Pointed Gromov-Hausdorff Topological Stability for Non-compact Metric Spaces

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Abstract
We combine the pointed Gromov-Hausdorff metric with the locally $C^0$-distance to obtain the pointed $C^0$-Gromov-Hausdorff distance between maps of possibly different non-compact pointed metric spaces. The latter is combined with Walters’s locally topological stability proposed by Lee–Nguyen–Yang, and $GH$-stability from Arbieto-Morales to obtain the notion of topologically $GH$-stable pointed homeomorphism. We give one example to show the difference between the distance when taking different base points in a pointed metric space.

Keywords
Pointed Gromov-Hausdorff metric · Pointed $C^0$-Gromov-Hausdorff distance · Locally topological stability · $C$-expansive · $C$-shadowing property

Mathematics Subject Classification
Primary 37Bxx · 37Dxx; Secondary 53C23

1 Introduction

One of the interests in Dynamical Systems theory is to determine conditions to guarantee the stability of a system, i.e., preservation of the dynamical structure by small perturbations. In this way, Walters [30] proposed a notion of stability where the behavior of the dynamics $C^0$-close to it is the same as its dynamic. The notion of topological stability to maps given by Walters was used by this author to prove that every Anosov...
diffeomorphism on compact manifolds is topologically stable. In 2017, Arbieto and Rojas [1] presented an understanding of classical Gromov-Hausdorff metrics extending it to maps between different metric spaces, and started an analysis of asymptotic behavior through the concept of usual topological stability. Then they combine the resulting distance $C^0$-Gromov-Hausdorff called with the Walters topological stability to obtain the notion of topological GH-stability. Other authors made another approach to this situation. We recommend studying the papers Chung [5], Khan et al. [15], Dong et al. [9], and references therein, where they describe how to extend Gromov-Hausdorff distance to be a distance between maps.

Since the theory of dynamical systems deals with dynamical properties of group actions, an extension of the distance $C^0$-GH to group actions was developed in Dong et al. [9], and several results can be found in Khan et al. [15]. On the other hand, Chulluncuy [4] defined $C^0$-GH distance, which measures distances between two continuous flows of possibly different metric spaces. An essential fact in these results was the compactness of the spaces.

Therefore, the natural question of how to develop a theory for a non-compact case arises. In this way, Das et al. [6] extended the result obtained by Walters for Anosov diffeomorphisms for first countable, locally compact, paracompact, Hausdorff spaces, and later Lee et al. [17] expanded the Walters’s stability theorem to homeomorphisms on locally compact metric spaces.

The purpose behind this work is to follow Arbieto and Morales to provide an understanding of the classical pointed Gromov-Hausdorff distance between non-compact metric spaces (see [24, p.209]) by extending it to maps between proper metric spaces. We call this distance the pointed $C^0$-Gromov-Hausdorff distance. Later, we combine the resulting distance with the topological stability of locally compact metric spaces [17] to obtain the notion of topological pointed GH-stability. The reader will also compare this approach with the definition of pointed Gromov-Hausdorff distance between the two pointed triples and Theorem 3.3 of [26].

The concept of topological stability of functions between different spaces has been studied from various perspectives. Some authors have investigated distance-preserving maps between Hausdorff, second countable topological spaces by considering a positive definite metric space, as seen in works such as [19, 20, 32, 33]. In this sense, one could develop a theory of stability in such cases, similar to the work of Rong [24] in 2020, where they studied the stability of almost submetries. In 2012, Rong and Shicheng [25] investigated the stability of $e^\epsilon$-Lipschitz and co-Lipschitz maps, while [18] demonstrated a diffeomorphic stability Theorem for $\delta$-Riemannian submersions. Similarly, we could define stability using the Gromov-Hausdorff distance for space-functions, as defined in David [7].

The paper is organized as follows:

In Sect. 2, we state the geometrical and dynamical preliminaries. In Sect. 3, we introduce the pointed $C^0$-Gromov-Hausdorff distance and prove some geometrical properties. In Sect. 4, we define topological stability in a pointed sense, mention our main results, and provide some examples and figures that illustrate those concepts.
2 Preliminaries

2.1 Gromov-Hausdorff Distance

The definitions given in this subsection can be found in Rong [24] and Burago et al. [3].

Let $(Z, d^Z)$ be a metric space. Let $A \subset Z$ and $\varepsilon > 0$. The $\varepsilon$-tube of $A$ is the open set defined by

$$N_\varepsilon(A) = \{ z \in Z : d^Z(z, A) < \varepsilon \} = \bigcup_{a \in A} B(a, \varepsilon),$$

where $d^Z(z, A) = \inf_{a \in A} d(z, a)$.

**Definition 2.1 (Hausdorff Distance)** Let $(Z, d)$ be a metric space, and let $A, B$ two bounded subsets of $Z$. The Hausdorff distance between $A$ and $B$, denoted by $d^Z_H(A, B)$, is defined by

$$d^Z_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \right\} = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\} = \inf \{ \varepsilon > 0 : A \subset N_\varepsilon(B), B \subset N_\varepsilon(A) \}.$$

The Gromov-Hausdorff distance via intrinsic Hausdorff distance is given as:

**Definition 2.2 (Intrinsic Gromov-Hausdorff distance)** Let $X$ and $Y$ be metric spaces of finite diameter. The Intrinsic Gromov-Hausdorff distance (briefly, $\hat{d}_{GH}$-distance) of $X$ and $Y$ is

$$\hat{d}_{GH}(X, Y) = \inf_{Z} \left\{ d^Z_H(\phi(X), \psi(Y)) : \text{exist isometric embedding } \phi : X \leftrightarrow Z, \psi : Y \leftrightarrow Z \right\}$$

where $Z$ varies over all compact metric spaces, and for each $Z$, $\phi$ and $\psi$ run over all possible isometric embeddings.

**Definition 2.3** Two maps $\varphi, \psi : X \to Y$ are $\alpha$-close if and only if for all $x \in X$

$$d^Y(\varphi(x), \psi(x)) \leq \alpha.$$

The following can be found in Definition 7.1.4. of [3].

**Definition 2.4** Let $(X, d^X)$ and $(Y, d^Y)$ be metric spaces and $i : X \to Y$ be an arbitrary map. The distortion of $i$ is defined by

$$\text{dis}(i) = \sup_{x_1, x_2 \in X} \left| d^Y(i(x_1), i(x_2)) - d^X(x_1, x_2) \right|.$$
When we restrict to $U \subset X$, we write

$$\text{dis}(i)|_U = \sup_{x_1, x_2 \in U} \left| d^Y(i(x_1), i(x_2)) - d^X(x_1, x_2) \right|. $$

For compact metric spaces $X$ and $Y$, a map $i : X \to Y$ is called an $\varepsilon$-GH approximation (briefly, $\varepsilon$-GHA) if $i$ satisfies the following two conditions:

1. (it almost preserves distances): $\text{dis}(i) < \varepsilon$.
2. (it is almost surjective): $N_\varepsilon(i(X)) = Y$.
   (i.e, $\forall y \in Y, \exists x \in X; d^Y(i(x), y) \leq \varepsilon$). In particular, $d^Y_H(i(X), Y) \leq \varepsilon$.

The set of $\varepsilon$-GHA between $X$ and $Y$ is called $\text{App}_\varepsilon(X, Y)$. We emphasize that an $i \in \text{App}_\varepsilon(X, Y)$ is not necessarily continuous, injective, or surjective.

**Definition 2.5** For compact metric spaces $X$ and $Y$,

$$d_{GH}(X, Y) = \inf \{ \varepsilon > 0: \exists i \in \text{App}_\varepsilon(X, Y) \text{ and } j \in \text{App}_\varepsilon(Y, X) \}$$

is called the Gromov-Hausdorff distance between $X$ and $Y$.

**Remark 2.6** To maintain the symmetry of $d_{GH}(X, Y)$, it is necessary to have maps $i$ and $j$. However, this requirement is not significant since for any $i \in \text{App}_\varepsilon(X, Y)$, there exists an approximate inverse $i' \in \text{App}_{4\varepsilon}(Y, X)$ that satisfies

$$d^X(i' \circ i(x), x) \leq 3\varepsilon, \quad d^Y(i \circ i'(y), y) \leq 3\varepsilon.$$ 

Such a map $i'$ is called an $\varepsilon$-inverse of $i$. A proof of this fact can be found in [14, Proposition 1.4] or see [29, pg 750].

The proof of the following lemma can be found in Rong [24] (Lemma 1.3.3 and 1.3.4)

**Lemma 2.7** Let $X$ and $Y$ be compact metric spaces, then

$$d_{GH}(X, Y) \leq 2\hat{d}_{GH}(X, Y) \leq 3d_{GH}(X, Y).$$

So, we get an estimate for $d_{GH}$ from $\hat{d}_{GH}$, but they are different.

### 2.2 Pointed Gromov-Hausdorff Distance

For non-compact spaces, the Gromov-Hausdorff distance is not very useful. Instead, working with the concept of pointed Gromov-Hausdorff distance between pointed metric spaces is more convenient. We suggest the reader to see [3, 12, 14, 24] and references therein.

**Definition 2.8** Let $(X, x)$ and $(Y, y)$ be pointed metric spaces, $\varepsilon > 0$. A pointed map $i : (X, x) \to (Y, y)$ is called an $\varepsilon$-pointed Gromov-Hausdorff approximation if

1. $i(x) = y$. 


(2) \( \text{dis}(i)|_{B^X(x, 1/\varepsilon)} < \varepsilon. \)
(3) \( B^Y(y, 1/\varepsilon) \subseteq N_\varepsilon(i(B^X(x, 1/\varepsilon))) = \bigcup_{y \in i(B^X(x, 1/\varepsilon))} B^Y(y, \varepsilon). \)

The set of all \( \varepsilon \)-pointed Gromov-Hausdorff approximations is denoted by

\[ \text{App}_\varepsilon((X, x), (Y, y)). \]

**Definition 2.9** The pointed GH-distance between \((X, x)\) and \((Y, y)\) is defined by

\[ d_{GH}^p((X, x), (Y, y)) = \inf\{\varepsilon > 0 : \exists i \in \text{App}_\varepsilon((X, x), (Y, y)) \text{ and } j \in \text{App}_\varepsilon((Y, y), (X, x))\}. \]

It follows from conditions (1) and (2) of the Definition 2.8 that

\[ i\left(B^X(x, r)\right) \subseteq B^Y(y, r + \varepsilon) \quad \text{for } 0 < \varepsilon < r \leq \frac{1}{\varepsilon}. \]  
(2.1)

Using (2.1) and (3), we have

\[ B^Y(y, r) \subseteq N_\varepsilon(i(B^X(x, r))) \subseteq B^Y(y, r + \varepsilon), \]
and again using (3), we get

\[ B^Y(y, r + \varepsilon) \subseteq N_{2\varepsilon}(i(B^X(x, r))). \]

Then \( i \in \text{App}_{2\varepsilon}(B^X(x, r), B^Y(y, r + \varepsilon)) \). This observation gives the following lemma.

**Lemma 2.10** Given \( \varepsilon > 0 \). If we have that \( d_{GH}^p((X, x), (Y, y)) < \varepsilon \), then there exist \( i \in \text{App}_{2\varepsilon}(B^X(x, 1/\varepsilon), B^Y(y, 1/\varepsilon + \varepsilon)) \) and \( j \in \text{App}_{2\varepsilon}(B^Y(y, 1/\varepsilon), B^X(x, 1/\varepsilon + \varepsilon)) \), with \( i(x) = y \) and \( j(y) = x \).

**Remark 2.11** Using a standard argument, it is easy to check that \( d_{GH}^p((X, x), (\hat{X}, x)) = 0 \), where \( \hat{X} \) denotes the completion of \( X \). Therefore, for any pointed spaces \((X, x)\) and \((Y, y)\), we have:

\[ d_{GH}^p((X, x), (Y, y)) = d_{GH}^p((\hat{X}, x), (\hat{Y}, y)). \]

**Definition 2.12** We say that two pointed metric spaces \((X, x)\) and \((Y, y)\) are isometric if there is an isometry \( h \) from \((X, x)\) to \((Y, y)\) such that \( h(x) = y \).

This result is given in [24, Proposition 1.6.3].

**Proposition 2.13** Let \((X, x)\) and \((Y, y)\) be proper pointed metric spaces. Then,

\[ d_{GH}^p((X, x), (Y, y)) = 0 \text{ if and only if } (X, x) \text{ is isometric to } (Y, y). \]

**Remark 2.14** By Proposition 2.13, we work with the isometric classes of pointed proper metric spaces.
Definition 2.15 (Convergence) We say that a sequence \((X_k, x_k)\) of pointed proper metric spaces converges to \((X, x)\), and write \((X_k, x_k) \overset{pGH}{\longrightarrow} (X, x)\) if
\[
\lim_{k \to \infty} d_{GH}^p ((X_k, x_k), (X, x)) = 0.
\]
That is, if there is a sequence of \(i_k \in \text{App}_{\varepsilon_k} ((X_k, x_k), (X, x))\) such that \(\varepsilon_k \to 0\) as \(k \to \infty\).

An inspiration to multi-point metric spaces is based on [21].

Definition 2.16 Let \((X, x_1, \ldots, x_k)\) and \((Y, y_1, \ldots, y_k)\) be multi-pointed metric spaces and \(\varepsilon > 0\). Suppose that there exists \(i : X \to Y\) such that for each \(n = 1, \ldots, k\), satisfies
\[
\begin{align*}
(1) & \quad i(x_n) = y_n, \\
(2) & \quad \text{dis}(i)_{B(x_n, 1/\varepsilon)} < \varepsilon, \\
(3) & \quad B^Y(y_n, 1/\varepsilon) \subset N_\varepsilon(i(B^X(x_n, 1/\varepsilon))).
\end{align*}
\]
A map \(i\) satisfying (1), (2) and (3) is called an \(\varepsilon\)-multi-pointed Gromov-Hausdorff approximation. The set of all \(\varepsilon\)-multi-pointed Gromov-Hausdorff approximations is denoted by
\[
\text{App}_\varepsilon ((X, x_1, \ldots, x_k), (Y, y_1, \ldots, y_k)).
\]

2.3 Gromov-Hausdorff Convergence of Maps

In Sinaei [28], we find the following definition and lemma (Definition 2.11 and Lemma 2.12), which are based on Appendix of [11].

Definition 2.17 Let \((X_k, x_k)\), \((X, x)\), \((Y_k, y_k)\) and \((Y, y)\) be pointed metric spaces such that \((X_k, x_k)\) converges to \((X, x)\) in the pointed Gromov-Hausdorff topology (resp. \((Y_k, y_k)\) converges to \((Y, y)\)). We say that a sequence of maps \(f_k : (X_k, x_k) \to (Y_k, y_k)\) converges to a map \(f : (X, x) \to (Y, y)\) if there exists a subsequence \(X_{k_j}\) such that if \(x_{k_j} \in X_{k_j}\) and \(x_{k_j} \to x\) in sense of the Definition 2.17, then \(f_{k_j}(x_{k_j}) \to f(x)\). (We then say \(f_k \to f\).)

A family of maps \(f_k : X_k \to Y_k\) is called equicontinuous if, for any \(\varepsilon > 0\), there is \(\delta > 0\) such that \(d^X_k(x_k, x'_k) < \delta\) implies that \(d^Y_k(f_k(x_k), f_k(x'_k)) < \varepsilon\) for all \(x_k, x'_k \in X_k\) and all \(k\).

Lemma 2.18 (Convergence of maps) Let \((X_k, p_k) \overset{d_{GH}}{\longrightarrow} (X, p)\) and \((Y_k, q_k) \overset{d_{GH}}{\longrightarrow} (Y, q)\), and let \(f_k : (X_k, p_k) \to (Y_k, q_k)\).

(1) If \(f_k\) are equicontinuous, then there is a uniform continuous map \(f : (X, p) \to (Y, q)\) and a subsequence \(X_{k_j}\) such that if \(x_{k_j} \in X_{k_j}\), \(x_{k_j} \to x\) in sense of the Definition 2.17, then \(f_{k_j}(x_{k_j}) \to f(x)\). (We then say \(f_k \to f\).)

(2) If \(f_k\) are isometries, then the limit map \(f : (X, p) \to (Y, q)\) is also an isometry.
But for our purpose, we don’t use the metric space $\bigsqcup X_{k_j} \bigsqcup X$. We use an alternative definition for Gromov-Hausdorff convergence of maps (compare with [31] for the non-compact case).

**Definition 2.19** If $(X_k, x_k) \xrightarrow{pGH} (X, x)$, via $i_k \in \text{App}_{\varepsilon_k}((X_k, x_k), (X, x))$, we say that points $z_k \in X_k$ converge to a point $z \in X$ if and only if $d^X(i_k(z_k), z) \to 0$. For this convergence we write $z_k \xrightarrow{pGH} z$.

This permits one to define the convergence of maps.

**Definition 2.20** If $f_k : (X_k, x_k) \to (Y_k, y_k)$ are maps, $(X_k, x_k) \xrightarrow{pGH} (X, x)$ and $(Y_k, y_k) \xrightarrow{pGH} (Y, y)$, then we say that $f_k$ converges in the sense of pointed Gromov-Hausdorff to a map $f : (X, x) \to (Y, y)$ if there exists a subsequence $(X_{k_j}, x_{k_j})$ such that for any $z_{k_j} \in (X_{k_j}, x_{k_j})$ with $z_{k_j} \xrightarrow{pGH} z$, then $f_{k_j}(z_{k_j}) \xrightarrow{pGH} f(z)$.

**Remark 2.21** According to Lemma 2.7, a sequence converges with respect to $\hat{d}_{GH}$ if and only if converges with respect to $d_{GH}$. Therefore, we obtain a result equivalent to Lemma 2.18 but using Definition 2.20.

**Remark 2.22** For a pointed metric space $(X, x)$, consider a constant sequence $g_k = f : (X, x) \to (X, x)$. Similar to [23, p.401], $g_k$ converges to $f$ in the sense of pointed Gromov-Hausdorff only if $f$ is continuous. For this reason, obtaining an alternative definition of convergence for maps in pointed metric spaces is convenient. That definition is proposed in Sect. 3.

### 2.4 Local Stability

It is not always possible to modify the definition of a global property to a local one, only paraphrasing it as “in some neighborhood of a given point” [13]. Our problem is to find a good and consistent definition to compare two dynamical systems on non-compact spaces that are close in the pointed Gromov-Hausdorff sense, and for which the behavior of their homeomorphisms is close in a $C^0$-topology local sense. Despite these observations, we find the straightforwardly adapted definitions useful.

**Definition 2.23** Let $f : X \to X$ and $g : Y \to Y$ be homeomorphisms of topological spaces $X$ and $Y$. A topological conjugacy from $f$ to $g$ is a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$.

**Definition 2.24** Let $X$ and $Y$ metric spaces and consider $f : X \to X$ and $g : Y \to Y$. We say that $f$ and $g$ are isometric (resp. homeomorphic) if there is an isometry (resp. homeomorphism) $h : Y \to X$ such that $f \circ h = h \circ g$. We say that $f$ is isometric (resp. homeomorphic) to $g$ if such a map $h$ exists.

**Definition 2.25** Two maps $f : (X, x) \to (X, x)$ and $g : (Y, y) \to (Y, y)$ are pointed isometric if there exists a map $h : X \to Y$ such that
1. $h$ is an isometry.
2. $h(x) = y$.
3. $h \circ f = g \circ h$.

$h$ is called a pointed isometry.

2.5 $C^0$-Gromov-Hausdorff

We recall the classical $C^0$-distance between maps to introduce a metric on the space of maps between metric spaces.

**Definition 2.26** Consider $f, g : X \to Y$, and define the $C^0$-distance between $f$ and $g$ by

$$d_{C^0}(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

A slight modification of the Gromov-Hausdorff distance, which incorporates the $C^0$-distance mentioned above, yields the following definition (See Definition 1.1 of [1]).

**Definition 2.27** Let $X, Y$ be compact metric spaces. The $C^0$-Gromov-Hausdorff distance between maps $f : X \to X$ and $g : Y \to Y$ is defined by

$$d_{GH^0}(f, g) = \inf\{\epsilon > 0 : \exists i \in \text{App}_\epsilon(X, Y) \text{ and } j \in \text{App}_\epsilon(Y, X) \text{ such that } d_{C^0}(g \circ i, i \circ f) < \epsilon \text{ and } d_{C^0}(j \circ g, f \circ j) < \epsilon\}.$$

**Remark 2.28** It is easy to prove that if $Y = \{y\}$, then $d_{GH}(X, Y) = \text{diam}(X)$. Also, if we take $f$ and $g$ such as the respective identities we have that $d_{GH^0}(f, g) = \text{diam}(X)$.

The basic properties of the map $d_{GH^0}$ are given below (Theorem 1 of [1]).

**Theorem 2.29** Let $(X, d_X)$ and $(Y, d_Y)$ be compact metric spaces. Let $f : X \to X$ and $g : Y \to Y$ be maps, then

1. If $X = Y$, then $d_{GH^0}(f, g) \leq d_{C^0}(f, g)$ (and the equality is not necessarily true).
2. $d_{GH}(X, Y) \leq d_{GH^0}(f, g)$ and $d_{GH}(X, Y) = d_{GH^0}(Id_X, Id_Y)$, where $Id_Z$ is the identity map of $Z$.
3. If $X$ and $Y$ are compact and $g$ continuous, then $d_{GH^0}(f, g) = 0$ if and only if $f$ and $g$ are isometric.
4. $d_{GH^0}(f, g) = d_{GH^0}(g, f)$.
5. For any map $r : Z \to Z$ of any metric space $Z$ one has

$$d_{GH^0}(f, g) \leq 2(d_{GH^0}(f, r) + d_{GH^0}(r, g)).$$

6. $d_{GH^0}(f, g) \geq 0$ and if $X$ and $Y$ are bounded, then $d_{GH^0}(f, g) < \infty$.
7. If $X$ is compact and there is a sequence of isometries $g_n : Y_n \to Y_n$ such that $d_{GH^0}(f, g_n) \to 0$ as $n \to \infty$, then $f$ is also an isometry.

**Remark 2.30** Note that in particular $d_{GH^0}(f, g) < \epsilon$ implies $\hat{d}_{GH}(X, Y) < \frac{3}{2} \epsilon$. 

3 Pointed $C^0$-Gromov-Hausdorff

The notion of pointed Gromov-Hausdorff distance has been extended to space-functions in different ways, for example Definition 2.2 of [7] and Definition 3.7 of [16] and [26, p.794]. We propose the following version.

**Definition 3.1** (Pointed $C^0$-Gromov-Hausdorff) Let $(X, x), (Y, y)$ be pointed metric spaces, and consider $f : X → X$ and $g : Y → Y$. We define

$$d_{GH}^0((X, x, f), (Y, y, g)) = \inf \{ \varepsilon > 0 : \exists i \in \text{App}_\varepsilon((X, x, f(x)), (Y, y, g(y))) \text{ and } j \in \text{App}_\varepsilon((Y, y, g(y)), (X, x, f(x))) \text{ such that}$$

$$d_{0,\varepsilon}^{x,y}(i \circ f, g \circ i) < \varepsilon, d_{0,\varepsilon}^{y,x}(j \circ g, f \circ j) < \varepsilon \} ,$$

where we call

$$d_{0,\varepsilon}^{x,y}(i \circ f, g \circ i) = \sup_{q \in B(x, 1/\varepsilon)} d_B^{g(y),1/\varepsilon}(i \circ f(q), g \circ i(q)),$$

$$d_{0,\varepsilon}^{y,x}(j \circ g, f \circ j) = \sup_{q \in B(y, 1/\varepsilon)} d_B^{f(x),1/\varepsilon}(j \circ g(q), f \circ j(q))$$

and $i, j$ are $\varepsilon$-multi-pointed Gromov-Hausdorff approximation such as in Definition 2.16. In this way, the pointed $C^0$-Gromov-Hausdorff distance between maps $f$ and $g$ is given by

$$d_{GH}^0((X, x, f), (Y, y, g)) = \min \left\{ \frac{1}{2} \right\} .$$

If we do not need to explicit the pointed spaces, we write $d_{GH}^0((X, x, f), (Y, y, g)) = d_{GH}^0(f, g)$.

**Remark 3.2** Following the Remark of [7], observe that $\tilde{d}_{GH}^0((X, x, f), (\tilde{X}, x, f)) = 0$, where $\tilde{X}$ is the metric completion of $X$ and $f$ is identified with its unique completion to $\tilde{X}$.

As in Remark 2.6, we have the same property of symmetry for the non-compact case in pointed metric spaces, i.e. for $i \in \text{App}_\varepsilon((X, x, f(x)), (Y, y, g(y)))$ there exists an $i' \in \text{App}_{4\varepsilon}((Y, y, g(y)), (X, x, f(x)))$.

Note that when restricting to compact metric spaces, the $d_{GH}$-convergence is stronger than the $d_{GH}^0$-convergence, and $d_{GH}^0$-convergence does not implies $d_{GH}$-convergence, in fact, let $(X_n, p_n) = ([0, n], 0)$ and $(X, 0) = ([0], 0)$ be pointed metric spaces. We have that $d_{GH}((X_n, p_n), (X, 0)) = \frac{1}{n}$, while by Remark 2.28 $d_{GH}(X_n, X) = \text{diam}(X_n) = n$, so $X_n$ does not converge in $d_{GH}$ to $X$. Now, in the same sense, for any $f_n : X_n → X_n, g = Id_X$ and $\frac{1}{\varepsilon} < n$, by Theorem 2.29 part 2,

$$d_{GH}^0((X_n, f_n), (X, g)) ≥ n,$$
while

\[ d_{GH}^p((X_n, p_n, f_n), (X, 0, g)) = \frac{1}{n}. \]

Therefore \( d_{GH}^0 \)-convergence is stronger than the \( d_{GH}^p \)-convergence, and \( d_{GH}^p \)-convergence does not imply \( d_{GH}^0 \)-convergence.

Now, with this concept, we give the pointed version of Theorem 2.29.

**Theorem 3.3** Let \((X, x)\) and \((Y, y)\) be proper pointed metric spaces. Let \( f : X \to X \) and \( g : Y \to Y \) be maps, then

1. If \((X, x) = (Y, y)\), then \( d_{GH}^p(f, g) \leq d_{0, \varepsilon}^{x,y}(f, g) \leq d_{C^0}(f, g) \) for all \( \varepsilon > 0 \) (and the equality is not necessarily true).
2. \( d_{GH}^p((X, x), (Y, y)) \leq d_{GH}^0(f, g) \) and
   \( d_{GH}^p((X, x), (Y, y)) = d_{GH}^0(Id_X, Id_Y) \) where \( Id_Z \) is the identity map of \( Z \).
3. \( d_{GH}^0(f, g) = 0 \) if and only if \( f \) and \( g \) are pointed isometric.
4. \( d_{GH}^0(f, g) = d_{GH}^0(g, f) \).
5. For \( f_m : X_m \to X_m \), we have
   \[
   d_{GH}^0(f_1, f_3) \leq 2 \left( d_{GH}^0(f_1, f_2) + d_{GH}^0(f_2, f_3) \right).
   \]
6. \( 0 \leq d_{GH}^0(f, g) < \infty \).
7. If \((X, x)\) is a proper pointed metric space and there is a sequence of isometries \( g_n : (Y_n, y_n) \to (Y_n, g_n(y_n)) \) such that \( d_{GH}^p(f, g_n) \to 0 \) as \( n \to \infty \), then \( f \) is also an isometry.

**Proof** (1) Consider \((X, x) = (Y, y)\). By Definition 3.1 the inequality \( d_{GH}^p(f, g) \leq d_{0, \varepsilon}^{x,y}(f, g) \) is valid, it is enough to take \( i = j = Id_X \in App^e((X, x), (X, x)) \). On the other hand

\[
 d_{0, \varepsilon}^{x,y}(f, g) = \sup_{p \in B(x, 1/\varepsilon)} d_{B(x, 1/\varepsilon)}(f(p), g(p)) \leq \sup_{p \in X} d_X(f(p), g(p)) = d_{C^0}(f, g).
\]

If we take to pointed isometric maps \( f \) and \( g \), but no equal, we get \( d_{GH}^0(f, g) = 0 \) by (3), but \( d_{0, \varepsilon}^{x,y}(f, g) > 0 \).

(2) It follows easily from the definitions that \( d_{GH}^p((X, x), (Y, y)) \leq d_{GH}^0(f, g) \). Now, set \( \varepsilon = d_{GH}^p((X, x), (Y, y)) \) and fix \( \delta > 0 \). Then \( \varepsilon < d_{GH}^p((X, x), (Y, y)) + \delta \). By Lemma 2.10, there exist \( i \in App_{2(\varepsilon + \delta)}((B(x, (\varepsilon + \delta)^{-1}), B(y, (\varepsilon + \delta)^{-1}) + \varepsilon + \delta), j \in App_{2(\varepsilon + \delta)}((B(y, (\varepsilon + \delta)^{-1}), B(x, (\varepsilon + \delta)^{-1}) + \varepsilon + \delta), \) with \( i(x) = y \) and \( j(y) = x \). On the other hand

\[
 d_{0, \varepsilon}^{x,y}(Id_X, Id_Y) = \sup_{p \in B(x, 1/\varepsilon)} d_{B(y, 1/\varepsilon)}(i \circ Id_X(p), Id_Y \circ i(p)) = 0 < \varepsilon,
\]
and similarly \(d_{0,\varepsilon}^{x,y}(I_{DY}, I_{DX}) = 0 < \varepsilon\). Then \(d_{GH}^{p}(I_{DX}, I_{DY}) \leq \varepsilon < d_{GH}^{p}((X, x), (Y, y)) + \delta\). As \(\delta\) is arbitrary, \(d_{GH}^{p}(I_{DX}, I_{DY}) \leq d_{GH}^{p}((X, x), (Y, y))\).

From this, we conclude that \(d_{GH}^{p}(I_{DX}, I_{DY}) = d_{GH}^{p}((X, x), (Y, y))\).

(3) If \(f\) and \(g\) are pointed isometric, then there exists a pointed isometry \(h : (X, x) \rightarrow (Y, y)\). In addition, \(h(f(x)) = g(h(x)) = g(y)\). Therefore, for all \(\varepsilon > 0\), \(h \in \text{App}_{\varepsilon}((X, x, f(x)), (Y, y, g(y)))\) and \(h^{-1} \in \text{App}_{\varepsilon}((Y, y, g(y)), (X, x, f(x)))\) satisfying

\[
\sup_{p \in B_{x,1/\varepsilon}} d_{B_{x,1/\varepsilon}}^{B(g(y), 1/\varepsilon)}(h \circ f(p), g \circ h(p)) = \sup_{q \in B_{x,1/\varepsilon}} d_{B_{x,1/\varepsilon}}^{B(f(x), 1/\varepsilon)}(g \circ h^{-1}(p), h^{-1} \circ f(p)) = 0 < \varepsilon.
\]

Hence \(d_{GH}^{p}(f, g) \leq \varepsilon\). As \(\varepsilon\) is arbitrary, \(d_{GH}^{p}(f, g) = 0\).

On the other hand, suppose that \(d_{GH}^{p}(f, g) = 0\), then there exist \(i_{n} \in \text{App}_{1/n}((X, x, f(x)), (Y, y, g(y)))\) and \(j_{n} \in \text{App}_{1/n}((Y, y, g(y)), (X, x, f(x)))\), such as

\[
d_{0,\varepsilon}^\frac{1}{n}(f, g) = \sup_{p \in B_{x,n}} d_{B_{x,n}}^{B(g(y), n)}(i_{n} \circ f(p), g \circ i_{n}(p)) < \frac{1}{n}
\]

and

\[
d_{0,\varepsilon}^\frac{1}{n}(g, f) = \sup_{q \in B_{y,n}} d_{B_{y,n}}^{B(f(x), n)}(j_{n} \circ g(q), f \circ j_{n}(q)) < \frac{1}{n}.
\]

We can extend \(i_{n}\) to a map from \(X\) to \(Y\) as a \(\text{App}_{2/n}((X, x, f(x)), (Y, y, g(y)))\), and similarly \(j_{n}\). Because of \(X\) and \(Y\) are proper (closed balls are compact), the sequence of maps \(i_{n}, j_{n}\) has a subsequence that converges uniformly on compact sets to an isometry from \(X\) to \(Y\) satisfying the conditions.

(4) Follows from the Definition 3.1.

(5) Now fix \(\delta > 0\), and from Definition 3.1, there exist

\(i_{mn} \in \text{App}_{\varepsilon_{mn}}((X_{m}, x_{m}, f_{m}(x_{m})), (X_{n}, x_{n}, f_{n}(x_{n})))\), with

\(\varepsilon_{mn} < d_{GH}^{p}(f_{m}, f_{n}) + \delta\),

and such that

\[
d_{0,\varepsilon}^{x_{m},x_{n}}(f_{m}, f_{n}) = \sup_{B_{x,m}^{\varepsilon_{mn}^{-1}}}(f_{n}(x_{n}), \varepsilon_{mn}^{-1}) \circ f_{n} \circ f_{m} \circ i_{mn} < \varepsilon_{mn}
\]

for \((m, n)\) equal to \((1, 2), (2, 3), (2, 1)\) or \((3, 2)\), and \(\varepsilon_{mn} = \varepsilon_{nm} < \frac{1}{2}\).

Similar to the proof of triangle inequality A.1, we can show that

\(i_{kl} \in \text{App}_{\varepsilon_{kl}}((X_{k}, x_{k}, f_{k}(x_{k})), (X_{l}, x_{l}, f_{l}(x_{l})))\),
for $i_{kl} = i_{2l} \circ i_{k2}$ for $(k, l) = (1, 3)$ or $(3, 1)$, and $\varepsilon_{kl} = \varepsilon_{lk} = \min\{2(\varepsilon_{k2} + \varepsilon_{2l}), \frac{1}{3}\}$.

And observe that

$$\frac{1}{2(\varepsilon_{k2} + \varepsilon_{2l})} \leq \min\left\{\frac{1}{\varepsilon_{k2}} - 2\varepsilon_{2l}, \frac{1}{\varepsilon_{2l}} - 2\varepsilon_{k2}\right\}$$

Then if $p_k \in B(x_k, 1/\varepsilon_{k2})$ we have

$$d^{X_3}(i_{kl} \circ f_k(p_k), f_l \circ i_{kl}(p_k))$$

$$\leq d^{X_3}(i_{kl} \circ f_k(p_k), i_{2l} \circ f_2 \circ i_{k2}(p_k))$$

$$+ d^{X_3}(i_{2l} \circ f_2 \circ i_{k2}(p_k), f_l \circ i_{2l}(i_{k2}(p_1)))$$

$$= d^{X_3}(i_{2l} (i_{k2} \circ f_k(p_k)), i_{2l} (f_2 \circ i_{k2}(p_k)))$$

$$+ d^{X_3}(i_{2l} \circ f_2 (i_{k2}(p_k)), f_l \circ i_{2l} (i_{k2}(p_k)))$$

$$\leq d^{X_2}(i_{k2} \circ f_k(p_k), f_2 \circ i_{k2}(p_k)) + \varepsilon_{2l} + \varepsilon_{2l}$$

$$\leq \varepsilon_{k2} + 2\varepsilon_{2l} < \varepsilon_{kl}.$$

Then we have

$$d^{r^k, r_l}(f_k, f_l) = \sup_{p_k \in B^k(x_k, \varepsilon_{kl}^{-1})} d^{B_l(f_l(x_l), \varepsilon_{kl}^{-1})}(i_{kl} \circ f_k(p_k), f_k \circ i_{kl}(p_k)) < \varepsilon_{kl}.$$  

This implies

$$d^{P}_0(f_1, f_3) \leq 2(\varepsilon_{12} + \varepsilon_{23}) \leq 2(d^{P}_0(f_1, f_2) + d^{P}_0(f_2, f_3) + \delta),$$

and since $\delta$ is arbitrary we obtain

$$d^{P}_0(f_1, f_3) \leq 2\left(d^{P}_0(f_1, f_2) + d^{P}_0(f_2, f_3)\right).$$

(6) Follows immediately by definition.

(7) Since $d^{P}_0(f, g_n) \to 0$, then for any $r > 0$, we can assume that $d^{P}_0(f|_{B(x, r)}, g_n|_{B(y_n, r)}) \to 0$. Since the balls are compact we can apply Theorem 2.29.7 to deduce that $f|_{B(x, r)}$ is an isometry. By arbitrariness of $r$ we conclude that $f$ is an isometry. \hfill \Box

4 Topological Stability

**Definition 4.1** A homeomorphism $f : X \to X$ of a compact metric space $X$ is topologically stable if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every homeomorphism $g : X \to X$ with $d_{C^0}(f, g) < \delta$ there is a continuous map $h : X \to X$ with

$$d_{C^0}(h, I_{dX}) < \varepsilon$$

such that $f \circ h = h \circ g$. 
Definition 4.2 A homeomorphism \( f : X \rightarrow X \) of a compact metric space \( X \) is topologically \( GH \)-stable if for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that for every homeomorphism \( g : Y \rightarrow Y \) of compact metric space \( Y \) satisfying \( d_{GH}(f, g) < \delta \) there is a continuous map \( h \in \text{App}_{\varepsilon}(Y, X) \) such that \( f \circ h = h \circ g \).

Definition 4.3 \( f \) has the POTP if for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for any \( \delta \)-pseudo orbit \( \{x_n\}_{n \in \mathbb{Z}} \) of \( f \); i.e., \( d(f(x_n), x_{n+1}) < \delta \) there is a point \( x \in X \) that \( \varepsilon \)-traces the pseudo orbit, that is, \( d(f^n(x), x_n) < \varepsilon \) for all \( n \in \mathbb{Z} \).

Definition 4.4 \( f \) is expansive if there exists \( \delta > 0 \) such that if \( x, y \in X \) and \( d(f^n(x), f^n(y)) < \delta \) for all \( n \in \mathbb{Z} \), then \( x = y \).

The below result gives sufficient conditions for \( GH \)-stability (Theorem 4 [1]).

Theorem 4.5 Every expansive homeomorphism with the POTP of a compact metric space is topologically \( GH \)-stable.

Following [17], we give the notions of expansiveness, shadowing property and topological stability for homeomorphisms on non-compact metric spaces. We denote \( C(X) \) the collection of continuous functions from \( X \) to \( (0, \infty) \).

Definition 4.6 Let \( X \) be a metrizable space and \( f \) be a homeomorphism of \( X \) onto itself. We say that

1. \( f \) is \( C \)-expansive if there exists a metric \( d \) for \( X \) and \( \delta \in C(X) \) such that \( d(f^n(x), f^n(y)) < \delta(f^n(x)) \) \( (x, y \in X) \) for all \( n \in \mathbb{Z} \) implies \( x = y \).
2. \( f \) has the \( C \)-shadowing property if for any \( \varepsilon \in C(X) \), there exists \( \delta \in C(X) \) such that for any \( \delta \)-pseudo orbit \( \{x_n\}_{n \in \mathbb{Z}} \) of \( f \); i.e., \( d(f(x_n), x_{n+1}) < \delta(f(x_n)) \) there is a point \( x \in X \) that \( \varepsilon \)-traces the pseudo orbit, that is, \( d(f^n(x), x_n) < \varepsilon(f^n(x)) \) for all \( n \in \mathbb{Z} \).
3. \( f \) is \( C \)-topologically stable if for any \( \varepsilon \in C(X) \), there is \( \delta \in C(X) \) such that if \( g \) is any homeomorphism of \( X \) with \( d(f(x), g(x)) < \delta(f(x)) \) for all \( x \in X \), then there is a continuous map \( h : X \rightarrow X \) with \( h \circ g = f \circ h \) and \( d(h(x), x) < \varepsilon(h(x)) \) for all \( x \in X \).

Some technical necessary results are the following:

Lemma 4.7 (Lemma 2.8 of [17]) For any \( \alpha \in C(X) \), there is \( \gamma \in C(X) \) such that \( \gamma(x) < \inf \{\alpha(y) : y \in B(x, \gamma(x))\} \) for all \( x \in X \).

Remark 4.8 (Remark 3.2 of [17]) For any \( \varepsilon \in C(X) \) there exists \( \gamma \in C(X) \) such that \( \gamma(x) < \inf \{\varepsilon(y) : y \in B(x, \gamma(x))\} \) for \( x \in X \), by Lemma 4.7. Hence, if \( d(x, y) < \max\{\gamma(x), \gamma(y)\} \) \( (x, y \in X) \), then \( d(x, y) < \varepsilon(x) \). Indeed, if \( \max\{\gamma(x), \gamma(y)\} = \gamma(x) \), then \( d(x, y) < \gamma(x) < \varepsilon(x) \); otherwise, we have \( x \in B(y, \gamma(y)) \) and so \( \gamma(y) < \varepsilon(x) \). Finally, we get \( d(x, y) < \varepsilon(x) \).

Lemma 4.9 (Lemma 3.3 of [17]) Let \( f \) be an expansive homeomorphism of a locally compact metric space \( X \) with an expansive function \( e \in C(X) \) with \( e < \alpha X \), where \( \alpha X \in C(X) \) such that \( B(x, \alpha X(x)) \) is compact. For any \( x_0 \in X \) and \( \lambda \in C(X) \), there is \( N > 0 \) such that if \( d(x_0, y) \geq \lambda(x_0) \), then \( d(f^n(x_0), f^n(y)) \geq e(f^n(x_0)) \) for some \( |n| < N \).
In [17, Theorem 1.1] they extend Walters’s stability theorem to homeomorphisms on locally compact spaces.

**Theorem 4.10** Let $X$ be a locally compact metric space and $f$ be a homeomorphism of $X$ onto itself. If $f$ is $C$-expansive and has the $C$-shadowing property then it is $C$-topologically stable.

**Remark 4.11** The POTP is the $C$-shadowing property by considering $\varepsilon \in C(X)$ and $\delta \in C(X)$ as constant functions in compact metric spaces.

Now we can define spaces topologically $GH$-stable in a locally pointed sense.

**Definition 4.12** A homeomorphism $f : (X, x) \to (X, f(x))$ of a proper metric space $X$ is topologically $pGH$-stable if for every $\varepsilon \in C(X)$, there is $\delta \in C(X)$ such that for every homeomorphism $g : Y \to Y$ of metric space $Y$ satisfying $d_{GH}^{p}(f, g) < \delta(f(x))$ there is a continuous map $h \in \text{App}_{\varepsilon}((Y, y, g(y)), (X, x, f(x)))$ such that $f \circ h = h \circ g$.

**Theorem 4.13** Let $(X, x)$ be a proper pointed metric space and $f$ be a homeomorphism of $X$ onto itself. If $f$ is $C$-expansive and has the $C$-shadowing property, then it is topologically $pGH$-stable.

The proof of Theorem 4.13 is a direct modification of the method used in Lee et al. [17] for variable spaces. It is also similar to the approach used in Arbieto and Rojas [1] for the compact setting.

**Proof** Let $f : X \to X$ be expansive with a $C$-expansive function $e \in C(X)$, let $\gamma \in C(X)$ be such that $\gamma(z) < \inf \{e(y) | y \in B(z, \gamma(z))\}$ for all $z \in X$.

Fix $\varepsilon \in C(X)$ and take $\tilde{\varepsilon} \in C(X)$ such that $0 < \tilde{\varepsilon} < \frac{1}{4} \min \{\varepsilon, \gamma\}$. For this $\tilde{\varepsilon}$ we choose $\delta$ from the $C$-shadowing, we can assume that $\delta < \tilde{\varepsilon}$.

First, we claim that any $\delta$-pseudo orbit is $\tilde{\varepsilon}$-traced by only one point in $X$. Indeed, we suppose that there are $p, q \in X$ that $\tilde{\varepsilon}$-trace a $\delta$-pseudo orbit $(p_{n})_{n \in \mathbb{Z}}$. Then we get

$$d\left(f^{n}(p), f^{n}(q)\right) \leq d\left(f^{n}(p), p_{n}\right) + d\left(p_{n}, f^{n}(q)\right) < \tilde{\varepsilon} \left(f^{n}(p)\right) + \tilde{\varepsilon} \left(f^{n}(q)\right)$$

$$< \frac{\gamma \left(f^{n}(p)\right)}{4} + \frac{\gamma \left(f^{n}(q)\right)}{4} < \max \left\{\gamma \left(f^{n}(p)\right), \gamma \left(f^{n}(q)\right)\right\}$$

$$< e \left(f^{n}(p)\right)$$

by Remark 4.8

for all $n \in \mathbb{Z}$. So, the expansivity implies $p = q$.

Now, for $\delta = \delta(f(x))$, there exists $\tilde{\delta} > 0$ such that $1/\delta = \tilde{\delta} + 1/\tilde{\delta}$. Let $g$ be a homeomorphism of $Y$ with $d_{GH}^{p}(f, g) < \tilde{\delta}$. As consequence of Lemma 2.10, there exists $j \in \text{App}_{\tilde{\delta}}((Y, y, g(y)), (X, x, f(x)))$ such that $d_{0,\delta}^{x,y} (j \circ g, f \circ j) < \delta(f(x))$.

For $q \in B(y, 1/\delta) \subset Y$, consider the sequence $(p_{n})_{n \in \mathbb{Z}}$ defined by $p_{n} = j(g^{n}(q))$ for $n \in \mathbb{Z}$. Condition (3) of Definition 2.8 implies $d^{X}(p_{n}, f(x)) < 1/\delta$ for all $n$. Therefore,

$$d^{B(f^{n}(x), 1/\delta)}(p_{n+1}, f(p_{n})) = d^{B(f^{n}(x), 1/\delta)}(j[g^{n+1}(q)], f(j[g^{n}(q)]))$$
Then \( (j \circ g^n(q))_{n \in \mathbb{Z}} \) is a \( \delta \)-pseudo orbit of \( f \) for each \( q \in B(y, 1/\delta) \subset Y \). Hence, we can define a map \( h : B(y, 1/\delta) \to X \) by \( h(q) \), the unique shadowing map of the \( \delta \)-pseudo orbit \( (j \circ g^n(q))_{n \in \mathbb{Z}} \). So, we have

\[
d^{B(f(x), 1/\delta)}(j \circ g^n(q)), f \circ j(g^n(q))) < \delta(f(x)).
\]

for all \( n \in \mathbb{Z} \). Then \( (j \circ g^n(q))_{n \in \mathbb{Z}} \) is a \( \delta \)-pseudo orbit of \( f \) for each \( q \in B(y, 1/\delta) \subset Y \). Hence, we can define a map \( h : B(y, 1/\delta) \to X \) by \( h(q) \), the unique shadowing map of the \( \delta \)-pseudo orbit \( (j \circ g^n(q))_{n \in \mathbb{Z}} \). So, we have

\[
d^{B(f(x), 1/\delta)}((f^n(h(q)), j(g^n(q)))) < \bar{\epsilon}(f^n(h(q)))
\]

for all \( q \in B(y, 1/\delta) \subset Y \) and \( n \in \mathbb{Z} \).

Taking \( n = 0 \) above, we get \( d^{B(f(x), 1/\delta)}(h(q), j(q)) < \bar{\epsilon} \) for all \( q \in B(y, 1/\delta) \subset Y \). Then \( d^{B(f(x), 1/\delta)}(h, j) \leq \bar{\epsilon} \).

Since \( j \in \text{App}_\delta((Y, y, g(y)), (X, x, f(x))) \), we have for all \( p \in B(f(x), 1/\delta) \subset X \)

\[
d(h(B(y, 1/\delta)), p) \leq d^{B(f(x), 1/\delta)}(h, j) + d^{B(f(x), 1/\delta)}(j(B(y, 1/\delta)), p) < \bar{\epsilon} + \delta < \epsilon,
\]

and for all \( p_1, p_2 \in B(y, 1/\delta) \)

\[
\left| d^X(h(p_1), h(p_2)) - d^Y(p_1, p_2) \right|
\]

\[
\leq \left| d^X(h(p_1), h(p_2)) - d^X(j(p_1), j(p_2)) \right| + \left| d^X(j(p_1), j(p_2)) - d^Y(p_1, p_2) \right|
\]

\[
\leq d^X(h(p_1), h(p_2)) - d^X(h(p_1), j(p_2))
\]

\[
+ \left| d^X(h(p_1), j(p_2)) - d^X(j(p_1), j(p_2)) \right| + \delta
\]

\[
\leq d^X(h(p_2), j(p_2)) + d^X(h(p_1), j(p_1)) + \delta
\]

\[
< 2\bar{\epsilon} + \delta < \frac{\epsilon}{4} + \frac{\epsilon}{8} < \epsilon.
\]

Then \( \text{dis}(h)\big|_{B(y, 1/\delta)} < \epsilon \), and this implies \( \text{dis}(h)\big|_{B(y, 1/\epsilon)} < \epsilon \), that is, \( h \in \text{App}_\epsilon((Y, y, g(y)), (X, x, f(x))) \). By a similar way we can prove that \( h \in \text{App}_\epsilon((Y, y, g(y)), (X, x, f(x))) \) implies \( h \in \text{App}_\epsilon((Y, y, g(y)), (X, x, f(x))) \).

On the other hand, since

\[
d(f^n(h(g(q))), j(g^n(q))) < \bar{\epsilon}
\]

and

\[
d\left(f^n(h(q)), j(g^{n+1}(q))\right) = d\left(f^{n+1}(h(x)), j(g^{n+1}(q))\right) < \epsilon\left(f^{n+1}(h(q))\right)
\]

for all \( n \in \mathbb{Z} \), we know that two points \( h(g(q)) \) and \( f(h(q)) \) \( \bar{\epsilon} \)-trace the \( \delta \)-pseudo orbit \( (g^{n+1}(q))_{n \in \mathbb{Z}} \). By the uniqueness, we have \( f \circ h = h \circ g \).

Now, we show \( h \) is continuous for each \( x_0 \in B(f(x), 1/\delta) \subset X \).

Indeed, fix \( \lambda > 0 \). By lemma 4.9, there exists \( N \in \mathbb{Z}^+ \) such that for any \( x_1 \in X \) if

\[
d(f^n(h(x_0)), f^n(h(x_1))) \leq \epsilon(f^n(h(x_0)))
\]
for all \(|n| < N\), then \(d(h(x_0), h(x_1)) < \lambda\).

By continuity of \(g\) and compactness of \(B(y, 1/\delta) \subset Y\), we have that \(g\) is uniformly continuous on \(B(y, 1/\delta)\). So, for \(\eta > 0\) such that \(d(x_0, x_1) < \eta\), it follows

\[
d(g^n(x_0), g^n(x_1)) < \frac{\gamma(h(g^n(x_0)))}{4}
\]

for all \(|n| < N\).

Then, if \(d(x_0, x_1) < \eta\), we have

\[
d^X(f^n(h(x_0)), f^n(h(x_1)))
= d^X(h(g^n(x_0)), h(g^n(x_1)))
\leq d^X(h(g^n(x_0)), j(g^n(x_0))) + d^X(j(g^n(x_0)), j(g^n(x_1)))
+ d^X(h(g^n(x_0)), j(g^n(x_1)))
\leq \bar{e}(hg^n(x_0)) + \frac{\gamma(hg^n(x_0))}{4} + \bar{e}(hg^n(x_1))
< \frac{\gamma(hg^n(x_0))}{4} + \frac{\gamma(hg^n(x_0))}{4} + \frac{\gamma(hg^n(x_0))}{4}
< \max\{\gamma(f^n(h(x_0))), \gamma(f^n(h(x_1)))\} < e(f^n(h(x_0)))
\]
by Remark 4.8

for all \(|n| < N\), and so \(d(h(x_0), h(x_1)) < \lambda\) by the choice of \(N\). Then, \(h\) is continuous. \(\square\)

Finally, we show some examples which exhibit the \(pGH\)-stability and the convergence in a pointed sense. For this, we remember Remark 2.14 and 2.11.

**Example 4.14** Let \(T_A : \mathbb{T}^2 \to \mathbb{T}^2\) be a hyperbolic toral automorphism, for instance, induced by the matrix

\[
A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}
\]
on the torus \(\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1\). It is known that \(T_A\) has the POTP and \(T_A\) is expansive. Then, \(T_A\) is \(GH\)-stable. In particular, \(T_A\) is \(pGH\)-stable. Generally, any homeomorphism \(f : X \to X\) on a compact metric space \(X\), with the POTP and the expansivity is \(pGH\)-stable.

For the second example, the following definitions and notations are introduced. Consider a metric space \(M\).

**Definition 4.15** The \(\omega\)-limit set of \(p \in M\) is the set

\[
\omega(x) = \left\{ y \in M : y = \lim_{n \to \infty} \varphi_{t_n}(x) \text{ for some } t_n \to \infty \right\}.
\]

The \(\alpha\)-limit set of \(p \in M\) is the set

\[
\alpha(x) = \left\{ z \in M : z = \lim_{n \to \infty} \varphi_{-t_n}(x) \text{ for some } t_n \to \infty \right\}.
\]
Definition 4.16 The stable set of a point $x$ is

$$W^s(x) = \left\{ y \in M : \lim_{k \to \infty} d(\varphi_{t_n}(x), \varphi_{t_n}(y)) = 0 \text{ for some } t_n \to \infty \right\}.$$ 

The unstable set of a point $x$ is

$$W^u(x) = \left\{ y \in M : \lim_{k \to \infty} d(\varphi_{-t_n}(x), \varphi_{-t_n}(y)) = 0 \text{ for some } t_n \to \infty \right\}.$$ 

Example 4.17 Let $\mathcal{X}$ be a $C^\infty$ vector field which induces a Cherry flow $\varphi_t$ on $\mathbb{T}^2$, the two-dimensional torus ([2, 22]).

The vector field $\mathcal{X}$ of the Cherry flow $\varphi_t$ has the following properties (Fig. 1):

1. $\mathcal{X}$ has two hyperbolic singularities, a saddle $\sigma$ and a sink $p_s$.
2. $\mathcal{X}$ is transverse to a meridian circle $\Sigma$ in $\mathbb{T}^2$.
3. One of the two orbits in $W^u(\sigma) \setminus \sigma$ intersect $\Sigma$ in a first point $c$.
4. There is an open interval $(a, b) \subset \Sigma$ such that the positive orbit of $y \in (a, b)$ goes directly to $p_s$.
5. A Poincare map $g : \Sigma \setminus [a, b] \to \Sigma$ associated to $\mathcal{X}$ is expanding.
6. The map $g$ in (5) is extended to the whole $\Sigma$ defining $g(y) = c$ for every $y \in [a, b]$. Moreover, $g$ has irrational rotation number.

The proof of the following lemma can be found in [22].

Lemma 4.18 If $\varphi_t$ is a Cherry flow, then

1. $\varphi_t$ has no periodic orbits.
2. $\Lambda = \mathbb{T}^2 \setminus W^s(p_s)$ is a transitive set of $\varphi_t$. 
To continue, define $f_0 : \mathbb{T}^2 \to \mathbb{T}^2$ by $f_0(x) = \varphi_1(x)$, the time one map of $\varphi_t$. By Lemma 4.18, $f_0$ has the $C$-shadowing property, and property (5.) implies $g$ expanding. So, $f_0|_\Lambda$ is expanding but $f_0$ is not. However, $f_0$ is $pGH$-stable by property (1.) and Lemma 4.18. Now, take $\epsilon > 0$ small and consider an open neighborhood $U$ of the sink $p_s$ such that $U$ not intersects $\Sigma_1$, diameter of $U$ is less than $\epsilon$ and $\bigcap_{n \geq 1} (f_0)^n(U) = \{p_s\}$. For each $n \in \mathbb{N}$, $\mathbb{T}^2$ is pulled in $U$ as it is shown in Fig. 2.

The deformation generates another metric space $X_n$, composed of a torus, a sphere, and a circular cylinder of length $n$ and radius $\epsilon/n$ connecting them, such as in Fig. 2. Let $i_n : \mathbb{T}^2 \to X_n$ be a homeomorphism which produces the deformation from $\mathbb{T}^2$ to $X_n$, with $i_n(p_s)$ in the sphere. Now, take $f_n : X_n \to X_n$ given by $f_n = i_n \circ f_0 \circ i_n^{-1}$.

Observe that $(X_n, p_n) \xrightarrow{pGH} (X, p)$, where $(X, p)$ can be: a torus $(\mathbb{T}^2 \setminus \{p_s\}, p)$, $(S^2 \setminus \{q_u, q_s\})$, or $(\mathbb{R}, p)$, depending the ubication of points $p_n$ [10, example 6.3].

Now, we take $\tilde{X} = \hat{X}$. So, $(X_n, p_n, f_n) \xrightarrow{d_{GH}^0} (\tilde{X}, p, f)$ (Fig. 3).

$(\tilde{X}, p, f)$ satisfies the following:

- $f|_{\mathbb{T}^2} = f_0$.
- $f|_{\mathbb{R}}$ is a homeomorphism and has no fixed points.
- $f|_{S^2}$ has two fixed points, a sink $q_s$ and a source $q_u$.
- If $p \in \mathbb{R}$, then $\alpha(p) = p_s$ and $\omega(p) = q_u$. Moreover, there exists a sequence $p_n \in X_n$ such that $(X_n, p_n) \xrightarrow{pGH} (\mathbb{R}, p)$.
- If $p \in S^2 \setminus \{q_u, q_s\}$, then $\alpha(p) = q_u$ and $\omega(p) = q_s$.

In this way, $f$ is $pGH$-stable.

**Example 4.19** Let $g : S^1 \to S^1$ be given by $g(\theta) = 2\theta$, with $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $d(\theta_1, \theta_2)$ is a minimal arc-length connecting $\theta_1$ and $\theta_2$ over $2\pi$. The homeomorphism $g$ is expansive and has the POTP [8], so $g$ is $pGH$-stable. For each $n \in \mathbb{N}$, consider the homeomorphism $i_n : S^1 \to Y_n$, which produces the deformation from $S^1$ to $Y_n$ such
that a ball centered in $\theta = \pi$ and radius $1/4n$ is transformed such as is described in Fig. 4 in the blue color section, where the neck has fixed length. Define $f_n : Y_n \to Y_n$ by $f_n = i_n \circ g \circ i_n^{-1}$. Theorem 2.3.6 in [27] follows that for all $n$, $f_n$ has the POTP. On the other hand, the expansiveness is directly guaranteed by the definition of $f_n$. Therefore, $f_n$ is $pGH$-stable for all $n$. In this case, $Y_n \rightharpoonup GH Y$, where $Y$ is the union between two circles and a line connects them. However, the sequence $\{f_n\}_n$ does not converge to a map $f : Y \to Y$ in $GH^0$-sense neither $pGH^0$-sense. Indeed, taking points $p_n, q_n \in Y_n$ such as Fig. 4, then $p_n \rightharpoonup pGH x$ and $q_n \rightharpoonup qGH x$ for some $x \in Y$ in the line. But $f(x)$ is not well defined. This example shows us that in certain phenomena, it is convenient to consider another way to measure the distance between metric spaces and, consequently, the distance between maps.

5 Conclusions

The pointed $C^0$-Gromov-Hausdorff distance expands the $GH$-distance proposed in Arbieto and Rojas [1] for the non-compact case of metric spaces, satisfying similar properties. Moreover, these definitions coincide when the space is compact.

The $pGH$-stability allows us to analyze when perturbations of maps defined in proper pointed metric spaces preserve dynamical properties. On the other hand, Theorem 4.13 gives sufficient conditions to guarantee $pGH$-stability of a map defined in a proper metric space, which includes the result given in Arbieto and Rojas [1].

The pointed $C^0$-Gromov-Hausdorff distance can be studied for some distance between maps mentioned in Preliminaries such as [5, 17] and [4].

Example 4.19 shows that in certain phenomena, such as collapsing, it is convenient to consider other alternatives measure for the distance between metric spaces and maps.

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Appendix A Appendix

Proposition A.1 (triangle inequality) Let \((X_m, x_m)\) be pointed metric spaces for \(m = 1, 2, 3\). If \(d^n_{GH}((X_m, x_m), (X_n, x_n)) \leq 1/2\) for \((m, n) = (1, 2)\) and \((2, 3)\), then

\[
d^n_{GH}((X_1, x_1), (X_3, x_3)) \leq 2 \left[d^n_{GH}((X_1, x_1), (X_2, x_2)) + d^n_{GH}((X_2, x_2), (X_3, x_3))\right]
\]

**Proof** Let \(i_{mn} \in \text{App}_{\epsilon_{mn}}((X_m, x_m), (X_n, x_n))\), where \((m, n)\) equal to \((1, 2)\), or \((2, 3)\). We use the notation \(B^m\) for \(B^{X_m}\), and \(d^m\) for \(d^{X_m}\), and take \(\epsilon_{mn} < \frac{1}{2}\). Then we have that \(i_{mn}(x_m) = x_n\),

\[
d(i_{mn})|_{B^m(x_m, \epsilon_{mn}^{-1})} < \epsilon_{mn}, \quad B^n(x_n, \epsilon_{mn}^{-1} - \epsilon_{mn}) \subset N_{\epsilon_{mn}}(i_{mn}(B^n(x_n, \epsilon_{mn}^{-1}))).
\]

We call \(i_{13} = i_{23} \circ i_{12}\), and let \(\epsilon_{13} = 2(\epsilon_{12} + \epsilon_{23})\). We want to show that \(i_{13} \in \text{App}_{\epsilon_{13}}((X_1, x_1), (X_3, x_3))\).

Observe that \(\epsilon_{mn} < 2 \epsilon_{mn} < \epsilon_{13} < 1\),

Let \(p \in B^1(x_1, \epsilon_{13}^{-1}) \subset B^1(x_1, \epsilon_{12}^{-1})\), and since \(d(i_{12})|_{B^1(x_1, \epsilon_{12}^{-1})} < \epsilon_{12}\) and \(i_{12}(x_1) = x_2\) we have that

\[
|d^1(p, x_1) - d^2(i_{12}(p), x_2)| \leq \epsilon_{12},
\]

then

\[
d^2(i_{12}(p), x_2) \leq d^1(p, x_1) + \epsilon_{12} \\
\leq \epsilon_{13}^{-1} + \epsilon_{12} \\
\leq \frac{1 + 2\epsilon_{12}(\epsilon_{12} + \epsilon_{23})}{2(\epsilon_{12} + \epsilon_{23})} \leq \frac{1}{\epsilon_{12} + \epsilon_{23}} \\
\leq \frac{1}{\epsilon_{23}}.
\]

Note that we use that \(\epsilon_{mn} < 1/2\).

Now if \(p_1, p_2 \in B^1(x_1, \epsilon_{13}^{-1}), i_{12}(p_1), i_{12}(p_2) \in B^2(x_2, \epsilon_{23}^{-1})\), then

\[
|d^1(p_1, p_2) - d^3(i_{13}(p_1), i_{13}(p_2))| \\
\leq |d^1(p_1, p_2) - d^2(i_{12}(p_1), i_{12}(p_2))| + |d^2(i_{12}(p_1), i_{12}(p_2)) - d^3(i_{13}(p_1), i_{13}(p_2))| \\
\leq \epsilon_{12} + \epsilon_{23} < \epsilon_{13}.
\]

Then \(d(i_{13})|_{B^1(x_1, \epsilon_{13}^{-1})} < \epsilon_{13}\).

For the third part, let \(p_3 \in B^3(x_3, \epsilon_{13}^{-1} - \epsilon_{13}) \subset B^3(x_3, \epsilon_{23}^{-1}) \subset N_{\epsilon_{23}}(i_{23}B^2(x_2, \epsilon_{23}^{-1}))\), then there exists \(p_2 \in B^2(x_2, \epsilon_{23}^{-1})\), such that \(d^3(i_{23}(p_2), p_3) < \epsilon_{23}\), and using the distortion of \(i_{23}\)
that is, \( p_2 \in B^2\left(x_2,\varepsilon_{12}^{-1} - \varepsilon_{12}\right) \subset N_{\varepsilon_{12}}\left(i_{12}B^1(x_1,\varepsilon_{12}^{-1})\right) \). So there is \( p_1 \in B_1\left(x_1,\varepsilon_{12}^{-1}\right) \) such that \( d^2(i_{12}(p_1), p_2) < \varepsilon_{12} \), and again by the dilation of \( i_{12} \)

\[
d^1(p_1, x_1) \leq d^2(i_{12}(p_1), p_2) + d^2(p_2, x_2) + \varepsilon_{12} < \varepsilon_{12} + \varepsilon_{13}^{-1} - \varepsilon_{13} + 2\varepsilon_{23} + \varepsilon_{12} = \varepsilon_{13}^{-1},
\]

that is \( p_1 \in B^1(x_1,\varepsilon_{13}^{-1}) \). Now using the distortion of \( i_{23} \) we have

\[
d^3(i_{13}(p_1), p_3) \leq d^3(i_{13}(p_1), i_{23}(p_2)) + d^3(i_{23}(p_2), p_3) = d^3(i_{23}(i_{12}(p_1)), i_{23}(p_2)) + d^3(i_{23}(p_2), p_3) \leq d^2(i_{12}(p_1), p_2) + \varepsilon_{23} + d^3(i_{23}(p_2), p_3) \leq \varepsilon_{12} + 2\varepsilon_{23} < \varepsilon_{13}.
\]

That is \( B^3(x_3,\varepsilon_{13}^{-1} - \varepsilon_{13}) \subset N_{\varepsilon_{13}}\left(i_{13}(B^1(x_1,\varepsilon_{13}^{-1}))\right) \).

Making \((m, n)\) equal to \((2, 1)\), or \((3, 2)\), and interchanging 1 by 3, we get that the same result for \( i_{31} \), and the proof is complete. \(\Box\)

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