Abstract

All 2nd order classical and quantum superintegrable systems in 3 dimensional conformally flat spaces with nondegenerate (i.e., 4-parameter) potentials have been classified and great progress has been made on the classification of semidegenerate (i.e., 3-parameter) potentials. By definition these admit 5 functionally linearly independent symmetry operators, i.e., they are not only linearly independent in the usual sense but also if the coefficients are allowed to depend on the spatial variables. However 2nd order superintegrable systems with at least 3-parameter potentials and 5 symmetry operators that are functionally linearly dependent have never been classified. The best known such example is the Calogero system with 3 bodies on a line. Here we work out the structure theory for such systems in conformally flat spaces and show, for example, that they always admit a 1st order symmetry. We give a complete classification for all such systems in 3-dimensional flat space.
1 Introduction

We recall some basic facts and results about conformally flat superintegrable systems. An $n$-dimensional complex Riemannian space is conformally flat if and only if it admits a set of local coordinates $\{x_1, \ldots, x_n\}$ such that the contravariant metric tensor takes the form $g^{ij} = \delta^{ij}/\lambda(x)$ \[1, 2\]. A classical superintegrable system $\mathcal{H} = \sum g^{ij}p_ip_j + V(x)$ on the phase space of this manifold is one that admits $2n-1$ functionally independent generalized symmetries (or constants of the motion) $S_k$, $k = 1, \ldots, 2n-1$ with $S_{(1)} = \mathcal{H}$ where the $S_k$ are polynomials in the momenta $p_j$. It is easy to see that $2n-1$ is the maximum possible number of functionally independent symmetries and, locally, such (in general nonpolynomial) symmetries always exist. The system is second order superintegrable if the $2n-1$ functionally independent symmetries can be chosen to be quadratic in the momenta. Second order superintegrable systems, though complicated, are tractable because standard orthogonal separation of variables techniques are associated with second order symmetries, and these techniques can be brought to bear. Thus we concentrate on second-order superintegrable systems in which the symmetries take the form $\mathcal{S} = \sum a^{ij}(x)p_ip_j + W(x)$, quadratic in the momenta.

For a classical 3D system on a conformally flat space (note that all 2D spaces are conformally flat) we can always choose local coordinates $\{x, y, z\}$, not unique, such that the Hamiltonian takes the form $\mathcal{H} = (p_1^2 + p_2^2 + p_3^2)/\lambda(x, y, z) + V(x, y, z)$. This system is second order superintegrable with potential $V = V(x, y, z, \alpha, \beta, \gamma) = \alpha V^\alpha(x) + \beta V^\beta(x) + \gamma V^\gamma(x)$ if it admits 5 functionally independent quadratic constants of the motion (i.e., generalized symmetries)

$$S_k = \sum a^{ij}_{(k)}p_ip_j + W_{(k)}(x, y, z, \alpha, \beta, \gamma) = S^0_k + W_k, \quad k = 1, \ldots, 5. \quad (1)$$

Here the set $\{V^\alpha, V^\beta, V^\gamma\}$ must be functionally independent and we ignore the additive constant. (We call this a 3-parameter potential.) Furthermore the 5 constants of the motion must be functionally linearly independent, i.e., the equation

$$\sum_{k=1}^{5} f_{(k)}(x)S^0_k(x) = 0 \quad (2)$$

is satisfied if and only if $f_{(k)}(x) = 0$ for $k = 1, \ldots, 5$. If equation (2) is satisfied for functions $f_{(k)}(x)$ not identically 0, the set of constants of the motion are functionally linearly dependent (FLD).
For 2nd order superintegrable systems in 3 dimensions that are functionally linearly independent, the 4-parameter systems have all been classified, \[6, 7\] and there has been considerable progress on the 3-parameter systems \[4, 8\]. However, little has been done to classify superintegrable systems in 3 dimensions that are FLD. The best known such system is the generalized 3-body Calogero system on the line. In this paper we derive structure results for all 2nd order superintegrable FLD systems on conformally flat real or complex spaces that have potentials that depend on 2 functionally independent variables (the maximum possible). (For the analogous 2nd order 2-dimensional FLD systems the answer is known: there is only one such family of systems, \[9\].)

The paper is organized as follows: in §2 and §3 we present the generalized Calogero system and a system on 3-dimensional Minkowski space as examples. In §4 we present structure results for all FLD systems on conformally flat spaces. The most important result is that all such systems admit a 1st order constant of the motion. In §5 we work out the classification of all 3-dimensional superintegrable FLD systems in flat space, including the structure of the symmetry algebras for these systems. In §6 we summarize the corresponding result for 3-dimensional FLD systems on the complex 3-sphere. In §7 we present some conclusions. Here all of our systems are classical. However the quantum analogs follow easily by symmetrization of the symmetry operators and there is a 1-1 matching of the Hamiltonians.

2 An FLD example: generalized Calogero potential

This potential takes the form \[10, 11, 12, 13, 14, 15, 16, 17\]

\[V = \frac{\alpha}{(x-y)^2} + \frac{\beta}{(y-z)^2} + \frac{\gamma}{(z-x)^2}. \tag{3}\]

Let us consider the system of symmetries defining the system with potential \(V\). A basis for the space of symmetries is (using \(J_{12} = xp_2 - yp_1, J_{23} = yp_3 - zp_2, J_{13} = zp_1 - xp_3\)),

\[
S_{(1)} = \mathcal{H} = p_1^2 + p_2^2 + p_3^2 + V, \quad S_{(2)} = (p_1 + p_2 + p_3)^2, \quad S_{(3)} = J_{12}^2 + J_{23}^2 + J_{13}^2 + W_3, \\
S_{(4)} = p_1(J_{13} - J_{12}) + p_2(J_{12} - J_{23}) + p_3(J_{23} - J_{13}) + W_4, \\
S_{(5)} = J_{12}J_{13} + J_{23}J_{12} + J_{13}J_{23} + W_5,
\]

where the potential terms \(W_i\) contain the parameters \(\alpha, \beta, \gamma\). In this case, the Bertrand-Darboux equations \[5, 6\] for each symmetry \(S_{(k)} = \sum_{ij} a_{(k)}^{ij} p_ip_j + \]

\[\sum_{ij} a_{(k)}^{ij} p_ip_j + \]

\[\]
$W_k$ of $\mathcal{H}$ are

\[ V_x + V_y + V_z = 0, \quad (x - y)V_{xy} + (z - y)V_{yz} - V_x + 2V_y - V_z = 0, \quad (x - z)V_{xz} + (y - z)V_{yz} - V_x - V_y + 2V_z = 0, \]

and their differential consequences. The complete system of equations is in involution and a particular solution is determined uniquely by choosing $V_y, V_z, V_{yz}$ at a regular point. Thus we have a 3-parameter potential.

What is important to notice here is the occurrence of the first order condition $V_x + V_y + V_z = 0$ for the potential as a consequence of the Bertrand-Darboux equations. Thus the potential is a function of only two variables, impossible for nondegenerate potentials. To understand this, observe the relation

\[ (x + y + z)^2 S_{(1)}^0 - (x^2 + y^2 + x^2)S_{(2)}^0 + 2S_{(3)}^0 - 2(x + y + z)S_{(4)}^0 - 2S_{(5)}^0 = 0 \]

obeyed by the purely quadratic terms in the symmetries, i.e., we have set $S_{(k)} = S_{(k)}^0 + W_i$. This means that the 5 functionally independent symmetries $S_{(k)}$ are functionally linearly dependent.

### 3 A Minkowski space FLD example

Here

\[ \mathcal{H} = p_1^2 + p_2^2 + p_3^2 + \alpha(x - z) + \beta(y + iz) + \gamma(y + iz)^2, \]

which admits the 1st order symmetry

\[ J = p_1 - ip_2 + p_3 \]

and the 2nd order symmetries [4]

\[ S_{(1)} = \mathcal{H} = p_1^2 + p_2^2 + p_3^2 + \alpha(x - z) + \beta(y + iz) + \gamma(y + iz)^2, \]

\[ S_{(2)} = J^2, \quad S_{(3)} = p_1^2 + \alpha x, \quad S_{(4)} = (-ip_2 + p_3)p_1 + (p_3 - ip_2)^2 + \frac{\alpha}{2}(iy - x - z), \]

\[ S_{(5)} = 2i xp_2 p_3 + z p_1^2 + x p_2^2 - x p_3^2 - i y p_1^2 + i x p_1 p_2 - y p_1 p_2 - i z p_1 p_2 - x p_1 p_3 + z p_1 p_3 \]

\[ - i y p_1 p_3 - \frac{i}{2} \alpha y z - \frac{i}{2} \alpha x y + \frac{1}{4} \alpha x^2 + \frac{1}{2} \alpha x z - \frac{1}{4} \alpha y^2 + \frac{1}{4} \alpha z^2, \]

The 5 generators are linearly independent and satisfy

\[ (iy - z)S_{(2)}^0 + (-iy + x + z)S_{(4)}^0 + S_{(5)}^0 = 0, \]

where as before $S_{(k)}^0$ is the quadratic momentum part of the symmetry $S_{(k)}$, so the system is FLD.
4 Some theory

Functional linear dependence of a functionally independent maximal set of symmetries is hard to achieve. We recall the following result where the system need not be superintegrable [3]:

**Theorem 1** Let the linearly independent set \( \{ \mathcal{H} = \mathcal{S}_{(1)}, \mathcal{S}_{(2)}, \ldots, \mathcal{S}_{(t)} \} \), \( t > 2 \) be a functionally linearly dependent basis of 2nd order symmetries for the system \( \mathcal{H} = (p_1^2 + p_2^2 + p_3^2)/\lambda(x) + V = \mathcal{H}^0 + V \) with nontrivial potential \( V \), i.e., there is a relation \( \sum h c^{(h)}(x) S_{(h)}^0 \equiv 0 \) in an open set, where not all \( c^{(h)}(x) \) are constants, and no such relation holds for the \( c^{(h)} \) all constant, except if the constants are all zero. (Here \( S_{(r)} = S_{(0)} + W_r \) where the \( W_r \) are the potential terms.) Then the potential must satisfy a first order relation \( AV_1 + BV_2 + CV_3 = 0 \) where not all of the functions \( A, B, C \) are zero.

**Proof:** By relabeling, we can express one of the quadratic parts of the constants of the motion \( S_{(0)}^{(0)} \) as a linear combination of a functionally independent subset \( \{ \mathcal{S}_{(r)}^{(0)}, \ldots, \mathcal{S}_{(r)}^{(r)}, 1 \leq r \leq 4 \} \): \( \mathcal{S}_{(r)}^{(0)} = \sum_{\ell=1}^{r} c^{(\ell)}(x) \mathcal{S}_{(0)}^{(0)} \). Taking the Poisson bracket of both sides of this equation with \( (p_1^2 + p_2^2 + p_3^2)/\lambda \) and using the fact that each of the \( \mathcal{S}_{(h)} \) is a constant of the motion, we obtain the identity

\[
\sum_{\ell=1}^{r} \sum_{i,j=1}^{3} (\partial_{x_k} c^{(\ell)}(x)) a^{ij}_{(\ell)} p_i p_j p_k = 0
\]

where \( (x, y, z) \equiv (x_1, x_2, x_3) \). It is straightforward to check that this identity can be satisfied if and only if the functions

\[
c^{ij}_{(r)} = \sum_{\ell=1}^{r} (\partial_{x_k} c^{(\ell)}(x)) a^{ij}_{(\ell)}, \quad 1 \leq i, j, k \leq 3
\]

satisfy the equations

\[
c^{ij}_{(r)} = 0, \quad c^{ij}_{(r)} + 2c^{ij}_{(r)} = 0, \quad (i \neq j), \quad c^{12}_{(r)} + c^{23}_{(r)} + c^{31}_{(r)} = 0.
\]

Note that \( c^{ij}_{(r)} = c^{ij}_{(r)} \). Corresponding to each of the basis symmetries \( \mathcal{S}_h \) there is a linear set \( \mathcal{C}_h \equiv 0 \) of Bertrand-Darboux equations. A straightforward substitution into the identity \( C_0 - \sum_{\ell=1}^{r} c^{(\ell)}(x) C_\ell = 0 \) yields the relation

\[
\begin{pmatrix}
c^{12}_{1} - c^{11}_{1} \\
c^{11}_{1} - c^{11}_{2} \\
c^{31}_{2} - c^{21}_{3}
\end{pmatrix} V_1 + \begin{pmatrix}
c^{22}_{1} - c^{21}_{1} \\
c^{32}_{1} - c^{12}_{3} \\
c^{32}_{2} - c^{22}_{3}
\end{pmatrix} V_2 + \begin{pmatrix}
c^{32}_{1} - c^{31}_{2} \\
c^{33}_{1} - c^{13}_{3} \\
c^{33}_{2} - c^{23}_{3}
\end{pmatrix} V_3 = 0.
\]
These first order differential equations for the potential cannot all vanish identically. Indeed if they did all vanish then we would have the conditions
\[ c_{12}^{1} = c_{21}^{1}, \quad c_{11}^{3} = c_{31}^{3}, \quad c_{31}^{2} = c_{21}^{2}, \quad c_{21}^{3} = c_{31}^{1}, \quad c_{22}^{3} = c_{32}^{2}, \quad c_{32}^{1} = c_{23}^{1}. \]
These conditions, together with conditions (7) show that \( c_{ij}^{jk} = 0 \) for all \( i, j, k \). Thus we have
\[ \sum_{\ell=1}^{r} (\partial_{x_k} c_{ij}^{(\ell)}) a_{ij}^{(\ell)} = 0, \quad 1 \leq i, j, k \leq 3. \]
Since the set \( \{ S_{(1)}, \ldots, S_{(r)} \} \), is functionally linearly independent, we have \( \partial_{x_k} c_{ij}^{(\ell)} = 0 \) for \( 1 \leq k \leq 3, \quad 1 \leq \ell \leq r. \) Hence the \( c_{ij}^{(\ell)} \) are constants, which means that \( S_{(0)} - \sum_{\ell=1}^{r} c_{ij}^{(\ell)} S_{(\ell)} = 0. \) Thus the set \( \{ S_{(0)}, \ldots, S_{(s)} \} \) is linearly dependent. This is a contradiction!

This shows that the potential function for any system, superintegrable or not, with a basis of symmetries that is functionally linearly dependent must satisfy at least one nontrivial first order partial differential equation
\[ AV_1 + BV_2 + CV_3 = 0 \]
where the functions \( A, B, C \) are parameter free. This means that all such potentials depend on either one or two coordinates.

**Lemma 1** Equations (7) imply
\[ \partial_{x_i} (c_{ij}^{ij} - c_{ji}^{ij}) = 0, \quad \partial_{x_i} (c_{ij}^{ik} - c_{ji}^{ik}) = 0. \]

A new result is:

**Theorem 2** Under the hypotheses of Theorem 1 there exists a 1st order symmetry \( \mathcal{J} \) for \( \mathcal{H} \), i.e., \( \{ \mathcal{J}, \mathcal{H} \} = \{ \mathcal{J}, V \} = 0. \)

**Proof:** Let
\[ \mathcal{J}_1 = (c_{12}^{12} - c_{12}^{11}) p_1 + (c_{22}^{12} - c_{22}^{12}) p_2 + (c_{13}^{23} - c_{13}^{23}) p_3 = \mathcal{J}_1^x p_1 + \mathcal{J}_1^y p_2 + \mathcal{J}_1^z p_3, \]
so that the first of equations (8) is \( \{ \mathcal{J}, V \} = 0. \) From equations (7) and Lemma 1 we can verify that
\[ \{ \mathcal{J}, \mathcal{H} \} = - \left[ (c_{12}^{12} - c_{12}^{11}) \lambda_1 + (c_{22}^{12} - c_{22}^{12}) \lambda_2 + (c_{13}^{23} - c_{13}^{23}) \lambda_3 \right] \mathcal{H}^0 = - \frac{1}{\lambda} \left[ 3c_{12}^{12} \lambda_1 - 3c_{22}^{12} \lambda_2 + (c_{13}^{23} - c_{13}^{23}) \lambda_3 \right] \mathcal{H}^0, \]
so either \( \mathcal{J}_1 = 0 \) or \( \mathcal{J}_1 \) is a conformal symmetry of \( \mathcal{H}^0 \). However, from Lemma 1 we see that
\[ \partial_x \mathcal{J}_1^x = \partial_y \mathcal{J}_1^y = \partial_z \mathcal{J}_1^z = 0. \]
The first order conformal symmetries of $\mathcal{H}^0$ are the same as for the case $\lambda = 1$, and the only such symmetries that satisfy the requirements (9) are linear combinations of $p_1, p_2, p_3$ and

$$J_{12} = xp_2 - yp_1, \quad J_{23} = yp_3 - zp_2, \quad J_{31} = zp_1 - xp_2$$

and these would be actual symmetries of $\mathcal{H}_0$ (True conformal symmetries, such as $xp_1 + yp_2 + zp_3$ fail this test.) Thus either $J_1$ vanishes or it is a 1st order symmetry of $\mathcal{H}$.

Analogous constructions and conclusions can be obtained for the 2nd and 3rd of equations (8). However, at least one of these equation is nonzero. □

**Corollary 1** Under the hypotheses of Theorem 4 there exists a 1st order symmetry $J$ of $\mathcal{H}$ that can be written in the form

$$a_1p_1 + a_2p_2 + a_3p_3 + a_4(xp_2 - yp_1) + a_5(yp_3 - zp_2) + a_6(zp_1 - xp_2)$$

for some constants $a_j$, not all zero.

Since any Euclidean coordinate transformation applied to the Hamiltonian $\mathcal{H}$ takes it into one of similar form

$$\tilde{\mathcal{H}} = \frac{\tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2}{\lambda} + \tilde{V},$$

without loss of generality, we can assume that, up to conjugacy, $J$ takes one of the five canonical forms:

$$p_1, \quad p_1 + ip_2, \quad xp_2 - yp_1, \quad (xp_2 - yp_1) + i(yp_3 - zp_2), \quad (xp_2 - yp_1) + i(yp_3 - zp_2) + p_3 + ip_1.$$

(10)

### 5 Euclidean space

We first study the possible FLD 2nd order superintegrable systems in 3D complex Euclidean space. By relabeling, we can express one of the quadratic parts of the constants of the motion $\mathcal{S}^0_{(0)}$ as a linear combination of the quadratic parts of the remaining 4 generators

$$\mathcal{S}^0_{(0)} = \sum_{\ell=1}^{4} c^{(\ell)}(x)\mathcal{S}^0_{(\ell)}.$$  

(11)
We limit ourselves to the maximal case where the expansion (11) is unique. The generators \( S_0^0, S_0^1, S_0^2, S_0^3, S_0^4 \) are polynomials in \( x, y, z \) of order at most 2 and are linearly independent. Thus we can solve for the expansion coefficients in the form \( e^{(\ell)}(x, y, z) = s^{(\ell)}(x, y, z), \ell = 1, \ldots, 4 \) where \( s^{(0)}, s^{(1)}, \ldots, s^{(4)} \) are polynomials in \( x, y, z \) of order at most 2. It follows that

\[
\sum_{a_1,a_2,a_3} A(a_1, a_2, a_3)x^{a_1}y^{a_2}z^{a_3} \equiv s^{(0)}S_{(0)}^{0} - \sum_{r=1}^{4} s^{(r)}S_{(r)}^{0} = 0, \quad (12)
\]

where each coefficient \( A(a_1, a_2, a_3) \) must vanish. In particular, the sum of all terms homogeneous of degree \( n \) must vanish for each \( n = 0, \ldots, 4 \):

\[
B(n) \equiv \sum_{a_1+a_2+a_3=n} A(a_1, a_2, a_3)x^{a_1}y^{a_2}z^{a_3} = 0.
\]

Each of the generators \( S_{(r)}^{0} \) is a linear combination of terms \( J_{ij}J_{kl} \), (order 2), \( J_{ij}p_k \), (order 1) and \( p_ip_j \), (order 0).

Since we have assumed that the expansion (11) is unique, there must be only 1 term \( B(N) \) that is not identically 0 and each \( S_{(r)}^{0} \) is homogeneous of degree 0, 1, or 2. Thus each \( s^{(r)} \) must be homogeneous of degree \( b \) and each \( S_{(r)}^{0} \) must be homogeneous of degree \( c = 0, 1, 2 \) where \( b + c = N \). This greatly restricts the possibilities for (12).

### 5.1 First case: \( J = p_1 \)

Here the centralizer of \( J \) is the group generated by translation in \( x, y, z \) and rotation about the \( x \)-axis. We can use this freedom to simplify the computation. Since \( p_1 \) is a symmetry the potential must be of the form \( V(y, z) \). Writing a 2nd order symmetry in the form

\[
S = F_{11}(x, y, z) p_1^2 + F_{22}(x, y, z) p_2^2 + F_{33}(x, y, z) p_3^2 + F_{12}(x, y, z) p_1 p_2 + F_{13}(x, y, z) p_1 p_3 + F_{23}(x, y, z) p_2 p_3 + F_0(x, y, z)
\]

and requiring that \( \{S, \mathcal{H}\} = 0 \) we can solve for the \( F_{jk} \) to get

\[
F_{11} = \frac{1}{2}c_4z^2 + (c_2y + c_5)z + \frac{1}{2}c_1y^2 + c_3y + c_6,
F_{12} = c_1z^2 + (-c_2x - c_7y + c_16)z + (-c_1y - c_3)x - c_8y + c_{17},
F_{13} = c_7y^2 + (-c_{12}z - c_2x + c_{18})y + (-c_4z - c_3)x - c_{13}z + c_{19},
F_{22} = \frac{1}{2}c_9z^2 + (c_7x + c_{10})z + \frac{1}{2}c_1x^2 + c_8x + c_{11},
F_{23} = c_2x^2 + (-c_{12}z - c_7y - c_{16} - c_{18})x + (-c_9z - c_{10})y - c_{14}z + c_{20},
F_{33} = \frac{1}{2}c_9y^2 + (c_{12}x + c_{14})y + \frac{1}{2}c_4x^2 + c_{13}x + c_{15} \quad (13)
\]
where the $c_j$ are constants to be determined. In addition we obtain a series of equations for the first derivatives $\partial_x F_0, \partial_y F_0, \partial_z F_0$, which lead to Bertrand-Darboux equations for $V(y, z)$. At the end we have to find 5 linearly independent solutions for $S$ and verify that they admit one functionally linearly dependent solution.

The adjoint action $S \to \{ p_1, S \} \equiv \text{Ad}_{p_1}S$ will map the 5-dimensional space of a solution set into itself. Since this action is essentially differentiation with respect to $x$, it is clear that $\text{Ad}_{p_1}^3 = 0$. Thus the possible actions on a generalized eigenbasis are

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

(14)

5.1.1 Form (14a)

We first look at the possibilities for form (14a). Restricting to symmetries that generate a chain of 3 we find

$$
\mathcal{L} = \left( \frac{1}{2}c_4z^2 + c_2yz + \frac{1}{2}c_1y^2 \right)p_1^2 + \left( (-c_1xy - c_2xz)p_1p_2 \\
+ (-c_2xy - c_4xz)p_1p_3 + \frac{1}{2}c_1x^2)p_2^2 + c_2x^2)p_2p_3 + \frac{1}{2}c_4x^2)p_3^2 + W,
\right.
$$

where we omit the expressions for the functions $W, W_1, W_2$.

The symmetries that are annihilated by $\text{Ad}_{p_1}$ take the form

$$
\mathcal{K} = k_6p_1^2 + \left( \frac{1}{2}k_9z^2 + k_{10}z + k_{11} \right)p_2^2 + \left( (-k_9yz - k_{10}y - k_{14}z + k_{20})p_2p_3 + \frac{1}{2}k_9y^2 + k_{14}y + k_{15} \right)p_3^2 + U
$$

(16)
where the \( k_j \), analogous to \( c_j \), are constants to be determined.

Case: \( \mathcal{L} \) of order 2. In this case we have \( c_{13} = c_{18} = c_5 = c_{19} = c_{10} = c_8 = c_{11} = c_{14} = c_{16} = c_{20} = c_3 = c_{15} = c_{17} = c_6 = 0 \), and \( \mathcal{L}_2 \) takes the form
\[
\mathcal{L}_2 = c_1 p_2^2 + 2c_2 p_2 p_3 + c_4 p_3^2 + W_2.
\]
A very special case is that where, by a rotation if necessary, \( \mathcal{L}_2 \) takes the form where \( c_4 = c_1 \neq 0 \), \( c_2 = 0 \). Thus we have
\[
\mathcal{L}_2 = c_1 (\mathcal{H} - \mathcal{J}^2).
\]
Always \( \mathcal{H} \) can be assumed to be a basis symmetry, so to achieve form (14a) we have to select a symmetry \( K \) that is linearly independent of the 4 forms already exhibited.

If we choose \( K \) of order 2, so
\[
K = k_{10} (zp_2 - yp_3)^2,
\]
we can verify FLD and find the solution
\[
V(y, z) = b y^2 + \frac{F(z/y)}{y^2}, \quad (17)
\]
for \( F \) an arbitrary function. This is the class to which the Calogero potential (3) belongs. Indeed, under the Jacobi transformation
\[
x = \frac{1}{\sqrt{3}} (r_1 + r_2 + r_3), \quad y = \frac{1}{\sqrt{2}} (r_2 - r_1), \quad z = \frac{1}{\sqrt{6}} (2r_3 - r_2 - r_1), \quad (18)
\]
we obtain the Calogero potential (3) in variables \( r_1, r_2, r_3 \) by choosing
\[
F(w) = \frac{\beta}{2(1 - \sqrt{3}w)^2} + \frac{\gamma}{2(1 + \sqrt{3}w)^2}
\]
and \( b = \alpha/2 \).

If we choose \( K \) of order 1, so that
\[
K = k_{10} z p_2^2 - (k_{10} y + k_{14} z) p_2 p_3 + k_{14} y p_3^2 + U
\]
where \( |k_{10}| + |k_{14}| > 0 \), we can verify FLD and find the solution
\[
V(y, z) = b_1 y^2 + \frac{1}{(k_{10} y - k_{14} z)^2} \left( b_3 + \frac{(k_{10} z + k_{14} y) b_2}{\sqrt{y^2 + z^2}} + \frac{(2y^2 - z^2) b_1 k_{14}^2 + 2b_1 k_{10} k_{14} y z}{y^2} \right), \quad (19)
\]
where \( b_1, b_2, b_3 \) are arbitrary parameters. Similarly, applying the Jacobi transformation (18) to (19) we can obtain a solution adapted to translation invariance.

If we choose \( K \) of order 0, there is no 3 parameter solution. The other possibilities for \( \mathcal{L} \) of order 2 are that 1) \( \mathcal{L}_2 \) can be transformed so that \( c_2 = c_4 = 0 \) and the one chains are \( \mathcal{H} \) and \( p_1^2 \), in which case there is no 3-parameter potential, and 2) \( \mathcal{L}_2 \) can be transformed so that \( c_2 = ic_1, c_4 = -c_1 \) and the one chains are \( \mathcal{H} \) and \( p_1^2 \), which is not FLD.

Case: \( \mathcal{L} \) of order 1 or 0. This is incompatible with form (14a).
5.1.2 Form (14b)

Here there is one chain of length 3 and one chain of length 2. The general form for the chain of length 3 is \((15)\) again. The general form for a chain of length 2 is

\[
L' = ((k_{12}z^2 - k_7yz + k_{16}z - k_8y)p_2 + (-k_{12}yz + k_7y^2 - k_{13}z + k_{18}y)p_3)p_1
\]

\[
+ (k_7x + k_8x)p_2' + (-k_{12}x + k_7xy - k_{16}x - k_{18}x)p_3p_2 + (k_{12}xy + k_{13}x)p_2' + W_3
\]

\[
L'_2 = \text{Ad}_{p_1}L'_1 = (-k_7z - k_8)p_2' + (k_{12}z + k_7y + k_{16} + k_{18})p_2p_3 + (-k_{12}y - k_{13})p_2' + W_4
\]

**Case:** \(L\) of order 2, \(L'_1\) of order 2. In this case we have

\[
c_{13} = c_{18} = c_5 = c_{19} = c_{10} = c_8 = c_{11} = c_{14} = c_{16} = c_{20} = c_3 = c_{15} = c_{17} = c_6 = 0,
\]

\[
L_2 = c_1p_2^2 + 2c_2p_2p_3 + c_4p_3^2 + W_2, \text{ and } k_{16} = k_8 = k_{17} = k_{13} = k_{18} = k_9 = k_{10} = k_{11} = k_{14} = k_{20} = k_{15} = 0, \text{ whereas } L'_1 \text{ takes the form } L'_1 = -k_7zp_2' + (k_{12}z + k_7y)p_2p_3 - k_{12}yp_2' + W_4. \text{ Since both } H \text{ and } p_2' \text{ are of order 0, and since they both must be included in form } (14b), \text{ this case cannot occur.}
\]

**Case:** \(L\) of order 2, \(L'_1\) of order 1. In this case we have

\[
c_{13} = c_{18} = c_5 = c_{19} = c_{10} = c_8 = c_{11} = c_{14} = c_{16} = c_{20} = c_3 = c_{15} = c_{17} = c_6 = 0,
\]

\[
L_2 \text{ takes the form } L_2 = c_1p_2^2 + 2c_2p_2p_3 + c_4p_3^2 + W_2, \text{ and } k_{12} = k_7 = k_{17} = k_9 = k_10 = k_11 = k_{12} = k_9 = k_{20} = k_{15} = 0, \text{ whereas } L'_1 \text{ vanishes.}
\]

Thus \(L'_1\) cannot be of order 1 and this case cannot occur.

**Case:** \(L\) of order 1 or 0. This case cannot occur since \(L_2\) vanishes.

Thus we conclude that form \((14b)\) does not occur.

5.1.3 Form (14c)

Now we have 2 chains of length 2 and one of length 1. The general form for a chain of length 2 is

\[
L_1 = ((c_{12}z^2 - c_7yz + c_{16}z - c_8y)p_2 + (-c_{12}yz + c_7y^2 - c_{13}z + c_{18}y)p_3)p_1 \tag{21}
\]

\[
+ (c_7x + c_8x)p_2' + (-c_{12}xz - c_7xy - c_{16}x - c_{18}x)p_3p_2 + (+c_{12}xy + c_{13}x)p_2' + W_1
\]

\[
L_2 = \text{Ad}_{p_1}L = (-c_7z - c_8)p_2' + (c_{12}z + c_7y + c_{16} + c_{18})p_2p_3 + (-c_{12}y - c_{13})p_2' + W_2
\]

The other chain of length 2 is

\[
L'_1 = ((k_{12}z^2 - k_7yz + k_{16}z - k_8y)p_2 + (-k_{12}yz + k_7y^2 - k_{13}z + k_{18}y)p_3)p_1 \tag{22}
\]

\[
+ (k_7x + k_8x)p_2' + (-k_{12}xz - k_7xy - k_{16}x - k_{18}x)p_3p_2 + (k_{12}xy + k_{13}x)p_2' + W_3
\]

\[
L'_2 = \text{Ad}_{p_1}L'_1 = (-k_7z - k_8)p_2' + (k_{12}z + k_7y + k_{16} + k_{18})p_2p_3 + (-k_{12}y - k_{13})p_2' + W_4
\]

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The general form for a chain of length 1 is

\[ L'' = \frac{1}{2} z(h_9 z + 2h_{10})p_2^2 + (-h_9 y z - h_{10} y - h_{14} z + h_{20})p_3 p_2 \quad (23) \]

+ \frac{h_0}{2} p_1^2 + h_1 p_2^2 + \frac{1}{2} h_0 y^2 + h_{14} y + h_{15})p_3^2 + W_5

It is not possible for both \( L_1 \) and \( L'_1 \) to be of order 2 since then there would only be one symmetry of order 0, not enough to contain both \( \mathcal{H} \) and \( p_1^2 \).

**Case:** \( L_1 \) of order 2, \( L'_1 \) of order 1. This implies that \( L'' \) must be of order 0, so that \( \mathcal{H} \) and \( p_1^2 \) can be contained in the spanning set. One of the symmetries is \(-c_7 z p_2^2 + (c_{12} z + c_7 y)p_2 p_3 - c_{12} y p_3^2\). By rotation of coordinates about the \( z \)-axis we can achieve one of the forms \( c_{12} \neq 0, c_7 = 0 \) or \( c_{12} \neq 0, c_7 = i c_{12} \). For the second form the basis is not FLD, so can be ruled out. For the first form the basis is FLD but fails the requirement of yielding a 3-parameter potential depending on 2 coordinates.

**Case:** Both \( L_1 \) and \( L'_1 \) are of order 1. Then, since \( p_1^2 \) and \( \mathcal{H} \) are always basis vectors, the remaining basis symmetry must be of order 0. It can be chosen as either \( p_2^2 \) or \((p_2 + i p_3)^2\). In the 1st case we determine all possible choices of basis vectors such that the set in FLD. There are only 4 general cases and we verify that none of them define a superintegrable system, i.e., yield a 2-parameter potential. In the 2nd case there are 9 possible FLD families, but they all fail the symmetry test.

### 5.1.4 Form \( (13d) \)

Here we have 1 chain of length 2 and 3 chains of length 1.

The general form for a chain of length 2 is

\[ L_1 = \frac{1}{2} (c_{12} z^2 - c_7 y z + c_{16} z - c_8 y + (c_{12} y z + c_7 y^2 - c_{13} z + c_{18} y + c_{19}) p_1) p_1 \]

+ \( c_7 x z + c_8 x + (c_{12} x z - c_7 x y - c_{16} x - c_{18} x) p_3 p_2 + c_{12} x y + c_{13} x p_3^2 + W_1 \)

\[ L_2 = A_{p_1} L \]

= \(-c_7 z - c_8)p_2^2 + (c_{12} z + c_7 y + c_{16} + c_{18})p_2 p_3 + (-c_{12} y - c_{13})p_3^2 + W_2 \)

The general form for a chain of length 1 is

\[ L'' = \frac{1}{2} z(h_9 z + 2h_{10})p_2^2 + (-h_9 y z - h_{10} y - h_{14} z + h_{20})p_3 p_2 \]

+ \( \frac{1}{2} h_0 y^2 + h_{14} y + h_{15})p_3^2 + h_0 p_1^2 + h_1 p_2^2 + W_5 \quad (25) \]

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There are 2 basic cases: 1) $L_1$ is of order 2, $L_2$ is of order 1 and $L''$ is of orders, 2,1, or zero; 2) $L_1$ is of order 1, $L_2$ is of order 0 and $L''$ is of orders, 2,1, or zero. We check all of the possibilities and find the Hamiltonian
\[ p_x^2 + p_y^2 + p_z^2 + V(y, z), \]
where
\[ V(y, z) = b(z - iy) + F(z + iy). \]  

(26)

Here $b$ is a free constant and $F$ is an arbitrary function. The Minkowski example in §3 is a special case of this. Indeed, under the complex orthogonal change of coordinates
\[ x = -2ir_1, \quad y = \frac{1}{2}(r_1 + r_2 - (1 - i)r_3), \quad z = \frac{i}{2}(r_1 - r_2 - (1 - i)r_3) \]
the Hamiltonian (26) becomes (3) when we choose $F(w) = \beta w + \gamma w^2$ and $b = \alpha$.

A second solution is
\[ V(y, z) = F(z) + \frac{b}{(y + q)^2}, \]  

(27)

where $F$ is an arbitrary function, $b$ is an arbitrary constant and $q$ is a nonzero parameter depending on the symmetry operators.

5.1.5 Form (14e)

Here we have 5 chains of length 1. The possibilities are 1) 1 symmetry of order 2, 2 symmetries of order 1 and 2 symmetries of order 0; 2) 1 symmetry of order 2, 1 symmetry of order 1 and 3 symmetries of order 0; 3) 2 symmetries of order 1 and 3 symmetries of order 0; 4) 1 symmetry of order 1 and 4 symmetries of order 0. In all cases the systems are FLD but they do not admit 3-parameter potentials.

5.1.6 Structure algebras

For the generalized Calogero system (17) a basis for the generators is
\[ J = p_1, \quad S_{(1)} = H = p_1^2 + p_2^2 + p_3^2 + \frac{F(y)}{y^2}, \quad S_{(2)} = p_1^2, \]
\[ S_{(3)} = \frac{(z^2 p_2^2 y^2 - 2z y p_2 p_3 + p_3^2 y^4 + F(y) y^2 + F(z) z^2 + b z^2)}{2y^2}, \]
\[ S_{(4)} = \frac{p_1^2 y^4 + p_1^2 y^2 z^2 - 2p_1 p_2 x y^3 - 2p_1 p_3 x y^2 z + x^2 p_2^2 y^2 + x^2 p_3^2 y^2 + x^2 F(y) + bx^2}{2y^2}, \]
\[ S_{(5)} = \frac{-p_1 p_3 y^2 z + x p_2^2 y^2 + x p_3^2 y^2 + x F(y) + bx}{y^2}, \]
The nonzero commutators of the generators are

\[ \{ \mathcal{J}, S_1 \} = -S_5, \quad \{ \mathcal{J}, S_5 \} = \mathcal{J}^2 - \mathcal{H}, \quad \{ S_4, S_5 \} = -2S_4\mathcal{J} - 2S_3\mathcal{J} - b\mathcal{J}, \]

and the functional relationship is

\[-\frac{1}{2}(x^2 + y^2 + z^2)\mathcal{J}^2 + S_{(4)} - xS_{(5)} + \frac{1}{2}x^2\mathcal{H} = 0.\]

Note that both \( \mathcal{H} \) and \( S_5 \) lie in the center of this algebra.

For the system \( \text{(19)} \) a basis for the 1st and 2nd order generators is

\[ \mathcal{J} = p_1, \quad S_{(1)} = \mathcal{H}, \quad S_{(2)} = \left( \frac{1}{2}y^2 + \frac{1}{4}z^2 \right)p_1^2 - (p_2xy - p_3xz)p_1 + \frac{1}{2}x^2p_2^2 + \]

\[ \frac{1}{2}p_3^2x^2 + \frac{1}{2}x^2 \left( \sqrt{y^2 + z^2}b_1k_{10}^2 + 2\sqrt{y^2 + z^2}b_1k_{14}^2 + b_2k_{10}z + b_2k_{14}y + \sqrt{y^2 + z^2}b_3 \right) \sqrt{y^2 + z^2}(k_{10}y - k_{14}z)^2, \]

\[ S_{(3)} = (-p_2y - p_3z)p_1 + xp_2 + p_3x + \]

\[ x\left( \sqrt{y^2 + z^2}b_1k_{10}^2 + 2\sqrt{y^2 + z^2}b_1k_{14}^2 + b_2k_{10}z + b_2k_{14}y + \sqrt{y^2 + z^2}b_3 \right) \sqrt{y^2 + z^2}(k_{10}y - k_{14}z)^2 \]

\[ S_{(4)} = \left( \frac{1}{2}y^2 + \frac{1}{4}z^2 \right)p_2 - yzp_3p_2 + \frac{1}{2}y^2p_3^2 + \]

\[ \frac{1}{2}\frac{1}{2}\sqrt{y^2 + z^2}b_1k_{10}^4z^2 + 2\sqrt{y^2 + z^2}b_1k_{10}^3k_{14}yz + \sqrt{y^2 + z^2}b_1k_{10}^2k_{14}^2z^2 + 4\sqrt{y^2 + z^2}b_1k_{10}k_{14}^3yz \]

\[ -2\sqrt{y^2 + z^2}b_1k_{14}^4z^2 + b_2k_{10}^3y^2z + b_2k_{10}^2z^3 + b_2k_{10}^2k_{14}^3y^3 + b_2k_{10}k_{14}^4y^2z \]

\[ \sqrt{y^2 + z^2}b_1k_{10}^2k_{14}^2z^2 + 2\sqrt{y^2 + z^2}b_1k_{10}k_{14}^3yz + \sqrt{y^2 + z^2}b_1k_{14}^4yz \]

\[ \frac{1}{2} \left( \frac{-(k_{10}y - k_{14}z)p_2p_1 + (k_{10}y - k_{14}z)yxp_3p_1 + k_{10}p_3^2p_2}{k_{10}} \right), \]

\[ \frac{1}{2} \left( \frac{1}{2} \frac{2\sqrt{y^2 + z^2}b_1k_{10}^3z + 2\sqrt{y^2 + z^2}b_1k_{10}^2k_{14}y + 4\sqrt{y^2 + z^2}b_1k_{10}k_{14}^2z}{k_{10}\sqrt{y^2 + z^2}(k_{10}y - k_{14}z)^2} \right) \]

\[ + 4\sqrt{y^2 + z^2}b_1k_{14}^3y + b_2k_{10}^3y^2 + 2b_2k_{10}^2z^2 + 2b_2k_{10}k_{14}yz + 2b_2k_{14}^3y^2 + b_2k_{14}^2z^2 \]

\[ \frac{1}{2} \left( \frac{2\sqrt{y^2 + z^2}b_1k_{10}^3z + 2\sqrt{y^2 + z^2}b_1k_{10}^2k_{14}y + 4\sqrt{y^2 + z^2}b_1k_{10}k_{14}^2z}{k_{10}\sqrt{y^2 + z^2}(k_{10}y - k_{14}z)^2} \right) \]

\[ + 2\sqrt{y^2 + z^2}b_3k_{10}z + 2\sqrt{y^2 + z^2}b_3k_{14}y \]

\[ \frac{1}{2} \left( \frac{2\sqrt{y^2 + z^2}b_1k_{10}^3z + 2\sqrt{y^2 + z^2}b_1k_{10}^2k_{14}y + 4\sqrt{y^2 + z^2}b_1k_{10}k_{14}^2z}{k_{10}\sqrt{y^2 + z^2}(k_{10}y - k_{14}z)^2} \right) \]

\[ + 2\sqrt{y^2 + z^2}b_3k_{10}z + 2\sqrt{y^2 + z^2}b_3k_{14}y \]

\[ \frac{1}{2} \left( \frac{2\sqrt{y^2 + z^2}b_1k_{10}^3z + 2\sqrt{y^2 + z^2}b_1k_{10}^2k_{14}y + 4\sqrt{y^2 + z^2}b_1k_{10}k_{14}^2z}{k_{10}\sqrt{y^2 + z^2}(k_{10}y - k_{14}z)^2} \right) \]

\[ + 2\sqrt{y^2 + z^2}b_3k_{10}z + 2\sqrt{y^2 + z^2}b_3k_{14}y \]

\[ \frac{1}{2} \left( \frac{2\sqrt{y^2 + z^2}b_1k_{10}^3z + 2\sqrt{y^2 + z^2}b_1k_{10}^2k_{14}y + 4\sqrt{y^2 + z^2}b_1k_{10}k_{14}^2z}{k_{10}\sqrt{y^2 + z^2}(k_{10}y - k_{14}z)^2} \right) \]

\[ + 2\sqrt{y^2 + z^2}b_3k_{10}z + 2\sqrt{y^2 + z^2}b_3k_{14}y \]

\[ \frac{1}{2} \left( \frac{2\sqrt{y^2 + z^2}b_1k_{10}^3z + 2\sqrt{y^2 + z^2}b_1k_{10}^2k_{14}y + 4\sqrt{y^2 + z^2}b_1k_{10}k_{14}^2z}{k_{10}\sqrt{y^2 + z^2}(k_{10}y - k_{14}z)^2} \right) \]

\[ + 2\sqrt{y^2 + z^2}b_3k_{10}z + 2\sqrt{y^2 + z^2}b_3k_{14}y \]

\[ \frac{1}{2} \left( \frac{2\sqrt{y^2 + z^2}b_1k_{10}^3z + 2\sqrt{y^2 + z^2}b_1k_{10}^2k_{14}y + 4\sqrt{y^2 + z^2}b_1k_{10}k_{14}^2z}{k_{10}\sqrt{y^2 + z^2}(k_{10}y - k_{14}z)^2} \right) \]

\[ + 2\sqrt{y^2 + z^2}b_3k_{10}z + 2\sqrt{y^2 + z^2}b_3k_{14}y \]
Since the potential-free parts of the generators satisfy $x H$ with $S$, the resulting algebra doesn’t close. However, if any linear combination of $bX$, $S_1 = 2S_2 + 2S_4$ plus $bY$, $F(Z)$, the generating symmetries are

$$\mathcal{J} = p_X, \quad S_1 = H = S_1 = p_X^2 + 4p_Y p_Z + bY + F(Z), \quad S_2 = \mathcal{J}^2,$$

$$S_3 = Z p_X^2 - 2X p_X p_Y - \frac{1}{2} bX^2, \quad S_4 = p_X p_Y + \frac{1}{2} bX, \quad S_5 = p_Y^2 + \frac{1}{2} bZ,$$

and the nonzero structure relations are

$$\{\mathcal{J}, S_3\} = 2S_4, \quad \{\mathcal{J}, S_4\} = -\frac{b}{2}, \quad \{S_3, S_4\} = -2\mathcal{J} S_5,$$

with $H$ in the center of the algebra. The potential-free parts of the generators satisfy $-z \mathcal{J}^2 + S_3^2 - 2x S_4^2 = 0$, so the system is FLD.

For the system (27) the generating symmetries are

$$\mathcal{J} = p_1, \quad S_1 = H = p_1^2 + p_2^2 + p_3^2 + F(z) + \frac{b}{(y + q)^2}, \quad S_2 = p_1^2,$$

$$S_3 = \left(\frac{1}{2} y^2 + qy\right)p_1^2 - x(y + q)p_1 p_2 + \frac{1}{2} x^2 p_2^2 + \frac{1}{2} \frac{b x^2}{(y + q)^2},$$

$$S_4 = -(y + q)p_1 p_2 + x p_2^2 + \frac{b x}{(y + q)^2}, \quad S_5 = p_2^2 + \frac{b}{(y + q)^2},$$

and the resulting algebra doesn’t close.
and the nonzero structure relations are
\[ \{ \mathcal{J}, S_{(3)} \} = -S_{(4)}, \{ \mathcal{J}, S_{(4)} \} = -S_{(5)}, \{ S_{(3)}, S_{(4)} \} = -\mathcal{J} (q^2 \mathcal{J}^2 + 2S_{(3)} - q^2 S_{(5)} - b), \]
\[ \{ S_{(3)}, S_{(5)} \} = -2S_{(4)} \mathcal{J}, \{ S_{(4)}, S_{(5)} \} = -2S_{(5)} \mathcal{J}. \]
The potential-free parts of the generators satisfy the relation \( -(\frac{1}{2} y^2 + g y) \mathcal{J}^2 + S_{(3)}^0 - x S_{(4)}^0 + x^2 S_{(5)}^0 = 0 \), so the system is FLD.

5.2 Second case: \( \mathcal{J} = p_1 + i p_2 \)

We introduce appropriate new coordinates \( \{ \xi, \eta, z \} \) where \( x = \frac{1}{2}(\xi + \eta), \ y = \frac{i}{2}(\xi - \eta), \ z = z \). In the new coordinates the 1st order symmetries for the potential-free case are:
\[ p_1 + i p_2 = 2 p_\eta = \mathcal{J}, \quad p_2 = i (p_\xi - p_\eta), \quad J_{12} = i (\xi p_\xi - \eta p_\eta), \quad J_{13} = \frac{1}{2} (\xi + \eta) p_z - z (\xi + \eta) p_z, \quad J_{23} = i ((\xi + \eta) p_z + z (\eta - \xi)). \]

In this case \( Ad_{p_1 + i p_2}^3 = 0 \). A generalized eigenbasis for the 20-dimensional space of symmetries is
\[ L_1 = \frac{1}{2} (\xi p_\xi - \eta p_\eta)^2, \quad L_2 = \frac{1}{2} (\xi p_\xi - \eta p_\eta)(2 z p_\xi - \eta p_z), \quad L_3 = \frac{1}{2} [2 z (p_\xi + p_\eta) - (\xi + \eta) p_z]^2, \]
\[ M_1 = p_\eta (\xi p_\xi - \eta p_\eta), \quad M_2 = p_\eta (xz p_\xi - \eta p_z) + \frac{\xi}{2} p_\xi p_z, \quad M_3 = p_\eta [2 z (p_\xi + p_\eta) - (\xi + \eta) p_z] \]
\[ M_4 = (2 z p_\xi - \eta p_z)(2 z p_\eta - \xi p_z), \quad M_5 = (\xi p_\xi - \eta p_\eta)(2 z p_\eta - \xi p_z) \]
\[ M_6 = (p_\xi - p_\eta)(2 z p_\xi - \eta p_z), \quad M_7 = 4 p_\xi (\xi p_\xi - \eta p_\eta) + 2 z p_\xi p_z - \eta^2, \]
\[ N_1 = p_\eta^2, \quad N_2 = p_\eta p_z, \quad N_3 = p_z^2, \quad N_4 = p_z (2 z p_\eta - \xi p_z), \quad N_5 = p_\eta (2 z p_\eta - \xi p_z) \]
\[ N_6 = p_z (p_\xi - p_\eta), \quad N_7 = H = 4 p_\xi p_\eta + p_z^2, \quad N_8 = (2 z p_\eta - \xi p_z)^2 \]
\[ N_9 = p_\xi (2 z p_\eta - \xi p_z), \quad N_{10} = p_z^2, \]
where the 3-chains and 2-chains are \( \{ L_1, M_1, N_1 \}, \{ L_2, M_2, N_2 \}, \{ L_3, M_3, N_3 \}, \{ M_4, N_4 \}, \{ M_5, N_5 \}, \{ M_6, N_6 \}, \) and \( \{ M_7, N_7 \} \). \( N_8, N_9, \) and \( N_{10} \) are 1-chains.
5.2.1 Form (14a)

Here we have a 3-chain and two 1-chains, one of which must be $H$. There are two cases to consider. Either the terminal element of the three chain or the second 1-chain must be $N_1 = p_7^2$.

In the first case, the 3-chain is $\{L_1 + \beta_1 M_4 + \beta_2 M_5 + \gamma N_8, M_1 + \beta_1 N_4 + \beta_2 N_5, N_1 = p_7^2\}$ and the 1-chain is one of $\mu_1 N_2 + \mu_2 N_3 + \mu_3 N_6 + \mu_4 N_{10}$, $\mu_1 N_4 + \mu_2 N_5 + \mu_3 N_9$, or $N_8$ (in which case we can take $\gamma = 0$ by a canonical form-preserving change of basis). The first subcase is FLD when $\beta_1 = -1/4$, $\mu_2 = \mu_3 = \mu_4 = 0$, or $\beta_1 = 0$, $\gamma = \beta_2^2/2$, $\mu_1 = 2\beta_2(2\mu_2 - 2\beta_2^2 - 1)$, $\mu_3 = -2\beta_2$, $\mu_4 = 1$ or $\beta_1 = \mu_4 = 0$, $\mu_4 = 4\beta_2\mu_2$, or $\beta_1 = \beta_2 = \mu_1 = \mu_3 = \gamma \mu_4 = 0$. The third subcase with $\gamma = \beta_2^2/2$ and the fourth subcase with $\gamma = \mu_4 = 0$ lead to the admissible potentials

$$V(\xi, z) = \frac{b}{\xi^2} + F(q\xi + z)$$

and

$$V(\xi, z) = \frac{b}{\xi^2} + F(z),$$

respectively. Note that (29) is special case of (28).

The second subcase is FLD when $\mu_1 = \mu_3 = 0$ and $\beta_1 = -1/4$ but does not lead to an admissible potential.

The third subcase is FLD when $\beta_1 = -1/2$ and $\beta_2 = 0$, leading to the admissible potential

$$V(\xi, z) = \frac{F(z/\xi)}{\xi^2}. \quad (30)$$

In the second case, the 3-chain is $\{\alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3 + \beta_1 M_4 + \beta_2 M_5 + \gamma N_8, \alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 + \beta_1 N_4 + \beta_2 N_5, \alpha_1 N_1 + \alpha_2 N_2 + \alpha_3 N_3\}$. This case is not FLD for any choice of parameters.

5.2.2 Form (14b)

Here we have one 3-chain and one 2-chain. The 3-chain must be $\{L_1 + \beta_1 M_4 + \beta_2 M_5 + \gamma N_8, M_1 + \beta_1 N_4 + \beta_2 N_5, N_1 = p_7^2\}$ and the 2-chain must be $\{M_7 + \mu_1 N_4 + \mu_2 N_5 + \mu_3 N_9, N_7 = H\}$. The symmetries are not FLD for any choice of parameters.

5.2.3 Form (14c)

Here we have two 2-chains and a single 1-chain. There are three cases to consider: the terminal elements of the 2-chains are $p_7^2$ and $H$, one 2-chain
terminates in $p_2^γ$ and the 1-chain is $H$, one 2-chain terminates with $H$ and the 1-chain is $p_2^γ$.

In the first case, the 2-chains are $\{M_1 + \beta_1 N_4 + \beta_2 N_5 + \beta_3 N_9, N_1\}$ and $\{M_7 + \gamma_1 N_4 + \gamma_2 N_5 + \gamma_3 N_9, N_7\}$ and the 1-chain is one of $N_8$, $\mu_1 N_4 + \mu_2 N_5 + \mu_3 N_9$, $\mu_1 N_2 + \mu_3 N_6 + \mu_4 N_{10}$. In the first subcase, the symmetries are FLD when $\beta_1 = -1/2$, $\beta_2 = \beta_3 = 0$, but this does not lead to an admissible potential. The second subcase is FLD when $\beta_1 = -1/4$, $\beta_3 = \mu_1 = \mu_3 = 0$, but this does not lead to an admissible potential. In the third subcase, the symmetries are FLD when either $\beta_1 = \beta_3 = \mu_3 = 0$, $\mu_1 = 4 \beta_2 \mu_2$ or $\beta_1 = -1/4$, $\beta_3 = \mu_2 = \mu_3 = \mu_4 = 0$, but neither corresponds to an admissible potential.

In the second case, one 2-chain is $\{M_1 + \beta_1 N_4 + \beta_2 N_5 + \beta_3 N_9, N_1\}$ and the second 2-chain is either $\{\gamma_1 M_1 + \gamma_2 M_2 + \gamma_3 M_3 + \gamma_4 M_6 + \gamma_5 M_7 + \delta_1 N_4 + \delta_2 N_5 + \delta_3 N_9, \gamma_1 N_1 + \gamma_2 N_2 + \gamma_3 N_3 + \gamma_4 N_6 + \gamma_5 N_7\}$ (we can take $\gamma_1 = 0$ by a canonical form-preserving change of basis) or $\{\gamma_1 M_4 + \gamma_2 M_5 + \delta N_8, \gamma_1 N_1 + \gamma_2 N_3\}$. To simplify the analysis, we observe that the symmetry $M_1 + \beta_1 N_4 + \beta_2 N_5 + \beta_3 N_9$ leads to an inadmissible potential unless $\beta_3 = 0$; similarly, if $\gamma_1 N_1 + \gamma_2 N_2 + \gamma_3 N_3 + \gamma_4 N_6 + \gamma_5 N_7$ is a symmetry of an admissible potential we must have $\gamma_4 = 0$. In the first subcase, we find three sets of FLD symmetries: $\beta_1 = \beta_3 = \gamma_1 = \gamma_4 = 0$, $\gamma_2 = 4 \beta_2 \gamma_3$; $\beta_1 = -1/4$, $\beta_3 = \gamma_1 = \gamma_3 = \gamma_4 = 0$; and $\beta_3 = \gamma_1 = \gamma_3 = \gamma_4 = \gamma_5 = 0$, $\gamma_2 = 2 \delta_3$, but none of these lead to admissible potentials. The second subcase is FLD when $\beta_1 = -1/4$, $\beta_3 = \gamma_1 = 0$, but this does not lead to an admissible potential.

In the third case, one 2-chain is $\{M_7 + \beta_1 N_4 + \beta_2 N_5 + \beta_3 N_9, N_7\}$ and the second 2-chain is either $\{\gamma_1 M_1 + \gamma_2 M_2 + \gamma_3 M_3 + \gamma_4 M_6 + \gamma_5 M_7 + \delta_1 N_4 + \delta_2 N_5 + \delta_3 N_9, \gamma_1 N_1 + \gamma_2 N_2 + \gamma_3 N_3 + \gamma_4 N_6 + \gamma_5 N_7\}$ (we can take $\gamma_5 = 0$ by a canonical form-preserving change of basis) or $\{\gamma_1 M_4 + \gamma_2 M_5 + \delta N_8, \gamma_1 N_1 + \gamma_2 N_3\}$. Using the requirement $\gamma_4 = 0$ from the second case, we find that the first subcase is not FLD for any choice of parameters. The second subcase is also not FLD for any choice of parameters.

5.2.4 Form (134)

Here we have a 2-chain and three 1-chains. There are again three cases to consider: $p_2^γ$ and $H$ are 1-chains, $p_2^γ$ is the terminal element of a 2-chain and $H$ is a 1-chain, and $H$ is the terminal element of a 2-chain and $p_2^γ$ is a 1-chain.

In the first case, the 2-chain is either $\{\alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 + \alpha_4 M_6 + \alpha_5 M_7 + \beta_1 N_4 + \beta_2 N_5 + \beta_3 N_9, \alpha_1 N_1 + \alpha_2 N_2 + \alpha_3 N_3 + \alpha_4 N_6 + \alpha_5 N_7\}$ or $\{\alpha_1 M_4 + \alpha_2 M_5 + \beta N_8, \alpha_1 N_1 + \alpha_2 N_2\}$ and the final 1-chain is one of $\mu_1 N_2 + \mu_2 N_3 + \mu_3 N_6 + \mu_4 N_{10}$, $\mu_1 N_4 + \mu_2 N_6 + \mu_3 N_9$, $N_8$. To simplify the analysis, it is sometimes useful to find conditions under which the nontrivial 1-chains are compatible.
(both correspond to the same admissible potential) before searching for FLD systems. In the first subcase where the final 1-chain is order-0, we have the conditions \( \alpha_2 = -2\alpha_3\mu_3/\mu_4 \) and \( \mu_2 = (\mu_3^3 - 2\mu_1\mu_2^2 + 2\mu_3\mu_4^2)/4\mu_3\mu_4 \) when \( \mu_4 \neq 0 \) (we must also assume \( \alpha_3\mu_3 \neq 0 \) to avoid linear dependence), but this does not lead to an FLD system with admissible potential. When \( \mu_4 = 0 \), the 1-chains are incompatible. In the first subcase where the final 1-chain is order-1, we have the compatibility conditions \( \alpha_3 = 0 \) or \( \alpha_2 = 2\alpha_3\mu_2/\mu_1 \). The first of these leads to an FLD system \((\alpha_3 = \alpha_4 = \alpha_5 = \mu_1 = \mu_3 = 0, \beta_3 = 3\alpha_2/2)\) with admissible potential

\[
V(\xi, z) = b \xi^2 + b_2(q\xi + z) \tag{31}
\]

and an FLD system \((\alpha_3 = \alpha_4 = \alpha_5 = \mu_1 = \mu_3 = 0, \beta_3 = -5\alpha_2/2)\) with admissible potential

\[
V(\xi, z) = b_1 \xi^{2/3} + \frac{b_2(q\xi + z)}{\xi^{1/3}} \tag{32}
\]

In the first subcase where the final 1-chain is order-2, the symmetries are FLD when \( \alpha_3 = \alpha_4 = \alpha_5 = 0 \) and \( \beta_3 = \alpha_2/2 \), but this does not lead to an admissible potential. In the second subcase where the final 1-chain is order-0, the symmetries are FLD when \( \alpha_1 = \mu_2 = \mu_3 = \mu_4 = 0 \), but this does not lead to an admissible potential. In the second subcase where the final 1-chain is order-1, imposing \( \mu_3 = 0 \) we find that the symmetries are not FLD for any choice of parameters. In the second subcase where the final 1-chain is order-2, the symmetries are not FLD for any choice of parameters.

In the second case, the 2-chain is \( \{M_1 + \beta_1 N_4 + \beta_2 N_5 + \beta_3 N_9, N_1\} \) and there are five subcases for the two remaining 1-chains: one order-2 and one order-1 1-chain, one order-2 and one order-0 1-chain, two order-1 1-chains, one order-1 and one order-0 1-chain, and two order-0 1-chains.

In the first subcase, the symmetries are FLD when \( \beta_1 = -1/4 \) and \( \beta_3 = \mu_1 = \mu_3 = 0 \), but this does not lead to an admissible potential.

In the second subcase, the symmetries are FLD when either \( \beta_1 = -1/2 \), \( \beta_2 = \beta_3 = 0 \) or \( \beta_3 = \mu_3 = \mu_4 = 0 \). From here we obtain three admissible potentials. When \( \beta_1 = -1/2, \beta_2 = \beta_3 = 0, \mu_3, \mu_4 \neq 0 \) and \( \mu_2 = (\mu_3^3 - 2\mu_1\mu_2^2 + 2\mu_3\mu_4^2)/4\mu_3\mu_4 \), we have the potential

\[
V(\xi, z) = b_1 \xi + \frac{b_2}{(q\xi + z)^2}, \tag{33}
\]

when \( \beta_2 = (\mu_1 + 2\beta_1\mu_1)/4\mu_2, \beta_3 = \mu_3 = \mu_4 = 0 \), we have the potential

\[
V(\xi, z) = \frac{b}{(q\xi + z)^2} + F(\xi), \tag{34}
\]

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and when $\beta_3 = \mu_2 = \mu_3 = \mu_4 = 0$, we have the potential

$$V(\xi, z) = \frac{b z}{\xi^3} + F(\xi).$$  \hspace{1cm} (35)$$

In the third subcase, we recall that $\mu_1 N_4 + \mu_2 N_5 + \mu_3 N_9$ only leads to an admissible potential when $\mu_3 = 0$. Then, by a canonical form-preserving change of basis, we see that $N_4$ and $N_5$ must be independent symmetries. The symmetries are FLD when $\beta_3 = 0$ and lead to an admissible potential

$$V(\xi, z) = \frac{b z}{\xi^3/2} + F(\xi).$$ \hspace{1cm} (36)$$

In the fourth subcase, we write $\mu_1 N_4 + \mu_2 N_5 + \mu_3 N_9 + \mu_4 N_6 + \nu_3 N_6 + \nu_4 N_{10}$ for the order-1 and order-2 1-chains, respectively. The symmetries are FLD when $\beta_1 = -1/4, \beta_3 = \mu_1 = \mu_3 = 0$ or $\beta_3 = \nu_3 = \nu_4 = 0$. There are two resulting FLD systems with admissible potentials: $\beta_2 = (\mu_2 + 2\beta_1 \mu_2)/2\mu_1, \nu_1 = 2\mu_2 \nu_2/\mu_1, \beta_3 = \mu_3 = \nu_3 = \nu_4 = 0$, we obtain a potential equivalent to (34) and $\beta_3 = \mu_1 = \mu_3 = \nu_3 = \nu_2 = \nu_4 = 0$ with

$$V(\xi, z) = b z \xi^a + F(\xi).$$ \hspace{1cm} (37)$$

In the fifth subcase, we $\mu_1 N_2 + \mu_2 N_3 + \mu_3 N_6 + \mu_4 N_{10}$ and $\nu_1 N_2 + \nu_2 N_3 + \nu_3 N_6 + \nu_4 N_{10}$ for the two order-0 1-chains. Assume first that $\mu_4$ and $\nu_4$ are not both zero. Without loss of generality we assume $\mu_4 \neq 0$, so we can take $\nu_4 = 0$ by a canonical form-preserving change of basis. It is then required that $\nu_3 = 0$ if we are to have an admissible potential. The 1-chains are incompatible unless $\mu_3 = -\nu_1 \mu_4/2\nu_2$. When additionally $\nu_1 = 4\beta_2 \nu_2, \beta_1 = \beta_3 = \nu_3 = \nu_4 = 0$, we find an FLD system with admissible potential

$$V(\xi, z) = b \xi F(q \xi + z).$$ \hspace{1cm} (38)$$

If $\mu_4 = \nu_1 = 0$, we must also have $\mu_3 = \nu_3 = 0$ and we can consider $N_2$ and $N_3$ as independent symmetries. The symmetries are FLD when $\beta_3 = 0$; when additionally $\beta_1 = -1/10$, we find the admissible potential

$$V(\xi, z) = b \xi z + F(\xi),$$ \hspace{1cm} (39)$$

and when additionally $\beta_1 = 0$, we find the admissible potential

$$V(\xi, z) = b z + F(\xi).$$ \hspace{1cm} (40)$$

In the third case, the 2-chain is $\{M_7 + \beta_1 N_4 + \beta_2 N_5 + \beta_3 N_9, N_1\}$ and there are five subcases for the two remaining 1-chains: one order-2 and one
order-1 1-chain, one order-2 and one order-0 1-chain, two order-1 1-chains, one order-1 and one order-0 1-chain, and two order-0 1-chains. The first three subcases are not FLD for any choice of parameters. In the fourth subcase, we write \( \mu_1 N_4 + \mu_2 N_5 + \mu_3 N_9 \) and \( \nu_1 N_2 + \nu_2 N_3 + \nu_3 N_6 + \nu_4 N_{10} \) for the order-1 and order-2 1-chains, respectively. The symmetries are FLD when \( \mu_1 = \mu_3 = \nu_2 = \nu_3 = \nu_4 = 0 \), but this does not lead to an admissible potential. In the fifth subcase, we write \( \mu_1 N_2 + \mu_2 N_3 + \mu_3 N_6 + \mu_4 N_{10} \) and \( \nu_1 N_2 + \nu_2 N_3 + \nu_3 N_6 + \nu_4 N_{10} \) for the two 1-chains. Compatibility of these 1-chains requires \( \mu_3 = \mu_4 = \nu_3 = \nu_4 = 0 \) and we may take \( N_2 \) and \( N_3 \) as independent symmetries. However, the simultaneous admissible potential of \( N_2 \) and \( N_3 \) is incompatible with the \( M_7 + \beta_1 N_4 + \beta_2 N_5 + \beta_3 N_9 \) for all choices of parameters.

5.2.5 Form (14e)

Here we have five 1-chains, two of which must be \( H \) and \( p_{\eta}^2 \). There are seven cases for the three additional 1-chains:

1. one order-2 1-chain and two order-1 1-chains
2. one order-2, one order-1, and one order-0 1-chain
3. one order-2 1-chain and two order-0 1-chains
4. three order-1 1-chains
5. two order-1 1-chains and one order-0 1-chain
6. one order-1 and two order-0 1-chains
7. three order-0 1-chains.

In the first case, we write \( N_8, \mu_1 N_4 + \mu_2 N_5 + \mu_3 N_9 \) and \( \nu_1 N_4 + \nu_2 N_5 + \nu_3 N_9 \) for the three 1-chains. The potential is admissible only if \( \mu_3 = \nu_3 = 0 \), so we may take \( N_4 \) and \( N_5 \) as independent symmetries. The symmetries are incompatible (do not have a simultaneous admissible potential).

In the second case, we write \( N_8, \mu_1 N_4 + \mu_2 N_5 + \mu_3 N_9 \) and \( \nu_1 N_2 + \nu_2 N_3 + \nu_3 N_6 + \nu_4 N_{10} \) for the three 1-chains. The symmetries are FLD when \( \mu_3 = \nu_3 = \nu_4 = 0 \). When also \( \mu_2 = \nu_1 = 0 \), we find the potential

\[
V(\xi, z) = \frac{b}{z^2} + F(\xi);
\]  

(41)

when also \( \nu_2 = \mu_1 \nu_1 / 2 \mu_2 \), we find the admissible potential

\[
V(\xi, z) = \frac{b_1 \xi^2 + b_2 z (\mu_1 z + \mu_2 \xi)}{\xi^2 (2 \mu_1 z + \mu_2 \xi)^2} + F(\xi),
\]  

(42)
which contains \ref{eq:11} as a special case.

In the third case, we write \( N_8, \mu_1 N_2 + \mu_2 N_3 + \mu_3 N_6 + \mu_4 N_{10} \) and \( \nu_1 N_2 + \nu_2 N_3 + \nu_3 N_6 + \nu_4 N_{10} \) for the 1-chains. We first assume that one of \( \mu_4, \nu_4 \) is nonzero. Without loss of generality we take \( \mu_4 \neq 0 \) so that we may take \( \nu_4 = 0 \) by a canonical form-preserving change of basis. We can only have an admissible potential if also \( \nu_3 = 0 \). The symmetries are not FLD for any choice of the remaining parameters. We then consider the case where \( \mu_3 = \mu_4 = \nu_3 = \nu_4 = 0 \). We can then take \( N_2 \) and \( N_3 \) as independent symmetries, but the symmetries are not FLD.

In the fourth case, we can make a canonical-form preserving change of basis and consider \( N_4, N_5 \) and \( N_9 \) as independent symmetries. These symmetries are incompatible (in particular, \( N_9 \) does not produce an admissible symmetry).

The fifth case is similar to the first case: we may take \( N_4 \) and \( N_5 \) as independent symmetries. We write \( \mu_1 N_2 + \mu_2 N_3 + \mu_3 N_6 + \mu_4 N_{10} \) for the remaining nontrivial 1-chain. The symmetries are FLD when \( \mu_3 = \mu_4 = 0 \); when also \( \mu_2 = 0 \), we find the admissible potential \ref{eq:36}.

In the sixth case, we write \( \mu_1 N_2 + \mu_2 N_3 + \mu_3 N_6 + \mu_4 N_{10}, \nu_1 N_2 + \nu_2 N_3 + \nu_3 N_6 + \nu_4 N_{10}, \) and \( \sigma_1 N_2 + \sigma_2 N_3 + \sigma_3 N_6 + \sigma_4 N_{10} \) for the three 1-chains. This case is similar to the third case: the two subcases reduce to \( \nu_4 \neq 0, \sigma_3 = \sigma_4 = 0 \) and \( \nu_2 = \nu_3 = \nu_4 = \sigma_1 = \sigma_3 = \sigma_4 = 0 \). The first subcase is FLD when also \( \mu_1 = \sigma_2 = 0 \), but we do not get an admissible potential. The second subcase is FLD and when also \( \mu_1 = 0 \), we find the admissible potential \ref{eq:11}.

In the seventh case, we write \( \mu_1 N_2 + \mu_2 N_3 + \mu_3 N_6 + \mu_4 N_{10}, \nu_1 N_2 + \nu_2 N_3 + \nu_3 N_6 + \nu_4 N_{10}, \) and \( \sigma_1 N_2 + \sigma_2 N_3 + \sigma_3 N_6 + \sigma_4 N_{10} \) for the three 1-chains. We assume that at least one of \( \mu_4, \nu_4, \sigma_4 \) is nonzero. Without loss of generality, we take \( \mu_4 \neq 0 \) so we can make a canonical form-preserving change of basis and take \( \nu_4 = \sigma_4 = 0 \). The second and third symmetries will only have an admissible potential if also \( \nu_3 = \sigma_3 = 0 \), so we may also take \( \nu_2 = \sigma_1 = 0 \): \( N_2 \) and \( N_3 \) are independent symmetries. The symmetries are incompatible unless \( \mu_4 = 0 \), a contradiction. We next assume \( \mu_4 = \nu_4 = \sigma_4 = 0 \). Then we may consider \( N_2, N_3, \) and \( N_6 \) as independent symmetries. These symmetries are incompatible.
5.2.6 Structure algebras

For the potential (28), we have the symmetries
\[ J = \frac{p_\eta}{2}, \quad S_\eta = \mathcal{H} = 4p_\xi p_\eta + p_z^2 + \frac{b_1}{\xi^2} + F(q_\xi + z), \quad S_2 = 4J^2 = p_\eta^2, \]
\[ S_3 = L_1 + qM_5 + \frac{q^2}{2}N_8 + \frac{b(2qz - \eta)}{2\xi}, \quad S_4 = M_1 + qN_5 + \frac{b}{2\xi}, \]
\[ S_5 = N_3 + 4qN_2 + F(qz + \xi). \]

They satisfy 16(2qz - \eta)J^2 + \xi S_\eta^0 - 4S_\eta^0 - \xi S_5^0 = 0 and their nonzero commutators are
\[ \{J, S_3\} = \frac{1}{2}S_4, \quad \{J, S_4\} = 2J^2, \quad \{S_3, S_4\} = -4JS_\eta - 2q^2bJ, \]
\[ \{S_3, S_5\} = -16q^2JS_4, \quad \{S_4, S_5\} = 64q^2J^3. \]

For the potential (29), the symmetries and their FLD relation and algebra are obtained from that of (28) in the limit \( q \to 0 \).

For the potential (30), we have the symmetries
\[ J = \frac{p_\eta}{2}, \quad S_\eta = \mathcal{H} = 4p_\xi p_\eta + p_z^2 + \frac{b_1}{\xi^2} + b_2(q_\xi + z), \quad S_2 = 4J^2 = p_\eta^2, \]
\[ S_3 = L_1 + qM_5 + \frac{q^2}{2}N_8 + \frac{b_1(2qz - \eta)}{2\xi}, \quad S_4 = M_1 + \frac{b_1}{2\xi} + \frac{qb_2\xi^2}{8}, \]
\[ S_5 = N_3 + 4qN_2 + F(qz + \xi). \]

They satisfy 16(\xi \eta + z^2)J^2 - \xi^2 S_\eta^0 - 4\xi S_\eta^0 - S_5^0 = 0, and their nonzero commutators are
\[ \{J, S_3\} = \frac{1}{2}S_4, \quad \{J, S_4\} = 2J^2, \quad \{S_3, S_4\} = -4JS_\eta. \]

For the potential (31), we have the symmetries
\[ J = \frac{p_\eta}{2}, \quad S_\eta = \mathcal{H} = 4p_\xi p_\eta + p_z^2 + \frac{b_1}{\xi^2} + b_2(q_\xi + z), \quad S_2 = 4J^2 = p_\eta^2, \]
\[ S_3 = L_1 + qM_5 + \frac{q^2}{2}N_8 + \frac{b_1(2qz - \eta)}{2\xi}, \quad S_4 = M_1 + \frac{b_1}{2\xi} + \frac{qb_2\xi^2}{8}, \]
\[ S_5 = M_2 - qN_4 + \frac{3}{2}N_9 + \frac{b_1z}{\xi^2} + \frac{b_2(2qz - \eta)\xi}{4}, \quad S_6 = N_2 + \frac{b_2\xi^2}{4}, \]
\[ S_7 = N_3 + b_2z, \quad S_8 = N_5 - \frac{b_2\xi^2}{8}. \]
They satisfy \(16\mathcal{J}^2 - \xi S_{(1)}^0 + 4S_{(4)}^0 + \xi S_{(7)}^0 = 8z\mathcal{J}^2 - \xi S_{(6)}^0 - S_{(8)}^0 = 0\). The subset \(\{\mathcal{J}, S_{(1)}, S_{(2)}, S_{(4)}, S_{(6)}, S_{(7)}, S_{(8)}\}\) generates a closed quadratic algebra with nonzero relations

\[
\{\mathcal{J}, S_{(4)}\} = 2\mathcal{J}^2, \quad \{S_{(4)}, S_{(6)}\} = -2\mathcal{J} S_{(6)}, \quad \{S_{(4)}, S_{(8)}\} = -4\mathcal{J} S_{(8)}, \\
\{S_{(6)}, S_{(7)}\} = -2b_3\mathcal{J}, \quad \{S_{(6)}, S_{(8)}\} = -16\mathcal{J}^3, \quad \{S_{(7)}, S_{(8)}\} = -8\mathcal{J} S_{(6)}.
\]

However, if any linear combination of \(S_{(3)}, S_{(5)}\) is added to the generators, a new 3rd order symmetry is produced that is not a polynomial in the generators, so the resulting algebra doesn’t close.

For the potential \([32]\), we have the symmetries

\[
\mathcal{J} = \frac{p_\eta}{2}, \quad S_{(1)} = \mathcal{H} = 4p_\xi p_\eta + p^2_z + \frac{b_1}{\xi\eta} + \frac{b_2(q\xi + z)}{\xi\eta}, \quad S_{(2)} = 4\mathcal{J}^2 = p^2_\eta, \\
S_{(3)} = M_1 + 2N_4 - \frac{b_1\xi^{1/3}}{2} - \frac{b_2(q\xi + 16z)}{8\xi^{4/3}}, \\
S_{(4)} = M_2 - qN_4 - \frac{5}{2}N_9 - \frac{b_1 z}{\xi^{2/3}} + \frac{b_2(2q\xi z + 3q\eta - 4z^2)}{4\xi^{4/3}}, \\
S_{(5)} = N_2 - \frac{3b_2}{4\xi^{1/3}}, \quad S_{(6)} = N_5 - \frac{3b_2\xi^{2/3}}{8}.
\]

They satisfy \(8z\mathcal{J}^2 - \xi S_{(5)}^0 - S_{(6)}^0 = 0\). The subset \(\{\mathcal{J}, S_{(1)}, S_{(2)}, S_{(3)}, S_{(5)}, S_{(6)}\}\) generates a closed quadratic algebra with nonzero relations

\[
\{\mathcal{J}, S_{(3)}\} = 2\mathcal{J}^2, \quad \{S_{(3)}, S_{(5)}\} = 6\mathcal{J} S_{(5)}, \\
\{S_{(3)}, S_{(6)}\} = -12\mathcal{J} S_{(6)}, \quad \{S_{(5)}, S_{(6)}\} = -16\mathcal{J}^3.
\]

However, if \(S_{(4)}\) is added to the generators, a new 3rd order symmetry is produced that is not a polynomial in the generators, so the resulting algebra doesn’t close.

For the potential \([33]\), we have the symmetries

\[
\mathcal{J} = \frac{p_\eta}{2}, \quad S_{(1)} = 4\mathcal{J}^2 = p^2_\eta, \quad S_{(2)} = \mathcal{H} = 4p_\xi p_\eta + p^2_z + \frac{b_1}{\xi\eta} + \frac{b_2}{(q\xi + z)^2}, \quad S_{(3)} = M_1 + qN_5 + \frac{b_1\xi^2}{8}, \\
S_{(4)} = M_3 + 4qM_2 + 2q(1 - q^2)N_5 + \frac{b_1 z + 2q\xi}{2} + \frac{b_2(2qz + 2q^2\xi - 2(\xi + \eta))}{(q\xi + z)^2}, \\
S_{(5)} = N_3 + 4qN_2 + \frac{b_2}{(q\xi + z)^2}, \quad S_{(6)} = N_4 + 2qN_5 - \frac{b_2\xi}{(q\xi + z)^2}, \quad S_{(7)} = N_8 + \frac{b_2}{(q\xi + z)^2}, \\
S_{(8)} = N_{10} - 2qN_6 - 2q(1 + q^2)N_2 - \frac{b_1(qz - \eta)}{2} - \frac{b_2q^2z}{(q\xi + z)^2}.
\]
They satisfy
\begin{align*}
16(qz-\eta)\mathcal{J}^2 + \xi S^{0}_{(1)} - 4S^{0}_{(3)} - \xi S^{0}_{(5)} &= 16(\xi \eta + z^2)\mathcal{J}^2 - \xi^2 S^{0}_{(1)} + 4\xi S^{0}_{(2)} - 2\xi S^{0}_{(6)} - S^{0}_{(7)} = 0.
\end{align*}

The subset \(\{\mathcal{J}, S_{(1)}, S_{(2)}, S_{(3)}, S_{(5)}, S_{(6)}, S_{(7)}\}\) generates a closed quadratic algebra with nonzero relations
\begin{align*}
\{\mathcal{J}, S_{(3)}\} &= 2\mathcal{J}^2, \quad \{S_{(3)}, S_{(5)}\} = 64q^2\mathcal{J}^3, \quad \{S_{(3)}, S_{(6)}\} = -2\mathcal{J} S_{(6)}, \\
\{S_{(3)}, S_{(7)}\} &= -4\mathcal{J} S_{(7)}, \quad \{S_{(5)}, S_{(6)}\} = -8\mathcal{J} S_{(5)} - 128q^2\mathcal{J}^3 \\
\{S_{(5)}, S_{(7)}\} &= -16\mathcal{J} S_{(6)}, \quad \{S_{(6)}, S_{(7)}\} = -8\mathcal{J} S_{(7)}.
\end{align*}

However, if any linear combination of \(S_{(4)}, S_{(8)}\) is added to the generators, a new 3rd order symmetry is produced that is not a polynomial in the generators, so the resulting algebra doesn’t close.

For the potential (34), we have the symmetries
\begin{align*}
\mathcal{J} &= \frac{p_n}{2}, \quad S_{(1)} = H = 4p_\xi p_\eta + p^2 + \frac{b}{(q\xi + z)^2} + F(\xi), \quad S_{(2)} = 4\mathcal{J}^2 = p^2_\eta, \\
S_{(3)} &= M_1 + qN_5 + \frac{1}{4} \int \xi F'(\xi) d\xi, \quad S_{(4)} = N_3 + 4qN_2 + \frac{b}{(q\xi + z)^2}, \\
S_{(5)} &= N_4 + 2qN_5 - \frac{b\xi}{(q\xi + z)^2}, \quad S_{(6)} = N_8 + \frac{b\xi^2}{(q\xi + z)^2}.
\end{align*}

They satisfy
\begin{align*}
16(2qz-\eta)\mathcal{J}^2 + \xi S^{0}_{(1)} - 4S^{0}_{(3)} - \xi S^{0}_{(4)} &= 16(z^2 + \xi \eta)\mathcal{J}^2 - \xi^2 S^{0}_{(1)} + 4\xi S^{0}_{(2)} - 2\xi S^{0}_{(6)} - S^{0}_{(7)} = 0
\end{align*}
and their nonzero commutators are
\begin{align*}
\{\mathcal{J}, S_{(3)}\} &= 2\mathcal{J}^2, \quad \{S_{(3)}, S_{(4)}\} = 64q^2\mathcal{J}^3, \quad \{S_{(3)}, S_{(5)}\} = -2\mathcal{J} S_{(5)}, \quad \{S_{(3)}, S_{(6)}\} = -4\mathcal{J} S_{(6)}, \\
\{S_{(4)}, S_{(5)}\} &= -8\mathcal{J} S_{(4)} - 128q^2\mathcal{J}^3, \quad \{S_{(4)}, S_{(6)}\} = -16\mathcal{J} S_{(5)}, \quad \{S_{(5)}, S_{(6)}\} = -8\mathcal{J} S_{(6)}.
\end{align*}

For the potential (35), we have the symmetries
\begin{align*}
\mathcal{J} &= \frac{p_n}{2}, \quad S_{(1)} = H = 4p_\xi p_\eta + p^2 + \frac{b\xi}{\xi^2} + F(\xi), \quad S_{(2)} = 4\mathcal{J}^2 = p^2_\eta, \\
S_{(3)} &= M_1 - \frac{1}{2}N_4 + \frac{b\xi}{2\xi^2} + \frac{1}{4} \int \xi F'(\xi) d\xi, \quad S_{(4)} = N_2 - \frac{b}{8\xi^2}, \\
S_{(5)} &= N_5 + \frac{b}{4\xi}, \quad S_{(6)} = N_8 + \frac{b\xi}{\xi}.
\end{align*}
They satisfy $16(\xi \eta + z^2)J^2 - \xi^2 S^0_{(1)} + 4\xi S^0_{(3)} - S^0_{(6)} = 8zJ^2 - \xi S^0_{(4)} - S^0_{(5)} = 0$ and their nonzero commutators are

$$\{J, S_{(3)}\} = 2J^2, \quad \{S_{(3)}, S_{(4)}\} = -4JS_{(4)}, \quad \{S_{(3)}, S_{(5)}\} = -2JS_{(5)}.$$

$$\{S_{(4)}, S_{(6)}\} = -8JS_{(5)}, \quad \{S_{(5)}, S_{(6)}\} = -2bJ.$$

The case of the potential (36) is treated as a special case of (37) (with $a = -3/2$) below.

We consider the potential (37):

$$V(\xi, z) = bz\xi^a + F(\xi), \quad a \neq -2, -3/2, -1;$$

we cover these exclusions as special cases below. Under our assumptions we have the symmetries

$$J = \frac{p_\eta}{2}, \quad S_{(1)} = \mathcal{H} = 4p_\xi p_\eta + p_z^2 + bz\xi^a + F(\xi), \quad S_{(2)} = 4J^2 = p_\eta^2,$$

$$S_{(3)} = M_1 - \frac{a}{2(2a + 3)} N_4 + \frac{2abz\xi^{a+1}}{4(2a + 3)} + \frac{1}{4} \int \xi F'(\xi) d\xi,$$

$$S_{(4)} = N_2 + \frac{b\xi^{a+1}}{4(1 + a)}, \quad S_{(5)} = N_5 - \frac{b\xi^{a+2}}{4(a + 2)}.$$

They satisfy

$$8zJ^2 - \xi S^0_{(4)} - S^0_{(5)} = 0 \quad (43)$$

and their nonzero commutators are

$$\{J, S_{(3)}\} = 2J^2, \quad \{S_{(3)}, S_{(4)}\} = -\frac{6(a + 1)}{2a + 3} JS_{(4)},$$

$$\{S_{(3)}, S_{(5)}\} = -\frac{6(a + 2)}{2a + 3} JS_{(5)}, \quad \{S_{(4)}, S_{(5)}\} = -16J^3.$$

In the case $a = -2$ we have the symmetries

$$J = \frac{p_\eta}{2}, \quad S_{(1)} = \mathcal{H} = 4p_\xi p_\eta + p_z^2 + \frac{bz}{\xi^2} + F(\xi), \quad S_{(2)} = 4J^2 = p_\eta^2,$$

$$S_{(3)} = M_1 - N_4 + \frac{bz}{\xi} + \frac{1}{4} \int \xi F'(\xi) d\xi, \quad S_{(4)} = N_2 - \frac{b}{4\xi}, \quad S_{(5)} = N_5 - \frac{b \log \xi}{4}.$$

They satisfy (43) and their nonzero commutators are

$$\{J, S_{(3)}\} = 2J^2, \quad \{S_{(3)}, S_{(4)}\} = -6JS_{(4)}, \quad \{S_{(3)}, S_{(5)}\} = -\frac{3b}{2} J, \quad \{S_{(4)}, S_{(5)}\} = -16J^3.$$

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In the case $a = -3/2$ we have the symmetries
\[
\mathcal{J} = \frac{p_{\eta}}{2}, \quad S_{(1)} = \mathcal{H} = 4p_{\xi}p_{\eta} + p_z^2 + \frac{bz}{\xi^{3/2}} + F(\xi), \quad S_{(2)} = 4\mathcal{J}^2 = p_{\eta}^2,
\]
\[
S_{(3)} = N_4 - \frac{bz}{\xi^{1/2}}, \quad S_{(4)} = N_2 - \frac{b}{2\xi^{1/2}}, \quad S_{(5)} = N_5 - \frac{b\xi^{1/2}}{2}.
\]
They satisfy (43) and their nonzero commutators are
\[
\{S_{(3)}, S_{(4)}\} = 4\mathcal{J} S_{(4)}, \quad \{S_{(3)}, S_{(5)}\} = -4\mathcal{J} S_{(5)}, \quad \{S_{(4)}, S_{(5)}\} = -16\mathcal{J}^3.
\]

In the case $a = -1$ we have the symmetries
\[
\mathcal{J} = \frac{p_{\eta}}{2}, \quad S_{(1)} = \mathcal{H} = 4p_{\xi}p_{\eta} + p_z^2 + \frac{bz}{\xi} + F(\xi), \quad S_{(2)} = 4\mathcal{J}^2 = p_{\eta}^2,
\]
\[
S_{(3)} = M_1 - \frac{1}{2} N_4 - \frac{bz}{2} + \int \xi F'(\xi) \, d\xi, \quad S_{(4)} = N_2 + \frac{b \log \xi}{4}, \quad S_{(5)} = N_5 - \frac{b\xi}{4}.
\]
They satisfy (43) and their nonzero commutators are
\[
\{\mathcal{J}, S_{(3)}\} = 2\mathcal{J}^2, \quad \{S_{(3)}, S_{(4)}\} = -\frac{3b}{2} \mathcal{J}, \quad \{S_{(3)}, S_{(5)}\} = -6\mathcal{J} S_{(5)}, \quad \{S_{(4)}, S_{(5)}\} = -16\mathcal{J}^3.
\]

For the potential (38), we have the symmetries
\[
\mathcal{J} = \frac{p_{\eta}}{2}, \quad S_{(1)} = \mathcal{H} = 4p_{\xi}p_{\eta} + p_z^2 + b\xi + F(q\xi + z), \quad S_{(2)} = 4\mathcal{J}^2 = p_{\eta}^2,
\]
\[
S_{(3)} = M_1 + q N_5 + \frac{bk^2}{8}, \quad S_{(4)} = 4q N_2 + N_3 + F(q\xi + z),
\]
\[
S_{(5)} = 2q(1 + 2q^2) N_2 + 2q N_6 - N_{10} - \frac{bn}{2} - qbz + q^2 F(q\xi + z).
\]
They satisfy
\[
16(2qz - \eta)\mathcal{J}^2 + \xi S_{(1)}^0 - 4S_{(3)}^0 - \xi S_{(4)}^0 = 0 \quad (44)
\]
and their nonzero commutators are
\[
\{\mathcal{J}, S_{(3)}\} = 2\mathcal{J}^2, \quad \{\mathcal{J}, S_{(5)}\} = \frac{b}{4}, \quad \{S_{(3)}, S_{(4)}\} = 64q^2 \mathcal{J}^3,
\]
\[
\{S_{(3)}, S_{(5)}\} = 64q^4 \mathcal{J}^3 + 2q^2 \mathcal{J} S_{(2)} - 6q^2 \mathcal{J} S_{(4)} + 4\mathcal{J} S_{(5)}, \quad \{S_{(4)}, S_{(5)}\} = -8q^2 b \mathcal{J}.
\]

The case of the potential (39) is obtained exactly as a special case of (37) (with $a = 1$) above.
For the potential (40) (a special case of (37) with \(a = 0\), but with an additional symmetry), we have the symmetries

\[
\mathcal{J} = \frac{p_\eta}{2}, \quad \mathcal{S}_{(1)} = \mathcal{H} = 4p_\xi p_\eta + p_z^2 + b z + F(\xi), \quad \mathcal{S}_{(2)} = 4\mathcal{J}^2 = p_\eta^2,
\]

\[
\mathcal{S}_{(3)} = M_1 + \frac{1}{4} \int \xi F'(\xi) \, d\xi, \quad \mathcal{S}_{(4)} = N_2 + \frac{b \xi}{4},
\]

\[
\mathcal{S}_{(5)} = N_3 - b z, \quad \mathcal{S}_{(6)} = N_5 - \frac{b \xi^2}{8}.
\]

They satisfy

\[
16\eta \mathcal{J}^2 - \xi \mathcal{S}_{(1)} + 4 \mathcal{S}_{(3)}^0 + \xi \mathcal{S}_{(5)}^0 = 8 z \mathcal{J}^2 - \xi \mathcal{S}_{(4)}^0 - \mathcal{S}_{(6)}^0 = 0
\]

and their nonzero commutators are

\[
\{\mathcal{J}, \mathcal{S}_{(3)}\} = 2 \mathcal{J}^2, \quad \{\mathcal{S}_{(3)}, \mathcal{S}_{(4)}\} = -2 \mathcal{J} \mathcal{S}_{(4)}, \quad \{\mathcal{S}_{(3)}, \mathcal{S}_{(6)}\} = -4 \mathcal{J} \mathcal{S}_{(6)}
\]

\[
\{\mathcal{S}_{(4)}, \mathcal{S}_{(5)}\} = -2 b \mathcal{J}, \quad \{\mathcal{S}_{(4)}, \mathcal{S}_{(6)}\} = -16 \mathcal{J}^2, \quad \{\mathcal{S}_{(5)}, \mathcal{S}_{(6)}\} = -8 \mathcal{J} \mathcal{S}_{(4)}
\]

For the potential (41), we have the symmetries

\[
\mathcal{J} = \frac{p_\eta}{2}, \quad \mathcal{S}_{(1)} = \mathcal{H} = 4p_\xi p_\eta + p_z^2 + b z + F(\xi), \quad \mathcal{S}_{(2)} = 4\mathcal{J}^2 = p_\eta^2,
\]

\[
\mathcal{S}_{(3)} = M_1 + \frac{1}{4} \int \xi F'(\xi) \, d\xi, \quad \mathcal{S}_{(4)} = N_2 + \frac{b \xi}{z^2},
\]

\[
\mathcal{S}_{(5)} = N_4 - \frac{b \xi}{z^2}, \quad \mathcal{S}_{(6)} = N_8 + \frac{b \xi^2}{z^2}.
\]

They satisfy

\[
16(\eta + z^2) \mathcal{J}^2 - \xi^2 \mathcal{S}_{(1)}^0 + 4 \xi \mathcal{S}_{(3)}^0 - 2 \xi \mathcal{S}_{(5)}^0 - \mathcal{S}_{(6)}^0 = 16 \mathcal{J}^2 - \xi \mathcal{S}_{(4)}^0 + 4 \mathcal{S}_{(3)}^0 + \xi \mathcal{S}_{(6)}^0 = 0
\]

and their nonzero commutators are

\[
\{\mathcal{J}, \mathcal{S}_{(3)}\} = 2 \mathcal{J}^2, \quad \{\mathcal{S}_{(3)}, \mathcal{S}_{(5)}\} = -2 \mathcal{J} \mathcal{S}_{(5)}, \quad \{\mathcal{S}_{(3)}, \mathcal{S}_{(6)}\} = -4 \mathcal{J} \mathcal{S}_{(6)}
\]

\[
\{\mathcal{S}_{(4)}, \mathcal{S}_{(5)}\} = -8 \mathcal{J}, \quad \{\mathcal{S}_{(4)}, \mathcal{S}_{(6)}\} = -16 \mathcal{J} \mathcal{S}_{(5)}, \quad \{\mathcal{S}_{(5)}, \mathcal{S}_{(6)}\} = -8 \mathcal{J} \mathcal{S}_{(4)}
\]

For the potential (42), we consider two cases. In the first case, \(\mu_2 = 0\) and (42) reduces to (41) after a redefinition of \(F(\xi)\). In the second case, we take \(\mu_2 \neq 0\), so we define \(q = \mu_1/\mu_2\) so that (42) reduces to

\[
V(\xi, z) = \frac{b z (\xi + q z)}{\xi^2 (\xi + 2 q z)^2} + F(\xi)
\]

after a redefinition of \(F(\xi)\) and introduction of a new free parameter \(b\). For
this potential we have the symmetries
\[ J = \frac{p_q}{2}, \quad S_{(1)} = \mathcal{H} = 4p_xp_y + p_z^2 + \frac{bz(\xi + qz)}{\xi^2(\xi + 2qz)^2} + F(\xi), \quad S_{(2)} = 4J^2 = p_{\eta}^2, \]
\[ S_{(3)} = M_1 - \frac{1}{2}N_4 + \frac{bz(\xi + qz)}{2\xi(\xi + 2qz)^2} + \frac{1}{4} \int \xi F(\xi) \, d\xi, \quad S_{(4)} = N_2 + \frac{q}{2}N_3 - \frac{b}{8(\xi + 2qz)^2}, \]
\[ S_{(5)} = N_5 + qN_4 + \frac{b\xi}{4(\xi + 2qz)^2}, \quad S_{(6)} = N_8 - \frac{b\xi^2}{4q(\xi + 2qz)^2}. \]
They satisfy
\[ 16(q\eta - z)J^2 - q\xi S_{(1)}^0 + 4qS_{(3)}^0 + 2\xi S_{(4)}^0 + 2S_{(5)}^0 = 16(\xi\eta + z^2)J^2 - \xi^2 S_{(1)}^0 + 4\xi S_{(3)}^0 - S_{(6)}^0 = 0 \]
and their nonzero commutators are
\[ \{ J, S_{(3)} \} = 2J^2, \quad \{ S_{(3)}, S_{(4)} \} = -4JS_{(4)}, \quad \{ S_{(3)}, S_{(5)} \} = -2JS_{(5)} \]
\[ \{ S_{(4)}, S_{(5)} \} = -8qJS_{(4)} - 16J^3, \quad \{ S_{(4)}, S_{(6)} \} = -8JS_{(5)}, \quad \{ S_{(5)}, S_{(6)} \} = -8qJS_{(6)}. \]

5.3 Third case: \( J = xp_2 - yp_1 \)

Here the centralizer of \( J \) is the group generated by translation in \( z \) and rotations about the \( z \)-axis. We can use this freedom to simplify the computation. Since \( J \) is a symmetry the potential must be of the form \( V(x^2 + y^2, z) \).

Exactly as for the first case, writing a 2nd order symmetry in the form
\[ S = F_{11}(x, y, z) \ p_1^2 + F_{22}(x, y, z) \ p_2^2 + F_{33}(x, y, z) \ p_3^2 + F_{12}(x, y, z) \ p_1p_2 + F_{13}(x, y, z) \ p_1p_3 + F_{23}(x, y, z) \ p_2p_3 + F_0(x, y, z) \]
and requiring that \( \{ S, \mathcal{H} \} = 0 \) we have
\[ F_{11} = \frac{1}{2}c_4z^2 + (c_2y + c_5)z + \frac{1}{2}c_1y^2 + c_3y + c_6, \]
\[ F_{12} = c_{12}z^2 + (-c_{22}x - c_{17}y + c_{16})z + (-c_{11}y - c_{3})x - c_{8}y + c_{17}, \]
\[ F_{13} = c_{13}y^2 + (-c_{14}z - c_{2}x + c_{18})y + (-c_{4}z - c_{5})x - c_{13}z + c_{19}, \]
\[ F_{22} = \frac{1}{2}c_9z^2 + (c_7x + c_{10})z + \frac{1}{2}c_1x^2 + c_8x + c_{11}, \]
\[ F_{23} = c_{23}x^2 + (-c_{12}z - c_{7}y - c_{16} - c_{18})x + (-c_{9}z - c_{10})y - c_{14}z + c_{20}, \]
\[ F_{33} = \frac{1}{2}c_9y^2 + (c_{12}x + c_{14})y + \frac{1}{2}c_4x^2 + c_{13}x + c_{15} \]  \( (46) \)

where the \( c_j \) are constants to be determined. In addition we obtain a series of equations for the first derivatives \( \partial_x F_0, \partial_y F_0, \partial_z F_0 \), which lead to Bertrand-Darboux equations for \( V(x^2 + y^2, z) \). At the end we have to find 5 linearly
independent solutions for $\mathcal{S}$ and verify that they admit one functionally linearly dependent solution.

The adjoint action $\mathcal{S} \rightarrow \{xp_2 - yp_1, \mathcal{S}\} \equiv \text{Ad}_{xp_1 - yp_2}\mathcal{S}$ will map the 5-dimensional space of a solution set into itself. This action preserves the order of symmetry operators that are homogeneous in Cartesian coordinates. However, it is also convenient to pass to cylindrical coordinates $\{r, \theta, z\}$ where $x = r \cos(\theta)$, $y = r \sin(\theta)$, and

$$p_1 = p_r \cos(\theta) - p_\theta \sin(\theta)/r, \quad p_2 = p_r \sin(\theta) + p_\theta \cos(\theta)/r, \quad p_3 = p_z.$$

A complex eigenbasis for the 6-dimensional space of symmetries of order 2 is:

$$L_{\text{quad}00} = p_\theta^2$$
$$L_{\text{quad}0} = \frac{(p_r^2 r^2 z^2 - 2 p_r p_z r^3 z + p_\theta^2 r^4 + p_\theta^2 z^2)}{r^2}$$
$$L_{\text{quad}p} = -\exp(-i\theta) \frac{(izp_\theta - p_r r z + r^2 p_z) p_\theta}{2r}$$
$$L_{\text{quad}m} = -\exp(i\theta) \frac{(izp_\theta + p_r r z - r^2 p_z) p_\theta}{2r}$$
$$L_{\text{quad}m2} = -i \frac{1}{8} \exp(2i\theta) \frac{(izp_\theta + p_r r z - r^2 p_z)^2}{r^2}$$
$$L_{\text{quad}p2} = -i \frac{1}{8} \exp(-2i\theta) \frac{(izp_\theta - p_r r z + r^2 p_z)^2}{r^2}.$$ 

A complex eigenbasis for the 8-dimensional space of symmetries of order 1 is:

$$L_{\text{linear}00} = -p_\theta p_z$$
$$L_{\text{linear}0} = \frac{p_r^2 r^2 z^2 - p_r p_z r^3 z + p_\theta^2 z}{r^2}$$
$$L_{\text{linear}mm} = \frac{p_z (izp_\theta p_r r z - r^2 p_z) \exp(i\theta)}{2r}$$
$$L_{\text{linear}m} = \frac{p_\theta (izp_\theta + p_r r) \exp(i\theta)}{2r}$$
$$L_{\text{linear}pp} = \frac{p_z (izp_\theta - p_r r z + r^2 p_z) \exp(-i\theta)}{2r}$$
$$L_{\text{linear}p} = \frac{p_\theta (izp_\theta - p_r r) \exp(-i\theta)}{2r}$$
$$L_{\text{linear}p2} = \frac{(izp_\theta - p_r r z + r^2 p_z) \exp(-2i\theta) (izp_\theta - p_r r)}{4r^2}$$
$$L_{\text{linear}m2} = \frac{(izp_\theta + p_r r z - r^2 p_z) \exp(2i\theta) (izp_\theta + p_r r)}{4r^2}.$$

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A complex eigenbasis for the 6-dimensional space of symmetries of order 0 is:

\[
L_{\text{scalar}} m^2 = -\frac{(ip_\theta + rp_r)^2 \exp(2i\theta)}{4r^2} \\
L_{\text{scalar}} p^2 = \frac{(ip_\theta - rp_r)^2 \exp(-2i\theta)}{4r^2} \\
L_{\text{scalar}} m = -\frac{(ip_\theta + rp_r)p_z \exp(i\theta)}{2r} \\
L_{\text{scalar}} p = -\frac{(ip_\theta - rp_r)p_z \exp(-i\theta)}{2r} \\
L_{\text{scalar}} 00 = \frac{p^2}{r^2} + \frac{p^2}{r^2} + \frac{p^2}{r^2}. \\
L_{\text{scalar}} 0 = \frac{p^2}{r^2} + \frac{p^2}{r^2} + \frac{p^2}{r^2}.
\]

The eigenvectors (61, 60, 61) correspond to eigenvalue \(-2i\), eigenvectors (50, 55, 60, 63) correspond to eigenvalue \(-i\), eigenvectors (52, 59, 62) correspond to eigenvalue \(2i\), eigenvectors (49, 57, 58, 64) correspond to eigenvalue \(i\), and eigenvectors (47, 48, 53, 54, 65, 66) span the 6-dimensional null space of \(\text{Ad}_{p_\theta}\).

Thus the possible actions of \(\text{Ad}_{p_\theta}\) on an eigenbasis are described by the canonical forms

\[
\begin{pmatrix}
\lambda_1 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
\lambda_1 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

where \(\lambda_j = \pm i, \pm 2i\).

5.3.1 Form (67a)

Since the eigenvalues for real Euclidean space must occur in complex-conjugate pairs, a system of this form is only possible for Minkowski space. We examine all such cases and find numerous FLD systems, but none are 3-parameter superintegrable.
5.3.2 Form (67b)

We find the following potentials (in each case $F$ is an arbitrary function of its argument and $b$ is an arbitrary parameter)

\[ V(r, z) = F(r^2 + z^2) + \frac{bz}{r(r^2 + z^2)}, \quad (68) \]

\[ V(r, z) = F(r^2 + z^2) + \frac{b}{z^2}, \quad (69) \]

\[ V(r, x) = br^2 + F(z), \quad (70) \]

and

\[ V(r, z) = \frac{b}{r} + F(z), \quad (71) \]

In addition, there is the potential

\[ V(r, z) = \frac{c_1(4b_5r^2 + b_5z^2 + 2b_3z)}{4a_1b_5} - \frac{c_2}{(2b_5(a_1 + 2k_1)(b_5z + b_3))^2}, \quad (72) \]

which is not strictly 3-parameter superintegrable, since it depends on only 2 arbitrary parameters $c_1$ and $c_2$, but also is a function of the parameters $b_5, b_3, a_1, k_1$ which depend on the symmetry algebra.

Also worth mentioning is the very strange case

\[ V(r, z) = -\frac{k_1}{2r^2} + k_4 + k_{10} + \frac{k_{19}}{(z + \frac{k_7}{k_4})^2}, \quad (73) \]

where there are no free parameters and the $k'$s depend on the symmetry algebra.

5.3.3 Form (67c)

Since the eigenvalues for real Euclidean space must occur in complex-conjugate pairs, a system of this form is only possible for Minkowski space. We examine all such systems and find that none are FLD.

5.3.4 Form (67d)

Checking over all possibilities for systems with this eigenvalue form, we find that none are FLD.
5.3.5 Symmetry algebras

For the potential (68), we have the symmetries
\[ J = p_\theta, \quad S_{(2)} = H = L_{\text{scalar}} 0 + F(r^2 + z^2) + \frac{b z}{r(r^2 + z^2)}, \quad S_{(4)} = J^2 = p_\theta^2, \]
\[ S_{(3)} = L_{\text{quad}} 0 + \frac{b z}{r}, \quad S_{(5)} = L_{\text{quad}} m - \frac{ib e^{-i\theta}}{4}, \quad S_{(5)} = L_{\text{quad}} p - \frac{ib e^{i\theta}}{4}. \]
They satisfy \( ie^{i\theta} J^2 + r e^{2i\theta} S_{(4)} + r S_{(5)} = 0 \) and their nonzero commutators are
\[ \{ J, S_{(4)} \} = i S_{(4)}, \quad \{ J, S_{(5)} \} = -i S_{(5)}, \quad \{ S_{(3)}, S_{(4)} \} = -2i J S_{(4)}, \]
\[ \{ S_{(3)}, S_{(5)} \} = 2i J S_{(5)}, \quad \{ S_{(4)}, S_{(5)} \} = \frac{i}{2} J S_{(3)} - \frac{i}{2} J^3. \]

For the potential (69), we have the symmetries
\[ J = p_\theta, \quad S_{(1)} = H = L_{\text{scalar}} 0 + F(r^2 + z^2) + \frac{b z}{z^2}, \quad S_{(2)} = J^2 = p_\theta^2, \]
\[ S_{(3)} = L_{\text{quad}} 0 + \frac{b r^2}{z^2}, \quad S_{(4)} = L_{\text{quad}} p 2 - \frac{ibr e^{2i\theta}}{8 z^2}, \quad S_{(5)} = L_{\text{quad}} m 2 - \frac{ib r e^{-2i\theta}}{8 z^2}. \]
They satisfy \( 2i e^{2i\theta} J^2 - r e^{2i\theta} S_{(3)} - 4r^2 S_{(4)} + 4r^2 e^{4i\theta} S_{(5)} = 0 \) and their nonzero commutators are
\[ \{ J, S_{(4)} \} = -2i S_{(4)}, \quad \{ J, S_{(5)} \} = 2i S_{(5)}, \quad \{ S_{(3)}, S_{(4)} \} = -4i J S_{(4)}, \]
\[ \{ S_{(3)}, S_{(5)} \} = -4i J S_{(5)}, \quad \{ S_{(4)}, S_{(5)} \} = \frac{i}{8} J S_{(3)} + \frac{b}{4} J^3. \]

For the potential (70), we have the symmetries
\[ J = p_\theta, \quad S_{(1)} = H = L_{\text{scalar}} 0 + br^2 + G(z), \quad S_{(2)} = J^2 = p_\theta^2, \]
\[ S_{(3)} = L_{\text{scalar}} p 2 - \frac{br^2 e^{2i\theta}}{4}, \quad S_{(4)} = L_{\text{scalar}} m 2 + \frac{br^2 e^{-2i\theta}}{4}, \quad S_{(5)} = L_{\text{scalar}} 0 0 + G(z). \]
They satisfy \( 2e^{2i\theta} J^2 - r^2 e^{2i\theta} S_{(2)} - 2r^2 S_{(3)} r^2 e^{4i\theta} S_{(4)} + r^2 e^{2i\theta} S_{(5)} = 0 \) and their nonzero commutators are
\[ \{ J, S_{(3)} \} = -2i S_{(3)}, \quad \{ J, S_{(4)} \} = 2i S_{(4)}, \quad \{ S_{(3)}, S_{(4)} \} = ib J. \]

For the potential (71), we have the symmetries
\[ J = p_\theta, \quad S_{(1)} = J^2 = p_\theta^2, \quad S_{(2)} = H = L_{\text{scalar}} 0 + F(z) + \frac{b}{r}, \]
\[ S_{(3)} = L_{\text{linear}} p + \frac{ib e^{i\theta}}{4}, \quad S_{(4)} = L_{\text{linear}} m + \frac{ib e^{-i\theta}}{4}, \quad S_{(5)} = L_{\text{scalar}} 0 0 + F(z). \]
They satisfy \(ie^i\theta J^2 - r S^0_{(3)} - re^{2i\theta} S^0_{(4)} = 0\) and their nonzero commutators are
\[
\{J, S_{(3)}\} = -iS_{(3)}, \quad \{J, S_{(4)}\} = iS_{(4)}, \quad \{S_{(3)}, S_{(4)}\} = \frac{i}{2} J (S_{(5)} - S_{(2)}).
\]

We write the potential (72) as
\[
V(r, z) = b_1 (4r^2 + z^2 + 2qz) + \frac{b_2}{(z + q)^2},
\]
where \(b_1\) and \(b_2\) are free parameters and \(q\) is related to the symmetry algebra.

We have the symmetries
\[
J = p\theta, \quad S_{(1)} = H = L_{scalar} 0 + p_x^2 + b_1 (4r^2 + z^2 + 2qz) + \frac{b_2}{(z + q)^2}, \quad S_{(2)} = J^2 = p_\theta^2
\]
\[
S_{(3)} = L_{linear}pp - qL_{scalar}p + \frac{re^{i\theta}(b_1(z + q)^4 - b_2)}{2(z + q)^2},
\]
\[
S_{(4)} = L_{linear}mm - qL_{scalar}m - \frac{re^{-i\theta}(b_1(z + q)^4 - b_2)}{2(z + q)^2},
\]
\[
S_{(5)} = L_{scalar} p2 - b_1 r^2 e^{2i\theta}, \quad S_{(6)} = L_{scalar} m2 + b_1 r^2 e^{-2i\theta},
\]
\[
S_{(7)} = L_{scalar} 00 + b_1 z(z + 2q) + \frac{b_2}{(z + q)^2}.
\]

They satisfy \(J^2 - r^2 S^0_{(1)} - 2r^2 e^{-2i\theta} S_{(5)} + 2r^2 e^{2i\theta} S_{(6)} + r^2 S_{(7)} = 0\). The subset \(\{J, S_{(1)}, S_{(2)}, S_{(5)}, S_{(6)}, S_{(7)}\}\) generates a closed quadratic algebra with nonzero relations
\[
\{J, S_{(5)}\} = -2iS_{(5)}, \quad \{J, S_{(6)}\} = 2iS_{(6)}, \quad \{S_{(5)}, S_{(6)}\} = 4ib_1J.
\]

However, if any linear combination of \(S_{(3)}, S_{(4)}\) is added to the generators, a new 3rd order symmetry is produced that is not a polynomial in the generators, so the resulting algebra doesn’t close.

### 5.4 Fourth case: \(J = J_{12} + iJ_{23}\)

We make the change of variables
\[
x = -\rho \left[e^{-\theta} + e^\theta \left(1/4 - r^2\right)\right], \quad y = -\rho r \exp(\theta), \quad z = \rho \left[e^{-\theta} - e^\theta \left(1/4 + r^2\right)\right]
\]
so that \(p_r = 2(J_{12} + iJ_{23})\). As before, we have the symmetries
\[
S = F_{11}(x, y, z) p_x^2 + F_{22}(x, y, z) p_y^2 + F_{33}(x, y, z) p_z^2 + F_{12}(x, y, z) p_1 p_2 +
\]

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Here, we have two separate chains: 

\[ F_{13}(x, y, z) p_1 p_3 + F_{23}(x, y, z) p_2 p_3 + F_0(x, y, z), \]

where

\[
\begin{align*}
F_{11} &= \frac{1}{2} c_4 z^2 + (c_2 y + c_3) z + \frac{1}{2} c_1 y^2 + c_3 y + c_6, \\
F_{12} &= c_{12} z^2 + (-c_2 x - c_7 y + c_{16}) z + (-c_1 y - c_3) x - c_8 y + c_{17}, \\
F_{13} &= c_7 y^2 + (-c_{12} z - c_2 x + c_{18}) y + (-c_4 z - c_3) x - c_{13} z + c_{19}, \\
F_{22} &= \frac{1}{2} c_9 z^2 + (c_7 x + c_{10}) z + \frac{1}{2} c_1 x^2 + c_8 x + c_{11}, \\
F_{23} &= c_2 x^2 + (-c_{12} z - c_7 y - c_16 - c_{18}) x + (-c_9 z - c_{10}) y - c_{14} z + c_{20}, \\
F_{33} &= \frac{1}{2} c_9 y^2 + (c_{12} x + c_{14}) y + \frac{1}{2} c_4 x^2 + c_{13} x + c_{15}. \quad (74)
\end{align*}
\]

A basis for the six-dimensional space of order-two symmetries is

\[
\begin{align*}
L_1 &= \frac{1}{384} \left[ (4 e^{-2 \theta} + 4 r^2 + 1) p_r - 8 r p_\theta \right]^2 \\
L_2 &= -\frac{1}{24} (r p_r - p_\theta) \left[ (4 e^{-2 \theta} + 4 r^2 + 1) p_r - 8 r p_\theta \right] \\
L_3 &= \frac{1}{24} (12 r^2 + 4 e^{-2 \theta} + 1) p_r^2 - 12 r p_r p_\theta + 4 p_\theta^2 \\
L_4 &= -r p_r^2 + p_r p_\theta, \quad L_5 = p_r^2, \quad L_6 = e^{-2 \theta} p_r^2 - p_\theta^2.
\end{align*}
\]

Here, \( \{L_1, L_2, L_3, L_4, L_5 = p_r^2\} \) form a chain and \( \{L_5 = p_r^2, L_6\} \subset \ker \text{Ad}_{p_r} \).

A basis for the eight-dimensional space of order-one symmetries is

\[
\begin{align*}
M_1 &= \frac{1}{384 \rho} (4 e^{-3 \theta} p_r + e^{-\theta} [(4 r^2 + 1) p_r - 8 r p_\theta]) \\
&\quad \times \left[ e^{\theta} (4 r^2 - 1) (\rho p_\rho - p_\theta) - 4 (\rho p_\rho - 2 r p_r + p_\theta) \right] \\
M_2 &= -\frac{4 e^{-3 \theta} p_r^2 + e^{\theta} ((12 r^2 - 1) p_\theta - 8 r^3 p_r) (\rho p_\rho - p_\theta)}{48 \rho} \\
&\quad - \frac{e^{-\theta} [(1 + 12 r^2) p_r^2 + 4 p_\theta (\rho p_\rho + p_\theta) - 24 r p_r p_\theta]}{48 \rho} \\
M_3 &= \frac{1}{2 \rho} (r p_r - p_\theta) (r e^{\theta} (\rho p_\rho - p_\theta) - e^{-\theta} p_r), \quad M_4 = \frac{e^{\theta} (\rho p_\rho - p_\theta) (p_\theta - 2 r p_r) - e^{-\theta} p_r^2}{2 \rho} \\
M_5 &= \frac{e^{\theta} p_r (\rho p_\rho - p_\theta)}{\rho}, \quad M_6 = \frac{-4 e^{-3 \theta} p_r^2 + e^{\theta} ((1 + 4 r^2) p_\theta - 2 r p_r) (\rho p_\rho - p_\theta)}{8 \rho} \\
&\quad - \frac{e^{-\theta} [(1 - 4 r^2) p_r^2 + 8 r p_r p_\rho - 4 p_\theta (\rho p_\rho + p_\theta)]}{8 \rho} \\
M_7 &= \frac{e^{\theta} (\rho p_\rho - p_\theta) (p_r - 4 r p_\theta) + 4 e^{-\theta} p_r (\rho p_\rho - r p_r)}{\rho}, \quad M_8 = \frac{e^{\theta} p_r (\rho p_\rho - p_\theta) + e^{-\theta} p_r^2}{\rho}
\end{align*}
\]

Here, we have two separate chains: \( \{M_1, M_2, M_3, M_4, M_5\} \) and \( \{M_6, M_7, M_8\} \): \( \{M_5, M_8\} \subset \ker \text{Ad}_{p_r} \).
A basis for the six-dimensional space of order-zero symmetries is

\[
N_1 = \frac{e^{-2\theta} [e^{2\theta} (1 - 4r^2)(p_\theta - pp_{r}) - 4(pp_{r} - 2rp_{r} + p_\theta)]^2}{384\rho^2},
\]

\[
N_2 = -\frac{e^{-2\theta}(p_r - e^{-2\theta}r(p_\theta - pp_{r})) (e^{2\theta} (1 - 4r^2)(p_\theta - pp_{r}) - 4(pp_{r} - 2rp_{r} + p_\theta))}{24\rho^2},
\]

\[
N_3 = \frac{8e^{-2\theta}p_r^2 + e^{2\theta} (12r^2 - 1)(pp_{r} - p_\theta)^2 - 4(pp_{r} - p_\theta)(pp_{r} - 6rp_{r} + p_\theta)}{24\rho^2},
\]

\[
N_4 = -\frac{(pp_{r} - p_\theta)(p\theta e^{2\theta} p_{r} - p_r - r e^{2\theta} p_\theta)}{\rho^2}, \quad N_5 = \frac{e^{2\theta}(pp_{r} - p_\theta)^2}{\rho^2}, \quad N_6 = p_r^2 + \frac{e^{-2\theta}p_r^2 - p_\theta^2}{\rho^2}.
\]

Here, \( \{N_1, N_2, N_3, N_4, N_5\} \) form a chain and \( \{N_5, N_6 = H\} \subset \ker \text{Ad}_{p_r} \).

The possible canonical forms are

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad (75)
\]

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

5.4.1 Form (75a)

Here we have a 5-chain. This form does not occur because it cannot not contain \( \mathcal{H} \) (moreover, \( \mathcal{H} \) and \( p_r^2 \) cannot be in the same chain).

5.4.2 Form (75b)

This form can (and must) contain both \( p_r^2 \) and \( \mathcal{H} \). Because \( \mathcal{H} \) is not in a nontrivial chain, the basis must be

\[ \{H, L_2 + \beta L_3 + \gamma L_4 + \delta_1 L_5 + \delta_2 L_6, L_3 + \beta L_4 + \gamma L_5, L_4 + \beta L_5, L_5 = p_r^2\} \]
but we can take $\beta = \gamma = \delta = 0$ by a canonical form-preserving change of basis. The chain \( \{L_2 + \delta L_6, L_3, L_4, L_5\} \) is FLD but does not correspond to an admissible potential.

### 5.4.3 Form (76c)

Here we have a 3-chain and a 2-chain. This form does not occur because it cannot contain $H$.

### 5.4.4 Form (76d)

Here we have a 3-chain and two 1-chains. One of the 1-chains is $H$. First suppose the second one-chain is $p_2 r$. Using canonical form-preserving changes of basis when necessary, the possible 3-chains are equivalent to one of \{\(N_3, N_4, N_5\)\}, \{\(\alpha M_3 + M_6 + \beta M_7 + \gamma M_8\), \(M_4 + \alpha M_7 + \beta M_8\), \(M_5 + \alpha M_8\)\}. The first case is FLD and provides the admissible potential

\[
V(\rho, \theta) = b\rho^2 + F(\rho e^{i\theta}).
\]  
(76)

The second case is not FLD. The third case is FLD when $\alpha = 1$, $\beta = \gamma = 0$, and $\delta = -1/2$ but does not yield an admissible potential.

### 5.4.5 Form (76e)

Here we have two 2-chains and a 1-chain, which must be $H$. One of the 2-chains must be \(\{L_4 + \mu L_6, L_5\}\). The possibilities for the other 2-chain are (after canonical form-preserving changes of basis) \(\{\alpha M_4 + M_7 + \gamma M_5, \alpha M_5 + M_8\}\) or \(\{M_4 + \beta M_7 + \delta M_8, M_5 + \beta M_8\}\). Only the latter (together with $L_5$) is FLD when $\alpha = 1$, $\beta = \gamma = 0$, and $\delta = -1/2$ but does not yield an admissible potential.
5.4.6 Form (76f)

Here we have a 2-chain and three 1-chains, one of which must be $\mathcal{H}$. We first assume that the 2-chain is $\{L_4 + \mu L_6, L_5\}$. There are then four ways to choose the remaining two one chains: $\{N_5, \alpha M_5 + \beta M_8, L_6\}$ (in which case we take $\mu = 0$), $\{N_5, L_6\}$ (again, $\mu = 0$), or $\{M_5, M_8\}$. The first case is FLD when $\alpha = 0$ but does yield an admissible potential. The second, third, and fourth cases are not FLD.

If the 2-chain is not $\{L_4 + \mu L_6, L_5\}$, one of the 1-chains must be $L_5 = p^2 r$. Then we have one 1-chain ($N_5$, $L_6$, or $\mu M_5 + \nu M_8$) and one 2-chain ($\{N_4, N_5\}$, $\{\alpha M_4 + M_7 + \gamma M_5, \alpha M_5 + M_8\}$, or $\{M_4 + \beta M_7 + \delta M_8, M_5 + \beta M_8\}$) to choose. There are several FLD bases but only one leads to an admissible potential:

$$V(\rho, \theta) = \frac{b e^{-3\theta}}{\rho} + F(\rho e^\theta).$$  \hspace{1cm} (78)

5.4.7 Form (76g)

This case consists of five 1-chains, two of which must be $\mathcal{H}$ and $p^2 r$. There are therefore three subcases to consider: the remaining symmetries are either $\{L_6, \alpha M_5 + \beta M_8, N_5\}$, $\{L_6, M_5, M_8\}$, or $\{M_5, M_8, N_5\}$. The first and third cases are FLD in certain cases but the corresponding potentials do not have 3 parameters.

5.4.8 Structure algebras

For the potential (76), we have the symmetries

$$J = \frac{p_r}{2}, \quad S_{(1)} = \mathcal{H} = N_6 + b \rho^2 + F(\rho e^\theta), \quad S_{(2)} = 4J^2 = p_r^2,$$

$$S_{(3)} = L_5 + \frac{-[4 + e^{28}(1 - 12\rho^2)]b \rho^2 - 4F(\rho e^\theta)}{24}, \quad S_{(4)} = N_4 - b \rho^2 e^{28}, \quad S_{(5)} = N_5 + b \rho^2 e^{28}.$$

They satisfy $\rho^2 e^{28}\left(4S_{(1)}^0 + 24S_{(3)}^0 + 24S_{(4)}^0 + (1 + 12\rho^2)S_{(5)}^0\right) - 48J^2 = 0$ and their nonzero commutators are

$$\{J, S_{(3)}\} = \frac{1}{2}S_{(4)}, \quad \{J, S_{(4)}\} = \frac{1}{2}S_{(5)}, \quad \{S_{(3)}, S_{(4)}\} = 2bJ.$$

For the potential (77), we have the symmetries

$$J = \frac{p_r}{2}, \quad S_{(1)} = \mathcal{H} = N_6 + \frac{e^{-29}(p_r^2 + b)}{\rho^2} + F(\rho), \quad S_{(2)} = 4J^2 + \nu_0 = p_r^2 + \nu_0,$$

$$S_{(3)} = L_3 + \frac{b\nu_0}{2} - \frac{be^{-29}}{3}, \quad S_{(4)} = L_4 - b\nu, \quad S_{(5)} = L_6 + be^{-29}.$$  

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They satisfy \((1 + 12e^{-2\theta} - 12r^2)J^2 - 6S_{(3)}^0 - 6rS_{(4)}^0 - 2S_{(5)}^0 = 0\) and their nonzero commutators are
\[
\{J, S_{(3)}\} = \frac{1}{2}S_{(4)}, \quad \{J, S_{(4)}\} = \frac{1}{2}S_{(2)},
\]
\[
\{S_{(3)}, S_{(4)}\} = -4JS_{(3)} + \frac{2}{3}JS_{(5)} + \frac{4}{3}J^3 + \frac{bJ}{6}.
\]

For the potential \((78)\), we have the symmetries
\[
J = \frac{p_\rho}{2}, \quad S_{(1)} = H = N_6 + \frac{be^{-3\theta}}{\rho} + F(e^\theta \rho), \quad S_{(2)} = 4J^2 = p_\rho^2,
\]
\[
S_{(3)} = M_4 - \frac{1}{2}M_8 - \frac{bpr^2}{2\rho}, \quad S_{(4)} = M_5 + br, \quad S_{(5)} = N_5 - \frac{2be^{-\theta}}{\rho}.
\]
They satisfy \(4J^2 + \rho e^{\theta}S_{(3)}^0 + \rho e^{-\theta}S_{(4)}^0 = 0\), and their nonzero commutators are
\[
\{J, S_{(3)}\} = \frac{1}{2}S_{(4)}, \quad \{J, S_{(4)}\} = -\frac{b}{2}, \quad \{S_{(3)}, S_{(4)}\} = -JS_{(4)}.
\]

### 5.5 Fifth case: \(J = -iJ_{12} + J_{23} - ip_1 + p_3\)

This case does not occur for complex Euclidean systems since the symmetry \(J\) is not homogeneous.

### 6 The complex 3-sphere

We choose a standardized Cartesian-like coordinate system \(\{x, y, z\}\) on the 3-sphere such that the Hamiltonian is
\[
\mathcal{H} = (1 + \frac{r^2}{4})^2(p_x^2 + p_y^2 + p_z^2) + V, \quad (79)
\]
where \(r^2 = x^2 + y^2 + z^2\). These coordinates can be related to the standard realization of the sphere via complex coordinates \(s = (s_1, s_2, s_3, s_4)\) such that \(\sum_{j=1}^{4} s_j^2 = 1\) and \(ds^2 = \sum_{j} ds_j^2\) via
\[
s_1 = \frac{4x}{4 + r^2}, \quad s_2 = \frac{4y}{4 + r^2}, \quad s_3 = \frac{4z}{4 + r^2}, \quad s_4 = \frac{4 - r^2}{4 + r^2}, \quad (80)
\]
with inverse \(x = 2s_1/(1 + s_4), y = 2s_2/(1 + s_4), z = 2s_3/(1 + s_4)\). A basis of Killing vectors for the zero potential system is \(J_h, K_h, h = 1, 2, 3\) where
\[
J_{23} = yp_z - zp_y, \quad J_{31} = xp_z - zp_x, \quad J_{12} = xp_y - yp_x, \quad (81)
\]

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\[ \begin{align*}
K_1 &= (1 + \frac{x^2 - y^2 - z^2}{4})p_x + \frac{xy}{2} p_y + \frac{xz}{2} p_z, \\
K_2 &= (1 + \frac{y^2 - x^2 - z^2}{4})p_y + \frac{xy}{2} p_x + \frac{yz}{2} p_z, \\
K_3 &= (1 + \frac{z^2 - x^2 - y^2}{4})p_z + \frac{xz}{2} p_x + \frac{yz}{2} p_y.
\end{align*} \]

The relation between this basis and the standard basis of rotation generators on the sphere \( I_{\ell m} = s_{\ell p_m} - s_m p_{\ell} = -I_{m\ell} \) is

\[ J_{23} = I_{23}, \quad J_{31} = I_{31}, \quad J_{12} = I_{12}, \quad K_1 = I_{41}, \quad K_2 = I_{42}, \quad K_3 = I_{43}. \quad (82) \]

To solve the classification problem on the complex 3-sphere we can use methods analogous to those for Euclidean space. From Corollary 1 applied to the 3-sphere we see that, up to conjugacy, there are just 2 cases to consider: \( \mathcal{J} = J_{12} \) and \( \mathcal{J} = J_{12} + iJ_{23} \). The details are complicated but we find that there are no Calogero-like superintegrable systems on the complex 3-sphere. To save space we do not provide the details here. They can be found in the arXiv paper [18].

7 Conclusions

This paper is part of a program to classify all 2nd order superintegrable classical and quantum systems on 3-dimensional conformally flat complex manifolds. We have worked out the basic structure theory for Calogero-like superintegrable systems on these manifolds and classified all such systems on constant curvature spaces. There turn out to be no such systems on the complex 3-sphere. For complex Euclidean space we find systems (17), (19), (26-42), (68-72), and (76-78). In most of the cases the potential depends on at least one arbitrary function. The key to the classification is a proof that all such systems admit a 1st order symmetry.

We note that for all of these systems we can find a complete integral for the Hamilton-Jacobi equation. For example, the system (37) with \( a = -3 \), has the Hamilton-Jacobi equation

\[ H = 4 \frac{\partial S}{\partial \xi} \frac{\partial S}{\partial \eta} + \left( \frac{\partial S}{\partial z} \right)^2 + \frac{bz}{\xi^3} + F(\xi) = E. \]

For this equation, we find the complete integral

\[ S(\xi, \eta, z) = \frac{b^2}{768c_1^3 \xi^3} + \frac{b(2c_1^2 z + c_2 \xi)}{16c_1^2 \xi^2} + c_1 \eta + \frac{c_1^2 (4c_2 z + Ez) - c_2^2 \xi}{4c_1^3} - \frac{1}{4c_1} \int F(\xi) \, d\xi \]

where \( c_1, c_2 \) are arbitrary constants and another constant appears in the integral of \( F \).
The symmetry algebras for these FLD superintegrable systems don’t always close. However the symmetries always provide some information about the classical trajectories of solutions of the Hamilton-Jacobi equation. If a superintegrable system is functionally linearly independent the trajectories are uniquely determined, If the system is FLD then we can solve for one of the constants of the motion in terms of the others. Thus a 2-parameter manifold can be computed from the symmetries such that the trajectories of solutions of the Hamilton-Jacobi equation must lie on this manifold.

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