Low temperature expansion for $p = 2$ soft spin dynamics

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In this letter, we address the low-temperature expansion issue for the closed equation arising from the asymptotic behavior of the $p = 2$ spin dynamics with sixtic confining potential. The dynamics are described by a Langevin equation for a real vector $q_i$ of size $N$, where disorder is materialized by a Wigner matrix. We identify the radius of convergence of the expansion as the critical temperature, and we investigate the large time behavior of the averaging square length $a(t) \equiv \sum_{i} q_i^2(t)/N$, depending on the values of couplings involved in the potential. Finally, our methods generalize for higher potential, and we provide a sketched derivation of the main relations in that case.

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I. INTRODUCTION

Glassy systems are usually characterized by their static properties, as replica symmetry breaking is the most famous example. Alternatively, they can be characterized by their dynamical aspects, and never reach equilibrium for experimental time scales below the “glass” transition temperature. As the transition point is reached, relaxation time increase and the decay toward equilibrium becomes slower than exponential law [1]. The soft $p$-spin model is a popular mathematical incarnation of such a glassy system [2]-[11] see also [15, 16] and references therein. It describes the dynamics of $N$ random variables $q_i \in \mathbb{R}$ through a Langevin-like equation where disorder is materialized by a rank $p$ random real and symmetric tensor $J_{i_1i_2\cdots i_p}$:

$$\frac{dq_i}{dt} = -\frac{\partial}{\partial q_i} V_{ij}[q(t)] - \ell(t)q_i(t) + \eta_i(t),$$

where:

$$V_{ij}[q] := \frac{1}{p} \sum_{i_1,\cdots,i_p} J_{i_1i_2\cdots i_p} q_{i_1} \cdots q_{i_p},$$

$$\eta_i(t)$$ is a Gaussian random field with Dirac delta correlations:

$$\langle \eta_i(t)\eta_j(t') \rangle = 2T \delta_{ij} \delta(t-t'),$$

and the function $\ell(t)$ avoids large values configurations for $q_i$’s. The parameter $T$ involved in the definition (3) identifies physically as the temperature regarding the equilibrium states. For the spherical model, $\sum_{i=1}^{N} q_i^2 = N$, and $\ell(t)$ is a Lagrange multiplier. Alternatively, $\ell(t)$ can be an $O(N)$ invariant polynomial function: $\ell(t) := 2\sigma + \sum_{n} h_n a^n(t)$ with $a(t) := \sum_i q_i^2(t)/N$. In the large $N$ limit, the spherical $p = 2$ spin dynamics has been investigated analytically twenty-five years ago [2], exploiting Wigner semi-circle law for the eigenvalue distribution for the disorder $J_{ij}$. As a result, even though the $p = 2$ spherical spin glass looks like a ferromagnet in disguise [1, 3] rather than a true spin glass regarding its statics properties, its dynamical aspects are however non-trivial. Indeed below the critical temperature $T_c$, the system never reaches equilibrium with exponential decay except for very special initial “staggered” configurations for $q_i(t = 0)$’s and ergodicity is weakly broken. As for the static limit, this behavior is reminiscent of the domain coarsening for a ferromagnet in the low-temperature phase [4], where equilibrium fails as the size of the domains with positive and negative magnetization grows in time.

About the case of a confining potential and relaxing the strict spherical constraint, the existent literature mainly focuses on the quartic potential $\ell(t) := 2\sigma + h_1 + h_2 a(t)$ [12, 13], but higher polynomials are difficult to address with standard methods. In this letter, we mainly focus on the sixtic issue for $p = 2$. We suggest a general formalism to investigate the asymptotic closed equation satisfied by $a(t)$ in the large $N$ limit. This equation arises in the quenching limit because $a(t)$ self-average for large $N$ is trapped by local stable minimums depending on the shape of the potential. Our methods are based on a low-temperature expansion, which has a finite radius of convergence that we identify with the critical temperature. Finally, we expect that the method discussed in this paper generalizes straightforwardly for higher potential, and we provide a sketched derivation for the general case.

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1 In particular, no replica symmetry breaking occurs.
II. CLOSED EQUATION FOR THE $p = 2$ SPIN DYNAMICS

A. Closed equation for large time

In eigenspace of $J_{ij}$, the Langevin equation (1) for $p = 2$ reads:

$$\frac{dq^\lambda}{dt} = -[\lambda + \ell(t)]q^\lambda(t) + \eta^\lambda(t), \quad \text{(4)}$$

where for large $N$, eigenvalues $\lambda$ are assumed to display accordingly with the Wigner semi-circle law [14] with variance $\sigma^2$:

$$\frac{1}{N} \sum_\lambda f(\lambda) \rightarrow \int_{-2\sigma}^{2\sigma} \mu(\lambda) f(\lambda) d\lambda, \quad \text{(5)}$$

where $\mu(\lambda)$ is the standard Wigner distribution:

$$\mu(\lambda) := \frac{\sqrt{4\sigma^2 - \lambda^2}}{2\pi\sigma^2}. \quad \text{(6)}$$

Equation (4) can be solved formally taking $t = 0$ as the initial condition:

$$q^\lambda(t) = q^\lambda(0) e^{-t(2\sigma + \lambda)} \rho(t)$$
$$+ \int_0^t dt' e^{-t(2\sigma + \lambda)(t-t')} \eta^\lambda(t') \frac{\rho(t)}{\rho(t')} \quad \text{(7)}$$

with:

$$\rho(t) := e^{2\sigma t - J_0^2 dt' \ell(t')} \quad \text{(8)}$$

We are aiming to investigate the large-time behavior of the Langevin equation (4), focusing on the function $g(t)$ defined as:

$$g(t) = \int_0^t dt' \ell(t'). \quad \text{(9)}$$

Assuming uniform initial condition for $q^\lambda$, namely $q^\lambda(0) = 1/\forall \lambda^2$, we get after the quench for the expectation value of $q^2(t)$:

$$\langle q^2 \rangle = e^{-2(\lambda t + g(t))} + 2T \int_0^t dt' e^{-2(\lambda t - t') - 2(g(t) - g(t'))} \quad \text{(10)}$$

where $\sigma(t)$ is assumed to be self-averaged. Now, using the definition for $a(t)$ given in the introduction and

$$\ell(t) := (h_0 + 2\sigma) + \sum_{n=1}^K h_n a^n(t), \quad \text{(11)}$$

one can easily obtain a closed equation for $g(t)$ from the observation that thermal fluctuations have the effect of precipitating the system to the equilibrium point of the potential. Hence, we expect that for a long time $t \rightarrow 2\sigma$. For the quartic case, namely $K = 1$, this closed equation reads:

$$G(t) = -\frac{h_1}{h_2} \left(H(t) + 2TF(t)\right), \quad \text{(12)}$$

where $F(t)$ is the convolution:

$$F(t) = \int_0^t dt' H(t-t')G(t'), \quad \text{(13)}$$

and:

$$G(t) := \exp \left(2g(t) - 4\sigma t\right), \quad \text{(14)}$$

$$H(t) := \int_{-2\sigma}^{2\sigma} \mu(\lambda)e^{-2\lambda t - 4\sigma t} d\lambda. \quad \text{(15)}$$

Note that by integrating over the Wigner distribution, the exact value of the Function $H(t)$ is

$$H(t) = \frac{e^{-4\sigma t} J_1(4\sigma t)}{2\sigma} \quad \text{(16)}$$

where $J_n(x)$ is the standard first kind of Bessel function. Explicitly, for $\sigma = 1$ and $t$ large enough:

$$H(t) \approx \frac{1}{4\sqrt{2\pi t^2}} \quad \text{(17)}$$

For the sixtic case ($K=2$), the closed equation reads:

$$\frac{h_0}{h_2} G^2(t) + \frac{h_1}{h_2} A(t, G) G(t) + A^2(t, G) = 0 \quad \text{(18)}$$

where

$$A(t, G) := H(t) + 2TF(t). \quad \text{(19)}$$

The quartic potential has been considered in the literature [12], and the asymptotic closed equation is quite similar to the closed equation arising for the spherical case, which has been mainly addressed in the literature -- see [1] and reference therein for a detailed treatment of the spherical model. The difficulty with the confining potential is that the closed equation is only an asymptotic relation, whereas it holds for all time in spherical dynamics. In this paper, and accordingly, with the treatment done in appendix B of [18] that we will shortly review in the next subsection, we assume that the solution of the closed equation provides the true asymptotic behavior for $G(t)$. This can be motivated by the observation that, for $t$ large enough, $H(t-t')$ suppresses low-time contributions provided that $G(t)$ has a finite limit for short times. We do the same assumption for higher-order potentials that we consider in this paper.
B. Solving closed equation for quartic potential

The convolution equation (12) can be easily solved using Laplace transform, defined for some function \( f(t) \) as:

\[
\tilde{f}(p) := \int_0^\infty \text{e}^{-pt} f(t) \, \text{d}t.
\]

In that way, equation (12) reads:

\[
\tilde{G}(p) = -\frac{1}{2T + \frac{h_0}{h_1} H^{-1}(p)},
\]

where the Laplace transform of \( H(t) \) reads explicitly:

\[
\tilde{H}(p) = \frac{4\sigma + p - \sqrt{(4\sigma + p)^2 - 16\sigma^2}}{8\sigma^2}.
\]

The function \( \tilde{G}(p) \) has to be positive, hence 1) we must have \( h_0 < 0 \) and 2) \( T \) must be smaller than the critical temperature \( T_c \) defined as:

\[
T_c = -\frac{h_0}{h_1} \frac{1}{2H(0)} \approx -\frac{h_0}{h_1} \sigma,
\]

at which the expression for \( G(p) \) is singular. Note that the critical temperature is nothing but the radius of convergence of the power expansion in \( T \) of \( G(0) \). To understand the large time behavior of \( G(t) \), we expand \( H(p) \) around \( p = 0 \), and because of the integral:

\[
\int_0^\infty \frac{du}{\sqrt{u}(u + 2u)} \sim \frac{\pi}{\sqrt{2}} \frac{1}{p^{1/2}},
\]

we get:

\[
\tilde{H}(p) \approx \tilde{H}(0) - \frac{\sqrt{\sigma}}{4\sqrt{2}} p^{1/2},
\]

and:

\[
\tilde{G}(p) \approx \frac{1}{2(T_c - T)} \left[ 1 + \frac{\tilde{A}(\sigma) T_c^{1/2}}{T_c - T} + \mathcal{O}(p) \right],
\]

where \( \tilde{A}(\sigma) := \frac{h_0 \sqrt{\sigma}}{4h_1 \sqrt{2}} \). The asymptotic expression for \( G(t) \) can be obtained, for small \( p \), from standard results about the asymptotic expression of inverse Laplace transform near the origin, and for \( t \) large enough we get \( G(t) \sim t^{-3/2} \), implying: \( \tilde{G}(t) \sim 2\sigma + \mathcal{O}(t^{-1}) \). Hence the relaxation of the system below the critical temperature behaves as a power rather than an exponential, i.e. has infinite relaxation time.

C. Sixth asymptotic dynamics issue

In the sixtic case, the closed equation is no longer linear in \( G(t) \), and a direct approach like the one we just described is more complicated to implement. One attempt would be to use an ansatz for \( G(t) \) below some critical temperature, i.e.:

\[
G(t) \sim \frac{b}{t^{\alpha}},
\]

for \( t \) large enough, and to solve the closed equation (18) for the convolution \( f(t) \) defined in (13). We can then hope to find the exact asymptotic value of the function \( G \) and thus determine the critical temperature from the standard theorems on the Laplace transform near the origin. In [17] for instance, we have the following statement:

Theorem 1. Let \( f(t) \) be a locally integrable function on \([0, \infty)\) such that \( f(t) \approx \sum_{m=0}^{\infty} c_m t^{r_m} \) as \( t \to \infty \) where \( r_m < 0 \). If the Mellin transformation of this function is defined and if no \( r_m = -1, -2, \ldots \) then the Laplace transform of \( f(t) \) is

\[
\tilde{f}(p) = \sum_{m=0}^{\infty} c_m \Gamma(r_m + 1) p^{-r_m - 1} + \sum_{n=0}^{\infty} Mf(n + 1) \frac{(-p)^n}{n!}
\]

where \( Mf(z) = \int_0^\infty t^{z-1} f(t) \, dt \) is the Mellin transform of the function \( f(t) \).

Unfortunately, if these theorems allow determining the dominant behavior of a large \( t \) by fixing the value of \( \alpha (= 3/2) \), they do not allow fixing the constant \( b \), and thus the critical temperature. In particular, Mellin’s transform cannot be computed because our ansatz (27) holds only for large \( t \). The low-temperature expansion that we discuss in the next section allows for overcoming these difficulties.

III. SMALL TEMPERATURE EXPANSION

In this section we consider a small \( T \) expansion for the function \( G(t) \), namely:

\[
G(t) = \sum_{n=0}^{\infty} T^n G^{(n)}(t),
\]

assumed to have a finite radius of convergence, which we identify with the critical temperature below. The functions \( G^{(n)}(t) \) can be constructed recursively from the closed equation (18). Let us investigate the quartic case as an illustration. It is straightforward to check that functions \( \{G^{(n)}(t)\} \) satisfy the following recurrence relations:

\[
G^{(0)}(t) = -\frac{h_1}{h_0} H(t),
\]

\[
G^{(n)}(t) = -\frac{2h_1}{h_0} \int_0^t dt' H(t - t') G^{(n-1)}(t'), \quad n > 0.
\]
Hence, the Laplace transform $\tilde{G}(p)$ of $G(t)$ reads:

$$\tilde{G}(p) = \frac{h_1}{h_0} \tilde{H}(p) \sum_{n=0}^{\infty} \left( - \frac{2h_1}{h_0} T \tilde{H}(p) \right)^n = \frac{h_1}{h_0} \frac{\tilde{H}(p)}{1 + \frac{2a_1}{h_0} T \tilde{H}(p)}. \quad (32)$$

Because $\tilde{H}(p)$ is a decreasing function of $p$ (see Figure (1), the radius of convergence $R$ is fixed by setting $p = 0$, and the series formally resumes as (21) for $T < R \equiv T_c$.

In the same way, for the sixth potential, we have the following statement:

**Proposition 1.** The functions $\{G^{(n)}\}$ involved in the low temperature expansion satisfy the recursion relation

$$G^{(n)}(t) = 2\gamma \int_0^t dt' H(t-t')G^{(n-1)}(t'), \quad (33)$$

with: $G^{(0)}(t) = \gamma H(t)$ and:

$$\gamma := -\frac{b}{2a} \left( 1 \pm \sqrt{1 - \frac{4a}{b^2}} \right), \quad (34)$$

where we set $a := \frac{h_0}{h_2}$ and $b := \frac{h_1}{h_2}$.

**Proof.** The proof follows a simple recurrence.

1) The zero-order of the $T$ expansion gives

$$a(G^{(0)}(t))^2 + bG^{(0)}(t)H(t) + H^2(t) = 0, \quad (35)$$

which leads to the solution $G^{(0)}(t) = \gamma H(t)$.

2) For order $T$ we have straightforwardly:

$$2aG^{(0)}(t)G^{(1)}(t) + 4H(t)F^{(0)}(t) + b\left( 2G^{(0)}(t)F^{(0)}(t) + G^{(1)}(t)H(t) \right) = 0 \quad (36)$$

where

$$F^{(n)}(t) := \int_0^t dt' H(t-t')G^{(n)}(t). \quad (37)$$

The solution to this first-order equation is given by

$$G^{(1)}(t) = \frac{2b\gamma_1 + 4}{2a\gamma_1 + b} F^{(0)}(t). \quad (38)$$

Now, because $\gamma_\pm$ is a solution of the equation $a\gamma_\pm^2 + b\gamma_\pm + 1 = 0$, it is easy to check the identity:

$$\frac{2b\gamma_\pm + 4}{2a\gamma_\pm + b} = -2\gamma_\pm. \quad (39)$$

Hence, $G^{(1)}(t) = 2\gamma H^{(0)}(t)$.

3) To prove the statement holds for order $n > 1$, we assume $G^{(n)}(t) = 2\gamma H^{(n-1)}(t)$ and shows that it holds at order $n + 1$. The term of order $T^{n+1}$ in the series expansion of the closed equation leads to:

$$\begin{align*}
& a \sum_{p=0}^{n+1} G^{(p)}(t)G^{(n+1-p)}(t) + bH(t)G^{(n+1)}(t) \\
& + 2b \sum_{p=0}^{n} F^{(p)}(t)G^{(n-p)}(t) + 4 \sum_{p=0}^{n-1} F^{(p)}(t)F^{(n-1-p)}(t) \\
& + 4H(t)F^{(n)}(t) = 0,
\end{align*} \quad (40)$$

which can be rewritten as:

$$\begin{align*}
& 2aG^{(0)}(t)G^{(n+1)}(t) + bH(t)G^{(n+1)}(t) + 2bH(t)G^{(n+1)}(t) \\
& + 4H(t)F^{(n)}(t) + \left\{ a \sum_{p=1}^{n} G^{(p)}(t)G^{(n+1-p)}(t) \\
& + \sum_{p=0}^{n-1} \left( 2bF^{(p)}(t)F^{(n-p)} + 4F^{(p)}(t)F^{(n-1-p)} \right) (t) \right\} = 0. \quad (41)
\end{align*}$$

The second term in bracket contains only $G^{(k)}(t)$ for $1 \leq k \leq n$, and we can use of the recursion assumption $G^{(k)}(t) = 2\gamma H^{(k-1)}(t)$ to rewrite it as:

$$(a\gamma_\pm^2 + b\gamma_\pm + 1) \sum_{p=0}^{n-1} F^{(p)}(t)F^{(n-1-p)}(t), \quad (42)$$

which vanishes because of the definition of $\gamma_\pm$. Using the explicit expression for $G^{(0)}(t)$ in the remaining terms of (41), we show that $G^{(n+1)}(t) = 2\gamma H^{(n)}(t)$. \(\square\)

The construction proposed for a sixth confining potential can be generalized for arbitrary potential: $l(t) = 2\sigma + h_0 + h_1 a(t) + \cdots + h_m a^m(t)$, for $m \in \mathbb{N}$. To be more precise, the closed equation of the function $G(t)$ is given by

$$\sum_{k=0}^{m} \frac{h_k}{h_m} G^{m-k}(t) A^k(t, G) = 0. \quad (43)$$

Then expanding $G(t)$ as a power series of the temperature as in (29) we get the zero-order equation

$$\sum_{k=0}^{m} \frac{h_k}{h_m} (G^{(0)}(t))^{m-k} H^k(t) = 0. \quad (44)$$
Then setting $a_k^{(m)} = h_k/h_m$, the above equation have a solution of the form $G^{(0)}(t) = \gamma H(t)$ where $\gamma$ satisfy the following equation:

$$
\sum_{k=0}^{m} a_k^{(m)} \gamma^{m-k} = 0. \quad (45)
$$

The first-order equation becomes

$$
\sum_{k=0}^{m} \left[ 2k(G^{(0)})^{m-k}H^{k-1}(t)F^{(0)}(t) \right. \\
\left. + (m-k)(G^{(0)})^{m-k-1}G^{(1)}(t)H^k(t) \right] = 0, \quad (46)
$$

which leads to the solution

$$
G^{(1)}(t) = -\frac{\sum_{k=0}^{m} 2a_k^{(m)} k\gamma^{m-k}}{\sum_{k=0}^{m} a_k^{(m)} (m-k)\gamma^{m-k-1}} F^{(0)}(t). \quad (47)
$$

Then by using (45) we can simply deduce that

$$
-\frac{\sum_{k=0}^{m} 2a_k^{(m)} k\gamma^{m-k}}{\sum_{k=0}^{m} a_k^{(m)} (m-k)\gamma^{m-k-1}} = 2\gamma \quad (48)
$$

and therefore $G^{(1)}(t) = 2\gamma F^{(0)}(t)$. The recurrence relation remains the same as for $K = 2$ and we get after a few computations

$$
G^{(i)}(t) = 2\gamma F^{(i-1)}(t), \quad i > 1. \quad (49)
$$

### IV. DISCUSSIONS AND CONCLUSION

From the previous statement see proposition (1) we deduce the low-temperature expansion for the Laplace transform $\tilde{G}(p)$, explicitly:

$$
\tilde{G}(p) = \gamma_{\pm} \tilde{H}(p) \sum_{n=0}^{\infty} (2T\gamma_{\pm} \tilde{H}(p))^n, \quad (50)
$$

which formally resumes as:

$$
\tilde{G}(p) \equiv \gamma_{\pm} \frac{\tilde{H}(p)}{1 - 2T\gamma_{\pm} \tilde{H}(p)}, \quad (51)
$$

provided that $T$ remains smaller than the radius of convergence:

$$
T < R_{\pm} \equiv \frac{\sigma}{|\gamma_{\pm}|}. \quad (52)
$$

Because $\tilde{G}(0)$ is formally a positive quantity, the low-temperature expansion exists only for $\gamma_{\pm} \geq 0$. The solution is furthermore defined for low temperatures, below the critical values $T_c = R_{\pm}$, at which the series fails to be summable. As for the quartic case, we thus have that, below the critical temperature $G(t) \sim t^{-3/2}$ and the system goes toward equilibrium states of the potential as $t(t) \sim 2\sigma + O(t^{-1})$. As $T > T_c$ however, outside of the radius of convergence of the low-temperature expansion, exponential decay is expected for the quartic and spherical model [1].

The number of solutions depends on the shape of the potential:

$$
\psi(x) := \frac{1}{3}h_2 x^3 + \frac{1}{2}h_1 x^2 + h_0 x, \quad (53)
$$

from which $\ell(t)$ derives setting $x \to a(t)$. Furthermore, the equation for $\gamma_{\pm}$ is nothing but:

$$
\ell(\gamma_{\pm}^{-1} - 2\sigma := \frac{\partial \psi}{\partial x}(\gamma_{\pm}^{-1}) \equiv 0, \quad (54)
$$

and explicitly:

$$
\gamma_{\pm}^{-1} = \frac{h_1}{2h_2} \pm \sqrt{h_2^2 - 4h_2h_0}. \quad (55)
$$

The coupling $h_2$ must be positive because of the stability requirement. Hence, if $h_1 > 0$, we have two configurations:

- For $h_0 < 0$, there is a single positive solution, $\gamma_{+}^{-1}$.
- For $h_0 > 0$, the two solutions are negative or imaginary, and the low $T$ expansion does not exist.

For $h_1 < 0$ on the other hands,
For $h_0 > 0$, as soon as $h_1^2 > 4h_2h_0$, there are two solutions for $\gamma_{\pm}^{-1}$. There are no solutions for $h_1^2 < 4h_2h_0$.

For $h_0 < 0$ finally, there are only one solution again, namely $\gamma_{+}^{-1}$.

Figure 2 illustrates this behavior for $h_1 < 0 (= -3)$ and $h_2 = 1$. For curves from (A) to (D) we decrease the value of the quadratic term $h_0$. For the curve labelled by (A), $h_0 > 0$ and the discriminant is negative. Hence, the solution set is empty, and the potential has a single minimum for $x = 0$. As $h_0$ decreases in $(B - C)$, the discriminant becomes positive, and two solutions appear. But as $h_0 < 0$, the first solution disappears and we have only one solution, as the transition regime with phase coexistence is passed.