The Peculiar–Velocity–Field in Structure Formation Theories with Cosmic Strings

Carsten van de Bruck¹

¹ Institut für Astrophysik und Extraterrestrische Forschung
Auf dem Hügel 71
53121 Bonn, Germany
cvdb@astro.uni-bonn.de
(today)

We investigate the peculiar velocity field due to long cosmic strings in several cosmological models and analyse the influence of a non-scaling behaviour of the string network, which is expected in open cosmological models or models with a cosmological constant. It is shown that the deviation of the probability distribution of the peculiar velocity field from the normal distribution is only weak in all models. It is further argued that one can not necessarily obtain the parameter \( \beta = \Omega_0^{1/6}/b \) from density and velocity fields, where \( \Omega_0 \) is the density parameter and \( b \) the linear biasing parameter, if cosmic strings are responsible for structure formation in the universe. An explanation for this finding is given.

Another possibility for testing structure formation theories was proposed by \[5\]. The probability distribution of the peculiar velocity field should be different in inflationary models and models with topological defects such as cosmic strings. However, as emphasized by \[8\], the probability distribution of the peculiar velocity field in cosmic string theories is Gaussian to high accuracy. This conclusion was based on the assumption that the string network reaches a scaling behaviour.

It was shown by several authors that the scaling solution is only expected in the Einstein–de Sitter model \[8\]; in open models, in flat models with a cosmological constant and in closed (loitering) models the behaviour of the network is different from scaling. Further, the transition to the matter scaling behaviour is much longer than previously estimated \[8\]. These possible sources of deviation from the string scaling solution should have interesting consequences. A first step was done in \[10\] and \[11\]. Whereas in \[10\] it was shown, that only a drastic departure from scaling could solve the problems of structure formation which cosmic strings, \[11\] have shown that there is only weak dependence of the density parameter \( \Omega_0 \) in open and flat models with cosmological constant on the normalisation of \( G\mu \) from COBE data. It might be, that the cosmic string scenario is successful also in the absence of the scaling behaviour, i.e. that it works well in open models or in models with a cosmological constant.

In this paper we investigate the influence of cosmic strings on the peculiar velocity field. Earlier investigations of the peculiar velocity field concentrated on the spectrum of the field, i.e. its dependence on the length scale \( L \), see e.g. \[5,12\]. We are interested in the effects of a departure from the scaling behaviour. We use an approximation, first introduced by \[6\] to calculate the effects of many strings.

The paper is organized as follows: In section II we discuss the influence of cosmic strings on the peculiar velocity field. Our calculations of the string network are based on the calculations by \[8\]. Our results for the peculiar velocities are presented in section III. In section IV we argue that if cosmic strings seed the structure in the universe then the peculiar velocity field and the density field is correlated but one cannot obtain information on the parameter \( \beta = \Omega_0^{1/6}/b \), where \( \Omega_0 \) is the density parameter and \( b \) is a linear bias parameter. In section V we summarize our results and give some conclusions. Throughout the paper we set \( c = 1 \).
II. THE PECULIAR VELOCITY FIELD DUE TO LONG COSMIC STRINGS

The space–time of a straight cosmic string is similar to the Minkowski space–time, except for a deficit angle $\Delta \phi$, given by \[\Delta \phi = 8\pi G \mu \gamma_s v_s.\] (2.1)

As a result, the matter gets a kick towards the plane swept out by the string ($v_s$ is the string velocity and $\gamma_s$ is the Lorentz factor). The velocity kick due to a wiggly string is given by

$$ u_s = 4\pi G \mu \gamma_s v_s f = 3.8(G\mu)a(\gamma_s v_s)f \text{ km/s}, \quad (2.2) $$

with

$$ f = 1 + \frac{1}{2v_s^2} \left(1 - \frac{T}{\mu}\right). \quad (2.3) $$

The term $f$ corresponds to the small scale structure on the string, where $\mu$ is the effective mass per unit length on the string and $T$ is the string tension.

A. The Zeldovich approximation

To calculate the peculiar velocity field, we use the Zeldovich approximation, in which the physical coordinates of a particle are written by

$$ \mathbf{r}(x, t) = a(t) \left[ \mathbf{x} + \psi(x, t) \right], \quad (2.4) $$

where $a(t)$ is the scale factor, $\mathbf{x}$ is the comoving coordinate and $\psi$ is the displacement vector due to inhomogeneities in the cosmic fluid, i.e. cosmic strings in our context \[14\]. The equation of motion is given by Newton's law

$$ \ddot{\mathbf{r}} = -\nabla \Phi. \quad (2.5) $$

The gravitational field is connected with the matter distribution (Poisson equation), which can be obtained from linearising Einstein's field equation:

$$ \nabla^2 \Phi = 4\pi G (\rho_b + \delta \rho) + \Lambda c^2. \quad (2.6) $$

In this equation $\rho_b$ is the matter density, $\delta \rho$ the matter density fluctuation and $\Lambda$ the cosmological constant. To first order one obtains

$$ \delta \equiv \frac{\rho - \rho_b}{\rho_b} = -\nabla \cdot \psi(\mathbf{x}, t), \quad (2.7) $$

Where $\rho$ is the total matter density. This leads to

$$ \nabla^2 \Phi = 4\pi G \rho_b (1 - \nabla \cdot \psi) + \Lambda c^2. \quad (2.8) $$

Integration of this equation and substitution of

$$ \ddot{\mathbf{r}} = \frac{\ddot{a}}{a} \mathbf{r} + 2 \frac{\dot{a}}{a} \mathbf{v} + a \mathbf{\ddot{v}} \quad (2.9) $$

and the second Friedmann equation

$$ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \rho_b + \frac{\Lambda c^2}{3}. \quad (2.10) $$

leads to the evolution equation for the displacement $\psi$ \[14\]:

$$ \ddot{\psi} + 2 \frac{\dot{a}}{a} \dot{\psi} - 4\pi G \rho_b \psi = 0. \quad (2.11) $$

For our purposes we have to calculate the peculiar velocity field, which can be obtained from eq. (2.4):

$$ \mathbf{v}_{pec} = \dot{a} \psi. \quad (2.12) $$

The effect of the cosmological constant on the evolution of a density perturbation is only due to the effect of $\Lambda$ on the evolution of the scale factor $a$.

B. The influence of cosmic strings

We use an analytical approximation (the so called multiple impulse approximation), first introduced by Vachaspati \[1\], which also was successfully applied to calculate the CMBR anisotropies \[1\]. We devide the time interval from $t_{eq}$ (at which structures starts to form) to $t_0$ in $N$ steps with $t_{eq} = 2t_i$. Between $t_i$ and $t_{i+1}$ the strings intercommute, form loops etc. so that (approximately) at $t_{i+1}$ the “new ordered state” of the network is uncorrelated with the “old state” at $t_i$. Again, at this time the network influences the matter within the horizon (due to scalar field radiation scales larger than the horizon are not affected). This is not true in vacuum dominated epochs. In this case the velocity of the strings decreases and therefore the probability of string interaction decreases. This means that the new state is correlated with the old one. However, in such epochs the number of strings within the Hubble horizon decreases rapidly and therefore our results aren’t changed significantly (see below).

At $t_1 = t_{eq}$ each string within the horizon gives the matter a kick in the direction of the surface swapped out by the string:

$$ \mathbf{v}_{1,i} = u_1 \mathbf{k}_{i,1}, \quad (2.13) $$

where $\mathbf{k}_{i,1}$ is a (random) unit vector in direction of the string $i$. The resulting peculiar velocity from all strings at $t_1$ is

$$ \mathbf{v}_1 = \sum_{i=1}^{n_{g,1}} u_i \mathbf{k}_{i,1}, \quad (2.14) $$

The sum is now taken over the number $n_{g,1}$ of all strings within the horizon at $t_1$. This peculiar velocity field grows between $t_1$ and $t_2$ by a factor $A(t_1, t_2)$ via eq. (2.11) and eq. (2.12):
\[ A(t_i, t_f) = \frac{a(t_f)}{a(t_i)} \frac{\dot{\psi}(t_f)}{\dot{\psi}(t_i)} \]  \hspace{1cm} (2.15)

At the time \( t_2 \) the peculiar velocity field is given by

\[ v_2 = A(t_1, t_2)v_1 + u_s \sum_{j=1}^{n_{g,2}} k_{2,j} \]

\[ = u_s \sum_{i=1}^{2} \sum_{j=1}^{n_{g,i}} A(t_2, t_i)k_{i,j} \]  \hspace{1cm} (2.16)

Here we have used that \( A(t_1, t_2)A(t_3, t_1) = A(t_2, t_1) \) and that the velocity of the strings is the same in every epoch. For our purposes this is a good approximation, because when the strings slow down the number of strings within the horizon also decreases.

Iteration leads to the peculiar velocity field at the present time on a scale \( L \):

\[ v_0(L) = u_s \sum_{i=1}^{N_L} \sum_{j=1}^{n_{g,i}} A(t_0, t_i)k_{i,j} \]  \hspace{1cm} (2.17)

In this equation \( N_L \) is the number of Hubble time steps during which a volume of comoving size \( L^3 \) experiences string impulses, \( n_{g,i} \) is the number of strings within the horizon at the time \( t_i \). We assume that the vectors \( k_{i,j} \) are random, that is

\[ <k_{i,j} \cdot k_{l,m}> = \delta_{i\ell} \delta_{jm}. \]  \hspace{1cm} (2.18)

From these equations we calculate the RMS velocity numerically on a scale \( L(t_{eq}) \). On scales smaller than \( L(t_{eq}) \) the peculiar velocity field depends only weakly on \( L \) whereas on scales larger than \( L(t_{eq}) \) the predicted velocity field scales as \( L^{-1} \) [2].

C. Network parameter

We use the calculation from [3] for the statistical properties of the string network. We set the number of strings within the horizon \( H^{-1} \) by

\[ n_s = 1 + (\xi \cdot H)^{-1}, \]  \hspace{1cm} (2.19)

where \( \xi \) is the characteristic length scale of the string network, defined by

\[ \rho_\infty = \frac{\mu}{\xi^2}, \]  \hspace{1cm} (2.20)

where \( \rho_\infty \) is the density of the long strings. In the radiation dominated epoch \( n_s \) is about 10, in the (late) matter dominated epoch (with scaling) this number is about 3. In more general cosmological models this number is a function of time [3]. Later we will discuss the influence of the ansatz [2.13].

III. RESULTS

We calculate the peculiar velocity field for four representative models, shown in Tab. 1. For the Einstein–de Sitter (E–dS) model we discuss the influence of the long transition between the radiation and matter scaling solution [5]. As a test, we include the case for an ideal scaling in the E–dS model. The probability distributions of the peculiar velocity fields at a scale \( L(t_{eq}) \) for the models are shown in Fig. 1–5. Each plot was obtained after 50000 realisations. For an exact scaling behaviour in the E–dS model this distribution was shown to be Gaussian [3]. We obtain the same result (see Fig.1). In the case of the long transition between the radiation and matter scaling behaviour the distribution remains nearly Gaussian. However, the distribution becomes broader and the peculiar velocity increases (Fig. 2). The Gaussian character of the probability distribution can be found in the other models, too. There is only a slight deviation at large and small velocities.

In the models, we obtain a peculiar velocity at a scale corresponding to \( L(t_{eq}) \) given by (the length scales are calculated with \( H_0 = 100\,\text{km/(s\,Mpc)} \)):

\[ v_{pec}(L_{eq}) \approx 70 \,\text{Mpc})_{\text{closed}} \]  \hspace{1cm} (3.1)
\[ = (460 \pm 200)(G\mu)_{\delta}(\gamma_s v_s) f \,\text{km/s}, \]

\[ v_{pec}(L_{eq}) \approx 1 \,\text{Mpc})_{\text{EdS, ni}} \]  \hspace{1cm} (3.2)
\[ = (1740 \pm 760)(G\mu)_{\delta}(\gamma_s v_s) f \,\text{km/s}, \]

\[ v_{pec}(L_{eq}) \approx 10 \,\text{Mpc})_{\text{EdS, id}} \]  \hspace{1cm} (3.3)
\[ = (1240 \pm 570)(G\mu)_{\delta}(\gamma_s v_s) f \,\text{km/s}, \]

\[ v_{pec}(L_{eq}) \approx 10 \,\text{Mpc})_{\text{A, flat}} \]  \hspace{1cm} (3.4)
\[ = (280 \pm 120)(G\mu)_{\delta}(\gamma_s v_s) f \,\text{km/s}, \]

\[ v_{pec}(L_{eq}) \approx 10 \,\text{Mpc})_{\text{open}} \]  \hspace{1cm} (3.5)
\[ = (80 \pm 35)(G\mu)_{\delta}(\gamma_s v_s) f \,\text{km/s}. \]

The length scale corresponding to the time \( t_{eq} \) is set to be 0.1\( H(t_{eq})^{-1} \) [5]:

\[ L_{eq} \approx 1.11 \frac{1}{H_0} h_0^{-2}\text{Mpc} \]  \hspace{1cm} (3.6)

Note, that in [5] the length scale was set to be 0.7\( t_{eq} \). Therefore, we assume a somewhat pessimistic view when strings could significantly influence the volume at \( t_{eq} \). In our picture, the volume must be within the typical length scale between all strings. However, the volume is influenced by the strings outside the volume and therefore we somewhat underestimate the peculiar velocities. However, this can be taken into account with including a parameter \( \zeta \), which modifies our ansatz [2.13] (see below, eq. [8.8]).
Within this length the peculiar velocity remains nearly constant, because a smaller length corresponds to times \( t < t_{eq} \) in which perturbation grow only weakly. This would imply that for the closed model we would expect nearly constant bulk flows on scales smaller than 70 Mpc, which is indeed observed. The situation in the other models is not so clear, because for scales larger that \( L_{eq} \) the velocity decreases as \( L \) increases (\( v \propto L^{-1} \)).

It is interesting to note that in all models the standard deviation is related to the mean value by

\[
\sigma \approx 0.45 v_{\text{mean}}. \tag{3.7}
\]

For our calculations we have used the ansatz (2.19) for the number of strings within the horizon. Although this should be a good approximation we could set \( n_s = \zeta(1 + (\xi \cdot H)^{-1}) \). The frequency distribution remains Gaussian, however, now the RMS velocity and the standard deviation is given by

\[
v_{pec,\zeta} = \sqrt{\zeta} v_{pec,\zeta=1} \tag{3.8}
\]

and

\[
\sigma_{\zeta} = \sqrt{\zeta} \sigma_{\zeta=1}. \tag{3.9}
\]

Here, \( v_{pec,\zeta=1} \) and \( \sigma_{\zeta=1} \) is given by eq. (3.3) for the cosmological models. The peculiar velocities therefore depend on the parameter

\[
\alpha = \sqrt{\zeta} \mu_0 (v_{s} \gamma_s) f. \tag{3.10}
\]

To conclude, the fluctuation of number of strings doesn’t change the shape of the probability distribution of the peculiar velocities and the amplitude depends on the same set of parameters \( \{3.1\} \) as in the case for a ideal scaling behaviour of the string network. However, the effective number of strings and the maximum length on which coherent bulk flows are expected, depends on the cosmological parameters \( \Omega_{m,0} \), \( \lambda_0 \) and \( H_0 \).

### IV. MATTER DISTRIBUTION, BULK FLOWS AND BIASING

The results presented in the last section imply that the parameter \( \beta = \Omega_{m,0}^6 / b \) could not be obtained from velocity-density reconstruction methods such as POTENT. To see this, we remember that the fundamental equation, on which these kinds of reconstruction methods are based, is given by (3.17)

\[
\nabla \cdot \mathbf{v} = -\beta H \delta. \tag{4.1}
\]

Here, \( \delta \) is the density fluctuation. On the other side, the continuity equation holds:

\[
\nabla \cdot \mathbf{v} = -\dot{\delta}. \tag{4.2}
\]

In fact, in linear approximation eq. (4.1) can be obtained from eq. (4.3). The important point is that if the ratio \( \delta / \delta \) is independent of space, an arbitrary application of eq. (4.1) can lead to an under– or overestimation of \( \beta \) if one applies eq. (4.1) arbitrarily to the data. This was shown by [13] in the context of the explosion scenario. To demonstrate this point we repeat their short analytical example:

Let us consider an empty universe with \( \Omega_0 = 0 \), filled with massless particles. At some time \( t_i \), the matter gets a kick due to a cosmic string (in the paper by Babul et al., the case of the explosion scenario was considered, but in the case of cosmic strings the analysis is identical). The linear Euler equation reads:

\[
\frac{\partial \mathbf{v}}{\partial t} + 2H(t) v = v_{\text{string}} \delta (t - t_i). \tag{4.3}
\]

The density contrast evolves according to

\[
\frac{\partial^2 \delta}{\partial t^2} + 2H(t) \frac{\partial \delta}{\partial t} = 0. \tag{4.4}
\]

This equation can be solved with the boundary conditions at the time \( t_i \), which are \( \delta(x, t_i) = 0 \) and \( \delta(x, t_i) = \zeta(x, t_i) = -\nabla x \cdot v_{\text{string}}(x, t_i) \). The solution of the equation (4.4) is given by \( \delta(t_i) = 1 \)

\[
\delta(x, t) = \zeta(x, t_i) t_i / a(t), \tag{4.5}
\]

\[
t^2 \dot{\delta}(x, t) = \zeta(x, t_i) t_i / a(t). \tag{4.6}
\]

One can see, that the ratio \( \dot{\delta} / \delta \) is independent of space. We can use the continuity equation (4.2) and equation (4.1) to get

\[
\beta_{\text{eff}} = \frac{1}{a(t) - 1}. \tag{4.7}
\]

Although the true value is zero, an observer, applying (4.1) to the data, will get a value that is different from the true one. Only at late times \( \beta_{\text{eff}} \) will approach the value \( \beta = 0 \).

We expect similar results for the models in table 1. The cosmological model changes the time dependence of \( \delta \) and \( \dot{\delta} \), but in general, if there were only one velocity kick on the matter, the ratio \( \dot{\delta} / \delta \) would be independent of space.

We have seen, that in cosmic string models of structure formation the observed peculiar velocity field is a vector sum of many contributions of strings. We write the general solutions of \( \delta \) and \( \dot{\delta} \), which arise from one kick as

\[
\delta_{i,j}(x, t_i) = \zeta(x, t_i) \gamma_{i,j} B(t_i), \tag{4.8}
\]

\[
\dot{\delta}_{i,j}(x, t_i) = \zeta(x, t_i) \gamma_{i,j} C(t_i). \tag{4.9}
\]

The fields \( \delta \) and \( \dot{\delta} \), which arise from all strings in the past are given by
The velocity drops with $L$ constant. However, on scales larger than $L_{\text{eq}}$ the velocity increases only weakly as $L$ decreases and remains nearly constant at 460 km/s (for $G\mu = 10^{-6}$ and $f_{\gamma_k}u_s \approx 1$) up to scales of about 200 Mpc.

The results imply that if cosmic strings seed the structure in the universe, one can not necessarily obtain the density parameter from the data. Comparison of density fields and velocity fields lead to an effective value, which is a measure of the deviation of an exact relation between the velocity and density field.

Further work should be done on structure formation with cosmic strings in order to investigate the effects on a non–scaling behaviour of the cosmic string network.

**Acknowledgements:** I thank Matthias Soika and Harald Giersche for discussions. This work was supported by the Deutsche Forschungsgemeinschaft (DFG).

**V. DISCUSSION & CONCLUSIONS**

In this paper we have considered the properties of the peculiar velocity field of galaxies in structure formation theories with cosmic strings. We considered the fact that the string network might not have developed a scaling behaviour (as is the case in open models or models with a cosmological constant) and showed that the probability distribution of the peculiar velocities is nearly Gaussian. The RMS peculiar velocity depends on the (effective) number of strings within the horizon and on the string parameter $G\mu$. The length, within the peculiar velocity is nearly independent of the scale, depends on the cosmological parameter $\Omega_0$, $\lambda_0$ and $H_0$.

Open models have more problems with the amplitude of the peculiar velocity field. Only an unphysical high value of $f$ could solve the problems ($f \approx 5$). The situation might be better in flat models with a cosmological constant. However, on scales larger than $L_{eq}$ the peculiar velocity drops with $L^{-1}$, that is on a scale of about 60 Mpc we expect in model 4 peculiar velocities of 50–100 km/s. This is not in agreement with the observed value of 350–450 km/s. The situation is very good in the closed model. On scales smaller than $L_{eq}$ the velocity increases only weakly as $L$ decreases and remains nearly constant at 460 km/s (for $G\mu = 10^{-6}$ and $f_{\gamma_k}u_s \approx 1$) up to scales of about 200 Mpc.

The results imply that if cosmic strings seed the structure in the universe, one can not necessarily obtain the density parameter from the data. Comparison of density fields and velocity fields lead to an effective value, which is a measure of the deviation of an exact relation between the velocity and density field.

Further work should be done on structure formation with cosmic strings in order to investigate the effects on a non–scaling behaviour of the cosmic string network.

**Acknowledgements:** I thank Matthias Soika and Harald Giersche for discussions. This work was supported by the Deutsche Forschungsgemeinschaft (DFG).
TABLE I. The four representative cosmological models. $K$ is the curvature parameter, $\Omega_0$ is the matter density parameter, $\lambda_0$ is the cosmological term, $H_0$ is the Hubble parameter (in km s$^{-1}$Mpc$^{-1}$). $N_{eq}$ is the number of Hubble steps between $t_{eq}$ and $t_0$. 

| Model | $K$ | $\Omega_0$ | $\lambda_0$ | $H_0$ | $N_{eq}$ |
|-------|-----|------------|-------------|-------|---------|
| 1     | $+1$| 0.014      | 1.08        | 90    | 13      |
| 2     | 0   | 1.0        | 0.0         | 60    | 20      |
| 3     | $-1$| 0.1        | 0.0         | 60    | 14      |
| 4     | 0   | 0.1        | 0.9         | 60    | 15      |
FIG. 1. The probability distribution of the peculiar velocity field at a scale corresponding to $t_{eq}$ in the E-dS model with ideal scaling. The solid curve is the normal distribution. The mean value of $v_{pec}/u_s$ is given by 325 and the standard deviation by 150.
FIG. 2. The probability distribution of the peculiar velocity field at a scale corresponding to $t_{eq}$ in the E-dS model with non-ideal scaling. The solid curve is the normal distribution. The mean value of $v_{pec}/u_s$ is given by 460 and the standard deviation by 200.
FIG. 3. The probability distribution of the peculiar velocity field at a scale corresponding to $t_{eq}$ in the closed model. The solid curve is the normal distribution. The mean value of $v_{pec}/u_s$ is given by 123 and the standard deviation by 55.
FIG. 4. The probability distribution of the peculiar velocity field at a scale corresponding to $t_{eq}$ in the flat model with cosmological constant. The solid curve is the normal distribution. The mean value of $v_{pec}/u_s$ is given by 73 and the standard deviation by 33.
FIG. 5. The probability distribution of the peculiar velocity field at a scale corresponding to $t_{eq}$ in the open model. The solid curve is the normal distribution. The mean value of $v_{pec}/u_s$ is given by 22 and the standard deviation by 9.8.