Study on Nonlinear Inverse Relations of Bell Polynomials via the Lagrange Inversion Formula

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Abstract

In this paper, by using the classical Lagrange inversion formula, we solve a general problem of finding nonlinear inverse relations proposed in the previous paper [J. Wang, Nonlinear inverse relations for Bell polynomials via the Lagrange inversion formula, J. Int. Seq., Vol. 22 (2019), Article 19.3.8.]. As applications of this inverse relation, we not only find a short proof of another nonlinear inverse relation due to Birmajer et al., but also set up a few convolution identities concerning the Mina polynomials.

Keywords: Formal power series; Bell polynomial; Mina polynomial; recurrence relation; Lagrange inversion formula; nonlinear inverse relation; \(t\)-coefficient method; convolution identity.

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1. Introduction

Throughout this paper, we shall adopt the same notation of Henrici [8]. For instance, we shall use \(\mathbb{C}[[x]]\) to denote the ring of formal power series (in short, fps) over the complex number field \(\mathbb{C}\).
and for any $f(x) = \sum_{n \geq 0} a_n x^n \in \mathbb{C}[[x]]$, the coefficient functional

$$[x^n]f(x) = a_n, n \geq 0.$$ 

For convenience, define

$$L_0 = \left\{ \sum_{n=0}^{\infty} a_n x^n | a_0 \neq 0 \right\}, \quad L_1 = \left\{ \sum_{n=0}^{\infty} a_n x^n | a_0 = 0, a_1 \neq 0 \right\}.$$ 

Moreover, for $f(x), g(x) \in \mathbb{C}[[x]]$, $g(x)$ is said to be the composite inverse of $f(x)$ if $f(g(x)) = g(f(x)) = x$. As conventions, we denote the composite inverse $g(x)$ of $f(x)$ by $f^{(-1)}(x)$.

**Lemma 1.1.** Given $f(x) \in \mathbb{C}[[x]]$, $f(x)$ has the composite inverse if and only if $f^{(-1)}(x) \in L_1$.

We also need the concept of the ordinary Bell polynomials (referenced by A263633 in the OEIS [14]).

**Definition 1.2.** For integers $n \geq k \geq 0$ and variables $(x_n)_{n \geq 1}$, the sums

$$\sum_{\sigma_k(n)} \frac{k!}{i_1! i_2! \cdots i_{n-k+1}!} x_1^{i_1} x_2^{i_2} \cdots x_{n-k+1}^{i_{n-k+1}}$$

(1.1)

are called the ordinary Bell polynomials in $x_1, x_2, \ldots, x_{n-k+1}$, where $\sigma_k(n)$ denotes the set of partitions of $n$ with $k$ parts, namely, all nonnegative integers $i_1, i_2, \ldots, i_{n-k+1}$ subject to

$$\left\{ \begin{array}{l}
i_1 + i_2 + \cdots + i_{n-k+1} = k \\
i_1 + 2i_2 + \cdots + (n-k+1)i_{n-k+1} = n.
\end{array} \right.$$

(1.2)

We write $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ for such the Bell polynomials as given by (1.1).

As of today, the Bell polynomials have played very important roles in analysis, combinatorics, and number theory. It should be pointed out here that the above Bell polynomials are in agreement with the exponential Bell polynomials [5, Definition, p. 133] with the specialization $x_n \to x_n/n!$ and multiplied by $n!/k!$. The exponential Bell polynomials are referenced by A111785 in the OEIS [14].

The ordinary generating function of the Bell polynomials $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ will be often used in our discussions.

**Lemma 1.3 ([16, Lemma 6]).** For any fps $f(x) = \sum_{n \geq 1} x_n x^n \in L_1$, it holds that

$$f^k(x) = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) x^n.$$  

(1.3)
Aside from the generating function of the Bell polynomials, it is necessary to study inverse relations lurking behind it. Hereafter, the word “inverse” means a pair of equivalent relations expressing \((x_n)_{n \geq 1}\) in terms of the Bell polynomials in variables \((y_n)_{n \geq 1}\) and vice versa. To the best of our knowledge, it is one of the most interesting problems first posed and solved by Riordan [13, Chaps. 2 and 3], and also investigated by Hsu et al. [4] and Mihoubi [12]. The reader may consult Riordan [13, Sect. 5.3] for further details and Mihoubi [12] for many of such inverse relations. It is especially noteworthy that in their paper [2], via the establishment of many interesting identities for the Bell polynomials, Birmajer et al. achieved the following somewhat unusual (essentially different from [4, 10, 13]) inverse relation.

**Theorem 1.4** ([2, Theorem 17]). Let \(B_{n,k}(x_1, x_2, \ldots, x_{n-k})\) denote the Bell polynomials as above. Then for any integers \(a, b, m\) with \(m \geq 1\), \(a^2 + b^2 \neq 0\), and any sequence \((x_m)_{m \geq 1}\), the system of nonlinear relations

\[
z_m(b) = \sum_{k=1}^{m} \frac{am + bk}{k(am + b)} \binom{-am - b}{k - 1} B_{m,k}(x_1, x_2, \ldots, x_{m-k+1})
\]

is equivalent to the system of nonlinear relations

\[
x_m = \sum_{k=1}^{m} \frac{1}{k} \binom{am + bk}{k - 1} B_{m,k}(z_1(b), z_2(b), \ldots, z_{m-k+1}(b)).
\]

In the above theorem and as follows, we use the notation \(\binom{x}{n}\) to denote the generalized binomial coefficients \((x)_n/n!\) and \((x)_n\) to the usual falling factorial \(x(x-1)\cdots(x-n+1)\).

Motivated by Birmajer et al.’s result, the first author found the following nonlinear inverse relation.

**Theorem 1.5** ([16, Theorem 5]). Let \(B_{n,k}(x_1, x_2, \ldots, x_{n-k})\) denote the Bell polynomials as above. For any integers \(m \geq 1\) and \(a, b \in \mathbb{C}\), and any sequence \((x_m)_{m \geq 1}\), the system of nonlinear relations

\[
y_m(b) = \frac{1}{am + b} \sum_{k=1}^{m} \left(-\frac{am - b}{k}\right) B_{m,k}(x_1, x_2, \ldots, x_{m-k+1})
\]

is equivalent to the system of nonlinear relations

\[
x_m = \frac{1}{am + 1} \sum_{k=1}^{m} \left(-\frac{(am + 1)b}{k}\right) B_{m,k}(y_1(b), y_2(b), \ldots, y_{m-k+1}(b)).
\]

Almost at the same time, Birmajer et al. [3] found another general and more beautiful nonlinear inverse relation for the Bell polynomials. We rephrase it as follows.
Theorem 1.6 ([3, Corollary 2.3]). Let \( c = r - q \neq 0, \ pqr \neq 0 \). Then
\[
x_n = \frac{1}{r-q} \sum_{k=1}^{n} \left( \frac{q}{q+np} \binom{r+np}{k} - \frac{r}{r+np} \binom{r+np}{k} \right) B_{n,k}(y_1, y_2, \ldots, y_{n-k+1})
\] (1.8)
if and only if
\[
y_n = - \sum_{k=1}^{n} \prod_{j=1}^{k-1} (np + kq + cj - 1) B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}).
\] (1.9)

This result inspires us to consider the following research problem.

Research problem 1.1. For any integers \( m \geq 1 \), let \( p, (a_k)_{k=1}^{m}, \) and \( (q_k)_{k=1}^{m} \) be \( 2m+1 \) complex numbers subject to
\[
p \neq 0, \ \sum_{k=1}^{m} a_k = 0, \ \sum_{k=1}^{m} a_k q_k \neq 0.
\]
Assume further that \( F(x) = \sum_{n \geq 1} x_n x^n \) and \( \phi(x) = 1 + \sum_{n \geq 1} y_n x^n \) satisfy
\[
F(x/\phi^p(x)) = \sum_{k=1}^{m} a_k \phi^{q_k}(x).
\] (1.10)

Find any relationship between the sequences \((x_n)_{n \geq 1}\) and \((y_n)_{n \geq 1}\).

In the sequel, the first author [16] offered a positive solution to this problem as follows but without any proof.

Theorem 1.7 ([16, Theorem 14]). With the same notation and assumptions as above. Then the system of nonlinear relations
\[
x_n = \sum_{k=1}^{n} \left( \sum_{l=1}^{m} \frac{a_l q_l}{np + q_l} \binom{np + q_l}{k} \right) B_{n,k}(y_1, y_2, \ldots, y_{n-k+1})
\] (1.11)
is equivalent to the system of nonlinear relations
\[
y_n = \sum_{k=1}^{n} \lambda_k \frac{(-1/p + n)}{1 - pm} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}),
\] (1.12)
where \( \lambda_n(s) \) is defined by
\[
(n+1) \lambda_{n+1}(s) \sum_{k=1}^{m} a_k q_k = -ps \lambda_n(s) - \sum_{k=1}^{m} a_k q_k \sum_{j=1}^{n-1} \lambda_{n+1-j}(-q_k/p) j \lambda_j(s).
\] (1.13)
The theme of the present paper is to show Theorem 1.7 in full details. Our main ingredient is the classical Lagrange inversion formula, restated as follows.

**Lemma 1.8 (The Lagrange inversion formula).** [5, p.150, Theorem C, Theorem D] Let $\phi(x) \in \mathcal{L}_0$. Then for any fps $F(x)$, it always holds that

$$F(x) = \sum_{n=0}^{\infty} a_n \left( \frac{x}{\phi(x)} \right)^n,$$

where

$$a_n = \frac{1}{n} [x^{n-1}] F'(x) \phi^n(x).$$

As further applications, we discuss the cases for $m = 2$ and $m = 3$. The former case reduces to Theorem 1.6 and the latter leads us to a new nonlinear inverse relation as below.

**Corollary 1.9.** With the same notation and assumptions as above, the system of nonlinear relations

$$x_n = \sum_{k=1}^{n} \left( \sum_{i=1}^{3} \frac{a_i q_i}{np + q_i} \binom{np + q_i}{k} \right) B_{n,k}(y_1, y_2, \ldots, y_{n-k+1})$$

is equivalent to the system of nonlinear relations

$$y_n = -\sum_{k=1}^{n} \frac{f_k(1-k/p)}{k!(c_1)^k} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}),$$

where we define

$$f_n(x) = \sum_{k=0}^{n-1} \frac{C_{n,k}(c_1, c_2, \ldots, c_{n-k})}{c_1^{n-1-k}} (px)^k$$

where $C_{n,k}(x_1, x_2, \ldots, x_{n-k})$ is the Mina polynomial (see Definition 3.6), and

$$c_k = a_1 q_1^k + a_2 q_2^k + a_3 q_3^k.$$

Our paper is planned as follows. The next section is devoted to the full proof for Theorem 1.7, wherein the coefficients $\lambda_n(s)$ is introduced and discussed in details. As further applications of Theorem 1.7 in the case $m = 2, 3$, the proofs of Theorem 1.6 and Corollary 1.9 are presented in Section 3. Further, some combinatorial identities for $\{\lambda_n(s)\}_{n \geq 0}$ are established.
2. Proofs of the main results

2.1. The proof of Theorem 1.7

Our proof of Theorem 1.7 is composed of the following lemma. At first, by using the Lagrange inversion formula (1.14), it is easy to express \( x_n \) in terms of \( \{y_n\}_{n \geq 1} \) directly.

**Lemma 2.1.** Under the assumptions of Research Problem 1.1. We have

\[
x_n = \sum_{k=1}^{n} \left( \sum_{i=1}^{m} \frac{q_i a_i}{np + q_i} \binom{np + q_i}{k} \right) B_{n,k}(y_1, y_2, \ldots, y_{n-k+1}).
\]  

(2.1)

**Proof.** It suffices to apply the Lagrange inversion formula (1.14) to (1.10), namely

\[
\sum_{n=1}^{\infty} x_n (x/\phi^p(x))^n = \sum_{k=1}^{m} a_k \phi^{q_k}(x).
\]

Thus we compute directly

\[
x_n = \frac{1}{n} \left[ x^{n-1} \right] \left( \phi^{np}(x) \left( \sum_{i=1}^{m} a_i \phi^{q_i}(x) \right)' \right)
\]

\[
= \frac{1}{n} \left[ x^{n-1} \right] \left( \sum_{i=1}^{m} q_i a_i \phi^{q_i-1}(x) \phi'(x) \phi^{np}(x) \right)
\]

\[
= \sum_{i=1}^{m} \frac{q_i a_i}{n} \left[ x^{n-1} \right] \phi^{np+q_i-1}(x) \phi'(x)
\]

\[
= \sum_{i=1}^{m} \frac{q_i a_i}{np + q_i} \left[ x^{n-1} \right] (\phi^{np+q_i}(x))'
\]

\[
= \sum_{i=1}^{m} \frac{q_i a_i}{np + q_i} \phi^{np+q_i}(x)
\]

\[
= \sum_{k=1}^{n} \left( \sum_{i=1}^{m} \frac{q_i a_i}{np + q_i} \binom{np + q_i}{k} \right) B_{n,k}(y_1, y_2, \ldots, y_{n-k+1}).
\]

As desired. \( \square \)

All remains to express \( y_n \) in terms of \( \{x_n\}_{n \geq 1} \). For that end, we have to establish a series of preliminaries. The first one is a general result about the composite inverse for any fps in \( L_1 \).

**Lemma 2.2.** Let \( g(t) = t/w(t) \) be the composite inverse of \( f(t) = t/\phi^p(t) \in L_1 \). Then it holds

\[
\phi(t/w(t)) = w^{-1/p}(t).
\]  

(2.2)
Proof. At first, from
\[ f(t) = t/\phi^p(t) \]
it follows
\[ \phi(t) = \left( \frac{t}{f(t)} \right)^{1/p}. \]  \hfill (2.3)
Replacing \( t \) with \( g(t) \) in (2.3) and noting \( f(g(t)) = g(f(t)) = t \), we further get
\[ \phi(g(t)) = \left( \frac{g(t)}{t} \right)^{1/p}. \]
It, after \( g(t) = t/w(t) \) inserted, turns out be (2.2).

By virtue of Lemma 2.2, we are able to express \( y_n \) in terms of \( \{x_n\}_{n \geq 1} \).

**Lemma 2.3.** Under the assumptions of Research Problem 1.1. We have
\[ y_n = \sum_{k=1}^{n} \frac{\lambda_k(-1/p + n)}{1 - pn} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}), \]  \hfill (2.4)
where \( \{\lambda_n(s)\}_{n \geq 0} \) is defined by
\[ w^s(t) = 1 + \sum_{n=1}^{\infty} \lambda_n(s)F^n(t) \quad (s \in \mathbb{C}). \]  \hfill (2.5)

Proof. Observe that (2.2) equals to
\[ 1 + \sum_{n=1}^{\infty} y_n(t/w(t))^n = w^{-1/p}(t). \]  \hfill (2.6)
In this form, by the Lagrange inversion formula (1.14), we obtain
\[ y_n = \frac{-1}{pn} \left[ t^{n-1} w^{-1/p-1}(t) \right] \frac{1}{w'(t)w^n(t)} \]
\[ = \frac{-1}{pn(-1/p + n)} \left[ t^{n-1} (w^{-1/p+n}(t))' \right] = \frac{1}{1 - pn} \left[ t^n \right] w^{-1/p+n}(t) \]
\[ = \sum_{k=1}^{n} \frac{\lambda_k(-1/p + n)}{1 - pn} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}). \]
The last identity is based on the definitions (1.3) and (2.5). The lemma is proved.  \[ \square \]
Actually, Lemma 2.1 and Lemma 2.3 together gives the complete proof of Theorem 1.7, except for a full characterization on $\lambda_n(s)$. It will be discussed below.

The following is a concrete case derived from Lemma 2.2.

**Example 2.4.** Let $\phi(t) = 1 - t$ and $p = 1$. Then from (2.2) we solve out $w(t) = 1 + t$. Then we get a combinatorial identity:

$$\sum_{k=1}^{n} \lambda_k(n-1)B_{n,k}(x_1, x_2, \cdots, x_{n-k+1}) = 0 \quad (n \geq 2),$$

(2.7)

where $\{x_n\}_{n \geq 1}$ and $\lambda_n(s)$ are given respectively by

$$x_n = (-1)^n \sum_{i=1}^{n} a_i \left( \frac{n - 1 + q_i}{n} \right),$$

(2.8)

$$(1 + t)^s = 1 + \sum_{n=1}^{\infty} \lambda_n(s) \left( \sum_{k \geq 1} x_k t^k \right)^n.$$

(2.9)

Furthermore, it follows from (2.9) a more general identity than (2.7):

$$\binom{s}{n} = \sum_{k=1}^{n} \lambda_k(s) B_{n,k}(x_1, x_2, \cdots, x_{n-k+1}).$$

(2.10)

This example shows that it seems difficult to find any closed-form expression for $\lambda_n(s)$, even for $p = 1, 2$. Thus, it is necessary to study possible recurrence relations of the sequence $\{\lambda_n(s)\}_{n \geq 0}$.

**2.2. Recurrence relations for $\lambda_n(s)$**

**Lemma 2.5.** Let $\{\lambda_n(s)\}_{n \geq 0}$ be given by (2.5) with $\lambda_0(s) = 1$. Then for any $a, b$, there must hold

$$\lambda_n(a + b) = \sum_{k=0}^{n} \lambda_k(a) \lambda_{n-k}(b),$$

(2.11)

$$n\lambda_n(a + b) = \frac{a + b}{a} \sum_{k=0}^{n} k \lambda_k(a) \lambda_{n-k}(b).$$

(2.12)

**Proof.** Evidently, (2.11) is a direct consequence of the basic relation

$$w^{a+b}(t) = w^a(t)w^b(t).$$

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We only need to show (2.12). For this end, we recall
\[ w^s(t) = \sum_{n=0}^{\infty} \lambda_n(s) F^n(t) \quad \text{and} \]
\[ sw^{s-1} w'(t) = \sum_{n=0}^{\infty} n\lambda_n(s) F^{n-1}(t) F'(t). \]
Therefore, for \( s = a, b, a+b \), we have
\[ w^b(t) = \sum_{j=0}^{\infty} \lambda_j(b) F^j(t); \]
\[ aw^{a-1} w'(t) = \sum_{i=0}^{\infty} i\lambda_i(a) F^{i-1}(t) F'(t); \]
\[ (a+b)w^{a+b-1} w'(t) = \sum_{n=0}^{\infty} n\lambda_n(a+b) F^{n-1}(t) F'(t). \]
It is obvious that
\[ (a+b)w^{a+b-1} w'(t) = \frac{a+b}{a} w^b(t) \times aw^{a-1}(t) w'(t), \]
we thereby get
\[ \sum_{n=0}^{\infty} n\lambda_n(a+b) F^n(t) = \frac{a+b}{a} \sum_{j=0}^{\infty} \lambda_j(b) F^j(t) \times \sum_{i=0}^{\infty} i\lambda_i(a) F^i(t). \tag{2.13} \]
By equating the coefficients of \( F^n(t) \) on the both sides of (2.13), we finally obtain
\[ n\lambda_n(a+b) = \frac{a+b}{a} \sum_{i+j=n} i\lambda_i(a)\lambda_j(b). \]
Thus (2.12) is confirmed. \( \blacksquare \)

**Lemma 2.6.** Let \( \{\lambda_n(s)\}_{n \geq 0} \) be given by (2.5) and all parameters \( \{a_k, q_k\}_{k=1}^{m} \) satisfy
\[ \sum_{k=1}^{m} a_k = 0, \quad c_1 = \sum_{k=1}^{m} a_k q_k \neq 0. \]
Then
\[ \lambda_n(s) = (n+1) \sum_{k=1}^{m} \frac{a_k q_k}{q_k - ps - \lambda_{n+1}(s - q_k/p)}. \tag{2.14} \]
In particular,
\[ (n+1)c_1\lambda_{n+1}(s) = -ps\lambda_n(s) - \sum_{k=1}^{m} a_k q_k \sum_{j=1}^{n-1} \lambda_{n+1-j}(-q_k/p) j \lambda_j(s). \tag{2.15} \]
Proof. Observe first that, since \( c_0 = 0 \) and \( c_1 \neq 0 \), \( \{ F^n(t) \}_{n=0}^{\infty} \) forms a base for the ring of formal power series \( \mathbb{C}[[t]] \). That means that \( \lambda_n(s) \) given by (2.5) is well-defined and unique. To show (2.14), we start with (1.10), namely

\[
F(t/\phi^p(t)) = \sum_{k=1}^{m} a_k \phi^{q_k}(t).
\]

According to Lemma 2.2, we see

\[
F(t) = \sum_{k=1}^{m} a_k \phi^{q_k}(t/w(t)) = \sum_{k=1}^{m} a_k w^{-q_k/p}(t).
\]

Evidently, differentiating with respect to \( t \) on both sides of (2.5) leads us to

\[
sw^{s-1}(t)w'(t) = \sum_{n=1}^{\infty} n \lambda_n(s) F^{n-1}(t) F'(t)
\]

\[
= \frac{1}{p} \sum_{n=1}^{\infty} n \lambda_n(s) F^{n-1}(t) \left( -\sum_{k=1}^{m} a_k q_k w^{-q_k/p-1}(t) \right) w'(t).
\]

Or equivalently

\[
psw^s(t) = \sum_{n=1}^{\infty} n \lambda_n(s) F^{n-1}(t) \left( -\sum_{k=1}^{m} a_k q_k w^{-q_k/p}(t) \right).
\] (2.16)

Upon taking (2.5) into account, we have

\[
ps \sum_{n=0}^{\infty} \lambda_n(s) F^n(t) = -\sum_{j=0}^{\infty} (j+1) \lambda_{j+1}(s) F^j(t) \times \left( \sum_{k=1}^{m} a_k q_k \sum_{i=0}^{\infty} \lambda_i(-q_k/p) F^i(t) \right).
\]

By equating the coefficients of \( F^n(t) \) on both sides, we get

\[
ps \lambda_n(s) = -\sum_{k=1}^{m} a_k q_k \sum_{i+j=n \atop i,j \geq 0} \lambda_i(-q_k/p)(j+1) \lambda_{j+1}(s).
\] (2.17)

At this stage, referring to (2.12), we are able to evaluate the sum on the right side of (2.17). The result is

\[
\sum_{i+j=n \atop i,j \geq 0} \lambda_i(-q_k/p)(j+1) \lambda_{j+1}(s) = \sum_{i+j=n+1 \atop i,j \geq 0} \lambda_i(-q_k/p) j \lambda_j(s)
\]

\[
= \sum_{i+j=n+1 \atop i,j \geq 0} \lambda_i(-q_k/p) j \lambda_j(s) = \frac{(n+1)ps}{ps-q_k} \lambda_{n+1}(s-q_k/p).
\]
Substituting this result into (2.17) gives rise to

$$\lambda_n(s) = (n + 1) \sum_{k=1}^{m} \frac{a_k q_k}{q_k - ps} \lambda_{n+1}(s - q_k/p).$$

Note that (2.15) is obtained by solving (2.17) for $\lambda_{n+1}(s)$. The proof is finished. \hfill \square

**Remark 2.7.** In view of (2.15), it is clear that $\lambda_n(s)$ is polynomial in $ps$ of degree $n$. This fact is useful for our forthcoming discussion.

3. Applications to the cases $m = 2, 3$

In this section, we will focus on the proofs of Theorem 1.6 and Corollary 1.9.

3.1. A short proof of Theorem 1.6

**Proposition 3.1.** Let $\{\lambda_n(s)\}_{n\geq 0}$ be given by (2.5) and $m = 2$. Then

$$\lambda_n(s) = \frac{ps}{n!} \prod_{i=1}^{n-1} (ps + qi + jr). \quad (3.1)$$

**Proof.** In this case, we take $a_1 = -a_2 = 1/(r-q)$ and $q_1 = q, q_2 = r$. In this situation, using (2.17), we come to

$$(ps + qn)\lambda_n(s) = \sum_{i+j\geq n \atop i,j \geq 0} \lambda_i(-r/p)(j + 1)\lambda_{j+1}(s)$$

$$= \frac{(n+1)s}{s - r/p} \lambda_{n+1}(s - r/p).$$

That is

$$\lambda_{n+1}(s - r/p) = \frac{(ps + qn)(ps - r)}{(n+1)ps} \lambda_n(s).$$

Next, replacing $s$ with $s + r/p$. It is easy to see that

$$\lambda_{n+1}(s) = \frac{(ps + qn + r)}{(n+1)(ps + r)} \lambda_n(s + r/p). \quad (3.2)$$
By iterating (3.2) \( n \) times and noting \( \lambda_0(s) = 1 \), we thereby show by induction on \( n \)

\[
\lambda_{n+1}(s) = \frac{\prod_{j=1}^{n+1} (ps + q(n+1 - j) + jr)}{(n+1)!} \frac{ps}{(ps + (n+1)r)} \lambda_0(s + (n+1)2r/p)
\]

\[
= \frac{ps}{(n+1)!} \prod_{j=1}^{n} (ps + q(n + 1 - j) + jr).
\]

As desired.

Proof of Theorem 1.6. Proposition 3.1 together with Lemma 2.1 gives the complete proof of Theorem 1.6.

3.2. The full proof of Corollary 1.9

Unlike the case \( m = 2 \), in order to show Corollary 1.9, we need a few of preliminaries.

**Proposition 3.2.** With the same assumption as in Lemma 2.6. Let \( \{\lambda_n(s)\}_{n \geq 0} \) be given by (2.5) and \( m = 3 \). Then

\[
\frac{ps + nq_3}{(n+1)ps} \lambda_n(s) = \frac{a_1(q_3 - q_1)}{ps - q_1} \lambda_{n+1}(s - q_1/p) + \frac{a_2(q_2 - q_1)}{ps - q_2} \lambda_{n+1}(s - q_2/p), \tag{3.3}
\]

\[
\frac{n}{(n+1)ps} \lambda_n(s) = \frac{a_1}{ps - q_1} \lambda_{n+1}(s - q_1/p) + \frac{a_2}{ps - q_2} \lambda_{n+1}(s - q_2/p) + \frac{a_3}{ps - q_3} \lambda_{n+1}(s - q_3/p). \tag{3.4}
\]

**Proof.** Under the assumption (1.10), it holds

\[
F(t/\phi^p(t)) = \sum_{k=1}^{3} a_k \phi^{q_k}(t).
\]

A direct application of Lemma 2.2 reduces it to

\[
F(t) = \sum_{k=1}^{3} a_k \phi^{q_k}(t/w(t)) = \sum_{k=1}^{3} a_k w^{-q_k/p}(t). \tag{3.5}
\]

Now, taking the derivative with respect to \( t \) of both sides of (3.5), we obtain

\[
sw^{s-1}(t)w'(t) = \sum_{n=1}^{\infty} n \lambda_n(s) F^{n-1}(t) F'(t)
\]

\[
= \frac{1}{p} \sum_{n=1}^{\infty} n \lambda_n(s) F^{n-1}(t) \left( -\sum_{k=1}^{3} a_k q_k w^{-q_k/p-1}(t) \right) w'(t).
\]
After simplification, it turns out to be

\[ psw^n(t) = \sum_{n=1}^{\infty} n\lambda_n(s)F^{n-1}(t) \left( -\sum_{k=1}^{3} a_kq_kw^{-q_k/p}(t) \right). \]

Since, by (1.19), \( c_0 = a_1 + a_2 + a_3 = 0 \), it holds

\[ a_3w^{-q_3/p}(t) = F(t) - a_1w^{-q_1/p}(t) - a_2w^{-q_2/p}(t). \]

Substituting this relation into the right side of this identity and replacing the left side with (2.5), we obtain

\[-ps\sum_{n=0}^{\infty} \lambda_n(s)F^n(t) = \sum_{n=1}^{\infty} n\lambda_n(s)F^{n-1}(t) \left( a_1q_1w^{-q_1/p}(t) + a_2q_2w^{-q_2/p}(t) \\
+ q_3(F(t) - a_1w^{-q_1/p}(t) - a_2w^{-q_2/p}(t)) \right) \]
\[= q_3 \sum_{n=1}^{\infty} n\lambda_n(s)F^n(t) \]
\[+ a_1(q_1 - q_3)w^{-q_1/p}(t)\sum_{n=1}^{\infty} n\lambda_n(s)F^{n-1}(t) \]
\[+ a_2(q_2 - q_3)w^{-q_2/p}(t)\sum_{n=1}^{\infty} n\lambda_n(s)F^{n-1}(t). \]

In this form, by equating the coefficients of \( F^n(t) \), we get

\[(nq_3 + ps)\lambda_n(s) = a_1(q_3 - q_1) \sum_{i+j=n} (j + 1)\lambda_{j+1}(s)\lambda_i(-q_1/p) \]
\[+ a_2(q_3 - q_2) \sum_{i+j=n} (j + 1)\lambda_{j+1}(s)\lambda_i(-q_2/p). \]

Recall that

\[\sum_{i+j=n, i,j \geq 0} (j + 1)\lambda_{j+1}(s)\lambda_i(-q_k/p) = \frac{(n + 1)ps}{ps - q_k} \lambda_{n+1}(s - q_k/p). \]

Now we are able to evaluate the sums on the right side of (3.5). The result is

\[\frac{ps + nq_3}{(n + 1)ps} \lambda_n(s) = \frac{a_1(q_3 - q_1)}{ps - q_1} \lambda_{n+1}(s - q_1/p) + \frac{a_2(q_3 - q_2)}{ps - q_2} \lambda_{n+1}(s - q_2/p). \]

On the other hand, on referring to (2.14), we have

\[ \frac{1}{n+1}\lambda_n(s) = \frac{a_1q_1}{q_1 - ps} \lambda_{n+1}(s - q_1/p) + \frac{a_2q_2}{q_2 - ps} \lambda_{n+1}(s - q_2/p) + \frac{a_3q_3}{q_3 - ps} \lambda_{n+1}(s - q_3/p). \]
Upon subtracting (3.8) from (3.7), we immediately get

\[
\frac{n}{(n+1)ps} \lambda_n(s) = \frac{a_1}{ps-q_1} \lambda_{n+1}(s-q_1/p) + \frac{a_2}{ps-q_2} \lambda_{n+1}(s-q_2/p) + \frac{a_3}{ps-q_3} \lambda_{n+1}(s-q_3/p).
\]

Thus, (3.4) is proved.

As a direct application of Proposition 3.2, we can show

Example 3.3. Let \( \{\lambda_n(s)\}_{n \geq 0} \) be given by (2.5) and \( m = 3, q_1 = q_2 \). Then

\[
\lambda_n(s) = \frac{ps}{n!(a_3(q_1-q_3))} \prod_{j=1}^{n-1} (ps + jq_1 + (n-j)q_3).
\] (3.9)

Proof. Note that the case \( q_1 = q_2 \) reduces (3.3) to

\[
\frac{ps + nq_3}{(n+1)ps} \lambda_n(s) = \frac{a_3(q_3-q_1)}{q_1-ps} \lambda_{n+1}(s-q_1/p).
\] (3.10)

From (3.10) it follows

\[
\lambda_{n+1}(s-q_1/p) = \frac{ps + nq_3}{(n+1)ps} \frac{q_1-ps}{a_3(q_3-q_1)} \lambda_n(s).
\]

Replace \( s \) with \( s + q_1/p \). The result is

\[
\lambda_{n+1}(s) = \frac{ps + q_1 + nq_3}{(n+1)(ps+q_1) a_3(q_1-q_3)} \lambda_n(s+q_1/p).
\] (3.11)

Iterating (3.11) in succession and then showing by induction leads us to

\[
\lambda_{n+1}(s) = \frac{ps}{(n+1)!(a_3(q_1-q_3))} \prod_{j=1}^n (ps + jq_1 + (n+1-j)q_3).
\]

The proof is finished.

Proposition 3.4. Let \( \{\lambda_n(s)\}_{n \geq 0} \) be given by (2.5) with \( m = 3 \) and write

\[
\lambda_n(s) = \frac{ps}{n!(c_1)^n} f_n(s).
\] (3.12)

Then \( f_0(s) = 1/(ps) \), \( f_1(s) = 1 \), and

\[
f_{n+1}(s) = ps f_n(s) + \frac{1}{c_1} \sum_{k=1}^3 a_k q_k \sum_{i=1}^n \binom{n}{i} f_i(-q_k/p) f_{n+1-i}(s).
\] (3.13)
Proof. From (2.17) it follows that
\[
ps\lambda_n(s) = -\sum_{k=1}^{3} a_k q_k \lambda_0(-q_k/p)(n + 1)\lambda_{n+1}(s) \\
- \sum_{k=1}^{3} a_k q_k \sum_{i+j=n \atop i \geq 1} \lambda_i(-q_k/p)(j+1)\lambda_{j+1}(s)
\]
\[
= -(n+1)c_1\lambda_{n+1}(s) - \sum_{k=1}^{3} a_k q_k \sum_{i+j=n \atop i \geq 1} \lambda_i(-q_k/p)(j+1)\lambda_{j+1}(s).
\]
It yields the recurrence relation
\[
(n+1)c_1\lambda_{n+1}(s) = -ps\lambda_n(s) - \sum_{k=1}^{3} a_k q_k \sum_{i+j=n \atop i \geq 1} \lambda_i(-q_k/p)(n + 1 - i)\lambda_{n+1-i}(s).
\]
Taking (3.12) into account and simplifying the last identity, we obtain
\[
f_{n+1}(s) = psf_n(s) + \frac{1}{c_1} \sum_{k=1}^{3} a_k q_k^2 \sum_{i=1}^{n} \binom{n}{i} f_i(-q_k/p)f_{n+1-i}(s).
\]
The conclusion is proved.

Example 3.5. Let \( \{c_n\}_{n \geq 0} \) be given by (1.19) and \( f_n(s) \) be given by (3.13). Then
\[
f_2(s) = ps + \frac{c_2}{c_1},
\]
\[
f_3(s) = (ps)^2 + 3\frac{c_2}{c_1}ps + \frac{3c_2^2 - c_1 c_3}{c_1^2},
\]
\[
f_4(s) = (ps)^3 + 6\frac{c_2}{c_1}(ps)^2 + \frac{15c_2^2 - 4c_1 c_3}{c_1^2} ps + \frac{15c_2^2 - 10c_1 c_2 c_3 + c_1^2 c_4}{c_1^4}.
\]
Proof. At first, by virtue of the definition (3.12), it is easy to check
\[
f_2(s) = ps + \frac{c_2}{c_1}.
\]
Based on this, by iterating, it is easy to get
\[
f_3(s) = psf_2(s) + \frac{1}{c_1} \sum_{k=1}^{3} a_k q_k^2 (2f_1(-q_k/p)f_2(s) + f_2(-q_k/p)f_1(s))
\]
\[
= ps\frac{psc_1 + c_2}{c_1} + \frac{1}{c_1} \sum_{k=1}^{3} a_k q_k^2 \left(2\frac{psc_1 + c_2}{c_1} + \frac{-q_k c_1 + c_2}{c_1}\right)
\]
\[
= (ps)^2 + 3\frac{c_2}{c_1}ps + \frac{3c_2^2}{c_1^2} - \frac{c_3}{c_1}.
\]
Start with $f_2, f_3$ and iterate further. We have

\[
f_4(s) = psf_3(s) + \frac{1}{c_1} \sum_{k=1}^{3} a_k q_k^2 \sum_{i=1}^{3} \binom{3}{i} f_i(-q_k/p) f_{4-i}(s)
\]

\[
= (ps)^3 + 3 \frac{c_2}{c_1} (ps)^2 + 3 \frac{c_2^2}{c_1^2} ps - \frac{c_3}{c_1} ps
\]

\[
+ 3 \frac{c_2}{c_1} f_3(s) + \frac{3}{c_1} \sum_{k=1}^{3} a_k q_k^2 f_2(-q_k/p) f_2(s) + \frac{1}{c_1} \sum_{k=1}^{3} a_k q_k^2 f_3(-q_k/p)
\]

\[
= (ps)^3 + 3 \frac{c_2}{c_1} (ps)^2 + 3 \frac{c_2^2}{c_1^2} ps - \frac{c_3}{c_1} ps + 3 \frac{c_2}{c_1} (ps)^2 + 9 \frac{c_2}{c_1} (ps) + 9 \frac{c_2}{c_1} - 3 \frac{c_2 c_3}{c_1^2}
\]

\[
+ \frac{3}{c_1} \sum_{k=1}^{3} a_k q_k^2 (-q_k ps + c_2/c_1 ps - q_k c_2/c_1 + c_2^2/c_1^2)
\]

\[
+ \frac{1}{c_1} \sum_{k=1}^{3} a_k q_k^2 \left( q_k^2 - 3 \frac{c_2}{c_1} q_k + 3 \frac{c_2^2}{c_1} - \frac{c_3}{c_1} \right)
\]

Thus

\[
f_4(s) = (ps)^3 + 6 \frac{c_2}{c_1} (ps)^2 + 15 \frac{c_2^2}{c_1^2} ps - 4 \frac{c_3}{c_1} ps + 15 \frac{c_2}{c_1} - 10 \frac{c_2 c_3}{c_1^2} + \frac{c_4}{c_1}.
\]

These computational results about $f_n(s)$ ($1 \leq n \leq 4$) inspire us to introduce

**Definition 3.6 (Mina polynomial).** The Mina polynomial $C_{n,k}(x_1, x_2, \ldots, x_{n-k})$ is defined by

\[
C_{n,k}(x_1, x_2, \ldots, x_{n-k}) = x_1^{n-1-k} [A_{n,0}^{-1} A_{n,1}^{-1} \cdots A_{n,n-2}^{-1}]_{n,k},
\]

where two $n$ by $n$ (blocked) matrices $A_{n,r}$ and $M_n$ are respectively given by

\[
A_{n,r} := \begin{pmatrix} E_r & 0 \\ 0 & M_{n-r} \end{pmatrix},
\]

\[
[M]_{m \times k} := \binom{k}{m} (-1)^{k-m} x_{k-m+1}.
\]

In terms of the Mina polynomials, we can give the full characterization on $f_n(s)$.

**Proposition 3.7.** Let $\lambda_n(s)$ and $f_n(s)$ be given by (3.12). Then we have

\[
f_n(s) = \sum_{k=0}^{n-1} \lambda_n(k)(ps)^k
\]
where $\chi_n(k)$ is given by

$$\chi_n(k) = \frac{1}{c_1^{n-1-k}} C_{n,k}(c_1, c_2, \ldots, c_{n-k})$$  \hspace{1cm} (3.17)

while $c_k$ is the same as in (1.19).

Proof. It suffices to substitute $\lambda_n(s)$ given by (3.12) into (2.14) and make a bit simplification, yielding

$$a_1q_1 \sum_{k=0}^{n} \chi_{n+1}(k)(ps - q_1)^k + a_2q_2 \sum_{k=0}^{n} \chi_{n+1}(k)(ps - q_2)^k$$

$$+ a_3q_3 \sum_{k=0}^{n} \chi_{n+1}(k)(ps - q_3)^k = c_1ps \sum_{k=0}^{n-1} \chi_n(k)(ps)^k,$$

That is

$$\sum_{k=0}^{n} \chi_{n+1}(k) \sum_{i=0}^{k} \binom{k}{i} (ps)^i (-1)^{k-i} c_{k-i+1} = c_1 \sum_{k=0}^{n} \chi_n(k-1)(ps)^k.$$

By equating the coefficients of $(ps)^m$, we obtain

$$\sum_{0 \leq k \leq n} \chi_{n+1}(k) \binom{k}{m} (-1)^{k-m} c_{k-m+1} = c_1 \chi_n(m-1).$$  \hspace{1cm} (3.18)

It amounts to a system of linear equations in $n + 1$ unknowns

$$M_{n+1} \chi_{n+1} = c_1 \begin{pmatrix} 0 \\ \chi_n \end{pmatrix},$$  \hspace{1cm} (3.19)

where, order $\chi(k) = 0$ for $k < 0$ and $M_{n+1}$ is given by (3.15b). Here we define the column vector

$$\chi_n^T := (\chi_n(0), \chi_n(1), \ldots, \chi_n(n-1)).$$

Obviously, $|M_{n+1}|_{m \times m} = c_1 \neq 0$, thus (3.19) has the unique solution

$$\chi_{n+1} = c_1 M_{n+1}^{-1} \begin{pmatrix} 0 \\ \chi_n \end{pmatrix}.$$  \hspace{1cm} (3.20)

In terms of $A_{n,r}$ given by (3.15a) and the vector $\beta_{n,r} := \begin{pmatrix} 0_r \\ \chi_{n-r} \end{pmatrix}$, we can reformulate (3.20) in the form

$$\beta_{n+1,0} = c_1 A_{n+1,0}^{-1} \beta_{n+1,1}.$$  \hspace{1cm} (3.21)
Iterating (3.21) \(n\) times yields
\[
\beta_{n+1,0} = c_1^n A_{n+1,0}^{-1} A_{n+1,1}^{-1} \cdots A_{n+1,n-1}^{-1} \beta_{n+1,n}.
\]

It is easy to check that \(\beta_{n+1,n}^T = (0, 0, \ldots, 0, 1)^T\). Finally, we arrive at
\[
\beta_{n+1,0} = c_1^n [A_{n+1,0}^{-1} A_{n+1,1}^{-1} \cdots A_{n+1,n-1}^{-1}]_{n+1}.
\]

Hence, (3.17) is proved.

Next is the detailed computation for \(n = 3\).

**Example 3.8.** Let \(n = 3\). Then the system of linear equations (3.19) reduces to
\[
\begin{pmatrix}
  c_1 & -c_2 & c_3 & -c_4 \\
  0 & c_1 & -2c_2 & 3c_3 \\
  0 & 0 & c_1 & -3c_2 \\
  0 & 0 & 0 & c_1
\end{pmatrix}
\begin{pmatrix}
  \chi_4(0) \\
  \chi_4(1) \\
  \chi_4(2) \\
  \chi_4(3)
\end{pmatrix}
= c_1
\begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & c_1 & -2c_2 & 3c_3 \\
  0 & 0 & c_1 & -3c_2 \\
  0 & 0 & 0 & c_1
\end{pmatrix}
\begin{pmatrix}
  0 \\
  \chi_3(0) \\
  \chi_3(1) \\
  \chi_3(2)
\end{pmatrix}.
\]

Iterating this linear relation, we may solve out
\[
\begin{pmatrix}
  \chi_4(0) \\
  \chi_4(1) \\
  \chi_4(2) \\
  \chi_4(3)
\end{pmatrix}
= c_1^2
\begin{pmatrix}
  c_1 & -c_2 & c_3 & -c_4 \\
  0 & c_1 & -2c_2 & 3c_3 \\
  0 & 0 & c_1 & -3c_2 \\
  0 & 0 & 0 & c_1
\end{pmatrix}
\begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & c_1 & -2c_2 & 3c_3 \\
  0 & 0 & c_1 & -3c_2 \\
  0 & 0 & 0 & c_1
\end{pmatrix}
\begin{pmatrix}
  0 \\
  \chi_3(0) \\
  \chi_3(1) \\
  \chi_3(2)
\end{pmatrix}
= c_1^3
\begin{pmatrix}
  c_1 & -c_2 & c_3 & -c_4 \\
  0 & c_1 & -2c_2 & 3c_3 \\
  0 & 0 & c_1 & -3c_2 \\
  0 & 0 & 0 & c_1
\end{pmatrix}
\begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & c_1 & -2c_2 & 3c_3 \\
  0 & 0 & c_1 & -3c_2 \\
  0 & 0 & 0 & c_1
\end{pmatrix}
\begin{pmatrix}
  0 \\
  \chi_2(0) \\
  \chi_2(1) \\
  \chi_2(2)
\end{pmatrix}.
\]

Finally
\[
\begin{pmatrix}
  \chi_4(0) \\
  \chi_4(1) \\
  \chi_4(2) \\
  \chi_4(3)
\end{pmatrix}
= c_1^3
\begin{pmatrix}
  \frac{1}{c_1} & 0 & 0 & \frac{3c_2^2-6c_2 c_3}{c_1^2} \\
  0 & \frac{1}{c_1} & \frac{3c_2}{c_1^2} & \frac{6c_2^2 - 4c_1 c_3 c_2}{c_1^3} \\
  0 & 0 & \frac{1}{c_1} & \frac{6c_2^2 - 4c_1 c_3 c_2}{c_1^3} \\
  0 & 0 & 0 & \frac{1}{c_1}
\end{pmatrix}
\begin{pmatrix}
  0 \\
  \frac{1}{c_1} \\
  \frac{6c_2^2 - 4c_1 c_3 c_2}{c_1^3} \\
  \frac{6c_2^2 - 4c_1 c_3 c_2}{c_1^3}
\end{pmatrix}
= \begin{pmatrix}
  \frac{15c_2^2 - 10c_1 c_3 c_2 + c_2^2 c_4}{c_1^3} \\
  \frac{15c_2^2 - 10c_1 c_3 c_2 + c_2^2 c_4}{c_1^3} \\
  \frac{15c_2^2 - 10c_1 c_3 c_2 + c_2^2 c_4}{c_1^3} \\
  \frac{15c_2^2 - 10c_1 c_3 c_2 + c_2^2 c_4}{c_1^3}
\end{pmatrix}.
\]

It yields
\[
f_4(s) = (ps)^3 + 6\frac{c_2}{c_1}(ps)^2 + \frac{15c_2^2 - 4c_1 c_3}{c_1^2} ps + \frac{15c_2^2 - 10c_1 c_3 c_2 + c_2^2 c_4}{c_1^3}.
\]

It is in agreement with all computations in Example 3.5.
We are in a good position to show

Proof of Corollary 1.9. Propositions 3.2, 3.7, and 3.12 together gives the complete proof of Corollary 1.9.

3.3. Convolution identities

Similar to the Bell polynomials $B_{n,k}(x_1, x_2, \ldots, x_{n-k})$, the polynomials $C_{n,k}(x_1, x_2, \ldots, x_{n-k})$ given by Definition 3.6 deserves further consideration.

Corollary 3.9. Let $\{c_n\}_{n \geq 0}$ be given by (1.19). Then for arbitrary integers $1 \leq m \leq n - 1$, we have

$$C_{n,m}(c_1, c_2, \ldots, c_{n-m}) = \frac{1}{2^{n+1} - 2} \sum_{i+j=m-1}^{n-1} \sum_{k=1}^{n-1} \binom{n}{k} C_{k,i}(c_1, c_2, \ldots, c_{k-i}) C_{n-k,j}(c_1, c_2, \ldots, c_{n-k-j}).$$

In particular,

$$2C_{n,1}(c_1, c_2, \ldots, c_{n-1}) = \sum_{k=1}^{n-1} \binom{n}{k} C_{k,0}(c_1, c_2, \ldots, c_k) C_{n-k,0}(c_1, c_2, \ldots, c_{n-k}),$$

$$3C_{n,2}(c_1, c_2, \ldots, c_{n-2}) = \sum_{k=1}^{n-1} \binom{n}{k} C_{k,1}(c_1, c_2, \ldots, c_{k-1}) C_{n-k,0}(c_1, c_2, \ldots, c_{n-k}).$$

Proof. By Proposition 3.7, we see that for $0 \leq k \leq n - 1$, it holds

$$\chi_n(k) = \frac{C_{n,k}(c_1, c_2, \ldots, c_{n-k})}{c_1^{n-1-k}}.$$ (3.25)

Using (2.11) of Lemma 2.5 and making the replacement $(a, b) \to (as, bs)$ in (2.11), we achieve

$$\lambda_n(as + bs) = \lambda_n(as) + \lambda_n(bs) + \sum_{k=1}^{n-1} \lambda_k(as) \lambda_{n-k}(bs).$$ (3.26)

Next, using Lemma 2.6 and substituting (3.16) into (3.26), we obtain

$$\frac{p(a+b)s}{n!c_1^n} \sum_{k=0}^{n-1} \chi_n(k)(p(a+b)s)^k = \frac{pas}{n!c_1^n} \sum_{k=0}^{n-1} \chi_n(k)(pas)^k - \frac{pbs}{n!c_1^n} \sum_{k=0}^{n-1} \chi_n(k)(pbs)^k$$

$$= \sum_{k=1}^{n-1} \left( \frac{pas}{k!c_1^k} \sum_{i=0}^{k-1} \chi_k(i)(pas)^i \right) \times \left( \frac{pbs}{(n-k)!c_1^{n-k}} \sum_{j=0}^{n-k-1} \chi_{n-k}(j)(pbs)^j \right).$$
A bit simplification reduces it to
\[(a + b) \sum_{k=0}^{n-1} \chi_n(k)(p(a + b)s)^k - a \sum_{k=0}^{n-1} \chi_n(k)(pas)^k - b \sum_{k=0}^{n-1} \chi_n(k)(pbs)^k\]
\[= abps \sum_{k=1}^{n-1} \binom{n}{k} \left( \sum_{i=0}^{k-1} \chi_k(i)(pas)^i \right) \times \left( \sum_{j=0}^{n-k-1} \chi_{n-k}(j)(pbs)^j \right).\]

By equating the coefficients of \((ps)^m\) (0 \(\leq m \leq n - 1\) on both sides, we have
\[((a + b)^{m+1} - a^{m+1} - b^{m+1})\chi_n(m) = ab \sum_{k=1}^{n-1} \binom{n}{k} \sum_{i+j=m-1}^{n-1} \chi_k(i)a^i b^j \chi_{n-k}(j)\]
\[= ab \sum_{i+j=m-1}^{n-1} a^i b^j \sum_{k=1}^{n-1} \binom{n}{k} \chi_k(i)\chi_{n-k}(j).\] (3.27)

Since the parameters \(a, b\) are arbitrary and \(\chi_n(k)\) are independent of \(a, b\), we are able to set \(a = b\) in (3.27) and then equate the coefficients of \(a^{m+1}\), obtaining
\[2(2^m - 1)\chi_n(m) = \sum_{i+j=m-1}^{n-1} \sum_{k=1}^{n-1} \binom{n}{k} \chi_k(i)\chi_{n-k}(j).\]

After (3.25) inserted, it becomes
\[
\frac{C_{n,m}(c_1, c_2, \ldots, c_{n-m})}{c_1^{n-1-m}} = \frac{1}{2^{m+1} - 2} \sum_{i+j=m-1}^{n-1} \sum_{k=1}^{n-1} \binom{n}{k} \frac{C_{k,i}(c_1, c_2, \ldots, c_{k-i})}{c_1^{k-1-i}} \times \frac{C_{n-k,j}(c_1, c_2, \ldots, c_{n-k-i})}{c_1^{n-k-1-j}}.
\]

A direct simplification gives (3.22). Evidently, the cases \(m = 1\) and \(m = 2\) specialize (3.22) to (3.23) and (3.24), respectively.  

\[\text{Corollary 3.10. Let } \{c_n\}_{n \geq 0} \text{ be given by (1.19). Then for arbitrary integers } 1 \leq m \leq n - 1, \text{ we have}
\]
\[(m + 1)C_{n,m}(c_1, c_2, \ldots, c_{n-m}) = \sum_{k=m}^{n-1} \binom{n}{k} C_{k,m}(c_1, c_2, \ldots, c_{k-m+1}) C_{n-k,0}(c_1, c_2, \ldots, c_{n-k}).\] (3.28)
Proof. It suffices to compare the coefficients of $a^m$ on both sides of (3.27). Then we have

$$\binom{m+1}{m} \chi_n(m) = \sum_{k=1}^{n-1} \binom{n}{k} \chi_k(m-1) \chi_{n-k}(0).$$

It is, after (3.17) inserted, equivalent to (3.28) \qed

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