PERFECT MATCHINGS IN RANDOM SUBGRAPHS OF REGULAR BIPARTITE GRAPHS

ROMAN GLEBOV, ZUR LURIA, AND MICHAEL SIMKIN

Abstract. Consider the random process in which the edges of a graph $G$ are added one by one in a random order. A classical result states that if $G$ is the complete graph $K_{2n}$ or the complete bipartite graph $K_{n,n}$, then typically a perfect matching appears at the moment at which the last isolated vertex disappears. We extend this result to arbitrary $k$-regular bipartite graphs $G$ on $2n$ vertices for all $k = \Omega(n)$.

Surprisingly, this is not the case for smaller values of $k$. We construct sparse bipartite $k$-regular graphs in which the last isolated vertex disappears long before a perfect matching appears.

1. Introduction

The study of the random graph model $G(n;p)$ began with two influential papers by Erdős and Rényi [7, 8]. In [7] and [9], they considered the range $p = \Theta(\log n/n)$ and the appearance of spanning structures in that regime. Later, several papers [1, 3, 14, 15, 16, 21] led to the following understanding. Consider a random graph process on $n$ vertices, in which edges are added one by one in a random order. Asymptotically almost surely\(^1\), the first edge that makes the minimum degree one connects the graph, and creates a perfect matching. Likewise, when the minimum degree becomes two, it immediately contains a Hamilton cycle. Philosophically, spanning structures appear once local obstructions disappear.

For a graph $G = (V, E)$ and $p \in [0, 1]$, let $G(p)$ denote the distribution on subgraphs of $G$ in which each edge is retained with probability $p$, all choices independent. Recently, a series of papers [17, 12, 18, 22] extended the above philosophy to $G(p)$ for various $G$. For example, in [17] it was shown that if $G$ is a Dirac graph, then the threshold for Hamiltonicity of $G(p)$ remains $\Theta(\log n/n)$. See [23] for a survey of these and related results.

In this paper we consider the threshold $p_0$ for the appearance of a perfect matching in $G(p)$ where $G$ is a $k$-regular bipartite graph on $2n$ vertices. The celebrated permanent inequalities of Bregman [5] and Egorychev–Falikman [6, 10] imply that the number of perfect matchings in $G$ is $(1 + o(1))\frac{k}{e} n^k$, and in particular, that this number depends little on the specific structure of $G$. It is therefore natural to conjecture that $p_0$ depends only on $n$ and $k$. Furthermore, the logical candidate is the threshold for the disappearance of isolated vertices in $G(p)$, which is $p = \log n/k$.

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\(^1\)An event occurs “asymptotically almost surely” (a.a.s.) if the probability of its occurrence tends to 1 as $n \to \infty$. We say that a property holds for “almost every” element of a set if it holds a.a.s. for a uniformly random element of the set.
Indeed, for \( k = \Omega(n) \), this is true. We prove the strongest possible form of this statement; namely, that if one reconstructs \( G \) by adding its edges one by one in a random order, then typically a perfect matching appears at the same moment that the last isolated vertex vanishes.

Formally, a **graph process** in \( G = (V,E) \) is a sequence of graphs 

\[
(V,\emptyset) = G_0, G_1, \ldots, G_{|E|} = G
\]
on the vertex set \( V \), where for each \( i \), \( G_i \) is obtained from \( G_{i-1} \) by adding a single edge of \( G \). The **hitting time** of a monotone graph property \( P \) with respect to a graph process is \( \min\{ t : G_t \in P \} \).

For a graph process \( \tilde{G} \), let \( \tau_M(\tilde{G}) \) and \( \tau_I(\tilde{G}) \) denote the hitting times for containing a perfect matching and having no isolated vertices, respectively. Clearly, for every graph process \( \tilde{G} \) we have \( \tau_M(\tilde{G}) \geq \tau_I(\tilde{G}) \). Our main result is that if \( G \) is dense and \( \tilde{G} \) is chosen uniformly at random, equality a.a.s. holds.

**Theorem 1.1.** Let \( k = \Omega(n) \), let \( G \) be a \( k \)-regular bipartite graph on \( 2n \) vertices, and let \( \tilde{G} \) be a uniformly random graph process in \( G \). Then, a.a.s. \( \tau_M(\tilde{G}) = \tau_I(\tilde{G}) \).

**Corollary 1.2.** For \( G \) and \( k \) as above,

- If \( p = \frac{\log n - \omega(1)}{k} \), then a.a.s. \( G(p) \) does not contain a perfect matching.
- If \( p = \frac{\log n + \omega(1)}{k} \), then a.a.s. \( G(p) \) contains a perfect matching.

Quite surprisingly, it turns out that these results fail when \( k \) is significantly smaller than \( n \). We present a series of counterexamples in which the threshold for a perfect matching is much larger than the threshold for the disappearance of isolated vertices. In particular, \( k \) may be as large as \( \Theta \left( \frac{n}{\log n \log \log n} \right) \).

Theorem 1.1 is almost a triviality if one assumes that \( G \) is pseudorandom (cf. [20, Lemma 3.2]). The main element needed in our proof is a way to control induced subgraphs of \( G \) with high discrepancy. To this end we prove a result on the structure of high discrepancy sets in dense, regular, bipartite graphs (Lemma 3.4).

The remainder of this paper is organized as follows. Section 1.1 introduces our notation. In Section 2 we construct the aforementioned counterexamples, and discuss connections with a conjecture of Kahn and Kalai. In Section 3 we prove Lemma 3.4 and in Section 4 we establish some probabilistic tools. Finally, in Section 5 we prove Theorem 1.1.

### 1.1. Notation

Throughout the paper, we disregard floor and ceiling signs to improve readability. Large real numbers should be rounded to the nearest integer. We denote by “log” the natural logarithm.

For an integer \( m \in \mathbb{N} \), we define \( [m] = \{1,2,\ldots,m\} \). Let \( X \) be a set and let \( f : |X| \to \mathbb{R} \). We sometimes abuse notation by writing

\[
\sum_{|S|=1}^{m} \binom{|X|}{|S|} f(|S|) = \sum_{S \subseteq X : |S| \in [m]} f(|S|).
\]

Let \( f, g : \mathbb{N} \to \mathbb{R} \). We write \( f = \tilde{O}(g) \) if, for some \( c > 0 \) and all large enough \( n \in \mathbb{N} \), \( f(n) \leq g(n) \log^c (g(n)) \).
Let $G = (V,E)$ be a graph. For $A,B \subseteq V$, denote by $E_G(A,B)$ the set of edges incident to both $A$ and $B$, and let $e_G(A,B) = |E_G(A,B)|$. Let $N_G(A)$ denote the set of neighbors of $A$, i.e., the set $\{v \in V : \exists a \in A \text{ s.t. } av \in E\} \setminus A$. We define $G \setminus A$ to be the induced graph on the vertex set $V(G) \setminus A$.

Suppose $G$ is a bipartite graph with the vertex partition $X,Y$. A vertex set $A$ is partite if $A \subseteq X$ or $A \subseteq Y$. We denote by $A'$ the complement of $A$ w.r.t. its own part, i.e., $X \setminus A$ if $A \subseteq X$ and $Y \setminus A$ if $A \subseteq Y$. If $A$ is empty, it will be clear from context whether $A' = X$ or $A' = Y$.

By a common abuse of notation, we speak of $G(p)$ as having a certain property, instead of saying that $G \sim G(p)$ has that property.

2. Counterexamples

It may be intuitive to think at first - as all three of us did - that the conclusion of Theorem 1.1 holds for all large regular bipartite graphs, i.e., the requirement $k = \Omega(n)$ is not necessary. In this section, we show that this is not true, and indeed for small values of $k$, $G(p)$ might not contain a perfect matching even for relatively large $p$.

The intuition for all of our counterexamples comes from the following simple construction.

**Definition 2.1.** A $k$-resistor between two vertices $x$ and $y$ is the following bipartite graph: The vertex set is $\{x,y\} \cup X' \cup Y'$, where $X'$ and $Y'$ have cardinality $k$. Let $x' \in X'$, $y' \in Y'$ be “special” vertices. The edge set is:

$$\{xx', yy'\} \cup \left(\{ab : a \in X', b \in Y'\} \setminus \{x'y'\}\right).$$

In other words, starting from the complete bipartite graph on $X'$ and $Y'$, the edge $x'y'$ is removed, and the edges $xx'$ and $yy'$ are added.

Notice that of the $2k + 2$ vertices of a $k$-resistor between $x$ and $y$, all but $x$ and $y$ have degree $k$. Furthermore, if a spanning subgraph of the resistor contains a perfect matching, both edges $xx'$ and $yy'$ are present. This leads to the following construction.

**Proposition 2.2.** Construct a $k$-regular, $n = (2k^2 + 2)$-vertex bipartite graph $G$ as follows. Let $x$ and $y$ be two initial vertices. Add $k$ distinct $k$-resistors between $x$ and $y$. Then, a.a.s. the random subgraph $G(p)$ does not contain a perfect matching for any $p = o\left(n^{-1/4}\right)$. On the other hand, a.a.s. $G(p)$ contains no isolated vertices for any $p = \omega\left((\log n)/\sqrt{n}\right)$.

**Proof.** Both conclusions follow from the first moment method.

Let $H \sim G(p)$. Note that $H$ contains a perfect matching only if for one of the resistors, both edges $xx'$ and $yy'$ are present. This occurs with probability $p^2$. As there are $k$ different resistors, and they are all edge-disjoint, the expected number of such pairs is $kp^2$. Since $k = \Theta(\sqrt{n})$, if $p = o\left(n^{-1/4}\right)$, a.a.s. there is no such pair in $H$.

The expected number of isolated vertices in $H$ is $n(1-p)^k \leq \exp\left(\log n - pk\right)$. When $p = \omega\left((\log n)/\sqrt{n}\right)$ this tends to zero, and a.a.s. there are no isolated vertices.

In this example we had $k = \Theta(\sqrt{n})$, leaving a large gap between it and the range $k = \Theta(n)$ in Theorem 1.1. We reduce this gap as follows.
Definition 2.3. A \((k, \ell, r)\)-series of resistors between two vertices \(x\) and \(y\) is constructed as follows. Let \(K_1, K_2, \ldots, K_\ell\) be \(\ell\) copies of the complete bipartite graph \(K_{k,k}\), with respective vertex sets \(X_1 \cup Y_1, X_2 \cup Y_2, \ldots, X_\ell \cup Y_\ell\). For each \(1 \leq i \leq \ell\), let \(x_i^1, x_i^2, \ldots, x_i^\ell \in X_i, y_i^1, y_i^2, \ldots, y_i^\ell \in Y_i\) be distinct. Remove all edges of the form \(x_i^j y_i^1\), and add all edges of the form \(y_i^j x_{i+1}^j\), as well as \(x x_i^j, y_i^\ell y\).

The following proposition uses a construction similar to the one in Proposition 2.2.

Proposition 2.4. For \(n = 2 + 20k \log k \log \log k\), construct a \(k\)-regular \(n\)-vertex bipartite graph \(G\) as follows. Starting with two vertices \(x\) and \(y\), add \(\log k\) distinct \((k, 10 \log \log k, \frac{1}{\log k})\)-series of resistors between \(x\) and \(y\). A.a.s. the random subgraph \(G(p)\) does not contain a perfect matching for any \(p \leq 2 \log n/k\). On the other hand, \(p = (\log n + \omega(1)) / k\) suffices for \(G(p)\) to contain no isolated vertices a.a.s.

Proof. For consistency with Definition 2.3, let \(\ell = 10 \log \log k\) and \(r = k / \log k\). For a spanning subgraph \(G' \subseteq G\) to contain a perfect matching, there must be at least one series of resistors containing at least one edge of the form \(xx_i^j\), at least one edge of the form \(y^j y\), and one edge of the form \(y^i x_{i+1}^j\) for each \(i\) between 1 and \(\ell - 1\). Therefore, applying the union bound over all \(k/r\) choices of the \((k, \ell, r)\)-series, we obtain

\[
\mathbb{P}[G(p) \text{ contains a perfect matching}] \leq \frac{k}{r} [1 - (1 - p)^r]^{\ell + 1}.
\]

Let \(p = 2 \log n/k\). Then \((1 - p)^r \sim e^{-2}\), and therefore

\[
\mathbb{P}[G(p) \text{ contains a perfect matching}] \leq \log k \left(1 - e^{-2}\right)^{10 \log \log k} = o(1).
\]

The statement about isolated vertices follows from an argument similar to the one in the proof of Proposition 2.2.

2.1. Connection to the Kahn–Kalai Conjecture. In their paper [13], Kahn and Kalai set forth a series of important conjectures regarding the connection between “thresholds” and “expectation thresholds”. Suppose \(X\) is a set of cardinality \(n\), and \(\mu(X;p)\) is the distribution on \(2^X\) where each \(x \in X\) is retained with probability \(p\), all choices independent. Let \(\mathcal{P} \subseteq 2^X\) be a non-trivial increasing property. The threshold for \(\mathcal{P}\), denoted \(\text{th}(\mathcal{P})\), is defined as the unique \(p \in (0, 1)\) s.t. \(\mathbb{P}_{A \sim \mu(X;p)} [A \in \mathcal{P}] = \frac{1}{7}\).

The expectation threshold for \(\mathcal{P}\), denoted \(\text{th}_E(\mathcal{P})\), is defined as the supremum over \(q \in [0, 1]\) for which there exists a family \(\mathcal{G} \subseteq 2^X\) satisfying:

- \(\sum_{A \in \mathcal{G}} q^{|A|} (1 - q)^{|X| - |A|} < \frac{1}{7}\).
- For every \(B \in \mathcal{P}\) there exists some \(A \in \mathcal{G}\) s.t. \(A \subseteq B\).

Kahn and Kalai conjectured [13, Conjecture 1.1] that there exists a universal constant \(K\) s.t. for all sets \(X\) and all (non-trivial increasing) properties \(\mathcal{P}\), \(\text{th}(\mathcal{P}) \leq K \text{th}_E(\mathcal{P}) \log n\).

As an example, if \(X = E(K_{n,n})\) and \(\mathcal{P}\) is the property of containing a perfect matching, then \(\text{th}(\mathcal{P}) = \Theta(\log n/n)\). In line with the conjecture, it also holds that \(\text{th}_E(\mathcal{P}) = \Omega(1/n)\), as may be seen by taking as \(\mathcal{G}\) either all edges adjacent to a fixed vertex or the collection of all perfect matchings. Interestingly, in our first construction (Proposition 2.2), \(\text{th}(\mathcal{P}) = \Omega(1/\sqrt{k})\), whereas the expectation threshold given by the
families above would be only $\Theta(1/k)$. However, a high enough expectation threshold is realized by taking $G$ to be all pairs of edges of the form $\{xx', yy'\}$.

3. A Structural Lemma

Throughout this section $G = (X \cup Y, E)$ is a $k$-regular bipartite graph on $2n$ vertices. A cut in $G$ is a pair $(S, T)$ where $S \subseteq X$ and $T \subseteq Y$. We call $(S, T)$ a Hall cut if $|S| > |T|$ and $N(S) \subseteq T$.

Recall that Hall’s marriage theorem states that a balanced bipartite graph contains a perfect matching if and only if it contains no Hall cuts. The main idea in the proof of Theorem 1.1 is to show that a.a.s. $G_{\tau_l}$ does not contain a Hall cut.

Let $(S, T)$ be a cut in $G$. The cross edges of $G$ with respect to $(S, T)$ are those in $E(S \cup T, S^c \cup T^c)$. We call the remaining edges parallel. For a vertex $x \in V(G)$, we denote by $\deg_{G,S,T}^\text{Par}(x)$ and $\deg_{G,S,T}^\text{Cr}(x)$ the number of parallel and cross edges incident to $x$, respectively. If the cut $(S, T)$ is clear from the context, we sometimes write $\deg_G^\text{Par}$ and $\deg_G^\text{Cr}$.

We define the following distance function on the set of cuts in $G$:

$$d((S_1, T_1), (S_2, T_2)) = |S_1 \setminus S_2| + |S_2 \setminus S_1| + |T_1 \setminus T_2| + |T_2 \setminus T_1|.$$  

For $c \in \mathbb{R}$, we say that two cuts are $c$-close if their distance is at most $c$.

**Observation 3.1.** Let $(S, T)$ be a cut in $G$. Then $e(S, T^c) = k \cdot (|S| - |T|) + e(S^c, T)$.

**Proof.** Since $G$ is $k$-regular we have:

$$e(S, T) + e(S, T^c) = k \cdot |S|$$

$$e(S, T) + e(S^c, T) = k \cdot |T|.$$  

Subtracting the second equation from the first yields the result. □

**Observation 3.2.** Let $(S, T)$ be a cut in $G$ with $|S| > |T|$. Let $C = e_G(S \cup T, S^c \cup T^c)$ be the number of cross edges in $G$ w.r.t. $(S, T)$. Then, for any $p \in (0, 1)$, it holds that:

$$\mathbb{P}[(S, T) \text{ is a Hall cut in } G(p) \leq (1 - p)^{C/2}].$$

**Proof.** We have $e_G(S \cup T, S^c \cup T^c) = e_G(S, T^c) + e_G(S^c, T)$. As $|S| > |T|$, Observation 3.1 implies that $e_G(S, T^c) > e_G(S^c, T)$ and therefore $e_G(S, T^c) > e_G(S \cup T, S^c \cup T^c)/2$. The probability that none of these cross edges are edges in $G(p)$ (and thus $(S, T)$ is a Hall cut) is therefore bounded above by $(1 - p)^{C/2}$, as desired. □

The following structural lemma is the heart of our proof. Observation 3.2 implies that if $G$ has almost no cuts with few cross edges, a union bound is enough to show that a.a.s. $G(p)$ contains no Hall cuts. This is the case, for example, in random regular graphs. However, in an arbitrary graph this need not hold. Therefore, we must understand the behavior of cuts with few cross edges, and hence a significant chance of being Hall cuts in $G(p)$. We show that in any dense, regular, bipartite graph, all such cuts can be grouped into a constant number of equivalence classes. This allows us to control the contribution of these cuts to the probability that $G(p)$ contains a Hall cut.

**Definition 3.3.** Let $c > 0$. A cut $(S, T)$ is $c$-internal if it has at most $4cn^2/\log n$ cross edges.
If a cut is 1-internal, we sometimes just say that it is internal. Note that \((S,T)\) is c-internal if and only if its complement \((S^c,T^c)\) is \(c\)-internal. Indeed, both cuts have the same cross edges.

**Lemma 3.4.** Fix \(\delta_0 > 0\) and let \(G = (X \cup Y, E)\) be a \(k\)-regular bipartite graph on \(2n\) vertices, with \(k \geq \delta_0 n\) and \(n\) sufficiently large. For every \(\varepsilon\) such that \(0 < \varepsilon \leq \delta_0\) there exists some \(m = 2^{\Theta(1/\delta_0)}\) and cuts \((S_1, T_1), \ldots, (S_m, T_m)\) with the following properties.

(a) Every internal cut \((S,T)\) with \(|S| > |T|\) is \(\varepsilon n\)-close to \((S_i, T_i)\) for some \(i \in [m]\).

(b) For every \(i \in [m]\) and \(x \in V(G)\), we have \(\deg_{G,S_i,T_i}^c(x) \leq (1 + \varepsilon)\frac{k}{2}\).

It is relatively straightforward to derive Lemma 3.4, albeit with a vastly larger bound on \(m\), from Szemerédi’s regularity lemma. In a similar fashion, one could also obtain Lemma 3.4 from the decomposition of dense regular graphs into “robust components” (induced subgraphs with good expansion properties) due to Kühn, Lo, Osthus, and Staden [19, Theorem 3.1]. However, we do not need the full power of [19]. For completeness, we provide a shorter, self-contained proof of Lemma 3.4, which also introduces concepts and definitions used later in the paper.

We begin by defining a lattice-like structure on the internal cuts. Let \(\delta = k/n \geq \delta_0\) be the density of \(G\).

**Claim 3.5.** Let \((S_1, T_1)\) and \((S_2, T_2)\) be two \(c\)-internal cuts for some \(c = o(\log n)\). Then
\[
d((S_1, T_1), (S_2, T_2)) \leq \frac{40cn}{\delta \log n} \text{ or } d((S_1, T_1), (S_2, T_2)) \geq k/10.
\]

In other words, the distance between two \(c\)-internal cuts is either very large or very small, assuming that \(c = o(\log(n))\). Throughout this section, \(c\) will always be a constant independent of \(n\).

**Proof.** Let \(d = d((S_1, T_1), (S_2, T_2))\). Without loss of generality assume that \(|S_2 \setminus S_1| \geq d/4\). As \((S_1, T_1)\) is \(c\)-internal, we have:
\[
e_G(S_2 \setminus S_1, T_1) \leq \frac{4cn^2}{\log n}.
\]

Similarly, since \((S_2, T_2)\) is \(c\)-internal, we have:
\[
e_G(S_2 \setminus S_1, T_2) = e_G(S_2 \setminus S_1, Y) - e_G(S_2 \setminus S_1, T_2^c) \geq e_G(S_2 \setminus S_1, Y) - \frac{4cn^2}{\log n}.
\]

Since \(G\) is \(k\)-regular, we have:
\[
e_G(S_2 \setminus S_1, Y) = k |S_2 \setminus S_1| \geq \frac{kd}{4}.
\]

Therefore:
\[
(1) \quad e_G(S_2 \setminus S_1, T_2 \setminus T_1) \geq e_G(S_2 \setminus S_1, T_2) - e_G(S_2 \setminus S_1, T_1) \geq \frac{kd}{4} - \frac{8cn^2}{\log n}.
\]

On the other hand, it is certainly true that \(e_G(S_2 \setminus S_1, T_2 \setminus T_1) \leq |S_2 \setminus S_1| |T_2 \setminus T_2|\).

As \(|S_2 \setminus S_1| + |T_2 \setminus T_2| \leq d\), we have:
\[
(2) \quad e_G(S_2 \setminus S_1, T_2 \setminus T_1) \leq \frac{d^2}{4}.
\]
Combining 1 and 2 and rearranging yields:

\[ d(k - d) \leq \frac{32cn^2}{\log n}. \]

Suppose \( d < k/10 \). Then \( k - d > 9k/10 \). We thus obtain the inequality:

\[ d \leq \frac{320cn^2}{9n \log n} < \frac{40cn}{\delta \log n}, \]

as desired. \( \square \)

We say that two \( c \)-internal cuts are equivalent if they are \((\varepsilon k/100)\)-close. The triangle inequality, together with Claim 3.5 implies that this is an equivalence relation. Let \( \mathcal{X} \) be the set of equivalence classes of \( c \)-internal cuts. We say that a cut is trivial if it is equivalent to the cut \((\emptyset, \emptyset)\).

We now define an intersection operation on equivalence classes. Note that if the cuts \((S_1, T_1)\) and \((S_2, T_2)\) are \(c_1\)-internal and \(c_2\)-internal, respectively, then the cut \((S_1 \cap S_2, T_1 \cap T_2)\) (their intersection) is \((c_1 + c_2)\)-internal. This follows from the fact that any cross edge of the intersection is a cross edge in at least one of the cuts.

Denote the equivalence class of a \( c \)-internal cut \((S, T)\) by \( [(S, T)]_c \). When the value of \( c \) is clear from the context, we omit the subscript \( c \).

**Definition 3.6.** The intersection between \([(S_1, T_1)] \in \mathcal{X}_{c_1}\) and \([(S_2, T_2)] \in \mathcal{X}_{c_2}\) is

\[ [(S_1, T_1)] \cap [(S_2, T_2)] = [(S_1 \cap S_2, T_1 \cap T_2)] \in \mathcal{X}_{c_1 + c_2}. \]

The fact that this is well-defined, in the sense that it does not depend on the choice of representatives, follows from Claim 3.5. Indeed, different choices of representatives may only change the intersection by at most \(80(c_1 + c_2)n/\log n\) vertices. This is much smaller than \(\varepsilon k/100\), and therefore the two intersections are equivalent.

**Definition 3.7.** Two classes \([(S_1, T_1)] \in \mathcal{X}_{c_1}\) and \([(S_2, T_2)] \in \mathcal{X}_{c_2}\) are disjoint if their intersection is trivial. We say that \([(S_1, T_1)]\) contains \([(S_2, T_2)]\) if \([(S_1, T_1)] \cap [(S_2, T_2)]\) is trivial.

Note that if \([(S, T)]\) is a non-trivial class, then, by Claim 3.5, \( |S \cup T| \geq k/10 \).

**Observation 3.8.** If \([(S_1, T_1)]\) is not contained in \([(S_2, T_2)]\), then for every \((S', T') \in [(S_1, T_1)] \cap [(S_2, T_2)]\), we have \( |S_1 \cup T_1| - |S' \cup T'| \geq k/20 \).

**Proof.** Since \([(S_1, T_1)]\) is not contained in \([(S_2, T_2)]\), we have that \((S_1 \setminus S_2, T_1 \setminus T_2)\) is non-trivial, and therefore \( |S_1 \setminus S_2| + |T_1 \setminus T_2| \geq k/10 \). Note that \( S_1 \) is the disjoint union of \( S_1 \setminus S_2 \) and \( S_1 \setminus S_2 \), and that a similar statement holds for \( T_1 \). This implies that \( |S_1 \cup T_1| \geq |S_1 \setminus S_2| + |T_1 \setminus T_2| + k/10 \). As \((S', T')\) is equivalent to \((S_1 \cap S_2, T_1 \cap T_2)\), the observation follows. \( \square \)

Consider the following process:

(a) Initialize \([(S_1, T_1)] \in \mathcal{X}_1\) to be an arbitrary non-trivial internal equivalence class.

(b) As long as there exists a class \([(S^*, T^*)] \in \mathcal{X}_1\) that is neither disjoint from nor containing \([(S_i, T_i)]\), set \([(S_{i+1}, T_{i+1})] = [(S_i, T_i)] \cap [(S^*, T^*)]\).

We call the equivalence classes that can be obtained at the end of this process atoms.
Observation 3.9.

(a) The process halts after at most $40/\delta$ steps.
(b) All of the atoms are $(40/\delta)$-internal.
(c) All of the atoms are non-trivial.
(d) The atoms are pairwise disjoint.
(e) There are at most $30/\delta$ atoms.

Proof. Item \((a)\) follows from the fact that $|S_i \cup T_i| \leq 2n$, and Observation \ref{obs:cut_bound} implies that for each $i$, $|S_i \cup T_i| \leq |S_{i-1} \cup T_{i-1}| - k/20$. Item \((b)\) follows from Item \((a)\) and the fact that the intersection of a $c$-internal cut with a $1$-internal cut is $(c + 1)$-internal. Item \((c)\) holds because at each stage of the process, $[(S_i, T_i)]$ and $[(S^*, T^*)]$ are not disjoint.

For Item \((d)\), assume that $[(S, T)]$ and $[(S', T')]$ are distinct atoms. Since they are not equivalent, $d((S, T), (S', T')) \geq \epsilon k/100$. Without loss of generality, we may assume that $|S \setminus S'| + |T \setminus T'| \geq \epsilon k/200$. Now, $[(S', T')]$ is the intersection of at most $40/\delta$ internal classes. Therefore at least one of these classes, $[(S^*, T^*)]$, has $|S \setminus S'| + |T \setminus T'| \geq (\epsilon k/200) \cdot (\delta/40)$. By Claim \ref{clm:intersection_bound}, this implies that $[(S, T)] \cap [(S^*, T^*)]$ is not trivial, and therefore $[(S, T)]$ is not contained in $[(S^*, T^*)]$. Suppose, for a contradiction, that $[(S, T)]$ and $[(S', T')]$ are not disjoint. Then $[(S, T)]$ and $[(S^*, T^*)]$ are not disjoint. Therefore $[(S, T)]$, by definition, is not an atom, which is a contradiction.

Item \((e)\) is true because there are $2n$ vertices, each atom contains at least $k/10$ vertices, and the atoms are pairwise disjoint. Therefore, all intersections have at most $\frac{2k}{10}$ vertices, so letting $a$ denote the number of atoms, we have

$$a \cdot \left(\frac{k}{10}\right) - \binom{a}{2} \left(\frac{\epsilon k}{100}\right) \leq 2n,$$

and the result follows. \hfill \Box

Proof of Lemma \ref{lem:atmost40}. We first construct the cuts $(S_1, T_1), \ldots, (S_m, T_m)$. Let $A$ be the set of atoms. For each $\alpha \in A$, fix a representative $(S_\alpha, T_\alpha)$. For $S \subseteq A$, let $(S'_S, T'_S)$ be the cut $(\cup_{\alpha \in S} S_\alpha, \cup_{\alpha \in S} T_\alpha)$. Finally, define the sets:

$$S_S := \{ x \in X : \deg_{G, S'_S, T'_S}^{\text{Par}}(x) \geq \frac{k}{2} \}, \quad T_S := \{ y \in Y : \deg_{G, S'_S, T'_S}^{\text{Par}}(y) \geq \frac{k}{2} \}.$$

Let $(S_1, T_1), \ldots, (S_m, T_m)$ be a list of the cuts $\{(S_S, T_S)\}_{S \subseteq A}$. By Observation 3.9, $m \leq 2^{|A|}$ does not depend on $n$. It remains to prove that in each of these cuts, every vertex is incident to few (i.e., less than $(1 + \epsilon)k/2$) cross edges, and that every internal cut $(S, T)$ with $|S| > |T|$ is $\epsilon n$-close to one of the $(S_i, T_i)$s.

Let $S \subseteq A$. Since, by Observation 3.9 every atom is $(40/\delta)$-internal, the number of cross edges w.r.t. $(S'_S, T'_S)$ is at most $O\left(\frac{n^2}{\log n}\right)$. Therefore there are at most $O\left(\frac{n}{\log n}\right)$ vertices $x \in V(G)$ s.t. $\deg_{G, S'_S, T'_S}^{\text{Cr}}(x) > k/2$. Thus $d((S'_S, T'_S), (S_S, T_S)) = O\left(\frac{n}{\log n}\right)$. Let $x \in V(G)$. By construction,

$$\deg_{G, S_S, T_S}^{\text{Cr}}(x) \leq \frac{k}{2} + d((S'_S, T'_S), (S_S, T_S)) \leq (1 + \epsilon)\frac{k}{2},$$

as desired.
For the second property, let \((S, T)\) be an internal cut. Let \(\mathcal{S}\) be the set of atoms contained in \([(S, T)]\). It suffices to show that \((S_{\mathcal{S}}, T_{\mathcal{S}})\) is equivalent to \((S, T)\).

Suppose that \((S_{\mathcal{S}}, T_{\mathcal{S}})\) is not equivalent to \((S, T)\). Define
\[
(S', T') = (S, T) \cap \bigcup_{\alpha \in \mathcal{S}} (S_{\alpha}, T_{\alpha})^c.
\]
Note that the cut \((S', T')\) is \([40/\delta) \cdot (30/\delta) + 1]\)-internal and non-trivial, and therefore \(|S| + |T| \geq k/10\) by Claim 3.5. Now, fix \(\alpha_0 \in \mathcal{S}\), and let \([(S_1, T_1)], \ldots, [(S_l, T_l)]\) be the 1-internal classes that participated in the process that resulted in \(\alpha_0\). Thus, \(\alpha_0 = \bigcap_{i=1}^l [(S_i, T_i)]\). Since \(\alpha_0\) is disjoint from \([(S', T')]\), by the pigeonhole principle there exists a \(j\) such that \([(S_j, T_j)^c] \cap [(S', T')]\) is nontrivial. We can now create a new atom contained in \([(S, T)]\) by initializing \([(S_1, T_1)]\) and taking the first 1-internal set to be \([(S^*, T^*)] = [(S_j, T_j)^c]\). This results in an atom contained in \((S', T')\), which is a contradiction to the definition of \(\mathcal{S}\).

\[
\square
\]

4. Properties of Random Subgraphs

Let \(k = \delta n\), with \(\delta = \Omega(1)\), and fix a \(k\)-regular bipartite graph \(G = (X \cup Y, E)\) on \(2n\) vertices. In this section we collect properties of random subgraphs of \(G\) that are essential for our proof.

Set
\[
p_1 = \frac{\log n - \log \log \log \log n}{k},
\]
\[
p_2 = \frac{\log n + \log \log \log \log n}{k}.
\]
We define the following random subgraphs of \(G\):
\[
G_2 \sim G(p_2),
\]
\[
G_1 \sim G_2 \left(\frac{p_1}{p_2}\right).
\]
Observe that \(G_1 \sim G(p_1)\). Furthermore, the same distribution on \(G_1, G_2\) can be obtained as follows. Let \(G_1 \sim G(p_1), G' \sim G \left(\frac{p_1}{p_2 - p_1}\right)\), and set \(G_2 = G_1 \cup G'\).

We will show presently that a.a.s. \(G_1\) contains isolated vertices, while \(G_2\) does not. Furthermore, the distance between any two vertices that are isolated in \(G_1\) is at least 2 in \(G_2\). This motivates the following construction: let \(G_1 \subseteq G_H \subseteq G_2\) be the random graph obtained by adding, for each isolated vertex \(v\) in \(G_1\), an edge drawn uniformly at random from \(\{e \in E(G_2) : v \in e\}\). If any of these sets are empty, or if there are two isolated vertices in \(G_1\) that are connected in \(G_2\), set \(G_H = G_1\). The next claim, a variation of Lemma 7.9 in [4], establishes that it is sufficient to prove that a.a.s. \(G_H\) contains a perfect matching.

**Claim 4.1.** Let \(Q\) be a monotone increasing property of subgraphs of \(G\). If \(Q\) holds a.a.s. for \(G_H\) then, in almost every graph process in \(G\), \(Q\) holds for \(G_{\tau_I}\), the first graph in which there are no isolated vertices.

We defer the proof until after establishing some properties of \(G_1\) and \(G_2\).
Claim 4.2. A.a.s. $G_1$ contains isolated vertices and $G_2$ does not. Furthermore, a.a.s. there is no pair $x, y$ of vertices that are isolated in $G_1$ and $xy \in E(G_2)$.

Proof. The probability that a specific vertex is isolated in $G$ is $(1-p)^k$. The expected number of isolated vertices in $G_2$ is therefore:

$$2n(1-p)^k \leq 2n \exp(-pk) = 2n \frac{1}{\omega(n)} = o(1).$$

Applying Markov’s inequality, a.a.s. $G_2$ contains no isolated vertices.

By a similar calculation, the probability that a specific vertex is isolated in $G_1$ is $\omega(1/n)$. Let the random variable $I$ be the number of vertices in $X$ that are isolated in $G_1$. Then $\mathbb{E}[I] = \omega(1)$. Furthermore, the events that two vertices $x, y \in X$ are each isolated in $G_1$ are independent. Thus $\text{Var}[I] \leq \mathbb{E}[I]$, and by Chebychev’s inequality, a.a.s. $I > 0$ and $G_1$ contains isolated vertices.

For the second part of the claim, observe that by the calculations above a.a.s. the number of isolated vertices in $G_1$ is $O(\log n)$. Therefore the expected number of edges between these vertices in $G_2$ is $o(1)$, and so by Markov’s inequality a.a.s. there are none.

Proof of Claim 4.2. We describe a coupling that relates $G_1, G_2$ and $G_H$ to $G_{\tau_1}$. Consider the following random process. For each edge $e$ of $G$, choose a real number $\alpha_e \sim U[0,1]$ uniformly at random from the interval $[0,1]$, all choices independent. Let $G'_1$ and $G'_2$ be the subgraphs of $G$ whose edges are $E(G'_1) = \{e \in E(G) : \alpha_e < p_i\}$ for $i \in \{1,2\}$. Let $G'_H$ be the random graph obtained by adding, for each isolated vertex in $G'_1$, the edge incident to it in $G'_2$ whose $\alpha$ value is minimal. If $G'_2$ contains isolated vertices or an edge between two vertices that are isolated in $G'_1$, set instead $G'_H = G'_1$.

Observe that the distributions of $(G_1, G_2, G_H)$ and $(G'_1, G'_2, G'_H)$ are identical. Furthermore, the distribution of the random graph process is identical to that of the process in which edges are revealed in increasing order of $\alpha$. With respect to this process, a.a.s. $G'_H$ is a subgraph of $G_{\tau_1}$. Therefore, if $Q$ holds a.a.s. for $G_H$, then $Q$ holds a.a.s. for $G'_H$, and as $Q$ is monotone increasing, a.a.s. $G_{\tau_1} \in Q$.

For a cut $(S, T)$ we define the set $$\Gamma(S, T) = \{ x \in V : \deg_{G_{S,T}^C}(x) \geq n^{-1/20}k \}$$ of vertices with high cross degree.

Lemma 4.3. Let $(S, T)$ be a cut. Suppose $R \subseteq V \setminus \Gamma(S, T)$ is a set of $O(\log n)$ isolated vertices in $G_1$. Then, a.a.s. for every $x \in R$, $\deg_{G_H^C}(x) = 0$.

Proof. It suffices to show that a.a.s. for every $x \in R$, $\deg_{G_{H}^C}(x) = 0$. Indeed, the expected number of cross edges incident to $x$ in $G_2$ is bounded above by $n^{-1/20}k \frac{p_2 - p_1}{1 - p_1} = \tilde{O}(n^{-1/20})$. The conclusion follows by applying Markov’s inequality and a union bound over the $O(\log n)$ vertices.

Lemma 4.4. Let $(S, T)$ be a cut, and let $V' = V \setminus \Gamma(S, T)$. Then, a.a.s. for every $x \in V'$, $\deg_{G_1^C}(x) \leq 30$. 

Proof. Observe that \( \deg_{G_1}^{cr}(x) \sim Bin \left( \deg_{G_1}^{cr}(x), p_1 \right) \). Therefore:

\[
\Pr \left[ \deg_{G_1}^{cr}(x) \geq 30 \right] \leq \left( \frac{n^{-1/20}k}{30} \right)^{30} = o \left( \frac{1}{n} \right).
\]

The lemma follows by applying a union bound over all \( O(n) \) vertices in \( V' \). \( \square \)

**Lemma 4.5.** Let \((S, T)\) be a cut, and let \( V_{low} \) be the set of vertices \( x \in V \setminus \Gamma(S, T) \) s.t. \( \deg_{G_1}^{par}(x) \leq \frac{\log n}{1000} \). A.a.s. the following hold:

(a) \( |V_{low}| \leq n^{0.01} \).

(b) For each \( x, y \in V_{low} \), the distance between \( x \) and \( y \) in \( G_H \) is at least 6.

Proof. We first show that a.a.s. \( |V_{low}| \leq n^{0.01} \). Indeed, suppose \( x \in V \) satisfies \( \deg_{G_1}^{cr}(x) < n^{-1/20}k \). The probability that \( x \in V_{low} \) is at most

\[
\sum_{i=0}^{\frac{\log n}{1000}} \Pr \left[ \deg_{G_1}^{cr} = i \right] \leq \log n \left( \frac{k}{\log n} \right) \left( \frac{1}{1000} \right) \log n \left( 1 - p_1 \right) \left( 1 - O(n^{-1/20}) \right) k
\]

\[
\leq \left( \frac{1.1e}{1000} \right) \frac{\log n}{1000} \cdot \tilde{O} \left( \frac{1}{n} \right) < n^{-0.991}.
\]

The result follows from Markov’s inequality and a union bound over \( G \)'s vertices.

The proof of [b] is similar to the proof of Claim 4.4 in [2] and property (P2) in Lemma 5.1.1 in [11]. Fix two distinct vertices \( u, w \in V(G) \) and consider a path \((u = v_0, \ldots, v_r = w)\) in \( G \), where \( 1 \leq r \leq 5 \). Denote by \( A \) the event that for every \( 0 \leq i \leq r - 1 \), we have \( \{v_i, v_{i+1}\} \in E(G_2) \), i.e., the path exists in \( G_2 \). Denote by \( B \) the event that \( u, w \in V_{low} \).

Clearly, \( \Pr[A] = p_2^r \), hence

\[
\Pr[B \land A] = p_2^r \cdot \Pr[B \mid A].
\]

Let \( X \) denote the random variable which counts the number of parallel edges in \( G_2 \) incident with \( u \) or \( w \) disregarding the pairs \( \{u, v_1\}, \{v_{r-1}, w\} \), and \( \{u, w\} \). Observing that \( X \sim Bin ((1 - o(1))2k, p_2) \) and using standard concentration inequalities, we have

\[
\Pr \left[ B \mid A \right] \leq \Pr \left[ X < 2 \frac{1}{1000} \log n \right] < n^{-1.8}.
\]

Fixing the two endpoints \( u, w \), the number of such sequences is at most \( k^{r-1} \). Applying a union bound over all pairs of vertices and possible paths between them, we conclude that the probability of a path in \( G_2 \) of length \( r \leq 5 \), connecting two distinct vertices of \( V_{low} \) is at most

\[
\sum_{r=1}^{5} n^2 \cdot k^{r-1} \cdot p_2^r \cdot n^{-1.8} = o(1).
\]

This completes the proof of the lemma. \( \square \)

We say that a set \( A \subset V \) is **partite** if \( A \subset X \) or \( A \subset Y \). A vertex is a **parallel neighbor** of \( A \) if it is connected to a vertex in \( A \) via a parallel edge.

**Lemma 4.6.** Let \((S, T)\) be a cut. Then a.a.s. the following holds. If \( A \subset V \) is a partite set satisfying

- \( |A| \leq n^{0.9} \), and
for every \( x \in A \), \( \deg^{Par}_{G_1}(x) \geq \frac{1}{1000} \log n \), then \( A \) has at least \( |A| \frac{\log n}{2000} \) parallel neighbors in \( G_1 \).

Proof. Let \( A \subseteq V \) be a partite set, and let \( t = t(A) = |A| \frac{1}{2000} \log n \). Let \( \mathcal{P}(A) \) be the event that the minimum parallel degree of a vertex in \( A \) is at least \( \frac{1}{1000} \log n \).

For any fixed set \( B \),

\[
\mathbb{P}[N_{G_1}(A) \subseteq B \land \mathcal{P}(A)] \leq \mathbb{P}[e_{G_1}(A, B) \geq 2t] \leq \left( \frac{e_{G_1}(A, B)}{2t} \right)^{2t} \leq \left( \frac{|A||B|}{2t} \right)^{2t} \leq \left( \frac{e \cdot |A||B|}{2t} \right)^{2t}.
\]

Applying a union bound, we have:

\[
\mathbb{P}[\exists A \text{ s.t. } |A| \leq n^{0.9} \land \mathcal{P}(A) \land |N_{G_1}(A)| \leq t(A)] \leq 2 \sum_{|A|=1}^{n^{0.9}} \left( \frac{n}{|A|} \right)^{t(A)} \left( \frac{|A||t(A)|}{2t} \right)^{2t(A)} \leq 2 \sum_{|A|=1}^{n^{0.9}} \left( \frac{ne}{|A|} \right)^{t(A)} \left( \frac{e^3|A|^2 \log^2(n)}{4\delta^2 t(A)n} \right)^{t(A)} \leq \sum_{|A|=1}^{n^{0.9}} n^{-t(A)/20} = o(1).
\]

\( \square \)

5. Proof of Theorem 1.1

5.1. Outline. As mentioned previously, we prove Theorem 1.1 by showing that a.a.s. \( G_H \) does not contain a Hall cut. This is similar to the approach used in [9] to show that \( p = \log n/n \) is the threshold for \( K_{n,n}(p) \) to contain a perfect matching. There, a union bound over all cuts \((S, T)\) (satisfying certain conditions) was sufficient for the result. In this regard, the crucial property of \( K_{n,n} \) is that every cut has many outgoing edges. Essentially the same approach was utilized in the proof of [20, Lemma 3.2] to show that if a \( k \)-regular, bipartite graph \( G \) satisfies a certain expansion property, then the threshold for \( G(p) \) to contain a perfect matching is \( p = \Theta(\log n/k) \). However, for arbitrary \( G \), there may be many cuts \((S, T)\) with few outgoing edges, potentially foiling the union bound. Indeed, this is the case in the counterexamples described in Section 2.

To overcome this we take a more delicate approach, wherein we group the various cuts in \( G \) into families that we treat separately. Informally, the steps are as follows:

(a) The first family contains all cuts that are not internal (in the sense of Definition 3.3), and therefore have many outgoing edges in \( G \). Here a simple union bound suffices to show that a.a.s. none of these cuts are Hall cuts in \( G_H \) (Claim 5.1).

(b) At this point we apply Lemma 5.4 to conclude that any cut not covered in the previous step is close to one of the cuts from the lemma. As there are only a constant number of these, we may fix one, say \((S', T')\), and consider only cuts that are close to it. For a cut \((S, T)\), we define the set of shifted vertices:

\[
\Delta = \Delta(S, T) = (S \setminus S') \cup (S' \setminus S) \cup (T \setminus T') \cup (T' \setminus T).
\]
We make use of the natural correspondence $\Delta \leftrightarrow (S, T)$, and interchange between them freely. We recall the definition of the set

$$
\Gamma = \Gamma(S', T') = \left\{ x \in V : \deg_{G, S', T'}(x) \geq n^{-1/20}k \right\}
$$

of vertices with many cross edges in $G$ w.r.t. $(S', T')$. We emphasize that $\Gamma$ is deterministic, i.e., depends only $G$ and $(S', T')$.

(c) The second family of cuts consists of those with $|\Delta| \geq n^{0.9}$. The insight here is that shifting a large number of vertices w.r.t. $(S', T')$ creates many cross edges. Thus, here too, a union bound suffices to show that a.a.s. none of these are Hall cuts in $G_H$ (Claim 5.2).

(d) The third family consists of cuts satisfying $|\Delta| \leq n^{-1/20} |\Gamma|$. As the vertices in $\Gamma$ have, by definition, many cross edges, if $\Delta$ is much smaller than $\Gamma$ then most of these cross edges are unaffected. This allows us to employ a union bound here as well (Claim 5.3).

(e) We now argue that, a.a.s. we may remove from $G_H$ a small matching $M$ covering all vertices that have low degree in $G_1$, leaving the residual graph $\tilde{G}_H = G_H \setminus V(M)$. We further argue that if $G_H$ contains a Hall cut that is $c$-close to $(S', T')$, then $\tilde{G}_H$ contains a Hall cut that is $c$-close to $(S' \setminus V(M), T' \setminus V(M))$ (Observation 5.8). It therefore suffices to consider cuts in $\tilde{G}_H$.

(f) It remains to consider $\Delta$ such that $n^{-1/20} |\Gamma| < |\Delta| < n^{0.9}$. We show that a.a.s. there is no such Hall cut if either $|\Delta| \geq \log \log n$ or $\Delta \cap \Gamma \neq \emptyset$ (Claim 5.11). Here we take advantage of the fact that once the low degree vertices have been removed from $G_1$, the remaining vertices satisfy an expansion property (Claim 5.9).

(g) Finally, we argue that a.a.s. there is no Hall cut satisfying $|\Delta| \leq \log \log n$ and $\Delta \cap \Gamma = \emptyset$ (Claim 5.12). Here we use the expansion property to show that for such a cut to exist, a.a.s. $(S', T')$ contains many outgoing edges in $G_1$, and that it is impossible to make all these edges parallel by shifting fewer than $\log \log n$ vertices.

5.2. The proof. We first show that if a cut has many outgoing edges, the probability that it is a Hall cut in $G_H$ is very small.

Claim 5.1. A.a.s. $G_H$ contains no Hall cut $(S, T)$ that is not internal.

Proof. Since $G_1 \subseteq G_H$ it suffices to prove the statement with $G_H$ replaced by $G_1$. Suppose $(S, T)$ is not internal. Then it has at least $4n^2/(\log n)$ cross edges. By Observation 3.2 the probability that it is a Hall cut is less than

$$(1 - p_1)^{c_{G(S, T)}} \leq \exp \left( -p_1 \frac{2nk}{\log n} \right) = \exp \left( -(1 - o(1))2n \right) = o \left( 4^{-n} \right).$$

The claim follows by applying a union bound over all $4^n$ cuts in $G$. \hfill \Box

We now apply Lemma 3.4 with $\varepsilon = \delta^2/100$ to obtain the cuts $(S_1, T_1), \ldots, (S_m, T_m)$. By Claim 5.1 and Lemma 3.4 it suffices to show that a.a.s. all cuts $(S, T)$ that are
\(\varepsilon n\)-close to \((S_i, T_i)\) for some \(i\) are not Hall cuts in \(G_H\). As \(m = O(1)\) it is enough to show that for arbitrary \(i \in [m]\), a.a.s. no cut \((S, T)\) that is \(\varepsilon n\)-close to \((S_i, T_i)\) is a Hall cut in \(G_H\).

Fix an index \(i \in [m]\), set \(S' = S_i, T' = T_i\), and define \(\Gamma\) and \(\Delta\) with respect to \((S', T')\) as in the outline. Henceforth, cross edges, parallel edges, cross degrees and parallel degrees are with respect to \((S', T')\).

**Claim 5.2.** A.a.s. for every \(\Delta\) such that \(n^{0.9} \leq |\Delta| \leq \varepsilon n\), \((S, T)\) is not a Hall cut in \(G_H\).

**Proof.** By Lemma 3.4, each \(x \in \Delta\) satisfies \(\text{deg}_{G_H}^{\text{Par}}(x) \geq (1 - \varepsilon)\frac{k}{2}\). Most of these parallel edges - all those with an endpoint not in \(\Delta\) - are cross edges w.r.t. \((S, T)\). Thus the number of cross edges satisfies:

\[
e_G(S, T^c) + e_G(S^c, T) \geq (1 - \varepsilon)|\Delta|\frac{k}{2} - |\Delta|^2 \geq |\Delta|^k/3.
\]

By Observation 3.2 the probability that \((S, T)\) is a Hall cut in \(G_1\) is at most:

\[
(1 - p_1)^{|\Delta|k/6} \leq \left(\frac{1}{n}\right)^{|\Delta|/7}.
\]

Applying a union bound, the probability that there exists such a Hall cut is at most:

\[
\sum_{|\Delta| = n^{0.9}} \left(\frac{2n}{|\Delta|}\right) \left(\frac{1}{n}\right)^{|\Delta|/7} = o(1).
\]

**Claim 5.3.** A.a.s. for all \(\Delta\) s.t. \(|\Delta| \leq \frac{n^{1/20}}{10} |\Gamma|\), the corresponding cut \((S, T)\) is not a Hall cut in \(G_H\).

**Proof.** Suppose \(\Delta\) satisfies the claim’s hypothesis. By Lemma 3.4 and the definition of \(\Gamma\), each \(x \in \Gamma\) satisfies

\[
\min \{\text{deg}_G^C(x), \text{deg}_G^{\text{Par}}(x)\} \geq n^{-1/20}k.
\]

Ignoring, for the moment, the possibility that \(N_G(x) \cap \Delta \neq \emptyset\), this means that every \(x \in \Gamma\) is incident to at least \(n^{-1/20}k\) cross edges w.r.t. \((S, T)\), regardless of whether \(x \in \Delta\). There are at most \(|\Delta| \min\{|\Gamma|, k\}\) edges between \(\Delta\) and \(\Gamma\). Accounting for possible double counting of the edges incident to \(\Gamma\), we obtain:

\[
e_G(S, T^c) + e_G(S^c, T) \geq \frac{|\Gamma|n^{-1/20}k}{2} - |\Delta| \min\{|\Gamma|, k\} \geq 4|\Delta|k.
\]

Applying Observation 3.2, the probability that \((S, T)\) is a Hall cut in \(G_1\) is at most:

\[
(1 - p_1)^{4|\Delta|k/2} \leq \left(\frac{1}{n}\right)^{1.9n}.
\]

We now observe that if \(\Delta = \emptyset\) (i.e., \((S, T) = (S', T')\)), then the probability that \((S, T)\) is a Hall cut in \(G_1\) is at most \((1 - p_1)^k = o(1)\). Let \(X\) be the number of cuts
satisfying the claim’s hypothesis that are Hall cuts in \( G_1 \). Applying a union bound, we have:

\[
P[X > 0] \leq o(1) + \sum_{1 \leq |\Delta| \leq \min \left\{ \frac{n - 1/20}{10}, |\Gamma| n^{0.9} \right\}} 2n |\Delta| \left( \frac{1}{n} \right)^{1.9|\Delta|} = o(1).
\]

\[ \square \]

**Remark 5.4.** As a consequence of Claim 5.3, if \( |\Gamma| \geq 10n^{19/20} \) then a.a.s. \( G_H \) does not contain a Hall cut. This is because Claim 5.2 covers all cases where \( |\Delta| \geq n^{0.9} \), and the previous claim covers all cases where \( |\Delta| \leq \frac{n - 1/20}{10} |\Gamma| \). Therefore, we proceed under the assumption that \( |\Gamma| < 10n^{19/20} \).

Before proceeding to steps (f) and (g), we modify \( G_H \) by removing a small matching covering the low degree vertices that are not in \( \Gamma \). Moreover, this matching contains only parallel edges. The following claim, together with Lemma 4.5, implies that a.a.s. such a matching exists.

**Claim 5.5.** A.a.s. every vertex in \( V \setminus \Gamma \) is incident to at least one parallel edge in \( G_H \).

**Proof.** Recall that a.a.s. there are no isolated vertices in \( G_H \). Assuming this, if there exists some \( v \in V \setminus \Gamma \) s.t. \( \deg_{G_H}^{\text{Par}}(v) = 0 \), then \( \deg_{G_1}^{\text{Par}}(v) = 0 \) and \( \deg_{G_2}^{\text{Cr}}(v) > 0 \). We use the first moment method to show that a.a.s. there are no vertices \( v \notin \Gamma \) for which this holds. Indeed, if \( v \notin \Gamma \) then the probability of this occurring is bounded from above by

\[
\frac{k}{n^{1/20}} p_2 (1 - p_1) k \left( 1 - n^{-1/20} \right) = \tilde{O} \left( \frac{1}{n^{21/20}} \right).
\]

Therefore the expected number of such vertices is \( \tilde{O} \left( n^{-1/20} \right) = o(1) \), and a.a.s. there are none.

Recall that

\[
V_{\text{low}} = \left\{ x \in V \setminus \Gamma : \deg_{G_1}^{\text{Par}}(x) \leq \frac{1}{1000} \log n \right\}.
\]

Conditioning on the conclusions of Lemma 4.5 and Claim 5.5 holding, there exists a matching \( M \subseteq G_H \) of size \( |V_{\text{low}}| \) consisting of parallel edges that contains \( V_{\text{low}} \).

**Claim 5.6.** A.a.s. \( N_{GH}(V_{\text{low}}) \cap \Gamma = \emptyset \).

**Proof.** By Remark 5.4 we may assume \( |\Gamma| < n^{0.96} \). Fix an arbitrary vertex \( x \notin \Gamma \). Then

\[
P \left[ x \in V_{\text{low}} \land N_{G_H}(x) \cap \Gamma \neq \emptyset \right] \leq \sum_{y \in \Gamma} P \left[ x \in V_{\text{low}} \land y \in N_{G_H}(x) \right]
\]

\[
\leq \sum_{y \in \Gamma} P \left[ |N_{G_1}^{\text{Par}}(x) \setminus \{y\}| \leq \frac{1}{1000} \log n \land y \in N_{G_2}(x) \right]
\]

\[
= \sum_{y \in \Gamma} P \left[ |N_{G_1}^{\text{Par}}(x) \setminus \{y\}| \leq \frac{1}{1000} \log n \right] \cdot P \left[ y \in N_{G_2}(x) \right]
\]

\[
\leq |\Gamma| \left( \frac{1}{n} \right)^{0.99} \cdot p_2 = o \left( \frac{1}{n} \right),
\]
where the probability of the first event is estimated as in the proof of Lemma 4.5. The equality between the second and third lines is due to the fact that the events \(|N_{G_1}^{\text{Par}}(x)\setminus\{y\}| \leq \frac{1}{1000} \log n\) and \(y \in N_{G_1}(x)\) are independent. The statement of the claim follows from a union bound over all \(O(n)\) choices of \(x\).

\[\square\]

**Claim 5.7.** A.a.s. the number of cross edges incident to \(V(M)\) in \(G\) is \(o(n^{0.99})\).

**Proof.** By Lemma 4.5 and Claim 5.6 we may assume that \(|M| = |V_{\text{low}}| < n^{0.01}\) and \(V(M) \cap \Gamma = \emptyset\). Therefore, each vertex in \(V(M)\) is incident to \(O(n^{0.95})\) cross edges, and the claim follows. \(\square\)

Observe that the identity of \(M\) depends only on the parallel edges of \(G_H\) w.r.t. \((S', T')\). This allows us to think of \(G_1\) as being exposed in two independent stages. In the first stage the parallel edges of \(G_1\) are exposed, and in the second the cross edges are exposed.

Set

\[\tilde{G} = G \setminus V(M),\]
\[
\tilde{G}_1 = G_1 \setminus V(M),
\]
\[
\tilde{G}_H = G_H \setminus V(M),
\]
\[
(S, \tilde{T}) = (S' \setminus V(M), T' \setminus V(M)),
\]
and
\[
\Gamma_{\text{bad}} = \left\{ x \in \Gamma : \deg_{G_1}^{\text{Par}}(x) < \frac{1}{1000} \log n \right\}.
\]

**Observation 5.8.** Suppose that in \(\tilde{G}_H\), there is no Hall cut that is \(c\)-close to \((\tilde{S}, \tilde{T})\) for some given \(c\). Then there is no Hall cut in \(G_H\) that is \(c\)-close to \((S', T')\).

**Proof.** We prove the contrapositive. Suppose there exists a Hall cut \((S, T)\) in \(G_H\) that is \(c\)-close to \((S', T')\). Observe that there is no edge connecting \(S\) and \(T\), and therefore \((S \setminus V(M), T \setminus V(M))\) is also a Hall cut in \(\tilde{G}_H\). Furthermore, since we only removed vertices, this cut is \(c\)-close to \((\tilde{S}, \tilde{T})\). \(\square\)

By Observation 5.8 it suffices to show that a.a.s. there is no Hall cut in \(\tilde{G}_H\) that is \(\min\left\{n^{0.9}, \frac{n^{-1/20}}{10}\right\}\)-close to \((\tilde{S}, \tilde{T})\). We first explore pseudo-random properties of \(\tilde{G}_H\).

**Claim 5.9.** A.a.s. every partite set \(A \subseteq V \setminus (V(M) \cup \Gamma_{\text{bad}})\) of size at most \(n^{0.9}\) satisfies

\[|N_{\tilde{G}_1}(A)| \geq |A| \frac{1}{3000} \log n.\]

**Proof.** Suppose the conclusion does not hold, i.e., there is a partite set \(A \subseteq V(\tilde{G}_H) \setminus \Gamma_{\text{bad}}\) with \(|A| \leq n^{0.9}\) s.t. \(|N_{\tilde{G}_1}(A)| < |A| \frac{1}{3000} \log n\). Then, for every \(x \in A\), \(\deg_{\tilde{G}_1}^{\text{Par}}(x) \geq \frac{1}{1000} \log n\). Further,

\[N_{\tilde{G}_1}(A) \subseteq N_{\tilde{G}_1}(A) \cup ((V_{\text{low}} \cup N_{G_H}(V_{\text{low}})) \cap N_{G_H}(A)).\]
However, by Lemma 4.3.5 \(|V_{low} \cup N_{G_H}(V_{low})| \cap N_{G_1}(A)| \leq |A|\), because if a vertex in 
A has two neighbors in \(V_{low} \cup N_{G_H}(V_{low})\), then there are two vertices in \(V_{low}\) whose 
distance in \(G_H\) is at most 4. Therefore:

\[
|N_{G_1}(A)| \leq |N_{\tilde{G}_1}(A)| + |A| \leq |A| \frac{1}{3000} \log n + |A| < |A| \frac{1}{2000} \log n.
\]

The set \(A\) does not satisfy the conclusion of Lemma 4.6 which holds a.a.s. Therefore 
the conclusion of our claim holds a.a.s. as well. □

Claim 5.10. A.a.s. \(|\Gamma_{bad}| \leq |\Gamma|/n^{0.4}\).

Proof. By Lemma 3.4, every vertex has parallel degree in \(G\) at least \((1 - \varepsilon)\frac{k}{2}\). Therefore 
the probability that a vertex’s parallel degree in \(G_1\) is less than \(\frac{1}{1000} \log n\) is bounded 
above by

\[
\frac{1}{1000} \log n \left( \frac{1 - \varepsilon}{2} \right)^{\frac{1}{1000} \log n} \left( 1 - p_1 \right)^{\frac{(1 - \varepsilon)k}{2} - \frac{1}{1000} \log n} = O \left( n^{-0.49} \right).
\]

The conclusion follows from an application of Markov’s inequality. □

Henceforth, unless otherwise specified, parallel degrees, cross degrees, etc., are with 
respect to the vertex set \(V \setminus V(M)\) and the cut \((\tilde{S}, \tilde{T})\).

Claim 5.11. The following holds a.a.s. Suppose \(\Delta\) of size \(\frac{n^{1/20}}{10} |\Gamma| \leq |\Delta| \leq n^{0.9}\) 
satisfies one of:

\(a\) |\(\Delta| \geq \log \log n.
\(b\) \(\Delta \cap \Gamma \neq \emptyset\).

Then \((S, T)\) is not a Hall cut in \(\tilde{G}_H\).

Proof. Set:

\[a = \tilde{S} \setminus S, b = \tilde{S}^c \setminus S^c, c = \tilde{T}^c \setminus T^c, d = \tilde{T} \setminus T.\]

We first observe that if either \(N_{\tilde{G}_1}(b) \not\subseteq c\) or \(N_{\tilde{G}_1}(d) \not\subseteq a\), then \((S, T)\) is not a Hall 
cut with probability 1. Thus we may assume that \(N_{\tilde{G}_1}(b) \subseteq c\) and \(N_{\tilde{G}_1}(d) \subseteq a\). The 
conclusion of Claim 5.9 then implies that

\[|a| \geq |d \setminus \Gamma_{bad}| \frac{1}{3000} \log n, |c| \geq |b \setminus \Gamma_{bad}| \frac{1}{3000} \log n.\]

As by the previous claim and the hypothesis \(|\Gamma_{bad}| \leq |\Gamma| = o (|\Delta|/\log n)\):

\[|b| + |d| = |b \setminus \Gamma_{bad}| + |b \cap \Gamma_{bad}| + |d \setminus \Gamma_{bad}| + |d \cap \Gamma_{bad}| \leq O \left( \frac{|a| + |c|}{\log n} \right) + |\Gamma_{bad}| = O \left( \frac{|\Delta|}{\log n} \right).\]

We may assume that |\(S| > |T|\), for otherwise \((S, T)\) is not a Hall cut by definition. It 
also holds that:

\[|S' - T'| = |\tilde{S}| - |\tilde{T}| = (|S| + |a| - |b|) - (|T| + |d| - |c|) > |a| + |c| - (|b| + |d|) = \left( 1 - O \left( \frac{1}{\log n} \right) \right) |\Delta|.\]
Since $G$ is $k$-regular,
\[ e_G(S, \tilde{T}^c) \geq \left( 1 - O\left( \frac{1}{\log n} \right) \right) |\Delta|k. \]

By Claim 5.7, the number of cross edges in $G$ that are not cross edges in $\tilde{G}$ is at most $n^{0.99}$. Thus:
\[ e_{\tilde{G}}(S, \tilde{T}^c) \geq \left( 1 - O\left( \frac{1}{\log n} \right) \right) |\Delta|k. \]

Set $\Delta_1 = \Delta \cap \Gamma, \Delta_2 = \Delta \setminus \Gamma$. We then have:
\[
\left| E_{\tilde{G}}(S, T^c) \cap E_{\tilde{G}}(\tilde{S}, \tilde{T}^c) \right| \geq e_{\tilde{G}}(\tilde{S}, \tilde{T}^c) - |\Delta_1|(1 + \varepsilon)\frac{k}{2} - |\Delta_2|\frac{k}{n^{1/20}} \\
\geq \left( \frac{1}{2} - \varepsilon \right) |\Delta_1|k + \left( 1 - O\left( \frac{1}{\log n} \right) \right) |\Delta_2|k.
\]

Therefore, the probability that none of the cross edges is in $\tilde{G}_1$ is at most:
\[
(1 - p_1)(\frac{1}{2} - \varepsilon)|\Delta_1|k(1 - p_1)\left( 1 - O\left( \frac{1}{\log n} \right) \right) |\Delta_2|k.
\]

Suppose (a) holds. Let $m = \max \left\{ \frac{n^{-1/20}}{10} |\Gamma|, \log \log n \right\}$. Therefore, applying a union bound over all choices of $\Delta_1 \subseteq \Gamma$ and $\Delta_2$:
\[
\alpha := \sum_{|\Delta| \in \{m, \ldots, n^{0.9}\}} \left( \frac{|\Gamma|}{|\Delta_1|} \right) \left( \frac{2n}{|\Delta_2|} \right) (1 - p_1)\left( \frac{1}{2} - \varepsilon \right)|\Delta_1|k(1 - p_1)\left( 1 - O\left( \frac{1}{\log n} \right) \right) |\Delta_2|k \\
\leq \sum_{|\Delta| \in \{m, \ldots, n^{0.9}\}} |\Delta_1|^{-|\Delta_1|} |\Delta_2|^{-|\Delta_2|} \left( e|\Gamma| \left( \frac{1}{n} \right)^{1/2 - 2\varepsilon} \right) \left( O(\log \log n) \right) |\Delta_2|.
\]

Now
\[
|\Delta_1|^{-|\Delta_1|} |\Delta_2|^{-|\Delta_2|} \leq \left( \frac{2}{|\Delta|} \right)^{|\Delta|} = \left( \frac{2}{|\Delta|} \right)^{|\Delta_1|} \left( \frac{2}{|\Delta|} \right)^{|\Delta_2|}.
\]

Thus:
\[
\alpha \leq \sum_{|\Delta| \in \{m, \ldots, n^{0.9}\}} \left( \frac{2e|\Gamma|}{|\Delta|} \left( \frac{1}{n} \right)^{1/2 - 2\varepsilon} \right)^{|\Delta_1|} \left( O(\log \log n / |\Delta|) \right) |\Delta_2| \\
\leq \sum_{|\Delta| \in \{m, \ldots, n^{0.9}\}} \left( \frac{1}{n^{2/5}} \right)^{|\Delta_1|} \left( O\left( \log |\Delta| / |\Delta| \right) \right) |\Delta_2| = o(1).
\]

Otherwise, (b) holds. Then $|\Delta_1| \geq 1$. By a similar application of a union bound, the probability that there exists any Hall cut $(S, T)$ in $\tilde{G}_1$ satisfying the hypothesis is
bounded above by:
\[
\sum_{|\Delta| \in \log \log n} \left( \frac{1}{n^{2/5}} \right)^{|\Delta_1|} O(\log \log n / |\Delta|)|\Delta_2| \leq \frac{(\log \log n)^2}{n^{2/5}} O(\log \log n)^{\log \log n} = o(1).
\]

It remains to show that a.a.s. there is no Hall cut with \(|\Delta| \leq \log \log n\) and \(\Delta \cap \Gamma = \emptyset\).

**Claim 5.12.** A.a.s. there exists no Hall cut \((S, T)\) in \(\tilde{G}_H\) with \(|\Delta| < \log \log n\) and \(\Delta \cap \Gamma = \emptyset\).

**Proof.** We first show that if \(|\tilde{S}| \leq |\tilde{T}|\) then a.a.s. there is no Hall cut satisfying the claim’s hypothesis. Suppose, for a contradiction, that \(|\Delta| < \log \log n\) and \(\Delta \cap \Gamma = \emptyset\) corresponds to a Hall cut. Since \(\Delta \cap \Gamma = \emptyset\), we have \(b, d \subseteq V \setminus (V(M) \cup \Gamma)\). Therefore for every \(x \in b \cup d\), \(N_{\tilde{G}_1}(x) \subseteq \Delta\). However, by Claim 5.9 a.a.s. for every such \(x\), \(|N_{\tilde{G}_1}(x) \subseteq \Delta| = \Omega(\log n)\). Assuming this, and since \(|\Delta| \leq \log \log n\), we conclude that \(b = d = \emptyset\). Thus:
\[
|S| - |T| = |\tilde{S}| - |\tilde{T}| - |a| + |b| - |c| + |d| = |\tilde{S}| - |\tilde{T}| - |a| - |c| \leq |\tilde{S}| - |\tilde{T}| \leq 0.
\]
Therefore, \((S, T)\) is not a Hall cut.

We now assume that \(|\tilde{S}| - |\tilde{T}| > 0\). We will show presently that a.a.s. \(e_{\tilde{G}_1}(S, T^c) = \Omega(\log n)\). Suppose \((S, T)\) is a cut satisfying the claim’s hypothesis. Then \(\Delta\) must contain a vertex cover of \(E_{\tilde{G}_1}(S, T^c)\). However, since \(\Delta \cap \Gamma = \emptyset\), by Lemma 4.3 a.a.s. each vertex in \(\Delta\) is incident to at most 30 cross edges in \(G_1\). Since \(|\Delta| \leq \log \log n\), \(\Delta\) does not contain a vertex cover of \(E_{\tilde{G}_1}(S, T^c)\), and so \((S, T)\) is not a Hall cut.

It remains to show that a.a.s. \(e_{\tilde{G}_1}(\tilde{S}, \tilde{T}^c) = \Omega(\log n)\). Since \(G\) is \(k\)-regular, we have \(e_G(S, T) \geq k \left( |\tilde{S}| - |\tilde{T}| \right) \geq k\). By Claim 5.7 the number of cross edges of \(G\) incident to \(V(M)\) is \(o(n^{0.99})\), so \(\tilde{G}\) has at least \((1 - o(1))k\) cross edges. Now, \(\mathbb{E} \left[ e_{\tilde{G}_1}(\tilde{S}, \tilde{T}) \right] \geq (1 - o(1)) \log n\), and by an application of Chebychev’s inequality, a.a.s. \(e_{\tilde{G}_1}(\tilde{S}, \tilde{T}) \geq (1 - o(1)) \log n\).

\[\Box\]

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Hebrew University of Jerusalem, Jerusalem 91904, Israel
E-mail address: roman.l.glebov@gmail.com

Israel Institute for Advanced Studies
E-mail address: zluria@gmail.com

Institute of Mathematics and Federmann Center for the Study of Rationality, The Hebrew University of Jerusalem, Jerusalem 91904, Israel
E-mail address: menahem.simkin@mail.huji.ac.il