Light-Cone Quantization of the Liouville Model

Jadwiga Bieńkowska†

Department of Physics and
Enrico Fermi Institute
The University of Chicago
5640 S. Ellis Ave Chicago, IL 60637

We present the quantization of the Liouville model defined in light-cone coordinates in (1,1) signature space. We take advantage of the representation of the Liouville field by the free field of the Backlund transformation and adapt the approach by Braaten, Curtright and Thorn [1]. Quantum operators of the Liouville field $\partial_+ \phi$, $\partial_- \phi$, $e^{g\phi}$, $e^{2g\phi}$ are constructed consistently in terms of the free field. The Liouville model field theory space is found to be restricted to the sector with field momentum $P_+ = -P_-$, $P_+ > 0$, which is a closed subspace for the Liouville theory operator algebra.

† jadwiga@yukawa.uchicago.edu
1. Introduction

The Liouville model has attracted attention recently for two main reasons. First the Liouville theory shows up in covariant quantization of the relativistic string \[2\]. Second, it describes a two dimensional gravity \[3\] \[4\]. The classical Liouville action

\[
S_L = \int d^2x \left( \frac{1}{2} \phi \partial^\mu \partial_\mu \phi - \frac{(2m)^2}{2g^2} e^{2g\phi} \right) \tag{1.1}
\]

with solutions given by the Liouville equation

\[
\partial_- \partial_+ \phi = \frac{(2m)^2}{g} e^{2g\phi}, \tag{1.2}
\]

where \( \phi \) is a Liouville field, describes the gravity theory of two-dimensional surfaces with constant curvature.

The classical theory (1.1) is conformally invariant and the improved stress-energy tensor is found to have the form \[5\] \[1\]

\[
T_{++} = (\partial_+ \phi)^2 - \frac{1}{g} \partial_+^2 \phi \tag{1.3a}
\]

\[
T_{--} = (\partial_- \phi)^2 - \frac{1}{g} \partial_-^2 \phi. \tag{1.3b}
\]

We expect that the construction of a quantum Liouville theory will help to understand quantum gravity at least in the case of simple two dimensional models \[3\]. Liouville type theories also show up in the CGHS model \[3\] \[7\] \[8\] \[9\] which tries to address the black hole evaporation problem.

E. Braaten, T. Curtright and C. Thorn \[1\] have constructed the quantum Liouville theory defined on Minkowski-type space with a spatial dimension compactified on a circle using the equal time quantization prescription. There have also been different approaches to quantize the Liouville model and some of them have developed in unexpected ways \[10\] \[11\] leading to the development of quantum groups theory.
The consistency of quantizing the Liouville theory on the affine Minkowski space in the light-cone coordinates was proven in [12]. However there has been no attempt to construct the quantum field theory operators for the affine Minkowski space Liouville model. The use of the light-cone quantization simplifies the analysis of the massless field theory and we expect that it will also help to simplify description of the Liouville model through Backlund transformation.

In this paper we quantize the Liouville theory in the light-cone coordinates and construct quantum operators using the regularized Backlund transformation. We require that quantum operators of the theory transform consistently under the conformal algebra of the stress-energy tensor. We check the consistency of the quantum equations of motion described in terms of quantum operators defined by the regularization of their classical analogs. The physical states space of the Liouville theory is restricted by the equations of motion to be only half the space of the free pseudoscalar field theory and is described by the vacuum states eigenvalues $P_+ = -P_-$ and $P_+ > 0$. This is in agreement with the result of Braaten, Curtright and Thorn [1] obtained for the Liouville model defined on a circle.

2. Classical Backlund Transformation

In light-cone coordinates the Backlund transformation for the Liouville equation reads:

\[
\partial_+ \phi = \partial_+ \psi - \frac{2m}{g} e^{g\phi} e^{g\psi} \tag{2.1a}
\]

\[
\partial_- \phi = -\partial_- \psi - \frac{2m}{g} e^{g\phi} e^{-g\psi}. \tag{2.1b}
\]

where $\psi(x^+, x^-) = \psi(x^+) + \psi(x^-)$ is a free field. The functions $\psi(x^+) \text{ and } \psi(x^-)$ are completely independent.

To obtain the integral representation of the Liouville field in terms of the free field $\psi$ we first integrate the equation (2.1a) along the $x^- = \text{const light cone}$ from $x^+_0 = -\infty$ to $x^+$.
The equation (2.1) can be integrated on the light cone defined by \( x^+ = x_0^+ = \text{const} \) from \( x_0^- = -\infty \) to \( x^- \). Combining these two integrated equations together we get the integral representation for the classical Liouville field:

\[
e^{-g\phi(x^+,x^-)} = 2m\left(\int_{x_0^+}^{x^+} dy^+ e^{2g\psi(y^+)} e^{g\psi(x^-) - g\psi(x^+)} + \int_{x_0^-}^{x^-} dy^- e^{-2g\psi(y^-)} e^{g\psi(x^-) - g\psi(x^+)}
+ e^{-g\phi(x_0^+,x_0^-)} e^{-g\psi(x_0^-)} e^{g\psi(x_0^+) - g\psi(x^+)}\right),
\]

(2.2)

To completely specify the solutions to the Backlund equations (2.1) we must impose the boundary conditions for the Liouville field at \((x_0^+, x_0^-)\). The boundary condition most appropriate for this problem is the one where the Liouville potential approaches zero at spatial and time negative infinity i.e. \( e^{2g\phi(x_0^+, x_0^-)} \to 0 \). This condition is equivalent to the statement that the Liouville field approaches the free scalar field in this limit and is given by \( \phi(x_0^+, x_0^-) \to -\psi(x_0^-) + \psi(x_0^+) \). With this boundary condition the classical representation of the Backlund transformation is

\[
e^{-g\phi(x^+,x^-)} = 2m\left(\int_{x_0^+}^{x^+} dy^+ e^{2g\psi(y^+)} e^{g\psi(x^-) - g\psi(x^+)} + \int_{x_0^-}^{x^-} dy^- e^{-2g\psi(y^-)} e^{g\psi(x^-) - g\psi(x^+)}
+ e^{g\psi(x^-) - g\psi(x^+)}\right),
\]

(2.3)

3. Light-Cone Quantization of the Liouville Model.

We can quantize the Liouville \( \phi \) and free \( \psi \) fields on the light cone plane defined by \( x^- = \text{const} \) using the Dirac prescription for quantization of systems with dynamical constraints \[13\] \[14\]. Following the standard procedure we get the commutation relations for the fields at fixed \( x^- \):

\[
[\phi(x^+), \phi(y^+)] = [\psi(x^+), \psi(y^+)] = -i \frac{1}{4} \epsilon(x^+ - y^+)
\]

(3.1)
where \( \epsilon(x) = \text{sign}(x) \).

The Fourier representations of the free field in terms of the annihilation and creation operators is

\[
\psi(x^+) = \int_0^\infty \frac{dp}{\sqrt{4\pi p}} (a(p)e^{-ipx^+} + a^\dagger(p)e^{ipx^+})
\]

where the annihilation and creation operators obey the commutation relations

\([a(p), a^\dagger(q)] = p\delta(p - q)\). For quantization at the \( x^+ = \text{const} \) plane, the \( \psi(x^-) \) part of the free field obeys the analogous commutation relation and has a similar Fourier expansion in terms of the \( b(p), b^\dagger(p) \) operators which commute with \( a(p), a^\dagger(p) \). The \( \phi \) field can be represented in the same manner

\[
\phi(x^+) = \int_0^\infty \frac{dp}{\sqrt{4\pi p}} (d(p)e^{-ipx^+} + d^\dagger(p)e^{ipx^+})
\]

In the representation (3.3) of the Liouville field the quantum stress energy tensor is defined as the normal ordered version of the classical (1.3) stress-energy tensor

\[
T^{\phi}_{++}(x^+) = : (\partial_+ \phi(x^+))^2 : - \frac{1}{\gamma} \partial_+^2 \phi(x^+)
\]

where the normal ordering is defined with respect to the \( d(p) \) and \( d^\dagger(p) \) annihilation and creation operators. It is easy to check that the stress-energy tensor (3.4) obeys the conformal algebra commutation relations

\[
[T(x^+), T(y^+)] = (T(x^+) + T(y^+)) i \delta'(x^+ - y^+) - C i \delta''(x^+ - y^+)
\]

where \( C \) is quantum improved conformal anomaly \( C = \frac{1}{2\gamma^2} + \frac{1}{16\pi} \). \( \frac{1}{\gamma} \) is the quantum improved conformal factor given by the renormalized coupling constant \( \gamma = g(1 + \frac{g^2}{2\pi})^{-1} \). The renormalization of the Liouville model coupling constant \( g \) is easily derived from general considerations (see [15]). This is in agreement with the results found before in [1] [3] [12].
4. Quantum Backländ Transformation

In order to determine how the quantum version of the classical Backländ transformation looks we will adapt the approach developed in [1]. The quantum operators corresponding to the classical ones defined in equation (2.3) are determined by the normal ordering prescription.

Using the representation of the Liouville field in terms of the creation and annihilation operators (3.3) we can define the normal ordering of an exponential of the Liouville field as

\[ N\phi(e^{\alpha\phi(x^+)} = e^{\alpha\phi^+(x^+)}e^{\alpha\phi^-(x^+)} \] (4.1)

where

\[ \phi^-(x^+) = \int_0^\infty \frac{dp}{\sqrt{4\pi p}}d(p)e^{-ipx^+} \] (4.2)

\[ \phi^+(x^+) = \int_0^\infty \frac{dp}{\sqrt{4\pi p}}d^{\dagger}(p)e^{ipx^+}. \]

It is easy to check [16] that the fields defined above obey the commutation relation

\[ [\phi^-(x^+), \phi^+(y^+)] = -\frac{1}{4\pi} \ln|x^+-y^+|. \] (4.3)

Any conformal field \( \Phi_{\Delta^+} \) with the left dimension \( \Delta^+ \) obeys the following commutation relation with the left stress energy tensor

\[ i[T^+\Phi_{\Delta^+}(y^+)] = \delta(x^+-y^+)\partial_y\Phi(y^+) - \Delta^+\delta'(x^+-y^+)\Phi_{\Delta^+}. \] (4.4)

The normal ordered with respect to \( d(p) \), \( d^{\dagger}(p) \) \( N\phi(e^{-g\phi}) \) operator has a dimension determined by its commutation relation with quantum stress energy tensor (3.4). Using the formula (4.4) it is straightforward to find that the dimension of the operator \( N\phi(e^{-\alpha\phi(x^+)}) \) is equal to \( \Delta^+ = \frac{\alpha}{2\gamma} - \frac{\alpha^2}{8\pi} \). From the consistent construction of the quantum Backländ transformation we require that the same operator expressed in terms of the free field \( \psi(x^+) \) would have the same conformal dimension with respect to the \( \psi \) field stress-energy tensor.
We postulate then that the quantum version of the Backlund transformation has the form

\[
N_\phi e^{-g\phi(x^+,x^-)} = 2m N_\psi \left[ e^{g\psi(x^-)-g\psi(x^+)} (1 + \int_{-\infty}^{x^+} dy^+ f(x^+ - y^+) e^{2g\psi(y^+)} + \int_{-\infty}^{x^-} dy^- e^{-2g\psi(y^-)}) \right]
\]  

(4.5)

and the right hand side of the equation (4.3) is normal ordered with respect to \( a(p), a^+(p) \) annihilation and creation operators.

The stress energy tensor for the \( \psi \) field is

\[
T^\psi_{++}(x^+) =: (\partial_+ \psi(x^+))^2: -\beta_+ \partial_+^2 \psi(x^+) \]  

(4.6)

The normal ordering prescription for exponentials is defined as for the Liouville field (4.1) and the creation and annihilation parts of the free field \( \psi(x^+) \) have the commutation relations

\[
[\psi^-(x^+), \psi^+(y^+)] = -\frac{1}{4\pi} \ln |x^+ - y^+|.
\]

(4.7)

The requirement that the quantum Liouville field exponential defined by the transformation (4.5) in the \( \psi \) representation is the conformal field of the same dimension as in the \( \phi \) field representation fixes \( f(x^+ - y^+) \) and \( \beta_+ \) conformal improvement factor for free field. It is straightforward to check that \( f(x^+ - y^+) = (x^+ - y^+) \frac{g^2}{2\pi} \) and \( \beta_+ = \frac{1}{\gamma} \), which is the same as the conformal improvement factor for the quantum Liouville field stress energy tensor (3.4).

The same line of argument can be followed for the \( x^- \) coordinate independently. Even if we do not know the commutation relations of the Liouville field \( \phi(x^+, x^-) \) itself for arbitrary points in the space-time (neither \( x^+ \) nor \( x^- \) held fixed), we know that the free fields \( \psi(x^+) \) and \( \psi(x^-) \) commute with each other and so we can construct the complete expression for the quantum \( e^{-g\phi} \) operator in terms of the free field \( \psi \). Repeating the same calculation for the right handed part of the operator \( N_\phi(e^{-g\phi}) \) we obtain that \( f(x^- - y^-) = \)}
\[(x^- - y^-)^{\frac{2}{2\pi}} \text{ and } \beta_- = -\frac{1}{\gamma}\]. This confirms the results obtained before [1] that field \(\psi\) has to be a pseudo-scalar free field.

The derivation of the quantum version of the Backlund transformation can be summarized by the formula

\[
N(e^{-g\phi(x^+,x^-)}) = 2mN_\psi[e^{g\psi(x^-)} - g\psi(x^+)] \left(1 + \int_{-\infty}^{x^+} dy^+(x^+-y^+)\frac{\sigma^2}{2\pi} e^{2g\psi(y^+)} + \int_{-\infty}^{x^-} dy^-(x^- - y^-)\frac{\sigma^2}{2\pi} e^{-2g\psi(y^-)}\right) \right].
\]

(4.8)

To completely define the set of quantum operators in this language we have to specify the normal ordering prescription for two exponential operators of the Liouville field. The natural definition is:

\[
N(e^{\alpha\phi}e^{\beta\phi}) = \lim_{x \to y} |x^+ - y^+| \frac{\sigma^2}{2\pi} |x^- - y^-| \frac{\sigma^2}{2\pi} N(e^{\alpha\phi(x^+,x^-)}N(e^{\beta\phi(y^+,y^-)}).
\]

(4.9)

Using the equation (4.9) with \(\alpha = g, \beta = -g\) we find the quantum operator

\[
N e^{g\phi(x^+,x^-)} = \frac{1}{2m} e^{g\psi^+(x^+)} e^{-g\psi^+(x^-)} X(x^+,x^-) e^{g\psi^-(x^+)} e^{-g\psi^-(x^-)}
\]

(4.10)

where

\[
X^{-1}(x^+,x^-) = N_\psi \left[1 + \int_{-\infty}^{x^+} dy^+(x^+-y^+)\frac{\sigma^2}{2\pi} e^{2g\psi(y^+)}
\]

\[
+ \int_{-\infty}^{x^-} dy^-(x^- - y^-)\frac{\sigma^2}{2\pi} e^{-2g\psi(y^-)}\right].
\]

(4.11)

The same way we find that

\[
N e^{2g\phi(x^+)} = \frac{1}{(2m)^2} e^{2g\psi^+(x^+)} e^{-2g\psi^+(x^-)} \tilde{X}^+(x^+,x^-) \tilde{X}(x^+,x^-) e^{2g\psi^-(x^+)} e^{-2g\psi^-(x^-)}
\]

(4.12)
where

\[ \tilde{X}^{-1}(x^+, x^-) = N_\psi \left[ 1 + \int_{-\infty}^{x^+} dy^+ (x^+ - y^+) - \frac{2}{\pi} e^{2g\psi(y^+)} \right. \]

\[ \left. + \int_{-\infty}^{x^-} dy^- (x^- - y^-) - \frac{2}{\pi} e^{-2g\psi(y^-)} \right]. \]

(4.13)

From definitions (4.11) and (4.13) it is clear that these formulas do not contain the integration singularities as long as \( \frac{a^2}{\pi} < 1 \). They are valid in the weak coupling limit of the Liouville model. This restriction is identical to the one obtained for the Liouville model with space dimension compactified on a circle [1].

The complete set of operators includes also \( \partial_+ \phi \) and \( \partial_- \phi \) which are defined by the classical Backlund transformations (2.3). Looking at the equations (2.3) and (4.10) we may guess the form of the quantum version of the classical operators \( e^{g\phi} e^{g\psi} \) and \( e^{-g\phi} e^{-g\psi} \)

\[ N(e^{g\phi} e^{g\psi}) = \frac{1}{2m} e^{2g\psi(x^+)} Y_+(x^+, x^-) e^{2g\psi(X^+)} \]

(4.14a)

\[ N(e^{g\phi} e^{-g\psi}) = \frac{1}{2m} e^{-2g\psi(x^-)} Y_-(x^+, x^-) e^{-2g\psi(x^-)}. \]

(4.14b)

Requiring that the equation of motion (1.2) hold also for the quantum operators we find that \( Y_+ \), \( Y_- \) are given by the expressions

\[ Y_+^{-1}(x^+, x^-) = N_\psi \left[ 1 + \int_{-\infty}^{x^+} dy^+ (x^+ - y^+) - \frac{2}{\pi} e^{2g\psi(y^+)} + \int_{-\infty}^{x^-} dy^- e^{-2g\psi(y^-)} \right] \]

(4.15a)

\[ Y_-^{-1}(x^+, x^-) = N_\psi \left[ 1 + \int_{-\infty}^{x^+} dy^+ e^{2g\psi(y^+)} + \int_{-\infty}^{x^-} dy^- (x^- - y^-) - \frac{2}{\pi} e^{-2g\psi(y^-)} \right]. \]

(4.15b)

We can also check, using the equation (4.14) and its right-handed counterpart, that the left and right conformal dimensions of the operators \( N(e^{g\phi} e^{g\psi}) \) and \( N(e^{g\phi} e^{-g\psi}) \) are respectively \( \Delta_+ = 1 \) and \( \Delta_- = 1 \) which proves the consistency of the definitions (4.14).
The quantum equations of motion are then described by

\[
\begin{align*}
\partial_+ \partial_- \phi &= \partial_- (\partial_+ \psi - \frac{2m}{g} N(e^{g\phi} e^{g\psi})) \\
\partial_+ (-\partial_- \psi - \frac{2m}{g} N(e^{g\phi} e^{-g\psi})) &= \frac{(2m)^2}{g} N(e^{2g\phi})
\end{align*}
\] (4.16)

and expressions (4.14), (4.12).

5. Liouville Equation on the Physical States Space

It was proven in \([1]\) that the physical states space for the Liouville model defined on a circle is half the space of the free field theory. The constraints of the Liouville theory space came from the requirement that quantum equations of motion are valid. It turns out that they hold only on half of the free field theory space. In this section we would like to establish if similar restrictions apply to our quantization procedure.

In order to do so we first have to define the Fock space of the theory. As usual the Fock space is defined by the states built up from the creation operators acting on the unique vacuum state which is annihilated by all the annihilation operators \(a(p)|0\rangle = b(p)|0\rangle = 0\). Special attention however has to be paid to the constant (coordinate independent) modes of the free field operator which as in the case of quantization on the time-like plane define different super-selection sectors of the vacuum. The constant modes of the free field operator are implicitly contained in the Fourier expansion (3.2). We can recover constant modes of the free field operator as

\[
\begin{align*}
Q^+ &= \lim_{p \to 0} \frac{1}{p \sqrt{4\pi}} (a(p) + a^\dagger(p)) \\
Q^- &= \lim_{p \to 0} \frac{1}{p \sqrt{4\pi}} (b(p) + b^\dagger(p)).
\end{align*}
\] (5.1)

These operators have their canonical conjugates defined by

\[
\begin{align*}
P^+ &= \lim_{p \to 0} i \sqrt{\pi} (-a(p) + a^\dagger(p)) \\
P^- &= \lim_{p \to 0} i \sqrt{\pi} (-b(p) + b^\dagger(p)).
\end{align*}
\] (5.2)
They obey the canonical commutation relations

\[ [Q^+, P_+] = i \quad [Q^-, P_-] = i \] (5.3)

With constant mode operators and their canonical conjugates given by the formulas (5.1) and (5.2) we may define normalized vacuum states of the theory as

\[ e^{iQ^+P_+}e^{iQ^-P_-}|0\rangle = |P'_+, P'_-, 0\rangle \] (5.4)

and

\[ \langle 0P'_+, P'_- | P''_+, P''_-, 0 \rangle = \delta(P'_+ - P''_+)\delta(P'_- - P''_-) \]

where zero in the \(|P'_+, P'_-, 0\rangle\) means a state annihilated by all \(a(p), b(p)\) with \(p \neq 0\).

We can also rewrite the Fourier expansion of the free field as

\[
\psi(x^+) = Q^+ + P_+ \frac{x^+}{2\pi} + \int_{0^+}^{\infty} \frac{dp}{\sqrt{4\pi p}} (a(p)e^{-ipx^+} + a^+(p)e^{ipx^+}) = \\
= Q^+ + P_+ \frac{x^+}{2\pi} + \tilde{\psi}(x^+) \\
\psi(x^-) = Q^- + P_- \frac{x^-}{2\pi} + \int_{0^+}^{\infty} \frac{dp}{\sqrt{4\pi p}} (b(p)e^{-ipx^-} + b^+(p)e^{ipx^-}) = \\
= Q^- + P_- \frac{x^-}{2\pi} + \tilde{\psi}(x^-). \] (5.5)

New fields \(\tilde{\psi}(x^+), \tilde{\psi}(x^-)\) contain only nonzero momentum creation, annihilation modes and have slightly modified commutation relations

\[
[\tilde{\psi}^-(x^+), \tilde{\psi}^+(y^+)] = -\frac{1}{4\pi} \ln|x^+ - y^+| + \frac{i}{4\pi} (x^+ - y^+) \\
[\tilde{\psi}^-(x^-), \tilde{\psi}^+(y^-)] = -\frac{1}{4\pi} \ln|x^- - y^-| + \frac{i}{4\pi} (x^- - y^-). \] (5.6)

We have to specify the normal ordering prescription for the new set of the operators including the “momenta” and “position” operators on the field space. For the nonzero momentum operators the normal ordering prescription is as usual. For the \(Q^+, P_+, Q^-, P_-\) operators we define the normal ordering by

\[
N(e^{2\alpha Q^+} F(P_+)) = e^{\alpha Q^+} F(P_+) e^{\alpha Q^+} \quad N(e^{2\alpha Q^-} F(P_-)) = e^{\alpha Q^-} F(P_-) e^{\alpha Q^-}. \] (5.7)
It is easy to check that this normal ordering prescription differs only by a coordinate
independent constant from the one given in equation (4.1).

The Liouville model operators defined by the modified ordering prescription have the
form
\[ \mathcal{N} e^{g \phi (x^+, x^-)} = \frac{1}{2m} e^{g \tilde{y}^+ (x^+)} e^{-g \tilde{y}^+ (x^-)} X_Q (x^+, x^-) e^{g \tilde{y}^- (x^+)} e^{-g \tilde{y}^- (x^-)} \]
\[ X_Q^{-1} (x^+, x^-) = \mathcal{N} \left[ e^{-g \psi_0 (x^+)} e^{g \psi_0 (x^-)} \right] \left( 1 + \int_{-\infty}^{x^+} dy^+ (x^+ - y^+) - \frac{2m}{\pi} e^{g \psi (y^+)} + \int_{-\infty}^{x^-} dy^- (x^- - y^-) - \frac{2m}{\pi} e^{-2g \psi (y^-)} \right] \]

\[ \hat{X}_Q^{-1} (x^+, x^-) = \mathcal{N} \left[ e^{-g \psi_0 (x^+)} e^{g \psi_0 (x^-)} \right] \left( 1 + \int_{-\infty}^{x^+} dy^+ (x^+ - y^+) - \frac{2m}{\pi} e^{g \psi (y^+)} + \int_{-\infty}^{x^-} dy^- (x^- - y^-) - \frac{2m}{\pi} e^{-2g \psi (y^-)} \right] \]

The operators involved in the expression of \( \partial_+ \phi \) and \( \partial_- \phi \) are given by

\[ \mathcal{N} (e^{g \phi} e^{g \psi}) = \frac{1}{2m} e^{2g \tilde{y}^+ (x^+)} Y_{Q_+} (x^+, x^-) e^{2g \tilde{y}^- (x^+)} \]
\[ \mathcal{N} (e^{g \phi} e^{-g \psi}) = \frac{1}{2m} e^{-2g \tilde{y}^+ (x^+)} Y_{Q_-} (x^+, x^-) e^{-2g \tilde{y}^- (x^+)} \]

and

\[ Y_{Q_+}^{-1} (x^+, x^-) = \mathcal{N} \left[ e^{-2g \psi_0 (x^+)} \right] \left( 1 + \int_{-\infty}^{x^+} dy^+ (x^+ - y^+) - \frac{2m}{\pi} e^{2g \psi (y^+)} + \int_{-\infty}^{x^-} dy^- e^{-2g \psi (y^-)} \right] \]
\[ Y_{Q_-}^{-1} (x^+, x^-) = \mathcal{N} \left[ e^{2g \psi_0 (x^-)} \right] \left( 1 + \int_{-\infty}^{x^+} dy^+ e^{2g \psi (y^+)} + \int_{-\infty}^{x^-} dy^- (x^- - y^-) - \frac{2m}{\pi} e^{-2g \psi (y^-)} \right) \]
In the above expressions we used the shorthand notations for the zero modes of the free field \( \psi_0(x^+) = Q^+ + P_+ \frac{x^+}{2\pi} \) and \( \psi_0(x^-) = Q^- + P_- \frac{x^-}{2\pi} \).

To establish the consistency of the quantum equation of motion to the lowest order in the \( g \) expansion we have to check if the equation

\[
\partial_- \langle 0, P'_+, P'_- | \partial_+ \phi | P''_+, P''_-, 0 \rangle = \langle 0, P'_+, P'_- | \frac{(2m)^2}{g} N e^{2g\phi} | P''_+, P''_-, 0 \rangle \tag{5.12}
\]

holds to the lowest order in the \( g \) expansion.

From the equations (5.9) and (5.10) we see that we need to evaluate expressions

\[
\partial_- \langle 0, P'_+, P'_- | - \frac{1}{g} Y_Q (x^+, x^-) | P''_+, P''_-, 0 \rangle
\]

and

\[
\langle 0, P'_+, P'_- | \frac{1}{g} \tilde{X}_Q \tilde{X}_Q | P''_+, P''_-, 0 \rangle. \tag{5.13}
\]

The evaluation of the equation of motion coming from the second equation of Backl"{u}nd transformation (2.17) can be reduced to the problem of calculating the matrix element \( \langle 0, P'_+, P'_- | - \frac{1}{g} Y_Q (x^+, x^-) | P''_+, P''_-, 0 \rangle \) which is similar to the (5.13) formula and we will concentrate on examining the consistency of the equation (5.12).

To the lowest order in \( g \) the following formulas hold

\[
Y^{-1}_{Q+} = (1 + O(g^2)) (e^{-gQ^+} e^{-gP_+} \frac{x^+}{2\pi} e^{-gQ^+} + \int^{x^+}_{-\infty} dy^+ e^{gP_+} \frac{(x^+-y^+)}{2\pi} + e^{-gQ^+} e^{-gP_+} \frac{x^+}{2\pi} e^{-gQ^+} \int^{x^-}_{-\infty} dy^- e^{-gP_+} \frac{y^+}{2\pi} e^{-gQ^-}) \tag{5.14}
\]

\[
\tilde{X}^{-1}_Q = (1 + O(g^2)) (e^{-\frac{g}{2}Q^+} e^{-gP_+} \frac{x^+}{2\pi} e^{-\frac{g}{2}Q^+} e^{\frac{g}{2}Q^-} e^{-gP_-} \frac{x^-}{2\pi} e^{\frac{g}{2}Q^-} + e^{\frac{g}{2}Q^-} e^{-gP_-} \frac{x^-}{2\pi} e^{\frac{g}{2}Q^+} e^{-gP_+} \frac{x^+}{2\pi} \int^{x^+}_{-\infty} dy^+ e^{gP_+} \frac{(x^+-y^+)}{2\pi} e^{\frac{g}{2}Q^+} + \int^{x^-}_{-\infty} dy^- e^{gP_-} \frac{(x^-y^-)}{2\pi} e^{\frac{g}{2}Q^-} e^{-gP_-} \frac{x^-}{2\pi} e^{\frac{g}{2}Q^-}) \tag{5.15}
\]
The integrations over $y^+$ and $y^-$ in the expressions (5.14), (5.15) give a finite result (which is equivalent to the nonzero matrix elements (5.13) ) provided we are restricted to the space with $P_+ > 0$ and $P_- < 0$ eigenvalues. After performing the integrations and using identities

$$e^{-gQ^+} e^{-gP_+ e^{\pm i \pi}} e^{-gQ^+} = e^{iP_+^2 + g e^{-2gQ^+} e^{-iP_+^2 + g}}$$  \hspace{1cm} (5.16a)$$

$$e^{gQ^+} P_+^{-1} e^{gQ^+} = \frac{1}{2i g} \Gamma (\frac{P_+ + i g}{2i g}) e^{2gQ^+} \Gamma^{-1} (\frac{P_+ + i g}{2i g})$$  \hspace{1cm} (5.16b)$$

$$e^{-gQ^-} (-P_-)^{-1} e^{-gQ^-} = \frac{1}{2i g} \Gamma^{-1} (\frac{P_- - i g}{2i g}) e^{-2gQ^-} \Gamma (\frac{P_- - i g}{2i g})$$  \hspace{1cm} (5.16c)$$

we get the following expressions for the matrix elements (5.13):

$$\partial_- \langle 0, P'_+ , P'_- | \frac{1}{g} Y_{+}(x^+, x^-) | P'_{+}, P'_{-}, 0 \rangle =$$

$$= -\frac{i g}{g \pi} \frac{(P'^2 - P''^2)}{4\pi} e^{\frac{(P'^2 - P''^2)x^+}{4\pi}} e^{\frac{(P'^2 - P''^2)x^+}{4\pi}} P'_+ \times \Gamma (\frac{P'_+}{2i g}) \Gamma (\frac{P'_- - i g}{2i g}) \Gamma^{-1} (\frac{P''^+}{2i g}) \Gamma^{-1} (\frac{P''^- - i g}{2i g}) \times$$

$$\langle 0, P'_+ , P'_- | e^{2gQ^+} e^{-2gQ^-} | P''^+ , P''^-, 0 \rangle (1 + O(g^2))$$  \hspace{1cm} (5.17)$$

and

$$\langle 0, P'_+ , P'_- | \frac{1}{g} \tilde{X}^+ \chi_Q^+ \tilde{X}^+ Q^+ | P''^+ , P''^-, 0 \rangle =$$

$$= \frac{1}{g} \frac{(g^2)^2}{2} e^{\frac{(P'^2 - P''^2)x^+}{4\pi}} e^{\frac{(P'^2 - P''^2)x^+}{4\pi}} \Gamma^{-1} (\frac{P'_+}{2i g}) \Gamma^{-1} (\frac{P'_+}{2i g}) \Gamma^{-1} (\frac{P''^+}{2i g}) \Gamma^{-1} (\frac{P''^+}{2i g}) \times$$

$$\langle 0, P'_+ , P'_- | e^{2gQ^+} e^{-2gQ^-} \Gamma \left(\frac{P'_+}{2i g}, \frac{1}{2} \right) \Gamma \left(\frac{P'_-}{2i g}, \frac{1}{2} \right) \Gamma \left(\frac{P''^+}{2i g}, \frac{1}{2} \right) \Gamma \left(\frac{P''^-}{2i g}, \frac{1}{2} \right)$$

$$\frac{e^{gQ^+} e^{-gQ^-}}{e^{2gQ^+} e^{-2gQ^-}} | P''^+ , P''^-, 0 \rangle (1 + O(g^2)).$$  \hspace{1cm} (5.18)$$

The expressions (5.17) and (5.18) can be further evaluated quite straightforwardly. We insert the complete set of states $\int \frac{dq^+}{2\pi} |q^\pm\rangle \langle q^\pm|$ or $\int \frac{dp^\pm}{2\pi} |p^\pm\rangle \langle p^\pm|$ and after tedious but standard integrations we obtain the following expressions:
\[
\partial_-(0, P'_+, P'_-| - \frac{1}{g} Y_{Q+}(x^+, x^-)|P''_+, P''_-) = \\
= \frac{1}{g} \frac{\left( P'^2_2 - P''^2_2 \right)}{4\pi} e^{i (P'^2_2 - P''^2_2) x^-} e^{i \left( P'^2 - P''^2 \right) x^+} P'_+ \times \\
\Gamma \left( \frac{P'_+}{2i g} \right) \Gamma \left( - \frac{P''_+ - ig}{2i g} \right) \Gamma^{-1} \left( \frac{P''_+ + ig}{2i g} \right) \Gamma^{-1} \left( - \frac{P''_- - ig}{2i g} \right) \times \\
\frac{1}{2g \, 2\text{sh} \pi \Delta P_+} \delta (\Delta P_+ - \Delta P_-)
\] (5.19)

and

\[
(0, P'_+, P'_-| - \frac{1}{g} \tilde{X}^i_+ \tilde{X}_Q|P''_+, P''_-) = \\
= \frac{-1}{g^2} \frac{\left( P'^2_2 - P''^2_2 \right)}{4\pi} e^{i (P'^2_2 - P''^2_2) x^-} e^{i \left( P'^2 - P''^2 \right) x^+} \Gamma^{-1} \left( - \frac{P'_+ + ig}{2i g} \right) \Gamma^{-1} \left( \frac{P''_- + ig}{2i g} \right) \times \\
\left( \frac{\pi P'_+}{2g \text{sh} P'_+} \frac{\pi P''_+}{2g \text{sh} P''_+} \frac{\pi (S - S_+)}{4g} + \\
\frac{2\Delta P_+ \left( \text{ch}^2 \frac{2\pi \Delta P_+}{4g} \text{sh} \pi S_+ \text{sh} \pi S_- + \text{sh}^2 \pi \Delta P_+ \text{ch} \pi S_+ \text{ch} \pi S_- \right)}{4g} + \\
\frac{\pi \Delta P_+ \text{sh} \pi S_- \text{sh} \pi (\Delta P_+ + S_+)}{2g} \frac{\pi \Delta P_+ + S_-}{4g} \text{sh} \pi (\Delta P_+ + S_-) \right) \times \delta (\Delta P_+ - \Delta P_-)
\] (5.20)

We have introduced the symbols: \( \Delta P_+ = P''_+ - P'_+ \), \( \Delta P_- = P''_- - P'_- \), \( S_+ = P'_+ + P'_+ \), \( S_- = P''_+ + P''_- \), \( \text{sh} = \sinh \) and \( \text{ch} = \cosh \).

Inspection of expressions (5.19) and (5.20) indicates this equivalence in the small \( g \) limit, if the field theory space is restricted to the vacuum states \( |P'_+, P'_-, 0 \rangle \) with momenta \( P'_+ = -P'_- \), \( P'_+ > 0 \). This is half of the field theory space for the pseudoscalar free field \( \psi \) defined by \( P'_+ = -P'_- \) condition. The leading \( g \) contribution to the expression (5.20) comes from the second term in the sum.

On this space the matrix elements are equal the previous matrix elements.

---

1 Useful identities for \( \Gamma \) function can be found in [17].
\[ \partial_- \langle 0, P'_{+, P'}^\prime | \partial_+ \phi | P''_{+, P''}^\prime, 0 \rangle = \langle 0, P'_{+, P'}^\prime | Ne^{2g \phi} | P''_{+, P''}^\prime, 0 \rangle = \frac{1}{\pi} \frac{P'^2}{(2\pi)^2} \delta(2\Delta P_+). \]

The restriction of the physical states space is consistent with the integrability condition for the equations (5.14) and (5.15) i.e. \( P'_{+, P'}^\prime > 0 \) and \( P'_{-, P'}^\prime < 0 \). The subspace defined above is closed under the operator algebra since acting by a Liouville vertex operator with a positive momentum on the state with positive momentum will always produce another state with positive momentum.

6. Conclusion

In this paper we have consistently constructed the quantized Liouville model defined on the affine Minkowski space. Aside from the fact that this procedure proposes a different way of quantizing the theory we find it interesting because it can be readily applied to the CGHS [6] model expressed by Liouville type fields [8] [9]. The CGHS model is believed to describe the properties of evaporating black holes and quantizing it may provide a new insight into the problem. Work on this problem is in progress.

I would like to thank Emil Martinec for many helpful discussions and comments, and Eric D’Hoker for discussions about the Liouville theory. This work is submitted in partial fulfillment of the requirements for a Ph.D. degree in physics at the University of Chicago. This work is supported in part by the DOE grant DE-FG02-90ER40560.
References

[1] E. Braaten, T. Curtright and C. Thorn, Ann. of Phys. 147 (1983) 365 ;
E. Braaten, T. Curtright, G. Ghandour and C. Thorn, Ann. of Phys. 153 (1984) 147.

[2] A. M. Polyakov, Phys. Lett. B (1981) 207.

[3] F. David, Mod. Phys. Lett. A3 (1988) 1651 ;
J. Distler and H. Kawai, Nucl. Phys. B321 (1989).

[4] N. Seiberg, Prog. Theor. Phys. Supp. 102 (1990) 319.

[5] E. D’Hoker and R. Jackiw Phys. Rev D 26 (1982) 3517.

[6] C. G. Callan, S. B. Giddings, J. A. Harvey and A. Strominger, Phys. Rev. D45 (1992)
R1005.

[7] J. G. Russo, L. Susskind, L. Thorlacius, preprints SU-ITP-92-4, SU-ITP-92-17 ;
L.Susskind, L. Thorlacius, preprint SU-ITP-92-12.

[8] S. P. de Alwis, preprints COLO-HEP-280. May 1992, COLO-HEP-284, June 1992.
COLO-HEP-288 July 1992.

[9] A. Bilal, C. G. Callan, preprint PUPT-1320, May 1992

[10] J. L. Gervais and A. Neveu, Nucl. Phys B199 (1982) 59. Nucl. Phys. B209 (1982) 125.
Nucl. Phys. B224 (1983) 329. Nucl. Phys. B238 (1984) 125, 396. Phys. Lett. 151 B
(1985) 271.

[11] E. D’Hoker, D. Z. Freedman and R. Jackiw, Phys. Rev D 28 (1983) 2583.

[12] P. Mansfield, Nucl. Phys B208 (1982), 277. Nucl. Phys. B222 (1983) 419 ;
H. C. Liao and P.Mansfield, Nucl. Phys. B344 (1990), 696.

[13] P. A. M. Dirac, “Lectures on Quantum Mechanics” (1964). Belfer Graduate School of
Science, Yeshiva University, New York ;
A. Hanson, T. Regge and C. Teitelboim, “Constrained Hamiltonian Systems”, lectures
(1976), Academia Nazionale dei Lincei, Roma.

[14] H. Leutwyler and J. Stern, Ann. of Phys. 112 (1978) 94.

[15] E. D’Hoker “Lecture Notes on 2D Quantum Gravity and Liouville Theory” , proceedings
of VI-th Swieca Summer School, Brazil, January 17-28, 1991.

[16] Yu. A. Brychkov and A. P. Prudnikov “Integral Transforms of Generalized Functions”,
ed. Gordon and Breach Science Publishers S.A., Amsterdam (1989).

[17] I. S. Gradshteyn and I. M. Ryzhik, “Tables of Integrals, Series and Products”, ed.
Academic Press, Inc. New York (1981).