Heat and work distributions for mixed Gauss–Cauchy process

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Abstract. We analyze energetics of a non-Gaussian process described by a stochastic differential equation of the Langevin type. The process represents a paradigmatic model of a nonequilibrium system subject to thermal fluctuations and additional external noise, with both sources of perturbations considered as additive and statistically independent forcings. We define thermodynamic quantities for trajectories of the process and analyze contributions to mechanical work and heat. As a working example we consider a particle subjected to a drag force and two statistically independent Lévy white noises with stability indices $\alpha = 2$ and $\alpha = 1$. The fluctuations of dissipated energy (heat) and distribution of work performed by the force acting on the system are addressed by examining contributions of Cauchy fluctuations ($\alpha = 1$) to either bath or external force acting on the system.

Keywords: transport processes/heat transfer (theory), fluctuations (theory), stochastic processes
1. Introduction

Superlinear and sublinear diffusional transport and nonexponential relaxation kinetics are ubiquitous in nature and have been observed and analyzed in a number of systems ranging from hydrodynamic flows and plasma physics to transport in a crowded environment inside cells and price variations in the economy [1]. Usually, a starting point of description of systems exhibiting anomalous transport properties involves use of continuous time random walk (CTRW) and fractional differential diffusion equations of the Fokker–Planck type (FFPE). Although both concepts turned out to be extremely powerful in applications, there is still lack of full understanding of their links with first principles of mechanics and thermodynamics [2–4].

The celebrated Smoluchowski–Fokker–Planck equation [5] provides a tool to quantify propagation of randomness in stochastic nonlinear dynamical systems and in a standard derived from the stochastic differential equations driven by Gaussian noises. However, by virtue of the generalized central limit theorem put forward by Paul Lévy [6], the Gaussian distribution is a special case of more fundamental ‘stable statistics’ which form the class of distributions possessing self-scaling properties under convolution. In particular, summing independent random variables having this distribution results in a random variable with a similar distribution to the original ones. This attraction property of stable distributions
is characterized by the stability index $\alpha$, ($0 < \alpha \leq 2$) which describes the exponent of the power-law of probability distribution tails: $p(\xi)d\xi \propto |\xi|^{-(1 + \alpha)}d\xi$. In analogy to the Wiener process representing realization of diffusive Brownian motion, the Lévy process $\{X(t), t \geq 0\}$ can be then described as a continuous, homogeneous stochastic process with independent increments sampled from the stable Lévy distribution and converging to the common Gaussian case for $\alpha = 2$. Accordingly, formal time derivatives of Lévy processes can be understood as examples of white (Markov) noises with (in general) non-Gaussian ($\alpha \neq 2$) distributions of jumps. In consequence, similarly to the correspondence between Smoluchowski–Fokker–Planck equation and stochastic differential equations driven by Gaussian noises, derivation of fractional Fokker–Planck equation [1, 7] establishes the link between the Langevin-type equation with non-Gaussian stable white noise and temporal evolution of probability distribution of the ensemble particles whose motion it describes.

Among various problems addressed in the field of Lévy-noise driven dynamics is the origin of superdiffusive transport in momentum space investigated in a number of studies [8–11]. In particular, based on the Langevin model with linear friction proportional to velocity and non-Gaussian noise, superdiffusive transport in velocity space for a magnetized plasma has been analyzed [8]. The resulting FFPE has been further shown to describe the evolution of the velocity distribution function of strongly nonequilibrium and hot plasmas obtained in tokamaks [10].

On the other hand, anomalous transport is also relevant for analysis of signal transmission and detection in biological systems [12, 13]. In this field Brownian-ratchet models provide mechanisms by which operation of biomolecular motors is investigated [14–16]. In recent analysis it has been shown that the minimal setups of simple ratcheting potentials and additive white symmetric Lévy (non-Gaussian) noise are sufficient to produce directional transport [17–20] and inversion of currents can be obtained by considering a time periodic modulation of the chirality of the Lévy noise [21–23].

Here we continue this line of research and analyze thermodynamic energetics of a linear system subject to Lévy fluctuations. We assume that the motion of a test particle confined by a harmonic potential is described by a Langevin equation

$$\dot{x}(t) = -a(x(t) - vt) + \sqrt{2}\sigma_G(t) + \gamma\xi_G(t).$$

(1)

In the above equation $x$ represents a position at time $t$, $a$ is a parameter related to the strength of the harmonic potential and friction coefficient, and $\xi_G(t)$, $\xi_C(t)$ stand for independent white Gaussian ($\alpha = 2$) and Cauchy ($\alpha = 1$) stable noises. Since the dynamics is analyzed in the overdamped limit, the particle has fully equilibrated (Maxwellian distributed) velocities and its internal energy is solely given by the potential energy which includes a comoving frame, i.e. $U(x, t) = (x(t) - vt)^2/2$. Models of that type with only Gaussian noise term entering equation (1) have been previously successfully applied to describe motion of colloidal particles in optical tweezers and unfolding of biopolymers [15, 24–26]. There have also been test arrangements used in asserting the character of fluctuation theorems for measurable quantities and their symmetry. For thermostated dynamical systems of that type, interacting with thermal reservoirs, generalization of the fundamental second law of thermodynamics has led to derivation of so-called fluctuation theorems (fluctuation relations) [27]. The latter express large-deviation symmetry properties of the probability density functions for physical observables considered in nonequilibrium situations. They have been thoroughly analyzed for
systems operating close to equilibrium, when fluctuations of thermodynamic variables follow the Gaussian law of statistics. Extensions to non-Gaussian white noise and correlated Gaussian fluctuations have been also recently addressed [27–30]. In particular, for linear systems perturbed by Lévy white noise, it was found that the ratio of the probabilities of positive and negative work fluctuations of equal magnitude behaves in an anomalous nonlinear way, developing a convergence on the one for large fluctuations [27, 29, 30]. Hence, negative fluctuations of the work performed on the particle are just as likely to happen as large positive work fluctuations of equal magnitude. This unusual property strongly departs from the Gaussian case, where the validity of a standard FR implies that positive fluctuations are exponentially more probable than negative ones. Our analysis further generalizes those findings and refers to a situation when the source of fluctuations is modeled by two independent white noises: Cauchy noise of intensity $\gamma$ and Gaussian noise of intensity $\sigma$, both entering the system as random additive forces.

The paper is organized as follows: section 2 provides a brief recapitulation of the thermodynamic interpretation of a standard Langevin equation, in which the source of random forces is represented by a generalized derivative of a Gaussian Wiener process (white noise). In section 3 we devise analysis of thermodynamic potentials based on the concept of the ensemble average of the system trajectories, i.e. utilizing the probability density function of a corresponding Fokker–Planck–Smoluchowski equation. The dynamic behavior of the nonequilibrium system is further investigated in sections 4 and 5 in terms of fluctuating thermodynamic components of energy such as heat, mechanical and dissipated work and internal energy. Crucial for this analysis is the statistical balance of total energy (section 6) and fluctuation relations for thermodynamic quantities, discussed in section 7. Section 8 brings about a summary of the results.

2. Thermodynamic description: the entropy production in a system subjected to external driving forces

Following derivation of de Groot and Mazur [31–33], we first note that the external field $\mathbf{F}$ acting on the system increases its internal energy $U$ by an amount $dU = \mathbf{F} \cdot dx$, where $x$ denotes a state variable coupled to the action of the $\mathbf{F}$ force. The differential of entropy can be expanded in variables $x$ and $U$ (both assumed to be independent variables) yielding

$$dS = \left( \frac{\partial S}{\partial U} \right)_x dU + \left( \frac{\partial S}{\partial x} \right)_U dx = \left( \frac{\mathbf{F}}{T} - g \cdot x \right) dx \quad (2)$$

In the linear regime of nonequilibrium thermodynamics we identify stationary states of the system with thermal equilibrium for which probability $p(x)dx$ that a system is in a state for which $x$ takes on values between $x$ and $x + dx$ can be expressed as $p(x)dx = C \exp[S(x)/k_B] = C \exp[-\frac{1}{2}k_B^{-1} \sum g_{ij} x_i x_j]$. Accordingly, $g$ stands for the matrix whose elements are $g_{ij} = (\partial^2 S/\partial x_i \partial x_j)_U$ and we introduce $X_i = \frac{\partial S}{\partial x_i} = -g \cdot x$, an intensive thermodynamic state variable (generalized thermodynamic force) coupled to an extensive variable $x$. For driving forces which are constant in time, the linear response [11] allows us to establish a relation for the mean value $\langle x(t) \rangle = \hat{X}(0) \mathbf{F}$ which is identified with the most probable value of $x$ in the stationary ensemble [31]. Based on equation (2), the entropy production in the system is described by
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\[ \frac{dS}{dt} = \left( \frac{F}{T} + X \right) dx. \]  

By use of the the first and second laws of thermodynamics expressed for quasi-stationary processes, the entropy production under constant temperature can be rewritten as

\[ \frac{dS}{dt} = \frac{1}{T} \frac{d}{dt} \left\{ [dU - DW + p dV - \mu dN] = \frac{1}{T} \frac{d}{dt} \{dU - DW\} \right\}_{V,N}, \]

where the symbol \(DW\) (and in the forthcoming paragraphs, \(DQ\)) has been used to denote the general Pfaffian differential forms (to be distinguished from the exact differential, which they can be turned into by finding an appropriate integrating factor). As an exemplary case, we consider further time variations of the variable \(x\) described by the Langevin equation for the one-dimensional damped harmonic oscillator of a unit mass:

\[ \ddot{x} = -\Gamma \dot{x} - kx + F + \xi_G(t). \]  

Here \(\Gamma\) stands for the friction constant related to the correlation function of Gaussian white noise

\[ \langle \xi_G(t) \xi_G(t') \rangle = 2\Gamma k_B T \delta(t - t'). \]  

In accordance with equation (3), the thermodynamic force coupled to the coordinate \(x\) is of the Hooke’s law type \(X = -kx/T\). Examination of time variations of mechanical energy of the system yields:

\[ \frac{d}{dt} \left( \frac{1}{2} x^2 + \frac{k}{2} \dot{x}^2 \right) = \left[ -\Gamma \dot{x} + F + \xi_G(t) \right] \frac{dx}{dt}. \]

By combining equations (7) and (5), we identify heat transferred from the system to the heat bath as

\[ -DQ \equiv \left( \Gamma \frac{dx}{dt} - \xi_G(t) \right) dx, \]

and work performed on the system \(DW \equiv Fdx\). In the overdamped limit, the equation of motion for variable \(x\) takes on the form

\[ \dot{x} = -\frac{k}{\Gamma} x + \frac{F}{\Gamma} + \frac{1}{\Gamma} \xi_G(t), \]

so that \(1/\Gamma = \tau\) represents the time scale of relaxation of velocities to the stationary state. Accordingly, the energy conservation law can be expressed by multiplying the forces in equation (9) by an incremental change of the state \(dx\) and identifying heat discarded by the system into the bath \(DQ\) (or heat transferred from the bath into the system, \([25, 34]\)), internal energy \(dU\) and work performed by the external force \(dW\):

\[ (\Gamma \dot{x} + \xi_G(t) - kx + F) dx = DQ - dU + dW = 0 \]
In the absence of additional Cauchy random force, equation (9) transforms ($\gamma = 0$) into (1) by identifying $a = k\Gamma^{-1}$, $\xi(t)\Gamma^{-1} = \sqrt{2}\sigma \xi(t)$ and $F\Gamma^{-1} = avt = k\Gamma^{-1}vt$. Now, the Hooke’s force stems from the potential pulled with a constant velocity $v$. In a comoving frame $y = x - vt$, the model system equation (1) can be further reduced to

$$\dot{y}(t) = -ay - v + \sqrt{2}\sigma \xi_C(t) + \gamma \xi_C(t)$$  
(11)

with the coefficient $a$ established by the relation $a = k/\Gamma$. Consequently, for negligible Cauchy noise contribution ($\gamma = 0$), in line with the definition equation (3) the instantaneous entropy production can be defined at the trajectory level and written in the form of

$$\dot{S} = -\frac{1}{T}[ay(\dot{y} + v)],$$  
(12)

with the heat transferred to the system identified as

$$Q = T\int dt \dot{S}(\dot{y}, y)$$  
(13)

where $T$ stands for the ambient temperature. When averaged over an ensemble of trajectories (i.e. with respect to the PDF $p(x, t) = \langle x(t) - x\rangle_{\xi_C(t)}$), equations (12) and (13) reflect an increase of the entropy in the medium $\dot{S}_m = -\frac{1}{T}\langle ay(\dot{y} + v)\rangle_{p(x, t)}$, in line with the definition of stochastic energetics [25, 34]. It has to be stressed that the problem of studying the action of additive Gaussian white noise on a Brownian particle, as described by a standard Langevin equation, has a well documented universal statistical physics approach which, by use of the fluctuation-dissipation theorem, relates thermal noise intensity to the friction coefficient of temperature of the system [25, 31, 34]. This is however, not the case of white noise with a heavy tailed distribution of impulse intensities which are qualitatively very different from standard noise characterized by probability distribution densities with finite cumulants (moments) [27–30]. In contrast to regular, Gaussian-like noise, the variance of more general Lévy noise is infinite and its power spectral density does not exist. Clearly, this feature violates the classical fluctuation-dissipation theorem and obscures thermodynamic interpretation of the stochastic Langevin equation by posing dubiety on the definition of the temperature of the heat bath and its relation to fluctuations imparted to the system [11, 25, 31, 35]. Such censure of Lévy fluctuations is clearly legitimate if the noise is treated as internal (arising from the environment directly surrounding the test particle). If, on the other hand, the Lévy noise is considered to be external, the time dependent driving force, the friction and external noise are decoupled and the fluctuation–dissipation relation equation (6) has to be fulfilled solely by the thermal (internal) part of the noise.

In what follows we will reconsider contributions to entropy production in a linear system driven by independent Gauss and Cauchy noises by analyzing, instead of direct use of the relation equation (13), the balance of energy with fluctuating work and heat being exchanged in the system.
3. Irreversible thermodynamics for continuous time stochastic Markov process

Time evolution of the probability density function (PDF) of the system described by equation (11) is associated with the fractional FFPE:

$$\frac{\partial p(y, t)}{\partial t} = -\frac{\partial}{\partial y}[-a y - v]p(y, t) + \sigma^2 \frac{\partial^2}{\partial y^2}p(y, t) + \gamma \frac{\partial}{\partial[y]}p(y, t), \tag{14}$$

which can be derived by use of the cumulant generating function of the motion defined by the Langevin equation [7, 8]. Here, the fractional derivative is understood in the Riesz-Weyl sense [7] and defined by its Fourier transform

$$\mathcal{F}\left[\frac{\partial}{\partial x^\alpha}f(y)\right] = -|k|^\alpha \mathcal{F}[f(y)].$$

Consequently, equation (14) has the following Fourier representation:

$$\frac{\partial \hat{p}(k, t)}{\partial t} = -ak \frac{\partial}{\partial k} \hat{p}(k, t) + ikv \hat{p}(k, t) - \sigma^2|k|^2 \hat{p}(k, t) - \gamma |k| \hat{p}(k, t), \tag{15}$$

where $\hat{p}(k, t) = \mathcal{F}[p(y, t)]$. Note, that in equation (14) the strength of the white Gaussian noise $\sigma^2$ is related to the temperature of the thermal bath via the usual relation $\langle \xi_G(t)\xi_G(t') \rangle = 2k_B T \delta(t - t') = 2\sigma^2 \delta(t - t')$ with the viscous drag coefficient $\Gamma = 1$ referring to the time units $t = 1/\Gamma$. By contrast, the intensity $\gamma$ of the external fluctuating force $\xi_C$ is an arbitrary parameter of the model.

Due to linearity of the equation (11), the PDF of the process attains the form of the convolution of Lévy distributions (with unknown, so far, time-dependent parameters). The corresponding characteristic function is then:

$$\hat{p}(k, t) = e^{ik\mu(t) - \sigma^2(t)k^2 - \gamma(t)k}. \tag{16}$$

Using this ansatz and FFPE (equation (14)) we can easily obtain evolution equations for parameters:

$$\dot{\mu}(t) = -a \mu(t) + v \tag{17}$$

$$\dot{\gamma}(t) = \gamma - a \gamma(t) \tag{18}$$

$$\frac{d\sigma^2(t)}{dt} = \sigma^2 - 2a\sigma^2(t). \tag{19}$$

Accordingly, the stationary solution to the FFPE subject to conditions $\dot{\mu}(t) = \dot{\gamma}(t) = 0$ for a constant $v$ attains the form [35]:

$$\mu_\infty = \lim_{t \to \infty} \mu(t) = \frac{v}{a} \tag{20}$$

$$\gamma_\infty = \lim_{t \to \infty} \gamma(t) = \frac{\gamma}{a} \tag{21}$$

$$\sigma^2_\infty = \lim_{t \to \infty} \sigma(t) = \frac{\sigma^2}{2a} \tag{22}$$
Stationary PDF can be expressed by a reverse Fourier transform of the characteristic function:

\[ \hat{p}_s(k) = e^{-\sigma_s^2 |k|^2 - \gamma_s |k| + ik \cdot y} \]

\[ p_s(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \hat{p}_s(k) e^{-iky}. \] (23)

Although it cannot be expressed in terms of elementary functions, for \( v = 0 \) it can be rewritten [11] in a closed form by use of the Faddeeva function \( w(x) = e^{-x^2} \text{erfc}(-ix) \) with \( \text{erfc}(x) \) standing for the complementary error function. Accordingly, the stationary PDF solution takes on the form

\[ p_s(y) = \frac{1}{2\sqrt{\pi} \sigma_\infty} \Re w \left( \frac{-y + i\gamma_\infty}{2\sigma_\infty} \right) \] (24)

and assumes a Gibbsian, equilibrium profile with well defined first and second moments only in the absence of the Cauchy-noise contribution, i.e. for \( \gamma = 0 \). Despite that observation, for the Markovian system described by equation (11), a general version of the H-theorem has been shown to hold [36], after identifying the H-functional with the relative entropy. Considering a correspondence with stochastic energetics under Gaussian noise [34, 37–39], one is prompted to define a fluctuating entropy function following the Shannon–Gibbs prescription \( S[p(y, t)] = -\int p(y, t) \log p(y, t) dy \). By an analogy to the Gibbsian equilibrium state the functional \( \Phi(y) = -T \log p_s(y) \) can be then interpreted as a nonequilibrium pseudo-potential [40, 41] with equivalents of internal and free energies of the system given by

\[ U = \langle \Phi(y) \rangle_{p(y, t)} = -T \int p(y, t) \log p_s(y) dy \] (25)

and

\[ F = U - TS = \langle \Phi(y) \rangle_{p(y, t)} - T \langle \log p(y, t) \rangle_{p(y, t)}, \] (26)

respectively. It follows then that all transient solutions to equation (14) tend towards the stationary nonequilibrium state described by \( p_s(x) \). It should be stressed however, that such thermodynamic analogy lacks full integrity with statistical interpretation of thermodynamic quantities and their relation to variations of entropy and precise meaning of the stationary solution.

The interplay of Gaussian and Lévy noises in the dynamics of the linear system equation (11) results in scaling properties of the PDF \( p(y, t) = 1/C(t) \tilde{p}(y/C(t)), \) cf [11]. Accordingly, the dynamic entropy

\[ S[p(y, t)] = -\int \tilde{p}(z) \log \tilde{p}(z) dz + \log C(t) \quad z = y/C(t) \] (27)

increases in time and its production attains the form

\[ \frac{dS}{dt} = \frac{\dot{C}(t)}{C(t)}. \] (28)
In particular, for systems driven by Lévy white noise described by the stability index $\alpha$, the variation of entropy is governed by a scaling function $C(t) \propto t^{1/\alpha}$, i.e., $S[p(y, t)] = 1/\alpha \log t + \text{const}$ and accordingly,

$$\frac{dS}{dt} = (\alpha t)^{-1},$$

so that for increasing $\alpha$ the entropy rate decreases [42]. This observation has been further used as a concept in diffusion entropy analysis, DEA applied among others, to characterize the statistical properties of the natural time series (as e.g. the dynamics of electroencephalogram recordings [43]) and maximizing information exchange in complex networks [44]. For Lévy jump diffusion$^3$ in a force field, Vlad et al [36] performed relaxation analysis of a Lyapunov function—equivalent to the entropy in equation (27)—and discussed existence and stability of the nonequilibrium steady state. In a different context, a general discussion on ‘measures of irreversibility’ and time evolution of relative Shannon and Tsallis entropies constructed for PDFs governed by space-fractional differential operators has been initiated by Prehl et al [42] who investigated the ‘entropy production paradox’ (i.e. aforementioned decrease of the entropy rate with increasing scaling parameter $\alpha$) for anomalous superdiffusive processes. All those studies point to the significant differences of statistical properties of systems driven by Brownian and Lévy noises while stressing uncertainty of rigorous thermodynamic interpretation of entropy.

Accordingly, in the forthcoming sections we do not adhere to a specific formulation of nonequilibrium entropy. As an alternative, we choose direct statistical analysis of fluctuations in work performed on the system and heat transferred to the environment, thus providing the insight into the analogue of entropy production in systems operating close to equilibrium.

4. Heat and work distribution in the presence of Gaussian noise ($\gamma = 0$)

Let us first review the heat and work distribution function for a generic Brownian particle system in a confining harmonic potential and in contact with a heat reservoir. Such a scenario is used as a typical model of manipulated nano-systems (e.g. motion of colloidal particles in optical tweezers) and biomolecules (unfolding of biopolymers) [15, 24, 25]. Analytical results can be easily obtained in a steady state regime when the harmonic potential is dragged for a long time $t_0 >> 1$ before any observables of interest have been measured. In such a case the position of the particle is a Gaussian random variable with corresponding mean and variance given by

$$\langle y_0 \rangle = -\frac{\nu}{a}$$

$^3$ Lévy white fluctuations have been represented in this case by a compound Poisson process $\xi(t) = \sum_{m}^{N} \eta_m(t)$ with the number of summands $N$ given by a Poisson distribution and selfsimilar statistics of impulse intensities $f(\eta)$

$$d\eta = \text{const} \times |\eta|^{1+\alpha} d\eta, 2 > \alpha > 0.$$
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\[ \langle y_0^2 \rangle - \langle y_0 \rangle^2 = \frac{\sigma^2}{2a}. \]  
\[ (31) \]

In the absence of dragging \((v = 0)\) no mechanical work is done on the system. Therefore, dissipated heat is equal to the internal energy change of the particle:

\[ Q_t = U_t - U_0 = \frac{a}{2}(y_t^2 - y_0^2). \]  
\[ (32) \]

In order to derive the formula for heat PDF we start from the characteristic function:

\[ G_Q(k) = \langle e^{ik} \rangle = \int_{-\infty}^{\infty} dy_t \int_{-\infty}^{\infty} dy_0 e^{ik(y_t^2 - y_0^2)/2} p(y_t, y_0) \]  
\[ (33) \]

where \(p(y_t, y_0) = p(y_t | y_0)p(y_0)\) is a joint PDF expressing probability of finding the particle at a position \(y_0\) at time \(t = 0\) and finding it at \(y_t\) at time \(t\). The propagator is given by

\[ p(y_t | y_0) = \frac{1}{\sqrt{2\pi a^2(1 - e^{-2at})}} e^{-\frac{(y_t - y_0 e^{-at})^2}{2a^2(1 - e^{-2at})}}. \]  
\[ (34) \]

After integration the closed-form formula for the heat characteristic function reads

\[ G_Q(k) = \frac{1}{\sqrt{1 + \sigma^2(1 - e^{-2at})}k^2} \]  
\[ (35) \]

and its inverse Fourier transform yields a final result

\[ P_Q(q) = \frac{1}{\pi \sigma^2 \sqrt{1 - e^{-2at}}} K_0\left(\frac{q}{\sigma^2 \sqrt{1 - e^{-2at}}}\right) \]  
\[ (36) \]

where \(K_0(x)\) stands for the zeroth order modified Bessel function of the second kind. This result is complementary to former derivations: in [45] the Fourier transform of the distribution of heat fluctuations is obtained for a constant velocity \(v\) by observing that \(W_t\) is Gaussian distributed, while \(Q_t = W_t - U(x, t)\), through definition of \(U(x, t)\) is quadratic in trajectory \(x\). In turn, works [46, 47] have presented long-time heat PDF under assumption of equilibrium (Boltzmann) distribution of initial positions and \(v = 0\). Here the time-dependent PDF for the heat equation (36) has been derived by assuming stationary Gaussian form of \(p(x_0)\) with parameters equations (30) and (31).

In the case when \(v \neq 0\) mechanical work performed on the system is given by the formula:

\[ W_t = -av \int_0^t ds (x_s - vs) = -av \int_0^t ds y_s. \]  
\[ (37) \]
Again, due to linearity of the Langevin equation governing evolution of $y(t)$ and the above definition, at any instant of time $W_t$ is a Gaussian random variable with parameters:

$$\langle W_t \rangle = v^2 t$$

$$\langle W_t^2 \rangle - \langle W_t \rangle^2 = \frac{2v^2 \sigma^2}{a} (at + e^{-at} - 1).$$

Figures 1 and 2 display congruence between analytical and numerical results, thus validating the numerical algorithm used in our evaluation of heat and work distributions under action of combined Gauss and Cauchy noises.

5. Analytical solutions for $\gamma \neq 0$

Mechanical work done by the potential dragging force is still given by equation (37). By using the method of characteristic functional [29, 30, 48–50] of the process $x_t$, we derive (for details, see appendix) the characteristic function of work distribution $G_W(w) \equiv \langle \exp(iwW_t) \rangle :$

$$\ln G_W(k) = iv^2kt - \sigma^2 v^2 k^2 \left(t + \frac{1}{a} (e^{-at} - 1)\right) - \gamma v|k|t.$$ (40)

We can also obtain a similar result for a non-steady state, e.g. for $x_0$ not fulfilling the requirements equations (30) and (31). For initial condition sampled from the delta distribution $p(x, t = 0) = \delta(x - x_0)$ one obtains
The asymptotic form of the work PDF $P(W_t)$ is governed by tails of the Cauchy distribution behaving like $|W_t|^{-2}$. Note that, as explained in the
following section, our calculations refer to the situation when additive Gaussian and Cauchy noises acting on the system are assumed here to be generated by the surroundings and do not contribute to the definition of mechanical work [11, 22, 23].

6. Numerical schemes used for generating distributions of heat and work

Unlike Gaussian noise, which is pertinent to situations close to equilibrium, where fluctuations of the components of the Gibbs entropy are governed by Gaussian law, more general infinitely divisible probability laws are known to account accurately for long-range interactions and anomalous fluctuations of physical observables as encountered in strictly non-equilibrium complex systems [3, 51, 52]. Typical examples are stochastic fields acting on charged particles in plasma and random gravitational forces in clustered systems [3, 9, 10], both described by Holtsmark distributions of energy and velocities exhibiting heavy long tails. In a recent work [11], we have shown that mixed (convoluted) Gauss–Lévy distributions are candidates for describing observed deviations from Maxwell distributions in plasmas and other systems [9, 10]. They are also proper distributions describing effects of energy transfer in radiation resulting in so called Voigt profile (convolution of Doppler and Lorenz line profiles) of spectral lines. As an addition to those works, in the forthcoming paragraphs we present evaluation of work and heat distributions constrained by the assumption that (1) Cauchy noise is an external driving random force acting on the system, otherwise subject to Gaussian (equilibrated bath) fluctuations and (2) Cauchy noise is a component of a nonequilibrium bath. In both cases our major concern is analysis of the balance of energy and energy exchanged between the system and its reservoir.
Following definitions of section 2 expressing stochastic energetics [25] at the level of trajectories described in terms of a Langevin equation, we further investigate thermodynamic quantities like work and dissipated heat by exploring their PDFs. That concept can be studied by formulating discretized versions of corresponding stochastic differential equations which we interpret according to Stratonovich. In particular, for a system subject to internal Cauchy noise, discretized formulas for heat and work read:

- Internal Cauchy noise

\[
\Delta Q_{\text{int}}^t = \frac{1}{\hbar} \sqrt{2\sigma} w_{t,t+h}^G + \gamma w_{t,t+h}^C - (x_{t+h} - x_t) \right) (x_{t+h} - x_t)
\]  

where \( w_{t,t+h}^G \) and \( w_{t,t+h}^C \) stand for (Gaussian- or Cauchy-distributed) increments of the (stationary) Wiener process. Alternatively, heat exchange can be obtained by direct use of the energy balance formula \( \Delta Q = \Delta U - \Delta W \) with the change in internal energy due to the mechanical work given by

\[
\Delta W_t = -k\lambda(x_t - \lambda t)h
\]  

Both methods of evaluation have been tested for the problem at hand and produced identical results. In turn, for a corresponding situation with external Cauchy–Lévy noise driving we refer to a discretization scheme:

- External Cauchy noise

\[
\Delta Q_{\text{ext}}^t = \frac{1}{\hbar} \sqrt{2\sigma} w_{t,t+h}^G - (x_{t+h} - x_t) \right) (x_{t+h} - x_t)
\]

\[
\Delta W_t = \Delta W_i + \gamma w_{t,t+h}^C \frac{x_{t+h} - x_t}{h}
\]  

Note, that in both cases the Cauchy random force does not contribute directly to evaluation of the mechanical work \( W_t \). When considering \( \xi_t(t) \) as an explicit random force acting on the system, we assess the ‘total work’ PDF by analyzing random quantity \( W_t^\xi = W_t + W_t^r \), i.e. we add extra (random) energy \( W_t^r \) from the Cauchy noise to the overall definition of work performed on the system.

In order to achieve convergence for estimated PDF for heat, a correct integration method requires that:

\[
0 = -\frac{dU(x^*)}{dx}h - (x_{t+h} - x_t) + w_{t,t+h}
\]  

where \( x^*_t = \frac{x_t + x_{t+h}}{2} \), i.e. the Stratononich rule of integration has to be applied [25].

Moreover, we assume that a proper non-trivial spatio-temporal structure of environmental fluctuations can be preferentially modeled by a smooth colored noise process which achieves the limit of a somewhat ill-defined idealization, namely the white noise, as the correlation time of the approximation tends to zero. Indeed, according to the Wong–Zakai theorem [53–57] in this limit the smoothed stochastic integral converges.
to the Stratonovich stochastic integral. Naively, one could think that e.g. in order to achieve a white noise limit of the Ornstein–Uhlenbeck process, one should scale the corresponding SDE

$$d\xi(t) = -\frac{\xi}{\epsilon^2}dt + \frac{\sigma}{\epsilon}dW(t) \equiv -\Gamma \xi dt = \delta dW(t)$$

(47)

with the parameter $\epsilon \to 0$. Such scaling yields formula for the correlation function of the process

$$C(|t - s|) \equiv \langle \xi(t)\xi(s) \rangle = \frac{\hat{\sigma}^2}{2\Gamma} \exp(-\Gamma|t - s|) = \frac{\sigma^2 \epsilon^2}{2} \exp\left(-\frac{|t - s|}{\epsilon^2}\right)$$

(48)

with the characteristic correlation time of the noise given by $\Gamma^{-1}$. The spectral density of the rescaled process is then

$$S(\nu) \equiv \frac{1}{2\pi} \int e^{-\nu\tau} C(\tau) d\tau = \frac{\hat{\sigma}^2}{2\pi(\nu^2 + \Gamma^2)} = \frac{\sigma^2 \epsilon^2}{2\pi(\nu^2 \epsilon^4 + 1)}$$

(49)

and tends to 0 for $\epsilon \to 0$ which indicates that we end up with the ‘noiseless’ limit. In contrast, the proper white noise limit can be achieved when at the same time $\sigma \to \infty$ and $\Gamma \to \infty$ in such a way that $\sigma^2 / 2\Gamma \to const$. Under these circumstances the power density $S(\nu)$ tends to a constant which is the signature of white noise. In other words, by taking e.g. the rescaled version of the Gaussian colored noise $\xi_G(t/\epsilon^2)$ in the Langevin equation with multiplicative noise term

$$\dot{x} = a(x) + \frac{f(x)}{\epsilon} \xi_G(t/\epsilon^2),$$

$$\langle \xi_G(t)\xi_G(s) \rangle = \frac{1}{2} \exp\left(-\frac{|t - s|}{\epsilon^2}\right)$$

(50)

in the limit of $\epsilon \to 0$ the result of integration converges weakly to $\hat{x}(t)$ which satisfies SDE

$$d\hat{x} = a(\hat{x}) dt + f(\hat{x}) \circ dw(t)$$

(51)

where $w(t)$ stands for a standard 1-dim Wiener process. Since our analysis of (stochastic) work and heat functionals relies on estimation of multiplicative white noise sources entering equations (42)–(45), we adjust to the aforementioned scheme, in which the white noise limit solutions are achieved as limits of integrals performed with respect to colored noise sources.

Figures 6 and 5 summarize our findings. First, by using the aforementioned definitions of stochastic heat and work (equations (44)–(46), the overall balance of energy at the level of a single realization of the stochastic process is accomplished. Second, distribution of energy at the ensemble level changes profoundly, depending on whether the random Cauchy noise is treated as external forcing contributing to the evaluation of the total work performed on the system, or included in the definition of heat. The mechanical part of the energy can be evaluated analytically and numerical results corroborate correctly with formulae derived in the appendix (cf upper panel of figure 5). On the other hand, estimation of heat or dissipated work can be achieved either by evaluation of stochastic integrals or by using the energy balance formula of the first
law. In both cases, for finite time integration, we obtain almost indistinguishable histograms of derived PDFs.

7. Fluctuation relations

Fluctuations around out-of-equilibrium steady states are frequently described in terms of the probability ratio of a time-integrated observable (like entropy production, heat absorbed by driven Brownian particles or the current of the zero-range processes) evaluated for the forward and reverse changes a system undergoes in course of its evolution. The ratio has been named fluctuation relation (FR) and by definition serves as a measure of the symmetry in the distribution of fluctuations [45, 58–60]. In particular, a typical arrangement of a Brownian particle dragged by a spring through a thermal environment modelled by Gaussian white noises has been exploited both experimentally and theoretically to deduce symmetry of FR. It has been shown [15, 45, 50] that the work performed on the particle satisfies a standard or conventional FR

$$P(W_\tau = w) / P(W_\tau = -w) \approx e^{\rho(w)\tau}$$

with $\rho(w)$ being a linear function of $w$. By contrast, the heat fluctuations follow and extended the fluctuation relation

$$P(Q_\tau = q) / P(Q_\tau = -q) \approx e^{\rho(q)\tau}$$

where the exponential

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Short time ($t = 0.1$, left panel) and long time ($t = 10$, right panel) heat PDFs for the system subject to mixed Gaussian–Cauchy fluctuations. Plots represent PDFs for different intensities $\gamma$ of the Cauchy additive noise. Parameters of the model: $v = 1$, $a = 1$, $t = 0.1$, $\sigma = 1$, correlation time of the noise $\epsilon = 0.01$. Histograms have been collected on $N = 10^6$ trajectories, time step of simulations $\Delta t = 10^{-3}$. Upper panels refer to Cauchy noise included in the bath, equation (42) whereas lower panels depict heat PDFs for external Cauchy noise, equation (44) treated as additional random force acting on the system.}
\end{figure}
weight that relates the probability of positive and negative fluctuations does not scale linearly with the heat variable, i.e. \( \rho(q) \) is a non-linear function of \( q \). As discussed elsewhere [27, 30, 45, 59], conventional or extended FRs are valid when the underlying probability distributions obey (for long times \( \tau \to \infty \)) a large deviation principle \( P(W_\tau = w) \approx e^{-rk(w)} \) with some rate function \( k(w) \). On the other hand, the large deviation principle is violated in self-similar systems, where power law distributions of fluctuations govern the statistics [27, 30]. In such systems the ratio of probabilities of positive and negative work fluctuations of equal magnitude shows anomalous behavior \( (P(W)/P(-W) \approx 1) \) exhibiting the same probability of occurrence for large positive and large negative fluctuations [27].

Figures 7 and 8 display work and heat fluctuations asymmetry for short and long time limit evaluated for internal Gaussian noise, representing fluctuations of the heat bath and an external Cauchy noise standing for additional random forcing. In the analysis, the aforementioned Cauchy noise is assumed as either an additional noise in the surroundings or a random force contributing to the definition of work \( \Delta W_t \) performed on the system, see equation (45).

For a linear system driven solely by Gaussian fluctuations represented by an additive white noise, conventional fluctuation relation is detected, in line with the large deviation probability [29, 45, 50]. By contrast, in the case when the system
Figure 7. Work fluctuation (a)symmetry for short time ($t = 0.1$, left panel) and long time ($t = 10$, right panel). Increments of the mechanical work have been evaluated according to equations (43) and (45). Histograms collected on $N = 10^6$ trajectories, $a = 1$, $v = 1$, time step of simulations $\Delta t = 10^{-2}$. Solid lines represent analytical results obtained by use of equation (40).

Figure 8. Heat fluctuation (a)symmetry for short time ($t = 0.1$, left panel) and long time ($t = 10$, right panel). Histograms collected on $N = 10^6$ trajectories, $a = 1$, $v = 1$, time step of simulations $\Delta t = 10^{-2}$.
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becomes perturbed by additional Cauchy noise, FR deviates from the symmetric form $P(W_t)/P(-W_t) \approx e^{4 rw}$ and the asymmetry increases with the strength of the Cauchy component in external forcing. For stronger Lévy noise, corresponding PDFs do not decay exponentially (reflecting loss of the large deviation principle) and the work fluctuation relation assumes the anomalous limit $P(W_t)/P(-W_t) \approx 1$. The location of the maximum of the $P(W)/P(-W)$ ratio depends on the intensity of the Cauchy noise and time of observation: the asymmetry is linear in $W$, it deviates from linearity for increasing values of $W$ and then saturates to a constant value. For longer times ($t = 10$), the turnover of the $P(W_t)/P(-W_t)$ ratio is observed at higher values of $W$ and the slope of the linear constituent in the ratio is bigger than for short times. Altogether, these observations stay in compliance with the character of the PDFs of increments $\Delta W$ where domination of very large fluctuations leads to the aforementioned asymmetry.

Inclusion of the Cauchy white noise into definition of work performed on the system equation (45) results in heavy tailed, asymmetric PDF of work (see figure 5) resembling the mirror-image of heat PDF for system subject to mixed Gaussian-Cauchy fluctuations in the bath (see figure 6). Also in this case, the fluctuation ratio $P(W)/P(-W)$ reflects the generic asymmetry of work PDF with violation of the exponential large deviation form and recovery of the transient fluctuation relation $P(W_t)/P(-W_t) \approx e^{4 rw}$ for $\gamma = 0$.

8. Summary

We have examined stochastic linear systems subject to action of two independent stable noises, one of which exhibits a power law decay of large deviations. In a comoving frame our system represents a generalization of the Ornstein–Uhlenbeck process with convoluted Gaussian-Cauchy fluctuations. Following standard interpretations of the thermodynamics at the level of the Langevin equation, we have derived formulae for heat and work distribution in such system for various interpretations of the Cauchy noise which enters dynamics either as a contribution to (nonequilibrated) thermal bath, or as an additional external random driving. In order to conform with Stratonovich rules of integration, for external Cauchy forces we have adopted white-noise limit equations (51) and (51) of a colored noise. For Gaussian additive noises, a standard fluctuation relation implies that positive fluctuations are exponentially more probable than negative ones [2, 29]. This scenario becomes severely affected by the presence of white but otherwise non-Gaussian noise. The interplay of the power-law distributed Lévy fluctuations causes a strong asymmetry of the heat PDF and, in line with previous studies [2, 27] leads to an anomalous structure of the $P(W_t)/P(-W_t)$ ratio.

The infinite power of the Cauchy noise spectrum results in a breakdown of the standard fluctuation-dissipation relationship (FDR) between friction force and correlation function of fluctuations. Nevertheless, when an appropriate conjugate (dynamic) variable is chosen [35] in the form of a derivative of the stochastic entropy in the stationary

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state $X_c = - T \frac{\partial \phi(x)}{\partial f} \bigg|_{f=0}$ with respect to external driving $f(t)$, the proper form of FDR can be recast for sufficiently weak operating noises. This issue, analyzed for the linear system perturbed by Gaussian and Cauchy noises will be the object of our forthcoming studies.

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**Appendix**

Let us recall (see equation (11)) that in the model $w(t) = \int_0^t dt' \xi_G(t')$ stands for a standard Brownian motion (Wiener process) and $w_a(t) = \int_0^t dt' \xi_C(t')$ is a Lévy $\alpha$-stable process with the stability index $\alpha = 1$. We assume that increments of both processes are statistically independent. Our aim is to calculate the characteristic function $G_W(k) = \langle e^{ikW} \rangle$ of the process:

$$W_t = -av \int_0^t ds (X_s - vs) = -av \int_0^t ds y_s. \quad (A.1)$$

We start with the observation that the characteristic functional of the total noise \cite{36, 48, 49} entering equation (11) is given by the relation:

$$G_{\xi}[k] \equiv G_{\sqrt{2} \sigma_\xi \xi_{\xi_G} + \gamma_\xi} = G_{\sqrt{2} \sigma_\xi \xi_{\xi_C}} G_{\gamma_\xi} = e^{-\sigma^2 \int_0^\infty dt |k(t)|^2 - \gamma \int_0^\infty dt |k(t)|}. \quad (A.2)$$

Next, we substitute $\xi = \dot{y}_t + ay_t + v$ in the definition of the characteristic function and integrate the expression with $\dot{y}_t$ by parts:

$$G_{\xi}[k] = \left\langle e^{i \int_0^\infty dt \dot{y}_t + (ay_t + v)} \right\rangle = \left\langle e^{i (k \sigma y_\infty - k y_0) + \gamma \int_0^\infty dt (ak(t) + (ak(t) - k(t))y_0) + \int_0^\infty dt (ak)} \right\rangle. \quad (A.3)$$

Assuming that $y_0 \equiv y_0$ is a number (i.e. $p_{y_0}(y) = \delta(y - y_0)$) and that the function $k(t)$ decays sufficiently fast (securing $k(\infty) = 0$), we arrive at:

$$G_{\xi}[k] = e^{-ik y_0 + \gamma \int_0^\infty dt (ak(t) + (ak(t) - k(t))y_0)} G_{y|y_0}[-k + ak]. \quad (A.4)$$

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In the next step we introduce a new function $m(t) \equiv - \dot{k}(t) + ak(t)$. It is easy to reverse this relation yielding:

$$k(t) = k(0)e^{at} - e^{at}\int_0^t dse^{-as}m(s) = e^{at}\int_t^\infty dse^{-as}m(s)$$

(A.5)

with $k(0) = \int_0^\infty dse^{-as}m(s)$. This identification together with equation (A.4) allows us to express the $G_{y|y_0}[m]$ in terms of the known noise characteristic functional:

$$G_{y|y_0}[m] = e^{i\int_0^\infty dse^{-as}m(s)z_0 - i\int_0^\infty dte^{at}\int_t^\infty dse^{-as}m(s)} \times G_{\zeta}[e^{at}\int_t^\infty dse^{-as}m(s)].$$

(A.6)

Accordingly,

$$\ln G_{y|y_0}[m] = i\int_0^\infty dse^{-as}m(s)y_0 - iv\int_0^\infty dte^{at}\int_t^\infty dse^{-as}m(s)$$

$$- \sigma^2\int_0^\infty dt|e^{at}\int_t^\infty dse^{-as}m(s)|^2 - \gamma\int_0^\infty dt|e^{at}\int_t^\infty dse^{-as}m(s)|$$

(A.7)

Work characteristic functional $G_{W|y_0}(k)$ can be now obtained by choosing a proper test function $m(t')$:

$$m(t') = - \text{avg}\Theta(t - t')$$

(A.8)

where $\Theta(\cdot)$ denotes the Heaviside step function. Indeed, by plugging it into formula $G_{y}[k] \equiv \langle \exp(i\int_0^\infty dsk(s) y_0) \rangle$ one can easily verify that this choice leads to $G_{W}(k)$.

The very last step is to calculate integrals in equation (A.8) with $m(\cdot)$ given by equation (A.8) which leads to the work characteristic function for the dynamics equation (11) with an initial condition $y_0 = x_0$:

$$\ln G_{W|x_0}(k) = iK\left[k^2t - v\left(x_0 + \frac{v}{a}\right)(1 - e^{-at})\right]$$

$$- \gamma v|k|^2\left[1 - \frac{1}{a}(1 - e^{-at})\right] - \sigma^2 v^2 k^2\left[t + \frac{1}{2a}(1 - e^{-2at}) - \frac{2}{a}(1 - e^{-at})\right]$$

(A.9)

In order to get the $G_{W}(k)$ in a steady state regime we have to average $G_{W|x_0}(k)$ over steady state initial conditions at time $t = 0$. It is worthy noticing that $G_{W|x_0}(k)$ can be rewritten as

$$G_{W|x_0}(k) = g(q)e^{-i\xi_0}$$

(A.10)
where \( b = v q (1 - e^{-at}) \) and \( g(q) \) does not depend on \( x_0 \). Thanks to that feature the final formula contains just the Fourier transform of the initial (steady state) PDF:

\[
G_W(k) = \int dx G_{W|x}(k)p_x(x) = g(q)\int dx e^{-ibx}p_x(x) = g(q)\tilde{p}_x(-b),
\]

(A.11)

and

\[
\tilde{p}_x(s) = e^{-\frac{vq}{\pi}s} - \frac{\gamma v}{\pi s} - \frac{\sigma^2 v^2}{2a}s^2.
\]

(A.12)

With the help of equations (A.10)–(A.12) we arrive at the final result, equation (40),

\[
\ln G_W(k) = iv^2kt - \gamma v|k|t - \frac{\sigma^2 v^2}{a}k^2(at + e^{-at} - 1).
\]

(A.13)

References

[1] Klafter J, Metzler R and Lim S C 2011 Fractional Dynamics: Recent Advances (Singapore: World Scientific)
[2] Klages R, Radons G and Sokolov I M 2008 Anomalous Transport: Foundations and Applications (Weinheim: Wiley)
[3] West B J, Geneston E L and Grigolini P 2008 V-Langevin equations, continuous time random walks and fractional diffusion Phys. Rep. 468 1–99
[4] Marconi U M B, Puglisi A, Rondoni L and Vulpiani A 2008 Fluctuation–dissipation: response theory in statistical physics Phys. Rep. 461 111–95
[5] Risken H 1989 The Fokker–Planck Equation: Methods of Solution and Applications (Berlin: Springer)
[6] Lévy P 1937 Théorie de l’addition des Variables Aléatoires (Paris: Gauthier-Villars)
[7] Schertzer D, Larchevêque M, Duan J, Yanovsky V V and Lovejoy S 2001 Fractional Fokker–Planck equation for nonlinear stochastic differential equations driven by non-Gaussian Lévy stable noises J. Math. Phys. 42 200–12
[8] Chechkin A V, Gonchar V Y and Szydlowski M 2002 Fractional kinetics for relaxation and superdiffusion in a magnetic field Phys. Plasmas 9 78–88
[9] Trigger S A 2009 Anomalous diffusion in velocity space Phys. Lett. A 374 134–8
[10] Ebeling W, Romanovsky M Y, Sokolov I M and Valuev I A 2010 Convoluted Gauss-Lévy distributions and exploding Coulomb clusters Eur. Phys. J. Spec. Top. 187 157–70
[11] Kuśmier L, Ebeling W, Sokolov I M and Gudowska-Nowak E 2013 Onsagers fluctuation theory and new developments including non-equilibrium lévy fluctuations Acta Phys. Pol. B 44 859–80
[12] Stauffer D, Schulze C and Heermann D W 2007 Superdiffusion in a model for diffusion in a molecularly crowded environment J. Biol. Phys. 33 305–12
[13] Matthäus F, Jagodić M and Dobnikar J 2009 E-coli superdiffusion and chemotaxis search strategy, precision, and motility Biophys. J. 97 946–57
[14] Bier M 2008 The energetics, chemistry, and mechanics of a processive motor protein BioSystems 93 23–8
[15] Ciliberto S, Joubaud S and Petrosyan A 2010 Fluctuations in out-of-equilibrium systems: from theory to experiment J. Stat. Mech. 2010 P12003
[16] Fiasconaro A, Ebeling W and Gudowska-Nowak E 2008 Active Brownian motion models and application to ratchets Eur. Phys. J. B 65 493–14
[17] Dybiec B and Gudowska-Nowak E 2004 Resonant activation driven by strongly non-Gaussian noises Fluctuat. Noise Lett. 4 L273–85
[18] Dybiec B, Gudowska-Nowak E and Sokolov I M 2007 Stationary states in Langevin dynamics under asymmetric Lévy noises Phys. Rev. E 76 041122
[19] del Castillo-Negrete D, Gonchar V Y and Chechkin A V 2008 Fluctuation-driven directed transport in the presence of Lévy flights Physica A 387 6693–704
[20] Pavlyukevich I, Dybiec B, Chechkin A V and Sokolov I M 2010 Lévy ratchet in a weak noise limit: theory and simulation Eur. Phys. J. Spec. Top. 191 223–37
Heat and work distributions for mixed Gauss–Cauchy process

[21] Ebeling W, Gudowska-Nowak E and Fiasconaro A 2008 Statistical distributions for Hamiltonian systems coupled to energy reservoirs and applications to molecular energy conversion Acta Phys. Pol. B 39 1251–72
[22] Dybiec B and Gudowska-Nowak E 2009 Lévy stable noise-induced transitions: stochastic resonance, resonant activation and dynamic hysteresis J. Stat. Mech. 2009 P05004
[23] Dybiec B 2008 Current inversion in the Lévy ratchet Phys. Rev. E 78 061120
[24] Gustamante C, Liphardt J and Ritort F 2005 The nonequilibrium thermodynamics of small systems Phys. Today 58 43–8
[25] Sekimoto K 2010 Stochastic Energetics (Berlin: Springer)
[26] Reimann P, Van den Broeck C, Linke H, Hänggi P, Rubi J M and Pérez-Madrid A 2001 Giant acceleration of free diffusion by use of tilted periodic potentials Phys. Rev. Lett. 87 010602
[27] Chechkin A V and Klages R 2009 Fluctuation relations for anomalous dynamics J. Stat. Mech. 2009 L03002
[28] Chechkin A V, Lenz F and Klages R 2012 Normal and anomalous fluctuation relations for Gaussian stochastic dynamics J. Stat. Mech. 2012 L11001
[29] Touchette H and Cohen E G D 2009 Anomalous fluctuation properties Phys. Rev. E 80 011114
[30] Touchette H and Cohen E G D 2007 Fluctuation relation for a Lévy particle Phys. Rev. E 76 020101
[31] De Groot S R and Mazur P 1969 Nonequilibrium Thermodynamics (Amsterdam: North Holland)
[32] Sabhapandit S 2011 Work fluctuations for a harmonic oscillator driven by an external random force Chaos Solitons Fractals 47 1057–73
[33] Garbaczewski P and Olkiewicz R 1999 Cauchy noise and affiliated stochastic processes J. Math. Phys. 40 1057–73
[34] Garbaczewski P 2005 Differential entropy and time Entropy 7 253–99
[35] Wang E and Zakai M 1965 On the convergence of ordinary integrals to stochastic integrals Ann. Math. Stat. 36 1560–4
[36] Kupferman R, Pavliotis G A and Stuart A M 2004 Itô versus Stratonovich white-noise limits for systems with inertia and colored multiplicative noise Phys. Rev. E 70 036120

DOI:10.1088/1742-5468/2014/09/P09002
Heat and work distributions for mixed Gauss–Cauchy process

[55] Sussmann H J et al 1978 On the gap between deterministic and stochastic ordinary differential equations Ann. Probab. 6 19–41
[56] Mannella R and McClintock P V E 2011 Comment on ‘influence of noise on force measurements’ Phys. Rev. Lett. 107 078901
[57] Kanazawa K, Sagawa T and Hayakawa H 2012 Stochastic energetics for non-Gaussian processes Phys. Rev. Lett. 108 210601
[58] Gallavotti G and Cohen E G D 1995 Dynamical ensembles in stationary states J. Stat. Phys. 80 931–70
[59] Kurchan J 1998 Fluctuation theorem for stochastic dynamics J. Phys. A: Math. Gen. 31 3719
[60] Jarzynski C and Mazonka O 1999 Feynman’s ratchet and pawl: an exactly solvable model Phys. Rev. E 59 6448