HAMILTONIAN CIRCLE ACTIONS WITH ISOLATED FIXED POINTS

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Abstract. Let the circle act in a Hamiltonian fashion on a compact $2n$-dimensional symplectic manifold $M$ with $n+1$ isolated fixed points. Let $M'$ be $\mathbb{CP}^n$ or $\tilde{G}_2(\mathbb{R}^{n+2})$ with $n \geq 3$ odd in the latter. We show that the following 4 conditions are equivalent: (1) $M$ has the same first Chern class as $M'$, (2) $M$ has the same integral cohomology ring as $M'$, (3) $M$ has the same total Chern class as $M'$, and (4) the $S^1$ representations at the fixed points are the same as those of a standard circle action on $M'$. If additionally $M$ is Kähler and the action is holomorphic, under any of the above 4 equivalent conditions, we show that $M$ is equivariantly biholomorphic and symplectomorphic to $M'$ equipped with a standard circle action.

1. Introduction

Let the circle act symplectically on a compact connected $2n$-dimensional symplectic manifold $(M, \omega)$. It is known that if the fixed point set is nonempty, then such an action has at least 2 fixed points, and has at least 3 fixed points if $\dim(M) \geq 8$ [11]. A necessary condition for the action to be Hamiltonian is that there are at least $n+1$ fixed points. This can be seen by using that the moment map is a perfect Morse-Bott function and that the even Betti numbers of $M$ satisfy $b_{2i}(M) \geq 1$ for all $0 \leq 2i \leq 2n$. There are many interesting studies of symplectic circle actions on compact symplectic manifolds, one of the questions is when such an action is Hamiltonian. In [9], McDuff showed that in dimension 4, the action is Hamiltonian if there is a fixed point, she also constructed a symplectic but not Hamiltonian circle action in dimension 6 with non-isolated fixed point set, and posed the interesting question whether a symplectic circle action on a compact symplectic manifold with isolated fixed points is Hamiltonian. For the case when the circle acts semifreely (i.e., the action is free outside fixed points) with isolated fixed points, Tolman and Weitsman [13] showed that the action has exactly $2^n$ fixed points and the action is Hamiltonian. An example of such a Hamiltonian $S^1$-manifold is the product of $n$ copies of $S^2$, equipped with the standard rotation by $S^1$ on each copy of $S^2$. Another case (in some sense the opposite case) is when the symplectic circle action has exactly $n+1$ fixed points, and the action is Hamiltonian.

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isolated fixed points. For this case, according to Hattori’s work [3, Lemmas 5.10 and 6.4], if $c_1(M) = (n+1)x$ or $c_1(M) = nx$, where $x \in H^2(M; \mathbb{Z})$ is a generator, then the action is Hamiltonian. Two standard examples of such Hamiltonian $S^1$-manifolds are $\mathbb{CP}^n$, and $\widetilde{G}_2(\mathbb{R}^{n+2})$ with $n \geq 3$ odd, equipped with standard circle actions, where the latter is the Grassmannian of oriented 2-planes in $\mathbb{R}^{n+2}$, also called the complex quadratic hypersurface in $\mathbb{CP}^{n+1}$. See Examples 1.4 and 1.5. Note that in these examples the actions are very “non-semifree”.

Motivated by the second case above, we now assume that $(M, \omega)$ is a compact Hamiltonian $S^1$-manifold of dimension $2n$ with exactly $n+1$ isolated fixed points. Then using that the moment map is a perfect Morse function, we see that the even Betti numbers of $M$ are minimal, i.e., $b_{2i}(M) = 1$ for all $0 \leq 2i \leq 2n$. In this paper, we study exactly the cases when $c_1(M) = (n+1)x$ or $c_1(M) = nx$, where $x \in H_2(M; \mathbb{Z})$ is a generator.

We show that in these cases, the first Chern class of $M$, the total Chern class of $M$, the integral cohomology ring of $M$, and the circle action on $M$ mutually determine each other; moreover, if $M$ is Kähler and the action is holomorphic, we show that $\mathbb{CP}^n$, and $\widetilde{G}_2(\mathbb{R}^{n+2})$ with $n \geq 3$ odd, equipped with the standard circle actions are the only examples in the complex and symplectic categories. These results are respectively stated in the following theorems.

**Theorem 1.1.** Let the circle act in a Hamiltonian fashion on a compact $2n$-dimensional symplectic manifold $(M, \omega)$ with $n+1$ isolated fixed points. Then the following conditions are equivalent:

1. $M$ has the same first Chern class as $\mathbb{CP}^n$, i.e., $c_1(M) = (n+1)x$, where $x \in H^2(M; \mathbb{Z})$ is a generator.
2. $M$ has the same integral cohomology ring as $\mathbb{CP}^n$.
3. $M$ has the same total Chern class as $\mathbb{CP}^n$.
4. The $S^1$ representations at the fixed points are the same as those of a standard circle action on $\mathbb{CP}^n$.

**Theorem 1.2.** Let the circle act in a Hamiltonian fashion on a compact $2n$-dimensional symplectic manifold $(M, \omega)$ with $n+1$ isolated fixed points, where $n \geq 3$ is odd. Then the following conditions are equivalent:

1. $M$ has the same first Chern class as $\widetilde{G}_2(\mathbb{R}^{n+2})$, i.e., $c_1(M) = nx$, where $x \in H^2(M; \mathbb{Z})$ is a generator.
2. $M$ has the same integral cohomology ring as $\widetilde{G}_2(\mathbb{R}^{n+2})$.
3. $M$ has the same total Chern class as $\widetilde{G}_2(\mathbb{R}^{n+2})$.
4. The $S^1$ representations at the fixed points are the same as those of a standard circle action on $\widetilde{G}_2(\mathbb{R}^{n+2})$.

**Theorem 1.3.** Let the circle act holomorphically and in a Hamiltonian fashion on a compact complex $n$-dimensional Kähler manifold $M$ with $n+1$ isolated fixed points. When one of (1), (2), (3) and (4) in Theorem 1.1 is satisfied, $M$ is equivariantly biholomorphic to $\mathbb{CP}^n$, and is equivariantly
symplectomorphic to $\mathbb{C}P^n$. When $n \geq 3$ is odd, and one of (1), (2), (3) and (4) in Theorem 1.2 is satisfied, $M$ is equivariantly biholomorphic to $\widetilde{G}_2(\mathbb{R}^{n+2})$, and is equivariantly symplectomorphic to $\widetilde{G}_2(\mathbb{R}^{n+2})$.

In [2], Godinho and Sabatini proved Theorem 1.1 for the particular case when $\dim(M) = 8$. Our ideas of proof are very different.

In particular, the manifold $M$ satisfying the assumption of Theorem 1.1 is simply connected (Lemma 5.4 or [5]). If $M$ has the same integral cohomology ring as $\mathbb{C}P^n$, then $M$ is homotopy equivalent to $\mathbb{C}P^n$. Theorem 1.1 says that in this case the total Chern class of $M$ agrees with those of $\mathbb{C}P^n$. Recall the classical Petrie’s conjecture: if a homotopy complex projective space admits a circle action, then its Pontryagin classes agree with those of a complex projective space [12]. Hence Theorem 1.1 says Petrie’s conjecture holds in the particular case of a Hamiltonian circle action with a minimal number of isolated fixed points.

As we have mentioned, a compact Hamiltonian $S^1$-manifold $(M, \omega)$ of dimension $2n$ with $n + 1$ isolated fixed points has minimal even Betti numbers, i.e., $b_{2i}(M) = 1$ for all $0 \leq 2i \leq 2n$. Recent research on compact Hamiltonian $S^1$-manifolds with minimal even Betti numbers include [13, 10, 8, 7, 2, 6]. These works mainly dealt with the cases when the manifold is not more than 8 dimensional, or the number of connected components of the fixed point set is not too large. For a compact Hamiltonian $S^1$-manifold $(M, \omega)$ with minimal even Betti numbers, in [8], we raised the following general questions: Is $b_{2i+1}(M) = 0$ for all $i$? Is $H^*(M; \mathbb{Z})$ torsion free? Is the total Chern class of $M$ determined by its integral cohomology ring? Do the circle representations on the normal bundles of the fixed point components agree with those of some “standard” circle action? Our Theorems 1.1 and 1.2 gave affirmative answers to these questions for the cases when the fixed points are isolated and when $c_1(M) = (n + 1)x$ or $c_1(M) = nx$, where $x \in H^2(M; \mathbb{Z})$ is a generator.

Now let us give the standard examples of $\mathbb{C}P^n$ and $\widetilde{G}_2(\mathbb{R}^{n+2})$.

**Example 1.4.** Consider the $S^1$ action on $\mathbb{C}P^n$ given by
\[ \lambda \cdot [z_0, z_1, \cdots, z_n] = [\lambda^{b_0} z_0, \lambda^{b_1} z_1, \cdots, \lambda^{b_n} z_n], \]
where the $b_i$’s are mutually distinct integers. This action has $n + 1$ isolated fixed points, $P_i = [0, \cdots, 0, z_i, 0, \cdots, 0]$, where $i = 0, 1, \cdots, n$, and the set of weights of the action at $P_i$ is
\[ \{w_{ij}\} = \{b_j - b_i\}_{j \neq i}. \]
Moreover, the action is Hamiltonian, and the moment map values of the fixed points are respectively $b_0, b_1, \cdots, b_n$ (we may assume that we chose the order of the $b_i$’s such that $b_0 < b_1 < \cdots < b_n$).
Example 1.5. Consider the $S^1$ action on $\tilde{G}_2(\mathbb{R}^{n+2})$, where $n \geq 3$ is odd, induced by the $S^1$ action on $\mathbb{R}^{n+2} = \mathbb{R} \times \mathbb{C}^{n+1}$ given by

$$\lambda \cdot \left(t, z_0, \cdots, z_{n-1}\right) = \left(t, \lambda^{b_0} z_0, \cdots, \lambda^{b_{n-1} - \frac{1}{2}} z_{n-1}\right),$$

where the $b_i$’s, with $i \in J = \{0, \cdots, \frac{n-1}{2}\}$, are mutually distinct non-zero integers. This action has $n + 1$ isolated fixed points $P_0, \cdots, P_{\frac{n+1}{2}}, P_{\frac{n+3}{2}}, \cdots, P_n$, where for each $i \in J$, $P_i$ and $P_{n-i}$ are given by the plane $(0, \cdots, 0, z_i, 0, \cdots, 0)$ respectively with two different orientations. The weights of the action at $P_i$ and at $P_{n-i}$ are respectively

$$\{w_{ij}\} = \{b_j + b_i, -b_j + b_i\}_{J \ni j \neq i} \cup b_i,$$

and

$$\{w_{n-i,j}\} = \{b_j - b_i, -b_j - b_i\}_{J \ni j \neq i} \cup (-b_i).$$

Moreover, the action is Hamiltonian, and the moment map values of the fixed points are respectively $-b_0, \cdots, -b_{\frac{n-1}{2}}, b_{\frac{n+1}{2}}, \cdots, b_n$.

A similar theorem to Theorem 1.2 holds for $n$ even. In this case, there are $n + 2$ fixed points. We will treat this case in a different work.

Finally we give an outline of proof of the theorems. For Theorems 1.1 and 1.2, clearly (3) $\implies$ (1); for (4) $\implies$ (2) and (3), we can directly use Tolman’s work; for (1) $\implies$ (4), we will show that our Hamiltonian $S^1$-manifolds satisfy the conditions of Hattori’s theorems, which allows us to use Hattori’s work to conclude; our new ideas primarily devote to the proof of (2) $\implies$ (4) and (1), for which, the gradient flow of the moment map and Morse theory play key roles. To prove Theorem 1.3 we use Theorems 1.1 1.2 and a result we proved in [6], Proposition 7.1.

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2. Preliminaries

In this section, we introduce the basic notions we will use, for compact Hamiltonian $S^1$-manifolds and compact Hamiltonian $S^1$-manifolds with a minimal number of isolated fixed points, we state and prove important facts which we will use for the proof of our main theorems.

We start with introducing the basic notions. Let us first introduce equivariant cohomology. Let $M$ be a smooth $S^1$-manifold. The equivariant cohomology of $M$ in a coefficient ring $R$ is $H^*_S(M; R) = H^*(S^\infty \times S^1, M; R)$, where $S^1$ acts on $S^\infty$ freely. If $p$ is a point, then $H^*_S(p; R) = H^*(\mathbb{C}P^\infty; R) = R[t]$, where $t \in H^2(\mathbb{C}P^\infty; \mathbb{Z})$ is a generator. If $S^1$ acts on $M$ trivially, i.e., it fixes $M$, then $H^*_S(M; R) = H^*(M; R) \otimes R[t] = H^*(M; R)[t]$. The projection map $\pi: S^\infty \times S^1 M \to \mathbb{C}P^\infty$ induces a pull back map

$$\pi^*: H^*(\mathbb{C}P^\infty) \to H^*_S(M),$$

so $H^*_S(M)$ is a $H^*(\mathbb{C}P^\infty)$-module.
Let \((M, \omega)\) be a compact symplectic manifold. There exists an almost complex structure \(J: TM \to TM\) which is compatible with \(\omega\), i.e., \(\omega(J(\cdot), \cdot)\) is a Riemannian metric. The set of compatible almost complex structures on \((M, \omega)\) is contractible, hence there is a well defined total Chern class
\[
c(M) = 1 + c_1(M) + \cdots + c_n(M) \in H^*(M; \mathbb{Z}),
\]
where \(c_i(M) \in H^{2i}(M; \mathbb{Z})\) is the \(i\)-th Chern class of \(M\). Similarly, if \((M, \omega)\) is a compact symplectic \(S^1\)-manifold (the action preserves the symplectic form \(\omega\)), then at each connected component \(F\) of the fixed point set, there is a well defined set of nonzero integers, called the (nonzero) weights of the action; moreover, the normal bundle of \(M\) at \(F\) naturally splits into subbundles, one corresponding to each weight. Furthermore, if \((M, \omega)\) is a compact Hamiltonian \(S^1\)-manifold with moment map \(\phi: M \to \mathbb{R}\), the map \(\phi\) is a perfect Morse-Bott function, and its critical set coincides with the fixed point set of the action. At each connected component \(F\) of the fixed point set, the negative normal bundle of \(F\) is the subbundle of the normal bundle of \(F\) with negative weights; if \(\lambda_F\) is the number of negative weights (counted with multiplicities) at \(F\), then the Morse index at \(F\) (the dimension of the negative normal bundle of \(F\)) is \(2\lambda_F\).

Let \((M, \omega)\) be a compact \(2n\)-dimensional Hamiltonian \(S^1\)-manifold with moment map \(\phi: M \to \mathbb{R}\). Assume the fixed points are isolated. Let \(P\) be a fixed point, and let \(\{w_1^-, \cdots, w_k^-, w_{k+1}^+, \cdots, w_n^+\}\) be the set of weights at \(P\), where the first \(k\) number of them are negative, and the rest are positive. Then for the Morse function \(\phi\), the Morse index of \(P\) is \(2k\). We denote the equivariant total Chern class of \(M\) as
\[
c^{S^1}(M) = 1 + c_1^{S^1}(M) + \cdots + c_n^{S^1}(M) \in H^{*}_{S^1}(M; \mathbb{Z}),
\]
where \(c_i^{S^1}(M) \in H^{2i}_{S^1}(M; \mathbb{Z})\) is the \(i\)-th equivariant Chern class of \(M\). The restriction of \(c^{S^1}(M)\) to \(P\) is
\[
c^{S^1}(M)|_P = 1 + \sum_{i=1}^{n} \sigma_i(w_1^-, \cdots, w_k^-, w_{k+1}^+, \cdots, w_n^+) t^i,
\]
where \(\sigma_i(w_1^-, \cdots, w_k^-, w_{k+1}^+, \cdots, w_n^+) = c_i^{S^1}(M)|_P\) is the \(i\)-th symmetric polynomial in the weights.

Next, for a compact Hamiltonian \(S^1\)-manifold \(M\), the following Lemmas \[2.1\] and \[2.2\] will be very useful for the proof of our theorems.

**Lemma 2.1.** \[6, Lemma 2.3\] Let the circle act on a connected compact symplectic manifold \((M, \omega)\) with moment map \(\phi: M \to \mathbb{R}\). Assume \(H^2(M; \mathbb{R}) = \mathbb{R}\). Then
\[
c_1(M) = \frac{\Gamma_F - \Gamma_{F'}}{\phi(F') - \phi(F)}[\omega],
\]
where \(F\) and \(F'\) are any two fixed components such that \(\phi(F') \neq \phi(F)\), and \(\Gamma_F\) and \(\Gamma_{F'}\) are respectively the sums of the weights at \(F\) and \(F'\).
Note that if we take $F$ to be the minimum and $F'$ the maximum of $\phi$, we have that $\Gamma_F > 0$, $\Gamma_{F'} < 0$, and $\phi(F') - \phi(F) > 0$. So $c_1(M) = C[\omega]$ for some constant $C > 0$.

For a compact Hamiltonian $S^1$-manifold, when the class of the symplectic form is integral, up to a translation, the moment map values of the fixed components are integers. When there exists a finite stabilizer group $\mathbb{Z}_k \subset S^1$, where $k > 1$, the set of points, $M_{Z_k} \subseteq M$, which is fixed by $\mathbb{Z}_k$ but not fixed by $S^1$, is a symplectic submanifold, called an isotropy submanifold. If an isotropy submanifold is a sphere, it is called an isotropy sphere.

**Lemma 2.2.** Let the circle act on a compact symplectic manifold $(M, \omega)$ with moment map $\phi: M \to \mathbb{R}$. Assume $[\omega]$ is integral. Then for any two fixed components $F$ and $F'$, $\phi(F) - \phi(F') \in \mathbb{Z}$. If $\mathbb{Z}_k$ fixes any point on $M$, then for any two fixed components $F$ and $F'$ on the same connected component of the isotropy submanifold $M_{Z_k}$, we have $k| (\phi(F') - \phi(F))$.

**Proof.** Since $M$ is compact, the action has at least two fixed components. Since $[\omega]$ is integral, there is an equivariant integral class $\tilde{\omega}$ such that $\int_{F'} = [\tilde{\omega}|_{F'}] + t (\phi(F_0) - \phi(F'))$, where $F_0$ and $F'$ are fixed components [8]. So $\phi(F') - \phi(F) \in \mathbb{Z}$ for any two fixed components $F$ and $F'$.

The isotropy submanifold $M_{Z_k}$ is compact and contains at least two fixed components. Consider the $S^1/\mathbb{Z}_k \approx S^1$ action on $M_{Z_k}$, whose moment map is $\phi' = \phi/k$. Since $[\omega]_{M_{Z_k}}$ is integral, by the first paragraph, for any two fixed components $F$ and $F'$ on the same connected component of $M_{Z_k}$, we have $\phi'(F') - \phi'(F) \in \mathbb{Z}$, i.e., $\frac{\phi(F')}{k} - \frac{\phi(F)}{k} \in \mathbb{Z}$. \hfill $\Box$

Finally, we consider the case of our concern — compact Hamiltonian $S^1$-manifolds with a minimal number of isolated fixed points. The facts in the following lemma will be important for our arguments in the next sections, most of these facts were mentioned in [13].

**Lemma 2.3.** Let the circle act on a compact $2n$-dimensional symplectic manifold $(M, \omega)$ with moment map $\phi: M \to \mathbb{R}$. Assume the fixed point set consists of $n + 1$ isolated points, $P_0, P_1, \ldots, P_n$. Then they respectively have Morse indices $0$, $2$, $\ldots$, and $2n$, and $\phi(P_0) < \phi(P_1) < \cdots < \phi(P_n)$. Moreover, $H^1(M; \mathbb{Z}) = H^1(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}$ as groups for all $i$, and the negative disk bundle of the fixed point $P_i$ contributes to $H^{2i}(M; \mathbb{Z})$.

**Proof.** The manifold $M$ is compact and symplectic implies that $\dim H^{2i}(M) \geq 1$ for all $0 \leq 2i \leq \dim(M)$. The moment map is a perfect Morse function, whose critical points $P_0$, $P_1$, $\ldots$, $P_n$ all have even indices. These indices must be respectively $0$, $2$, $\ldots$, $2n$ to make $\dim H^{2i}(M) = 1$ for all $0 \leq 2i \leq \dim(M)$. The fact $\phi(P_0) < \phi(P_1) < \cdots < \phi(P_n)$ follows from the following Lemma 2.4.

Since the fixed points are isolated and have no torsion cohomology, the moment map $\phi$ is a perfect Morse function for cohomology groups in integer
coefficients (see for example [8] Sect. 2), i.e.,

\[ H^j(M; \mathbb{Z}) = \sum_{i=0}^{n} H^{j-2\lambda_i}(P_i; \mathbb{Z}). \]

Then using the above indices of the fixed points, we get that

\[ H^i(M; \mathbb{Z}) = \mathbb{Z} = H^i(\mathbb{C}P^n; \mathbb{Z}) \]

as groups. Moreover, by Morse theory, \( M \) has a CW-structure consisting of a unique cell in each even dimension — the negative disk bundle of \( P_i \) for all \( i \).

\[ \Box \]

**Lemma 2.4.** [13 Lemma 3.1] Let the circle act on a compact symplectic manifold \((M, \omega)\) with moment map \( \phi : M \to \mathbb{R} \). Given any fixed component \( F \), we have

\[ 2\lambda_F \leq \sum_{\phi(F') < \phi(F)} (\dim(F') + 2), \]

where the sum is over all fixed components \( F' \) such that \( \phi(F') < \phi(F) \).

3. **Proof of Theorems 1.1 and 1.2 for the implications (4) \( \implies \) (2) and (3)**

Tolman’s work will directly give us the implications (4) \( \implies \) (2) and (3) for Theorems 1.1 and 1.2.

First, let us cite the following result by Tolman on the generators of \( H^*(M; \mathbb{Z}) \). We will also use this in Sections 5 and 6.

**Proposition 3.1.** [13 Corollaries 3.14 and 3.19] Let the circle act on a compact \( 2n \)-dimensional symplectic manifold \((M, \omega)\) with moment map \( \phi : M \to \mathbb{R} \). Assume that there exists a unique fixed point \( P_i \) of index \( 2i \) for all \( 0 \leq i \leq n \). Then \( H^{2i}(M; \mathbb{Z}) \) is freely generated by \( \alpha_i \), \( i = 0, \ldots, n \), where

\[ \alpha_i = \Lambda_i^- c_1(M)^i \prod_{j=0}^{i-1} (\Gamma_i - \Gamma_j) \in H^{2i}(M; \mathbb{Z}). \]

Here, \( \Lambda_i^- \) is the product of the negative weights at \( P_i \), and \( \Gamma_j \) is the sum of the weights at \( P_j \).

In [13 Corollary 3.19], Tolman expressed the generators \( \alpha_i \in H^{2i}(M; \mathbb{Z}) \) with \( i \geq 2 \) in terms of the weights at the fixed points and \( (\alpha_1)^i \), she also expressed the Chern classes of \( M \) in terms of the weights at the fixed points and the \( \alpha_i \)'s. More precisely,

\[ \alpha_i = \frac{\Lambda_i^- (\Gamma_1 - \Gamma_0)^i}{\prod_{j=0}^{i-1} (\Gamma_i - \Gamma_j)} (\alpha_1)^i, \quad \text{and} \]

\[ c_i(M) = \frac{1}{\Lambda_i^+} \prod_{j=0}^{i-1} (\Gamma_i - \Gamma_j) \left( \sum_{k=0}^{i} \frac{c^S_{i-k}(M)|_{P_k} \prod_{j=i+1}^{n} (\Gamma_k - \Gamma_j)}{\prod_{j=i+1}^{n} (\Gamma_k - \Gamma_j)} \right) \alpha_i \]

\[ = \prod_{j=i+1}^{n} (\Gamma_i - \Gamma_j) \left( \sum_{k=0}^{i} \frac{c^S_{i-k}(M)|_{P_k} \prod_{j=i+1}^{n} (\Gamma_k - \Gamma_j)}{\prod_{j=i+1}^{n} (\Gamma_k - \Gamma_j)} \right) \alpha_i, \]
where \( -\Lambda_i \) is the product of the negative weights at \( P_i \), \( \Lambda_i^+ \) is the product of the positive weights at \( P_i \), and \( \Gamma_i \) is the sum of the weights at \( P_i \). For Theorems 1.1 and 1.2 by Lemma 2.3, there exists a unique fixed point \( P_i \) of index 2 for each \( 0 \leq i \leq n \). Hence the implications (4) \( \Rightarrow \) (2) and (3) of the two theorems immediately follow from this.

4. Proof of Theorems 1.1 and 1.2 for the implications (1) \( \Rightarrow \) (4)

The idea of the proof is that we change our set up into the set up of almost complex \( S^1 \)-manifolds admitting quasi-ample complex line bundles. Then Hattori’s work \([3]\) will give the implications (1) \( \Rightarrow \) (4) of Theorems 1.1 and 1.2.

Let \( M \) be a smooth manifold, it is called an almost complex \( S^1 \)-manifold if it is almost complex and admits a smooth \( S^1 \) action preserving the almost complex structure.

Let \( M \) be an almost complex \( S^1 \)-manifold, with fixed point set nonempty and consisting of isolated points. A complex line bundle \( L \) over \( M \) with a compatible \( S^1 \) action is called quasi-ample if

- the restrictions of \( L \) at the fixed points are mutually distinct \( S^1 \)-modules, and
- \( c_1(L)^n \neq 0 \), where \( 2n = \dim(M) \).

Now let \((M, \omega)\) be a compact Hamiltonian \( S^1 \)-manifold of dimension \( 2n \) with moment map \( \phi \). Choose an \( S^1 \)-invariant almost complex structure compatible with \( \omega \). Then \( M \) is an almost complex \( S^1 \)-manifold. If \([\omega]\) is an integral class, then it has an integral equivariant extension \([\omega - \phi \xi] \in H^2_{S^1}(M; \mathbb{Z}) \) ([8 Lemma 2.3]). Then \( M \) admits an \( S^1 \)-equivariant complex line bundle \( L \) whose equivariant first Chern class is \([\omega - \phi \xi] \) (see for example [4] Appendix C). Assume that the fixed point set of the Hamiltonian \( S^1 \)-manifold \((M, \omega)\) consists of \( n + 1 \) isolated points, \( P_0, P_1, \ldots, P_n \). By Lemma 2.3 \( H^2(M; \mathbb{R}) = \mathbb{R} \), so by rescaling \( \omega \), we may assume that \([\omega]\) is a primitive integral class and hence we have the equivariant complex line bundle over \( M \) as above. Note that the ordinary first Chern class of \( L \) is \( c_1(L) = [\omega] \), and \( S^1 \) acts on \( L|_{P_i} \) with weight \(-\phi(P_i)\). By Lemma 2.3 \( \phi(P_i) \neq \phi(P_j) \) if \( i \neq j \). Moreover, \( c_1(L)^n = [\omega]^n \neq 0 \) since \( M \) is symplectic. Consequently, the Hamiltonian \( S^1 \)-manifold \((M, \omega)\) is an almost complex \( S^1 \)-manifold admitting a quasi-ample complex line bundle. Together with Lemmas 2.3 and 2.4 we obtain the following proposition.

**Proposition 4.1.** Let \((M, \omega)\) be a compact Hamiltonian \( S^1 \)-manifold of dimension \( 2n \) with moment map \( \phi : M \to \mathbb{R} \). Assume that the fixed point set consists of \( n + 1 \) isolated points, \( P_0, P_1, \ldots, P_n \), and \( \omega \) is scaled to represent a primitive integral class. Then \( M \) is an almost complex \( S^1 \)-manifold with Euler characteristic \( \chi(M) = n + 1 \), and \( M \) admits a quasi-ample complex line bundle whose equivariant first Chern class is \([\omega - \phi \xi] \) (ordinary first Chern class is \([\omega] \)) and \( c_1(M) = C[\omega] \) for some positive integer \( C \).
Remark 4.2. Suppose we are in the case of Proposition 4.1. Since $H^2(M; \mathbb{Z}) = \mathbb{Z}$ and $[\omega]$ is a generator, $t$ and $[\omega - \phi t]$ generate $H^2_{S^1}(M; \mathbb{Z})$. If $c_1(M) = C[\omega]$, then we may write the equivariant first Chern class of $M$ as

$$c_1^{S^1}(M) = C[\omega - \phi t] + dt,$$  

for some $d \in \mathbb{Z}$.

If $\{w_k\}_{1 \leq k \leq n}$ is the set of weights of the $S^1$ action on $M$ at $P_i$, then restricting (4.3) to any fixed point $P_i$, we get

$$\sum_{k=1}^{n} w_{ik} = C(-\phi(P_i)) + d.$$  

As we have mentioned, $-\phi(P_i)$ is the weight of the $S^1$ action on the restriction of the quasi-ample complex line bundle to the fixed point $P_i$. Hattori called a condition like (4.3) as condition D, and stated his main theorems using this condition. We note that under our assumptions (as in Proposition 4.1), this condition is a natural consequence of the condition $c_1(M) = C[\omega]$. So we just use the simpler condition $c_1(M) = C[\omega]$ here.

Proof of Theorems 1.1 and 1.2 for the implications (1) $\implies$ (4).

By Lemma 2.3 by rescaling, we may assume that $[\omega]$ is primitive integral.

For Theorem 1.1 assume we have (1), i.e., $c_1(M) = (n+1)x$, where $x$ is a degree 2 generator. By Lemma 2.1 we may take $x = [\omega]$. Together with Proposition 4.1 $M$ is an almost complex $S^1$-manifold with Euler characteristic $n+1$, and $M$ admits a quasi-ample complex line bundle $L$ with $c_1(L) = [\omega]$ and $c_1(M) = (n+1)[\omega]$. By Hattori’s work, [3, Proposition 3.18 and Theorem 5.7] (or [3, Corollaries 3.15 and 5.8]), we have (4).

For Theorem 1.2 assume we have (1), i.e., $c_1(M) = nx$, where $x$ is a degree 2 generator. Similar to the above, using Proposition 4.1 we have that $M$ is an almost complex $S^1$-manifold with Euler characteristic $n+1$, and $M$ admits a quasi-ample complex line bundle $L$ with $c_1(M) = nc_1(L)$. So the manifold $M$ satisfies the conditions of [3, Corollaries 3.15 and 6.2], which allows us to conclude (4).  

5. Proof of Theorem 1.1 for the implication (2) $\implies$ (4)

Our main result of this section is Proposition 5.10 which corresponds to (2) $\implies$ (4) of Theorem 1.1.

Let us first introduce the following notation, which we will use in this and the next sections.

Definition 5.1. Let the circle act on a compact $2n$-dimensional symplectic manifold $(M, \omega)$ with moment map $\phi: M \to \mathbb{R}$. Assume the fixed point set consists of $n+1$ isolated points, $P_0, P_1, \ldots, P_n$. By Lemma 2.3 we may take $a_0, a_1, \ldots, a_n \in \mathbb{R}$ such that

$$\phi(P_0) < a_0 < \phi(P_1) < a_1 < \cdots < \phi(P_{n-1}) < a_{n-1} < \phi(P_n) < a_n.$$
Define
\[ M_i = \{ m \in M \mid \phi(m) < a_i \}, \quad \text{and} \quad M'_i = \{ m \in M \mid \phi(m) > a_i \}. \]

In this and the next sections, we may implicitly use the order of the moment map values of the fixed points as above without referring to Lemma 2.3.

The proof of Proposition 5.10 goes in two steps, first, for each fixed point, we express the product of the negative weights (and the product of the positive weights) as a product in terms of the moment map values of certain fixed points (Lemma 5.2), then, we show that at each fixed point, the weights are exactly the factors on the right hand side of the expression.

Lemma 5.2. Let the circle act on a compact 2n-dimensional symplectic manifold \((M, \omega)\) with moment map \(\phi: M \to \mathbb{R}\). Assume the fixed point set consists of \(n + 1\) isolated points, \(P_0, P_1, \ldots, P_n\). If for some \(i\), the manifold \(M_i\) has the same integral cohomology ring as \(\mathbb{CP}^d\), i.e., \(H^*(M_i; \mathbb{Z}) = \mathbb{Z}[x]/x^{i+1}\), where \(x \in H^2(M_i; \mathbb{Z})\) is a generator, then for each \(0 \leq k < i\), \(M_k\) has the same integral cohomology ring as \(\mathbb{CP}^k\). If moreover \([\omega]\) is a primitive integral class, and \(x = [\omega]|_{M_i}\), then at each fixed point \(P_k\) with \(0 \leq k \leq i\),
\[ \Lambda^-_k = \prod_{j=0}^{k-1} (\phi(P_j) - \phi(P_k)), \]
where \(\Lambda^-_k\) is the product of the negative weights at \(P_k\).

Proof. By Lemma 2.3, the fixed points \(P_0, \ldots, P_i\) respectively have Morse indices 0, \ldots, and \(2n\), and the negative disk bundles of \(P_0, \ldots, P_i\) respectively contribute to \(H^0(M; \mathbb{Z}), \ldots, \) and \(H^{2n}(M; \mathbb{Z})\). If the manifold \(M_i\), which contains only \(P_0, \ldots, P_i\), has the same integral cohomology ring as \(\mathbb{CP}^d\), then the manifold \(M_k\), where \(0 \leq k < i\), which contains only the fixed points \(P_0, \ldots, P_k\), has the same integral cohomology ring as \(\mathbb{CP}^k\).

Now assume \([\omega]\) is a primitive integral class, then by Lemma 2.2, \(\phi(P_i) - \phi(P_j) \in \mathbb{Z}\) for any \(i, j\). Proposition 3.1 gives the generators, 1, \(\alpha_1, \ldots, \alpha_i\), of \(H^0(M; \mathbb{Z}), H^2(M; \mathbb{Z}), \ldots, \) and \(H^{2n}(M; \mathbb{Z})\). By Lemma 2.3, their restrictions to \(M_i\), still denoted 1, \(\alpha_1, \ldots, \alpha_i\), are the generators of \(H^0(M_i; \mathbb{Z}), H^2(M_i; \mathbb{Z}), \ldots, \) and \(H^{2n}(M_i; \mathbb{Z})\). If we still denote the restriction of \([\omega]\) to \(M_i\) as \([\omega]\) and let \([\omega]\) = \(\alpha_1\), then \(\alpha_2 = [\omega]^2, \ldots, \) and \(\alpha_i = [\omega]^i\). Together with Proposition 3.1 and Lemma 2.1, we have
\[ \alpha_1 = \Lambda_1 \frac{c_1(M)}{\Gamma_1 - \Gamma_0} = \frac{\Lambda^-_1}{\phi(P_0) - \phi(P_1)} [\omega] = [\omega]. \]
Hence
\[ \Lambda^-_1 = \phi(P_0) - \phi(P_1). \]

Similarly, for each \(k\) with \(0 \leq k \leq i\), by Proposition 3.1 and Lemma 2.1
\[ \alpha_k = \Lambda^-_k \frac{c_1(M)^k}{\prod_{j=0}^{k-1} (\Gamma_k - \Gamma_j)}, \quad \text{and} \quad c_1(M)^k = \prod_{j=0}^{k-1} \frac{\Gamma_k - \Gamma_j}{\phi(P_j) - \phi(P_k)} [\omega]^k. \]
So
\[ \alpha_k = \Lambda_k^- \prod_{j=0}^{k-1} (\phi(P_j) - \phi(P_k)) [\omega]^k = [\omega]^k. \]

Hence
\[ \Lambda_k^- = \prod_{j=0}^{k-1} (\phi(P_j) - \phi(P_k)). \]

Lemma 5.4. Let the circle act on a compact $2n$-dimensional symplectic manifold $(M, \omega)$ with moment map $\phi: M \to \mathbb{R}$. Assume the fixed point set consists of $n + 1$ isolated points, $P_0, P_1, \ldots, P_n$. Then for each $0 \leq i \leq n$, the manifold $M_i$ is simply connected.

Proof. First, $M_0$ is a $2n$-disk bundle over $P_0$, clearly it is simply connected.

Assume that for some $i \geq 1$, $M_{i-1}$ is simply connected. By Morse theory, $M_i$ is obtained by gluing the negative disk bundle of $P_i$, a $2i$-disk (since $P_i$ has index $2i$ by Lemma 2.3, along its boundary $S^{2i-1}$ to $M_{i-1}$. Since $M_{i-1}$ and the $2i$-disk are both simply connected, by the Van-Kampen theorem, $M_i$ is simply connected. 

□

The following corollary follows from Lemmas 5.2, 5.4 and the Whitehead theorem.

Corollary 5.5. Let the circle act on a compact $2n$-dimensional symplectic manifold $(M, \omega)$ with moment map $\phi: M \to \mathbb{R}$. Assume the fixed point set consists of $n + 1$ isolated points, $P_0, P_1, \ldots, P_n$. If for some $i$, the manifold $M_i$ has the same integral cohomology ring as $\mathbb{C}P^i$, then for each $k$ with $0 \leq k \leq i$, $M_k$ has the homotopy type of $\mathbb{C}P^k$.

If we choose an $S^1$-invariant compatible almost complex structure $J$ on $M$, we have an $S^1$-invariant Riemannian metric on $M$. If $X_M$ is the vector field generated by the circle action, then the gradient vector field of the moment map $\phi$ is
\[ \text{grad}(\phi) = JX_M. \]

The $S^1$ action and the gradient flow of the moment map together give a $\mathbb{C}^*$-action. The closure of a nontrivial $\mathbb{C}^*$-orbit contains two fixed points, and it is a topological sphere, called a gradient sphere. A free gradient sphere is one whose generic point has stabilizer $1 \subset S^1$, and a non-free gradient sphere is one whose generic point has stabilizer $\mathbb{Z}_k \subset S^1$ for some $k > 1$. A non-free gradient sphere is an isotropy sphere. If a gradient sphere originates from the negative (or positive) normal bundle of a fixed point $P$, following the flow of $-\text{grad}(\phi)$ (or the flow of $+\text{grad}(\phi)$), ends at a fixed point $Q$, we say that there is a gradient sphere from $P$ to $Q$.

From now on, when we speak about the flow of $\pm\text{grad}(\phi)$, we are implicitly assuming that we have chosen an $S^1$-invariant compatible almost complex structure and hence an invariant metric on $M$. 


Lemma 5.6. Let the circle act on a compact 2n-dimensional symplectic manifold \((M, \omega)\) with moment map \(\phi: M \to \mathbb{R}\). Assume the fixed point set consists of \(n + 1\) isolated points, \(P_0, P_1, \ldots, P_n\). Assume \([\omega]\) is an integral class. Let \(\{w_{ij}\}_{0 \leq j \leq i - 1}\) be the set of negative weights at \(P_i\). If the flow of \(-\text{grad}(\phi)\) from the negative disk bundle of \(P_i\) directly hits all the fixed points \(P_0, P_1, \ldots, P_{i-1}\), then there is a gradient sphere from \(P_i\) to \(P_j\) and up to reordering the weights, \(w_{ij}^- (\phi(P_j) - \phi(P_i))\) for each \(0 \leq j \leq i - 1\).

**Proof.** Since \([\omega]\) is integral, by Lemma 2.2, \(\phi(P_i) - \phi(P_j)\) is an integer for any \(i, j\).

By assumption, there exist complex coordinates \((z_0, z_1, \ldots, z_{i-1})\) in the negative disk bundle \(D^{2i}\) of \(P_i\) such that \(S^1\) acts on them as follows:

\[
\lambda \cdot (z_0, \ldots, z_{i-1}) = (\lambda^{w_{0i}z_0, \ldots, \lambda^{w_{i-1}i-1}z_{i-1}}).
\]

Suppose the flow of \(-\text{grad}(\phi)\) from \(D^{2i}\) directly hits all the fixed points \(P_0, P_1, \ldots, P_{i-1}\). Since the flow is \(S^1\)-equivariant and continuous, up to reordering the coordinates and the weights, the \(i\) number of closed \(S^1\)-orbits \(S^1 \cdot (0, \ldots, 0, z_j, 0, \ldots, 0)\) in \(D^{2i}\) must respectively flow to the \(i\) number of distinct fixed points \(P_0, P_1, \ldots, P_{i-1}\) below \(P_i\). So there is a \(\mathbb{Z}_{[w_{ij}^-]}\) gradient sphere from \(P_i\) to \(P_j\) for each \(0 \leq j \leq i - 1\). If \(w_{ij}^- = -1\), then naturally \(w_{ij}^- (\phi(P_j) - \phi(P_i))\). If \(w_{ij}^- \neq -1\), then the \(\mathbb{Z}_{[w_{ij}^-]}\) gradient sphere from \(P_i\) to \(P_j\) is contained in a \(\mathbb{Z}_{[w_{ij}^-]}\) isotropy submanifold, by Lemma 2.2, \(w_{ij}^- (\phi(P_j) - \phi(P_i))\). \(\square\)

In the following lemma, we obtain the single negative weights at each fixed point.

**Lemma 5.7.** Let the circle act on a compact 2n-dimensional symplectic manifold \((M, \omega)\) with moment map \(\phi: M \to \mathbb{R}\). Assume the fixed point set consists of \(n + 1\) isolated points, \(P_0, P_1, \ldots, P_n\). If for some \(i\), the manifold \(M_i\) has the same integral cohomology ring as \(\mathbb{C}P^2\), and \([\omega|_{M_i}] \in H^2(M_i; \mathbb{Z})\) is a generator, then for each \(k\) with \(0 \leq k \leq i\), there is a gradient sphere from \(P_k\) to \(P_j\) for each \(0 \leq j \leq k - 1\), and the set of negative weights at the fixed point \(P_k\) is

\[
\{w_{kj}^-\} = \{\phi(P_j) - \phi(P_k)\}_{j < k}.
\]

**Proof.** By Lemma 5.2, \((5.8)\) holds for \(P_0\) and \(P_1\). We may assume that \(i \geq 2\).

First note that, for each \(i \geq 1\), since the fixed points outside \(M_i\) have Morse indices bigger than 2, by Morse theory, the restriction map \(H^2(M_i; \mathbb{Z}) \to H^2(M_i; \mathbb{Z})\) is an isomorphism. Hence \([\omega]\) is primitive integral if and only if \([\omega|_{M_i}]\) is primitive integral.

By Corollary 5.5, for each \(k\) with \(0 \leq k \leq i\), \(M_k\) has the homotopy type of \(\mathbb{C}P^k\). Since the flow of \(-\text{grad}(\phi)\) induces a deformation retract of \(M_k\) to some space, this space must be (a topological) \(\mathbb{C}P^k\).
By Morse theory, \(M_k\) is obtained by (equivariantly) gluing the negative disk bundle \(D^{2k}\) of \(P_k\) along its boundary \(S^{2k-1}\) to \(M_{k-1}\). Since the flow of \(-\text{grad}(\phi)\) induces an equivariant deformation retract of \(M_{k-1}\) to \(\mathbb{C}P^{k-1}\), the flow induces an equivariant gluing map from \(S^{2k-1}\) to \(\mathbb{C}P^{k-1}\). Let us look at this process inductively: \(M_0\) is an \(S^1\)-invariant disk bundle over \(P_0\), which equivariantly deformation retracts to \(P_0\). The space \(M_1\) is obtained by equivariantly gluing a 2-disk \(D^2\) over \(P_1\) along its boundary \(S^1\) to \(M_0\), which is taken equivariantly to \(P_0\) by the flow, so \(M_1\) has the homotopy type of \(\mathbb{C}P^1 \approx S^2\), which has \(P_0\) and \(P_1\) as south and north poles. The space \(M_2\) is obtained by equivariantly gluing a 4-disk over \(P_2\) along its boundary \(S^3\) to \(\mathbb{C}P^1\), and etc. For each \(k\) with \(2 \leq k \leq i\), the fact that \(M_k\) has the homotopy type of \(\mathbb{C}P^{k}\) implies that the flow of \(-\text{grad}(\phi)\) induces an equivariant surjective gluing from \(S^{2k-1} = \partial(D^{2k})\) to \(\mathbb{C}P^{k-1}\), where in \(\mathbb{C}P^{k-1}\), the closed \(S^1\)-orbit type consists of the fixed points \(P_0, \cdots, P_{k-1}\). Let \(w_{k0}, w_{k1}, \cdots, w_{k,k-1}\) be the negative weights of the \(S^1\) action at \(P_k\), by Lemma 5.6 there is a gradient sphere from \(P_k\) to \(P_{j}\) and \(w_{kj}^{-} \cdot (\phi(P_j) - \phi(P_k))\) for each \(0 \leq j \leq k - 1\). Now (5.3) in Lemma 5.2 gives the claim (5.8) for \(P_k\). □

If we use \(-\phi\) as a perfect Morse function, completely analogous to Lemmas 5.4, 5.2, and Corollary 5.5, we can prove the following claims.

**Remark 5.9.** Let the circle act on a compact \(2n\)-dimensional symplectic manifold \((M, \omega)\) with moment map \(\phi: M \to \mathbb{R}\). Assume the fixed point set consists of \(n + 1\) isolated points, \(P_0, P_1, \cdots, P_n\). Let \(c_0, c_1, \cdots, c_n\) be real numbers such that\[
\begin{align*}
c_0 < \phi(P_0) < c_1 < \phi(P_1) < \cdots < \phi(P_{n-1}) < c_n < \phi(P_n).
\end{align*}
\]
Let \(M''_i = \{m \in M \mid \phi(m) > c_i\}\). Then for each \(0 \leq i \leq n\), \(M''_i\) is simply connected.

If for some \(i\), \(M''_i\) has the same integral cohomology ring as \(\mathbb{C}P^{n-i}\), then for each \(k\) with \(i < k \leq n\), \(M''_k\) has the same integral cohomology ring as \(\mathbb{C}P^{n-k}\). If moreover \(|\omega|\) is primitive integral and \(|\omega|_{M''_i} \in H^2(M''_i; \mathbb{Z})\) is the generator, then at each fixed point \(P_k\) with \(i \leq k \leq n\), \(\Lambda_k^+ = \prod_{j=k+1}^n (\phi(P_j) - \phi(P_k))\), where \(\Lambda_k^+\) is the product of the positive weights at \(P_k\). Moreover, each \(M''_k\) has the homotopy type of \(\mathbb{C}P^{n-k}\).

Finally, we come to the conclusion of the weights at the fixed points.

**Proposition 5.10.** Let the circle act on a compact \(2n\)-dimensional symplectic manifold \((M, \omega)\) with moment map \(\phi: M \to \mathbb{R}\). Assume the fixed point set consists of \(n + 1\) isolated points, \(P_0, P_1, \cdots, P_n\). Assume \(M\) has
the same integral cohomology ring as \(\mathbb{CP}^n\), i.e., \(H^*(M;\mathbb{Z}) = \mathbb{Z}[x]/x^{n+1}\), where \(x = [\omega]\). Then the set of weights of the action at each fixed point \(P_i\) is
\[
\{w_{ij}\} = \{\phi(P_j) - \phi(P_i)\}_{j \neq i}.
\]
That is, the set of weights coincide with those of \(\mathbb{CP}^n\) with the standard circle action as in Example 1.4.

**Proof.** The set of negative weights at \(P_i\) follows from Lemma 5.7.

Similar to Lemma 5.7, using \(-\phi\) and Remark 5.9, we get that the set of positive weights at \(P_i\) is
\[
\{w_{ij}^+\} = \{\phi(P_j) - \phi(P_i)\}_{j > i}.
\]

\(\square\)

6. **Proof of Theorem 1.2 for the implication (2) \(\Rightarrow\) (1) or (4)**

Our main results of this section are Propositions 6.21 and 6.22, which correspond to (2) \(\Rightarrow\) (1) and (2) \(\Rightarrow\) (4) of Theorem 1.2.

First, we look at the integral cohomology rings of the manifolds \(M_i\)’s, and express the product of the negative weights at each fixed point in terms of the moment map values of the fixed points in \(M_i\).

**Lemma 6.1.** Let the circle act on a compact 2n-dimensional symplectic manifold \((M, \omega)\) with moment map \(\phi: M \to \mathbb{R}\). Assume the fixed point set consists of \(n + 1\) isolated points, \(P_0, P_1, \ldots, P_n\). Assume \(M\) has the same integral cohomology ring as \(\tilde{G}_2(\mathbb{R}^{n+2})\) with \(n \geq 3\) odd, i.e.,
\[
H^*(M;\mathbb{Z}) = \mathbb{Z}(x, y)/(x^{(n+1)/2}y^2),
\]
where \(\deg(x) = 2\), and \(\deg(y) = n + 1\). Then for each \(0 \leq i \leq \frac{n-1}{2}\), the integral cohomology groups of the manifold \(M_i\) are generated by 1, \(x, \ldots, x^i\) (i.e., \(M_i\) has the same integral cohomology ring as \(\mathbb{CP}^i\)), and for each \(\frac{n+1}{2} \leq i \leq n\), the integral cohomology groups of \(M_i\) are generated by 1, \(x, \ldots, x^\frac{n-1}{2}, \frac{1}{2}x^\frac{n+1}{2}, \ldots, \frac{1}{2}x^i\).

If moreover \([\omega]\) is primitive integral, and \(x = [\omega]\), then at each fixed point \(P_i\), the product of the negative weights is
\[
(6.2) \quad \Lambda_i^- = \begin{cases} 
\prod_{j=0}^{i-1} (\phi(P_j) - \phi(P_i)), & \text{when } 0 \leq i \leq \frac{n-1}{2}, \\
\frac{1}{2} \prod_{j=0}^{i-1} (\phi(P_j) - \phi(P_i)), & \text{when } \frac{n+1}{2} \leq i \leq n. 
\end{cases}
\]

**Proof.** Similar to the proof of Lemma 5.2, using \(\phi\) as a perfect Morse function, by Lemma 2.3, the fixed points \(P_0, P_1, \ldots, P_n\) respectively have indices 0, 2, \ldots, and \(2n\). Their negative disk bundles respectively contribute to \(H^0(M;\mathbb{Z})\), \(H^2(M;\mathbb{Z})\), \ldots, and \(H^{2n}(M;\mathbb{Z})\). By Proposition 6.1, the generators of these groups are \(1, \alpha_1, \ldots, \alpha_n\). The assumption on the cohomology ring of \(M\) implies that \(\alpha_1 = x, \ldots, \alpha_{\frac{n-1}{2}} = x^{\frac{n-1}{2}}, \alpha_{\frac{n+1}{2}} = \frac{1}{2}x^\frac{n+1}{2}, \ldots, \alpha_n = \frac{1}{2}x^n\). The manifold \(M_i\) only contains the fixed points \(P_0, P_1, \ldots, P_i\), and
Hamiltonian circle actions with isolated fixed points

\[ \cdots, \text{and } P_i. \] The claim on the generators of the cohomology groups of \( M_i \) follows from these facts.

Now assume \([\omega]\) is primitive integral and \( x = [\omega] \). Then

\[ \alpha_i = \begin{cases} [\omega]^i, & \text{when } 0 \leq i \leq \frac{n-1}{2}, \\ \frac{1}{2}[\omega]^i, & \text{when } \frac{n+1}{2} \leq i \leq n. \end{cases} \]

Similar to the proof of Lemma 5.2 using the expression of \( \alpha_i \) in Proposition 3.1, the expression of \( c_1 \) in Lemma 2.1, we can get \( \Lambda_i^- \) as claimed. □

By Lemmas 5.4, 6.1, and the Whitehead theorem, we have the following corollary.

**Corollary 6.3.** Under the assumptions of Lemma 6.1, for each \( 0 \leq i \leq \frac{n-1}{2} \), the manifold \( M_i \) has the homotopy type of \( \mathbb{C}P^i \).

Using \(-\phi\) as a perfect Morse function, we can similarly prove the following claims.

**Remark 6.4.** Under the assumptions of Lemma 6.1, we have a similar corresponding statement for the integral cohomology ring of each manifold \( M'_i \), and if \([\omega] = x\), then the product of the positive weights at each fixed point \( P_i \) is

\[ \Lambda_i^+ = \begin{cases} \frac{1}{2} \prod_{j=i+1}^{n} (\phi(P_j) - \phi(P_i)), & \text{when } 0 \leq i \leq \frac{n-1}{2}, \\ \prod_{j=i+1}^{n} (\phi(P_j) - \phi(P_i)), & \text{when } \frac{n+1}{2} \leq i \leq n. \end{cases} \]

Moreover, for each \( \frac{n-1}{2} \leq i \leq n-1 \), \( M'_i \) has the homotopy type of \( \mathbb{C}P^{n-1-i} \).

Using Lemmas 6.1 and 5.7, and using \(-\phi\) and Remark 6.4, we have the following corollary.

**Corollary 6.5.** Let the circle act on a compact \( 2n \)-dimensional symplectic manifold \( (M, \omega) \) with moment map \( \phi: M \to \mathbb{R} \). Assume the fixed point set consists of \( n+1 \) isolated points, \( P_0, P_1, \cdots, P_n \). Assume \( M \) has the same integral cohomology ring as \( G_2(\mathbb{R}^{n+2}) \) with \( n \geq 3 \) odd and \([\omega] \in H^2(M; \mathbb{Z})\) being the degree 2 generator. Then for any \( 0 \leq i \leq \frac{n-1}{2} \), the set of negative weights at \( P_i \) is

\[ \{w^-_{ij}\} = \{\phi(P_j) - \phi(P_i)\}_{j<i}, \]

and for any \( \frac{n+1}{2} \leq i \leq n \), the set of positive weights at \( P_i \) is

\[ \{w^+_{ij}\} = \{\phi(P_j) - \phi(P_i)\}_{j>i}. \]

Next, we propose a new argument to obtain the rest of the weights. We start with the following observation.

**Lemma 6.6.** Let the circle act on a compact \( 2n \)-dimensional symplectic manifold \( (M, \omega) \) with moment map \( \phi: M \to \mathbb{R} \). Assume the fixed point set consists of \( n+1 \) isolated points, \( P_0, P_1, \cdots, P_n \). Then for each fixed point \( P_i \), the flow of \(-\text{grad}(\phi)\) from the negative disk bundle of \( P_i \) cannot miss the center point of the \( 2i - 2 \)-cell of \( M_{i-1} \), viewed as a CW-complex (consisting
of a unique cell in each even dimension). In particular, the flow cannot miss the fixed point \( P_{i-1} \).

Proof. By Lemma 2.3, each manifold \( M_i \) has the same rational cohomology ring as \( \CP^i \):

\[
H^*(M_i; \mathbb{Q}) \cong H^*(\CP^i; \mathbb{Q}) = \mathbb{Q}[x]/x^{i+1},
\]

and \( M_i \) has a CW structure consisting of a unique 2k-cell, the negative disk bundle \( D^{2k} \) of \( P_k \), for each \( 0 \leq k \leq i \), with the attaching maps induced by the flow of \( -\text{grad}(\phi) \). This CW-structure is given by Morse theory. Suppose \( M_{i-1} \) has a CW-structure, which may be different from the one above, but consisting of a unique cell in each 2k-dimension for \( 0 \leq k \leq i-1 \). The flow of \( -\text{grad}(\phi) \) induces a gluing of the 2i-cell \( D^{2i} \) to \( M_{i-1} \). If the flow from \( D^{2i} \) misses the center point of the 2i-2-cell in \( M_{i-1} \), then up to homotopy, the 2i-cell \( D^{2i} \) is attached to the 2i-4-skeleton, so the generator of \( H^{2i}(M_i; \mathbb{Q}) \) has no relation with the generator of \( H^{2i-2}(M_i; \mathbb{Q}) \), a contradiction.

In particular, for the CW-structure given by Morse theory, the center point of the 2i-2-cell in \( M_{i-1} \) is the fixed point \( P_{i-1} \). So the flow cannot miss the fixed point \( P_{i-1} \). □

A gradient sphere from one fixed point to another (following the flow of \( +\text{grad}(\phi) \) or \( -\text{grad}(\phi) \)) is smooth at the fixed point it originates, but may not be smooth at the fixed point it ends. When a gradient sphere is smooth at both fixed points, we say that there is a gradient sphere between the two fixed points, in which case, \( w \) is a weight at one fixed point if and only if \( -w \) is a weight at the other fixed point.

Lemma 6.7. Let the circle act on a compact 2n-dimensional symplectic manifold \((M, \omega)\) with moment map \( \phi: M \to \mathbb{R} \). Assume the fixed point set consists of \( n + 1 \) isolated points, \( P_0, P_1, \ldots, P_n \). If in some \( M_{i-1} \), there is a gradient sphere between any two fixed points, then the flow of \( -\text{grad}(\phi) \) from the negative disk bundle of \( P_i \) cannot miss any of the fixed points, \( P_0, P_1, \ldots, P_{i-1} \) in \( M_{i-1} \).

Proof. The topology of \( M_{i-1} \) only depends on its cell structure. By Morse theory, \( M_{i-1} \) has a cell structure consisting of a unique cell in each even dimension — the negative disk bundle \( D^{2j} \) of \( P_j \), where \( 0 \leq j \leq i-1 \), glued by the maps induced by the flow of \( -\text{grad}(\phi) \). If there is a gradient sphere between any two fixed points in \( M_{i-1} \), then we can change the cell structure: with any \( P_j \) (\( 0 \leq j \leq i-1 \)) being the 0-cell, with a gradient sphere connecting \( P_j \) minus \( P_j \) being the 2-cell, \( \cdots \), and with the flow out of \( \pm\text{grad}(\phi) \) from one initially chosen fixed point being the 2(i-1)-cell. If the flow of \( -\text{grad}(\phi) \) from the negative disk bundle of \( P_i \) misses any one of the fixed points in \( M_{i-1} \), then it misses the center point of the 2(i-1)-cell for one of these cell structures on \( M_{i-1} \), which contradicts Lemma 6.6. □

Let us prove a stronger claim to that of Lemma 6.7.
Lemma 6.8. Let the circle act on a compact $2n$-dimensional symplectic manifold $\left(M, \omega\right)$ with moment map $\phi: M \to \mathbb{R}$. Assume the fixed point set consists of $n + 1$ isolated points, $P_0, P_1, \cdots, P_n$. If for some $i$, the manifold $M_i$ has the same integral cohomology ring as $\mathbb{C}P^n$, with $[\omega_{\mid M_i}]$ being the generator, then in $M_i$, there is a gradient sphere between any two fixed points, and for any $0 \leq j < k \leq i$, $\phi(P_k) - \phi(P_j)$ is a positive weight at $P_j$.

Proof. By Lemma 5.7, for each fixed point $P_k$, where $0 \leq k \leq i$, there is a gradient sphere from $P_k$ to $P_j$ for each $0 \leq j \leq k - 1$, and the negative weights at $P_k$ are all not $-1$ except possibly $\phi(P_{k-1}) - \phi(P_k)$. So the gradient spheres are non-free except possibly the one from $P_k$ to $P_{k-1}$. The non-free gradient spheres are isotropy spheres, hence smooth at the poles, and $\phi(P_k) - \phi(P_j)$ is a positive weight at $P_j$. If the gradient sphere from $P_k$ to $P_{k-1}$ is free, then using Lemma 6.6 for $-\phi$, we know that the flow of $+\text{grad}(\phi)$ from the positive disk bundle of $P_{k-1}$ cannot miss $P_k$, so there is a gradient sphere from $P_{k-1}$ to $P_k$, and there is a positive weight 1 at $P_{k-1}$. Since the flow of $-\text{grad}(\phi)$ from the negative disk bundle $D^2k$ of $P_k$ surjects to the fixed points $P_0, P_1, \cdots, P_{k-1}$, there cannot be more than one gradient spheres from $P_k$ to $P_{k-1}$, so the one from $P_k$ to $P_{k-1}$ and the one from $P_{k-1}$ to $P_k$ coincide and hence it is smooth at the poles, and $\phi(P_{k-1}) - \phi(P_k)$ is a positive weight at $P_{k-1}$. \hfill \Box

Similarly, using $-\phi$, we can prove the following “symmetric claim”.

Lemma 6.9. Let the circle act on a compact $2n$-dimensional symplectic manifold $\left(M, \omega\right)$ with moment map $\phi: M \to \mathbb{R}$. Assume the fixed point set consists of $n + 1$ isolated points, $P_0, P_1, \cdots, P_n$. If for some $i$, $M'_i$ has the same integral cohomology ring as $\mathbb{C}P^{n-i-1}$, with $[\omega_{\mid M'_i}]$ being the generator, then in $M'_i$, there is a gradient sphere between any two fixed points, and for any $i + 1 \leq k < j \leq n$, $\phi(P_k) - \phi(P_j)$ is a negative weight at $P_j$.

Next, we obtain the set of negative weights at $P_{\frac{n+1}{2}}$, and symmetrically, the set of positive weights at $P_{\frac{n-1}{2}}$.

Lemma 6.10. Let the circle act on a compact $2n$-dimensional symplectic manifold $\left(M, \omega\right)$ with moment map $\phi: M \to \mathbb{R}$. Assume the fixed point set consists of $n + 1$ isolated points, $P_0, P_1, \cdots, P_n$. Assume $M$ has the same integral cohomology ring as $\mathbb{G}_2(\mathbb{R}^{n+2})$ with $n \geq 3$ odd and $[\omega] \in H^2(M; \mathbb{Z})$ being the degree 2 generator. Then there is a gradient sphere from $P_{\frac{n+1}{2}}$ to $P_j$ for any $0 \leq j \leq \frac{n-1}{2}$, and the set of negative weights at $P_{\frac{n+1}{2}}$ is

\begin{equation}(6.11) \{ w_{\frac{n+1}{2}, j} \} = \left\{ \phi(P_j) - \phi(P_{\frac{n+1}{2}}) \right\}_{j < \frac{n+1}{2}, j \neq (\frac{n+1}{2})} \cup \left\{ \frac{1}{2} \left( \phi(P_{(\frac{n+1}{2})}) - \phi(P_{\frac{n+1}{2}}) \right) \right\}, \end{equation}
where \((\frac{n + 1}{2})' \in \{0, 1, \ldots, \frac{n - 1}{2}\}\). Similarly, there is a gradient sphere from \(P_{\frac{n + 1}{2}}^{-} \rightarrow P_j\) for any \(\frac{n + 1}{2} \leq j \leq n\), and the set of positive weights at \(P_{\frac{n + 1}{2}}^{-}\) is

\[\{w^{+}_{\frac{n + 1}{2}, j}\} = \left\{ \phi(P_j) - \phi(P_{\frac{n + 1}{2}}^-) \right\}_{j > \frac{n + 1}{2}, j \neq (\frac{n-1}{2})'} \cup \left\{ \frac{1}{2} \left( \phi(P_{\frac{n + 1}{2}}^-) - \phi(P_{\frac{n - 1}{2}}) \right) \right\},\]

where \((\frac{n - 1}{2})' \in \{\frac{n + 1}{2}, \ldots, n\}\).

**Proof.** First, by Lemma 2.2 \([\omega]\) is integral implies \(\phi(P_i) - \phi(P_j) \in \mathbb{Z}\) for any \(i, j\).

Consider \(M_{\frac{n + 1}{2}}\). By Lemmas 6.1 and 6.8 there is a gradient sphere between any two fixed points in \(M_{\frac{n + 1}{2}}\). By Lemma 6.7 the flow of \(-\text{grad}(\phi)\) from the negative disk bundle of \(P_{\frac{n + 1}{2}}\) cannot miss any one of the fixed points \(P_0, P_1, \ldots, P_{\frac{n + 1}{2}}\). By Lemma 5.6 there is a gradient sphere from \(P_{\frac{n + 1}{2}}\) to \(P_j\) for any \(0 \leq j \leq \frac{n - 1}{2}\), and if \(\{w^{'}_{\frac{n + 1}{2}, j}\}_{0 \leq j \leq \frac{n - 1}{2}}\) is the set of negative weights at \(P_{\frac{n + 1}{2}}\), then up to reordering them, \(w^{'}_{\frac{n + 1}{2}, j}\) for each \(0 \leq j \leq \frac{n - 1}{2}\). Then (6.2) in Lemma 6.1 allows us to obtain the negative weights at \(P_{\frac{n + 1}{2}}\) as claimed.

Similarly, using \(-\phi\), we obtain the claims for \(P_{\frac{n - 1}{2}}\). \(\square\)

Before we proceed to get the \((\frac{n + 1}{2})'\) and \((\frac{n - 1}{2})'\) in Lemma 6.10 for any \(n \geq 3\) odd, we first look at the case \(n = 3\). The following fact follows from [13, Theorem 1]. Here we use our method to prove it.

**Lemma 6.13.** Let the circle act on a compact 6-dimensional symplectic manifold \((M, \omega)\) with moment map \(\phi: M \rightarrow \mathbb{R}\). Assume the fixed point set consists of 4 isolated points, \(P_0, P_1, P_2, P_3\). Assume \(M\) has the same integral cohomology ring as \(G_2(\mathbb{R}^5)\) with \([\omega] \in H^2(M; \mathbb{Z})\) being the degree 2 generator. Then \(c_1(M) = 3[\omega]\).

**Proof.** By Lemma 6.10 and Corollary 6.5 the sets of weights at \(P_2\) and \(P_1\) have two possibilities:

**Case 1.**

\[w_2 = \left\{ \phi(P_0) - \phi(P_2), \frac{1}{2} (\phi(P_1) - \phi(P_2)), \phi(P_3) - \phi(P_2) \right\},\]

\[w_1 = \left\{ \phi(P_0) - \phi(P_1), \frac{1}{2} (\phi(P_2) - \phi(P_1)), \phi(P_3) - \phi(P_1) \right\}.\]

**Case 2.**

\[w_2 = \left\{ \frac{1}{2} (\phi(P_0) - \phi(P_2)), \phi(P_1) - \phi(P_2), \phi(P_3) - \phi(P_2) \right\},\]

\[w_1 = \left\{ \phi(P_0) - \phi(P_1), \phi(P_2) - \phi(P_1), \frac{1}{2} (\phi(P_3) - \phi(P_1)) \right\}.\]

For Case 1, using Lemma 2.1 we get \(c_1(M) = 3[\omega]\).

Now assume we have Case 2.
First assume $|\frac{1}{2}(\phi(P_0) - \phi(P_2))| > 1$. Then the gradient sphere from $P_2$ to $P_0$ is an isotropy sphere, so $\frac{1}{2}(\phi(P_2) - \phi(P_0))$ is a weight at $P_0$, and symmetrically $\frac{1}{2}(\phi(P_1) - \phi(P_3))$ is a weight at $P_3$. Together with Lemmas 6.10 and 6.8 (and Remark 6.4 and Lemma 6.9), we get the set of weights at $P_0$ and $P_3$:

$$w_0 = \left\{ \phi(P_1) - \phi(P_0), \frac{1}{2}(\phi(P_2) - \phi(P_0)), \phi(P_3) - \phi(P_0) \right\},$$

and

$$w_3 = \left\{ \phi(P_0) - \phi(P_3), \frac{1}{2}(\phi(P_1) - \phi(P_3)), \phi(P_2) - \phi(P_3) \right\}.$$ 

Using Lemma 2.11 and respectively the pair of fixed points $\{P_1, P_2\}$ and $\{P_0, P_3\}$ to compute $c_1(M)$, we have a contradiction.

Now assume $|\frac{1}{2}(\phi(P_0) - \phi(P_2))| = 1$. Then $|\phi(P_0) - \phi(P_1)| = 1$ and $|\phi(P_1) - \phi(P_2)| = 1$. By symmetry, $|\phi(P_2) - \phi(P_3)| = 1$. Together with (6.2) of Lemma 6.1, we get the set of weights at $P_3$, and by symmetry the set of weights at $P_0$:

$$w_3 = \{-1, -1, 3\}, \text{ and } w_0 = \{1, 1, 3\}.$$ 

Consider the $Z_3$ isotropy sphere between $P_0$ and $P_3$, we should have that $w_3 = w_0 \mod 3$ (see for example [13] Lemma 2.6), a contradiction. □

**Lemma 6.14.** In Lemma 6.10 $(\frac{n+1}{2})' \neq 0, 1, \cdots, \frac{n-5}{2}$ and $(\frac{n-1}{2})' \neq n, n - 1, \cdots, \frac{n+5}{2}$.

**Proof.** By Lemma 6.10 and Corollary 6.5 the sets of weights at $P_{\frac{n+1}{2}}$ and $P_{\frac{n+1}{2}}$ are respectively (6.15)

$$\{ w_{\frac{n+1}{2}}, j \} = \left\{ \phi(P_j) - \phi(P_{\frac{n+1}{2}}) \right\}_{j \neq \frac{n+1}{2}, (\frac{n+1}{2})'} \cup \left\{ \frac{1}{2} \left( \phi(P_{\frac{n+1}{2}}) - \phi(P_{\frac{n+1}{2}}) \right) \right\},$$

where $(\frac{n+1}{2})' \in \{0, 1, \cdots, \frac{n-1}{2}\}$, and

(6.16)

$$\{ w_{\frac{n+1}{2}}, j \} = \left\{ \phi(P_j) - \phi(P_{\frac{n+1}{2}}) \right\}_{j \neq \frac{n+1}{2}, (\frac{n-1}{2})'} \cup \left\{ \frac{1}{2} \left( \phi(P_{\frac{n+1}{2}}) - \phi(P_{\frac{n-1}{2}}) \right) \right\},$$

where $(\frac{n-1}{2})' \in \{\frac{n+1}{2}, \cdots, n - 1\}$. By Lemma 2.1 we obtain (6.17)

$$c_1(M) = \frac{\Gamma_{\frac{n-1}{2}} - \Gamma_{\frac{n+1}{2}}}{\phi(P_{\frac{n+1}{2}}) - \phi(P_{\frac{n-1}{2}})} = \left( n + \frac{1}{2} - \frac{1}{2} \frac{\phi(P_{\frac{n+1}{2}}) - \phi(P_{\frac{n+1}{2}})}{\phi(P_{\frac{n-1}{2}}) - \phi(P_{\frac{n-1}{2}})} \right) \left[ \omega \right].$$

If $n = 3$, by Lemma 6.13 $c_1(M) = 3[\omega]$. So $2' \neq 0$ and $1' \neq 3$. Now assume $n > 3$. Suppose $(\frac{n+1}{2})' = i$ with $0 \leq i \leq \frac{n-5}{2}$, and by symmetry, $(\frac{n-1}{2})' = n - i$.

Recall that $\phi(P_k)$ are integers for all $k$. So $|\frac{1}{2}(\phi(P_{\frac{n+1}{2}}) - \phi(P_{\frac{n-1}{2}}))| > 1$. Observe (6.11) that the negative weights at $P_{\frac{n+1}{2}}$ all have absolute values bigger than 1 except possibly $\phi(P_{\frac{n-1}{2}}) - \phi(P_{\frac{n+1}{2}})$, or, the gradient sphere.
from $P_{\frac{n+1}{2}}$ to $P_j$, where $0 \leq j \leq \frac{n-1}{2}$, is non-free except possibly the one from $P_{\frac{n+1}{2}}$ to its neighbor point $P_{\frac{n-1}{2}}$. Similar to the proof of Lemma 6.8, we can prove that there is a gradient sphere between $P_{\frac{n+1}{2}}$ and $P_j$ for any $0 \leq j \leq \frac{n-1}{2}$. Together with Lemma 6.8, we have that in $M_{\frac{n+1}{2}}$, there is a gradient sphere between any two fixed points. By Lemmas 6.7, 5.6 and (6.2) in Lemma 6.1, we obtain that there is a gradient sphere from a gradient sphere between any two fixed points. By Lemmas 6.7, 5.6 and (6.2) in Lemma 6.1, we obtain that there is a gradient sphere from $P_{\frac{n+1}{2}}$ to $P_j$ for any $0 \leq j \leq \frac{n+1}{2}$, and the set of negative weights at $P_{\frac{n+1}{2}}$ is

$$\{w_{\frac{n+1}{2},j}\} = \{\phi(P_j) - \phi(P_{\frac{n+1}{2}})\}_{j \neq \frac{n+1}{2}, \frac{n-1}{2}} \cup \left\{\frac{1}{2} \left(\phi(P_{\frac{n+1}{2}}) - \phi(P_{\frac{n+1}{2}})\right)\right\}.$$

By Lemma 6.9 and Remark 6.4, $\phi(P_{\frac{n+1}{2}}) - \phi(P_{\frac{n-1}{2}})$ is a negative weight at $\phi(P_{\frac{n+1}{2}})$. Hence $(\frac{n+3}{2})' \in \{0, \cdots, \frac{n-1}{2}\}$. Together with Corollary 6.5, we get the set of weights at $P_{\frac{n+3}{2}}$:

$$\{w_{\frac{n+3}{2},j}\} = \{\phi(P_j) - \phi(P_{\frac{n+3}{2}})\}_{j \neq \frac{n+3}{2}, \frac{n-1}{2}} \cup \left\{\frac{1}{2} \left(\phi(P_{\frac{n+3}{2}}) - \phi(P_{\frac{n+3}{2}})\right)\right\},$$

where $(\frac{n+3}{2})' \in \{0, \cdots, \frac{n-1}{2}\}$. Similarly using $-\phi$, we get the set of weights at $P_{\frac{n-3}{2}}$:

$$\{w_{\frac{n-3}{2},j}\} = \{\phi(P_j) - \phi(P_{\frac{n-3}{2}})\}_{j \neq \frac{n-3}{2}, \frac{n+1}{2}} \cup \left\{\frac{1}{2} \left(\phi(P_{\frac{n-3}{2}}) - \phi(P_{\frac{n-3}{2}})\right)\right\},$$

where $(\frac{n-3}{2})' \in \{\frac{n+1}{2}, \cdots, n\}$. By Lemma 2.1, we obtain

(6.18)

$$c_1(M) = \frac{\Gamma_{\frac{n-3}{2}} - \Gamma_{\frac{n+3}{2}}}{\phi(P_{\frac{n+3}{2}}) - \phi(P_{\frac{n-3}{2}})}[\omega] = \left(n + 1 \right) \left[ \frac{1}{2} \left(\phi(P_{\frac{n+3}{2}}) - \phi(P_{\frac{n-1}{2}})\right) \right] [\omega].$$

Comparing (6.17) and (6.18), we have

$$\phi(P_{\frac{n+3}{2}}) - \phi(P_{\frac{n-1}{2}}) > \phi(P_{\frac{n+1}{2}}) - \phi(P_{\frac{n+1}{2}}).$$

So $(\frac{n+3}{2})' < (\frac{n+1}{2})'$. Then $P_{\frac{n+1}{2}}$ satisfies the condition as $P_{\frac{n+1}{2}}$ does. Repeat the argument above, we obtain the set of weights at $P_{\frac{n+3}{2}}$, $\cdots$, and $P_n$, and we have $n' < \cdots < (\frac{n+3}{2})' < (\frac{n+1}{2})'$, which is impossible if $(\frac{n+1}{2})' = i$ with $0 \leq i \leq \frac{n-5}{2}$. \hfill \square

In the next lemma, with only the weights data at $P_{\frac{n+1}{2}}$ and $P_{\frac{n-1}{2}}$, we get $(\frac{n+1}{2})' \neq \frac{n-3}{2}$ and $(\frac{n-1}{2})' \neq \frac{n+3}{2}$.

**Lemma 6.19.** Let the circle act on a compact $2n$-dimensional symplectic manifold $(M, \omega)$ with moment map $\phi: M \to \mathbb{R}$. Assume $n \geq 3$ is odd, the fixed point set consists of $n + 1$ isolated points, $P_0, \cdots, P_{\frac{n+1}{2}}, P_{\frac{n-1}{2}}, \cdots, P_n$, and the sets of weights at $P_{\frac{n+1}{2}}$ and $P_{\frac{n-1}{2}}$ are respectively as in (6.17) and (6.16). Then $(\frac{n+1}{2})' \neq \frac{n-3}{2}$ and $(\frac{n-1}{2})' \neq \frac{n+3}{2}$. 
Proof. Assume \((\frac{n+1}{2})' = \frac{n-3}{2}\), then by symmetry, \((\frac{n-1}{2})' = \frac{n+3}{2}\). By (6.17),

\[
c_1(M) = \left( n + \frac{1}{2} - \frac{1}{2} \phi(P_{n+1}) - \phi(P_{n-1}) \right) [\omega].
\]

First, assume \(n = 3\). Then by Tolman’s result in dimension 6 on the 4 kinds of cohomology ring and Chern classes [13, Theorem 1], only when \(M\) has the ring of \(\mathbb{R}^5\), the weights at \(P_1\) and \(P_2\) are as we assumed. In this case, \(c_1(M) = 3[\omega]\). Hence the claim has to hold.

Now assume \(n > 3\). Consider

\[
M - \{P_0, P_n\},
\]

which contains the fixed points \(P_1, \cdots, P_{n-1}, P_{n+1}, \cdots, P_n\), the weights at \(P_{\frac{n+1}{2}}\) and \(P_{\frac{n-1}{2}}\) “relative” to these fixed points are the same as in a compact \(2(n-2)\)-dimensional Hamiltonian \(S^1\)-manifold \(M_{2(n-2)}\), containing \(P_1, \cdots, P_{n-1}, P_{n+1}, \cdots, P_n\) as fixed points (\(P_1\) being the minimum and \(P_{n-1}\) being the maximum) and satisfying our assumptions. Similar to (6.20), using Lemma [2.1] for \(M_{2(n-2)}\), we get

\[
c_1(M_{2(n-2)}) = \left( n - 2 + \frac{1}{2} - \frac{1}{2} \phi(P_{n+1}) - \phi(P_{n-1}) \right) [\omega] = c_1(M) - 2[\omega].
\]

We continue this process inductively, until we have

\[
M - \{P_0, P_n\} - \{P_1, P_{n-1}\} - \cdots - \{P_{\frac{n-1}{2}}, P_{\frac{n+1}{2}}\},
\]

which contains 4 fixed points, \(P_{\frac{n+1}{2}}, P_{\frac{n-1}{2}}, P_{\frac{n-1}{2}}, \text{ and } P_{\frac{n+1}{2}}\), and we have

\[
c_1(M_0) = \left( 3 + \frac{1}{2} - \frac{1}{2} \phi(P_{n+1}) - \phi(P_{n-1}) \right) [\omega] \neq 3[\omega].
\]

This contradicts to the 6-dimensional case we proved.

Proposition 6.21. In Lemma [6.10] \((\frac{n+1}{2})' = \frac{n-1}{2}\) and \((\frac{n-1}{2})' = \frac{n+1}{2}\). Moreover,

\[
c_1(M) = n[\omega].
\]

Proof. By Lemma [6.10] and Corollary [6.5] \(M\) satisfies the conditions of Lemma [6.19]. The first claim follows from Lemmas [6.10] [6.14] and [6.19] and the claim for \(c_1(M)\) then follows from (6.17).

By the result \(1) \implies \(4)\) we proved, Proposition [6.21] on \(c_1(M)\) implies Proposition [6.22]. However, using the result \((\frac{n+1}{2})' = \frac{n-1}{2}\) and \((\frac{n-1}{2})' = \frac{n+1}{2}\) of Proposition [6.21] we can also argue directly, following the method of proof of Lemma [6.14] to get the set of negative weights at each \(P_i\) for \(\frac{n+1}{2} \leq i \leq n\), and using \(-\phi\) to get the set of positive weights at each \(P_i\) for \(0 \leq i \leq \frac{n-3}{2}\), and hence prove Proposition [6.22].
Proposition 6.22. Under the assumptions of Lemma 6.10, the set of weights of the action at each fixed point $P_i$ is

$$\{w_{ij}\} = \{\phi(P_j) - \phi(P_i)\}_{j \neq i, n-i} \cup \left\{\frac{1}{2}(\phi(P_{n-i}) - \phi(P_i))\right\}.$$ 

That is, the set of weights coincides with those of $\widetilde{G}_2(\mathbb{R}^{n+2})$ with the standard circle action as in Example 1.3.

Remark 6.23. Note that we can also use Lemmas 6.6, 6.7 and 6.8 to prove Proposition 5.10.

Remark 6.24. Let $(M, \omega)$ be a compact 6-dimensional Hamiltonian $S^1$-manifold with 4 isolated fixed points, $P_0$, $P_1$, $P_2$ and $P_3$. In [13, Theorems 1 and 2], besides the two cases corresponding to the integral cohomology rings of $\mathbb{C}P^3$ and $\widetilde{G}_2(\mathbb{R}^5)$, Tolman also obtained the following two cases (which exist by [10]):

1. $H^*(M; \mathbb{Z}) = \mathbb{Z}[x, y]/(x^2 - 5y, y^2)$. The weights at $P_0, P_1, P_2,$ and $P_3$ are respectively $(1, 2, 3), (-1, 1, 4), (-1, -4, 1)$ and $(-1, -2, -3)$.
2. $H^*(M; \mathbb{Z}) = \mathbb{Z}[x, y]/(x^2 - 22y, y^2)$. The weights at $P_0, P_1, P_2,$ and $P_3$ are respectively $(1, 2, 3), (-1, 1, 5), (-1, -5, 1)$ and $(-1, -2, -3)$.

In both cases, $x$ is of degree 2 and $y$ is of degree 4. In these cases, we can check that the flow of $-\text{grad}(\phi)$ from $P_3$ does not surject to the fixed points in $M_2$.

7. Proof of Theorem 1.3

With the results of Theorems 1.1 and 1.2, we can prove Theorem 1.3 as follows.

Using a theorem by Kobayashi and Ochiai [4], and by incorporating the circle action, we proved the following proposition in [6].

Proposition 7.1. Let $(M, \omega, J)$ be a compact Kähler manifold of complex dimension $n$, which admits a holomorphic Hamiltonian circle action. Assume that $[\omega]$ is an integral class. If $c_1(M) = (n+1)[\omega]$, then $M$ is $S^1$-equivariantly biholomorphic to $\mathbb{C}P^n = \mathbb{P}(H^0(M; L))$, and if $c_1(M) = n[\omega]$, then $M$ is $S^1$-equivariantly biholomorphic to a quadratic hypersurface in $\mathbb{C}P^{n+1} = \mathbb{P}(H^0(M; L))$, where $L$ is a holomorphic line bundle over $M$ with first Chern class $[\omega]$ and $H^0(M; L)$ is its space of holomorphic sections.

Since we have $H^2(M; \mathbb{R}) = \mathbb{R}$, by rescaling, we may assume that $[\omega]$ is a primitive integral class. Then by Theorems 1.1 and 1.2, when one of the 4 conditions is satisfied, the condition of Proposition 7.1 is satisfied. So we have the claimed equivariant biholomorphism:

$$f: (M, \omega, J) \to (M', \omega', J'),$$

where $(M', \omega', J')$ stands for the Kähler manifold $\mathbb{C}P^n$ with $n \geq 1$ or $\widetilde{G}_2(\mathbb{R}^{n+2})$ with $n \geq 3$ odd, with the standard structures and the standard $S^1$ action.
By rescaling $\omega'$, we may assume that $\omega$ and $f^*\omega'$ represent the same cohomology class. We consider the family of forms $\omega_t = (1 - t)\omega + tf^*\omega'$ on $M$, where $t \in [0, 1]$. Each $\omega_t$ is nondegenerate: for any point $x \in M$, suppose $X \in T_xM$ is such that $\omega_t(X,Y) = 0$ for all $Y \in T_xM$. In particular, if $Y = JX$, then $\omega_t(X,JX) = 0$. Using the facts $\omega(X,JX) \geq 0$, $f^*(JX) = J^tf^*X$, and $\omega'(f^*X,J^tf^*X) \geq 0$, we get $X = 0$. So $\omega_t$ is a family of symplectic forms in the same cohomology class. Then by Moser’s method, we obtain an equivariant symplectomorphism between $M$ and $M'$.

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