ENSEMBLE TIMESTEPPING ALGORITHMS FOR NATURAL CONVECTION

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Abstract. This paper presents two algorithms for calculating an ensemble of solutions to laminar natural convection problems. The ensemble average is the most likely temperature distribution and its variance gives an estimate of prediction reliability. Solutions are calculated by solving two coupled linear systems, each involving a shared coefficient matrix, for multiple right-hand sides at each timestep. Storage requirements and computational costs to solve the system are thereby reduced. Stability and convergence of the method are proven under a timestep condition involving fluctuations. A series of numerical tests, including predictability horizons, are provided which confirm the theoretical analyses and illustrate uses of ensemble simulations.

1. Introduction. Ensemble calculations are essential in predictions of the most likely outcome of systems with uncertain data, e.g., weather forecasting [12], ocean modeling [14], turbulence [11], etc. Ensemble simulations classically involve J sequential, fine mesh runs or J parallel, coarse mesh runs of a given code. This leads to a competition between ensemble size and mesh density. We develop linearly implicit timestepping methods with shared coefficient matrices to address this issue. For such methods, it is more efficient in both storage and solution time to solve J linear systems with a shared coefficient matrix than with J different matrices.

Prediction of thermal profiles is essential in many applications [1, 7, 16, 17]. Herein, we extend [6] from isothermal flows to temperature dependent natural convection. We consider two natural convection problems enclosed in mediums with: non-zero wall thickness [3] and zero wall thickness; Figure 1 illustrates a typical setup. The latter problem is often utilized as a thin wall approximation.

Consider the Thick wall problem. Let $\Omega_f \subset \Omega$ be polyhedral domains in $\mathbb{R}^d (d = 2, 3)$ with boundaries $\partial \Omega_f$ and $\partial \Omega$, respectively, such that $\text{dist}(\partial \Omega_f, \partial \Omega) > 0$. The boundary $\partial \Omega$ is partitioned such that $\partial \Omega = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $|\Gamma_1| > 0$. Given $u(x, 0; \omega_j) = u_0(x; \omega_j)$ and $T(x, 0; \omega_j) = T_0(x; \omega_j)$ for $j = 1, 2, ..., J$, let $u(x, t; \omega_j) : \Omega \times (0, t^*) \rightarrow \mathbb{R}$, $p(x, t; \omega_j) : \Omega \times (0, t^*) \rightarrow \mathbb{R}$, and $T(x, t; \omega_j) : \Omega \times (0, t^*) \rightarrow \mathbb{R}$ satisfy

\begin{align}
\tag{1}
&u_t + u \cdot \nabla u - Pr \Delta u + \nabla p = Pr Ra \gamma T + f \quad \text{in } \Omega_f, \\
\tag{2}
&\nabla \cdot u = 0 \quad \text{in } \Omega_f, \\
\tag{3}
&T_i + u \cdot \nabla T - \nabla \cdot (\kappa \nabla T) = g \quad \text{in } \Omega, \\
\tag{4}
&u = 0 \quad \text{on } \partial \Omega_f, \quad u = 0 \quad \text{in } \Omega - \Omega_f, \quad T = 0 \quad \text{on } \Gamma_1 \text{ and } n \cdot \nabla T = 0 \quad \text{on } \Gamma_2.
\end{align}

Here $n$ denotes the usual outward normal, $\gamma$ denotes the unit vector in the direction of gravity, $Pr$ is the Prandtl number, $Ra$ is the Rayleigh number, and $\kappa = \kappa_f$ in $\Omega_f$ and $\kappa = \kappa_s$ in $\Omega - \Omega_f$ is the thermal conductivity of the fluid or solid medium, further, $f$ and $g$ are the body force and heat source, respectively.

Let $<u>_n := \frac{1}{J} \sum_{j=1}^{J} u^n$ and $u'^n = u^n - <u>_n$. To present the idea, suppress the spatial discretization for the moment. We apply an implicit-explicit time-discretization to the system (1) - (4), while keeping the coefficient matrix independent of the ensemble members. This leads to the following timestepping method:

\begin{align}
\tag{5}
&\frac{u^{n+1} - u^n}{\Delta t} + <u>_n \cdot \nabla u^{n+1} + u'^n \cdot \nabla u^n - Pr \Delta u^{n+1} + \nabla p^{n+1} = Pr Ra \gamma T^{n+1} + f^{n+1}, \\
\tag{6}
&\nabla \cdot u^{n+1} = 0, \\
\tag{7}
&\frac{T^{n+1} - T^n}{\Delta t} + <u>_n \cdot \nabla T^{n+1} + u'^n \cdot \nabla T^n - \kappa \Delta T^{n+1} = g^{n+1},
\end{align}

Consider the Thin wall problem. The main difference is a “$u_1$” term on the r.h.s of the temperature equation (10) absent in (3). This apparently small difference in the model produces a significant difference in the stability of the approximate solution. In particular, a discrete Gronwall inequality is used which allows

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for the loss of long-time stability; see Section 4 below. Consider:

\[
    u_t + u \cdot \nabla u - Pr \Delta u + \nabla p = Pr Ra_\gamma T + f \quad \text{in } \Omega, \\
    \nabla \cdot u = 0 \quad \text{in } \Omega, \\
    T_t + u \cdot \nabla T - \nabla \cdot (\kappa \nabla T) = u_1 + g \quad \text{in } \Omega, \\
    u = 0 \quad \text{on } \partial \Omega, \quad T = 0 \quad \text{on } \Gamma_1, \quad n \cdot \nabla T = 0 \quad \text{on } \Gamma_2,
\]

where \( u_1 \) is the first component of the velocity. If we again momentarily disregard the spatial discretization, our timestepping method can be written as:

\[
    \frac{u^{n+1} - u^n}{\Delta t} + \nabla u^{n+1} + u^n \cdot \nabla u^n - Pr \Delta u^{n+1} + \nabla p^{n+1} = Pr Ra_\gamma T^n + f^{n+1}, \\
    \nabla \cdot u^{n+1} = 0, \\
    \frac{T^{n+1} - T^n}{\Delta t} + \nabla T^{n+1} + u^n \cdot \nabla T^n - \kappa \Delta T^{n+1} = u_1^n + g^{n+1},
\]

By lagging both \( u' \) and the coupling terms in the method, the fluid and thermal problems uncouple and each sub-problem has a shared coefficient matrix for all ensemble members.

**Remark:** The formulation (5) - (7) arises, e.g., in the study of natural convection within a unit square or cubic enclosure with a pair of differentially heated vertical walls. In particular, the temperature distribution is decomposed into \( \theta(x,t) = T(x,t) + \phi(x) \), where \( \phi(x) = 1 - x_1 \) is the linear conduction profile and \( T(x,t) \) satisfies homogeneous boundary conditions on the corresponding pair of vertical walls.

In Section 2, we collect necessary mathematical tools. In Section 3, we present algorithms based on (5) - (7) and (12) - (14). Stability and error analyses follow in Section 4. We end with numerical experiments and conclusions in Sections 5 and 6. In particular, two stable, convergent ensemble algorithms are presented. These algorithms can be used to efficiently compute an ensemble of solutions to (1) - (4) and (8) - (11) and estimate predictability horizons. The ensemble average is shown to produce a better estimate of the energy in the system, for a test problem, than any member of the ensemble.

**2. Mathematical Preliminaries.** The \( L^2(\Omega) \) inner product is \( \langle \cdot, \cdot \rangle \) and the induced norm is \( \| \cdot \| \).

Define the Hilbert spaces,

\[
    X := H^1_0(\Omega)^d = \{ v \in H^1(\Omega)^d : v = 0 \text{ on } \partial \Omega \}, \quad Q := L^2_0(\Omega) = \{ q \in L^2(\Omega) : \int_\Omega q dx = 0 \}, \\
    W := \{ S \in H^1(\Omega) : S = 0 \text{ on } \Gamma_1 \}, \quad V := \{ v \in X : (q, \nabla \cdot v) = 0 \forall q \in Q \}.
\]
The explicitly skew-symmetric trilinear forms are denoted:

\[
b(u, v, w) = \frac{1}{2}(u \cdot \nabla v, w) - \frac{1}{2}(u \cdot \nabla w, v) \quad \forall u, v, w \in X,
\]

\[
b^\ast(u, T, S) = \frac{1}{2}(u \cdot \nabla T, S) - \frac{1}{2}(u \cdot \nabla S, T) \quad \forall u \in X, \forall T, S \in W.
\]

They enjoy the following continuity results and properties.

**Lemma 1.** There are constants \(C_1, C_2, C_3, C_4, C_5,\) and \(C_6\) such that for all \(u, v, w \in X\) and \(T, S \in W, b(u, v, w)\) and \(b^\ast(u, T, S)\) satisfy

\[
b(u, v, w) = \int_\Omega u \cdot \nabla v \cdot wdx + \frac{1}{2} \int_\Omega (\nabla \cdot u)v \cdot wdx,
\]

\[
b^\ast(u, T, S) = \int_\Omega u \cdot \nabla TSdx + \frac{1}{2} \int_\Omega (\nabla \cdot u)TSdx,
\]

\[
b(u, v, w) \leq C_1\|\nabla u\|\|\nabla v\|\|\nabla w\|,
\]

\[
b^\ast(u, T, S) \leq C_2\sqrt{\|u\|\|\nabla u\|\|\nabla v\|\|\nabla w\|},
\]

\[
b(u, v, w) \leq C_3\|\nabla u\|\|\nabla T\|\|\nabla S\|,
\]

\[
b^\ast(u, T, S) \leq C_4\sqrt{\|u\|\|\nabla u\|\|\nabla T\|\|\nabla S\|},
\]

\[
b(u, v, w) \leq C_5\|\nabla u\|\|\nabla v\|\sqrt{\|w\|\|\nabla w\|},
\]

\[
b^\ast(u, T, S) \leq C_6\|\nabla u\|\|\nabla T\|\sqrt{\|S\|\|\nabla S\|}.
\]

**Proof.** The proof of the first two identities is a calculation. The next four results follow from applications of Hölder and Sobolev embedding inequalities; see Lemma 2.2 on p. 2044 of [13]. We will prove the last two results for \(d = 3\); for \(d = 2\) they are improvable. For all \(u, v, w \in X,\)

\[
|(u \cdot \nabla v, w)| \leq C\|u\|_{L^6}\|\nabla v\|\|w\|_{L^3}
\]

\[
\leq C\|\nabla u\|\|\nabla v\|\sqrt{\|w\|\|\nabla w\|},
\]

where Hölder, Ladyzhenskaya and Gagliardo-Nirenberg inequalities were used, respectively. Using the above result and inequalities and the first identity in Lemma 1,

\[
|b(u, v, w)| = |(u \cdot \nabla v, w) + \frac{1}{2} \int_\Omega (\nabla \cdot u)v \cdot wdx|
\]

\[
\leq |(u \cdot \nabla v, w)| + \frac{1}{2} \int_\Omega (\nabla \cdot u)v \cdot wdx
\]

\[
\leq C\|\nabla u\|\|\nabla v\|\sqrt{\|w\|\|\nabla w\|} + C\|\nabla \cdot u\|\|v\|_{L^6}\|w\|_{L^3}
\]

\[
\leq C\|\nabla u\|\|\nabla v\|\sqrt{\|w\|\|\nabla w\|} + C\|\nabla u\|\|\nabla v\|\sqrt{\|w\|\|\nabla w\|}
\]

\[
\leq C\|\nabla u\|\|\nabla v\|\sqrt{\|w\|\|\nabla w\|}.
\]

In similar fashion, there is a \(C = C(\Omega)\) such that

\[
|b^\ast(u, T, S)| \leq |(u \cdot \nabla T, S)| + \frac{1}{2} \int_\Omega (\nabla \cdot u)TSDx
\]

\[
\leq C\|\nabla u\|\|\nabla T\|\sqrt{\|S\|\|\nabla S\|} + C\|\nabla \cdot u\|\|T\|_{L^6}\|S\|_{L^3}
\]

\[
\leq C\|\nabla u\|\|\nabla T\|\sqrt{\|S\|\|\nabla S\|} + C\|\nabla u\|\|\nabla T\|\sqrt{\|S\|\|\nabla S\|}
\]

\[
\leq C\|\nabla u\|\|\nabla T\|\sqrt{\|S\|\|\nabla S\|}.
\]
The weak formulation of system (1) - (4) is: Find \( u : [0, t^*] \to X, p : [0, t^*] \to Q, T : [0, t^*] \to W \) for a.e. \( t \in (0, t^*) \) satisfying for \( j = 1, \ldots, J \):

\[
\begin{align*}
(u, v) + b(u, u, v) + Pr(\nabla u, \nabla v) - (p, \nabla \cdot v) &= PrRa(\gamma T, v) + (f, v) \quad \forall v \in X, \\
(q, \nabla \cdot u) &= 0 \quad \forall q \in Q, \\
(T_t, S) + b^*(u, T, S) + \kappa(\nabla T, \nabla S) &= (g, S) \quad \forall S \in W.
\end{align*}
\]

Similarly, the weak formulation of system (8) - (11) is: Find \( u : [0, t^*] \to X, p : [0, t^*] \to Q, T : [0, t^*] \to W \) for a.e. \( t \in (0, t^*) \) satisfying for \( j = 1, \ldots, J \):

\[
\begin{align*}
(u, v) + b(u, u, v) + Pr(\nabla u, \nabla v) - (p, \nabla \cdot v) &= PrRa(\gamma T, v) + (f, v) \quad \forall v \in X, \\
(q, \nabla \cdot u) &= 0 \quad \forall q \in Q, \\
(T_t, S) + b^*(u, T, S) + \kappa(\nabla T, \nabla S) &= (u_1, S) + (g, S) \quad \forall S \in W.
\end{align*}
\]

### 2.1. Finite Element Preliminaries.

Consider a regular, quasi-uniform mesh \( \Omega_h = \{ K \} \) of \( \Omega \) with maximum triangle diameter length \( h \). Further, for the system (1) - (4), suppose that \( \partial \Omega_f \) and \( \partial \Omega - \partial \Omega_f \) lie along the meshlines of the triangulation of \( \Omega \). Let \( X_h \subset X, Q_h \subset Q, \) and \( W_h \subset W \) be conforming finite element spaces consisting of continuous piecewise polynomials of degrees \( j, l, k, \) and \( j \), respectively. Moreover, assume they satisfy the following approximation properties \( \forall 1 \leq j, l \leq k, m \):

\[
\begin{align*}
\inf_{v_h \in X_h} \left\{ \| u - v_h \| + h\| \nabla (u - v_h) \| \right\} &\leq Ch^{k+1} \| u \|_{k+1}, \\
\inf_{q_h \in Q_h} \| p - q_h \| &\leq Ch^m \| p \|_m, \\
\inf_{S_h \in W_h} \left\{ \| T - S_h \| + h\| \nabla (T - S_h) \| \right\} &\leq Ch^{k+1} \| T \|_{k+1},
\end{align*}
\]

for all \( u \in X \cap H^{k+1}(\Omega)^d, p \in Q \cap H^m(\Omega), \) and \( T \in W \cap H^{k+1}(\Omega) \). Furthermore, we consider those spaces for which the discrete inf-sup condition is satisfied,

\[
\inf_{q_h \in Q_h} \sup_{v_h \in X_h} \frac{(q_h, \nabla \cdot v_h)}{\|q_h\\| \| \nabla v_h \|} \geq \beta > 0,
\]

where \( \beta \) is independent of \( h \). The space of discretely divergence free functions is defined by

\[
V_h := \{ v_h \in X_h : (q_h, \nabla \cdot v_h) = 0, \forall q_h \in Q_h \}.
\]

The space \( V_h^* \), dual to \( V_h \), is endowed with the following dual norm

\[
\| w \|_{V_h^*} := \sup_{v_h \in V_h} \frac{(w, v_h)}{\| \nabla v_h \|}
\]

The discrete inf-sup condition implies that we may approximate functions in \( V \) well by functions in \( V_h \).

**Lemma 2.** Suppose the discrete inf-sup condition (24) holds, then for any \( v \in V \)

\[
\inf_{v_h \in V_h} \| \nabla (v - v_h) \| \leq C(\beta) \inf_{v_h \in X_h} \| \nabla (v - v_h) \|.
\]

**Proof.** See Chapter 2, Theorem 1.1 on p. 59 of [8].

We will also assume that the mesh and finite element spaces satisfy the standard inverse inequality [5]:

\[
\| \nabla \chi_{1,2} \| \leq C_{inv,1,2}(\alpha_{min})h^{-1} \| \chi_{1,2} \| \quad \forall \chi_1 \in X_h, \forall \chi_2 \in W_h,
\]

where \( \alpha_{min} \) denotes the minimum angle in the triangulation. A discrete Gronwall inequality will play a role in the upcoming analysis.
Lemma 3. (Discrete Gronwall Lemma). Let $\Delta t$, $H$, $a_n$, $b_n$, $c_n$, and $d_n$ be finite nonnegative numbers for $n \geq 0$ such that for $N \geq 1$

$$a_N + \Delta t \sum_{0}^{N} b_n \leq \Delta t \sum_{0}^{N-1} d_n a_n + \Delta t \sum_{0}^{N} c_n + H,$$

then for all $\Delta t > 0$ and $N \geq 1$

$$a_N + \Delta t \sum_{0}^{N} b_n \leq \exp(\Delta t \sum_{0}^{N-1} d_n)(\Delta t \sum_{0}^{N} c_n + H).$$

Proof. See Lemma 5.1 on p. 369 of [10]. □

The discrete time analysis will utilize the following norms $\forall 1 \leq k \leq \infty$:

$$\|v\|_{\infty,k} := \max_{0 \leq n \leq N} \|v^n\|_k, \quad \|v\|_{p,k} := (\Delta t \sum_{n=0}^{N} \|v^n\|_k^p)^{1/p}.$$

3. Numerical Scheme. Denote the fully discrete solutions by $u_h^n$, $p_h^n$, and $T_h^n$ at time levels $t^n = n\Delta t$, $n = 0, 1, \ldots, N$, and $t^* = N\Delta t$. Given $(u_h^n, p_h^n, T_h^n) \in (X_h, Q_h, W_h)$, find $(u_h^{n+1}, p_h^{n+1}, T_h^{n+1}) \in (X_h, Q_h, W_h)$ satisfying, for every $n = 0, 1, \ldots, N$, the fully discrete approximation of the Thick wall problem:

$$\begin{align*}
(25) \quad & \left(\frac{u_h^{n+1} - u_h^n}{\Delta t}, v_h\right) + b(< u_h >^n, u_h^{n+1}, v_h) + b(u_h^n, u_h^n, v_h) + Pr(\nabla u_h^{n+1}, \nabla v_h) - (p_h^{n+1}, \nabla \cdot v_h) = PrRa(\gamma T_h^{n+1}, v_h) + (f^{n+1}, v_h) \quad \forall v_h \in X_h, \\
(26) \quad & (q_h, \nabla \cdot u_h^{n+1}) = 0 \quad \forall q_h \in Q_h, \\
(27) \quad & \left(\frac{T_h^{n+1} - T_h^n}{\Delta t}, S_h\right) + b^*(< u_h >^n, T_h^{n+1}, S_h) + b^*(u_h^n, T_h^n, S_h) + \kappa(\nabla T_h^{n+1}, \nabla S_h) = (g^{n+1}, S_h) \quad \forall S_h \in W_h.
\end{align*}$$

Thin wall problem:

$$\begin{align*}
(28) \quad & \left(\frac{u_h^{n+1} - u_h^n}{\Delta t}, v_h\right) + b(< u_h >^n, u_h^{n+1}, v_h) + b(u_h^n, u_h^n, v_h) + Pr(\nabla u_h^{n+1}, \nabla v_h) - (p_h^{n+1}, \nabla \cdot v_h) = PrRa(\gamma T_h^n, v_h) + (f^{n+1}, v_h) \quad \forall v_h \in X_h, \\
(29) \quad & (q_h, \nabla \cdot u_h^{n+1}) = 0 \quad \forall q_h \in Q_h, \\
(30) \quad & \left(\frac{T_h^{n+1} - T_h^n}{\Delta t}, S_h\right) + b^*(< u_h >^n, T_h^{n+1}, S_h) + b^*(u_h^n, T_h^n, S_h) + \kappa(\nabla T_h^{n+1}, \nabla S_h) = (u_h^n, S_h) + (g^{n+1}, S_h) \quad \forall S_h \in W_h.
\end{align*}$$

Remark: The treatment of the nonlinear terms in the time discretizations (5) - (7) and (12) - (14) leads to a shared coefficient matrix independent of the ensemble members.

4. Numerical Analysis of the Ensemble Algorithm. We present stability results for the aforementioned algorithms under the following timestep condition:

$$\begin{align*}
(31) \quad & \frac{C_1 \Delta t}{h} \max_{i,j \leq J} \|\nabla u_h^n\|^2 \leq 1,
\end{align*}$$

where $C_1 \equiv C_1(|\Omega|, \alpha_{min}, \kappa, Pr)$. In Theorems 4 and 5, the nonlinear stability of the velocity, temperature, and pressure approximations are proven under condition (31) for the thick wall (25) - (27) and thin wall problems (28) - (30), respectively.

Remark: Stability of the numerical approximations can also be proven under: $\frac{JC_1 \Delta t}{h} < \|\nabla u_h^n\|^2 \leq 1$. If $C_1/J \geq 1$, then $JC_1$ can be replaced with $C_1$. 

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4.1. Stability Analysis.

**Theorem 4.** Consider the Thick wall problem (25) - (27). Suppose \( f \in L^\infty(0, t^*; H^{-1}(\Omega)^d) \), \( g \in L^\infty(0, t^*; H^{-1}(\Omega)) \). If (25) - (27) satisfy condition (31), then

\[
\|T_h^N\|^2 + \|u_h^N\|^2 + \frac{1}{2} \sum_{n=0}^{N-1} \left( \|T_h^{n+1} - T_h^n\|^2 + \|u_h^{n+1} - u_h^n\|^2 \right) + \kappa \Delta t \|\nabla T_h^N\|^2 + P \Delta t \|\nabla u_h^N\|^2 \\
\leq 2 \Delta t P \eta^2 C_{PF,1}^2 \sum_{n=0}^{N-1} \left( \frac{\Delta t}{\eta} \sum_{k=0}^{n} \|g^{k+1}\|^2 + \|T_h^n\|^2 + \kappa \Delta t \|\nabla T_h^0\|^2 \right) + \frac{2 \Delta t}{P \eta} \sum_{n=0}^{N-1} \|f^{n+1}\|^2 + \|u_0\|^2 \\
+ P \Delta t \|\nabla u_h^0\|^2 + \|T_h^0\|^2 + \kappa \Delta t \|\nabla T_h^0\|^2.
\]

Further,

\[
\beta \Delta t \sum_{n=0}^{N-1} \|p_h^{n+1}\| \leq 2 \Delta t \sum_{n=0}^{N-1} \left( C_1 \|\nabla < u_h > \| \|\nabla u_h^{n+1}\| + C_1 \|\nabla u_h^n\| \|\nabla u_h^n\| \\
+ P \|\nabla u_h^{n+1}\| + P \eta \eta R \|T_h^0\| + \|f^{n+1}\| - 1 \right).
\]

**Proof.** Let \( \bar{T}_h = T_h^{n+1} \) in equation (27) and use the polarization identity. Multiply by \( \Delta t \) on both sides and rearrange. Then,

\[
\frac{1}{2} \left( \|T_h^{n+1}\|^2 - \|T_h^n\|^2 + \|T_h^{n+1} - T_h^n\|^2 \right) + \kappa \Delta t \|\nabla T_h^{n+1}\|^2 = \Delta t (g^{n+1}, T_h^{n+1}) - \Delta t b^*(u_h^n, T_h^n, T_h^{n+1}).
\]

Use Cauchy-Schwarz-Young on \( \Delta t (g^{n+1}, T_h^{n+1}) \),

\[
\Delta t (g^{n+1}, T_h^{n+1}) \leq \frac{\Delta t}{2 \kappa} \|g^{n+1}\|^2 - 1 + \frac{\Delta t \kappa}{2} \|\nabla T_h^{n+1}\|^2.
\]

Consider \(- \Delta t b^*(u_h^n, T_h^n, T_h^{n+1})\). Add and subtract \( - \Delta t b^*(u_h^n, T_h^n, T_h^n) \), use skew-symmetry, Lemma 1, the inverse inequality, and the Cauchy-Schwarz-Young inequality. Then,

\[
| - \Delta t b^*(u_h^n, T_h^n, T_h^{n+1})| = | - \Delta t b^*(u_h^n, T_h^n, T_h^{n+1} - T_h^n)| \\
\leq \Delta t C_6 \|\nabla u_h^n\| \|\nabla T_h^n\| \sqrt{\|T_h^{n+1} - T_h^n\| \|\nabla (T_h^{n+1} - T_h^n)\|} \\
\leq \Delta t C_6 \|\nabla u_h^n\| \|\nabla T_h^n\| \|\nabla T_h^{n+1} - T_h^n\| \\
\leq \frac{C_6^2 C_{inv,2} \Delta t^2}{h} \|\nabla u_h^n\| \|\nabla T_h^n\| + \frac{1}{4} \|T_h^{n+1} - T_h^n\|^2.
\]

Using (33) and (34) in (32) leads to

\[
\frac{1}{2} \left( \|T_h^{n+1}\|^2 - \|T_h^n\|^2 + \|T_h^{n+1} - T_h^n\|^2 \right) + \kappa \Delta t \|\nabla T_h^{n+1}\|^2 \leq \frac{\Delta t}{2 \kappa} \|g^{n+1}\|^2 - 1 \\
+ \frac{\Delta t \kappa}{2} \|\nabla T_h^{n+1}\|^2 + \frac{C_6^2 C_{inv,2} \Delta t^2}{h} \|\nabla u_h^n\|^2 \|\nabla T_h^n\|^2 + \frac{1}{4} \|T_h^{n+1} - T_h^n\|^2.
\]

Let \( \epsilon = \kappa \), add and subtract \( \frac{\kappa \Delta t}{2} \|\nabla T_h^n\|^2 \) to the l.h.s. Regrouping terms leads to

\[
\frac{1}{2} \left( \|T_h^{n+1}\|^2 - \|T_h^n\|^2 \right) + \frac{\kappa \Delta t}{2} \|\nabla T_h^{n+1}\|^2 - \|\nabla T_h^n\|^2 \\
+ \frac{\kappa \Delta t}{2} \|\nabla T_h^n\|^2 \left[ 1 - \frac{2 C_6^2 C_{inv,2} \Delta t}{\kappa h} \|\nabla u_h^n\|^2 \right] \leq \frac{\Delta t}{2 \kappa} \|g^{n+1}\|^2 - 1.
\]
By hypothesis, $\frac{2C_2^{\text{inv},2}\Delta t}{\kappa h} \|u^n_{n+1}\|^2 \leq 1$. Thus,

$$\frac{1}{2} \left\{ \|T_{n+1}^h\|^2 - \|T_n^h\|^2 \right\} + \frac{\kappa \Delta t}{2} \left\{ \|\nabla T_{n+1}^h\|^2 - \|\nabla T_n^h\|^2 \right\} \leq \frac{\Delta t}{2\kappa} \|g^{n+1}\|^2.$$

Sum from $n = 0$ to $n = N - 1$ and put all data on the right hand side. This yields

$$\frac{1}{2} \|T_N^h\|^2 + \frac{\kappa \Delta t}{2} \|\nabla T_N^h\|^2 \leq \frac{\Delta t}{2\kappa} \sum_{n=0}^{N-1} \|g^{n+1}\|^2 + \frac{1}{2} \|T_0^h\|^2 + \frac{\kappa \Delta t}{2} \|\nabla T_0^h\|^2.$$

Therefore, the l.h.s. is bounded by data on the r.h.s. The temperature approximation is stable.

We follow an almost identical form of attack for the velocity as we did for the temperature. Let $v_h = u_h^{n+1} \in V_h$ in (25) and use the polarization identity. Multiply by $\Delta t$ on both sides and rearrange terms. Then,

$$\frac{1}{2} \left\{ \|u_h^{n+1}\|^2 - \|u_h^n\|^2 + \|u_h^{n+1} - u_h^n\|^2 \right\} + \Delta t Pr\|\nabla u_h^{n+1}\|^2$$

$$= -\Delta t b(u_h^n, u_h^n, u_h^{n+1}) + \Delta t Pr Ra(\gamma T_{n+1}^h, u_h^{n+1}) + \Delta t (f^{n+1}, u_h^{n+1}).$$

Use the Cauchy-Schwarz-Young inequality on $\Delta t Pr Ra(\gamma T_{n+1}^h, u_h^{n+1})$ and $\Delta t (f^{n+1}, u_h^{n+1})$ and note that $|\gamma| = 1$,

$$\Delta t Pr Ra(\gamma T_{n+1}^h, u_h^{n+1}) \leq \frac{\Delta t Pr^2 Ra^2 C_{PF,1}^2}{2} \|T_{n+1}^h\|^2 + \frac{\Delta t}{2} \|\nabla u_h^{n+1}\|^2,$$

$$\Delta t (f^{n+1}, u_h^{n+1}) \leq \frac{\Delta t}{2\epsilon} \|f^{n+1}\|^2 + \frac{\Delta t}{2} \|\nabla u_h^{n+1}\|^2.$$

Using skew-symmetry, Lemma 1, the inverse inequality, and the Cauchy-Schwarz-Young inequality on $\Delta t b(u_h^n, u_h^n, u_h^{n+1})$ leads to

$$| -\Delta t b(u_h^n, u_h^n, u_h^{n+1})| \leq \frac{C_2^{\text{inv},2}\Delta t^2}{h} \|\nabla u_h^n\|^2 \|\nabla u_h^n\|^2 + \frac{1}{4} \|u_h^{n+1} - u_h^n\|^2.$$

Using (37), (38), and (39) in (36) leads to

$$\frac{1}{2} \left\{ \|u_h^{n+1}\|^2 - \|u_h^n\|^2 + \|u_h^{n+1} - u_h^n\|^2 \right\} + Pr \Delta t \|\nabla u_h^{n+1}\|^2 + \frac{Pr \Delta t}{2} \left\{ \|\nabla u_h^{n+1}\|^2 - \|\nabla u_h^n\|^2 \right\}$$

$$+ \frac{Pr \Delta t}{2} \|\nabla u_h^n\|^2 \left[ 1 - \frac{2C_2^{\text{inv},1}\Delta t}{Pr h} \|\nabla u_h^n\|^2 \right] \leq \Delta t Pr Ra^2 C_{PF,1}^2 \|T_{n+1}^h\|^2 + \frac{\Delta t}{Pr} \|f^{n+1}\|^2.$$
Together with the stability of the temperature approximation, the l.h.s. is bounded above by data; that is, the velocity approximation is stable. Adding (35) and (40) and multiplying by 2 yields the result. We now prove stability of the pressure approximation. We first form an estimate for the discrete time derivative.

\[ (u_h^{n+1} - u_h^n, v_h) = -\Delta t(b(u_h^n, u_h^n, v_h) - \Delta tPr(\nabla u_h^{n+1}, v_h) + \Delta tPrRa(\gamma T_h^{n+1}, v_h) + \Delta t(f^{n+1}, v_h). \]

Applying Lemma 1 to the skew-symmetric trilinear terms and the Cauchy-Schwarz and Poincaré-Friedrichs inequalities to the remaining terms yields

\[ | - \Delta t(b(u_h^n, u_h^n, v_h)) | \leq C_1 \Delta t \||\nabla < u_h^n ||| \nabla u_h^{n+1} ||| \nabla v_h||, \]

\[ | - \Delta tPr(\nabla u_h^{n+1}, v_h)) | \leq C_1 \Delta t \||\nabla u_h^n ||| \nabla u_h^{n+1} ||| \nabla v_h||, \]

\[ \Delta tPrRa(\gamma T_h^{n+1}, v_h) | \leq Pr RaRaT_h^{n+1} ||v_h\| \leq PrRaCPF, 1 \Delta t \||T_h^{n+1} ||| \nabla v_h||, \]

Apply the above estimates in (41), divide by the common factor \||\nabla v_h|| on both sides, and take the supremum over all \( 0 \neq v_h \in V_h \). Then,

\[ \|u_h^{n+1} - u_h^n\| \leq C_1 \Delta t\||\nabla < u_h^n \| \||u_h^{n+1}\| + C_1 \Delta t\||\nabla u_h^n \| \||u_h^{n+1}\| + Pr RaRaCPF, 1 \Delta t \||T_h^{n+1} ||| \nabla v_h||. \]

Reconsider equation (25). Multiply by \( \Delta t \) and isolate the pressure term,

\[ \Delta t(p_h^{n+1}, \nabla \cdot v_h) = (u_h^{n+1} - u_h^n, v_h) + \Delta t(b(u_h^n, u_h^n, v_h) + \Delta tPr(\nabla u_h^{n+1}, v_h) - Pr RaRaT_h^{n+1}, v_h) - f^{n+1}, v_h). \]

Apply (42) - (46) on the r.h.s terms. Then,

\[ \Delta t(p_h^{n+1}, \nabla \cdot v_h) \leq (u_h^{n+1} - u_h^n, v_h) + \left( C_1 \Delta t\||\nabla < u_h^n \| \||u_h^{n+1}\| + C_1 \Delta t\||\nabla u_h^n \| \||u_h^{n+1}\| + Pr RaRaCPF, 1 \Delta t \||T_h^{n+1} ||| \nabla v_h|| \right). \]

Divide by \||\nabla v_h|| and note that \( \frac{(u_h^{n+1} - u_h^n, v_h)}{||\nabla v_h||} \leq ||u_h^{n+1} - u_h^n||_{V_h^*} \). Take the supremum over all \( 0 \neq v_h \in X_h \),

\[ \Delta t \sup_{0 \neq v_h \in X_h} \frac{(p_h^{n+1}, \nabla \cdot v_h)}{||\nabla v_h||} \leq 2 \left( C_1 \Delta t\||\nabla < u_h^n \| \||u_h^{n+1}\| + C_1 \Delta t\||\nabla u_h^n \| \||u_h^{n+1}\| + Pr RaRaCPF, 1 \Delta t \||T_h^{n+1} ||| \nabla v_h|| \right). \]

Use the discrete inf-sup condition (24),

\[ \beta \Delta t ||p_h^{n+1}|| \leq 2 \left( C_1 \Delta t\||\nabla < u_h^n \| \||u_h^{n+1}\| + C_1 \Delta t\||\nabla u_h^n \| \||u_h^{n+1}\| + Pr RaRaCPF, 1 \Delta t \||T_h^{n+1} ||| \nabla v_h|| \right). \]

Summing from \( n = 0 \) to \( n = N - 1 \) yields stability of the pressure approximation, built on the stability of the temperature and velocity approximations.
Theorem 5. Consider the Thin wall problem (28) - (30). Suppose \( f \in L^\infty(0,t^*;H^{-1}(\Omega)^d) \) and \( g \in L^\infty(0,t^*;H^{-1}(\Omega)). \) If (28) - (30) satisfy condition (31), then

\[
\begin{align*}
\|T_h^n\|^2 + \|u_h^n\|^2 &+ \frac{1}{2} \sum_{n=0}^{N-1} \left( \|T_h^{n+1} - T_h^n\|^2 + \|u_h^{n+1} - u_h^n\|^2 \right) + \kappa \Delta t \|\nabla T_h^n\|^2 + Pr \Delta t \|\nabla u_h^n\|^2 \\
&\leq \exp(2Ct^*) \left\{ \Delta t \sum_{n=0}^{N-1} \left( \frac{1}{Pr} \|f^{n+1}\|_{-1}^2 + \frac{1}{\kappa} \|g^{n+1}\|_{-1}^2 \right) + \|u_h^0\|^2 + \|T_h^0\|^2 \\
&\quad + Pr \Delta t \|\nabla u_h^0\|^2 + \kappa \Delta t \|\nabla T_h^0\|^2 \right\}.
\end{align*}
\]

Further,

\[
\beta \Delta t \sum_{n=0}^{N-1} \|p_{h,n+1}^n\| \leq 2 \sum_{n=0}^{N-1} \left( C_1 \Delta t \|\nabla < u_h >^n\| \|\nabla u_{h,n}^n\| + C_1 \Delta t \|\nabla u_{h,n}^n\| \|\nabla u_{h,n}^n\| \\
&\quad + Pr \Delta t \|\nabla u_{h,n}^n\| + Pr Ra C_{PF,1} \Delta t \|T_h^n\| + \Delta t \|f^{n+1}\|_{-1} \right) \right\}
\]

Proof. Add equations (28) and (30), let \( S_h = T_h^{n+1} \in W_h \) and \( v_h = u_h^{n+1} \in V_h \) and use the polarization identity. Then,

\[
\begin{align*}
\frac{1}{2 \Delta t} \left\{ \|T_h^{n+1}\|^2 - \|T_h^n\|^2 + \|T_h^{n+1} - T_h^n\|^2 \right\} &+ \frac{1}{2 \Delta t} \left\{ \|u_h^{n+1}\|^2 - \|u_h^n\|^2 + \frac{1}{2} \|u_h^{n+1} - u_h^n\|^2 \right\} \\
+ \kappa \|\nabla T_h^{n+1}\|^2 + Pr \|\nabla u_{h,n}^n\|^2 + b(u_h^n, u_h^n, u_h^n) + b^*(u_h^n, T_h^n, T_h^{n+1}) = Pr Ra (\gamma T_h^n, u_h^n) \\
+ (u_h^n, T_h^n) + (f^{n+1}, u_h^n) + (g^{n+1}, T_h^{n+1}).
\end{align*}
\]

Apply similar techniques and estimates as in the proof of Theorem 4,

\[
\begin{align*}
\frac{1}{2} \left\{ \|T_h^{n+1}\|^2 - \|T_h^n\|^2 + \frac{1}{2} \|T_h^{n+1} - T_h^n\|^2 \right\} &+ \frac{1}{2} \left\{ \|u_h^{n+1}\|^2 - \|u_h^n\|^2 + \frac{1}{2} \|u_h^{n+1} - u_h^n\|^2 \right\} \\
+ \kappa \Delta t \|\nabla T_h^n\|^2 \left\{ 1 - \frac{2 \Delta t C_{PF,2} \kappa}{\kappa h} \|\nabla u_h^n\|^2 \right\} + \frac{Pr \Delta t}{2} \|\nabla u_h^n\|^2 \left\{ 1 - \frac{2 \Delta t C_{PF,1} \kappa}{Pr h} \|\nabla u_h^n\|^2 \right\} \\
&\leq \Delta t Pr Ra C_{PF,2} \|T_h^n\|^2 + \frac{\Delta t C_{PF,2}}{\kappa} \|u_h^n\|^2 + \frac{\Delta t}{Pr} \|f^{n+1}\|_{-1}^2 + \frac{\Delta t}{\kappa} \|g^{n+1}\|_{-1}^2.
\end{align*}
\]

Using the timestep condition, multiplying by 2, taking a maximum over constants in the first two terms on the r.h.s. and summing from \( n = 0 \) to \( n = N - 1 \) leads to,

\[
\begin{align*}
\|T_h^n\|^2 + \|u_h^n\|^2 &+ \frac{1}{2} \sum_{n=0}^{N-1} \left( \|T_h^{n+1} - T_h^n\|^2 + \|u_h^{n+1} - u_h^n\|^2 \right) + \kappa \Delta t \|\nabla T_h^n\|^2 + Pr \Delta t \|\nabla u_h^n\|^2 \\
&\leq C \Delta t \sum_{n=0}^{N-1} \left\{ \|T_h^n\|^2 + \|u_h^n\|^2 \right\} + 2 \Delta t \sum_{n=0}^{N-1} \left\{ \frac{1}{Pr} \|f^{n+1}\|_{-1}^2 + \frac{1}{\kappa} \|g^{n+1}\|_{-1}^2 \right\} + \|u_h^0\|^2 + \|T_h^0\|^2 \\
&\quad + Pr \Delta t \|\nabla u_h^0\|^2 + \kappa \Delta t \|\nabla T_h^0\|^2.
\end{align*}
\]

Lastly, apply Lemma 3. Then,

\[
\begin{align*}
\|T_h^n\|^2 + \|u_h^n\|^2 &+ \frac{1}{2} \sum_{n=0}^{N-1} \left( \|T_h^{n+1} - T_h^n\|^2 + \|u_h^{n+1} - u_h^n\|^2 \right) + \kappa \Delta t \|\nabla T_h^n\|^2 + Pr \Delta t \|\nabla u_h^n\|^2 \\
&\leq \exp(Ct^*) \left\{ 2 \Delta t \sum_{n=0}^{N-1} \left( \frac{1}{Pr} \|f^{n+1}\|_{-1}^2 + \frac{1}{\kappa} \|g^{n+1}\|_{-1}^2 \right) + \|u_h^0\|^2 + \|T_h^0\|^2 \\
&\quad + Pr \Delta t \|\nabla u_h^0\|^2 + \kappa \Delta t \|\nabla T_h^0\|^2 \right\}.
\end{align*}
\]
Thus, numerical approximations of velocity and temperature are stable. Stability of the pressure approximation follows by similar arguments as in Theorem 4.

**Remark:** Theorem 4 implies long-time stability of the approximate solutions. Application of Lemma 3 in Theorem 5 leads to the loss of long-time stability due to the exponential growth factor, in $t^*$. 

### 4.2. Error Analysis.

Denote $u^n$, $p^n$, and $T^n$ as the true solutions at time $t^n = n\Delta t$. Assume the solutions satisfy the following regularity assumptions:

\[
\begin{align*}
    u &\in L^\infty(0, t^*; X \cap H^{k+1}(\Omega)),
    T &\in L^\infty(0, t^*; W \cap H^{k+1}(\Omega)), \\
    u_t, T_t &\in L^\infty(0, t^*; H^{k+1}(\Omega)),
    u_{tt}, T_{tt} &\in L^\infty(0, t^*; L^2(\Omega)), \\
    p &\in L^\infty(0, t^*; Q \cap H^m(\Omega)).
\end{align*}
\]

The errors are denoted

\[
e^{n}_u = u^n - u^n_h, \quad e^{n}_T = T^n - T^n_h, \quad e^{n}_p = p^n - p^n_h.
\]

**Definition 6. (Consistency error).** The consistency errors are defined as

\[
\tau_u(u^n; v_h) = \left(\frac{u^n - u^{n-1}}{\Delta t} - u^n_h, v_h\right), \quad \tau_T(T^n; S_h) = \left(\frac{T^n - T^{n-1}}{\Delta t} - T^n_h, S_h\right).
\]

**Lemma 7.** Provided $u$ and $T$ satisfy the regularity assumptions (56), then $\forall r > 0$

\[
\begin{align*}
    |\tau_u(u^n; v_h)| &\leq \frac{C^2_{PF,1}C_r\Delta t^2}{\epsilon}||u_{tt}||_2^2L_{(t^n-1,t^n;L^2(\Omega))} + \frac{\epsilon}{r}||\nabla v_h||^2, \\
    |\tau_T(T^n; S_h)| &\leq \frac{C^2_{PF,2}C_r\Delta t^2}{\epsilon}||T_{tt}||_2^2L_{(t^n-1,t^n;L^2(\Omega))} + \frac{\epsilon}{r}||\nabla S_h||^2.
\end{align*}
\]

**Proof.** These follow from the Cauchy-Schwarz-Young inequality, Poincaré-Friedrichs inequality, and Taylor’s theorem with integral remainder.

**Theorem 8.** For $(u, p, T)$ satisfying (1) - (5), suppose that $(u^n_h, p^n_h, T^n_h) \in (X_h, Q_h, W_h)$ are approximations of $(u^0, p^0, T^0)$ to within the accuracy of the interpolant. Further, suppose that condition (31) holds. Then there exists a constant $C$ such that

\[
\begin{align*}
    ||e^N_T||^2 + ||e^N_p||^2 &+ \frac{1}{2} \sum_{n=0}^{N-1} (||e^{n+1}_T - e^n_T||^2 + ||e^{n+1}_u - e^n_u||^2) + \frac{\kappa\Delta t}{2}||\nabla e^N||^2 + \frac{P\Delta t}{2}||\nabla e^N||^2 \\
    &\leq C \{\Delta t \inf_{v_h \in X_h} \left( ||\nabla (u - v_h)||^2_{\infty,0} + ||(u - v_h)_t||^2_{\infty,0} \right) + \Delta t \inf_{S_h \in W_h} \left( ||\nabla (T - S_h)||^2_{\infty,0} + ||(T - S_h)_t||^2_{\infty,0} \right) \\
    &+ \Delta t \inf_{q_h \in Q_h} ||p - q_h||^2_{\infty,0} + \Delta t^3 + \Delta t||\nabla q^0||^2 + \Delta t||\nabla e^0||^2 + ||q^0||^2 \\
    &+ ||e^0||^2 + ||e^0||^2 + ||e^0||^2 + \Delta t||\nabla e^0||^2 + \Delta t||\nabla e^0||^2 \}.
\end{align*}
\]

**Proof.** The true solutions satisfy for all $n = 0, 1, …, N$:

\[
\begin{align*}
    \left(\frac{u^{n+1} - u^n}{\Delta t}, v_h\right) + b(u^{n+1}, u^{n+1}, v_h) + Pr(\nabla u^{n+1}, \nabla v_h) - (p^{n+1}, \nabla \cdot v_h) = PrRa(\gamma T^{n+1}, v_h) + (f^{n+1}, v_h) + \tau_u(u^{n+1}; v_h) \forall v_h \in X_h, \\
    (q_h, \nabla \cdot u^{n+1}) = 0 \forall q_h \in Q_h, \\
    \left(\frac{T^{n+1} - T^n}{\Delta t}, S_h\right) + b^*(u^{n+1}, T^{n+1}, S_h) + \kappa(\nabla T^{n+1}, \nabla S_h) = (g^{n+1}, S_h) + \tau_T(T^{n+1}; S_h) \forall S_h \in W_h.
\end{align*}
\]
Subtract (59) and (27), then the error equation for temperature is

\begin{equation}
\left( \frac{e^{n+1} - e^n}{\Delta t}, S_h \right) + b^*(u^{n+1}, T^{n+1}, S_h) - b^*(u_h^n - u_h^{n+1}, T_h^{n+1}, S_h) - b^*(u_h^n, T_h^n, S_h) + \kappa(\nabla e_{T}^{n+1}, \nabla S_h) = \tau_T(T^{n+1}, S_h) \quad \forall S_h \in W_h.
\end{equation}

Letting \( e_T^n = (T^n - \hat{T}^n) - (T_h^n - \hat{T}^n) = \zeta^n - \psi_h^n \) and rearranging give,

\begin{equation}
\left( \frac{\psi_h^{n+1} - \psi_h^n}{\Delta t}, S_h \right) + \kappa(\nabla \psi_h^{n+1}, \nabla S_h) = \left( \frac{\zeta^{n+1} - \zeta^n}{\Delta t}, S_h \right) + \kappa(\nabla \zeta^{n+1}, \nabla S_h) - \tau_T(T^{n+1}, S_h) + b^*(u^{n+1}, T^{n+1}, S_h) - b^*(u_h^n - u_h^{n+1}, T_h^{n+1}, S_h) - b^*(u_h^n, T_h^n, S_h) \quad \forall S_h \in W_h.
\end{equation}

Set \( S_h = \psi_h^{n+1} \in W_h \). This yields

\begin{equation}
\frac{1}{2\Delta t} \left\{ \| \psi_h^{n+1} \|^2 - \| \psi_h^n \|^2 + \| \psi_h^{n+1} - \psi_h^n \|^2 \right\} + \kappa \| \nabla \psi_h^{n+1} \|^2 = \frac{1}{\Delta t} \left( \zeta^{n+1} - \zeta^n, \psi_h^{n+1} \right) + \kappa(\nabla \zeta^{n+1}, \nabla \psi_h^{n+1}) - \tau_T(T^{n+1}, \psi_h^{n+1}) + b^*(u^{n+1}, T^{n+1}, \psi_h^{n+1}) - b^*(u_h^n - u_h^{n+1}, T_h^{n+1}, \psi_h^{n+1}) - b^*(u_h^n, T_h^n, \psi_h^{n+1}).
\end{equation}

Add and subtract \( b^*(u_h^{n+1}, T_h^{n+1}, \psi_h^{n+1}) \), \( b^*(u^n, T_h^{n+1}, \psi_h^{n+1}) \), and \( b^*(u_h^n, T_h^n, \psi_h^{n+1}) \). Then,

\begin{equation}
\frac{1}{2\Delta t} \left\{ \| \psi_h^{n+1} \|^2 - \| \psi_h^n \|^2 + \| \psi_h^{n+1} - \psi_h^n \|^2 \right\} + \kappa \| \nabla \psi_h^{n+1} \|^2 = \frac{1}{\Delta t} \left( \zeta^{n+1} - \zeta^n, \psi_h^{n+1} \right) + \kappa(\nabla \zeta^{n+1}, \nabla \psi_h^{n+1}) + b^*(u^{n+1}, \zeta^{n+1}, \psi_h^{n+1}) + b^*(u^n, u_h^{n+1}, \psi_h^{n+1}) + \kappa(\nabla \zeta^{n+1}, \nabla \psi_h^{n+1}) + b^*(\hat{\phi}_h^n, T_h^{n+1}, \psi_h^{n+1}) + b^*(u_h^n, \zeta^{n+1}, \psi_h^{n+1}) - b^*(u_h^n, \zeta^{n+1}, \psi_h^{n+1}) + b^*(u_h^n, T_h^{n+1}, \psi_h^{n+1}) + \kappa(\nabla \zeta^{n+1}, \nabla \psi_h^{n+1}) + b^*(u_h^n, T_h^{n+1} - T^n, \psi_h^{n+1}) - \tau_T(T^{n+1}, \psi_h^{n+1}).
\end{equation}

Follow analogously for the velocity error equation. Subtract (57) and (25), split the error into \( e_u^n = (u^n - \hat{u}^n) - (u_h^n - \hat{u}^n) = \eta^n - \phi_h^n \), let \( v_h^n = \phi_h^{n+1} \in V_h \), add and subtract \( b(u^{n+1}, u_h^{n+1}, \phi_h^{n+1}) \), \( b(u^n, u_h^{n+1}, \phi_h^{n+1}) \), and \( b(u_h^n, u_h^{n+1} - u^n, \phi_h^{n+1}) \). Then,

\begin{equation}
\frac{1}{2\Delta t} \left\{ \| \phi_h^{n+1} \|^2 - \| \phi_h^n \|^2 + \| \phi_h^{n+1} - \phi_h^n \|^2 \right\} + Pr\| \nabla \phi_h^{n+1} \|^2 = \frac{1}{\Delta t} \left( \eta^{n+1} - \eta^n, \phi_h^{n+1} \right) + Pr(\nabla \phi_h^{n+1}, \nabla \phi_h^{n+1}) - \left( \eta^n - \phi_h^{n+1}, \nabla \phi_h^{n+1} \right) + Pr Ra(\zeta^{n+1}, \phi_h^{n+1}) - Pr Ra(\zeta^{n+1}, \phi_h^{n+1}) + b^*(u^{n+1}, \eta^{n+1}, \phi_h^{n+1}) + b(u^n - u_h^{n+1}, \phi_h^{n+1}) + b(\eta^n, u_h^{n+1}, \phi_h^{n+1}) - b^*(\hat{\phi}_h^n, \phi_h^{n+1}) + b(u_h^n, \eta^{n+1}, \phi_h^{n+1}) - b(u_h^n, \eta^{n+1}, \phi_h^{n+1}) + b(u_h^n, \phi_h^{n+1}, \phi_h^{n+1}) + b(u_h^n, u_h^{n+1} - u^n, \phi_h^{n+1}) - \tau_u(u_h^{n+1} - u^n, \phi_h^{n+1}).
\end{equation}

Our goal now is to estimate all terms on the r.h.s. in such a way that we may hide the terms involving unknown pieces \( \psi_h^n \) into the l.h.s. The following estimates are formed using Lemma 1 in conjunction with the Cauchy-Schwarz-Young inequality,

\begin{align}
&|b^*(u^{n+1}, \zeta^{n+1}, \psi_h^{n+1})| \leq C_3 \| \nabla u^{n+1} \| \| \nabla \zeta^{n+1} \| \| \nabla \psi_h^{n+1} \| \leq \frac{C_r C^2_6}{\epsilon^3_3} \| \nabla u^{n+1} \|^2 \| \nabla \zeta^{n+1} \|^2 + \frac{\epsilon^3_3}{r} \| \nabla \psi_h^{n+1} \|^2, \\
&|b^*(\eta^n, T_h^{n+1}, \psi_h^{n+1})| \leq C_3 \| \nabla \eta^n \| \| \nabla T_h^{n+1} \| \| \nabla \psi_h^{n+1} \| \leq \frac{C_r C^2_6}{\epsilon^5_5} \| \nabla \eta^n \|^2 \| \nabla T_h^{n+1} \|^2 + \frac{\epsilon^5_5}{r} \| \nabla \psi_h^{n+1} \|^2, \\
&|b^*(u_h^n, \zeta^{n+1}, \psi_h^{n+1})| \leq C_3 \| u_h^n \| \| \nabla \zeta^{n+1} \| \| \nabla \psi_h^{n+1} \| \leq \frac{C_r C^2_6}{\epsilon^7_7} \| \nabla u_h^n \|^2 \| \nabla \zeta^{n+1} \|^2 + \frac{\epsilon^7_7}{r} \| \nabla \psi_h^{n+1} \|^2, \\
&|b^*(u_h^n, \zeta^n, \psi_h^{n+1})| \leq C_3 \| \nabla u_h^n \| \| \nabla \zeta^n \| \| \nabla \psi_h^{n+1} \| \leq \frac{C_r C^2_6}{\epsilon^8_8} \| \nabla u_h^n \|^2 \| \nabla \zeta^n \|^2 + \frac{\epsilon^8_8}{r} \| \nabla \psi_h^{n+1} \|^2.
\end{align}
Applying Lemma 1, the Cauchy-Schwarz-Young inequality, and Taylor’s theorem yields,

\begin{equation}
|b^*(u^{n+1} - u^n, T_h^{n+1}, \psi_h^{n+1})| \leq C_3 \|\nabla (u^{n+1} - u^n)\| \|\nabla T_h^{n+1}\| \|\nabla \psi_h^{n+1}\|
\leq \frac{C_r C_3^2}{\epsilon_4} \|\nabla (u^{n+1} - u^n)\|^2 \|\nabla T_h^{n+1}\|^2 + \frac{\epsilon_4}{r} \|\nabla \psi_h^{n+1}\|^2.
\end{equation}

Similarly,

\begin{equation}
|b^*(u^n_h, T^{n+1} - T^n, \psi_h^{n+1})| \leq C_3 \|\nabla T_h^n\| \|\nabla (T^{n+1} - T^n)\| \|\nabla \psi_h^{n+1}\|
\leq \frac{C_r C_3^2 \Delta t^2}{\epsilon_4} \|\nabla u_h^n\|^2 \|\nabla T_h^n\|^2 \|\nabla \psi_h^{n+1}\|^2.
\end{equation}

Apply Lemma 1 and the Cauchy-Schwarz-Young inequality twice. This yields

\begin{equation}
|\Delta t b^*(u_h^n, \psi_h^n, \psi_h^{n+1})| = |\Delta t b^*(u_h^n, \psi_h^n, \psi_h^{n+1} - \psi_h^n)|
\leq \Delta t C_4 \|\nabla u_h^n\| \|\nabla \psi_h^n\| \sqrt{\|\psi_h^{n+1} - \psi_h^n\|^2 \|\nabla (\psi_h^{n+1} - \psi_h^n)\|^2}
\leq \Delta t C_4 C_{inv,2}/h^{1/2} \|\nabla u_h^n\| \|\nabla \psi_h^n\| \|\psi_h^{n+1} - \psi_h^n\|
\leq \frac{C_4 C_{inv,2} \Delta t}{2 \epsilon_4} \|\nabla u_h^n\|^2 \|\nabla \psi_h^n\|^2 + \frac{\epsilon_4}{2} \|\psi_h^{n+1} - \psi_h^n\|^2.
\end{equation}

The Cauchy-Schwarz-Young inequality, Poincaré-Friedrichs inequality and Taylor’s theorem yield

\begin{equation}
\frac{1}{\Delta t} (\zeta^{n+1} - \zeta^n, \psi_h^{n+1}) \leq \frac{C_2^2 P_{F,2} C_r}{\epsilon_1} \|\nabla \zeta^n\|^2 \|\nabla \psi_h^{n+1}\|^2 + \frac{\epsilon_1}{r} \|\nabla \psi_h^{n+1}\|^2.
\end{equation}

Lastly, use the Cauchy-Schwarz-Young inequality,

\begin{equation}
|\kappa (\nabla \zeta^{n+1}, \nabla \psi_h^{n+1})| \leq \frac{C_r C_2^2}{\epsilon_2} \|\nabla \zeta^n\|^2 + \frac{\epsilon_2}{r} \|\nabla \psi_h^{n+1}\|^2.
\end{equation}

Similar estimates follow for the r.h.s. terms in (63), however, we must treat an additional pressure term and error term,

\begin{equation}
|-(p^{n+1} - q_h^{n+1}, \nabla \phi_h^{n+1})| \leq \sqrt{d} \|p^{n+1} - q_h^{n+1}\| \|\nabla \phi_h^{n+1}\| \leq \frac{d C_r}{\epsilon_{14}} \|p^{n+1} - q_h^{n+1}\|^2 + \frac{\epsilon_{14}}{r} \|\nabla \phi_h^{n+1}\|^2.
\end{equation}

\begin{equation}
|Pr Ra (\nabla \zeta^{n+1}, \phi_h^{n+1})| \leq \frac{Pr^2 Ra^2 C_2 P_{F,1} C_{P,F,2} C_r}{\epsilon_{15}} \|\nabla \zeta^{n+1\|2} + \frac{\epsilon_{15}}{r} \|\nabla \phi_h^{n+1}\|^2.
\end{equation}

\begin{equation}
|Pr Ra (\nabla \psi_h^{n+1}, \phi_h^{n+1})| \leq \frac{Pr^2 Ra^2 C_2 P_{F,1} C_{P,F,2} C_r}{\epsilon_{16}} \|\nabla \psi_h^{n+1}\|^2 + \frac{\epsilon_{16}}{r} \|\nabla \phi_h^{n+1}\|^2.
\end{equation}

Applying the estimates and Lemma 7 into the temperature and velocity error equations (62), (63) and...
multiplying by $\Delta t$:

\[
\frac{1}{2} \left\{ \| \psi_h^{n+1} \|^2 - \| \psi_h^n \|^2 + \| \psi_h^{n+1} - \psi_h^n \|^2 \right\} + \kappa \Delta t \| \nabla \psi_h^{n+1} \|^2 + \\
\frac{\Delta t C_P C_{PF,2} \| \psi_h \|_{L^\infty(t^n,t^{n+1};L^2(\Omega))}}{\epsilon_1} \| \nabla \psi_h^{n+1} \|^2 + \frac{C_{r} \kappa^2 \Delta t}{\epsilon_2} \| \nabla \xi^{n+1} \|^2 + \frac{\Delta t e_2}{r} \| \nabla \psi_h^{n+1} \|^2 + \\
\frac{C_P^2 C_{\Delta t} \| \nabla \psi_h^{n+1} \|^2}{\epsilon_3} \| \nabla \psi_h^{n+1} \|^2 + \frac{C_{r} \kappa C_{\Delta t}^3}{\epsilon_4} \| \nabla T_h^{n+1} \|^2 \| \nabla u_t \|_{L^\infty(t^n,t^{n+1};L^2(\Omega))} \| + \\
\frac{\Delta t e_4}{r} \| \nabla \psi_h^{n+1} \|^2 + \frac{C_P C_{\Delta t} \| \nabla \psi_h^{n+1} \|^2}{2} + \\
\frac{C_P C_{\Delta t} \| \nabla \psi_h^{n+1} \|^2}{46} \| \nabla \psi_h^{n+1} \|^2 + \frac{C_{r} \kappa C_{\Delta t}^3}{\epsilon_5} \| \nabla \psi_h^{n+1} \|^2 + \\
\frac{C_{r} \kappa C_{\Delta t}^3}{\epsilon_6} \| \nabla \psi_h^{n+1} \|^2 + \frac{C_{r} \kappa C_{\Delta t}^3}{\epsilon_7} \| \nabla \psi_h^{n+1} \|^2 + \\
\frac{C_{r} \kappa C_{\Delta t}^3}{\epsilon_8} \| \nabla \psi_h^{n+1} \|^2 + \frac{C_{r} \kappa C_{\Delta t}^3}{\epsilon_9} \| \nabla \psi_h^{n+1} \|^2 + \\
\frac{C_{r} \kappa C_{\Delta t}^3}{\epsilon_{10}} \| \nabla \psi_h^{n+1} \|^2 + \frac{C_{r} \kappa C_{\Delta t}^3}{\epsilon_{11}} \| \nabla \psi_h^{n+1} \|^2, \\
\frac{\Delta t e_{12}}{r} \| \nabla \phi_h^{n+1} \|^2 + \frac{\Delta t e_{13}}{r} \| \nabla \phi_h^{n+1} \|^2 + \frac{\Delta t e_{14}}{r} \| \nabla \phi_h^{n+1} \|^2 + \\
\frac{\Delta t e_{15}}{r} \| \nabla \phi_h^{n+1} \|^2 + \frac{\Delta t e_{16}}{r} \| \nabla \phi_h^{n+1} \|^2 + \\
\frac{\Delta t e_{17}}{r} \| \nabla \phi_h^{n+1} \|^2 + \frac{\Delta t e_{18}}{r} \| \nabla \phi_h^{n+1} \|^2 + \frac{\Delta t e_{19}}{r} \| \nabla \phi_h^{n+1} \|^2 + \\
\frac{\Delta t e_{20}}{r} \| \nabla \phi_h^{n+1} \|^2 + \frac{\Delta t e_{21}}{r} \| \nabla \phi_h^{n+1} \|^2 + \\
\frac{\Delta t e_{22}}{r} \| \nabla \phi_h^{n+1} \|^2 + \frac{\Delta t e_{23}}{r} \| \nabla \phi_h^{n+1} \|^2 + \\
\frac{\Delta t e_{24}}{r} \| \nabla \phi_h^{n+1} \|^2 + \frac{\Delta t e_{25}}{r} \| \nabla \phi_h^{n+1} \|^2 + \\
\frac{\Delta t e_{26}}{r} \| \nabla \phi_h^{n+1} \|^2.
\]

and

\[
\frac{1}{2} \left\{ \| \phi_h^{n+1} \|^2 - \| \phi_h^n \|^2 + \| \phi_h^{n+1} - \phi_h^n \|^2 \right\} + Pr \Delta t \| \nabla \phi_h^{n+1} \|^2 + \\
\frac{\Delta t C_P C_{PF,1} \| \phi_h \|_{L^\infty(t^n,t^{n+1};L^2(\Omega))}}{\epsilon_{12}} \| \nabla \phi_h^{n+1} \|^2 + \frac{\Delta t \epsilon_{12}}{r} \| \nabla \phi_h^{n+1} \|^2 + \frac{C_{r} Pr^2 \Delta t}{\epsilon_{13}} \| \nabla \eta^{n+1} \|^2 + \\
\frac{\Delta t \epsilon_{13}}{r} \| \nabla \phi_h^{n+1} \|^2 + \frac{dC_P \Delta t}{\epsilon_{14}} \| p^{n+1} - q_h^{n+1} \|^2 + \frac{\Delta t \epsilon_{14}}{r} \| \nabla \phi_h^{n+1} \|^2 + \\
\frac{\Delta t \epsilon_{15}}{r} \| \nabla \phi_h^{n+1} \|^2 + \frac{1}{\epsilon_{15}} \| \nabla \phi_h^{n+1} \|^2 + \frac{C_{r} \kappa C_{\Delta t}}{\epsilon_{16}} \| \nabla \phi_h^{n+1} \|^2 + \\
\frac{C_{r} \kappa C_{\Delta t}}{\epsilon_{17}} \| \nabla \phi_h^{n+1} \|^2 + \frac{\Delta t \epsilon_{17}}{r} \| \nabla \phi_h^{n+1} \|^2 + \\
\frac{C_{r} \kappa C_{\Delta t}}{\epsilon_{18}} \| \nabla u_h^{n+1} \|^2 \| \nabla \phi_h^{n+1} \|^2 + \frac{\Delta t \epsilon_{18}}{r} \| \nabla \phi_h^{n+1} \|^2 + \\
\frac{C_{r} \kappa C_{\Delta t}}{\epsilon_{19}} \| \nabla \phi_h^{n+1} \|^2 + \frac{C_{r} \kappa C_{\Delta t}}{\epsilon_{20}} \| \nabla \phi_h^{n+1} \|^2 + \\
\frac{C_{r} \kappa C_{\Delta t}}{\epsilon_{21}} \| \nabla \phi_h^{n+1} \|^2 + \frac{\Delta t \epsilon_{21}}{r} \| \nabla \phi_h^{n+1} \|^2 + \\
\frac{C_{r} \kappa C_{\Delta t}}{\epsilon_{22}} \| \nabla \phi_h^{n+1} \|^2 + \frac{\Delta t \epsilon_{22}}{r} \| \nabla \phi_h^{n+1} \|^2 + \\
\frac{C_{r} \kappa C_{\Delta t}}{\epsilon_{23}} \| \nabla \phi_h^{n+1} \|^2 + \frac{\Delta t \epsilon_{23}}{r} \| \nabla \phi_h^{n+1} \|^2 + \\
\frac{C_{r} \kappa C_{\Delta t}}{\epsilon_{24}} \| \nabla \phi_h^{n+1} \|^2 + \frac{\Delta t \epsilon_{24}}{r} \| \nabla \phi_h^{n+1} \|^2.
\]

Combine (77) and (78), choose free parameters appropriately, use condition (31), and take the maximum
over all constants on the r.h.s. Then,

\[(79) \quad \frac{1}{2} \left(\|\psi_h^{n+1}\|^2 - \|\psi_h^n\|^2 + \frac{1}{4} \|\phi_h^{n+1} - \phi_h^n\|^2 + \frac{\kappa \Delta t}{4} \left(\|\nabla \psi_h^{n+1}\|^2 - \|\nabla \psi_h^n\|^2\right)\]

\[+ \frac{1}{2} \left(\|\phi_h^{n+1}\|^2 - \|\phi_h^n\|^2\right) + \frac{1}{4} \|\phi_h^{n+1} - \phi_h^n\|^2 + \frac{P r \Delta t}{4} \left(\|\nabla \phi_h^{n+1}\|^2 - \|\nabla \phi_h^n\|^2\right)\]

\[\leq C \left\{ \|\nabla \psi_h\|_{L^2(\Omega)}^2 + \Delta t \|\nabla \phi_h\|_{L^2(\Omega)}^2 + \|\nabla \nabla \psi_h\|_{L^2(\Omega)}^2 + \Delta t \|\nabla \nabla \phi_h\|_{L^2(\Omega)}^2 + \Delta t^3 \|\nabla \nabla \nabla \psi_h\|_{L^2(\Omega)}^2 + \Delta t^3 \|\nabla \nabla \nabla \phi_h\|_{L^2(\Omega)}^2 + \Delta t^3 \|\nabla \nabla \nabla \nabla \psi_h\|_{L^2(\Omega)}^2 + \Delta t^3 \|\nabla \nabla \nabla \nabla \phi_h\|_{L^2(\Omega)}^2\right\}.

Multiply by 2, sum from \(n = 0\) to \(n = N - 1\), apply Lemma 3, and reform. Then,

\[\|\psi_h^n\|^2 + \|\phi_h^n\|^2 + \frac{1}{2} \sum_{n=0}^{N-1} \left(\|\psi_h^{n+1} - \psi_h^n\|^2 + \|\phi_h^{n+1} - \phi_h^n\|^2\right) + \frac{\kappa \Delta t}{2} \|\nabla \psi_h^n\|^2 + \frac{P r \Delta t}{2} \|\nabla \phi_h^n\|^2\]

\[\leq C \left\{ \|\nabla \nabla \nabla \psi_h\|_{L^2(\Omega)}^2 + \|\nabla \nabla \nabla \phi_h\|_{L^2(\Omega)}^2 + \|\nabla \nabla \nabla \nabla \psi_h\|_{L^2(\Omega)}^2 + \|\nabla \nabla \nabla \nabla \phi_h\|_{L^2(\Omega)}^2 + \|\nabla \nabla \nabla \nabla \nabla \psi_h\|_{L^2(\Omega)}^2 + \|\nabla \nabla \nabla \nabla \nabla \phi_h\|_{L^2(\Omega)}^2\right\}.

Take infimums over \(X_h, Q_h,\) and \(W_h\). Apply the triangle inequality, then

\[\|e_T^N\|^2 + \|e_N^N\|^2 + \frac{1}{2} \sum_{n=0}^{N-1} \left(\|e_T^{n+1} - e_T^n\|^2 + \|e_N^{n+1} - e_N^n\|^2\right) + \frac{\kappa \Delta t}{2} \|\nabla e_T^n\|^2 + \frac{P r \Delta t}{2} \|\nabla e_N^n\|^2\]

\[\leq C \left\{ \Delta t \inf_{v_h \in X_h} \left(\|\nabla (u - v_h)\|_{L^2(\Omega)}^2 + \|\nabla (u - v_h)\|_{H^1(\Omega)}^2\right) + \Delta t \inf_{S_h \in W_h} \left(\|\nabla (T - S_h)\|_{L^2(\Omega)}^2 + \|T - S_h\|_{H^1(\Omega)}^2\right)\]

\[+ \frac{\kappa \Delta t}{2} \|\nabla e_T^n\|^2 + \|e_T^n\|^2 + \|e_N^n\|^2 + \Delta t \|\nabla e_T^0\|^2 + \|e_T^0\|^2\]

\[\leq C \left\{ \|\nabla (u - v_h)\|_{L^2(\Omega)}^2 + \|\nabla (u - v_h)\|_{H^1(\Omega)}^2 + \|\nabla (T - S_h)\|_{L^2(\Omega)}^2 + \|T - S_h\|_{H^1(\Omega)}^2\right\}.

The same result holds, with a different constant, for the thin wall problem.

**Theorem 9.** For \((u, p, T)\) satisfying (9) - (13), suppose that \((u_h^0, p_h^0, T_h^0)\) is \((X_h, Q_h, W_h)\) approximations of \((u^0, p^0, T^0)\) to within the accuracy of the interpolant. Further, suppose that condition (31) holds. Then there exists a constant \(C\) such that

\[\|e_T^N\|^2 + \|e_N^N\|^2 + \frac{1}{2} \sum_{n=0}^{N-1} \left(\|e_T^{n+1} - e_T^n\|^2 + \|e_N^{n+1} - e_N^n\|^2\right) + \frac{\kappa \Delta t}{2} \|\nabla e_T^n\|^2 + \frac{P r \Delta t}{2} \|\nabla e_N^n\|^2\]

\[\leq C \left\{ \Delta t \inf_{v_h \in X_h} \left(\|\nabla (u - v_h)\|_{L^2(\Omega)}^2 + \|\nabla (u - v_h)\|_{H^1(\Omega)}^2 + \|\nabla (T - S_h)\|_{L^2(\Omega)}^2 + \|T - S_h\|_{H^1(\Omega)}^2\right)\]

\[+ \frac{\kappa \Delta t}{2} \|\nabla e_T^n\|^2 + \|e_T^n\|^2 + \|e_N^n\|^2 + \Delta t \|\nabla e_T^0\|^2 + \|e_T^0\|^2\]

\[+ \|e_N^0\|^2 + \|e_T^0\|^2 + \|e_N^0\|^2 + \Delta t \|\nabla e_T^0\|^2 + \|\nabla e_T^0\|^2\].
Proof. We follow the same methodology as in Theorem 8. The error equations for velocity and temperature are

\begin{equation}
\frac{e^{n+1} - e^n}{\Delta t}, v_h \right) - b(u^n_h - u^n_{n+1}, v_h) + Pr(\nabla e^{n+1}_u, \nabla v_h) - (e^{n+1}_p, \nabla \cdot v_h) = PrRa\left(\gamma T^{n+1}, v_h \right) - (\gamma T^n, v_h) + \tau_u(u^{n+1}, v_h) \forall v_h \in X_h,
\end{equation}

\begin{equation}
\frac{e^{n+1} - e^n}{\Delta t}, S_h \right) + b^*(u^{n+1}, T^{n+1}, S_h) - b^*(u^n_h, T^n_h, S_h) - b^*(u^n_h, T^n_h, S_h) + \kappa(\nabla e^{n+1}_T, \nabla S_h) = (u^{n+1}_1, S_h) - (u^n_1, S_h) + \tau_T(T^{n+1}, S_h) \forall S_h \in W_h.
\end{equation}

Add and subtract $PrRa(\gamma T^n, v_h)$ in (80) and $(u^n_h, S_h)$ in (81). Then,

\begin{equation}
\frac{e^{n+1} - e^n}{\Delta t}, v_h \right) - b(u^n_h - u^n_{n+1}, v_h) + Pr(\nabla e^{n+1}_u, \nabla v_h) - (e^{n+1}_p, \nabla \cdot v_h) = PrRa\left(\gamma (T^{n+1} - T^n), v_h \right) - (\gamma e^n_T, v_h) + \tau_u(u^{n+1}, v_h) \forall v_h \in X_h,
\end{equation}

\begin{equation}
\frac{e^{n+1} - e^n}{\Delta t}, S_h \right) + b^*(u^{n+1}, T^{n+1}, S_h) - b^*(u^n_h, T^n_h, S_h) - b^*(u^n_h, T^n_h, S_h) + \kappa(\nabla e^{n+1}_T, \nabla S_h) = (u^{n+1}_1 - u^n_1, S_h) - (e^n_1, S_h) + \tau_T(T^{n+1}, S_h) \forall S_h \in W_h.
\end{equation}

Estimate the new terms using similar techniques as in Theorem 8:

\begin{equation}
\left|PrRa(\gamma (T^{n+1} - T^n), v_h)\right| \leq \frac{Pr^2 Ra^2 C^2_{PF,1} C_T}{\epsilon^2_26} \|T^{n+1} - T^n\|^2 + \frac{\epsilon^2_{26}}{r} \|\nabla v_h\|^2 \leq \frac{Pr^2 Ra^2 C^2_{PF,1} C_T \Delta t^2}{\epsilon^2_26} \|T^n_L\|_{L^\infty(L^1, L^2(\Omega))} + \frac{\epsilon^2_{26}}{r} \|\nabla v_h\|^2,
\end{equation}

\begin{equation}
\left|PrRa(\gamma e^n_T, v_h)\right| = \left|PrRa(\gamma \zeta^n, v_h) - PrRa(\gamma \psi^n_h, v_h)\right| \leq \frac{Pr^2 Ra^2 C^2_{PF,1} C_T}{\epsilon^2_27}(\|\zeta^n\|^2 + \|\psi^n_h\|^2) + \frac{2\epsilon^2_{27}}{r} \|\nabla v_h\|^2,
\end{equation}

\begin{equation}
\left|(u^{n+1}_1 - u^n_1, S_h)\right| \leq \frac{C^2_{PF,2} C_T}{\epsilon^2_28} \|u^{n+1}_1 - u^n_1\|^2 + \frac{\epsilon}{r} \|\nabla S_h\|^2 \leq \frac{C^2_{PF,2} C_T \Delta t^2}{\epsilon^2_28} \|u^n\|_L^{\infty}(L^1, L^2(\Omega)) + \frac{\epsilon^2_28}{r} \|\nabla S_h\|^2,
\end{equation}

\begin{equation}
\left|(e^n_1, S_h)\right| = \left|\gamma e^n_T, v_h\right| - (\phi^n_{1h}, S_h) \leq \frac{Pr^2 Ra^2 C^2_{PF,1} C_T}{\epsilon^2_29}(\|e^n\|^2 + \|\phi^n_{1h}\|^2) + \frac{2\epsilon^2_{29}}{r} \|\nabla S_h\|^2.
\end{equation}

Apply estimates similar to those in Theorem 8 as well as the above estimates, multiply by $2\Delta t$, sum from $n = 0$ to $n = N - 1$. Further, apply Lemma 3, triangle inequality and arrive at the result. \qed

**Corollary 10.** Suppose the assumptions of Theorem 4 hold. Further suppose that the finite element spaces $(X_h, Q_h, W_h)$ are given by P2-P1-P2 (Taylor-Hood), then the errors in velocity and temperature satisfy

\[
\|e^n_T\|^2 + \|e^n_T\|^2 + \frac{1}{2} \sum_{n=0}^{N-1} (\|e^{n+1}_T - e^n_T\|^2 + \|e^{n+1}_u - e^n_u\|^2) + \frac{\kappa \Delta t}{2} \|\nabla e^n_T\|^2 + \frac{Pr \Delta t}{2} \|\nabla e^n_T\|^2 \leq C(\Delta t^4 + \Delta t^6 + \Delta t^3 + \Delta t^2 \|\nabla \xi^n\|^2 + \Delta t^2 \|\nabla \xi^n\|^2 + \|\phi^n\|^2)
\]

\[
+ \|e^n\|^2 + \|e^n_T\|^2 + \|e^n_u\|^2 + \Delta t \|\nabla e^n_T\|^2 + \Delta t \|\nabla e^n_u\|^2).
\]
The bred vector (BV) algorithm of Toth and Kalnay [18] is used to generate perturbations in the double pane window problem and in exploring predictability. The BV algorithm (28) - (30), denote the control and perturbed numerical approximations \( \chi \), that is, a pair of initial perturbations for each component of velocity and temperature. Utilizing the scheme problem with \( Ra \) never increased. The condition is violated three times during the computation of the double pane window problem appearing below. We set \( C \) for the double pane window problem appearing below. We set \( C \).

5. Numerical Experiments. In this section, we illustrate the stability and convergence of the numerical scheme described by (28) - (30) using Taylor-Hood (P2-P1-P2) elements to approximate the average velocity, pressure, and temperature. The numerical experiments include the double pane window benchmark problem of de Vahl Davis [19], a convergence experiment and predictability exploration with an analytical solution adopted from [22] devised through the method of manufactured solutions. The software used for all tests is FreeFem++ [9].

5.1. Stability condition. The constant appearing in condition (31) is estimated by pre-computations of the double pane window benchmark problem of de Vahl Davis [19], a convergence experiment and predictability exploration with an analytical solution adopted from [22] devised through the method of manufactured solutions. The software used for all tests is FreeFem++ [9].

5.2. Perturbation generation. The bred vector (BV) algorithm of Toth and Kalnay [18] is used to generate perturbations in the double pane window problem and in exploring predictability. The BV algorithm produces a perturbation with maximal separation rate. We set \( J \) and \( \delta t \) are given.

\[
\|
\begin{align*}
&\| e_T^N \|^2 + \|e_T^N \|^2 + \frac{1}{2} \sum_{n=0}^{N-1} (\| e_T^{n+1} - e_T^n \|^2 + \| e_u^{n+1} - e_u^n \|^2) + \frac{k \Delta t}{2} \| \nabla e_T^N \|^2 + \frac{Pr \Delta t}{2} \| \nabla e_u^N \|^2 \\
\leq & C(\Delta th^2 + \Delta th^3 + \Delta t \| \nabla \eta^0 \|^2 + \Delta t \| \nabla \xi^0 \|^2 + \| \eta^0 \|^2 \\
& + \| \xi^0 \|^2 + \| e_T^0 \|^2 + \| e_u^0 \|^2 + \Delta t \| \nabla e_T^0 \|^2 + \Delta t \| \nabla e_u^0 \|^2).
\end{align*}
\]

Step one: Given \( \chi_h^0 \) and \( \epsilon_i \), put \( \chi_{p,h}^0 = \chi_h^0 + \epsilon_i \). Select time reinitialization interval \( \delta t \geq \Delta t \) and let \( t^k = k \delta t \) with \( 0 \leq k \leq k^* \leq N \).

Step two: Compute \( \chi_h^k \) and \( \chi_{p,h}^k \). Calculate \( bv(\chi^k; \epsilon_i) = \frac{e_i}{\| \chi_{p,h}^k - \chi_h^k \|} (\chi_{p,h}^k - \chi_h^k) \).

Step three: Put \( \chi_{p,h}^k = \chi_h^k + bv(\chi^k; \epsilon_i) \).

Step four: Repeat Step two with \( k = k + 1 \).

Step five: Put \( bv(\chi^k; \epsilon_i) = bv(\chi^k; \epsilon_i) \).

A positive/negative perturbed initial condition pair is generated via \( \chi_{\pm} = \chi^0 + bv(\chi; \pm \epsilon_i) \). We let \( \delta t = \Delta t = 0.001 \) and \( k^* = 5 \).

Fig. 2: Variation of the local Nusselt number at the hot (left) and cold walls (right).
5.3. The double pane window problem. The first numerical experiment is the benchmark problem of de Vahl Davis [19]. The problem is the two-dimensional flow of a fluid in an unit square cavity with $Pr = 0.71$ and $\kappa = 1.0$. Both velocity components (i.e. $u = 0$) are zero on the boundaries. The horizontal walls are insulated and the left and right vertical walls are maintained at temperatures $T(0, y, t) = 1$ and $T(1, y, t) = 0$, respectively; see Figure 1b. We let $10^3 \leq Ra \leq 10^6$. The initial conditions for velocity and temperature are generated via the BV algorithm in Section 5.2,

$$u_\pm(x, y, 0) := u(x, y, 0; \omega_{1,2}) = (1 + bv(u; \pm \epsilon_1), 1 + bv(u; \pm \epsilon_2))^T,$$

$$T_\pm(x, y, 0) := T(x, y, 0; \omega_{1,2}) = 1 + bv(T; \pm \epsilon_3).$$

Both $f(x, t; \omega_j)$ and $g(x, t; \omega_j)$ are identically zero for $j = 1, 2$. The finite element mesh is a division of $[0, 1]^2$ into $64^2$ squares with diagonals connected with a line within each square in the same direction. The stopping condition is

$$\max_{0 \leq n \leq N-1} \left\{ \frac{\|u_h^{n+1} - u_h^n\|}{\|u_h^{n+1}\|}, \frac{\|T_h^{n+1} - T_h^n\|}{\|T_h^{n+1}\|} \right\} \leq 10^{-5}$$

and initial timestep $\Delta t = 0.001$. The timestep was halved three times to 0.000125 to maintain stability for $Ra = 10^6$. Several quantities are compared with benchmark solutions in the literature. These include the maximum vertical velocity at $y = 0.5$, $\max_{x \in \Omega_h} u_2(x, 0.5, t^*)$, and maximum horizontal velocity at $x = 0.5$, $\max_{y \in \Omega_h} u_1(0.5, y, t^*)$. We present our computed values for the mean flow in Tables 1 and 2 alongside several of those seen in the literature. Figures 3 and 4 present the velocity streamlines and temperature isotherms for the averages. All results are seen to be in good agreement with the benchmark values in the literature [4, 15, 19, 20, 22].

| Ra $\times 10^3$ | Present study | Ref. [19] | Ref. [15] | Ref. [20] | Ref. [4] | Ref. [22] |
|------------------|---------------|-----------|-----------|-----------|-----------|-----------|
| $10^4$           | 16.18 (64×64) | 16.18 (41×41) | 16.10 (71×71) | 16.10 (101×101) | 15.90 (11×11) | 16.18 (64×64) |
| $10^5$           | 34.72 (64×64) | 34.81 (81×81) | 34 (71×71) | 34 (101×101) | 33.51 (21×21) | 34.74 (64×64) |
| $10^6$           | 64.80 (64×64) | 65.33 (81×81) | 65.40 (71×71) | 65.40 (101×101) | 65.52 (32×32) | 64.81 (64×64) |

Table 1: Comparison of maximum horizontal velocity at $x = 0.5$ together with mesh size used in computation for the double pane window problem.

| Ra $\times 10^3$ | Present study | Ref. [19] | Ref. [15] | Ref. [20] | Ref. [4] | Ref. [22] |
|------------------|---------------|-----------|-----------|-----------|-----------|-----------|
| $10^4$           | 19.60 (64×64) | 19.51 (41×41) | 19.90 (71×71) | 19.79 (101×101) | 19.91 (11×11) | 19.62 (64×64) |
| $10^5$           | 68.53 (64×64) | 68.22 (81×81) | 70 (71×71) | 70.63 (101×101) | 70.60 (21×21) | 68.48 (64×64) |
| $10^6$           | 215.96 (64×64) | 216.75 (81×81) | 228 (71×71) | 227.11 (101×101) | 228.12 (32×32) | 220.44 (64×64) |

Table 2: Comparison of maximum horizontal velocity at $y = 0.5$ together with mesh size used in computation for the double pane window problem.
5.4. Numerical convergence study. In this section, we illustrate the convergence rates for the proposed algorithm (28) - (30). The unperturbed solution is given by

\[ u(x, y, t) = (10x^2(x - 1)^2y(y - 1)(2y - 1)\cos(t), -10x(x - 1)(2x - 1)y^2(y - 1)^2\cos(t))^T, \]

\[ T(x, y, t) = u_1(x, y, t) + u_2(x, y, t), \]

\[ p(x, y, t) = 10(2x - 1)(2y - 1)\cos(t), \]

with \( \kappa = Pr = 1.0, Ra = 100, \) and \( \Omega = [0, 1]^2. \) The perturbed solutions are given by

\[ u(x, y, t; \omega_1) = (1 + \epsilon_1,2)u(x, y, t), \]

\[ T(x, y, t; \omega_1) = (1 + \epsilon_1,2)T(x, y, t), \]

\[ p(x, y, t; \omega_1) = (1 + \epsilon_1,2)p(x, y, t), \]

where \( \epsilon_1 = 1 - 2 = -\epsilon_2 \) and both forcing and boundary terms are adjusted appropriately. The perturbed solutions satisfy the following relations,

\[ < u >= 0.5\{u(x, y, t; \omega_1) + u(x, y, t; \omega_2)\} = u(x, y, t), \]

\[ < T >= 0.5\{T(x, y, t; \omega_1) + T(x, y, t; \omega_2)\} = T(x, y, t), \]

\[ < p >= 0.5\{p(x, y, t; \omega_1) + p(x, y, t; \omega_2)\} = p(x, y, t). \]

The finite element mesh is a Delaunay triangulation generated from 4, 8, 16, 32, 64, and 128. Table 5 confirms first order convergence in velocity, second order convergence in velocity and temperature in the \( L^\infty(0, t^*; L^2(\Omega)) \) norm and both forcing and boundary terms are adjusted appropriately. The perturbed solutions satisfy the following relations,

\[ < u >= 0.5\{u(x, y, t; \omega_1) + u(x, y, t; \omega_2)\} = u(x, y, t), \]

\[ < T >= 0.5\{T(x, y, t; \omega_1) + T(x, y, t; \omega_2)\} = T(x, y, t), \]

\[ < p >= 0.5\{p(x, y, t; \omega_1) + p(x, y, t; \omega_2)\} = p(x, y, t). \]

Table 3: Comparison of average Nusselt number on the vertical boundary at \( x = 0 \) together with mesh size used in computation for the double pane window problem.

| \( m \) | \( L^\infty(0, t^*; L^2(\Omega)) \) norm | \( L^\infty(0, t^*; H^1(\Omega)) \) norm | \( L^\infty(0, t^*; H^2(\Omega)) \) norm |
|---|---|---|---|
| 4 | 1.21E-04 | 1.21E-04 | 1.21E-04 |
| 8 | 6.35E-05 | 6.35E-05 | 6.35E-05 |
| 16 | 3.17E-05 | 3.17E-05 | 3.17E-05 |
| 32 | 1.59E-05 | 1.59E-05 | 1.59E-05 |
| 64 | 7.94E-06 | 7.94E-06 | 7.94E-06 |
| 128 | 3.97E-06 | 3.97E-06 | 3.97E-06 |

Table 4: Errors and rates for average velocity, temperature, and pressure in corresponding norms.

| \( m \) | \( L^\infty(0, t^*; L^2(\Omega)) \) norm | \( L^\infty(0, t^*; H^1(\Omega)) \) norm | \( L^\infty(0, t^*; H^2(\Omega)) \) norm |
|---|---|---|---|
| 4 | 0.00134087 | 0.00134087 | 0.00134087 |
| 8 | 3.68E-04 | 3.68E-04 | 3.68E-04 |
| 16 | 1.87E-04 | 1.87E-04 | 1.87E-04 |
| 32 | 9.38E-05 | 9.38E-05 | 9.38E-05 |
| 64 | 4.69E-05 | 4.69E-05 | 4.69E-05 |
| 128 | 2.35E-05 | 2.35E-05 | 2.35E-05 |

The temporal convergence is illustrated by choosing a fixed \( \Delta t = 0.0001 \) and setting the final time \( t^* = 0.001 \). The parameter \( m \) is varied between 4, 8, 16, 32, 64, and 128. Results are presented in Table 4. Third order convergence is observed in velocity and temperature and second order convergence in pressure in the \( L^\infty(0, t^*; L^2(\Omega)) \) norm and second order convergence in velocity and temperature in the \( L^\infty(0, t^*; H^1(\Omega)) \) norm.

Temporal convergence is illustrated by choosing a fixed \( m = 64 \) and setting the final time \( t^* = 1.0 \). The timestep is varied between 4, 8, 16, 32, 64, 128. Table 5 confirms first order convergence in velocity, temperature, and pressure in the \( L^\infty(0, t^*; L^2(\Omega)) \) norm and in velocity and temperature in the \( L^\infty(0, t^*; H^1(\Omega)) \) norm.
5.5. Exploration of predictability. Consider the problem with manufactured solution in Section 5.4. However, instead of specifying the perturbations on the initial conditions, the BV algorithm in Section 5.2 yields

\[ u_\pm(x, y, 0) := u(x, y, 0; \omega_1, \omega_2) = u_1(x, y, 0) + bv(u; \pm \epsilon_1), \]
\[ T_\pm(x, y, 0) := T(x, y, 0; \omega_1, \omega_2) = T(x, y, 0) + bv(T; \pm \epsilon_3). \]

The forcing functions and boundary conditions are left unperturbed. Further, the Rayleigh number is varied between $10^2$ and $10^4$. The initial timestep is 0.001 and final time $t^* = 0.5$. Herein, we will define energy, variance, average effective Lyapunov exponent [2], and $\delta$-predictability horizon [2].

**Definition 12.** The energy is given by

\[ \text{Energy} := \|T\| + \frac{1}{2}\|u\|^2. \]

The variance of $\chi$ is

\[ V(\chi) :=< \|\chi\|^2 > - \|\chi\|^2 = < \|\chi'\|^2 > . \]

The relative energy fluctuation is

\[ r(t) := \frac{\|\chi_+ - \chi_-\|^2}{\|\chi_+\|\|\chi_-\|}, \]

and the average effective Lyapunov exponent over $0 < \tau \leq t^*$ is

\[ \gamma_\tau(t) := \frac{1}{2\tau} \log \left( \frac{r(t + \tau)}{r(t)} \right), \]

with $0 < t + \tau \leq t^*$. The $\delta$-predictability horizon is

\[ t_p := \frac{1}{\gamma_\tau(0)} \log \left( \delta \frac{1}{\|\chi_+ - \chi_-(0)\|} \right). \]

Figure 3 presents the energy and variance of the approximate solutions with $Ra = 10^4$. The variance of the perturbed solutions indicates that they do not deviate much from the mean and therefore not much from each other. This seems to explain, in part, why the energy associated with these solutions is similar. Interestingly, the energy associated with the unperturbed and mean computed solutions sit atop of one another; that is, the mean leads to a superior estimate than either member of the ensemble. It seems that the BV algorithm generated a positive/negative initial condition pair leading to two solutions whose average approximates the unperturbed solution well.

Figures 4 and 5 present $\gamma_\tau(t)$ and $t_p$ for mean temperature and velocity approximations for $10^2 \leq Ra \leq 10^4$ and $\|\chi_+ - \chi_-\| \leq \delta \leq 0.15$. The approximated effective Lyapunov exponent $\gamma_\tau(0)$ and $t_p$ are negative for both velocity and temperature for all Rayleigh numbers indicating a predictable flow. However, $\gamma_\tau(t)$ changes sign for temperature and velocity at approximately $t = 0.11$ for $Ra = 10^4$ indicating a loss of predictability.

6. Conclusion. We presented two algorithms for calculating an ensemble of solutions to two laminar natural convection problems. These algorithms addressed the competition between ensemble size and resolution in simulations. In particular, both algorithms required the solution of a single matrix equation, at each time step, with multiple right hand sides. Stability and convergence were proven and numerical experiments were performed to illustrate these properties.
Fig. 3: Comparison of the energy in the system (left) and variance of each velocity and temperature ensemble member (right).

Fig. 4: Comparison of average effective Lyapunov exponent for temperature (left) and velocity (right).

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Fig. 5: Comparison of $\delta$-predictability horizons for temperature (left) and velocity (right).

Fig. 6: Streamlines for $Ra = 10^3, 10^4, 10^5$, and $10^6$, from left to right, respectively.

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Fig. 7: Isotherms for $Ra = 10^3, 10^4, 10^5$, and $10^6$, from left to right, respectively.