Dynamics of rotationally reciprocating stirred tank with planetary actuator

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Abstract. The article investigates the dynamics of rotationally reciprocating stirred tank (RRST), whose actuator is the original planetary mechanism with elliptical gears. The dynamic model is constructed by reduction of driving forces, masses and moments to the reduction link (the input shaft of the actuator). The study of the resulting dynamic model was carried out by energy-mass method. As a result of the dynamic analysis we determined the necessary moment of driven forces and found the reduction link law of motion. The flywheel has been designed to ensure the required coefficient of rotation irregularity. Resulting dynamic model can be used for development and research of rotationally reciprocating stirred tanks.

1. Introduction
Mechanically stirred vessels are widely used in machinery, petrochemical, chemical, food and many other industries [1, 2]. Currently, the most common and investigated stirred tanks are agitators with a rotational movement of impellers, because they are highly reliable, easy to manufacture and use. Angular velocity of impeller is constant in such vessels, so over time stirred liquid velocity and impeller velocity are equalized, which leads to low intensity of heat and mass transfer. For better performance of stirred tanks it is necessary to change velocity and direction of impeller rotation [3–5]. In [6, 7] RRST is investigated, and rotationally reciprocating motion of its impeller is provided by reversing motion of the stepping motor. The developed machine is simple, due to the wide range of the rotation angle it allows to carry out various experiments in the small volume reactors. However, the use of such device is impractical on an industrial scale, since the stepping motor has a low efficiency and can not be used at high oscillation frequency of impeller. Therefore, [8, 9] proposed a planetary converter of rotational motion into reciprocating rotational motion, which can be used as an actuator of the stirred tank. It is a double row planetary gear with two external gearing, in which one pair of cylindrical gears is replaced by elliptical ones. Angle of the output shaft rotationally reciprocating motion is defined by eccentricity of elliptical gears. Mechanism with two pairs of elliptical gears will increase the output shaft rotation angle at the same size (Fig. 1).

Planetary gear (Fig. 1) consists of a rack 0, the input shaft 1, the carrier 2, the output shaft 3, the sun elliptic gear 4, elliptic gear 5, satellite elliptical gears 6 and 7, a shaft 8, which connects the satellite gears. Reciprocating rotational motion is provided by a rational sizing of actuator links.
Connecting the input shaft of investigated mechanism with a motor and output shaft with impeller, we obtain rotationally reciprocating stirred tank (RRST) (Fig. 2).

The considered vessel achieves high gradient of stirred liquid velocities, leading to increase in the heat and mass transfer intensity. Selection of the optimal amplitude and frequency of vibrations will reduce the flow time of many processes by 1.5-2 times, and operating costs by 1.2-1.8 times [10].

One of the most important steps in the design of new machines is the study of dynamic processes. To investigate the rotationally reciprocating stirred tank it is necessary to build dynamic model and conduct its analysis.

2. Dynamic model of the actuator

Since the planetary actuator has one degree of freedom [11] and its links are rigidly interconnected, for the rational solution it is advisable to take the input shaft 1 as the reduction link (Fig. 1). Then single-mass dynamic model of RRST (Fig. 2) takes the following form (Fig. 3).

To find the law of reduction link motion, it is necessary to define the following parameters of the dynamic model: reduced moment of inertia $I_r$, and the reduced moment of resistance forces $M_r$. The required drive moment $M_d$ is calculated during the analysis of the dynamic model.

2.1. Determination of the reduced moment of inertia

According to [12-14] reduced moment of inertia given by:

$$I_r = \sum_{i=1}^{n} m_i S_i^2 + \sum_{i=1}^{n} I_i \varphi_i^2,$$  

(1)
where \( n \) is the number of movable links, which masses and moments of inertia are known; \( m_i \) is the mass of the \( i \)-th link; \( I_{Si} \) is the moment of inertia of the \( i \)-th link about an axis passing through the center of mass; \( S'_i = \frac{dS_i}{d\varphi_i} \) is the velocity analogue of the \( i \)-th link center of mass; \( \varphi'_i = \frac{d\varphi_i}{d\varphi_i} \) is the angular velocity analogue of the \( i \)-th link.

For the investigated mechanism the equation (1) takes the following form:

\[
I_r = (I_d + I_1 + I_2) \cdot \varphi'_1^2 + m_6 \cdot S'_6^2 + m_7 \cdot S'_7^2 + m_8 \cdot S'_8^2 + (I_6 + I_7 + I_8) \cdot \varphi'_8^2 + (I_3 + I_5 + I_{im}) \cdot \varphi'_3^2, \tag{2}
\]

where \( I_d \) is the motor moment of inertia, \( I_{im} \) is the impeller moment of inertia. The moments of inertia and velocity analogues of the actuator links identified in accordance with Figure 1.

Differentiating (2) according to the generalized coordinate, we get:

\[
\frac{dI_r}{d\varphi_i} = 2[m_6 \cdot S'_6 \cdot S'_6^* + m_7 \cdot S'_7 \cdot S'_7^* + m_8 \cdot S'_8 \cdot S'_8^* + (I_6 + I_7 + I_8) \cdot \varphi'_8 \cdot \varphi'_8^* + (I_3 + I_5 + I_{im}) \cdot \varphi'_3 \cdot \varphi'_3^*]. \tag{3}
\]

As seen from equations (2), (3), to find the reduced moment of inertia and its derivative it is necessary to find kinematic characteristics of planetary actuator – angular velocity and angular acceleration analogues of mechanism links, as well as linear velocity and linear acceleration analogues of link’s centers of mass.

2.2. Kinematic analysis of planetary actuator

To carry out the kinematic analysis we shall represent the kinematic scheme of the mechanism in one of the positions and construct a plan of linear velocities (Fig. 4).

Angular velocity analogue of the output shaft, in accordance with Figure 4, is determined as:

\[
\varphi'_3 = \frac{\omega_3}{\omega_1} = \frac{\nu_D \cdot AC}{\nu_C \cdot DE} = \frac{BD \cdot AC}{BC \cdot DE}. \tag{4}
\]

To determine the distances included in the equation (4), let’s consider the equation of the ellipse in polar coordinates. The focus of the driving ellipse will be taken as the pole, and the major axis will be taken as the polar axis (Fig. 5), then we will get the following ellipse equation [15]:

\[
\rho(\varphi) = \frac{p}{1 - e \cos \varphi}; \tag{5}
\]

\[
p = a(1-e^2), \tag{6}
\]
where $\phi$ is a drive gear rotation angle; $p$ is an ellipse focal parameter; $e$ is eccentricity of the ellipse; $a$ is a semi-major axis of the ellipse.

![Figure 5. Elliptical gear.](image)

According to Figure 6, taking into account (6), the equation (5) for the ellipses 4 and 7 will have the following form:

$$
\rho_4(\phi_4) = \frac{a(1-e_1^2)}{1-e_1 \cos \phi_4};
$$

$$
\rho_7(\phi_7) = \frac{a(1-e_2^2)}{1-e_2 \cos \phi_7},
$$

(7)

where $\phi_4$, $\phi_7$ are angles of ellipses rotation; $e_1$ is eccentricity of the first ellipse gear pair (4, 6); $e_2$ is eccentricity of the second ellipse gear pair (5, 7).

![Figure 6. Mechanism scheme.](image)

According to [16] transmission function of elliptical gear can be written as:

$$
u_{21} = \frac{1-e^2}{1+e^2 - 2e \cos \phi}.
$$

Taking into account (8) we established the relation between the rotation angles of elliptical gearwheels (Fig. 6):
\[ \varphi_4 = \varphi_1 + \pi; \]
\[ \varphi_5 = \int u_{64} d\varphi_4 = \int \frac{1 - e_1^2}{1 + e_1^2 - 2e_1 \cos(\varphi_1 + \pi)} d(\varphi_1 + \pi); \]  \hfill (9)
\[ \varphi_7 = \pi - \varphi_6 = \pi - \left( \int \frac{1 - e_1^2}{1 + e_1^2 - 2e_1 \cos(\varphi_1 + \pi)} d(\varphi_1 + \pi) \right). \]

It follows from Figure 6 that the distances in the equation (4) are defined as:
\[ BD = 2a - \rho_4 - \rho_7; \]  \hfill (10)
\[ BC = 2a - \rho_4; \]  \hfill (11)
\[ AC = 2a; \]  \hfill (12)
\[ DE = 2a - \rho_7. \]  \hfill (13)

Substituting (10)…(13) into equation (4) we find the angular velocity analogue of the planetary mechanism output shaft 3:
\[ \varphi'_5 = \frac{(2a - \rho_4 - \rho_7) \cdot 2a}{(2a - \rho_4) \cdot (2a - \rho_7)}. \]  \hfill (14)

Figure 6 indicates that the satellite 8 and elliptical gearwheels 6 and 7 make the plane-parallel motion, and point B for satellite will be instantaneous center of velocity. Figure 6 also shows that the satellite shaft center of mass \( (C_8) \) velocity \( v_8 \) and the angular velocity \( \omega_8 \) of the satellite are determined as:
\[ v_8 = \omega_8 \cdot AC_8 = \omega_8 \cdot 2a; \]  \hfill (15)
\[ \omega_8 = \frac{v_8}{BC_8} = \frac{\omega_8 \cdot 2a}{2a - \rho_4}. \]  \hfill (16)

Taking into account (15) and (16) velocity analogues \( \varphi'_8 \) and \( S'_8 \) will take the following form:
\[ \varphi'_8 = \frac{2a}{2a - \rho_4}; \]  \hfill (17)
\[ S'_8 = 2a. \]  \hfill (18)

Velocities of elliptical gearwheels mass centers \( C_6 \) and \( C_7 \) are determined as:
\[ v_6 = \omega_6 \cdot BC_6; \]  \hfill (19)
\[ v_7 = \omega_7 \cdot BC_7. \]  \hfill (20)

Distances \( BC_6 \) and \( BC_7 \) (Fig. 5) are determined by the law of cosines from \( BC_8C_6 \) and \( BC_8C_7 \) triangles:
\[ BC_6 = \sqrt{C_6C_8^2 + BC_6^2 - 2C_6C_8 \cdot BC_6 \cdot \cos \varphi_6}; \]  \hfill (21)
\[ BC_7 = \sqrt{C_7C_8^2 + BC_7^2 - 2C_7C_8 \cdot BC_8 \cdot \cos \varphi_7}, \]  \hfill (22)
where \( C_6C_8 = a \cdot e_1 \), \( C_7C_8 = a \cdot e_2 \) are the focal distances of elliptical gears, \( BC_8 \) is determined by (11), angles \( \varphi_6 \) and \( \varphi_7 \) are determined from equation (9).

Then, considering (16), (19)…(22), the velocity analogues \( S'_6 \) and \( S'_7 \), are determined as:
\[ S_0' = \frac{2a \cdot \sqrt{(a - e_1)^2 + (2a - \rho_a)^2 - 2ae_1(2a - \rho_a) \cdot \cos \phi_0}}{(2a - \rho_a)}; \]  

\[ S_\gamma' = \frac{2a \cdot \sqrt{(a - e_2)^2 + (2a - \rho_a)^2 - 2ae_2(2a - \rho_a) \cdot \cos \phi_\gamma}}{(2a - \rho_a)}. \]

Substituting (14) and (17), (18), (23), (24) to (2), (3), we obtain the reduced moment of inertia and its derivative at any rotation angle of reduction link.

2.3. Determination of the fluid resistance moment

A useful fluid resistance acts on stirred tank impeller, which determines motion laws of the mechanism links. To construct a dynamic model it is necessary to find a moment of force, which arises at interaction between fluid and impeller, and reduces it to the input shaft of the planetary actuator.

The stirred product is a liquid, so in general, the viscous resistance force will act on the impeller, and it can be either linear or quadratic function of the velocity:

\[ F_1 = k_1 \cdot \nu; \]

\[ F_2 = k_2 \cdot \nu^2, \]

where \( k_1, k_2 \) are coefficients of resistance, \( \nu \) is the impeller linear velocity.

The linear velocity of different impeller areas is variable; therefore, the variable force of fluid resistance will act on the RRST impeller (Fig. 7).

In [17] there is obtained the following equation for determining the resistance moment of impeller that perform rotationally reciprocating motion:
\[ M = \frac{k}{3} B'_{lin} \cdot \omega \cdot l_{im} \cdot h_i^3 + \frac{k}{4} B'_{quad} \cdot \omega^2 \cdot l_{im} (h_{im}^4 - h_i^4) \cdot \text{sign}(\omega), \]  

(27)

where \( B'_{lin} \), \( B'_{quad} \) are reduced coefficients of linear and quadratic resistance, \( \omega \) is the angular velocity of impeller, \( l_{im} \) is an impeller length, \( h_{im} \) is the blade width, \( h_i \) is the distance from the rotation axis to the boundary between the laminar and turbulent regimes, \( k \) is the number of impeller blades (in investigated stirred tank \( k = 2 \)).

For most technological processes there we can observe turbulent mode [1], so (27) takes the form:

\[ M = \frac{1}{2} B'_{quad} \cdot \omega^2 \cdot l_{im} \cdot h_{im}^4 \cdot \text{sign}(\omega). \]  

(28)

Denote \( B = 0.5 B'_{quad} \cdot l_{im} \cdot h_{im}^3 \), as a result we get:

\[ M = B \cdot \omega^2 \cdot \text{sign}(\omega). \]  

(29)

According to [10], reduced resistance moment is generally defined as follows:

\[ M_r = \sum_{i=1}^{n} \left( \sum_{i=1}^{m} F_i l_i' + \sum_{i=1}^{q} M_i \varphi_i' \right), \]  

(30)

where \( n \) is total number of mobile links; \( m \) is number of forces \( F \), acting on the \( i \)-th link; \( l_i' \) is velocity analogue of force application point; \( q \) is the number of moments \( M \), acting on the \( i \)-th link.

Taking into account that only the impeller resistance moment acts in the stirred tank, then (30) to determine the reduced resistance moment takes the form:

\[ M_r = M \cdot \varphi_1'. \]  

(31)

Given that \( \omega_i = \omega_1 \cdot \varphi_i' \), we substitute (29) to (31) and obtain:

\[ M_r = B \cdot \omega_1^2 \cdot \varphi_3'^3 \cdot \text{sign}(\varphi_3'). \]  

(32)

Equation (32) allows to find resistance moment, reduced to the input shaft 1.

### 3. Investigation of the dynamic model

As an example, we consider a stirring device with the following parameters (link numbers correspond to Fig. 1): \( I_2 = 100 \text{ g} \cdot \text{cm}^2 \) (motor); \( I_1 = 9.8 \text{ g} \cdot \text{cm}^2 \); \( I_2 = 1233 \text{ g} \cdot \text{cm}^2 \); \( I_3 = 22.4 \text{ g} \cdot \text{cm}^2 \); \( I_5 = 627 \text{ g} \cdot \text{cm}^2 \); \( I_6 = 564 \text{ g} \cdot \text{cm}^2 \); \( m_6 = 0.09 \text{ kg} \); \( I_7 = 564 \text{ g} \cdot \text{cm}^2 \); \( m_7 = 0.09 \text{ kg} \); \( I_8 = 19.2 \text{ g} \cdot \text{cm}^2 \); \( m_8 = 0.04 \text{ kg} \); \( I_{im} = 15 \text{ g} \cdot \text{cm}^2 \); \( h_{im} = 0.045 \text{ m} \); \( l_{im} = 0.12 \text{ m} \) (impeller); \( B = 6 \times 10^{-7} \); allowable coefficient of rotation irregularity \( [\delta] = 0.05 \). The eccentricities of the first and second pairs of elliptical wheels are respectively \( e_1 = 0.3 \) and \( e_2 = 0.4 \), semi-major axes of elliptical wheels \( a = 25 \text{ mm} \). Input shaft of actuator driven by a motor, which rotating speed \( \omega_1 = 52 \text{ rad} / \text{s} \) (\( n_1 = 500 \text{ rpm} \)).

A study of the dynamic model is performed using the energy-mass method [12, 13], which is widely used in the dynamic analysis of machines. In accordance with the selected method we will find the increment of the kinetic energy \( \Delta T \):

\[ \Delta T = A_d - A_r, \]  

(33)

where \( A_d \) is work of driving forces, \( A_r \) is work of resistant forces. Works in (33) are determined as:

\[ A_d = M_d \cdot \varphi_1', \]  

(34)
\[ M_d = \frac{1}{2\pi} \int_0^{2\pi} M_\varphi d\varphi, \]
(35)
\[ A_r = \int_0^\varphi M_\varphi d\varphi, \]
(36)

For investigated RRST we construct graphs of \( A_r, A_d, \Delta T \) (Fig. 8).

Angular velocity of reduction link is determined as [13]:
\[ \omega_\Lambda = \sqrt{\frac{2\Delta T - C_{\text{max}} - C_{\text{min}}}{I_r}}, \]
(37)
where:
\[ C_{\text{max}} = \Delta T - \frac{1}{2} I_r \omega_{\text{avr}}^2(1 + [\delta]); \]
\[ C_{\text{min}} = \Delta T - \frac{1}{2} I_r \omega_{\text{avr}}^2(1 - [\delta]). \]

Using (37) and the calculation results (Fig. 8), we construct a graph \( \omega_\Lambda(t) \) (Fig. 9).
Figure 9. Graph of function $\omega_a(t)$. 

The graph shows that the reduction link angular velocity is not constant and varies around the average value. Velocity oscillations are determined by the intracyclic changes in gear ratio of mechanism with elliptical gearwheels and force changes on impeller. Since the angular velocity of reduction link is variable, then we define coefficient of rotation irregularity $\delta$ [13]:

$$\delta = \frac{\omega_{\text{max}} - \omega_{\text{min}}}{\omega_{av}}.$$  \hspace{1cm} (38)

It is seen from Figure 9 that $\omega_{\text{max}} = 65.8 \text{rad/s}$, $\omega_{\text{min}} = 43.7 \text{rad/s}$, $\omega_{av} = 52 \text{rad/s}$. Then coefficient of irregularity $\delta = 0.42$. It can be seen that the found coefficient does not satisfy the previously allowable coefficient $|\delta| = 0.05$. Therefore, it is necessary to install the flywheel in the investigated RRST.

According to [13], flywheel moment of inertia $I_f$ will be defined as:

$$I_f = \frac{C_{\text{max}} - C_{\text{min}}}{[\delta]} \cdot \omega_{av}^2.$$  \hspace{1cm} (39)

Substituting in (39) $C_{\text{max}}$, $C_{\text{min}}$, $[\delta]$ and $\omega_{av}$, we find the required flywheel moment of inertia. Knowing the flywheel, we find the angular velocity of reduction link [13]:

$$\omega_i = \sqrt{\frac{I_f \cdot \omega_{av}^2 (1 + [\delta]) - 2(C_{\text{max}} - \Delta T)}{I_f + I_r}}.$$  \hspace{1cm} (40)

Angular velocity of reduction link for RRST is shown in Figure 10.
Figure 10. Graphs of functions $\omega(t)$ with and without flywheel.

As can be seen from the graphs, the installation of a flywheel reduced rotation irregularity of reduction link. Coefficient of rotation irregularity decreased to allowable value $[\delta] = 0.05$.

4. Conclusion
In this paper we construct and investigate dynamic model of the stirred tank with a rotationally reciprocating motion of impellers. As the actuator of such machines, the new planetary mechanism for converting rotational motion into reciprocating rotational with elliptic gears was proposed. The investigations produced the following results:
- the reduced link (input shaft of the mechanism) law of motion was found;
- we calculated the value of the drive moment, which is required for the stirred tank operation;
- we determined flywheel moment of inertia, which is necessary to reduce coefficient of rotation irregularity to allowable value.

The resulting mathematical model can be used in the calculations, design and investigation of stirred tanks with rotationally reciprocating motion of impellers.

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