A NOTE ON THE STRONG FELLER PROPERTY OF DIFFUSION PROCESSES

TIMUR YASTRZHEMBSKIY

Abstract. In this note we prove the strong Feller property of a strong Markov quasi diffusion process corresponding to an elliptic operator with merely bounded measurable coefficients. We also prove Hölder continuity of harmonic functions associated with the quasi diffusion process and Harnack inequality. As an application, we show that for such diffusion processes the probabilistic definition of a regular boundary point coincides with the 'analytic' one. The parabolic counterparts of these results are presented as well. The proofs are adaptations of arguments from [8] and [11].

1. Introduction and Main Results

Let \( d \geq 1 \), \( \mathbb{R}^d \) be the Euclidean space of points \( x = (x^1, \ldots, x^d) \). Let \( \mathcal{B}^d \) be the Borel sigma-algebra on \( \mathbb{R}^d \). For \( A \in \mathcal{B}^d \) let \( \partial A \) be its boundary, \( \bar{A} \) be its closure, and \( |A| \) be its Lebesgue measure. For \( r > 0 \) and \( x \in \mathbb{R}^d \) denote
\[
B_r = \{ y \in \mathbb{R}^d : |y| < r \}, \quad B_r(x) = x + B_r.
\]

For an open set \( G \) we denote by \( B(G) \) the set of bounded Borel measurable functions. Let \( C(\mathbb{R}^d) \) be the space of bounded continuous functions on \( \mathbb{R}^d \), and \( C^2_0(\mathbb{R}^d) \) be the space of twice continuously differentiable functions with compact support.

Let \( \nu, K > 0 \) be numbers and \( a(t, x) = (a^{ij}(t, x), i, j = 1, \ldots, d), b(t, x) = (b^i(t, x), i = 1, \ldots, d) \) be Borel measurable functions such that, for every \((t, x) \in \mathbb{R}^{d+1}, \xi \in \mathbb{R}^d, i, j \)
\[
\nu|\xi|^2 \leq a^{kl}(t, x)\xi^k\xi^l \leq \nu^{-1}|\xi|^2, \quad a^{ij} \equiv a^{ji}, \quad |b^i(t, x)| < K. \tag{1.1}
\]

Denote
\[
L = a^{ij}D_{x^i x^j} + b^i D_{x^i}, \tag{1.2}
\]
and let \( L_{\nu, K} \) be the set of all such operators with coefficients satisfying (1.1). By \( N(\cdot) \) we denote a positive constant depending only on the parameters listed inside the parenthesis.

Results in the elliptic case. First, we consider the case when \( a \) and \( b \) are independent of \( t \).

Let \( \Omega \) be the set of all continuous \( \mathbb{R}^d \)-valued functions \( x \), \( N_t^\nu = \sigma(x_\eta, \eta \in [0, t]), N_\infty = \sigma(x_t, t \geq 0) \). Let \( X = (x_t, \infty, N_t, P_x) \) be a Markov process in
the terminology of [3]. We say that $X$ is a quasi diffusion process corresponding to $L$ and denote it by $X(L)$ if for any $\phi \in C^2_0(\mathbb{R}^d), t \in, x \in \mathbb{R}^d,$

$$\phi(x) = E_x \phi(x_t) - E_x \int_0^t L \phi(x_\eta) \, d\eta.$$ (1.3)

Here by $E_x$ we mean the integral over $\Omega$ with respect to $P_x$. Note that we may replace (1.3) by the following condition:

$$\zeta_t = \phi(x_t) - \phi(x) - \int_0^t L \phi(x_\eta) \, d\eta$$ (1.4)

is a centered martingale relative to $(\mathcal{N}_t, t \geq 0)$ on the probability space $(\Omega, \mathcal{N}_\infty, P_x)$. The fact that $\zeta_t$ is a martingale with zero mean implies (1.4). The other implication follows from the Markov property.

It is a classical fact that there exists a strong Markov process $X(L)$. This result was first proved by N.V. Krylov in [5]. A different proof was later given by R.F. Bass in [2]. It is shown in [14, 15] that such strong Markov process is generally not unique. However, under certain regularity assumptions on the leading coefficients, uniqueness does hold. An interested reader can find some classical results in [16, 2] and some recent developments in [12]. Strong Markov quasi diffusion processes corresponding to elliptic operators with discontinuous coefficients arise naturally in stochastic optimal control as Markov processes controlled by an optimal Markov policy (see Remark 1.1 of [6]).

**Definition 1.1.** We say that a quasi diffusion process $X(L)$ is strong Feller if for any number $t > 0$ and $f \in B(\mathbb{R}^d)$, the function $T_t f(x) := E_x f(x_t)$ is continuous on $\mathbb{R}^d$.

In the sequel, the process $X(L)$ is strong Markov.

**Theorem 1.1.** For any $f \in B(\mathbb{R}^d), r \in (0,1], t > 0, x_0 \in \mathbb{R}^d,$ and $x_i \in B_r(x_0), i = 1, 2$

$$|T_t f(x_1) - T_t f(x_2)| \leq N r^{-\alpha} |x_1 - x_2|^\alpha \sup_{B_{2r}(x_0)} |T_t f|,$$

where $N, \alpha$ depend only on $d, \nu, K$. If $b \equiv 0$, then this conclusion holds for $r > 1$.

To state the corollary of this theorem we denote for $G \subset \mathbb{R}^d$

$$\tau_G = \inf\{t \geq 0 : x_t \not\in G\}, \inf \emptyset = \infty.$$ (1.5)

We call $\tau_G$ the first exit time from $G$. It is well known that if $G$ is an open set, then $\tau_G$ is a stopping time relative to $(\mathcal{N}_t, t \geq 0)$. If, in addition, $G$ is bounded, and $x \in G$ then, $\tau_G < \infty$ $P_x$ a.s. (see, for example, Proposition 1.8.2 of [2]). Similarly, we define the first exit time from $G$ after $+0$ as follows:

$$\tau'_G = \inf\{t > 0 : x_t \not\in G\}.$$
Further, in the terminology of [3] a point \( x \in \partial G \) is called regular if
\[
P_x(\tau'_G = 0) = 1,
\]
and, the same point is called almost regular if
\[
\lim_{G \ni y \to x} E_y g(x_{\tau_G}) = g(x)
\]
for any \( g \in B(\mathbb{R}^d) \) that is continuous at the point \( x \). Further, if the coefficients \( a^{ij} \) are regular enough, say, continuous everywhere, then, \( E_y g(x_{\tau_G}) \) coincides with the Perron solution to the Dirichlet problem
\[
Lu = 0 \text{ a.e. in } G, \quad u = g \text{ on } \partial G.
\]

The following assertions are used in [10] to explain the idea behind the construction of operators with discontinuous coefficients.

**Corollary 1.2.** Let \( G \subset \mathbb{R}^d \) be a bounded domain. Then, the following assertions hold.

(i) The probability \( P_x(\tau'_G = 0) \) is either 0 or 1.

(ii) A boundary point is regular if and only if it is almost regular.

**Proof.** (i) By Theorem 1.1 the process \( X(L) \) is strong Feller. Then, by Theorem 3.5 of [3] \( x_t, N_t, P_x \) is a Markov process. Note that \( \tau'_G \) is an \( N_{0+} \) measurable random variable, and, hence, the claim holds by the zero-one law (see Corollary 1 in Section 3 of Chapter 3 of [3] ).

(ii) Since \( X(L) \) is strong Feller, by Theorem 13.3 of [3] a regular point is almost regular. The other implication is proved in Corollary 1 of [4].

**□**

**Results for harmonic functions.**

**Definition 1.2.** We say that \( u \) is a harmonic function for \( X(L) \) on a bounded domain \( G \) if \( u \in B(G) \), and for any domain \( D \subset G \), and any \( x \in D \), one has
\[
u(x) = E_x u(x_{\tau_D}).
\]

**Example 1.3.** One common example of harmonic functions is an exit distribution for the process \( X(L) \) defined as
\[
\pi_G(x, A) := P_x(x_{\tau_G} \in A \cap \partial G), \quad A \in \mathcal{B}^d.
\]

By the strong Markov property the function \( x \to \pi_G(x, A) \) is harmonic for \( X(L) \) on \( G \). Note that the next two results hold for this function.

**Theorem 1.4.** Let \( r \in (0, 1] \), \( x_0 \in \mathbb{R}^d \), and \( u \) be a harmonic function for \( X(L) \) on \( B_{2r}(x_0) \). Then, for any \( x, y \in B_{r}(x_0) \)
\[
|u(x) - u(y)| \leq N r^{-\alpha} |x - y|^\alpha \sup_{B_{2r}(x_0)} |u|
\]
with \( N, \alpha \) depending only on \( d, \nu, \) and \( K \). If \( b \equiv 0 \), then the conclusion also holds for \( r > 1 \).
Theorem 1.5 (Harnack inequality). Invoke the assumptions of Theorem 1.4 and assume additionally that $u$ is a nonnegative function on $B_{2r}(x_0)$. Then, for any $x \in B_{r/2}(x_0)$ we have

$$u(x_0) \leq N(d, \nu, K) u(x).$$

If $b \equiv 0$, the assertion also holds for $r > 1$.

Remark 1.6. Let us call the assumption that $u$ is harmonic for $X(L)$ on $D = B_r(x_0)$ by (A) and the one that $u \in B(D)$ and $u(x_{t \wedge \tau_D}), t \geq 0$ is a martingale relative to $(\mathcal{N}_t, t \geq 0)$ by $(A')$.

In [2] it is shown that Theorems 1.3 and 1.4 holds with the assumption $(\tilde{A})$ replaced by $(A')$ and the additional constraint $b \equiv 0$ (see Theorems 5.7.5 and 5.7.6 of [2]).

Actually, $(A')$ and (A) are very similar. Indeed, by Doob’s sampling theorem $(A')$ implies (A). Further, assume that (A) holds and let $\bar{G}$ be an open set such that $\bar{G} \subset D$, and $\tau$ be a stopping time. Then, by the strong Markov property

$$u(x) = E_x E_{x_t} u(x_{t \wedge \tau}) I_{t \wedge \tau} + E_x u(x_{t \wedge \tau}) I_{t \wedge \tau} = E_x u(x_{t \wedge \tau}) I_{t \wedge \tau} = E_x u(x_{t \wedge \tau}).$$

Since the equality between the extreme terms hold for any bounded stopping time, $u(x_{t \wedge \tau}), t \geq 0$ is a martingale.

Results in the parabolic case.

Let $\omega_t$ be continuous $\mathbb{R}^{d+1}$-valued function, and $x_t^0$ be its first component, and $x_t$ be the last $d$ components. Denote $\Omega = C([0, \infty), \mathbb{R}^{d+1})$, $\mathcal{N}_t = \sigma(\omega_t, r \in [0, t])$, $\mathcal{N}_\infty = \sigma(\omega_t, t \geq 0)$. Let $Y = (\omega_t, \infty, I_t, P_{s,x})$ be a time-homogeneous Markov process in the terminology of [3], where $s \in \mathbb{R}$, $x \in \mathbb{R}^d$. We say that $Y(D_t + L) := Y$ is a quasi diffusion process corresponding to $D_t + L$ if

$$x_t^0 = s + t, \quad P_{s,x} \text{ a.s.,}$$

and for any $\phi \in C^2_0(\mathbb{R}^{d+1})$

$$\phi(s, x) = E_{s,x} \phi(\omega_t) - E_{s,x} \int_0^t (D_t + L) \phi(\omega_\eta) \, d\eta$$

$$= E_{s,x} \phi(s + t, x_t) - E_{s,x} \int_0^t (D_t + L) \phi(s + t, x_\eta) \, d\eta. \quad (1.6)$$

One can show that a strong Markov quasi diffusion process $Y(D_t + L)$ exists by repeating the argument of [3] and replacing the Alexandrov estimate by its parabolic counterpart (see Section 2.2 of [7]). The existence of such process also follows from a more general result of [1].

In the sequel, we assume that $Y(D_t + L)$ is strong Markov.

Parabolic analogue of the strong Feller property.

For $(t_i, x_i), i = 1, 2$ and $T, r > 0$ we denote

$$\rho((t_1, x_1), (t_2, x_2)) = |t_1 - t_2|^{1/2} + |x_1 - x_2|.$$
Here is the parabolic counterpart of Theorem \[1.1\] This time, however, our result does not imply that $Y(D_t + L)$ is strong Feller (see Remark \[1.8\]).

**Theorem 1.7.** For $T \in \mathbb{R}$, $f \in B(\mathbb{R}^d)$ denote

$$H(s, x) = E_{s,x} f(x_{T-s}), \quad s \in (-\infty, T), \, x \in \mathbb{R}^d. \quad \text{(1.7)}$$

Then, there exist constants $N, \alpha$ depending only on $d, \nu, K$ such that for any $r \in (0, 1], (s_0, x_0) \in \mathbb{R}^{d+1}$, and any $(s_i, x_i) \in Q_r(s_0, x_0)$, $i = 1, 2$,

$$|H(s_1, x_1) - H(s_2, x_2)| \leq N r^{-\alpha} \rho^0((s_1, x_1), (s_2, x_2)) \sup_{Q_{2r}(s_0, x_0)} |H|.$$

**Remark 1.8.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function discontinuous at 1. Then, the function $(s, x) \to E_{s,x} f(\omega_1) = f(s + 1)$ is discontinuous on $\{0\} \times \mathbb{R}^d$, and, then, $Y(D_t + L)$ is not a strong Feller process.

The next theorem states that $Y(D_t + L)$ is a Feller process.

**Theorem 1.9.** For any $f \in C(\mathbb{R}^{d+1})$ and $T > 0$ the function $u(s, x) = E_{s,x} f(s + T, x_T)$ is of class $C(\mathbb{R}^{d+1})$.

To state the next results we introduce some notation. For a set $A \subset \mathbb{R}^{d+1}$ we denote by $\partial_p A$ the parabolic boundary of $A$, that is, a subset of $\partial A$ of all points $(s_0, x_0)$ such that there exists a function $x. \in C([0, \infty), \mathbb{R}^d)$ and a number $\varepsilon \in (0, s_0]$ such that $(t, x_t) \in A$ for all $t \in [s_0 - \varepsilon, s_0)$. In particular,

$$\partial_p Q_r(t, x) = ((t, t + r^2) \times \partial B_r(x)) \cup (\{t + r^2\} \times B_r(x)).$$

For a nonempty open set $Q \subset \mathbb{R}^{d+1}$ and $s \in \mathbb{R}$ we denote

$$\tau(Q) = \inf \{t \geq 0 : \omega_t \notin Q\},$$

$$\tau_s(Q) = \inf \{t \geq 0 : (s + t, x_t) \notin Q\}. \quad \text{(1.8)}$$

We make a few observations. First, $\tau(Q)$ is a stopping time relative to $\mathcal{N}_t, t \geq 0$, and $\tau_s(Q)$ is a stopping time relative to $\mathcal{N}_t, t \geq 0$. Second, for any $s, x$,

$$\tau_s(Q) = \tau(Q) \text{ P}_{s,x} \text{ a.s.}$$

Third, for any $(s, x) \in Q$ we have

$$\tau_s(Q) = \inf \{t \geq 0 : (s + t, x_t) \in \partial_p Q\}.$$

We say that $u$ is a harmonic function for $Y(D_t + L)$ on $Q$ if $u \in B(Q)$ and for any domain $G$ such that $G \subset Q$ and $(s, x) \in G$, one has

$$u(s, x) = E_{s,x} u(\omega_{\tau(G)}) = E_{s,x} u(s + \tau_s(G), x_{\tau_s(G)}).$$

**Example 1.10.** The first example of a harmonic function is an exit distribution given by

$$\pi_Q(s, x, A) = P_{s,x}((s + \tau_s(Q), x_{\tau_s(Q)}) \in A \cap \partial_p Q), \quad A \in \mathcal{B}^{d+1}.$$

Indeed, by the strong Markov property the function $(s, x) \to \pi_Q(s, x, A)$ is harmonic for $Y(D_t + L)$ on $Q$. 
The second example is the function $H(s, x)$ given by (1.7). First, we prove that $H(s, x)$ is a measurable function. It suffices to show that a function $h(s, x, t) = E_{s, x} f(x_t)$ is measurable. By the standard approximation argument we may assume that $f$ is bounded and continuous. Due to continuity of $x_t$, $h(s, x, t)$ is a continuous function of $t$. On the other hand, $h(s, x, t)$ is a measurable function of $(s, x)$ because so is the transition function $P_{s, x} (\omega_t \in A), A \in \mathfrak{B}^{d+1}$. Combining these two facts, we conclude that $h(s, x, t)$ is measurable.

Next, we show that $H$ is harmonic for $Y(D_t + L)$ on $(T_0, T) \times \mathcal{D}$, where $\mathcal{D} \subset \mathbb{R}^d$ is a bounded set. Let $s \in (T_0, T)$. By the strong Markov property for any stopping time $\tau \leq T_2 - s$ relative to $(\hat{N}_t, t \geq 0)$

$$E_{s, x} f(x_{T-s}) = E_{s, x} E_{s+\tau, x, \tau} f(x_{T-s-\tau}) = E_{s, x} H(s + \tau, x_\tau).$$

This implies the validity of the claim.

Most of the theorems stated above will be derived from the next two theorems.

**Theorem 1.11.** Let $r \in (0, 1], (s_0, x_0) \in \mathbb{R}^{d+1}$, and $u$ be a harmonic function for $Y(D_t + L)$ on $Q_{2r}(s_0, x_0)$. Then, there exist constants $N, \alpha$ depending only on $d, \nu, K$ such that for any $(s_i, x_i) \in \mathcal{Q}_r(t_0, x_0), i = 1, 2$

$$|u(s_1, x_1) - u(s_2, x_2)| \leq N r^{-\alpha} \rho^\alpha((s_1, x_1), (s_2, x_2)) \sup_{Q_{2r}(s_0, x_0)} |u|. \quad (1.9)$$

**Theorem 1.12** (Harnack inequality). Let $(t_0, x_0) \in \mathbb{R}^{d+1}, r \in (0, 1], \varepsilon \in (0, 1)$, and $u$ be a nonnegative harmonic function for $Y(D_t + L)$ on $Q_{8r^2}(t_0, x_0)$. Then, there exists a constant $N$ depending only on $d, \nu, K$ and $\varepsilon$ such that

$$u(r^2, x_0) \leq N \inf_{x \in B(2-\varepsilon)r(x_0)} u(t_0, x).$$

If $b \equiv 0$, then the claim holds for $r > 1$.

**Remark 1.13.** In the case when $a$ and $b$ are independent of $t$ and $b \in L_d$ in \[13\] N.V. Krylov proves the existence of a strong Markov quasi diffusion process corresponding to $L$. Further, he shows that this process has strong Feller property (see Theorem 4.12 of \[13\]). In fact, it is shown that $T_t f(x)$ is continuous on $(0, T) \times \mathcal{D}^d$. Further, under the aforementioned conditions, in Section 6 of \[13\] the author proves the appropriate versions of Theorems 1.4, 1.5, 1.11 and 1.12 of the present paper.

The rest of the note is organized as follows. In Section \[2\] we prove some lemmas including a probabilistic version of the Krylov-Safonov estimate. In Section \[3\] we prove Theorem 1.11 first, then we obtain Theorems 1.4, 1.7, and 1.9 as corollaries. In Section \[4\] we prove the parabolic Harnack inequality and derive the elliptic one from it.

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2. Auxiliary results

The following lemma is a version of Theorem 4.5.1 of [16].

**Lemma 2.1.** (i) Let $Y(D_t + L)$ be a quasi diffusion process. Then, for any $(s, x)$ there exists a Wiener process $w_t$ on the probability space $(\Omega, \mathcal{N}_\infty, P_{s,x})$ such that $x$ solves the following stochastic differential equation:

$$x_t = x + \int_0^t b(s + \eta, x_\eta) \, d\eta + \int_0^t \sqrt{2a(s + \eta, x_\eta)} \, dw_\eta$$

(2.1)

on the same probability space.

(ii) Assume that $a$ and $b$ are independent of $t$ and let $X(L)$ be a quasi diffusion process. Then, on $(\Omega, \mathcal{N}_\infty, P_x)$ there exists a Wiener process $w_t$ such that $x$ satisfies

$$x_t = x + \int_0^t b(x_\eta) \, d\eta + \int_0^t \sqrt{2a(x_\eta)} \, dw_\eta.$$

**Proof.** Denote

$$b_\eta = b(s + \eta, x_\eta), \quad a_\eta = a(s + \eta, x_\eta).$$

First, note that by the Markov property, for any $v \in C_0^2(\mathbb{R}^d)$ and (1.6) the process

$$v(x_t) - v(x) - \int_0^t Lv(x_\eta) \, d\eta$$

is a centered martingale relative to $(\mathcal{N}_t, t \geq 0)$ on $(\Omega, \mathcal{N}_\infty, P_{s,x})$. By the standard approximation argument, for $v = x^i$ and $v = x^i x^j$, the above process is a local martingale. Then, by what was just said

$$\xi_t^i := x_t^i - \int_0^t b_{\eta}^i \, d\eta,$$

$$\xi_t^{ij} := x_t^i x_t^j - \int_0^t (b_{\eta}^i x_{\eta}^j + b_{\eta}^j x_{\eta}^i) \, d\eta - 2 \int_0^t a_{\eta}^{ij} \, d\eta$$

are local martingales.

Next, denote

$$\phi_t = \int_0^t b_{\eta} \, d\eta.$$

We claim that

$$\xi_t^{ij} := \xi_t^i \xi_t^j - 2 \int_0^t a_{\eta}^{ij} \, d\eta$$

is a local martingale. To prove this, first, we rewrite $\xi_t^{ij}$ as follows:

$$\xi_t^{ij} = \xi_t^i \xi_t^j + \xi_t^i \phi_t^j + \xi_t^j \phi_t^i + \phi_t^i \phi_t^j - \int_0^t (b_{\eta}^i \xi_t^j + b_{\eta}^j \xi_t^i) \, d\eta - \int_0^t (b_{\eta}^i \phi_t^j + b_{\eta}^j \phi_t^i) \, d\eta - 2 \int_0^t a_{\eta}^{ij} (s + \eta, x_\eta) \, d\eta.$$
Due to integration by parts the sum of the fourth and the sixth terms on the right hand side of the above expression is zero. Further, thanks to Lemma 3.6.15 of [9], the process
\[ \xi_t^i \phi_t^j - \int_0^t \xi_{t \eta}^i b_{\eta}^j \, d\eta \]
is a local martingale. Hence, \( \xi_t^i \) is a local martingale. By this, the fact that \( \xi_t^i \) is a local martingale, and Theorem 3.10.8 of [9] \( w_t = \int_0^t (2a_\eta)^{-1/2} \, d\xi_\eta \) is a Wiener process, and \( P_{s,x} \) a.s. for all \( t \geq 0 \)
\[ \xi_t - x = \int_0^t \sqrt{2a_\eta} \, dw_\eta. \]

**Lemma 2.2.** Let \( (s_0, x_0) \in \mathbb{R}^{d+1}, r > 0 \) be a number. Denote
\[ a^{ij}(t, x) = a^{ij}(s_0 + r^2 t, x_0 + r x), \quad b(t, x) = r b^i(s_0 + r^2 t, x_0 + r x) \]
and let \( \hat{L} \) be an operator given by [12] with \( a \) and \( b \) replaced by \( \hat{a} \) and \( \hat{b} \).

For any open set \( A \subset \mathbb{R}^{d+1} \) we denote
\[ \hat{A} = \{ ((t - s_0)/r^2, (x - x_0)/r) : (t, x) \in A \}. \]
Let \( u \) be a harmonic function for \( Y(D_t + \hat{L}) \) on \( Q_r(s_0, x_0) \) if and only if
\[ \hat{u}(s, x) = u(s_0 + r^2 s, x_0 + r x) \]
is harmonic for some strong Markov process \( Y(D_t + \hat{L}) \) on \( Q_1(0, 0) \).

**Proof.** For any \( \omega = (x^0, x, \ldots) \in \hat{\Omega} \) we denote
\[ \hat{x}_t^0 = r^{-2}(x^0_{r t} - s_0), \quad \hat{x}_t = r^{-1}(x_{r t} - x_0), \quad \hat{\omega}_t = (\hat{x}_t^0, \hat{x}_t), \]
\[ \hat{N}_t = \sigma(\hat{\omega}_{t \eta}, \eta \in [0, t]), \quad \hat{P}_{s,x} = P_{s_0 + r^2 s, x_0 + r x}. \]
It follows that the process \( \hat{X} := (\hat{\omega}_t, \hat{N}_t, \hat{P}_{s,x}) \) is strong Markov. Below we will show that \( \hat{X} \) is a quasi diffusion process corresponding to \( \hat{L} \).

Note that by Lemma 2.1 \( \hat{x}_t \) satisfies the equation
\[ \hat{x}_t = r^{-1}(x - x_0) + r^{-1} \int_0^{r t^2} b(s + \eta, x_{\eta}) \, d\eta + r^{-1} \int_0^{r t^2} \sqrt{2a(s + \eta, x_{\eta})} \, dw_\eta \]
on \( (\Omega, \hat{N}_\infty, \hat{P}_{s,x}) \). Due to the scaling property of a Wiener process \( w_\eta := r^{-1} w_{r^2 \eta} \) is a Wiener process on the same probability space, and
\[ \hat{x}_t = r^{-1}(x - x_0) + r \int_0^t b(s + r^2 \eta, x_{r^2 \eta}) \, d\eta + \int_0^t \sqrt{2a(s + r^2 \eta, x_{r^2 \eta})} \, dw_\eta \]
\[ = r^{-1}(x - x_0) + \int_0^t b(r^{-2}(s - s_0) + \eta, \hat{x}_{\eta}) + \int_0^t \sqrt{2\hat{a}(r^{-2}(s - s_0) + \eta, \hat{x}_{\eta})} \, dw_\eta. \]
By Itô’s formula applied to the process \( r^{-2}(s - s_0) + t, \hat{x}_t \), for any \( \phi \in C^2_0(\mathbb{R}^{d+1}) \) and \( t \geq 0 \),
\[ \phi(r^{-2}(s - s_0), r^{-1}(x - x_0)) = E_{s,x} \phi(r^{-2}(s - s_0) + t, \hat{x}_t) \]
\[-E_{s,x} \int_0^t (D_t + \hat{L}) \phi(r^{-2}(s - s_0) + \eta, \hat{x}_\eta) \, d\eta \]
\[= \hat{E}_{r^{-2}(s - s_0), r^{-1} (x - x_0)} \phi(\hat{\omega}_t) - \hat{E}_{r^{-2}(s - s_0), r^{-1} (x - x_0)} \int_0^t (D_t + \hat{L}) \phi(\hat{\omega}_\eta) \, d\eta.\]

Hence, $\hat{X}$ is a quasi diffusion process corresponding to $\hat{L}$.

Let $u$ be any harmonic function for $X(L)$ on $Q_r(s_0, x_0)$, and $A \subset Q_r(s_0, x_0)$ be an open set. Then, $P_{s,x}$ a.s. we have
\[\tau_s(A) := \inf\{t \geq 0 : (s + t, x_t) \not\in A\} \]
\[= \inf\{t \geq 0 : (r^{-2}(s - s_0) + r^{-2}t, r^{-1}(x_t - x_0)) \in \hat{A}\} \]
\[= r^2 \inf\{t \geq 0 : (r^{-2}(s - s_0) + t, \hat{x}_t) \in \hat{A}\} := r^2 \hat{\tau}.\]

On the other hand, $\hat{\tau}$ coincides $\hat{P}_{r^{-2}(s - s_0), r^{-1} (x - x_0)}$ a.s. with the first exit time from $A$ for the trajectory $\hat{\omega}_t$. By what was just said we get
\[E_{s,x} u(s + \tau_s(A), x_{\tau_s(A)}) \]
\[= E_{s,x} \hat{u}(r^{-2}(s - s_0) + \hat{\tau}, r^{-1}(x_{r^2 \hat{\tau}} - x_0)) \]
\[= \hat{E}_{r^{-2}(s - s_0), r^{-1} (x - x_0)} \hat{u}(\hat{\omega}_t).\]

This combined with the fact that $u$ is harmonic for $X(L)$ on $A$ yields
\[\hat{u}(r^{-2}(s - s_0), r^{-1}(x - x_0)) = \hat{E}_{r^{-2}(s - s_0), r^{-1} (x - x_0)} \hat{u}(\hat{\omega}_t).\]

\[\Box\]

The following lemma is a version of the so-called oblique cylinder lemma (see Lemma 9.2.1 of [11]).

**Lemma 2.3.** Let $u$ be a nonnegative harmonic function on $Q_{8,2}$. Let $r \in (0, 2], \gamma \in (0, 1], \varepsilon, \kappa \in (0, 1)$, $T \in [\kappa, \kappa^{-1}]$ be numbers and $x_0 \in \mathbb{R}^d$ be a point such that $|x_0| \leq \kappa^{-1}$. Denote $V = \{(t, x) : t \in [0, T], |x - x_0 t / T| \leq r\}$ and assume additionally $V \subset Q_{8,2}$. Let $u$ be a nonnegative harmonic function for $Y(D_t + L)$ on $V$. Then, there exist constants $n, N > 0$ depending only on $d, \nu, K, \kappa$ and $\varepsilon$ such that
\[\inf_{|x| \leq (1 - \varepsilon)r} u(0, x) \geq N \gamma^n \inf_{|x - x_0| \leq \gamma r} u(T, x).\]

If $b \equiv 0$, then the estimate holds for $r > 2$.

**Proof.** We start by making several simplifications. First, by Lemma 2.2 we may assume that $r = 1$. Second, we may assume that $m := \inf_{|x - x_0| \leq \gamma} u(T, x) = 1$. Indeed, if $m = 0$ then the desired assertion trivially holds. Otherwise, we replace $u$ by $u/m$. Third, replacing $\gamma$ by $\gamma \varepsilon / 2$ we may assume that $\gamma \leq \varepsilon / 2$. Fourth, it suffices to prove the estimate for $|x| < 1 - \varepsilon / 2$, and, hence, we may replace $\varepsilon$ by $\varepsilon / 2$.

Next, let $\mu = \varepsilon^2 - \gamma^2$, $y \in B_{1 - \varepsilon}$, $\xi = y - x_0$. Denote
\[\phi(t, x) = \mu(1 - t / T) - |x - y + \xi t / T|^2 + \gamma^2,\]
\[Q = \{(t, x) : t \in [0, T], \phi(t, x) > 0\}.\]
Observe that $Q \subset Q_{8.2}$ is an open set bounded by the slanted paraboloid $\phi \equiv 0$ and two planes $t = 0$ and $t = T$. The cross sections with these planes are $B_\varepsilon(y)$ and $B_\nu(x_0)$ respectively. Next, let $n > 0$ be a number which we will choose later and denote

\[
\phi(t, x) = \mu(1 - t/T) - |x - y + \xi t/T|^2 + \gamma^2,
\]

\[
v(t, x) = \phi^2(t, x)(\mu(1 - t/T) + \gamma^2)^{-n},
\]

By direct calculations

\[
(\mu(1 - t/T) + \gamma^2)^n(D_t + L)v(t, x) = A(t)\phi^2(t, x)
\]

\[
-B(t, x)\phi(t, x) + 8\alpha^i(t, x)(x^i - y^i + \xi t/T)(x^i - y^i + \xi t/T),
\]

where

\[
A(t) = n\mu/T(\mu(1 - t/T) + \gamma^2)^{-1},
\]

\[
B(t, x) = 2\mu/T + (4/T)\xi^i(x^i - y^i + \xi t/T)
\]

\[
+ 4\alpha^i + 4\xi^j(t, x)(x^i - y^i + \xi t/T).
\]

Since $|\xi^i| \leq 1 - \varepsilon + \kappa^{-1}$, and $\phi \leq \varepsilon^2$ in $Q$, we have

\[
|B(t, x)| \leq N_1(d, \nu, K, \varepsilon, \kappa) \forall (t, x) \in Q.
\]

Then, by what was just said, if $t \in [0, T]$ and

\[
|x - y + \xi t/T|^2 \geq (8\nu)^{-1}N_1\phi(t, x),
\]

then one has $(D_t + L)v(t, x) \geq 0$. In the case

\[
t \in [0, T], \quad |x - y + \xi t/T|^2 \leq (8\nu)^{-1}N_1\phi(t, x)
\]

we have

\[
\mu(1 - t) + \gamma^2 \leq (N_1(8\nu)^{-1} + 1)\phi(t, x),
\]

and, then, since $\mu > 3\varepsilon^2/4$,

\[
A(t, x)\phi(t, x) - B(t, x) > (3/4) n \varepsilon^2(\kappa + \kappa N_1(8\nu)^{-1})^{-1} - N_1.
\]

The last expression is positive for

\[
n = 2 + (4/3) \varepsilon^{-2} N_1(\kappa + \kappa N_1(8\nu)^{-1}).
\]

Hence, $(D_t + L)v \geq 0$ in $Q$.

Further,

\[
\max_{|x - x_0| \leq \gamma} v(T, x) \leq \gamma^{4 - 2n}, \quad v(t, x) = 0, \text{ if } \phi(t, x) = 0, t \in (0, T),
\]

so that $v \leq \gamma^{4 - 2n}u$ on $\partial_\nu Q$. Then, by Itô’s formula and the fact that $u$ is harmonic for $Y(D_t + L)$ on $Q$ we get

\[
\varepsilon^{4 - 2n} = v(0, y) \leq E_{0, y}v(\tau_0(Q), x_{\tau_0(Q)})
\]

\[
\leq \gamma^{4 - 2n} E_{0, y}u(\tau_0(Q), x_{\tau_0(Q)}) = \gamma^{4 - 2n}u(0, y).
\]

Replacing $n$ by $n - 2$ we prove the desired assertion. \qed
Lemma 2.4 (A variant of Krylov-Safonov inequality). Let $r, q, \kappa \in (0, 1]$ be numbers. Fix some $(s, x) \in \overline{Q}_{q, \kappa r^2, x_r}$ and let $\Gamma$ be any compact subset of $Q_r$ such that $|\Gamma| > q|Q_r|$. 

Let $a$ and $b$ be Borel functions satisfying (1.1) and $x$ be a solution of (2.1) on some probability space $(\Omega, \mathcal{F}, P)$. Denote $\tau_s = \tau_s(Q_r), \gamma_s = \tau_s(Q_r \setminus \Gamma)$.

Then, there exists a constant $\delta = \delta(d, \nu, K, q, \kappa) > 0$ such that

$$P(\gamma_s < \tau_s) > \delta.$$ 

If $b \equiv 0$ then the assertion holds for $r > 1$.

Proof. Let $\mu_{s,x}$ be a positive finite Borel measure on $Q_r$ given by

$$\mu_{s,x}(A) = E_{s,x} \int_0^{\tau_s} I_{(s+\tau, x_t) \in A} \, ds.$$ 

First, we will show that

$$\mu_{s,x}(\Gamma) > \delta r^2 \quad (2.2)$$

for some $\delta > 0$.

Let $\Lambda_{t,K} \subset \mathbb{L}^{t,K}_{\nu,K}$ be the subset of operators with constant coefficients. We consider the following Bellman’s equation:

$$\inf_{L \in \Lambda_{t,K}} (D_t + L)v = -I_{\Gamma} \quad \text{in} \ Q_r, \quad v = 0 \text{ on } \partial_p Q_r. \quad (2.3)$$

By Theorem 15.1.4 of [11] there exists a unique solution $v \in W^{1,2}_{d+3}(Q_r)$. By the Sobolev embedding theorem we conclude that $v \in C(\overline{Q}_r)$.

Further, by Lemma 4.1.5 of [11] there exist measurable functions $\tilde{a}, \tilde{b}$ satisfying (1.1) such that

$$D_t v + \tilde{a}^{ij} D_{x^j} v + \tilde{b}^i D_{x^i} v = -I_{\Gamma} \quad \text{in} \ Q_r, \quad v = 0 \text{ on } \partial_p Q_r \quad (2.4)$$

Then, by the maximum principle (see Theorem 3.1.5 of [11]) $v \geq 0$ a.e. in $Q_r$. Next, for $(t, y) \in Q_1$ we set

$$\hat{v}_n(t, y) = r^{-2} v_n(r^2 t, ry), \quad \hat{a}(t, y) = \tilde{a}(r^2 t, ry), \quad \hat{b}(t, y) = r \tilde{b}(r^2 t, ry)$$

and note that $\hat{v}_n$ satisfies Eq. (2.4) with $\tilde{a}, \tilde{b}$ and $Q_r$ replaced with $\hat{a}, \hat{b}$ and $Q_1$ respectively. Observe that $\hat{b}$ satisfies (1.1). Then, by Lemma 9.1.4 and Theorem 9.1.2 of [11] there exists a constant $\delta = \delta(d, \nu, K, q, \kappa) > 0$ such that

$$v(s, x) \geq \delta r^2. \quad (2.5)$$

Next, by Itô’s formula (see Theorem 2.10.1 of [7])

$$v(s, x) = -E_{s,x} \int_0^{\tau_s} (D_t + L)v(s + t, x_t) \, dt.$$ 

Due to (2.3) we have

$$(D_t + L)v \geq -I_{\Gamma}, \text{ a.e. in } Q, \quad (2.6)$$
and, then, by this and the parabolic Alexandrov estimate (see Theorem 2.2.4 of [7]) we conclude
\[ v(s, x) \leq \mu_{s, x}(\Gamma). \]

Now (2.2) follows from (2.5).

Further, observe that \( \tau_s \leq r^2 \). By this and (2.2) we obtain
\[ \delta r^2 < \mu_{s, x}(\Gamma) = E_{s, x} I_{\gamma_s < \tau_s} \int_0^{\tau_s} I_{(s+t, x) \in \Gamma} dt \leq r^2 P_{s, x}(\gamma_s < \tau_s), \]
and this proves the claim.

□

3. Proofs of Theorems 1.1, 1.4, 1.7, 1.9, and 1.11

We prove the theorems in the following order: 1.11, 1.7, 1.9, 1.1, 1.4.

Proof of Theorem 1.11. We follow the argument of [8].

Denote
\[ \rho((s_1, x_1), (s_2, x_2)) = |s_1 - s_2|^{1/2} + |x_1 - x_2|. \]
By Lemma 2.2 we may assume that \( r = 1 \) and \((t_0, x_0) = (0, 0)\).

We will show that for any \( R \in (0, 1/2] \) and a cylinder \( Q_{2R}(\tilde{s}_0, \tilde{x}_0) \subset Q_2 \),
\[ |u(s_1, x_1) - u(s_2, x_2)| \leq NR^\alpha \sup_{Q_{2R}(\tilde{s}_0, \tilde{x}_0)} |u|, \quad \forall (s_i, x_i) \in Q_{2R}(\tilde{s}_0, \tilde{x}_0), \quad (3.1) \]
where \( N \) and \( \alpha \) depend only on \( d, \nu, K \). After that one finishes the argument as follows. If \( \rho((s_1, x_1), (s_2, x_2)) \leq 1/2 \) we obtain (1.9) by using (3.1) with \( R = \rho((s_1, x_1), (s_2, x_2)), \quad \tilde{s}_0 = s_1 \wedge s_2, \quad \tilde{x}_0 = (x_1 + x_2)/2 \).

Otherwise, (1.9) trivially holds with \( N = 2^{1/2} \).

Next, by Lemma 2.2 we may assume that \((\tilde{s}_0, \tilde{x}_0) = 0\). Replacing \( u \) with \( c_1 u + c_2, c_1, c_2 \in \mathbb{R} \) if necessary, we may assume that
\[ \sup_{(s, x) \in \bar{Q}_R} u(s, x) = 1, \quad \inf_{(s, x) \in \bar{Q}_R} u(s, x) = -1. \quad (3.2) \]
\[ |\{(s, x) \in Q_{2R} : u(s, x) \leq 0\}| \geq |Q_{2R}|/2. \quad (3.3) \]

Let
\[ \Gamma \subset \{(s, x) \in Q_{2R} : u(s, x) \leq 0\} \]
be a compact set such that
\[ |\Gamma| > (3/8)|Q_{2R}|. \]

Let \( \delta(d, \nu, K, q, \kappa) \) be the constant from Lemma 2.4 with \( q = 3/8, \kappa = 3/4, r = 2R \) and let
\[ \hat{Q} = [0, 9R^2/8] \times \hat{B}_{3R/2} \]
be the cylinder \( Q_{qR^2, \kappa r} \) from Lemma 2.4. Then, there exists a point \((s, x) \in \hat{Q}_R\) such that
\[ u(s, x) > 1 - \delta/2. \quad (3.4) \]
Further, for the sake of convenience, we denote $\gamma = \tau_s(Q_{2R} \setminus \Gamma)$. Thanks to the fact that $u$ is harmonic for $Y(D_t + L)$ on $Q_{2R}$ we have

$$u(s, x) = E_{s,x} u(s + \gamma, x_\gamma) (I_{\gamma = \tau_s(Q_{2R})} + I_{\gamma < \tau_s(Q_{2R})}).$$  

(3.5)

Since $\Gamma$ is a closed set, $u(s + \gamma, x_\gamma I_{\gamma < \tau_s(Q_{2R})} \leq 0$ $P_{s,x}$ a.s. Using this and (3.4), we obtain

$$1 - \delta/2 < u(s, x) \leq P_{s,x} (\gamma = \tau_s(Q_{2R})) \sup_{Q_{2R}} u \leq (1 - \delta) \sup_{Q_{2R}} u. \quad (3.6)$$

The last inequality follows from Lemma 2.4 because $(s, x) \in \bar{Q}_R \subset \hat{Q}$. By (3.6) combined with (3.2) we get

$$\text{osc}_{Q_{2R}} u \geq N \text{osc}_{Q_R} u,$$

where

$$N = \frac{1 - 3\delta/4}{1 - \delta} > 1, \quad \text{osc}_{Q_R} u = 2.$$  

By an iteration argument this implies (1.9).

**Proof of Theorem 1.7.**

Theorem 1.7 is a direct corollary of Example 1.10 and Theorem 1.11.

**Proof of Theorem 1.9** By the continuity theorem for characteristic functions it suffices to prove that for any $\xi \in \mathbb{R}^d$

$$\phi(s, x) = E_{s,x} \exp(i\xi \cdot X_t)$$

is continuous at every point $(s, x) \in \mathbb{R}^{d+1}$.

Let $|s_1 - s_2|, |x_1 - x_2| < 1$. Denote $h = s_2 - s_1$ and assume that $h \in (0, T)$. By the triangle inequality we only need to estimate $I_1 = |\phi(s_1, x_1) - \phi(s_1, x_2)|$ and $I_2 = |\phi(s_1, x_2) - \phi(s_2, x_2)|$. First, by Theorem 1.7

$$I_1 \leq N(d, \nu, K, K) |x_1 - x_2|^\alpha.$$ 

Next, for any $(s, x)$

$$\psi(s, x) = |E_{s,x} \exp(i\xi j \cdot X_{T-h}) - E_{s,x} \exp(i\xi j \cdot X_T)|$$

$$\leq N(d, \xi)(E_{s,x} |X_T - X_{T-h}|^2)^{1/2}.$$ 

By Lemma 2.1 and isometry of stochastic integral

$$E_{s,x} |X_T - X_{T-h}|^2 \leq N(d, \nu, K)(h + h^2). \quad (3.7)$$

Further, by the Markov property

$$\phi(s_1, x) = E_{s_1, x} E_{s_2, x_h} \exp(i\xi j \cdot X_{T-h})$$

and, hence,

$$I_2 \leq |E_{s_2, x} (E_{s_2, x_h} \exp(i\xi j \cdot X_{T-h}) - E_{s_2, x_h} \exp(i\xi j \cdot X_{T-h}))|$$

where

$$\tilde{I}_2 = |E_{s_1, x} (E_{s_2, x_h} \exp(i\xi j \cdot X_{T-h}) - E_{s_2, x_h} \exp(i\xi j \cdot X_{T-h}))|$$

Note that by Theorem 1.7

$$\tilde{I}_2 \leq N(d, \nu, K, \xi)(E_{s_1, x_h} |x_h - x_2|^\alpha + P_{s_1, x_h}(|x_h - x_2| \geq 1),$$
and, then, by Chebyshev’s inequality and (3.7)

\[ \tilde{I}_2 \leq N(h^{\alpha/2} + h^{1/2} + h). \]

This combined with the estimate of \( I_1 \) and \( \psi(s_2, x_2) \) proves the validity of the assertion.

**Proof of Theorem 1.1.** For \( s \in [0, T) \) and \( x \in \mathbb{R}^d \) we denote

\[ u(s, x) = E_x f(x_{T-s}). \]

Thanks to the strong Markov property for any stopping time \( \tau \leq T \) relative to \((N_t, t \geq 0)\)

\[ u(s, x) = E_x u(s + \tau, x_{\tau}). \]

Repeating word-for-word the argument of the proof of Theorem 1.11 we conclude that the function satisfies the estimate (1.9). The theorem is proved.

**Proof of Theorem 1.4.** Let \( \tilde{P}_{s, x} \) be the distribution of the process \((s + t, x_t), t \geq 0\) on \((\tilde{\Omega}, \tilde{\mathcal{N}}_\infty)\). By Lemma 2.1 and Itô’s formula for any \( \phi \in C^2_0(\mathbb{R}^{d+1}) \) (1.6) holds.

Next, by by Theorem 1.1 the function \( \tilde{E}_{s, x} \exp(i \xi^j x^i_T) = E_x \exp(i \xi^j x^i_T) \) is continuous on \( \mathbb{R}^{d+1} \) for any \( \xi \in \mathbb{R}^d \), and, hence, by the continuity theorem \( \tilde{P}_{s, x} \) is a Feller family of probability measures. Then, by Theorem 3.10 of [3] \( \tilde{Y} = (\omega_t, \infty, \tilde{N}_t, \tilde{P}_{s, x}) \) is a strong Markov process. Thus, we constructed a strong Markov quasi diffusion process \( Y(D_t + L) = \tilde{Y}. \)

Now we derive the desired assertion from Theorem 1.11. Note that for any \( x \in G := B_{3r/2}, \)

\[ u(x) = \int_{\partial G} u(y) \pi_G(x, dy), \quad (3.8) \]

where \( \pi_G(x, A) \) is defined in Example 1.3. Hence, we only need to prove the claim for \( u(x) = \pi_G(x, A), \ A \in \mathcal{B}_d^e. \) Let \( T > 4 \) and denote

\[ Q_T = [0, T) \times G, \quad A_T = [0, T) \times (A \cap \partial G). \]

Due to Example 1.10 the function

\[ v_T(s, x) := \tilde{P}_{s, x}((s + \tau_s(Q_T), x_{\tau_s(Q_T)}) \in A_T) = P_x(x_{\tau_G} \in A \cap \partial G, \tau_G < T) \]

is harmonic for \( Y(D_t + L) \) on \( Q_T. \) Then, by Theorem 1.11

\[ |v_T(0, x) - v_T(0, y)| \leq N|x - y|^\alpha, \ x, y \in \tilde{B}_r/2, \quad (3.9) \]

where \( N \) and \( \alpha \) depend only on \( d, \nu \) and \( K. \) Recall that for any \( x \in G \) we have \( \tau_G < \infty P_x \) a.s., and, then, for any \( s \geq 0, \)

\[ \lim_{T \to \infty} v_T(s, x) = \pi_G(x, A), \ x \in B_R. \quad (3.10) \]

Passing to the limit in (3.9) we prove the assertion.
4. Proofs of Theorems [1.5] and [1.12]

First we prove an auxiliary result which first assertion is a variant of Theorem 9.5.1 of [11], and the second one is a version of Lemma 9.5.3 of [11]. We follow the argument of [11] very closely.

**Lemma 4.1.** Let \( q \in (0, 1] \), and \( U_q(Q_1) \) be the set of nonnegative Borel measurable functions \( u \) such that

- there exists a cylinder \( Q_R(s, x) \supset \tilde{Q}_1 \), an operator \( L \in \mathbb{L}_{\nu,K}^1 \), and a strong Markov process \( Y(D_t + Y) \) such that \( u \) is harmonic for \( Y(D_t + L) \) on \( Q_R(s, x) \),
- \(|\{(t, x) \in Q_1 : u(t, x) \geq 1\}| \geq q|Q_1|\).

We introduce a quantity

\[
p(q) = \inf_{u \in U_q(Q_1)} |u(0, 0)|
\]

which is akin to capacity from the classical potential theory. Then, the following assertions hold.

(i) \( p(q) \in (0, 1] \) and \( p(q) \to 1 \) as \( q \to 1 \).

(ii) Let \( (t_i, x_i) \in \mathbb{R}^{d+1}, i = 0, 1, r \in (0, 1], R > 0, \kappa > 0 \), and \( \tilde{Q}_r(t_0, x_0) \subset Q_R(t_1, t_2) \) and \( L \in \mathbb{L}_{\nu,K}^1 \). Let \( v \) be a nonnegative function harmonic for \( X(L) \) on \( Q_R(t_1, x_1) \) such that \(|\{(t, x) \in Q_r(t_0, x_0) : v(t, x) \leq \kappa\}| \geq q|Q_r|\). Then,

\[
v(t_0, x_0) \leq p(q)\kappa + (1 - p(q)) \max_{\tilde{Q}_r(t_0, x_0)} v.
\]

**Proof.** (i) By definition, \( p(q) \geq 0 \). Since \( u \equiv 1 \) belongs to \( U_q(Q_1) \), we have \( p(q) \leq 1 \).

Fix any \( u \in U_q(Q_1) \). To prove the second claim we only need to show that, for any \( \varepsilon > 0 \) there exists \( q \) such that \( u(0, 0) \geq 1 - \varepsilon \). We may assume that \( u(0, 0) < 1 \). Denote

\[
\phi(t, x) = 1 - t - |x|^2, \quad \tilde{Q} = \{(t, x) \in Q_1 : \phi(t, x) > u(t, x)\}.
\]

By Theorem 1.11 \( u \) is a continuous function on \( \tilde{Q}_1 \), and, then, \( \tilde{Q} \) is an open set. Since, \( 1 = \phi(0, 0) > u(0, 0) \), we have \( \tilde{Q} \neq \emptyset \). Further, since \( \phi \leq 0 \) outside of \( Q_1 \) and \( u \) is nonnegative, we conclude that \( \partial_p \tilde{Q} \subset Q_1 \), and

\[
u = \phi \text{ on } \partial_p \tilde{Q}.
\]

By this, \( \{(t, x) \in Q_1 : u(t, x) \geq 1\} \subset Q_1 \setminus \tilde{Q} \) and, hence, \(|\tilde{Q}| \leq (1 - q)|Q_1|\).

Next, note that

\[
L\phi(t, x) = -1 - 2b^i(t, x)x^i - 2\text{tr } a \geq -1 - Kd - 2d\nu^{-1}, (t, x) \in Q_1.
\]

This combined with Itô’s formula yields

\[
1 = \phi(0, 0) = E_{0,0}\phi(\tau_Q(\tilde{Q}), x_{\tau_Q(\tilde{Q})}) - E \int_0^{\tau_Q(\tilde{Q})} L\phi(t, x_t) dt
\]

\[
\leq E_{0,0}\phi(\tau_Q(\tilde{Q}), x_{\tau_Q(\tilde{Q})}) + NE \int_0^{\tau_Q(\tilde{Q})} I_{t,\xi} \in \tilde{Q} dt.
\]

(4.2)
By Theorem 2.2.4 of [7] the last integral is less than
\[ N\tilde{Q}\frac{1}{1/(d+1)} \leq N(1-q)^{1/(d+1)}|Q_1|^{1/(d+1)}. \]
By (4.1) and the fact that \( u \) is harmonic for \( Y(D_t + L) \) on \( Q_1 \) we have
\[ E_{0,0}\phi(\tau_0(Q), x_{\tau_0(Q)}) = u(0, 0). \]
Combining this (4.2) with we obtain
\[ u(0, 0) \geq 1 - N(1-q)^{1/(d+1)}|Q_1|^{1/(d+1)}, \]
and this proves the assertion (i).

(ii) By virtue of Lemma 2.2 we may assume that \( r = 1, (t_0, x_0) = (0, 0), \) and then, \( v \in U_q(Q_1) \). Obviously, the claim holds if \( \kappa \geq \max Q_1 v \), and, hence, we may assume that \( \kappa < \max Q_1 v \).

Next, denote \( \tilde{v} = (\max Q_1 v - \kappa)^{-1}(\max Q_1 v - v) \) and observe that \( \tilde{v} \) is a nonnegative function harmonic for \( Y(D_t + L) \) on \( A \) such that \( \{(t, x) \in Q_1 : v(t,x) \leq \kappa\} = \{(t, x) \in Q_1 : \tilde{v}(t,x) \geq 1\} \). Since \( \tilde{v} \in U_q(Q_1) \), we have \( \tilde{v}(0, 0) \geq p(q) \), and this finishes the proof.

**Proof Theorem 1.12** We repeat the proof of Theorem 9.6.1 almost verbatim. First, we exclude the trivial case \( u(4,0) = 0 \). By Lemma 2.2 we may assume that \( t_0 = 0, x_0 = 0, \) and \( r = 1 \). Next, let \( n \) be the constant from Lemma 2.3 with \( \kappa = 1/4 \) and \( \varepsilon \in (0, 1] \). By Lemma 4.1 it is possible to find \( q \in (0, 1) \) such that
\[ p(q) > (2^n - 1)(2^n - 1/2)^{-1}. \] (4.3)
We denote \( Q_0(4, 0) = (4, 0) \), and for \( r \in [0, 1] \) we set
\[ m(r) = \max_{(t,x) \in \bar{Q}_r(4,0)} u(t,x), \quad f(r) = (1 - r)^{-n}u(4, 0). \]
Obviously, \( f \) is continuous on \([0,1]\), and \( f(r) \to \infty \) as \( r \uparrow 1 \). The function \( m \) is a continuous function on \([0,1]\), because so is \( u \) on \( \bar{Q}_{3/2}(4,0) \) thanks to Theorem 1.1. This combined with the fact that \( m(0) = f(0) \) implies the existence of the greatest root of the equation \( m(r) = f(r) \) which we denote by \( r_0 \).

Due to the continuity of \( u \), there exists a point \((t_1, x_1) \in \bar{Q}_{r_0}(4,0)\) such that \( m(r_0) = u(t_1, x_1) \). Observe that \( Q := \bar{Q}_{(1-r_0)/2}(t_1, x_1) \subset \bar{Q}_{(1+r_0)/2}(4,0) \), and, since \((1 + r_0)/2 > r_0, \) we get
\[ \max_{Q} u < 2^n u(4, 0)(1-r_0)^{-n} = 2^n f(r_0). \] (4.4)
Denote \( \Gamma = \{(t,x) \in Q : u(t,x) \geq 2^{-1}f(r_0)\} \). We claim that
\[ |\Gamma| \geq (1-q)|Q|. \] (4.5)
Assume that the contrary holds. Since \( u \) is harmonic for \( Y(D_t + L) \) on \( Q_{8/2} \), by Lemma 4.1 (ii) with \( r = (1-r_0)/2, \kappa = 2^{-1}f(r_0) \) combined with (4.4) we have
\[ f(r_0) = u(t_1, x_1) \leq p(q)2^{-1}f(r_0) + (1-p(q))\max_{Q} u \]
This contradicts (4.3), and, hence, (4.5) holds.

Next, denote $\tau = \tau_1(Q_{8,2} \setminus \Gamma)$. Using the fact that $u$ is a nonnegative harmonic function combined with (4.4), for any $x \in B_{(1-r_0)/2}(x_1)$ we get

$$u(t_1, x) \geq E_{t_1,x}u(t_1 + \tau, x_\tau)\mathbb{I}_{\tau < \tau_1(Q_{8,2})} \geq 2^{-1}f(r_0)P_{t_1,x}(\tau < \tau_1(Q_{8,2}))$$.

Combining this with Lemmas 2.1 and 2.4 we get

$$\inf_{x \in B_{(1-r_0)/4}(x_1)} u(t_1, x) \geq \delta(1-r_0)^{-n}2^{-1}u(4,0).$$

Now the desired assertion is derived from Lemma 2.3 in the following way:

$$\inf_{x \in B_{2-\varepsilon}} u(0,x) \geq N4^{-n}(1-r_0)^n \inf_{x \in B_{(1-r_0)/4}(x_1)} u(t_1, x) \geq N\delta 2^{-3n-1}u(4,0).$$

**Proof of Theorem 1.5** It follows from (3.8) that it suffices to prove the theorem for $u(x) = \pi_G(x, A)$, $G = B_{3r/2}$, $A \in \mathcal{B}^d$.

Next, let $T > 2(3r/2)^2$ be a number and $d\nu_T(s, x)$ be a function defined in the proof of Theorem 1.4. Observe that by Theorem 1.12 we have

$$v_T(r^2, x_0) \leq N(d, \nu, K) \inf_{x \in B_{r/4}(x_0)} v_T(0, x).$$

Now the theorem follows from (3.10).

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E-mail address: yastr002@umn.edu

127 Vincent Hall, University of Minnesota, Minneapolis, MN, 55455