Coarse-graining and self-similarity of price fluctuations

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Abstract

We propose a new approach for analyzing price fluctuations in their strongly correlated regime ranging from minutes to months. This is done by employing a self-similarity assumption for the magnitude of coarse-grained price fluctuation or volatility. The existence of a Cramér function, the characteristic function for self-similarity, is confirmed by analyzing real price data from a stock market. We also discuss the close interrelation among our approach, the scaling-of-moments method and the multifractal approach for price fluctuations.

Scaling and self-similarity play an important role not only in physics but also in various socio-economic systems including market fluctuations, internet traffic, growth rates of firms and social networks. If phenomenological theory is shown to be applicable to observations, it would guide one to meaningful models and possibly to universal and specific description of such systems without any apparent “Hamiltonian”.

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Price fluctuations in speculative markets from physical viewpoint have recently received great attention [1, 2, 3, 4]. Advances have been aided by concepts and techniques derived from statistical physics and also by the new feasibility of analyzing high-frequency financial data. Considerable statistical features have been uncovered that, interestingly, are quite independent of asset type (stock/foreign-exchange) and of markets (New York/Tokyo). In fact, we shall describe below that price fluctuation is spectrally white, but is not statistically independent since its magnitude exhibits a strong correlation in the form of a power law autocorrelation [5, 6, 7] (see also references in [7]). Moreover, some have suggested that price fluctuations might have similar properties in different time-scales, that is partially supported by the multifractal analysis of price fluctuations [8, 9, 10]. In this paper we propose a new approach to characterizing self-similarity in the strongly correlated regime. We actually show the existence of self-similarity and characterize it both qualitatively and quantitatively from a new statistical physical viewpoint. Specifically, we directly evaluate the invariant scaling relation for fluctuations under coarse-graining of asset price returns.

Time-scale of our interest is related to the temporal features of the price fluctuations known so far. Let \( P(s) \) be the asset price at time-step \( s \), then the (logarithmic) return at \( s \) for duration \( t \) is defined by

\[
\mathcal{r}_t(s) = \log\left(\frac{P(s)}{P(s - t)}\right).
\]

Return is equal to the fractional change of price after duration \( t \) when the change is small. We are interested in the fluctuation \( \mathcal{r}_t \) for different values of \( t \). Each transaction is recorded for every stock in a particular market at the finest time-scale. For the dataset from the New York Stock Exchange (NYSE) used below, the typical time interval between successive transactions was about 10 seconds for stocks with large (trading) volumes.

In a regime with time-scales of minutes and longer, considerable statistical features are known to exist [3, 8, 9]. For an appropriate small time-scale \( \Delta t \) (hereafter chosen as one minute), (i) \( \mathcal{r}_{\Delta t} \) has an autocorrelation function (ACF) that almost vanishes beyond several minutes of time lag, whereas (ii) \( |\mathcal{r}_{\Delta t}| \) has a long correlation ranging from several minutes to more than several months or even a year. In fact, the ACF for \( |\mathcal{r}_{\Delta t}| \) has a power-law decay that is valid for such a long range of time lags. While (i) is interpretable as the absence of arbitrage using linear prediction, the long memory (ii), known as volatility clustering in economic literature, is not completely understood in its origin.
We assume from these observations that self-similarity property which will be formulated shortly holds for time-scales between \( t_s \) and \( T \), where \( t_s \) is the small-, and \( T \) the large-scale cut-off. The value of \( t_s \) is typically one minute or less, while \( T \) may range from several months to one year being regarded as a parameter. Using an appropriate minimal time-scale \( \Delta t \), we focus on the magnitude of the coarse-grained return,\
\[
y_t(s) \equiv \left| \sum_{k=0}^{t/\Delta t-1} r_{\Delta t}(s - k\Delta t) \right| = |r_t(s)|. \tag{2}
\]
The variable \( t \) is the coarse-graining time-scale. Fig. 1 exemplifies in an actual data set the additional property of (iii) self-similarity and intermittency for values of \( y_t \) with different time-scales \( t \). The main aim of this work is to propose a new approach for analyzing this self-similarity.

First, we introduce the coarse-graining step \( n \) as
\[
t = t_n \equiv T e^{-n} \tag{3}
\]
for \( n = 0, 1, \ldots, N \) \( (N \equiv \log(T/t_s) \gg 1) \), and \( y_n \equiv y_{tn} \). Next, we assume that the ratio \( y_{n+1}/y_n \) is statistically independent of \( n \), i.e., statistically steady for \( 1 \ll n \ll N \). This ansatz is the expression of the self-similarity assumption for price fluctuations. That is, by using the equation
\[
\frac{y_{n+1}}{y_n} = c^{z_n}, \tag{4}
\]
\( z_n \) is statistically steady with respect to \( n \) for the similarity region \( 1 \ll n \ll N \). If we introduce the exponent \( \bar{z}(t) \) as
\[
y_t \sim t^{-\bar{z}(t)}, \tag{5}
\]
\( \bar{z}(t) \) corresponds to the local (Hölder) exponent in multifractal theory. Except for the \( t \)-dependence of the magnitude \( y_t \), the similar temporal evolution of \( y_t \) for different values of \( t \) (Fig. 1) is the origin of the self-similarity ansatz.

It follows immediately from eq. (4) that
\[
y_n = y_T \exp \left( \sum_{k=0}^{n-1} z_k \right) = y_T \exp(n \bar{z}_n), \tag{6}
\]
where \( \bar{z}_n \equiv (1/n) \sum_{k=0}^{n-1} z_k \). We assume that \( y_T = y_{n=0} \) has no fluctuation in the sense that one can neglect fluctuation if the coarse-graining scale is
beyond the time-scale $T$. Large deviation theory (e.g. [1]) in probability theory tells us that this self-similarity hypothesis, the statistical independence of exponents $z_k$ in the coarse-graining step $k$, leads us to express the probability density function (PDF) $Q_n(z)$ of the average $\bar{z}_n$ as

$$Q_n(z) \propto e^{-nS(z)}, \quad (7)$$

for $1 \ll n \ll N$. Precisely speaking, eq. (7) is valid for $n \gg n_c$, where $n_c$ is the correlation step of $z_n$ (e.g. defined as the decay step of the correlation function of $\langle \delta z_n \delta z_0 \rangle$, where $\delta z_n \equiv z_n - \langle z_n \rangle$). Here the positive function $S(z)$, being independent of $n$ for $n \gg n_c$, is known as a fluctuation spectrum or Cramér function, whose functional form characterizes the self-similar fluctuations and can be immediately determined by observing the price fluctuations. Under the variable transformation given by Eqs. (3) and (6), $z_t(y) = \log(y/y_T)/\log(T/t)$ (denoting $\bar{z}_n(y_n)$ as $z_t$ for short), the PDF $Q_n(z)$ is transformed to the PDF of $y_t$, $P_t(y)$, as

$$P_t(y) \propto \frac{1}{y} \left( \frac{T}{t} \right)^{-S(z_t(y))}. \quad (9)$$

As a dataset for investigating this self-similarity hypothesis, we used the Trade and Quote (TAQ) database of intraday transactions provided by NYSE, Inc. The prices of a particular stock recorded between Jan. 4, 1993 and June 30, 1999 were used. The median of the time differences between quotes was 12 sec. The timing of the quotes was used to construct a time-series of returns $r_{\Delta t}$ for $\Delta t = 1$ min, excluding midnights, i.e., the returns corresponding to pairs of closing price and next day’s open. The total dataset is comprised of 638,901 records. Next we obtained an ensemble for $y_t$ with different time-scales $t$ by using eq. (2). Other stocks with high volumes were analyzed and yielded similar results to those given below.

Removing the intraday periodicity from the time-series is a vital but difficult task. We used essentially the same method as that in [6] (see also [12]). Note that a daily period for the NYSE from open to close is approximately 390 minutes. The observables adjusted with the intraday periodicity will be referred to as $r_t$ and $y_t$.

Let us now calculate how the self-similarity ansatz formulated above actually holds in the range of $t_s \ll t \ll T$. This can be done by directly
examining the scaling relation, eq. (9), rewritten as

$$\log[yP_t(y)]/\log(T/t) = -S(z_t(y)),$$

where we ignored an additional constant term which corresponds to the normalization of the PDF. The result is summarized in Fig. 2. It shows that the curves for different coarse-graining time-scales \( t \) collapse. The value of \( T \) is taken to be 40960 min, \( t = 10, 20, 40, \ldots \) and 1280 min, and \( y_T \) is set to be the mean of the data for the cut-off time-scale \( T \). The whole set of curves are shifted vertically so that the minimums are at the origin of the vertical axis (this corresponds to the additional constant ignored in eq. (10)). This invariant, i.e. the \( t \)-independent scaling relation is the main result of the present work. It supports the self-similarity ansatz and yields important information about the functional form of \( S(z) \). Note that the minimum \( z_m \) for \( S'(z_m) = 0 \) is approximately \(-0.5\) demonstrating the diffusive nature of \( \langle y_t \rangle \sim \sqrt{t} \) in eq. (9).

It is easily shown that if the Cramér function is a linear function, \( S(z) = a(z - z_0) \), in a certain region of \( z \), the PDF for \( y_t \) takes the power law \( P_t(y) \sim y^{-a-1} \) in the corresponding region of \( y \). Data analysis shows that the region \( dS(z)/dz < 0 \) can be closely approximated by using \( a = -1 \). This corresponds to the fact that \( P_t(y) \) has a constant value for small fluctuations in \( y \). It should be noted that this region, and that for even smaller fluctuations, is subject to the fact that price value is actually discrete (not shown in Fig. 2), which is of limited interest. On the other hand, the region \( dS(z)/dz > 0 \) describes large fluctuations, which is related to previous work on the scaling of moments reported in Refs. [8, 9, 10]. That is, the moments \( \langle y_t^q \rangle \) of order \( q \) scales with \( t \) as \( \langle y_t^q \rangle \propto t^{\phi(q)} \). Such scaling is related to the approach presented here. Actually, the relation between \( \phi(q) \) and \( S(z) \) can be found if we note that

$$\langle y_t^q \rangle \propto y_T^q \int_{-\infty}^{\infty} (T/t)^{qz-S(z)} dz,$$

(11)

by using eq. (3). If we assume that the scaling is of the form, \( \langle y_t^q \rangle \propto t^{\phi(q)} \), and \( S''(z) > 0 \), evaluating eq. (11) by using the steepest-descent method yields

$$\phi(q) = \min_z [S(z) - qz].$$

(12)

Thus \( \phi(q) \) and \( S(z) \) are related to each other by the Legendre transformation of eq. (12).
Our self-similarity formulation originates from Kolmogorov’s lognormal theory in the statistical theory of turbulence [13], and has been recently applied, from the statistical viewpoint of large deviation theory, to analyze the shell model of turbulence [14] and the long-time correlation in on-off intermittency [15]. Indeed the return \( r_t \) for duration \( t \) corresponds to the velocity difference across a spatial distance arbitrarily chosen in the inertial subrange in turbulence, where the self-similar energy cascade process is observed [13, 11]. In the study of on-off intermittency which is typically observed right after the breakdown of synchronization of coupled chaos [15], the difference of dynamical variables of chaotic elements also shows self-similar characteristics. In connection with this phenomenon, \( r_t \) corresponds to the difference of dynamical variables which exhibits on-off intermittency. In the latter analogy, we note that the corresponding variable possesses similar properties as the above (i)–(iii) for the price fluctuations. The present analysis is useful for these systems which show self-similarity due to strongly correlated fluctuations either in space or in time.

Fig. 3 compares \( S(z) \) with the result numerically obtained from \( \phi(q) \) by the Legendre transformation in the region of \( dS(z)/dz > 0 \). Order \( q \) of the moment is increased from 0 to 8, near and beyond which the scaling of moments, \( \langle y_t^q \rangle \propto t^{\phi(q)} \), becomes unclear. This might be related to an ill definition of higher moments caused by the heavy-tail behavior of the PDF for \( y_t \) [16]. Near the minimum of \( S(z) \) \( (z < -0.3) \), the curve obtained from \( \phi(q) \) can be closely approximated by a quadratic polynomial [8], and fits reasonably to the curve of \( S(z) \). However, it deviates significantly from \( S(z) \) for the region of larger fluctuations, where \( S''(z) \) approaches zero. This indicates a breakdown in the relation between \( \phi(q) \) and \( S(z) \), which is based on the concavity of \( S(z) \). Actually, if we approximate \( S(z) \) in the region \(-0.3 < z < -0.1 \) by using the linear relation, \( S(z) = a(z - z_0) \), with \( a \sim 3 \), the result corresponds to the power-law \( P_t(y) \sim y^{-4} \). This is compatible with the heavy-tail behavior found in [10], although more detailed study is needed to clarify the functional form \( S(z) \) for this region and those of even larger fluctuations.

The Cramér function \( S(z) \) is related to the multifractal spectrum investigated in [8]. Suppose the time-series with span \( T \) is divided into \((T/t)\) intervals each of which has a time span \( t \). By evaluating the exponent \( \bar{z}(t) \), eq. (3), in each interval, one obtains the number density \( N_t(z) \) for the intervals such that \( \bar{z}(t) \) is between \( z \) and \( z + dz \). The fractal dimension \( f(z) \) of
the support for those intervals is defined by

$$N_t(z) \sim (t/T)^{-f(z)}.$$  \hspace{1cm} (13)

Noting that $Q_t(z) \sim N_t(z)/(T/t)$ results in the relation

$$S(z) = 1 - f(z).$$  \hspace{1cm} (14)

Thus the multifractal spectrum $f(z)$ is essentially equivalent to the present Cramér function $S(z)$. It should be mentioned, however, that the “dimension” $f(z)$ can take negative values because, in principle, $S(z)$ can take any positive value. We also stress that the function $S(z)$ can be directly calculated from data without resorting to the scaling of moments and the Legendre transformation as is necessary to evaluate $f(z)$. These facts may suggest that the concept of $S(z)$ is more useful than $f(z)$.

In summary, we have found that self-similarity of volatility at different time-scales in the strongly correlated regime can be characterized by direct evaluation of the Cramér function. The scaling relation eq. (10) is our main result, which was a consequence of the self-similarity and large deviation theory. We have also shown the relations of the Cramér function $S(z)$ to the scaling-of-moments method and to the multifractal spectrum. The functional form of $S(z)$ is compatible with the power-law (“cubic law”) behavior at extremely large deviation where $S(z)$ deviates from quadratic form.

Finally we would like to remind the reader that price fluctuates inevitably near the critical point of the apparent equilibrium between demand and supply [17]. The critical point is not observable to any market participants whose expectations and memory give rise to the interesting statistical features of price fluctuations. It is indeed intriguing that the critical fluctuations in market can be well characterized by methods in non-equilibrium statistical physics. While we have not reached an understanding of how market dynamics gives rise to the scaling and self-similarity found here, we believe that our finding and analysis will provide a good basis for phenomenology challenging any models used to understand market dynamics.

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References

[1] B. B. Mandelbrot, *Fractals and Scaling in Finance: Discontinuity, Concentration, Risk* (Springer-Verlag, New York, 1997).

[2] R. N. Mantegna and H. E. Stanley, *Introduction to Econophysics: Correlations and Complexity in Finance* (Cambridge Univ. Press, Cambridge, 1999).

[3] J. P. Bouchaud and M. Potters, *Theory of Financial Risk* (Cambridge Univ. Press, Cambridge, 2000).

[4] J. D. Farmer, Computing in Science & Engineering 1 (1999) 26.

[5] M. M. Dacorogna, U. A. Müller, R. J. Nagler, R. B. Olsen and O. V. Pictet et al., J. Inter. Money and Finance 12 (1993) 413.

[6] Y. Liu, P. Gopikrishnan, P. Cizeau, M. Meyer, C. -K. Peng and H. E. Stanley, Phys. Rev. E 60 (1999) 1390.

[7] P. Gopikrishnan, V. Plerou, Y. Liu, L.A.N. Amaral, X. Gabaix and H.E. Stanley, Physica A 287 (2000) 362.

[8] B. B. Mandelbrot, A. Fisher and L. Calvet, Cowles Foundation Discussion Papers 1164, 1165, 1166, Yale University (1997).

[9] A. Arneodo, J.-F. Muzy, D. Sornette, Eur. Phys. J. B 2 (1998) 277.

[10] F. Schmitt, D. Schertzer and S. Lovejoy, Applied Stochastic Models and Data Analysis 15 (1999) 29.

[11] U. Frisch, *Turbulence: the legacy of A. N. Kolmogorov* (Cambridge Univ. Press, Cambridge, 1995).

[12] Briefly, each day in NYSE is divided into bins with an interval of 5 minutes, for each of which an average of $y_t(s)$ is calculated for the entire dataset with $s$ being in the bin. Then, $y_t(s)$ is normalized by the average according to the bin of $s$.

[13] A. N. Kolmogorov, J. Fluid Mech. 13 (1962) 82.

[14] T. Watanabe, Y. Nakayama and H. Fujisaka, Phys. Rev. E 61 (2000) R1024.
[15] H. Fujisaka, H. Suetani and T. Watanabe, Progr. Theor. Phys. Suppl.,
No. 139 (2000) 70.

[16] V. Plerou, P. Gopikrishnan, L. A. N. Amaral, M. Meyer and H. E.
Stanley, Phys. Rev. E 60 (1999) 6519.

[17] D. Sornette, D. Stauffer and H. Takayasu, in workshop “Facets of Uni-
versality: climate, biodynamics and stock markets” (Giessen University,
1999).
Figure 1: Time-series of price fluctuations magnitudes $y_t = |r_t(s)|$ (normalized by standard deviation of each after removing intraday periodicity) for time-scales $t = 10$, 20 and 40 min. Similar temporal evolutions are visible.
Figure 2: Variable $z$ vs. $S(z)$ (Eqs. (8) and (10)) for time-scales $t = 10, 20, 40, \ldots$ and 1280 min, when $T = 40960$ min. Plots lie on similar curve, which represents functional form of Cramér function $S(z)$. Note that $S(z)$ has minimum at $z \approx -0.5$. 
Figure 3: Details of Fig. 2 in region $dS(z)/dz \geq 0$ for time-scales up to $t = 320$ min. Compare with the result (heavy line) obtained from scaling of moments (of order $0 \leq q \leq 8$) by Legendre transformation. Broken line is quadratic-polynomial fitting for solid line and an extrapolation (lognormal approximation).