On the generality of certain predictions for quark mixing*

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Abstract

The relations  \(\frac{|V_{ub}|}{|V_{cd}|} = \sqrt{\frac{m_u}{m_c}}\)  and  \(\frac{|V_{td}|}{|V_{ts}|} = \sqrt{\frac{m_d}{m_s}}\)  are significant successes of some specific models for quark masses.  We show that these relations are more general, resulting from a much wider class of models.  Consequences of these predictions for CP violating asymmetries in neutral B meson decays are discussed.

* This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY90-21139.
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Motivation

A theory of fermion masses should explain both the values of the quark and lepton masses and the sizes of the four independent parameters of the Kobayashi-Maskawa (KM) mixing matrix [1]. In the standard model these quantities appear as free Yukawa coupling parameters and must be determined from experiment. While we are far from a fundamental understanding of fermion masses, theories which go beyond the standard model can possess symmetries which reduce the number of free parameters of these Yukawa coupling matrices, giving relationships between the KM matrix elements and the quark masses. The first relationship so obtained in a gauge theory was the very successful prediction for the Cabibbo angle: $|V_{us}| = \sqrt{\frac{m_u}{m_s}}$ [2], where $|V_{us}| = 0.221 \pm 0.002$ and $\sqrt{\frac{m_u}{m_s}} = 0.226 \pm 0.009$ [3]. Much interest has also centered around the relation $|V_{cb}| = \sqrt{\frac{m_c}{m_t}}$ obtained by Harvey, Ramond and Reiss [4] working with the form for the Yukawa matrices written down by Georgi and Jarlskog [5]. If this relation were valid at the weak scale the top quark would be predicted to be too heavy [6]. However, inclusion of renormalization group (RG) corrections show that such a relation in a supersymmetric grand unified theory leads to a prediction of $130 < m_t < 195$ GeV [7,8,9].

We choose $|V_{us}|$, $|V_{cb}|$, $\frac{|V_{us}|}{|V_{cb}|}$ and $\frac{|V_{td}|}{|V_{ts}|}$ as the the four independent parameters of the KM matrix. Of these $|V_{us}|$ and $|V_{cb}|$ are the two which are best measured. In this letter we concentrate on predictions for $\frac{|V_{us}|}{|V_{cb}|}$ and $\frac{|V_{td}|}{|V_{ts}|}$. These are predicted in several schemes for fermion masses in terms of ratios of quark masses[10,11,6,7]

$$\frac{|V_{ub}|}{|V_{cb}|} = \sqrt{\frac{m_u}{m_c}} \approx 0.061 \pm 0.009,$$ (1)

and

$$\frac{|V_{td}|}{|V_{ts}|} = \sqrt{\frac{m_d}{m_s}} \approx 0.226 \pm 0.009,$$ (2)
where mass values from reference [3] have been used, keeping in mind that the values in the ratios must be taken at the same renormalization scale $\mu$. In this letter we make two comments about these relations: they are very successful, and they are quite generic, following from a simple pattern for the Yukawa matrices.

The success of these relations has been magnified by last year’s announcement by the CLEO collaboration [12] of lower values for $|V_{ub}|/|V_{cb}|$. They find central values of $|V_{ub}|/|V_{cb}|$ of 0.053, 0.062, 0.065 and 0.095 in four phenomenological models used to analyze the data. The experimental uncertainty is about ±0.020. Also the value of the top quark mass obtained from precision electroweak data from LEP [13]: $m_t = 145 \pm 25 \text{GeV}$ is relevant because $|V_{td}|$ is probed experimentally via the $B^0 - \bar{B}^0$ mixing parameter $x_d$ which is strongly dependent on $m_t$:

$$
x_d = \frac{G_F^2}{6\pi^2} \left( \sqrt{B f_B} \right)^2 m_B \eta_B m_t^2 S(y_t) R e(V_{td}^* V_{tb})^2
= 0.69 \left( \frac{\sqrt{B f_B}}{0.17 \text{GeV}} \right)^2 \left( \frac{\eta_B}{0.85} \right)^2 \left( \frac{m_t}{145 \text{GeV}} \right)^2 \left( \frac{S(y_t)}{0.59} \right) \left( \frac{|V_{td}|}{|V_{cb}|} \right)^2 \left( \frac{|V_{cb}|}{0.043} \right)^2,
$$

(3)

where $y_t = m_t^2/M_W^2$, $S(y_t) = 1 - 3 \left( \frac{y_t}{1+y_t} \right)^2 \left[ 1 + \frac{2y_t}{1-y_t} \ln(y_t) \right]$ and $\eta_B$ is the QCD correction factor.

From this it can be seen that by using central values for $m_t$ and other parameters, together with the experimental result that $x_d = 0.70 \pm 0.10$, the prediction of equation (2) is highly successful.

Given the success of these two predictions, it is interesting to ask whether they result from just a few specific models, or whether they are generic features of a wide class of theories [14]. In the rest of this letter we show that predictions (1) and (2) occur whenever two conditions on the elements of the Yukawa matrices are satisfied. We also show that CP violation measurements with neutral B mesons will provide a test of whether the relations (1) and (2) provide a correct understanding of $|V_{ub}|$ and $|V_{td}|$. 
General constraint on the Yukawa matrices

What conditions must the Yukawa matrices \( Y \) (\( Y = U \) or \( D \)) satisfy in order to get relations (1) and (2)? The observed hierarchy of quark masses and mixing angles leads us to the assumption that the entries in the Yukawa matrices have a hierarchical structure, with \( Y_{33} \) being the largest. We first take \( Y_{ij} \) to be real and later consider how the analysis is modified by CP violating phases. The matrices \( Y \) can be diagonalized by three successive rotations in the (2,3), (1,3) and (1,2) sectors (denoted by \( s_{23}, s_{13} \) and \( s_{12} \)):

\[
\begin{pmatrix}
\tilde{Y}_{11} & 0 & 0 \\
0 & \tilde{Y}_{22} & 0 \\
0 & 0 & Y_{33}
\end{pmatrix} = \begin{pmatrix}
1 & -s_{12}^Y & 0 \\
s_{12}^Y & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & -s_{13}^Y \\
0 & 1 & 0 \\
0 & 1 & -s_{23}^Y
\end{pmatrix} \times \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

(4)

The small rotation angles are given to leading order by

\[
s_{23}^Y \approx \frac{Y_{23}}{Y_{33}} + \frac{Y_{32}Y_{22}}{Y_{33}^2}, \quad s_{23}^Y \approx \frac{Y_{23}}{Y_{33}} + \frac{Y_{23}Y_{22}}{Y_{33}^2},
\]

(5)

\[
s_{13}^Y \approx \frac{Y_{13}}{Y_{33}} + \frac{Y_{31}Y_{11}}{Y_{33}^2}, \quad s_{13}^Y \approx \frac{Y_{13}}{Y_{33}} + \frac{Y_{13}Y_{11}}{Y_{33}^2},
\]

(6)

\[
s_{12}^Y \approx \frac{Y_{12}}{Y_{22}} + \frac{Y_{21}Y_{11}}{Y_{22}^2}, \quad s_{12}^Y \approx \frac{Y_{12}}{Y_{22}} + \frac{Y_{12}Y_{11}}{Y_{22}^2}.
\]

(7)

The successive rotations produce elements

\[
\tilde{Y}_{11} \simeq \tilde{Y}_{11} - \frac{\tilde{Y}_{12}\tilde{Y}_{21}}{Y_{22}}, \quad \tilde{Y}_{11} \simeq Y_{11} - \frac{\tilde{Y}_{13}\tilde{Y}_{31}}{Y_{33}}, \quad \tilde{Y}_{22} \simeq Y_{22} - \frac{Y_{23}Y_{32}}{Y_{33}},
\]

(8)

and

\[
\tilde{Y}_{12} = Y_{12} - Y_{13}s_{23}^Y, \quad \tilde{Y}_{21} = Y_{21} - Y_{31}s_{23}^Y,
\]

(9)
\[ Y_{13} = Y_{13} + Y_{12}s_{23}^{Y}, \quad Y_{31} = Y_{31} + Y_{21}s_{23}^{Y}. \] (10)

The KM matrix which results from these rotations is

\[
V = \begin{pmatrix}
1 & s_{12} + s_{13}^{U}s_{23} & s_{13} - s_{12}^{U}s_{23} \\
-s_{12} - s_{13}^{D}s_{23} & 1 & s_{23} + s_{12}^{U}s_{13} \\
-s_{13} + s_{12}^{D}s_{23} & -s_{23} - s_{12}^{D}s_{13} & 1
\end{pmatrix}, \quad (11)
\]

where \( s_{23} = s_{23}^{D} - s_{23}^{U}, \quad s_{13} = s_{13}^{D} - s_{13}^{U} \) and \( s_{12} = s_{12}^{D} - s_{12}^{U} \).

To get relations (1) and (2) it is sufficient to have:

• \( \frac{|V_{ub}|}{|V_{cb}|} = |s_{12}^{U}| \) and \( \frac{|V_{td}|}{|V_{ts}|} = |s_{12}^{D}| \) which is obtained by:

\[
|s_{13}| << |s_{12}^{U}s_{23}| \quad \text{and} \quad |s_{13}| << |s_{12}^{D}s_{23}|. \] (12)

• \( |s_{12}^{U}| = \sqrt{\frac{m_{u}}{m_{c}}} \) and \( |s_{12}^{D}| = \sqrt{\frac{m_{d}}{m_{s}}} \) which is obtained by:

\[
|\tilde{Y}_{11}| << |\frac{\tilde{Y}_{12}\tilde{Y}_{21}}{Y_{22}}| \quad \text{and} \quad |\tilde{Y}_{12}| = |\tilde{Y}_{21}|. \] (13)

The conditions (12) and (13) on the Yukawa matrices \( U \) and \( D \) allow for a wide class of mass ansatzes. It is possible that there are some other cases which will lead to (1) and (2) but we believe that they would be quite special, involving for example nontrivial cancellations. All proposed ansatzes that we know of which lead to (1) and (2) satisfy the conditions (12) and (13).

Now consider redoing the analysis with \( Y_{ij} \) complex. The sequence of rotations in (4) will now be interspersed with various diagonal rephasing matrices. This will change the above equations in several ways. For example, in (5), (6), (7) and (8) the right hand sides of the equations must be replaced by their absolute values. In equations (9) and (10) there will be relative phases between the terms on the right-hand sides. Finally the phase rotations will affect the KM matrix. While the phase transformations cannot induce any new terms in \( V_{ij} \),
they can multiply any of the existing ones by phases. However, it is clear that even in this case equations (12) and (13) are the correct conditions for yielding the predictions (1) and (2).

The conditions (12) and (13) are very simple, however, when expressed in terms of $U_{ij}$ and $D_{ij}$ via equations (5) - (10), they appear quite cumbersome. Nevertheless, a simple heuristic way of stating the conditions on $U_{ij}$ and $D_{ij}$ is as follows:

- $Y_{11}, Y_{13}$ and $Y_{31}$ must be small.

While conditions (12) and (13) are the precise statement on the smallness of these elements, a feel for their meaning can be grasped as follows. The smallness and hierarchy of fermion masses and mixings can be restated in terms of approximate chiral and flavor symmetries which act on each fermion type [15]. From these approximate symmetries alone one finds that the inequalities of (12) and (13) become approximate equalities, thus $|s_{13}| \approx |s_{12}s_{23}|$ etc. Hence these approximate chiral and flavor symmetries are not sufficient to guarantee results (1) and (2). These results follow only if the 11, 13 and 31 entries of the Yukawa matrices are constrained by some more powerful means, for example by some new exact symmetry. In many specific ansatzes, family symmetries force these to vanish [10,11,6,7].

- $|\tilde{Y}_{12}| = |\tilde{Y}_{21}|$ usually results, to sufficient accuracy, whenever $|Y_{12}| = |Y_{21}|$.

While the 12 entries must be symmetric, other entries need not have any symmetry. None of the models discussed in reference [16] has symmetric Yukawa matrices, but they all have predictions (1) and (2).

The general conditions (12) and (13) are satisfied by many special forms for the Yukawa matrices, so that it is not possible to use them to derive a definite hierarchical structure for $U$.
and D. However, this can be done for the subclass of theories in which the Yukawa matrices are also symmetric $Y_{ij} = Y_{ji}$, and the resulting hierarchical patterns are given in the appendix.

The KM matrix

We now study the KM matrix which results from Yukawa couplings which satisfy the conditions (12) and (13). In particular, the 11, 13 and 31 entries are found to be sufficiently small that they give only negligible corrections to the diagonalization of $U$ and $D$

$$L_U^\dagger U R_U = U_d, \quad L_D^\dagger D R_D = D_d. \quad (14)$$

by the unitary matrices

$$L_U^\dagger = \begin{pmatrix} 1 & -s_2 & 0 \\ s_2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\phi_U} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -s_U \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s_U & 1 \end{pmatrix}, \quad (15)$$

$$L_D^\dagger = \begin{pmatrix} 1 & -s_1 & 0 \\ s_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\phi_D} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s_D & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & e^{-i\phi_D} & 0 \end{pmatrix}, \quad (16)$$

where $s_U$ and $s_D$ are $s_{23}$ rotations in the $U$ and $D$, respectively, and $\phi_U, \phi_D$ and $\phi$ are the necessary phase redefinitions. This gives the KM matrix

$$V = L_U^\dagger L_D = \begin{pmatrix} e^{i\psi} & s_1 e^{i\phi} & -s_2(s_D e^{i\phi} - s_U) \\ s_2 e^{i\psi} - s_1 e^{i\phi} & e^{i\phi} & (s_D e^{i\phi} - s_U) \\ -s_1(s_U e^{i\phi} - s_D) & (s_U e^{i\phi} - s_D) & 1 \end{pmatrix}, \quad (17)$$

where $\psi = \phi_U - \phi_D$. For this matrix to yield the predictions (1) and (2) we impose constraint (13), which implies $|s_1| = \sqrt{m_u/m_s}$ and $|s_2| = \sqrt{m_c/m_t}$. After an additional phase redefinition, the KM matrix can be brought into the form introduced in reference [7]:

$$V = \begin{pmatrix} 1 & s_1 + s_2 e^{-i\phi} & s_2 s_3 \\ -s_2 - s_1 e^{-i\phi} & e^{-i\phi} & s_3 \\ s_1 s_3 & -s_3 & e^{i\phi} \end{pmatrix}, \quad (18)$$
where $\phi \to \psi - \phi$ and $s_2 \to -s_2$. In (18) we do not loose any generality by choosing the corresponding angles $\theta_1$, $\theta_2$ and $\theta_3$ to lie in the first quadrant.

This form is now seen to result directly from the straightforward diagonalization of a large class of Yukawa textures. It has the appealing feature that to a very good approximation $s_1, s_2$ and $\phi$ are renormalization group invariants, while $s_3$ obeys a simple scaling law. Note that while $s_1 = \sqrt{m_s/m_u}$ and $s_2 = \sqrt{m_s/m_c}$ are given by quark masses and directly yield the predictions (1) and (2), $s_3$ and $\phi$ are determined from $|V_{cb}|$ and $|V_{us}|$, giving $s_3 = 0.043 \pm 0.007$ and, using the numbers quoted in (1) and (2), $\sin \phi = 0.98^{+0.02}_{-0.07}[17]$. Hence we find that if the Yukawa matrices satisfy (12) and (13), the entire KM matrix can be determined quite accurately.

Since all four independent parameters of this KM matrix have now been specified by CP conserving magnitudes $|V_{ij}|$, a crucial question is whether the resulting prediction for CP violation agrees with data. Since $V_{us}$ in (18) is not real, we use a rephase invariant result for the CP violating kaon parameter $\epsilon$ [18]:

$$|\epsilon| = \frac{G_F^2}{12\pi^2} (\sqrt{B_K f_K})^2 \frac{m_K}{\sqrt{2\Delta m_K}} m_i^2 \frac{1}{|\xi_u|^2} \sum_{ij} S(y_i, y_j) \text{Im}(\xi_i \xi_j (\xi_u^*)^2) \eta_{ij},$$  

(19)

where $y_i = m_i^2/m_K^2$, $\xi_i = V_{is}V_{id}^*$,

$$S(y_i, y_j) = \frac{y_i y_j}{y_t} \left\{ \left[ \frac{1}{4} + \frac{3}{2} (1-y_j)^{-1} \right] - \frac{3}{4} (1-y_j)^{-2} \right\} \frac{\ln y_j}{y_j - y_i} + (y_j \to y_i) - \frac{3}{4} [(1-y_j)^{-1}] \right\},$$  

(20)

and $\eta_{tt} = 0.63, \eta_{ct} = 0.34$ are the QCD correction factors. Note that $S(y_i) = S(y_t, y_t)$. Using central values we find

$$|\epsilon| = 2.26 \cdot 10^{-3} \sin \phi \left( \frac{\sqrt{B_K f_K}}{0.16 \text{GeV}} \right)^2 \left( \frac{m_t}{145 \text{GeV}} \right)^2 \left( \frac{0.221}{|\xi_u|} \right)^2 \times$$

$$\left[ \frac{S(y_i)}{0.59} \left( \frac{s_1}{0.226} \right)^3 \frac{s_2}{0.061} \left( \frac{s_3}{0.043} \right)^3 \frac{\eta_{tt}}{0.63} + 0.12 \frac{S(y_c, y_t)}{0.24 \cdot 10^{-3} \left( \frac{s_1}{0.226} \right)^3 \frac{s_2}{0.061} \left( \frac{s_3}{0.043} \right)^3 \frac{\eta_{ct}}{0.34}} \right].$$  

(21)

Here we used $\text{Im}(\xi_u^2 (\xi_u^*)^2) \approx 2s_1^3 s_2^2 s_3^3 \sin \phi$ and $\text{Im}(\xi_c \xi_t (\xi_u^*)^2) \approx s_1^3 s_2^2 s_3^2 \sin \phi$. We see that the experimental results indicate a large (CP violating) phase $\phi$, consistent with its determination
from $|V_{us}|$. An alternative way of stating the prediction for CP violation is via the quantity $J$ [19]:

$$J = Im(V_{ud}V_{ub}^*V_{td}^*V_{tb}) = s_1 s_2 s_3^2 sin \phi = \sqrt{\frac{m_d}{m_s}} \sqrt{\frac{m_u}{m_c}} V_{cb}^2 sin \phi = (2.6 \pm 0.9) \times 10^{-5} sin \phi. \quad (22)$$

**CP asymmetries in B decays**

A good test of this KM matrix comes from looking at the allowed values for the CP asymmetries in B decays [20]. The asymmetries, given by $sin 2\alpha$ (coming from $B_d \to \pi^+\pi^-$) and $sin 2\beta$ (coming from $B_d \to \psi K_S$), can be expressed in terms of the Cabibbo angle $s_c \equiv |V_{us}|$, $s_1$ and $s_2$[21]

$$sin 2\alpha = -2 cos \phi sin \phi, \quad (23)$$

$$sin 2\beta = \frac{2 s_1 s_2 sin \phi}{s_c^2} (1 + \frac{s_2 cos \phi}{s_1}). \quad (24)$$

In the figure we plot the allowed region for $sin 2\alpha$ and $sin 2\beta$. The dotted region is the region allowed by the standard model [22]. The $sin 2\alpha$ variation comes mainly from the uncertainty in $s_1$ (i.e. from the uncertainties in d and s masses), while the $sin 2\beta$ variation comes mainly from the uncertainty in $s_2$ (i.e. from the uncertainties in u and c masses)[3]. Precise measurements of $sin 2\alpha$ and $sin 2\beta$ will reduce the experimental uncertainties on $s_1$ and $s_2$, thereby providing a stringent test of (1) and (2).

We have shown that $|V_{ub}|/|V_{cb}| = \sqrt{m_u/m_c}$ and $|V_{td}|/|V_{ts}| = \sqrt{m_d/m_s}$ are highly successful relations which result from a wide range of models: the Yukawa matrices $U$ and $D$ need only satisfy the constraints (12) and (13). This typically means that these matrices have small 11, 13, and 31 entries, and symmetric 12 entries. Given the generality of these results, one might question whether $|V_{ub}/V_{cb}|$ and $|V_{td}/V_{ts}|$ can be used as a probe of specific mass matrix ansatzes in future B-
physics experiments. The answer is that they can, but only if these schemes are able to predict \( m_d/m_s \) and \( m_u/m_c \) more accurately than they are currently extracted from experiment.

**Acknowledgement**

One of us (A.R.) would like to thank Uri Sarid for help in preparation of the figure. L.J.H. acknowledges partial support from the NSF Presidential Young Investigator Program.
Appendix. Forms of U and D with the additional constraint $|Y_{ij}| = |Y_{ji}|$

Let us try to find the most general forms of symmetric $U$ and $D$ which lead to (1) and (2). We neglect phases for simplicity. As mentioned before, we assume no accidental cancelations, so if a sum of two elements is small it is because they are both small. First, because of symmetry, expressions (5)-(10) simplify: $s^{Y}_{23} = s^{Y'}_{23} = \frac{Y_{23}}{Y_{33}}$ and therefore

$$\tilde{Y}_{12} = \tilde{Y}_{21} = Y_{12} - Y_{13} \frac{Y_{23}}{Y_{33}},$$  \hfill (25)

$$\tilde{Y}_{13} = \tilde{Y}_{31} = Y_{13} + Y_{12} \frac{Y_{23}}{Y_{33}},$$  \hfill (26)

$$s^{Y}_{13} = s^{Y'}_{13} = \frac{\tilde{Y}_{13}}{Y_{33}},$$  \hfill (27)

$$s^{Y}_{12} = s^{Y'}_{12} = \frac{\tilde{Y}_{12}}{Y_{33}}.$$  \hfill (28)

and

$$\tilde{Y}_{11} \simeq \tilde{Y}_{11} - \frac{\tilde{Y}_{12}^2}{Y_{22}}, \quad \tilde{Y}_{11} \simeq Y_{11} - \frac{\tilde{Y}_{13}^2}{Y_{33}}, \quad \tilde{Y}_{22} \simeq Y_{22} - \frac{Y_{23}^2}{Y_{33}}.$$  \hfill (29)

We will express all mass matrix elements in terms of their eigenvalues (recall that $m_1 = \tilde{Y}_{11}$, $m_2 = \tilde{Y}_{22}$ and $m_3 = Y_{33}$). Because of our assumption of no accidental cancelations we divide possible forms into two categories: either $Y_{22} << m_2$ and $Y_{23} = \sqrt{m_2 m_3}$, or $Y_{22} = m_2$ and $Y_{23} << \sqrt{m_2 m_3}$.

Now we use conditions (12) and (13). Let us first use condition (13) because it does not depend on whether $Y$ is $U$ or $D$. It tells us that

$$\tilde{Y}_{12} = \sqrt{m_1 m_2},$$  \hfill (30)

$$Y_{11} << m_1,$$  \hfill (31)

$$\tilde{Y}_{13} << \sqrt{m_1 m_3}.$$  \hfill (32)
Since $Y_{23} \leq \sqrt{m_2m_3}$ (from $m_2 = Y_{22}$), it follows from equations (30) and (32) that $Y_{12} = \sqrt{m_1m_2}$ and $Y_{13} \ll \sqrt{m_1m_3}$. Therefore the symmetric $U$ and $D$ that obey (13) must take one of the following forms:

$$
\begin{pmatrix}
(Y_{11} \ll m_1) & \sqrt{m_1m_2} & (Y_{13} \ll \sqrt{m_1m_3}) \\
\sqrt{m_1m_2} & (Y_{22} \ll m_2) & \sqrt{m_2m_3} \\
(Y_{13} \ll \sqrt{m_1m_3}) & \sqrt{m_2m_3} & m_3 \\
\end{pmatrix} ,
$$

(33)

where $Y_{22} \ll \frac{Y_{22}^2}{Y_{33}}$, or

$$
\begin{pmatrix}
(Y_{11} \ll m_1) & \sqrt{m_1m_2} & (Y_{13} \ll \sqrt{m_1m_3}) \\
\sqrt{m_1m_2} & m_2 & (Y_{23} \ll \sqrt{m_2m_3}) \\
(Y_{13} \ll \sqrt{m_1m_3}) & (Y_{23} \ll \sqrt{m_2m_3}) & m_3 \\
\end{pmatrix} ,
$$

(34)

where $Y_{22} \gg \frac{Y_{22}^2}{Y_{33}}$.

Let us now use the constraint (12) which will further constrain some of the bracketed elements in (33) or (34). Using reasonable values for quark masses we see that in (12) the more stringent constraint is

$$
s_{13} \ll \sqrt{\frac{u_1}{u_2} s_{23}} ,
$$

(35)

where $s_{13} = s_{13}^D - s_{13}^U$ and $s_{23} = s_{23}^D - s_{23}^U$. We use $u_1 = m_u$, $u_2 = m_c$, etc. In particular, both $s_{13}^U$ and $s_{13}^D$ must be less than $\sqrt{\frac{u_1}{u_2} s_{23}}$:

$$
\left| \frac{U_{13}}{u_3} + \sqrt{\frac{u_1 u_2}{u_3}} \frac{U_{23}}{u_3} \right| \ll \sqrt{\frac{u_1}{u_2}} | \frac{D_{23}}{d_3} - \frac{U_{23}}{u_3} | ,
$$

(36)

$$
\left| \frac{D_{13}}{d_3} + \sqrt{\frac{d_1 d_2}{d_3}} \frac{D_{23}}{d_3} \right| \ll \sqrt{\frac{u_1}{u_2}} | \frac{D_{23}}{d_3} - \frac{U_{23}}{u_3} | .
$$

(37)

Notice that consistency of solutions is automatically obeyed since $\sqrt{\frac{d_1 d_2}{d_3}} \ll \sqrt{\frac{u_1}{u_2}}$ and $\sqrt{\frac{u_1 u_2}{u_3}} \ll \sqrt{\frac{u_1}{u_2}}$ for reasonable quark masses. Therefore, we conclude that limits on $U_{13}$ and $D_{13}$ may be somewhat stringent

$$
U_{13} \ll \min\{ \sqrt{\frac{u_1 u_3}{u_2}}, \sqrt{\frac{u_1}{u_2}} u_3 s_{23} \} \equiv a ,
$$

(38)
\[ D_{13} \ll \min\{\sqrt{d_1d_3}, \sqrt{\frac{u_1}{u_2}}d_3s_{23}\} \equiv b. \quad (39) \]

If \( U \) or \( D \) is of type (34) somewhat stringent limits on \( U_{23} \) and \( D_{23} \) are also possible

\[ U_{23} \ll \min\{\sqrt{u_2u_3}, \sqrt{\frac{u_3}{u_2}}s_{23}\} \equiv c, \quad (40) \]

\[ D_{23} \ll \min\{\sqrt{d_2d_3}, \sqrt{\frac{u_1}{u_2}}\frac{d_3^2}{\sqrt{d_1d_2}}s_{23}\} \equiv d. \quad (41) \]

We can now write possible forms of symmetric \( U \) and \( D \) which lead to successful predictions (1) and (2). There are four possibilities depending on whether \( U \) or \( D \) take on the form (33) or (34)[23]

1) \[
U = \begin{pmatrix}
(U_{11} \ll u_1) & \sqrt{u_1u_2} & (U_{13} \ll \sqrt{u_1u_3}) \\
\sqrt{u_1u_2} & (U_{22} \ll u_2) & \sqrt{u_2u_3} \\
(U_{13} \ll \sqrt{u_1u_3}) & \sqrt{u_2u_3} & u_3
\end{pmatrix}, \quad (42)
\]

\[
D = \begin{pmatrix}
(D_{11} \ll d_1) & \sqrt{d_1d_2} & (D_{13} \ll \sqrt{d_1d_3}\sqrt{\frac{u_1d_2}{u_2d_1}}) \\
\sqrt{d_1d_2} & (D_{22} \ll d_2) & \sqrt{d_2d_3} \\
(D_{13} \ll \sqrt{d_1d_3}\sqrt{\frac{u_1d_2}{u_2d_1}}) & \sqrt{d_2d_3} & d_3
\end{pmatrix}. \quad (43)
\]

2) \[
U = \begin{pmatrix}
(U_{11} \ll u_1) & \sqrt{u_1u_2} & (U_{13} \ll \sqrt{u_1u_3}) \\
\sqrt{u_1u_2} & u_2 & (U_{23} \ll \sqrt{u_2u_3}) \\
(U_{13} \ll \sqrt{u_1u_3}) & (U_{23} \ll \sqrt{u_2u_3}) & u_3
\end{pmatrix}, \quad (44)
\]

\[
D = \begin{pmatrix}
(D_{11} \ll d_1) & \sqrt{d_1d_2} & (D_{13} \ll \sqrt{d_1d_3}\sqrt{\frac{u_1d_2}{u_2d_1}}) \\
\sqrt{d_1d_2} & (D_{22} \ll d_2) & \sqrt{d_2d_3} \\
(D_{13} \ll \sqrt{d_1d_3}\sqrt{\frac{u_1d_2}{u_2d_1}}) & \sqrt{d_2d_3} & d_3
\end{pmatrix}. \quad (45)
\]

3) \[
U = \begin{pmatrix}
(U_{11} \ll u_1) & \sqrt{u_1u_2} & (U_{13} \ll a) \\
\sqrt{u_1u_2} & (U_{22} \ll u_2) & \sqrt{u_2u_3} \\
(U_{13} \ll a) & \sqrt{u_2u_3} & u_3
\end{pmatrix}, \quad (46)
\]

\[
D = \begin{pmatrix}
(D_{11} \ll d_1) & \sqrt{d_1d_2} & (D_{13} \ll b) \\
\sqrt{d_1d_2} & d_2 & (D_{23} \ll d) \\
(D_{13} \ll b) & (D_{23} \ll d) & d_3
\end{pmatrix}. \quad (47)
\]
\[ \mathbf{U} = \begin{pmatrix} (U_{11} << u_1) & \sqrt{u_1 u_2} & (U_{13} << a) \\ \sqrt{u_1 u_2} & u_2 & (U_{23} << c) \\ (U_{13} << a) & (U_{23} << c) & u_3 \end{pmatrix}, \quad (48) \]

\[ \mathbf{D} = \begin{pmatrix} (D_{11} << d_1) & \sqrt{d_1 d_2} & (D_{13} << b) \\ \sqrt{d_1 d_2} & d_2 & (D_{23} << d) \\ (D_{13} << b) & (D_{23} << d) & d_3 \end{pmatrix}, \quad (49) \]

where a, b, c and d are given in equations (38)-(41).

In the above it is understood that \( U_{23} \) and \( D_{23} \) cannot be simultaneously =0 since they are constrained by the condition \( V_{cb} = s_{23} \).

Some specific mass matrix ansatzes can be recovered by setting bracketed elements to zero. For example, 1) contains the Fritzsch scheme[10], while 2) is the generalization of the Harvey,Reiss and Ramond form[4]. Nevertheless, it is important to notice that although the bracketed elements can in many cases be set to zero they need not to be. As long as they obey the limits, relations (1) and (2) will follow. For example, \( Y_{13} \) can be as big as \( Y_{12} \)!
(a) On leave of absence from the Ruđer Bošković Institute, Zagreb, Croatia.

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[23] Notice that when $D$ is of type (33) then $s_{23} = \sqrt{d_2/d_3}$ and when reasonable quark masses are
    used, it is easy to see that the additional constraint (12) affects only $D_{13}$ ($U_{13}, U_{23}$ and $D_{23}$
    are more constrained by (13)).