Stationary BGK Models for Chemically Reacting Gas in a Slab

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Abstract
We study the boundary value problem of two stationary BGK-type models—the BGK model for fast chemical reaction and the BGK model for slow chemical reaction—and provide a unified argument to establish the existence and uniqueness of stationary flows of reactive BGK models in a slab. For both models, the main difficulty arise in the uniform control of the auxiliary parameters from above and below, since, unlike the BGK models for non-reactive gases, the auxiliary parameters for the reactive BGK models are defined through highly nonlinear relations. To overcome this difficulty, we introduce several nonlinear functionals that capture essential structures of such nonlinear relations such as the monotonicity in specific variables, that enable one to derive necessary estimates for the auxiliary parameters.

Keywords Kinetic theory of gases · BGK model · Boltzmann equation · Chemically reacting gases · Gas mixtures · Stationary problems

1 introduction

The classical BGK model [8] describes the relaxation process of the Boltzmann equation in a simpler setting. Due to its reliable performances in reproducing qualitative features of the Boltzmann equation in a numerically amenable way, the BGK model has been popularly used in place of the Boltzmann equation in many fields of rarefied gas dynamics. As a model equation of the Boltzmann equation, the BGK model also inherits various modeling assumptions of the Boltzmann equation: The gas molecules are assumed to be non-ionized, monatomic, elastic and non-reactive. Efforts to remove any of these assumptions usually involve many complications and difficulties. And for each such removal of the assumptions, relevant BGK...
models were proposed. Regarding the removal of the non-reactiveness assumption, which is practically very important since the chemical reaction of gases arises in various physical situations such as combustion processes, hypersonic flows around space vehicles, many efforts have been made in a series of works.

The first relaxation type model for the system of reacting gases was suggested by Monaco and Pandolfi in [31], providing the relaxational approximation of the reactive Boltzmann equation of Rossani and Spiga [34]. The consistent BGK model for mixture problem derived in [1] was extended in [24] for gas systems undergoing slow chemical reactions, existence of which is studied in the current work. Brull derived a reactive BGK model for which the relaxation operator is split into the elastic part and the chemical reaction part in [15]. Extension to polyatomic reacting gases can be found in [10], and the relaxational model for an irreversible reactive chemical transformation is considered in [9]. For the study of shock problems of reactive gases using BGK type models, see [26].

To the best knowledge of authors, the existence issue of any of such reactive BGK models has never been considered in the literature, which is the main motivation of the current work. In this paper, we study the stationary problem for reactive BGK models involving the bimolecular fast reaction or slow chemical reactions. More precisely, we consider the stationary problems in a slab for reactive BGK models proposed in [24] (slow reaction) and [23] (fast reaction). The term “slow reaction” and “fast reaction” are coined to compare the time scale of the chemical reactions measured up against the time scale of elastic collisions. Slow reaction denotes the case in which the chemical reaction occurs over a time scale longer than the elastic collisions, and the fast reaction denotes the opposite case.

The paper is organized as follows: In Sect. 2, we introduce the reactive BGK models we are studying in this paper. In Sect. 3, we present our main result. Section 4 is devoted to estimate the macroscopic parameters. In Sect. 5, we define our solution spaces and formulate our problem as a fixed point problem in the solution space. In Sect. 6, we show that our solution map is invariant in the solution space under the assumption of Theorems 3.3 and 3.4. In Sect. 7, we finish the proof by establishing Cauchy estimates for the solution maps.

2 BGK Models for Chemically Reacting Gases

In this section, we introduce two reactive BGK models we are considering in this work, for slow and fast chemical reaction respectively. We first define various coefficients and quantities shared by both models, and set up notational conventions:

- **Velocity distribution function** In the following, the velocity distribution function \( f_i(x, v) \) \((i = 1, 2, 3, 4)\) represents the number density of \(i\)th molecule at the position \(x \in [0, 1]\) with velocity \(v = (v_1, v_2, v_3) \in \mathbb{R}^3\).

- **Physical and chemical constants**
  \begin{itemize}
  \item (a) \(\tau\) is the Knudsen number defined by the ratio of the mean free path and the characteristic length of the system. It measures how rarefied the system is.
  \item (b) Mass: \(m_i\) represents the mass for each species and \(M\) denotes the total mass involved in the reaction process: \(M = m_1 + m_2 = m_3 + m_4\). And we use \(\mu_{ij} := \frac{m_i m_j}{m_i + m_j} \) \((i, j = 1, 2, 3, 4)\) for the reduced mass.
  \item (c) Energy: \(E_i\) denotes the energy of the chemical bond and \(\Delta E = -\sum_{i=1}^{4} \lambda_i E_i\) is the energy threshold, where we denote \(\lambda_1 = \lambda_2 = -\lambda_3 = -\lambda_4 = 1\).
  \item (d) Interactions: \(\chi_{ij}\) denotes the interaction coefficient, and the microscopic collision frequency is denoted by \(\nu_{ij}\). \(\chi_{ij}\) and \(\nu_{ij}\) must satisfy the relation: \(\chi_{ij} \leq \nu_{ij}\) (See [1]) (We
mention that this condition is necessary in the proof of Lemma 4.5). We use \( v_{12}^{34} \) and \( v_{34}^{12} \) to denote the chemical microscopic collision frequencies for the first and second model is respectively. See (2.7).

- **Macroscopic fields** We now define the macroscopic fields to construct the auxiliary fields for the slow and fast reactive models below.

(a) Single component macroscopic fields:

\[
\rho^{(i)} := m_i n^{(i)} := m_i \int_{\mathbb{R}^3} f_i dv, \\
\rho^{(i)} U^{(i)} := m_i \int_{\mathbb{R}^3} v f_i dv, \\
3k \rho^{(i)} T^{(i)} := m_i^2 \int_{\mathbb{R}^3} |v - U^{(i)}|^2 f_i dv.
\]

(2.1)

(b) Global macroscopic fields:

\[
n = \sum_{i=1}^{4} n^{(i)}, \quad \rho = \sum_{i=1}^{4} \rho^{(i)}, \quad U = \frac{1}{\rho} \sum_{i=1}^{4} \rho^{(i)} U^{(i)}, \\
nkT = \sum_{i=1}^{4} n^{(i)} kT^{(i)} + \frac{1}{3} \sum_{i=1}^{4} \rho^{(i)} (|U^{(i)}|^2 - |U|^2).
\]

(2.2)

Now, we are ready to derive the parameters determining reactive Maxwellian \( M_i \), and present our models.

**2.1 BGK Model for Slow Chemical Reaction**

Our first model is proposed in [24] and describes the dynamics for Maxwellian molecules with slow chemical reactions. The stationary problem in a slab for this model reads

\[
v_1 \frac{\partial f_i}{\partial x} = \frac{v_i}{\tau} (M_i - f_i) \quad \text{on} \quad [0, 1] \times \mathbb{R}^3, \quad (i = 1, 2, 3, 4)
\]

(2.3)

subject to the boundary data:

\[
f_i(0, v) = f_{i,L}(v), \quad \text{on} \quad v_1 > 0, \quad f_i(1, v) = f_{i,R}(v), \quad \text{on} \quad v_1 < 0,
\]

with the reactive mawellians \( M_i \) defined by

\[
M_i = n_i \left( \frac{m_i}{2\pi kT_i} \right)^{3/2} \exp \left( -\frac{m_i |v - U_i|^2}{2kT_i} \right),
\]

\( \mathcal{M}_i \)

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where the auxiliary fields \( n_i, U_i, T_i \) are determined as follows: First, we define the collision frequencies \( \nu_i \) by

\[
\nu_1 = \sum_{j=1}^{4} v_{1j} n^{(j)} + 2 \sqrt{\frac{\pi}{\Gamma}} \left( \frac{3}{2} \frac{\Delta E}{kT} \right) v_{12} n^{(2)},
\]
\[
\nu_2 = \sum_{j=1}^{4} v_{2j} n^{(j)} + 2 \sqrt{\frac{\pi}{\Gamma}} \left( \frac{3}{2} \frac{\Delta E}{kT} \right) v_{12} n^{(1)},
\]
\[
\nu_3 = \sum_{j=1}^{4} v_{3j} n^{(j)} + 2 \sqrt{\frac{\pi}{\Gamma}} \left( \frac{3}{2} \frac{\Delta E}{kT} \right) \left( \frac{\mu_{12}}{\mu_{34}} \right)^{3/2} e^{\Delta E/kT} v_{12} n^{(4)},
\]
\[
\nu_4 = \sum_{j=1}^{4} v_{4j} n^{(j)} + 2 \sqrt{\frac{\pi}{\Gamma}} \left( \frac{3}{2} \frac{\Delta E}{kT} \right) \left( \frac{\mu_{12}}{\mu_{34}} \right)^{3/2} e^{\Delta E/kT} v_{12} n^{(3)}.
\] (2.4)

Then we define the auxiliary fields \( n_i, U_i \) and \( T_i \) as

\[
n_i = n^{(i)} + \frac{\lambda_i}{\nu_i} S,
\]
\[
m_i n_i U_i = m_i n^{(i)} U^{(i)} + \frac{2}{\nu_i} \sum_{j=1}^{4} \chi_{ij} \mu_{ij} n^{(i)} n^{(j)} (U^{(j)} - U^{(i)}) + \frac{\lambda_i}{\nu_i} m_i U S,
\]
\[
\frac{3}{2} n_i k T_i = \frac{3}{2} n^{(i)} k T^{(i)} - \frac{1}{2} m_i [n_i |U_i|^2 - n^{(i)} |U^{(i)}|^2] + 6 k \frac{m_i}{\nu_i} \sum_{j=1}^{4} \chi_{ij} \mu_{ij} m_i + m_j n^{(i)} n^{(j)} (T^{(j)} - T^{(i)})
\]
\[
+ \frac{2}{\nu_i} \sum_{j=1}^{4} \chi_{ij} \mu_{ij} m_i + m_j n^{(i)} n^{(j)} (m_i U^{(i)} + m_j U^{(j)}) (U^{(j)} - U^{(i)}) + \frac{\lambda_i}{\nu_i} S \left[ \frac{1}{2} m_i |U|^2 + \frac{3}{2} k T + \frac{M - m_i}{M} k T (\Delta E/kT)^{3/2} e^{-\Delta E/kT} \Gamma \left( \frac{3}{2}, \frac{\Delta E}{kT} \right) \right]
\]
\[
- \frac{1 - \lambda_i}{2} \frac{M - m_i}{M} \Delta E,
\] (2.5)

where the quantity \( S \) is defined by

\[
S = v_{12}^{34} \left( \frac{3}{2} \frac{\Delta E}{kT} \right) \left[ n^{(3)} n^{(4)} \left( \frac{m_1 m_2}{m_1 m_2} \right)^{3/2} e^{\Delta E/kT} - n^{(1)} n^{(2)} \right].
\]

We note that the collision frequencies are given by the combination of non-reactive part, which takes the same form with the original BGK model for non-reacting gases [1,8], and the chemical reaction part.
2.2 BGK Model for Fast Chemical Reaction

Our second model is proposed in [23], and it represents gas mixtures with fast chemical reactions:

\[
v_1 \frac{\partial f_i}{\partial x} = \frac{\tilde{v}_i}{\tau} (\tilde{M}_i - f_i) \quad \text{on} \quad [0, 1] \times \mathbb{R}^3, \quad (i = 1, 2, 3, 4) \tag{2.6}
\]

subject to boundary data:

\[
f_i(0, v) = f_{i,L}(v) \quad \text{on} \quad v_1 > 0, \quad f_i(1, v) = f_{i,R}(v) \quad \text{on} \quad v_1 < 0.
\]

The reactive maxwellian \(\tilde{M}_i\) is defined by

\[
\tilde{M}_i := \tilde{n}^i \left( \frac{m_i}{2\pi kT} \right)^{3/2} \exp \left( -\frac{m_i |v - \tilde{U}|^2}{2kT} \right).
\]

The auxiliary parameters for this model are determined implicitly through the following procedure. Note that the auxiliary field is defined in the order: \(\tilde{U} \rightarrow \tilde{n}_1 \rightarrow \tilde{n}_2, \tilde{n}_3, \tilde{n}_4\) and \(\tilde{T}\).

(a) First, we define the collision frequencies \(\tilde{v}_i\) as follows:

\[
\begin{align*}
\tilde{v}_1 &= \sum_{j=1}^{4} v_{1j} n^{(j)} + \left( \frac{\mu^{34}}{\mu^{12}} \right)^{3/2} e^{-\Delta E/kT} v^{12}_{34} n^{(2)}, \\
\tilde{v}_2 &= \sum_{j=1}^{4} v_{2j} n^{(j)} + \left( \frac{\mu^{34}}{\mu^{12}} \right)^{3/2} e^{-\Delta E/kT} v^{12}_{34} n^{(1)}, \\
\tilde{v}_3 &= \sum_{j=1}^{4} v_{3j} n^{(j)} + v^{12}_{34} n^{(4)}, \\
\tilde{v}_4 &= \sum_{j=1}^{4} v_{4j} n^{(j)} + v^{12}_{34} n^{(3)}.
\end{align*}
\tag{2.7}
\]

(b) Now we define the auxiliary fields \(\tilde{U}\) and \(\tilde{n}_1\): First, we define \(\tilde{U}\) by

\[
\tilde{U} := \sum_{i=1}^{4} \tilde{v}_i m_i n^{(i)} U^{(i)} \left/ \sum_{i=1}^{4} \tilde{v}_i m_i n^{(i)} \right.
\]

We then set a function \(F(x)\) to be

\[
F(x) := \left\{ \sum_{i=1}^{4} \tilde{v}_i n^{(i)} \left[ \frac{3}{2} m_i (|U^{(i)}|^2 - |\tilde{U}|^2) + \frac{3}{2} kT^{(i)} \right] + \Delta E \tilde{v}_1 (x - n^{(1)}) \right\} \left/ \left( \frac{3}{2} k \sum_{i=1}^{4} \tilde{v}_i n^{(i)} \right) \right.
\]

\[
\tag{2.8}
\]

and define \(\tilde{n}_1\) as the unique root of the equation

\[
\frac{\tilde{v}_3 \tilde{v}_4}{\tilde{v}_1 \tilde{v}_2} \left[ \tilde{v}_3 n^{(3)} - \tilde{v}_1 (x - n^{(1)}) \right] \exp \left( -\frac{\Delta E}{k F(x)} \right) = \left( \frac{\mu^{12}}{\mu^{34}} \right)^{3/2}
\tag{2.9}
\]
or, equivalently,
\[
\frac{\tilde{v}_3\tilde{v}_4}{\tilde{v}_1\tilde{v}_2} \left[ \tilde{v}_3 n^{(3)} - \tilde{v}_1(\tilde{n}_1 - n^{(1)}) \right] \exp \left( - \frac{\Delta E}{k F(\tilde{n}_1)} \right) = \left( \frac{\mu^{12}}{\mu^{34}} \right)^{3/2}
\]
(2.10)
in the domain defined by the constraint of positivity for density and temperature fields, i.e.,
\[
\tilde{n}_1 > 0, \quad \tilde{n}_1 > n^{(1)} - \frac{\tilde{v}_2}{\tilde{v}_1} n^{(2)}, \quad \tilde{n}_1 < n^{(1)} + \frac{\tilde{v}_3}{\tilde{v}_1} n^{(3)}, \quad \tilde{n}_1 < n^{(1)} + \frac{\tilde{v}_4}{\tilde{v}_1} n^{(4)}, \quad \tilde{n}_1 > n^{(1)} - \frac{1}{\tilde{v}_1} \frac{1}{\Delta E} \sum_{i=1}^{4} \tilde{v}_i n^{(i)} \left[ \frac{1}{2} m_i (|U^{(i)}|^2 - |\tilde{U}|^2) + \frac{3}{2} k T^{(i)} \right].
\]
Since the left-hand-side of (2.9) is a strictly increasing function of $x$ with its range $(0, \infty)$, the root of (2.9) always uniquely exists. (See Lemma 4.8)
(c) With such $\tilde{n}_1$ and $\tilde{U}$, we define the remaining auxiliary fields $\tilde{n}_2, \tilde{n}_3, \tilde{n}_4$ and $\tilde{T}$ as follows:
\[
\tilde{n}_i := n^{(i)} + \lambda_i \frac{\tilde{v}_1}{\tilde{v}_i} (\tilde{n}_1 - n^{(1)}), \quad i = 2, 3, 4,
\]
(2.11)
and
\[
\tilde{T} := F(\tilde{n}_1) = \left\{ \frac{\sum_{i=1}^{4} \tilde{v}_i n^{(i)} \left[ \frac{1}{2} m_i (|U^{(i)}|^2 - |\tilde{U}|^2) + \frac{3}{2} k T^{(i)} \right] + \Delta E \tilde{v}_1 (\tilde{n}_1 - n^{(1)})}{\left( \frac{3}{2} k \sum_{i=1}^{4} \tilde{v}_i n^{(i)} \right)} \right\}.
\]
(2.12)

3 Main Result

Before we state our main result, we need to define notations and norms:

- Every constant denoted by $C$ will be generically defined. The values of $C$ may differ line by line.
- We use $C_{l,u}$ to denote a positive constant depending only on the given constants and the quantities defined in (3.1), (3.2) and (3.3). The value of $C_{l,u}$ may differ line by line.
- We define the norm $|| \cdot ||_{L^1}$ by
  \[
  ||f||_{L^1} = \int_{\mathbb{R}^3} |f(x, v)|(1 + |v|^2) dv.
  \]
- We define the following quantities for brevity $(i = 1, 2, 3, 4)$.
  \[
  a_{i,u} = 2 \int_{\mathbb{R}^3} f_{i,L,R} dv, \quad a_{i,s} = \int_{\mathbb{R}^3} \frac{1}{|v_1|} f_{i,L,R} dv, \quad a_{i,l} = \frac{1}{8} a_{i,u},
  \]
  \[
  c_{i,u} = 2 \int_{\mathbb{R}^3} f_{i,L,R} |v|^2 dv, \quad c_{i,s} = \int_{\mathbb{R}^3} \frac{1}{|v_1|} f_{i,L,R} |v|^2 dv, \quad c_{i,l} = \frac{1}{8} c_{i,u},
  \]
where we used the notation:
\[
f_{i,L,R}(v) = f_{i,L}(v) 1_{v_1 > 0} + f_{i,R}(v) 1_{v_1 < 0}.
\]
From this, we define
\[ a_u = \max_{1 \leq i \leq 4} \{ a_i, u \}, \quad a_l = \min_{1 \leq i \leq 4} \{ a_i, l \}, \quad c_u = \max_{1 \leq l \leq 4} \{ c_i, u \}, \quad c_l = \min_{1 \leq l \leq 4} \{ c_i, l \}. \tag{3.2} \]

- We also define the following quantity, which will serve as a lower bound for the temperature.
\[ \gamma_{\iota, l} = \frac{1}{16} \left( \int_{v_1 > 0} f_{i, L} |v_1| dv \right) \left( \int_{v_1 < 0} f_{i, R} |v_1| dv \right) \tag{3.3} \]
and
\[ \gamma_l = \min_{1 \leq i \leq 4} \{ \gamma_{\iota, i} \}. \]
Now, we define the mild solution of (2.3):

**Definition 3.1** A pair of functions \( f = (f_1, f_2, f_3, f_4) \in (L^\infty([0, 1]; L^1_2(\mathbb{R}^3_v)))^4 \) is said to be a mild solution for (2.3) if \( f_i \) satisfies the following equation:
\[
f_i(x, v) = \left( e^{-\frac{1}{\tau |v_1|} \int_0^x \tilde{v}_i(y) dy} f_{i, L}(v) + \frac{1}{\tau |v_1|} \int_0^x e^{-\frac{1}{\tau |v_1|} \int_0^y \tilde{v}_i(z) dz} v_i M_i dy \right) 1_{v_1 > 0} + \left( e^{-\frac{1}{\tau |v_1|} \int_0^1 \tilde{v}_i(y) dy} f_{i, R}(v) + \frac{1}{\tau |v_1|} \int_x^1 e^{-\frac{1}{\tau |v_1|} \int_0^y \tilde{v}_i(z) dz} v_i M_i dy \right) 1_{v_1 < 0},
\]
for each \( i = 1, 2, 3, 4 \).

We also define the mild solution to (2.6):

**Definition 3.2** A pair of functions \( f = (f_1, f_2, f_3, f_4) \in (L^\infty([0, 1]; L^1_2(\mathbb{R}^3_v)))^4 \) is said to be a mild solution for (2.6) if \( f_i \) satisfies the following equation:
\[
f_i(x, v) = \left( e^{-\frac{1}{\tau |v_1|} \int_0^x \tilde{v}_i(y) dy} f_{i, L}(v) + \frac{1}{\tau |v_1|} \int_0^x e^{-\frac{1}{\tau |v_1|} \int_0^y \tilde{v}_i(z) dz} v_i \tilde{M}_i dy \right) 1_{v_1 > 0} + \left( e^{-\frac{1}{\tau |v_1|} \int_0^1 \tilde{v}_i(y) dy} f_{i, R}(v) + \frac{1}{\tau |v_1|} \int_x^1 e^{-\frac{1}{\tau |v_1|} \int_0^y \tilde{v}_i(z) dz} v_i \tilde{M}_i dy \right) 1_{v_1 < 0},
\]
for each \( i = 1, 2, 3, 4 \).

The main results of this paper are as follows:

**Theorem 3.3** Suppose \( f_{i, L R}, \frac{1}{|v_1|} f_{i, L R} \in L^1_2(\mathbb{R}^3_v) \). Assume that the inflow data does not induce vertical flows on the boundary:
\[
\int_{\mathbb{R}^2} f_{i, L} v_j dv_2 dv_3 = \int_{\mathbb{R}^2} f_{i, R} v_j dv_2 dv_3 = 0. \quad (j = 2, 3)
\]
Then there exist two constants \( \epsilon, L > 0 \), depending only the constants defined in (3.1), (3.2) and (3.3), such that if \( \epsilon > \nu_1^2 \geq 0 \) and \( \tau > L \), then there exists a unique mild solution \( f = (f_1, f_2, f_3, f_4) \) for (2.3) satisfying
\[
a_{i, l} \leq \int_{\mathbb{R}^3} f_i(x, v) dv \leq a_{i, u}, \quad c_{i, l} \leq \int_{\mathbb{R}^3} |v|^2 f_i(x, v) dv \leq c_{i, u}
\]
and
\[
\left( \int_{\mathbb{R}^3} f_i dv \right) \left( \int_{\mathbb{R}^3} |v|^2 f_i dv \right) - \left( \int_{\mathbb{R}^3} v_i f_i dv \right)^2 \geq \gamma_i.
\]
Unlike the corresponding results for the BGK model for fast reactions below, we impose smallness on the reaction frequency in the slow reactive case. Whether this is an intrinsic problem, or is a technical problem which can be overcome by developing more sophisticated analysis, is not clear for now. The result in [29] says that there is a possibility that the auxiliary fields \( T_i \) may take negative values without such smallness assumption, which indicates that this restriction may be unavoidable.

**Theorem 3.4** Suppose \( f_{i,LR}, \frac{1}{|v_1|} f_{i,LR} \in L^1_2(\mathbb{R}^3_+) \). Assume that the inflow data does not induce vertical flows on the boundary:

\[
\int_{\mathbb{R}^2} f_{i,L} v_j dv_2 dv_3 = \int_{\mathbb{R}^2} f_{i,R} v_j dv_2 dv_3 = 0. \quad (j = 2, 3)
\]

Then there exists a constant \( L > 0 \), depending only on the constants defined in (3.1), (3.2) and (3.3), such that if \( \tau > L \), then there exists a unique mild solution \( f = (f_1, f_2, f_3, f_4) \) for (2.6) satisfying

\[
a_{i,l} \leq \int_{\mathbb{R}^3} f_i(x, v) dv \leq a_{i,u}, \quad c_{i,l} \leq \int_{\mathbb{R}^3} |v|^2 f_i(x, v) dv \leq c_{i,u}
\]

and

\[
\left( \int_{\mathbb{R}^3} f_i dv \right) \left( \int_{\mathbb{R}^3} |v|^2 f_i dv \right) - \left( \int_{\mathbb{R}^3} v_1 f_i dv \right)^2 \geq \gamma_i.
\]

The existence results are obtained only when the Knudsen number is sufficiently small since, among others, the contractivity of the solution map is realized only in this case. Development of existence theory in the framework of Banach fixed point theorem that works for small Knudsen numbers will be an interesting problem. The assumption that the gas on the wall does not flow vertically at the boundary is crucially used in the proof of Lemma 6.7 in Sect. 6, which is eventually used to prove the positivity of the auxiliary fields \( T_i \). We believe that this can be relaxed to a suitable smallness condition of the vertical flow, which is left for future work. We also mention that this condition is physically relevant in that, if a large amount of gas flows vertically at the boundary, the gas flow between the two boundary may not be created.

The key difficulty and novelty, along with being able to treat two different types of reactive model in a unified manner, arise from the way in which the auxiliary fields are estimated. The macroscopic fields for non-reactive BGK models, which corresponds to the auxiliary fields of the reactive BGK models, are defined from velocity distribution functions in an explicit manner through simple integral relation, and the relevant lower and upper bounds for the fields follows directly from the definition once suitable upper and lower bounds are known for the moments of the distribution function. The auxiliary parameters for the reactive system on the other hand, are defined through highly nonlinear relations as was given in Sect. 2 above. Therefore, determining various necessary a priori estimates of them cannot be treated in a straightforward manner as in the non-reactive case.

To overcome this difficulty, we introduce several nonlinear functionals which capture important structures of such nonlinear relations and carefully analyze those functionals to derive the desired results for auxiliary parameters. For example, to estimate \( T_i \) of the first reactive model of our paper, we introduce the following nonlinear functional:
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4 Estimates of the Macroscopic Parameters

Throughout this section, we assume that the velocity distribution \( F = (f_1, f_2, f_3, f_4) \) satisfies the following inequalities:

\[
\begin{align*}
  a_{i,l} \leq \int_{\mathbb{R}^3} f_i(x, v) dv & \leq a_{i,u}, \\
  c_{i,l} \leq \int_{\mathbb{R}^3} |v|^2 f_i(x, v) dv & \leq c_{i,u}
\end{align*}
\]

and show that \( p \) satisfies \( p(t) \geq -C|t| \) near zero. This observation enables us to estimate \( T_i \) from above and below (See Lemma 4.5).

On the other hand, to obtain the uniform lower and upper bound of \( \tilde{n}_1 \) of the second model, we devise the following nonlinear functional:

\[
F_{x,y,\mu,\eta,\alpha,\beta}(z) = \log \frac{\mu_3 \mu_4}{\eta_2} + \log z + \log(\mu_2 x_2 + \mu_1 z - \eta_1 y_1)
\]

\[
- \log(\eta_3 y_3 - \mu_1 z + \eta_1 y_1)
\]

\[
- \log(\eta_4 y_4 - \mu_1 z + \eta_1 y_1)
\]

\[
- \frac{3}{2} \Delta E \sum_{i=1}^{4} \eta_i y_i
\]

\[
\sum_{i=1}^{4} \mu_i x_i \left[ \frac{1}{2} m_i (\beta_i^2) + \frac{3}{2} k \alpha_i \right] + \Delta E (\mu_1 z - \eta_1 y_1)
\]

and use the fact that for each fixed \( z \), the function \( (x, y, \mu, \eta, \alpha, \beta) \mapsto F_{x,y,\mu,\eta,\alpha,\beta}(z) \) is monotone in \( x_i, \mu_i, \alpha_i, |\beta_i|, y_i, \eta_i \) (See Lemma 4.8).

Before moving on to the proof, a brief review on the relevant analytical results on stationary problems are in order. The stationary problem for the BGK model in a bounded interval was first studied by Ukai in [33] using a Schauder type fixed point theorem. Nouri studied the existence of weak solutions for a quantum BGK model with a discretized condensation ansatz in [32]. The existence of unique mild solutions were obtained in [7], using classical Banach fixed point argument. This argument were then applied to the ES-BGK model with the correct Prandtl number [16], the relativistic BGK models of Marle and Anderson-Witting type [27,28] and to the quantum BGK model for non-saturated Fermion system and the Boson system without condensation [6].

For the Boltzmann equation, Arkeryd et al. considered the slab problem in the framework of measure-valued solutions [2]. Arkeryd and Nouri studied the existence of weak solutions in a series of papers [3–5], which were extended to gas mixture problems by Brull [12,13]. Ghomesi considered the existence and uniqueness of the Boltzmann equation in a slab in [22] (See also [30]). Esposito et al. [18] studied hydrodynamic limits in a slab. For the existence and stability of stationary solutions near equilibrium, see [19]. All the literature reviewed on the existence is for slab problems. For stationary problems in general domains, we refer to [19–21].
and
\[
\left( \int_{\mathbb{R}^3} f_i dv \right) \left( \int_{\mathbb{R}^3} |v|^2 f_i dv \right) - \left( \int_{\mathbb{R}^3} v_1 f_i dv \right)^2 \geq \gamma_i.
\]

### 4.1 Single Component Macroscopic Parameters

To prove the main theorems, we first estimate the macroscopic parameters.

**Lemma 4.1** The single component parameters satisfy
\[
|U^{(i)}| \leq \frac{a_{i,u} + c_{i,u}}{2a_{i,l}}
\]

and
\[
\frac{m_i \gamma_l}{3ka_{i,u}^2} \leq T^{(i)} \leq \frac{m_i c_{i,u}}{3ka_{i,l}}.
\]

**Proof** Firstly, \(|U^{(i)}|\) can be written as follows:
\[
|U^{(i)}| = \frac{\rho^{(i)} \cdot |U^{(i)}|}{\rho^{(i)}} = \frac{\int_{\mathbb{R}^3} v f_i dv}{\int_{\mathbb{R}^3} f_i dv}.
\]

By Young’s inequality, we have
\[
\left| \int_{\mathbb{R}^3} v f_i dv \right| \leq \frac{\int_{\mathbb{R}^3} f_i dv + \int_{\mathbb{R}^3} |v|^2 f_i dv}{2} \leq \frac{a_{i,u} + c_{i,u}}{2},
\]
so that
\[
|U^{(i)}| \leq \frac{a_{i,u} + c_{i,u}}{2a_{i,l}}.
\]

Secondly, \(T^{(i)}\) can be expressed as
\[
T^{(i)} = \frac{(3kn^{(i)} T^{(i)} + \rho^{(i)} |U^{(i)}|^2) - \rho^{(i)} |U^{(i)}|^2 (\rho^{(i)})^{-1}}{3kn^{(i)} - \frac{m_i \int_{\mathbb{R}^3} |v|^2 f_i dv - m_i \int_{\mathbb{R}^3} v f_i dv |^2 (\int_{\mathbb{R}^3} f_i dv)^{-1}}{3k \int_{\mathbb{R}^3} f_i dv}.
\]

For the upper bound, we see that
\[
T^{(i)} = \frac{m_i \int_{\mathbb{R}^3} |v|^2 f_i dv - m_i \int_{\mathbb{R}^3} v f_i dv |^2 (\int_{\mathbb{R}^3} f_i dv)^{-1}}{3k \int_{\mathbb{R}^3} f_i dv} \leq \frac{m_i \int_{\mathbb{R}^3} |v|^2 f_i dv}{3k \int_{\mathbb{R}^3} f_i dv} \leq \frac{m_i c_{i,u}}{3ka_{i,l}}.
\]

For the lower bound, we have
\[
T^{(i)} = \frac{m_i (\int_{\mathbb{R}^3} f_i dv) (\int_{\mathbb{R}^3} |v|^2 f_i dv) - m_i \int_{\mathbb{R}^3} v f_i dv |^2 (\int_{\mathbb{R}^3} f_i dv)^{-1}}{3k (\int_{\mathbb{R}^3} f_i dv)^2} \geq \frac{m_i \gamma_l}{3ka_{i,u}}.
\]

\(\square\)
4.2 Global Macroscopic Parameters

Lemma 4.2 Global macroscopic parameters $U$ and $T$ satisfy

$$|U| \leq \max_{1 \leq i \leq 4} \left\{ \frac{a_{i,u} + c_{i,u}}{2a_{i,l}} \right\}$$

and

$$T_l \leq T \leq T_u,$$

where $T_l := \min_{1 \leq i \leq 4} \left\{ \frac{m_{iy}}{5ka_{i,u}} \right\}$ and $T_u := \frac{c_u}{12ka_l} \sum_{i=1}^{4} m_i$.

Proof For the bound of $U$, we estimate as follows:

$$|U| \leq \frac{1}{\rho} \sum_{i=1}^{4} \rho^{(i)} |U^{(i)}| \leq \max_{1 \leq i \leq 4} |U^{(i)}| \leq \max_{1 \leq i \leq 4} \left\{ \frac{a_{i,u} + c_{i,u}}{2a_{i,l}} \right\}.$$

For the lower bound of $T$, we observe

$$4 \sum_{i=1}^{4} \rho^{(i)} (|U^{(i)}|^2 - |U|^2) = 4 \sum_{i=1}^{4} \rho^{(i)} (|U^{(i)} - U|^2) \geq 0,$$

which implies

$$T \geq \sum_{i=1}^{4} \frac{n^{(i)}}{n} T^{(i)} \geq \min_{1 \leq i \leq 4} T^{(i)} \geq T_l.$$

For the upper bound of $T$, we estimate as follows:

$$T \leq \sum_{i=1}^{4} \frac{n^{(i)}}{n} T^{(i)} + \frac{1}{3nk} \sum_{i=1}^{4} \rho^{(i)} |U^{(i)}|^2 = \frac{1}{3nk} \sum_{i=1}^{4} \frac{m_i}{\rho} \int_{\mathbb{R}^3} |v|^2 f_i dv \leq \frac{c_u}{12ka_l} \sum_{i=1}^{4} m_i.$$

\[\Box\]

4.3 Macroscopic Parameters for First Model

Lemma 4.3 There exists a positive lower bound for $n_i$ depending only on the quantities given in (3.1), (3.2) and (3.3).

Proof We consider only the case with $i = 1$. By the definition

$$n_1 = n^{(1)} - \frac{1}{\nu_1} v_{12}^{34} \frac{2}{\sqrt{\pi}} \Gamma \left( \frac{3}{2}, \frac{\Delta E}{kT} \right) n^{(1)} n^{(2)}$$

$$+ \frac{1}{\nu_1} v_{12}^{34} \frac{2}{\sqrt{\pi}} \Gamma \left( \frac{3}{2}, \frac{\Delta E}{kT} \right) n^{(3)} n^{(4)} \left( \frac{m_1}{m_3 m_4} \right)^{3/2} e^{\Delta E/kT}.$$

Since

$$\left( \sum_{j=1}^{4} v_{ij} + \frac{2}{\sqrt{\pi}} \Gamma \left( \frac{3}{2}, \frac{\Delta E}{kT} \right) v_{12}^{34} \right) a_u \geq v_{12}^{34} \frac{2}{\sqrt{\pi}} \Gamma \left( \frac{3}{2}, \frac{\Delta E}{kT} \right) n^{(2)}.$$
we obtain

\[
n_1 \geq \frac{1}{v_1} \frac{v_{12}^{34}}{\sqrt{\pi}} \Gamma \left( \frac{3}{2} \right) \frac{\Delta E}{kT} n^{(3)} \frac{n^{(4)}}{m_3 m_4} \frac{3}{2} e^{\Delta E/kT} \geq \frac{2 v_{12}^{34}}{\sqrt{\pi} \left( \sum_{j=1}^{4} v_{1j} + \frac{2}{\sqrt{\pi}} \Gamma \left( \frac{3}{2} \right) \frac{\Delta E}{kT} v_{12}^{34} a_u \right)} \Gamma \left( \frac{3}{2} \right) \frac{\Delta E}{kT} a_u^2 \left( \frac{m_1 m_2}{m_3 m_4} \right)^{3/2} e^{\Delta E/kT}.
\]

Lemma 4.4 There exist positive upper bounds for \( n_i \), \(|U_i|\), and \( T_i \) depending only on the quantities given in (3.1), (3.2) and (3.3).

Proof We have

\[
|S| \leq v_{12}^{34} \left[ \frac{\mu_{12}^{12}}{\mu_{34}^{34}} e^{\Delta E/kT_l} + 1 \right] a_u^2, \tag{4.1}\]

and

\[
v_i \geq \frac{4}{\sum_{j=1}^{4} v_{ij} a_l}, \tag{4.2}\]

which directly implies

\[
n_i \leq a_{i,u} + \frac{1}{\sum_{j=1}^{4} v_{ij} a_l} v_{12}^{34} \left[ \frac{\mu_{12}^{12}}{\mu_{34}^{34}} e^{\Delta E/kT_l} + 1 \right] a_u^2.
\]

For \( U_i \), the triangle inequality gives

\[
|U_i| \leq \frac{n^{(i)}}{n_i} |U^{(i)}| + \frac{2}{m_i n_i v_i} \sum_{j=1}^{4} \chi_{ij} \frac{\mu_{ij}}{m_i + m_j} n^{(j)} (|U^{(j)}| + |U^{(i)}|) + \frac{1}{n_i v_i} |U||S|.
\]

Thus, Lemma 4.1 and 4.2, together with (4.1) and (4.2) give

\[
|U_i| \leq C_{i,u}.
\]

And for \( T_i \), we have from Lemma 4.2 that

\[
T_i \leq \frac{n^{(i)}}{n_i} T^{(i)} + \frac{m_i n^{(i)}}{3 k n_i} |U^{(i)}|^2 + \frac{4}{n_i v_i} \sum_{j=1}^{4} \chi_{ij} \frac{\mu_{ij}}{m_i + m_j} n^{(j)} T^{(j)} + \frac{4}{3 k n_i v_i} \sum_{j=1}^{4} \chi_{ij} \frac{\mu_{ij}}{m_i + m_j} n^{(j)} (m_i |U^{(i)}| + m_j |U^{(j)}|) (|U^{(j)}| + |U^{(i)}|) + \frac{2}{3 k n_i v_i} |S| \left[ \frac{1}{2} m_i |U|^2 + \frac{3}{2} k T + \frac{M - m_i}{M} \frac{k T (\Delta E / kT)^{3/2} e^{-\Delta E / kT}}{\Gamma \left( \frac{3}{2}, \frac{\Delta E}{kT} \right)} + \frac{M - m_i}{M} \Delta E \right] \leq C_{i,u}.
\]

Lemma 4.5 There exists a positive number \( \epsilon \) depending only on the quantities defined in (3.1), (3.2) and (3.3) such that if \( \epsilon > v_{12}^{34} \geq 0 \), then \( T_i \) has a positive lower bound depending only on the quantities given in (3.1), (3.2) and (3.3).
Remark 4.6 The smallness of $v_{12}^{34}$ in Theorem 3.3 comes from this lemma.

**Proof** We define $I_i$ and $II_i$ ($i = 1, 2, 3$) by

$$\frac{3}{2} n_i k T_i = \frac{3 k n_i^{(i)}}{2} \left(1 - \frac{4}{v_i} \sum_{j=1}^{4} \chi_{ij} \frac{m_j m_j}{(m_i + m_j)^2} n_j^{(j)}\right) T^{(i)} + \frac{6 k}{v_i} \sum_{j=1}^{4} \chi_{ij} \frac{\mu_{ij}}{m_i + m_j} n_j^{(i)} n_j^{(j)} T^{(j)} + \frac{\lambda_i}{v_i} S \left[\frac{3}{2} k T + \frac{M - m_i}{M} \Delta E \left(\frac{(\Delta E/k T)^{1/2} e^{-\Delta E/k T}}{\Gamma(\frac{3}{2}, \Delta E/k T)} - \frac{1}{2} \right)\right]$$

$$+ 2 \frac{\sum_{j=1}^{4} \chi_{ij} \mu_{ij} n_j^{(i)} n_j^{(j)} (m_j U^{(i)} + m_j U^{(j)}) (U^{(j)} - U^{(i)})}{v_i} - \frac{1}{2} m_i n_i |U_i|^2 - n_i |U^{(i)}|^2 \right] + \frac{\lambda_i}{v_i} S \frac{1}{2} m_i |U|^2$$

$$= I_1 + I_2 + I_3 + II_1 + II_2 + II_3$$

(1) Estimate of $I_1 + I_2 + I_3$: Since

$$v_i \geq \sum_{j=1}^{4} v_{ij} n_j^{(j)} \geq \sum_{j=1}^{4} \chi_{ij} n_j^{(j)} \geq 4 \sum_{j=1}^{4} \chi_{ij} \frac{m_j m_j}{(m_i + m_j)^2} n_j^{(j)},$$

we have $I_1 \geq 0$.

For the estimate of $I_2 + I_3$, we observe from Lemma 4.2 that

$$I_3 \geq - \frac{1}{v_i} S \left[\frac{3}{2} k T + \frac{M - m_i}{M} \Delta E \left(\frac{(\Delta E/k T)^{1/2} e^{-\Delta E/k T}}{\Gamma(\frac{3}{2}, \Delta E/k T)} + 1\right)\right]$$

$$\geq - \frac{1}{v_i} v_{12}^{34} \left[\frac{\mu_{12}}{\mu_{34}} e^{\Delta E/k T} + 1\right] a_u^2 \left[\frac{3}{2} k T + \frac{M - m_i}{M} \Delta E \left(\frac{(\Delta E/k T)^{1/2} e^{-\Delta E/k T}}{\Gamma(\frac{3}{2}, \Delta E/k T)} + 1\right)\right],$$

so that, for sufficiently small $v_{12}^{34}$, we have

$$I_2 + I_3$$

$$\geq 6 \frac{k}{v_i} \sum_{j=1}^{4} \chi_{ij} \frac{\mu_{ij}}{m_i + m_j} (a_l)^2 \frac{m_j n_j}{(a_u)^2}$$

$$- \frac{1}{v_i} v_{12}^{34} \left[\frac{\mu_{12}}{\mu_{34}} e^{\Delta E/k T} + 1\right] a_u^2 \left[\frac{3}{2} k T + \frac{M - m_i}{M} \Delta E \left(\frac{(\Delta E/k T)^{1/2} e^{-\Delta E/k T}}{\Gamma(\frac{3}{2}, \Delta E/k T)} + 1\right)\right]$$

$$\geq 3 \frac{k}{v_i} \sum_{j=1}^{4} \chi_{ij} \frac{\mu_{ij}}{m_i + m_j} (a_l)^2 \frac{m_j n_j}{(a_u)^2}.$$

(2) Estimate of $II_1 + II_2 + II_3$: By straightforward computations, we get

$$II_1 + II_2 + II_3 = \frac{2}{v_i} \sum_{j=1}^{4} \chi_{ij} \frac{\mu_{ij}}{m_i + m_j} n_j^{(i)} n_j^{(j)} m_j (U^{(j)} - U^{(i)})^2$$

$$- \frac{m_j n_j}{2} |U_i - U^{(i)}|^2 + \frac{\lambda_i}{v_i} S \frac{1}{2} m_j |U^{(i)} - U|^2.$$

(4.3)
To estimate this, we note that the second term in (4.3)
\[
\frac{m_i n_i}{2} |U_i - U^{(i)}|^2 = \frac{1}{2m_i n_i} |m_i n_i U_i - m_i n_i U^{(i)}|^2
\]
\[
= \frac{1}{2m_i n_i} \left| m_i n_i U_i - m_i n^{(i)} U^{(i)} - m_i \frac{\lambda_i}{v_i} S U^{(i)} \right|^2
\]
\[
= \frac{1}{2m_i n_i} \left| \frac{\lambda_i}{v_i} S (U - U^{(i)}) + \frac{2}{v_i} \sum_{j=1}^{4} \chi_{ij} \mu_{ij} n^{(i)} n^{(j)} (U^{(j)} - U^{(i)}) \right|^2,
\]
(4.4)

where (2.5)_1 and (2.5)_2 were used in the second and the third line. In view of (4.3) and (4.4), we define a function \( p \) by
\[
p(t) = \frac{2}{v_i} \sum_{j=1}^{4} \chi_{ij} \frac{\mu_{ij}}{m_i + m_j} n^{(i)} n^{(j)} m_j |U^{(j)} - U^{(i)}|^2 + \frac{m_i}{2} |U^{(i)} - U|^2
\]
\[
- \frac{1}{2m_i (n^{(i)} + t)} m_i t (U - U^{(i)}) + \frac{2}{v_i} \sum_{j=1}^{4} \chi_{ij} \mu_{ij} n^{(i)} n^{(j)} (U^{(j)} - U^{(i)})^2.
\]

Then we employ Theorem 3.1 in [1] to find \( p(0) \geq 0 \). Moreover, by differentiating this function, we get
\[
p'(t) = \frac{m_i}{2} |U^{(i)} - U|^2 + \frac{1}{2m_i (n^{(i)} + t)^2} \left| m_i t (U - U^{(i)}) + A \right|^2
\]
\[
- \frac{U - U^{(i)}}{(n^{(i)} + t)} \cdot \left( m_i t (U - U^{(i)}) + A \right)
\]
\[
= \frac{1}{2m_i (n^{(i)} + t)^2} |m_i n^{(i)} (U - U^{(i)}) + A|^2.
\]

Since \( |p'(t)| \leq C_{l,a} \) on \(-a_{i,l}/2 \leq t \leq a_{i,l}/2\), we obtain
\[
p(t) \geq p(0) - C_{l,a} |t| \geq -C_{l,a} |t|.
\]

Therefore,
\[
\frac{3}{2} n_j k \tau_i \geq \frac{3k}{v_i} \sum_{j=1}^{4} \chi_{ij} \frac{\mu_{ij}}{m_i + m_j} \frac{a_i}{(a_i)^2} + \left( \frac{\lambda_i S}{v_i} \right)
\]
\[
\geq \frac{3k}{v_i} \sum_{j=1}^{4} \chi_{ij} \frac{\mu_{ij}}{m_i + m_j} \frac{a_i}{(a_i)^2} - C_{l,a} \left| \frac{\lambda_i S}{v_i} \right|
\]
\[
\geq \frac{3k}{2v_i} \sum_{j=1}^{4} \chi_{ij} \frac{\mu_{ij}}{m_i + m_j} \frac{a_i}{(a_i)^2}.
\]

where we used that (4.1), (4.2) imply that there exist a positive number \( \epsilon \) depending only on the quantities defined in (3.1), (3.2) and (3.3) such that if \( \epsilon > v_{12}^0 > 0 \), then
\[
\left| \frac{\lambda_i S}{v_i} \right| \leq \min \left\{ \frac{a_{i,l}}{2} \cdot \frac{3k}{2C_{l,a} v_i} \sum_{j=1}^{4} \chi_{ij} \frac{\mu_{ij}}{m_i + m_j} \frac{a_i}{(a_i)^2} \right\}.
\]
Finally, note that if \( \epsilon > v_{12}^{34} \geq 0 \), then for \( i = 1, 2 \),
\[
v_i \leq \sum_{j=1}^{4} v_{1j} a_u + \frac{2}{\sqrt{\pi}} \Gamma \left( \frac{3}{2}, \frac{\Delta E}{kT_u} \right) v_{12}^{34} a_u \leq C_{i,u},
\]
and for \( i = 3, 4 \),
\[
v_i \leq \sum_{j=1}^{4} v_{3j} a_u + \frac{2}{\sqrt{\pi}} \Gamma \left( \frac{3}{2}, \frac{\Delta E}{kT_u} \right) \left( \frac{\mu^{12}}{\mu^{34}} \right)^{3/2} e^{\Delta E/kT_u} v_{12}^{34} a_u \leq C_{i,u},
\]
which completes the proof. \( \square \)

4.4 Macroscopic Parameters for the Second Model

**Lemma 4.7** There exist positive bounds for \( |\bar{U}| \) and \( \bar{v}_i \) depending only on the quantities given in (3.1), (3.2) and (3.3).

**Proof** We obtain from Lemma 4.1 that
\[
|\bar{U}| \leq \sum_{i=1}^{4} v_i m_i n^i |U^i| / \sum_{i=1}^{4} v_i m_i n^i \leq \max_{1 \leq i \leq 4} |U^i| \leq \max_{1 \leq i \leq 4} \left\{ \frac{a_{i,u} + c_{i,u}}{2a_{i,u}} \right\} =: R,
\]
and
\[
v_1^M := \sum_{j=1}^{4} v_{1j} a_{1,j} \leq \bar{v}_1 \leq \sum_{j=1}^{4} v_{1j} a_{1,u} + \left( \frac{\mu^{34}}{\mu^{12}} \right)^{3/2} v_{12}^{34} a_{2,u} =: v_1^M,
\]
\[
v_3^M := \sum_{j=1}^{4} v_{3j} a_{3,j} \leq \bar{v}_3 \leq \sum_{j=1}^{4} v_{3j} a_{3,u} + v_{34}^{12} a_{4,u} =: v_3^M.
\]
By the same way, we have \( v_2^M \leq \bar{v}_2 \leq v_2^M \) and \( v_4^M \leq \bar{v}_4 \leq v_4^M \). \( \square \)

**Lemma 4.8** There exist positive lower and upper bounds for \( \bar{n}_1 \) depending only on the quantities given in (3.1), (3.2) and (3.3).

**Proof** We first set
\[
x = (x_1, x_2, x_3, x_4), \quad y = (y_1, y_2, y_3, y_4), \quad \mu = (\mu_1, \mu_2, \mu_3, \mu_4),
\]
\[
v = (v_1, v_2, v_3, v_4), \quad \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4), \quad \beta = (\beta_1, \beta_2, \beta_3, \beta_4).
\]
Then, for each \( (x, y, \mu, \eta, \alpha, \beta) \in (\mathbb{R}_+)^{20} \times \mathbb{R}^4 \), we define a map \( F_{x,y,\mu,\eta,\alpha,\beta} : \Omega_{x,y,\mu,\eta,\alpha,\beta} \rightarrow (\mathbb{R}_+)^{20} \times \mathbb{R}^4 \) by
\[
F_{x,y,\mu,\eta,\alpha,\beta}(z) = \log \frac{\mu_3 \mu_4}{\eta_2} + \log z + \log (\mu_2 x_2 + \mu_1 z - \eta_1 y_1) - \log (\eta_3 y_3 - \mu_1 z + \eta_1 y_1) - \log (\eta_4 y_4 - \mu_1 z + \eta_1 y_1) - \frac{3}{2} \Delta E \sum_{i=1}^{4} \eta_i y_i - \sum_{i=1}^{4} \mu_i x_i \left[ \frac{1}{2} m_i (\beta_i^2) + \frac{3}{2} k \alpha_i \right] + \Delta E (\mu_1 z - \eta_1 y_1).
\]
where the domain $\Omega_{x,y,\mu,\eta,\alpha,\beta}$ is given by
\[
\Omega_{x,y,\mu,\eta,\alpha,\beta} = \{ z > 0 \} \cap \{ \mu_2 x_2 + \mu_1 z - \eta_1 y_1 > 0 \} \cap \{ \eta_4 y_4 - \mu_1 z + \eta_1 y_1 > 0 \}
\]
\[
\cap \left\{ \sum_{i=1}^{4} \mu_i x_i \left[ \frac{1}{2} m_i (\beta_i^2) + \frac{3}{2} k \alpha_i \right] + \Delta E (\mu_1 z - \eta_1 y_1) > 0 \right\}.
\]
We mention that $\Omega_{x,y,\mu,\eta,\alpha,\beta}$ is always non-empty since $\eta_1 y_1 / \mu_1$ always belongs to $\Omega_{x,y,\mu,\eta,\alpha,\beta}$.

We note that, for each $(x, y, \mu, \eta, \alpha, \beta) \in (\mathbb{R}_+)^2 \times \mathbb{R}^4$, $F_{x,y,\mu,\eta,\alpha,\beta}$ is a strictly increasing surjective function with respect to $z$ on $\Omega_{x,y,\mu,\eta,\alpha,\beta}$. Also, for fixed $z$, the function $(x, y, \mu, \eta, \alpha, \beta) \mapsto F_{x,y,\mu,\eta,\alpha,\beta}(z)$ is increasing in $x_i, \mu_i, \alpha_i, |\beta_i|$ $(i = 1, 2, 3, 4)$, and decreasing in $y_i, \eta_i$ $(i = 1, 2, 3, 4)$, as long as the function is well-defined.

Now, taking logarithms on both sides of (2.9) yields (with $x$ replaced by $\bar{n}_1$)
\[
F_{n,n,\bar{v},\bar{v},T,V}(\bar{n}_1) = \frac{3}{2} \log \left( \frac{\mu_{12}^{12}}{\mu_{34}} \right),
\]
where we used the following notations:
\[
\begin{align*}
\mathbf{n} &= (n^{(i)})_{i=1,2,3,4}, \quad \bar{\mathbf{v}} = (\bar{v}_i)_{i=1,2,3,4}, \quad \mathbf{T} = (T^{(i)})_{i=1,2,3,4}, \quad \mathbf{V} = ((U^{(i)} - \bar{U})^2)_{i=1,2,3,4}.
\end{align*}
\]
Therefore, there exists the unique function $G : (\mathbb{R}_+)^2 \times \mathbb{R}^4 \to \mathbb{R}_+$ satisfying
\[
G(x, y, \mu, \eta, \alpha, \beta) = F_{x,y,\mu,\eta,\alpha,\beta}^{-1} \left( \frac{3}{2} \log \left( \frac{\mu_{12}^{12}}{\mu_{34}} \right) \right)
\]
and furthermore, $G$ is decreasing in $x_i, \mu_i, \alpha_i, |\beta_i|$ $(i = 1, 2, 3, 4)$, and increasing in $y_i, \eta_i$ $(i = 1, 2, 3, 4)$. Therefore, we obtain
\[
0 < G(a_i, a_i, \mathbf{v}, \mathbf{m}, T_\mathbf{t}, 2R1) \leq \bar{n}_1 = G(\mathbf{n}, \mathbf{n}, \bar{\mathbf{v}}, \bar{\mathbf{v}}, \mathbf{T}, \mathbf{V}) \leq G(a_i, a_i, \mathbf{v}_m, \mathbf{v}, T_\mathbf{t}, 1, 0)
\]
where we used the following notations:
\[
\begin{align*}
a_i &= (a_i, i)_{i=1,2,3,4}, \quad a_{u} = (a_{i,u}, i)_{i=1,2,3,4}, \quad \mathbf{v} = (\mathbf{v})_{i=1,2,3,4}, \quad \mathbf{v}_m = (\mathbf{v}_m)_{i=1,2,3,4}, \quad \mathbf{v}^M = (\mathbf{v}^M)_{i=1,2,3,4},
\end{align*}
\]
\[
\begin{align*}
\mathbf{n} &= (n^{(i)})_{i=1,2,3,4}, \quad \bar{\mathbf{v}} = (\bar{v}_i)_{i=1,2,3,4}, \quad \mathbf{T} = (T^{(i)})_{i=1,2,3,4}, \quad \mathbf{1} = (1, 1, 1, 1), \quad \mathbf{0} = (0, 0, 0, 0), \quad \mathbf{V} = ((U^{(i)} - \bar{U})^2)_{i=1,2,3,4}.
\end{align*}
\]
\[\square\]

**Corollary 4.1** There exist positive lower and upper bounds for $\bar{n}_2$ depending only on the quantities given in (3.1), (3.2) and (3.3).

**Proof** By the definition (2.11), we know $\bar{v}_2(\bar{n}_2 - n^{(2)}) = \bar{v}_1(\bar{n}_1 - n^{(1)})$, which implies the following equality:
\[
\frac{\bar{v}_3 \bar{v}_4}{\bar{v}_1 \bar{v}_2} [\bar{v}_1 n^{(1)} + \bar{v}_2(\bar{n}_2 - n^{(2)})] \exp \left( -\Delta E \frac{k T(\bar{n}_2)}{k T(\bar{n}_2)} \right) = \left( \frac{\mu_{12}^{12}}{\mu_{34}} \right)^{3/2}
\]
and
\[
\bar{T}(\bar{n}_2) = \left\{ \sum_{i=1}^{4} \bar{v}_1 n^{(i)} \left[ \frac{1}{2} m_i (|U^{(i)}|^2 - |\bar{U}|^2) + \frac{3}{2} k T^{(i)} \right] + \Delta E \bar{v}_2(\bar{n}_2 - n^{2}) \right\} / \left( \frac{3}{2} k \sum_{i=1}^{4} \bar{v}_i n^{(i)} \right).
\]
Repeating the argument used in Lemma 4.8, we obtain the desired result. □

**Corollary 4.2** There exist positive lower and upper bounds for \( \tilde{n}_3 \) and \( \tilde{n}_4 \) depending only on the quantities given in (3.1), (3.2) and (3.3).

**Proof** We rewrite (2.11) as
\[
\tilde{v}_3(n^{(3)}) = -\tilde{v}_1(n^{(1)})
\]
and plug this into (2.10) to get
\[
\frac{\tilde{v}_3 \tilde{v}_4}{\tilde{v}_1} [\tilde{v}_1 n^{(1)} - \tilde{v}_3(n^{(3)})] \frac{\tilde{v}_2 n^{(2)} - \tilde{v}_3(n^{(3)})}{\tilde{v}_2 \tilde{n}_3} \frac{\tilde{v}_4 n^{(4)} + \tilde{v}_3(n^{(3)})}{\tilde{v}_3(n^{(3)})} \exp \left[ -\frac{\Delta E}{kT(\tilde{n}_3)} \right] = \left( \frac{\mu^{12}}{\mu^{34}} \right)^{3/2},
\]
where \( \tilde{T}(\tilde{n}_3) \) is defined by
\[
\tilde{T}(\tilde{n}_3) := \left\{ \sum_{i=1}^{4} \tilde{v}_i n^{(i)} \left[ \frac{1}{2} m_i (|U^{(i)}|^2 - |\tilde{U}|^2) + \frac{3}{2} k T^{(i)} \right] - \Delta E \tilde{v}_3(n^{(3)}) \right\} \left( \frac{3}{2} k \sum_{i=1}^{4} \tilde{v}_i n^{(i)} \right).
\]
Note that \( \tilde{T}(\tilde{n}_3) \) is obtained by inserting (4.5) into (2.12).

Now, for each \( (x, y, \mu, \eta, \alpha, \beta) \in (\mathbb{R}^+)^{20} \times \mathbb{R}^4 \), we define a map \( H_{x, y, \mu, \eta, \alpha, \beta} : \Omega_{x, y, \mu, \eta, \alpha, \beta} \rightarrow (-\infty, \infty) \) by
\[
H_{x, y, \mu, \eta, \alpha, \beta}(z) = \log \frac{\mu_4}{\eta_1 \eta_2} - \log z + \log(\mu_1 x_1 - \eta_3 z + \mu_3 x_3)
+ \log(\mu_2 x_2 - \eta_3 z + \mu_3 x_3)
- \log(\eta_4 y_4 + \eta_3 z - \mu_3 x_3)
- \frac{3}{2} \Delta E \sum_{i=1}^{4} \eta_i y_i
- \frac{4}{3} \sum_{i=1}^{4} \mu_i x_i \left[ \frac{1}{2} m_i (\beta_i^2) + \frac{3}{2} k \alpha_i \right] - \Delta E(\eta_3 z - \mu_3 x_3),
\]
where
\[
\Omega_{x, y, \mu, \eta, \alpha, \beta}' = \left\{ z > 0 \right\} \cap \left\{ \mu_1 x_1 - \eta_3 z + \mu_3 x_3 > 0 \right\} \cap \left\{ \mu_2 x_2 - \eta_3 z + \mu_3 x_3 > 0 \right\}
\cap \left\{ \eta_4 y_4 + \eta_3 z - \mu_3 x_3 > 0 \right\}
\cap \left\{ \sum_{i=1}^{4} \mu_i x_i \left[ \frac{1}{2} m_i (\beta_i^2) + \frac{3}{2} k \alpha_i \right] - \Delta E(\eta_3 z - \mu_3 x_3) > 0 \right\}.
\]
For each \( (x, y, \mu, \eta, \alpha, \beta) \in (\mathbb{R}^+)^{20} \times \mathbb{R}^4 \), \( H_{x, y, \mu, \eta, \alpha, \beta} \) is a strictly decreasing surjective function on \( \Omega_{x, y, \mu, \eta, \alpha, \beta}' \). Hence there exists the unique function \( J : (\mathbb{R}^+)^{20} \times \mathbb{R}^4 \rightarrow \mathbb{R}^+ \) satisfying
\[
J(x, y, \mu, \eta, \alpha, \beta) = H_{x, y, \mu, \eta, \alpha, \beta}^{-1} \left( \frac{3}{2} \log \left( \frac{\mu^{12}}{\mu^{34}} \right) \right).
\]
As in the proof of Lemma 4.8, \( J \) is increasing in \( x_i, \mu_i, \alpha_i, |\beta_i| \) \( (i = 1, 2, 3, 4) \), and decreasing in \( y_i, \eta_i \) \( (i = 1, 2, 3, 4) \). Hence the following inequality holds:
\[
0 \leq J(a_l, a_u, v^m, v^M, T_l, 1, 0) \leq \tilde{n}_3 \leq J(a_u, a_l, v^M, v^m, T_u, 1, 2R1).
\]
The proof for \( \tilde{n}_4 \) is similar. We omit it. □
Lemma 4.9 There exist positive lower and upper bounds for $\tilde{T}$ depending only on the quantities given in (3.1), (3.2) and (3.3).

Proof From (2.12) we get the following relation:

$$\tilde{T} \times A - B = \tilde{v}_1 (\tilde{n}_1 - n^{(1)})$$

(4.6)

where

$$A = \frac{3}{2} k \sum_{i=1}^{4} \tilde{v}_i n^{(i)} / \Delta E,$$

$$B = \sum_{i=1}^{4} \tilde{v}_i n^{(i)} \left[ \frac{1}{2} m_i (|U^{(i)} - \tilde{U}|^2) + \frac{3}{2} k T^{(i)} \right] / \Delta E.$$

Plugging (4.6) into (2.10), we have

$$\tilde{v}_1 \tilde{v}_4 \left[ \tilde{v}_1 n^{(1)} + \tilde{T} \times A - B \right] \exp \left( - \frac{\Delta E}{k \tilde{T}} \right) = \left( \frac{\mu_{12}}{\mu_{34}} \right)^{3/2}.$$

For each $(x_1, x_2, y_3, y_4, \mu, \eta, a, b) \in (\mathbb{R}_+)^4$, we define a strictly increasing function $K : \Lambda_{x_1, x_2, y_3, y_4, \mu, \eta, a, b} \to (0, \infty)$ by

$$K_{x_1, x_2, y_3, y_4, \mu, \eta, a, b}(z) := \frac{\mu_3 \mu_4 (\mu_1 x_1 + az - b)(\mu_2 x_2 + az - b)}{\eta_1 \eta_2 (\eta_3 y_3 - az + b)(\eta_4 y_4 - az + b)} \exp \left( - \frac{\Delta E}{kz} \right).$$

where

$$\Lambda_{x_1, x_2, y_3, y_4, \mu, \eta, a, b} = \{ x > 0 | \{ \mu_1 x_1 + az - b \} \cap \{ \mu_2 x_2 + az - b \} \cap \{ \eta_3 y_3 - az + b \} \cap \{ \eta_4 y_4 - az + b \} \}.$$

Note that there exist positive constants $A_l, A_u, B_l, B_u$ depending only on the parameters given in (3.1), (3.2) and (3.3) such that $A_l \leq A \leq A_u$ and $B_l \leq B \leq B_u$. In the same way as in the proof of Lemma 4.8, we have the following inequality:

$$0 < K^{-1}_{a_l, a_u, a_l, a_u, \mu, \eta, A_u, B_l} \left( \frac{\mu_{12}}{\mu_{34}} \right)^{3/2} \leq \tilde{T} \leq K^{-1}_{a_l, a_u, a_l, a_u, \mu, \eta, A_l, B_u} \left( \frac{\mu_{12}}{\mu_{34}} \right)^{3/2}.$$

5 Fixed Point Set-up

We prove our main theorem applying Banach fixed point theorem to a solution operator defined from the mild form in an appropriately constructed solution space. We define our solution space as follows:

$$\Omega = \left\{ f = (f_1, f_2, f_3, f_4) \in (L^\infty([0, 1]^3; L^2_1(\mathbb{R}_+^3)))^4 | f_i \text{ satisfies } (A), (B), (C) \right\}$$

with the metric $d(f, g) = \sup_{x \in [0, 1]} \| f_i - g_i \|_{L^2_1}$, where $(A), (B),$ and $(C)$ denote

$(A)$ $f_i$ are non-negative.
(B) The macroscopic quantities satisfy the followings:

\[ a_{i,l} \leq \int_{\mathbb{R}^3} f_i(x, v) dv \leq a_{i,u}, \quad c_{i,l} \leq \int_{\mathbb{R}^3} |v|^2 f_i(x, v) dv \leq c_{i,u}. \]

(C) The following lower bound holds:

\[ \left( \int_{\mathbb{R}^3} f_i dv \right) \left( \int_{\mathbb{R}^3} |v|^2 f_i dv \right) - \left( \int_{\mathbb{R}^3} v f_i dv \right)^2 \geq \gamma_i. \]

And we define our solution map for the slow reaction model (2.3) \( \Phi : \Omega \rightarrow \Phi(\Omega) \) by \( \Phi(f_1, f_2, f_3, f_4) = (\phi_1, \phi_2, \phi_3, \phi_4) \) where \( \Phi_i \) is defined as below

\[
\Phi_i(x, v) = \left( e^{-\frac{1}{\tau_1} \int_0^x v_1(y) dy} f_{i,L}(v) + \frac{1}{\tau |v_1|} \int_0^x e^{-\frac{1}{\tau_1} \int_y^x v_1(z) dz} v_1 M_i dy \right) 1_{v_1 > 0} \\
+ \left( e^{-\frac{1}{\tau} \int_x^1 v(y) dy} f_{i,R}(v) + \frac{1}{\tau |v_1|} \int_x^1 e^{-\frac{1}{\tau} \int_y^1 v(z) dz} v_1 M_i dy \right) 1_{v_1 < 0}.
\]

For simplicity, we denote \( \Phi_i = \Phi_i^+ + \Phi_i^- \), where

\[
\Phi_i^+(x, v) = e^{-\frac{1}{\tau_1} \int_0^x v_1(y) dy} f_{i,L}(v) + \frac{1}{\tau |v_1|} \int_0^x e^{-\frac{1}{\tau_1} \int_y^x v_1(z) dz} v_1 M_i dy, \\
\Phi_i^-(x, v) = e^{-\frac{1}{\tau} \int_x^1 v(y) dy} f_{i,R}(v) + \frac{1}{\tau |v_1|} \int_x^1 e^{-\frac{1}{\tau} \int_y^1 v(z) dz} v_1 M_i dy.
\]

In a similar manner, we define our solution map for the fast reaction model (2.6) \( \tilde{\Phi} : \Omega \rightarrow \tilde{\Phi}(\Omega) \) by \( \tilde{\Phi}(f_1, f_2, f_3, f_4) = (\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3, \tilde{\phi}_4) \) where \( \tilde{\Phi}_i \) is defined as below

\[
\tilde{\Phi}_i(x, v) = \left( e^{-\frac{1}{\tau_1} \int_0^x \tilde{v}_1(y) dy} f_{i,L}(v) + \frac{1}{\tau |v_1|} \int_0^x e^{-\frac{1}{\tau_1} \int_y^x \tilde{v}_1(z) dz} \tilde{v}_1 \tilde{M}_i dy \right) 1_{v_1 > 0} \\
+ \left( e^{-\frac{1}{\tau} \int_x^1 \tilde{v}(y) dy} f_{i,R}(v) + \frac{1}{\tau |v_1|} \int_x^1 e^{-\frac{1}{\tau} \int_y^1 \tilde{v}(z) dz} \tilde{v}_1 \tilde{M}_i dy \right) 1_{v_1 < 0}.
\]

For simplicity, we denote \( \tilde{\Phi}_i = \tilde{\Phi}_i^+ + \tilde{\Phi}_i^- \), where

\[
\tilde{\Phi}_i^+(x, v) = e^{-\frac{1}{\tau_1} \int_0^x \tilde{v}_1(y) dy} f_{i,L}(v) + \frac{1}{\tau |v_1|} \int_0^x e^{-\frac{1}{\tau_1} \int_y^x \tilde{v}_1(z) dz} \tilde{v}_1 \tilde{M}_i dy, \\
\tilde{\Phi}_i^-(x, v) = e^{-\frac{1}{\tau} \int_x^1 \tilde{v}(y) dy} f_{i,R}(v) + \frac{1}{\tau |v_1|} \int_x^1 e^{-\frac{1}{\tau} \int_y^1 \tilde{v}(z) dz} \tilde{v}_1 \tilde{M}_i dy.
\]

To apply the Banach fixed point theorem and conclude our main results, we need to prove that \( \Phi(\tilde{\Phi}) \) maps \( \Omega \) into \( \Omega \), and \( \Phi(\tilde{\Phi}) \) is a contraction on \( \Omega \), under the assumption of Theorem 3.3 (Theorem 3.4). These are proved respectively in Proposition 6.1 in Sect. 6 and Proposition 7.1 in Sect. 7.

### 6 \( \Phi \) Maps \( \Omega \) into Itself

The main goal of this section is stated in the following proposition. Since the arguments are similar, we provide detail mainly for the solution operator \( \Phi \) for the slow reaction model 2.3.

**Proposition 6.1** (1) Assume the assumptions in Theorem 3.3 are satisfied. Let \( f \in \Omega \). Then, \( \Phi(f) \in \Omega \) for sufficiently large \( \tau \).
(2) Assume the assumptions in Theorem 3.4 are satisfied. Let \( f \in \Omega \). Then, \( \tilde{\Phi}(f) \in \Omega \) for sufficiently large \( \tau \).

**Remark 6.1** We only consider the slow reaction model. The proof for the fast reaction model is identical.

**Proof** The proof is divided into Lemmas 6.3, 6.5, 6.6, and 6.8 below.

**Lemma 6.2** Let \( f \in \Omega \). Then there exist positive constants \( C_{l,u} \) such that

\[
M_i (1 + |v|^2) \leq C_{l,u} \exp\left(-C_{l,u}|v|^2\right).
\]

**Proof** Lemmas 4.4 and 4.5 imply that

\[
M_i = n_i \left(\frac{m_i}{2\pi k T_i}\right)^{3/2} \exp\left(-\frac{m_i |v - U_i|^2}{2k T_i}\right)
\]

\[
\leq C_{l,u} \exp\left(-\frac{m_i |v - U_i|^2}{2k T_i}\right)
\]

\[
\leq C_{l,u} \exp\left(\frac{m_i |U_i|^2}{2k T_i}\right) \exp\left(-\frac{m_i |v|^2}{4k T_i}\right)
\]

\[
\leq C_{l,u} \exp\left(-C_{l,u}|v|^2\right).
\]

And for \( |v|^2 M_1 \), we know

\[
|v|^2 M_1 \leq C_{l,u} \exp\left(-C_{l,u}|v|^2\right)|v|^2 \leq C_{l,u} \exp\left(-C_{l,u}|v|^2\right),
\]

where we use \( x^2 e^{-x^2} < C \) for some \( C > 0 \).

**Lemma 6.3** Let \( f \in \Omega \). Assume \( f_{i,L} \) and \( f_{i,R} \) satisfy all assumptions in Theorem 3.3. Then \( \phi_i \geq 0 \).

**Proof** By Lemma 4.4, we have

\[
\mathcal{M}_i = n_i \left(\frac{m_i}{2\pi k T_i}\right)^{3/2} \exp\left(-\frac{m_i |v - U_i|^2}{2k T_i}\right)
\]

\[
\geq a_{i,L} \left(\frac{m_i}{2\pi k T_u}\right)^{3/2} \exp\left(-\frac{m_i |v - U_i|^2}{2k T_i}\right)
\]

\[
> 0.
\]

Hence,

\[
\phi_i \geq e^{-\frac{1}{\tau_1} \int_0^1 v(y) dy} f_{i,L}(v) 1_{v_1 > 0} + e^{-\frac{1}{\tau_1} \int_0^1 v(y) dy} f_{i,R}(v) 1_{v_1 < 0} \geq 0.
\]
Lemma 6.4 Assume \( f_{i,L} \) and \( f_{i,R} \) satisfy all assumptions in Theorem 3.3. Then, for sufficiently large \( \tau \), we have

\[
\int_{v_1 > 0} e^{-\frac{1}{\tau |v_1|} \int_0^1 v_1(y) dy} f_{i,L}(v) \left( \frac{1}{|v_1|} \right) dv \geq \frac{1}{4} \int_{v_1 > 0} f_{i,L}(v) \left( \frac{1}{|v_1|} \right) dv
\]

and

\[
\int_{v_1 < 0} e^{-\frac{1}{\tau |v_1|} \int_0^1 v_1(y) dy} f_{i,R}(v) \left( \frac{1}{|v_1|} \right) dv \geq \frac{1}{4} \int_{v_1 < 0} f_{i,R}(v) \left( \frac{1}{|v_1|} \right) dv.
\]

**Proof** Take \( r > 0 \) small enough so that

\[
\int_{v_1 \geq r} f_{i,L}(v) dv \geq \frac{1}{2} \int_{v_1 > 0} f_{i,L}(v) dv.
\]

Then for sufficiently large \( \tau \),

\[
\int_{v_1 > 0} e^{-\frac{1}{\tau |v_1|} \int_0^1 v_1(y) dy} f_{i,R}(v) dv \geq e^{-\frac{C_{i,u}}{\tau r}} \int_{v_1 > r} f_{i,L}dv \geq \frac{1}{4} \int_{v_1 > 0} f_{i,L}dv = a_{i,L}.
\]

Other estimates can be proved by the same argument. We omit it. \( \Box \)

Lemma 6.5 Assume \( f \in \Omega \) and \( f_{i,L} \) and \( f_{i,R} \) satisfy the assumptions in Theorem 3.3. Then we have

\[
\begin{align*}
  a_{i,l} &\leq \int_{\mathbb{R}^3} \phi_i dv, & c_{i,l} &\leq \int_{\mathbb{R}^3} |v|^2 \phi_i dv.
\end{align*}
\]

**Proof** We know

\[
\phi_i \geq e^{-\frac{1}{\tau |v_1|} \int_0^1 v_1(y) dy} f_{i,L}(v)1_{v_1 > 0} + e^{-\frac{1}{\tau |v_1|} \int_1^0 v_1(y) dy} f_{i,R}(v)1_{v_1 > 0}.
\]

Intergrating with respect to \( dv \) and \( |v|^2 dv \), we obtain from Lemma 6.4 that

\[
\int_{\mathbb{R}^3} \phi_i dv \geq a_{i,l}
\]

and

\[
\int_{\mathbb{R}^3} |v|^2 \phi_i dv \geq c_{i,l}.
\]

\( \Box \)

Lemma 6.6 Let \( f \in \Omega \). Assume \( f_{i,L} \) and \( f_{i,R} \) satisfy the assumptions in Theorem 3.3. Then for \( \tau > 0 \) sufficiently large, we have

\[
\begin{align*}
  \int_{\mathbb{R}^3} \phi_i dv &\leq a_{i,u}, & \int_{\mathbb{R}^3} |v|^2 \phi_i dv &\leq c_{i,u}.
\end{align*}
\]

**Proof** We define \( I \) and \( II \) from

\[
\int_{\mathbb{R}^3} \phi_i^\pm dv = \int_{v_1 > 0} e^{-\frac{1}{\tau |v_1|} \int_0^1 v_1(y) dy} f_{i,L}(v) dv
\]

\[
+ \int_{v_1 > 0} \int_0^x \frac{1}{\tau |v_1|} e^{-\frac{1}{\tau |v_1|} \int_y^1 v_1(z) dz} v_1 M_i dy dv.
\]
= I + II.

For $I$, we have,

$$
\int_{v_1 > 0} e^{-\frac{1}{\tau|v_1|} \int_0^t v(y)dy} f_{1,L}(v)dv \leq \int_{v_1 > 0} f_{1,L}(v)dv.
$$

(6.1)

By Lemma 6.2, we compute $II$ as

$$
\int_{v_1 > 0} \int_0^x \frac{1}{\tau|v_1|} e^{-\frac{1}{\tau|v_1|} \int_0^t v(y)dy} v_i \mathcal{M}_i dv
\leq C_{l,u} \int_{v_1 > 0} \int_0^x \frac{1}{\tau|v_1|} e^{-\sum_{r=1}^4 v_r a_l (x-y)/\tau|v_1|} e^{-C_{l,u}|v_1|^2} dy dv
\leq C_{l,u} \left( \int_0^x \int_{v_1 > 0} \frac{1}{\tau|v_1|} e^{-\sum_{r=1}^4 v_r a_l (x-y)/\tau|v_1|} e^{-C_{l,u}|v_1|^2} dv_1 dy \right) \times \left( \int_{\mathbb{R}^2} e^{-C_{l,u}(|v_2|^2+|v_3|^2)} dv_2 dv_3 \right)
\leq C_{l,u} \int_0^x \int_{v_1 > 0} \frac{1}{\tau|v_1|} e^{-\sum_{r=1}^4 v_r a_l (x-y)/\tau|v_1|} e^{-C_{l,u}|v_1|^2} dv_1 dy
=: C_{l,u} II.

We divide the domain of integration as follows:

$$
II = \left\{ \int_0^x \int_{0<v_1<\frac{1}{\tau}} + \int_0^x \int_{\frac{1}{\tau}<v_1<\tau} + \int_0^x \int_{v_1 > \tau} \right\}
\frac{1}{\tau|v_1|} e^{-\sum_{r=1}^4 v_r a_l (x-y)/\tau|v_1|} e^{-C_{l,u}|v_1|^2} dv_1 dy
=: A + B + C.
$$

For $A$, we compute

$$
A = \int_{0<v_1<\frac{1}{\tau}} \int_0^x \frac{1}{\tau|v_1|} e^{-\sum_{r=1}^4 v_r a_l (x-y)/\tau|v_1|} e^{-C_{l,u}|v_1|^2} dy dv_1
\leq \frac{1}{\sum_{r=1}^4 v_r a_l} \int_{0<v_1<\frac{1}{\tau}} \left(1 - e^{-\sum_{r=1}^4 v_r a_l x/\tau|v_1|} \right) e^{-C_{l,u}|v_1|^2} dv_1
\leq \frac{1}{\sum_{r=1}^4 v_r a_l} \frac{1}{\tau},
$$

where we used $1 - e^{-\frac{x}{|v_1|}} \leq 1$ and $e^{-C_{l,u}|v_1|^2} \leq 1$. Similarly we estimate $B$ as

$$
B \leq \frac{1}{\sum_{r=1}^4 v_r a_l} \int_{\frac{1}{\tau}<v_1<\tau} \left(1 - e^{-\sum_{r=1}^4 v_r a_l x/\tau|v_1|} \right) e^{-C_{l,u}|v_1|^2} dv_1
\leq \int_{\frac{1}{\tau}<v_1<\tau} \frac{1}{\tau|v_1|} dv_1
\leq \frac{2}{\tau} \ln \tau,
$$
where we used $1 - e^{-x} \leq x$. 

Finally, we compute

$$C \leq \int_{\tau < v_1} \int_0^\infty \frac{1}{\tau |v_1|} e^{-\sum_{i=1}^4 v_i, a_i (x-y)/\tau |v_1|} e^{-C_l,u |v_1|^2} dy dv_1$$

$$\leq \frac{1}{\tau^2} \int_\mathbb{R} e^{-C_l,u |v_1|^2} dv_1$$

$$\leq C_l,u \tau^2.$$

Summarizing the estimates for $A$, $B$ and $C$, we obtain

$$II \leq C_l,u \left\{ \frac{1}{\tau} + \frac{\ln \tau}{\tau} + \frac{1}{\tau^2} \right\} \leq C_l,u \left\{ \frac{\ln \tau + 1}{\tau} \right\}. \quad (6.3)$$

Combining (6.1) with (6.3), we have

$$\int_{\mathbb{R}^3} \phi_i^+ dv \leq \int_{v_1 > 0} f_{i, L}(v) dv + C_l,u \left\{ \frac{\ln \tau + 1}{\tau} \right\}.$$

We can derive similar estimate for $\phi_i^-$:

$$\int_{\mathbb{R}^3} \phi_i^- dv \leq \int_{v_1 < 0} f_{i, R}(v) dv + C_l,u \left\{ \frac{\ln \tau + 1}{\tau} \right\},$$

and hence

$$\int_{\mathbb{R}^3} \phi_i dv \leq \frac{a_{i,u}}{2} + C_l,u \left\{ \frac{\ln \tau + 1}{\tau} \right\}.$$

By choosing sufficiently large $\tau > 0$, we get the desired result. The proof for the second estimate is almost identical. We omit it. $\square$

**Lemma 6.7** Let $f \in \Omega$. Assume $f_{i, L}$ and $f_{i, R}$ satisfy the assumptions in Theorem 3.3. Then for $j = 2, 3$, we have

$$\left| \int_{\mathbb{R}^3} \phi_i v_j dv \right| \leq C_l,u \left( \frac{\ln \tau + 1}{\tau} \right).$$

**Proof** We consider this only for $\phi_i^+$ because the other case can be proved by similar ways. Integrating $\phi_i^+$ with respect to $v_2 dv_2 dv_3$, we have

$$\int_{\mathbb{R}^2} \phi_i^+ v_2 dv_2 dv_3 = e^{-\frac{1}{\tau |v_1|} \int_0^x v_i(y) dy} \int_{\mathbb{R}^2} f_{i, L}(v) v_2 dv_2 dv_3$$

$$+ \frac{1}{\tau |v_1|} \int_0^x e^{-\frac{1}{\tau |v_1|} \int_0^y v_i(z) dz} v_i(y) \left( \int_{\mathbb{R}^2} \mathcal{M}_i v_2 dv_2 dv_3 \right) dy.$$

By our assumption on $f_{i, L}$, it can be reduced to the following:

$$\int_{\mathbb{R}^2} \phi_i^+ v_2 dv_2 dv_3 = \frac{1}{\tau |v_1|} \int_0^x e^{-\frac{1}{\tau |v_1|} \int_0^y v_i(z) dz} v_i(y) \left( \int_{\mathbb{R}^2} \mathcal{M}_i v_2 dv_2 dv_3 \right) dy. \quad (6.4)$$

As in the computation in (6.2), we see
\[
\mathcal{M}_i v_2 d v_2 d v_3 \leq C_{l,u} e^{-C_{l,u} |v_1|^2} \int_{\mathbb{R}^2} e^{-C_{l,u} (|v_2|^2 + |v_3|^2)} |v_2| d v_2 d v_3 \\
\leq C_{l,u} e^{-C_{l,u} |v_1|^2}.
\]

Substituting this in (6.4) and then integrating on \( v_1 > 0 \), we get
\[
\int_{\mathbb{R}^2} \phi_1^+ v_2 d v \leq C_{l,u} \int_0^\infty \int_{v_1 > 0} \frac{1}{\tau |v_1|} e^{-\frac{1}{\tau |v_1|} \int_0^\infty \phi_1^+ v_1(z) dz} e^{-C_{l,u} |v_1|^2} d v_1 d y \\
\leq C_{l,u} \left\{ \ln \tau + \frac{1}{\tau} \right\}
\]
where we had the last inequality from (6.2) and (6.3).

\[\square\]

**Lemma 6.8** Let \( f \in \Omega \). Assume \( f_{i,L} \) and \( f_{i,R} \) satisfy the assumptions in Theorem 3.3. Then, for sufficiently large \( \tau > 0 \), we have
\[
\left( \int_{\mathbb{R}^3} \phi_i d v \right) \left( \int_{\mathbb{R}^3} \phi_i |v|^2 d v \right) - \left| \int_{\mathbb{R}^3} \phi_i v d v \right|^2 \geq \gamma_i.
\]

**Proof** Applying the Cauchy–Schwarz inequality, we have
\[
\left( \int_{\mathbb{R}^3} \phi_i d v \right) \left( \int_{\mathbb{R}^3} \phi_i |v|^2 d v \right) - \left| \int_{\mathbb{R}^3} \phi_i v d v \right|^2 \\
\geq \left( \int_{\mathbb{R}^3} \phi_i |v| d v \right)^2 - \left| \int_{\mathbb{R}^3} \phi_i v d v \right|^2 \\
\geq \left( \int_{\mathbb{R}^3} \phi_i |v_1| d v \right)^2 - \left| \int_{\mathbb{R}^3} \phi_i v_1 d v \right|^2.
\]

And we decompose the last term as
\[
\left( \int_{\mathbb{R}^3} \phi_i |v_1| d v \right)^2 - \left| \int_{\mathbb{R}^3} \phi_i v_1 d v \right|^2 \\
= \left( \int_{\mathbb{R}^3} \phi_i |v_1| d v \right)^2 - \left[ \left( \int_{\mathbb{R}^3} \phi_i v_1 d v \right)^2 + \left( \int_{\mathbb{R}^3} \phi_i v_2 d v \right)^2 + \left( \int_{\mathbb{R}^3} \phi_i v_3 d v \right)^2 \right].
\]

In view of Lemma 6.7, we have
\[
R \leq C_{l,u} \left( \frac{\ln \tau + 1}{\tau} \right).
\]

On the other hand, since
\[
\left( \int_{\mathbb{R}^3} \phi_i |v_1| d v \right)^2 - \left( \int_{\mathbb{R}^3} \phi_i v_1 d v \right)^2 \\
\geq \left( \int_{\mathbb{R}^3} \phi_i (|v_1| + v_1) d v \right) \left( \int_{\mathbb{R}^3} \phi_i (|v_1| - v_1) d v \right) \\
= 4 \left( \int_{v_1 > 0} \phi_i |v_1| d v \right) \left( \int_{v_1 < 0} \phi_i |v_1| d v \right).
\]
Lemma 6.4 implies that
\[
\left( \int_{\mathbb{R}^3} \phi_i |v_1| dv \right)^2 - \left( \int_{\mathbb{R}^3} \phi_i v_1 dv \right)^2 \\
\geq 4 \left( \int_{v_1 > 0} e^{-\frac{1}{4v_1^2} \int_0^1 v(y) dy} f_{i,L}(v) |v_1| dv \right) \left( \int_{v_1 < 0} e^{-\frac{1}{4v_1^2} \int_0^1 v(y) dy} f_{i,R}(v) |v_1| dv \right) \\
\geq \frac{1}{4} \left( \int_{v_1 > 0} f_{i,L}(v) |v_1| dv \right) \left( \int_{v_1 < 0} f_{i,R}(v) |v_1| dv \right) \\
= 4\gamma_l.
\]

In conclusion, for sufficiently large \( \tau > 0 \), we obtain
\[
\left( \int_{\mathbb{R}^3} \phi_i dv \right) \left( \int_{\mathbb{R}^3} \phi_i |v|^2 dv \right) - \left| \int_{\mathbb{R}^3} \phi_i v dv \right|^2 \\
\geq 4\gamma_l - C_{l,u} \left( \ln \frac{\tau + 1}{\tau} \right) \\
\geq \gamma_l.
\]

7 \( \Phi \) is Contractive in \( \Omega \)

It remains to show the solution map \( \Phi \) and \( \tilde{\Phi} \) are contraction maps. We start with the estimates for the single component macroscopic fields and global macroscopic fields, which holds commonly for the first and second model.

Lemma 7.1 Let \( f = (f_1, f_2, f_3, f_4) \), \( g = (g_1, g_2, g_3, g_4) \) \( \in \Omega \). Then we have:

(1) The single component macroscopic parameters satisfy
\[
|n_f^{(i)} - n_g^{(i)}|, |U_f^{(i)} - U_g^{(i)}|, |T_f^{(i)} - T_g^{(i)}| \leq C_{l,u} \sup_{x \in [0,1]} \|f_i - g_i\|_{L_2^1}.
\]

(2) The global macroscopic parameters satisfy
\[
|n_f - n_g|, |U_f - U_g|, |T_f - T_g| \leq C_{l,u}d(f, g).
\]

Proof (1) The first estimate is straightforward:
\[
|n_f^{(i)} - n_g^{(i)}| = \int_{\mathbb{R}^3} |f_i - g_i| dv \leq \sup_{x \in [0,1]} \|f_i - g_i\|_{L_2^1}.
\]

For the second estimate, we use \( \rho_f^{(i)} \geq m_i a_{i,1} \) to get
\[
|U_f^{(i)} - U_g^{(i)}| \leq \frac{1}{\rho_f^{(i)}} |\rho_f^{(i)} U_f^{(i)} - \rho_g^{(i)} U_g^{(i)}| + \frac{1}{\rho_f^{(i)}} |\rho_f^{(i)} - \rho_g^{(i)}| \|U_g^{(i)}\| \\
\leq \frac{m_i}{\rho_f^{(i)}} \int_{\mathbb{R}^3} |f_i - g_i| |v| dv + \frac{m_i |U_g^{(i)}|}{\rho_f^{(i)}} \int_{\mathbb{R}^3} |f_i - g_i| dv \\
\leq C_{l,u} \sup_{x \in [0,1]} \|f_i - g_i\|_{L_2^1}.
\]

□
For the third estimate, we decompose

\[
|T_f^{(i)} - T_g^{(i)}| \leq \frac{1}{n_f} |n_f^{(i)} T_f^{(i)} - n_g^{(i)} T_g^{(i)}| + \frac{1}{n_f} |n_f^{(i)} - n_g^{(i)}| |T_f^{(i)}|
\]

\[
\leq \frac{m_i}{3k n_f^{(i)}} \int_{\mathbb{R}^3} \left| f_i |v - U_f^{(i)}|^2 - g_i |v - U_g^{(i)}|^2 \right| dv + \frac{m_i |T_g^{(i)}|}{n_f} \int_{\mathbb{R}^3} |f_i - g_i| dv
\]

\[= I + II.\]

Then, \( n_f^{(i)} \geq a_{i,i} \) and Lemma 4.1 gives

\[II \leq C_{l,u} \sup_{x \in [0,1]} ||f_i - g_i||_{L^1_2},\]

and

\[I \leq \frac{m_i}{k a_{i,i}} \int_{\mathbb{R}^3} \left| f_i |v - U_f^{(i)}|^2 - g_i |v - U_g^{(i)}|^2 \right| dv
\]

\[\leq \frac{m_i}{k a_{i,i}} \int_{\mathbb{R}^3} \left| (f_i - g_i) |v - U_f^{(i)}|^2 + g_i (|v - U_f^{(i)}|^2 - |v - U_g^{(i)}|^2) \right| dv
\]

\[= \frac{m_i}{k a_{i,i}} \int_{\mathbb{R}^3} \left| (f_i - g_i) |v - U_f^{(i)}|^2 + g_i (2v - U_f^{(i)} - U_g^{(i)}) (U_f^{(i)} - U_g^{(i)}) \right| dv
\]

\[\leq C_{l,u} \int_{\mathbb{R}^3} |f_i - g_i| (1 + |v|^2) + |g_i| (1 + |v|) |U_f^{(i)} - U_g^{(i)}| dv
\]

\[\leq C_{l,u} \sup_{x \in [0,1]} ||f_i - g_i||_{L^1_2}.\]

(2) The estimates for the global macroscopic parameters follows directly from (1). We omit the proof.

\[\square\]

And then, we consider the estimates for the auxiliary parameters and the collision frequencies for the first model (2.3).

**Lemma 7.2** Let \( f = (f_1, f_2, f_3, f_4), \ g = (g_1, g_2, g_3, g_4) \in \Omega \). Then we have

\[|n_{f,i} - n_{g,i}|, |U_{f,i} - U_{g,i}|, |T_{f,i} - T_{g,i}|, |v_{f,i} - v_{g,i}| \leq C_{l,u} \sum_{j=1}^{4} \sup_{x \in [0,1]} ||f_j - g_j||_{L^1_2}.\]

**Proof** We first establish the following claim:

\[|S_f - S_g| \leq C_{l,u} d(f, g), \quad (7.1)\]

and

\[|v_{f,i} - v_{g,i}| \leq C_{l,u} \sum_{j=1}^{4} \sup_{x \in [0,1]} ||f_j - g_j||_{L^1_2}. \quad (7.2)\]
For simplicity, we denote $K(x) = \Gamma(3/2, x)e^x$ and compute

$$|S_f - S_g| \leq C_{l,u} \left[ K\left(\frac{\Delta E}{kT_f}\right) - K\left(\frac{\Delta E}{kT_g}\right)\right] n_f^{(3)} n_f^{(4)} + K\left(\frac{\Delta E}{kT_g}\right) n_f^{(3)} - n_g^{(3)} n_f^{(4)}$$

$$+ C_{l,u} \left[ \Gamma\left(\frac{3}{2}, \frac{\Delta E}{kT_f}\right) - \Gamma\left(\frac{3}{2}, \frac{\Delta E}{kT_g}\right)\right] n_f^{(3)} n_f^{(4)} + \Gamma\left(\frac{3}{2}, \frac{\Delta E}{kT_f}\right) n_f^{(3)} - n_g^{(3)} n_f^{(4)}$$

$$+ \Gamma\left(\frac{3}{2}, \frac{\Delta E}{kT_g}\right) n_f^{(3)} n_f^{(4)} - n_f^{(4)} \right].$$

(7.3)

Since $f, g \in \Omega$, $T_f$ and $T_g$ are bounded from below and above by constants defined in terms of constants given in (3.1), (3.2) and (3.3) (Lemma 4.1). Therefore, since $K$ and $\Gamma$ are continuously differentiable, we derive

$$K\left(\frac{\Delta E}{kT_f}\right), \Gamma\left(\frac{3}{2}, \frac{\Delta E}{kT_f}\right) \leq C_{l,u}$$

and

$$\left| K\left(\frac{\Delta E}{kT_f}\right) - K\left(\frac{\Delta E}{kT_g}\right) \right|, \left| \Gamma\left(\frac{3}{2}, \frac{\Delta E}{kT_f}\right) - \Gamma\left(\frac{3}{2}, \frac{\Delta E}{kT_g}\right) \right| \leq C_{l,u} \left| \frac{\Delta E}{kT_f} - \frac{\Delta E}{kT_g} \right|$$

by the Mean value theorem

$$\leq C_{l,u} \left| \frac{\Delta E(T_g - T_f)}{kT_f T_g} \right|$$

$$\leq C_{l,u} |T_f - T_g|$$

Combining these with (7.3) and Lemma 7.1 proves the first estimate of the claim. Once (7.1) is established, the second estimate of the claim follows directly from the definition of $v_j$ and Lemma 7.1.

From (7.1) and (7.2), we have

$$|n_{f,i} - n_{g,i}| \leq |n_f^{(i)} - n_g^{(i)}| + |S_f| - S_g| \left| \frac{v_{f,i} - v_{g,i}}{v_{f,i} v_{g,i}} \right| + \frac{1}{v_{f,i}} |S_f - S_g|$$

$$\leq C_{l,u} \sum_{j=1}^4 \sup_{x \in [0,1]} \| f_j - g_j \|_{L^2_x}.$$

We recall the definition of $U_i$

$$U_i = \frac{n_i^{(i)}}{n_i} U^{(i)} + \frac{2}{m_i v_i} \sum_{j=1}^4 \chi_{ij} \mu_{ij} n_i^{(j)} (U^{(j)} - U^{(i)}) + \frac{\lambda_i}{n_i v_i} US,$$

and use (7.1), (7.2) and Lemma 4.1 to get

$$|U_{f,i} - U_{g,i}| \leq C_{l,u} \sum_{j=1}^4 \sup_{x \in [0,1]} \| f_j - g_j \|_{L^2_x}.$$

The proof for the remaining estimates are almost identical, we omit it.
And then, we consider the estimates for the auxiliary parameters and the collision frequency for the second model (2.6).

**Lemma 7.3** Let \( f = (f_1, f_2, f_3, f_4), \ g = (g_1, g_2, g_3, g_4) \in \Omega. \ Then we have

\[
|\tilde{n}_{f,i} - \tilde{n}_{g,i}|, |\tilde{U}_f - \tilde{U}_g|, |\tilde{T}_f - \tilde{T}_g|, |\tilde{v}_{f,i} - \tilde{v}_{g,i}| \leq C_{l,u} \sum_{j=1}^{4} \sup_{x \in [0,1]} ||f_j - g_j||_{L^1_2}.
\]

**Proof** We recall from the proof of (4.8) that \( \tilde{n}_{f,1} = G(n_f, n_f, \tilde{v}_f, \tilde{v}_f, T_f, V_f) \) and \( \tilde{n}_{g,1} = G(n_g, n_g, \tilde{v}_g, \tilde{v}_g, T_g, V_g) \), and for each \( (x, y, \mu, \eta, \alpha, \beta) \in (\mathbb{R}_+)^{20} \times \mathbb{R}^4 \) we have

\[
\frac{d}{dz} F_{x,y,\mu,\eta,\alpha,\beta}(z) = \frac{1}{z} + \frac{\mu_1}{\mu_2 x_2 + \mu_1 z - \eta_1 y_1} + \frac{\mu_1}{\eta_3 y_3 - \mu_1 z + \eta_1 y_1}
\]

\[
+ \frac{3}{2} (\Delta E)^2 \frac{\mu_1}{\eta_4 y_4 - \mu_1 z + \eta_1 y_1} \left[ \sum_{i=1}^{4} \mu_i x_i \left( \frac{1}{2} m_i (\beta_i^2 + 3 k \alpha_i) + \Delta E (\mu_1 z - \eta_1 y_1) \right) \right]^2 > 0.
\]

By Implicit function theorem, \( G \) is continuously differentiable at all \( (x, y, \mu, \eta, \alpha, \beta) \in (\mathbb{R}_+)^{20} \times \mathbb{R}^4 \). Hence the gradient of \( G \) is bounded on the following compact set:

\[
K := [a_1, a_2]^8 \times [v_m, v_M]^8 \times [T_1, T_3]^4 \times [-R, R]^4.
\]

Therefore, by the mean value theorem, we get

\[
|\tilde{n}_{f,1} - \tilde{n}_{g,1}| \leq \max_{(x, y, \mu, \eta, \alpha, \beta) \in K} |\nabla G(x, y, \mu, \eta, \alpha, \beta)|
\times |(n_f, n_f, \tilde{v}_f, \tilde{v}_f, T_f, V_f) - (n_g, n_g, \tilde{v}_g, \tilde{v}_g, T_g, V_g)|.
\]

These, together with Lemma 7.1 imply \( |\tilde{n}_{f,1} - \tilde{n}_{g,1}| \leq C_{l,u} \sum_{j=1}^{4} \sup_{x \in [0,1]} ||f_j - g_j||_{L^1_2}. \) The proof for the other estimates are, albeit more tedious, essentially same. We omit it. \( \square \)

**Lemma 7.4** Let \( f = (f_1, f_2, f_3, f_4) \in \Omega \) and \( g = (g_1, g_2, g_3, g_4) \in \Omega. \ Then, we have

\[
|M(f_i) - M(g_i)| \leq C_{l,u} d(f, g),
\]

and

\[
|\tilde{M}(f_i) - \tilde{M}(g_i)| \leq C_{l,u} d(f, g).
\]

**Proof** Since the proof is identical, we only consider the first estimate. We denote \( M(f_i) := M(m_i, n_{f,i}, U_{f,i}, T_{f,i}), M(g_i) := M(m_i, n_{g,i}, U_{g,i}, T_{g,i}) \) and apply Taylor expansion to write \( M(f_i) - M(g_i) \) as

\[
M(f_i) - M(g_i) = (n_{f,i} - n_{g,i}) \int_0^1 \frac{\partial M(\theta)}{\partial n} d\theta
\]

\[
+ (U_{f,i} - U_{g,i}) \int_0^1 \frac{\partial M(\theta)}{\partial U} d\theta
\]

\[
+ (T_{f,i} - T_{g,i}) \int_0^1 \frac{\partial M(\theta)}{\partial T} d\theta
\]

\[
= A + B + C.
\]
where
\[
\frac{\partial M(\theta)}{\partial X} = \frac{\partial M(\theta)}{\partial X}(m_i, n_\theta, U_\theta, T_\theta)
\]
for \((n_\theta, U_\theta, T_\theta) = (1 - \theta)(n_{f,i}, U_{f,i}, T_{f,i}) + \theta(n_{g,i}, U_{g,i}, T_{g,i})\). For \(A\), we observe
\[
\frac{\partial M(\theta)}{\partial n} = \frac{1}{n_\theta} M(\theta),
\]
so that
\[
\left| \frac{\partial M(\theta)}{\partial \rho} \right| \leq C_{l,u} e^{-C_{l,u}|v|^2},
\]
from Lemma 6.2.

For \(B\), we similarly observe
\[
\frac{\partial M(\theta)}{\partial U} = \frac{m_i(v - U_\theta)}{kT_\theta} M(\theta),
\]
which implies
\[
\left| \frac{\partial M(\theta)}{\partial U} \right| \leq C_{l,u} (1 + |v|) M(\theta)
\]
\[
\leq C_{l,u} e^{-C_{l,u}|v|^2},
\]
by Lemma 6.2 and Lemma 4.4.

Finally, we compute the derivative w.r.t \(T\) as
\[
\frac{\partial M(\theta)}{\partial T} = \left\{ -\frac{3}{2T_\theta} + \frac{m_i|v - U_\theta|^2}{2kT_\theta^2} \right\} M(\theta),
\]
and apply Lemma 6.2, Lemma 4.4 and Lemma 4.5 to get
\[
\left| \frac{\partial M(\theta)}{\partial T} \right| \leq C_{l,u} (1 + |v|^2) e^{-C_{l,u}|v|^2} \leq C_{l,u} e^{-C_{l,u}|v|^2}.
\]
Combining all these estimates, we obtain
\[
|M(f_i) - M(g_i)| \leq C_{l,u} \left\{ |n_{f,i} - n_{g,i}| + |U_{f,i} - U_{g,i}| + |T_{f,i} - T_{g,i}| \right\} e^{-C_{l,u}|v|^2}.
\]
This, together with Lemma 7.1, gives the desired result. \(\square\)

**Proposition 7.1** Assume \(f_{i,L}\) and \(f_{i,R}\) satisfy the assumptions in Theorem 3.3 or 3.4. Let \(f = (f_1, f_2, f_3, f_4), g \in (g_1, g_2, g_3, g_4) \in \Omega\). Then there exists a \(\alpha \in (0, 1)\) such that
\[
d(\Phi(f), \Phi(g)) \leq \alpha d(f, g),
\]
and
\[
d(\tilde{\Phi}(f), \tilde{\Phi}(g)) \leq \alpha d(f, g),
\]
if \(\tau\) is taken sufficiently large.
**Proof** The proof is almost identical for both case. We only consider the first estimate. Also, we only compute $|\phi^+(f_i) - \phi^+(g_i)|$ because the argument for $|\phi^-(f_i) - \phi^-(g_i)|$ is same. Consider

\[
\phi^+(f_i) - \phi^+(g_i) = \left\{ e^{-\frac{1}{|v_1|} \int_0^x v_{f,i}(z)dz} - e^{-\frac{1}{|v_1|} \int_0^x v_{g,i}(z)dz} \right\} f_{i,L}(v) + \frac{1}{|v_1|} \left( \int_0^x e^{-\frac{1}{|v_1|} \int_0^1 f_y v_{f,i}(z)dz} v_{f,i}(y) \mathcal{M}(f_i)dy - \int_0^x e^{-\frac{1}{|v_1|} \int_0^1 f_y v_{g,i}(z)dz} v_{g,i}(y) \mathcal{M}(g_i)dy \right).
\]

By the mean value theorem, there exists $0 < \theta < 1$ such that

\[
I = \left\{ e^{-\frac{1}{|v_1|} \int_0^x v_{f,i}(z)dz} - e^{-\frac{1}{|v_1|} \int_0^x v_{g,i}(z)dz} \right\} f_{i,L}(v) + \frac{1}{|v_1|} e^{-\frac{1}{|v_1|} \int_0^1 (1-\theta)v_{f,i}(y)+\theta v_{g,i}(y)dy} \left( \int_0^x \{ v_{f,i}(y) - v_{g,i}(y) \} dy \right) f_{i,L}(v).
\]

(7.4)

Therefore, by $n_f^{(i)} - n_g^{(i)} \geq a_i$, and Lemma 7.1, we obtain

\[
|I| \leq \frac{1}{|v_1|} \left( e^{-\frac{1}{|v_1|} \int_0^x \sum_{i=1}^4 v_{i,j}q_i dy} \int_0^x |v_{f,i} - v_{g,i}| dy \right) f_{i,L}(v)
\]

\[
\leq \frac{C_{i,u}}{|v_1|} e^{-\frac{\sum_{i=1}^4 v_{i,j}q_i}{|v_1|}} f_{i,L}(f, g).
\]

(7.5)

We divide the estimate of $II$ into the following three parts. First, by a similar way as in the proof for $I$, we estimate the difference of integrating factor as

\[
\frac{1}{|v_1|} \left| \int_0^x e^{-\frac{1}{|v_1|} \int_0^1 f_y v_{f,i}(z)dz} v_{f,i}(y) \mathcal{M}(f_i)dy - \int_0^x e^{-\frac{1}{|v_1|} \int_0^1 f_y v_{g,i}(z)dz} v_{g,i}(y) \mathcal{M}(f_i)dy \right|
\]

\[
\leq \frac{1}{|v_1|} \left| \int_0^x \frac{1}{|v_1|} e^{-\frac{1}{|v_1|} \int_0^1 (1-\theta)v_{f,i}(y)+\theta v_{g,i}(y)dz} \int_0^x |v_{f,i}(z) - v_{g,i}(z)| dz v_{f,i}(y) \mathcal{M}(f_i)dy \right|
\]

\[
\leq \left( \frac{C_{i,u}}{|v_1|} \right) \int_0^x \frac{1}{|v_1|} e^{-\frac{\sum_{i=1}^4 v_{i,j}q_i}{|v_1|}} \mathcal{M}(f_i)dy \right) d(f, g).
\]

(7.6)

where we used that $xe^{-x} < C$ for some $C > 0$. Secondly we use (7.2) to estimate the difference of the collision frequency:

\[
\frac{1}{|v_1|} \left| \int_0^x e^{-\frac{1}{|v_1|} \int_0^1 f_y v_{g,i}(z)dz} |v_{f,i}(y) - v_{g,i}(y)| \mathcal{M}(f_i)dy \right|
\]

\[
\leq \frac{C_{i,u}}{|v_1|} \int_0^x e^{-\frac{\sum_{i=1}^4 v_{i,j}q_i}{|v_1|}(x-y)} \mathcal{M}(f_i)dy \cdot d(f, g).
\]

(7.7)
Finally, by Lemma 7.4, we estimate the difference of the Maxwellians:

\[
\frac{1}{\tau |v_1|} \int_0^x e^{-\frac{\tau}{|v_1|} \int_0^y \phi(z) dz} \phi(g)(z) \{\mathcal{M}(f_i) - \mathcal{M}(g_i)\} \, dy
\]

(7.8)

\[
\leq C_{l,u} \frac{1}{\tau |v_1|} \int_0^x e^{-\frac{\sum_{j=1}^L v_{ij}^{\eta_q}}{\tau |v_1|} (y-x)} e^{-C_{l,u}|v|^2} \, dy \cdot d(f, g).
\]

Combining (7.5), (7.6), (7.7), and (7.8), we obtain

\[
|\phi^+(f_i) - \phi^+(g_i)| \leq C_{l,u} d(f, g) \left( \int_{v_1 > 0} \int_0^x \frac{1}{\tau |v_1|} f_{i,L}(1 + |v|^2) \, dy \, dv 
+ \int_{v_1 > 0} \int_0^x \frac{1}{\tau |v_1|} e^{-\frac{\sum_{j=1}^L v_{ij}^{\eta_q}}{\tau |v_1|} (y-x)} \mathcal{M}(f_i)(1 + |v|^2) \, dy \, dv 
+ \int_{v_1 > 0} \int_0^x \frac{1}{\tau |v_1|} e^{-\frac{\sum_{j=1}^L v_{ij}^{\eta_q}}{\tau |v_1|} (y-x)} e^{-C_{l,u}|v|^2} (1 + |v|^2) \, dy \, dv \right).
\]

Therefore,

\[
||\phi^+(f_i) - \phi^+(g_i)||_{L^2} \leq C_{l,u} d(f, g) \left( \int_{v_1 > 0} \int_0^x \frac{1}{\tau |v_1|} f_{i,L}(1 + |v|^2) \, dy \, dv 
+ \int_{v_1 > 0} \int_0^x \frac{1}{\tau |v_1|} e^{-\frac{\sum_{j=1}^L v_{ij}^{\eta_q}}{\tau |v_1|} (y-x)} \mathcal{M}(f_i)(1 + |v|^2) \, dy \, dv 
+ \int_{v_1 > 0} \int_0^x \frac{1}{\tau |v_1|} e^{-\frac{\sum_{j=1}^L v_{ij}^{\eta_q}}{\tau |v_1|} (y-x)} e^{-C_{l,u}|v|^2} (1 + |v|^2) \, dy \, dv \right).
\]

Applying Lemma 6.2, we have

\[
||\phi^+(f_i) - \phi^+(g_i)||_{L^2} \leq C_{l,u} d(f, g) \left( \int_{v_1 > 0} \int_0^x \frac{1}{\tau |v_1|} f_{i,L}(1 + |v|^2) \, dy \, dv 
+ \int_{v_1 > 0} \int_0^x \frac{1}{\tau |v_1|} e^{-\frac{\sum_{j=1}^L v_{ij}^{\eta_q}}{\tau |v_1|} (y-x)} e^{-C_{l,u}|v|^2} (1 + |v|^2) \, dy \, dv \right).
\]

Then, from the same computation as in the estimate of \( \mathcal{I} \mathcal{I} \) in (6.2), we obtain

\[
||\phi^+(f_i) - \phi^+(g_i)||_{L^2} \leq C_{l,u} \left[ a_{i,s} + \frac{c_{i,s}}{\tau} + \left( \frac{\ln \tau + 1}{\tau} \right) \right] d(f, g)
\]

\[
\leq C_{l,u} \left( \frac{\ln \tau + 1}{\tau} \right) d(f, g).
\]

By a similar argument, we have

\[
||\phi^-(f_i) - \phi^-(g_i)||_{L^2} \leq C_{l,u} \left( \frac{\ln \tau + 1}{\tau} \right) d(f, g).
\]

Hence,

\[
||\phi(f_i) - \phi(g_i)||_{L^2} \leq C_{l,u} \left( \frac{\ln \tau + 1}{\tau} \right) d(f, g).
\]

Taking supremum on both sides and choosing sufficiently large \( \tau > 0 \), we get the desired result.

\[\square\]

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