QUANTITATIVE HOMOGENIZATION OF ELLIPTIC PDE WITH RANDOM OSCILLATORY BOUNDARY DATA

WILLIAM M. FELDMAN, INWON KIM AND PANAGIOTIS E. SOUGANIDIS

Abstract. We study the averaging behavior of nonlinear uniformly elliptic partial differential equations with random Dirichlet or Neumann boundary data oscillating on a small scale. Under conditions on the operator, the data and the random media leading to concentration of measure, we prove an almost sure and local uniform homogenization result with a rate of convergence in probability.

1. Introduction

In this article we investigate the averaging behavior of the solutions to nonlinear uniformly elliptic partial differential equations with random Dirichlet or Neumann boundary data oscillating on a small scale. Under conditions on the operator, the data and the random media leading to concentration of measure, we prove an almost sure and local uniform homogenization result with a rate of convergence in probability.

In particular, we consider the Dirichlet and Neumann boundary value problems

\begin{equation}
\begin{cases}
F(D^2 u_\varepsilon) = 0 & \text{in } U, \\
u_\varepsilon = g(\cdot, \cdot, \omega) & \text{on } \partial U,
\end{cases}
\end{equation}

and

\begin{equation}
\begin{cases}
F(D^2 u_\varepsilon) = 0 & \text{in } U \setminus K, \\
\partial_\nu u_\varepsilon = g(\cdot, \cdot, \omega) & \text{on } \partial U, \\
u_\varepsilon = f & \text{on } \partial K,
\end{cases}
\end{equation}

where $U$ is a smooth bounded domain in $\mathbb{R}^d$ with $d \geq 2$, $K$ is a compact subset of $U$, $\nu$ is the inward normal, $F$ is positively homogeneous of degree one and uniformly elliptic, $f$ is continuous on $K$ and $g = g(x, y, \omega)$ is bounded and Lipschitz continuous in $x, y$ uniformly in $\omega$ belonging to a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and, for each fixed $x \in U$, stationary with respect to the translation action of $\mathbb{R}^d$ on $\Omega$ and strongly mixing with respect to $(y, \omega)$ (the precise assumptions are given in Section 2).

We show that there exist a deterministic continuous functions $\overline{F}_D, \overline{F}_N : \partial U \rightarrow \mathbb{R}$ such that, as $\varepsilon \rightarrow 0$, the solutions $u_\varepsilon = u_\varepsilon(\cdot, \omega)$ of (1.1) and (1.2) converge almost surely and locally uniformly in $U$ (with a rate in probability) to the unique solution $\overline{u}$ of respectively

\begin{equation}
\begin{cases}
F(D^2 \overline{u}) = 0 & \text{in } U, \\
\overline{u} = \overline{F}_D & \text{on } \partial U,
\end{cases}
\end{equation}

and

\begin{equation}
\begin{cases}
F(D^2 \overline{u}) = 0 & \text{in } U \setminus K, \\
\partial_\nu \overline{u} = \overline{F}_N & \text{on } \partial U, \\
\overline{u} = f & \text{on } \partial K.
\end{cases}
\end{equation}

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The homogenized boundary condition \( \mathcal{g} \) (here and in the rest of the paper we omit the subscript and always denote the homogenized boundary condition by \( \mathcal{g} \)) depend on \( F, \nu, d \) and the random field \( g \).

The rate of convergence depends on the regularity of \( U \), the continuity and mixing properties of \( g \), the dimension \( d \), the ellipticity ratio of \( F \) and, in the case of the Neumann problem, the bounds of \( f \).

We discuss next heuristically what happens as \( \varepsilon \to 0 \) in the Dirichlet problem (1.1). It follows from the up to the boundary continuity of the solutions to (1.1) that, close to the boundary, \( u^\varepsilon \) typically has unit size oscillations over distances of order \( \varepsilon \). Therefore any convergence to a deterministic limit must be occurring outside of some shrinking boundary layer, where the solution remains random and highly oscillatory even as \( \varepsilon \to 0 \). In order to analyze the behavior of the \( u^\varepsilon \) near a point \( x_0 \in \partial U \) with inner normal \( \nu \) we “blow up” \( u^\varepsilon \) to scale \( \varepsilon \), that is we consider

\[
v^\varepsilon(y, \omega) = u^\varepsilon(x_0 + \varepsilon y, \omega).
\]

If homogenization holds, then \( v^\varepsilon(R\nu, \omega) \) should converge to \( \mathcal{g}(x_0) \) for \( R > 0 \) sufficiently large to escape the boundary layer. Noting that the random function \( v^\varepsilon(\cdot, \omega) \) is uniformly continuous, as \( \varepsilon \to 0 \), we can approximate \( v^\varepsilon(\cdot, \omega) \) by the solution of the half-space problem, obtained after “blowing up” in the tangent half-space at \( x_0 \),

\[
\begin{align*}
F(D^2v) &= 0 & \text{in} & \{ y \in \mathbb{R}^d : y \cdot \nu > 0 \} \\
v(\cdot, \omega) &= \psi(\cdot, \omega) & \text{on} & \{ y \in \mathbb{R}^d : y \cdot \nu = 0 \}.
\end{align*}
\]  

(1.5)

Formally, based on the problem satisfied by \( v^\varepsilon(\cdot, \omega) \), we expect that,

\[
|v^\varepsilon(y, \omega) - v(y, \omega)| \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{when} \quad \psi(y, \omega) = g(x_0, y + \varepsilon^{-1}x_0, \omega).
\]  

(1.6)

With the expectation that \( u^\varepsilon(x + \varepsilon R\nu, \omega) \) can be approximated, for small \( \varepsilon \) and large \( R \), by \( v(R\nu, \omega) \), we are led to consider, for random fields \( \psi \) satisfying assumptions similar to \( g \) for fixed \( x \), the existence of an almost sure limit, as \( y \cdot \nu \to \infty \), of \( v(y, \omega) \). This is the analogue, in our setting, of the cell problem in classical homogenization theory.

We say that the cell problem (1.5) has a solution, if there exists a constant \( \mu = \mu(\nu, F, \psi) \), often referred to as the ergodic constant, such that

\[
\lim_{R \to \infty} v(R\nu, \omega) = \mu \quad \text{almost surely.}
\]  

(1.7)

As indicated above there are two main steps in the argument. The first is to show the existence of the limit (1.7) for the cell problem. This is where all the assumptions on \( F \) and the randomness come in (see Section 2). The second is to show that the approximation of \( u^\varepsilon \) near the boundary by the cell problem (1.5) holds with quantitative estimates so that the convergence results for the cell problem can be used to identify the boundary condition for the general domain problem.

Similar heuristic arguments lead to the Neumann cell problem, which is to show that there exists a deterministic (ergodic) constant \( \mu = \mu(\nu, F, \psi) \) such that, if \( v_R(\cdot, \omega) = v_{R, \nu} \) is the unique bounded solution of

\[
\begin{align*}
F(D^2v_R) &= 0 & \text{in} & \{ y \in \mathbb{R}^d : 0 < y \cdot \nu < 2R \}, \\
\partial_y v_R(\cdot, \omega) &= \psi(\cdot, \omega) & \text{on} & \{ y \in \mathbb{R}^d : y \cdot \nu = 0 \}, \\
v_R(\cdot, \omega) &= 0 & \text{on} & \{ y \in \mathbb{R}^d : y \cdot \nu = 2R \},
\end{align*}
\]  

(1.8)

then

\[
\lim_{R \to \infty} \frac{v_R(R\nu, \omega)}{R} = \mu \quad \text{almost surely.}
\]  

(1.9)
In the Neumann problem, it is the gradient $Du$ of the solution to (1.2) that has an oscillatory boundary layer, outside of which it should be approaching a deterministic constant far from the Neumann part of the boundary.

In order to have any hope for the convergence described in (1.7) and (1.9), it is necessary to impose assumptions on the randomness. To motivate them, we consider the Dirichlet cell problem (1.5) in the linear case where we can represent $v(R\nu, \omega)$, using the Poisson kernel, as

$$v(R\nu, \omega) = \int_{\{y \in \mathbb{R}^d : y \cdot \nu = 0\}} P(R\nu, y)\psi(y, \omega)dy.$$ 

Since $P(R\nu, y) \sim R^{1-d}$ on $B_R \cap \{y \in \mathbb{R}^d : y \cdot \nu = 0\}$, $v(R\nu, \omega)$ is essentially the average of the boundary values on $B_R \cap \{y \in \mathbb{R}^d : y \cdot \nu = 0\}$. In this case the convergence, as $R \to \infty$, is a consequence of the ergodic theorem as long as the boundary data satisfies two important assumptions, namely stationarity and ergodicity, which we describe next.

Firstly the distribution of $\psi(y, \omega)$ should not depend on $y$, in other words $\psi(y, \omega)$ must be stationary with respect to translations parallel to the hyperplane $\{y \in \mathbb{R}^d : y \cdot \nu = 0\}$. On the other hand, in view of (1.6), we actually need to consider the cell problem for translations of the boundary data parallel to $\nu$ as well and so we actually require stationarity of $\psi$ with respect to all $\mathbb{R}^d$ translations. Without some assumption of this form one can easily construct examples for which the limit will not exist.

For reference it is useful to consider the periodic version of our problem. In this case, there is a $\mathbb{Z}^d$ translation action under which the underlying probability space, which is the torus $\mathbb{T}^d$, is stationary and ergodic. When the normal direction $\nu$ is rational, the boundary values are not stationary with respect to the translations parallel to $\nu$.

One possible way to address this problem (see Section 2 for more discussion of this version of the problem) is to consider data defined on a hyperplane which is stationary with respect to an $\mathbb{R}^{d-1}$ or $\mathbb{Z}^{d-1}$ action, and then define data on pieces of the boundary of the general domain by “lifting up from the hyperplane”. This is in contrast to the setting described already which we refer to as assigning boundary data by “restricting from the whole space”. More specifically, one could take $\psi(y, \omega)$ on $\mathbb{R}^{d-1}$ and a diffeomorphism $\zeta$ from an open subset of $\mathbb{R}^{d-1}$ to an open subset of $\partial U$ and then define the boundary data, for the general domain problem, by

$$g^\varepsilon(x, \omega) = \psi(\varepsilon^{-1}\zeta^{-1}(x), \omega).$$

(1.10)

Secondly, we need to assume that the action of the translations of $\mathbb{R}^d$ on $\Omega$ is ergodic. Indeed some assumption on the long range decorrelation of the values of $\psi(y, \omega)$, ergodicity being the weakest is always necessary in to prove a law of large numbers/ergodic theorem-type of result. The exact form of the ergodic behavior actually turns out to be quite a delicate issue for boundary data homogenization because it does not necessarily restrict to lower dimensional subspaces.

An instructive way to understand this difficulty is again to consider the periodic version of the problem. In this case, the translations parallel to $\{y \in \mathbb{R}^d : y \cdot \nu = 0\}$ are not ergodic when the normal direction is rational; for more discussion see Choi and Kim [12] and Feldman [17]. It turns out, however, that it is enough that most (in an appropriate sense) directions yield an ergodic action.

It is not at all clear to the authors what kind of generalization of the periodicity assumption would yield this kind of homogenization for almost every direction. Again if we take boundary data on the general domain by “lifting up from hyperplanes” as in (1.10), the concern above is not an issue. On the other hand, if one assumes a more quantitative decay of correlations like strong mixing, the translation action of the $d-1$ dimensional subspaces $\partial P_\nu$ on $\Omega$ will be ergodic and the resulting rate of homogenization is uniform in the normal direction.
We also remark that, as is always the case in random homogenization, we lack the compactness of the periodic setting. All the major assumptions are used to overcome the above difficulties, that is stationarity and ergodicity on hyperplanes and lack of compactness.

Qualitative homogenization results in the stochastic setting for elliptic equations with oscillations in the interior of the domain typically rely on identifying a quantity which controls, via Alexandrov-Bakelman-Pucci (ABP for short)-type inequalities, the asymptotic behavior of the solution and whose ergodic properties can be studied using the sub-additive ergodic theorem; see Caffarelli, Souganidis and Wang [11] and, later, Armstrong and Smart [10].

It is not clear to the authors whether such a quantity exists for boundary homogenization. This is related to the fact that it is not known whether there exists an estimate analogous to the ABP-inequality controlling solutions to boundary value problems for linear elliptic equations with bounded measurable coefficients in terms of a measure theoretic norm of the boundary data. Relatedly it is known that the harmonic measure for linear non-divergence form operators with only bounded measurable coefficients may be singular with respect to the surface measure on the boundary and, in fact, may be supported on a set of lower Hausdorff dimension (see Wu [35] or Caffarelli, Fabes and Kenig [9] for the divergence form case.)

On the other hand, an argument, which is more in the spirit of the linear problem, works well here since it turns out that the value of the solutions of the cell problems can still be thought of as Lipschitz “nonlinear averages” of the boundary values. This observation lends itself to using tools from the theory of concentration of measure, which, generally speaking, provide estimates in probability for the concentration of Lipschitz functions of many independent random variables about their mean. There is more detailed discussion about this later in the paper.

To give an idea of the type of results we obtain, we state informally the main theorems without any technical assumptions. Exact statements are given in the next section. In the case that $F$ is either convex or concave, there is a very powerful concentration inequality due to Talagrand which allows the homogenization result to hold without any additional assumptions on $F$. Without convexity/concavity, the available concentration inequalities either lead to a restriction on the ellipticity ratio of $F$ or depend on stronger assumptions on the random media. We state the results separately depending on this property of $F$.

**Theorem A.** Let $F$ be positively homogeneous of degree one, uniformly elliptic and either convex or concave. Assume that the stationary random field $\psi$ is bounded, uniformly Hölder continuous and satisfies a strong mixing condition with sufficient decay. Then the Dirichlet and Neumann cell problems (1.5) and (1.8) homogenize.

**Theorem B.** Let $F$ be positively homogeneous of degree one and uniformly elliptic and $\psi$ be a bounded and uniformly Hölder continuous stationary random field.

(i) Under a restriction on the ellipticity ratio of $F$ and if $\psi$ satisfies a strong mixing condition with sufficient decay, the cell problems (1.5) and (1.8) homogenize.

(ii) If the random media is of “random checkboard”-type, stationary with respect to the $\mathbb{Z}^{d-1}$ translation action on \( \{ y \in \mathbb{R}^d : y \cdot \nu = 0 \} \), and has a log-Sobolev inequality, then the cell problems (1.5) and (1.8) homogenize without any assumption on the ellipticity ratio of $F$.

**Theorem C.** Under the assumptions of either Theorem A or Theorem B part (i), the general domain Dirichlet and Neumann boundary value problems (1.1) and (1.2) homogenize to (1.3) and (1.4), with the homogenized boundary condition determined by the corresponding cell problems (1.5) and (1.8).

See Section 2 for more precise results. We expect that methods very similar to the ones presented here will also show that the homogenization of the cell problems (1.5) and (1.8) for boundary data
as, for example in Theorem 13 part (ii), will yield a corresponding version of Theorem C for general domains with boundary data defined by “lifting up from the hyperplane” as in (1.10) (see Section 2 for more details).

Our approach can be extended to equations with nonhomogeneous of degree one nonlinearities and spatial oscillations inside the domain as long as the latter do not “interact” with the second-order derivatives, because the lower order terms scale out during the “blow up” procedure that leads to the cell problems.

Understanding the homogenization in the presence of random oscillations in both the second order term and the boundary data is still open. The main obstacle appears to be the fact that the methods applied to study the homogenization in the interior and on the boundary do not provide sufficient control to deal with the additional boundary layer created by the combined oscillations.

We present next a short review of the existing literature. Since there is a large body of work concerning the homogenization of elliptic pde, we organize this review around oscillatory and non-oscillatory boundary value problems. In each case we discuss general qualitative homogenization results and quantitative error estimates.

Classical references for homogenization of linear (divergence and non-divergence form) operators in periodic media are the books Bensoussan, Lions, Papanicolaou [6] and Jikov, Kozlov, Oleinik [23], while for fully nonlinear problems we refer to Evans [16] and Caffarelli [7]. The most general nonlinear result, without any convexity assumptions on $F$, is due to Caffarelli and Souganidis [10] using $\delta$ viscosity solutions.

The first results in random media for linear divergence and non-divergence form operators are due to Papanicolaou and Varadhan [31,32] and Kozlov [25]. Nonlinear variational problems were studied by Dal Maso and Modica [15]. The first general nonlinear result was obtained by Caffarelli, Souganidis and Wang [11], who introduced a sub-additive quantity, based on a family of auxiliary obstacle problems, which controls the behavior of the solutions using the ABP-inequality and, in view of the sub-additive ergodic theorem, has an ergodic behavior. Error estimates for linear divergence and non-divergence form equations under Cordes-type assumptions were obtained by Yurinskii [30,37], while Gloria and Otto [21], Gloria, Neukamm and Otto [20] and Marahrens and Otto [28] proved optimal rates of convergence in the discrete setting for linear divergence form elliptic pde. For nonlinear non-divergence form equations the first error estimate (logarithmic rate) was obtained by [10] in the strong mixing setting; under some assumptions this was upgraded recently to an algebraic error by Armstrong and Smart [3].

Much less is known about the homogenization of oscillatory boundary data. In the linear divergence form case with co-normal Neumann boundary data the homogenization is proved in the book [6]. In the periodic setting, some special cases were discussed in Arisawa [1], the Neumann problem in a special half-space setting with periodic boundary data was studied by Barles, Da Lio, Lions and Souganidis [11] (some results were obtained earlier by Tanaka [34] using probabilistic methods), and recently, for general domains and rotationally invariant equations, by Choi and Kim [12] and Choi, Kim and Lee [13]. Qualitative results for the oscillatory Dirichlet problems in periodic media were obtained by Barles and Mironescu [5] in the half-space setting and, recently, by Feldman [17] in general domains. In the periodic case, the analysis for general domains requires either a careful geometric analysis, such as in [12], to obtain a uniform modulus of continuity, or an argument as in [17] that ignores discontinuities of the data in small parts of the boundary. We also point out the recent results of Garet-Varet and Masmoudi [18,19] about systems of divergence-form operators with oscillatory Dirichlet data in periodic media with error estimates. In a similar vein are the results of Kenig,
Lin and Shen [24] on the rate of convergence for interior homogenization with Dirichlet or Neumann boundary conditions in Lipschitz domains.

As far as we know, the results of this paper are the first concerning homogenization of nonlinear, and, for general domains, even of linear Dirichlet and Neumann problems with random oscillatory data. Our approach is based on measuring, using appropriate barriers, the fluctuation of the interior values of the solutions to the cell problems in terms of the randomness on the boundary. Once such control has been established, we use tools from the theory of concentration inequalities to prove estimates on the deviation of the interior values of the solutions from their mean. Estimates from the elliptic theory and the maximum principle are then used to show that the means converge to a deterministic constant. The different versions of our results in the convex and non-convex cases are due to the nature of the available concentration estimates. Once the homogenization of the cell problem for half spaces in any direction has been established, we employ estimates from the elliptic theory to make rigorous the “blow up” argument described earlier to show that homogenization occurs for the general domain problem. We also discuss separately the cell problems of, what we called above, the “lifting up from the hyperplane” problem. The arguments in this case are similar but, since the probabilistic setting is simpler, we are able to obtain general results as far as $F$ is concerned.

The paper is organized as follows: In Section 2 we introduce the notation and some conventions, the assumptions on the equations, the probabilistic setting, the concentration inequalities we use in the paper and some useful technical facts about uniformly elliptic pde. Then we present the precise statements of the results. In Section 3 we begin with an outline of the proof of the homogenization of the Dirichlet cell problem, then we continue with the details which consist of two steps. The first is the Lipschitz estimates of solutions of the cell problem in terms of the boundary data and the second is the probabilistic arguments based on the aforementioned concentration results. The presentation is divided into two parts depending on the assumptions on $F$ and the randomness. We also prove the continuity of the homogenized boundary condition and the homogenization of the cell problem in the “lifting up from the hyperplane” setting. Section 4 contains the proof of homogenization for the Dirichlet problem in a general domain. The Neumann problem is discussed in Section 5. The structure of the proof mirrors that of the Dirichlet problem.

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2. Preliminaries, Assumptions and Statements of Results

Notation and some terminology/conventions. We denote by $\mathcal{M}^d$, tr$M$ and $I_d$ the class of $d \times d$ symmetric matrices with real entries, the trace of $M \in \mathcal{M}^d$ and the the $d \times d$ identity matrix respectively. We write $Q$ for the unit cube $[0,1)^{d-1}$. For each $\nu \in S^{d-1}$, the sphere in $\mathbb{R}^d$, $P_\nu := \{ x \in \mathbb{R}^d : x \cdot \nu > 0 \}$, $\Pi_\nu := \{ x \in \mathbb{R}^d : 0 < x \cdot \nu < 1 \}$ and for $r > 0$ we call $\Pi_{\nu,r} = r \Pi_\nu$. We call $B_r(x_0) = \{ x \in \mathbb{R}^d : |x-x_0| < r \}$ and $B_r = B_r(0)$ and $B_r^+ = B_r(0) \cap P_{nd}$ respectively. Given $\nu \in S^{d-1}$, we also write $x' = x - x \cdot \nu$, and, for $L > 0$, we use the cylinders $\text{Cyl}_{\nu,L} = \{ x \in \mathbb{R}^d : |x'| \leq L \} \times \{ x \in \mathbb{R}^d : 0 \leq x \cdot \nu \leq L \}$. Moreover, $2^n$ is the set of dyadic numbers, $1_A$ is the indicator function of the set $A$ and $C^{0,\alpha}(D)$, $C^{0,1}(D)$, $C^{1,\alpha}(D)$ and $C^2(D)$ are the spaces of $\alpha$-Hölder continuous, Lipschitz continuous, continuously differentiable with $\alpha$-Hölder continuous derivatives and twice continuously differentiable functions $f : D \to \mathbb{R}$ with norms respectively $\| f \|_{C^{0,\alpha}(D)}$, $\| f \|_{C^{0,1}(D)}$, $\| f \|_{C^{1,\alpha}(D)}$ and $\| f \|_{C^2(D)}$. We write $\text{osc}_D f = \sup_D f - \inf_D f$ for the oscillation over $D$ of the continuous function $f$ and $\| f \|_{L^\infty,D}$ for the $L^\infty$ norm of the bounded function $f : D \to \mathbb{R}$; if there is no ambiguity for the domain, for simplicity, we only write $\| f \|_\infty$. If $U$ is an open subset of $\mathbb{R}^d$, then, for $r > 0$, $U_r := \{ x \in U : \text{dist}(x, \partial U) > r \}$, where dist is the usual Euclidean distance. For $a \in \mathbb{R}$, $a_+$ and
a_− are respectively the positive and negative parts of a. Given a metric space \( X, \mathcal{B}(X) \) is the \( \sigma \)-algebra of the Borel sets of \( X \) associated with the metric. On product spaces, for example \( \Xi = X^{\mathbb{Z}^n} \), the notation \( \mathcal{B}(\Xi) \) will refer to the cylinder \( \sigma \)-algebra of Borel sets. For bounded linear operators \( L : \ell^2(\mathbb{Z}^n) \to \ell^2(\mathbb{Z}^n) \) we write \( \|L\|_\sigma \) for the usual operator norm.

We say that constants are universal, if they only depend on the underlying parameters of the problem, which are the ellipticity ratio of \( F \), the dimension \( d \), the constants associated with the \( C^2 \)-property of \( U \) and the mixing conditions. Also note that constants may change, without explicitly said, from line to line. Given two quantities \( A \) and \( B \) we write \( A \lesssim B \) when \( A \leq CB \) for some universal \( C \). If \( A \lesssim B \) and \( B \lesssim A \), then we write \( A \sim B \).

Throughout the paper subsolutions, supersolutions and solutions should be interpreted in the Crandall-Lions viscosity sense. We refer to Crandall, Ishii and Lions [14] for the facts we use throughout the paper.

The probabilistic setting and concentration inequalities. We are given a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). If \((\Xi, \mathcal{G})\) is a measurable space, \( \xi : \Omega \to \Xi \) is a random variable, if it is a measurable mapping in the sense that \( \xi^{-1}(E) \in \mathcal{F} \) for all \( E \in \mathcal{G} \). When \( \Xi = \mathbb{R} \) we write \( \mathcal{E} = \mathcal{F} \). If \( f : \mathbb{R} \to \mathbb{R} \), we write \( \mathbb{E} f(x) \) for \( \int \mathbb{1}_E f(x, \omega) d\mathbb{P}(\omega) \). If \( \Xi \) has a metric space structure and \( \mathcal{G} = \mathcal{B}(\Xi) \), then the composition of a continuous function \( f : \Xi \to \mathbb{R} \) with a random variable \( \xi \), \( f \circ \xi : \Omega \to \mathbb{R} \), is still measurable mapping with respect to \( \mathcal{B}(\mathbb{R}) \). This justifies the fact that the various functions on \( \Omega \) we consider throughout the paper are indeed random variables. Finally, \( \sigma(O_n : a \in A) \) is the smallest \( \sigma \)-algebra containing the collection \( \{O_n\}_{a \in A} \subset \mathcal{F} \).

We work on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), which is endowed with the action of the group of \( \mathbb{R}^d \)-translations \((\tau_y)_{y \in \mathbb{R}^d}\) such that, for each \( y \in \mathbb{R}^d \), \( \tau_y : \Omega \to \Omega \) is measurable and measure preserving. More specifically, for every measurable set \( A \in \mathcal{F} \) and every \( y, z \in \mathbb{R}^d \),

\[
\mathbb{P}(\tau_y(A)) = \mathbb{P}(A) \quad \text{and} \quad \tau_z \tau_y = \tau_{y+z}, \quad \tau_0 = \text{id}.
\]

We say that the action is ergodic if, for any \( E \in \mathcal{F} \),

\[
\text{if } \tau_x E = E \text{ for all } x \in \mathbb{R}^d, \text{ then } \mathbb{P}(E) \in \{0, 1\}.
\]

We will also use the above notions in the context of a \( \mathbb{Z}^n \)-action on a probability space and the definitions are analogous.

Given a bounded subset \( O \) of \( \mathbb{R}^d \) and a random field \( f : \mathbb{R}^d \times \Omega \to \mathbb{R} \), \( \mathcal{G}(O) \) is the \( \sigma \)-algebra

\[
\mathcal{G}(O) := \sigma(\{f(y, \cdot) : y \in O\}).
\]

The \( \phi \)-mixing rate function

\[
\phi(r) := \sup\{\|\mathbb{P}(E|F) - \mathbb{P}(E)\| : \text{dist}(O, V) \geq r, E \in \mathcal{G}(O), F \in \mathcal{G}(V) \text{ with } \mathbb{P}(F) \neq 0\} \tag{2.1}
\]

measures of the decorrelation of the values of the random field \( f \). If we want to emphasize the dependence on the random field \( f \), we write \( \phi_f \).

We say that \( f \) is \( \phi \)-mixing if \( \phi(r) \to 0 \) as \( r \to \infty \). \( \tag{2.2} \)

We will also use the \( \phi \)-mixing condition in the context of random fields on a \( \mathbb{Z}^n \)-lattice; the definition is completely analogous.

Let \( A \) be a complete separable metric space and \( \mathcal{B}(A) \) the associated Borel \( \sigma \)-algebra. Fix \( n \in \mathbb{N} \), let \( m \) be a probability measure on \( \Xi := A^{\mathbb{Z}^n} \) with the cylinder \( \sigma \)-algebra and assume that the random field, \( (X_j)_{j \in \mathbb{Z}^n} \), on \( \mathbb{Z}^n \) given by the coordinate maps satisfies the \( \mathbb{Z}^n \)-lattice version of the \( \phi \)-mixing condition \( \text{(2.2)} \).
We say that \( f : \Xi \to \mathbb{R} \) is Lipschitz with respect to the \( \alpha \)-weighted Hamming distance with weight \( \alpha \in \ell^2(\mathbb{Z}^n) \) if
\[
|f(X) - f(Y)| \leq \sum_{i \in \mathbb{Z}^n} \alpha_i 1_{X_i \neq Y_i}.
\] (2.3)

We will be making use of the following concentration inequalities in the strong mixing setting due to Marton [29] and Samson [33]. This first is about functions which are Lipschitz with respect to the Hamming distance (2.3).

**Theorem 2.1.** There exist positive constants \( c, C \) depending only on \( n \) such that, for any \( f : \Xi \to \mathbb{R} \) which is \( 1 \)-Lipschitz with respect to the \( \alpha \)-weighted Hamming distance,
\[
m(\{|f - \int f \, dm| > t\}) \leq C \exp\left(-\frac{ct^2}{|\alpha|_{\ell^2}(\sum_{i \in \mathbb{Z}^n} \phi(|i|)1/2)^2}\right).
\]

Suppose additionally that \( A \) is a closed convex subspace of some separable Banach space with norm \( \| \cdot \| \) and, for \( 1 \leq p < \infty \), define the \( \ell^p \)-distances on \( \Xi^{\mathbb{Z}^n} \) by
\[
|X - Y|_{\ell^p} := \left( \sum_{j \in \mathbb{Z}^n} \|X_j - Y_j\|^p \right)^{1/p} \text{ for } X, Y \in \Xi^{\mathbb{Z}^n}.
\]

Concentration inequalities with respect to the Hamming, or \( \ell^0 \), distance on a product space (which is just \( d_{\ell^0}(X, Y) = \sum 1_{X_i \neq Y_i} \)) are often suboptimal when the underlying space also has a Euclidean, that is \( \ell^2 \), distance. This is essentially due to the fact that being \( 1 \)-Lipschitz with respect to \( \ell^0 \)-distance on the Hamming cube \( \{0, 1\}^n \) is worse by a factor of \( \sqrt{n} \) than being \( 1 \)-Lipschitz with respect to \( \ell^2 \)-distance. For functions which are \( 1 \)-Lipschitz with respect to the \( \ell^2 \)-distance and convex, there is the following result of Samson [33], which is an extension of Talagrand’s concentration inequality.

**Theorem 2.2.** There exist positive constants \( c, C \) that depend only on \( n \) such that, for any convex and \( 1 \)-Lipschitz with respect to \( \ell^2 \)-metric \( f : \Xi \to \mathbb{R} \),
\[
m(\{|f - \int f \, dm| > t\}) \leq C \exp\left(-\frac{ct^2}{(\sum_{i \in \mathbb{Z}^n} \phi(|i|)1/2)^2}\right).
\] (2.4)

Talagrand’s inequality is very powerful but it requires convexity. This assumption appears to be, in general, necessary. However, the known counterexamples without convexity do involve some irregularity of the underlying measure.

A concentration inequality like (2.4) does hold for Gaussian measures without the convexity assumption on the Lipschitz function \( f \). A less stringent assumption, which also yields concentration for general Lipschitz functions, is the so-called log-Sobolev inequality that we describe next.

A probability measure \( m \) on \( \mathcal{B}(\mathbb{R}^n) \) is said to have a log-Sobolev inequality (LSI for short), if there exists \( \rho > 0 \) so that, for any locally continuously differentiable \( f : \mathbb{R}^n \to \mathbb{R} \),
\[
\int f^2 \log f^2 \, d\mu \, dm \leq \frac{1}{2\rho} \sum_{j=1}^n |\partial_j f|^2 \, dm.
\] (2.5)

The LSI can be defined on infinite product spaces as well, for example, \( \mathbb{R}^{\mathbb{Z}^n} \). It turns out that measures which have log-Sobolev inequality satisfy the following dimension independent concentration property; for a proof we refer to Ledoux [26].
Theorem 2.3. Suppose that $m$ has LSI with constant $\rho$. For any 1-Lipschitz with respect to the $\ell^2$-metric $f : \mathbb{R}^n \to \mathbb{R}$,
$$m(\{|f - \int f \, dm| > t\}) \leq 2 \exp(-2\rho t^2).$$
The same result holds on infinite product spaces like $\mathbb{R}^{\mathbb{Z}^n}$ as well.

We list next as lemmas two important properties of the log-Sobolev inequalities which are used to construct large classes of measures with LSI. The first is the so-called tensorization property, which says that LSI is preserved under taking product measures; we refer again to [26] for more explanations.

Lemma 2.4. Fix $N \in \mathbb{N}$ and assume that, for $1 \leq j \leq N$, the measures $m_j$ on $\mathbb{R}^n_j$ have LSI with constants $\rho_j$. Then the product measure $m = m_1 \otimes \cdots \otimes m_N$ on the product space $\mathbb{R}^{n_1+\cdots+n_N}$ also has LSI with constant $\rho = \min \rho_j$.

The second is that the pushforward $\phi \# m$ of a measure $m$ with LSI under a Lipschitz transformation $\phi$ still has a log-Sobolev inequality; recall that $\phi \# m(E) := m(\phi^{-1}E)$ for any $E \in \mathcal{B}(\mathbb{R}^n)$.

Lemma 2.5. Let $m$ be a measure on $\mathbb{R}^n$ with LSI and $\phi : \mathbb{R}^n \to \mathbb{R}^n$ a (globally) Lipschitz and locally continuously differentiable function. Then the pushforward measure $\phi \# m$ of $m$ by $\phi$ also has log-Sobolev inequality.

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally continuously differentiable. Writing, for notational convenience, $\tilde{f} := f \circ \phi$, we find
$$\int f^2 \log \frac{f^2}{\int f^2 \, dm} \, d\phi \# m = \int \tilde{f}^2 \log \frac{\tilde{f}^2}{\int \tilde{f}^2 \, dm} \, d\phi \# m \leq \frac{1}{2\rho} \int |D\tilde{f}|^2 \, dm \leq \frac{\sup |D\phi|^2}{2\rho} \int |Df| \, dm \leq \frac{\sup |D\phi|^2}{2\rho} \int |Df|^2 \, d\phi \# m.$$ 

Combining Lemma 2.4 and Lemma 2.5 it is possible to construct large classes of measures with LSI starting from some simple examples of such measures, like the uniform and Gaussian measures on the unit interval and $\mathbb{R}$ respectively, which are the prototypical examples of measures with log-Sobolev inequality; see [26].

For example, the distribution of $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$, with the $X_i$’s i.i.d. Gaussians with mean 0 and variance 1, has LSI with the same constant $\rho$ as the distribution of the $X_i$’s. Moreover, given a $n \times n$ matrix $\sigma$, the random variable $Y = \sigma X$ is also Gaussian with covariance matrix $\Sigma = (E(Y_i Y_j))_{1 \leq i,j \leq N}$ and its distribution $m_\sigma$ on $\mathbb{R}^n$ has LSI, since
$$\int f^2 \log \frac{f^2}{\int f^2 \, dm_\sigma} \, dm_\sigma \leq \frac{|\Sigma|_{\ell^2}^2}{2\rho} \int |Df|^2 \, dm_\sigma. \quad (2.6)$$

Based on the discussion above we can construct now some random fields on $\mathbb{Z}^{d-1}$ with LSI.

Let $X = (X_j)_{j \in \mathbb{Z}^{d-1}}$ be i.i.d. and such that law$(X_0)$ has log-Sobolev inequality on $\mathbb{R}$. Then, by Lemma 2.4 the law, $m$, of $X$ on $\mathbb{R}^{\mathbb{Z}^{d-1}}$ has LSI.

In the Gaussian case we can accommodate a more general mixing condition. Let $Y_j$ be Gaussian random variables and $Y' = \phi(Y)$, where $\phi$ is Lipschitz with respect to $| \cdot |_{\ell^2}$ and $|\phi|_{lip} \leq 1$. If the covariance $\Sigma$ of the $Y_j$’s, with elements $\Sigma_{ij} = E(Y_i Y_j)$, has a finite operator norm as a map from $\ell^2$ to $\ell^2$, then the law, $m$, of $Y'$ has LSI on $\mathbb{R}^{\mathbb{Z}^{d-1}}$. The finiteness of $|\Sigma|_{\ell^2}$ follows from the summability in $\mathbb{Z}^{d-1}$ of the covariances, since
$$|\Sigma|_{\ell^2} \leq (\max_i \sum_j |\Sigma_{ij}|).$$
We note that from the above example we may infer, at least heuristically, that the assumption of a log-Sobolev inequality amounts to some regularity of the 1-dimensional distributions combined with a summable decay of correlations.

The assumptions. We present next our assumptions. For the convenience of the reader we divide them into separate groups.

The domain. We assume that

\[ \mathbf{U} \subset \mathbb{R}^d \] has \( C^2 \)-boundary, (2.7)

and

\[ K \] is a compact subset of \( \mathbf{U} \). (2.8)

The regularity of \( \partial \mathbf{U} \) means, in particular, that, for each \( x \in \partial \Omega \), there exists an inward normal \( \nu_x \), \( r_0 > 0 \) and \( M > 0 \) such that, for all \( 0 < r < r_0 \) and all \( x \in \partial \mathbf{U} \),

\[ d((x + \partial P_r) \cap \overline{B_r}, \partial \mathbf{U} \cap \overline{B_r}) \leq Mr^2. \] (2.9)

The nonlinearity. We assume that \( F : \mathcal{M}^d \to \mathbb{R} \) is uniformly elliptic, that is there exist \( \Lambda > \lambda > 0 \) such that

\[ \lambda \text{tr}(N) \leq F(M) - F(M + N) \leq \Lambda \text{tr}(N) \] for all \( M,N \in \mathcal{M}^d \) such that \( N \geq 0 \), (2.10)

and homogeneous, that is

\[ F(0) = 0. \] (2.11)

Typical examples of \( F \)'s satisfying (2.10) and (2.11) are the convex Hamilton-Jacobi-Bellman and the non convex Hamilton-Jacobi-Isaacs nonlinearities

\[ F(M) = \sup_{\alpha \in \mathcal{A}} [ -\text{tr}(A^\alpha M) ] \] with \( \lambda I \leq A^\alpha \leq \Lambda I \) for all \( \alpha \in \mathcal{A} \)

and

\[ F(M) = \inf_{\beta \in \mathcal{B}} \sup_{\alpha \in \mathcal{A}} [ -\text{tr}(A^{\alpha \beta} M) ] \] where \( \lambda I \leq A^{\alpha \beta} \leq \Lambda I \) for all \( (\alpha, \beta) \in \mathcal{A} \times \mathcal{B} \),

which arise respectively in the theories of stochastic control of diffusion processes and zero-sum stochastic differential games in both cases without running cost.

In fact all \( F \)'s satisfying (2.10) and (2.11) have a max-min representation as above and, hence, are one-positively homogeneous, that is

\[ F(tM) = tF(M) \] for all \( t > 0 \) and \( M \in \mathcal{M}^d \). (2.12)

The boundary data for the Neumann problem. As far as the deterministic (non-random) boundary condition on \( K \) is concerned we assume what is sufficient for the Neumann problems (1.2) and (1.4) to have unique solutions, that is

\[ f : K \to \mathbb{R} \] is bounded and continuous. (2.13)

The random media and data. We make different assumptions depending on whether the boundary data is the restriction of a function defined on the whole space or is obtained by lifting from a hyperplane. To avoid repetition we state some properties for functions defined on \( \mathbb{R}^n \) with \( n = d \) in the former and \( n = d - 1 \) in the latter cases.

We begin with the boundary data \( \psi : \mathbb{R}^n \times \Omega \to \mathbb{R} \) arising in the cell problems. We assume that \( \psi \) is measurable with respect to \( \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{F} \), stationary in \((y, \omega)\) with respect to the group action \((\tau_y)_y \in \mathbb{R}^d\), that is, for all \( y \in \mathbb{R}^n \),

\[ \psi(y, \omega) = \psi(0, \tau_y \omega), \] (2.15)
QUANTITATIVE HOMOGENIZATION OF ELLIPTIC PDE WITH RANDOM OSCILLATORY BOUNDARY DATA

and bounded uniformly in \( \omega \in \Omega \), that is there exists \( C > 0 \) such that
\[
|\psi(y, \omega)| \leq C \quad \text{for all } y \in \mathbb{R}^n \text{ and } \omega \in \Omega. \tag{2.16}
\]

When \( n = d \), which is the case of the cell problem \([1,3]\), we also assume that \( \psi \) is Lipschitz continuous uniformly in \( \omega \), that is there exists \( C > 0 \) such that, for all \( y, w \in \mathbb{R}^n \) and \( \omega \in \Omega \),
\[
|\psi(y, \omega) - \psi(w, \omega)| \leq C|y - w|; \tag{2.17}
\]
we remark that, with only minor changes, we may assume that \( \psi \) is Hölder or even just uniformly continuous in \( y \). We leave it up to the interested reader to fill in the details.

Furthermore, as mentioned in the introduction, we need some mixing assumption on the random field \( \psi \). We assume that
\[
\psi(\cdot, \cdot) \text{ is } \phi\text{-mixing with a rate } \rho = \int_0^\infty \phi(r)^{1/2} r^{d-2} dr < +\infty; \tag{2.18}
\]
the value of this integral, \( \rho \), will be considered a universal constant in the following sections.

In the case that the boundary data is assigned by lifting from a hyperplane, that is when \( n = d - 1 \), the condition on the boundary data is based on LSI.

We assume that there exists a collection of identically distributed \((X_j)_{j \in \mathbb{Z}^{d-1}}\) such that, if \( X = (X_j)_{j \in \mathbb{Z}^{d-1}} \), then
\[
\text{the measure law}(X) \text{ on } \mathbb{R}^{2d-1} \text{ has LSI with constant } \rho \tag{2.19}
\]
and
\[
\psi(y, \omega) = X_j(\omega) \text{ when } y \in j + [0,1)^{d-1}; \tag{2.20}
\]
this is just the random checkerboard boundary data. Note that in this case we only have \( \mathbb{Z}^{d-1} \) stationarity.

Next we state the conditions on the random boundary data \( g : \partial U \times \mathbb{R}^d \times \Omega \to \mathbb{R} \). We assume that \( g \) is measurable with respect to \( \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F} \),
\[
\text{and bounded and Lipschitz continuous uniformly in } \omega \in \Omega, \text{ that is there exists } C > 0 \text{ such that, for all } (x, y), (z, w) \in \partial U \times \mathbb{R}^d \text{ and } \omega \in \Omega,
\[
|g(x, y, \omega)| \leq C \quad \text{and} \quad |g(x, y, \omega) - g(z, w, \omega)| \leq C(|x - z| + |y - w|). \tag{2.22}
\]

We remark again that, with only minor changes, we may assume that \( g \) is Hölder or even just uniformly continuous in \((x, y)\); we leave it up to the interested reader to fill in the details.

Finally we assume that, for each fixed \( x_0 \in \partial U \),
\[
g(x_0, \cdot, \cdot) \text{ is stationary in } (y, \omega) \tag{2.23}
\]
and
\[
g(x_0, \cdot, \cdot) \text{ satisfies } \text{(2.18)} \text{ with constants uniform in } x_0. \tag{2.24}
\]
Some pde background. We recall and prove some background results from the theory of fully nonlinear uniformly elliptic equations that we will need for the proofs in the sequel. First we discuss (1.3) and then (1.8). Then we introduce a selection criterion to identify uniquely a particular solutions to (1.5) and then (1.8) when \( \psi \) is discontinuous.

In what follows it is convenient to use the extremal Pucci operators \( \mathcal{P}^{\pm}_{\lambda, \Lambda} \) associated with \( F \) defined by

\[
\mathcal{P}^{+}_{\lambda, \Lambda}(N) := \lambda \text{tr}(N_+) - \lambda \text{tr}(N_-) \quad \text{and} \quad \mathcal{P}^{-}_{\lambda, \Lambda}(N) := \lambda \text{tr}(N_+) - \lambda \text{tr}(N_-);
\]

it follows from (2.10) and (2.11) that, for all \( M, N \in \mathcal{M}^d \),

\[
- \mathcal{P}^{+}_{\lambda, \Lambda}(M - N) \leq F(M) - F(N) \leq - \mathcal{P}^{-}_{\lambda, \Lambda}(M - N).
\]

The Dirichlet Problem (1.5). The first two results are an oscillation decay lemma, which is a consequence of the interior Lipschitz estimates for solutions to elliptic pdes, and a boundary Lipshitz estimate; for the proofs we refer to Caffarelli and Cabre [8].

**Lemma 2.6.** (Oscillation decay/Interior Lipschitz estimate) Assume (2.10), (2.11), (2.16) and (2.22) and let \( v \) be the unique bounded solution of (1.5). Then, for \( R > 1 \) and \( y, z \in \partial P_v + R \nu \),

\[
\sup_{|y - z| \leq 1} |v(y, \omega) - v(z, \omega)| \leq CR^{-1} \text{osc}_{P_v} \psi.
\]

**Lemma 2.7.** (Boundary Lipshitz) Assume (2.11) and (2.16). If \( u \in \mathcal{C}(B^+_1) \) solves \( F(D^2 u) = 0 \) in \( B^+_1 \) and \( u = h \in C^{0,1}(\partial P_v \cap B_1) \), then \( u \in C^{0,1}(B^+_1) \) and there exists \( C = C(n, \lambda, \Lambda) > 0 \) such that

\[
\|u\|_{C^{0,1}(B^+_1)} \leq C(\sup_{B^+_1} |u| + \|h\|_{C^{0,1}(\partial P_v \cap B_1)}).
\]

The next lemma is an important tool for the proofs later in the paper, because it quantifies the error introduced by solving the Dirichlet problem in a half space with a cut-off applied to the boundary data.

**Lemma 2.8.** (Localization) Fix \( \nu \in S^{d-1} \), let \( M > 0 \) and \( L > 1 \) and suppose that \( v \) is a solution of

\[
\begin{cases}
-\mathcal{P}^{+}_{\lambda, \Lambda}(D^2 v) \leq 0 & \text{in } P_v \cap \text{Cyl}_{\nu, L}, \\
v \leq 0 & \text{on } \partial P_v \cap \text{Cyl}_{\nu, L}, \\
v \leq M & \text{on } \partial \text{Cyl}_{\nu, L} \cap \overline{P_v}.
\end{cases}
\]

Then

\[
v \leq 2\frac{\Lambda}{\lambda}dML^{-1} \quad \text{in } \text{Cyl}_{\nu, 1}.
\]

**Proof.** Since the Pucci maximal operators are rotation invariant, without loss of generality, we assume that \( \nu = e_d \) and, for notational simplicity, write \( \text{Cyl}_L \) for \( \text{Cyl}_{e_d, L} \). Consider the barrier \( \psi \)

\[
\psi(x) := ML^{-2}(|x'|^2 - 2\frac{\Lambda}{\lambda}(d - 1)x_d^2) + (2\frac{\Lambda}{\lambda}(d - 1) + 1)ML^{-1}x_d.
\]

It is straightforward to check that \( \psi \geq v \) on \( \partial \text{Cyl}_L \) and that \( \psi \) is a smooth supersolution of \( -\mathcal{P}^{+}_{\lambda, \Lambda}(D^2 \psi) \geq 0 \) in \( P_{x_d} \). From the comparison of viscosity solution we have \( v \leq \psi \) in \( \text{Cyl}_L \) and the claim follows, since

\[
\psi \leq 2\frac{\Lambda}{\lambda}(d - 1) + 1)ML^{-1} \quad \text{in } \text{Cyl}_1.
\]

Next we recall a particular case of Theorem 1 from Armstrong, Sirakov and Smart [2] which yields the existence of singular solutions to nonlinear uniformly elliptic equations.
It will be useful for later to estimate, in the half-space case, the decay of \( \Phi \) in the directions \( e \) are orthogonal to \( \nu \).

We first note that Lemma 2.7 and a simple covering argument yield that \( \Phi \) is Lipschitz with universal constant in \( (\Lambda, \lambda, \gamma) \).

Next we show that \( \Phi(\frac{x}{\nu} \partial K_0) \geq \min \{ \frac{d}{\lambda}, \frac{d}{\lambda} \} \) and observe that the maximum principle yields \( \Phi(\frac{x}{\nu}) = 1 \) and the Harnack inequality. Then, for each \( \gamma \), let \( x' = x - x_0 e_d \) and \( y = \frac{3}{\Lambda} e_d + \frac{x'}{|x'|} \), consider the radially symmetric barrier \( v \) satisfying

\[
\begin{cases}
-\mathcal{P}_{\Lambda}^+(D^2 v) = 0 & \text{in } B_{1/2}(y) \setminus B_{1/4}(y), \\
v = 0 & \text{on } \partial B_{1/2}(y), \\
v = A & \text{on } \partial B_{1/4}(y),
\end{cases}
\]

and observe that the maximum principle yields \( \Phi \geq v \) in \( B_{1/2}(y) \setminus B_{1/4}(y) \). Meanwhile, an explicit computation shows that

\[
v(\frac{x}{|x|}) \geq cA \frac{x_d}{|x|},
\]

and, hence,

\[
\Phi(\frac{x}{|x|}) \geq v(\frac{x}{|x|}) \geq \frac{x_d}{|x|},
\]

which proves both directions of (2.30).

**The Neumann Problem** (1.8). The first result is a localization result analogous to Lemma 2.8.

**Lemma 2.10.** (Localization) Assume (2.10) and (2.11), fix \( \nu \in \mathbb{S}^{d-1} \) and, for \( L > 0 \), let \( u : \Pi_\nu \to \mathbb{R} \) satisfy

\[
\begin{cases}
-\mathcal{P}_{\Lambda}(D^2 u) \leq 0 & \text{in } \Pi_\nu, \\
\partial_\nu u \leq 0 & \text{on } \partial P_\nu \cap \partial B_L, \\
u \leq 0 & \text{on } \partial P_\nu + \nu \cap \partial B_L, \\
u \leq M & \text{on } \Pi_\nu \cap \partial B_L.
\end{cases}
\]
Then
\[ |u| \leq (1 + (d - 1)\frac{A}{A}) \frac{M}{L^2} \text{ on } \Pi_{\nu} \cap \{|x'| \leq 1\} \]

Proof. The claim follows from the comparison principle using the barrier
\[ \varphi(x) = \frac{M}{L^2} \left(|x'|^2 + (d - 1)\frac{A}{A}(1 - (x \cdot \nu)^2)\right). \]

The next result is about up to the boundary regularity of solutions of the Neumann problem. Its proof can be found in Milakis and Silvestre [30].

Lemma 2.11. (Boundary C^{1,\alpha}-regularity) Assume (2.10) and (2.11). For every \( \alpha \in (0,1) \), there exists \( C = C(d, \lambda, A, \alpha) > 0 \) such that, if \( v \) solves the Neumann problem
\[
\begin{cases}
F(D^2v) = 0 & \text{in } B^+_{1}, \\
\partial_{\nu}v = h & \text{on } \partial P_{ed} \cap B_1,
\end{cases}
\]
then
\[ ||v||_{C^{0,\alpha}(B^+_{1/2})} \leq \sup_{B_{1}}|v| + C \max_{\partial P_{ed} \cap B_1} |h|. \]

Furthermore, there is some \( \alpha = \alpha(d, \lambda, A) \in (0,1) \) such that,
\[ ||v||_{C^{1,\alpha}(B^+_{1/2})} \leq \sup_{B_{1}}|v| + C\|h\|_{C^{0,\alpha}(\partial P_{ed} \cap B_1)}. \]

As for the decay of oscillations, the estimate is slightly better than for the Dirichlet problem, since the oscillations decay up to the boundary.

Lemma 2.12. (Oscillation decay) Assume (2.10) and (2.11) and fix \( \nu \in S^{d-1} \). If \( v_R \) is the solution to (1.5), then,
\[ \sup_{|y-z| \leq 1} R^{-1}|v_R(y, \omega) - v_R(z, \omega)| \leq CR^{-1} \sup_{\partial P_{\nu}} |\psi| \text{ if } y \in \Pi_{\nu} + \frac{1}{2}R\nu \]
and there exists \( 0 < \alpha = \alpha(d, \lambda, A) \) such that
\[ \sup_{|y-z| \leq 1} |\partial_{\nu}(v_R(y, \omega) - v_R(z, \omega))| \leq CR^{-\alpha} \text{osc}_{\partial P_{\nu}} \psi(y) \text{ if } y \in \Pi_{\nu} + \frac{1}{2}R\nu. \]

Proof. To prove the first inequality we consider the rescaled function \( \tilde{v}(y, \omega) := \frac{1}{R}v_R(Ry, \omega) \) and apply Lemma 2.11 to the Neumann problem in \( B_2 \cap P_{\nu} \) for \( |y-z| \leq R^{-1} \).

For the second inequality we apply the interior C^{1,\alpha}-regularity result for solutions to uniformly elliptic operators (see [30]) to \( \tilde{v} \) and use the fact that osc_{\Pi_{\nu}} \tilde{v} \leq 2 \text{osc}_{\partial P_{\nu}} \psi. \)

Finally we introduce the analogue of the singular solutions Theorem 2.10 in the Neumann case. Let
\[ \Phi(x) = |x|^{-\beta} \text{ with } \beta = \frac{A}{A}(d - 1) - 1. \]

An explicit calculation shows that \( \phi(x) = \Phi(x + e_d) \) satisfies
\[
\begin{cases}
-P_{\lambda, \nu}^+(D^2\phi) = 0 & \text{in } P_{e_d}, \\
-\partial_{e_d}\phi \geq c_d(1 + |x|)^{\frac{d}{d-1}} & \text{on } \partial P_{e_d}.
\end{cases}
\]

Discontinuous Boundary Data. We discuss the Dirichlet problem here, but similar arguments apply to the Neumann problem. In the course of the proof of the homogenization of the cell problem, we
will have to consider (1.5) with bounded but discontinuous boundary data. Such boundary value problems may not have, in general, a unique solution unless $F$ has some additional structure. It was shown, for example, in [17] that uniqueness holds if the discontinuities are concentrated on a subset of $\partial D$ of Hausdorff dimension less than $d - 1$ and if $|\lambda /\Lambda - 1|$ is small depending on the dimension of the discontinuity set.

We describe next a selection mechanism we will be using throughout the paper, which allows to talk about a unique solution to (1.5) satisfying the comparison principle. Since the argument is more general than (1.5), we consider the boundary value problem

$$
\begin{cases}
F(D^2 u) = 0 \text{ in } D, \\
u = \phi \text{ on } \partial D;
\end{cases}
$$

(2.33)

here $D$ is a general domain, which typically in this paper will be $P_\nu$ for some $\nu \in S^{d-1}$, and $\phi : \partial D \to \mathbb{R}$ is bounded but possibly discontinuous. In what follows we write $u(\cdot; \phi)$ to denote the solution of the boundary value problem with data $\phi$.

A maximal/minimal solution to (2.33) can be constructed by the classical Perron’s method (see, for example, [13]). Here we describe an alternative method to select in a unique fashion a solution $u(\cdot; \phi)$, which is defined as the pointwise in $D$ limit of $u(\cdot; \phi_\delta)$ of (2.33), for some appropriately defined regularization $\phi_\delta$ of $\phi$. Moreover, $u(\cdot; \phi)$ satisfies the contraction property, that is, for any two bounded possibly discontinuous $\phi, \phi'$,

$$
\sup_D (u(\cdot; \phi) - u(\cdot; \phi')) \leq \sup_{\partial D} (\phi - \phi')_{\pm} ;
$$

(2.34)

in what follows we say that the so defined $u(\cdot; \phi)$ satisfies the maximum principle.

To this end, given a bounded possibly discontinuous $\phi : \partial D \to \mathbb{R}$, for $\delta > 0$, we define

$$
\phi_\delta(y) := \max_{z \in \partial D} [\phi(z) - \frac{1}{\delta} |y - z|].
$$

It is immediate that $\phi_\delta$ is Lipschitz continuous, $\phi_\delta \geq \phi$ and the $\phi_\delta$’s decrease, as $\delta \to 0$, to $\phi$ at every point of continuity of $\phi$. The standard comparison principle for bounded uniformly continuous solutions to (2.33) yields that the solutions $u(\cdot; \phi_\delta)$’s are decreasing in $\delta$ and, hence, for each $y \in \overline{D}$, we may define $u(y; \phi)$ by $u(y; \phi) := \lim_{\delta \to 0} u(y; \phi_\delta)$ pointwise in $\overline{D}$ and locally uniformly in $D$. Since the regularization of $\phi$ is contractive, that is, for any $\phi, \phi'$ as above, $\sup_{\partial D} (\phi_\delta - \phi_\delta')_{\pm} \leq \sup_{\partial D} (\phi - \phi')_{\pm},$ it follows, again from the comparison principle, that

$$
\sup_D (u(\cdot; \phi_\delta) - u(\cdot; \phi_\delta')) \leq \sup_{\partial D} (\phi_\delta - \phi_\delta')_{\pm} \leq \sup_{\partial D} (\phi - \phi')_{\pm};
$$

letting $\delta \to 0$ on the left hand side above yields (2.34).

The results. Now that we have set up the assumptions, we give the precise statements of the main results, Theorems [A], [B] and [C] from the Introduction.

The Dirichlet cell problem (1.5). The first result assumes convexity/concavity and imposes no restrictions on the ellipticity ratio of $F$, while the second applies to general $F$’s but requires a dimension dependent upper bound on $\Lambda / \lambda$.

In what follows $\beta(\lambda, \Lambda)$ is the homogeneity exponent of the singular solution for the ellipticity class of $F$ in the half plane (see (2.21)).
Theorem 2.13. Suppose that, in addition to (2.10) and (2.11), $F$ is either convex or concave and $\psi$ satisfies (2.14), (2.15), (2.16), (2.17) and (2.18). Then, for each $\nu \in S^{d-1}$, the cell problem (1.5) has a unique solution $v_{\nu}$ which concentrates about its mean with rate
\begin{equation}
\mathbb{P}(\{\omega : |v_{\nu}(y' + R\nu, \omega) - \mathbb{E}v_{\nu}(y' + R\nu)| > t\}) \leq C \exp(-cR^{2}\beta/t^2) \quad \text{for} \quad y' \in \partial P_{\nu} \text{ and } t > 0.
\end{equation}
where $\hat{\beta} = \beta(\lambda, \Lambda)$. Moreover, the limit $\mu(\psi, F, \nu) := \lim_{R \to \infty} \mathbb{E}v_{\nu}(R\nu)$ exists and equals the almost sure limit of $v_{\nu}(R\nu, \omega)$, as $R \to \infty$, and, furthermore, for some universal constant $C = C(d, \lambda, \Lambda, \rho) = C(d, \lambda, \Lambda)\rho^2$ and all $y' \in \partial P_{\nu}$,
\begin{equation}
|\mathbb{E}v_{\nu}(y' + R\nu) - \mu(\psi, F, \nu)| \leq C(\log R)^{1/2}R^{-\hat{\beta}/2}.
\end{equation}

Theorem 2.14. Assume that $F$ and $\psi$ satisfy (2.10), (2.11), (2.14), (2.15), (2.16), (2.17) and (2.18) and assume that $2\beta(\lambda, \Lambda) - (d - 1) > 0$. Then the results of Theorem 2.13 hold with $\hat{\beta} = 2\beta(\lambda, \Lambda) - (d - 1)$.

Under the assumptions that lead to the homogenization of the cell problem, the ergodic constant (homogenized boundary condition) is continuous with respect to the normal direction and the boundary data. This is a very important property to establish homogenization in general domains. Its proof relies in a critical way on having a rate of convergence for the cell problem.

Theorem 2.15. Assume (2.10), (2.11) and $2\beta(\lambda, \Lambda) - (d - 1) > 0$ if $F$ is neither convex nor concave.

(i) If $\psi$ and $\psi'$ satisfy (2.14), (2.15), (2.16), (2.17) and (2.18) and fix $\nu \in S^{d-1}$, then
\begin{equation}
|\mu(\psi, F, \nu) - \mu(\psi', F, \nu)| \leq \|\psi - \psi'|_{\infty,R^{d} \times \Omega}.
\end{equation}

(ii) There exists $C = C(d, \lambda, \Lambda) > 0$ such that, for every $\nu, \nu' \in S^{d-1}$ and $\alpha \in (0, \alpha')$ with $\alpha' = \alpha'(\hat{\beta}) := \frac{\hat{\beta}}{2(1+\beta)}$, if $\psi$ satisfies (2.14), (2.15), (2.16), (2.17) and (2.18), then
\begin{equation}
|\mu(\psi, F, \nu) - \mu(\psi, F, \nu')| \leq C(\text{osc } \psi)(1 + \|D\psi\|_{\infty})^{\alpha'}|\nu - \nu'|^{\alpha}.
\end{equation}

Lastly we have the following homogenization result for the Dirichlet problem in general domains with an algebraic rate in probability.

Theorem 2.16. Assume (2.7), (2.11) and (2.11) and define $\hat{\beta}$ as in either Theorem 2.13 or Theorem 2.14 depending on whether $F$ is convex/concave or not. In addition suppose that the boundary data $g$ satisfies (2.21), (2.22), (2.23) and (2.24). Let $u^\varepsilon$ and $\overline{u}$ be respectively the solutions to (1.1) and (1.3) with Dirichlet boundary data $g(x, x/\varepsilon, \omega)$ and $\overline{g}(x) := \mu(g(x, \cdot, \cdot), F, \nu_2)$ respectively. The $u^\varepsilon$ concentrates about its mean in the sense that, for $\alpha_0 := \frac{\hat{\beta}}{4 + 3\beta}$ and every $p > 0$, there exists a sufficiently large universal constant $M_p$ such that
\begin{equation}
\mathbb{P}(\{\omega : \sup_{\{x : d(x, \partial U) > \varepsilon^{1 - 2\alpha_0}\}} |u^\varepsilon(x, \omega) - \mathbb{E}u^\varepsilon(x)| > M_p(\log \frac{1}{\varepsilon})^{1/2}\varepsilon^{\alpha_0}\}) \lesssim \varepsilon^p,
\end{equation}
and the expected value $\mathbb{E}u^\varepsilon$ converges, as $\varepsilon \to 0$, to $\overline{u}$ with the rate
\begin{equation}
\sup_{\{x : d(x, \partial U) > \varepsilon^{1 - 2\alpha_0}\}} |\mathbb{E}u^\varepsilon(x) - \overline{u}(x)| \lesssim (\log \frac{1}{\varepsilon})^{1/2}\varepsilon^{\alpha_0}.
\end{equation}

The lifting from the hyperplane plane. We describe the general setting for the Dirichlet cell problem arising when assigning boundary data by “lifting from the hyperplane” as in (1.10) and then state the result which asserts that the cell problem homogenizes for general $F$ without either a convexity/concavity assumption or a restriction on the ellipticity ratio.
We assume that
\[ \zeta_0 \] is a diffeomorphism from an open subset of \( \mathbb{R}^{d-1} \) to a relatively open subset of \( \partial U \), \hspace{2cm} (2.39)
and we extend \( \zeta_0 \) to a local diffeomorphism \( \zeta : \mathbb{R}^d \rightarrow \mathbb{R}^d \) by
\[ \zeta(x) = (\zeta_0(x'), x_d \nu_{\zeta(x')}). \] \hspace{2cm} (2.40)
Given a random field \( \psi : \mathbb{R}^{d-1} \times \Omega \rightarrow \mathbb{R} \), we define the boundary data on the general domain \( U \) by
\[ g^\varepsilon(x,\omega) := \psi(\varepsilon^{-1}\zeta^{-1}(x),\omega) \hspace{1cm} \text{for} \hspace{1cm} x \in \partial U; \]
\( g^\varepsilon \) satisfies the boundary value problem \( \partial \partial U \)
\[ \frac{\partial v^\varepsilon}{\partial x^i} = 0 \hspace{1cm} \text{in} \hspace{1cm} \varepsilon^{-1}(U - x_0) \]
\[ v^\varepsilon(y,\omega) = \psi(\varepsilon^{-1}\zeta^{-1}(x_0) + (D\zeta^{-1})(x_0)y + w^\varepsilon(y),\omega) \hspace{1cm} \text{on} \hspace{1cm} \varepsilon^{-1}\partial(U - x_0); \]
with \( |w^\varepsilon(y)| \leq C\varepsilon|y|^2 \) for some \( C = C(\|\zeta^{-1}\|_{C^2}) \).
The corresponding cell problem is then to solve
\[ \begin{cases} F(D^2v_T) = 0 & \text{in} \hspace{1cm} TP_{e_d}, \\ v_T(y,\omega) = \psi(T^{-1}y,\omega) & \text{on} \hspace{1cm} T\partial P_{e_d}, \end{cases} \] \hspace{2cm} (2.42)
where \( T : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is an invertible linear map, and to show that there exists an ergodic constant \( \tilde{\mu}(\psi,F,T) \) such that, almost surely,
\[ \tilde{\mu}(\psi,F,T) = \lim_{R \rightarrow \infty} v_T(R\nu,\omega). \]
We can further normalize \( T \) so that \( Te_d \) is a unit vector orthogonal to \( T\partial P_{e_d} \) as this will not change the definition of \( v_T \). We remark that, at the expense of changing \( F \), we may reduce to the case of \( T = I \) by changing variables to \( z = T^{-1}y \). Notice that the ellipticity constants will remain bounded as long as we work with a class of maps with \( T \) and \( T^{-1} \) bounded. For example, if \( T = D\zeta(\zeta^{-1}(x_0)) \), the bound will depend only on the properties of the diffeomorphism \( \zeta \). We emphasize that such a transformation need not change \( \beta \), since, in this case, we will use \( \Phi(T^*) \), which has the same homogeneity as \( \Phi \), as a supersolution barrier for the transformed problem.
The result for \( (2.42) \) is the following theorem.

**Theorem 2.17.** Assume that \( F \) and \( \psi \) satisfy \( (2.10), (2.11), (2.14), (2.15), (2.16) \), and the log-Sobolev assumptions \( (2.19) \) and \( (2.20) \). Then the conclusions of Theorem 2.13 holds for \( v_T \) with \( \tilde{\beta} = \min\{\beta(\lambda, \Lambda), 2\} \) and constants that depend on \( \lambda, \Lambda \) and the bounds for \( T \) and \( T^{-1} \).

The proof of Theorem 2.17 is based on the concentration inequalities for measures with log-Sobolev inequality and does not depend on the concentration inequalities for the strong-mixing setting Theorems 2.1 and 2.2. As a result no restrictions are imposed on the ellipticity ratio of \( F \). We expect the proof very similar to the one of Theorem 2.10 will also give a homogenization for general domains with data locally constructed by lifting from hyperplanes. Note that in order to prove the continuity of \( \tilde{\mu} \) with respect to \( T \), which is required for the general domain result, we actually need to take a regularized version of the random checkerboard boundary data \( (2.20) \) so that \( (2.17) \) holds. We leave the details to the reader.

The *Neumann cell problem* \( (1.5) \). The result when \( F \) either convex or concave is:
Theorem 2.18. Assume that, in addition to (2.10) and (2.11), $F$ is either convex or concave and $\psi$ satisfies (2.14), (2.15), (2.16), (2.17) and (2.18). Then, for each $\nu \in S^{d-1}$, the cell problem (1.8) has a unique solution $v_{\nu,R}$ which concentrates about its mean with rate

$$
P(\{\omega : R^{-1}|v_{\nu,R}(y' + R\nu, \omega) - \mathbb{E}v_{\nu,R}(y' + R\nu)| > t\}) \leq C \exp(-cR^{\hat{\beta}} t^2) \text{ on } \partial P_\nu \text{ and } t > 0. \quad (2.43)$$

with $\hat{\beta} = \frac{1}{d - 1}$. Moreover, the limit $\mu(\psi, F, \nu) := \lim_{R \to \infty} R^{-1}\mathbb{E}v_{\nu,R}(R\nu)$ exists and equals the almost sure limit of $R^{-1}v_{\nu,R}(R\nu, \omega)$, as $R \to \infty$. Furthermore, for some universal constant $C = C(d, \lambda, \lambda, \rho) = C(d, \lambda, \rho)^2$ and all $y' \in \partial P_\nu$,

$$\|\mathbb{E}v_{\nu}(y' + R\nu) - \mu(\psi, F, \nu)\| \leq C(\log R)^{1/2} R^{-\hat{\beta}/2}. \quad (2.44)$$

In the nonconvex/nonconcave case the result is:

Theorem 2.19. Assume that $F$ and $\psi$ satisfy (2.10), (2.11), (2.14), (2.15), (2.16), (2.17) and (2.18) and $\frac{1}{2} > \frac{1}{d}$. Then the conclusions of Theorem 2.18 hold with $\hat{\beta} = (\frac{a}{d} - \frac{1}{2})(d - 1)$.

As for the Dirichlet problem, under the assumptions that lead to the homogenization of the cell problem, the ergodic constant is continuous with respect to the normal direction and the boundary data. This an very important property to establish homogenization in general domains. Its proof relies in a critical way on the rate of convergence for the cell problem.

Theorem 2.20. Assume (2.10), (2.11) and $\frac{1}{2} > \frac{1}{d}$ if $F$ is neither convex nor concave.

(i) If $\psi$ and $\psi'$ satisfy (2.14), (2.15), (2.16), (2.17) and (2.18), then, for every $\nu \in S^{d-1}$,

$$|\mu(\psi, F, \nu) - \mu(\psi', F, \nu)| \leq \|\psi - \psi'\|_{\infty, B^d \times \Omega}. \quad (2.45)$$

(ii) There exists $C = C(d, \lambda, \lambda) > 0$ such that, for every $\nu, \nu' \in S^{d-1}$ and $\alpha \in (0, \alpha')$ with $\alpha'(\hat{\beta}) := \frac{\hat{\beta}}{2(1 + \hat{\beta})}$, if $\psi$ satisfies (2.14), (2.15), (2.16), (2.17) and (2.18), then

$$|\mu(\psi, F, \nu_1) - \mu(\psi, F, \nu_2)| \leq C\|\psi\|_{C^{\alpha', \alpha}}\|\nu_1 - \nu_2\|^\alpha. \quad (2.46)$$

Finally we have the following homogenization result for the general domain problem whenever the cell problem homogenizes.

Theorem 2.21. Assume (2.8) and (2.14) and suppose that the assumptions of either Theorem 2.16 or Theorem 2.17 hold and define $\hat{\beta}$ accordingly. Let $u^\varepsilon$ and $\mathbb{u}$ be respectively the solutions to (1.2) and (1.4), with Neumann data $g(x, x/\varepsilon, \omega)$ and $g(x) := \mu(g(x, x, \cdot), F, \nu_2)$. The $u^\varepsilon$ concentrates about its mean in the sense that, there exists $\alpha_0(\hat{\beta}) > 0$ and every $p > 0$, there exists a sufficiently large universal constant $M_p$ such that

$$P(\left\{\omega : \sup_{x \in U \setminus K} |u^\varepsilon(x, \omega) - \mathbb{E}u^\varepsilon(x)| > M_p(\log \frac{1}{\varepsilon})^{1/2} \varepsilon^{\alpha_0}\right\}) \lesssim \varepsilon^p, \quad (2.45)$$

and the expected value $\mathbb{E}u^\varepsilon$ converges, as $\varepsilon \to 0$, to $\mathbb{u}$ with rate

$$\sup_{x \in U \setminus K} |\mathbb{E}u^\varepsilon(x) - \mathbb{u}(x)| \lesssim (\log \frac{1}{\varepsilon})^{1/2} \varepsilon^{\alpha_0}. \quad (2.46)$$

We remark that the analogue of Theorem 2.17 will hold in the Neumann case as well although we do not provide the proof as it is a natural adaptation of the other proofs presented. Again the extension to a result in general domains with Neumann data given by “lifting up from the hyperplane” should also follow with similar arguments.
3. The Dirichlet Cell Problem

Here we present the proofs of Theorem 2.13, Theorem 2.14, Theorem 2.15 and Theorem 2.17 which are about the existence and properties of the homogenized boundary condition or ergodic constant, that is the asymptotic behavior, as \( R \to \infty \), of the solutions to Dirichlet cell problem (1.5) and the continuity with respect to the normal directions.

In many places throughout the section we will consider \( 1.5 \) with discontinuous data. In this case, when we talk about the solution satisfying the maximum/comparison principle we refer to the one constructed in the previous section.

Since the arguments are rather long, we have divided the section into three major parts. The first is about Theorem 2.13 and Theorem 2.14, the second deals with Theorem 2.15 and the third concerns Theorem 2.17.

The proofs of Theorem 2.13 and Theorem 2.14. There are two main steps here. The first is to consider, for \( R \) large, \( v_R(R\nu,\omega) \) as a function of the boundary data and prove a Lipschitz estimate in order to apply one of the concentration inequalities given in Theorem 2.1 and Theorem 2.2. The argument is essentially deterministic and will be the focus of one of the subsections below. In the second step we obtain a quantitative estimate for the concentration of \( v_R(y,\omega) \) and show the convergence of the means \( \mathbb{E} v_R(y) \), as \( y \cdot \nu \to \infty \), to a deterministic constant \( \mu(\psi,F,\nu) \). To motivate the arguments we begin with an outline of the proof.

The outline of the proof. We consider the cell problem \( 1.5 \). For simplicity we take \( \nu = e_d \) and boundary data \( \psi(y,\omega) = \xi_{y \mod \mathbb{Z}^{d-1}}(\omega) \), where \( (\xi_k)_{k \in \mathbb{Z}^{d-1}} \) is a stationary ergodic field on \( \mathbb{Z}^{d-1} \) with \( \mu = \mathbb{E} \xi_0 \) and \( \sigma^2 = \text{var}(\xi_0) \). Let \( v(\cdot;\psi) \) be the solution of the cell problem \( 1.5 \) (note that for notational simplicity we write \( v \) instead of \( v_{e_d} \)). Then, for large \( R \), we consider the value \( v_R(\psi) = v(Re_d;\psi) \) as a function \( v_R: [-1,1]^{d-1} \to \mathbb{R} \).

To keep the ideas simple, we first describe the argument when the interior equation is just the Laplacian, \( -\Delta \). Then, as we already said in the introduction, we write \( v_R(\psi) \) as

\[
 v_R(\psi) = \int_{\partial P_{e_d}} P(Re_d,y)\psi(y)dy = \sum_{k \in \mathbb{Z}^{d-1}} a_k \xi_k,
\]

where \( P: P_{e_d} \times \partial P_{e_d} \to \mathbb{R} \) is the Poisson kernel for the upper half space and

\[
 a_k = a_k(R) := \int_{k+\{0,1\}^{d-1}} P(Re_d,y) \, dy,
\]

and observe that

\[
 \mathbb{E} v_R(\psi) = (\mathbb{E} \xi_0) \int_{\partial P_{e_d}} P(Re_d,y) \, dy = \mu,
\]

and

\[
 a_k \sim R(R^2 + |k|^2)^{-\frac{d}{2}}.
\]

In particular we see that \( v_R(\psi) \) has the form of an ergodic average and, hence, we can apply the ergodic theorem to get

\[
 v_R(\psi) - \mu \to 0 \quad \text{almost surely as} \quad R \to \infty.
\]

When the \( \xi_k \)'s are i.i.d., a standard variance estimate yields

\[
 \text{var}(v_R(\psi)) = \sigma^2 \sum_{k \in \mathbb{Z}^{d-1}} a_k^2 \leq \int_{\partial P_{e_d}} P(Re_d,y)^2 \, dy \leq C_d R^{1-d},
\]

which already implies, by Chebyshev’s inequality, homogenization in probability.
This is not, however, optimal, since it is possible to obtain, using Hoeffding’s inequality (see [22]), the following Gaussian-type concentration about the mean:

\[ P(\{ \omega : |v_R(\psi) - \mu| > t \}) \leq 2 \exp \left( -\frac{t^2}{2 \sum_{k \in \mathbb{Z}^d} a_k^2} \right) \leq 2 \exp \left( -c_d R^{d-1} t^2 \right). \]  

(3.1)

Next we describe a way to generalize the main components of the above argument to the nonlinear setting.

Arguing for the moment heuristically, we suppose that we can linearize \( v_R \) around \( \psi \equiv 0 \) and write

\[ v_R(\psi) = \sum_{k \in \mathbb{Z}^d} a_k \xi_k + o(|\xi|_\infty), \]

where \( a_k := \partial \xi_k v_R(0) \) and \( |\xi|_\infty \) is small.

Fix \( j \in \mathbb{Z}^{d-1} \). Taking as boundary data \( \xi_k = 1_{\{j\}} \) and using the strong maximum principle, the positive homogeneity of \( F \) and Taylor’s expansion, we find

\[ 0 < v_R(\psi) = \eta^{-1} v_R(\eta \psi) = a_j + \eta^{-1} o(\eta), \]

and, after sending \( \eta \to 0 \), \( a_j > 0 \).

Taking next \( \psi \equiv 1 \), in which case \( v_R(\psi) = 1 \), the above argument yields, after letting \( \eta \to 0 \),

\[ |a|_{\ell^1} = \sum_{k \in \mathbb{Z}^{d-1}} a_k = 1. \]

To show homogenization then, it is enough to obtain a vanishing, as \( R \to \infty \), bound on the \( \ell^2 \)-norm of \( a_k \), and, since,

\[ \sum_{k \in \mathbb{Z}^{d-1}} a_k^2 \leq |a|_\infty |a|_{\ell^1} \leq |a|_\infty, \]

it suffices to show that, as \( R \to \infty \), \( |a|_\infty \to 0 \).

Of course this depends on a concentration estimate of the form (3.1) holding in the nonlinear case. This is a delicate issue related to the difference between the various concentration results stated in Section 2.

This is where the structure of the pde is needed to complete the argument. It is, however, evident from the heuristics above, that, to obtain the result, it is enough to get bounds on \( v_R(\psi) \) for boundary data of the form \( 1_{k+([0,1])^{d-1}} \).

For nonlinear equations we are not aware of any quantity other than \( v_R(\psi) \) which controls the homogenization and for which we can prove satisfactory estimates. Here \( v_R(\psi) \) is no longer necessarily a linear or even a subadditive average of the \( \xi_k \) and, hence, it is not possible, as far as we can tell, to apply any version of the ergodic/subadditive ergodic theorem.

We are, however, able to prove that \( v_R(\psi) \) is a Lipschitz function of the variables \( \xi_k \) which, as we show, does not put too much weight on any of them in the sense that \( |\partial \xi_k v_R|_\ell^\infty \to 0 \) as \( R \to \infty \).

In this case, the heuristic argument above suggests that the problem will homogenize. We do need, however, to deal with the nonlinearity of the equation reflected in the fact that \( v_R(\psi) \) is only a Lipschitz, instead of linear, average of the \( \xi_k \). This is where we use the tools of concentration inequalities to show that the \( v_R \)’s concentrate around their mean.

**The discretized cell problem.** We present here some estimates for the solution of the cell problem in terms of the boundary data, which can be seen as the analogue to the classical \( L^p \)-estimates for the Poisson kernel of the boundary data in the linear case.
We prove Lipshitz estimates for $f$, but the proof will apply in general.
Recall that $Q = [0, 1)^{d-1}$ is the unit cube on $\partial P_{e_d}$ and note that
$$\partial P_{e_d} = \bigcup_{i \in \mathbb{Z}^{d-1}} (i + Q).$$

Instead of using $Q$, it is possible to cover $\partial P_{e_d}$ using $rQ$ for some $r > 0$ which can be chosen later based on the mixing rate function to optimize the estimates. Since this would only change constants and not the rate of convergence, we just take $r = 1$ for simplicity.

For $X \in \Xi := C(Q)^{\mathbb{Z}^{d-1}}$, we consider the solution $u : P_{e_d} \times \Xi \to \mathbb{R}$ to
$$\begin{cases}
F(D^2_u) = 0 \\
u(y; X) = \sum_{i \in \mathbb{Z}^{d-1}} X_i(y - i)1_{i+Q}(y)
\end{cases} \text{ in } P_{e_d},$$
$$\text{on } \partial P_{e_d}. \tag{3.2}$$

We remark that, if $-F$ is convex (an analogous claim is true if $-F$ is concave), then $u(\cdot; X)$ is convex with respect to $X$. This follows from the classical fact (see [14]) that, when $-F$ is convex, linear combinations of supersolutions of (3.2) are also supersolutions.

In Lemma 3.1 below, we show that, for each fixed $y \in P_{e_d}$, by the maximum principle, $u(y; X)$ is continuous in the sup-norm with respect to each component $X_j$, and, hence $u(y; \cdot) : (\Xi, \mathcal{B}(\Xi)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a measurable mapping.

We think of the value of $u(Re_d; X)$ as a function of $R$ and $X$ and write
$$f_R(X) := u(Re_d; X). \tag{3.3}$$

We prove Lipshitz estimates for $f_R$ with respect to the various norms $\ell^p$-norms on $C(Q)^{\mathbb{Z}^{d-1}}$, which we define next. For $X \in C(Q)^{\mathbb{Z}^{d-1}}$, the $\ell^p(\mathbb{Z}^{d-1}; C(Q))$ norms are given for $1 \leq p < \infty$ and $p = \infty$, by
$$|X|_{\ell^p} = \left( \sum_{i \in \mathbb{Z}^{d-1}} \sup_{y \in Q} |X_i(y)|^p \right)^{1/p}, \tag{3.4}$$
and
$$|X|_{\ell^\infty} = \sup_{i \in \mathbb{Z}^{d-1}} \sup_{y \in Q} |X_i(y)|.$$

**Lemma 3.1.** Assume (2.10) and (2.11). The map $f_R : C(Q)^{\mathbb{Z}^{d-1}} \to \mathbb{R}$ has the following continuity properties for $X, Y \in C(Q)^{\mathbb{Z}^{d-1}}$ and some universal constant $C := C(d, \lambda, \Lambda) > 0$:

(i) $|f_R(X) - f_R(Y)| \leq |X - Y|_{\ell^\infty}.$ \hspace{1cm} (3.5)
(ii) $|f_R(X) - f_R(Y)| \leq CR^{-\beta}|X - Y|_{\ell^1}.$ \hspace{1cm} (3.6)
(iii) $|f_R(X) - f_R(Y)| \leq CR^{-\beta/2}|X - Y|_{\ell^2}.$ \hspace{1cm} (3.7)

**Proof.** The first estimate is an immediate consequence of the definition of $f_R$ and the comparison principle.

The proof of the $\ell^1$-continuity consists of constructing a barrier which controls how much $u(Re_d; X)$ can change when the value of $X$ is altered at a single site $k \in \mathbb{Z}^{d-1}$.
The comparison principle and the homogeneity of \( \Phi \) yield, for some universal and convexity of \( m \) with \( \max \),

\[
\begin{align*}
\mathcal{P}_{\lambda, \Lambda}^+(D^2\Phi) &= 0 \quad \text{in} \quad \mathcal{P}_{e_d}, \\
\Phi &= 0 \quad \text{on} \quad \partial \mathcal{P}_{e_d} \setminus \{0\},
\end{align*}
\]

with \( \max_{\partial B_1 \cap \mathcal{P}_{e_d}} \Phi = 1 \), which is \( \beta \) homogeneous with \( \beta = \beta(d, \lambda, \Lambda) \) as in (2.29). Let

\[
\tilde{\phi}(x) := 2m^{-1}\Phi(x + e_d),
\]

with \( m := \min\{\Phi(x) : x = (x', 1) \text{ with } x' \in Q\} > 0 \), and observe that, in view of the uniform ellipticity and convexity of \( \mathcal{P}_{\lambda, \Lambda}^+ \), \( \tilde{\phi} \in C^2(\mathcal{P}_{e_d}) \) (see §), and moreover,

\[
\tilde{\phi} \geq 1 \quad \text{on} \quad \partial \mathcal{P}_{e_d} \cap Q.
\]

Then, for \( X, Y \in \ell^1(\mathbb{Z}^{d-1}) \), let,

\[
\phi(x) := \sum_{j \in \mathbb{Z}^{d-1}} (\sup_Q |X_j - Y_j|)\tilde{\phi}(x - j);
\]

note that the sum, which is only over sites where \( X_j \neq Y_j \), converges in \( \mathcal{P}_{e_d} \) since \( X, Y \in \ell^1(\mathbb{Z}^{d-1}; C(\overline{Q})) \) and , in view of the regularity of \( \tilde{\phi} \), \( \phi \in C^{2,\alpha}(\mathcal{P}_{e_d}) \).

Using the definition and regularity of \( \phi \) and the subadditivity of the maximal operator \( \mathcal{P}_{\lambda, \Lambda}^+ \) we get

\[
F(D^2u(y; X) + D^2\phi, y) \geq 0 \quad \text{in} \quad \mathcal{P}_{e_d} \quad \text{and} \quad u(y; X) + \phi(y) \geq u(y; Y) \quad \text{on} \quad \partial \mathcal{P}_{e_d}.
\]

The comparison principle and the homogeneity of \( \Phi \) yield, for some universal \( C \),

\[
u(Re_d; Y) \leq \nu(Re_d; X) + \phi(Re_d) \leq \nu(Re_d; X) + m^{-1}C \sum_{j \in \mathbb{Z}^{d-1}} R(|j|^2 + R^2)^{-\beta + 1/2} \sup_Q |X_j - Y_j|;
\]

it then follows that

\[
|f_R(X) - f_R(Y)| \leq C \sum_{j \in \mathbb{Z}^{d-1}} R(|j|^2 + R^2)^{-\beta + 1/2} \sup_Q |X_j - Y_j|.
\]

Finally, employing Hölder’s inequality, we find for any \( p \in [1, \frac{1}{1-\frac{\beta}{d-1}}) \) or \( p \in [1, \infty] \), if \( \beta > d - 1 \),

\[
|f_R(X) - f_R(Y)| \leq R^{-\beta + (d-1)/p'} |X - Y|_{\ell^p(\mathbb{Z}^{d-1})},
\]

where \( p' \) is the Hölder dual exponent of \( p \); note that, using the Poisson kernel for the upper half space, it is possible to derive the same result as in the linear case with \( \beta = d - 1 \).

For the \( \ell^2 \)-continuity, we consider \( X \in C(\overline{Q})^{\mathbb{Z}^{d-1}} \) and \( Z \in C(\overline{Q})^{\mathbb{Z}^{d-1}} \) with \( |Z|_{\ell^2} < +\infty \) and, for some \( t > 0 \) to be chosen later, we write

\[
Z = Z_{> t} + Z_{\leq t} \quad \text{where} \quad (Z_{> t})_j = Z_j 1_{\{\sup_Q |Z_j| > t\}} \quad \text{and} \quad Z_{\leq t} = Z - Z_{> t}.
\]

Then

\[
|f_R(X + Z) - f_R(X)| \leq |f_R(X + Z) - f_R(X_{> t})| + |f_R(X + Z_{> t}) - f_R(X)|
\]

\[
\leq |Z_{\leq t}|_{\ell^\infty} + CR^{-\beta} |Z_{> t}|_{\ell^t}
\]

\[
\leq t + CR^{-\beta - 1} |Z|_{\ell^2}^2,
\]

and, after choosing \( t = R^{-\beta/2} |Z|_{\ell^2} \),

\[
|f_R(X + Z) - f_R(X)| \leq CR^{-\beta/2} |Z|_{\ell^2}^2.
\]

\[\-boxed{}\]
The homogenization of the cell problem \(1.5\). We apply the estimates of Lemma \(3.1\) to the random problem and establish the homogenization of the cell problem \(1.5\).

As we did before, in what follows, for simplicity, we assume that \(\nu = e_d, |\psi| \leq 1\) almost surely, and we work with the solution \(v_{e_d}\) of the cell problem

\[
\begin{aligned}
F(D^2v_{e_d}) &= 0 & \text{in } & P_{e_d}, \\
v_{e_d}(\cdot,\omega) &= \psi(\cdot,\omega) & \text{on } & \partial P_{e_d}.
\end{aligned}
\tag{3.9}
\]

We now transform to the setting of the previous section to apply the concentration results Theorem \(2.1\) and Theorem \(2.2\). More precisely, we consider the Banach space \(C(\bar{Q})\) with the sup-norm and the associated Borel \(\sigma\)-algebra \(\mathcal{B}(C(\bar{Q}))\) and the measure space \((\Xi,\mathcal{B}(\Xi))\), where, as before \(\Xi := C(\bar{Q})^{d-1}\), and we define the measurable map \(\Psi : \Omega \to \Xi\) given, for each \(j \in \mathbb{Z}^{d-1}\), by

\[
\Psi_j(\omega)(\cdot) := \psi(j + \cdot,\omega),
\]

which induces a probability measure \(P\) on \((\Xi,\mathcal{B}(\Xi))\), the pushforward of \(\mathbb{P}\) by \(\Psi\) or, equivalently, the law of the random variable \(\Psi\). Recall that, by the definition of the pushforward measure, for any measurable \(f : \Xi \to \mathbb{R}\), \(\int f(X) dP(X) = \int f(\Psi) d\mathbb{P}\).

It is immediate, in view of the properties of \(\psi\) and \(\mathbb{P}\), that \(P\) is stationary with respect to the natural \(\mathbb{Z}^{d-1}\) action on \(\Xi\). Furthermore, the random field \(\Psi\) on \(\mathbb{Z}^{d-1}\) has the \(\phi\)-mixing property \(2.2\) adapted to the lattice with the same rate function as \(\psi\) up to a dimensional constant, and, in view of \(2.18\),

\[
\sum_{i \in \mathbb{Z}^{d-1}} \phi(|i|)^{1/2} \lesssim \rho.
\tag{3.10}
\]

We also remark that, in view of the definition of \(\Psi\), \(u(\cdot; \Psi) = v_{e_d}(\cdot,\omega)\) on \(\partial P_{e_d}\). Since \(\psi(\cdot,\omega)\) is continuous for each \(\omega\), in view of the uniqueness of the solutions to the boundary value problem,

\[
u(\cdot; \Psi) = v_{e_d}(\cdot,\omega) \quad \text{on} \quad \partial P_{e_d}.
\]

Furthermore, given the definition of \(P\) and in view of the relationship between \(u(\cdot; \Psi)\) and \(v_{e_d}(\cdot,\omega)\), the concentration of \(u(\cdot; \Psi)\) with respect to \(P\) is the same as concentration of \(v_{e_d}(\cdot,\omega)\) with respect to \(\mathbb{P}\). In other words any probability estimates and expectations on \(u(\cdot; \Psi)\) with respect to \(P\) are the same as probability estimates and expectations on \(v_{e_d}(\cdot,\cdot)\) with respect to \(\mathbb{P}\).

We apply next Theorem \(2.1\) or Theorem \(2.2\) depending on whether \(F\) and, hence, \(u(Re_d; \cdot)\) are convex or concave.

When \(F\) is neither convex nor concave, we need to assume that the ellipticity constants \((\lambda, \Lambda)\) of \(F\) are such that

\[
2\beta(\lambda, \Lambda) > d - 1,
\tag{3.11}
\]

where again \(\beta\) is the exponent of the upward pointing half-space singular solution for the maximal operator. In view of \(2.29\), a sufficient condition for \(3.11\) is

\[
\frac{\lambda}{\Lambda} > \frac{1}{2} \frac{d + 1}{d}.
\tag{3.12}
\]

In this case, recalling that \(|\psi| \leq 1\) (and therefore \(|\Psi|_{\infty} \leq 1\) almost surely), we have, from the proof of Lemma \(3.1\) the following weighted Hamming distance continuity of \(f_R = u(Re_d; \cdot)\):

\[
|f_R(X) - f_R(Y)| \lesssim \sum_{k \in \mathbb{Z}^{d-1}} R(|k|^2 + R^2)^{-(\beta+1)/2} 1_{\{X_k \neq Y_k\}} \quad \text{for all} \quad X, Y \in \text{supp}(P).
\]
Since we can bound the $\ell^2$-norm of the coefficients in the above inequality as follows
\[
\sum_{k \in \mathbb{Z}^{d-1}} R^2(|k|^2 + R^2)^{-(\beta + 1)} \lesssim R^{d-1-2\beta} \int_{\mathbb{R}^{d-1}} (1 + |x|^2)^{-(\beta + 1)} \, dx,
\]
applying Theorem 2.1 and using (3.11) we find
\[
\mathbb{P}(\omega : |v_{e_d}(R\omega_2, \omega) - \mathbb{E}v_{e_d}(R\omega_2, \cdot)| \geq t) = \mathbb{P}(\{X \in \Xi : |f_R(X) - \mathbb{E}_P f_R| \geq t\}) \leq C \exp\left(-cR^{2\beta - (d-1)\gamma}\right),
\]
with $c = c'/\rho^2$ and $c' = c'(d, \lambda, \lambda) > 0$.

When $-F$ is convex, we use the convexity of the solutions with respect to the boundary data, the $\ell^2$-continuity of Lemma 3.1 and Theorem 2.2 to obtain
\[
\mathbb{P}(\omega : |v_{e_d}(R\omega_2, \omega) - \mathbb{E}v_{e_d}(R\omega_2, \cdot)| \geq t) = \mathbb{P}(\{X \in \Xi : |f_R(X) - \mathbb{E}_P f_R| \geq t\}) \leq C \exp\left(-cR^2 \gamma^2\right),
\]
with, as before, $c = c'/\rho^2$ and $c' = c'(d, \lambda, \lambda) > 0$. The same applies in the concave case.

For brevity we write from now on
\[
\hat{\beta} := \beta(\lambda, \Lambda) \quad \text{when } F \text{ is either convex or concave and, otherwise, } \hat{\beta} := 2\beta(\lambda, \Lambda) - (d - 1) \quad (3.13)
\]
assuming, of course, for the latter case (3.12).

We proceed now with the proofs of Theorem 2.13 and Theorem 2.14.

Proof of Theorem 2.13 and Theorem 2.14. It suffices to consider the case $v = e_d$ since the proof for general $v$ is analogous, and, for simplicity, we write $v = v_{e_d}$. Although $v$ is a random function, in what follows, to keep the notation simple, we sometimes omit $\omega$.

Let $N \in 2^\mathbb{N}$. The first step in the proof is to show that the expected average at height $N$
\[
\mu_N := \mathbb{E}u(Ne_d; \Psi) = \mathbb{E}v(Ne_d) = \mathbb{E}v(Ne_d + y') \quad \text{for all } y' \in \partial P_{e_d}
\]
is Cauchy.

For a large constant $A = A(d, \lambda, \Lambda) >> (1 + \hat{\beta}/2)(d - 1)$ and $k \in \mathbb{Z}^{d-1}$, we define the events
\[
E^N_k = \{\omega \in \Omega : |v(Ne_d + N^{1-\hat{\beta}/2}k, \omega) - \mu_N| \geq A^{1/2}(\log N)^{1/2}N^{-\hat{\beta}/2}\}. \quad (3.14)
\]
In view of the assumed stationarity and either Theorem 2.1 or 2.2 we have
\[
\mathbb{P}(E^N_k) = \mathbb{P}(E^N) = \mathbb{P}(\{\omega : |\mu(Ne_d; \Psi) - \mu_N| \geq A^{1/2}(\log N)^{1/2}N^{-\hat{\beta}/2}\}) \leq C \exp(-cA \log N).
\]

It is then immediate, from a simple union bound, that, if $E^N := \bigcup_{|k| \leq N^{\hat{\beta}}} E^N_k$, then
\[
\mathbb{P}(E^N) \leq CN^{\hat{\beta}(d-1)} \exp(-cA \log N).
\]

We fix some large $M \in 2^\mathbb{N}$ and observe that, as long as $A > \hat{\beta}(d - 1)/c$, if $E^{\geq M} := \bigcup_{N \geq M} E^N$, then
\[
\mathbb{P}(E^{\geq M}) \leq C \sum_{N \geq M} N^{\hat{\beta}(d-1) - cA} < +\infty.
\]

It follows that, for some universal sufficiently large $M$, $P(E^{\geq M}) < 1$, which, of course, implies that $P(\Omega \setminus E^{\geq M}) > 0$; note that on $\Omega \setminus E^{\geq M}$,
\[
|v(Ne_d + N^{1-\hat{\beta}/2}k, \omega) - \mu_N| \lesssim (\log N)^{1/2}N^{-\hat{\beta}/2} \quad \text{for all } N \geq M \text{ and } |k| \leq N^{\hat{\beta}}.
\]
Then, using the oscillation decay estimates (Lemma 2.6), for every \( N \geq M \), we have
\[
|v(y' + Ne_d, \omega) - \mu_N| \lesssim N^{1-\hat{\beta}/2}N^{-1} + (\log N)^{1/2}N^{-\hat{\beta}/2} \lesssim (\log N)^{1/2}N^{-\hat{\beta}/2}
\]
for \( |y'| \leq N^{1+\hat{\beta}/2} \).

Moreover the localization Lemma 2.8 gives
\[
|v(2Ne_d, \omega) - \mu_N| \lesssim \sup_{|y'| \leq N^{1+\hat{\beta}/2}} |v(y' + Ne_d, \omega) - \mu_N| + N^{1-(1+\hat{\beta}/2)} \lesssim (\log N)^{1/2}N^{-\hat{\beta}/2}
\]
Combining the previous two estimates we get
\[
|\mu_2N - \mu_N| \lesssim (\log 2N)^{1/2}(2N)^{-\hat{\beta}/2} + |v(2Ne_d, \omega) - \mu_N| \lesssim (\log N)^{1/2}N^{-\hat{\beta}/2},
\]
and, for every \( N, L \geq M \) in \( 2^N \),
\[
|\mu_N - \mu_L| \lesssim \sum_{K \geq M} (\log K)^{1/2}K^{-\hat{\beta}/2} \lesssim (\log M)^{1/2}M^{-\hat{\beta}/2}.
\]
Thus the sequence \( (\mu_N)_{N \in 2^\mathbb{N}} \) is Cauchy sequence and, therefore, has a limit \( \mu \).

Next we use the above facts to prove (2.36) for \( R > \nu \) not necessarily dyadic and \( y' \in \partial P_{e_d} \).

Let \( N \in 2^\mathbb{N} \) such that \( N \leq R \leq 2N \). The above arguments yield that, if \( R \) and, hence, \( N \) are sufficiently large depending only universal constants, then for any \( y' \in \partial P_{e_d} \)
\[
\mathbb{P}(\tau_{-y'}E^{\geq N}) = \mathbb{P}(E^{\geq N}) < 1/2.
\]
The convergence rate of \( \mu_N \) to \( \mu \) and the same localization argument as above yield that for \( \omega \notin \tau_{-y'}E^{\geq N} \),
\[
|v(y' + Re_d, \omega) - \mu| = |v(Re_d, \tau_{y'}\omega) - \mu| \leq |v(Re_d, \tau_{y'}\omega) - \mu_N| + C(\log R)^{1/2}R^{-\hat{\beta}/2} \leq C(\log R)^{1/2}R^{-\hat{\beta}/2},
\]
(3.15)
since \( \tau_{y'}\omega \notin E^{\geq N} \). On the other hand, since
\[
\mathbb{P}(\{|\omega : |v(y' + Re_d, \omega) - \mathbb{E}v(y' + Re_d)| > t\}) \leq \mathbb{E}\exp(-cR^{\hat{\beta}/2}t^2),
\]
taking \( t = AR^{-\hat{\beta}/2} \), where \( A \) is sufficiently large universal, gives
\[
\mathbb{P}(\{|\omega : |v(y' + Re_d, \omega) - \mathbb{E}v(y' + Re_d)| > t\}) \leq 1/2,
\]
and thus,
\[
\{v(y' + Re_d, \omega) - \mathbb{E}v(y' + Re_d) \mid C \neq 0 \} \leq A(\log R)^{1/2}R^{-\hat{\beta}/2} \cap (\tau_{-y'}E^{\geq N})^C \neq \emptyset.
\]
Evaluating (3.15) for an \( \omega \) in this intersection, we get
\[
|\mathbb{E}v(y' + Re_d) - \mu| \lesssim (\log R)^{1/2}R^{-\hat{\beta}/2}.
\]

Arguments along the same lines also imply that the limit \( \lim_{R \to \infty} v(Re_d, \omega) = \mu \) holds pointwise on the set \( \Omega_0 = \Omega \setminus \bigcap_{M \geq 1}E^{\geq M} \) which has probability 1. We do not go into the details, since this fact will not be used in the general domain case.

For the proof of the homogenization of the general domain Dirichlet problem we need an additional estimate which we state and prove next. Essentially, because the concentration has exponential rate, we can combine the oscillation decay with the concentration to get uniform estimates on a polynomially large (in \( R \)) subset of \( P_{e_d} \).
Lemma 3.2. Under the assumptions of Theorem 2.13 and Theorem 2.14 and for \( \nu \in S^{d-1} \) and \( R > 1 \), the solution \( v_\nu(\cdot, \omega) \) of the cell problem \( (1.9) \) satisfies the spatially uniform concentration estimate
\[
\mathbb{P}\left\{ \omega : \sup_{y \in K(R, \nu)} |v_\nu(y, \omega) - \mathbb{E}v_\nu(y, \cdot)| > t \right\} \leq CR^{d/2} \exp(-cR^{\hat{\beta}}t^2), \tag{3.16}
\]
where
\[
K(R, \nu) := \{ y \in \mathbb{R}^d : R/2 \leq y \cdot \nu \leq 2R, |y - (y \cdot \nu)\nu| \leq 3R \}. \tag{3.17}
\]

Proof. Again we take \( \nu = e_d \) and write \( v = v_{e_d} \); the general case follows similarly.

We divide \( K(R, e_d) \) into \( O(\varepsilon^{-d}) \) disjoint cubes \( \hat{Q}_j \) centered at \( c(\hat{Q}_j) \) of size \( O(\varepsilon R) \). On each of the \( \hat{Q}_j \)'s, we use the interior Lipschitz estimates along with the fact that \( d(\hat{Q}_j, \partial P_{e_d}) \sim R \) to derive that, almost surely,
\[
\sup_{y \in \hat{Q}_j} |v(y, \omega) - v(c(\hat{Q}_j), \omega)| \lesssim \varepsilon,
\]
and, hence,
\[
\sup_{y \in K(R, e_d)} |v(y, \omega) - \mathbb{E}v(y)| \leq \sup_j |v(c(\hat{Q}_j), \omega) - \mathbb{E}v(c(\hat{Q}_j), \cdot)| + C_0 \varepsilon.
\]
Then, for \( t > 2C_0 \varepsilon \), we have
\[
\mathbb{P}\left\{ \omega : \sup_{y \in K(R, e_d)} |v(y, \omega) - \mathbb{E}v(y)| > t \right\} \leq \mathbb{P}\left\{ \omega : \sup_j |v(c(\hat{Q}_j), \omega) - \mathbb{E}v(c(\hat{Q}_j))| + C_0 \varepsilon > t \right\}
\leq \mathbb{P}\left\{ \omega : \sup_j |v(c(\hat{Q}_j), \omega) - \mathbb{E}v(c(\hat{Q}_j))| > t/2 \right\}
\leq C\varepsilon^{-d} \mathbb{P}\left\{ \omega : |v(c(\hat{Q}_j), \omega) - \mathbb{E}v(c(\hat{Q}_j))| > t/2 \right\}
\leq C\varepsilon^{-d} \exp(-cR^{\hat{\beta}}t^2).
\]

On the other hand, when \( t \leq 2C_0 \varepsilon \),
\[
\mathbb{P}\left\{ \omega : \sup_{y \in K(R, e_d)} |v(y, \omega) - \mathbb{E}v(y)| > t \right\} \leq 1 \leq [C^{-1}\varepsilon^d \exp(cC_0 R^{\hat{\beta}}\varepsilon^2)] [C\varepsilon^{-d} \exp(-cR^{\hat{\beta}}t^2)].
\]
Choosing \( \varepsilon^2 \sim R^{-\hat{\beta}} \) and combining the two cases yields the claim.

- The continuity properties of the homogenized boundary condition. We discuss the continuity properties of \( \mu(\nu, F, \psi) \) and, in particular, we prove Theorem 2.15. Both parts of the theorem are used to establish the continuity of the homogenized boundary condition and the homogenization in general domains. In particular, (i) yields the continuity of \( \overline{\mathbb{F}}(x) = \mu(g(x, \cdot, \cdot), F, \nu) \) with respect to the large scale \( x \)-dependence of \( g \), while (ii) implies continuity of \( \overline{\mathbb{F}} \) with respect to changing normal directions.

Proof of Theorem 2.15 Part (i) is a direct consequence of the comparison principle.

To show (ii), we fix \( \nu, \nu' \in S^{d-1} \) and, without loss of generality, we assume that \( \nu \cdot \nu' \geq 1/2 \) and, after a rescaling, that \( |\nu| \leq 1 \).

We estimate \( |\mu - \mu'| \), where \( \mu = \mu(\nu, F, \psi) \) and \( \mu' = \mu(\nu', F, \psi) \), by comparing the solutions \( v = v_\nu \) and \( v' = v_{\nu'} \) of the corresponding cell problems in the intersection of the half spaces \( P_\nu \) and \( P_{\nu'} \). Here, for simplicity, we do not display in most places the dependence of \( v \) and \( v' \) on \( \omega \).

Let us now show that \( v(\cdot, \omega) \) and \( v'(\cdot, \omega) \) are close in \( B_R \), if their respective domains are close in \( B_L \) for \( L \gg R \). Note that,
\[
\sup_{y \in B_L(0) \cap \partial P_{\nu'}} y \cdot \nu \leq L|\nu' - \nu|.
\]
Choosing \(L\) continuity of \(\psi\) discussion before its statement in the previous section, for simplicity, we assume that

The "lifting from the hyperplane" cell problem.

Let \(C\) be defined as in (3.2). The comparison principle yields

\[
\psi(x, \omega) = \sum_{i \in \mathbb{Z}^{d-1}} X_i(\omega) 1_{i + Q}(\cdot).
\]

Let \(u : P_{e_d} \times \mathbb{R}^{d-1}\) be defined as in (3.18). The comparison principle yields

\[
v(\cdot, \omega) = u(\cdot; X(\omega)) \quad \text{in } P_{e_d}.
\]

We note that Lemma 3.1 still applies and yields a universal constant \(C\) such that, for all \(Y, Z \in \mathbb{R}^{d-1}\)

\[
|u(R_{e_d}; Y) - u(R_{e_d}; Z)| \leq CR^{-\beta/2}|Y - Z|^{\alpha_2}.
\]

Finally, we recall, again because of (2.20), that the law of \(X = (X_j)_{j \in \mathbb{Z}^{d-1}}\) on \(\mathbb{R}^{d-1}\), which we call \(P\) in analogy with the previous subsection, has LSI and, hence, from Theorem 2.8 we have the concentration estimate

\[
P(|u(R_{e_d}; X(\omega)) - \mathbb{E}u(R_{e_d}; X)| > t) = P(|Y \in \mathbb{R}^{d-1} : |u(R_{e_d}; Y) - \int_{\mathbb{R}^{d-1}} u(R_{e_d}; Z) dP(Z)| > t|) \leq C \exp(-cR^2t^2),
\]
which is as good as the convex case.

**Proof of Theorem 2.17** The proof is almost the same as Theorem 2.13 and Theorem 2.14 with one minor difference. Here, instead of \( \mathbb{R}^{d-1} \), we only have \( \mathbb{Z}^{d-1} \) translation invariance of the distribution of the boundary data. The only consequence of this is that the rate of convergence of \( \mathbb{E}[v(R_d)] \) is limited also by the interior oscillation decay.

## 4. Homogenization in General Domains

Since we can only expect homogenization to hold on every compact subset of the domain \( U \), we will consider here the rate of convergence in a subdomain \( U_{R\varepsilon} \) and we will make use of the additional free parameter \( R \).

We prove next a slightly more general version of Theorem 2.16 leaving \( R \) free and then explain the choice of \( R \) that leads to Theorem 2.16.

**Theorem 4.1.** Suppose that all the assumptions of Theorem 2.16 hold. Then, as \( \varepsilon \to 0 \), almost surely and for every \( x \in U \), \( u^\varepsilon(x,\omega) \to u(x) \). Furthermore, for any \( 1 < R \lesssim \varepsilon^{-\frac{2}{d+3}} \), \( u^\varepsilon \) concentrates about its mean with the estimate

\[
P(\{ \omega : \sup_{x \in U_{R\varepsilon}} |u^\varepsilon(x,\omega) - \mathbb{E}u^\varepsilon(x)| > t \} \leq C \exp(-cR^{-\beta}t^2 + C\log \frac{1}{t})), \tag{4.1}
\]

and the expected value converges to \( \overline{u} \) with the estimate

\[
\sup_{x \in U_{R\varepsilon}} |\mathbb{E}u^\varepsilon(x) - \overline{u}(x)| \lesssim \varepsilon^{\frac{1}{2}} R^\beta + (\log \frac{1}{\varepsilon})^{1/2} R^{-\beta/2} + (\log \frac{1}{\varepsilon})^{1/2}(\varepsilon R)^{-\frac{\beta}{2(d+3)}}, \tag{4.2}
\]

with constants that depend on \( \lambda, \Lambda, d, \rho \), and the upper bound for \( R \varepsilon^{-\frac{2}{d+3}} \). In particular, choosing \( R = \varepsilon^{-\frac{2}{d+3}} \) and \( t = (\log \frac{1}{\varepsilon})^{1/2} \varepsilon^{\frac{1}{d+3}} \) yields Theorem 2.16.

If we assigned boundary data by “lifting up from the hyperplane”, then the homogenized boundary condition would be

\[
\overline{u}(x) := \tilde{\mu}(\psi, F, D\zeta(\zeta^{-1}(x))),
\]

where \( \tilde{\mu} \) comes from the alternate cell problem (2.42). The only difference in the estimates obtained would be in the continuity of the homogenized boundary condition, which corresponds to the third term in (4.2).

We give first an outline of the strategy of the proof of Theorem 4.1 which follows from a series of Lemmas. We begin by rescaling at a point \( x_0 \in \partial U \) to \( u^\varepsilon(x_0 + \varepsilon y) \) and prove an estimate on the difference, in a large box of size \( R \) around the origin, between the rescaled solution in the general domain and the solution of the corresponding cell problem in \( P_{\nu_{x_0}} \). Then we use one of the cell problem homogenization results, that is either Theorem 2.13 or Theorem 2.14 to prove an estimate of the concentration of \( u^\varepsilon(x_0 + \varepsilon y) \) about its mean and the convergence of \( \mathbb{E}u^\varepsilon(x_0 + \varepsilon y) \) to \( \overline{u}(x_0) \) on a strip of size \( \sim R \), which is \( \sim R \) away from \( \partial P_{\nu_{x_0}} \). Rescaling back to the \( \varepsilon \) scale, we use this estimate on a finite subset of \( \partial U \), which is “\( R \varepsilon \) dense” in \( \partial U \). This implies that \( u^\varepsilon \) concentrates about its mean on the boundary of the slightly smaller domain \( U_{R\varepsilon} \). Then the comparison principle gives the concentration in the interior of \( U_{R\varepsilon} \). Finally, the proof of the almost sure pointwise convergence involves a careful use of the Borel-Cantelli Lemma.

We turn now to the full details. We fix \( x_0 \in \partial U \) and define the local rescaling of \( u^\varepsilon \) near \( x_0 \) by

\[
v^\varepsilon_{x_0}(y) := u^\varepsilon(x_0 + \varepsilon y),
\]
Figure 1. Comparing with the solution of the cell problem.

which solves
\[
\begin{cases}
F(D^2 v^\varepsilon_{x_0}) = 0 & \text{in } \varepsilon^{-1}(U - x_0), \\
v^\varepsilon_{x_0}(y, \omega) = g(x_0 + \varepsilon y, y, \tau_{x_0/\varepsilon}) & \text{on } \varepsilon^{-1}\partial(U - x_0).
\end{cases}
\] (4.3)

Let \(v_\nu\), with \(\nu = \nu_{x_0}\), be the solution to (1.5) with boundary data
\[
\psi(y, \omega) := g(x_0, y, \omega) \text{ on } \partial P_\nu,
\]
and observe that, in view of the assumptions our on \(g\), \(\psi\) satisfies the hypotheses of Theorem 2.13 or Theorem 2.14 and Lemma 3.2.

The next lemma provide an estimate for the difference between \(v^\varepsilon_{x_0}\) and \(v_\nu\); for its statement recall the definition (3.17) of the sets \(K(R, \nu_{x_0})\).

Lemma 4.2. Assume the hypotheses of Theorem 2.16. For any \(R > 1\) and almost surely
\[
|v^\varepsilon_{x_0}(\cdot, \omega) - v_{\nu_{x_0}}(\cdot, \tau_{x_0/\varepsilon})| \leq C\varepsilon^{1/3} R^{2/3} \text{ in } K(R, \nu_{x_0}) \cap \varepsilon^{-1}(U - x_0).
\] (4.4)

We remark that this is the main place where the proof of Theorem 2.16 would change, if we assigned boundary data in the general domain by “lifting up from the hyperplane”, in which case the estimate, with the notation of the previous section, would instead be
\[
|v^\varepsilon_{x_0}(\cdot, \omega) - v_{\nu_{\partial U}(\cdot, \tau_{x_0/\varepsilon}) \mod Z_d} \tau_{x_0/\varepsilon}| \leq C\varepsilon^{1/3} R^{2/3} \text{ in } K(R, \nu_{x_0}) \cap \varepsilon^{-1}(U - x_0).
\]

Proof. We omit reference to \(x_0\), since it is fixed for now.

We estimate the difference between \(v^\varepsilon\) and \(v_\nu\) by a localization argument in a region where \(\partial U\) is close to its tangent hyperplane at \(x_0\). For a succinct presentation we call \(U^\varepsilon = \varepsilon^{-1}(U - x_0)\).

It follows from (2.17) that there exists some sufficiently large \(C_0 = C_0(U)\) depending on the \(C^2\)-regularity of \(\partial U\) such that, for any \(L > R\),
\[
\{x \in \mathbb{R}^d : x \cdot \nu \geq C_0 L^2 \varepsilon\} \subset U^\varepsilon \cap B_L(0) \subset \{x \in \mathbb{R}^d : x \cdot \nu \geq -C_0 L^2 \varepsilon\}.
\]

Note that, due to the up to the boundary Lipschitz continuity of \(v^\varepsilon\), we have
\[
|v^\varepsilon(y, \omega) - g(x_0 + \varepsilon y, y, \tau_{x_0/\varepsilon})| \leq C L^2 \varepsilon \text{ on } \{x \in \mathbb{R}^d : x \cdot \nu = C_0 L^2 \varepsilon\},
\] (4.4)
and
\[
|v_\nu(y, \tau_{x_0/\varepsilon}) - \psi(y, \tau_{x_0/\varepsilon})| \leq C L^2 \varepsilon \text{ on } \{x \in \mathbb{R}^d : x \cdot \nu = C_0 L^2 \varepsilon\}.
\] (4.5)
Using the previous two inequalities as well as, the, uniform in in \((y, \omega), \text{ Lipshitz continuity of } g(\cdot, y, \omega)\) and the definition of \(\psi\) we obtain
\[
|v_\nu(\cdot, \tau_{x_0/\varepsilon}\omega) - v_\nu^\varepsilon(\cdot, \omega)| \leq CL^2\varepsilon + C\varepsilon \lesssim L^2\varepsilon \quad \text{on} \quad C_0 L^2\Pi \cap U_\varepsilon. \tag{4.6}
\]

Fix \(R > 0\) with \(R \ll L\). Then using (4.6) and the localization result (Lemma 2.13), we conclude that
\[
|v_\nu(\cdot, \omega) - v_\nu(\cdot, \tau_{x_0/\varepsilon}\omega)| \lesssim L^2\varepsilon + \frac{R^2}{\varepsilon} \quad \text{in} \quad K(R, \nu) \cap U^\varepsilon,
\]
and choosing \(L = R^{\frac{1}{2}}\varepsilon^{-\frac{1}{2}},\) so that both terms in the above estimate are the same size, we find
\[
|v_\nu(\cdot, \omega) - v_\nu(\cdot, \tau_{x_0/\varepsilon}\omega)| \lesssim \varepsilon^{\frac{1}{2}} R^\frac{5}{2} \quad \text{in} \quad K(R, \nu) \cap U^\varepsilon. \tag{4.7}
\]

Next we combine (4.7) with the estimate for the cell problem given in either Theorem 2.16 or Theorem 2.14 depending on our assumptions.

From the previous estimates we have, for \(y \in K(R, \nu_{x_0}) \cap \varepsilon^{-1}(U - x_0),\)
\[
|v_\nu(\cdot, y, \omega) - \mathbb{E}v_\nu(\cdot, y)| \leq C_1 \varepsilon^{\frac{1}{2}} R^\frac{5}{2} + |v_\nu(\cdot, \tau_{x_0/\varepsilon}\omega) - \mathbb{E}v_\nu(y)| \tag{4.8}
\]
and
\[
|\mathbb{E}v_\nu(\cdot, y) - \mathbb{E}v_\nu(\cdot, x_0)| \leq |\mathbb{E}v_\nu(y) - \mathbb{E}v_\nu(y)| + |\mathbb{E}v_\nu(y) - \mathbb{E}v_\nu(y)| \lesssim \varepsilon^{\frac{1}{2}} R^\frac{5}{2} + (\log R)^{1/2} R^{-\frac{3}{2}}. \tag{4.9}
\]
Using (4.8) we derive, for \(t > 2C_1\varepsilon^{1/3} R^{2/3},\) the following uniform in \(K(R, \nu_{x_0})\) concentration estimate of \(v_\nu\) about its mean:
\[
\mathbb{P}(\{\omega : \sup_{y \in K(R, \nu_{x_0})} |v_\nu(\cdot, y, \omega) - \mathbb{E}v_\nu(\cdot, y)| > t\}) \leq \mathbb{P}(\{\omega : \sup_{y \in K(R, \nu_{x_0})} |v_\nu(\cdot, \tau_{x_0/\varepsilon}\omega) - \mathbb{E}v_\nu(\cdot, y)| > t/2\})
\leq CR \frac{d\beta}{t^2} \exp(-cR^{\frac{5}{2}}).
\]

On the other hand, if \(t \leq 2C_1\varepsilon^{1/3} R^{2/3},\) then, for \(R \leq M\varepsilon^{-\frac{2}{d+3}},\) we have
\[
C \exp(-cR^{\frac{5}{2}} t^2) \geq C \exp(-4cC_1^2 R^{d+4/3}) \geq C \exp(-cC_1^2 R^{d+4/3}) \mathbb{P}(\sup_{y \in K(R, \nu_{x_0})} |v_\nu(\cdot, y) - \mathbb{E}v_\nu(\cdot, y)| > t).
\]
Combining the two last inequalities gives the following Lemma.

**Lemma 4.3.** Assume the hypotheses of Theorem 2.16. For every \(x_0 \in \partial U\) and \(1 < R \leq M\varepsilon^{-\frac{2}{d+3}},\) \(v_\nu\) concentrates about its mean uniformly in \(K(R, \nu_{x_0})\) with rate
\[
\mathbb{P}(\{\omega : \sup_{y \in K(R, \nu_{x_0})} |v_\nu(\cdot, y, \omega) - \mathbb{E}v_\nu(\cdot, y)| > t\}) \leq C(M, \alpha, \lambda, \Lambda) R^{d\beta/2} \exp(-cR^{\frac{5}{2}} t^2),
\]
and its expectation \(\mathbb{E}v_\nu\) converges, again uniformly in \(K(R, \nu_{x_0})\), to \(\mathbb{E}(x_0)\) with rate
\[
\sup_{y \in K(R, \nu_{x_0})} |\mathbb{E}v_\nu(y) - \mathbb{E}(x_0)| \lesssim \varepsilon^{\frac{1}{2}} R^\frac{5}{2} + (\log R)^{1/2} R^{-\frac{3}{2}}.
\]

The next step is to put a finite net of points on \(\partial U\) and use the concentration estimate of the cell problem along with (4.8) and a union bound to get a concentration estimate on the entire boundary of a subdomain of \(U\).

For each \(\delta > 0\), choose a finite subset \(\Gamma_\delta\) of \(\partial U\) such that
\[
\partial U \subset \cup_{x \in \Gamma_\delta} B(x, \delta) \quad \text{and} \quad |\Gamma_\delta| \lesssim \delta^{-(d-1)}. \tag{4.10}
\]
The following lemma provides a cover of \(\partial U_\delta\) consisting of sets centered at points on \(\Gamma_\delta\).
Let \( \delta \Gamma \) as well as the up to the boundary modulus of continuity of \( \Gamma \).

Then using Lemma 4.3 as well as the up to the boundary modulus of continuity of Lemma 4.4.

To complete the proof, it is only necessary to replace in the above two estimates \( \pi \Rightarrow \delta \), we find

\[
\begin{align*}
\sup_{x \in \partial U_\delta} & |u^\varepsilon(x, \omega) - \bar{u}^\varepsilon(x)| \\
& \leq C \left( \delta \varepsilon^2 + (\log R)^{1/2} R^{-\hat{\beta}/2} + (\log \varepsilon R)^{1/2} \right)^{\alpha'}.
\end{align*}
\]

recall that \( \pi \) is any point \( x \in \Gamma_\delta \) such that \( x \in x_0 + \delta K(1, \nu_{x_0}) \), which, in view of Lemma 4.4, is well defined on \( \partial U_\delta \).

We consider next the boundary value problem

\[
\begin{align*}
F(D^2 \bar{u}^\varepsilon) &= 0 \quad \text{in} \quad U_\delta, \\
\bar{u}^\varepsilon(x) &= \bar{u}^\varepsilon(x, \omega) \quad \text{on} \quad \partial U_\delta.
\end{align*}
\]

its solution \( \bar{u}^\varepsilon \) can be thought as the “extension by \( F \)” of \( \bar{u}^\varepsilon \) to the interior of \( U_\delta \).

Employing the comparison principle in \( U_\delta \) yields

\[
P\{ \omega : \sup_{x \in U_\delta} |u^\varepsilon(x, \omega) - \bar{u}^\varepsilon(x)| > t \} \leq P\{ \omega : \sup_{x \in \partial U_\delta} |u^\varepsilon(x, \omega) - \bar{u}^\varepsilon(x, \omega)| > t \} \lesssim (R \varepsilon)^{-(d-1)} R^{d\hat{\beta}/2} \exp(-cR^\hat{\beta}/2),
\]
and

\[
\sup_{x \in U_\delta} |\bar{u}^\varepsilon(x) - \bar{u}(x)| \lesssim \varepsilon^{1/2} R^{\hat{\beta}/2} + (\log R)^{1/2} R^{-\hat{\beta}/2} + (\log \varepsilon R)^{1/2} \varepsilon R^{\hat{\beta}/2}.
\]

To complete the proof, it is only necessary to replace in the above two estimates \( \bar{u}^\varepsilon(x) \) by \( \bar{u}^\varepsilon(x) \).
For this we argue as follows: For each $x \in U_\delta$ and $A > 1$ to be chosen, we have
\[
|E u^\epsilon(x) - \bar{u}^\epsilon(x)| \leq E|u^\epsilon(x) - \bar{u}^\epsilon(x)| = \int_0^\infty \mathbb{P}(\{\omega : |u^\epsilon(x, \omega) - \bar{u}^\epsilon(x)| > t\}) dt
\]
\[
\lesssim \int_0^\infty [(R\epsilon)^{-(d-1)} R^{\beta/2} \exp(-cR^{\beta/2} t^2)] \wedge 1 dt
\]
\[
\lesssim AR^{-\beta/2}(\log \frac{1}{\epsilon})^{1/2} + \int_0^\infty AR^{-\beta/2}(\log \frac{1}{\epsilon})^{1/2} (R\epsilon)^{-(d-1)} R^{\beta/2} \exp(-cR^{\beta/2} t^2) dt
\]
\[
\lesssim AR^{-\beta/2}(\log \frac{1}{\epsilon})^{1/2} + \int_0^\infty A(\log \frac{1}{\epsilon})^{1/2} (R\epsilon)^{-(d-1)} R^{\beta/2} \exp(-cR^{\beta/2} t^2) dt
\]
\[
\lesssim AR^{-\beta/2}(\log \frac{1}{\epsilon})^{1/2} + (R\epsilon)^{-(d-1)} R^{\beta/2} \exp(cA)
\]
\[
\lesssim R^{-\beta/2}(\log \frac{1}{\epsilon})^{1/2},
\]
where for the last inequality to hold we chose $A$ large depending only on universal constants.

Finally we discuss the almost sure convergence.

**Lemma 4.5.** Assume the hypotheses of Theorem 2.10. There exists a measurable $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ such that, as $\epsilon \to 0$, $u^\epsilon(x, \omega) \to \pi(x)$ for all $x \in U$ and all $\omega \in \Omega_0$.

**Proof.** In view of the fact that we already know that $E u^\epsilon \to \pi$, it suffices to show that $|u^\epsilon(x, \omega) - E u^\epsilon| \to 0$ almost surely, a fact that typically follows from a Borel-Cantelli-type argument.

Since, however, we can apply the latter only along sequences $\epsilon_n \to 0$, we first need to measure the dependence of $u^\epsilon$ on $\epsilon$.

The assumptions on $g$ and $U$ yield a universal constant $C$ such that, for any $\epsilon, \epsilon' > 0$ and for all $x \in \partial U$,
\[
|g(x, \frac{x}{\epsilon}, \omega) - g(x, \frac{x}{\epsilon'}, \omega)| \leq C |x| \frac{1}{\epsilon} - \frac{1}{\epsilon'} |x|,
\]
and, hence, using the comparison principle, we find that, for all $x \in U$,
\[
|u^\epsilon(x) - u^\epsilon'(x)| \leq C \frac{1}{\epsilon} - \frac{1}{\epsilon'}.
\]

Let $\delta(\epsilon) = \epsilon^{1-2\alpha_0/\beta}$. Then, as long as $\frac{1}{\epsilon} - \frac{1}{\epsilon'} \leq c\epsilon^{\alpha_0}$ and $\frac{\epsilon}{\epsilon'} \geq (2/3)^{1/\alpha_0}$ with $c$ universal and independent of $p$, and, without loss of generality, $M_p > 1$, the estimate proved above yields
\[
P(\{\omega : \sup_{U_\delta} |u^\epsilon(x, \omega) - E u^\epsilon(x)| > 3M_p \epsilon^{\alpha_0}\}) \leq P(\{\omega : \sup_{U_\delta} |u^\epsilon(x, \omega) - E u^\epsilon(x)| > 2M_p \epsilon^{\alpha_0} - C \frac{1}{\epsilon} - \frac{1}{\epsilon'}\})
\]
\[
\leq P(\{\omega : \sup_{U_\delta} |u^\epsilon(x, \omega) - E u^\epsilon(x)| > M_p \epsilon^{\alpha_0}\}) \lesssim \epsilon^p.
\]

Now we just choose a sequence $\epsilon_k \to 0$ such that $\epsilon_{k+1}^{-1} - \epsilon_k^{-1} \leq c\epsilon_k^{\alpha_0}$. For example, we take $\epsilon_k = k^{-\gamma}$, since, as long as $\gamma \leq \min\{c, \frac{1}{1+\alpha_0}\}$,
\[
\epsilon_{k+1}^{-1} = (k+1)^{-\gamma} \leq k^{-\gamma} + \gamma k^{-1} = \epsilon_k^{-1} + \gamma \epsilon_k^{-1} \leq \epsilon_k^{-1} + c\epsilon_k^{\alpha_0} \text{ and } k^\gamma/(k+1)^{-\gamma} \to 1 \text{ as } k \to \infty,
\]
and thus $\epsilon_{k+1}/\epsilon_k > (2/3)^{1/\alpha_0}$ for $k$ large enough universal.

Then, for $p > \frac{1}{\gamma}$, we find
\[
\sum_k P(\{\omega : \sup_{U_\delta} |u^{\epsilon_k}(x, \omega) - E u^{\epsilon_k}(x)| > M_p \epsilon_k^{\alpha_0}\}) \leq \sum_k k^{-\gamma p} < +\infty,
\]
which, by the Borel-Cantelli lemma, yields
\[
P(\{ \omega : \sup_{U_\delta} |u^{\varepsilon_k}(x, \omega) - \mathbb{E}u^{\varepsilon_k}(x)| > M_p\varepsilon_\alpha^k \text{ for infinitely many } k \} ) = 0,
\]
and, by the previous estimates,
\[
P(\{ \omega : \sup_{U_\delta} |u^{\varepsilon_k}(x, \omega) - \mathbb{E}u^{\varepsilon_k}(x)| > 3M_p\varepsilon_\alpha^k \text{ for arbitrarily small } \varepsilon \} ) \leq
\]
\[
P(\{ \omega : \sup_{U_\delta} |u^{\varepsilon_k}(x, \omega) - \mathbb{E}u^{\varepsilon_k}(x)| > M_p\varepsilon_\alpha^k \text{ for infinitely many } k \} ) = 0.
\]
Note that, in fact, we have now proven the following stronger result: There exists an almost surely positive random variable \( \varepsilon_0 : \Omega \rightarrow \mathbb{R}_+ \) such that
\[
\sup_{U_\delta} |u^{\varepsilon}(x, \omega) - \mathbb{E}u^{\varepsilon}(x)| \leq 3M_p\varepsilon_\alpha^k \text{ for } \varepsilon \leq \varepsilon_0(\omega).
\]
\square

5. Neumann Problem

Here we prove the results about the Neumann problem with random oscillatory data, that is the existence of the ergodic constant (Theorem 2.18 and Theorem 2.19), its continuity (Theorem 2.20) and the homogenization in general domains (Theorem 2.21). The methods are very similar to those used for the Dirichlet problem, but there are a few differences. Where the proofs parallel or the same as the Dirichlet case, we simply outline the arguments or refer to the previous sections.

We note that throughout the section we take \( \beta(\lambda, \Lambda) = \frac{d}{2}(d-1) - 1 \) to be the homogeneity exponent of the fundamental solution of the Neumann problem (2.31).

The Discretized Cell Problem. Similarly to the Dirichlet cell problem, we consider the solution of the Neumann cell problem (1.8) as a function of the boundary data and prove Lipschitz estimates in the appropriate norms and for discretized data.

For simplicity we take again \( \nu = \epsilon_d \), and, for \( X \in \Xi = \mathcal{C}(\overline{Q})^{d-1} \), let \( u_R(y) = u_R(y; X) \) be the solution of
\[
\begin{cases}
F(D_y^2u_R) = 0 & \text{in } \Pi^{2R}_{\epsilon_d}, \\
\partial_{\epsilon_d}u_R = \sum_{i \in \mathbb{Z}^{d-1}} X_i(-i)X_{i+Q} & \text{on } \partial P_{\epsilon_d}, \\
u R = 0 & \text{on } \partial P_{\epsilon_d} + 2Re_d;
\end{cases}
\]
(5.1)
notice that since, in general, (5.1) does not have a unique solution due to the discontinuity of the boundary data, here we choose a unique \( u_R \), using an approximation procedure similar to the one in Section 2, so that it satisfies a comparison principle.

With the \( u_R \) as above, let \( f_R : \Xi \rightarrow \mathbb{R} \) be given by
\[
f_R(X) := \frac{1}{R}u_R(Re_d; X).
\]
(5.2)
In the next lemma, which similar to Lemma 3.1, we discuss the continuity properties of \( f_R \).

Lemma 5.1. Assume (2.10) and (2.11) and let \( \beta \) as in (2.31). Then \( f_R \) has the following continuity properties for \( X, Y \in \mathcal{C}(\overline{Q})^{d-1} \) and some universal constant \( C := C(d, \lambda, \Lambda) > 0 \):

(i) \[ |f_R(X) - f_R(Y)| \leq |X - Y|_{\ell^\infty}. \]
(5.3)
(ii) \[ |f_R(X) - f_R(Y)| \leq CR^{-(1+\beta)}|X - Y|_{\ell^1}. \]
(5.4)
(iii) \[ |f_R(X) - f_R(Y)| \leq CR^{-(1+\beta)/2}|X - Y|_{\ell^2}. \]
(5.5)
Proof. The first estimate follows from the maximum principle, since the difference $u_R(y; X) - u_R(y, Y)$ can be bounded by the linear profiles with slopes $\pm |X - Y| e^\infty$. The third claim follows from interpolating of (i) and (ii) as in the proof of Lemma 5.1 and thus to conclude we need to prove (ii).

Let 

$$m = \min [-\partial_d \Phi(x + e_d) : x \in Q] > 0$$

where $\Phi$ is given by (3.31), and define the barrier

$$\phi_0(x) = 2m^{-1}\Phi(x + e_d),$$

so that $-\partial_e \phi_0 \geq 1$ on $Q$.

Then for $X, Y \in \ell^1(Z^{d-1}; C(Q))$ we sum over the sites where $X_j \neq Y_j$,

$$\phi(x) = \sum_{j \in Z^{d-1}} (\sup_Q |X_j - Y_j|) \phi_0(x - j).$$

Then, as in the proof of Lemma 5.1,

$$F(D^2 u_R(; X) + D^2 \phi) \geq F(D^2 u_R(; Y) - P_{\Lambda}(D^2 \phi) = 0 \text{ in } P_{\Lambda}^R,$$

and, since $\phi > 0$,

$$u_R(; X) + \phi \geq u_R(; Y) \text{ on } \partial P_{\Lambda} + 2Re_d.$$

The comparison principle then yields $u_R(; Y) \leq u_R(; X) + \phi$ in $P_{\Lambda}^R$, and, in particular, for some universal $C > 0$,

$$u_R(Re_d; Y) \leq u_R(Re_d; X) + \phi(Re_d) \leq u_R(Re_d; X) + m^{-1}C \sum_{j \in Z^{d-1}} (|j|^2 + R^2)^{-\beta/2} \sup_Q |X_j - Y_j|,$$

and the desired estimate for $f_R = R^{-1}u_R$ follows. \hfill \Box

Solving the Cell Problem. We use the above estimates to solve the Neumann cell problem.

As before we take $\nu = e_d$, assume $|\psi| \leq 1$ almost surely and consider the solution $v_{e_d, R}$ to the cell problem,

$$\begin{cases} 
F(D^2 v_{e_d, R}(\cdot, \omega)) = 0 & \text{in } P_{\Lambda}^{2R}, \\
v_{e_d, R}(\cdot, \omega) = 0 & \text{on } \partial P_{\Lambda} + 2Re_d, \\
\partial_d v_{e_d, R}(\cdot, \omega) = \psi(\cdot, \omega) & \text{on } \partial P_{\Lambda}.
\end{cases} \quad (5.6)$$

We transform to the setting of the previous subsection by considering $\Psi : \Omega \to \Xi$ defined, for each $j \in Z^{d-1}$ as an element of $C(Q)$,

$$\Psi_j(\omega)(\cdot) := \psi(x + \cdot, \omega),$$

and using the uniqueness of solutions we can identify,

$$v_{e_d, R}(\cdot, \omega) = u(\cdot, \Psi).$$

Next we sketch the argument leading to the concentration inequalities as it was described in more detail before. Since the $u(Re_d; \cdot)$ is Lipschitz on $\Xi$ and, in view of (2.18), $\Psi$ is a $\phi$-mixing random field on $Z^{d-1}$ with

$$\sum_{j \in Z^{d-1}} \phi_\Psi(|j|^{1/2} \lesssim \rho,$$

we apply Theorem 2.1 or Theorem 2.2 to get concentration of $R^{-1}u(Re_d; \cdot)$ and, hence, of $R^{-1}v(Re_d, \cdot)$ as well about their means.
In the convex case, we have
\[ \mathbb{P}(\{\omega : R^{-1}|v_{e_d}(R\omega) - E v_{e_d}(R\omega)| \geq t\}) \leq C \exp\left(-cR^{\tilde{\beta}/2}t^2\right) \text{ for all } t > 0, \]
while in the non-convex case,
\[ \mathbb{P}(\{\omega : R^{-1}|v_{e_d}(R\omega) - E v_{e_d}(R\omega)| \geq t\}) \leq C \exp\left(-cR^{2\tilde{\beta}-(d-1)/2}t^2\right) \text{ for all } t > 0. \]

Define
\[ \tilde{\beta} := \frac{\lambda}{\Lambda}(d-1) \text{ if } F \text{ is convex or concave and, otherwise, } \tilde{\beta} := 2(\frac{\lambda}{\Lambda} - \frac{1}{2})(d-1), \quad (5.7) \]
and note that, if we assume \( \frac{\lambda}{\Lambda} > 1/2 \), then \( \tilde{\beta} > 0 \) in the non-convex case; this corresponds to (3.11) for the Dirichlet problem.

Following the arguments in the Dirichlet case, we prove Theorem 2.18 and Theorem 2.19

Proof of Theorem 2.18 and Theorem 2.19 For \( N \in 2^\mathbb{N} \), a large universal constant \( A = A(d, \lambda, \Lambda) > 1 \) and \( k \in \mathbb{Z}^d \) we consider the events
\[ E_k^N = \{\omega \in \Omega : |N^{-1} v_N(N\nu + N^{1-\tilde{\beta}/2}k, \omega) - \mu_N| \geq A^{1/2}(\log N)^{1/2} N^{-\tilde{\beta}/2}\}. \quad (5.8) \]
It follows from the stationarity and either Theorem 2.1 or 2.2 that
\[ \mathbb{P}(E_k^N) = \mathbb{P}(E_0^N) = \mathbb{P}(\{\omega : |N^{-1} v_N(N\nu, \omega) - \mu_N| \geq A^{1/2}(\log N)^{1/2} N^{-\tilde{\beta}/2}\}) \leq C \exp(-cA \log N). \]

If \( E_{\geq M} = \bigcup_{N \geq M} E_N \), a simple union bound yields
\[ \mathbb{P}(E_N) \leq C N^{3\tilde{\beta}(d-1)/4} \exp(-cA \log N). \]

Let \( E_{\geq M} = \bigcup_{N \geq M} E_N \). It follows that, for some large \( M \in 2^\mathbb{N} \) and as long as \( A > 3\tilde{\beta}(d-1)/(4c) \),
\[ \mathbb{P}(E_{\geq M}) \leq C \sum_{N \geq M} N^{3\tilde{\beta}(d-1)/4 - cA} < +\infty. \]

Then, for \( M \) sufficiently large in a universal way, \( P(\Omega \setminus E_{\geq M}) > 0 \), and, on \( \Omega \setminus E_{\geq M} \),
\[ |N^{-1} v_N(N\nu + N^{1-\tilde{\beta}/2}k, \omega) - \mu_N| \lesssim (\log N)^{1/2} N^{-\tilde{\beta}/2} \text{ for all } N \geq M \text{ and } |k| \leq N^{3\tilde{\beta}/4}. \]

We work with \( \omega \in \Omega \setminus E_{\geq M} \). It follows from the interior oscillation decay estimates (Lemma 2.12) that, for every \( N \geq M \) and \( y' \in \partial P_\nu \cap B_{N^{1+\tilde{\beta}/4}} \),
\[ |N^{-1} v_N(N\nu + y', \omega) - \mu_N| \lesssim N^{1-\tilde{\beta}/2} N^{-1} + (\log N)^{1/2} N^{-\tilde{\beta}/2} \lesssim (\log N)^{1/2} N^{-\tilde{\beta}/2}, \]
while the localization estimate in Lemma 2.10 gives, for \( 0 \leq t \leq N \),
\[ |v_N(t\nu, \omega) - \frac{1}{2} \mu_N(2N - t)| \lesssim (\log N)^{1/2} N^{1-\tilde{\beta}/2} + N N^{2(1-(1+\tilde{\beta}/4))} \lesssim (\log N)^{1/2} N^{1-\tilde{\beta}/2}. \quad (5.9) \]

Next we use again a localization estimate to compare \( v_N \) with \( v_2N \) in their common domain. To this end, notice that \( 2N\mu_{2N} + v_N \) and \( v_{2N} \) have the same Neumann boundary data on \( \partial P_\nu \), while, since \( v_N(y' + 2N\nu, \omega) = 0 \) for \( y' \in \partial P_\nu \),
\[ |2N\mu_{2N} + v_N(y' + 2N\nu, \omega) - v_{2N}(y' + 2N\nu, \omega)| \lesssim (\log N)^{1/2} N^{1-\tilde{\beta}/2} \text{ on } (P_\nu + 2N\nu) \cap \overline{B_{N^{1+\tilde{\beta}/4}}}. \]
It then follows from Lemma 2.10 that
\[
|2N\mu_{2N} + v_N(N\nu,\omega) - v_{2N}(N\nu,\omega)| \leq \sup_{|y'| \leq N^{1+\beta/2}} |2N\mu_{2N} + v_N(y' + 2N\nu,\omega) - v_{2N}(y' + 2N\nu,\omega)| + N^{-\beta/2}
\]
\[
\lesssim (\log N)^{1/2}N^{1-\beta/2},
\]
(5.10)
Combining the previous estimates, using (5.10) and (5.9) for \(v_{2N}\) to estimate the three terms below and with the choice of \(\omega\) above, for every \(N, L \geq M\) in \(2^\mathbb{N}\), we get
\[
|\mu_N - \mu_{2N}| \leq |N^{-1}v_N(N\nu,\omega) - \mu_N| + |2\mu_{2N} + N^{-1}v_N(N\nu,\omega) - N^{-1}v_{2N}(N\nu,\omega)|
\]
\[
+ 2|(2N)^{-1}v_{2N}(N\nu,\omega) - \frac{1}{2}\mu_{2N}| \lesssim (\log N)^{1/2}N^{-\beta/2}.
\]
Therefore, for every \(N, L \geq M\) in \(2^\mathbb{N}\),
\[
|\mu_N - \mu_L| \lesssim \sum_{K \geq M} (\log K)^{1/2}K^{-\beta/2} \lesssim (\log M)^{1/2}M^{-\beta/2}.
\]
It follows that \((\mu_N)_{2^\mathbb{N}}\) is a Cauchy sequence and, therefore, has a limit \(\mu\).
The extension to an estimate of \(R^{-1}E^R(R\nu) - \mu\), for all \(R > 1\), is omitted as it is just a combination of the above arguments with the ideas from the Dirichlet case.
For the proof for general domains, we need the following spatially uniform concentration estimate. Since the notation and proof parallel that of Lemma 3.2 we omit the details.

**Lemma 5.2.** Let \(v_R\) be as given above. Then, for any \(t > 0\),
\[
\mathbb{P}(\{\omega : \sup_{\mathbb{R}^d \cap \{|y'| \leq 3R\}} R^{-1}|v_R(y,\omega) - \mathbb{E}v_R(y)| > t\}) \leq CR^{d\beta/2}\exp(-CR^{\beta}t^2),
\]

**The continuity of the homogenized boundary condition.** We sketch here the proof of Theorem 2.20.

**Proof.** The first assertion is a direct consequence of the comparison principle. To prove the second, we first assume, without any loss of generality, that \(|\psi| \leq 1\).
Fix \(v_1, v_2 \in S^{d-1}\) and let \(v_1, v_2\) and \(\mu_1, \mu_2\) be respectively the solutions to the corresponding Neumann cell problems and the associated ergodic constants.
Similarly to the proof of Lemma 2.11, we define
\[
E = \{y \in \mathbb{R}^d : L|v_1 - v_2| < y \cdot v_1 < 2R - L|v_1 - v_2|, \quad |y - (y \cdot v_1)v_1| \leq L\} \subset P_{v_1} \cap P_{v_2},
\]
and using the up to the boundary \(C^{1,\alpha}\) regularity of the \(v_1\) and \(v_2\) (Lemma 2.11), we find that, for every \(\omega\),
\[
\sup_{y \in \partial E \cap \{y \cdot v=L|v_1-v_2|\}} |\partial_{v_1} v_1(y,\omega) - \psi(y,\omega)| \lesssim L^\alpha |v_1 - v_2|^\alpha
\]
and
\[
|\partial_{v_1} v_1 - \partial_{v_2} v_2| \leq \sup |Dv_1||v_1 - v_2| \leq C||\psi||_{C^{1,\alpha}}|v_1 - v_2|.
\]
Therefore, for \(i = 1, 2\), we have
\[
\sup_{y \in \partial E \cap \{y \cdot v_1=|v_1-v_2|\}} |\partial_{v_1} v_1(y,\omega) - \psi(y,\omega)| \leq C(||\psi||_{C^{1,\alpha}}|v_1 - v_2| + L^\alpha |v_1 - v_2|^\alpha),
\]
and, moreover, since \(|\psi| \leq 1\), \(|v_i| \leq C|v_i - v_\psi|L\) on \(\{y \in \mathbb{R}^d : y \cdot v_1 = 2R - L|v_1 - v_2|\}\) and \(|v_i| \leq 2R\) in \(E\).
Finally the localization Lemma 2.10 that, for $|y'| \leq R$,
\[
\frac{1}{R}|v_1(y, \omega) - v_2(y, \omega)| \leq C(|\psi|_{C^\infty} |v_1 - v_2| + L^\alpha |v_1 - v_2| + CR^{-1}L|v_1 - v_2| + R^2L^{-2}).
\]
From here the proof follows that of Lemma 2.11. Without loss we can assume that $L|v_1 - v_2| \leq 1$, since otherwise $\frac{1}{R}|v_1(y, \omega) - v_2(y, \omega)| \leq 2$ is a better bound.

Using this observation to consolidate terms and the cell problem homogenization result after taking expectations on both sides, we obtain
\[
|\mu_1 - \mu_2| \leq C(1 + |\psi|_{C^\infty})L^\alpha |v_1 - v_2| + R^2L^{-2} + (\log R)^{1/2}R^{-\beta/2}.
\]
Choosing $R, L$ in terms of $|v_1 - v_2|$ to optimize the bound above gives the desired result. □

**The homogenization in general domains.** The proof and statement of Theorem 2.21 are a bit easier than that of Theorem 2.16. In contrast to the Dirichlet case the convergence rate of $u^\varepsilon$ to $\overline{u}$ is uniform in $U$. In particular the boundary layer, represented by the parameter $R$ in Theorem 2.16, does not appear in the Neumann setting.

In spite of this difference, the proof of Theorem 2.21 parallels the one of Theorem 2.16 and consists of two main steps, namely approximating $u^\varepsilon$ in the general domain with the solution in half-space and in a local neighborhood of the “base points”, and then using the results on the half-space solutions to get the homogenization in each neighborhood.

We do not calculate the optimal convergence rate allowed by the method in this case. It will be evident from what follows that a more careful analysis, as in the case of the Dirichlet problem, will give the full statement of Theorem 2.21 with explicit exponents.

The goal is to show that, if $u^\varepsilon$ and $\overline{u}$ are respectively the solution of the general domain Neumann problem 1.2 and the homogenized equation 1.4 with boundary data as in the statement of Theorem 2.21 then, for every $p > 0$ and any $k < k := \min(\frac{2}{3}, \frac{\alpha}{\alpha + 1}, \frac{\beta}{\beta + 1})$, with $\alpha < \alpha'(\beta, \lambda, \Lambda)$ from Lemma 2.20 there exists $C$, which depends on universal constants $p$ and $k'$, such that,
\[
P\left(\sup_{x \in U \setminus \Omega} |u^\varepsilon(x, \omega) - \overline{u}(x)| > \varepsilon^{k'}\right) \leq C\varepsilon^p. \tag{5.11}
\]

**Proof of Theorem 2.21.** Fix $\varepsilon > 0$ and $t > 0$, select a set $\Gamma_{2/3} \subset \partial U$ of at most $C\varepsilon^{-(2/3)(d-1)}$ boundary points such that every $\varepsilon^{2/3}$-neighborhood of a point on $\partial U$ contains at least one point in $\Gamma_{2/3}$, let
\[
\Omega_{\varepsilon,t} := \bigcup_{x \in \Gamma_{2/3}} E_{\varepsilon}^t(\nu_x), \tag{5.12}
\]
where
\[
E_{\varepsilon}^t(\nu) := \{\omega : R^{-1} \sup_{y \in U^\varepsilon \cap \{y' \leq \varepsilon R\}} |v_{R,\nu}(y) - \mathbb{E}v_{R,\nu}(y)| > t\} \text{ with } R = R_{\varepsilon} = \varepsilon^{-1/3}.
\]
and note that, in view of Lemma 5.2
\[
P(\Omega_{\varepsilon,t}) \leq C\varepsilon^{-(2/3)(d-1)+d\beta/6} \exp(-c\varepsilon^{-\beta/3}t^2). \tag{5.13}
\]
Choose $t_{\varepsilon} = c_0\varepsilon^{k'}$, with $k' < k$ and a universal $c_0$ to be chosen small. In particular $k' < k \leq \beta/6$ and, thus,
\[
P(\Omega_{\varepsilon}) \leq \varepsilon^p \text{ for every } p < \infty \text{ as } \varepsilon \to 0. \tag{5.14}
\]
We prove (5.11) by showing that, for $\omega \in \Omega \setminus \Omega_{\varepsilon}$,
\[
\phi^- \leq u^\varepsilon(\cdot, \omega) \leq \phi^+,
\]
where \( \phi^\pm \) solve (1.2) with the modified Neumann boundary data \( \mu(g(x, \cdot), F, \nu_x) \pm c_k \varepsilon^{k'} \) for \( k' < k \) given in the statement of the theorem. Here \( c_k \) can be chosen universally small so that \( \phi^+ - \phi^- \leq \varepsilon^{k'} \), since \( \phi^+ - \phi^- \leq \varepsilon^{k'} \), where \( h \) is the solution of \( \frac{\partial^2 u}{\partial x^2} = 0 \) in \( U \setminus K, \ h = 0 \) on \( \partial K \) and Neumann data identically 1 on \( \partial U \), and, therefore, has a universal upper bound.

The concentration estimate (5.11) then follows since, for \( \omega \in \Omega \setminus \Omega_\varepsilon \),

\[
\sup_{x \in U \setminus K} |u^\varepsilon(x, \omega) - \mathbb{E}(x)| \leq \sup_{x \in U \setminus K} |\phi^+ - \phi^-| \leq \varepsilon^{k'},
\]

or, in other words,

\[
\mathbb{P}(\{\omega : \sup_{x \in U \setminus K} |u^\varepsilon(x, \omega) - \mathbb{E}(x)| > \varepsilon^{k'}\}) \leq \mathbb{P}(\Omega_\varepsilon) \lesssim \varepsilon^p.
\]

Below we only prove that \( u^\varepsilon \leq \phi^+ \), since the proof of \( u^\varepsilon \geq \phi^- \) is similar. We argue by contradiction observing that, if not, then \( m := \max_U (u^\varepsilon - \phi^+) > 0 \). By the maximum principle the maximum must be attained at a boundary point \( x_0 \in \partial U \), which must belong to the \( \varepsilon^{2/3} \) neighborhood of one of “grid points” \( x_0 \in \Gamma_{\varepsilon^{2/3}} \) on \( \partial U \).

Let \( \nu = \nu_0 \) and, for \( z_0 = x_0 + \varepsilon^{2/3} \nu \) and \( z' := z - (z \cdot \nu) \nu \),

\[
T(x) := \phi(z_0) + D\phi(z_0) \cdot (x - z_0).
\]

Note that, in view of Lemma 2.20 and the fact that \( g(\cdot, y, \omega) \in C^{0,1}(\mathbb{R}^d) \), for any \( \alpha < \alpha'(\hat{\beta}) \) from Lemma 2.20 we have

\[
g(x) = \mu(g(x, \cdot), F, \nu_x) \in C^{0,\alpha}(\partial U).
\]

Then Lemma 2.11 yields that \( \phi^+ \in C^{1,\alpha}(\bar{U} \setminus K) \), and, in particular,

\[
|\phi^+(x) - T(x)| \leq C|(x - z_0) \cdot \nu| + C|(x - z_0)'|^{1+\alpha} \quad \text{in } U \setminus K.
\]

Let \( U^\varepsilon := \{(x - x_0) \cdot \nu \leq \varepsilon^{2/3}\} \cap U \) and fix \( L > 1 \) to be chosen later in terms of \( \varepsilon \) and consider the solution \( w_\varepsilon \) to

\[
\begin{align*}
F(D^2 w_\varepsilon) &= 0 & \text{in } U^\varepsilon \cap B_{L\varepsilon^{2/3}}, \\
\partial_\nu w_\varepsilon(\cdot, \omega) &= g(x_0, \cdot, \varepsilon, \omega) & \text{on } \partial U \cap B_{L\varepsilon^{2/3}}(x_0), \\
w_\varepsilon(\cdot, \omega) &= 0 & \text{on } (\partial P_\nu + z_0) \cap B_{L\varepsilon^{2/3}}(x_0), \\
w_\varepsilon(\cdot, \omega) &= 0 & \text{on } U^\varepsilon \cap \partial B_{L\varepsilon^{2/3}}(x_0).
\end{align*}
\]

It then follows from (5.15) and the zero Dirichlet condition for \( w_\varepsilon \) that

\[
u^\varepsilon \leq m + \phi^+ \leq m + T + w_\varepsilon + CL^{1+\alpha}\varepsilon^{2/3(1+\alpha)} \quad \text{on } (\partial P_\nu + z_0) \cap U,
\]

and, similarly,

\[
u^\varepsilon \leq m + T + w_\varepsilon + C\varepsilon^{2/3} \quad \text{on } U^\varepsilon \cap \partial B_{L\varepsilon^{2/3}}(x_0).
\]

Choose \( L \) so that \( L^{1+\alpha}\varepsilon^{2/3(1+\alpha)} \leq \varepsilon^{2/3} \) holds and observe that, in view of the continuity of \( g \),

\[
\sup_{x \in B_{L\varepsilon^{2/3}}} |g(x, x/\varepsilon, \omega) - g(x_0, x/\varepsilon, \omega)| \leq C\varepsilon^{2/3}.
\]

Arguing as in Lemma 2.11 we estimate the difference of \( u^\varepsilon \) and \( m + T(x) + w_\varepsilon(x) \) using a rotated version of the barrier

\[
\varphi(x) = CL^{1+\alpha}\varepsilon^{2/3(1+\alpha)} + CL\varepsilon^{2/3}(\varepsilon^{2/3} - x_d) + CL^{-2}\varepsilon^{-4/3}\varepsilon^{2/3}(|x'|^2 + (d - 1) \frac{\Lambda}{\varepsilon}(1 - x_d^2)),
\]

and we get

\[
u^\varepsilon \leq m + T + w_\varepsilon + CL^{1+\alpha}\varepsilon^{2/3(1+\alpha)} + L^{-2}\varepsilon^{2/3} + L\varepsilon^{4/3} \quad \text{on } U \cap B_{L\varepsilon^{2/3}}(x_0).
\]
Choosing $L = \varepsilon^{-\frac{2\alpha}{9+3\alpha}}$ and, hence, $L\varepsilon = \varepsilon^{\frac{2\alpha}{9+3\alpha}}$ gives
\[ u^\varepsilon \leq m + T + w_\varepsilon + C\varepsilon^{\frac{2}{3} + k} \quad \text{on} \quad U \cap B_{\varepsilon^{2/3}}(x_0), \] (5.16)
since $k \leq \frac{3\alpha}{3+\alpha}$.

Next we compare a rescaled version of $w_\varepsilon$ and the solution $v_R = v_{R,\nu}$ of the cell problem (1.8) with boundary data
\[ \psi(y, \omega) := g(x_0, y, \tau_{x_0}/\varepsilon \omega). \]
Since, in view of (2.7), for $c$ sufficiently small depending on $C^2$-regularity of $\partial U$,
\[ d(\partial U - x_0, \partial P_\nu \cap B_{\varepsilon^{\frac{2}{3} + \alpha}}(x_0)) \leq \varepsilon^{\frac{2\alpha}{9+3\alpha}}, \]
it follows from the up to the boundary Hölder regularity of $v_R$ and $w_\varepsilon$, Lemma 2.11 that
\[ |\partial_\nu w_\varepsilon(x_0 + \cdot, \omega) - \partial_\nu v_R(\cdot, \omega)| \leq C\varepsilon^{\frac{2}{3}} \quad \text{on} \quad (\partial P_\nu + \varepsilon^{-\frac{2}{3}}\nu) \cap B_{\varepsilon^{2/3}}. \] (5.17)

Now we rescale to
\[ \tilde{w}_\varepsilon(y, \omega) := \varepsilon^{-1} w_\varepsilon(x_1 + \varepsilon y, \omega), \]
and observe that, due to the facts that $|g| \leq 1$, $R = \varepsilon^{-1/3}$ and (5.17), $h := \tilde{w}_\varepsilon - v_R$ solves
\[
\begin{align*}
-P_{\lambda,\Lambda}^+ (D^2 h) &\leq 0 \quad \text{in} \quad \Pi^R \cap B_{R \frac{3(1+\alpha)}{3+\alpha}}, \\
|\partial_\nu h| &\leq C\varepsilon^{\frac{2}{3} + \alpha} \quad \text{on} \quad \partial P_\nu \cap B_{R \frac{3(1+\alpha)}{3+\alpha}}, \\
h &= 0 \quad \text{on} \quad (\partial P_\nu + R\nu) \cap B_{R \frac{3(1+\alpha)}{3+\alpha}}, \\
h(x) &\leq 2R \quad \text{on} \quad \partial B_{R \frac{3(1+\alpha)}{3+\alpha}} \cap \Pi^R_{\nu}.
\end{align*}
\]
Using a rotated version of the barrier
\[ \varphi(x) = C\varepsilon^{\frac{2\alpha}{3+\alpha}} (R - x_d) + 2R^{1 - \frac{3(1+\alpha)}{3+\alpha}} \left( (|x'|^2 - (d-1)\alpha((x_d)^2 - R^2)) \right), \]
we conclude that, for some $C > 0$ which is independent of $\varepsilon$ and $x_0$,
\[ |\tilde{w}_\varepsilon - v_R| \leq CR(\varepsilon^{\frac{2\alpha}{3+\alpha}} + R^{-\frac{2\alpha}{3+\alpha}}) = CR(\varepsilon^{\frac{2\alpha}{3+\alpha}} + \varepsilon^{\frac{2\alpha}{3+\alpha}}) \leq CR\varepsilon^{\alpha} \quad \text{in} \quad U \cap P_\nu \cap B_R. \] (5.18)

Note that, since $\omega \notin \Omega_\varepsilon$, the definition of $\Omega_\varepsilon$ yields that, for $\mu = \mu(g(x_0, \cdot), F, \nu),
\[ |R^{-1} v_R(\cdot, \omega) + \mu(\frac{y}{R}, \nu - 1)| \leq c_0\varepsilon^{k'} \quad \text{in} \quad \Pi^R \cap \{|y'| \leq 3R\}. \] (5.19)

Rewriting (5.18) and (5.19) in terms of the original variable yields that, for $\varepsilon$ sufficiently small depending on $k - k'$ and $c_0$,
\[ \varepsilon^{-2/3}|w_\varepsilon + \mu((-x_0) \cdot \nu - \varepsilon^{2/3})| \leq 3c_0\varepsilon^{k'} \quad \text{in} \quad U \cap B_{\varepsilon^{2/3}}(x_0). \] (5.20)

Finally, combining (5.16) and (5.20), we obtain, again for $\varepsilon$ sufficiently small, the estimate
\[ u^\varepsilon \leq m + T - \mu((-x_1) \cdot \nu - \varepsilon^{2/3}) + 4c_0\varepsilon^{2/3+k'} \quad \text{in} \quad U \cap B_{\varepsilon^{2/3}}. \]

Recall that $-\partial_\nu \phi^+ \geq \mu + c_1 \varepsilon^{k'}$. Then, for sufficiently small $\varepsilon > 0$ and $c_0 < c_1/8$, for $x \in \partial U \cap B_{\varepsilon^{2/3}}(x_0)$ we have
\[
\begin{align*}
m + \phi^+(x) &\geq m + T(x) - \mu(x \cdot \nu - \varepsilon^{2/3}) + c_1 \varepsilon^{2/3+k'} - C\varepsilon^{2/3(1+\alpha)} \\
&\geq m + T(x) - \mu(x \cdot \nu - \varepsilon^{2/3}) + \frac{1}{2}c_1 \varepsilon^{2/3+k'} \\
&> u^\varepsilon(x),
\end{align*}
\]
which is a contradiction.
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Department of Mathematics, UCLA, Los Angeles, CA 90024, USA

E-mail address: wfeldman10@math.ucla.edu, ikim@math.ucla.edu

Department of Mathematics, The University of Chicago, Chicago, IL 60637, USA

E-mail address: souganidis@math.uchicago.edu