Determination of the stress state of anisotropic hollow bar with eccentricity for the elastoplastic torsion problem

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Abstract. The article is devoted to the problem of elastoplastic torsion of anisotropic hollow bar with eccentricity. The influence of mass forces is not taken into account. The bar is twisted about the symmetry axis by equal and opposite pairs of forces. The value of the moment has such a value that in some parts of the cross section the material passes into a plastic state and plastic zones are formed. The propagation of plastic flow comes from the outer contour to the inside. It’s assumed that the value of the moment is such that the plastic region entirely covers the outer contour of the cross section and an elastoplastic boundary is formed. It is assumed that the bar material is anisotropic and in special cases there are the kinematic properties of the anisotropy and anisotropy according to Hill. With using perturbation method, stress state and elastoplastic boundary at first approximate is defined.

1. Introduction

The work [1] uses well-known formulation of the torsion problem in complex variable theory and used to obtain an expression for shear stresses, angle of twist and cross-sectional warping. The obtained mapping is used to achieve a solution for torsion problem of bars with specially shaped hollow cross-section. Calculated results are compared with the finite-element method solution. By using the meshless regularized integral equation method (MRIEM), the solution of elastic torsion problem of a uniform bar with arbitrary cross-section is presented by the first kind Fredholm integral equation on an artificial circle, which just encloses the bar’s cross-section [2]. The paper considers [3] a parabolic partial differential equation involving the infinity Laplace operator and a Leray-Lions operator with no coercitive assumption. The existence and uniqueness of the corresponding approached problem is proved and shows that the limit the solution solves the parabolic variational inequality arising in the elastoplastic torsion problem. For the first time, the problem of determining the stress-strain state of an eccentric hollow bar under the influence of internal pressure for an elastic perfectly plastic material was solved in [4] with using perturbation method. In the work [5] with application of the perturbation method, a stress state is determined in a cylindrical hollow bar subjected to external and internal pressures, with boundaries of the cross section close to circular. The question of the existence and uniqueness of the solution of the problem is considered on the basis of the implicit function theorem. The work [6] is devoted to the investigation of the stress-strain state of a hollow bar of a non-circular cross section.
2. Model

In the presented work with application of the perturbation method [4], [7] the stress state and elastoplastic boundary is determined for elastoplastic torsion [8] of hollow bar with eccentricity including anisotropy properties of material.

![Cylindrical hollow bar with small eccentricity](image)

**Figure 1.** Cylindrical hollow bar with small eccentricity.

We consider a cylindrical hollow bar (figure 1). \( a \) and \( b \) radius of outer and inner circular cylinders respectively. Suppose that the lateral surface of the bar is free of loads. Eccentricity is equal \( e \). It is assumed that the plastic zone covers completely the outer contour of the pipe. The case of small eccentricity is considered. The solution uses a cylindrical coordinate system. The origin of coordinate system is aligned with the center of the inner contour. In this case, the equation of the outer contour of the hollow bar has the form

\[
(x - e)^2 + y^2 = b^2.
\]

Passing to the polar coordinate system, we have

\[
\rho^2 - 2\rho\delta\cos\theta + \delta^2 - \beta^2 = 0,
\]

where \( \rho = \rho/r_s \), \( \delta = e/r_s \), \( \beta = b/r_s \), where \( r_s \) radius of the elastoplastic boundary in the zero approximation. From equation (1) we find

\[
\rho = \delta\cos\theta + \sqrt{\beta^2 - \delta^2\sin^2\theta}.
\]

Expanding this expression in powers of the parameter \( \delta \) for the case of small eccentricity \( \delta \ll 1 \), we obtain the equation of the outer contour linearized with respect to the small parameter.

\[
\rho = \beta + \delta\cos\theta - \frac{\delta^2}{2\beta} \sin^2\theta + \ldots.
\]
The yield criteria for the torsion problem can be written in the form [9]

\[ A(\tau_{xc} - k_1)^2 + B(\tau_{yc} - k_2)^2 = k_0^2 \]  

(3)

where \( A, B, k_1, k_2 \) is anisotropy constant; where \( k_0 \) is yield strength.

Next, we pass to the cylindrical coordinate system \( \rho, \theta, z \), where the axis \( z \) coincides with the axis of the hollow bar, and we use dimensionless quantities. Divide quantities having the dimensionality of the stresses by the yield strength \( k_0 \) and the quantities having the dimensionality of length divide into the radius of the elastoplastic boundary in the zero approximation \( \rho_0 \).

Taking into account the formulas for the transition of stress components from Cartesian coordinates to cylindrical coordinates,

\[ \tau_{xc} = \tau_{\rho c} \cos \theta - \tau_{\theta c} \sin \theta, \]

\[ \tau_{yc} = \tau_{\rho c} \sin \theta + \tau_{\theta c} \cos \theta, \]

We write down the yield criterion (3) in the form.

\[ A(\tau_{\rho c} \cos \theta - \tau_{\theta c} \sin \theta - k_1)^2 + B(\tau_{\rho c} \sin \theta + \tau_{\theta c} \cos \theta - k_2)^2 = k_0^2 \]  

(4)

The equation of equilibrium reduces to a single differential equation

\[ \frac{\partial \tau_{\rho c}}{\partial \rho} + \frac{1}{\rho} \frac{\partial \tau_{\theta c}}{\partial \theta} + \tau_{\rho c} = 0. \]  

(5)

In the elastic region, the Cauchy relationships

\[ \varepsilon_{\rho c}^e = \frac{\omega}{2} \frac{\partial \omega^e}{\partial \rho}, \varepsilon_{\theta c}^e = \frac{\omega}{2} \left( \frac{1}{\rho} \frac{\partial \omega^e}{\partial \theta} + \rho \right). \]  

(6)

The Cauchy relations connecting components of the strain tensor with the function \( \omega^e \). \( \omega^e \) characterizes the cross-sectional warping, where \( \omega \) is twist.

Hooke’s Law relationships

\[ \tau_{\rho c}^e = 2G \varepsilon_{\rho c}^e, \tau_{\theta c}^e = 2G \varepsilon_{\theta c}^e, \]  

(7)

where \( G \) is shear modulus.

Since the side surfaces of the hollow bar are free from the load, the boundary conditions are satisfied with the outer and inner contours of the cross section

\[ \left( \tau_{\rho c} n_{\rho} + \tau_{\theta c} n_{\theta} \right)_{L} = 0, \]  

(8)

\[ \tau_{\rho c} \bigg|_{\rho = a} = 0, \]

where \( L \) is outer contour, and \( n_{\theta}, n_{\rho} \) is the components of the unit normal. The components of the unit normal are determined by the formulas

\[ n_{\rho} = \frac{1}{\rho} \frac{\partial \phi}{\partial \theta} \left( \frac{\partial \phi}{\partial \rho} \right)^2 + \left( \frac{1}{\rho} \frac{\partial \phi}{\partial \theta} \right)^2, \]  

(9)

\[ n_{\rho} = \frac{\partial \phi}{\partial \rho} \left( \frac{\partial \phi}{\partial \rho} \right)^2 + \left( \frac{1}{\rho} \frac{\partial \phi}{\partial \theta} \right)^2. \]
where \( \phi(\rho, \theta) = 0 \) is equation of the outer contour of the hollow bar.

On the elastoplastic boundary the stress-continuity condition

\[
[r_{ij}]_{L_e} = 0,
\]

where \( L_e \) is elastoplastic boundary.

In the elastic region from the equilibrium equation (5), taking equations (6) and (7) into account, we obtain

\[
\Delta w^{(e)} = 0,
\]

where \( \Delta = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \) is Laplace operator.

On the elastoplastic boundary \( L_e \) the displacement continuity condition is fulfilled.

\[
[w]_{L_e} = 0.
\]

We seek solutions in the form of the decomposition with respect to a small dimensionless parameter \( \delta \) \( (\delta << 1) \). The translational anisotropy and anisotropy constants according to Hill [10], stresses, cross-sectional warping and elastoplastic boundary are representable in the form

\[
A = 1 + \delta \alpha_1, k_1 = \delta k_1^{(1)}, k_2 = \delta k_2^{(1)}, B = 1 + \delta \beta_1, \tau_{ij} = \tau_{ij}^{(0)} + \delta \tau_{ij}^{(1)} + \ldots,
\]

\[
w^{(e)} = w^{(e)(0)} + \delta w^{(e)(1)} + \ldots, \quad \rho_s = \rho_0 + \delta \rho_1 + \ldots,
\]

\[
\Phi(\rho, \theta) = \rho - \beta \delta \cos \theta - \frac{\delta^2}{2\beta} \sin^2 \theta + \ldots,
\]

where \( \rho_s \) is elastoplastic boundary.

Substitute decomposition (13) into relationships (4) – (12) and equate the terms at the same powers. We obtain the system of equations for each approximation.

In the zero approximation, the known problem of elastoplastic torsion of a homogeneous hollow rod with a ring cross-section takes place. The solution of this problem has the form

\[
\tau_{\theta \rho}^{(0)} = 1, \quad \tau_{\rho \theta}^{(0)} = 0, \quad w^{(0)} = 0, \quad \tau_{e \rho}^{(0)} = G \omega \rho, \quad \tau_{e \theta}^{(0)} = 0, \quad \rho_s = 1.
\]

Consider only the solution of the problem in the first approximation. Since the relationships (5) and (7) are linear in \( \delta \), they preserve their form for each approximation.

Substituting decomposition (13) into equation (4) in the plastic region, we obtain the yield criteria in the first approximation and express from it the stress component \( \tau_{\theta \rho}^{(1)} \) as a coordinate function \( \theta \) containing the anisotropy constants. This expression takes the form

\[
\tau_{\theta \rho}^{(1)} = k_2^{(1)} \cos \theta - k_1^{(1)} \sin \theta - \frac{1}{2}(a_1^{(1)} \sin^2 \theta + b_1^{(1)} \cos^2 \theta).
\]

To determine the stress component \( \tau_{\rho \theta}^{(1)} \), we obtain a system by substituting solution (15) into the equilibrium equation (5) and the boundary condition on the outer contour (8), taking into account the decomposition (13).
\[
\begin{aligned}
\frac{\partial}{\partial \rho} \left( \rho \tau_{\rho \rho}^{(1)} \right) &= - \frac{\partial \tau_{\theta \rho}^{(1)}}{\partial \theta}, \\
\tau_{\rho \rho}^{(1)} |_{\rho = \beta} &= - \frac{\sin \theta}{\beta}.
\end{aligned}
\] (17)

Solving system of equations (17), we obtain the component \( \tau_{\rho \rho}^{(1)} \) as a function of the coordinates, the anisotropy constants, and the radius of the outer contour in the zero approximation. Write this in the form

\[
\tau_{\rho \rho}^{(1)} = \left( 1 - \frac{\beta}{\rho} \right) \left( k_2^{(1)} \sin \theta + k_1^{(1)} \cos \theta - \frac{b_1^{(1)} - a_1^{(1)}}{2} \sin 2\theta \right) - \frac{\beta}{\rho} \sin \theta.
\] (18)

From the relationships (6) and (7) in the elastic region, taking into account the decomposition (13), we obtain the stress components in the first approximation

\[
\tau_{\rho \rho}^{(1)} = G_\omega \frac{\partial w^{(1)}}{\partial \rho}, \quad \tau_{\theta \rho}^{(1)} = G_\omega \frac{1}{\rho} \frac{\partial w^{(1)}}{\partial \theta}.
\] (19)

The stress-continuity condition on the elastoplastic boundary (10), taking into account the decomposition (13), in the first approximation will take the form

\[
\left. \frac{\partial \tau_{\rho \rho}^{(0)}}{\partial \rho} \rho_1 + \tau_{\rho \rho}^{(1)} \right|_{\rho=1} = \left. \frac{\partial \tau_{\theta \rho}^{(0)}}{\partial \rho} \rho_1 + \tau_{\theta \rho}^{(1)} \right|_{\rho=1},
\] (20)

\[
\left. \frac{\partial \tau_{\rho \rho}^{(0)}}{\partial \rho} \rho_1 + \tau_{\rho \rho}^{(1)} \right|_{\rho=1} = \left. \frac{\partial \tau_{\theta \rho}^{(0)}}{\partial \rho} \rho_1 + \tau_{\theta \rho}^{(1)} \right|_{\rho=1}.
\] (21)

After the substitution of solution (15) and relationships (19) the relationship (2) is transformed into

\[
\left. \frac{\partial w^{(1)}}{\partial \rho} \right|_{\rho=1} = \frac{1}{G_\omega} \tau_{\rho \rho}^{(1)} \left|_{\rho=1}.
\] (22)

For the displacement function \( w^{(1)} \) in the elastic region, combining boundary conditions (8), equation (11), relationships (19) and relationship (22), we obtain the Neumann problem for the ring. The necessary and sufficient condition of this problem for the existence of a solution is satisfied.

\[
\begin{aligned}
\Delta w^{(1)} &= 0, \alpha < \rho < 1, \theta \in [0; 2\pi], \\
\left. \frac{\partial w^{(1)}}{\partial \rho} \right|_{\rho=1} &= \frac{1}{G_\omega} \tau_{\rho \rho}^{(1)} \left|_{\rho=1}, \\
\left( \frac{\partial w^{(1)}}{\partial \rho} \right)_{\rho=2} &= 0.
\end{aligned}
\] (23)

3. Results and discussion

Solving the problem, we obtain \( w^{(1)} \) in the elastic region as a function of the coordinates containing the anisotropy constants and the radius of inner and outer contours in the zero approximation. We represent this in the form.
Figure 2. Graph of the dependence of displacement function in elastic region on the radius of hollow bar.

This graph (Figure 2) indicates the dependence of the displacement function in elastic region, characterizes the cross-sectional warping on the radius of hollow bar. The graph is constructed for a fixed value of \( \theta = \frac{\pi}{4} \). It can be seen from the graph that the maximum displacement occurs near the outer contour.

In the elastic region, the stress state in the first approximation is determined by substituting relationship (24) in relationships (19).

\[
w^{(1)}(r) = \left( k_2^{(1)} \sin \theta + k_1^{(1)} \cos \theta \right) \left( \beta - 1 \right) \frac{\rho^2 + \alpha^2}{\rho \alpha (\alpha^2 - 1)} + \sin \theta \frac{\rho^2 + \alpha^2}{\rho \alpha (\alpha^2 - 1)} - \frac{1}{2} \sin 2\psi \frac{\beta - 1}{\rho \alpha (\alpha^2 - 1)}.
\]

Define the elastoplastic boundary in the first approximation using condition (21) with allowance for solution (15)

\[
\rho_1 = \left[ r_1^{(1)} - r_2^{(1)} \right]_{	ext{pl}}.
\]

Substituting relationship (25) and relationship (16) into equation (27), we obtain the required expression for an elastoplastic boundary in the first approximation.
\[
\rho^{(1)} = \frac{1}{G\omega} \left( (k_2^{(1)} \cos\theta - k_1^{(1)} \sin\theta) \frac{\alpha^2 - 1}{\alpha^2 - 1} \beta - 1 \frac{1 + \alpha^2}{\alpha^2 - 1} \beta - 1 \left( a_i^{(1)} \sin^2\theta + b_i^{(1)} \cos^2\theta \right) \right. \\
+ \frac{1}{2} \left( b_i^{(1)} - a_i^{(1)} \right) \frac{\beta - 1}{\alpha^4 - 1} \cos 2\theta \\
\left. \right) 
\]

Figure 3. Elastoplastic boundary of the elastoplastic torsion problem of an anisotropy elastoplastic cylindrical hollow bar with eccentricity.

The results of the numerical experiment are presented in the following figures, where \( \delta = 0.025, k_1 = 0.5, k_2 = 0.6, a_i = 0.3, b_i = 0.4, G\omega = 1, G = 323.67 \). In the figure 3 the gray represents the elastoplastic boundary. The dash line is an elastoplastic boundary in the zero approximation. For analyzing the obtained solution and estimating the influence of the anisotropy constants on the expression for the elastoplastic boundary, it is necessary to compare the obtained relationship (28) with the solution of the elastoplastic torsion problem of an elastic perfectly plastic cylindrical hollow bar with eccentricity. The elastoplastic boundary in the first approximation of the elastoplastic torsion problem of an elastic perfectly plastic cylindrical hollow bar with eccentricity has the form

\[
\rho^{(1)} = \frac{1 + \alpha^2}{G\omega(1 - \alpha^2)} \cos \theta. 
\]
Figure 4. Elastoplastic boundary of the elastoplastic torsion problem of an elastic perfectly plastic cylindrical hollow bar with eccentricity.

This graph (figure 4) indicates the dependence of the elastoplastic boundary (gray), on the components of the polar coordinate system of hollow bar. The dash line is an elastoplastic boundary in the zero approximation. The graph shows that the deviation between the elastoplastic boundary and the elastoplastic boundary in the zero approximation (dash line) is substantially less than the deviation between the elastoplastic boundary for the problem with allowance for the anisotropy and the elastoplastic boundary in the zero approximation (figure 3). The zero approximation corresponds to the problem of elastoplastic torsion of a homogeneous hollow rod with a ring cross-section.

Then we calculate the maximum difference between relationship (29) and relationship (28) in absolute value and plot the dependence of the given deviation on $k_1^{(1)}$, running through the values from 0..0.1, fixing the remaining parameters of the anisotropy.
Figure 5. Graph of the dependence of the maximum deviation on the anisotropy constant $k^{(1)}_i$.

This graph (figure 5) indicates the character of the dependence of the anisotropy constant on the maximum deviation between the solutions of the problems of elastoplastic torsion of an eccentric anisotropic hollow bar and the elastoplastic torsion of an eccentric hollow bar with elastic perfectly plastic material. Similar relationships can also be created for other anisotropy constants $a^{(1)}_i, b^{(1)}_i, k^{(2)}_i$.

4. Conclusion
In the course of solving the problem posed, the dependencies of the stress components in the elastic and plastic regions, the displacement function in the elastic region, characterizing the cross-section warping in the elastic region and the expressions for the elastoplastic boundary have been obtained. The dependence of the radius of the elastoplastic boundary on the anisotropy constant has been analyzed. The plot for the maximum deviation of the radius of the elastoplastic boundary for the solution of the problem of elastoplastic torsion of an eccentric anisotropic hollow bar from elastoplastic torsion of an eccentric hollow bar of an elastic perfectly plastic material has been created.

References
[1] Bazehhour BG and Rezaeepazhand J 2014 Mathematics and mechanics of solids 19 pp 260–276
[2] Chein–Shan Liu 2007 Computers, Materials and Continua 5 pp 31–42
[3] Messelmi F 2017 Analysis and Mathematical Physics 7 pp 437–447
[4] Ivlev D D and Ershov L V 1978 Perturbing approximation in theory of elastoplastic body (Moscow: Science) p 208
[5] Zadorozhniy V G, Kovalev A V and Sporykhin A N 2008 Russian Academy of Sciences: Mechanics of solid body 1 pp 138–146
[6] Kovalev A V, Shcheglova Yu D and Sviridov I E 2016 Bulletin of the Yakovlev Chuvash State Pedagogical University: Mechanics of Limit State 4(30) pp 42–54
[7] Sporykhin A N, Kovalev A V and Shcheglova Yu D 2004 Nonuniform elastic-viscoplasticity problems with an unknown boundary (Voronezh: Voronezh State University Press) p 219
[8] Kachanov L M 1969 Basics of the theory of plasticity (Moscow: Science) p 420
[9] Fominyh S O 2013 Bulletin of the Yakovlev Chuvash State Pedagogical University: Mechanics
of Limit State 2(16) pp 150–153

[10] Hill R 1950 *The Mathematical theory of plasticity* (Oxford: The Clarendon Press) p 407

[11] Ivlev D D and Mironov B G 2010 *Bulletin of the Yakovlev Chuvash State Pedagogical University: Mechanics of Limit State* 3 pp 560–600 (in Russian)

[12] Mironov B G and Mitrofanova T V 2011 *Bulletin of the Yakovlev Chuvash State Pedagogical University: Mechanics of Limit State* 1(9) pp 150–155