A transport coefficient: the electrical conductivity

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Abstract. I describe the lattice determination of the electrical conductivity of the quark gluon plasma \[6\]. Since this is the first extraction of a transport coefficient with a degree of control over errors, I next use this to make estimates of other transport related quantities using simple kinetic theory formulæ. The resulting estimates are applied to fluctuations, ultra-soft photon spectra and the viscosity. Dimming of ultra-soft photons is exponential in the mean free path, and hence is a very sensitive probe of transport.

1. Introduction
Transport coefficients have recently been computed to leading logarithmic order in weak coupling perturbation theory \[1, 2\]. However, at temperatures of interest to heavy-ion collisions experiments, the gauge coupling \(g = \mathcal{O}(1)\) and this computation is ineffective. At the other extreme of \(g \to \infty\), the AdS/CFT correspondence has been exploited to bound the shear viscosity \(\eta/S \geq 1/4\pi \[3\]\), where \(S\) is the extropy density. This result is for a highly supersymmetric version of QCD and needs to be corrected in order to connect with experiments.

There is clearly a case for a lattice computation, if one can have control over the process of analytic continuation from Euclidean to Minkowski space-time. Direct computations of the shear viscosity were tried \[4, 5\], but were inconclusive because statistical problems beset even the Euclidean computations of the correlator of the energy-momentum tensor, thus making it hard to check the analytic continuation.

In this work \[6\] I shift the focus slightly and compute the electrical conductivity, \(\sigma\), of the quark-gluon plasma. This involves a lattice computation of the correlator of the electromagnetic current. Since this is no harder than the extraction of the \(\rho\)-meson mass, statistical errors in the Euclidean computation are under control, and effort can be concentrated on the harder job of analytical continuation. We are able to demonstrate control over this process, and thereby extract \(\sigma\).

This is interesting in itself for heavy-ion collisions because several properties of soft photons in media can be inferred once \(\sigma\) is known. Since this is the first statistically reliable extraction of a transport coefficient from lattice QCD, one can use this to estimate many different quantities which depend on transport, and are important for heavy-ion physics. With \(\sigma\) in hand, and using two well-specified approximations, we are able to predict the value of \(\eta\), and compare the lattice results with others.

The lattice computations described in Section 2 are from \[6\]. The material in Section 3 is partly new.
2. The lattice computation

All lattice computations of transport coefficients are performed in the context of linear response theory. The response, \( A \), of a system to a time-dependent force \( F \) is given in the frequency domain by \( A(\omega) = \chi(\omega)F(\omega) \). The response function \( \chi(\omega) \) can then be written in terms of the retarded correlation function of \( A \), and the transport coefficient can be extracted using Kubo formulæ. For \( \sigma \), the Kubo formula is

\[
\sigma(T) = \frac{1}{6} \frac{\partial}{\partial \omega} \rho^{\mu\nu}_{EM}(\omega, 0, T) \bigg|_{\omega=0},
\]

where the sum is over spatial polarisations of the imaginary part of the retarded photon propagator, i.e., the spectral density, \( \rho^{\mu\nu}_{EM} \), of the electromagnetic current correlator. The usual QED Ward identity gives \( \rho^{00}(\omega, 0, T) = 2\pi\chi_Q\delta(\omega) = 0 \), where \( \chi_Q \) is the usual charge susceptibility.

A lattice computation proceeds from the spectral representation for Euclidean current correlators—

\[
G_{EM}(t, T) = \int_0^\infty \frac{d\omega}{2\pi} K(\omega, t, T)\rho_{EM}(\omega, T),
\]

where \( K \) is the free propagator, \( G_{EM} \) is the product of the zero (spatial) momentum vector correlator summed over all spatial polarisations, \( G_V \), and the EM vertex factor \( C_{EM} = 4\pi\alpha \sum_f e_f^2 \), where \( e_f \) is the charge of a quark of flavour \( f \). For later use we note that \( C_{EM} \approx 1/20 \) for two flavours.

The main problem in this determination is that \( \rho_{EM} \) is needed at an infinite number of points and \( G_{EM} \) is known only at \( N_t \) different values. The solution is to constrain the function \( \rho_{EM} \) through an informed guess, and use a Bayesian method to extract it.

We note a second complication. A free field theory computation shows that \( \rho_{EM}(\omega, T) \) increases as \( \omega^3 \) at large \( \omega \). This means that a naive application of the Kubo formulæ would violate causality [7]. We try to solve this problem by using \( \Delta G_{EM}(t, T) \) which is the difference of the measured and free-field theory propagators to extract \( \Delta \rho_{EM} \).

**Figure 1.** The spectral density \( \Delta \rho_{EM} \) for \( T = 2T_c \) on a 12 \( \times \) 26 \( \times \) 48 lattice with quark mass 0.03\( T_c \). See the text for a discussion of statistical and systematic errors.

**Figure 2.** The propagator \( \Delta G_{EM} \) obtained through the fit of \( \Delta \rho_{EM} \) compared to the data for \( T = 2T_c \) and \( N_t = 8 \) (circles), 10 (squares) and 12 (pentagons) with quark mass 0.03\( T_c \).

Using a variety of Bayesian methods [6] to extract \( \Delta G_{EM}(t, T) \) we found a peak at \( \omega/(2\pi T) \approx 0.15 [6] \) and a smooth drop to zero at \( \omega = 0 \) (see Figure 1). The checks which have been performed on this extraction are—
The statistical errors are negligible, as seen in Figures 1 and 2.

The position of the peak seen in Figure 1 is not strongly dependent on the lattice spacing, as checked by using \( N_t = 8, 10 \) and 12.

Variations in the results are investigated due to algorithmic changes in the Bayesian prior and the truncation and discretization of eq. (2). The width of the band of systematic errors shown in Figure 3 is thrice the width due to such variation.

It would be useful to have similar tests of robustness on all uses of Bayesian methods including the maximum entropy method (MEM).

The goodness of fit to the Euclidean data can be judged from Figure 2. We draw attention to the small error bars on this data when compared to the errors on typical measurements of Euclidean correlators of energy momentum tensor.

Having established the existence of the peak, further information is extracted using a parametrization of the peak in \( \Delta \rho_{EM} \) as the ratio of two polynomials, with that in the numerator being of lower order than the denominator, so that the integrals used in the Kramers-Krönig relation converge. The denominator has to be at least a quartic polynomial in order to smoothly connect to the weak-coupling picture of a pinch singularity giving rise to the peak along real \( \omega \) \( [8] \). (We note in this connection that earlier attempts to extract transport quantities have worked with the full \( \rho_{EM} \) rather than with \( \Delta \rho_{EM} \). This error can strongly suppress estimates of the transport coefficients. Since \( \rho_{EM} \) increases with \( \omega \), this biases the fitted peak towards larger \( \omega \), thus decreasing the slope at the origin for any fit form with one peak.) Using these fits our estimate of the electrical conductivity is—

\[
\frac{\sigma(T)}{T} = C_{EM} \times \begin{cases} 
7.5 \pm 0.8 & (T = 1.5T_c) \\
7.7 \pm 0.6 & (T = 2T_c) \\
7.0 \pm 0.4 & (T = 3T_c)
\end{cases}
\] (3)

3. Applications

3.1. Viscosity

In kinetic theory one can write the Drude formula for the electrical conductivity—

\[
\sigma = \frac{C_{EM} S_q \tau_q}{m},
\] (4)

where \( S_q \) is the entropy density in quarks, \( m \) their screening mass and \( \tau_q \) their mean free time. Using the lattice result for \( \sigma \) one gets \( \tau_q = 0.30 \pm 0.03 \) fm. While charge is transported through quarks, momentum is transported mainly through gluons. Due to colour factors the mean free time for gluons is about half the mean free time for quarks. Then using a kinetic theory formula for the viscosity along with such an estimate of \( \tau_q \) one gets

\[
\frac{\eta}{S} \approx 0.21 \pm 0.02 > \frac{1}{4\pi}.
\] (5)

Only lattice errors are shown in this estimate. Errors on the kinetic theory are harder to extract. The lattice + kinetic theory result is 2.5 times the AdS/CFT limit.

3.2. Photon dimming

Non-zero electrical conductivity leads to screening of photons with energy below the peak of the spectral function found earlier, i.e., for \( \omega < 300 \) MeV. This is a direct consequence of Maxwell’s equations. This range of screened energies lies far below the excess which may have been seen by the CERN NA49 experiment. It seems possible that direct photons at such low-energy could...
be distinguished from photons coming from decays of pions at RHIC-II. One method for using an observation of photon dimming to extract the conductivity has been suggested [9].

The mean free path, $\ell$, of these ultra-soft photons is given by the formula

$$\ell = \frac{\tau_q}{C_{\text{EM}}} \approx 6 \text{ fm.}$$

Since the fireball at RHIC is about 7 fm in size, one could expect dimming by a factor of $\exp(-7/6)$, i.e., by a factor of 3.2. There is a self-consistency argument here involving only the data. If the viscosity is small, then mean free times of quarks and gluons are small, and the mean free path of photons can only be about 20 ($= 1/C_{\text{EM}}$) times larger. However, dimming is exponential in the mean free path. For example, the mean free path as predicted in AdS/CFT would lead to dimming by a factor of 18.8. Thus, the use of photon dimming to extract $\sigma$ is an important cross check of values of $\eta$ extracted by other means.

3.3. Fluctuations

The study of event-to-event fluctuations in limited rapidity bins is expected to give information on the phase diagram of QCD as well as make contact between experiments and basic lattice QCD computations. However, fluctuations of conserved quantities in a limited region of space are expected to even out by transport processes such as diffusion. The computation of $\sigma$ is relevant not only to the diffusion of charge, but also of baryon number and isospin.

The diffusion constant can be obtained using the kinetic theory formula [10]

$$D = \frac{\tau_q c_s}{s} \approx 0.1 \text{ fm},$$

where we have taken the square of the speed of sound $c_s^2 = 0.30 \pm 0.01$ at $T = 2T_c$, as shown by recent lattice computations [11]. If the plasma is highly dissipative, then $D$ must be small. When $D$ decreases, the fluctuation are more localized at the same time. Small viscosity implies smaller mean free path, and hence longer persistence of the fluctuation signal. In the approach of [10] this would mean that the rapidity width of a fluctuation is $\Delta Y \leq 2D/\tau_0 \approx 0.4$ for fluctuations created at time $\tau_0 = 0.5 \text{ fm}$.

3.4. A brave new world

When weak coupling was the only tool to investigate transport properties, it was felt that the QGP would be weakly dissipative. Now with other tools at our disposal, the question seems more open, and preliminary evidence is that dissipation is strong. The first lattice results on transport [6] pin down many aspects of transport, as outlined in this section, and lead to the hope that multiple experimental constraints on transport are feasible.

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