Nonisomorphic Ordered Sets with Arbitrarily Many Ranks That Produce Equal Decks

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Abstract

We prove that for any $n$ there is a pair $(P^n_1, P^n_2)$ of nonisomorphic ordered sets such that $P^n_1$ and $P^n_2$ have equal maximal and minimal decks, equal neighborhood decks, and there are $n + 1$ ranks $k_0, \ldots, k_n$ such that for each $i$ the decks obtained by removing the points of rank $k_i$ are equal. The ranks $k_1, \ldots, k_n$ do not contain extremal elements and at each of the other ranks there are elements whose removal will produce isomorphic cards. Moreover, we show that such sets can be constructed such that only for ranks 1 and 2, both without extremal elements, the decks obtained by removing the points of rank $r_i$ are not equal.

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1 Introduction

The reconstruction problem for ordered sets asks if it is possible to reconstruct the isomorphism type of a given ordered set from its collection (the “deck”) of one-point-deleted subsets. In [10], Sands asked if ordered sets might even be reconstructible from the the collection of subsets obtained by erasing single maximal elements (the “maximal deck”). The negative answer to Sands’ question in [10] together with the paper [3] were the starting point for serious investigation of the reconstruction problem for ordered sets. (Reconstruction of graphs and other relations has a longer history, see [1, 6, 7, 16].)

Since then, results that show reconstructibility given certain types of information (for example, see [3]) as well as results on reconstructibility of certain classes of ordered sets (for example, see [9]) have been proved. For a more comprehensive survey of results and references available to date, consider [8, 12].

Recently in [13] it has been shown that even the maximal and minimal decks together are not sufficient to reconstruct ordered sets. Moreover, it was shown in [13] that there are families of $2^{O(\sqrt{n})}$ pairwise nonisomorphic ordered sets of size $n$ that all have equal maximal and minimal decks.

The desire to rely on only a limited, focused amount of information in a reconstruction proof, and the fact that almost all ordered sets are reconstructible from two identifiable maximal cards (see [14], Corollary 3.10) motivates extensions of Sands’ question. What subsets of the deck have a reasonable chance to effect reconstruction? In [15] it was shown that the maximal deck plus the minimal deck plus one deck obtained by removing points of a rank $k$ that contains no extremal elements are not sufficient for reconstruction. The examples in [15] are somewhat limited. They could not be extended to an ambiguity with more than two sets. It also appeared as if they could not be extended to more than two maximal elements, two minimal elements and two elements of rank $k$ or to more than one middle rank producing equal decks. Immediately two questions arise.

- Are ordered sets reconstructible from the maximal deck, the minimal deck and a deck obtained by removing points of rank $k$ if one of these decks has at least three cards?

- Are ordered sets reconstructible from the maximal deck, the minimal deck and two decks that were obtained by removing points of ranks $k$ and $l$ with $k \neq l$?
In this paper these two questions are answered negatively, even if the neighborhood decks are equal and for any two isomorphic cards the neighborhoods of the removed elements are also isomorphic. The present examples provide new guidance as to what kind of partial information is at least needed to reconstruct ordered sets. In particular, they show that information derived from “small” ranks, and even from many small ranks, is not sufficient to effect reconstruction. Analysis of the examples also leads to results that underscore the role of rigidity in order reconstruction (cf. Section 6). Ideas on what types of information to consider next are given in the conclusion.

2 Basic Definitions and Preliminaries

An ordered set is a set $P$ equipped with a reflexive, antisymmetric and transitive relation $\leq$, the order relation. Throughout this paper we will assume that all ordered sets involved are finite. Elements $x, y \in P$ are called comparable iff $x \leq y$ or $y \leq x$. An antichain is an ordered set in which each element is only comparable to itself. A chain is an ordered set in which any two elements are comparable. The length of a chain is its number of elements minus 1. An element $m \in P$ is called maximal iff for all $x$ comparable to $m$ we have $x \leq m$. Minimal elements are defined dually. The rank of an element $x \in P$ is the length of the longest chain that has a minimal element as its smallest element and $x$ as its largest element.

The dual $P^d$ of an ordered set $P$ is the ordered set obtained by reversing all comparabilities. The dual rank of an element $x \in P$ is the length of the longest chain that has a maximal element as its largest element and $x$ as its smallest element.

A function $f : P \to Q$ from the ordered set $P$ to the ordered set $Q$ is called order-preserving iff for all $x, y \in P$ we have that $x \leq y$ implies $f(x) \leq f(y)$. The function $\varphi : P \to Q$ is called an (order) isomorphism iff $\varphi$ is bijective, order-preserving and $\varphi^{-1}$ is order-preserving, too. An order isomorphism with equal domain and range is called an (order) automorphism. An ordered set with exactly one order automorphism (the identity) is called rigid.

For precise overall reconstruction terminology, cf. [12, 14, 15]. For the purposes of this paper, a card of an ordered set is a subset with one point deleted. If the deleted element is of rank $k$ we shall also call the card a rank $k$ card. The set of all cards obtained by erasing elements of rank $k$ is called
the rank $k$ deck. The set of all rank $k$ decks is called the ranked deck. A rank $k$ card is marked iff there is a function that indicates the rank of each element in the original set. The set of all marked rank $k$ cards is the marked rank $k$ deck and the set of all marked rank $k$ decks is the marked ranked deck. A maximal card is a subset in which a maximal element is erased and a minimal card is a subset in which a minimal element is erased. The sets of maximal and minimal cards respectively are called the maximal deck and the minimal deck. Marked maximal cards are maximal cards for which there is a function that indicates which elements are maximal in the original set. The set of all marked maximal cards is called the marked maximal deck. Marked minimal cards and the marked minimal deck are defined dually. Isomorphic cards will also be called equal cards, because their isomorphism classes are equal. Decks will be called equal iff there is a bijection such that each card is isomorphic to its image. Marked cards will be called equal iff there is an isomorphism that preserves the marked property (rank in the original set or maximality/minimality in the original set). Marked decks will be called equal iff there is a bijection such that any set is isomorphic to its image and each isomorphism also preserves the marked property. The up-set of an element $x$ is the set $\uparrow x = \{ p \in P : p \geq x \}$ and the down-set is $\downarrow x = \{ p \in P : p \leq x \}$. The neighborhood of an element $x$ is the set $\uparrow \downarrow x = \uparrow x \cup \downarrow x$. The set of all neighborhoods of points of rank $k$ is called the rank $k$ neighborhood deck.

We are concerned with results that show what type of information is not sufficient to effect reconstruction. Therefore, throughout we construct nonisomorphic ordered sets such that between some of their cards there are isomorphisms with certain properties.

3 Pairs of Nonisomorphic Ordered Sets with Equal Maximal and Minimal Decks and $n + 1$ Non-Extremal Ranks for Which the Rank $k$ Decks are Equal

In this section we describe the fundamental construction used to build the examples. Lemma 3.1 gives the overall idea, which is an extension of the work in [15]. Lemma 3.1 is also quite similar to the examples in [7]. In a
way, Lemma 3.1 is reminiscent of a “vertical Möbius strip”.

Lemma 3.2 then shows that sets as needed in the construction in Lemma 3.1 actually exist. In the following, we explore some features of the construction as well as variations that lead to examples with other properties.

**Lemma 3.1** Let $Q$ be an ordered set such that

1. $Q$ has exactly two maximal elements $d$ and $p$,
2. $d$ and $p$ have the same rank,
3. There is an isomorphism $\psi : Q \setminus \{d\} \rightarrow Q \setminus \{p\}$ with $\psi(p) = d$,
4. $Q$ has two minimal elements $a$ and $b$,
5. $Q \setminus \{a\}$ has an automorphism $\psi^a$ with $\psi^a(p) = d$, $\psi^a(d) = p$, and $\psi^a(b) = b$,
6. $Q \setminus \{b\}$ has an automorphism $\psi^b$ with $\psi^b(p) = d$ and $\psi^b(d) = p$, and $\psi^b(a) = a$,
7. $Q$ is rigid.

Let $R$ be an ordered set such that

8. $R$ has exactly two maximal elements $d$ and $p$,
9. $d$ and $p$ have the same rank,
10. $R$ has exactly two minimal elements $\overline{d}$ and $\overline{p}$,
11. $R \setminus \{\overline{p}\}$ has an automorphism $\psi^{\overline{p}}$ such that $\psi^{\overline{p}}(d) = p$, $\psi^{\overline{p}}(p) = d$, and $\psi^{\overline{p}}(\overline{d}) = \overline{d}$,
12. $R \setminus \{\overline{d}\}$ has an automorphism $\psi^{\overline{d}}$ such that $\psi^{\overline{d}}(d) = p$, $\psi^{\overline{d}}(p) = d$, and $\psi^{\overline{d}}(\overline{p}) = \overline{p}$,
13. $R$ has an automorphism $\varphi$ with $\varphi(d) = p$, $\varphi(p) = d$, $\varphi(\overline{d}) = \overline{p}$, $\varphi(\overline{p}) = \overline{d}$,
14. $R$ has no automorphism that is the identity on the minimal elements and not the identity on the maximal elements.
Let $\tilde{Q}$ be the dual of $Q$ and let $R_1, \ldots, R_n$ be isomorphic copies of $R$ such that $Q, \tilde{Q}, R_1, \ldots, R_n$ are all mutually disjoint. Let the elements of $R_i$ be distinguished by subscripts $i$, that is, the maximal and minimal elements of $R_i$ are $d_i, p_i$ and $\overline{d}_i, \overline{p}_i$. Let the elements of $\tilde{Q}$ similarly be distinguished by tildes.

Define $P_1$ to be the ordered set obtained from $Q, R_1, \ldots, R_n, \tilde{Q}$ as follows (also cf. Figure 1).

i. All non-maximal elements of $Q$ are below all non-minimal elements of $R_1$.

ii. The element $d$ is identified with the element $\overline{d}_1$ and the element $p$ is identified with the element $\overline{p}_1$. Call the thus obtained elements $d_0$ and $p_0$, respectively.

iii. For $i = 1, \ldots, n - 1$, all non-maximal elements of $R_i$ are below all non-minimal elements of $R_{i+1}$.

iv. For $i = 1, \ldots, n - 1$, the element $d_i$ is identified with the element $\overline{d}_{i+1}$ and the element $p_i$ is identified with the element $\overline{p}_{i+1}$. Call the thus obtained elements $d_i$ and $p_i$, respectively.

v. All non-maximal elements of $R_n$ are below all non-minimal elements of $\tilde{Q}$.

vi. The element $d_n$ is identified with the element $\overline{d}$ and the element $p_n$ is identified with the element $\overline{p}$. Call the thus obtained elements $d_n$ and $p_n$, respectively.

vii. Plus all comparabilities forced by transitivity.

Define $P_2$ to be the ordered set obtained from $Q, R_1, \ldots, R_n, \tilde{Q}$ in the same way as $P_1$ except that $\overline{d}_i$ is replaced with the following (also cf. Figure 1).

vi'. The element $d_n$ is identified with the element $\overline{p}$ and the element $p_n$ is identified with the element $\overline{d}$. Call the thus obtained elements $d_n$ and $p_n$, respectively.

Then

a. $P_1$ and $P_2$ are not isomorphic,
b. $P_1$ and $P_2$ have equal marked maximal and minimal decks,

c. For $i = 0, \ldots , n$ the card $P_1 \setminus \{d_i\}$ is isomorphic to the card $P_2 \setminus \{d_i\}$ and the card $P_1 \setminus \{p_i\}$ is isomorphic to the card $P_2 \setminus \{p_i\}$, and for all isomorphisms $\Phi$ of cards and all elements $x \in P_1$ we have that $\text{rank}_{P_2}(\Phi(x)) = \text{rank}_{P_1}(x)$.

d. For all the above mentioned isomorphic cards $P_1 \setminus \{x\}$ and $P_2 \setminus \{x\}$ the neighborhoods $\downarrow_P x$ and $\uparrow_P x$ are isomorphic.

**Proof.** To prove (a) we assume that $P_1$ and $P_2$ are isomorphic. So suppose that $\Phi : P_1 \to P_2$ is an isomorphism. Then, because $Q$ is rigid, we have that $\Phi(a) = a, \Phi(b) = b, \Phi(d_0) = d_0$ and $\Phi(p_0) = p_0$. By property (c) this implies that $\Phi(d_1) = d_1, \Phi(p_1) = p_1, \ldots , \Phi(d_n) = d_n, \Phi(p_n) = p_n$. But then, because in $P_1$ we have $d_n = d$ and $p_n = \tilde{p}$, while in $P_2$ we have $d_n = \tilde{d}$ and $p_n = \tilde{d}$, $\Phi|_\tilde{Q}$ would be an automorphism of $\tilde{Q}$ with $\Phi|_\tilde{Q}(d) = \tilde{p}$ and $\Phi|_\tilde{Q}(\tilde{p}) = \tilde{d}$. This is a contradiction to the rigidity of $\tilde{Q}$. Therefore, $P_1$ and $P_2$ cannot be isomorphic.

To show (b) first note that $P_1 \setminus \{a\}$ is isomorphic to $P_2 \setminus \{a\}$. To see this let $\varphi_i$ denote the automorphism for $R_i$ guaranteed by property (b). We define

$$\Phi(x) := \begin{cases} 
\psi^a(x); & \text{if } x \in Q \setminus \{a\}, \\
\varphi_i(x); & \text{if } x \in R_i, \\
x; & \text{if } x \in \tilde{Q} \setminus \{\tilde{d}, \tilde{p}\}.
\end{cases}$$

The function $\Phi$ is well-defined and bijective between $P_1 \setminus \{a\}$ and $P_2 \setminus \{a\}$ and it maps minimal elements of $P_1$ to minimal elements of $P_2$. To see that $\Phi$ is order-preserving both ways, let $x < y$ in $P_1$. It is trivial to infer that $\Phi(x) < \Phi(y)$ is equivalent to $x < y$ unless $x \in \{\tilde{d}, \tilde{p}\}$. Assume without loss of generality that $x = \tilde{d} = d_n$. Then $\Phi(x) = p_n = \tilde{d} = x$. Since $y > x$ we have $y \in \tilde{Q}$ and thus $\Phi(y) = y$. The other direction, as well as the proof for $x = \tilde{p} = p_n$ is similar.

We have shown that $P_1 \setminus \{a\}$ is isomorphic to $P_2 \setminus \{a\}$ and that the isomorphism preserves the marked property “minimality”. Similarly, $P_1 \setminus \{b\}$ is isomorphic to $P_2 \setminus \{b\}$ (and minimal elements of $P_1$ are mapped to minimal elements of $P_2$) via

$$\Phi(x) := \begin{cases} 
\psi^b(x); & \text{if } x \in Q \setminus \{b\}, \\
\varphi_i(x); & \text{if } x \in R_i, \\
x; & \text{if } x \in \tilde{Q} \setminus \{\tilde{d}, \tilde{p}\}.
\end{cases}$$
Figure 1: Ordered sets $P_1$ and $P_2$ as constructed in Lemma 3.1. The mentioned symmetry is symmetry along the vertical axis.
The proof that $P_1$ and $P_2$ have equal marked maximal decks is similar. The set $P_1 \setminus \{\tilde{a}\}$ is isomorphic to $P_2 \setminus \{\tilde{a}\}$ via

$$
\Phi(x) := \begin{cases} 
\psi^{\tilde{a}}(x); & \text{if } x \in \tilde{Q} \setminus \{\tilde{a}\}, \\
x; & \text{if } x \notin \tilde{Q}, 
\end{cases}
$$

where $\psi^{\tilde{a}}$ denotes the automorphism of $\tilde{Q} \setminus \{\tilde{a}\}$ that is guaranteed by the dual of property 5. Clearly $\Phi$ maps maximal elements of $P_1$ to maximal elements of $P_2$. The set $P_1 \setminus \{\tilde{b}\}$ is isomorphic to $P_2 \setminus \{\tilde{b}\}$ (and maximal elements of $P_1$ are mapped to maximal elements of $P_2$) via

$$
\Phi(x) := \begin{cases} 
\psi^{\tilde{b}}(x); & \text{if } x \in \tilde{Q} \setminus \{\tilde{b}\}, \\
x; & \text{if } x \notin \tilde{Q}, 
\end{cases}
$$

where $\psi^{\tilde{b}}$ denotes the automorphism of $\tilde{Q} \setminus \{\tilde{b}\}$ that is guaranteed by the dual of property 6.

In regards to $\square$ note that the above isomorphisms show that for $x \in \{a, b, \tilde{a}, \tilde{b}\}$ the neighborhoods $\uparrow_{P_1} x$ and $\uparrow_{P_2} x$ are isomorphic. (For example, the isomorphism between $P_1 \setminus \{b\}$ and $P_2 \setminus \{\tilde{b}\}$ provides an isomorphism between $\uparrow_{P_1} a = \uparrow_{P_1} a$ and $\uparrow_{P_2} a = \uparrow_{P_2} a$.)

For $\lozenge$ let it be stated here that it is easy to see that all isomorphisms $\Phi$ constructed in the following satisfy $\text{rank}_{P_2}(\Phi(x)) = \text{rank}_{P_1}(x)$ for all elements $x \in P_1$.

Now first notice that the set $P_1 \setminus \{d_n\}$ is isomorphic to $P_2 \setminus \{d_n\}$ via

$$
\Phi(x) := \begin{cases} 
\tilde{\psi}(x); & \text{if } x \in \tilde{Q} \setminus \{\tilde{d}\}, \\
x; & \text{if } x \notin \tilde{Q}, 
\end{cases}
$$

where $\tilde{\psi}$ is the isomorphism guaranteed by the dual of property 3. The set $P_1 \setminus \{p_n\}$ is isomorphic to $P_2 \setminus \{p_n\}$ via

$$
\Phi(x) := \begin{cases} 
\left(\tilde{\psi}\right)^{-1}(x); & \text{if } x \in \tilde{Q} \setminus \{\tilde{p}\}, \\
x; & \text{if } x \notin \tilde{Q}, 
\end{cases}
$$

Finally let $i \in \{0, \ldots, n - 1\}$. For $R_i$ denote the automorphisms $\psi^{\tilde{p}}$ and $\psi^{\tilde{d}}$ of the respective cards of $R$ guaranteed by properties $\blacksquare$ and $\blacklozenge$ by $\psi_i^{\tilde{p}}$ and $\psi_i^{\tilde{d}}$. Then the set $P_1 \setminus \{p_i\}$ is isomorphic to $P_2 \setminus \{p_i\}$ via

$$
\Phi(x) := \begin{cases} 
\left(\psi_i^{\tilde{p}}\right)^{-1}(x); & \text{if } x \in \tilde{Q} \setminus \{\tilde{p}\}, \\
x; & \text{if } x \notin \tilde{Q}, 
\end{cases}
$$
A easy to verify, properties of the sets above isomorphisms show that for cards on which the respective "other" element of that rank has been erased. 

\( R \) is a set as desired in the description of the set neighborhoods \( \uparrow \). There is an ordered set dually isomorphic to \( B \) \( \downarrow \). This implies that the automorphism must be the identity on \( \{ \} \) and then it must be the identity on \( \{ \} \) to \( \{ \} \). This isomorphism maps \( C \) to itself and the sets \( \{ \} \) to \( \{ \} \). Therefore, properties proved for \( B \) and \( C \) will hold dually for \( A \) and \( D \). The set \( B \cup \{ c, d, p \} \) is rigid. This is because any automorphism of \( B \cup \{ c, d, p \} \) must map \( c \) to itself and the sets \( B_1, B_2 \) and \( B_{1,2} \) to themselves, respectively. This implies that the automorphism must be the identity on \( B \) and then it must be the identity on \( \{ d, p \} \) also.

Moreover, there is exactly one isomorphism from \( B \cup \{ c, d, p \} \) to \( C \). This isomorphism maps \( d \) to \( p \), \( p \) to \( d \), \( c \) to \( c \) and \( B_{1,2} \) to \( C_{1,2} \), \( B_1 \) to \( C_1 \) and \( B_2 \) to \( C_2 \). Furthermore, there is exactly one isomorphism from \( B \cup \{ c, d \} \) to \( C \). This isomorphism maps \( d \) to \( c \), \( c \) to \( c \), \( B_{1,2} \) to \( C_1 \), \( B_2 \) to \( C_{1,2} \) and \( B_1 \) to \( C_2 \). Similarly, there is exactly one isomorphism from \( B \cup \{ c, p \} \) to \( C \). This isomorphism maps \( p \) to \( c \), \( c \) to \( c \), \( B_{1,2} \) to \( C_2 \), \( B_2 \) to \( C_1 \) and \( B_1 \) to \( C_{1,2} \). These facts and their duals will be used freely in the following.

For Claim \( \{ \} \) define \( \psi \) to be

\[
\Phi(x) := \begin{cases} 
  x; & \text{if } x \in \bar{Q} \setminus \{ \bar{d}, \bar{p} \} \text{ or } x \in R_j \text{ for } j \leq i \text{ or } x \in Q, \\
  \varphi_i(x); & \text{if } x \in R_j \text{ for } j > i + 1, \\
  \psi^d_i(x); & \text{if } x \in R_{i+1} \setminus \{ p_i \}.
\end{cases}
\]

All parts of the definition of isomorphism are readily verified. Similarly, the set \( P_1 \setminus \{ d_i \} \) is isomorphic to \( P_2 \setminus \{ d_i \} \) via

\[
\Phi(x) := \begin{cases} 
  x; & \text{if } x \in \bar{Q} \setminus \{ \bar{d}, \bar{p} \} \text{ or } x \in R_j \text{ for } j \leq i \text{ or } x \in Q, \\
  \varphi_i(x); & \text{if } x \in R_j \text{ for } j > i + 1, \\
  \psi^d_i(x); & \text{if } x \in R_{i+1} \setminus \{ d_i \}.
\end{cases}
\]

To finish the proof of \( \{ \} \) similar to what was said after the proof of \( \{ \} \) the above isomorphisms show that for \( x \in \{ d_0, p_0, \ldots, d_{n-1}, p_{n-1} \} \) the neighborhoods \( \uparrow \) \( x \) and \( \downarrow \) \( x \) are isomorphic. Just use the isomorphism between the cards on which the respective “other” element of that rank has been erased.

\[ \blacksquare \]

**Lemma 3.2** There is an ordered set \( R \) as described in Lemma 3.1.

**Proof.** Let \( R \) be the ordered set indicated in Figure 4. We claim that \( R \) is a set as desired in the description of the set \( R \) in Lemma 3.1.

Claims \( \{ \} \) and \( \{ \} \) are trivial. The rest of the proof relies on the following, easy to verify, properties of the sets \( A \), \( B \), \( C \) and \( D \). First, \( A \cup \{ c, d, p \} \) is dually isomorphic to \( B \cup \{ c, d, p \} \) and \( D \cup \{ c, d, p \} \) is dually isomorphic to \( C \cup \{ c, d, p \} \). Therefore, properties proved for \( B \) and \( C \) will hold dually for \( A \) and \( D \). The set \( B \cup \{ c, d, p \} \) is rigid. This is because any automorphism of \( B \cup \{ c, d, p \} \) must map \( c \) to itself and the sets \( B_1, B_2 \) and \( B_{1,2} \) to themselves, respectively. This implies that the automorphism must be the identity on \( B \) and then it must be the identity on \( \{ d, p \} \) also.

For Claim \( \{ \} \) define \( \psi \) to be

\[ 10 \]
Figure 2: An ordered set $R$ as needed in Lemma 3.1 and constructed in Lemma 3.2. The middle levels form an ordered set of height 1. The maximal element $d$ is above all maximal elements of the middle levels except the circled maximal elements of the middle levels. The maximal element $p$ is above all maximal elements of the middle levels except the boxed maximal elements of the middle levels. Similarly, the minimal element $\overline{d}$ is below all minimal elements of the middle levels except the circled minimal elements and the minimal element $\overline{p}$ is below all minimal elements of the middle levels except the boxed minimal elements. Within the middle levels, the connected dotted arches indicate that the elements immediately above the upper arch and the elements immediately below the lower arch form a complete bipartite.
1. $\psi(p) := \overline{d}$, $\psi(d) := p$, $\psi(p) := d$,

2. On $A \cup \{c_t\}$, $\psi$ is the restriction of the unique isomorphism from $A \cup \{c_t, \overline{d}\}$ to $D \cup \{c_t, \overline{d}\}$,

3. On $D \cup \{c_t\}$, $\psi$ is the restriction of the unique isomorphism from $D \cup \{c_t, \overline{d}\}$ to $A \cup \{c_t, d\}$,

4. On $B \cup \{c_b\}$, $\psi$ is the restriction of the unique isomorphism from $B \cup \{c_b, d, p\}$ to $C \cup \{c_b, d, p\}$,

5. On $C \cup \{c_b\}$, $\psi$ is the restriction of the unique isomorphism from $C \cup \{c_b, d, p\}$ to $B \cup \{c_b, d, p\}$.

Then $\psi$ is as desired. Claim 12 is proved similarly.

For Claim 13 define $\varphi$ to be

1. $\varphi(\overline{d}) := \overline{p}$, $\varphi(\overline{p}) := \overline{d}$, $\varphi(d) := p$, $\varphi(p) := d$,

2. On $A \cup \{c_t\}$, $\varphi$ is the restriction of the unique isomorphism from $A \cup \{c_t, \overline{d}, \overline{p}\}$ to $D \cup \{c_t, \overline{d}, \overline{p}\}$,

3. On $D \cup \{c_t\}$, $\varphi$ is the restriction of the unique isomorphism from $D \cup \{c_t, \overline{d}, \overline{p}\}$ to $A \cup \{c_t, d, p\}$,

4. On $B \cup \{c_b\}$, $\varphi$ is the restriction of the unique isomorphism from $B \cup \{c_b, d, p\}$ to $C \cup \{c_b, d, p\}$,

5. On $C \cup \{c_b\}$, $\varphi$ is the restriction of the unique isomorphism from $C \cup \{c_b, d, p\}$ to $B \cup \{c_b, d, p\}$.

Finally, for Claim 14 suppose the automorphism $\Psi : R \to R$ is the identity on the minimal elements. Then, because the unique isomorphism between $A \cup \{c_t, \overline{d}, \overline{p}\}$ and $D \cup \{c_t, \overline{d}, \overline{p}\}$ switches $\overline{d}$ and $\overline{p}$, $\Psi$ must map $A \cup \{c_t\}$ to $A \cup \{c_t\}$ and $D \cup \{c_t\}$ to $D \cup \{c_t\}$. Therefore, $\Psi$ must map $B \cup \{c_b\}$ to $B \cup \{c_b\}$ and $C \cup \{c_b\}$ to $C \cup \{c_b\}$. Since $B \cup \{c_b, d, p\}$ is rigid, this means that $\Psi$ must fix $d$ and $p$.

The set $R$ in Figure 2 has an additional property that will allow us to prove further properties of our examples.

**Lemma 3.3** The sets $R \setminus \{c_b\}$ and $R \setminus \{c_t\}$ with $R$ as in Figure 2 each have an automorphism $\Psi$ with $\Psi(\overline{d}) = \overline{d}$, $\Psi(\overline{p}) = \overline{p}$, $\Psi(d) = p$ and $\Psi(p) = d$. 

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Proof. For $R \setminus \{c_t\}$ we define $\Psi(d) := \overline{d}$, $\Psi(\overline{p}) := \overline{p}$, $\Psi(p) := p$ and $\Psi(d) := d$. We let $\Psi$ map $A_{1,2}$ to $D_{1,2}$, $A_2$ to $D_1$, $A_1$ to $D_2$ and vice versa. (Visually, in each case the map is obtained by sliding one wedge horizontally onto the other.)

On $B \cup \{c_b, d, p\}$ we define $\Psi$ to be the unique isomorphism from $B \cup \{c_b, d, p\}$ to $C \cup \{c_b, d, p\}$. Finally, on $C \cup \{c_b, d, p\}$ we define $\Psi$ to be the unique isomorphism from $C \cup \{c_b, d, p\}$ to $B \cup \{c_b, d, p\}$.

For $R \setminus \{c_t\}$ we let $\Psi$ be the identity on $\{d, p, c_t\} \cup A \cup D$. We let $\Psi(d) = p$ and $\Psi(p) = d$. Finally $\Psi$ maps $B_{1,2}$ to itself, $B_1$ to $B_2$, $B_2$ to $B_1$, $C_{1,2}$ to itself, $C_1$ to $C_2$, $C_2$ to $C_1$, each in such a way that the comparabilities with the appropriate maximal elements are preserved.

With Lemmas 3.1 and 3.2 proved, we can state our first main result. Aside from insights on ranks and maximal and minimal cards, we also see that our construction yields pairs of nonisomorphic sets for which a significant number of cards is isomorphic.

Definition 3.4 For two ordered sets $P_1$ and $P_2$ with $n$ elements each, define the equal card ratio $ECR(P_1, P_2)$ to be the number of isomorphic cards divided by the size of the set.

Theorem 3.5 There is a sequence of pairs of ordered sets $(P_1^n, P_2^n)$ such that

1. $P_1^n$ is not isomorphic to $P_2^n$,
2. $P_1^n$ and $P_2^n$ have equal marked maximal and minimal decks,
3. There are ranks $k_0, \ldots, k_n$ such that $P_1^n$ and $P_2^n$ have equal marked rank $k_i$ decks,
4. For any pair of isomorphic cards as in parts 2 and 3, the neighborhoods of the respective removed elements are isomorphic,
5. There are only four ranks that do not produce any isomorphic cards,
6. $\liminf_{n \to \infty} ECR(P_1^n, P_2^n) \geq \frac{1}{10}$,
7. For every rank $k$, the rank $k$ neighborhood decks of $P_1^n$ and $P_2^n$ are equal.
Proof. To construct sets as indicated, with notation as in Lemma 3.1 we make the following choices.

1. For all \( n \), as the set \( Q \), use a fixed set \( Q \) as in the proof of Theorem 5.3 of [15]. These sets have height 3.

2. For all \( n \), as the set \( R \), use a fixed set \( R \) as guaranteed by Lemma 3.2.

3. The ranks \( k_0, \ldots, k_n \) are \( k_i = \text{height}(Q) + i \cdot \text{height}(R) \).

This construction, independent of the choices for \( Q \) and \( R \), yields a sequence of sets that satisfy all parts of this theorem except possibly parts 5, 6 and 7. For these parts the construction must be done with the set \( R \) indicated in the proof of Lemma 3.2.

For part 5 note that by Lemma 3.3 the erasure of an element \( c_t \) or \( c_b \) at corresponding ranks produces isomorphic marked cards for \( P^n_1 \) and \( P^n_2 \). This means that only at the ranks 1 and 2 and at the two ranks immediately below the maximal elements will there be no elements whose removal produces isomorphic cards.

For part 6, first note that because each pair of sets has at least \( 4n + 6 \) equal cards we have

\[
ECR(P^n_1, P^n_2) \geq \frac{4n + 6}{n(|R| - 2) + 2|Q| - 2} \xrightarrow{n \to \infty} \frac{4}{|R| - 2}.
\]

That is, our lower bound on \( \liminf_{n \to \infty} ECR(P^n_1, P^n_2) \) will solely depend on the number of elements that \( R \) has.

With the set \( R \) as given in the proof of Lemma 3.2 we have \( |R| = 42 \) and so \( \liminf_{n \to \infty} ECR(P^n_1, P^n_2) \geq \frac{4}{42 - 2} = \frac{1}{10} \).

For part 7 we know that the neighborhoods \( \uparrow_{P^n_1} x \) and \( \downarrow_{P^n_2} x \) are isomorphic for \( x \in \{a, b, \tilde{a}, \tilde{b}, d_0, p_0, \ldots, d_n, p_n\} \). This leaves the non-extremal elements of the \( R_i \) and the non-extremal elements of \( Q \) and \( \tilde{Q} \).

We first consider the non-extremal elements of the \( R_i \). For \( x \in A \cup D \) we have that \( \uparrow_{R} x \setminus \{x\} \) is a four-crown if \( x \) is minimal and a two-antichain if \( x \) is maximal. Similarly, \( \uparrow_{R} x \setminus \{x\} \) is a two-antichain for the maximal elements of \( C \) and \( D \) that are below both \( d \) and \( p \) as well as for \( c_t \). This means that \( \downarrow_{R} x \) has an automorphism that fixes \( \mathcal{D} \) and \( \mathcal{P} \) and that switches \( d \) and \( p \), which means that for these elements (in any of the \( R_i \)) we have that \( \uparrow_{P^n_1} x \) and \( \downarrow_{P^n_2} x \) are isomorphic. For the minimal elements of \( B_{1,2} \) and \( C_{1,2} \), call
them $b_{1,2}$ and $c_{1,2}$, the set $\updownarrow R x \setminus \{x\}$ is the disjoint union of two 2-chains and for these elements (in any $R_i$) we have $\downarrow P_i b_{1,2}$ is isomorphic to $\downarrow P_2 c_{1,2}$ and $\downarrow P_1 c_{1,2}$ is isomorphic to $\downarrow P_2 b_{1,2}$. For the minimal elements of $B_1$ and $B_2$, call them $b_1$ and $b_2$, the set $\uparrow R x \setminus \{x\}$ is an “$N$” and for these elements (in any $R_i$) we have $\uparrow P_1 b_1$ is isomorphic to $\uparrow P_2 b_2$. The corresponding elements of $C$ are handled similarly. The remaining maximal elements of $B$ and $C$ are below exactly one of $d$ and $p$, call this element $m$. For these elements (in any $R_i$) we prove that $\uparrow P_1 m$ is isomorphic to $\uparrow P_2 m$ as follows. In an $R_i$ with $i < n$ this is because $R_{i+1} \setminus \{m\}$ has an automorphism that switches the maximal elements of $R_{i+1}$. In $R_n$ this is because $\tilde{Q} \setminus \{\tilde{d}\}$ is isomorphic to $\tilde{Q} \setminus \{\tilde{p}\}$. Finally, $\uparrow R c_b \setminus \{c_b\}$ is made up of two sets “M” with their maximal elements identified. This means that $\downarrow P_i c_b$ is isomorphic to $\downarrow P_2 c_b$.

To consider the non-extremal elements of $Q$, we need to use sets $Q$ as in [15]. For all non-extremal elements $x$ of $Q$ the same ideas as above show that $\downarrow P_1 x$ and $\downarrow P_2 x$ are isomorphic. If the strict upper bounds of $x$ in $Q$ are a set (such as a two antichain, a four crown or the disjoint union of two 2-chains) that allows interchanging the maximal elements, we get an isomorphism. If not, the strict upper bounds in $Q$ are a singleton, in which case the switch can be done in $R_1$, or the strict upper bounds form an “$N$”. By choosing the appropriate permutations in the construction of the set $Q$, this last situation can be avoided. This takes care of the non-extremal elements of $Q$. The argument for the non-extremal elements of $\tilde{Q}$ is the dual of the above.

Remark 3.6 It may be considered unsatisfying that the ordered sets in Lemma 3.1 are such that the ranks $k$ that yield equal rank $k$ decks are separated from each other. A small modification in the construction can also produce adjacent ranks such that the rank $k$ decks of two nonisomorphic sets are equal. The ordered set in Figure 3 has the same symmetry properties as the sets $R$ in Lemma 3.1 except that $d$ and $p$ are not minimal.

If these sets are now “stacked” as indicated in Figure 4 with $d$’s merged with $d$’s and $p$’s merged with $p$’s we obtain a tower structure as in Lemma 3.1 without $Q$ and $\tilde{Q}$ attached. The only difference is that the points that eventually yield equal cards are concentrated in the four crown tower. Replacing stretches of sets $R$ in a set as in Lemma 3.1 with sets as in Figure 4 introduces stretches in which arbitrarily many consecutive ranks produce equal rank $k$ decks. Also note that the elements of the four crown tower in Figure 4 are the only elements with rank exceeding 2 and that we can again show that the resulting sets will have equal rank $k$ neighborhood decks for
Figure 3: Another construction for middle level sets “$R$”, which allows adjacent ranks $k, k + 1$ with equal rank $k$ and rank $k + 1$ decks. Here, $d$ is above all maximal elements of $B \cup C$, except those that are circled. $p$ is above all maximal elements of $B \cup C$, except those that are boxed. By transitivity, $d$ and $p$ are above all elements of $B \cup C$. Moreover, $d$ is above all maximal elements of $A \cup D$, except those that are circled. $p$ is above all maximal elements of $A \cup D$, except those that are boxed.
4 A Folding Operation

The construction that leads to Theorem 3.5 produces ordered sets that are “tall” in the sense that their height can exceed their width by an arbitrary factor. Moreover, the ranks that produce equal rank $k$ decks are, with 2 elements, as small as can be. It is possible to produce “wider” examples by folding the examples in Theorem 3.5 appropriately. This idea is explored in this section.

**Definition 4.1** Let $P$ be a connected finite ordered set. An antichain $A \subseteq P$ will be called a **seam** iff the removal of

1. All comparabilities $x < y$ such that there are $a, b \in A$ with $x < a$ and $b < y$, and of

2. The antichain $A$ and all comparabilities involving points of $A$, disconnects the ordered set $P$. Call the resulting ordered set the **$A$-separation of $P$**. A seam will be called **foldable** iff

   F) For all $x, y \in P$ we have that if $x < y$ and there are $a, b \in A$ such that $x < a$ and $b < y$, then there is a $c \in A$ such that $x < c < y$.

   A seam will be called **breakable** iff

   B) For all $x, y \in P$ we have that if there are $a, b \in A$ such that $x < a$ and $b < y$, then $x < y$.

Let $A$ be a foldable or breakable seam in $P$, and let $F$ (the part to be folded) and $S$ (the part that will stay as is) be nonempty unions of components of the $A$-separation of $P$ such that in $P$ no element of $S$ is above any element of $F$. We define $P_{F,S}$ (also cf. Figure 5) to be the ordered set obtained from $P$ by

1. Erasing all comparabilities $x < y$ in $P$ such that there are $a, b \in A$ with $x < a$ and $b < y$,

2. Keeping the remaining comparabilities in $S \cup A$ as is,
Figure 4: A construction to combine sets as in Figure 3 to obtain sets in which many consecutive ranks have equal rank \( k \) decks. Fine structure of the involved sets is only indicated roughly by showing with arrows which boxes (boxes are stand ins for the sets \( A, B, C, D \)) go with which points. Crossovers “point on left to box on right” and symmetrically, as well as points \( c_b \) and \( c_t \) are omitted. The bottom set (oval with some structure inside) can be a set \( R \) or a set \( Q \). The two bottom points are lower bounds of all elements in the boxes. None of the boxes have any comparabilities between them unless they are connected, and the connection signifies comparability between the minimal and the maximal elements in the indicated direction. This implies that the elements of the four crown tower are the only elements in this structure whose rank exceeds 2.
3. Reversing all comparabilities in $F \cup A$.

(It is easy to see that this “folding operation” produces an ordered set.)

The folding operation of Definition 4.1 is well behaved with respect to isomorphism, isomorphism of cards and isomorphism of neighborhoods as the next lemma shows.

**Lemma 4.2** Let $P, P'$ be connected ordered sets and let $k \in \mathbb{N}$ be such that $A := \{a \in P : \text{rank}_P(a) = k\} \subseteq P$ and $A' := \{a' \in P' : \text{rank}_{P'}(a') = k\} \subseteq P'$ are both foldable seams (or both breakable seams) in $P$ and $P'$ respectively. Moreover assume that we can choose $F, S, F'$ and $S'$ as follows.

1. $F$ and $S$ are nonempty unions of components of the $A$-separation of $P$ such that $F$ contains no points that are below any element of $A$ and $S$ contains no points that are above any element of $A$.

2. $F'$ and $S'$ are nonempty unions of components of the $A'$-separation of $P'$ such that $F'$ contains no points that are below any element of $A'$ and $S'$ contains no points that are above any element of $A'$.

3. All elements of $P$ that have dual rank $\geq k$ are in $S \cup A$ and all elements of $S$ have rank less than $k$.

4. All elements of $P'$ that have dual rank $\geq k$ are in $S' \cup A'$ and all elements of $S'$ have rank less than $k$.

5. No component of $F \cup A$ is isomorphic to the dual of a component of $S' \cup A'$.

Figure 5: The folding operation in Definition 4.1
6. No component of $F' \cup A'$ is isomorphic to the dual of a component of $S' \cup A$. 

Then the following hold.

1. $P$ is isomorphic to $P'$ iff $P_{F,S}$ is isomorphic to $P_{F',S'}'$.

2. For all points $p \in P$ and $p' \in P'$ such that $P \setminus \{p\}$ is isomorphic to $P' \setminus \{p'\}$ via an isomorphism that maps $A \setminus \{p\}$ to $A' \setminus \{p'\}$, $S \setminus \{p\}$ to $S' \setminus \{p'\}$ and $F \setminus \{p\}$ to $F' \setminus \{p'\}$, we have that $P_{F,S} \setminus \{p\}$ is isomorphic to $P_{F',S'}' \setminus \{p'\}$.

3. For all points $p \in P$ and $p' \in P'$ such that $\text{rank}_P(p) = \text{rank}_{P'}(p')$ and the neighborhood $\downarrow_P p$ is isomorphic to the neighborhood $\downarrow_{P'} p'$, we have that $\text{rank}_{P_{F,S}}(p) = \text{rank}_{P_{F',S'}}'(p')$ and the neighborhood $\downarrow_{P_{F,S}} p$ is isomorphic to the neighborhood $\downarrow_{P_{F',S'}}' p'$.

**Proof.** For part 1, first let $P_{F,S}$ be isomorphic to $P_{F',S'}'$ via the isomorphism $\Phi : P_{F,S} \rightarrow P_{F',S'}'$. By hypotheses 3 and 4 we have that $A = \{p \in P_{F,S} : \text{rank}_{P_{F,S}}(p) = k\}$ and $A' = \{p' \in P_{F',S'}' : \text{rank}_{P_{F',S'}'}(p') = k\}$. Therefore $\Phi[A] = A'$. Then by hypotheses 5 and 6 we have that $\Phi|_{S \cup A}$ is an isomorphism between $S \cup A$ and $S' \cup A'$ and $\Phi|_{F \cup A}$ is an isomorphism between $F \cup A$ and $F' \cup A'$, where the respective sets carry the orders that are induced by $P_{F,S}$ and $P_{F',S'}'$, respectively. Since the folding construction only reverses the orders on $F \cup A$ and $F' \cup A'$, $\Phi|_{S \cup A}$ is an isomorphism between $S \cup A$ and $S' \cup A'$ and $\Phi|_{F \cup A}$ is an isomorphism between $F \cup A$ and $F' \cup A'$, where the respective sets carry the orders that are induced by $P$ and $P'$, respectively. Now if both $A$ and $A'$ are foldable seams, then all comparabilities between elements of $F$ and $S$ (and of $F'$ and $S'$) are induced by transitivity through an element of $A$ ($A'$ respectively). This implies that $\Phi$ is an isomorphism between $P$ and $P'$. If both $A$ and $A'$ are breakable seams, then all comparabilities between elements of $F$ and $S$ (and of $F'$ and $S'$) that are related to any element of $A$ ($A'$ respectively) are present in $P$ ($P'$ respectively) and these are all comparabilities between elements of $F$ and $S$ (and of $F'$ and $S'$). Again $\Phi$ is an isomorphism between $P$ and $P'$.

For the converse, let $P$ be isomorphic to $P'$ via the isomorphism $\Psi : P \rightarrow P'$. Then $\Psi|_{S \cup A}$ is an isomorphism between $S \cup A$ and $S' \cup A'$ and $\Psi|_{F \cup A}$ is an isomorphism between $F \cup A$ and $F' \cup A'$, where the respective sets carry
the orders that are induced by $P$ and by $P'$, respectively. Since there are no comparabilities between elements of $F$ and $S$ ($F'$ and $S'$ respectively) in $P_{F,S}$ ($P'_{F',S'}$), this means that $\Psi$ is also an isomorphism between $P_{F,S}$ and $P'_{F',S'}$.

For part 2 let $\Psi : P \setminus \{p\} \rightarrow P \setminus \{p'\}$ be an isomorphism that maps $A \setminus \{p\}$ to $A' \setminus \{p'\}$, $S \setminus \{p\}$ to $S' \setminus \{p'\}$ and $F \setminus \{p\}$ to $F' \setminus \{p'\}$. Then $\Psi$ is a bijection between $P_{F,S} \setminus \{p\}$ and $P'_{F',S'} \setminus \{p'\}$ that is an isomorphism between $(F \cup A) \setminus \{p\}$ and $(F' \cup A') \setminus \{p'\}$ and between $(S \cup A) \setminus \{p\}$ and $(S' \cup A') \setminus \{p'\}$, respectively. Thus $\Psi$ is an isomorphism between $P_{F,S} \setminus \{p\}$ and $P'_{F',S'} \setminus \{p'\}$.

For part 3 first note that because the neighborhoods are isomorphic, the dual rank of $p$ in $P$ is equal to the dual rank of $p'$ in $P'$. Since $p$ and $p'$ have the same rank in their respective unfolded sets, their ranks in the folded sets is either their original rank or their original dual rank. Either way, their ranks in the folded sets are equal. If the original rank of $p$ and $p'$ is less than $k$, then their neighborhoods in the folded sets are obtained by discarding all elements of rank greater than $k$ from the original neighborhoods. If the original rank of $p$ and $p'$ is greater than $k$, then their neighborhoods in the folded sets are obtained by discarding all elements of rank less than $k$ from the original neighborhoods and dualizing the order. If $p \in A$ and $p' \in A'$, then their neighborhoods in the folded sets are obtained from the original neighborhoods by folding at $p$ or $p'$, respectively. In all cases the neighborhoods in the folded sets are isomorphic.

Theorem 4.3 For every sequence of natural numbers $\{s_n\}_{n \geq 1}$, there is a sequence of pairs of ordered sets $(Q^n_1, Q^n_2)$ such that

1. $Q^n_1$ is not isomorphic to $Q^n_2$,

2. $Q^n_1$ and $Q^n_2$ have equal maximal and minimal decks,

3. There are ranks $r_0, \ldots, r_n$ such that $Q^n_1$ and $Q^n_2$ have equal marked rank $r_i$ decks, and the ranks $r_1, \ldots, r_n$ do not contain extremal elements,

4. For each rank $k$ there is at least one element of rank $k$ in $Q^n_1$ and $Q^n_2$ such that removal of these elements produces isomorphic marked rank $k$ cards,

5. For all the above mentioned isomorphic cards $Q^n_1 \setminus \{x\}$ and $Q^n_2 \setminus \{x\}$ the neighborhoods $\uparrow_{Q^n_1} x$ and $\uparrow_{Q^n_2} x$ are isomorphic.
6. $\liminf_{n \to \infty} ECR(Q_1^n, Q_2^n) \geq \frac{1}{10}$.

7. For every rank $k$, the rank $k$ neighborhood decks of $Q_1^n$ and $Q_2^n$ are equal.

8. $Q_1^n$ and $Q_2^n$ have at least $s_n$ maximal elements, at least $s_n$ minimal elements and for each $k_i$ at least $s_n$ elements of rank $k_i$.

**Proof.** This result is a direct consequence of Theorem 3.5 and Lemma 4.2. Without loss of generality, assume that $s_n$ is even (otherwise replace it with $s_n + 1$). Now let $t_n := \frac{s_n}{2}(n + 2)$, consider the pair of sets $(H^{\frac{t_n}{2}}, K^{\frac{t_n}{2}}) := (P_{t_1}, P_{t_2})$ from Theorem 3.5 and let $k_0, \ldots, k_n$ be the ranks mentioned in Theorem 3.5.

Now for $i \in \{s_n/2, \ldots, 1\}$, obtain $(H^{i-1}, K^{i-1})$ from $(H^i, K^i)$ by folding the sets $H^i, K^i$ as indicated in Lemma 4.2 at rank $[\text{height}(Q) + i(n+2)\text{height}(R)]$. At each fold, the height of the part $S$ that stays as is is greater than the height of the part $F$ that is folded and both parts are connected. Thus there cannot be any isomorphisms between duals and Lemma 4.2 applies without a problem.

In a last step, apply the dual of Lemma 4.2 at rank $[\text{height}(Q)]$ (which is possible because the height of $Q$ can be chosen to be equal to the height of $R$).

Let $(Q_1^n, Q_2^n)$ be the pair of ordered sets thus obtained. By Lemma 4.2 the pair $(Q_1^n, Q_2^n)$ is as desired. The last step of folding up the bottom set $Q$ guarantees part 4.

**Remark 4.4** While the examples in Theorem 4.3 are not counterexamples to the reconstruction conjecture, they show that the subtlety of order reconstruction reaches beyond tools available today. All parameters that we know to be reconstructible are equal for these sets. Moreover, several parameters, such as maximal and minimal decks as well as rank $k$ decks, plus appropriate markings, are equal also. These parameters have not yet been proven to be reconstructible.

**Remark 4.5** Using sets as in Remark 3.6 and their duals and folding appropriately, using Lemma 4.2 and its dual, it is now possible to produce pairs of nonisomorphic ordered sets of arbitrary height for which all ranks except ranks 1 and 2 produce equal rank $k$ decks, for which all ranks have arbitrarily
Figure 6: An ordered set $R$ as needed in Lemma 3.1. Connected dotted arches again indicate a complete bipartite structure. The sets in this figure can be used to construct sets with equal maximal and minimal decks for which the ratio of the number of extremal elements to the size of the set is as large as possible to date.

many elements and for which even in ranks 1 and 2 there are arbitrarily many elements whose removal produces isomorphic cards. The equal card ratio will approach at least $\frac{1}{10}$ with the constructions available in this paper.

Remark 4.6 Aside from results in this paper, the only construction to obtain sets with equal maximal decks with $k > 3$ cards is due to Ille and Rampon (cf. [8], Section 8.2.2). The size of their sets is exponential in the number of maximal elements. The size of the sets in Theorem 4.3 is linear in the number of maximal elements. In this construction, to date the largest ratio of extremal elements to the size of the set is achieved with sets $R$ as in Figure 6. The ratio approaches $\frac{1}{19}$ and it is achieved by folding at every merge of sets $R$ and at the merge of the top set $R$ with $\tilde{Q}$. With sets as presented earlier, the ratio approaches $\frac{1}{20}$. 

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5 Further Consequences of the Folding Construction

Aside from giving access to examples on reconstruction, the folding construction of Definition 4.1 also allows access to some results that can simplify the start of a reconstruction proof.

Recall that two elements $x, y$ of an ordered set are called adjacent iff $x < y$ and for all $z \in P$ with $x \leq z \leq y$ we have $z \in \{x, y\}$ (or the dual of this statement). An ordered set $P$ is called graded iff there is a function $g : P \to \mathbb{N}$ such that if $x, y \in P$ are adjacent then $|g(x) - g(y)| = 1$. If the function $g$ can be chosen to be the function that assigns each element its rank, we say $P$ is graded by the rank function.

**Proposition 5.1** If ordered sets of height 2 are reconstructible, then all ordered sets that are graded by the rank function are reconstructible from their marked ranked decks.

**Proof.** Suppose $P$ and $P'$ were two nonisomorphic ordered sets that are graded by their rank functions and which have equal ranked decks. Note that in an ordered set that is graded by the rank function, every rank $P_k$ is a foldable seam. Repeatedly folding $P$ and $P'$ as indicated in Lemma 4.2 at the rank that is one less than the height of the set and stopping when the resulting sets have height 2, would create two nonisomorphic ordered sets of height 2 with the same deck. (Equality of the marked ranked decks is needed here to make sure that all hypotheses in part 2 of the conclusion of Lemma 4.2 are satisfied.) We have a contradiction to the assumption that ordered sets of height 2 are reconstructible.

**Remark 5.2** Note that except for extremely symmetric sets, we could actually fold until we have an ordered set of height 1 in the proof of Proposition 5.1. This observation underscores once more the importance of understanding more about the reconstruction of ordered sets of height 1.

**Proposition 5.3** If all ordered sets with at most one foldable or breakable seam are reconstructible from their marked ranked decks, then all ordered sets are reconstructible from their marked ranked decks.

**Proof.** Suppose $P_1$ and $P_2$ are two nonisomorphic ordered sets that have equal marked ranked decks. Find the two consecutive seams for which the
difference in the ranks is maximal. Then fold all seams except these two. In a final step, one of these two can be folded also. The result are two nonisomorphic ordered sets with equal marked ranked decks and at most one foldable or breakable seam.

Remark 5.4 Proposition 5.3 is reminiscent of the result in [17]. Removal of a foldable seam disconnects the covering graph, while removal of a breakable seam is a step towards disconnecting it (cross-connections still need to be erased). This means for reconstruction work, some type of “vertical connectivity” could be assumed in any attempt to reconstruct ordered sets. Unlike the result in [17], there is no restriction on the size of the separating set.

Remark 5.5 Similar to Remark 5.2 in the proof of Proposition 5.3 unless we encounter a highly symmetrical situation, both seams can be folded at the end of the proof, leaving ordered sets without seams.

6 The Role of Rigidity

We have seen that the rigid bottom and top sets $Q$ and $\tilde{Q}$ play a crucial role in the development of the examples presented here. Without these rigid “anchors” the sets $P_1$ and $P_2$ would “untwist” and be isomorphic to each other. Thus it is reasonable to shed some light on the role of rigidity in reconstruction. The following results show that isomorphism between various types of rigid substructures immediately leads to an isomorphism between the sets. Such rigid substructures are often recognizable from a maximal card or would be recognizable if the marked ranked deck is available. Thus the presence of such structures leads to reconstructibility. Consequently, if there is a counterexample to the order reconstruction conjecture, then it must be made up largely of non-rigid structures. For other properties that a counterexample must have, cf. [14, 15].

In particular, the results in this section seem to indicate that any development of counterexamples based on the present examples will need to

- Remove the rigid “anchors” without letting the two sets become isomorphic,
- Avoid the introduction of rigid structures in middle levels.

In particular, the results in this section seem to indicate that any development of counterexamples based on the present examples will need to
To state the results in this section, for an ordered set $P$ we denote

\[ P_k := \{ x \in P : \text{rank}(x) = k \} \]
\[ P_{k \downarrow} := \{ x \in P : \text{rank}(x) \leq k \} \]
\[ P_{k \uparrow} := \{ x \in P : \text{rank}(x) \geq k \} \].

**Proposition 6.1** Let $k \geq 0$ and let $P, Q$ be ordered sets with equal marked rank $k$ decks. If

1. $|P| > 1$ and $|Q| > 1$, and

2. No two elements of rank $k$ have the same strict upper and lower bounds in either $P$ or $Q$, and

3. $P \setminus P_k$ and $Q \setminus Q_k$ are both rigid,

then $P$ is isomorphic to $Q$.

**Proof.** Let $P \setminus \{p\}$ and $Q \setminus \{q\}$ be isomorphic rank $k$ cards and let $\psi : P \setminus \{p\} \to Q \setminus \{q\}$ be an isomorphism that preserves the original rank of each element. Let $P \setminus \{p'\}$ and $Q \setminus \{q'\}$ be isomorphic rank $k$ cards with $p \neq p'$, $q \neq q'$ and let $\phi : P \setminus \{p'\} \to Q \setminus \{q'\}$ be an isomorphism that preserves the original rank of each element. Then $\phi|_{P \setminus P_k} = \psi|_{P \setminus P_k}$.

If $\psi(x) = q'$ and $x \neq p'$, then $\psi(p') \neq q'$, which means $y := \phi^{-1}(\psi(p')) \neq p'$. This means that $y$ and $p'$ have the same sets of strict upper and lower bounds, a contradiction. Thus $\psi(p') = q'$ and symmetrically $\phi(p) = q$.

Now for any element $x \in P_k \setminus \{p, p'\}$, the inequality $\psi(x) \neq \phi(x)$ would imply $\phi^{-1}(\psi(x)) \neq x$ has the same strict upper and lower bounds as $x$, a contradiction. Thus for all $x \in P_k \setminus \{p, p'\}$, we have $\psi(x) = \phi(x)$.

Via $\phi|_{P \setminus P_k} = \psi|_{P \setminus P_k}$ we immediately conclude that

\[ \Phi(x) := \begin{cases} 
\psi(x); & \text{for } x \neq p, \\
\phi(p); & \text{for } x = p,
\end{cases} \]

is an isomorphism between $P$ and $Q$. \hfill \blacksquare

**Definition 6.2** Let $P$ be an ordered set and let $0 < k < l$. The subset $P_{k \uparrow} \cap P_{l \downarrow}$ of $P$ is called a rigid separator iff $P_{k \uparrow} \cap P_{l \downarrow}$ is rigid, there are elements in $P$ of rank $> l$ and no element of rank $< k$ is a lower cover of an element of rank $> l$.

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Proposition 6.3  Let $P$ and $Q$ be ordered sets and let $0 < k < l$. If

1. $P_k \uparrow \cap P_{l \downarrow}$ and $Q_k \uparrow \cap Q_{l \downarrow}$ are rigid separators,

2. There is an isomorphism $\phi : P \setminus \{M_P\} \to Q \setminus \{M_Q\}$, where $M_P$ and $M_Q$ denote maximal elements of rank $r > l$ in $P$ and $Q$, respectively, and where $\text{rank}_Q(\phi(x)) = \text{rank}_P(x)$ for all $x$,

3. There is an isomorphism $\psi : P \setminus \{m_P\} \to Q \setminus \{m_Q\}$, where $m_P$ and $m_Q$ denote minimal elements in $P$ and $Q$, respectively, and where $\text{rank}_Q(\psi(x)) = \text{rank}_P(x)$ for all $x$,

then $P$ is isomorphic to $Q$.

Proof. Because we must have $\phi|_{P_k \uparrow \cap P_{l \downarrow}} = \psi|_{P_k \uparrow \cap P_{l \downarrow}}$ and because there are no adjacencies that “cross” $P_k \uparrow \cap P_{l \downarrow}$ the function

$$\Phi(x) := \begin{cases} \psi(x); & \text{if rank}(x) > l, \\ \phi(x); & \text{if rank}(x) \leq l, \end{cases}$$

is an isomorphism between $P$ and $Q$. \qed

7  Conclusion

The examples presented in this paper show that even substantial partial information on the deck of an ordered set is not sufficient to effect reconstruction. In particular (see Theorem 4.3), the maximal deck plus the minimal deck plus $n + 1$ rank $k$ decks, plus the rank $k$ neighborhood decks are not sufficient for reconstruction even if all these decks have substantially more than 2 cards. Moreover, by Remark 4.5 even all rank $k$ decks except for the ones for rank 1 and rank 2 are not sufficient to effect reconstruction. Again the absolute number of cards within these ranks is immaterial.

Future reconstruction research has to take these facts into account. Proof attempts that do not consider enough information from the deck will not succeed. On the other hand, the examples presented here show what kind of information might effect reconstruction or lead to a counterexample.

1. All examples of ordered sets with equal maximal, minimal and some equal rank $k$ decks so far have “small waists”. That is, the ranks that
have equal decks all have fewer elements (by at least a factor 6, even if we use sets $R$ as in Figure 4) than other ranks nearby. It thus should be instructive to investigate what can be concluded from equality of rank $k$ decks, where $k$ is such that no other rank has more elements than the $k$th rank.

2. Alternatively, the construction of examples could be expanded to the point where the overlap between the decks of two nonisomorphic sets becomes so large that the decks would indeed have to be equal. The present examples have been shown to be quite malleable. They might point the way towards examples with similarly strong properties and larger equal card ratios or maybe even a counterexample overall. The largest equal card ratios in ordered sets observed so far slightly exceed 50% (see [2]), but the examples do not have the equal subdecks that the examples presented here have.

3. Along these lines, would it be true that if there is a sequence of pairs of nonisomorphic ordered sets $(P^n_1, P^n_2)$ such that $ECR(P^n_1, P^n_2) \to 1$ as $n \to \infty$, then there is a counterexample to the reconstruction conjecture?

4. On p. 185 of [6], P. Stockmeyer is quoted to have said only half in jest that “The reconstruction conjecture [for graphs] is not true, but the smallest counterexample has 87 vertices and will never be found.” The present examples seem to show that if there is a counterexample to the order reconstruction conjecture, it would have to be of substantial size and complexity. At the same time, the approach of analyzing macrostructure as in Lemma 3.1 and microstructure as in Lemma 3.2 separately may allow to break down the complexity to manageable stages.

5. Finally, Proposition 5.3 shows that we can concentrate on ordered sets that have a certain type of “vertical connectivity”.

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