Noncommutative complex geometry of the quantum projective space

Masoud Khalkhali and Ali Moatadelro
Department of Mathematics, University of Western Ontario
London, Ontario, Canada

Abstract

We define holomorphic structures on canonical line bundles of the quantum projective space $\mathbb{C}P_q^\ell$ and identify their space of holomorphic sections. This determines the quantum homogeneous coordinate ring of the quantum projective space. We show that the fundamental class of $\mathbb{C}P_q^\ell$ is naturally presented by a twisted positive Hochschild cocycle. Finally, we verify the main statements of Riemann-Roch formula and Serre duality for $\mathbb{C}P_q^1$ and $\mathbb{C}P_q^2$.

Contents

1 Introduction

2 Preliminaries on $U_q(\mathfrak{su}(\ell + 1))$ and $\mathcal{A}(SU_q(\ell + 1))$

2.1 The quantum enveloping algebra $U_q(\mathfrak{su}(\ell + 1))$ ............................................. 2
2.2 The quantum group $\mathcal{A}(SU_q(\ell + 1))$ ................................................................. 3
2.3 Irreducible representations of $U_q(\mathfrak{su}(\ell + 1))$ and the related Gelfand-Tsetlin tableaux ................................................................. 3

3 The complex structure of $\mathbb{C}P_q^\ell$

3.1 Noncommutative complex structures ................................................................. 5
3.2 Holomorphic connections ................................................................. 5
3.3 Holomorphic structures on bimodules ................................................................. 6

4 $\mathbb{C}P_q^\ell$ and the associated quantum line bundles

4.1 Holomorphic line bundles ................................................................. 7

5 Bimodule connections ................................................................. 11

6 Existence of a twisted positive Hochschild cocycle for $\mathbb{C}P_q^\ell$

6.1 A twisted positive Hochschild cocycle on $\mathbb{C}P_q^\ell$ ................................................................. 15

7 The Riemann-Roch theorem for $\mathbb{C}P_q^\ell$, $\ell = 1, 2$

7.1 The case of $\mathbb{C}P_q^1$ ................................................................. 20
7.2 Serre duality for $\mathbb{C}P_q^2$ ................................................................. 21
1 Introduction

In this paper we continue a study of complex structures on quantum projective spaces that was initiated in [8] for $CP_1^q$ and was further continued in [9] for the case $CP_2^q$. In the present paper we consider a natural holomorphic structure on the quantum projective space $CP_t^q$, already presented in [5, 6], and define holomorphic structures on canonical quantum line bundles on it. The space of holomorphic sections of these line bundles then will determine the quantum homogeneous coordinate ring of $CP_t^q$.

In Section 2, we review the preliminaries on irreducible representations of quantum groups $U_q(su(\ell + 1))$ and the Gelfand-Tsetlin basis for these representations. In Section 3 we recall the definition of a complex structure, holomorphic line bundles and bimodule connections. In Section 4 we recall the definition of the quantum projective space $CP_t^q$, and endow its canonical line bundles with holomorphic connections. We also identify the space of holomorphic sections of these line bundles. In Section 5 we define bimodule connections on canonical line bundles. This enables us to define the quantum homogeneous coordinate ring of $CP_t^q$ and identify this ring with the ring of twisted polynomials. In Section 6 we introduce a twisted positive Hochschild cocycle $2\ell$-cocycle on $CP_t^q$, by using the complex structure of $CP_t^q$, and show that it is cohomologous to its fundamental class which is represented by a twisted cyclic cocycle. This certainly provides further evidence for the belief, advocated by Alain Connes [3, 4], that holomorphic structures in noncommutative geometry should be represented by (extremal) positive Hochschild cocycles within the fundamental class. Finally in the last Section we verify directly that the main statements of Riemann-Roch formula and Serre duality theorem hold for that the $CP^1_q$ and $CP^2_q$.

2 Preliminaries on $U_q(su(\ell + 1))$ and $A(SU_q(\ell + 1))$

2.1 The quantum enveloping algebra $U_q(su(\ell + 1))$

Let $0 < q < 1$. We use the following notation

$$[a, b]_q = ab - q^{-1}ba, \quad [z] = \frac{q^z - q^{-z}}{q - q^{-1}}, \quad [n]! = [n][n-1]\cdots[1],$$

$$[n]_m = \frac{[n]!}{[m]![n-m]!}, \quad [j_1, j_2, \ldots, j_k]! = q^{-\sum_{1 \leq i < j \leq k} [j_1 + j_2 + \cdots + j_k]!} \frac{[j_1]![j_2]!\cdots[j_k]!}{[j_1]![j_2]!\cdots[j_k]!}.$$

The quantum enveloping algebra $U_q(su(\ell + 1))$, as a $*$-algebra, is generated by elements $K_i, K_i^{-1}, E_i, F_i$, $i = 1, 2, \ldots, \ell$, with $K_i^* = K_i$ and $E_i^* = F_i$, subject to the following relations for $0 \leq i, j \leq \ell$ [10],

$$K_iK_j = K_jK_i \quad E_iK_i = q^{-1}K_iE_i$$
$$E_iK_j = q^{1/2}K_jE_i \quad \text{if} \quad |i - j| = 1$$
$$E_iK_j = K_jE_i \quad \text{if} \quad |i - j| > 1$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i^2 - K_i^{-2}}{q - q^{-1}}$$
$$E_iE_j = E_jE_i \quad \text{if} \quad |i - j| > 1,$$

and the Serre relation

$$E_i^2E_j - (q + q^{-1})E_iE_jE_i + E_jE_i^2 = 0 \quad \text{if} \quad |i - j| = 1.$$
The coproduct, counit and antipode of this Hopf algebra is given by

\[ \Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i + K_i^{-1} \otimes E_i, \]
\[ \epsilon(K_i) = 1, \quad \epsilon(E_i) = 0, \quad S(K_i) = K_i^{-1}, \quad S(E_i) = -qE_i. \]

2.2 The quantum group \( \mathcal{A}(SU_q(\ell + 1)) \)

As a \(*\)-algebra, \( \mathcal{A}(SU_q(\ell + 1)) \) is generated by \((\ell + 1)^2\) elements \(u^i_j\), where \(i, j = 1, 2, \ldots, \ell + 1\) subject to the following commutation relations

\[ u^i_k u^j_k = q u^j_k u^i_k, \quad u^i_k u^j_l = q u^j_l u^i_k \quad \forall i < j, \]
\[ [u^i_l, u^j_k] = 0, \quad [u^i_k, u^j_l] = (q - q^{-1})u^i_l u^j_k \quad \forall i < j, k < l, \]
and

\[ \sum_{\pi \in S_{\ell+1}} (-q)^{||\pi||} u^1_{\pi(1)} u^2_{\pi(2)} \cdots u^{\ell+1}_{\pi(\ell+1)} = 1, \]

where the sum is taken over all permutations of the \(\ell + 1\) elements and \(||\pi||\) is the number of simple inversions of the permutation \(\pi\). The involution is given by

\[ (u^i_j)^* = (-q)^{j-i} \sum_{\pi \in S_{\ell}} (-q)^{||\pi||} u^{k_1}_{\pi(n_1)} u^{k_2}_{\pi(n_2)} \cdots u^{k_\ell}_{\pi(n_\ell)} \]

with \(\{k_1, \ldots, k_\ell\} = \{1, 2, \ldots, \ell + 1\} \setminus \{i\}\) and \(\{n_1, \ldots, n_\ell\} = \{1, 2, \ldots, \ell + 1\} \setminus \{j\}\) as ordered sets, and the sum is over all permutations \(\pi\) of the set \(\{n_1, \ldots, n_\ell\}\). The Hopf algebra structure is given by

\[ \Delta(u^i_j) = \sum_k u^i_k \otimes u^j_k, \quad \epsilon(u^i_j) = \delta^i_j, \quad S(u^i_j) = (u^i_j)^*. \]

2.3 Irreducible representations of \( U_q(\mathfrak{su}(\ell + 1)) \) and the related Gelfand-Tsetlin tableaux

The finite dimensional irreducible \(*\)-representations of \( U_q(\mathfrak{su}(\ell + 1)) \) are indexed by \(\ell\)-tuples of non-negative integers \(n := (n_1, n_2, \ldots, n_\ell)\). We denote this representation by \(V_n\). A basis for \(V_n\) is given by Gelfand-Tsetlin (GT) tableaux that we denote here by

\[ |m\rangle := \begin{bmatrix} m_{1,\ell+1} & m_{2,\ell+1} & \cdots & m_{\ell,\ell+1} & m_{\ell+1,\ell+1} \\ m_{1,\ell} & m_{2,\ell} & \cdots & m_{\ell,\ell} \\ \vdots & \vdots \\ m_{1,2} & m_{2,2} \\ m_{1,1} \end{bmatrix} \]

where \(n_i = m_{i,\ell+1} - m_{i+1,\ell+1}\) for \(i = 1, 2, \ldots, \ell\), which fixes \(m_{ij}\) up to an additive constant.

The action of generators on this basis is given by [10], \(K_k |m\rangle = q^{\frac{k}{2}} |m_k\rangle\), where

\[ a_k = \sum_{i=1}^k m_{i,k} - \sum_{i=1}^{k-1} m_{i,k-1} - \sum_{i=1}^{k+1} m_{i,k+1} + \sum_{i=1}^k m_{i,k} \]
\[ = 2 \sum_{i=1}^k m_{i,k} - \sum_{i=1}^{k-1} m_{i,k-1} - \sum_{i=1}^{k+1} m_{i,k+1} \] (2)
and the action of $E_k$ is given by
\[
E_k |m\rangle = \sum_{j=1}^{k} A^j_k |m^j\rangle,
\]
where $|m^j\rangle$ is obtained from $|m\rangle$ when $m_{j,k}$ is replaced by $m_{j,k} + 1$ and
\[
A^j_k = \left(-\frac{\Pi_{i=1}^{k+1}[l_{i,k+1} - l_{j,k}][\Pi_{i=1}^{k-1}[l_{i,k} - l_{j,k} - 1]}{\Pi_{i \neq j}[l_{i,k} - l_{j,k}][l_{i,k} - l_{j,k} - 1]}\right)^{1/2}.
\]
Here $l_{i,j} = m_{i,j} - i$, and the positive square root is taken. For the inner product $\langle i | j \rangle := \delta_{i,j}$, this will be a $*$-representation and the matrix coefficients of $\rho^n : U_q(\mathfrak{su}(\ell + 1)) \to \text{End}(V_n)$ will be $\rho^n_{ij}(h) = \langle i | h | j \rangle$. Note that the basic representation of $U_q(\mathfrak{su}(\ell + 1))$ is given by $\sigma : U_q(\mathfrak{su}(\ell + 1)) \to M_{\ell+1}(\mathbb{C})$ where
\[
\sigma^i_j(K_r) = \delta^i_j q^{\frac{1}{2}(\delta_{r+1,i} - \delta_{r,i})}, \quad \sigma^i_j(E_r) = \delta^i_{r+1} - \delta^i_j,
\]
and the Hopf pairing $\langle , \rangle : U_q(\mathfrak{su}(\ell + 1)) \times A(SU_q(\ell + 1)) \to \mathbb{C}$ is defined by $\langle h, u_j^i \rangle := \sigma^i_j(h)$. Therefore
\[
\langle K_r, u_j^i \rangle = \sigma^i_j(K_r) = \delta^i_j q^{\frac{1}{2}(\delta_{r+1,i} - \delta_{r,i})}, \quad \langle E_r, u_j^i \rangle = \sigma^i_j(E_r) = \delta^i_{r+1} - \delta^i_j.
\]
Using Peter-Weyl theorem, a basis $\{t^n_{\mu,\nu}\}$ for $A(SU_q(\ell + 1))$ is implicitly given by $\langle h, t^n_{\mu,\nu} \rangle = \rho^n_{\mu,\nu}(h)$. For later use it is worth mentioning here that for $n = (0, 0, \ldots, 0, 1)$ these basis $t^n_{\mu,\nu}$ are just generators $u^i_j$. In order to show this, it is enough to compute $\rho^n_{\mu,\nu}(h)$ for generators of $U_q(\mathfrak{su}(\ell + 1))$. Indeed for $n = (0, 0, \ldots, 0, 1)$ a basis element $|m\rangle$ takes the following form
\[
|m\rangle :=
\begin{bmatrix}
m & m & \ldots & m & m & m - 1 \\
m & m & \ldots & m & m_l \\
: & : & \\
m & m_2 \\
m & m_1
\end{bmatrix}
\]
where each of the $m_i$'s is either $m$ or $m - 1$ such that $m_1 \geq m_2 \geq \ldots \geq m_l$. So $|m\rangle$ can be parametrized just by one integer $i$. Let us denote $|m\rangle$ by $|i\rangle$ when $m_j = m$ for $j \leq i - 1$ and $m_j = m - 1$ for $j \geq i$.
\[
\rho^n_{\mu,\nu}(K_r) = \langle i | K_r | j \rangle = q^{\frac{a_r}{2}} \langle i | j \rangle = q^{\frac{a_r}{2}} \delta_{i,j}.
\]
where
\[
a_r = 2 \sum_{i=1}^{r} m_{i,r} - \sum_{i=1}^{r-1} m_{i,r-1} - \sum_{i=1}^{r+1} m_{i,r+1}.
\]
So for our case we will end up with
\[
\rho^n_{i,j}(K_r) = \langle i | K_r | j \rangle = q^{\frac{a_r}{2}} \langle i | j \rangle = q^{\frac{a_r}{2}} \delta_{i,j},
\]
Definition 3.1. A complex structure on an algebra $A$, equipped with a differential calculus $(\Omega^\bullet(A), d)$, is a bigraded differential $*$-algebra $\Omega^{(p,q)}(A)$ and two differential maps $\partial : \Omega^{(p,q)}(A) \to \Omega^{(p+1,q)}(A)$ and $\overline{\partial} : \Omega^{(p,q)}(A) \to \Omega^{(p,q+1)}(A)$ such that:

$$\Omega^n(A) = \bigoplus_{p+q=n} \Omega^{(p,q)}(A), \quad \partial a^* = (\overline{\partial} a)^*, \quad d = \partial + \overline{\partial}. \tag{6}$$

Also, the involution $*$ maps $\Omega^{(p,q)}(A)$ to $\Omega^{(q,p)}(A)$.

We will use the simple notation $(A, \overline{\partial})$ for a complex structure on $A$.

Definition 3.2. Let $(A, \overline{\partial})$ be an algebra with a complex structure. The space of holomorphic elements of $A$ is defined as

$$\mathcal{O}(A) := \text{Ker}\{\overline{\partial} : A \to \Omega^{(0,1)}(A)\}.$$
3.2 Holomorphic connections

Suppose we are given a differential calculus \((\Omega^\bullet(\mathcal{A}), d)\). We recall that a connection on a left \(\mathcal{A}\)-module \(\mathcal{E}\) for the differential calculus \((\Omega^\bullet(\mathcal{A}), d)\) is a linear map \(\nabla : \mathcal{E} \to \Omega^1(\mathcal{A}) \otimes_A \mathcal{E}\) with left Leibniz property:

\[
\nabla(a\xi) = a\nabla\xi + da \otimes_A \xi, \quad \forall a \in \mathcal{A}, \forall \xi \in \mathcal{E}.
\]

(7)

By the graded Leibniz rule, i.e.

\[
\nabla(\omega \xi) = (-1)^n \omega \nabla\xi + d\omega \otimes_A \xi, \quad \forall \omega \in \Omega^n(\mathcal{A}), \forall \xi \in \Omega(\mathcal{A}) \otimes_A \mathcal{E},
\]

(8)

this connection can be uniquely extended to a map, which will be denoted again by \(\nabla\), \(\nabla : \Omega^\bullet(\mathcal{A}) \otimes_A \mathcal{E} \to \Omega^{\bullet+1}(\mathcal{A}) \otimes_A \mathcal{E}\).

The curvature of such a connection is defined by \(F_\nabla = \nabla \circ \nabla\). One can show that, \(F_\nabla\) is an element of \(\text{Hom}_A(\mathcal{E}, \Omega^2(\mathcal{A}) \otimes_A \mathcal{E})\).

**Definition 3.3.** Suppose \((\mathcal{A}, \overline{\partial})\) is an algebra with a complex structure. A holomorphic structure on a left \(\mathcal{A}\)-module \(\mathcal{E}\) with respect to this complex structure is given by a linear map \(\nabla_{\overline{\partial}} : \mathcal{E} \to \Omega^{(0,1)}(\mathcal{A}) \otimes_A \mathcal{E}\) such that

\[
\nabla_{\overline{\partial}}(a\xi) = a\nabla_{\overline{\partial}}\xi + \overline{\partial} a \otimes_A \xi, \quad \forall a \in \mathcal{A}, \forall \xi \in \mathcal{E},
\]

(9)

and such that \(F_{\nabla_{\overline{\partial}}} = (\nabla_{\overline{\partial}})^2 = 0\).

Such a connection will be called a flat \(\overline{\partial}\)-connection. In the case which \(\mathcal{E}\) is a finitely generated \(\mathcal{A}\)-module, \((\mathcal{E}, \nabla_{\overline{\partial}})\) will be called a holomorphic vector bundle.

Associated to a flat \(\overline{\partial}\)-connection, there exists a complex of vector spaces

\[
0 \to \mathcal{E} \to \Omega^{(0,1)}(\mathcal{A}) \otimes_A \mathcal{E} \to \Omega^{(0,2)}(\mathcal{A}) \otimes_A \mathcal{E} \to ...\]

(10)

Here \(\nabla_{\overline{\partial}}\) is extended to \(\Omega^{(0,q)}(\mathcal{A}) \otimes_A \mathcal{E}\) by the graded Leibniz rule. The zeroth cohomology group of this complex is called the space of holomorphic sections of \(\mathcal{E}\) and will be denoted by \(H^0(\mathcal{E}, \nabla_{\overline{\partial}})\).

3.3 Holomorphic structures on bimodules

**Definition 3.4.** Let \(\mathcal{A}\) be an algebra with a differential calculus \((\Omega^\bullet(\mathcal{A}), d)\). A bimodule connection on an \(\mathcal{A}\)-bimodule \(\mathcal{E}\) is given by a connection \(\nabla\) which satisfies a left Leibniz rule as in formula (7) and a right \(\sigma\)-twisted Leibniz property with respect to a bimodule isomorphism \(\sigma : \mathcal{E} \otimes_A \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes_A \mathcal{E}\). i.e.

\[
\nabla(\xi a) = (\nabla\xi)a + \sigma(\xi \otimes da), \quad \forall \xi \in \mathcal{E}, \forall a \in \mathcal{A}.
\]

(11)

The tensor product connection of two bimodule connections \(\nabla_1\) and \(\nabla_2\) on two \(\mathcal{A}\)-bimodules \(\mathcal{E}_1\) and \(\mathcal{E}_2\) with respect to the bimodule isomorphisms \(\sigma_1\) and \(\sigma_2\) is a map \(\nabla : \mathcal{E}_1 \otimes_A \mathcal{E}_2 \to \Omega^1(\mathcal{A}) \otimes_A \mathcal{E}_1 \otimes_A \mathcal{E}_2\) defined by

\[
\nabla := \nabla_1 \otimes 1 + (\sigma_1 \otimes 1)(1 \otimes \nabla_2).
\]

It can be checked that, \(\nabla\) has the right \(\sigma\)-twisted property with \(\sigma : \mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes \mathcal{E}_1 \otimes \mathcal{E}_2\) given by \(\sigma = (\sigma_1 \otimes 1) \circ (1 \otimes \sigma_2)\).
4 $\mathbb{CP}_q^\ell$ and the associated quantum line bundles

We recall the definition of the quantum projective space $\mathbb{CP}_q^\ell$ as the quantum homogeneous space of the quantum group $SU_q(\ell+1)$ and its quantum subgroup $U_q(\ell)$ from [3]. Let $K := (K_1 K_2^2 \cdots K_\ell^2)/2^{\ell+1}$ and $\mathcal{L}_a := a \triangleleft S^{-1}(h)$. Then we define the quantum $2\ell+1$ sphere as

$$\mathcal{A}(S_q^{2\ell+1}) := \{ a \in \mathcal{A}(SU_q(\ell+1))| \mathcal{L}_h(a) = \epsilon(h)a, \quad \forall h \in U_q(\mathfrak{su}(\ell)) \}. $$

The invariant elements of this space under the action of $K$ will provide the coordinate functions of the quantum projective space

$$\mathcal{A}(\mathbb{CP}_q^\ell) := \{ a \in \mathcal{A}(S_\ell^{2\ell+1})| \mathcal{L}_K a = a \}. $$

The space of sections of the canonical line bundles $L_N$, $N \in \mathbb{Z}$, are defined by

$$L_N := \{ a \in \mathcal{A}(S_\ell^{2\ell+1})| \mathcal{L}_K a = q^{N/\ell}a \}. $$

Let $M_{jk} := [E_j, [E_{j+1}, ..., [E_{k-1}, E_k]_q]_q]_q$ for $1 \leq j < k \leq \ell$, and

$$N_{jk} := (K_j K_{j+1} ... K_\ell). (K_{k+1} K_{k+2} ... K_\ell). K^{-1} \quad \text{for } 1 \leq j < k \leq \ell. $$

Let $X_i := N_i M_i^*$. We will also use a right black action instead of left action by $h \triangleright a := a \triangleleft \theta(h)$, where $\theta : U_q(\mathfrak{su}(\ell+1)) \to U_q(\mathfrak{su}(\ell+1))^\text{op}$ is the Hopf $*$-algebra isomorphism which is defined on generators as

$$\theta(K_i) = K_i, \quad \theta(E_i) = F_i, \quad \theta(F_i) = E_i, $$

and satisfying $\theta^2 = \text{id}$.

For any $r$-dimensional $*$-representation of $U_q(\mathfrak{su}(\ell))$ like $\sigma$, we define the $\mathcal{A}(\mathbb{CP}_q^\ell)$-bimodule $\mathfrak{M}(\sigma) := \{ v \in \mathcal{A}(SU_q(\ell+1))^r | v \triangleleft h = \sigma(h)v, \quad \forall h \in U_q(\mathfrak{su}(\ell)) \}$. Suppose that $\sigma_N^{i}$ is obtained from the basic representation $\sigma_1 : U_q(\mathfrak{su}(\ell)) \to \text{End}(\mathcal{C}^\ell)$ lifted to a representation of $U_q(\mathfrak{su}(\ell))$ by $\sigma_N^{i} := q^{\frac{i}{\ell}} \sigma_N^{\ell}$. Then the space of anti-holomorphic 1-forms is given by $\Omega^{\ell+1}_{(0,1)} := \mathfrak{M}(\sigma_1^{0})$. Hence, any anti-holomorphic 1-form is a $\ell$-tuple $v := (v_1, ..., v_\ell)$ such that $v \triangleleft h = \sigma_1^{0}(h)v$. The complex structure of $\mathbb{CP}_q^\ell$ is given by

$$\mathcal{J} := \sum \mathcal{L}_{KX_i} \otimes e_i^\ell. $$

Here $e_i$’s are elements of the standard basis and $e_i^\ell$ is the left exterior product by $e_i$. We show that on $\mathcal{A}(\mathbb{CP}_q^\ell)$ we have

$$\overline{\partial} a = - \left( a \triangleleft F_\ell F_{\ell-1} ... F_1, a \triangleleft F_\ell F_{\ell-1} ... F_2, ..., a \triangleleft F_\ell F_{\ell-1}, a \triangleleft F_\ell \right). $$

(13) In fact,

$$\mathcal{L}_{X_i}a = a \triangleleft S^{-1}(K K_i K_{i+1} \cdots K_\ell K_i^{-1} \cdots K_{i+1}^{-1} K_\ell^{-1} K_i^{-1} K_{i+1} ... K_\ell^{-1} K_i^{-1} K_{i+1} ... K_\ell^{-1} K_i^{-1} K_{i+1} ... K_\ell^{-1} F_\ell F_{\ell-1} ... F_i) \right)$$

$$\triangleleft (-q^{-1})^{\ell-i} (-q)^{\ell-i+1} a \triangleleft F_\ell F_{\ell-1} ... F_i K K_i^{-1} K_{i+1}^{-1} ... K_\ell^{-1} K_i^{-1} K_{i+1} ... K_\ell^{-1} F_\ell F_{\ell-1} ... F_i$$

$$= a \triangleleft F_\ell F_{\ell-1} ... F_i$$
Proposition 4.1. For any non-negative integer $N$, equations (14) force that $|m\rangle$, as a second component of $a = |m'\rangle \otimes |m\rangle$, to be of the form

$$
\begin{bmatrix}
m_{1,\ell+1} & m & \ldots & m & 2m - m_{1,\ell+1} - N \\
m & m & \ldots & m \\
\vdots & \vdots & & \vdots \\
m & m & m & \ldots & m
\end{bmatrix}.
$$

Proof. $K_1 \triangleright a = a$ and $E_1 \triangleright a = 0$ give the equality for $m_{11} = m_{12} = m_{22}$. We know that $K_k \triangleright a = q^{\frac{a_k}{2}}a$, where $a_k$ is given by (2). For instance $a_1 = 2m_{11} - m_{12} - m_{22}$ and $a_2 = 2(m_{12} + m_{22}) - m_{11} - (m_{13} + m_{23} + m_{33})$ and so on. By (3) and (4) we have

$$
E_1|m\rangle = \left( -[m_{11} - m_{12}][m_{11} - m_{22} + 1] \right)^{1/2} |m_1\rangle,
$$

$$
E_2|m\rangle = \left( \frac{\left[m_{13} - m_{12} \right] \left[m_{23} - m_{12} - 1\right] \left[m_{33} - m_{12} - 2\right] \left[m_{12} - m_{11} + 1\right]}{\left[m_{12} - m_{22} + 1\right] \left[m_{12} - m_{22} + 2\right]} \right)^{\frac{1}{2}} |m_2\rangle,
$$

and

$$
F_2|m\rangle = \left( \frac{\left[m_{13} - m_{12} + 1\right] \left[m_{23} - m_{12}\right] \left[m_{33} - m_{12} - 1\right] \left[m_{11} - m_{12} - 1\right]}{\left[m_{12} - m_{22}\right] \left[m_{12} - m_{22} + 1\right]} \right)^{\frac{1}{2}} |m_2^{-1}\rangle
$$

$$
+ \left( \frac{\left[m_{13} - m_{22} + 2\right] \left[m_{23} - m_{22} + 1\right] \left[m_{33} - m_{22}\right] \left[m_{11} - m_{22} - 2\right]}{\left[m_{12} - m_{22} + 1\right] \left[m_{12} - m_{22} + 2\right]} \right)^{\frac{1}{2}} |m_2^{-2}\rangle.
$$

Now it is not difficult to see that $K_1|m\rangle = |m\rangle$ and $E_1|m\rangle = 0$, imposing

$$
2m_{11} - m_{12} - m_{22} = 0,
$$

$$
m_{11} - m_{12} = 0.
$$

So $m_{11} = m_{12} = m_{22}$. In the same manner $K_2|m\rangle = |m\rangle$, $E_2|m\rangle = 0$ and $F_2|m\rangle = 0$ give

$$
2m_{12} + 2m_{22} - m_{11} - m_{13} - m_{23} - m_{33} = 0,
$$

$$
m_{12} - m_{13} = 0,
$$

$$
m_{22} - m_{33} = 0.
$$

So we have $m_{11} = m_{12} = m_{22} = m_{13} = m_{23} = m_{33}$. Suppose that rows 1 to $k$ with $k + 1 < \ell + 1$, have been found equal to $m$. Let us prove that $E_k|m\rangle = 0$ and $F_k|m\rangle = 0$
will make the equality of all elements up to and including row \( k + 1 \). First note that in row \( k + 1 \), we have \( m_{2,k+1} = \ldots = m_{k,k+1} = m \). Let us look at \( A_k^1 \):

\[
A_k^1 = \left( - \frac{\prod_{i=1}^{k+1} [l_i,k+1 - l_i,k]}{\prod_{i \neq j} [l_i,k - l_j,k]} \right)^{1/2} \left( - \frac{\prod_{i=1}^{k+1} [l_i,k+1 - l_i,k]}{\prod_{i \neq j} [l_i,k - l_j,k]} \right)^{1/2}.
\]

It is not hard to see that \( A_k^1 = 0 \) if \( [l_{1,k+1} - l_{1,k}] = [m_{1,k+1} - m_{1,k}] = 0 \). So \( m_{1,k+1} = m_{1,k} = m \) and by a similar observation the action of \( F_k \) gives the equality \( m_{k,k+1} = m_{kk} = m \). But to get to the very top row we need to use the action of \( K_i \). We have

\[
a_\ell = 2 \sum_{i=1}^{\ell} m_{i,\ell} - \sum_{i=1}^{\ell-1} m_{i,\ell-1} - \sum_{i=1}^{\ell+1} m_{i,\ell+1} = 2\ell m - (\ell - 1)m - m_{1,\ell+1} - (\ell - 1)m - m_{\ell+1,\ell+1} = 2m - m_{1,\ell+1} - m_{\ell+1,\ell+1}.
\]

Since \( \ell a_\ell/2 = N\ell/2 \), we see that \( m_{1,\ell+1} = 2m - m_{1,\ell+1} - N \).

So we will find a basis for line bundles \( L_N \) as \( \langle t_n \rangle \), where \( n = (n_1,0,\ldots,0,n_1+N) \) and

\[
|\emptyset\rangle = \begin{bmatrix} m_{1,\ell+1} & m & \ldots & m & 2m - m_{1,\ell+1} - N \\ m & m & \ldots & m \\ \vdots & \vdots & & \vdots \\ m & m & & m \end{bmatrix}.
\]

Note that for a negative integer \( N \), the basis elements of \( L_N \) are of the form of \( t_n \), where \( n = (n_1 - N, 0, \ldots, 0, n_1) \).

**Theorem 4.1.** Let \( N \) be a non-negative integer. Then \( \dim \ker E_\ell \big|_{L_N} = \binom{N+\ell}{\ell} \) and \( \dim \ker E_\ell \big|_{L_{-N}} = 0 \).

**Proof.** The proof for \( L_{-N} \) is easy, so we just consider the case \( L_N \). A similar argument as the previous proposition shows that the vanishing of the action of \( E_\ell \) on \( |\emptyset\rangle \) gives \( m_{1,\ell+1} = m_{1,\ell} = m \). So we get the required tableaux form. Now we count the free entries in the first component \( |m_\ell\rangle \).

\[
\begin{bmatrix} m \ m \ldots \ m \\ x_{1,\ell} x_{2,\ell} \ldots \ x_{\ell,\ell} \\ \vdots \vdots \\ x_{1,2} x_{2,2} \\ x_{1,1} \end{bmatrix} = \begin{bmatrix} m \ m \ldots \ m \\ m \ m \ldots \ x_\ell \\ \vdots \vdots \\ m \ m \ldots \ x_2 \\ x_1 \end{bmatrix}
\]

with \( x_i = x_{i,i} \). The question turns into a simple combinatorial problem of counting the number of non-decreasing sequences \( m \geq x_1 \geq x_2 \geq \ldots \geq x_\ell \geq m - N \), which is \( \binom{N+\ell}{\ell} \). \( \square \)
Corollary 4.1. There is no non-constant holomorphic polynomials in $\mathcal{A}(\mathbb{C}P^\ell_q)$.

Proof. By (13) it is obvious that $\overline{\partial}a = 0$ iff $E_{\ell} \triangleright a = 0$. Now the previous lemma for $N = 0$ gives the result. \hfill \Box

4.1 Holomorphic line bundles

An anti-holomorphic connection on the line bundle $L_N$ is given by

$$\nabla^\ell_N : L_N \to \Omega^{0,1} \otimes \mathcal{A}(\mathbb{C}P^\ell_q) L_N$$

$$\nabla^\ell_N(\xi) := q^{-N} \Psi^*_N \overline{\partial} \Psi_N,$$

where $\Psi_N$ is a column vector $[5]$, given by $\Psi_N := (\psi_N^{j_1, ..., j_{\ell+1}})$ with

$$\psi_N^{j_1, ..., j_{\ell+1}} := [j_1, ..., j_{\ell+1}]^{1/2}(z_1^{j_1} ... z_{\ell+1}^{j_{\ell+1}})^*, \quad \forall j_1 + ... + j_{\ell+1} = N.$$

This is a flat connection as can be verified directly like [9]. This gives us the following Dolbeault complex

$$0 \to L_N \to \Omega^{0,1} \otimes \mathcal{A}(\mathbb{C}P^\ell_q) L_N \to \cdots \to \Omega^{0,\ell} \otimes \mathcal{A}(\mathbb{C}P^\ell_q) L_N \to 0.$$

The structure of the zeroth cohomology group $H^0(L_N, \nabla^\ell_N)$ of this complex which is called the space of holomorphic sections of $L_N$, is best described by the following theorem.

Corollary 4.2. For any positive integer $N$, the space of holomorphic sections of the canonical line bundles of $\mathbb{C}P^\ell_q$ is

$$H^0(L_N, \nabla^\ell_N) \simeq \mathbb{C}^{(N+\ell)}_i,$$

$$H^0(L_{-N}, \nabla_{-N}) = 0.$$

Proof. It is not difficult to see that the kernel of $\nabla^\ell_N$ coincides with the kernel of $E_{\ell} \triangleright (.)$. Now the result is an obvious consequence of (4.1). \hfill \Box

Here we would like to establish the fact that for any integers $N$ and $M$ we have a bimodule isomorphism $L_N \otimes \mathcal{A}(\mathbb{C}P^\ell_q) L_M \simeq L_{N+M}$. The multiplication map from left to right is an injective $\mathcal{A}(\mathbb{C}P^\ell_q)$-bilinear map. To see that this map is a surjection, we use a PBW-basis for $\mathcal{A}(S^2_{q^{\ell+1}})$ generated by

$$\{z_1^{s_1} z_2^{s_2} \cdots z_\ell^{s_\ell} (z_1^*)^{t_1} (z_2^*)^{t_2} \cdots (z_{\ell-1}^*)^{t_{\ell-1}} z_1^{s_1} z_2^{s_2} \cdots z_{\ell-1}^{s_{\ell-1}} (z_1^*)^{t_1} (z_2^*)^{t_2} \cdots (z_{\ell-1}^*)^{t_{\ell-1}} \},$$

for non-negative integers $s_i$ and $t_i$. Since

$$K_j \triangleright z_i = z_i, \quad K_j \triangleright z_i^* = z_i^* \text{ for } j < \ell$$

and

$$K_{\ell} \triangleright z_i = q^{1/2} z_i, \quad K_{\ell} \triangleright z_i^* = q^{-1/2} z_i^*,$$

we have

$$K_1 K_2^2 \cdots K_{\ell} \triangleright Z = q^{\ell/2(\sum s_i - \sum t_i)} Z,$$
where
\[ Z = z_1^{s_1} z_2^{s_2} \cdots z_\ell^{s_\ell} (z_1^{*})^1 (z_2^{*})^{t_2} \cdots (z_{\ell-1}^{*})^{t_{\ell-1}} \]
or
\[ Z = z_1^{s_1} z_2^{s_2} \cdots z_{\ell-1}^{s_{\ell-1}} (z_1^{*})^1 (z_2^{*})^{t_2} \cdots (z_{\ell-1}^{*})^{t_{\ell-1}}. \]
It is obvious that \( Z \in L_N \) iff \( \sum s_i - \sum t_i = N \).

Now suppose that \( Z = z_1^{s_1} z_2^{s_2} \cdots z_\ell^{s_\ell} (z_1^{*})^1 (z_2^{*})^{t_2} \cdots (z_{\ell-1}^{*})^{t_{\ell-1}} \in L_{N+M} \) and suppose \( k \) is the first positive integer such that \( \sum_{i=1}^k s_i > N \). Then take a partition of \( N \) as \( \sum_{i=1}^k r_i = N \), such that \( s_i - r_i \geq 0 \). Now the following is a preimage of \( Z \).
\[ q^R (z_1^{r_1} z_2^{r_2} \cdots z_k^{r_k}) = z_1^{s_1} z_2^{s_2} \cdots z_k^{s_k} \cdots z_{\ell-1}^{s_{\ell-1}}. \]

By the above discussion it is obvious that \( Z_1 := z_1^{r_1} z_2^{r_2} \cdots z_k^{r_k} \in L_N \) and
\[ Z_2 := z_1^{s_1-r_1} z_2^{s_2-r_2} \cdots z_k^{s_k-r_k} \cdots z_{\ell-1}^{s_{\ell-1}} \in L_M. \]
The result is obtained by noting that the product \( Z_1 Z_2 = q^{-R} Z \).

For later use we would like to mention here that \( \Omega^{(0,1)} \otimes A(\mathcal{CP}_q^\ell) L_N \simeq L_{\ell+1} \otimes A(\mathcal{CP}_q^\ell) L_N \simeq L_{N+\ell+1} \). In order to see this we recall the definition of \( \Omega^{(0,1)} := M(\sigma^0) \), where \( \sigma^0 \) is obtained from the representation \( \sigma_k : U_q(\mathfrak{su}(\ell)) \rightarrow \text{End}(W_k) \) lifted to a representation of \( U_q(\mathfrak{u}(\ell)) \) by \( \sigma^0_k(K) = q^k Id_{W_k} \). We define the \( A(\mathcal{CP}_q^\ell) \)-bimodule \( M(\sigma) := \{ v \in A(U_q(\ell+1)) \mid v \triangleleft h = \sigma(h)v, \forall h \in U_q(\mathfrak{h}(\ell)) \} \), where \( \sigma \) is an \( r \)-dimensional \( * \)-representation of \( U_q(\mathfrak{h}(\ell)) \). So in our case \( \sigma^0_k \) will be a \( 1 \)-dimensional \( * \)-representation of \( U_q(\mathfrak{u}(\ell)) \). Hence, any anti-holomorphic \( \ell \)-form is an element like \( v \) such that \( v \triangleleft h = \sigma^0_\ell(h)v \). The conditions that must hold are:
\[ K_i \triangleright a = a, \quad E_i \triangleright a = F_i \triangleright a = 0, \quad i = 1, 2, \ldots, \ell - 1. \]
\[ K_1 K_2 \cdots K_\ell \triangleright a = q^{(\ell+1)/2} a. \]
This gives us \( \Omega^{(0,1)} \simeq L_{\ell+1} \).

## 5 Bimodule connections

In this section we would like to show that line bundles \( L_N \) accept a bimodule connection in the sense of [S]. This means that there exists an isomorphism \( \lambda_N : L_N \otimes A(\mathcal{CP}_q^\ell) \Omega^{(0,1)} \rightarrow \Omega^{(0,1)} \otimes A(\mathcal{CP}_q^\ell) L_N \) such that
\[ \nabla_N(\xi a) := (\nabla_N \xi)a + \lambda_N(\xi \otimes a). \]
Let us first check the case \( N = 1 \). We will define \( \lambda_1 := \alpha_1^{-1} \beta_1 \) where,
\[ \alpha_1 : \Omega^{(0,1)} \otimes A(\mathcal{CP}_q^\ell) L_1 \rightarrow A(SU_q(\ell + 1))^\ell, \]
\[ \alpha_1((v_1, \ldots, v_\ell) \otimes \xi) := q^{1/2} (v_1 \xi, \ldots, v_\ell \xi) \]
\begin{align*}
\beta_1 : L_1 \otimes \mathcal{A}(\mathbb{C}P^\ell_1) \Omega^{(0,1)} & \rightarrow \mathcal{A}(SU_q(\ell + 1))^{\ell}, \\
\beta_1(\xi \otimes (v_1, \ldots, v_\ell)) & := q^{-1/2}(\xi v_1, \ldots, \xi v_\ell).
\end{align*}

Note that on the generators \( p_{jk} := z^*_j z_k \) of \( \mathcal{A}(\mathbb{C}P^\ell_1) \) we will have

\[ \overline{\partial} p_{jk} = \left( (-1)\ell q^{1/2-(\ell-1)}(u_j^1)^*, \ldots, q^{-3/2}(u_j^{\ell-1})^*, -q^{-1/2}(u_j^\ell)^* \right) u_k^{\ell+1}, \]

and a basis element of \( L_1 \) is of the form of \( t^n_{\underline{\omega}_{\underline{\Delta}}} \), where \( n = (n_1, 0, \ldots, 0, n_1 + 1) \),

\[ |0\rangle = \begin{bmatrix} m_1, e_{\ell+1} & m \ldots & m & m_1, e_{\ell+1} - 1 \\
m & m \ldots & m \\
& & \ddots & \ddots \\
m & & & m \end{bmatrix}, \]

and

\[ |i\rangle = \begin{bmatrix} m_1, e_{\ell+1} & m \ldots & m & m_1, e_{\ell+1} - 1 \\
x_\ell & m \ldots & m & y_\ell \\
& & \ddots & \ddots \\
x_2 & & & y_2 \\
x_1 & & & \end{bmatrix}. \]

Any typical element of \( L_1 \otimes \mathcal{A}(\mathbb{C}P^\ell_1) \Omega^{(0,1)} \) is a linear combination of \( t^n_{\underline{\omega}_{\underline{\Delta}}} \otimes \mathcal{A}(\mathbb{C}) \mathcal{R} \). We claim that

\[ v := t^n_{\underline{\omega}_{\underline{\Delta}}} \mathcal{R} \left( (-1)\ell q^{1/2-(\ell-1)}(u_j^1)^*, \ldots, q^{-3/2}(u_j^{\ell-1})^*, -q^{-1/2}(u_j^\ell)^* \right) \in \Omega^{(0,1)} \]

and \( u_k^{\ell+1} \in L_1 \). The latter can be easily obtained from the following observation,

\[ u_k^{\ell+1} \in (L_1 K_{2}^\ell \cdots K_{\ell}^\ell)^{2/\ell+1} = q^{\ell/\ell+1} u_k^{\ell+1}. \]

So we have to show that \( v \prec h = \sigma_1^0(h) v \), for all \( h \in U_q(\mathfrak{u}(\ell)) \). We will check this on generators. We need to know the following actions

\[ t^n_{\underline{\omega}_{\underline{\Delta}}} \prec h \quad \text{and} \quad (u_j^r)^* \prec h \]

for \( h = E_i, F_i, K_i \) and \( i = 1, \ldots, \ell - 1 \).

For \( \sigma_1^0 : U_q(\mathfrak{su}(\ell + 1)) \rightarrow \text{End}(W_1(\simeq \mathbb{C}^l)) \) we have [5]

\[ (\sigma_1^0(K_r) w)_I = q^{1/2r#I} w_I \]

\[ (\sigma_1^0(F_r) w)_I = \delta_{r#I+1} w_{r+} \]

\[ (\sigma_1^0(F_r) w)_I = \delta_{r#I-1} w_{r-} \]

where \( I = 1, \ldots, \ell \). Here \( r#I = 1 \) if \( r = I, r#I = -1 \) if \( r = I + 1 \) and \( r#I = 0 \) otherwise.

For \( K_1 \) we have

\[ \sigma_1^0(K_1) v = \sigma_1^0(K_1) \left( t^n_{\underline{\omega}_{\underline{\Delta}}} \left( (-1)\ell q^{-1/2-(\ell-1)}(u_j^1)^*, \ldots, q^{-3/2}(u_j^{\ell-1})^*, -q^{-1/2}(u_j^\ell)^* \right) \right) = t^n_{\underline{\omega}_{\underline{\Delta}}} \left( (-1)\ell q^{-1/2-(\ell-1)}(u_j^1)^*, \ldots, q^{-3/2}(u_j^{\ell-1})^*, -q^{-1/2}(u_j^\ell)^* \right). \]
Note that just a factor of $q^{1/2}$ and $q^{-1/2}$ contributed in the first and second component respectively. But on the other hand
\[
\left\{ t^n_{\mathcal{U}_{\ell}} \left( (-1)^{\ell} q^{1/2-(\ell-1)} (u_j^*)^*, \ldots, q^{-3/2} (u_j^*)^*, -q^{-1/2} (u_j^*)^* \right) \right\} \ll K_1
\]
\[
= t^n_{\mathcal{U}_{\ell}} K_1 \left( (-1)^{\ell} q^{1/2-(\ell-1)} (u_j^*)^*, \ldots, q^{-3/2} (u_j^*)^*, -q^{-1/2} (u_j^*)^* \right) \ll K_1
\]
\[
= t^n_{\mathcal{U}_{\ell}} \left( (-1)^{\ell} q^{-(\ell-1)} (u_j^*)^*, (-1)^{\ell-1} q^{-(\ell-2)} (u_j^*)^*, \ldots, -q^{-1/2} (u_j^*)^* \right).
\]
Here we used the fact that $t^n_{\mathcal{U}_{\ell}} K_1 = t^n_{\mathcal{U}_{\ell}}$ and $u_j^* K_1 = q^{1/2(\delta_2, i, -\delta_1, i)} u_j^*$, which follows from
\[
u_j^* K_1 = \sum_k (K_1, u_k^*) u_j^k = \sum_k \delta_k q^{1/2(\delta_2, i, -\delta_1, i)} u_j^k = q^{1/2(\delta_2, i, -\delta_1, i)} u_j^*.
\]
Now we have
\[
u_j^* K_1^{-1} = q^{1/2(\delta_2, i, -\delta_1, i)} u_j^*
\]
and
\[
(u_j^* K_1^{-1})^* = (u_j^*)^* \ll S(K_1^{-1})^* = q^{1/2(\delta_2, i, -\delta_1, i)} (u_j^*)^*.
\]
Therefore
\[
(u_j^*)^* K_1 = q^{1/2(\delta_2, i, -\delta_1, i)} (u_j^*)^*.
\]
The same proof works for other $K_r$'s.

Now let us prove the equality for $E_1$.
\[
\sigma_1^0(E_1) v = \sigma_1^0(E_1) \left\{ t^n_{\mathcal{U}_{\ell}} \left( (-1)^{\ell} q^{1/2-(\ell-1)} (u_j^*)^*, \ldots, q^{-3/2} (u_j^*)^*, -q^{-1/2} (u_j^*)^* \right) \right\}
\]
\[
= t^n_{\mathcal{U}_{\ell}} \left( (-1)^{\ell-1} q^{1/2-(\ell-2)} (u_j^*)^*, 0, 0, \ldots, 0 \right)
\]
On the other hand
\[
\left\{ t^n_{\mathcal{U}_{\ell}} \left( (-1)^{\ell} q^{1/2-(\ell-1)} (u_j^*)^*, \ldots, q^{-3/2} (u_j^*)^*, -q^{-1/2} (u_j^*)^* \right) \right\} \ll E_1
\]
\[
= t^n_{\mathcal{U}_{\ell}} K_1^{-1} \left( (-1)^{\ell} q^{1/2-(\ell-1)} (u_j^*)^*, \ldots, q^{-3/2} (u_j^*)^*, -q^{-1/2} (u_j^*)^* \right) \ll E_1
\]
\[
= t^n_{\mathcal{U}_{\ell}} \left( (-1)^{\ell-1} q^{1/2-(\ell-2)} (u_j^*)^* (u_j^*)^*, 0, 0, \ldots, 0 \right)
\]
Note that
\[
u_j^* \ll E_1 = \sum_k (E_1, u_k^*) u_j^k = \sum_k \delta_k \delta_k^1 u_j^k = \delta_k^1 u_j^1
\]
and $(u_j^*)^* \ll E_1 = -q\delta_1^1 (u_j^1)^*$. The same argument will work for other $E_r$'s.

**Lemma 5.1.** With the above notation $\text{Im } \alpha_1 = \text{Im } \beta_1$.

**Proof.** By the discussion before the lemma, the proof is clear. \qed
Taking the isomorphism $\lambda_1 := \alpha_1^{-1} \beta_1 : L_1 \otimes_{A(\mathbb{C}P_q^d)} \Omega(0,1) \to \Omega(0,1) \otimes_{A(\mathbb{C}P_q^d)} L_1$, since $L_N = L_1^\otimes N$, we can define the isomorphism $\lambda_N : L_N \otimes_{A(\mathbb{C}P_q^d)} \Omega(0,1) \to \Omega(0,1) \otimes_{A(\mathbb{C}P_q^d)} L_N$ by

$$\lambda_N := (\lambda_1 \otimes 1^{N-1}) \circ (1 \otimes \lambda_1 \otimes 1^{N-2}) \circ \cdots \circ (1^{N-1} \otimes \lambda_1).$$

Now, we prove that $\nabla^\partial_N$ has the right $\lambda_N$-twisted Leibniz property.

**Proposition 5.1.** Taking $\lambda_N$ as above, the following holds

$$\nabla^\partial_N(\xi a) = (\nabla^\partial_N\xi) a + \lambda_N(\xi \otimes \partial a), \quad \forall a \in A(\mathbb{C}P_q^d), \quad \forall \xi \in L_N,$

i.e. $\nabla^\partial_N$ is a bimodule connection on $L_N$.

**Proof.** Let us compute first the last (i.e. the $\ell$ th component) of the left hand side. Since $\xi \triangleleft K_\ell = q^{-N/2} \xi$ and $\Psi_N \triangleleft K_\ell = q^{-N/2} \Psi_N$, we will find that

$$q^{-N} \Psi_N^\dagger(\Psi_N \xi a \triangleleft F_\ell) = q^{-N} \Psi_N^\dagger(\Psi_N \xi a \triangleleft F_\ell)(\xi a \triangleleft K_\ell) + (\Psi_N \triangleleft K_\ell^{-1})(\xi a \triangleleft F_\ell)$$

$$= q^{-N/2}(\xi \triangleleft F_\ell)a + q^{-N} \xi(\triangleright F_\ell).$$

For the $\ell$ th component of the right hand we have

$$q^{-N/2}(\xi \triangleleft F_\ell)a + \lambda_N(\xi \otimes a \triangleleft F_\ell).$$

The previous lemma says that $q^{-N}$ will appear after acting by $\lambda_N$ on the second term. It can be seen that $\alpha_N$ of both sides coincides. For $i$ th component of the left hand side we have

$$q^{-N} \Psi_N^\dagger(\Psi_N \xi a \triangleleft F_i F_{i-1} \cdots F_1)$$

$$= q^{-N} \Psi_N^\dagger(\Psi_N \xi a \triangleleft F_i)(\xi a \triangleleft K_\ell) + (\Psi_N \triangleleft K_\ell^{-1})(\xi a \triangleleft F_\ell) F_{i-1} \cdots F_i$$

$$= q^{-N/2}(\xi \triangleleft F_\ell)a + q^{-N} \xi(\triangleright F_\ell) F_{i-1} \cdots F_i$$

$$= q^{-N/2}(\xi \triangleleft F_\ell F_{i-1} \cdots F_i)a + q^{-N} \xi(\triangleright F_\ell \cdots F_i).$$

For the right hand side we get the same result. Other components will be computed similarly. $\square$

Now we can prove that the two holomorphic structures on $L_N \otimes_{A(\mathbb{C}P_q^d)} L_M$ and $L_{N+M}$ are identical after the canonical isomorphism of these two spaces.

**Proposition 5.2.** The tensor product connection $\nabla^\partial_N \otimes 1 + (\lambda_N \otimes 1)(1 \otimes \nabla^\partial_M)$ coincides with the holomorphic structure on $L_N \otimes_{A(\mathbb{C}P_q^d)} L_M$ when identified with $L_{N+M}$.

**Proof.** We will look at the last component first.

$$\nabla^\partial_{N+M}(\xi_1 \xi_2) \bigg|_{\ell} = q^{-(N+M)} \Psi_{N+M}^\dagger \Psi_{N+M}(\xi_1 \xi_2)$$

$$= q^{-(N+M)} \Psi_{N+M}^\dagger(\Psi_{N+M} \xi_1 \xi_2) \triangleleft F_\ell$$

$$= q^{-(N+M)} \Psi_{N+M}^\dagger(\Psi_{N+M} \triangleleft F_\ell)(\xi_1 \xi_2) \triangleleft K_\ell$$

$$+ q^{-(N+M)} \Psi_{N+M}^\dagger(\Psi_{N+M} \triangleleft K_\ell^{-1})(\xi_1 \xi_2) \triangleleft F_\ell$$

$$= q^{-N+M} \xi_1 \xi_2 \triangleleft F_\ell$$

$$= q^{-N}(\xi_1 \triangleleft F_\ell) \xi_2 + q^{-N-M/2} \xi_1(\xi_2 \triangleleft F_\ell).$$
On the other hand
\[ \left\{ (\nabla_N^\ell \otimes 1) + (\lambda_N \otimes 1)(1 \otimes \nabla_M^\ell)) (\xi_1 \otimes \xi_2) \right\}_\ell = q^{-N/2}\xi_1 \trianglelefteq F_\ell \otimes \xi_2 + (\lambda_N \otimes 1)(\xi_1 \otimes q^{-M/2} \trianglelefteq F_\ell). \]

Interpreting this expression as an element of \( \Omega^{(0,1)} \otimes L_{N+M} \), after applying the map \( \lambda_N \), which gives us \( q^{-N} \) on the second summand, we will get the same result. The same argument as previous proposition gives the result for other components. \( \square \)

Now the quantum homogeneous coordinate ring \( R := \bigoplus_{n \geq 0} H^0(L_N, \nabla_N^\ell) \) of the quantum projective space can be described as follows. This result was first obtained for \( \ell = 1, 2 \) in [8, 9] where its relation with the work in [1, 2] is also explained.

**Theorem 5.1.** We have the algebra isomorphism
\[ \bigoplus_{n \geq 0} H^0(L_N, \nabla_N^\ell) \simeq \frac{\mathbb{C} \langle z_1, z_2, ..., z_\ell \rangle}{\langle z_i z_j - q z_j z_i : 1 \leq i < j \leq \ell \rangle} \]

**Proof.** The ring structure on \( R \) is coming from the tensor product \( L_{N_1} \otimes_{A(\mathbb{C}P^1_q)} L_{N_2} \simeq L_{N_1+N_2} \). Since the basis elements \( h_{0,j}^{(0,0,1)} \) of \( H^0(L_1, \nabla_1^\ell) \), as shown in section 2 are \( z_j \) for \( j = 1, 2, ..., \ell \), one can easily see that \( H^0(L_1, \nabla_1^\ell) = \mathbb{C} z_1 \oplus \mathbb{C} z_2 \oplus \cdots \oplus \mathbb{C} z_\ell \). Now the isomorphism follows from the identities \( z_i \otimes_{A(\mathbb{C}P^1_q)} z_j - q z_j \otimes_{A(\mathbb{C}P^1_q)} z_i = 0 \) in \( L_2 \), which is obvious. \( \square \)

6 **Existence of a twisted positive Hochschild cocycle for \( \mathbb{C}P^\ell_q \)**

In [3], Section VI.2, Connes shows that extremal positive Hochschild cocycles in the sense of [4] on the algebra of smooth functions on a compact oriented 2-dimensional manifold encode the information needed to define a holomorphic structure on the surface. There is a similar result for holomorphic structures on the noncommutative two torus (cf. Loc cit.). In particular the positive Hochschild cocycle is defined via the holomorphic structure and represents the fundamental cyclic cocycle. In [3] a notion of twisted positive Hochschild cocycle is introduced and a similar result is proved for the holomorphic structure of \( \mathbb{C}P^1_q \) and \( \mathbb{C}P^2_q \) in [8, 9]. Although the corresponding problem of characterizing holomorphic structures on higher dimensional (commutative or noncommutative) manifolds via positive Hochschild cocycles is still open, nevertheless these results suggest regarding (twisted) positive Hochschild cocycles as a possible framework for holomorphic noncommutative structures. In this section we prove an analogous result for \( \mathbb{C}P^\ell_q \) for all \( \ell \).

First we recall the notion of twisted Hochschild and cyclic cohomologies. Let \( A \) be an algebra and \( \sigma \) an automorphism of \( A \). For each \( n \geq 0 \), \( C^n(A) := \text{Hom}(A^{\otimes (n+1)}, \mathbb{C}) \) is the space of \( n \)-cochains on \( A \). Define the space of *twisted Hochschild* \( n \)-cochains as \( C^n_{\sigma}(A) := \text{Ker}\{(1 - \lambda_{\sigma}^{n+1}) : C^n(A) \to C^n(A)\} \), where the twisted cyclic map \( \lambda_{\sigma} : C^n(A) \to C^n(A) \) is defined as
\[ (\lambda_{\sigma}\phi)(a_0, a_1, ..., a_n) = (-1)^n\phi(\sigma(a_n), a_0, a_1, ..., a_{n-1}). \]
The twisted Hochschild coboundary map \( b_\sigma : C^n(A) \to C^{n+1}(A) \) is given by
\[
b_\sigma \phi(a_0, a_1, \ldots, a_{n+1}) = \sum_{i=0}^{n} (-1)^i \phi(a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1} \phi(\sigma(a_{n+1})a_0, \ldots, a_{n}).
\]

The cohomology of the complex \((C^*(A), b_\sigma)\) is called the **twisted Hochschild cohomology** of \(A\). We also need the notion of **twisted cyclic cohomology** of \(A\). It is by definition the cohomology of the complex \((C^*_{\sigma, \lambda}(A), b_\sigma)\), where
\[
C^n_{\sigma, \lambda} := \text{Ker}\{(1 - \lambda) : C^n(A) \to C^{n+1}(A)\}.
\]

Now we come back to the case of our interest, that is \(\mathbb{C}P^\ell_q\). Let \(\tau\) be the fundamental class on \(\mathbb{C}P^\ell_q\) defined as in [6] by a twisted cyclic cocycle
\[
\tau(a_0, a_1, a_2, \ldots, a_{2\ell}) := \int_h a_0 da_1 da_2 \cdots da_{2\ell}, \quad \forall a_i \in \mathcal{A}(\mathbb{C}P^\ell_q). \quad (15)
\]

Here \(h\) stands for the Haar state functional of the quantum group \(\mathcal{A}(SU_q(\ell + 1))\) which has a twisted tracial property \(h(xy) = h(y|x(x))\). Here the algebra automorphism \(\sigma\) is defined by
\[
\sigma : \mathcal{A}(SU_q(\ell + 1)) \to \mathcal{A}(SU_q(\ell + 1)), \quad \sigma(x) = K \triangleright x \triangleleft K,
\]
where \(K = (K_1^\ell K_2^{2(\ell - 1)} \cdots K_{j}^{j(\ell-j+1)} \cdots K_{\ell}^{\ell})^2\), see [5]. The map \(\sigma\), restricted to the algebra \(\mathcal{A}(\mathbb{C}P^\ell_q)\) is given by \(\sigma(x) = K \triangleright x\). Non-triviality of \(\tau\) has been shown in [3]. Now we recall the definition of a twisted positive Hochschild cocycle as given in [8].

**Definition 6.1.** A twisted Hochschild 2n-cocycle \(\phi\) on a *-algebra \(A\) is said to be **twisted positive** if the following map defines a positive sesquilinear form on the vector space \(\mathcal{A}^{\otimes (n+1)}\):
\[
\langle a_0 \otimes a_1 \otimes \ldots \otimes a_n, b_0 \otimes b_1 \otimes \ldots \otimes b_n \rangle = \phi(\sigma(b^*_n)a_0, a_1, \ldots, a_n, b^*_n, \ldots, b^*_1).
\]

### 6.1 A twisted positive Hochschild cocycle on \(\mathbb{C}P^\ell_q\)

We recall that the set of \((\ell, \ell)\)-shuffles (denoted by \(S_{\ell, \ell}\)) is set of all permutations \(\pi \in S_{2\ell}\) such that \(\pi(1) < \pi(2) < \cdots < \pi(\ell)\) and \(\pi(\ell + 1) < \pi(\ell + 2) < \cdots < \pi(2\ell)\). Here we would like to look at a shuffle \(\pi\) as an increasing function from \(\{\ell+1, \ldots, 2\ell\}\) to \(\{1, 2, \ldots, 2\ell\}\). Let us define \(\theta^\pi : \{1, 2, \ldots, 2\ell\} \to \{\pm\}\) by \(\theta^\pi|_{\text{Im } \pi} = -1\) and \(\theta^\pi|_{\text{Im } \pi^c} = +1\). For any \(\pi \in S_{\ell, \ell}\) define
\[
\varphi_\pi(a_0, a_1, \cdots, a_{2\ell}) := \int_h a_0(\partial^{\varphi^\pi} a_1)(\partial^{\varphi^\pi} a_2) \cdots (\partial^{\varphi^\pi} a_{2\ell}). \quad (16)
\]

Here \(\partial^+ = \partial^* \partial\) and \(\theta^\pi = \theta^\pi(\cdot)\). Now suppose that \(\pi\) and \(\pi'\) are two shuffles that are just different in their values on a single value \(i\) such that \(|\pi'(i) - \pi(i)| = 1\). We define a cochain \(\psi_{\pi, \pi'}\) by
\[
\psi_{\pi, \pi'}(a_0, a_1, a_2, \ldots, a_{2\ell-1}) := \int_h a_0(\partial^{\psi^\pi} a_1)(\partial^{\psi^\pi} a_2) \cdots (\partial^{\psi^\pi} a_{j}) (\partial^{\psi^\pi} a_{j+1}) \cdots (\partial^{\psi^\pi} a_{2\ell-1}).
\]

Here \(j = \text{min}\{\pi(i), \pi'(i)\}\). It is then easy to prove that \(b_\sigma \psi_\pi = \pm (\varphi_\pi - \varphi_{\pi'})\). The proof is based on the following easy observation.
\[
\partial \bar{\partial}(ab) = a\partial \bar{b} + \partial(a \bar{b}) - \bar{a}a\partial b + (\partial \bar{\partial}a)b.
\]
The term $\partial^\pi \partial^{\pi'}$ is either $\partial^\pi \partial^\pi$ or $\overline{\partial} \partial$ simply because of our choice of $\pi$ and $\pi'$.

Now we recall an easy combinatorial fact. The number of permutations of $2\ell$ letters including $\ell$ letter $A$ and $\ell$ letter $B$ is $(2\ell) \ell = (2\ell)!$. All permutations can be grouped in two groups and in each group there exists an order on permutations $\{\pi_1, \ldots, \pi_r\}$ and $\{\pi'_1, \ldots, \pi'_r\}$ with $r = \frac{1}{2}(2\ell)$, such that $\pi_{i+1}$ (respectively $\pi'_{i+1}$), can be obtained from $\pi_i$ (resp. $\pi'_i$) just with replacing the two letters in the spots $j$ and $j+1$ where $1 \leq j \leq r-1$. In addition we can always choose $\pi_1 = AA \cdots ABB \cdots B$ and $\pi'_1 = BB \cdots BAA \cdots A$. The permutation $\pi_r$ has the above mentioned property with respect to one of $\pi'_i$.

Now we come back to the case $\mathbb{C}P^\ell_q$. We consider a complex structure $(\Omega^{(\ell, \ell)}(A), \partial, \overline{\partial})$ on the $*$-algebra $\mathcal{A}(\mathbb{C}P^\ell_q)$ with $\ast : \Omega^{(\ell, \ell)}(A) \to \Omega^{(\ell, \ell)}(A)$ such that $\overline{\partial} a^\ast = (\partial a)^\ast$. We have seen that $\Omega^0(\mathfrak{su}(\ell))$, where $\sigma^{0,1}$ restricted to $\mathcal{U}_q(\mathfrak{su}(\ell))$ is the fundamental representation of $\mathcal{U}_q(\mathfrak{su}(\ell))$ in $\mathbb{C}^\ell$ and $\sigma^{0,1}(K^1 K_2 \cdots K^\ell) = q^{\ell+1} I$. The representation $\sigma^{1,0}$ can be obtained from $\sigma^{0,1}$ by conjugation. Define

$$\partial a := \alpha(E_\ell, E_\ell E_{\ell-1}, \ldots, E_\ell \cdots E_2 E_1), \quad \overline{\partial} a := \alpha(F_\ell \cdots F_2 F_1, \cdots F_\ell F_{\ell-1}, F_\ell).$$

For an anti-holomorphic 1-form $\omega = (\omega_1, \omega_2, \ldots, \omega_\ell)$ we define

$$\omega^\ast := (-q \omega_1^\ast, q^2 \omega_{\ell-1}^\ast, \ldots, (-q)^{\ell-1} \omega^\ast_2, (-q)^{\ell} \omega^\ast_1).$$

The property $\overline{\partial} a^\ast = (\partial a)^\ast$ holds simply because

$$(a^\ast \alpha F_\ell F_{\ell-1} \cdots F_1)^\ast = a \alpha S(F_\ell F_{\ell-1} \cdots F_1)^\ast = (-q)^{(\ell+1)}a \alpha E_\ell E_{\ell-1} \cdots E_1.$$

One can define $\ast$ on anti-holomorphic forms such that $(\omega \wedge_q \omega')^\ast = (-1)^{\deg(\omega) \deg(\omega')} \omega^\ast \wedge_q \omega^\ast$, then extend it to all holomorphic and anti-holomorphic forms with $\overline{\partial} a^\ast = (\partial a)^\ast$. Note that we can extend $\wedge_q$ on holomorphic forms as in [5]. One can see that

$$\partial a_1 \partial a_2 \cdots \partial a_\ell \overline{\partial} a_\ell^\ast \cdots \overline{\partial} a_2 \overline{\partial} a_1^\ast = \partial a_1 \partial a_2 \cdots \partial a_\ell (\overline{\partial} a_\ell)^\ast \cdots (\overline{\partial} a_2)^\ast (\overline{\partial} a_1)^\ast = -\partial a_1 \partial a_2 \cdots \partial a_\ell (\partial a_1 \partial a_2 \cdots \partial a_\ell)^\ast.$$

We will need the following simple lemma for future computations.

**Lemma 6.1.** For any $a_0, a_1, a_2, \cdots, a_{2\ell+1} \in \mathcal{A}(\mathbb{C}P^\ell_q)$ the following identities hold:

$$\int h_0 \partial a_1 \cdots \partial a_\ell \overline{\partial} a_\ell \cdots \overline{\partial} a_2 \overline{\partial} a_1 + = \int h_0 \sigma(a_{2\ell+1}) a_0 \partial a_1 \cdots \partial a_\ell \overline{\partial} a_\ell \cdots \overline{\partial} a_2 \overline{\partial} a_1.$$

**Proof.** The space of $\Omega^{(\ell, \ell)}$ is a rank one free $\mathcal{A}(\mathbb{C}P^\ell_q)$-module. Let $\omega$ be the central basis element for the space of $\Omega^{(\ell, \ell)}$ and let $\partial a_1 \cdots \partial a_\ell \overline{\partial} a_\ell \cdots \overline{\partial} a_2 = x\omega$. Then

$$\int h_0 \left\{ a_0(\partial a_1 \cdots \partial a_\ell \overline{\partial} a_\ell \cdots \overline{\partial} a_2)n_{2\ell+1} - \sigma(a_{2\ell+1}) a_0 \partial a_1 \cdots \partial a_\ell \overline{\partial} a_\ell \cdots \overline{\partial} a_2 \right\}$$

$$= \int h_0 (a_0 x \omega a_{2\ell+1} - \sigma(a_{2\ell+1}) a_0 x \omega)$$

$$= \int h_0 (a_0 x a_{2\ell+1} \omega - \sigma(a_{2\ell+1}) a_0 x \omega)$$

$$= h(a_0 x a_{2\ell+1} - \sigma(a_{2\ell+1}) a_0 x) = 0.$$

The last equality comes from the twisted property of the Haar state. \qed
Using $d = \partial + \overline{\partial}$, we have

$$\tau = \sum_{\pi \in S_{\ell, \ell}} \varphi_\pi,$$

where $\varphi_\pi$ is given by (16). Let $\pi_1 = \text{id}$, i.e. $\pi_1$ is the shuffle that keeps every letter at the same spot. Define the Hochschild cocycle

$$\varphi := -2r \varphi_{\pi_1},$$

where $r = \frac{1}{2} \binom{2\ell}{\ell}$.

**Theorem 6.1.** The $2\ell$-cocycle $\varphi$ defined by (17), is a twisted positive Hochschild cocycle and it is cohomologous to the fundamental twisted cyclic cocycle $\tau$.

**Proof.** We first verify the twisted cocycle property.

$$\varphi(\sigma(a_0), \sigma(a_1), \sigma(a_2), \ldots, \sigma(a_{2\ell}))$$

$$= 2r \int_h \sigma(a_0) \partial \sigma(a_1) \cdots \partial \sigma(a_\ell) \overline{\partial} \sigma(a_{\ell+1}) \cdots \overline{\partial} \sigma(a_{2\ell})$$

$$= 2r \int_h K \triangleright (a_0 \partial a_1 \cdots \partial a_\ell \overline{\partial} a_{\ell+1} \cdots \overline{\partial} a_{2\ell})$$

$$= 2r \epsilon(K) \int_h a_0 \partial a_1 \cdots \partial a_\ell \overline{\partial} a_{\ell+1} \cdots \overline{\partial} a_{2\ell}$$

$$= \varphi(a_0, a_1, a_2, \ldots, a_{2\ell}).$$

For positivity one can see that

$$\varphi(\sigma(a_0^*) a_0, a_1, a_2, \ldots, a_\ell, a_\ell^*, \ldots, a_2^*, a_1^*) = -2r \int_h \sigma(a_0^*) a_0 \partial a_1 \partial a_2 \cdots \partial a_\ell \overline{\partial} a_\ell^* \cdots \overline{\partial} a_2^* \overline{\partial} a_1^*$$

$$= -2r \int_h a_0 \partial a_1 \partial a_2 \cdots \partial a_\ell \overline{\partial} a_\ell^* \cdots \overline{\partial} a_2^* \overline{\partial} a_1^* a_0$$

$$= 2r \int_h (a_0 \partial a_1 \partial a_2 \cdots \partial a_\ell) (a_0 \partial a_1 \partial a_2 \cdots \partial a_\ell)^*.$$

One can take $\partial a_i = (v_i^1, v_i^2, \ldots, v_i^\ell)$, then using the multiplication rule of type $(1,0)$ forms (for $(0,1)$ forms c.f. 5), we find that $(a_0 \partial a_1 \partial a_2 \cdots \partial a_3) (a_0 \partial a_1 \partial a_2 \cdots \partial a_3)^* = \mu \mu^*$, where

$$\mu = a_0 \sum_{\pi \in S_\ell} (-q^{-1})^{||\pi||} v_{\pi(1)}^1 v_{\pi(2)}^2 \cdots v_{\pi(\ell)}^\ell.$$

Hence

$$\varphi(\sigma(a_0^*) a_0, a_1, a_2, \ldots, a_\ell, a_\ell^*, \ldots, a_2^*, a_1^*) = 2r h(\mu \mu^*) \geq 0.$$

Here we used the positivity of the Haar functional $h$.

Now we would like to find the coefficients $m, k$ such that $m \tau - k \varphi_\pi = b_\pi \psi$ for a suitable $(2\ell - 1)$-cocycle $\psi$. Here we order all $\varphi_\pi$’s as explained at the beginning of the section, i.e. we use the order for permutations of $\partial$ and $\overline{\partial}$ to make two sets $\{\varphi_{\pi_1}, \varphi_{\pi_2}, \ldots, \varphi_{\pi_r}\}$ and $\{\varphi_{\pi'_1}, \varphi_{\pi'_2}, \ldots, \varphi_{\pi'_s}\}$, where $r = \frac{1}{2} \binom{2\ell}{\ell}$. For instance we give the formula for one choice of $\varphi_{\pi_2}$.

$$\varphi_{\pi_2}(a_0, a_1, \ldots, a_{2\ell}) := \int_h a_0 \overline{\partial} a_1 \overline{\partial} a_2 \cdots \overline{\partial} a_{\ell-1} \overline{\partial} a_\ell \overline{\partial} a_{\ell+1} \partial a_{\ell+2} \cdots \partial a_{2\ell}. $$
One can show that there exist $2r - 1$ twisted cochains $\psi_{i, j}$ such that

\[
\begin{align*}
  b_{\sigma} \psi_{i, 1, 2} &= \varphi_{i, 1} - \varphi_{i, 2}, \\
  b_{\sigma} \psi_{i, 2, 3} &= \varphi_{i, 2} - \varphi_{i, 3}, \\
  \vdots \\
  b_{\sigma} \psi_{i, r - 1, r} &= \varphi_{i, r - 1} - \varphi_{i, r}, \\
  b_{\sigma} \psi_{i, r, r} &= \varphi_{i, r} - \varphi_{i, r}', \\
  b_{\sigma} \psi_{i, r, r}' &= \varphi_{i, r}' - \varphi_{i, r}'', \\
  b_{\sigma} \psi_{i, r', r} &= \varphi_{i, r} - \varphi_{i, r}', \\
  \vdots \\
  b_{\sigma} \psi_{i, r', r'} &= \varphi_{i, r} - \varphi_{i, r}'.
\end{align*}
\]

For instance $\psi_{1, 2}$ (up to a $\pm$ sign) is defined by

\[
\psi_{1, 2}(a_0, a_1, ..., a_{2\ell - 1}) := \int_h a_0 \partial a_1 \cdots \partial a_{\ell - 1} (\partial \overline{a}_0 \partial \overline{a}_1 \partial \overline{a}_{\ell - 1} \cdots \partial \overline{a}_{2\ell - 1}).
\]

Define

\[
\psi := \sum_{i=1}^{r-1} x_i \psi_{i, i+1} + x_r \psi_{i, i}' + \sum_{i=1}^{r-1} x_{r+i} \psi_{i, i}'',
\]

with constants $x_i$’s $i = 1, 2, \cdots, 2r - 1$ have to be determined. We find the following linear system of equations for $m\tau - k\varphi_{1} = b_{\sigma} \psi$.

\[
\begin{align*}
  m - k - x_1 &= 0, \\
  m + x_1 - x_2 &= 0, \\
  \vdots \\
  m + x_{r-1} - x_r &= 0, \\
  m + x_r + 1 &= 0, \\
  m + x_{r+1} - x_{r+2} &= 0, \\
  \vdots \\
  m + x_{r+k-1} - x_{r+k} &= 0, \\
  m + x_{r+k} - x_{r+k+1} &= 0, \\
  \vdots \\
  m + x_{2r-2} - x_{2r-1} &= 0, \\
  m + x_{2r-1} &= 0.
\end{align*}
\]

This system has the one parameter family of solutions given by

\[
x_i = -(2r - i)m \quad \text{for} \quad i \in \{1, 2, \cdots, 2r - 1\} - \{r + 1\}, \quad x_{r+1} = -m, \quad k = 2rm.
\]

For $m = 1$, we have $\tau - 2r\varphi_{1} = b_{\sigma} \psi$. Note that $\psi_i$’s are defined up to sign. \hfill \Box
7 The Riemann-Roch theorem for $\mathbb{C}P^\ell_q$, $\ell = 1, 2$

First recall that, for classical projective space $\mathbb{C}P^m$, its sheaf (or equivalently Dolbeault) cohomology with coefficients in the sheaf of holomorphic sections of line bundles $\mathcal{O}(m)$ are given by

$$H^i(\mathbb{C}P^m, \mathcal{O}(m)) = \begin{cases} \mathbb{C}[z_0, z_1, \ldots, z_n]_m & \text{if } i = 0, m \geq 0, \\ 0 & \text{if } 0 < i < n \\ H^0(\mathbb{C}P^m, \mathcal{O}(-m - n - 1))^* & \text{if } i = n, m > -n - 1. \end{cases}$$

Therefore for the holomorphic Euler characteristic of $\mathcal{O}(m)$, we get

$$\chi(\mathbb{C}P^1, \mathcal{O}(m)) : = \dim H^0(\mathbb{C}P^1, \mathcal{O}(m)) - \dim H^1(\mathbb{C}P^1, \mathcal{O}(m)) = m + 1.$$

7.1 The case of $\mathbb{C}P^1_q$

This last formula has an analog in the case of $\mathbb{C}P^1_q$. The zeroth cohomology has been computed in [3], but for completeness we recall it here again. First let us recall that finite dimensional irreducible representations of $U_q(\mathfrak{su}(2))$ are given by vector spaces $V_l$, where $2l \in \mathbb{N}$ with basis $|l, m\rangle$, $m \in \{-l, \ldots, l\}$. The action on generators are given by

$$K|l, m\rangle = q^m|l, m\rangle, \quad E|l, m\rangle = \sqrt{[l - m + 1][l + m]} |l, m - 1\rangle, \quad F|l, m\rangle = \sqrt{[l + m + 1][l - m]} |l, m + 1\rangle.$$

We will have the isomorphism $A(SU_q(2)) = \bigoplus V_l \otimes V_l^*$ and under this isomorphism the space of canonical quantum line bundle $L_N : = \{ a \in A(SU_q(2)) \mid h \triangleright a = q^{N/2} a \}$ corresponds to $\{|l, N/2\otimes |l, m\rangle \mid l \geq |N/2|, m = -2l, \ldots, 2l\}$. From now on we will use the notation $|l, n, m\rangle = |l, n\rangle \otimes |l, m\rangle$.

The anti-holomorphic part of the connection on $L_N$ is given by $\nabla^\overline{\partial}|l, \frac{N}{2}, n\rangle := E|l, \frac{N}{2}, n\rangle$. Consider the Dolbeault complex of $\mathbb{C}P^1_q$

$$0 \to L_N \to \Omega^{(0,1)} \otimes L_N \to 0,$$

or equivalently

$$0 \to L_N \to L_{N-2} \to 0.$$

One can easily see that $\nabla^\overline{\partial} = E|l, \frac{N}{2}, m\rangle = \sqrt{[l - \frac{N}{2} + 1][l + \frac{N}{2}]} |l, \frac{N}{2} - 1, m\rangle$. To find the holomorphic Euler characteristic $\chi(\mathbb{C}P^1_q, L_N)$, we will consider the following three cases.

- $N \geq 2$.

In this case, the kernel of $\nabla^\overline{\partial}$ is zero, simply because $l + \frac{N}{2}$ cannot be zero and $l - \frac{N}{2} + 1$ is zero only if $l = \frac{N}{2} - 1$, which is impossible in this case, since by assumption $l \geq \frac{N}{2}$. The Image of $\nabla^\overline{\partial}$ will be generated by the basis elements $|l, \frac{N}{2} - 1, m\rangle$ with $l \geq \frac{N}{2}$. But it differs from basis of $L_{N-2}$ by elements $|l, \frac{N}{2} - 1, \frac{N}{2} - 1, m\rangle$ which can be counted as $N - 1$ elements.
• $N = 1$.

Here we have $\nabla^\partial \xi = \sqrt{[l - \frac{1}{2} + 1][l + \frac{1}{2}][l, \frac{1}{2} - 1, m]}$. So $E[l, \frac{1}{2}, m] = [l + \frac{1}{2}][l, -\frac{1}{2}, m]$ and it is not hard to see that $\text{Im} \nabla^\partial = L_{N-2}$. The same argument as case $N \geq 2$ shows that $\text{Ker} \nabla^\partial = 0$. Hence $\chi(\mathbb{C}P^1_q, L_N) = 0$.

• $N \leq 0$.

If $N \leq 0, l + \frac{N}{2} = 0$ when $l = -\frac{N}{2}$ and this gives the set $\{-\frac{N}{2}, \frac{N}{2}, m\}$ as a basis for the space of holomorphic sections of $\mathcal{L}_N$. So $\dim \text{Ker} \nabla^\partial = \lfloor N \rfloor + 1$. In a similar manner to case $N = 1$ one can show that the map $\nabla^\partial$ is surjective. Therefore we will come to the following result

$$\chi(\mathbb{C}P^1_q, L_N) = -N + 1.$$  

Note that there is a switch between $N$ and $-N$ with respect to the classical case.

### 7.2 Serre duality for $\mathbb{C}P^2_q$

There exists a non-degenerate pairing $\langle \cdot, \cdot \rangle : L_N \times L_{-N} \to \mathbb{C}$, given by

$$\langle \xi, \eta \rangle := h(\xi \eta), \quad \forall \xi \in L_N, \quad \forall \eta \in L_{-N}. \quad (19)$$

Here $h$ is the Haar state of the quantum group $A(SU_q(3))$. The map is obviously bilinear and the nondegeneracy comes from the facts that $L^*_N \subset L_{-N}$ and $h$ is faithful. Now consider the $(0, q)$-Dolbeault complex of $\mathbb{C}P^2_q$

$$0 \to L_N \to \Omega^{(0,1)} \otimes L_N \to \Omega^{(0,2)} \otimes L_N \to 0. \quad (20)$$

We would like to state an analogue of Serre duality theorem for this complex as

**Proposition 7.1.** There exists a non-degenerate pairing defined by

$$\langle \cdot, \cdot \rangle : H^2(\nabla, L_N) \times H^0(\nabla, L_{-N-3}) \to \mathbb{C}$$

$$\langle [\xi], [\eta] \rangle := h(\xi \eta), \quad \forall \xi \in L_{N+3}, \quad \forall \eta \in L_{-N-3}.$$  

**Proof.** First note that $H^2(\nabla, L_N)$ is a quotient of $L_{N+3}$ and $H^0(\nabla, L_{-N-3})$ is a subspace of $L_{-N-3}$. We show that this map is well defined. For this, suppose that $\xi$ and $\xi'$ are in the same cohomology class. Hence $h(\xi \eta) - h(\xi' \eta) = h((\xi - \xi') \eta) = h(\partial \alpha \eta) = h(\overline{\partial}(\alpha \eta) - \alpha \overline{\partial} \eta) = 0$, by noting that $\eta \in \text{Ker} \overline{\partial}$ and $h$ has invariance property with respect to the map $\overline{\partial}$. Now non-degeneracy is obvious by the above discussion.

\[\square\]

The above result easily can be lifted to the general case of $\mathbb{C}P^k_q$ in the following way. The pairing

$$\langle \xi, \eta \rangle := h(\xi \eta), \quad \forall \xi \in L_N, \forall \eta \in L_{-N}. \quad (21)$$

is a nondegenerate pairing and hold true passing to the cohomology

$$\langle \cdot, \cdot \rangle : H^1(\nabla, L_N) \times H^0(\nabla, L_{-N-\ell-1}) \to \mathbb{C}$$

$$\langle [\xi], [\eta] \rangle := h(\xi \eta), \quad \forall \xi \in L_{N+\ell+1}, \quad \forall \eta \in L_{-N-\ell-1}.$$
In the following we will compute the \((0, q)\)-Dolbeault cohomology of \(\mathbb{C}P^2_q\). The result is analog of the classical case. i.e.

**Theorem 7.1.** With the above notations

\[
H^i(\nabla^0, L_N) = \begin{cases} 
\mathbb{C}\langle z_1, z_2, z_3 \rangle_N & \text{if } i = 0, N \geq 0, \\
0 & \text{if } i = 0, N < 0, \\
\mathbb{C}\langle z_1, z_2, z_3 \rangle_{N-3}^* & \text{if } i = 2, N > -3 \\
\mathbb{C}\langle z_1, z_2, z_3 \rangle_{N+3} & \text{if } i = 2, N \leq -3.
\end{cases}
\]

**Proof.** The zeroth-cohomology has been computed in [2] and the second cohomology comes from the Serre duality. So we just have to prove that the triviality of the first cohomology. In order to do so, we will calculate the \(\text{Im} \, \bar{\mathcal{D}}_1\) and the Ker \(\mathcal{D}_2\) and show the equality.

For \(\text{Ker} \, \mathcal{D}_2\) we will use the \(\bar{\mathcal{D}}_2(v_+, v_-) = -E_2 \triangleright v_+ - E_2E_1 + 2\,[2^{-1}]E_1E_2 \triangleright v_-\). Applying \(v_+ = t(n, n + 3)_{\frac{1}{2}}^{1,0,1/2}\) and \(v_- = t(n, n + 3)_{\frac{1}{2}}^{1,0,-1/2}\) we will have

\[
-E_2 \triangleright t(n, n + 3)_{\frac{1}{2}}^{1,0,1/2} = -\sqrt{\frac{n(n+3+2)}{2\,[3]}} t(n, n+3)_{\frac{1}{2}}^{1,1,0} - \sqrt{\frac{n+2\,[n+3]}{2}} t(n, n+3)_{\frac{1}{2}}^{0,1,0}.
\]

and

\[
-E_2E_1 \triangleright t(n, n + 3)_{\frac{1}{2}}^{1,0,-1/2} = -\sqrt{\frac{n(n+3+2)}{2\,[3]}} t(n, n+3)_{\frac{1}{2}}^{1,1,0} - \sqrt{\frac{n+2\,[n+3]}{2}} t(n, n+3)_{\frac{1}{2}}^{0,1,0}.
\]

and

\[
2\,[2^{-1}]E_1E_2 \triangleright t(n, n + 3)_{\frac{1}{2}}^{1,0,-1/2} = \sqrt{\frac{n\,[n+5]}{2\,[3]}} t(n, n+3)_{\frac{1}{2}}^{1,1,0}.
\]

Hence

\[
\bar{\mathcal{D}}_2(t(n, n + 3)_{\frac{1}{2}}^{1,0,1/2}, t(n, n + 3)_{\frac{1}{2}}^{1,0,-1/2}) = -2 \sqrt{\frac{n+2\,[n+3]}{2}} t(n, n+3)_{\frac{1}{2}}^{0,1,0}.
\]

This shows that \(H^1 = \frac{\text{Ker} \, \bar{\mathcal{D}}_2}{\text{Im} \, \bar{\mathcal{D}}_1} = 0\) in the case of \(N = 0\).

By a similar but lengthier calculation, one can prove that \(H^0(\nabla^0, L_N) = 0\) for all \(N \neq 0\).

**Acknowledgments**

We are much obliged and thankful to Francesco D’Andrea for kindly and promptly answering many questions about the subject of [6, 7] and suggesting improvements on the first draft of the current paper.
References

[1] M. Artin, M. van den Bergh, Some algebras associated to automorphisms of elliptic curves. The Grothendieck Festschrift, Vol. I, 33-85, Progr. Math. 86, Birkhäuser, Boston, MA, 1990.

[2] M. Artin, M. van den Bergh, Twisted homogeneous coordinate rings. J. Algebra 133 (1990), 249-271.

[3] A. Connes, Noncommutative geometry, Academic Press, 1994.

[4] A. Connes, J. Cuntz, Quasi homomorphismes, cohomologie cyclique et positivite, Comm. Math. Phys. 114 (1988) 515-526.

[5] F. D’Andrea, L. Dabrowski, Dirac operators on quantum projective spaces. arXiv:0901.4735v1

[6] F. D’Andrea, G. Landi, Anti-selfdual connections on the quantum projective plane: Monopoles. Commun. Math. Phys. 297(2010) 841-893; arXiv:0903.3551v1.

[7] F. D’Andrea, L. Dabrowski, G. Landi, The noncommutative geometry of the quantum projective plane. arXiv:0712.3401v2. Rev. Math. Phys. 20 (2008), 979-1006.

[8] M. Khalkhali, G. Landi, W. van Suijlekom, Holomorphic structures on the quantum projective line. Int. Math. Res Notices, doi:10.1093/imrn/rnq097. arXiv:0907.0154v2.

[9] M. Khalkhali, A. Moatadelro, The quantum homogeneous coordinate ring of projective plane, J. Geom. Phys. Volume 61, Issue 1, January 2011, Pages 276-289, arXiv:1007.3255

[10] A. Klimyk, K. Schmüdgen, Quantum groups and their representations, Springer, 1997.

Department of Mathematics, University of Western Ontario, London, Ontario, N6A5B7, Canada.

Email: masoud@uwo.ca

Department of Mathematics, University of Western Ontario, London, Ontario, N6A5B7, Canada.

Email: amotadel@uwo.ca