A NEW REPRESENTATION OF HANKEL OPERATORS AND ITS SPECTRAL CONSEQUENCES

D. R. YAFAEV

To the memory of Mikhail Zakharovich Solomyak

Abstract. We describe a new representation of Hankel operators $H$ as pseudo-differential operators $A$ in the space of functions defined on the whole axis. The amplitudes of such operators $A$ have a very special structure: they are products of functions of a one variable only. This representation has numerous spectral consequences both for compact Hankel operators and for operators with the continuous spectrum.

1. Introduction

1.1. This is a short survey based on the talk given by the author at the 9th Saint-Petersburg Spectral Theory Conference held in Euler Institute (Saint-Petersburg, Russia) during 3-6 July 2017.

Among numerous papers of M. Sh. Birman and M. Z. Solomyak on spectral theory of self-adjoint operators, their study (summarized in [2]) of the Weyl asymptotics of eigenvalues of differential operators plays a distinguished role. The methods developed in their papers on this subject were extended by the authors to pseudo-differential and integral operators in [1] and [3]. We directly use the results of [1, 3] in this article.

1.2. Hankel operators can be defined as integral operators

$$(Hu)(t) = \int_0^\infty h(t + s)u(s)ds \quad (1.1)$$

in the space $L^2(\mathbb{R}_+)$ with kernels $h$ that depend on the sum of variables only. If necessary, we write $H = H(h)$ for the operator (1.1). We refer to the books [7, 8] for basic information on Hankel operators. Of course $H$ is symmetric if $h(t) = h(t)$. There are very few cases when
Hankel operators can be explicitly diagonalized. The simplest and most important example $h(t) = t^{-1}$ was considered by T. Carleman. The corresponding Hankel operator $(1.1)$ is bounded but not compact; actually, it has the absolutely continuous spectrum $[0, \pi]$ of multiplicity 2. It follows that a Hankel operator $H$ is compact if, for example, $h \in L^\infty_{\text{loc}}(\mathbb{R}^+)$ and $h(t) = o(t^{-1})$ as $t \to \infty$ and as $t \to 0$. It turns out that singular values $s_n(H)$ of $H$ (and its eigenvalues in the self-adjoint case) have power asymptotics as $n \to \infty$ if the kernel $h(t)$ is close to $t^{-1}$ in the logarithmic scale (both for large and small $t$), that is, if $h(t)$ behaves as $\kappa_\infty t^{-1}|\ln t|^{-\alpha}$ with some $\alpha > 0$ for $t \to \infty$ and as $\kappa_0 t^{-1}|\ln t|^{-\alpha}$ for $t \to 0$. On the contrary, $H$ is unbounded if $h(t)t \to \infty$ as $t \to \infty$ or as $t \to 0$.

Our first goal is to describe in Section 2 a procedure suggested in [16, 19] reducing an arbitrary Hankel operator $H$ by an explicit unitary transformation $M$ (essentially, by the Mellin transform) to a special integral, or pseudo-differential, operator $A$ in the space $L^2(\mathbb{R})$:

$$H = M^{-1}AM.$$  

(1.2)

In many cases, the spectral properties of the operators $A$ are easier to study than those of the original Hankel operators $H$. We emphasize that the identity $(1.2)$ does not require that the operators $H$ be symmetric, but in our spectral applications $H$ are self-adjoint (except Section 6).

The operator $A$ can be defined as follows. Put

$$(Xu)(x) = xu(x), \quad (Du)(x) = -iu'(x).$$

Then

$$A = v(X)s(D)v(X)$$  

(1.3)

where the standard function

$$v(x) = \frac{\sqrt{\pi}}{\sqrt{\cosh(\pi x)}}$$  

(1.4)

is quite explicit; it is the same for all Hankel operators. The function $s(\xi)$ depends of course on $h(t)$, and it can be constructed in the following way.

Let us formally define the so-called sigma-function by the equation

$$h(t) = \int_0^\infty e^{-t\lambda} \sigma(\lambda)d\lambda;$$  

(1.5)

in general, $\sigma(\lambda)$ is a distribution. The function $s(\xi)$ (it is called the sign-function of a Hankel operator $H$ in [16]) differs from $\sigma(\lambda)$ by a
change of variables only:

\[ s(\xi) = \sigma(e^{-\xi}). \]  

Thus the operator \( A \) can be considered in the space \( L^2(\mathbb{R}) \) either as a \( \Psi DO \) (pseudo-differential operator) with the amplitude

\[ a(x, y; \xi) = v(x)s(\xi)v(y), \quad x, y, \xi \in \mathbb{R}, \]

or as an integral operator with kernel

\[ (2\pi)^{-1/2}v(x)(\Phi^* s)(x - y)v(y) \]

where \( \Phi \),

\[ (\Phi f)(\xi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)dx, \]

is the Fourier transform. Of course a correct definition of the operator \( A \) requires some assumptions on \( h(t) \); this will be discussed later.

**1.3.** We apply this construction to two essentially different classes of Hankel operators \( H \) which lead to two essentially different classes of \( \Psi DO A \).

In Section 3, we consider generalized Carleman operators with kernels

\[ h(t) = P(\ln t)t^{-1} \]  

where

\[ P(\xi) = \sum_{m=0}^{n} p_m \xi^m, \quad p_n = 1, \]

is an arbitrary real polynomial. Obviously, kernels (1.7) have two singular points \( t = \infty \) and \( t = 0 \). For \( n \geq 1 \), such Hankel operators are unbounded but are well defined as self-adjoint operators.

For kernels (1.7), the sign-function \( s(\xi) \) is a real polynomial

\[ Q(\xi) = \sum_{m=0}^{n} q_m \xi^m \]

(1.9)
determined by \( P(\xi) \). In this case

\[ A = v(X)Q(D)v(X) \]  

is a differential operator. The polynomials \( P(\xi) \) and \( Q(\xi) \) have the same degree, and their coefficients are linked by an explicit formula (see formula (3.2) below); in particular, \( q_n = (-1)^n \). If \( n = 0 \), then \( Q(\xi) = P(\xi) = 1 \) so that \( A \) is the multiplication operator by \( v(x)^2 \). This yields the familiar diagonalization of the Carleman operator.

Observe that the highest order term of the operator \( A \) equals \( v^2(x)D^n \) where \( v^2(x) \) tends to zero (exponentially) as \( |x| \to \infty \). Apparently such
differential operators were never studied before, and we are led to fill in this gap. Studying differential operators (1.10) in Section 3, we do not make specific assumption (1.4) and consider sufficiently arbitrary real functions \( v(x) \) tending to zero as \( |x| \to \infty \). The essential spectrum of differential operators (1.10) was localized in [15] where it was shown that \( \text{spec}_{\text{ess}}(A) = \text{spec}(A) = \mathbb{R} \) if \( n \) is odd, and \( \text{spec}_{\text{ess}}(A) = [0, \infty) \) if \( n \) is even. The last result should be compared with the fact that \( \text{spec}_{\text{ess}}(A) = [\min Q(\xi), \infty) \) if \( v(x) = 1 \). Thus, even in this relatively simple question, the degeneracy of \( v(x) \) at infinity significantly changes spectral properties of differential operators \( A \). The detailed spectral structure, in particular, the absolutely continuous spectrum, of differential operators (1.10) and hence of the Hankel operators with kernels (1.7) was described in [20].

1.4. In Section 4, we are interested in compact self-adjoint Hankel operators (1.1) with power-like asymptotics of eigenvalues \( \lambda_{\pm n}^+(H) \) as \( n \to \infty \). Let us denote by \( \{\lambda_{\pm n}^+(H)\}_{n=1}^{\infty} \) the non-increasing sequence of positive eigenvalues of a compact self-adjoint operator \( H \) (with multiplicities taken into account), and set \( \lambda_{-n}^+(H) = \lambda_{n}^+(\sqrt{H^*H}) \). Sharp estimates of \( \lambda_{\pm n}^+(H) \) (and, more generally, of singular values \( s_n(H) = \lambda_{n}^+(\sqrt{H^*H}) \) in the non-self-adjoint case) are very well known. Thus V. V. Peller found (see Chapter 6 of his book [8]) necessary and sufficient conditions for the validity of the estimates \( s_n(H) = O(n^{-\alpha}) \).

At the same time, there are practically no results on the asymptotic behaviour of eigenvalues of Hankel operators. This state of affairs is in a sharp contrast with the case of differential operators, where the Weyl type asymptotics of eigenvalues is established in a very large variety of situations. Our goal here is to fill in this gap by describing classes of Hankel operators where the leading term of eigenvalue asymptotics can be found explicitly.

In general, the study of eigenvalue asymptotics for any class of operators involves two steps: construction of an appropriate model problem where the eigenvalue asymptotics can be determined sufficiently explicitly, and using eigenvalue estimates (or variational methods) to extend the asymptotics to a wider class of operators. Apparently, for a given Hankel operator \( H \), there is no natural model operator in the class of Hankel operators. So, the crucial step of our approach is a construction of the model operator \( \Psi \text{DO } A_{\ast} \). The spectral asymptotics of \( A_{\ast} \) and hence of the corresponding Hankel operators \( H_{\ast} \) is given by the Weyl law. This result is then extended to the initial operator \( H \).

1.5. Hankel operators can also be naturally realized in the space \( \ell^2(\mathbb{Z}_+) \) of sequences. Namely, for sequences \( g = \{g(j)\}_{n \in \mathbb{Z}_+} \), Hankel
operators \( G = G(g) \) are defined by infinite matrices:

\[
(Gu)(j) = \sum_{k=0}^{\infty} g(j+k)u(k).
\]

As is well known, the operators \( H \) and \( G \) give two different representations of the same object. Indeed, let us introduce a unitary transformation \( U : \ell^2(\mathbb{Z}_+) \to L^2(\mathbb{R}_+) \) by the formula

\[
(Uu)(t) = \sum_{j=0}^{\infty} L_j(t)u(j)e^{-t/2}, \quad u = \{u(j)\}_{j \in \mathbb{Z}_+} \in \ell^2(\mathbb{Z}_+)
\]

where \( L_j = L_j^0 \) are the Laguerre polynomials. The Hankel operators \( H \) and \( G \) are linked by this transformation, that is

\[
UGU^* = H,
\]

provided

\[
h(t) = \sum_{j=0}^{\infty} L_j^1(t)g(j)e^{-t/2}
\]

where \( L_j^1 \) are the generalized Laguerre polynomials.

Although the operators \( H \) and \( G \) are unitary equivalent, it is convenient to study eigenvalues asymptotics of \( G \) independently of the results on \( H \). This is done in Section 5.

In the discrete case the role of the Carleman operator is played by the Hankel operator \( G \) (the Hilbert matrix) with the matrix elements \( g(j) = (j+1)^{-1} \). This operator is bounded but not compact; actually, it has the simple absolutely continuous spectrum \([0, \pi]\). Note that the asymptotic behavior of a kernel \( h(t) \) like \( t^{-1}\ln t|^{-\alpha} \) as \( t \to \infty \) (resp. as \( t \to 0 \)) is essentially equivalent to the asymptotic behavior of the matrix elements of the corresponding Hankel operator \( G \) like \((-1)^j j^{-1}(\ln j)^{-\alpha}\) (resp. like \( j^{-1}(\ln j)^{-\alpha}\)) as \( j \to \infty \). It is often useful to keep in mind that Hankel operators \( G \) and \( \tilde{G} \) are unitarily equivalent provided their matrix elements are linked by the relation \( \tilde{g}(j) = (-1)^j g(j) \).

It is natural to expect that a faster rate of convergence to zero as \( j \to \infty \) of the sequence \( g(j) \) results in a faster convergence to zero as \( n \to \infty \) of the eigenvalues \( \lambda_n^+(G) \). Indeed, there is a deep result of H. Widom who showed in [14] that for \( \gamma > 1 \) the Hankel operator corresponding to the sequence \( g(j) = (j+1)^{-\gamma} \) is non-negative and its eigenvalues converge to zero as

\[
\lambda_n^+(G) = \exp(-\pi\sqrt{2\gamma n} + o(\sqrt{n})), \quad n \to \infty.
\]
By the way, for the proof of this result, H. Widom also used a reduction of Hankel operators he considered to \( \PsiDO s(D)^{1/2}v(X)^2s(D)^{1/2} \). Such a reduction is possible if \( s(\xi) \geq 0 \).

Note that both Hankel operators \( H \) and \( G \) can be realized in Hardy spaces of analytic functions where these operators are determined by their symbols. We do not discuss the representations of Hankel operators in Hardy spaces but emphasize that their symbols briefly mentioned in Section 6 and sigma-functions are completely different objects — see Section 3 in [19].

1.6. Finally, in Section 6 we discuss more general results on singular values and eigenvalues of Hankel operators with kernels \( h(t) \) oscillating as \( t \to \infty \) and with matrix elements \( g(j) \) oscillating as \( j \to \infty \).

The results on power-like asymptotics of singular values have direct applications to rational approximations of functions with logarithmic singularities. A result of such type is stated in Section 6.

2. Main identity

A detailed presentation of the results of this section can be found in the papers [16, 19].

2.1. For a given Hankel operator \( H \), let the sigma-function \( \sigma(\lambda) \) be formally defined by equation (1.5), and let \( \Sigma \) be the operator of multiplication by \( \sigma \), that is,

\[
(\Sigma f)(\lambda) = \sigma(\lambda)f(\lambda), \quad \lambda > 0.
\]

We will show that

\[
H = L^*\Sigma L \quad (2.2)
\]

where \( L \) is the Laplace transform:

\[
(Lu)(\lambda) = \int_0^\infty e^{-\lambda t}u(t)dt. \quad (2.3)
\]

A formal proof of the identity (2.2) is quite simple. Indeed, the integral kernel of the operator in the right-hand side of (2.2) equals

\[
\int_0^\infty e^{-\lambda t}\sigma(\lambda)e^{-\lambda s}d\lambda = h(t+s)
\]

if \( \sigma(\lambda) \) and \( h(t) \) are linked by formula (1.5). Thus it equals the integral kernel of the operator defined by (1.1).

The precise sense of formula (2.2) needs of course to be clarified. Observe that, by its definition (1.5), \( \sigma(\lambda) \) can be a regular function only for kernels \( h(t) \) satisfying some specific analytic assumptions. Without such very restrictive assumptions, \( \sigma \) is necessarily a distribution. Even
for very good kernels $h(t)$ (and especially for them), $\sigma(\lambda)$ may be a highly singular distribution. For example, for $h(t) = t^k e^{-\alpha t}$ where $\Re \alpha > 0$ ($\alpha$ may be complex) and $k = 0, 1, \ldots$, the sigma-function $\sigma(\lambda) = \delta^{(k)}(\lambda - \alpha)$ is a derivative of the delta-function. On the contrary, singular kernels $h(t)$ may yield sigma-functions $\sigma(\lambda)$ smooth on $\mathbb{R}_+$. For example, if $h(t) = t - q$ where $q > 0$ may be arbitrary large, then $\sigma(\lambda) = \Gamma(q) - 1/\lambda^1 - q$; here and below $\Gamma(\cdot)$ is the gamma function.

Thus, we replace (2.2) by the identity

$$\langle Hf_1, f_2 \rangle = \langle \Sigma Lf_1, Lf_2 \rangle$$

for arbitrary test functions $f_1, f_2 \in C_0^\infty(\mathbb{R}_+)$. With respect to $h$ we require only that $h \in C_0^\infty(\mathbb{R}_+)'$. Let us introduce the Laplace convolution

$$(\bar{f}_1 \ast f_2)(t) = \int_0^\infty \bar{f}_1(s)f_2(t - s)ds.$$ 

Then formally

$$(Hf_1, f_2) = \langle h, \bar{f}_1 \ast f_2 \rangle.$$ 

By one of the versions of the Paley-Wiener theorem, the Laplace transform $L$ is an isomorphism of $C_0^\infty(\mathbb{R}_+)$ onto the space $\mathcal{Y}$ of analytic functions $g(\lambda)$ of $\lambda \in \mathbb{C}$ exponentially decaying as $\Re \lambda \to +\infty$, exponentially bounded as $\Re \lambda \to -\infty$ and decaying faster than any power of $|\lambda|^{-1}$ as $|\Im \lambda| \to \infty$ (see [19] for details). By duality, $L^* : \mathcal{Y}' \to C_0^\infty(\mathbb{R}_+)'$ is also an isomorphism, and hence according to definition (1.5),

$$\sigma = (L^*)^{-1}h \in \mathcal{Y}'$$

if $h \in C_0^\infty(\mathbb{R}_+)'$. This yields a one-to-one correspondence between kernels $h \in C_0^\infty(\mathbb{R}_+)'$ of Hankel operators and their sigma-functions $\sigma \in \mathcal{Y}'$ and makes the theory self-consistent. Note that instead of operators, we consequently work with quadratic forms which is both more general and more convenient. For $g \in \mathcal{Y}$, we set $g^*(\lambda) = \overline{g(\lambda)}$.

Now we are in a position to state our main identity (2.2) precisely.

**Theorem 2.1.** Let $h \in C_0^\infty(\mathbb{R}_+)'$, and let $\sigma \in \mathcal{Y}'$ be defined by formula (2.4). Then the identity

$$\langle h, \bar{f}_1 \ast f_2 \rangle = \langle \sigma, (Lf_1)^*Lf_2 \rangle$$

holds for arbitrary $f_1, f_2 \in C_0^\infty(\mathbb{R}_+)$.

The identity (2.2) does not of course give a diagonalization of Hankel operators because the operator $L$ is not unitary. However it is continuously invertible as a mapping $L : C_0^\infty(\mathbb{R}_+) \to \mathcal{Y}$ so that equality (2.2) plays the same role as Sylvester’s inertia theorem which states that two Hermitian matrices $H$ and $\Sigma$ related by equation (2.2) have the
same total numbers of positive and negative eigenvalues. In partic-
ular, \( \pm H \geq 0 \) if and only if \( \pm \Sigma \geq 0 \). According to Theorem 2.1 the same assertion is true for Hankel operators \( H \) and operators of multi-
plication \( \Sigma \). Now the operators \( H \) and \( \Sigma \) are of a completely different
nature and \( \Sigma \) (but not \( H \)) admits an explicit spectral analysis. As an
example of this approach, we show in Section 4 of [18] that if \( \sigma(\lambda) > 0 \)
(or \( \sigma(\lambda) < 0 \)) on a set of positive Lebesgue measure, then the Hankel
operator \( H \) has infinite positive (or negative) spectrum. On the other
hand, singularities of \( \sigma(\lambda) \) at some isolated points produce finite num-
bers (depending on the order of the singularity) of positive or negative
eigenvalues (see Section 4 of [16]). In particular, this approach enables
us [17] to give an explicit formula for total numbers of positive and
negative eigenvalues of finite-rank Hankel operators.

2.2. To perform the spectral analysis of Hankel operators, we will
transform the identity (2.2) using the factorization of the operator \( L \).
Let us introduce the Mellin transform \( M : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}) \),
\[
(Mu)(x) = (2\pi)^{-1/2} \int_0^\infty u(t)t^{-1/2-it}dt,
\]
the reflection operator \( J \), \((Ju)(x) = u(-x)\), and set
\[
(\Gamma u)(x) = \Gamma(1/2 + ix)u(x).
\]
We use the following elementary fact.

**Lemma 2.2.** For the Laplace transform defined by (2.3) the identity
\[
(Lf)(\lambda) = (M^{-1}JMf)(\lambda), \quad \lambda > 0,
\]
holds for all \( f \in L^2(\mathbb{R}_+) \).

By the way, factorization (2.5) enables one to invert the Laplace
transform:
\[
L^{-1} = M^{-1} \Gamma^{-1} J M.
\]
We use (2.5) to establish the unitary equivalence of the operators \( H \)
and \( A \). Observe that
\[
|\Gamma(1/2 + ix)| = \frac{\sqrt{\pi}}{\sqrt{\cosh(\pi x)}} = v(x)
\]
and set
\[
(Mf)(x) = \frac{\Gamma(1/2 - ix)}{|\Gamma(1/2 - ix)|}(Mf)(-x).
\]
Then (2.5) can be rewritten as
\[
L = M^{-1} v(X) M.
\]
Note also that $M = \Phi^{-1}W$ where the unitary operator $W : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ is defined by $(Wf)(\xi) = e^{-\xi/2}f(e^{-\xi})$, and hence it follows from (1.6), (2.1) that

$$s(D) = M^{-1}\Sigma M.$$  

In view of (2.6) this yields the identity

$$H = M^{-1}v(X)s(D)v(X)M. \quad (2.7)$$

Let us state a simple sufficient condition for the validity of this representation which has been derived above rather formally.

**Theorem 2.3.** Suppose that

$$\sigma \in L^\infty(\mathbb{R}_+),$$

so that the operator $s(D)$ is bounded. Then the identity (2.7) holds.

2.3. The above results on integral operators (1.1) can be extended to Hankel operators $G$ defined by formula (1.11) in the space $\ell^2(\mathbb{Z}_+)$. In the discrete case, the role of the sigma-function $\sigma(\lambda)$ of $\lambda \in \mathbb{R}_+$ is played by the function $\eta(\mu)$ defined on the interval $(-1,1)$ and linked to $\sigma(\lambda)$ by the relation

$$\sigma(\lambda) = \eta \left( \frac{2\lambda - 1}{2\lambda + 1} \right). \quad (2.8)$$

Let

$$g(j) = \int_{-1}^{1} \eta(\mu)\mu^j d\mu, \quad j = 0, 1, 2, \ldots, \quad (2.9)$$

be the sequence of moments of $\eta$, and let $G$ be the Hankel operator with the matrix elements $g(j)$. We emphasize that equations (2.9) play the role of (1.5). It can be easily shown that relation (1.12) is satisfied if the kernel $h(t)$ of $H$ is given by (1.5). Therefore an analogue of Theorem 2.3 is stated as follows.

**Theorem 2.4.** [19, Theorem 7.7] Let $\eta \in L^\infty(-1,1)$. Then the Hankel operator $G$ in $\ell^2(\mathbb{Z}_+)$ with the matrix elements (2.9) is unitarily equivalent to the $\Psi DO$ (1.3) in $L^2(\mathbb{R})$ with the sign-function $s(\xi)$ defined by (1.6), (2.8).

3. **Generalized Carleman operators**

A detailed presentation of the results of this section can be found in the papers [15, 20].
3.1. Our goal here is to study spectral properties of generalized Carleman operators with kernels (1.7) where \( P(\xi) \) is an arbitrary real polynomial (1.8). In this case, we have

\[
h(t) = \int_0^\infty e^{-\lambda t} \sum_{m=0}^n q_m \ln^m \lambda d\lambda
\]

where the coefficients

\[
q_m = (-1)^m \sum_{j=m}^n \binom{j}{m} \gamma(j-m)(0)p_j, \quad m = 0, \ldots, n, \quad \gamma(z) = \Gamma(1-z)^{-1}.
\]

For example, \((-1)^n q_n = p_n\) and \((-1)^n q_{n-1} = -p_{n-1} + \Gamma'(1)n p_n\) for all \( n \) (recall that \(-\Gamma'(1)\) is the Euler constant). Of course formulas (3.2) enable one to recover the coefficients \( p_n, p_{n-1}, \ldots, p_0 \) given the coefficients \( q_n, q_{n-1}, \ldots, q_0 \). It follows from (3.1) that \( s(D) =: Q(D) \) is a differential operator given by formula (1.9).

For Hankel operators (1.1) with kernels (1.8), the identity (2.7) yields the following result.

**Theorem 3.1.** [15, Theorem 3.2] Let \( Q(\xi) \) be polynomial (1.9) with the coefficients \( q_m \) defined by formulas (3.2), and let \( A \) be the differential operator (1.10). Then for all functions \( u_j, j = 1, 2 \), such that their Mellin transforms \( \mathcal{M}u_j \) belong to \( C_0^\infty(\mathbb{R}) \), the identity

\[
(Hu_1, u_2) = (\mathcal{A}\mathcal{M}u_1, \mathcal{M}u_2)
\]

holds.

A large part of our results on generalized Carleman operators can be summarized by the following assertion. Below we denote by \( \langle x \rangle \) the operator of multiplication by the function \((1 + x^2)^{1/2}\).

**Theorem 3.2.** Let \( H \) be the self-adjoint Hankel operator defined by formula (1.1) where \( h(t) \) is function (1.7) and \( P(\xi) \) is a real polynomial (1.8) of degree \( n \geq 1 \). Then

(i) The spectrum of the operator \( H \) is absolutely continuous except eigenvalues that may accumulate to zero and infinity only.

(ii) The absolutely continuous spectrum of the operator \( H \) covers \( \mathbb{R} \) and is simple for \( n \) odd. It coincides with \([0, \infty)\) and has multiplicity 2 for \( n \) even.

(iii) If \( n \) is odd, then the multiplicities of eigenvalues of the operator \( H \) are bounded (from above) by \((n-1)/2\). If \( n \) is even, then the multiplicities of positive eigenvalues are bounded by \(n/2 - 1\), and the multiplicities of negative eigenvalues are bounded by \(n/2\).
(iv) For any $\delta > 1/2$, the operator-valued function
\[ \langle \ln t \rangle^{-\delta} (H - z)^{-1} \langle \ln t \rangle^{-\delta}, \quad \text{Im} \ z \neq 0, \] 
(3.3)
is Hölder continuous with exponent $\alpha < \delta - 1/2$ (and $\alpha < 1$) up to the real axis, except the eigenvalues of the operator $H$ and the point zero.

Clearly, this assertion is similar in spirit to the corresponding results for differential operators of Schrödinger type. Statement (iv) is known as the limiting absorption principle.

3.2. In view of Theorem [3.1] the proof of Theorem [3.2] reduces to a proof of the corresponding results for the differential operator $A$ defined by (1.10). However the standard results on differential operators are not applicable in this case because of a strong degeneracy of $v(x)$ at infinity. Fortunately, operators (1.10) can be reduced by an explicit unitary transformation $L$ (the generalized Liouville transformation) to standard differential operators. Set
\[ (Lu)(x) = y'(x)^{1/2}u(y(x)) \]
where the variables $x$ and $y$ are linked by the relation
\[ y = y(x) = \int_0^x v(s)^{-2/n} ds \]
so that $y'(x) = v(x)^{-2/n}$. Then $B = L^{-1}AL$ is also a differential operator in the space $L^2(\mathbb{R})$, and it is given by the formula
\[ B = D^n + \sum_{m=0}^{n-1} b_m(y)D^m, \quad D = D_y = -id/dy. \] 
(3.4)

Our crucial observation is that the coefficients $b_m(y), m = 0, 1, \ldots, n - 1$, of the operator $B$ decay at infinity. Moreover, all coefficients $b_0(y), \ldots, b_{n-2}(y)$ of the operator $B$ are short-range, that is, they decay faster than $|y|^{-1}$ as $|y| \to \infty$. The coefficient $b_{n-1}(y)$ can be removed by a gauge transformation $\mathcal{J}$ defined by
\[ (\mathcal{J}u)(y) = e^{i\phi(y)}u(y) \quad \text{where} \quad \phi(y) = -\frac{1}{n} \int_0^y b_{n-1}(s) ds. \]
This means that the operator $\tilde{B} = \mathcal{J}^* B \mathcal{J}$ has again the form (3.4) with $\tilde{b}_{n-1}(y) = 0$. The coefficients $\tilde{b}_0(y), \ldots, \tilde{b}_{n-2}(y)$ remain short-range.

Using fairly standard methods of scattering theory we obtain assertions (i), (ii) and (iii) of Theorem [3.2] for the operator $B$. Since
\[ A = LBL^{-1} \quad \text{and} \quad H = M^{-1}AM, \] 
(3.5)
these results remain true for the operators $A$ and $H$. Recall that for differential operators $B$, one defines their (generalized) eigenfunctions as special solutions $\psi(y, k)$, $k \in \mathbb{R}$, of the equation $B\psi = k^n\psi$ satisfying some asymptotic conditions as $y \to \infty$ and $y \to -\infty$ and then establishes the expansion theorem over these eigenfunctions. Relation (3.5) allows us to carry over these results to the operators $A$ and $H$. According to (3.5) one can define the eigenfunctions of the operator $H$ by the equality

$$\theta(t, k) = (M^{-1}L\psi(k))(t) = (2\pi)^{-1/2}t^{-1/2} \int_{-\infty}^{\infty} e^{-ix(y)\ln t}e^{i\eta(x(y))} x'(y) \psi(y, k) dy$$

where $\eta(x) = \arg \Gamma(1/2 + ix)$. This integral converges although not absolutely. Applying the stationary phase method one can deduce asymptotics of the eigenfunctions $\theta(t, k)$ as $t \to 0$ and as $t \to \infty$ from this representation. In particular, we obtain the uniform in $k$ (on compact subsets of $\mathbb{R} \setminus \{0\}$, away from the eigenvalues of $H$) estimate

$$|\theta(t, k)| \leq Ct^{-1/2}$$

and a similar estimate on differences $\theta(t, k') - \theta(t, k)$. Using these estimates one can prove assertion (iv) of Theorem 3.2. This result looks similar to the limiting absorption principle for differential operators. The difference, however, is that the weight is $\langle \ln t \rangle^{-\delta}$ in (3.3) while it is $\langle x \rangle^{-\delta}$ (also with $\delta > 1/2$) for the resolvents of differential operators. Thus the power scale for differential operators corresponds to the logarithmic scale for Hankel operators.

3.3. Actually, the specific expression (1.4) for the function $v(x)$ in definition (1.10) of the operator $A$ is inessential. It is noteworthy that if $v(x)$ tends to zero exponentially as $|x| \to \infty$, then the coefficients $b_0(y), \ldots, b_{n-2}(y)$ of the operator $B$ decay faster than $|y|^{-1}$ as $|y| \to \infty$. On the contrary, for slower decay of $v(x)$, these coefficients decay slower than (or as) $|y|^{-1}$. Thus, somewhat counter-intuitively, a stronger degeneracy of the operator (1.10) yields better properties of the operator $B = L^{-1}AL$.

We also note that our approach applies to sufficiently arbitrary differential operators of order $n$ with a degeneracy of the coefficient in front of $D^n$. 
4. Compact operators. Asymptotics of singular values and eigenvalues

4.1. In this subsection we collect necessary auxiliary results. First we recall Weyl asymptotics of ΨDO. For \( x \in \mathbb{R} \), we use the standard notation \( x_\pm = \max\{0, \pm x\} \).

**Theorem 4.1.** Let \( s \in C^\infty(\mathbb{R}) \) be a real-valued function such that

\[
s(\xi) = \begin{cases} 
s_\infty \xi^{-\alpha} (1 + o(1)), & \xi \to \infty, \\
s_{-\infty} |\xi|^{-\alpha} (1 + o(1)), & \xi \to -\infty,
\end{cases}
\]

for some \( \alpha > 0 \) and some constants \( s_\infty \) and \( s_{-\infty} \). Assume that \( v(x) = v(x) \) and

\[
|v(x)| \leq C \langle x \rangle^{-\rho}, \quad x \in \mathbb{R},
\]

for some \( \rho > \alpha / 2 \). Put

\[
a_\pm = (2\pi)^{-\alpha} \left( (s_{-\infty})^{1/\alpha} + (s_\infty)^{1/\alpha} \right)^{\alpha} \left( \int_{-\infty}^\infty |v(x)|^{2/\alpha} dx \right)^{\alpha}.
\]

Then for the \( \Psi \)DO (1.3) in \( L^2(\mathbb{R}) \) one has

\[
\lambda_n^\pm(A) = a_\pm n^{-\alpha} + o(n^{-\alpha}), \quad n \to \infty.
\]

For compactly supported \( v \), Theorem 4.1 was proven by M. Sh. Birman and M. Z. Solomyak in [1]; actually, the multi-dimensional case was considered in [1]. Their result can be easily extended to arbitrary functions \( v \) satisfying (4.2) (see, e.g., Appendix in [10]). For general \( \Psi \)DO (acting in a bounded domain) with amplitudes asymptotically homogeneous at infinity, Weyl type formula for the asymptotics of the spectrum was obtained in [3].

We need also some estimates on singular values \( s_n(H) \) of Hankel operators (1.1). They are stated in the next assertion established in [9]. For \( \alpha \geq 1/2 \), the proof of these estimates relies heavily on deep results by V. V. Peller (see Chapter 6 of his book [8]). For an arbitrary \( \alpha > 0 \), we set

\[
N(\alpha) = [\alpha] + 1 \text{ if } \alpha \geq 1/2 \quad \text{and} \quad N(\alpha) = 0 \text{ if } \alpha < 1/2.
\]

**Theorem 4.2.** Let \( \alpha > 0 \) and let \( h \in L^\infty_{\text{loc}}(\mathbb{R}_+) \) be a complex valued function; if \( \alpha \geq 1/2 \), suppose also that \( h \in C^N(\alpha)(\mathbb{R}_+) \). Assume that \( h \) satisfies the conditions

\[
h^{(m)}(t) = o(t^{-1-m}(\log t)^{-\alpha}) \quad \text{as } t \to 0 \text{ and as } t \to \infty
\]

for all \( m = 0, 1, \ldots, N(\alpha) \). Then \( s_n(H) = o(n^{-\alpha}) \) as \( n \to \infty \).
Theorem 4.2 remains true if $o(\cdot)$ (on both occasions above) is replaced by $O(\cdot)$.

We need also the following standard result (see, e.g., [4, Section 11.6]) in spectral perturbation theory, which asserts the stability of eigenvalue asymptotics.

**Lemma 4.3.** Let $K_0$ and $K$ be compact self-adjoint operators and let $\alpha > 0$. Suppose that, for both signs “±”,
\[
\lambda_n^\pm(K_0) = a^\pm n^{-\alpha} + o(n^{-\alpha}), \quad \text{and} \quad s_n(K) = o(n^{-\alpha}), \quad n \to \infty.
\]
Then
\[
\lambda_n^\pm(K_0 + K) = a^\pm n^{-\alpha} + o(n^{-\alpha}), \quad n \to \infty.
\]

**4.2.** Our main result on asymptotics of eigenvalues of Hankel operators (1.1) can be stated as follows. Let us set
\[
\tau(\alpha) = \frac{2^{-\alpha} \pi^{1/2}}{\Gamma(1/2 + \alpha)}
\]
where $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}$ is the Beta function.

**Theorem 4.4.** Let $\alpha > 0$ and let the integer $N(\alpha)$ be given by (4.3). Let $h$ be a real valued function in $L^\infty_{\text{loc}}(\mathbb{R}_+)$; if $\alpha \geq 1/2$, assume also that $h \in C^{N(\alpha)}(\mathbb{R}_+)$. Suppose that
\[
\begin{align*}
\left(\frac{d}{dt}\right)^m (h(t) - \kappa_0 t^{-1}(\log(1/t))^{-\alpha}) &= o(t^{-1-m}(\log t)^{-\alpha}), \quad t \to 0, \\
\left(\frac{d}{dt}\right)^m (h(t) - \kappa_\infty t^{-1}(\log t)^{-\alpha}) &= o(t^{-1-m}(\log t)^{-\alpha}), \quad t \to \infty,
\end{align*}
\]
for some $\kappa_0, \kappa_\infty \in \mathbb{R}$ and all $m = 0, \ldots, N(\alpha)$. Then the eigenvalues of the corresponding Hankel operator $H$ have the asymptotic behaviour
\[
\lambda_n^\pm(H) = a^\pm n^{-\alpha} + o(n^{-\alpha}), \quad n \to \infty,
\]
where
\[
a^\pm = \tau(\alpha) \left((\kappa_0)^{1/\pm}\Gamma(1/\pm) + \kappa_\infty^{1/\pm}\right). \tag{4.6}
\]

Our proof of this result relies on the following three ingredients:

(i) Theorem 2.3 allows us to replace Hankel operators (1.1) by the $\Psi DO A$ defined by (1.3).

(ii) Weyl type spectral asymptotics for $\Psi DO$ of this type stated in Theorem 4.1.

(iii) Estimates on singular values of Hankel operators of Theorem 4.2.
4.3. Let us sketch the proof of Theorem 4.4. The first and the most important step is to construct a model operator. To that end, we introduce an auxiliary explicit function by the formula

$$\sigma^*_\lambda = \kappa_\infty \lambda^{-\alpha} \chi_0(\lambda) + \kappa_0 \lambda^{-\alpha} \chi_\infty(\lambda), \quad \lambda > 0,$$

where the cut-off functions $\chi_0, \chi_\infty \in C^\infty(\mathbb{R}_+)$ satisfy

$$\chi_0(t) = \begin{cases} 
1 & \text{for } t \leq 1/4, \\
0 & \text{for } t \geq 1/2,
\end{cases} \quad \chi_\infty(t) = \begin{cases} 
0 & \text{for } t \leq 2,
1 & \text{for } t \geq 4.
\end{cases}$$

It turns out that the functions $h(t)$ and $h^*_\lambda = L\sigma^*(t)$ have the same asymptotics as $t \to \infty$; a similar relation holds as $t \to 0$. For the proof, we need an elementary technical result about the Laplace transform of functions with logarithmic singularities at $\lambda = 0$ and $\lambda = \infty$.

**Lemma 4.5.** Let $\alpha > 0$, $m \in \mathbb{Z}_+$,

$$I^\infty_m(t) = \int_c^\infty (\log \lambda)^{-\alpha} \lambda^m e^{-\lambda t} d\lambda, \quad c \in (0, 1),$$

and

$$I^0_m(t) = \int_c^\infty (\log \lambda)^{-\alpha} \lambda^m e^{-\lambda t} d\lambda, \quad c > 1.$$

Then

$$I^\infty_m(t) = m! t^{-1-m} \log t^{-\alpha} (1 + O(\log t^{-1}))$$

as $t \to \infty$, and $I^0_m(t)$ has the same asymptotic behaviour (4.9) as $t \to 0$.

This result is well known; see, e.g., Lemmas 3 and 4 in [5]. Its simple straightforward proof can be found in [10].

**Corollary 4.6.** Let the function $\sigma_\lambda$ be given by (4.7), and let $h_\lambda = L\sigma_\lambda$ be its Laplace transform. Then

$$h_\lambda = \kappa_0 h_0 + b_\infty \kappa_\infty + \tilde{h}_\lambda,$$

where the model kernels $h_0, h_\infty$ are defined by

$$h_0(t) = t^{-1} \log t^{-\alpha} \chi_0(t), \quad h_\infty(t) = t^{-1} \log t^{-\alpha} \chi_\infty(t)$$

and the error term $\tilde{h} \in C^\infty(\mathbb{R}_+)$ satisfies the estimates

$$|\tilde{h}_m(t)| \leq C_m t^{-1-m} \log t^{-\alpha-1}, \quad t > 0,$$

for all integers $m \geq 0$.

Our model operator is the Hankel operator $H_\lambda = H(h_\lambda)$ with kernel $h_\lambda = L\sigma_\lambda$. 

Lemma 4.7. The eigenvalues of the operator $H_*$ obey the asymptotic relation
\begin{equation}
\lambda_{n}^\pm(H_*) = a^\pm n^{-\alpha} + o(n^{-\alpha}), \quad n \to \infty,
\end{equation}
where the coefficients $a^\pm$ are given by (4.6).

Indeed, according to Theorem 2.3 the Hankel operator $H_*$ is unitarily equivalent to the \PsiDO $A_* = v(X)s_*(D)v(X)$ in $L^2(\mathbb{R})$. As usual, $v(x)$ is the standard function (1.4), and it follows from (1.6) and (4.7) that
\begin{equation}
s_*(\xi) = \sigma_*(e^{-\xi}) = \kappa_\infty|\xi|^{-\alpha}\chi_0(e^{-\xi}) + \kappa_0|\xi|^{-\alpha}\chi_\infty(e^{-\xi}), \quad \xi \in \mathbb{R}.
\end{equation}

This function belongs to $C^\infty(\mathbb{R})$ and has the asymptotic behaviour (4.1) with $s_\infty = \kappa_\infty$ and $s_{-\infty} = \kappa_0$. Therefore Theorem 4.1 (Weyl spectral asymptotics of \PsiDO) applies to the operator $A_*$. This yields the asymptotic formula
\begin{equation}
\lambda_{n}^\pm(A_*) = a^\pm n^{-\alpha} + o(n^{-\alpha}), \quad n \to \infty,
\end{equation}
where
\begin{equation}
a^\pm = (2\pi)^{-\alpha}\left((\kappa_0)^{1/\alpha} + (\kappa_\infty)^{1/\alpha}\right)^\alpha\left(\int_{-\infty}^{\infty}(\pi(\cosh(\pi x))^{-1})^{1/\alpha}dx\right)^\alpha.
\end{equation}

Using the change of variables $y = (\cosh(\pi x))^2$, it is easy to check that the coefficients $a^\pm$ here and in (4.6) coincide. In view of Theorem 2.4 the operators $H_*$ and $A_*$ are unitarily equivalent so that relation (4.12) yields (4.11). \qed

Now we are in a position to conclude the proof of Theorem 4.4. By its hypotheses, we have the representation
\begin{equation}
h = \kappa_0h_0 + \kappa_\infty h_\infty + \tilde{h},
\end{equation}
where $h_0$ and $h_\infty$ are given by (4.10) and $\tilde{h}$ satisfies the assumptions of Theorem 4.2 (singular value estimates). Therefore it follows from Corollary 4.6 that the difference
\begin{equation}
h - h_* = \tilde{h} - \tilde{h}_*
\end{equation}
also satisfies the hypothesis of Theorem 4.2 and hence
\begin{equation}
s_n(H - H_*) = o(n^{-\alpha}), \quad n \to \infty.
\end{equation}

In view of the abstract Lemma 4.3 the asymptotic formula (4.5) is a direct consequence of (4.11) and (4.13). \qed

4.4. We emphasize that the asymptotics of the spectrum of integral Hankel operators (1.11) is determined by the behavior of $h(t)$ as $t \to 0$ and $t \to \infty$ as well as by local singularities of $h(t)$. Following [16], let us
consider Hankel operators whose integral kernels (or their derivatives) have jumps of continuity at some positive point.

**Theorem 4.8.** Let \( l \in \mathbb{Z}_+, t_0 > 0 \), and let \( h(t) = h_0(t_0 - t)^l \) for \( t \leq t_0 \) and \( h(t) = 0 \) for \( t > t_0 \). Then eigenvalues of the Hankel operator \( H \) have the asymptotics

\[
\lambda_n^\pm(H) = |h_0| l!(2\pi)^{-l-1} t_0^{l+1} n^{-l-1}(1 + O(n^{-1})) \tag{4.14}
\]

as \( n \to \infty \).

Of course, the exact expression for \( h(t) \) is inessential. Indeed, if a real function \( v(t) \) satisfies the assumptions of Theorem 4.2 with \( \alpha = l \), then singular numbers \( s_n(V) \) of the Hankel operator \( V \) with kernel \( v(t) \) satisfy the bound \( s_n(V) = o(n^{-l-1}) \). Therefore, in view of Lemma 4.3 asymptotics \((4.14)\) remains true for the eigenvalues of the Hankel operator \( H + V \); however, in this case the remainder \( O(n^{-1}) \) in \((4.14)\) should be replaced by \( o(1) \).

We emphasize that, according to \((4.14)\), the leading terms of the asymptotics of positive and negative eigenvalues of the Hankel operator \( H \) are the same. Of course, if \( h(t) \) becomes smoother \( (l \) increases), then eigenvalues of \( H \) decrease faster as \( n \to \infty \). Observe that for \( l = 0 \) (when the kernel itself is discontinuous), the Hankel operator \( H \) does not belong to the trace class. We finally note that under the assumptions of Theorem 4.8 the asymptotics of singular values of the operator \( H \) was found long ago in [6].

## 5. Discrete Case

5.1. In the discrete case, the role of derivatives \( h^{(m)}(t) \) of a function \( h(t) \) is played by iterated differences \( g^{(m)}(j) \) of a sequence \( g(j) \). Those are defined iteratively by setting \( g^{(0)}(j) = g(j) \) and

\[
g^{(m)}(j) = g^{(m-1)}(j+1) - g^{(m-1)}(j), \quad j \geq 0.
\]

The following result plays the role of Theorem 4.2.

**Theorem 5.1.** [9] Let \( \alpha > 0 \) and let \( g \) be a sequence of complex numbers that satisfies

\[
g^{(m)}(j) = o(j^{-1-m}(\log j)^{-\alpha}), \quad j \to \infty,
\]

for all \( m = 0, 1, \ldots, N(\alpha) \) with \( N(\alpha) \) defined by \((4.3)\). Then \( s_n(G) = o(n^{-\alpha}) \) as \( n \to \infty \).

Theorem 5.1 remains true if \( o(\cdot) \) (on both occasions above) is replaced by \( O(\cdot) \).

Below is our main result in the discrete case.
Theorem 5.2. Let \( \alpha > 0, \kappa_1, \kappa_{-1} \in \mathbb{R} \), and let \( g(j) \) be a sequence of real numbers given (for \( j \geq 2 \)) by
\[
g(j) = (\kappa_1 + (-1)^j \kappa_{-1}) j^{-1} (\log j)^{-\alpha} + \tilde{g}_1(j) + (-1)^j \tilde{g}_{-1}(j) \tag{5.1}
\]
where the error terms \( \tilde{g}_{\pm 1} \) satisfy the conditions of Theorem 5.1. Then the eigenvalues of the corresponding Hankel operator \( G \) have the asymptotic behaviour
\[
\lambda_n^{\pm}(G) = b^{\pm} n^{-\alpha} + o(n^{-\alpha}), \quad n \to \infty,
\]
where
\[
b^{\pm} = \tau(\alpha) \left( (\kappa_1)^{1/\alpha} + (\kappa_{-1})^{1/\alpha} \right)^{\alpha} \tag{5.2}
\]
and \( \tau(\alpha) \) is given by (4.4).

5.2. Let us describe the plan of the proof of Theorem 5.2. We follow the same steps as in Section 4, but instead of the Laplace transform \( h_* = L \sigma_* \) of the function \( \sigma(\lambda), \lambda > 0 \), we consider the sequence of moments
\[
g_*(j) = \int_{-1}^1 \eta_*(\mu) \mu^j d\mu, \quad j \geq 0, \tag{5.3}
\]
of the function:
\[
\eta_*(\mu) = \left| \log \frac{1+\mu}{2(1-\mu)} \right|^{-\alpha} \left( \kappa_1 \chi_\infty \left( \frac{1+\mu}{2(1-\mu)} \right) + \kappa_{-1} \chi_0 \left( 2 \frac{1+\mu}{1-\mu} \right) \right) \tag{5.4}
\]
where the smooth cut-off functions \( \chi_\infty \) and \( \chi_0 \) are given by equalities (4.8). Note that the function \( \eta_* \) belongs to the class \( C^\infty(-1, 1) \) and has the following asymptotic behaviour:
\[
\eta_*(\mu) = \kappa_1 |\log(1-\mu)|^{-\alpha} + o(|\log(1-\mu)|^{-\alpha}), \quad \mu \to 1,
\]
\[
\eta_*(\mu) = \kappa_{-1} |\log(1+\mu)|^{-\alpha} + o(|\log(1+\mu)|^{-\alpha}), \quad \mu \to -1.
\]

These relations allow us to obtain the asymptotics of the sequence \( g(j) \) as \( j \to \infty \). We use again Lemma 4.5, but one needs to replace the continuous parameter \( t \) with the discrete one \( j \). The following assertion plays the role of Corollary 4.6.

Lemma 5.3. The sequence \( g_*(j) \) defined by (5.3), (5.4) has the asymptotics
\[
g_*(j) = (\kappa_1 + (-1)^j \kappa_{-1}) j^{-1} (\log j)^{-\alpha} + \tilde{g}_1(j) + (-1)^j \tilde{g}_{-1}(j) \tag{5.5}
\]
where the error terms \( \tilde{g}_{\pm 1}(j) \) satisfy the estimates
\[
\tilde{g}^{(m)}_{\pm 1}(j) = O(j^{-1-m}(\log j)^{-\alpha-1}), \quad j \to \infty,
\]
for all \( m = 0, 1, 2, \ldots \).
Let \( v(x) \) be the function (1.4) and

\[
s_*(\xi) = \eta_*(\frac{2e^{-\xi} - 1}{2e^{-\xi} + 1}), \quad \xi \in \mathbb{R}.
\]

Theorem 2.4 implies that the Hankel operator \( G_* \) with matrix elements (5.3) is unitarily equivalent to the \( \Psi DO \) \( A_* = v(X)s_*(D)v(X) \) acting in \( L^2(\mathbb{R}) \).

By the definition (5.4) of \( \eta_*(\mu) \), we have

\[
s_*(\xi) = |\xi|^{-\alpha}(\kappa_1\chi_\infty(e^{-\xi}) + \kappa_1\chi_\infty(4e^{-\xi})), \quad \xi \in \mathbb{R}.
\]

Applying Theorem 4.1 to the \( \Psi DO \) \( A_* \), we see that

\[
\lambda_\pm^n(G_*) = \lambda_\pm^n(A_*) = b_\pm n^{-\alpha} + o(n^{-\alpha}), \quad n \to \infty,
\]  

where the numbers \( b_\pm \) are given by (5.2).

5.3. Now we can conclude the proof of Theorem 5.2. Comparing (5.1) and (5.5), we see that

\[
g(j) - g_*(j) = f_1(j) + (-1)^j f_{-1}(j)
\]

where the error terms \( f_{\pm 1}(j) = g_{\pm 1}(j) - \tilde{g}_{\pm 1}(j) \) satisfy the condition

\[
f_{\pm 1}^{(m)}(j) = o(j^{-1-m}(\log j)^{-\alpha}), \quad j \to \infty,
\]

for all \( m = 0, 1, \ldots, N(\alpha) \). According to Theorem 5.1 (singular value estimates) we have \( s_n(G(f_{\pm 1})) = o(n^{-\alpha}) \) as \( n \to \infty \).

Put \( \tilde{f}_{-1}(j) = (-1)^j f_{-1}(j) \). Then \( s_n(G(\tilde{f}_{-1})) = s_n(G(f_{-1})) \) and hence

\[
s_n(G(f_1 + \tilde{f}_{-1})) = o(n^{-\alpha}), \quad n \to \infty.
\]  

(5.7)

In view of (5.6) and (5.7), we can apply the abstract Lemma 4.3 to the operators \( K_0 = G(g_*) \) and \( K = G(f_1 + \tilde{f}_{-1}) \). This yields the eigenvalue asymptotics (5.2) for the operator \( G = G(g) = K_0 + K \). \( \square \)

6. Generalizations and applications

6.1. In this subsection we state results on asymptotic behavior of singular values of Hankel operators. Now the operators are not assumed to be self-adjoint. Recall that the numerical coefficient \( \tau(\alpha) \) is given by formula (4.4).

It is convenient to start with the discrete case.

**Theorem 6.1.** [11, Theorem 3.1] Let \( \alpha > 0 \), let \( \zeta_1, \ldots, \zeta_L \in \mathbb{T} \) be pairwise distinct numbers, and let \( \kappa_1, \ldots, \kappa_L \in \mathbb{C} \). Let \( g(j) \) be a sequence of complex numbers such that

\[
g(j) = \sum_{\ell=1}^L (\kappa_\ell j^{-1}(\log j)^{-\alpha} + \tilde{g}_\ell(j))\zeta_\ell^{-j}, \quad j \geq 2,
\]  

(6.1)
where the error terms $\tilde{g}_\ell$, $\ell = 1, \ldots, L$, satisfy the estimates
\[
\tilde{g}_\ell^{(m)}(j) = o(j^{-1-m}(\log j)^{-\alpha}), \quad j \to \infty,
\]
for all $m = 0, 1, \ldots, N(\alpha)$ with $N(\alpha)$ given by (4.3). Then the singular values of the Hankel operator $G$ defined in $\ell^2(\mathbb{Z}_+)$ by formula (1.11) satisfy the asymptotic relation
\[
s_n(G) = bn^{-\alpha} + o(n^{-\alpha}), \quad n \to \infty,
\]
where
\[
b = \tau(\alpha) \left( \sum_{\ell=1}^{L} |\kappa_\ell|^{1/\alpha} \right)^{\alpha}.
\]

The plan of the proof of Theorem 6.1 is the following. For $L = 1$, Theorem 6.1 is a consequence of Theorem 5.2 for the particular case $\kappa_{-1} = 0$. To pass to the general case, one needs the notion of the symbol $\omega(\mu)$ of a Hankel operator $G$. The function $\omega(\mu)$ can be defined by the relation
\[
g(j) = \int_{\mathbb{T}} \omega(\mu) \mu^{-n} d\text{m}(\mu)
\]
where $d\text{m}(\mu)$ is the normalized Lebesgue measure on the unit circle $\mathbb{T}$. Of course, the function $\omega(\mu)$ satisfying (6.5) is not unique.

Consider the leading part
\[
g_{\text{lead}}(j) = \sum_{\ell=1}^{L} \kappa_\ell j^{-1}(\log j)^{-\alpha} \zeta_\ell^{-j}, \quad j \geq 2,
\]
of sequence (6.1). It can be easily checked that for every $\alpha > 0$ the function
\[
\omega_0(\mu) = \sum_{j=2}^{\infty} j^{-1}(\log j)^{-\alpha}(\mu^j - \bar{\mu}^j), \quad \mu \in \mathbb{T},
\]
is bounded and $\omega_0 \in C^\infty(\mathbb{T} \setminus \{1\})$. This means that its singular support is $\text{sing supp} \omega_0 = \{1\}$. Therefore the singular support of the symbol $\omega_{\text{lead}}(\mu)$ corresponding to $g_{\text{lead}}(j)$ consists of the points $\zeta_1, \ldots, \zeta_L$. It can be deduced from this property that the singular value counting function $\# \{ n : s_n(G_{\text{lead}}) > \varepsilon \}$ of the Hankel operator $G_{\text{lead}}$ with the matrix elements $g_{\text{lead}}(j)$ is asymptotically (as $\varepsilon \to +0$) the sum of such functions for separate terms in (6.6). This fact is called the localization principle in [11]. In terms of singular values the result on counting functions is equivalent to relations (6.3), (6.4) for the operator $G_{\text{lead}}$. The singular values of the operator $G - G_{\text{lead}}$ can be easily estimated with the help of Theorem 6.1

In the continuous case, we have the following result.
Theorem 6.2. [11] Theorem 5.1] Let $\alpha > 0$, let $\rho_1, \ldots, \rho_L \in \mathbb{R}$ be pairwise distinct numbers and let $\kappa_0, \kappa_1, \ldots, \kappa_L \in \mathbb{C}$. Let the number $N(\alpha)$ be given by (4.3). Suppose that $h \in L^\infty_{\text{loc}}(\mathbb{R}_+)$ if $\alpha < 1/2$ and $h \in C^{N(\alpha)}(\mathbb{R}_+)$ if $\alpha \geq 1/2$. Assume that

$$h(t) = \sum_{\ell=1}^L (\kappa_\ell t^{-1}(\log t)^{-\alpha} + \bar{h}_\ell(t)) e^{-i\rho_\ell t}, \quad t \geq 2,$$

$$h(t) = \kappa_0 t^{-1}(\log(1/t))^{-\alpha} + \bar{h}_0(t), \quad t \leq 1/2,$$

where the error terms $\bar{h}_\ell$ and their derivatives $\bar{h}^{(m)}_\ell$ satisfy the estimates

$$\bar{h}^{(m)}_\ell(t) = o(t^{-1-m}(\log t)^{-\alpha}), \quad m = 0, \ldots, N(\alpha),$$

as $t \to \infty$ for $\ell = 1, \ldots, L$ and as $t \to 0$ for $\ell = 0$. Then the singular values of the integral Hankel operator $H$ with kernel $h(t)$ in $L^2(\mathbb{R}_+)$ satisfy the asymptotic relation

$$s_n(H) = a n^{-\alpha} + o(n^{-\alpha}), \quad n \to \infty,$$

where

$$a = \tau(\alpha) \left( \sum_{\ell=0}^L |\kappa_\ell|^{1/\alpha} \right)^\alpha.$$

The proof of Theorem 6.2 follows the same general outline as that of Theorem 6.1. In the continuous case the symbol may be defined by the relation

$$h(t) = \int_{-\infty}^{\infty} \Omega(x)e^{-ixt}dx.$$

Compared with the proof of Theorem 6.2 the only essential difference is that the singularity of the kernel $h(t)$ at $t = 0$ has to be treated separately. It corresponds to the singularity of the symbol $\Omega(x)$ at infinity.

6.2. Here we find asymptotics of eigenvalues of self-adjoint Hankel operators. The first result generalizes Theorem 5.2. Now we consider the real sequences of the form (6.1).

Theorem 6.3. [12] Theorem 5.7] Let $\alpha > 0$, $p = 1/\alpha$; let $\zeta_1, \ldots, \zeta_L \in \mathbb{T}$ be pairwise distinct points with $\text{Im} \zeta_\ell > 0$, and let $\kappa_1, \kappa_{-1} \in \mathbb{R}$, $\kappa_1, \ldots, \kappa_L \in \mathbb{C}$. Let $g(j)$ be a sequence of real numbers such that

$$g(j) = \kappa_1 j^{-1}(\log j)^{-\alpha} + \bar{g}_1(j) + (-1)^j \left( \kappa_{-1} j^{-1}(\log j)^{-\alpha} + \bar{g}_{-1}(j) \right)$$

$$+ 2 \operatorname{Re} \sum_{\ell=1}^L \zeta_\ell^{-j} (\kappa_\ell j^{-1}(\log j)^{-\alpha} + \bar{g}_\ell(j)), \quad j \geq 2,$$

(6.8)
where all error terms $\tilde{g}_1, \tilde{g}_{-1}, \tilde{g}_1, \ldots, \tilde{g}_l$ obey condition (6.2) for $m = 0, 1, \ldots, N(\alpha)$ ($N(\alpha)$ is given by (4.3)). Then the eigenvalues of the Hankel operator $G$ with matrix elements (6.8) satisfy the asymptotic relation (5.2) with the coefficient $b^\pm$ defined by

$$b^\pm = \tau(\alpha)((\kappa_{-1})^{1/\alpha}_\pm + (\kappa_1)^{1/\alpha}_\pm + \sum_{\ell=1}^L |\kappa_\ell|^{1/\alpha})^\alpha.$$  

Compared to the proof of Theorem 6.1 one has to additionally use the so called symmetry principle (see [12]). It states that if the singular support of the symbol of a compact self-adjoint Hankel operator $G$ does not contain the points 1 and $-1$, then the spectrum of $G$ is asymptotically symmetric with respect to the point zero.

In the continuous case, we consider real kernels $h(t)$ that are singular at $t = 0$ and contain several oscillating terms at infinity. The assertion below plays the role of Theorem 6.3, and its proof follows essentially the same lines. Similarly to the proof of Theorem 6.2, the contributions of the points $t = \infty$ and $t = 0$ should be considered separately.

**Theorem 6.4.** [12, Theorem 6.6] Let $\alpha > 0$, let $\rho_1, \ldots, \rho_L \in \mathbb{R}_+$ be pairwise distinct numbers, and let $\kappa_0, \kappa_\infty \in \mathbb{R}$, $\kappa_1, \ldots, \kappa_L \in \mathbb{C}$. Suppose that $h \in L^\infty_{\text{loc}}(\mathbb{R}_+)$ if $\alpha < 1/2$ and $h \in C^{N(\alpha)}(\mathbb{R}_+)$ if $\alpha \geq 1/2$ where the number $N(\alpha)$ is given by (4.3). Assume that

$$h(t) = \kappa_\infty t^{-1}(\log t)^{-\alpha} + \tilde{h}_\infty(t) + 2 \text{Re} \sum_{\ell=1}^L (\kappa_\ell t^{-1}(\log t)^{-\alpha} + \tilde{h}_\ell(t)) e^{-i\rho_\ell t}$$

for $t \geq 2$ and

$$h(t) = \kappa_0 t^{-1}(\log(1/t))^{-\alpha} + \tilde{h}_0(t), \quad t \leq 1/2,$$

where the error terms $\tilde{h}_\infty, \tilde{h}_1, \ldots, \tilde{h}_L$ obey the estimates (6.7) as $t \to \infty$ and $\tilde{h}_0$ obeys these estimates as $t \to 0$. Then the eigenvalues of the integral Hankel operator $H$ with kernel $h(t)$ satisfy asymptotic relation (4.5) where

$$a^\pm = \tau(\alpha)((\kappa_0)^{1/\alpha}_\pm + (\kappa_\infty)^{1/\alpha}_\pm + \sum_{\ell=1}^L |\kappa_\ell|^{1/\alpha})^\alpha.$$  

6.3. As an application of Theorem 6.1 we state here a result on rational approximations in the BMO-norm of functions $\omega(z)$ (of bounded mean variation on the unit circle $\mathbb{T}$) analytic on the unit disc $D$ and acquiring some singularities on $\mathbb{T}$. We are interested in singularities of logarithmic type. We study the asymptotic behavior as $n \to \infty$ of the distance $\text{dist}_{\text{BMO}}\{\omega, R_n\}$ in the BMO-norm between $\omega$ and the set $R_n$. 


of all rational functions of degree \( \leq n \) without poles on \( \overline{D} \). A short description of relevant results in this vast domain can be found in [13]. In view of the Adamyan-Arov-Kreĭn theorem the problem considered is equivalent to the study of the asymptotic behaviour of singular values of the Hankel operator with symbol \( \omega(z) \).

Let us describe the class of admissible functions \( \omega(z) \). Let \( u(z) \) be analytic in \( D \), \( u \in C^\infty(D) \); fix some \( \zeta = e^{i\varphi} \in \mathbb{T} \) and assume that

\[
- \log(\zeta - z) + u(z) \neq 0, \quad z \in \overline{D}.
\]

(6.9)

Define

\[
\omega(z) = (- \log(\zeta - z) + u(z))^{1-\alpha}, \quad z \in D, \quad \alpha > 0.
\]

We have introduced \( u(z) \) to avoid irrelevant singularities of \( \omega(z) \) inside \( D \). The branch of the analytic function \( \log(\zeta - z) \) is fixed by the condition \( \log(\zeta - z) = \log(1 - r) + i\varphi \) if \( z = r\zeta \) for \( r \in (0, 1) \). We fix \( \arg(- \log(\zeta - z) + u(z)) \) by the condition that it tends to zero as \( z = re^{i\varphi} \) and \( r \to 1 - 0 \). Obviously, the function \( \omega(z) \) is analytic in the unit disc \( D \) and is smooth up to the boundary \( \mathbb{T} \), except at the point \( z = \zeta \).

Theorem 6.1 allows us to consider \( \omega(z) \) as well as finite sums of such functions.

**Theorem 6.5.** [13, Theorem 3.8] Let \( \zeta_1, \zeta_2, \ldots, \zeta_L \in \mathbb{T} \) be pairwise distinct points, and let functions \( v_\ell, u_\ell, \ell = 1, \ldots, L \), be analytic in \( D \) and \( v_\ell, u_\ell \in C^\infty(D) \). Assume that (6.9) is satisfied for all \( u_\ell, \zeta_\ell \) and set

\[
\omega(z) = \sum_{\ell=1}^{L} v_\ell(z)(- \log(\zeta_\ell - z) + u_\ell(z))^{1-\alpha}, \quad \alpha > 0.
\]

Then there exists the limit

\[
\lim_{n \to \infty} n^\alpha \text{dist}_{\text{BMO}}\{\omega, R_n\} = |1 - \alpha|\tau(\alpha)\left(\sum_{\ell=1}^{L} |v_\ell(\zeta_\ell)|^{1/\alpha}\right)^\alpha.
\]

Note that for \( \alpha < 1 \), the functions \( \omega(\zeta) \) are unbounded as \( \zeta \in \mathbb{T} \) tends to one of the points \( \zeta_\ell \) so that their approximation in the norm of \( C(\mathbb{T}) \) is a priori impossible.

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Univ Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, FRANCE and SPGU, Univ. Nab. 7/9, Saint Petersburg, 199034 RUSSIA

E-mail address: yafaev@univ-rennes1.fr