Subcritical branching processes in random environment with immigration stopped at zero

Doudou Li†, Vladimir Vatutin‡ and Mei Zhang§

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Abstract
We consider the subcritical branching processes with immigration which evolve under the influence of a random environment and study the tail distribution of life periods. We prove that, the tail distribution decays at an exponential rate or more quickly, according to the solution of an equation. The main tools are the change of measure and some limit theorems for the random walks.

Keywords: branching processes; random environment; immigration; life period

1 Introduction and statement of main results
We consider branching processes allowing immigration and evolving in a random environment. Individuals in such processes reproduce independently of each other according to offspring distributions which vary in a random manner from one generation to the other. In addition, a number of immigrants join each generation independently on the development of the population and according to the laws varying at random from generation to generation. A formal definition of the process looks as follows. Let \( \Delta = (\Delta_1, \Delta_2) \) be the space of all pairs of probability measures on \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} \). Supplying \( \Delta \) with the component-wise metric of total variation we obtain a Polish space. Let \( \mathbf{Q} = \{F, G\} \) be a two-dimensional random vector with independent components

\[
F := (F(\{j\}), j = 0, 1, \ldots), \quad G := (G(\{j\}), j = 0, 1, \ldots)
\]

†School of Mathematical Sciences & Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875, P.R. China. Email: lidoudou@mail.bnu.edu.cn
‡Steklov Mathematical Institute, 8 Gubkin St., Moscow, 119991, Russia. Email: vatutin@mi-ras.ru
§School of Mathematical Sciences & Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875, P.R. China. Email: meizhang@bnu.edu.cn
taking values in $\Delta$, and let $Q_n = \{F_n, G_n\}, n = 1, 2, \ldots$, be a sequence of independent copies of $Q$. The infinite sequence $E = \{Q_1, Q_2, \ldots\}$ is called a random environment.

A sequence of $N_0$-valued random variables $Y = \{Y_n, n \in N_0\}$ specified on the respective probability space $(\Omega, \mathcal{F}, P)$ is called a branching process with immigration in the random environment (BPIRE), if $Y_0$ is independent of $E$ and, given $E$ the process $Y$ is a Markov chain with

$$
\mathcal{L}(Y_n|Y_{n-1} = y_{n-1}, E = \{q_1, q_2, \ldots\}) = \mathcal{L}(\xi_1 + \ldots + \xi_{ny_{n-1}} + \eta_n)
$$

for every $n \in \mathbb{N} := \mathbb{N}_0 \setminus \{0\}, y_{n-1} \in \mathbb{N}_0$ and $q_1 = (f_1, g_1), q_2 = (f_2, g_2), \ldots \in Q$, where $\xi_{n1}, \xi_{n2}, \ldots$ are i.i.d. random variables with distribution $f_n$ and independent of the random variable $\eta_n$ with distribution $g_n$. In the language of branching processes $Y_{n-1}$ is the $(n-1)$th generation size of the population, $f_n$ is the distribution of the number of children of an individual at generation $n-1$ and $g_n$ is the law of the amount of immigrants joining generation $n$.

Along with the process $Y$ we consider a branching process $Z = \{Z_n, n \in N_0\}$ in the random environment (BPRE) $E_1 = \{F_1, F_2, \ldots\}$ which, given $E_1$ is a Markov chain with $Z_0 = 1$ and, for $n \in \mathbb{N}$

$$
\mathcal{L}(Z_n|Z_{n-1} = z_{n-1}, E_1 = \{f_1, f_2, \ldots\}) = \mathcal{L}(\xi_1 + \ldots + \xi_{nz_{n-1}}).
$$

An important role in studying BPRE and BPIRE is played by the so-called associated random walk $S = \{S_0, S_1, \ldots\}$. This random walk has initial state $S_0$ and increments $X_n = S_n - S_{n-1}, n \geq 1$, defined as

$$
X_n := \log m(F_n).
$$

Thus, the increments are i.i.d. copies of the logarithmic mean offspring number $X := \log m(F)$ with

$$
m(F) := \sum_{j=0}^{\infty} jF(\{j\}).
$$

We suppose that $X$ is a.s. finite.

We call a BPIRE $Y$ supercritical if $E[X] > 0$, subcritical if $E[X] < 0$, and critical if either $E[X] = 0$ or $E[X]$ does not exist.

It will be convenient to assume that if $Y_{n-1} = y_{n-1} > 0$ is the population size of the $(n-1)$th generation of $Y$ then first $\xi_{n1} + \ldots + \xi_{ny_{n-1}}$ individuals of the nth generation are born and than $\eta_n$ immigrants join the population.

This agreement allows us to consider a modified version $W = \{W_n, n \in N_0\}$ of the process $Y$ specified as follows. Assume, without loss of generality, that $Y_0 > 0$. Let $W_0 = Y_0$ and for $n \geq 1$,

$$
W_n := \begin{cases} 
0, & \text{if } T_n := \xi_{n1} + \ldots + \xi_{ny_{n-1}} = 0, \\
T_n + \eta_n, & \text{if } T_n > 0.
\end{cases}
$$

We call $W$ as a branching process with immigration stopped at zero and evolving in the random environment $E$. 

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The aim of the present paper is to study, under the annealed approach, the tail distribution of the random variable
\[ \zeta := \min \{ n \geq 1 : W_n = 0 \} \]
for subcritical BPIRE.

With each pair of measures \((F, G)\) we associate the respective probability generating functions
\[ F(s) := \sum_{j=0}^{\infty} F(\{j\}) s^j, \quad G(s) := \sum_{j=0}^{\infty} G(\{j\}) s^j. \]

Given the environment \(E = \{(F_n, G_n), n \in \mathbb{N}\}\), we construct the i.i.d. sequence of pairs of generating functions
\[ F_n(s) := \sum_{j=0}^{\infty} F_n(\{j\}) s^j, \quad G_n(s) := \sum_{j=0}^{\infty} G_n(\{j\}) s^j, \quad s \in [0, 1], \]
and use below the convolutions of the generating functions \(F_1, ..., F_n\) specified for \(0 \leq i \leq n - 1\) by the equalities
\[ F_{i,n}(s) := F_{i+1}(F_{i+2}(\ldots F_n(s) \ldots)), \]
\[ F_{n,i}(s) := F_n(F_{n-1}(\ldots F_{i+1}(s) \ldots)) \quad \text{and} \quad F_{n,n}(s) := s. \]

We assume for convenience that \(W_0 = Y_0 > 0\) has the (random) probability generating function
\[ N(0; s) := \frac{G_0(s) - G_0(0)}{1 - G_0(0)} \]
where \(G_0(s) \stackrel{d}{=} G(s)\). Other cases of initial distributions may be considered in a similar way.

Denote
\[ H_n := \mathbb{E} \left[ \left( 1 - G_0(F_{0,n+1}(0)) \right) \prod_{i=1}^{n} G_i(F_{i,n+1}(0)) \right], \]
\[ H^*_n := \mathbb{E} \left[ \frac{1 - G_0(F_{0,n+1}(0))}{1 - G_0(0)} \prod_{i=1}^{n} G_i(F_{i,n+1}(0)) \right], \]
\[ R_n := \mathbb{P} (\zeta > n), \]
and let
\[ \mathcal{H}(s) := \sum_{n=0}^{\infty} H_n s^n, \quad \mathcal{H}^*(s) := \sum_{n=1}^{\infty} H^*_n s^n \quad \text{and} \quad \mathcal{R}(s) := \sum_{n=1}^{\infty} R_n s^n. \]

It is known (see, Lemma 1 in [6]) that \(\mathcal{R}(s)\) can be calculated by the formula
\[ \mathcal{R}(s) = \frac{s \mathcal{H}^*(s) + s R_1}{1 - s \mathcal{H}(s)}. \]
The following restrictions are imposed on the distributions of $F$ and $G$.

**Hypothesis A1.** The BPRE is subcritical, i.e.

$$E[X] < 0.$$  

Note that subcritical BPRE’s may be additionally split into several other classes with essentially different properties (see [5] and [10] for more detail). In particular, a subcritical BPRE is called strongly subcritical if $E[Xe^X] < 0$, intermediate subcritical if $E[Xe^X] = 0$, and weakly subcritical if there is a number $0 < \beta < 1$ such that

$$E[Xe^{\beta X}] = 0.$$  

One of the main tools in analyzing properties of BPRE and BPIRE is a change of measure. We follow this approach and introduce a new measure $P$ by setting, for any $n \in \mathbb{N}$ and any measurable bounded function $\psi : \Delta_n \times \mathbb{N}_n^{n+1} \to \mathbb{R}$

$$E[\psi(Q_1, \cdots, Q_n, W_0, \cdots, W_n)] := \gamma^{-n}E[\psi(Q_1, \cdots, Q_n, W_0, \cdots, W_n)e^{\delta S_n}],$$  

with

$$\gamma := E[e^{\delta X}].$$  

Here $\delta = 1$ for strongly and intermediate subcritical BPIRE and $\delta = \beta$ for weakly subcritical BPIRE.

Observe that $E[Xe^{\delta X}] = 0$ translates into

$$E[X] = 0.$$  

We assume that under the new measure the following set of conditions holds true.

**Hypothesis A2.** The distribution of $X$ is nonlattice. If a BPIRE is either intermediate or weakly subcritical then the distribution of $X$ has finite variance with respect to $P$ or (more generally) belongs to the domain of attraction of a two-sided stable law with index $\alpha \in (1, 2]$.

Since $E[X] = 0$, Hypothesis A2 provides existence of an increasing sequence of positive numbers

$$a_n = n^{1/\alpha}l_1(n)$$  

with slowly varying sequence $l_1(1), l_1(2), \ldots$ such that the distribution law of $S_n/a_n$ converges weakly, as $n \to \infty$ to the mentioned two-sided stable law.

Our next assumption concerns the standardized truncated second moment of $F$,

$$\vartheta(a) := \sum_{j=a}^{\infty} j^2 F(j)/m(F)^2, \quad a \in \mathbb{N}.$$  

Define $\log^+ x := \log(\max(1, x))$.

**Hypothesis A3.**
1) If the BPRE is intermediate subcritical, then
\[ E \left[ (\log^+ \vartheta(a))^{n+\epsilon} \right] < \infty \]
for some \( \epsilon > 0 \) and some \( a \in \mathbb{N} \).

2) If the BPRE is strongly subcritical, then
\[ E[\log^+ \vartheta(a)] < \infty \]
for some \( a \in \mathbb{N} \).

Now we introduce the last condition which imposes restrictions on the immigration component.

**Hypothesis A4.**
\[ E \left[ \frac{G'_0(1)}{1 - G_0(0)} \right] < \infty. \]

With Hypotheses A1-A4 in hands we are ready to formulate the main result of this note.

**Theorem 1.** Let Hypotheses A1-A4 be satisfied. Then, as \( n \to \infty \)

1) if the equation \( r\mathcal{H}(r) = 1 \) has a root \( 1 < r < \gamma^{-1} \), then
\[ P(\zeta > n) \sim \frac{r\mathcal{H}^*(r) + rR_1}{\mathcal{H}(r) + r\mathcal{H}'(r)} r^{-n-1}; \]

2) if the BPIRE is weakly subcritical and \( \gamma^{-1}\mathcal{H}(\gamma^{-1}) < 1 \), then,
\[ P(\zeta > n) \sim C \frac{s^n}{b_n}, \quad C \in (0, \infty) \]
with \( b_n := na_n \);

3) if the BPIRE is weakly subcritical and \( \gamma^{-1}\mathcal{H}(\gamma^{-1}) = 1 \), then
\[ P(\zeta > n) = o(\gamma^n). \]

**Remark 2.** We show below that, under our conditions, the equation \( r\mathcal{H}(r) = 1 \)
always has a root \( r \in (1, \gamma^{-1}) \) for strongly and intermediate subcritical BPIRE.

We note that Zubkov [18] considered a similar problem for a branching process with immigration \( \{Y_c(n), n \geq 0\} \) evolving in a constant environment with \( G(0) > 0 \) and investigated the distribution of the so-called life period \( \zeta_c \) of such a process initiated at time \( N \) and defined as
\[ Y_c(N - 1) = 0, \quad \min_{N \leq k < N + \zeta_c} Y_c(k) > 0, \quad Y_c(N + \zeta_c) = 0. \]

The same problem for other models of branching processes with immigration evolving in a constant environment was analysed, for instance, in [7], [13], [15] and [17].
Various properties of BPIRE were investigated by several authors (see, for instance, [1], [9], [11], [12], [14], [16]). Life periods of the critical BPIRE stopped at zero were considered in [6].

In the sequel if no otherwise is stated, we write $h_n \sim Ck_n, C > 0$ if $\lim_{n \to \infty} h_n/k_n = C$, $h_n = O(k_n)$ if $\lim \sup_{n \to \infty} h_n/k_n < \infty$, and $h_n = o(k_n)$ if $\lim_{n \to \infty} h_n/k_n = 0$. We also denote by $C, C_1, ...$ positive constants which may vary from place to place.

2 Auxiliary results

Our goal is to investigate the asymptotic properties of $H_n$ and $H_n^*$ and, having the asymptotics in hands, to find an asymptotic representation for $R_n$ as $n \to \infty$. Observing that
\[ H_n = \gamma_n + 1 \mathbb{E} \left[ (1 - G_0(F_{0,n+1}(0))) \prod_{i=1}^{n} G_i(F_{i,n+1}(0)) e^{-\delta S_{n+1}} \right] := \gamma_n H_n, \]
we reduce the first problem to considering the asymptotic behavior of $H_n^*$. Set
\[ M_n := \max (S_1, ..., S_n), L_n := \min (S_0, S_1, ..., S_n) \]
and denote
\[ \tau(n) := \min \{ i \geq 0 : S_i = L_n \}. \]

**Lemma 3** Let Hypotheses A1-A2 be satisfied. If the process is weakly subcritical, then for each $\varepsilon > 0$, there exists $p = p(\varepsilon)$ such that
\[ \mathbb{E}[(1 - F_{0,n}(0)) e^{-\beta S_n}; \tau(n) \in [p, n - p]] < \frac{\varepsilon}{b_n} \]
for all sufficiently large $n$.

**Proof.** Note that
\[ 1 - F_{0,n}(0) = \mathbb{P}(Z_n > 0 | \mathcal{E}) \leq \min_{0 \leq k \leq n} \mathbb{P}(Z_k > 0 | \mathcal{E}) \leq \min_{0 \leq k \leq n} \mathbb{E}[Z_k | \mathcal{E}] = e^{\min_{0 \leq k \leq n} S_k}. \]
Therefore, for each $p \in [1, n/2]$
\[ \mathbb{E}[(1 - F_{0,n}(0)) e^{-\beta S_n}; \tau(n) \in [p, n - p]] \leq \mathbb{E}[e^{-\beta S_n} \cdot e^{\min_{0 \leq k \leq n} S_k}; \tau(n) \in [p, n - p]]. \]
We fix $k \in [p, n - p]$, set $S_j^* := S_{k+j} - S_k$, $j = 0, 1, ..., n - k$ and denote $L_{n-k} := \min_{0 \leq j \leq n-k} S_j^*$. The duality property of random walks gives
\[ \mathbb{E}[e^{-\beta S_n} \cdot e^{S_k}; \tau(n) = k] = \mathbb{E}[e^{(1-\beta)S_k} \cdot e^{-\beta S_{n-k}}; \tau(k) = k; L_{n-k} \geq 0] = \mathbb{E}[e^{(1-\beta)S_k}; M_k < 0] \mathbb{E}[e^{-\beta S_{n-k}}; L_{n-k} \geq 0]. \quad (3) \]
According to Proposition 2.1 in [3], for each $\theta > 0$ there exist positive constants $K_i = K_i(\theta), i = 1, 2$ such that, as $n \to \infty$

$$
E[e^{\theta S_n}; M_n < 0] \sim \frac{K_1}{b_n}, \quad E[e^{-\theta S_n}; L_n \geq 0] \sim \frac{K_2}{b_n}. \tag{4}
$$

We know by (2) that $b_n$ is a regularly varying sequence. Therefore, for any $\varepsilon > 0$ there exists an integer number $p = p(\varepsilon)$ such that

$$
E[(1 - F_{0,n}(0)) e^{-\beta S_n}; \tau(n) \in [p, n - p] \leq C \sum_{k=p}^{n-p} \frac{1}{b_k} \frac{1}{b_{n-k}} \leq \frac{C_1}{b_n} \sum_{k=p}^{\infty} \frac{1}{b_k} \leq \frac{\varepsilon}{b_n}
$$

for all sufficiently large $n$.

The lemma is proved.

**Lemma 4** Let Hypotheses A1-A2 be satisfied. If the process is intermediate subcritical, then for each $\varepsilon > 0$, there exists $p = p(\varepsilon)$ such that, for all sufficiently large $n$

$$
E[(1 - F_{0,n}(0)) e^{-\beta S_n}; \tau(n) \in [0, n - p] < \frac{\varepsilon}{n^{1-\alpha}l_2(n)},
$$

where $l_2(1), l_2(2), ...$ is a sequence slowly varying at infinity.

**Proof.** It follows from Lemma 2.2 in [3] that, as $n \to \infty$

$$
\mathbb{P}[M_n < 0] \sim \frac{1}{n^{1-\alpha}l_2(n)}.
$$

Setting $\beta = 1$ in (3) and using the arguments of the preceding lemma we see that for any $\varepsilon > 0$

$$
E[(1 - F_{0,n}(0)) e^{-S_n}; \tau(n) \in [0, n - p] \leq K_3 \sum_{k=1}^{n-p} \frac{1}{k^{1-\alpha}l_2(k)} \frac{1}{b_{n-k}}
\leq \frac{CK_3}{b_n} \sum_{1 \leq k \leq n/2} \frac{1}{k^{1-\alpha}l_2(k)} + \frac{CK_3}{n^{1-\alpha}l_2(n)} \sum_{j=p}^{\infty} \frac{1}{b_j}
= O\left(\frac{1}{nl_1(n)l_2(n)}\right) + \frac{1}{n^{1-\alpha}l_2(n)} O\left(\frac{1}{p^{1/\alpha}l_1(p)}\right)
$$

completing the proof.

To go further we need to perform two more changes of measure using the right-continuous functions $U : \mathbb{R} \to [0, \infty)$ and $V : \mathbb{R} \to [0, \infty)$ specified by

$$
U(x) := 1 + \sum_{n=1}^{\infty} \mathbb{P}(S_n \geq -x, M_n < 0), \quad x \geq 0,
$$
\[ V(x) := 1 + \sum_{n=1}^{\infty} \mathbb{P}(S_n < -x, L_n \geq 0), \quad x \leq 0. \]

It is known (see, for instance, [2] and [3]) that for any oscillating random walk
\[
\mathbb{E}[U(x + X); X + x \geq 0] = U(x), \quad x \geq 0, \tag{5}
\]
\[
\mathbb{E}[V(x + X); X + x < 0] = V(x), \quad x \leq 0. \tag{6}
\]

Let \( \mathcal{E} = \{Q_1, Q_2, \ldots\} \) be a random environment and let \( \mathcal{F}_n, n \geq 1, \) be the \( \sigma \)-field of events generated by the random vectors \( Q_1, Q_2, \ldots, Q_n \) and the sequence \( W_0, W_1, \ldots, W_n. \) These \( \sigma \)-fields form a filtration \( \mathcal{F}. \) Now the increment \( X_n, n \geq 1, \) of the random walk \( S \) are measurable with respect to the \( \sigma \)-field \( \mathcal{F}_n. \) Using the martingale property (5)-(6) of \( U, V \) we introduce in now a standard way (see, for instance, [10], Chapter 7) a sequence of probability measures \( \{\mathbb{P}^+(n), n \geq 1\} \) on the \( \sigma \)-field \( \mathcal{F}_n \) by means of the densities
\[
d\mathbb{P}^+(n) := U(S_n)I\{L_n \geq 0\} d\mathbb{P}.
\]

This and Kolmogorov’s extension theorem show that, on a suitable probability space there exists a probability measure \( \mathbb{P}^+ \) on the \( \sigma \)-field \( \mathcal{F} \) such that
\[
\mathbb{P}^+|\mathcal{F}_n = \mathbb{P}^+(n), \quad n \geq 1. \tag{7}
\]

In the sequel we allow for arbitrary initial value \( S_0 = x. \) Then, we write \( \mathbb{P}_x \) and \( \mathbb{E}_x \) for the corresponding probability measures and expectations. Thus, \( \mathbb{P} = \mathbb{P}_0. \) Using this agreement we rewrite (7) as
\[
\mathbb{E}^+_x [O_n] := \frac{1}{U(x)} \mathbb{E}_x [O_n U(S_n); L_n \geq 0], \quad x \geq 0,
\]
for every \( \mathcal{F}_n \)-measurable random variable \( O_n. \)

Similarly, \( V \) gives rise to probability measures \( \mathbb{P}^-_x, x \leq 0, \) and
\[
\mathbb{E}^-_x [O_n] := \frac{1}{V(x)} \mathbb{E}_x [O_n V(S_n); M_n < 0], \quad x \leq 0.
\]

Now we come back to branching processes. To have a unified approach in studying the asymptotic behavior of \( H_n \) and \( H^*_n \) as \( n \to \infty \) we consider the sequence
\[
B_n(s) := \mathbb{E} \left[ (1 - B(F_{0,n}(s))) \prod_{i=1}^{n} G_i(F_{i,n}(s)) e^{-sS_n} \right], \quad n \geq 1,
\]
where \( B(s) \) is a (random) probability generating function which is independent of the sequence \( Q_n, n \geq 1, \) and satisfies the restriction
**Hypothesis A4*. \[
\mathbb{E}[B'(1)] < \infty.
\]
Taking \( B(s) = G_0(s) \) and \( s = F_{n+1}(0) \) leads to \( \gamma^{-n-1}H_n \), while \( B(s) = (G_0(s) - G_0(0))/(1 - G_0(0)) \) with the same \( s \) gives \( \gamma^{-n-1}H_n^* \).

Our plan is to find asymptotic representations of \( B_n(s) \) for all types of sub-critical BPIRE. To this aim we will use a decomposition

\[
B_n(s) = \sum_{k=0}^{n} B_{k,n}(s)
\]

where

\[
B_{k,n}(s) := \mathbb{E} \left[ (1 - B(F_{0,n}(s))) \prod_{i=1}^{n} G_i(F_{i,n}(s)) e^{-\delta S_n}; \tau(n) = k \right].
\]

### 2.1 Weakly subcritical case

In this subsection we prove the following statement.

**Theorem 5** Let Hypotheses A1-A2 and A4* be satisfied. If the process is weakly subcritical with parameter \( \beta \in (0,1) \), then for each \( s \in (0,1) \)

\[
B_n(s) \sim \frac{C_\beta(s)}{b_n}, C_\beta(s) > 0,
\]

as \( n \to \infty \).

The idea of proving Theorem 5 looks as follows. We show that, for a fixed \( k \) and \( n \to \infty \)

\[
B_{k,n}(s) \sim C_k(s) \mathbb{E} \left[ e^{-\beta S_n}; L_n \geq 0 \right], \quad B_{n-k,n}(s) \sim \tilde{C}_k(s) \mathbb{E} \left[ e^{(1-\beta)S_n}; \tau(n) = n \right]
\]

for some positive constants \( C_k(s) \) and \( \tilde{C}_k(s) \), while \( \sum_{k=p}^{n} B_{k,n}(s) \) is negligible in comparison with \( 1/b_n \) if \( p \) is sufficiently large.

The proof of the asymptotic representations above is based on several important statements established in [3]. To check the applicability of the statements we need to prove several preparatory lemmas.

Let \( Z(k,n) \) be the number of particles at moment \( n \) in a branching process initiated at time \( k \) by a single particle and \( Z_i(k,n) \), \( i = 1, 2, \ldots \) be independent probabilistic copies of \( Z(k,n) \).

Put

\[
Y(k,n) := Z_1(k-1,n) + \ldots + Z_{\eta_{k-1}}(k-1,n), \quad \Xi(n) := \sum_{k=1}^{n} Y(k,n) + \eta_n,
\]

where we assume (with a slight abuse of notation) that \( B(s) \) is the probability generating function of \( \eta_0 \).
From now on we let \( m := \lfloor n/2 \rfloor \), where \( x \) stand for the integer part of \( x \) and write
\[
B_n(s) = \mathbb{E} \left[ (1 - B(F_{0,n}(s))) \prod_{i=1}^{m} G_i(F_{i,n}(s)) \prod_{i=m+1}^{n} G_i(F_{i,n}(s)) e^{-\beta S_n} \right] 
\]
\[
= \mathbb{E} \left[ (1 - F_{m,n}^{(1,m)}(s)) F_{m,n}^{\Xi(m) - Y(1,m)}(s) \prod_{i=m+1}^{n} G_i(F_{i,n}(s)) e^{-\beta S_n} \right].
\]

Introduce two-dimensional random variables
\[
U_n = (U_{n1}, U_{n2}) := (e^{-S_m}Y(1,m), e^{-S_m}(\Xi(m) - Y(1,m))),
\]
\[
\tilde{V}_n(s) = \left( \tilde{V}_1(s), \tilde{V}_2(s) \right) := \left( (F_{m,n}(s))^{\exp(-S_n-S_m)}, \prod_{i=m+1}^{n} G_i(F_{i,n}(s)) \right),
\]
and
\[
V_n(s) = (V_{n1}(s), V_{n2}(s)) := \left( (F_{n-m,0}(s))^{\exp(-S_n-S_m)}, \prod_{j=1}^{n-m} G_j(F_{j-1,0}(s)) \right).
\]

**Lemma 6** If a BPIRE is weakly subcritical and Hypotheses A2 and A4* are valid then, for each \( x \geq 0 \)
\[
U_n \to U_\infty := (U_{\infty1}, U_{\infty2}) \quad \mathbb{P}_x^+ - \text{a.s.}
\]
as \( n \to \infty \), where \((U_{\infty1}, U_{\infty2})\) is a random vector whose components are positive with positive probabilities.

**Proof.** Since the measure \( \mathbb{P}_x^+ \) imposes restriction on the offspring probability laws of particles but not on the reproduction of particles themselves, one can check that the random sequences
\[
e^{-S_m}Y(1,m), \quad e^{-S_m}(\Xi(m) - Y(1,m)), \quad m = 1, 2, ...
\]
form, correspondingly, a non-negative martingale and a submartingale with respect to the filtration \( \mathfrak{F} \). Hence, there exists a random variable \( U_{\infty1} \) such that, as \( n \to \infty \)
\[
e^{-S_m}Y(1,m) \to U_{\infty1}, \quad \mathbb{P}_x^+ - \text{a.s.}
\]
Since,
\[
\mathbb{E}_x^+ [e^{-S_m}Y(1,m)] = e^{-x}\mathbb{E}[B'(1)] \in (0, \infty),
\]
the random variable \( U_{\infty1} \) is positive with a positive probability. Next, we claim that
\[
\sup_m \mathbb{E}_x^+ [e^{-S_m}\Xi(m)] < \infty.
\]


If we prove this statement, then we may conclude that, as $m \to \infty$

$$e^{-S_m}(\Xi(m) - Y(1, m)) \to \mathcal{U}_\infty^+ \text{ P}_x - a.s.$$ 

where the random variable $\mathcal{U}_\infty^+$ is positive with a positive probability in view of $e^{-S_m}(\Xi(m) - Y(1, m)) \geq e^{-S_m}Y(2, m)$.

To establish the desired estimate recall that according to our change of measure,

$$I_m := E_x^+ [e^{-S_m}(\Xi(m))] = \frac{1}{U(x)} E_x [e^{-S_m}(\Xi(m))U(S_m)I\{L_m \geq 0\}]$$

$$= \frac{1}{U(x)} \sum_{k=1}^{m} E_x [e^{-S_m}Y(k, m)U(S_m)I\{L_m \geq 0\}] + \frac{1}{U(x)} E_x [\eta_m e^{-S_m}U(S_m)I\{L_m \geq 0\}].$$

Conditioning first on the environment $E$ and then on $\eta_{k-1}, S_0, S_1, \ldots S_m$ and observing that, for any $k < m$

$$E_x [e^{-S_k}U(S_m)I\{L_m \geq 0\}] = E_x [e^{-S_k}U(S_k)I\{L_k \geq 0\}]$$

in view of (4), we obtain

$$I_m = \frac{1}{U(x)} \sum_{k=1}^{m} E_x [\eta_{k-1} e^{-S_k-1}U(S_m)I\{L_m \geq 0\}] + \frac{1}{U(x)} E_x [\eta_m e^{-S_m}U(S_m)I\{L_m \geq 0\}]$$

$$\leq \frac{E[\eta_k + \eta_0]}{U(x)} \sum_{k=0}^{m} E_x [e^{-S_k}U(S_m)I\{L_m \geq 0\}] = \frac{E[G'(1) + B'(1)]}{U(x)} \sum_{k=0}^{m} E_x [e^{-S_k}U(S_k)I\{L_k \geq 0\}].$$

Since $U(y)$ is a renewal function, there exists a constant $C$ such that $U(y) \leq C(1 + y)$ for all $y \geq 0$. Combining this estimate with the inequality

$$(1 + y)e^{-y} \leq 2e^{-y/2}, \quad y \geq 0,$$

we see that

$$I_m \leq \frac{C}{U(x)} \sum_{k=0}^{m} E_x [e^{-S_k}(1 + S_k)I\{L_k \geq 0\}] \leq \frac{2C}{U(x)} \sum_{k=0}^{m} E_x [e^{-S_k/2}I\{L_k \geq 0\}].$$

Recalling (4), it follows that, for all $m \in \mathbb{N}$

$$I_m \leq \frac{2C}{U(x)} \sum_{k=0}^{\infty} E_x [e^{-S_k/2}I\{L_k \geq 0\}] < \infty,$$

as desired.

The lemma is proved.

Denote

$$\Xi_z(n) = \sum_{j=1}^{z} Z_j(0, n),$$

where $Z_j(0, n)$ is the $j$-th zero of $\Xi_z(n)$.
and introduce the random vector
\[ \mathcal{U}_n(z) = (\mathcal{U}_n^1(z), \mathcal{U}_n^2) := \left( e^{-S_n} \Xi_z(m), e^{-S_n}(\Xi(m) - Y(1,m)) \right). \]

Setting \( B(s) = s^z \) we obtain the following statement.

**Corollary 7** Under the conditions of Lemma 2, for each \( z \in \mathbb{N} \) and \( x \geq 0 \)
\[ \mathcal{U}_n(z) \to \mathcal{U}_\infty(z) := (\mathcal{U}_\infty^1(z), \mathcal{U}_\infty^2) \quad \mathbb{P}_x^+ \text{ a.s.} \]
as \( n \to \infty \), where \((\mathcal{U}_\infty^1(z), \mathcal{U}_\infty^2)\) is a random vector whose components are positive with positive probabilities.

Now we deal with measure \( \mathbb{P}^- \).

**Lemma 8** If a BPIRE is weakly subcritical and Hypotheses A2 and A4* are valid then, for each fixed \( s \in (0,1) \) and \( x \leq 0 \)
\[ \mathcal{V}_n(s) = (\mathcal{V}_n^1(s), \mathcal{V}_n^2(s)) \to \mathcal{V}_\infty(s) := (\mathcal{V}_\infty^1(s), \mathcal{V}_\infty^2(s)) \quad \mathbb{P}_x^- \text{ a.s.} \]
as \( n \to \infty \), where \( \mathcal{V}_\infty^1(s) \) and \( \mathcal{V}_\infty^2(s) \) are proper positive random variables.

**Proof.** The fact that \( \mathcal{V}_n^1(s) \to \mathcal{V}_\infty^1(s) \quad \mathbb{P}_x^- \text{ a.s.} \) as \( n \to \infty \) is a particular case of Lemma 3.2 in [3]. To prove convergence of \( \mathcal{V}_n^2(s) \) note that given Hypotheses A2 and A4*
\[ \sum_{j=1}^{n-m} (1-G_j(F_{j-1,0}(s))) \leq \sum_{j=1}^{\infty} G_j'(1)(1-F_{j-1,0}(0)) \leq \sum_{j=1}^{\infty} G_j'(1)e^{S_j-1} < \infty \quad \mathbb{P}_x^- \text{ a.s.} \]
for every \( s \in [0,1] \). Hence, for each \( s > 0 \)
\[ \prod_{j=1}^{n-m} G_j(F_{j-1,0}(s)) \to \mathcal{V}_\infty^2(s) := \prod_{j=1}^{\infty} G_j(F_{j-1,0}(s)) > 0 \quad \mathbb{P}_x^- \text{ a.s.} \]
The lemma is proved.

For \( u_i \geq 0, 0 \leq u_i \leq 1, t \geq 0 \) (\( i = 1, 2 \)) introduce the function
\[ \varphi(u, v, t) = \varphi((u_1, u_2), (v_1, v_2), t) := v_1^{(u_1+u_2)e^t} v_2. \]
One may check that \( \varphi \) is bounded and continuous within the specified range of variables. For \( z \in \mathbb{N} \), let
\[ J_\nu(s; z) := \int_{t \in (-\infty,0]} \int_{u \in \mathbb{R}^2} \int_{v \in \mathbb{R}^2} \varphi(u, v, -t) \mathbb{P}^+(\mathcal{U}_\infty(z) \in du) \mathbb{P}^-(\mathcal{V}_\infty(s) \in dv) \nu_\beta(dt), \]
where
\[ \nu_\beta(dt) := K_1 e^{\beta t} V(t) I\{t < 0\} dt \]
with scaling constant
\[ K_1^{-1} := \int e^{\beta t} V(t) I\{t < 0\} dt. \]
Lemma 9 If a BPIRE is weakly subcritical and Hypotheses A2 and A4* are valid then, for each \( z \in \mathbb{N} \)

\[
\lim_{n \to \infty} \frac{\mathbb{E} \left[ F_{0,n}(s)^n \prod_{i=1}^{n} G_i(F_{i,n}(s)) e^{-\beta S_n}; L_n \geq 0 \right]}{\mathbb{E} \left[ e^{-\beta S_n}; L_n \geq 0 \right]} = J_{\nu}(s; z).
\]

Proof. We write

\[
\mathbb{E} \left[ F_{0,n}(s)^n \prod_{i=1}^{n} G_i(F_{i,n}(s)) e^{-\beta S_n}; L_n \geq 0 \right] = \mathbb{E} \left[ \varphi(U_n(z), \tilde{V}_n(s), S_n) e^{-\beta S_n}; L_n \geq 0 \right],
\]

where \( \tilde{V}_n(s) \) is the same as in (8) and using Theorem 2.7 in [3] we complete the proof of the lemma.

For \( a > 0 \) and \( u_i \geq 0, 0 \leq v_i \leq 1, z \leq 0 \) \((i = 1, 2)\) let

\[
\phi_a(u,v,z) = \phi_a((u_1, u_2), (v_1, v_2), z) := \left( 1 - v_1 u_1 e^z \right) v_1 u_2 e^z I\{z \geq -a\}.
\]

Clearly, \( \phi_a \) is bounded and continuous in the specified domain. By means of \( \phi_a \) we specify, for \( s \in [0,1] \) the function

\[
J_{\mu}(s; a) := \int_{t \in [0, \infty]} \int_{u \in \mathbb{R}^2} \int_{v \in \mathbb{R}^2} \phi_a(u,v,-t) P^+_t(U_{\infty} \in du) P^-_t(V_{\infty} \in dv) \mu_{(1-\beta)}(dt),
\]

where

\[
\mu_{(1-\beta)}(dt) := K_2 e^{-(1-\beta)t} U(t) I\{t \geq 0\} dt
\]

with scaling constant

\[
K_2^{-1} := \int e^{-(1-\beta)t} U(t) I\{t \geq 0\} dt.
\]

Lemma 10 If a BPIRE is weakly subcritical and Hypotheses A2 and A4* are valid then, for each \( s \in [0,1] \)

\[
\lim_{n \to \infty} \frac{B_{a,n}(s)}{e^{(1-\beta)S_n}; \tau(n) = n} = J_{\mu}(s; \infty).
\]
Proof. We write

\[
B_{n,n}(s) = \mathbb{E} \left[ \frac{1 - B(F_{0,n}(s))}{e^{S_n}} \prod_{i=1}^{n} G_i(F_{i,n}(s)) e^{(1-\beta)S_n}; \tau(n) = n \right]
\]

\[
= \mathbb{E} \left[ \frac{1 - B(F_{0,n}(s))}{e^{S_n}} \prod_{i=1}^{n} G_i(F_{i,n}(s)) e^{(1-\beta)S_n} I_{\{S_n < -a\}}; \tau(n) = n \right]
\]

\[
+ \mathbb{E} \left[ \phi_a(U_n, \bar{V}_n(s), S_n) e^{(1-\beta)S_n}; \tau(n) = n \right] := g(n; s; 1) + g(n; s; 2).
\]

According to (4),

\[
g(n; s; 1) \leq \mathbb{E} \left[ B'(1) \frac{1 - F_{0,n}(s)}{e^{S_n}} e^{(1-\beta)S_n} I_{\{S_n < -a\}}; \tau(n) = n \right]
\]

\[
\leq \mathbb{E}[B'(1)] \mathbb{E} \left[ e^{(1-\beta)S_n} I_{\{S_n < -a\}}; \tau(n) = n \right]
\]

\[
\leq \mathbb{E}[B'(1)] e^{-(1-\beta)\frac{a}{2}} \mathbb{E} \left[ e^{(1-\beta)S_n}; M_n < 0 \right] \leq C \mathbb{E}[B'(1)] e^{-(1-\beta)\frac{a}{2}/b_n}
\]

for all \( n \in \mathbb{N} \) and \( a > 0 \).

Further, we know from Lemmas 6–8 that the conclusion of Theorem 2.8 in [3] holds for \( \phi_a(u, v, z) \), i.e.

\[
\lim_{n \to \infty} \mathbb{E} \left[ \frac{\phi_a(U_n, \bar{V}_n(s), S_n) e^{(1-\beta)S_n}; \tau(n) = n}{\mathbb{E} \left[ e^{(1-\beta)S_n}; \tau(n) = n \right]} \right] = J_\mu(s; a).
\]

Hence, letting \( a \) to infinity we prove the lemma.

Proof of Theorem 5 Since

\[
(1 - B(F_{0,n}(s))) \prod_{i=1}^{n} G_i(F_{i,n}(s)) \leq B'(1)(1 - F_{0,n}(s)),
\]

it follows from Lemma 8 that for any \( \varepsilon > 0 \)

\[
\sum_{k=p}^{n-p} B_{k,n}(s) \leq \mathbb{E}[B'(1)] \mathbb{E} \left[ (1 - F_{0,n}(s)) e^{-\beta S_n}; \tau(n) \in [p, n - p] \right] \leq \frac{\varepsilon \mathbb{E}[B'(1)]}{b_n}
\]

for all sufficiently large \( n \) and \( p = p(\varepsilon) \).

Further, for fixed \( k \leq p \), we take the expectation with respect to the \( \sigma \)-algebra \( F_k \) and obtain

\[
\mathbb{E} \left[ (1 - B(F_{0,n}(s))) \prod_{i=1}^{n} G_i(F_{i,n}(s)) e^{-\beta S_n}; \tau(n) = k \right]
\]

\[
= \mathbb{E} \left[ e^{-\beta S_n} \Theta_1(n-k; Y(1,k), \Xi(k)) I \{\tau(k) = k\} \right],
\]
where

$$\Theta_1(n; z_1, z_2) := E \left[ (1 - F_{0,n}(s)) F_{0,n}^{s-z_1}(s) \prod_{i=1}^{n} G_i(F_{i,n}(s)) e^{-\beta S_n}; L_n \geq 0 \right]$$

$$= E \left[ F_{0,n}^{s-z_1}(s) \prod_{i=1}^{n} G_i(F_{i,n}(s)) e^{-\beta S_n}; L_n \geq 0 \right] - E \left[ F_{0,n}^{s}(s) \prod_{i=1}^{n} G_i(F_{i,n}(s)) e^{-\beta S_n}; L_n \geq 0 \right].$$

Using Lemma 9, applying the dominated convergence theorem and recalling (4), we conclude that

$$\lim_{n \to \infty} B_{k,n}(s) = E \left[ e^{-\beta S_k} \prod_{i=1}^{j} G_i(F_{i,j}(s)) I\{L_j \geq 0\} \right].$$

Finally, we fix $j \geq 0$ and consider the expectation

$$B_{n-j,n}(s) = E \left[ (1 - B(F_{0,n}(s))) \prod_{i=1}^{n} G_i(F_{i,n}(s)) e^{-\beta S_n}; \tau(n) = n - j \right].$$

Denote $S_k := S_{n-j+k} - S_{n-j}$ ($k = 1, \ldots, j$) and let $\mathcal{F}_j$ be the $\sigma$-algebra generated by $Q_{n-j+1}, \ldots, Q_n$. Taking the internal expectation with respect to $\mathcal{F}_j$ and supplying the respective variables with bars $-$ we see that

$$B_{n-j,n}(s) = E \left[ e^{-\beta S_j} B_{n-j,n-j}(\mathcal{F}_{0,j}(s)) \prod_{i=1}^{j} G_i(F_{i,j}(s)) I\{L_j \geq 0\} \right].$$

Using Lemma 10 and the dominated convergence theorem we conclude that

$$\lim_{n \to \infty} E \left[ e^{-\beta S_j} B_{n-j,n-j}(\mathcal{F}_{0,j}(s)) \prod_{i=1}^{j} G_i(F_{i,j}(s)) I\{L_j \geq 0\} \right] = E \left[ e^{-\beta S_j} J_\mu(\mathcal{F}_{0,j}(s); \infty) \prod_{i=1}^{j} G_i(F_{i,j}(s)) I\{L_j \geq 0\} \right].$$

Combining (9)-(10) with Proposition 2.1 in [3], we complete the proof.

### 2.2 Intermediate and strongly subcritical cases

In this subsection we find the asymptotics of $B_n(s)$ for intermediate and strongly subcritical BPIRE.

**Theorem 11** Let Hypotheses A1-A3 and A4* be satisfied. If the process is intermediate subcritical then, as $n \to \infty$

$$B_n(s) \sim \frac{C}{n^{1+1-\alpha} I_2(n)}.$$
Proof. Recalling that
\[ B_{k,n}(s) = \mathbb{E} \left[ (1 - B(F_{0,n}(s))) \prod_{i=1}^{n} G_i(F_{i,n}(s)) e^{-S_n}; \tau(n) = k \right] \]
we have, for fixed \( j \geq 0 \)
\[ B_{n-j,n}(s) = \mathbb{E} \left[ e^{-S_j} B_{n-j,n-j}(F_{0,j}(s)) \prod_{i=1}^{j} G_i(F_{i,j}(s)) I\{L_j \geq 0\} \right]. \]

Using the duality property of random walks we see that
\[ B_{n,n}(s) = \mathbb{E} \left[ (1 - B(F_{n,0}(s))) \prod_{i=0}^{n-1} \hat{G}_i(F_{i,0}(s)) e^{-S_n}; M_n < 0 \right], \]
where \( \hat{G}_i \) are independent probabilistic copies of \( G_i \).

Then, for any \( \kappa > 0 \),
\[ B_{n,n}(s) = \mathbb{E} \left[ (1 - B(F_{n,0}(s))) \prod_{i=0}^{n-1} \hat{G}_i(F_{i,0}(s)) e^{-S_n} I\{B'(1) \leq \kappa\}; M_n < 0 \right] + \mathbb{E} \left[ (1 - B(F_{n,0}(s))) \prod_{i=0}^{n-1} \hat{G}_i(F_{i,0}(s)) e^{-S_n} I\{B'(1) > \kappa\}; M_n < 0 \right] := h(n; s; 1) + h(n; s; 2). \]

First observe that, as \( n \to \infty \)
\[ \frac{1 - B(F_{n,0}(s))}{1 - F_{n,0}(s)} \to B'(1) \quad \mathbb{P} - a.s. \]
and, by monotonicity and Hypothesis A3
\[ \frac{1 - F_{n,0}(s)}{e^{S_n}} \geq \left( \frac{1}{1 - s} + \sum_{j=1}^{\infty} \vartheta_j(1)e^{S_j} \right)^{-1} > 0 \quad \mathbb{P} - a.s. \]
where \( \vartheta_j, j = 1, 2, \ldots \) are i.i.d. copies of \( \vartheta \). Hence, there exists a positive random variable \( \Theta(s) \) such that
\[ \frac{1 - F_{n,0}(s)}{e^{S_n}} \to \Theta(s) \quad \mathbb{P} - a.s. \]

Using the arguments similar to those applied to prove Lemma \( \mathbb{Q} \), we conclude that
\[ \xi_n(s) := \prod_{i=0}^{n-1} \hat{G}_i(F_{i,0}(s)) \to \xi_\infty(s) := \prod_{i=0}^{\infty} \hat{G}_i(F_{i,0}(s)) > 0 \quad \mathbb{P} - a.s. \]
Moreover,

\[
(1 - B(F_{n,0}(s))) \prod_{i=0}^{n-1} \hat{G}_i(F_{i,0}(s)) e^{-S_n} \leq \frac{1 - B(F_{n,0}(s)) - F_{n,0}(s)}{1 - F_{n,0}(s)} e^{S_n} \leq B'(1).
\]

Hence it follows that

\[
\frac{h(n; s; 2)}{\mathbb{P}[M_n < 0]} \leq \frac{\mathbb{E}[B'(1)I\{B'(1) > \kappa\}; M_n < 0]}{\mathbb{P}[M_n < 0]} = \mathbb{E}[B'(1)I\{B'(1) > \kappa\}]
\]

for all \( n \) and, according to Lemma 2.5 in [2], as \( n \to \infty \)

\[
\frac{h(n; s; 1)}{\mathbb{P}[M_n < 0]} \to \mathbb{E}^- [\Theta(s)B'(1)I\{B'(1) \leq \kappa\} \xi_\infty(s)].
\]

By these estimates and the arguments similar to those applied to check the validity of Lemma 10 we conclude that, as \( n \to \infty \)

\[
\frac{B_{n,n}(s)}{\mathbb{P}[M_n < 0]} \to \mathbb{E}^- [B'(1)\Theta(s)\xi_\infty(s)].
\]

Combining this result with Lemma 4 we complete the proof.

**Theorem 12** Let Hypotheses A1-A3 and A4* be satisfied. If the BPIRE is strongly subcritical then, for each \( s \in (0, 1) \)

\[
B_n(s) \sim C(s) > 0
\]

as \( n \to \infty \).

**Proof.** The proof is based on the transformed measure \( \mathbb{P} \). In this case the inequality \( \mathbb{E}[X e^X] < 0 \) translates into

\[
\mathbb{E}[X] < 0.
\]

Hence, the process is still subcritical under the probability measure \( \mathbb{P} \).

This fact, the equality

\[
\mathbb{E} \left[ (1 - B(F_{0,n}(s))) \prod_{i=1}^{n} G_i(F_{i,n}(s)) e^{-S_n} \right] = \mathbb{E} \left[ (1 - B(F_{n,0}(s))) \prod_{i=0}^{n-1} \hat{G}_i(F_{i,0}(s)) e^{-S_n} \right],
\]

the estimates

\[
(1 - B(F_{n,0}(s))) \prod_{i=0}^{n-1} \hat{G}_i(F_{i,0}(s)) e^{-S_n} \leq \frac{1 - B(F_{n,0}(s)) - F_{n,0}(s)}{1 - F_{n,0}(s)} e^{S_n} \leq B'(1)
\]

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and convergence

\[
(1 - B(F_{n,0}(s))) \prod_{i=0}^{n-1} \tilde{G}_i(F_{i,0}(s)) e^{-S_n} \to B'(1)\Theta(s)\xi_\infty(s) \quad P - a.s.
\]

as \( n \to \infty \) allow us to apply the dominated convergence theorem to conclude that

\[
\lim_{n \to \infty} B_n(s) = \mathbb{E}[B'(1)\Theta(s)\xi_\infty(s)] > 0.
\]

The theorem is proved.

\section{Proof of Theorem \ref{thm:main}}

Our proof of Theorem \ref{thm:main} essentially uses the following technical lemma.

\begin{lemma} \label{lem:technical}\hspace{1em}
(see Theorem 1.4.6 in \cite{19}) Let

\[
T(s) = \sum_{n=0}^{\infty} T_n s^n
\]

be a function with \( T_n \geq 0 \) for all \( n \). Assume that there exist a number \( \varrho > 1 \) and a function \( l_0(n) \) slowly varying at infinity such that

\[
T_n \sim \frac{l_0(n)}{n^\varrho}
\]

as \( n \to \infty \). If \( C(t) \) is an analytical function in a domain containing the circle

\[
|t| \leq T(1) = \sum_{n=0}^{\infty} T_n,
\]

then

\[
C(T(s)) = \sum_{n=0}^{\infty} c_n s^n, \quad \sum_{n=0}^{\infty} |c_n| < \infty
\]

and

\[
c_n \sim C'(T(1)) T_n.
\]

Now everything is ready for proving Theorem \ref{thm:main}. We know that

\[
\mathcal{R}(s) = \frac{sH^*(s) + sR_1}{1 - sH(s)}.
\]

Note that according to Theorems \ref{thm:conv1} \ref{thm:conv2} \ref{thm:conv3} and the change of measure \ref{thm:change} there is a positive constant \( C(\delta) \) such that

\[
\lim_{n \to \infty} \frac{H_n^*}{H_n} = C(\delta). \quad (11)
\]
Besides, for $\gamma = E \left[ e^{AX} \right]$

$$H_n \sim \begin{cases} 
\frac{C_{\alpha}}{b_n} \gamma^{n+1} & \text{if the process is weakly subcritical}, \\
\frac{C\gamma^{n+1}}{n^{1-\alpha^{-1}I_4(n)}} & \text{if the process is intermediate subcritical}, \\
C\gamma^{n+1} & \text{if the process is strongly subcritical}.
\end{cases} \quad (12)$$

**Proof of Point 1)** Note that $H(\gamma^{-1}) = \infty$ for the strongly and intermediate subcritical cases. Hence a solution of the equation $rH(r) = 1$ within the interval $(1, \gamma^{-1})$ always exists for these cases and $H^*(r) < \infty$. The same is true if $\gamma^{-1}H(\gamma^{-1}) > 1$ for the weakly subcritical case. Taking these facts into account and recalling point 3) of Theorem 1 in ([8], XIII.10) we conclude that under the conditions of point 1) of Theorem 1, as $n \to \infty$

$$P(\zeta > n) \sim \frac{rH^*(r) + rR_1}{H(r) + rH'(r)} r^{-n-1}.$$ 

**Proof of Point 2)** Setting

$$T(s) := \frac{s\gamma}{\gamma} H\left(\frac{s}{\gamma}\right) := \sum_{n=1}^{\infty} T_n s^n,$$

we see that

$$T_n \sim \frac{C\beta}{b_n}.$$ 

If $T(1) = \gamma^{-1}H(\gamma^{-1}) < 1$ then, taking

$$C(t) := \frac{1}{1-t}$$

in Lemma 13 and writing

$$C(T(s)) = \frac{1}{1-T(s)} = \sum_{n=0}^{\infty} c_n s^n,$$

we conclude that

$$c_n \sim C'(T(1)) T_n \sim \frac{C\beta}{(1-\gamma^{-1}H(\gamma^{-1}))^2 b_n} \frac{1}{b_n}.$$ 

Observing that

$$R\left(\frac{s}{\gamma}\right) = \frac{sH^*(\frac{s}{\gamma}) + \frac{s}{\gamma}R_1}{1 - \frac{s}{\gamma}H\left(\frac{s}{\gamma}\right)} = \left(\frac{sH^*(\frac{s}{\gamma}) + \frac{s}{\gamma}R_1}{1 - \frac{s}{\gamma}H\left(\frac{s}{\gamma}\right)}\right) C(T(s)).$$
and using (11)–(12) we deduce, after evident estimates that

\[
\frac{R_n}{\gamma^n} = \sum_{k=1}^{n-1} \frac{H^*_k}{\gamma^k} c_{n-k-1} + \frac{R_1}{\gamma} c_{n-1} \sim \frac{C}{b_n}
\]

as \( n \to \infty \) as desired.

**Proof of Point 3)** Assume that \( \mathcal{T}(1) = \gamma^{-1} \mathcal{H}(\gamma^{-1}) = 1 \). Then

\[
\mathcal{R}(s) := R \left( \frac{s}{\gamma} \right) = \frac{\frac{s}{\gamma} H^*_1(\frac{s}{\gamma}) + \frac{s}{\gamma} R_1}{1 - \mathcal{T}(s)} := \frac{\mathcal{G}(s)}{1 - \mathcal{T}(s)}.
\]

By (11) and (12)

\[
\mathcal{G}(1) < \infty, \quad \mathcal{T}(1) = 1, \quad \mathcal{T}'(1) = \infty.
\]

Hence, applying to the recurrent sequence \( \{\gamma^{-n} R_n, n \geq 1\} \) point 2) of Theorem 1 in ([8], XIII.10) we conclude that

\[
\lim_{n \to \infty} R_n = \frac{\mathcal{G}(1)}{\mathcal{T}'(1)} = 0.
\]

Theorem is proved.

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