TENSOR ALGEBRAS AND DISPLACEMENT STRUCTURE. III.
ASYMPTOTIC PROPERTIES

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Abstract. We continue to investigate some classes of Szegö type polynomials in several
variables. We focus on asymptotic properties of these polynomials and we extend several
classical results of G. Szegö to this setting.

Keywords: Spectral factorization, polynomials on several variables, asymptotic properties

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1. Introduction

An extension to several non-commuting variables of the Szegö orthogonal polynomi-
als on the unit circle was considered in [6]. Some of the basic algebraic results on these
polynomials were also obtained, including recurrence equations, Christoffel-Darboux
formulae, and a Favard type result. Also, it was explained their connection with dis-
placement structure theory. Our main goal is to continue to investigate this kind of
polynomials and in this paper we focus on some of their asymptotic properties. There
are several fundamental results of G. Szegö involving asymptotic properties of the or-
thogonal polynomials on the unit circle. Thus, let \( T \) be the unit circle and let \( \mu \) be a
positive Borel measure on \( T \) with \( \log \mu' \in L^1 \). Also let \( \{\varphi_n\}_{n \geq 0} \) be the family of or-
thogonal polynomials associated to \( \mu \) and \( \varphi^\#_n(z) = z^n \varphi_n(1/\overline{z}) \), \( n \geq 0 \). It is well-known
(see [16]) that

\[
\varphi_n \to 0
\]

and

\[
\frac{1}{\varphi^\#_n} \to \Theta_\mu,
\]

where \( \Theta_\mu \) is the spectral factor of \( \mu \) and the convergence is uniform on the compact
subsets of the unit disk \( \mathbb{D} \). The second limit (1.2) is related to the so-called Szegö limit
theorems concerning the asymptotic behaviour of Toeplitz determinants. Thus,

\[
\frac{\det T_n}{\det T_{n-1}} = \frac{1}{|\varphi^\#_n(0)|^2},
\]

where \( T_n = [s_{i-j}]_{i,j=0}^n \) and \( \{s_k\}_{k \in \mathbb{Z}} \) is the set of the Fourier coefficients of \( \mu \). As a
consequence of the previous relation and (1.2) we deduce Szegö’s first limit theorem,

\[
\lim_{n \to \infty} \frac{\det T_n}{\det T_{n-1}} = |\Theta_\mu(0)|^2 = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log \mu'(t) dt\right).
\]
The second (strong) Szegö limit theorem improves \((1.3)\) by showing that
\[
(1.4) \quad \lim_{n \to \infty} \det T_n = \exp \left( \frac{1}{\pi} \int \int_{|z| \leq 1} |\Theta'_\mu(z)/\Theta_\mu(z)|^2 d\sigma(z) \right),
\]
where \(g(\mu)\) is the limit in \((1.3)\) and \(\sigma\) is the planar Lebesgue measure. These two limits \((1.3)\) and \((1.4)\) have an useful interpretation in terms of asymptotics of angles in the geometry of a stochastic process associated to \(\mu\) (see \([12]\)).

Our goal in this paper is to extend these results to the class of orthogonal polynomials on several non-commuting variables introduced in \([6]\). The paper is organized as follows. In Section 2 we review notation and a framework for studying orthogonal polynomials associated to polynomial relations on several non-commuting variables. Thus the paper can be read independently of \([5]\) and \([6]\). In Section 3 we analyse the case of no relation in dimension one. It turns out that this is, in fact, the most general situation, and for this reason we treat this case separately. The main result is Theorem 3.3, which extends \((1.1)\) and \((1.2)\). Theorem 3.4 contains extensions of \((1.3)\) and \((1.4)\). In Section 4 we discuss a few examples. First, we show how to recapture the classical setting of orthogonal polynomials on the unit circle and on the real line. Then, we turn our attention to the orthogonal polynomials considered in \([6]\).

2. Preliminaries

We introduce some necessary terminology and notation. Especially, we briefly review a rather familiar setting for orthogonal polynomials associated to relations on several variables (for some details, see \([4]\)).

2.1. Tensor Algebras. Let \(\mathbb{F}_N^+\) be the unital free semigroup on \(N\) generators \(1, \ldots, N\) with lexicographic order \(\prec\). The empty word is the identity element and the length of the word \(\sigma\) is denoted by \(|\sigma|\). The length of the empty word is 0 and \(l(\sigma)\) denotes the number of words \(\tau \preceq \sigma\).

The tensor algebra over \(\mathbb{C}^N\) is defined by the algebraic direct sum
\[
T_N = \bigoplus_{k \geq 0} (\mathbb{C}^N)^{\otimes k},
\]
where \((\mathbb{C}^N)^{\otimes k}\) denotes the \(k\)-fold tensor product of \(\mathbb{C}^N\) with itself. The addition is the componentwise addition and the multiplication is defined by juxtaposition:
\[
(x \otimes y)_n = \sum_{k+l=n} x_k \otimes y_l.
\]
If \(\{e_1, \ldots, e_N\}\) is the standard basis of \(\mathbb{C}^N\), then \(\{e_{i_1} \otimes \cdots \otimes e_{i_k} \mid 1 \leq i_1, \ldots, i_k \leq N\}\) is an orthonormal basis of \(T_N\). If \(\sigma = i_1 \ldots i_k\) then we write \(e_\sigma\) instead of \(e_{i_1} \otimes \cdots \otimes e_{i_k}\), so that any element of \(T_N\) can be uniquely written in the form \(x = \sum_{\sigma \in F_N^+} c_\sigma e_\sigma\), where only finitely many of the complex numbers \(c_\sigma\) are different from 0.

Another construction of \(T_N\) is given by the algebra \(\mathcal{P}_N\) of polynomials in \(N\) non-commuting indeterminates \(X_1, \ldots, X_N\) with complex coefficients. Each element \(P \in \mathcal{P}_N\) can be uniquely written in the form \(P = \sum_{\sigma \in F_N^+} c_\sigma X_\sigma\), with \(c_\sigma \neq 0\) for finitely many
\( \sigma \)'s and \( X_{\sigma} = X_{i_1} \ldots X_{i_k} \) where \( \sigma = i_1 \ldots i_k \in \mathbb{F}^+_N \). The linear extension \( \Phi_1 \) of the mapping \( e_\sigma \to X_\sigma, \sigma \in \mathbb{F}^+_N \), gives an isomorphism of \( \mathcal{T}_N \) with \( \mathcal{P}_N \).

Another known realization of the tensor algebra was used in [6] in order to establish a connection with the displacement structure theory. This was useful since many results for the tensor algebra could be seen just as particular instances of more general results in the triangular algebra. Thus, let \( \mathcal{E} \) be a Hilbert space and define: \( \mathcal{E}_0 = \mathcal{E} \) and for \( k \geq 1, \)

\[
\mathcal{E}_k = \mathcal{E}_{k-1} \oplus \ldots \oplus \mathcal{E}_{k-1} = \mathcal{E} \oplus N \text{ terms}.
\]

For \( \mathcal{E} = \mathbb{C} \) we have that \( \mathbb{C}_k \) can be identified with \( (\mathbb{C}^N)^{\otimes k} \) and \( \mathcal{T}_N \) is isomorphic to the algebra \( \mathcal{L}_N \) of lower triangular operators \( T = [T_{k,j}] \in \mathcal{L}(\oplus_{k \geq 0} \mathbb{C}_k) \) with the property

\[
T_{k,j} = T_{k-1,j-1} \oplus \ldots \oplus T_{k-1,j-1} = T \oplus N \text{ terms}.
\]

for \( k \leq j, k, j \geq 1 \), and \( T_{j,0} = 0 \) for all sufficiently large \( j \)'s. The isomorphism is given by the map \( \Phi_2 \) defined as follows: let \( x = (x_0, x_1, \ldots) \in \mathcal{T}_N \) \( (x_p \in (\mathbb{C}^N)^{\otimes p} \) is the \( p \)th homogeneous component of \( x \)); then \( x_p = \sum_{|\sigma| = p} c_\sigma e_\sigma \) and for \( j \geq 0 \), \( T_{j,0} \) is given by the column matrix \( \{c_\sigma\}_{|\sigma| = j} \), where "\( t \)" denotes the matrix transpose. Then \( T_{j,0} = 0 \) for all sufficiently large \( j \)'s and we can define \( T \in \mathcal{L}(\oplus_{k \geq 0} \mathbb{C}_k) \) by using (3.1). Finally, set \( \Phi_2(x) = T \).

2.2. Spectral Factorization. We briefly review the spectral factorization of positive definite kernels on the set \( \mathbb{N}_0 \) of nonnegative integers. For more details, see [4]. Let \( \mathcal{E} \) be a Hilbert space and let \( \mathcal{P}_+(\mathcal{E}) \) be the set of positive definite kernels on \( \mathbb{N}_0 \) with values in \( \mathcal{L}(\mathcal{E}) \). The order on \( \mathcal{P}_+(\mathcal{E}) \) is: \( K_1 \leq K_2 \) if \( K_2 - K_1 \) belongs to \( \mathcal{P}_+(\mathcal{E}) \). Next consider a family \( \mathcal{F} = \{\mathcal{F}_n\}_{n \geq 0} \) of Hilbert spaces and call lower triangular array a family \( \Theta = \{\Theta_{k,j}\}_{k,j \geq 0} \) of operators \( \Theta_{k,j} \in \mathcal{L}(\mathcal{E}, \mathcal{F}_k) \) such that \( \Theta_{k,j} = 0 \) for \( k < j \) and each column \( c_j(\Theta) = [\Theta_{k,j}]_{k \geq 0}, \ j \geq 0 \), belongs to \( \mathcal{L}(\mathcal{E}, \oplus_{k \geq j} \mathcal{F}_k) \). Denote by \( H^2(\mathcal{E}, \mathcal{F}) \) the set of all lower triangular arrays as above. A lower triangular array is called outer if the set \( \{c_j(\Theta)\mathcal{E} \mid j \geq k\} \) is total in \( \oplus_{j \geq k} \mathcal{F}_j \) for all \( k \geq 0 \). If \( \Theta \) is an outer triangular array, then the formula

\[
K_{\Theta}(k, j) = c_k(\Theta)^* c_j(\Theta)
\]

defines an element of \( \mathcal{P}_+(\mathcal{E}) \). For the proof of the following result see [4], Chapter 5.

**Theorem 2.1.** Let \( K \) be an element of \( \mathcal{P}_+(\mathcal{E}) \). Then there exists a family \( \mathcal{F} = \{\mathcal{F}_n\}_{n \geq 0} \) of Hilbert spaces and an outer triangular array \( \Theta \in H^2(\mathcal{E}, \mathcal{F}) \) such that

(a) \( K_{\Theta} \leq K \).

(b) For any other family \( \mathcal{F}' = \{\mathcal{F}'_n\}_{n \geq 0} \) of Hilbert spaces and any outer triangular array \( \Theta' \in H^2(\mathcal{E}, \mathcal{F}') \) such that \( K_{\Theta'} \leq K \), we have \( K_{\Theta'} \leq K_{\Theta} \).

(c) \( \Theta \) is uniquely determined by (a) and (b) up to a left unitary diagonal factor.
2.3. Orthogonal Polynomials. Let $\mathcal{P}_{2N}$ be the algebra of polynomials in $2N$ non-commuting indeterminates $X_1, \ldots, X_N, X_{N+1}, \ldots, X_{2N}$ with complex coefficients. An involution $J$ can be introduced on $\mathcal{P}_{2N}$ as follows:

\[
J(X_k) = X_{N+k}, \quad k = 1, \ldots, N,
\]
\[
J(X_l) = X_{l-N}, \quad l = N+1, \ldots, 2N;
\]
on monomials,

\[
J(X_{i_1} \ldots X_{i_k}) = J(X_{i_k}) \ldots J(X_{i_1}),
\]
and finally, if $Q = \sum_{\sigma \in F_{2N}^+} c_\sigma X_\sigma$, then $J(Q) = \sum_{\sigma \in F_{2N}^+} \overline{c}_\sigma J(X_\sigma)$. Thus, $\mathcal{P}_{2N}$ is a unital, associative, $*$-algebra over $\mathbb{C}$ and we notice that $\mathcal{P}_N$ is a subalgebra of $\mathcal{P}_{2N}$.

We say that $\mathcal{A} \subset \mathcal{P}_{2N}$ is $J$-symmetric if $P \in \mathcal{A}$ implies $cJ(P) \in \mathcal{A}$ for some $c \in \mathbb{C} - \{0\}$. We construct an associative algebra $\mathcal{T}_N(\mathcal{A})$ as the quotient of $\mathcal{P}_{2N}$ by the two-sided ideal $\mathcal{E}(\mathcal{A})$ generated by $\mathcal{A}$. We notice that $\mathcal{T}_N(\emptyset) = \mathcal{P}_{2N}$. We let $\pi = \pi_\mathcal{A} : \mathcal{P}_{2N} \to \mathcal{T}_N(\mathcal{A})$ be the quotient map and since $\mathcal{A}$ is $J$-symmetric,

\[
\mathcal{T}_A(\pi(P)) = \pi(J(P))
\]
gives an involution on $\mathcal{T}_N(\mathcal{A})$. We will be interested in linear functionals $\phi$ on $\mathcal{T}_N(\mathcal{A})$ with the property that $\phi(\mathcal{T}_A(\pi(P))\pi(P)) \geq 0$ for all $P \in \mathcal{P}_N$ and we will say that $\phi$ is a positive functional on $\mathcal{T}_N(\mathcal{A})$. We notice that $\phi(\mathcal{J}_A(\pi(P))) = \overline{\phi(\pi(P))}$ for $P \in \mathcal{P}_N$ and

\[
|\phi(\mathcal{J}_A(\pi(P_1))\pi(P_2))|^2 \leq \phi(\mathcal{J}_A(\pi(P_1))\pi(P_1))\phi(\mathcal{J}_A(\pi(P_2))\pi(P_2))
\]
for $P_1, P_2 \in \mathcal{P}_N$.

We now consider the GNS construction associated to $\phi$. Thus, we define on $\pi(\mathcal{P}_N)$,

\[
\langle \pi(P_1), \pi(P_2) \rangle_\phi = \phi(\mathcal{J}_A(\pi(P_2))\pi(P_1)),
\]
and factor out the subspace $\mathcal{N}_\phi = \{ \pi(P) | P \in \mathcal{P}_N, \langle \pi(P), \pi(P) \rangle_\phi = 0 \}$. Completing this quotient with respect to the norm induced by (2.4) we obtain a Hilbert space $\mathcal{H}_\phi$.

From now on we will assume that $\phi$ is strictly positive, that is, $\phi(\mathcal{T}_A(\pi(P))\pi(P)) > 0$ for all $P \in \mathcal{P}_N - \mathcal{E}(\mathcal{A})$, so that $\mathcal{N}_\phi = \{ 0 \}$ and $\pi(\mathcal{P}_N)$ can be viewed as a subspace of $\mathcal{H}_\phi$. The index set of $\mathcal{A}$, $G \subset F_{2N}^+$, is chosen as follows: let $\emptyset \in G$; if $\alpha \in G$, choose the next element in $G$ to be the least $\beta \in F_{2N}^+$ such that the set of elements $\pi(X_{\alpha'})$, $\alpha' \preceq \alpha$, and $\pi(X_\beta)$ is linearly independent. Define $F_\alpha = \pi(X_\alpha)$ for $\alpha \in G$ and set $B = \{ F_\alpha \}_{\alpha \in G}$. Let $G_n = \{ \alpha \in G \mid |\alpha| = n \}$, then $G_0 = \{ \emptyset \}$ and $\{ G_n \}_{n \geq 0}$ is a partition of $G$.

Since $\phi$ is strictly positive it follows that $B$ is a linearly independent family in $\mathcal{H}_\phi$ and the Gram-Schmidt procedure gives a family $\{ \varphi_\alpha \}_{\alpha \in G}$ of elements in $\pi(\mathcal{P}_N) \subset \mathcal{T}_N(\mathcal{A})$ such that

\[
\varphi_\alpha = \sum_{\beta \preceq \alpha} a_{\alpha, \beta} F_\beta, \quad a_{\alpha, \alpha} > 0;
\]
\[
\langle \varphi_\alpha, \varphi_\beta \rangle_\phi = \delta_{\alpha, \beta}, \quad \alpha, \beta \in G.
\]
The elements $\varphi_\alpha$, $\alpha \in G$, will be called the orthogonal polynomials associated to $\phi$. Typically, the theory of orthogonal polynomials deals with the study of algebraic and asymptotic properties of the orthogonal polynomials associated to strictly positive functionals on $T_N(A)$. We also notice that the use of the Gram-Schmidt process depends on the order that we have chosen on $\mathbb{F}_N^+$. A different order would yield a different family of orthogonal polynomials. Due to the natural grading on $\mathbb{F}_N^+$ it is possible to develop a base free approach to orthogonal polynomials. In the case of orthogonal polynomials on several commuting variables this is presented in [8]. However, in this paper we stick to the lexicographic order on $\mathbb{F}_N^+$ (and on the index set $G$).

An explicit formula for the orthogonal polynomials can be obtained in the same manner as in the classical (one scalar variable) case. Define

\begin{equation}
(2.7) \quad s_{\alpha,\beta} = \phi(J_A(F_\alpha)F_\beta) = \langle F_\beta, F_\alpha \rangle_\phi, \quad \alpha, \beta \in G,
\end{equation}

and

\begin{equation}
(2.8) \quad D_\alpha = \det \left[ s_{\alpha',\beta'} \right]_{\alpha',\beta' \preceq \alpha} > 0, \quad \alpha \in G.
\end{equation}

We notice that $\phi$ is a positive functional on $T_N(A)$ if and only if $K_\phi(\alpha, \beta) = s_{\alpha,\beta}$, $\alpha, \beta \in G$, is a positive definite kernel on $G$. While the kernel $K_\phi$ characterizes the positivity of the functional $\phi$ and contains the basic information for the construction of the orthogonal polynomials, in general it does not determine $\phi$ uniquely. We will occasionally say that the orthogonal polynomials are associated to the kernel $K_\phi$ rather than $\phi$ itself. One typical situation when $K_\phi$ determines $\phi$ is when $\{J(X_k) - X_k \mid k + 1, \ldots, N \} \subset A$; another example is provided by the Wick polynomials,

\[ X_iJ(X_j) - \delta_{ij} + \sum_{k,l=1}^N T_{ij}^{kl}J(X_l)X_k, \quad i, j = 1, \ldots, N, \]

where $T_{ij}^{kl}$ are complex numbers and $\delta_{ij}$ is the Kronecker symbol (see [14]).

From now on $\tau - 1$ denotes the predecessor of $\tau$ with respect to the lexicographic order on $\mathbb{F}_N^+$, while $\sigma + 1$ denotes the successor of $\sigma$. It is showed in [4] that $\varphi_\emptyset = s_{\emptyset,\emptyset}^{-1/2}$ and for $\emptyset \prec \alpha$,

\begin{equation}
(2.9) \quad \varphi_\alpha = \frac{1}{\sqrt{D_{\alpha - 1}D_\alpha}} \det \left[ \begin{array}{c|c} \left[ s_{\alpha',\beta'} \right]_{\alpha' \prec \alpha; \beta' \leq \alpha} \\ \hline F_\emptyset & \ldots & F_\alpha \end{array} \right],
\end{equation}

with an appropriate interpretation of the determinant. In most of the cases, the formula (2.9) is not very useful for the actual computation of the orthogonal polynomials or for their study. Instead there are used recurrence formulae. We discuss several examples in the next sections.

3. The case $A = \emptyset$, $N = 1$

It turns out that this is, in fact, the most general situation. For this reason we treat this case separately. For $A = \emptyset$ and $N = 1$, the index set is $G = \mathbb{N}_0$ and a linear functional on $P_2$ is positive if and only if $K_\phi(n,m) = \phi(J(X_1^n)X_1^m)$, $n, m \in \mathbb{N}_0$, is a
positive definite kernel on \( \mathbb{N}_0 \). However, \( K_\phi \) does not completely determine \( \phi \). Thus, there is no way to deduce \( \phi(X_1 J(X_1)) \) from \( K_\phi \) in general. Still, we notice that there is no other restriction on \( K_\phi \), in the sense that given a positive definite kernel \( K \) on \( \mathbb{N}_0 \), there exist positive functionals \( \phi \) on \( P_2 \) such that \( K_\phi = K \). This is done simply by the linearization of any function \( \phi_0 : \mathbb{F}^+_2 \to \mathbb{C} \) such that \( \phi_0(J(X_\alpha)X_\beta) = K(\alpha, \beta) \) for \( \alpha, \beta \in \mathbb{F}^+_2 \).

Let \( \{ \gamma_{k,j} \}_{0 \leq k < j} \) be the parameters associated to \( K_\phi \) by \([4]\), Theorem 1.5.3. Assuming that \( \phi \) is strictly positive means \( |\gamma_{k,j}| < 1 \) for all \( 0 \leq k < j \). Define \( d_{k,j} = (1-|\gamma_{k,j}|^2)^{1/2} \).

An explicit connection between \( K_\phi \) and \( \{ \gamma_{k,j} \}_{0 \leq k < j} \) is given by formula (1.4.6) in \([4]\),

\[
(3.1) \quad s_{k,j} = s_{k,k} \begin{bmatrix} 1 & 0 & \ldots \end{bmatrix} U_{k,j} \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix} \frac{1}{s_{j,j}},
\]

where \( U_{k,j} \) is defined by (1.5.5) in \([4]\). For a better understanding of this formula it could be useful to write it explicitly for a few particular indices and to answer a related combinatorial question. Thus,

\[
s_{01} = s_{00}^{1/2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \gamma_{01} & d_{01} \\ d_{01} & -\overline{\gamma}_{01} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} s_{11}^{1/2} = s_{00}^{1/2} \gamma_{01} s_{11}^{1/2};
\]

\[
s_{02} = s_{00}^{1/2} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_{01} & d_{01} & 0 \\ d_{01} & -\overline{\gamma}_{01} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma_{02} & d_{02} \\ 0 & d_{02} & -\overline{\gamma}_{02} \end{bmatrix} \begin{bmatrix} \gamma_{12} & d_{12} & 0 \\ d_{12} & -\overline{\gamma}_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} s_{22}^{1/2} = s_{00}^{1/2} (\gamma_{01} \gamma_{12} + d_{01} \gamma_{02} d_{12}) s_{22}^{1/2};
\]

\[
s_{03} = s_{00}^{1/2} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_{01} & d_{01} & 0 & 0 \\ d_{01} & -\overline{\gamma}_{01} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_{02} & d_{02} & 0 \\ 0 & d_{02} & -\overline{\gamma}_{02} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_{13} & d_{13} & 0 \\ 0 & d_{13} & -\overline{\gamma}_{13} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s_{33}^{1/2} = s_{00}^{1/2} (\gamma_{01} \gamma_{12} \gamma_{23} + \gamma_{01} d_{12} \gamma_{13} d_{23} + d_{01} \gamma_{02} d_{12} \gamma_{23} + d_{01} \gamma_{02} \overline{\gamma}_{12} \gamma_{13} d_{23} + d_{01} d_{02} \gamma_{03} \gamma_{13} d_{23}) s_{33}^{1/2}.
\]
A natural combinatorial question would be to calculate the number $N(s_{k,j})$ of additive terms in the expression of $s_{k,j}$. Thus, for $k \geq 0$,

$$N(s_{01}) = N(s_{k,k+1}) = 1,$$

$$N(s_{02}) = N(s_{k,k+2}) = 2,$$

$$N(s_{03}) = N(s_{k,k+3}) = 5.$$ 

The general formula is given by the following result.

**Theorem 3.1.** $N(s_{k,k+l})$ is given by the Catalan number $\frac{1}{l+1} \binom{2l}{l}$.

**Proof.** The first step of the proof considers the realization of $s_{k,j}$ through a time varying transmission line (or lattice) (see [4], Chapter 4, for more details). For illustration we consider the case of $s_{03}$ in Figure 1.

![Figure 1. Lattice representation for $s_{03}$](image)

Each box in Figure 1 represents the action of the unitary matrix

$$\begin{bmatrix} \gamma_{k,j} & d_{k,j} \\ d_{k,j} & -\gamma_{k,j} \end{bmatrix}$$

and we see that the number of additive terms in the formula of $s_{03}$ is given by the number of paths from $A$ to $B$ in Figure 1. In it clear that to each path from $A$ to $B$ in Figure 1 it corresponds a “good” path from $C$ to $D$ in Figure 2, that is, a path that never steps below the diagonal and goes only to the right or downward.

![Figure 2. A “good” path from $C$ to $D$](image)
More precisely, each box in Figure 1 corresponds to a point strictly above the diagonal in Figure 2. Once this one-to-one correspondence is established, we can use the well-known fact that the number of “good” paths like the one in Figure 2 is given exactly by the Catalan numbers.

Returning to orthogonal polynomials, we notice that they obey the following recurrence relations (see [6], formulae (3.10) and (3.11)):

\begin{equation}
\varphi_0(X, l) = \varphi_0^\sharp(X, l) = s^{-1/2}_{l, l}, \quad l \in \mathbb{N}_0,
\end{equation}

and for \( n \geq 1, l \in \mathbb{N}_0, \)

\begin{equation}
\varphi_n(X, l) = \frac{1}{d_{l, n+1}} \left( X \varphi_{n-1}(X, l+1) - \gamma_{l, n+l} \varphi_{n-1}^\sharp(X, l) \right),
\end{equation}

\begin{equation}
\varphi_n^\sharp(X, l) = \frac{1}{d_{l, n+1}} \left( -\gamma_{l, n+l} X \varphi_{n-1}(X, l+1) + \varphi_{n-1}^\sharp(X, l) \right),
\end{equation}

where \( \varphi_n(X) = \varphi_n(X, 0) \) and \( \varphi_n^\sharp(X) = \varphi_n^\sharp(X, 0) \). While it is clear how to recover the coefficients \( \gamma_{k, j} \) from \( K_{\varphi} \), it appears to be also useful to recover these parameters from the orthogonal polynomials. It follows from the proof of Theorem 3.2 in [6] that \( \{ \varphi_n(X, l) \}_{n \geq 0} \) is the family of orthogonal polynomials associated to the kernel \( K_{\varphi}^l(\alpha, \beta) = s_{\alpha+l, \beta+l}, \alpha, \beta \in \mathbb{N}_0 \). Let \( k_n^l \) be the leading coefficient of \( \varphi_n(X, l) \). We obtain the following formula for the parameters \( \gamma_{k, j} \).

**Theorem 3.2.** For \( l \in \mathbb{N}_0 \) and \( n \geq 1, \)

\[ \gamma_{l, n+l} = -s^{-1/2}_{0, 0} s^{-1/2}_{l+1, l+1} \varphi_n(0, l) \frac{k_{l+1}^1 \cdots k_{n-1}^{l+1}}{k_1^l \cdots k_n^l}. \]

**Proof.** We deduce from [3] that

\[ \varphi_n(0, l) = -\gamma_{l, n+l} \frac{1}{d_{l, n+1}} \varphi_{n-1}(0, l), \]

and from formula [3.4] we deduce

\[ \varphi_n^\sharp(0, l) = \frac{1}{d_{l, n+1}} \varphi_{n-1}^\sharp(0, l) = \cdots = s_{0, 0}^{-1/2} \prod_{p=1}^n \frac{1}{d_{l, p+l}}, \]

hence

\[ \varphi_n(0, l) = -s_{0, 0}^{-1/2} \gamma_{l, n+l} \prod_{p=1}^n \frac{1}{d_{l, p+l}}. \]

Using Theorem 1.5.10 in [4], we deduce that

\[ \prod_{p=1}^n d_{l, p+l}^2 = s_{l, l}^{-1} \frac{D_{l, l+n}}{D_{l+1, l+n}} \]
so that,

\begin{equation}
\gamma_{l,n+1} = -s_{l+1/2} s_{l+1/2}^{-1/2} \varphi_n(0, l) \sqrt{D_{l,l+n} / D_{l+1,l+n}}.
\end{equation}

On the other hand, (3.3) gives that

\[ k^l_n = \prod_{p=1}^{n-1} \frac{1}{d_{l+p,l+n}}. \]

and using once again Theorem 1.5.10 in [4], we deduce

\[ k^l_n = \sqrt{D_{l,l+n-1} / D_{l,l+n}}. \]

This implies that

\[ k^l_1 \cdots k^l_n = \frac{s_{l/2}^1}{\sqrt{D_{l,l+n}}}, \]

and this can be used in (3.5) in order to conclude the proof. \(\square\)

We now develop an analogue of (1.1) and (1.2). The formulae (3.3) and (3.4) suggest that it is more convenient to work in a larger algebra. Thus, we consider the set \(\mathcal{R}_1\) of lower triangular arrays

\[ a = \begin{bmatrix} a_{k,j} \\ \end{bmatrix} \]

with complex entries. No boundedness assumption is made on these arrays. The addition in \(\mathcal{R}_1\) is defined by entry-wise addition and the multiplication is defined as follows: for \(a = \begin{bmatrix} a_{k,j} \\ \end{bmatrix}, b = \begin{bmatrix} b_{k,j} \\ \end{bmatrix}\) two elements of \(\mathcal{R}_1\),

\[ (ab)_{k,j} = \sum_{l \geq 0} a_{k,l} b_{l,j}, \]

(the sum is finite since both \(a\) and \(b\) are lower triangular). Thus, \(\mathcal{R}_1\) becomes an associative, unital algebra.

Next we associate the element \(\Phi_n\) of \(\mathcal{R}_1\) to the polynomials \(\varphi_n(X, l) = \sum_{k=0}^n a^l_{n,k} X^k\), \(n, l \geq 0\), by the formula

\begin{equation}
(\Phi_n)_{k,j} = \begin{cases} a^l_{n,k-j} & k \geq j \\ 0 & k < j \end{cases}.
\end{equation}

Similarly, the element \(\Phi^\sharp_n\) of \(\mathcal{R}_1\) is associated to the family of polynomials \(\varphi^\sharp_n(X, l) = \sum_{k=0}^n b^l_{n,k} X^k\), \(n, l \geq 0\), by the formula

\begin{equation}
(\Phi^\sharp_n)_{k,j} = \begin{cases} b^l_{n,k-j} & k \geq j \\ 0 & k < j \end{cases}.
\end{equation}

We notice that since \(K_\phi\) is a scalar-valued kernel, the Hilbert spaces \(\mathcal{F}_n, n \geq 0\), given by Theorem 2.1 are at most one-dimensional (see [4], Section 5.1 for details). This implies that we can uniquely determine the spectral factor \(\Theta_\phi\) of \(K_\phi\) by the requirement that \((\Theta_\phi)_{n,n} \geq 0\) for all \(n \geq 0\). Also, \(\Theta_\phi \in \mathcal{R}_1\). From now on we assume that \((\Theta_\phi)_{n,n} > 0\) for all \(n \geq 0\) and we say that in this case \(\phi\) (or \(K_\phi\)) belongs to the
Szegő class. By formula (5.1.5) in [4], it follows that \( \phi \) belongs to the Szegő class if and only if

\[
(3.8) \quad s_{k,k} \prod_{n > k} d_{k,n} > 0
\]

for all \( k \geq 0 \). This implies, in particular, that \( \Phi_n^\# \) is invertible in \( R_1 \) for all \( n \geq 0 \). Finally, we say that a sequence \( \{a_n\} \subset R_1 \) converges to \( a \in R_1 \) if \( \{(a_n)_{k,j}\} \) converges to \( a_{k,j} \) for all \( k, j \geq 0 \) (and we write \( a_n \to a \)). We now obtain the following generalization of (1.1) and (1.2).

**Theorem 3.3.** Let \( \phi \) belong to the Szegő class. Then

\[
(3.9) \quad \Phi_n \to 0
\]

and

\[
(3.10) \quad (\Phi_n^\#)^{-1} \to \Theta_\phi.
\]

**Proof.** First we show (3.9). It is convenient to consider the natural derivation on \( P_1 \); for \( P = \sum_{k=0}^n a_k X^k \in P_1 \),

\[
P^{(1)} = \sum_{k=1}^n k a_k X^{k-1},
\]

and then, for \( k \geq 1 \),

\[
P^{(k)} = (P^{(k-1)})^{(1)}.
\]

We see that (3.9) is equivalent to

\[
\varphi_n^{(k)}(0, l) \to 0
\]

for each fixed \( k, l \geq 0 \). We claim that

\[
(3.11) \quad \sum_{n \geq 0} |\varphi_n^{(k)}(0, l)|^2 < \infty
\]

and for each \( l, k \geq 0 \),

\[
(3.12) \quad \lim_{n \to \infty} (\varphi_n^\#)^{(k)}(0, l) \quad \text{exists and is finite.}
\]

We prove these statements by induction on \( k \geq 0 \). For \( k = 0 \) we use the formula

\[
\varphi_n(0, l) = -s_{0,0}^{-1/2} \gamma_{l,n+l} \prod_{p=1}^n \frac{1}{d_{l,p+l}}.
\]

obtained in the proof of Theorem 3.2 in order to deduce that

\[
\sum_{n \geq 0} |\varphi_n^{(k)}(0, l)|^2 = s_{0,0}^{-1} \sum_{n \geq 0} |\gamma_{l,n+l}|^2 \prod_{p=1}^n \frac{1}{d_{l,p+l}^2}.
\]
Since $\phi$ belongs to the Szegö class, we have that 
\[ g_l = s_{l,t} \prod_{n > l} d_{l,n} > 0, \]

hence \( \prod_{p=1}^{n} \frac{1}{d_{l,p+l}} \leq c_l \) for some \( c_l > 0 \) and all \( n \geq 0 \). Also, for all \( n \geq 0 \),
\[ \sum_{n \geq 0} |\gamma_{l,n+l}|^2 < \infty. \]

In particular, \( \gamma_{l,n+l} \to 0 \) as \( n \to \infty \); all of these give (3.11) and (3.12) for \( k = 0 \) and all \( l \geq 0 \).

We now proceed to prove the general case. From (3.3) and (3.4) we also deduce (3.13)
\[ \varphi_n^{(k)}(0, l) = \frac{1}{d_{l,n+l}} \left( k\varphi_{n-1}^{(k-1)}(0, l + 1) - \gamma_{l,n+l}(\varphi_{n-1}^{\sharp}(k)(0, l)) \right), \]

(3.14)
\[ (\varphi_n^{\sharp})^{(k)}(0, l) = \frac{1}{d_{l,n+l}} \left( -k\varphi_{l,n+l}(\varphi_{n-1}^{(k-1)}(0, l + 1) + (\varphi_{n-1}^{\sharp})(k)(0, l) \right) \]

for \( k \geq 1 \).

Since \( k \geq 1 \), \( (\varphi_n^{\sharp})^{(k)}(0, l) = 0 \), and we deduce from (3.14) that
\[ (\varphi_n^{\sharp})^{(k)}(0, l) = -k \left( \prod_{p=1}^{n} \frac{1}{d_{l,p+l}} \right) \sum_{j=1}^{n} \gamma_{l,j+l} \left( \prod_{q=1}^{j} d_{l,q+l} \right) \varphi_{j-1}^{(k-1)}(0, l + 1). \]

By Schwarz inequality,
\[ \sum_{j \geq 1} |\gamma_{l,j+l} \left( \prod_{q=1}^{j} d_{l,q+l} \right) \varphi_{j-1}^{(k-1)}(0, l + 1)| \]
\[ \leq \left( \sum_{j \geq 1} |\gamma_{l,j+l}|^2 \prod_{q=1}^{j} d_{l,q+l}^2 \right)^{1/2} \left( \sum_{j \geq 1} |\varphi_{j-1}^{(k-1)}(0, l + 1)|^2 \right)^{1/2}. \]

Again, since \( \phi \) belongs to the Szegö class and \( \prod_{q=1}^{j} d_{l,q+l}^2 \leq 1 \), we deduce that
\[ \sum_{j \geq 1} |\gamma_{l,j+l} \varphi_{j-1}^{(k-1)}(0, l + 1) \prod_{q=1}^{j} d_{l,q+l}| \]
\[ \leq C \left( \sum_{j \geq 1} |\varphi_{j-1}^{(k-1)}(0, l + 1)|^2 \right)^{1/2}. \]

This and the induction hypothesis give that the series
\[ \sum_{j \geq 1} \gamma_{l,j+l} \left( \prod_{q=1}^{j} d_{l,q+l} \right) \varphi_{j-1}^{(k-1)}(0, l + 1) \]
converges absolutely and since
\[ \lim_{n \to \infty} \prod_{p=1}^{n} \frac{1}{d_{l,p+l}} = \frac{s_{l,t}}{g_l} < \infty, \]

we deduce that \( \lim_{n \to \infty} (\varphi_n^{\sharp})^{(k)}(0, l) \) exists and is finite.
Using (3.13),
\[ \sum_{n \geq 1} |\varphi_n^{(k)}(0, l)|^2 \leq k^2 \sum_{n \geq 1} \frac{1}{d_{l,n+l}} |\varphi_{n-1}^{(k-1)}(0, l + 1)|^2 \]
\[ + 2k \sum_{n \geq 1} \frac{1}{d_{l,n+l}} |\varphi_{n-1}^{(k-1)}(0, l + 1) \gamma_{l,n+l}^{(l)}(\varphi_{n-1}^{(k)})(0, l)| \]
\[ + \sum_{n \geq 1} \frac{1}{d_{l,n+l}} |\gamma_{l,n+l}|^2 |(\varphi_{n-1}^{(k)})^2(0, l)|^2. \]

Since for sufficiently large \( n \), \( d_{l,n+l} \geq C_l > 0 \) and \( |(\varphi_{n-1}^{(k)})^2(0, l)| \leq C'_l \), another application of the Schwarz inequality, the fact that \( \phi \) belongs to the Szeg"o class, and the induction hypothesis give that \( \sum_{n \geq 1} |\varphi_n^{(k)}(0, l)|^2 < \infty \). In particular, \( \varphi_n^{(k)}(0, l) \to 0 \) as \( n \to \infty \), concluding the proof of (3.9).

A convenient proof of (3.10) can be based on the so-called Toeplitz embedding, systematically used in [9]. This approach would also explain the meaning of the elements \( \Phi_n, \Phi_n^\sharp \) of \( R_1 \). Define, for \( n \geq 1 \),
\[ (\Gamma_n)_{k,j} = \begin{cases} \gamma_{k,j} & j = k + n \\ 0 & \text{otherwise}. \end{cases} \]

Then let \( A = [A_{j-k}]_{k,j \geq 0} \) be the positive definite Toeplitz kernel associated by Proposition 1.5.6 in [4] to \( \{\Gamma_n\}_{n \geq 1} \) and \( A_0 = [s_{l,l}] \). By Proposition 1.6.10 (a) in [4], \( K_\phi \) is just a compression of the kernel \( A \). By Proposition 1.6.10 (b) and Theorem 5.1.2 in [4], the spectral factor \( \Theta_\phi \) of \( K_\phi \) is a corresponding compression of the spectral factor of \( A \). The key point of the proof is the connection between \( \Phi_n \) and the right orthogonal polynomials of \( A \). For details on operator-valued orthogonal polynomials see [1]. Thus, let \( \{R_n\}_{n \geq 0} \) be the set of the right orthogonal polynomials of \( A \),
\[ R_n(z) = \sum_{k=0}^{n} R_{n,k} z^k, \quad R_{n,n} \geq 0, \]
and define \( R^*_n = z^n R_n(1/z)^* = \sum_{k=0}^{n} n R_{n,n-k} z^k \). Also, define
\[ \rho_n = [R^*_{n,n}, \ldots, R^*_{n,0}]^t \]
and let \( \tilde{\rho}_n \) be obtained by canonical reshuffle of \( \rho_n \) ([15], Chapter 7). Then,
\[ \Phi^\sharp_n = \begin{bmatrix} I_n & 0 \\ 0 & \tilde{\rho}_n \end{bmatrix}, \]
where \( I_n \) denotes the \( n \times n \) identity matrix. This relation can be easily checked by using the characterization of
\[ [R_{n,0}, \ldots, R_{n,n}]^t \]
as the unique solution of
\[ [A_{j-k}]_{0 \leq k,j \leq n} [R_{n,0}, \ldots, R_{n,n}]^t = [0, \ldots, 0, D_n]_0 \]
where $D_n$ is a positive operator, and the orthogonality properties of $\{\phi_n\}_{n \geq 0}$. Finally, an application of Theorem 4.37 in [1] concludes the proof.

In order to provide generalizations of (1.3) and (1.4) in this setting we consider first their geometrical interpretation. Thus, by a result of Kolmogorov, $K_\phi$ is the covariance kernel of a stochastic process $\{f_n\}_{n \geq 0} \subset L^2(\mu)$ for some probability space $(X, \mathcal{M}, \mu)$. That is,

$$K_\phi(m, n) = \int_X f_n f_m d\mu.$$

We can suppose, without loss of generality, that $\{f_n\}_{n \geq 0}$ is total in $L^2(\mu)$ and for $p \leq q$ we introduce the subspaces $\mathcal{E}_{p,q}$ given by the closure in $L^2(\mu)$ of the linear span of $\{f_k\}_{k=p}^q$.

The operator angle between two spaces $\mathcal{E}_1$ and $\mathcal{E}_2$ of $L^2(\mu)$ is defined by

$$B(\mathcal{E}_1, \mathcal{E}_2) = P_{\mathcal{E}_1}P_{\mathcal{E}_2}P_{\mathcal{E}_1},$$

where $P_{\mathcal{E}_1}$ is the orthogonal projection of $L^2(\mu)$ onto $\mathcal{E}_1$. Also define

$$\Delta(\mathcal{E}_1, \mathcal{E}_2) = I - B(\mathcal{E}_1, \mathcal{E}_2).$$

We associate to the process $\{f_n\}_{n \geq 0}$ a family of subspaces $\mathcal{H}_{r,q}$ of $L^2(\mu)$ such that $\mathcal{H}_{r,q}$ is the closure of the linear space generated by $f_k$, $r \leq k \leq q$.

The geometric interpretation of the limits (1.3) and (1.4) is discussed in [12] and nonstationary extensions are presented in [4], Chapter 6. The interpretation of the second Szegö limit theorem in [4] required a stochastic process indexed by the set of integers, which is not the case in our situation. So, we need a modification of that interpretation that fits into our setting. Thus, we consider first the scale of limits:

$$s - \lim_{r \to \infty} \Delta(\mathcal{H}_{0,n}, \mathcal{H}_{n+r,0}, \mathcal{H}_{n+1,r}) = \Delta(\mathcal{H}_{0,n}, \mathcal{H}_{n+1,\infty})$$

for $n \geq 0$, and then we let $n \to \infty$ and deduce

$$s - \lim_{n \to \infty} \Delta(\mathcal{H}_{0,n}, \mathcal{H}_{n+1,\infty}) = \Delta(\mathcal{H}_{0,\infty}, \cap_{n \geq 0} \mathcal{H}_{n,\infty}),$$

where $s - \lim$ denotes the strong operatorial limit.

We then deduce analogues of the Szegö limit theorems by expressing these limits in terms of the determinants $D_{r,q} = \det [K_\phi(r', q')]_{r', q' \leq q}$, $r \leq q$.

**Theorem 3.4.** Let $\phi$ belong to the Szegö class. Then

$$\frac{D_{r,q}}{D_{r+1,q}} = \frac{s_{r,r} \det \Delta(\mathcal{H}_{r,r}, \mathcal{H}_{r+1,q})}{|\phi_q^2(0, r)|^2},$$

$$\lim_{q \to \infty} \frac{D_{r,q}}{D_{r+1,q}} = \frac{s_{r,r} \det \Delta(\mathcal{H}_{r,r}, \mathcal{H}_{r+1,\infty})}{|\Theta_\phi(r, r)|^2} = s_{r,r} \prod_{j \geq 1} d_{r,r+j}^2.$$

If we denote the above limit by $g_r$ and

$$L = \lim_{n \to \infty} \prod_{0 \leq k \leq n \leq j} d_{k,j}^2 > 0,$$
then

\[
\lim_{n \to \infty} \frac{D_{0,n}}{\prod_{l=0}^{n} g_l} = \frac{1}{\det \Delta(\mathcal{H}_{0,\infty}, \cap_{n \geq 0} \mathcal{H}_{n,\infty})} = \frac{1}{L}.
\]

**Proof.** The connection between the operator angles and determinants of type $D_{r,q}$ is given by the following formula which is a consequence of Lemma 6.4.1 in [4]: for $r \leq l \leq q,$

\[
\det \Delta(\mathcal{H}_{r,l}, \mathcal{H}_{l+1,q}) = \frac{D_{r,q}}{D_{r,l}D_{l+1,q}}.
\]

Then Theorem 1.5.10 in [4] allows the computation of $D_{r,q}$ in terms of the parameters $\gamma_{i,j}.$ Noticing that $D_{r,r} = s_{r,r}$ and using the formula $\varphi^\sharp_q(0,r) = \prod_{i=1}^{q} \frac{1}{d_{r,r+i}}$ obtained in the proof of Theorem 3.3, we deduce that

\[
\frac{D_{r,q}}{D_{r+1,q}} = s_{r,r} \det \Delta(\mathcal{H}_{r,r}, \mathcal{H}_{r+1,q})
\]

\[
= s_{r,r} \frac{\prod_{r \leq k < j \leq q} d^2_{k,j}}{\prod_{r+1 \leq k \leq j \leq q} d^2_{k,j}}
\]

\[
= s_{r,r} \prod_{j=1}^{q} d^2_{r+r+j} = \frac{s_{r,r}}{|\varphi^\sharp_q(0,r)|^2},
\]

which is (3.17). This relation and Theorem 6.2.2 in [4] imply (3.18).

Using again (3.20), we deduce for $n < r,$ that

\[
\det \Delta(\mathcal{H}_{0,\infty}, \mathcal{H}_{n+1,r}) = \frac{D_{0,r}}{D_{0,n}D_{n+1,r}}
\]

\[
= \prod_{0 \leq k \leq n < j \leq r} d^2_{k,j},
\]

hence

\[
\det \Delta(\mathcal{H}_{0,\infty}, \mathcal{H}_{n+1,\infty}) = \lim_{r \to \infty} \det \Delta(\mathcal{H}_{0,\infty}, \mathcal{H}_{n+1,r}) = \prod_{0 \leq k \leq n < j} d^2_{k,j}.
\]

On the other hand,

\[
\frac{\prod_{l=0}^{n} g_l}{D_{0,n}} = \frac{\prod_{l=0}^{n} \prod_{j \geq 1} d^2_{l,l+j}}{\prod_{0 \leq k \leq j \leq n} d^2_{k,j}}
\]

\[
= \prod_{0 \leq k \leq n < j} d^2_{k,j},
\]

which shows that

\[
\det \Delta(\mathcal{H}_{0,\infty}, \mathcal{H}_{n+1,\infty}) = \frac{\prod_{l=0}^{n} g_l}{D_{0,n}},
\]

hence (3.19). □
Formula (3.18) would represent an analogue of (1.3), while (3.19) is an analogue of (1.4). It would be of interest to express the limit in (3.19) in terms of the spectral factor $\Theta$.

4. Some examples

4.1. Polynomials on the unit circle. Consider $A = \{1 - J(X_1)X_1\}$. In this case the index set is $\mathbb{N}_0$ and if $\phi$ is a linear functional on $\mathcal{T}_1(A)$, then

$$K_\phi(n + k, m + k) = K_\phi(n, m), \quad m, n, k \in \mathbb{N}_0,$$

which means that $K_\phi$ is a Toeplitz kernel. It turns out that the parameters $\{\gamma_{k,j}\}$ also satisfy the Toeplitz condition, $\gamma_{n+k,m+k} = \gamma_{n,m}, \ n < m, \ k \geq 1$. The orthogonal polynomials associated to $\phi$ are then the orthogonal polynomials on the unit circle and (3.3), (3.4) reduce to the classical recurrence equations in [16]. Also, Theorem 3.3 and Theorem 3.4 reduce to the classical results of Szegö, [16].

4.2. Polynomials on the real line. Consider $A = \{X_1 - J(X_1)\}$. In this case the index set is still $\mathbb{N}_0$, and if $\phi$ is a linear functional on $\mathcal{T}_1(A)$, this time the kernel $K_\phi$ has the Hankel property, that is

$$K_\phi(n, m + k) = K_\phi(n + k, m), \quad m, n, k \in \mathbb{N}_0.$$

The parameters $\{\gamma_{k,j}\}$ do not necessarily satisfy a similar Hankel property. In fact, it might be interesting to find a characterization of those families of parameters $\{\gamma_{k,j}\}$ producing Hankel forms. Traditionally, there are other parameters, usually called canonical moments, that are used. The canonical moments of $\phi$ can be calculated by using a $Q-D$ (quotient-difference) algorithm (see [13]). Also, the recurrence formulas of type (3.3), (3.4) are replaced by a three-term recurrence equation,

$$(4.1) \quad x \varphi_n(x) = b_n \varphi_{n+1}(x) + a_n \varphi_n(x) + b_{n-1} \varphi_{n-1}(x),$$

with initial conditions $\varphi_{-1} = 0, \varphi_0 = 1$ ([16]).

Still, parameters $\{\gamma_{k,j}\}$ can be associated such that (3.3), (3.4) hold. Also, Theorem 3.3 and Theorem 3.4 provide asymptotic properties of the orthogonal polynomials and, respectively, Hankel determinants in the corresponding Szegö class.

We consider an example computing the parameters $\gamma_{k,j}$ of the Hilbert matrix,

$$H = \begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{3} & \ldots \\
\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \ldots \\
\frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}$$

and notice that the associated orthogonal polynomials satisfy the three-term recurrence equation (4.1) with

$$b_{n-1} = \frac{n}{2\sqrt{4n^2 - 1}}, \quad n \geq 1,$$
and 

\[ a_n = \frac{1}{2}, \quad n \geq 0. \]

For example, the first 5 polynomials are:

\[ \varphi_{-1} = 0, \quad \varphi_0 = 1, \quad \varphi_1(x) = \sqrt{3}(2x - 1), \]

\[ \varphi_2(x) = \sqrt{5}(6x^2 - 6x + 1), \quad \varphi_3(x) = \sqrt{7}(20x^3 - 30x^2 + 12x - 1). \]

The canonical moments \( \{p_n\}_{n \geq 0} \) can be calculated from the continued fraction expansion of the Stieltjes transform of the uniform measure on \([0, 1]\),

\[
\int_0^1 \frac{dx}{z - x} = \frac{1}{z - \frac{\frac{\frac{\frac{\frac{2}{3}}{}{}{}\cdot \cdot \cdot}{1}2}{1}1}{1}1 - \ldots}
\]

which gives

\[ p_{2k-1} = \frac{1}{2}, \quad p_{2k} = \frac{k}{2k + 1}, \quad k \geq 1. \]

We deduce that, for \( n \geq 1 \),

\[
\det \begin{bmatrix}
1 & \frac{1}{2} & \cdots & \frac{1}{n} \\
\frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n}
\end{bmatrix} = \left( \prod_{k=1}^{n+1} \frac{1}{2k-1} \right) \prod_{l=0}^{n-1} \prod_{k=1}^{n-l} \left( \frac{k}{k+2l+1} \right)^2.
\]

This formula, (3.3), and (4.1) give that

\[ \gamma_{0,l} = (-1)^{l-1} \frac{\sqrt{2l+1}}{l+1}, \quad l \geq 1. \]

Extending this argument (based on results from [16]), we deduce that

\[ \gamma_{k,k+l} = (-1)^{l-1} \frac{\sqrt{(2k+1)(2k+2l+1)}}{2k+l+1}, \quad k \in \mathbb{N}_0, l \geq 1; \]

hence

\[ d_{k,k+l} = \frac{l}{2k+l+1}. \]

These formulae show that the uniform measure on \([0, 1]\) does not belong to the Szegö class.

We can obtain explicit computation of \( \{\gamma_{k,j}\} \) for other classes of classical orthogonal polynomials. The main point is to notice that if \( \{\varphi\}_{n \geq 0} \) is the family of orthogonal polynomials associated to a certain weight \( w(x) \), then \( \{\varphi_n(x, l)\}_{n \geq 0} \) is the family of orthogonal polynomials associated to the weight \( x^{2l}w(x) \). The polynomials associated to
4.3. Szegö polynomials on several non-commuting variables. The next examples are motivated in part by multiscale processes. These are stochastic processes indexed by the nodes of a tree. Isotropic processes on homogeneous trees were systematically studied, see [3] and the references therein. An extension to chordal graphs was recently given in [10]. Some classes of stochastic processes associated to the full binary (Cayley) tree were also considered (see, for instance, [11]).

We discuss here this last example. The vertices of the Cayley tree are indexed by \( \mathbb{F}_N^+ \). Let \((X, \mathcal{M}, \mu)\) be a probability space and let \(\{v_\sigma\}_{\sigma \in \mathbb{F}_N^+} \subset L^2(\mu)\) be a family of random variables. Its covariance kernel is

\[
K(\sigma, \tau) = \int_X \tau_\sigma v_\tau dP.
\]

The processes is called stationary (see [11]), if

\[
(4.2) \quad K(\tau \sigma, \tau' \sigma') = K(\sigma, \sigma'), \quad \tau, \sigma, \sigma' \in \mathbb{F}_N^+,
\]

\[
(4.3) \quad K(\sigma, \tau) = 0 \quad \text{if there is no } \alpha \in \mathbb{F}_N^+ \text{ such that } \sigma = \alpha \tau \text{ or } \tau = \alpha \sigma.
\]

Let \( \mathcal{A}_S = \{1 - J(X_k)X_k \mid k = 1, \ldots, N\} \cup \{J(X_k)X_l, k, l = 1, \ldots, N, k \neq l\} \) and note that the index set of \( \mathcal{A}_S \) is \( \mathbb{F}_N^+ \). We see that \( \phi \) is a positive functional on \( T_N(\mathcal{A}_S) \) if and only if \( K_\phi \) is the covariance of a stationary process as above. It was noticed in [6] that this happens if and only if

\[
(4.4) \quad \gamma_{\tau \sigma, \tau' \sigma'} = \gamma_{\sigma, \sigma'}, \quad \tau, \sigma, \sigma' \in \mathbb{F}_N^+,
\]

\[
(4.5) \quad \gamma_{\sigma, \tau} = 0 \quad \text{if there is no } \alpha \in \mathbb{F}_N^+ \text{ such that } \sigma = \alpha \tau \text{ or } \tau = \alpha \sigma,
\]

where \( \{\gamma_{\sigma, \tau} \mid \sigma, \tau \in \mathbb{F}_N^+, \sigma \leq \tau\} \) is the family of parameters associated to \( K_\phi \) by Theorem 1.5.3 in [4]. The main consequence of these relations is that we can define \( \gamma_{\sigma} = \gamma_{\emptyset, \sigma}, \sigma \in \mathbb{F}_N^+, \) and \( \{\gamma_{\sigma, \tau} \mid \sigma, \tau \in \mathbb{F}_N^+, \sigma \leq \tau\} \) is uniquely determined by \( \{\gamma_{\sigma}\}_{\sigma \in \mathbb{F}_N^+} \) due to the relation

\[
(4.6) \quad [\gamma_{\sigma, \tau}]_{|\sigma| = j, |\tau| = k} = ([\gamma_{\sigma', \tau'}]_{|\sigma'| = j-1, |\tau'| = k-1})^{\oplus N}, \quad j, k \geq 1.
\]

From now on we assume that \( \phi \) is unital, \( \phi(1) = 1 \). Then we can show that the recurrence equations (3.3) and (3.4) simplify to \( \varphi_\emptyset = 1 \) and for \( k \in \{1, \ldots, N\}, \sigma \in \mathbb{F}_N^+ \),

\[
(4.7) \quad \varphi_{k\sigma} = \frac{1}{d_{k\sigma}}(X_k \varphi_{\sigma} - \gamma_{k\sigma} \varphi_{k\sigma - 1}),
\]
where \( \varphi\sigma_{\emptyset} = 1 \) and for \( k \in \{1, \ldots, N\}, \sigma \in \mathbb{P}^+_N \),

\[
(4.8) \quad \varphi\sigma_{k\sigma} = \frac{1}{d_{k\sigma}}(-\tau_{k\sigma}X_k\varphi\sigma + \varphi\sigma_{k\sigma-1}).
\]

We also notice that the algebra \( \mathcal{T}_N \) is naturally embedded into \( \mathcal{R}_1 \) and \( \Phi_n, \Phi^\sharp_n \in \mathcal{T}_N \). Then, Theorem 3.3 implies that \( \Theta_\phi \) belongs to \( \mathcal{T}_N \) (but this is also seen directly), and through the isomorphisms mentioned in Section 2.1, \( \Theta_\phi \) can be identified with an element of the full Fock space over \( \mathbb{C}^N \), and therefore with a formal power series on variables \( X_1, \ldots, X_N \). Similarly, \( (\Phi^\sharp_n)^{-1} \) can be identified with a formal power series, denoted \( (\varphi_n^\sharp)^{-1} \), on variables \( X_1, \ldots, X_N \). Finally, we notice that \( \phi \) belongs to the Szegö class if and only if

\[
\prod_{\sigma \in \mathbb{P}^+_N} d_\sigma > 0.
\]

For two formal series on variables \( X_1, \ldots, X_N \), the sign \( \to \) means coefficient-wise convergence. The next result is a consequence of Theorem 3.3.

**Theorem 4.1.** Let \( \phi \) be a functional on \( \mathcal{T}_N(\mathcal{A}_S) \) and belonging to the Szegö class. Then

\[
(4.9) \quad \phi_n \to 0
\]

and

\[
(4.10) \quad (\phi^\sharp_n)^{-1} \to \Theta_\phi.
\]

As a consequence of Theorem 3.4 we obtain the following result (which, aside the new geometrical interpretation, whould be also a direct consequen ce of Theorem 6.4.5 in [4]).

**Theorem 4.2.** Let \( \phi \) be a functional on \( \mathcal{T}_N(\mathcal{A}_S) \) and belonging to the Szegö class. Then

\[
(4.11) \quad \lim_{|\tau| \to \infty} \frac{D_{\emptyset,\tau}}{D_{1,\tau}} = |\Theta_\phi(0)|^2 = \prod_{\sigma \in \mathbb{P}^+_N} d_{\sigma}^2.
\]

If we denote the above limit by \( g \) and

\[
L = \prod_{\sigma \in \mathbb{P}^+_N} d_{\sigma}^{2|\sigma|} > 0,
\]

then

\[
(4.12) \quad \lim_{|\tau| \to \infty} \frac{D_{\emptyset,\tau}}{g(\tau)} = \frac{1}{L}.
\]

Finally, we mention that similar results can be obtained for the commutative case, \( \mathcal{A}_C = \mathcal{A}_S \cup \{X_kX_l - X_lX_k \mid k, l = 1, \ldots, N\} \). Positive functionals on \( \mathcal{A}_C \) correspond to positive definite functions on \( \mathbb{Z}^N \). Details as well as other examples will be given in [2].

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