A Latent Class Modeling Approach for Generating Synthetic Data and Making Posterior Inferences from Differentially Private Counts

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Abstract

Several algorithms exist for creating differentially private counts from contingency tables, such as two-way or three-way marginal counts. The resulting noisy counts generally do not correspond to a coherent contingency table, so that some post-processing step is needed if one wants the released counts to correspond to a coherent contingency table. We present a latent class modeling approach for post-processing differentially private marginal counts that can be used (i) to create differentially private synthetic data from the set of marginal counts, and (ii) to enable posterior inferences about the confidential counts. We illustrate the approach using a subset of the 2016 American Community Survey Public Use Microdata Sets and the 2004 National Long Term Care Survey.

1 Introduction

National statistical organizations and other data curators, henceforth all called agencies, often seek to share collected information with outside researchers. Doing so requires methods that provide privacy protection to data subjects while maintaining statistical relationships in the data. Many agencies use statistical disclosure control (SDC) methods that blur the private data in a prescribed way, with the aim to provide some level of privacy protection. Traditional SDC methods include data swapping, cell suppression, top- or bottom- coding, and the addition of random noise [Hundepool et al., 2012]. Generally, traditional SDC techniques have been applied with low intensity, so as not to degrade the quality of the information in the data. However, with the growth in available data and powerful computing, many agencies have become concerned that such SDC methods are not adequately protective.

To address this issue, several agencies have turned to using synthetic data methods [Rubin, 1993, Little, 1993, Reiter, 2005a]. The agency releases data that are generated from

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some statistical model, estimated with the private data. With a full synthesis, agencies can ensure that no sensitive values are released on the file, which can reduce disclosure risks. Further, the agency can offer researchers access to record-level data instead of only summary statistics or other high-level information. Many different methods have been proposed to generate synthetic data [e.g., Reiter, 2005b, Woodcock and Benedetto, 2009, Mateo-Sanz et al., 2004, Slavković and Lee, 2010] and assess the utility and disclosure risk associated with the data release [Drechsler, 2011, Hundepool et al., 2012, Karr et al., 2006, Reiter et al., 2014, Snoke et al., 2018]. The U.S. Census Bureau has released synthetic data for the Longitudinal Business Database [Kinney et al., 2011], Survey of Income and Program Participation [Benedetto et al., 2013], and the LEHD Origin-Destination Employment Statistics (with the data product being known as OnTheMap) [Machanavajjhala et al., 2008]. Affiliated researchers have proposed methodologies for other national surveys (e.g., federal administrative data from the Office of Personnel Management [Barrientos et al., 2018]), and other statistical agencies have developed synthetic data products such as the Scottish Longitudinal Survey [SLS-DSU, 2018] and the German IAB Establishment Data [Drechsler, 2009].

Most SDC methods, including many synthetic data techniques, do not provide formally quantifiable privacy guarantees. Instead, agencies evaluate disclosure risks based on assumptions of intruder knowledge and behavior [Hundepool et al., 2012]. An alternative approach is to design and use SDC methods that satisfy differential privacy [Dwork et al., 2006b]. Differentially private algorithms exist for releasing counts [e.g., Dwork et al., 2006b, Ghosh et al., 2012], k-way marginals [Barak et al., 2007, Yang et al., 2012, Li et al., 2018], regression coefficients [Zhang et al., 2012, Snoke and Slavković, 2018, Awan and Slavkovic, 2021], and many other quantities that can be viewed as answers to user-specified queries. Several statistical agencies are interested in combining the privacy guarantees from differential privacy with the flexibility afforded by synthetic data. For example, the U.S. Census Bureau uses differentially-private synthetic data as the backbone of its 2020 population census data releases [Abowd, 2018]. However, it can be challenging to generate differentially private synthetic data with low error [Garfinkel et al., 2018].

In this article, we propose methodology to generate differentially private synthetic data sets for multivariate count data, i.e., data from contingency tables. Our approach can be viewed as a post-processing method for generating synthetic data, a strategy also suggested in McKenna et al. [2021a]. Specifically, we first assume that an agency has selected a set of marginal counts and has used some existing differentially private algorithm to add noise to those counts. For example, the agency could select the margins for which accurate values are especially important. Alternatively, the agency could select a set of margins that covers many types of analyses done by users of the data, e.g., all k-way margins. Second, we specify a Bayesian latent class model for the underlying confidential data [Dunson and Xing, 2009]. We write the agency-selected margins as functions of the model parameters. Third, we estimate the parameters in the functions using only the differentially private counts and a composite likelihood-based approach [Lindsay, 1988]. Finally, we sample from the estimated model to generate record-level, synthetic data. Agencies can generate multiple copies of the synthetic datasets, without any extra privacy loss, in order to enable secondary data analysts to estimate uncertainty and make inferences [Reiter, 2003]. Generating differentially-private synthetic data from a user-specified list of summaries can be particularly advantageous for contingency tables with complex structures, e.g., data with structural zeros [Manrique-Vallier and Reiter, 2014, Li et al., 2018] or nested data such as individuals in households [Hu et al., 2018]. We illustrate the latter in the online supplementary material.
The latent class modeling approach also can be viewed as an engine for approximate posterior inferences for the underlying confidential counts given a collection of differentially private counts, under the assumptions implied by the latent class model. Specifically, analysts can use parameters draws from the Markov chain Monte Carlo (MCMC) sampler to obtain posterior inferences for any functional of the parameters, i.e., any count in the table.

The remainder of the paper is organized as follows. In Section 1.1 we review related literature. In Section 2 we present preliminary information on data privacy and latent class models. In Section 3 we discuss the proposed approach and implementation details. In Section 4, we illustrate the approach by making posterior inferences from differentially private counts with a small set of variables from the 2016 American Community Survey (ACS) Public Use Microdata Sets (PUMS), and compare it to several other approaches from the literature. In Section 5, we illustrate the approach for both posterior inference and synthetic data generation using the 2004 National Long Term Care Survey, which has larger dimensions than the ACS example. For both illustrations, we use the Geometric Mechanism [Ghosh et al., 2012] to ensure differential privacy in the first stage counts. Finally, in Section 6 we end with a discussion.

1.1 Related Literature

Protection of counts and contingency tables has been the subject of many papers in the privacy literature. Some of the first examples of differentially-private (DP) data were counts and histograms perturbed by the Laplace mechanism [Dwork et al., 2006b]. Barak et al. [2007] proposed methodology to produce DP marginal counts by using Fourier bases. Yang et al. [2012] extended this work to multi-dimensional contingency tables and noted some theoretical and practical shortcomings to both approaches. Machanavajjhala et al. [2008] proposed constructing DP contingency tables via a Dirichlet-Multinomial synthesizer, and the Hardt-Ligett-McSherry algorithm [Hardt et al., 2012] used a multiplicative weights approach.

Park and Ghosh [2014] proposed methodology for synthesis of DP contingency tables by disintegrating the data into suitable building blocks, injecting noise to these blocks, and using a Gibbs sampler to draw synthetic samples. Li et al. [2018] proposed to generate synthetic data by constructing an empirical hashed conditional distribution from the whole histogram and applying a Stability Based Algorithm to these empirical distributions. McKenna et al. [2019] proposed constructing DP synthetic data where “suitable building blocks” are a set of selected marginal counts; they add Laplace noise to the selected counts, fit a graphical model to the perturbed counts, and sample synthetic data from this model. Similarly, PrivBayes [Zhang et al., 2017] constructs DP synthetic data sets via Bayesian networks. These authors use networks to approximate the underlying data distribution through lower order marginals, add noise to these marginals, and approximate a full data distribution through the noisy counts and constructed Bayesian network. We note that these two methods placed first [McKenna et al., 2021b] and third [Bao et al., 2021], respectively, in the 2018-2019 National Institute of Standards and Technology Public Safety Communications Research division’s Differential Privacy Synthetic Data Challenge [National Institute of Standards and Technology, 2021]; see Bowen and Snoke [2019] for a discussion of the performance of these methods in the context of the challenge. Similarly, the CIPHER method [Eugenio and Liu, 2018] estimates the joint distribution of a table based on a lower-order set of DP marginals. Their approach can be viewed as a post-processing technique for generating synthetic data.
Hay et al. [2010] showed that accuracy can be greatly improved by accounting for the constraints that a contingency table must satisfy in post-processing steps, although some very recent papers argue for post-processing to be part of posterior modeling (e.g., see Seeman et al. [2020], and references therein). Lee et al. [2015] extended this result by also accounting for the noise distribution and developed a fast generic approach for solving the resulting optimization problem.

Bowen and Liu [2016] compared many parametric and nonparametric methods for differentially private data synthesis, beyond those discussed above, and Charest [2012] showed that requiring greater privacy can degrade inferential results. Rinott et al. [2018] discussed several issues related to contingency table release under differential privacy, including the potential effects of rounding and other post-processing steps. They compared several simple mechanisms and offered comments on utility in private data releases. Similarly, Raab [2019] discussed some practical limitations to contingency table protection under formal privacy guarantees and presented a new method to create differentially private contingency tables from a subset of marginals via the Laplace mechanism and iterative proportional fitting.

All of these methods (ours included) are subject to the results of Ullman and Vadhan [2020]. They showed that, even in simple cases, it is impossible to construct a polynomial time differentially private algorithm that preserves all two-way marginals.

Accounting for the additional privacy-preserving noise in inference also has been discussed in the literature. Charest [2012] used a simple Bayesian model to account for noise added under the Beta-Binomial synthesizer. Karwa et al. [2015] accounted for noise by treating the original data as missing. As the resulting likelihoods are often intractable, they developed a technique relying on a variational approximation for estimation. Seeman et al. [2020] showed that naive post-processing (such as direct modification of counts to meet constraints) can result in loss of information and proposed a Bayesian sampling scheme to post-process counts based on the noisy counts and outside constraining information. Gong [2019] showed that approximate Bayesian computing can be used to account for DP noise to obtain draws from the correct posterior distribution. Karwa et al. [2017] investigated methods to release DP synthetic networks while also accounting for the additional noise by treating the original private data as missing and using a likelihood-based approach. While not dealing with DP tables, Woo and Slavković [2012] proposed several EM algorithms to obtain correct logistic regression estimates for tables that are protected using the Post Randomization Method (PRAM) which could be extended to DP settings.

Turning to latent class models, Si and Reiter [2013] used latent class specifications for missing data imputation in contingency tables of Dunson and Xing [2009]. This latent specification belongs to the family of Bayesian nonparametric techniques, which are well-known for being a flexible modeling choice to capture complex data patterns. Manrique-Vallier and Reiter [2014] refined these latent class models to incorporate cases with structural zeros, e.g., a person age five cannot have a college degree. The full table latent class approach was further extended to handle nested variables (e.g., individuals nested in households) [Hu et al., 2018], structural zeros [Akande et al., 2019a], and missing data [Akande et al., 2019b]; see the Supplementary Material for a discussion of the model presented in Hu et al. [2018] and how our proposed approach can be augmented for nested data structures.
2 Preliminaries

2.1 Data Privacy Methods

2.1.1 Synthetic Data

Let $X = \{(x_{i1}, \ldots, x_{ip})\}_{i=1}^{n}$ be an observed data set comprising $n$ individuals measured on $p$ confidential categorical variables. For $i = 1, \ldots, n$ and $j = 1, \ldots, p$, each $x_{ij} \in \{1, \ldots, d_j\}$. We refer to the variables in $X$ using $X_j$ where $j = 1, \ldots, p$. Synthetic data approaches aim to preserve the joint distribution $f(X)$ of these variables by modeling them simultaneously, e.g., as in Hu et al. [2018] and Akande et al. [2019b], or by using sequential modeling of the form,

$$f(X) = f(X_1)f(X_2|X_1)\cdots f(X_p|X_1, \ldots, X_{p-1}).$$

(1)

Different types of statistical models can be used for the conditional distributions in (1), including classification and regression trees (CART) [Reiter, 2005b], random forests [Caiola and Reiter, 2010], kernel density estimators [Woodcock and Benedetto, 2009], and combinations thereof [Barrientos et al., 2018]. Choice of model depends on both the data structure and desired privacy levels.

2.1.2 Differential Privacy

Differential privacy (DP, [Dwork et al., 2006b, 2014]) is a formal privacy framework that guarantees privacy protection by bounding the ratio of output densities for all neighboring data sets. Intuitively, it protects against attackers seeking to learn whether or not any specific individual’s data were included (or excluded) in the data set over which the output was calculated. That is, the inclusion (or exclusion) of a single person’s information does not change the output of the algorithm greatly. In this paper we focus on its strongest form, $\epsilon$-DP, where $\epsilon$ is a privacy-loss parameter, with smaller values indicating stronger protection, but our model could be modified for other forms of DP.

Definition 2.1. $\epsilon$-Differential Privacy [Dwork et al., 2006b]: A randomized algorithm $A$ satisfies $\epsilon$-differential privacy if for all data sets $X = \{(x_{i1}, \ldots, x_{ip})\}_{i=1}^{n}$ and $X'$ differing on at most one row, and $S \subseteq \text{Range}(A)$,

$$\frac{\Pr[A(X) \in S]}{\Pr[A(X') \in S]} \leq \exp(\epsilon).$$

(2)

Differential privacy is a property of the algorithm itself. It is achieved by randomness. Many different algorithms have been proposed to release differentially private statistics or data sets (e.g., the Laplace [Dwork et al., 2006b] or Geometric [Ghosh et al., 2012] mechanisms, which are the most relevant to our setting). Several relaxations and extensions of differential privacy have been proposed such as $(\epsilon, \delta)$-differential privacy [Dwork et al., 2006a] and Rényi differential privacy [Mironov, 2017]; for more, see Dwork et al. [2014].

Many DP mechanisms (ours included) rely on the useful properties of post-processing and sequential composition. These properties outline how DP privacy guarantee can be preserved and how the privacy-loss parameter is propagated through multiple data releases. Theorem 2.2 shows that DP is preserved through post-processing, and is key to our proposed approach: the augmented Bayesian latent class model and measurement error model is a
post-processing technique for a noisy, privacy-preserving count. Theorem 2.3 shows how the privacy-loss parameter $\epsilon$ propagates through multiple data releases on the same individual.

**Theorem 2.2.** Post-processing [Dwork et al., 2006b]: Let $A$ be any randomized algorithm such that $A(X)$ is $\epsilon$-differentially private, and let $g$ be any function. Then, $g(A(X))$ also satisfies $\epsilon$-differential privacy.

**Theorem 2.3.** Sequential composition [McSherry, 2009]: Let $A_i$ each provide $\epsilon_i$-differential privacy. The sequence of $A_i(X)$ provides $(\sum_i \epsilon_i)$-differential privacy.

### 2.1.3 Geometric Mechanism

The Geometric Mechanism [Ghosh et al., 2012] adds noise from a two-sided geometric distribution to achieve differential privacy. Properties of this distribution, including simulation techniques, are discussed in Inusah and Kozubowski [2006] and its references.

**Definition 2.4.** A random variable $Y$ distributed as a two-sided geometric distribution has probability mass function

$$P(Y = k) = \frac{1 - \alpha}{1 + \alpha} \alpha^{|k|}$$

where $0 \leq \alpha \leq 1$.

**Theorem 2.5.** Geometric Mechanism [Ghosh et al., 2012]: For $f : D \to \mathbb{R}^d$, the mechanism $A$ that adds independently drawn noise from a two-sided-Geom($\exp\{\frac{-\epsilon}{\Delta f}\}$) distribution to each of the $d$ terms of satisfies $\epsilon$-differential privacy.

This mechanism has several appealing properties for protecting count data. First, the added noise values are integers, which eliminates the need for a post-processing step to deal with decimals. Second, Ghosh et al. [2012] show that this mechanism is optimal for every potential user regardless of the side information that they posses when releasing DP count queries using a Bayesian framework.

### 2.2 Bayesian Latent Class Models

Dunson and Xing [2009] developed a flexible methodology for modeling unordered categorical data using a Bayesian nonparametric mixture model. Their methods are directly applicable to modeling complete contingency tables.

Suppose each observation $i = 1, \ldots, n$ belongs to a latent class denoted by $z_i \in \{1, 2, \ldots\}$. The probability for each unique combination of variable levels is specified according to these latent classes. For ease of notation, we allow $x_{ij}$ to stand for random variables as well as the observed data. The latent class model is of the form,

$$x_{ij} | z_i, \{\Psi_h^{(j)}\}_{h=1}^{\infty} \overset{ind}{\sim} \text{Multinomial}\{1, \Psi_{z_i,1}^{(j)}, \ldots, \Psi_{z_i,d_j}^{(j)}\}, i = 1, \ldots, n, j = 1, \ldots, p,$$

$$z_i | \{\pi_h\}_{h=1}^{\infty} \overset{ind}{\sim} \text{Discrete}\{1, \ldots, \infty, (\pi_1, \ldots, \pi_{\infty})\},$$

$$\pi_h = V_h \prod_{i<h}(1 - V_i), \quad V_h \sim \beta(1, \alpha),$$

$$\Psi_h^{(j)} \sim \text{Dirichlet}(a_{j1}, \ldots, a_{jd_j}),$$

(4)
where \( \Psi^{(j)}_h = (\Psi^{(j)}_{h1}, \ldots, \Psi^{(j)}_{hd_j}) \), \( \alpha > 0 \), and \((a_{j1}, \ldots, a_{jd_j})\) is a vector with positive components. Under model (4), for any feasible set of values \((c_1, \ldots, c_p)\) of the categorical variables, one has

\[
Pr(x_{i1} = c_1, \ldots, x_{ip} = c_p | \{\Psi^{(j)}_h\}_{h=1}^{\infty}, \{\pi_h\}_{h=1}^{\infty}) = \sum_{h=1}^{\infty} \pi_h \prod_{j=1}^{p} \Psi^{(j)}_{hc_j},
\]

which corresponds to a mixture of product multinomials with a Dirichlet process [Ferguson, 1973] as a mixing distribution.

To facilitate posterior computation, we use a finite dimensional approximation of model (4), which can be estimated using Gibbs sampling. The approximation relies on the assumption that \( z_i \in \{1, \ldots, k\} \), where \( k < \infty \), which is equivalent to truncating the mixture model (5) to \( k \) components; see Ishwaran and James [2001] for a discussion of the truncated Dirichlet processes. Similar approximations along with Gibbs sampling have been successfully used for private data synthesis, including in Hu et al. [2018], Manrique-Vallier and Reiter [2014] and Akande et al. [2019b], among others.

### 2.3 Composite Likelihood Methods

Composite likelihood methods allow analysts to circumvent the specification of the full joint likelihood function [Lindsay, 1988, Varin et al., 2011]. Instead, analysts can use an approximation to the likelihood function, often with the assumption of some form of independence, to simplify computation.

We use composite likelihood methods to approximate the joint likelihood of the latent class model given an agency-specified set of marginal counts. Let \((M_1, \ldots, M_T)\) represent the list of marginal counts of \(X\) with \( \theta = \left( \{\Psi^{(j)}_h\}_{h=1,j=1}^{k,p}, \{\pi_h\}_{h=1}^{k} \right) \). Let \( \mathcal{L}_t(\theta; M_t) \) be the likelihood function corresponding to \( M_t \) induced by Model (4). We approximate the joint likelihood of \( \theta \) given \((M_1, \ldots, M_T)\) using

\[
\mathcal{L}_C(\theta; M_1, \ldots, M_T) \approx \prod_{t=1}^{T} \mathcal{L}_t(\theta; M_t).
\]

The composite likelihood \( \mathcal{L}_C \) allows for inference about \( \theta \) only from the marginal counts [Varin et al., 2011]. We note that, while computationally expedient, inference based on only a subset of counts of \( X \) could result in inaccurate estimates of \( \theta \) [Walker, 2013], particularly for regions of its distribution that describe the counts excluded from the set of selected marginals. Furthermore, not all combinations of marginals, and possibly conditional distributions fully and uniquely specify the joint distribution (e.g., see Fienberg and Slavkovic [2005], Slavkovic et al. [2015] within the context of confidentiality protection, and more general related references therein).

### 3 Proposed Methods

#### 3.1 Overview

We assume the agency adds noise to the true counts for a set of agency-specified marginals, \((M_1, \ldots, M_T)\), computed from \( X \) using the Geometric mechanism. In our simulation examples, we use the definition of sensitivity based on changing one row; hence, the sensitivity
for each $M_t$, where $t = 1, \ldots, T$, equals two.\footnote{Under other definitions of sensitivity, the sensitivity equals 1 since adding or deleting one row changes at most one marginal count.} We assume the agency has determined appropriate values of $\epsilon$ to control the overall privacy budget. See Section 4 for a discussion in the context of our simulations.

When estimating the parameters of the Bayesian latent class model, we account for the additional noise due to DP using a measurement error model [Fuller, 2009]. Specifically, we treat the true underlying counts as unknown and incorporate the privacy preserving mechanism into the model. Let $\tilde{M}$ denote the observed noisy marginal counts. We have

$$\tilde{M} = (M_1 + \varepsilon_1, \ldots, M_T + \varepsilon_T),$$

$$\varepsilon_t \overset{\text{ind}}{\sim} \text{two-sided-Geom}_{r_t}\left(\exp\left\{\frac{-\epsilon}{\Delta M_t T}\right\}\right), t = 1, \ldots, T,$$

$$M_t|\pi_k, \Psi_k \overset{\text{ind}}{\sim} \text{Multinomial}_{r_t}(n, P_t(\pi_k, \Psi_k)), t = 1, \ldots, T,$$

$$\pi_k = \{\pi_{kh}\}_{h=1}^k, \Psi_k = \{\Psi^{(j)}_{kh}\}_{h=1,j=1}^k,$$

where each $M_t$ comprises $r_t$ counts, $\varepsilon_t$ is a random vector with $r_t$ independent components distributed as two-sided-Geom $\left(\exp\left\{\frac{-\epsilon}{\Delta M_t T}\right\}\right)$, and $\Delta M_t = 2$ is the sensitivity of $M_t$. We complete the model specification by assuming the prior distributions for $\pi_k$ and $\Psi_k$ used in (4). Since the summaries $M_t$ are assumed to be marginal counts, the distribution induced by $X \mapsto M_t$ is multinomial with parameters $n$ and $P_t(\pi_k, \Psi_k)$, where $P_t(\pi_k, \Psi_k)$ represents the probabilities of the counts in $M_t$ and is computed using expression (5).

The third line of (7) represents our “augmented” step [Lindsay, 1988, Varin et al., 2011]: each marginal count is assumed to be independently drawn from multinomial distributions. We rely on this assumption as it is not obvious how to characterize the joint distribution for $(M_1, \ldots, M_T)|\pi_k, \Psi_k$. Although the premise of independence leads to a product of multinomial distributions that places positive probability outside the range of $(M_1, \ldots, M_T)$, we expect such probability to be small since $(P_1(\pi_k, \Psi_k), \ldots, P_T(\pi_k, \Psi_k))$ is a coherent system of probabilities that accounts for the underlying constraints among $M_1, \ldots, M_T$. Hence, in order to generate synthetic tables that consider such constraints, our strategy is to sample from the distribution of $\pi_k, \Psi_k|\tilde{M}$, and approximate the underlying table cell probabilities by

$$Pr(x_{(n+1)p} = c_1, \ldots, x_{(n+1)p} = c_p|\tilde{M}) = \int \int Pr(x_{(n+1)p} = c_1, \ldots, x_{(n+1)p} = c_p|\pi_k, \Psi_k)Pr(\pi_k, \Psi_k|\tilde{M})d\pi_kd\Psi_k.$$ 

Here, $Pr(x_{(n+1)p} = c_1, \ldots, x_{(n+1)p} = c_p|\pi_k, \Psi_k)$ is defined as in (5).

Regardless of which marginal counts are used in estimation, probabilities corresponding to any cell of the contingency table or marginal count can be estimated through $(\pi_k, \Psi_k)$. At each iterate, the desired probabilities can be tabulated using the sampled values of $(\pi_k, \Psi_k)$ at that iterate. This results in a posterior distribution for each desired probability. Synthetic data can be created from the full cell counts either by straightforward post-processing (e.g., multiply full table probabilities by the desired table size and expand) or by multinomial sampling of the cells weighted by the full table probabilities.

Any arbitrary set of summary statistics can be fed into the model. However, to increase the statistical usefulness of the estimated table, we suggest using sets of summary statistics
that include at least one instance of every variable and its corresponding levels, which we do here. In addition, when privacy budgets are constrained, we recommend using few rather than many marginal counts. Using a smaller set of marginals allocates more of the privacy budget to each marginal table, which can improve accuracy for those counts. We note that the dimension of parameters \((\pi, \psi)\) to be estimated does not depend on the number of tables passed to the model.

In general, we expect these results to improve as the accuracy of the formally private counts improves. However, difficulties can arise with the measurement error step of our model if it is hard to specify the DP noise generating process.

This approach to differentially private synthesis does not suffer from privacy loss from running the estimation algorithm. Noise is injected to the marginal counts prior to the MCMC sampling [Chaudhuri et al., 2013]. The algorithm uses these noisy marginals and knowledge of the noise distribution (which does not violate differential privacy) to estimate the underlying contingency table through \((\pi_k, \Psi_k)\).

### 3.2 Illustrative Implementation

To illustrate implementations, suppose the agency has generated all 40 two-way marginal counts from a 2\(^5\) table. We can write each two-way marginal probability as a sum over all combinations of the remaining three variables, for example,

\[
Pr(x_{i1} = 0, x_{i2} = 0 | \pi_k, \Psi_k) = \sum_{j=0}^{k} \sum_{l=0}^{m} \sum_{m=0}^{n} \Pr(x_{i1} = 0, x_{i2} = 0, x_{i3} = j, x_{i4} = l, x_{i5} = m | \pi_k, \Psi_k)
\]

\[
= \sum_{k} \sum_{h=1}^{k} \pi_h \Psi_{h0}^{(1)} \Psi_{h0}^{(2)} \sum_{j=0}^{k} \sum_{l=0}^{m} \sum_{m=0}^{n} \Psi_{hj}^{(3)} \Psi_{hl}^{(4)} \Psi_{hm}^{(5)} .
\]

(8)

Similarly, we have

\[
Pr(x_{i1} = 0, x_{i2} = 1 | \pi_k, \Psi_k) = \sum_{k} \sum_{h=1}^{k} \pi_h \Psi_{h1}^{(1)} \Psi_{h0}^{(2)} \sum_{j=0}^{k} \sum_{l=0}^{m} \sum_{m=0}^{n} \Psi_{hj}^{(3)} \Psi_{hl}^{(4)} \Psi_{hm}^{(5)}
\]

(9)

\[
Pr(x_{i1} = 1, x_{i2} = 0 | \pi_k, \Psi_k) = \sum_{k} \sum_{h=1}^{k} \pi_h \Psi_{h1}^{(1)} \Psi_{h0}^{(2)} \sum_{j=0}^{k} \sum_{l=0}^{m} \sum_{m=0}^{n} \Psi_{hj}^{(3)} \Psi_{hl}^{(4)} \Psi_{hm}^{(5)}
\]

\[
Pr(x_{i1} = 1, x_{i2} = 1 | \pi_k, \Psi_k) = \sum_{k} \sum_{h=1}^{k} \pi_h \Psi_{h1}^{(1)} \Psi_{h1}^{(2)} \sum_{j=0}^{k} \sum_{l=0}^{m} \sum_{m=0}^{n} \Psi_{hj}^{(3)} \Psi_{hl}^{(4)} \Psi_{hm}^{(5)}.
\]

We denote the probabilities in (8) and (9) by \(P_i(\pi_k, \Psi_k)\) for each corresponding count in Model 7.

After writing all the two-way marginal counts as functions of the latent class parameters, we can estimate the posterior distribution of all parameters using a MCMC sampler; see Supplementary Section 2 for details. We developed an R package to fit the proposed model given an arbitrary set of two-way marginals from a 2\(^p\) contingency table of any size\(^2\)

\[^2\text{This package, along with all code used in this paper can be found on Github at https://github.com/michellepistner/BayesLCM.}\]
4 Application to American Community Survey Data

In this section, we illustrate the proposed approach using data from the ACS PUMS and a small number of variables; see Table 1. In this illustration, we use the latent class model as a post-processing algorithm to obtain posterior inferences from a set of released noisy counts.

4.1 Data Description

For each simulation run, we randomly select a subset of 10,000 individuals from the 2016 one-year ACS PUMS. We use five variables on each individual, namely sex, age, race, citizenship, and income. We recode the latter four variables to binary variables using the rules in Table 1. Table 2 displays all the 2-way marginal tables from one such sample, to give a sense of the typical counts.

Table 1: Data description for the ACS recoded variables. Original data was collected from the 2016 1-year ACS PUMS and recoded according to the guidelines below.

| Variable | Label | Levels |
|----------|-------|--------|
| Sex      | SEX   | 0: males, 1: females |
| Age      | AGE   | 0: under age 18, 1: over age 18 |
| Race     | RACE  | 0: non-white, 1: white |
| Citizenship | CIT | 0: non-U.S. citizen, 1: U.S. citizen |
| Income   | INC   | 0: income under federal poverty line, 1: income above federal poverty line |

4.2 Simulation Details

We base our example on the assumptions that the agency has decided that all two-way marginal tables are useful to the end data user and will create differentially private versions of them. To make the differentially private counts, we add independent noise drawn from the Geometric mechanism with $\epsilon \in \{0.25, 0.5, 1.0\}$. We invoke sequential composition theorems to adhere to the specified values of $\epsilon$. In particular, each of the 40 two-way marginal counts belongs to one of $\binom{5}{2} = 10$ two-way tables. Thus, each individual contributes to only one cell per table but contributes across all ten tables. Accordingly, each marginal table should be perturbed with one-tenth of the privacy budget.

After generating the noisy counts in each simulation run, we use a MCMC sampler to iteratively sample the unknown parameters, $(M_1, \ldots, M_T, \pi_k, \Psi_k)$. Unless otherwise noted, we set the number of latent classes equal to $k = 10$. In fitting, generally at least four classes have non-trivial mass (as determined by $\pi_k$); see Supplementary Section 2 for model implementation details.

One source of computational overhead for the MCMC sampler is computing all the marginal probabilities from the model parameters, i.e., $P_t(\pi_k, \Psi_k)$. The complexity of this calculation depends on the size of the contingency table, the number of latent classes, and the total number of marginals used as input. In simulations with our R package, we find that computing each two-way marginal probability is extremely fast ($< 2$ milliseconds) when the dimension is ten or less regardless of the number of latent classes. However, run time greatly increases for higher dimensions, leading to slower sampling for the overall model for higher-dimensional tables. Full results are presented in Section 3 of the supplementary material.
Table 2: All two-way marginal tables for one randomly sampled subset of 10,000 observations from the 2016 ACS PUMS.

| Age | Citizenship 0 | Citizenship 1 |
|-----|--------------|--------------|
| 0   | 11           | 596          |
| 1   | 443          | 8,950        |

| Race | Citizenship 0 | Citizenship 1 |
|-----|--------------|--------------|
| 0   | 299          | 308          |
| 1   | 1,731        | 7,662        |

| Sex | Citizenship 0 | Citizenship 1 |
|-----|--------------|--------------|
| 0   | 273          | 354          |
| 1   | 4,505        | 4,888        |

| Income | Citizenship 0 | Citizenship 1 |
|--------|--------------|--------------|
| 0      | 294          | 313          |
| 1      | 2,916        | 6,477        |

| Race | Age 0 | Age 1 |
|-----|------|------|
| 0   | 110  | 344  |
| 1   | 1,920| 7,626|

| Sex | Age 0 | Age 1 |
|-----|------|------|
| 0   | 239  | 215  |
| 1   | 4,539| 5,007|

| Income | Age 0 | Age 1 |
|--------|------|------|
| 0      | 445  | 9    |
| 1      | 2,765| 6,781|

| Sex | Race 0 | Race 1 |
|-----|-------|-------|
| 0   | 945   | 1,085 |
| 1   | 3,833 | 4,137 |

| Income | Race 0 | Race 1 |
|--------|-------|-------|
| 0      | 827   | 1,203 |
| 1      | 2,382 | 5,587 |

| Sex | Income 0 | Income 1 |
|-----|---------|---------|
| 0   | 1,281   | 3,497   |
| 1   | 1,929   | 3,293   |

We run the algorithm for 5,000 iterations, discarding the first 2,000 iterations as burn-in. We monitor convergence of model probabilities using trace plots, a standard practice in Bayesian data analysis [Gelman et al., 2013, Roy, 2020]. In each iteration, we compute the probabilities for all two-way margins and cells in the full table according to the latent class specification using the sampled parameters \((\pi_k, \Psi_k)\) at that iterate.

We compare outputs from the latent class model to output generated using PrivBayes [Zhang et al., 2017], following the architecture for adapting PrivBayes to private data releases described in Ping et al. [2017]. To do so, we adapt their publicly-available Python code, accessible at https://github.com/DataResponsibly/DataSynthesizer. We also compare to a graphical model-based approach [McKenna et al., 2019] by adapting their publicly-available Python code, accessible at https://github.com/ryan112358/private-pgm. We chose these methods due to their prevalence, performance capabilities, and similarities to our proposed approach in terms of inputs in the sense that all comparison methods are created from marginal counts instead of the full, underlying table.

The codes for these two methods return different outputs. PrivBayes returns a synthetic data set, which we convert into probabilities for the full table. The graphical model-based

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*3Due to potential identifiability issues due to label-switching Gelman et al. [2013], monitoring convergence of \((\pi_k, \Psi_k)\) is infeasible. Instead, we monitor estimated full and marginal probabilities. The codes for these two methods return different outputs. PrivBayes returns a synthetic data set, which we convert into probabilities for the full table. The graphical model-based
approach returns a vector of numbers (not necessarily counts) summing to the sample size that corresponds to “counts” for each of the full table cells. We normalize these numbers to create probabilities for the full table. We note that, unlike our proposed latent class approach, neither of these methods generate uncertainty estimates.

4.3 Results

Before turning to the results using differentially private marginals, we first study the usefulness of the composite likelihood approach and the corresponding assumption of independent counts absent privacy concerns, i.e., with $\epsilon = \infty$.

4.3.1 Performance Absent Privacy Concerns

We estimate the augmented latent class model for the underlying table without adding privacy-preserving noise to the marginal counts. To do, we fit model (7) using the true $(M_1, \ldots, M_T)$ and without the measurement error components in the first two lines of that model.

Figure 1 displays posterior modes from one run of the simulation for the marginal and full table probabilities. Results from additional runs are very similar. Both marginal and full cell probabilities are estimated accurately. The independence assumption appears to have minimal negative effects on estimation of the marginal probabilities, while also returning reasonable estimates for the full table cell probabilities. We note that this latter fact stems from the absence of strong three-way (and higher) interaction effects among the variables. In general, one should not expect accurate estimates for the full table; rather, one should expect accurate estimates of the marginal counts used in the estimation routines.

4.3.2 Performance with Noisy Counts

We next generate differentially-private counts and corresponding posterior inferences for the data described in Section 4.1, using the algorithm described in Section 3.

Figure 2 displays posterior means for the estimated marginal and full cell probabilities modes from one run of the simulation. Results from additional runs are very similar. The results exhibit some expected trends. As the privacy loss budget $\epsilon$ increases, the amount of added noise decreases, resulting in better estimates of marginal and full cell probabilities. In the case of marginal probabilities, results for all values of the privacy loss budget return estimates that coincide well with the underlying probabilities, as shown in Figure 2. The results demonstrate this behavior for all values of $\epsilon$. For the full table probabilities, the proposed approach reasonably reconstructs the underlying probabilities even though full table counts are not used in estimation. In general, there is more variation from the truth when compared to the estimates of the marginal probabilities. This also corresponds with the privacy budget: lower values of $\epsilon$ display more disagreement from the true values compared to higher values. This is reasonable to expect as only the marginal counts were used directly in estimation.

We also conducted repeated simulations to assess how well the procedure captures the underlying marginal counts for each value of $\epsilon$. To do so, we repeat the entire procedure 100 times. Each time we sample 10,000 observations from the full original data, compute all two-way marginals, apply the Geometric Mechanism, and implement our method. For each marginal count, we examine the posterior distribution of the corresponding marginal probability and construct 95% credible intervals using the 2.5th and 97.5th quantiles of the
Figure 1: True versus estimated two-way marginal (top) and full table (bottom) probabilities for the ACS simulation with no noise added for privacy. True counts without added noise used as input. The number of latent classes set to 10. As two-way margins were used as inputs, we see more accurate estimation of the marginal probabilities. Accurate estimation of the full cell probabilities is not guaranteed, but can occur.
Figure 2: Plots of true marginal probabilities versus estimated marginal probabilities (top) and true full table probabilities versus estimated probabilities (bottom) for $\epsilon \in \{0.25, 0.5, 1.0\}$. Estimated marginal probabilities denote the mean of the posterior distribution for the corresponding marginal probability. As expected, estimation accuracy visually increase as the $\epsilon$ grows.
posterior distribution. We record the coverage of the true probability, that is, whether or not the population count is inside the posterior interval.

Table 3 displays the coverage probabilities and average interval length for each marginal probability. In general, the credible intervals are close to or exceed the nominal coverage rate when \( \epsilon = \infty \), confirming the usefulness of the latent class approach absent privacy concerns. The coverage rates stay high for \( \epsilon = 1 \), with a few coverage rates for small probabilities dipping to around 70\%. The coverage rates degrade noticeably when \( \epsilon = 0.25 \), especially for the small probabilities. Overall, we notice that coverage probabilities for cells corresponding to statistically insignificant terms in the log-linear model typically perform worse than other counts (e.g., cells 0..0. and 1..1.). This is because the noise has greater impact on small counts, and in a sample of size 10,000 many of these low probability cells have very few sampled cases. We conjecture that the undercoverage also relates, in part, to the sample size in our simulation design. In fact, we repeated the simulation using samples of 100,000 records and found coverage rates at or exceeding the nominal rates for all cell probabilities and all four values of \( \epsilon \).

4.3.3 Comparisons to Other Methods

We next compare the performance of our approach to PrivBayes and the graphical models approach. We ensure that all methods satisfy \( \epsilon \)-DP with the same \( \epsilon \) and use consistent definitions of sensitivity. For the graphical models approach, we use all two way marginals as inputs. For PrivBayes, we set the maximum degree of the Bayesian network to two. Results are shown in Figure 3 for one run of each method. Results for other runs are similar. In general, all approaches return reasonable estimates for the marginal probabilities. Again, quality is dependent of the level of privacy. For smaller values of \( \epsilon \), variability is most pronounced. While all methods perform extremely accurately for \( \epsilon = 1.00 \), one benefit of the latent class modeling approach is that it generates posterior inferences that account for the measurement error introduced by the privacy protection.

PrivBayes and the graphical models approach, rather than competitors, can complement our approach. For example, one can use PrivBayes to determine which marginal tables should be released and how to approximate the full table. Then, one could take the noisy and incoherent marginal tables obtained through PrivBayes and apply the graphical models approach to obtain a coherent version of these tables. Thus, PrivBayes and graphical models techniques may be used to select which marginal tables to consider in our proposed latent class approach and to provide a starting point for \((M_1, \ldots, M_T)\) in the MCMC algorithm. We discuss some of these complementary uses in the next section.

5 Application to the National Long-Term Care Survey

In this section we illustrate the latent class approach on a large-scale dataset from the National Long Term Care Survey (NLTCS).

5.1 Data Description

The NLTCS is a longitudinal survey sponsored by the U.S. National Institute on Aging\(^4\). First developed in 1982, the survey selects participants from Medicare enrollment files. All

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\(^4\)The NLTCS (National Long Term Care Study) is sponsored by the National Institute of Aging and was conducted by the Duke University Center for Demographic Studies under Grant No. U01-AG007198.
Table 3: Coverage probabilities ("Cov.") and average credible interval length ("Length") for the forty two-way marginal counts.

| Cell | Pop. Value | \( \epsilon = 0.25 \) | \( \epsilon = 0.50 \) | \( \epsilon = 1.00 \) | \( \epsilon = \infty \) |
|------|------------|-----------------|-----------------|-----------------|-----------------|
| 00...| 0.002      | 0.10            | 0.10            | 0.70            | 0.97            |
| 01...| 0.061      | 0.97            | 0.97            | 1.00            | 1.00            |
| 10...| 0.044      | 0.71            | 0.71            | 1.00            | 1.00            |
| 11...| 0.894      | 0.49            | 0.89            | 1.00            | 0.99            |
| 0.0..| 0.031      | 0.83            | 0.89            | 1.00            | 0.98            |
| 0.1..| 0.031      | 0.16            | 0.37            | 1.00            | 0.74            |
| 1.0..| 0.174      | 0.88            | 0.89            | 1.00            | 0.96            |
| 1.1..| 0.763      | 0.57            | 0.79            | 1.00            | 0.95            |
| ...0.| 0.031      | 1.00            | 1.00            | 1.00            | 1.00            |
| ...1.| 0.353      | 0.98            | 1.00            | 1.00            | 1.00            |
| 1...1| 0.298      | 1.00            | 1.00            | 1.00            | 1.00            |
| .00..| 0.011      | 0.94            | 0.96            | 1.00            | 1.00            |
| .01..| 0.035      | 0.77            | 0.95            | 1.00            | 1.00            |
| .10..| 0.195      | 0.99            | 1.00            | 1.00            | 1.00            |
| .11..| 0.759      | 0.89            | 0.97            | 1.00            | 1.00            |
| .0.0..| 0.024     | 0.95            | 1.00            | 1.00            | 1.00            |
| .0.1..| 0.022     | 0.78            | 0.84            | 1.00            | 0.93            |
| .10..| 0.461      | 1.00            | 1.00            | 1.00            | 1.00            |
| .11..| 0.493      | 0.98            | 0.99            | 1.00            | 1.00            |
| .0.0..| 0.045     | 0.60            | 0.67            | 1.00            | 0.93            |
| .0.1..| 0.001     | 0.00            | 0.00            | 1.00            | 0.93            |
| .10..| 0.280     | 0.93            | 0.93            | 1.00            | 1.00            |
| .11..| 0.674     | 0.21            | 0.43            | 1.00            | 0.93            |
| .0..0.| 0.098     | 1.00            | 1.00            | 1.00            | 1.00            |
| .0.1..| 0.108     | 1.00            | 1.00            | 1.00            | 1.00            |
| .10..| 0.386     | 1.00            | 1.00            | 1.00            | 1.00            |
| .11..| 0.408     | 1.00            | 1.00            | 1.00            | 1.00            |
| .0..0.| 0.086     | 1.00            | 1.00            | 1.00            | 1.00            |
| .0.1..| 0.120     | 0.99            | 1.00            | 1.00            | 1.00            |
| .10..| 0.239     | 1.00            | 1.00            | 1.00            | 1.00            |
| .11..| 0.555     | 1.00            | 1.00            | 1.00            | 1.00            |
| .0.0..| 0.127     | 1.00            | 1.00            | 1.00            | 1.00            |
| .0.1..| 0.357     | 1.00            | 1.00            | 1.00            | 1.00            |
| .10..| 0.198     | 1.00            | 1.00            | 1.00            | 1.00            |
| .11..| 0.318     | 1.00            | 1.00            | 1.00            | 1.00            |

Average: 0.839 0.049 0.896 0.041 0.945 0.038 0.969 0.038

The positions of the variables are, in order, Citizenship, Age, Race, Sex, Income.

Thus, cell ...11 represents females (SEX = 1) with income above the federal poverty line (INC=1).
Figure 3: Plots of true full table probabilities versus estimated full table probabilities for $\epsilon \in \{0.25, 0.5, 1.0\}$. Comparison of latent class modeling approach with a graphical models-based approach and PrivBayes for $\epsilon = 0.25$ (top), 0.50 (middle), and 1.00 (bottom). For the proposed approach, estimated full cell probabilities denote the mean of the corresponding posterior distribution.
participants are over the age of 65, and new participants are added to the survey every five years [National Institute on Aging, 2021]. Access to the data is restricted but can be obtained through an approved protocol [Manton, 2010]. For this analysis, we restricted ourselves to the 2004 wave of the NLTCS. In addition, we focused on sixteen variables representing markers of daily living as defined in Table 4. Participants with missing or unknown values for any of these variables were dropped, resulting in a $2^{16}$ contingency table comprised of $n = 15,636$ observations.

Table 4: Data description for the NLTCS variables. Data was collected from the 2004 wave of the NLTCS. All variables had two levels: “yes” and “no”.

| Variable      | Description                                      | Counts by Level     |
|---------------|--------------------------------------------------|---------------------|
| SCN.15_A.Y04  | Problem eating by self                           | “yes”: 416; “no”: 15,220 |
| SCN.15_B.Y04  | Problem getting in/out of bed by self            | “yes”: 862; “no”: 14,774 |
| SCN.15_C.Y04  | Problem getting in/out of chairs by self         | “yes”: 1,054; “no”: 14,582 |
| SCN.15_D.Y04  | Problem walking inside without help              | “yes”: 1,763; “no”: 13,873 |
| SCN.15_E.Y04  | Problem going outside without help               | “yes”: 2,435; “no”: 13,201 |
| SCN.15_F.Y04  | Problem dressing without help                    | “yes”: 929; “no”: 14,707 |
| SCN.15_G.Y04  | Problem bathing without help                     | “yes”: 1,358; “no”: 14,278 |
| SCN.15_H.Y04  | Problem going to the bathroom without help       | “yes”: 842; “no”: 14,794 |
| SCN.15_I.Y04  | Incontinence                                     | “yes”: 1,355; “no”: 14,281 |
| SCN.17_A.Y04  | Prepare meals without help                       | “yes”: 13,670; “no”: 1,966 |
| SCN.17_B.Y04  | Do laundry without help                          | “yes”: 13,465; “no”: 2,171 |
| SCN.17_C.Y04  | Light housework without help                     | “yes”: 13,853; “no”: 1,783 |
| SCN.17_D.Y04  | Shop for groceries without help                  | “yes”: 12,678; “no”: 2,958 |
| SCN.17_E.Y04  | Manage money without help                        | “yes”: 13,737; “no”: 1,899 |
| SCN.17_F.Y04  | Take medicine without help                       | “yes”: 14,031; “no”: 1,605 |
| SCN.17_G.Y04  | Make phone calls without help                    | “yes”: 14,251; “no”: 1,385 |

5.2 Implementation Details

The 16 binary variables make $\binom{16}{2} = 120$ two-way marginal tables and $120 \times (2 \times 2) = 480$ marginal counts. We selected important margins using the first half of the PrivBayes routine using a quarter of our privacy budget. While this allocates some of our privacy budget, it greatly reduces the number of marginals used in estimating model parameters. This resulted in 29 two-way margins. All counts were used for each pair, corresponding to $29 \times (2 \times 2) = 116$ marginal counts.

As before, we used a Metropolis-within-Gibbs algorithm to sample from the posterior distribution of $\pi_k, \psi_k$ and $(M_1, \ldots, M_k)$. To avoid overparametrization of the model, we set the number of latent classes to seven, ensuring that the number of estimated parameters (118) is roughly equal to the number of input marginals (116). We follow a similar implementation strategy as our ACS example; see Supplementary Section 2 for details. We run the chain for 12,000 iterations with the first 2,000 discarded as burn-in. We use four values of $\epsilon$: 0.5, 1.0, 2.0, and 5.0. We estimate the two-way marginal probabilities at each iteration using $\pi_k$ and $\psi_k$. 

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5.3 Results

For each of the 116 marginal counts, we compare the estimated and original (non-noisy) two-way probabilities. Figure 4 displays the log ratio (or difference e.g., $\log(P_{est}/P_{true})$) by $\epsilon$. The plotted marginal counts are ranked in order according to the true count with the smallest counts being on the left. The average log difference across all counts ranged from 0.173 for $\epsilon = 0.5$ to 0.050 for $\epsilon = 5$. The estimates with $\epsilon = 0.5$ are improved over using the noisy marginal counts directly: the noisy marginal counts had an average log difference of -0.297 for $\epsilon = 0.5$. However, when $\epsilon = 5$, the results from the noisy count are better, having an average log difference of 0.002.

Two observations are readily apparent from Figure 4. First, log differences appear to decrease as a function of $\epsilon$ with higher values corresponding to greater similarity between the estimated and true marginal probabilities. Second, log differences appear to be greater for smaller marginal probabilities regardless of $\epsilon$.5 This result is somewhat unsurprising when considering the structure of our model: the variance of the binomial distribution directly corresponds to the probability. In fact, at the sample size of our data, the variance at the minimum marginal probability is 46 times smaller than the corresponding variance at 25%. This effect is not as extreme at larger probabilities as the maximum marginal probability is approximately 92.5%.

Furthermore, the vast majority of the log differences at these lower counts are positive, indicating that the estimated marginal probability is greater than the true marginal probability. We believe that this result is amplified by the inherent non-negativity of the counts and sampling from the simplex.

The main benefit of the latent class modeling approach is the capability to facilitate posterior inferences and to generate synthetic data. For the former, we computed 95% credible intervals for each marginal probability by selecting the 2.5th and 97.5th quantiles of the posterior draws. For the latter, we randomly selected 20 iterates of the MCMC chain, and used each set of parameter draws to sample 15,636 synthetic records. This creates 15,636 synthetic records that can summed to estimate probabilities in the full table. We computed the marginal probabilities in each synthetic data set and combined the estimates using the approach in Reiter [2003], resulting in 95% confidence intervals. Tables 5 and 6 present results for three randomly selected marginal tables for the credible intervals and synthetic data inferences, respectively. The posterior intervals and synthetic data intervals, which are quite similar, typically capture the true non-noisy estimate of the marginal probabilities, especially for larger values of $\epsilon$.

6 Concluding Remarks

We present a novel method for posterior inferences and creating differentially private synthetic data for contingency tables based on marginal counts. The simulation results indicate that the latent class modeling approach can offer reasonably accurate estimates of the counts used as inputs to construct the underlying tables (in our example, 2-way marginals). While it also did reasonably well in preserving counts from the full table in our illustrative data, analysts generally should not expect this to be the case. Rather than as a mechanism for generating data coherent with the full table probabilities, we view the approach as a means to make posterior inferences and generate synthetic data that are coherent with the marginal

5In the true data, the smallest marginal probability is roughly 0.5%, and about half of all marginal probabilities are smaller than 5%.
Figure 4: Plot of average log distance by marginal count for $\epsilon \in \{0.5, 1.0, 2.0, 5.0\}$. A log distance of zero indicates that the true and estimated probabilities are identical. Values greater than zero imply that the estimated probability is larger than the truth. Marginal counts are arranged in ascending order with the smallest counts on the left.

Table 5: Posterior intervals for selected marginal counts in the NLTCS example. Results also include the noisy counts input to the latent class model.

| Variable | Cell | Pop. Est. | Noisy Est. | 95% CI | Noisy Est. | 95% CI | Noisy Est. | 95% CI |
|----------|------|-----------|------------|--------|------------|--------|------------|--------|
| SCN_15_E | 00   | 0.041     | 0.033      | (0.053, 0.065) | 0.041      | (0.044, 0.052) | 0.041 | (0.040, 0.048) |
| SCN_17_D | 01   | 0.114     | 0.112      | (0.093, 0.107) | 0.117      | (0.106, 0.113) | 0.114 | (0.107, 0.116) |
| SCN_17_D | 10   | 0.769     | 0.776      | (0.762, 0.783) | 0.770      | (0.764, 0.774) | 0.768 | (0.768, 0.780) |
| SCN_17_D | 11   | 0.075     | 0.076      | (0.062, 0.080) | 0.075      | (0.070, 0.077) | 0.076 | (0.065, 0.076) |
| SCN_17_E | 10   | 0.853     | 0.846      | (0.839, 0.851) | 0.849      | (0.842, 0.849) | 0.854 | (0.853, 0.858) |
| SCN_17_B | 01   | 0.008     | 0.007      | (0.016, 0.033) | 0.010      | (0.013, 0.028) | 0.008 | (0.007, 0.011) |
| SCN_17_E | 10   | 0.033     | 0.039      | (0.032, 0.059) | 0.033      | (0.035, 0.043) | 0.033 | (0.030, 0.034) |
| SCN_17_C | 10   | 0.106     | 0.102      | (0.067, 0.102) | 0.107      | (0.089, 0.102) | 0.107 | (0.102, 0.106) |
| SCN_15_C | 01   | 0.021     | 0.018      | (0.013, 0.025) | 0.021      | (0.011, 0.019) | 0.021 | (0.015, 0.021) |
| SCN_17_B | 01   | 0.046     | 0.046      | (0.024, 0.044) | 0.036      | (0.034, 0.045) | 0.046 | (0.045, 0.050) |
| SCN_17_A | 10   | 0.853     | 0.864      | (0.846, 0.861) | 0.854      | (0.855, 0.866) | 0.852 | (0.853, 0.859) |
| SCN_17_E | 11   | 0.080     | 0.084      | (0.079, 0.105) | 0.085      | (0.079, 0.093) | 0.080 | (0.076, 0.081) |
Table 6: Interval estimates based on 20 synthetic data sets for selected marginal counts in the NLTCS example. Point estimates, standard errors, and multipliers were computed using the combining rules of Reiter [2003].

| Variable | Cell | Pop. Est. | Point Est. | 95% CI | Point Est. | 95% CI | Point Est. | 95% CI |
|----------|------|-----------|------------|--------|------------|--------|------------|--------|
| SCN_15_E | 01   | 0.114     | 0.08       | (0.076, 0.085) | 0.091     | (0.086, 0.096) | 0.107     | (0.102, 0.112) |
| SCN_17_D | 10   | 0.769     | 0.806      | (0.798, 0.815) | 0.793     | (0.786, 0.8)  | 0.779     | (0.772, 0.786) |
| SCN_17_D | 11   | 0.075     | 0.056      | (0.05, 0.061)  | 0.072     | (0.067, 0.076) | 0.07      | (0.066, 0.074) |
| SCN_17_B | 00   | 0.853     | 0.879      | (0.873, 0.886) | 0.869     | (0.864, 0.875) | 0.86      | (0.854, 0.865) |
| SCN_17_C | 01   | 0.008     | 0.017      | (0.014, 0.019) | 0.012     | (0.01, 0.015)  | 0.008     | (0.007, 0.01)  |
| SCN_17_C | 10   | 0.033     | 0.034      | (0.03, 0.039)  | 0.03      | (0.027, 0.033) | 0.03      | (0.028, 0.033) |
| SCN_17_C | 11   | 0.106     | 0.07       | (0.061, 0.078) | 0.088     | (0.083, 0.093) | 0.102     | (0.097, 0.106) |
| SCN_15_C | 01   | 0.046     | 0.023      | (0.018, 0.028) | 0.032     | (0.029, 0.036) | 0.046     | (0.042, 0.049) |
| SCN_17_A | 10   | 0.853     | 0.886      | (0.88, 0.892)  | 0.882     | (0.876, 0.887) | 0.86      | (0.855, 0.866) |
| SCN_17_A | 11   | 0.080     | 0.076      | (0.07, 0.081)  | 0.075     | (0.071, 0.08)  | 0.077     | (0.073, 0.081) |

counts used as inputs. Thus, we advise agencies to tell data users what counts are used as inputs, so that data users do not expect the synthetic data to be accurate for other counts.

More generally, our approach is to define summary statistics as functions of model parameters, and use composite likelihood approximations to estimate those parameters. This general strategy can be extended to more complex data structures. For example, and as illustrated in the online supplement, the strategy could be used to generate synthetic data for people nested within households, using the latent class model of Hu et al. [2018] as the underlying model. Developing methods for practical implementation in such complex models is an area for future research.

We have shown in the NLTCS example that not all marginals are required to estimate our model. We leave it to future research to study methods for selecting the marginals, and, similarly, methods for allocating the privacy budget across these marginals. For both of our examples, we equally allocated the privacy budget. However, equal allocation may not be optimal in all problems.

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Supplementary Materials for *A Latent Class Modeling Approach for Generating Synthetic Data and Making Posterior Inferences from Differentially Private Counts*

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1 Nested Data Example

In this section, we illustrate the latent class modeling strategy for nested categorical data, based on the model of Hu et al. [2018].

1.1 Modeling strategy

To facilitate understanding, we describe the method in the context of a subset of data from the 2012 American Community Survey (ACS). The ACS data have a nested structure: the data file has information on individuals nested within households. For simplicity, we restrict our focus to households of size three, i.e., a head of household and two other occupants. Thus, the data comprise information about the household head, each individual, relationships of the individuals to the household head (e.g., spouse, child), and about the household as a unit (e.g., is the house owned or rented). Table 1 includes a description of the variables. We treat data for the household head as part of the household-level information. We treat data available for the two remaining family members as individual-level information.

Before describing the approach, we introduce some notation for ownership ($X_0$); the household head’s gender and race ($X_1, X_2$); the first other individual’s gender, race, and relationship to household head ($X_{11}, X_{12}, X_{13}$); and, the second other individual’s gender, race, and relationship to household head ($X_{21}, X_{22}, X_{23}$).

We model the available data using the nested data Dirichlet process mixture of multinomial distributions employed in Hu et al. [2018] as well as in Akande et al. [2019]. We model the data-generating mechanism of $X = (X_0, X_1, X_2, X_{11}, X_{12}, X_{13}, X_{21}, X_{22}, X_{23})$ using

$$P(X = x|\theta) = \sum_g \pi_g \lambda^{(0)}_{gx_0} \lambda^{(1)}_{gx_1} \lambda^{(2)}_{gx_2} \left( \sum_m w_{gm} \phi^{(1)}_{gmx_{11}} \phi^{(2)}_{gmx_{12}} \phi^{(3)}_{gmx_{13}} \right) \left( \sum_m w_{gm} \phi^{(1)}_{gmx_{21}} \phi^{(2)}_{gmx_{22}} \phi^{(3)}_{gmx_{23}} \right).$$

(1)

Here, $x = (x_0, x_1, x_2, x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23})$, and $\theta$ represents the collection of parameters defining (1), i.e., $\pi_g, \lambda^{(j)}_{gx}, w_{gm}$, and $\phi^{(j)}_{gmx}$. 

1
Table 1: Variable description of nested ACS data. “HH” denotes head of household. See Akande et al. [2019] for remaining variable definitions.

| Variable          | Levels                                                                 |
|-------------------|-------------------------------------------------------------------------|
| **Household-level** |                                                                         |
| Household Index   | Household index                                                         |
| Ownership         | 1: owned or being bought, 0: rented                                       |
| Gender of HH      | 1: male, 2: female                                                       |
| Race of HH        | 1: white, 2: black                                                       |
|                   | 3: American Indian or Alaska native, 4: Chinese, 5: Japanese,            |
|                   | 6: other Asian/Pacific Islander, 7: other race, 8: two major races, 9:  |
|                   | three or more major races                                                |
| **Individual-level** |                                                                   |
| Within Household Index | Person indicator nested within household index.                        |
| Relationship to HH| 2: spouse, 3: biological child 4: adopted child, 5: stepchild, 6: sibling, |
|                   | 7: parent 8: grandchild, 9: parent-in-law, 10: child-in-law, 11: other  |
|                   | relative, 12: boarder, roommate or partner, 13: other non-relative or   |
|                   | foster child                                                            |
| Gender of Individual | Same as Gender of HH variable.                                           |
| Race of Individual | Same as Race of HH variable.                                             |

Table 2 displays the components of $\theta$. These components satisfy the following constraints. We enforce $\sum_m w_{gm} = 1$ with $w_{gm} \geq 0$; for $j = 0, 1, 2,$ and each $g$; $\sum_{x \in \text{Range} (X_{ij})} \lambda_{gx}^{(j)} = 1$ and $\lambda_{gx}^{(j)} \geq 0$; for each $g$; $\sum_m w_{gm} = 1$ and $w_{gm} \geq 0$; for $j = 1, 2, i = 1, 2,$ and each $g$ and $m$; $\sum_{x \in \text{Range} (X_{ij})} \phi_{gmx}^{(j)} = 1$ and $\phi_{gmx}^{(j)} \geq 0$.

We complete the specification of the model in (1) by assuming a prior distribution for $\theta$ and considering a truncated version of the mixture components. Specifically, for $\pi_g$ and $w_{gm}$, we consider a prior distribution induced by a stick-breaking representation as in Dunson and Xing [2009]. Since the remaining parameters (i.e., $\lambda_{gx}^{(j)}$ and $\phi_{gmx}^{(j)}$) lie on the simplex space, we consider Dirichlet distributions as prior distributions for these parameters.

Table 2: Description of parameters for nested data model

| Parameter          | Description                                                                 |
|--------------------|-----------------------------------------------------------------------------|
| $\pi_g$            | Mixing probability for the household-level latent class                     |
| $w_{gm}$           | Mixing probability for the individual-level latent class                    |
| $\lambda_{gx}^{(1)}$ | Latent class probability of ownership of house for HH = $x_1$              |
| $\lambda_{gx}^{(2)}$ | Latent class probability of HH’s race = $x_2$                            |
| $\phi_{gmx_{11}}^{(1)}$ | Latent class probability of individual 1’s gender = $x_{11}$         |
| $\phi_{gmx_{12}}^{(2)}$ | Latent class probability of individual 1’s race = $x_{12}$            |
| $\phi_{gmx_{13}}^{(3)}$ | Latent class probability of individual 1’s relationship to HH = $x_{13}$ |
| $\phi_{gmx_{21}}^{(1)}$ | Latent class probability of individual 2’s gender = $x_{21}$         |
| $\phi_{gmx_{22}}^{(2)}$ | Latent class probability of individual 2’s race = $x_{22}$            |
| $\phi_{gmx_{23}}^{(3)}$ | Latent class probability of individual 2’s relationship to HH = $x_{23}$ |
1.2 Data and Sampling Scheme

To demonstrate the proposed latent class modeling approach under nested data, we use a sample of \( N = 602 \) households (representing 1,806 individuals) from the 2012 ACS obtained from Akande et al. [2019]. We focus on five summaries of household-level and individual-level characteristics. See Table 3 for a description and total of each count.

Table 3: Description and counts of selected summaries.

| Description                          | Count |
|--------------------------------------|-------|
| \( S_1 \): households with all members white | 427   |
| \( S_2 \): households owned by HH     | 440   |
| \( S_3 \): households with all members of same race | 552   |
| \( S_4 \): households with spouse present | 401   |
| \( S_5 \): households with same race couple present | 376   |

1.2.1 Implementation Details

Recall from Section 3 in the main text that we treat each count as an independent draw from a binomial distribution with parameters \( N \) and \( P(\theta) \). Using (1), we can compute directly the probability for each individual combination of data and for each of the selected summaries in Table 3. We have:

\[
P(S_1|\theta) = \sum_g \pi_g \lambda_g^{(2)} \left( \sum_m w_{gm}\phi_{gm1}^{(2)} \right)^2
\]

\[
P(S_2|\theta) = \sum_g \pi_g \lambda_g^{(1)}
\]

\[
P(S_3|\theta) = \sum_{r \in \text{Races}} \sum_g \pi_g \lambda_{gr}^{(2)} \left( \sum_m w_{gm}\phi_{gm1}^{(2)} \right)^2
\]

\[
P(S_4|\theta) = 2 \sum_g \pi_g \left( \sum_m w_{gm}\phi_{gm2}^{(3)} \right) - \sum_g \pi_g \left( \sum_m w_{gm}\phi_{gm2}^{(3)} \right)^2
\]

\[
P(S_5|\theta) = 2 \sum_{r \in \text{Races}} \sum_g \pi_g \lambda_{gr}^{(2)} \left( \sum_m w_{gm}\phi_{gm1}^{(2)} \phi_{gm2}^{(3)} \right) - \sum_{r \in \text{Races}} \sum_g \pi_g \lambda_{gr}^{(2)} \left( \sum_m w_{gm}\phi_{gm1}^{(2)} \phi_{gm2}^{(3)} \right)^2.
\]

Since we have access to the noisy versions of \( S_1, \ldots, S_5 \), we use the same model specification as in our proposed approach of the main text, where the probability of success for each count is given by the corresponding \( P(S_i|\theta) \). We use a Metropolis-Hastings-within-Gibbs sampling algorithm was used to estimate model parameters. We set the number of household and individual latent classes (or truncated levels) equal to two. The prior distributions for the different components of \( \theta \) correspond to uniform distributions on the corresponding simplex spaces. We draw proposals for each of the probabilities from a Dirichlet distribution with concentration parameter centered on the last value of the chain. We run the algorithm
for 20,000 iterations, discarding the first 10,000 iterations as burn-in. We thin the truncated chain by every five iterations. We monitor convergence of model parameters using trace plots of probabilities for each count.

We run the model under two separate schemes. First, we use the true counts without added noise to assess the performance of the estimation method without concerns about privacy. Second, we add independent noise drawn from the Geometric mechanism to each count with $\epsilon \in \{0.01, 0.1, 1.0\}$.

1.3 Results

We first assess the effect of the assumption of independence between counts. Figure 1 displays the results for simulations. In general, the assumption of independence appears not to be problematic for the small number of counts used in parameter estimation.

![Figure 1: True versus estimated probabilities for the augmented nested model with no additional noise added.](image)

We next use the approach to generate differentially private synthetic nested data. We add privacy-preserving noise from the Geometric mechanism directly to each count. We allot one fifth of the privacy budget to each count. Results for this simulation are shown in Figure 2. For large values of $\epsilon$, the synthetic data provide estimates that mimic those in the underlying cell probabilities. As expected, this result degrades as privacy guarantees are made stronger.

Further investigation is needed to determine the quality of inference of the synthetic data, how the choice of included counts affects the estimation of other non-included counts, and applicability to higher dimensional data sets. These are important topics for future research.
2 Sampling Details

In this section, we discuss the sampling strategy. The parameters \((M_1, \ldots, M_t, \pi_k, \Psi_k)\) are drawn from a posterior distribution of the form:

\[
f(M_1, \ldots, M_t, \pi_k, \Psi_k) | \tilde{M} \propto f(\tilde{M} | M_1, \ldots, M_t) f(M_1, \ldots, M_t | \pi_k, \Psi_k) f(\pi_k) f(\Psi_k).
\] (3)

Many Bayesian latent class models specify a Dirichlet process prior for \(\pi_k\) and Dirichlet priors for \(\Psi_k\). They rely on conjugacy to sample from the posterior distributions of \(\pi_k\) and \(\Psi_k\). Two intricacies complicate such a sampling strategy with our proposed approach. First, since the full data are not available, it is not feasible to assign latent classes for every observation. These latent class assignments are typically used in the sampling of \(\pi_k\). Second, the distribution of \(f(M_1, \ldots, M_t | \pi_k, \Psi_k)\) is parameterized by functions of \((\pi_k, \Psi_k)\) and not \(\Psi_k\) directly as is typically the case.

To sample from the posterior distributions, we use a Metropolis-Hastings-within-Gibbs sampling algorithm. We iteratively update between \(M_1, \ldots, M_t, \pi_k,\) and \(\Psi_k\) by sampling from each of the respective complete conditional distributions. At each iteration \(m = 1, \ldots, N \text{runs}\), we proceed as follows.

**S1: Sampling for \(M_1, \ldots, M_t:\)**

1. For \(t = 1, \ldots, T:\)
   
   (a) Propose \(M_t^*\) from a truncated two-sided geometric distribution centered at \(\tilde{M}_t\) and scale parameter equal to \(\exp\left\{ -\epsilon \Delta \tilde{M}_t T \right\}\).
(b) Calculate the acceptance probability \( \alpha = \frac{f(M^*_t)}{f(M_{t - 1})} \) where \( f(M_t | M, \pi_k, \Psi_k) \propto f(M_t | \pi_k, \Psi_k) \).

\[
\alpha = \frac{f(M^*_t | \tilde{M}, \pi_k, \Psi_k) g(M_{t - 1}^m | M^*_t)}{f(M_{t - 1} | M, \pi_k, \Psi_k) g(M^*_t | M_{t - 1}^m)},
\]

where \( g() \) denotes the truncated two-sided geometric proposal distribution.

(c) Generate \( Y \sim \text{Unif}(0, 1) \). Set \( M_t^m = M_t^* \) if \( Y \leq \alpha \), otherwise set \( M_t^m = M_{t - 1}^m \).

**S2: Sampling for \( \pi_k \):**

**Strategy 1:**

1. Propose \( \pi_k^* \) from a Dirichlet distribution parameterized by \( \pi_k^{m - 1} \).
2. Calculate the acceptance probability \( \alpha = \frac{f(\pi_k^*_t | \pi_k^{m - 1}, \pi_k)}{f(\pi_k | \pi_k^{m - 1}, \pi_k)} \) where \( f(\pi_k | M, \Psi_k) \propto f(M | \pi_k, \Psi_k) \).

\[
f(M_t | \pi_k, \Psi_k) f(\pi_k | \pi_k^{m - 1}, \pi_k) \prod_{t=1}^T \text{Multinomial}(n, P_t(\pi_k, \Psi_k)) f(\pi_k)
\]

3. Generate \( Y \sim \text{Unif}(0, 1) \). Set \( \pi_k^m = \pi_k^* \) if \( Y \leq \alpha \), otherwise set \( \pi_k^m = \pi_k^{m - 1} \).

**Strategy 2:**

1. Reparameterize \( \pi_k \) as \( \pi_i = \eta_i / \sum_i \eta_i \) for \( i = 1, \ldots, k \) for \( \eta_1, \ldots, \eta_k \in [0, \infty] \).
2. Propose \( \eta_i^* \) for \( i = 1, \ldots, k \) from a Normal distribution with mean \( \eta_i^{m - 1} \) and variance tuned for an appropriate acceptance rate. Reject any negative proposals by returning \( \eta_i^{m - 1} \).
3. Calculate the acceptance probability \( \alpha = \frac{f(\eta_k^*_t | M, \Psi_k) g(\eta_k^{m - 1} | \eta_k)}{f(\eta_k | M, \Psi_k) g(\eta_k^{m - 1} | \eta_k)} \) where \( f(\eta_k | M, \Psi_k) \propto f(M_t | \eta_k, \Psi_k) f(\eta_k) \propto \prod_{t=1}^T \text{Multinomial}(n, P_t(\pi_k, \Psi_k)) f(\eta_k) \). Note that the finite representation of the stick-breaking prior on \( \pi_k \) implies a Gamma prior with constant scale on \( \eta_i \) [Ishwaran and Zarepour, 2000, Ishwaran and James, 2001].
4. Generate \( Y \sim \text{Unif}(0, 1) \). Set \( \pi_k^m = \pi_k^* \) if \( Y \leq \alpha \), otherwise set \( \pi_k^m = \pi_k^{m - 1} \).

**S3: Sampling for \( \Psi_k = \{\Psi_{h}^{(j)}\}_{h=1,j=1}^{k,p} \):**

1. For \( j = 1, \ldots, p \):
   a. For \( h = 1, \ldots, k \):
      i. Propose \( \Psi_{h}^{(j)}^* \) from a Dirichlet distribution parameterized by \( \Psi_{h}^{(j),m-1} \).
      ii. Calculate the acceptance probability \( \alpha = \frac{f(\Psi_{h}^{(j)*} | \Psi_{h}^{(j),m-1}, \Psi_{h}^{(j),m-1})}{f(\Psi_{h}^{(j)} | \Psi_{h}^{(j),m-1}, \Psi_{h}^{(j),m-1})} \) where \( f(\Psi_{h}^{(j)} | M, \pi_k) \propto f(M_t | \pi_k, \Psi_k) f(\Psi_{h}^{(j)} | \Psi_{h}^{(j),m-1}) \prod_{t=1}^T \text{Multinomial}(n, P_t(\pi_k, \Psi_k)) f(\Psi_{h}^{(j)}) \) and \( g() \) denotes the Dirichlet proposal distribution.
      iii. Generate \( Y \sim \text{Unif}(0, 1) \). Set \( \Psi_{h}^{(j),m} = \Psi_{h}^{(j)*} \) if \( Y \leq \alpha \), otherwise set \( \Psi_{h}^{(j),m} = \Psi_{h}^{(j),m-1} \).
Our algorithm returns all parameter estimates for each iteration. With these estimates, we can calculate corresponding marginal and full table probabilities. If desired, synthetic data of a desired size can be created using the full table probability estimates either through post-processing or sampling.

Our package supports both sampling strategies for $\pi_k$. In practice, however, we have found that Strategy 2 offers better mixing performance.

### 3 Run Time Complexity

Most of the computational complexity in our proposed approach stems from calculating the two-way marginal probabilities, i.e., $P(\pi, \psi)$. This complexity is effected by both the dimension of the table and the number of latent classes. For computational efficiency, functions for evaluating marginal probabilities were written in base R. Functions were vectorized and parallelized using the `parallel` [R Core Team, 2021] and `Rfast` [Papadakis et al., 2021] packages where possible.

To show how computational cost changes as a function of dimension and the number of latent classes, we simulated a two-way table with a given size and calculated how long it took to compute a single two-way marginal probability. Code was run on a Windows laptop (Intel i7 processor with 16 GB RAM) using 7 cores. Results are shown in Figure 3. Results were averaged over 100 replicates. As shown in the figure, this function performs well for moderately sized tables. However, it increases in as dimension increases, regardless of the number of latent classes.

![Figure 3: Average running time in milliseconds for computing a single two-way marginal probability as a function of dimension and the number of latent classes. Average computed over 100 iterations.](image)

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