Locally Optimal Routes for Route Choice Sets

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Abstract

Route choice is often modelled as a two-step procedure in which travellers choose their routes from small sets of promising candidates. Many methods developed to identify such choice sets rely on assumptions about the mechanisms behind the route choice and require corresponding data sets. Furthermore, existing approaches often involve considerable complexity or perform many repeated shortest path queries. This makes it difficult to apply these methods in comprehensive models with numerous origin-destination pairs. In this paper, we address these issues by developing an algorithm that efficiently identifies locally optimal routes. Such paths arise from travellers acting rationally on local scales, whereas unknown factors may affect the routes on larger scales. Though methods identifying locally optimal routes are available already, these algorithms rely on approximations and return only few, heuristically chosen paths for specific origin-destination pairs. This conflicts with the demands of route choice models, where an exhaustive search for many origins and destinations would be necessary. We therefore extend existing algorithms to return (almost) all admissible paths between a large number of origin-destination pairs. We test our algorithm and its applicability in route choice models on the road network of the Canadian province British Columbia and empirical data collected in this province.

Keywords: alternative paths; choice set; computer algorithm; local optimality; road network; route choice.

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1 Introduction

Route choice models have important applications in transportation network planning (Yang & Bell, 1998), traffic control (Mahmassani, 2001), and even epidemiology and ecology (Fischer et al., 2020). Route choice models can be classified as either perfect rationality models or bounded rationality models (Di & Liu, 2016). In perfect rationality models (Sheffi, 1984), travellers are assumed to have complete information and choose their routes optimally according to some goodness criterion, whereas bounded rationality models (Simon, 1957) take information constraints and the complexity of the optimization process into account. Though both perfect rationality models and bounded rationality models have been used in route choice modelling, bounded rationality models have been found to fit observed data better (Di & Liu, 2016).

Many bounded rationality models consider route choice as a two-stage process: first, a so-called “choice set” of potentially good routes is generated, and second, a route from the choice set is chosen according to some goodness measure (Ben-Akiva et al., 1984). This approach is motivated through travellers’ limited ability to consider all possible paths. Instead, they may heuristically identify a small set of routes from which they choose the seemingly best. Besides this conceptual reasoning, the two-step model has computational advantages, as the choice sets can be generated based on simple heuristics, while complex models may be applied to determine travellers’ preferences for the identified routes. Therefore, the two-stage process is widely used in route choice modelling (Prato, 2009).

Most of the approaches to identify route choice sets are based on a combination of the optimality assumption, the constraint assumption, and the stochasticity assumption.

- According to the optimality assumption, travellers choose routes optimally according to some criterion, which could be based on route characteristics (e.g. travel costs and travel time), or on scenarios (e.g. that the travel time on the shortest route increases). Examples include the link labelling approach (Ben-Akiva et al., 1984), link elimination (Azevedo et al., 1993), and link penalty (De La Barra et al., 1993).

- According to the constraint assumption, travellers consider all paths whose quality exceeds a
certain minimal value (e.g. acyclic paths not more than 25% longer than the shortest route). This assumption motivates constrained enumeration methods (Prato & Bekhor, 2006).

• The stochasticity assumption accounts for the possibility of stochastic fluctuations of route characteristics (e.g. through traffic jams or accidents) or error-prone information. Often, stochastic route choice sets are computed based on the optimality principle applied to a randomly perturbed graph (see Bovy, 2009).

Though each of the assumptions mentioned above has a sound mechanistic justification, they require that the heuristic that travellers use to identify potentially suitable paths is known and that corresponding data are available. However, if travellers choose a route for unknown reasons, e.g. because they desire to drive via some intermediate destination, their routes would be difficult to consider with the common methods. The natural solution would be to increase the set of generated routes by relaxing constraints or modelling more mechanisms explicitly. However, in comprehensive and large-scale route choice models, many origin-destination pairs may have to be considered, making it costly or even infeasible to work with large choice sets. Thus, it would be desirable to characterize choice sets based on a more general but sufficiently restrictive criterion that does not require knowledge or data of the specific mechanism behind route choices.

A potentially suitable criterion is local optimality. A route is locally optimal if all its short (“local”) subsections are optimal, respectively, according to a given measure. For example, if travel time is the applied goodness criterion, a locally optimal route would not contain local detours.

The rationale behind the principle of local optimality is that the factors impacting travellers’ routing decisions may differ dependent on the spatial scale. Tourists, for example, may want to drive along the shortest route locally but plan their trip globally to include a number of sights. Other travellers may want to drive along the quickest routes locally while minimizing the overall fuel consumption. Yet others may have a limited horizon of perfect information and act rationally within this horizon only. Independent of the specific mechanism behind travellers’ route choices on the large scale, it is possible to characterize many choice candidates as locally optimal routes.

A potential problem with considering locally optimal routes is that the set of locally optimal routes between an origin and a destination can be very large and include zig-zag routes, which may
seem unnatural. A possible solution is to focus on so-called single-via paths. A single-via path is the shortest path via a given intermediate location.

Since not all locally optimal paths are single-via paths, restricting the focus on single-via paths excludes some potentially suitable paths from the choice set. However, single-via paths have a reasonable mechanistic justification through travellers choosing intermediate destinations, and the reduced choice sets are likely to include most of the routes that travellers would reasonably choose. Since the reduced sets contain relatively few elements, sophisticated models can be used for the second decision stage, in which a route is chosen from the choice set. Therefore, constraining the search for locally optimal routes on single-via paths may lead to overall better fitting route choice models.

To date, methods identifying locally optimal single-via paths have been developed with the objective to suggest multiple routes to travellers (Abraham et al., 2013; Delling et al., 2015; Luxen & Schieferdecker, 2015; Bast et al., 2016). Such suggestions of alternative routes are a common feature in routing software, such as Google Maps or Bing Maps. However, route choice models have different demands than routing software, as travellers’ decisions shall be modelled or predicted rather than facilitated.

Route planning software seeks to compute a small number of high-quality paths that travellers may want to choose. Here, computational speed is more important than rigorous application of specific criteria characterizing the returned paths. In contrast, route choice models should consider all routes that travellers may take, and rigorous application of modelling assumptions is key to allow mechanistic inference and to make models portable. In addition, route choice models may consider multiple origins and destinations. Therefore, many algorithms designed to facilitate route planning cannot be directly applied to identify route choice sets.

In this paper, we bridge this gap by extending an algorithm originally designed for route planning. The algorithm REV by Abraham et al. (2013) searches a small number of “good” locally optimal paths between a single origin-destination pair. To this end, the algorithm uses an approximation causing some locally optimal paths to be misclassified as suboptimal.

Our extended algorithm overcomes these limitations. Unlike REV, our algorithm returns (al-
most) all admissible paths between a set of origins and a set of destinations. Therefore, we call our algorithm REVC, the “C” emphasizing the attempted complete search. REVC identifies locally optimal routes with arbitrarily high precision. That is, the algorithm may falsely reject some locally optimal single-via routes, but the error can be arbitrarily reduced by cost of computational speed. As the execution time of REVC depends mostly on the number of distinct origins and destinations rather than the number of origin-destination pairs, the algorithm is an effective tool to build traffic models on comprehensive scales.

This paper is structured as follows: first, we introduce helpful definitions and notation, review concepts we build on, and provide a clear definition of our objective. Then we give an overview of REVC, before we describe each step in detail. After describing the algorithm, we present results of numerical and empirical tests proving the algorithm’s applicability and efficiency in real-world problems. Finally, we discuss the test results and the limitations and benefits of our approach.

2 Algorithm

2.1 Preliminaries

In this section, we specify our goal and introduce helpful notation and concepts. First, we provide definitions and notation, which we then use to characterize the routes we are seeking. Afterwards, we recapitulate Dijkstra’s algorithm and briefly describe the method of reach based pruning, two basic concepts that our work builds on.

2.1.1 Problem statement and notation

Suppose we are given a graph $G = (V,E)$ that represents a road network. The set of vertices $V$ models intersections of roads as well as the start and end points of interest. The directed edges $e \in E$ represent the roads of the road network and are assigned non-negative weights $c_e$, denoting the costs for driving along the roads. To ease notation, we will refer to the cost of an edge or path as its length without loss of generality. In practice, other cost metrics, such as travel time, may be used. Our goal is to find locally optimal paths between all combinations of origin locations
s ∈ O ⊆ V and destination locations t ∈ D ⊆ V.

To specify the desired paths more precisely, we introduce convenient notation and make some definitions:

\( P_{st} \) is the shortest path from s to t.

\( P_{uv} \) is the subpath of P from \( u \in P \) to \( v \in P \).

\( l(P) \) is the length of the path P. That is, \( l(P) = \sum_{e \in P} c_e \).

\( d(u, v) := l(P_{uv}) \) is the length of the shortest path from the vertex u to the vertex v.

\( d_P(u, v) := l(P_{uv}) \) is the length of the subpath of P from vertex u to vertex v.

With this notation, we introduce the notion of single-via paths.

**Definition 1.** A single-via path (or short v-path) \( P_{svt} \) via a vertex v is the shortest path from a vertex s to a vertex t via v. We say v represents the single-via path \( P_{svt} \) with respect to the origin-destination pair \((s, t)\).

For simplicity, we assume that \( P_{st} \) and \( P_{svt} \) are always uniquely defined. In practise, the paths are concatenations of shortest paths found by algorithms outlined below, which are responsible for breaking ties.

We proceed with a precise definition of local optimality following Abraham et al. (2013). Generally speaking, a path is \( T \)-locally optimal if each subpath of P with a length of at most \( T \) is a shortest path. However, because paths are concatenations of discrete elements, we need a more technical definition.

**Definition 2.** Consider a subpath \( P' \subseteq P \) and let \( P'' \subseteq P' \) be \( P' \) after removal of its end points. We say \( P' \) is a \( T \)-significant subpath of P if \( l(P'') < T \). A path P is \( T \)-locally optimal if all its \( T \)-significant subpaths \( P' \) are shortest paths. We say P is \( \alpha \)-relative locally optimal if it is \( T \)-locally optimal with \( T = \alpha \cdot l(P) \).

We want to identify locally optimal paths between many origin and destination locations. However, there may be an excessive number of such paths. Therefore, we apply slightly stronger constraints on the searched paths, which we will call admissible below.
Definition 3. Let $\alpha \in (0, 1]$ and $\beta \geq 1$ be constants. A v-path $P_{svt}$ from vertex $s \in O$ to vertex $t \in D$ via vertex $v \in V$ is called admissible if

1. $P_{svt}$ is $\alpha$-relative locally optimal.

2. $P_{svt}$ is longer than the shortest path by no more than factor $\beta$, i.e. $l(P_{svt}) \leq \beta \cdot l(P_{st})$.

Objective. The objective of this paper is to identify (close to) all admissible single-via paths between each origin $s \in O$ and each destination $t \in D$.

2.1.2 Dijkstra’s algorithm

Large parts of our algorithm are based on modifications of Dijkstra’s algorithm (Dijkstra, 1959; Dantzig, 1998). Dijkstra’s algorithm is a frequently used method to find the shortest paths from an origin $s$ to all other vertices in a graph with non-negative edge weights. Though the algorithm is well-known to a large audience, we briefly recapitulate the algorithm to establish some notation that we will use later.

- In Dijkstra’s algorithm, every vertex $v$ is assigned a specific cost denoted $\text{cost}(v)$. Eventually, this cost shall be equal to the distance between the origin vertex $s$ and vertex $v$. Initially, however, the cost of each vertex is $\infty$. An exception is the origin $s$, for which the initial cost is 0.

- We say that a vertex $v$ is scanned if we are certain that $\text{cost}(v) = d(s, v)$. Furthermore, we say that a not yet scanned vertex $v$ is labelled if $\text{cost}(v) < \infty$. All other vertices are called unreached. In line with our notion of scanned vertices, we call edges $e = (u, v)$ scanned if we know that $e \in P_{sv}$ for some scanned vertex $v$.

Dijkstra’s algorithm is outlined in Algorithm 1. Initially, all vertices are in a container that allows us to determine the least-cost vertex efficiently. Dijkstra’s algorithm consecutively removes the least-cost vertex $v$ from the container and scans it. That is, the algorithm iterates over $v$’s successors $w$ and updates their costs if the distance from the origin $s$ to $w$ via $v$ is smaller than the current cost of $w$. In this case, $v$ is saved as the parent of $w$. 
Algorithm 1: Dijkstra’s algorithm.

1. while container is not empty do
   2. Take the vertex with the lowest cost from the container and remove it;
   3. Scan the vertex $v$:
      4. forall successors of $v$ that have not been scanned yet do
         5. Label $w$:
            6. if $\text{cost}(w) < \text{cost}(v) + c_{vw}$ then
               7. Set $\text{cost}(w) := \text{cost}(v) + c_{vw}$; \hfill // $c_{vw}$ is the length of the edge from $v$ to $w$
               8. Set $\text{parent}(w) := v$;

After execution of Dijkstra’s algorithm, shortest paths can be reconstructed by following the trace of the computed parent vertices, starting at the destination vertex and ending at the origin. The edges $(\text{parent}(v), v)$ for all scanned vertices $v \in V$ form a shortest path tree. Hence, we call the procedure described above “growing a shortest path tree”. The distance (measured in cost units) from the start vertex to its farthest descendant is called the height of the shortest path tree. As we will see below, it can be beneficial to stop the tree growth when the tree has reached a certain height.

When the shortest path between a specific pair of vertices $s$ and $t$ is sought, the bidirectional Dijkstra algorithm is more efficient than the classic algorithm (compare Figures 1 (a) and (b)). The bidirectional Dijkstra algorithm grows two shortest path trees: one in forward direction starting at the origin $s$ and one in backward direction starting at the destination $t$. The trees are grown simultaneously; i.e., the respective tree with smaller height is grown until its height exceeds the other tree’s height. The search terminates if a vertex $v$ is included in both trees, i.e., scanned from both directions. The shortest path is the concatenation of the $s$-$v$ path in the first shortest path tree and the $v$-$t$ path in the second tree.
2.1.3 Reach-based pruning

Dijkstra’s algorithm is not efficient enough to find shortest paths in large networks within reasonable time. Therefore, multiple methods have been developed to identify and prune vertices that cannot be on the shortest path. One of these approaches is reach-based pruning (RE; Goldberg et al., 2006), which we introduce below.

Let us start by introducing the notion of a vertex’s reach.

**Definition 4.** The reach of a vertex $v$ is defined as

$$
\text{reach}(v) := \max_{u, w \in V : v \in P_{uw}} \min \{d(u, v), d(v, w)\}.
$$

That is, if we consider all shortest paths that include $v$, split each of these paths at $v$, and consider the shorter of the two ends, then the reach of $v$ is the maximal length of these sections. The reach of $v$ is high if $v$ is at the centre of a long shortest path. Typically, vertices on highways have a high reach, since many long shortest paths include highways.

Disregarding vertices with small reaches can speed up shortest path searches. Suppose we use the bidirectional Dijkstra algorithm to find the shortest path between the vertices $s$ and $t$ and have already grown shortest path trees with heights $h$. Let $v \in P_{st}$ be a vertex that is located on the shortest path between $s$ and $t$ but has not been scanned yet. Then $d(s, v) > h$ and $d(v, t) > h$, since $v$ would have been included in one of the shortest path trees otherwise. Therefore, we know
Figure 2: Optimizations that REVC employs to efficiently identify admissible $v$-paths between origin-destination pairs $(s,t)$. (a) Shortest path trees (depicted as shaded areas) are grown to a tight bound only and exclude low-reach vertices, which cannot be on long locally optimal paths. (b) U-turn paths (e.g. $s \rightarrow v \rightarrow t$) are excluded by requiring that an edge adjacent to the via vertex is included both in the shortest path tree around the origin (black arrows) and the shortest path tree around the destination (blue arrows). Edges satisfying this constraint are highlighted with red background. Note that arrows with different directions depict distinct edges. (c) If $v$-paths via different vertices $v_1$ and $v_2$ are identical, only one of these vertices is chosen to represent the path. (d) If $v$-paths for different origin-destination pairs (here: $(s_1,t)$ and $(s_2,t)$) are represented by the same via vertex $v$ and share a subpath (highlighted red), the local optimality of this section is tested only once for all origin-destination pairs.

that $\text{reach}(v) \geq \min(d(s,v), d(v,t)) > h$. Thus, when adding further vertices to our shortest path trees, we can neglect all vertices with a reach less or equal to $h$. This speeds up the shortest path search.

Computing the precise reaches of all vertices is expensive, as this would require an extremely large number of shortest path queries. However, Goldberg et al. (2006) developed an algorithm to compute upper bounds on vertices’ reaches efficiently. These upper bounds can be used in the same way as exact vertex reaches.

### 2.2 Outline of the algorithm

After specifying our goal and introducing necessary notation and concepts, we can now proceed with an overview of our algorithm. The main idea of REVC is (1) to grow shortest path trees in forward direction from all origins and in backward direction from all destinations and (2) to check the admissibility of the $v$-paths via the vertices that have been scanned in both forward and backward direction (see Figure 1c). For each vertex $v$ that is scanned both from an origin $s$ and a destination $t$, the $v$-path $P_{svt}$ can be reconstructed easily from the information contained in the shortest path trees. Therefore, the only remaining step is to check whether $P_{svt}$ is admissible, i.e. locally optimal and not much longer than the shortest path $P_{st}$. 
As each vertex $v \in V$ could serve as via vertex for many origin-destination combinations, checking the admissibility of all possible v-paths may be infeasible. Therefore, it is important to identify and exclude vertices that cannot represent admissible v-paths. The following observations can be exploited: (1) v-paths via vertices that are very far from an origin or destination cannot fulfill the length requirement. (2) Some vertices represent intersections of minor roads, which can be bypassed on close-by major roads. Thus, these vertices cannot be part of locally optimal paths. (3) Some v-paths may include a u-turn at the via vertex (see Figure 2b). That is, travellers driving on such a path would need to drive back and forth along the same road. This is not locally optimal behaviour. (4) Some via vertices may represent the same v-paths. That is, the v-paths corresponding to distinct via vertices may be identical, and only one of these via vertices needs to be considered.

Our algorithm REVC makes use of the observations listed above. (1) When shortest path trees are grown around each origin and destination, the trees are grown up to a tightly specified height only. That way, many vertices that are too far off will not be scanned. (2) When the shortest path trees are grown, reach based pruning is applied to exclude vertices that are not on any sufficiently locally optimal path (see Figure 2a). (3) Instead of considering all v-paths via vertices scanned in forward and backward direction, REVC considers only v-paths in which an edge adjacent to the via vertex has been scanned forward and backward. This excludes paths involving u-turns (see
Figure 2b). (4) Before checking the admissibility of the remaining v-paths, the algorithm ensures that each v-path is represented by one vertex only (see Figure 2c).

After these steps, REVC excludes v-paths that are exceedingly long and checks which v-paths are sufficiently locally optimal. Testing whether all v-paths $P_{svt}$ via a specific vertex $v$ are locally optimal would be expensive if each origin-destination pair $(s, t) \in O \times D$ were considered individually. Therefore, REVC checks the admissibility of many paths simultaneously, reusing earlier results and applying approximations. That way, the algorithm becomes much more efficient than individual pair-wise searches for admissible paths (see Figure 2d). In Figure 3, we provide an overview of REVC.

Before the actual algorithm can be started, some preparational work and preprocessing is required. We will provide a detailed description of the preprocessing procedure after introducing the algorithm in detail.

### 2.3 Step 1: Growing shortest path trees

The algorithm REVC starts by growing forward shortest path trees out of each origin and backward shortest path trees into each destination. For each admissible v-path $P$, we need to scan at least one vertex $v$ with $P = P_{svt}$ from both the origin $s$ and the destination $t$. In addition, we want to scan one edge $e \in P$ adjacent to $v$ from both directions if possible. These edges will be used to exclude u-turn paths. For each vertex $v$ included in a shortest path tree, we note $v$’s predecessor and height in the tree. Furthermore, we memorize from which origins and destinations each edge has been scanned.

#### 2.3.1 Tree bound

To save the work of scanning vertices inadmissibly far away from the origins and destinations, we aim to stop the tree growth as soon as possible. We need to scan at least one vertex $v$ for each admissible path $P_{svt}$ with a length $l(P_{svt}) = d(s, v) + d(v, t) \leq \beta \cdot l(P_{st})$. Since either of $d(s, v)$ and $d(v, t)$ could be arbitrarily small, the algorithm REV by Abraham et al. (2013) grows the trees up to a height of $\beta \cdot l(P_{st})$. Nevertheless, we can terminate the search earlier if we take into account
that we are searching for locally optimal paths.

To derive a tighter tree bound, note that for an \( \alpha \)-relative locally optimal path \( P \), each subsection with length \( \alpha \cdot l(P) \) is a shortest path. This is in particular true for the subsection \( P' \subseteq P \) starting at the origin. Since \( P' \) is a shortest path, the end point \( x_s \) of this subsection will be included in the origin’s shortest path tree. Therefore, it suffices to grow the destination’s shortest path tree until \( x_s \) is reached, which is closer to \( t \) than \( \beta \cdot l(P_{st}) \). The same applies in the reverse direction.

To specify the tree bound, define \( x_s \in P \) more precisely to be the first vertex that is farther away from the origin than \( \alpha \cdot l(P) \). If this vertex is located in the second half of the path, change \( x_s \) to be the last vertex in the first half of \( P \). Choose \( x_t \) accordingly in relation to the destination. Our observations from above are formalized in the following lemma and corollary, which we prove in Appendix A.

**Lemma 1.** With \( s, t, x_s, x_t, \) and \( P \) defined as above, there is at least one vertex \( v \in P \) with

1. \( d_P(s, v) = d(s, v) \leq d_P(s, x_t) \) and

2. \( d_P(v, t) = d(v, t) \leq d_P(x_s, t) \).

**Corollary 1.** For each admissible \( v \)-path between an origin-destination pair \((s, t)\), a via vertex will be scanned from both directions if the shortest path trees are grown up to a height of

\[
h_{\text{max}} := \max \left\{ (1 - \alpha) \beta l(P_{st}), \frac{1}{2} \beta l(P_{st}) \right\}.
\]  

(2)

In Corollary 1, we consider a single origin-destination pair. However, we want to identify admissible paths between multiple origins and destinations and have to adjust the tree bound accordingly. The tree around each origin and destination shall be large enough to include via vertices for all paths starting at the respective endpoint. Hence, if we grow a tree out of origin \( s \), we grow it to a height of \( \max \{ (1 - \alpha) \beta M_s, \frac{1}{2} \beta M_s \} \) with \( M_s = \max_{t \in \mathcal{D}} l(P_{st}) \). We proceed with destinations similarly.

Note that the tree bounds above can only be determined if the shortest distances between the origins and destinations are known. Though these distances can be determined while the shortest
path trees are grown, we will see in the next section that the shortest distances can also be used to speed up the tree growth itself. Therefore, it is beneficial to determine the shortest distances in a preprocessing stage. This also makes it easy to grow the trees in parallel.

2.3.2 Pruning the trees

The search for admissible paths can be significantly sped up if vertices with small reach values are ignored when the shortest paths are grown. Consider a vertex \( v \) on an admissible \( s \)-\( t \) path \( P \). Let us regard the subpath \( P' \) that is centred at \( v \) and has a length just greater than \( \alpha \cdot l(P) \). Since \( P \) is \( \alpha \)-relative locally optimal, we know that \( P' \) is a shortest path. Furthermore, \( P' \) is roughly split in half by \( v \), unless \( v \) is close to one of the end points of \( P \). Thus,

\[
\text{reach}(v) \geq \min \left\{ \frac{\alpha}{2} l(P), d(s,v), d(v,t) \right\}
\]

(see Lemma 5.1 in Abraham et al., 2013).

If we are growing the tree out of origin \( s \), we can use (3) to prune the successors of vertices \( v \) with \( \text{reach}(v) < \min \left\{ \frac{\alpha}{2} l(P), d(s,v) \right\} \). Pruning the successors but not \( v \) itself ensures that at least one vertex per admissible path is scanned from both directions, even if (3) is dominated by \( d(v,t) \).

Since \( l(P) \) is unknown when the shortest path trees are grown, the length of \( P \) must be bounded with known quantities. Abraham et al. (2013) use the triangle inequality

\[
l(P) \geq d(s,v) + d(v,t) \geq \text{cost}(v).
\]

(4)

However, we can also determine shortest distances before we search admissible paths and exploit that \( P \geq d(s,t) \) or, if we are considering multiple origins and destinations, \( l(P) \geq L_s := \min_{\tilde{t} \in D} d(s,\tilde{t}) \). Therefore, we may prune the successors of vertices \( v \) with

\[
\text{reach}(v) < \min \left\{ \text{cost}(v), \frac{\alpha}{2} \max \{\text{cost}(v), L_s\} \right\}
\]

(5)

when we grow the shortest path tree out of origin \( s \).
We can prune even more vertices if we grow the trees in forward and backward direction in separate steps. The idea is to use data collected in the first step to derive a sharper pruning bound for the second step. Whether we grow the forward or the backward trees in the first step depends on whether there are more destinations or more origins to process. Below we assume without loss of generality that we consider more destinations than origins, $|D| \geq |O|$.

We proceed as follows: we start by growing the forward trees out of the origins. In this phase, we prune vertices’ successors according to inequality (5). After growing the forward trees, we determine for each scanned vertex $v$ the distance $d_{\min}(v) := \min_{s \in O; v \text{ scanned from } s} d(s, v)$ to the closest origin it has been scanned from. If $v$ has not been scanned, we set $d_{\min}(v) := \infty$. Now we grow the backward trees and use $d_{\min}(v)$ as a lower bound for $d(s, v)$ for all origins $s \in O$. Hence, we can prune all vertices with

$$\text{reach}(v) < \min \left \{ \text{cost}(v), \frac{\alpha}{2} \max \{ \text{cost}(v), L_t \}, d_{\min}(v) \right \}. \quad (6)$$

In contrast to criterion (5), we can apply criterion (6) directly to each vertex $v$ and not only to its successors. This decreases the number of considered vertices. We provide pseudo code for the tree growth procedures in Algorithms 2 and 3.

### 2.3.3 Determining potential via vertices

With the shortest path trees, we can determine which vertices may potentially represent admissible $v$-paths. Each vertex scanned in forward and backward direction could be such a via vertex. However, since some of the resulting paths could include u-turns, we consider the scanned edges rather than the vertices. This excludes paths with u-turns (see Figure 4).

We proceed as follows: we determine for each scanned edge $e$ the sets $O_e$ and $D_e$ of origins and destinations that $e$ has been scanned from. We discard all edges that have not been scanned from at least one origin and one destination. Let $E_{\text{via}}$ be the resulting set of edges. The set of considered via vertices $V_{\text{via}} := \{v \in V \mid \exists w \in V : (v, w) \in E_{\text{via}}\}$ is given by the starting points of the edges in $E_{\text{via}}$.

Note that though the procedure above eliminates paths with u-turns, some admissible $v$-paths
Algorithm 2: Growing a forward shortest path tree out of origin $s$.

1. while container is not empty do
2.    Take the vertex $v$ with the lowest cost from the container and remove it;
3.    Mark edge leading to $v$ as visited from origin $s$;
4.    Include $v$ in the shortest path tree;
5.    if $d_{\text{min}}(v) > \text{cost}(v)$ then
6.        $d_{\text{min}}(v) := \text{cost}(v)$;
7.    if reach($v$) $\geq \min(\text{cost}(v), \frac{\alpha}{2} \max(\text{cost}(v), L_s))$ then
8.        Scan the vertex $v$;  // see Algorithm 1

Algorithm 3: Growing a forward shortest path into destination $t$.

1. while container is not empty do
2.    Take the vertex $v$ with the lowest cost from the container and remove it;
3.    Mark edge leading to $v$ as visited from destination $t$;
4.    if reach($v$) $\geq \min(\text{cost}(v), \frac{\alpha}{2} \max(\text{cost}(v), L_t))$ then
5.        Include $v$ in the shortest path tree;
6.        Scan the vertex $v$ with early pruning:
7.        forall neighbors $w$ of $v$ that have not been scanned yet do
8.            newCost := cost($v$) + $d(v, w)$;
9.            if reach($v$) $\geq \min(\text{newCost}, \frac{\alpha}{2} \max(\text{newCost}, L_t), d_{\text{min}}(v))$ then
10.               Label $w$;  // see Algorithm 1

Figure 4: Advantages of considering via edges instead of via vertices. Arrows highlighted in dark blue depict the forward shortest path tree grown from the origin $s$, and arrows highlighted in light red represent the backward tree grown into the destination $t$. Edges that are scanned from both directions are potential via edges and drawn as solid black lines. The remaining edges are drawn as dashed black lines. All vertices are scanned both from $s$ and $t$ and would therefore considered potential via vertices. However, paths via the two topmost vertices would require a u-turn. Restricting the focus on v-paths via vertices adjacent to the solid lines excludes these u-turn paths.
may be rejected as well. However, this issue will rarely occur in realistic road networks, since the problem arises only at specific merging points of very long edges. We provide details in Appendix B.

2.4 Step 2: Identifying vertices representing identical v-paths

Some of the vertices in $V_{\text{via}}$ may represent identical v-paths. Since we want to save the effort of checking the admissibility of the same path multiple times and, similarly importantly, we do not want to return multiple identical paths, we need to ensure that each admissible path is represented by one via vertex only.

To identify vertices representing identical paths, we have to compare the v-paths corresponding to all $v \in V_{\text{via}}$ for each origin-destination pair. This requires $O(|V_{\text{via}}| |O| |D|)$ steps. However, for some vertices, identical paths can be identified more quickly, as adjacent vertices typically represent similar sets of v-paths. Therefore, we proceed in two steps: first, we reduce $V_{\text{via}}$ by eliminating vertices whose via paths are also represented by their respective neighbours, and second, we check which of the remaining vertices represent identical v-paths. Below we describe the two steps in greater detail.

2.4.1 Eliminating vertices that represent the same v-paths as their neighbours

The endpoints of an edge can be neglected as via vertices if the edge has been scanned from the same origins and destinations as a neighbouring edge. Consider for example an edge $(v, w)$ that has been scanned from both an origin $s$ and a destination $t$. Then $P_{sw} = P_{svw}$ and $P_{vt} = P_{vwt}$. It follows that $v$ and $w$ represent the same v-path with respect to $(s, t)$: $P_{svt} = P_{swt}$. Now consider an adjacent edge $(u, v)$ that has been scanned from $s$ and $t$ as well. Clearly, it is $P_{sut} = P_{svt}$ and $P_{svt} = P_{vut}$, which implies that the v-paths via $u$, $v$, and $w$ are identical. Therefore, only one of these vertices has to be considered.

To introduce an algorithm that efficiently detects such configurations, let $O_e$ be the set of origins and $D_e$ the set of destinations that edge $e$ has been scanned from. For each edge $e \in E_{\text{via}}$, we check whether one directly preceding edge $e' \in E_{\text{via}}$ has been scanned from a superset of origins
and destinations, i.e. $O_e \subseteq O_{e'}$ and $D_e \subseteq D_{e'}$. If such an edge exists and one of the set inequalities holds strictly, i.e. $O_e \subset O_{e'}$ or $D_e \subset D_{e'}$, we may disregard edge $e$, as all v-paths via $e$ are also v-paths via $e'$.

Things become more complicated if $O_e = O_{e'}$ and $D_e = D_{e'}$, as we may either reject $e$, $e'$, or both edges. The latter case may occur if $e'$ has another directly preceding edge $e'' \in E_{\text{via}}$ with $O_{e'} \subseteq O_{e''}$ and $D_{e'} \subseteq D_{e''}$. If one of these inequalities is strict, we disregard both $e$ and $e'$. Otherwise, we continue traversing the edges in $E_{\text{via}}$ until either (1) an edge is found whose origin and destination sets supersede the sets of all previous edges or (2) no further predecessor with sufficiently large origin and destination sets is found. In the second case, we may disregard all traversed edges but $e$. We apply the same approach to the successors of $e$ and repeat this procedure until all edges in $E_{\text{via}}$ have been processed.

The updated set $V_{\text{via}}$ of via vertices consists of the starting vertices of the edges in the reduced edge set $E_{\text{via}}$. We provide pseudo code for the outlined algorithm in Algorithm 4. An efficient implementation may compare the origin and destination sets of the edges in $E_{\text{via}}$ before the traverse is started. This makes it easy to implement the most expensive parts of the algorithm in parallel.

### 2.4.2 Identifying remaining identical v-paths

The method outlined above identifies vertices that represent the same v-paths as their neighbours. However, two vertices may represent the same v-path with respect to one origin-destination pair but different v-paths with respect to another origin-destination pair. Consequently, these vertices could not be rejected in the step described above, and a second procedure is required to eliminate the remaining identical v-paths.

We identify the remaining identical v-paths by comparing path lengths. To this end, we assume that $P_{svt} = P_{swt}$ if and only if $l(P_{svt}) = l(P_{swt})$. Though it can happen that distinct paths have the same length, this case is usually not of greater concern in practical applications. The issue can be reduced by introducing a small random perturbation for the lengths of edges. We examine this limitation further in the discussion section.

With the above assumption, identical paths can be identified efficiently. Since for each origin-
Algorithm 4: Eliminating vertices that represent the same v-paths as their neighbours.

1 Function has_superior_predecessor(e):
   2  Remove $e$ from $E_{via}$;
   3  forall directly preceding edges $e'$ of $e$ do
   4   if $O_e \subseteq O_{e'}$ and $D_e \subseteq D_{e'}$ then
   5     if $O_e = O_{e'}$ and $D_e = D_{e'}$ then
   6       return has_superior_predecessor($e'$)
   7     else
   8       return True;
   9   else
  10   return False;

10 Function has_superior_successor(e):
  11  Remove $e$ from $E_{via}$;
  12  forall directly succeeding edges $e'$ of $e$ do
  13   if $O_e \subseteq O_{e'}$ and $D_e \subseteq D_{e'}$ then
  14     if $O_e = O_{e'}$ and $D_e = D_{e'}$ then
  15       return has_superior_successor($e'$)
  16     else
  17       return True;
  18   else
  19   return False;

19 $E'_{via} := \emptyset$;
20 while $E_{via} \neq \emptyset$ do
21   Set $e :=$ next entry in $E'_{via}$;
22   if not has_superior_predecessor($e$) and not has_superior_successor($e$) then
23     Add $e$ to $E'_{via}$;
24   $E_{via} := E'_{via}$;
destination pair \((s, t)\) and each potential via vertex \(v \in V_{\text{via}}\) the distances \(d(s, v)\) and \(d(v, t)\) are known, the v-path lengths can be computed easily. For each origin-destination pair, a comparison of the lengths of the v-paths corresponding to all \(v \in V_{\text{via}}\) can be conducted in linear average time via hash maps. Note that the path lengths must be compared with an appropriate tolerance for machine imprecision.

In later steps it will be of benefit if most v-paths are represented by a small set of via vertices. If there are multiple vertices representing the same v-paths, we therefore choose the via vertex \(v\) that has been scanned from the most origin-destination combinations \(O_v \times D_v\). This makes it easier to reuse partial results when we check whether the v-paths are locally optimal.

### 2.5 Step 3: Excluding long paths

Before we check whether paths are sufficiently locally optimal, we exclude the paths that exceed the length allowance. That is, we disregard all paths \(P_{svt}\) with \(l(P_{svt}) > \beta \cdot l(P_{st})\) with origin-destination pairs \((s, t)\) and via vertices \(v \in V_{\text{via}}\). Since this step involves a simple comparison only, it is computationally cheaper than identifying identical paths. Therefore, it is efficient to conduct this step just before identical paths are eliminated (section 2.4.2). This also reduces the memory required to store potentially admissible combinations \((s, v, t)\) of origin-destination pairs and via vertices.

### 2.6 Step 4: Excluding locally suboptimal paths

The most challenging part of the search for admissible paths is to check whether paths are sufficiently locally optimal. To test whether a subpath is optimal, we need to find the shortest alternative, which is computationally costly. Therefore, we apply an approximation to limit the number of necessary shortest path queries.

Our method generalizes the approximate local optimality test by Abraham et al. (2013). They noted that v-paths are concatenations of two optimal paths. Hence, v-paths are locally optimal everywhere except in a neighbourhood of the via vertex. More precisely, a v-path \(P_{svt}\) from \(s\) to \(t\) via \(v\) is guaranteed to be \(T\)-locally optimal everywhere except in the section that begins \(T\)
distance units before $v$ and ends $T$ distance units after $v$. Therefore, Abraham et al. (2013) suggest to perform a shortest path query between the end points $x$ and $y$ of this section to check whether it is optimal. Abraham et al. (2013) call this procedure the T-test.

The T-test does not return false positives. That is, a path that is not $T$-locally optimal will never be misclassified as locally optimal. However, the T-test may return false negatives: paths that are $T$-locally optimal but not $2T$-locally optimal may be rejected. In modelling applications, a more precise local optimality test may be desired.

It is possible to increase the precision of the T-test. Instead of checking whether the whole potentially suboptimal subpath is optimal, we may test multiple subsections to gain a higher accuracy. While this procedure ensures that fewer admissible paths are falsely rejected, the gain in accuracy comes with an increase in computational cost. Therefore, it is desirable to use the results of earlier local optimality checks to test the admissibility of other paths.

There are two situations in which local optimality results can be reused. First, if a subsection of a path is found to be suboptimal, other paths that include this section can be rejected as well. Second, if a subpath of a path is found to be locally optimal, other paths including this subpath may be classified as locally optimal as well. That way, many paths can be processed all at once.

When reusing partial results, it is important to note that even though we require all paths to be $\alpha$-relative locally optimal, the absolute lengths of the subsections that need to be optimal depend on how long the considered paths are. Therefore, paths must be considered in an order dependent on their lengths. We provide details below.

2.6.1 Preparation

Before we can start testing whether the remaining $v$-paths are locally optimal, a preparation step is needed to identify the subpaths that may be suboptimal and thus need to be assessed more closely. To reuse partial results efficiently, we furthermore need to determine subsections that different paths have in common. We describe the preparation procedure below.

We start by introducing helpful notation. Suppose we want to test whether the $v$-paths via vertex $v$ are locally optimal. Let $\tilde{O} := \{s \in O \mid \exists t \in D : l(P_{st}) \leq \beta \cdot l(P_{st})\}$ be the origins for
which at least one destination can be reached via \( v \) without violating the length constraint. Let \( \tilde{D} \) be defined accordingly for the destinations. Define \( \tilde{D}_s : = \left\{ t \in \tilde{D} \mid l(P_{svt}) \leq \beta \cdot l(P_{st}) \right\} \) as the set of destinations that can be reached from the origin \( s \) via \( v \) without violating the length constraint.

In the preparation step, we determine for each origin \( s \in \tilde{O} \) the destination \( t_s : = \arg\max_{t \in \tilde{D}_s} l(P_{svt}) \) for which the potentially suboptimal section is longest. Furthermore, we search for the vertex \( x_s : = \arg\min_{\tilde{x} \in P_{sv}; d(\tilde{x}, v) \geq \alpha l(P_{svs})} d(\tilde{x}, v) \), which is the last vertex on \( P_{sv} \) with \( d(x_s, v) \geq \alpha \cdot l(P_{svs}) \), and we determine \( x_t \) defined accordingly. Now we fill the arrays

\[
A_{us} := \begin{cases} 
\text{True} & \text{if } u \in P_{sv} \\
\text{False} & \text{else,}
\end{cases} \quad A_{ut} := \begin{cases} 
\text{True} & \text{if } u \in P_{vt} \\
\text{False} & \text{else}
\end{cases}
\]

for all vertices \( u \in P_{xs,v} \) and \( u \in P_{ext} \), respectively.

The information saved in the shortest path trees are suitable to find paths from scanned vertices to the origins and destinations. However, the trees contain no information on the reverse paths starting at the end points. That is, while it is easy to find the backward shortest path from \( v \) to \( x_s \), it is hard to follow the path in the opposite direction starting at \( x_s \). We gather the necessary information in the preparation step: for each origin \( s \in \tilde{O} \), we save the successors of each relevant vertex \( u \in P_{sv} \).

In Algorithm 5, we provide pseudo code for the described procedures. The pseudo-code considers the origins only. The algorithm for the destinations is similar. The preparation phase ends with sorting all origin-destination pairs with respect to the lengths of the respective \( v \)-paths via \( v \).

### 2.6.2 Testing local optimality for one origin-destination pair

We use an approximation approach with flexible precision to check whether paths are locally optimal. For a parameter \( \delta \in [1, 2] \), we call this procedure the \( T_\delta \)-test. The parameter \( \delta \) is a measure for the test’s precision.

To outline the \( T_\delta \)-test, let us consider a \( v \)-path \( P := P_{svt} \) from \( s \) to \( t \) via the vertex \( v \). Let \( S_s := \{ u \in P_{sv} \mid d(u, v) < T \} \) be the set of vertices that are on the path \( P_{sv} \) and have a distance less
Algorithm 5: Filling the array $A$ for the origins and finding successors. The algorithm for the destinations is similar.

1. **foreach** destination $s \in \tilde{O}$ **do**
   2. $t_s := \text{argmax}_{t \in \tilde{D}_s} (d(s, v) + d(v, t))$;
   3. $u := \text{parent}_s(v)$;
   4. $\text{successor}_s(u) := v$;
   5. $\text{stop} := False$;
   6. **while** not **stop** **do**
      7. if $u \notin A$ **then**
         8. Initialize $A_{us} := False$ for all $\tilde{s} \in \tilde{O}$;
         9. $A_{us} := True$;
        10. $\text{successor}_s(\text{parent}(u)) := u$;
        11. **if** $d(v, u) > \alpha (d(s, v) + d(v, ts))$ **then**
            12. **stop** := True;
        **else**
            13. $u := \text{parent}(u)$;

than $T$ to the vertex $v$. Furthermore, add to $S_s$ the vertex $x := \text{argmin}_{\tilde{x} \in P_s; d(\tilde{x}, v) \geq T} d(\tilde{x}, v)$ that is closest to $v$ but has $d(x, v) \geq T$ if such a vertex exists. Choose $S_t$ accordingly with respect to the destination vertex $t$. Let $\text{partner}_t(u; \tau) := \text{argmin}_{\tilde{w} \in S_t; d_P(u, \tilde{w}) \geq \tau} d_P(u, \tilde{w})$ for $u \in S_s$ be the vertex $w \in S_t$ that is closest to $u$ but has $d_P(u, w) \geq \tau$. If no such vertex exists in $S_t$, set $\text{partner}_t(u; \tau) = y := \text{argmax}_{\tilde{w} \in S_t} d_P(u, \tilde{w})$.

Define accordingly $\text{partner}_s(w; \tau)$ for $w \in S_t$ as the vertex $u \in S_s$ that is closest to $w$ but has $d_P(u, w) \geq \tau$.

The $T_\delta$-test proceeds as follows: the algorithm starts at the vertex $u_1 := x$ and checks whether the subpath $P^{u_1w_1}$ between $u_1$ and $w_1 := \text{partner}_t(u_1; \delta T)$ is a shortest path. If so, the algorithm progresses searching $u_2 := \text{partner}_s(u_1; T)$ in backward direction and repeats the steps formerly applied to $u_1$ now with $u_2$. This procedure repeats until $u_n = v$ for some $n \in \mathbb{N}$. If all the shortest path queries yield subpaths of $P$, the path is deemed approximately $T$-locally optimal. Otherwise, it is classified as not locally optimal. We depict the algorithm in Figure 5 and provide pseudo-code in Algorithm 6.
Figure 5: $T_\delta$-test with $\delta = 1.4$. The three subfigures depict the steps of the $T_\delta$-test for a path $P_{svt}$ connecting origin-destination pair $(s, t)$ via vertex $v$. The vertices $x$ and $y$ are the end points of the potentially locally suboptimal section. The edge lengths are given by the Euclidean distance except for the edges with an indicated gap. (a) In a first step, the test determines the vertex $w_1$ that is at least $\delta T$ units along the path away from $u_1 := x$ (the distance is depicted as blue arrow). (b) If the shortest path query between $u_1$ and $w_1$ indicates that the subsection $P_{u_1w_1}$ is optimal, the test continues by determining the first vertex $u_2$ that is at least $T$ units away from $w_1$ in backwards direction. (c) From $u_2$, the algorithm searches the vertex $w_2$ that is at least $\delta T$ units along the path beyond $u_2$ and conducts a shortest path query between $u_2$ and $w_2$. If all the shortest path queries yield subpaths of $P_{svt}$, the path is deemed approximately $T$-locally optimal. Note that a $T_2$-test would have misclassified the path as not locally optimal, provided the shortest path from $x$ to $y$ includes the horizontal edge.

Similar to the $T$-test, the $T_\delta$ test does not return false positives. However, paths that are $T$-locally optimal but not $\delta T$-locally optimal might be rejected. Hence, the $T_1$-test is exact, whereas the “classical” $T$-test by Abraham et al. (2013) is the $T_2$-test. An increase in precision comes with a computational cost. The $T_\delta$-test requires at most $2 \left\lceil \frac{1}{\delta - 1} \right\rceil$ shortest path queries if $\delta > 1$. However, query numbers around $\frac{1}{\delta - 1}$ are more common. Either way, the number of required queries is bounded by a constant independent of the graph, unless $\delta = 1$.

### 2.6.3 Using test results to check local optimality for multiple origin-destination pairs

The $T_\delta$-test is a suitable procedure to check whether a single $v$-path is locally optimal. However, if many $v$-paths shall be tested, the required number of shortest path queries may exceed a feasible limit. Therefore, we show below how negative test results can be used to reject multiple paths at once. Afterwards we describe a method to use positive test results to classify many paths as locally optimal.

#### 2.6.3.1 Rejecting paths

Suppose that in order to test whether $P_{sv}$ is admissible, we have checked whether the subpath $P_{uw}$ between some vertices $u$ and $w$ is a shortest path, and suppose we have obtained a negative
Algorithm 6: $T_\delta$-test.

1. Search for the vertex $x \in S_s$ with maximal distance to $v$;
2. Set $u := x$;
3. Set $w := v$;
4. while $u \neq v$ and $w \neq y$ do
   5. Set $w' := \text{partner}_t(u, \delta T)$;
   6. if $w = w'$ then
      7. Set $w := \text{next farthest vertex to } v \text{ in } S_t$;
   else
      8. Set $w := w'$;
   9. Check whether the $u$-$w$ subpath is optimal
      10. if $d(u, w) < d(u, v) + d(v, w)$ then
          11. return "Not locally optimal"
       else
          12. Set $u' := \text{partner}_s(w, T)$;
          13. if $u = u'$ then
              14. Set $u := \text{next closest vertex to } v \text{ in } S_s$;
          else
              15. Set $u := u'$;
      16. return "Locally optimal"

result, i.e. we have found that $d(u, w) < d(u, v) + d(v, w)$. We can not only conclude that the path $P_{svt}$ is not locally optimal but also reject other $v$-paths that include the subpath $P_{uw}$ (see Figure 6).

To see which paths can be rejected, let $\Omega_u := \{ \tilde{s} \in O | d(\tilde{s}, v) = l(P_{suv}) \}$ be the set of origins for which $u$ is on the shortest path to $v$ and define $\Delta_w := \{ \tilde{t} \in D | d(v, \tilde{t}) = l(P_{vwt}) \}$ accordingly for the destinations. Let furthermore $P := \left\{ (s, t) \in \tilde{O} \times \tilde{D} | l(P_{svt}) \leq \beta \cdot l(P_{st}) \right\}$ be the set of all origin-destination pairs with a potentially admissible $v$-path via $v$, and let $P_{uw} := P \cap (\Omega_u \times \Delta_w)$ denote the respective set of origin-destination pairs for which the $v$-path via $v$ also includes $u$ and $w$. The following lemma shows which paths can be rejected as approximately inadmissible.

**Lemma 2.** Suppose the $T_\delta$-test is applied to check whether a path $P_{svt}$ is $\alpha$-relative locally optimal and that the test fails, because $d(u, w) < d(u, v) + d(v, w)$ for some vertices $u$ and $w$. Then, for
Since we continue with the pair \((\alpha, \beta)\), it is approximately locally optimal as well.

By construction of \(P\), it is \(P_{xy} \subseteq P_{svt}\) for any origin-destination pair \((\tilde{s}, \tilde{t}) \in P_{uw}\). Therefore, \(P_{svt}\) is at most \(T\)-locally optimal with \(T < l(P_{xy})\). Hence, the local optimality factor \(\alpha_{svt}\) for \(P_{svt}\) satisfies

\[
\alpha_{svt} = \frac{T}{l(P_{svt})} < \frac{l(P_{xy})}{l(P_{svt})} \leq \frac{l(P_{xy})}{l(P_{svt})} \leq \alpha \delta \frac{l(P_{svt})}{l(P_{svt})} = \alpha \delta.
\]

\(\Box\)

Following Lemma 2, we can reject all pairs \((\tilde{s}, \tilde{t}) \in P_{uw}\) with \(P_{svt} \geq l(P_{svt})\). The origin-destination pairs in question can be determined by considering the array \(A\) constructed in the preparation phase (equation (7)). Let \(\tilde{A}_u := \{ s \in \tilde{O} \mid A_{us} = \text{True} \}\) and \(\tilde{A}_w := \{ t \in \tilde{D} \mid A_{wt} = \text{True} \}\). Then, \(\tilde{A}_{uw} := \tilde{A}_u \times \tilde{A}_w \subseteq P_{uw}\), and \(P_{uw} \setminus A_{uw}\) contains only pairs \((\tilde{s}, \tilde{t})\) with \(l(P_{svt}) < l(P_{svt})\). It follows that all pairs \((\tilde{s}, \tilde{t}) \in P_{uw}\) with \(P_{svt} \geq l(P_{svt})\) are also in \(A_{uw}\).

As \(A_{uw}\) may also contain pairs \((\tilde{s}, \tilde{t})\) with \(l(P_{svt}) < l(P_{svt})\), we process the origin-destination pairs in the order of increasing via-path length. Then the pairs \((\tilde{s}, \tilde{t}) \in A_{uw}\) with \(l(P_{svt}) < l(P_{svt})\) will be processed before \((s, t)\). If we label these pairs as “processed” and exclude them from \(A_{uw}\), then we can reject all remaining pairs in \(A_{uw}\).
2.6.3.2 Accepting paths

The procedure outlined in the previous section allows us to reject many inadmissible paths with a single shortest distance query. However, the procedure may yield limited performance gain if many of the considered paths are admissible. Therefore, we introduce a second relaxation of our local optimality condition: we classify paths as (approximately) admissible if they are \((\alpha \gamma)\)-relative locally optimal with some constant \(\gamma \in (0, 1]\).

To see how this relaxation can be exploited, suppose that we are considering an origin-destination pair \((s, t)\) and that we have already confirmed that the path \(P_{svt}\) is \(\alpha\)-relative locally optimal. Let \(x := \arg\min_{\tilde{x} \in P_{sv}; d(\tilde{x}, v) \geq \alpha l(P_{sv})} d(\tilde{x}, v)\) be the last vertex on \(P_{sv}\) with a distance to \(v\) of at least \(\alpha l(P_{svt})\). Let \(y := \arg\min_{\tilde{y} \in P_{vt}; d(v, \tilde{y}) \geq \alpha l(P_{svt})} d(v, \tilde{y})\) be defined accordingly for the destination branch.

During the \(T_\delta\)-test we have ensured that the section \(P_{xvy}\) is approximately \(T\)-locally optimal with \(T = \alpha \cdot l(P_{svt})\).

In the lemma below, we identify the paths that can be classified as approximately admissible after a successful \(T_\delta\)-test. In line with the notation in the previous section, let \(\Omega_x := \{\hat{s} \in O \mid d(\hat{s}, v) = l(P_{\hat{s}sv})\}, \Delta_y := \{\hat{t} \in D \mid d(v, \hat{t}) = l(P_{vy\hat{t}})\}\), and \(P_{xy} := P \cap (\Omega_x \times \Delta_y)\).

**Lemma 3.** Let \((s, t) \in P\) be an origin-destination pair. If the \(T_\delta\)-test applied to \(P_{svt}\) considered the vertices on \(P_{xvy} \subseteq P_{svt}\) and confirmed that the path is \(\alpha\)-relative locally optimal, then all paths \(P_{\hat{s}\hat{t}}\) with \((\hat{s}, \hat{t}) \in P_{xy}\) and \(l(P_{\hat{s}\hat{t}}) \leq \frac{1}{\gamma} l(P_{svt})\) are at least \((\alpha \gamma)\)-relative locally optimal.

**Proof.** The \(T_\delta\)-test for \(P_{svt}\) assured that \(P_{svt}\) is \(T\)-locally optimal with \(T = \alpha \cdot l(P_{svt})\). Therefore, all paths \(P_{\hat{s}\hat{t}}\) with \((\hat{s}, \hat{t}) \in P_{xy}\) are also \(T\)-locally optimal with \(T = \alpha \cdot l(P_{svt})\). The local optimality factor \(\alpha_{\hat{s}\hat{t}}\) of paths \(P_{\hat{s}\hat{t}}\) with \((\hat{s}, \hat{t}) \in P_{xy}\) and \(l(P_{\hat{s}\hat{t}}) \leq \frac{1}{\gamma} l(P_{svt})\) is therefore at least

\[
\alpha_{\hat{s}\hat{t}} = \frac{T}{l(P_{\hat{s}\hat{t}})} \geq \frac{T}{\frac{1}{\gamma} l(P_{svt})} = \gamma \alpha l(P_{svt}) = \alpha \gamma.
\]

That is, the paths \(P_{\hat{s}\hat{t}}\) are at least \((\alpha \gamma)\)-relative locally optimal. \(\square\)

Following Lemma 3, we can accept all pairs \((\hat{s}, \hat{t}) \in P_{uw}\) with \(l(P_{\hat{s}\hat{t}}) \leq \frac{1}{\gamma} l(P_{svt})\). We do this in the same manner as we rejected paths. Let \(A_{xy} \subseteq P_{xy}\) be defined as in the previous section.
Since $P_{xy} \setminus A_{xy}$ contains only pairs $(\tilde{s}, \tilde{t})$ with $l(P_{\tilde{s} \tilde{t}}) < l(P_{svt})$, which have been processed before $P_{svt}$, we only need to consider the pairs in $A_{xy}$ and classify all not yet processed v-paths $P_{\tilde{s} \tilde{t}}$ with $(\tilde{s}, \tilde{t}) \in A_{xy}$ and $l(P_{\tilde{s} \tilde{t}}) \leq \frac{1}{\gamma} l(P_{svt})$ as admissible. The described procedure to reject and accept multiple paths at once is outlined in Algorithm 7.

Algorithm 7: Testing whether the potentially admissible paths are approximately $\alpha$-relative locally optimal.

1. $R := \emptyset$; // set of approximately admissible paths
2. foreach vertex $v \in V_{via}$ do
   3. Let $\mathcal{P}$ be the set of all origin-destination combinations for which $v$ is a potential via vertex;
   4. Sort the pairs in $\mathcal{P}$ in increasing order of the lengths of their v-paths;
   5. while $\mathcal{P} \neq \emptyset$ do
      6. $(s, t) :=$ next origin-destination pair in $\mathcal{P}$;
      7. Do a $T_\delta$-test for the path $P_{svt}$ via $v$;
      8. if the test fails and finds a suboptimal section $P_{uvw} \subseteq P_{svt}$ then
         9. foreach pair $(s', t') \in \mathcal{P}$ do
            10. if $P_{uvw} \subseteq P_{s'v't'}$ then
               11. Remove $(s', t')$ from $\mathcal{P}$;
            else
               12. Add $P_{svt}$ to $R$;
               13. Let $P_{xvy} \subseteq P_{svt}$ be the subsection of $P_{svt}$ that has been checked for local optimality;
               14. foreach pair $(s', t') \in \mathcal{P}$ do
                  15. if $P_{xvy} \subseteq P_{s'v't'}$ and $\gamma \cdot l(P_{s'v't'}) \leq l(P_{svt})$ then
                     16. Add $P_{s'v't'}$ to $R$;
                     17. Remove $(s', t')$ from $\mathcal{P}$;
      18. return $R$;

2.6.4 Optimization: using previous shortest path queries to determine locally optimal subsections

The outlined speedups become even more effective if the results of individual shortest path queries are reused. Therefore, we save all vertex pairs \((u, w)\) for which we know that \(P_{uvw} = P_{uw}\). Note that we do not have to save unsuccessful shortest path tests, because all v-paths \(P_{sv}\) with \(P_{uvw} \subseteq P_{sv}\) will be rejected right after \(P_{uvw}\) has been found to be suboptimal (see section 2.6.3).

The gain obtained from reusing shortest path results decreases as the considered paths become longer. Since we are considering paths in increasing order of lengths, the lengths of the subsections that are required to be optimal increase as well. Therefore, the results of earlier shortest path queries are of limited value if they are only used as a lookup table.

However, we can exploit that due to the \(\delta\)-approximation, the shortest path queries in the \(T_\delta\)-test typically consider sections longer than required. The \(T_\delta\)-test conducts shortest path queries between vertices \(u\) and their partners \(w := \text{partner}_t(u; \delta T)\). Choosing \(\delta > 1\) not only reduces the number of necessary shortest path queries but also makes the algorithm reject admissible paths. Therefore, a test that sets \(w := \text{partner}_t(u; \tau)\) for some \(\tau \in [T, \delta T]\) will perform at least as well as the original algorithm.

With this observation, we can reuse previous shortest path results as follows: when we search for the partner \(w := \text{partner}_t(u; \delta T)\) of a vertex \(u\), we test for all intermediate visited vertices \(\tilde{w} := \text{partner}_t(u; \tau)\) with \(\tau \leq \delta T\) whether the subpath \(P_{uw\tilde{w}}\) is known to be optimal. If such a vertex \(\tilde{w}\) is found and \(\tau \geq T\), we accept \(\tilde{w}\) as the partner of \(u\) and progress as usual.

2.7 Preprocessing

Before REVC can be applied, a preprocessing step is required. If the set of origins and destinations of interest is known a priori, we may start by reducing the graph by deleting dead ends that do not lead to any of the considered origins and destinations. In a second step, we may add a random perturbation to the edge lengths to make it easier to identify identical paths based on their length. As the road costs (length, travel time, or other) are usually known with limited precision, small perturbations will typically not change the results significantly.
After these preparation steps, we can follow the preprocessing algorithm by Goldberg et al. (2006). The algorithm determines upper bounds on the reaches of vertices. To gain efficiency, the algorithm introduces shortcut edges, which may bias the results so that admissible paths are falsely rejected. However, it is easy to impose a length constraint on the shortcut edges to reduce the introduced error. If REVC is applied to a set of origins and destinations known in the preprocessing phase, vertices bypassed by shortcut edges can be removed completely from the graph. This increases the efficiency further.

The preprocessing step concludes with computing the shortest distances between all origins and destinations. This can either be done with individual shortest path queries for all origin-destination combinations or in a single effort involving only one shortest path tree per origin-destination pair. Either way, this step usually does not add significantly to the algorithm’s overall runtime. If the origins and destinations are not known at the reprocessing time, this step can be postponed to the execution of REVC.

3 Tests

To test the performance of REVC and to assess how input parameters and the introduced optimizations affect the results and the computational efficiency, we applied REVC to random route finding scenarios. To gain insights into the algorithms’ validity in modelling applications, we tested how well the resulting paths are suited to predict observed traveller behaviour. Below we provide details about our implementation of REVC and the applied test procedures. Afterwards we present the test results.

3.1 Implementation

We implemented REVC in the high-level programming language Python (version 3.7) in combination with the numerical computing library Numpy (version 1.16) and the software Cython (version 0.29), which we used in particular to build a C extension for the shortest path search. Despite our efforts to reduce bottle necks via C extensions, a low-level implementation of REVC can be
expected to be faster by orders of magnitude. We computed shortest paths with the algorithm RE (Goldberg et al., 2006). The code used in this paper can be retrieved as package “lopaths” from the Python Package Index (see pypi.org/project/lopaths). We executed our code in parallel on a Linux server with an Intel Xeon E5-2689 CPU (20 cores with 3.1 GHz) and with 512 GB RAM.

3.2 Test methods

3.2.1 Test graph

We tested REVC by applying it to a road network modelling the Canadian province British Columbia (BC). The graph had 1.36 million vertices and 3.16 million edges weighted by travel time. When we preprocessed the graph, we limited the length of shortcut edges to 20 min, which was less than 3% of the mean shortest travel time between the considered origins and destinations. For the empirical tests, we joined the British Columbian road network with a graph representation of the North American highway network. This additional network had 2 thousand vertices and 5.6 thousand edges.

3.2.2 The effect of input parameters on the results and computation time

We used a Monte Carlo approach to assess the effect of different input parameters on the performance and the results of REVC. Specifically, we considered the local optimality constant $\alpha$, the length factor $\beta$, the approximation parameters $\gamma$ and $\delta$, and the numbers of origins and destinations. We randomly generated 10 route finding scenarios (20 for tests on $\gamma$ and $\delta$) and computed the mean and standard deviation of the results.

For each of these scenarios, we selected the origin and destination locations randomly from the graph’s vertices. We generated 10 (+10 for tests on $\gamma$ and $\delta$) sets of origins and destinations, which we reused for each assessed parameter combination to reduce random influences on the results. When we varied the numbers of origins and destinations, we increased the origin and destination sets as necessary.

To measure the performance of the algorithm, we noted its total execution time and the execution time per resulting path. Furthermore, we determined the slowdown factor (see Abraham
..., 2013), denoting the ratio between the execution time of REVC and the corresponding pair-wise shortest path search. In contrast to the execution time, the slowdown factor is not strongly affected by the implementation and hardware, since both REVC and the shortest path queries are run with the same software on the same machine. Therefore, the slowdown factor may be a more meaningful performance measure than the execution time.

Note that it is possible to execute shortest path queries between many origin-destination pairs in linear time of the origins and destinations (Bast et al., 2016). However, the pair-wise approach used to compute the slowdown factor provides a better comparison to pair-based algorithms used in route choice modelling. Therefore, we applied the pair-wise approach.

For a general assessment of the resulting paths, we determined the average number and distribution of identified approximately admissible paths and the mean length of these paths. These metrics may provide hints on which parameter combinations are suitable in different modelling applications.

### 3.2.3 Assessment of optimizations

To assess the importance of the different optimizations we introduced to make REVC computationally efficient, we executed the algorithm repeatedly with different optimization steps disabled. We examined the optimizations of (1) the growth bound for the shortest path trees, (2) the tree pruning procedure, (3) the elimination of identical paths, (4) the joint local optimality tests for multiple paths, and (5) reusing shortest path query results. We applied the same randomized test procedure as outlined in the previous section and executed the algorithm with local optimality constant \( \alpha = 0.2 \), length factor \( \beta = 1.5 \), and optimization constants \( \gamma = 0.9 \) and \( \delta = 1.1 \). We determined the algorithm’s execution time after disabling one optimization at a time and computed the resulting relative changes in computation time as compared to the fully optimized algorithm. To examine the importance of the optimizations on different problem scales, we repeated the tests with different numbers of origins and destinations.

Disabling the respective considered optimizations was done as follows. (1) We examined the role of the optimized shortest path tree growth bound by resetting the bound to the naive value...
\[ \max_{t \in D} \beta \cdot l(P_{st}) \] with origin set \( O \), destination set \( D \), and \((s, t) \in O \times D \). (2) We tested the importance of the optimized pruning procedure in two steps. First, we pruned only vertices \( v \) with \( \text{reach}(v) < \alpha \cdot \text{cost}(v)/2 \) (see Abraham et al., 2013). Second, we used our stronger pruning condition (equation (5)) but refrained from pruning even more vertices when growing backward shortest path trees (equation (6)). (3) We examined a simplified algorithm to eliminate identical paths as well as skipping this step completely. To simplify the algorithm, we skipped the step of eliminating vertices that represent the same v-paths as their neighbours (section 2.4.1). (4) To test the significance of joint local optimality tests of multiple paths, we tested each route individually. (5) We examined the gain from reusing shortest path query results by running the algorithm without saving these results.

### 3.2.4 Empirical tests

To test the empirical validity of the generated route choice sets, we used data from road-side surveys in which travellers were surveyed for their origins and destinations. Based on these data, we determined which of the survey locations were passed frequently by travellers driving between certain origins and destinations. By this means, we obtained for each considered origin-destination pair a set of intermediate locations where travellers were observed (called observed positive) and a set of locations where travellers were not observed (called observed negative). If travellers choose admissible routes as hypothesized, then the “observed positive” locations will be on admissible routes for some reasonable parameters \( \alpha \) and \( \beta \), and the “observed negative” locations will be on inadmissible routes.

To see whether this was the case for our empirical observations, we applied REVC to compute admissible routes between the considered origins and destinations and classified all survey locations on admissible routes as predicted positives. The remaining survey locations were considered predicted negatives. Then, we determined (1) the true positive rate, i.e. the fraction of “observed positive” survey locations that were also “predicted positive”, and (2) the false positive rate, the fraction of “observed negative” locations that were “predicted positive”. We repeated this procedure for different values of the local optimality constant \( \alpha \) and plotted the true positive rates.
against the false positive rates. The resulting curve is the so-called receiver operating characteristic (ROC), which is a widely used tool to assess the performance of classification algorithms (Hosmer et al., 2013). The area under the curve (AUC) is a measure for the overall performance of the classifier (Hanley & McNeil, 1982). Large AUC values correspond to large true positive rates and small false positive rates and thus indicate a good discrimination of positive and negative observations. Since the set of admissible routes depends not only on the local optimality constant $\alpha$ but also on the length factor $\beta$, we computed the ROC and AUC for different values of $\beta$.

In addition to the ROC and AUC, we also determined how small local optimality constant $\alpha$ must be chosen to cover 95% or 100% of the positive observations. This result is of particular interest if the admissible routes are filtered further before they are used in route choice models. In this case, the false positive rate is of minor concern, and the goal is to identify as many used routes as feasible.

We based our analysis on survey data collected at watercraft inspection stations in British Columbia in the years 2015 and 2016. These inspection stations are set up to prevent human-mediated spread of aquatic invasive species, and all road travellers transporting watercraft are required to stop at these locations. We considered all inspection locations where more than 50 survey shifts were conducted in total. The mean survey shift length at these 12 locations was about 7 hours. Travellers were surveyed for the origin and destination waterbody of their watercraft and nearby cities. To ensure that the traffic was sufficiently dense to distinguish frequently used routes from others, we considered origin-destination pairs for which more than 50 travellers were observed in total. This were 13 pairs with 5 different origins and 7 different destinations. The origins, destinations, and survey locations are displayed in Figure 7.

To discern which survey locations were located on commonly used routes, we used a threshold value for the mean number of observed travellers per survey shift. Locations were classified as “observed positive” for an origin-destination pair if and only if the corresponding mean traveller count exceeded the threshold value. Using a threshold value has two advantages, namely (1) to reduce the potential bias resulting from differing survey effort at different survey locations, and (2) to reduce noise due to possible sampling error and travellers with highly uncommon behaviour.
To assess the impact of the threshold value on the results, we considered different threshold values ranging between a small positive value $\epsilon > 0$ (any traveller observation results in a positive classification) and 3 observations per 100 inspection shifts. Depending on the threshold value, the number of “observed positive” survey locations per origin-destination pair ranged between 2.23 and 4.15, and the mean count of distinct origin-destination pairs observed per survey location ranged between 2.42 and 4.5, with one survey location not being passed by any traveller of interest.

### 3.3 Test results

Below we provide the results of our tests. First, we focus on the general results from the randomized experiments before we describe the results of the tests involving empirical data.

#### 3.3.1 The effect of input parameters on the results and computation time

The results from the tests investigating the impact of the input parameters on the algorithm’s speed and results are displayed in Figure 8. The constant $\alpha$, controlling the local optimality requirement, had a strong influence both on the algorithm’s running time and the number of resulting paths.
Figure 8: Test results. Different performance measures and result characteristics are plotted against parameters. The whiskers depict the estimated standard deviation. The line colours in column C correspond to different values of the approximation constant $\gamma$. The line colours in column D correspond to different ratios of origin number and destination number.

(Parameters unless specified otherwise: $\alpha = 0.2$, $\beta = 1.5$, $\gamma = 0.9$, $\delta = 1.1$, $|O| = |D| = 100$)
The effect of \( \alpha \) on the execution time levelled off at high values of \( \alpha \). Decreasing \( \alpha \) from 0.3 to 0.05 doubled the total execution time and reduced the execution time per identified path by about factor 15. In comparison, increasing \( \alpha \) from 0.3 to 0.5 had a minor effect only. The mean number of paths followed a power law in \( \alpha \) (exponent \(-1.84\)). The length of the resulting paths decreased gradually as \( \alpha \) increased. An increase from 0.05 to 0.5 decreased the mean length of admissible paths by about a quarter.

The parameter \( \beta \), limiting the length of admissible paths, affected the number and length of identified admissible paths but not the execution time. The number of admissible paths increased almost linearly with \( \beta \); an increase of 0.1 resulted in about 0.8 additional paths being found per origin-destination pair. Consequently, the execution time per resulting path decreased with increasing \( \beta \). The mean lengths of the identified paths increased with their number. Raising \( \beta \) from 1 to 2 increased the mean path length by about 40%.

The approximation parameters \( \gamma \) and \( \delta \) had little effect on the execution time but a notable impact on the results. An increase of \( \gamma \) (increase in precision) consistently lengthened execution times slightly. However, a decrease of \( \delta \) (again, increase in precision) reduced the execution time per resulting path and led to an optimal overall execution time at intermediate values of \( \delta \).

The number of identified paths varied more strongly than the execution time when \( \gamma \) and \( \delta \) were changed. Dependent on the value of \( \delta \), decreasing \( \gamma \) from 1 to 0.6 increased the number of identified routes by 40%-85%. Conversely, an increase of \( \delta \) from 1 to 2 decreased the number of identified paths by more than 50%. The lengths of the resulting paths decreased gradually both in \( \gamma \) and \( \delta \).

Changing the number of origins and destinations affected the execution time but not the characteristics of the admissible paths. The execution time increased almost linearly with the origin and destination number; the slope depended on the origin to destination ratio. With a ratio of 1 : 1, the execution time increased by 80 s per 100 origins and destinations. With a ratio of 1 : 4, the average increase was 48 s per 100 origins and destinations. The time per identified path and the slowdown factor decreased as more origin and destination locations were added.

Figure 9 displays the distribution of paths per origin-destination pair dependent on the local
optimality constant $\alpha$ and the length constant $\beta$. Many origin-destination pairs are connected by numerous admissible paths if $\alpha$ is smaller than 0.2. For example, with $\alpha = 0.1$ and $\beta = 1.5$, about three quarters of the origin-destination pairs were connected by more than 20 routes. In contrast, with $\alpha = 0.3$, less than 0.7% of the pairs were connected by more than 5 paths, and 22% of the pairs were connected by the shortest path only. The latter fraction increased to 72% for $\alpha = 0.5$.

The distribution of paths per origin-destination pair changed more gradually with $\beta$. With $\alpha = 0.2$, a large value of $\beta = 2$ resulted in 99% of the pairs being connected by multiple admissible paths; 22% were connected by more than 10 paths. On the other end of the spectrum, with $\beta = 0.1$, 40% of the origin-destination pairs were connected by 1 admissible path only and 0.6% were connected by more than 5 admissible paths.

3.3.2 Assessment of optimizations

Assessing the role of the different optimizations we introduced to make REVC more efficient, we obtained a broad spectrum of results, shown in Table 1.

First, changing the growth bound for the shortest path trees had only a small effect on the computation time. The changes in computation time were smaller than the corresponding standard deviations.

Second, growing the shortest path trees with less strict pruning procedures increased the com-
putation time by 11%-56%. This the change was particularly high when the number of origins and destinations was imbalanced. There was only a small difference between disabling all pruning optimizations and solely refraining from earlier pruning in backward direction.

Third, omitting the step of identifying vertices that represent the same paths as their neighbours decreased the computation time by 2%-8%. This effect was smaller the less balanced the numbers of origins and destinations were. In contrast, refraining from any identification of identical paths increased the computation time by more than 70% with a larger effect in scenarios with strongly differing origin and destination numbers.

Fourth, disabling the joint tests for local optimality led to large changes in computation time (increase by factor 3 up to factor 11.8). The increase was stronger the more origin-destination pairs were considered.

Lastly, stopping to reuse shortest path query results increased the computation time moderately by 3%-16%. The effect was strongest when the number of origins was small and the number of destinations large.

### 3.3.3 Empirical tests

The ROC curves that we obtained for different length factors $\beta$ and classification thresholds are displayed in Figure 10. Quantitative results are given in Table 2. For large admissible length factors ($\beta \geq 2$), the area under the curve (AUC) constantly exceeded 0.9. For smaller length factors ($1.3 \leq \beta < 2$), the AUC values were smaller but never below 0.78.

A moderate local optimality requirement of $\alpha = 0.25$ sufficed to cover 95% of the positive observations regardless of how many traveller observations were required for positive classification of survey locations. In the scenario in which any observation sufficed for positive classification of survey locations, one triplet of origin, intermediate destination, and final destination was not covered by any admissible path for the tested parameter values. In the remaining scenarios, all “observed positive” locations were on admissible routes for some $\alpha$ value. For a classification threshold of 1 observation per 100 survey shifts, $\alpha$ had to be chosen as low as 0.07 to cover all positive observations. When the classification threshold was large ($\geq 2$ observations per 100 survey shifts), $\alpha$ had to be chosen as high as 0.5 to cover all positive observations.
| Disabled optimization | Execution time [s] (standard deviation) | % increase (standard deviation) |
|-----------------------|----------------------------------------|---------------------------------|
|                       | \(|O| \times |D|\) | 50 \times 100 | 100 \times 100 | 200 \times 500 | 50 \times 1000 | 50 \times 100 | 100 \times 100 | 200 \times 500 | 50 \times 1000 |
| None                  |                                      |                                |                  |                  |                  |                  |                  |                  |                  |
| Optimized shortest path tree height |                            | 135 (4.0) | 193 (5.8) | 467 (10.6) | 323 (11.4) | 1.8 (4.3) | 0.9 (4.6) | −1.9 (3.8) | 1.8 (5.1)                |
| Earlier pruning in backward direction |                                | 158 (4.3) | 213 (4.3) | 558 (10.4) | 493 (11.3) | 18.5 (4.8) | 11.4 (4.5) | 17.3 (4.3) | 52.5 (6.4)              |
| All pruning optimizations |                                    | 158 (4.5) | 216 (5.1) | 556 (11.8) | 505 (14.6) | 18.8 (4.9) | 13.3 (4.7) | 16.9 (4.5) | 56.3 (7.1)              |
| Identifying neighbouring vertices representing identical paths |                              | 123 (4.2) | 177 (4.2) | 452 (11.7) | 317 (11.7) | −7.6 (4.2) | −7.1 (3.9) | −5.0 (3.9) | −2.1 (5.0)             |
| Identifying identical paths |                                  | 237 (14.5) | 331 (13.4) | 887 (31.7) | 683 (42.9) | 77.9 (12.2) | 73.4 (9.2) | 86.4 (8.9) | 111.5 (15.2)          |
| Joint tests for local optimality |                                | 400 (9.8) | 735 (17.3) | 5615 (105.3) | 2849 (66.3) | 200.2 (11.7) | 284.8 (16.1) | 1079.9 (43.4) | 781.6 (37.3)         |
| Reusing shortest path query results |                                  | 140 (4.9) | 200 (9.3) | 492 (9.9) | 374 (13.0) | 5.5 (4.9) | 4.5 (6.1) | 3.4 (3.9) | 15.7 (5.7)             |

Table 1: The impact that different optimizations introduced with REVC have on the the algorithm’s running time. For each given optimization, the table displays (1) the running time of REVC if this optimization were disabled and (2) the corresponding relative increase in running time. The results are given for four scenarios with different numbers of origins and destinations. The standard deviations of the results are displayed in parenthesis.
Figure 10: Receiver operating characteristic (ROC) curves for different length factors $\beta$. In Subfigure (a), any location with observed travellers was classified as “observed positive”, whereas in Subfigure (b), only locations with 2 traveller observations per 100 shifts were classified “observed positive”. In both scenarios, the locations on admissible routes coincided strongly with the locations with positive observations. This can be seen from the high true positive rates achieved at the same time as small false positive rates. The dashed line shows the performance of a hypothetical random classifier. A point with true positive and false positive rate of 1 was added to complete the curve though these values did not occur in practice.

shifts), all “observed positive” locations were on 0.4-relative locally optimal paths. That is, these paths were optimal on all subsections shorter than 40% of the entire path.

4 Discussion

We have introduced an algorithm that efficiently identifies locally optimal paths between many origin-destination pairs and tested both the algorithm’s computational performance and its ability to predict empirical traffic observations. Our algorithm REVC identifies all approximately admissible routes between the origins and destinations, and its execution time is driven by the number of distinct origins and destinations rather than the number of origin-destination pairs. The empirical tests suggest that locally optimal routes are a suitable tool to predict where travellers between specific origins and destination are likely to be observed. These results combined indicate that REVC is applicable in large-scale traffic models.

Our tests examining the impact of different input parameters on the results and computation
Observed travellers per 100 shifts required for positive classification

| AUC ($\beta = 1.5$) | AUC ($\beta = 3$) | $\alpha$ required to cover 95% of observed positives ($\beta = 3$) | $\alpha$ required to cover all observed positives ($\beta = 3$) | Fraction of “observed positive” locations on shortest routes |
|----------------------|-------------------|-------------------------------------------------|-------------------------------------------------|--------------------------------------------------|
| Any                  | 0.80              | 0.92                                            | 0.25                                            | –                                                |
| 1                    | 0.84              | 0.94                                            | 0.3                                             | 0.07                                             | 0.41                                             |
| 2                    | 0.96              | 0.97                                            | 0.4                                             | 0.4                                              | 0.61                                             |
| 3                    | 0.97              | 0.97                                            | 0.4                                             | 0.4                                              | 0.66                                             |

Table 2: Classification results for different classification thresholds. The AUC values are generally high and increase as more traveller observations are required to classify a survey location as “observed positive”. The first column shows the traveller counts per 100 survey shifts required to classify a location as “observed positive”. The second and third column display AUC values obtained with different path length constraints. The fourth and fifth column contain the maximal value of the local optimality constant $\alpha$ for which 95% or 100% of the “observed positive” locations were on admissible routes, respectively. The right-most column indicates how many locations classified as “observed positive” were located on the shortest routes between the respective origins and destinations.

time show that REVC’s performance depends mostly on the local optimality constant $\alpha$ and the number of origins and destinations. While the total execution time increases with the number of considered origins and destinations and with reduced $\alpha$, the execution time per identified path decreases. That is, REVC becomes more efficient compared to repeated path queries the more paths are generated.

The length bound $\beta$ had only a minor effect on the execution time. This may be surprising, as an increase in $\beta$ allows more vertices to be included in the shortest path trees. However, the impact of $\beta$ is reduced by our pruning technique, which is most effective for long paths. Furthermore, large parts of the graph had been scanned for small values of $\beta$ already, since the considered origins and destinations were distributed over the entire graph. Therefore, few additional vertices were considered with increased $\beta$.

The effect of $\beta$ may be larger if all origin and destination locations are located within a small subsection of the graph. Nonetheless, in many modelling applications, the origin and destination locations will be distributed over the whole considered road network. For example, when the traffic from the outskirts of a city to downtown is modelled, it is unlikely that travellers leave the greater metropolitan area. Therefore, it is reasonable to consider an accordingly constrained graph.
REVC applies approximations to gain efficiency. However, the approximation constants had relatively small effects on the algorithm’s performance in our tests. This suggests that approximations may not always be necessary. However, the benefit of the approximations will become larger if the origin and/or destination vertices are not randomly spread over the whole graph but located in constrained areas. Then, partial results can be reused more effectively. As the admissibility checks were responsible for a limited portion of the overall execution time only, the gain of the approximations will also become more significant if more paths have to be checked for local optimality.

An interesting observation is that intermediate values of the approximation constant $\delta$ led to lower execution times than large values. This is surprising, because smaller values of $\delta$ increase the number of shortest path queries required in the $T_\delta$-test. However, small values of $\delta$ have the advantage that the subsections checked for local optimality get shorter. This makes it more likely that test results can be reused to reject many inadmissible paths at once. In point to point queries, the $T_2$-test (used by Abraham et al., 2013) may still be superior.

The tests evaluating the importance of the different optimizations introduced with REVC resulted in a heterogeneous picture. The most important innovation of REVC was the joint local optimality test of many paths. This result was expected, since separate tests must consider each origin-destination pair individually, thus making the algorithm’s runtime strongly dependent on the number of origin-destination pairs.

Another significant speedup was obtained by rejecting identical paths prior to local optimality checks. However, identifying and neglecting vertices that represent the same v-paths as their neighbours decreased the algorithm’s efficiency despite having a positive effect on the asymptotic runtime. This was due to our implementation of REVC, where the computation time required to identify paths with identical lengths was dominated by the number of origin-destination pairs rather than the number of paths. Though omitting the first path comparison step can apparently speed up the algorithm, the procedure can prove useful if the origins and destinations are spatially separated, which allows more vertices to be rejected in this step.

A moderate speedup was gained by improving the pruning procedure applied during the shortest
path tree growth. Here, earlier pruning in backward direction turned out to be the most important optimization. Disabling this improvement only had almost the same effect as disabling all pruning optimizations. This is because early pruning reduces shortest path trees by a complete layer of leaf vertices that may need to be considered in computationally expensive $T_\beta$-tests otherwise.

Reusing the results of shortest path queries had a small but notable effect on computational efficiency. The efficiency gain is highest if many v-paths via a vertex share subsections. This happens if origins and destinations are spatially separated or if the numbers of origins and destinations are imbalanced. Note, however, that reusing shortest path query results increases the number of optimality checks and thus facilitates the accuracy of the results.

The optimization of the shortest path tree growth bound had a minor effect only. This result is in line with the small impact that the length factor $\beta$ had on the computation time, and the explanation for the result is similar. Consequently, the optimized tree growth bound will become more important if all origins and destinations are located in a small part of the considered graph.

Besides assessing the computational performance of REVC, we also tested the empirical validity of the computed routes. The tests showed that the paths returned by REVC allow precise predictions of where individuals travelling between given origins and destinations can be observed. Typically, predictors with AUC values exceeding 0.8 are considered excellent and those with AUC values exceeding 0.9 outstanding (Hosmer et al., 2013). The large AUC values we obtained, consistently greater than 0.9 for $\beta = 3$, suggest that local optimality can be a helpful criterion to discriminate used roads from unused roads – and thus to characterize route choice sets. Though no tracking data were available to us that would have allowed us to assess the overlap between observed and computed routes, our survey points and data were sufficiently heterogeneous to give significant insight into the validity of locally optimal routes in modelling applications.

### 4.1 Significance

Determining multiple paths between an origin and a destination based on a local optimality criterion is a well established approach in route planning research (Abraham et al., 2013; Delling et al., 2015; Luxen & Schieferdecker, 2015; Bast et al., 2016). An obstacle hindering the application of
these algorithms in route choice models was that these algorithms return only few heuristically chosen paths rather than the complete set of admissible paths. Furthermore, these algorithms are based on an inflexible approximation whose impact on the result was not exactly known. Our algorithm REVC solves these issues. Though REVC may not be competitive in point to point queries, the algorithm efficiently exploits redundancies occurring when many origin-destination pairs are considered.

Generating route choice sets based on local optimality has multiple advantages. The underlying principle is simple and has a sound mechanistic justification. The optimality principle is applied on a local scale, whereas the mechanisms governing travellers’ overall route choices do not need to be known. Therefore, no extensive data sets are needed to generate choice sets. In addition, our empirical test results suggest that local optimality is indeed a suitable criterion to distinguish used roads from unused roads, yielding a high coverage of actual observations and a low rate of false positive predictions.

Fitting the choice set parameters to data is a discrete optimization problem and can therefore be challenging. REVC permits two free variables: the local optimality parameter $\alpha$ and the length parameter $\beta$. As the latter does not have a strong impact on the execution time, $\beta$ can be chosen liberally, leaving $\alpha$ as the only remaining free parameter. Optimizing $\alpha$, in turn, is comparatively easy, as this is a one-dimensional problem.

Choice sets consisting of locally optimal v-paths are typically relatively small while still covering a broad spectrum of different routes (see Abraham et al., 2013). This agrees with our empirical tests, where a low rate of false positive predictions was achieved at the same time as a high true positive rate. The high specificity of local optimal routes allows for sophisticated models for the second route choice step, in which travellers select routes from the choice sets. The option to use sophisticated metrics to measure the quality of the route candidates may improve the overall model fit. In addition, using small choice sets also reduces a bias observed in route choice models when many insignificant routes are present in choice sets (Bliemer & Bovy, 2008).

The favourable quality to quantity ratio of locally optimal v-paths and the practically linear relationship between execution time and origin and destination numbers make REVC particularly
useful in comprehensive traffic models. In such applications, many origin-destination pairs have to be considered, and the computed choice sets need to be kept in memory for further processing. This makes it difficult to apply methods based on point to point queries, such as link elimination (Azevedo et al., 1993), link penalty (De La Barra et al., 1993), or constrained enumeration methods (Prato & Bekhor, 2006). Similar challenges face algorithms that need to generate many paths, such as stochastic approaches or methods that include a filtering step to select admissible paths from a large number of candidates (see Bovy, 2009). Therefore, REVC may be of specific use in comprehensive models.

The results of REVC provide insights into the distribution and properties of locally optimal routes in real road networks. In our tests, the number of admissible paths decreased with $\alpha$ in a power law relationship, whereas it increased linearly in $\beta$. Such experimental results could be the starting point for a more in-depth theoretical analysis of the distribution of locally optimal routes in road networks. The resulting insights may facilitate the development of new algorithms.

The experimental results are also valuable as benchmarks for existing algorithms searching locally optimal v-paths for route planning purposes (Abraham et al., 2013; Kobitzsch, 2013; Luxen & Schieferdecker, 2015). Some of these algorithms apply approximations to gain efficiency. The presented results can help to assess the impact of these approximations. Our results suggest that the applied $T_2$-approximation falsely rejects half of the admissible paths.

In addition to assessing the accuracy of faster algorithms, the complete sets of admissible paths generated with REVC can also be used to evaluate the success rate and the quality of the paths generated with these algorithms. Note, however, that our definition of admissible paths deviates slightly from the definition applied in earlier papers. Refer to Appendix C for details.

REVC contains several optimizations that can be directly applied to make the family of algorithms based on REV more efficient. These optimizations include the improved bounds for tree growth and pruning as well as the idea to exclude u-turn paths by considering via edges. Similarly, the $T_\delta$-test can be directly applied to increase the accuracy of all algorithms using the T-test. Our randomized tests can be used to assess the benefit gained from the different optimizations. Hence, this paper may also contribute to make route planning software more efficient. We provide a more
in-depth discussion in Appendix C.

4.2 Limitations

REVC focuses on single-via paths. A complete search for locally optimal routes should not limit the set of considered paths. However, considering v-paths can be justified by assuming that travellers may drive via an intermediate destination. Furthermore, the focus on v-paths excludes zig-zag routes, which may be deemed unrealistic. Therefore, a criterion limiting the set of admissible paths may not only be a computational necessity but also beneficial in route choice models.

Nonetheless, REVC may be extendable to include paths via two intermediate destinations. Road networks usually have a small set $W$ of vertices so that every sufficiently long shortest path includes at least one of these vertices (Abraham et al., 2010). If $W$ could be identified efficiently, REVC could be applied to compute v-paths from the origins to the vertices in $W$ and from the vertices in $W$ to the destinations. Concatenating these v-paths to admissible “double-via” paths would be comparable to the admissibility checks described in this paper.

REVC seeks to identify all admissible paths between the given origins and destinations. However, even if we do not apply approximations (i.e. choose $\gamma = \delta = 1$), some admissible paths may be falsely rejected. This limitation is due to the preprocessing step, in which shortcut edges are added to the graph, and the requirement that an edge adjacent to the via vertex must be scanned in forward and backward direction. However, we have already noted that the effect of the shortcut edges can be arbitrarily reduced by imposing length constraints on shortcut edges. Furthermore, most admissible paths will satisfy the mentioned edge requirement (see Appendix B). Therefore, these limitations generally have minor effects on the results.

REVC, as introduced in this paper, identifies identical paths based on their lengths. Alternative approaches exist but might be less efficient. In practice, distinct paths may have identical lengths, and REVC may therefore falsely reject some admissible paths. Paths with equal lengths occur most frequently in cities whose roads form a grid structure. Nevertheless, since the roads may have distinct speed limits and traffic volumes, and because turns take additional time, paths with identical lengths may not occur frequently in practice. Since ties are even less likely in long paths,
we argue that it is reasonable to distinguish paths based on their lengths.

Misclassifications of distinct paths with equal lengths can be reduced by adding small random perturbations to the lengths of all edges. Though this procedure makes it unlikely that admissible paths with similar lengths are considered identical, the perturbation term randomly defines an optimal path in grid networks. Therefore, the random perturbation is of limited help in these networks. Note, however, that regardless of how we identify identical paths, REVC and similar shortest-path-based methods are not well suited to work in grid networks, as ties must be broken when the shortest path trees are grown.

An important feature of REVC is to reject u-turn paths by considering via edges instead of via vertices. In undirected graphs, this procedure also ensures that the returned routes do not contain cycles. However, in directed graphs it is possible that locally optimal paths include a cycle, and REVC may return such paths. Though it is unlikely that locally optimal routes with cycles occur in realistic road networks, it is possible to check paths for cycles before returning them. Confirming that no vertices appear twice in a path can be done efficiently.

From a modelling perspective, it may be desirable to restrict admissible paths not only by excluding paths with cycles but to impose a more general requirement instead. Abraham et al. (2013) suggest to apply the relative length bound not only to entire paths but also to their subsections (see Appendix C). This would exclude paths with subsections for which much better shortcuts exist. However, testing this constraint involves relatively high computational complexity, and there may also be situations in which the additional requirement may not be of benefit in models. In any event, the route sets returned by REVC can serve as a starting point before further restrictions are applied.

In this paper, we presented performance measurements to assess the efficiency of REVC and applied optimization procedures. When evaluating these results, it is important to note the limitations of our implementation. For example, our parallel implementation comes with scheduling overheads. Some parts of the algorithm were not parallelized at all, leaving room for further speedups. Furthermore, the slowdown factors we measured can be considered as upper bounds, since we compared a highly optimized shortest path search with a high-level implementation of
REVC. Despite these limitations, the most important timing result remains visible: the performance of REVC scales well with the numbers of routes and end points.

We conducted empirical tests showing that locally optimal paths can be used to discern which roads are used by travellers of interest. Though this is a strong indicator that local optimality criteria can be successfully applied in route choice models, our test does not provide final proof. On the one hand, we have surveyed traveller behaviour at a small set of locations only and are thus unable to know how these travellers behaved elsewhere. On the other hand, even if we knew that travellers use only roads that are part of locally optimal routes, we would not know how the travellers combine these roads. In addition to the conceptual arguments and empirical results presented in this paper, a more thorough analysis of empirical tracking data (see e.g. Bekhor et al., 2006) would be worthwhile. This will remain a task for future research.

5 Conclusion

Generating route choice sets with locally optimal single-via paths has a sound mechanistic justification, leads to small choice sets with reasonable alternatives, and requires minimal data. We presented an algorithm that efficiently generates such choice sets for large numbers of origin-destination pairs. The algorithm is able to identify (almost) all locally optimal single-via paths up to a specified length between the origins and destinations. Therefore, the algorithm extends earlier methods based on local optimality and makes the approach a valuable method to generate route choice sets.

We confirmed that predictions made based on the algorithm’s results matched empirical traffic observations. Furthermore, we assessed the algorithm’s performance dependent on the input parameters. The results provide insights into the effect of approximation parameters and the distribution of locally optimal paths in real road networks. Therefore, our study provides the necessary prerequisites to construct route choice sets based on local optimality in large-scale traffic simulation applications.
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Appendix

A  Proofs

In this Appendix, we prove Lemma 1 and Corollary 1 (main text). We adjust the statement of Lemma 1 to recall notation from the main text.

Lemma 1. Consider an arbitrary admissible single-via path $P$ from $s$ to $t$. With $x'_s = \arg\min_{x \in P; d_P(s,x) \geq \alpha l(P)} d_P(s,x)$, let

$$x_s := \begin{cases} x'_s & \text{if } d_P(s,x'_s) \leq \frac{1}{2} l(P) \\ \arg\max_{x \in P; d_P(s,x) \leq \frac{1}{2} l(P)} d_P(s,x) & \text{else.} \end{cases} \quad (A10)$$

Choose $x_t$ accordingly. Then there is at least one vertex $v \in P$ with

1. $d_P(s,v) = d(s,v) \leq d_P(s,x_t)$ and
2. $d_P(v,t) = d(v,t) \leq d_P(x_s,t)$.

Proof. Since $P$ is a single-via path, $P$ contains at least one vertex $v'$ such that $d_P(s,v') = d(s,v')$ and $d_P(v',t) = d(v',t)$. That is, $v'$ splits $P$ into two shortest paths. Now choose a vertex $v$ as follows:

$$v := \begin{cases} v' & \text{if } d_P(s,v') \leq d_P(s,x_t) \text{ and } d_P(v',t) \leq d_P(x_s,t), \\ x_t & \text{if } d_P(s,v') > d_P(s,x_t), \\ x_s & \text{if } d_P(v',t) > d_P(x_s,t). \end{cases} \quad (A11)$$

We show that $v$ satisfies the lemma’s requirements by regarding the different possible choices of $v$:

1. If $d_P(s,v') \leq d_P(s,x_t)$ and $d_P(v',t) \leq d_P(x_s,t)$, then the conditions 1 and 2 are clearly satisfied for $v := v'$.
2. If \( d_P(s,v') > d_P(s,x_t) \), then inserting \( v := x_t \) yields \( d_P(s,v') > d_P(s,v) \). Therefore, the subpath \( P^{sv} \) from \( s \) to \( v \) is a subpath of the subpath \( P^{sv'} \) from \( s \) to \( v' \). Since \( v' \) splits \( P \) into two shortest paths, \( P^{sv'} \) is a shortest path. Therefore, \( P^{sv} \) must be a shortest path, too. Thus, \( d_P(s,v) = d(s,v) = d_P(s,x_t) \), and condition 1 is satisfied.

To show that condition 2 holds as well, observe that
\[
d_P(v,t) = d_P(x_t,t) \leq \frac{1}{2} l(P) \leq l(P) - d_P(s,x) = d_P(x_s,t).
\]
It remains to be shown that \( d_P(v,t) = d(v,t) \). Since \( P \) is \( \alpha \)-relative locally optimal, each subpath whose length after removal of one end point would be smaller than \( \alpha l(P) \) is a shortest path. By construction, this applies to the subpath from \( x_t \) to \( t \). Hence, it is \( d_P(v,t) = d(v,t) \) and condition 2 is satisfied.

3. The proof for the case \( d_P(v',t) > d_P(x_s,t) \) is analogous to the argument presented under point 2.

\[\square\]

**Corollary 1.** For each admissible \( v \)-path between an origin-destination pair \((s,t)\), a via vertex will be scanned from both directions if the shortest path trees are grown up to a height of

\[
h_{\text{max}} := \max \left\{ (1 - \alpha) \beta l(P_{st}), \frac{1}{2} \beta l(P_{st}) \right\}.
\]

**Proof.** Let \( P \) be an admissible path, which implies that \( l(P) \leq \beta l(P_{st}) \). Recall that

\[
x_t' = \arg\min_{x \in P; d_P(x,t) \geq \alpha l(P)} d_P(x,t)
\]

\[
= \arg\min_{x \in P; l(P) - d_P(s,x) \geq \alpha l(P)} (l(P) - d_P(s,x))
\]

\[
= \arg\max_{x \in P; d_P(s,x) \leq (1 - \alpha) l(P)} d_P(s,x).
\]

Therefore, \( x_t \) is either the last vertex in \( P \) with \( d_P(s,x) \leq (1 - \alpha) l(P) \leq (1 - \alpha) \beta l(P_{st}) \) or the last vertex with \( d_P(s,x) \leq \frac{1}{2} l(P) \leq \frac{1}{2} \beta l(P_{st}) \) (see equation (A10)). Either way, \( x_t \) will be included in the shortest path tree if we grow the tree to a height of just above \( \max \{(1 - \alpha) \beta l(P_{st}), \frac{1}{2} \beta l(P_{st})\} \).

The same argument holds in backward direction for \( x_s \). From Lemma 1 we know that \( P \) is a \( v \)-path via a vertex \( v \in P^{xsx_t} \) located between \( x_s \) and \( x_t \). Since both \( x_s \) and \( x_t \) are scanned from both
sides, the vertex $v$ will be scanned from both sides as well.

\[ \square \]

B Admissible paths excluded by requiring that a neighbouring edge of the via vertex has been scanned from both directions

Requiring that a neighbouring edge of the via vertex has been scanned in both directions excludes u-turns without reducing the number of found admissible paths significantly. However, there is exactly one scenario in which an admissible v-path is not found if we impose this constraint. The situation is depicted in figure A1.

Suppose the v-path $P$ from $s$ to $t$ via the vertex $v$ is admissible but falsely rejected by the exact version of REVC ($\gamma = \delta = 1$). Suppose furthermore that $u \in P$ is the predecessor of $v$ and $w \in P$ the successor. Then there must be a vertex $x \in P^{su}$ and a vertex $y \in P^{uv}$ such that the following conditions hold:

1. The shortest path from $x$ to $w$ does not include $v$: $d(x, v) + d(v, w) > d(x, w)$.

2. The shortest path from $u$ to $y$ does not include $v$: $d(u, v) + d(v, y) > d(u, y)$.

3. Let $x'$ be the direct successor of $x$ in $P$. It must be $d(x', v) > \alpha \cdot l(P)$.

4. Let $y'$ be the direct predecessor of $y$ in $P$. It must be $d(v, y') > \alpha \cdot l(P)$.

5. The shortest path from $u$ to $w$ must include $v$: $d(u, w) = d(u, v) + d(v, w)$.

If the first two conditions were not satisfied, at least one edge on $P$ adjacent to $v$ would be scanned from both directions and $P$ would be found. If the last three conditions were not satisfied, $P$ would not be admissible.

Though it is possible that all of these conditions are satisfied, we believe that such a scenario is unlikely in real road networks.

Remark 1. It can be shown that pruning does not weaken these conditions.
Figure A1: Scenario in which an admissible path is excluded due to the requirement that an edge adjacent to the via vertex is scanned in both directions. Blue lines depict the edges included in the forward shortest path tree grown from the origin $s$ and orange lines the edges of the backward tree grown into the destination $t$. Lines that may represent multiple edges are indicated with a gap. As the edges adjacent to $v$ are included in one shortest path tree only, the path $P_{svt}$ would be rejected by REVC.

\section{Comparison of REV and REVC}

In this Appendix, we compare our algorithm REVC to the algorithm REV \cite{Abraham2013} that it is based on. To a large extent, REVC uses the same ideas as REV: shortest path trees are grown around the origin and destination, and v-paths via vertices scanned from both directions are checked for admissibility using an approximate test for local optimality. However, REV and REVC differ in (1) the admissibility definition (2) the choice of the returned paths, and (3) technical optimizations that REVC introduces. Below we discuss each of these points.

\subsection{Admissibility definition}

The admissibility definition by Abraham \textit{et al.} \cite{Abraham2013} includes three requirements. They say a v-path $P_{svt}$ is admissible if

1. $P_{svt}$ has limited overlap with previously identified admissible paths $P_{swt}$ between $s$ and $t$. That is, $l\left(P_{svt} \cap \bigcup_{w} P_{swt}\right) \leq \eta \cdot l(P_{st}).$

2. $P_{svt}$ is $T$-locally optimal with $T = \alpha \cdot l(P_{st}).$

3. $P_{svt}$ has $\beta$-uniformly bounded stretch. That is, for all $u, w \in P_{svt}$, it is $l(P_{uw}^{st}) \leq \beta \cdot l(P_{uw}).$

None of these requirements coincides exactly with the constraints we imposed in our paper.

Requirement 1 does not appear in our admissibility definition. The constraint requires that the admissible paths have a clearly specified order. However, though Abraham \textit{et al.} \cite{Abraham2013} suggest a
reasonable ordering, this introduces another degree of freedom whose impact on the results may be oblique. Furthermore, we were interested in identifying all routes that satisfy certain criteria and leave it to the second modelling stage, in which a route is chosen from the choice set, to take route overlaps into account (see e.g. Cascetta et al., 1996). Lastly, the local optimality criterion naturally limits the pair-wise overlap of paths. Therefore, we dropped this constraint.

Requirement 2 differs from our local optimality constraint, because the length \( T \) of the subsections required to be optimal depends on the shortest distance between \( s \) and \( t \) rather than the length of the via path. This allows for more admissible paths. We changed this requirement for two reasons: (1) the spatial scale at which travellers' decision routines change is likely dependent on the path they actually choose rather than the shortest alternative, which may – dependent on the global quality metric – not even be a favourable option. Travellers on a long trip may have a higher incentive to choose a route with long optimal subsections. (2) The adjusted local optimality criterion allows for more effective pruning with simpler bounds when considering many origin-destination pairs. Using a pair-wise static local optimality criterion as Abraham et al. (2013) would require us to choose the pruning bound dependent on the origin-destination pair closest together. For these reasons, we introduced the notion of relative local optimality. Note that REVC can also be used to identify all paths satisfying requirement 2 if the constant \( \alpha \) is adjusted accordingly and the resulting paths are filtered so that suboptimal paths are excluded.

Requirement 3 is relaxed in our admissibility definition. Abraham et al. (2013) do not introduce an efficient algorithm to identify paths satisfying requirement 3. Instead of bounding the lengths of all subpaths, they consider the complete path only, as we do in this paper. Nonetheless, uniformly bounded stretch is a valuable characteristic for choice set elements. However, since REVC will return a moderate number of paths in many applications, paths could be checked for uniformly bounded stretch after execution of REVC. Consequently, we have used the relaxed constraint directly.

C.2 Returned paths

Abraham et al. (2013) aim to compute a small number of high-quality paths between an origin
and a destination efficiently. To save computation time, they do not assess the admissibility of all path candidates. Instead, REV processes the potentially admissible paths in an order dependent on some objective function, estimating the quality of the paths. REV returns the first $n$ processed approximately admissible paths.

Since we are interested in an exhaustive search for admissible paths, we do not process the paths in a specific order. We return all approximately admissible paths and leave the assessment of their quality, if desired, to a second, independent algorithm.

### C.3 Optimizations

REVC introduces multiple optimization to REV. First, REVC uses a tighter bound for the tree growth and the pruning stage. Though our pruning bound would have to be adjusted to comply with the admissibility definition applied by Abraham et al. (2013) (see section C.1), the ideas introduced in this paper are still applicable.

Second, REVC excludes u-turns by considering via edges rather than via vertices. Furthermore, REVC identifies vertices representing identical paths before assessing their admissibility. Both optimizations could be directly applied to speed up REV. However, REV processes the paths in an order given by some objective function (see section C.2). It is possible to construct this objective function so that u-turn paths are not processed before the admissible paths.

Third, to control the accuracy of the results, REVC uses the $T_\delta$-test instead of the T-test to check whether a path is locally optimal. This optimization could also be applied in REV, though it may effect the performance of REV more strongly than the performance of REVC.

Lastly, REVC is optimized to process many origin-destination pairs at once. Though the idea to grow each shortest path three only once per origin and destination is straightforward, the main innovation of REVC is in the efficient local optimality checks of many v-paths via one via vertex.