ESSENTIAL NORM OF THE PULL BACK OPERATOR

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Abstract. We obtain some estimations of the essential norm of a pull back operator induced by quasi-symmetric homeomorphisms. As a corollary, we deduce the compactness criterion of this operator.

1. Introduction

Let $\Delta = \{z : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$ and $\Delta^* = \mathbb{C}\setminus \Delta$. A homeomorphism $h$ is said to be quasi-symmetric if there is some $M > 0$, called the quasi-symmetric constant of $h$, such that

$$\frac{1}{M} \leq \frac{|h(e^{i(\theta+t)}) - h(e^{i\theta})|}{|h(e^{i\theta}) - h(e^{i(\theta-t)})|} \leq M$$

for all $\theta$ and $t > 0$. Denote by $QS(S^1)$ the group of quasi-symmetric homeomorphisms of the unit circle $S^1$. Beurling and Ahlfors [1] proved that a sense preserving self-homeomorphism $h$ is quasi-symmetric if and only if there exists some quasi-conformal homeomorphism of $\Delta$ onto itself which has boundary value $h$. Later Douady and Earle [3] gave a quasi-conformal extension of $h$ to the unit disk which is also conformally

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invariant. Let \( \text{M"ob}(S^1) \) be the group of M"obius transformations mapping \( \Delta \) onto itself. The universal Teichm"uller space is the right coset space \( T = QS(S^1)/\text{M"ob}(S^1) \).

A quasisymmetric homeomorphism \( h \) is said to be symmetric if

\[
\lim_{t \to 0^+} h(e^{i(\theta + t)}) - h(e^{i\theta}) \over h(e^{i\theta}) - h(e^{i(\theta - t)}) = 1
\]

for all \( \theta \) and \( t > 0 \). Let \( S(S^1) \) denote the set of all symmetric homeomorphisms of the unit circle. Then \( S(S^1) \) is a subgroup of \( QS(S^1) \). The little universal Teichm"uller space is defined as \( T_0 = S(S^1)/\text{M"ob}(S^1) \).

The class of symmetric homeomorphisms has several equivalent definitions and has been much investigated in classical complex analysis [6].

For any quasi-conformal homeomorphism \( f \) of the unit disk \( \Delta \) onto itself with Beltrami coefficient \( \mu(z) \), define \( b^*(f) \) to be the infimum taken over all compact subset \( F \) contained in \( \Delta \) of the essential supremum norm of \( \mu(z) \) as \( z \) varies over \( \Delta \setminus F \). We say a quasisymmetric homeomorphism \( f \) is asymptotically conformal if \( b^*(f) = 0 \). Define the boundary dilatation \( b(h) \) of a quasisymmetric homeomorphism \( h \) to be the infimum of \( b^*(f) \) taken over all quasi-conformal mapping \( f \) with the boundary value \( f|_{S^1} = h \). The following results are well known.

**Proposition 1.1.** [4] A quasisymmetric homeomorphism \( h \) is symmetric if and only if \( b(h) = 0 \).

Actually, Gardiner and Sullivan [4] proved that for a symmetric homeomorphism, the Beurling-Ahlfors extension is asymptotically conformal. The Douady-Earle extension also has this property (see [2] and [5]).

Hu and Shen [5] introduced some pull-back operators and functions induced by quasisymmetric homeomorphism to study the universal Teichm"uller space and some subspaces of the universal Teichm"uller space.

We recall some notations and definitions.

The Bergman space \( A^2 \) consists of all holomorphic functions \( \phi \) in the unit disk \( \Delta \) with finite norm

\[
\| \phi \| = \left( \frac{1}{\pi} \int_{\Delta} |\phi(z)|^2 dx dy \right)^{\frac{1}{2}} < \infty.
\]

This is a Hilbert space with inner product defined as

\[
\langle \phi, \psi \rangle = \frac{1}{\pi} \int_{\Delta} \phi(z) \overline{\psi(z)} dx dy.
\]
Let $h$ be a quasisymmetric homeomorphism in the unit circle. The following two kernel functions induced by $h$ were introduced in [5],

$$
\phi_h(\zeta, z) = \frac{1}{2\pi i} \int_{S^1} \frac{h(w)}{(1 - \zeta w)^2(1 - zh(w))} dw, \quad (\zeta, z) \in \Delta \times \Delta,
$$

(3)

$$
\psi_h(\zeta, z) = \frac{1}{2\pi i} \int_{S^1} \frac{h(w)}{(\zeta - w)^2(1 - zh(w))} dw, \quad (\zeta, z) \in \Delta \times \Delta.
$$

(4)

Both $\phi_h$ and $\psi_h$ are holomorphic functions. It is noted that the function $\phi_h$ was also appeared in Cui [2]. The two kernel functions induce the following two operators from Bergman space $A^2$ into itself respectively,

$$
T_h^- \psi(\zeta) = \frac{1}{\pi} \int \int_{\Delta} \phi_h(\zeta, z) \psi(\bar{z}) dxdy, \quad \zeta \in \Delta,
$$

(5)

$$
T_h^+ \psi(\zeta) = \frac{1}{\pi} \int \int_{\Delta} \psi_h(\zeta, z) \psi(\bar{z}) dxdy, \quad \zeta \in \Delta.
$$

(6)

The two kernel functions also induce two functions,

$$
\phi_h(z) = \left( \frac{1}{\pi} \int \int_{\Delta} |\phi_h(\zeta, z)|^2 d\xi d\eta \right)^{\frac{1}{2}}, \quad z \in \Delta,
$$

(7)

$$
\psi_h(z) = \left( \frac{1}{\pi} \int \int_{\Delta} |\psi_h(\zeta, z)|^2 d\xi d\eta \right)^{\frac{1}{2}}, \quad z \in \Delta.
$$

(8)

These pull-back operators and functions play an important role in the Teichmüller theory (see [5], [8], [9]). They were used in [5] and [8] to characterize when a quasisymmetric homeomorphism is symmetric or belongs to the Weil-Petersson class. They were also used to study the BMO-Teichmüller theory in [9].

**Proposition 1.2.** [5] $T_h^- : A^2 \to A^2$ is bounded operator if and only if $h$ is quasisymmetric and $\|T_h^-\| \leq \frac{\|\mu\|_{\infty}}{\sqrt{1 - \|\mu\|_{\infty}^2}}$, where $\mu$ is the Beltrami coefficient of a quasi-conformal extension of $h$.

**Proposition 1.3.** [5] Let $h$ be a quasisymmetric homeomorphism. Then the following statements are equivalent:

1. $T_h^- : A^2 \to A^2$ is a compact operator;
2. $h$ is symmetric;
3. $\lim_{|z| \to 1} \phi_h(z)(1 - |z|^2) = 0$. 

A nature problem is how to estimate the essential norm of the pull-back operator $T_h^-$. In this note, we will give some estimations of the essential norm of the pull back operator $T_h^-$ and then deduce the compactness criterion of this operator. We first recall the definition of essential norm of an operator in Banach space. Let $X$ and $Y$ be Banach spaces. For a bounded linear operator $T : X \to Y$, the essential norm $\|T\|_e$ is defined to be the distance from $T$ to the set of the compact operators $K : X \to Y$, precisely

$$\|T\|_e = \inf \|T - K\|$$

where the infimum is taken over all compact operators $K$ from $X$ into $Y$ and $\|\cdot\|$ denotes the usual operator norm. Note that $T$ is compact if and only if $\|T\|_e = 0$.

We first give a representation formula for the essential norm of the pull-back operator by means of the degenerating sequence (see the definition in the next section). Then we obtain our main estimation of the essential norm of the pull back operator $T_h^-$ as follows.

**Theorem 1.4.** Let $h$ be a quasisymmetric homeomorphism. Then there is a constant $C > 0$ which depends only on the quasisymmetric constant of $h$ such that

$$\lim_{|a| \to 1} (1 - |a|^2) \phi_h(a) \leq \|T_h^-\|_e \leq C \lim_{|a| \to 1} (1 - |a|^2) \phi_h(a).$$

By means of Proposition 1.1, Proposition 1.3 and Theorem 1.4, we have the following result.

**Corollary 1.5.** Let $h$ be a quasisymmetric homeomorphism. Then the following statements are equivalent:

1. $T_h^- : A^2 \to A^2$ is a compact operator;
2. $h$ is symmetric;
3. $\lim_{|z| \to 1} \phi_h(z)(1 - |z|^2) = 0$;
4. $b(h) = 0$.

The formula of the essential norm of $T_h^-$ will be given in the second section and the proofs of Theorems 3.1 and 1.4 will be presented in the next two sections.
2. A formula of the essential norm of $T_h$

Let $\phi$ be an analytic function in $\Delta$ with Taylor expansion

$$\phi(z) = \sum_{k=0}^{\infty} a_k z^k.$$  

For any positive integer $n \geq 1$, define two operators as following

$$\mathbb{R}_n \phi(z) = \sum_{k=n}^{\infty} a_k z^k,$$

and

$$\mathbb{K}_n = I - \mathbb{R}_n,$$

where $I$ is the identity operator. We need the following unified estimation for the operator $\mathbb{R}_n$.

**Lemma 2.1.** For any $\epsilon > 0$ and $0 < r < 1$, there is a positive integer $N_0$ which depends only on $r$ such that for any $n > N_0$, $|a| < r$ and $f \in A^2$,

$$\sup_{\|f\| \leq 1} |\mathbb{R}_n f(a)| < \epsilon.$$

**Proof.** The reproducing kernel function in $A^2$ is

$$K_a(z) = \frac{1}{(1 - \bar{a}z)^2}, \quad a \in \Delta, z \in \Delta.$$  

For fixed $a \in \Delta$, the function $K_a(z)$ is a bounded analytic function. It is easy to see that the operator $\mathbb{R}_n$ is a self-adjoint operator in $A^2$. Therefore, for any $f \in A^2$,

$$\langle \mathbb{R}_n f, K_a \rangle = \langle f, \mathbb{R}_n K_a \rangle.$$

This yields

$$|\mathbb{R}_n f(a)| = \frac{1}{\pi} |\langle \mathbb{R}_n f, K_a \rangle| = \frac{1}{\pi} |\langle f, \mathbb{R}_n K_a \rangle| \leq \frac{1}{\pi} \|f\| \|\mathbb{R}_n K_a\|_\infty.$$

It is noted that $K_a(z) = \sum_{n=0}^{\infty} (n+1)\bar{a}^n z^n$, therefore for any $|a| < r$,

$$|\mathbb{R}_n K_a(z)| = |\sum_{k=n}^{\infty} (k+1)\bar{a}^k z^k| \leq \sum_{k=n}^{\infty} (k+1)r^k.$$
For any $\epsilon > 0$, take $N_0$ such that $\frac{1}{\pi} \sum_{k=N_0}^{\infty} (k + 1)r^k < \epsilon$, then when $n > N_0$ and $|a| < r$,

$$\sup_{\|f\| \leq 1} \|R_n f(a)\| < \epsilon.$$ 

The proof of Lemma 2.1 is completed.

We say a sequence $\{\varphi_n\} \in A^2$ is degenerating, if $\|\varphi_n\| \leq 1$ and $\{\varphi_n\}$ converges uniformly to zero uniformly on any compacted subset of $\Delta$. The following result gives an expression of the essential norm of the operator $T_h^-$.

**Theorem 2.2.** Let $h$ be a quasisymmetric homeomorphism. Then

$$\|T_h^-\|_e = \sup_{\{\varphi_n\}} \left\{ \lim_{n \to \infty} \|T_h^- \varphi_n\| \right\},$$

where the supremum is taken over all degenerating sequence $\{\varphi_n\} \subset A^2$.

**Proof.** Note that the degenerating sequence $\{\varphi_n\} \in A^2$ weakly converges to zero. Therefore, for any compact operator $K : A^2 \to A^2$, we have $\|K(\varphi_n)\| \to 0$ as $n \to \infty$. We deduce that

$$\|T_h^- - K\| \geq \lim_{n \to \infty} \|(T_h^- - K)(\varphi_n)\|$$

$$\geq \lim_{n \to \infty} \|T_h^- (\varphi_n)\| - \lim_{n \to \infty} \|K(\varphi_n)\|$$

$$= \lim_{n \to \infty} \|T_h^- (\varphi_n)\|.$$

Take the supremum over all degenerated sequence $\{\varphi_n\} \in A^2$, and then take the infimum over all compact operator $K : A^2 \to A^2$, we have

$$\|T_h^-\|_e \geq \sup_{\{\varphi_n\}} \left\{ \lim_{n \to \infty} \|T_h^- \varphi_n\| \right\}.$$ 

Noting that $h$ is a quasisymmetric homeomorphism, by Proposition 1.2, we know that $T_h^-$ is a bounded operator in $A^2$. It is noted that for each $n$, $K_n$ is a compact operator, which implies that $T_h^- K_n$ is also a compact operator for all $n$. Therefore, we have

$$\|T_h^-\|_e = \|T_h^- R_n + T_h^- K_n\|_e \leq \|T_h^- R_n\| \leq \lim_{n \to \infty} \sup_{\|\phi\| \leq 1} \|T_h^- R_n(\phi)\|.$$ 

For each $n$, there is a sequence $\{\varphi^*_m\} \subset A^2$ with $\|\varphi^*_m\| \leq 1$ such that

$$\|T_h^- R_n\| = \lim_{m \to \infty} \|T_h^- R_n(\varphi^*_m)\|.$$
We choose a sequence \( \{ \phi_n \} \subset A^2 \) with \( \| \phi_n \| \leq 1 \) such that
\[
\| T^{-}_h \mathbb{R}_n \| \leq \| T^{-}_h \mathbb{R}_n(\phi_n) \| + \frac{1}{n}.
\]
Denote the sequence \( \{ \mathbb{R}_n(\phi_n) \} \) by \( \{ \varphi_n \} \). Note that for each \( n \), \( \mathbb{R}_n \) is a projection operator from \( A^2 \) to \( A^2 \), therefore \( \| \mathbb{R}_n \| = 1 \) and \( \| \varphi_n \| \leq 1 \).

It follows from Lemma 2.1 that the sequence \( \{ \varphi_n \} \) converges uniformly to zero on any compact subset of \( \Delta \). Thus the sequence \( \{ \varphi_n \} \) is a degenerating sequence and
\[
\| T^{-}_h \|_e \leq \lim_{n \to \infty} \| T^{-}_h \mathbb{R}_n \| \leq \lim_{n \to \infty} \| T^{-}_h \varphi_n \|.
\]
Therefore we have
\[
\| T^{-}_h \|_e \leq \sup_{\{ \varphi_n \}} \left\{ \lim_{n \to \infty} \| T^{-}_h \varphi_n \| \right\},
\]
where the supremum is taken over all degenerating sequence \( \{ \varphi_n \} \subset A^2 \).

The proof of Theorem 2.2 is completed. \( \square \)

3. Estimations in terms of boundary distortion

In this section, an estimation of the essential norm of the pull-back operator by means of the boundary distortion will be given. We prove the following result.

**Theorem 3.1.** Let \( h \) be a quasymmetric homeomorphism. Then
\[
\lim_{|a| \to 1} (1 - |a|^2)|\phi_h(a)| \leq \| T^{-}_h \|_e \leq \frac{b(h)}{\sqrt{1 - b(h)^2}},
\]
where \( b(h) \) is the boundary dilatation of \( h \).

To prove the theorem, we need the following results.

**Lemma 3.2.** [5] [9] Let \( h \) be a quasymmetric homeomorphism.

1. For any \( \psi \in A^2 \), choosing \( \phi \) such that \( \phi' = \psi \), we have
\[
T^{-}_h \psi(\zeta) = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{\phi(h(w))}{(1 - \zeta w)^2} dw.
\]

2. Let \( a \in \Delta \) and \( K_a(\zeta) = \frac{1 - |a|^2}{(1 - a\zeta)^2} \). Then we have
\[
T^{-}_h (K_a(\zeta)) = (1 - |a|^2)\phi_h(\zeta, a).
\]
We first estimate the lower bound. Let \( a \in \Delta \). Consider the function 
\[
K_a(\zeta) = \frac{1 - |a|^2}{(1 - a\zeta)^2},
\]
it is known that \( K_a(\zeta) \in A^2 \) and \( \|K_a\| = 1 \). Notice that the family \( K_a \) in \( A^2 \) converges uniformly to zero locally in \( \Delta \) as \( |a| \to 1 \). Therefore, for any compact operator \( K : A^2 \to A^2 \), we have \( \|K(K_a)\| \to 0 \) as \( |a| \to 1 \). By Theorem 2.2, we deduce that
\[
\|T_h^{-}\|_e \geq \lim_{|a| \to 1} \|T_h^{-}(K_a)\|
\]
By Lemma 3.2,
\[
T_h^{-}(K_a(\zeta)) = (1 - |a|^2)\phi_h(\zeta, a).
\]
Therefore, we have
\[
\|T_h^{-}\|_e \geq \lim_{|a| \to 1} (1 - |a|^2)|\phi_h(a)|.
\]

We now estimate the upper bound. From the proof of Theorem 2.2, we have
\[
\|T_h^{-}\|_e = \|T_h^{-}R_n + T_h^{-}K_n\|_e \leq \|T_h^{-}R_n\|.
\]
Thus, we will proceed to estimate the norm \( \|T_h^{-}R_n\| \) and obtain the upper bound estimation of \( \|T_h^{-}\|_e \). Let \( f \) be a quasi-conformal extension of the quasisymmetric homeomorphism \( h \) into \( \Delta \). By Lemma 3.2, for any \( \psi \in A^2 \), choosing \( \phi \) such that \( \phi' = \psi \), we have
\[
T_h^{-}\psi(\zeta) = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{\phi(h(w))}{(1 - \zeta w)^2} \, dw.
\]
The Green formula yields
\[
T_h^{-}\psi(\zeta) = \frac{1}{\pi} \int_{\Delta} \frac{\psi(f(w))\overline{f(w)}}{(1 - \zeta w)^2} \, dudv.
\]
Noting that the Hilbert transformation is isometry on $L^2(\mathbb{C})$, we deduce that

$$
\|T_h^{-1} \mathbb{R}_n \psi\|^2 = \frac{1}{\pi} \int_{\Delta} \left| \frac{1}{\pi} \int_{\Delta} \frac{\mathbb{R}_n \psi(f(w)) \overline{\partial f(w)}}{(\frac{1}{\xi} - w)^2} dudv \right|^2 \frac{1}{\xi^4} d\xi d\eta
$$

$$
= \frac{1}{\pi} \int_{\Delta^*} \left| \frac{1}{\pi} \int_{\Delta} \frac{\mathbb{R}_n \psi(f(w)) \overline{\partial f(w)}}{(\xi - w)^2} dudv \right|^2 d\xi d\eta
$$

$$
\leq \frac{1}{\pi} \int_{\Delta} \left| \mathbb{R}_n \psi(f(w)) \overline{\partial f(w)} \right|^2 dudv
$$

$$
\leq \frac{1}{\pi} \int_{\Delta} \frac{|\mu(w)|^2}{1 - |\mu(w)|^2} \left| \mathbb{R}_n \psi(w) \right|^2 dudv.
$$

where $\mu(w)$ is the Beltrami coefficient of $f^{-1}$.

Let $0 < r_0 < 1$, $A_{r_0} = \{ z \in \Delta : |z| > r_0 \}$ and $\Delta_{r_0} = \Delta \setminus A_{r_0}$. We divide the integral above into two parts,

$$
\|T_h^{-1} \mathbb{R}_n \psi\|^2 \leq \frac{1}{\pi} \int_{\Delta_{r_0}} \frac{|\mu(w)|^2}{1 - |\mu(w)|^2} \left| \mathbb{R}_n \psi(w) \right|^2 dudv
$$

$$
+ \frac{1}{\pi} \int_{A_{r_0}} \frac{|\mu(w)|^2}{1 - |\mu(w)|^2} \left| \mathbb{R}_n \psi(w) \right|^2 dudv
$$

$$
= J_1 + J_2.
$$

We first estimate the term $J_2$. Let $M = \sup_{|w| > r_0} \frac{|\mu(w)|^2}{1 - |\mu(w)|^2}$. Noting that $\sup\{\|\mathbb{R}_n\| : n \geq 1\} = 1$, we have

$$
J_2 = \frac{1}{\pi} \int_{A_{r_0}} \frac{|\mu(w)|^2}{1 - |\mu(w)|^2} \left| \mathbb{R}_n \psi(w) \right|^2 dudv
$$

$$
\leq M \|\mathbb{R}_n \psi\|^2
$$

$$
\leq M \sup\{\|\mathbb{R}_n\| : n \geq 1\} \|\psi\|^2
$$

$$
\leq M \|\psi\|^2.
$$

Next, we estimate the term $J_1$. Let $k = \|\mu\|_\infty$, we have

$$
J_1 = \frac{1}{\pi} \int_{\Delta_{r_0}} \frac{|\mu(w)|^2}{1 - |\mu(w)|^2} \left| \mathbb{R}_n \psi(w) \right|^2 dudv
$$

$$
\leq \frac{k^2}{1 - k^2} \frac{1}{\pi} \int_{\Delta_{r_0}} \left| \mathbb{R}_n \psi(w) \right|^2 dudv.
$$
Therefore we deduce that
\[
\sup_{\|\psi\| \leq 1} \|T_n^{-R_n}\psi\| \leq \left( M + \frac{k^2}{1-k^2} \frac{1}{\pi} \int_{\Delta_{r_0}} \sup_{\|\psi\| \leq 1} |R_n\psi(w)|^2 dudv \right)^{\frac{1}{2}}
\]
Thus, it follows from Lemma 2.1 that
\[
\lim_{n \to \infty} \sup_{\|\psi\| \leq 1} \|T_n^{-R_n}\psi\| \leq M^\frac{1}{2}.
\]
Let \( r_0 \to 1 \) and then take the infimum over all Beltrami coefficient \( \mu \) of quasi-conformal extension of the quasi-symmetric homeomorphism \( h \), we have
\[
\|T_h^{-\infty}\|_e \leq \lim_{n \to \infty} \|T_h^{-R_n}\| \leq \frac{b(h)}{\sqrt{1-b(h)^2}}.
\]
The proof of Theorem 3.1 follows.

4. Proof of Theorem 1.4

From Theorem 3.1, we need only estimate the upper bound. Recall that the Douady-Earle extension \( w = E(h)(z) \) of the quasi-symmetric homeomorphism \( h \) is defined as the equation, for \( z, w \in \Delta \),
\[
F(z, w) = \frac{1}{2\pi} \int_{S^1} \frac{(h(t) - w)(1 - |w|^2)}{(1 - \overline{w}h(t)) |z - t|^2} dt = 0,
\]
(see [3]). Let \( \mu(w) \) be the Beltrami coefficient of the inverse mapping \( E(h)^{-1} \) of the Douay-Earle extension \( E(h) \) of quasi-symmetric homeomorphism \( h \). It follows from [2] and [5] that there is a constant \( C > 0 \) which depends only on the quasi-symmetric constant of \( h \) such that
\[
\frac{|\mu(w)|^2}{1 - |\mu(w)|^2} \leq C(1 - |w|^2)^2 \phi_h^2(\bar{w}).
\]
Therefore, from (15) and (17), we have
\[
\lim_{n \to \infty} \|T_h^{-R_n}\|^2 \leq C \sup_{|w| > r_0} (1 - |w|^2)^2 \phi_h^2(\bar{w}).
\]
Let \( r_0 \to 1 \). We get
\[
\|T_h^{-}\|_e \leq C \lim_{|w| \to 1} (1 - |w|^2) \phi_h(w).
\]
We complete the proof of Theorem 1.4.
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