MASS BOUNDS FOR MULTIDIMENSIONAL CHARGED DILATONIC BLACK HOLES

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Abstract

The multidimensional charged dilatonic black hole solution with $n$ internal Ricci-flat spaces is considered. The bound on the mass of the black hole is obtained. In the strong dilatonic coupling limit the critical mass becomes zero. The case $n = \infty$ is also considered.

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1 Introduction

In [1] Myers and Perry obtained the multidimensional $O(d + 1)$-symmetric analogue of the well-known Reissner-Nordström charged black hole solution. In [2,3] the generalization of the Myers-Perry solution to the case of $n$ Ricci-flat internal spaces was obtained. Special cases of the solution [2,3] were considered earlier in the following publications: [4,5] ($d = 2$, $\lambda = 0$), [6] ($d = 2$, $n = 1$), [7] ($d \geq 2$, $\lambda = 0$). (In [8] the special case of solution [2,3] with zero electric and scalar charges was obtained.)

The model considered in [2,3] is described by the action

$$S = \int d^D x \sqrt{|g|} \left\{ -\frac{1}{2\kappa^2} R[g] - \frac{1}{2\kappa^2} \partial_M \varphi \partial_N \varphi g^{MN} - \frac{1}{4} \exp(2\lambda \varphi) F_{MN} F^{MN} \right\}, (1.1)$$

where $g = g_{MN} dx^M \otimes dx^N$ is the metric, $F = \frac{1}{2} F_{MN} dx^M \wedge dx^N = dA$ is the strength of the electromagnetic field and $\varphi$ is the scalar field (dilatonic field). Here $\lambda$ is the dilatonic coupling constant and $\kappa$ denotes Einstein’s gravitational constant. The action (1.1) describes for certain values of the coupling constant $\lambda$ and space-time dimension $D$ a lot of interesting physical models including standard Kaluza-Klein theory, dimensionally reduced Einstein-Maxwell theory, certain sectors of supergravity theories etc. For

$$\lambda^2 = \frac{1}{2}, \quad D = 10, \quad (1.2)$$

the action (1.1) describes a part of the bosonic sector for the $N = 1$ ten-dimensional Einstein-Yang-Mills supergravity that occurs in the low energy limit of superstring theory [9].

In this paper we consider as special case of the solution [2,3] a charged dilatonic black hole with $n$ internal Ricci-flat spaces. Objects of such sort are very popular in the literature (see for example [2,6,10-14]). Here we present bounds on the mass of this black hole. For $D = 4$, $d = 2$ such a bound was obtained by Gibbons and Wells in [14]. We also consider an interesting special
case of the charged black hole solution with infinite number of internal spaces \( n = \infty \). We note that to our knowledge the Einstein equations in the infinite-dimensional space were considered first by Kalitzin [15]. Infinite-dimensional spherically-symmetric and cosmological solutions were also presented in [3] and [16] respectively.

\section{Spherically Symmetric Solutions}

The field equations, corresponding to the action (1.1), have the following form

\begin{align}
R_{MN} - \frac{1}{2}g_{MN}R &= \kappa^2 T_{MN}, \quad (2.1) \\
\Box \varphi - \frac{\kappa^2}{2} \lambda \exp(2\lambda \varphi) F_{MN} F^{MN} &= 0, \quad (2.2) \\
\nabla_M (\exp(2\lambda \varphi) F^{MN}) &= 0, \quad (2.3)
\end{align}

where

\[ T_{MN} = \frac{1}{\kappa^2} (\partial_M \varphi \partial_N \varphi - \frac{1}{2} g_{MN} \partial_P \varphi \partial^P \varphi) + \exp(2\lambda \varphi)(F_{MP} F^{NP} - \frac{1}{4} g_{MN} F_{PQ} F^{PQ}). \quad (2.4) \]

Here we consider the spherically \( O(d+1) \)-symmetric solutions of the field equations (2.1)-(2.3) obtained in [2,3]. In this paper the notations of ref. [3] are used. The solution [4] is defined on the manifold

\[ M = M^{(2+d)} \times M_1 \times \ldots \times M_n, \quad (2.5) \]

and has the following form

\begin{align}
g &= -f_1^{(D-3)/A(\lambda)} f_\varphi^{2\lambda} d\bar{t} \otimes d\bar{t} \\
&\quad + f_1^{-1/A(\lambda)} (f_2^{-1} f_\varphi^{2\lambda} f_2^{2(1-d)/D}) [f_2 du \otimes du + d\Omega_d^2] \\
&\quad + \sum_{i=1}^n f_1^{-1/A(\lambda)} \exp(2A_i u + 2D_i) g^{(i)}, \quad (2.6)
\end{align}
\[ F = Q f_1 du \wedge d\bar{t}, \quad (2.7) \]
\[ \exp \varphi = f_1^{(2-D)\lambda/2A(\lambda)} f_{\phi}. \quad (2.8) \]

where \( M^{(2+d)} \) is a \((2 + d)\)-dimensional space-time \((d \geq 2)\), the \((M_i, g^{(i)})\) are Ricci-flat manifolds \((g^{(i)}\) is the metric on \(M_i)\), \(\dim M_i = N_i(i = 1, \ldots, n)\), \(d\Omega^2_d\) is the canonical metric on the \(d\)-dimensional sphere \(S^d\). In (2.6)-(2.8)

\[ f_1 = f_1(u) = C_1(D - 2)/\kappa^2 Q^2 A(\lambda) \sinh^2(\sqrt{C_1}(u - u_1)), \quad (2.9) \]
\[ f_2 = f_2(u) = C_2/(d - 1)^2 \sinh^2(\sqrt{C_2}(u - u_2)), \quad (2.10) \]
\[ f_{\phi} = f_{\phi}(u) = \exp(Bu + D_{\phi}), \quad (2.11) \]
\[ f = f(u) = \exp[\sum_{i=1}^{n} N_i(A_i u + D_i)], \quad (2.12) \]
\[ A(\lambda) = D - 3 + \lambda^2(D - 2), \quad (2.13) \]

and \(Q \neq 0, D_1, D_\phi, u_1, u_2\) are constants and the parameters \(C_1, C_2, B, A_i\) satisfy the relation

\[ \frac{C_2d}{d-1} = \frac{C_1(D - 2)}{D - 3 + \lambda^2(D - 2)} + B^2(1 + \lambda^2) \]
\[ + \frac{1}{d-1}(\lambda B + \sum_{i=1}^{n} A_i N_i)^2 + \sum_{i=1}^{n} A_i^2 N_i. \quad (2.14) \]

### 3 Dilatonic Charged Black Hole

Now, we are looking for a black hole solution, that means for the special case of a configuration with external horizon. To do so, we take the solution (2.6)-(2.14) with the parameters

\[ C_1 = C_2 = C > 0, \quad u_2 = 0, \quad u_1 = -u_0 < 0, \quad (3.1) \]
\[ A_i/\sqrt{C} = -1/A(\lambda), \quad B/\sqrt{C} = -\lambda(D - 2)/A(\lambda). \quad (3.2) \]
Introducing the parameters

\[ B_{\pm} = \frac{\kappa |Q| \sqrt{A(\lambda)}}{(d-1) \sqrt{D-2}} \exp(\pm \sqrt{C}u_0) \]  

(3.3)

and reparametrizing the time and radial coordinates

\[ \bar{t} = \frac{\kappa |Q| \sqrt{A(\lambda)} \sinh(\sqrt{C}u_0)}{\sqrt{C(D-2)}} t, \]  

(3.4)

\[ r^{d-1} = \frac{\kappa |Q| \sqrt{A(\lambda)} \sinh(\sqrt{C}(u + u_0))}{(d-1) \sqrt{(D-2)} \sinh(\sqrt{C}u)} \]  

(3.5)

we get the following formulas for the solution \((\lambda \neq 0)\)

\[ g = -f_+ f^{1+2\alpha_t} \, dt \otimes dt + f_2^{2\alpha_r} \left[ \frac{dr \otimes dr}{f_+ f_-} + r^2 d\Omega^2_d + f_-^{-2/A(\lambda)} \sum_{i=1}^{n} g^{(i)} \right] + f_-^{2/4} \]  

(3.6)

\[ F = Q r^{-d} dt \wedge dr, \]  

(3.7)

\[ \exp(2\lambda \varphi) = f_-^{2\alpha_r}. \]  

(3.8)

Here

\[ f_{\pm} = f_\pm(r) = 1 - \frac{B_{\pm}}{r^{d-1}}, \]  

(3.9)

\[ \alpha_t = -\lambda^2 (D-2)/A(\lambda), \quad \alpha_r = \frac{1}{d-1} - \frac{1}{A(\lambda)}, \]  

(3.10)

and the constants \(B_{\pm}\) and \(Q\) satisfy the relations

\[ B_+ B_- = \frac{\kappa^2 Q^2 A(\lambda)}{(d-1)^2(D-2)}. \]  

(3.11)

We remind that we consider the case \(Q \neq 0\). Due to eq. (3.3) we have

\[ B_+ > B_- > 0. \]  

(3.12)

In this case the \((2 + d)\)-dimensional section of the metric (3.6) has a horizon at \(r^{d-1} = B_+\). For \(r^{d-1} = B_-\) the horizon is absent \((\lambda \neq 0)\). In the limit
\( \lambda \to 0 \) and \( D \to 2 + d \) we get the Myers-Perry \( O(d+1) \)-symmetric charged black hole solution [1]. For \( d = 2, \ D = 4 \) the solution coincides (up to redefinitions of the field variables) with the 4-dimensional dilatonic charged black hole solution [10,11].

4 Mass Bounds

The solution (3.6)-(3.11) describes an \( O(d+1) \)-symmetric charged dilatonic black hole with a chain of internal Ricci-flat spaces. The charge of the black hole is \( Q \) and the mass \( M \) is found from the time component of the metric (3.6) to be

\[
2GM = B_+ + B_- \beta(\lambda),
\]

where

\[
\beta(\lambda) = 1 + 2\alpha = \frac{D - 3 - \lambda^2(D - 2)}{D - 3 + \lambda^2(D - 2)}
\]

and

\[
G = S_D \kappa^2
\]

is the effective gravitational constant (\( S_D \) is defined in [1]). It is clear that

\[-1 < \beta(\lambda) < 1\]

and \( \beta(\lambda) = 0 \) for \( \lambda^2 = (D - 3)/(D - 2) \). Using the relation (4.1) and the inequalities (3.12) and (4.4) we get

\[
M > M_c,
\]

where

\[
M_c = \frac{\kappa |Q|(D - 3)}{G(d-1)\sqrt{A(\lambda)(D - 2)}}.
\]

This can be easily deduced from Fig. 1.
The critical case corresponds to

$$B_+ = B_- = \frac{\kappa |Q| \sqrt{A(\lambda)}}{(d - 1) \sqrt{(D - 2)}} \quad (4.7)$$

Formula (4.6) agrees with the corresponding relation from ref. [14] in the case $D = 4$.

It is not difficult to verify that in the critical case (4.7) we have a horizon, if and only if

$$\lambda^2 \leq \frac{1}{d - 1} - \frac{D - 3}{D - 2}. \quad (4.8)$$

For these values of $\lambda$ the inequality (4.5) should be replaced by

$$M \geq M_c. \quad (4.9)$$

In the strong coupling limit $\lambda^2 \to +\infty$ the critical mass (4.6) tends to zero. A possible interpretation of this effect seems to be screening by dilatonic field of the electric charge.

**Infinite-dimensional case.** Now we consider the interesting special case when $n \to +\infty$. In this limit the exact solution of the field equations taken from (3.6)-(3.11) reads

$$g = -f_+ f_-^{\frac{1 - \lambda^2}{1 + \lambda^2}} dt \otimes dt + f_-^{\frac{2}{d - 1}} \left[ \frac{dr \otimes dr}{f_+ f_-} + r^2 d\Omega_d^2 \right] + \sum_{i=1}^n g^{(i)}, \quad (4.10)$$

$$F = Qr^{-d} dt \wedge dr, \quad (4.11)$$

$$\exp(2\lambda \varphi) = f_-^{\frac{1 - \lambda^2}{1 + \lambda^2}}, \quad (4.12)$$

where

$$B_+ B_- = \frac{\kappa^2 Q^2 (1 + \lambda^2)}{(d - 1)^2}. \quad (4.13)$$

In this case the internal space scale-factors do not depend on the radial coordinate but the information about the presence of internal dimensions is
contained in the $(2 + d)$-dimensional part of the metric: this part does not coincide with the $(D = 2 + d)$-dimensional solution without internal spaces. The critical mass is non-zero in this limit:

$$M_c = \frac{\kappa |Q|}{G(d - 1) \sqrt{1 + \lambda^2}}.$$  \hfill (4.14)

## 5 Conclusion

We considered the $O(d + 1)$-symmetric charged dilatonic black hole solution with $n$ Ricci-flat internal spaces. We obtained the bound on the mass of a black hole for all (non-zero) values of the coupling constant. We found that the critical mass tends to zero in the strong coupling limit. We also considered the case of infinite number of internal spaces: $n = \infty$. In this case we obtained a non-trivial solution of the field equations and a non-zero value for the critical mass.

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References

[1] R. C. Myers and M. J. Perry, Ann. of Phys., 172 (1986) 304.

[2] U. Bleyer, K. A. Bronnikov, V. N. Melnikov and S. B. Fadeev, On black hole stability in multidimensional gravity, AIP preprint (Potsdam) 94-01, 1994.

[3] V. D. Ivashchuk and V. N. Melnikov, Multitemporal generalization of the Tangherlini solution, submitted to Class. and Quantum Grav.; submitted to Izvestiya Vuzov (in Russian).

[4] S. B. Fadeev, V. D. Ivashchuk and V. N. Melnikov, Chinese Phys.Lett., 8 (1991) 439; In: Gravitation and Modern Cosmology, Plenum Publ. NY, (1991) 37.

[5] K. A. Bronnikov, Ann. der Phys., 48 (1991) 527.

[6] K. A. Bronnikov, Izvestija Vuzov, No 1 (1992) 106 [in Russian].

[7] S. B. Fadeev, V. D. Ivashchuk and V. N. Melnikov, On Generalization of Charged Version of the Tangherlini Solution on the Case of n Internal Ricci-flat Spaces, Nuclear Safety Institute preprint, Moscow, 1994; submitted to Izvestiya Vuzov (in Russian).

[8] S. B. Fadeev, V. D. Ivashchuk and V. N. Melnikov, Phys. Lett., A 161 (1991) 98.

[9] M. B. Green, J. H. Schwarz and E. Witten, Superstring Theory, Cambridge Univ. Press, 1986, p. 527.

[10] G. W. Gibbons and K. Maeda, Nucl. Phys., B298 (1988) 741.
[11] D. Garfinkle, G. Horowitz and A. Strominger, Phys. Rev., D43 (1991) 3140; D45 (1992) 3888(E).

[12] G. W. Gibbons and P. K. Townsend, Vacuum Interpolation in Supergravity via Super p-branes, DAMTP preprint R-93/19, 1993.

[13] K. Shiraishi, Mod. Phys. Lett., A 7 (1992) 3569.

[14] G. W. Gibbons and C. G. Wells, Anti-Gravity Bounds and the Ricci Tensor, DAMTP preprint R93/25, 1993; to be published in Commun. Math. Phys.

[15] N. S. Kalitzin, Wissenschaftliche Zeitschrift der Humboldt-Universität zu Berlin, Jg. VII Nr 2 207 (1957/58).

[16] V. D. Ivashchuk, Phys. Lett., A 170 (1992) 16.
Figures:

a)

b)

Fig. 1: Lines of $M = \text{const}$ for a) $\beta > 0$ and b) $\beta < 0$
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9405018v1