Secure and Efficient Compressed Sensing Based Encryption With Sparse Matrices

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Abstract—In this paper, we study the security of a compressed sensing (CS) based cryptosystem called a sparse one-time sensing (S-OTS) cryptosystem, which encrypts a plaintext with a sparse measurement matrix. To construct the secret matrix and renew it at each encryption, a bipolar keystream and a random permutation pattern are employed as cryptographic primitives, which can be obtained by a keystream generator of stream ciphers. With a small number of nonzero elements in the measurement matrix, the S-OTS cryptosystem achieves efficient CS encryption in terms of memory and computational cost. In security analysis, we show that the S-OTS cryptosystem can be indistinguishable as long as each plaintext has constant energy, which formalizes computational security against ciphertext only attacks (COA). In addition, we consider a chosen plaintext attack (CPA) against the S-OTS cryptosystem, which consists of two sequential stages, keystream and key recovery attacks. Against keystream recovery under CPA, we demonstrate that the S-OTS cryptosystem can be secure with overwhelmingly high probability, as an adversary needs to distinguish a prohibitively large number of candidate keystreams. Finally, we conduct an information-theoretic analysis to show that the S-OTS cryptosystem can be resistant against key recovery under CPA by guaranteeing that the probability of success is extremely low. In conclusion, the S-OTS cryptosystem can be computationally secure against COA and the two-stage CPA, while providing efficiency in CS encryption.

Index Terms—Compressed encryption, stream ciphers, indistinguishability, plaintext attacks.

I. INTRODUCTION

COMPRESSED sensing (CS) allows to recover a sparse signal from a much smaller number of measurements than the signal dimension. A signal \(\mathbf{x} \in \mathbb{R}^N\) is called \(K\)-sparse with respect to an orthonormal sparsifying basis \(\Psi\) if \(\alpha = \Psi^T \mathbf{x}\) has at most \(K\) nonzero entries, where \(K \ll N\). The sparse signal \(\mathbf{x}\) is linearly measured by \(\mathbf{y} = \Phi^T \mathbf{x} + \mathbf{n} = \Phi^T \alpha + \mathbf{n}\), where \(\Phi = \mathbf{A} \Psi^T\) obeys the restricted isometry property (RIP). A stable and robust reconstruction of \(\alpha\) can be guaranteed from the incomplete measurement \(\mathbf{y}\). The CS reconstruction can be accomplished by solving an \(l_1\)-minimization problem with convex optimization or greedy algorithms. With efficient measurement and stable reconstruction, the CS technique has been of interest in a variety of research fields, e.g., communications, sensor networks, image processing, radar, etc.

The CS principle can be applied in a symmetric-key cryptosystem for information security. The CS-based cryptosystem can simultaneously compress and encrypt a plaintext \(\mathbf{x}\) through a CS measurement process by keeping the measurement matrix \(\Phi\) secret. With the knowledge of \(\Phi\), the ciphertext \(\mathbf{y}\) can then be decrypted by a legitimate recipient through a CS reconstruction process. The CS-based cryptosystem can be suitable for security of real-world applications such as multimedia, smart grid, and the Internet of Things (IoT) where plaintexts of interest can be modeled to be sparse in a proper basis. Readers are referred to [29] for a comprehensive review of CS in the field of information security.

Racchi and Baron proved that the CS-based cryptosystem cannot be perfectly secure, but might be computationally secure. In [26], Orsdeim et al. showed that it is computationally secure against a key search technique via an algebraic approach. In [27–29], CS-based cryptosystems have been studied in the framework of physical layer security by exploiting the randomness of wireless channels. To avoid plaintext attacks, a CS-based cryptosystem can employ the secret measurement matrix in a one-time sensing (OTS) manner, where the matrix is renewed at each encryption. In a CS-based cryptosystem using the OTS concept, a sender and a legitimate recipient can use a secure random number generator (SRNG) to construct the secret matrices efficiently, by sharing only the initial seed of SRNG as a secret key.

Using random Gaussian measurement matrices in the OTS manner, Bianchi et al. showed that the Gaussian-OTS (G-OTS) cryptosystem can be perfectly secure, as long as each plaintext has constant energy. In [32], the authors made a similar security analysis for a CS-based cryptosystem which employs circulant matrices for efficient CS processes. It has also been studied for wireless security in [34], while a CS-based cryptosystem with general partial unitary matrices embedding a keystream has been investigated in [35]. In [36], Cambareri et al. employed random Bernoulli matrices with the OTS concept to encrypt plaintexts sparse with respect to a non-identity orthonormal basis, which we call the Bernoulli-OTS (B-OTS) cryptosystem in this paper. With the notion of asymptotic spherical secrecy, they analyzed the security of the B-OTS cryptosystem and its class dependent variations can be resistant against known key recovery attacks in [37]. In [38], the security of the asymptotically Gaussian-OTS (AG-OTS) cryptosystem, which employs random Bernoulli matrices multiplied by a unitary matrix, has been discussed in the presence of wireless channels. In addition, the indistinguishability of the G-OTS and the B-OTS cryptosystems has been studied in [39], which turned out to be highly sensitive to energy variation of plaintexts.
Although the OTS concept is necessary for security against plaintext attacks, it may cause complexity issues in practical implementation. In processing large-size signals, renewing the measurement matrix at each encryption would require massive data storage and computing resources. To resolve this issue in CS imaging field, a series of works have applied the technique of parallel CS (PCS) \cite{41} to CS-based cryptosystems in the OTS manner, where the PCS framework can significantly reduce the size of a measurement matrix at each encryption. The CS-based cryptosystem proposed in \cite{42} encrypts each image column-by-column, renewing its measurement matrix at each encryption with the counter mode of operation \cite{39} in block ciphers. In this scheme, however, the plaintexts with unequal energy result in information leakage, which can be a cryptographic weakness. To overcome this issue, each plaintext should be normalized, which requires a secure auxiliary channel to transfer the energy information to a recipient \cite{43}. In \cite{43}, Hu et al. applied an additional cryptographic diffusion process after each CS encryption of \cite{42} in order to prevent the information leakage.

In this paper, we propose the sparse-OTS (S-OTS) cryptosystem, which employs sparse measurement matrices in the OTS manner, to pursue efficiency and security simultaneously. Since only a few entries of its measurement matrix take bipolar values and all the others are zero, the S-OTS cryptosystem can save the data storage and reduce the computational cost required for encryption. To renew the secret matrix at each encryption, we employ a linear feedback shift register (LFSR) based keystream generator. In the S-OTS cryptosystem, we show that a reliable CS decryption is theoretically guaranteed for a legitimate recipient.

For security against ciphertext only attacks (COA), we investigate the indistinguishability of the S-OTS cryptosystem. If each plaintext has constant energy, we show that the S-OTS cryptosystem can be indistinguishable, which formalizes the notion of computational security against COA. Then, we analyze security against chosen plaintext attacks (CPA), which can be more threatening. Against the S-OTS cryptosystem, this paper considers a CPA of two sequential stages, keystream and key recovery attacks. At the first stage, we verify that the S-OTS cryptosystem can be secure against keystream recovery under CPA with high probability, by showing that the number of candidate keystreams is tremendously large. At the second stage, conducting an information-theoretic analysis, we show that the success probability of key recovery is extremely low.

To sum up, the S-OTS cryptosystem can be computationally secure against COA and the two-stage CPA, while providing efficient CS encryption by using sparse measurement matrices. Implemented in parallel, the encryption process of the S-OTS cryptosystem can also be fast. Although CS decryption requires high complexity to solve an $l_1$-minimization problem, a legitimate recipient of potential applications, e.g., control center in IoT systems, may have sufficiently high computing power for CS decryption. Due to its fast and efficient encryption process, the S-OTS cryptosystem can be a good alternative to conventional encryption schemes, e.g., AES \cite{44}, for delay-sensitive lightweight devices.

This paper is organized as follows. First of all, Section II presents the system model, reliability analysis, and complexity benefits of the S-OTS cryptosystem. The indistinguishability of the S-OTS cryptosystem against COA is investigated in Section III. Section IV discusses adversary’s CPA strategies and the corresponding security measures. Then, the security of the S-OTS cryptosystem against CPA is analyzed in Section V. Section VI numerically analyzes the security of the S-OTS cryptosystem and demonstrates image encryption examples. Finally, concluding remarks will be given in Section VII.

II. SYSTEM MODEL

A. Notations

$u_{i,j}$, $u^{(i)}$, $u$, and $U^T$ are the entry of a matrix $U \in \mathbb{R}^{M \times N}$ in the $i$-th row and the $j$-th column, the $i$-th row vector, the $j$-th column vector, and the transpose of $U$, respectively, where $1 \leq i \leq M$ and $1 \leq j \leq N$. An identity matrix is denoted by $I$, by whose dimension is determined in the context. For a vector $x = (x_1, \ldots, x_N)^T$, the $l_p$-norm of $x$ is denoted by $||x||_p = \left( \sum_{k=1}^{N} |x_k|^p \right)^{1/p}$ for $1 \leq p < \infty$. Also, $||x||_0$ denotes the number of nonzero elements of $x$. If the context is clear, $||x||$ denotes the $l_2$-norm of $x$. For an index set $\Lambda$, $|\Lambda|$ denotes the number of elements in $\Lambda$. Finally, a vector $n \sim N(0, \sigma^2 I)$ is a Gaussian random vector with mean $0 = (0, \ldots, 0)^T$ and covariance $\sigma^2 I$.

B. Sparse One-Time Sensing (S-OTS) Cryptosystem

1) Mathematical Formulation: Let $x \in \mathbb{R}^N$ be a $K$-sparse plaintext with respect to an orthonormal sparsifying basis $\Psi$, i.e., $x = \Psi^T \alpha$ with $||\alpha||_0 \leq K$. The S-OTS cryptosystem employs a secret measurement matrix $\Phi = \frac{1}{\sqrt{Mr}} SP$, where $S \in \{-1, 0, 1\}^{M \times N}$ is a sparse matrix containing $q$ nonzero elements in each row, $P \in \{0, 1\}^{N \times N}$ is a matrix for permuting the columns of $S$, and $r = \frac{1}{\sqrt{q}}$ is the row-wise sparsity. With $\Phi$, the S-OTS cryptosystem encrypts the plaintext $x$ to provide the corresponding ciphertext

$$y = \frac{1}{\sqrt{Mr}} SPx + n = \frac{1}{\sqrt{Mr}} SP\Psi^T \alpha + n, \quad (1)$$

where $n \sim N(0, \sigma^2 I)$ is the measurement noise. We assume $\frac{N}{q} \leq q \ll \frac{N}{q}$ for efficient encryption, where $q$ is known to an adversary. For convenient analysis, we assume that $\eta = \frac{N}{q}$ and $Mr$ are integers throughout this paper.

2) Keystream Generation: To construct its secret matrix fast and efficiently in the OTS manner, the S-OTS cryptosystem may generate nonzero elements of $S$ and a permutation pattern $P$ with SRNG. In this paper, we employ the self-shrinking generator (SSG) \cite{45} to continuously generate a secure pseudorandom keystream fast and efficiently based on LFSR. The initial state of LFSR, or the key, should be kept secret between a sender and a legitimate recipient, while the structure of the keystream generator can be publicly known. It is noteworthy that LFSR-based keystream generators are more friendly to fast hardware implementation than other keystream generators, e.g., chaos-based keystream generators \cite{46,47}.

**Definition 1:** \cite{45} Assume that a $k$-stage LFSR generates a binary $m$-sequence \cite{48} of $a = (a_1, a_2, \ldots)$, where $a_i \in \{0, 1\}$.
{0, 1}. With a clock-controlled operation, the self-shrinking generator outputs $d_t = a_{2t}$ if $a_{2t-1} = 1$, and discards $a_{2t}$ if $a_{2t-1} = 0$. Then, we obtain a bipolar keystream of $b = (b_1, b_2, \cdots)$, where $b_t = (-1)^{a_{t+1}}$ for $t = 1, 2, \cdots$.

The SSG has a simple structure of a $k$-stage LFSR along with a clock-controlled operator. Meier and Staffelbach [45] showed that the SSG keystream is balanced, and has a period of at least $2^k$ and a linear complexity of at least $2^{k-1}$. With the nice pseudorandomness properties, we assume that each keystream bit takes $\pm 1$ independently and uniformly at random, which facilitates our security analysis of the S-OTS cryptosystem by modeling the keystream bits to be truly random Bernoulli distributed. In [49], numerical results demonstrated the good statistical properties of the SSG keystream, which supports our assumption. It is noteworthy that any other LFSR-based keystream generators can be used to construct each measurement matrix efficiently, as long as their keystream bits can be modeled to be Bernoulli distributed.

3) Secret Matrix Construction: For given $q$ and $N$, let

$$
\Lambda_i = \left\{ (i-1) \mod q + l \mid l = 1, \cdots, q \right\}
$$

be an index set of nonzero entries in the $i$-th row of $S$. Then, $c_s = qM$ bits of the SSG keystream are embedded in $S$, where

$$
s_{i,j} = \begin{cases} 0, & \text{if } j \in \Lambda_i, \\ b_{i,j}^{-1} \cdot \frac{N}{2}, & \text{otherwise}. \end{cases}
$$

(3)

After the S-OTS cryptosystem constructs $S$, next $c_p$ bits of the SSG output sequence can be used to generate a permutation pattern $P$, where a number of algorithms that generate random permutations from coin-tossing (0 and 1) have been studied in [50] and [51]. To the best of our knowledge, $c_p \approx N \log_2 N$ on average and the computational cost of its generation is approximately $N \log_2 N$ [51]. To sum up, $S$ and $P$ can be constructed from consecutive $c_s + c_p \approx qM + N \log_2 N$ bits of the SSG output sequence at each encryption.

A list of notations and variables, and a description of the S-OTS cryptosystem can be found in Tables I and II respectively.

### C. Recovery Guarantee for CS Decryption

In CS decryption, reliability and stability must be guaranteed for a legitimate recipient of ciphertext $y$, who knows the secret matrix $\Phi$. In the S-OTS cryptosystem, the sensing matrix $A = \frac{1}{\sqrt{Mr}}SP\Psi^T$ can be interpreted as a structurally-subsampled unitary matrix [52], as $\Psi$ is unitary, i.e., $\Psi^T \Psi = I$. In the following, Theorem I gives a sufficient condition for $A$ to obey the RIP [3] - [5] with high probability, which guarantees a reliable and stable CS decryption.

**Theorem 1:** [52] Let $\mu_\Psi = \sqrt{N} \max_{i,j \in \{1, \cdots, N\}} |\psi_{i,j}|$ for $\Psi$. With positive constants $c_a$, $c_b$, $\varepsilon_1 \in (0, 1)$, and $\delta_K \in (0, 1)$, the sensing matrix $A = \frac{1}{\sqrt{Mr}}SP\Psi^T$ of the S-OTS cryptosystem satisfies the RIP of order $K$ with probability exceeding $1 - 20 \max \left\{ \exp \left( -c_1 \frac{\delta_K}{\varepsilon_1^2} \right), N^{-1}\right\}$, as long as

$$
M \geq c_a \mu_\Psi^2 K \log^2 N \log^3 N \cdot \varepsilon_1^{-2}.
$$

(4)

**Remark 1:** For a legitimate recipient, Theorem I shows that the S-OTS cryptosystem can theoretically guarantee a stable CS decryption with a proper choice of $\Psi$, i.e., $\mu_\Psi = O(1)$. Since the sufficient condition of (4) is irrelevant to $q$, changing $q$ to meet the security requirement does not affect the recovery guarantee in CS decryption.
Figure 1 illustrates the phase transition diagrams of the B-OTS and the S-OTS cryptosystems in noiseless condition, respectively, where \( \Psi \) is the discrete cosine transform (DCT) basis and \( q = 32 \). Applying the orthogonal matching pursuit (OMP) [53] for CS decryption, we tested 10^3 different plaintexts at each test point, where the step sizes of \( M \) and \( N \) are 2^\text{−}5 \text{ and } 2^{-2} \text{, respectively. For each encryption, the plaintext } \mathbf{x} = \Psi^\top \alpha \text{ is randomly generated, where nonzero entries of } \alpha \text{ are Gaussian distributed and their positions are chosen uniformly at random. In the region below each phase transition curve, the corresponding CS-based cryptosystem successfully decrypts ciphertexts with probability exceeding 99%, where a decryption is declared as a success if the decrypted plaintext } \hat{\mathbf{x}} \text{ achieves } \|x - \hat{x}\|^2 \text{ < } 10^{-2}. \text{ The figure shows that the CS decryption performance of the S-OTS cryptosystem with } q \ll N \text{ is similar to that of the B-OTS cryptosystem with } q = N \text{ over a wide range of } M, \text{ which implies that the theoretical guarantee of the S-OTS cryptosystem is a bit pessimistic.}

Using the S-OTS cryptosystem, we also encrypt an \( n \times n \) bit grayscale image in noiseless condition. For CS encryption, all the columns of the image are stacked into a vector \( \mathbf{x} \in \mathbb{R}^N \), where \( N = n^2 \). To examine the decryption performance with the plaintext \( \mathbf{x} \), we employ SPGLI [54] to obtain the decrypted plaintext \( \hat{\mathbf{x}} \) and then measure the peak signal-to-reconstruction noise ratio (PSNR) averaged over 10^2 different \( \Phi \), where PSNR = 10 \cdot \log_{10} \left( \frac{N \cdot 255^2}{||x - \hat{x}||^2} \right). \text{ To obtain } \alpha = \Psi^\top \mathbf{x}, \text{ 2D versions of the DCT, the Daubechies 4 (D4) wavelet transform, and the Haar wavelet transform bases are employed as } \Psi = \Psi_{n} \otimes \Psi_{n}, \text{ where } \Psi_{n} \in \mathbb{R}^{n \times n} \text{ is an 1D sparsifying basis and } \otimes \text{ is the Kronecker product. Using the test image “Lena” with } n = 256 \text{ and } \rho = \frac{M}{N} = 0.5, \text{ Table III shows the average PSNR (APSNR) of a legitimate recipient in the S-OTS cryptosystem for various } q \text{ and } \Psi. \text{ The S-OTS cryptosystem guarantees a reliable CS decryption with the DCT basis having } \mu_\Psi = \mathcal{O}(1). \text{ Also, we empirically found that CS decryption in the S-OTS cryptosystem can be reliable with the D4 and the Haar wavelet bases, which have much higher } \mu_\Psi = \mathcal{O}(\sqrt{N}). \text{ As predicted by Remark 1, the decryption performance turns out to be irrelevant to } q.

### D. Complexity Benefits

The B-OTS cryptosystem can be computationally more efficient than the G-OTS cryptosystem, since the bipolar entries take less data storage and make matrix-vector multiplications simpler. Nevertheless, the B-OTS cryptosystem requires \( MN \) keystream bits and \( MN \) operations at each encryption, which can be a burden to lightweight systems. By embedding fewer nonzero entries in its measurement matrix, the S-OTS cryptosystem can reduce the number of keystream bits and computations. Moreover, it may have the benefit of fast encryption by conducting the matrix-vector multiplication row-wise in parallel. Table IV briefly compares the S-OTS and the B-OTS cryptosystems in terms of reliability and complexity. Although they have different \( M \) in theoretical recovery guarantees, Table III and Figure 1 demonstrate that the S-OTS and the B-OTS cryptosystems empirically guarantee similar decryption performance with the same \( M \). Furthermore, Section VI will demonstrate that the S-OTS cryptosystem can be secure with \( q \ll N \). Thus, the S-OTS cryptosystem enjoys a significant benefit in complexity, compared to the B-OTS cryptosystem, while guaranteeing its reliability and security.

### III. Security Analysis Against COA

#### A. Security Measures

In ciphertext only attacks (COA), an adversary tries to figure out a plaintext by only observing the corresponding ciphertext. We consider the indistinguishability [39] to formalize the notion of computational security against COA. In Table V, the indistinguishability experiment [39] is described for a CS-based cryptosystem in the presence of an eavesdropper. If no adversary passes the experiment with probability significantly better than that of random guess, the cryptosystem is said to have the indistinguishability. In other words, if a cryptosystem has the indistinguishability, an adversary is unable to learn any partial information of the plaintext in polynomial time from a given ciphertext.

In Table VI let \( d_{TV}(p_1, p_2) \) be the total variation (TV) distance [55] between probability distributions \( p_1 = \Pr(y|\mathbf{x}_1) \) and \( p_2 = \Pr(y|\mathbf{x}_2) \). Then, it is readily checked from [56] that the probability that an adversary can successfully distinguish the plaintexts by a binary hypothesis test \( \mathcal{D} \) is bounded by

\[
p_d \leq \frac{1}{2} + \frac{d_{TV}(p_1, p_2)}{2},
\]

where \( d_{TV}(p_1, p_2) \in [0, 1] \). Therefore, if \( d_{TV}(p_1, p_2) \) is zero, the probability of success is at most that of a random guess, which leads to the indistinguishability [39].

Since computing \( d_{TV}(p_1, p_2) \) directly is difficult [57], we will employ an alternative distance metric to bound the TV distance. In particular, the Hellinger distance [55], denoted by \( d_{H}(p_1, p_2) \), is useful by giving both upper and lower bounds on the TV distance [58], i.e.,

\[
d_H(p_1, p_2) \leq d_{TV}(p_1, p_2) \leq d_H(p_1, p_2) \sqrt{2 - d_H^2(p_1, p_2)},
\]

Table III

| CS-based cryptosystem | S-OTS | B-OTS |
|-----------------------|-------|-------|
| No. of nonzero entries in a row (\( q \)) | 16    | 32    | 64    | 128   | 256   | 16384 |
| DCT                   | 29.8  | 29.7  | 29.8  | 29.6  | 29.7  | 29.7  |
| D4 Wavelet            | 32.2  | 32.3  | 32.3  | 32.4  | 32.3  | 32.4  |
| Haar Wavelet          | 30.5  | 30.5  | 30.7  | 30.6  | 30.7  | 30.6  |

Table IV

| CS-based cryptosystem | S-OTS | B-OTS |
|-----------------------|-------|-------|
| Measurements for recovery guarantee | \( \Omega(M^2 \log^2 N) \) | \( O(K \log N) \) |
| Keystream bits per encryption | \( qM + N \log_2 N \) | \( MN \) |
| Computational cost per encryption | \( qM + N \log_2 N \) | \( MN \) |

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\]
where \( d_H(p_1, p_2) \in [0, 1] \). For formal definitions and properties of the distance metrics, see [53]–[57].

To analyze the security of the S-OTS cryptosystem against COA, we examine the success probability of \( \mathbb{E} \) as a security measure.

B. Indistinguishability Analysis

In the indistinguishability experiment for the S-OTS cryptosystem, we examine adversary’s success probability with the TV distance \( d_{TV}(p_1, p_2) \). In the following, Theorem 2 gives upper and lower bounds on \( d_{TV}(p_1, p_2) \).

**Theorem 2:** In the S-OTS cryptosystem, let \( p_1 = \Pr(y|x_1) \) and \( p_2 = \Pr(y|x_2) \) in Table V. For a plaintext \( x_k \), let \( \theta_h = \frac{c}{\|x_h\|} \) and \( c_h = N\|\theta_h\|_2^2 \) for \( h = 1 \) and 2, respectively. Assuming that \( x_{min} \) and \( x_{max} \) are the plaintexts of minimum and maximum possible energies, respectively, \( \gamma = \frac{\|x_{min}\|_2}{\|x_{max}\|_2} \) is the minimum plaintext energy ratio and \( \gamma_{max} = \frac{\|x_{max}\|_2}{\|x_{max}\|_2} \) is the maximum plaintext-to-noise power ratio (PNR) of the cryptosystem. Then, the worst-case lower and upper bounds on \( d_{TV}(p_1, p_2) \) are given by

\[
d_{TV, low} \approx 1 - \left( \frac{4\gamma e}{(\gamma e + 1)^2} \right)^{\frac{M}{2}} \cdot \left( 1 - \frac{c}{8q} \left( \frac{\gamma e - 1}{\gamma e + 1} \right)^2 \right)^M,
\]

\[
d_{TV, up} \approx \left( 1 - \left( \frac{4\gamma e}{(\gamma e + 1)^2} \right)^{\frac{M}{2}} \cdot \left( 1 - \frac{c}{8q} \left( \frac{\gamma e - 1}{\gamma e + 1} \right)^2 \right)^{2M} \right)^{\frac{1}{2}},
\]

respectively, where

\[
c = \frac{c_{max}}{(1 + PNR_{max})^2} \cdot \left( \frac{\gamma}{\gamma - 1} \right)^2 + 1,
\]

\[
c_{max} = \max_{x_1, x_2}(c_1, c_2), \text{ and } \gamma_{e} = \frac{1 + \gamma PNR_{max}}{1 + PNR_{max}}.
\]

**Proof:** In the S-OTS cryptosystem, we can compute \( d_H(p_1, p_2) \) by replacing \( N \) by \( q \) in the proof of [40] Theorem 4, where

\[
d_H^2(p_1, p_2) = 1 - \left( \frac{4\gamma e}{(\gamma e + 1)^2} \right)^{\frac{M}{2}} \cdot \left( 1 - \frac{c}{8q} \left( \frac{\gamma e - 1}{\gamma e + 1} \right)^2 \right)^M.
\]

Then, the lower and upper bounds on \( d_{TV}(p_1, p_2) \) can be given by \( d_H(p_1, p_2) \) and \( \mathbb{E} \), which completes the proof. \( \square \)

**Corollary 1:** In the S-OTS cryptosystem, the success probability of an adversary in the indistinguishability experiment is bounded by

\[
pd \leq \frac{1}{2} + \frac{1}{2} \sqrt{1 - \left( \frac{4\gamma e}{(\gamma e + 1)^2} \right)^{\frac{M}{2}} \cdot \left( 1 - \frac{c}{8q} \left( \frac{\gamma e - 1}{\gamma e + 1} \right)^2 \right)^{2M}}.
\]

In particular, if \( PNR_{max} = \infty \) and \( \gamma_e = \gamma \),

\[
pd \leq \frac{1}{2} + \frac{1}{2} \sqrt{1 - \left( \frac{4\gamma e}{(\gamma e + 1)^2} \right)^{\frac{M}{2}} \cdot \left( 1 - \frac{c_{max}}{4q} \left( \frac{\gamma - 1}{\gamma + 1} \right)^2 \right)^{2M}}.
\]

**Remark 2:** In (7), \( (\gamma e)^2 + 1 \leq 2 \) for \( \gamma \in (0, 1] \). Therefore, we need \( q \geq \frac{c_{max}}{4(1 + PNR_{max})^2} \) to guarantee \( \frac{c}{8q} \leq 1 \) for all possible \( \gamma \), which makes the upper bound of (8) valid when \( PNR_{max} \) is given. In addition, \( q \geq \frac{c_{max}}{\gamma_e} \) ensures that the bound is valid for any \( PNR_{max} \). As a result, the S-OTS cryptosystem can be indistinguishable with any choice of \( q \geq \frac{c_{max}}{\gamma_e} \), as long as each plaintext has constant energy, i.e., \( \gamma = 1 \). Note that the indistinguishability can be achieved asymptotically, due to the Gaussian approximation [40] Remark 2.

**Remark 3:** The upper bound of \( pd \) in (8) converges to \( \frac{1}{2} \) as \( \gamma \) goes to 1, where the convergence speed depends on \( c_{max} \) for given \( PNR_{max} \). To achieve faster convergence, it is necessary to have lower \( c_{max} \) in the S-OTS cryptosystem. We can obtain lower \( c_{max} \) if the energy of \( x \) is distributed as uniformly as possible in each element [40]. When \( c_{max} \) is given, we need to increase \( q \) to obtain a lower \( \frac{c_{max}}{q} \).

IV. CPA AGAINST THE S-OTS CRYPTOSYSTEM

A. Two-stage CPA

Chosen plaintext attacks (CPA) against the S-OTS cryptosystem aim to retrieve the initial state of SSG, or the key \( k \) of length \( k \), from the pairs of a deliberately chosen plaintext and the corresponding ciphertext. However, the SSG has a remarkable resistance against known cryptanalytic attacks [59]–[63], even if a consecutive SSG keystream is observed. Since an adversary needs to make further efforts to reconstruct \( k \) after restoring a consecutive SSG keystream, this paper considers a CPA of two sequential stages against the S-OTS cryptosystem, keystream and key recovery attacks.

To retrieve \( k \) at the second stage, an adversary may deploy the key search algorithm in [63] requiring \( O(2^{0.161k}) \) consecutive SSG keystream bits, where the search complexity is \( O(2^{0.556k}) \). However, the probability to successfully recover such a long keystream at the first stage will be extremely low, since the keystream can be placed across multiple encryptions with different permutation patterns. Thus, we assume that an adversary tries to observe a short keystream of length \( O(k) \) at the first stage, and then employs the key search algorithms in [60]–[62] at the second stage with the search complexity of \( O(2^k) \), where \( \lambda \in (0, 1) \). To the best of our knowledge, \( \lambda_{min} \approx 0.66 \) from [62], where \( \lambda_{min} \) is the smallest \( \lambda \) among the algorithms.

---

**TABLE V**

**INDISTINGUISHABILITY EXPERIMENT FOR A CS-BASED CRYPTOSYSTEM**

| Step | Action |
|------|--------|
| 1.   | An adversary produces a pair of plaintexts \( x_1 \) and \( x_2 \) of the same length, and submits them to a CS-based cryptosystem. |
| 2.   | The CS-based cryptosystem encrypts a plaintext \( x_h \) by randomly selecting \( h \in \{1, 2\} \), and the corresponding ciphertext \( y = \Phi x_h + n \) is given to the adversary. |
| 3.   | Given the ciphertext \( y \), the adversary carries out a polynomial time test \( D : y \rightarrow h' \in \{1, 2\} \), to figure out which plaintext was encrypted. |

**Decision:** The adversary passes the experiment if \( h' = h \), or fails otherwise.
To analyze the security of the S-OTS cryptosystem against the two-stage CPA, we impose some mild assumptions without loss of generality.

A1) The measurement noise is negligibly small, so noiseless ciphertexts are available for an adversary.

A2) In cryptanalysis, an adversary with bounded computing power cannot execute any detection algorithm with complexity greater than $2^L$, where $L > 0$.

A3) The key length is sufficiently large, i.e., $k > L$.

A4) To deploy key search algorithms, it is sufficient for an adversary to observe $k$ consecutive keystream bits.

A5) In $\Phi$, $N \geq k$, which allows an adversary to recover $k$ consecutive keystream bits from a single measurement matrix by guaranteeing $qM \geq N \geq k$.

Under the assumptions, Table VI describes the two-stage CPA against the S-OTS cryptosystem, which exploits a single plaintext-ciphertext pair. At the first stage (Step 3 of Table VI), an adversary tries to deduce the true key $\hat{k}$ at the second stage (Step 4 of Table VI).

### B. Stage 1: Keystream Recovery Attacks

Since $S$ has $q$ consecutive keystream bits in each row, a true consecutive SSG keystream $b^k$ can be obtained by recovering $\tau = \lceil \frac{k}{q} \rceil$ consecutive rows of $S$. Therefore, an adversary can attempt to obtain an estimated keystream $\hat{b}^k$ satisfying

$$y_i = \sum_{j=1}^{N} \phi_{i,j} x_j$$

for $i = 1, \cdots, \tau$. In the S-OTS cryptosystem, the permutation pattern of $P$ requires additional complexity for an adversary to reconstruct $b^k$ from (9), by diffusing a consecutive keystream across $\Phi$. To obtain $b^k$ in the presence of $P$, the plaintexts of an adversary’s choice can be classified into two classes.

- The first class includes plaintexts each of which enables an adversary to bypass $P$ by $Px = x$. Obviously, the plaintexts of this class have the form of $x = (a, \cdots, a)^T$ with a nonzero constant $a$.

- The second class contains plaintexts each of which recovers all the entries of $\Phi$ by a single CPA. In particular, if $x = (2^0, 2^1, \cdots, 2^{N-1})^T$, all the entries of $\Phi$ can be restored, while $S$ and $P$ are unknown.

We believe that bypassing $P$ or recovering $\Phi$, the two classes may require lower complexity for keystream recovery under CPA than any other selection of plaintext. Thus, we investigate the security of the S-OTS cryptosystem by applying plaintexts chosen from the above two classes. A security analysis employing a more efficient selection of plaintext is left open for future research.

In the first class, if an adversary applies a plaintext $x = (a, \cdots, a)^T$ with $a = \sqrt{Mr}$, (9) becomes

$$y_i = \sum_{j \in \Lambda_i} s_{i,j}$$

for $i = 1, \cdots, \tau$, where $\Lambda_i$ is defined by (2) and $s_{i,j}$ takes $\pm 1$ for $j \in \Lambda_i$. Let $q_i^+ = |\Lambda_i^+|$ and $q_i^- = |\Lambda_i^-|$, respectively, where $\Lambda_i^+ = \{j \mid s_{i,j} = +1, j \in \Lambda_i\}$ and $\Lambda_i^- = \{j \mid s_{i,j} = -1, j \in \Lambda_i\}$. Then, $q_i^+ = \frac{1}{2}(q + y_i)$ and $q_i^- = \frac{1}{2}(q - y_i)$ from (10), where the adversary obtains the numbers of $+1$’s and $-1$’s in the $i$-th row of $S$.

With a plaintext $x = (2^0, 2^1, \cdots, 2^{N-1})^T$, all the entries of a secret measurement matrix in the B-OTS cryptosystem can be easily recovered by a single CPA (54). Similarly, by choosing the plaintext $x$ in the second class, an adversary can successfully restore all the entries of $\Phi = \frac{1}{\sqrt{Mr}}SP$ in the S-OTS cryptosystem. If $N \geq k$ under A5), the adversary can obtain $q_i^+$ and $q_i^-$ for $i = 1, \cdots, \tau$, from the entries of $\Phi$, but with no perfect knowledge of $P$.

**Remark 4:** From an attack with the second class plaintext, an adversary can reduce the number of possible candidates of $P$ by observing the positions of nonzero entries of $\Phi$, which is not possible by an attack with the first class plaintext. Also, if $P$ is restored completely, the keystream bits embedded in $S$ can be directly obtained from $\Phi$ and $P$. However, we assume that an adversary makes no attempt to recover $P$ in the attack with the second class plaintext, since retrieving $P$ may still require an extremely large number of computations, as shown in an example attack of Appendix A. To analyze the effect of $P$ on the security of the S-OTS cryptosystem thoroughly, a further analysis is necessary.

In summary, this paper assumes that an adversary trying to bypass $P$ or recover $\Phi$ exploits the information of $q_i^+$ and $q_i^-$, which can be obtained by applying either of the two classes of plaintexts. Thus, we count the number of $\hat{b}^k$ satisfying (10) for $i = 1, \cdots, \tau$, as a security measure against keystream recovery under CPA, which can be applicable to the attacks with both classes of plaintexts.

### C. Stage 2: Key Recovery Attacks

After estimating a consecutive SSG keystream $\hat{b}^k$, an adversary tries to recover the key $k$ of the S-OTS cryptosystem. Once a true SSG keystream $b^k$ has been successfully recovered, or $b^k = b^k$, the adversary is able to reconstruct $k$ via the key search algorithms in (50)–(52). As long as $\lambda_{\text{min}}k \leq L$, if $\lambda_{\text{min}}k > L$ from a sufficiently long key, we assume that no adversary is able to exploit such key search algorithms, even when $\hat{b}^k = b^k$. Then, we conduct an information-theoretic analysis to develop an upper bound on the success probability of key recovery $P_{\text{key}} = \Pr[k = \hat{k}]$, which will be used as a security measure against key recovery under CPA.

### V. Security Analysis Against CPA

#### A. Stage 1: Keystream Recovery Attacks

At the first stage of CPA, we assumed in Section IV.B that an adversary attempts to obtain an estimated keystream $\hat{b}^k$ satisfying (10) for $i = 1, \cdots, \tau$, by exploiting the numbers of $+1$’s and $-1$’s in each row of $S$. In what follows, Theorem 3 gives a lower bound on the number of possible $\hat{b}^k$.

**Theorem 3:** Let $S_{\text{CPA}}$ be the number of possible keystreams of length $k$, when an adversary attempts to reconstruct a true
keystream $b^k$ with the knowledge of the numbers of +1’s and −1’s in each row of $S$. Then,

$$S_{\text{CPA}} \geq \left( \frac{q}{2^{1/2}} \right)^{\tau} \triangleq S_{\text{CPA,low}} \quad (11)$$

with probability exceeding $1 - \varepsilon_2$ for small $\varepsilon_2 \in (0, 1)$, where $\tau = \left\lceil \frac{k}{q} \right\rceil$ and $t = \frac{2q \cdot \log \frac{1}{1 - (1 - \varepsilon_2)^\tau}}{1 - (1 - \varepsilon_2)^{\tau+1}} \in [0, q]$.

**Proof:** See Appendix [B]

In Theorem [3] if $k$ and $\varepsilon_2$ are fixed, $S_{\text{CPA,low}}$ is irrelevant to $N$ and only depends on $q$. Therefore, we can easily adjust the lower bound of (11) by changing $q$. In the two-stage CPA, if $S_{\text{CPA}} > 2^L$ with a large $q$, the intractability of keystream recovery at the first stage prohibits an adversary from finding the true key at the second stage. In what follows, Theorem 4 gives a sufficient condition on $q$ to guarantee $S_{\text{CPA}} > 2^L$ with high probability, which makes keystream recovery under CPA infeasible.

**Theorem 4:** The S-OTS cryptosystem guarantees $S_{\text{CPA}} > 2^L$ with probability exceeding $1 - \varepsilon_2$, if $k \geq L \cdot \log 2$ and

$$q \geq \frac{1}{2} \left( 2 + \frac{4}{\beta - 2} \right)^2 \log \frac{2}{1 - (1 - \varepsilon_2)^{\tau+\beta}} \triangleq q_{\text{CPA}}. \quad (12)$$

where $\rho = \frac{M}{N}$, $\beta = -\frac{k}{\log 2} W_1 \left( -\frac{L \cdot \log 2}{k} \right)$, and $W_1(\cdot)$ is the lower branch of Lambert $W$ function [63].

**Proof:** See Appendix [C]

Theorem 4 demonstrates that if each row of $\Phi$ takes more nonzero entries than $q_{\text{CPA}}$, keystream recovery under CPA is theoretically infeasible with high probability. In what follows, Corollary 2 gives the largest possible value of $q_{\text{CPA}}$.

**Corollary 2:** In Theorem 4 if $k \geq L \cdot \log 2$, then $\beta \geq e$, which leads to

$$q_{\text{CPA}} \leq \frac{1}{2} \left( 2 + \frac{4}{e - 2} \right)^2 \log \frac{2}{1 - (1 - \varepsilon_2)^{\tau+\beta}} \triangleq q_{\text{CPA,up}}. \quad (13)$$

Corollary 2 implies that if we choose $q > q_{\text{CPA,up}}$, then $S_{\text{CPA}} > 2^L$ with high probability for every $L \leq \log 2$. By Theorem 4 and Corollary 2 the security parameter $q$ should be as large as possible to ensure that the S-OTS cryptosystem can be secure against keystream recovery under CPA.

If the S-OTS cryptosystem has $S_{\text{CPA}} \leq 2^L$, an adversary may be able to obtain a true keystream $b^k$. In Corollary 3 we derive an upper bound on the probability that the adversary successfully recovers $b^k$ with its computing power, where the proof is straightforward from (10) and Appendix [C].

**Corollary 3:** Let $P_{\text{suc}} = \Pr \left[ S_{\text{CPA}} \leq 2^L \right]$ be the probability that an adversary may succeed in keystream recovery under CPA with the bounded computing power of $2^L$. If the S-OTS cryptosystem satisfies $k \geq L \cdot \log 2$, we have

$$P_{\text{suc}} \leq 1 - \left( 1 - 2e^{-\frac{\beta}{2}} \right)^{\tau} \triangleq P_{\text{suc,up}},$$

where $\tau = \left\lceil \frac{k}{q} \right\rceil$ and $\beta = -\frac{k}{L \cdot \log 2} W_1 \left( -\frac{L \cdot \log 2}{k} \right)$.

**Remark 5:** As long as $k \geq L \cdot \log 2$, we have $\beta \geq e$, which implies that $P_{\text{suc,up}} < \tau \cdot 2e^{-\frac{\beta}{2}} \triangleq \Phi$ in Corollary 3.

Note that the bound exponentially decays as $q$ grows larger. Thus, if $k \geq L \cdot \log 2$, the success probability of keystream recovery under CPA disappears exponentially over $q$, and is negligible [49], regardless of $L$. In summary, the S-OTS cryptosystem is computationally secure against keystream recovery under CPA in an asymptotic manner.

**B. Stage 2: Key Recovery Attacks**

At the second stage of CPA against the S-OTS cryptosystem, an adversary attempts to recover the key $k$ from an estimated keystream $b^k$, which is obtained at the first stage. If $b^k = b^k$, the adversary can successfully reconstruct $k$ via the key search algorithms in [60–62] with the complexity of $O(2^{\lambda k})$, as long as $\lambda_{\min} k \leq L$. With a long key satisfying $\lambda_{\min} k > L$, no known key search algorithms for SSG can succeed even when $b^k = b^k$. Note that $k \geq L \cdot \log 2$ in Theorem 4 is sufficient to achieve $\lambda k > L$ for all $\lambda \geq \lambda_{\min} = 0.66$ in [62]. Alternatively, we employ an information-theoretic tool to investigate the key recovery performance for the S-OTS cryptosystem. Under the condition that an adversary has obtained $b^k$ at the first stage, we consider a hypothesis testing that the adversary chooses a candidate key $\hat{k}$ among $2^{	heta k}$ hypotheses, where $\frac{1}{2} \leq \delta \leq 1$. In what follows, Theorem 5 gives an upper bound of the success probability of key recovery, or $P_{\text{key}} = \Pr[\hat{k} = k]$.

**Theorem 5:** Based on $b^k$, an adversary chooses a candidate key $\hat{k}$ among $2^{	heta k}$ hypotheses, where $\frac{1}{2} \leq \delta \leq 1$. Let $P_{\text{key}}$ be the probability that an adversary successfully recovers the key of the S-OTS cryptosystem at the second stage of CPA. Then, $P_{\text{key}} \leq 2^{\log 2} \left[ 1 - 2^{k - \delta} - \delta + \frac{1}{2} \right] \cdot P_{\text{suc,up}} \triangleq P_{\text{key,up}}, \quad (14)$

where $P_{\text{suc,up}}$ is the upper bound on the success probability of keystream recovery under CPA in Corollary 3.

**Proof:** See Appendix [D]

**Remark 6:** In (14), $P_{\text{key,up}}$ only depends on $P_{\text{suc,up}}$ when $k$ and $\delta$ are given. Therefore, reducing the success probability of keystream recovery with a proper $q$ can yield a low success probability of key recovery, which leads to the security of the S-OTS cryptosystem against the two-stage CPA.
C. Key Refresh Time

When an adversary attempts the two-stage CPA repeatedly, one can renew the key in every $T_{\text{ref}}$ encryptions to keep the S-OTS cryptosystem secure, where $T_{\text{ref}}$ is the key refresh time. In what follows, Theorem 6 provides a sufficient condition on $T_{\text{ref}}$ to guarantee security against the two-stage CPA with high probability.

**Theorem 6:** For small $\varepsilon_3 \in (0, 1)$, the S-OTS cryptosystem guarantees security against the two-stage CPA within $T_{\text{ref}}$ encryptions with probability exceeding $1 - \varepsilon_3$, if

$$T_{\text{ref}} \leq \frac{\log(1 - \varepsilon_3)}{\log(1 - P_{\text{key, up}})} \triangleq T_{\text{ref, up}},$$

where $P_{\text{key, up}}$ is the upper bound on the success probability of key recovery under CPA in Theorem 5.

**Proof:** Since the S-OTS cryptosystem renews its measurement matrix at each CS encryption, the probability that the S-OTS cryptosystem is secure against $T_{\text{ref}}$ repeated CPA is given by $(1 - P_{\text{key}})^{T_{\text{ref}}}$, which yields (13) immediately from (14) $\square$

In the proof of Theorem 6 we assumed that an adversary has no benefits by applying multiple plaintext-ciphertext pairs to the S-OTS cryptosystem, due to the usage of keystreams in a one-time manner. However, this assumption does not take into account the potential of a more elaborate strategy of an adversary. More research efforts will be necessary to analyze the security of the S-OTS cryptosystem against repeated CPA with multiple ciphertext-plaintext pairs, which is left open.

As the SSG keystream has a period of at least $2^{127}$, one needs $(c_s + c_p) \cdot T_{\text{ref}} < 2^{127}$ must be satisfied to prevent reuse of keystream bits, where $c_s$ and $c_p$ are the lengths of SSG output sequences required to construct $S$ and $P$ in each encryption, respectively.

VI. NUMERICAL RESULTS

This section presents numerical results of the indistinguishability and the security against the two-stage CPA of the S-OTS cryptosystem. Also, we demonstrate digital image encryption examples.

A. Indistinguishability

Table VII shows the empirical values of $c_{\text{max}}$ in Theorem 2 for various $\Psi$ and $N$. Each $c_{\text{max}}$ is measured over $10^6$ plaintext pairs $x = \Psi \alpha$, where $\alpha$ has Gaussian distributed nonzero entries with $K = 8$. We can notice that the DCT and the Walsh-Hadamard transform (WHT) bases, which have no zero entries, yield low $c_{\text{max}}$ with dense $x$. On the other hand, the D4 and the Haar wavelet transform bases, which have many zero entries, yield high $c_{\text{max}}$. According to Remarks 2 and 3, one needs $\frac{c_{\text{max}}}{q} \leq 4$ for a valid upper bound of $p_d$, which should be as low as possible for fast convergence of the bound. Therefore, the S-OTS cryptosystem may require a large $q$ when we employ a basis $\Psi$ with high $c_{\text{max}}$, like the D4 and the Haar wavelet bases, which compromises its efficiency.

Figure 2 depicts the upper bounds of the success probability of an adversary in the indistinguishability experiment over $\gamma$ for the S-OTS and the G-OTS cryptosystems, respectively, where $N = 1024$, $M = 256$, and $\text{PNR}_{\text{max}} = \infty$. Even though the upper bound of 8 cannot be smaller than that of the G-OTS cryptosystem presented in [40, Corollary 1], the figure implies that we can make them closer to each other with lower $c_{\text{max}}$. If $c_{\text{max}} = 4$ for the DCT or the WHT basis, numerical results revealed that the difference of the upper bounds between the S-OTS and the G-OTS cryptosystems is less than $10^{-2}$ for $q \geq 48$. If $c_{\text{max}} = 684.4$ for the D4 wavelet transform basis, it is necessary to have $q \geq 172$ to make the upper bound valid. In this case, the figure demonstrates that the S-OTS cryptosystem with such a high $c_{\text{max}}$ cannot make the upper bound close to that of the G-OTS cryptosystem even with $q = \frac{N}{2}$.

B. Security Against CPA

Figure 3 sketches $\log_2 S_{\text{CPA, low}}$ of Theorem 3 over $q$ for $k = 128$ and 256, where $\varepsilon_2 = 10^{-5}$. The figure shows that $S_{\text{CPA, low}}$ does not monotonically increase over $q$, but drops whenever $\tau = \left\lceil \frac{k}{q} \right\rceil$ changes its value. Thus, one needs to choose $q$ carefully, to avoid such drops and get a higher $S_{\text{CPA, low}}$. Moreover, if a selection of $q$ yields $\log_2 S_{\text{CPA, low}} > k$, the keystream recovery attack can be computationally more expensive than a brute-force key search. For example, if $k = 256$ and $q \in \{108, \ldots, 127, 151, \ldots, 255, 279, \ldots\}$, $S_{\text{CPA}} > 2^k$ with probability exceeding $1 - 10^{-5}$. With such $q$, a brute-force key search would be a better strategy, which demonstrates the security of the S-OTS cryptosystem against
Figure 3. $\log_2 S_{\text{CPA,low}}$ over $q$ for $k = 128$ and $256$, where $\varepsilon_2 = 10^{-5}$.

Figure 4. $\log_2 q_{\text{CPA}}$ and $\log_2 q_{\text{CPA,up}}$ over $\varepsilon_2$ with various $L$, where $k = 256$ and $\rho = 0.5$.

Figure 5. $P_{\text{suc,up}}$ and $P_{\text{key,up}}$ over $\log_2 q$ with various $L$, where $k = 256$.

Figure 6. $T_{\text{ref,up}}$ over $\log_2 q$ with various $\varepsilon_3$, where $\delta = 0.5$.

Keystream recovery under CPA.

Figure 4 depicts $\log_2 q_{\text{CPA}}$ of Theorem 4 over $\varepsilon_2$ for $\rho = 0.5$, $k = 256$, and $L \leq 128$, where $k \geq L \cdot \varepsilon \log 2$ is met. The figure shows that if $q \geq 137$, then $S_{\text{CPA}} > 2^{128}$ with probability exceeding $1 - 10^{-5}$, which suggests that the S-OTS cryptosystem with such $q$ can be secure against keystream recovery under CPA from an adversary with computing power of at most $2^{128}$. In addition, $q_{\text{CPA,up}} < 512$ at $\varepsilon_2 = 10^{-5}$, which implies that $q \geq 512$ ensures $S_{\text{CPA}} > 2^{L}$, or the infeasibility of keystream recovery under CPA, for any $L < \frac{k}{\varepsilon \log 2}$ with probability exceeding $1 - 10^{-5}$.

Figure 5 sketches $P_{\text{suc,up}}$ of Corollary 3 and $P_{\text{key,up}}$ of Theorem 5 over $\log_2 q$ to demonstrate the security of the S-OTS cryptosystem against the two-stage CPA, where $\delta = 0.5$.

When $k = 256$ and $L \leq 128$, the figure shows that $q \geq 128$ guarantees $P_{\text{suc}} < 10^{-6}$. In addition, Figure 6 depicts the key refresh time $T_{\text{ref}}$ over $\log_2 q$ with $\delta = 0.5$. When $q = 256$ and $\varepsilon_3 = 10^{-5}$, we have $T_{\text{ref}} > 10^{8}$, which implies that the S-OTS cryptosystem can be secure against the two-stage CPA using the same key for $10^8$ encryptions by keeping $P_{\text{key}} < 10^{-5}$.

To sum up, Figures 3-6 show that if $k = 256$, $L \leq 128$, and $\rho = 0.5$, the S-OTS cryptosystem with $q \geq 512$ has sufficient resistance against the two-stage CPA with probability exceeding $1 - 10^{-5}$, regardless of $N$. Recall from Remark 2 that the S-OTS cryptosystem must satisfy $q \geq \frac{1}{\varepsilon_{\max}}$ for the indistinguishability, where $\varepsilon_{\max}$ can be given with respect to $\Psi$ and $N$. In the end, we need to carefully choose $q$ by taking into account both the constraints for indistinguishability and CPA security.
C. Digital Image Encryption

Recall the simulation setup for digital image encryption in Section II.C. With 256 × 256 images “Lena”, “Boat”, “Plane”, “Peppers”, and “Barbara”, we encrypt each image, and then decrypt it with SPGL1 [54] in noiseless condition, where 2D version of the D4 wavelet transform basis is employed as Ψ. We select q = 512, which meets the requirements for indistinguishability and CPA security. Figure 7 visualizes the Φρ32 where PSNR values are 32.2 dB, 29.6 dB 31.1 dB, 31.6 dB, and 29.5 dB, respectively. In the examples of Figure 7 the measurement matrix Φ is highly sparse with the row-wise sparsity r = q/N = 2−7 ≈ 0.78%. Using the sparse matrix Φ, the S-OTS cryptosystem requires c_s + c_p ≈ 224 bits of the SSG output sequence for each encryption process, while the B-OTS cryptosystem uses MN = 231 bits. This implies that the S-OTS cryptosystem can save a significant amount of keystream bits while guaranteeing security against the two-stage CPA.

VII. Conclusion

In this paper, we proposed the S-OTS cryptosystem, which employs sparse measurement matrices for secure and efficient CS encryption. With a small number of nonzero elements in the measurement matrix, the S-OTS cryptosystem has complexity benefits in terms of memory and computing resources. In addition, the S-OTS cryptosystem can present theoretically guaranteed CS recovery performance for a legitimate recipient. In the presence of an adversary, we analyzed the security of the S-OTS cryptosystem against COA and CPA. Against COA, we exhibited that the S-OTS cryptosystem can asymptotically achieve the indistinguishability, as long as each plaintext has constant energy. To investigate its security against CPA, we consider an adversary’s strategy that consists of two sequential stages, keystream and key recovery attacks. We then showed that the keystream recovery can be infeasible with overwhelmingly high probability. Also, we conducted an information-theoretic analysis to demonstrate that the success probability of following key recovery can be extremely low with a proper selection of parameters. Through numerical results, we demonstrated that the S-OTS cryptosystem guarantees its reliability and security, while providing computational efficiency.

APPENDIX

A. Example of Permutation Recovery Attack

In Section IV.B, a keystream recovery attack with the second class plaintext can provide an adversary with the positions of nonzero entries in Φ, which can be exploited to recover P. Let Λφ be an index set of nonzero entries in the i-th row of Φ. Given Λφ for i = 1, · · · , M, an adversary may attempt a known plaintext attack against the permutation-only cipher [66] to find a true P, where a plaintext p_i = (p_{i,1}, · · · , p_{i,N}) and the corresponding ciphertext c_i = (c_{i,1}, · · · , c_{i,N}) with respect to Λ_i and Λφ are given by

\[
p_{i,j} = \begin{cases} 1, & \text{if } j \in \Lambda_i, \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad c_{i,j} = \begin{cases} 1, & \text{if } j \in \Lambdaφ, \\ 0, & \text{otherwise.} \end{cases}
\]

Let \( C \in \{0,1\}^{M \times N} \) be a matrix having \( c_i \) as its i-th row for \( i = 1, \cdots , M \) and \( c \in \mathbb{R}^N \) be a composite representation \[66\] of \( C \), where the j-th element of \( c \) is given by

\[
\tilde{c}_j = \sum_{i=1}^{M} c_{i,j} \cdot 2^{i-1}.
\]

According to \[66\] Proposition 2, the permutation pattern can be uniquely determined if and only if all the entries of \( c \) are distinct. In the S-OTS cryptosystem, however, \( c \) consists of \( \eta = N \) distinct integers \( \{ zi_1, \cdots , zi_\eta \} \), where \( zi \) appears \( q \) times in \( c \) for all \( i = 1, \cdots , \eta \), due to the structure of \( S \) defined in \[2\] and \[3\]. Then, the number of possible permutation patterns is

\[ S_p = (q!)^\eta, \]

which is smaller than \( N! \), but still extremely large even for small \( q \) and \( N \). Although we can reduce the number of possible permutations by observing the nonzero entries of \( \Phi \), retrieving \( P \) can be intractable to an adversary with bounded computing power. Note that this example does not take into account the potential of a more elaborate attack exploiting the respective positions of +1’s and −1’s in \( \Phi \), which is left open for future research.

B. Proof of Theorem 3

Without loss of generality, let \( \Gamma = \{1, \cdots , \tau \} \) be a set of the indices of the first \( \tau \) rows in \( S \). Then, an adversary can obtain \( \tilde{b}^i \) by estimating the i-th row of \( S \), or \( s^{(i)} \), for every \( i \in \Gamma \). Given \( q_i^+ = \frac{1}{2}(q + y_i) \) and \( q_i^- = \frac{1}{2}(q - y_i) \) from (10), the number of possible solutions for \( s^{(i)} \) is given by

\[ S_i = \left( \frac{q}{q_i^+} \right) = \left( \frac{q}{q_i^-} \right), \quad (16) \]

Assuming that each nonzero entry of \( S \) takes ±1 independently with probability 0.5, the sum of \( q \) independent random variables yields \( y_i = q_i^+ - q_i^- = 2q_i^+ - q \in \{-q, \cdots , q\} \), which can be considered as a binomial random variable. Therefore, the Hoeffding’s inequality \[67\] yields

\[ \Pr [ |y_i| < t ] = \Pr [ \frac{q - t}{2} < q_i^+ - q_i^- < \frac{q + t}{2} ] \geq 1 - 2e^{-\frac{t^2}{2q}}. \quad (17) \]

Since \( q_i^+ \) takes an integer value,

\[ \Pr [ \frac{q - t}{2} < q_i^+ < \frac{q + t}{2} ] = \Pr \left[ \left. \frac{q - t}{2} \leq q_i^+ \leq \frac{q + t}{2} \right] \right]. \]

From \[16\] and \[17\], \( S_i \) can be bounded by

\[ S_i = \left( \frac{q}{q_i^+} \right) \geq \left( \frac{q}{q_i^-} \right), \]

with probability exceeding \( 1 - 2e^{-\frac{t^2}{2q}} \). Finally, the number of all possible solutions for \( \tau \) consecutive rows is \( S_{\text{CPA}} = \prod_{i \in \Gamma} S_i \).
where
\[
\Pr \left[ S_{\text{CPA}} \geq \left( \frac{q}{2^L} \right)^\tau \right] \geq \prod_{i \in T} \Pr \left[ S_i \geq \left( \frac{q}{2^L} \right)^\tau \right] \geq \left( 1 - 2e^{\frac{-t^2}{2q}} \right)^\tau.
\]

Letting \( t = 1 - \varepsilon_2, t = 2q \cdot \log \frac{2}{1 - (1 - \varepsilon_2)^{\tau}} \), which completes the proof. \( \square \)

**C. Proof of Theorem 2**

In (11), let \( \frac{q}{\alpha} = \alpha \), then \( \tau = \frac{1}{\alpha} \geq \frac{b}{q} \) yields
\[
S_{\text{CPA}} \geq \left( \frac{q}{\alpha} \right)^\tau \geq \left( \frac{q}{\alpha} \right)^{\frac{b}{q}} > \left( \frac{q}{\alpha} \right)^{\frac{b}{q}}.
\]
Therefore, \( S_{\text{CPA}} > 2^L \) if \( q \) satisfies \( \left( \frac{q}{\alpha} \right)^{\frac{b}{q}} \geq 2^L \), which is equivalent to
\[
\frac{\alpha}{q} \log \frac{\alpha}{q} \leq -L \log_2 \frac{2}{k}.
\]
Since \( \frac{1}{2^L} \log \frac{1}{x} \geq -\frac{1}{e^x} \) for \( x > 0 \), (19) is valid as long as
\[
k \geq L \cdot \log_2 2.
\]

By taking \( W_{-1}(\cdot) \), (19) yields \( q \leq \alpha \beta \), where \( \alpha = \frac{2^L}{\log 2} \), \( \beta = -\frac{k}{L \log_2 2} \).

From \( q \leq \alpha \beta \), we have
\[
q \geq \frac{1}{2} \left( 2 + \frac{4}{\beta - 2} \right)^2 \log \frac{2}{1 - (1 - \varepsilon_2)^{\tau}},
\]
which is a sufficient condition for \( S_{\text{CPA}} > 2^L \). Furthermore, since \( \tau = \frac{q}{\alpha} < \frac{kM}{N} + 1 \) from \( \frac{N}{M} \leq q \), we have \( 1 - (1 - \varepsilon_2)^{\tau} > 1 - (1 - \varepsilon_2)^{\tau + 1} \), where \( \rho = \frac{M}{N} \) and the sufficient condition of (21) becomes (12), which completes the proof. \( \square \)

**D. Proof of Theorem 3**

Based on \( b^k \), an adversary attempts to recover the true key \( k \) by choosing a candidate key \( \hat{k} \). Let \( P_{\text{err}} \) be the error probability of key recovery, where
\[
P_{\text{err}} = \Pr \left[ \hat{k} \neq k \mid \hat{b}^k = b^k \right] \cdot \Pr \left[ \hat{b}^k = b^k \right]
\]
\[
+ \Pr \left[ \hat{k} \neq k \mid \hat{b}^k \neq b^k \right] \cdot \Pr \left[ \hat{b}^k \neq b^k \right].
\]
When \( \hat{b}^k = b^k \), the data processing and the Fano’s inequalities 68 yield
\[
H( k \mid b^k ) \leq H \left( k \mid \hat{k} \right) \leq H_b(p_e) + p_e \cdot \log_2 |K|,
\]
where \( H(\cdot) \) is the entropy of a random vector, \( p_e = \Pr[\hat{k} \neq k \mid \hat{b}^k = b^k] \), \( H_b(p_e) = -p_e \log_2 p_e - (1 - p_e) \log_2 (1 - p_e) \), and \( K \) is a set of all possible candidates of \( k \), i.e., \( |K| = 2^k \). Given \( b^k \), assume that \( 2^{\delta k} \) candidates of \( k \) are uniformly distributed, i.e., \( H( k \mid b^k ) = \delta k \). Using \( H_b(p_e) \leq 1 \), (23) yields
\[
p_e = \Pr \left[ \hat{k} \neq k \mid \hat{b}^k = b^k \right] \geq \delta - \frac{1}{k}.
\]

When the adversary has a wrong keystream, i.e., \( \hat{b}^k \neq b^k \), we assume that the success probability of key recovery becomes
\[
\Pr \left[ \hat{k} = k \mid \hat{b}^k \neq b^k \right] = 2^{-k}.
\]

With (24), (25), and \( P_{\text{suc}} = \Pr[\hat{b}^k = b^k] \), (22) yields
\[
P_{\text{err}} \geq \left( \delta - \frac{1}{k} \right) \cdot P_{\text{suc}} + (1 - 2^{-k}) \cdot (1 - P_{\text{suc}})
\]
\[
\geq 1 - 2^{-k} - \left( 1 - 2^{-k} - \delta + \frac{1}{k} \right) \cdot P_{\text{suc,up}},
\]
which completes the proof by \( P_{\text{key}} = 1 - P_{\text{err}} \). \( \square \)
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