A closed form for Consistent Anomalies in Gauge Theories

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Abstract

The new method for solving the descent equations for gauge theories proposed in [1] is shown to be equivalent with that based on the "Russian formula". Moreover it allows to obtain in a closed form the expressions of the consistent anomalies in any space-time dimension.

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1 Introduction

Consistent anomalies in gauge field theories occur when the gauge invariance cannot be maintained at the quantum level.

In this case the variation of the connected vacuum functional in the presence of external gauge fields does not vanish. This implies that the gauge currents are no longer covariantly conserved but have the anomalies as their divergence. As a direct consequence of its definition the anomaly must satisfy the well known Wess-Zumino consistency conditions [2]. These conditions, when formulated in terms of the BRST transformations [3, 4], yield a cohomology problem for the nilpotent BRST operator $s$

$$s\Delta = 0$$  \hspace{1cm} (1)

where $\Delta$ is the integral of a local polynomial in the fields and their derivatives. An useful and powerful method to find the non-trivial solutions of Eq.(1) is given by the descent-equations technique [5, 6, 7, 8].

Writing $\Delta = \int A$ Eq.(1) translates into the local condition

$$sA + dQ = 0$$  \hspace{1cm} (2)

where $d$ is the exterior differential on the flat space-time $M$, $Q$ is a local polynomial in the fields and

$$s^2 = d^2 = sd + ds = 0$$  \hspace{1cm} (3)

It is easy to check that Eq. (2), due to the vanishing of the cohomology of $d$, generates a tower of descent equations:

$$sA + dQ = 0$$
$$sQ + dQ^1 = 0$$
$$\quad \cdots$$
$$\quad \cdots$$
$$sQ^{k-1} + dQ^k = 0$$
$$sQ^k = 0$$  \hspace{1cm} (4)
with $Q^i$ local polynomials in the fields.

These equations, as it is well known since several years, can be solved by means of a transgression procedure generated by the so-called "Russian formula" [3, 9, 11, 14, 15].

More recently a new way of finding nontrivial solutions of the tower (4) has been proposed by one of the authors and successfully applied to the study of the Yang-Mills cohomology [1] and of the gravitational anomalies [16].

The method is based on the introduction of an operator $\delta$ which allows to express the exterior derivative $d$ as a BRST commutator

$$d = - [s, \delta] \quad (5)$$

It is easy to prove that, once the decomposition (5) has been found, repeated applications of the operator $\delta$ on the polynomial $Q^k$ which solves the last equation of Eqs.(4) give an explicit solution for the other cocycles $Q^i$ and for the anomaly $A$.

It is interesting to observe that the decomposition (5) occurs also in topological field theories [17] and in the bosonic string [18]. For these models the operator $\delta$ is the generator of the topological vector supersymmetry and allows for a complete classification of anomalies and counterterms.

Let us emphasize that solving the last equation of the tower (4) is a problem of local cohomology instead of a modulo-$d$ one. It is apparent that, thanks to the operator $\delta$, the characterization of the cohomology of $s$ modulo $d$ is essentially reduced to a problem of local cohomology. Let us recall also that the latter has been solved in the case of the gauge theories [10, 11, 14, 19] and, more generally, it can be systematically studied by using several methods as, for instance, the spectral sequences technique [20, 21, 22]. We have thus an alternative tool for an algebraic characterization of the cohomology of $s$ modulo $d$. It remains, however, to prove the complete equivalence of the expressions obtained by the decomposition (5) with the ones given by the "Russian formula". This is the aim of this paper, to give an explicit proof of the equivalence of the two methods.

Actually, as we shall see in details, the proof of the equivalence turns
out to be a direct consequence of the vanishing of the cohomology of 
\((d + s)\) and of an elegant and quite simple formula which allows to 
collect into a unique closed equation the solution of the tower \((\mathbb{H})\) for 
any space-time dimension and ghost number. In the following we will 
take as explicit example the case of the Yang-Mills gauge theories, the 
result being easily extended to the gravitational anomalies and to the 
topological models.

The paper is organized as follows. In Sect. 2 we introduce the differen-
tials \(d\) and \(s\) and we briefly recall some properties concerning their 
cohomologies. In Sect. 3 we introduce the operator \(\delta\) and we prove the 
equivalence with the ”Russian formula”. In Sect. 4 we present a simple 
closed expression for the solution of the descent equations. Finally, 
Sect. 5 is devoted to some examples.

## 2 Notations and Conventions

To fix the notations let us introduce the local space \(\mathcal{V}\) of form-
polynomials \([10]\) in the variables \((A, dA, c, dc)\); \(A\) and \(c\) being re-
spectively the one-form gauge connection \(A = A_\mu dx^\mu\) and the zero 
form ghost field. All the fields are Lie algebra valued; \(A_\mu = T_a A^a_\mu\) and 
\(c = T_a c^a\) with \(\{T_a\}\) the antihermitian generators of a semisimple Lie 
group \(G\) in some finite representation. The BRST transformations are

\[
\begin{align*}
  sA &= Dc = dc + [A, c], \\
  sc &= c^2 = \frac{1}{2} [c, c].
\end{align*}
\]

where \([a, b] = ab - (-1)^{|a||b|} ba\) denotes the graded commutator and 
\(|a|\) is the degree of \(a\) defined as the sum of its ghost number and of 
its form degree. \(A\) and \(c\) have both degree one.

The two-form field strength \(F\) reads

\[
F = dA + A^2
\]

and

\[
dF = [F, A]
\]
is its Bianchi identity.

To study the cohomology of $s$ and $d$ on the space of form-polynomials we switch, following [10], from $(A, dA, c, dc)$ to the more convenient set of variables $(A, F, c, ξ = dc)$, i.e. we replace everywhere $dA$ with $F$ by using Eq. (7) and we introduce the variable $ξ = dc$ to emphasize the local character of descent equations (6). On the local space $\mathcal{V}(A, F, c, ξ)$ the action of the differentials $s$ and $d$ is given by

$$sA = ξ + [c, A], \quad sF = [c, F];$$
$$sc = c^2, \quad sξ = [c, ξ]$$

(9)

and

$$dA = F - A^2, \quad dF = [F, A],$$
$$dc = ξ, \quad dξ = 0$$

(10)

In particular, from Equations (9) and (10) it follows that on $\mathcal{V}(A, F, c, ξ)$ both $d$ and $(d + s)$ have vanishing cohomology [10, 13, 23].

This is easily proved by introducing the counting (filtering) operator $\mathcal{N}$

$$\mathcal{N}A = A, \quad \mathcal{N}F = F$$
$$\mathcal{N}c = 2c, \quad \mathcal{N}ξ = 2ξ$$

(11)

according to which the exterior derivatives $d$ and $\hat{d} = (d + s)$ decompose as

$$d = d^{(0)} + d^{(1)},$$

$$[\mathcal{N}, d^{(\nu)}] = νd^{(\nu)} \quad ν = 0, 1$$

(12)

and

$$\hat{d} = \hat{d}^{(0)} + \hat{d}^{(1)} + \hat{d}^{(2)},$$

$$[\mathcal{N}, \hat{d}^{(\nu)}] = ν\hat{d}^{(\nu)} \quad ν = 0, 1, 2$$

(13)

with

$$d^{(0)}A = \hat{d}^{(0)}A = F$$
$$d^{(0)}c = \hat{d}^{(0)}c = ξ$$

(14)
and
\[ d^{(0)} d^{(0)} = 0 \quad \quad \dot{d}^{(0)} \dot{d}^{(0)} = 0 \] (15)

From Eqs. (14) it is apparent that \( d^{(0)} \) and \( \dot{d}^{(0)} \) have vanishing cohomology, it then follows that also \( d \) and \((d + s)\) have trivial cohomology due to the fact that the cohomology of \( d \) (resp.\((d + s)\)) is isomorphic to a subspace of the cohomology of \( d^{(0)} \) (resp.\((d + s)^{(0)}\)) [20, 21, 22]. For what concerns the cohomology of \( s \) it turns out [10, 11, 13, 14, 15, 19] that it is spanned by polynomials in the variables \((c, F)\) generated by elements of the form
\[
\left( \text{Tr} \frac{c^{2m+1}}{(2m+1)!} \right) \cdot P_{2n+2}(F) \quad m, n = 1, 2, \ldots \] (16)

with \( P_{2n+2}(F) \) the invariant monomial of degree \((2n + 2)\):
\[
P_{2n+2}(F) = \text{Tr} F^{n+1}. \] (17)

The Bianchi identity (8) implies that \( P_{2n+2}(F) \) is \( d \)-closed
\[
P_{2n+2}(F) = d\omega_{2n+1}^{0}. \] (18)

Using the triviality of the cohomology of \( d \) one immediately gets the tower of descent equations:
\[
\begin{align*}
sw_{2n+1} + d\omega_{2n}^{1} & = 0 \\
sw_{2n}^{1} + d\omega_{2n-1}^{2} & = 0 \\
\vdots \\
sw_{1}^{2n} + d\omega_{0}^{2n+1} & = 0 \\
sw_{0}^{2n+1} & = 0
\end{align*} \] (19)

where, as usual, \( \omega^{q}_{p} \) denotes a \( p \)-form with ghost number \( q \).

In particular, from Eq. (16) one has that the nontrivial solution of the last equation in (19) corresponding to \( P_{2n+2}(F) \) is given by the ghost monomial of degree \((2n + 1)\)
\[
\omega_{0}^{2n+1} = \text{Tr} \frac{c^{2n+1}}{(2n+1)!}. \] (20)

In what follows we shall assume that the descent equations (19) refer always to the monomials of the basis (16).
3 Equivalence with the Russian formula

In order to solve the descent equations (19) we proceed as in [1] and we introduce two differential operators δ and G defined by

\[
\begin{align*}
\delta A &= 0, \quad \delta F = 0 \\
\delta c &= -A, \quad \delta \xi = F + A^2,
\end{align*}
\]

and

\[
\begin{align*}
G A &= 0, \quad GF = 0 \\
G c &= -F, \quad G \xi = FA - AF.
\end{align*}
\]

The operators δ and G are respectively of degree zero and one and obey the following algebraic relations:

\[
\begin{align*}
d &= - [s, \delta] \\
2G &= [d, \delta] \\
\{d, G\} &= \{s, G\} = 0 \\
\{G, G\} &= [G, \delta] = 0
\end{align*}
\]

In particular Eq. (23) shows that the operator δ decomposes the exterior derivative \(d\) as a BRST commutator.

Equations (23)-(26) define an algebraic setup which, as we shall see later, gives a systematic procedure for solving the tower (19).

To this purpose let us make use of the following identity

\[
e^\delta se^{-\delta} = s + d - G
\]

which is a direct consequence of (23)-(24) and of the elementary formula

\[
e^A Be^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \cdots
\]

Eq. (27) can be rewritten in a more useful way as

\[
e^\delta s = (s + d)e^\delta - e^\delta G
\]
Let us apply now Eq. (28) to the ghost monomial of Eq. (20). Taking into account that \( \omega_0^{2n+1} \) belongs to the cohomology of \( s \) one gets

\[
(s + d)e^\delta \omega_0^{2n+1} - e^\delta G \omega_0^{2n+1} = 0 \quad (29)
\]

From Eq. (25) one has

\[
sG \omega_0^{2n+1} + G s \omega_0^{2n+1} = 0 \quad (30)
\]

which shows that \( G \omega_0^{2n+1} \) is \( s \)-invariant. However, from Eq. (22) and from the general result (16), it follows that \( G \omega_0^{2n+1} \) cannot belong to the cohomology of \( s \). Hence it is trivial, i.e.

\[
G \omega_0^{2n+1} = s \Omega_2^{2n-1} \quad (31)
\]

As explained in [11], this equation yields a chain of forms \((\Omega_2^{2n-2}, \Omega_4^{2n-3}, \cdots, \Omega_1^{2n})\) which obey a tower of descent equations involving the operators \( s \) and \( G \):

\[
\begin{align*}
G \Omega_2^{2n-1} + s \Omega_4^{2n-3} &= 0 \\
G \Omega_4^{2n-3} + s \Omega_6^{2n-5} &= 0 \\
G \Omega_6^{2n-5} + s \Omega_8^{2n-7} &= 0 \\
&\quad \cdots \\
G \Omega_{2n-2}^{3} + s \Omega_{2n}^{1} &= 0 \\
\end{align*}
\]

and

\[
G \Omega_1^{2n} = (\text{const.}) P_{2n+2}(F) \quad (33)
\]

where \( P_{2n+2}(F) \) is the invariant monomial of degree \((2n + 2)\). The expression for the \( \Omega \)-cocycles and for the constant factor of eq. (33) will be computed exactly in the next section. Let us focus then, for the time being, on the proof of the equivalence with the "Russian formula".

Using the algebraic relations (23) and (24) and the ladder (32)-(33) it is easy to prove that the equation (29) can be iterated to give

\[
(s + d) \omega_S + (\text{const.}) P_{2n+2}(F) = 0 \quad (34)
\]

with

\[
\omega_S = e^\delta (\omega_0^{2n+1} - \Omega_2^{2n-1} - \cdots - \Omega_1^{2n}) \quad (35)
\]
Equation (34) represents our main result. It summarizes in a unique closed equation the whole solution of the descent equation (19). Indeed, projecting out from $\omega_S$ the terms with a given ghost number one gets:

$$\omega_{2n+1}^0 = Tr \frac{c^{2n+1}}{(2n+1)!};$$  \hspace{1cm} (36)

$$\omega_{2p}^{2n+1-2p} = \frac{\delta^{2p}}{(2p)!} \omega_0^{2n+1} - \sum_{j=0}^{p-1} \frac{\delta^{2j}}{(2j)!} \Omega_{2p-2j}^{2n+1-2p+2j},$$  \hspace{1cm} (37)

for the even space-time form sector and

$$\omega_1^{2n} = \delta \omega_0^{2n+1};$$  \hspace{1cm} (38)

$$\omega_{2p+1}^{2n-2p} = \frac{\delta^{2p+1}}{(2p+1)!} \omega_0^{2n+1} - \sum_{j=0}^{p-1} \frac{\delta^{2j+1}}{(2j+1)!} \Omega_{2p-2j}^{2n+1-2p+2j},$$  \hspace{1cm} (39)

for the odd sector and $p = 1, 2, \cdots, n$.

Equations (36)-(39) are nothing but the explicit solution of the ladder (19) obtained in ref.[1].

It is easy now to compare Eq.(34) with the corresponding expression given by the "Russian formula"; this will allow to establish the cohomological equivalence of the two methods.

To this purpose let us recall that, using the "Russian formula" and the transgression procedure of [3, 6, 9, 10, 11, 15], we can rewrite the invariant monomial $\mathcal{P}_{2n+2}(F)$ as

$$\mathcal{P}_{2n+2}(F) = (d + s) \tau(A - c, F)$$  \hspace{1cm} (40)

where the Chern-Simons term $\tau(A - c, F)$ is defined by Eq. (18), i.e.

$$\mathcal{P}_{2n+2}(F) = d \tau(A, F)$$  \hspace{1cm} (41)

Using Eq.(40), the expression (34) takes the form

$$(s + d)(\omega_S + (\text{const.}) \tau(A - c, F)) = 0$$  \hspace{1cm} (42)

from which, using the triviality of the cohomology of $(d + s)$ (see Sect.2) it follows that $\omega_S$ and $\tau(A - c, F)$ differ only by a trivial $s$ or $d$-coboundary. We have thus established the equivalence with the "Russian formula".

$^{1}$The unusual relative minus sign of the combination $(A - c)$ in the "Russian formula" (40) is due to the convention adopted for the BRST transformations (6).
4 Solution of the descent equations

In this section we give a simple procedure for finding the explicit expressions for the $\Omega$-cocycles of the tower (32). We will be able then to give an elegant and very useful expression for the cocycle $\omega_S$ of Eq. (35). This will give us a straightforward way of computing the anomalies for any space-time dimension.

Let us begin by recalling some important algebraic properties concerning the symmetrized trace of a set of $n$ matrices $(T_{a_1}, T_{a_2}, \cdots, T_{a_n})$ belonging to the Lie algebra of G. Following ref. [5, 6, 8] it is given by

$$\sum_{\pi} \text{Tr}(T_{a_{\pi(1)}}, T_{a_{\pi(2)}}, \cdots, T_{a_{\pi(n)}}) = \frac{1}{n!} \sum_{\pi} \text{Tr}(T_{a_{\pi(1)}}, T_{a_{\pi(2)}}, \cdots, T_{a_{\pi(n)}})$$ (43)

where the sum is over all permutations $(\pi(1), \pi(2), \cdots, \pi(n))$ of $(1, 2, \cdots, n)$. The symmetric invariant polynomial $P(F_1, F_2, \cdots, F_n)$ in the Lie algebra-valued forms $(F_1, F_2, \cdots, F_n)$ is defined as

$$P(F_1, F_2, \cdots, F_n) = F_{a_1}^1 \cdots F_{a_n}^n \text{Str}(T_{a_1}, \cdots, T_{a_n})$$ (44)

with $F_j = F_j^a T_a$ ($j = 1, \cdots, n$). The symmetrized trace has the remarkable property

$$\sum_{i=1}^{n} \text{Str}(T_{a_1}, \cdots, [\theta, T_{a_i}], \cdots, T_{a_n}) = 0$$ (45)

with $\theta$ an arbitrary element of the Lie algebra.

Now let $\Theta$ and $(\Lambda_i, i = 1, \cdots, n)$ be Lie algebra-valued forms of degree $d_\Theta$ and $d_i$. Then from Eq. (43), one gets the useful identity:

$$\sum_{i=1}^{n} (-1)^{d_1 + \cdots + d_{i-1} + d_\Theta} P(\Lambda_1, \cdots, [\Theta, \Lambda_i], \cdots, \Lambda_n) = 0$$ (46)

where the extra sign in each term accounts for the exchange of $\Theta$ with the forms $(\Lambda_1, \cdots, \Lambda_{i-1})$.

Using the symmetrized trace we can rewrite the invariant ghost monomial $\omega^{2n+1}_0(c)$ of Eq. (21) as

$$\omega^{2n+1}_0(c) = \frac{1}{(2n+1)!} P(c, c^2, \cdots, c^2) = \frac{1}{(2n+1)!} P(c, (c^2)^n)$$ (47)
where in the last equation we have used Zumino’s convention \[5, 6, 8\]

\[
P(F_1, F_2, F_3, F, \ldots, F) = P(F_1, F_2, F_3, F^n)
\]

We turn now to solve the ladder (31)-(32). Let us start by considering the first equation. Taking into account the form of \(\omega^{2n+1}_0\) and the definition of the differential \(G\) of Eq. (22) one has:

\[
G\omega^{2n+1}_0 = \frac{1}{(2n+1)!} \left[ P(Gc, (c^2)^n) - nP(c, Gc^2, (c^2)^{n-1}) \right]
\]

\[
= \frac{1}{(2n+1)!} \left[ -P(F, (c^2)^n) - nP(c, sF, (c^2)^{n-1}) \right]
\]

(48)

since \(Gc^2 = sF\).

The left hand side of this equation can be written as an exact \(s\)-cocycle. Indeed, the use of Eq.(46) with \(\Theta = c, \Lambda_1 = c, \Lambda_2 = F, (\Lambda_3 = \cdots = \Lambda_n = c^2)\) implies

\[
P([c, c], F, (c^2)^{n-1}) - P(c, [c, F], (c^2)^{n-1}) - (n-1)P(c, F, [c, c^2], (c^2)^{n-2}) = 0
\]

(49)

or

\[
2P(F, (c^2)^n) + P(sF, c, (c^2)^{n-1}) = 0
\]

(50)

due to the Jacobi identity.

From

\[
sP(c, F, (c^2)^{n-1}) = P(F, (c^2)^n) - P(c, sF, (c^2)^{n-1}).
\]

(51)

one easily gets, using Eq. (50),

\[
sP(c, F, (c^2)^{n-1}) = -\frac{1}{2}P(c, (sF), (c^2)^{n-1})
\]

(52)

Eq.(48) takes the form

\[
G\omega^{2n+1}_0 = s \left( \frac{1}{(2n)!} P(F, c, (c^2)^{n-1}) \right)
\]

(53)
so that $\Omega^{2n-1}_2$ can be identified with
\[
\Omega^{2n-1}_2 = \frac{1}{(2n)!} P(F, c, (c^2)^{n-1}).
\] (54)

We proceed now by induction. Let us suppose that the form of the cocycles $\Omega^{2n-2p+1}_{2p}$ in Eqs. (32) is given by
\[
\Omega^{2n-2p+1}_{2p} = \frac{(-1)^{p+1}}{(2n-p+1)!p!} P(F^p, c, (c^2)^{n-p}).
\] (55)

For $p = 1$ Eq. (55) reduces just to expression (54). Acting with $G$ on Eq. (55) one has:
\[
G \Omega^{2n-2p+1}_{2p} = \frac{(-1)^{p+1}}{(2n-p+1)!p!} \left( - P(F^{p+1}, (c^2)^{n-p}) - (n-p) P(F^p, c, sF, (c^2)^{n-p-1}) \right).
\] (56)

Repeating the same arguments of Eqs. (49), (50) it is not difficult to get the identity
\[
2P(F^{p+1}, (c^2)^{n-p}) + (p + 1) P(sF, F^p, c, (c^2)^{n-p-1}) = 0
\] (57)
which combined with
\[
sP(F^{p+1}, c, (c^2)^{n-p-1}) = (p + 1) P(sF, F^p, c, (c^2)^{n-p-1}) + P(F^p, (c^2)^{n-p})
\] (58)
leads to the equation
\[
G \Omega^{2n-2p+1}_{2p} = -s \Omega^{2n-2(p+1)+1}_{2(p+1)}
\] (59)
with
\[
\Omega^{2n-2(p+1)+1}_{2(p+1)} = \frac{(-1)^{p+2}}{(2n-p)!(p+1)!} P(F^{p+1}, c, (c^2)^{n-p-1}).
\] (60)

Therefore we can conclude that the expressions for the $\Omega$-cocycles given in Eq. (59) are indeed solutions of the tower (31)-(32). In particular, for $p = n$ one finds
\[
\Omega^{1}_{2n} = \frac{(-1)^{n+1}}{(n+1)!n!} P(F^n, c)
\] (61)
and Eqs. (33), (34) take now a perfect defined form

\[ \mathcal{G} \Omega_{2n}^1 = \frac{(-1)^n}{(n + 1)!n!} P(F^{n+1}) \]  

(62)

without any unspecified constant.

Let us close this section with a very important remark. As one can see from Eq.(21) the operator \( \delta \) acts as a translation on the ghost \( c \) with an amount given by \( -A \). Then if \( e^\delta \) acts on a quantity which depends only on \( A, F \) and \( c \) it has the simple effect of translating \( c \). This implies that the cocycle \( \omega_S \) of Eq.(35) can be written as

\[ \omega_S = \omega_0^{2n+1}(c - A) - \Omega_2^{2n-1}(c - A, F) - \cdots - \Omega_{2n}^1(c - A, F) \]  \[ = \sum_{p=0}^{n} \frac{(-1)^p}{(2n - p + 1)!p!} P(F^p, c - A, [(c - A)^2]^{n-p}). \]  

(63)

This remarkable equation collects in a very elegant and simple form the solution of the descent equations (19). In particular, as we will see in next section, we can immediately write down the expressions for the gauge anomalies for any space-time dimension.

5 Some examples

This section is devoted to use the closed expression (64) to discuss some explicit examples.

5.1 The case \( n=1 \)

In this case, relevant for the two-dimensional anomaly and for the three dimensional Chern-Simons term, the \( \omega_S \) cocycle reads

\[ \omega_S(F, c - A) = \frac{1}{3!} P(c - A, (c - A)^2) - \frac{1}{2!} P(F, c - A) \]  

(65)

and by expanding in power of \( c \) we get

\[ \omega_0^3 = \frac{1}{3!} P(c, c^2) = \frac{1}{3!} Tr(c^3), \]  

(66)
\[
\omega_1^2 = -\frac{1}{3!}P(A, c^2) - \frac{1}{3!}P(c, [c, A]) = -Tr(Ac^2) = Tr(\xi c) - sTr(Ac), \quad (67)
\]

\[
\omega_2^1 = \frac{1}{3!}P(c, A^2) + \frac{1}{3!}P(A, [c, A]) - \frac{1}{2!}P(c, F) = \frac{1}{2}(Tr(cA^2) - Tr(cF)) = -\frac{1}{2}Tr[c(dA)] \quad (68)
\]

\[
\omega_0^3 = -\frac{1}{3!}P(A, A^2) + \frac{1}{2!}P(F, A) = \frac{1}{2}Tr(AdA + \frac{2}{3}A^3). \quad (69)
\]

One can easily recognize that the expressions (67)-(69) coincide, modulo coboundaries, with the solution given by Zumino, Wu and Zee [6]. In particular \(\omega_1^2\), \(\omega_2^1\) and \(\omega_0^3\) give respectively the two dimensional gauge anomaly, the Schwinger term and the three dimensional Chern-Simons action.

### 5.2 The case n=2

For \(n=2\) expression (64) takes the form

\[
\omega_S = \frac{1}{5!}P(c - A, [(c - A)^2]^2) - \frac{1}{4!}P(F, c - A, (c - A)^2) + \frac{1}{3!2!}P(F^2, c - A). \quad (70)
\]

In particular \(\omega_5^0\) and \(\omega_4^1\) are computed to be

\[
\omega_5^0 = -\frac{1}{12}Tr\left(\frac{1}{10}A^5 - \frac{1}{2}FA^3 + F^2A\right), \quad (71)
\]

\[
\omega_4^1 = \frac{1}{4!}Tr\left(c(A^4 - FA^2 - A^2F - AFA + 2F^2)\right) \quad (72)
\]

and give respectively the generalized five-dimensional Chern-Simons term and the four-dimensional gauge anomaly. Again, they coincide, modulo coboundaries, with that of ref.[6].
5.3 The general case

It is straightforward now to generalize the previous examples and find out from the formula (64) the general form for the cocycles $\omega_{2p+1}^{n-2p}$ for any $n$ and $p$. We shall give here only the expressions of the generalized $(2n+1)$-dimensional Chern-Simons term and of the $2n$-dimensional anomaly:

$$\omega_{2n+1}^0 = \sum_{p=0}^{n} \frac{(-1)^{p+1}}{(2n-p+1)!p!} P(F^p, A, (A^2)^{n-p})$$  \hspace{1cm} (73)

and

$$\omega_{2n}^1 = \sum_{p=0}^{n} \frac{(-1)^{p}}{(2n-p+1)!p!} P(c, F^p, (A^2)^{n-p}) + \sum_{p=0}^{n} \frac{(-1)^{p}(n-p)}{(2n-p+1)!p!} P([c, A], F^p, A, (A^2)^{n-p-1})$$  \hspace{1cm} (74)

Let us conclude by emphasizing that, actually, expression (74) represents one of the most closed algebraic formula for the gauge anomaly in any space-time dimension.

5.4 Conclusions

We have proved that the method proposed by [1] for solving the descent equations associated with the Wess-Zumino consistency condition is completely equivalent to that based on the well-known "Russian formula" [2, 4, 5, 9, 10, 11, 15]. Moreover, it naturally extends to the case of the gravitational anomalies as well as to the recently proposed topological field theories.

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