ON VECTOR BUNDLES OF FINITE ORDER

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Introduction

Bundles of finite order were first introduced in [7] and were studied systematically in [5]. Griffiths’ “working hypothesis” is that on an affine variety (smooth, over $\mathbb{C}$ and satisfying some extra conditions, cf. §1) every holomorphic vector bundle has a finite order structure. This he calls the “Oka principle with growth conditions” by analogy with the Oka-Grauert principle which states that on a Stein manifold every topological vector bundle has a unique holomorphic structure.

Griffiths and Cornalba prove the Oka principle for line bundles only. See (4.4) for the precise statement. For bundles of rank greater than 1 their theory is somewhat insufficient. To understand why we review here the four definitions they gave for what should mean that a holomorphic vector bundle $E$ on $X$ has finite order:

(I) $E$ has a Hermitian metric whose holomorphic bisectional curvature has polynomial growth, see (4.1);
(II) there is a holomorphic map $f$ of finite order from $X$ to some Grassmannian such that $E \simeq f^*U$, where $U$ is the universal bundle on the Grassmannian;
(III) $E$ has Schubert cycles of finite order;
(IV) $E$ has transition matrices of finite order relative to a suitable covering of $X$ with punctured polycylinders.
We refer to section 2 for the notions of holomorphic map of finite order, resp.
analytic subset of finite order. The polycylinders from (IV) are as in §1. In the case of a line bundle $L$ the above conditions take a simpler form:

- (I) $L$ has a Hermitian metric whose first Chern form has polynomial growth;
- (II) there is a holomorphic map of finite order $f : X \to \mathbb{P}^N$ such that $L \simeq f^*\mathcal{O}_{\mathbb{P}^N}(1)$;
- (III) $L = [Z]$ with $Z \subset X$ an analytic divisor of finite order;
- (IV) $L$ has transition functions of finite order relative to a suitable covering
of $X$.

As shown in [5] the four conditions from above are equivalent in the case
of a line bundle. The implication “(I)$\Rightarrow$(II)” is true for any rank and follows
from the vanishing theorem (4.6).

The implication “(II)$\Rightarrow$(III)” fails in general because of the so-called “unsolvability of the Bezout problem in codimension greater than 1”. In a form
tailored to our purposes, the Bezout problem asks whether intersection of sets
of finite order is of finite order, too. It has negative answer because of the
counter-example from [6]. Shiffman and Cornalba give two analytic sets of
order zero in $\mathbb{C}^2$ whose intersection has infinite order.

The implication “(III)$\Rightarrow$(IV)” cannot be carried out if the rank is greater
than 1 because of technical reasons: when $k \geq 2$ one does not know whether an
analytic set of codimension $k$ in $\mathbb{C}^n$ can be given as the zero-set of $k$ functions
of finite order. The case $k = 1$ has positive answer, cf. [16]. In the case $k > 1$
Skoda showed that one can find $n+1$ defining functions of finite order, cf. [17],
but this is of little help.

In view of these difficulties we propose the following approach: one should
give the conditions for finite order growth not in terms of $E$ but in terms of the
hyperplane bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$. This approach is inspired from classical complex
gometry: one knows that a holomorphic vector bundle $E$ over a compact
complex manifold is ample (in the sense that the zero-section of its dual can
be collapsed to a point, or in the sense that Cartan’s Theorem A or Theorem
B hold) if and only if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample. Denoting $\mathcal{L} = \mathcal{O}_{\mathbb{P}(E)}(1)$ we propose
the following definition for what should mean that the bundle $E^*$ have finite
order:

- (I') $E$ has a Finsler metric of finite order, cf. (5.5).

One is forced to think about Finsler metrics because giving a Finsler metric
on $E$ is, roughly speaking, equivalent to giving a Hermitian metric on $\mathcal{L}$. For
Finsler metrics one has the notion of holomorphic bisectional curvature, cf.
(3.21), and horizontal holomorphic bisectional curvature, cf. (3.22). We define
finite order Finsler metrics by imposing estimates on the curvature in a very
similar manner to the Hermitian case.

Once we are given a Finsler metric on $E$ there is a natural way of doing
Nevanlinna theory on $\mathbb{P}(E)$ which we explain in section 7. Thus we can formu-
late a definition analogous to (II):

- (II') there is a holomorphic map of finite order to some projective space
$f : \mathbb{P}(E) \to \mathbb{P}^N$ such that $f^*\mathcal{O}_{\mathbb{P}^N}(1) \simeq \mathcal{L}$.
The crucial step in the proof of “(I)⇒(II)” is a vanishing theorem for the sheaf of sections of finite order of $E$. At (5.12) we prove the corresponding statement in a Finsler context:

**Theorem 1:** Let $X$ be a special affine variety and $E$ a holomorphic vector bundle on $X$ equipped with a Finsler metric of finite order. Then, for any $q \geq 1$, the $q^{th}$ cohomology of $O_{f.o.}(E^*)$ vanishes.

As a consequence we get at (7.10) the implication “(I')⇒(II')”:

**Theorem 2:** Let $X$ be a special affine variety of dimension $n$. Let $E$ be a holomorphic vector bundle of rank $r + 1$ on $X$ equipped with a Finsler metric of finite order. Then, for any integer $N > 2(n + r)$, there is a holomorphic immersion $f : \mathbb{P}(E) \to \mathbb{P}^N$ of finite order satisfying $f^*O_{\mathbb{P}^N}(1) \simeq \mathcal{L}$.

In §1 we introduce the spaces on which we are working. They come equipped with complete Kähler metrics of finite volume and bounded Ricci curvature. In §2 we present just the amount of Nevanlinna theory that we need. In §3 we define Finsler metrics and, following [3], we compute the curvature of $\mathcal{L}$. In §4 we recall the main results of Griffiths and Cornalba on Hermitian metrics of finite order. We begin to make the transition to Finsler metrics by giving an equivalent definition for sections of finite order, cf. (4.12). §5 and §7 contain our two main results as stated above. They should be understood as a very small step in attempting to elucidate the Oka principle with growth conditions. In §6 we apply the classical theory of embeddings of Stein manifolds to show that the map from theorem 2 can be made to be an immersion. Notice that we do not claim that the image of $f$ in $\mathbb{P}^N$ is closed and, in fact, it is not.

1. Preliminaries

We begin by introducing the spaces on which we will be working. Let $\overline{X}$ be a complex projective manifold of dimension $n$. Let $D_1, \ldots, D_\nu$ be effective smooth ample divisors on $\overline{X}$. We put $D = D_1 + \cdots + D_\nu$. We assume that $D$ has simple normal crossings. This means that around each point $x \in \overline{X}$ there is a coordinate chart $(z_1, \ldots, z_n)$ in which $D = \{z_1 \cdots z_k = 0\}$ for some $0 \leq k \leq n$. We put $X = \overline{X} \setminus D$ and call $X$ a special affine variety. Note that, by Hironaka’s resolution of singularities, a smooth affine variety always admits a compactification such that the divisor at infinity have simple normal crossings. Thus the word “special” refers to our requirement that each $D_i$ be ample.

Let $\Delta = \{z \in \mathbb{C}, |z| \leq 1\}$ be the unit disc and $\Delta^* = \{z \in \mathbb{C}, 0 < |z| \leq 1\}$ be the punctured unit disc. A $k$-fold punctured polycylinder is of the form $P^* = (\Delta^*)^k \times \Delta^{n-k}$, $0 \leq k \leq n$. The compact manifold $\overline{X}$ can be covered with finitely many polycylinders of the form $P = \Delta^n$. They can be chosen in such a manner that the intersections $P^* = P \cap X$ be $k$-fold punctured
polygons. If \( k = 0 \) then we have a polycylinder that is entirely included in \( X \). If \( k > 0 \) we call \( P^* \) a neighbourhood at infinity.

On the punctured unit disc we have the Poincaré metric

\[
\omega_{\Delta^*} = \frac{\sqrt{-1}}{\pi} \frac{dz \wedge d\bar{z}}{|z|^2 \log(\frac{|z|^2}{c})^2} = -dd^c \log(\log(\frac{|z|^2}{c})^2), \quad c > 1,
\]

which has constant negative Gauss curvature, is complete and has finite volume. By the Poincaré metric \( \omega_{P^*} \) on a punctured polycylinder we mean the product of Poincaré metrics on the punctured components and Euclidean metrics on the non-punctured components. For a positive (1,1)-form \( \varphi \) on \( X \) we write

\[
\varphi \sim \omega_{P^*}
\]

if on each polycylinder at infinity \( \varphi \) is equivalent to the Poincaré metric.

Using the assumption that \( D_i \) are ample Griffiths and Cornalba construct an exhaustive function \( \tau \) on \( X \) with the following properties:

(i) \( \tau \) is strictly plurisubharmonic;
(ii) the Levi-form \( \varphi = dd^c \tau \) induces a complete metric on \( X \);
(iii) the Ricci curvature of this metric is bounded:

\[
|\text{Ric}(\varphi)| = O(\varphi); \quad (1.2)
\]

(iv) \( \varphi \) has finite volume.

The construction of \( \tau \) goes as follows: Consider the line bundles \([D_i]\) on \( \overline{X} \) associated to \( D_i \). They are ample, so on each of them we can choose a Hermitian metric with positive Chern form. We choose global sections \( \sigma_i \) of \([D_i]\) whose zero-sets coincide with \( D_i \) and which have norm less than 1 at every point. We put \( \sigma = \sigma_1 \otimes \cdots \otimes \sigma_\nu \), which is a section of \([D] = \bigotimes_{i=1}^\nu [D_i]\), and define

\[
\tau = c \log \frac{1}{|\sigma|^2} - \log \left\{ \log(\log|\sigma_1|^2)^2 \cdots \log(\log|\sigma_\nu|^2)^2 \right\}. \quad (1.3)
\]

Notice that \( dd^c \log \frac{1}{|\sigma|^2} = c_1([D]) \) is a positive (1,1)-form on \( \overline{X} \), while \( dd^c \log(\log|\sigma_i|^2)^2 \) is the Poincaré metric in the vicinity of \( D_i \). Thus, choosing \( c \) large enough, we can make sure that the Levi-form \( \varphi = dd^c \tau \) be positive, i.e. (i) holds. In addition \( \varphi \sim \omega_{P^*} \) which tells us that (ii) and (iv) hold. The remaining property (iii) was proven in [5] and amounts to the fact mentioned above that \( \omega_{\Delta^*} \) has bounded curvature. We shall also make use of the exhaustive function \( \rho = e^{\tau/2} \).

From now on \( X \) will be a special affine variety. We fix a compactification \( \overline{X} \) and an exhaustive function \( \tau \) as above. We assume that \( X \) is embedded in some \( \mathbb{C}^m \) and \( \overline{X} \) is the closure of \( X \) in \( \mathbb{P}^m \). Such an embedding exists because the line bundle \([D]\) associated to the divisor at infinity is ample. In addition, we may assume that \([D]\) is the restriction of \( O_{\mathbb{P}^m}(1) \) to \( X \). We choose affine coordinates \( z = (z_1, \ldots, z_m) \) on \( \mathbb{C}^m \) and homogeneous coordinates \( (z_0; \ldots; z_m) \) on \( \mathbb{P}^m \). As global section of \([D]\) in (1.3) we can take \( \sigma = z_0 \). We have \( |\sigma|^2 = (1 + ||z||^2)^{-1} \) and \( \rho \sim (1 + ||z||^2)^{c/2} \) outside a compact subset of \( X \). Hence

\[
\rho \sim ||z||^c \quad (1.4)
\]

outside a compact subset of \( X \).
Next we introduce a semipositive \((1,1)\)-form \(\psi\) on \(X\) which is suitable for doing Nevanlinna theory. We fix a projection \(p : X \rightarrow \mathbb{C}^n\) onto an \(n\)-dimensional subspace of \(\mathbb{C}^m\) such that \(p\) is a branched covering. We put
\[
\tau' = \log ||p(x)||^2, \quad \rho' = e^{\tau'}/2, \quad \psi = dd^c\tau'.
\]
We have \(\psi \geq 0\), \(\psi^{n-1} \neq 0\) and \(\psi^n = 0\). Thus \(\psi\) is not positive definite. We notice that outside a compact subset we have
\[
\tau \sim \tau'.
\]
(1.5)
We finish this section with several estimates from [12] that we will need later:
\[
\psi \leq \rho^c\varphi,
\]
(1.6)
\[
d\tau' \wedge d^c\tau' \leq \rho^c\varphi,
\]
(1.7)
\[
d\tau \wedge d^c\tau \leq \rho^c\varphi
\]
(1.8)
for some positive constant \(c\).

2. NEVANLINNA THEORY ON SPECIAL AFFINE VARIETIES

In this section we will recall several standard notions from Nevanlinna theory: the characteristic function measuring the growth of a holomorphic map from a special affine variety \(X\) to projective space, the counting function measuring the growth of an analytic subset of \(X\). We will then formulate basic relationships between these functions. As general reference we point out the elegantly written [14].

Relative to the exhaustive function \(\rho' = e^{\tau'}/2\) introduced in §1 we put
\[
X[r] = \{x \in X, \rho'(x) \leq r\}, \quad X < r > = \{x \in X, \rho'(x) = r\}.
\]
By Sard’s theorem the sets \(X < r >\) are smooth for all \(r\) outside a set of measure zero. In the sequel, each time we integrate over the set \(X < r >\), it will tacitly be assumed that the latter is smooth.

Given a holomorphic map \(f : X \rightarrow \mathbb{P}^N\) we define its characteristic function
\[
T_f(r,s) = \int_s^r dt \int_{X[t]} f^*\omega \wedge \psi^{n-1}
\]
and its higher characteristic functions
\[
T_f^{(k)}(r,s) = \int_s^r dt \int_{X[t]} f^*\omega^k \wedge \psi^{n-k},
\]
where \(r > s > 0\) are real numbers, \(1 \leq k \leq n\) and \(\omega\) is the Fubini-Study form on \(\mathbb{P}^N\). Given an analytic subset \(Z \subset X\) of pure dimension \(k\) we define its counting function
\[
N_Z(r,s) = \int_s^r dt \int_{Z[t]} \psi^k,
\]
where $Z[t] = Z \cap X[t]$. In this definition and hereafter $Z$ does not have to be reduced, in other words its components may have multiplicities. Note that if the image of $Z$ under the projection $p : X \to \mathbb{C}^n$ does not contain the origin then, the quantity $N_Z(r) = N_Z(r,0)$ is well-defined. We call it “counting function” because in the case $k = 0$, i.e. when $Z$ is a sequence of points, $N_Z(r)$ equals the logarithmic average of the number of points of $Z$ inside the ball $X[r]$. Given a global section $\sigma$ of $\mathcal{O}_{\mathbb{P}^n}(1)$ we define the proximity function of $f$ to the zero-set of $\sigma$ by

$$m_\sigma(r) = \int_{X < r>} \log \frac{1}{|\sigma \circ f|} d^* \tau' \wedge \psi^{n-1}.$$ 

Here $| \cdot |$ is the canonical norm of $\mathcal{O}_{\mathbb{P}^n}(1)$ given at a point $[v]$ by $|\sigma|_v = \frac{|<\sigma,v>|}{|v|}$. Notice that $d^* \tau' \wedge \psi^{n-1}$ is a volume form on $X < r>$. We call $m_\sigma(r)$ “proximity function” because it takes large values whenever the image of $X < r>$ under $f$ is close to the zero-set of $\sigma$. By choosing $\sigma$ to have norm less than 1 at all points we can arrange that $m_\sigma(r)$ be non-negative.

\textbf{(2.1) First Main Theorem:} Let $f : X \to \mathbb{P}^n$ be a holomorphic map. Let $\sigma$ be a global section of $\mathcal{O}_{\mathbb{P}^n}(1)$ and $Z = f^* \{ \sigma = 0 \}$. Assume that the image of $f$ is not contained in $\{ \sigma = 0 \}$. Then, for $r > s > 0$, we have

$$N_Z(r,s) + m_\sigma(r) - m_\sigma(s) = T_f(r,s).$$

\textbf{(2.2) Corollary} (Nevanlinna’s Inequality): Fix $s > 0$. Then, under the hypotheses of the previous theorem, we have for $r > s$

$$N_Z(r,s) \leq T_f(r,s) + O(1).$$

Here $O(1)$ is a constant that may depend on $s$.

Thus the growth of the preimage of a hyperplane is bounded by the growth of $f$. The corresponding statement for planes of codimension greater than 1 is false: Cornalba and Shiffman give in [6] an example of a holomorphic map of order zero from $\mathbb{C}^2$ to $\mathbb{C}^2$ for which the preimage of the origin has infinite order. This falls into the circle of ideas known as “Bezout problem”. More about the Bezout problem we said in the introduction. We now state a converse to (2.2) in an “average” sense. To explain it we introduce the Grassmannian $G(N,k)$ of planes of codimension $k$ in $\mathbb{P}^n$. There is a unique measure $\mu$ on $G(N,k)$ which is invariant under the action of the unitary group $U(n + 1)$ and which is so normalized that the measure of the total space be 1.

\textbf{(2.3) Crofton Formula:} Let $f : X \to \mathbb{P}^n$ be a holomorphic map which is non-degenerate in the sense that the preimage of a plane $P$ of codimension $k$ in $\mathbb{P}^n$ is an analytic subset of codimension $k$ in $X$. Then, for $r > s > 0$, we have

$$T_f^{(k)}(r,s) = \int_{P \in G(N,k)} N_{f^* P}(r,s) d\mu(P).$$
(2.4) Definition: Let \( f : X \rightarrow \mathbb{P}^N \) be a holomorphic map. We say that \( f \) has finite order if there is \( \lambda \geq 0 \) such that

\[
T_f(r, s) = O(r^\lambda).
\]

Likewise, we say that an analytic subset \( Z \subseteq X \) of pure dimension \( k \) has finite order if there is \( \lambda \geq 0 \) such that

\[
N_Z(r, s) = O(r^\lambda).
\]

By an abuse of language we will say that \( \lambda \) is the order of \( f \) or of \( Z \) if the above estimates hold. Technically, we would have to say that “\( Z \) has order at most \( \lambda \)” but we want to avoid obstinate repetitions.

(2.5) Remark: Assume that \( f \) has finite order and is linearly non-degenerate. Then (2.2) tells us that for any hyperplane \( H \subseteq \mathbb{P}^N \) the preimage \( f^*-H \) has finite order. Conversely, assume that the preimages \( f^*-H \) have finite order in a uniform fashion, i.e. there are \( r_o, \kappa, \lambda \geq 0 \) such that for \( r \geq r_o \) and all \( H \) we have \( N_{f^*-H}(r, s) \leq \kappa r^\lambda \). Then, by (2.3), also \( f \) has finite order.

The order of growth of an analytic subset \( Z \subseteq X \) depends on the embedding \( X \subseteq \mathbb{C}^m \) and the projection \( p : X \rightarrow \mathbb{C}^n \) that we fixed in \( \S 1 \). However, the notion of \( Z \) having finite order is intrinsic. We see this as follows: assume that \( Z \) has pure dimension \( k \) and consider the counting function of \( Z \) computed using \( \varphi \) instead of \( \psi \):

\[
\hat{N}_Z(r, s) = \int_s^r \frac{dt}{t} \int_{\{s \leq t \}} \varphi^k.
\]

Using (1.5), (1.7) and the technique from [4] we can show that \( N_Z(r, s) = O(r^\lambda) \) if and only if there is \( \mu \) depending only on \( \lambda \) such that \( \hat{N}_Z(r, s) = O(r^\mu) \). See [12] for the details. Now, if \( \tau_1 \) and \( \tau_2 \) are constructed starting from two different embeddings of \( X \) then it is an easy matter to see that \( \tau_1 \sim \tau_2 \) and \( dd^c \tau_1 \sim dd^c \tau_2 \). Thus \( \hat{N}_1 \) and \( \hat{N}_2 \) have polynomial growth at the same time.

By (2.3) the same considerations apply to holomorphic maps \( f : X \rightarrow \mathbb{P}^N \). The order of growth of \( f \) is not intrinsic however, the notion of \( f \) having finite order does not depend on any of the choices made.

Finally, we would like to mention the sheaf of holomorphic functions of finite order. First note that a holomorphic function \( f : X \rightarrow \mathbb{C} \) can be regarded as a map to \( \mathbb{P}^1 \) by sending \( x \) to \((1; f(x)) \). Its characteristic function takes the form

\[
T_f(r, s) = \int_s^r \frac{dt}{t} \int_X dd^c \log(1 + |f|^2) \wedge \psi^n - 1.
\]

Let us introduce one last growth function, the maximum modulus function

\[
M_f(r) = \log \max\{|f(x)|, \rho(x) = r\}.
\]
Using the fact that log(1 + |f|^2) is plurisubharmonic one can show that \( T_f(r, s) \) has roughly the same growth as \( M_f(r) \). This point of view allows us to localize the notion of finite order function:

\[(2.6) \text{Definition: We define the sheaf } \mathcal{O}_\lambda \text{ of germs of holomorphic functions of order } \lambda \text{ on } X \text{ as follows: } \mathcal{O}_\lambda \text{ is a sheaf on } X; \text{ for each open set } U \subset X \text{ the space of sections } \mathcal{O}_\lambda(U) \text{ consists of those holomorphic functions on } U \cap X \text{ with the property that around each point at infinity } x \in (X \setminus X) \cap U \text{ there is a neighbourhood } W \subset U \text{ and a constant } \kappa \geq 0 \text{ such that the estimate } |f(w)| \leq \exp(\kappa r^\lambda) \text{ with } w \in W, r = \rho(w) \text{ holds.}

Similarly we define the sheaf \( \mathcal{O}_{\text{f.o.}} \) of germs of holomorphic functions of finite order on \( X \) by the requirement that the estimate \( |f(w)| \leq \exp(\kappa r^\lambda) \) hold with \( \kappa, \lambda \) depending on the point at infinity \( x \).

It is easily seen that \( \mathcal{O}_\lambda \) and \( \mathcal{O}_{\text{f.o.}} \) are sheaves on \( X \). In fact they are \( \mathcal{O}_X \)-modules. Their restrictions to \( X \) coincide with the sheaf \( \mathcal{O}_X \) of germs of holomorphic functions on \( X \). By the compactness of the divisor at infinity it is also clear that the spaces of global sections \( \mathcal{O}_\lambda(X) \) and \( \mathcal{O}_{\text{f.o.}}(X) \) coincide with the space of holomorphic functions of order \( \lambda \) on \( X \), respectively the space of holomorphic functions of finite order on \( X \).

These sheaves were studied in [5] and [13]. Griffiths and Cornalba showed that \( \mathcal{O}_\lambda \) and \( \mathcal{O}_{\text{f.o.}} \) are acyclic, i.e. their cohomology groups, apart from \( H^0 \), vanish. Wong et al. proved that \( \mathcal{O}_{\text{f.o.}} \) is flat over \( \mathcal{O}_X \) and as a corollary obtained the acyclicity of \( \mathcal{O}_{\text{f.o.}} \).

3. Finsler metrics and the geometry of \( \mathbb{P}(E) \)

Let \( X \) be a special affine variety of dimension \( n \). Let \( E \) be a holomorphic vector bundle of rank \( r + 1 \) on \( X \). We begin by recalling some facts about Hermitian metrics on \( E \). We will then introduce Finsler metrics and see how the notion of holomorphic bisectional curvature generalizes from the Hermitian case to the Finsler case. Most of our calculations are taken from [3]. We also refer to [1].

We will denote by \( \mathcal{O}_X, \mathcal{A}_X, \mathcal{A}^k_X, \mathcal{A}^{p,q}_X \) the sheaves of holomorphic functions, of smooth \( C \)-valued functions, of smooth \( \mathbb{C} \)-valued \( k \)-forms and of \( (p,q) \)-forms on \( X \). We will denote by \( \mathcal{O}_X(E), \mathcal{A}_X(E), \mathcal{A}^k_X(E), \mathcal{A}^{p,q}_X(E) \) the corresponding sheaves of \( E \)-valued functions or forms.

Let \( U \subset X \) be an open coordinate set with coordinates \((z^1, \ldots, z^n)\). We assume that \( E \) is trivial over \( U \) and we choose a holomorphic frame \( \{e_0, \ldots, e_r\} \) for \( E \) over \( U \). Relative to this frame vectors \( v \) of \( E \) can be written

\[
v = \sum_{i=0}^r v^i e_i
\]
and $E$-valued $(p,q)$-forms can be written

$$u = \sum_{i=0}^{r} \sum_{|I|=p, |J|=q} u_{i,J} dz^I \wedge d\bar{z}^J \otimes e_i$$

where $dz^I = dz^{i_1} \wedge \ldots \wedge dz^{i_p}$ and $J, I = \{i_1, \ldots, i_p\}$ are increasing multiindices.

Let $h$ be a Hermitian metric on $E$. We represent it by a Hermitian matrix

$$h = (h^i_{\bar{j}})_{0 \leq i,j \leq r}, \quad h^i_{\bar{j}} = <e_i, e_j>.$$  

Associated to it there is the Chern connection $\nabla : \mathcal{A}(E) \longrightarrow \mathcal{A}^1(E)$ which can be represented by a matrix of 1-forms

$$(\theta^i_j)_{0 \leq i,j \leq r}, \quad \nabla e_i = \sum_{j=0}^{r} \theta^i_j \otimes e_j.$$  

The Christoffel symbols of the first kind $\Gamma^j_{ik}$ are given by the relations

$$\theta^i_j = \sum_{k=1}^{n} \Gamma^i_{jk} dz^k.$$  

We have

$$\theta = (\partial h) \cdot h^{-1}, \quad \Gamma^i_{jk} = \sum_{s=0}^{r} \frac{\partial h^i_{\bar{s}}}{\partial z^k} \cdot h^{s\bar{j}}.$$  

Here $(h^{s\bar{j}})_{s,j}$ is the inverse of the matrix $h$. The connection $\nabla$ induces unique $\mathbb{C}$-linear maps $\nabla : \mathcal{A}^k(E) \longrightarrow \mathcal{A}^{k+1}(E)$ by enforcing the Leibnitz rule: $\nabla(u \otimes v) = du \otimes v + (-1)^{|u|} v \wedge \nabla u$. The composition $\nabla^2 : \mathcal{A}(E) \longrightarrow \mathcal{A}^2(E)$ is called the curvature of $\nabla$ and has the remarkable property that it is a tensor, i.e. it is a morphism of $\mathcal{A}$-modules. $\nabla^2$ can be represented by a matrix of $(1,1)$-forms

$$(\Theta^i_j)_{0 \leq i,j \leq r}, \quad \nabla^2 e_i = \sum_{j=0}^{r} \Theta^i_j \otimes e_j.$$  

The Christoffel symbols of the second kind $K^j_{ikl}$ are given by the relations

$$\Theta^i_j = \sum_{k,l=1}^{n} K^j_{ikl} dz^k \wedge d\bar{z}^l.$$  

We have

$$\Theta = d\theta - \theta \wedge \theta = \bar{\partial} \theta,$$  

$$K^j_{ikl} = -\sum_{s=0}^{r} \frac{\partial^2 h^i_{\bar{s}}}{\partial z^k \partial \bar{z}^l} h^{s\bar{j}} + \sum_{p,q,s=0}^{r} \frac{\partial h^p_{\bar{s}}}{\partial \bar{z}^l} \cdot h^{q\bar{p}} h^{s\bar{j}}.$$  

Given $v \in E_x$ we construct the $(1,1)$-form

$$\Theta(v) = \frac{\sqrt{-1}}{2\pi} \frac{\langle \nabla^2 v, v \rangle_h}{||v||^2_h} = \frac{\sqrt{-1}}{2\pi} \frac{1}{||v||^2_h} K^j_{ikl} v^i h_{j\bar{s}} \bar{v}^s d\bar{z}^k \wedge \bar{d}z^l. $$
In the case of a line bundle $L$ the form $\Theta(v)$ is nothing but the first Chern form of the metric defined by
\[
c_1(L, h) = -dd^c \log h.
\]

Given $\xi \in T_xX$ and $v \in E_x$ we define the holomorphic bisectional curvature of $h$ along $\xi$ and $v$ by
\[
(3.3) \quad k_x(\xi, v) = \frac{\langle \nabla^2_{\xi} v, v \rangle_h}{||\xi||^2_h ||v||^2_h} = \frac{1}{||\xi||^2_h ||v||^2_h} \sum_{i,j=0}^{r} \sum_{k,l=1}^{n} K_{ij}^k \xi^k v^i h_{js} \bar{v}^s.
\]

**Definition:** Let $X$ be a complex manifold of dimension $n$ and $E$ a holomorphic vector bundle of rank $r+1 \geq 2$ on $X$. A Finsler metric $h$ on $E$ is a function $h : E \to [0, \infty)$ satisfying the following conditions:

(i) $h$ is continuous on $E$ and smooth on the complement of the zero-section;
(ii) $h(\lambda v) = |\lambda|h(v)$ for $v \in E$, $\lambda \in \mathbb{C}$;
(iii) $h(v) > 0$ if $v$ is non-zero;
(iv) $h|_{E_x \setminus \{0\}}$ is a strictly plurisubharmonic function for all $x \in X$.

The norm associated to a Hermitian metric is thus a particular case of Finsler metric. We will see at (3.11) that a Finsler metric $h$ comes from a Hermitian metric if and only if $h^2$ is of class $C^2$ on $E$.

Let $\mathbb{P}(E)$ be the fiber bundle with fibers $\mathbb{P}(E_x) =$projective space of lines through the origin in $E_x$, $x \in X$. We denote by $\pi : \mathbb{P}(E) \to X$ the projection onto the base. The tautological line bundle $L^{-1} = \mathcal{O}_{\mathbb{P}(E)}(-1)$ is the subbundle of $\pi^*E$ whose fiber at $(x, [v])$ consists of the line generated by $v$ inside $E_x$. The bundle space of $L^{-1}$ is the blow-up of $E$ along the zero-section; let $\beta : |L^{-1}| \to E$ be the blowing-up map. Outside the zero-sections $\beta$ is an isomorphism. To give $h$ satisfying properties (i), (ii) and (iii) from above is equivalent to giving a Hermitian metric $\tilde{h}$ on $L^{-1}$ via the correspondence $\tilde{h}^{-1} = h \circ \beta$.

In the sequel we will denote $E_o = E \setminus \{\text{zero-section}\}$. We denote $p : E \to X$ the projection onto the base and by $q : E_o \to \mathbb{P}(E)$ the quotient map. Thus
\[
\begin{array}{ccc}
E_o & \xrightarrow{q} & \mathbb{P}(E) \\
\downarrow p & & \downarrow \pi \\
X & & \\
\end{array}
\]
is a commutative diagram. We will consider the function $G = h^2$ which is continuous on $E$ and smooth on $E_o$.

As before, let $U$ be a coordinate set which trivializes $E$. On $p^{-1}U$ we have coordinates $(z^1, \ldots, z^n, v^0, \ldots, v^r)$. We will consider the following smooth functions on $p^{-1}U \cap E_o$:
\[
(3.5) \quad G_{ij} = \frac{\partial^2 G}{\partial v^i \partial \bar{v}^j}, \quad 0 \leq i, j \leq r.
\]
By property (ii) for all $\lambda \in \mathbb{C}$ we have

$$G(z, \lambda v) = |\lambda|^2 G(z, v). \tag{3.6}$$

Differentiating with respect to $v^i$ and $\bar{v}^j$ we obtain

$$G_{ij}(z, \lambda v) = G_{ij}(z, v). \tag{3.7}$$

Differentiating this relation with respect to $\lambda$ we obtain

$$\sum_{p=0}^{r} v^p \frac{\partial G_{ij}}{\partial v^p} = 0 = \sum_{q=0}^{r} \bar{v}^q \frac{\partial G_{ij}}{\partial \bar{v}^q}. \tag{3.8}$$

Differentiating (3.6) with respect to $\lambda$ and $\bar{\lambda}$ we obtain

$$G(z, v) = \sum_{i,j=0}^{r} G_{ij}(z, \lambda v)v^i \bar{v}^j \tag{3.9}$$

which, in view of (3.7), takes the form

$$G(z, v) = \sum_{i,j=0}^{r} G_{ij}(z, v)v^i \bar{v}^j. \tag{3.10}$$

If $h$ were a Hermitian metric with matrix $(h_{ij})$ we would have

$$G = \sum_{i,j=0}^{r} h_{ij} v^i \bar{v}^j$$

and $G_{ij}(z, v) = h_{ij}(z)$. Thus $h$ comes from a Hermitian metric if and only if the functions $G_{ij}$ are constant along the fibers. In fact, more can be said:

**Remark:** $h$ comes from a Hermitian metric on $E$ if and only if the function $G = h^2$ is of class $C^2$ on $E$. Indeed, if this were the case, we could take limit as $\lambda$ tends to 0 in (3.9) and we would obtain $G(z, v) = \sum_{i,j=0}^{r} G_{ij}(z, 0)v^i \bar{v}^j$. Differentiating again with respect to $v^i$ and $\bar{v}^j$ we would get $G_{ij}(z, v) = G_{ij}(z, 0)$, q.e.d.

Let us denote by $p^*: TE \rightarrow TX$ the differential of $p$. The kernel of $p_*$ is a holomorphic subbundle $\mathcal{V} \subset TE$ called the *vertical tangent bundle*. Notice that, relative to coordinates $(z^1, \ldots, z^n, v^0, \ldots, v^r)$ on $p^{-1}U$, a frame of $\mathcal{V}$ is given by $\{\frac{\partial}{\partial v^0}, \ldots, \frac{\partial}{\partial v^r}\}$. The transformation rule for this frame is the same as for the frame $\{e_0, \ldots, e_r\}$. This shows that there is a canonical isomorphism $\mathcal{V} \simeq p^*E$ identifying $\frac{\partial}{\partial v^i}$ with $p^*e_i$. A vector $V$ of $\mathcal{V}$ can be written

$$V = \sum_{i=0}^{r} V^i \frac{\partial}{\partial v^i}.$$ 

A section of $\mathcal{V}$ of particular interest is the *position vector field*

$$P(z, v) = \sum_{i=0}^{r} v^i \frac{\partial}{\partial v^i}.$$ 

Notice that $q^* \mathcal{L}^{-1}$ is the subbundle of $p^*E$ generated by $P$. Property (iv) tells us that the matrix $(G_{ij})_{0 \leq i,j \leq r}$ is positive definite at each point of $E_o$. This
allows us to define a Hermitian metric on \( V|_{E_o} \) by putting
\[
< V, W >_V = \sum_{i,j=0}^{r} G_{ij}(z,v) V^i W^j.
\]
This metric is nothing but pull-back of \( h \) in the case when \( h \) is Hermitian. Notice that, by virtue of (3.10), we have \( G = ||P||_V^2 \). Also notice that, by virtue of (3.9), \( <,> \) is constant on the fibers of \( q \) hence it descends to a Hermitian metric of \( \pi^* E \).

The Chern connection \( \nabla^V \) and its curvature make sense just as for any Hermitian metric. \( \nabla^V \) has connection matrix
\[
\theta^j_i = \sum_{k=1}^{n} \Gamma^j_{ik} dz^k + \sum_{p=0}^{r} \gamma^j_{ip} dv^p
\]
where
\[
\Gamma^j_{ik} = \sum_{s=0}^{r} \frac{\partial G_{is}}{\partial z^k} \cdot G^{sj}, \quad \gamma^j_{ip} = \sum_{s=0}^{r} \frac{\partial G_{is}}{\partial v^p} \cdot G^{sj}.
\]
The curvature matrix \( \Theta \) has now horizontal, vertical and mixed components:
\[
\Theta^j_i = \sum_{k,l=1}^{n} K^j_{ikl} dz^k \wedge d\bar{z}^l + \sum_{p,q=0}^{r} \kappa^j_{ipq} dv^p \wedge d\bar{v}^q
\]
where
\[
K^j_{ikl} = -\sum_{s=0}^{r} \frac{\partial^2 G_{is}}{\partial z^k \partial \bar{z}^l} \cdot G^{sj} + \sum_{p,q,s=0}^{r} \frac{\partial G_{iq}}{\partial z^k} \frac{\partial G_{ps}}{\partial \bar{z}^l} \cdot G^{qp} G^{sj},
\]
\[
\kappa^j_{ipq} = -\frac{\partial \gamma^j_{ip}}{\partial v^q}, \quad \mu^j_{ikq} = -\frac{\partial \Gamma^j_{ik}}{\partial v^q}, \quad \nu^j_{ipl} = -\frac{\partial \gamma^j_{ip}}{\partial \bar{z}^l}.
\]
Using (3.8) we obtain the following relations:
\[
\sum_{i=0}^{r} \gamma^j_{ip} v^i = 0, \quad (3.14)
\]
\[
\sum_{i=0}^{r} \nu^j_{ipl} v^i = \sum_{i=0}^{r} \kappa^j_{ipq} v^i = \sum_{j,s=0}^{r} \mu^j_{ikq} G_{js} \bar{v}^s = 0. \quad (3.15)
\]
For any tangent vector \( \zeta \in T E_o \) of the form
\[
\zeta = \sum_{k=1}^{n} a^k \frac{\partial}{\partial z^k} + \sum_{i=0}^{r} b^i \frac{\partial}{\partial v^i}
\]
we have

\[
\nabla^V \xi P = \sum_{i=0}^{r} (b^i + \sum_{j=0}^{r} \sum_{k=1}^{n} \Gamma^i_{jk} v^j a^k + \sum_{j,p=0}^{r} \gamma^i_{jp} v^j b^p) \frac{\partial}{\partial v^i} \\
= \sum_{i=0}^{r} (b^i + \sum_{j=0}^{r} \sum_{k=1}^{n} \Gamma^i_{jk} v^j a^k) \frac{\partial}{\partial v^i} \quad \text{by (3.14)}.
\]

This calculation shows that the linear map of bundles

\[
\nabla^V P : T E_o \rightarrow \mathcal{V}, \quad \xi \rightarrow \nabla^V \xi P
\]

is a surjection. Thus the kernel of \(\nabla^V P\) is a smooth subbundle \(\mathcal{H} \subset T E_o\) which we call the horizontal tangent bundle. We have the smooth decomposition

\[
T E_o = \mathcal{H} \oplus \mathcal{V}
\]

and at any point \((x,v) \in E_o\) the differential \(p_x : \mathcal{H}_{(x,v)} \rightarrow T_x X\) is an isomorphism. Under this isomorphism a vector \(\xi \in T_x X\) corresponds to a vector \(\xi^H \in \mathcal{H}_{(x,v)}\) called the horizontal lift of \(\xi\). The vertical tangent fields together with the horizontal lifts of the tangent fields \(\partial_k = \partial/\partial z^k\) give a smooth frame of \(T E_o\):

\[
\begin{aligned}
\frac{\partial}{\partial v^i}, & \quad 0 \leq i \leq r, \\
\partial_k^H = \frac{\partial}{\partial z^k} - \sum_{i,j=0}^{r} \Gamma^i_{jk} v^j \frac{\partial}{\partial v^i}, & \quad 1 \leq k \leq n.
\end{aligned}
\]

The dual basis is

\[
\begin{aligned}
dz^k, & \quad 1 \leq k \leq n, \\
\zeta^i = dv^i + \sum_{k=1}^{n} \sum_{j=0}^{r} \Gamma^i_{jk} v^j dz^k, & \quad 0 \leq i \leq r.
\end{aligned}
\]

Just as in the Hermitian case, for any point \((x,v) \in E_o\) there is a holomorphic frame of \(E\) on a neighbourhood of \(x\) with respect to which we have

\[
G_{ij}(x,v) = \delta_{ij}, \quad \frac{\partial G_{ij}}{\partial z^k}(x,v) = 0, \quad 0 \leq i, j \leq r, \quad 1 \leq k \leq n.
\]

Here \(\delta_{ij}\) is the Kronecker symbol. We call such a frame normal at \((x,v)\).

Relative to such a frame we have \(\partial_k^H = \partial/\partial z^k\), \(\zeta^i = dv^i\) at \((x,v)\).

We claim that for any non-zero vector \(v \in E_x\) the following expression gives us a well-defined \((1,1)\)-form on \(T_x X\):

\[
<K(\cdot, \cdot)v,v>_y = \sum_{i,j,p=0}^{r} \sum_{k,l=1}^{n} K^p_{i k l} v^i G_p v^j dz^k \wedge d\bar{z}^l.
\]

To see this we will show that for any \(\xi_1, \xi_2 \in T_x X\) we have

\[
<K(\xi_1, \xi_2)v,v>_y = <\nabla^2 \xi_1^H, \xi_2^H>_y P, P>_y
\]

where \(\nabla^2\) is the curvature of \(\nabla^V\). Indeed, adopting Einstein’s summation convention, we have

\[
<\nabla^2 P, P>_y = K^i_{j k l} v^i G_{j k} \bar{v}^s dz^k \wedge d\bar{z}^l + \mu^i_{j k l} v^i G_{j k} \bar{v}^s dv^p \wedge d\bar{v}^q
\]

\[
+ \nu^i_{j k l} v^i G_{j k} \bar{v}^s dv^p \wedge d\bar{z}^l + \nu^i_{j k l} v^i G_{j k} \bar{v}^s dz^k \wedge d\bar{v}^q.
\]
By (3.15) the last three terms on the right-hand-side vanish, yielding (3.20). As a byproduct we get the relation

$$\Theta(P) = \frac{\sqrt{-1}}{2\pi} \frac{1}{\|P\|^2} \sum_{i,j,s=0}^r K^j_{ikl} v^i G_{j \bar{s}} \bar{v}^s dz^k \wedge d\bar{z}^l.$$

In order to define the holomorphic bisectional curvature of $h$ we need to first put a metric on $TE_o$. This is done by making the decomposition $TE_o = H \oplus V$ an orthogonal decomposition. Namely, any tangent vector $\zeta \in TE_o$ can be written $\zeta = \xi^H + V$ and we put $||\zeta||^2 = ||\xi||^2 + ||v||^2_V$. We then define the \textit{holomorphic bisectional curvature of $h$ along $\zeta$ and $v$ just as at (3.3)}:

$$(3.21) \quad k(\zeta, v) = \frac{< \nabla^{\xi^H}_\zeta v, v >_V}{||\xi||^2 ||v||^2_h}$$

$$= \frac{1}{||\xi||^2 ||v||^2_h} \sum_{i,j,s=0}^r \left\{ \sum_{k,l=1}^r K^j_{ikl} s^k_a^l + \sum_{p,q=0}^r \kappa^j_{ipq} b^p \bar{b}^q \right. $$

$$\left. + \sum_{k=1}^r \sum_{q=0}^r \mu^j_{ikq} a^k \bar{b}^q + \sum_{l=1}^r \sum_{p=0}^r \nu^j_{ipl} b^p \bar{a}^l \right\} v^i G_{j \bar{s}} \bar{v}^s.$$

The knowledge that (3.19) is well defined allows us to introduce the notion of \textit{horizontal holomorphic bisectional curvature of $h$ along $\zeta$ and $v$ in $E_x$}:

$$(3.22) \quad k_x(\zeta, v) = k(\zeta, v)(\xi^H, v) = \frac{< \nabla^{\xi^H}_\zeta v, v >_V}{||\xi||^2 ||v||^2_h}$$

$$= \frac{1}{||\xi||^2 ||v||^2_h} \sum_{i,j,s=0}^r \sum_{k,l=1}^r K^j_{ikl} \xi^k v^i G_{j \bar{s}} \bar{v}^s.$$

In the remaining part of this section we will compute $c_1(\mathcal{L}, \tilde{h})$. Recall that, under the isomorphism $\mathcal{V} \cong p^* E$, we can identify $q^* \mathcal{L}^{-1}$ with the sub-bundle of $\mathcal{V}$ spanned by $P$. Relation (3.10) tells us that the pull-back metric $q^* \tilde{h}^{-1}$ is nothing but the induced metric from $\mathcal{V}$. Therefore we have

$$q^* c_1(\mathcal{L}, \tilde{h}) = \text{dd}^c \log ||P||^2_V.$$

We claim that

$$\text{dd}^c \log ||P||^2_V = \frac{\sqrt{-1}}{2\pi} \frac{1}{G^2} \left\{ \sum_{i,j=0}^r G G_{ij} \xi^i \wedge \bar{\xi}^j - \sum_{i,j,p,q=0}^r G_{ij} G_{pq} \bar{v}^p \bar{v}^q \xi^i \wedge \bar{\xi}^j \right\}$$

$$- \frac{\sqrt{-1}}{2\pi} \frac{1}{G} \sum_{i,j,s=0}^r \sum_{k,l=1}^r K^j_{ikl} v^i G_{j \bar{s}} \bar{v}^s dz^k \wedge d\bar{z}^l.$$

(3.23)

The part involving terms $\xi^i \wedge \bar{\xi}^j$ will be called the \textit{vertical component of $c_1(\mathcal{L}, \tilde{h})$} and will be denoted $c_1(\mathcal{L}, \tilde{h})^V$. The part involving terms $dz^k \wedge d\bar{z}^l$ will be called the \textit{horizontal component of $c_1(\mathcal{L}, \tilde{h})$} and will be denoted $c_1(\mathcal{L}, \tilde{h})^H$. Notice
that $c_1(L, \tilde{h})^{\mathcal{V}}$ is semi-positive definite and its restriction to the fibers of $\mathbb{P}(E)$ is positive-definite. Indeed, relative to a normal frame for $E$ at $(x, v)$ we have

$$c_1(L, \tilde{h})^{\mathcal{V}} = \frac{\sqrt{-1}}{2\pi} \cdot \frac{1}{||v||^2} \sum_{i=0}^{r} dv^i \wedge d\bar{v}^i - \frac{\sqrt{-1}}{2\pi} \cdot \frac{1}{||v||^4} \sum_{i,j=0}^{r} \bar{v}^i v^j dv^i \wedge d\bar{v}^j$$

with $||v||^2 = \sum_{i=0}^{r} |v_i|^2$. The above is nothing but the Fubini-Study form on projective space which is known to be positive-definite. Notice that

$$c_1(L, \tilde{h})^{\mathcal{H}} = -\Theta(P).$$

In normal coordinates at $(x, v)$ we have

$$c_1(L, \tilde{h}) = \omega - \Theta(P)$$

where $\omega$ is the Fubini-Study form on the first component of

$$T_{(x, [v])}\mathbb{P}(E) = T_{[v]}\mathbb{P}(E_x) \oplus T_x X.$$ 

Thus $L$ equipped with $\tilde{h}$ is a positive line bundle if and only if the horizontal holomorphic bisectional curvature of the Finsler metric $h$ is negative! This has an important consequence: assume $X$ to be compact; then $E^*$ is ample if and only if there is a Finsler metric (with only properties (i), (ii) and (iii)) on $E$ having negative horizontal holomorphic bisectional curvature. We will not need this fact, but it is worth mentioning because it illustrates the philosophy we advertized in the introduction: geometric properties of $E^*$ should be defined by means of $\mathcal{O}_{\mathbb{P}(E)}(1)$.

4. Hermitian metrics of finite order

Let $E$ be a holomorphic vector bundle on the special affine variety $X$. We recall from [5] the notion of Hermitian metric on $E$ of finite order. It is defined by means of an estimate on the holomorphic bisectional curvature. The theorem of Griffiths and Cornalba (4.4) states that a holomorphic line bundle on $X$ admits a unique finite order structure. In fact, as we explained in the introduction, Griffiths and Cornalba gave four definitions for the notion of finite order vector bundle which are equivalent in the case of line bundles. One of the crucial steps in the proof of the four equivalences is their vanishing theorem (4.6) for the sheaf of sections of finite order.

(4.1) Definition: A Hermitian holomorphic vector bundle $(E, h)$ on $X$ is said to have order $\lambda$ if its holomorphic bisectional curvature is of order $\rho^\lambda$: there is $\kappa > 0$ such that

$$|k_x(\xi, v)| \leq \kappa \rho^\lambda$$

for all $x \in X$, $\xi \in T_x X$, $v \in E_x$.

Notice that, adopting the notation following (3.2), the above estimate is equivalent to the inequality

$$|\Theta(v)| \leq \kappa \rho^\lambda \varphi.$$
Also notice that a line bundle \((L, h)\) has finite order if and only if
\[ |c_1(L, h)| \leq \kappa \rho^\lambda \varphi. \]

**Remark:** The above definition is preserved under standard operations with vector bundles: direct sum, tensor product, dualization, symmetric power, exterior power etc. In particular, if \(E_1\) and \(E_2\) are Hermitian vector bundles of finite order then so is \(\mathcal{H}om(E_1, E_2) \simeq E_1^* \otimes E_2^*\).

Sections of finite order of a Hermitian vector bundle are defined the same way as functions: for a global section \(\sigma\) of \(E\) we introduce the maximum modulus function
\[ M_\sigma(r) = \log \max \{|\sigma(x)|_h, \rho(x) \leq r\}. \]

We say that \(\sigma\) has order \(\lambda\) in the sup-norm sense if \(M_\sigma(r) = O(r^\lambda)\). In view of the previous remark this gives us a notion of finite order morphism between two Hermitian vector bundles. The composite of morphisms of finite order is of finite order, too. Thus we have constructed the category of Hermitian holomorphic vector bundles of finite order. In this category a vector bundle of rank \(r\) is trivial, i.e. it is isomorphic to the trivial vector bundle \(\mathbb{C}^r\), if and only if it possesses \(r\) global sections of finite order which generate the fiber at every point. We will denote by \(\text{Vect}_{f.o.}(X)\) the set of all Hermitian holomorphic vector bundles of rank \(r\) on \(X\) modulo isomorphisms in the finite order category. \(\text{Vect}_{f.o.}(X)\) together with the tensor product forms a group which we call the Picard group of finite order line bundles on \(X\) and denote \(\text{Pic}_{f.o.}(X)\). As we explained in the introduction, the motivation for our work is the following:

**Question:** Is the canonical map \(\text{Vect}_{f.o.}(X) \to \text{Vect}_{hol}(X)\) a bijection?

The question was answered in the affirmative for the case of line bundles.

**Theorem** (Griffiths and Cornalba [5]): The canonical map
\[ \text{Pic}_{f.o.}(X) \to \text{Pic}_{hol}(X) \]
is an isomorphism of groups. In other words, every holomorphic line bundle on \(X\) admits a metric of finite order and a Hermitian holomorphic line bundle of finite order is trivial in the holomorphic category if and only if it is trivial in the finite order category.

**Definition:** Let \((E, h)\) be a holomorphic Hermitian vector bundle. A global section \(\sigma\) of \(E\) is said to have order \(\lambda\) if there is \(\kappa \geq 0\) such that
\[ \int_X |\sigma|^2 e^{-\kappa \rho^\lambda \Phi} < \infty. \]
Here \(\Phi = \varphi^n\) is the volume form of \(\varphi\). We define the sheaf \(\mathcal{O}_\lambda(E)\) of germs of holomorphic sections of \(E\) of order \(\lambda\) as follows: \(\mathcal{O}_\lambda(E)\) is a sheaf on \(X\); for each open set \(U \subset X\) the space of sections \(\mathcal{O}_\lambda(U, E)\) consists of those holomorphic sections \(\sigma\) of \(E\) defined on \(U \cap X\) with the property that around each point
at infinity \( x \in (\overline{X} \setminus X) \cap U \) there is a neighbourhood \( W \subset U \) and a constant \( \kappa > 0 \), depending on \( x \) and \( \sigma \), such that
\[
\int_{W \cap X} |\sigma|^2 \cdot e^{-\kappa r^\lambda} \Phi < \infty.
\]
Similarly, we define the sheaf \( \mathcal{O}_{\text{f.o.}}(E) \) of germs of holomorphic sections of \( E \) of finite order by the same estimate as above with the additional requirement that \( \lambda \) depend on \( \sigma \) and on the point at infinity \( x \).

It is easily seen that \( \mathcal{O}_\lambda(E) \) and \( \mathcal{O}_{\text{f.o.}}(E) \) are modules over \( \mathcal{O}_\lambda \), respectively over \( \mathcal{O}_{\text{f.o.}} \). Their restrictions to \( X \) coincide with the sheaf \( \mathcal{O}(E) \) of germs of sections of \( E \) because in (4.5) there are no conditions at the points \( x \in X \) away from infinity. By the compactness of the divisor at infinity it is also transparent that the spaces of global sections \( \mathcal{O}_\lambda(X, E) \) and \( \mathcal{O}_{\text{f.o.}}(X, E) \) are nothing but the spaces of global sections of \( E \) of order \( \lambda \), respectively of finite order.

Let us now assume that \( E \) has order \( \lambda \). Then a global section \( \sigma \) of \( E \) has finite order in the sup-norm sense if and only if it has finite order in the sense of (4.5). One direction is immediate: if \( M_\sigma(r) = O(r^\lambda) \) then \( |\sigma|_h \leq e^{\kappa r^\lambda} \) for some \( \kappa \geq 0 \) and the integral
\[
\int_X |\sigma|^2 \cdot e^{-2\kappa r^\lambda} \Phi
\]
is finite because \( \Phi \) has finite volume. For the other direction we refer to [5].

**Theorem 4.6**: Let \( E \) be a Hermitian holomorphic vector bundle of order \( \lambda \) on a special affine variety \( X \). Then, for all \( q \geq 1 \), we have
\[
H^q(X, \mathcal{O}_\lambda(E)) = 0, \quad H^q(X, \mathcal{O}_{\text{f.o.}}(E)) = 0.
\]

We now prepare to move from the Hermitian to the Finsler case. We begin by translating definition (4.5) “upstairs” on \( \mathbb{P}(E) \). There is a canonical isomorphism of vector bundles \( \pi_*(\mathcal{L}) \simeq E^* \) which associates to a section \( \sigma \) of \( E^* \) the section \( \tilde{\sigma} \) of \( \mathcal{L} \) defined by
\[
\tilde{\sigma}_{(x,[v])}(tv) = \sigma(tv),
\]
for all \( x \in X, \ v \in E_x \setminus \{0\}, \ t \in \mathbb{C} \). Recall that the Hermitian metric \( h \) on \( E \) (which in particular is a Finsler metric) induces a metric \( \bar{h} \) on \( \mathcal{L} \). Denote by \( h^* \) the dual metric on \( E^* \). It happens that
\[
|\sigma(x)|_{h^*} = \max\{||\tilde{\sigma}(x,[v])||_{\bar{h}}, \ v \in E_x \setminus \{0\}\}.
\]

Assume that \( E \) has a Hermitian metric \( h \) of order \( \lambda \). We claim that there is a positive constant \( \kappa \) such that the expression
\[
\tilde{\varphi} = c_1(\mathcal{L}, \bar{h}) + dd^c(\kappa r^\lambda)
\]
gives a positive (1,1)-form on \( \mathbb{P}(E) \). We check this at a point \( (x,[v]) \). Since positivity is preserved under change of coordinates we can work with a frame
of $E$ which is normal at $x$. By our hypothesis (4.1) we have $|\Theta(v)| \leq \kappa \rho^{\lambda} \varphi$.

We also have

$$\frac{\kappa \lambda}{2} \rho^{\lambda} \varphi = \frac{\kappa \lambda}{2} \rho^{\lambda} dd^{c} \tau \leq \frac{\kappa \lambda}{2} \rho^{\lambda} dd^{c} \tau + \frac{\kappa \lambda^{2}}{4} \rho^{\lambda} d\tau \wedge d^{c} \tau$$

$$= \kappa dd^{c} e^{\lambda \tau / 2}$$

$$= dd^{c}(\kappa \rho^{\lambda})$$

$$\leq \frac{\kappa \lambda}{2} \rho^{\lambda} dd^{c} \tau + \frac{\kappa \lambda^{2}}{4} \rho^{\lambda+\epsilon} \varphi$$

by (1.8).

From this calculation we get the estimate

\[(4.10) \quad \frac{\kappa \lambda}{2} \rho^{\lambda} \varphi \leq -\Theta(v) + dd^{c}(\kappa \rho^{\lambda}) \leq 2\kappa \rho^{\lambda+\epsilon} \varphi.\]

Combining (3.24) with (4.10) we get the following estimate at $(x, [v])$:

\[(4.11) \quad \omega + \frac{\kappa \lambda}{2} \rho^{\lambda} \varphi \leq \bar{\varphi} \leq \omega + 2\kappa \rho^{\lambda+\epsilon} \varphi.\]

This insures the positivity of $\bar{\varphi}$. In the sequel $\mathbb{P}(E)$ will be considered equipped with the Kähler metric induced by $\bar{\varphi}$. Since $\rho^{\lambda} \circ \pi$ is a plurisubharmonic exhaustive function, we can arrange, possibly by choosing a larger $\kappa$, that this metric on $\mathbb{P}(E)$ be complete.

\[(4.12) \quad \text{Claim: Assume that } E \text{ is a holomorphic Hermitian vector bundle of order } \lambda. \text{ Let } U \subset X \text{ be an open subset and } \sigma \text{ a section of } E^{\ast} \text{ over } U. \text{ Then there exists } \kappa_{1} \geq 0 \text{ such that}

\[\int_{U} |\sigma|^{2}_{\tilde{h}} \cdot e^{-\kappa_{1}\rho^{\lambda}} \Phi \leq \infty\]

\[\text{if and only if there exists } \kappa_{2} \geq 0 \text{ such that}

\[\int_{\pi^{-1}(U)} |\tilde{\sigma}|^{2}_{\tilde{h}} \cdot e^{-\kappa_{2}\rho^{\lambda}} \tilde{\Phi} \leq \infty.\]

\[\text{Proof: Let us denote by } \tilde{\varphi}^{V} \text{ and } \tilde{\varphi}^{H} \text{ the vertical, respectively the horizontal part of } \tilde{\varphi}, \text{ as defined at the end of §3. From (4.10) we have}

\[(4.13) \quad \frac{\kappa \lambda}{2} \rho^{\lambda} \varphi \leq \tilde{\varphi}^{H} \leq 2\kappa \rho^{\lambda+\epsilon} \varphi\]

where, by an abuse of notation, we write $\varphi$ instead of $\pi^{\ast} \varphi$. We have

$$\tilde{\Phi} = \tilde{\varphi}^{n+r} = \binom{n+r}{n} (\tilde{\varphi}^{H})^{n} \wedge (\tilde{\varphi}^{V})^{r}$$

which, combined with (4.13), gives

\[(4.14) \quad C^{-1} \rho^{n \lambda} \varphi^{n} \wedge (\tilde{\varphi}^{V})^{r} \leq \tilde{\Phi} \leq C \rho^{n \lambda+nc} \varphi^{n} \wedge (\tilde{\varphi}^{V})^{r}\]

for a fixed positive constant $C$. This estimate is useful because one can apply Fubini’s theorem to $|\tilde{\sigma}|^{2}_{\tilde{h}} \cdot e^{-\kappa \rho^{\lambda}} \varphi^{n} \wedge (\tilde{\varphi}^{V})^{r}$. More precisely, one can integrate this form first vertically along the fibers of $\mathbb{P}(E)$ and then horizontally along
This finishes the proof of the claim because there is a positive constant $A$ depending only on $r$ such that for all $x \in X$

\begin{equation}
\int_{[v] \in \mathbb{P}(E_x)} |\tilde{\sigma}(x,[v])|_R^2 (\tilde{\phi}_{\mathbb{P}(E_x)})^r = A |\sigma(x)|_{\mathbb{R}}^2.
\end{equation}

5. FINSLER METRICS OF FINITE ORDER. THE VANISHING THEOREM

We explained in the introduction the difficulty one encounters in trying to generalize (4.4) and the four equivalences to bundles of rank greater than 1. It seems to us that the correct antidote is to translate the definition of finite order "upstairs" on $\mathbb{P}(E)$. So we define Finsler metrics of finite order by a very similar estimate on the holomorphic bisectional curvature. We then define sections of finite order and we prove that they span at every point, cf. (5.23). This we achieve by means of the vanishing theorem (5.12) which generalizes (4.6).

Traditionally, vanishing theorems are given either on Stein manifolds or on compact Kähler manifolds. Our case here is a hybrid: we will have to solve the $\bar{\partial}$-equation on $\mathbb{P}(E)$ whose base is Stein (because it is affine) while its fibers are compact.

The original proof of Kodaira’s Vanishing Theorem makes use of Hodge theory: one knows that on a compact Kähler manifold cohomology classes can be represented by harmonic forms and one argues, using the a priori estimate, that such forms do not exist. For non-compact manifolds this argument does not work because we do not know if the Hodge representation theorem holds. Instead we will use a very potent technique developed by Hörmander in [8] involving some rudiments of functional analysis. We synthetize this technique in proposition (5.3) and theorem (5.4) from below. We refer to [9], [8], [2] and [10].

Let $Y$ be a complex manifold of dimension $m$ equipped with a positive Kähler form $\omega$ which induces a complete metric. Relative to local holomorphic coordinates $(z^1, \ldots, z^m)$ we write

$$\omega = \sqrt{-1} \sum_{i,j=1}^{m} g_{ij} \, dz^i \wedge d\bar{z}^j.$$ 

The condition that $\omega$ be positive definite means that the matrix $(g_{ij})_{1 \leq i,j \leq m}$ is Hermitian and positive-definite. The condition that $\omega$ be Kähler means that $d\omega = 0$. This is a very natural condition to consider because it is equivalent to saying that the complex structure on the real tangent bundle of $Y$ is compatible with the Levi-Civita connection associated to the induced Riemannian metric on the real tangent bundle of $Y$.

Let $\text{Ric}(\omega)$ be the Ricci curvature of $\omega$. It is a (1,1)-form given in local coordinates by

$$\text{Ric}(\omega) = dd^c \log(\det(g_{ij})).$$

It is nothing but the first Chern form of the canonical line bundle $K_Y := \wedge^m T^*Y$ equipped with the metric induced by $\omega$. 
Recall from §3 that the first Chern form of a holomorphic line bundle $L$ equipped with a Hermitian metric $h$ is given by

$$c_1(L,h) = -dd^c\log(h).$$

Let us choose an orthonormal frame $\{\xi_1, \ldots, \xi_m\}$ for the tangent space at a point $y \in Y$ and a unitary vector $e \in L_y$. Relative to the dual frame $\{d\xi_1, \ldots, d\xi_m\}$ for $T^*Y$ we write

$$\text{Ric}(\omega) = \sqrt{-1} \sum_{i,j=1}^m R_{ij} \, d\xi^i \wedge d\bar{\xi}^j,$$

$$c_1(L) = \sqrt{-1} \sum_{i,j=1}^m K_{ij} \, d\xi^i \wedge d\bar{\xi}^j.$$

For an $L$-valued $(0,q)$-form $u = \sum_{|I|=q} u_I \, d\bar{\xi}^I \otimes e$ we define the pointwise operators

$$<Ru,u> = q \sum_{i,j=1}^m \sum_{|I|=q-1} R_{ij} \ u_{iI} \bar{u}_{jI},$$

$$<Ku,u> = q \sum_{i,j=1}^m \sum_{|I|=q-1} K_{ij} \ u_{iI} \bar{u}_{jI},$$

and their integrated versions

$$(Ru, u) = \int_Y <Ru, u> \, dV,$$

$$(Ku, u) = \int_Y <Ku, u> \, dV.$$ 

Here $u$ is assumed to have compact support and $dV = \omega^m$ is the volume form of $\omega$. In the sequel we will denote by $\mathcal{D}^{p,q}(Y,L)$ the space of smooth $L$-valued $(p,q)$-forms with compact support. The Hermitian inner products on $T^*Y$ and on $L$ induce a Hermitian inner product $\langle \cdot, \cdot \rangle$ on $\wedge^{p,q}T^*Y \otimes L$. Given $u, v \in \mathcal{D}^{p,q}(Y,L)$ we put

$$(u, v) = \int_Y \langle u, v \rangle \, dV.$$ 

Clearly $(\cdot, \cdot)$ defines a Hermitian inner product on $\mathcal{D}^{p,q}(Y,L)$. Its associate norm is given by $\|u\|^2 = (u, u)$. The operator

$$\bar{\partial} : \mathcal{D}^{p,q}(Y,L) \longrightarrow \mathcal{D}^{p,q+1}(Y,L)$$

has a formal adjoint

$$\partial : \mathcal{D}^{p,q+1}(Y,L) \longrightarrow \mathcal{D}^{p,q}(Y,L).$$
given by the condition \((\bar{\partial} u, v) = (u, \vartheta v)\) for all \(u \in \mathcal{D}^p,q(Y, L),\ v \in \mathcal{D}^{p,q+1}(Y, L)\).

We mention in passing that \(\vartheta = -\star D'\star\) where \(\star\) is the Hodge-\(\star\) operator while \(D'\) is the \((1,0)\)-component of the Chern connection of \(L\).

Our key ingredient towards proving vanishing theorems is the following inequality which we state in a particular case that is of interest to us. See [10] p. 124 for the full statement and p. 126 for the statement from below.

\[(5.1)\] **Weitzenböck Inequality:** Let \(Y\) be a Kähler manifold. Let \(L\) be a holomorphic Hermitian line bundle on \(Y\) and \(u\) a smooth \(L\)-valued \((0,q)\)-form with compact support. Then we have:

\[||\bar{\partial} u||^2 + ||\vartheta u||^2 \geq (Ku, u) - (Ru, u).\]

\[(5.2)\] **Definition:** We say that \(L\) is \((p,q)\)-elliptic if there is a positive constant \(\epsilon\), called ellipticity constant, such that for all \(u \in \mathcal{D}^{p,q}(Y, L)\) we have the estimate

\[||\bar{\partial} u||^2 + ||\vartheta u||^2 \geq \epsilon ||u||^2.\]

\[(5.3)\] **Proposition:** Assume \(L\) is \((p,q)\)-elliptic and that the metric on \(Y\) is complete and Kähler. Then for any \(u \in L^{p,q}(Y, L)\) with \(\bar{\partial} u = 0\) there is \(v \in L^{p,q-1}(Y, L)\) with \(\bar{\partial} v = u\). In addition \(v\) is smooth.

Here \(L^{p,q}(Y, L)\) is the completion of \(\mathcal{D}^{p,q}(Y, L)\) relative to the norm \(||u||^2\).

Notice here the crucial assumption that \(Y\) be complete. An equivalent condition is that any ball in \(Y\) be relatively compact. We need this for a density argument to make sure that (5.1) holds not only for smooth forms but also for square-integrable forms. See [2], p. 92, lemma 4.

\[(5.4)\] **Theorem:** Let \(Y\) be a complex manifold equipped with a complete Kähler metric. Let \(L\) be a holomorphic line bundle on \(Y\) equipped with a Hermitian metric \(h\). We denote by \(\omega\) the Kähler form of \(Y\). Assume that there is a positive constant \(\epsilon\) such that

\[c_1(L, h) - \text{Ric}(\omega) \geq \epsilon \omega.\]

Let \(q\) be a positive integer. Then for any \(u \in L^{0,q}(Y, L)\) with \(\bar{\partial} u = 0\) there is \(v \in L^{0,q-1}(Y, L)\) with \(\bar{\partial} v = u\). In addition \(v\) is smooth.

**Proof:** By hypothesis for any \(u \in D^{0,q}(Y, L)\) we have

\[(Ku, u) - (Ru, u) \geq q\epsilon||u||^2.\]

Combining this with the Weitzenböck Inequality we obtain

\[||\bar{\partial} u||^2 + ||\vartheta u||^2 \geq q\epsilon||u||^2.\]

Hence \(L\) is \((0,q)\)-elliptic with ellipticity constant \(q\epsilon\). The theorem now follows from (5.3).

\[(5.5)\] **Definition:** Let \(E\) be a holomorphic vector bundle on \(X\) and \(h\) a Finsler
metric on $E$. We say that $h$ has order $\lambda$ if its holomorphic bisectional curvature is of order $\rho^\lambda$: there is $\kappa > 0$ such that

$$|k(\zeta, v)| \leq \kappa \rho^\lambda$$

for all $y \in E_o$, $\zeta \in T_y E_o$, $v \in E_x$ where $p(y) = x$.

Notice that this definition encompasses (4.1). For a line bundle a Finsler metric is the same thing as a Hermitian metric, so (5.5) is relevant only when the rank of $E$ is at least 2, which we will assume henceforth.

Notice that if we take $\zeta$ to be the horizontal lift of a tangent vector from $X$ and $v$ to be the position vector field we obtain that the horizontal holomorphic bisectional curvature is also of order $\rho^\lambda$:

$$|k_x(\xi, v)| \leq \kappa \rho^\lambda$$

for all $x \in X$, $\xi \in T_x X$, $v \in E_x$. Adopting the notation below (3.20) the above estimate is equivalent to the inequality

$$|\Theta(P)| \leq \kappa \rho^\lambda \varphi.$$  

In view of (3.24) and using the same argument as in the Hermitian case we can find a positive constant $\kappa$ such that the expression

$$\tilde{\varphi} = c_1(\mathcal{L}, \tilde{h}) + dd^c(\kappa \rho^\lambda)$$

defines a Kähler form on $\mathbb{P}(E)$ which induces a complete metric. Estimate (4.13) also holds:

$$\frac{\kappa \lambda}{2} \rho^\lambda \varphi \leq \tilde{\varphi}^\mathcal{H} \leq 2\kappa \rho^\lambda + \varphi.$$  

Moreover, the constant $\kappa$ can be so chosen that

$$|c_1(G_{ij})| \leq \kappa \rho^\lambda \tilde{\varphi}.$$  

This is so because an estimate on the holomorphic bisectional curvature of a vector bundle $E$ gives an estimate of the curvature of the induced metric on the determinant of $E$. From the proof of (5.12) it transpires that the Ricci curvature of $\mathbb{P}(E)$ is, in essence, equal to $c_1(G_{ij})$ plus some other terms that are controllable. Thus, roughly speaking, the above estimate says that the Ricci curvature of $\mathbb{P}(E)$ has polynomial growth. Taking horizontal and vertical parts we obtain the following inequalities:

$$|c_1(G_{ij})^H| \leq \kappa \rho^\lambda \varphi,$$

$$|c_1(G_{ij})^V| \leq \kappa \rho^\lambda c_1(\mathcal{L}, \tilde{h})^V.$$  

In fact, (5.6), (5.9) and (5.10) are all that we will need in the sequel. It is not clear to us if these estimates imply (5.5), in other words it seems to us that (5.5) is stronger than (5.6), (5.9) and (5.10). We could have chosen the three latter estimates as our definition of finite order Finsler metric, but then the similarity with the Hermitian case would have been obscured.

Our Finsler metric is on $E$ but we will be concerned with and we will prove a vanishing theorem for the sections of order $\lambda$ of $E^*$. We define the latter in the spirit of (4.12), by using the one-to-one correspondence (4.7) between sections
of $E^*$ and sections of $\mathcal{L}$. Before doing that we fix a Hermitian metric $g$ of finite order on $\det(E)$. Such a metric exists by (4.4). Under the isomorphism

$$\mathcal{L} \simeq \mathcal{L} \otimes \pi^*(\det(E^*)) \otimes \pi^*(\det(E))$$

we consider the following metric on $\mathcal{L}$:

$$\tilde{l} = \tilde{h} \cdot \det(G_i) - 1 \cdot \pi^*(g).$$

(5.11) Definition: Let $E$ be a holomorphic vector bundle on $X$ equipped with a Finsler metric of order $\lambda$. A global section $\sigma$ of $E^*$ is said to have order $\lambda$ if there is $\kappa > 0$ such that

$$\int_{\mathbb{P}(E)} |\tilde{\sigma}|^2 \cdot e^{-\kappa \rho^k} \Phi < \infty.$$ 

We define the sheaf $\mathcal{O}_\lambda(E^*)$ of germs of holomorphic sections of $E^*$ of order $\lambda$ as follows: $\mathcal{O}_\lambda(E^*)$ is a sheaf on $\overline{X}$; for each open set $U \subset \overline{X}$ the space of sections $\mathcal{O}_\lambda(U, E^*)$ consists of those holomorphic sections $\sigma$ of $E^*$ defined on $U \cap X$ with the property that around each point at infinity $x \in (\overline{X} \setminus X) \cap U$ there is a neighbourhood $W \subset U$ and a constant $\kappa > 0$, depending on $X$ and $\sigma$, such that

$$\int_{\pi^{-1}(W \cap X)} |\tilde{\sigma}|^2 \cdot e^{-\kappa \rho^k} \Phi < \infty.$$

Similarly, we define the sheaf $\mathcal{O}_{f.o.}(E^*)$ of germs of holomorphic sections of $E^*$ of finite order by the same estimate as above with the additional requirement that $\lambda$ depend on $\sigma$ and on the point at infinity $x$.

It is easily seen that $\mathcal{O}_\lambda(E^*)$ and $\mathcal{O}_{f.o.}(E^*)$ are modules over $\mathcal{O}_\lambda$, respectively over $\mathcal{O}_{f.o.}$. Their restrictions to $X$ coincide with the sheaf $\mathcal{O}(E^*)$ of germs of sections of $E^*$ because in (5.11) there are no conditions at the points $x \in X$ away from infinity. By the compactness of the divisor at infinity it is also transparent that the spaces of global sections $\mathcal{O}_\lambda(\overline{X}, E^*)$ and $\mathcal{O}_{f.o.}(\overline{X}, E^*)$ are the space of global sections of $E^*$ of order $\lambda$, respectively of finite order which is independent of the choice of compactification $\overline{X}$. Finally, the use of $\tilde{l}$ instead of $\tilde{h}$ may seem awkward but, in doing so, we do not deviate from our aim of studying finite order objects. Our choice of metric is dictated by technical reasons which will become clear in the proof of the next theorem. So here is the main result of this section:

(5.12) Theorem: Let $X$ be a special affine variety and $E$ a holomorphic vector bundle on $X$ equipped with a Finsler metric $h$ of order $\lambda$. Then there is $\mu \geq \lambda$ such that for all $q \geq 1$, we have

$$H^q(\overline{X}, \mathcal{O}_{\mu}(E^*)) = 0, \quad H^q(\overline{X}, \mathcal{O}_{f.o.}(E^*)) = 0.$$ 

Proof: Let $\mu$ be the largest between $\lambda$ and the order of $g$. For conciseness of notation we write $Y$ instead of $\mathbb{P}(E)$. It is known, see for instance [15], that we have the isomorphism

$$K_Y \simeq \mathcal{L}^{-r-1} \otimes \pi^*(\det(E^*)) \otimes \pi^*(K_X).$$
On the canonical line bundle $K_X$ we have the metric $k$ induced by $\varphi$ as follows: if, relative to a local coordinates system $(z^1, \ldots, z^n)$, we can write

$$\varphi = \sqrt{-1} \sum_{i,j=1}^n \varphi_{ij} \, dz^i \wedge d\bar{z}^j$$

then, relative to the frame $dz^1 \wedge \ldots \wedge dz^n$ of $K_X$ we have

$$k = \det(\varphi_{\bar{z}z})^{-1}.$$ 

Notice that $c_1(K_X, k) = \text{Ric}(\varphi)$. Likewise, let $\tilde{k}$ be the metric on $K_Y$ induced by $\tilde{\varphi}$. Using the above isomorphism we put another metric on $K_Y$ by

$$\tilde{k}' = \tilde{h}^{-r-1} \cdot \det(G_{\tilde{ij}})^{-1} \cdot \pi^*(k).$$

We claim that $\tilde{k}$ and $\tilde{k}'$ are almost equivalent: there is $C > 0$ such that

$$(5.13) \quad C^{-n\lambda-nc} \tilde{k}' \leq \tilde{k} \leq C^{-n\lambda} \tilde{k}'.$$ 

We check this at a point $(x, [v])$. We may assume that the frame of $E$ is normal at $(x, v)$. Taking into account that $\det(G_{\tilde{ij}}) = 1$ at $(x, [v])$ and taking determinants in (4.11) we obtain

$$\det(\omega)\det(G_{ij})2^{-n} \kappa^n \rho^{\lambda+nc} \det(\varphi) \leq \det((\tilde{\varphi})) \leq \det(\omega)\det(G_{ij})2^{n} \kappa^n \rho^{n\lambda+nc} \det(\varphi).$$

Recall that $\omega$ is the Fubini-Study form on $T_{[v]}\mathbb{P}^r$. It is known that the metric $\det(\omega)$ induced by $\omega$ on $\wedge^r T\mathbb{P}^r \simeq \mathcal{O}_{\mathbb{P}^r}(r+1)$ coincides with the canonical metric of $\mathcal{O}_{\mathbb{P}^r}(r+1)$. Therefore, under the identification $\mathcal{L}^{(r+1)} \simeq \wedge^r T_{[x,v]}\mathbb{P}(E_x)$, we have $\tilde{h}^{r+1} = \det(\omega)$. Taking inverse in the above inequalities we obtain (5.13).

Under the isomorphism $\mathcal{L} \simeq \mathcal{L} \otimes K_Y^{-1} \otimes K_Y$ we put a metric $\tilde{l}$ on $\mathcal{L}$ by

$$\tilde{l} = \tilde{l} \cdot (\tilde{k}')^{-1} \cdot \tilde{k}.$$ 

By (5.13) the two metrics $\tilde{l}$ and $\tilde{l}'$ on $\mathcal{L}$ are almost equivalent:

$$C^{-n\lambda} \tilde{l}' \leq \tilde{l} \leq C^{-n\lambda+nc} \tilde{l}'$$

Thus, in definition (5.11) we can replace $\tilde{l}$ by $\tilde{l}'$. The latter has the advantage that its Chern form can be bounded from below. Indeed, denoting $\mathcal{L}_n$ the line bundle $\mathcal{L}$ equipped with the metric $\tilde{l}_n = \tilde{l}' \cdot e^{-n\rho^\mu}$, we have:

$$c_1(\mathcal{L}_n) - \text{Ric}(\tilde{\varphi}) = c_1(\mathcal{L}_n, \tilde{l}') + dd^c(\kappa \rho^\mu) - c_1(\mathcal{L}_n, \tilde{k})$$

$$= c_1(\mathcal{L}_n, \tilde{l}) - c_1(\mathcal{L}_n, \tilde{k}) + dd^c(\kappa \rho^\mu)$$

$$= c_1(\mathcal{L}, \tilde{h}) - c_1(\pi^*(\det(E)), \det(G_{\tilde{ij}})) + c_1(\det(E), g)$$

$$+ (r+1)c_1(\mathcal{L}, \tilde{h}) + c_1(\pi^*(\det(E)), \det(G_{\tilde{ij}}))$$

$$- \pi^*\text{Ric}(\varphi) + dd^c(\kappa \rho^\mu)$$

$$= (r+2)c_1(\mathcal{L}, \tilde{h}) + c_1(\det(E), g)$$

$$- \pi^*\text{Ric}(\varphi) + dd^c(\kappa \rho^\mu).$$
From (1.2) and from the hypothesis that \( g \) have order \( \mu \) we conclude that there is \( \kappa_0 > 0 \) such that for all \( \kappa \geq \kappa_0 \) we have

\[
(5.14) \quad c_1(L_\kappa) - \text{Ric}(\varphi) \geq \varphi.
\]

The abstract de Rham theorem tells us that the cohomology of a sheaf can be computed by taking an acyclic resolution. We recall that a sheaf is said to be \textit{acyclic} if all its cohomology groups, beside \( H^0 \), vanish. Therefore, in order to show that the higher cohomology of \( \mathcal{O}_\mu(E^*) \) vanishes, we will construct an acyclic resolution of this sheaf which is exact at the level of global sections. We define the sheaves \( \mathcal{A}_{\mu,q}^0(E^*) \) as follows: at each \( x \in X \) the stalk \( \mathcal{A}_{\mu,q}^0(E^*)_x \) is the space of germs of smooth \( L \)-valued \((0,q)\)-forms defined on \( \pi^{-1}(U) \), where \( U \) is an open neighbourhood of \( x \) in \( X \). If \( x \in \overline{X} \setminus X \) is a point at infinity, the stalk \( \mathcal{A}_{\mu,q}^0(E^*)_x \) is the space of germs of smooth \( L \)-valued \((0,q)\)-forms defined on \( \pi^{-1}(U \cap X) \), with \( U \) some open neighbourhood of \( x \) in \( \overline{X} \), such that both \( u \) and \( \partial u \) have order \( \mu \) in the \( L^2 \)-sense. This means that there is \( \kappa > 0 \) such that

\[
\int_{\pi^{-1}(U \cap X)} |u|^2 \cdot e^{-\kappa|u|^2} \Phi < \infty, \quad \int_{\pi^{-1}(U \cap X)} |\partial u|^2 \cdot e^{-\kappa|\partial u|^2} \Phi < \infty.
\]

Here \( | \cdot | \) is taken with respect to \( \tilde{l} \). Clearly \( \mathcal{A}_{\mu,q}^0(E^*) \), \( 0 \leq q \leq n + r \), are sheaves on \( \overline{X} \). In fact, they are modules over the sheaf \( \mathcal{A}_{\overline{X}} \) of smooth \( \mathbb{C} \)-valued functions on \( \overline{X} \). As such they are soft, because any \( \mathcal{A}_{\overline{X}} \)-module is a soft sheaf. This is due to the fact that we can find smooth partitions of the unity on \( \overline{X} \). We recall that a sheaf is said to be \textit{soft} if any section over a closed subset can be extended to a global section. One knows that soft sheaves are acyclic. Notice that we have a complex

\[
(5.15) \quad 0 \longrightarrow \mathcal{O}_\mu(E^*) \longrightarrow \mathcal{A}_{\mu,0}^0(E^*) \xrightarrow{\partial} \mathcal{A}_{\mu,1}^0(E^*) \xrightarrow{\partial} \ldots \xrightarrow{\partial} \mathcal{A}_{\mu,q}^0(E^*) \xrightarrow{\partial} \ldots
\]

which is clearly exact at the level of \( \mathcal{A}_{\mu,q}^0(E^*) \). Thus the theorem will be proven once we manage to establish that

(i) the complex (5.15) is exact,

(ii) the complex (5.15) is exact at the level of global sections.

We begin with the latter. Choose \( u \in \mathcal{A}_{\mu,q}^0(\overline{X}, E^*) \) such that \( \partial u = 0 \). There is \( \kappa \geq \kappa_0 \) so large that \( u \) be square integrable with respect to the metric of \( L_\kappa \). But (5.14) tells us that \( L_\kappa \) satisfies the hypotheses of theorem (5.4). We can find a smooth \( v \in \mathcal{L}^{0,q-1}(Y, L_\kappa) \) such that \( \partial v = u \). By definition \( v \in \mathcal{A}_{\mu,q-1}^0(\overline{X}, E^*) \). This proves (ii).

We now turn to (i). Exactness of a complex of sheaves means exactness at the level of stalks. Fix an arbitrary point \( x \in \overline{X} \) and a germ \( u_x \in \mathcal{A}_{\mu,q}^0(E^*)_x \) represented by some \( u \in \mathcal{A}_{\mu,q}^0(U, E^*) \). Here \( U \) is a small open neighbourhood of \( x \) in \( \overline{X} \). Our aim is to find \( v \in \mathcal{A}_{\mu,q}^0(W, E^*) \), defined over a possibly smaller neighbourhood \( W \) of \( x \), such that \( \partial v = u \) on \( \pi^{-1}(W \cap X) \).

Let us choose a Stein neighbourhood of \( x \), say an open polycylinder \( P \subset U \). After possibly shrinking \( P \) we may assume that there is \( \kappa \geq \kappa_0 \) such that \( u \) is square integrable relative to \( l_\kappa \), i.e. \( u \in \mathcal{L}^{0,q}(\pi^{-1}(P \cap X), L_\kappa) \). We will fix this
κ for the remainder of this proof. However, we cannot yet use (5.4) because the metric \( \hat{\varphi} \) on \( \pi^{-1}(U \cap X) \) is not complete. We correct this by adding a horizontal term to \( \hat{\varphi} \).

Let \( \chi \) be a strictly plurisubharmonic exhaustive function of \( P \). Such a function exists because \( P \) is Stein. We put \( Z = \pi^{-1}(P \cap X) \). We claim that the (1,1)-form 

\[
\hat{\varphi}_Z = \hat{\varphi} + \partial \overline{\partial} (\chi \circ \pi)
\]
determines a complete metric on \( Z \). To justify this we need to show that any ball \( B \) in \( Z \) is relatively compact. Here \( B \) is a ball relative to the geodesic distance induced by \( \hat{\varphi}_Z \). Since \( \hat{\varphi}_Z \) dominates \( \hat{\varphi} \) it is clear that \( B \) is included in a ball relative to the geodesic distance induced by \( \hat{\varphi} \). Thus \( \pi(B) \) stays away from the divisor at infinity \( X \setminus X \). Also \( \pi(B) \) is included in a ball relative to the geodesic distance induced by \( \partial \overline{\partial} \chi \) on \( P \). But \( \partial \overline{\partial} \chi \) induces a complete metric on \( P \) because \( \chi \) is exhaustive. Thus \( \pi(B) \) stays away from the boundary of \( P \).

We conclude that the closure of \( B \) in \( Z \) is compact, which justifies the claim.

Let \( \tilde{k}_Z \) be the metric on \( K_Z \) induced by \( \hat{\varphi}_Z \). It is easy to see that there are smooth functions \( \alpha, \beta : P \rightarrow (0, \infty) \) such that on \( P \) we have 

\[
\alpha \rho^\lambda \varphi - \frac{\alpha \rho^\lambda \varphi + \partial \overline{\partial} \chi}{2 \kappa \rho^\lambda + \partial \overline{\partial} \chi} \leq \beta \rho^\lambda \varphi.
\]

From this and (4.10) we get 

\[
\alpha \rho^\lambda \varphi - \frac{\alpha \rho^\lambda \varphi + \partial \overline{\partial} \chi}{2 \kappa \rho^\lambda + \partial \overline{\partial} \chi} \leq \beta \rho^\lambda \varphi.
\]

From this we obtain the analogue of (4.11):

\[
\omega + \alpha \rho^\lambda \varphi = \hat{\varphi}_Z \leq \omega + \beta \rho^\lambda \varphi.
\]

This can be used to get the analogue of (5.13):

\[
(5.16) \quad \beta^{-n} \rho^{-n\lambda - nc} \tilde{k}' \leq \tilde{k}_Z \leq \alpha^{-n} \rho^{-n\lambda} \tilde{k}'.
\]

Under the isomorphism \( L|Z \simeq L_Z \otimes K_Z^{-1} \otimes K_Z \) we put a metric \( \tilde{l}_Z \) on \( L|Z \) by 

\[
\tilde{l}_Z = (\tilde{k}')^{-1} \cdot \tilde{k}_Z.
\]

By (5.16) the two metrics \( \tilde{l} \) and \( \tilde{l}_Z \) on \( L_Z \) satisfy

\[
(5.17) \quad \alpha^n \rho^{n\lambda} \tilde{l}_Z \leq \tilde{l} \leq \beta^n \rho^{n\lambda + nc} \tilde{l}_Z.
\]

The metric \( \tilde{l}_Z \) has the advantage that we can make it “satisfy” the hypothesis of theorem (5.4): let \( \gamma : P \rightarrow (0, \infty) \) be a strictly plurisubharmonic exhaustive function and let \( L_\gamma \) be the line bundle \( L_Z \) equipped with the metric 

\[
\tilde{l}_\gamma = \tilde{l}_Z \cdot e^{-\kappa \rho^\mu} \cdot e^{-\gamma}.
\]

We have

\[
c_1(L_\gamma) - \text{Ric}(\hat{\varphi}_Z) = (r + 2)c_1(L, \hat{h}) + c_1(\det(E), g)
\]

\[
- \pi^* \text{Ric}(\varphi) + \partial \overline{\partial} (\kappa \rho^\mu) + \partial \overline{\partial} \gamma
\]

\[
\geq \hat{\varphi} + \partial \overline{\partial} \gamma.
\]
for \( \kappa \geq \kappa_o \) so that (5.14) hold. Hence for any \( \gamma \) growing faster than \( \chi \) we will have
\[
(5.18) \quad c_1(\mathcal{L}_\gamma) - \text{Ric}(\hat{\varphi}_Z) \geq \hat{\varphi}_Z.
\]

Let now \( Q \) be a polycylinder containing \( x \) and with \( \overline{Q} \subset P \). From (5.17) and the fact that \( u \) is square integrable with respect to the metrics \( \hat{I} \cdot e^{-\kappa l^\rho} \) on \( \mathcal{L} \) and \( \hat{\varphi} \) on \( Z \) we see that \( u \) is square integrable on \( \pi^{-1}(Q \cap X) \) relative to the metrics \( \hat{I}_Z \cdot e^{-\kappa l^\rho} \) on \( \mathcal{L} \) and \( \hat{\varphi}_Z \) on the manifold.

Choosing a sequence of polycylinders \( \{Q\} \) which exhaust \( P \) we prove that we can choose \( \gamma \) growing so fast that \( u \) be square integrable on \( Z \) relative to the metric \( \hat{I}_\gamma \) on \( \mathcal{L} \) and \( \hat{\varphi}_Z \) on \( Z \). In other words \( u \) belongs to \( L^{0,q}(Z, \mathcal{L}_\gamma) \).

The hypotheses of theorem (5.4) are fulfilled: the metric on \( Z \) is exact at every point and \( \hat{\kappa} \) and \( \hat{\varphi} \) on \( Z \) are Kähler and \( F \cdot O \)

\[
\text{Proof:} \quad \text{By Hilbert's syzygy theorem } F \text{ has a finite resolution.}
\]

We claim that tensoring the above with \( O_{\mu}(E^*) \) we obtain a resolution for the sheaf \( F \otimes_{O_X} O_{\mu}(E^*) \). Indeed, the complex
\[
(5.20) \quad 0 \rightarrow O_X^{a_n} \xrightarrow{\alpha_n} O_X^{a_{n-1}} \xrightarrow{\alpha_{n-1}} \ldots \rightarrow O_X^{a_1} \xrightarrow{\alpha_1} F \rightarrow 0.
\]

We claim that tensoring the above with \( O_{\mu}(E^*) \) we obtain a resolution for the sheaf \( F \otimes_{O_X} O_{\mu}(E^*) \). Indeed, the complex
\[
(5.20) \quad 0 \rightarrow O_X^{a_n} \otimes O_{\mu}(E^*) \rightarrow O_X^{a_{n-1}} \otimes O_{\mu}(E^*) \rightarrow \ldots \rightarrow F \otimes O_{\mu}(E^*) \rightarrow 0
\]

is exact at every point \( x \in X \) by the mere fact that \( O_{\mu}(E^*)_x \simeq O_{X,x}^{a_n} \) is a free, hence flat, \( O_{X,x} \)-module. When \( x \in X \setminus X \) is a point at infinity the hypothesis that \( F_x \) be flat and standard arguments in homological algebra ensure that \( K\text{er}(\alpha_i)_x \) is flat for all \( 1 \leq i \leq n \). Assembling the exact sequences

\[
0 = \text{Tor}_1^C(\pi^*_x(K\text{er}(\alpha_{i-1})_x, O_{\mu}(E^*)_x) \rightarrow K\text{er}(\alpha_i)_x \otimes O_{\mu}(E^*)_x \rightarrow O_{\pi^*_x,x}^{a_n} \otimes O_{\mu}(E^*)_x \rightarrow K\text{er}(\alpha_{i-1})_x \otimes O_{\mu}(E^*)_x \rightarrow 0
\]

we conclude that (5.20) is exact. By the previous theorem we have the vanish-

ment of cohomology
\[
H^q(X, O_X^{a_n} \otimes O_{\mu}(E^*)) = 0, \quad q \geq 1.
\]

The corollary now follows from standard long exact sequences in cohomology induced by (5.20).

(5.21) Corollary: Assume that \( E \) satisfies the conditions from theorem (5.12).
Then $E^*$ is spanned at every point by global sections of order $\mu$. More precisely, given $x_1, \ldots, x_N \in X$ and $e_1 \in E^*_{x_1}, \ldots, e_N \in E^*_{x_N}$, there is a global section $\sigma$ of $E^*$ of order $\mu$ such that $\sigma(x_i) = e_i$ for $1 \leq i \leq N$.

**Proof:** Let $\mathcal{I} \subset \mathcal{O}_{\overline{X}}$ be the ideal sheaf of $\{x_1, \ldots, x_N\}$. It is a coherent algebraic sheaf on $\overline{X}$ which is flat at all $x \in X \setminus X$. In fact, it is locally free there. By (5.19) we have

\begin{equation}
H^1(\overline{X}, \mathcal{I} \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{O}_\mu(E^*)) = 0.
\end{equation}

Tensoring the exact sequence

\[ 0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\overline{X}} \longrightarrow \mathcal{O}_{\{x_1, \ldots, x_N\}} \longrightarrow 0 \]

with $\mathcal{O}_\mu(E^*)$ we obtain a sequence

\[ 0 \longrightarrow \mathcal{I} \otimes \mathcal{O}_\mu(E^*) \longrightarrow \mathcal{O}_\mu(E^*) \twoheadrightarrow \mathcal{O}(E^*)_{\{x_1, \ldots, x_N\}} \longrightarrow 0 \]

which is exact by the same discussion as in the proof of (5.19). Its long exact sequence in cohomology together with (5.22) show that $\alpha$ is surjective at the level of global sections. This finishes the proof because $\mathcal{O}(E^*)_{\{x_1, \ldots, x_N\}}$ is a skyscraper sheaf with stalks $E^*_{x_i}$ at $x_i$ and zero outside $\{x_1, \ldots, x_N\}$.

**(5.23) Proposition:** Assume that $E$ satisfies the conditions from theorem (5.12). Then there are finitely many global sections $\sigma_1, \ldots, \sigma_N$ of $E^*$ of order $\mu$ which span $E^*$ at every point. Moreover, these sections can be chosen in such a manner that for each $x \in X$ their differentials at $x$, written relatively to a trivialization of $E$ at $x$, span $E^*_x \otimes T^*_x X$.

**Proof:** We repeat here the arguments from §11 in [5]. By (5.21) there exist linearly independent global sections $\sigma_1, \ldots, \sigma_r$ of $E^*$ of order $\mu$. In this proof $r$ is the rank of $E$. Then $\sigma_1 \wedge \ldots \wedge \sigma_r$ is a nontrivial section of $\det(E^*)$. Its zero-set $Z$ is a proper analytic subset of $X$. We choose points $\{x_i\}_{i \geq 1}$ on the relative interiors of each component of $Z$. By (11.6) in [5] we can find a global section $\sigma_{r+1}$ of $E^*$ of order $\mu$ such that $\sigma_{r+1}(x_i) \notin \text{span}\{\sigma_1(x_i), \ldots, \sigma_r(x_i)\}$ for all $i$.

Let us make this explicit: given a diverging sequence of points $\{x_i\}_{i \geq 1}$ in $X$ and subspaces $F_i \neq E_{x_i}^*$, there exists a global section $\eta$ of $E^*$ of order $\mu$ such that $\eta(x_i) \notin F_i$ for all $i$. To see this choose $e_i \in E_{x_i}^* \setminus F_i$ and global sections $\eta_i$ of $E^*$ satisfying:

\[ \eta_i(x_j) = 0 \text{ for } j < i, \quad \eta_i(x_i) = e_i, \quad \int_{\mathbb{P}(E)} |\eta_i|^2 \cdot e^{-\kappa \rho^\alpha \Phi} < \infty. \]

Such sections exist by (5.21). The constant $\kappa$ can be assumed to be the same for all $i$ because the Vanishing Theorem and its corollary (5.21) hold with fixed large $\kappa$. We skip the details. Inductively on $i$ choose nonzero constants $c_i$ such that

\[ \sum_{j \leq i} c_j \eta_j(x_i) \notin F_i, \quad \int_{\mathbb{P}(E)} |c_i \eta_i|^2 \cdot e^{-\kappa \rho^\alpha \Phi} < 2^{-i}, \quad |c_i \eta_i| < 2^{-i} \text{ on } K_i. \]
Here $K_i, \ i \geq 1,$ is an exhaustion of $X$ by compact subsets. The norm $| \cdot |$ is taken relative to $l.\text{ Clearly,}$

$$\eta = \sum_{i \geq 1} c_i \eta_i$$

is well defined and satisfies our requirements.

Repeating this procedure we construct sections $\sigma_{r+1}, \ldots, \sigma_{2r}$ of $E^*$ of order $\mu$ such that $\sigma_1, \ldots, \sigma_{2r}$ span $E^*$ at all points $x_i.$ Thus $\sigma_1, \ldots, \sigma_{2r}$ span $E^*$ at all points outside an analytic subset included in $Z$ and of strictly smaller dimension. Repeating this procedure we can make $Z$ to be empty proving the first part of the proposition.

What we have proven is that for all $x \in X$ the images of $\sigma_1, \ldots, \sigma_N$ inside the vector space $\mathcal{O}(E^*)_x \otimes \mathcal{O}_x / m_x$ span.

We turn now to the second part of the proposition. We notice that there is no canonical map $E^* \rightarrow E^* \otimes T^* X.$ However, saying that $\sigma_1, \ldots, \sigma_N$ span $E^*$ at $x$ and $d\sigma_1, \ldots, d\sigma_N$ span $E^*_x \otimes T^*_x X$ is equivalent to saying that the images of $\sigma_1, \ldots, \sigma_N$ inside $\mathcal{O}(E^*)_x \otimes \mathcal{O}_x / m_x$ span. Indeed, we have a non-cannonical decomposition (depending on the coordinate system)

$$\mathcal{O}_x / m_x^2 = \mathbb{C} \oplus m_x / m_x^2 \oplus \mathbb{C} \oplus T^*_x X.$$ 

To prove the second assertion we need only repeat the arguments from above with $E^*_{x_i}$ replaced by $\mathcal{O}(E^*)_{x_i} \otimes \mathcal{O}_{x_i} / m_{x_i}.$

6. An Immersion

The sections $\sigma_0, \ldots, \sigma_N$ from (5.23) are associated in a canonical fashion to sections $\tilde{\sigma}_0, \ldots, \tilde{\sigma}_N$ of $L.$ The latter span $L$ at every point, cf. (6.1). Therefore they induce a holomorphic map $f : \mathbb{P}(E) \rightarrow \mathbb{P}^N, \ f(y) = (\tilde{\sigma}_0(y); \ldots; \tilde{\sigma}_N(y)).$

In the next section we will show that $f$ is of finite order. Our goal in the present section is to show that the sections from (5.23) can be chosen in such a manner that $f$ be an immersion. A holomorphic map $f$ is said to be an immersion if it is one-to-one and its Jacobian has maximal rank at every point. Equivalently, $f$ is an immersion if its image is a complex submanifold and $f$ is a biholomorphism onto its image. We note that the image of $f$ does not have to be closed and, in fact, in our situation, it will not be closed.

We begin by noting at (6.1) that already $f$ is a local immersion. However, to get the injectivity of $f,$ the arguments from (5.23) are not sufficient. We will, instead, use arguments from the theory of Stein spaces, more precisely, the classical way of proving that a Stein manifold of dimension $n$ can be embedded into $\mathbb{C}^{2n+1}.$ Our reference will be 5.3 in [9]. Hörmander’s lemmas can be applied, quite generally, to a line bundle whose sections separate points and provide holomorphic coordinates at each point. We have checked at (5.23) that sections of finite order of $L$ satisfy these two properties. One then uses the Baire category theorem to show that the set of sections giving a “good” map
$f$ is dense in a certain topology. We will not be able to apply directly the
category theorem because the topology on the space of sections of finite order
of $L$ is not complete. However, Baire’s method of proof still works, cf. (6.5).

(6.1) Remark: The sections $\tilde{\sigma}_0, \ldots, \tilde{\sigma}_N$ span $L$ at every point, i.e. for every
$y \in \mathbb{P}(E)$ some $\tilde{\sigma}_i(y)$ is nonzero. Indeed, if $y = (x, [v]), v \in E_x \setminus \{0\}$, then
$< \tilde{\sigma}_i, v > = < \sigma_i(x), v >$. But $\sigma_0(x), \ldots, \sigma_N(x)$ span $E_x$, hence some $< \tilde{\sigma}_i, v >$
must be nonzero.

Thus, as already said above, there is an induced holomorphic map

$$f = (\tilde{\sigma}_0, \ldots, \tilde{\sigma}_N) : \mathbb{P}(E) \longrightarrow \mathbb{P}^N.$$  

We claim that $f$ is a local immersion, i.e. for all $y \in \mathbb{P}(E)$ the Jacobian of $f$
at $y$ has maximal rank $n + r$. The accepted terminology is also “$f$ is regular
at $y$”.

(6.2) Remark: Let $s_0, \ldots, s_m$ be global sections of $L$ inducing a map $f$
from $Y$ to $\mathbb{P}^m$. Then $f$ is regular at a point $y \in Y$ if and only if the images of
$s_0, \ldots, s_m$ under the canonical map $L_y \longrightarrow L_y \otimes \mathcal{O}_y / \mathfrak{m}_y^2$
generate the latter as a vector space.

Let us denote by $\Sigma$ the space of global sections $\sigma \in \Gamma(Y, L)$ of order $\mu$
in the $L^2$-sense:

$$||\sigma||^2 := \int_{\mathbb{P}(E)} |\sigma|^2 \cdot e^{-\kappa \rho^\Phi} < \infty.$$  

We choose $\kappa$ and $\mu$ so large that the Vanishing Theorem (5.12) hold and its
corollary (5.21) apply to $\Sigma$. In addition, we choose $\kappa$ so large that the sections
$\tilde{\sigma}_0, \ldots, \tilde{\sigma}_N$ from (5.23) belong to $\Sigma$. We will fix these $\kappa$ and $\mu$ for the remainder
of this section. For the proofs of the following two lemmas we refer to [9] (5.3).

(6.3) Lemma: Let $K \subset Y$ be a compact subset. Then we can find an integer
$m$ and sections $s_0, \ldots, s_m \in \Sigma$ inducing a map $f : Y \longrightarrow \mathbb{P}^m$ which regular
and one-to-one on $K$.

(6.4) Lemma: Let $K \subset Y$ be a compact subset. Assume that some global
sections $s_0, \ldots, s_{m+1}$ of $L$ induce a map from $Y$ to $\mathbb{P}^{m+1}$ which is regular
and one-to-one on $K$. Then, if $m \geq 2(n + r) + 1$, we can find $(a_0, \ldots, a_m) \in \mathbb{C}^{m+1}$
arbitrarily close to the origin such that $s_0 - a_0 s_{m+1}, \ldots, s_m - a_m s_{m+1}$ induce a
map $f : Y \longrightarrow \mathbb{P}^m$ which is regular and one-to-one on $K$. In fact, this is true
for all $a \in \mathbb{C}^{m+1}$ outside a set of measure zero.

We are nearing our goal. The last step is to put a topology on $\Sigma$ and to show
that the set of sections giving a regular one-to-one map from $Y$ to projective
space is dense in $\Sigma^m$ equipped with the product topology. Let $\{K_p\}_{p \geq 1}$ be
an exhaustion of $Y$ by compact subsets. We equip $\Sigma$ with the topology of
convergence on compact subsets: a sequence $\{\sigma_q\}_{q \geq 1}$ converges to $\sigma$ if for each
compact subset $K$ of $Y$ we have

$$\lim_{q \to \infty} |\sigma_q - \sigma|_K = 0,$$

where

$$|\sigma_q - \sigma|_K = \text{sup}\{ |\sigma_q(y) - \sigma(y)|_I, y \in K \}.$$

This topology is given by a metric invariant under translations:

$$d(\sigma', \sigma'') = \sum_{p \geq 1} \frac{1}{2^p} \cdot \frac{|\sigma' - \sigma''|_{K_p}}{1 + |\sigma' - \sigma''|_{K_p}}.$$

It is not clear whether $(\Sigma, d)$ is a complete metric space; it is not clear that all Cauchy sequences in $\Sigma$ converge. However, Cauchy sequences which are bounded in $|| \cdot ||$ are convergent. Indeed, assume that $\{\sigma_q\}_{q \geq 1}$ is Cauchy relative to the distance $d$ and $||\sigma_q|| \leq M$ for some $M > 0$ and all $q \geq 1$. There is a global section $\sigma$ of $L$ such that $\{\sigma_q\}_{q \geq 1}$ converges pointwise in $|| \cdot ||_I$ to $\sigma$, uniformly on compact sets. We have

$$\int_{K_p} |\sigma_q|^2 \cdot e^{-n \rho^\alpha \hat{\Phi}} < ||\sigma_q||^2 \leq M^2.$$

Taking limit as $q \to \infty$ we get

$$\int_{K_p} |\sigma|^2 \cdot e^{-n \rho^\alpha \hat{\Phi}} \leq M^2.$$

Taking limit as $p \to \infty$ we obtain $||\sigma|| \leq M$ forcing $\sigma \in \Sigma$. Thus $\{\sigma_q\}_{q \geq 1}$ converges to $\sigma$ in $(\Sigma, d)$.

(6.5) **Theorem:** Let $m \geq 2(n + r) + 1$ be a given integer. Then the set of $(m+1)$-tuples $(\sigma_0, \ldots, \sigma_m) \in \Sigma^{m+1}$ which induce a regular injective map $f : Y \to \mathbb{P}^m$ is dense in $\Sigma^{m+1}$.

**Proof:** Let $G_p$ denote the set of $(m+1)$-tuples $(\sigma_0, \ldots, \sigma_m) \in \Sigma^{m+1}$ which induce a map from $Y$ to $\mathbb{P}^m$ that is regular and one-to-one on $K_p$. Clearly $G_p$ is open. We claim that $G_p$ is dense. To see this choose an arbitrary $(m+1)$-tuple $\sigma = (\sigma_0, \ldots, \sigma_m) \in \Sigma^{m+1}$. Lemma (6.3) provides us with sections $s_0, \ldots, s_N \in \Sigma$ which induce a map from $Y$ to $\mathbb{P}^N$ that is regular and one-to-one on $K_p$. Then $(\sigma_0, \ldots, \sigma_m, s_0, \ldots, s_N)$ induce a map from $Y$ to $\mathbb{P}^{m+N+1}$ that is regular and one-to-one on $K_p$. Applying repeatedly lemma (6.4) to this $(m+N+1)$-tuple we deduce that we can find $a_{ij}$, $0 \leq i \leq m$, $0 \leq j \leq N$ arbitrarily close to the origin such that $\sigma'_i := \sigma_i - \sum_{j=0}^N a_{ij}s_j$, $0 \leq i \leq m$, induce a map from $Y$ to $\mathbb{P}^m$ that is regular and one-to-one on $K_p$. Thus $\sigma'$ belongs to $G_p$ and it can be made arbitrarily close to $\sigma$ relative to the distance $d$. We conclude that $G_p$ is dense.

Denoting $||\sigma|| = \max\{||\sigma_j||, 0 \leq j \leq m\}$ we also notice that $\sigma'$ can be made arbitrarily close to $\sigma$ relative to $|| \cdot ||$. 


We claim that \( \cap_{p \geq 1} G_p \) is dense in \( \Sigma^{m+1} \). Had we known that \((\Sigma, d)\) is complete this would have been guaranteed by Baire’s category theorem. Nevertheless, in our situation, the arguments from the proof of Baire’s theorem can be carried out:

Choose arbitrary \( \sigma \in \Sigma^{m+1}, \varepsilon > 0 \). Choose \( \sigma_1 \in G_1 \) such that \( d(\sigma, \sigma_1) < \varepsilon \) and \( ||\sigma - \sigma_1|| < 1 \). Choose \( \varepsilon_1 \in (0, \varepsilon/2) \) such that the closed ball \( B[\sigma_1, 2\varepsilon_1] \) is contained in \( G_1 \). Inductively choose \( \sigma_p \in G_1 \cap \ldots \cap G_p \) and \( \varepsilon_{p+1} \in (0, \varepsilon_p/2) \) such that \( d(\sigma_p, \sigma_{p+1}) < \varepsilon_p, \quad ||\sigma_{p+1} - \sigma_p|| < 2^{-p} \) and \( B[\sigma_p, 2\varepsilon_p] \subset G_1 \cap \ldots \cap G_p \). Then \( \{\sigma_p\}_{p \geq 1} \) is a Cauchy sequence in \( \Sigma^{m+1} \). Also \( ||\sigma_p|| \leq ||\sigma|| + 2 \) for all \( p \). By the discussion preceding the theorem \( \{\sigma_p\}_{p \geq 1} \) converges to some \( \sigma' \in \Sigma^{m+1} \).

For \( q > p \) we have \( d(\sigma_p, \sigma_q) < \varepsilon_p + \ldots + \varepsilon_{q-1} < 2\varepsilon_p \). Taking limit as \( q \to \infty \) we obtain \( d(\sigma_p, \sigma') \leq 2\varepsilon_p \). By construction the closed ball of center \( \sigma_p \) and radius \( 2\varepsilon_p \) is contained in \( G_1 \cap \ldots \cap G_p \). Thus \( \sigma' \in G_1 \cap \ldots \cap G_p \). Since \( p \) is arbitrary we conclude that \( \sigma' \in \cap_{p \geq 1} G_p \). We have \( d(\sigma, \sigma') \leq 2\varepsilon \). Since \( \varepsilon > 0 \) is arbitrary we conclude that \( \cap_{p \geq 1} G_p \) is dense in \( \Sigma^{m+1} \). Q.e.d.

7. NEVANLINNA THEORY ON \( \mathbb{P}(E) \)

Let \( X \) be a special affine variety of dimension \( n \) and \( E \) a holomorphic vector bundle of rank \( r + 1 \) equipped with a Finsler metric of order \( \lambda \) as defined at (5.5). Our goal is to provide means for measuring growth of holomorphic maps from \( \mathbb{P}(E) \) to projective space or growth of analytic subsets of \( \mathbb{P}(E) \). We will develop a theory along the same lines as in section 2.

For brevity we will write \( Y = \mathbb{P}(E) \). We also write \( \rho, \varphi, \psi \) etc. instead of \( \rho \circ \pi, \pi^* \varphi, \pi^* \psi \). Given \( r \in (0, \infty) \) we put

\[
Y[r] = \{(x, [v]) \in Y, \quad \rho'(x) \leq r\}, \quad Y < r = \{(x, [v]) \in Y, \quad \rho'(x) = r\}.
\]

By Sard’s theorem the sets \( Y < r > \) are smooth for all \( r \) outside a set of measure zero. In the sequel, each time we integrate over \( Y < r > \), it will be tacitly assumed that the latter is smooth.

We will consider \( \mathbb{P}(E) \) equipped with the Kähler metric induced by \( \tilde{\varphi} \). Our first observation is that the volume of the balls \( Y[r] \) grows at most polynomially in \( r \). Indeed,

\[
\text{vol}(Y[r]) = \int_{Y[r]} \varphi^{n+r} = \int_{Y[r]} \binom{n+r}{r} (\tilde{\varphi}^H)^n \wedge (\tilde{\varphi}^V)^r \leq \int_{Y[r]} \binom{n+r}{r} (2\kappa^{\lambda+c})^n \varphi^n \wedge (\tilde{\varphi}^V)^r \quad \text{by (5.8)}.
\]

But \( \varphi^n \) is a volume form on \( X \). Thus we can apply Fubini’s theorem: the integral from above can be computed by first integrating vertically along the fibers of \( \mathbb{P}(E) \) and then horizontally along \( X \). We get

\[
\text{vol}(Y[r]) \leq \int_{x \in X[r]} \binom{n+r}{n} (2\kappa^{\lambda+c})^n \varphi^n \int_{\mathbb{P}(E_x)} (\tilde{\varphi}^V)_{|\mathbb{P}(E_x)}^r \leq r^n.
\]
for some positive constant $a$. The above expression has polynomial growth because of (1.5) and of the fact that $\varphi$ has finite volume.

Given a holomorphic map $f : Y \to \mathbb{P}^N$ we define its characteristic function

$$T_f(r, s) = \int_s^r \frac{dt}{t} \int_{Y[t]} f^* \omega \wedge \psi^{n-1} \wedge \tilde{\varphi}^r,$$

and, for $1 \leq k \leq n$, the higher characteristic functions

$$T_f^{(k)}(r, s) = \int_s^r \frac{dt}{t} \int_{Y[t]} f^* \omega^k \wedge \psi^{n-k} \wedge \tilde{\varphi}^r.$$

Here $r > s > 0$ are real numbers and $\omega$ is the Fubini-Study form on $\mathbb{P}^N$. Given an analytic subset $Z \subset Y$ of pure dimension $k \geq r$ we define its counting function

$$N_Z(r, s) = \int_s^r \frac{dt}{t} \int_{Z[t]} \psi^{k-r} \wedge \tilde{\varphi}^r,$$

where $Z[t] = Z \cap Y[t]$. If the image of $Z$ under the projection $p \circ \pi : Y \to \mathbb{C}^n$ does not contain the origin then $N_Z(r, s)$ is well defined and we write $N_Z(r) = N_Z(r, 0)$. Given a global section $\sigma$ of $\mathcal{O}_{\mathbb{P}^N}(1)$ we define the proximity function of $f$ to the zero-set of $\sigma$

$$m_\sigma(r) = \int_{Y < r>} \log \frac{1}{|\sigma \circ f|^r} d^c \tau' \wedge \psi^{n-1} \wedge \tilde{\varphi}^r.$$

Here $| \cdot |$ is taken relative to the canonical metric of $\mathcal{O}_{\mathbb{P}^N}(1)$. Notice that $d^c \tau' \wedge \psi^{n-1} \wedge \tilde{\varphi}^r$ is a volume form on $Y < r >$. By choosing $\sigma$ to have norm less than 1 at all points we can arrange that $m_\sigma(r)$ be non-negative.

The First Main Theorem and the Crofton Formula hold also in this context with virtually the same proofs as in the classical case:

**Theorem:** Let $f : \mathbb{P}(E) \to \mathbb{P}^N$ be a holomorphic map. Let $H \subset \mathbb{P}^N$ be a hyperplane defined as the zero-set of a global section $\sigma$ of $\mathcal{O}_{\mathbb{P}^N}(1)$ and $Z = f^* H$. Assume that the image of $f$ is not contained in $H$. Then, for $r > s > 0$, we have

$$N_Z(r, s) + m_\sigma(r) - m_\sigma(s) = T_f(r, s).$$

**Corollary:** Fix $s > 0$. Then, for $r > s$, and under the hypotheses of the previous theorem, we have

$$N_Z(r, s) \leq T_f(r, s) + O(1).$$

Here $O(1)$ is a constant that may depend on $s$.

**Crofton Formula:** Let $f : \mathbb{P}(E) \to \mathbb{P}^N$ be a holomorphic map which is non-degenerate in the sense that the preimage of a plane $P$ of codimension $k$ in $\mathbb{P}^N$ is an analytic subset of pure codimension $k$ in $\mathbb{P}(E)$. Then, for $r > s > 0$,
we have
\[ T_f^{(k)}(r, s) = \int_{P \in G(N, k)} N_{f^* P}(r, s) \, d\mu(P). \]

Here \( G(N, k) \) is the Grassmannian of planes of codimension \( k \) in \( \mathbb{P}^N \) while \( \mu \) is the measure defined before (2.3).

**Definition (7.4):** Let \( \lambda \) be a non-negative real number. Let \( f : \mathbb{P}(E) \rightarrow \mathbb{P}^N \) be a holomorphic map. We say that \( f \) has order \( \lambda \) if there is \( \kappa \geq 0 \) such that for some \( s \) and all \( r > s \) we have
\[ T_f(r, s) \leq \kappa r^\lambda. \]
Likewise, we say that an analytic subset \( Z \subset \mathbb{P}(E) \) of pure dimension \( k \geq r \) has order \( \lambda \) if
\[ N_Z(r, s) \leq \kappa r^\lambda. \]

**Remark (7.5):** Assume that \( f \) has finite order and is linearly non-degenerate. Then (7.2) tells us that the preimage under \( f \) of any hyperplane \( H \subset \mathbb{P}(E) \) is a divisor of finite order. Conversely, assume that the pull-backs \( f^*(H) \) have finite order in a uniform manner, i.e. there are \( r_0, \kappa, \lambda \geq 0 \) such that for all \( r \geq r_0 \) and \( s < r \) we have \( N_{f^* (H)}(r, s) \leq \kappa r^\lambda \). Then, by (7.3), \( f \) has finite order, too.

Before proceeding further let us notice that for an analytic subset \( Z \subset \mathbb{P}(E) \) of pure dimension \( k \geq r \) we can define a counting function
\[ \hat{N}_Z(r, s) = \int_{s}^{r} \frac{dt}{t} \int_{Z[t]} \varphi^{k-r} \wedge \varphi^r \]
by replacing \( \psi \) with \( \varphi \). One can see that \( \hat{N}_Z \) and \( N_Z \) have roughly the same growth, cf. [12]. What we mean is that there are constants \( a, b \geq 1 \) independent of \( Z \) such that
\[ N_Z(r) \leq r^a \hat{N}_Z(r^b), \quad \hat{N}_Z(r) \leq r^a N_Z(r^b). \]
In particular, both counting functions have polynomial growth at the same time. The reader may wonder why we didn’t use \( \hat{N}_Z \) to measure growth in the first place. The problem with this is that it doesn’t satisfy the First Main Theorem.

**Remark (7.6):** The theory we build in this section encompasses the theory we built in section 2. What we claim is the following: let \( Z \subset X \) be a pure \( k \)-dimensional analytic subset and \( \tilde{Z} = \pi^{-1}Z \). Then \( Z \) and \( \tilde{Z} \) have roughly the same growth. In particular, both have finite order at the same time. Indeed, by Wirtinger’s theorem the restriction of \( \varphi^k \) to \( Z \) is a volume form on \( Z \). Thus we can apply Fubini’s theorem in the definition of \( \hat{N}_{\tilde{Z}} \) and then horizontally along \( Z \). We get \( \hat{N}_{\tilde{Z}}(r, s) = \hat{N}_{\tilde{Z}}(r, s) \). From §2 we know that \( \hat{N}_Z \) has roughly the same growth at \( N_{\tilde{Z}} \). From the comments preceding this
remark we know that $\hat{N}_Z$ has roughly the same growth at $N_Z$. This finishes the proof of the claim.

**Proposition (7.7):** Assume that $E$ is equipped with a Finsler metric of order $\lambda$. Let $\sigma$ be a global section of $E^*$ of order $\lambda$ and $Z \subset \mathbb{P}(E)$ the zero-set of $\tilde{\sigma}$. Then there is $\mu \geq \lambda$ depending only on $\lambda$ and $E$ such that $Z$ has order $\mu$.

**Proof:** Integrating the Poincaré-Lelong formula

$$[Z] = c_1(L, \tilde{\sigma}) + dd^c \log|\tilde{\sigma}|^2$$

and taking logarithmic average we get

$$N_Z(r, s) = \int_s^r \frac{dt}{t} \int_{Z[t]} \psi^{n-1} \wedge \tilde{\varphi}^r$$

$$= \int_s^r \frac{dt}{t} \int_{Y[t]} \left[c_1(L, \tilde{\sigma}) + dd^c \log|\tilde{\sigma}|^2\right] \wedge \psi^{n-1} \wedge \tilde{\varphi}^r$$

$$= \int_s^r \frac{dt}{t} \int_{Y[t]} c_1(L, \tilde{\sigma}) \wedge \psi^{n-1} \wedge \tilde{\varphi}^r$$

$$- \int_s^r \frac{dt}{t} \int_{Y[t]} c_1(\pi^* \det(E), \det(G_{ij})) \wedge \psi^{n-1} \wedge \tilde{\varphi}^r$$

$$+ \int_s^r \frac{dt}{t} \int_{Y[t]} c_1(\det(E), g) \wedge \psi^{n-1} \wedge \tilde{\varphi}^r$$

$$+ \int_s^r \frac{dt}{t} \int_{Y[t]} dd^c \log|\tilde{\sigma}|^2 \wedge \psi^{n-1} \wedge \tilde{\varphi}^r.$$

Let us denote by (i), (ii), (iii) and (iv) the integrals from the right-hand-side above. First we notice that (i) has polynomial growth: by (5.7) and (1.6)

$$(i) \leq \int_s^r \frac{dt}{t} \int_{Y[t]} \psi^{n-1} \wedge \tilde{\varphi}^r$$

$$\leq \int_s^r \frac{dt}{t} \int_{Y[t]} c\chi(n-1) \tilde{\varphi}^r$$

$$\leq \int_s^r \frac{dt}{t} \int_{Y[t]} c\chi(n-1) \tilde{\varphi}^{r+1}$$

$$\leq r^{c(n-1)c'} \text{vol}(Y[\eta]) \leq r^{a+c(n-1)c'}.$$

Let $\mu$ be the largest between $\lambda$ and the order of $g$. By hypothesis $|c_1(\det(E), g)| \leq \rho^n \varphi$ yielding

$$(iii) \leq \int_s^r \frac{dt}{t} \int_{Y[t]} \rho^n \varphi \wedge \psi^{n-1} \wedge \tilde{\varphi}^r$$

$$\leq \int_s^r \frac{dt}{t} \int_{Y[t]} \rho^{\mu+c(n-1)} \tilde{\varphi}^{n+r}$$

$$\leq r^{a+c(n-1)c'}.$$


By type considerations $\psi^{n-1} \wedge (\tilde{\varphi})^2 = 0$ forcing
\[
\psi^{n-1} \wedge \tilde{\varphi}^r = \psi^{n-1} \wedge (\tilde{\varphi}^H + \tilde{\varphi}^V)^r
= \psi^{n-1} \wedge (\tilde{\varphi}^V)^r + r \psi^{n-1} \wedge \tilde{\varphi}^H \wedge (\tilde{\varphi}^V)^{r-1},
\]
\[
c_1(G_{ij}) \wedge \psi^{n-1} \wedge \tilde{\varphi}^r = c_1(G_{ij})^H \wedge \psi^{n-1} \wedge (\tilde{\varphi}^V)^r
+ r c_1(G_{ij})^V \wedge \psi^{n-1} \wedge \tilde{\varphi}^H \wedge (\tilde{\varphi}^V)^{r-1}.
\]

By (5.9), (5.10) and (5.8) we obtain
\[
(ii) \leq \int_{s}^{r} \frac{dt}{t} \int_{Y[t]} \left[ \kappa \rho^\lambda \varphi \wedge \psi^{n-1} \wedge (\tilde{\varphi}^V)^r + \kappa \rho^\lambda (\tilde{\varphi})^V \wedge \psi^{n-1} \wedge \tilde{\varphi}^H \wedge (\tilde{\varphi}^V)^{r-1} \right]
\leq 2\kappa r^{a+c' + c(n-1)c'}.
\]

Performing the standard “integration twice” procedure from Nevanlinna theory we obtain
\[
(iv) = \int_{Y(<r)} \log |\tilde{\sigma}| \cdot d^c \tau' \wedge \psi^{n-1} \wedge \tilde{\varphi}^r
- \int_{Y(<s)} \log |\tilde{\sigma}| \cdot d^c \tau' \wedge \psi^{n-1} \wedge \tilde{\varphi}^r.
\]

We fix $s$. By using the concavity of the logarithmic function we obtain the estimate
\[
(7.8) \quad \int_{Y(<r)} \log |\tilde{\sigma}| \cdot d^c \tau' \wedge \psi^{n-1} \wedge \tilde{\varphi}^r \leq \text{vol}(r) \cdot \log \left\{ \frac{1}{\text{vol}(r)} \int_{Y(<r)} |\tilde{\sigma}| \cdot d^c \tau' \wedge \psi^{n-1} \wedge \tilde{\varphi}^r \right\}
\]
where
\[
\text{vol}(r) = \int_{Y(<r)} d^c \tau' \wedge \psi^{n-1} \wedge \tilde{\varphi}^r.
\]

But $d^c \tau' \wedge \psi^{n-1}$ is a volume form on $X < r >$ away from the branching set of the projection $p: X \to \mathbb{C}^n$. This set has measure zero so we ignore it. Therefore, we can apply Fubini’s theorem on $Y(<r)$. We get
\[
(7.9) \quad \text{vol}(r) = \int_{X(<r)} d^c \tau' \wedge \psi^{n-1} = \int_{\mathbb{C}^n(<r)} d^c \log |z|^2 \wedge (dd^c \log |z|^2)^{n-1} = 1.
\]

It remains to estimate the integral
\[
v(r) = \int_{Y(<r)} |\tilde{\sigma}| \cdot d^c \tau' \wedge \psi^{n-1} \wedge \tilde{\varphi}^r.
\]

We will first estimate the integral
\[
w(r) = \int_{1}^{r} \frac{v(t)}{t} dt.
\]
We have
\[ w(r) = \int_{Y[1,r]} |\tilde{\sigma}| \cdot dt' \wedge d^c t' \wedge \psi^{n-1} \wedge \tilde{\varphi}^r \]
(by Fubini’s theorem)
\[ \leq \int_{Y[1,r]} |\tilde{\sigma}| \cdot \rho^c \varphi \wedge (\rho^c \varphi)^{n-1} \wedge \tilde{\varphi}^r \]
(by (1.6) and (1.7))
\[ \leq r^{cn \epsilon} \int_{Y[1,r]} |\tilde{\sigma}| \cdot \varphi^n \wedge \tilde{\varphi}^r \]
(here \( c' \) is such that \( \tau \leq c' \tau' \), cf. (1.5))
\[ \leq r^{cn \epsilon} \int_{Y[1,r]} \left( \int_{Y[r]} |\tilde{\sigma}|^2 \cdot e^{-\kappa \rho^c} \cdot \tilde{\Phi} \right)^{1/2} \left( \int_{Y[r]} e^{\kappa \rho^c} \cdot \tilde{\Phi} \right)^{1/2} \]
(by Hölder’s Inequality)
\[ \leq r^{cn \epsilon} C^{1/2} e^{\kappa \rho^c / 2} \text{vol}(Y[r])^{1/2} \]
\[ \leq r^{cn \epsilon} C^{1/2} e^{\kappa \rho^c / 2 + \alpha / 2}. \]

Here \( \kappa \) is so large that \( \sigma \) be square integrable with respect to the metric \( \tilde{l} \cdot e^{-\kappa' \rho^c} \)
on \( L \) and \( \tilde{\phi} \) on \( \mathbb{P}(E) \). Next we notice that \( w(r) \) is an increasing function of class \( C^1 \) with \( w'(r) = \frac{v(r)}{r} \). Without loss of generality we may assume that \( \lim_{r \to \infty} w(r) = \infty \). We claim that for all \( r \geq 1 \) outside a set of finite Lebesgue measure we have
\[ w'(r) \leq w(r)^2. \]
Indeed, let \( B \) be the set of “bad” \( r \) for which \( w'(r) > w(r)^2 \). Let \( r_0 \) be such that \( w(r_0) = 1 \). For \( r \geq r_0 \) we have
\[ \text{measure}(B \cap [1, w(r)]) < \int_1^{w(r)} \frac{w'(t)}{w(t)^2} \, dt = \int_{r_0}^r \frac{dt}{t^2} < \frac{1}{r_0}. \]
By letting \( r \) converge to infinity we get that the measure of \( B \) is at most \( 1/r_0 \) which proves the claim.

We conclude that for all \( r > 0 \) outside a set of finite Lebesgue measure we have
\[ v(r) \leq r^{1 + 2cn \epsilon + a} C \cdot e^{\kappa \rho^c}. \]

Finally, taking logarithm and using (7.8) and (7.9) we arrive at the estimate
\[ (iv) \leq O(\log(r)) + \kappa r^\lambda. \]

We deduce the proposition holds with \( \mu \) depending on the order of \( g \), on \( \lambda \) and some constants but with \( r \) outside a set of Lebesgue measure \( 1/r_0 \). But \( T_f(r) \) is increasing hence \( T_f(r) = O((r + 1/r_0)^\mu) = O(r^{\mu + \epsilon}) \) with arbitrary positive \( \epsilon \). Q.e.d.
(7.10) **Theorem:** Let $X$ be a special affine variety of dimension $n$. Let $E$ be a holomorphic vector bundle of rank $r + 1$ on $X$. Assume that $E$ is equipped with a Finsler metric of order $\lambda$. Then there is $\mu \geq \lambda$ such that for any integer $N \geq 2(n + r) + 1$ there is a holomorphic immersion

$$f : \mathbb{P}(E) \longrightarrow \mathbb{P}^N$$

of order $\mu$ satisfying

$$f^* \mathcal{O}_{\mathbb{P}^N}(1) \simeq L.$$

**Proof:** Let $(\sigma_0, \ldots, \sigma_N)$ be as in (6.5). Let $\mu$ be as in (7.7). For any linear combination $\sigma = a_0\sigma_0 + \ldots + a_N\sigma_N$, $|a_i| \leq 1$, the zero-set of $\bar{\sigma}$ has order $\mu$. In fact, going through the estimates of (7.7) one sees that these zero-sets have order $\mu$ in a uniform manner. The theorem now follows from remark (7.5).
List of Notations

\begin{itemize}
  \item $X$ special affine variety of dimension $n$
  \item $\mathcal{O}_{f.o.}$ sheaf of holomorphic functions on $X$ of finite order, cf. (2.6)
  \item $\tau, \rho = e^{\tau/2}$ strictly plurisubharmonic exhaustive functions on $X$, cf. (1.3)
  \item $\tau', \rho' = e^{\tau'/2}$ plurisubharmonic exhaustive functions on $X$
  \item $\bar{d}$ the twisted derivative $\frac{1}{4\pi \sqrt{-1}} (\bar{\partial} - \partial)$
  \item $dd^c$ the complex Hessian (also known as Levi form) $\frac{1}{2\pi \sqrt{-1}} \partial \bar{\partial}$
  \item $\varphi = dd^c \tau$ Kähler form on $X$
  \item $\Phi$ the volume-form $\varphi^n$ of $\varphi$
  \item $\psi = dd^c \tau'$ semipositive-definite form on $X$ satisfying $\psi^n = 0$
  \item $p$ generic projection from $X$ to $\mathbb{C}^n$
  \item $E$ holomorphic vector bundle of rank $r + 1$ on $X$
  \item $E^*$ the dual of $E$
  \item $\mathcal{O}_{f.o.}(E^*)$ sheaf of holomorphic sections of finite order of $E^*$, cf. (5.11)
  \item $\mathbb{P}(E)$ projectivization of $E$
  \item $E_o$ complement of the zero-section in $E$
  \item $\mathcal{V}$ the vertical tangent bundle inside $TE_o$
  \item $\mathcal{H}$ the horizontal tangent bundle inside $TE_o$
  \item $\pi$ projection from $\mathbb{P}(E)$ onto the base $X$
  \item $\mathcal{L}$ the hyperplane line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$
  \item $\hat{\sigma}$ section of $\mathcal{L}$ corresponding to section $\sigma$ of $E^*$, cf. (4.7)
  \item $g$ metric of finite order on $\text{det}(E)$
  \item $h$ Hermitian or Finsler metric on $E$
  \item $\tilde{h}$ Hermitian metric on $\mathcal{L}$ induced by $h$
  \item $(G_{ij})$ Hermitian metric on $\mathcal{V}$ induced by $h$
  \item $l$ the Hermitian metric $\tilde{h} \cdot \det(G_{ij})^{-1} \cdot \pi^* g$ on $\mathcal{L}$
  \item $\tilde{\varphi}$ Kähler form on $\mathbb{P}(E)$, cf. (5.7)
  \item $\tilde{\Phi}$ the volume-form $\tilde{\varphi}^{n+r}$ of $\tilde{\varphi}$
  \item $\tilde{\varphi}^V, \tilde{\varphi}^H$ the vertical, resp. the horizontal part of $\tilde{\varphi}$, cf. paragraph after (3.23)
\end{itemize}

References

[1] Marco Abate and Giorgio Patrizio. Fisler Metrics- A Global Approach, Lecture Notes in Mathematics 1591, Springer Verlag, Berlin, 1994.
[2] Aldo Andreotti and Eduardo Vesentini, Carleman Estimates for the Laplace-Beltrami Equation on Complex Manifolds, Publ. Math. I.H.E.S. 25 (1965), 81-130.
[3] Jianguo Cao and Pit-Mann Wong, Geometry of Projectivized Vector Bundles, Journal of Math. of Kyoto Univ. 43, no. 2, (2003), 369-410.
[4] James Carlson and Phillip Griffiths, The Order Function for Entire Holomorphic Mappings, Value Distribution Theory, Part A, R. O. Kujala, A. L. Vitter (eds.), Marcel Dekker Inc., New York, 1974.
[5] Maurizio Cornalba and Phillip Griffiths, Analytic Cycles and Vector Bundles on Non-compact Algebraic Varieties, Invent. Math. 28 (1975), 1-106.
[6] Maurizio Cornalba and Bernard Shiffman, A Counterexample to the Transcendental Bezout Problem, Ann. of Math. 96 (1972), 402-406.
[7] Phillip Griffiths, Function Theory of Finite Order on Algebraic Varieties, Journal of Diff. Geom. 6 (1972), 285-306 and 7 (1972), 45-66.
[8] Lars Hörmander, $L^2$-estimates and Existence Theorems for the $\bar{\partial}$-operator, Acta. Math. 113 (1965), 89-52.
[9] Lars Hörmander, An Introduction to Complex Analysis in Several Variables, North-Holland Publ. Comp., Amsterdam-London, 1973.
[10] Kunihiko Kodaira and James Morrow, Complex Manifolds, Holt, Rinehart and Winston, Inc., 1971.
[11] Pierre Lelong and Lawrence Gruman, Entire Functions of Several Complex Variables, Springer Verlag, Berlin, 1986.
[12] Mario Maican, Vector Bundles of Finite Order on Affine Manifolds, Thesis at the University of Notre Dame, July 2005.
[13] James Mulflur, Albert Vitter, and Pit-Mann Wong, Holomorphic Functions of Finite Order on Affine Varieties, Duke Math. Journal 48, no. 2, (1981), 389-399.
[14] Boris Shabat, Distribution of Values of Holomorphic Mappings, Translations of Mathematical Monographs 61, American Mathematical Society, Providence, RI, 1985.
[15] B. Shiffman, A. Sommese, Vanishing Theorems on Complex Manifolds, Progress in Mathematics 56, Birkhäuser Boston, Inc., 1985.
[16] Henri Skoda, Solution a la croissance du second problème de Cousin dans $\mathbb{C}^n$, Ann. Inst. Fourier, Grenoble 21, no. 1, (1971), 11-23.
[17] Henri Skoda, Sous-ensembles analytiques d’ordre fini ou infini dans $\mathbb{C}^n$, Bull. Soc. Math. France 100 (1972), 353-408.

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