Scoped Capabilities for Polymorphic Effects

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Type systems usually characterize the shape of values but not their free variables. However, many desirable safety properties could be guaranteed if one knew the free variables captured by values. We describe $\text{CC}_{\star}$, a calculus where such captured variables are succinctly represented in types, and show it can be used to safely implement effects and effect polymorphism via scoped capabilities. We discuss how the decision to track captured variables guides key aspects of the calculus, and show that $\text{CC}_{\star}$ admits simple and intuitive types for common data structures and their typical usage patterns. We demonstrate how these ideas can be used to guide the implementation of capture checking in a practical programming language.

1 INTRODUCTION

Effects are aspects of computation that go beyond describing shapes of values and that we still want to track in types. What exactly is modeled as an effect is a question of language or library design. Some possibilities are: reading or writing to mutable state outside a function, throwing an exception to signal abnormal termination of a function, I/O including file operations, network access, or user interaction, non-terminating computations, suspending a computation e.g., waiting for an event, or using a continuation for control operations.

Despite hundreds of published papers there is comparatively little adoption of static effect checking in programming languages. The few designs that are widely implemented (for instance Java’s checked exceptions or monadic effects in some functional languages) are often critiqued for being both too verbose and too rigid. The problem is not lack of expressiveness – systems have been proposed and implemented for many quite exotic kinds of effects. Rather, the problem is simple lack of usability and flexibility, with particular difficulties in describing polymorphism. This leads either to overly complex definitions, or to the requirement to duplicate large bodies of code.

Classical type-systematic approaches fail since effects are inherently transitive along the edges of the dynamic call-graph: A function’s effects include the effects of all the functions it calls, transitively. Traditional type and effect systems have no lightweight mechanism to describe this behavior. The standard approach is either manual specialization along specific effect classes, which means large-scale code duplication, or quantifiers on all definitions along possible call graph edges to account for the possibility that some call target has an effect, which means large amounts of boilerplate code. Arguably, it is this problem more than any other that has so far hindered wide scale application of effect systems.

A promising alternative that circumvents this problem is to model effects via capabilities [Brachthäuser et al. 2020a; Gordon 2020; Liu 2016; Marino and Millstein 2009; Osvald et al. 2016]. Capabilities exist in many forms, but we will restrict the meaning here to simple object capabilities represented as regular program variables. For instance, consider the following four formulations of a method in Scala\(^1\) which are all morally equivalent:

\(^1\)Scala 3.1 with language import saferExceptions enabled.
def f(): T throws E
def f(): throws[T, E]  
def f(): CanThrow[E] ?=> T  
def f()(using ct: CanThrow[E]): T

The first version looks like it describes an effect: Function $f$ returns a $T$, or it might throw exception $E$. The second version expresses this information in a type and the third version expands the type to a kind of function type. These transitions are enabled by the following definition of type `throws`:

```
infix type throws[T, E <: Exception] = CanThrow[E] ?=> T
```

The type `CanThrow[E] ?=> T` is a context function type. It represents functions from `CanThrow[E]` to `T` that are applied implicitly to arguments synthesized by the compiler. The definition is morally equivalent to the fourth version above, which employs a `using` clause to declare an implicit parameter $ct$. The mechanics of context function types and their expansion to implicit parameters was described by Odersky et al. [2018].

In the fourth version, the parameter $ct$ acts as an (object) capability. A capability of type `CanThrow[E]` is required to perform the effect of throwing exception $E$.

An important benefit of this switch from effects to capabilities is that it gives us polymorphism for free. For instance, consider the `map` function in class `List[A]`. If we wanted to take effects into account, it would look like this:

```
def map[B, E](f: A -> B eff E): List[B] eff E
```

Here, $A -> B eff E$ is hypothetical syntax for the type of functions from $A$ to $B$ that can have effect $E$. While looking reasonable in the small, this scheme quickly becomes unmanageable, if we consider that every higher-order function has to be expanded that way and, furthermore, that in an object-oriented language almost every method is a higher-order function [Cook 2009]. Indeed, many designers of programming languages with support for effect systems agree that programmers should ideally not be confronted with explicit effect quantifiers [Brachthäuser et al. 2020a; Leijen 2017; Lindley et al. 2017].

On the other hand, here is the type of `map` if we represent effects with capabilities.

```
def map(f: A => B): List[B]
```

Interestingly, this is exactly the same as the type of `map` in current Scala, which does not track effects! In fact, compared to effect systems, we now decompose the space of possible effects differently: `map` is classified as pure since it does not produce any effects in its own code, but when analyzing an application of `map` to some argument, the capabilities required by the argument are also capabilities required by the whole expression. In that sense, we get effect polymorphism for free.

The reason this works is that in an effects-as-capabilities discipline, the type $A => B$ represents the type of `impure` function values that can close over arbitrary effect capabilities. (Alongside, we also define a type of pure functions $A => B$ that are not allowed to close over capabilities.)

This seems almost too good to be true, and indeed there is a catch: It now becomes necessary to reason about capabilities captured in closures, ideally by representing such knowledge in a type system.

To see why, consider that effect capabilities are often scoped and therefore have a limited lifetime. For instance a `CanThrow[E]` capability would be generated by a `try` expression that catches $E$. It is valid only as long as the `try` is executing. Figure 1 shows an example of capabilities for checked exceptions, both as source syntax on the left and with compiler-generated implicit capability arguments on the right. The following slight variation of this program would throw an unhandled...
class TooLarge extends Exception

def f(x: Int): Int throws TooLarge =
    if x < limit then x * x
    else throw TooLarge()

val xs: List[Int]
try xs.map(x => f(x))
catch case TooLarge => Nil

Fig. 1. Exception handling: source (left) and compiler-generated code (right)

exception since the function f is now evaluated only when the iterator’s next method is called, which is after the try handling the exception has exited.

val it =
    try xs.iterator.map(f)
    catch case TooLarge => Iterator.empty
    it.next()

A question answered in this paper is how to rule out Iterator’s lazy map statically while still allowing List’s strict map. A large body of research exists that could address this problem by restricting reference access patterns. Relevant techniques include linear types [Wadler 1990], rank 2 quantification [Lauchbury and Sabry 1997], regions [Grossman et al. 2002; Tofte and Talpin 1997], uniqueness types [Barendsen and Smetsers 1996], ownership types [Clarke et al. 1998; Noble et al. 1998], and second class values [Osvald et al. 2016]. A possible issue with many of these approaches is their relatively high notational overhead, in particular when dealing with polymorphism.

The approach we pursue here is different. Instead of restricting certain access patterns a priori, we focus on describing what capabilities are possibly captured by values of a type. At its core there are the following two interlinked concepts:

- A capturing type is of the form \{c₁, \ldots, cₙ\} \text{T} where \text{T} is a type and \{c₁, \ldots, cₙ\} is a capture set of capabilities.
- A capability is a parameter or local variable that has as type a capturing type with non-empty capture set. We call such capabilities tracked variables.

Every capability gets its authority from some other, more sweeping capabilities which it captures. The most sweeping capability, from which ultimately all others are derived, is “*”, the universal capability.

As an example how capabilities are defined and used, consider a typical try-with-resources pattern:

```scala
def usingFile[T](name: String, op: (\{*\} OutputStream) => T): T =
    val f = new FileOutputStream(name)
    val result = op(f)
    f.close()
    result

val xs: List[Int] = ...
def good = usingFile("out", f => xs.foreach(x => f.write(x)))
def fail =
    val later = usingFile("out",
        f => (y: Int) => xs.foreach(x => f.write(x + y)))
```
The `usingFile` method runs a given operation `op` on a freshly created file, closes the file, and returns the operation’s result. The method enables an effect (writing to a file) and limits its validity (to until the file is closed). Function `good` invokes `usingFile` with an operation that writes each element of a given list `xs` to the file. By contrast, function `fail` represents an illegal usage: It invokes `usingFile` with an operation that returns a function that, when invoked, will write list elements to the file. The problem is that the writing happens in the application `later(1)` when the file has already been closed.

We can accept the first usage and reject the second by marking the output stream passed to `op` as a capability. This is done by prefixing its type with `{*}`. Our prototype implementation of capture checking will then reject the second usage with an error message.

This example used a capability parameter that was directly derived from `{*}`. But capabilities can also be derived from other non-universal capabilities. For instance:

```scala
def usingLogFile[T](f: {*} OutputStream, op: ({f} Logger) => T): T = op(Logger(f))
```

The `usingLogFile` method takes an output stream (which is a capability) and an operation, which gets passed a `Logger`. The `Logger` capability is derived from the output stream capability, as can be seen from its type `{f} Logger`.

This paper develops a capture calculus, $\text{CC}_{<:\text{□}}$, as a foundational type system that allows reasoning about scoped capabilities. By sketching a prototype language design based on this calculus, we argue that it is expressive enough to support a wide range of usage patterns with very low notational overhead. The paper makes the following specific contributions.

- We define a simple yet expressive type system for tracking captured capabilities in types. The calculus extends $\text{F}_{<:\text{□}}$ with capture sets of capabilities.
- We prove progress and preservation of types relative to a small-step evaluation semantics. We also prove a capture prediction lemma that states that capture sets in types over-approximate captured variables in runtime values.
- We report on a prototype language design in the form of an extension of Scala 3 and demonstrate its practicality in a number of examples that discuss both the embedding in fundamental datatypes and effect checking.

The presented design is at the same time simple in theory and concise and flexible in its practical application. We demonstrate that the following elements are essential for achieving good usability:

- Use reference-dependent typing, where a formal function parameter stands for the potential references captured by its argument [Brachthäuser et al. 2022; Odersky et al. 2021]. This avoids the need to introduce separate binders for capabilities or effects. Technically, this means that references (but not general terms) can form part of types as members of capture sets. A similar approach is taken in the path-dependent typing discipline of DOT [Amin et al. 2016; Rompf and Amin 2016] and by reachability types for alias checking [Bao et al. 2021].
- Employ a subtyping discipline that mirrors subsetting of capabilities and that allows capabilities to be refined or abstracted. Subtyping of capturing types relies on a new notion of `subcapturing` that encompasses both subsetting (smaller capability sets are more specific than larger ones) and derivation (a capability singleton set is more specific than the capture set of the capability’s type). Both dimensions are essential for a flexible modelling of capability domains.
• Limit propagation of capabilities in instances of generic types where they cannot be accessed directly. This is achieved by boxing types when they enter a generic context and unboxing on every use site [Brachthäuser et al. 2022].

The version of CC\textsubscript{\textless;\texttrade}\ presented here is similar to a system that was originally proposed to make exception checking safe [Odersky et al. 2021]. Their system conjectured progress and preservation properties, but did not have proofs. Compared to their system, the version in this paper presents a fully worked-out meta theory with proofs of type soundness as well as a semantic characterization of capabilities. There are also some minor differences in the operational semantics, which were necessary to make a progress theorem go through. Finally, we present several use cases outside of exception handling, demonstrating the broad applicability of the calculus.

Whereas many of our motivating examples describe applications in effect checking, the formal treatment presented here does not mention effects. In fact, the effect domains are intentionally kept open since they are orthogonal to the aim of the paper. Effects could be exceptions, file operations or region allocations, but also algebraic effects, IO, or any sort of monadic effects. To express more advanced control effects, one usually needs to add continuations to the operational semantics, or use an implicit translation to the continuation monad. In short, capabilities can delimit what effects can be performed at any point in the program but they by themselves don’t perform an effect [Brachthäuser et al. 2020a; Gordon 2020; Liu 2016; Marino and Millstein 2009; Osvald et al. 2016]. For that, one needs a library or a runtime system that would be added as an extension of CC\textsubscript{\textless;\texttrade}. Since CC\textsubscript{\textless;\texttrade} is intended to work with all such effect extensions, we refrain from adding a specific extension to its operational semantics.

The rest of this paper is organized as follows. Section 2 explains and motivates the core elements of our calculus. Section 3 presents CC\textsubscript{\textless;\texttrade}. Section 4 lays out its meta-theory. Section 5 illustrates the expressiveness of typing disciplines based on the calculus in examples. Section 6 discusses related work and Section 7 concludes.

2 INFORMAL DISCUSSION

This section motivates and discusses some of the key aspects of capture checking. All examples are written in an experimental language extension of Scala 3 and were compiled with our prototype implementation of a capture checker [Odersky et al. 2022].

2.1 Capability Hierarchy

We have seen in the intro that every capability except ∗ is created from some other capabilities which it retains in the capture set of its type. Here is an example that demonstrates this principle:

```scala
class FileSystem

class Logger(fs: {∗} FileSystem):
  def log(s: String): Unit = ... // Write to a log file, using `fs`

def test(fs: {∗} FileSystem): {fs} LazyList[Int] =
  val l: {fs} Logger = new Logger(fs)
  l.log("hello_world!")
  val xs: {l} LazyList[Int] =
  LazyList.from(1)
    .map { i =>
      l.log(s"computing_elem_#{i}"")
      i * i
```

```
Here, the test method takes a FileSystem as a parameter. fs is a capability since its type has a non-empty capture set. The capability is passed to the Logger constructor and retained as a field in class Logger. Hence, the local variable l has type {fs} Logger; it is a Logger which retains the fs capability.

The second variable defined in test is xs, a lazy list that is obtained from LazyList.from(1) by logging and mapping consecutive numbers. Since the list is lazy, it needs to retain the reference to the logger l for its computations. Hence, the type of the list is {1} LazyList[Int]. On the other hand, since xs only logs but does not do other file operations, it retains the fs capability only indirectly. That’s why fs does not show up in the capture set of xs.

Capturing types come with a subtype relation where types with “smaller” capture sets are subtypes of types with larger sets (the subcapturing relation is defined in more detail below). If a type T does not have a capture set, it is called pure, and is a subtype of any capturing type that adds a capture set to T.

2.2 Function Types

The function type A => B stands for a function that can capture arbitrary capabilities. We call such functions impure. By contrast, the new single arrow function type A -> B stands for a function that cannot capture any capabilities, or otherwise said, is pure. One can add a capture set in front of an otherwise pure function. For instance, {c, d} A -> B would be a function that can capture capabilities c and d, but no others.

The impure function type A => B is treated as an alias for {*} A -> B. That is, impure functions are functions that can capture anything.

Function types and captures both associate to the right, so {c} A -> {d} B -> C is the same as {c} (A -> {d} (B -> C)).

Contrast with ({{c} A} -> {{d} B} -> C which is a curried pure function over argument types that can capture c and d, respectively.

Note. The distinctions between pure vs impure function types do not apply to methods (which are defined using def). In fact, since methods are not values in Scala, they never capture anything directly. References to capabilities in a method are instead counted in the capture set of the enclosing object.

2.3 Capture Checking of Closures

If a closure refers to capabilities in its body, it captures these capabilities in its type. For instance, consider:

```scala
def test(fs: FileSystem): {fs} String -> Unit =
  (x: String) => Logger(fs).log(x)
```

Here, the body of test is a lambda that refers to the capability fs, which means that fs is retained in the lambda. Consequently, the type of the lambda is {fs} String -> Unit.

Note. On the term level, function values are always written with => (or ?=> for context functions). There is no syntactic distinction for pure vs impure function values. The distinction is only made in their types.

A closure also captures all capabilities that are captured by the functions it calls. For instance, in

```scala
def test(fs: FileSystem) =
```
\begin{verbatim}
def f() = (x: String) => Logger(fs).log(x)
def g() = f()
g
the result of test has type \{fs\} String -> Unit even though function g itself does not refer to fs.
\end{verbatim}

2.4 Subtyping and Subcapturing
Capturing influences subtyping. As usual we write $T_1 <: T_2$ to express that the type $T_1$ is a subtype of the type $T_2$, or equivalently, that $T_1$ conforms to $T_2$. An analogous \textit{subcapturing} relation applies to capture sets. If $C_1$ and $C_2$ are capture sets, we write $C_1 <: C_2$ to express that $C_1$ is \textit{covered by} $C_2$, or, swapping the operands, that $C_2$ \textit{covers} $C_1$.

Subtyping extends as follows to capturing types:
\begin{itemize}
    \item Pure types are subtypes of capturing types. That is, $T <: CT$, for any type $T$, capturing set $C$.
    \item For capturing types, smaller capture sets produce subtypes: $C_1 T_1 <: C_2 T_2$ if $C_1 <: C_2$ and $T_1 <: T_2$.
\end{itemize}

A subcapturing relation $C_1 <: C_2$ holds if $C_2$ accounts for every element $c$ in $C_1$. This means one of the following two conditions must be true:
\begin{itemize}
    \item $c \in C_2$,
    \item $c$'s type has capturing set $C$ and $C_2$ accounts for every element of $C$ (that is, $C <: C_2$).
\end{itemize}

Example. Given
\begin{verbatim}
fs: {*} FileSystem
cr: {*} CanThrow[Exception]
l : {fs} Logger
we have
{l} <: {fs} <: {*}
{fs} <: {fs, cr} <: {*}
{cr} <: {fs, cr} <: {*}
\end{verbatim}
The set consisting of the root capability {*} covers every other capture set. This is a consequence of the fact that, ultimately, every capability is created from *.

2.5 Capture Tunneling
Next, we discuss how type-polymorphism interacts with reasoning about capture. To this end, consider the following simple definition of a \texttt{Pair} class:
\begin{verbatim}
class Pair[{+A, +B}](x: A, y: B):
def fst: A = x
def snd: B = y
\end{verbatim}

What happens if we pass arguments to the constructor of \texttt{Pair} that capture capabilities?
\begin{verbatim}
def x: {cr} Int -> String
def y: {fs} Logger
def p = Pair(x, y)
\end{verbatim}

Here the arguments $x$ and $y$ close over different capabilities $ct$ and $fs$, which are assumed to be in scope. So what should the type of $p$ be? Maybe surprisingly, it will be typed as:
\begin{verbatim}
def p: {} Pair[{{cr} Int -> String, {fs} Logger} = Pair(x, y)
\end{verbatim}
That is, the outer capture set is empty and does neither mention \(ct\) nor \(fs\), even though the value \(Pair(x, \ y)\) does capture them. So why don’t they show up in its type at the outside?

While assigning \(p\) the capture set \(\{ct, \ fs\}\) would be sound, types would quickly grow inaccurate and unbearably verbose. To remedy this, \(CC_{<\circlearrowleft}\) performs capture tunneling. Once a type variable is instantiated to a capturing type, the capture is not propagated beyond this point. On the other hand, if the type variable is instantiated again on access, the capture information “pops out” again.

Even though \(p\) is technically untracked because its capture set is empty, writing \(p.f\) would record a reference to the captured capability \(ct\). So if this access was put in a closure, the capability would again form part of the outer capture set. E.g.,

\[
() \Rightarrow p.f : \{ct\} () \Rightarrow \{ct\} \text{Int} \Rightarrow \text{String}
\]

In other words, references to capabilities “tunnel through” generic instantiations—from creation to access; they do not affect the capture set of the enclosing generic data constructor applications. As mentioned above, this principle plays an important part in making capture checking concise and practical. To illustrate, let us take a look at the following example:

\[
def \text{mapFirst}[A,B,C](p: \text{Pair}[A,B], f: A \Rightarrow C): \text{Pair}[C,B] = \\
\text{Pair}(f(p.x), \ p.y)
\]

Relying on capture tunneling, neither the types of the parameters to \text{mapFirst}, nor its result type need to be annotated with capture sets. Intuitively, the capture sets do not matter for \text{mapFirst}, since parametricity forbids it from inspecting the actual values inside the pairs. If not for capture tunneling, we would need to annotate \(p\) as \(\{\ast\} \text{Pair}[A,B]\), since both \(A\) and \(B\) and through them, \(p\) can capture arbitrary capabilities. In turn, this means that for the same reason, without tunneling we would also have \(\{\ast\} \text{Pair}[C,B]\) as the result type. This is of course unacceptably inaccurate.

Section 3 describes the foundational theory on which capture checking is based. It makes tunneling explicit through so-called box and unbox operations. Boxing hides a capture set and unboxing recovers it. The capture checker inserts virtual box and unbox operations based on actual and expected types similar to the way the type checker inserts implicit conversions. Boxing and unboxing has no runtime effect, so the insertion of these operations is only simulated, but not kept in the generated code.

### 2.6 Escape Checking

Following the principle of object capabilities, the universal capability \(\ast\) should conceptually only be available as a parameter to the main program. Indeed, if it was available everywhere, capability checking would be undermined since one could mint new capabilities at will. In line with this reasoning, some capture sets are restricted and must not contain the universal capability.\(^2\) There is a connection to tunneling here. The capture set of a type has to be present in the environment when a type is instantiated from a type variable. But \(\ast\) is not itself available as a global entity in the environment. Hence, this should result in an error.

Using this principle, we can show why the introductory example in Section 1 reported an error. To recall, function usingFile was declared like this:

\[
def \text{usingFile}[T](name: \text{String}, op: (\{\ast\} \text{FileOutputStream}) \Rightarrow T): T = ...
\]

The capture checker rejects the illegal definition of later

\(^2\)This follows since type variables range over pure types, so \(\ast\) must appear under a box. But rule \((\text{Box})\) in Figure 3 restricts variables in boxed capture sets to be declared in the enclosing environment, which does not hold for \(\ast\).
\begin{verbatim}
val later = usingFile("out",
f => (y: Int) => xs.foreach(x => f.write(x + y)))
\end{verbatim}

with the following error message

```
| val later = usingFile("out", f => (y: Int) => xs.foreach(x => f.write(x + y)))
| ^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^ ^^^^^^^^^^^^^^^^ 
|The expression's type {*} Int -> Unit is not allowed to capture the root capability "*" 
|This usually means that a capability persists longer than its allowed lifetime.
```

This error message was produced by the following reasoning steps:

- Parameter \( f \) has type \({*} \) FileOutputStream, which makes it a capability.
- Therefore, the type of the expression
  \[(y: \text{Int}) \Rightarrow \text{xs.foreach}(x \Rightarrow \text{f.write}(x + y))\]
is \({f} \) Int -> Unit.
- Consequently, we assign the whole closure passed to usingFile the dependent function type \( \text{(f: \{*\} FileOutputStream) \Rightarrow \{f\} () \Rightarrow \text{Unit}.} \)
- The expected type of the closure is a simple, parametric, impure function type \( \{\{\} \} \) FileOutputStream \Rightarrow \{\} \Int \Rightarrow \text{Unit.} \)
- We cannot instantiate \( \mathcal{T} \) with \( \{f\} () \Rightarrow \text{Unit} \) since the expected function type is non-dependent. The smallest supertype that matches the expected type is thus \( \{\{\} \} \) FileOutputStream \Rightarrow \{\*\} \Int \Rightarrow \text{Unit.} \)
- Hence, the type variable \( \mathcal{T} \) is instantiated to \(*\) () \Rightarrow Unit, which is not allowed and causes the error.

### 2.7 Escape Checking of Mutable Variables

Another way one could try to undermine capture checking would be to assign a closure with a local capability to a global variable. For instance like this:\footnote{Stable variables are not covered by the formal treatment of \( \text{CC}_{<\Sigma} \). We include the discussion anyway to show that escape checking can be generalized to scope extrusions separate from result values.}

```
var loophole: {*} () -> Unit = () => ()
usingFile("tryEscape", f =>
  loophole = () => f.write(0)
}
loophole()
```

We prevent such scope extrusions by imposing the restriction that mutable variables cannot have types with universal capture sets.

One also needs to prevent returning or assigning a closure with a local capability in an argument of a parametric type. For instance, here is a slightly more refined attack:

```
val sneaky = usingLogFile { f => Pair(() => f.write(0), 1) }
sneaky.fst()
```

At the point where the \emph{Pair} is created, the capture set of the first argument is \{f\}, which is OK. But at the point of use, it is \{*\}: since \( f \) is no longer in scope we need to widen the type to a supertype that does not mention it (\emph{c.f.} the explanation of avoidance in Section 3.3). This causes an error, again, as the universal capability is not permitted to be in the unboxed form of the return type (\emph{c.f.} the precondition of \emph{(Unbox)} in Figure 3).
3 THE $C_{<\Box}$ CALCULUS

The syntax of $C_{<\Box}$ is given in Figure 2. In short, it describes a dependently typed variant of System $F_{<\Box}$ in monadic normal form (MNF) with capturing types and boxes.

**Dependently typed:** Types may refer to term variables in their capture sets, which introduces a simple form of (variable-)dependent typing. As a consequence, a function’s result type may now refer to the parameter in its capture set. To be able to express this, the general form of a function type $\forall(x : U)T$ explicitly names the parameter $x$. We retain the non-dependent syntax $U \rightarrow T$ for function types as an abbreviation if the parameter is not mentioned in the result type $T$.

Dependent typing is attractive since it means that we can refer to object capabilities directly in types, instead of having to go through auxiliary region or effect variables. We thus avoid clutter related to quantification of such auxiliary variables.

**Monadic normal form:** The term structure of $C_{<\Box}$ requires operands of applications to be variables. This does not constitute a loss of expressiveness, since a general application $t_1 \, t_2$ can be expressed as $\text{let } x_1 = t_1 \text{ in } \text{let } x_2 = t_2 \text{ in } x_1 \, x_2$. This syntactic convention has advantages for variable-dependent typing. In particular, typing function application in such a calculus requires substituting actual arguments for formal parameters. If arguments are restricted to be variables, these substitutions are just variable-variable renamings, which keep the general structure of a type. If arguments were arbitrary terms, such a substitution would in general map a type to something that was not syntactically a type. Monadic normal form [Hatcliff and Danvy 1994] is a slight generalization of the better-known A-normal form (ANF) [Sabry and Felleisen 1993] to allow arbitrary nesting of let expressions. We use a here a variant of MNF where applications are over variables instead of values.

A similar restriction to MNF was employed in DOT [Amin et al. 2016], the foundation of Scala’s object model, for the same reasons. The restriction is invisible to source programs, which can still be in direct style. For instance, the Scala compiler selectively translates a source expression in direct style to MNF if a non-variable argument is passed to a dependent function. Type checking then takes place on the translated version.

**Capturing Types:** The types in $C_{<\Box}$ are stratified as shape types $S$ and regular types $T$. Regular types can be shape types or capturing types $\{x_1, \ldots, x_n\} \, S$. Shape types are made up from the
usual type constructors in $F_{\prec}$ plus boxes. We freely use shape types in place of types, assuming the equivalence $\{\} S \equiv S$.

**Boxes:** Type variables $X$ can be bounded or instantiated only with shape types, not with regular types. To make up for this restriction, a regular type $T$ can be encapsulated in a shape type by prefixing it with a box operator $\Box T$. On the term level, $\Box x$ injects a variable into a boxed type. A variable of boxed type is unboxed using the syntax $C \rightsquigarrow \! x$ where $C$ is the capture set of the underlying type of $x$. We have seen in Section 2 that boxing and unboxing allow a kind of capability tunneling by omitting capabilities when values of parametric types are constructed and charging these capabilities instead at use sites.

**System $F_{\prec}:$** We base $CC_{\prec}$ on a standard type system that supports the two principal forms of polymorphism, subtyping and universal.

Subtyping comes naturally with capabilities in capture sets. First, a type capturing fewer capabilities is naturally a subtype of a type capturing more capabilities, and pure types are naturally subtypes of capturing types. Second, if capability $x$ is derived from capability $y$, then a type capturing $x$ can be seen as a subtype of the same type but capturing $y$.

Universal polymorphism poses specific challenges when capture sets are introduced which are addressed in $CC_{\prec}$ by the stratification into shape types and regular types and the box/unbox operations that map between them.

Note that the only form of term dependencies in $CC_{\prec}$ relate to capture sets in types. If we omit capture sets and boxes, the calculus is equivalent to standard $F_{\prec}$, despite the different syntax. We highlight in the figures the essential additions wrt $F_{\prec}$ with a grey background.

$CC_{\prec}$ is intentionally meant to be a small and canonical core calculus that does not cover higher-level features such as records, modules, objects, or classes. While these features are certainly important, their specific details are also somewhat more varied and arbitrary than the core that’s covered. Many different systems can be built on $CC_{\prec}$, extending it with various constructs to organize code and data on higher-levels.

**Capture Sets.** Capture sets $C$ are finite sets of variables of the form $\{x_1, \ldots, x_n\}$. We understand $\ast$ to be a special variable that can appear in capture sets, but cannot be bound in $\Gamma$. We write $C \setminus x$ as a shorthand for $C \setminus \{x\}$.

Capture sets of closures are determined using a function $cv$ over terms.

**Definition (Captured Variables).** The captured variables $cv(t)$ of a term $t$ are given as follows.

\[
\begin{align*}
\text{cv}(\lambda (x : T)t) & = \text{cv}(t) \setminus x \\
\text{cv}(\lambda [X \ll S]t) & = \text{cv}(t) \\
\text{cv}(x) & = \{x\} \\
\text{cv}(\text{let } x = v \text{ in } t) & = \text{cv}(t) \quad \textbf{if } x \notin \text{cv}(t) \\
\text{cv}(\text{let } x = s \text{ in } t) & = \text{cv}(s) \cup \text{cv}(t) \setminus x \\
\text{cv}(x \ y) & = \{x, y\} \\
\text{cv}(x[S]) & = \{x\} \\
\text{cv}(\Box x) & = \{\} \\
\text{cv}(C \rightsquigarrow x) & = C \cup \{x\}
\end{align*}
\]

The definitions of captured and free variables of a term are very similar, with the following three differences:

1. Boxing a term $\Box x$ obscures $x$ as a captured variable.
2. Dually, unboxing a term $C \rightsquigarrow x$ counts the variables in $C$ as captured.
(3) In an evaluated let binding \( \text{let } x = v \text{ in } t \), the captured variables of \( v \) are counted only if \( x \) is a captured variable of \( t \).

The first two rules encapsulate the essence of box-unbox pairs: Boxing a term obscures its captured variable and makes it necessary to unbox the term before its value can be accessed; unboxing a term presents variables that were obscured when boxing.

Figure 3 presents typing and evaluation rules for \( \text{CC}_{<:□} \). There are four main sections on subcapturing, subtyping, typing, and evaluation. These are explained in the following.

### 3.1 Subcapturing

Subcapturing establishes a preorder relation on capture sets that gets propagated to types. Smaller capture sets with respect to subcapturing lead to smaller types with respect to subtyping.

The relation is defined by three rules. The first two rules (sc-set) and (sc-elem) establish that subsets imply subcaptures. That is, smaller capture sets subcapture larger ones. The last rule (sc-var) is the most interesting since it reflects an essential property of object capabilities. It states that a variable \( x \) of capturing type \( C \ S \) generates a capture set \( \{x\} \) that subcaptures the capabilities \( C \) with which the variable was declared. In a sense, (sc-var) states a monotonicity property: a capability refines the capabilities from which it is created. In particular, capabilities cannot be created from nothing. Every capability needs to be derived from some more sweeping capabilities which it captures in its type.

The rule also validates our definition of capabilities as variables with non-empty capture sets in their types. Indeed, if a variable is defined as \( x : \{\} \ S \), then by (sc-var) we have \( \{x\} <: \{\} \). This means that the variable can be disregarded in the formation of \( cv \), for instance. Even if \( x \) occurs in a term, a capture set with \( x \) in it is equivalent (with respect to mutual subcapturing) to a capture set without. Hence, \( x \) can safely be dropped without affecting subtyping or typing.

Rules (sc-set) and (sc-elem) mean that if set \( C \) is a subset of \( C' \), we also have \( C <: C' \). But the reverse is not true. For instance, with (sc-var) we can derive the following relationship assuming lambda-bound variables \( x \) and \( y \):

\[
  x : \{\ast\}, y : \{x\}, \Gamma \vdash \{y\} <: \{x\}
\]

Intuitively this makes sense, as \( y \) can capture no more than \( x \). However, we cannot derive \( \{x\} <: \{y\} \), since arguments passed for \( y \) may in fact capture less than \( x \), e.g. they could be pure.

While there are no subcapturing rules for top or bottom capture sets, we can still establish:

**Proposition 3.1.** If \( C \) is well-formed in \( \Gamma \), then \( \Gamma \vdash \{\} <: C <: \{\ast\} \).

A proof is enclosed in the appendix.

**Proposition 3.2.** The subcapturing relation \( \Gamma \vdash _{-} <: _{-} \) is a preorder.

**Proof.** We can show that transitivity and reflexivity are admissible. \( \square \)

### 3.2 Subtyping

The subtyping rules of \( \text{CC}_{<:□} \) are very similar to those of System \( \text{F}_{<:} \), with the only significant addition being the rules for capturing and boxed types. Note that as \( S \equiv \{\} \ S \), both transitivity and reflexivity apply to shape types as well. (capt) allows comparing types that have capture sets, where smaller capture sets lead to smaller types. (boxed) propagates subtyping relations between types to their boxed versions.
Subcapturing

\[\Gamma \vdash C <: C\]
\[\Gamma \vdash \{x\} <: C\]  \hspace{1cm} (sc-elem)
\[\Gamma \vdash \{x_1\} <: C \ldots \Gamma \vdash \{x_n\} <: C\]  \hspace{1cm} (sc-set)
\[\Gamma \vdash \{x_1, \ldots, x_n\} <: C\]  \hspace{1cm} (sc-var)
\[x : C' S \in \Gamma \quad \Gamma \vdash C' <: C\]

\[\Gamma \vdash \{x\} <: C\]

Subtyping

\[\Gamma \vdash T <: T\]  \hspace{1cm} (refl)
\[\Gamma \vdash T_1 <: T_2 \quad \Gamma \vdash T_2 <: T_3\]  \hspace{1cm} (trans)
\[\Gamma \vdash X <: S \in \Gamma\]  \hspace{1cm} (tv)
\[\Gamma \vdash S <: \top\]  \hspace{1cm} (top)
\[\Gamma \vdash U_2 <: U_1 \quad \Gamma, x : U_2 + T_1 <: T_2\]  \hspace{1cm} (fun)
\[\Gamma \vdash \forall(x : U_1)T_1 \vdash \forall(x : U_2)T_2\]
\[\Gamma \vdash S_2 <: S_1 \quad \Gamma, x : S_2 + T_1 <: T_2\]  \hspace{1cm} (tfun)
\[\Gamma \vdash \forall[X <: S_1]T_1 \vdash \forall[X <: S_2]T_2\]
\[\Gamma \vdash C_1 <: C_2 \quad \Gamma \vdash S_1 <: S_2\]  \hspace{1cm} (capt)
\[\Gamma \vdash C_1 S_1 <: C_2 S_2\]
\[\Gamma \vdash \square T_1 <: \square T_2\]  \hspace{1cm} (boxed)

Typing

\[\Gamma \vdash t : T\]  \hspace{1cm} (var)
\[\Gamma \vdash t : T \quad \Gamma \vdash T <: U\]  \hspace{1cm} (sub)
\[\Gamma \vdash U \text{ wf}\]  \hspace{1cm} (wf)
\[\Gamma \vdash \lambda(x : U)t : \text{cv}(t)\backslash x \forall(x : U)T\]  \hspace{1cm} (abs)
\[\Gamma \vdash \lambda[X <: S]t : \text{cv}(t) \forall[X <: S]T\]  \hspace{1cm} (tabs)
\[\Gamma \vdash x : C \forall(z : U)T\]  \hspace{1cm} (app)
\[\Gamma \vdash x [z := y]T\]  \hspace{1cm} (app)
\[\Gamma \vdash x : C S \quad C \subseteq \text{dom}(\Gamma)\]  \hspace{1cm} (box)
\[\Gamma \vdash \square x : \square C S\]  \hspace{1cm} (box)
\[\Gamma \vdash \text{let } x = s \text{ in } t : U\]  \hspace{1cm} (let)
\[\Gamma \vdash s : T \quad \Gamma, x : T + t : U\]  \hspace{1cm} (let)
\[x \notin \text{fv}(U)\]

Evaluation

\[\sigma[e[x y]] \rightarrow \sigma[e[z := y]t]\]  \hspace{1cm} (apply)
\[\sigma[e[X := S]t] \quad \text{if } \sigma(x) = \lambda[Z : T]t\]  \hspace{1cm} (apply)
\[\sigma[e[y]] \quad \text{if } \sigma(x) = \square y\]  \hspace{1cm} (rename)
\[\sigma[e[let x = y in t]] \rightarrow \sigma[e[x := y]t]\]  \hspace{1cm} (open)
\[\sigma[e[let x = v in t]] \rightarrow \sigma[let x = v in e[t]]\]  \hspace{1cm} (lift)
\[\text{where} \quad \text{Store context } \sigma ::= [] | \text{let } x = v \text{ in } \sigma\]
\[\text{Eval context } e ::= [] | \text{let } x = e \text{ in } t\]

Fig. 3. Typing and Evaluation Rules of System CC<:□
3.3 Typing

Typing rules are again close to System $F_{<>}$, with differences to account for capture sets.

Rule (VAR) is the basis for capability refinements. If $x$ is declared with type $C S$, then the type of $x$ has $\{x\}$ as its capture set instead of $C$. The capturing set $\{x\}$ is more specific than $C$, in the subcapturing sense. Therefore, we can recover the capture set $C$ through subsumption.

Rules (ABS) and (tabs) augment the abstraction’s type with a capture set that contains the captured variables of the term. Through subsumption and rule (sc-var), untracked variables can immediately be removed from this set.

The (app) rule substitutes references to the function parameter with the argument to the function. This is possible since arguments are guaranteed to be variables. The function’s capture set is disregarded, reflecting the principle that the function closure is consumed by the application. Rule (tapp) is analogous.

Aside: A more conventional version of (tapp) would be

$$
\Gamma \vdash x : C \forall [X <: S']T \quad \Gamma \vdash S <: S' \\
\Gamma \vdash x[S] : [X := S]T
$$

(TAPP′)

That formulation is equivalent to (tapp) in the sense that either rule is derivable from the other, using subsumption and contravariance of type bounds.

Rules (box) and (unbox) map between boxed and unboxed types. They require all members of the capture set under the box to be bound in the environment $\Gamma$. Consequently, while one can create a boxed type with $\{*\}$ as its capture set through subsumption, one cannot unbox values of this type. This property is fundamental for ensuring scoping of capabilities.

Avoidance. As is usual in dependent type systems, Rule (let) has as a side condition that the bound variable $x$ does not appear free in the result type $U$. This so called avoidance property is usually attained through subsumption. For instance consider an enclosing capability $e : T_1$ and the term

$$\begin{align*}
\text{let } x &= \lambda (y : T_2). c \text{ in } \lambda (z : \{x\} T_3). z
\end{align*}
$$

The most specific type of $x$ is $\{c\}$ ($T_2 \rightarrow T_1$) and the most specific type of the body of the let is $\forall (z : \{x\} T_3). \{z\} T_3$. We need to find a supertype of the latter type that does not mention $x$. It turns out the most specific such type is $T_3 \rightarrow \{c\} T_3$, so that is a possible type of the let, and it should be the inferred type.

In general there is always a most specific avoiding type for a (let):

**Proposition 3.3.** Consider a term $\text{let } x = s \text{ in } t$ in an environment $\Gamma$ such that $\Gamma \vdash s : T_1$ and $\Gamma, x : T_1 \vdash t : T_2$. Then there exists a minimal (wrt $<:$) type $T_3$ such that $T_2 <: T_3$ and $x \notin \text{fv}(T_3)$

**Proof.** Without loss of generality, assume that $\Gamma \vdash s : C_S U$ is the most specific typing for $s$ in $\Gamma$ and $\Gamma, x : C_S U \vdash t : T$ is the most specific typing for $t$ in the context of the body of the let, namely $\Gamma, x : C_S U$. Let $T'$ be constructed from $T$ by replacing $x$ with $C_S$ in covariant capture set positions and by replacing $x$ with the empty set in contravariant capture set positions. Then for every type $V$ avoiding $x$ such that $\Gamma, x : C_S S \vdash T <: V, \Gamma \vdash T' <: V$. This is shown by a straightforward structural induction, which we give in the appendix in Section A.51.
3.5 Evaluation

Evaluation is defined by a small-step reduction relation. This relation is quite different from usual reduction via term substitution. Substituting values for variables would break the monadic normal form of a program. Instead, we reduce the right hand sides of let-bound variables in place and lookup the bindings in the environment of a redex.

Every redex is embedded in an outer store context and an inner evaluation context. These represent orthogonal decompositions of let bindings. An evaluation context e always puts the focus [] on the right-hand side \( t_1 \) of a let binding \( \text{let} x = t_1 \text{ in } t_2 \). By contrast, a store context \( \sigma \) puts the focus on the following term \( t_2 \) and requires that \( t_1 \) is evaluated.

The first three rules — (APPLY), (TAPPLY), (OPEN) — rewrite simple redexes: applications, type applications and unboxings. Each of these rules looks up a variable in the enclosing store and proceeds based on the value that was found.

The last two rules are administrative in nature. They both deal with evaluated \textbf{lets} in redex position. If the right hand side of the \textbf{let} is a variable, the \textbf{let} gets expanded out by renaming the bound variable using (RENAME). If it is a value, the \textbf{let} gets lifted out into the store context using (LIFT).

**Proposition 3.4.** Evaluation is deterministic. For any term \( t_1 \) there is at most one term \( t_2 \) such that \( t_1 \rightarrow t_2 \).

**Proof.** By a straightforward inspection of the reduction rules and definitions of contexts.

4 METATHEORY

We prove that \( \text{CC}_{\triangleleft} \) is sound through the standard progress and preservation theorems. The proofs for all the lemmas and theorems stated in this section are provided in the appendix.

In order to prove both progress and preservation, we need technical lemmas that allow manipulation of typing judgments for terms under store and evaluation contexts. To state these lemmas, we first need to define what it means for typing and store contexts to match, which we do in Figure 4.

\[
\begin{align*}
\Gamma, x : T &\vdash \sigma \sim \Delta \\
\Gamma &\vdash v : T \\
&x \not\in \text{fv}(T) \\
\Gamma &\vdash \text{let } x = v \text{ in } \sigma \sim x : T, \Delta
\end{align*}
\]

Fig. 4. Matching environment \( \Gamma \vdash \sigma \sim \Delta \)

Having \( \Gamma \vdash \sigma \sim \Delta \) lets us know that \( \sigma \) is well-typed in \( \Gamma \) if we use \( \Delta \) as the types of the bindings. Using this definition, we can state the following four lemmas, which also illustrate how the store and evaluation contexts interact with typing:

**Definition 4.1 (Evaluation context typing \( (\Gamma \vdash e : U \Rightarrow T) \)).** We say that \( \Gamma \) types an evaluation context \( e \) as \( U \Rightarrow T \) if \( \Gamma, x : U \vdash e[x] : T \).

**Lemma 4.2 (Evaluation context typing inversion).**
\( \Gamma \vdash e[s] : T \) implies that for some \( U \) we have \( \Gamma \vdash e : U \Rightarrow T \) and \( \Gamma \vdash s : U \).
With this definition, we can now state the progress theorem:

**Theorem 4.6 (Preservation).** If we have \( \Gamma \vdash \sigma \sim \Delta \) and \( \Gamma, \Delta \vdash t : T \), then \( \sigma[ t ] \rightarrow \sigma[ t' ] \) implies that \( \Gamma, \Delta \vdash t' : T \).

To neatly state the progress theorem, we first need to define canonical store-plug splits, which simply formally express that the store context wrapping a term is as large as possible:

**Definition 4.7 (Canonical store-plug split).** We say that a term of the form \( \Gamma \vdash e[ t ] : T \) is a canonical split (of the entire term into store context \( \sigma \) and the plug \( t \)) if \( t \) is not of the form \( \text{let } x : T = v \text{ in } t' \).

With this definition, we can now state the progress theorem:

**Theorem 4.8 (Progress).** If \( \Gamma \vdash \sigma[ e[ t ] ] : T \) and \( \sigma[ e[ t ] ] \) is a canonical store-plug split, then either \( e[ t ] = a \), or there exists \( \sigma[ t' ] \) such that \( \sigma[ e[ t ] ] \rightarrow \sigma[ t' ] \).

The lemmas needed to prove progress and preservation are for the most part standard. As our calculus is term-dependent, we also need to account for term substitution affecting both environments and types, not only terms. For instance, the lemma stating that term substitution preserves typing is expressed as follows:

**Lemma 4.9 (Term substitution preserves typing).**

If \( \Gamma, x : U, \Delta \vdash t : T \) and \( \Gamma \vdash y : U \), then \( \Gamma, [ x \mapsto y ] \Delta \vdash [ x \mapsto y ] t : [ x \mapsto y ] T \).

In this statement, we can also see that we only consider substituting one term variable for another, due to MNF. Using MNF affects other parts of the proof as well – in addition to typical canonical forms lemmas, we also need to show that looking up the value bound to a variable in a store preserves the types we can assign to the variable:

**Lemma 4.10 (Variable lookup inversion).**

If we have both \( \Gamma \vdash \sigma \sim \Delta \) and \( x : C R \in \Gamma, \Delta \), then \( \sigma(x) = v \) implies that \( \Gamma, \Delta \vdash v : C R \).

**Capture sets and captured variables.** Our typing rules use \( cv \) to calculate the capture set that should be assigned to terms. With that in mind, we can ask the question: what is the exact relationship between captured variables and capture sets we use to type the terms?

Because of subcapturing, this relationship is not as obvious as it might seem. For fully evaluated terms (of the form \( \sigma[a] \)), their captured variables are the most precise capture set they can be assigned. The following lemma states this formally:

**Lemma 4.11 (Capture prediction for answers).** If \( \Gamma \vdash \sigma[a] : C S \), then \( \Gamma \vdash cv(\sigma[a]) \ll C \).

If we start with an unreduced term \( \sigma[t] \), then the situation becomes more complex. It can mention and use capabilities that will not be reflected in the capture set at all – for instance, if \( t = x y \), the capture set of \( x \) is irrelevant to the type assigned to \( t \) by \( \text{app} \). However, if \( \sigma[t] \) reduces fully
to a term of the form $\sigma[\sigma'[a]]$, the captured variables of $\sigma'[a]$ will correspond to capture sets we could assign to $t$.

In other words, the capture sets we assign to unevaluated terms under a store context predict variables that will be captured by the answer those terms reduce to. Formally we can express this as follows:

**Lemma 4.12 (Capture prediction for terms).**

Let $\vdash \sigma \leadsto \Delta$ and $\vdash t : C R$. Then $\sigma[t] \rightarrow^* \sigma[\sigma'[a]]$ implies that $\Delta \vdash cv(\sigma'[a]) <: C$.

### 4.1 Correctness of boxing

Boxing capabilities allows temporarily hiding them from capture sets. Given that we can do so, one may be inclined to ask: what is the correctness criterion for our typing rules for box and unbox forms?

Intuitively, we should not be able to “smuggle in” a capability by using boxes: all the capabilities of a term should be somehow accounted for. A basic criterion our typing rules should satisfy is that a term that boxes and immediately unboxes a capability should have a cv at least as wide than that of the capability, i.e. a cv that accounts for the capability. Formally we can state this property as follows:

**Proposition 4.13.** Let $\vdash \sigma \vdash \Delta$ and let $t = \text{let } y = \Box x \text{ in } C \leadsto y$ such that for some $e, T$, we have $\Delta \vdash \sigma[ e[ t ] ] : T$. Then we also have:

$$\Delta \vdash \{ x \} <: C = cv(t)$$

It is straightforward to verify that this holds based on the progress theorem and by induction on the typing derivation. Ideally, our correctness criterion would speak about arbitrary terms, by characterising their captured variables alongside reductio paths. Since this clearly doesn’t make sense for closed terms, we will need one additional definition: a way of introducing well-behaved capabilities. A **platform** $\Psi$ is an outer portion of the store which formally represents primitive capabilities. It is defined as follows:

$$\Psi ::= \text{let } x = v \text{ in } \Psi \quad \text{if } \text{fv}(v) = \{ \}$$

Since a platform $\Psi$ is meant to introduce primitive capabilities, it doesn’t make sense to type them with any other capture set than $\{\ast\}$. The following definition captures this idea:

**Definition (Well-typed program).** We say that a term $\Psi[t]$ is a well-typed program if for some typing context $\Delta$ such that $\vdash \Psi \leadsto \Delta$ we have $\Delta \vdash t : T$, and for all $x \in \text{dom } \Psi$ we have $x : \{\ast\} S \in \Delta$.

The capabilities introduced by the platform of a well-typed program are well-behaved in the sense that their values cannot reference any other variable, and their capture sets are $\{\ast\}$. Given $cv$ and platform, we can state our desired lemma as follows:

**Lemma 4.14 (Program authority preservation).** Let $\Psi[t]$ be a well-typed program such that $\Psi[t] \rightarrow \Psi'[t']$. Then $cv(t')$ is a subset of $cv(t)$.

The name of this lemma hints at the property we want to demonstrate next, which offers another perspective at what it means for boxes to be correctly typed. This property will capture the connection between the $cv$ of a term and the capabilities it may use during evaluation. We make intuitive usage of this connection when we say, for instance, that a function $f$ typed as $\{\}$ Unit $\rightarrow$ Unit cannot perform any effects when called. Formally, it cannot do so because when we reduce a term of the form $fx$, based on (abs) we know it will be replaced by some term such that $cv(t) = \{\}$. 

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Here is where we make an intuitive jump: since \( cv(t) \) is empty and we do not pass anything to \( t \), it should not use any capabilities during evaluation.

To state this formally, first we need to define which capabilities are used during reduction. All formal capabilities are abstractions, so a natural definition is to say that a capability \( x \) was used if it was applied, i.e. if a term of the form \( x \mathcal{Y} \) was reduced. To speak about this, we will define \( \text{used}(t \rightarrow s) \) as function from reduction derivations to sets of variables, as follows:

\[
\text{used}(t_1 \rightarrow t_2 \rightarrow \cdots \rightarrow t_n) = \text{used}(t_1 \rightarrow t_2) \cup \text{used}(t_2 \rightarrow \cdots \rightarrow t_n)
\]

\[
\text{used}(\sigma[e[x \mathcal{Y}]] \rightarrow \sigma[t]) = \{x\}
\]

\[
\text{used}(\sigma[e[x \mathcal{T}]] \rightarrow \sigma[t]) = \{x\}
\]

\[
\text{used}(t_1 \rightarrow t_2) = \{} \quad \text{(otherwise)}
\]

The last case applies to rules (OPEN), (RENAME), (LIFT).

Now, let us consider how exactly the \( cv \) of a term is connected to the capabilities it will use during evaluation. If we disregard boxes, then clearly a term \( t \) can only use a capability \( x \) if it is free in \( t \), which also means that it is in \( cv(t) \). However, boxing a capability hides it from the \( cv \). If we disregard unbox forms, this is again not a problem: a capability under a box cannot be used unless the box is opened first. Now, if we consider unbox forms \( C \leftarrow x \), we encounter the essence of the problem: \( cv(C \leftarrow x) = C \). On the term level, we have no guarantee that the capture sets of \( C \) and \( x \) are connected. The (box) and (unbox) typing rules ensure that if a term gains access to a capability by opening a box, then the “key” used to open the box (the \( C \) in \( C \leftarrow x \)) must account for the boxed capability. Thanks to them, we can say that \( cv(t) \) is the authority of the term \( t \): during reduction, \( t \) can only use capabilities contained in \( cv(t) \). We formally state this property as follows:

**Lemma 4.15 (Used Capability Prediction).** Let \( \Psi[t] \) be a well-typed program that reduces in zero or more steps to \( \Psi[t'] \). Then the primitive capabilities used during the reduction are a subset of the authority of \( t \).

### 5 EXAMPLES

We have implemented a type checker for \( \text{CC}<\square \) as an extension of the Scala 3 compiler to enable experimentation with larger code examples. Notably, our extension infers which types must be boxed, and automatically generates boxing and unboxing operations when values are passed to and returned from instantiated generic datatypes, so none of these technical details appear in the actual user-written Scala code. In this section, we present examples that demonstrate the usability of the language. In Section 5.1, we remain close to the core calculus by encoding lists using only functions; here, we still show the boxed types and boxing and unboxing operations that the compiler infers in gray, though they are not in the source code. In Sections 5.2 and 5.3, we use additional Scala features in larger examples, to implement stack allocation and polymorphic data structures. For these examples, we present the source code without cluttering it with the boxing operations inferred by the compiler.

#### 5.1 Church-Encoded Lists

Using the Scala prototype implementation of \( \text{CC}<\square \), the Böhm-Berarducci encoding [Böhm and Berarducci 1985] of a linked list data structure can be implemented and typed as follows. Here, a list is represented by its right fold function:

\[
\text{type Op}[T <: \square (* Any, C <: \square (* Any] = (v: T) => (s: C) => C
\]

type List[T <: □ {∗} Any] =
[C <: □ {∗} Any] => (op: Op[T, C]) => {op} (s: C) => C

def nil[T <: □ {∗} Any]: List[T] =
[C <: □ {∗} Any] => (op: Op[T, C]) => (s: C) => s

def cons[T <: □ {∗} Any](hd: T, tl: List[T]): List[T] =
[C <: □ {∗} Any] => (op: Op[T, C]) => (s: C) => op(hd)(tl(op)(s))

A list inherently captures any capabilities that may be captured by its elements. Therefore, naively,
one may expect the capture set of the list to include the capture set of the type T of its elements.
However, boxing and unboxing enables us to elide the capture set of the elements from the capture
set of the containing list. When constructing a list using cons, the elements must be boxed:
cons(□ 1, cons(□ 2, cons(□ 3, nil)))

A map function over the list can be implemented and typed as follows:

def map[A <: □ {∗} Any, B <: □ {∗} Any](xs: List[A])(f: {∗} A -> B): List[B] =
{} ◦ xs(List[B])(() => (hd: A) => (tl: List[B]) => cons(f(hd), tl))(nil)

The mapped function f may capture any capabilities, as documented by the capture set {∗} in its
type. However, this does not affect the type of map or its result type List[B], since the mapping is
strict, so the resulting list does not capture any capabilities captured by f. If a value returned by
the function f were to capture capabilities, this would be reflected in its type, the concrete type
substituted for the type variable B, and would therefore be reflected in the concrete instantiation
of the result type List[B] of map.

5.2 Stack Allocation

Automatic memory management using a garbage collector is convenient and prevents many er-
rors, but it can impose significant performance overheads in programs that need to allocate large
numbers of short-lived objects. If we can bound the lifetimes of some objects to coincide with a
static scope, it is much cheaper to allocate those objects on a stack as follows:\footnote{For simplicity, this example is neither thread nor exception safe.}
class Pooled

val stack = mutable.ArrayBuffer[Pooled]()
var nextFree = 0

def withFreshPooled[T](op: Pooled => T): T =
if nextFree >= stack.size then stack.append(new Pooled)
val pooled = stack(nextFree)
nextFree = nextFree + 1
val ret = op(pooled)
nextFree = nextFree - 1
ret

The withFreshPooled method calls the provided function op with a freshly stack-allocated
instance of class Pooled. It works as follows. The stack maintains a pool of already allocated
instances of Pooled. The nextFree variable records the offset of the first element of stack that
is available to reuse; elements before it are in use. The withFreshPooled method first checks
whether the stack has any available instances; if not, it adds one to the stack. Then it increments nextFree to mark the first available instance as used, calls op with the instance, and decrements nextFree to mark the instance as freed. In the fast path, allocating and freeing an instance of Pooled is reduced to just incrementing and decrementing the integer nextFree.

However, this mechanism fails if the instance of Pooled outlives the execution of op, if op captures it in its result. Then the captured instance may still be accessed while at the same time also being reused by later executions of op. For example, the following invocation of withFreshPooled returns a closure that accesses the Pooled instance when it is invoked on the second line, after the Pooled instance has been freed:

```scala
val pooledClosure = withFreshPooled(pooled => () => pooled.toString)
pooledClosure()
```

Using capture sets, we can prevent such captures and ensure the safety of stack allocation just by marking the Pooled instance as tracked:

```scala
def withFreshPooled[T](op: (pooled: {*} Pooled) => T): T =
```

Now the pooled instance can be captured only in values whose capture set accounts for `{pooled}`. The type variable T cannot be instantiated with such a capture set because pooled is not in scope outside of withFreshPooled, so only `{*}` would account for `{pooled}`, but we disallowed instantiating a type variable with `{*}`. With this declaration of withFreshPooled, the above pooledClosure example is correctly rejected, while the following safe example is allowed:

```scala
withFreshPooled(pooled => pooled.toString)
```

### 5.3 Collections

In the following examples we show that a typing discipline based on $\text{CC}_{<\tau}$ can be lightweight enough to make capture checking of operations on standard collection types practical. This is important, since such operations are the backbone of many programs. All examples compile with our current capture checking prototype [Odersky et al. 2022].

We contrast the APIs of common operations on Scala’s standard collection types List and Iterator when capture sets are taken into account. Both APIs are expressed as Scala 3 extension methods [Odersky and Martres 2020] over their first parameter. Here is the List API:

```scala
extension [A](xs: List[A])
  def apply(n: Int): A
  def foldLeft[B](z: B)(op: (B, A) => B): B
  def foldRight[B](z: B)(op: (A, B) => B): B
  def foreach(f: A => Unit): Unit
  def iterator: Iterator[A]
  def drop(n: Int): List[A]
  def map[B](f: A => B): List[B]
  def flatMap[B](f: A => {*} IterableOnce[B]): List[B]
  def ++[B >: A](xs: {*} IterableOnce[B]): List[B]
```

Notably, these methods have almost exactly the same signatures as their versions in the standard Scala collections library. The only differences concern the arguments to `flatMap` and `++` which now admit an `IterableOnce` argument with an arbitrary capture set. Of course, we could have left out the `{*}` but this would have needlessly restricted the argument to non-capturing collections.

Contrast this with some of the same methods for iterators:
extension [A](it: Iterator[A])
  def apply(n: Int): A
  def foldLeft[B](z: B)(op: (B, A) => B): B
  def foldRight[B](z: B)(op: (A, B) => B): B
  def foreach(f: A => Unit): Unit
  def drop(n: Int): Iterator[A]
  def map[B](f: A => B): Iterator[B]
  def flatMap[B](f: A => IterableOnce[B]): Iterator[B]
  def ++[B >: A](xs: IterableOnce[B]): Iterator[B]

Here, methods apply, foldLeft, foldRight, foreach again have the same signatures as in the current Scala standard library. But the remaining four operations need additional capture annotations. Method drop on iterators returns the given iterator it after skipping n elements. Consequently, its result has {it} as capture set. Methods map and flatMap lazily map elements of the current iterator as the result is traversed. Consequently they retain both it and f in their result capture set. Method ++ concatenates two iterators and therefore retains both of them in its result capture set.

The examples attest to the practicality of capture checking. Method signatures are generally concise. Higher-order methods over strict collections by and large keep the same types as before. Capture annotations are only needed for capabilities that are retained in closures and are executed on demand later. Where such annotations are needed they match the developer’s intuitive understanding of reference patterns and signal information that looks relevant in this context.

6 RELATED WORK

The results presented in this paper did not emerge in a vacuum and many of the underlying ideas appeared individually elsewhere in similar or different form. We follow the structure of the informal presentation in Section 2 and organize the discussion of related work according to the key ideas behind \( \text{CC}_{<c> \Box} \).

Effects as Capabilities. Establishing effect safety by moderating access to effects via term-level capabilities is not a new idea [Marino and Millstein 2009]. It has been proposed as a strategy to retrofit existing languages with means to reason about effect safety [Choudhury and Krishnaswami 2020; Liu 2016; Osvald et al. 2016]. Recently, it also has been applied as the core principle behind a new programming language featuring effect handlers [Brachthäuser et al. 2020a]. Similar to the above prior work, we propose to use term-level capabilities to restrict access to effect operations and other scoped resources with a limited lifetime. Representing effects as capabilities results in a good economy of concepts: existing language features, like term-level binders can be reused, programmers are not confronted with a completely new concept of effects or regions.

Making Capture Explicit. Having a term-level representation of scoped capabilities introduces the challenge to restrict use of such capabilities to the scope in which they are still live. To address this issue, effect systems have been introduced [Biernacki et al. 2020; Brachthäuser et al. 2020b; Zhang and Myers 2019] but those can result in overly verbose and difficult to understand types [Brachthäuser et al. 2020a]. A third approach, which we follow in this paper, is to make capture explicit in the type of functions.
Hannan [1998] proposes a type-based escape analysis with the goal to facilitate stack allocation. The analysis tracks variable reference using a type-and-effect system and annotates every function type with the set of free variables it captures. The authors leave the treatment of effect polymorphism to future work. In a similar spirit, Scherer and Hoffmann [2013] present Open Closure Types to facilitate reasoning about data flow properties such as non-interference. They present an extension of the simply typed lambda calculus that enhances function types $\Gamma_0(\tau) \rightarrow \tau$ with the lexical environment $\Gamma_0$ that was originally used to type the closure.

Brachthäuser et al. [2022] show System C, which mediates between first- and second-class values with boxes. In their system, scoped capabilities are second-class values. Normally, second-class values cannot be returned from any scope, but in System C they can be boxed and returned from some scopes. The type of a boxed second-class value tracks which scoped capabilities it has captured and accordingly, from which scopes it cannot be returned. System C tracks second-class values with a coeffect-like environment and uses an effect-like discipline for tracking captured capabilities, which can in specific cases be more precise than $cv$. In comparison, $CC_{<:\square}$ does not depend on a notion of second-class values and deeply integrates capture sets with subtyping.

Recently, Bao et al. [2021] have proposed to qualify types with reachability sets. Their reachability types allow reasoning about non-interference, scoping and uniqueness by tracking for each reference what other references it may alias or (indirectly) point to. Their system formalizes subtyping but not universal polymorphism. However, it relates reachability sets along a different dimension than $CC_{<:\square}$. Whereas in $CC_{<:\square}$ a subtyping relationship is established between a capability $c$ and the capabilities in the type of $c$, reachability types assume a subtyping relationship between a variable $x$ and the variable owning the scope where $x$ is defined. Reachability types track detailed points-to and aliasing information in a setting with mutable variables, while $CC_{<:\square}$ is a more foundational calculus for tracking references and capabilities that can be used as a guide for an implementation in a complete programming language. It would be interesting to explore how reachability types can be expressed in $CC_{<:\square}$.

**Capture Polymorphism.** Combining effect tracking with higher-order functions immediately gives rise to effect polymorphism, which has been a long-studied problem.

Similar to the usual (parametric) type polymorphism, the seminal work by Lucassen and Gifford [1988] on type and effect systems featured (parametric) effect polymorphism by adding language constructs for explicit region abstraction and application. Similarly, work on region based memory management [Tofte and Talpin 1997] supports region polymorphism by explicit region abstraction and application. Recently, languages with support for algebraic effects and handlers, such as Koka [Leijen 2017] and Frank [Lindley et al. 2017], feature explicit, parametric effect polymorphism.

It has been observed multiple times, for instance by Osvald et al. [2016] and Brachthäuser et al. [2020a], that parametric effect polymorphism can become verbose and results in complicated types and confusing error messages. Languages sometimes attempt to hide the complexity – they "simplify the types more and leave out 'obvious' polymorphism" [Leijen 2017]. However, this solution is not satisfying since the full types resurface in error messages. In contrast, we support polymorphism by reusing existing term-level binders and support simplifying types by means of subtyping and subcapturing.

The problem of how to prevent capabilities from escaping in closures is also addressed by second-class values that can only be passed as arguments but not be returned in results or stored in mutable fields. Siek et al. [2012] enforce second-class function arguments using a classical polymorphic effect discipline whereas Osvald et al. [2016] and Brachthäuser et al. [2020a] present a specialized type discipline for this task. Second-class values cannot be returned or closed-over by first-class functions. On the other hand, second-class functions can freely close over capabilities, since they...
are second-class themselves. This gives rise to a convenient and light-weight form of contextual effect polymorphism [Brachthäuser et al. 2020a]. While this approach allows for effect polymorphism with a simple type system, it is also restrictive because it also forbids local returns and retentions of capabilities; a problem solved by adding boxing and unboxing [Brachthäuser et al. 2022].

Foundations of Boxing. Contextual modal type theory (CMTT) [Nanevski et al. 2008] builds on intuitionistic modal logic. In intuitionistic modal logic, the graded propositional constructor $\Psi$ (pronounced box) witnesses that $A$ can be proven only using true propositions in $\Psi$. Judgements in CMTT have two contexts: $\Gamma$, roughly corresponding to $\text{CC}_{<\Box}$ bindings with $\{\ast\}$ as their capture set, and a modal context $\Delta$ roughly corresponding to bindings with concrete capture sets. Bindings in the modal context are necessarily boxed and annotated with a modality $x :: A[\Psi] \in \Delta$. Just like our definition of captured variables in $\text{CC}_{<\Box}$, the definition of free variables by Nanevski et al. [2008] assigns the empty set to a boxed term (that is, $f_\sigma(\text{box}(\Psi, M)) = \{\}$). Similar to our unboxing construct, using a variable bound in the modal context requires that the current context satisfies the modality $\Psi$, mediated by a substitution $\sigma$. Different to CMTT, $\text{CC}_{<\Box}$ does not introduce a separate modal context. It also does not annotate modalities on binders, instead these are kept in the types. Also different to CMTT, in $\text{CC}_{<\Box}$ unboxing is annotated with a capture set and not a substitution.

Comonadic type systems were introduced to support reasoning about purity in existing, impure languages [Choudhury and Krishnaswami 2020]. Very similar to the box modality of CMTT, a type constructor ‘Safe’ witnesses the fact that its values are constructed without using any impure capabilities. The type system presented by Choudhury and Krishnaswami [2020] only supports a binary distinction between pure values and impure values, however, the authors comment that it might be possible to generalize their system to graded modalities.

In the present paper, we use boxing as a practical tool, necessary to obtain concise types when combining capture tracking with parametric type polymorphism.

Coeffect Systems. Coeffect systems also attach additional information to bindings in the environment, leading to a typing judgment of the form $\Gamma @ C \vdash e : \tau$. Such systems can be seen as similar in spirit to $\text{CC}_{<\Box}$, where additional information is available about each variable in the environment through the capture set of its type. Petricek et al. [2014] show a general coeffect framework that can be instantiated to track various concepts such as bounded reuse of variables, implicit parameters and data access. This framework is based on simply typed lambda calculus and its function types are always coeffect-monomorphic. In contrast, $\text{CC}_{<\Box}$ is based on System $F_\omega$ (thus supporting type polymorphism and subtyping) and supports capture-polymorphic functions.

Object Capabilities. The (object-)capability model of programming [Boyland et al. 2001; Crary et al. 1999; Miller 2006], controls security critical operations by requiring access to a capability. Such a capability can be seen as the constructive proof that the holder is entitled to perform the critical operation. Reasoning about which operations a module can perform is reduced to reasoning about which references to capabilities a module holds.

The Wyvern language [Melicher et al. 2017] implements this model by distinguishing between stateful resource modules and pure modules. Access to resource modules is restricted and only possible through capabilities. Determining the authority granted by a module amounts to manually inspecting its type signature and all of the type signatures of its transitive imports. To support this analysis, Melicher [2020] extends the language with a fine-grained effect system that tracks access of capabilities in the type of methods.
In CC_{<:□}, one can reason about authority and capabilities simply by inspecting capture sets of types. If we model modules via function abstraction, the function’s capture set directly reflects its authority. Importantly, CC_{<:□} does not include an effect system and thus tracks mention rather than use.

7 CONCLUSION

We introduced a new type system CC_{<:□} to track captured references of values. Tracked references are restricted to capabilities, where capabilities are references bootstrapped from other capabilities, starting with the universal capability. Implementing this simple principle then naturally suggests a chain of design decisions:

1. Because capabilities are variables, every function must have its type annotated with its free capability variables.
2. To manage the scoping of those free variables, function types must be dependently-typed.
3. To prevent non-variable terms from occurring in types, the programming language is formulated in monadic normal form.
4. Because of type dependency, the let-bindings of MNF have to satisfy the avoidance property, to prevent out-of-scope variables from occurring in types.
5. To make avoidance possible, the language needs a rich notion of subtyping on the capture sets.
6. Because the capture sets represent object capabilities, the subcapture relation cannot just be the subset relation on sets of variables – it also has to take into account the types of the variables, since the variables may be bound to values which themselves capture capabilities.
7. To keep the size of the capture sets from ballooning out of control, the paper introduces a box connective with box and unbox rules to control when free variables are counted as visible.

We showed that the resulting system can be used as the basis for lightweight polymorphic effect checking, without the need for effect quantifiers. We also identified three key principles that keep notational overhead for capture tracking low:

- Variables are tracked only if their types have non-empty capture sets. In practice the majority of variables are untracked and thus do not need to be mentioned at all.
- Subcapturing, subtyping and subsumption mean that more detailed capture sets can be subsumed by coarser ones.
- Boxed types stop propagation of capture information in enclosing types which avoids repetition in capture annotations to a large degree.

Our experience so far indicates that the presented calculus is simple and expressive enough to be used as a basis for more advanced effect and resource checking systems and their practical implementations.

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A PROOFS

A.1 Proof devices

We extend type well-formedness to environments as $\vdash \Gamma \text{wf}$ in the typical way – empty environment is well-formed, and $\Gamma, x : T$ or $\Gamma, X <: T$ is well-formed if both $\vdash \Gamma \text{wf}$ and $\Gamma \vdash T \text{wf}$.

We implicitly assume all variables in an environment are unique, for instance whenever we write $\Gamma, x : T$ we assume that $x \notin \text{dom}(\Gamma)$.

In order to make proving Preservation (Theorem A.38) possible, we need to relate the derivation of a type for a term of the form $\sigma[ t ]$ to the derivation of the type of the plug term $t$ inside the store $\sigma$. To do so, we need to define what environment matches the store.

Matching environment

\[
\frac{\Gamma, x : T \vdash \sigma \sim \Delta \quad \Gamma \vdash v : T \quad x \notin \text{fv}(T)}{\Gamma \vdash \text{let } x = v \text{ in } \sigma \sim x : T, \Delta}
\]

Definition A.1 (Evaluation context typing ( $\Gamma \vdash e : U \Rightarrow T$)). We say that $\Gamma$ types an evaluation context $e$ as $U \Rightarrow T$ if $\Gamma, x : U \vdash e[ x ] : T$.

Fact A.1. If $\sigma[ t ]$ is a well-typed term in $\Gamma$, then there exists a $\Delta$ matching $\sigma$ (i.e. such that $\Gamma \vdash \sigma \sim \Delta$), finding it is decidable, and $\Gamma, \Delta$ is well-formed.

Fact A.2. The analogous holds for $e[ t ]$.

A.2 Properties of Evaluation Contexts and Stores

In the proof, we will be using $D$ as another metavariable for capture sets.

We will also be writing $\text{cv}(T)$ to extract the capture set of a type (which is defined simply as $\text{cv}(C R) = C$).

We assume that all environments in the premises of our lemmas and theorems are well-formed.

Lemma A.2 (Inversion of typing under an evaluation context). If $\Gamma \vdash e[ u ] : T$, then $u$ is well-typed in $\Gamma$.

Proof. Proceed by induction on $e$. If $e$ is empty then the result directly holds, so consider when $s = \text{let } x = e'[ u ] \text{ in } s$. By inverting the typing derivation we know that $\Gamma \vdash e'[ u ] : T'$, and hence by induction we conclude that $\Gamma \vdash s : U$ for some type $U$.

Lemma A.3 (Evaluation context typing inversion).

$\Gamma \vdash e[ s ] : T$ implies that for some $U$ we have $\Gamma \vdash e : U \Rightarrow T$ and $\Gamma \vdash s : U$.

Proof. First, we show that for some $U$, we have $\Gamma, x : U \vdash \sigma[ e ] : T$. This can be done by a straightforward induction on the derivation of typing for $e[ s ]$.

By definition, this then gives us $\Gamma \vdash e : U \Rightarrow T$.

Lemma A.4 (Evaluation context reification).

If both $\Gamma \vdash e : U \Rightarrow T$ and $\Gamma \vdash s : U$, then $\Gamma \vdash e[ s ] : T$.

Proof. A corollary of Lemma.

Lemma A.5 (Replacement of term under an evaluation context).

If $\Gamma \vdash e[ s ] : U \Rightarrow T$ and $\Gamma \vdash s' : U$, then $\Gamma \vdash e[ s' ] : T$.

Proof. A corollary of .
Lemma A.6 (Store context reification). \( \text{If } \Gamma, \Delta \vdash t : T \text{ and } \Gamma \vdash \sigma \sim \Delta \text{ then } \Gamma \vdash \sigma[t] : T. \)

Proof. By induction on \( \sigma \).

Case \( \sigma = [ ] \). Immediate.

Case \( \sigma = \sigma'[\text{let } x = v \text{ in } \Gamma] \). Then \( \Delta = \Delta', x : U \) for some \( U \). Since \( x \notin \text{fv}(T) \) as \( \Gamma \vdash T \text{ wf} \), by (let), we have that \( \Gamma, \Delta' \vdash \text{let } x = v \text{ in } t \) and hence by the induction hypothesis we have, for some \( U \), that \( \Gamma, x : U \vdash \sigma'[t] : T \). The result follows directly.

The above lemma immediately gives us:

Corollary A.7 (Replacement of term under a store context). \( \text{If } \Gamma \vdash \sigma[t] : T \text{ and } \Gamma \vdash \sigma \sim \Delta \text{ and } \Delta, \Delta' \vdash t : T \), then for all \( t' \) such that \( \Gamma, \Delta \vdash t' : T \) we have \( \Gamma \vdash \sigma[t'] : T \).

A.3 Properties of Subcapturing

Lemma A.8 (Top capture set). \( \text{Let } \Gamma \vdash C \text{ wf. Then } \Gamma \vdash C <: \{ \} \).

Proof. By induction on \( \Gamma \). If \( \Gamma \) is empty, then \( C \) is either empty or \( * \in C \), so we can conclude by (sc-set) or (sc-elem) correspondingly. Otherwise, \( \Gamma = \Gamma', x : DS \) and since \( \Gamma \) is well-formed, \( \Gamma' \vdash D \text{ wf} \). By (sc-set), we can conclude if for all \( y \in C \) we have \( \Gamma \vdash \{ y \} <: \{ * \} \). If \( y = x \), by IH we derive \( \Gamma' \vdash D <: \{ * \} \), which we then weaken to \( \Gamma \) and conclude by (sc-var). If \( y \neq x \), then \( \Gamma' \vdash \{ y \} \text{ wf} \), so by IH we derive \( \Gamma' \vdash \{ y \} <: \{ * \} \) and conclude by weakening.

Corollary A.9 (Effectively top capture set). \( \text{Let } \Gamma \vdash C, D \text{ wf such that } * \in D. \text{ Then we can derive } \Gamma \vdash C <: D \).

Proof. We can derive \( \Gamma \vdash C <: \{ * \} \) by Lemma A.8 and then we can conclude by Lemma A.12 and (sc-elem).

Lemma A.10 (Star subcapture inversion). \( \text{Let } \Gamma \vdash C <: D. \text{ If } * \in C, \text{ then } * \in D \).

Proof. By induction on subcapturing. Case (sc-elem) immediate, case (sc-set) by repeated IH, case (sc-var) contradictory.

Lemma A.11 (Subcapturing distributivity). \( \text{If } \Gamma \vdash C <: D \text{ and } x \in C, \text{ then } \Gamma \vdash \{ x \} <: D \).

Proof. By inspection of the last subcapturing rule used to derive \( C <: D \). All cases are immediate. If the last rule was (sc-set), we have our goal as premise. Otherwise, we have that \( C = \{ x \} \), and the result follows directly.

Lemma A.12 (Subcapturing transitivity). \( \text{If } \Gamma \vdash C_1 <: C_2 \text{ and } \Gamma \vdash C_2 <: C_3 \text{ then } \Gamma \vdash C_1 <: C_3 \).

Proof. By induction on the first derivation.

Case (sc-elem). \( C_1 = \{ x \} \) and \( x \in C_2 \), so by Lemma A.11 \( \Gamma \vdash \{ x \} <: C_3 \).

Case (sc-var). Then \( C_1 = \{ x \} \) and \( x : C_1 R \in \Gamma \) and \( \Gamma \vdash C_4 <: C_2 \). By IH \( \Gamma \vdash C_4 <: C_3 \) and we can conclude by (sc-var).

Case (sc-set). By repeated IH and (sc-set).

Lemma A.13 (Subcapturing reflexivity). \( \text{For all } C, \text{ if } \Gamma \vdash C \text{ wf, then } \Gamma \vdash C <: C \).

Proof. By (sc-set) and (sc-elem).

Lemma A.14 (Subtyping implies subcapturing). \( \text{If } \Gamma \vdash C_1 R_1 <: C_2 R_2, \text{ then } \Gamma \vdash C_1 <: C_2 \).

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PROOF. By induction on the subtyping derivation. If (capt), immediate. If (trans), by IH and subcapturing transitivity Lemma A.12. Otherwise, \( C_1 = C_2 = \{\} \) and we can conclude by (sc-set).

A.3.1 Subtype inversion. While it may be unusual to show these lemmas so early, we in fact need one of them to show narrowing.

FACT A.3. Both subtyping and subcapturing are transitive.

PROOF. Subtyping is intrinsically transitive through (trans), while subcapturing admits transitivity as per Lemma A.12.

FACT A.4. Both subtyping and subcapturing are reflexive.

PROOF. Again, this is an intrinsic property of subtyping (by (refl)) and an admissible property of subcapturing per Lemma A.13.

LEMMA A.15 (Subtyping inversion: type variable). If \( \Gamma \vdash U <: C X \), then \( U \) is of the form \( C' X' \) and we have \( \Gamma \vdash C' <: C \) and \( \Gamma \vdash X' <: X \).

PROOF. By induction on the subtyping derivation.

Case (tvar), (refl). Follow from reflexivity (A.4).

Case (capt). Then we have \( U = C' R \) and \( \Gamma \vdash C' <: C \) and \( \Gamma \vdash R <: X \).

This relationship is equivalent to \( \Gamma \vdash \{\} R \vdash \{\} X \), on which we invoke the IH.

By IH we have \( \{\} R = \{\} Y \) and we can conclude with \( U = C' Y \).

Case (trans). Then we have \( \Gamma \vdash U <: U \) and \( \Gamma \vdash U <: C X \). We proceed by using the IH twice and conclude by transitivity (A.3).

Other rules are impossible.

LEMMA A.16 (Subtyping inversion: capturing type). If \( \Gamma \vdash U <: C R \), then \( U \) is of the form \( C' R' \) such that \( \Gamma \vdash C' <: C \) and \( \Gamma \vdash R' <: R \).

PROOF. We take note of the fact that subtyping and subcapturing are both transitive (A.3) and reflexive (A.4). The lemma then follows from a straightforward induction on the subtyping judgment.

LEMMA A.17 (Subtyping inversion: function type). If \( \Gamma \vdash U <: C \forall(x : T_1)T_2 \), then \( U \) either is of the form \( C' X \) and we have \( \Gamma \vdash C' <: C \) and \( \Gamma \vdash X <: \forall(x : T_1)T_2 \), or \( U \) is of the form \( C' \forall(x : U_1)U_2 \) and we have \( \Gamma \vdash C' <: C \) and \( \Gamma \vdash T_1 <: U_1 \) and \( \Gamma, x : T_1 \vdash U_2 <: T_2 \).

PROOF. By induction on the subtyping derivation.

Case (tvar). Immediate.

Case (fun), (refl). Follow from reflexivity (A.4).

Case (capt). Then we have \( \Gamma \vdash C' <: C \) and \( \Gamma \vdash R <: \forall(x : T_1)T_2 \).

This relationship is equivalent to \( \Gamma \vdash \{\} R \vdash \{\} \forall(x : T_1)T_2 \), on which we invoke the IH.

By IH \( \{\} R \) might have two forms. If \( \{\} R = \{\} X \), then we can conclude with \( U = C' X \).

Otherwise we have \( \{\} R = \{\} \forall(x : U_1)U_2 \) and \( \Gamma \vdash T_1 <: U_1 \) and \( \Gamma, x : T_1 \vdash U_2 <: T_2 \). Then, \( U = C' \forall(x : U_1)U_2 \) lets us conclude.

Case (trans). Then we have \( \Gamma \vdash U <: U' \) and \( \Gamma \vdash U <: C \forall(x : T_1)T_2 \). By IH \( U \) may have one of two forms. If \( U = C' X \), we proceed with Lemma A.15 and conclude by transitivity (A.3).
Otherwise $U = C' \forall(x : U_1)U_2$ and we use the IH again on $\Gamma \vdash U' <: C' \forall(x : U_1)U_2$. If $U = C'' X$, we again can conclude by (A.3). Otherwise if $U = C'' \forall(x : U_1)U_2$, the IH only gives us $\Gamma, x : U_1 \vdash U_2 <: U_2$, which we need to narrow to $\Gamma, x : T_1$ before we can similarly conclude by transitivity (A.3).

Other rules are not possible. □

\textbf{Lemma A.18 (Subtyping inversion: type function type).} If $\Gamma \vdash U <: C \forall[X : T_1]T_2$, then $U$ either is of the form $C' X$ and we have $\Gamma \vdash C' <: C$ and $\Gamma \vdash X <: \forall[X : T_1]T_2$, or $U$ is of the form $C' \forall[X : U_1]U_2$ and we have $\Gamma \vdash C' <: C$ and $\Gamma \vdash T_1 <: U_1$ and $\Gamma, X <: T_1 \vdash U_2 <: T_2$.

\textbf{Proof.} Analogous to the proof of Lemma A.17. □

\textbf{Lemma A.19 (Subtyping inversion: boxed type).} If $\Gamma \vdash U <: C \Box T$, then $U$ either is of the form $C' X$ and we have $\Gamma \vdash C' <: C$ and $\Gamma \vdash X <: \Box T$, or $U$ is of the form $C' \Box U'$ and we have $\Gamma \vdash C' <: C$ and $\Gamma \vdash U <: T$.

\textbf{Proof.} Analogous to the proof of Lemma A.17. □

\textbf{A.3.2 Permutation, weakening, narrowing.}

\textbf{Lemma A.20 (Permutation).} Permutating the bindings in the environment up to preserving environment well-formedness also preserves type well-formedness, subcapturing, subtyping and typing.

Let $\Gamma$ and $\Delta$ be the original and permuted context, respectively. Then:

(1) If $\Gamma \vdash T \text{ wf}$, then $\Delta \vdash T \text{ wf}$.

(2) If $\Gamma \vdash C_1 <: C_2$, then $\Delta \vdash C_1 <: C_2$.

(3) If $\Gamma \vdash U <: T$, then $\Delta \vdash U <: T$.

(4) If $\Gamma \vdash t : T$, then $\Delta \vdash t : T$.

\textbf{Proof.} As usual, order of the bindings in the environment is not used in any rule. □

\textbf{Lemma A.21 (Weakening).} Adding a binding to the environment such that the resulting environment is well-formed preserves type well-formedness, subcapturing, subtyping and typing.

Let $\Gamma$ and $\Delta$ be the original and extended context, respectively. Then:

(1) If $\Gamma \vdash T \text{ wf}$, then $\Delta \vdash T \text{ wf}$.

(2) If $\Gamma \vdash C_1 <: C_2$, then $\Delta \vdash C_1 <: C_2$.

(3) If $\Gamma \vdash U <: T$, then $\Delta \vdash U <: T$.

(4) If $\Gamma \vdash t : T$, then $\Delta \vdash t : T$.

\textbf{Proof.} As usual, the rules only check if a variable is bound in the environment and all versions of the lemma are provable by straightforward induction. For rules which extend the environment, such as (ABS), we need permutation. All cases are analogous, so we will illustrate only one.

\textit{Case (ABS).} WLOG we assume that $\Delta = \Gamma, x : T$. We know that $\Gamma \vdash \lambda(y : U) t' : \forall(y : U)U$. and from the premise of (ABS) we also know that $\Gamma, y : U \vdash t' : U$.

By IH, we have $\Gamma, y : U, x : T \vdash t' : U$. $\Gamma, x : T, y : U$ is still a well-formed environment (as $T$ cannot mention $y$) and by permutation we have $\Gamma, x : T, y : U \vdash t' : U$. Then by (ABS) we have $\Gamma, x : T \vdash \lambda(y : U) t' : \forall(y : U)U$, which concludes.

\textbf{Lemma A.22 (Type binding narrowing).}

(1) If $\Gamma \vdash S' <: S$ and $\Gamma, X <: S, \Delta \vdash T \text{ wf}$, then $\Gamma, X <: S', \Delta \vdash T \text{ wf}$. 30
(2) If $\Gamma \vdash S' <: S$ and $\Gamma, X <: S, \Delta \vdash C_1 <: C_2$, then $\Gamma, X <: S', \Delta \vdash C_1 <: C_2$.
(3) If $\Gamma \vdash S' <: S$ and $\Gamma, X <: S, \Delta \vdash U <: T$, then $\Gamma, X <: S', \Delta \vdash U <: T$.
(4) If $\Gamma \vdash S' <: S$ and $\Gamma, X <: S, \Delta \vdash t : T$, then $\Gamma, X <: S', \Delta \vdash t : T$.

Proof. By straightforward induction on the derivations. Only subtyping considers types to which type variables are bound, and the only rule to do so is (tvar), which we prove below. All other cases follow from IH or other narrowing lemmas.

Case (tvar). We need to prove $\Gamma, X <: S', \Delta \vdash X <: S$, which follows from weakening the lemma premise and using (trans) together with (tvar).

\[ \square \]

Lemma A.23 (Term binding narrowing).

(1) If $\Gamma \vdash U' <: U$ and $\Gamma, x : U, \Delta \vdash T \text{ wf}$, then $\Gamma, x : U', \Delta \vdash T \text{ wf}$.
(2) If $\Gamma \vdash U' <: U$ and $\Gamma, x : U, \Delta \vdash C_1 <: C_2$, then $\Gamma, x : U', \Delta \vdash C_1 <: C_2$.
(3) If $\Gamma \vdash U' <: U$ and $\Gamma, x : U, \Delta \vdash T_1 <: T_2$, then $\Gamma, x : U', \Delta \vdash T_1 <: T_2$.
(4) If $\Gamma \vdash U' <: U$ and $\Gamma, x : U, \Delta \vdash t : T$, then $\Gamma, x : U', \Delta \vdash t : T$.

Proof. By straightforward induction on the derivations. Only subcapturing and typing consider types to which term variables are bound. Only (sc-var) and (var) do so, which we prove below. All other cases follow from IH or other narrowing lemmas.

Case (var). We know that $U = C \ R$ and $\Gamma, x : C \ R, \Delta \vdash x : \{x\} \ R$. As $\Gamma \vdash U' <: U$, from Lemma A.16 we know that $U' = C' \ R'$ and that $\Gamma \vdash R' <: R$. We need to prove that $\Gamma, x : C' \ R', \Delta \vdash x : \{x\} \ R$. We can do so through (var), (sub), (capt), (sc-elem) and weakening $\Gamma \vdash R' <: R$.

Case (sc-var). Then we know that $C_1 = \{y\}$ and that $y : T \in \Gamma, x : U, \Delta$ and that $\Gamma, x : U, \Delta \vdash \text{cv}(T) <: C_2$.

If $y \neq x$, we can conclude by IH and (sc-var).

Otherwise, we have $T = U$. From Lemma A.16 we know that $\Gamma \vdash \text{cv}(U') <: \text{cv}(U)$, and from IH we know that $\Gamma, x : U', \Delta \vdash \text{cv}(U) <: C_2$. By (sc-var) to conclude it is enough to have $\Gamma, x : U', \Delta \vdash \text{cv}(U') <: C_2$, which we do have by connecting two previous conclusions by weakening and Lemma A.12.

\[ \square \]

A.4 Substitution

A.4.1 Term Substitution.

Lemma A.24 (Term substitution preserves subcapturing). If $\Gamma, x : U, \Delta \vdash C_1 <: C_2$ and $\Gamma \vdash D <: \text{cv}(U)$, then $\Gamma, [x \mapsto D] \Delta \vdash [x \mapsto D] C_1 <: [x \mapsto D] C_2$.

Proof. Define $\theta \triangleq [x \mapsto D]$. By induction on the subcapturing derivation.

Case (sc-elem). Then $C_1 = \{y\}$ and $y \in C_2$. Inspect if $y = x$. If no, then our goal is $\Gamma, \theta \Delta \vdash \{y\} <: \theta C_2$. In this case, $y \in \theta C_2$, which lets us conclude by (sc-elem). Otherwise, we have $\theta C_2 = (C_2 \setminus \{x\}) \cup D$, as $x \in C_2$. Then our goal is $\Gamma, \theta \Delta \vdash D : (C_2 \setminus \{x\}) \cup D$, which can be shown by (sc-set) and (sc-elem).

Case (sc-var). Then $C_1 = \{y\}$ and $y : C_3 P' \in \Gamma, x : U, \Delta$ and $\Gamma, x : U, \Delta \vdash C_3 <: C_2$.

Inspect if $y = x$. If yes, then our goal is $\Gamma, \theta \Delta \vdash D : \theta C_2$. By IH we know that $\Gamma, \theta \Delta \vdash \theta C_3 : \theta C_2$. As $x = y$, we have $U = C_3 P'$ and therefore based on an initial premise of the lemma we have $\Gamma \vdash D : C_3$. Then by weakening and IH, we know that $\Gamma, \theta \Delta \vdash \theta D : \theta C_3$, which means we can conclude by Lemma A.12.

Otherwise, $x \neq y$, and our goal is $\Gamma, \theta \Delta \vdash C_1 : \theta C_2$. We inspect where $y$ is bound.
Case $y \in \text{dom}(\Gamma)$. Then $y \notin C_3$, as $\Gamma, x : U, \Delta \not\vdash x : C_3$. By IH we have $\Gamma, \theta \Delta \vdash \theta C_3 \iff \theta C_2$.
We can conclude by (sc-var) as $[x \mapsto D] C_3 = C_3$ and $y : C_3 \in \Gamma, \theta \Delta$.

Case $y \in \text{dom}(\Delta)$. Then $y : \theta(C_3 P) \in \Gamma, \theta \Delta$ and we can conclude by IH and (sc-var).

Case (sc-set). Then $C_1 = \{y_1, \ldots, y_n\}$ and we inspect if $x \in C_1$.

If no, then for all $y \in C_1$ we have $\theta(y) = \{y\}$ and so we can conclude by repeated IH on our premises and (sc-set).

If yes, then we know that: $\forall y \in C_1, \Gamma, x : U, \Delta \vdash \{y\} : \iff C_2$. We need to show that $\Gamma, \theta \Delta \vdash \theta C_1 \iff \theta C_2$. By (sc-set), it is enough to show that if $y' \in \theta C_1$, then $\Gamma, \theta \Delta \vdash \{y'\} : \iff \theta C_2$.

For each such $y'$, there exists $y \in C_1$ such that $y' \in \theta(y)$. For this $y$, from a premise of (sc-set) we know that $\Gamma, x : U, \Delta \vdash \{y\} : \iff \theta C_2$ and so by IH we have $\Gamma, \theta \Delta \vdash \theta \{y\} \iff \theta C_2$.

Based on that, by Lemma A.12 we also have $\Gamma, \theta \Delta \vdash \{y'\} \iff \theta C_2$, which is our goal.

\[\square\]

\textbf{Lemma A.25 (Term Substitution Preserves Subtyping).} If $\Gamma, x : U, \Delta \vdash x : T$ and $\Gamma \vdash y : U$, then $\Gamma, [x \mapsto y] \Delta \vdash [x \mapsto y]U : \iff [x \mapsto y]T$.

\textbf{Proof.} Define $\theta \triangleq [x \mapsto y]$. Proceed by induction on the subtyping derivation.

\textit{Case (refl)}: By same rule.

\textit{Case (capt)}: By IH and Lemma A.27 and (capt).

\textit{Case (trans)}: (boxed), (fun), (tfun). By IH and re-application of the same rule.

\textit{Case (tvar)}: Then $U = Y$ and $T = C R$ and $Y \iff C R \in \Gamma, x : U, \Delta$ and our goal is $\Gamma, x : U, \Delta \vdash \theta Y \iff \theta (C R)$. Note that $x \neq Y$ and inspect where $Y$ is bound. If $Y \in \text{dom}(\Gamma)$, we have $Y \iff C R \in \Gamma, \theta \Delta$ and since $X \notin \text{fv}(R)$ (as $\Gamma, X : R, \Delta \not\vdash C R = C R$. Then, we can conclude by (tvar). Otherwise if $Y \in \text{dom}(\Delta)$, we have $Y \iff C \theta R \in \Gamma, \theta \Delta$ and we can conclude by (tvar).

\[\square\]

\textbf{Lemma A.26 (Term Substitution Preserves Typing).} If $\Gamma, x : U, \Delta \vdash \langle t \rangle : T$ and $\Gamma \vdash y : U$, then $\Gamma, [x \mapsto y] \Delta \vdash \langle [x \mapsto y]t \rangle : [x \mapsto y]T$.

\textbf{Proof.} Define $\theta \triangleq [x \mapsto y]$. Proceed by induction on the typing judgement.

\textit{Case (var)}: Then $t = z$ and $z : C R \in \Gamma, x : U, \Delta$ and $T = \{z\} R$ and our goal is $\Gamma, \theta \Delta \vdash z : \theta(\{z\} R)$.

If $z = x$, then $U = C R$ and $\theta(\{z\} R) = \{y\} R$ and by (var) $\Gamma, \theta \Delta \vdash y : \{y\} R$.

Otherwise, $z \neq x$ and we inspect $z$ it is bound.

If $z \in \text{dom}(\Gamma)$, then $x \notin \text{fv}(C R)$ and so $\theta(\{z\} R) = \{z\} R$ and we can conclude by (var).

Otherwise, $z \in \text{dom}(\Delta)$ and $\theta(\{z\} R) \in \Gamma, \theta \Delta$ and we can conclude by (var).

\textit{Case (sub)}: By IH, Lemma A.25 and (sub).

\textit{Case (abs)}: Then $\Gamma, x : U, \Delta, x' : U' \vdash \langle t \rangle : T$.

By IH, we have that $\Gamma, \theta \Delta, x' : U' \vdash \theta \langle t \rangle$. If we note that term substitution does not affect cv, the result immediately follows.

\textit{Case (tabs)}: Similar to previous rule.

\textit{Case (app)}: Then $t = z z'$ and $\Gamma, x : U, \Delta \vdash z : C V(x' : U')T' \text{ and } \Gamma, x : U, \Delta \vdash z' : U' \text{ and } T = [x' \mapsto z']T'$.

By IH we have $\Gamma, \theta \Delta \vdash \theta z : \theta(C V(x' : U')T') \text{ and } \Gamma, \theta \Delta \vdash \theta z' : \theta U'$.

Then by (app) we have $\Gamma, \theta \Delta \vdash \theta(z z') : [x' \mapsto \theta z'] \theta T'$.

As $x' \neq x$, we have $[x' \mapsto \theta z'] \theta T' = \theta([x' \mapsto z']T')$, which concludes.

\textit{Case (tapp)}: Similar to previous rule.
Case (box). Then $t = \square z$ and $\Gamma, x : U, \Delta \vdash z : C R$ and $T = \square C R$.
By IH, we have $\Gamma, \theta \Delta \vdash \theta z : \theta C \theta R$. If $x \notin C$, we have $\theta C = C$ and $C \subseteq \text{dom}(\Gamma, \theta \Delta)$ which lets us conclude by (box). Otherwise, $\theta C = (C \setminus \{x\}) \cup \{y\}$ As $\Gamma \vdash y : U, \theta C \subseteq \text{dom}(\Gamma, \theta \Delta)$, which again lets us conclude by (box).
Case (unbox). Analogous to the previous rule. Note that we just swap the types in the premise and the conclusion.

\[ \square \]

A.4.2 Type Substitution.

**Lemma A.27** (Type substitution preserves subcapturing). If $\Gamma, X <: R, \Delta \vdash C <: D$ and $\Gamma \vdash P <: R$ then $\Gamma, [X \mapsto P] \Delta \vdash C <: D$.

**Proof.** Define $\theta \triangleq [X \mapsto R]$. Proceed by induction on the subcapturing judgement.
- **Case (sc-set), (sc-elem).** By IH and same rule.
- **Case (sc-var).** Then $C = \{y\}, y : C' R' \in \Gamma, X <: R, \Delta, y \neq X$. Inspect where $y$ is bound. If $y \in \text{dom}(\Gamma)$, we have $y : C' R' \in \Gamma, \theta \Delta$. Otherwise, by definition of substitution we have $y : C' \theta R' \in \Gamma, \theta \Delta$. In both cases we can conclude by (sc-var).

\[ \square \]

**Lemma A.28** (Type substitution preserves subtyping). If $\Gamma, X <: R, \Delta \vdash U <: T$ and $\Gamma \vdash P <: R$, then $\Gamma, [X \mapsto P] \Delta \vdash [X \mapsto P]U <: [X \mapsto P]T$.

**Proof.** Define $\theta \triangleq [X \mapsto P]$. Proceed by induction on the subtyping derivation.
- **Case (refl), (top).** By same rule.
- **Case (capt).** By IH and Lemma A.27 and (capt).
- **Case (trans), (boxed), (fun), (tfun).** By IH and re-application of the same rule.
- **Case (tvar).** Then $U = Y$ and $T = C R'$ and $Y <: C R' \in \Gamma, X <: R, \Delta$ and our goal is $\Gamma, X <: R, \Delta \vdash \theta Y <: \theta(C R')$. If $Y = X$, by premise and weakening. Otherwise, inspect where $Y$ is bound. If $Y \in \text{dom}(\Gamma)$, we have $Y <: C' R' \in \Gamma, \theta \Delta$ and since $X \notin \text{fv}(R')$ (as $\Gamma, X <: R, \Delta \text{ wf}$), $\theta(C R') = C R'$. Then, we can conclude by (tvar). Otherwise if $Y \in \text{dom}(\Delta)$, we have $Y <: C \theta R' \in \Gamma, \theta \Delta$ and we can conclude by (tvar).

\[ \square \]

**Lemma A.29** (Type substitution preserves typing). If $\Gamma, X <: R, \Delta \vdash t : T$ and $\Gamma \vdash P <: R$, then $\Gamma, [X \mapsto P] \Delta \vdash [X \mapsto P]t : [X \mapsto P]T$.

**Proof.** Define $\theta \triangleq [X \mapsto P]$. Proceed by induction on the typing derivation.
- **Case (var).** Then $t = y, y : C' R' \in \Gamma, X <: R, \Delta, y \neq X$, and our goal is $\Gamma, \theta \Delta \vdash x : \{x\} \theta R'$. Inspect where $y$ is bound. If $y \in \text{dom}(\Gamma)$, then $y : C' R' \in \Gamma, \theta \Delta$ and $X \notin \text{fv}(R')$ (as $\Gamma, X <: R, \Delta \text{ wf}$). Then, $\theta(C' R') = C' R'$ and we can conclude by (var). Otherwise, $y : C' \theta R' \in \Gamma, \theta \Delta$ and we can directly conclude by (var).
- **Case (abs), (tabs).** In both rules, observe that type substitution does not affect cv and conclude by IH and rule re-application.
- **Case (app).** Then we have $t = x \cdot x$ and $\Gamma, X <: R, \Delta \vdash x : C \forall(z : U)T_0$ and $T = [z \mapsto y]T_0$.
  We observe that $\theta[z \mapsto y]T_0 = [z \mapsto y]T_0$ and $\theta t = t$ and conclude by IH and (app).
- **Case (tapp).** Then we have $t = x[R']$ and $\Gamma, X <: R, \Delta \vdash x : C \forall[Z <: R']T_0$ and $T = [Z \mapsto R']T_0$.
  We observe that $\theta[Z \mapsto R']T_0 = [Z \mapsto \theta R']T_0$. By IH, $\Gamma, \theta \Delta \vdash x : C \forall[Z : \theta R']T_0$.
Then, we can conclude by (tapp).
Case (box). Then \( t = \Box y \) and \( \Gamma, X : C R \vdash y : C R \) and \( T = \Box C R \), and our goal is 
\( \Gamma, \Delta \vdash y : \Theta(C' R') \).

Inspect where \( y \) is bound. If \( y \in \text{dom}(\Gamma) \), then \( y : C' R' \in \Gamma, \Delta \) and \( X \notin \text{fv}(R') \) (as \( \Gamma, X : C R \not\vdash \Delta, \text{wf} \)). Then, \( \Theta(C' R') = C' R' \) and we can conclude by (box). Otherwise, \( y : C' \Theta R' \in \Gamma, \Delta \) and we can directly conclude by (box).

Case (unbox). Proceed analogously to the case for (box) – we just swap the types in the premise and in the consequence.

Case (sub). By IH and A.28.

\[ \square \]

A.5 Main Theorems – Soundness

A.5.1 Preliminaries. As we state Preservation (Theorem A.38) in a non-empty environment, we need to show canonical forms lemmas in such an environment as well. To do so, we need to know that values cannot be typed with a type that is a type variable, which normally follows from the environment being empty. Instead, we show the following lemma:

**Lemma A.30 (Value typing).** If \( \Gamma \vdash v : T \), then \( T \) is not of the form \( C X \).

**Proof.** By induction on the typing derivation.

For rule (sub), we know that \( \Gamma \vdash v : U \) and \( \Gamma \vdash U : T \). Assuming \( T = C X \), we have a contradiction by Lemma A.15 and IH.

Rules (box), (abs), (tabs) are immediate, and other rules are not possible. \[ \square \]

**Lemma A.31 (Canonical forms: term abstraction).** If \( \Gamma \vdash v : C \forall(x : U)T \), then \( v = \lambda (x : U') \, t \) and \( \Gamma \vdash U : U' \) and \( \Gamma, x : U \vdash t : T \).

**Proof.** By induction on the typing derivation.

For rule (sub), we observe that by Lemma A.17 and by Lemma A.30, the subtype is of the form \( C' \forall(y : U'')T' \) and we have \( \Gamma \vdash U : U'' \). By IH we know that \( v = \lambda (x : U') \, t \) and \( \Gamma \vdash U' \) and \( \Gamma, x : U'' \vdash t : T \). By (trans) we have \( \Gamma \vdash U < : U' \) and by narrowing we have \( \Gamma, x : U \vdash t : T \), which concludes.

Rule (abs) is immediate, and other rules cannot occur. \[ \square \]

**Lemma A.32 (Canonical forms: type abstraction).** If \( \Gamma \vdash v : C \forall[X : U]T \), then \( v = \Lambda [X : U'] \, t \) and \( \Gamma \vdash U : U' \) and \( \Gamma, X < : U \vdash t : T \).

**Proof.** Analogous to the proof of Lemma A.31. \[ \square \]

**Lemma A.33 (Canonical forms: boxed term).** If \( \Gamma \vdash v : C \Box T \), then \( v = \Box x \) and \( \Gamma \vdash x : T \).

**Proof.** Analogous to the proof of Lemma A.31. \[ \square \]

**Lemma A.34 (Variable lookup inversion).** If we have both \( \Gamma \vdash \sigma \sim \Delta \) and \( x : C R \in \Gamma, \Delta \), then \( \sigma(x) = v \) implies that \( \Gamma, \Delta \vdash v : C R \).

**Proof.** By structural induction on \( \sigma \). It is not possible for \( \sigma \) to be empty. Otherwise, \( \sigma = \sigma'[\boxed{\text{let}} y = v \boxed{\text{in}} ] \) and for some \( U \) we have both \( \Delta = \Delta', y : U \) and \( \Gamma, \Delta' \vdash v : U \).

If \( y \neq x \), we can proceed by IH as \( x \) can also be typed in \( \Gamma, \Delta' \), after which we can conclude by weakening. Otherwise, we proceed by induction on the typing derivation.

Case (var). Then \( C = \{x\} \). Using subsumption and (sc-var), we can conclude with \( U = C' R \).

Case (sub). Then we have \( \Gamma, \Delta \vdash x : C'' R' \) and \( \Gamma, \Delta \vdash C'' R' : C R \). By Lemma A.16 we have both \( \Gamma, \Delta \vdash C'' : C \) and \( \Gamma, \Delta \vdash R' \vdash: R \). Then we proceed by using the IH on \( \Gamma, \Delta \vdash x : C'' R' \) and conclude through subsumption and transitivity (A.3).
Other rules are impossible.

Lemma A.35 (Term Abstraction Lookup Inversion). If \( \Gamma \vdash \sigma \sim \Delta \) and \( \Delta \vdash x : C \forall (z : U) T \) and \( \sigma(x) = \lambda (z : U') t \), then \( \Gamma, \Delta \vdash U <: U' \) and \( \Gamma, \Delta, z : U \vdash t : T \).

Proof. A corollary of Lemma A.34 and Lemma A.31.

Lemma A.36 (Type Abstraction Lookup Inversion). If \( \Gamma \vdash \sigma \sim \Delta \) and \( \Delta \vdash x : C \forall [Z <: U] \rightarrow T \) and \( \sigma(x) = \Lambda [Z <: U'] t \), then \( \Gamma, \Delta \vdash U <: U' \) and \( \Gamma, \Delta, Z <: U \vdash t : T \).

Proof. A corollary of Lemma A.34 and Lemma A.32.

Lemma A.37 (Box Lookup Inversion). If \( \Gamma \vdash \sigma \sim \Delta \) and \( \sigma(x) = \square y \) and \( \Gamma, \Delta \vdash x : \square T \), then \( \Gamma, \Delta \vdash y : T \).

Proof. A corollary of Lemma A.34 and Lemma A.33.

A.5.2 Soundness. In this section, we show the classical soundness theorems.

Theorem A.38 (Preservation). If we have \( \Gamma \vdash \sigma \sim \Delta \) and \( \Delta \vdash t : T \), then \( \sigma[ t ] \longrightarrow \sigma[ t' ] \) implies that \( \Gamma, \Delta \vdash t' : T \).

Proof. We proceed by inspecting the rule used to reduce \( \sigma[ t ] \).

Case (Apply). Then we have \( t = e[ x \ y ] \) and \( \sigma(x) = \lambda (z : U) s \) and \( t' = e[ z \mapsto y ] s \).

By Lemma A.2, \( x \ y \) is well-typed in \( \Gamma, \Delta \) and we proceed by induction on its typing derivation.

Only two rules could apply: (sub) and (app). In the first case, we can conclude by IH and subsumption. In the second case, we have: \( \Gamma, \Delta \vdash x : \forall (z : U_0) T_0 \) and \( \Gamma, \Delta \vdash y : U_0 \). By Lemma A.35 and narrowing, we have \( \Gamma, \Delta, z : U_0 \vdash \rightarrow \delta_0 \). By Lemma A.26, we can type the substituted \( s \) as \( \Gamma, \Delta \vdash [z \mapsto y] s : [z \mapsto y] T_0' \). We can now conclude by restoring the contexts \( \sigma \) and \( e \) with Lemmas A.5 and A.6.

Case (Tapapply), (Unbox). As above.

Case (Rename). Then we have \( t = e[ \text{let } x = y \text{ in } t_0 ] \) and \( t' = e[ x \mapsto y ] t_0 \).

As \( \Gamma, \Delta \vdash \rightarrow \delta_0 \) and by Barendregt \( x \notin \text{dom} \Gamma, \Delta \), we have \( x \notin \text{fv}(T_0) \). We again proceed by induction on the typing derivation of the plug inside \( t \), that is \( \text{let } x = y \text{ in } t_0 \). Only two rules could apply: (sub) and (let). In the first case, we can conclude by IH and subsumption.

In the second case for some \( U \) we have \( \Gamma, \Delta, x : U \vdash t_0 : T_0 \) and \( \Gamma, \Delta \vdash y : U \). Then by Lemma A.26, \( \Gamma, \Delta \vdash [x \mapsto y] t_0 : [x \mapsto y] T_0 \). Since \( x \notin \text{fv}(T_0) \) we have \( x \mapsto y \) is a canonical split.

Case (Lift). Then we have \( t = e[ \text{let } x : U = v \text{ in } t_0 ] \) and \( t' = e[ \text{let } x = v \text{ in } t_0 ] \).

Like in the previous case, we proceed by induction on the typing derivation of the plug inside \( t \). The (sub) is, again, trivial. The only other case is again (let), where for some \( U \) we have \( \Gamma, \Delta, x : U \vdash t_0 : T_0 \). By Lemma ??, we have \( \Gamma, \Delta \vdash e[ t ] : T \). As \( t \) and \( t_0 \) have the same types in \( \Gamma, \Delta \), we can replug \( e \) (Lemma A.5) and weaken the typing context to obtain \( \Gamma, \Delta, x : U \vdash e[ t_0 ] : T \). From the Barendregt convention, we know that \( x \notin \text{fv}(T) \) and so by (let) we have \( \Gamma, \Delta \vdash \text{let } x = v \text{ in } e[ t_0 ] : T \). To conclude, we restore \( \sigma \) with Lemma A.6.

Definition A.39 (Canonical store-plug split). We say that a term of the form \( \sigma[ t ] \) is a canonical split (of the entire term into store context \( \sigma \) and the plug \( t \)) if \( t \) is not of the form \( \text{let } x : T = v \text{ in } t' \).

Fact A.5. Every term has a unique canonical store-plug split, and finding it is decidable.
LEMMA A.40 (Extraction of bound value). If $\Gamma, \Delta \vdash x : T$ and $\vdash \sigma \sim \Delta$ and $x \in \text{dom}(\Delta)$, then $\sigma(x) = v$.

Proof. By structural induction on $\Delta$. If $\Delta$ is empty, we have a contradiction. Otherwise, $\Delta = \Delta', z : T'$ and $\sigma = \sigma'[\text{let } z : T' = v \text{ in } []]$ and $\Gamma, \Delta', z : T' \vdash v : T'$. Note that $\Delta$ is the environment matching $\sigma$ and can only contain term bindings. If $z = x$, we can conclude immediately, and otherwise if $z \neq x$, we can conclude by IH.

THEOREM A.41 (Progress). If $\vdash \sigma[e[t]] : T$ and $\sigma[e[t]]$ is a canonical store-plug split, then either $e[t] = a$, or there exists $\sigma[t']$ such that $\sigma[e[t]] \rightarrow \sigma[t']$.

Proof. Since $\sigma[e[t]]$ is well-typed in the empty environment, there clearly must be some $\Delta$ such that $\vdash \sigma \sim \Delta$ and $\Delta \vdash e[t] : T$. By Lemma A.2, we have that $\Delta \vdash t : U$ for some $U$. We proceed by induction on the derivation of this judgement.

Case (VAR). Then $t = x$.

If $e$ is non-empty, $e[x] = e'[\text{let } y : T = x \text{ in } t]$ and we can step by (RENAME); otherwise, immediate.

Case (ABS), (TABS), (BOX). Then $t = v$.

If $e$ is non-empty, $e[v] = e'[\text{let } x : T = v \text{ in } t]$ and we can step by (LIFT); otherwise, immediate.

Case (APP). Then $t = x \ y$ and $x : C \forall(z : U_0)T_0 \in \Delta$ and $\Delta \vdash y : U_0$.

By Lemmas A.40 and A.31, $\sigma(x) = \lambda(z : U_1)\ t'$, which means we can step by (APPLY).

Case (TAPP). Then $t = x [R]$ and $x : C \forall[z < : R]T_0 \in \Delta$.

By Lemmas A.40 and A.32, $\sigma(x) = \Delta[z < : R].\ t'$, which means we can step by (TAPPLY).

Case (UNBOX). Then $t = C \ o x$ and $x : C \ni C \ R \in \Delta$.

By Lemmas A.40 and A.33, $\sigma(x) = \Box y$, which means we can step by (OPEN).

Case (LET). Then $t = \text{let } x : T = s \text{ in } t'$ and we proceed by IH on $s$, with $e[\text{let } x : T = [ ] \text{ in } t']$ as the evaluation context.

Case (SUB). By IH.

A.5.3 Consequences.

LEMMA A.42 (Capture prediction for answers). If $\Gamma \vdash \sigma[a] : C \ R$, then $\Gamma \vdash \sigma[a] : \text{cv}(\sigma[a]) \ R$ and $\Gamma \vdash \text{cv}(\sigma[a]) < : C$.

Proof. By induction on the typing derivation.

Case (SUB). Then $\Gamma \vdash a : C' R'$ and $\Gamma \vdash C' R' < : C \ R$. By IH, $\Gamma \vdash \sigma[a] : \text{cv}(\sigma[a]) \ R'$ and $\Gamma \vdash \text{cv}(\sigma[a]) \ R'$ and $\Gamma \vdash \text{cv}(\sigma[a]) < : C$. By Lemma A.16, we have that $\Gamma \vdash C' < : C$ and $\Gamma \vdash R' < : R$.

To conclude we need $\Gamma \vdash \sigma[a] : \text{cv}(\sigma[a]) \ R$ and $\Gamma \vdash \text{cv}(\sigma[a]) < : C$, which we respectively have by subsumption and Lemma A.12.

Case (VAR), (ABS), (TABS), (BOX). Then $\sigma$ is empty and $C = \text{cv}(a)$. One goal is immediate, other follows from Lemma A.13.

Case (LET). Then $\sigma = \text{let } x = v \in \sigma'$ and $\Gamma, x : U \vdash \sigma'[a] : C \ R$ and $x \notin C$.

By IH, $\Gamma, x : U \vdash \sigma'[a] < : \text{cv}(\sigma'[a]) \ R$ and $\Gamma, x : U \vdash \text{cv}(\sigma'[a]) < : C$.

By Lemma A.24, we have $\Gamma \vdash [x \mapsto \text{cv}(v)](\text{cv}(\sigma'[a])) < : [x \mapsto \text{cv}(v)]C$.

By definition, $[x \mapsto \text{cv}(v)](\text{cv}(\sigma'[a])) = \text{cv}(\text{let } x = v \text{ in } \sigma'[a])$, and we also already know that $x \notin C$.

This lets us conclude, as we have $\Gamma \vdash \text{cv}(\text{let } x = v \text{ in } \sigma'[a]) < : C$.

Other rules cannot occur.
**Lemma A.43 (Capture prediction for terms).** Let \( \vdash \sigma \sim \Delta \) and \( \Delta \vdash t : C \ R \). Then \( \sigma[t] \rightarrow^* \sigma[\sigma'[a]] \) implies that \( \Delta \vdash cv(\sigma'[a]) :< C \).

**Proof.** By preservation, \( \vdash \sigma'[a] : C \ R \), which lets us conclude by Lemma A.42. \( \square \)

### A.6 Correctness of boxing

#### A.6.1 Relating cv and stores.

We want to relate the \( cv \) of a term of the form \( \sigma[t] \) with \( cv(\sigma) \). We are after a definition like this:

\[
cv(\sigma[t]) = \text{resolve}(\sigma, cv(t))
\]

Let us consider term of the form \( \sigma[t] \) and a store of the form \( \text{let } x = \sigma \text{ in } \sigma' \). There are two rules that could be used to calculate \( cv(\text{let } x = \sigma \text{ in } \sigma') \):

\[
\begin{align*}
\text{cv(let } x = \sigma \text{ in } t) &= cv(t) & \text{if } x \notin cv(t) \\
\text{cv(let } x = s \text{ in } t) &= cv(s) \cup cv(t) \setminus x
\end{align*}
\]

Observe that since we know that \( x \) is bound to a value, we can reformulate these rules as:

\[
\text{cv(let } x = \sigma \text{ in } t) = \{x \mapsto cv(\sigma)\} \cdot cv(t)
\]

Which means that we should be able to define \( \text{resolve} \) with a substitution. We will call this substitution a **store resolver**, and we define it as:

\[
\begin{align*}
\text{resolver(let } x = \sigma \text{ in } \sigma) &= [x \mapsto cv(\sigma)] \circ \text{resolver}(\sigma) \\
\text{resolver([])} &= id
\end{align*}
\]

Importantly, note that we use **composition** of substitutions. We have:

\[
\text{resolve(let } x = a \text{ in let } y = x \text{ in [])} = [x \mapsto \{a\}, y \mapsto \{a\}]
\]

With the above, we define \( \text{resolve} \) as:

\[
\text{resolve}(\sigma, C) = \text{resolver}(\sigma)(C)
\]

This definition satisfies our original desired equality with \( cv \).

#### A.6.2 Relating cv and evaluation contexts.

Now to connect with evaluation contexts. We can extend \( cv \) to evaluation contexts like this:

\[
\begin{align*}
\text{cv(let } x = e \text{ in } t) &= cv(e) \cup cv(t) \\
\text{cv([])} &= \{
\end{align*}
\]

Now, the problem is that we can have “degenerate” evaluation contexts syntactically. For instance we can split a term like \( \text{let } x = \sigma \text{ in } z \) into a form like \( \text{(let } x = [ ] \text{ in } z)[\sigma] \), even though this term is in normal form. The problem with such a term is that \( cv \) may or may not count \( cv(\sigma) \) as the \( cv \) of the overall term, which makes the relationship too complicated. Instead, we will depend on reduction, like this:

**Fact A.6.** Let \( \sigma[ e \{ t \} ] \) be a term that can reduce. Then:

\[
\text{cv}(\sigma[ e \{ t \} ]) = \text{resolve}(\sigma, cv(e) \cup cv(t))
\]

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A.6.3 Correctness of cv.

Definition A.44 (Platform environment).
Γ is a platform environment if for all \( x \in \text{dom} \Gamma \) we have \( x : \{\ast\} S \in \Gamma \) for some \( S \).

Lemma A.45 (Inversion of subcapturing under platform environment).
If \( \Gamma \) is a platform environment and \( \Gamma \vdash C <: D \), then either \( C \subseteq D \) or \( \ast \in D \).

Proof. By induction on the subcapturing relation. Case \( \Gamma \) holds by repeated IH. In case \( \text{sc-elem} \) holds, we have \( C = \{x\} \) and \( x : C' S \in \Gamma \). Since \( \Gamma \) is a platform environment, we have \( C' = \{\ast\} \), which means that the other premise of \( \text{sc-var} \) is \( \Gamma \vdash \{\ast\} <: D \). Since \( \Gamma \) is well-formed, \( \ast \notin \text{dom} \Gamma \), which means that we must have \( \ast \in D \). □

Lemma A.46 (Strengthening of subcapturing). If \( \Gamma, \Gamma' \vdash C <: D \) and \( C \subseteq \text{dom} \Gamma \), then we must have \( \Gamma \vdash C <: D \).

Proof. First, we consider that if \( \ast \in D \), we trivially have the desired goal. If \( \ast \notin D \), we proceed by induction on the subcapturing relation. Case \( \text{sc-elem} \) trivially holds and case \( \text{sc-set} \) holds by repeated IH.

In case \( \text{sc-var} \), we have \( C = \{x\} \) and \( x : C' S \in \Gamma, \Gamma' \). This implies that \( \Gamma = \Gamma_1, x : C' S, \Gamma_2 \) (as \( x \notin \text{dom} \Gamma \)). Since \( \Gamma, \Gamma' \) is well-formed, we must have \( \Gamma_1 \vdash C' \text{wf} \). Since we already know \( \ast \notin D \), then we must also have \( \ast \notin C' \), which then leads to \( C' \subseteq \text{dom} \Gamma_1 \). This in turn means that by IH and weakening we have \( \Gamma \vdash C' <: D \), and since we also have \( x : C' S \in \Gamma \), we can conclude by (sc-var).

Then we will need to connect it to subcapturing, because the keys used to open boxes are supercaptures of the capability inside the box. We want:

Lemma A.47. Let \( \Gamma \) be a platform environment, \( \Gamma \vdash \sigma \sim \Delta \) and \( \Delta \vdash C_1 <: C_2 \). Then \( \text{resolve}(\sigma, C_1) \subseteq \text{resolve}(\sigma, C_2) \).

Proof. By induction on \( \sigma \). If \( \sigma \) is empty, we have \( \text{resolve}(\sigma, C_1) = C_1 \), likewise for \( C_2 \), and we can conclude by Lemma A.45.

Otherwise, \( \sigma = \sigma'[\{x : \nu \text{ in} \}] \) and \( \Delta = \Delta', x : D_x S_x \) for some \( S_x \). Let \( \theta = \text{resolve}(\sigma) \).

We proceed by induction on the subcapturing derivation. Case \( \text{sc-elem} \) trivially holds and case \( \text{sc-set} \) holds by repeated IH.

In case \( \text{sc-var} \), we have \( C_1 = \{y\} \) and \( y : D_y S_y \in \Gamma, \Delta \) for some \( S_y \), and \( \Gamma, \Delta \vdash D_y <: C_2 \). We must have \( \Delta' \vdash D_y \text{wf} \) and so we can strengthen subcapturing to \( \Delta' \vdash D_y <: C_2 \), which by IH gives us \( \text{resolve}(\sigma')(D_y) \subseteq \text{resolve}(\sigma')(C_2) \). By definition we have \( \theta = \text{resolve}(\sigma) = \text{resolve}(\sigma') \circ [x \mapsto \text{cv}(\nu)] \). Since by well-formedness \( x \notin D_y \), we now have:

\[ \theta D_y \subseteq \theta C_2 \]

By Lemma A.34 and Lemma A.42, we must have \( \Gamma, \Delta \vdash \text{cv}(\nu) <: D_y \). Since \( \Gamma, \Delta \vdash \text{cv}(\nu) \text{wf} \), we can strengthen this to \( \Gamma, \Delta \vdash \text{cv}(\nu) <: D_y \). By outer IH this gives us \( \text{resolve}(\sigma')(\text{cv}(\nu)) \subseteq \text{resolve}(\sigma')(D_y) \).

Which means we have \( \theta \text{cv}(\nu) \subseteq \theta C_2 \) and we can conclude by \( \theta \text{cv}(\nu) = \theta \{x\} \), since:

\[ \theta \{x\} = (\text{resolve}(\sigma') \circ [x \mapsto \text{cv}(\nu)])(\{x\}) = \text{resolve}(\sigma')(\text{cv}(\nu)) \]

\[ \theta \text{cv}(\nu) = \text{resolve}(\sigma')(\text{cv}(\nu)) \text{ (since } x \notin \text{cv}(\nu)) \]

□
A.6.4 Core lemmas.

**Lemma A.48 (Program Authority Preservation).** Let $\Psi[t]$ be a well-typed program such that $\Psi[t] \rightarrow \Psi[t']$. Then $\text{cv}(t') \subseteq \text{cv}(t)$.

**Proof.** By inspection of the reduction rule used.

- **Case (APPLY).** Then $t = \sigma[e[x y]]$ and $t' = \sigma[e[z \mapsto y]s]$. Note that we have both $\text{cv}(t) = \text{resolver}(\sigma)(\text{cv}(e) \cup \text{cv}(x y))$ and $\text{cv}(t') = \text{resolver}(\sigma)(\text{cv}(e) \cup \text{cv}([z \mapsto y]s))$.

  If we have $x \in \text{dom} \Psi$, then $\Psi(x) = \lambda(z : U) s$. By definition of platform, the lambda is closed and we have $\text{fv}(s) \subseteq \{z\}$, which in turn means that $\text{cv}([z \mapsto y]s) \subseteq \{y\} \subseteq \text{cv}(x y)$.

  Otherwise, we have $x \in \text{dom} \sigma$ and $\sigma(x) = \square z$, since all values bound in a platform must be closed and a box form cannot be closed. Since $\Psi[t]$ is a well-typed program, there must exist some $\Gamma, \Delta$ such that $\Gamma$ is a platform environment and $\vdash \Psi[\sigma] \sim \Gamma, \Delta$.

  If $z \in \text{dom} \sigma$, then by Lemma A.34 and Lemma A.37 we have $\Gamma, \Delta \vdash z : C S_z$ for some $S_z$. By straightforward induction on the typing derivation, we then must have $\Gamma, \Delta \vdash \{z\} \vdash : C$. Then by Lemma A.47 we have $\text{resolver}(\sigma)(\{z\}) \subseteq \text{resolver}(\sigma)(C)$, which lets us conclude by an argument similar to the (APPLY) case.

  Otherwise, $z \in \text{dom} \Psi$. Here we also have $\Gamma, \Delta \vdash \{z\} \vdash : C$, which implies we must have $z \in C$, so we have $\text{cv}(z) \subseteq \text{cv}(C \vdash x)$ and can conclude by a similar argument as in the (APPLY) case.

- **Case (RENAME), (LIFT).** The lemma is clearly true since these rules only shift subterms of $t$ to create $t'$.

**Lemma A.49 (Single-step Program Trace Prediction).** Let $\Psi[t]$ be a well-typed program such that $\Psi[t] \rightarrow \Psi[t']$. Then the primitive capabilities used during this reduction are a subset of $\text{cv}(t)$.

**Proof.** By inspection of the reduction rule used.

- **Case (APPLY).** Then $t = \sigma[e[x y]]$. If $x \in \text{dom} \sigma$, the lemma vacuously holds. Otherwise, $x \in \text{dom} \Psi \setminus \text{dom} \sigma$. From the definition of $\text{cv}$, we have $\{x\} \setminus \text{dom} \sigma \subseteq \text{cv}(t)$. Since $x$ is bound in $\Psi$, we then have $x \in \text{cv}(t)$, which concludes.

- **Case (TAPPLY).** Analogous reasoning.

- **Case (OPEN), (RENAME), (LIFT).** Hold vacuously, since no capabilities are used by reducing using these rules.

**Lemma A.50 (Program Trace Prediction).** Let $\Psi[t]$ be a well-typed program such that $\Psi[t] \rightarrow^* \Psi[t']$. Then the primitive capabilities used during this reduction are a subset of $\text{cv}(t)$.

**Proof.** By IH, Single-step program trace prediction and authority preservation.

A.7 Avoidance

Here, we restate Lemma 3.3 and prove it.

**Lemma A.51.** Consider a term $\text{let} x = s \text{ in } t$ in an environment $\Gamma$ such that $E \vdash s : C_s U$ is the most specific typing for $s$ in $\Gamma$ and $\Gamma, x : C_s U \vdash t : T$ is the most specific typing for $t$ in the context.
of the body of the let, namely $\Gamma, x : C_s U$. Let $T'$ be constructed from $T$ by replacing $x$ with $C_s$ in covariant capture set positions and by replacing $x$ with the empty set in contravariant capture set positions. Then for every type $V$ avoiding $x$ such that $\Gamma, x : C_s S \vdash T <: V, \Gamma \vdash T' <: V$.

**Proof.** We will construct a subtyping derivation showing that $T' <: V$. Proceed by structural induction on the subtyping derivation for $T <: V$. Since $T'$ has the same structure as $T$, most of the subtyping derivation carries over directly except for the subcapturing constraints in (CAPT).

In this case, in covariant positions, whenever we have $C_T <: C_V$ for a capture set $C_T$ from $T$ and a capture set $C_V$ from $V$, we need to show that that $\vdash C_T[x \rightarrow C_s] <: C_V$. Conversely, in contravariant positions, whenever we have $C_V <: C_T$, we need to show that $C_V <: C_T[x \rightarrow \{}\}$. For the covariant case, since $x \in C_T$ but not in $C_V$, by inverting the subcapturing relation $C_T <: C_V$, we obtain $C_s <: C_V$. Hence $C_T[x \rightarrow C_s] <: C_V$, as desired.

The more difficult case is the contravariant case, when we have $C_V <: C_T$. Here, however, we have that $C_V <: C_T[x \rightarrow \{}\}]$ by structural induction on the subcapturing derivation as $x$ never occurs on the left hand side of the subcapturing relation as $V$ avoids $x$. □