CLASSICAL DEFOCUSING OF WORLD LINES IN HIGHER DIMENSIONS

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ABSTRACT

A five-dimensional gravity theory, motivated by the brane-world picture, with Kaluza scalar in the 5-dimensional metric as $g_{55}(r) = \sqrt{x^2 + y^2 + z^2}$, is considered near the possible singularity (small distance scales where gravity is strong) and is shown to give rise to a positive contribution to the Raychaudhuri equation. This inhibits the focusing of world lines and contributes to non-focusing of the worldlines in the 5-dimensional space. It is also shown that the results extend to time dependent cases such as those relevant for black hole interiors and cosmology.

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1. Introduction

An interesting question in the field of gravitation is whether there is a simple way to avoid the singularities present in certain classical solutions. The Standard Cosmological Model based on Einstein’s theory of gravitation, for example, implies that the universe began with a big bang singularity. The fascinating arena of black holes also possess singularities. Within the framework of Einstein’s theory, the above singularities cannot be avoided without imposing exotic matter of some sort. This can be understood, for instance, by the Raychaudhuri equation \[1\] which in the absence of torsion, exhibits focusing of geodesics converging to the singularities \[2\]. In order to avoid such singularities without imposing exotic matter, one has to go beyond Einstein’s theory of gravitation. This has been investigated in the brane-world scenario and the recent studies indicate possible avoidance of the big bang singularity \[3\] - \[5\].

Studies in string theory motivated non-singular cosmologies \[6\], \[7\] can avoid the big bang singularity. It has also been shown that in effective loop quantum gravity theories singularities can be avoided \[8\], \[9\]. Ellis and Maartens \[10\], and Ellis, Murugan and Tsagas \[11\] proposed the emergent scenario in which the universe stays in a static past eternally and then evolves to a subsequent inflationary era, suggesting that the universe originates from Einstein static state rather than a big bang singularity. An emergent scenario has been made possible in the modified theories of gravity such as \(f(R)\) gravity, loop quantum gravity \[12\] and in Einstein - Cartan theory \[13\]. *These studies motivate the consideration of higher dimensional gravity as a possible candidate for avoiding a singularity.* Further motivation is provided by the trace anomaly of Conformal Field Theory dual to a 5-dimensional Schwarzschild AdS geometry \[14\] in which \(H^4\) (\(H\) being the Hubble parameter) terms are present in the equation for \(\dot{H}\) and which leads to an infinite age of the universe, avoiding the singularity. Similar resolution of the singularity comes from the corrections to Raychaudhuri equation in the brane world scenario \[15\], in the approach using the ‘generalized uncertainty principle’ of quantum gravity \[16\] and in the quantum corrected Friedmann equations \[17\], \[18\]. An interesting analysis from black hole-brane interactions has been done in \[19\].

While a theory of quantum gravity is far from being realized, a quan-
tum corrected Raychaudhuri equation has been proposed by Das [20] and this was the basis to obtain the corrected Friedmann equation for $\dot{H}$ by Ali and Das [21] which avoids the big bang singularity, predicting infinite age for the universe. The said corrections to the Raychaudhuri equation cause defocussing of the geodesics thereby avoiding the singularity. We find in this manuscript that there is also a simple classical mechanism to produce a similar defocussing term in 5-dimensional world.

From the above studies, it is clear that in order to avoid the singularity, one needs to modify gravity such that defocussing of the geodesics occurs. One way to modify Einstein’s theory of gravity is to consider five dimensional gravity (without electromagnetic fields) near the singularity (small scales where gravity is strong), a minimum modification.

A direct way to understand the possible avoidance of certain space-like singularities is to consider the Raychaudhuri equation. This equation in 4-dimensional gravity is

$$\frac{d\Theta}{ds} = -\frac{\Theta^2}{3} - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}u^\mu u^\nu + (\dot{u})_{\mu},$$

where $\Theta = u^\mu_{;\mu}$ characterizing the volume of the collection of particles with 4-velocity $u^\mu$ as they fall under gravity. That is, $\Theta$ provides a description of the expansion or contraction of a material body containing streamlines. The quantity $\sigma_{\mu\nu}$ is the symmetric tensor representing the shear, $\omega_{\mu\nu}$ is the antisymmetric tensor representing the vorticity and the last term $(\dot{u})_{\mu}$ vanishes when the particles travel on their geodesics in 4-d theory. The vorticity causes expansion while the shear contraction. In the absence of vorticity or exotic matter, the geodesics contract or focus causing the universe to have a beginning a finite time ago, creating the big bang singularity or black hole spacelike singularity [2]. One could obtain solutions with shear and no vorticity; but not with vorticity and no shear [22] in Einstein’s theory. The singularity theorems of Penrose and Hawking use this feature to state that there is an inevitable spacetime singularity [2], [22]. In the absence of vorticity and shear, if the last term $(\dot{u}_{\mu})_{;\mu}$ exists and is positive, then defocussing of the world lines occurs thereby softening, or potentially avoiding the singularity.

Thus, attempts to avoid the singularity require either use of complicated field theoretic models of matter or modified gravity. The quantum corrections to the Raychaudhuri equation in [15], [16], [20] achieve this, preventing focusing of geodesics. This feature in cosmological considerations led to the
avoiding of the big bang singularity with the universe without a beginning. Since certain black hole interiors have similar causal structure, the arguments may also apply to those singularities.

It is worthwhile to examine whether a non-focusing term of the world lines similar to the above studies could emerge classically in higher dimensional gravity with minimum modifications.

The aim of this paper is to show a defocussing term of world lines arises by considering five-dimensional Kaluza theory with fifth dimension at small scales (near the singularity) where gravity is expected to be strong - a minimal modification of the 4-d gravity. We hasten to add that we interpret our results modestly, in that we show that a defocussing term appears similar to what is found in the models cited above, but from the consideration of classical general relativity without exotic or quantum matter. The price to pay in this case is the introduction of higher dimensions. The defocussing of world lines occurs in the 5-dimensional world.

In Section 2, we show this by considering static and spherically symmetric 5-d space time. The '55' part of the 5-d metric, a 4-d scalar, is shown to be responsible for the non-focusing feature. We also extend the analysis to time-dependent domains of the type applicable to some black hole interiors.

2. Effect of Kaluza scalar on Raychaudhuri equation

We consider gravity in 5 - d spacetime as in Kaluza-Klein theory with a non-compact or compact fifth dimension. (If the fifth dimension is compact, one will be dealing with specific discrete modes.) Non-compact Kaluza-Klein theory has been considered by Wesson [23] and in [24]-[27] as Space-Time-Matter theory. The 5 - dimensional metric chosen corresponds to (without electromagnetism)

\[
(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu - g_{55}(r) (dx^5)^2,
\]

where \( g_{\mu\nu} \) is the 4-d metric and \( g_{55} = g_{55}(r) \); \( r^2 = x^2 + y^2 + z^2 \) the Kaluza scalar. The line element \( (1) \) corresponds to a 5-d spacetime which can be viewed as 4-d gravity with Kaluza scalar \( g_{55}(r) \). The quantity \( dS \) is the full 5-d element in all that follows. That is, it is an invariant under the full 5-d diffeomorphism group. The 5-d spacetime will correspond to a solution of
the 5-d vacuum Einstein equations $\tilde{R}_{AB} = 0$; $A, B = 0, 1, 2, 3, 5$, where $\tilde{R}_{AB}$ is the Ricci tensor in the full 5-d spacetime. We study the strong gravity regime by this 5-d gravity.

A remarkable consequence of this metric in (1), is that 5-d world line equation, restricting to 4-d coordinates, has an ‘acceleration’ term from $g_{55}(r)$ (by decomposing $A, B, C$ to $\mu, \nu, \lambda$ and separating out the $A, B, C = 5$ components), namely

$$\frac{d^2 x^\mu}{dS^2} + \Delta^\mu_{\nu\lambda} \frac{dx^\nu}{dS} \frac{dx^\lambda}{dS} = \frac{1}{2} a^2 g^\mu\lambda_{55} (\partial_\lambda g_{55}),$$

(2)

where $a$ is a constant along the world line [28], a consequence of the independence of the metric components in (1) on the fifth coordinate $x^5$. This can be seen as: The world line equation in 5-dimensional theory is

$$\frac{d^2 z^A}{dS^2} + \Delta^A_{BC} \frac{dz^B}{dS} \frac{dz^C}{dS} = 0,$$

where $\Delta$ is the 5-dimensional connection coefficients. The above world line equation can be rewritten as

$$\frac{d}{dS} \left( g_{AB} \frac{dz^B}{dS} \right) - \frac{1}{2} (\partial_A g_{CD}) \frac{dz^C}{dS} \frac{dz^D}{dS} = 0.$$

The $A = 5$ part of this equation gives

$$\frac{d}{dS} \left( g_{5B} \frac{dz^B}{dS} \right) = 0,$$

since the 5-dimensional metric are chosen to be independent of $x^5$ the fifth coordinate. So we have

$$g_{5B} \frac{dz^B}{dS} = a,$$

where $a$ is a constant along the 5-d world line. In our case as $g_{5\mu} = 0$ as we are not considering electromagnetic fields and so $a = g_{55} \frac{dx^5}{dS}$. The ‘acceleration’ stated above is the acceleration with respect to the full $dS$ in the 5-dimensional world.
In order to see how the scheme works in the case of a non-compact fifth
dimension, let us consider the 5-d action

\[ S_5 = \frac{1}{16\pi G_5} \int d^4x \, dx^5 \sqrt{G} \hat{R} \]

(3)

where \( G_5 \) is the 5-d gravitational constant, \( G \) is 5-d metric determinant and
\( \hat{R} \) is the 5-d Ricci scalar. As none of the components of the metric \( G_{AB} \) and
hence \( \hat{R} \) depend on \( x^5 \), we see the split (4+1) as

\[ S = \frac{1}{16\pi G_5} \int dx^5 \int d^4x \sqrt{-G} \hat{R}, \]

\[ \equiv \frac{1}{16\pi G_4} \int d^4x \sqrt{-G} \hat{R}, \]

where we define the effective 4-dimensional gravitational Newton constant
\( G_4 \) as

\[ \frac{1}{G_4} \equiv \frac{1}{G_5} \int dx^5. \]

Then the fifth coordinate does not appear in the effective action. This leads
to 4-dimensional action, which given the structure of \( G_{AB} \) and \( \hat{R} \) corresponds
in 4-dimensions gravitation with scalar \( g_{55}(r) \) coupling. The non compact \( x^5 \)
is made small so that \( G_4 \) remains finite. The fifth coordinate due to its
small dimensions is not an observable at present day experiments. In the
resulting expressions we only use \( G_4 \) and so the motion in the fifth coordinate
becomes unobservable. Of course, one may also pick that the fifth dimension
is compact, in which case as long as the pitch (in the 4-d hypersurface) due to
the motion in the fifth dimension is small, the motion in the fifth dimension
will be unobservable.

The occurrence of the ‘acceleration’ term (right side in (2)) in writing
down the 4-d worldline equation from 5-d Kaluza theory, has been realized
earlier Schmutzer [29], Kovacs [30], Gegenberg and Kunstatter [31], Mash-
hoon, Liu and Wesson [32], Wesson, Mashhoon, Liu and Sajko [33] and in
the brane world scenario by Youm [34]. This author studied the effect of this
acceleration to restore causality in brane world [35].

The Raychaudhuri equation in 5-d spacetime, from 5-d theory, describing
the evolution of a collection of particles following their world line [2] in the
5-dimensional world, is given by

\[ \dot{\Theta} = -\frac{\Theta^2}{4} - 2\sigma^2 + 2\omega^2 - \frac{R(4)}{2} + \langle \dot{u}^\mu \rangle_{,\mu} \] (4)

where \( u^\mu = \frac{dx^\mu}{ds}, \) \( 2\sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu}, \) \( 2\omega^2 = \omega_{\mu\nu}\omega^{\mu\nu}, \) \( R(4) \) is the 4-dimensional Ricci scalar. The subscript ; stands for covariant derivative (the explicit form of this is given below). For simplicity the cosmological constant \( \Lambda \) is set to zero.

We have considered the 5-dimensional Raychaudhuri equation from the 5-dimensional theory, in terms of 4-dimensional quantities and \( dS \) refers to the 5-dimensional line element. This has the factor \( \frac{1}{4} \) in front of \( \Theta^2 \) instead of \( \frac{1}{3} \). This numeric has very little effect in the discussion. In (4), the last term involves \( \dot{u}^\mu = u^\mu_{,\nu}u^\nu \), the possible ‘acceleration’ (orthogonal to \( u^\mu \)) of the collection of particles. In view of (2), this term exists now. This term is:

\[ \langle \dot{u}^\mu \rangle_{,\mu} = \frac{a^2}{2} g^{\mu\rho} (\partial_\rho \frac{1}{g_{55}}) \] (5)

where \( D_\mu \) stands for the covariant derivative, namely \( D_\mu (\partial_\rho \frac{1}{g_{55}}) = \partial_\mu \partial_\rho \frac{1}{g_{55}} - \Delta_\mu^\sigma (\partial_\sigma \frac{1}{g_{55}}) \). Thus (4) is

\[ \dot{\Theta} = -\frac{\Theta^2}{4} - 2\sigma^2 + 2\omega^2 - \frac{3}{c^4} (\rho c^2 + 3p) - \frac{a^2}{2} g^{\mu\rho} D_\mu (\partial_\rho \frac{1}{g_{55}}). \] (6)

In this equation, the presence of the last term will be shown to induce expansion using \( \tilde{R}_{AB} = 0 \).

When compared with the quantum corrected Raychaudhuri equation of Das [20], we see that the role of the quantum correction \( \frac{h^2}{m^2} h^{ab} R_{a,b} \) of [20] (the other correction term in [20] has been set to zero in [20]) is played by

\[ -\frac{a^2}{2} g^{\mu\rho} D_\mu (\partial_\rho \frac{1}{g_{55}}), \] which is classical in origin.

The ‘acceleration’ term \( -\frac{a^2}{2} g^{\mu\rho} D_\mu (\partial_\rho \frac{1}{g_{55}}) \) can possibly induce expansion in the 5-dimensional space if this term turns out to be positive. We examine this for a static spherically symmetric ansatz for the 5-dimensional metric in (1) as

\[ (dS)^2 = e^\mu e^\nu (dt)^2 - e^\nu (dr)^2 - r^2 (d\theta)^2 + \sin^2 \theta (d\phi)^2 - \psi(r)(dx^5)^2, \] (7)
where $\mu, \nu$ are functions of $r = \sqrt{x^2 + y^2 + z^2}$ only and $\psi(r) = g_{55}(r)$ for ease of notation. In (7), $\mu(r), \nu(r)$ and $\psi(r)$ are unknown functions of $r$ to be determined by the 5 - d Einstein vacuum (Ricci flat) equations, $\tilde{R}_{AB} = 0$, $A, B = 0, 1, 2, 3, 5$. The choice of static ansatz (7) is partially motivated by [9] (proposing an emergent scenario in which the universe stays in a static past eternally - universe originating from Einstein static state).

For the metric in (7), $\tilde{R}_{tt}, \tilde{R}_{rr}, \tilde{R}_{\theta\theta}, \tilde{R}_{\phi\phi}, \tilde{R}_{55}$ can be evaluated and the results are given in the Appendix. Now, we show that the additional term in (5) due to 'acceleration' from $g_{55}(r)$ is always positive, that is

The additional (acceleration) term in (5), namely $-\frac{a^2}{2}g^{\mu\rho}D_\mu(\partial_\rho \frac{1}{g_{55}})$, is always positive for a spherically symmetric general solution with $\tilde{R}_{AB} = 0$.

Expanding the covariant derivative $D_\mu$ and setting $g_{55}(r) = -\psi(r)$ (for ease of notation), we have

$$-\frac{a^2}{2}g^{\mu\rho}D_\mu(\partial_\rho \frac{1}{g_{55}}) = -\frac{a^2}{2}g^{\mu\rho}\partial_\mu(\partial_\rho \frac{1}{g_{55}}) + \frac{a^2}{2}g^{\mu\rho} \triangle^\lambda_\mu \partial_\lambda (\partial_\rho \frac{1}{g_{55}}),$$

$$= -\frac{a^2}{\psi^3}g^{\mu\rho}\partial_\mu \psi \partial_\rho \psi + \frac{a^2}{2\psi^2}g^{\mu\rho}\partial_\mu \partial_\rho \psi - \frac{a^2}{2\psi^2}g^{\mu\rho} \triangle^\lambda_\mu \partial_\lambda (\partial_\rho \psi).$$

(8)

Since $\psi = \psi(r)$, we have the above expression as

$$-\frac{a^2}{2}g^{\mu\rho}D_\mu(\partial_\rho \frac{1}{g_{55}}) = -\frac{a^2}{\psi^3}g^{rr}(\psi')^2 + \frac{a^2}{2\psi^2}g^{rr} \psi'' - \frac{a^2}{2\psi^2}g^{\mu\rho} \triangle^r_{\mu\rho} \psi',$$

(9)

where $\psi' = \frac{d\psi}{dr}$; $\psi'' = \frac{d^2\psi}{dr^2}$. For the line element (metric) in (7), $g^{rr} = -e^{-\nu}$ and the connection coefficients $\triangle$ for (6), via (8) and (9), are evaluated in the Appendix. Then we have, using them

$$g^{\mu\rho} \triangle^r_{\mu\rho} = \frac{1}{2}e^{-\nu} \mu' - \frac{1}{2}e^{-\nu} \nu' + \frac{2}{r}e^{-\nu},$$

(10)

where $\mu' = \frac{d\mu}{dr}$; $\nu' = \frac{d\nu}{dr}$. Substituting (10) in (9), we find

$$-\frac{a^2}{2}g^{\mu\rho}D_\mu(\partial_\rho \frac{1}{g_{55}}) = \frac{a^2}{2\psi^2}e^{-\nu} \{ \frac{(\psi')^2}{\psi} - \frac{\psi''}{2} - \frac{\mu' \psi'}{4} + \frac{\nu' \psi'}{4} - \frac{\psi'}{r} \}.$$

(11)
The equation $\tilde{R}_{55} = 0$ in the Appendix, gives

$$\frac{-\psi''}{2} - \frac{\mu'\psi'}{4} + \frac{\nu'\psi'}{4} - \frac{\psi'}{r} = -\frac{(\psi')^2}{4\psi},$$

and substituting this in $\{...\}$ of (11), we obtain

$$-\frac{a^2}{2} g^{\mu\rho} D_\mu (\partial_\rho \frac{1}{g_{55}}) = \frac{3a^2}{4} e^{-\nu} \frac{(\psi')^2}{\psi^3}. \quad (12)$$

Since $\psi(r) > 0$ (so as to preserve the sign convention for the metric in (7)) and $e^{-\nu}$ is always positive, it follows that

$$\frac{3a^2}{4} e^{-\nu} \frac{(\psi')^2}{\psi^3} > 0. \quad (13)$$

The above result proves that

$$-\frac{a^2}{2} g^{\mu\rho} D_\mu (\partial_\rho \frac{1}{g_{55}}) > 0, \quad (14)$$

for a general spherically symmetric situation. Thus, the additional term in the Raychaudhuri equation is always positive for a general spherically symmetric situation and therefore is effectively repulsive in the 4-d spacetime, as the world lines tend to defocus. Further, the defocussing can be seen also from the following considerations. The effective 4-dimensional action after using $G_{AB}$ and $\tilde{R}$ becomes explicitly

$$S = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} \sqrt{g_{55}} \left( R - \frac{1}{g_{55}(r)} g^{\mu\nu}(\partial_\mu g_{55});_\nu \right), \quad (15)$$

where $R$ is 4-dimensional Ricci scalar and the second term is the kinetic energy term for $g_{55}$. This has wrong sign and hence eventually leads to defocussing.

With the presence of the last term in (5) whose contribution has been shown to be positive in general, the focusing of the world lines in 5 - dimensions is inhibited and could cause non-focusing of world lines in 5-dimensional world. Before proceeding to cosmological implications, the ‘geodesic deviation equation’ gets modified by the acceleration term in (2) as

$$\frac{D^2(\delta x^\mu)}{Ds^2} = -R^\mu_{\lambda\rho} \frac{dx^\lambda}{ds} \frac{dx^\rho}{ds} + f^\rho_{\nu} \delta x^\nu, \quad (16)$$
where \( f^\mu_{\nu} = -\frac{a^2}{2} g^{\mu\lambda} D_\nu (\partial_\lambda \frac{1}{g_{55}}) \), showing the deviation \( \delta x^\mu \neq 0 \). The observed deviation acceleration would remain finite. When \( \delta x^\mu \neq 0 \), the world lines are inhibited to converge, supporting the defocussing feature from the considerations of Raychaudhuri equation.

We can similarly consider the analogous spherically symmetric time-dependent domain (also sometimes known as the T-domain) by extending the study to the following line element:

\[
\frac{d^2 x^\alpha}{dS^2} + \Delta^\alpha_{\beta\gamma} \frac{dx^\beta}{dS} \frac{dx^\gamma}{dS} = \frac{1}{2} (\partial_\beta g_{55}) g^{\alpha\beta} \frac{a^2}{(g_{55})^2},
\]

(18)

where the constant \( a \) is defined as in the previous section. Note though that different metric functions from before will appear from the index sum due to the time dependence of the metric. Also, an analogous result to the expression just before equation (5) is also calculated:

\[
u^\alpha_{;\beta} \nu^\beta = -\frac{a^2}{2} g^{\alpha\rho} \left( \partial_\rho \frac{1}{g_{55}} \right).
\]

(19)

This leads to the investigation of the positivity (following a proof similar to the previous one) of the following:

\[-\frac{a^2}{2} g^{\alpha\beta} D_\beta \left( \partial_\rho \frac{1}{g_{55}} \right) = \frac{3}{4} \frac{a^2}{\psi^3} e^{-\mu(t)},
\]

(20)

which is indeed positive since \( \psi > 0 \) due to the Lorentzian structure demanded of the metric. (Here the over-dot represents \( \partial_t \).)

3. Concluding remarks
We have shown that five dimensional world line defocussing may be achieved by considering five dimensional spacetime with the '55' component of $G_{AB}$ a Kaluza scalar. Qualitatively this is similar to the defocussing which occurs when considering some effective quantum corrected models. This shows that such a defocussing effect can also have a purely classical origin which is completely in five dimensional general relativity.

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Appendix

For the metric in (7), the non - vanishing five dimensional Christoffel connections are:

\[ \nabla^t_{rt} = \frac{\mu'}{2}; \quad \nabla^r_{rr} = \frac{\nu'}{2}; \quad \nabla^r_{ut} = \frac{1}{2} e^{(\mu - \nu)} c^2 \mu'; \]

\[ \nabla^r_{\theta\theta} = -e^{-\nu} r; \quad \nabla^r_{\phi\phi} = -e^{-\nu} r \sin^2 \theta; \quad \nabla^r_{55} = -\frac{1}{2} e^{-\nu} \psi'; \]

\[ \nabla^\theta_{r\theta} = \frac{1}{r}; \quad \nabla^\theta_{\phi\phi} = -\sin \theta \cos \theta; \quad \nabla^\phi_{r\phi} = \frac{1}{r}; \quad \nabla^\phi_{\theta\phi} = \cot \theta; \quad \nabla^5_{r5} = \frac{\psi'}{2\psi}, \]

(A1)

where the prime \( ' \) stands for differentiation with respect to \( r \). The Ricci tensor

\[ \tilde{R}_{AB} = \partial_C \Delta^C_{AB} - \partial_B \Delta^C_{AC} + \Delta^C_{DC} \Delta^D_{AB} - \Delta^C_{DB} \Delta^D_{CA}, \]

gives the components as

\[ \tilde{R}_{tt} = \frac{1}{2} e^{(\mu - \nu)} c^2 \left\{ \mu'' - \frac{\mu' \nu'}{2} + \frac{\mu'^2}{2} + \frac{2 \mu'}{r} + \frac{\mu' \psi'}{2 \psi} \right\}, \]

\[ \tilde{R}_{rr} = -\frac{\mu''}{2} + \frac{\mu' \nu'}{4} + \frac{\nu'}{4} - \frac{\mu'^2}{4} - \frac{\psi'}{2 \psi} + \frac{\nu' \psi'}{4 \psi} + \frac{\psi'^2}{4 \psi^2}, \]

\[ \tilde{R}_{\theta\theta} = 1 - e^{-\nu} - re^{-\nu} \left\{ \frac{\mu'}{2} - \frac{\nu'}{2} + \frac{\psi'}{2 \psi} \right\}, \]

\[ \tilde{R}_{\phi\phi} = \sin^2 \theta \ \tilde{R}_{\theta\theta}, \]

\[ \tilde{R}_{55} = e^{-\nu} \left\{ -\frac{\psi''}{2} - \frac{\mu' \psi'}{4} + \frac{\nu' \psi'}{4} - \frac{\psi'}{r} + \frac{\psi'^2}{4 \psi} \right\}, \]

(A2)
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