Weighted $L^2$–cohomology of Coxeter groups based on barycentric subdivisions

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Abstract

Associated to any finite flag complex $L$ there is a right-angled Coxeter group $W_L$ and a contractible cubical complex $\Sigma_L$ (the Davis complex) on which $W_L$ acts properly and cocompactly, and such that the link of each vertex is $L$. It follows that if $L$ is a generalized homology sphere, then $\Sigma_L$ is a contractible homology manifold. We prove a generalized version of the Singer Conjecture (on the vanishing of the reduced weighted $L^2_q$–cohomology above the middle dimension) for the right-angled Coxeter groups based on barycentric subdivisions in even dimensions. We also prove this conjecture for the groups based on the barycentric subdivision of the boundary complex of a simplex.

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1 Introduction

A construction of Davis ([1], [2], [4]), associates to any finite flag complex \( L \), a “right-angled” Coxeter group \( W_L \) and a contractible cubical cell complex \( \Sigma_L \) on which \( W_L \) acts properly and cocompactly. \( W_L \) has the following presentation: the generators are the vertices of \( L \), each generator has order 2, and two generators commute if the span an edge in \( L \). The most important feature of this construction is that the link of each vertex of \( \Sigma_L \) is isomorphic to \( L \).

A simplicial complex \( L \) is a generalized homology \( m \)-sphere (for short, a \( \text{GHS}^m \)) if it is a homology \( m \)-manifold having the same homology as a standard sphere \( S^m \) (the homology is with real coefficients.) It follows, that if \( L \) is a \( \text{GHS}^{n-1} \), then \( \Sigma_L \) is a homology \( n \)-manifold.

If \( L \) is a simplicial complex, \( bL \) will denote the barycentric subdivision of \( L \). \( bL \) is a flag simplicial complex. Let \( \partial \Delta^n \) denote the boundary complex of the standard \( n \)-dimensional simplex.

We study a certain weighted \( L^2 \)-cohomology theory \( L^2_\mathbf{q} \mathcal{H}^* \), described in [7], [5]. Suppose, for each vertex of \( v \in L \) we are given a positive real number \( q_v \), and let \( \mathbf{q} \) denote the vector with components \( q_v \). Given a minimal word \( w = v_1 \ldots v_n \in W_L \), let \( \mathbf{q}^w = q_{v_1} \ldots q_{v_n} \). For each \( W_L \)-orbit of cubes pick a representative \( \sigma_0 \) and let \( w(\sigma) = w \) if \( \sigma = w\sigma_0 \). (The ambiguity in the choices will not matter in our discussion.) Let \( L^2_\mathbf{q} C^i(\Sigma_L) = \{ \Sigma c_\sigma \sigma \mid \Sigma^2_\mathbf{q}^w(\sigma) < \infty \} \) be the Hilbert space of infinite \( i \)-cochains, which are square-summable with respect to the weight \( \mathbf{q}^w \). The usual coboundary operator \( d \) is then a bounded operator, and we define the reduced weighted \( L^2_\mathbf{q} \)-cohomology to be \( L^2_\mathbf{q} \mathcal{H}^i(\Sigma_L) = \ker(d^i)/\text{Im}(d^{i-1}) \).

Similarly, one can define the reduced weighted \( L^2_\mathbf{q} \)-homology, except, instead of the usual boundary operator one uses the adjoint of \( d \). It follows from the Hodge decomposition that the resulting homology and cohomology spaces are naturally isomorphic. These spaces are Hilbert modules over the Hecke–von Neumann algebra \( \mathcal{N}_\mathbf{q} \) (an appropriately completed Hecke algebra of \( W_L \).) This allows us to introduce the weighted \( L^2_\mathbf{q} \) Betti numbers — the dimension of \( L^2_\mathbf{q} \mathcal{H}^i \) over \( \mathcal{N}_\mathbf{q} \). If \( \mathbf{q} = 1 = (1, \ldots, 1) \), we obtain the usual reduced \( L^2 \)-cohomology, and we omit the index \( \mathbf{q} \). We write \( \mathbf{q} \leq 1 \), if each component of \( \mathbf{q} \) is \( \leq 1 \).

The following conjecture, attributed to Singer, goes back to 1970’s.

**The Singer Conjecture** If \( M^n \) is a closed aspherical manifold, then

\[ L^2 \mathcal{H}^i(\tilde{M}^n) = 0 \text{ for all } i \neq n/2. \]

As explained in [5, Section 14], the appropriate generalization of the Singer Conjecture to the weighted case is the following conjecture:
The Generalized Singer Conjecture  Suppose $L$ is a flag $\text{GHS}^{n-1}$. Then
$L^2_q \mathcal{H}^i(\Sigma_L) = 0$ for $i > n/2$ and $q \leq 1$.

(Poincaré duality shows that for $q = 1$ this conjecture implies the Singer Conjecture for $\Sigma$.)

This conjecture holds true for $n \leq 4$ by [5]. One of the main results of this paper is a proof of this conjecture for barycentric subdivisions in even dimensions. The proof uses a reduction to a very special case $L = b\partial \Delta^{2k-1}$.

We prove this case as Theorem 5.2. It turns out (Theorem 5.3), that this result implies the vanishing of the $L^2_q$-cohomology in a certain range for arbitrary right-angled Coxeter groups based on barycentric subdivisions. (For $q = 1$, this implication is proved in [5].) In particular, it follows that the Generalized Singer Conjecture is true for all barycentric subdivisions in even dimensions (Theorem 5.4), and for $b\partial \Delta^n$ in all dimensions (Theorem 5.6).

This paper relies very heavily on [5]. In the inductive proofs we mostly omit the first steps, they are easy exercises using [5].

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2 Vanishing conjectures

We will follow the notation from [5]. Given a flag complex $L$ and a full subcomplex $A$, set:

\begin{align*}
h^q_i(L) &= L^2 \mathcal{H}_i(\Sigma_L) \\
h^q_i(A) &= L^2 \mathcal{H}_i(W_L \Sigma_A) \\
h^q_i(L, A) &= L^2 \mathcal{H}_i(\Sigma_L, W_L \Sigma_A) \\
b^i_q(L) &= \dim_{\mathbb{N}_q}(h^q_i(L)) \\
b^i_q(L, A) &= \dim_{\mathbb{N}_q}(h^q_i(L, A))
\end{align*}

The dimension of $\Sigma_L$ is one greater than the dimension of $L$. Hence, $b^i_q(L) = 0$ for $i > \dim L + 1$.

We will use the following three properties of $L^2_q$-homology.

**Proposition 2.1**  (See [5] Section 15)
The Mayer–Vietoris sequence  If $L = L_1 \cup L_2$ and $A = L_1 \cap L_2$, where $L_1$ and $L_2$ (and therefore, $A$) are full subcomplexes of $L$, then
\[ \rightarrow h^q_i(A) \rightarrow h^q_i(L_1) \oplus h^q_i(L_2) \rightarrow h^q_i(L) \rightarrow \]
is weakly exact.

The Künneth Formula  The Betti numbers of the join of two complexes are given by:
\[ b^k_q(L_1 \ast L_2) = \sum_{i+j=k} b^i_q(L_1) b^j_q(L_2). \]

Poincaré Duality  If $L$ is a flag GHS$^{n-1}$, then $b^i_q(L) = b^{n-i}_q(L)$.

If $\sigma$ is a simplex in $L$, let $L_\sigma$ denote the link of $\sigma$ in $L$. To simplify notation we will write $bL_\sigma$ instead of $(bL)_\sigma$ to denote the link of the vertex $v$ in $bL$. Let $C$ be a class of GHS’s closed under the operation of taking link of vertices, i.e. if $S \in C$ and $v$ is a vertex of $S$ then $S_v \in C$. Following Section 15 of [5] we consider several variations of the Generalized Singer Conjecture for the class $C$.

I(n)  If $S \in C$ and dim $S = n - 1$, then $b^i_q(S) = 0$ for $i > n/2$ and $q \leq 1$.

III'$^{(2k+1)}$  Let $S \in C$ and dim $S = 2k$. Let $v$ be a vertex of $S$. Then the map $i_\ast: h^q_k(S_v) \rightarrow h^q_k(S)$, induced by the inclusion, is the zero homomorphism for $q \geq 1$.

V(n)  Let $S \in C$ and dim $S = n - 1$. Let $A$ be a full subcomplex of $S$.
- If $n = 2k$ is even, then $b^i_q(S, A) = 0$ for all $i > k$ and $q \leq 1$.
- If $n = 2k + 1$ is odd, then $b^i_q(A) = 0$ for all $i > k$ and $q \leq 1$.

The argument in Section 16 of [5] goes through without changes if we consider only GHS’s from a class $C$ to give the following:

Theorem 2.2  (Compare [5 Section 16]) If we only consider GHS’s from a class $C$, then the following implications hold.

1. $I(2k+1) \implies III'(2k+1)$.
2. $V(n) \implies I(n)$.
3. $V(2k-1) \implies V(2k)$.
4. $[V(2k) \text{ and III'}(2k+1)] \implies V(2k+1)$. 

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Let $\mathcal{JD}$ denote the class of finite joins of the barycentric subdivisions of the boundary complexes of standard simplices:

$$\mathcal{JD} = \{ b\partial \Delta^{n_1} \ast \cdots \ast b\partial \Delta^{n_j} \}.$$ 

**Lemma 2.3** The class $\mathcal{JD}$ is closed under the operation of taking link of vertices.

**Proof** Let $S = b\partial \Delta^{n_1} \ast \cdots \ast b\partial \Delta^{n_j}$ and $v \in S$. We can assume that $v \in b\partial \Delta^{n_1}$. Then $S_v = b\partial \Delta^{n_1}_v \ast b\partial \Delta^{n_2} \ast \cdots \ast b\partial \Delta^{n_j} = b\partial \Delta^{\text{dim}(v)} \ast b\partial \Delta^{n_1-\text{dim}(v)-1} \ast b\partial \Delta^{n_2} \ast \cdots \ast b\partial \Delta^{n_j}$, and therefore $S \in \mathcal{JD}$. \hfill $\Box$

Next, consider the following statement:

**III''(2k + 1)** Let $v$ be a vertex of $b\partial \Delta^{2k+1}$. Then the map $i_* : h^q_k(b\partial \Delta^{2k+1}_v) \to h^q_k(b\partial \Delta^{2k+1})$, induced by the inclusion, is the zero homomorphism for $q \geq 1$.

**Lemma 2.4** III''(2k + 1) $\implies$ III'(2k + 1) for the class $\mathcal{JD}$.

**Proof** By induction, we can assume that the lemma holds for all odd numbers $< 2k + 1$. Then it follows from the Theorem 2.2 that $V(m)$ and therefore $I(m)$ hold for all $m < 2k + 1$.

Let $S = b\partial \Delta^{n_1} \ast \cdots \ast b\partial \Delta^{n_j}$ with $n_1 + \cdots + n_j = 2k + 1$ and $v \in S$. We assume that $v \in b\partial \Delta^{n_1}$. Then $S_v = b\partial \Delta^{n_1}_v \ast b\partial \Delta^{n_2} \ast \cdots \ast b\partial \Delta^{n_j}$ and, by the Künneth formula, the map in question decomposes as the direct sum of maps of the form

$$(h^q_{k_1}(b\partial \Delta^{n_1}_v) \to h^q_{k_1}(b\partial \Delta^{n_1})) \otimes \bigotimes_{i=2}^j (h^q_{k_i}(b\partial \Delta^{n_i}) \to h^q_{k_i}(b\partial \Delta^{n_i}))$$

where $k_1 + \cdots + k_j = k$. Since $n_1 + \cdots + n_j = 2k + 1$ it follows that $k_i < n_i/2$ for some index $i$. If $n_i < 2k + 1$, then the range of the corresponding map in the above tensor product is 0 by $I(n_i)$ and Poincaré duality, and therefore the tensor product map is 0. If $n_i = 2k + 1$ then, in fact, $i = 1$ (the join is a trivial join) and the result follows from III''(2k + 1). \hfill $\Box$

Thus, it follows from Theorem 2.2, Lemmas 2.3 and 2.4 and induction on dimension, that in order to prove the Generalized Singer Conjecture for the class $\mathcal{JD}$ all we need is to prove III''(2k + 1).

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3 Removal of an odd-dimensional vertex

Let $L$ be a simplicial complex and $bL$ be its barycentric subdivision. The vertices of $bL$ are naturally graded by "dimension": each vertex $v$ of $bL$ is the barycenter of a unique cell (which we still denote $v$) of the complex $L$, and we call the dimension of this cell the dimension of the vertex $v$. Let $E_L$ denote the subcomplex of $bL$ spanned by the even dimensional vertices. Let $A_L$ denote the set of full subcomplexes $A$ of $L$ containing $E_L$, which have the following property: if $A$ contains a vertex of odd dimension $2j + 1$, then $A$ contains all vertices of $bL$ of dimensions $\leq 2j$. In other words, any such $A$ can be obtained inductively from $bL$ by repeated removal of an odd-dimensional vertex of the highest dimension.

If $L = \partial \Delta^n$ we will use the notation $E_n = E_L$ and $A_n = A_L$.

**Lemma 3.1** Assume $\text{III}''(2m + 1)$ holds for $2m + 1 < n$. Then for any $(n - 1)$-dimensional simplicial complex $L$ and any complex $A \in A_L$ we have:

\[ b_i^q(A) = b_i^q(bL) = 0 \text{ for } i > (n + 1)/2 \text{ and } q \leq 1. \]

**Proof** By induction, we can assume that the lemma holds for all $m < n$. First, we claim that removal of odd-dimensional vertices does not change the homology above $(n + 1)/2$. Let $A \in A_L$ and let $B = A - v$ where $v$ is a vertex of the highest odd dimension of $A$. We let $\dim(v) = 2d - 1$, $1 \leq d \leq k$. We want to prove that $b_i^q(A) = b_i^q(B)$ for $i > (n + 1)/2$. Consider the Mayer–Vietoris sequence of the union $A = B \cup A_v$:

\[ \cdots \rightarrow b_i^q(A_v) \rightarrow b_i^q(B) \oplus b_i^q(CA_v) \rightarrow b_i^q(A) \rightarrow b_{i-1}^q(A_v) \rightarrow \cdots \]

Suppose $i > (n + 1)/2$. Since $A_v = B \cap bL_v = B \cap (b\partial \Delta^{2d-1} \ast b(L_v))$, and since $B \in A_L$, it follows, by construction, that $A_v$ splits as the join:

\[ A_v = b\partial \Delta^{2d-1} \ast A_1, \]

with $A_1 \in A_{(L_v)}$. By inductive assumption the lemma holds for $L_v$, i.e. $b_i^q(A_1) = b_i^q(b(L_v)) = 0$ for $i > (n + 1)/2 - d$.

Since $\text{III}''(2d-1)$ holds by hypothesis, by Lemma 2.1 and Theorem 2.2 $I(2d-1)$ holds for the class $\mathcal{F}D$, and, thus, $b_i^q(b\partial \Delta^{2d-1}) = 0$ for $i \geq d$.

Then, by the Künneth formula, $b_i^q(A_v) = 0$ for $i - 1 \geq (n + 1)/2$, i.e. for $i > (n + 1)/2$. By [5 Proposition 15.2(d)], $b_i^q(CA_v) = \frac{1}{b_i^q(A_v)}$. Therefore in the above sequence the terms corresponding to $A_v$ and $CA_v$ are 0, and...
the claim follows. Then it follows by induction, that $b^i_q(A) = b^i_q(bL)$ for all $A \in A_L$ and $i > (n + 1)/2$.

To prove the vanishing we note that, in particular, $b^i_q(E_L) = b^i_q(bL)$ for $i > (n + 1)/2$. Since $E_L$ is spanned by the even-dimensional vertices of $bL$ and since a simplex in $bL$ has vertices of pairwise different dimensions, we have $\dim(E_L) = [(n + 1)/2] - 1$. Therefore, $b^i_q(E_L) = 0$ for $i > (n + 1)/2$ and we have proved the lemma.

In the special case $L = \Delta^{2k+1}$ this lemma admits the following strengthening:

**Lemma 3.2** Let $n = 2k + 1$. Assume $\text{III}''(2m + 1)$ holds for $2m + 1 < n$. Then for any complex $A \in A_n$, $A \subset b\partial \Delta^n$, we have:

$$b^i_q(A) = b^i_q(b\partial \Delta^n) \text{ for } i > k \text{ and } q \leq 1.$$ 

**Proof** We proceed as in the previous proof. As before, we have $B = A - v$, $\dim(v) = 2d - 1$ and $A_v = b\partial \Delta^{2d-1} \ast A_1$, where now $A_1 \subset b\partial \Delta^{2k+1-2d}$. Therefore, the inductive assumption and the hypothesis on $\text{III}''(2d-1)$ imply that $b_i(A_1) = 0$ for $i > k + d$. The lemma follows as before.

As explained in [6], when $q = 1$, the removal of the odd-dimensional vertex does not change homology in all dimensions. We record this result below.

**Lemma 3.3** Assume $\text{III}''(2m + 1)$ holds for $2m + 1 < n$ and $q = 1$. Then for any $(n - 1)$-dimensional simplicial complex $L$ and for any complex $A \in A_L$, obtained by the repeated removal of highest odd-dimensional vertices, we have:

$$b^*(A) = b^*(bL).$$

**Proof** Again we repeat the proof of Lemma 3.1. As before, we have the splitting $A_v = b\partial \Delta^{2d-1} \ast A_1$. The point now is that for $q = 1$, $I(2d-1)$ and Poincaré duality imply $b^*(b\partial \Delta^{2d-1}) = 0$ and therefore $b^*(A_v) = 0$ by the Künneth formula.

## 4 Intersection form

**Lemma 4.1** Let $L$ be a GHS$^{2k}$ and let $v$ be a vertex of $L$. Then the image of the restriction map on $L^2$-cohomology $i^*: L^2\mathcal{H}^k(\Sigma_L) \to L^2\mathcal{H}^k(\Sigma_{L_v})$ is an isotropic subspace of the intersection form of $\Sigma_{L_v}$.
Proof  Note that the cup product of two $L^2$–cocycles is an $L^1$–cocycle. The intersection form is the result of evaluation of the cup product of two middle-dimensional cocycles on the fundamental class, which is $L^\infty$. Since $\Sigma_L$ bounds a half-space in $\Sigma_L$, $i_*([\Sigma_L]) = 0$ in $L^\infty$–homology of $\Sigma_L$. Thus, if $a, b \in L^2 H^n(\Sigma_L)$, then $\langle i^*(a) \cup i^*(b), [\Sigma_L] \rangle = \langle a \cup b, i_*([\Sigma_L]) \rangle = 0$.

Lemma 4.2  Let $G$ be a group and let $A$ be a bounded $G$–invariant (with respect to the diagonal action) non-degenerate bilinear form on a Hilbert submodule $M \subset \ell^2(G)$. Then $A$ has no nontrivial $G$–invariant isotropic subspaces.

Proof  Let us consider the case $M = \ell^2(G)$ first. $G$–invariance and continuity of the form $A$ implies that $A$ is completely determined by its values $a_g = (g A 1)$, $g \in G$. It is convenient to think of the form as given by $(x A y) = \langle x, Ay \rangle$, where $\langle , \rangle$ is the inner product and $A = \Sigma_{g \in G} a_g g$ is a bounded $G$–equivariant operator on $\ell^2(G)$. Non-degeneracy of $A$ means that $Ax = 0$ only if $x = 0$. $A$ is the limit of the group ring elements, and $Ax$ is the limit of the corresponding linear combinations of $G$–translates of $x$, i.e. $Ax = \lim \Sigma_{g \in G_n} a_g (gx)$, where $G_n$ is some exhaustion of $G$ by finite sets. It follows that if $x$ belongs to $G$–invariant isotropic subspace, then $Ax$ belongs to the closure of this subspace. Thus, we have $\langle Ax, Ax \rangle = (Ax A x) = 0$ by isotropy and continuity, therefore $x = 0$.

The case of general submodule $M \subset \ell^2(G)$ reduces to the above, since the bilinear form $A$ can be extended to $\ell^2(G)$, for example, by taking the orthogonal sum $A \oplus \langle , \rangle$ of $A$ on $M$ and the inner product on the orthogonal complement of $M$.

5 Vanishing theorems

Our main technical results are the following two theorems.

Theorem 5.1  III$''(2k + 1)$ is true for all $k > 0$ and $q = 1$.

Proof  The proof is by induction on $k$. Suppose the theorem is true for all $m < k$.

Let $v$ be a vertex of $b\partial \Delta^{2k+1}$. We need to show that the restriction map $i^*: \mathfrak{h}^k(b\partial \Delta^{2k+1}) \to \mathfrak{h}^k(b\partial \Delta^{2k+1})$ is the 0–map.

First let us suppose that $v$ is a vertex of dimension 0, i.e. a vertex of $\Delta^{2k+1}$.
Consider the action of the symmetric group $S_{2k+1}$ on $\Delta^{2k+1}$ which fixes the vertex $v$ and permutes other vertices. This action gives a simplicial action of $S_{2k+1}$ on $b\partial \Delta^{2k+1}$ and therefore, after choosing a base point, lifts to a cubical action of $S_{2k+1}$ on $\Sigma_{b\partial \Delta^{2k+1}}$ stabilizing $\Sigma_{b\partial \Delta^{2k+1}}$. Let $G'$ be the group of cubical automorphisms of $\Sigma_{b\partial \Delta^{2k+1}}$ generated by this action and the standard action of $W_{b\partial \Delta^{2k+1}}$, and let $G$ be the orientation-preserving subgroup of $G'$. Similarly, let $G''_v$ be the group of cubical automorphisms of $\Sigma_{b\partial \Delta^{2k+1}}$ generated by this action and the standard action of $W_{b\partial \Delta^{2k+1}}$, and let $G''_v$ be the orientation-preserving subgroup of $G''_v$.

We claim that, as a Hilbert $G_v$–module $L^2H^k(\Sigma_{b\partial \Delta^{2k+1}})$, is a submodule of $\ell^2(G_v)$. Note that $b\partial \Delta^{2k+1}$ is naturally isomorphic to $b\partial \Delta^{2k}$.

Using the inductive assumption and Lemma 3.3, we can remove from $b\partial \Delta^{2k}$ all odd-dimensional vertices without changing the $L^2$-cohomology: $h^*(E_{2k}) = h^*(b\partial \Delta^{2k})$. Since the action of $S_{2k+1}$ on $b\partial \Delta^{2k+1} = b\partial \Delta^{2k}$ preserves the dimension of the vertices, we have isomorphism $L^2H^*(G_v \Sigma_{E_{2k}}) = L^2H^*(\Sigma_{b\partial \Delta^{2k}})$ as Hilbert $G_v$–modules.

The complex $E_{2k}$ is spanned by the even-dimensional vertices of $b\partial \Delta^{2k}$, which correspond to the proper subsets of vertices of $\Delta^{2k}$ of odd cardinality. Thus, the dimension of $E_{2k}$ is $k - 1$, and its top-dimensional simplices are chains $v_0 \prec v_1 \prec v_2 \prec \ldots \prec v_0 \ldots v_{2k-2}$ of length $k$ of distinct vertices of $\Delta^{2k}$. Therefore $S_{2k+1}$ acts transitively on $(k - 1)$–dimensional simplices of $E_{2k}$ and it follows that $G_v$ acts transitively on $k$–dimensional cubes of $G_v \Sigma_{E_{2k}}$. Therefore the space of $k$–cochains is a Hilbert $G_v$–submodule of $\ell^2(G_v)$, and the claim follows from the Hodge decomposition.

We have, by construction, $G_v = \text{Stab}_G(\Sigma_{b\partial \Delta^{2k+1}})$. Then the restriction map $i^*: L^2H^k(\Sigma_{b\partial \Delta^{2k+1}}) \to L^2H^k(\Sigma_{b\partial \Delta^{2k+1}})$ is $G_v$–equivariant and therefore its image is a $G_v$–invariant subspace of $L^2H^k(\Sigma_{b\partial \Delta^{2k+1}})$. Since $G_v$ acts preserving orientation, the intersection form is $G_v$–invariant. By Lemma 4.1 the image is isotropic, thus by Lemma 4.2 it is 0. Thus, the map $i^*: h^k(b\partial \Delta^{2k+1}) \to h^k(b\partial \Delta^{2k+1}) = L^2H^k(W_{b\partial \Delta^{2k+1}} \Sigma_{b\partial \Delta^{2k+1}})$ is the 0–map.

For vertices of the other even dimensions the argument is similar. If $\dim(v) = 2d$, then its link is $b\partial \Delta^{2d} * b\partial \Delta^{2k-2d}$. Again, using Lemma 3.3 we remove, without changing the $L^2$–cohomology, all odd-dimensional vertices from each factor to obtain $E_{2d} * E_{2k-2d}$. The group $S_{2d+1} \times S_{2k-2d+1}$ acts naturally on $b\partial \Delta^{2k+1}$ fixing the vertex $v$ and stabilizing both the link and $E_{2d} * E_{2k-2d}$. This action is again transitive on the top-dimensional simplices of $E_{2d} * E_{2k-2d}$, and the rest of the argument goes through.
Finally, if \( v \) is an odd-dimensional vertex, \( \dim(v) = 2d + 1 \), then we have \( b\partial \Delta^2_{v+1} = b\partial \Delta^{2d+1} \ast b\partial \Delta^{2k-2d-1} \). The hypothesis on \( III'' \) and Theorem \ref{thm2.2} and Lemma \ref{lem2.4} imply that both \( I(2d+1) \) and \( I(2k-2d-1) \) hold. Therefore, by the Künneth formula \( \gamma^k(b\partial \Delta^2_{v+1}) = 0 \) in this case.

**Theorem 5.2** The Generalized Singer Conjecture holds true for \( b\partial \Delta^{2k+1} \):
\[ b^i_q(b\partial \Delta^{2k+1}) = 0 \text{ for } i > k \text{ and } q \leq 1. \]

**Proof** We proceed by induction on \( k \). Using the inductive assumption and Lemma \ref{lem3.2}, we can remove all odd-dimensional vertices without changing the weighted \( L^2 \)-homology above \( k \). Thus, since the remaining part \( E_{2k+1} \) is \( k \)-dimensional, the problem reduces to showing that \( h^q_{k+1}(E_{2k+1}) = 0 \) for \( q \leq 1 \). Since \( E_{2k+1} \) is \( k \)-dimensional, the natural map \( h^q_{k+1}(E_{2k+1}) \to h^q_{k+1}(E_{2k+1}) \) is injective and the result follows from the Theorem \ref{thm5.1}.

Next, we list some consequences. Lemma \ref{lem3.1} implies:

**Theorem 5.3** Let \( bL \) be the barycentric subdivision of an \((n-1)\)-dimensional simplicial complex \( L \). Then
\[ b^i_q(bL) = 0 \text{ for } i > (n+1)/2 \text{ and } q \leq 1. \]

Taking \( L \) to be a GHS\(^{2k-1} \), we obtain:

**Theorem 5.4** The Generalized Singer Conjecture holds true for the barycentric subdivision of a GHS\(^{n-1} \) for all even \( n \).

For odd \( n \) we obtain a weaker statement:

**Theorem 5.5** Let \( bL \) be the barycentric subdivision of a GHS\(^{2k} \). Then
\[ b^i_q(bL) = 0 \text{ for } i > k + 1 \text{ and } q \leq 1. \]

In particular,
\[ b^i(bL) = 0 \text{ for } i \neq k, k + 1. \]

Specializing further, and combining with Theorem \ref{thm5.2}, we obtain:

**Theorem 5.6** The Generalized Singer Conjecture holds true for \( b\partial \Delta^n \):
\[ b^i_q(b\partial \Delta^n) = 0 \text{ for } i > n/2 \text{ and } q \leq 1, \]

and, therefore, for the class \( JD \).

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Finally, let us mention an application of the above result to a more analytic object. Let $T_n$ denote the space of all symmetric tridiagonal $(n+1) \times (n+1)$--matrices with fixed generic eigenvalues, the so-called Tomei manifold. It is proved in [8] that $T_n$ is an $n$–dimensional closed aspherical manifold.

**Theorem 5.7** The Singer Conjecture holds true for Tomei manifolds $T_n$.

**Proof** The space $T_n$ can be identified with a natural finite index orbifoldal cover of $\Sigma_{b\partial \Delta^n}/W_{b\partial \Delta^n}$ [3]. Thus $\Sigma_{b\partial \Delta^n}$ is the universal cover of $T^n$, and the claim follows from the previous theorem. □

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