A CHARACTERIZATION OF THE MENER
PROPERTY BY MEANS OF ULTRAFILTER
CONVERGENCE

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Abstract. We characterize various Menger/Rothberger-related
properties, and discuss their behavior with respect to products.

Motivated by classical arguments in the theory of ultrafilter conve-
gerence, we give a characterization of the Menger property and of the
Rothberger property by means of ultrafilters and filters, respectively.
In this vein, we discuss the behavior with respect to products of the
above notions, and of some weaker variants.

A summary of the paper follows. In Section 1 we briefly recall the
notion of filter convergence, together with some classical examples,
which furnish the main motivation for the present paper.

In Section 2 we show that the Menger property allows a charac-
terization in terms of ultrafilter convergence. Actually, our methods
work also for the notion in which we fix a bound for the cardinality
of the covers under consideration. Even more generally, we can also
consider more than countably many families of covers, and even allow
infinite subsets to be selected. We are thus led to consider a general-
ed Rothberger notion \( R(\lambda, \mu; < \kappa) \) which depends on three cardinals,
and generalizes simultaneously the Menger property, the Rothberger prop-
erty, the Menger and the Rothberger properties for countable covers, as
well as \([\kappa, \mu]-\)compactness (in particular, countable compactness, ini-
tial \( \mu\)-compactness, Lindel"{o}fness and final \( \kappa\)-compactness). See Definition
2.1.

In Section 3 we study preservation and non preservation under pro-
ducts of the generalized Rothberger notion \( R(\lambda, \mu; < \kappa) \), establishing a
strong connection with preservation/non preservation of \([\kappa, \mu]-\)com-
 pactness. For example, we get that every product of members of some

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family $\mathcal{F}$ of topological spaces satisfies the Menger property for countable covers if and only if every product of members of $\mathcal{F}$ is countably compact.

Finally, in Section 4 we deal with particular cases strongly resembling the Rothberger property (in the sense that only one element can be selected from each cover). In this case, the characterization in terms of convergence involves filters which are not maximal, and this fact can be seen as the one "responsible" for non preservation under products.

The sections of the paper are rather independent and, apart from some comments and with the exceptions mentioned below, can be read in any order. All sections use the (classical) definitions recalled in the first two paragraphs of Section 1. Sections 2-4 rely on Definition 2.1. Finally, Section 4 depends on Lemma 2.2 and Proposition 3.1.

1. Some facts about ultrafilter convergence

Many topological properties have been characterized in terms of ultrafilter convergence. Recall that if $I$ is a set, $(x_i)_{i \in I}$ is an $I$-indexed sequence of elements of some topological space $X$, and $F$ is a filter over $I$, then a point $x \in X$ is said to be an $F$-limit point of the sequence $(x_i)_{i \in I}$ if $\{i \in I \mid x_i \in U\} \in F$, for every open neighborhood $U$ of $x$ (Choquet [Ch, Section IV], Katetov [Ka]). Notice that if $X$ is $T_2$, then a sequence has at most one $F$-limit point; this is the reason for the alternative expression $F$-convergent sequence. In this note we shall not actually need unicity of $F$-limits, therefore we shall assume no separation axiom.

Among the many properties which have been considered, definable in terms of ultrafilter convergence, a classical one asks that every sequence has an $F$-limit point, for some given $F$. In details, a topological space $X$ is $F$-compact, for some filter $F$ over $I$, if every $I$-indexed sequence of elements of $X$ has some limit point in $X$ (Bernstein [B] for ultrafilters over $I = \omega$, strongly motivated by Robinson non-standard analysis [R]; Saks [Sa] for ultrafilters over cardinals). $F$-compactness has the pleasant property of being preserved under taking products.

Interesting topological properties arise from conditions asserting that a topological space is $D$-compact, for all ultrafilters $D$ in some particular class of ultrafilters. For example, a regular topological space $X$ is $\lambda$-bounded (i.e., the closure of every subset of cardinality $\leq \lambda$ is compact) if and only if $X$ is $D$-compact, for every ultrafilter $D$ over $\lambda$. In particular, a topological space is compact if and only if it is $D$-compact, for every ultrafilter $D$. See [Sa, Theorem 2.9, and Section 5].
A much broader range of applications of ultrafilter convergence has been discovered shortly after the appearance of Bernstein paper. For example, it follows from Ginsburg and Saks [GS, p. 404] that a topological space is countably compact if and only if, for every sequence \((x_i)_{i\in\omega}\) of elements of \(X\), there exists some ultrafilter \(D\) uniform over \(\omega\) such that the sequence has some \(D\)-limit point in \(X\). Here the ultrafilter is not fixed in advance, but, in general, it depends on the sequence. The above characterization of countable compactness is the key to the result, also due to Ginsburg and Saks [GS, Theorem 2.6], that all powers of some space \(X\) are countably compact if and only if \(X\) is \(D\)-compact, for some ultrafilter \(D\) uniform over \(\omega\). Similar characterizations, and product theorems as well, appear in [Sa] for general types of accumulation properties, including, in particular, initial \(\mu\)-compactness, and, in equivalent form, \([\mu, \mu]\)-compactness, for \(\mu\) regular. Saks’ results have been subsequently extended to \([\kappa, \mu]\)-compactness by Caicedo [Ca]. Caicedo’s treatment has also the advantage of using a single ultrafilter, rather than a family of ultrafilters.

A comprehensive survey of earlier results on the subject can be found in Vaughan [V] and Stephenson [St], together with many related notions and results. See Kočinac and García-Ferreira [GK] in particular, Section 3, for a survey of additional results, and Lipparini [L4, L5, L6] for even more general treatments of the subject.

2. The Menger property in terms of ultrafilter convergence

In this section we present a characterization of the Menger property along the lines described in the above section. See, e.g., Scheepers [Sc], Tsaban [Ts] and Kočinac [Ko] for information and references about the Menger and related properties.

Our proofs become cleaner if we explicitly settle the cardinalities of the families of covers under consideration. Thus we are lead to consider the following general property which depends on 3 cardinal parameters (and which might also have independent interest).

**Definition 2.1.** For \(\lambda, \mu, \kappa\) nonzero cardinals, let us say that a topological space \(X\) satisfies \(R(\lambda, \mu; <\kappa)\), short for \(X\) satisfies the \(<\kappa\)-Rothberger property for sequences of \(\lambda\) covers of cardinality \(\leq \mu\), if, for every sequence \((U_\alpha)_{\alpha\in\lambda}\) of open covers of \(X\), such that \(|U_\alpha| \leq \mu\), for every \(\alpha \in \lambda\), there are subsets \(V_\alpha \subseteq U_\alpha\) (\(\alpha \in \lambda\)) such that \(\bigcup_{\alpha\in\lambda} V_\alpha\) is a cover of \(X\), and \(|V_\alpha| < \kappa\), for every \(\alpha \in \lambda\).

We shall write \(R(\lambda, \infty; <\kappa)\) if we put no restriction on the cardinality of the \(U_\alpha\)’s, and we shall write \(R(\lambda, \mu; \kappa)\) for \(R(\lambda, \mu; <(\kappa^+))\).
Thus, in the above notations, the Menger property is \( R(\omega, \infty; <\omega) \), and the Rothberger property is \( R(\omega, \infty; 1) \). Notice that in [L5] we used a nonstandard terminology, calling a space “Menger” if it satisfies \( R(\omega, \omega, <\omega) \), and “Rothberger” if it satisfies \( R(\omega, \omega, 1) \). Of course, the terminology agrees, say, in the class of Lindelöf spaces. Actually, a space is Menger (Rothberger) if and only if it is Lindelöf and satisfies \( R(\omega, \omega, <\omega) \) (\( R(\omega, \omega, 1) \), respectively).

Here we shall call \( R(\omega, \omega, <\omega) \) the Menger property for countable covers and \( R(\omega, \omega, 1) \) the Rothberger property for countable covers. When \( \mu < \infty \), these appear to be the most interesting particular cases of \( R(\lambda, \mu; <\kappa) \). The relevance of \( R(\omega, \omega, <\omega) \) and of \( R(\omega, \omega, 1) \) has been pointed out, for example, in [T3].

Notice that, in the above terminology, the Lindelöf property can be written as \( R(1, \infty; \omega) \). There are some trivial relations between \( R(\lambda, \mu; <\kappa) \) and \( R(\lambda', \mu'; <\kappa') \), for various cardinals \( \lambda, \lambda', \mu, \ldots \), but we shall not need these here.

For cardinals \( \lambda, \mu \), and \( \kappa \), we let \( [\mu]^{<\kappa} \) denote the set of all subsets of \( \mu \) of cardinality \( <\kappa \), and we let \( \lambda([\mu]^{<\kappa}) \) denote the set of all functions from \( \lambda \) to \( [\mu]^{<\kappa} \). To avoid complex formulas in subscripts, we sometimes shall denote a sequence \( (x_i)_{i \in I} \) as \( \{x_i \mid i \in I\} \).

**Lemma 2.2.** For every topological space \( X \), and \( \lambda, \mu \), and \( \kappa \) nonzero cardinals, the following conditions are equivalent.

1. \( X \) satisfies \( R(\lambda, \mu; <\kappa) \).
2. For every sequence \( (C_{\alpha})_{\alpha \in \lambda} \) of families of closed sets of \( X \), each family having cardinality \( \leq \mu \), if, for every choice of subfamilies \( D_{\alpha} \subseteq C_{\alpha} \) in such a way that each \( D_{\alpha} \) has cardinality \( <\kappa \), it happens that \( \bigcap\{C \mid C \in D_{\alpha}, \alpha \in \lambda\} \neq \emptyset \), then there is \( \bar{\alpha} \in \lambda \) such that \( \bigcap C_{\bar{\alpha}} \neq \emptyset \).
3. For every sequence \( \{f \mid f : \lambda \rightarrow [\mu]^{<\kappa}\} \) of elements of \( X \), there is \( \bar{\alpha} \in \lambda \) such that \( \bigcap_{\beta \in \mu} E_{\bar{\alpha}, \beta} \neq \emptyset \), where, for \( \alpha \in \lambda \) and \( \beta \in \mu \), we put \( E_{\alpha, \beta} = \{x_f \mid f \in I, \beta \in f(\alpha)\} \).

**Proof.** The equivalence of (1) and (2) is trivial by taking complements.

(2) \( \Rightarrow \) (3) Let the \( E_{\alpha, \beta} \)'s be defined as in (3), and let \( E_{\alpha} = \{E_{\alpha, \beta} \mid \beta \in \mu\} \), for \( \alpha \in \lambda \). For each \( \alpha \in \lambda \), choosing a subfamily \( D_{\alpha} \subseteq E_{\alpha} \) with \( |D_{\alpha}| < \kappa \) corresponds to choosing some \( Z_{\alpha} \in [\mu]^{<\kappa} \) in such a way that \( D_{\alpha} = \{E_{\alpha, \beta} \mid \beta \in Z_{\alpha}\} \). If we make such a choice for each \( \alpha \in \lambda \), and we let \( f : \lambda \rightarrow [\mu]^{<\kappa} \) be defined by \( f(\alpha) = Z_{\alpha} \), for every \( \alpha \in \lambda \), then \( x_f \in E_{\alpha, \beta} \), for \( \alpha \in \lambda \) and \( \beta \in f(\alpha) \), hence \( \bigcap\{C \mid C \in D_{\alpha}, \alpha \in \lambda\} \neq \emptyset \). Applying (2) to the family \( (E_{\alpha})_{\alpha \in \lambda} \), we get that there is \( \bar{\alpha} \in \lambda \) such that \( \bigcap E_{\alpha} \neq \emptyset \), that is, (3) holds.
Proof. The proof combines ideas from [Ca, Section 3] and [L5, Theorem 5.8].

(3) $\Rightarrow$ (2) Suppose that (3) holds, and that $(C_\alpha)_{\alpha \in \lambda}$ satisfies the premise in (2). We can write $C_\alpha = \{C_{\alpha,\beta} \mid \beta \in \mu\}$ (using repetitions of members of $C_\alpha$, when $|C_\alpha| < \mu$). For every $f : \lambda \to [\mu]^{<\kappa}$, we have that $C_f = \bigcap\{C_{\alpha,\beta} \mid \alpha \in \lambda, \beta \in f(\alpha)\} \neq \emptyset$, by the premise of (2). Pick some $x_f \in C_f$, for each $f$, and let $E_\alpha$ and $E_{\alpha,\beta}$ be defined as above. By (3), there is $\bar{\alpha} \in \lambda$ such that $\bigcap E_{\bar{\alpha}} \neq \emptyset$. But this implies that $\bigcap C_{\bar{\alpha}} \neq \emptyset$, since, by construction, $C_{\alpha,\beta} \supseteq E_{\alpha,\beta}$, for every $\alpha \in \lambda$ and $\beta \in \mu$. $\square$

We say that an ultrafilter $D$ over $\lambda([\mu]^{<\kappa})$ is functionally regular if there is $\bar{\alpha} \in \lambda$ such that, for every $\beta \in \mu$, $A_{\alpha,\beta} = \{f \in \lambda([\mu]^{<\kappa}) \mid \beta \in f(\bar{\alpha})\} \in D$.

Theorem 2.3. Suppose that $\mu$ and $\kappa$ are infinite cardinals, $\lambda$ is a nonzero cardinal, and let $I = \lambda([\mu]^{<\kappa})$. Then, for every topological space $X$, the following conditions are equivalent.

1. $X$ satisfies $R(\lambda,\mu; <\kappa)$.
2. For every sequence $\langle x_f \mid f \in I \rangle$ of elements of $X$, there is $\bar{\alpha} \in \lambda$ such that $\bigcap C_{\bar{\alpha}} \neq \emptyset$, where $C_\alpha = \{C_{\alpha,s} \mid s \in [\mu]^{<\omega}\}$, for $\alpha \in \lambda$, and $C_{\alpha,s} = \{x_f \mid f \in I, s \subseteq f(\alpha)\}$, for $s \in [\mu]^{<\omega}$.
3. For every sequence $\langle x_f \mid f \in I \rangle$ of elements of $X$, there is a functionally regular ultrafilter $D$ over $I$ such that $\langle x_f \mid f \in I \rangle$ has some $D$-limit point in $X$.

Proof. The proof combines ideas from [Ca, Section 3] and [L5, Theorem 5.8].

Suppose that (1) holds, and let the sequence $\langle x_f \mid f \in I \rangle$ be given. Since $\mu$ is infinite, then $|[\mu]^{<\omega}| = \mu$, thus $|C_\alpha| \leq \mu$, for every $\alpha \in \lambda$. Suppose that, for every $\alpha \in \lambda$, $D_\alpha \subseteq C_\alpha$, and $|D_\alpha| < \kappa$. This means that, for every $\alpha \in \lambda$, there is $Z_\alpha \subseteq [\mu]^{<\omega}$ such that $|Z_\alpha| < \kappa$, and $D_\alpha = \{C_{\alpha,s} \mid s \in Z_\alpha\}$. Define a function $f : \lambda \to [\mu]^{<\kappa}$ by $f(\alpha) = \bigcup Z_\alpha$. Notice that $|\bigcup Z_\alpha| < \kappa$, since $|Z_\alpha| < \kappa$, $\kappa$ is infinite, and each member of $|Z_\alpha|$ is finite. If $\alpha \in \lambda$, and $s \in Z_\alpha$, then $x_f$ belongs to $C_{\alpha,s}$, thus $x_f \in \bigcap\{C \mid C \in D_\alpha, \alpha \in \lambda\}$, hence this last set is nonempty. Since, by assumption, $X$ satisfies $R(\lambda,\mu; <\kappa)$, then, by Lemma 2.2(1), there is $\bar{\alpha} \in \lambda$ such that $\bigcap C_{\bar{\alpha}} \neq \emptyset$. Thus (2) is proved.

Now suppose that (2) holds, and that $\langle x_f \mid f \in I \rangle$ is a sequence of elements of $X$. For the $\bar{\alpha}$ given by (2), we can pick $x \in \bigcap C_{\bar{\alpha}}$. Thus $x \in C_{\alpha,s}$, for every $s \in [\mu]^{<\omega}$. For any neighborhood $U$ of $x$ in $X$, let $B_U = \{f \in I \mid x_f \in U\}$, and let $F = \{B_U \mid U \text{ a neighborhood of } x\}$. Let $G = \{A_{\alpha,\beta} \mid \beta \in \mu\}$, where, as in the definition of functional regularity, for every $\beta \in \mu$, we let $A_{\alpha,\beta} = \{f \in I \mid \beta \in f(\bar{\alpha})\}$. We want to show that $F \cup G$ has the finite intersection property. Notice that $B_U \cap B_V = B_{U \cup V}$, for every pair $U, V$ of neighborhoods of $x$. Moreover,
\[ A_{\bar{\alpha},\beta_1} \cap \cdots \cap A_{\bar{\alpha},\beta_m} = \{ f \in I \mid s \subseteq f(\bar{\alpha}) \}, \text{ for } s = \{ \beta_1, \ldots, \beta_m \}. \]

Hence, in order to prove that \( \mathcal{F} \cup \mathcal{G} \) has the finite intersection property, it is enough to show that \( B_U \cap A_{\bar{\alpha},s} \neq \emptyset \), for every neighborhood \( U \) of \( x \), and every \( s \in [\mu]^{<\omega} \), where we have put \( A_{\bar{\alpha},s} = \{ f \in I \mid s \subseteq f(\bar{\alpha}) \} \).

Indeed, for every \( U \) and \( s \) as above, since \( x \in C_{\bar{\alpha},s} \), there is \( f \in I \) such that \( s \subseteq f(\bar{\alpha}) \), and \( x_f \in U \), and this means exactly that \( x_f \) belongs to \( B_U \cap A_{\bar{\alpha},s} \), thus this set is not empty. We have proved that \( \mathcal{F} \cup \mathcal{G} \) has the finite intersection property.

Thus \( \mathcal{F} \cup \mathcal{G} \) can be extended to an ultrafilter \( D \) over \( I \). Since \( D \supseteq \mathcal{G} \), then \( D \) is functionally regular. Since \( D \supseteq \mathcal{F} \), then \( x \) is a \( D \)-limit point of \( \langle x_f \mid f \in I \rangle \). The implication \( (2) \Rightarrow (3) \) is thus proved.

Now assume that \( (3) \) holds. We shall prove that Condition (3) in Lemma 2.2 holds, thus \( X \) satisfies \( R(\lambda,\mu;\kappa) \). Suppose that \( \langle x_f \mid f \in I \rangle \) is a sequence of elements of \( X \). Condition (3) in the present theorem furnishes an element \( x \in X \), an ultrafilter \( D \) over \( I \), and an \( \bar{\alpha} \in \lambda \) such that \( x \) is a \( D \)-limit point of \( \langle x_f \mid f \in I \rangle \), and, for every \( \beta \in \mu \), \( \{ f \in I \mid x_f \in U \} \subseteq D \).

We will show that \( x \in E_{\bar{\alpha},\beta} \), for every \( \beta \in \mu \), where, as in Lemma 2.2,

\[ E_{\bar{\alpha},\beta} = \{ x_f \mid f \in I, \beta \in f(\bar{\alpha}) \}. \]

If, by contradiction, \( x \notin E_{\bar{\alpha},\beta} \), for some \( \beta \in \mu \), then there is a neighborhood \( U \) of \( x \) such that \( U \cap \{ x_f \mid f \in I, \beta \in f(\bar{\alpha}) \} = \emptyset \), hence \( \{ f \in I \mid x_f \in U \} \cap \{ f \in I \mid \beta \in f(\bar{\alpha}) \} = \emptyset \).

Since \( x \) is a \( D \)-limit point of \( \langle x_f \mid f \in I \rangle \), then \( \{ f \in I \mid x_f \in U \} \subseteq D \) and this contradicts \( \{ f \in I \mid \beta \in f(\bar{\alpha}) \} \subseteq D \), since \( D \) is a proper filter. □

**Corollary 2.4.** For every topological space \( X \), the following conditions are equivalent.

1. \( X \) satisfies the Menger property.
2. For every infinite cardinal \( \mu \), and for every sequence \( \langle x_f \mid f \in \omega([\mu]^{<\omega}) \rangle \) of elements of \( X \), there is a functionally regular ultrafilter \( D \) over \( \omega([\mu]^{<\omega}) \) such that the sequence has some \( D \)-limit point in \( X \).
3. \( X \) is Lindelöf, and, for every sequence \( \langle x_f \mid f \in \omega([\omega]^{<\omega}) \rangle \) of elements of \( X \), there is a functionally regular ultrafilter \( D \) over \( \omega([\omega]^{<\omega}) \) such that the sequence has some \( D \)-limit point in \( X \).

In order to avoid trivial exceptions, when dealing with products, we shall always assume that all topological spaces under consideration are non empty.

**Corollary 2.5.** Suppose that \( \mu \) and \( \kappa \) are infinite cardinals, \( \lambda \) is a nonzero cardinal, and let \( I = \lambda([\mu]^{<\kappa}) \). Then, for every product \( X = \prod_{j \in I} X_j \) of topological spaces, the following conditions are equivalent.
(1) \( X \) satisfies \( R(\lambda, \mu; <\kappa) \).

(2) For every choice of sequences \( \langle x_{f,j} \mid f \in I \rangle \) in \( X_j \) (one sequence for each \( j \in J \)), there is a functionally regular ultrafilter \( D \) over \( I \) such that, for every \( j \in J \), the sequence \( \langle x_{f,j} \mid f \in I \rangle \) has some \( D \)-limit point in \( X_j \) (here \( D \) is the same for all sequences).

(3) Every subproduct of \( X \) with \( \leq 2^{2^{I}} \) factors (that is, every product \( Y = \prod_{j \in J'} X_j \) with \( J' \subseteq J \) and \( |J'| \leq 2^{2^{I}} \)) satisfies \( R(\lambda, \mu; <\kappa) \).

Proof. The equivalence of (1) and (2) is immediate from Theorem 2.3 and the fact that a sequence in a product has a \( D \)-limit point if and only if each projection onto each factor has a \( D \)-limit point.

The equivalence of (1) and (3) is similar to [Sa, Theorem 2.3]. Cf. also [Co]. First, notice that (1) \( \Rightarrow \) (3) is trivial. We shall complete the proof by showing that if (2) fails, then (3) fails. Hence suppose that we can choose sequences \( \langle x_{f,j} \mid f \in I \rangle \) in \( X_j \), for every \( j \in J \), such that, for every functionally regular ultrafilter \( D \) over \( I \), there is \( j \in J \) such that the sequence \( \langle x_{f,j} \mid f \in I \rangle \) has no \( D \)-limit point in \( X_j \). Pick one such \( j \in J \) for each functionally regular ultrafilter \( D \) over \( I \), and let \( J' \subseteq J \) be the set of the \( j \)'s chosen in this way. Since there are \( \leq 2^{2^{I}} \) functionally regular ultrafilters over \( I \), we have that \( |J'| \leq 2^{2^{I}} \). Now applying the already proved equivalence of (1) and (2) to \( Y = \prod_{j \in J'} X_j \) we get that \( Y \) does not satisfy \( R(\lambda, \mu; <\kappa) \). \( \Box \)

It is not clear whether \( 2^{2^{I}} \) is the best possible bound in Condition (3) in Corollary 2.5 in general. As we shall see in Corollary 3.2 below, in the parallel situation in which we allow arbitrary repetitions of factors from a given family, we can indeed get a lower bound. Of course, in certain particular cases, a better bound in Corollary 2.5 can be obtained. For example, it is immediate from Proposition 3.1 below that a product is Menger if and only if all but finitely many factors are compact, and the product of the noncompact factors (if any) is Menger. In particular, a product is Menger if and only if all countable subproducts are Menger. As another example, it follows easily from [L1] that a product is Lindelöf if and only if all countable subproducts with \( \leq \omega_1 \) factors are Lindelöf.

Some of the results from [Sa, Ca] mentioned in the introduction can be obtained as a particular case of Theorem 2.3 by taking \( \lambda = 1 \). In fact, condition \( R(1, \mu; <\kappa) \) is nothing but a restatement of \( [\kappa, \mu] \)-compactness. Moreover, when \( \lambda = 1 \), an ultrafilter \( D \) over \( \lambda(\mu)^{<\kappa} \) \( \cong [\mu]^{<\kappa} \) is functionally regular if and only if it covers \( \mu \), that is, \( \{ z \in [\mu]^{<\kappa} \mid \beta \in z \} \in D \), for every \( \beta \in \mu \). An ultrafilter \( D \) over \( H \) is \( (\kappa, \mu) \)-regular if and only if there is a function \( f : H \to [\mu]^{<\kappa} \) such that \( f(D) \).
covers \( \mu \), where \( f(D) = \{ z \in [\mu]^{<\kappa} \mid f^{-1}(z) \in D \} \). Thus the results in \( \text{Ca}, \text{Section 3} \) are the particular case of Theorem 2.3 when \( \lambda = 1 \). See \( \text{L2} \) for a survey on regularity of ultrafilters and applications (notice that \( [\mu]^{<\kappa} \) is denoted by \( S_\kappa(\mu) \) in \( \text{L2} \)).

3. Product theorems

From Theorem 2.3 and the general theory developed in \( \text{L6} \) about preservation of properties under products, we get that all powers of some space \( X \) satisfy \( R(\lambda, \mu; <\kappa) \) if and only if \( X \) is \( D \)-compact, for some functionally regular ultrafilter over \( \lambda([\mu]^{<\kappa}) \). However, in this particular case, a stronger result can be obtained in a direct way.

We first need an easy proposition. As in Definition 2.1, it is sometimes convenient to allow the possibility that \( \mu = \infty \). Notice that, in this sense, \( R(1, \infty; <\kappa) \) is what is usually called final \( \kappa \)-compactness, which we shall also call \( [\kappa, \infty] \)-compactness.

If \( X = \prod_{\alpha \in \lambda} X_\alpha \), we shall denote by \( \pi_\alpha \) the natural projection onto the \( \alpha \)th component.

Proposition 3.1. Let \( \lambda \) and \( \kappa \) be nonzero cardinals, and \( \mu \) be a nonzero cardinal or \( \infty \). If, for every \( \alpha \in \lambda \), \( X_\alpha \) is a space which is not \( [\kappa, \mu] \)-compact, then \( X = \prod_{\alpha \in \lambda} X_\alpha \) does not satisfy \( R(\lambda, \mu; <\kappa) \).

Proof. By assumption, for every \( \alpha \in \lambda \), there is an open cover \( \mathcal{W}_\alpha \) of \( X_\alpha \) such that \( |\mathcal{W}_\alpha| \leq \mu \), and no subset of \( \mathcal{W}_\alpha \) of cardinality \( < \kappa \) is a cover of \( X_\alpha \). For every \( \alpha \in \lambda \), let \( \mathcal{U}_\alpha \) be the open cover of \( X \) consisting of all the products of the form \( \prod_{\delta \in \lambda} Y_\delta \), where all \( Y_\delta \)'s equal \( X_\delta \), except for \( \delta = \alpha \), in which case we require that \( Y_\delta \in \mathcal{W}_\alpha \). The family \( (\mathcal{U}_\alpha)_{\alpha \in \lambda} \) is clearly a counterexample to \( R(\lambda, \mu; <\kappa) \). We have to show that, for every choice of subfamilies \( \mathcal{V}_\alpha \subseteq \mathcal{U}_\alpha \) such that \( |\mathcal{V}_\alpha| < \kappa \) for every \( \alpha \in \lambda \), we have that \( \bigcup_{\alpha \in \lambda} \mathcal{V}_\alpha \) fails to be a cover of \( X \). Indeed, for every \( \alpha \in \lambda \), since no subset of \( \mathcal{W}_\alpha \) of cardinality \( < \kappa \) is a cover of \( X_\alpha \), there is \( x_\alpha \in X_\alpha \) such that, whenever \( x \in X \), and \( \pi_\alpha(x) = x_\alpha \), then \( x \) belongs to no member of \( \mathcal{V}_\alpha \). By choosing an \( x_\alpha \) as above, for each \( \alpha \in \lambda \), and by taking \( x = (x_\alpha)_{\alpha \in \lambda} \in X \), we get that \( x \) belongs to no member of \( \bigcup_{\alpha \in \lambda} \mathcal{V}_\alpha \); thus this last family is not a cover of \( X \). \( \square \)

In the statement of the next corollary the expression “product of members of a family \( F \)” is always intended in the sense that repetitions are allowed in the product, that is, the same space can occur multiple times.

For \( \mu \) a cardinal, we let \( \mu^{<\kappa} = \sup_{\kappa' < \kappa} \mu^{\kappa'} \), that is, \( \mu^{<\kappa} = |[\mu]^{<\kappa}|. \)
Corollary 3.2. Suppose that \( \mu \) and \( \kappa \) are infinite cardinals, \( \lambda \) is a nonzero cardinal, and let \( \nu = \max\{\lambda, 2^{\omega}^{<\kappa}\} \). For every family \( \mathcal{F} \) of topological spaces, the following conditions are equivalent.

1. All products of members of \( \mathcal{F} \) satisfy \( R(\lambda, \mu; <\kappa) \).
2. All products of \( \leq \nu \) members of \( \mathcal{F} \) satisfy \( R(\lambda, \mu; <\kappa) \).
3. All products of \( \leq \nu \) members of \( \mathcal{F} \) are \([\kappa, \mu]\)-compact.
4. There is a \((\kappa, \mu)\)-regular ultrafilter \( \mathcal{D} \) (which can be chosen over \([\mu]^{<\kappa}\)) such that every member of \( \mathcal{F} \) is \( \mathcal{D} \)-compact.
5. All products of members of \( \mathcal{F} \) are \([\kappa, \mu]\)-compact.
6. All products of members of \( \mathcal{F} \) satisfy \( R(\lambda', \mu; <\kappa) \), for every nonzero cardinal \( \lambda' \).

In particular, every product of members of \( \mathcal{F} \) satisfies the Menger property for countable covers if and only if so does any product of \( \leq 2^{\omega} \) factors, if and only if every product of members of \( \mathcal{F} \) is countably compact.

Proof. The implications (1) \( \Rightarrow \) (2) and (6) \( \Rightarrow \) (1) are trivial.

(2) \( \Rightarrow \) (3) Let \( X = \prod_{j \in J} X_j \) be a product of \( \leq \nu \) members of \( \mathcal{F} \). Since \( \lambda \leq \nu \), and \( \nu \) is infinite, then also \( X^\lambda \) is (can be reindexed as) a product of \( \leq \nu \) members of \( \mathcal{F} \). By (2), \( X^\lambda \) satisfies \( R(\lambda, \mu; <\kappa) \), and, by Proposition 3.1 with all \( X_\alpha \)'s equal to \( X \), we get that \( X \) is \([\mu, \kappa]\)-compact. We have proved that (3) holds.

The equivalence of (3), (4) and (5) comes from [Ca, Section 3]. It can be also obtained from [L6], and the particular case of Theorem 2.3 when \( \lambda = 1 \), using the remarks at the end of Section 2.

(5) \( \Rightarrow \) (6) follows from the trivial fact that \([\kappa, \mu]\)-compactness implies \( R(\lambda', \mu; <\kappa) \), for every nonzero cardinal \( \lambda' \). \( \square \)

As in [Ca] or [L3, Remark 2.4], there are cases in which the value of \( \nu \) in Corollary 3.2 can be improved. Just to state a simple example, we can have \( \nu = \max\{\lambda, 2^{\omega}\} \), when \( \mu = \kappa \) is a regular cardinal or, more generally, when \( \text{cf} \mu \geq \kappa \).

It is trivial to see that if \( X \) and \( Y \) are topological spaces, \( f : X \to Y \) is a continuous and surjective function, and \( X \) satisfies \( R(\lambda, \mu; <\kappa) \), then \( Y \) satisfies \( R(\lambda, \mu; <\kappa) \), too. In particular, if a product satisfies \( R(\lambda, \mu; <\kappa) \), then all subproducts and all factors satisfy it. Thus from Proposition 3.1 we get that if some product \( X = \prod_{j \in J} X_j \) satisfies \( R(\lambda, \mu; <\kappa) \), then all but at most \( \lambda \) factors are \([\kappa, \mu]\)-compact. In particular, if a product satisfies the Menger property for countable covers, then all but finitely many factors are countably compact (this extends [A, Proposition 1], where a different terminology is used, and those spaces we call Menger here are called Hurewicz there).
The above remarks can be improved in combination with results from [L1], as we shall show in the next proposition.

We let \( \kappa^{+n} \) be the \( n \)th iterated successor of \( \kappa \), that is, \( \kappa^{+0} = \kappa \), and \( \kappa^{+n+1} = (\kappa^{+n})^+ \).

**Proposition 3.3.** Suppose that \( \lambda \) and \( \kappa \) are infinite cardinals, \( \kappa \) is regular, \( \mu \) is either an infinite cardinal or \( \infty \), \( n \in \omega \), and \( \mu \geq \kappa^{+n+1} \).

If some product \( X = \prod_{j \in J} X_j \) satisfies \( R(\lambda, \mu; \kappa^{+n}) \), then \( \| \{ j \in J \mid X_j \) is not \([\kappa, \mu]\)-compact\} \) \( < \sup \{ \lambda, \kappa^{+n+1} \} \).

**Proof.** Since, for \( \lambda' \geq \lambda \), \( R(\lambda, \mu; \kappa) \) implies \( R(\lambda', \mu; \kappa) \), it is no loss of generality to assume that \( \lambda \geq \kappa^{+n+1} \).

By the remarks before the statement of the proposition, it is enough to prove that if \( Y = \prod_{\gamma \in \lambda} Y_\gamma \) is a product of \( \lambda \)-many spaces, and no \( Y_\gamma \) is \([\kappa, \mu]\)-compact, then \( Y \) does not satisfy \( R(\lambda, \mu; \kappa^{+n}) \). So suppose that \( Y \) and the \( Y_\gamma \)'s are as above. Since a topological space is \([\kappa, \mu]\)-compact if and only if it is both \([\kappa, \kappa^{+n}]\)-compact and \([\kappa^{+n+1}, \mu]\)-compact, and since \( \lambda \) is infinite, then either \( \| \{ \gamma \in \lambda \mid Y_\gamma \) is not \([\kappa, \kappa^{+n}]\)-compact\} \) \( = \lambda \), or \( \| \{ \gamma \in \lambda \mid Y_\gamma \) is not \([\kappa^{+n+1}, \mu]\)-compact\} \) \( = \lambda \).

In the latter case we have that \( Y \) does not satisfy \( R(\lambda, \mu; \kappa^{+n}) \), by Proposition 3.1 and recalling that \( R(\lambda, \mu; \kappa^{+n}) \) is the same as \( R(\lambda, \mu; < \kappa^{+n+1}) \).

In the former case, for simplicity and without loss of generality, we can suppose that no \( Y_\gamma \) is \([\kappa, \kappa^{+n}]\)-compact, hence not \([\kappa, \kappa^{+n+1}]\)-compact. Partition \( \lambda \) into \( \lambda \)-many classes, each of cardinality \( \kappa^{+n+1} \) (this is possible, since \( \lambda \geq \kappa^{+n+1} \)). Say, \( \lambda = \bigcup_{\delta \in \lambda} W_\delta \). Then \( Y = \prod_{\gamma \in \lambda} Y_\gamma \cong \prod_{\delta \in \lambda} (\prod_{\gamma \in W_\delta} Y_\gamma) \). By [L1] Theorem 23 with \( \aleph_\alpha = \kappa \) (here we are using the assumption that \( \kappa \) is regular), we get that, for each \( \delta \in \lambda \), \( Z_\delta = \prod_{\gamma \in W_\delta} Y_\gamma \) is not \([\kappa^{+n+1}, \kappa^{+n+1}]\)-compact, hence, in particular, not \([\kappa^{+n+1}, \mu]\)-compact, since \( \mu \geq \kappa^{+n+1} \). By applying Proposition 3.1 to the product \( \prod_{\delta \in \lambda} Z_\delta \), we get that \( Y \cong \prod_{\delta \in \lambda} Z_\delta \) does not satisfy \( R(\lambda, \mu; \kappa^{+n}) \). \( \Box \)

**Corollary 3.4.** If \( \mu \geq \aleph_{n+1} \), and some product satisfies \( R(\aleph_{n+1}, \mu; \aleph_n) \), then all but at most \( \aleph_n \) factors are initially \( \mu \)-compact.

4. **The Rothberger property**

We now compare the Menger and the Rothberger properties, as far as filter convergence is concerned.

Notice that in the proof of Theorem 2.3 we made an essential use of the assumption that \( \kappa \) is infinite. Indeed, for \( \kappa = 2 \), say, for the Rothberger property \( R(\omega, \infty; 1) \), a similar characterization is not possible, at least, not using ultrafilters. However, some of the arguments in the
proof of Theorem 2.3 can be carried over, furnishing a characterization in terms of filters. The results from [L6] do apply also in this more general situation, but the fact that the characterization involves only filters which are not maximal implies that the behavior with respect to products is entirely different. See Corollary 4.2 below.

**Proposition 4.1.** Suppose that \( \mu, \kappa \) and \( \lambda \) are nonzero cardinals, and let \( I = \lambda([\mu]^{<\kappa}) \). Then, for every topological space \( X \), the following conditions are equivalent.

1. \( X \) satisfies \( R(\lambda, \mu; <\kappa) \).
2. For every sequence \( \langle x_f \mid f \in I \rangle \) of elements of \( X \), there is a filter \( F \) over \( I \) such that
   a. there is \( \bar{\alpha} \in \lambda \) such that, for every \( A \in F \) and every \( \beta \in \mu \), \( A \cap A_{\bar{\alpha},\beta} \neq \emptyset \), where \( A_{\bar{\alpha},\beta} = \{ f \in I \mid \beta \in f(\bar{\alpha}) \} \).
   b. \( \langle x_f \mid f \in I \rangle \) has some \( F \)-limit point in \( X \).

**Proof.** If \( X \) satisfies \( R(\lambda, \mu; <\kappa) \), and \( \langle x_f \mid f \in I \rangle \) is a sequence of elements of \( X \), then, by Lemma 2.2(3), there are \( \bar{\alpha} \in \lambda \), and \( x \in \bigcap_{\beta \in \mu} \{ x_f \mid f \in I, \beta \in f(\bar{\alpha}) \} \). For any neighborhood \( U \) of \( x \) in \( X \), let \( B_U = \{ f \in I \mid x_f \in U \} \); then the filter \( F \) generated by \( \{ B_U \mid U \text{ a neighborhood of } x \} \) witnesses (2). (Notice that \( B_U \cap B_V = B_{U \cap V} \), hence any \( A \in F \) contains some \( B_U \), hence (2)(a) holds)

Conversely, if (2) holds, then any \( \bar{\alpha} \) given by (a) and any point \( x \) given by (b) are such that \( x \in \bigcap_{\beta \in \mu} \{ x_f \mid f \in I, \beta \in f(\bar{\alpha}) \} \), thus Condition (3) in Lemma 2.2 holds. \( \square \)

Of course, Proposition 4.1 holds also when \( \mu \) and \( \kappa \) are infinite. The main point in Theorem 2.3 is the non trivial result that, in the case when \( \mu \) and \( \kappa \) are infinite, we can equivalently restrict ourselves to ultrafilters satisfying Condition (2)(a) in Proposition 4.1. Indeed, if some ultrafilter \( D \) over \( \lambda([\mu]^{<\kappa}) \) satisfies (2)(a), for \( \bar{\alpha} \in \lambda \), then each \( A_{\bar{\alpha},\beta} \) belongs to \( D \), since otherwise \( A = I \setminus A_{\bar{\alpha},\beta} \in D \), contradicting (2)(a). Thus an ultrafilter \( D \) satisfies (2)(a) if and only if it is functionally regular. That is, Condition (2) in Proposition 4.1, when restricted to ultrafilters, becomes Condition (3) in Theorem 2.3.

On the other hand, if \( \kappa = 2 \) and \( \mu \geq 2 \), in particular, when dealing with approximations \( R(\omega, \mu; 1) \) to the Rothberger property, then there exists no ultrafilter which satisfies Condition (2)(a) in Proposition 4.1. Indeed, arguing as above, such an ultrafilter should contain \( A_{\bar{\alpha},\beta} \), for every \( \beta \in \mu \). However, when \( \kappa = 2 \), we have that \( A_{\bar{\alpha},\beta_1} \cap A_{\bar{\alpha},\beta_2} = \emptyset \), for \( \beta_1 \neq \beta_2 \), contradicting the property of being a proper filter. A similar argument applies when \( \kappa \) is finite and \( \mu \geq \kappa \). In other words, for such
µ and κ, and arbitrary λ, there is no functionally regular ultrafilter over \(\lambda([\mu]<\kappa)\).

The notion of \(F\)-compactness is usually given only for ultrafilters, since no \(T_1\) space with more than one point is \(F\)-compact, when \(F\) is a filter which is not maximal [L6]. However, as Proposition 4.1 shows, the notion of an \(F\)-limit point has some interest even when \(F\) is a filter not maximal. Another example in which general filters are necessary is sequential compactness, dealt with in [L6].

In [L6] we have studied those classes \(K\) of topological spaces which can be characterized by means of filter convergence, in the sense that there is a set \(I\) and a family \(\mathcal{P}\) of filters over \(I\) such that, for every \(I\)-indexed sequences of elements from any space \(X \in K\), there is some \(F \in \mathcal{P}\) such that the sequence \(F\)-converges to some element of \(X\) (here examples of such classes \(K\) are furnished by Theorem 2.3 and Proposition 4.1). It has been proved in [L6] that if \(K\) allows a characterization as above, and the corresponding family \(\mathcal{P}\) contains exclusively filters which are not maximal (e. g., as in Proposition 4.1 but not as in Theorem 2.3), then any space \(X\) with the property that all powers of \(X\) belong to \(K\) must be ultraconnected (a topological space is ultraconnected if it does not contain a pair of disjoint nonempty closed sets).

In particular, if all powers of some space \(X\) are Rothberger, then \(X\) is ultraconnected, and thus satisfies very few separation properties (unless it is a one-element space). In fact, a stronger result follows directly from Proposition 3.1.

**Corollary 4.2.** If \(\lambda\) is a nonzero cardinal and, for every \(\alpha \in \lambda\), \(X_\alpha\) is a space which is not ultraconnected, then \(X = \prod_{\alpha \in \lambda} X_\alpha\) does not satisfy \(R(\lambda, 2; 1)\).

**Proof.** Immediate from Proposition 3.1, noticing that ultraconnectedness is the same as \([2, 2]\)-compactness (every two-elements open cover has a one-element subcover).

The last statement follows from the remarks before Proposition 3.3.

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