TIdentity module for the reconstruction of the moments of multiplicity distributions

Mesut Arslandok\textsuperscript{a}, Anar Rustamov\textsuperscript{a,b,c}

\textsuperscript{a}Physikalisches Institut, University of Heidelberg
\textsuperscript{b}GSI Helmholtzzentrum für Schwerionenforschung, Darmstadt, Germany
\textsuperscript{c}National Nuclear Research Center, Baku, Azerbaijan

Abstract

In this report a new software module for the reconstruction of the moments of multiplicity distributions of identified particles, the TIdentity module, is presented. The module exploits the Identity Method, which allows to circumvent the issues of incomplete particle identifications caused by unavoidable overlapping particle identification signals. After demonstrating the performance of the module in a number of simulations, we provide a user’s guide with a detailed description of its functionality. The module can be used in high energy nuclear interactions aiming at the determination of the moments of multiplicity distributions of identified particles.

Keywords: Event-by-event fluctuations, Higher moments, Identity Method, Particle identification, Unfolding

1. Introduction

The study of the multiplicity distributions of identified particles, in terms of their moments, is one of the major topics in high energy nuclear collision experiments. For a system in statistical equilibrium, such as strongly interacting matter created in relativistic nuclear collisions, specific combinations of the moments of multiplicity distributions, referred to as event-by-event fluctuation signals, are directly related to the equation of state of strongly interacting matter [1, 2, 3]. The standard approaches of the determination of e-by-e fluctuations signals are to count the number of particles in each collision (event). However, in experiments, it is often challenging to identify uniquely the type of every detected particle due to overlapping particle identification signals. For this reason, analyses of identified particle fluctuations are usually performed in a limited kinematic acceptance, where particle
identification is relatively reliable, or exploiting additional detectors. These procedures, however, reduce the overall experimental acceptance and/or particle identification efficiencies, which ultimately pushes the fluctuations of interest to the Poisson limit.

Although it is usually difficult to uniquely identify every detected particle in a given event, to each particle one can still assign a probability of being of a given type. The Identity Method eliminates the effect of incomplete particle identification using these probabilities, and allows for the calculation of pure and mixed moments of otherwise unknown particle multiplicity distributions.

The paper is organized as follows. In Section 2 a short introduction to the Identity Method and its application to net-particle cumulants are given. In Section 3.1 we demonstrate the performance of the TIdentity module in Monte-Carlo simulations up to the second order moments. Finally, in Section 4 we discuss the implementation of the method in the "TIdentity" module and provide a user’s guide.

2. The Identity Method

The Identity Method was proposed in Ref. [4] as a solution to the aforementioned misidentification problem for the analysis of events with two different particle species. The method was extended to calculate the second moments of the multiplicity distributions of more than two particle species in Ref. [5]. Finally, in Ref. [6], it was generalized to the first and higher moments of multiplicity distributions for an arbitrary number of particle species and further reexamined in Refs. [7, 8]. The first experimental results using the Identity Method were published by the NA49 [9] and ALICE [10, 11] collaborations at CERN.

Instead of identifying every detected particle event-by-event, the Identity Method calculates the moments of particle multiplicity distributions by means of an unfolding procedure using only two basic experimentally measurable track-by-track and event-by-event quantities, \( \omega \) and \( W \), respectively. They are defined as

\[
\omega_j(x_i) = \frac{\rho_j(x_i)}{\rho(x_i)} \in [0, 1], \quad \rho(x_i) = \sum_j \rho_j(x_i), \quad (1)
\]

\[
W_j \equiv \sum_{i=1}^{N(n)} \omega_j(x_i). \quad (2)
\]
where $x_i$ stands for the $dE/dx$ of a given track $i$, $\rho_j(x)$ is the $dE/dx$ distribution of particle type $j$ within a given phase-space bin and $N(n)$ is the number of tracks in the $n^{th}$ event. We note that, $x$ can refer to any experimentally measured particle property related to its identity. The $\omega_j$ quantity is the probability for a given track to be of particle type $j$ and $W_j$ represents a proxy for its multiplicity in a given event. Thus, in case of a perfect identification one gets $W_j = N_j$ with $N_j$ denoting the number of particle type $j$ in a given event.

As the $W_j$ quantities are calculated for each event, by straightforward averaging over all events the moments of $W_j$ distributions, including mixed moments between different particle types can be easily reconstructed. The Identity Method provides a set of linear equations, which relates the moments of the $W_j$ distributions to the looked for moments of multiplicity distributions [6].

3. The TIdentity Module

The TIdentity module is developed for the calculation of the moments of particle multiplicity distributions for an arbitrary number of particle species using the Identity Method described in the previous section. The code works within the ROOT framework [12] (versions 5 and 6) and is accessible online via:

git clone https://github.com/marslandALICE/TIdentityModule.git

The TIdentity module requires two inputs: (i) the track-by-track measurements in each event, such as $x_i$ quantities in Eq. 1 (ii) the distribution of the identification signal for each particle type, like $\rho_j(x)$ of Eq. 1. The current version of the module contains calculations of all pure and mixed moments of particle multiplicity distributions up to the second order. To calculate the $n^{th}$ moment of multiplicity distributions, the TIdentity module uses all pure and mixed $(n-1)^{th}$ moments and $\rho_j(x)$ functions. The details on practical usage of the TIdentity module are given in the user’s guide, provided in section 1.

3.1. Test on simulated data

Here we demonstrate the usage of the TIdentity module on a simulated data sample of 10 million events. We assume that particle identification is achieved by measuring the specific energy loss $dE/dx$. In each event we
generate five different particle species; electron (e), pion (π), proton (p), kaon (K) and deuteron (d), as well as their anti-particles. The multiplicity distributions of these particles are simulated according to the Poisson distribution with different mean values.

The input information to the simulation is a $dE/dx$ distribution for each particle type (see Fig. 1) and mean multiplicities of particles (anti-particles), which are taken to be 8 (6), 14 (10), 8 (6), 10 (8) and 6 (5) for electrons, pions, kaons, protons and deuterons, respectively. The mean multiplicity of kaons (anti-kaons) is taken to be the same as for electrons (positrons) in each event in order to demonstrate that the method functions also for correlated particle production.

The simulation process comprises the following steps: (i) in each event we randomly generate particle multiplicities from the Poisson distribution using their mean values; (ii) we generate a $dE/dx$ value for each particle in an event from the inclusive distribution $\rho_j(x)$ with Gaussian shape for the corresponding particle type $j$. Using this simulated data we reconstruct all the first and second moments of the multiplicity distributions with the TIdentity module.

Figure 1: (Color online) A Monte-Carlo simulation of particle energy losses for electrons, pions, kaons, protons and deuterons. Anti-particle and particle distributions are shown in the left and right panels, respectively.
Figure 2: (Color online) The distribution of track variables $\omega$ (first column), event variables $W$ (second column) and the true multiplicity distributions $N$ (third column) of electrons, pions, kaons, protons and deuterons.

For each simulated particle in an event we calculate the identity variables $\omega_j(x)$ according to Eq. 1, where $j$ stands for the particle type. Next, for each particle type, we determine the event variable $W_j$ (cf. Eq. 2). The
left panel of Fig. 2 shows the distributions of the identity variables $\omega_j$ for different particle species. In the middle and right columns of Fig. 2 the distributions of the $W_j$ quantities and the true multiplicity distributions $N_j$ are plotted, respectively.

The mean values of $W_j$ and $N_j$ distributions are the same, which is reflected in the middle and right panels of Fig. 2. Moreover, the $W$ distributions of electrons and kaons have a smooth behavior, since their $W$ values are dominated by the overlapping regions of the $dE/dx$ distributions, thus they predominantly take non-integer values. The distributions of pions, protons and deuterons exhibit evident structures because the $dE/dx$ distributions of these particles are relatively well separated from those of electrons and kaons as illustrated in the middle panel of Fig. 2. These structures are nothing but the smeared original multiplicity distributions because of the non-ideal particle identification, i.e., because of the overlapping $dE/dx$ distributions of different particle species.

![Figure 3: Reconstructed values of $\langle K \rangle$ for 25 subsamples. The dashed green lines indicate the corresponding averaged values of $\langle K \rangle$ over subsamples and the black solid line indicates the generated value of $\langle K \rangle$.](image)

Using Eq. 16 of Ref [6], we reconstruct all first moments of the particles from the corresponding moments of the $W$ quantities. The statistical uncertainties of the moments are evaluated with the subsample method. We first divided the data set into $n = 25$ random subsamples, and reconstructed the moments for each subsample, as illustrated in Fig. 3. Next, we calculated
the statistical error for a given moment $M$ with the following formula:

$$
\sigma_{\langle M \rangle} = \frac{\sigma}{\sqrt{n}}
$$

where

$$
\langle M \rangle = \frac{1}{n} \sum M_i, \quad \sigma = \sqrt{\frac{\sum (M_i - \langle M \rangle)^2}{n - 1}}.
$$

Finally, as shown in Fig. 4, we compare the reconstructed moments calculated with the TIdentity module to the generated ones. The precision of the agreement is better than 0.1%. We note that, the observed tiny deviations from unity are due to the systematic uncertainties stemming from the finite bin width of the $dE/dx$ distributions and the precision of the $\rho_j$ fit functions.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{ratio_reconstructed_generated.png}
\caption{Ratio of the reconstructed moments to the generated ones obtained with the TIdentity module.}
\end{figure}

### 3.2. Net-particle cumulants

Net-particle cumulants are in particular important to address fluctuations of conserved charges [2]. In this section we demonstrate a procedure for reconstructing net-particle cumulants of the second order. The higher order cumulants of net-particles can be reconstructed in a similar way. Given the probability distribution $P(n_j)$ of particle type $j$, its $r^{th}$ raw moment is defined as:

$$
\langle n_j^r \rangle = \sum_{n_j=0}^{\infty} n_j^r P(n_j).
$$
The second cumulant of net-particles \( \Delta n_j = n_j - \bar{n}_j \), with \( n_j \) and \( \bar{n}_j \) referring to the number of particles and anti-particles, reads:

\[
\kappa_2(\Delta n_j) = \langle \Delta n_j^2 \rangle - \langle \Delta n_j \rangle^2,
\]

which can be represented as a sum of corresponding cumulants for particles plus the correlation term for the joint probability distributions of particles and antiparticles \( P(n_j, \bar{n}_j) \)

\[
\kappa_2(\Delta n_j) = \kappa_2(n_j) + \kappa_2(\bar{n}_j) - 2 \langle n_j \bar{n}_j \rangle - \langle n_j \rangle \langle \bar{n}_j \rangle,
\]

where the mixed cumulant \( \langle n_j \bar{n}_j \rangle \) is defined as

\[
\langle n_j \bar{n}_j \rangle = \sum_{n_j=0}^{\infty} \sum_{\bar{n}_j=0}^{\infty} n_j \bar{n}_j P(n_j, \bar{n}_j).
\]

Here, we discuss two distinct ways of calculating this correlation term. In the first case, we run the program for 3 different settings: only for particles, only for antiparticles and for the sum of particles and antiparticles. The latter means that the probabilities calculated according to Eq. 1 are summed up for particles and antiparticles. Hence, using the reconstructed second order moments, \( \langle n_j^2 \rangle \), \( \langle \bar{n}_j^2 \rangle \) and \( \langle (n_j + \bar{n}_j)^2 \rangle \), the correlation term can be easily calculated:

\[
\langle n_j \bar{n}_j \rangle = \frac{1}{2} \left( \langle (n_j + \bar{n}_j)^2 \rangle - \langle n_j^2 \rangle - \langle \bar{n}_j^2 \rangle \right).
\]

Alternatively one can directly calculate the second moment of the \( n_j - \bar{n}_j \) distribution entering Eq. 6:

\[
\langle \Delta n_j^2 \rangle = 2 \langle n_j^2 \rangle + 2 \langle \bar{n}_j^2 \rangle - \langle (n_j + \bar{n}_j)^2 \rangle.
\]

We note that this approach was used in the ALICE net-proton analysis [11]. In the second case, the correlation term \( \langle n_j \bar{n}_j \rangle \) can be obtained directly by using the particles and antiparticles simultaneously. This means that the probabilities defined in Eq. 1 are calculated separately for particles and antiparticles. Consequently this yields the moments of both \( W_j \) and \( \bar{W}_j \) distributions including the mixed moments of their joint distributions. This allows for the determination of the correlation term \( \langle n_j \bar{n}_j \rangle \) mentioned above. We note that both of these methods should yield identical results. However, their performance may depend on the number of analyzed particles, because in the second case the effective number of particles doubles, which in turn may lead to computational difficulties.
4. User’s Guide

The current version of the TIdentity module is composed of the following classes:

- **TIdentityBase** is the base class, from which the other classes inherit.
- **TIdentityFunctions** is used to process the line-shapes.
- **TIdentity2D** is the main steering class for the calculation of the second order moments.

An instance of the TIdentity2D class can be created via its constructor:

```
TIdentity2D::InitIden2D(Int_t size)
```

where “size” is the number of particle species used in a given analysis. Below, important functions that are involved in the TIdentity2D class are introduced in detail.

The TIdentity module works within the ROOT framework [12] (versions 5 and 6) and is accessible online via:

```
git clone https://github.com/marslandALICE/TIdentityModule.git
```

In the first step, the module has to be compiled as it is specified in the provided README file. The following input files are needed: (i) a file containing a TTree object with the track-by-track information of each event; (ii) functional form of the \( dE/dx \) distributions of each particle species.

Next, the following three functions in the test macros need to be modified according to the needs of a given analysis:

- **InitializeObjects()** is used to initialize an output tree that contains the moments and other relevant information, as well as the pointer arrays, which keep the line-shapes in the memory.

- **ReadFitParamsFromLineShapes(TString)** is used to read the input lookup table that contains the line-shapes. The line-shapes can be stored in either TF1 or TH1D objects, where the latter is advantageous in terms of CPU time but requires an optimum selection of the histogram binning.

- **EvalFitValue(Int_t particle, Double_t x)** is used to calculate \( \rho_j(x) \) values for each track. The pointer to this function is set by the “SetFunctionPointers” function.
The calculation of the moments is carried out by means of the following public member functions of the TIdentity2D class:

- **TIdentity2D::SetFileName(TString *dataTreeFilePath)** sets the path to the input data tree that contains track-by-track information.
- **TIdentity2D::GetTree(Long_t &n, TString)** reads the input tree-branches and provides access to them via the corresponding “Get” functions.
- **TIdentity2D::SetBranchNames(const Int_t tmpNBranches, TString tmpBranchNameArr[])** is used to introduce the data tree structure to the TIdentity code.
- **TIdentity2D::SetFunctionPointers(fptr)** sets a pointer to the “EvalFitValue” function.
- **TIdentity2D::GetEntry(Int_t entry)** retrieves the entries of the data tree.
- **TIdentity2D::GetBins(const Int_t nExtraBins, Double_t *)** allows access to the branches of the input tree.
- **TIdentity2D::Reset()** resets all counters to zero.
- **TIdentity2D::SetUseSign(Int_t sign)** sets the particle type to be analysed; particle\((sign = 1)\) or anti-particle\((sign = -1)\). One can also run the code for the sum of particles and anti-particles by setting \(sign = 0\).
- **TIdentity2D::AddParticles()** accumulates the \(\omega\) quantities which are used to calculate \(W\) quantities.
- **TIdentity2D::Finalize()** calculates the moments of the \(W\) quantities.
- **TIdentity2D::AddIntegrals(Int_t sign)** calculates the elements of the Identity matrix.
- **TIdentity2D::CalcMoments()** calculates all moments up to the second order for each particle type.
- **TIdentity2D::GetMean(Int_t j)** is used to retrieve the first moment of particle type \(j; \langle N_j \rangle\).
• TIdentity2D::GetSecondMoment(Int_t j) is used to retrieve the second moment of particle type j; \( \langle N_j^2 \rangle \)

• TIdentity2D::GetMixedMoment(Int_t i, Int_t j) is used to retrieve the mixed moment of particle types i and j; \( \langle N_i N_j \rangle \)

For further details two example macros are provided in the module: (i) testIdentity_Sign.C calculates the moments for particles and antiparticles separately or together, (ii) testIdentity_NetParticles.C calculates the moments of particles and anti-particles simultaneously. Further details are given in a README file.

5. Conclusions

In summary, we developed the TIdentity module to reconstruct the moments of particle multiplicity distributions. The method was tested in Monte-Carlo simulations with ten million events containing five particle (anti-particle) species with different multiplicities. In order to account for particle misidentification, we allowed for overlaps of the individual particle energy loss distributions. The obtained moments are consistent with the generated ones with a precision better than 0.1\%.

Even though the Identity Method was developed specifically for the analysis of event-by-event fluctuations of particle multiplicities, in particular using \( dE/dx \) as the identity variable, it can be used for different studies such as reconstruction of particle spectra. Moreover, instead of \( dE/dx \), other observables e.g. invariant mass or time-of-flight measurements can be used as the identity variable.

The current version of the code unfolds the moments up to the second order, which will be extended to higher-order moments in the future releases. We further note that, the method can also be exploited in different fields facing similar problems.

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