Abstract

Supersymmetry between bosons and fermions is modeled within $\mathcal{PT}$–symmetric quantum mechanics. A non-Hermitian alternative to the Witten’s supersymmetric quantum mechanics is obtained.
1 Introduction

Final chapters of the current textbooks on quantum mechanics usually mention its natural extensions towards relativistic kinematics and towards the quantization of fields (cf., e.g., [1]). Recently, fundamental concepts like symmetries and interpretations have been re-considered within quantum mechanics itself. For example, Witten [2] proposed the so called supersymmetric (SUSY) quantum mechanics (cf. its review [3]) and Bender and Boettcher [4] initiated a study of an innovative formalism of \( \mathcal{PT} \) symmetric quantum mechanics (cf. also [5]). In what follows, we intend to discuss and further develop a certain overlap or combination of the the latter two schemes in the direction indicated recently in our letter [6].

Before moving to the deeper technical details, let us start from an overall setting of the stage. Indeed, there exists a palpable contrast between the well developed formalism of the traditional Hermitian quantum mechanics of ref. [1] and the striking incompleteness of interpretation of the alternative non-Hermitian \( \mathcal{PT} \)–symmetric formalism of ref. [4]. As mentioned by the referee of this paper, the gross mathematical consistency of the former theory is already well understood since H. Weyl. The essence of this theory is most clearly presented in the language of the self-adjoint operators and/or their essentially self-adjoint extensions [7]. In contrast, the unanswered open questions abound in the latter context. Even the fundamental conjecture of the connection between the reality of the spectrum and the related “un-broken” \( \mathcal{PT} \) symmetry is just a more or less intuitive hypothesis at present. Many of its illustrative manifestations are puzzling [8]. At the same time, quite recently there emerged several indications that its “natural” formulation may be provided by resurgence theory (cf. subject number 34M37 within the Mathematics Subject Classification 2000 scheme) rather than by functional analysis (ref. [9] can be recalled for an updated review of the related literature).

Similarly, in the language of physics, the multi-faceted character of the possi-
ble physical interpretations of the Witten’s scheme [2] cannot be compared, at the present stage of development at least, with the first-step nature of our fairly simple-minded proposals in ref. [6]. At the same time, it is necessary to imagine that our latter representation of supersymmetry “lives” in an unusual space. In a way depending on its explicit physical interpretation (which is very often being related to the zero-dimensional field theory [10] and, hence, need not necessarily coincide with the traditional versions of quantum mechanics), one often speaks about the “states” which are manifestly regularized by their $\mathcal{PT}$ symmetrization. These states need not necessarily survive in the appropriate Hermitian limit. This is a crucial aspect of our forthcoming considerations [11] and has to be kept in mind whenever one tries to understand properly the relations between the non-equivalent formalisms as depicted in the following diagram

\[
\begin{array}{ccc}
\text{Hermitian QM} & \longrightarrow & \text{Hermitian QM} \\
\text{of ref. [1]} & \downarrow & \text{with SUSY [2]} \\
\downarrow & * & \downarrow \\
\text{non–Hermitian} & \longrightarrow & \text{non–Hermitian} \\
\mathcal{PT} \text{ QM [4]} & \downarrow & \mathcal{PT} \text{ SUSY QM [3]}
\end{array}
\]

In our forthcoming text we shall solely pay attention to the $*$–marked correspondence. We shall outline a consistent approach to supersymmetry after the weakening of the Hermiticity to the mere $\mathcal{PT}$ symmetry.

For the sake of clarity, our considerations will be presented in an elementary language inspired by the possible $\mathcal{PT}$ symmetrization of the Calogero two-particle solvable model [12]. In combination with the methodical analysis of the $\mathcal{PT}$ symmetric quartic anharmonic oscillators of ref. [4] this will enable us to modify slightly the current interpretation of the $\mathcal{PT}$ symmetry itself. We shall also analyze anew
some of its possible consequences. This material will be split in the sketchy description of the Witten’s supersymmetry in section 2, similar sketchy outline of the \( \mathcal{PT} \) symmetry in section 3 and their synthesis in section 4.

## 2 Hermitian SUSY

Energies \( E_n^{(HO)} = 2n + 1 \) of bound states generated by the one-dimensional harmonic oscillator Hamiltonian \( H^{(HO)} = p^2 + x^2 \) belong to the wave functions \( \psi_n(x) \) with the definite parity, \( \mathcal{P}\psi_n(x) = \psi_n(-x) = (-1)^n\psi_n(x) \), \( n = 0, 1, \ldots \). Let us now employ this particular example and recollect its re-interpretation within supersymmetric quantum mechanics.

### 2.1 Harmonic oscillator in supersymmetric picture

In accord with the review of the supersymmetric quantum mechanics by Cooper et al \[3\] we can consider the left-shifted and right-shifted harmonic oscillator Hamiltonians

\[
H^{(L)} = p^2 + x^2 - 1,
\]

\[
H^{(R)} = p^2 + x^2 + 1
\]

and notice their almost complete isospectrality,

\[
E_m^{(L)} = 2m, \quad m = 0, 1, \ldots, \quad E_k^{(R)} = 2k + 2, \quad k = 0, 1, \ldots
\]

Wave functions of the same energy become arranged in doublets,

\[
\psi_{n+1}^{(L)}(x) \leftrightarrow \psi_n^{(R)}(x), \quad n = 0, 1, \ldots
\]

Formally this can be interpreted as a consequence of an underlying supersymmetry \( sl(1/1) \) since a representation of this algebra working with both commutators and anticommutators can be generated by the three two-by-two matrices one of which is defined in terms of our two toy Hamiltonians,

\[
\mathcal{H} = \begin{bmatrix} H^{(L)} & 0 \\ 0 & H^{(R)} \end{bmatrix} = \begin{bmatrix} BA & 0 \\ 0 & AB \end{bmatrix}.
\]
The well known factorization of the Hamiltonian is then defined in terms of the harmonic oscillator superpotential $W^{(HO)}(x) = x$,

$$A = A^{(HO)} = \frac{d}{dx} + W^{(HO)}(x), \quad B = B^{(HO)} = -\frac{d}{dx} + W^{(HO)}(x).$$

The other two generators or “supercharges” read

$$Q = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$$

and close the required superalgebra,

$$[\mathcal{H}, Q] = [\mathcal{H}, \tilde{Q}] = 0, \quad \{Q, Q\} = \{\tilde{Q}, \tilde{Q}\} = 0, \quad \{Q, \tilde{Q}\} = \mathcal{H}.$$

Formally the whole scheme with its Riccati equation background (cf. ref. [3] for more details) is firmly rooted in the nineteen century mathematics [13] and offers an explanation of the exact solvability of many potentials after a suitable change of the superpotential $W(x)$.

### 2.2 The model of Calogero in supersymmetric picture

In the present setting let us note that the supersymmetrized pairs of states of the same energy are characterized by their parity,

$$\mathcal{P}\psi^L_m(x) = -\mathcal{P}\psi^R_k(x), \quad m = k + 1, \quad k = 0, 1, \ldots.$$

This feature finds an interesting interpretation within the so called Calogero model of the two particles interacting via the harmonic oscillator forces complemented by a short-term “repulsion” [12]. The complete two-body Hamiltonian reads

$$H^{(Cal)} = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + \left[\frac{1}{2} (x_1 - x_2)^2 + \frac{g}{(x_1 - x_2)^2}\right].$$

After a routine elimination of the centre-or-mass coordinate $R = (x_1 + x_2)/\sqrt{2}$ one is left with the radial Schrödinger equation

$$\left[-\frac{d^2}{dr^2} + r^2 + \frac{1}{2} \frac{g}{r^2} - E\right] \psi(r) = 0$$

(3)
in the relative coordinate \( r = (x_1 - x_2)/\sqrt{2} \). This is a singular ordinary differential equation and \( E \) is the energy in the centre-of-mass system. The singularity acquires the common centrifugal form whenever we introduce the (in general, non-integer) “angular momentum”

\[
\ell = \ell(g) = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{g}{2}}.
\] (4)

The strongly repulsive centrifugal-like core can be perceived as impenetrable. The normalizability of the wave function at \( g \in [3/2, \infty) \) implies that \( \psi^{(admissible)}(r) \) vanishes at \( r = 0 \). By “brute force”, Calogero extended such an impenetrability feature to all the couplings \( g > -1/2 \) by demanding that it takes place for the weak repulsion (or, if necessary, attraction) as well. The related explicit boundary conditions

\[
\begin{align*}
\lim_{r \to 0} \psi^{(admissible)}(r) &= 0, & g \in (0, 3/2), \\
\lim_{r \to 0} [r^{-1/2}\psi^{(admissible)}(r)] &= 0, & g \in (-1/2, 0)
\end{align*}
\] (5)

were more thoroughly discussed in the related recent comment [14] containing the discussion of such a regularization interpreted as a selection of the “admissible” or “physical” solutions at any \( g > -1/4 \).

It may be useful [11] to re-emphasize here that the choice of the regularization is fairly conventional, indeed. In the various modern applications of the formalism of quantum mechanics, the whole family of the “nonstandard” models may prove equally useful. The formal freedom of choosing any essentially self-adjoint extension as our “physical” Hamiltonian operator in the interval of \( g \in (-1/2, 3/2) \) enables us to modify accordingly its desirable physical meaning [15].

Under our present conventions we may summarize that the Calogero’s “non-communication rules” [5] make the Calogero’s equation selfconsistently defined on both the half-axes \( r \in (0, \infty) \) and \( r \in (-\infty, 0) \). In a way dictated by the pure physics one can join these separate branches in both the symmetric (= bosonic) and
antisymmetric (= fermionic) manner,

\[ \mathcal{P}\psi^{(\text{bosonic})}(x) = +\psi^{(\text{bosonic})}(x), \quad \mathcal{P}\psi^{(\text{fermionic})}(x) = -\psi^{(\text{fermionic})}(x) \]  

observing that

- the operator \( \mathcal{P} \) of parity can be re-interpreted as an exchange of particles, within the framework of the sufficiently non-harmonic Calogero model at least;

- tentatively, we can try to move to the smaller couplings \( g > -1/2 \) and pick up the \( g = 0 \) special case. Then, within the above-mentioned supersymmetric arrangement (6), we obtain a new interpretation of supersymmetry as a very natural mapping between the Calogerian bosons and fermions (6).

We can add that the Hermitian supersymmetric transformation unavoidably fails in all the \( g \neq 0 \) cases where the superpotential \( W(x) \) becomes singular. More details can be found in chapter 12 of the review [3]. Here we shall propose a systematic remedy in a way inspired by our letter [6].

3 \( \mathcal{PT} \) symmetric Hamiltonians

In the usual Hermitian conjugation \( H = H^\dagger \) the superscript \( ^\dagger \) means the transposition \( H \rightarrow H' \) combined with the complex conjugation, \( H^+ = \mathcal{T}H'\mathcal{T} \). Within the \( \mathcal{PT} \) symmetric quantum mechanics of ref. [5] another type of conjugation is necessary.

3.1 Complexifications of smooth oscillators

Let us briefly review the main features of the \( \mathcal{PT} \) symmetric quantum mechanics. Its basic idea is simple and dates in fact back to the old paper by Caliceti et al [10]. In this work the complex forces of a “minimally” perturbed type \( V(x) = x^2 + i\lambda x^3 \) have been shown to posses the real spectrum at the sufficiently small and real couplings \( \lambda \).
Using the symbols $\mathcal{P}$ (parity) and $\mathcal{T}$ (complex conjugation) the probable relevance of the $\mathcal{PT}$ invariance $V(x) = \mathcal{PT}V(x)\mathcal{PT}$ of this model has been emphasized and re-emphasized in different contexts \cite{17,18}.

The recent explicit formulation of the hypothesis connecting the reality of spectrum to the unbroken $\mathcal{PT}$ symmetry of wave functions belongs to Bender et al \cite{4,5}. It has been verified on several different solvable examples. During this verification an important role has been played by the partially solvable systems \cite{19,20} and by the systematic $\mathcal{PT}$ symmetrization of the so called shape invariant family of the polynomially solvable models \cite{21,22}. After a return to their Hermitian limit, their classification is offered by the supersymmetric quantum mechanics \cite{3}.

A key to the relevance of the $\mathcal{PT}$ symmetrization can be seen in the ambiguity of quantization resulting from a complex deformation of the coordinate axis. In this way the analytic potentials $V(x)$ admit a change of the spectrum caused by a mere deformation of the integration path (i.e., asymptotic boundary conditions) in the complex plane. This idea first appeared in the context of field theory \cite{10}. Its most elementary illustrations have been mediated by the regular harmonic oscillator \cite{4} and by its partially solvable anharmonic modifications of the sextic type \cite{23}. A decisive progress in the study of the $\mathcal{PT}$ symmetric systems of this type has been achieved by Buslaev and Grecchi \cite{24} who discovered an unexpected byproduct of the elementary $\mathcal{PT}$ symmetric complexifications which lies in a facilitated regularization of the centrifugal singularities.

### 3.2 $\mathcal{PT}$ symmetric regularization

In the spirit of paper \cite{24} the range $r \in (0,\infty)$ of many solvable radial Schrödoinger equations can be extended to the whole complex contour $r = r(t)$ parameterized by a real $t \in (-\infty,\infty)$. Quite often, the most elementary implementation of the latter
idea is represented by the real line shifted slightly downwards,

\[ r(t) = t - i \varepsilon, \quad \varepsilon > 0. \]

One can further deform this curve \( r(t) \) with \( |r(t)| \to \infty \) for \( t \to \pm \infty \) to many other suitable shapes. For the harmonic oscillator example the curve only has to stay within the asymptotic wedges in which the dominant part \( \exp(-r^2/2) \) of our explicit wave functions decreases, \( |\text{Im} \, r(t)| < |\text{Re} \, r(t)|, |t| \gg 1 \) (cf. [4]).

Thorough repetition of a \( \mathcal{PT} \) symmetric analysis of the Calogerian singular example (3) in [25] (and of its quartic [24] and decadic [20] modifications) revealed that a core of the difference between the regular and regularized cases lies in the necessity of introduction of the (say, upward-running) cut in our complex plane of \( r \) in the majority of cases [18]. In the other words, the quantization (and spectrum) can be influenced by the continuation of \( r(t) \) on the other Riemannian sheets. Such a type of the subtle non-equivalence deserves a due care. Its exactly solvable example was constructed by Cannata et al and can be found, slightly hidden and implicit, in sec. 5 of their letter [27].

4 \( \mathcal{PT} \) symmetric supersymmetry

We are now prepared to find a way towards a synthesis of the separate supersymmetric and \( \mathcal{PT} \) symmetric modifications of the traditional quantum mechanics. Our main idea is that after a regularization of the centrifugal-like singularities many difficulties encountered in the classical SUSY in connection with the singular superpotentials could be in principle avoided via the \( \mathcal{PT} \) symmetric regularization. This returns us once more to the general experience with quantum theory where our choice or specification of the physical meaning of the Hamiltonians can remain, sometimes, quite flexible and adapted to our specific phenomenological needs [13].
4.1 Anharmonic oscillators with the $x^{-2}$ core

Teaching by example once more, let us pick up the first nontrivial (viz., quartic anharmonic) example. In a way close to our present discussion this example has been first studied by the purely non-perturbative means by Flessas [28]. For the purposes of inserting this example in the new supersymmetric scheme let us also recall the results of the papers [13] and [26]. We recollect that the potentials

$$V^{(sm)}(x) = -4i(x - i\varepsilon) - (x - i\varepsilon)^4,$$

$$V^{(sp)}(x) = \frac{2}{(x + i\varepsilon)^2} - (x + i\varepsilon)^4$$

possess the respective exact zero-energy solutions

$$\psi^{(sm)}(x) = (x - i\varepsilon) \exp \left(-i \frac{(x - i\varepsilon)^3}{3}\right) \in L_2(-\infty, \infty),$$

$$\psi^{(sp)}(x) = \frac{1}{x + i\varepsilon} \exp \left(+i \frac{(x + i\varepsilon)^3}{3}\right) \in L_2(-\infty, \infty).$$

We can, therefore, introduce the usual superpotentials

$$W^{(sp/sm)}(x) = -\left[\frac{d}{dx} \psi^{(sp/sm)}(x)\right]/\psi^{(sp/sm)}(x) = \pm \left[\frac{1}{x \pm i\varepsilon} - i (x \pm i\varepsilon)^2\right].$$

In a way proposed in our recent study [3] we employ the standard definition of the supercharges (2) containing the nonstandard creation- and annihilation-type operators

$$A = A^{(PT)} = T \cdot \left[\frac{d}{dx} + W^{(sm)}(x)\right], \quad B = B^{(PT)} = \left[-\frac{d}{dx} + W^{(sm)}(x)\right] \cdot T.$$

This preserves all the underlying supersymmetric algebra $sl(1/1)$ and extends the concept of the supersymmetric partners to the case of our non-Hermitian Hamiltonians.
4.2 \( \mathcal{PT} \) symmetrized Calogero model at \( A = 2 \)

All the Hermitian harmonic oscillators with a central symmetry (i.e., real potentials \( V(\vec{r}) = |\vec{r}|^2 \) in \( D \) dimensions) are described by the ordinary differential “radial” Schrödinger equation of the form (3) with \( g/2 = \ell(\ell + 1) \). This equation is solvable in terms of the confluent hypergeometric functions and its integer of half-integer parameter \( \ell = (D - 3)/2 + L \) depends on an integer quantum number \( L \) of the free angular motion. This implies that we can write all the exact normalizable solutions in terms of Laguerre polynomials in a way which is fully parallel to the well known Hermitian case.

On a move to the singular and \( \mathcal{PT} \) symmetrized Calogeroian harmonic oscillator (3) with, in general, non-integer 2\( \ell \) (and with the straight line \( r(t) = t - i \varepsilon \) for definiteness), we can recollect the results obtained in our paper [25]. There, the well known “regular” harmonic oscillator bound state solutions \( \psi^{(\text{reg})}(r) \) which behave like \( r^{\ell+1} \) near the origin were complemented by their “irregular-like” partners \( \psi^{(\text{irreg})}(r) = O(r^{-\ell}) \). The latter states remain formally acceptable whenever \( \varepsilon \neq 0 \). At the same time, they just play a purely formal role and “naturally” disappear in the “standard” or “physical” non-\( \mathcal{PT} \) Hermitian limit \( \varepsilon \to 0 \). Hence, no contradictory predictions can emerge. The physics remains unchanged while merely its mathematical presentation is modified. In fact the overall picture becomes simplified in a way resembling the simplification of algebraic equations after one moves from the real line to complex plane.

- Our two sets of solutions form the non-equidistant spectrum \( \{ E^{(\text{reg})}_n, E^{(\text{irreg})}_n \} \)

where

\[
E^{(\text{reg})}_n = 4n + 2\ell + 3, \quad E^{(\text{irreg})}_n = 4n - 2\ell + 1, \quad n = 0, 1, \ldots .
\]

These “energies” exhibit an \( \alpha \leftrightarrow -\alpha \) symmetry with \( \alpha = \ell + 1/2 \).
The related wave functions remain bounded, normalizable and proportional to the Laguerre polynomials,

\[
\psi^{(\text{reg})}(r) = \text{const} \cdot r^{\ell+1} e^{-r^2/2} L^{\ell+1/2}_n(r^2), \\
\psi^{(\text{irreg})}(r) = \text{const} \cdot r^{-\ell} e^{-r^2/2} L^{-\ell-1/2}_n(r^2).
\]

They exhibit the same \(\alpha \leftrightarrow -\alpha\) symmetry.

- At \(g = 0\) (i.e., \(\ell = 0\)) we return to the well known one-dimensional spectrum,

\[
E_n^{(\text{reg})} = 4n + 3, \quad E_n^{(\text{irreg})} = 4n + 1, \quad \ell = 0, \quad n = 0, 1, \ldots
\]

The seemingly formal \(\alpha \leftrightarrow -\alpha\) symmetry degenerates to the parity.

- The latter feature (parity) survives smoothly the transition to the singular systems. At all the sufficiently small \(g\) the \(\mathcal{PT}\) symmetric regularization preserves the continuity and reality of the energies as well as their numbering by the integer \(n\) and by the sign of the parameter \(\alpha = \ell + 1/2 = \sqrt{1/4 + g/2}\).

The latter smoothness is in a sharp contrast with the abrupt loss of the irregular half of the spectrum at any \(g \neq 0\) in the traditional Hermitian formalism. Summarizing the situation, we can now start from the \(g = 0\) or \(\ell = 0\) bound states with the well defined values of parity \(= \pm 1\). A smooth continuation in \(g\) transfers the label \(\pm 1\) to all the energies. The complexified bosonic and fermionic symmetry becomes re-established in this manner. This enables us to speak about the bosons and fermions defined by the following rule

\[
\psi^{(\text{bosonic})}(-r) = (-1)^{-\ell} \psi^{(\text{bosonic})}(r), \\
\psi^{(\text{fermionic})}(-r) = (-1)^{\ell+1} \psi^{(\text{fermionic})}(r).
\]

The new statistics (10) resembles strongly some aspects of its Hermitian alternative. In essence, the transition from the new fermions to bosons is just a change of sign of \(\alpha = \ell + 1/2\). Such a “complexified supersymmetry” differs from its older form in [6].
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