} is equivalent to:

$$(e t \partial_t)^2 \Psi(t) + (m_+ m_- (t \partial_t)^2 \log \vartheta(t; q) + u_2 - u_1^2 / 4) \Psi(t) = 0$$  \hspace{1cm} \text{(7.39)}$$

7.2.3. $D$ and $E$ quivers. For the $\mathfrak{g}_r = D_r$, the quantum determinant equation, or the linear QT-equation like \textbf{(6.1)} was found in \textbf{[210]}, but for the exceptional series $E_6, E_7, E_8$ the linearization problem seems to be open. The $E_r$-series $(q, t)$-characters were computed by Nakajima \textbf{[133]} on supercomputer using sophisticated optimization.

\section*{Appendix A. Quantum affine algebras}

\textbf{A.1. Quantum affine algebra.} In this section we collect the conventions and the definitions related to quantum affine algebras \textbf{[101, 116]} and the $q$-characters \textbf{[49]}.

\textbf{A.1.1. Conventions.} Literature differs in the conventions for the $q$-parameter of quantum groups. The parameter $q$ of the present note relates to the definitions of $q$-characters by Frenkel and Reshetikhin \textbf{[49]} and Drinfeld \textbf{[116]} and Jimbo \textbf{[101]} and [211]

$$e^{i \ell \epsilon_1} = q_{\text{present}} = e^{118} = e^{h_{118}} = t_{101}^4 = q_{49}^2$$  \hspace{1cm} \text{(A.1)}$$

where $\ell$ is the circumference of the circle $S^1$ in the compactification of the five dimensional theory on the twisted bundle $\mathbb{R}^4 \times S^1_{(\epsilon_1, \epsilon_2; \ell)}$.

Our multiplicative spectral parameter $\xi$ corresponds to the variable $z$ of Nakajima \textbf{[131]} which corresponds to the variable $u^{-1}$ of Frenkel and Reshetikhin \textbf{[49]}, see more details in the appendix

$$\xi_{\text{present}} = u_{49}^{-1} = z_{130}$$  \hspace{1cm} \text{(A.2)}$$

We mostly use the conventions of \textbf{[49, 109, 131]}, with the dictionary of conventions \textbf{(A.1)} \textbf{(A.2)}. 

Let $a_{ij} = 2\delta_{ij} - I_{ij}$, where $I_{ij}$ is the incidence matrix, be the Cartan matrix $\mathfrak{g}_r$. The indices $i, j \in I_r$ label the nodes of the Dynkin diagram $\Gamma$. We restrict our definitions to simply laced ADE Lie algebras $\mathfrak{g}_r$. Let $(\alpha_i)_{i \in I_r}$ denote the simple roots. Then $a_{ij} = (\alpha_i, \alpha_j)$ where $(\cdot, \cdot)$ is the standard bilinear form on the dual Cartan algebra in normalization where the squared length simple roots is 2.

By $[n]_q$ we denote a $q$-number,

$$[n]_q = \frac{q^n - q^{-n}}{q^2 - q^{-2}},$$

(A.3)

by $[n!]_q$ the $q$-factorial,

$$[n!]_q = [n]_q[n - 1]_q \cdots [1]_q,$$

(A.4)

and by $[\frac{s}{k}]_q$ the $q$-binomial coefficient:

$$\left[ \begin{array}{c} s \\ k \end{array} \right]_q = \frac{[s]_q}{[(s - k)]_q[k]_q}$$

(A.5)

A.1.2. Chevalley-Drinfeld-Jimbo construction. Let $\mathfrak{g}$ be a generalized Kac-Moody algebra with symmetrizable Cartan matrix $\bar{a}_{ij}$. In particular, $\mathfrak{g}$ could be a finite-dimensional simple Lie algebra of affine Kac-Moody Lie algebra. Then we can define the quantum algebra $U_q(\mathfrak{g})$ by a $q$-deformation of Serre relation for Chevalley generators of $\mathfrak{g}$ \[50, 101, 116, 211\]. If $\mathfrak{g}_r$ is finite-dimensional, then applying such construction for $\mathfrak{g} = \hat{\mathfrak{g}}_r$ we obtain quantum affine algebra $U_q(\mathfrak{g}) = U_q(\hat{\mathfrak{g}}_r) \cong U^{\text{aff}}(\mathfrak{g}_r)$.

To simplify presentation, we assume that $\mathfrak{g}$ is simply laced

$$a_{ij} = a_{ji}$$

(A.6)

The algebra $U_q(\mathfrak{g})$ is an associative algebra generated by the Chevalley generators $(h_i, e_i^+, e_i^-)_{i \in I_\mathfrak{g}}$, called the coroots, the raising and the lowering elements, respectively, where $i$ runs over the set of nodes $I_\mathfrak{g}$ on Dynkin graph of $\mathfrak{g}$. The relations are the $q$-deformation of the Chevalley relations:\[3\]

$$h_i h_j = h_j h_i, \quad q^{h_i^+} e_j^\pm q_{h_j} = q^{h_i} e_j^\pm e_j^\pm, \quad [e_i^+, e_i^-] = \delta_{ij} [h_i]_q$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \begin{array}{c} 1 - a_{ij} \\ r \end{array} \right]_q (e_i^+)^{1-a_{ij}-r} e_j^\pm (e_i^\pm)^r = 0, \quad i \neq j$$

(A.7)

The *co-multiplication* $\Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, the *antipod* $\gamma : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ and the *counit* $\varepsilon : U_q(\mathfrak{g}) \to \mathbb{C}$ are given by

$$\Delta(q^{h_i^+}) = q^{h_i^+} \otimes q^{h_i^+}, \quad \Delta(e_i^+) = e_i^+ \otimes 1 + q^{h_i^+} \otimes e_i^+, \quad \Delta(e_i^-) = e_i^- \otimes q^{-h_i^+} + 1 \otimes e_i^+$$

$$\gamma(e_i^+) = -e_i^+ q^{h_i^+}, \quad \gamma(e_i^-) = -q^{h_i^+} e_i^-, \quad \gamma(h_i) = 1$$

$$\varepsilon(e_i^+) = 0, \quad \varepsilon(q^{h_i^+}) = 1$$

(A.8)

---

3Here $q^{h_i^+} = k_i \left[ \begin{array}{c} 4 \\ 4 \end{array} \right] \left[ \begin{array}{c} 101 \\ 211 \end{array} \right] = (e^{h_i}) \left[ \begin{array}{c} 116 \\ 211 \end{array} \right]$ and $q^{h_i} = q^{e_i}$.

4Different sign choices are possible and different conventions exist in the literature.
The Borel subalgebra \( \mathbf{U}_q(\mathfrak{g}^+) \subset \mathbf{U}_q(\mathfrak{g}) \) is a subalgebra generated by the generators \((h_i, e_i^\pm)\) in this realization \((A.7)\).

### A.2. Drinfeld current realization

In the Drinfeld current realization \([47, 109]\), applicable for arbitrary symmetrizable Kac-Moody \(\mathfrak{g}_r\), the quantum affine algebra \(\mathbf{U}^{\text{aff}}(\mathfrak{g}_r)\) is generated by the loop roots \((h_{i,n})_{i \in I_r, n \in \mathbb{Z}}\), loop raising and lowering generators \((x_{i,n}^\pm)_{i \in I_r, n \in \mathbb{Z}}\), the center \(c\) and the energy \(d\), where \(i \in I_r\) runs over the nodes of \(\mathfrak{g}_r\), with commutation relations \([49, 50, 134]6\):

\[
[h_i, e_{j,n}^\pm] = \pm a_{ij} e_{j,m}^\pm, \quad [h_i, e_{j,m}^\pm] = \pm \frac{1}{n} [na_{ij}]_q q^{\mp|n|/4} e_{j,n+m}^\pm
\]

\((A.9)\)

\[
[h_i, h_j] = 0, \quad [h_i, h_0] = \delta_{n+m,0} [na_{ij}]_q [nc]_q
\]

\((A.10)\)

\[
[e_{i,n}^+, e_{j,m}^-] = \delta_{ij} q^{(n-m)\epsilon/4} \psi_{i,n+m}^+ - q^{-(n-m)\epsilon/4} \psi_{i,n+m}^-
\]

\((A.11)\)

\[
e_{i,n+1}^+ e_{j,m}^+ - q^{\pm \epsilon} a_{ij} e_{i,n+1}^+ e_{j,m}^+ = q^{\pm \epsilon} a_{ij} e_{i,n+1}^+ e_{j,m}^+ - e_{i,j,m+1}^+ e_{i,n}^-
\]

\((A.12)\)

\[
\sum_{\sigma \in S_n} (-1)^k \left[ \begin{array}{c} a \\ k \end{array} \right] q^{\sigma_{i,n}(1)} \ldots q^{\sigma_{i,n}(k)} e_{i,n_{(1)}}^\pm \ldots e_{i,n_{(k+1)}}^\pm = 0, \quad (i \neq j, a = 1 - a_{ij})
\]

\((A.13)\)

where \((\psi_{i,n}^\pm)_{i \in I_r, n \in \mathbb{Z}}\) are loop modes\(^5\) of the current \(\psi_{i}^\pm(\xi) \in \mathbf{U}^{\text{aff}}(\mathfrak{g}_r)\)

\[
\psi_{i}^\pm(\xi) := \sum_{n=0}^{\infty} \psi_{i,n}^\pm \xi^n := q^{\pm \frac{1}{2} h_i} \exp \left( \pm (q^{\frac{1}{2}} - q^{\frac{1}{2}}) \sum_{n=1}^{\infty} h_{i,\pm n} \xi^n \right)
\]

\((A.14)\)

The level zero specialization \(\mathbf{U}^{\text{aff}}(\mathfrak{g}_r)|_{c=0}\) is a specialization to the center \(c = 0\). At \(c = 0\) the operators \((h_{i,n})_{i \in I_r, n \in \mathbb{Z}}\) generate the commutative subalgebra of \(\mathbf{U}^{\text{aff}}(\mathfrak{g}_r)|_{c=0}\), and the currents \(\psi_{i}^\pm(\xi)\) are commutative. In the limit \(q \to 1\), the \(\mathbf{U}^{\text{aff}}(\mathfrak{g}_r)|_{c=0}\) becomes algebra \(\mathbf{U}(L\mathfrak{g}_r)\) where \(L\mathfrak{g}_r\) is the loop algebra of \(\mathfrak{g}_r\).

### A.2.1. Drinfeld currents

We can write the defining relations of \(\mathbf{U}^{\text{aff}}(\mathfrak{g}_r)\) in terms of currents \(x_{i,n}^\pm(\xi)\), see \([134]\) and \([121]\),

\[
e_{i,n}^\pm(\xi) = \sum_{n \in \mathbb{Z}} e_{i,n}^\pm \xi^{-n}
\]

\((A.15)\)

In terms of the currents \((\psi_{i}^\pm(\xi), e_{i}^\pm(\xi))_{i \in I_r}\), the relations \([A.9], [A.10], [A.11]\) are respectively packed into\(^7\):

\(^5\)Here \(q_\frac{1}{2} = q_{49}^{1/2} 50^{1/2} 134^{1/2}\) and \(q_{\pm c} = c_{49}^{1/2} 50^{1/2}\) and \(h_{i,n} q_{-|n|/4} = a_{i,n}\) \([122]\)

\(^6\)Here \(\xi = z_{49}^{1/2} 50^{1/2} 134^{1/2}\) and \(z_{134}^{1/2} = z_{134}^{1/2}\)

\(^7\)It is convenient to compute \([\log \psi_{i}^\pm(\xi), e_{j}^\pm(\xi')]\) from \([A.9]\) and then use the fact that for any operators \(h, x, \) the commutator \([h, x] = \lambda x\), where \(\lambda\) is a number, implies \(e^{\lambda} x e^{-\lambda} = e^{\lambda} x\). If operators \(a, b\) commute as \([a, b] = \lambda\), where \(\lambda\) is a number, then \(e^{a} e^{b} e^{-a} = e^{\lambda} e^{b}\)
\[ \psi_i^\pm(\xi)e_j^\pm(\xi') = q^{\pm h_{i,j}} \frac{1 - q^{\pm a_{ij}} q^{\mp \frac{1}{2} \xi' / \xi}}{1 - q^{\pm a_{ij}} q^{\mp \frac{1}{2} \xi' / \xi}} \psi_i^\pm(\xi)e_j^\pm(\xi'), \quad \# \in \{+, -\} \]  \hfill (A.16)

where

\[ \delta(z) := \sum_{n \in \mathbb{Z}} z^n \]  \hfill (A.21)

and

\[ \sum_{k=0}^{a} (-1)^k \left[ \frac{a}{k} \right] \text{Sym}_{\xi_1, \ldots, \xi_a} e_i^\pm(\xi_1) \ldots e_i^\pm(\xi_k)e_j^\pm(\xi')e_i^\pm(\xi_{k+1}) \ldots e_i^\pm(\xi_a) = 0, \quad (i \neq j, a = 1 - a_{ij}) \]  \hfill (A.22)

We will also use the current

\[ \psi_i^\pm(\xi) := \sum_{n=0}^{\infty} \psi_i^{\pm n} \xi^{\mp n} := \exp \left( \pm (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \sum_{n=1}^{\infty} h_{i,\pm n} \xi^{\mp n} \right) \]  \hfill (A.23)

and

\[ p_i^\pm(\xi) := \exp \left( - \sum_{n=1}^{\infty} h_{i,\pm n} \xi^{\mp n} \right) \]  \hfill (A.24)

The currents \( \psi_i^\pm \in U_q(\mathfrak{g}_\tau) \) and \( p_i^\pm \in U_q(\mathfrak{g}_\tau) \) are related

\[ \psi_i^\pm(\xi) = q^{\pm \frac{1}{2} h_{i}} p_i^\pm(q^{\frac{1}{2}} \xi) p_i^\pm(q^{-\frac{1}{2}} \xi). \]  \hfill (A.25)

A.2.2. Yangian and elliptic version. Here we define the algebras \( Y_c(\mathfrak{g}) \), \( U_{q,\tau}(\mathfrak{g}) \) and \( U_{q,\tau}^{\text{aff}}(\mathfrak{g}) \) uniformly called \( U_q(\mathfrak{g}_{\tau}(\mathfrak{C}) \) for a Kac-Moody Lie algebra \( \mathfrak{g}_{\tau} \) with symmetric Cartan matrix \( a_{ij} \) using Drinfeld currents on \( \mathfrak{C}_{x} \), where \( \mathfrak{C}_{x} = \mathbb{C} \) for Yangian \( Y_c(\mathfrak{g}) \), \( \mathfrak{C}_{x} = \mathbb{C}/(2\pi\mathbb{Z}) \) for quantum affine \( U_{q,\tau}(\mathfrak{g}) \), and \( \mathfrak{C}_{x} = \mathbb{C}/(2\pi\mathbb{Z} \tau) \) for quantum elliptic \( U_{q,\tau}^{\text{aff}}(\mathfrak{g}) \). Instead of multiplicative variable

\[ \xi = e^{ix} \]  \hfill (A.26)

we use additive variable \( x \in \mathfrak{C}_{x} \) in the domain \( \mathfrak{C}_{x} \) which might have 0, 1, or 2 periods.
The basic (quasi)-periodic function \( s(x) \), with 0, 1 or 2 periods would be given by

\[
s(x) = \begin{cases} 
  x, & C_x = \mathbb{C} \\
  \frac{2}{\ell} \sin \frac{\ell x}{2}, & C_x = \mathbb{C}/\frac{2\pi}{\ell} \mathbb{Z} \\
  \frac{\theta_1(e^{i\pi \ell p})}{\partial_x \theta_1(e^{i\pi \ell p})}_{x=0}, & C_x = \mathbb{C}/\frac{2\pi}{\ell}(\mathbb{Z} + \tau_p \mathbb{Z})
\end{cases}
\]  

(A.27)

where \( \theta_1(\xi; p) \) for \( p = e^{2\pi i\tau_p} \) is defined in (2.82). We have the hierarchy of degenerations: elliptic \( \rightarrow \) trigonometric \( \rightarrow \) linear

\[
s_{\mathbb{C}/\frac{2\pi}{\ell}(\mathbb{Z} + \tau_p \mathbb{Z})}(x) \xrightarrow{p \to 0} s_{\mathbb{C}/\frac{2\pi}{\ell} \mathbb{Z}}(x) \xrightarrow{\ell \to 0} s_{\mathbb{C}}(x)
\]  

(A.28)

Then we would simply replace the defining relations (A.9) (A.10) (A.11) (A.12) by

\[
\psi^\pm_j(\xi) \epsilon^\pm_j(\xi') = \frac{s(x - x' \pm \frac{1}{2} a_{ij} \epsilon \pm \frac{\# \epsilon}{2})}{s(x - x' \mp \frac{1}{2} a_{ij} \epsilon \mp \frac{\# \epsilon}{2})} \epsilon^\pm_j(\xi') \psi^\pm_j(\xi), \quad \# \in \{+, -\}
\]  

(A.29)

\[
\psi^+_i(\xi) \psi^-_j(\xi') = \frac{s(x - x' - \frac{1}{2} a_{ij} \epsilon - \frac{\# \epsilon}{2} \ell)}{s(x - x' + \frac{1}{2} a_{ij} \epsilon + \frac{\# \epsilon}{2} \ell)} s(x - x' + \frac{1}{2} a_{ij} \epsilon + \frac{\# \epsilon}{2} \ell) s(x - x' - \frac{1}{2} a_{ij} \epsilon - \frac{\# \epsilon}{2} \ell) \psi^-_j(\xi') \psi^+_i(\xi);
\]  

(A.30)

\[
\psi^+_i(\xi) \psi^-_j(\xi') = \psi^+_j(\xi') \psi^+_i(\xi)
\]  

(A.31)

\[
[e^+_i(\xi), e^-_j(\xi')] = \frac{\delta_{ij} \epsilon}{s(\epsilon)} \left( \psi^+_i(x' + \frac{\# \epsilon}{2} \ell) \delta(x - x' - \frac{\# \epsilon}{2} \ell) - \psi^-_i(x' - \frac{\# \epsilon}{2} \ell) \delta(x - x' + \frac{\# \epsilon}{2} \ell) \right)
\]  

(A.32)

\[
e^\pm_i(\xi) e^\pm_j(\xi') = \frac{s(x - x' \pm \frac{1}{2} a_{ij} \epsilon)}{s(x - x' \mp \frac{1}{2} a_{ij} \epsilon)} \epsilon^\pm_j(\xi') e^\pm_i(\xi)
\]  

(A.33)

See more on elliptic currents in [212].

### A.3. Level zero representations.

#### A.3.1. Level zero representations and generalized weights.

For illustration, we first consider a fundamental evaluation module \( L_{1,q^{-\mu}} \) for \( U_{q^{-\mu}}^{\text{aff}}(\mathfrak{sl}_2) \). Let \( \mu \in \mathbb{C}^\times \) be an evaluation parameter, and \( |\omega_0|, |\omega_1| \) be basis vectors in \( L_{1,q^{-\mu}} \) so that \( |\omega_0| \) is the highest vector. In this basis the currents \( \psi(\xi), e^\pm(\xi) \) are represented by matrices

\[
\psi^\pm(\xi) = \begin{pmatrix}
q^{\pm \frac{1}{2} - \frac{1}{\mu} \omega - \frac{1}{\ell} / \xi} & 0 \\
0 & q^{\pm \frac{1}{2} - \frac{1}{\mu} \omega / \xi}
\end{pmatrix}^\pm
\]  

(A.34)

and

\[
e^+(\xi) = \begin{pmatrix}
0 & \delta(\mu/\xi) \\
\delta(\mu/\xi) & 0
\end{pmatrix}, \quad e^-(\xi) = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\]  

(A.35)

where symbols \( f(\xi) \) for a function \( f(\xi) \) denotes expansions near \( \xi = \infty \) and \( \xi = 0 \) respectively

\[
f(\xi)_+ = \sum_{n \geq n_+} f_n \xi^{-n}, \quad f(\xi)_- = \sum_{n \leq n_-} f_n \xi^n
\]  

(A.36)
The key property of these expansions is localization to the poles, such as
\[
\left(\frac{1}{1 - 1/\xi}\right)_+ - \left(\frac{1}{1 - 1/\xi}\right)_- = \sum_{n \geq 0} \xi^{-n} + \sum_{n > 0} \xi^n = \delta(\xi) \quad (A.37)
\]

For example, checking the commutation relation (A.20), in the right hand side we find the matrix elements such as
\[
\left( q^\frac{1}{2} \frac{1 - \mu q^{-1/2}}{1 - \mu/\xi} \right)_+ - \left( q^\frac{1}{2} \frac{1 - \mu q^{-1/2}}{1 - \mu/\xi} \right)_- = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \sum_{n \in \mathbb{Z}} (\mu/\xi)^n = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \delta(\mu/\xi) \quad (A.38)
\]
so that indeed
\[
[e^+(\xi), e^-(\xi')] = \delta(\mu/\xi) \delta(\mu/\xi')\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\delta(\xi/\xi') \delta(\mu/\xi)}{q^\frac{1}{2} - q^{-\frac{1}{2}}} \begin{pmatrix} q^{\frac{1}{2}} - q^{-\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} - q^{\frac{1}{2}} \end{pmatrix} = \frac{1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \delta(\xi/\xi')(\psi^+(\xi) - \psi^-(\xi)) \quad (A.39)
\]

The diagonal elements in (A.34) are called the current weights or loop weights of an operator \(\psi(\xi)\).

The highest weight of the evaluation representation \(L_{1,\mu q^{-1/2}}\) is encoded by the Drinfeld polynomial
\[
P_1 = (1 - \mu q^{-\frac{1}{2}}/\xi)
\]
The operator \(\psi(\xi) \in \mathcal{U}^{\text{aff}}(\mathfrak{sl}_2)\) acts by
\[
\psi(\xi)|\omega_0\rangle = q^{\frac{1}{2}} \frac{1 - \mu q^{-1/2}}{1 - \mu/\xi} |\omega_0\rangle
\]
\[
\psi(\xi)|\omega_1\rangle = q^{-\frac{1}{2}} \frac{1 - \mu q^{1/2}}{1 - \mu/\xi} |\omega_1\rangle \quad (A.40)
\]
Hence, by definition, the \(q\)-character of the \(\mathcal{U}^{\text{aff}}(\mathfrak{sl}_2)\) fundamental module \(V_{1,\mu q^{-1/2}}\) is
\[
\chi_q[V_{1,\mu q^{-1/2}}] = Y_{1,\mu q^{-1/2}} + \frac{1}{Y_{1,\mu q^{-1/2}}} \quad (A.41)
\]
Notice that the sum of eigenvalues of \(\psi(\xi)\) does not encode as much information about the \(\mathcal{U}^{\text{aff}}(\mathfrak{sl}_2)\) module \(V_{1,\mu q^{-1/2}}\) as it turns out to be simply
\[
\text{tr}_V \psi_1(z) = q^{\frac{1}{2}} + q^{-\frac{1}{2}} \quad (A.42)
\]
In terms of the current modes \(\psi_{1,n}^\pm, e_{1,n}^\pm\) for the same example of the evaluation representation \(L_{1,\mu q^{-1/2}}\) of \(\mathcal{U}^{\text{aff}}(\mathfrak{sl}_2)\) with evaluation parameter \(\mu q^{-1/2}\) we find the explicit
action by the $U_q^{\text{aff}}(\mathfrak{sl}_2)$ generators on $V_{1,\mu q^{-\frac{1}{2}}}$ is given by

$$
e_{1,n}^-(\omega_0) = \mu^n|\omega_1\rangle \quad e_{1,n}^+(\omega_0) = \mu^n|\omega_0\rangle$$

$$h_{1,n}|\omega_0\rangle = \mu^n q^{-\frac{n}{2}}[n]\frac{[n]_q}{n}|\omega_0\rangle \quad (n > 0), \quad h_{1,0}|\omega_0\rangle = |\omega_0\rangle$$

$$h_{1,n}|\omega_1\rangle = \mu^n(q^{-\frac{n}{2}}[n]_q - [2n]_q)|\omega_1\rangle \quad (n > 0), \quad h_{1,0}|\omega_1\rangle = -|\omega_1\rangle$$

or equivalently by the matrices

$$e_{1,n} = \mu^n \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad e_{1,n}^+ = \mu^n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$h_{1,n\neq 0} = \frac{\mu^n}{n} \begin{pmatrix} 1 & 0 \\ \frac{1-q^n}{q^{1/2}-q^{-1/2}} & 0 \end{pmatrix} \quad h_{1,0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\psi^\pm(\xi) = \begin{pmatrix} q^{-\frac{1}{2}}(1-\mu q^{-1}/\xi) \\ 1-\mu/\xi \end{pmatrix}^\pm \Rightarrow \psi_{1,n\in\mathbb{Z}_{>0}}^\pm = \pm \left( \mu^n(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right)$$

which obviously satisfy the commutation relations \((A.9)(A.10)(A.11)(A.12)\) of $U_q^{\text{aff}}(\mathfrak{sl}_2)$ at $c = 1$

$$[h_{1,n\neq 0}, e_{1,m}^\pm] = \frac{[2n]_q}{n} e_{m+n}^\pm \quad [e_{1,m}^+, e_{1,n}^-] = \frac{1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} (\psi_{1,m+n}^+ - \psi_{1,m+n}^-)$$

\(A.45\)

### A.3.2. Level zero fundamental highest weight modules.

The basic level zero fundamental representations of quantum affine algebra $U_q^{\text{aff}}(\mathfrak{g})$ are highest weight modules $L_{i,\mu}$, labeled by a node $i \in I_\Gamma$ of Dynkin diagram of $\mathfrak{g}$ and an evaluation parameter $\mu \in \mathbb{C}^\times$. A module $L_{i,\mu}$ is a highest weight $U_q^{\text{aff}}(\mathfrak{g})$-module with the current weights of $(\psi_i^\pm)_{i \in I_\Gamma}$ on the highest vector $|\omega_{i,\mu}\rangle$ given by

$$\psi_i^\pm(\xi)|\omega_{i,\mu}\rangle = \begin{cases} 1, & j \neq i \\ q^{\frac{1}{2}} \frac{1-\mu q^{-\frac{1}{2}}}{1-\mu q^{\frac{1}{2}}} \frac{1}{\xi}, & j = i \end{cases}$$

\(A.46\)

The fundamental module $L_{i,\mu}$ is a quantum affine algebra analogue of the $i$-th fundamental highest weight evaluation module at $\xi = \mu$ for the loop algebra $\mathfrak{g}(\langle \xi \rangle)$.

### A.4. Loop weights.

Following \([50],[49]\) we call a vector $|w\rangle \in W$, for a $U_q^{\text{aff}}(\mathfrak{g}_\Gamma)$-module $W$, a generalized eigenvector of the operators $\psi_i^\pm(\xi) \in U_q^{\text{aff}}(\mathfrak{g}_\Gamma)$ with the generalized eigenvalues $\Psi_i^\pm(\xi) \in \mathbb{C}[|\xi^\pm|]$ if there exists a positive integer $n$, such that

$$\left(\psi_i^\pm(\xi) - \Psi_i^\pm(\xi)\right)^n|w\rangle = 0 \quad \text{for all } i \in I_\Gamma.$$ \(A.47\)

in other words it belongs to the Jordan block of uniformly bounded size $n$ for all $\psi_i^\pm(\xi)$’s. The values $\Psi_i^+(\infty)$ and $\Psi_i^-(0)$ are the eigenvalues of $q^{\frac{1}{2}h_i}$ and $q^{-\frac{1}{2}h_i}$ since

$$q^{\frac{1}{2}h_i}|w\rangle = \Psi_i^+(\infty)|w\rangle$$

$$q^{-\frac{1}{2}h_i}|w\rangle = \Psi_i^-(0)|w\rangle.$$ \(A.48\)
All integrable finite-dimensional modules of $U_q^{\text{aff}}(\mathfrak{g}_\Gamma)$ with highest weight have been classified [50] (theorem 3.3). The generalized eigenvalue $\Psi^\pm(\xi)$ of the operator $\psi^\pm(\xi)$ on the highest vector $|v\rangle$ of an integrable finite-dimensional module $V$ of $U_q^{\text{aff}}(\mathfrak{g}_\Gamma)$ always has the form

$$\Psi^\pm_i(\xi) = q^{\frac{1}{2}\text{deg} P_i} \left(\frac{P_i(q^{\frac{1}{2}} \xi)}{P_i(q^{-\frac{1}{2}} \xi)}\right)^\pm$$ (A.49)

where $P_i^+(\xi) := \prod_{k=1}^{\text{deg} P_i}(1 - \xi_k/\xi)$ is a polynomial in $\xi^{-1}$ with $P_i(\infty) = 1$ and $()^\pm$ denotes the expansion at $\xi = \infty$ and $\xi = 0$ respectively. The collection of polynomials $P_i(\xi)_{i \in I_{\Gamma}}$, defining an integrable highest weight module of $U_q^{\text{aff}}(\mathfrak{g}_\Gamma)$ is called Drinfeld polynomial.

The equation (A.24) implies that the highest vector $|v_0\rangle$ in the $i$-th fundamental representation with Drinfeld polynomial $P_j(\xi) = 1$ for $j \neq i$ and $P_i(\xi) = 1 - 1/\xi$, such that the eigenvalue of $p_j^+(\xi)$ on $|v_0\rangle$ is equal to $P_j(\xi)$, is a common eigenvector of the generators $h_{j,n}$ with the corresponding eigenvalues

$$h_{j,n} = \begin{cases} 0, & j \neq i \\ 1, & j = i, n = 0 \\ \frac{[n]_q}{n}, & j = i, n \neq 0 \end{cases}$$ (A.50)

E. Frenkel and N. Reshetikhin [49] have shown that for any finite-dimensional $U_q^{\text{aff}}(\mathfrak{g}_\Gamma)$ module $V$ the generalized eigenvalues of the operators $\psi^\pm_i(\xi) \in U_q(\hat{\mathfrak{g}}_\Gamma)$ always have the form

$$\text{ev}_{\psi(\xi)} \prod_{\xi_k}(Y_{i,\xi_k})^\pm$$ (A.51)

where evaluation on $\psi(\xi)$ is defined multiplicatively by

$$\text{ev}_{\psi(\xi)}(Y_{i,\xi}) := q^{\frac{1}{2}} \frac{1 - q^{\frac{1}{2}} \xi/\xi}{1 - q^{\frac{1}{2}} \xi/\xi}$$ (A.52)

Similarly, for stripped element $\hat{\psi}(\xi)$

$$\text{ev}_{\hat{\psi}(\xi)}(Y_{i,\xi}) := \frac{1 - q^{\frac{1}{2}} \xi/\xi}{1 - q^{\frac{1}{2}} \xi/\xi}$$ (A.53)

The $q$-character $\chi_q(V)$ of $U_q(\hat{\mathfrak{g}}_\Gamma)$ finite-dimensional module $V$ is defined as a sum of $\dim V$ monomials of the form

$$\chi_q(V) = \sum_{|v\rangle} \prod_{i,\xi_i \in \Xi_{|v\rangle}} (Y_{i,\xi_i})^\pm$$ (A.54)

one monomial for each generalized eigenvector $|v\rangle$ in $U_q^{\text{aff}}(\mathfrak{g}_\Gamma)$ module $V$ for the commuting set of operators $\psi_i(\xi) \in U_q^{\text{aff}}(\mathfrak{g}_\Gamma)$, such that each such monomial encodes the generalized eigenvalue of the operator

$$\psi(\xi) = \prod_{i \in I_{\Gamma}} \psi_i(\xi) \in U_q(\hat{\mathfrak{g}}_\Gamma)$$ (A.55)

by the evaluation (A.53).
**A.5. Universal R-matrix.** Abstractly, quantum affine algebra \( A = U_q^\text{aff}(\mathfrak{g}_\text{aff}) \) is a Hopf algebra, and hence is equipped with the *comultiplication* operation \( \Delta : A \to A \otimes A \). The comultiplication operation is what allows to define tensor products of representations for associative algebras. If \( \rho : A \to \text{End}(V) \) are two representations, then the tensor product \( \rho : A \to \text{End}(V_1 \otimes \text{End}(V_2) \) is defined by \( x \mapsto \rho_1 \otimes \rho_2(\Delta(x)) \).

If an abstract Hopf algebra \( A \) is co-commutative, then \( A \)-modules \( V_1 \otimes V_2 \) and \( V_2 \otimes V_1 \) are naturally isomorphic, but there is no natural reason for such isomorphism if \( A \) is not co-commutative. Quantum affine algebras have extra structure in addition to the structure of generic Hopf algebras, called *quasi-triangular structure* that ensures that tensor products \( V_1 \otimes V_2 \) and \( V_2 \otimes V_1 \) are isomorphic. Such isomorphism is not a simple permutation of the factors, but is given by a certain linear map that depends on representations \( V_1, V_2 \). Namely, by definition, a quasi-triangular structure on a Hopf algebra \( A \) is an element \( \mathcal{R} \in A \otimes A \), called *universal R-matrix* such that

\[
\mathcal{R} \Delta(x) = \Delta^{op}(x) \mathcal{R} \quad \quad (\Delta \otimes \text{Id}_A)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{23} = a_i \otimes a_j \otimes b_i b_j \quad \quad (A.56)
\]

\[
(\text{Id}_A \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{12} = a_i a_j \otimes b_j \otimes b_i \quad \quad (A.57)
\]

where \( x \in A \) and if \( \mathcal{R} = \sum_i a_i \otimes b_i \) then \( \mathcal{R}_{12} = \sum_i a_i \otimes b_i \otimes 1, \mathcal{R}_{13} = \sum_i a_i \otimes 1 \otimes b_i, \mathcal{R}_{23} = \sum_i 1 \otimes a_i \otimes b_i \), and \( \Delta^{op} \) denotes the co-multiplication in the opposite order. The map \( \hat{\mathcal{R}}_{V_1, V_2} = PR_{V_1, V_2} \), where \( P : V_1 \otimes V_2 \to V_2 \otimes V_1 \) is a permutation of factors, gives isomorphism \( \hat{\mathcal{R}}_{V_1, V_2} : V_1 \otimes V_2 \to V_2 \otimes V_1 \). The relations (A.56) imply that the universal R-matrix \( \mathcal{R} \in A \otimes A \) satisfies universal Yang-Baxter equation

\[
\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12} \quad \quad (A.57)
\]

The axioms on R-matrix ensure that the category of representations of \( A \) is a *braided tensor category* (which means this tensor category is equipped with a commutativity isomorphism \( \sigma_{V_1, V_2} : V_1 \otimes V_2 \to V_2 \otimes V_1 \), an associativity isomorphism \( \alpha_{V_1, V_2, V_3} : (V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3) \) and unit morphisms \( \lambda_V : 1 \otimes V \to V, \rho_V : V \otimes 1 \to V \), which satisfy certain compatibility axioms. For more details see \([213]\).

For any finite-dimensional Hopf algebra \( A \), Drinfeld constructed \([116]\) structure of quasi-triangular Hopf algebra on *Drinfeld double*

\[
D(A) = A \otimes A^{op}
\]

If \( a_i \) is a basis in \( A \) and \( a^i \) is a dual basis in \( A^* \) then \( D(A) \) is a quasi-triangular Hopf algebra with the R-matrix

\[
\mathcal{R} = \sum_i (a_i \otimes 1_{A^*}) \otimes (1_A \otimes a^i) \quad \quad (A.58)
\]

The quasi-triangular structure on \( U_q^\text{aff}(\mathfrak{g}_\text{aff}) \) (for \( \mathfrak{g}_\text{aff} \) of finite type) can be found using the fact that \( U_q^\text{aff}(\mathfrak{g}_\text{aff}) \) is almost Drinfeld double of it Borel subalgebra \( U_q^\text{aff}(\mathfrak{g}^{+}_\text{aff}) \) \([213, 214]\). The structure of Hopf algebra on \( U_q^\text{aff}(\mathfrak{g}^{+}_\text{aff}) \) naturally induces the structure of Hopf algebra on its dual \( U_q^\text{aff}(\mathfrak{g}^{-}_\text{aff})^* \). Moreover, V. Drinfeld has shown \([116]\) that there exists an isomorphism of Hopf algebras \( \theta : U_q^\text{aff}(\mathfrak{g}_\text{aff}^-) \to U_q^\text{aff}(\mathfrak{g}_\text{aff}^{+})^{op} \) such that for \( h \) in the extended Cartan subalgebra of \( \mathfrak{g}_\text{aff} \) we have \( \theta(q^h) = q^h \) if we identify the extended
Cartan subalgebra with its dual using the standard Cartan-Killing quadratic form. The isomorphism $\theta$ provides a non-degenerate bilinear form (called Drinfeld pairing) $U_q^\text{aff}(g_{\Gamma}^+) \otimes U_q^\text{aff}(g_{\Gamma}^-) \to \mathbb{C}$. Using Drinfeld pairing we present the Drinfeld double of the Borel $U_q^\text{aff}(g_{\Gamma}^\pm)$ as

$$D(U_q^\text{aff}(g_{\Gamma}^\pm)) = U_q^\text{aff}(g_{\Gamma}^+) \otimes U_q^\text{aff}(g_{\Gamma}^-)$$  \hspace{1cm} (A.59)

The Drinfeld double $D(U_q^\text{aff}(g_{\Gamma}^\pm))$ contains two-sided ideal $H$ generated by $q^h \otimes 1 - 1 \otimes q^h$ for $h$ in the extended Cartan of $\hat{g}_{\Gamma}$. One can check that the map $f : D(U_q^\text{aff}(g_{\Gamma}^\pm))/H \to U_q^\text{aff}(g_{\Gamma})$ is isomorphism. The pushforward by $f$ of the universal $R$-matrix on $D(U_q^\text{aff}(g_{\Gamma}^\pm))/H$ will give a universal $R$-matrix on $U_q^\text{aff}(g_{\Gamma})$. An explicit formula can be found in [215].

A.6. Dual interpretation. The $q$-character $\chi_q(V)$ for a representation $V$ of $U_q^\text{aff}(g_{\Gamma})$ can be interpreted not only as a formal expression encoding generalized eigenvalues of diagonal generators of $U_q^\text{aff}(g_{\Gamma})$ but also as a certain element of $U_q^\text{aff}(g_{\Gamma})$, in fact, that is how the $q$-character was originally introduced in [49]. The tensor product $U_q(g_{\Gamma}) \otimes U_q(g_{\Gamma})$ contains a special element $\hat{g}_{\Gamma}$.

Let $(V, \pi_V)$ denote a finite-dimensional representation of $U_q(g_{\Gamma})$. Let $V(\xi)$ denote the twist of representation $V$ by spectral parameter $\xi$. Define the transfer matrix $t_V(\xi)$ associated to $V$ as an element of $U_q^\text{aff}(g_{\Gamma})$ given by the trace over $V$:

$$t_V(\xi) = \text{tr}_V q^\rho(\pi_V(\xi) \otimes \text{id})\mathcal{R}$$  \hspace{1cm} (A.60)

where $\rho = \sum \tilde{h}_i$ with $\tilde{h}_i = \tilde{a}_{ij} h_j$.

The explicit formula for universal $R$-matrix is given in [216] (eq 42), [217]

$$\mathcal{R} = \mathcal{R}^+ \mathcal{R}^{-0} \mathcal{R}^{-1} T$$  \hspace{1cm} (A.61)

where

$$\mathcal{R}^0 = \exp \left( -(q^{1/2} - q^{-1/2}) \sum_{n>0} \frac{n}{|n|_q} \tilde{a}_{ij}(q^n) h_{i,n} \otimes h_{i,-n} \right)$$  \hspace{1cm} (A.62)

$$T = q^{-\frac{1}{2} \sum \tilde{h}_i \otimes \tilde{h}_j}$$

and the factors $\mathcal{R}^\pm \in U_q^\text{aff}(g_{\Gamma}^\pm) \otimes U_q(\hat{g}_{\Gamma})$.

The $q$-character $\chi_q(V(\xi))$ is essentially the diagonal projection of $t_V(\xi)$, i.e.

$$\chi_q(V(\xi)) = \text{tr}_V (q^\rho \Gamma_V(\pi_V(\xi) \otimes 1)(T))$$

$$\Gamma_V = \exp \left( -(q^{1/2} - q^{-1/2}) \sum_{n>0} \frac{n}{|n|_q} \tilde{a}_{ij}(q^n) \xi^{-n} \pi_V(h_{i,n}) \otimes h_{j,-n} \right)$$  \hspace{1cm} (A.63)

Now we introduce special elements $\hat{Y}_{i,a}$ of $U_q^\text{aff}(g_{\Gamma})$ by definition

$$\hat{Y}_{i,a} = q^{(\rho, \omega_i)} q^{-\frac{1}{2} \tilde{h}_i} \exp \left( -(q^{1/2} - q^{-1/2}) \sum_{n>0} \tilde{h}_{i,-n} \xi^n \xi^{-n} \right)$$  \hspace{1cm} (A.64)

where $\omega_i$ are fundamental weights.

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8If $g_{\Gamma}$ of finite type has rank $r$, the extended Cartan subalgebra of $\hat{g}_{\Gamma}$ has rank $r+2$ and is generated by the coroots of $g_{\Gamma}$, the center and the energy element.
The $q$-character is a sum of $\dim V$ terms. Each term in $\chi_q(V(\xi))$ corresponds to an eigenvector of $V$. The eigenvalue of a given eigenvector of $V$ is encoded by Drinfeld polynomial loop-weights which can be factorized into a product of basic monomials $(P) = (1, \ldots, 1, 1 - \zeta/\xi, 1, \ldots, 1)$ with $1 - \zeta/\xi$, say, in $i$-th position. Each such factor produces a factor $\hat{Y}_{i,\zeta}$ in the given term in the $q$-character associated with this eigenvalue.

A.6.1. **Twisted $q$-characters.** For our purposes it is natural to consider the twisted transfer matrix defined by the formula similar to the (A.60) in which $q\sum_{i} \hat{h}_i$ is replaced by $\prod_i q^{-\hat{h}_i}$:

$$t_{q,V(\xi)} = \text{tr}_{V(\xi)} \prod_{i} q^{-\hat{h}_i} (\pi_{V(\xi)} \otimes \text{id}) \mathcal{R}$$  \hspace{1cm} (A.65)

and then set

$$\hat{Y}_i(\xi) = q^{-\frac{1}{2} \hat{h}_i} \exp \left( - (q^\frac{1}{2} - q^{-\frac{1}{2}}) \sum_{n>0} \hat{h}_{i,-n} \xi^{-n} \right)$$ \hspace{1cm} (A.66)

The $q$-twisted $q$-character is a sum of monomials associated to eigenvectors. Each monomial is a product of factors $\hat{Y}_{i,a}^\pm$ encoding the generalized eigenvalues with an additional $q_i$-dependent factor precisely like in (4.7) for the quiver theory without fundamental matter, with $\mathcal{P}_i = q_i$. For example, the $q$-twisted $q$-character associated to the $A_1$ quiver and the fundamental evaluation module is

$$t_{q,L_1} = q^{-\frac{1}{2}} \hat{Y}_1(\xi) + q^\frac{1}{2} \frac{1}{\hat{Y}_1(q^{-1}\xi)}$$ \hspace{1cm} (A.67)

**Proposition.** The equations

$$\text{q-twisted } q\text{-characters } \chi_i = c\text{-number polynomials } T_i(\xi)$$ \hspace{1cm} (A.68)

can be interpreted as follows: there exists a special eigenvector of the $U_q^{\text{aff}}(\mathfrak{g}_\Gamma)$ elements $\hat{Y}_i(\xi)$ with the generalized eigenvalues equal to our $c$-number functions $Y_i(\xi)$ and the eigenvalue of the $q$-twisted $q$-character viewed as an element of $U_q^{\text{aff}}(\mathfrak{g}_\Gamma)$ equal to the polynomial $T_i(\xi)$.

It looks like our formulation is $T$-$Q$ dual (in a sense of [201]) to the results of the recent paper [94] on the category $O$ of representations of the quantum affine algebras, at least for the finite dimensional $\mathfrak{g}_\Gamma$.

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$^9$It is not clear whether this eigenvector belongs to a natural $U_q^{\text{aff}}(\mathfrak{g}_\Gamma)$-module though
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