Geodesics in the static Mallett spacetime

Ken D. Olum

Institute of Cosmology, Department of Physics and Astronomy, Tufts University, Medford, MA 02155

Abstract

Mallett has exhibited a cylindrically symmetric spacetime containing closed timelike curves produced by a light beam circulating around a line singularity. I analyze the static version of this spacetime obtained by setting the intensity of the light to zero. Some null geodesics can escape to infinity, but all timelike geodesics in this spacetime originate and terminate at the singularity. Freely falling matter originally at rest quickly attains relativistic velocity inward and is destroyed at the singularity.

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*Electronic address: kdo@cosmos.phy.tufts.edu
I. INTRODUCTION

A few years ago, Mallett [1] exhibited a spacetime involving light rays circulating around an infinitely long cylinder in the manner of the current in a solenoid. He showed that this spacetime contains timelike paths that move purely azimuthally and so close on themselves after going once around the light cylinder. Although Ref. [1] discusses photonic crystals as a possible means of keeping the light in its circular path, in fact so no such mechanism is necessary. Rather, as shown in Ref. [2], the light rays travel along geodesics orbiting a central line singularity.

Ref. [1] does not discuss the singularity, but in Ref. [3], Mallett says that he introduced it to confine the light rays to the cylinder in a way leading to simpler calculations, and that the spacetime of Ref. [1] is thus the combination of effects due to the circulating light and those due to the singular source. However, Ref. [2] argues that this means that the spacetime of Ref. [1] is unlike the spacetime one would get by introducing circulating light into a flat background. To further investigate this question, we will examine the static spacetime resulting from the line singularity without any circulating light. This spacetime does not have causality violation [3], but it does have unusual properties, as I will discuss below.

The full Mallett spacetime cannot be constructed, because the cylinder and the singularity are infinitely long. But with advanced technology that would permit us to produce singularities, we could presumably set out to construct a finite approximation to this spacetime. We could first produce the background and then introduce a very intense beam of light in an effort to produce closed timelike paths. But even the background is quite problematic. As I show below, the orbiting geodesics discovered by Mallett are the only complete geodesics in this spacetime. Every other null geodesic perpendicular to the singular line either originates or terminates at the singularity; every null geodesic not perpendicular to the singular line both originates and terminates there, as does every timelike geodesic.

II. GEODESICS

In Ref. [1], the intensity of the circulating light beam is a free parameter $\epsilon$. By setting $\epsilon = 0$, we get the static spacetime without the light. The metric then becomes

$$ds^2 = -(\rho/\alpha)dt^2 + \rho d\phi^2 + \sqrt{\alpha/\rho}(d\rho^2 + dz^2).$$  \hspace{1cm} (1)

I have used here the metric signature $(-+++)$, opposite to that of Ref. [1]. The parameter $\alpha$ is a constant with the dimensions of length, presumably related to the radius of the light cylinder. To simplify the computation, we can go to units in which $\alpha = 1$, giving

$$ds^2 = \rho(-dt^2 + d\phi^2) + \rho^{-1/2}(d\rho^2 + dz^2).$$  \hspace{1cm} (2)

It is straightforward to compute the connection and write out the components of the
geodesic equation. For a geodesic \( x(\lambda) \), we find

\[
\ddot{t} + \frac{1}{\rho} \dot{\rho} \dot{t} = 0, \tag{3}
\]
\[
\ddot{\phi} + \frac{1}{\rho} \dot{\rho} \dot{\phi} = 0, \tag{4}
\]
\[
\ddot{z} - \frac{1}{2 \rho} \dot{\rho} \dot{z} = 0, \tag{5}
\]
\[
\ddot{\rho} + \frac{1}{4 \rho} \left( \dot{z}^2 - \dot{\rho}^2 \right) + \frac{\sqrt{\rho}}{2} \left( \dot{t}^2 - \dot{\phi}^2 \right) = 0, \tag{6}
\]

where a dot denotes differentiation with respect to \( \lambda \). From Eq. (3–5), we can write 3 constants of the motion,

\[
T = \rho \dot{t} = \text{constant} \tag{7}
\]
\[
L = \rho \dot{\phi} = \text{constant} \tag{8}
\]
\[
Z = \rho^{-1/2} \dot{z} = \text{constant} \tag{9}
\]

which make Eq. (6)

\[
\ddot{\rho} = -\frac{1}{4 \rho} \dot{\rho}^2 - \frac{3}{4} Z^2 - \frac{1}{2 \sqrt{\rho}} V \leq 0 \tag{12}
\]

This is utterly unlike the normal motion in polar coordinates. Since \( \ddot{\rho} \leq 0 \), there is no geodesic which passes by the singularity, attaining a minimum distance and then traveling away again. If a geodesic is directed inward (\( \dot{\rho} < 0 \)), it eventually collides with the singularity.

What about an outward-directed geodesic? Could it escape to infinity? Consider the right hand side of Eq. (11) as \( \rho \to \infty \). If \( Z \neq 0 \), it goes to \(-\infty\), clearly impossible. If \( Z = 0 \), it goes to 0, which is possible only if \( V = 0 \). So no timelike geodesic, nor any geodesic that moves in the \( z \) direction, can escape to infinity.

Now consider Eq. (12). If \( Z \neq 0 \), then \( \ddot{\rho} \) is bounded by a negative number, so eventually \( \dot{\rho} \) becomes negative: the geodesic turns around and moves inward to the singularity. Since we know that \( \rho \) cannot grow to \( \infty \), the same argument applies for any timelike \((V = 1)\) geodesic. Thus all timelike geodesics and all geodesics with any motion in \( z \) originate and terminate in the singularity.

What about null geodesics perpendicular to the singularity, which have \( Z = V = 0 \)? In that case, if we set \( \dot{\rho} = 0 \) we find \( \ddot{\rho} = 0 \), so a photon can orbit at any fixed \( \rho \). This is the path found by Mallett. More generally, Eq. (11) becomes \( \dot{\rho}^2 = \rho^{-1/2}(T^2 - L^2) \). If \( T^2 < L^2 \)
there are no solutions. If $T^2 = L^2$, we get the circular orbit. If $T^2 > L^2$ the general solution is

$$\rho = [c(\lambda - \lambda_0)]^{4/5}$$

where

$$c = \pm \frac{5}{4}\sqrt{T^2 - L^2}.$$  

By choice of parameterization of the null geodesic, we can eliminate both $\lambda_0$ and $c$, resulting in simply

$$\rho = \pm \lambda^{4/5}.$$  

The upper sign in Eq. (15) represents a geodesic starting from the singularity at parameter $\lambda = 0$ and going out to infinity. The lower represents a geodesic coming in from infinity and terminating at the singularity at $\lambda = 0$.

If $L = 0$, this geodesic is radial. Otherwise, it winds around the singularity. The angular velocity of the outgoing geodesic is $\dot{\phi} = L\lambda^{-4/5}$. Thus

$$\phi(\lambda) - \phi(0) = 5L\lambda^{1/5}.$$  

The geodesic winds a finite number of times near the singularity, and an infinite number of times as it goes out to infinity.

Now let us return to timelike geodesics and consider the fate of a particle initially at rest at some position $\rho_0$. It will always have $L = Z = 0$, so Eq. (11) becomes

$$1 = -\rho^{-1/2}\dot{\rho}^2 + \rho^{-1}T^2$$

At $\rho_0$, $\dot{\rho} = 0$, so $T^2 = \rho_0$, and we have

$$\dot{\rho}^2 = \rho_0\rho^{-1/2} - \rho^{1/2}.$$  

Relabeling the proper time parameter $\tau$, and choosing the inward-going path in the future of the initial time, we find

$$\frac{d\tau}{d\rho} = -\frac{\rho^{1/4}}{\sqrt{\rho_0 - \rho}}.$$  

If we start at rest at $\tau = 0$, integration gives $\rho = 0$ at

$$\tau = B\left(\frac{1}{2}, \frac{5}{4}\right)\rho_0^{3/4},$$  

where $B$ is the Euler beta function; $B(1/2, 5/4) = 1.748 \ldots$.

The proper distance from the singularity to the position with $\rho = \rho_0$ is given by

$$R = \int_{\rho_0}^{\rho_0} \rho^{-1/4}d\rho = \frac{4}{3}\rho_0^{3/4}.$$  

We conclude that a particle initially at rest at proper distance $R$ will be destroyed at the singularity after proper time $(3/4)B(1/2, 5/4)R \approx 1.3R$. 

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III. CONCLUSION

I have shown that nearly every geodesic in the Mallett spacetime originates and terminates at the singular line. The only exceptions are null geodesics perpendicular to the line singularity. Such geodesics with any outward motion originate at the singularity and go to infinity, those with inward motion originate at infinity and terminate at the singularity, and those with no radial motion orbit at fixed radius.

Even a single causal path going from a singularity to some point $p$ makes it impossible to predict what will happen at $p$, because the information coming from the singularity cannot be known. But one might perhaps finesse this issue if $p$ is far from the singularity and its influence is diluted by distance. But in the static Mallett spacetime considered here, from any point the singularity fills the entire sky except for an infinitesimally thin strip, so the loss of predictability is much more severe.

Furthermore, all timelike geodesics terminate in the singularity, so any freely falling object will eventually reach the singular line. If the object begins at rest, its remaining proper lifetime is of order its proper distance to the singularity. It therefore appears that any attempt to build a “time machine” along the lines described by Mallett would have a very unfortunate effect on nearby objects.

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