Statistical description of short pulses in long optical fibers:

Effects of nonlocality

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We present a statistical description of the propagation of short pulses in long optical fibers, taking into account the Kerr and nonlocal nonlinearities on an equal footing. We use the Wigner approach on the modified nonlinear Schrödinger equation to obtain a wave kinetic equation and a nonlinear dispersion relation. The latter exhibit that the optical pulse decoherence reduces the growth rate of the modulational instability, and thereby contribute to the nonlinear stability of the pulses in long optical fibers. It is also found that the interaction between spectral broadening and nonlocality tends to extend the instability region.

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The nonlinear propagation of pulses in optical fibers has attracted a great deal of interest since the early seventies\(^1,2\), and research into the application and theory of this field is still growing\(^3\). The evolution of coherent weakly nonlinear optical pulse envelopes is given by the cubic nonlinear Schrödinger equation (CNLSE)\(^1,3\) involving the Kerr nonlinearity. The CNLSE admits bright, dark and gray solitons, which are used for ultrahigh-speed optical communications without pulse broadening and spectral dilution\(^4\). Even so, it has become clear that the effects of noise in fibers and amplifiers could alter the pulse properties in significant ways. Thus, it is of crucial importance to obtain qualitative and quantitative estimates of the effects of different types of incoherent perturbations\(^5,6\) on the optical pulse propagation. Recently, the Wigner transform technique\(^7,8\) in nonlinear dispersive media has been used to analyze Landau-like damping\(^9,10\), partially coherent higher order dispersive effects\(^11\), the importance of the incoherence spectrum\(^12\), and the influence of incoherence on the modulational instability\(^13,14\) for cases involving the cubic Kerr nonlinearity.

However, there are other important nonlinearities\(^15,16,17\) (e.g. saturation and higher order nonlocal nonlinearities) which can compete with the cubic nonlinearity in optical fibers. The combined influence of the cubic and nonlocal nonlinearities on the modulational instability of a constant amplitude coherent optical pulse has been examined by Shukla and Rasmussen\(^17\). In this Letter, we present a statistical description of partially incoherent pulses in long optical fibers, taking into account the Kerr and nonlocal nonlinearities on an equal footing. We use the Wigner approach and deduce a wave kinetic equation from which a nonlinear dispersion relation (NDR) has been derived. The NDR is then analyzed to demonstrate the effect of random noise on the modulational instability of incoherent optical pulses. It is found that
the optical pulse decoherency can contribute to the nonlinear stability of pulses in optical fibers.

Given the electric field $E(z,t) \exp(ik_0z - i\omega_0 t)$ of the optical pulses, the evolution the pulse envelope $E$ in the slowly varying envelope limit, i.e. $k_0 \gg (\partial_z - 2k_0'\partial_t)$, is governed by

$$i(\partial_z + \Gamma)E + \alpha \partial_t^2 E + \beta I E + i\gamma \partial_t(IE) = 0,$$

where we have introduced the parameters $\alpha = -k''_0/2$, $\beta = n_2k_0/n_0$, $\gamma = 2n_2/c$, and $\Gamma = k_0\chi_0/n_0$. Moreover, the prime denotes differentiation with respect to $\omega_0$, the intensity parameter is given by $I = |E|^2$, the refractive index is $n(\omega_0, I) = n_0 + i\chi_0 + n_2 I$, $n_0 = n(\omega_0)$, and and $\chi_0 = \chi(\omega_0)$ represents losses in the medium.

In order to take the effects of partial coherence into account, we define the space-time correlation function for the electric field as

$$C(z_+, z_-, t_+, t_-) = E^*(z_+, t_+)E(z_-, t_-),$$

where $z_\pm = z \pm \zeta/2$ and $t_\pm = t \pm \tau/2$. Then, the Wigner distribution function of the optical pulse is given by

$$F(z, t, k, \omega) = \frac{1}{(2\pi)^2} \int d\zeta d\tau e^{i(k\zeta - \omega\tau)}C(z_+, z_-, t_+, t_-),$$

such that

$$I(z, t) = \frac{1}{(2\pi)^2} \int dk d\omega F(z, t, k, \omega).$$

Thus, from Eq. (1) the evolution equation for the Wigner function (2) corresponding to the envelope field $E$ becomes (see also Ref. [8])

$$2\omega\alpha \partial_t F - \partial_z F + 2\beta I \sin \left(\frac{1}{2} \partial_t \partial_\omega \right) F + \gamma \left\{ -\partial_t \left[ I \cos \left(\frac{1}{2} \partial_t \partial_\omega \right) F \right] + 2\omega I \sin \left(\frac{1}{2} \partial_t \partial_\omega \right) F \right\} = 2\Gamma F,$$

where we have performed the Wigner transformation over the time domain. Here the arrows
denotes direction of operation, and the operator functions are defined in terms of their respective Taylor expansion. The system of equations (3) and (4) determines the evolution of short partially coherent optical pulses in nonlinear media.

In order to analyse the modulational instability and the effects of the terms due to a nonzero $\gamma$, we make the ansatz $F(z, t, \omega) = F_0(\omega) + F_1(\omega) \exp(iKz - i\Omega t) + c.c.$, where $c.c.$ denotes the complex conjugate, and $|F_1| \ll F_0$. Moreover, since we are interested in the short-pulse effects, we will for simplicity neglect the loss term $\Gamma$ in Eq. (4), in order to obtain clearly interpretable results.\(^1\) Expanding Eq. (4) in terms of this ansatz, and using Eq. (3), we obtain

$$1 = \frac{1}{2\alpha \Omega} \int d\omega \left[ \frac{\beta + \gamma(\omega + \Omega/2)}{\omega + (K - \gamma \Omega I_0)/2\alpha \Omega} \right] F_0(\omega - \Omega/2) - \left[ \frac{\beta + \gamma(\omega - \Omega/2)}{\omega + (K - \gamma \Omega I_0)/2\alpha \Omega} \right] F_0(\omega + \Omega/2),$$

where $I_0 = \int d\omega F_0(\omega)$. Equation (5) represents the NDR for a short optical pulse, where the pulse may have spectral broadening and partial coherence.

In the case of a mono-energetic pulse, we have $F_0(\omega) = I_0 \delta(\omega - \Omega_0)$, where $\Omega_0$ corresponds to a frequency shift of the background plane wave solution, and the NDR (5) gives:\(^7\)

$$K = 2(\gamma I_0 - \alpha \Omega_0)\Omega \pm \left[ \gamma^2 I_0^2 \Omega^2 + \alpha^2 \Omega^4 - 2 \alpha I_0(\beta + \gamma \Omega_0)\Omega^2 \right]^{1/2}. \quad (6)$$

In practice however, the wave envelope will always suffer perturbations due to various noise sources, e.g. fiber and amplifier noise. A noisy environment may cause the pulse field to attain a random component in its phase. Thus, if the phase $\varphi(x)$ of the electric field varies stochastically, such that the ensemble average of the phase satisfies\(^{20,21}\) \(\exp[-i\varphi(t + 1) It should be stressed that in certain applications, the losses may not be small, and the $\Gamma$ term should under these circumstances be kept. As noted by Shukla and Rasmussen\(^7\) the effect of the loss term is to damp the pulse according to $\exp(-2\Gamma z)$ as it propagates through the fiber.
\[ \frac{\tau}{2} \] \exp[i\varphi(t - \tau/2)] = \exp(-\Omega_T|\tau|), \] the background Wigner distribution is given by the Lorentzian spectrum

\[ F_0(\omega) = \frac{I_0}{\pi} \frac{\Omega_T}{(\omega - \Omega_0)^2 + \Omega_T^2}, \] (7)

where \( \Omega_T \) corresponds to the width of the spectrum. Then, the NDR (5) takes the form

\[ 1 = I_0 \Omega^2 \gamma [K - \gamma I_0 \Omega + \alpha \Omega(\Omega_0 - i\Omega_T)] - 2\alpha \beta \Omega \]

\[ \frac{(K - \gamma I_0 \Omega + \Omega_0 - i\Omega_T)^2 - \alpha^2 \Omega^4}{(K - \gamma I_0 \Omega + \Omega_0 - i\Omega_T)^2 - \alpha^2 \Omega^4}, \] (8)

which has the solution

\[ K = 2 [\gamma I_0 - \alpha(\Omega_0 - i\Omega_T)] \Omega \pm [\gamma^2 I_0^2 \Omega^2 + \alpha^2 \Omega^4 - 2\alpha I_0(\beta + \gamma(\Omega_0 - i\Omega_T))\Omega^2]^{1/2}. \] (9)

This solution generalizes the result (6) to the case of a random phase background envelope field. Equation (9) clearly shows that the width gives a nontrivial contribution to the NDR.

We note that when \( \gamma = 0 \), we may define the growth rate \( \kappa \) according to \( K = -2\alpha \Omega_0 \Omega - i\kappa \), and the width \( \Omega_T \) then gives rise to a Landau like damping from Eq. (9).

When \( \gamma \) is non-zero, the growth/damping behavior becomes considerably more complex, with new instability regions. Letting \( f = \gamma^2 I_0^2 + \alpha^2 \Omega^2 - 2\alpha I_0(\beta + \gamma \Omega_0) \), and assuming \( \Omega_T \ll f/\alpha \gamma I_0 \), we obtain the approximate expression

\[ K/\Omega \approx 2\gamma I_0 - 2\alpha \Omega_0 \pm f^{1/2} + 2i\alpha \Omega_T \pm i\alpha \gamma I_0 \Omega_T/f^{1/2} \] (10)

from Eq. (9). When \( \alpha > 0 \), and \( 2\alpha I_0(\beta + \gamma \Omega_0) > \gamma^2 I_0^2 + \alpha^2 \Omega^2 \), we have \( f < 0 \). Denoting the growth rate by \( \kappa = -\text{Im}(K) \), we obtain \( \kappa = |f|^{1/2} - 2i\alpha \Omega_T \). Thus, as expected, the coherence spread \( \Omega_T \) gives rise to a smaller growth rate for the modulational instability of incoherent optical pulses. We note that this instability occurs also when \( \gamma = 0 \).

On the other hand, if \( \alpha > 0 \), but \( 2\alpha I_0(\beta + \gamma \Omega_0) < \gamma^2 I_0^2 + \alpha^2 \Omega^2 \), or \( \alpha < 0 \), so that \( f > 0 \), a new novel effect is present due to a nonzero \( \gamma \). We have \( \kappa = (\gamma I_0/f^{1/2} - 2) \alpha \Omega_T \). Thus,
a short pulse in conjunction with a finite statistical spread $\Omega_T$ could give rise to a shift in the damping due to the decoherency of the pulse, which hence implies a shift also in the growth rate. This effect can be seen in Fig. 1 where we have plotted $\kappa$ as given by the full dispersion relation (9) for the frequency shift $\Omega_0 = 0$. We have used the rescaling $I_0 \to \beta I_0$, $\Omega_T \to \sqrt{\alpha} \Omega_T$, $\Omega \to \sqrt{\alpha} \Omega$, and $\gamma \to \gamma/(\beta \sqrt{\alpha}) \equiv \sqrt{2} n_0/(ck_0|k_0''|^{1/2})$. We note that not only is the damping shifted, but the instability regions is also extended, and quite significantly for higher values of $\gamma/(\beta \sqrt{\alpha})$. Since $\gamma/(\beta \sqrt{\alpha}) \propto |D|^{-1/2}$, where $D$ is the dispersion parameter commonly used in fiber-optics, the value of the normalized non-locality strength may become large, as $D$ can be designed to be very close to zero for certain wavelengths. Thus, the novel coupling between spectral broadening and nonlocality should be possible to measure using a suitable setup.

To summarize, we have presented an investigation of the modulational instability of incoherent optical pulses in a nonlinear optical medium that contains the Kerr and higher order nonlocal nonlinearities on an equal footing. By using the Wigner transform, we have derived a wave kinetic equation for incoherent pulses from the generalized nonlinear Schrödinger equation. The wave kinetic equation is further exploited to obtain a nonlinear dispersion relation, which exhibits new features of the modulational instability. We find that the decoherence of the optical pulses reduce the modulational instability growth rate due to a spatial damping caused by the broad optical pulse spectrum. However, the combined effect of a random phase and a non-local nonlinearity is to extend the instability region as compared to the case of a monochromatic spectrum. Thus, the present result thus contribute to the nonlinear stability of incoherent optical pulses in long optical fibers.
References

1. A. Hasegawa and T. Tappert, Appl Phys. Lett. 23, 142 (1973).
2. A. Hasegawa and T. Tappert, Appl Phys. Lett. 23, 171 (1973).
3. Yu. Kivshar and G. P. Agrawal, Optical Solitons, From fibers to Photonic Crystals (Academic, San Diego, 2003).
4. A. Hasegawa, Phys. Plasmas 8, 1763 (2001).
5. C. C. Jeng, M. F. Shih, K. Motzek, and Y. S. Kivshar, Phys. Rev. Lett. 92, 043904 (2004).
6. T. S. Ku, M. F. Shih, A. A. Sukhorukov, and Y. S. Kivshar, Phys. Rev. Lett. 94, 063904 (2005).
7. E. Wigner, Phys. Rev. 40, 749 (1932).
8. J. T. Mendonça and N. L. Tsintsadze, Phys. Rev. E 62, 4276 (2000).
9. R. Fedele and D. Anderson, J. Opt. B: Quantum Semiclass. Opt. 2, 207 (2000).
10. B. Hall, M. Lisak, D. Anderson, R. Fedele, and V.E. Semenov, Phys. Rev. E 65, 035602(R) (2002).
11. L. Helczynski, M. Lisak and D. Anderson, Phys. Rev. E 67, 026602 (2003).
12. D. Anderson, L. Helczynski-Wolf, M. Lisak, and V. Semenov, Phys. Rev. E 69, 025601 (2004).
13. D. Anderson, L. Helczynski-Wolf, M. Lisak, and V. Semenov, Phys. Rev. E 70, 026603 (2004).
14. B. Hall, M. Lisak, D. Anderson, and V. E. Semenov, Phys. Lett. A 321, 255 (2004).
15. Y. Kodama and A. Hasegawa, IEEE J. Quantum Electron. 23, 510 (1987).
16. A. Hasegawa and Y. Kodama, *Solitons and Optical Communications* (Oxford University Press, Oxford, 1995).

17. P.K. Shukla and J. Juul Rasmussen, Opt. Lett. **11**, 171 (1986).

18. I.M. Besieris and F.D. Tappert, J. Math. Phys. **17**, 734 (1976).

19. J.T. Mendonça, *Theory of Photon Acceleration* (IOP Publishing, Bristol, 2001).

20. R. Loudon, *The Quantum Theory of Light* (Oxford University Press, Oxford, 2000).

21. D. Anderson, B. Hall, M. Lisak, and M. Marklund, Phys. Rev. E **65**, 046417 (2002).

22. G.P. Agrawal, *Nonlinear Fiber Optics* (Academic Press, San Diego, 2001).
Fig. 1. The effects of spectral broadening and non-locality. Normalizing the variables $K$ and $\Omega$, as well as the parameters $\Omega_T$ and $\gamma$, such that $\alpha = \beta = 1$, we have plotted the imaginary part $\kappa = -\text{Im}(K)$ as a function of $\Omega$, when the frequency shift $\Omega_0$ is put to zero. From the peaks of the curves downwards, we have used $I_0 = 0.5$, and the full curve represents $\Omega_T = \gamma = 0$, and shows the regular modulational instability growth rate. The next curve (dashed) gives $\kappa$ for $\Omega_T = 0.1$ and $\gamma = 0$, while the third (dashed-dotted) curve uses $\Omega_T = 0$ and $\gamma = 1$, and the fourth (dashed-dotted) curve has $\Omega_T = 0.1$ and $\gamma = 1$. The last two curves (dashed and dotted, respectively), where $\Omega_T = 0.1$, $\gamma = 1.9$, and $\Omega_T = 0$, $\gamma = 1.9$, respectively, clearly shows the character of the combined effect of broadening and non-locality, namely a widening instability region.