SOME $L^2$ RESULTS FOR $\overline{\partial}$ ON PROJECTIVE VARIETIES WITH GENERAL SINGULARITIES

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Abstract. Let $X$ be an irreducible $n$-dimensional projective variety in $\mathbb{CP}^N$ with arbitrary singular locus. We prove that the $L^2$-$\overline{\partial}$-$(p,1)$-cohomology groups (with respect to the Fubini-Study metric) of the regular part of $X$ are finite dimensional.

1. Introduction

This paper is the third in a series of papers in which we discuss $L^2$ existence results for the Cauchy-Riemann operator on the regular part of complex spaces with non-isolated singularities. In [7] we considered the case of a relatively compact, open, Stein subset $\Omega$ of a reduced Stein complex space $X$ and studied the equation $\overline{\partial}u = f$ on $\Omega^* = \Omega \setminus A$ where $A$ was a lower dimensional complex analytic set with empty interior containing the singular locus of $X$ and $f$ was a $\overline{\partial}$-closed, square integrable form vanishing to high order on $A$. In [25] we proved some $L^2$ existence theorems for $\overline{\partial}$ in the case of product singularities.

In this paper we consider an irreducible $n$-dimensional projective variety $X$ in $\mathbb{CP}^N$ with arbitrary singular locus $\text{Sing}X$. The regular part of $X$, denoted by $\text{Reg}X$, inherits a metric from the restriction of the Fubini-Study metric, which we call the ambient metric. We denote by $\langle, \rangle_{FS}$, $| \cdot |_{FS}$, $dV_{FS}$ respectively the pointwise inner product, norm on multi-(co) vectors and volume element on $\text{Reg}X$ induced by the ambient metric and by $L^2_{p,q}(\text{Reg}X,dV_{FS})$ the space of $(p,q)$ forms on $\text{Reg}X$ that are square-integrable with respect to the ambient metric. We consider the weak $\overline{\partial}$-operator acting on forms on $\text{Reg}X$. Let $Z^{p,q}_{(2)}(\text{Reg}X) := \{ f \in L^2_{p,q}(\text{Reg}X,dV_{FS}) | \overline{\partial}f = 0 \text{ on } \text{Reg}X \}$ and let $f \in Z^{p,q}_{(2)}(\text{Reg}X)$ where $0 \leq p \leq n$ and $1 \leq q \leq n$. We address the question of whether we can solve $\overline{\partial}u = f$ on $\text{Reg}X$ with $u \in L^2_{(p,q-1)}(\text{Reg}X,dV_{FS})$. Our main result is the following theorem:

Theorem 1.1. There exists a closed subspace $\mathcal{H}$ of finite codimension on the space $Z^{p,1}_{(2)}(\text{Reg}X)$ such that for every $f \in \mathcal{H}$ there exists a $u \in L^2_{(p,0)}(\text{Reg}X,dV_{FS})$ such that $\overline{\partial}u = f$ on $\text{Reg}X$.

Pardon and Stern proved in [27] that the $L^2$-$(n,q)$-$\overline{\partial}$-cohomology groups of $n$-dimensional projective varieties with general singularities (with respect to the ambient metric) are finite dimensional for all $q \geq 0$ (more precisely they showed that they are isomorphic to the $(n,q)$-cohomology groups of a desingularization). To our knowledge Theorem 1.1 is the first result on finite dimensionality of some $L^2$-$(p,q)$-$\overline{\partial}$-cohomology groups (with respect to the ambient metric) with $p \neq n$ of such varieties. The case of projective varieties with isolated singularities has been studied by Nagase [17], Ohsawa [18, 19, 20, 21, 22], Pardon [26] and Pardon and Stern [27, 28, 29]. A local analogue of this question for varieties with isolated singularities was
considered by Fornæss [5], Diederich, Fornaess and Vassiliadou [8] and Fornæss, Øvrelid and Vassiliadou [11] (see [3, 7] for extended references). Ohsawa constructed in [23] a complete Kähler metric on a small, Stein, deleted neighborhood of a singular point of a complex space with arbitrary singularities and showed that with respect to that metric, certain $L^2$-$\partial$-cohomology groups with support conditions vanish there. Few years later, Grant and Milman constructed in [8] a complete Kähler metric on the regular part of a compact complex space with arbitrary singular locus. Grant and Milman’s construction was motivated in part by the hope that this metric might be useful for proving the existence of a Hodge structure on the intersection cohomology of projective varieties with arbitrary singularities. It is not clear to us at this moment whether the local $L^2$ results for $\partial$ obtained in [23] (with respect to the specific complete Kähler metric) or Grant and Milman’s complete Kähler metric can be used to understand the global $L^2$-$\partial$-cohomology groups (with respect to the Fubini-Study metric) we consider in this paper.

Our proof is inspired by the methods employed in [8] and [5]. It is based on the observation that Hörmander’s $L^2$-theory carries over unchanged to Stein Riemann domains $Y$ over $\mathbb{C}^n$ when we equip $Y$ with the pull back Euclidean metric from $\mathbb{C}^n$. We look at the affine pieces $X_0, X_1, \ldots, X_N$ of $X$ and for each piece we choose a family of non-degenerate projections (for the definition see Section 2) $\pi_j: X_j \to \mathbb{C}^n$ along with subvarieties $\Sigma_{\pi_j}$ such that $\pi_j: X_j\setminus\Sigma_{\pi_j} \to \mathbb{C}^n$ is a local biholomorphism and $\bigcup (X_j\setminus\Sigma_{\pi_j}) = \text{Reg } X$. Now the set $Z_j := X_j\setminus\Sigma_{\pi_j}$ inherits a metric from the pull back of the Euclidean metric in $\mathbb{C}^n$ and let us denote the pointwise norm with respect to this metric by $| \cdot |_j$ and volume element $dV_j$. Let $(\pi_j, Z_j, \Sigma_{\pi_j})^{M}_{j=1}$ be an enumeration of the various projections of the affine pieces. When $f \in L^2(\text{Reg } X, dV_{FS})$ we can show that $f$ satisfies estimates of the following form:

$$\int_{Z_j} |f|_j^2 e^{-\psi_j} dV_j \leq C \int_{\text{Reg } X} |f|_{FS}^2 dV_{FS},$$

where $\psi_j$ are specific plurisubharmonic functions and $C$ some positive constant independent of $j$. Then, we may use Hörmander’s $L^2$-theory to obtain $L^2$ solutions $v_j$ (with respect to the metric induced on $Z_j$ by the pull back of the Euclidean metric in $\mathbb{C}^n$) to $\partial v = f$ on $Z_j$.

The delicate part is to use Łojasiewicz’s inequalities to obtain good control of the boundary behaviour of $| v_j |_j, dV_j$ on $Z_j$ in terms of the Fubini-Study metric. The forms $h_{jj'} := v_j - v_{j'}$ are holomorphic on $Z_j \cap Z_{j'}$. From what we know about their boundary behaviour we can deduce from sheaf-theoretic results that they lie in a finite dimensional vector space of holomorphic $p$-forms. Since the map $f \to \{h_{jj'}\}_{1 \leq j < j' \leq M}$ is linear, we have that $h_{jj'} = 0$ for all $j, j'$, whenever $f$ lies in a finite codimensional subspace of $Z_{(2)}^{p,1}(\text{Reg } X)$. In this case, we can show that the $v_j$ define an $L^2$ solution $v$ (with respect to the ambient metric) to $\partial v = f$ on $\text{Reg } X$.

As a by-product of the techniques used in the paper we obtain a weighted $L^2$-estimate for $\partial$ on irreducible affine algebraic subvarieties of $\mathbb{C}^N$. More precisely we prove the following theorem:

**Theorem 1.2.** Let $X$ be an irreducible, $n$-dimensional affine subvariety of $\mathbb{C}^N$ and let $\psi$ be a strictly plurisubharmonic function on $\text{Reg } X$ with at most logarithmic growth (i.e. $\psi(z) \leq A \log(1 + ||z||^2) + B$ for some $A, B \geq 0$) and not necessarily bounded from below. Let $Z_{\psi}^{(p,1)} := \{f \in L^2_{\partial,loc}^{(p,1)}(\text{Reg } X, dV_E); \partial f = 0 \text{ on }\text{Reg } X; \int_{\text{Reg } X} |f|^2_E e^{-\psi} dV_E < \infty\}$. Then, there exists a subspace $\mathcal{H} \subset Z_{\psi}^{(p,1)}$ of finite codimension such that for all $f \in \mathcal{H}$ there exists a $u \in L^2_{\partial,loc}^{(p,0)}(\text{Reg } X, dV_E)$ with $\partial u = f$ on $\text{Reg } X$.
\[ \int_{\text{Reg}X} |u|^2_E (1 + \|z\|^2)^{-2} e^{-\psi} \, dV_E \leq C \int_{\text{Reg}X} |f|^2_E e^{-\psi} \, dV_E, \]

where \( C \) is some positive constant.

Our main theorem can also be used to prove finite dimensionality results for some local \( L^2-J \)-cohomology groups of varieties with isolated singularities. In particular, let \( X \) be an irreducible \( n \)-dimensional analytic set in \( \mathbb{C}^N \) with an isolated singularity at 0. In [5] (section 9) a question was raised about understanding the \( L^2-J(p, q) \)-cohomology groups (with respect to the Euclidean metric) of the regular part of a small Stein neighborhood of 0 in \( X \) when \( p + q = n \) and \( p, q > 0 \). With the aid of Theorem 1.1 we can prove that when \( p = n - 1, q = 1 \) these groups are finite dimensional.

The paper is organized as follows: In section 2 we describe some geometric facts about affine algebraic varieties and recall some basic concepts about projective varieties. Section 3 deals with the comparison between the Fubini-Study metric and the pull-back metrics \(|\cdot|_J\) that were defined earlier in the introduction. Section 4 contains the estimates for the solutions \( v_j \) to \( \partial v = f \) on \( Z_j \). In Section 5, we show that \( h_{jj'} \) lie in a finite dimensional subspace of the space of holomorphic \((p, 0)\) forms. In Section 6 we prove Theorem 1.2 and finally in section 7 we outline the proof for the finite dimensionality of the local \( L^2-J(n - 1, 1) \)-cohomology groups of a small Stein neighborhood of an isolated singular point of an irreducible \( n \)-dimensional variety of \( \mathbb{C}^N \).

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2. Some geometric facts about varieties

2.1. Non-degenerate projections. Let \( Y \) be an irreducible, \( n \)-dimensional, affine algebraic variety in \( \mathbb{C}^N \). Let \( L \) be an \( n \)-dimensional linear subspace of \( \mathbb{C}^N \) and \( \pi: \mathbb{C}^N \to L \) be an orthogonal projection.

Definition 2.1. We shall say that \( \pi|_Y : Y \to L \) is a non-degenerate projection on \( Y \), if there exists a point \( p \in \text{Reg}Y \) such that the induced map on tangent spaces \((\pi|_Y)^*: T_pY \to L\) is an isomorphism.

Remark: Every projection \( \pi: \mathbb{C}^N \to L \), where \( L \) is an \( n \)-dimensional subspace of \( \mathbb{C}^N \) can be made non-degenerate on \( Y \) after a slight perturbation.

Definition 2.2. The ramification locus \( \Sigma \) of the above projection consists of all points \( p \in \text{Sing}Y \) as well as those \( p \in \text{Reg}Y \) for which \((\pi|_Y)^*: T_pY \to L\) fails to be an isomorphism.

Another way to think about the non-degenerate projections and their ramification loci is the following: The map \( \pi|_Y : Y \to L \) induces for all \( y \in Y \), a linear map \((\pi|_Y)^*: T_yY \to L\), where \( T_yY \) is the Zariski tangent space at \( y \). As a linear map between vector spaces it has a rank and a corank (the dimension of the kernel of the linear mapping \((\pi|_Y)^*\)).
exists a regular point $p$ in $Y$ such that $\text{rank}((\pi|_Y)_{*,p}) = n$. Under this light we can define the ramification locus of a projection $\pi|_Y : Y \to L$ as follows:

**Definition 2.3.** The ramification locus of the above projection is the set

$$\Sigma = \{ y \in Y; \text{ corank}((\pi|_Y)_{*,y}) \geq 1 \}.$$

**Proposition 2.4.** The ramification locus $\Sigma$ is an algebraic subvariety of $Y$. When the projection $\pi|_Y : Y \to L$ is non-degenerate on $Y$ then $\Sigma$ is a proper subvariety of $Y$.

**Proof.** The proof of the Proposition is similar to that of Theorem 4 (page 136, Volume II) in Gunning [10] (we just replace holomorphic subvarieties by algebraic and holomorphic map by regular map). The fact that $\Sigma$ is a proper subvariety of $Y$ when the projection is non-degenerate on $Y$ follows trivially from the definitions. $\square$

**Remark:** For a non-degenerate projection on $Y$, the set $A := \{ y \in \text{Reg}Y; \text{ rank}((\pi|_Y)_{*,y}) = \text{n} \}$ is a non-empty open Zariski dense set in $Y$. Thus, for most points $y \in \text{Reg}Y$ the projection $\pi|_Y$ is non-degenerate.

### 2.2. Projections onto the $n$-coordinate planes.

Let $I := (i_1, \cdots, i_n)$, $1 \leq i_1 < \cdots < i_n \leq N$ be an increasing $n$-tuple and let $I'$ be its complement in $\{1, 2, \cdots, N\}$. Let $L_I$ be an $n$-dimensional subspace of $\mathbb{C}^N$ defined by $L_I := \{ (z_1, \cdots, z_N) \in \mathbb{C}^N; \ z_j = 0 \text{ for all } j \in I' \}$ and such that $\pi_I : \mathbb{C}^N \to L_I$, $(z_1, \cdots, z_N) \to (z_{i_1}, \cdots, z_{i_n})$, the projection onto the $I$ coordinates is non-degenerate on $Y$ (we can always assume that after a slight perturbation). Since $Y$ is an irreducible affine variety in $\mathbb{C}^N$ there exists a prime ideal $\mathcal{B} \subset \mathbb{C}[z_1, \cdots, z_N]$ such that $Y = Z(\mathcal{B})$, the zero locus of $\mathcal{B}$. Let $p_1, \cdots, p_s$ be generators of $\mathcal{B}$. It is a well-known fact that the germs $p_{1,z}, \cdots, p_{s,z}$ generate $\mathcal{I}_{Y,z}$, the stalk at $z$ of the ideal sheaf of $Y$, when the latter is viewed as a holomorphic variety; see for example Proposition 13.3.3 in [30]. Let $\Sigma_I$ denote the ramification locus of $\pi_I$ and let $y \in \Sigma_I$. The induced map on tangent spaces $(\pi_I|_Y)_{*,y} : T_y Y \to L_I$ sends an element $T_y Y \ni v = (v_1, \cdots, v_N) \to (v_{i_1}, \cdots, v_{i_n})$. We know that the kernel of $((\pi_I|_Y)_{*,y})$ is at least one complex dimensional if and only if all $(N-n) \times (N-n)$ minors of the matrix

$$\begin{pmatrix} \frac{\partial p_i}{\partial z_r}(y) \\ 1 \leq i \leq s, \ i \in I', \ 1 \leq r \leq N \end{pmatrix}$$

have zero determinants. Points $y \in \text{Sing}Y$ are in $\Sigma_I$ since the rank of the Jacobian $\left( \frac{\partial p_i}{\partial z_r}(y) \right)_{1 \leq i \leq s, \ 1 \leq r \leq N}$ at these points is less that $N - n$.

**Proposition 2.5.** Let $Y$ be an irreducible, $n$-dimensional, affine algebraic variety in $\mathbb{C}^N$. There exist finitely many, complex linear, orthogonal projections $\pi_j : \mathbb{C}^N \to L_j$, with $n$-dimensional images $L_j$ and a constant $c > 0$ such that:

i) The restriction of $\pi_j$'s on $Y$ are non-degenerate on $Y$ with ramification locus $\Sigma_j$. 
ii) For every $z \in \text{Reg} Y$ there exists a $j$ such that $z \notin \Sigma_j$ and such that $(\pi_j)_{\ast, z} : T_z Y \to L_j$ satisfies $\| (\pi_j)_{\ast, z} v \| \geq c \| v \|$ for all $v \in T_z Y$. (Here the norms are the induced Euclidean norms).

In particular ii) implies that $\cap \Sigma_j = \text{Sing} Y$.

Proof. After a slight perturbation we can take as $\pi_j$’s the projections onto the (newly defined) $n$-dimensional coordinate planes in $\mathbb{C}^N$. Let $I = \{i_1, \cdots, i_n\}$, $1 \leq i_1 < \cdots < i_n \leq N$ be an increasing $n$-tuple. Let $\pi_I : C^N \to L_I$ be the projection onto the $I$ coordinates. To prove part ii) we need the following lemma.

Lemma 2.6. There exists an absolute constant $c > 0$ such that for each $z \in \text{Reg} Y$ there exists a multi-index $I_0$ such that $\| (\pi_{I_0})_{\ast, z} v \| \geq c \| v \|$ for all $v \in T_z Y$

Proof. Let $z \in \text{Reg} Y$ and $v_1, \cdots, v_n$ be an orthonormal basis of $T_z Y$. Let $\{e' := e_{i_1} \wedge \cdots \wedge e_{i_n} ; |I| = n\}$ be an orthonormal basis of $\wedge^n C^N$. Consider the expansion of $v_1 \wedge v_2 \cdots \wedge v_n$ in terms of $e'$. In what follows for abbreviation we shall write $\pi_{I_0}$ instead of $(\pi_{I_0})_{\ast, z}$. We have

$$1 = \| v_1 \wedge \cdots \wedge v_n \|^2 = \sum_{|I| = n}^\prime \| \pi_{I_0} (v_1) \wedge \cdots \wedge \pi_{I_0} (v_n) \|^2 = \sum_{|I| = n}^\prime D_I$$

where $D_I := G(\pi_{I_0} (v_1), \cdots, \pi_{I_0} (v_n))$ is the Gram determinant of the vectors $\pi_{I_0} (v_1), \cdots, \pi_{I_0} (v_n)$.

Recall that on a unitary space $E$ endowed with a hermitian inner product $(\ , \ )$ the Gram determinant of vectors $x_1, \cdots, x_p$ in $E$ is described by:

$$G(x_1, \cdots, x_p) := \text{det} \begin{pmatrix} (x_1, x_1) & \cdots & (x_1, x_p) \\ \vdots & \ddots & \vdots \\ (x_p, x_1) & \cdots & (x_p, x_p) \end{pmatrix}$$

In general $G(x_1, \cdots, x_p) \geq 0$ and equality holds if the vectors $x_1, \cdots, x_p$ are linearly dependent.

Since there are $N \choose n$ terms on the right hand side of equation (2.2.1) there should exist a multi-index $I_0$ such that $D_{I_0} \geq (N \choose n)^{-1}$. Now if we let $S := \pi_{I_0} \circ \pi_{I_0}: T_z Y \to T_z Y$ we obtain

$$D_{I_0} = \text{det} \left( (Sv_1, v_j) \right) = \text{det} S.$$ 

But $S$ is a positive, symmetric form that has eigenvalues $\{l_j\}$, $0 < l_1 \leq l_2 \cdots \leq l_n \leq 1$ since $\| S \| \leq 1$. Then for all $v \in T_z Y$ we have

$$\| \pi_{I_0} v \|^2 = (Sv, v) \geq l_n \| v \|^2 \geq (\prod_j l_j) \| v \|^2 = D_{I_0} \| v \|^2 \geq (N \choose n)^{-1} \| v \|^2.$$ 

Remarks: a) Part ii) of the above Proposition can be thought of as a statement about $n$-planes in $\mathbb{C}^N$. Recall that the Grassmannian $\text{Gr}(n, N)$ can be covered by open affine sets $U_\Gamma$ where $\Gamma$ is a $(N - n)$-dimensional subspace of $\mathbb{C}^N$. Each $U_\Gamma$ is defined to be the subset of planes $\Lambda \subset \mathbb{C}^N$ complementary to $\Gamma$. Fixing any subspace $\Lambda \subset U_\Gamma$ a subspace $\Lambda' \subset U_\Gamma$ is the graph of a homomorphism $\phi : \Lambda \to \Gamma$, so that $U_\Gamma = \text{Hom}(\Lambda, \Gamma)$ (for more information on this...
the interested reader may look at Lecture 16 in [11]). We shall consider a slightly different covering of the Grassmanian \( \text{Gr}(n,N) \). Let \( L \) be an \( n \)-dimensional subspace as before and let \( L^\perp \) denote its orthogonal complement in \( \mathbb{C}^N \). Let \( B(L,L^\perp) \) denote the set of bounded linear maps from \( L \) to \( L^\perp \). Consider the following local parametrizations:

\[
\phi_L : B(L,L^\perp) \rightarrow \text{Gr}(n,N)
\]

given by \( \phi_L(T) = \text{Graph}(T) \). Let \( \epsilon > 0 \). The set \( \phi_L(\{T; \|T\| < \epsilon\}) \) is an open neighborhood of \( L \) in \( \text{Gr}(n,N) \). Since the latter variety is compact there exist finitely many \( L_1, \cdots, L_K \in \text{Gr}(n,N) \) and open sets \( U_j := \phi_{L_j}(\{T; \|T\| < \epsilon\}) \) such that \( \text{Gr}(n,N) = \bigcup_{j=1}^K U_j \). When \( L \in U_j \), the orthogonal projection \( \pi_j : L \rightarrow L_j \) is bounded from below by \( (1 + \epsilon^2)^{-\frac{1}{2}} \). Choosing \( \epsilon \) small enough we can make the constant \( c \) that appears in part ii) of the above proposition to be as close to 1 as we like, using sufficiently many projections. \( \square \)

b) Part ii) of the above Proposition guarantees that \( \bigcap_j \Sigma_j = \text{SingY} \). Hence, \( \{Y \setminus \Sigma_j\}_j \) will cover \( \text{RegY} \).

### 2.3. Projective spaces and Fubini-Study metric.

A point in \( \mathbb{CP}^N \) is usually written as a homogeneous vector \([Z_0, \cdots, Z_N]\) by which we mean the line spanned by \((Z_0, \cdots, Z_N) \in \mathbb{C}^{N+1} \setminus \{0\}\). For \( i = 0, \cdots, N \) we define

\[
\phi_i : \mathbb{C}^N \rightarrow U_i \subset \mathbb{CP}^N
\]

given by \( \phi_i(z_1, \cdots, z_N) = [z_1 : \cdots : z_i : 1 : z_{i+1} : \cdots : z_N] \).

In particular \( \phi_0(z_1, \cdots, z_N) = [1 : z_1 : \cdots : z_N] \). We set for \( 0 \leq i \leq N \),

\[
H_i := \mathbb{CP}^N \setminus \phi_i(\mathbb{C}^N) = \{[Z]; Z_i = 0\}.
\]

On \( \phi_i^{-1}(U_i) \), using the affine coordinates \( z_1, \cdots, z_N \) the Fubini-Study metric takes the form

\[
\left( \sum h_{\mu\nu}(z)dz_{\mu} \otimes d\overline{z}_{\nu} \right) (1 + \|z\|^2)^{-2}
\]

where \( h_{\mu\nu}(z) = (1 + \|z\|^2)\delta_{\mu\nu} - \overline{z}_{\mu} z_{\nu} \), \( \mu, \nu = 1, \cdots, n \). The associated \((1,1)\) Kähler form is described by

\[
\omega = \frac{i}{2} \sum h_{\mu\nu} dz_{\mu} \wedge d\overline{z}_{\nu}.
\]

Let \( \lambda_1, \cdots, \lambda_N \) be the eigenvalues of the restriction of the Fubini-Study metric on the affine piece \( \phi_i^{-1}(U_i) \) with respect to the Euclidean metric. A direct calculation shows that

\[
(1 + \|z\|^2)^{-2} = \lambda_1 \leq \lambda_2 = \cdots = \lambda_N = (1 + \|z\|^2)^{-1}.
\]
Let $X$ be an irreducible $n$-dimensional projective variety in $\mathbb{CP}^N$. Set $X_i := \phi^{-1}_i(X \cap U_i) \subset \mathbb{C}^N$. For each $i$, $i \in \{0, \cdots, N\}$ we shall choose $\{L^i_k\}_{k=1}^{M_i}$, families of $n$-dimensional complex subspaces of $\mathbb{C}^N$ and orthogonal linear projections $\pi_k^i : \mathbb{C}^N \to L^i_k$ such that $\pi_k^i|_{X_i}$ is non-degenerate on $X_i$ with ramification locus $\Sigma_k^i$ and such that part ii) of Proposition 2.4 holds for each $i$. Set $W_k^i := X_i \setminus \Sigma_k^i$. To reduce the number of indices we choose an ordering of the set $\{(i, k); i \in \{0, \cdots, N\}, k = 1, \cdots, M_i\}$ and of the corresponding objects $L^i_k$, $\pi_k^i$, $W_k^i$ such that we have a bijection

$$\Theta : \{1, \cdots, M\} \to \{(i, k) \in \mathbb{N}^2; i \in \{0, \cdots, N\}, 1 \leq k \leq M_i\},$$

$$j \to (i(j), k(j))$$

Let $L_1, \cdots, L_M$ be this ordering of all the $n$-dimensional subspaces $\{L^i_k\}$ and let $\pi_1, \cdots, \pi_M$ be the corresponding projections. In what follows the index $i(j)$ will determine the affine variety $X_i$ that contains $W_j$, $\Sigma_j$.

Let $<, >$, $| |$, $dV$ denote the pointwise inner product, norm on muti-(co)-vectors and volume element on Reg$X_i$ induced by the Fubini-Study metric, and let $< >_E$, $| |_E$, $dV_E$ those induced on Reg$X_i$ from the Euclidean metric in $\mathbb{C}^N$ and $<, >_j$, $| |_j$, $dV_j$ those pull-backed on $W_j \subset$ Reg$X_{i(j)}$ via $\pi_j$ from the Euclidean metric on $L_j$. By the min-max principle we know that the eigenvalues of the restriction of the Fubini-Study metric with respect to the Euclidean metric on Reg$X_i$ satisfy

$$(1 + \|z\|^2)^{-2} \leq \lambda_1 \leq \lambda_2 = \cdots = \lambda_n = (1 + \|z\|^2)^{-1}.$$

For $f \in L^2_{p,q,loc}\left(\text{Reg}X_i, dV\right)$ we have

$$(3.0.1) \quad (1 + \|z\|^2)^{-\alpha} f^2_E dV_E \leq |f|^2 \, dV$$

while for a form $u \in L^2_{p,q-1,loc}\left(\text{Reg}X_i, dV_E\right)$ we have

$$(3.0.2) \quad (1 + \|z\|^2)^{-\beta} |u|^2 \, dV \leq |u|^2_E \, dV,$$

where $\alpha$, $\beta$ are some non-negative constants that depend only on $p, q$.

3.1. Comparison between $dV_j$, $dV_E$ on $W_j$. The set $W_j$ inherits two metrics. One from the restriction of the Euclidean metric in $\mathbb{C}^N$ and another from the pull-back of the Euclidean metric on $L_j$ via the map $\pi_j$. We begin this section by relating the volume elements of these two metrics on $W_j$.

Lemma 3.1. There exists a smooth function $m_j$ defined on $W_j$ such that

i) $dV_E = m_j \, dV_j$ on $W_j$.

ii) The function $\log m_j$ is plurisubharmonic on $W_j$. 
Proof. Part i) can be taken as the definition of the function $m_j$. To prove part ii) of the lemma we need a local description of the function $m_j$. Without loss of generality we can assume that the $n$-dimensional subspace $L_j$ corresponds to the $n$-coordinate plane $L_j$ in $\mathbb{C}^N$ where $I = (1, \ldots, n)$ is an increasing $n$-tuple. Then $m_j : X_{i(j)} \to L_j$ is the projection onto the first $n$-coordinates. The ramification locus of this projection is characterized by the vanishing of the determinants of all $(N - n) \times (N - n)$ minors of the matrix
\[
\begin{pmatrix}
\frac{\partial p_i}{\partial z_j} \\
1 \leq i \leq n
\end{pmatrix}
\]
where $p_1, \ldots, p_s$ are the generators of the ideal of the variety $X_{i(j)}$. We can describe the set $W_j$ as
\[W_j = \{ z_0 \in X_{i(j)} : \Delta_K(z_0) \neq 0 \text{ for some multi-index } K : 1 \leq k_1 < \cdots < k_{N-n} \leq s \},\]
where
\[\Delta_K(z_0) := \text{det} \left( \begin{array}{c}
n \times n \\
\frac{\partial p_{k_1}}{\partial z_{n+1}}, \ldots, \frac{\partial p_{k_{N-n}}}{\partial z_N}
\end{array} \right) (z_0).
\]
Let $z_0 \in W_j$. Then there exists a neighborhood $V$ of $z_0$ in $X_{i(j)}$ that is parametrized as $(z', g_1(z'), \ldots, g_{N-n}(z'))$ for some functions $g_j$ and with $z' = (z_1, \ldots, z_n) \in \pi_{i(j)}(V)$ (by the implicit function theorem). For all $k \in K$ we have that $p_k(z', g_1(z'), \ldots, g_{N-n}(z')) = 0$. The implicit function theorem allows us to compute for all $l$ with $1 \leq l \leq N - n$ and $\nu$ with $1 \leq \nu \leq n$,
\[
(3.1.1) \quad \frac{\partial g_k}{\partial z_\nu} (\pi_{i(j)}(z)) = \frac{A_{\nu K_l}(z)}{\Delta_K(z)}
\]
where $A_{\nu K_l}$ is the determinant of the $(N - n) \times (N - n)$ matrix $\left( \frac{\partial p_{k_1}}{\partial z_{n+1}}, \ldots, \frac{\partial p_{k_{N-n}}}{\partial z_N} \right)$ where the $l$-th column has been replaced by $\top (-\frac{\partial p_{k_1}}{\partial z_{n+1}}, \ldots, -\frac{\partial p_{k_{N-n}}}{\partial z_N})$.

For $i = 1, \ldots, n$ and $z \in V$ we let
\[\zeta_i := e_i + \sum_{k=1}^{N-n} \frac{\partial g_k}{\partial z_i} (z') e_{n+k},\]
where $\{e_i\}_{i=1}^N$ is the standard basis of $\mathbb{C}^N$. It is not hard to show that $\{\zeta_i\}_{i=1}^n$ form a basis of $T_zW_j$. Moreover for $1 \leq i \leq n$ we have $(\pi_{i(j)})_* z_i \zeta_i = \tilde{e}_i$, where $\tilde{e}_i$ is the standard basis in $L_j$. Clearly
\[
(3.1.2) \quad dV_E = \text{det} B \, dV_j,
\]
where $B = (b_{kl})$ is the $n \times n$ matrix with entries $b_{kl} := \langle \zeta_k, \zeta_l \rangle_E$ and $\langle, \rangle_E$ is the pointwise Euclidean inner product on elements of $T_zW_j$.

Let us look at $\| \Lambda_{i=1}^n \zeta_i \|^2_E$. It follows from the definition of the Euclidean inner product on vectors in $\Lambda_{i=1}^n T_zW_j$ that $\| \Lambda_{i=1}^n \zeta_i \|^2_E = \text{det} B$. Using (3.1.2) we have the following local description of $m_j(z) = \| \Lambda_{i=1}^n \zeta_i \|^2_E$. 


Let us also consider the following \( n \times N \) matrix:

\[
C = \begin{pmatrix}
1 & \cdots & 0 & \frac{\partial g_1}{\partial z_1} & \cdots & \frac{\partial g_N}{\partial z_1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & \frac{\partial g_1}{\partial z_N} & \cdots & \frac{\partial g_N}{\partial z_N}
\end{pmatrix}
\]

To prove ii) we notice that \( m_j(z) = \sum'_{H} |C_H|^2 \), where the summation runs over all multi-indices \( H = (h_1, \ldots, h_n) \) with \( 1 \leq h_1 < \cdots < h_n \leq N \) and \( C_H \) are determinants from the \( n \times N \) matrix \( C \) with columns \( h_1, \ldots, h_n \). The \( C_H \)'s are holomorphic functions. Hence \( \log m_j \) is a plurisubharmonic function in a neighborhood of \( z_0 \). \( \square \)

We have seen that \( \{\zeta_j\}_{j=1}^n \) form an orthonormal basis for \( T_z W_j \) with respect to the pull back metric. Let \( \{\lambda_j\} \) be the eigenvalues of the Kähler form \( \omega_E \) of the Euclidean metric on \( W_j \) with respect to the pull-back metric. We have \( 1 \leq \lambda_1 \leq \cdots \leq \lambda_n \). For a \( (p,q) \) form \( f \) on \( W_j \) we have:

\[
|f|^2_E = \sum_{I,J} \prod_{i \in I} \lambda_i \prod_{j \in J} \lambda_j |f_{IJ}|^2.
\]

Taking into account part i) of the previous lemma and the fact that the eigenvalues \( \{\lambda_j\} \) are greater or equal to 1 we have the following estimate for \( (p,q) \)-forms \( f \) with \( p > 0, q > 0 \)

\[
m_j^{-1} |f_j|^2 dV_j \leq |f|^2_E dV_E
\]

In the special case where \( p = 0, q > 0 \) we have a stronger estimate

\[
|f_j|^2 dV_j \leq |f|^2_E dV_E.
\]

3.2. Global Lojasiewicz inequality. In the next section we shall need an upper bound for the function \( m_j \) that was defined in the previous lemma. The upper bound will be obtained by applying a global Lojasiewicz-type inequality obtained first by Brownawell [1] and later improved by Ji, Kollár and Shiffman [14].

**Corollary 3.2.** (Corollary 6 in [14]) Let \( f_1, \ldots, f_k \in \mathbb{C}[z_1, \ldots, z_n] \) and let \( d_i = \deg f_i \). Let \( Z \subset \mathbb{C}^n \) be the common zero set of these polynomials. Then there is a constant \( C > 0 \) such that

\[
\left( \frac{\text{dist}(z,Z)}{1 + \|z\|^2} \right)^n \leq C \max_i |f_i(z)|
\]

holds for all \( z \in \mathbb{C}^n \). Here \( C \) and \( \overline{B}(n, d_1, \ldots, d_k) \) are positive constants that depend on the \( f_i \) and \( \|z\|^2 := |z_1|^2 + \cdots + |z_n|^2 \).
Recall that \( m_j(z) = \det (\langle \zeta_k, \zeta_i \rangle)_{k,i} \) and \( \langle \zeta_k, \zeta_i \rangle = \delta_{ki} + \sum_{\mu=1}^{N-n} \frac{\partial \rho_k}{\partial x_{2\mu}} (\frac{\partial \rho_k}{\partial x_{2\mu}}) \). Using (3.2.1) we can obtain the following upper bound for \( m_j(z) \):

\[
(3.2.2) \quad m_j(z) \leq \left( 1 + |\Delta_K|^{-2} \sum_{\nu=1}^{n} \sum_{l=1}^{N-n} |A_{\nu K_l}|^2 \right)^n
\]

The ramification locus \( \Sigma_j \) is the zero locus of the following set of polynomials \( \{p_1, \ldots, p_s\} \cup \{\Delta_K; \text{for all increasing multi-indices } K, |K| = N-n\} \). Applying (3.2.1) we have for \( z \in W_j \)

\[
(3.2.3) \quad \max K |\Delta_K(z)| \geq C^r_{i(j)} d_E(z, \Sigma_j)^{-\frac{n}{2}} \left( 1 + \|z\|^2 \right)^{-\frac{n}{2}}
\]

If we choose for each \( z \in W_j \) the multi-index \( K \) such that \( \Delta_K(z) \) is maximal then using (3.2.3) inequality (3.2.2) will become

\[
(3.2.4) \quad m_j(z) \leq \left( 1 + C^{r_2}_{i(j)} d_E(z, \Sigma_j)^{-2\frac{n}{2}} \left( 1 + \|z\|^2 \right)^{\frac{n}{2}} \sum_{\nu=1}^{n} \sum_{l=1}^{N-n} |A_{\nu K_l}|^2 \right)^n
\]

Finally there exist constants \( C_{i(j)}, E_{i(j)}, D_j > 0 \) such that

\[
(3.2.5) \quad m_j(z) \leq C_{i(j)} (1 + \|z\|^2)^{D_j} (\min \{1, d_E(z, \Sigma_j)\})^{-E_{i(j)}}
\]

4. \( L^2 \)-solvability for \( \overline{\partial} \) on \( W_j \)

4.1. \( L^2 \)-existence theorem. Hörmander’s \( L^2 \) theory for Stein domains in \( \mathbb{C}^n \) extends naturally to Stein Riemann domains over \( \mathbb{C}^n \) when these are given the pull back metric. In our case \( (W_j, \pi_j, L_j) \) are Riemann domains over \( L_j \). Since the sets \( X_i \) are singular though, the \( W_j \)'s are not necessarily Stein. We shall need the following variant of Hörmander’s theory:

**Proposition 4.1.** Let \( q > 0 \) and \( f \in L^2_{p,q} \) \( (W_j, dV_E) \) with \( \overline{\partial} f = 0 \) on \( W_j \). Then there exists a solution \( u \) to \( \overline{\partial} u = f \) on \( W_j \) satisfying the following estimate:

\[
(4.1.1) \quad \int_{W_j} |u_j|^2 (1 + \|\pi_j(z)\|^2)^{q-2} e^{-\psi} dV_j \leq \int_{W_j} |f_j|^2 e^{-\psi} dV_j,
\]

whenever the RHS is finite and \( \psi \) is a plurisubharmonic function on \( W_j \).

**Proof.** We can choose polynomials \( Q_j(z) \) such that they vanish on \( \Sigma_j \) but do not vanish identically on \( X_i(j) \). Let \( Z(Q_j) \) denote the zero set of \( Q_j \). Then \( (X_i(j) \setminus Z(Q_j), \pi_j, L_j) \) is a Stein Riemann domain. Theorem 4.4.2 in [12] (or Theorems 2.2.1 and 2.2.1' in [13]) carries (resp. carry) over almost verbatim to \( W'_j := X_i(j) \setminus Z(Q_j) \) and we obtain the existence of a solution \( u \in L^2_{p,q-1} \) \( (W'_j, dV_j) \) to \( \overline{\partial} u = f \) on \( W'_j \) satisfying (4.1.1) with \( W_j \) being replaced by \( W'_j \). We want to show that the solution \( u \) extends to \( W_j \) and satisfies a similar estimate there. Let us look at the set \( W_j \setminus W'_j = W_j \cap Z(Q_j) \). It is a hypersurface in the complex manifold \( W_j \). If we could show that \( u \) is locally in \( L^2 \) (with respect to \( dV_j \)) near \( W_j \setminus W'_j \) then the result would follow from the following lemma:
Lemma 4.2. Let \( \Omega \) be an open subset of \( \mathbb{C}^n \) and \( Y \) an analytic subset of \( \Omega \). Assume that \( v \) is a \((p,q-1)\) form with \( L^2,\text{loc} \) coefficients and \( w \) a \((p,q)\)-form with \( L^1,\text{loc} \) coefficients such that \( \overline{\partial} v = w \) on \( \Omega \setminus Y \) (in the sense of distributions). Then \( \overline{\partial} v = w \) on \( \Omega \).

Proof. This is Lemma 6.9 (page 485) in [2]. \( \square \)

Remark: To be more precise we need to prove a similar lemma where \( \Omega \) is an open set of a complex manifold. But, Lemma 4.2 generalizes quite easily in this case.

Now for every \( z_0 \in W_j \setminus W_j' \) there exists a neighborhood \( U \) of \( z_0 \) in \( X_{i(j)} \) such that \( \pi_j : U \to \pi_j(U) \subset \mathbb{C}^n \) is a biholomorphism and there exists a positive constant \( c_1 \) such that

\[
e^{-\psi(z)} (1 + \|\pi_j(z)\|^2)^{-2} \geq c_1 \quad \text{for all } z \in U.
\]

Hence \( u \in L^2(U \setminus Z(Q_j), dV_j) \) and \( \overline{\partial} u = f \) on \( U \setminus Z(Q_j) \). But then \( \overline{\partial} u = f \) on \( U \) and hence \( \overline{\partial} u = f \) on \( W_j \). \( \square \)

4.2. Estimates for the solution to \( \overline{\partial} u = f \) on \( W_j \). In what follows, we shall use the notation \( L^2_{p,q}(W, \phi, dV_h) \) to denote the Hilbert space of \((p,q)\) forms on a complex hermitian manifold \((W, h)\) for which \( \|f\|^2 := \int_W |f|^2 e^{-\phi} dV_h < \infty \). We shall also use the notation \( a \preceq b \) (resp. \( a \succeq b \)) if there exists an absolute positive constant \( c \) such that \( a \leq c b \) (resp. \( a \geq c b \)), \( a \approx b \) if there exist absolute positive constants \( c, c' \) such that \( a \leq c b \), \( b \leq c' a \).

We shall choose as \( \psi_j(z) := \log m_j(z) + \alpha \log(1 + \|z\|^2) \) where \( \alpha \) is the positive constant that appears in (3.0.1). Then \( e^{-\psi_j} = m_j^{-1} (1 + \|z\|^2)^{-2} \). Let \( f \in L^2_{p,q}(\text{Reg} X, dV_{FS}) \) with \( \overline{\partial} f = 0 \) there. Then \( \phi_{i(j)}^* f \in L^2_{p,q}(\text{Reg} X_{i(j)}, dV) \). Using (3.0.1), (3.1.4) we obtain

\[
m_j^{-1} (1 + \|z\|^2)^{-2} |\phi_{i(j)}^* f_j|^2 dV_j \leq |\phi_{i(j)}^* f|^2 dV.
\]

But this last inequality implies that \( \phi_{i(j)}^* f \in L^2_{p,q}(W_j, \psi_j, dV_j) \). Then we can apply Proposition 4.1 to obtain a solution \( u'_j \in L^2,\text{loc}(W_j) \) that satisfies

\[
\int_{W_j} |u'_j|^2 (1 + \|\pi_j(z)\|^2)^{-2} e^{-\psi_j} dV_j \leq \int_{W_j} |\phi_{i(j)}^* f_j|^2 e^{-\psi_j} dV_j.
\]

Taking into account (3.0.2), Lemma 3.1 i) and the fact that \( \|\pi_j(z)\| \leq \|z\| \) we obtain

\[
(4.2.1) \quad \int_{W_j} m_j^{-2} (1 + \|z\|^2)^{-2-\alpha-\beta} |u'_j|^2 dV \leq \int_{\text{Reg} X} |f|_{FS}^2 dV_{FS}
\]

Using (3.2.3) the last inequality becomes

\[
(4.2.2) \quad \int_{W_j} (1 + \|z\|^2)^{-2-\alpha-\beta-D_j} \left( \min\{1, d_E(z, \Sigma_j)\} \right)^{E_{i(j)}} |u'_j|^2 dV \leq \int_{\text{Reg} X} |f|_{FS}^2 dV_{FS}.
\]

We set \( \Sigma^*_j := \phi_{i(j)}(\Sigma_j) \cup (X \cap H_{i(j)}) \). Then \( \Sigma^*_j \) is a projective subvariety of \( X \). Recall that \( d_E(z, \Sigma_j) > d(\phi_{i(j)}(z), \Sigma^*_j) \) and \( d(\phi_{i(j)}(z), H_{i(j)}) \approx (1 + \|z\|^2)^{-\frac{1}{2}} \) where by \( d(\bullet, \bullet) \) we denote the projective distance. Taking into account these inequalities we obtain

\[
(4.2.3) \quad \int_{\phi_{i(j)}^{-1}(\phi_{i(j)}(W_j))} d(\phi_{i(j)}(z), H_{i(j)})^{D_j} \left( \min\{1, d(\phi_{i(j)}(z), \Sigma^*_j)\} \right)^{E_{i(j)}} |u'_j(z)|^2 dV(z) \leq \int_{\text{Reg} X} |f|_{FS}^2 dV_{FS}
\]
where $D_j' := 2(2 + \alpha + \beta + D_j)$. To relate $d(\phi_{i(j)}(z), H_{i(j)})$ to $d(\phi_{i(j)}(z), \Sigma_j^*)$ we need to recall the notion of regular separation. Following Lojasiewicz (page 242 in [16]) we define:

**Definition 4.3.** Let $(E, F)$ be closed subsets of a manifold $M$. We say that $(E, F)$ satisfy the condition of regular separation if for each point $w \in E \cap F$ the following inequality holds true in a neighborhood of the point $w$

$$\rho(z, E) + \rho(z, F) \geq c \rho(z, E \cap F)^p$$

where $c, p$ are some positive constants.

**Theorem 4.4.** Every pair of analytic sets of a complex manifold satisfies the condition of regular separation.

**Proof.** This theorem is proved in [16] (page 244). \qed

Since the sets $X$, $H_i$ are regularly separated we have for $z \in X_i$

$$d(\phi_{i(j)}(z), H_{i(j)}) \geq c d(\phi_{i(j)}(z), H_{i(j)} \cap X)^p \geq c d(\phi_{i(j)}(z), \Sigma_j^*)^{p(j)}.$$  

where $c, p$ are the positive constants that appear in the definition of regular separation.

Pulling back to $X$ via the biholomorphism $\phi_i^{-1} : U_i \to \mathbb{C}^N$ we obtain from (4.2.3)

$$(4.2.4) \quad \int_{\phi_{i(j)}(W_j)} (\min\{1, d(z, \Sigma_j^*)\})^{b_j} |u_j|^2_{FS} dV_{FS} \leq \int_{Reg_X} |f|^2_{FS} dV_{FS},$$

where $b_j := p_{i(j)} D_j' + E_{i(j)}$ and $u_j := (\phi_{i(j)}^{-1})^* u_j'$. Now, since $\mathbb{CP}^N$ has a finite diameter with respect to the Fubini-Study metric we have that $\min\{1, d(z, \Sigma_j^*)\} \approx d(z, \Sigma_j^*)$. Hence $\bar{\partial} u_j = f$ on $\phi_{i(j)}(W_j)$ and from (4.2.4) we see that $u_j$ satisfies

$$(4.2.5) \quad \int_{\phi_{i(j)}(W_j)} d(z, \Sigma_j^*)^{b_j} |u_j|^2_{FS} dV_{FS} \leq \int_{Reg_X} |f|^2_{FS} dV_{FS}.$$  

To summarize: We have found solutions $u_j$ that satisfy $\bar{\partial} u_j = f$ in $\phi_{i(j)}(W_j)$ and the estimate $\ref{1.2.25}$. For each $j, j'$ with $1 \leq j < j' \leq M$ we have $\bar{\partial} (u_j - u_{j'}) = 0$ on $\phi_{i(j)}(W_j) \cap \phi_{i(j')}(W_{j'})$. Now, the sets $\phi_{i(j)}(W_j)$ are nonempty, Zariski open in $X$ whose complement in $X$ is $\Sigma_j^*$. Set $\Sigma := \cup_{j=1}^M \Sigma_j^*$. We can restrict $u_j$ on $X \setminus \Sigma$. Let $h_{j,j'} := (u_j - u_{j'}) \mid (X \setminus \Sigma)$. Using (4.2.5) we can show that the $h_{j,j'}$’s satisfy the following estimates:

$$(4.2.6) \quad \int_{X \setminus \Sigma} d(w, \Sigma)^{b_{j,j'}} |h_{j,j'}|^2_{FS} dV_{FS} \lesssim \int_{Reg_X} |f|^2_{FS} dV_{FS} < \infty$$

where $b_{j,j'}$ are some positive constants that depend on $b_j, b_{j'}$. 


4.3. Construction of the global solution. Let us define the linear mapping
\[
T : Z_{(2)}^{p,1}(\text{Reg}X) \rightarrow (\Omega^p(X \setminus \Sigma))^{(M)}_2
\]
\[
f \rightarrow (h_{j,j'})_{1 \leq j < j' \leq M}
\]

Suppose we could show that the range of \( T \) were finite dimensional. Then \( \mathcal{H} := \text{kern} T \) would be a subspace of \( Z_{(2)}^{p,1}(\text{Reg}X) \) of finite codimension. For an \( f \in \mathcal{H} \) we shall have that \( u_j = u_j' \) on \( X \setminus \Sigma \) which is non-empty, Zariski open subset of \( \phi_i(j)(W_j) \cap \phi_i(j')(W_{j'}) \). But then \( u_j = u_j' \) on \( \phi_i(j)(W_j) \cap \phi_i(j')(W_{j'}) \). To finish the proof of the main theorem it would suffice to show that the \( u_j \)'s determine a global, square-integrable (with respect to the ambient metric) \( (p,0) \)-form on \( \text{Reg}X \).

Let \( R \) be a positive real number with \( R > \sqrt{N + 1} \). Then \( \mathbb{CP}^N = \cup_{i=0}^{n} \phi_i(B(0, R)) \), where \( B(0, R) \) is the Euclidean ball centered at the origin in \( \mathbb{C}^N \) and having radius \( R \). Consider the sets \( F_j := \{ z \in X_i(j) \setminus \Sigma_j; \| z \| < R \text{ and } \| (\pi_j)_a z v \| \geq c\| v \| \text{ for all } v \in T_z X_i(j) \} \). Let \( V_j := \phi_i(j)(F_j) \subset \phi_i(j)(W_j) \).

**Lemma 4.5.** The sets \( V_j \) cover \( \text{Reg}X \).

*Proof.* Obvious from the definition of the sets \( V_j \). \( \Box \)

When \( z \in F_j \) we have the following upper bound for \( m_j(z); m_j(z) \leq c^{-2n} \). Hence by (4.2.1) we obtain that the solutions \( u_j' \) satisfy the following \( L^2 \)-estimate on \( F_j \)’s:
\[
\int_{F_j} c^{-2n} (1 + R^2)^{-2-\alpha-\beta} |u_j'|^2 dV \leq \int_{\text{Reg}X} |f|^2_{FS} dV_{FS}
\]

Pulling back to \( X \) via \( \phi_i^{-1} \) we see that the solutions \( u_j \) satisfy an \( L^2 \)-estimate (with respect to the ambient metric) on \( V_j \). Since \( u_j = u_j' \) on \( V_j \cap V_{j'} \) and \( \{ V_j \}_j \) cover \( \text{Reg}X \) we obtain a global \( (p,0) \) form \( u \) that satisfies \( \overline{\mathcal{F}} u = f \) on \( \text{Reg}X \) and
\[
\int_{\text{Reg}X} |u|^2_{FS} dV_{FS} \leq C \int_{\text{Reg}X} |f|^2_{FS} dV_{FS}.
\]

The next section is devoted to proving that the \( \text{Rang}(T) \) is finite dimensional.

5. **Finite dimensionality of certain holomorphic \( p \)-forms**

The main goal in this section is to prove the following lemma:

**Lemma 5.1.** Let \( A \) be a nonnegative real number. Let
\[
\mathcal{E}_A := \{ h \in \Omega^p(X \setminus \Sigma); \int_{w \in X \setminus \Sigma} d^A(w, \Sigma) |h(w)|^2_{FS} dV_{FS}(w) < \infty \}.
\]
For any \( A \geq 0 \), \( \mathcal{E}_A \) is a finite dimensional complex vector space.
Proof. The idea of the proof is to “kill” the singularity of \( h \in \mathcal{E}_A \) by tensoring it with a section of an invertible sheaf on \( X \) vanishing to high order on \( \Sigma \) and then use classical finiteness results for \( \Gamma(X, S_k) \) where \( S_k \) is a suitable coherent analytic \( \mathcal{O}_X \)-module.

Let \( p : \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{C}P^N \) be the standard projection map. In what follows we shall think mostly of \( X \) as a complex compact space. The sheaves on \( X \) that we will consider shall be analytic sheaves. By \( \mathcal{O}_X \) we shall denote the sheaf of holomorphic functions on \( X \). For every positive integer \( l \) we define the twisting sheaf \( \mathcal{O}(l) \) on \( \mathbb{C}P^N \) as follows: if \( U \) is an open subset of \( \mathbb{C}P^N \) in the Euclidean topology then \( \mathcal{O}(l)(U) \) consists of the space of holomorphic functions on \( p^{-1}(U) \) which are homogeneous of degree \( l \). The global holomorphic sections of this sheaf can be naturally identified with homogeneous polynomials \( Q(Z_0, \cdots, Z_N) \) of degree \( l \) in \( \mathbb{C}^{N+1} \). Over \( U_i, \mathcal{O}(l) \) has a trivialization with transition functions \( \left( \frac{Z}{Z_i^l} \right)^l \) on \( U_i \cap U_j \).

We choose a homogeneous polynomial \( P(Z) \) of degree \( d \) that vanishes on \( \Sigma \) but does not vanish identically on \( X \). Let \( k \in \mathbb{N} \) and let \( \sigma^k \) be the section of \( \mathcal{O}(kd) \) that corresponds to \( P(Z)^k \). We shall prove that when \( h \in \mathcal{E}_A \) and \( k \) is a sufficiently large positive integer, \( h \otimes \sigma^k \) extends to a global section \( h \otimes \sigma^k \) of some suitably chosen coherent analytic \( \mathcal{O}_X \)-module \( S_k \). More precisely we shall choose as \( S_k := R^0 \pi_* (\Omega^p_X) \otimes \mathcal{O}_X(\mathcal{O}(kd)) \) where \( \pi : \tilde{X} = \pi^{-1}(X) \) is a desingularization of \( X \) such that \( \Sigma := \pi^{-1}(\Sigma) \) is a divisor with normal crossings, \( \Omega^p_X \) is the sheaf of holomorphic \( p \)-forms on \( \tilde{X} \) and \( \mathcal{O}(kd)|_{\tilde{X}} = i^{-1}(\mathcal{O}(kd)) \) with \( i : X \hookrightarrow \mathbb{C}P^N \).

Let us cover \( X \) by finitely many open affine balls \( \mathcal{U}_\nu \) with \( \mathcal{U}_\nu \subset X \times U_i \) where \( i = i(\nu) \in \{0, \cdots, N\} \). We shall first prove that \( h \otimes \sigma^k \mid_{\mathcal{U}_\nu} \) extends to a section \( S_k(\mathcal{U}_\nu) \). Let us work with inhomogeneous coordinates. Then the section \( \sigma^k \) is represented by \( p_i(z) = Z_r^{-kd} P(Z)^k \) for \( z \in U_i \) and we have \( |p_i(z)|^k \leq C d_E(z, \Sigma)^k \) for \( z \in U_\nu \). When \( h \in \mathcal{E}_A \), \( h \otimes \sigma^k \) is represented over \( \mathcal{U}_\nu \) by \( h p_i^k \) and

\[
\int_{z \in \mathcal{U}_\nu \setminus \Sigma} |h(z) p_i(z)|^2 d_E(z, \Sigma)^{A-2k} dV(z) < \infty
\]

The following pointwise estimates were proven in Lemma 3.1 in [7]:

\[
(5.0.1) \quad c' \ d^c(x, \tilde{\Sigma}) \leq d(\pi(x), \Sigma) \leq C' d(x, \tilde{\Sigma}),
\]

\[
(5.0.3) \quad c \ d^M(x, \tilde{\Sigma}) |v|_{x, \sigma} \leq |\pi_*(v)|_{\pi(x)} \leq C |v|_{x, \sigma}.
\]

for some positive constants \( c', c, C', C, t, M \), where \( c, C, M \) may depend on \( r \) and \( \sigma \) is a real analytic hermitian metric on \( \pi^{-1}(X) \).

For an \( r \)-form \( a \) in \( \mathcal{U}_\nu \setminus \Sigma \) set \( |\pi^* a|_{x, \sigma} := \max \left\{ |< a_{\pi(x)}, \pi_* v > | \mid v \in \wedge^r T_{\pi^{-1}(\mathcal{U}_\nu)}(\tilde{\Sigma}) \right\} \), where \( <,> \) we denote the pairing of an \( r \)-form with a corresponding multi-vector. Using (5.0.3) we obtain:

\[
(5.0.4) \quad c \ d^M(x, \tilde{\Sigma}) |a|_{\pi(x)} \leq |\pi^* a|_{x, \sigma} \leq C |a|_{\pi(x)}
\]
on \( \pi^{-1}(\mathcal{U}_\nu) \), for some positive constant \( M \).
Using the above lemma and choosing $k$ to be sufficiently large we can show that $\pi^*(h p^k) \in L^2_{(p,0)}(\pi^{-1}(U_\nu \setminus \Sigma))$ and thus extends to a holomorphic $p$-form on $\pi^{-1}(U_\nu)$. But then $h \otimes \sigma^k$ extends uniquely to a section in $S_k(U_\nu)$. The local extensions fit together to a unique global section $h \otimes \sigma^k \in \Gamma(X, S_k)$ extending $h \otimes \sigma^k$. Since the map $h \to (h \otimes \sigma^k)$ is injective and $\Gamma(X, S_k)$ is finite dimensional, the space $E_A$ is finite dimensional as well.

**Remark:** H. Flenner showed in [3] that holomorphic $p$-forms on the regular part of a projective variety with general singularities are the push-forward of holomorphic $p$-forms on $\tilde{X}$, a desingularization of $X$, provided that $0 \leq p < \text{codim Sing } X - 1$. In our case we do not need to put any restriction on the range of $p$ since we are tensoring our section $h$ with sections of an invertible sheaf on $X$. $\square$

5.1. **Conclusion of proof of Theorem 1.** Clearly we can find an $A > 0$ such that for all $1 \leq j < j' \leq M$, $h_{j,j'} \in E_A$. Combining this with lemma 5.1 we have shown that the $\text{Rang}(T)$ is finite dimensional. Then the main theorem follows by the argument used in Section 4.3.

6. **Proof of Theorem 1.2**

One interesting application of the techniques used in this paper is a derivation of a weighted $L^2$-estimate for solutions to $\partial u = f$ on the regular part of affine varieties. Our goal in this section is to prove Theorem 1.2.

**Proof.** According to Proposition 2.4 there exist $\{L_j\}_{j=1}^M$ a finite collection of $n$-dimensional subspaces of $\mathbb{C}^N$ and orthogonal projections $\pi_j : \mathbb{C}^N \to L_j$ such that:

i) $\pi_j|_X$ is non-degenerate on $X$ with ramification locus $\Sigma_j$ and

ii) For every $z \in \text{Reg } Y$ there exists a $j$ such that $z \notin \Sigma_j$ and such that $(\pi_j)_{*,z} : T_z Y \to L_j$ satisfies $\|((\pi_j)_{*,z} v)\| \geq c \|v\|$ for all $v \in T_z Y$. (Here the norms are the induced Euclidean norms).

Let us denote by $W_j := X \setminus \Sigma_j$. Then $(W_j, \pi_j, L_j)$ are Riemann domains and we would like to apply Hörmander’s $L^2$ theory for $\mathcal{F}$ (in particular Proposition 4.1) to our $f|_{(W_j)}$. Let us define on $W_j$ the function $\psi_j := \psi + \log m_j$. Then $\psi_j$ is plurisubharmonic on $W_j$ and using (3.1.4) we can show that

$$\int_{W_j} m_j^{-1}|f|_j^2 e^{-\psi} \, dV_j \leq \int_{W_j} |f|_E^2 e^{-\psi} \, dV_E < \infty$$

Noticing that $m_j^{-1} e^{-\psi} = e^{-\psi_j}$ we conclude that $f \in L^2_{(p,1)}(W_j, \psi_j, dV_j)$. Hence we can apply Proposition 4.1 to each $W_j$ and obtain a solution $v_j$ to $\partial v = f$ on $W_j$ and $v_j$ satisfies the following estimate:

$$\int_{W_j} |v_j|_j^2 (1 + \|\pi_j(z)\|^2)^{-2} e^{-\psi_j} \, dV_j \leq C \int_{W_j} |f|_E^2 e^{-\psi} \, dV_E < \infty$$

(6.0.1)

Noticing that $\|\pi_j(z)\| \leq \|z\|$ we derive the following estimate from (6.0.1)

$$\int_{W_j} |v_j|_j^2 (1 + \|z\|^2)^{-2} e^{-\psi_j} \, dV_j \leq C \int_{W_j} |f|_E^2 e^{-\psi} \, dV_E < \infty$$
We want to estimate \( v_j \) using the restriction of the Fubini-Study metric on \( \text{Reg}X \). Recall that \( |v_j|_E \geq |v_j|_E, dV_j = m_j^{-1} dV_E \) and \( m_j \leq C_j (1 + \|z\|^2)^{D_j} \min\{1, d_E(z, \Sigma_j)\}^{-E_j} \).

Hence \( |v_j|^2 dV_j \geq C_j^{-1} (1 + \|z\|^2)^{-D_j} \min\{1, d_E(z, \Sigma_j)\} |v_j|^2 dV_E \). Taking into account that \( |v_j|^2 dV_E \geq (1 + \|z\|^2)^{-\beta} |v_j|^2 dV \) we obtain

\[
(6.0.2) \quad e^{-\phi_j} |v_j|^2 dV_j \geq e^{-\psi} C_j^{-2} (1 + \|z\|^2)^{-2D_j-\beta} \min\{1, d_E(z, \Sigma_j)\}^2 E_j |v_j|^2 dV.
\]

We can consider the projective closure \( \text{PCl}(X) \) of \( X \) in \( \mathbb{CP}^N \) and let \( \phi_0 : \mathbb{C}^N \to U_0 \subset \mathbb{CP}^N \) be the corresponding affine chart. Let \( \Sigma_j^* := \phi_0(\Sigma_j) \cup (\text{PCl}(X) \cap H_0) \) and let \( \Sigma := \bigcup \Sigma_j^* \).

Recall that \((1 + \|z\|^2)^{-\frac{1}{2}} \approx d(\phi_0(z), H_0) \geq C' d(\phi_0(z), \Sigma_j^*)^p \) and \( d_E(z, \Sigma_j) > d(\phi_0(z), \Sigma_j^*) \). As in section 4 we can show that

\[
(1 + \|z\|^2)^{-2D_j-\beta} \min\{1, d_E(z, \Sigma_j)\}^2 E_j \geq C'' d(\phi_0(z), \Sigma)^p. \]

Hence the \( v_j \) satisfy the following estimate on \( W_j \)

\[
(6.0.3) \quad \int_{W_j} d(\phi_0(z), \Sigma)^{N_j} e^{-\psi} |v_j|^2 dV \leq C \int_{W_j} |f|^2_E e^{-\psi} dV
\]

where \( N_j \) are some positive constants. Since \( \psi \) has at most logarithmic growth we can bound from below \( e^{-\psi} \geq e^{-D} (1 + \|z\|^2)^{-A} \geq d(\phi_0(z), H_0)^{2A} \geq d(\phi_0(z), \Sigma_j^*)^{2Ap} \). Hence we have

\[
(6.0.4) \quad \int_{W_j} d(\phi_0(z), \Sigma)^{N_j} |v_j|^2 dV \leq C \int_{W_j} |f|^2_E e^{-\psi} dV
\]

where \( N_j \) some positive constants that depend on \( N_j', A \).

Let us look at \( u_j := (\phi_0^{-1})^* v_j \). Then, \( \overline{\partial}(u_j - u_k) = 0 \) on \( \phi_0(W_j) \cap \phi_0(W_k) \). Now, each of the sets \( \phi_0(W_j) \) is a non-empty, Zariski open in \( \text{PCl}(X) \) whose complement is \( \Sigma_j^* \). Let us denote by \( h_{jk} := (u_j - u_k) \mid_{\text{PCl}(X) \setminus \Sigma} \). Then \( \{h_{jk}\}_{1 \leq j < k \leq M} \) are holomorphic p-forms on \( \text{PCl}(X) \setminus \Sigma \) and from (6.0.4) we can see that they satisfy the following estimates

\[
(6.0.5) \quad \int_{w \in \text{PCl}(X) \setminus \Sigma} d(w, \Sigma)^K |h_{jk}(w)|^2_{FS} dV_{FS}(w) < \infty,
\]

where \( K \) is a positive constant that depends on \( N_j, N_k \). But then, lemma 5.1 tells us that the space \( E_K \) of such forms is a finite dimensional complex vector space. Hence we can repeat the argument in section 4.3 and construct a linear operator \( T : Z_{\psi}^{(p,1)} \ni f \mapsto (h_{jk})_{1 \leq j < k \leq M} \in E_K^{(M)} \). Then \( \mathcal{H} := \ker T \) is a finite codimensional subspace of \( Z_{\psi}^{(p,1)} \). For an \( f \in Z_{\psi}^{(p,1)} \), the local solutions \( \{u_j\} \) agree on \( \text{PCl}(X) \setminus \Sigma \), thus they agree on \( \phi_0(W_j) \cap \phi_0(W_k) \).

Hence \( v_j = v_k \) on \( W_j \cap W_k \) and we can define a \((p,0)\)-form \( u \) on \( \bigcup W_j = \text{Reg}X \) that would satisfy \( \overline{\partial} u = f \) on \( \text{Reg}X \). Repeating a similar argument as in the end of section 4.3 we can show that this \( u \) satisfies the desired estimate. \( \square \)

**Remarks:**

1) The codimension of \( \mathcal{H} \) in \( Z_{\psi}^{(p,1)} \) will increase with \( A \) the constant that appears in the logarithmic growth for \( \psi \) (and in general it will not be finite). Indeed, let us consider the variety \( X := Y \times \mathbb{C} \) where \( Y \) is an irreducible surface in \( \mathbb{CP}^3 \) described by a homogeneous polynomial of degree \( d \geq 3 \). In [3], Fornæss constructed finitely many \( \overline{\partial} \)-closed \((0,1)\) forms...
Let $\lambda$ on the $\text{Reg} Y$ satisfying: 1) $\lambda \in L^2_{0,1}(\text{Reg} Y \cap B_1(0), dV_E)$ and 2) $\overline{\partial} v = \lambda$ is not solvable in $L^2(\text{Reg} Y \cap B_1(0))$ (here $B_1(0)$ is the unit ball in $\mathbb{C}^N$ centered around 0). For one of these forms $\lambda$ one can further show that

$$\int_{z \in \text{Reg} Y} |\lambda(z)|^2_E (1 + \|z\|^2)^{-a} dV_E(z) < \infty$$

when $a > 1$. We also have

$$\int_{w \in \mathbb{C}} |w|^{2k} (1 + |w|^2)^{-b} dA(w) < \infty$$

when $b > k + 1$. 

Let $m \in \mathbb{N}$. On $\text{Reg} X$ we consider forms $\lambda(z) \otimes p_m(w)$ where $p_m(w)$ is a polynomial in $w$ of degree $m$ and $\lambda$ is the non-solvable $(0,1)$-form on $\text{Reg} Y$ as above that satisfies (6.0.6). Let us define the function $\psi := A \log (1 + \|z, w\|^2)$ where $A := a + \beta$ and where $a$ is as above and $\beta$ is chosen such that $\beta > m + 1$ (hence $A > m + 2$). Clearly $\lambda(z) \otimes p_m(w) \in Z^{(0,1)} \otimes \mathcal{H}$ (otherwise the equation $\overline{\partial} v = \lambda$ would have an $L^2$ solution in a deleted neighborhood of 0 in $Y$ which would contradict the choice of $\lambda$). Hence the codimension of $\mathcal{H}$ in $Z_{\psi}^{(0,1)}$ will be greater or equal to $m + 1$ and it will be infinite whenever $\psi$ has faster growth than a logarithmic one.

ii) The following question was posed to the first author by Henkin and Zeriahi: Could it be possible to take as $\mathcal{H}$ in Theorem 1.2 the whole space $Z_{\psi}^{(p,1)}$ when $X$ is non-singular?

7. Another Application of Theorem 1.1

Let $X$ be an irreducible $n$-dimensional variety in $\mathbb{C}^N$ with an isolated singularity at 0. Let $\Omega$ be a small Stein neighborhood of 0 with $\partial \Omega$ smooth. In this section we shall prove that the $L^2_{\overline{\partial}}(n-1,1)$-cohomology group of $\text{Reg} \Omega$ (with respect to the Euclidean metric) is finite dimensional. In what follows by $L^2_{p,q}(\text{Reg} \Omega)$ we denote the space of $(p,q)$ forms on $\text{Reg} \Omega$ that are square-integrable with respect to the restriction of the Euclidean metric on $\text{Reg} \Omega$.

We shall need the following general result:

**Proposition 7.1.** Under the above assumptions and for any $p, q \in \mathbb{N}$ with $q > 0$ there exists a finite codimensional subspace $E_0$ of $L^2_{p,q}(\text{Reg} \Omega) \cap \ker \overline{\partial}$ and a linear operator $S : E_0 \ni f \to u_0 \in L^2_{p,q-1}(\Omega \setminus \{0\})$ such that $\overline{\partial} u_0 = f$ on $\text{Reg} \Omega$.

Suppose for the moment that Proposition 7.1 were true. We choose $\chi \in C^\infty(\Omega)$ such that $\chi = 1$ near $\partial \Omega$ and $\chi = 0$ near 0. For an $f \in E_0$ we write $f = \overline{\partial}(\chi u_0) + \overline{\partial}((1-\chi) u_0)$. Let $f_0 := \overline{\partial}((1-\chi) u_0) = (1-\chi) f - \overline{\partial} \chi \wedge u_0$. Then $f_0 \in L^2_{p,q-1}(\text{Reg} \Omega)$ and is $\overline{\partial}$-closed there. Now $\Omega$ can be embedded as a subdomain of an irreducible $n$-dimensional projective variety $X'$ and we may extend $f_0$ by zero to $\tilde{f}_0$, a globally defined form on $\text{Reg} X'$ that is $\overline{\partial}$-closed on $\text{Reg} X'$ and square-integrable (with respect to the Fubini-Study metric) there. Let us consider the case where $q = 1$. According to Theorem 1.1 there exists a subspace $\mathcal{H}$ of $Z_{\psi}^{(1)}(\text{Reg} X')$ of finite codimension such that whenever $g \in \mathcal{H}$ we can solve $\overline{\partial} v = g$ with $L^2$ estimates (with
respect to the Fubini-Study metric) on $\text{Reg} X'$. We apply Theorem 1.1 to $\bar{f}_0$ and we obtain a solution $v$ to $\bar{\partial} v = \bar{f}_0$ on $\text{Reg} X'$.

Let us consider the map $T : E_0 \to Z^{p,1}_{\{2\}}(\text{Reg} X')$ sending an element $E_0 \ni f \to \bar{f}_0$. Clearly $T$ is a linear map, hence $\text{codim}_{E_0}(T^{-1}(\mathcal{H})) \leq \text{codim}Z^{p,1}_{\{2\}}(\text{Reg} X')(\mathcal{H})$. Setting $E := T^{-1}(\mathcal{H})$ we see that $E$ is of finite codimension in $L^2_{p,1}(\text{Reg} \Omega) \cap \ker(\bar{\partial})$ and when $f \in E$ we can find a $u := \chi u_0 + v_{f|_{\text{Reg} \Omega}} \in L^2_{p,0}(\text{Reg} \Omega)$ satisfying $\bar{\partial} u = f$ on $\text{Reg} \Omega$. Taking $p = n - 1$ we obtain the finite dimensionality of the $L^2_{p,q}(n - 1,1)$-cohomology group of $\text{Reg} \Omega$.

We return now to the proof of Proposition 7.1.

**Proof.** We consider a desingularization $\pi : \bar{X} \to X$ with exceptional divisor $D$ and let $\sigma$ be a hermitian metric on $\bar{X}$ and $d\nu_{\sigma}$ the volume element induced by this metric. Let $\mathcal{O}(kD)$ denote the holomorphic line bundle on $\bar{X}$ associated to the divisor $kD$ and let $\Omega := \pi^{-1}(\Omega)$. Choose a hermitian metric $h$ on $\mathcal{O}(kD)$. When $f \in L^2_{p,q}(\text{Reg} \Omega)$, $\pi^* f$ does not necessarily belong to $L^2_{p,q}(\Omega, d\nu_{\sigma})$. However it gives rise to a section $\xi_f \in L^2_{p,q}(\Omega, \mathcal{O}(kD)|_\Omega)$ for some sufficiently large integer $k$. Moreover, $\bar{\partial} \xi_f = 0$ on $\bar{\Omega}$. Since $\bar{\Omega}$ has strictly pseudoconvex boundary the $L^2_{p,q}$ cohomology groups $H^q_{p,q}(\bar{\Omega}, \mathcal{O}(kD))$ for $q > 0$ are finite dimensional (see for example Theorem 5.11 in [15]). Hence there exists a finite codimensional subspace $E$ of $L^2_{p,q}(\Omega, \mathcal{O}(kD)) \cap \ker(\bar{\partial})$ such that whenever $\xi_f \in E$ there exists $\tau_f \in L^2_{p,q-1}(\bar{\Omega}, \mathcal{O}(kD))$ such that $\bar{\partial} \tau_f = \xi_f$. Choosing $\tau_f$ to be the minimal solution we make the map $\xi_f \to \tau_f$ linear. Now $\tau_f|_{\Omega \setminus D}$ determines a form $\bar{\nu}_f \in L^2_{p,q-1}(\bar{\Omega} \setminus \{D\})$. Setting $u_0 := (\pi^{-1})^* \bar{\nu}_f$ we obtain a $u_0 \in L^2_{p,q-1}(\bar{\Omega} \setminus \{0\})$ satisfying $\bar{\partial} u_0 = f$ on $\text{Reg} \Omega$. Moreover the set $E_0 := \{ f \in L^2_{p,q}(\text{Reg} \Omega) \cap \ker(\bar{\partial}) : \xi_f \in E \}$ is a finite codimensional subspace of $L^2_{p,q}(\text{Reg} \Omega) \cap \ker(\bar{\partial})$, since the map $f \to \xi_f$ is injective. 

**Remark:** Arguments similar to the one used in the proof of Proposition 7.1 have already appeared in the works of Nagase [17], Pardon [26] and Pardon and Stern [29] (sections 3,4 of their 1997 preprint).

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