Abstract. A system of three quantum particles on the three-dimensional lattice $\mathbb{Z}^3$ with arbitrary “dispersion functions” having non-compact support and interacting via short-range pair potentials is considered. The energy operators of the systems of the two-and three-particles on the lattice $\mathbb{Z}^3$ in the coordinate and momentum representations are described as bounded self-adjoint operators on the corresponding Hilbert spaces. For all sufficiently small nonzero values of the two-particle quasi-momentum $k \in (-\pi, \pi]^3$ the finiteness of the number of eigenvalues of the two-particle discrete Schrödinger operator $h_{\alpha}(k)$ below the continuous spectrum is established. A location of the essential spectrum of the three-particle discrete Schrödinger operator $H(K), K \in (-\pi, \pi]^3$ the three-particle quasi-momentum, by means of the spectrum of $h_{\alpha}(k)$ is described. It is established that the essential spectrum of $H(K), K \in (-\pi, \pi]^3$ consists of a finitely many bounded closed intervals.

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1. Introduction

Location of the essential spectrum of $N$-body Schrödinger operators for particles moving in $\mathbb{R}^3$ has been extensively studied in many works (See, e.g., [2, 3, 6, 7, 19, 20, 22, 23] and references therein).

We recall that for the three-particle Schrödinger operators the three-particle continuum of the essential spectrum coincides with the semi-axis $[0, \infty)$. “Two-particle branches” fill the interval $(\kappa, \infty)$ where $\kappa \leq 0$ is the lowest eigenvalue of the two-particle Subhamiltonians. Thus, there are no gaps in the essential spectrum.

In models of solid state physics [5, 17, 18, 19, 21] and also in lattice field theory [8, 16] discrete Schrödinger operators are considered, which are lattice analogs of the continuous three-particle Schrödinger operator.

In [2, 10, 11, 12, 13, 14, 15] for the Hamiltonians of systems of three quantum particles moving on the three-dimensional lattice $\mathbb{Z}^3$ and interacting via zero-range attractive pairs potentials the location and structure of the essential spectrum has been investigated.

In particular in [15] it is shown that the essential spectrum of $H(K)$ consists of no more than four bounded closed intervals. The existence of infinitely many eigenvalues of

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$H(0)$ is proven. It is found that for the number $N(0, z)$ of eigenvalues of $H(0)$ lying below $z < 0$ the following limit exists

$$\lim_{z \to 0^-} \frac{N(0, z)}{\log |z|} = U_0$$

with $U_0 > 0$. Moreover, for all sufficiently small nonzero values of the three-particle quasi-momentum $K$ the finiteness of the number $N(K, \tau_{\text{ess}}(K))$ of eigenvalues of $H(K)$ below the essential spectrum is established and the asymptotics for the number $N(K, 0)$ of eigenvalues lying below zero is given.

The fundamental difference between the discrete and continuous multiparticle Schrödinger operators is that in the discrete case the kinetic energy operator is not rotationally invariant. The absence of this property impedes the use of the technique of separation of variables (which was essential for considering non-interacting clusters of particles in the continuum case). In lattice terms the "center-of-mass separation" corresponds to a realization of Hamiltonian as a "fibered operator", that is, as the "direct integral of a family of operators" $H(K)$ depending on the values of the total quasi-momentum $K \in T^3 = (-\pi, \pi]^3$ (see [5, 19]). In this case a "bound state" is an eigenvector of the operator $H(K)$ for some $K \in T^3$. Typically, this eigenvector depends continuously on $K$.

In the present work we consider the system of three quantum particles on the three-dimensional lattice $\mathbb{Z}^3$ with arbitrary "dispersion functions" having non-compact support and interacting via short-range pair potentials. We describe the energy operators (Hamiltonians) for the two-and three-particles on the lattice $\mathbb{Z}^3$ in the coordinate and momentum representations as bounded self-adjoint operators on the corresponding Hilbert spaces. Then we decompose the energy operators into von Neumann direct integrals and introduce the two-and three-particle quasimomenta and relative coordinate systems. We show that the two-and three-particle fiber operators $\tilde{h}_\alpha(k), k \in T^3,$ and $\tilde{H}(K), K \in T^3,$ are unitarily equivalent to the "two-and three-particle discrete Schrödinger operators $h_\alpha(k), k \in T^3,$ and $H(K), K \in T^3,"$ on the Hilbert spaces $L_2(T^3)$ and $L_2((T^3)^2)$.

We also obtain a generalization of the Birman-Schwinger principle for the two-particle discrete Schrödinger operators $h_\alpha(k)$ (Theorem 5.4) and, using this, we establish that for all sufficiently small values of the two-particle quasi-momentum $k \in (-\pi, \pi]^3$ the number of eigenvalues of $h_\alpha(k)$ below the continuous spectrum is finite (Theorem 4.1).

We describe a location of the essential spectrum for the three-particle discrete Schrödinger operator $H(K)$ by means of the eigenvalues of the two-particle operators $h_\alpha(k)$ (Theorem 4.2).

Our main results is that the essential spectrum of the discrete three-particle Schrödinger operator $H(K), K \in T^3$ consists of only a finitely many bounded closed intervals (Theorem 4.3). Our proof is based on the fact that for each $\alpha = 1, 2, 3$ and some $k \in T^3$ the operator $h_\alpha(k)$ has finitely many eigenvalues below the bottom of the continuous spectrum.

The plan of the paper is as follows:

Section 1 is an introduction.

In section 2 the Hamiltonians of systems of two and three-particles in coordinate and momentum representations are described as bounded self-adjoint operators in the corresponding Hilbert spaces.
In section 3 we introduce the total quasi-momentum and decompose the energy operators into von Neumann direct integrals. We show, choosing relative coordinate systems, that the "two"-and "three"-particle fiber operators are unitarily equivalent to the family of bounded self-adjoint operators acting in the Hilbert spaces $L_2(T^3)$ and $L_2((T^3)^3$).

In section 4 we state the main results of the paper.

In section 5 we study spectral properties of the two-particle discrete Schrödinger operators $h_\alpha(k)$, $k \in T^3$, $\alpha = 1, 2, 3$ on the three-dimensional lattice $T^3$.

In section 6 we introduce the "channel operators" and prove that the spectrum of "channel operator" is the union of a finite number of bounded closed intervals.

In section 7 applying the Faddeev type system of integral equations we establish the location of the essential spectrum of $H(K)$ (Theorem 4.2) and prove the main result (Theorem 4.3).

Throughout the paper we adopt the following conventions: For each $\delta > 0$ the notation $U_\delta(0) = \{ K \in T^3 : |K| < \delta \}$ stands for a $\delta$-neighborhood of the origin. The subscript $\alpha$ (and also $\beta$ and $\gamma$) always equal to 1 or 2 or 3 and $\alpha \neq \beta, \beta \neq \gamma, \gamma \neq \alpha$.

2. ENERGY OPERATORS OF SYSTEMS OF TWO AND THREE ARBITRARY PARTICLES ON THE LATTICE $Z^3$ IN THE COORDINATE AND MOMENTUM REPRESENTATIONS.

Let $Z^d$-dimensional lattice.

The free Hamiltonian $\hat{H}_0$ of a system of three quantum mechanical particles on the three-dimensional lattice $Z^3$ is defined in terms of three functions $\hat{\varepsilon}_\alpha(\cdot)$ corresponding to the particles $\alpha = 1, 2, 3$ (called "dispersion functions" in the physical literature, see, e.g. [17]). The operator $\hat{H}_0$ usually associated with the following bounded self-adjoint operator on the Hilbert space $\ell_2((Z^3)^3)$:

$$\hat{H}_0 = \hat{D}_{x_1} + \hat{D}_{x_2} + \hat{D}_{x_3},$$

with $\hat{D}_{x_1} = D_1 \otimes I \otimes I$, $\hat{D}_{x_2} = I \otimes D_2 \otimes I$ and $\hat{D}_{x_3} = I \otimes I \otimes D_3$, where $I$ is identity operator in $\ell_2(Z^3)$ and

$$(D_\alpha \hat{\psi})(x) = \sum_{s \in Z^3} \hat{\varepsilon}_\alpha(s) \hat{\psi}(x + s), \quad \hat{\psi} \in \ell_2(Z^3).$$

Here $\hat{\varepsilon}_\alpha(\cdot)$, $\alpha = 1, 2, 3$ are assumed to be real-valued bounded functions having non compact support in $Z^3$ and describing the dispersion low of the corresponding particles (see, e.g. [17]).

The three-particle Hamiltonian $\hat{H}$ of the quantum-mechanical three-particles systems with two-particle pair interactions $\hat{\varepsilon}_{\beta \gamma}$, $\beta \gamma = 12, 23, 31$ is a bounded perturbation of the free Hamiltonian $\hat{H}_0$

$$(\hat{H} = \hat{H}_0 - \hat{V}_1 - \hat{V}_2 - \hat{V}_3,$$

where $\hat{V}_{\alpha}, \alpha = 1, 2, 3$ are multiplication operators on $\ell_2((Z^3)^3)$

$$(\hat{V}_{\alpha} \hat{\psi})(x_1, x_2, x_3) = \hat{\varepsilon}_{\beta \gamma}(x_\beta - x_\gamma) \hat{\psi}(x_1, x_2, x_3), \quad \hat{\psi} \in \ell_2((Z^3)^3),$$

and $\hat{\varepsilon}_{\beta \gamma}$ is bounded real-valued function.

Throughout this paper we assume the following additional Hypothesis.
Hypothesis 2.1. The functions \( \hat{\varepsilon}_\alpha(s) \), \( \alpha = 1, 2, 3 \) satisfy the following conditions:

(i) \( \hat{\varepsilon}_\alpha(s) \) depends only on \( |s| = |s^{(1)}| + |s^{(2)}| + |s^{(3)}| \), \( s = (s^{(1)}, s^{(2)}, s^{(3)}) \in \mathbb{Z}^3 \);

(ii) there exist numbers \( a, C > 0 \) such that \( |\hat{\varepsilon}_\alpha(s)| \leq C \exp(-a|s|), s \in \mathbb{Z}^3 \);

(iii) \( \hat{\varepsilon}_\alpha(s) < 0 \), \( |s| = 1 \) and \( \hat{\varepsilon}_\alpha(s) \leq 0 \), \( |s| > 1 \), \( s \in \mathbb{Z}^3 \).

Remark 2.2. The number \( m_\alpha = 3(-\sum_{s \in \mathbb{Z}^3} [(s^{(1)})^2 + (s^{(2)})^2 + (s^{(3)})^2] \hat{\varepsilon}_\alpha(s))^{-1} > 0 \) means the (effective) mass of the particle \( \alpha \).

Hypothesis 2.3. Assume that \( \hat{v}_{\beta \gamma}(s), \beta \gamma = 12, 23, 31 \) are real even nonnegative functions on \( \mathbb{Z}^3 \) and verifying

\[
\lim_{|s| \to \infty} |s|^{3+\kappa} \hat{v}_{\beta \gamma}(s) = 0, \quad \kappa > 0.
\]

It is clear that under Hypothesis 2.1 and 2.3 the three-particle Hamiltonian is a bounded self-adjoint operator on the Hilbert space \( \ell_2((\mathbb{Z}^3)^3) \).

Similarly as we introduced \( \hat{H} \), we shall introduce the corresponding two-particle Hamiltonians \( \hat{h}_\alpha, \alpha = 1, 2, 3 \) as bounded self-adjoint operators on the Hilbert space \( \ell_2((\mathbb{Z}^3)^2) \)

\[
\hat{h}_\alpha = \hat{h}_0^\alpha - \hat{\varepsilon}_\alpha,
\]

where

\[
\hat{h}_0^\alpha = \mathcal{D}_{x_\beta} + \mathcal{D}_{x_\gamma},
\]

with \( \mathcal{D}_{x_\beta} = D_\beta \otimes I, \mathcal{D}_{x_\gamma} = I \otimes D_{x_\gamma} \) and

\[(\hat{\varepsilon}_\alpha, \hat{\varphi})(x_\beta, x_\gamma) = \hat{\varphi}_{\beta \gamma}(x_\beta - x_\gamma) \hat{\varphi}(x_\beta, x_\gamma), \quad \hat{\varphi} \in \ell_2((\mathbb{Z}^3)^2).\]

Let us rewrite our operators in the momentum representation. Let \( \mathcal{F}_m : L_2((\mathbb{T}^3)^m) \to \ell_2((\mathbb{Z}^3)^m) \) denote the standard Fourier transform, where \((\mathbb{T}^3)^m, m \in \mathbb{N}\) denotes the Cartesian \( m \)-th power of the set \( \mathbb{T}^3 = (-\pi, \pi]^3 \).

Remark 2.4. The operations addition and multiplication by real number of elements of \( \mathbb{T}^3 \subset \mathbb{R}^3 \) should be regarded as operations on \( \mathbb{R}^3 \) modulo \( 2\pi \mathbb{Z}^3 \). For example, let

\[
a = \left( \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{11\pi}{12} \right), \quad b = \left( \frac{2\pi}{3}, \frac{\pi}{2}, \frac{5\pi}{6} \right) \in \mathbb{T}^3
\]

then

\[
a + b = \left( -\frac{2\pi}{3}, -\frac{3\pi}{4}, -\frac{\pi}{4} \right) \in \mathbb{T}^3, \quad 12a = (0, \pi, \pi) \in \mathbb{T}^3.
\]

The three resp. two-particle Hamiltonians (in the momentum representation) are given by the bounded self-adjoint operators on the Hilbert spaces \( L_2((\mathbb{T}^3)^3) \) resp. \( L_2((\mathbb{T}^3)^2) \) as follows

\[
H = \mathcal{F}_3^{-1} \hat{H} \mathcal{F}_3
\]

resp.

\[
\hat{h}_\alpha = \mathcal{F}_2^{-1} \hat{h}_\alpha \mathcal{F}_2, \quad \alpha = 1, 2, 3.
\]

One has

\[
H = H_0 - V_1 - V_2 - V_3,
\]
where

\[ H_0 = \mathcal{D}_{k_1} + \mathcal{D}_{k_2} + \mathcal{D}_{k_3}, \]

with \( \mathcal{D}_{k_1} = \hat{D}_1 \otimes I \otimes I, \mathcal{D}_{k_2} = I \otimes \hat{D}_2 \otimes I, \mathcal{D}_{k_3} = I \otimes I \otimes \hat{D}_3 \)

and \( \hat{D}_\alpha, \alpha = 1, 2, 3 \) is the multiplication operator by the function \( \varepsilon_\alpha(k) \)

\[ \hat{D}_\alpha f(k) = \varepsilon_\alpha(k) f(k), \quad f \in L_2(\mathbb{T}^3) \]

and \( V_\alpha, \alpha = 1, 2, 3 \) are integral operators of convolution type

\[ (V_\alpha f)(k_1, k_2, k_3) = (2\pi)^{-\frac{3}{2}} \int_{(\mathbb{T}^3)^3} v_\alpha \left( \frac{k_\beta - k_\gamma - k'_\beta + k'_\gamma}{2} \right) \delta(k_\alpha - k'_\alpha) \delta(k_\beta + k_\gamma - k'_\beta - k'_\gamma) f(k'_1, k'_2, k'_3) d\beta d\gamma, \]

\[ f \in L_2((\mathbb{T}^3)^3), \]

where \( \delta(\cdot) \) denotes the Dirac delta-function at the origin.

Here the functions \( \varepsilon_\alpha(k), v_\alpha(k), \alpha = 1, 2, 3 \) are given by the Fourier series \( \mathcal{F}^{-1}_1 \) and are of the form

\[ \varepsilon_\alpha(k) = \sum_{s \in \mathbb{Z}^3} \tilde{\varepsilon}_\alpha(s) e^{i(k,s)}, \quad v_\alpha(k) = (2\pi)^{-3/2} \sum_{s \in \mathbb{Z}^3} \tilde{v}_{\beta\gamma}(s) e^{i(k,s)}, \]

\[ \beta\gamma = 12, 23, 31, \alpha \neq \beta \neq \gamma \]

with

\[ (k, s) = \sum_{j=1}^{3} k^{(j)} s^{(j)}, \quad k = (k^{(1)}, k^{(2)}, k^{(3)}) \in \mathbb{R}^3, \quad s = (s^{(1)}, s^{(2)}, s^{(3)}) \in \mathbb{Z}^3. \]

For the two-particle Hamiltonians \( h_\alpha, \alpha = 1, 2, 3 \) we have:

\[ h_\alpha = h_\alpha^0 - v_\alpha, \]

where

\[ h_\alpha^0 = \mathcal{D}_{k_\beta} + \mathcal{D}_{k_\gamma}, \]

with \( \mathcal{D}_{k_\beta} = \hat{D}_{k_\beta} \otimes I, \mathcal{D}_{k_\gamma} = I \otimes \hat{D}_{k_\gamma} \) and

\[ (v_\alpha f)(k_\beta, k_\gamma) = (2\pi)^{-\frac{3}{2}} \int_{(\mathbb{T}^3)^2} v_\alpha \left( \frac{k_\beta - k_\gamma - k'_\beta + k'_\gamma}{2} \right) \delta(k_\beta + k_\gamma - k'_\beta - k'_\gamma) f(k'_\beta, k'_\gamma) d\beta d\gamma, \]

\[ f \in L_2((\mathbb{T}^3)^2). \]

3. Decomposition of the energy operators into von Neumann direct integrals. Quasimomentum and coordinate systems

Given \( m \in \mathbb{N} \), denote by \( \hat{U}_s^m, s \in \mathbb{Z}^3 \) the unitary operators on the Hilbert space \( \ell_2((\mathbb{Z}^3)^m) \) defined as:

\[ (\hat{U}_s^m f)(n_1, n_2, ..., n_m) = f(n_1 + s, n_2 + s, ..., n_m + s), \quad f \in \ell_2((\mathbb{Z}^3)^m). \]

We easily see that

\[ \hat{U}_{s+p}^m = \hat{U}_s^m \hat{U}_p^m, \quad s, p \in \mathbb{Z}^3. \]
that is, $\hat{U}^m_s, s \in \mathbb{Z}^3$ is a unitary representation of the abelian group $\mathbb{Z}^3$.

Via the Fourier transform $\mathcal{F}_m$ the unitary representation of $\mathbb{Z}^3$ in $\ell_2((\mathbb{Z}^3)^m)$ induces a representation of the group $\mathbb{Z}^3$ in the Hilbert space $L_2((\mathbb{S}^1)^m)$ by unitary (multiplication) operators $U^m_s = \mathcal{F}_m^{-1}\hat{U}^m_s\mathcal{F}_m, s \in \mathbb{Z}^3$ given by:

\begin{equation}
(U^m_s f)(k_1, k_2, \ldots, k_m) = \exp(-i(s, k_1 + k_2 + \ldots + k_m)) f(k_1, k_2, \ldots, k_m),
\end{equation}

\begin{equation}
f \in L_2((\mathbb{S}^1)^m).
\end{equation}

Decomposing the Hilbert space $L_2((\mathbb{S}^1)^m)$ into the direct integral

\[ L_2((\mathbb{S}^1)^m) = \int_{K \in \mathbb{T}^3} \oplus L_2(\mathbb{F}^m_K) dK, \]

where

\[ \mathbb{F}^m_K = \{(k_1, k_2, \ldots, k_m) \in (\mathbb{S}^1)^m : k_1 + k_2 + \ldots + k_m = K\}, \quad K \in \mathbb{T}^3, \]

we obtain the corresponding decomposition of the unitary representation $U^m_s, s \in \mathbb{Z}^3$ into the direct integral

\[ U^m_s = \int_{K \in \mathbb{T}^3} \oplus U_s(K) dK, \]

where

\[ U_s(K) = \exp(-i(s, K))I \quad \text{on} \quad L_2(\mathbb{F}^m_K) \]

and $I = I_{L_2(\mathbb{F}^m_K)}$ denotes the identity operator on the Hilbert space $L_2(\mathbb{F}^m_K)$.

The above Hamiltonians $\hat{H}$ and $\hat{h}_\alpha, \alpha = 1, 2, 3$ obviously commute with the groups of translations $\hat{U}^3_s$ and $\hat{U}^2_s, s \in \mathbb{Z}^3$, respectively, that is,

\[ \hat{U}^3_s \hat{H} = \hat{H} \hat{U}^3_s, \quad s \in \mathbb{Z}^3 \]

and

\[ \hat{U}^2_s \hat{h}_\alpha = \hat{h}_\alpha \hat{U}^2_s, \quad s \in \mathbb{Z}^3, \quad \alpha = 1, 2, 3. \]

Correspondingly, the Hamiltonians $H$ and $h_\alpha, \alpha = 1, 2, 3$ (in the momentum representation) commute with the groups $U^m_s, s \in \mathbb{Z}^3$ given by (3.1) for $m = 3$ and $m = 2$, respectively.

Hence, the operators $H$ and $h_\alpha, \alpha = 1, 2, 3$, can be decomposed into the direct integrals

\[ H = \int_{K \in \mathbb{T}^3} \oplus \hat{H}(K) dK \quad \text{and} \quad h_\alpha = \int_{k \in \mathbb{T}^3} \oplus \hat{h}_\alpha(k) dk, \quad \alpha = 1, 2, 3, \]

with respect to the decompositions

\[ L_2((\mathbb{S}^1)^3) = \int_{K \in \mathbb{T}^3} \oplus L_2(\mathbb{F}^3_K) dK \quad \text{and} \quad L_2((\mathbb{S}^1)^2) = \int_{k \in \mathbb{T}^3} \oplus L_2(\mathbb{F}^2_K) dk, \]

respectively.

For any permutation $\alpha \beta \gamma$ of 123 we set:

\[ l_{\beta \gamma} = \frac{m_{\alpha}}{m_{\beta} + m_{\gamma}}, \quad M \equiv \sum_{\alpha=1}^{3} m_{\alpha}, \quad l_{\alpha} = \frac{m_{\alpha}}{M}. \]
where the quantity $m_\alpha$ entered in Remark 2.

Given a cyclic permutation $\alpha \beta \gamma$ of 123 we introduce the mappings

$$\pi^{(3)}_\alpha : (\mathbb{T}^3)^3 \rightarrow (\mathbb{T}^3)^2, \quad \pi^{(3)}_\alpha ((k_\alpha, k_\beta, k_\gamma)) = (q_\alpha, p_\alpha)$$

and

$$\pi^{(2)}_\alpha : (\mathbb{T}^3)^2 \rightarrow \mathbb{T}^3, \quad \pi^{(2)}_\alpha ((k_\beta, k_\gamma)) = q_\alpha,$$

where

$$q_\alpha = l_\beta k_\beta - l_\gamma k_\gamma \quad \text{and} \quad p_\alpha = l_\alpha (k_\beta + k_\gamma) - (l_\beta + l_\gamma) k_\alpha.$$

Denote by $\pi^{(3)}_K, K \in \mathbb{T}^3$ resp. $\pi^{(2)}_k, k \in \mathbb{T}^3$ the restriction of $\pi^{(3)}_\alpha$ resp. $\pi^{(2)}_\alpha$ onto $\mathbb{F}_K^3 \subset (\mathbb{T}^3)^3$ resp. $\mathbb{F}_k^2 \subset (\mathbb{T}^3)^2$, that is,

$$\pi^{(3)}_K = \pi^{(3)}_\alpha|_{\mathbb{F}_K^3} \quad \text{and} \quad \pi^{(2)}_k = \pi^{(2)}_\alpha|_{\mathbb{F}_k^2}.$$

At this point it is useful to remark that

$$\mathbb{F}_K^3 = \{(k_\alpha, k_\beta, k_\gamma) \in (\mathbb{T}^3)^2 : k_\alpha + k_\beta + k_\gamma = K\} \quad K \in \mathbb{T}^3$$

and

$$\mathbb{F}_k^2 = \{(k_\beta, k_\gamma) \in (\mathbb{T}^3)^2 : k_\beta + k_\gamma = k\}, \quad k \in \mathbb{T}^3$$

are six and three-dimensional manifolds isomorphic to $(\mathbb{T}^3)^2$ and $\mathbb{T}^3$, respectively.

**Lemma 3.1.** The mappings $\pi^{(3)}_K, K \in \mathbb{T}^3$ and $\pi^{(2)}_k, k \in \mathbb{T}^3$ are bijective from $\mathbb{F}_K^3 \subset (\mathbb{T}^3)^3$ and $\mathbb{F}_k^2 \subset (\mathbb{T}^3)^2$ onto $(\mathbb{T}^3)^2$ and $\mathbb{T}^3$ with the inverse mappings given by

$$(\pi^{(3)}_K)^{-1}(q_\alpha, p_\alpha) = (l_\alpha K - p_\alpha, l_\beta K + l_\gamma p_\alpha + q_\alpha)$$

and

$$(\pi^{(2)}_k)^{-1}(q_\alpha) = (l_\gamma k + q_\alpha, l_\beta k - q_\alpha).$$

**Proof.** We obviously have that

$$(l_\alpha K - p_\alpha) + (l_\beta K + l_\gamma p_\alpha + q_\alpha) + (l_\gamma K + l_\beta p_\alpha - q_\alpha) = K$$

and

$$(l_\gamma k + q_\alpha) + (l_\beta k - q_\alpha) = k.$$

Therefore, the images of the mappings $(\pi^{(3)}_K)^{-1}$ and $(\pi^{(2)}_k)^{-1}$ are the subsets of $\mathbb{F}_K^3$ and $\mathbb{F}_k^2$, respectively.

Conversely, given

$$(k_\alpha, k_\beta, k_\gamma) \in \mathbb{F}_K^3 \subset (\mathbb{T}^3)^3 \quad \text{and} \quad (k_\beta, k_\gamma) \in \mathbb{F}_k^2 \subset (\mathbb{T}^3)^2$$

one computes that

$$(\pi^{(3)}_K)^{-1}(q_\alpha, p_\alpha) = (k_\alpha, k_\beta, k_\gamma) \quad \text{and} \quad (\pi^{(2)}_k)^{-1}(q_\alpha) = (k_\beta, k_\gamma),$$

where

$$q_\alpha = l_\beta k_\beta - l_\gamma k_\gamma \quad \text{and} \quad p_\alpha = l_\alpha (k_\beta + k_\gamma) - (l_\beta + l_\gamma) k_\alpha.$$
Let the operator $H(K), K \in \mathbb{T}^3$ act on the Hilbert space $L_2((\mathbb{T}^3)^2)$ as follows:

$$H(K) = H_0(K) - V_1 - V_2 - V_3.$$  

The operators $H_0(K)$ and $V_\alpha$ in the coordinates $(q_\alpha, p_\alpha)$ are defined by

$$(H_0(K)f)(q_\alpha, p_\alpha) = E_{\alpha\beta}(K; q_\alpha, p_\alpha)f(q_\alpha, p_\alpha), \quad f \in L_2((\mathbb{T}^3)^2),$$

$$(V_\alpha f)(q_\alpha, p_\alpha) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{T}^3} v_\alpha(q_\alpha - q'_\alpha)f(q'_\alpha, p_\alpha) dq'_\alpha, \quad f \in L_2((\mathbb{T}^3)^2),$$

where

$$E_{\alpha\beta}(K; q_\alpha, p_\alpha) = \varepsilon_\alpha(l_\alpha K - p_\alpha) + \varepsilon_\beta(l_\beta K + l_\gamma p_\alpha + q_\alpha) + \varepsilon_\gamma(l_\alpha K + l_\beta p_\alpha - q_\alpha).$$

Let the operators $h_\alpha(k), \alpha = 1, 2, 3, k \in \mathbb{T}^3$ acts on the Hilbert space $L_2(\mathbb{T}^3)$ as follows:

$$(h_\alpha^0(k)f)(q_\alpha) = E^{(\alpha)}_k(q_\alpha)f(q_\alpha), \quad f \in L_2(\mathbb{T}^3),$$

$$(v_\alpha f)(q_\alpha) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{T}^3} v_\alpha(q_\alpha - q'_\alpha)f(q'_\alpha) dq'_\alpha, \quad f \in L_2(\mathbb{T}^3)$$

and

$$(E^{(\alpha)}_k(q_\alpha) = \varepsilon_\beta(l_\gamma k + q_\alpha) + \varepsilon_\gamma(l_3 k - q_\alpha).$$

Let us consider the unitary operators

$$U_K : L_2(\mathbb{F}_K^3) \longrightarrow L_2((\mathbb{T}^3)^2), \quad U_K f = f \circ (\pi^{(3)}_K)^{-1}, \quad K \in \mathbb{T}^3,$$

and

$$u_k : L_2(\mathbb{F}_k^3) \rightarrow L_2(\mathbb{T}^3), \quad u_k g = g \circ (\pi^{(2)}_k)^{-1}, \quad k \in \mathbb{T}^3,$$

where $\pi^{(3)}_K$ and $\pi^{(2)}_k$ are defined by (3.2).

For the fiber operators $\tilde{H}(K)$ resp. $\tilde{h}_\alpha(k)$ the following equalities

$$H(K) = U_K \tilde{H}(K)U_K^{-1}, \quad h_\alpha(k) = u_k \tilde{h}_\alpha(k) u_k^{-1}, \quad \alpha = 1, 2, 3$$

hold.

All of our further calculations will be carried out in the “momentum representation” in a system of coordinates connected with the fixed center of inertia of the system of three particles. We order 1, 2, 3 by the conditions $1 \prec 2, 2 \prec 3$ and $3 \prec 1$. Sometimes instead of the coordinates $(q_\alpha, p_\alpha)$ (if it does not lead to any confusion we will write $(q, p)$ instead of $(q_\alpha, p_\alpha)$) it is convenient to choose some pair of the three variables $p_\alpha$. The connection between the various coordinates is given by the relations

$$(p_1 + p_2 + p_3 = 0, \pm q_\alpha = l_{\gamma\beta} p_\alpha + p_\beta, \quad l_{\gamma\beta} = \frac{m_\beta}{m_\beta + m_\gamma}, \quad (\alpha \neq \beta, \beta \neq \gamma, \gamma \neq \alpha),$$
where the plus sign corresponds to the case $\beta < \alpha$, the minus sign corresponds to the case $\alpha < \beta$. Expressions for the variables $q_\alpha$ in terms of $p_\alpha$ and $p_\beta$ can be written in the form $q_\alpha = d_{\alpha\beta}p_\alpha + e_{\alpha\beta}p_\beta$ and explicit formulas for the coefficients $d_{\alpha\beta}$ and $e_{\alpha\beta}$ are obtained by combining the latter equation with (3.5).

4. Statement of the main results

For each $K \in \mathbb{T}^3$ the minimum and the maximum taken over $(q, p)$ of the function $E_{\alpha\beta}(K; q, p)$ are independent of $\alpha$, $\beta = 1, 2, 3$. We set:

$$E_{\min}(K) \equiv \min_{q,p} E_{\alpha\beta}(K, q, p), \quad E_{\max}(K) \equiv \max_{q,p} E_{\alpha\beta}(K, q, p).$$

The main results of the paper are given in the following theorems, which will be proven in section 5, 7.

**Theorem 4.1.** Assume Hypothesis 2.1 and 2.3 Then for any $\alpha = 1, 2, 3$ and for all $k \in U_\delta(0)$, $\delta > 0$ sufficiently small, the operator $h_\alpha(k)$ has a finite number of eigenvalues outside of the essential spectrum $\sigma_{\text{ess}}(h_\alpha(k))$.

**Theorem 4.2.** Assume Hypothesis 2.1 and 2.3 For the essential spectrum $\sigma_{\text{ess}}(H(K))$ of $H(K)$ the following equality

$$\sigma_{\text{ess}}(H(K)) = \bigcup_{\alpha=1}^{3} \bigcup_{p \in \mathbb{T}^3} \sigma_q(h_\alpha((l_\beta + l_\gamma)K + p)) + \alpha(l_\alpha K - p) \cup \sigma_{\text{min}}(K), E_{\text{max}}(K)$$

holds, where $\sigma_q(h_\alpha(k))$ is the discrete spectrum of the operator $h_\alpha(k), k \in \mathbb{T}^3$.

**Theorem 4.3.** Assume Hypothesis 2.1 and 2.3 The essential spectrum $\sigma_{\text{ess}}(H(K))$ of $H(K)$ consists of the union of a finite number of bounded closed intervals (segments).

5. Spectral properties of the two-particle operator $h_\alpha(k)$

In this section we study the spectral properties of the two-particle discrete Schrödinger operator $h_\alpha(k), \alpha = 1, 2, 3$, $k \in \mathbb{T}^3$ defined by (3.3).

By the Weyl theorem the continuous spectrum $\sigma_{\text{cont}}(h_\alpha(k))$ of the operator $h_\alpha(k), k \in \mathbb{T}^3$ coincides with the spectrum $\sigma(h_{\alpha}^n(k))$ of $h_{\alpha}^n(k)$. More specifically,

$$\sigma_{\text{cont}}(h_\alpha(k)) = [E^{(\alpha)}_{\text{min}}(k), E^{(\alpha)}_{\text{max}}(k)],$$

where

$$E^{(\alpha)}_{\text{min}}(k) \equiv \min_{p \in \mathbb{T}^3} E^{(\alpha)}_k(p), \quad E^{(\alpha)}_{\text{max}}(k) \equiv \max_{p \in \mathbb{T}^3} E^{(\alpha)}_k(p)$$

and $E^{(\alpha)}_k(p)$ is defined by (3.4).

**Lemma 5.1.** Assume Hypothesis 2.1 Then the functions $\varepsilon_\alpha(p), \alpha = 1, 2, 3$ defined on $\mathbb{R}^3$ are even, real-analytic and the point $p = 0$ is its unique non-degenerate minimum in $\mathbb{T}^3$.

**Proof.** The conditions (i) and (ii) of Hypothesis 2.1 and the properties of Fourier transform implies that the function $\varepsilon_\alpha(p)$ is even and real-analytic.

By (i) of Hypothesis 2.1 the function $\varepsilon_\alpha(p)$ is represented as

$$\varepsilon_\alpha(p) = \sum_{s \in \mathbb{Z}^3} \hat{\varepsilon}_\alpha(s)e^{i(s \cdot p)} = \sum_{s \in \mathbb{Z}^3} \hat{\varepsilon}_\alpha(s)\cos(s^{(1)} p^{(1)})\cos(s^{(2)} p^{(2)})\cos(s^{(3)} p^{(3)}).$$
From the representation (5.2) we obtain that for the second-order partial derivatives of \( \varepsilon_\alpha(p) \) at the point \( p = 0 \) the equalities

\[
\frac{\partial^2 \varepsilon_\alpha}{\partial p^i \partial p^j}(0) = 0, \quad i \neq j,
\]

\[
\frac{\partial^2 \varepsilon_\alpha}{\partial p^i \partial p^j}(0) = \frac{1}{m_\alpha}, \quad i, j = 1, 2, 3,
\]

hold, where the number \( m_\alpha > 0 \) is defined in Remark 2.2. Hence the Taylor series expansion of \( \varepsilon_\alpha(p) \) at the point \( p = 0 \) gives us

\[
\varepsilon_\alpha(p) = \varepsilon_\alpha(0) + \frac{p^2}{2m_\alpha} + \tilde{\varepsilon}_\alpha(p), \quad \varepsilon_\alpha(p) = O(|p|^4) \quad \text{as} \quad p \to 0.
\]

The equality (5.4) yields that the point \( p = 0 \) is a non-degenerated minimum of the function \( \varepsilon_\alpha(p) \).

Therefore according to (5.2) we get

\[
\varepsilon_\alpha(p) - \varepsilon_\alpha(0) = - \sum_{s \in \mathbb{Z}^3} \tilde{\varepsilon}_\alpha(s)[1 - \cos(s(1)p^{(1)}) \cos(s(2)p^{(2)}) \cos(s(3)p^{(3)})],
\]

The condition (iii) of Hypothesis 2.1 and (5.5) implies that \( p = 0 \) is the unique non-degenerated minimum of the function \( \varepsilon_\alpha(p) \) in \( \mathbb{T}^3 \).

Lemma 5.2. There exist an analytical function \( p_\alpha(k) \) defined on \( \delta \)-neighborhood \( U_\delta(0) \) of the point \( p = 0 \) such that for any \( k \in U_\delta(0) \) the point \( p_\alpha(k) \) is an unique non-degenerate minimum of the function \( E_k^{(\alpha)}(p) \).

Proof. Since the function \( \varepsilon_\alpha(p), \alpha = 1, 2, 3 \) has a unique non degenerate minimum at the point \( p = 0 \), the gradient \( \nabla \varepsilon_\alpha(p) \) is equal to zero at the point \( p = 0 \), i.e.

\[
\nabla \varepsilon_\alpha(p)|_{p=0} = (\frac{\partial \varepsilon_\alpha(p)}{\partial p^{(1)}}, \frac{\partial \varepsilon_\alpha(p)}{\partial p^{(2)}}, \frac{\partial \varepsilon_\alpha(p)}{\partial p^{(3)}})|_{p=0} = 0.
\]

Therefore by (5.3) the matrix

\[
B_\alpha(p)|_{p=0} = \left(\frac{\partial^2 \varepsilon_\alpha(0)}{\partial p^i \partial p^j}\right)|_{i,j=1,2,3} = m_\alpha^{-1}I_3, \quad \alpha = 1, 2, 3,
\]

is positive, where \( I_3 \) is the \( 3 \times 3 \) unit matrix.

From here it follows that \( \nabla E_0^{(\alpha)}(0) = 0 \) and the matrix \( B(0) = \left(\frac{\partial^2 E_0^{(\alpha)}(0)}{\partial p^i \partial p^j}\right)|_{i,j=1} = (m_\beta^{-1} + m_\gamma^{-1})I_3 \) is positive definite. Now we will apply the implicit function theorem to the equation \( \nabla E_k^{(\alpha)}(p) = 0, \quad k, p \in \mathbb{T}^3 \). Then there exists a vector function \( p_\alpha(k) \) defined and analytic in some \( \delta \)-neighborhood \( U_\delta(0) \) of the point \( k = 0 \), and for any \( k \in U_\delta(0) \) the equality \( \nabla E_k^{(\alpha)}(p_\alpha(k)) = 0 \) holds.

Denote by \( B(k) \) the matrix of the second order partial derivatives of the function \( E_k^{(\alpha)}(p) \) at the point \( p_\alpha(k) \).

Since the matrix \( B(0) \) is positive and \( B(k) \) is continuous in \( U_\delta(0) \), we conclude that for any \( k \in U_\delta(0) \) the matrix \( B(k) \) is positive definite. Thus \( p_\alpha(k), \quad k \in U_\delta(0) \) is the unique non-degenerated minimum point of \( E_k^{(\alpha)}(p) \).
Let $\mathbb{C}$ be the complex plane. Denote by $r_0^\alpha(k, z)$ the resolvent of the operator $h_0^\alpha(k)$, $\alpha = 1, 2, 3$. For any $k \in \mathbb{T}^3$ denote by $\Delta_\alpha(k, z)$ the Fredholm determinant of the operator
\begin{equation}
I - v_\alpha r_0^\alpha(k, z), \quad z \in \mathbb{C} \setminus [E_{\min}^{(\alpha)}(k), E_{\max}^{(\alpha)}(k)],
\end{equation}
where $I$ is the identity operator on $L_2(\mathbb{T}^3)$.

Lemma 5.3. For any $k \in \mathbb{T}^3$ the number $z \in \mathbb{C} \setminus [E_{\min}^{(\alpha)}(k), E_{\max}^{(\alpha)}(k)]$ is an eigenvalue of the operator $h_\alpha(k)$, $\alpha = 1, 2, 3$ if and only if
\begin{equation}
\Delta_\alpha(k, z) = 0.
\end{equation}

Proof. By the Birman-Schwinger principle the number $z \in \mathbb{C} \setminus [E_{\min}^{(\alpha)}(k), E_{\max}^{(\alpha)}(k)]$ is an eigenvalue of the operator $h_\alpha(k)$, $\alpha = 1, 2, 3$, $k \in \mathbb{T}^3$ if and only if the equation
\begin{equation}
g = v_\alpha r_0^\alpha(k, z)g
\end{equation}
has a nontrivial solution $\hat{g} \in L_2(\mathbb{T}^3)$.

By Fredholm’s theorem the equation (5.7) has nontrivial solutions if and only if
\begin{equation}
\Delta_\alpha(k, z) = 0.
\end{equation}

Now we obtain a generalization of the Birman-Schwinger principle for the discrete two-particle Schrödinger operators and hence we prove Theorem 4.1.

Let $N(k, z)$ denote the number of eigenvalues of the operator $h_\alpha(k)$, $k \in \mathbb{T}^3$ below $z \leq E_{\min}^{(\alpha)}(k)$. For any bounded self-adjoint operator $A$ acting in the Hilbert space $\mathcal{H}$ not having any essential spectrum on the right of the point $z$ we denote by $\mathcal{H}_A(z)$ the subspace such that $(Af, f) > z(f, f)$ for any $f \in \mathcal{H}_A(z)$ and set $n(z, A) = \sup_{\mathcal{H}_A(z)} \dim \mathcal{H}_A(z)$.

For any $k \in U_3(0)$ and $z \leq E_{\min}^{(\alpha)}(k)$ we define the integral operator $G_\alpha(k, z)$ with the kernel
\begin{equation}
G_\alpha(k, z; p, q) = (2\pi)^{-\frac{3}{2}} \frac{v_\alpha(p - q)}{(E_k^{(\alpha)}(p) - z)^{\frac{3}{2}} (E_k^{(\alpha)}(q) - z)^{\frac{3}{2}}}.
\end{equation}

The following theorem is a realization of the well known Birman-Schwinger principle for the two-particle Schrödinger operators on lattice $\mathbb{Z}^3$.

Theorem 5.4. The operator $G_\alpha(k, z)$, $k \in U_3(0)$ acts in $L_2(\mathbb{T}^3)$, is positive, belongs to the Hilbert-Schmidt class $\Sigma_2$ and is continuous in $z$ up to $z = E_{\min}^{(\alpha)}(k)$. In addition the equality
\begin{equation}
N(k, z) = n(1, G_\alpha(k, z)), \quad z \leq E_{\min}^{(\alpha)}(k)
\end{equation}
holds.

Proof. We note that for all $z < E_{\min}^{(\alpha)}(k)$ the equality $G_\alpha(k, z) = (r_0^\alpha(k, z))^{\frac{3}{2}} v_\alpha(r_0^\alpha(k, z))^\frac{3}{2}$ holds.

The quantity $N(k, z)$ coincides with $n(1, G_\alpha(k, z))$ by the Birman-Schwinger principle, i.e.,
\begin{equation}
N(k, z) = n(1, G_\alpha(k, z)), \quad z < E_{\min}^{(\alpha)}(k).
\end{equation}
By Hypothesis 2.3 the function \( v_\alpha(p) \) is continuous on \( \mathbb{T}^3 \). Since for any \( k \in U_\delta(0) \) the function \( E^{(\alpha)}_k(q) \) has a unique non-degenerate minimum at the point \( p = p_\alpha(k) \) the kernel \( G_\alpha(k, E^{(\alpha)}_\min(k); p, q) \) is square integrable on \( (\mathbb{T}^3)^2 \), that is, the operator \( G_\alpha(k, E^{(\alpha)}_\min(k)) \) belongs to the Hilbert-Schmidt class \( \Sigma_2 \).

Then the dominated convergence theorem implies that \( G_\alpha(k, z) \) is continuous from the left up to \( E^{(\alpha)}_\min(k) \).

Let us show \( N(k, E^{(\alpha)}_\min(k)) = n(1, G_\alpha(k, E^{(\alpha)}_\min(k))) \). Since \( G_\alpha(k, E^{(\alpha)}_\min(k)) \) is a compact operator the number \( n(1 - \gamma, G_\alpha(k, E^{(\alpha)}_\min(k))) \) is finite for any \( \gamma < 1 \).

Then according to the Weyl inequality \( n(\lambda_1 + \lambda_2, A_1 + A_2) \leq n(\lambda_1, A_1) + n(\lambda_2, A_2) \) for all \( z < E^{(\alpha)}_\min(k) \) and \( \gamma \in (0, 1) \) we have
\[
N(k, z) = n(1, G_\alpha(k, z)) = n(1 - \gamma, G_\alpha(k, z)) + n(\gamma, G_\alpha(k, z) - G_\alpha(k, E^{(\alpha)}_\min(k))).
\]

Since \( G_\alpha(k, z) \) is continuous from the left up to \( E^{(\alpha)}_\min(k) \) we obtain
\[
\lim_{z \to E^{(\alpha)}_\min(k)^-} N(k, z) = N(k, E^{(\alpha)}_\min(k)) \leq n(1 - \gamma, G_\alpha(k, E^{(\alpha)}_\min(k))) \text{ for all } \gamma \in (0, 1)
\]
and so
\[
N(k, E^{(\alpha)}_\min(k)) \leq \lim_{\gamma \to 0} n(1 - \gamma, G_\alpha(k, E^{(\alpha)}_\min(k))) < \infty.
\]

Since \( N(k, E^{(\alpha)}_\min(k)) \) is finite we have \( N(k, E^{(\alpha)}_\min(k) - \gamma) = N(k, E^{(\alpha)}_\min(k)) \) for all small enough \( \gamma \in (0, 1) \). Therefore using the continuity of \( N(k, z) \) from the left we derive the equality
\[
\lim_{\gamma \to 0} n(1, G_\alpha(k, E^{(\alpha)}_\min(k) - \gamma)) = \lim_{\gamma \to 0} N(k, E^{(\alpha)}_\min(k) - \gamma) = N(k, E^{(\alpha)}_\min(k)).
\]

\[\square\]

**Proof of Theorem 5.1** Since \( v_\alpha \) is positive we conclude that \( h_\alpha(k) \) has no eigenvalue lying on the r.h.s of \( \sigma_{cont}(h_\alpha(k)) \). Then the finiteness of the discrete spectrum \( \sigma_d(h_\alpha(k)) \) of \( h_\alpha(k) \) follows from the compactness of \( G_\alpha(k, E^{(\alpha)}_\min(k)) \) and Lemma 5.4. \[\square\]

6. Spectrum of ”channel operator”

In this section we introduce a ”channel operator” and prove that its spectrum consists of only a finitely many segments.

The ”channel operator” \( H_\alpha(K), K \in \mathbb{T}^3 \) acts in the Hilbert space \( L_2((\mathbb{T}^3)^2) \) as

\[
H_\alpha(K) = H_0(K) - V_\alpha.
\]

The decomposition of the space \( L_2((\mathbb{T}^3)^2) \) into the direct integral

\[
L_2((\mathbb{T}^3)^2) = \int_{p \in \mathbb{T}^3} \oplus L_2(\mathbb{T}^3) dp
\]
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yields for the operator $H_\alpha(K)$ the decomposition into the direct integral

$$H_\alpha(K) = \int_{p \in T^3} \oplus H_\alpha(K, p) dp.$$

The fiber operator $H_\alpha(K, p)$ acts in the Hilbert space $L^2(T^3)$ and has the form

$$(6.1) \quad H_\alpha(K, p) = h_\alpha((l_\beta + l_\gamma)K + p) + \varepsilon_\alpha(l_\alpha K - p)I,$$

where $h_\alpha(k)$ is the two-particle operator defined by (3.3).

Set

$$\Delta_\alpha(K, p, z) = \Delta_\alpha((l_\beta + l_\gamma)K + p, z - \varepsilon_\alpha(l_\alpha K - p)),$$

$$E^{(\alpha)}_{\min}(K, p) = E^{(\alpha)}_{\min}((l_\beta + l_\gamma)K + p) + \varepsilon_\alpha(l_\alpha K - p),$$

$$E^{(\alpha)}_{\max}(K, p) = E^{(\alpha)}_{\max}((l_\beta + l_\gamma)K + p) + \varepsilon_\alpha(l_\alpha K - p),$$

where $\Delta_\alpha(k, z)$ is the Fredholm determinant of the operator $I - \nu_\alpha r^{(0)}_\alpha(k, z)$ and $E^{(\alpha)}_{\min}(k)$ and $E^{(\alpha)}_{\max}(k)$ are defined in (5.1).

**Lemma 6.1.** For any $p \in T^3$ the number $z \in \mathbb{C} \setminus [E^{(\alpha)}_{\min}(K, p), E^{(\alpha)}_{\max}(K, p)]$ is an eigenvalue of the operator $H_\alpha(K, p)$ if and only if

$$\Delta_\alpha(K, p, z) = 0, \quad \alpha = 1, 2, 3.$$

The proof of Lemma similarly to that Lemma5.3.

The representation of the operator $H_\alpha(K, p)$ implies the equality

$$\sigma(H_\alpha(K, p)) = \{\sigma_d(h_\alpha((l_\beta + l_\gamma)K + p)) + \varepsilon_\alpha(l_\alpha K - p)\}$$

$$\cup [E^{(\alpha)}_{\min}(K, p), E^{(\alpha)}_{\max}(K, p)],$$

where $\sigma_d(h_\alpha(k))$ is the discrete spectrum of the operator $h_\alpha(k)$. Set

$$\sigma_{\text{two}}(H_\alpha(K)) = \bigcup_{p \in T^3} \{\sigma_d(h_\alpha((l_\beta + l_\gamma)K + p)) + \varepsilon_\alpha(l_\alpha K - p)\}, \quad \alpha = 1, 2, 3.$$

The theorem (see e.g., [19]) on the spectrum of decomposable operators and above obtained structure for the spectrum of $H_\alpha(K, p)$ gives

**Lemma 6.2.** The equality holds

$$\sigma(H_\alpha(K)) = \sigma_{\text{two}}(H_\alpha(K)) \cup [E^{(\alpha)}_{\min}(K), E^{(\alpha)}_{\max}(K)].$$

**Theorem 6.3.** The set

$$\hat{\sigma}_\alpha = \sigma_{\text{two}}(H_\alpha(K)) \setminus [E^{(\alpha)}_{\min}(K), E^{(\alpha)}_{\max}(K)]$$

consists of an union of a finite number of segments.
Proof. Let the set \( \sigma_\alpha \) be represented as the union of a finite or infinite number of disjoint segments \( S_\omega \) represented in the form

\[
\sigma_\alpha = \bigcup_{\omega \in W} S_\omega,
\]

where \( W \) is a subset of the real axis.

Denote by \( d_\omega \) the distance between of the segments \( S_\omega \) and \([E_{\text{min}}(K), E_{\text{max}}(K)]\).

Lemma 6.4. Let \( d_\omega > 0 \) for some \( \omega \in W \). Then for any \( p \in \mathbb{T}^3 \) the operator \( H_\alpha(K, p) \) has eigenvalues in \( S_\omega \).

Proof. Let \( d_\omega > 0 \) for some \( \omega \in W \). Denote by \( G_\omega \) the set of all \( p \in \mathbb{T}^3 \) such that the operator \( H_\alpha(K, p) \) has an eigenvalue lying in \( S_\omega \). We show that \( G_\omega = \mathbb{T}^3 \). Let \( p_0 \in G_\omega \) then by Lemma 6.2 and 6 there is \( z_0 \in S_\omega \) such that \( \Delta_\alpha(K, p_0, z_0) = 0 \).

For the following considerations in this proof we shall consider in \( \mathbb{T}^3 = (−\pi, \pi]^3 \) as equipped with the topology of the corresponding 3-dimensional torus, and vice versa. For any \( p \in \mathbb{T}^3 \) the function \( \Delta_\alpha(K, p, z) \) is analytic by \( z \) in some region containing \( S_\omega \) and nonzero. Therefore there is natural number \( n \) such that the inequality \( \frac{\partial^{2n}}{\partial z^{2n}} \Delta_\alpha(K, p_0, z_0) \neq 0 \) holds.

By the implicit function theorem there exist a neighborhood \( U(p_0) \) of \( p_0 \) and a continuous function \( z(p) \in S_\omega \) defined on \( U(p_0) \) such that the identity \( \Delta_\alpha(K, p, z(p)) = 0 \) is valid. According to Lemma 6 the number \( z(p) \in S_\omega \) is an eigenvalue of \( H_\alpha(K, p) \) for \( p \in U(p_0) \subset G_\omega \). This means that the set \( G_\omega \) is open.

Now we prove that \( G_\omega \) is a closed set.

Indeed, let \( \{p_n\} \subset G_\omega \) converge to \( p_0 \in \mathbb{T}^3 \) and let \( \{z(p_n)\} \subset S_\omega \) be eigenvalues of \( H_\alpha(K, p_n) \). Let \( z_0 \in S_\omega \) be a limit point of \( \{z(p_n)\} \).

The function \( \Delta_\alpha(K, p, z) \) is continuous in \( (p, z) \in \mathbb{T}^3 \times S_\omega \). Therefore

\[
0 = \lim_{n \to \infty} \Delta_\alpha(K, p_n, z(p_n)) = \Delta_\alpha(K, p_0, z_0)
\]

and hence \( p_0 \in G_\omega \). So the set \( G_\omega \) is closed. Since \( G_\omega \) is an open and closed set we have \( G_\omega = \mathbb{T}^3 \).

Now we prove that the set \( W \) is finite. Assume in fact, ad absurdum that the set \( W \) is infinite, i.e., for infinitely many elements \( \omega \in W \) one has \( d_\omega > 0 \). Then by Lemma 6.3 for any \( p \in \mathbb{T}^3 \) the operator \( H_\alpha(K, p) \) has an eigenvalue in \( S_\omega \), \( \omega \in W \).

Therefore by the equality \( \sigma(H_\alpha(K, p)) = \sigma_\alpha(h_\alpha((l_3 + l_\gamma)K + p)) + \varepsilon_\alpha(l_\gamma(K − p)) \) for any \( p \in \mathbb{T}^3 \) the set \( \sigma_\alpha(h_\alpha(p)) \) is infinite. By Theorem 4.3 for all \( k \in U_\delta(0) \) the operator \( h_\alpha(k) \) has finitely many eigenvalues. This is in contradiction with our assumption \( \Box \)

7. Essential spectrum of the discrete three-particle Schrödinger operator \( H(K) \)

In this section we prove Theorem 4.2 and 4.3 using the Faddeev type system of integral equations.

Proof of Theorem 4.2 Set

\[
\Sigma(K) = \bigcup_{\alpha=1}^3 \sigma_{\text{two}}(H_\alpha(K)) \cup [E_{\text{min}}(K), E_{\text{max}}(K)].
\]
We first show that \( \Sigma(K) \subset \sigma_{\text{ess}}(H(K)) \). Let \( z_0 \in \Sigma(K) \) be arbitrary point. We construct an orthogonal sequence of functions \( \{ f_n \}_n^{\infty} \) such that
\[
\|(H(K) - z_0)f_n\| \to 0 \quad \text{as} \quad n \to \infty.
\]

For any \( z_0 \in [E_{\text{min}}(K), E_{\text{max}}(K)] \) there exists \((q_0, p_0) \in (\mathbb{T}^3)^2 \) such that \( z_0 = E_{\alpha\beta}(K; q_0, p_0) \).

For any \( n \in \mathbb{N} \) and \( p, q \in \mathbb{T}^3 \) we introduce the notations:
\[
(7.1) \quad U_n := U_n(p) = \{ x \in \mathbb{T}^3 : \frac{1}{n} < |x - p| < \frac{1}{n} \} \quad \text{and} \quad W_n := W_n(q, p) = U_n(q) \times U_n(p).
\]

Let
\[
f_n(q, p) = \begin{cases} \sqrt{\mu(W_n)} & \text{if} \quad (q, p) \in W_n(q_0, p_0) \\ 0 & \text{if} \quad (q, p) \notin W_n(q_0, p_0), \end{cases}
\]

where \( \mu(W_n) \) is the Lebesgue measure of \( W_n(q_0, p_0) \).

For any \( n \neq m \) the equality \( W_n \cap W_m = \emptyset \) holds, hence \( \{ f_n \}_n^{\infty} \) is an orthogonal system.

The norm \( \|(H(K) - z_0)f_n\| \) is estimated by
\[
(7.2) \quad \|(H(K) - z_0)f_n\|^{2} \leq 2 \left( \|(H_0(K) - z_0)f_n\|^{2} + \sum_{\alpha=1}^{3} \|V_{\alpha}f_n\|^{2} \right).
\]

We shall prove that each item on the r.h.s. of (7.2) tends to zero as \( n \to \infty \).

Since \( E_{\alpha\beta}(K; q, p) \) is continuous and \( \sup_{(q, p) \in W_n} |p - q| \to 0 \) as \( n \to \infty \), we have
\[
\|(H_0(K) - z_0)f_n\|^{2} \leq \sup_{(q, p) \in W_n} |E_{\alpha\beta}(K; q, p) - z_0|^{2} \to 0 \quad \text{as} \quad n \to \infty.
\]

By the Schwarz inequality and the absolutely continuity of the Lebesgue integral we have
\[
\int_{(\mathbb{T}^3)^2} |(V_{\alpha} f_n)(q, p)|^{2} dq \, dp \leq (2\pi)^{-3} \int_{\mathbb{T}^3} \int_{U_n} |v_{\alpha}(t - p)|^{2} dt \, dp \to 0 \quad \text{as} \quad n \to \infty.
\]

By Weyl’s criterion \( z_0 \in \sigma_{\text{ess}}(H(K)) \).

Since \( z_0 \in [E_{\text{min}}(K), E_{\text{max}}(K)] \) is arbitrary, we have \([E_{\text{min}}(K), E_{\text{max}}(K)] \subset \sigma_{\text{ess}}(H(K)) \).

Let us show that \( \sigma_{\text{two}}(H_{\alpha}(K)) \subset \sigma_{\text{ess}}(H(K)) \). Let \( z_0 \in \sigma_{\text{two}}(H_{\alpha}(K)) \). By Lemma 6 and Lemma 6.2 there exists \( p_0 \in \mathbb{T}^3 \) such that \( \Delta_{\alpha}(K, p_0; z_0) = 0 \).

By the implicit function theorem there are neighborhoods \( U(p_0) \subset \mathbb{T}^3 \) and \( V(z_0) \subset \mathbb{R}^l \) of \( p_0 \) resp. \( z_0 \) and continuous function \( z : U(p_0) \to V(z_0) \), satisfying the condition \( \Delta(K, p, z(p)) = 0 \) and \( z(p) \in V(z_0) \). It is clear \( z(p) \) is an eigenvalue of the operator \( H_{\alpha}(K, p) \) for \( p \in U(p_0) \).

Let for \( p \in U(p_0) \) the function \( f_p(q) \) be an eigenfunction of \( H_{\alpha}(K, p) \) corresponding to the eigenvalue \( z(p) \in V(z_0) \), that is,
\[
H_{\alpha}(K, p)f_p(q) = z(p)f_p(q), \quad p \in U(p_0).
\]
Let \( \chi_{U_n} (p) \) be the characteristic function of the set \( U_n'(p_0) = U(p_0) \cap U_n(p_0) \), where \( U_n(p_0) \) is defined by (7.1).

Define the function \( f(q, p) \) on \((\mathbb{T}^3)^2\) by

\[
    f(q, p) = \begin{cases} 
        f_p(q), & p \in U(p_0), \ q \in \mathbb{T}^3 \\
        0, & p \in \mathbb{T}^3 \setminus U(p_0), \ q \in \mathbb{T}^3
    \end{cases}
\]

It is obvious that \( f \in L_2((\mathbb{T}^3)^2) \). Let \( f_n(q, p) = C_n \chi_{U_n} (p) f(q, p) \), where

\[
    C_n = \left( \int_{\mathbb{T}^3} |\chi_{U_n}(p) f(q, p)|^2 \, dp \, dq \right)^{-\frac{1}{2}} > 0
\]

for a sufficiently large \( n \in \mathbb{N} \). Then \( \|f_n\| = 1 \).

Since

\[
    ||(H - z_0)f_n|| \leq ||(H_\alpha(K) - z_0)f_n|| + ||V_\beta f_n|| + ||V_\gamma f_n||,
\]

we will show that each term on the r.h.s. of (7.3) tends to zero as \( n \to \infty \). Since

\[
    \| (H_\alpha(K) - z_0)f_n \| \leq \sup_{p \in U_\alpha'(p_0)} |z(p) - z_0|^2
\]

and \( z(p) \) is continuous in \( U(p_0) \), we obtain that \( ||(H_\alpha(K) - z_0)f_n|| \to 0 \) as \( n \to \infty \).

Using the Schwarz inequality and applying the absolutely continuity of the Lebesgue integral we have

\[
    ||V_\beta f_n|| \leq (2\pi)^{-3} \int_{\mathbb{T}^3} \int_{U_n} |v_\beta(p - p')|^2 \, dp \, dp' \to 0 \quad \text{as} \quad n \to \infty.
\]

Thus \( z_0 \in \sigma_{ess}(H(K)) \). Since \( z_0 \in \sigma_{two}(H_\alpha(K)) \) is arbitrary, we have \( \sigma_{two}(H_\alpha(K)) \subset \sigma_{ess}(H) \).

So we proved the inclusion \( \Sigma(K) \subset \sigma_{ess}(H(K)) \).

Now we prove the following inclusion \( \sigma_{ess}(H(K)) \subset \Sigma(K) \).

Let \( R_\alpha(K, z) \) resp. \( R_0(K, z) \) be the resolvents of the operators \( H_\alpha(K) \) resp. \( H_0(K) \).

Denote by \( v_\alpha^\frac{1}{2} \) the positive square root of the integral operator \( v_\alpha \) and by \( v_\alpha^\frac{1}{2} (p - p') \) the kernel of the integral operator \( v_\alpha^\frac{1}{2} \).

Let us consider the operator acting in \( L_2((\mathbb{T}^3)^2) \) as

\[
    (V_\alpha^\frac{1}{2} f)(q_\alpha, p_\alpha) = (v_\alpha^\frac{1}{2} \otimes I) f(q_\alpha, p_\alpha),
\]

where \( \otimes \) denotes the tensor product of operators.

Let \( W_\alpha(K, z), \alpha = 1, 2, 3 \) be the operators on \( L_2((\mathbb{T}^3)^2) \) defined as

\[
    W_\alpha(K, z) = I + V_\alpha^\frac{1}{2} R_\alpha(K, z) V_\alpha^\frac{1}{2},
\]

where \( I \) is the identity operator on \( L_2((\mathbb{T}^3)^2) \). One can check that

\[
    W_\alpha(K, z) = (I - V_\alpha^\frac{1}{2} R_0(K, z) V_\alpha^\frac{1}{2})^{-1}.
\]

Denote by \( L_2(\mathbb{T}^3)^2 \) the space of vector functions \( w \), with components \( w_\alpha \in L_2((\mathbb{T}^3)^2), \alpha = 1, 2, 3 \).
Let $T(K, z)$, $z \in \mathbb{C} \setminus \Sigma(K)$ be the operator on $L^2_\mathbb{T}((\mathbb{T}^3)^2)$ with the entries
\begin{align*}
T_{\alpha\alpha}(K, z) &= 0, \\
T_{\alpha\beta}(K, z) &= W_{\alpha}(K, z)V_{\alpha}^\frac{1}{2}R_0(K, z)V_{\beta}^\frac{1}{2}, \quad \alpha \neq \beta, \alpha, \beta = 1, 2, 3.
\end{align*}

**Lemma 7.1.** For any $z \in \mathbb{C} \setminus \Sigma(K)$ the operator $T(K, z)$ is an Hilbert-Schmidt operator.

**Proof.** Recall that the kernel function $v_{\alpha}^\frac{1}{2}(p)$ of $V_{\alpha}^\frac{1}{2}$ is the inverse Fourier transform of the function $v_{\beta}^\frac{1}{2}(s)$ and belongs to $L^2_\mathbb{T}((\mathbb{T}^3)^2)$. Then one can check that for any $z \in \mathbb{C} \setminus \Sigma(K)$ the operator $V_{\alpha}^\frac{1}{2}R_0(K, z)V_{\beta}^\frac{1}{2}$, $\alpha \neq \beta$ belongs to the Hilbert-Schmidt class $\Sigma_2$. Since for any $z \in \mathbb{C} \setminus \Sigma(K)$ the operator $W_{\alpha}(K, z)$ is bounded, the operator $T_{\alpha\beta}(K, z)$ also belongs to $\Sigma_2$. \qed

Denote by $R(K, z) = (H(K) - zI)^{-1}$ the resolvent of the operator $H(K)$. We consider the well known resolvent equation
\begin{equation}
R(K, z) = R_0(K, z) + R_0(K, z)\sum_{\alpha=1}^{3} V_{\alpha}R(K, z). \tag{7.4}
\end{equation}

Multiplying (7.4) from the left side to $V_{\alpha}^\frac{1}{2}$ and setting $R_{\alpha}(K, z) \equiv V_{\alpha}^\frac{1}{2}R(K, z)$ we get the equation
\begin{equation}
R_{\alpha}(K, z) = V_{\alpha}^\frac{1}{2}R_0(K, z) + V_{\alpha}^\frac{1}{2}R_0(K, z)\sum_{\beta=1}^{3} V_{\beta}^\frac{1}{2}R_{\beta}(K, z), \tag{7.5}
\end{equation}
i.e., the following system of three equations
\begin{equation}
(I - V_{\alpha}^\frac{1}{2}R_0(K, z)V_{\alpha}^\frac{1}{2})R_{\alpha}(K, z) = V_{\alpha}^\frac{1}{2}R_0(K, z) + \sum_{\beta=1, \beta \neq \alpha}^{3} V_{\alpha}^\frac{1}{2}R_0(K, z)V_{\beta}^\frac{1}{2}R_{\beta}(K, z). \tag{7.6}
\end{equation}

Multiplying the equality (7.5) from the left by the operator
\begin{equation}
W_{\alpha}(K, z) = (I - V_{\alpha}^\frac{1}{2}R_0(K, z)V_{\alpha}^\frac{1}{2})^{-1}
\end{equation}
we get the Faddeev type equation
\begin{equation}
R(K, z) = R_0(K, z) + R_0(K, z)T(K, z)R(K, z), \tag{7.7}
\end{equation}
where $R(K, z) = (R_1(K, z), R_2(K, z), R_3(K, z))$ and
\begin{align*}
R_0(K, z) &= (W_1(K, z)V_1^\frac{1}{2}R_0(K, z), W_2(K, z)V_2^\frac{1}{2}R_0(K, z), W_3(K, z)V_3^\frac{1}{2}R_0(K, z))
\end{align*}
are vector operators.

From (7.4) we have the following representation for the resolvent
\begin{equation}
R(K, z) = R_0(K, z) + R_0(K, z)\sum_{\alpha=1}^{3} V_{\alpha}^\frac{1}{2}R_{\alpha}(K, z). \tag{7.7}
\end{equation}

Let $I$ be the identity operator in $L^2_\mathbb{T}((\mathbb{T}^3)^2)$. The operator $T(K, z)$ is a compact operator-valued function on $\mathbb{C} \setminus \Sigma(K)$ and $I - T(K, z)$ is invertible if $z$ is real and either
very negative or very positive. The analytic Fredholm theorem (see, e.g., Theorem VI.14 in [19]) implies that there is a discrete set $S \subset \mathbb{C} \setminus \Sigma(K)$ so that $(I - T(K, z))^{-1}$ exists and is analytic in $\mathbb{C} \setminus (\Sigma(K) \cap S)$ and meromorphic in $\mathbb{C} \setminus \Sigma(K)$ with finite rank residues. Thus $(I - T(K, z))^{-1} R_0(K, z) \equiv F(K, z)$ is analytic in $\mathbb{C} \setminus (\Sigma(K) \cup S)$ with finite rank residues at the points of $S$.

Let $z \notin S$, $\text{Im } z \neq 0$, then by (7.6), (7.7) we have $F(K, z) = R(K, z)$. In particular,

$$R(K, z)(H(K) - zI) = (R_0(K, z) + R_0(K, z) \sum_{\alpha=1}^{3} V_{\alpha}^2 R_\alpha(K, z))(H(K) - zI) = I.$$

By analytic continuation, this holds for any $z \notin \Sigma(K) \cup S$. We conclude that, for any such $z$, $H(K) - zI$ has a bounded inverse. Thus $\sigma(H(K)) \setminus \Sigma(K)$ consists of isolated points and only the frontier points of $\Sigma(K)$ are possible their limit points. Finally, since $R(K, z)$ has finite rank residues at any point $z \in S$, we conclude that $\sigma(H(K)) \setminus \Sigma(K)$ belongs to the discrete spectrum $\sigma_d(H(K))$ of $H(K)$.

**Proof of Theorem 4.3** Theorem 4.3 follows immediately from Theorem 6.3 and Theorem 4.2.

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