Partitions of Matrix Spaces
With an Application to $q$-Rook Polynomials

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Abstract

We study the row-space partition and the pivot partition on the matrix space $\mathbb{F}^{n\times m}_q$. We show that both these partitions are reflexive and that the row-space partition is self-dual. Moreover, using various combinatorial methods, we explicitly compute the Krawtchouk coefficients associated with these partitions. This establishes MacWilliams-type identities for the row-space and pivot enumerators of linear rank-metric codes. We then generalize the Singleton-like bound for rank-metric codes, and introduce two new concepts of code extremality. The latter are both preserved by trace-duality, and generalize the notion of an MRD code. Moreover, codes that are extremal according to either notion satisfy strong rigidity properties, analogous to those of MRD codes. As an application of our results to algebraic combinatorics, we give closed formulas for the $q$-rook polynomials associated with Ferrers diagram boards. Moreover, we exploit connections between matrices over finite fields and rook placements to prove that the number of rank $r$ matrices over $\mathbb{F}_q$ supported on a Ferrers diagram is a polynomial in $q$ whose degree is strictly increasing in $r$. We close the paper by investigating the natural analogues of the MacWilliams Extension Theorem for the rank, the row-space and the pivot partitions.

Introduction

The emergence of linear network coding in information theory almost 20 years ago by Ahlswede et al. [1] and the algebraic framework introduced by Kötter and Kschischang [25] has led to intensive research efforts on rank-metric codes and subspace codes.

Rank-metric codes are subspaces of an ambient space of full rectangular matrices $\mathbb{F}^{n\times m}$ over a finite field $\mathbb{F} = \mathbb{F}_q$ with the property that the rank of any pairwise difference is not too small. This is made precise by the rank distance, see (4.1). In [36, 35] it was shown that rank-metric codes can be efficiently utilized for error correction in coherent single-source networks with an adversarial noise. See [8, 32] for the use of rank-metric codes in multi-source networks.

Another important application of rank-metric codes is the construction of subspace codes. These are collections of subspaces of a fixed ambient vector space over a finite field, with the property that...
the pairwise intersections are not too big (which is quantified by, for instance, the subspace distance). Subspace codes find applications to error correction in the context of random linear network coding [20, 25]. Most of the research focuses on constant-dimension subspace codes, that is, all subspaces have the same dimension. The simplest methods for obtaining a subspace code from a rank-metric code are the lifting construction [36] and the multilevel construction [11], which in fact has led to the best codes known at that time. For further details we refer to the vast literature on subspace codes and their properties.

All of this has led to great interest in matrix spaces with good properties, such as large rank distance [4, 7, 9, 18, 33, 34, 36] and possibly additional Ferrers diagram support [11, 10, 2, 26].

The best known class of rank-metric codes are the MRD codes, maximum rank distance codes. Fixing the desired matrix size, field size, and rank distance, they have the largest cardinality meeting these parameters, see (4.2) and the paragraph thereafter. Various constructions of such codes are known [7, 13, 33, 40], and their properties have been studied intensively, see [6, 22, 29] and the references already mentioned.

MRD codes have remarkable rigidity properties: (a) the dual of an MRD code, with respect to a natural bilinear form, is an MRD code again; (b) the rank distribution of an MRD code is fully determined by the parameters of the code and thus does not depend on the choice of the code. This shows the analogy between MRD codes in the rank metric case and MDS codes in the classical case of block codes with the Hamming distance, which enjoy similar properties.

The rank distribution is a special instance of a partition distribution. Partitioning the entire ambient space into subsets according to some property, such as the rank, gives rise to the partition enumerator of a code, which simply encodes the number of codewords in any partition block. This paper is devoted to the study of the row-space partition $\mathcal{P}_{rs}$ and the pivot partition $\mathcal{P}_{piv}$, and to the connections between these partitions and $q$-rook theory.

In $\mathcal{P}_{rs}$ matrices in $\mathbb{F}_q^{n \times m}$ are in the same partition block if they have the same row space, while in $\mathcal{P}_{piv}$ they are grouped according to their pivot indices after row reduction. Thus $\mathcal{P}_{rs}$ is finer than $\mathcal{P}_{piv}$, which is finer than the rank partition, $\mathcal{P}_{rk}$. Continuing the analogy from above, where the rank weight distribution is the analogue of the Hamming weight distribution, the row space distribution may be considered the analogue of the support distribution (counting the number of codewords with a given support set). The terminology “support” is consistent with [31].

In this paper, we show that that the row-space partition and the pivot partition are both reflexive, and that the row-space partition is also self-dual. We then compute the Krawtchouk coefficients of the row-space partition using a combinatorial approach based on Möbius inversion. This leads to an explicit MacWilliams identity for the row-space enumerator. We then introduce $U$-extremal codes (which generalize MRD codes) and show that they satisfy natural rigidity properties: (a) $U$-extremality is preserved by trace-duality; (b) for codes that are $U$-extremal for all $U$ of a fixed dimension and below a fixed subspace, say $T$, the partial row-space distribution below $T$ only depends on the specified parameters, but not on the code or $T$ itself (Theorem 4.10).

In the second part of the paper we study the dual of the pivot partition, showing that it can be naturally identified with the reverse pivot partition $\mathcal{P}_{piv}$, where matrices are grouped according to their pivot indices after row reduction from the right. We then express the Krawtchouk coefficients of the partition pair $(\mathcal{P}_{piv}, \mathcal{P}_{piv})$ in terms of the rank distribution of matrices supported on certain Ferrers diagrams (see Section 5 for the precise definition of Ferrers diagram), establishing a MacWilliams identity in this context. We also provide both a recursive and an explicit formula for such rank distributions. Then we define pivot-extremal codes and show that they satisfy rigidity properties analogous to (a) and (b) described above for $U$-extremal codes.

Following work by Haglund [19], in the third part we explore new connections between $q$-rook poly-
nomials and the rank distribution of matrices supported on Ferrers diagrams. More precisely, as an application of our results we give explicit expressions for the $q$-rook polynomials associated with a Ferrers board, and show that the number of rank $r$ matrices over $\mathbb{F}_q$ supported on a Ferrers diagram is a polynomial in $q$ whose degree strictly increases with $r$.

In the last part of the paper we characterize the linear maps on $\mathbb{F}^{n \times m}$ that preserve the rank, the row-space, or the pivot partition. We then give examples to show that in neither situation a MacWilliams Extension Theorem holds.

Outline. The paper is organized as follows. In Section 1 we recall the main definitions and results on partitions of finite abelian groups, Krawtchouk coefficients, rank-metric codes and MacWilliams identities. In Section 2 we introduce and establish the first properties of the row-space partition, the pivot partition and the reverse-pivot partition on the matrix space $\mathbb{F}^{n \times m}$. We devote Section 3 to the computation of the Krawtchouk coefficients of the row-space partition. In Section 4 we define $U$-extremal codes and establish their rigidity properties. We compute the Krawtchouk coefficients of the pivot partition in Section 5, expressing them in terms of the rank-distribution of matrices having a Ferrers diagram shape. Pivot-extremal codes are studied in Section 6. In Section 7 we give both a recursive and an explicit formula for the rank-distribution of matrices supported on a Ferrers diagram. As a corollary, we show that such distribution is a polynomial in $q$. We then use these results to give a closed formula for the $q$-rook polynomials associated with Ferrers diagrams. In Section 8 we study, for each of the three partitions, the partition-preserving linear maps.

1 Partitions and MacWilliams Identities

In this section we introduce partitions on matrix spaces and their character-theoretic dual. We also define the Krawtchouk coefficients, which then determine the MacWilliams identities.

Throughout this paper, $q$ denotes a prime power and $\mathbb{F} = \mathbb{F}_q$ is the finite field with $q$ elements. We also fix integers $n, m \geq 1$, and denote by $\mathbb{F}^{n \times m}$ the space of $n \times m$ matrices over $\mathbb{F}$. We denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the natural numbers. For $i \in \mathbb{N}$, we let $[i] := \{1, \ldots, i\}$.

Recall that the trace product of matrices $M, N \in \mathbb{F}^{n \times m}$ is

$$\langle A, B \rangle := \text{Tr}(AB^\top),$$

(1.1)

where $\text{Tr}$ denotes the matrix trace. Identifying $\mathbb{F}^{n \times m}$ with $\mathbb{F}^{nm}$ via row concatenation, the trace product becomes the classical inner product of $\mathbb{F}^{nm}$. Thus $(A, B) \mapsto \langle A, B \rangle$ defines a symmetric and non-degenerate bilinear form on $\mathbb{F}^{n \times m}$.

Definition 1.1. Let $(G, +)$ be a group. The character group of $G$ is the set of all group homomorphisms $G \rightarrow \mathbb{C}^\ast$ endowed with point-wise multiplication. It is denoted by $\hat{G}$.

It is well known that if $G$ is a finite abelian group, then $G$ and $\hat{G}$ are isomorphic (though not canonically so). This is not the case for more general classes of groups. Note also that if $G$ is finite, then $|\chi(g)| = 1$ for all $g \in G$. The character $\chi$ given by $\chi(g) = 1$ for all $g \in G$ is called the trivial character. For an $\mathbb{F}$-vector space $V$ we simply write $\hat{V}$ for the character group of $(V, +)$. Note that in this case $\hat{V}$ carries a natural $\mathbb{F}$-vector space structure via

$$(c\chi)(v) := \chi(cv) \text{ for all } c \in \mathbb{F}, \chi \in \hat{V}, v \in V.$$  

(1.2)
Let \( \chi : F \rightarrow \mathbb{C}^* \) be a non-trivial character of \((F, +)\). The trace-product on \( \mathbb{F}^{n \times m} \) induces via \( \chi \) an isomorphism of \( \mathbb{F} \)-vector spaces

\[
\mathbb{F}^{n \times m} \rightarrow \mathbb{F}^{n \times m}, \quad B \mapsto \left\{ \begin{array}{ll}
\mathbb{F}^{n \times m} & \mapsto \mathbb{C}^* \\
A & \mapsto \chi(\langle A, B \rangle).
\end{array} \right.
\]

This isomorphism allows us to identify \( \mathbb{F}^{n \times m} \) with its character group via the chosen character. We will implicitly make this identification in the sequel.

**Definition 1.2.** Let \( \mathcal{P} = (P_i)_{i \in \mathcal{I}} \) be a partition of \( \mathbb{F}^{n \times m} \), and let \( \chi \) be a non-trivial character of \( F \). The **dual** of \( \mathcal{P} \) with respect to \( \chi \) is the partition \( \mathcal{P}' \) of \( \mathbb{F}^{n \times m} \) defined via the equivalence relation

\[
B \sim_{\mathcal{P}} B' \iff \sum_{A \in P_i} \chi(\langle A, B \rangle) = \sum_{A \in P'_i} \chi(\langle A, B' \rangle) \quad \text{for all } i \in \mathcal{I}.
\]

We say that \( \mathcal{P} \) is **reflexive** if \( \mathcal{P} = \mathcal{P}' \) and **self-dual** if \( \mathcal{P} = \mathcal{P}' \). Note that self-duality implies reflexivity.

One should be aware of the fact that the dual partition may depend on the choice of the non-trivial character \( \chi \); for an example see, for instance, [16, Ex. 2.2]. Therein, it is shown that even self-duality of a partition depends in general on the choice of the character. Reflexivity, however, is independent of this choice. This is a consequence of [3, Prop. 4.4].

The just described dependence will not occur for the partitions studied in this paper, which (as we will see) belong to the following special class.

**Definition 1.3.** A partition \( \mathcal{P} = (P_i)_{i \in \mathcal{I}} \) of \( \mathbb{F}^{n \times m} \) is called **invariant** if \( uP_i = P_i \) for all \( u \in \mathbb{F}^* \) and \( i \in \mathcal{I} \), that is, all blocks of the partition are invariant under multiplication by non-zero scalars.

**Remark 1.4.** Suppose \( \mathcal{P} = (P_i)_{i \in \mathcal{I}} \) is an invariant partition of \( \mathbb{F}^{n \times m} \).

1. The dual partition \( \mathcal{P}' \) does not depend on the choice of the non-trivial character \( \chi \). This follows from the fact that every other non-trivial character of \( F \) is of the form \( u\chi \) for some \( u \in \mathbb{F}^* \); see (1.2) for \( u\chi \). Hence

\[
\sum_{A \in P_i} (u\chi)(\langle A, B \rangle) = \sum_{A \in P_i} \chi(u\langle A, B \rangle) = \sum_{A \in P_i} \chi(\langle uA, B \rangle) = \sum_{A \in P_i} \chi(\langle A, B \rangle),
\]

from which the statement follows.

2. The partition \( \mathcal{P}' \) is invariant as well.

Now we are ready to introduce some fundamental invariants of an invariant partition.

**Definition 1.5.** Fix a non-trivial character \( \chi \) of \( F \). Let \( \mathcal{P} = (P_i)_{i \in \mathcal{I}} \) be an invariant partition of \( \mathbb{F}^{n \times m} \) and let \( \mathcal{P}' = (Q_j)_{j \in \mathcal{J}} \) be its dual partition. For all \((i, j) \in \mathcal{I} \times \mathcal{J}\), the complex number

\[
K(\mathcal{P}; i, j) := \sum_{A \in P_i} \chi(\langle A, B \rangle), \quad \text{where } B \in Q_j,
\]

is called the **Krawtchouk coefficient** of \( \mathcal{P} \) with index \((i, j)\). Note that, thanks to (1.4), the Krawtchouk coefficients do not depend on the choice of \( \chi \).

We now introduce the main objects studied in this paper.
**Definition 1.6.** A (matrix) code is a linear subspace $C \leq \mathbb{F}^{n \times m}$. The dual of $C$ is the matrix code

$$C^\perp := \{ B \in \mathbb{F}^{n \times m} \mid \langle A, B \rangle = 0 \text{ for all } A \in C \}.$$  

Observe that $\dim(C^\perp) = mn - \dim(C)$, and that $C^{\perp \perp} = C$.

**Definition 1.7.** Given a partition $\mathcal{P} = (P_i)_{i \in \mathcal{I}}$ of $\mathbb{F}^{n \times m}$ and a code $C \leq \mathbb{F}^{n \times m}$, we define

$$\mathcal{P}(C, i) := \{ C \cap P_i \}, \quad i \in \mathcal{I}.$$  

We call the collection $(\mathcal{P}(C, i))_{i \in \mathcal{I}}$ the $\mathcal{P}$-distribution of $C$.

Now we can formulate a general version of the MacWilliams identities. Such identities have been established various times for different settings: for general subgroups of finite abelian groups in [5, Thm. 4.72, Prop. 5.42] and [17, Thm. 2.7], for discrete subgroups of locally compact abelian groups in [12, p. 94], and for $W$-admissible pairs over Frobenius rings in [21, Thm. 21].

**Theorem 1.8 (MacWilliams Identities).** Let $\mathcal{Q} = (Q_j)_{j \in \mathcal{J}}$ be an invariant partition of $\mathbb{F}^{n \times m}$ and let $\mathcal{P} = (P_i)_{i \in \mathcal{I}} = \mathcal{Q}$. For all codes $C \leq \mathbb{F}^{n \times m}$ and all $j \in \mathcal{J}$ we have

$$\mathcal{Q}(C^\perp, j) = \frac{1}{|\mathcal{Q}|} \sum_{i \in \mathcal{I}} K(\mathcal{Q}, j, i) \mathcal{P}(C, i).$$

Note that in the above formulation $\mathcal{Q}$ is the primal partition and $\mathcal{P}$ its dual. The result tells us that the $\mathcal{Q}$-distribution of $C^\perp$ is fully determined by the $\mathcal{P}$-distribution of $C$. The converse is not true in general. However, if $\mathcal{Q}$ is reflexive, thus $\mathcal{Q} = \mathcal{P}$, then the two distributions mutually determine each other.

The MacWilliams identities give rise to the task to determine the Krawtchouk coefficients explicitly. We will do so for various invariant partitions of $\mathbb{F}^{n \times m}$, which we introduce in the next section.

## 2 The Row Space Partition and the Pivot Partition

In this section we introduce the partitions mentioned in the title along with their character-theoretic duals. Before doing so, we will briefly discuss the rank partition.

**Definition 2.1.** For $0 \leq i \leq m$ set $P_i^{\text{rk}} := \{ A \in \mathbb{F}^{n \times m} \mid \text{rk}(A) = i \}$. Then $\mathcal{P}^{\text{rk}} := (P_r^{\text{rk}})_{0 \leq r \leq m}$ is a partition of $\mathbb{F}^{n \times m}$ of size $m + 1$, called the rank partition of $\mathbb{F}^{n \times m}$.

This partition, which is clearly invariant, has been well studied in the past. Self-duality is well-known but will also follow from our more general considerations later; see Corollary 2.6. MacWilliams identities for additive codes endowed with the rank partition were first discovered by Delsarte [7, Thm. 3.3] along with explicit expressions for the Krawtchouk coefficients [7, Thm. A2]; see also [31, Ex. 39] for a shorter proof using lattice theory. They are given by

$$K(\mathcal{P}^{\text{rk}}, r, s) = \sum_{i=0}^{n} (-1)^{r-i} q^{mi + \binom{r-i}{2}} \binom{n-i}{n-s} \binom{n-r}{i}$$

for all $0 \leq r, s \leq m$. (2.1)

Here $[^d_b]$ denotes the $q$-binomial coefficient. It counts the number of $b$-dimensional subspaces of $\mathbb{F}^q$.  

We now turn to the partitions that will be the main subject of our investigation later on. Let $\mathcal{L}$ be the set of all subspaces of $\mathbb{F}^m$. We have $\mathcal{L} = \bigcup_{l=0}^{m} \mathcal{G}_q(m, l)$, where $\mathcal{G}_q(m, l)$ is the Grassmannian of $l$-dimensional subspaces of $\mathbb{F}^m$. Then $\mathcal{L}$ is a lattice with respect to inclusion.
Definition 2.2. For a matrix $A \in \mathbb{F}^{n \times m}$ we define $\text{rs}(A) := \{ uA \mid u \in \mathbb{F}^m \}$ to be the row space of $A$. For $U \in \mathcal{L}$ set $\mathcal{P}_U^\text{rs} := \{ A \in \mathbb{F}^{n \times m} \mid \text{rs}(A) = U \}$. Then $\mathcal{P}^\text{rs} := (\mathcal{P}_U^\text{rs})_{U \in \mathcal{L}}$ is a partition of $\mathbb{F}^{n \times m}$, called the row-space partition of $\mathbb{F}^{n \times m}$.

Definition 2.3. Define $\Pi = \{ (j_1, \ldots, j_r) \mid 1 \leq r \leq m, 1 \leq j_1 < \ldots < j_r \leq m \} \cup \{ () \}$, where () denotes the empty list. For a list $\lambda \in \Pi$ we define $|\lambda| \in \{0, \ldots, m\}$ as its length. For a matrix $A \in \mathbb{F}^{n \times m}$ we denote by $\text{RREF}(A)$ the reduced row echelon form of $A$, and define

$$\text{piv}(A) := (j_1, \ldots, j_r) \in \Pi, \text{ where } 1 \leq j_1 < \ldots < j_r \leq m \text{ are the pivot indices of RREF}(A).$$

Then $\text{piv}(0) := ()$ and $|\text{piv}(A)| = \text{rk}(A)$ for all $A \in \mathbb{F}^{n \times m}$. Matrices $A, B \in \mathbb{F}^{n \times m}$ are called pivot-equivalent if $\text{piv}(A) = \text{piv}(B)$. This defines an equivalence relation on $\mathbb{F}^{n \times m}$. The equivalence classes form the pivot partition of $\mathbb{F}^{n \times m}$, denoted by $\mathcal{P}^{\text{piv}}$.

Obviously, $\Pi$ is bijective to the set of all subsets of $[m]$. For us it will be helpful to record pivots as ordered lists, as introduced above. We will use set-theoretical operations in the obvious way for pivot lists.

The three partitions defined above ($\mathcal{P}^{\text{rk}}, \mathcal{P}^{\text{rs}}$, and $\mathcal{P}^{\text{piv}}$) arise as the collection of orbits with respect to suitable group actions on $\mathbb{F}^{n \times m}$. Indeed, consider the general linear groups of order $n$ and $m$ as well as the group $\mathcal{U}_n(\mathbb{F}) = \{ S \in \text{GL}_m(\mathbb{F}) \mid S \text{ is upper triangular} \}$. Define the actions

$$\begin{align*}
\rho_1 : & \text{GL}_n(\mathbb{F}) \times \mathbb{F}^{n \times m} & \to & \mathbb{F}^{n \times m}, & (S, A) & \mapsto & SA, \\
\rho_2 : (\text{GL}_n(\mathbb{F}) \times \mathcal{U}_n(\mathbb{F})) & \times \mathbb{F}^{n \times m} & \to & \mathbb{F}^{n \times m}, & (S, U, A) & \mapsto & SAU^{-1}, \\
\rho_3 : (\text{GL}_n(\mathbb{F}) \times \text{GL}_m(\mathbb{F})) & \times \mathbb{F}^{n \times m} & \to & \mathbb{F}^{n \times m}, & (S, T, A) & \mapsto & SAT^{-1}.
\end{align*}$$

Denote by $\mathcal{O}_i$ the partition of $\mathbb{F}^{n \times m}$ consisting of the orbits of $\rho_i$. We summarize some important properties of these partitions.

Proposition 2.4. 1. $\mathcal{P}^{\text{rs}} \leq \mathcal{P}^{\text{piv}} \leq \mathcal{P}^{\text{rk}}$, that is, the row-space partition is finer than the pivot partition, which is finer than the rank partition.

2. $|\mathcal{P}^{\text{rk}}| = m + 1, |\mathcal{P}^{\text{rs}}| = |\mathcal{L}| = \sum_{i=0}^{m} \binom{m}{i}$, and $|\mathcal{P}^{\text{piv}}| = |\Pi| = \sum_{r=0}^{m} \binom{m}{r} = 2^m$.

3. $\mathcal{P}^{\text{rs}} = \mathcal{O}_1$, $\mathcal{P}^{\text{piv}} = \mathcal{O}_2$, and $\mathcal{P}^{\text{rk}} = \mathcal{O}_3$.

4. $\mathcal{P}^{\text{rk}}, \mathcal{P}^{\text{rs}}$ and $\mathcal{P}^{\text{piv}}$ are invariant partitions.

Proof. Property (1) is clear and (4) is immediate from (3). Property (2) follows from the fact that for every possible rank $r \in \{0, \ldots, m\}$ we have $\binom{m}{r}$ possibilities for the pivot indices of a rank $r$ matrix in $\mathbb{F}^{n \times m}$. The other two statements are clear.

Let us show (3). The identities concerning $\mathcal{P}^{\text{rs}}$ and $\mathcal{P}^{\text{rk}}$ are basic Linear Algebra. It remains to show $\mathcal{P}^{\text{piv}} = \mathcal{O}_2$. Consider a matrix $A \in \mathbb{F}^{n \times m}$ and denote its columns by $A_1, \ldots, A_m$. Then for any $j \in [m]$ we have

$$j \in \text{piv}(A) \iff A_j \text{ is not in the span of the columns } A_1, \ldots, A_{j-1}. \quad (2.3)$$

Let now $B = SAU^{-1}$ for some $S \in \text{GL}_n(\mathbb{F})$ and $U \in \mathcal{U}_n(\mathbb{F})$. Then (2.3) immediately implies that $j \in \text{piv}(A) \iff j \in \text{piv}(B)$ for any $j \in [m]$. This proves $\mathcal{O}_2 \leq \mathcal{P}^{\text{piv}}$. For the converse let $A, B \in \mathbb{F}^{n \times m}$ such that $\text{piv}(A) = \text{piv}(B) := (j_1, \ldots, j_r)$. Let $\hat{A}, \hat{B}$ be the RREF’s of $A, B$, respectively. Then $\hat{A} = XA$ and $\hat{B} = YB$ for some $X, Y \in \text{GL}_n(\mathbb{F})$. Denote by $e_1, \ldots, e_n$ the standard basis (column) vectors of $\mathbb{F}^n$. Define the matrix $M = (M_1, \ldots, M_m) \in \mathbb{F}^{n \times m}$ via

$$M_i = \begin{cases} e_{j_\ell}, & \text{if } i = j_\ell \text{ for some } \ell \in \{1, \ldots, r\}, \\ 0, & \text{otherwise} \end{cases}$$
In other words, $M$ is obtained from $\hat{A}$ (hence $\hat{B}$) by keeping the pivot columns and erasing the others. Now (2.3) implies
\[ \hat{A} = MV, \quad \hat{B} = MW \text{ for some } V, W \in \mathcal{U}_m(F). \]
Hence $B = Y^{-1}XAV^{-1}W$, and since $V^{-1}W$ is in $\mathcal{U}_m(F)$ we conclude that the matrices $A, B$ are in the same orbit of $\mathcal{O}_2$.

We now turn to the duals of these partition. The following more general result will be helpful. It is a special case of [3, Prop. 4.6], where partitions induced by group actions are considered for arbitrary finite Frobenius rings instead of finite fields. For the sake of self-containment we provide a short proof.

**Proposition 2.5.** Let $\mathcal{S} \leq \text{GL}_n(F)$ and $\mathcal{T} \leq \text{GL}_m(F)$ be subgroups and define their transposes as $\mathcal{S}' = \{ S^\top \mid S \in \mathcal{S} \}$ and $\mathcal{T}' = \{ T^\top \mid T \in \mathcal{T} \}$. Consider the group actions
\[ \rho : \mathcal{S} \times \mathcal{T} \times \mathbb{P}^{n \times m} \longrightarrow \mathbb{P}^{n \times m}, \quad (S, T, A) \longmapsto SAT^{-1}, \]
\[ \rho' : \mathcal{S}' \times \mathcal{T}' \times \mathbb{P}^{n \times m} \longrightarrow \mathbb{P}^{n \times m}, \quad (S', T', A) \longmapsto SAT^{-1}. \]
Let $\mathcal{O}$ and $\mathcal{O}'$ be the orbit partitions of $\rho$ and $\rho'$, respectively. Then $\hat{\mathcal{O}} = \mathcal{O}'$ and $\hat{\mathcal{O}'} = \mathcal{O}$. Thus, the partitions are reflexive and $|\mathcal{O}| = |\mathcal{O}'|$. 

**Proof.** We show that $\mathcal{O}' \leq \mathcal{O}$. Let $B, B' \in \mathbb{P}^{n \times m}$ be in the same orbit of $\mathcal{O}'$, hence $B' = SBT$ for some $S \in \mathcal{S}'$ and $T \in \mathcal{T}'$. For any orbit $O$ of $\mathcal{O}$ we have $S^\top OT^\top = O$ and therefore
\[
\sum_{A \in O} \chi(\langle A, B' \rangle) = \sum_{A \in O} \chi(\langle A, SBT \rangle) = \sum_{A \in O} \chi(\text{tr}(AT^\top B^\top S^\top)) = \sum_{A \in O} \chi(\text{tr}(S^\top AT^\top B^\top))
\]
\[ = \sum_{A \in O} \chi(\text{tr}(AB^\top)) = \sum_{A \in O} \chi(\langle A, B \rangle). \]
Hence $\mathcal{O}' \leq \mathcal{O}$. By symmetry we also have $\mathcal{O} \leq \mathcal{O}'$ and thus $\mathcal{O} \leq \mathcal{O}'$. Since by [17, Thm. 2.4] the converse is true for any partition, we conclude $\mathcal{O} = \mathcal{O}'$. Furthermore, any partition $\mathcal{P}$ satisfies $|\mathcal{P}| \leq |\mathcal{P}'|$, see again [17, Thm. 2.4], and thus we obtain $|\mathcal{O}'| \leq |\mathcal{O}'| \leq |\mathcal{O}| \leq |\mathcal{O}'|$, where the middle step follows from $\mathcal{O} \leq \mathcal{O}'$. Now the relation $\mathcal{O}' \leq \mathcal{O}$ implies $\mathcal{O}' = \mathcal{O}$. The rest follows from symmetry. 

The following is now immediate with Proposition 2.4(3).

**Corollary 2.6.** $\mathcal{P}_R = \mathcal{P}_R'$ and $\mathcal{P}_S = \mathcal{P}_S'$, that is, the rank partition and the row-space partition are self-dual.

In order to describe the dual of the pivot partition we need the reverse pivot indices. They are defined by performing Gaussian elimination on a matrix from right to left. This is most conveniently defined using the matrix
\[
S = \begin{pmatrix}
1 & 1 & \cdot & \cdot \\
0 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot
\end{pmatrix} \in \text{GL}_m(F). \tag{2.4}
\]
Obviously, right multiplication of a matrix $A$ by $S$ reverses the order of the columns of $A$. 

\[
\]
**Definition 2.7.** Let $A \in \mathbb{F}^{n \times m}$ be a matrix and set $\hat{A} := AS$. Let $\lambda = \text{piv}(\hat{A}) = (j_1, \ldots, j_r) \in \Pi$. Then we define the **reverse pivot indices** of $A$ as

$$rpiv(A) = (m + 1 - j_r, \ldots, m + 1 - j_1).$$

We call RREF($\hat{A}$)S the **reverse reduced row echelon form** of $A$. Its pivot indices are $rpiv(A)$. Matrices $A, B \in \mathbb{F}^{n \times m}$ are called **reverse-pivot-equivalent** if $rpiv(A) = rpiv(B)$. The resulting equivalence classes form the **reverse-pivot partition** of $\mathbb{F}^{n \times m}$, denoted by $\mathscr{P}^{rpiv}$.

Note that $rpiv(A) \in \Pi$, which means that the indices are ordered increasingly. They satisfy the reverse analogue of (2.3), i.e., for all $j \in [m]$

$$j \in rpiv(A) \iff A_j \text{ is not in the span of } A_{j+1}, \ldots, A_m.$$  \quad (2.5)

In analogy to Proposition 2.4(3), $\mathscr{P}^{rpiv}$ is the orbit partition of the group action $\rho_2$ if we replace $\mathcal{M}_m(F)$ by the group of lower triangular invertible matrices.

**Corollary 2.8.** We have $\hat{\mathcal{P}}^{piv} = \mathscr{P}^{rpiv}$ and $\hat{\mathcal{P}}^{rpiv} = \mathcal{P}^{piv}$. In particular, the partitions $\mathcal{P}^{piv}$ and $\mathcal{P}^{rpiv}$ are reflexive, but not self-dual.

The above tells us that the pivot indices and the reverse pivot indices encode partitions that are mutually dual with respect to the trace inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{F}^{n \times m}$ as in (1.1). In the remainder of this section we show how these indices reflect duality of subspaces in $\mathbb{F}^n$ with respect to the standard inner product on $\mathbb{F}^m$.

For $V \in \mathcal{L}$ denote by $V^\perp$ its orthogonal with respect to the standard inner product. Furthermore, thanks to the uniqueness of the reduced row echelon form we can extend both the pivot partition and the reverse pivot partition to the lattice $\mathcal{L}$ of all subspaces of $\mathbb{F}^m$: define $\text{piv}(V) = \text{piv}(A)$, where $A \in \mathbb{F}^{r \times m}$ is any matrix of full rank with row space $V$ and define $rpiv(V)$ similarly. We need the following notion.

**Definition 2.9.** Let $\lambda = (\lambda_1, \ldots, \lambda_r) \in \Pi$. We denote by $\hat{\lambda} \in \Pi$ the **dual pivot list** of $\lambda$, that is, $\hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_{m-r}) \in \Pi$ such that $\{\lambda_1, \ldots, \lambda_r, \hat{\lambda}_1, \ldots, \hat{\lambda}_{m-r}\} = [m]$.

Now we can show that for any subspace $V \in \mathcal{L}$ the list of reverse pivot indices of the dual subspace $V^\perp$ is the dual of the list of pivot indices of $V$. We will need this result later in Section 6. Even though this is an entirely basic result from Linear Algebra, we were not able to find it in the literature and thus provide a proof.

**Proposition 2.10.** Let $V \in \mathcal{L}$ and $\text{piv}(V) = \lambda$. Then $rpiv(V^\perp) = \hat{\lambda}$.

**Proof.** Throughout this proof, for any matrix $M \in \mathbb{F}^{r \times m}$ we denote by $M_t$ the $t$th column of $M$. Furthermore, we let $e_1, \ldots, e_m$ denote the standard basis vectors in $\mathbb{F}^m$ and also use $e_1, \ldots, e_{m-r}$ as the standard basis vectors in $\mathbb{F}^{m-r}$. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_{m-r})$.

Let $\dim(V) = r$ and let $A = (A_{ij}) \in \mathbb{F}^{r \times m}$ be in RREF (reduced row echelon form) and such that $\text{rs}(A) = V$. Define the permutation matrix $P = (e_{\lambda_1}, \ldots, e_{\lambda_r}, e_{\hat{\lambda}_1}, \ldots, e_{\hat{\lambda}_{m-r}}) \in \text{GL}_m(\mathbb{F})$. Then

$$AP = (I_r \mid B), \text{ where } B = (B_{a,b}) = (A_{a,\hat{\lambda}_b}) \in \mathbb{F}^{r \times (m-r)} \text{ satisfies } B_{a,\beta} = 0 \text{ whenever } \hat{\lambda}_\beta < \lambda_a.$$ \quad (2.6)

In other words, the pivot columns have been sorted to the front and the remaining columns appear in their original order in the matrix $B$. It follows that

$$V^\perp = \text{rs}(M), \text{ where } M = ((-B)^T \mid I_{m-r})P^{-1}$$
We show now that $M$ is in reverse reduced row-echelon form with $\text{rpiv}(M) = \hat{\lambda}$ (see Definition 2.7).

Condition (2.6) implies for the columns of $C := (-B)^\top \in \mathbb{F}^{(m-r) \times r}$

$$ C_\alpha \in \text{span}\{e_\beta \mid \hat{\lambda}_\beta > \lambda_\alpha\}. $$

Hence the columns $M_t$ are given by

$$ M_t = \begin{cases} C_\alpha, & \text{if } t = \lambda_\alpha \text{ for some } \alpha = 1, \ldots, r \\ e_\beta, & \text{if } t = \hat{\lambda}_\beta \text{ for some } \beta = 1, \ldots, m-r. \end{cases} $$

Thus (2.7) reads as $M_{\lambda_\alpha} \in \text{span}\{M_{\hat{\lambda}_\beta} \mid \hat{\lambda}_\beta > \lambda_\alpha\}$, and this means that $\lambda_\alpha$ is not a reverse pivot index of $M$; see (2.5). As this is true for all $\alpha \in \{1, \ldots, r\}$ and $M$ has rank $m-r$, we arrive at $\text{rpiv}(V^\perp) = \text{rpiv}(M) = (\hat{\lambda}_1, \ldots, \hat{\lambda}_{m-r}) = \hat{\lambda}$.

\[\Box\]

3 \hspace{1em} The Krawtchouk Coefficients of the Row-Space Partition

In this section we explicitly determine the Krawtchouk coefficients of the row-space partition. Recall that $\mathcal{L}$ denotes the lattice of all subspaces of $\mathbb{F}^m$.

\textbf{Definition 3.1.} Let $\mathcal{C} \leq \mathbb{F}^{n \times m}$ be a code. For $U \in \mathcal{L}$ define $\mathcal{C}(U) = \{A \in \mathcal{C} \mid \text{rs}(A) \leq U\}$. Then $\mathcal{C}(U)$ is a code as well (i.e., it is a linear subspace of $\mathcal{C}$).

Note that we consider two kinds of dual spaces: the dual $\mathcal{C}^\perp$ of a matrix code $\mathcal{C} \leq \mathbb{F}^{n \times m}$ with respect to the trace product (see Definition 1.6) and the dual $U^\perp$ of a subspace $U \in \mathcal{L}$ with respect to the standard inner product on $\mathbb{F}^m$. These two kinds of dual spaces are related as follows.

\textbf{Lemma 3.2 ([30, Lem. 28]).} Let $U \in \mathcal{L}$ with $\text{dim}U = u$. Then

$$ |\mathcal{C}(U)| = \frac{|\mathcal{C}|}{q^{n(m-u)}} |\mathcal{C}^\perp(U^\perp)|. $$

Now we obtain the following explicit formulas for the Krawtchouk coefficients of $\mathcal{P}^{\text{rs}}$.

\textbf{Theorem 3.3.} For all $U, V \in \mathcal{L}$ we have

$$ K(\mathcal{P}^{\text{rs}}; U, V) = \sum_{t=0}^{m} (-1)^{\text{dim}(U)-t} q^{m+\left(\frac{\text{dim}(U)-t}{2}\right)} \left[\text{dim}(U \cap V^\perp)\right]_t. $$

\textbf{Proof.} Fix a subspace $V \in \mathcal{L}$ and let $M \in \mathbb{F}^{n \times m}$ be any matrix with $\text{rs}(M) = V$. Fix any non-trivial character $\chi$ of $\mathbb{F}$. Let $f, g : \mathcal{L} \rightarrow \mathbb{C}$ be the functions defined, for all $U \in \mathcal{L}$, by

$$ f(U) := \sum_{N \in \mathbb{F}^{n \times m}} \chi(\text{Tr}(MN^t)), \quad g(U) := \sum_{U' \leq U} f(U'). $$

Therefore $f(U) = K(\mathcal{P}^{\text{rs}}; U, V)$ for all $U \in \mathcal{L}$; see Definition 1.5. By Definition 3.1 we have $\mathbb{F}^{n \times m}(U) = \{N \in \mathbb{F}^{n \times m} \mid \text{rs}(N) \leq U\}$. It follows that $\mathbb{F}^{n \times m}(U)^\perp = \mathbb{F}^{n \times m}(U^\perp)$ by [30, Lem. 27] and that $|\mathbb{F}^{n \times m}(U)^\perp| = q^{n\text{dim}(U)}$ by [30, Lem. 26]. Thus for all $U \in \mathcal{L}$ we have

$$ g(U) = \sum_{N \in \mathbb{F}^{n \times m}} \chi(\text{Tr}(MN^t)) = \sum_{N \in \mathbb{F}^{n \times m}} \chi(\text{Tr}(MN^t)) = \begin{cases} q^{n\text{dim}(U)} & \text{if } M \in \mathbb{F}^{n \times m}(U^\perp), \\
0 & \text{otherwise}, \end{cases} $$

where $\chi$ is the character.
where the last equality follows from the orthogonality relations of characters. Denote by \( \mu_\mathcal{L} \) the Möbius function of the lattice \( \mathcal{L} \). From [38, Ex. 3.10.2] we know
\[
\mu(W, V) = \begin{cases} 
(-1)^{v-w}q^{(v-w)/2} & \text{if } W \leq V, \\
0 & \text{otherwise,}
\end{cases}
\]  
(3.1)
where \( \dim W = w \) and \( \dim V = v \). Using that \( M \in \mathbb{F}^{n \times m}(U'\perp) \) iff \( U' \leq V'\perp \), we thus obtain from Möbius inversion
\[
f(U) = \sum_{U' \leq U} g(U') \mu_\mathcal{L}(U', U) = \sum_{U' \leq U \cap V'} q^{\dim(U')}(1 - 1)^{u - \dim U'} q^{(u - \dim U')/2}
\]
for all subspaces \( U \in \mathcal{L} \) with \( \dim(U) = u \). As a consequence,
\[
f(U) = \sum_{t=0}^{m} \sum_{U' \leq U \cap V'} q^{mt}(1)^{u-t} q^{(u-t)/2} = \sum_{t=0}^{m} (-1)^{u-t}q^{n-t} \left[ \dim(U \cap V') \right].
\]
This gives the desired formula. \( \square \)

Combining Theorem 1.8 with Theorem 3.3 one immediately obtains MacWilliams-type identities for the row-space partition.

**Corollary 3.4.** Let \( \mathcal{C} \leq \mathbb{F}^{n \times m} \) be a code. Then for all \( V \in \mathcal{L} \) we have
\[
\mathcal{P}_\mathcal{S}(\mathcal{C} \perp, V) = \frac{1}{|\mathcal{C}|} \sum_{U \in \mathcal{L}} \mathcal{P}_\mathcal{S}(\mathcal{C}, U) \sum_{t=0}^{m} (-1)^{\dim(V) - t} q^{n-t} \left[ \dim(V \cap U') \right].
\]

In the remainder of this section we provide different relations between the row-space partition distribution of a code \( \mathcal{C} \) and that of \( \mathcal{C} \perp \).

**Proposition 3.5.** Let \( \mathcal{C} \leq \mathbb{F}^{n \times m} \) be a matrix code. Then for all \( U \in \mathcal{L} \) we have
\[
\sum_{V \leq U} \mathcal{P}_\mathcal{S}(\mathcal{C}, V) = \frac{|\mathcal{C}|}{q^{\dim U}} \sum_{W \leq U'} \mathcal{P}_\mathcal{S}(\mathcal{C} \perp, W).
\]

**Proof.** Using Lemma 3.2 we obtain
\[
\sum_{V \leq U} \mathcal{P}_\mathcal{S}(\mathcal{C}, V) = |\mathcal{C}(U)| = \frac{|\mathcal{C}|}{q^{\dim U}} |\mathcal{C}(U')| = \frac{|\mathcal{C}|}{q^{\dim U}} \sum_{W \leq U'} \mathcal{P}_\mathcal{S}(\mathcal{C} \perp, W). \square
\]

The last proposition gives \( N \) linear relations, where \( N = |\mathcal{L}| \). They may be written as a linear system as follows. Define the row vectors
\[
\mathcal{P}_\mathcal{S}(\mathcal{C}) = (\mathcal{P}_\mathcal{S}(\mathcal{C}, V))_{V \in \mathcal{L}}, \quad \mathcal{P}_\mathcal{S}(\mathcal{C} \perp) = (\mathcal{P}_\mathcal{S}(\mathcal{C} \perp, V))_{V \in \mathcal{L}} \in \mathbb{C}^N,
\]
which describe the partition distribution of the codes \( \mathcal{C} \) and \( \mathcal{C} \perp \), respectively; see Definition 1.7. Then Proposition 3.5 reads as
\[
\mathcal{P}_\mathcal{S}(\mathcal{C}) \cdot A = |\mathcal{C}| \cdot \mathcal{P}_\mathcal{S}(\mathcal{C} \perp) \cdot B \cdot D,
\]
where \( A, B, D \in \mathbb{C}^{N \times N} \) are defined as
\[
A(V, U) := \begin{cases} 
1 & \text{if } V \leq U, \\
0 & \text{otherwise,}
\end{cases} \quad B(V, U) := A(V, U'), \quad D := \text{diag}\left(1/q^{\dim(U')}\right)_{U \in \mathcal{L}}.
\]
The matrix \( A \) may be regarded as the \( \zeta \)-function of the subspace lattice \( \mathcal{L} \). Thus its inverse is the Möbius function, which shows that \( A \) is invertible. The same is true for the matrix \( B \). Therefore we have
\[
\mathcal{P}(\mathcal{L}) = \frac{1}{|\mathcal{C}|} \cdot \mathcal{P}(\mathcal{C}) \cdot M, \quad \text{where } M := A \cdot \text{diag} \left( q^{\dim(U)} \right)_{U \in \mathcal{L}} \cdot B^{-1}.
\]

This provides us with a different method to compute the enumerators \( \mathcal{P}(\mathcal{C}, V) \) from the enumerators \( \mathcal{P}(\mathcal{C}, V) \) for \( V \in \mathcal{L} \). The entries of the matrix \( M \in \mathbb{C}^{N \times N} \) are the Krawtchouk coefficients of the row-space partition \( \mathcal{P} \). This follows, for instance, from [17, Thm. 2.7].

We close this section by presenting the binomial moments of the row-space distribution. They consist of \( m + 1 \) identities and form the analogue to those for the Hamming weight in \( \mathbb{F}^n \) (see [23, (M_2) on p. 257]) and for the rank weight (see [14, Prop. 4] for \( \mathbb{F}_q \)-linear rank-metric codes and [30, Thm. 31] for \( \mathbb{F}_q \)-linear rank-metric codes).

**Proposition 3.6.** Let \( \mathcal{C} \subseteq \mathbb{F}^{n \times m} \) be a matrix code. Then for all integers \( 0 \leq v \leq m \) we have
\[
\sum_{V \in \mathcal{L}} \left[ \begin{array}{c} m - \dim V \\ v \end{array} \right] \mathcal{P}(\mathcal{C}, V) = \frac{|\mathcal{C}|}{q^v} \sum_{W \in \mathcal{L}} \left[ \begin{array}{c} m - \dim W \\ m - v \end{array} \right] \mathcal{P}(\mathcal{C}^\perp, W).
\]

**Proof.** By [18, Eq. (8)], for all \( \mathcal{C} \subseteq \mathbb{F}^{n \times m} \) and all \( 0 \leq v \leq m \) we have
\[
\sum_{U \in \mathcal{L}} |\mathcal{C}(U)| = \sum_{i=0}^{m-v} \left[ \begin{array}{c} m - i \\ v \end{array} \right] \mathcal{P}^{\text{rk}}(\mathcal{C}, i).
\]

Therefore
\[
\sum_{V \in \mathcal{L}} \left[ \begin{array}{c} m - \dim V \\ v \end{array} \right] \mathcal{P}(\mathcal{C}, V) = \sum_{i=0}^{m} \left[ \begin{array}{c} m - i \\ v \end{array} \right] \sum_{V \in \mathcal{L}, \dim(V) = i} \mathcal{P}(\mathcal{C}, V) = \sum_{i=0}^{m} \left[ \begin{array}{c} m - i \\ v \end{array} \right] \mathcal{P}^{\text{rk}}(\mathcal{C}, i) = \sum_{\dim U = m - v} |\mathcal{C}(U)|.
\]

Similarly,
\[
\sum_{W \in \mathcal{L}} \left[ \begin{array}{c} m - \dim W \\ m - v \end{array} \right] \mathcal{P}(\mathcal{C}^\perp, W) = \sum_{\dim U = v} |\mathcal{C}(U)|.
\]

Using Lemma 3.2 we obtain
\[
\sum_{V \in \mathcal{L}} \left[ \begin{array}{c} m - \dim V \\ v \end{array} \right] \mathcal{P}(\mathcal{C}, V) = \frac{|\mathcal{C}|}{q^v} \sum_{\dim U = m - v} |\mathcal{C}(U)^\perp| = \frac{|\mathcal{C}|}{q^v} \sum_{\dim U = v} |\mathcal{C}^\perp(U)|
\]
\[
= \frac{|\mathcal{C}|}{q^v} \sum_{W \in \mathcal{L}} \left[ \begin{array}{c} m - \dim W \\ m - v \end{array} \right] \mathcal{P}(\mathcal{C}^\perp, W),
\]
for all \( 0 \leq v \leq m \), which is the desired equation.

\[ \square \]

### 4 \( U \)-Extremal Codes

Recall that MRD codes are maximal in size as codes in \( \mathbb{F}^{n \times m} \) with prescribed distance \( d \). It is well-known that they enjoy various properties. We will briefly list these properties and then generalize the concept to matrix codes with respect to the row-space partition.
Notation 4.1. Throughout this section, we assume without loss of generality that \( n \geq m \). This choice will simplify the statements and the notation.

Recall that the (minimum rank) distance of a non-zero code \( \mathcal{C} \subseteq \mathbb{F}^{n \times m} \) is defined as
\[
d_{rk}(\mathcal{C}) := \min\{\text{rk}(A - B) \mid A, B \in \mathcal{C}, A \neq B\} = \min\{\text{rk}(M) \mid M \in \mathcal{C} \setminus \{0\}\}. \tag{4.1}
\]
The Singleton-like bound for rank-metric codes \([7, \text{Thm. 5.4}]\) tells us that if \( \mathcal{C} \subseteq \mathbb{F}^{n \times m} \) is a non-zero code of distance \( d \) and \( m \leq n \), then
\[
|\mathcal{C}| \leq q^{n(m-d+1)}. \tag{4.2}
\]

A code \( \mathcal{C} \subseteq \mathbb{F}^{n \times m} \) is an MRD code if \( \mathcal{C} = \{0\} \) or if \( \mathcal{C} \neq \{0\} \) and \( |\mathcal{C}| = q^{n(m-d+1)} \), where \( d = d_{rk}(\mathcal{C}) \). In other words, MRD codes are extremal with respect to the Singleton-like bound.

Remark 4.2. Let \( \mathcal{C} \subseteq \mathbb{F}^{n \times m} \) be an MRD code. The following hold.

1. The dual code \( \mathcal{C}^\perp \) is MRD as well. Moreover, if \( \mathcal{C} \neq \{0\} \) has minimum distance \( d \), then \( \mathcal{C}^\perp \) has minimum distance \( m - d + 2 \); see \([7, \text{Thm. 5.5}]\) or also \([30, \text{Cor. 41}]\).

2. The rank-distribution of \( \mathcal{C} \) only depends only on the parameters \( q, n, m, d \) of the code; see \([7, \text{Thm. 5.6}]\) or \([30, \text{Cor. 44}]\).

MRD codes are even more rigid than stated in (2) above: even their row-space distribution depends only on their parameters, as the following result shows. This fact also appears from the proof of \([18, \text{Thm. 8}]\). Recall the notation \( \mathcal{L} \) for the lattice of subspaces in \( \mathbb{F}^{m} \) as well as the notation for partition distributions in Definition 1.7.

Theorem 4.3. Let \( \mathcal{C} \subseteq \mathbb{F}^{n \times m} \) be a non-zero MRD code of minimum distance \( d \), and let \( V \in \mathcal{L} \) with \( \dim(V) = v \). Then
\[
\mathcal{P}^{rs}(\mathcal{C}, V) = \sum_{i=0}^{d-1} \binom{v}{i} (-1)^{v-i} q^{\binom{v-i}{2}} + \sum_{i=d}^{v} \binom{v}{i} q^{n(i-d+1)}(-1)^{v-i} q^{\binom{v-i}{2}}.
\]
In particular, the row-space distribution of \( \mathcal{C} \) depends only on the parameters \( q, n, m, d \).

One may note that the above expression actually does not explicitly depend on \( m \). This parameter only enters via the lattice \( \mathcal{L} \).

Proof. Fix \( V \in \mathcal{L} \) with \( \dim(V) = v \). It follows from \([31, \text{Lem. 48}]\) (see also \([18, \text{Lem. 25}]\)) that
\[
|\mathcal{C}(V)| = \begin{cases} 
1 & \text{if } 0 \leq v \leq d - 1, \\
q^{n(v-d+1)} & \text{if } v \geq d.
\end{cases} \tag{4.3}
\]
Define functions \( f, g : \mathcal{L} \longrightarrow \mathbb{R} \) by
\[
f(V) = \mathcal{P}^{rs}(\mathcal{C}, V) \quad \text{and} \quad g(V) = \sum_{U \subseteq V} f(U)
\]
for all \( V \in \mathcal{L} \). Then \( g(V) = |\mathcal{C}(V)| \) by definition. Using Möbius inversion in the lattice \( \mathcal{L} \) and (3.1) we compute
\[
f(V) = \sum_{U \subseteq V} g(U) \mu_{\mathcal{L}}(U, V) \\
= \sum_{\dim(U) \leq d - 1} (-1)^{v-\dim(U)} q^{\binom{v-\dim(U)}{2}} + \sum_{\dim(U) \geq d} q^{n(\dim(U) - d + 1)} (-1)^{v-\dim(U)} q^{\binom{v-\dim(U)}{2}}.
\]
The desired identity follows from the fact that \( V \) contains \( \left\lfloor \frac{v}{d} \right\rfloor \) subspaces of dimension \( i \).
Note that using the \( q \)-binomial theorem [38, p. 74] one easily confirms that \( \mathcal{P}^3(\mathcal{C}, V) = 0 \) whenever \( \dim(V) \in \{1, \ldots, d-1\} \).

We now propose a generalization of the Singleton-type bound for matrix codes. This will lead to a refined notion of extremality.

**Proposition 4.4.** Let \( \mathcal{C} \leq \mathbb{F}^{n \times m} \) and \( U \in \mathcal{L} \) with \( u := \dim(U) \). Assume \( \mathcal{C}(U) = \{0\} \). Then we have \( |\mathcal{C}| \leq q^{n(m-u)} \).

**Proof.** From [30, Lem. 26] we know that \( \dim(\mathbb{F}^{n \times m}(U)) = nu \). Therefore \( 0 = \dim(\mathcal{C}(U)) = \dim(\mathcal{C} \cap \mathbb{F}^{n \times m}(U)) \geq \dim(\mathcal{C}) + nu - nm \), which results in the stated bound.

The following generalization of MRD codes is natural from the previous result. We will see that these codes satisfy similar rigidity properties as listed for MRD codes in Remark 4.2.

**Definition 4.5.** Let \( \mathcal{C} \leq \mathbb{F}^{n \times m} \) and \( U \in \mathcal{L} \) with \( u := \dim(U) \). We say \( \mathcal{C} \) is \( U \)-extremal if \( \mathcal{C}(U) = \{0\} \) and \( |\mathcal{C}| = q^{n(m-u)} \).

Clearly, \( \mathbb{F}^{n \times n} \) is the only \( \{0\} \)-extremal code and, dually, \( \{0\} \) is the only \( \mathbb{F}^{m \times 1} \)-extremal code. The Singleton-like bound (4.2) implies that if \( \mathcal{C} \) is \( U \)-extremal, then \( \dim(U) \geq d_k(\mathcal{C}) - 1 \). This immediately leads to the following observation.

**Remark 4.6.** Let \( \mathcal{C} \leq \mathbb{F}^{n \times m} \) be a non-zero code of minimum distance \( d \). The following are equivalent.

1. \( \mathcal{C} \) is an MRD code,
2. \( \mathcal{C} \) is \( U \)-extremal for all \( U \in \mathcal{L} \) with \( \dim(U) = d-1 \),
3. \( \mathcal{C} \) is \( U \)-extremal for some \( U \in \mathcal{L} \) with \( \dim(U) = d-1 \).

There exist \( U \)-extremal codes that are not MRD.

**Example 4.7.** Write \( m = m_1 + m_2 \) with \( m_1, m_2 \neq 0 \). Let \( \mathcal{C}_1 \leq \mathbb{F}^{n \times m_1} \) be a non-zero MRD code of minimum distance \( d \), say. Define \( \mathcal{C} = \{ (A \mid 0) \in \mathbb{F}^{n \times (m_1 + m_2)} \mid A \in \mathcal{C}_1 \} \). Then \( \mathcal{C} \leq \mathbb{F}^{n \times (m_1 + m_2)} \) has cardinality \( q^{n(m_1-d+1)} \) and minimum distance \( d \), thus \( \mathcal{C} \) is not MRD.

Choose any subspace \( U_1 \leq \mathbb{F}^{m_1} \) of dimension \( d-1 \) and set \( U = U_1 \times \mathbb{F}^{m_2} \). Then \( \dim(U) = m_2 + d - 1 \) and thus \( |\mathcal{C}| = q^{n(m_1+m_2-d)} \). In order to see that the code \( \mathcal{C} \) is \( U \)-extremal, let \( (A \mid 0) \in \mathcal{C} \) such that \( rs(A \mid 0) \leq U \). Then \( rs(A) \leq U_1 \) and thus \( \text{rk}(A) \leq d - 1 \). But then \( A = 0 \), and all of this shows that \( \mathcal{C}(U) = \{0\} \). Hence \( \mathcal{C} \) is \( U \)-extremal, but not MRD.

Extremality is preserved under dualization.

**Proposition 4.8.** Let \( \mathcal{C} \leq \mathbb{F}^{n \times m} \) and \( U \in \mathcal{L} \). The following are equivalent.

1. \( \mathcal{C} \) is \( U \)-extremal,
2. \( \mathcal{C}^\perp \) is \( U^\perp \)-extremal.

**Proof.** We only need to show one direction. If \( \mathcal{C} \) is \( U \)-extremal then \( |\mathcal{C}(U)| = 1 \) and \( |\mathcal{C}| = q^{n(m-u)} \). By Lemma 3.2 we have \( |\mathcal{C}(U^\perp)| = 1 \). Therefore \( \mathcal{C}(U^\perp) = \{0\} \). The result now follows from the fact that \( |\mathcal{C}^\perp| = q^{nm} / |\mathcal{C}| = q^{nu} = q^{n(m - \dim(U^\perp))} \).
Next we turn to the row-space distribution of $U$-extremal codes. It cannot be expected that the entire distribution depends only on the parameters of the code and the dimension of $U$. The following example illustrates this.

**Example 4.9.** Consider again Example 4.7, taking $m_2 = 1$. Then we have $\dim(U) = d$ and, of course, $\mathcal{P}^s(\mathcal{C}, U) = 0$. On the other hand, choose any $d$-dimensional subspace $\tilde{V} \leq \mathbb{F}^{m_1}$ and set $V = \tilde{V} \times \{0\}$. Then $\mathcal{P}^s(\mathcal{C}, V) = \mathcal{P}^s(\tilde{C}, \tilde{V}) = |\tilde{C}(\tilde{V})| - 1 = q^d - 1$ thanks to (4.3). Thus $\mathcal{P}^s(\mathcal{C}, V) \neq \mathcal{P}^s(\mathcal{C}, U)$ even though $\dim(V) = \dim(U)$.

However, we do obtain a rigidity result in the case where $\mathcal{C}$ is $U$-extremal for all subspaces $U$ of a fixed dimension contained in a given $T \in \mathcal{L}$. The following result generalizes Theorem 4.3.

**Theorem 4.10** (Rigidity of extremality). Let $\mathcal{C} \leq \mathbb{F}^{n \times m}$ and $T \in \mathcal{L}$. Let $0 \leq u \leq \dim(T)$ be an integer, and suppose that $\mathcal{C}$ is $U$-extremal for all $U \leq T$ of dimension $u$. Then for all $V \in \mathcal{L}$ with $V \leq T$ we have

$$\mathcal{P}^s(\mathcal{C}, V) = \sum_{i=0}^{u} \binom{v}{i} (-1)^{v-i} q^{\binom{v-i}{2}} + \sum_{i=u+1}^{v} q^{\dim(\tilde{C})} q^{\binom{v-i}{2}} (-1)^{v-i} q^{\binom{v-i}{2}},$$

(4.4)

where $v = \dim(V)$. Hence the partial row space distribution $(\mathcal{P}^s(\mathcal{C}, V))_{V \leq T}$ depends only on $n, q, u$.

Note the extreme case where $u = \dim(T)$, in which the assumptions simply mean that $\mathcal{C}$ is $T$-extremal. This clearly implies $\mathcal{P}^s(\mathcal{C}, V) = 0$ for all $0 < V \leq T$, which also follows from (4.4) along with the $q$-binomial formula. More interestingly, we also recover Theorem 4.3: choose $T = \mathbb{F}^m$ and $u = d - 1$. Then the above assumption means that $\mathcal{C}$ is MRD, see Remark 4.2, and (4.4) coincides with Theorem 4.3.

**Proof.** Let $V \leq T$ have dimension $v$. We show first that

$$|\mathcal{C}(V)| = \begin{cases} 1 & \text{if } 0 \leq v \leq u, \\ q^{\dim(V)} & \text{if } v > u. \end{cases}$$

(5.5)

Indeed, if $v \leq u$ then there exists $U \in \mathcal{L}$ with $V \leq U \leq T$. Since $\mathcal{C}$ is $U$-extremal, we have $|\mathcal{C}(V)| = 1$. Therefore $|\mathcal{C}(V)| = 1$. Now suppose that $v > u$, and fix a $u$-dimensional space $U \in \mathcal{L}$ with $U \leq V$. By Proposition 4.8, $\mathcal{C}^\perp$ is $U^\perp$-extremal. Therefore $\mathcal{C}^\perp(U^\perp) \leq \mathcal{C}^\perp(U^\perp) = \{0\}$. Thus by Lemma 3.2 we conclude $|\mathcal{C}(V)| = \prod_{U \leq T} = q^{\dim(V)}$. This establishes (4.5).

We can now proceed as in the proof of Theorem 4.3, this time using Möbius inversion on the interval $[0, T]$ of $\mathcal{L}$. \hfill \Box

We conclude this section by observing that there are indeed non-MRD codes that satisfy the assumptions of Theorem 4.10. The example also shows that the result just proven does not extend to subspaces that are not contained in $T$.

**Example 4.11.** 1. Let $n, m_1, m_2 \geq 1$ be integers with $n \geq m_1 + m_2 \geq u + 1 \geq 2$. Set $m := m_1 + m_2$, and let $\mathcal{C}_1 \leq \mathbb{F}^{n \times m_1}$ be an MRD code with minimum distance $u + 1$. Then $\mathcal{C}_1$ has dimension $n(m_1 - u)$. Moreover, $\mathcal{C}_1(U) = \{0\}$ for all $U \leq \mathbb{F}^{m_1}$ of dimension $u$. Construct the code

$$\mathcal{C} := \{ (A \mid B) \in \mathbb{F}^{n \times m} \mid A \in \mathcal{C}_1, B \in \mathbb{F}^{n \times m_2} \}.$$

Let $T := \mathbb{F}^{m_1} \times 0^{m_2} \leq \mathbb{F}^m$. Clearly, $\mathcal{C}(U) = \{0\}$ for all $U \leq T$ of dimension $u$. Moreover, $\dim(\mathcal{C}) = n(m - u)$. Thus $\mathcal{C}$ is $U$-extremal for all $U \leq T$ of dimension $u$. Note that $\mathcal{C}$ is not MRD, as its rank distance is 1 and $\dim(\mathcal{C}) = n(m - u) < nm$. 

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2. Consider the code from (1). By Theorem 4.10 we know that \( \mathcal{P}^{rs}(C, V) \) only depends \( \dim(V) \) for all \( V \leq T \). Note however that this is not the case in general for the spaces \( V \) that are not contained in \( T \). Let e.g. \( V_1 = \langle e_1, \ldots, e_u, e_m \rangle \) and \( V_2 = \langle e_m, e_{m-1}, \ldots, e_{m-u} \rangle \), where \( \{e_1, \ldots, e_m\} \) is the canonical basis of \( \mathbb{F}^m \). The spaces \( V_1 \) and \( V_2 \) have the same dimension, \( u+1 \), and neither of them is contained in \( T \). Suppose \( m_2 \geq u+1 \). Then \( \mathcal{P}^{rs}(C, V_1) = 0 \) and \( \mathcal{P}^{rs}(C, V_2) = \prod_{i=0}^{u}(q^n - q^i) \).

## 5 The Krawtchouk Coefficients of the Pivot Partition

This section is devoted to obtaining explicit expressions for the Krawtchouk coefficients of the pivot partition, introduced in Definition 2.3. They will be expressed in terms of the rank-distribution of matrices that are supported on a Ferrers diagram. We therefore start by introducing the needed notation and terminology.

**Definition 5.1.** An \( n \times m \) Ferrers diagram (or Ferrers board) \( \mathcal{F} \) is a subset of \( [n] \times [m] \) that satisfies the following properties:

1. if \( (i,j) \in \mathcal{F} \) and \( j < m \), then \( (i,j+1) \in \mathcal{F} \) (right aligned),

2. if \( (i,j) \in \mathcal{F} \) and \( i > 1 \), then \( (i-1,j) \in \mathcal{F} \) (top aligned).

For \( j = 1, \ldots, m \) let \( c_j = \left| \{(i,j) \mid (i,j) \in \mathcal{F}, 1 \leq i \leq n \} \right| \). Then we may identify the Ferrers diagram \( \mathcal{F} \) with the tuple \([c_1, \ldots, c_m]\). It satisfies \( 0 \leq c_1 \leq c_2 \leq \ldots \leq c_m \leq n \).

The Ferrers diagram \( \mathcal{F} = [c_1, \ldots, c_m] \) can be visualized as an array of top-aligned and right-aligned dots where the \( j \)-th column has \( c_j \) dots. Just like for matrices, we index the rows from top to bottom and the columns from left to right. For instance, \( \mathcal{F} = [1,2,4,4,5] \) is given by

\[
\begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \\
\bullet & \\
\end{array}
\]

We expressly allow \( c_1 = 0 \) or \( c_m < n \). This has the consequence that for all \( \tilde{n} \leq n \) and \( \tilde{m} \leq m \) an \( \tilde{n} \times \tilde{m} \) Ferrers diagram is also an \( n \times m \) Ferrers diagram. Moreover, the empty Ferrers diagram is given by \( \mathcal{F} = [0, \ldots, 0] \) of any length.

**Definition 5.2.** The \( (\text{Hamming}) \) support of a matrix \( M = (M_{ij}) \in \mathbb{F}^{n \times m} \) is defined as the index set \( \text{supp}(M) := \{(i,j) \mid M_{ij} \neq 0 \} \). The subspace of \( \mathbb{F}^{n \times m} \) of all matrices with support contained in \( \mathcal{F} \) is denoted by \( \mathbb{F}^\mathcal{F} \). For \( r \in \{0, \ldots, m\} \) we set \( P_r(\mathcal{F}) = \mathcal{P}_{rk}(\mathbb{F}^\mathcal{F}, r) \), that is,

\[
P_r(\mathcal{F}) = \left| \left\{ M \in \mathbb{F}^\mathcal{F} \mid \text{rk}(M) = r \right\} \right|.
\]

We call \( (P_r(\mathcal{F}))_{0 \leq r \leq m} \) the \textbf{rank-weight distribution} of \( \mathbb{F}^\mathcal{F} \). Clearly, \( P_0(\mathcal{F}) = 1 \) for any Ferrers diagram \( \mathcal{F} \), including the empty one.

The following result provides an explicit formula for the rank-weight distribution of the space \( \mathbb{F}^\mathcal{F} \) for any Ferrers diagram \( \mathcal{F} \). We postpone the proof to Section 7, where we will describe connections between the rank-weight distribution of \( \mathbb{F}^\mathcal{F} \) and \( q \)-rook polynomials.
Theorem 5.3. Let $\mathcal{F} = [c_1, \ldots, c_m]$ be an $n \times m$ Ferrers diagram, and let $r \geq 1$ be an integer. Define the set $\mathcal{I}_{rm} := \{(i_1, \ldots, i_r) \mid 1 \leq i_1 < \ldots < i_r \leq m\}$. Then

$$P_r(\mathcal{F}) = \sum_{(i_1, \ldots, i_r) \in \mathcal{I}_{rm}} q^{r_m - \sum_{j=1}^r i_j} \prod_{j=1}^r (q^{c_{i_j} - j + 1} - 1).$$

(5.1)

We also need the following technical result.

Lemma 5.4. Let $\sigma = (\sigma_1, \ldots, \sigma_a) \in \Pi$ and $B \in \mathbb{R}^{b \times m}$ be the matrix with columns

$$B_j = \begin{cases} e_{\alpha} & \text{if } j = \sigma_{\alpha}, \\ 0 & \text{else.} \end{cases}$$

Thus $B$ is in RREF with $\text{piv}(B) = \sigma$, and where all non-pivot columns are zero. Let $\lambda = (\lambda_1, \ldots, \lambda_\alpha) \in \Pi$ and set

$$\lambda \cap \sigma = (\lambda_{\alpha_1}, \ldots, \lambda_{\alpha_a})$$

and

$$\hat{\sigma} \setminus \lambda = (\hat{\sigma}_{\beta_1}, \ldots, \hat{\sigma}_{\beta_b}).$$

Furthermore, for $j \in [a]$ set $z_j = |\{i \mid \lambda_\alpha < \hat{\sigma}_\beta\}|$. Then for any $r \in \{0, \ldots, a\}$ we have

$$\{|A \in \mathbb{R}^{a \times m} \mid A \text{ is in RREF, } \text{piv}(A) = \lambda, \text{rk}(A) = b + r\} = P_{r-a+x}(\mathcal{F}),$$

where $\mathcal{F}$ is the $x \times y$ Ferrers diagram $\mathcal{F} = [z_1, \ldots, z_y]$ and $P_{r}(\mathcal{F})$ is the rank-weight distribution of $\mathbb{F}[\mathcal{F}]$ from Theorem 5.3.

From $\hat{\sigma}_{\beta_1} < \ldots < \hat{\sigma}_{\beta_b}$ we conclude $z_1 < \ldots < z_y \leq x$. Hence $\mathcal{F}$ is indeed an $x \times y$ Ferrers diagram. We may have $z_1 = 0$ and the Ferrers diagram could be shortened by removing empty columns. Precisely, let $t'$ be minimal such that $\hat{\sigma}_{\beta_t} > \lambda_\alpha$. Then $z_{t'} \neq 0 = z_{t'-1}$. Note also that for the given matrix $B$ and any matrix $A$ as specified above we have $\text{rk}(A) \geq b + |\lambda \setminus \sigma| = b + (a - x)$. Hence only $r \geq a - x$ matters in above formula. Before giving the proof of Lemma 5.4, we illustrate the count by an example.

Example 5.5. Let $m = 7$ and $\sigma = (3, 4, 6)$, $\lambda = (1, 4, 6)$. Then $\hat{\sigma} = (1, 2, 5, 7)$ and

$$\lambda \cap \sigma = (4, 6) = (\lambda_2, \lambda_3)$$

and $\hat{\sigma} \setminus \lambda = (2, 5, 7)$.

Using $\ast$ for the unspecified entries of the matrix $A$ in RREF we observe

$$\text{rk} \begin{pmatrix} B \\ A \end{pmatrix} = \text{rk} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & * & 0 & * & 0 & * \\ 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \end{pmatrix} = \text{rk} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & * & 0 & * & 0 & * \\ 0 & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix},$$

where we applied row operations to clear the columns of $A$ using the pivot positions of $B$. Clearing the rows of $A$ that still contain pivots shows that

$$\text{rk} \begin{pmatrix} B \\ A \end{pmatrix} = 3 + 1 + \text{rk} \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \end{pmatrix}.$$
Proof of Lemma 5.4. For any matrix $M$ denote by $M_{(i_1, \ldots, i_j)}^{(j_1, \ldots, j_k)}$ the $x \times y$-submatrix of $M$ consisting of the rows indexed by $i_1, \ldots, i_j$ and the columns indexed by $j_1, \ldots, j_k$. Following the idea of the example, we can clear in the matrix $(\beta \Big| \alpha)$ the columns of $A$ in the pivot positions of $B$ and observe that $\text{rk}(B) - b = \text{rk}(\sigma) - \text{rk}(\beta)$. Making use of the remaining pivots in $A$ to clear their respective rows, we see that the rank of $\sigma$ equals $|\lambda \setminus \sigma| + \text{rk}(M)$, where

$$M = \sigma^{(x,\ldots,x)}.$$

Since $|\lambda \setminus \sigma| = a - x$, we conclude that

$$\text{rk}(B) = b + r \iff \text{rk}(M) = r - a + x.$$

Now we obtain the desired result once we show that $M$ is in $\mathbb{F}[\mathcal{F}]$ for the stated Ferrers diagram $\mathcal{F}$. From the construction it is clear that the matrix $M$ is supported by a (top and right aligned) Ferrers diagram. Thus we just have to count the number of potentially nonzero entries in each column. The $j$th column of $M$ originates from the column of $A$ indexed by $\alpha$, which has the form

$$(*)^j, 0, \ldots, 0 \top$$

with a zero at position $i$ iff $\lambda_i > \alpha_i$. Hence the number of potentially nonzero entries in the $j$th column of $M$ is given by $z_j = |\{i \in [x] | \lambda_i < \alpha_i\}|$. All of this shows that $M \in \mathbb{F}[\mathcal{F}]$, and this concludes the proof. \hfill $\Box$

Now we are ready to present explicit expressions for the Krawtchouk coefficients of the pivot partition and its dual. From Corollary 2.8 we know that $\mathcal{P}^{\text{pivot}}$ and $\mathcal{P}^{\text{pivot}}_\mu$ are mutually dual, where $\mathcal{P}^{\text{pivot}}_\mu$ is the reverse-pivot partition. Denote the blocks of the partitions by $\mathcal{P}^{\text{pivot}}_\lambda$ and $\mathcal{P}^{\text{pivot}}_\mu$, respectively. Thus

$$\mathcal{P}^{\text{pivot}}_\lambda = \{A \in \mathbb{F}^{n \times m} | \text{pr}(A) = \lambda\} \quad \text{and} \quad \mathcal{P}^{\text{pivot}}_\mu = \{A \in \mathbb{F}^{n \times m} | \text{pr}(A) = \lambda\}.$$

Theorem 5.6. Let $\lambda, \mu \in \Pi$. Set $\sigma = \hat{\mu}$, $\lambda \setminus \sigma = (\lambda_1, \ldots, \lambda_m)$ and $\sigma \setminus \lambda = (\sigma_{\beta_1}, \ldots, \sigma_{\beta_n})$. Furthermore, for $j \in [y]$ set $z_j = |\{i \in [x] | \lambda_i < \sigma_{\beta_j}\}|$ and let $\mathcal{F}$ be the $x \times y$ Ferrers diagram $\mathcal{F} = [z_1, \ldots, z_y]$. Then

$$K(\mathcal{P}^{\text{pivot}}_\lambda; \lambda, \mu) = \sum_{i=0}^{m} (-1)^i \binom{\lambda - t}{\mu - t} \sum_{r=0}^{\text{rk}(\mathcal{F})} \left[ \binom{\lambda \cap \mu}{r} \right]^{-t}.$$

where $(\text{pr}(\mathcal{F}))_r$ is the rank-weight distribution of $\mathbb{F}[\mathcal{F}]$ given in Theorem 5.3.

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\mu = (\mu_1, \ldots, \mu_n)$, and $\hat{\mu} = (\sigma_1, \ldots, \sigma_n)$. By Definition 1.5

$$K(\mathcal{P}^{\text{pivot}}_\lambda; \lambda, \mu) = \sum_{A \in \mathcal{P}^{\text{pivot}}_\mu} \chi((A, C)),$$

where $C$ is any matrix in $\mathcal{P}^{\text{pivot}}_\mu$.

We may use for $C$ the reverse reduced row echelon form (see Definition 2.7) with reverse pivot indices $\mu$ and all unspecified entries equal to zero. Thus, using the standard basis vectors $e_i \in \mathbb{F}^n$ we may choose

$$C = (C_1, \ldots, C_m) \in \mathbb{F}^{n \times m} \quad \text{where} \quad C_j = \begin{cases} e_\alpha & \text{if } j = \mu (\alpha) \text{ for some } \alpha \in \{1, \ldots, c\}, \\ 0 & \text{else.} \end{cases}$$
Set \( V = \text{rs}(C) \). We will need \( V^\perp \), which is given by \( V^\perp = \text{rs}(B) \), where

\[
B = (B_1, \ldots, B_m) \in \mathbb{F}^{n \times m} \text{ where } B_j = \begin{cases} 
    e_\beta & \text{if } j = \sigma_\beta, \\
    0 & \text{else.}
\end{cases}
\]

Note that \( \dim V^\perp = b = m - c \). In the following computation we will be able to make use of the Krawtchouk coefficients for the row-space partition, which have been determined in Theorem 3.3. Using that any subspace \( U \) with \( \text{piv}(U) = \hat{\lambda} \) satisfies \( \dim(U) = |\hat{\lambda}| = a \) we compute

\[
K(\D, \hat{\lambda}, \mu) = \sum_{U \in \mathcal{L}} \sum_{A \in \mathbb{F}^{n \times m}_{\text{piv}(U) = \hat{\lambda} \text{ rs}(A) = U}} \chi(#\langle A, C \rangle) \\
= \sum_{U \in \mathcal{L}} K(\D, U, V) \\
= \sum_{U \in \mathcal{L}} \sum_{t = 0}^m (-1)^{a-t} q^{nt+\left(\frac{a-t}{2}\right)} \left[ \frac{\dim(U \cap V^\perp)}{t} \right] \\
= \sum_{t = 0}^m (-1)^{a-t} q^{nt+\left(\frac{a-t}{2}\right)} \sum_{U \in \mathcal{L}} \left[ \frac{\dim(rs(A) \cap V^\perp)}{t} \right].
\]

It remains to determine the inner sum. Since \( V^\perp = \text{rs}(B) \), we conclude that \( \dim(rs(A) \cap V^\perp) = a + b - \dim(rs(A) + V^\perp) = a + b - \text{rk}(A^\perp) \). As mentioned after Lemma 5.4, for any matrix \( A \) as specified we have \( \text{rk}(A^\perp) \in \{b + r \mid r = a - x, \ldots, a\} \), where \( x = |\hat{\lambda} \cap \hat{\mu}| \). Thus, thanks to the lemma the inner sum turns into

\[
\sum_{t = 0}^a \sum_{A \in \mathbb{F}^{n \times m}_{\text{piv}(A) = \hat{\lambda}, \text{rk}(A^\perp) = b + r}} \left[ \frac{a-r}{t} \right] = \sum_{t = 0}^a \text{P}_{r-a+x}(\mathcal{F}) \left[ \frac{a-r}{t} \right] = \sum_{r = 0}^x \text{P}_r(\mathcal{F}) \left[ \frac{x-r}{t} \right]
\]

with \( \mathcal{F} \) as in the theorem. This concludes the proof. \( \square \)

6 Pivot-Extremal Codes

In this section we generalize the notion of extremality to the pivot partition. To do so, we need to introduce a partial order on the set \( \Pi \) of all possible pivot lists for matrices in \( \mathbb{F}^{n \times m} \). This is done in the obvious way: for \( \lambda, \mu \in \Pi \) define \( \lambda \leq \mu \) if \( \lambda \leq \mu \), where for the latter we simply interpret pivot lists as sets. Then \( (\Pi, \leq) \) is a lattice, which of course is isomorphic to the subset lattice of \([m]\).

**Notation 6.1.** Throughout this section we assume without loss of generality that \( n \geq m \). This choice allows us to use the results from Section 4.

The following results from basic Linear Algebra will be crucial.
**Lemma 6.2.** 1. Let \( U, V \in \mathcal{L} \) such that \( U \leq V \). Then \( \text{piv}(U) \leq \text{piv}(V) \).

2. Let \( \lambda, \mu \in \Pi \) such that \( \mu \leq \lambda \) and let \( V \in \mathcal{L} \) be such that \( \text{piv}(V) = \lambda \). Then there exists \( U \in \mathcal{L} \) such that \( \text{piv}(U) = \mu \) and \( U \leq V \).

3. Let \( \lambda, \mu \in \Pi \) such that \( \mu \leq \lambda \) and let \( U \in \mathcal{L} \) be such that \( \text{piv}(U) = \mu \). Then there exists \( V \in \mathcal{L} \) such that \( \text{piv}(V) = \lambda \) and \( U \leq V \).

**Proof.** 1) We may write \( U = \text{rs}(A) \) and \( V = \text{rs}(M) \), where \( M = (\lambda) \). Then \( \text{piv}(U) \leq \text{piv}(V) \) follows from (2.3) applied to \( A \) and \( M \).

2) Let \( V = \text{rs}(A) \), where \( A \in \mathbb{F}^{n \times m} \) is in RREF. Hence \( \text{piv}(A) = \lambda \supseteq \mu \). Let \( B \) be the submatrix of \( A \) consisting of the rows of \( A \) that contain the pivots in \( \mu \). Then \( U : = \text{rs}(B) \leq V \) and \( \text{piv}(U) = \mu \).

3) Let \( A \in \mathbb{F}^{n \times m} \) be in RREF such that \( U = \text{rs}(A) \). Then \( \text{piv}(A) = \mu \). Let \( \lambda \setminus \mu = (\sigma_1, \ldots, \sigma_t) \). Consider the matrix \( M = (\lambda) \), where

\[
B = \begin{pmatrix}
e_{\sigma_1} \\
\vdots \\
e_{\sigma_t}
\end{pmatrix},
\]

where \( e_t \) denotes the standard basis row vectors in \( \mathbb{F}^m \). Then \( \text{piv}(M) = \mu \cup (\lambda \setminus \mu) = \lambda \) and \( V = \text{rs}(M) \) is the desired subspace. \( \square \)

We now define the analogue of \( \mathcal{C}(U) \) from Definition 3.1 for the pivot partition and reverse-pivot partition. The following properties are immediate with Lemma 6.2 and 4.4.

**Proposition 6.3.** Let \( \mathcal{C} \leq \mathbb{F}^{n \times m} \) be a code and \( \lambda \in \Pi \). Then

\[
\mathcal{C}(\lambda, \text{piv}) := \{ A \in \mathcal{C} \mid \text{piv}(A) \leq \lambda \} = \bigcup_{U \in \mathcal{L}} \mathcal{C}(U) \setminus \text{piv}(U) = \lambda
\]

In particular, \( \mathcal{C}(\lambda, \text{piv}) = \{0\} \) if and only if \( \mathcal{C}(U) = \{0\} \) for all \( U \in \mathcal{L} \) with \( \text{piv}(U) = \lambda \). Thus, if \( \mathcal{C}(\lambda, \text{piv}) = \{0\} \), then \( |\mathcal{C}| \leq \mathcal{q}^{n(m-|\lambda|)} \). Note that \( \mathcal{C}(\lambda, \text{piv}) \) is not a subspace in general. Likewise we define

\[
\mathcal{C}(\lambda, \text{rpiv}) := \{ A \in \mathcal{C} \mid \text{rpiv}(A) \leq \lambda \} = \bigcup_{U \in \mathcal{L}} \mathcal{C}(U) \setminus \text{rpiv}(U) = \lambda
\]

which has the analogous properties.

This gives naturally rise to the following notion of extremal codes.

**Definition 6.4.** Let \( \lambda \in \Pi \). A code \( \mathcal{C} \leq \mathbb{F}^{n \times m} \) is called \( (\lambda, \text{piv}) \)-extremal if \( \mathcal{C}(\lambda, \text{piv}) = \{0\} \) and \( |\mathcal{C}| = \mathcal{q}^{n(m-|\lambda|)} \). A code that is \( (\lambda, \text{piv}) \)-extremal for some \( \lambda \in \Pi \) is called \( \lambda \)-extremal. According definitions are in place for \( (\lambda, \text{rpiv}) \).

Therefore

\[
\mathcal{C} \text{ is } (\lambda, \text{piv}) \text{-extremal } \iff \mathcal{C} \text{ is } U \text{-extremal for all } U \in \mathcal{L} \text{ with } \text{piv}(U) = \lambda.
\]  \hfill (6.1)

**Remark 6.5.** Let \( \mathcal{C} \) be a nonzero code. Then

\( \mathcal{C} \) is MRD with minimum distance \( d \) \iff \( \mathcal{C} \) is \( (\lambda, \text{piv}) \)-extremal for all \( \lambda \) such that \( |\lambda| = d - 1 \).

The forward direction is immediate with Remark 4.6. For the backward direction note that \( |\mathcal{C}| = \mathcal{q}^{n(m-d+1)} \) by assumption, and the distance is clearly not smaller than \( d \).
Just like for the rank-weight and the subspace distribution, extremality is preserved under dualization. This is an immediate consequence of Propositions 4.8 and 2.10.

**Proposition 6.6.** Let \( \mathcal{C} \leq \mathbb{F}^{n \times m} \) and \( \lambda \in \Pi \). Then \( \mathcal{C} \) is \((\lambda, \text{piv})\)-extremal iff \( \mathcal{C}^\perp \) is \((\hat{\lambda}, \text{r piv})\)-extremal.

As for \( U \)-extremal codes, the partial pivot partition distribution of pivot-extremal codes satisfies some rigidity properties. Its values depend on the cardinality of the blocks \( P_{\mu}^{\text{piv}} \) of the pivot partition, which we will therefore compute first.

**Proposition 6.7.** Let \( \mu = (\mu_1, \ldots, \mu_r) \in \Pi \). Define

\[
c(\mu) = \sum_{i=1}^{r} (m - \mu_i - r + i).
\]

Then \( |P_{\mu}^{\text{piv}}| = q^{c(\mu)} \prod_{i=0}^{r-1} (q^n - q^i) \). Note also that \(|P_{\mu}^{\text{piv}}| = 1\) if \( \mu = (\cdot) \), the empty list.

**Proof.** Consider a matrix in reduced row echelon form with pivot list \( \mu \). The number of unspecified entries in the \( i \)-th row is given by \( m - \mu_i - (r - i) \). This shows that there exist \( c(\mu) \) matrices \( A \in \mathbb{F}^{n \times m} \) in RREF with \( \text{piv}(A) = \mu \). Denote the set of these matrices by \( R(\mu) \). Then \( |R(\mu)| = c(\mu) \).

The partition set \( P_{\mu}^{\text{piv}} \) is the set of all matrices in \( \mathbb{F}^{n \times m} \) with pivot list \( \mu \). It thus forms the disjoint union of the orbits of the matrices in \( R(\mu) \) under the group action

\[
\text{GL}_n(\mathbb{F}) \times \mathbb{F}^{n \times m} \rightarrow \mathbb{F}^{n \times m}, \quad (X, A) \mapsto XA.
\]

In order to determine the orbit size of any \( A \in R(\mu) \), we use the orbit-stabilizer theorem. For \( A \in R(\mu) \) we have \( A = (\hat{A} \ 0) \), where \( \hat{A} \in \mathbb{F}^{r \times m} \) has full row rank. This tells us that for any matrix

\[
X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \in \text{GL}_n(\mathbb{F}), \text{ where } X_1 \in \mathbb{F}^{r \times r},
\]

we have \( XA = A \) iff \( X_3 = 0 \) and \( X_1 = I_r \). Hence \( X_2, X_4 \) are free and thus the stabilizer has cardinality \( q^{(n-r)}|\text{GL}_{n-r}(\mathbb{F})| \). Now we arrive at

\[
|P_{\mu}^{\text{piv}}| = q^{c(\mu)} \frac{|\text{GL}_n(\mathbb{F})|}{q^{(n-r)}|\text{GL}_{n-r}(\mathbb{F})|} = q^{c(\mu)} \frac{\prod_{i=0}^{r-1} (q^n - q^i)}{q^{(n-r)} \prod_{i=0}^{n-r-1} (q^n - q^i)} = q^{c(\mu)} \prod_{i=0}^{r-1} (q^n - q^i),
\]

as desired. \(\Box\)

Now we can formulate the main result of this section.

**Theorem 6.8.** Let \( \hat{\lambda} \in \Pi \) and let \( 0 \leq u \leq |\hat{\lambda}| \) be an integer. Suppose that a code \( \mathcal{C} \) is \((\hat{\lambda}', \text{piv})\)-extremal for all \( \hat{\lambda}' \leq \hat{\lambda} \) with \( |\hat{\lambda}'| = u \). Then for all \( \mu \leq \hat{\lambda} \) we have

\[
\mathcal{P}^{\text{piv}}(\mathcal{C}, \mu) = q^{c(\mu)} \prod_{i=0}^{u-1} (q^n - q^i) \left( \sum_{i=0}^{u} \left\lfloor \frac{\mu}{i} \right\rfloor (-1)^{|\mu| - i} q^{\frac{|\mu| - i}{2}} \right) + \sum_{i=u+1}^{r} \left\lfloor \frac{\mu}{i} \right\rfloor q^{(n-u)(-1)^{|\mu| - i} q^{\frac{|\mu| - i}{2}}},
\]

where \( c(\mu) \) is defined as in Proposition 6.7.
Proof. Note first that by (6.1) the assumptions imply that $\mathcal{C}$ is $U$-extremal for all subspaces $U \in \mathcal{L}$ such that $\dim(U) = u$ and $\text{piv}(U) \leq \lambda$. Next, by definition, we have

$$\mathcal{P}^{\text{piv}}(\mathcal{C}, \mu) = \left| \left\{ A \in \mathcal{C} \mid \text{piv}(A) = \mu \right\} \right| = \left| \bigcup_{\text{piv}(T) = \mu} \{ A \in \mathcal{C} \mid \text{rs}(A) = T \} \right| = \sum_{\text{piv}(T) = \mu} \mathcal{P}_{\text{rs}}(\mathcal{C}, T).$$

Fix any $\mu$ such that $\mu \leq \lambda$. The case $\mu = ()$ is trivial. If $0 < |\mu| \leq u$, then the right hand side of the formula in the theorem is 0. This is indeed $\mathcal{P}^{\text{piv}}(\mathcal{C}, \mu)$ because, thanks to Lemma 6.2(3), any subspace $T$ with $\text{piv}(T) = \mu$ is contained in a subspace $U$ such that $\mu \leq \text{piv}(U) \leq \lambda$ and $\dim(U) = u$. Thus $\mathcal{C}(T) \leq \mathcal{C}(U)$ and $U$-extremality implies $\mathcal{P}_{\text{rs}}(\mathcal{C}, T) = 0$.

Let now $|\mu| > u$. Fix a subspace $T$ such that $\text{piv}(T) = \mu$. Let $U \leq T$ be an arbitrary subspace of dimension $u$ and let $\lambda' = \text{piv}(U)$. Clearly, $|\lambda'| = u$. Since $U \leq T$, Lemma 6.2(1) implies $\lambda' \leq \text{piv}(T) = \mu \leq \lambda$. Thus $\mathcal{C}$ is $U$-extremal.

All of this shows that $\mathcal{C}$ is $U$-extremal for any subspace $U \leq T$ of dimension $u$. In other words, $\mathcal{C}$ satisfies the assumptions of Theorem 4.10. Since this is the case for any subspace $T$ such that $\text{piv}(T) = \mu$ we conclude

$$\mathcal{P}^{\text{piv}}(\mathcal{C}, \mu) = \sum_{\text{piv}(T) = \mu} \left( \sum_{i=0}^{u} \left[ \frac{|\mu|}{i} \right] \left( -1 \right)^{|\mu|-i} q^{\left( \frac{|\mu|-i}{2} \right)} \right) = \sum_{i=0}^{u} \left[ \frac{|\mu|}{i} \right] \left( -1 \right)^{|\mu|-i} q^{\left( \frac{|\mu|-i}{2} \right)}.$$

Since the inner sum does not depend on the specific choice of $V$, we arrive at

$$\mathcal{P}^{\text{piv}}(\mathcal{C}, \mu) = \left\{ \sum_{i=0}^{u} \left[ \frac{|\mu|}{i} \right] \left( -1 \right)^{|\mu|-i} q^{\left( \frac{|\mu|-i}{2} \right)} \right\}.$$

and Proposition 6.7 concludes the proof. \qed

We conclude this section with an example of a code $\mathcal{C}$ that satisfies the assumptions of above theorem, but is not MRD.

Example 6.9. Let $m = m_1 + m_2$ with $m_1 \geq 1$ and $m_2 \geq 2$. Let $n \geq m$ and $\lambda = (m_1 + 1, \ldots, m_1 + m_2)$. Fix $1 \leq u \leq m_2 - 1$. Let $\mathcal{C}_2 \subseteq \mathbb{F}^{n \times m}$ be an MRD code of minimum distance $u + 1$. Construct the code $\mathcal{C} := \{ (A \mid B) \in \mathbb{F}^{n \times m} \mid A \in \mathbb{F}^{n \times m_1}, B \in \mathcal{C}_2 \}$.

Then $\mathcal{C}$ has minimum distance 1 and cardinality $|\mathcal{C}| = q^{(n-u)}$. In particular, $\mathcal{C}$ is not MRD. We claim that $\mathcal{C}$ is $(\lambda', \text{piv})$-extremal for all $\lambda' \leq \lambda$ with $|\lambda'| = u$. Fix any $\lambda' \leq \lambda$ with $|\lambda'| = u$, and let $U \subseteq \mathbb{F}^m$ be any space with $\text{piv}(U) = \lambda'$. There is only one space $V \subseteq \mathbb{F}^m$ with $\text{piv}(V) = \lambda$, namely, $V = \{ e_{m_1+1}, \ldots, e_m \}$, where $\{ e_1, \ldots, e_m \}$ is the canonical basis of $\mathbb{F}^m$. It is easy to see that $U \leq V$. Since $\mathcal{C}_2$ is MRD with minimum distance $u + 1$, we have $\mathcal{C}(U) = \{ 0 \}$. As $|\mathcal{C}| = q^{(n-u)}$, $\mathcal{C}$ is $(\lambda', \text{piv})$-extremal for all $\lambda' \leq \lambda$ with $|\lambda'| = u$, as claimed.

7 Matrices Supported on Ferrers Diagrams and $q$-Rook Polynomials

In this section we explicitly compute the rank-distribution of matrices supported on an arbitrary Ferrers diagram $\mathcal{F}$, establishing Theorem 5.3. In particular, we prove that $P_r(\mathcal{F})$ is a polynomial in $q$ for every value of $r$ and every diagram $\mathcal{F}$. We then exploit connections between the rank-distribution of matrices
supported on a Ferrers diagram and \( q \)-rook polynomials, giving explicit formulas for these and establishing the monotonicity in \( r \) of \( \text{deg}(P_r(\mathcal{F})) \). We follow the notation of Definitions 5.1 and 5.2.

Let \( c_1, \ldots, c_m \) be integers with \( c_{i+1} \geq c_i \) for all \( i \). For all \( r \in \mathbb{N} \) we let \( P_r(c_1, \ldots, c_m) := P_r(\mathcal{F}) \), where \( \mathcal{F} = [c_1, \ldots, c_m] \) is the Ferrers diagram whose columns lengths are \( c_1, \ldots, c_m \). Moreover, for all \( r \geq 1 \) we define the set 
\[
\mathcal{I}_{r,m} := \{(i_1, \ldots, i_r) \mid 1 \leq i_1 < \ldots < i_r \leq m\}.
\]

**Theorem 7.1.** Let \( c_1, \ldots, c_m \) be integers with \( c_{i+1} \geq c_i \) for all \( i \).

1. For all \( r \geq 1 \) we have the recursion 
\[
P_r(c_1, \ldots, c_m) = P_{r-1}(c_1, \ldots, c_{m-1})(q^{c_m} - q^{r-1}) + P_r(c_1, \ldots, c_{m-1})q^r
\]
with initial conditions

\[
P_0(c_1, \ldots, c_s) = 1 \quad \text{for all } s, \quad P_1(c_1) = q^{c_1} - 1, \quad P_r(c_1) = 0 \quad \text{for } r > 1.
\]

2. Let \( r \geq 1 \). Then \( P_r(c_1, \ldots, c_m) \) is given by the explicit formula

\[
P_r(c_1, \ldots, c_m) = \sum_{(i_1, \ldots, i_r) \in \mathcal{I}_{r,m}} q^{r(m-\sum_{j=1}^{r} i_j)} \prod_{j=1}^{r} (q^{c_{i_j}} - j+1 - 1). \tag{7.1}
\]

**Proof.** 1) The initial conditions are clear. For the recursion consider a matrix \( M \in \mathbb{F}[\mathcal{F}] \) of rank \( r \). Denote the submatrix of \( M \) consisting of the first \( m-1 \) columns by \( \tilde{M} \). If \( \tilde{M} \) has rank \( r-1 \), then the last column of \( M \) can be any choice outside the column span of \( \tilde{M} \). That results in \( q^{c_m} - q^{r-1} \) options. If \( \tilde{M} \) has rank \( r \), then the last column of \( M \) has to be in the column span of \( \tilde{M} \). Since \( c_i \leq c_m \) for all \( i \), this results in \( q^r \) options. This proves the desired recursion.

2) We induct on \( r \). Let \( r = 1 \). In this case, (7.1) reads as 

\[
P_1(c_1, \ldots, c_m) = \sum_{i=1}^{m} q^{m-1-i}(q^{c_i} - 1).
\]

We induct on \( m \) to prove this identity. Clearly \( P_1(c_1) = q^{c_1} - 1 \), as desired, because this is simply the number of nonzero vectors of length \( c_1 \). Now the recursion in (1) along with the induction hypothesis yields

\[
P_1(c_1, \ldots, c_m) = (q^{c_m} - 1) + \sum_{i=1}^{m-1} q^{m-1-i}(q^{c_i} - 1)q = \sum_{i=1}^{m} q^{m-i}(q^{c_i} - 1),
\]
as desired. We assume now (7.1) for all ranks at most \( r-1 \) and all \( m \), and want to show the identity for rank \( r \). For \( m < r \) we clearly have \( P_r(c_1, \ldots, c_m) = 0 \), and also the right hand side of (7.1) is zero because \( \mathcal{I}_{r,m} \) is empty. Thus we may again induct on \( m \). Denote the right hand side of (7.1) by \( Q \). We show that \( Q - P_{r-1}(c_1, \ldots, c_{m-1})(q^{c_m} - q^{r-1}) = P_r(c_1, \ldots, c_{m-1})q^r \), which then by the recursion establishes

\[
Q = P_r(c_1, \ldots, c_m).
\]

We compute \( Q - P_{r-1}(c_1, \ldots, c_{m-1})(q^{c_m} - q^{r-1}) = \)

\[
= \sum_{i \in \mathcal{I}_{r,m}} q^{r(m-\Sigma_{j=1}^{r} i_j)} \prod_{j=1}^{r} (q^{c_{i_j}} - j+1 - 1) - \sum_{i \in \mathcal{I}_{r-1,m-1}} q^{(r-1)(m-1)-\Sigma_{j=1}^{r-1} i_j} \prod_{j=1}^{r-1} (q^{c_{i_j}} - j+1 - 1) - (q^{c_{i_r}} - r+1 - 1)q^r
\]

\[
= \sum_{i \in \mathcal{I}_{r-1,m-1}} \prod_{j=1}^{r-1} (q^{c_{i_j}} - j+1 - 1) \sum_{i_r = i_r+1}^{m} q^{r(m-\Sigma_{j=1}^{r} i_j)} (q^{c_{i_r}} - r+1 - 1) - \sum_{i \in \mathcal{I}_{r-1,m-1}} \prod_{j=1}^{r-1} (q^{c_{i_j}} - j+1 - 1) (q^{c_{i_r}} - r+1 - 1)
\]

\[
= \sum_{i \in \mathcal{I}_{r-1,m-1}} \prod_{j=1}^{r-1} (q^{c_{i_j}} - j+1 - 1) \sum_{i_r = i_r+1}^{m} q^{r(m-\Sigma_{j=1}^{r} i_j)} (q^{c_{i_r}} - r+1 - 1) - \sum_{i \in \mathcal{I}_{r-1,m-1}} \prod_{j=1}^{r-1} (q^{c_{i_j}} - j+1 - 1) (q^{c_{i_r}} - r+1 - 1)
\]
\[ \sum_{i \in \mathcal{I}_{rm-1}} \prod_{j=1}^{r-1} (q^{i_j-j+1} - 1) \left[ \sum_{i_{l+1}}^{m-1} q^{m-\Sigma_{j=1}^r (q^{i_j-j+1} - 1)} \right] = \sum_{i \in \mathcal{I}_{rm-1}} q^{m-\Sigma_{j=1}^r i_j} \prod_{j=1}^{r} (q^{i_j-j+1} - 1) = P_r(c_1, \ldots, c_{m-1})q^r. \]

This establishes (7.1) and concludes the proof. □

**Corollary 7.2.** Let \( \mathcal{F} = [c_1, \ldots, c_m] \) be a Ferrers diagram, and let \( r \geq 0 \). Then \( P_r(\mathcal{F}) \) is a polynomial in \( q \).

For \( r \geq 1 \) define
\[ \mathcal{I}_{rm}(\mathcal{F}) := \{ i \in \mathcal{I}_{rm} | c_{ij} \neq j-1 \text{ for all } 1 \leq j \leq r \}. \]

Then
\[ \deg(P_r(\mathcal{F})) = \begin{cases} \infty & \text{if } \mathcal{I}_{rm}(\mathcal{F}) = \emptyset, \\ r m - \binom{r}{2} + \max \{ \Sigma_{j=1}^r (c_{ij} - j) | i \in \mathcal{I}_{rm}(\mathcal{F}) \} & \text{if } \mathcal{I}_{rm}(\mathcal{F}) \neq \emptyset. \end{cases} \]

**Proof.** We have \( P_r(1, \ldots, c_m) = \sum_{(i_1, \ldots, i_r) \in \mathcal{I}_{rm}(\mathcal{F})} q^{m-\Sigma_{j=1}^r (q^{i_j-j+1} - 1)}. \) For any \( i \in \mathcal{I}_{rm}(\mathcal{F}) \) the degree of the corresponding term is \( rm - \Sigma_{i=1}^r i_j + \Sigma_{i=1}^r (c_{ij} - j + 1) = rm - \binom{r}{2} + \Sigma_{i=1}^r (c_{ij} - i_j). \) □

**Remark 7.3.** We wish to point out that for certain matrix spaces \( \mathbb{F}[\mathcal{F}] \) where \( \mathcal{F} \subseteq [n] \times [m] \) is not a Ferrers diagram, the rank-weight functions \( P_r(\mathcal{F}) \) are not necessarily polynomials in \( q \). The smallest known case is for \( n = m = 7 \) and where \( \mathcal{F} \) is the support of the point-line incidence matrix of the Fano plane, see [24, Sec. 1] and [39, p. 381].

The formula in Theorem 7.1 takes a simpler form for some highly regular diagrams. This is the case, for example, for the upper-triangular board. The following is easily verified.

**Corollary 7.4.** Let \( \mathcal{F} = [1, 2, \ldots, m] \) be the \( m \times m \)-upper triangle. Then
\[ P_r(1, \ldots, m) = \sum_{i \in \mathcal{I}_{rm}} \prod_{j=1}^{r} (q^{m-i_j} - q^{m-i_j}) \text{ for all } r \geq 1. \]

**Remark 7.5.** Let \( \mathcal{F} = [n, \ldots, n] \) be the \( n \times m \) rectangle. For all \( r \geq 1 \) we have
\[ P_r(n, \ldots, n) = \sum_{i \in \mathcal{I}_{rm}} q^{m-\Sigma_{j=1}^r i_j} \prod_{j=1}^{r} (q^{m-i_j} - 1) = q^{-\binom{r}{2}} \sum_{i \in \mathcal{I}_{rm}} q^{-\Sigma_{j=1}^r i_j} \prod_{j=0}^{r-1} (q^{n} - q^{i_j}). \]

Comparing coefficients in the \( q \)-binomial identity \( \sum_{r=0}^{m} q^{inom{r}{2}} \binom{m}{r} t^r \prod_{j=0}^{r-1} (1 + q^{j+1}) \), one easily verifies that the last expression above simplifies to \( \sum_{1 \leq t_1 < \cdots < t_r \leq m-1} q^{-\Sigma_{j=1}^r t_j} \prod_{j=0}^{r-1} (q^{n} - q^{t_j}). \), which is indeed known as the number of matrices in \( \mathbb{F}^{n \times m} \) of rank \( r \).

Following work by Solomon [37], Haglund in [19, Section 2] establishes an interesting connection between \( P_r(\mathcal{F}) \) and the \( q \)-rook polynomial \( R_r(\mathcal{F}) \in \mathbb{Z}[q] \) for an arbitrary Ferrers board \( \mathcal{F} = [c_1, \ldots, c_m] \).

The latter has been introduced by Garsia and Remmel [15] and is defined as follows.
**Definition 7.6.** The $q$-rook polynomial associated with $\mathcal{F}$ and $r \geq 0$ is

$$R_r(\mathcal{F}) = \sum_{C \in \text{NAR}_r(\mathcal{F})} q^{\text{inv}(C,\mathcal{F})} \in \mathbb{Z}[q],$$

where $\text{NAR}_r(\mathcal{F})$ is the set of all placements of $r$ non-attacking rooks on $\mathcal{F}$ (non-attacking means that no two rooks are in the same column, and no two are in the same row), and $\text{inv}(C,\mathcal{F}) \in \mathbb{N}$ is computed as follows. For a placement $C$, cross out all dots which either contain a rook, or are above or to the right of any rook. The number of dots of $\mathcal{F}$ not crossed out is $\text{inv}(C,\mathcal{F})$.

For instance, placing on $\mathcal{F} = [1,2,4,4,5]$ the following three rooks (R) results in $\text{inv}(C,\mathcal{F}) = 7$.

- • × R ×
- × • R
- × • •
- R × ×
- •

Note that $|\mathcal{F}|$ is the number of dots in $\mathcal{F}$. Thus, if $\mathcal{F} = [c_1, \ldots, c_m]$, then $|\mathcal{F}| = \sum_{j=1}^m c_j$. By definition, $R_0(\mathcal{F}) = q^{|\mathcal{F}|}$ for any Ferrers diagram $\mathcal{F}$, including the empty diagram.

The connection between $q$-rook polynomials and the distribution of matrices supported on $\mathcal{F}$ lies in the following elegant formula.

**Theorem 7.7** ([19, Thm. 1]). For any Ferrers diagram $\mathcal{F}$ and any $r \geq 0$ we have

$$P_r(\mathcal{F}) = (q-1)^r q^{|\mathcal{F}|-r} R_r(\mathcal{F})_{q^{-1}}$$

in the ring $\mathbb{Z}[q,q^{-1}]$.

Combining Theorems 7.1 and 7.7 we obtain an explicit formula for the $q$-rook polynomials. Examples of $R_r(\mathcal{F})$ for some Ferrers diagrams are listed in [15, pp. 273].

**Corollary 7.8.** For any Ferrers diagram $\mathcal{F} = [c_1, \ldots, c_m]$ and for any $r \geq 1$ we have

$$R_r(\mathcal{F}) = \frac{q^{\sum_{j=1}^m c_j} - q^{r m}}{\sum_{\mathcal{F} \subseteq \mathcal{J}} \prod_{j=1}^m (q^{c_j} - q^j)} \frac{\prod_{j=1}^m (q^{c_j} - q^j)}{(1-q)^r}.$$

**Remark 7.9.** Corollary 7.8 can be used to derive an explicit formula for the $q$-Stirling number of the second kind. The latter are defined via the recursion

$$S_{m+1,r} = q^r S_{m,r-1} + \frac{q^r - 1}{q-1} S_{m,r}$$

with initial conditions $S_{0,0}(q) = 1$ and $S_{m,r}(q) = 0$ for $r < 0$ or $r > m$.\(^1\) It is known [15, p. 248] that for all $m$ and $r$ we have

$$S_{m+1,m+1-r} = R_r(\mathcal{F}),$$

where $\mathcal{F} = [1,\ldots,m]$ is the upper-triangular $m \times m$ Ferrers board. Therefore applying Corollary 7.8 we obtain

$$S_{m+1,m+1-r} = \frac{q^{\left(m+1\right)} - r m}{\sum_{\mathcal{F} \subseteq \mathcal{J}} \prod_{j=1}^m (q^{c_j} - q^j)} \frac{\prod_{j=1}^m (q^{c_j} - q^j)}{(1-q)^r} \text{ for } 1 \leq r \leq m+1.$$

\(^1\)In the combinatorics literature $q$-Stirling number of the second kind are often defined via the recursion $\hat{S}_{m+1,r}(q) = \hat{S}_{m,r-1}(q) + (q^r - 1)/(q-1)\hat{S}_{m,r}(q)$. It is easily seen that $S_{m,r}(q) = q^r \hat{S}_{m,r}(q)$.
As a second application of Theorem 7.1, we recover the recursion shown in [15] for the $q$-rook polynomials $R_r(\mathcal{F})$.

**Corollary 7.10** (see also [15, Thm. 1.1]). Let $\mathcal{F} = [c_1, \ldots, c_m]$ be a Ferrers diagram, and let $\mathcal{F}' = [c_1, \ldots, c_{m-1}]$. For all $r \geq 1$ we have

$$R_r(\mathcal{F}) = R_r(\mathcal{F}') q^{c_m - r} + R_{r-1}(\mathcal{F}') \frac{q^{c_m - r + 1} - 1}{q - 1}.$$  

**Proof.** By Theorem 7.7 we have

$$R_r(\mathcal{F})|_{q^{-1}} = P_r(\mathcal{F}) q^{-|\mathcal{F}|} (q - 1)^{-r}. $$

Using the recursion for $P_r(\mathcal{F})$ established in Theorem 7.1 we obtain

$$R_r(\mathcal{F})|_{q^{-1}} = \left( P_{r-1}(\mathcal{F}') (q^m - q^{-1}) + P_r(\mathcal{F}') q^{c_m - r} \right) q^{-|\mathcal{F}|} (q - 1)^{-r}. $$

Using that $|\mathcal{F}'| = \sum_{j=1}^{m-1} c_j$ and $|\mathcal{F}| = |\mathcal{F}'| + c_m$ and applying Theorem 7.7 twice we arrive at

$$R_r(\mathcal{F})|_{q^{-1}} = (q - 1)^{-1} q^{-c_m + 1} R_{r-1}(\mathcal{F}')|_{q^{-1}} (q^m - q^{-1}) + q^{-c_m} R_r(\mathcal{F}')|_{q^{-1}} q^|. $$

Applying the transformation $q \mapsto q^{-1}$ yields the desired result. \hfill \Box

We conclude this section by studying the degree (in $q$) of the polynomials $P_r(\mathcal{F})$. We will show that, for any given diagram $\mathcal{F}$, the function $r \mapsto \deg(P_r(\mathcal{F}))$ is strictly increasing as long as $P_r(\mathcal{F}) \neq 0$. This fact does not seem obvious from the explicit expression for $\deg(P_r(\mathcal{F}))$ given in Corollary 7.2. We will therefore take a different approach based on rook placements. This will also give us the chance to establish new connections between $P_r(\mathcal{F})$ and $R_r(\mathcal{F})$.

Recall that the **trailing degree** of a Laurent polynomial

$$P = \sum_i a_i q^i \in \mathbb{Z}[q, q^{-1}]$$

is defined as $\text{tdeg}(P) = \min\{i \mid a_i \neq 0\}$. The trailing degree of the zero polynomial is $+\infty$ by definition. Moreover, for any (possibly zero) Laurent polynomial $P \in \mathbb{Z}[q, q^{-1}]$ one has

$$\deg \left( P|_{q^{-1}} \right) = -\text{tdeg}(P). \tag{7.2}$$

We can relate the degree of $P_r(\mathcal{F})$ and the trailing degree of $R_r(\mathcal{F})$ as follows.

**Proposition 7.11.** Let $\mathcal{F}$ be a Ferrers diagram, and let $r \geq 0$. We have

$$\deg(P_r(\mathcal{F})) = |\mathcal{F}| - \text{tdeg}(R_r(\mathcal{F})).$$

In particular, $P_r(\mathcal{F})$ is the zero polynomial if and only if $R_r(\mathcal{F})$ is the zero polynomial.

**Proof.** By Theorem 7.7 we have the identity

$$q^r P_r(\mathcal{F}) = (q - 1)^r q^{|\mathcal{F}|} R_r(\mathcal{F})|_{q^{-1}}$$

in $\mathbb{Z}[q, q^{-1}]$. Taking degrees and using (7.2) we obtain $r + \deg(P_r(\mathcal{F})) = r + |\mathcal{F}| - \text{tdeg}(R_r(\mathcal{F})).$ \hfill \Box

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We can finally show that the function \( r \mapsto \deg(P_r(\mathcal{F})) \) is strictly increasing on \([0, \mathfrak{t}]\), where \( \mathfrak{t} \) is the maximum \( r \) with \( P_r(\mathcal{F}) \neq 0 \). The proof relies on Proposition 7.11 and on the following preliminary result.

**Lemma 7.12.** Let \( \mathcal{F} \) be a Ferrers diagram, and let \( r \geq 1 \). If \( \deg(R_r(\mathcal{F})) = 0 \), then \( R_{r+1}(\mathcal{F}) = 0 \).

**Proof.** We proceed by induction on \( r \). If \( r = 1 \) and \( \deg(R_1(\mathcal{F})) = 0 \), then \( \mathcal{F} \) consists of either a single column or a single row. Therefore \( R_2(\mathcal{F}) = 0 \). Now assume \( r \geq 2 \) and the statement is true for all \( 1 \leq r' < r - 1 \). Suppose that \( \deg(R_r(\mathcal{F})) = 0 \), and denote by \( \mathcal{F}' \) the Ferrers diagram obtained from \( \mathcal{F} \) by deleting the last column. We distinguish two cases.

**Case 1:** \( \deg(R_{r-1}(\mathcal{F}')) = 0 \). By induction hypothesis we have \( R_r(\mathcal{F}') = 0 \), and so it must be that \( R_{r+1}(\mathcal{F}) = 0 \) as well.

**Case 2:** \( \deg(R_{r-1}(\mathcal{F}')) > 0 \). By assumption there exists a placement \( C \) of \( r \) rooks on \( \mathcal{F} \) such that \( \text{inv}(C, \mathcal{F}) = 0 \). Then all the rooks of \( C \) must lie on \( \mathcal{F}' \) (as otherwise we would have \( \text{inv}(C', \mathcal{F}) = 0 \), where \( C' \) is obtained from \( C \) by removing the rook lying on \( \mathcal{F} \setminus \mathcal{F}' \), and this contradicts \( \deg(R_{r-1}(\mathcal{F}')) > 0 \)). Since \( \text{inv}(C, \mathcal{F}) = 0 \), every dot in the last column of \( \mathcal{F} \) is to the right of one of the \( r \) rooks. But this means that \( \mathcal{F} \) has exactly \( r \) non-empty rows. This in turn implies, \( R_{r+1}(\mathcal{F}) = 0 \), as desired. \( \square \)

**Theorem 7.13.** Let \( \mathcal{F} \) be a Ferrers diagram, and let \( r \geq 2 \). If \( P_r(\mathcal{F}) \) is not the zero polynomial, then

\[
\deg(P_r(\mathcal{F})) > \deg(P_{r-1}(\mathcal{F})).
\]

**Proof.** By Proposition 7.11, it suffices to show that \( \deg(R_{r-1}(\mathcal{F})) > \deg(R_r(\mathcal{F})) \). The result is immediate if \( R_{r-1}(\mathcal{F}) = 0 \). We henceforth assume that both \( R_r(\mathcal{F}) \) and \( R_{r-1}(\mathcal{F}) \) are non-zero. Let \( t = \deg(R_{r-1}(\mathcal{F})) \). By Lemma 7.12, if \( t = 0 \) then \( R_r(\mathcal{F}) \) must be the zero polynomial, contradicting our assumptions. Therefore we have \( t \geq 1 \). Let \( C \) be a placement of \( r-1 \) non-attacking rooks on \( \mathcal{F} \) such that \( \text{inv}(C, \mathcal{F}) = t \). Since \( t \geq 1 \), we can extend \( C \) to a placement \( C' \) of \( r \) non-attacking rooks on \( \mathcal{F} \). Clearly, we have \( \text{inv}(C', \mathcal{F}) < \text{inv}(C, \mathcal{F}) = t \). By the definition of \( R_r(\mathcal{F}) \), this implies \( \deg(R_r(\mathcal{F})) < t = \deg(R_{r-1}(\mathcal{F})) \), and this concludes the proof. \( \square \)

## 8 Partition-Preserving Maps

In this section we pursue another question that arises naturally in the context of coding theory with emphasis on various weight functions. For any linear codes in a common ambient space, endowed with a distance function, one may ask as to what the distance-preserving linear maps between such codes are. For the classical case of codes in \( \mathbb{F}^n \) endowed with the Hamming metric this has been answered by MacWilliams [27] and is known as the MacWilliams Extension Theorem. It states that (1) the Hamming-weight preserving maps \( \mathbb{F}^n \rightarrow \mathbb{F}^n \) are given by monomial matrices (i.e., matrices that have exactly one nonzero entry in each row and column), and (2) for any code \( \mathcal{C} \leq \mathbb{F}^n \) each Hamming-weight-preserving map \( \mathcal{C} \rightarrow \mathbb{F}^n \) extends to a Hamming-weight-preserving map on \( \mathbb{F}^m \). In short, the Hamming-isometries between codes in \( \mathbb{F}^n \) are monomial maps. This fully describes these maps. We refer to [23, Thm. 7.9.4] for further details. In this section, we study the analogous question for the rank, row space, and pivot partition.

**Definition 8.1.** Let \( \mathcal{C} \leq \mathbb{F}^{n \times m} \) be a subspace and \( f : \mathcal{C} \rightarrow \mathbb{F}^{n \times m} \) be a linear map.

1. \( f \) is **rank-preserving** if \( \text{rk}(f(A)) = \text{rk}(A) \) for all \( A \in \mathcal{C} \).
2. \( f \) is **row-space-preserving** if \( \text{rs}(f(A)) = \text{rs}(A) \) for all \( A \in \mathcal{C} \).
3. \( f \) is **pivot-preserving** if \( \text{piv}(f(A)) = \text{piv}(A) \) for all \( A \in \mathcal{C} \).
Note that rank-preserving maps preserve the rank partition in the sense that \( A \) and \( f(A) \) are in the same block of \( \mathcal{B}^n \) for all \( A \in \mathcal{C} \). Similar reformulations are true for row-space-preserving or pivot-preserving maps. Thus we may call maps **partition-preserving** if they are of the corresponding type above.

The question arises how to describe such maps explicitly. The simplest case is when the code \( \mathcal{C} \) is the entire space. In this situation the question is almost entirely answered by the following result due to Marcus/Moyls [28].

**Theorem 8.2** ([28, Thm. 1]). Let \( f : \mathbb{F}^{n \times m} \rightarrow \mathbb{F}^{n \times m} \) be a rank-preserving map. Then there exist matrices \( U \in \text{GL}_n(\mathbb{F}) \) and \( V \in \text{GL}_m(\mathbb{F}) \) such that

\[
    f(A) = UAV \quad \text{for all} \quad A \in \mathbb{F}^{n \times m}
\]

or, only in the case \( n = m \),

\[
    f(A) = UA^\top V \quad \text{for all} \quad A \in \mathbb{F}^{m \times m}.
\]

Clearly, any map \( f \) of such a form is rank-preserving.

Let us briefly comment on this result for the case where \( n \neq m \). From the rank-preserving property it is clear that for every \( A \) in \( \mathbb{F}^{n \times m} \) there exist \( U_A \in \text{GL}_n(\mathbb{F}) \) and \( V_A \in \text{GL}_m(\mathbb{F}) \) such that \( f(A) = U_AAV_A \). The strength of the above theorem lies in the fact that these matrices are **global**, that is, they do not depend on \( A \).

Now we can easily describe the row-space-preserving and the pivot-preserving maps on \( \mathbb{F}^{n \times m} \).

**Corollary 8.3.** Let \( f : \mathbb{F}^{n \times m} \rightarrow \mathbb{F}^{n \times m} \) be a linear map.

1. \( f \) is row-space-preserving iff there exists \( U \in \text{GL}_n(\mathbb{F}) \) such that \( f(A) = UA \) for all \( A \in \mathbb{F}^{n \times m} \).

2. \( f \) is pivot-preserving iff there exists \( U \in \text{GL}_n(\mathbb{F}) \) and \( V \in \mathcal{B}_m(\mathbb{F}) \) such that \( f(A) = UAV \) for all \( A \in \mathbb{F}^{n \times m} \), where \( \mathcal{B}_m(\mathbb{F}) = \{ V \in \text{GL}_m(\mathbb{F}) \mid V \text{ is upper triangular} \} \).

**Proof.** It is clear that maps of the form described in (1), resp. (2) are row-space-preserving, resp. pivot-preserving (see also Proposition 2.4(3)). Let us now turn to the other implications.

1) Let \( f \) be row-space-preserving. Then \( f \) is also rank-preserving and we may apply Theorem 8.2.

**Case 1:** There exist \( U \in \text{GL}_n(\mathbb{F}) \) and \( V \in \mathcal{B}_m(\mathbb{F}) \) such that \( f(A) = UAV \) for all \( A \in \mathbb{F}^{n \times m} \). Assume \( V \neq \alpha I_m \) for any \( \alpha \in \mathbb{F}^* \). Then there exists \( x \in \mathbb{F}^m \) such that \( xV \not\in \text{span}\{x\} \). Let \( A \in \mathbb{F}^{n \times m} \) be such that

\[
    UA = \begin{pmatrix} x \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\]

Then \( rs(A) = rs(UA) = \text{span}\{x\} \neq rs(UAV) \), a contradiction. Thus \( V = \alpha I_m \) for some \( \alpha \in \mathbb{F}^* \) and \( f(A) = (\alpha U)A \) for all \( A \in \mathbb{F}^{n \times m} \), as desired.

**Case 2:** Let \( m = n > 1 \) and suppose \( U \in \text{GL}_m(\mathbb{F}) \) and \( V \in \mathcal{B}_m(\mathbb{F}) \) are such that \( f(A) = UA^\top V \) for all \( A \in \mathbb{F}^{m \times m} \). Write

\[
    V = \begin{pmatrix} V_1 \\ \vdots \\ V_m \end{pmatrix}. \quad (8.1)
\]
Consider the standard basis matrices $E_{ij} \in \mathbb{F}^{m \times m}$ which have entry 1 at position $(i, j)$ and are zero elsewhere. Then $\text{span}(e_j) = rs(E_{ij}) = rs(UE_{ij}V) = rs(E_{ji}V) = \text{span}(V_i)$ for all $i \in [m]$. This contradicts the invertibility of $V$. Hence this case does not occur.

2) Let $f$ be pivot-preserving. Then $f$ is also rank-preserving, and we may proceed as in (1).

Case 1: There exist $U \in \text{GL}_m(\mathbb{F})$ and $V \in \text{GL}_m(\mathbb{F})$ such that $f(A) = UAV$ for all $A \in \mathbb{F}^{n \times m}$. Suppose $V = (v_{ij})$ is not upper triangular. Then there exists a smallest $j \in [m]$ and $i > j$ such that $v_{ij} \neq 0$. With $V$ as in (8.1) we arrive at $(i) = \text{piv}(E_{ii}) = \text{piv}(UE_{i1}V) = \text{piv}(E_{i1}V) = \text{piv}(V_i) = (j)$, which is a contradiction. Thus $V$ is upper triangular, as desired.

Case 2: Let $m = n > 1$ and suppose $U \in \text{GL}_m(\mathbb{F})$ and $V \in \text{GL}_m(\mathbb{F})$ are such that $f(A) = U A^\top V$ for all $A \in \mathbb{F}^{m \times m}$. Fix some $j > 1$. With $V$ as in (8.1) we obtain $(j) = \text{piv}(E_{ij}) = \text{piv}(UE_{ij}V) = \text{piv}(V_i)$ for all $i \in [m]$. This means that the first $j - 1$ columns of $V$ are zero, a contradiction to the invertibility of $V$. Hence, again, this case cannot occur. \(\square\)

We now conclude with examples showing that for any of the partitions $\mathcal{P}_{\text{rk}}, \mathcal{P}_{\text{rs}}, \mathcal{P}_{\text{piv}}$ partition-preserving maps between codes in $\mathbb{F}^{n \times m}$ do not in general extend to such maps on the entire space $\mathbb{F}^{n \times m}$.

**Example 8.4.** Let $\mathbb{F} = \mathbb{F}_2$.

1. In [3, Ex. 2.9(a)] it is shown that for the code $\mathcal{C} = \{(A | 0) \in \mathbb{F}_2^{2 \times 3} \mid A \in \mathbb{F}_2^{2 \times 2}\}$ the rank-preserving map 
$$
  f : \mathcal{C} \longrightarrow \mathcal{C}, \quad (A | 0) \longmapsto (A^\top | 0)
$$
does not extend to a rank-preserving map on $\mathbb{F}_2^{2 \times 3}$.

2. In $\mathbb{F}_2^{3 \times 3}$ consider the subset $\mathcal{C} = \mathbb{F}[P] = \{0, I, P, \ldots, P^6\}$, where
$$
  P = \begin{pmatrix}
    0 & 0 & 1 \\
    1 & 0 & 1 \\
    0 & 1 & 0
  \end{pmatrix}.
$$
Then $P$ is the companion matrix of the primitive polynomial $x^3 + x + 1 \in \mathbb{F}[x]$ and thus $\mathcal{C}$ is actually the field $\mathbb{F}_8$. In particular, $A \in \text{GL}_3(\mathbb{F})$ for all $A \in \mathcal{C} \setminus \{0\}$. As a consequence, the map
$$
  f : \mathcal{C} \longrightarrow \mathbb{F}_2^{3 \times 3}, \quad A \longmapsto A^\top
$$
is trivially row-space-preserving and pivot-preserving. We show that $f$ does not extend to a pivot-preserving map on $\mathbb{F}_2^{3 \times 3}$. Assume to the contrary that it does extend. Then Corollary 8.3(2) tells us that there exist $U \in \text{GL}_3(\mathbb{F})$ and $V \in \mathcal{U}_3(\mathbb{F})$ such that $f(A) = UAV$ for all $A \in \mathbb{F}_2^{3 \times 3}$. Since $I \in \mathcal{C}$ we have $I = I^\top = f(I) = UIV$, and thus $U = V^{-1}$ is upper triangular. Now $P^T = f(P) = UPU^{-1}$ implies $UP = P^\top U$. One easily verifies that no matrix $U \in \mathcal{U}_3(\mathbb{F})$ satisfies this identity. Hence $f$ does not extend to a pivot-preserving map on $\mathbb{F}_2^{3 \times 3}$ and thus also not to a row-space-preserving map.

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