Introduction

Holomorphic differential forms play an important role in the global study of complex projective or compact Kähler manifolds. Forms of degree 1 are well understood: there is a universal map, the Albanese map, to a complex torus and every 1-form is a pull-back. Forms of top degree also play a special role because they are sections in a line bundle, the canonical bundle. Forms of degree \( d \) with \( 2 \leq d \leq \dim X - 1 \) are however much harder to study. Especially 2-forms are very interesting, e.g. in symplectic geometry, or by the fundamental theorem of Kodaira describing them as obstruction for Kähler manifolds to be projective. In this paper we want to study the "first interesting case" of 2-forms on 3-folds. As general guideline we ask whether there is some kind of analogue of the Albanese for 2-forms. This means that we should try to find some universal object from which all 2-forms arise. This is impossible to some extend because we can take products of surfaces with curves and then would have to consider 2-forms on surfaces which are "wild". So instead we ask whether every 2-forms is produced by some canonical procedure. In sect. 1, which is partially written as an extended introduction, this is explained in great detail. Cum grano salis, we hope that every 2-form is induced by a meromorphic map to a surface, to a torus or to a symplectic manifold. We will explain in sect. 1, what we mean by "induced" (not just pull-back, of course).

In sections 3,4 and 5 we try to get a picture on 2-forms on 3-folds in terms of the Kodaira dimension. We describe this briefly - a more detailed account on the results of this paper is given in sect. 2. Let \( \omega \) be a 2-form on \( X \).

(a) If \( \kappa(X) = -\infty \) and if \( X \) is not simple (i.e. there is a compact subvariety of positive dimension through every point), then there is a meromorphic map onto a surface and \( \omega \) is a pull-back.

(b) If \( \kappa(X) = 0 \) and if \( X \) is projective, then after finite cover, étale in codimension 2, \( X \) is birationally to either a product of a K3-surface and an elliptic curve or a torus and \( \omega \) is induced in an obvious way. The same holds also in the Kähler case (sect. 8) unless \( X \) is simple and not covered by torus, a case which is expected not to exist.

(c) If \( 1 \leq \kappa(X) \leq 2 \), we consider the Iitaka reduction \( f : X \to B \) and relate \( \omega \) to \( f \). We have to distinguish the cases that \( \omega \) is vertical to \( f \) or not. In many cases we prove that \( \omega \) is induced.

(d) If \( X \) is of general type, we cannot say anything.

We also have more precise results when the zero set of \( \omega \) is finite which is in some sense the "generic" case.
Sections 6-9 study non-algebraic Kähler threefolds $X$. These carry automatically a 2-form. We prove that $X$ is uniruled if $\kappa(X) = -\infty$, unless $X$ is simple (which case should not exist). This is the Kähler version of the fundamental result of Miyaoka and Mori in the projective setting.

1. Examples and basic notions

In this section we give basic examples for projective and Kähler threefolds $X$ admitting a non-zero 2-form $\omega$. Our main concern is to try to understand how 2-forms on a Kähler manifold are created, i.e whether the 2-forms come from certain natural manifolds in analogy to the 1-forms which come from the Albanese torus. The most basic manifolds admitting 2-forms (in dimension at least 3) are tori and symplectic manifolds, i.e. simply connected manifolds with an everywhere non-degenerate holomorphic 2-form. 2-forms on surfaces play a completely different role since they are sections in a line bundle, so their geometry is easy to understand and there are plenty of examples.

1.1 General Method

Let $X$ be a compact Kähler threefold, $f : X \to S$ a dominant meromorphic map to a surface $S$ with $H^0(K_S) \neq 0$ or a meromorphic map to a torus or a symplectic manifold $S$. Let $\eta$ be a 2-form on $S$. Then $\omega = f^*(\eta)$ is a non-zero 2-form on $X$. We say that $\omega$ is induced from $S$ or $f$. More generally we can build $\omega = \sum \omega_i$ where the $\omega_i$ are induced from possibly different maps.

1.2 Vague Question

Does every 2-form on a Kähler threefold arise in this way; i.e. is every 2-form the sum of induced 2-forms, at least after some finite cover?

Remarks. (1) One might think of constructing 2-forms as wedge products of 1-forms, say $\omega = \eta_1 \wedge \eta_2$. This is in fact a special case of (1.1), i.e. $\omega$ is induced. Namely, let $\alpha : X \to \text{Alb}(X)$ be the Albanese map of $X$, then $\eta_i = \alpha_i^*(\lambda_i)$, so that $\omega = \alpha^*(\lambda_1 \wedge \lambda_2)$.

(2) (1.2) has to be interpreted in the following way in order to avoid obvious counterexamples. Consider an elliptic fiber bundle $f : X \to S$ over the surface $S$. Then we can have 2-forms $\omega$ arising from sections of $\Omega^1_{X|S} \otimes f^*(\Omega^2_S)$. If $X = E \times S$, then $\omega$ is a sum of wedge products of 1-forms on $E$ and $S$, however if $X$ is not a product, this is certainly false.

1.3 Examples

(1) Pairs $(X, \omega)$ with $X$ a Kähler threefold and $\omega$ a non-zero 2-form exist in every Kodaira dimension. We just take $X = S \times C$, the product of a surface $S$ admitting a 2-form with a curve $C$. We can even achieve that $\omega$ has no zeroes unless $X$ is of general type. For $\kappa \leq 1$, this is obvious: just take $S$ to be a torus. For $\kappa = 2$, take a surface $S$ of general type admitting a 1-form $\eta$ and a 2-form $\omega'$ without common zeroes. Let $C$ be elliptic with 1-form $\nu$. If $p_i$ denote the projections of $X = S \times C$, we put $\omega = p_1^*(\omega') + p_1^*(\eta) \wedge p_2^*(\mu)$. Then $\omega$ has no zeroes.
(2) We now consider fiber products. Let $S_i$ be projective surfaces with maps $f_i : S_i \rightarrow B$ to the smooth curve $B$. We assume that the fiber product $S_1 \times_B S_2$ is smooth. This holds if the singular fibers of $f_1$ are over different points than the singular fibers of $f_2$. Let $\omega_i$ be 2-forms on $S_i$ and $\alpha_i$ be 1-forms on $S_i$. Then we obtain a 2-form on $X$ by setting

$$\omega = p_1^*(\omega_1) + p_2^*(\omega_2) + p_1^*(\alpha_1) \wedge p_2^*(\alpha_2).$$

We shall see in (5.5) that every 2-form on $X$ is of this type. The condition for $\omega$ to be without zeroes is the following:

(a) $\omega_i$ and $\alpha_i$ have no common zeroes, $i = 1, 2$

(b) if $f_2^{-1}(b)$ is singular, then $\omega_1$ and $\alpha_1$ have no zeroes along $f_2^{-1}(b)$, and vice versa for $\omega_2$ and $\alpha_2$ over $f_1^{-1}(b)$.

(3) Let $S$ be a symplectic 4-fold and $X \subset S$ a smooth threefold. Let $w$ be the symplectic form on $S$. Then $\omega = w|X$ has no zeroes. In fact, choose local coordinates such that $w = dz_1 \wedge dz_2 + dz_3 \wedge dz_4$. Now $X$ is locally given by $z_4 = f(z_1, z_2, z_3)$. Hence

$$w|X = dz_1 \wedge dz_2 + dz_3 \wedge \frac{\partial f}{\partial z_1} dz_1 + dz_3 \wedge \frac{\partial f}{\partial z_2} dz_2$$

has no zeroes.

(4) Let $X$ be a smooth projective threefold and $f : X \rightarrow B$ a surjective fibration to a curve $B$. Assume that the general fiber of $f$ is an abelian surface and $K_X$ is nef. Then $X$ carries 2-forms, if $q(X) + h^0(K_X) \geq 2$. In particular this holds if $g(B) \geq 2$. In fact, let $F$ be a general smooth fiber of $f$. First observe

$$c_2(X) \cdot F = c_2(F) = 0.$$

On the other hand, $mK_X$ is generated by global sections, $K_X$ being nef. Then either $f$ is the Iitaka fibration of $X$ or $mK_X = \mathcal{O}_X$. In both cases $K_X \equiv \lambda F$ with some $\lambda \geq 0$. Thus in total

$$K_X \cdot c_2(X) = 0,$$

hence $\chi(X, \mathcal{O}_X) = 0$ by Riemann-Roch and the claim on the existence of 2-forms follows.

(5) (Beauville) Let $S$ be a K3-surface, $S^{(2)}$ its symmetric product and $S^{[2]}$ its canonical desingularisation, i.e. the blow-up of the image of the diagonal. Then $S^{[2]}$ is a symplectic 4-fold, call $w$ the symplectic form. If $S$ is projective, Markushevitch [Ma86] proved the following. If there is a surjective map $\pi : S^{[2]} \rightarrow \mathbb{P}_2$ with general fiber $F$ an abelian surface, then $S$ admits an elliptic fiber space structure over $\mathbb{P}_1$. Consequently $F$ is a product of elliptic curves. Now let $l \subset \mathbb{P}_2$ be a line, put $X = \pi^{-1}(l)$ and let $\omega = w|X$. In this situation $\omega$ is induced. In fact, $X$ is birational to a fiber product of two elliptic surfaces over $l$ and then $\omega$ is induced as explained in (2). Therefore one might ask whether all abelian fibrations $X$ over a curve $B$ with $h^2(\mathcal{O}_X) \neq 0$ are birational to a fiber product of two elliptic surfaces. This is however not the case as explained in the next example.
(6) (Markushevitch) There is a symplectic 4-fold $M$ (constructed as birational model of the symmetric product of a K3-surface which is a double covering of $\mathbb{P}_2$) admitting a surjective map $\pi : M \to \mathbb{P}_2$, whose general fiber $F$ is an abelian surface and actually birational to the symmetric product $C^{(2)}$ of a hyperelliptic curve of genus 2. In particular $F$ cannot be a product. As in (5) we let $X = \pi^{-1}(l)$ with $l \subset \mathbb{P}_2$ a general line, $X = \pi^{-1}(l)$ and let $\omega = w|X$. Then $\pi : X \to \mathbb{P}_1$ is an abelian fibration over $\mathbb{P}_1$ with a nowhere vanishing 2-form whose general fiber does not split. So $X$ is not birational to a product of two elliptic surfaces over $\mathbb{P}_1$.

1.4 Remarks

(a) In order to construct further examples where the 2-form is not a priori induced, one might think of deforming fiber products (1.3(2)) in non-fiber products. However it is not possible to get interesting examples (with $\kappa(X) \geq 0$, as shown in (1.5) below).

(b) If $f : X \to B$ is an abelian fibration of the projective threefold $X$, then without assuming $K_X$ to be nef, we can still produce 2-forms under the condition $q(X) + h^0(K_X) \geq 2$. Suppose first $g(B) \geq 1$. Let $X'$ be a minimal model of $X$. Then $f$ induces an abelian fibration $f' : X' \to B$. We still have $K_{X'} \cdot c_2(X') = 0$. However $X'$ might have non-Gorenstein singularities in which case $\chi(X, \mathcal{O}_X) > 0$, the positivity coming from contribution from the non-Gorenstein singularities, see [Fl87]. Hence $\chi(X, \mathcal{O}_X) = \chi(X', \mathcal{O}_{X'}) \geq 0$ and we conclude. If $B = \mathbb{P}_1$, we may assume $\kappa(X) \geq 0$; because if $\kappa(X) = -\infty$, we have $q(X) \geq 2$ by our assumption $q(X) + h^0(K_X) \geq 2$, and therefore we see immediately that the Albanese map of $X$ is a generic $\mathbb{P}_1$–fibration over a 2-dimensional torus, hence we get 2-forms on $X$. If $\kappa(X) = 0$, then it is clear by the same arguments as above that $\chi(X, \mathcal{O}_X) \geq 0$ and finally if $\kappa(X) = 1$, we pass to a minimal model $X'$ and consider its Iitaka fibration $g : X' \to C$.

1.5 Proposition

Let $g_i : S_i \to C$ be elliptic fibrations of the smooth compact Kähler surfaces $S_i$ onto the smooth curve $B$ such that the fiber product $X = S_1 \times_C S_2$ is smooth. Assume $\kappa(S_i) \geq 0$. Assume furthermore that $g(C) \geq 2$ or that the $j$–invariants of $f_i$ are not constant. Then every small deformation of $X$ is again a fiber product of elliptic surfaces.

Proof. (1) First assume that $g(C) \geq 2$. Our claim comes down to show that

$$H^1(T_X) = H^1(T_{S_1}) \oplus H^1(T_{S_2})$$

(\ast)

canonically. Let $p_i : S_1 \times S_2 \to S_i$ be the projections and $f_i = p_i|X$. Let $f : X \to C$ be the canonical map $(f = g_i \circ f_i)$. The normal bundle sequence associated to the embedding $X \subset S_1 \times S_2$ reads

$$0 \to T_X \to p_1^*(T_{S_1}) \oplus p_2^*(T_{S_2})|X \to f^*(T_C) \to 0.$$ 

This sequence induces a map

$$\alpha : H^1(T_X) \to H^1(f_1^*(T_{S_1})) \oplus H^1(f_2^*(T_{S_2}))$$

\text{Proof. (2)} First assume that $g(C) \geq 2$. Our claim comes down to show that

$$H^1(T_X) = H^1(T_{S_1}) \oplus H^1(T_{S_2})$$

(\ast)

canonically. Let $p_i : S_1 \times S_2 \to S_i$ be the projections and $f_i = p_i|X$. Let $f : X \to C$ be the canonical map $(f = g_i \circ f_i)$. The normal bundle sequence associated to the embedding $X \subset S_1 \times S_2$ reads

$$0 \to T_X \to p_1^*(T_{S_1}) \oplus p_2^*(T_{S_2})|X \to f^*(T_C) \to 0.$$ 

This sequence induces a map

$$\alpha : H^1(T_X) \to H^1(f_1^*(T_{S_1})) \oplus H^1(f_2^*(T_{S_2}))$$

\text{Proof. (2)} First assume that $g(C) \geq 2$. Our claim comes down to show that

$$H^1(T_X) = H^1(T_{S_1}) \oplus H^1(T_{S_2})$$

(\ast)

canonically. Let $p_i : S_1 \times S_2 \to S_i$ be the projections and $f_i = p_i|X$. Let $f : X \to C$ be the canonical map $(f = g_i \circ f_i)$. The normal bundle sequence associated to the embedding $X \subset S_1 \times S_2$ reads

$$0 \to T_X \to p_1^*(T_{S_1}) \oplus p_2^*(T_{S_2})|X \to f^*(T_C) \to 0.$$ 

This sequence induces a map

$$\alpha : H^1(T_X) \to H^1(f_1^*(T_{S_1})) \oplus H^1(f_2^*(T_{S_2}))$$
and we want to prove that $\alpha$ is injective with image $H^1(T_{S_i}) \oplus H^1(T_{S_2})$. The injectivity of $\alpha$ is immediately clear by $H^0(f^*(T_C)) = H^0(T_C)$ by virtue of our assumption $g(C) \geq 2$. For the description of the image of $\alpha$ we first compute by the Leray spectral sequence

$$H^1(T_X) = H^1(f_{i*}(T_X)) \oplus H^0(R^1f_{i*}(T_X))$$

and

$$H^1(f_{i*}(T_{S_i})) = H^1(T_{S_i}) \oplus H^0(T_{S_i} \otimes R^1f_{i*}(\mathcal{O}_X)).$$

Therefore we must only prove

$$H^0(T_{S_i} \otimes R^1f_{i*}(\mathcal{O}_X)) = 0. \quad (***)$$

By base change we have

$$R^1f_{i*}(\mathcal{O}_X) = g_{i*}R^1g_{j*}(\mathcal{O}_{S_j}),$$

where $j = 2$, if $i = 1$ and $j = 1$ if $i = 2$. Now $L_j = R^1g_{j*}(\mathcal{O}_{S_j})$ is a seminegative line bundle, i.e. its dual is nef. In order to prove (**), take a general hyperplane section $B \subset S_i$. By [Mi87], the vector bundle $\mathcal{O}_{S_i}|B$ is nef, since $\kappa(S_i) \geq 0$. So (***) follows unless possibly deg$L_j = 0$ and if $S_i$ is a torus. This means however that $S_j$ is zero j-invariant. Hence by our assumption $g(C) \geq 2$, which contradicts the fact that $S_i$ maps onto $C$.

(2) If $g(C) \leq 1$, we make the same arguments with the following modification. The map $\alpha$ is no longer injective, however its kernel comes from $H^0(T_C)$, and these elements in $H^1(T_X)$ just correspond to deformations of the morphisms $f_i : S_i \longrightarrow C$ by automorphisms of $C$, so they also correspond to deformations of $X$ as fibered product.

Remark. Of course (1.5) is false in the case of tori: take $S_i = F_1 \times C$ with elliptic curves $F_1, C$ so that $X = F_1 \times F_2 \times C$. Then the general deformation of $X$ is no longer a fiber product. We shall ignore the case $\kappa(S_i) = -\infty$ in the situation (1.5).

1.6 Problem Let $f : X \longrightarrow B$ be an abelian fibration of the projective 3-fold $X$ to the smooth curve $B$. Assume possibly $K_X$ nef. Is it true that $c_3(X) \geq 0$? It seems possible to construct examples with $c_3(X) = 0$ by choosing $B$ general in a suitable moduli space, say the moduli space $A(1,p)$ of $(1,p)$–polarised abelian varieties with level structures. Note that $c_3(X) = 0$ holds if all fibers of $f$ are of the type $C \times E$, with $C$ a singular rational fiber of an elliptic surface and $E$ an elliptic curve.

In the rest of this section we prepare the study of 2-forms on fibered threefolds.

1.7 Identification Let $X$ be a smooth threefold. Then there is a canonical isomorphism

$$\Omega_X^2 \simeq T_X \otimes K_X,$$

where $T_X$ is the tangent bundle of $X$ and $K_X$ its canonical bundle. Therefore any 2-form $\omega$ corresponds to a section $s \in H^0(T_X \otimes K_X)$ and vice versa and we will often switch from $\omega$ to $s$ and back.
In the following we fix a compact threefold $X$ and a surjective map $f : X \rightarrow B$ to a smooth curve or a normal surface $B$.

1.8 Notation Let $T_f : T_X \rightarrow f^*(T_B)$ denote the differential with kernel $T_X|_B$. Let $s \in H^0(T_X \otimes K_X)$. Then we define

$$f_*(s) = f_*(T_f \otimes \text{id})(s) \in H^0(B, f_*(f^*(T_B) \otimes K_X)).$$

If $B_0$ is the smooth part of $B$, then notice that $f_*(s)|B_0 \in H^0(B_0, T_{B_0} \otimes f_*(K_X))$.

If $b \in B$ and if $F_b$ denotes the analytic fiber over $b$, then $f_*(f^*(T_B) \otimes K_X)|\{b\} = f^*(T_B) \otimes K_X|F_b$ and therefore we have

$$s|F_b \in H^0(f^*(T_B) \otimes K_X|F_b).$$

By virtue of the exact sequence

$$0 \rightarrow T_{F_b} \rightarrow T_X|F_b \rightarrow N_{F_b} = f^*(T_B)|F_b$$

we deduce that if $f_*(s)(b) = 0$, then $s|F_b \in H^0(F, T_F \otimes K_X|F_b)$. There is also a morphism

$$\lambda_b : T_X \otimes K_X|F_{\text{red}} \rightarrow N_{\text{red}|F} \otimes K_X|\text{red}F.$$}

for $F = F_b$. The condition that $\lambda_b(s) = 0$ is slightly stronger than $f_*(s)(b) = 0$.

1.9 Definition Let $s \in H^0(T_X \otimes K_X)$. Let $b \in B$. Then we say that $s$ (or $\omega$) is vertical at $b$, if $f_*(s)(b) = 0$. $s$ is vertical if it is vertical at every point.

In other words, $s$ is vertical if and only if $f_*(s) = 0$, which is the same as to say that $s \in H^0(X, T_X|_B \otimes K_X)$. Or, that for $x$ general in $X$, the annihilator of $\omega$ in $T_{X,x}$ is tangential at $x$ to the fiber of $f$ at $x$.

We make the following observation.

Suppose $s$ is not vertical at $b$. Then $f_*(s)$ is a non-zero section of $T_B \otimes f_*(K_X)$, which is a torsion free sheaf. Hence $f_*(s) \neq 0$ generically, i.e. $s$ is not vertical at the general $b \in B$.

So $s$ is either vertical everywhere or $s$ is not vertical at the general $b$.

We now investigate the following situation. $X$ is a minimal projective 3-fold, i.e. $X$ has only terminal singularities, is $\mathbb{Q}$-Gorenstein and $K_X$ is nef. Then by abundance $mK_X$ is generated by global sections for large $m$ and the associated map $f : X \rightarrow B$ is the Iitaka fibration of $X$. We consider again $s \in H^0(X, T_X \otimes K_X)$.

1.10 Lemma Let $F$ be a fiber of $f$. Then there exists a neighborhood $V$ of $f(F)$ in $B$ and a covering $h : \tilde{X}_V \rightarrow X_V = f^{-1}(V)$, étale outside a finite set (contained in $\text{Sing}(X_V)$), such that $K_{\tilde{X}_V} = \mathcal{O}_{\tilde{X}_V}$ and such that $\tilde{s} = h^*(s) \in H^0(T_{\tilde{X}_V})$. 
Proof. We have \( mK_X = f^*(L) \). Hence \( mK_{X_V} = \mathcal{O}_{X_V} \), if \( L|V = \mathcal{O}_V \). Now take the canonical cover \( h : \tilde{X}_V \to X_V \) with respect to a nowhere vanishing section of \( mK_{X_V} \), see e.g. [KMM87]. Since \( K_{\tilde{X}_V} = \mathcal{O}_{\tilde{X}_V} \), we have \( \tilde{s} \in H^0(T_{\tilde{X}_V}) \).

1.11 Notation In the situation of (1.10) we let \( \tilde{h} : \tilde{X}_V \to \tilde{V} \) be the Stein factorisation of \( h \).

1.12 Corollary Assume the situation of (1.10). Let \( b \in B \). Then we are in exactly one of the two following cases.

1. \( \omega \) (resp. \( s \)) is vertical at \( b \)

2. there exists a one parameter group \( (g_t)_{t \in \mathbb{C}} \) (after possibly shrinking \( V \)) which acts fiber-preserving on \( \tilde{h} : \tilde{X}_V \to \tilde{V} \) and which does not fix \( b \in V \), where \( b \) is any preimage point on \( b \) in \( \tilde{V} \). In particular there is a local curve \( C_b \subset B \) through \( b \) such that \( F_{b'} \cong F_b \) for all \( b' \in C_b \).

Proof. Local integration of \( s \) near \( \tilde{h}^{-1}(b) \).

2. Statement of the Main Results

In this section we give precise statements of the results proved in this paper.

2.0 Notations By \( X \) we denote a compact connected threedimensional Khler manifold. A pair \((X, \omega)\) denotes a non-zero holomorphic 2-form on the manifold \( X \).

2.1 Definition The compact complex threefold \( X \) is said to be simple if it contains only finitely many irreducible divisors and only countably many compact irreducible curves not contained in any of these divisors (in particular, \( a(X) = 0 \)).

In other words, \( X \) is simple if there is no positive dimensional compact subspace through the very general point of \( X \).

2.2 Definition The compact complex threefold \( X \) is said to be Kummer if there exists a surjective meromorphic map \( \varphi : T \to X \) with \( T \) a torus.

2.3 Remark. The only known examples of simple Khler threefolds \( X \) are Kummer. It can be shown that these should actually be the only ones if a minimal model program exists in the Khler case when \( n = 3 \) ([Pe96]). Therefore we have the

2.4 Conjecture Every simple Khler threefold \( X \) is Kummer.

Since we are far from being able to prove this conjecture, we introduce the
2.5 Hypothesis (H) All compact Kähler manifolds are assumed not to be both simple and non-Kummer.

2.6 Theorem. Let $X$ be a compact Kähler threefold with $\kappa(X) = -\infty$ (assuming (H)). Then $X$ is uniruled. (see (8.1)).

This extends the famous projective case ([Mi88],[Mo88]) to the Kähler case.

2.7 Corollary Let $(X,\omega)$ be as in (2.0) with $\kappa(X) = -\infty$. There exists a surjective connected meromorphic map $\rho : X \to S$ to a smooth surface such that $h^{0,2}(S) > 0$ and $\rho^* : H^{0,2}(S) \to H^{0,2}(X)$ is bijective.

One just needs to take the rational quotient ([Ca81,92]) of $X$ to get $\rho$, see (3.1) for details. Recall also that non-algebraic Kähler threefolds carry a holomorphic 2-form by Kodaira’s theorem. A classification of non-algebraic uniruled threefolds $X$, according to $a(X)$, is given in sect. 9.

2.8 Theorem Let $X$ be a compact Kähler threefold with $\kappa(X) = 0$ and $h^{0,2}(X) > 0$ (assuming (H)). Then $X$ admits a bimeromorphic model $X'$ with a covering $\pi : \tilde{X}' \to X'$, tame in codimension one, such that $\tilde{X}'$ is either a torus or a product $S \times E$ with $S$ a K3-surface and $E$ an elliptic curve (see 8.1).

Again, this extends to the Kähler case a known result ([Ka81]) in the projective case.

2.9 Corollary In the situation of (2.8) let $\tilde{\omega}$ be the form on $\tilde{X}'$, induced by $\omega$. Then $\tilde{\omega}$ is induced from maps to either tori or K3-surfaces.

When $\kappa \geq 1$, the situation becomes more complicated, and our results are very partial. When $\kappa(X) = 3$, we shall not say anything. On the other hand, we don’t need any longer the hypothesis (H), since $a(X) \geq \kappa(X) \geq 1$. We shall denote in 2.10 and 2.11 below by $f : X \to B$ the (meromorphic) Iitaka reduction of $X$.

2.10 Theorem Let $(X,\omega)$ be as in 2.0 with $\kappa(X) = 1$. Then:
(i) If $\omega$ is not $f$-vertical, then $f$ is generically over $B$ a meromorphic fiber bundle (i.e. two generic fibers are bimeromorphic). When $X$ is projective, more precise conclusions can be drawn (see 4.1, 4.2).
(ii) If $\omega$ is $f$-vertical, and $X$ is minimal projective, the generic fiber $F$ of $f$ is either abelian or bielliptic. If $F$ is not a simple abelian surface, then after a finite base change $\beta : \overline{B} \to B$, the space $\overline{X} := X \times_B \overline{B}$ is a fiber product $\overline{X} = X_1 \times S_2$ of two elliptic surfaces and $\overline{\omega}$ (the lift of $\omega$ to $\overline{X}$) is induced.

See (4.4), (4.5) for details and further conclusions in even more special cases.
2.11 Theorem. Let \((X, \omega)\) be as in 2.0, and assume that \(\kappa(X) = 2\) with \(X\) a projective \(\mathbb{Q}\)-factorial threefold with only terminal singularities and \(K_X\) nef (so that \(f\) is holomorphic with \(B\) a normal surface with only quotient singularities).

(i) If \(\omega\) is \(f\)-vertical, then \(\omega = f^*(\eta)\) where \(\eta\) is a 2-form on \(B\) (see 5.1).
(ii) If \(\omega\) is not \(f\)-vertical and the invariant \(j : B \to \mathbb{P}_1\) is not constant, then \(X\) is birational to \(S \times B\) where \(J : B \to C\) is the Stein factorisation of \(j\) and \(\varphi : S \to C\) a suitable elliptic fibration. Moreover, \(\omega\) is induced from \(B\) and \(S\).

Let us now briefly explain the proofs of our main results 2.6 and 2.8: the results are known when \(a(X) = 3\) by the results on the minimal model program (Kawamata, Miyaoka, Mori). The case \(a(X) = 0\) is again very easy (under assumption (H)). We proceed by considering the algebraic reduction \(r : X \to S\) in the remaining cases \(a(X) = 2, 1\): the generic fiber \(F\) or \(r\) is either an elliptic curve; a ruled surface (the \(X\) is clearly uniruled) or a surface with \(\kappa(F) = 0\), in fact either K3 or torus (bimeromorphically). The difficult cases are the cases where \(F\) is a torus (either an elliptic curve or 2-dimensional).

Our main tools are first the solutions of Iitaka conjectures \(C_{3,1}, C_{3,2}\) ([Fu78], [Ka81]) and secondly the “untwisted model” (see 6) \(r_0 : X_0 \to S\) of \(r\), whose generic fiber is \(\text{Aut}^0(F)\), the group of translations of \(F\). The fiber \(F_0\) of \(r_0\) corresponding to \(F\) is (non-canonically) isomorphic to \(F\), and \(r_0\) has moreover a meromorphic canonical zero section. When \(\dim X = 2 = 1 + \dim S\), this is the classical Jacobian fibration used by K. Kodaira in his classification theory of surfaces.

We show that \(h^p(\mathcal{O}_{X_0}) = h^p(\mathcal{O}_X)\) (\(p \geq 0\)) and \(\kappa(X_0) \leq \kappa(X)\) (in special cases, sufficient for our purposes, but this should be a general fact).

To illustrate the method, let us explain how it works when \(a(X) = 2, \kappa(X) = -\infty\). If \(\kappa(S) \geq 0\), we are finished by \(C_{3,2}\) which shows that the generic fiber of \(r\) is \(\mathbb{P}_1\). Otherwise, we consider the untwisted model \(r_0 : X_0 \to S\). Since \(r_0\) has a section, \(X_0\) is projective. By general theory (6.5) we have \(\kappa(X_0) = -\infty\), therefore \(X_0\) is uniruled and we consider its rational quotient \(X_0 \to \Sigma_0\) whose image turns out to be a surface. From this map we construct another map \(\sigma : X \to \Sigma\) with \(\Sigma\) a surface with \(\kappa(\Sigma) \geq 0\) (since \(h^{0,2}(X) > 0\)) (see 8).

It might be noted that most (not all) of our arguments have an inductive nature on \(\dim X\), and also it seems quite plausible that if the implication “\(\kappa(X) = -\infty\)” implies “\(X\) uniruled” can be proved in the projective case, it should also hold in the Kähler case as well (provided \(C_{n,m}\) holds true, too).

3. The Case of non-positive Kodaira Dimension

For all of this section we fix a compact Kähler threefold \(X\) and a non-zero 2-form \(\omega\) on \(X\). We shall investigate the structure of \((X, \omega)\) in case \(\kappa(X) \leq 0\).

3.1 Theorem If \(\kappa(X) = -\infty\), then there exists a meromorphic dominant map \(f : X \to S\) to a Kähler surface \(S\) such that \(\omega = f^*(\eta)\), unless \(X\) is simple.
Recall that $X$ is simple, if there is no covering family of proper positive-dimensional subvarieties, in particular $X$ has non non-constant meromorphic function. It is expected that there are no simple threefolds of negative Kodaira dimension, see sect. 2.

**Proof.** (a) First note that $X$ is uniruled. In the algebraic situation this is a central result in minimal model theory; if $X$ is Kähler with $a(X) < 3$, this is Theorem 8.1, since $X$ is supposed not to be simple.

(b) Since $X$ is uniruled, there exists a covering family $(C_t)$ of rational curves and we can consider an associated rational quotient $f : X \to S$, i.e. $f$ contracts just the curves $C_t$. Since the general fiber of $f$ is rational, we find a 2-form $\eta$ with $\omega = f^*(\eta)$. In fact, choose a sequence of blow-ups $\hat{X} \to X$ such that the induced map $\hat{f} : \hat{X} \to S$ is a morphism. Let $S_0$ be maximal such that $\hat{f}$ is a submersion over $S_0$, i.e. a $\mathbb{P}_1$-bundle. Then we find a 2-form $\eta_0$ on $S_0$ such that $\omega = \hat{f}^*(\eta_0)$ over $X_0 = \hat{f}^{-1}(S_0)$. Since $f^*(\eta_0)$ extends as a 2-form on $X$, it is easily checked that \[
\int_{S_0} \eta_0 \wedge \eta_0 < \infty,
\]hence $\eta_0$ extends to all of $S$.

**3.2 Theorem** Assume that $X$ is projective. If $\kappa(X) = 0$, then after possibly a finite cover, étale in codimension 2, either $X$ is bimeromorphic to a torus, $\omega$ coming from this torus, or $X$ is birationally equivalent to $E \times S$, with $E$ an elliptic curve and $S$ a K3 surface, $\omega$ coming from the K3 surface.

**Proof.** (a) First assume that $q(X) > 0$. By passing to a minimal model we may assume from the beginning that $K_X \equiv 0$. Of course, $X$ is now singular in general. Let $\alpha : X \to A$ be the Albanese map. After performing a base change, étale in codimension 2, $\tilde{A} \to A$, $X$ is by [Ka85] birationally a product $F \times A$ with $F$ a K3 surface or empty.

(b) We now prove that the existence of a 2-form implies the existence of a 1-form if $K_X = \mathcal{O}_X$ which can be achieved after finite cover, étale in codimension 2. In fact, by Riemann-Roch [Fl87], we have $\chi(X, \mathcal{O}_X) = 0$. So $q(X) > 0$ and (a) applies. Compare [Pe94,sect.5] for an argument without using Riemann-Roch (and which applies directly also in the Kähler case).

We are now turning to the special case that the 2-form $\omega$ has only finitely many zeroes.

**3.3 Theorem** Let $X$ be a projective threefold and $\omega$ a 2-form on $X$ with finite zero set $Z(\omega)$.

1. Assume $\kappa(X) = -\infty$. Then $X$ is a $\mathbb{P}_1$-bundle over a K3-surface or a torus and $\omega$ is a pull-back.

2. If $\kappa(X) \geq 0$, then $K_X$ is nef.

**Proof.** Assume $K_X$ not to be nef. If $\kappa(X) = -\infty$, this is automatically satisfied. Let $\varphi : X \to Y$ be the contraction of an extremal ray. Let $d = \dim Y$. Since $H^2(X, \mathcal{O}_X) = H^2(Y, \mathcal{O}_Y)$, we must have $d \geq 2$. 

We first treat the case $d = 2$. Here $X \rightarrow Y$ is a conic bundle over the smooth surface $Y$. Let $\Delta \subset Y$ be the discriminant locus. If $\Delta = \emptyset$, then $\varphi$ is analytically a $\mathbb{P}_1$-bundle and there is a 2-form $\eta$ on $S$ with $\omega = \varphi^*(\eta)$. Since $\eta$ has no zeroes, $S$ is a torus or a K3-surface as claimed. So suppose $\Delta \neq \emptyset$. Pick a smooth point $y \in \Delta$. Then $\varphi^{-1}(y)$ is a reducible conic in $\mathbb{P}_2$, i.e. consists of 2 lines $L$ and $L'$ with normal bundles $(\mathcal{O} \oplus \mathcal{O}(-1))$. By taking $\bigwedge^2$ of the exact sequence $\mathcal{N}^*_L|_X \rightarrow \Omega^1_X \rightarrow \Omega^1_L \rightarrow 0$ we obtain the exact sequence

$$0 \rightarrow \bigwedge^2 \mathcal{N}^*_L|_X \rightarrow \Omega^2_X | L \rightarrow \Omega^1_L \otimes \mathcal{N}^*_L|_X \rightarrow 0,$$

which reads

$$0 \rightarrow \mathcal{O}_L(1) \rightarrow \Omega^2_X | L \rightarrow \mathcal{O}_L(-2) \oplus \mathcal{O}_L(-1) \rightarrow 0.$$

Consequently $\omega$ vanishes at some point of $L$. Varying $y$ we conclude $\dim Z(\omega) \geq 1$, contradiction. Hence $\Delta = \emptyset$.

(3) We finally consider the case $d = 3$, so that $\varphi$ is birational. Here we use Mori’s classification [Mo82]. If $\varphi$ is the blow-up of a smooth curve in the smooth threefold $Y$, then the abover arguments applied to a non-trivial fiber of $\varphi$ give the same contradiction. Let $E$ be the exceptional divisor of $\varphi$. Then we conclude $\dim(\varphi(E)) = 0$. If $E = \mathbb{P}_1 \times \mathbb{P}_1$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, then we have a factorisation $\varphi = \rho \circ \psi$, where $\psi : X \rightarrow Z$ is the blow-down of $X$ along one of the two rulings of $E$ and $\rho$ is a small contraction. Then we apply the above arguments to $\psi$ and exclude that case.

Next suppose that $E = \mathbb{P}_2$ with normal bundle $\mathcal{O}(-1)$, i.e. $\varphi$ is the blow-up of a smooth point in $Y$. We have the exact sequences

$$0 \rightarrow \mathcal{N}^*_E|_X \rightarrow \Omega^1_X | E \rightarrow \Omega^1_E \rightarrow 0$$

and

$$0 \rightarrow \mathcal{N}^*_E|_X \otimes \Omega^1_E \rightarrow \Omega^2_E \rightarrow 0.$$
at least on $E \setminus \{x_0\}$. Here $\Omega^2_E := \bigwedge^2 \Omega^1_E$/torsion. Actually it can be seen that $\bigwedge^2 \Omega_E$ is torsion free, but we do not need that. Now the maps $\alpha$ and $\beta$ extend to maps

$$N^*_E \otimes \Omega^1_E \to \Omega^1_X|$$

and

$$\Omega^1_X|E \to \Omega^2_E.$$

The second extension being obvious, the first comes from the reflexivity of $\Omega^1_E$. It is now immediately checked that the second sequence remains exact on all of $E$. Since $\Omega^2_E \subset \omega_E$, the dualising sheaf of $E$ we have $H^0(\Omega^2_E) = 0$, hence it suffices to prove

$$H^0(E, \Omega^1_E(1)) = 0$$

to obtain the desired contradiction. This is however obvious from the embedding $E \subset \mathbb{P}_3$.

In the non-algebraic case we can prove

3.4 Theorem Let $X$ be a compact Kähler threefold with $\kappa(X) = -\infty$. Assume that $X$ is not simple and that there is a 2-form $\omega$ on $X$ with $Z(\omega)$ finite. Then $X$ is a $\mathbb{P}_1$-bundle over a torus or a $K3$-surface.

Proof. By Theorem 8.1, $X$ is uniruled. Therefore [Pe96] applies and there is a contraction $\varphi : X \to Y$. Hence we can conclude as in 3.3.

In the proof of 3.3 the assumption $\kappa(X) = -\infty$ was only used to ensure the existence of a contraction. Therefore we can state more generally :

3.5 Corollary Let $X$ be a projective threefold with a 2-form having at most finitely many zeroes. If $K_X$ is not nef, then $X$ is a $\mathbb{P}_1$-bundle over a torus or a $K3$-surface.

3.6 Theorem Let $X$ be a projective threefold with $\kappa(X) = 0$ and $\omega$ a 2-form with finite zero set. Then $X$ is a torus or a product of a $K3$-surface and an elliptic curve up to finite étale cover.

Proof. $K_X$ is nef by 3.5. Since $\kappa(X) = 0$, it follows $K_X \equiv 0$. Then the result follows from the decomposition theorem for manifolds with $c_1(X) = 0$, observing that Calabi-Yau threefolds have no 2-forms.

In the non-algebraic case we cannot conclude since we do not know the existence of a contraction when $K_X$ is not nef. However it should be possible to prove (3.6) for non-simple Kähler threefolds with $\kappa(X) = 0$. By [Fu83], we have $a(X) \geq 1$. Then one should analyse the algebraic reduction.
4. Case $\kappa = 1$.

Let $X$ be a compact Kähler 3-fold with 2-form $\omega$. Throughout this section we assume $\kappa(X) = 1$. Let $f : X \to B$ be the Iitaka fibration to the smooth curve $B$. By possibly blowing up $X$, we may assume $f$ holomorphic. Anyway $f$ is automatically holomorphic if $g(B) \geq 1$. We consider the exact sequence

$$0 \to T_{X|B} \otimes K_X \to T_X \otimes K_X \xrightarrow{\kappa} f^*(T_B) \otimes K_X.$$ 

Then $\omega$ resp. $s$ is vertical iff $\kappa(\omega) = 0$. Equivalently, consider

$$f^*(\Omega^1_B) \otimes \Omega^1_{X|B} \to \Omega^2_X \xrightarrow{\mu} \Omega^2_{X|B} \to 0.$$ 

Then $\omega$ is vertical iff $\mu(\omega) = 0$ generically. So if $\omega$ is not vertical, then it defines an element $\overline{s} \in H^2(X, \mathcal{O}_X)$, and since $\mu(\omega) \neq 0$, we have $0 \neq \overline{s}|x_b \in H^2(X_b, \mathcal{O})$, at least for general $b$. Therefore we obtain a section

$$t \in H^0(B, R^2f_*(\mathcal{O}_X)).$$ 

Since $f_*(\omega_{X|B}) \simeq R^2f_*(\mathcal{O}_X)^*$ is nef, we conclude

$$f_*(\omega_{X|B}) \simeq \mathcal{O}_B,$$

thus by Torelli (we may assume all fibers minimal, see e.g. [Ue87,1.11]), $f : X_U \to U$ is an analytic fiber bundle, where $U$ is the largest open set over which $f$ is smooth.

We conclude

4.1 Proposition Let $X$ be a compact Kähler 3-fold with 2-form $\omega$ and with holomorphic Iitaka fibration $f : X \to B$. Assume that $\omega$ is not vertical with respect to $f$. Then the smooth part $f : X_U \to U$ is an analytic fiber bundle.

For further investigation let $X$ be projective and $X'$ be a minimal model of $X$. Let $f' : X' \to B$ be the Iitaka fibration, i.e. the fiber space defined by $|mK_{X'}|$ for suitable $m$. Again we let $s \in H^0(X, T_X \otimes K_X)$ be the section associated to $\omega$. Then the section in $H^0(f'_*(\omega_{X'|B}))$ associated to $\omega$ has no zeroes as seen above, hence this sheaf is trivial. Therefore $s$ does not vanish on full fibers of $f'$. Now pass to $\tilde{X}_V' \to \tilde{V}$ according to (1.10) and (1.12). Then actually $\tilde{s} \in H^0(T_{X'_V})$. Since $\tilde{s}$ is not vertical at every $\tilde{b} \in \tilde{B}$ by the above non-vanishing, (1.12) implies that all fibers of $\tilde{f}$ are smooth (singular fibers would be move horizontally to other singular fibers). We conclude that $f' : X' \setminus \text{Sing}(X) \to B$ is almost smooth, i.e. the only singular fibers are multiples of smooth surfaces. Now let $b_0 \in B$ such that

$$F_{b_0} \cap \text{Sing}(X') \neq \emptyset,$$
where \( F_{b_0} = f'^{-1}(b_0) \). Then \( S = \text{red}F_{b_0} \) is a normal surface with a certain fiber multiplicity \( \lambda \geq 1 \). If \( \varpi : \tilde{V} \to V \) is the canonical map as in (1.12), we take \( \tilde{b}_0 \in \tilde{V} \) such that \( \varpi(\tilde{b}_0) = b_0 \). Then \( \tilde{S} = \text{red}F_{\tilde{b}_0} \) is smooth, and we have a birational finite map \( \rho : \tilde{S} \to S \). Since \( S \) is normal, \( \rho \) is biholomorphic, and \( S \) is smooth. Therefore \( f' \) is almost smooth. Moreover there is a finite base change \( \tilde{B} \to B \) such that the induced fiber space \( \tilde{f} : \tilde{X} \to \tilde{B} \) is smooth, hence an analytic fiber bundle. Note that the typical fiber must be a K3 surface or abelian, since \( F \) is minimal with \( K_F \) numerically trivial; it cannot be hyperelliptic or an Enriques surface because of our 2-form \( \omega \). If \( F \) is a K3 surface, then, \( \text{Aut}(F) \) being discrete, \( \tilde{X} = F \times \tilde{B} \) after possibly another finite étale cover. So assume \( F \) abelian. It is well known (see e.g. [Fu83]) that after passing to a finite étale cover of \( \tilde{B} \), we have

\[
q(\tilde{X}) = q(F) + q(\tilde{B}).
\]

Now consider the Albanese map \( \tilde{\alpha} : \tilde{X} \to \tilde{B} \). After a last étale base change we will have \( \tilde{X} = F \times \tilde{B} \), see [Ue75] (note that since \( \kappa(\tilde{X}) \geq 1 \), we must have \( q(\tilde{B}) \geq 2 \)).

In total we have proved

**4.2 Theorem** Let \( X \) be a smooth projective threefold with 2-form \( \omega \). Assume that \( \omega \) is not vertical with respect to the Iitaka fibration. Let \( X' \) be a minimal model with Iitaka fibration \( f' : X' \to B \). Then \( f' \) is almost smooth. There is a finite base change \( \tilde{B} \to B \) with induced fiber space \( \tilde{f} : \tilde{X} \to \tilde{B} \) such that \( \tilde{X} \simeq F \times \tilde{B} \), where \( F \) is abelian or K3. If \( \omega' \) is the induced 2-form on \( X' \) and \( \tilde{\omega} \) the pull-back to \( \tilde{X} \), then \( \tilde{\omega} = p^*(\eta) + p^*(\eta) \wedge \tilde{f}^*(u) \), where \( p : \tilde{X} \to F \) is the projection and \( u, v \) are 1-forms.

Of course (4.2) holds in the Kähler case as well since also in this case the existence of a minimal model is known (Nakayama [Na95]).

**4.3** In case that \( \omega \) is vertical with respect to the Iitaka fibration we cannot say so much. The example 1.3(6) shows that the Iitaka map is not almost smooth in general for smooth minimal threefolds. We make this more precise and take over the notations of 1.3(6). We let \( f = \pi|X \). Then \( K_X = f^*(\mathcal{O}_l(2)) \), hence \( \kappa(X) = 1 \), \( K_X \) is nef and \( f \) is the Iitaka fibration. Assume \( f \) almost smooth. Then

\[
f_*(\omega_{X|l}) = \mathcal{O}_l,
\]

so dually \( R^2f_*(\mathcal{O}_X) = \mathcal{O}_l \). From the Leray spectral sequence we deduce \( h^2(\mathcal{O}_X) \geq 2 \). On the other hand we have the exact sequence

\[
\mathcal{C} \simeq H^2(\mathcal{O}_M) \to H^2(\mathcal{O}_X) \to H^3(\mathcal{I}_X) \to H^3(\mathcal{O}_M) = 0.
\]

Now \( H^3(\mathcal{I}_X) = H^1(\mathcal{O}_M(X)) = H^1(\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))) \). By the Leray spectral sequence

\[
H^1(\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))) \subset H^1(\mathcal{O}_{\mathbb{P}^2}(1)) \oplus H^0(R^1\pi_*(\mathcal{O}_M) \otimes \mathcal{O}_{\mathbb{P}^2}(1)).
\]
Since $q(M) = 0$, the Leray spectral sequence immediately implies that $R^1 \pi_* (\mathcal{O}_M)$ is a negative vector bundle, hence $H^3(\mathcal{I}_X) = 0$ and $h^2(\mathcal{O}_X) = 1$, contradiction. So $f$ is not almost smooth.

4.4 Proposition Let $X$ be a minimal Kähler threefold with $\kappa(X) = 1$. Let $\omega$ be a 2-form on $X$ which is vertical with respect to the Iitaka fibration $f : X \rightarrow B$. Let $F$ be the general fiber of $f$.

(1) $F$ is birationally an abelian surface or hyperelliptic.

(2) If $F$ is hyperelliptic or a non-simple abelian surface, then there exists a finite cover $\tilde{X} \rightarrow X$, unramified over the largest open set $B_0$ over which $f$ is smooth, such that $\tilde{f} : \tilde{X} \rightarrow \tilde{B}$ is birational to a fiber product $S_1 \times_{\tilde{B}} S_2$.

(3) Assume $q(X) > q(B)$. Then $F$ is hyperelliptic or a non-simple abelian surface, hence (2) holds, or $X$ is birational to an torus fiber bundle over a curve of genus $g \geq 2$ which gets a product after finite étale cover.

(4) If $B$ is rational, then $q(X) = 1$ and the general fiber $G$ of the Albanese map has $\kappa(G) = 1$.

Proof. (1) Since $\omega$ is vertical, it defines a 1-form on every smooth fiber $F$ of $f$. Since $\kappa(F) = 0$, $F$ must be abelian or hyperelliptic.

(2) If $F$ is hyperelliptic, then $F$ becomes a product after finite étale cover. The same holds if $F$ is a non-simple abelian surface by Poincaré complete reducibility, since then $F$ has the structure of an elliptic fiber bundle over an elliptic curve. Now consider the family of elliptic curves from the projection maps of the smooth fibers $F$ and take closure in the Chow scheme. After a finite cover of the base, unramified over $B_0$, we obtain a second family of elliptic curves and therefore the cover $\tilde{X} \rightarrow X$ is birationally a fiber product over $B$. The technical details are left to the reader.

(3) Suppose now that $q(X) > q(B)$. We have only to show that if $F$ is abelian, then either it is not simple or we are birationally in a product situation after some étale cover. Let $\alpha : X \rightarrow \text{Alb}(X)$ be the Albanese map of $X$ and $\beta : B \rightarrow \text{Alb}(B)$ that one of $B$. We obtain a commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & \text{Alb}(X) \\
\downarrow & & \downarrow \\
B & \longrightarrow & \text{Alb}(B)
\end{array}
$$

If $\dim \alpha(F) = 0$, then $\alpha$ would factor over $f$ which contradicts $q(X) > q(B)$. If $\dim \alpha(F) = 1$, then $F$ is not simple. So suppose $\dim \alpha(F) = 2$. Then $\alpha(F)$ is a subtorus of $\text{Alb}(X)$. If $\alpha(X) = \alpha(F)$, we have $q(X) = 2$, both 2-forms coming from $F$. Hence $q(B) = 0$. This case is excluded in (4). So we are reduced to the case $\dim \alpha(X) = 3$. Now $\alpha|F$ is étale, hence an isomorphism after an étale base change. This implies by the diagram that $\alpha$ is birational. Therefore we may substitute for our purpose $X$ by $Y$. Since $\kappa(X) = 1$, the map $Y \rightarrow \beta(B)$ is an étale fiber bundle [Ue75] and hence a product after finite étale cover.

(4) Assume that $B$ is rational. Let $\alpha : X \rightarrow Y \subset A$ denote again the Albanese map of $X$. Let $G$ be a general fiber of $\alpha$. Suppose $\dim Y = 1$. Then $C_{3,1}$ yields $\kappa(G) = 0$ unless $Y = A$
is an elliptic curve. If $\kappa(G) = 0$, then by nefness of $K_X$, we deduce $K_G \equiv 0$ and $K_X \cdot G = 0$. Since $f$ is the Iitaka fibration, $\dim f(G) = 0$, hence there is a factorisation $X \to Y \to B$. Therefore $f$ does not have connected fibers, contradiction. So $Y$ is elliptic and $\kappa(G) = 1$ as claimed.

Thus we may assume $\dim Y \geq 2$. If $\dim \omega(F) = 2$, then we obtain a 2-form on $X$ which is not vertical with respect to $f$, hence $f$ is almost holomorphic by (4.2). Thus $\kappa(X) \neq 1$ (or argue as follows: the sheaves $R^i f_* (\mathcal{O}_X)$ are trivial, hence the Leray spectral sequence gives $q(X) = 2$ which excludes all cases except $Y = \text{Alb}(X)$). It remains to treat the case that $\dim \omega(F) = 1$. If $Y = \text{Alb}(X)$, then $A$ is not simple, we have a bundle structure $A \to C$ to an elliptic curve, contracting all the curves $\alpha(F)$, and $f$ factors as $X \to Y \to C \to B$. Therefore the fiber of $f$ cannot be connected, contradiction. If $Y \neq A$, then there is a map $Y \to C$ to a curve of general type, contracting again all $\alpha(F)$, and we argue as before.

We finally have a short look at the case that $\omega$ has only finitely many zeroes.

4.5 Proposition Let $X$ be a smooth Kähler threefold with $\kappa(X) = 1$. Let $\omega$ be a 2-form with only finitely many zeroes, not vertical with respect to the Iitaka fibration. Then $K_X$ is nef, therefore we have a holomorphic Iitaka fibration $f : X \to B$. $f$ is almost holomorphic. There is a finite cover $\tilde{B} \to B$ such that the induced fiber space $\tilde{f} : \tilde{X} \to \tilde{B}$ splits: $\tilde{X} \simeq \tilde{B} \times S$ with $S$ a $K3$ surface or abelian, $\tilde{\omega}$ coming from $S$. In particular $\omega$ has no zeroes.

Proof. $K_X$ is nef by (3.3). Therefore $mK_X$ is generated by global sections for suitable $m$, and the Iitaka fibration is holomorphic. The rest comes from the proof of (4.2).

5. Case $\kappa = 2$.

We now turn to the case that $\kappa(X) = 2$. Throughout this section we assume that $X$ is a Kähler $\mathbb{Q}$–factorial threefold with only terminal singularities such that $K_X$ is nef; so $X$ is a minimal model. Then the Iitaka fibration is automatically a holomorphic elliptic fibration, and the surface $B$ has at most quotient singularities. We let $S \subset B$ be the closure of the set of all $b \in B$ such that the fiber $f^{-1}(b)$ is a multiple elliptic curve. Let $S_i$ be the irreducible components of $S_i$.

For any space $Z$ we let $\Omega^i_Z = \bigwedge^i \Omega^1_Z$ and if $Z$ is normal we define $\tilde{\Omega}^i_Z = j_*(\Omega^i_{\text{reg}Z})$.

5.1 Lemma Assume that the 2-form $\omega$ is vertical with respect to $f$. Then $\omega = f^*(\eta)$ with $\eta \in \omega_B(\sum k_iS_i) = \Omega^2_B(\sum k_iS_i)$ with $k_i \geq 0$.

Proof. Let $B_0$ be the largest subset of $B$ such that $f$ is a submersion over $B_0$. Let $X_0 = f^{-1}(B_0)$. Then it is clear that there is a 2-form $\eta_0$ on $B_0$ such that $\omega|B_0 = f^*(\eta_0)$. 


Let $\tilde{S}$ be the closure of the set of all $b \in B$ such that $f^{-1}(b)$ is a multiple elliptic or a singular rational (possibly reducible) curve. Let $\tilde{S}_i$ be its irreducible components. Since everything is algebraic, $\eta_0$ has a meromorphic extension to $B$. Since $B \setminus B_0 = \tilde{S} \cup E$ with a finite set $E$, this means that $\eta \in \Omega^2_B(\sum k_i \tilde{S}_i)$. To be more precise, we have first that $\eta|\text{reg}B \in H^0(\text{reg}B, \Omega^2_B(\sum k_i \tilde{S}_i))$, and then $\eta$ extends to all of $B$, since $\Omega^2_B(\sum k_i \tilde{S}_i)$ is reflexive. However $\eta$ cannot have poles along components $T$ of $\tilde{S}$ which are not in $S$ because the fiber over general points in $T$ are generically smooth; therefore the poles cannot cancel.

\textit{Remark.} Actually we have also $\omega \in \Omega^2_B(kS)$. In fact, let $b \in B$ be a singular point. Then locally around $b = 0$, we have $B = \mathbb{C}^2/G$ with a finite group $G \subset \text{Gl}(2)$. Let $p : \mathbb{C} \rightarrow B$ be the projection. Then $p^*(\eta|B \setminus \{0\})$ extends around $p^{-1}(0)$ to a meromorphic form $\tilde{\eta} \in \Omega^2(k\tilde{S})$ with $\text{hat}S = p^{-1}(S)$. Since $p^*(\eta)$ is $G$–invariant, $\tilde{\eta}$ is $G$–invariant, our claim follows.

\textbf{5.2 Lemma} Assume that the $j$-invariant of $f$ is constant. Then there is a finite set $E \subset B$ such that $f : X_0 \rightarrow B_0 = B \setminus E$ is almost smooth. There is a finite cover $\tilde{B} \rightarrow B$, such that the pull-back $\tilde{X} \rightarrow \tilde{B}$ is an elliptic bundle over $\tilde{B}_0$.

\textbf{Proof.} Let $C \subset B$ be an ample divisor and consider $X_C = f^{-1}(C)$. Then the $j$-invariant of $X_C$ is constant so that $f|X_C$ is almost smooth ([BPV84] e.g.). This proves the existence of $E$. The remaining part is standard.

\textbf{(5.2.a)} If in (5.2) $\omega$ is $f$–vertical, then we conclude by (5.1) that $\omega = p^*(\eta)$ with a \textit{holomorphic} 2-form $\eta$ on $B$. In the general case we can say the following; possibly after some cover. Choose a point $b \in B$ outside the finite set described in (5.2) and let $U$ be an open neighborhood such $X_U = f^{-1}(U) \simeq U \times E$ with $E$ the typical fiber of $f$. Let $p_i$ denote the obvious projections. Then

$$\omega|X_U = p^*_1(\omega_1) + p^*_1(\tau_1) \wedge p^*_2(\tau_2)$$

with a 2-form $\omega_1$ on $U$ and 1-forms $\tau_i$ on $U$ and $E$, respectively. This is also described by the exact sequence

$$0 \rightarrow f^*(\Omega^2_B) \rightarrow \Omega^2_X \rightarrow \Omega^1_{X|B} \otimes f^*(\Omega^1_B) \rightarrow 0$$

on $B_0$. Thus $\omega$ is induced in the sense of (1.2) and Remark (2) after (1.2).

Concerning non-vertical 2-forms we first show

\textbf{5.3 Lemma} Assume that the $j$-invariant $J : B \rightarrow \mathbb{P}_1$ of $f$ is non-constant. Let $\hat{B} \rightarrow \mathbb{P}_1$ be a holomorphic model of $J$. Then there exists a surface $g : S \rightarrow \mathbb{P}_1$ and a meromorphic dominant map $X \rightarrow S \times_{\mathbb{P}_1} B$. 


Proof. To keep notations easy, we assume from the beginning that $J$ is holomorphic (note that a priori $J$ is almost holomorphic, i.e. proper and holomorphic over an open set of $\mathbb{P}_1$). Let $g = J \circ f$. Let $G$ be the general fiber of $g$, say $G = g^{-1}(c)$. Then we have an elliptic fibration $\tau_c : G \rightarrow J^{-1}(c)$ whose $j$–invariant is constant by construction. Since $K_X$ is nef, $K_G = K_X|G$ is nef, hence $\tau_c$ is relatively minimal and therefore the only singular fibers if $\tau$ are multiples of smooth elliptic curves.

We now construct the surface $S$ birationally. Vaguely, we take the collection of the general fiber $E_c$ of each $\tau_c$ for general $c$ and then take closure. To be more precise, we consider the meromorphic map $h : \mathbb{P}_1 \rightarrow C$, associating to the general point $c$ the elliptic curve $E_c$ in the cycle space $C$. Let $A$ be the closure of the image of $h$ and $p : T \rightarrow A$ the associated family so that the general fiber of $p$ is of type $E_c$. After normalising $A$, the map $h : \mathbb{P}_1 \rightarrow A$ is holomorphic. Now let $S' = T \times_A \mathbb{P}_1$ and let $S$ be a desingularisation of $S'$. We clearly have a meromorphic dominant map $X \rightarrow S \times_{\mathbb{P}_1} B$.

5.4 Corollary In the situation of (5.3) let $\tilde{J} : \tilde{B} \rightarrow C$ be the Stein factorisation of $J : B \rightarrow \mathbb{P}_1$. Then $S \rightarrow \mathbb{P}_1$ induces a base change plus desingularisation $\tilde{S} \rightarrow C$. Then there is a birational rational map $\rho : X \rightarrow \tilde{S} \times_C \tilde{B}$. We can choose $\tilde{S}$ such that the fiber product $\tilde{S} \times_C \tilde{B}$ is smooth.

Proof. Everything is clear except for the smoothness statement. Here we go back to the proof of (5.3) and remark that applying a general automorphism of $\mathbb{P}_1$, we may assume that the singular fibers of $S \rightarrow \mathbb{P}_1$ and $J$ are over different points of $\mathbb{P}_1$, hence $S \times_{\mathbb{P}_1} B$ is smooth. Therefore also $\tilde{S} \times_C \tilde{B}$ is smooth.

(5.5) By (5.4) every elliptic fibration with $\kappa = 2$ and non-constant $j$-invariant is birational to a smooth fiber product. Therefore we are reduced to the case of a fiber product $X = S_1 \times_C S_2$ with maps $g_i : S_i \rightarrow C$. Let $f_i : X \rightarrow S_i$ be the projections. Then every 2-form $\omega$ on $X$ is of the form

$$\omega = f_1^*(\omega_1) + f_2^*(\omega_2) + f_1^*(\eta_1) \wedge f_2^*(\eta_2)$$

with 2-forms $\omega_i$ on $S_i$ and 1-forms $\eta_i$ on $S_i$. In fact, we compute $R^2f_1_* (\mathcal{O}_X)$ via the Leray spectral sequence:

$$H^2(\mathcal{O}_X) = H^2(\mathcal{O}_{S_2}) \oplus H^1(S_2, R^1f_1_* (\mathcal{O}_X)).$$

(1)

Note here that $R^2f_1_* (\mathcal{O}_X) = 0$ since all fibers of $f_1$ are 1-dimensional. It remains to compute the second term in (1). We apply the Leray spectral sequence for $g_2$ and use

$$R^1f_1_* (\mathcal{O}_X) \simeq g_1^* R^1g_2_* (\mathcal{O}_{S_2})$$

(compare (1.5)). Therefore

$$H^1(R^1f_1_* (\mathcal{O}_X)) = H^1(R^1g_2_* (\mathcal{O}_{S_2})) \oplus H^0(R^1g_1_* (\mathcal{O}_{S_1}) \otimes R^1g_2_* (\mathcal{O}_{S_2})).$$
Since the first term is clearly $H^2(O_{S_2})$ and the second term corresponds to wedge products of 1-forms, our claim follows.

We discuss now the situation when $\omega$ has only finitely many zeroes, i.e. $Z = \{\omega = 0\}$ is finite. If we start with a projective smooth threefold $X$ with $\kappa(X) = 2$, then by (3.3) $K_X$ is nef, so we again assume now $X$ to be a minimal model, possibly singular.

5.6 Proposition If $\omega$ is vertical, then $f$ has only finitely many singular fibers which are not multiple and these singular fibers have only finitely many singularities, all contained in $Z$.

**Proof.** Our claim is local in $B$. Since $B$ has only quotient singularities, it is thus clear that we may assume $B$ smooth. Let

$$T = \{x \in X | \text{rk} \Omega^1_{X|B,x} \geq 2\}.$$

Let $S$ be the union of the multiple fibers. Then we must show that $T \setminus S \subset N$. Take $b \in B$ such that there is some $x \in T \setminus S$ with $f(x) = b$. Since $\omega$ is vertical, we have $\omega = f^*(\eta)$ with $\eta \in H^0(\omega_B(\sum k_iS_i))$ where the $S_i$ are the irreducible components of the closure of $S$. First suppose that $x$ is not contained in the closure of $S$. Then locally we can write

$$\eta = \sum \alpha_i \wedge \beta_i$$

with holomorphic 1-forms $\alpha_i$ and $\beta_i$. Since the kernel of

$$\Omega^1_X|\{x\} \rightarrow \Omega^1_{X|B}|\{x\}$$

is 1-dimensional, we conclude

$$F^*(\alpha_i)(x) \wedge f^*(\beta_i)(x) = 0,$$

what had to be proved. If $x \in \overline{S}$, then we perform a (local) base change in $B$ to kill the multiplicities and apply the previous argument on $\tilde{X} \rightarrow \tilde{B}$.

6. Torus fibrations

(6.1) Let $r : X \rightarrow S$ be a surjective holomorphic map with connected fibers between compact complex connected manifolds. We assume throughout this section that $X$ is Kähler with generic fiber $F = F_s = X_s := r^{-1}(s)$ a complex torus.

For the relevant notions on relative cycle (or Douady) spaces we refer to [Ca85] and [Fu83] where the constructions used here have been introduced with more details.
(6.2) Let \( r_* : \text{Aut}_S(X) \to S \) be the relative automorphism space of \( S \) over \( S \); it consists of the closure in \( \mathcal{C} \left( \frac{X \times X}{S} \right) \) of the graphs of automorphisms \( \alpha_S : F_s \to F_s \) of some smooth fiber of \( r \), where these graphs are considered in \( \mathcal{C}(F_s \times F_s) \subset \mathcal{C} \left( \frac{X \times X}{S} \right) \) of automorphisms \( \alpha_S : F_S \to F_S \) of some smooth fiber of \( r \).

By the Khler assumption on \( X \), \( \text{Aut}_S(X) \) has compact irreducible components ([Li78]). There is one distinguished component \( r'_0 : X'_0 \to S \) of \( \text{Aut}_S(X) \), which lies surjectively on \( S \) and has a canonical meromorphic zero-section \( \zeta'_0 : S \to X'_0 \), namely the component containing \( \text{id}_{F_S} \) for \( s \) generic in \( S \). We denote by \( r_0 : X_0 \to S \) (and \( \zeta_0 : S \to X_0 \)) any smooth model of \( X'_0 \), isomorphic to \( X'_0 \) over some non-empty Zariski open subset \( S^* \) of \( S \).

6.3 Definition We call \( r_0 : X_0 \to S \) (equipped with \( \zeta_0 \)) the untwisted model of \( X \).

6.4 Proposition Let \( S' \) be a Zariski dense open subset of \( S \) over which \( r \) (and hence \( r_0 \)) is smooth. Let \( r' : X' \to S' \) be the restriction of \( r \) over \( S' \) to \( X' := r^{-1}(S') \); use similar notations for \( r_0 \).

For \( p \geq 0 \), there are canonical isomorphisms

\[
\psi^p_* : r'_* (\Omega^p_{X'/S'}) \to (r'_0)_* (\Omega^p_{X'_0/S'}),
\]

and by Hodge symmetry:

\[
\phi^p_* : R^p r'_* (\mathcal{O}_{X'/S'}) \to (R^p r'_0)_* (\mathcal{O}_{X'_0/S'}).
\]

Proof. The assertion is of local nature in \( S' \). It is sufficient to check it over any sufficiently small open neighborhood \( S'' \) of any \( s \in S' \). But then one has isomorphisms \( \psi : r^{-1}(S'') \to r^{-1}_0(S''), \) unique up to translations in the fibers, which define the asserted isomorphisms.

6.5 Theorem Assume that \( r \) and \( r_0 \) are locally projective. Then \( h^p(\mathcal{O}_X) = h^p(\mathcal{O}_{X_0}) \) for all \( p \geq 0 \) and in particular \( \chi(\mathcal{O}_X) = \chi(\mathcal{O}_{X_0}) \).

Proof. We can assume that \( S - S' \) is a divisor with only normal crossings. Then the sheaves \( R^p r_* \mathcal{O}_{X/S} \) are locally free and coincide with the canonical extensions of their restrictions to \( S' \) ([Ko86], [Mw87], [Na86]; observe that the assertion is of local nature over \( S \)).

The same properties hold for \( r_0 \), so that \( \phi^p_* \) extend to isomorphisms \( \phi^p_0 \). These isomorphisms define naturally an isomorphism of the Leray spectral sequences for \( \mathcal{O}_X \) and \( \mathcal{O}_{X_0} \) with respect to the maps \( r \) and \( r_0 \). Hence

\[
H^*(\mathcal{O}_X) \simeq H^*(\mathcal{O}_{X_0}),
\]

establishing our claim.

Remark. The assertions of (6.5) hold true certainly in the Khler case as well; unfortunately the necessary tools seem to have been only written up in the projective case.
6.6 Corollary Assume that the generic fiber of $r$ is an abelian variety. Then the conclusions of 6.5 hold.

Proof. Since $X$ and $X_0$ are Kähler and $F$ is abelian, $r$ and $r_0$ are locally Moishezon over $S$. Hence suitable bimeromorphic models are locally projective over $S$ and 6.5 applies.

6.7 Proposition Let $r : X \to S$ be as in 6.1. Assume moreover that the smooth fibers of $r$ are all isomorphic to a fixed torus $T$. Then:

(i) There exists a finite Galois cover $\beta : \tilde{S} \to S = \tilde{S}/G$ with Galois group $G$ such that the induced map $\tilde{r} : \tilde{X} := X \times \tilde{S} \to \tilde{S}$ is bimeromorphically a principal fiber bundle with fiber $T$.

(ii) Let $\beta$ be as in (a) and $\tilde{r}_0 : \tilde{X}_0 \to \tilde{S}$ be the untwisted model of $\tilde{r}$ (also obtained from $r_0$ by the base change $\beta$). Then $\tilde{r}_0$ is bimeromorphic to the second projection $\rho : T_0 \times \tilde{S} \to \tilde{S}$ where $T_0 = \text{Aut}^0(T)$.

(iii) $h^0(\tilde{X}, \Omega^p) = \sum_{q+t=p} h^0(\tilde{S}, \Omega^q) \times h^0(T, \Omega^t) = h^0(\tilde{X}_0, \Omega^p)$ ($p \geq 0$).

(iv) $h^0(X, \Omega^p) = h^0(X_0, \Omega^p)$ ($p \geq 0$).

Proof. (i) Let

$$ r_* : J := \text{Isom}_S(T \times S, X) \to S $$

be the space of relative isomorphisms over $S$ of $T \times S$ and $X$. Over a smooth fiber $F$ of $r$ it consists of all isomorphisms of $T$ and $F$, each identified with its graph and considered as a point in

$$ \mathcal{C}(F \times T) \subset \mathcal{C}(T \times S) \times X = T \times X. $$

Again see [Ca85] or [Fu83] for further details. Let $r_* : J \to S$ be some irreducible component of $J$ lying surjectively over $S$ (such a component exists by the Kähler assumption on $X$ and the fact that $F = F$ is isomorphic to $T$ for $s$ generic in $S$).

Let $\rho : J \to \overline{S}$ be the Stein factorisation of $r_* : J \to S$. We have a natural meromorphic composition map $\tau : T_0 \times J \to J$ over $\overline{S}$, which generically over $\overline{S}$ maps $(j_s : T \to F_s) \in \rho^{-1}(\overline{S})$ to $\tau(t, j_s) := j_s \circ t$. Here $t \in T_0 = \text{Aut}^0(T)$.

The conclusion now follows from the facts that (as just shown) $\rho : J \to \overline{S}$ is bimeromorphically a principal fiber bundle with fiber $T_0$, and that the natural evaluation map over $\overline{S}$ at $t' \in J \to \overline{S} := (X \times S)$ is an isomorphism (bimeromorphically), for any fixed $t' \in T$, where $\varepsilon$ sends $j_s$ to $j_s(t')$.

We conclude by taking a Galois base change dominating $\overline{S} \to S$.

(ii) This immediately follows from the fact that $\tilde{r}_0 : \tilde{X}_0 := X \times \tilde{S} \to \tilde{S}$ is the untwisted model of $\tilde{r} : X \times \tilde{S} \to \tilde{S}$, and is thus bimeromorphically a principal bundle of fiber $T_0$ with a (meromorphic) section.

(iii) This follows from the results of A. Blanchard ([Bl58]) about the degeneration of the Leray spectral sequence.
(iv) Observe that we do not assume here $r$ to be locally projective, otherwise (iv) follows from (6.5). The Galois group $G$ of the cover $\beta : \tilde{S} \to S'$ acts in a natural way on $H^0(T, \Omega^p)$ ($p \geq 0$) as follows: let $\pi \in \tilde{S}$ and $g \in G$. Choose $j_\pi : T \to F_S$ and $j_{\gamma \pi} : T \to F_S$ arbitrarily and define $\tilde{g} := (j^{-1}_{\gamma \pi} \circ j_\pi) \in \text{Aut}(T)$. $\tilde{g}$ is well defined up to a translation in $T$. Its action $g^*$ on $H^0(F, \Omega^p)$ is thus well-defined, independent on the choices $(\tilde{S}, j_{\tilde{S}}, j_\gamma)$ made, as one checks easily. Observe that this action is the same for $\tilde{X}$ and for $\tilde{X}_0$. We then just deduce the claim (iv) from the fact that $H^0(X, \Omega^p) = H^0(\tilde{X}, \Omega^p)^G$, where the action is defined via the decomposition $H^0(\tilde{X}, \Omega^p) = \bigoplus_{q+t=r} H^0(\tilde{S}, \Omega^q) \otimes H^0(T, \Omega^t)$, and similarly for $X_0$.

Remark. The equality $q(\tilde{X}) = q(\tilde{S}) + q(T)$ was obtained in [Fu83] using Deligne’s mixed Hodge structures in the Kähler situation. We gave here a more elementary proof, avoiding the Kähler version of the theory of Deligne (which we were not able to find explicitly written in the literature).

6.8 Corollary Let $r : X \to S$ be a surjective connected holomorphic map from a Kähler threefold $X$ to a curve $S$. Assume that its generic fiber $F$ is either a torus or a K3-surface and that $a(F) \leq 1$. Then any two smooth fibers of $r$ are isomorphic.

Proof. We follow here an argument of [Fu83]. Assume first that $F$ is a K3-surface. Since $X$ is not projective, we have $h^{0,2}(X) > 0$. If $0 \neq \omega \in H^0(\Omega^2_X)$, then $\omega|_F \neq 0$ since $q(F) = 0$, and $d\omega = 0$, since $X$ is Kähler. Hence the periods of $\omega$ on $F = F_s$ are constant in a local marking, and the conclusion follows from the Torelli theorem for K3-surfaces.

If $F$ is a torus, we replace $X$ by $X_0$ and consider the $\mathbb{Z}_2$-quotient $p' : Y_0' := (X_0/\pm 1) \to S$ of $r_0 : X_0 \to S$ by the Kummer involution in the fibers of $r_0$, relative to the canonical section (an easy cycle space argument shows that this involution extends meromorphically over all of $S$). Then replace $p'$ by a smooth model $p : Y_0 \to S$ with generic fiber $Y_{0,s}$, the Kummer surface of $X_{0,s}$. Because $a(F) \leq 1$, we still have $h^{0,2}(Y_0) > 0$; let $0 \neq \omega \in H^{0,2}(Y_0)$ and let $\varpi$ be its lift to $X_0$. Then argue as above, using the Torelli theorem for 2-forms on 2-dimensional complex tori.

We now consider the Kodaira dimension of $X_0$:

6.9 Proposition Let $r : X \to S$ be as in 6.1 with $\dim X = \dim S + 1$, i.e. $r$ is an elliptic fibration. Then $\kappa(X_0) \leq \kappa(X)$.

Proof. By [Ue87],[Na88] we have a canonical bundle formula:

$$K_X = r^*(K_S \otimes r_*\omega_{X/S}) \otimes O_X(\Sigma(m_i - 1)D_i) \otimes O_X(\Delta),$$

where:

- $\Delta$ is an effective divisor supported on the 2-dimensional fibers of $r$. 

• \(r_*(\omega_{X/S})\) is locally free (assuming \(S - S'\) to have only normal crossings).
• The divisors \(D_i\) are those having generically over their image \(\Delta_i\) in \(S\) multiple fibers of multiplicity \(m_i \geq 2\).

Thus: \(\kappa(X) = \kappa(S, D)\) where \(D\) is the \(\mathbb{Q}\)-divisor defined by: \(D = K_S \otimes r_*(\omega_{X/S}) \otimes \mathcal{O}_S \left( \sum (\frac{m_i - 1}{m_i})\Delta_i \right)\).

We perform the same arguments on \(X_0\). Since \(r_0\) has a meromorphic section, it has no multiple fibers, and thus we get

\[\kappa(X_0) = \kappa(S, K_S \otimes (r_0)_*(\omega_{X_0|S})).\]

Since \(r_*(\omega_{X|S}) \simeq (r_0)_*(\omega_{X_0|S})\) by (6.4), our claim follows.

**Remarks.**
(i) One may have strict inequality: if \(X\) is an Enriques surface, then \(X_0\) is a rational surface (a classical fact). Indeed: we have \(q(X) = q(X_0) = 0\) and \(\kappa(X_0) = -\infty\) since \(\deg(r_0^*\omega_{X/S}) = \chi(\mathcal{O}_S) = 1\).
(ii) The inequality 6.8 is probably true in general, i.e. for higher-dimensional fibers, but we are able to show this only in special cases.

**6.10 Lemma**

Let \(X\) be a normal compact threefold and \(h : X \to \mathbb{P}_1\) and \(g : X \to S\) be connected holomorphic surjective maps. Assume that \(S\) is a normal surface with \(H^0(K_S) \neq 0\), and that every fiber \(F\) of \(h\) is a finite quotient of either a torus or a K3-surface. Assume moreover that \(h\) has a section and that \(g(F) = S\).

Then \(h \times g : X \to \mathbb{P}_1 \times S\) is bimeromorphic.

**Proof.** It will be sufficient to show that if \(G\) is a generic fiber of \(g\), then \(h|_G : G \to \mathbb{P}_1\) has degree \(d\) equal to one.

Let \(0 \neq \omega \in H^0(K_S)\) and \(\eta = g^*(\omega)\) is at least a 2-form on the regular part of \(X\) but it defines also restrictions \(\eta|F \in H^0(K_F)\). Moreover \(\eta\) defines a section

\[s \in H^0(K_X|\mathbb{P}_1).\]

For the general reduced fiber \(F\) of \(h\), \(\eta|_F\) is everywhere non-zero (as 2-form on the smooth part of \(F\) or as section of \(K_F\)). On the other hand, we have

\[(K_{X/\mathbb{P}_1}) \cdot G = 2\gamma - 2 + 2d\]

where \(\gamma\) is the genus of \(G\), because \(K_{X/\mathbb{P}_1} \simeq K_X \otimes h^*\mathcal{O}_{\mathbb{P}_1}(2)\). If \(d \geq 2\), then \(s|_G\) has zeroes, so that \(s|_F\) has zeroes, hence vanishes identically for general \(F\). This is a contradiction.
7. Algebraic reductions of Kähler threefolds

We shall collect in this section some facts about algebraic reductions of compact Kähler manifolds, in particular in dimension 3. In the threefold case, most are due to [Fu83]; we will provide here short and elementary proofs.

7.1 Theorem ([Ue75]). Let \( r : X \to S \) be the algebraic reduction of a compact complex manifold, and \( F = X_s \) a general fiber of \( r \). Then \( \kappa(F) \leq 0 \).

7.2 Theorem ([Ca85], [Fu83] when \( Y \) is projective) Let \( f : X \to Y \) be a surjective connected map between compact complex manifolds with \( X \) Kähler. Assume that the generic fiber \( F \) of \( f \) is projective. Then:

(i) If \( q(F) = 0 \), then \( a(X) = a(Y) + \dim F \).

(ii) If \( a(X) = a(Y) \), then \( F \) is almost-homogeneous.

(iii) If \( X \) contains a compact analytic set \( Z \) with \( \dim Z = \dim Y \) and \( f(Z) = Y \), then \( a(X) = a(Y) + \dim F \).

7.3 Corollary If \( r : X \to S \) is an algebraic reduction of \( X \). If its generic fiber \( F \) is projective, then we have:

(i) \( q(F) > 0 \); \( F \) is almost-homogeneous and there does not exist any \( Z \subset X \) dominating \( S \).

(ii) In particular, if \( \dim F = 2 \), then \( F \) is either an abelian surface or a ruled surface over an elliptic curve \( E \), isomorphic to \( \mathbb{P}(\mathcal{O}_E \otimes \mathcal{L}) \), where \( \mathcal{L} \in \text{Pic}^0(E) \) is not torsion.

Proof. (7.3) follows directly from (7.2) and the classification of surfaces.

7.4 Corollary. Let \( r : X \to S \) be the algebraic reduction of a compact Kähler threefold with \( a(X) = 1 \). Then the generic fiber \( F \) of \( r \) is either

1. bimeromorphic to a torus or a K3-surface of algebraic dimension zero,
2. a ruled surface \( \mathbb{P}(\mathcal{O}_E \otimes \mathcal{L}) \) as in 7.3.(ii).

Proof. This follows from the classification of surfaces with \( \kappa \leq 0 \) again: Enriques and rational surfaces are excluded because projective with \( q = 0 \); bielliptic (hyperelliptic) surfaces do not occur because they are projective but not almost-homogeneous. The almost-homogeneous birationally ruled surfaces with \( q > 0 \) are exactly those of 7.3.(ii).

Finally if \( a(F) \leq 1 \), it has to be bimeromorphic to either a torus or a K3-surface. In this last case (K3), \( a(F) = 1 \) is excluded, because taking a relative algebraic reduction ([Ca81]) we obtain a meromorphic factorisation \( X \to S \to C \) with \( S \to C \) being a \( \mathbb{P}_1 \)-fibration, so that \( a(S) = 2 \) contradicting \( a(X) = 1 \).

The special case 7.4.(i) has been analyzed more closely in [Fu83]:

7.5 Lemma ([Fu83]) Let \( r : X \to S \) be the algebraic reduction of a compact Kähler threefold with \( a(X) = 1 \). and assume that \( F \) is bimeromorphic to either a torus or a K3-surface. Then \( F \) is either a torus or a K3-surface (after taking a suitable bimeromorphic model of \( X \)).
**Proof.** In the torus case, we just replace \( r : X \to S \) by its Albanese reduction ([Ca85]). In the K3-surface case, let \( 0 \neq \omega \in H^{0,2}(X) \). By the arguments used to prove 6.8, we see that any smooth \( F_s \) is bimeromorphic to a fixed K3-surface \( F_0 \).

Consider now any irreducible component \( \mu : M \to S \) of the space of \( S \)-morphisms \( \text{Mor}_S(F_0 \times S, X) \subset \mathcal{C}(F_0 \times X) \) which lies over \( S \) and consists generically over \( S \) of graphs of bimeromorphic maps from \( F_0 \) to \( F_s = X_s \). \( \mu : M \to S \) be the canonical map.

Then \( \mu : M \to S \) is onto and generically finite, and there is a natural evaluation map \( \varepsilon : F_0 \times M \to \hat{X} := X \times M \), which is bimeromorphic. By a further finite base change, we can assume that \( \mu : M \to S \) is Galois cover with Galois group \( G \).

We can now define, using the construction of the proof of 6.7.(iv), an action of \( M \) on \( F_0 \times M/G \) such that \( F_0 \times M/G \) is bimeromorphic to \( X \) over \( S = M/G \).

### 7.6 Corollary ([Fu83])

Let \( X \) be a non-projective compact Kähler threefold. Then \( X \) is bimeromorphic to one of the following, according to its algebraic dimension \( a(X) \):

1. \( a(X) = 0 \) : (i.a) simple non-Kummer.
   (i.b) simple Kummer.
   This fibration is unique.

Let \( F \) be a generic fiber of \( r : X \to S \), the algebraic reduction of \( X \).

1. \( a(X) = 1 \) : (ii.a) \( a(F) = 0 \) and \( F \) is a fixed torus.
   (ii.b) \( a(F) = 0 \) and \( F \) is a fixed K3-surface.
   Then \( X = F \times \tilde{S}/G \) for some Galois cover \( \tilde{S} \) of \( S = \tilde{S}/G \).
   (ii.c) \( a(F) = 1 \) and \( F \) is a fixed torus.
   (ii.d) \( a(F) = 2 \) and \( F \) is a (possibly varying) abelian surface.
   (ii.e) \( a(F) = 2 \) and \( F = \mathbb{P}(O_E \oplus L) \), as in 7.3.(ii).

1. \( a(X) = 2 \) : \( F \) is an elliptic curve (possibly varying).

**Proof.** Everything has been proved (using 6.8 for \( a(X) = 1, a(F) = 0, 1 \), and additionally the proof of 7.4 if \( F = K3 \) with \( a(F) = 0 \)), except that if \( a(X) = 0 \) and \( X \) is not simple, then there exists \( f : X \to \Sigma \) as in (i.c). Because \( X \) is not simple, it is covered by a family of curves. Only one curve goes through the generic point of \( X \), otherwise \( X \) were covered by divisors, contrary to \( a(X) = 0 \). The curves on \( X \) thus define a (unique) meromorphic map \( f : X \to \Sigma \) to a surface \( \Sigma \) with \( a(\Sigma) = 0 \), since \( 0 = a(X) \geq a(\Sigma) \). Let \( g \) be the genus of a generic fiber of \( f \); because \( a(\Sigma) = 0 \), any two smooth fibers of \( f \) are isomorphic to a fixed curve \( C \). If \( g \geq 2 \), then there exists a finite cover \( \tilde{\Sigma} \to \Sigma \) such that \( \tilde{X} := (X \times \tilde{\Sigma})_\Sigma \) is bimeromorphic to a torus since \( a(\tilde{X}) = a(\tilde{X}) = 0 \).

If \( g = 1 \), first notice that \( f \) cannot have moduli, since otherwise \( \kappa(X) > 0 \) by \( C_{2,1} \) contradicting \( a(X) = 0 \). Hence \( q(\tilde{X}) = q(\tilde{\Sigma}) + 1 \) by 6.7, and thus \( q(\tilde{\Sigma}) = 0 \) or \( 2 \) since \( a(\tilde{\Sigma}) = a(\Sigma) = 0 \), i.e. bimeromorphically a torus or K3. If \( q(\tilde{\Sigma}) = 0 \), we get \( q(\tilde{X}) = 1 \) contradicting \( a(\tilde{X}) = 0 \); if \( q(\tilde{\Sigma}) = 2 \), then \( \tilde{X} \) is bimeromorphic to a torus since \( a(\tilde{X}) = 0 \), and \( X \) is Kummer contradicting our assumption. Thus \( g = 0 \) and \( C = \mathbb{P}_1 \), as claimed.
The fact that we can choose \( f \) holomorphic is easily obtained by contracting some curves in \( \Sigma \).

8. Kähler threefolds with \( \kappa = 0, 1 \).

(8.0) **Convention** \( X \) will always denote a compact Kähler connected manifold with \( \dim X = 3 \) and we assume that \( X \) is not both simple and not Kummer. Let \( r : X \to S \) be an algebraic reduction of \( X \) and \( r_0 : X_0 \to S \) be its untwisted model. Both maps may assumed to be holomorphic.

8.1 **Theorem** Let \( X \) be a compact Kähler threefold which is not both simple and non-Kummer. Then :

(i) If \( \kappa(X) = -\infty \), \( X \) is uniruled.
(ii) If \( \kappa(X) = 0 \) and \( h^{0,2}(X) \neq 0 \), then \( X \) is bimeromorphic to some threefold \( X' \) (possibly with quotient singularities) which has a finite cover \( \tilde{X}' \) tale in codimension one such that \( \tilde{X}' \) either a torus or a product of an elliptic curve by a K3-surface.

**Proof.** Because the results are known when \( X \) is projective, due to [Mi88], [Mo87], [Ka85] and [Vi80], we consider only the non-algebraic case and proceed according to \( a(X) = 0, 2, 1 \) successively in the increasing order of difficulty.

(8.2) \( a(X) = 0 \). Then by (7.6) \( X \) is either Kummer with \( \kappa(X) = 0 \) so that 8.1.(ii) holds, or \( X \) is a generic \( \mathbb{P}_1 \)-bundle over a surface \( \Sigma \), and 8.1.(i) holds.

(8.3) \( a(X) = 2 \). Notice then that (by 7.2.(iii)) \( X_0 \) is projective since \( r_0 \) has a section. Moreover, \( \kappa(X_0) \leq \kappa(X) \) by (6.9) and \( h^{0,2}(X_0) = h^{0,2}(X) > 0 \) (by 6.5 and the assumption that \( X \) is not projective).

We thus distinguish the following possible cases :

- \( \kappa(X) = \kappa(X_0) = -\infty \) (8.4),
- \( \kappa(X) = 0 > \kappa(X_0) = -\infty \) (8.5)
- and \( \kappa(X) = \kappa(X_0) = 0 \) (8.6).

(8.4) \( a(X) = 2 ; \kappa(X) = \kappa(X_0) = -\infty \). Then \( X_0 \) is uniruled, \( X \) being projective.

Let \( \sigma_0 : X_0 \to \Sigma_0 \) be its rational quotient ([Ca92], [KoMiMo92]). Because \( h^{0,2}(X_0) = h^{0,2}(X) > 0 \), \( X \) being non-algebraic, \( \Sigma_0 \) is a surface with \( h^{0,2}(\Sigma_0) > 0 \), and the generic fiber of \( \sigma_0 \) is \( \mathbb{P}_1 \). We may assume \( \Sigma_0 \) smooth and \( \sigma_0 \) holomorphic.

Let \( F_0 \) be the generic fiber of \( r_0 \), and \( E := \sigma_0(F_0) \). Then \( E \) is a member of a 1-dimensional family of elliptic curves on \( \Sigma_0 \). Notice that \( E \) is elliptic and not \( \mathbb{P}_1 \) because \( \kappa(\Sigma_0) \geq 0 \). Since \( \kappa(\Sigma_0) \geq 0 \), there exists a connected meromorphic surjective map \( \tau_0 : \Sigma_0 \to C \) to some curve \( C \) contracting all the elliptic curves \( E \). In fact, one can introduce an equivalence relation such that two general points of \( \Sigma_0 \) are equivalent if and only if they can bejoined by a chain of these elliptic curves, see [Ca81]. Then not all points are equivalent,
otherwise $\kappa(\Sigma_0) = -\infty$. Hence the meromorphic map $\tau_0$ is just given by the quotient of the equivalence relation [Ca81].

We conclude the existence of a unique map $\rho : S \to C$ such that $\tau_0 \circ \sigma_0 = \rho \circ r_0$.

We thus have constructed a commutative diagram of surjective connected maps (which may assume to be holomorphic):

\[
\begin{array}{ccc}
S & \xrightarrow{r_0} & \Sigma_0 \\
\downarrow & & \downarrow \rho \\
X_0 & \xrightarrow{f_0} & C \\
\downarrow \sigma_0 & & \uparrow \tau_0 \\
\Sigma & & \\
\end{array}
\]

(8.4.i)

Let $G_0$ be a generic fiber of the composed map $f_0 := \rho \circ r_0$. We have $q(G_0) = 1$ (because $\tau_0$ is an elliptic fibration and $\sigma_0$ a $\mathbb{P}_1$-fibration); also observe that $r_0 : G_0 \to P := r_0(G_0) \subset S$ is the untwisted model of $r : G := r^{-1}(P) \to P$, where $P$ is the generic fiber of $\rho$.

Hence $q(G) = q(G_0) = 1$ by (6.5). Let now $X \xrightarrow{\sigma} \Sigma \xrightarrow{\tau} C$ be the Albanese reduction of $f := \rho \circ r$. We thus get a second commutative diagram:

\[
\begin{array}{ccc}
S & \xrightarrow{r} & \Sigma \\
\downarrow & & \downarrow \rho \\
X & \xrightarrow{f} & C \\
\downarrow \sigma & & \uparrow \tau \\
\Sigma & & \\
\end{array}
\]

(8.4.ii)

Here $\Sigma$ is not algebraic, otherwise $S \times \Sigma$ is algebraic and then $X$ would be projective. Thus $h^{0,2}(\Sigma) > 0$, and $\kappa(\Sigma) \geq 0$. We then apply $C_{3,2}$ ([Ue87]) to conclude that the generic fiber of $\Sigma$ has $\kappa = -\infty$, and is thus $\mathbb{P}_1$. So $X$ is uniruled, as claimed.

(8.5) $a(X) = 2 ; \kappa(X) = 0 ; \kappa(X_0) = -\infty$. We shall exclude this case.

Considering $r_0 : X_0 \to S$ and the same arguments as in 8.4, we get the diagrams (8.4.i) and (8.4.ii), with the same properties. From $C_{3,2}$ we get that $\kappa(\Sigma) = 0$, and that $\sigma : X \to \Sigma$ and $\tau : \Sigma \to C$ are elliptic fibrations. Moreover, the generic fiber of $\sigma_0$ is $\mathbb{P}_1$ and $\tau_0$ is an
elliptic fibration since \( h^{0,2}(X_0) = h^{0,2}(X) > 0 \), so that the rational quotient \( \Sigma_0 \) of \( X_0 \) is a surface with \( h^{0,2} > 0 \).

As above in (8.4), the generic fiber \( G_0 \) of \( f_0 \) has the following properties: \( q(G_0) = 1 \) and \( \kappa(G_0) = -\infty \). The generic fiber of \( \rho \) is therefore \( \mathbb{P}_1 \) since it is the image of the generic fiber of \( \sigma_0 \) by \( r_0 \). Notice that \( \kappa(G) \geq 0 \) since \( X \) is not uniruled and that \( G \) is an elliptic fibration over \( \mathbb{P}_1 \). So \( \kappa(G) \leq 1 \). We show that \( \kappa(G) = 0 \). Indeed: \( G_0 \) is bimeromorphic to a ruled surface over an elliptic curve \( E_0 \). This implies that \( r_0 : G_0 \to \mathbb{P}_1 = r_0(G_0) \subseteq S \) is a generically (over \( \mathbb{P}_1 \)) locally trivial bundle with fiber \( E'_0 \) isogeneous to \( E_0 \); in particular, it is a principal fiber bundle over \( \mathbb{P}_1 \) with fiber \( E'_0 \). But \( r_0 \) is the untwisted model of \( r : G \to \mathbb{P}_1 \); the corresponding fiber of \( f \) which is therefore also a principal fiber bundle with fiber \( E'_0 \). Hence \( \kappa(G) = -\infty \), because \( \chi(O_{G_0}) = \chi(O_G) = 0 \). This is a contradiction, since \( \kappa(X) = 0 \).

**8.6** \( a(X) = 2 ; \kappa(X) = \kappa(X_0) = 0 \). Consider again the untwisted model \( r_0 : X_0 \to S \). We shall establish the claim successively in the cases \( \kappa(S) \geq 0 \); \( \kappa(S) = -\infty \), \( q(S) > 0 \); and finally exclude the case \( \kappa(S) = -\infty \), \( q(S) = 0 \).

(1) Assume first that \( \kappa(S) \geq 0 \), so that \( \kappa(S) = 0 \) and \( S \) is (up to cover and bimeromorphy) either a K3-surface or an abelian surface. Moreover \( X_0 \) is projective since \( r_0 \) has a section. Let \( r'_0 : X'_0 \to S \) be a minimal model of \( X_0 \). Since \( X'_0 \) has a 2-form \( \eta \), there exists by [Pe94] a 1-form \( \omega \) such that \( \eta \wedge \omega \neq 0 \) generically. In particular \( q(X'_0) > 0 \). Now by [Ka85] there exist a cover, stable in codimension one, say

\[
\tilde{r}'_0 : \tilde{X}'_0 = X'_0 \times _S \tilde{S} \to \tilde{S}
\]

which is a product \( \tilde{X}'_0 = E \times \tilde{S} \) for some elliptic curve \( E \), and \( \tilde{S} \) either K3 or abelian. But \( \tilde{X}'_0 \) is the untwisted model of \( \tilde{r} : \tilde{X} := X \times _S \tilde{S} \to \tilde{S} \) (see the argument, exposed in the case \( \kappa(S) = -\infty \) and \( q(S) = 0 \) below). This shows the claim in that case.

(2) Let us now assume that \( \kappa(S) = -\infty \) and \( q(S) > 0 \).

Let \( \alpha : S \to E \) be the Albanese map for \( S \). Because \( \kappa(X) = 0 \), we get from \( C_{3,1} \) that \( E \) is an elliptic curve. Note that \( S \) is (bimeromorphically) ruled over \( E \). Again from \( C_{3,1} \) we deduce that \( F \), the generic fiber of \( f = \alpha \circ r : X \to E \), has \( \kappa(F) = 0 \). \( F \) has an elliptic fibration over \( \mathbb{P}_1 \). Hence \( q(F) \neq 2 \). If \( q(F) = 1 \), \( F \) is bimeromorphically a hyperelliptic surface. Then \( q(F_0) = 1 \) by (6.4) so that \( F_0 \) is hyperelliptic or a ruled surface over an elliptic curve. Since \( F_0 \) has a section, the first alternative cannot occur. But then \( X_0 \) is uniruled, contradicting \( \kappa(X_0) = 0 \). So the case \( q(F) = 1 \) does not occur.

If \( q(F) = 0 \), then \( F \) is K3 or Enriques. In the latter case \( h^0(O_F) = 0 \) for \( q = 1, 2 \), so that \( R^qf_*(O_X) = 0 \) for \( q = 1, 2 \) (both sheaves a priori being locally free). Hence \( H^2(O_X) = 0 \), contradiction. If \( F \) is K3, we can again apply [Ka85] as above to get our claim.

(3) We now exclude the remaining case \( \kappa(S) = -\infty \) and \( q(S) = 0 \) (i.e. \( S \) is rational). Assume that \( S \) is rational; then, arguing as in (1), we have, after changing \( X_0 \) birationally,
a finite cover, étale in codimension 2, say \( \bar{X}_0 \to X_0 \) such that \( \bar{X}_0 \) is a product \( \bar{E} \times \bar{\Sigma} \) where \( \bar{E} \) is an elliptic curve and the surface \( \bar{\Sigma} \) is either K3 or abelian. If we do not insist that \( \bar{X}_0 \) is really a product, i.e. if we blow up the product, then we may assume that \( g \) is holomorphic. Let \( \bar{r}_0 : \bar{X}_0 \to \bar{S} \to S \) with \( \sigma \circ \bar{r}_0 = r_0 \circ g \) be the Stein factorisation of \( r_0 \chi \). Let \( \bar{p}_1 : \bar{X}_0 \to \bar{E}, \bar{p}_2 : \bar{X}_0 \to \bar{\Sigma} \) the projections.

We first claim that \( \bar{r}_0 \) is an elliptic fibration. Indeed:

\[
K_{\bar{X}_0} = g^*K_{X_0} + B, \quad (\star)
\]

where where \( B \) is the exceptional set of the generically finite map \( g \) (recally that there is no ramification in codimension 1). Let \( \bar{F} \) be a general fiber of \( \bar{r}_0 \) and \( F \) a general fiber of \( r_0 \). Since \( K_{X_0} \cdot F = 0 \) we conclude by (\( \star \)) that \( K_{\bar{X}_0} \cdot \bar{F} = 0 \), hence \( \bar{F} \) is elliptic.

Now blow up \( \bar{X}_0 \) in order to make \( \bar{p}_2 \) holomorphic, if necessary (\( \bar{p}_1 \) is anyway holomorphic, since \( \bar{E} \) is elliptic).

We next claim the existence of a map \( \pi : \bar{S} \to \bar{E} \) such that \( \pi \circ \bar{r}_0 = \bar{p}_1 : \bar{X}_0 \to \bar{E} \). This comes down to show that, if \( \bar{F}_0 \) is a generic fiber of \( \bar{r}_0 \) then \( \bar{p}_1(\bar{F}_0) \neq \bar{E} \). Otherwise, every such \( \bar{F}_0 \) is a finite tale cover of \( \bar{E} \), and thus every fiber \( F_0 \) of \( r_0 \) is isogeneous to \( \bar{E} \). This would imply, that for \( P \subset S \) a rational curve, the preimage \( r_0^{-1}(P) \) is a birationally ruled elliptic surface, and so the rationality of \( S \) yields \( \kappa(X_0) = -\infty \), contrary to our hypothesis.

Now observe that \( \sigma : \bar{S} \to S \) is finite with finite ramification locus, i.e. étale in codimension 2. This implies easily that \( \bar{S} \) is rational, too, e.g. by observing the following commutative diagram:

\[
\begin{array}{ccc}
\bar{S} & \longrightarrow & S \\
\downarrow \pi & & \downarrow \pi' \\
\bar{E} & \overset{u}{\longrightarrow} & E
\end{array}
\]

with \( u \) tale and \( E \) elliptic. Or by observing that there are a lot of rational curves on \( \bar{S} \) by lifting rational curves from \( S \). In total we obtain a contradiction to the existence of \( \pi \), which yields a 1-form on \( \bar{S} \).

(8.7) \( a(X) = 1 \). We again consider the algebraic reduction \( r : X \to C \) of \( X \), where \( C \) is now a curve. Let \( F \) be a general fiber of \( r \). When \( F \) is a torus, we let \( r_0 : X_0 \to C \) again be the untwisted model of \( r \). Finally we let \( a(r) := a(F) \).

We consider successively the cases \( a(r) = 2, 1, 0 \) in 8.8, 8.9 and 8.10 respectively.

(8.8) \( a(X) = 1 ; a(r) = 2 \). Again we have two cases: \( F \) is either a torus or \( \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}) \) as in (7.3.ii). In the last case, we have \( \kappa(X) = -\infty \) and \( X \) is uniruled, as claimed. So we have only to concentrate on the first case.
(8.8.1) Claim: $\chi(O_X) = 0; q(X) > 0$.

**Proof.** Let $D \subset X_0$ be the canonical section of $r_0$. Let $r_0' : X_0' \to C$ be a relative minimal model, so $K_{X_0'}$ is $r_0'$-nef and moreover $mK_{X_0'}$ is $r_0'$-generated [KMM87,6-1-13], i.e.

$$r_0'^* r_0^*(mK_{X_0'}) \to mK_{X_0'}$$

is surjective. Since $r_0'^*(mK_{X_0'})$ is a line bundle, we conclude

$$r_0'^* r_0^*(mK_{X_0'}) \simeq mK_{X_0'},$$

so that $mK_{X_0'} \simeq r_0'^*(L')$ for some line bundle $L'$ on $C$. Since $r_0$ has a section $D$, it follows that $r_0'$ cannot have multiple fibers, all fibers have to be reduced. Then we conclude immediately that $K_{X_0'} \simeq r_0'^*(L)$ for some line bundle $L$ on $C$. So $X_0'$ is Gorenstein and by Riemann-Roch [Fl87], we obtain

$$\chi(X_0', O_{X_0'}) = -\frac{1}{24} K_{X_0'} \cdot c_2(X_0').$$

Let $G$ be the general fiber of $r_0'$. Then $K_{X_0'} \equiv kG$ for some integer $k$. Therefore $c_2(X_0') \cdot G = c_2(G) = 0$, since $G$ is a torus. Thus $\chi(X_0') = 0$, so

$$\chi(X, O_X) = \chi(X_0, O_{X_0}) = 0.$$

Since $h^2(O_X) > 0$ and since $\kappa(X) = 0$, we finally conclude $q(X) \geq 1$, proving the claim (8.8.1).

(8.8.2) Subcase: $\kappa(X) = -\infty$. Consider the Albanese map $\alpha : X \to \text{Alb}(X)$ of $X$ and conclude from either $C_{3,2}$ or $C_{3,1}$ that its fibers have $\kappa = -\infty$, so that $X$ is uniruled.

(8.8.3) Subcase: $\kappa(X) = 0$. In this case we may have $q(X) = 1, 2, 3$. We subdivide again in three subcases successively:

(8.8.3.1) $q(X) = 1$.

Notice that $r$ is locally Moishezon. On the other hand, $X$ is Kähler and therefore $r$ is locally projective [10.1]. By [Ka88], $r$ admits locally a relative minimal model via relative contractions and flips. Since the general fiber of $r$ is a torus, all local contractions and flips glue to global ones whence $r$ admits globally a relative minimal model $r' : X' \to C$. We claim that $mK_{X'} \simeq O_{X'}$. Write $kK_{X'} \simeq O_{X'}(E)$ for some positive $k$. Let $F'$ be a general fiber of $r'$. By adjunction we have $K_{X'}|F' = 0$. Thus $E \cap F' = \emptyset$ and therefore $\dim r'(E) = 0$. Since $E$ is relatively nef, $E \equiv rF'$ for some positive rational $r$. This means that a multiple of $E$ consists only of fibers of $r'$, hence $mK_{X'} \simeq r'^*(L)$. Since $\kappa(X') = 0$, the line bundle $L$ must be trivial. So $mK_{X'} \simeq O$. Take a canonical cover of $X'[KMM87]$; this is a covering $\tilde{X} \to X'$ which is étale outside the singularities of $X'$, hence étale in codimension 2. Take a non-zero 2-form $\eta$ on $\tilde{X}$ (= image of a 2-form on a desingularisation). Then by [Pe94] there is a 1-form $\omega$ on $X'$ such that $\omega \wedge \eta$ is a non-zero, hence nowhere
vanishing 3-form on $X'$, i.e. a section in $K_{X'}$. In particular $\omega$ has no zeroes on the regular part $\hat{X}_0$. Hence $\hat{r} : \hat{X}_0 \to C$ is a submersion. Now consider a singular fiber $D$ of $\hat{r}$. Since $D$ has only finitely many singularities, it is normal. Since $\omega_D \simeq \mathcal{O}_D$; we have $H^1(\mathcal{O}_D) = 0$ by [Um81]. Since $h^1(\mathcal{O}_F) = 2$ for the general fiber $\hat{F}$ of $\hat{F}$, this contradicts the semi-continuity theorem, $\hat{r}$ being flat. Thus $\hat{r}$ is a submersion, and $\hat{X}$ is smooth. Now it is classical (e.g. [Be83]) that $\hat{X}$ is a product, possibly after another finite étale cover.

(8.8.3.2) $q(X) = 2$. 
In (8.8.3.1) we used the assumption $q(X) = 1$ only in order to conclude that $C$ is elliptic (since $\kappa(X) = 0$, we cannot have $g(C) \geq 2$). So suppose $C$ rational. Then the Albanese map $\alpha : X \to A$ must be surjective and $a(A) = 0$. On the other hand the fibers of $r$ are projective and map onto $A$. This is a contradiction. Thus $C$ is elliptic and we can apply the same arguments as in (8.8.3.1).

(8.8.3.3) $q(X) = 3$. Because $\kappa(X) = 0$, $\alpha : X \to \text{Alb}(X)$ is bimeromorphic.

(8.9) $a(X) = a(r) = 1$. This case is very similar to the next one treated in 8.10 below. We give however in case $\kappa(X) = -\infty$ an alternative short proof:
Let $g : X \to Y$, $h : Y \to C$ with $h \circ g = r$ the relative algebraic reduction of $r$. Then $Y$ is a surface with $a(Y) = a(X) = 1$ and with algebraic reduction $h$. In particular, $\kappa(Y) \geq 0$ and we conclude from $C_{3,2}$ that $X$ is uniruled.

(8.10) $a(X) = 1 ; a(r) = 0$. Then $F$, the generic fiber of $r$, is isomorphic to a fixed surface which is either a torus or a K3-surface (use 7.5). Assume first that $F$ is a torus and consider the untwisted model $r_0 : X_0 \to C$. By (6.7) we may assume that $r_0 : X_0 = F \times \tilde{C})/G \to C = \tilde{C}/G$ for some finite group $G$ acting diagonally on $F \times \tilde{C}$.

We therefore have two maps

\[
\begin{CD}
X_0 @>r_0>> C \\
@VgVV \\
Z := (F/G)
\end{CD}
\]

Notice that $h^0(\Omega^2_Z) \neq 0$ since $a(Z) = a(F) = 0$. First suppose $\kappa(X) = -\infty$. By (6.10), the canonical map $X_0 \to C \times Z$ is bimeromorphic. Hence the restriction $F \to Z$ is bimeromorphic, and thus $q(X_0) > 0$. Hence $q(X) > 0$ by (6.7), and $X$ is uniruled via the Albanese map.
Assume now that $\kappa(X) = 0$; then $q(C) \leq 1$ by $C_{3,1}$. Assume first that $C = \mathbb{P}_1$. Argiumf as before, we have $q(X) = 2$. Then the Albanese map $\alpha : X \to \text{Alb}(X)$ is a connected surjective elliptic fibration.
As in (8.8.3.2) we conclude that there is a finite cover \( X' \to X \), étale in codimension 1, such that \( X' \) is bimeromorphically a torus. But this is only possible if \( C \) is elliptic. So this situation does not occur.

Assume now that \( \kappa(X) = 0 \) and that \( C \) is elliptic. The fibers of \( g \) are elliptic curves since \( \kappa(Z) = \kappa(X_0) = 0 \). The fibers of \( g \) are thus mapped to \( C \) in an étale way by \( r_0 \). Via the finite étale base change \( \beta : C \to C \), the map \( r_0 \times g : X_0 \to Z \times C \) is bimeromorphic; here \( X_0 = X_0 \times C \). But then \( q(X_0) = 3 \) and \( X_0 \) is bimeromorphic to a torus. Considering \( X := X \times C \) (which is étale over \( X \)), we get \( q(X) = 3 \) and the conclusion.

We still have to treat the case where \( F \) is a K3-surface (with \( a(X) = 1 \), \( a(r) = 0 \)). By (7.6) we have a meromorphic map to a K3-surface. Therefore \( X \) is uniruled by \( C_3 \) if \( \kappa(X) = -\infty \). For the case \( \kappa(X) = 0 \), we proceed as in (8.8.2.2).

This concludes the proof of 8.1.

9. Classification of non-algebraic threefolds with \( \kappa = -\infty \).

9.1 Theorem Let \( X \) be a compact connected non-algebraic Kähler threefold with \( \kappa(X) = -\infty \). Then \( X \) is bimeromorphic to exactly one of the following:

(i) \( a(X) = 0 \) : \( X \) is simple but not Kummer.

(ii) \( a(X) = 0 \) : \( X \) is not simple. Then it has a unique map \( \rho : X \to S \) to a surface \( S \) with generic fiber \( \mathbb{P}_1 \). Moreover \( S \) is bimeromorphically a torus or a K3-surface with \( a(S) = 0 \).

(iii) \( a(X) = 1 ; a(r) = 2 \) : The generic fiber of \( r : X \to C \) (the algebraic reduction of \( X \)) is \( \mathbb{P}(O_E \oplus L) \) as in (7.3.i). Let \( f : X \to S \), \( g : S \to C \) with \( g \circ f = r \) be the relative Albanese map of \( r \). Then \( a(S) = 1 \) and \( g \) is the algebraic reduction of \( S \). The generic fiber of \( f \) is \( \mathbb{P}_1 \).

(iv) \( a(X) = 1 ; a(r) = 0 \) : \( X \) is bimeromorphically \( \mathbb{P}_1 \times F \) with \( F \) either a torus or a K3-surface with \( a(F) = 0 \).

(v) \( a(X) = 2 \) : There exists a (unique) map \( \sigma : X \to \Sigma \) with generic fiber \( \mathbb{P}_1 \) and \( a(\Sigma) = 1 \). Moreover there is a ruling \( \rho : S \to C \) and an algebraic reduction \( \tau : \Sigma \to C \) such that the product map \( \sigma \times \tau : X \to \Sigma \times C \) is onto.

Proof. This has been established during the proof of 8.1; for (iv) use also (7.6) and (6.10).

10. A projectivity criterion for Moishezon morphisms

In this last section we prove the following result which was already used in section 8. Recall that a proper morphism is Moishezon if it is bimeromorphically equivalent to a projective morphism.
10.1 Theorem Let $f : X \rightarrow S$ be a surjective Moishezon morphism between complex manifolds $X$ and $S$. If $X$ is Kähler, $f$ is locally (over $S$) projective.

If $f : X \rightarrow S$ is a proper morphism of complex manifolds, an $f$-curve is by definition an irreducible compact curve which is contained in some fiber of $f$. We let $N_1(X/S) \subset H_2(X,\mathbb{Q})$ denote the subspace generated by the classes of $f$-curves and $N_1^*(X/S)$ its dual (over $\mathbb{Q}$). In this terminology Theorem 10.1 will be a consequence of the following

10.2 Theorem Let $f : X \rightarrow S$ be a surjective Moishezon morphism of complex manifolds. Let $s \in S$ and $\Lambda \in N_1^*(X/S)$. Then - possibly after shrinking $S$ around $s$ - there exists a line bundle $L \in \text{Pic}(X)$ such that a positive rational number $m$ such that $m\Lambda = L$, i.e. $$m\Lambda(C) = L \cdot C$$

for all $f$-curves $C$.

In the absolute case ($S$ a point) both statements are due to Moishezon [Ms67]. Usually $N_1(X/S)$ is defined in another way, by taking numerical equivalence using holomorphic line bundles, see e.g. [KMM87]. It follows from (10.2) that both notions coincide.

(10.3) Here we show how to derive (10.1) from (10.2).

Fix $s \in S$ and a Kähler form on $X$. We define $\Lambda \in N_1^*(X/S)$ by $$\Lambda(C) = \int_C \omega$$

for all irreducible $f$-curves $C \subset X$, hence all $C \in N_1(X/S)$. Then for given $\epsilon > 0$ we can find $\Lambda_\epsilon \in N_1^*(X/S)$ such that $$|\Lambda(C) - \Lambda_\epsilon(C)| \leq \epsilon\Lambda(C).$$

Applying Theorem 10.2 to $\Lambda_\epsilon$ we find $m > 0$ and a divisor $L_\epsilon$ such that $m\Lambda_\epsilon = L_\epsilon$, hence $$|m\Lambda(C) - L_\epsilon \cdot C| \leq m\epsilon\Lambda(C) \quad (*)$$

for $C \in N_1(X/S)$. We shall show that $L_\epsilon$ is $f$-ample, provided $\epsilon > 0$ is sufficiently small. From (*) we deduce that $L_\epsilon$ is $f$-nef if $\epsilon \leq 1$ and that $L_\epsilon|X_t$ is ample for all $t \in S$ by Kleiman’s criterion. Now the claim follows from

10.4 Lemma Let $X$ and $S$ be reduced complex spaces, $f : X \rightarrow S$ a proper surjective map. Let $L$ be a line bundle on $X$ such that $L|X_s$ is ample for every $s$. Then $f$ is locally (over $S$) projective.
Proof. Since the claim is local over \( S \), we may assume \( S \) Stein. We proceed by induction over \( d = \dim X \). If \( d = 0 \), the claim is obvious; so suppose \( d > 0 \). By substituting \( L \) by a multiple we may assume \( f_*(L) \neq 0 \), hence, \( S \) being Stein, \( H^0(L) \neq 0 \). We check the relative ampleness of \( L \) (locally over \( S \)) by the relative Serre vanishing

\[
R^p f_*(\mathcal{F} \otimes L^m) = 0
\]

for \( p > 0 \), every coherent sheaf \( \mathcal{F} \) and \( m \geq m_0(\mathcal{F}) \). So fix a coherent sheaf \( \mathcal{F} \) on \( X \) and denote \( S_m \) the support of the coherent sheaf \( R^p f_*(\mathcal{F} \otimes L^m) \) for a fixed \( p \). We claim that

\[
S_{m+1} \subset S_m \tag{\ast}
\]

for sufficiently large \( m \). In fact, take \( D \subset |L^m| \). Then we have a sequence

\[
0 \to L^m \to L^{m+1} \to L^{m+1}|D| \to 0.
\]

Now, taking direct images and applying induction, claim \((\ast)\) is clear. Fix \( s_0 \in S \). It follows that for all large \( m \), the point \( s_0 \) is not contained in \( S_m \). Moreover, since \( S_{m+1} \subset S_m \), there is a neighborhood \( U \) of \( s_0 \) disjoint from all these \( S_m \). Substituting \( S \) by \( U \) we have the relative vanishing we were looking for.

Proof of 10.2.

(1) First we notice that it is sufficient to prove the theorem for \( f \) projective; of course we may always assume \( S \) Stein. To reduce to the projective case, let \( f : X \to S \) be a Moishezon morphism. Let \( g : \hat{X} \to X \) be a sequence of blow-ups with compact smooth centers (in fibers of \( f \)) such that the induced morphism \( \hat{f} : \hat{X} \to S \) is projective. Let

\[
\hat{\Lambda} = \Lambda \circ g_* : N_1(\hat{X}/S) \to \mathbb{Q}.
\]

Then by assumption there exists \( \hat{D} \in \text{Pic}(\hat{X}) \) and \( m > 0 \) such that \( m\hat{\Lambda} = \hat{D} \). Let \( D = g_*(\hat{D}) \). Then it is immediately checked that \( m\Lambda = D \).

So from now on we shall assume that \( f \) is projective and \( S \) is Stein. Since our claim is local we fix a point \( s_0 \in S \) and may always shrink around \( s_0 \).

(2) We assume here that \( R^2 f_*(\mathcal{O}_X) \) is torsion free. This is e.g. guaranteed if there is a normal crossings divisor \( H \subset S \) such that \( f \) is smooth over \( S \setminus H \), see [Ko86],[Mw87],[Na86]. Fix \( \Lambda' \in N_1^+(X/S) \). Then there is an element \( \Lambda \in H^2(X, \mathbb{Q}) \) such that \( \Lambda \cdot C = \Lambda'(C) \) for all curves \( C \) contracted by \( f \). After multiplying \( \Lambda' \) by a suitable positive integer, we may assume that \( \Lambda \in H^2(X, \mathbb{Z}) \). We consider the following commutative diagram, all maps being canonically defined

\[
\begin{array}{ccc}
H^2(X, \mathbb{Z}) & \xrightarrow{\psi} & H^2(X, \mathcal{O}_X) \\
\downarrow u & & \downarrow v \\
H^0(S, R^2 f_*(\mathbb{Z})) & \xrightarrow{\psi'} & H^0(S, R^2 f_*(\mathcal{O}_X))
\end{array}
\]
Since $S$ is Stein, $v$ is an isomorphism. We construct a divisor $D' \in \text{Pic}(X)$ such that over a Zariski open dense set $S' \subset S^*$ we have
\[ v \circ \psi(\Lambda - c_1(D')) = 0. \] (\ast)
Given (\ast), it follows $v\psi(\Lambda - c_1(D')) = 0$ over $S$, since $R^2f_*(\mathcal{O}_X)$ is torsion free. Therefore $\psi(\Lambda) = \psi(\Lambda - c_1(D')) = 0$ and hence $\Lambda = c_1(L)$ on $X$ for some $L \in \text{Pic}(X)$. To prove (\ast) notice first that on by [Ms67] there is $D_s \in \text{Pic}(X_s)$ such that $c_1(D_s) = \Lambda$ for all $s \in S^*$. Fix $H \in \text{Pic}(X)$ $f$–ample. Then for every $s \in S^*$ we find positive integers $N_s$ and $\mu_s$ such that
\[ \mu_s(N_s H - D_s) \] is very ample on $X_s$. Moreover we can choose $\mu$ and $N$ independent on $s$ for $s$ in a non-empty Zariski-open subset $S' \subset S^*$. Let $m = \dim X - \dim S$. Choose an irreducible component $G \subset C_{m-1}(X/S)$ of the relative cycle space containing $m - 1$–cycles of $X$ contained in some fiber of $f$ and such that for generic $Z \in G$ the Cartier divisor $Z \subset X_s$ satisfies
\[ c_1(Z) = c_1(\mu(NH - D_s)). \]
Let $f_* : G \rightarrow S$ be the map associating to a cycle $Z$ the point $s$ with $Z \subset f^{-1}(s)$. Then we can choose $G$ in such a way that $f_*$ is surjective and proper and moreover that there is a closed analytic set $\tilde{S} \subset G$ satisfying
(a) $f_*(\tilde{S}) = S$
(b) $f_*|\tilde{S} \rightarrow S$ is generically finite. Possibly we have to shrink $S$ around $s_0$. Let $\tilde{D}$ be the graph of the family of $f$–cycles parametrised by $\tilde{S}$ and let $D$ be its image in $X$. Then $D$ is a divisor in $X$ such that
\[ D \cdot C_s = \mu(NH - D_s) \cdot C_s \]
for all $s \in S'$ and all irreducible curves $C_s \subset X_s$. In other words:
\[ \Lambda \cdot C_s = D' \cdot C_s. \]
with $D' = \mu(NH_D)$. Therefore $u(\Lambda - c_1(D')) = 0$ over $S'$ and our commutative diagram above gives claim (\ast).

(3) In general the arguments in (2) show that we can still construct $D'$ such that $D' \cdot C_s = \Lambda \cdot C_s$ for all $s \in S'$. Let
\[ \sigma = \psi'\lambda(\Lambda - c_1(D')) = \psi'\lambda(\Lambda) \in H^0(S, R^2f_*(\mathcal{O}_X)). \]
Let $T = \text{Supp}(\sigma)$. If $T = \emptyset$, we conclude by the old arguments. So suppose $T \neq \emptyset$; this case has to be ruled out.
(4) Let $A = f^{-1}(T)$ and $U = X \setminus A$. We have $\psi' u(\Lambda) = 0$ on $S \setminus T$. Therefore $\nu \psi(\Lambda) = 0$ on $S \setminus T$. It follows from the functoriality of the Leray spectral sequence that actually $\psi(\Lambda) = 0$ on $X \setminus A$. So the restriction map

$$H^2(X, \mathbb{Z}) \to H^2(U, \mathbb{Z})$$

maps $\Lambda$ to 0 and thus we obtain

$$\Lambda \in \text{Im}(H^2(X, U) \to H^2(X)).$$

By Lemma 10.5 below we see that, computing over $\mathbb{Q}$,

$$H^2(X, U) \cong H_{2n-2}(A) \cong \bigoplus \mathbb{Q}[A_i],$$

where the $A_i$ vary over the irreducible components of $A$ of codimension 1 in $X$. It follows $r\Lambda \in \text{Pic}(X)$ for a suitable positive integer, hence $\lambda \in \text{Pic}(X)$ (showing also that $T$ is empty). This ends the proof of (10.2).

**10.5 Lemma** Let $f : X \to S$ be a proper surjective map of complex manifolds with connected fibers. Let $n = \dim X$. Let $T \subset S$ be an analytic set, $A = f^{-1}(T)$ and $U = X \setminus A$. Then

$$H^2(X, U) \cong H_{2n-2}(A) \cong \bigoplus \mathbb{Q}[A_i],$$

where the $A_i$ vary over the irreducible components of $A$ of codimension 1 in $X$.

**Proof.** Let $B \subset A$ be the singular locus of $A$, $\pi : \hat{A} \to A$ a desingularisation and $\hat{B} = \pi^{-1}(B)$. First notice that

$$H^{2n-2}_c(A, B) \cong H^{2n-2}_c(\hat{A}, \hat{B}).$$

(1)

In fact, this follows from

$$H^{2n-2}_c(A, B) \cong H_0(A \setminus B)$$

(and analogously for $(\hat{A}, \hat{B})$) which is a consequence of [Sp66,thm19,p.297] via one point compactification. Since $H^i_c(\hat{B}) = H^i_c(B) = 0$ for $i \geq 2n - 1$, we conclude from (1):

$$H^{2n-2}_c(\hat{A},) \cong H^{2n-2}_c(A).$$

(2)

Since

$$H^{2n-2}_c(\hat{A}) = \bigoplus \mathbb{Q}[A_i],$$

the sum taken over all $(n - 1)$–dimensional components of $\hat{A}$, (2) gives

$$H^{2n-2}_c(A) \cong \bigoplus \mathbb{Q}[A_i].$$

(3)

Finally, we have

$$H^{2n-2}_c(A) \cong H^2(X, U)$$

by [Sp66,thm10,p.342], hence the universal coefficient theorem yields

$$H^{2n-2}_c(A) \cong H^2(X, U),$$

proving in combination with (3) our claim.
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