Enclosure and non-existence theorems for area stationary currents and currents with mean curvature vector

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Abstract. We discuss certain geometric properties for area stationary currents and currents with integrable mean curvature, so called “enclosure theorems”. As a consequence, we obtain non-existence results for currents with connected support. Finally, we extend these results to currents in submanifolds and state a non-existence result for stationary currents in spheres.

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1. Introduction. Let \( \Omega \subset \mathbb{R}^2 \) denote a domain and let \( X: \Omega \to \mathbb{R}^3 \) be a minimal surface, i.e. a harmonic and conformal mapping of class \( C^2 \). For detailed information on minimal surfaces, we refer to the monographs of Dierkes, Hildebrandt, Sauvigny, and Tromba [4] and [5].

It is well-known that \( X(\bar{\Omega}) \) is contained in the convex hull of its boundary components \( X(\partial \Omega) \). This result holds true for every harmonic mapping due to the maximum principle.

Together with the conformality conditions, Hildebrandt [10] obtained stronger results. In fact, he proved that a minimal surface \( X(\bar{\Omega}) \) is enclosed by the hyperboloid \( H(R) := \{(x, y, z) \in \mathbb{R}^3: x^2 + y^2 - z^2 \leq R \} \) for all \( R \in \mathbb{R} \) provided \( X(\partial \Omega) \subset H(R) \). For \( R = 0 \), one gets a cone and the theorem still holds true. Furthermore one has that the minimal surface cannot pass through the vertex of the cone. Hildebrandt [10] also stated analogous results for two dimensional \( H \)-surfaces.

Again, by using the maximum principle for elliptic equations, Dierkes [3] and Dierkes and Schwab [6] extended these results to compact \( n \)-dimensional \( C^2 \)-submanifolds \( M \) in \( \mathbb{R}^{n+k} \). By regularity, these results immediately lead
to non-existence theorems. In fact, there are no smooth connected minimal submanifolds with boundary components in both disjoint parts of a special cone.

The question arises whether this cone can be enlarged or not. For two dimensional minimal surfaces, this was answered by Osserman and Schiffer [13]. They give the optimal “non-existence” cone. Dierkes [3] proved the corresponding theorem for $n$-dimensional smooth submanifolds in $\mathbb{R}^{n+1}$. See also the monograph Dierkes et al. [5, Ch. 4] for a complete survey of these results.

We mention that there are more general results of this type in various situations, cf. [1, 9, 19], and [2].

Here we want to address the following question: Do these theorems also generalize to currents? We show in the sequel that this is basically the case. The classical maximum principle will be replaced by a maximum principle of Solomon and White [16]. Maximum principles for singular surfaces in general situations were studied by different authors in the last years. We mention [11, 14], and recently [18] for codimension 1 and [17] where certain varieties of arbitrary codimension are considered.

We use the notation of Simon [15]. Let $n \geq 2, k \geq 1$ be natural numbers and $U \subset \mathbb{R}^{n+k}$ be an open set. We write $T = \tau(M, \theta, \xi) \in \mathcal{R}_n(U)$ for the set of $n$-dimensional rectifiable currents in $U$. Our notation slightly differs from most authors as we do not require integer multiplicity $\theta$. As usual, we define the associated Radon measure $\mu_T := H^n \cdot \theta$ and $\|\delta T\|$ is the total variation measure. A current with mean curvature $H = -D\mu_T \parallel \delta T\parallel \sigma$ is given by

**Definition 1.1.** Let $T = \tau(M, \theta, \xi) \in \mathcal{R}_n(U)$ be a current and $H \in L^1_{\text{loc}}(M \cap U, \mathbb{R}^{n+k}; \mu_T)$. Then $T$ has mean curvature vector $H$ in $U$ if

$$ \int_U \text{div}_M X \, d\mu_T = -\int_U \langle H, X \rangle \, d\mu_T $$

whenever $X \in C^1_c(U \setminus \text{spt} \partial T, \mathbb{R}^{n+k})$.

An important special case is $H(x) \equiv 0$ leading to the definition of “stationary currents”.

2. Stationary currents.

2.1. Enclosure theorem. The following convex hull property for stationary currents is well-known, cf. [15, Thm. 19.2, Rmk. 34.2].

**Theorem 2.1.** Let $T \in \mathcal{R}_n(\mathbb{R}^{n+k})$ be a stationary current in $\mathbb{R}^{n+k}$ with compact support. Then $\text{spt} \, T \subset \text{conv}(\text{spt} \, \partial T)$.
**Theorem 2.2** (Enclosure theorem). Let $\mathcal{H}_j(R)$ be a generalized hyperboloid for $R \in \mathbb{R}$ and $j = 1, \ldots, n - 1$ congruent to $H_j(R)$. Let $T \in \mathcal{R}_n(\mathbb{R}^{n+k})$ be a stationary current in $\mathbb{R}^{n+k}$ with compact support $\text{spt} \, T$ and let the boundary values satisfy $\text{spt} \, \partial T \subset \mathcal{H}_j(R)$. Then we have $\text{spt} \, T \subset \mathcal{H}_j(R)$.

**Proof.** W.l.o.g. assume $\mathcal{H}_j(R) = H_j(R)$. Let $\epsilon > 0$ be arbitrary and $\gamma \in C^1(\mathbb{R})$ be non-negative and non-decreasing with $\gamma(t) \equiv 0$, $t \leq R + \epsilon$, and $\gamma(t) > 0$, $\gamma'(t) > 0$ for $t > R + \epsilon$. Define $\hat{x} : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ by

$$\hat{x}(x) := \left( x_1, \ldots, x_{n+k-j}, -\frac{n-j}{j} x_{n+k-j+1}, \ldots, -\frac{n-j}{j} x_{n+k} \right)$$

and consider $X(x) := \Psi_{\text{spt} \, T}(x) \gamma(q_j(x)) \hat{x}(x) \in C^1_c(\mathbb{R}^{n+k} \setminus H_j(R), \mathbb{R}^{n+k})$, where $\Psi_{\text{spt} \, T}(x) \equiv 1$ in a neighborhood of $\text{spt} \, T$ is a smooth cut-off function.

Let $T_xM$ denote the approximate tangent space of $T$ in $x \in M$ (which exists $\mathcal{H}^n$-a.e.) and $\mathcal{P}_{T_xM} : \mathbb{R}^{n+k} \to T_xM$ the orthogonal projection with matrix representation $(p_{ij})_{i,j=1,\ldots,n+k}$ w.r.t. the canonical basis of $\mathbb{R}^{n+k}$. We often abbreviate the projection by $(\cdot)^\top$.

For this vector field $X$, we have $\text{div}_M X = \langle \nabla_M \gamma, \hat{x} \rangle + \gamma \text{div}_M \hat{x}$ and we calculate the different expressions. Firstly, $\nabla_M \gamma(q_j) = \gamma'(q_j) (Dq_j)^\top = 2\gamma'(q_j) \hat{x}^\top$ which gives $\langle \nabla_M \gamma(q_j), \hat{x} \rangle = 2\gamma'(q_j) \langle \hat{x}^\top, \hat{x} \rangle = 2\gamma'(q_j) |\hat{x}|^2 \geq 0$ and secondly,

$$\text{div}_M \hat{x} = \sum_{i=1}^{n+k} \langle \nabla_M \hat{x}_i, e_i \rangle$$

$$= \sum_{i=1}^{n+k-j} \langle \nabla_M x_i, e_i \rangle - \frac{n-j}{j} \sum_{i=n+k-j+1}^{n+k} \langle \nabla_M x_i, e_i \rangle$$

$$= \sum_{i=1}^{n+k} p_{ii} - \sum_{i=n+k-j+1}^{n+k} p_{ii} - \frac{n-j}{j} \sum_{i=n+k-j+1}^{n+k} p_{ii}$$

$$= n - \frac{n}{j} \sum_{i=n+k-j+1}^{n+k} p_{ii} \geq 0.$$

We have used $\text{tr}(\mathcal{P}_{T_xM}) = n$ in the last equation. Plugging these into (1.1) yields

$$0 = \int_{\mathbb{R}^{n+k}} \gamma(q_j) \left( n - \frac{n}{j} \sum_{i=n+k-j+1}^{n+k} p_{ii} \right) + 2 \gamma'(q_j) |\hat{x}|^2 \, d\mu_T.$$
Otherwise it is strictly positive. Define the set \( E := \{ x \in M : T_x M \text{ exists and } e_{n+k-j+1}, \ldots, e_{n+k} \in T_x M \} \). We have on the one hand,

\[
\int_{\mathbb{R}^{n+k} \setminus E} \gamma(q_j) \left\{ n - \frac{n}{j} \sum_{i=n+k-j+1}^{n+k} p_{ii} \right\} \, d\mu_T = 0
\]

where the expression \( \{ \ldots \} \) is positive and due to the definition of \( \gamma \), this means \( \text{spt} \, \mu_T \cap (\mathbb{R}^{n+k} \setminus E) \subset \{ x \in \mathbb{R}^{n+k} : q_j(x) \leq R + \varepsilon \} = H_j(R + \varepsilon) \). On the other hand, notice that \( \{ \ldots \} = 0 \) in \( E \), thus

\[
0 = \int_E \gamma(q_j) \left\{ n - \frac{n}{j} \sum_{i=n+k-j+1}^{n+k} p_{ii} \right\} + \gamma'(q_j) |\hat{x}^T|^2 \, d\mu_T = \int_E \gamma'(q_j) |\hat{x}^T|^2 \, d\mu_T.
\]

Now notice that \( |\hat{x}^T|^2 = 0 \) iff \( \hat{x} \perp T_x M \), this means in particular \( x_{n+k-j+1} = \cdots = x_{n+k} = 0 \) in \( E \). In other words, if \( x_{n+k-j+i} \neq 0 \) for (at least) some \( l = 1, \ldots, j \), then \( |\hat{x}^T|^2 > 0 \) in \( E \). Again, because of \( \gamma \), this establishes

\[
\text{spt} \, \mu_T \cap (E \cap \{ x \in \mathbb{R}^{n+k} : x_{n+k-j+l} \neq 0 \text{ for one } l = 1, \ldots, j \}) \subset H_j(R + \varepsilon)
\]

and combining both results, we arrive at

\[
\text{spt} \, \mu_T \subset H_j(R + \varepsilon) \cup (E \cap \{ x \in \mathbb{R}^{n+k} : x_{n+k-j+1} = \ldots, x_{n+k} = 0 \}). \tag{2.1}
\]

Finally, we claim that \( E \cap \{ x \in \mathbb{R}^{n+k} : x_{n+k-j+1} = \ldots, x_{n+k} = 0 \} \subset H_j(R + \varepsilon) \). Assume there exists \( \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_{n+k}) \in \text{spt} \, \mu_T \) such that \( \tilde{x} \in E \) and \( \tilde{x}_{n+k-j+1} = \cdots = \tilde{x}_{n+k} = 0 \) but \( \tilde{x} \not\in H_j(R + \varepsilon) \). Choose a function \( f \in C^0_c(\mathbb{R}^{n+k}) \) which satisfies \( f \equiv 0 \) in \( \{ x \in \mathbb{R}^{n+k} : x_{n+k-j+1} = \ldots, x_{n+k} = 0 \} \) and is strict positive anywhere else in some sufficiently large ball. Then, since \( \tilde{x} \) is off the closed hyperboloid together with the fact (2.1), we have that in a small neighborhood points of \( \text{spt} \, T \) fulfill \( x_{n+k-j+1} = 0, \ldots, x_{n+k} = 0 \). By definition of \( f \), this means in particular

\[
\lim_{\lambda \searrow 0} \int_{\eta_{\tilde{x}, \lambda}(M)} f(y) \theta(\tilde{x} + \lambda y) \, dH^n(y) = 0.
\]

Notice that, because of \( \tilde{x} \in E \), we have \( e_{n+k-j+1}, \ldots, e_{n+k} \in T_{\tilde{x}} M \). In other words, \( T_{\tilde{x}} M \not\subset \{ x \in \mathbb{R}^{n+k} : x_{n+k-j+1} = 0, \ldots, x_{n+k} = 0 \} \).

This leads to a contradiction to the definition of the approximate tangent space \( \theta(\tilde{x}) \int_{T_{\tilde{x}} M} f(y) \, dH^n(y) > 0 \). This gives \( \text{spt} \, T = \text{spt} \, \mu_T \subset H_j(R + \varepsilon) \) and since \( \varepsilon > 0 \) was arbitrary, the theorem is established.

\[ \square \]

2.2. Non-existence theorem.

**Definition 2.3.** A current \( T \in \mathcal{R}_n(U) \) is called connected if \( \text{spt} \, T \) is a connected set.

For \( j = 1 \) and \( R = 0 \), we get the cone \( H_1(0) = K := \{ (x_1, \ldots, x_{n+k}) \in \mathbb{R}^{n+k} : x_1^2 + \cdots + x_{n+k-1}^2 \leq (n-1)x_{n+k}^2 \} \). Let \( K^\pm := K \cap \{ \pm x_{n+k} > 0 \} \) and \( T^\pm := T \cap \{ x \in \mathbb{R}^{n+k} : \pm x_{n+k} > 0 \} \in \mathcal{R}_n(\mathbb{R}^{n+k}) \). From Theorem 2.2, we deduce that \( \text{spt} \, T^+ \subset K^+ = K^+ \cup \{ 0 \} \) and \( \text{spt} \, T^- \subset K^- = K^- \cup \{ 0 \} \).
Lemma 2.4. Let $T \in \mathcal{R}_n(\mathbb{R}^{n+k})$ be a stationary current in $\mathbb{R}^{n+k}$ and $\text{spt} T \subset K$. Then both currents $T^+ \in \mathcal{R}_n(\mathbb{R}^{n+k})$ and $T^- \in \mathcal{R}_n(\mathbb{R}^{n+k})$ are also stationary in $\mathbb{R}^{n+k}$.

Proof. We consider $T^+$. Let $\varepsilon > 0$ be arbitrary and $\phi_\varepsilon(t) \in C^1(\mathbb{R})$ be non-negative such that $\phi_\varepsilon(t) \equiv 0$, $t \leq 0$, and $\phi_\varepsilon(t) \equiv 1$, $t > \varepsilon$, as well as $0 \leq \phi'_\varepsilon(t) \leq c/\varepsilon$ for all $t \in \mathbb{R}$ for some $c > 0$. Because $K^+$ is contained in $\{x \in \mathbb{R}^{n+k}: x_{n+k} > 0\}$, it follows $K^+ \cap \{x \in \mathbb{R}^{n+k}: x_{n+k} < \varepsilon\} \subset B_{c\varepsilon}(0)$ for some (other) $c > 0$, which depends only on the angle of the cone, i.e. on the dimension $n$. From monotonicity for stationary currents, the estimate $\mu_T(B_{c\varepsilon}(0)) \leq c\varepsilon^n$ holds true for small $\varepsilon > 0$. Let $X$ be an arbitrary compactly supported $C^1$ vector field in $\mathbb{R}^{n+k}$ and $\text{spt} \mathcal{X} \cap \text{spt} \partial T^+ = \emptyset$. In view of the stationarity of $T$, we can test $\phi_\varepsilon(x_{n+k})X$. Thus

$$0 = \int_{K^+ \cap \{x_{n+k} < \varepsilon\}} \phi'_\varepsilon(x_{n+k}) (e^\top_{n+k}, X) \, d\mu_T + \int_{\mathbb{R}^{n+k}} \phi_\varepsilon(x_{n+k}) \, \text{div}_M X \, d\mu_T$$

$$\leq \frac{c}{\varepsilon} \sup_{e_{n+k}^\top} |X| \int_{B_{c\varepsilon}(0)} \, d\mu_T + \int_{\mathbb{R}^{n+k}} \phi_\varepsilon(x_{n+k}) \, \text{div}_M X \, d\mu_T$$

$$\leq \frac{c}{\varepsilon} \mu_T(B_{c\varepsilon}(0)) + \int_{\mathbb{R}^{n+k}} \phi_\varepsilon(x_{n+k}) \, \text{div}_M X \, d\mu_T$$

$$\leq c\varepsilon^{n-1} + \int_{\mathbb{R}^{n+k}} \phi_\varepsilon(x_{n+k}) \, \text{div}_M X \, d\mu_T$$

$$\varepsilon \to 0 \int_{\mathbb{R}^{n+k}} \chi_{\{x_{n+k} > 0\}} \, \text{div}_M X \, d\mu_T = \int_{\mathbb{R}^{n+k}} \, \text{div}_M X \, d\mu_T^+.$$

The same inequality holds true for $-X$ and establishes the stationarity of $T^+$.

Applying Theorem 2.1 to $T^+$ and $T^-$, respectively, together with the fact that $\text{spt} \partial T^\pm \subset K^\pm$, we get $0 \notin \text{spt} T$. Since all calculations are invariant under translations, we have proved the following theorem.

Theorem 2.5 (Non-existence theorem). Let $\mathcal{K} = K^+ \cup \{p_0\} \cup K^- \subset \mathbb{R}^{n+k}$ be a cone with vertex $p_0$ which is congruent to $K$. Then there exists no stationary current $T \in \mathcal{R}_n(\mathbb{R}^{n+k})$ such that $\text{spt} T$ is compact and connected with boundary values $\text{spt} \partial T \subset \mathcal{K}^+ \cup \mathcal{K}^-$ such that both $\text{spt} \partial T \cap K^+$ and $\text{spt} \partial T \cap K^-$ are non-empty.

Remark. As $n$ increases, the angle of aperture $\beta$ of $K$ is increasing as well. Precisely we have $\beta = \arctan(\sqrt{n-1})$ and therefore $\beta \to 90^\circ$ as $n \to \infty$. The enclosure and non-existence theorems extend the existence theorem for currents with integer multiplicity, cf. [15, Lem. 34.1], as we get more information of the solution, e.g., the shape or the disconnectedness in at least two parts.

The proof of the following is the same as in [3, Cor.3].
Corollary 2.6 (Necessary conditions). Let $B_1, B_2 \subset \mathbb{R}^{n+k}$ be two closed sets and suppose there exists a connected, stationary current $T \in \mathcal{R}_n(\mathbb{R}^{n+k})$ in $\mathbb{R}^{n+k}$ with $\text{spt} \partial T \subset B_1 \cup B_2$ and that both $\text{spt} \partial T \cap B_1 \neq \emptyset$ and $\text{spt} \partial T \cap B_2 \neq \emptyset$. Then we have:

(i) If $B_i := \{x \in \mathbb{R}^{n+k}: |x - y_i| \leq \delta_i\}, i = 1, 2$, are closed balls with centers $y_i$ and radii $\delta_i$ and $R := |y_1 - y_2|$, then $R \leq \left(\frac{n-1}{n}\right)^{1/2} (\delta_1 + \delta_2)$.

(ii) If $B_1$ and $B_2$ are arbitrary compact sets of diameter $d_1$ and $d_2$, separated by a slab of width $r > 0$, then $r \leq \frac{1}{2} \left(\frac{2(n+k)}{(n-1)(n+k-1)}\right)^{1/2} (d_1 + d_2)$.

2.3. Optimal results in codimension one. The question arises if it is possible to “enlarge” the cone and still prove non-existence. We restrict ourselves to codimension $k = 1$. We use Dierkes’ [3] construction of $n$-dimensional catenoids enclosing a cone with a larger angle of aperture. Therefore we show that the enclosing procedure also holds true for currents.

Consider a curve $(x, y(x))$ in $\mathbb{R}^2$ and its rotational symmetric $n$-dimensional graph $\{(x, y(x)) \in \mathbb{R} \times \mathbb{R}^n: x \in [x_0, x_1], \omega \in S^{n-1}\}$. Its area is proportional to the one dimensional integral $\int_{x_0}^{x_1} y^n(x) \sqrt{1+y'(x)^2} \, dx$. Stationary solutions of this variational integral correspond to $n$-dimensional minimal submanifolds in $\mathbb{R}^{n+1}$. Results from calculus of variation give for a solution $y(x)$, the family of inverse functions $x(y) = a \int_{y}^{y_0} \frac{1}{\sqrt{x^2(y_0) - a^2}} \, d\xi$ for all $a > 0, c \in \mathbb{R}$, and for all $y \geq n\sqrt{a}$. Then every member of the one-parameter family $x = g(y, a), a > 0$, is tangent to the half line $y = \tau_0 x, x > 0$ (independent of $a$), where $\tau_0 := (z_0^{2(n-1)} - 1)^{1/2}$ and $z_0$ is the unique solution of

$$\frac{z}{\sqrt{z^2 - 1}} = \int_{1}^{z} \frac{d\xi}{\sqrt{\xi^2 - 1}}.$$ 

Furthermore every point of the half line is contained in exactly one member of this family.

Let $f(\cdot, a)$ be the family of inverse functions and extend $f$ by even reflection, $f(x, a) = f(-x, a)$ for $x \leq 0$, then we have a smooth function $f(\cdot, a): \mathbb{R} \to \mathbb{R}$. Define $\rho := (x_1^2 + \cdots + x_n^2)^{1/2}$, then $\mathcal{M}_a := \{x \in \mathbb{R}^{n+1}: \rho = f(x_{n+1}, a)\}, a > 0$, is a family of smooth $n$-dimensional minimal submanifolds in $\mathbb{R}^{n+1}$. Furthermore $C_a := \{x \in \mathbb{R}^{n+1}: \rho \leq f(x_{n+1}, a)\}, a > 0$, “encloses” by construction the cone $K^* := K \cup \{0\} := \{x \in \mathbb{R}^{n+1}: \rho < \pm \tau_0 x_{n+1}\} \cup \{0\}$.

Proposition 2.7 (Enclosure result.) Let $T \in \mathcal{R}_n(\mathbb{R}^{n+1})$ be a stationary current in $\mathbb{R}^{n+1}$ with compact support and $\text{spt} \partial T \subset K = \{x \in \mathbb{R}^{n+1}: \rho < \pm \tau_0 x_{n+1}\}$. Then we have $\text{spt} T \subset \bar{K}$.

Proof. We will show $\text{spt} T \subset C_a$ for all $a > 0$. Because of $\bigcap_{a>0} C_a = \bar{K}$, the proposition is then established. Let us assume $\text{spt} T \not\subset C_a$ for some $a > 0$ and consider $\eta_0, \lambda \# T \in \mathcal{R}_n(\mathbb{R}^{n+1})$ with minimal $\lambda > 1$ s.t. $\text{spt}(\eta_0, \lambda \# T) \subset C_a$. Then there exists $p \in \text{spt}(\eta_0, \lambda \# T)$ with $p \in \partial C_a$. The contracted current lies completely on one side of the smooth submanifold $\partial C_a$ and touches it at least
in \( \rho \). We want to apply the maximum principle of Solomon and White [16]. Therefore let us abbreviate for the fixed \( \lambda : \eta(x) := \eta_{0,\lambda}(x) \) and \( \tilde{T} := \eta_{0,\lambda\# T} = \eta_{# T} \).

**Claims:**

(i) If \( \text{spt} \partial T \subset K \), then we have \( \text{spt} \partial \tilde{T} \subset K \).

(ii) \( T_x M \) exists in \( x \) iff \( T_{\eta(x)} \eta(M) \) exists in \( \eta(x) \) and we have equality.

(iii) Let \( T \) be stationary in \( \mathbb{R}^{n+1} \), then so is \( \tilde{T} = \tau(\eta(M), \theta \tilde{T}, \xi \tilde{T}) \).

Claim (i) and (ii) are proved by direct calculations. For the third statement, note that we have for the Jacobian, \( J M \eta = \lambda^{-n} \), cf. [15, Ch. 12]. Let \( Y \in C^1_c(\mathbb{R}^{n+1} \setminus \text{spt}(\partial(\eta_{0,\lambda\# T}))) \) be arbitrary. Then \( X(x) := Y \circ \eta(x) \) is of class \( C^1_c(\mathbb{R}^{n+1} \setminus \text{spt}(\partial T), \mathbb{R}^{n+1}) \). We calculate the variation of the contracted current

\[
\delta \tilde{T}(Y) = \int_{\mathbb{R}^{n+1}} \text{div}_{\eta(M)} Y(y) \, d\mu_T(y)
\]

\[
= \int_{\eta(M)} \text{div}_{\eta(M)} Y(y) \, \theta \tilde{T}(y) \, d\mathcal{H}^n(y)
\]

\[
= \int_{\eta(M)} \text{div}_{\eta(M)} (X \circ \eta^{-1}(y)) \, \theta_T \circ \eta^{-1}(y) \, d\mathcal{H}^n(y).
\]

We have for the first factor

\[
\text{div}_{\eta(M)}(X \circ \eta^{-1}(y)) = \sum_{i=1}^{n+1} \left< \mathcal{P}_{T_y \eta(M)}[D(X_i \circ \eta^{-1}(y))], e_i \right>
\]

\[
= \sum_{i=1}^{n+1} \left< \mathcal{P}_{T_y \eta(M)}[DX_i \circ \eta^{-1}(y)\|_{n+1} \lambda], e_i \right>
\]

\[
= \lambda \sum_{i=1}^{n+1} \left< \mathcal{P}_{T_{\eta(x)} \eta(M)}[DX_i \circ \eta^{-1}(y)], e_i \right>
\]

\[
= \lambda \sum_{i=1}^{n+1} \left< \mathcal{P}_{T_{\eta(x)} \eta(M)}[DX_i], e_i \right> \circ \eta^{-1}(y)
\]

\[
= \lambda \left( \text{div}_M X \right) \circ \eta^{-1}(y)
\]

where we have used the second claim. Finally, we get by applying the area formula

\[
\delta \tilde{T}(Y) = \lambda \int_{\eta(M)} (\text{div}_M X \theta_T) \circ \eta^{-1} \, d\mathcal{H}^n = \lambda \int_M \text{div}_M X \theta_T \, J_M \eta \, d\mathcal{H}^n
\]

\[
= \lambda \lambda^{-n} \int_M \text{div}_M X \theta_T \, d\mathcal{H}^n = \lambda^{-n+1} \int_M \text{div}_M X \, d\mu_T = 0.
\]

Thus \( \tilde{T} \) is stationary in \( \mathbb{R}^{n+1} \) because \( Y \) was arbitrary. Now we are able to apply the maximum principle [16] since the support of \( T \) is compact. Therefore
we get coincidence of \( \text{spt} \, \hat{T} \) and \( \partial C_a \) in an open subset. This is a contradiction to claim (i). \( \square \)

With exactly the same argument as in Section 2.2, we can prove the non-existence theorem. Furthermore it is optimal because the cone is enclosed by a family of minimal submanifolds.

**Theorem 2.8** (Optimal non-existence theorem). Let \( \mathcal{K} = \mathcal{K}^+ \cup \{p_0\} \cup \mathcal{K}^- \subset \mathbb{R}^{n+1} \) be a cone with vertex \( p_0 \) congruent to \( \mathcal{K}^\pm \cup \{0\} = \{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 - \tau_0 x_{n+1}^2 < 0, \pm x_{n+1} > 0\} \cup \{0\} \). Then there is no stationary current \( T \in \mathcal{R}_n(\mathbb{R}^{n+1}) \) with compact support such that \( \text{spt} \, T \) is connected and the boundary fulfills \( \text{spt} \, \partial T \subset \mathcal{K}^\pm \cup \mathcal{K}^- \) such that \( \text{spt} \, \partial T \cap \mathcal{K}^+ \) as well as \( \text{spt} \, \partial T \cap \mathcal{K}^- \) is non-empty.

### 3. Currents with mean curvature vector.

#### 3.1. Enclosure theorem.

The enclosure Theorem 2.2 naturally extends to currents with mean curvature vector \( H \in L^1_{\text{loc}}(\mu_T) \) under appropriate conditions. To this end, we define the quadratic function \( q_j(x) \) with an additional parameter \( b \in [0,1] \). Thus \( q_j(x) := r_j(x) - (n-j)/j b s_j(x) \) and again \( H_j(R) := \{x \in \mathbb{R}^{n+k} : q_j(x) \leq R\} \).

**Theorem 3.1** (Enclosure theorem). Let \( T \in \mathcal{R}_n(\mathbb{R}^{n+k}) \) be a current in \( \mathbb{R}^{n+k} \) with mean curvature vector \( H(x) \) and compact support \( \text{spt} \, T \). It satisfies \( \text{spt} \, \partial T \subset H_j(R) \) for \( R \in \mathbb{R}, j = 1, \ldots, n-1, \) and \( b \in [0,1] \). The mean curvature vector \( H \) fulfills

\[
b + |H(x)| \left[ \frac{r_j(x)}{(n-j)^2} + \frac{b^2}{j^2} s_j(x) \right]^{1/2} \leq 1 \text{ for } \mu_T - \text{a.e. } x \in \mathbb{R}^{n+k} \setminus H_j(R).
\]

(3.1)

Then \( \text{spt} \, T \subset H_j(R) \).

**Proof.** The proof is similar to the case \( H \equiv 0 \). Define

\[
\hat{x}(x) := \left(x_1, \ldots, x_{n+k-j}, -\frac{n-j}{j} b x_{n+k-j+1}, \ldots, -\frac{n-j}{j} b x_{n+k}\right)
\]

and consider \( X(x) := \Psi_{\text{spt} \, T}(x) \gamma(q_j(x)) \hat{x}(x) \). Note that the additional term can be estimated directly with the Cauchy–Schwarz inequality \( \langle H(x), \hat{x}(x) \rangle \geq -|H(x)||\hat{x}(x)| = -|H(x)| \left[ r_j(x) + \left(\frac{n-j}{j}\right)^2 b^2 s_j(x) \right]^{1/2} \). Plugging all terms into (1.1) gives

\[
0 = \int_{\mathbb{R}^{n+k}} \langle \nabla M \gamma(q_j), \hat{x} \rangle + \text{div}_M \hat{x} \gamma(q_j) + \gamma(q_j) \langle \hat{x}, H \rangle \, d\mu_T
\]

\[
\geq \int_{\mathbb{R}^{n+k}} \gamma(q_j) \left\{ n - \left(1 + \frac{n-j}{j} b\right) \sum_{i=n+k-j+1}^{n+k} p_{ii} - |H| \left[ r_j + \left(\frac{n-j}{j}\right)^2 b^2 s_j \right]^{1/2} \right\}
\]

\[
+ 2\gamma'(q_j) |\hat{x}|^2 \, d\mu_T.
\]
We can estimate the expression \{\ldots\} pointwise

\[
\begin{aligned}
&n - \left(1 + \frac{n-j}{j}b\right) \sum_{i=n+k-j+1}^{n+k} p_{ii} - |H| \left[ r_j + \left(\frac{n-j}{j}\right)^2 b^2 s_j \right]^{1/2} \\
&\geq n - \left(1 + \frac{n-j}{j}b\right) j - |H| \left[ r_j + \left(\frac{n-j}{j}\right)^2 b^2 s_j \right]^{1/2} \\
&= (n-j) \left((1-b) - |H| \left[ \frac{r_j}{(n-j)^2} + \left(\frac{b}{j}\right)^2 s_j \right]^{1/2} \right)
\end{aligned}
\]

and now notice that the resulting term is non-negative in \(\mathbb{R}^{n+k}\setminus H_j(R)\) for \(\mu_T\)-a.e. \(x\). Next we integrate over the sets \(E\) and \(\mathbb{R}^{n+k}\setminus E\) where \(E\) is defined exactly as above. In the latter case, we evidently have strict inequality in \((3.2)\) as not all \(p_{ii}\) are equal to one. Thus \(\text{spt} \mu_T \cap (\mathbb{R}^{n+k}\setminus E) \subset H_j(R)\). The proof is now finished as before. \(\square\)

We mention two sufficient characterizations for the mean curvature vector \(H\) such that the condition \((3.1)\) is satisfied. Both are easier to check.

**Lemma 3.2.** The condition \((3.1)\) is fulfilled provided

(i) \(q := |x||H(x)| < 1\) for \(\mu_T\)-a.e. \(x \in \mathbb{R}^{n+k}\setminus H_j(R)\) and \(b \leq 1 - q\);

(ii) \(q := |x||H(x)| < n-j\) for \(\mu_T\)-a.e. \(x \in \mathbb{R}^{n+k}\setminus H_j(R)\) and \(b \leq \min \left\{\frac{1}{n-1}, 1 - \frac{q}{n-j}\right\}\).

**Proof.** A calculation gives

\[
b + |H| \left[ \frac{r_j(x)}{(n-j)^2} + \frac{b^2}{j^2 s_j(x)} \right]^{1/2} \leq b + |H||x| = b + q \leq 1
\]

thus i) is established. Estimations of \(b\) yield the second claim

\[
b + |H| \left[ \frac{r_j(x)}{(n-j)^2} + \frac{b^2}{j^2 s_j(x)} \right]^{1/2} = b + |H| \frac{1}{n-j} \left[ r_j(x) + \left(\frac{n-j}{j}\right)^2 b^2 s_j(x) \right]^{1/2} \leq 1 - \frac{1}{n-j} q + |H| \frac{1}{n-j} \left[ r_j(x) + \left(\frac{n-j}{j}\right)^2 \left(\frac{1}{n-1}\right)^2 s_j(x) \right]^{1/2} \leq 1 - \frac{1}{n-j} q + |H| \frac{1}{n-j} |x| = 1 - \frac{1}{n-j} q + \frac{1}{n-j} q = 1.
\]

\(\square\)

**3.2. Non-existence theorem.** Now let \(T \in \mathcal{R}_n(\mathbb{R}^{n+k})\) be a current with \(0 \in \text{spt} T \subset K := H_1(0)\). Notice, for any current \(T = \tau(M, \theta, \xi)\), we have an associated rectifiable varifold \(V = v(M, \theta)\) just by dropping the orientation. In fact, we have as in [15].

**Proposition 3.3.** Let \(V\) be a rectifiable varifold in some neighborhood of the origin. Suppose the density exists in \(0\) and for \(V_j := \eta_{0,\lambda_j}V, \lambda_j \searrow 0\), we have \(\mu_{V_j} \rightarrow \mu_C\) in \(\mathbb{R}^{n+k}\) such that \(\mu_C\) is associated to a rectifiable varifold \(C\) which is stationary in \(\mathbb{R}^{n+k}\). Then \(C\) is a cone.
Theorem 3.4. Let $V = v(M, \theta)$ and $C = v(N, \vartheta)$ be two varifolds as above. If $\mathrm{spt} V \subset K := \{ x \in \mathbb{R}^{n+k} : x_1^2 + \cdots + x_{n+k-1}^2 \leq (n-1) b x_{n+k}^2 \}$, then we have $\mathrm{spt} C \subset K$ as well.

Lemma 3.5. Let $(\mu_j)_{j \in \mathbb{N}}, \mu$ be Radon measures with $\mu_j \to \mu, j \to \infty$, in $\mathbb{R}^{n+k}$ (weak convergence of Radon measures). Assume for some closed set $A$, we have $\mathrm{spt} \mu_j \subset A$ for all $j \in \mathbb{N}$. Then $\mathrm{spt} \mu \subset A$.

Proof. We show $x \not\in A$ implies the existence of a $\rho > 0$ such that $\mu(B_\rho(x)) = 0$. From the hypothesis, we have $\mathrm{dist}(x, A) > \varepsilon > 0$ and therefore define $\rho := \frac{1}{3} \mathrm{dist}(x, A)$. Notice, because of $\mathrm{spt} \mu_j \subset A$ for all $j \in \mathbb{N}$, it evidently follows $B_{2\rho}(x) \cap \mathrm{spt} \mu_j = \emptyset$ for all $j \in \mathbb{N}$. Now choose a continuous function $\phi : \mathbb{R}^{n+k} \to [0,1]$ satisfying $\phi|_{B_\rho(x)} \equiv 1$ and $\phi|_{\mathbb{R}^{n+k}\setminus B_{2\rho}(x)} \equiv 0$. Since

$$0 \leq \mu(B_\rho(x)) \leq \liminf_{j \to \infty} \mu_j(B_\rho(x)) \leq \liminf_{j \to \infty} \int_{\mathbb{R}^{n+k}} \phi \mathrm{d}\mu_j = \liminf_{j \to \infty} \int_{\mathrm{spt} \mu_j} \phi \mathrm{d}\mu_j = 0,$$

this completes the proof. \hfill \square

Proof of Theorem 3.4. Again, define $V_j := \eta_{0, \lambda_j} \# V$; then we know for every $j$ that $\mathrm{spt} V_j = \mathrm{spt}(\eta_{0, \lambda_j} \# V) \subset \eta_{0, \lambda_j} (\mathrm{spt}(V)) \subset \eta_{0, \lambda_j} (K) = K$. From Proposition 3.3, we obtain $\mu V_j \to \mu C, j \to \infty$, and Lemma 3.5 gives $\mathrm{spt} C = \mathrm{spt} \mu C \subset K$ as required.

We consider the two restricted varifolds $C^\pm := C \cap \{ x : \pm x_{n+k} > 0 \} = v(N \cap \{ x : \pm x_{n+k} > 0 \}, \vartheta|_{N \cap \{ x : \pm x_{n+k} > 0 \}})$ satisfying $\mathrm{spt} C^+ \subset K^+ = K^+ \cup \{ 0 \} \subset \{ x \in \mathbb{R}^{n+k} : x_{n+k} \geq 0 \}$ and $\mathrm{spt} C^-$ is obviously contained in $\{ x \in \mathbb{R}^{n+k} : x_{n+k} \leq 0 \}$. Both varifolds are still cones and literally as in the proof of Lemma 2.4, it follows that both $C^+$ and $C^-$ are stationary in $\mathbb{R}^{n+k}$. Hence we can apply the following lemma to conclude a contradiction. For a proof, see Simon [15, Thm. 36.5 & Rmk. 36.6].

Lemma 3.6. Let $C = v(N, \vartheta)$ be a rectifiable varifold stationary in $\mathbb{R}^{n+k}$ and satisfying $\eta_{0, \lambda} \# C = C$ for all $\lambda > 0$. Furthermore assume $\mathrm{spt} C \subset \mathcal{H}$ where $\mathcal{H}$ is some open half space in $\mathbb{R}^{n+k}$ with $0 \in \partial \mathcal{H}$. Then $\mathrm{spt} C \subset \partial \mathcal{H}$.

This gives $\mathrm{spt} C^\pm \subset \{ x_{n+k} = 0 \}$ in contradiction to $\mathrm{spt} C^\pm \subset K = K$ because by construction evidently $C \neq 0$. Notice in particular this means that the support of a stationary tangent cone cannot pass through the vertex of the cone $K$ and therefore the origin cannot be contained in the support of the current $T$. In order to fulfill the assumptions of Proposition 3.3, we assume $H \in L^p_{\mathrm{loc}}(U, \mathbb{R}^{n+k}; \mu_T)$ for some $p > n$ and $\theta \geq 1$ $\mu_T$-a.e. in $U$ for some small neighborhood $U$ of the origin with $U \cap \mathrm{spt} \partial T = \emptyset$. This gives the existence of an area stationary tangent cone in every $x \in \mathrm{spt} T \cap U$, see [15, Ch. 8]. Finally, this establishes

Theorem 3.7 (Non-existence theorem). Let $T \in \mathcal{R}(\mathbb{R}^{n+k})$ be a current with compact support $\mathrm{spt} T$ and mean curvature vector $H$. Assume for some $b \in [0,1]$,

$$\mathrm{spt} \partial T \subset K^\pm := \left\{ x \in \mathbb{R}^{n+k} : \sum_{i=1}^{n+k-1} x_i^2 \leq (n-1) b x_{n+k}^2, \pm x_{n+k} > 0 \right\}.$$


such that both \( \text{spt} \partial T \cap K^+ \) and \( \text{spt} \partial T \cap K^- \) are non-empty. Furthermore define \( r_1(x) := x_1^2 + \cdots + x_{n+k-1}^2 \) and suppose the mean curvature vector \( \mathbf{H} \) satisfies on the one hand

\[
b + |\mathbf{H}| \left[ \frac{r_1(x)}{(n-1)^2} + b^2 x_{n+k}^2 \right]^{1/2} \leq 1 \text{ for } \mu_T \text{-a.e. } x \in \mathbb{R}^{n+k} \setminus K \tag{3.3}
\]

and on the other hand, in some neighborhood \( U \) of the origin, \( \mathbf{H} \in L^p_{\text{loc}}(U, \mathbb{R}^{n+k}; \mu_T) \) for some \( p > n \) and \( \theta(x) \geq 1 \) for \( \mu_T \)-a.e. \( x \in U \). Then \( \text{spt} T \) cannot be connected.

**Remark.** Instead of condition (3.3), one of the requirements of Lemma 3.2 with \( j = 1 \) and \( R = 0 \) can also be fulfilled. The condition \( \theta \geq 1 \) \( \mu_T \)-a.e. is automatically given for currents with integer multiplicity.

4. **Enclosure and non-existence theorems for currents in submanifolds.** Here we discuss an important modification of stationarity in Euclidean spaces. More generally we let \( \mathcal{N} \) be an \((n + l)\)-dimensional \( C^2 \)-submanifold of \( \mathbb{R}^{n+k} \) for \( 0 \leq l \leq k \). We denote by \( B_y : T_y \mathcal{N} \times T_y \mathcal{N} \rightarrow (T_y \mathcal{N})^\perp \) the second fundamental form of \( \mathcal{N} \) at \( y \). Then \( T \) is called stationary in \( \mathcal{N} \) if the first variational formula holds true with \( \mathbf{H}_M := \sum_{i=1}^n B_x(\tau_i, \tau_i) = \text{tr}(B_x)|_{T_x M} \) and \( \{\tau_1, \ldots, \tau_n\} \) is any orthonormal basis for the approximate tangent space \( T_x M \) of \( T \) in \( x \), cf. Simon [15, Def. 16.4].

**Theorem 4.1** (General enclosure and non-existence theorem). Let \( \mathcal{N} \) be an \((n + l)\)-dimensional embedded \( C^2 \)-submanifold of \( \mathbb{R}^{n+k} \) for \( 0 \leq l \leq k \). Let \( T = \tau(M, \theta, \xi) \in \mathcal{R}_n(\mathbb{R}^{n+k}) \), \( M \subset \mathcal{N} \) be a stationary current in \( \mathcal{N} \) with compact support. Furthermore for some \( R \in \mathbb{R} \), assume

\[
b + |\mathbf{H}_M(x)| \left[ \frac{r_2(x)}{(n-j)^2} + b^2 s_j(x) \right]^{1/2} \leq 1 \text{ for } \mu_T \text{-a.e. } x \in \mathcal{N} \setminus H_j(R). \tag{4.1}
\]

If \( \text{spt} \partial T \subset H_j(R) \), then \( \text{spt} T \subset H_j(R) \).

Let (4.1) be true for \( j = 1 \) and \( R = 0 \) and assume for \( \varepsilon > 0 \), on the one hand, \( \theta \geq 1 \) \( \mu_T \)-a.e. in \( B_\varepsilon(0) \) and on the other hand, \( \mathbf{H}_M \in L^p_{\text{loc}}(B_\varepsilon(0), \mathbb{R}^{n+k}; \mu_T) \) for some \( p > n \). Furthermore let \( \text{spt} \partial T \subset K^\pm \) such that both \( \text{spt} \partial T \cap K^+ \neq \emptyset \) and \( \text{spt} \partial T \cap K^- \neq \emptyset \). Then \( \text{spt} T \) cannot be a connected set.

We want to estimate the abstract curvature \( \mathbf{H}_M \) against some quantity depending only on the submanifold \( \mathcal{N} \). We restrict ourselves to the case \( l = k - 1 \). As usual, we can define the principal curvatures \( \kappa_1, \ldots, \kappa_{n+k-1} \) of \( \mathcal{N} \) which we order with respect to their absolute value: \( |\kappa_1| \geq \cdots \geq |\kappa_{n+k-1}| \). We define \( \Lambda_n := |\kappa_1| + \cdots + |\kappa_n| \) what we call the \( n \)-mean curvature of \( \mathcal{N} \). This number slightly differs from the usual \( n \)-mean curvature used in barrier principles of higher codimension, cf. [6,7]. We are now able to proceed similarly as in [12] and arrive at the important estimate: For \( \mu_T \)-a.e. \( x \in M \), we have \( |\mathbf{H}_M(x)| = \left| \text{tr}(B_x)|_{T_x M} \right| \leq \Lambda_n(x) \). This is just a conclusion of the following general result where we skip the proof.
Lemma 4.2. Let $Q: \mathcal{W} \to \mathbb{R}$ be a quadratic form of an $N$-dimensional Euclidean vector space $\mathcal{W}$ with ordered eigenvalues $|\kappa_1| \geq \cdots \geq |\kappa_N|$. Then we have for every $n$-dimensional subspace $\mathcal{W} \subset \mathcal{V}$, the estimate $|\text{tr}(Q)|_{\mathcal{W}} \leq |\kappa_1| + \cdots + |\kappa_n|$.

So we get results under conditions just depending on the ambient space.

Corollary 4.3 (Enclosure and non-existence theorem in hypersurfaces). Let $\mathcal{N}$ be an $(n+k-1)$-dimensional embedded $C^2$-submanifold of $\mathbb{R}^{n+k}$. Let $T = \tau(M, \theta, \xi) \in \mathcal{R}_n(\mathbb{R}^{n+k})$, $M \subset \mathcal{N}$ be a stationary current in $\mathcal{N}$ with compact support. Furthermore for some $R \in \mathbb{R}$, assume

$$b + \Lambda_n(x) \left[ \frac{r_j(x)}{(n-j)^2} + \frac{b^2}{j^2} s_j(x) \right]^{1/2} \leq 1 \text{ for } \mu_T \text{-a.e. } x \in \mathcal{N} \setminus H_j(R). \quad (4.2)$$

If $\text{spt} \partial T \subset H_j(R)$, then $\text{spt} T \subset H_j(R)$.

Let (4.2) be true for $j = 1$ and $R = 0$ and assume for $\varepsilon > 0$, on the one hand, $\theta \geq 1 \mu_T$-a.e. in $B_\varepsilon(0)$ and on the other hand, $\Lambda_n \in L^p_{\text{loc}}(B_\varepsilon(0), \mathbb{R}^{n+k}; \mu_T)$ for some $p > n$. Furthermore let $\text{spt} \partial T \subset K^\pm$ such that both $\text{spt} \partial T \cap K^+ \neq \emptyset$ and $\text{spt} \partial T \cap K^- \neq \emptyset$. Then $T$ cannot be a connected set.

Example 4.4 (Non-existence of stationary currents in spheres). Define the shifted sphere $S = S^{n+k-1}(x_0, R) := \{ x \in \mathbb{R}^{n+k} : |x-x_0| = R \}$ with $0 \in S$. Let $T = \tau(M, \theta, \xi) \in \mathcal{R}_n(\mathbb{R}^{n+k})$ with integer multiplicity and $M \subset S$ be a stationary current in $S$. Assume $\text{spt} T$ is compact, $q := \sup_{x \in \text{spt} T} |x| \frac{n}{R} < 1$, and suppose $\text{spt} \partial T \subset K := \{ x \in \mathbb{R}^{n+k} : x_1^2 + \cdots + x_{n+k-1}^2 \leq (n-1)(1-q) x_{n+k}^2 \}$ such that both $\text{spt} \partial T \cap (K \cap \{ x_{n+k} > 0 \}) \neq \emptyset$ and $\text{spt} \partial T \cap (K \cap \{ x_{n+k} < 0 \}) \neq \emptyset$. Then $T$ cannot be connected.

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