Energy and momentum associated with a Static Axially Symmetric Vacuum Space-Time

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Abstract
We use the Einstein and Papapetrou energy-momentum complexes to calculate the energy and momentum densities of Weyl metric as well as Curzon metric. We show that these two different definitions of energy-momentum complexes do not provide the same energy density for Weyl metric, although they give the same momentum density. We show that, in the case of Curzon metric, these two definitions give the same energy only when $R \to \infty$. Furthermore, we compare these results with those obtained using Landau and Lifshitz, Bergmann and Møller.

1 Introduction
One of the most interesting and intricate problems which remains unsolved since the outset of the general theory of relativity is the energy-momentum localization. After Einstein’s energy-momentum complex (E) [1], used for calculating energy and momentum in a general relativistic systems, many physicists, such as, Tolman (T) [2], Landau and Lifshitz (LL) [3], Papapetrou (P) [4], Bergmann (B) [5] and Weinberg (W) [6] (abbreviated to (ETLLPBW), in the sequel) had given different definitions for the energy-momentum complexes. These definitions were restricted to the use of quasi-Cartesian coordinates. Møller (M) [7] introduced a consistent expression which enables one to evaluate energy and momentum in any coordinate system. Some interesting results obtained recently lead to the conclusion that these prescriptions give the same energy distribution for a given space-time [8]-[15]. Aguirregabiria, Chamorro and Virbhadra [16] showed that the five different\(^2\) energy-momentum complexes (ELLPBW) give the same result for the energy distribution for any Kerr-Schild metric. Recently, Virbhadra

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\(^2\)Virbhadra [17] has shown that Tolman’s and Einstein’s are exactly the same
[17] investigated whether or not these definitions (ELLPBW) lead to the same result for the most general non-static spherically symmetric metric and found they disagree. He noted that the energy-momentum complexes (LLPW) give the same result as in the Einstein prescription if the calculations are performed in Kerr-Schild Cartesian coordinates. However, the complexes (ELLPW) disagree if computations are done in “Schwarzschild Cartesian coordinates”. 

Some interesting results [18]-[25] led to the conclusion that in a given space-time, such as: the Reissner-Nordstörm, the de Sitter-Schwarzschild, the charged regular metric, the stringy charged black hole and the Gödel-type space-time, the energy distribution according to the energy-momentum complex of Einstein is different from of Møller. But in some specific case [7, 18, 20, 26, 17] (the Schwarzschild, the Janis-Newman-Winicour metric) have the same result.

The scope of this paper is to evaluate the energy and momentum densities for the Weyl as well as Curzon metrics using the Einstein and Papapetrou energy-momentum complexes. Through this paper we use $G = 1$ and $c = 1$ units and follow the convention that Latin indices take value from 0 to 3 and Greek indices take value from 1 to 3.

The general static axially symmetric vacuum solution of Einstein’s field equations is given by the Weyl metric [27]

$$ds^2 = e^{2\lambda}dt^2 - e^{2(\nu-\lambda)}(dr^2 + dz^2) - r^2 e^{-2\lambda}d\phi^2$$

where

$$\lambda_{rr} + \lambda_{zz} + r^{-1}\lambda_r = 0$$

and

$$\nu_r = r(\lambda_r^2 - \lambda_z^2), \quad \nu_z = 2r\lambda_r\lambda_z.$$ 

It is well known that if the calculations are performed in quasi-Cartesian coordinates, all the energy-momentum complexes give meaningful results. According to the following transformations

$$r = \sqrt{x^2 + y^2}, \quad \phi = \arctan\left(\frac{y}{x}\right),$$

the line element (1) written in terms of quasi-Cartesian coordinates reads:

$$ds^2 = e^{2\lambda}dt^2 - \frac{1}{r^2}(x^2 e^{2(\nu-\lambda)} + y^2 e^{-2\lambda})dx^2 - \frac{2xy}{r^2}(e^{2(\nu-\lambda)} - e^{-2\lambda})dxdy - \frac{2y}{x^2 + y^2}.$$  

\(^3\text{Schwarzschild metric in “Schwarzschild Cartesian coordinates” is obtained by transforming this metric (in usual Schwarzschild coordinates \(\{t, r, \theta, \phi\}\)) to \(\{t, x, y, z\}\) using}\)

\(x = r \sin \theta \cos \phi, x = r \sin \theta \sin \phi, z = r \cos \theta.\)
\[
\frac{1}{r^2}(y^2 e^{2(\nu-\lambda)} + x^2 e^{-2\lambda})dy^2 - e^{2(\nu-\lambda)}dz^2,
\]  
\[\text{(2)}\]

where

\[
x^2 \lambda_{xx} + y^2 \lambda_{yy} + 2xy \lambda_{xy} + r^2 \lambda_{zz} + x \lambda_x + y \lambda_y = 0,
\]

\[
x \nu_x + y \nu_y - (x \lambda_x + y \lambda_y)^2 + r^2 \lambda_z = 0
\]

and

\[\nu_z = 2 \lambda_z (x \lambda_x + y \lambda_y).\]

For the above metric the determinant of the metric tensor and the contravariant components of the tensor are given, respectively, as follows

\[
\begin{align*}
det g &= -e^{4(\nu-\lambda)}, \\
g^{00} &= e^{-2\lambda}, \\
g^{11} &= -\frac{e^{2\lambda}}{r^2}(y^2 + x^2 e^{-2\nu}), \\
g^{12} &= \frac{xe^{2\lambda}}{r^2}(1 - e^{-2\nu}), \\
g^{22} &= -\frac{e^{2\lambda}}{r^2}(x^2 + y^2 e^{-2\nu}), \\
g^{33} &= -e^{2(\lambda-\nu)}.
\end{align*}
\]  
\[\text{(3)}\]

2 Energy-Momentum Complexes

The conservation laws of matter plus non-gravitational fields for physical system in the special theory of relativity are given by

\[T_{\nu,\mu}^\mu \equiv \frac{\partial T_{\nu}^\mu}{\partial x^\mu} = 0,\]
\[\text{(4)}\]

where \(T_{\nu}^\mu\) denotes the symmetric energy-momentum tensor in an inertial frame.

The generalization of equation (4) in the theory of general relativity is written as

\[T_{\nu,\mu}^\mu = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} (\sqrt{-g} T_{\nu}^\mu) - \Gamma_{\nu,\lambda}^\mu T_{\lambda}^\mu = 0,\]
\[\text{(5)}\]

where \(g\) is the determinant of the metric tensor \(g_{\mu\nu}(x)\).

The conservation equation may also be written as

\[\frac{\partial}{\partial x^\mu} (\sqrt{-g} T_{\nu}^\mu) = \xi_{\nu},\]
\[\text{(6)}\]

where

\[\xi_{\nu} = \sqrt{-g} \Gamma_{\nu,\lambda}^\mu T_{\lambda}^\mu\]
is a non-tensorial object and it can be written as

\[ \xi_\nu = -\frac{\partial}{\partial x^\mu}(\sqrt{-g}t^\mu_\nu). \]  

where \( t^\mu_\nu \) are certain functions of the metric tensor and its first derivatives.

Now combining equation (7) with equation (6) we get the following equation expressing a local conservation law:

\[ \Theta^\mu_{\nu,\mu} = 0, \]  

where

\[ \Theta^\mu_\nu = \sqrt{-g}(T^\mu_\nu + t^\mu_\nu) \]

which is called energy-momentum complex since it is a combination of the tensor, \( T^\mu_\nu \), of matter and all non-gravitational fields, and a pseudotensor \( t^\mu_\nu \) which describes the energy and momentum of the gravitational field\(^4\).

Equation (9) can be written as

\[ \Theta^\mu_\nu = \chi^{\mu\lambda}_{\nu,\lambda}, \]

where \( \chi^{\mu\lambda}_{\nu,\lambda} \) are called superpotentials and are functions of the metric tensor and its first derivatives.

### 3 Energy-momentum Complex in Einstein’s Prescription

The energy-momentum complex as defined by Einstein [1] is given by

\[ \theta^k_i = \frac{1}{16\pi}H^{kl}_{i,l}, \]

where the Einstein’s superpotential \( H^{kl}_{i,l} \) is of the form

\[ H^{kl}_{i} = -H^{lk}_{i} = \frac{g_{lm}}{\sqrt{-g}}[ -g(g^{kn}g^{lm} - g^{ln}g^{km})]_{,m}. \]

\( \theta^0_0 \) and \( \theta^0_\alpha \) are the energy and momentum density components, respectively. The energy-momentum complex \( \theta^k_i \) satisfies the local conservation law

\[ \frac{\partial \theta^k_i}{\partial x^k} = 0 \]

\(^4\)By introducing a local system of inertia, the gravitational part \( t^\mu_\nu \) can always be reduced to zero for any given space-time.
In order to evaluate the energy and momentum densities in Einstein’s prescription associated with the Weyl metric, we evaluate the non-zero components of $H^{kl}_{i}$

\[
H^{01}_{0} = \frac{1}{r^2} \left[ xye^{2\nu}(4\nu_y - 4\lambda_y) - xy(2\nu_y - 4\lambda_y) - y^2e^{2\nu}(4\nu_x - 4\lambda_x) - x^2(2\nu_x - 4\lambda_x) + x(e^{2\nu} - 1) \right] \\
H^{02}_{0} = \frac{1}{r^2} \left[ xye^{2\nu}(4\nu_x - 4\lambda_x) - xy(2\nu_x - 4\lambda_x) - x^2e^{2\nu}(4\nu_y - 4\lambda_y) - y^2(2\nu_y - 4\lambda_y) + y(e^{2\nu} - 1) \right] \\
H^{03}_{0} = 4\lambda_z - 2\nu_z.
\]

(13)

Using these components in equation (11), we get the energy and momentum densities as following

\[
\theta^0 = \frac{1}{16\pi} \left[ (2\nu_x - 4\lambda_x)(xe^{2\nu} + 2ye^{2\nu}(x\nu_y - y\nu_x)) + (2\nu_y - 4\lambda_y)(ye^{2\nu} + 2xe^{2\nu}(y\nu_x - x\nu_y)) + 4e^{2\nu}(y\nu_y + x\nu_x) - 4e^{2\nu}(x\nu_y - y\nu_x)^2 - 4xy\nu_{xy} - y^2e^{2\nu}(4\nu_y - 4\lambda_y) - 2x^2\nu_{xx} - 2y^2\nu_{yy} - 2(x\nu_x + y\nu_y) + 2xye^{2\nu}(4\nu_x - 4\lambda_x) - 2r^2\nu_{zz} \right], \\
\theta^\nu = 0.
\]

The momentum components are vanishing everywhere.

We now restrict our selves to the particular solutions of Curzon metric [28] obtained by setting

\[
\lambda = -\frac{m}{R} \quad \text{and} \quad \nu = -\frac{m^2r^2}{2R^4}, \quad R = \sqrt{r^2 + z^2}
\]

in equation (1).

For this solution it is found from equation (13) that the non-zero components of $H^{kl}_{i}$ take the form

\[
H^{01}_{0} = x \left[ \frac{2m^2}{R^6} - \frac{4m^2r^2}{R^4} + \frac{4m}{R^3} + \frac{1}{r}(e^{2\nu} - 1) \right] \\
H^{02}_{0} = y \left[ \frac{2m^2}{R^6} - \frac{4m^2r^2}{R^4} + \frac{4m}{R^3} + \frac{1}{r}(e^{2\nu} - 1) \right] \\
H^{03}_{0} = \frac{4mz}{R^3} \left[ 1 - \frac{m^2}{R^2} \right].
\]

(14)

Using these components the energy and momentum densities for the Curzon solution become

\[
\theta^0 = \frac{1}{16\pi} \left[ -\frac{4m^2r^2}{R^6} + \frac{4m^2}{R^4} + 2e^{2\nu} \left( -\frac{m^2}{R^4} + \frac{2m^2r^2}{R^6} \right) \right], \\
\theta^\nu = 0.
\]

(15)

(16)

The momentum components are vanishing everywhere.
4 The Energy-Momentum Complex of Papapetrou

The symmetric energy-momentum complex of Papapetrou [4] is given by

$$\Omega^{ij} = \frac{1}{16\pi} \gamma^{ijkl}_{,kl}$$

(17)

where

$$\gamma^{ijkl}_{,kl} = \sqrt{-g} (g^{ij} \eta^{kl} - g^{ik} \eta^{jl} + g^{jl} \eta^{ij} - g^{jl} \eta^{ik}),$$

(18)

and $\eta^{ik}$ is the Minkowski metric with signature $-2$.

$\Omega^{00}$ and $\Omega^{\alpha0}$ are the energy and momentum density components. In order to calculate the energy and momentum density components for Weyl metric, using the symmetric energy momentum complexes of Papapetrou, we require the following non-vanishing components of $\gamma^{ijkl}$

$$\gamma^{0011} = - (e^{2\nu} - 4\lambda x - \frac{y^2 e^{2\nu}}{r^2} + \frac{2 e^{2\nu}}{r^2}),$$

(19)

$$\gamma^{0012} = \frac{xy e^{2\nu}}{r^2},$$

$$\gamma^{0022} = - (e^{2\nu} - 4\lambda y - \frac{x^2 e^{2\nu}}{r^2} - \frac{2 e^{2\nu}}{r^2}),$$

$$\gamma^{0033} = - e^{2\nu} + 1.$$

Using these components in (17), we get the energy and momentum densities in the following form

$$\Omega^{00} = \frac{1}{16\pi} \left[ - e^{2\nu} - 4\lambda \right] \left( 2\nu x - 4\lambda x \right)^2 + 2\nu x - 4\lambda x + \left( 2\nu y - 4\lambda y \right)^2 + 2\nu y - 4\lambda y +$$

$$\left( 2\nu z - 4\lambda z \right)^2 + 2\nu z - 4\lambda z + \frac{4 e^{2\nu}}{r^2} \left( y \nu x - x \nu y \right)^2 - \frac{2 y^2 e^{2\nu}}{r^2} \nu xx - \frac{2 x^2 e^{2\nu}}{r^2} \nu yy +$$

$$\frac{4 y e^{2\nu}}{r^2} \nu y + \frac{4 x e^{2\nu}}{r^2} \nu x + \frac{4 x y e^{2\nu}}{r^2} \nu xy \right].$$

(20)

$$\Omega^{0\alpha} = 0.$$

(21)

For the Curzon solution the energy and momentum densities become

$$\Omega^{00} = \frac{1}{16\pi} \left[ - e^{2\nu} - 4\lambda \right] \left( \frac{4 m^4 r^2}{R^8} + \frac{12 m^2}{R^4} - \frac{16 m^2}{R^6} + \frac{4 m^2}{R^4} \right) +$$

$$2 e^{2\nu} \left( \frac{2 m^2 r^2}{R^6} - \frac{m^2}{R^4} \right)$$

(22)

$$\Omega^{0\alpha} = 0.$$

(23)

In the following table we summarize our results obtained (see, [29]) of the energy and momentum densities for Curzon metric, using Landau and Lifshitz, Bergmann and Møller.
Discussion

The problem of energy-momentum localization associated with much debate. Misner et al. [30] argued that the energy is localizable only for spherical systems. There are other opinions which contradict their viewpoint. Cooperstock and Sarracino [31] argued that if the energy localization is meaningful for spherical systems then it is meaningful for all systems. Bondi [32] expressed that a non-localizable form of energy is inadmissible in relativity and its location can in principle be found.

Using different definitions of energy-momentum complex, several authors studied the energy distribution for a given space-time. Most of them restricted their intention to the static and non-static spherically symmetric space-times. Rosen and Virbhadra [33] calculated the energy and momentum densities of non-static cylindrically symmetric empty space-time.

In this paper, we calculated the energy and momentum density components for Weyl metric as well as Curzon metric using Einstein and Papapetrou prescriptions.

We found that for both considered Weyl and Curzon metrics, the Einstein and Papapetrou give exactly the same momentum density but do not provide the same energy density, except only at $R \to \infty$, in the case of Curzon metric, where the energy density tends to zero.

Furthermore, we have made a comparison of our results with those calculated [29] using (LLBM) prescriptions. We obtained that the five prescriptions (ELLPBM) give the same result regarding the momentum density associated with Weyl as well as Curzon metrics. Concerning the energy density associated with both two metrics under consideration, we found that

| Prescription | Energy density | Momentum density |
|--------------|----------------|------------------|
| Landau and Lifshitz | $L^{00} = \frac{1}{8\pi} e^{4\nu - 4\lambda} \left[ \frac{m^2}{R^2} - \frac{4m^2r^2}{R^3} - \frac{2m^2r^2}{R^4} + \frac{8m^2r^2}{R^5} + \frac{2m^2}{R^6} \right]$ | $L^{00} = 0$ |
| Bergmann | $B^{00} = \frac{me^{-2\lambda}}{8\pi R^3} \left[ \frac{2mR^2}{R^2} - \frac{2m^2r^2}{R^3} - (e^{2\nu} - 1) - \frac{2m^2}{R^4} + \frac{2m^2r^2}{R^5} \right]$ | $B^{00} = 0$ |
| Møller | $\mathcal{E}_0 = \frac{m}{4\pi R^3} \left[ 2(r + z) - \frac{3}{R^2}(r^3 + z^3) \right]$ | $\mathcal{E}_0 = 0$. |

Table 1: The energy and momentum densities, using (LLBM), for the Curzon metric
these prescriptions (ELLPBM) do not give the same result except when $R \to \infty$, in the case of Curzon metric, where the energy in the all prescriptions (ELLPBM) tends to zero.

Finally, in the case of Curzon metric we see that the energy in the all prescriptions (ELLPBM) diverge at the singularity ($R = 0$).

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