Physics of Renormalization Group Equation in QED

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I. INTRODUCTION

The concept of renormalization is originated from the perturbative treatment in quantum field theory. In QED, one cannot calculate physical observables in a non-perturbative fashion, and therefore, one should employ the perturbation theory. In QED, one can well construct the free fermion and free gauge field Fock spaces (free QED space) which can be characterized by the number of degrees of freedom \( \mathcal{N} \) and the system size \( L \).

Free QED space : \( (L, \Lambda) \) with \( \Lambda = \frac{2\pi}{L} \). 

Therefore, one can develop the perturbation theory in which one treats perturbatively the interaction term which is gauge invariant together with the fermion current conservation. In this case, all the physical quantities must be described in terms of the free QED space terminology.

In the perturbation theory, the self-energy diagrams become divergent, and therefore one should employ the renormalization scheme which is very successful in QED. All of the infinities arising from the self-energy diagrams can be well renormalized into the redefinition of the fermion mass, coupling constant and wave functions. In addition, finite contributions are controlled and evaluated precisely, and some of them are compared with experiments, and one finds that the renormalization scheme is all consistent with experiments \([1, 2]\). After the renormalization, one realizes that the renormalized charge \( e \) should depend on the global quantity \( L \) which characterizes the unperturbed QED space, that is

\[ e = e(L). \]

This is reasonable since one evaluates the renormalization constant in the free QED space terminology, and therefore calculated quantities should depend on the properties of the unperturbed QED space in some way or the other. If one makes this dependence into the differential equation, then it becomes the RG equation which can give the description of the finite size effects of QED.

This must be the basic story of the RG equation which should be given by old field theory experts with the fundamental renormalization scheme of QED \([3]\).

Now, the understanding of the renormalization group equation in recent years is quite different and somewhat puzzling. This difference should be originated from the dimensional regularization scheme \([4, 5]\) even though the dimensional regularization itself has no problem in the renormalization procedure. Indeed, it can give just the same renormalization scheme as the cutoff momentum method, and furthermore it has some advantage since it is simpler and the calculation can be carried out in a covariant way.

However, when one wishes to derive the RG equation, then it becomes problematic. In the dimensional regularization, one introduces a new scale \( \lambda_s \) when one evaluates the momentum integral

\[
\int \frac{d^4k}{(2\pi)^4} \rightarrow \lambda_s^{4-D} \int \frac{d^Dk}{(2\pi)^D}
\]

where \( \lambda_s \) is a parameter which has a mass dimension in order to compensate the unbalance of the momentum integral dimension. \( D \) is set to be

\[
D = 4 - \epsilon
\]

where \( \epsilon \) is an infinitesimally small constant. In this dimensional regularization, the momentum integral is cut out by the four dimensional Euclidean space which is a compact space. This is in contrast to the normal way of regularization with the cutoff \( \Lambda \) confined in the box \( V = L^3 \).

What is \( \lambda_s \)? As long as one employs the perturbation theory, one has to find out a corresponding quantity of \( \lambda_s \) in free QED space, and otherwise one would discover new physics without doing any physics! The corresponding quantity of \( \lambda_s \) in free QED space must be a global quantity, and the only possible candidate should be

\[
\lambda_s \rightarrow \frac{1}{L}
\]

In fact, one may find some correspondence between the
two different regularizations, for example
\[ \Lambda \sim \lambda_s \exp \left( \frac{1}{\epsilon} \right), \quad L \sim \frac{1}{\lambda_s}, \quad N \sim \exp \left( \frac{1}{\epsilon} \right). \] (1.4)

In this way, one can construct the dimensional regularization scheme in terms of the original free QED space terminology.

Therefore, it is clear that one should not apply the RG equation to the treatment of the continuum limit in the lattice gauge theory [5, 6]. In the path integral formulation, the coupling constant \( \epsilon \) is just a constant like the bare charge since it is a non-perturbative treatment, and therefore unphysical results which are obtained by Wilson cannot be remedied by any means.

II. RENORMALIZATION SCHEME

Before coming to the renormalization group equations, we should review the renormalization procedure in QED so as to clarify where the problem comes about.

A. Free QED space

First, we start from the QED Lagrangian density \( L \) which is composed of the unperturbed Lagrangian density \( L_0 \) and the interaction term \( L_I \)
\[ L_0 = \bar{\psi}(\not{p} - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \] (2.1)
\[ L_I = -e A^\mu \bar{\psi} \gamma^\mu \psi. \] (2.2)

In this case, the unperturbed Hamiltonian \( H_0 \) can be constructed from the Lagrangian density \( L_0 \). The Hilbert space of the quantized Hamiltonian \( H_0 \) can be well constructed since one finds the exact eigenvalues and eigenstates of the \( H_0 \). In this case, the QED space can be specified by the box length \( L \) and the cutoff momentum \( \Lambda \) as well as by the energies and momenta of the free fermion and free gauge field states.

Free QED space with \( (L, \Lambda) \)

Fermions: \( E_p = \pm \sqrt{p_n^2 + m^2}, \quad p_n = \frac{2\pi n}{L} \) (2.3a)

Gauge fields: \( \omega_k = |k_n|, \quad k_n = \frac{2\pi n}{L} \) (2.3b)

where \( n_i \) runs as
\[ n_i = 0, \pm 1, \cdots, \pm N \quad \text{with} \quad \Lambda = \frac{2\pi N}{L}. \] (2.3c)

The maximum number of freedom \( N \) is taken to be the same between the fermion and the gauge fields. The perturbative evaluation can be made within this Hilbert space, and one can calculate physical quantities in terms of the expansion of the coupling constant \( \epsilon \). In other words, all the physical observables should be expressed in terms of the free QED space. This simple but important fact has been overlooked in deriving the renormalization group equation.

B. Mass Renormalization

The fermion self-energy \( \Sigma(p) \) can be evaluated with the dimensional regularization as
\[ \Sigma(p) = -ie^2 \frac{L}{4-D} \frac{1}{(2\pi)^D} \frac{1}{\gamma^\mu (\not{p} - k - m\gamma^\mu) k^2} \]
\[ = \frac{e^2}{8\pi^2\epsilon} (-\not{p} + 4m) + \text{finite terms} \] (2.4)

Therefore, the Lagrangian density of the free fermion part
\[ L_F = \bar{\psi}(\not{p} - m)\psi \] (2.5)

should be modified, up to one loop contributions, by the counter term \( \delta L_F \)
\[ \delta L_F = \bar{\psi} \left[ \frac{e^2}{8\pi^2\epsilon} (-\not{p} + 4m) \right] \psi. \] (2.6)

In this case, the total Lagrangian density of fermion becomes
\[ L_F' = \bar{\psi}_b (\not{p} - m_0) \psi_b + \text{finite terms} \] (2.7)

where one introduces the wave function renormalization and the bare mass \( m_0 \)
\[ \psi_b \equiv \sqrt{Z_2} \psi \] (2.8a)
\[ m_0 = m \left( 1 + \frac{e^2}{8\pi^2\epsilon} \right) \left( 1 - \frac{e^2}{2\pi^2\epsilon} \right) \simeq m - \frac{3m e^2}{8\pi^2\epsilon} \] (2.8b)

where one should always keep up to order of \( e^2 \). Here, one defines \( Z_2 \) as
\[ Z_2 = 1 - \frac{e^2}{8\pi^2\epsilon} = 1 - \frac{e^2}{8\pi^2\epsilon} \ln \left( \frac{\Lambda}{m} \right) \] (2.9)

where we also show the calculation of the cutoff momentum scheme. Therefore, the total Lagrangian density has just the same shape as the original one, and thus it is renormalizable.
C. Vacuum Polarization

The divergent contributions to the self-energy of photon can be described in terms of the vacuum polarization

\[ \Pi^{\mu\nu}(k) = i\lambda_s^{4-D} e^2 \int \frac{d^Dk}{(2\pi)^D} Tr \left[ \gamma^{\mu} \frac{1}{\not p - m - \not k - m} \gamma^{\nu} \frac{1}{\not p - \not k - m} \right] \]

\[ = \frac{e^2}{6\pi^2\epsilon} (k^\mu k^\nu - g^{\mu\nu} k^2) + \text{finite terms.} \quad (2.10) \]

Defining \( Z_3 \) and the vector field renormalization by

\[ Z_3 = 1 - \frac{e^2}{6\pi^2\epsilon} = 1 - \frac{\epsilon^2}{6\pi^2} \ln \left( \frac{\Lambda}{m} \right) \]

\[ A_b^\mu = \sqrt{Z_3} A^\mu \quad (2.12) \]

one can rewrite the Lagrangian density of the gauge field as

\[ \mathcal{L}'_{GF} = -\frac{Z_3}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} (\partial^\mu A_b^\nu - \partial^\nu A_b^\mu)^2 + \cdots. \quad (2.13) \]

D. Vertex Corrections

The vertex corrections can be evaluated as

\[ \Lambda^\mu(p', p) = -i\lambda_s^{4-D} e^2 \int \frac{d^Dk}{(2\pi)^D} \]

\[ \times \left[ \gamma^\nu \frac{1}{\not p' - \not k - m} \gamma^\mu \frac{1}{\not p - \not k - m} \gamma^\rho \frac{1}{k^2} \right] \]

\[ = \frac{e^2}{8\pi^2\epsilon} \gamma^\mu + \text{finite terms.} \quad (2.14) \]

Therefore, the counter term of the interaction Lagrangian density \( \delta \mathcal{L}_I \) becomes

\[ \delta \mathcal{L}_I = \epsilon \lambda_s^2 \left( \frac{e^2}{8\pi^2\epsilon} \right) A^\mu \bar{\psi} \gamma^\mu \psi. \quad (2.15) \]

In this case, the total interaction Lagrangian density can be written as

\[ \mathcal{L}'_I = -Z_1 e\lambda_s^2 A^\mu \bar{\psi} \gamma_\mu \psi + \text{finite terms} \quad (2.16) \]

where \( Z_1 \) is defined as

\[ Z_1 = 1 - \frac{e^2}{8\pi^2\epsilon} = 1 - \frac{\epsilon^2}{8\pi^2} \ln \left( \frac{\Lambda}{m} \right). \quad (2.17) \]

The interaction Lagrangian density can be rewritten in terms of the bare quantities

\[ \psi_b = \sqrt{Z_2} \psi, \quad A_b^\mu = \sqrt{Z_3} A^\mu \quad (2.18) \]

as

\[ \mathcal{L}'_I = -Z_1 e\lambda_s^2 A^\mu \bar{\psi} \gamma_\mu \psi = -Z_1 e\lambda_s^2 \frac{1}{Z_2 \sqrt{Z_3}} A_b^\mu \bar{\psi}_b \gamma_\mu \psi_b \]

\[ = -e_b A_b^\mu \bar{\psi}_b \gamma_\mu \psi_b + \text{finite terms} \quad (2.19) \]

where the bare charge \( e_b \) is defined as

\[ e_b = \epsilon \lambda_s^2 \frac{1}{\sqrt{Z_3}}. \quad (2.20) \]

Therefore, all the infinite quantities are renormalized into the physical constants as well as the wave functions.

III. RENORMALIZATION GROUP EQUATION

Now, one sees that the renormalized charge \( e \) depends on the properties which characterize the unperturbed system.

A. Dimensional regularization

In the dimensional regularization scheme, \( e \) depends on the momentum scale \( \lambda_s \) as

\[ e_b = \epsilon \lambda_s^2 \left( 1 - \frac{e^2}{6\pi^2\epsilon} \right)^{-\frac{1}{2}} \approx \epsilon \lambda_s^2 \left( 1 + \frac{e^2}{12\pi^2\epsilon} \right). \quad (3.1) \]

Since the bare charge \( e_b \) should not depend on the system, one finds the following RG equation

\[ \lambda_s \frac{\partial e}{\partial \lambda_s} = \frac{1}{12\pi^2\epsilon} e^3 + O(e^5). \quad (3.2) \]

This equation can be easily solved for \( e \). The expression for the running coupling constant \( \alpha(\lambda_s) \equiv \frac{\epsilon^2}{4\pi} \) is given as

\[ \alpha(\lambda_s) = \frac{\alpha(\lambda_s^0)}{1 - 2\lambda_s \alpha(\lambda_s^0) \ln \left( \frac{\lambda_s}{\lambda_s^0} \right)} \quad (3.3) \]

where \( \lambda_s^0 \) denotes the renormalization point for the coupling constant. This is the standard procedure to obtain the behavior of the coupling constant as the function of \( \lambda_s \). However, the \( \lambda_s \) does not appear in the Hilbert space of the unperturbed Hamiltonian \( H_0 \), and therefore one should find out the physical quantity corresponding to the momentum scale \( \lambda_s \) itself as we discussed above.

One can see that the only possible quantity for the \( \lambda_s \) in the unperturbed Hilbert space should be the inverse of the box length \( L \), that is \( \lambda_s \sim \frac{1}{L} \) as discussed in eq.(1.4). In this case, one should always take the thermodynamic limit at the end of calculation, and therefore, one should make the limit of

\[ \lambda_s \rightarrow \lambda_s^0 \approx 0 \quad (3.4) \]

where \( \lambda_s^0 \) corresponds to the thermodynamic limit.
B. Cutoff momentum regularization

The RG equation can be obtained in the case of the cutoff momentum treatment. In this case, the bare charge \( e_b \) can be written
\[
e_b = e + \frac{e^3}{12\pi^2} \ln \left( \frac{\Lambda}{m} \right) = e + \frac{e^3}{12\pi^2} \ln \left( \frac{2\pi N}{mL} \right). \tag{3.5}
\]
The bare charge \( e_b \) should not depend on the box length \( L \), and therefore one can derive the constraint equation for \( e \)
\[
L \frac{\partial e}{\partial L} = \frac{1}{12\pi^2} e^3 + O(e^5). \tag{3.6}
\]
Thus, one obtains for the running coupling constant \( \alpha(L) \)
\[
\alpha(L) = \frac{\alpha(L_\infty)}{1 - \frac{2\alpha(L_\infty)}{3\pi} \ln \left( \frac{L}{L_\infty} \right)}. \tag{3.7}
\]
where \( L_\infty \) denotes the value which corresponds to the thermodynamic limit, and \( \alpha(L_\infty) \) should be fit to the observed value of the fine structure constant. In normal circumstances, one should always take the thermodynamic limit of \( L \to \infty \) in order to obtain any physical observables. However, in case one wishes to examine the finite size effects in the model field theory, then one can make use of the RG equation of eq.(3.7).

C. Difference in Renormalization Group Equations

It should be interesting to note that the behaviors between eqs.(3.3) and (3.7) are opposite to each other. The different behavior basically originates from the wave functions in \( D \) dimensions in the dimensional regularization scheme. The dimensions of the fields \( \psi \) and \( A^\mu \) become in the dimensional regularization
\[
[\psi] \sim \lambda_s^{\frac{D-1}{2}}, \quad [A^\mu] \sim \lambda_s^{\frac{D-2}{2}}. \tag{3.8}
\]
Therefore, the dimension of the interaction Lagrangian density must be modified by hand as
\[
\mathcal{L}_I = -e\lambda_s^{\frac{D-3}{2}} A^\mu \bar{\psi} \gamma_\mu \psi \tag{3.9}
\]
since the dimension of the Lagrangian density must be \( [\mathcal{L}] \sim \lambda_s^D \). The factor \( \lambda_s^{\frac{D-3}{2}} \) plays an important role for the sign in front of RG equation, and indeed it causes the different RG equations from the cutoff regularization scheme. But the physical significance of the different RG equations between the two regularization schemes is unclear.

IV. CONCLUSIONS

The concept of the renormalization is originated from the perturbative treatment in quantum field theory. Since it is practically impossible to find the exact eigenstates of the quantized Hamiltonian in quantum field theory, it is natural that the theoretical framework is based on the perturbative approach.

After the renormalization, one realizes that the renormalized charge \( e \) should depend on the momentum scale \( \lambda_s \) since the bare charge should not depend on the system, and this is a reasonable condition. In this case, however, one should understand what the \( \lambda_s \) indicates in terms of physical observables. The Hilbert space of the unperturbed Hamiltonian is well constructed, and therefore one must find a quantity corresponding to the \( \lambda_s \) in this Hilbert space as long as one employs the perturbation theory. The only reasonable candidate for the \( \lambda_s \) must be the inverse of the box length \( L \) as shown in eq.(1.4). Therefore, one sees that the RG equation gives the finite size behavior of the renormalized coupling constant \( e \).

In this sense, one should always be careful for applying the result of the RG equation to other physical processes. The renormalization scheme itself is perfectly well constructed, but the RG group equation itself cannot be more than the perturbation theory, and it shows how the renormalized coupling constant \( e \) may respond to the change of the system size \( L \).

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