Multidimensional Quadrilateral Lattices are Integrable

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Abstract

The notion of multidimensional quadrilateral lattice is introduced. It is shown that such a lattice is characterized by a system of integrable discrete nonlinear equations. Different useful formulations of the system are given. The geometric construction of the lattice is also discussed and, in particular, it is clarified the number of initial–boundary data which define the lattice uniquely.

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1 Introduction

During the last three decades a new branch of mathematical physics, the theory of integrable systems, has been developed [1]–[8]. Besides its mathematical beauty, the theory has found also an enormous variety of applications in physics.

A lot of attention has been dedicated also to discrete (difference) integrable systems [1][3][5]–[7][9]–[16]. They share with their differential counterparts many common properties, like for example an integrability scheme, constants of motion, soliton solutions, etc.

To construct an integrable discretisation of the given soliton equation one can follow several different approaches. The standard ones are:

i) a discrete version of the Lax pair (see, for example [1], [9])
ii) the Hirota method via a bilinear form [10][11]
iii) extensions of the Zakharov – Shabat dressing method [12][13]
iv) direct linearization using linear integral equations [14]–[16].

Recently, another discretisation approach has been applied to those soliton equations which describe geometrically meaningful objects, like curves and surfaces. It is well known that many integrable PDE’s allow for such an interpretation (see, for example [17]–[20] and references cited there). Given a geometric integrable system, the geometry provides its integrability scheme (the associated linear problem) and associates with such an integrable system other integrable ones, which describe various objects "living” on the same submanifold [17] (see, for instance, the connection between the integrable vortex motion and the Heisenberg spin model).

It may be interesting to note that many of the important integrable PDE’s were studied by the prominent geometers of the XIXth century [21][22]. They have found also tools which allow to construct explicit formulas for the immersed submanifolds, like the Darboux transformation [21].

In the process of discretisation of such systems, as a working principle, the discrete analogues of the relevant geometric properties must be found. In this way the discrete curves [23] were “incorporated” into the soliton theory, and some classes of surfaces, including the pseudospherical, minimal, isothermal and constant mean curvature surfaces were discretised [24]–[26]. Recently a new example of integrable discrete surface, which contains in principle all the ones known before as a special reductions, has been found and related to the fully discrete Toda system (Hirota equation) [27].
In this paper we discretise the geometrically meaningful integrable multidimensional system which describes submanifolds parametrized by multi-conjugate coordinates. The associated differential equations were derived by Darboux [21] and rediscovered (and solved), in the matrix case, about ten years ago by Zakharov and Manakov [28], using a $\bar{\partial}$ approach; for this reason, they are now called Darboux–Zakharov–Manakov (DZM) equations. More precisely, we do the following.

i) We show that the proper geometric discretisation of the notion of (multi-dimensional) conjugate net is the notion of (multidimensional) quadrilateral lattice (MQL), meaning that all the elementary quadrilaterals of the lattice are planar.

ii) We characterize MQL’s in terms of a system of nonlinear integrable discrete equations in $N$ dimensions, $N \geq 3$ (equations (11) of Section 2).

iii) We show that the lattice is defined uniquely by $\frac{N(N-1)}{2}$ ”initial” surfaces or, equivalently, by $N$ ”initial” curves plus $N(N-1)$ arbitrary ”initial-boundary” data, which are functions of two variables.

We remark that the equations which characterize the MQL have been recently obtained by Bogdanov and Konopelchenko [13] as a natural discrete integrable analogue of the DZM equations, but their geometric meaning was unknown.

The MQL introduced in this paper allows one to generate, through a systematic geometrically meaningful reduction procedure, not only all the known geometric integrable lattices previously mentioned in this introduction, but also new ones. For this reason, a systematic investigation of the reductions of the MQL has recently started [30] and the list of integrable geometric lattices already obtained through it includes the following.

i) The fully discrete Toda lattice (Hirota equation) which, as it was shown in [27], describes the Laplace sequence of 2-dimensional quadrilateral lattices.

ii) The discrete analogue of the Lamé equations for orthogonal nets, corresponding to the case when the elementary quadrilaterals are inscribed in circles.

iii) Some symmetric reductions of the MQL including, for example, a discrete analogue of the $(-1)$ element of the Nizhnik–Veselov–Novikov hierarchy of $2 + 1$ dimensional integrable systems.

The integrability of the equations associated with the MQL was established in [13] using the $\bar{\partial}$ approach. A more geometric way of constructing
solutions of the MQL equations has been recently obtained in [31], using a
discrete vectorial variant of the Darboux transformations. These transformat-
tions allow to construct, from a given MQL, a new one, which is, in general,
topologically more complicated. In particular, dromionic and rational MQL’s
have been obtained.

The paper is organized as follows. In Section 2 we give the definition of the
multidimensional quadrilateral lattice and derive the underlying integrable
system of discrete equations. In Section 3 we present different forms of the
system. In Section 4 we discuss the initial value problem for the MQL’s.

2 Multidimensional Quadrilateral Lattices

As it was mentioned in the introduction, the notion of quadrilateral lattice
is the discrete generalization of the notion of conjugate nets.

In the continuous case, conjugate coordinates on a surface are defined as
follows [32]. Given a point on a surface and a second point which belongs
to the coordinate line passing through the first one, the tangent planes to
the surface at those two points intersect along one straight line. In the limit
when these two points coincide, the direction of the straight line is the tangent
direction to the second coordinate line (see Fig. 1).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Figure 1.}
\end{figure}

On a discrete level, a discrete surface can be defined as a mapping from
\( \mathbb{Z}^2 \) to the \( M \) dimensional space \( \mathbb{R}^M \)
\[ x : \mathbb{Z}^2 \to \mathbb{R}^M. \]
If we consider the two neighbouring "tangent planes" along the first coordinate line, defined by the triples \( \langle x, T_1 x, T_2 x \rangle \), \( \langle T_1 x, T_2^2 x, T_1 T_2 x \rangle \), where \( T_i \) is the shift operator in the \( i \)-th direction of the lattice, their intersection line coincides with the direction of the second coordinate line if and only if the elementary quadrilateral \( \{ x, T_1 x, T_2 x, T_1 T_2 x \} \) is planar (see Fig. 2).

![Figure 2.](image)

Therefore the natural discrete analogue of the notion of 2-dimensional conjugate net is given by the notion of 2-dimensional quadrilateral lattice \[3\].

**Definition 1** A 2-dimensional quadrilateral lattice is a mapping \( x : \mathbb{Z}^2 \to \mathbb{R}^M, M \geq 2 \) such that all the elementary quadrilaterals \( \{ x, T_1 x, T_2 x, T_1 T_2 x \} \) are planar.

A multidimensional conjugate net is characterized by the property that any surface made by varying two parameters forms a 2-dimensional conjugate net with respect to these two coordinates. The discrete analogue of the above notion is therefore given by the notion of multidimensional quadrilateral lattice.

**Definition 2** By an \( N \)-dimensional quadrilateral lattice we mean a mapping from an \( N \)-dimensional integer lattice to the \( M \)-dimensional linear space

\[
x : \mathbb{Z}^N \to \mathbb{R}^M, \quad M \geq N
\]
such that all the elementary quadrilaterals with vertices

\[ \{x, T_i x, T_j x, T_i T_j x\} \quad i, j = 1 \ldots N \quad i \neq j \]  \hspace{1cm} (3)

are planar.

Since the planarity of four points is preserved during the projective transformations of the ambient space, quadrilateral lattices are objects which actually should be described within the projective geometry approach. In the above definition, therefore, it is very convenient to replace the linear space \( \mathbb{R}^M \) by the projective space \( \mathbb{P}^M \):

\[ x : \mathbb{Z}^N \to \mathbb{P}^M , M \geq N . \]  \hspace{1cm} (4)

To perform the calculations it is natural then to use homogeneous coordinates \( x \in \mathbb{R}^{M+1} \setminus \{0\} \) such that \( x = [x] \in \mathbb{P}^M \) are the corresponding directions.

In terms of the homogeneous coordinates the linear relations between the vectors representing vertices of the elementary quadrilaterals can be written as a system of \( \frac{N(N-1)}{2} \) discrete Laplace equations

\[ D_i D_j x = (T_i A_{ij}) D_i x + (T_j A_{ji}) D_j x + C_{ij} x \quad i \neq j, \quad i, j = 1, \ldots, N \]  \hspace{1cm} (5)

where \( D_i = T_i - 1 \) is the discrete partial derivative, \( A_{ij} \) and \( C_{ij} = C_{ji} \) are scalar functions of the lattice variable \( \mathbf{n} = (n_1, \ldots, n_N) \in \mathbb{Z}^N \):

\[ A_{ij}, C_{ij} : \mathbb{Z}^N \to \mathbb{R} \quad i \neq j, \quad i, j = 1, \ldots, N \]  \hspace{1cm} (6)

The projective picture does not change if we multiply the homogeneous coordinates by a scalar function

\[ x \to \tilde{x} = \frac{1}{\rho} x \]  \hspace{1cm} (7)

In fact, after such a gauge transformation, the new coordinates satisfy the Laplace equation (5) with the coefficients

\[ \tilde{A}_{ij} = \frac{1}{T_j \rho} (A_{ij} \rho - D_j \rho) \quad i \neq j, \quad i, j = 1, \ldots, N \]  \hspace{1cm} (8)

\[ \tilde{C}_{ij} = \frac{1}{T_i T_j \rho} (D_i D_j \rho + (T_i A_{ij}) D_i \rho + (T_j A_{ji}) D_j \rho + C_{ij} \rho) . \]
Taking as gauge function (say) the last coordinate of $x$, i.e. $\rho = x^{M+1}$ (the non-homogeneous gauge), we "place" our net in the $M$-dimensional affine subspace of $\mathbb{R}^{M+1}$ characterized by $x^{M+1} = 1$. Then we get the special Laplace equations

$$D_i D_j x = (T_i A_{ij}) D_i x + (T_j A_{ji}) D_j x, \quad i \neq j, \quad i, j = 1, \ldots, N,$$

where we can identify also the affine subspace with $\mathbb{R}^M$, i.e. in the above formula, $x = (x^1, \ldots, x^M) \in \mathbb{R}^M$.

For $N = 2$ we have only one linear equation and no restrictions on its coefficients. The solution of the Laplace equation represents then a two-dimensional quadrilateral lattice (or discrete conjugate net) [27].

Starting from $N = 3$, we must repeat the two-dimensional construction in every pair of directions and, to make it possible, we should take into account the following compatibility condition between the Laplace equations

$$D_i A_{ij} = A_{ij} T_j A_{jk} + A_{ik} T_k A_{kj} - A_{ik} T_k A_{ij}, \quad i \neq j \neq k \neq i,$$

$$D_i C_{ij} = D_i C_{kj}, \quad i \neq j \neq k \neq i.$$

In the non-homogeneous gauge ($C_{ij} = 0$) the above equations reduce to $N(N-1)(N-2)$ equations for the $N(N-1)$ coefficients $A_{ij}$

$$D_i A_{ij} = A_{ij} T_j A_{jk} + A_{ik} T_k A_{kj} - A_{ik} T_k A_{ij}, \quad i \neq j \neq k \neq i.$$

Therefore the system of equations (11) is overdetermined for $N > 3$ and determined only for $N = 3$.

The continuous limit of the equations (9) and (11) gives rise to the classical Laplace equations

$$x_{ij} = a_{ij} x_i + a_{ji} x_j, \quad i \neq j,$$

and to the DZM equations for conjugate multidimensional nets [21]

$$a_{ij,k} = a_{ij} a_{jk} + a_{ik} a_{kj} - a_{ik} a_{ij}, \quad i \neq j \neq k \neq i,$$

where $f_i = \frac{\partial f}{\partial u_i}$.

We conclude this Section by pointing out that multidimensional quadrilateral lattices are characterized by the set of integrable equations (11), and
can be therefore called *integrable lattices*. Their integrability scheme is governed by the *linear* (discrete) Laplace equations (5) (or (9)) which express the planarity of the elementary quadrilaterals.

In the rest of the paper we will refer to equations (11) (or (10)) as to the multidimensional quadrilateral lattice equations.

### 3 Alternative Formulations of the Multidimensional Quadrilateral Lattice Equations

In this Section we discuss, for completeness, other forms of the MQL equations which will be useful in deriving integrable reductions [30] and in the construction of the Darboux transformations [31] for the lattice.

Equations (10b), together with the symmetry condition $C_{ij} = C_{ji}$, imply the existence of a function $C: \mathbb{Z}^N \to \mathbb{R}$ such that

$$C_{ij} = D_i D_j C \quad i \neq j. \quad (14)$$

On the other hand from equations (10a) one can derive the following constraint

$$\frac{T_k(A_{ji} + 1)}{A_{ji} + 1} = \frac{T_i(A_{jk} + 1)}{A_{jk} + 1}, \quad i \neq j \neq k \neq i \quad (15)$$

which is automatically satisfied expressing $A_{ij}$ in terms of the ”logarithmic potentials” $\gamma_i: \mathbb{Z}^N \to \mathbb{R}$

$$A_{ij} = \frac{D_j \gamma_i}{\gamma_i}, \quad i \neq j. \quad (16)$$

Finally, the nonlinear system (10) can be rewritten as a system of $\frac{N(N-1)(N-2)}{2}$ equations

$$D_k D_j \gamma_i = \left( T_j \frac{D_k \gamma_j}{\gamma_j} \right) D_j \gamma_i + \left( T_k \frac{D_j \gamma_k}{\gamma_k} \right) D_k \gamma_i + \gamma_i D_i D_j C, \quad i \neq j \neq k \neq i. \quad (17)$$

We remark that the last term vanishes when working in the non-homogeneous coordinates.

To see the geometric meaning of the function $\gamma_i$ defined in (16) let us note that, when taken as a gauge function, it removes all the corresponding coefficients $A_{ij}, (j = 1, \ldots, N, j \neq i)$ from the Laplace equations (5).
Of course, the definition of the functions $\gamma_i$ depends on the particular gauge we start with. From now on, we fix the non-homogeneous gauge (9) and define the functions $\gamma_i$ with respect to that gauge. However, we still have the freedom of multiplying $\gamma_i$ by an arbitrary function of the single variable $n_i$.

Let us rewrite the special Laplace equations (9) using the functions $\gamma_i$

$$D_i D_j x = \left( T_i \frac{D j \gamma_i}{\gamma_i} \right) D_i x + \left( T_j \frac{D j \gamma_j}{\gamma_j} \right) D_j x \ , \ i \neq j .$$

(18)

Following Darboux [21], one can define suitably normalized tangent vectors

$$X_i = \frac{D_i x}{T_i \gamma_i}$$

(19)

which possess the property that their variation in the second direction $j \neq i$ is proportional to $X_j$ only:

$$D_j X_i = (T_j \beta_{ji}) X_j \ , \ i \neq j$$

(20)

where

$$\beta_{ji} = \frac{D_i \gamma_j}{T_i \gamma_i} , \ i \neq j \ .$$

(21)

The compatibility condition written in terms of the new fields $\beta_{ij}$ takes the following form

$$D_j \beta_{ik} = \beta_{ij} (T_j \beta_{jk}) , \ i \neq j \neq k \neq i .$$

(22)

As it was mentioned in the introduction, the matrix version of equations (17)–(22), for $N = 3$, has been derived in [13], through a $\tilde{\partial}$ dressing approach, without a geometric interpretation, as an integrable discrete generalization of the DZM equations.

### 4 Geometric Construction of the Lattice

We conclude the Letter with some remarks about the constructability of the quadrilateral lattices. We discuss also the meaning of some objects introduced in the last Section.
Given two discrete curves $x_1(n_1)$ and $x_2(n_2)$ in $\mathbb{R}^M$ meeting in a point $x_1(0) = x_2(0)$, and given two functions $A_{12}$ and $A_{21}$ defined on $\mathbb{Z}^2$, these data allow to construct uniquely the discrete quadrilateral surface. Equivalently, one can use instead of $A_{12}$ and $A_{21}$ the functions $\gamma_1$ and $\gamma_2$. Once the functions $\gamma_i$, $i = 1, 2$ are given, the vectors $X_i$ are also defined in the points of the discrete quadrilateral surface according to formula (19).

The construction of the vectors $X_i$ for a given 2-dimensional quadrilateral lattice (2DQL) is shown in the Fig. 3. For example, $D_2X_1$ (the change of $X_1$ in the second direction) is proportional to $X_2$ only (the dashed line connecting the arrows of $X_1$ and $T_2X_1$ is parallel to $D_2X$).

Let us consider now three discrete curves $x_1(n_1)$, $x_2(n_2)$ and $x_3(n_3)$ in $\mathbb{R}^M$ ($M \geq 3$) meeting in a point $x_1(0) = x_2(0) = x_3(0)$. For any pair of curves: $x_i(n_i)$, $x_j(n_j)$, assigning two arbitrary functions $A_{ij}^{(0)} : \mathbb{Z}^2 \to \mathbb{R}$ we construct a 2-dimensional (initial) QL. It turns out that, starting from these three initial discrete surfaces, one constructs uniquely the three dimensional quadrilateral lattice. The construction is based on the simple fact that three different, non parallel, planes in a 3-dimensional space intersect, in general, in a single point. It may happen that the intersection point is at infinity,
but we still have non ambiguous construction looking at the lattice in the projective space \( \mathbb{P}^M = \mathbb{R}^M \cup \mathbb{P}^{M-1} \).

In particular, the point \( T_1T_2T_3x \) is the point of intersection of the three planes \( \langle T_1x, T_1T_2x, T_1T_3x \rangle \), \( \langle T_2x, T_1T_2x, T_2T_3x \rangle \) and \( \langle T_3x, T_1T_3x, T_2T_3x \rangle \) in the three dimensional subspace \( \langle x, T_1x, T_2x, T_3x \rangle \) of \( \mathbb{R}^M \) (see Fig. 4).

![Figure 4](image)

The above construction involves only linear operations and can be pro-longed to build, out of the three initial quadrilateral surfaces, the whole 3-dimensional lattice.

Fixing the vectors \( X_i \) (or the functions \( \gamma_i \)) on the initial curves allows to find their values in the points of the initial discrete quadrilateral surfaces, and then in the points of the whole lattice. Because of the planarity condition there is no contradiction in the construction of the vectors \( X_i \). For example, the vector \( T_2T_3X_1 \) is obtained by the intersection of the line \( \langle T_2T_3x, T_1T_2T_3x \rangle \) with the plane parallel to \( \langle x, T_2x, T_3x \rangle \) passing through the arrow of \( X_1 \) (see Fig. 5).
To summarise, in order to construct uniquely the 3-dimensional quadrilateral lattice, one has to give three arbitrary intersecting quadrilateral surfaces or, equivalently, three arbitrary intersecting curves plus six arbitrary functions of two discrete variables

$$A_{ij}^{(0)}(n_i, n_j), A_{ji}^{(0)}(n_i, n_j), 1 \leq i < j \leq 3.$$  \hspace{1cm} (23)

The construction outlined above allows then to build uniquely the whole 3-dimensional lattice or, equivalently, the functions

$$A_{ij}(n_1, n_2, n_3), i \neq j, i, j = 1, 2, 3$$  \hspace{1cm} (24)

which solve the MQL equation and satisfy the following boundary conditions:

$$A_{12}(n_1, n_2, 0) = A_{12}^{(0)}(n_1, n_2), \ldots, A_{32}(0, n_2, n_3) = A_{32}^{(0)}(n_2, n_3).$$  \hspace{1cm} (25)

If, instead of the $A_{ij}$'s, we use the data $\gamma_i$'s, then our arbitrary initial functions are $\gamma_1^{(0)}(n_1, n_2), \gamma_1^{(0)}(n_1, n_3), \gamma_2^{(0)}(n_1, n_2), \gamma_2^{(0)}(n_2, n_3), \gamma_3^{(0)}(n_1, n_3)$ and $\gamma_3^{(0)}(n_2, n_3)$.

Let us apply the same procedure to the case $N = 4$. Given four discrete curves and six, constructed from them, arbitrary initial 2DQL's, they give rise
to the four unique 3DQL’s $x(n_1, n_2, n_3, 0)$, $x(n_1, n_2, 0, n_4)$, $x(n_1, 0, n_3, n_4)$ and $x(0, n_2, n_3, n_4)$ in the way described above. Then the point $x(1, 1, 1, 1)$ can be constructed, for example, out of the surfaces $x(1, n_2, n_3, 0)$, $x(1, n_2, 0, n_4)$ and $x(1, 0, n_3, n_4)$, i.e. it is the intersection point of the three planes $T_1T_2P_{34}$, $T_1T_3P_{24}$ and $T_1T_4P_{23}$ in the three dimensional subspace $T_1P_{234}$ of $\mathbb{R}^M$. Here $P_{ij}$ ($i \neq j$) denotes the plane passing through the points $x$, $T_ix$ and $T_jx$, i.e. $P_{ij} = P_{ij}(x) = \langle x, T_ix, T_jx \rangle$, e. g., $T_1T_2P_{34} = \langle T_1T_2x, T_1T_2T_3x, T_1T_2T_4x \rangle$; analogously $P_{ijk} = \langle x, T_ix, T_jx, T_kx \rangle$ (all the indices are distinct). The same point could be constructed, however, also in three other ways. In the following we show that the results coincide.

First, we note that the plane $T_1T_3P_{24}$ is the intersection of the two three dimensional subspaces $T_1P_{234}$ and $T_3P_{124}$ in the four dimensional space $P_{1234} = \langle x, T_1x, T_2x, T_3x, T_4x \rangle$. Similarly $T_1T_2P_{34} = T_1P_{234} \cap T_2P_{134}$ and $T_1T_4P_{23} = T_1P_{234} \cap T_4P_{123}$.

Figure 6.

The point $T_1T_2T_3T_4x$ is then the unique intersection point of the four three dimensional subspaces $T_iP_{1\ldots4}$, $i = 1, \ldots, 4$ (the symbol $1\ldots4$ denotes the
sequence of the natural numbers from 1 to 4 with the \( i \)th element removed) of the four dimensional space \( P_{1234} \); consequently, it is the intersection point of the four triplets \( T_k T_i P_{1,i,k,i} \), \( k = 1, \ldots, 4 \) (where the index \( i, i = 1, \ldots, 4 \), \( i \neq k \), indicates each element of the \( k \)th triplet) of the two dimensional spaces in the corresponding three dimensional spaces \( T_k P_{1,k,i} \).

\[
P_{1234} \ni T_1 T_2 T_3 T_4 x = \bigcap_{i=1}^{4} T_i P_{1,i,4} = \bigcap_{i=1, i\neq k}^{4} T_k T_i P_{1,i,k,i} . \tag{26}
\]

The same argument can be used to prove the compatibility of the construction for an arbitrary dimension \( N \) of the lattice. In the natural notation inherited from the example \( N = 4 \)

\[
P_{1\ldots N} \ni T_1 \ldots T_N x = \bigcap_{i=1}^{N} T_i P_{1,i,N} = \bigcap_{i=1, i\neq k}^{N} T_k T_i P_{1,i,k,N} , \quad k = 1, \ldots, N \tag{27}
\]

which allows to apply the mathematical induction.

To summarise all the geometric considerations of this Section, we conclude it stating the general initial boundary–value problem for MQL’s.

\textit{In order to construct uniquely the \( N \)-dimensional quadrilateral lattice, one has to give \( \frac{N(N-1)}{2} \) arbitrary intersecting quadrilateral surfaces or, equivalently, \( N \) arbitrary intersecting curves (in general position) in \( \mathbb{R}^M \) plus \( N(N-1) \) arbitrary functions of two discrete variables}

\[
A_{ij}^{(0)}(n_i, n_j) , \quad A_{ji}^{(0)}(n_i, n_j) , \quad 1 \leq i < j \leq N . \tag{28}
\]

\textit{The linear construction} outlined above allows then to build uniquely the whole \( N \)-dimensional lattice or, equivalently, the functions

\[
A_{ij}(n_1, n_2 \ldots, n_N) , \quad i \neq j , \quad i, j = 1, \ldots, N \tag{29}
\]

\textit{which solve the MQL equation} \( [14] \) \textit{and satisfy the following boundary conditions:}

\[
A_{ij}(0, \ldots, n_i, \ldots, n_{j}, \ldots, 0) = A_{ij}^{(0)}(n_i, n_j) , \quad 1 \leq i < j \leq N . \tag{30}
\]

\[
A_{ji}(0, \ldots, n_i, \ldots, n_{j}, \ldots, 0) = A_{ji}^{(0)}(n_i, n_j) , \quad 1 \leq i < j \leq N .
\]
Note that the number of the arbitrary functions of two variables entering into the general solution of the MQL equation agrees with that of the continuous case \[21\].

**Remark** All the geometric considerations of this Section can be reformulated in terms of linear equations (for example, a 3-dimensional subspace of a 5-dimensional space is equivalent to a system of (5-3) linear equations for 5 unknowns) and the corresponding theorems about their solutions.

## 5 Final Remarks

It is important to emphasize that the MQL is, *by construction*, an *integrable lattice*. Indeed, the linear constraints (9) (the planarity constraints) provide a well defined way to construct the lattice; therefore they characterize completely the MQL and its integrability properties.

We are convinced that the simple geometric meaning of the MQL equations presented in this paper will allow to find for them many applications in all the branches of physics in which studies of multidimensional lattices are of importance.

We would like to point out that the projective picture used in this paper should not be considered as a mathematical curiosity. It allows to treat the points of the MQL which belong to the hyperplane at infinity on an equal footing with the "usual" points of the affine space. The presence of the points at infinity corresponds to singularities in the solution of the MQL equation (11). It may be interesting to develop this point of view in connection with the singularity confinement property of integrable discrete systems [34].

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