Signs and Stability in Higher-Derivative Gravity

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Abstract

Perturbatively renormalizable higher-derivative gravity in four space-time dimensions with arbitrary signs of couplings has been considered. Systematic analysis of the action with arbitrary signs of couplings in lorentzian flat space-time for no-tachyons, fixes the signs. Feynman $+i\epsilon$ prescription for these sign further grants necessary convergence in path-integral, suppressing the field modes with large action. This also leads to a sensible wick rotation where quantum computation can be performed. Running couplings for these sign of parameters makes the massive tensor ghost innocuous leading to a stable and ghost-free renormalizable theory in four space-time dimensions. The theory has a transition point arising from renormalisation group (RG) equations, where the coefficient of $R^2$ diverges without affecting the perturbative quantum field theory. Redefining this coefficient gives a better handle over the theory around the transition point. The flow equations pushes the flow of parameters across the transition point. The flow beyond the transition point is analysed using the one-loop RG equations which shows that the regime beyond the transition point has unphysical properties: there are tachyons, the path-integral loses positive definiteness, Newton's constant $G$ becomes negative and large, and perturbative parameters become large. These shortcomings indicate a lack of completeness beyond the transition point and need of a non-perturbative treatment of the theory beyond the transition point.
I. INTRODUCTION

Quantum field theory (QFT) is a beautiful framework established to address some of the mysteries of nature. Its success lies in the fact that it elegantly explains most of the observation seen in the accelerator experiments and condensed matter systems. On the other hand general relativity (a theory of gravity) has been widely used to understand and explain the mysteries at large scales. It comes to a puzzling situation when the two can’t be easily combined in a single theory without leading to problems. One of the most important problem that arises when methods of QFT are applied to Einstein-Hilbert gravity is that the resulting theory is plagued with ultraviolet (UV) divergences [1–7], leading to non-renormalizablity.

It has been noticed that the sickness of non-renormalizability can be cured by inclusion of higher-derivative terms [8, 9], when the QFT of modified theory becomes renormalizable to all loops in four space-time dimensions. The path-integral of this theory is given by,

$$Z = \int D\gamma_{\mu\nu} e^{iS_{GR}},$$

where

$$S_{GR} = \int d^4x \sqrt{-\gamma} \left[ 2\Lambda - aR + \frac{\omega R^2}{6M^2} - \frac{R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2}{M^2} \right].$$

Here $R$ is the Ricci scalar and $R_{\mu\nu}$ is the Ricci tensor for the corresponding quantum metric $\gamma_{\mu\nu}$. The last term is also proportional to square of Weyl tensor ($C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$) in $3 + 1$ space-time dimensions (modulo total-derivative), $\omega$ is dimensionless and $M$ has dimension of mass, $G$ is the gravitational Newton’s constant and $\Lambda$ is the cosmological constant term. The dimensionless parameter $a$ is for moment kept arbitrary but can take value either +1 or −1. It is a redundant parameter but is helpful in fixing some signs as will be seen later. In principle the path-integral in eq. (1) will also contain ghost action which enters due to gauge-fixing of the diffeomorphism invariant measure of theory, but for the purpose of this paper will not be considered explicitly, though their contributions has been taken into account for the renormalisation group running of parameters [11–15]. The interesting thing to note here is that the UV renormalizability of the theory doesn’t restrict the sign of various coefficients of terms in action [8].

Higher-derivative theory has been studied many times in past. This theory is notorious for its unitarity issues created by presence of ghost and tachyons [10]. These were first studied in [16–19] where proposal for avoiding them was made. After few years a euclidean version of the theory became popular as it was shown to have asymptotic freedom [12–15]. Over the years its coupling with matter has been investigated in euclidean framework [20–26]. Later effect of Gauss-Bonnet was also studied [27, 28]. But in general higher-derivative theories should not always be thought of having unitarity problems as has been shown in example considered in [29–31].

Recently, higher-derivative gravity has been studied in four space-time dimensions in lorentzian signature for issues of ghost and tachyons [32, 33], while gauge-field coupling was investigated in detail in [34, 35]. In these papers it was shown that unitarity problem can be tackled in the fully quantum theory by keeping the ghost mass always above the energy thus not allowing it to enter physical spectrum. The scale-invariant analog of this theory has been studied from phenomenological perspective in [36–38], where a scale dynamically arises in Einstein-frame of the theory. Fourth-order quantum-mechanical system were studied for
unitarity issues in [39, 40], where the authors argued that field theory when constructed in similar fashion for complicated gravitational systems might resolve unitarity problem. The idea of dimensional transmutation in scale-invariant theories where the scale arises via symmetry breaking due to quantum corrections ala Coleman-Weinberg has been studied in [41–44]. By making an analogy with QCD these systems were studied to get a resolution for the ghost problem using the wisdom acquired from non-perturbative sector of QCD [45, 46]. Recently an interesting proposal has been made in [47] where the scale-invariant higher-derivative theory studied directly in lorentzian was shown to break scale-symmetry via quantum corrections and in turn resolve problem of ghost by choosing RG trajectories where the induced ghost mass is always above energy.

These studies were conducted in perturbative framework (euclidean and lorentzian), however interesting developments have taken place in the field of asymptotic safety scenario [48] where the theory was considered using functional renormalisation group in euclidean signature [49–57]. It was found that the theory admits spectral positivity [58], where ghost can be tackled [49, 50] at the nontrivial fixed point [59, 60]. However these analysis differ from the past ones in the sense that the theory is analysed at the non-gaussian fixed point in euclidean framework.

The starting point for having a well-defined QFT is the existence of a stable vacuum on which tower of states can be constructed and the states to have a positive norm so to have a unitary evolution. A QFT satisfying these basic requirements along with renormalizability is a well-defined QFT in which testable predictions can be made and meaningful computation can be performed. The existence of a stable vacuum is guaranteed if the theory doesn’t have any tachyons and stays in a regime where vacuum never become unstable. In case of higher-derivative gravity both these basic requirements gets challenged. If the parameters in the theory doesn’t have appropriate signs they it is noticed that the theory will have tachyons indicating that the vacuum of this theory is unstable. Therefore a QFT constructed with this is not reliable. However, there exits certain signs of parameters where there are no tachyons in four space-time dimensions in lorentzian signature (see for a similar study in three dimensions [61]). In this short paper it is shown how one obtains these set of signs of parameters by carefully analysing the propagator for the tachyons. For these set of signs it is further noticed that the QFT constructed following feynman \(+i\epsilon\) prescription has a necessary convergence suppressing field modes with large action.

The renormalisation group flow of parameters is analysed in detail. It is seen that the flow of the coefficient of \(R^2\) terms goes to infinity, where the energy-scale reaches a maximum value below which there are no tachyons and theory remains ghost free [32–35]. This is a transition point. In this paper the theory around and beyond this transition point is explored to understand the true nature of this transition point and nature of the regime beyond.

The paper is organised as follows: section II deals with the analysis of propagator of theory in flat space-time where the tachyon analysis is done, section III deals with a short introduction of the idea of wick rotation and \(i\epsilon\) prescription where it is shown how the chosen sign of coupling allows convergence of lorentzian path-integral, section IV deals with beta-functions and their analysis, finally conclusions with discussions are presented in section V.
II. PROPAGATOR

Here we consider the propagator of theory given by the action in eq. (2). For this we consider the fluctuations of metric around flat spacetime $\gamma_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ (with $\eta_{\mu\nu} = \{+, -, -, -\}$). Flat space-time is not a solution of equation of motion for the action of theory given in eq. (2), so in the following we will put $\Lambda = 0$ in the action. This is also justified as the renormalizability of the theory has been only demonstrated rigorously for $\Lambda = 0$ case only [9], where it is possible to give a sensible particle content and use the methods/techniques of flat space-time quantum field theory. Moreover this is further justified when one is interested in investigating high-energy behaviour, where the usage of flat space-time is rightfully advocated. Such flat space-time analysis have limitation which become relevant in deep infrared and are not a concern at short distances.

In the Landau gauge ($\partial_\mu h^{\mu\nu} = 0$), a physical gauge allowing only transverse modes to propagate, the propagator of metric fluctuation field $h_{\mu\nu}$ is given by [32–35],

$$\Delta^{\mu\alpha\beta\gamma} = \left(\frac{i16\pi G}{a}\right)\left[ -\frac{2M^2 P_{\mu\alpha\beta\gamma}}{q^4 - aM^2 q^2} + \frac{M^2/\omega P_{\mu\alpha\beta\gamma}}{q^4 - aM^2/\omega q^2} \right],$$

(3)

where $q$ is the four-momentum of fluctuating field $h_{\mu\nu}$. Various spin projectors are $P_{\mu\alpha\beta\gamma}^2 = \frac{1}{2}[T_{\mu\alpha}T_{\nu\beta} + T_{\mu\beta}T_{\nu\alpha}] - \frac{1}{3}T_{\mu\nu}T_{\alpha\beta}$, $P_{\mu\alpha\beta\gamma}^s = \frac{1}{3}T_{\mu\nu}T_{\alpha\beta}$, where $T_{\mu\nu} = \eta_{\mu\nu} - q_\mu q_\nu/q^2$. They project the spin-2 and spin-0 component of the $h_{\mu\nu}$ field. Each of the denominator is a quadratic polynomial in $q^2$ (=$\eta_{\mu\nu}q^\mu q^\nu$), implying two roots (real or complex depending on sign of parameters). One can do partial fraction after factorisation of denominators to write the propagator in simple form.

$$\Delta^{\mu\alpha\beta\gamma} = \left(\frac{i16\pi G}{a}\right)\left[ \frac{2P_{\mu\alpha\beta\gamma}^2 - P_{s\mu\alpha\beta\gamma}}{q^2} - \frac{2P_{\mu\alpha\beta\gamma}^2}{q^2 - aM^2} + \frac{P_{s\mu\alpha\beta\gamma}}{q^2 - aM^2/\omega} \right].$$

(4)

So far the propagator is just a function of $q^2$ where the metric dependence enters implicitly. This propagator consist of just three parts: the usual massless graviton, massive tensor mode, and massive scalar mode. The massless graviton and massive scalar has the same sign of propagator while the massive tensor has the opposite sign. By comparing with low-energy, known Einstein-Hilbert gravity, one can fix the sign of parameter $a = +1$ (where $G$ is taken to be positive for attractive gravity). This is mandatory otherwise the gravitational-wave in flat space-time will have negative norm, but this can be actually absorbed in the definition of $G$ which decides how gravity will couple with matter. The parameter that will decide how at low energy matter couples with gravity is $a/G$, which should be positive for attractive nature of gravity. For consistency we will keep $a = +1$. Once this is fixed we have freedom to fix the sign of $M^2$ and $\omega$. This is done by requiring that the propagator shouldn’t have any tachyons. For the signature given by $\eta_{\mu\nu}$ the sign of $M^2$ and $\omega$ should be positive to have no tachyons. Reversing the signature sign will result in change in sign of terms where to avoid tachyons $M^2 \rightarrow -M^2$ while $\omega$ remains positive. This information can be neatly written in a tabular form. This is given in table. 4.

An alternative possibility is considering $a = -1$ and reversing the sign of $G$, so that $a/G$ remains unchanged. This will give rise to tachyons in massive tensor and massive scalar mode, thereby demanding $M^2 \rightarrow -M^2$ for tachyon elimination. However the action of two cases $a = \pm 1$ is same. In this sense parameter $a$ is redundant. In either case these are different from the version of lorentzian theories considered in [45, 46], where the propagator has tachyons.
TABLE I. Tachyon analysis for the signature and sign of couplings. Here yes and no refers to tachyon presence and absence respectively.

III. WICK ROTATION

Here in this section we discuss about standard wick-rotation and feynman $i\epsilon$ prescription. The first part of which is a small review of textbook material covered here to make the paper more clear. The sign of the $i\epsilon$-prescription gets fixed from the definition of the starting point path-integral. For the integrand in path-integral $e^{\pm iS}$ the prescription is $\pm i\epsilon$ to make the path-integral convergent. This allows for suppression of those field modes for which the action is large. Systematically this is achieved by doing the following $\int dt L \rightarrow \int dt(1 + i\epsilon)L = S + i\epsilon S$ ($0 < \epsilon < 1$), where the Lagrangian $L = \int d^3x \mathcal{L}$ and $\mathcal{L}$ is the Lagrangian density. Here the convergence is achieved by doing transformation $t \rightarrow t + i\epsilon$, which is a standard practice in defining lorentzian path-integrals which also respects all the symmetry of the original action. Moreover, this also leads to propagator with shifted poles in a natural manner, a standard thing covered in most textbooks on quantum field theory.

In the path-integral the $+i\epsilon$ offers a suppression factor for field modes for which the action is large. In our particular case this will imply

$$Z = \int D\gamma_{\mu\nu} e^{iS_{\text{GR}} - \epsilon S_{\text{GR}}},$$

where $S_{\text{GR}}$ is given by eq. (2) and $\gamma_{\mu\nu}$ is the quantum metric. To have the required convergence and well-defined path-integral it is required that the coefficient of the $R^2$ term to be positive i.e. $\omega > 0$. In the $+i\epsilon$ prescription this will give rise to an additional term in exponent: $-\epsilon\omega R^2$. For field modes with large $R^2$, such a term will heavily suppress that mode resulting in convergence of the lorentzian path-integral (in euclidean path-integral a positive coefficient of $R^2$ is needed for positive definiteness [13, 62, 63]). Interestingly $\omega > 0$ is also the regime which avoids tachyons in lorentzian signature (both for $\pm\eta_{\mu\nu}$).

In momentum space the prescription shifts the locations of poles in the flat space-time propagator[1] In the higher-derivative case for $a = +1$, $\Lambda = 0$ and signature $\eta_{\mu\nu}$ the propagator will be,

$$\Delta^{\mu\alpha\beta} = (i16\pi G) \left[ \frac{(2P_2^{\mu\alpha\beta} - P_s^{\mu\alpha\beta})}{q^2 + i\epsilon} - \frac{2P_2^{\mu\alpha\beta}}{q^2 - M^2 + i\epsilon} + \frac{P_s^{\mu\alpha\beta}}{q^2 - M^2/\omega + i\epsilon} \right].$$

In a complex $q_0$ plane the $+i\epsilon$ prescription will shift the poles on the real axis to second and fourth quadrant. When performing loop-computations for the quantum corrections, this will allow to choose contour which doesn’t enclose these shifted poles. The integral along the contour in the end will reduce to replacement of $q_0 \rightarrow iq_0$. This is the standard wick rotation in flat space-time usually covered in textbooks. It shows how the wick rotation is tied to $i\epsilon$ prescription whose sign is chosen to provide appropriate convergence in the path-integral.

\[1\] In momentum space the quadratic part of the any action will look like $(1 + i\epsilon)(q^2 - m^2) = q^2 - m^2 + i\epsilon'$, where $\epsilon' = \epsilon(q^2 - m^2)$ is the new parameter. This will result in standard $+i\epsilon$ prescription.
In the case when the poles lie on imaginary axis, which happens when there are tachyons, the \(+i\epsilon\) prescription shifts the poles in second and fourth quadrant. Here the same contour is fine and usual wick rotation can be done as the shifted poles are outside contour. In the present case however where appropriate choice of signs evades tachyons such a thing is not needed.

IV. BETA-FUNCTIONS AND RG ANALYSIS

The perturbative field theory is constructed in parameters \(M^2G\) and \(M^2G/\omega\), where loop computation are performed and beta-functions are computed. These beta-functions have been computed earlier [15, 64] in the case of euclidean field theory. In [32, 33] the beta-functions were translated to lorentzian signature in Landau gauge. Here we take over those beta-functions for further analysis. These are given by,

\[
\frac{d}{dt} \left( \frac{1}{M^2G} \right) = -\frac{133}{10\pi},
\]

\[
\frac{d}{dt} \left( \frac{\omega}{M^2G} \right) = \frac{5}{3\pi} \left( \omega^2 + 3\omega + \frac{1}{2} \right),
\]

\[
\frac{d}{dt} \left( \frac{1}{G} \right) = \frac{5M^2}{3\pi} \left( \omega - \frac{7}{40\omega} \right).
\]

where \(t = \ln(\mu/\mu_0)\) and the r.h.s. for all beta-function contain the leading contribution in \(G\) (\(M^2G\) is also taken to be small), with higher powers coming from higher loops being neglected. It should be mentioned that the RG flow of coupling \(G\) is gauge-fixing dependent.

Here its flow is given in Landau gauge [32, 33, 64]. Also, we have ignored the cosmological constant term and its running, which is done so that flat space-time exists as a solution to equation of motions on whose background quantum field theory analysis can be trustfully performed. From this we first extract the flow of parameter \(\omega\), which is the coefficient of the \(R^2\) term and for no-tachyons needs to be positive. Its running is given by,

\[
\frac{d\omega}{dt} = \frac{5M^2G}{3\pi} \left( \omega^2 + \frac{549}{50} \omega + \frac{1}{2} \right) = \frac{5M^2G}{3\pi} (\omega + \omega_1)(\omega + \omega_2)
\]

where \(\omega_{1,2} = (549 \pm \sqrt{6049})/100\). Both the roots \(-\omega_1\) and \(-\omega_2\) lie in the tachyonic regime. The fixed point given by \(-\omega_1\) is repulsive while \(-\omega_2\) is attractive. Since the r.h.s. of beta-function of \(\omega\) is positive and increase with \(\omega\), therefore \(\omega\) is a monotonic increasing function of RG time \(t\). No-tachyon condition puts a constraint on the allowed range of \(\omega\) to be \(0 \leq \omega \leq \infty\). One can solve the flow of \(\omega\) in terms of RG time \(t\) and obtains

\[
t = T \left[ 1 - \left( \frac{\omega + \omega_2}{\omega + \omega_1} \cdot \frac{\omega_0 + \omega_1}{\omega_0 + \omega_2} \right)^\alpha \right],
\]

where \(T = 10\pi/(133M_0^2G_0)\) and \(\alpha = 399/(50(\omega_2 - \omega_1)) > 0\), with subscript 0 meaning that the coupling parameters are evaluated at \(t = 0\) or \(\mu = \mu_0\), which will be decided later. The bound on \(\omega\) translates to a lower and upper bound on value of RG time \(t\) within which the parameter \(\omega\) remains positive and hence avoid tachyons. The lower bound \(t_{\text{min}}\) and upper
bound $t_{\text{max}}$ are given by,
\[ t_{\text{min}} = 1 - \left( \frac{\omega_2 \omega_0 + \omega_1}{\omega_1 \omega_0 + \omega_2} \right)^\alpha, \quad t_{\text{max}} = 1 - \left( \frac{\omega_0 + \omega_1}{\omega_0 + \omega_2} \right)^\alpha, \] (12)
respectively. Here $t_{\text{min}}$ is in infrared while $t_{\text{max}}$ is in UV. As $t \to t_{\text{max}}$, $\omega \to \infty$. The emergence of $t_{\text{max}}$ is puzzling as the theory has been studied without imposing any cutoff. This bound arises from the RG flow equations due the restriction imposed by stability of vacua (no-tachyons) in lorentzian signature. Although at this point the parameter $\omega \to \infty$ the perturbation theory remains valid as the parameters $M_2^2$ and $M_2^2/\omega$ remain small, with the amplitudes computed using them remains well-defined. It is therefore natural to go beyond this bound and curiously explore the regime beyond $t_{\text{max}}$. In this paper we do this exploration by extrapolating the beta-functions in this regime and the findings are presented in this paper.

A good way to explore this large $\omega$ regime is to define a new parameter $\theta = 1/\omega$. Unlike $\omega$, this new parameter $\theta$ is continuous near $\theta = 0$ ($\omega = \infty$). This new parametrisation allows to study the flow equation near this singular-point in a systematic manner. In particular the behaviour of flows on both sides of $\theta = 0$ point is different. The regime dictated by $t < t_{\text{max}}$ is $\theta > 0$, while the regime for $t > t_{\text{max}}$ is given by $\theta < 0$. In the following subsections both these regimes will be studied, and will be realised that $\theta = 0$ is a transition point where a smooth crossover happens from one regime to another.

A. $\theta > 0$ Regime

In this regime the beta-function of $\theta$ is given by following,
\[ \frac{d\theta}{dt} = -\frac{5M_2^2G}{3\pi} \left( 1 + \frac{549}{50} \theta + \frac{\theta^2}{2} \right). \] (13)
This also has two fixed points $-\theta_1 = -1/\omega_1$ and $-\theta_2 = -1/\omega_2$. Both of these lie in tachyonic regime (negative $\omega$). But now $-\theta_1$ is repulsive while $-\theta_2$ is attractive fixed point. The parameter $\theta$ has a continuous behaviour around the point $\theta = 0$, where its smooth flow is dictated by the first order ODE given in eq. (13). At $\theta = 0$ the beta-function of $\theta$ is negative thereby forcing $\theta$ to decrease further. It is expected as $\theta = 0$ is not a fixed point for the flow of $\theta$.

Other couplings parameters $M_2^2$ and $G$ can then be solved in terms of continuos parameter $\theta$ to study their behaviour near and across the $\theta = 0$ point. Writing the ODE for $M_2^2G$ in terms of $\theta$ quickly shows the solution of $M_2^2G$ in terms of $\theta$ to be,
\[ M_2^2G = M_0^2G_0 \left( \frac{\theta + \theta_1 \theta_0 + \theta_2}{\theta_0 + \theta_1 \theta + \theta_2} \right)^{-\alpha}, \] (14)
where $\alpha$ is the same number appearing in eq. (11) and $M_0^2G_0$ is the initial value of the parameter $M_2^2G$ at the reference point $\theta_0$ which will be fixed next. As $\theta \to 0$, $M_2^2G$ increase and goes to a fixed value $M_0^2G_0(\theta_0(\theta_0 + \theta_2)/\theta_2(\theta_0 + \theta_1))^{-\alpha}$.

The flow of $G$ can be similarly solved. The beta-function of $G$ can be expressed in terms of $\theta$ as,
\[ \frac{dG}{d\theta} = \frac{2G (1 - \frac{7}{40} \theta^2)}{\theta(\theta + \theta_1)(\theta + \theta_2)}. \] (15)
The flow of parameter $G$ is interesting as it has various fixed points. From its beta-function we notice that it has three fixed points: two at $\theta = \pm \sqrt{40/7}$ and one where $G/\theta$ vanishes. The fixed point $\theta = -\sqrt{40/7}$ lies in the unphysical domain so can be ignored. The fixed point given by $\theta = \sqrt{40/7}$ is attractive fixed point, we call this a reference point $\theta_0$ (the point $\theta = -\sqrt{40/7}$ is also attractive in nature but lies in unphysical regime). The third fixed point where simultaneously both $G$ and $\theta$ vanishes is a little tricky to analyse. For this analysis we first solve analytically $G$ in terms of $\theta$. This is easily achieved as eq. (15) is a first order ODE with r.h.s. being ratio of polynomials. This is given by,

$$
\frac{G}{G_0} = \left(\frac{\theta}{\theta_0}\right) \left(\frac{\theta + \theta_1}{\theta_0 + \theta_1}\right)^{A_1} \left(\frac{\theta + \theta_2}{\theta_0 + \theta_2}\right)^{A_2},
$$

(16)

where

$$
A_1 = \frac{2(1 - \frac{\theta}{\theta_0})}{\theta_1(\theta_1 - \theta_2)}, \quad A_2 = -\frac{2(1 - \frac{\theta}{\theta_0})}{\theta_2(\theta_2 - \theta_1)}.
$$

(17)

From this we see that as $\theta \to 0$, $G/G_0 \sim \theta$ and approaches zero therefore this point $\theta = 0$ is a fixed point for the flow of $G$, where $G$ also vanishes. Taking second derivative of eq. (15) it is seen that in the limit $\theta \to 0$ it has negative value thereby implying that it is an attractive fixed point for flow of $G$. However it is not a fixed point for the parameter $\theta = 1/\omega$. The RG flow of $\theta$ will therefore push $\theta$ beyond this point and take it in the negative regime. The point $\theta = 0$ is then just a crossover point where the transition from $\theta > 0$ to $\theta < 0$ happens. Interestingly during this transitory phase perturbation theory remains valid but $G$ changes sign becoming a repulsive interaction. Although $G = 0$ (at $\theta = 0$) is an attractive fixed point for $G$, but it is dragged away from this point as it is not a fixed point for $\theta$. As $\theta$ is pushed into negative regime it forces $G$ to change sign. Physically this will imply that the gravitational interaction between two bodies beyond a certain length scale becomes repulsive. In next subsection we study the theory after cross over.

**B. $\theta < 0$ regime**

The parameter space beyond the crossover point $\theta = 0$ is dictated by $\theta$ becoming negative where the Newton’s constant $G$ goes to negative values. This regime lies at ultra-high energies. To explore this regime systematically we write $\theta = -\beta$. The beta-function of the parameter $\beta$ is given by,

$$
\frac{d\beta}{dt} = \frac{5M^2G}{3\pi} \left(1 - \frac{549}{50}\beta + \beta^2\right).
$$

(18)

This regime is governed by the parameter $\beta$. The flow equation of $\beta$ shows that it has two fixed point $\beta_1 = \theta_1$ and $\beta_2 = \theta_2$ where $\beta_1 < \beta_2$. It is seen that $\beta_1$ is attractive and $\beta_2$ is repulsive. Moving over the cross-over point it is expected that due to continuity the flow of $\beta$ will move towards the attractive fixed point located at $\beta_1$.

The flow of other parameters can be studied in terms of $\beta$, where it is expected that running of $\beta$ to the attractive fixed point $\beta_1$ will have consequences for the flows of other coupling parameters. To investigate this throughly first the beta-function of $G$ is re-expressed in terms of $\beta$. This is given by,

$$
\frac{dG}{d\beta} = \frac{2G \left(1 - \frac{\beta}{\beta_0}\right)}{\beta(\beta - \beta_1)(\beta - \beta_2)}.
$$

(19)
This first order ODE can be solved easily as before giving the solution to be,

\[
\frac{G}{G_0} = -\left(\frac{\beta}{\theta_0}\right) \left(\frac{\beta_1 - \beta}{\theta_0 + \beta_1}\right)^{A_1} \left(\frac{\beta_2 - \beta}{\theta_0 + \beta_2}\right)^{A_2},
\]

where \(-\theta_0\) is the previous reference point and \(G_0\) is the value of the corresponding point. This gives the evolution of \(G\) in the regime \(\theta < 0\). The flow shows that at the crossover point \(\theta = 0\) the Newton’s constant \(G\) is zero and make the transition smoothly. Beyond the crossover point Newton’s constant \(G\) becomes negative. This means that in this domain gravitational interaction will be repulsive. It is further seen that after the crossover as the flow of \(\beta\) approaches the attractive fixed point at \(\beta_1\), the flow of \(G\) diverges \((G \rightarrow -\infty)\), implying that at this point the gravitational repulsive force will be infinite.

The flow of parameter \(M^2G\) can be solved in terms of \(\beta\) using the running of \(M^2G\) and \(\beta\) given in eq. (7) and (18) respectively. This is given by,

\[
M^2G = M_0^2G_0 \left(\frac{\beta - \beta_1 \theta_0 + \beta_2}{\theta_0 + \beta_1 \beta - \beta_2}\right)^{-\alpha}.
\]

From this flow we see that at the crossover point \(M^2G\) makes a smooth transition and remains small. However as \(\beta\) approaches the attractive fixed point \(\beta_1\), the parameter \(M^2G\) diverges as \(\alpha > 0\).

V. CONCLUSION

In this short paper we investigated UV renormalizable fourth-order higher-derivative gravity in four space-time dimensions in lorentzian signature with arbitrary parameters. We study the nature of flat space-time propagator for various signs of parameters and find the set of signs for which there exist no-tachyons. Interestingly in this domain of parameters with these signs it is noticed that in the feynman \(+i\epsilon\) prescription, the path-integral has the required convergence suppressing field modes with large action. This is the same regime where there are no tachyons. The RG flows of the parameters are analysed extensively. The parameter \(\omega\) (coefficient of \(R^2\)) has a range between zero and infinity within which the theory remains tachyon free. This bound translates into a bound on the allowed energy domain where theory remains tachyon-free. In this paper we analyse what happens beyond the upper bound of the RG time \(t_{\text{max}}\), where the parameter \(\omega\) runs to infinity. The beta-functions are analysed for large \(\omega\) regime by defining the parameter \(\theta = 1/\omega\). The parameter \(\theta\) has a well-defined behaviour in the large \(\omega\) regime. The regime below \(t_{\text{max}}\) is then given by \(\theta > 0\), while the regime beyond is dictated by \(\theta < 0\). The two regimes are analysed separately.

In the regime \(\theta > 0\) it is seen that the flow of \(\theta\) approaches zero logarithmically (in terms of energy). Moreover, the flow of \(G\) also approaches zero as \(\theta \rightarrow 0\). But the ratio \(G/\theta\) remains finite at this point. This point is an attractive fixed point for \(G\), however it is not a fixed point for parameter \(\theta\). As a result the flow is dragging away from this point. The non-zero beta-function of \(\theta\) at the point \(\theta = 0\) pushes \(\theta\) to negative values, making the point \(\theta = 0\) a transition point where crossover happens. At this crossover perturbation theory is well-defined. This crossover also changes the sign of \(G\), thereby making it negative in the regime \(\theta < 0\).
The regime beyond $t_{\text{max}}$, given by $\theta < 0$ has different flow of couplings. This regime is studied in terms of $\beta = -\theta$. In this regime it is seen that once beyond the crossover point, the flow of $\beta$ quickly approaches the attractive fixed point $\beta_1$. The Newtons’ constant $G$ beyond the crossover point becomes negative and is dragged to the attractive fixed point of $\beta$ where it diverges. Negative $G$ implies that the gravitational force becomes repulsive in this regime and this repulsion is infinite at the fixed point.

The theory considered and analysed here in this paper (and previous works [32, 33, 47]) is different from the studies conducted by other authors [12–15, 20–28], which were done in euclidean signature where the signs of couplings were taken to get positive definiteness for euclidean path-integral. Those choice of signs further gives them asymptotic freedom for flow of couplings but theory contains tachyons. This was mentioned in their work [12]. However, positive-definite asymptotically free euclidean theories considered by them are is correct in those signatures (modulo the tachyon problem which is important in infrared) but results obtained from them should not be expected to hold true for lorentzian theories.

The current work (and previous works [32, 33, 47]) differ from them in the sense we constructed the theory directly in lorentzian signature and signs are chosen so that tachyons are avoided. This choice of signs also differ from the other work done in scale-invariant lorentzian theories where the signs were chosen to have asymptotic freedom [36]. The choice of signs in this paper (and past ones [32, 33, 47]) further allows convergence in path-integral following feynman $i\epsilon$ prescription. Once the basic requirement of signs and stability is satisfied, the one-loop RG flows are investigated and conclusions are obtained.

In this lorentzian theory however the regime beyond $t_{\text{max}}$ is unphysical and the theory goes in a non-perturbative domain. This is inevitable as at the turning point the beta-function of $\theta$ is non-zero pushing it to negative values. In this regime the parameter $\theta$ becomes negative indicating that tachyons are present implying the instability of vacuum. The change of sign of gravitational coupling $G$ is an interesting outcome leading to repulsive gravity, but it arises in a regime where there are tachyons and attachment of any useful meaning to it should be done with skepticism. Moreover in this regime the perturbative parameters of QFT, $M^2G$ and $M^2G/\omega$ grows large. They becomes infinite at the attractive fixed point of $\beta$, thereby implying that this regime require a non-perturbative treatment. It will be interesting to see how these issues changes when the same analysis is performed with non-perturbative flow equations obtained through functional renormalisation group within the asymptotic safety scenario picture.

The shortcomings beyond $t_{\text{max}}$, witnessed in the one-loop analysis might indicate the following: (a) The one-loop analysis is not reliable beyond $t_{\text{max}}$ and a higher-loop study is required where $t_{\text{max}}$ might be pushed to infinity. (b) The theory needs some extra ingredient (like Strings etc) to make it UV complete, in which sense the theory is an effective theory. In fact the low energy limit of string theory does contain an infinite number of curvature-dependent terms (local and non-local) that goes beyond quadratic gravity, which hints that perhaps the addition of further higher-derivative terms might bring better stability and convergence in path-integral. (c) Beyond $t_{\text{max}}$ the effects of non-locality should be taken into account along the lines of super-renormalizable theories [65–68]. (d) This might also be hinting that beyond the transition point the theory become entirely conformal [69, 70]. The terms $R$ and $R^2$ are induced in low energy via spontaneous symmetry breaking of weyl-invariance due to radiative corrections [71, 72]. These possibilities are worthy of exploration in future.

Although the shortcomings in this regime are worrisome but nevertheless within the
large allowed energy range ($t_{\text{min}} \leq t \leq t_{\text{max}}$) the theory remains renormalizable to all loops, tachyon and ghost-free [32, 33]. It describes a well-defined renormalizable and unitary QFT in this energy range. This regime is large enough to address length scales ranging from Planck to current cosmological scales. Moreover, the existence of $t_{\text{max}}$ is a one-loop effect which might possible go away (or $t_{\text{max}} \to \infty$) in higher-loop studies as mentioned in [32, 33]. Even in one-loop, by appropriate choice of parameters one can push $t_{\text{max}}$ all the way to Planck-scale (and beyond). This is a relief as the renormalizable theory where ghosts and tachyons are eradicated can be trusted all the way to ultra high energies.

Acknowledgements

I would like to thank Nirmalya Kajuri and Tuhin Mukherjee for useful discussions at various stages of this work. I am grateful to KITPC and Prof. Tianjun Li for support. I would like to thank Ghanashyam Date and IMSc, Chennai for support where a part of work was done.

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