ON THE HOMEOMORPHISM AND DIFFEOMORPHISM TYPES
OF MANIFOLDS ADMITTING ROUND FOLD MAPS

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Abstract. (Stable) fold maps are fundamental tools in a generalization of
the theory of Morse functions on smooth manifolds and its application to
studies of geometric properties of smooth manifolds. Round fold maps were
introduced as stable fold maps with singular value sets, which are defined as
the set consisting of all the singular values, of concentric spheres by the author
in 2013-4 and topological information of such maps and their source manifolds
such as homology and homotopy groups of manifolds have been studied under
appropriate conditions by the author.

In this paper, as more precise information of manifolds admitting round
fold maps, we study the homeomorphism and diffeomorphism types of man-
ifolds admitting such maps under appropriate differential topological condi-
tions.

1. Introduction and fundamental notation and terminologies

Fold maps are fundamental tools in a generalization of the theory of Morse
functions on smooth manifolds and its application to studies of smooth manifolds
and studies of geometric (algebraic and differential topological) properties of fold
maps and their source manifolds have been important. Such studies were started
by Whitney ([27]) and Thom ([26]) in the 1950’s. A fold map from an m-dimensional
closed smooth manifold into an n-dimensional smooth manifold without boundary
(m ≥ n ≥ 2) is defined as a smooth map whose singular points are of the form

\[(x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_{n-1}, \sum_{k=n}^{m-i} x_k^2 - \sum_{k=m-i+1}^{m} x_k^2)\]

for an integer 0 ≤ i ≤ \( \frac{m-n+1}{2} \) (the integer i is uniquely determined and it is called
the index of the singular point). A Morse function is naturally regarded as a fold
map (n = 1). For such a map, the followings hold.

(1) The set consisting of all the singular points (the singular set) is a closed
smooth submanifold of dimension n − 1 of the source manifold.

(2) The restriction map to the singular set is a smooth immersion of codimen-
sion 1.

We also note that if the restriction map to the singular set is an immersion with
normal crossings, then it is stable (stable maps are important in the theory of global
singularity; see [9] for example).

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Since around the 1990’s, fold maps with additional conditions have been actively studied. For example, in [2], [8], [19], [20], [21] and [23], special generic maps, which are defined as fold maps such that the indices of singular points are always 0, were studied. A Morse function on a homotopy sphere with just two singular points is regarded as a special generic map; every homotopy sphere of dimension $k \neq 4$ and the 4-dimensional standard sphere $S^4$ admits such a function and a manifold admitting such a function is homeomorphic to a sphere (see [15] and [16] and see also [17]).

We obtain a special generic map from any standard sphere of dimension $k_1 \geq 2$ into the $k_2$-dimensional Euclidean space $\mathbb{R}^{k_2}$ by a natural projection under the assumption that $k_1 \geq k_2 \geq 1$ holds. On the other hand, it was shown that a homotopy sphere of dimension $k_1$ admitting a special generic map into $\mathbb{R}^{k_2}$ is diffeomorphic to the standard sphere $S^{k_1}$ under the assumption that $1 \leq k_1 - k_2 \leq 3$ holds (see [19] and Example 1 of this paper). In addition, in [19] and [21], the diffeomorphism types of manifolds admitting special generic maps into $\mathbb{R}^2$ and $\mathbb{R}^3$ are completely or partially classified. From a Morse function and its singular points, we can know homology groups and some information on homotopy of the source manifold and from a special generic map and its singular points, we can know more precise information such as the homeomorphism and diffeomorphism type of the source manifold.

Later, in [24], Sakuma studied simple fold maps, which are defined as fold maps such that fibers of singular values do not have any connected component with more than one singular points (see also [18]). In [13], Kobayashi and Saeki investigated topological properties of stable maps including fold maps which are stable into the plane. In [22], Saeki and Suzuoka found good topological properties of manifolds admitting stable maps whose regular fibers, which are defined as the inverse images of regular values, are disjoint unions of spheres.

Furthermore, in [11], round fold maps, which will be mainly studied in this paper, were introduced. A round fold map is defined as a fold map satisfying the followings.

1. The singular set is a disjoint union of standard spheres.
2. The restriction map to the singular set is an embedding.
3. The set consisting of all the singular values (the singular value set) of the map is a disjoint union of spheres embedded concentrically.

For example, some special generic maps on homotopy spheres are examples of round fold maps whose singular sets are connected. Any standard sphere whose dimension is $m > 1$ admits such a map into $\mathbb{R}^n$ with $m \geq n \geq 2$ and any homotopy sphere whose dimension is larger than 1 and not 4 admits such a map into the plane (see also [19] and Example 1 of this paper).

Homology and homotopy groups, which are algebraic topological objects, of manifolds admitting round fold maps were studied in [11] under appropriate conditions. Now, how about the homeomorphism and diffeomorphism types, which are more geometric objects, of manifolds admitting round fold maps? In this paper, we study such problems under appropriate conditions.

This paper is organized as the following.

In section 2, first, we review the Reeb space of a smooth map, which is defined as the space consisting of all the connected components of the fibers of the map. Second, we review the definition and fundamental terminologies of round fold maps
in [11]. For example, we recall axes, which are rays originating from points in the connected components of the sets consisting of all the regular values located in the centers of the target Euclidean spaces and proper cores, which are closed discs embedded in the previous connected components of the sets of regular values (FIGURE 1).

In section 3, we first introduce known results on round fold maps on spheres and we construct round fold maps on $m$-dimensional closed smooth manifolds having the structures of bundles over $S^n$ whose fibers are smooth manifolds and whose structure groups consist of diffeomorphisms with $m \geq n \geq 2$ such that the inverse images of axes are diffeomorphic to cylinders (Theorems 1-4).

In section 4, we study the diffeomorphism types of manifolds whose dimensions are $m$ admitting round fold maps into $\mathbb{R}^n \ (n \geq 2)$ with $m \geq 2n$ under additional conditions. First, we recall a result on homology and homotopy groups of manifolds admitting round fold maps whose regular fibers are always disjoint unions of spheres in [10] or [11] (Proposition 1). In subsection 4.1, under appropriate conditions, we construct a new round fold map on a manifold represented as a connected sum of two manifolds admitting a round fold map (Proposition 2 and Theorems 5 and 6). Conversely, in subsection 4.2, we decompose a round fold map on a closed and connected manifold into two round fold maps so that a connected sum of the resulting source manifolds is the original source manifold (Proposition 4 and Theorem 7). In these subsections, we apply generalizations of surgery operations on stable maps from closed and simply-connected smooth manifolds into the plane which do not change the diffeomorphism types of the source manifolds in [13] (R-operations) and their inverse operations. Last, in subsection 4.3, by applying the obtained results and their proofs, we determine the diffeomorphism types of manifolds admitting round fold maps whose regular fibers are disjoint unions of spheres satisfying Theorem 3 of [11] or Proposition 1 of this paper under a few additional differential topological conditions. More precisely, through Theorems 8, 10 and 11, we give a characterization of the family of closed and connected manifolds represented as connected sums of finite numbers of smooth homotopy spheres whose dimensions are $m$ admitting round fold maps into $\mathbb{R}^n$ with connected singular sets and finite numbers of manifolds having the structures of bundles over $S^n$ with fibers diffeomorphic to $S^{m-n}$ and structure groups consisting of diffeomorphisms by a class of round fold maps mentioned in the previous sentence. We also show Theorems 9 and 11, which give characterizations of some families of manifolds by certain classes of round fold maps.

This work is based on the doctoral dissertation by the author [11], in which algebraic and differential topological properties of manifolds admitting round fold maps are studied and the contents of the papers [10] and [12] by the author are included.

We note about notation and terminologies on spaces and maps in this paper. On a topological space $X$, we denote the identity map on $X$ by $\text{id}_X$. If a topological space $X$ is a topological manifold, then we denote the interior of $X$ by $\text{Int}X$ and the boundary of $X$ by $\partial X$. We denote the disjoint union of a family of topological spaces $\{X_j\}_{j=1}^l$ by $X_1 \sqcup \cdots \sqcup X_l$. For a map $c : X_1 \to X_2$ and subspaces $Y_1 \subset X_1$ and $Y_2 \subset X_2$ such that $c(Y_1) \subset Y_2$ holds, $c|_{Y_1} : Y_1 \to Y_2$ is the restriction
map of \( c \) to \( Y_1 \). Moreover, for a map and a point in the target space of the map, we call the inverse image of the point the fiber (of the point).

For a homeomorphism \( \phi : Y_2 \rightarrow Y_1 \) in the same situation, by gluing \( X_1 \) and \( X_2 \) together by \( \phi \), we obtain a new topological space and denote the space by \( X_1 \sqcup_{\phi} X_2 \). We often omit \( \phi \) of \( X_1 \sqcup_{\phi} X_2 \) and denote it by \( X_1 \sqcup X_2 \) in case we consider a natural identification.

For a smooth map \( c \), we define the singular set of \( c \) as the set consisting of all the singular points of \( c \) as in the presentation of the fundamental properties of fold maps before and denote it by \( S(c) \). For the smooth map \( c \), we call \( c(S(c)) \) the singular value set of \( c \) as in the presentation of the definition of a round fold map before. Moreover, for \( c \), we call the fiber of a regular value a regular fiber of \( c \).

We also note on (homotopy) spheres. In this paper, an almost-sphere of dimension \( k \) means a homotopy sphere given by gluing two \( k \)-dimensional standard closed discs together by a diffeomorphism between the boundaries.

We often use terminologies on (fiber) bundles in this paper (see also [25]). For a topological space \( X \), an \( X \)-bundle is a bundle whose fiber is \( X \) and a section of an \( X \)-bundle whose projection from the total space \( Y \) onto the base space \( Z \) is \( \pi : Y \rightarrow Z \) is a continuous map \( s : Z \rightarrow Y \) such that the composition \( \pi \circ s \) of the two maps \( s \) and \( \pi \) is the identity map on \( Z \). A bundle whose structure group is \( G \) is said to be a trivial bundle if it is equivalent to the product bundle as a bundle whose structure group is \( G \). Especially, a trivial bundle whose structure group is a subgroup of the homeomorphism group of the fiber is said to be a topologically trivial bundle. In this paper, a smooth (PL) bundle means a bundle whose fiber is a smooth (resp. PL) manifold and whose structure group is a subgroup of the diffeomorphism group (resp. PL homeomorphism group) of the fiber. A linear bundle is a smooth bundle whose fiber is a standard disc or a standard sphere and whose structure group consists of linear transformations on the fiber.

Throughout this paper, we assume that \( M \) is a closed smooth manifold of dimension \( m \), that \( N \) is a smooth manifold of dimension \( n \) without boundary, that \( f : M \rightarrow N \) is a smooth map and that \( m \geq n \geq 1 \). Manifolds are smooth and of class \( C^\infty \) and smooth maps between manifolds are also of class \( C^\infty \) unless otherwise stated in the proceeding sections.

### 2. Reeb spaces and round fold maps

#### 2.1. Reeb spaces

Definition 1. Let \( X, Y \) be topological spaces. For \( p_1, p_2 \in X \) and for a map \( c : X \rightarrow Y \), we define as \( p_1 \sim_c p_2 \) if and only if \( p_1 \) and \( p_2 \) are in the same connected component of \( c^{-1}(p) \) for some \( p \in Y \). \( \sim_c \) is an equivalence relation.

We denote the quotient space \( X/\sim_c \) by \( W_c \). We call \( W_c \) the Reeb space of \( c \).

We denote the induced quotient map from \( X \) into \( W_c \) by \( q_c \). We define \( \tilde{c} : W_c \rightarrow Y \) so that \( c = \tilde{c} \circ q_c \).

\( W_c \) is often homeomorphic to a polyhedron. For example, for a Morse function, the Reeb space is a graph and for a simple fold map, the Reeb space is homeomorphic to a polyhedron which is not so complex (see Proposition 1 later). For a special generic map, the Reeb space is homeomorphic to a smooth manifold (see section 2
In this section, we review round fold maps. See also [10] and [11].

2.2. Fundamental terms on round fold maps and constructions. First, we introduce the definition of a round fold map. Before that, we recall $C^\infty$ equivalence (see also [9] for example). For two $C^\infty$ maps $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, we say that they are $C^\infty$ equivalent if there exist $C^\infty$ diffeomorphisms $\phi_X : X_1 \to X_2$ and $\phi_Y : Y_1 \to Y_2$ such that the following diagram commutes.

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\phi_X} & X_2 \\
\downarrow f_1 & & \downarrow f_2 \\
Y_1 & \xrightarrow{\phi_Y} & Y_2
\end{array}
$$

Definition 2 (round fold map, [11], [12]). $f : M \to \mathbb{R}^n$ ($m \geq n \geq 2$) is said to be a round fold map if $f$ is $C^\infty$ equivalent to a fold map $f_0 : M_0 \to \mathbb{R}^n$ on a closed manifold $M_0$ such that the followings hold.

1. The singular set $S(f_0)$ is a disjoint union of standard spheres whose dimensions are $n-1$ and consists of $l \in \mathbb{N}$ connected components.
2. The restriction map $f_0|_{S(f_0)}$ is an embedding.
3. Let $D^n_r := \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid \sum_{k=1}^n x_k^2 \leq r\}$. Then, $f_0(S(f_0)) = \cup_{k=1}^l \partial D^n_k$ holds.

We call $f_0$ a normal form of $f$. We call a ray $L$ from $0 \in \mathbb{R}^n$ an axis of $f_0$ and $D^n_+$ the proper core of $f_0$. Suppose that for a round fold map $f$, its normal form $f_0$ and diffeomorphisms $\Phi : M \to M_0$ and $\phi : \mathbb{R}^n \to \mathbb{R}^n$, $\phi \circ f = f_0 \circ \Phi$ holds. Then for an axis $L$ of $f_0$, we also call $\phi^{-1}(L)$ an axis of $f$ and for the proper core $D^n_+$ of $f_0$, we also call $\phi^{-1}(D^n_+)$ a proper core of $f$.

Let $f$ be a normal form of a round fold map and $P := D^n_+$. We set $E_1 := f^{-1}(P)$ and $E_2 := M - f^{-1}(\text{Int}P)$. We set $F := f^{-1}(p)$ for a point $p \in \partial P$. 

![Figure 1. An axis and a proper core of a round fold map](image-url)
We put $Q := \mathbb{R}^n - \text{Int}P$. Let $f_1 := f|_{E_1} : E_1 \to P$ if $F$ is non-empty and let $f_2 := f|_{E_1} : E_2 \to Q$.

$f_1$ gives the structure of a trivial smooth bundle over $P$ and $f_1|_{\partial E_1} : \partial E_1 \to \partial P$ gives the structure of a trivial smooth bundle over $\partial P$ if $F$ is non-empty. $f_2|_{E_2} : \partial E_2 \to \partial Q$ gives the structure of a trivial smooth bundle over $\partial Q$.

We can give $E_2$ the structure of a bundle over $\partial Q$ as follows. Since for $\pi_P(x) := \frac{1}{2} \frac{x}{|x|} (x \in Q)$, $\pi_P \circ f|_{E_2}$ is a proper submersion, this map gives $E_2$ the structure of a smooth $f^{-1}(L)$-bundle over $\partial Q$ (apply Ehresmann’s fibration theorem [5]).

We call this bundle the surrounding bundle of $f$. Note that the structure group of this bundle is regarded as the group of diffeomorphisms on $f^{-1}(L)$ preserving the function $f|_{f^{-1}(L)} : f^{-1}(L) \to L(\subset \mathbb{R})$, which is naturally regarded as a Morse function.

For a round fold map $f$ which is not a normal form, we can consider similar maps and bundles. We call bundles naturally corresponding to the surrounding bundle of a normal form of $f$ a surrounding bundle of $f$.

We can define the following condition for a round fold map.

Definition 3. Let $f : M \to \mathbb{R}^n$ be a round fold map. If a surrounding bundle of $f$ as above is a topologically trivial bundle, then $f$ is said to be normally topologically trivial. If the bundle is a trivial PL bundle, then $f$ is said to be normally PL trivial. If the bundle is a trivial smooth bundle, then $f$ is said to be normally $C^\infty$ trivial.

We can construct a normally $C^\infty$ trivial round fold map as in the following manner.

Let $M$ be a compact manifold with non-empty boundary $\partial M$. Then, there exists a Morse function $\tilde{f} : M \to [a, +\infty]$ such that the following holds.

1. $a$ is the minimum of $\tilde{f}$.
2. $\tilde{f}^{-1}(a) = \partial M$.
3. Singular points of $\tilde{f}$ are always in the interior $\text{Int}M$ and at distinct singular points, the values are always distinct.

Let $\Phi : \partial(\tilde{M} \times \partial(\mathbb{R}^n - \text{Int}D^n)) \to \partial(\tilde{M} \times D^n)$ and $\phi : \partial(\mathbb{R}^n - \text{Int}D^n) \to \partial D^n$ be diffeomorphisms. Let $p_1 : \partial M \times \partial(\mathbb{R}^n - \text{Int}D^n) \to \partial(\mathbb{R}^n - \text{Int}D^n)$ and $p_2 : \partial M \times D^n \to \partial D^n$ be the canonical projections. Suppose that the following diagram commutes.

$$
\begin{array}{ccc}
\partial(\tilde{M} \times (\mathbb{R}^n - \text{Int}D^n)) & \xrightarrow{\Phi} & \partial(\tilde{M} \times D^n) \\
\downarrow p_1 & & \downarrow p_2 \\
\partial(\mathbb{R}^n - \text{Int}D^n) & \xrightarrow{\phi} & \partial D^n
\end{array}
$$

By using the diffeomorphism $\Phi$, we construct a manifold $M := (\partial \tilde{M} \times D^n) \bigcup_{\phi}(\tilde{M} \times \partial(\mathbb{R}^n - \text{Int}D^n))$. By using the diffeomorphism $\phi$, we can construct $D^n \bigcup_{\phi}(\mathbb{R}^n - \text{Int}D^n)$, which is diffeomorphic to $\mathbb{R}^n$. Let $p : \partial M \times D^n \to D^n$ be the canonical projection. Then, by gluing the two maps $p$ and $\tilde{f} \times \text{id}_{\partial D^n}$ together by the pair of diffeomorphisms $(\Phi, \phi)$, a round fold map $f : M \to \mathbb{R}^n$ is obtained; in this situation, we regard $\mathbb{R}^n - \text{Int}D^n$ as $D^n \times [a, +\infty)$ by a diffeomorphism between the spaces.

If $M$ is a compact manifold without boundary, then there exists a Morse function $\tilde{f} : M \to [a, +\infty)$ such that $\tilde{f}(M) \subset (a, +\infty)$ and that at distinct singular points, the values are distinct. We are enough to consider $\tilde{f} \times \text{id}_{S^{n-1}}$ and embed
$[a, +\infty) \times S^{n-1}$ into $\mathbb{R}^n$ to construct a round fold map whose source manifold is $\bar{M} \times S^{n-1}$.

We call the construction of a round fold map $f$ here a \textit{trivial spinning construction}.

3. Round fold maps on spheres and bundles over standard spheres

In this section, about the homeomorphism and diffeomorphism types of manifolds admitting round fold maps, we study about round fold maps on homotopy spheres and manifolds having the structures of bundles over standard spheres.

First, the following example is from fundamental discussions of [19].

Example 1. (1) Let $M$ be a closed manifold of dimension $m$. Let $n$ be an integer such that $m > n \geq 2$ and $n \neq 4, 5$ hold. A round fold map $f : M \to \mathbb{R}^n$ whose singular set is connected exists if and only if $M$ is a homotopy sphere admitting a special generic map into $\mathbb{R}^n$ whose Reeb space is homeomorphic to $D^n$.

(2) Any standard sphere of dimension $m$ admits a map into $\mathbb{R}^n$ as above if $m \geq n \geq 2$ holds. Furthermore, any homotopy sphere of dimension $m > 1$ admits a map into the plane as above unless $m = 4$ according to a discussion in section 5 of [19].

(3) Let $m$ be an integer larger than 3 and $n$ be an integer satisfying $m - n = 1, 2, 3$. In section 4 of [19] and [20], it is shown that if a homotopy sphere of dimension $m$ admits a special generic map into $\mathbb{R}^n$, then the homotopy sphere is a standard sphere.

Thus, in the situation of this part, if on a smooth sphere of dimension $m$, a round fold map into $\mathbb{R}^n$ whose singular set is connected exists, then the sphere is diffeomorphic to $S^m$.

Note also that round fold maps in Example 1 are normally topologically trivial. They are also normally PL trivial and normally $C^\infty$ trivial (see also Example 3 (1) of [11]).

The following theorem has been partially proven in [10] as introduced in Example 2 later.

Theorem 1. Let $M$ be a closed manifold of dimension $m$. Let $n \in \mathbb{N}$ and $m \geq n \geq 2$.

(1) Let $M$ have the structure of a smooth bundle over $S^n$ whose fiber is a closed manifold $F(\neq \emptyset)$. Then, there exists a normally $C^\infty$ trivial round fold map $f : M \to \mathbb{R}^n$ such that the fiber of a point in a proper core of $f$ is diffeomorphic to a disjoint union of two copies of $F$, which are regarded as fibers of the $F$-bundle over $S^n$, and that $f^{-1}(L)$ is diffeomorphic to $F \times [0, 1]$ for an axis $L$ of $f$.

(2) Suppose that a normally topologically trivial round fold map $f : M \to \mathbb{R}^n$ exists and that for an axis $L$ of $f$ and a closed manifold $F$ of dimension $m - n$, $f^{-1}(L)$ is diffeomorphic to $F \times [0, 1]$. Then, $M$ has the structure of an $F$-bundle over $S^n$. If $f$ is normally PL trivial, then $M$ has the structure of a PL $F$-bundle over $S^n$ and if $f$ is normally $C^\infty$ trivial, then $M$ has the structure of a smooth $F$-bundle over $S^n$.

Proof. We prove the first part.
We may represent $S^n$ as $(D^n \sqcup D^n) \sqcup (S^{n-1} \times [0,1])$, where we identify $\partial(D^n \sqcup D^n) = S^{n-1} \sqcup S^{n-1}$ and $\partial(S^{n-1} \times [0,1]) = S^{n-1} \sqcup S^{n-1}$. For a diffeomorphism $\Phi$ from $S^{n-1} \times (F \sqcup F)$ onto $\partial D^n \times (F \sqcup F)$ which is a bundle isomorphism between the trivial $F$-bundles over $\partial(D^n \sqcup D^n) = \partial(S^{n-1} \times [0,1]) = S^{n-1} \sqcup S^{n-1}$ inducing the identification between the base spaces, we may represent $M$ as $((D^n \sqcup D^n) \times F) \sqcup \Phi(S^{n-1} \times [0,1] \times F) = (D^n \times (F \sqcup F)) \sqcup \Phi(S^{n-1} \times [0,1] \times F)$.

There exists a good Morse function $f : F \times [0,1] \rightarrow [\alpha, +\infty)$, where $\alpha \in \mathbb{R}$ is the minimal value. We consider a map $\Phi$ from $S^{n-1} \times (F \sqcup F)$ onto $\partial D^n \times (F \sqcup F)$ which is a bundle isomorphism between the trivial $F$-bundles over $\partial(D^n \sqcup D^n)$, where $\Phi$ and $\partial$ are normally topologically trivial and replace it by the followings.

$$F \times ([0] \sqcup [1]) \times \partial(\mathbb{R}^n - \text{Int} D^n) \xrightarrow{\Phi} F \times ([0] \sqcup [1]) \times \partial D^n$$

Then, on $M$, by gluing the two maps $p$ and $\tilde{f} \times \text{id}_{\mathbb{R}^n - \text{Int} D^n}$ together by using diffeomorphisms $\Phi$ and $\phi$ or using a trivial spinning construction introduced just after Definition 3, we have a normally $C^\infty$ trivial round fold map $f : M \rightarrow \mathbb{R}^n$. We see that $f$ is a round fold map satisfying the given conditions.

We now prove the second part.

Suppose that there exists a normally topologically trivial round fold map $f : M \rightarrow \mathbb{R}^n$ and that for an axis $L$ of $f$ and a closed manifold $F$, $f^{-1}(L)$ is diffeomorphic to $F \times [0,1]$. Then, for a diffeomorphism $\Phi$ from $S^{n-1} \times (F \sqcup F)$ onto $\partial D^n \times (F \sqcup F)$ which is a bundle isomorphism between the trivial $(F \sqcup F)$-bundles over $S^{n-1} = \partial D^n$ inducing a diffeomorphism between the base spaces, $M$ is regarded as $(D^n \times (F \sqcup F)) \sqcup \Phi(S^{n-1} \times [0,1] \times F)$ in the topology category. Thus, $M$ has the structure of an $F$-bundle over $S^n$.

Moreover, suppose that there exists a normally PL ($C^\infty$) trivial round fold map $f : M \rightarrow \mathbb{R}^n$ and that for an axis $L$ of $f$ and a closed manifold $F$, $f^{-1}(L)$ is diffeomorphic to $F \times [0,1]$. In this case, in the PL (resp. $C^\infty$) category, for a diffeomorphism $\Phi$ from $S^{n-1} \times (F \sqcup F)$ onto $\partial D^n \times (F \sqcup F)$ which is a bundle isomorphism between the trivial $(F \sqcup F)$-bundles over $S^{n-1} = \partial D^n$ inducing a diffeomorphism between the base spaces, $M$ is regarded as $(D^n \times (F \sqcup F)) \sqcup \Phi(S^{n-1} \times [0,1] \times F)$.

This completes the proof of both parts of the theorem.

Example 2 ([10]). Let $M$ be a closed manifold of dimension $m$ and let $M$ have the structure of a smooth bundle over $S^n$ whose fiber is an almost-sphere $\Sigma$ of dimension $m - n$ with $m > n \geq 2$. Then, there exists a normally $C^\infty$ trivial round fold map $f : M \rightarrow \mathbb{R}^n$ such that the fiber of a point in a proper core of $f$ is diffeomorphic to a disjoint union of two copies of $\Sigma$ and that $S(f)$ consists of 2 connected components and is the disjoint union of the set of all the fold points whose indices are 0 and the set of all the fold points whose indices are 1.

Remark 1. In the situation of Theorem 1 (2), we may weaken the condition that $f$ is normally topologically trivial and replace it by the followings.
(1) For a proper core $P$ and an axis $L$ of $f$, $f^{-1}(\mathbb{R}^n - \text{Int}P)$ has the structure of a topologically trivial bundle over $\partial P$ whose fiber is diffeomorphic to $f^{-1}(L)$.

(2) $f|_{f^{-1}(\partial P)} : f^{-1}(\partial P) \to \partial P$ gives the structure of a subbundle of the previous bundle $f^{-1}(\mathbb{R}^n - \text{Int}P)$.

In the cases where $f$ is normally PL trivial and normally $C^\infty$ trivial, we can replace the conditions similarly. Such maps are said to be topologically trivial, PL trivial and $C^\infty$ trivial, respectively.

We have the following theorem by virtue of the fact that there exist homotopy spheres homeomorphic but not diffeomorphic to $S^7$ and having the structures of linear $S^3$-bundles over $S^4$ (see [4] and [?] and Example 2, which is mentioned in [10].

**Theorem 2 ([10]).** There exist homotopy spheres that are homeomorphic but not diffeomorphic to $S^7$ and they admit $C^\infty$ trivial round fold maps into $\mathbb{R}^4$ whose singular sets consist of 2 connected components. The 7-dimensional standard sphere $S^7$, which admits the structure of a linear $S^3$-bundle over $S^4$, also admits such a map.

Remark 2. If there exists a special generic map from a homotopy sphere homeomorphic to $S^7$ into $\mathbb{R}^4$, then the homotopy sphere is diffeomorphic to $S^7$ from Example 1 (3). This means that we cannot reduce the numbers of connected components of the singular sets of round fold maps in Theorem 2 for homotopy spheres not diffeomorphic to $S^7$.

In the case where $n = 2$ holds, we have the following theorems.

**Theorem 3.** Let $M$ be a closed manifold of dimension $m \geq 7$, $m = 3$ or $m = 4$. If $m \geq 7$ is assumed, let $F \neq \emptyset$ be a closed and simply-connected manifold of dimension $m - 2$ and if $m = 3, 4$ is assumed, let $F$ be the $(m - 2)$-dimensional standard sphere $S^{m-2}$. Then, the followings are equivalent.

1. $M$ has the structure of a smooth $F$-bundle over $S^2$.
2. $M$ admits a round fold map $f : M \to \mathbb{R}^2$ satisfying the followings.
   a. The regular fiber of a point in a proper core of $f$ is diffeomorphic to a disjoint union of two copies of $F$.
   b. For an axis $L$ of $f$, $f^{-1}(L)$ is diffeomorphic to $F \times [0,1]$.

**Proof.** If $M$ admits a round fold map $f : M \to \mathbb{R}^2$ as in the condition (2), then since for a proper core $P$ of $f$, $f|_{f^{-1}(P)} : f^{-1}(P) \to P$ gives the structure of a trivial bundle, $f$ is $C^\infty$ trivial by virtue of the pseudoisotopy theorem [3] ($m \geq 7$) or the fact that a smooth $(S^{m-2} \times [0,1])$-bundle over $S^1$ whose subbundle with the fiber $S^{m-2} \times \{0,1\} \subset S^{m-2} \times [0,1]$ is trivial is also trivial ($m = 4$).

From Theorem 1, this completes the proof. 

We also have the following theorem, which is a generalization of Theorem 1 (1).

**Theorem 4.** Let $M$ be a manifold obtained by the following steps.

1. Let $n$ be an integer larger than 1. Let $F_1 \neq \emptyset$ be a closed manifold and let $M_1$ be a closed manifold having the structure of a smooth $F_1$-bundle over $S^n$. 

   (1)
(2) Let \( F_2 \neq \emptyset \) be a closed manifold. Let \( M_2 \) be a closed manifold having the structure of a smooth \( F_2 \)-bundle over \( M_1 \) such that the restriction to any fiber of the previous bundle \( M_1 \) is a trivial smooth bundle.

(3) For \( M_k \) (\( k \geq 2 \)), let \( F_{k+1} \neq \emptyset \) be a closed manifold and we define a closed manifold \( M_{k+1} \) as a manifold having the structure of a smooth \( F_{k+1} \)-bundle over \( M_k \) such that the restriction of the bundle to the product of every fiber appearing in a family of fibers \( \{ F_j \}_{j=1}^{k} \) of bundles \( \{ M_j \}_{j=1}^{k} \) is a trivial smooth bundle.

(4) Let \( M := M_k \) (\( k \geq 2 \)).

In this situation, \( M \) admits a \( C^\infty \) trivial round fold map \( f : M \rightarrow \mathbb{R}^n \) such that the inverse image of an axis of \( f \) is diffeomorphic to the product of the manifolds of \( k \) manifolds belonging to the family \( \{ F_j \}_{j=1}^{k} \) and the closed interval \([-1, 1]\).

**Proof.** In the case where \( M = M_1 \) holds, the result follows from Theorem 1 (1).

Assume that \( M_k \) admits a round fold map \( f_k : M_k \rightarrow \mathbb{R}^n \) mentioned in the statement. Note that by Theorem 1 (2), \( M_k \) has the structure of a smooth bundle over \( S^n \) whose fiber is diffeomorphic to the product of the manifolds of \( k \) manifolds belonging to the family \( \{ F_j \}_{j=1}^{k} \).

We consider \( M = M_{k+1} \) having the structure of a smooth \( F_{k+1} \)-bundle over \( M_k \) as in the mentioned steps. If we restrict the \( F_{k+1} \)-bundle to the inverse image \( f_k^{-1}(P) \) of a proper core \( P \) of \( f_k \), which has the structure of a smooth trivial bundle over \( P \) with a fiber diffeomorphic to the product of the manifolds of \( k \) manifolds belonging to the family \( \{ F_j \}_{j=1}^{k} \), then it is a trivial smooth bundle by a condition on the structure of the bundle \( M_{k+1} \) over \( M_k \). If we restrict the \( F_{k+1} \)-bundle to the inverse image \( f^{-1}(\mathbb{R}^n - \text{Int}P) \), which is diffeomorphic to the product of \( \partial P \) or \( S^{n-1} \), all the \( k \) manifolds belonging to the family \( \{ F_j \}_{j=1}^{k} \) and the closed interval \([-1, 1]\), then the resulting bundle is trivial by the same condition on the structure of the bundle \( M_{k+1} \) over \( M_k \).

From this discussion, \( M_{k+1} \) has the structure of a smooth bundle over \( S^n \) whose fiber is diffeomorphic to the product of all the \( k+1 \) manifolds belonging to the family \( \{ F_j \}_{j=1}^{k+1} \). Thus, we can construct a desired round fold map \( f_{k+1} : M_{k+1} \rightarrow \mathbb{R}^n \) by using a method of the proof of Theorem 1 (1).

By the induction, this completes the proof. \( \square \)

4. **Manifolds admitting round fold maps with additional topological conditions**

As in the previous section, we can sometimes know the homeomorphism and diffeomorphism types of manifolds admitting round fold maps. In general, it seems to be difficult to know the homeomorphism types and diffeomorphism types of these manifolds strictly. Here, we consider such problems in more general situations and give some answers. In this section, we mainly consider round fold maps from manifolds whose dimensions are \( m \) into \( \mathbb{R}^n \) with \( n \geq 2 \) and \( m \geq 2n \) assumed.

Before the study, we review a result on homology and homotopy groups of manifolds admitting round fold maps whose regular fibers are always disjoint unions of homotopy spheres. Maps satisfying the assumption of the proposition often appear as important examples in discussions of this section.
Proposition 1 ([10], [11], [12]). Let $M$ be a closed and connected manifold of dimension $m$. Let $f : M \to \mathbb{R}^n$ be a round fold map $(m \geq n \geq 2)$ satisfying the followings and let $m > n$ also hold.

1. For each regular value $p$, $f^{-1}(p)$ is a disjoint union of almost-spheres.
2. Indices of fold points are 0 or 1.

Then, the quotient map $q_f : M \to W_f$ induces an isomorphism of homotopy groups $\pi_k(M) \cong \pi_k(W_f)$ for $0 \leq k \leq m - n - 1$.

Furthermore, let $M$ be simply-connected and let $m \geq 2n$ also hold. Let the fiber of a point in a proper core of $f$ consist of $l \in \mathbb{N}$ connected components. Then, we have $\pi_k(M) \cong \{0\}$ for $0 \leq k \leq n - 1$. We also have $\pi_n(M) \cong H_n(M; \mathbb{Z}) \cong \mathbb{Z}^{l-1}$ in the case where $m > 2n$ holds and $\pi_n(M) \cong H_n(M; \mathbb{Z}) \cong \mathbb{Z}^{2(l-1)}$ in the case where $m = 2n$ holds.

Remark 3. The former part of Proposition 2 is not only for round fold maps satisfying the given two restrictions but also for simple fold maps from $m$-dimensional closed and connected manifolds into $n$-dimensional manifolds without boundaries satisfying these restrictions with $m - n \geq 2$ assumed or with $m - n = 1$ and the orientability of the manifold $M$ assumed ([11] and [12]).

4.1. Round fold maps on connected sums of two connected oriented manifolds admitting round fold maps. In this subsection, we construct a round fold map on a manifold represented as a connected sum of two closed and connected manifolds admitting round fold maps under additional conditions.

We introduce an operation of constructing a new round fold map from given two round fold maps.

Let $m, n \in \mathbb{N}$ and let $m > n \geq 2$. Let $M_1$ and $M_2$ be closed and connected manifolds whose dimensions are $m$. Let there exist round fold maps $f_1 : M_1 \to \mathbb{R}^n$ and $f_2 : M_2 \to \mathbb{R}^n$ such that the followings hold. See also FIGURE 2.

1. The fiber of a point in a proper core $P_1$ of $f_1$ has a connected component diffeomorphic to $S^{m-n}$.
2. For the boundary $C$ of the unbounded connected component of $\mathbb{R}^n - \text{Int} f_2(M_2)$ and a small closed tubular neighborhood $P_2$, $f_2^{-1}(P_2)$ has the structure of a trivial linear $D^{m-n+1}$-bundle over the connected component $\partial P_2 \cap f_2(M_2)$ of $\partial P_2$ such that the map $f_2|_{f_2^{-1}(\partial P_2 \cap f_2(M_2))}$ gives the structure of a subbundle of the bundle $f_2^{-1}(P_2)$.

Let $V_1$ be a connected component of $f_1^{-1}(P_1)$ such that $f_1|_{V_1} : V_1 \to P_1$ gives the structure of a trivial smooth $S^{m-n}$-bundle over $D^n$, which exists by the assumption on $f_1$, and $V_2 := f_2^{-1}(P_2)$. $V_2$ is a small closed tubular neighborhood of $f_2^{-1}(C) \subset M_2$. $V_2$ has the structure of a trivial linear $D^{m-n+1}$-bundle from the latter assumption. Moreover, from the same assumption $f_2|_{\partial V_2}$ gives the structure of a subbundle of the bundle $V_2$.

We can consider two maps $f_1|_{M_1 - \text{Int} V_1}$ and $f_2|_{M_2 - \text{Int} V_2}$ and glue them together by to obtain a new round fold map from a new manifold into $\mathbb{R}^n$.

We call the operation a canonical combining operation to the pair $(f_1, f_2)$.

Proposition 2. Let $M_1$ and $M_2$ be closed and connected manifolds whose dimensions are $m$. Let there exist a round fold map $f_1 : M_1 \to \mathbb{R}^n$ $(m \geq n \geq 2)$ such that the fiber of a point in a proper core of $f_1$ has a connected component diffeomorphic
to $S^{m-n}$ and a round fold map $f_2 : M_2 \to \mathbb{R}^n$ such that for the boundary $C$ of the unbounded connected component of $\mathbb{R}^n - \text{Int} f_2(M_2)$, the inclusion of $f_2^{-1}(C)$ into $M_2$ is null-homotopic. We also assume that $m \geq 2n$ holds.

Then, on each connected sum $M$ of $M_1$ and $M_2$, by a canonical combining operation to the pair $(f_1, f_2)$ we obtain a round fold map $f : M \to \mathbb{R}^n$.

**Proof.** Let $P_1$ be a proper core of $f_1$ and $P_2$ be a small closed tubular neighborhood of the connected component $C$ of $f_2 : S(f_2)$. Let $V_1$ be a connected component of $f_1^{-1}(P_1)$ such that $f|_{V_1} : V_1 \to P_1$ gives the structure of a trivial smooth $S^{m-n}$-bundle over $D^n$, which exists by the assumption on $f_1$, and $V_2 := f_2^{-1}(P_2)$. $V_2$ is a closed tubular neighborhood of $f_2^{-1}(C) \subset M_2$. Since $m \geq 2n = 2(n - 1) + 2$ is assumed and the inclusion of $f_2^{-1}(C)$ into $M_2$ is assumed to be null-homotopic, $V_2$ has the structure of a trivial linear $D^{m-n+1}$-bundle. More precisely, $f_2|_{\partial V_2}$ gives the structure of a subbundle of the bundle.

Since $m \geq 2n = 2(n - 1) + 2$ is assumed, we may regard that the following holds for each diffeomorphism $\Psi : \partial D^m \to \partial D^m$ extending to some diffeomorphism on $D^m$ or from $M_2 - (M_2 - D^m)$ onto $M_1 - (M_1 - D^m)$ and for some diffeomorphism $\Phi : \partial V_2 \to \partial V_1$ regarded as a bundle isomorphism between the two trivial smooth $S^{m-n+1}$-bundles over $S^{n-1}$ inducing a diffeomorphism between the base spaces, where for two manifolds $X_1$ and $X_2$, $X_1 \cong X_2$ means that $X_1$ and $X_2$ are diffeomorphic.

\[
\begin{align*}
(M_1 - \text{Int} V_1) \bigcup_\Phi (M_2 - \text{Int} V_2) &
\cong (M_1 - \text{Int} V_1) \bigcup_\Phi ((D^m - \text{Int} V_2) \bigcup (M_2 - \text{Int} D^m)) \\
&
\cong (M_1 - \text{Int} V_1) \bigcup_\Phi ((S^m - (\text{Int} V_2 \cup \text{Int} D^m)) \bigcup (M_2 - \text{Int} D^m)) \\
&
\cong (M_1 - \text{Int} D^m) \bigcup_\Phi (M_2 - \text{Int} D^m)
\end{align*}
\]

This means that the resulting manifold $M$ is represented as a connected sum of the two manifolds $M_1$ and $M_2$ and that $M$ admits a round fold map $f : M \to \mathbb{R}^n$. 

**Figure 2.** $P_1, P_2, C \subset \mathbb{R}^n$ ($P_1$ is the bounded region bounded by the dotted line in the left figure and $P_2$ is the region bounded by the disjoint union of the two dotted lines in the right figure.)
$\mathbb{R}^n$. More precisely, $f$ is obtained by a canonical combining operation to the pair $(f_1, f_2)$.

Example 3. Let $M_1$ and $M_2$ be closed and connected oriented manifolds whose dimensions are $m$. Let there exist a round fold map $f_1 : M_1 \to \mathbb{R}^n$ ($n \geq 2$) such that the fiber of a point in a proper core of $f_1$ has a connected component diffeomorphic to $S^{m-n}$. We also assume that $m \geq 2n$ holds. If $\pi_{n-1}(M_2) \cong \{0\}$ holds and a round fold map $f_2 : M \to \mathbb{R}^n$ exists, then the pair of the maps $f_1$ and $f_2$ satisfies the assumption of Proposition 2. For example, if $M_2$ is simply connected and admits a round fold map $f_2 : M_2 \to \mathbb{R}^n$ satisfying the assumption of Proposition 1, then the pair of the two maps satisfies the assumption of Proposition 2.

We introduce classes of round fold maps satisfying some algebraic and differential topological conditions.

Let $f$ be a normal form of a round fold map such that the singular set $S(f)$ consists of $l$ connected components and $P := D^n_{\frac{1}{2}}$ as defined just before Definition 3.

Let $k_1$ and $k_2$ be integers satisfying $1 \leq k_1 < k_2 \leq l + 1$. We can give $f^{-1}(D^{k_2 - \frac{1}{2}} - D^{k_1 - \frac{1}{2}})$ the structure of a bundle over $\partial P$ as follows.

Since for $\pi_P(x) := \frac{1}{2}(x \in P)$, $\pi_P \circ f|_{f^{-1}(D^{k_2 - \frac{1}{2}} - D^{k_1 - \frac{1}{2}})}$ is a proper submersion, this map gives $f^{-1}(D^{k_2 - \frac{1}{2}} - D^{k_1 - \frac{1}{2}})$ the structure of a smooth bundle over $\partial P$. The structure group of this bundle is regarded as the group of diffeomorphisms on the fiber preserving a Morse function.

For example, if $k_2 = k_1 + 1$ holds and the inverse image $f^{-1}(D^{k_2 - \frac{1}{2}} - D^{k_1 - \frac{1}{2}})$ of the set $D^{k_2 - \frac{1}{2}} - D^{k_1 - \frac{1}{2}}$ of $f$ has a connected component containing fold points of index 0, then the connected component has the structure of a linear $D^{m-n+1}$-bundle which is also a subbundle of the bundle $f^{-1}(D^{k_2 - \frac{1}{2}} - D^{k_1 - \frac{1}{2}})$ mentioned before, which follows from a fundamental discussion in section 2 of [18].

For a round fold map $f$ which is not a normal form, we can consider similar maps and bundles.

Definition 4. In the situation above, the space $f^{-1}(D^{k_2 - \frac{1}{2}} - D^{k_1 - \frac{1}{2}})$ has the structures of isomorphic smooth bundles over $\partial D^{k_1 - \frac{1}{2}}$ and $\partial D^{k_2 - \frac{1}{2}}$ such that the bundle $f|_{f^{-1}(\partial D^{k_1 - \frac{1}{2}})} : f^{-1}(\partial D^{k_1 - \frac{1}{2}}) \to \partial D^{k_1 - \frac{1}{2}}$ and $f|_{f^{-1}(\partial D^{k_2 - \frac{1}{2}})} : f^{-1}(\partial D^{k_2 - \frac{1}{2}}) \to \partial D^{k_2 - \frac{1}{2}}$ give the structures of subbundles. We call such bundles $(k_1, k_2)$-part bundles of $f$ and the fibers the corresponding fibers of the bundles. Furthermore, the subbundle given by $f|_{f^{-1}(\partial D^{k_1 - \frac{1}{2}})} : f^{-1}(\partial D^{k_1 - \frac{1}{2}}) \to \partial D^{k_1 - \frac{1}{2}}$ is said to be the inner part of the $(k_1, k_2)$-part bundles and the subbundle given by $f|_{f^{-1}(\partial D^{k_2 - \frac{1}{2}})} : f^{-1}(\partial D^{k_2 - \frac{1}{2}}) \to \partial D^{k_2 - \frac{1}{2}}$ is said to be the outer part of the $(k_1, k_2)$-part bundles.

If we can give $f^{-1}(D^{k_2 - \frac{1}{2}} - D^{k_1 - \frac{1}{2}})$ the structures of such bundles which are topologically trivial, then $f$ is said to be topologically trivial, then $f$ is said to be topologically $(k_1, k_2)$-trivial. If the bundles are trivial PL bundles, then $f$ is said to be PL $(k_1, k_2)$-trivial and if the bundles are trivial smooth bundles, then $f$ is said to be $C^\infty (k_1, k_2)$-trivial.

Definition 5. Let $l, l' \in \mathbb{N}$ and let $l, l' \geq 2$. Let $\{k_j\}_{j=1}^{l'-1}$ be a sequence of integers such that $k_1 = 1$ and $k_l = l$ hold and that $k_j < k_{j+1}$ holds for any integer $1 \leq j \leq l' - 1$. If $f$ is topologically $(k_j, k_{j+1})$-trivial for all the integers $1 \leq j \leq l' - 1$, then $f$ is said...
to be \textit{almost topologically trivial about the sequence} \( \{k_j\}_{j=1}^{l-1} \). If \( f \) is PL \( (k_j, k_{j+1}) \)-trivial for all the integers \( 1 \leq j \leq l' - 1 \), then \( f \) is said to be \textit{almost PL trivial about the sequence} and if \( f \) is \( C^\infty \) \((k_j, k_{j+1})\)-trivial for all the integers \( 1 \leq j \leq l' - 1 \), then \( f \) is said to be \textit{almost }\( C^\infty \text{ trivial about the sequence}.\)

We can construct a round fold map almost \( C^\infty \) trivial about a sequence \( \{k_j\}_{j=1}^{l'} \) as in this definition easily by an analogy of a trivial spinning construction as in the following.

Let \( l' \in \mathbb{N} \) and \( l' \geq 2 \). Let \( \{E_j\}_{j=1}^{l'-1} \) be a family of compact manifold of dimension \( m - n + 1 \) whose fiber is a disjoint union of \( F_j \neq \emptyset \) and \( F_{j+1} \neq \emptyset \). There exist a positive integer \( l \) and a sequence of integers \( \{k_j\}_{j=1}^{l'} \) of integers such that \( k_1 = 1 \) and \( k_l = l \) hold and that \( k_j < k_{j+1} \) holds for any integer \( 1 \leq j \leq l' - 1 \). We can construct a Morse function \( \tilde{f}_j : E_j \to [k_j - \frac{1}{2}, k_{j+1} - \frac{1}{2}] \) satisfying the followings.

1. On \( F_j \), \( \tilde{f}_j \) is constant and minimal and on \( F_{j+1} \), \( \tilde{f}_j \) is constant and maximal.
2. The minimum and maximum of \( \tilde{f}_j \) are \( k_j - \frac{1}{2} \) and \( k_{j+1} - \frac{1}{2} \), respectively.
3. Singular points of \( \tilde{f}_j \) are always in the interior \( \text{Int} E_j \) of \( E_j \) and at distinct singular points, the values are always distinct. Furthermore, the set of all the values of singular points consists of all the integers larger than \( k_j - \frac{1}{2} \) and smaller than \( k_{j+1} - \frac{1}{2} \).
4. \( F' \) is diffeomorphic to \( S^{m-n} \).

We obtain a family of maps \( \{\tilde{f}_j \times \text{id}_{S^{n-1}} : E_j \times S^{n-1} \to (k_j - \frac{1}{2}, k_{j+1} - \frac{1}{2}) \times S^{n-1}\}_{j=1}^{l'-1} \).

We easily obtain a Morse function \( \tilde{f}_0 : D^{m-n+1} \to [l - \frac{1}{2}, +\infty) \) satisfying the followings.

1. \( \tilde{f}_0 \) is constant and \( l - \frac{1}{2} \) on the boundary.
2. \( \tilde{f}_0(\text{Int} D^{m-n+1}) \subset (l - \frac{1}{2}, +\infty) \) holds.
3. \( \tilde{f}_0 \) has just one singular point and it is maximal at this point.

By gluing the family of maps before, the map \( \tilde{f}_0 \times \text{id}_{S^{n-1}} \) and the projection \( p : D^{\frac{m}{2}} \times F_1 \to D^{\frac{m}{2}} \) together properly, we obtain a desired round fold map; for a non-negative real number \( t \), we regard \( \{t\} \times S^{n-1} \) as \( \partial D^n \) by identifying \((t, x) \in \{t\} \times S^{n-1} \) with \( \left( \frac{1}{|x|} x \right) \in D^n \).

We call such a construction of a round fold map a \textit{piecewise trivial spinning construction}.

We have the following theorem.

\textbf{Theorem 5.} Let \( m, n \in \mathbb{N} \) and \( m \geq n \geq 2 \) hold. Let \( M_1 \) and \( M_2 \) be closed and connected \( m \)-dimensional manifolds. Let there exist a round fold map \( f_1 : M_1 \to \mathbb{R}^n \) such that the fiber of a point in a proper core \( P_1 \) of \( f_1 \) has a connected component diffeomorphic to \( S^{m-n} \). Let there exist a round fold map \( f_2 : M_2 \to \mathbb{R}^n \) such that \( f_2(M_2) \) is diffeomorphic \( D^n \). Then, for an axis \( L \) of \( f_2 \), \( f_2^{-1}(L) \), which is the fiber of a surrounding bundle of \( f_2 \), is connected.

Let \( l, l' \in \mathbb{N} \) and \( l \geq 2 \). Let \( 1 < l' \leq l \). Let \( \{k_j\}_{j=1}^{l'} \) be a sequence of integers such that \( k_1 = 1 \) and \( k_l = l \) hold and that \( k_j \leq k_{j+1} \) for any integer \( 1 \leq j \leq l' - 1 \).

Then, under the assumptions that \( m \geq 2n \) holds and that at least one of the following four conditions is satisfied, on a manifold represented as a connected sum
$M$ of $M_1$ and $M_2$, we can always construct a round fold map $f : M \to \mathbb{R}^n$ by a canonical combining operation to $(f_1, f_2)$.

1. $f_2$ is normally topologically trivial and for a connected component $F$ of the boundary of the fiber $f_2^{-1}(L)$ of a surrounding bundle of $f_2$, $\pi_{n-1}(F) \cong \{0\}$ holds.

2. $f_2$ is almost topologically trivial about the sequence $\{k_j\}_{j=1}^{l'}$ and for any integer $1 \leq j \leq l'$, we can take trivial $(k_j, k_{j+1})$-part bundles of $f$ so that for any connected component $F$ of the fiber of the outer (inner) part of the $(k_j, k_{j+1})$-part bundles, $\pi_{n-1}(F) \cong \{0\}$ holds.

3. For the fiber $f_2^{-1}(L)$ of a surrounding bundle of $f_2$, $\pi_{n-1}(f_2^{-1}(L)) \cong \{0\}$ holds.

4. For any integer $1 \leq j \leq l' - 1$, we can take $(k_j, k_{j+1})$-part bundles of $f$ so that for any connected component $E_j$ of the fibers, $\pi_{n-1}(E_j) \cong \{0\}$ holds, that for any connected component $F$ of the fiber of the outer (inner) part of the $(k_j, k_{j+1})$-part bundles, $\pi_{n-1}(F) \cong \{0\}$ holds and that any connected component of the total space of the outer (inner) part of the $(k_j, k_{j+1})$-part bundles has the structure of a subbundle of the outer (resp. inner) part of the $(k_j, k_{j+1})$-part bundles and admits a section.

Proof of Theorem 5. In the proof, we denote a proper core of $f_2$ by $P_2(\subset \mathbb{R}^n)$. First, note that $f_2|_{f_2^{-1}(P_2)} : f_2^{-1}(P_2) \to P_2$ gives $f_2^{-1}(P_2)$ the structure of a trivial smooth bundle over $P_2$ and $f_2|_{f_2^{-1}(\partial P_2)} : f_2^{-1}(\partial P_2) \to \partial P_2$ gives $f_2^{-1}(\partial P_2)$ the structure of a trivial smooth bundle over $\partial P_2$.

Since $M_2$ is connected, two manifolds $f_2^{-1}(\mathbb{R}^n - \text{Int} P_2)$ and $f_2^{-1}(L)$ are connected.

We prove (1). Since $f_2$ is normally topologically trivial and $f_2^{-1}(\partial f_2(M_2))$ is the image of a section of the surrounding bundle and regarded as a subbundle of the surrounding bundle with a fiber consisting of just one point, we have a section of the trivial smooth bundle $f_2|_{f_2^{-1}(\partial P_2)} : f_2^{-1}(\partial P_2) \to \partial P_2$ which is also a section of the subbundle of this bundle with a fiber homeomorphic to $F$ and which is homotopic to the inclusion of $f_2^{-1}(\partial f_2(M_2))$ into $M_2$ as a continuous map into $M_2$. Since $f_2(M_2)$ is assumed to be diffeomorphic to $D^n$ and $f_2|_{f_2^{-1}(P_2)} : f_2^{-1}(P_2) \to P_2$ gives $f_2^{-1}(P_2)$ the structure of a trivial smooth bundle, $\pi_{n-1}(F) \cong \{0\}$ means that the inclusion of $f_2^{-1}(\partial f_2(M_2))$ into $M_2$ and the section of the trivial smooth $F$-bundle $f_2|_{f_2^{-1}(\partial P_2)} : f_2^{-1}(\partial P_2) \to \partial P_2$ in the previous sentence are null-homotopic in $M_2$.

From Proposition 2, this completes the proof of (1).

Also for other cases, we have only to show that the inclusion of $f_2^{-1}(\partial f_2(M_2))$ into $M_2$ is null-homotopic by virtue of Proposition 2.

We prove (2). $f_2^{-1}(\partial f_2(M_2))$ is the images of sections of $(l, l+1)$-part bundles of $f_2$ whose fibers are diffeomorphic to $D^{m-n+1}$. The section is homotopic to every section of the inner part of the $(l, l+1)$-part bundles as maps from a space, which is diffeomorphic to the $(n-1)$-dimensional standard sphere, into a space diffeomorphic to the total spaces of the $(l, l+1)$-part bundles since $\pi_{n-1}(D^{m-n+1}) \cong \{0\}$ holds. By the assumption, for any integer $j$ satisfying $1 \leq j \leq l'-1$, the inner part and the outer part of the given $(k_j, k_{j+1})$-bundles are trivial smooth bundles. Furthermore, by the assumption that $f_2$ is almost topologically trivial about the sequence $\{k_j\}_{j=1}^{l'}$ and an extra assumption on the homotopy groups of connected components of fibers of the outer part and inner part of the $(k_j, k_{j+1})$-part bundles, for any integer $j$
satisfying $1 \leq j \leq l' - 1$, these sections are always homotopic as maps from a space, which is diffeomorphic to the $(n - 1)$-dimensional standard sphere, into a space diffeomorphic to a connected component of the total spaces of the $(k_j, k_{j+1})$-part bundles if their images are in a same connected component of the total space of the $(k_j, k_{j+1})$-part bundles. From these discussions and the assumptions that $M_2$ is connected and that $f_2(M_2)$ is diffeomorphic to $D^n$, the inclusion of $f_2^{-1}(\partial f_2(M_2))$ is null-homotopic. More precisely, we can homotope the inclusion inductively to a section of the trivial smooth bundle given by $f_2|_{f_2^{-1}(\partial P_2)} : f_2^{-1}(\partial P_2) \to \partial P_2$. This completes the proof of (2).

We prove (3). $f_2^{-1}(\partial f_2(M_2))$ is the image of a section on a surrounding bundle of $f_2$. The fiber of the surrounding bundle is diffeomorphic to $f_2^{-1}(L)$ and $\pi_{n-1}(f_2^{-1}(L)) \cong \{0\}$ holds. Thus, the $(n-1)$-th homotopy group of the total space of the surrounding bundle is isomorphic to $\mathbb{Z}$ and all the sections of the surrounding bundle of $f_2$ are homotopic. From these discussions and the assumptions that $M_2$ is connected and that $f_2(M_2)$ is diffeomorphic to $D^n$, the inclusion of $f_2^{-1}(\partial f_2(M_2))$ is null-homotopic. This completes the proof of (3).

We prove (4). $f_2^{-1}(\partial f_2(M_2))$ is the images of sections of $(l, l+1)$-part bundles of $f_2$ whose fibers are diffeomorphic to $D^{m-n+1}$. The section is homotopic to every section of the inner part of the $(l, l+1)$-part bundles as maps from a space, which is diffeomorphic to the $(n-1)$-dimensional standard sphere, into a space diffeomorphic to the total space of the $(l, l+1)$-part bundles since $\pi_{n-1}(D^{m-n+1}) \cong \{0\}$ holds. By the assumption, for any integer $j$ satisfying $1 \leq j \leq l' - 1$, every connected component of the inner part and the outer part of the given $(k_j, k_{j+1})$-bundles is a smooth bundle and admits a section. Moreover, by an extra assumption on the $(n-1)$-th homotopy groups of connected components of fibers of $(k_j, k_{j+1})$-part bundles, these sections are always homotopic as maps from a space, which is diffeomorphic to the $(n-1)$-dimensional standard sphere, into a space diffeomorphic to a connected component of the total space of the $(k_j, k_{j+1})$-part bundles if the images of the sections are in a same connected component of the total space of the $(k_j, k_{j+1})$-part bundles. From these discussions and the assumptions that $M_2$ is connected and that $f_2(M_2)$ is diffeomorphic to $D^n$, the inclusion of $f_2^{-1}(\partial f_2(M_2))$ is null-homotopic. More precisely, we can homotope the inclusion inductively to a section of the trivial smooth bundle given by $f_2|_{f_2^{-1}(\partial P_2)} : f_2^{-1}(\partial P_2) \to \partial P_2$. This completes the proof of (4).

Remark 4. In the situation of Definition 5, if $m \geq 2n$ and $l' = 2$ hold and $f$ is almost topologically trivial about the sequence $\{k_j\}_{j=1}^2$ $(k_1 = 1$ and $k_2 = l$ hold), then it is topologically trivial. Similarly, if $m \geq 2n$ and $l' = 2$ hold and $f$ is almost PL trivial about the sequence $\{k_j\}_{j=1}^2$ $(k_1 = 1$ and $k_2 = l$ hold), then it is PL trivial.

Example 4. Let $M_1$ be a closed and connected manifold whose dimension is $m$. Let there exist a round fold map $f_1 : M_1 \to \mathbb{R}^n$ $(n \geq 2)$ such that the fiber of a point in a proper core of $f_1$ has a connected component diffeomorphic to $S^{m-n}$. We also assume that $m \geq 2n$ holds.

(1) Let $\tilde{M}$ be a compact and connected manifold whose dimension is $m$ such that for some connected component $F_0$ of $\partial \tilde{M}$, $\pi_{m-1}(F_0) \cong \{0\}$ holds. If we perform a trivial spinning construction mentioned just after Definition
3, then we have a round fold map \( f_2 : M_2 \to \mathbb{R}^n \) such that the pair of the map \( f_1 \) and \( f_2 \) satisfies the assumption of Theorem 5 (1).

(2) Let \( M_2 \) be a closed and connected manifold whose dimension is \( m \) and let \( f_2 : M_2 \to \mathbb{R}^n \) be a round fold map such that \( f_2(M_2) \) is diffeomorphic \( D^n \) and that all the regular fibers of \( f_2 \) are closed and \((n-1)\)-connected manifolds. Let \( L \) be an axis of \( f_2 \). Then, we may regard \( f_2|_{f_2^{-1}(L)} : f_2^{-1}(L) \to L \) as a Morse function and by the theory of handle attachments on this Morse function, the pair of the maps \( f_1 \) and \( f_2 \) satisfies the assumption of Theorem 5 (3).

(3) Let \( m - 2n + 2 > n - 1 \) hold. Let \( M_2 \) be a closed and connected manifold whose dimension is \( m \) and let \( f_2 : M_2 \to \mathbb{R}^n \) be a round fold map such that for an axis \( L \) of \( f_2 \), \( f_2^{-1}(L) \) is homeomorphic to \( S^{m-2n+2} \times D^n \). Then, the pair of the maps \( f_1 \) and \( f_2 \) does not satisfy the assumptions of Theorem 5 (1) and (2) but satisfies the assumption of Theorem 5 (3).

(4) Let \( m - 2n + 2 > n - 1 \) hold. Let \( M_2 \) be a closed and connected manifold whose dimension is \( m \) and let \( f_2 : M_2 \to \mathbb{R}^n \) be a round fold map such that the singular set \( S(f_2) \) of \( f_2 \) consists of three connected components and that for an axis \( L \) of \( f_2 \), \( f_2^{-1}(L) \) is homeomorphic to \( S^{n-1} \times S^{m-2n+2} \) with the interior of an \((m - n + 1)\)-dimensional standard closed disc removed. Assume also that the followings hold.

(a) We can take \((1,2)\)-part bundles and \((2,3)\)-part bundles of \( f \) so that the corresponding fibers to these bundles are homeomorphic to \( D^{n-1} \times S^{m-2n+2} \) with the interior of an \((m - n + 1)\)-dimensional standard closed disc smoothly embedded in the interior of \( D^{n-1} \times S^{m-2n+2} \) removed.

(b) The outer (inner) part of the previous \((1,2)\)-part (resp. \((2,3)\)-part) bundles is a bundle with a fiber homeomorphic to \( S^{n-2} \times S^{m-2n+2} \) and admits a section. The outer part of the \((2,3)\)-part bundles is a smooth \( S^{m-n} \)-bundle and admits a section.

Then, the pair of the maps \( f_1 \) and \( f_2 \) satisfies the assumption of Theorem 5 (4), but does not satisfy the assumption of Theorem 5 (3). Furthermore, if \( f_2 \) is topologically trivial, then the pair of the two maps \( f_1 \) and \( f_2 \) satisfies the assumption of Theorem 5 (1).

As a more explicit application, we have the following theorem in the case where the map is from a 5-dimensional manifold into the plane.

**Theorem 6.** Let \( M \) be a closed and simply-connected manifold of dimension 5. Then, \( M \) admits a round fold map \( f : M \to \mathbb{R}^2 \) satisfying the assumption of Proposition 1 if and only if \( M \) is represented as a connected sum of a finite number of manifolds having the structures of smooth \( S^3 \)-bundles over \( S^2 \).

For the proof of this theorem, we need the following proposition.

**Proposition 3** (Barden, [1]). Let \( M \) be a closed and simply-connected manifold of dimension 5. Then, \( H_3(M; \mathbb{Z}) \) is torsion-free if and only if \( M \) is represented as a connected sum of a finite number of manifolds having the structures of smooth \( S^3 \)-bundles over \( S^2 \).

**Proof of Theorem 6.** By Theorem 1 (1) or Example 2 and by Proposition 2 together with Example 3, a manifold represented as a connected sum of a finite number of manifolds having the structures of smooth \( S^3 \)-bundles over \( S^2 \) admits a round fold
map into \( \mathbb{R}^2 \) satisfying the assumption of Proposition 1. This completes the proof of the "if" part. Conversely, we show the "only if" part.

From Proposition 1, \( \pi_1(M) \cong \{0\} \) holds and \( \pi_2(M) \cong H_2(M; \mathbb{Z}) \) is torsion-free. Hence, a closed and simply-connected manifold \( M \) of dimension 5 such that \( H_2(M; \mathbb{Z}) \) is not torsion-free does not admit a round fold map into \( \mathbb{R}^2 \) satisfying the assumption of Proposition 1. From Proposition 3, \( H_2(M; \mathbb{Z}) \) is not torsion-free if \( M \) is not a connected sum of a finite number of oriented manifolds having the structures of smooth \( S^3 \)-bundles over \( S^2 \). This completes the proof of both part of the theorem. \( \square \)

4.2. Decompositions of a manifold admitting a round fold map into a connected sum of two manifolds. Conversely, in this subsection, we decompose a manifold admitting a round fold map into a connected sum of two manifolds admitting round fold maps under additional conditions.

Let \( M \) be a closed and connected manifold of dimension \( m \). Let \( f : M \to \mathbb{R}^n \) be a round fold map (\( n \geq 2 \)).

Let \( C \) be a submanifold of \( \mathbb{R}^n - f(S(f)) \). Suppose that \( C \) is diffeomorphic to \( S^{n-1} \) and that \( C \) is a deformation retract of the closure of a connected component of \( \mathbb{R}^n - f(S(f)) \) diffeomorphic to \( S^{n-1} \times (0, 1) \) or is in a proper core of \( f \). We denote the closure of the bounded domain of \( \mathbb{R}^n - C \) by \( R \) and for a connected component \( \bar{M} \) of \( f^{-1}(R) \), \( f|_{\partial \bar{M}} : \partial \bar{M} \to C \) gives the structure of a trivial smooth bundle over \( C \) and the fiber is a standard sphere.

We can glue the map \( f|_{\bar{M} - \text{Int} \bar{M}} : \bar{M} - \text{Int} \bar{M} \to \mathbb{R}^n \) and the natural projection \( p : R \times S^{m-n} \to R \) by a diffeomorphism \( \Phi_1 : \partial \bar{M} \to \partial R \times S^{m-n} \) regarded as a bundle isomorphism between the trivial \( S^{m-n} \)-bundles and a diffeomorphism between the base spaces induced from the isomorphism to obtain a round fold map \( f : M_1 \to \mathbb{R}^n \).

We can glue \( \bar{M} \) and \( V := S^{n-1} \times D^{m-n+1} \) on the boundaries by any bundle isomorphism \( \Phi_2 \) between the two trivial smooth bundles \( V := S^{n-1} \times \partial D^{m-n+1} \) and \( \partial \bar{M} \) over the \((n - 1)\)-dimensional standard spheres inducing a diffeomorphism between the base spaces to obtain a manifold \( M_2 \).

Let \( f_0 : D^{m-n+1} \to [0, +\infty) \) be a Morse function satisfying the following three as introduced just after Definition 5.

1. \( f_0 \) is constant and 0 on the boundary.
2. \( f_0(\text{Int} D^{m-n+1}) \subset (0, +\infty) \) holds.
3. \( f_0 \) has just one singular point and it is maximal at this point.

We can also glue the maps \( f|_{\bar{M}} : \bar{M} \to R \) and the product of the Morse function \( \bar{f} \) and the identity map \( \text{id}_{S^{n-1}} \) on \( S^{n-1} \) by the diffeomorphism \( \Phi_2 \) and a diffeomorphism between the base spaces induced from the isomorphism to obtain a round fold map \( f_2 : M_2 \to \mathbb{R}^n \).

We call this operation of obtaining the pair of two round fold maps \( f_1 \) and \( f_2 \) a canonical decomposing operation to the map \( f \).

Canonical decomposing operations are regarded as extensions of \( R \)-operations introduced in [13], which are regarded as surgery operations on stable maps from closed manifolds whose dimensions are larger than 2 into the plane; in the paper, the operations are introduced as surgery operations to pseudo quotient maps, which are regarded as continuous maps from closed manifolds whose dimensions are larger
than 2 into 2-dimensional polyhedra having local topological structures similar to those of maps induced from stable maps from the closed manifolds into the plane by considering natural quotient maps into the Reeb spaces. In [13], Neither canonical combining operations, which are regarded as inverse operations of the operations, nor their specific cases, are defined.

We introduce and show Proposition 4. Compare this with arguments on R-operations shown in [13].

**Proposition 4.** Let $M$ be a closed and connected manifold of dimension $m$. Let $f : M \to \mathbb{R}^n$ be a round fold map ($n \geq 2$). Suppose also that $m \geq 2n$ holds. Let $C$ be a submanifold of $\mathbb{R}^n - f(S(f))$. Suppose that $C$ is diffeomorphic to $S^{n-1}$ and that $C$ is a deformation retract of the closure of a connected component of $\mathbb{R}^n - f(S(f))$ diffeomorphic to $S^{n-1} \times (0,1)$ or is in a proper core of $f$. We denote the closure of the bounded domain of $\mathbb{R}^n - C$ by $R$ and let there exist a connected component $M$ of $f^{-1}(R)$ satisfying the followings.

1. $f|_{\partial M} : \partial M \to C$ gives the structure of a trivial smooth bundle over $C$ and the fiber is a standard sphere.
2. Let $p$ be a point in $D^{m-n+1}$. If we glue $\bar{M}$ and $V := S^{n-1} \times D^{m-n+1}$ on the boundaries by a bundle isomorphism $\Phi$ between the two trivial smooth bundles $V := S^{n-1} \times \partial D^{m-n+1}$ and $\partial M$ over the $(n-1)$-dimensional standard spheres inducing a diffeomorphism between the base spaces, then the natural inclusion $S^{n-1} \times \{p\} \subset V = S^{n-1} \times D^{m-n+1} \subset \bar{M} \cup \Phi V$ is null-homotopic.

Then, $M$ is represented as a connected sum of two connected manifolds $M_1$ and $M_2 := \bar{M} \cup \Phi V$ such that $M_i$ admits a round fold map $f_i : M_i \to \mathbb{R}^n$ ($i = 1,2$) and that the pair $(f_1, f_2)$ is obtained by a canonical decomposing operation to $f$.

**Proof.** Let $M_1$ be a closed manifold given by gluing $M - \text{Int} \bar{M}$ and $V' := D^n \times S^{m-n}$ by some diffeomorphism $\Phi' : \partial D^n \times S^{m-n} \to \partial M$ regarded as a bundle isomorphism between the natural two trivial smooth $S^{n-1}$-bundles inducing a diffeomorphism between the base spaces.

Since $m \geq 2n = 2(n-1) + 2$ is assumed and the natural inclusion $S^{n-1} \times \{p\} \subset V = S^{n-1} \times D^{m-n+1} \subset M \cup \Phi V$ is assumed to be null-homotopic, we may regard that the following holds for an orientation reversing diffeomorphism $\Psi : \partial D^m \to \partial D^m$ extending to a diffeomorphism on $D^m$ or from $M_2 - (M_2 - D^m)$ onto $M_1 - (M_1 - D^m)$, where for two manifolds $X_1$ and $X_2$, $X_1 \cong X_2$ means that $X_1$ and $X_2$ are diffeomorphic.

\[
\begin{align*}
(M - \text{Int} \bar{M}) \cup \bar{M} & \cong (M_1 - \text{Int} V') \cup (M_2 - \text{Int} V) \\
& \cong (M_1 - \text{Int} V') \cup ((D^m - \text{Int} V) \cup (M_2 - \text{Int} D^m)) \\
& \cong (M_1 - \text{Int} V') \cup ((S^m - (\text{Int} V \sqcup \text{Int} D^m)) \cup (M_2 - \text{Int} D^m)) \\
& \cong (M_1 - \text{Int} D^m) \cup (M_2 - \text{Int} D^m)
\end{align*}
\]

This means that $M$ is represented as a connected sum of the two connected manifolds $M_1$ and $M_2$ and that $M_i$ admits a round fold map $f_i : M_i \to \mathbb{R}^n$ satisfying the mentioned conditions ($i = 1,2$).
As an application, we have the following theorem.

**Theorem 7.** Let \( M \) be a closed manifold of dimension \( m \). Let \( f : M \to \mathbb{R}^n \) be a round fold map (\( m \geq n \geq 2 \)). Assume also that \( M \) is \((n-1)\)-connected and that \( m \geq 2n \) holds.

Let \( C \) be a submanifold of \( \mathbb{R}^n - f(S(f)) \). Suppose that \( C \) is diffeomorphic to \( S^{n-1} \) and that \( C \) is a deformation retract of the closure of a connected component of \( \mathbb{R}^n - f(S(f)) \) diffeomorphic to \( S^{n-1} \times (0, 1) \) or is in a proper core of \( f \). We denote the closure of the bounded domain of \( \mathbb{R}^n - C \) by \( R \). Let there exist a connected component \( \tilde{M} \) of \( f^{-1}(R) \) such that the followings hold.

1. \( f|_{\partial \tilde{M}} : \partial \tilde{M} \to C \) gives the structure of a trivial smooth bundle over \( C \) and the fiber is a standard sphere.
2. \( f(M - \text{Int} \tilde{M}) \) is diffeomorphic to \( S^{n-1} \times [0, 1] \).

Let \( \Phi \) be a bundle isomorphism \( \Phi \) between the two trivial smooth bundles \( \partial M \) and \( V := S^{n-1} \times \partial D^{m-n+1} \) over the \((n-1)\)-dimensional standard spheres inducing a diffeomorphism between the base spaces. Then, \( M \) is represented as a connected sum of two connected manifolds \( M_1 \) and \( M_2 := M \cup_\Phi V \) such that \( M_i \) admits a round fold map \( f_i : M_i \to \mathbb{R}^n \) (\( i = 1, 2 \)) and that the pair \( (f_1, f_2) \) is obtained by a canonical decomposing operation to \( f \).

We owe the idea of the proof of Theorem 7 to the proofs of Lemmas 4.7 and 4.8.

**Proof of Theorem 7.** \( \partial \tilde{M} \) is \((n-2)\)-connected and \( M \) is \((n-1)\)-connected. From these assumptions on the two manifolds, two manifolds \( \tilde{M} \) and \( M - \text{Int} \tilde{M} \) are \((n-2)\)-connected. We have the following homology exact sequence.

\[
\to H_{n-1}(\partial \tilde{M}; \mathbb{Z}) \oplus H_{n-1}(M - \text{Int} \tilde{M}; \mathbb{Z}) \to H_{n-1}(\tilde{M}; \mathbb{Z}) \to H_{n-1}(M; \mathbb{Z}) \cong \{0\}
\]

\( f(M - \text{Int} \tilde{M}) \) is assumed to be diffeomorphic to \( S^{n-1} \times [0, 1] \). We may regard that \( M - \text{Int} \tilde{M} \) is a submanifold of the total space of a surrounding bundle of \( f \). So, the restriction map of the submersion giving the surrounding bundle of \( f \) to \( M - \text{Int} \tilde{M} \) gives the structure of a subbundle of the surrounding bundle. The subbundle also admits a section: in fact \( f|_{\partial \tilde{M}} : \partial \tilde{M} \to C \) gives the structure of a trivial smooth bundle over \( C \) and as a result, we obtain a section. As a result, the projection of this subbundle induces a surjection from \( \pi_k(M - \text{Int} \tilde{M}) \) onto \( \pi_k(S^{n-1}) \) for any positive integer \( k \).

Let \( n \geq 3 \). Since \( M - \text{Int} \tilde{M} \) is \((n-2)\)-connected, \( \pi_{n-1}(M - \text{Int} \tilde{M}) \cong H_{n-1}(M - \text{Int} \tilde{M}; \mathbb{Z}) \) holds by virtue of Hurewicz theorem and they are not zero. By the homology exact sequence in the beginning of the proof, we have \( H_{n-1}(\tilde{M}; \mathbb{Z}) \cong \{0\} \). \( M \) is \((n-1)\)-connected by the fact that \( \tilde{M} \) is \((n-2)\)-connected together with Hurewicz theorem.

Let \( n = 2 \). \( \pi_{n-1}(M - \text{Int} \tilde{M}) \cong \pi_1(M - \text{Int} \tilde{M}) \) is also not zero. \( \pi_{n-1}(\partial \tilde{M}) \cong \pi_1(\partial \tilde{M}) \cong \mathbb{Z} \) holds. Moreover, the homomorphism from \( \pi_1(\partial \tilde{M}) \) into \( \pi_1(M - \text{Int} \tilde{M}) \) induced by the inclusion \( i_1 \) of the manifold \( \partial \tilde{M} \) giving the structure of a bundle over \( C \) into the manifold \( M - \text{Int} \tilde{M} \) having the structure of a bundle over \( C \) is injective and there exists a homomorphism \( \tau_1 \) from the group \( \pi_1(M - \text{Int} \tilde{M}) \) into \( \pi_1(\partial \tilde{M}) \) such that the composition \( \tau_1 \circ i_1 \) is the identity map on \( \pi_1(\partial \tilde{M}) \). In fact, we can take \( \tau_1 \) as the homomorphism induced from the composition of the submersion from \( M - \text{Int} \tilde{M} \) into \( C \) naturally defined as explained in the first paragraph of this proof and a section of the bundle \( \partial \tilde{M} \) over \( C \). By virtue of
van Kampen’s theorem, \( \pi_1(M) \) is isomorphic to \( \pi_1(M - \text{Int} \bar{M}) \ast \pi_1(\partial \bar{M}) \ast \pi_1(\bar{M}) \) and we obtain a natural homomorphism \( \tau_2 \) from \( \pi_1(M) \) into \( \pi_1(\bar{M}) \) such that for the natural inclusion \( i_2 : \bar{M} \rightarrow M \), the composition of the two homomorphism \( i_2 \) and \( \tau_2 \) is the identity map. Thus, we have \( \pi_{n-1}(\bar{M}) \cong \pi_1(\bar{M}) \cong \{0\} \).

Since \( \bar{M} \) is proven to be \((n-1)\)-connected for \( n \geq 2 \), by attaching \( V := S^{n-1} \times D^{m-n+1} \) to \( \bar{M} \) by any bundle isomorphism \( \Phi \) between the two trivial smooth bundles \( \partial \bar{M} \) and \( \partial V = S^{n-1} \times \partial D^{m-n+1} \) over the \((n-1)\)-dimensional standard spheres inducing a diffeomorphism between the base spaces, we obtain a closed and \((n-1)\)-connected manifold \( M \cup_\Phi V \) and for any point \( p \in D^{m-n+1} \), the natural inclusion \( S^{n-1} \times \{p\} \subset V = S^{n-1} \times D^{m-n+1} \subset M \cup_\Phi V \) is null-homotopic. We may apply Proposition 4 to complete the proof. \( \Box \)

4.3. Applications. In this subsection, by applying the previous results and their proofs, we determine the diffeomorphism types of manifolds admitting round fold maps under appropriate conditions. More precisely, as a work, we characterize the family of all the \( m \)-dimensional manifolds represented as connected sums of homotopy spheres admitting round fold maps into \( \mathbb{R}^n \) whose singular sets are connected and finite numbers of manifolds having the structures of smooth \( S^{m-n} \) bundles over \( S^n \) with \( m \geq 2n \) by a class of round fold maps including a map in Example 2.

**Theorem 8.** Let \( m, n \in \mathbb{N} \), \( n \geq 2 \) and \( m \geq 2n \). Let \( M \) be a closed and connected manifold of dimension \( m \). Suppose that there exists a round fold map \( f : M \rightarrow \mathbb{R}^n \). Let \( N((S(f))) \) be a small closed tubular neighborhood of \( f(S(f)) \) and for any connected component \( R \) of \( \mathbb{R}^n - \text{Int} N(f(S(f))) \), \( f|_{f^{-1}(R)} : f^{-1}(R) \rightarrow R \) gives the structure of a trivial smooth bundle whose fiber is a disjoint union of standard spheres. Assume also that the number of connected components of the fiber of a point in a proper core of \( f \) equals the number of connected components of \( S(f) \) and that the numbers are both larger than 2.

Then, \( M \) is represented as a connected sum of closed and connected manifolds admitting round fold maps with singular sets consisting of 2 connected components and with fibers of points in proper cores consisting of two standard spheres. More precisely, they are obtained by using a finite time of canonical decomposing operations to round fold maps inductively.

Conversely, such a manifold admits a round fold map into \( \mathbb{R}^n \) satisfying the assumption mentioned above. More precisely, it is obtained by using a finite time of canonical combining operations to round fold maps inductively, starting from a family of round fold maps with singular sets consisting of 2 connected components and with fibers of points in proper cores consisting of two standard spheres.

**Proof.** All but one connected components of the singular set \( S(f) \) of \( f \) consist of fold points whose indices are 1 by the assumption that the number of connected components of the fiber of a point in a proper core of \( f \) equals the number of connected components of \( S(f) \) and by the theory of attachments of handles. Furthermore, \( f \) satisfies the assumption of Proposition 1.

We define the following objects to represent the Reeb space \( W_f \) of \( f \).

1. \( A \) is a disjoint union of finite copies of \( D^n \) and \( B := S^{n-1} \times L \), where \( L \) is a compact and connected graph with no loops.
2. \( \psi : S^{n-1} \times \Lambda \rightarrow \partial A \) is a PL homeomorphism, where \( \Lambda \) is a set consisting of a finite number of degree 1 vertices of the graph \( L \).
Then, the Reeb space $W_f$ is PL homeomorphic to a polyhedron obtained by attaching $B$ to $A$ by a PL homeomorphism $\psi$. Note that $\pi_{n-1}(M) \cong \pi_{n-1}(W_f) \cong \{0\}$ holds from Proposition 1.

By Proposition 4 or Theorem 7, if the singular set of $f$ consists of more than 2 connected components, then by a canonical decomposing operation, we obtain 2 round fold maps; one is a map satisfying the assumption of the theorem such that the number of the singular set of the map is smaller than that of the original map $f$ by one and the other map satisfies the assumption of this theorem such that the number of the singular set of the map is two. Of course, $M$ is represented as a connected sum of the resulting source manifolds. By induction, it also follows that $M$ is represented as a connected sum of closed and connected manifolds admitting round fold maps with singular sets consisting of 2 connected components and with fibers of points in proper cores consisting of two standard spheres.

Conversely, if $M$ is represented as a connected sum of closed and connected manifolds admitting round fold maps with singular sets consisting of 2 connected components and with fibers of points in proper cores consisting of disjoint unions of two standard spheres, then we obtain a round fold map $f : M \to \mathbb{R}^n$ satisfying the assumption by using canonical combining operations inductively to the previous maps with singular sets consisting of two connected components by virtue of Proposition 2 or Theorem 5 together with Proposition 1 or Example 3. □

Definition 6. We denote the subgroup by $\Theta_{k_1, k_2} \subset \Theta_{k_1}$ and call it the $(k_1, k_2)$ round special generic group.

We have the following two theorems.

**Theorem 9.** Let $m, n \in \mathbb{N}$, $n \geq 2$ and $m \geq 2n$. Let $M$ be a closed and connected manifold of dimension $m$. Let $E$ be a manifold of dimension $m - n + 1$ whose boundary is non-empty and includes a $(n - 1)$-connected manifold or a manifold satisfying $\pi_{n-1}(E) \cong \{0\}$ whose boundary is non-empty.

Let $M$ admit a round fold map $f : M \to \mathbb{R}^n$ such that the singular set $S(f)$ consists of $l > 1$ connected components and that $f$ is almost $C^\infty$ trivial about the sequence $\{k_j\}_{j=1}^{l} \ (k_1 = 1$ and $k_2 = l$ hold). In addition, we assume that the corresponding fiber of the $(1, l)$-part bundles of $f$ is the manifold $E$ with the interior of a standard closed disc $D^{m-n+1}$ smoothly embedded in the interior $\text{Int}E$ removed.

Then, by a canonical decomposing operation to $f$, we have a pair of round fold maps such that one of the maps is a round fold map into $\mathbb{R}^n$ whose singular set is connected from a manifold in a class of $\Theta_{(m,n)}$ and the other is a $C^\infty$ trivial round fold map and that $M$ is represented as a connected sum of the resulting two source manifolds. In addition, we can take $(1, l + 1)$-part bundles of the latter map such that the corresponding fiber of the $(1, l + 1)$-part bundles is diffeomorphic to $E$.

**Proof.** Recall discussions in the proofs of Theorem 5 (1) and (3). In each proof, it is shown that the embedding of the inverse image $f_2^{-1}(\partial f_2(M_2))$ of the boundary...
Thus, from the assumption on the topology of manifold $E$ is almost $\partial f_2(M_2)$ of the image $f_2(M_2)$ of the map $f_2$ into the manifold $M_2$ is null-homotopic. Thus, from the assumption on the topology of manifold $E$ and the assumption that $f$ is almost $C^\infty$ trivial about the sequence $\{k_j\}_{j=1}^\infty$ with Proposition 4, the result follows.

\begin{theorem}
Let $m, n \in \mathbb{N}$, $n \geq 2$ and $m \geq 2n$. Let $M$ be a closed and connected manifold of dimension $m$ and let $\Sigma$ be an almost-sphere of dimension $m-n$. Then, the following two are equivalent.

\begin{enumerate}
\item A round fold map $f : M \to \mathbb{R}^n$ satisfying the followings exists.
  \begin{enumerate}
  \item $S(f)$ consists of 2 connected components.
  \item We can take $(1,2)$-part bundles of $f$ being trivial smooth bundles with fibers diffeomorphic to the cylinder $\Sigma \times [-1,1]$ with the interior of a standard closed disc $D^{m-n+1}$ smoothly embedded in the interior of the manifold removed.
  \end{enumerate}
\item $M$ is represented as a connected sum of a manifold in a class of $\Theta_{(m,n)}$ and a manifold having the structure of a smooth bundle over $S^n$ with a fiber diffeomorphic to the almost-sphere $\Sigma$.
\end{enumerate}
\end{theorem}

\begin{proof}
Assume that a round fold map $f : M \to \mathbb{R}^n$ satisfying the condition (1) exists. By virtue of Theorem 9, we can represent $M$ as a connected sum of two closed and connected manifolds $M_1$ and $M_2$ such that the followings hold.

\begin{enumerate}
\item $M_1$ admits a round fold map $f_1 : M_1 \to \mathbb{R}^n$ such that $S(f_1)$ is connected.
\item $M_2$ admits a $C^\infty$ trivial round fold map $f_2 : M_2 \to \mathbb{R}^n$ such that $S(f_2)$ consists of two connected components and that the inverse image of an axis of $f_2$ is diffeomorphic to $\Sigma \times [-1,1]$. $f_2$ satisfies the assumption of Proposition 1.
\end{enumerate}

Furthermore, the pair $(f_1, f_2)$ is obtained by a canonical decomposing operation to the map $f$. Then, $M$ is represented as a connected sum of a manifold in a class of $\Theta_{(m,n)}$ and a smooth bundle over $S^n$ with a fiber diffeomorphic to an almost-sphere by Theorem 1 (2).

Conversely, if $M$ is such a manifold, then by Proposition 2, Example 3 or Theorem 5 together with Theorem 1 (1) or Example 2, $M$ admits a round fold map $f : M \to \mathbb{R}^n$ satisfying the condition (1).

This completes the proof.
\end{proof}

Now we have the following theorem.

\begin{theorem}
Let $n \in \mathbb{N}$, $n \geq 2$ and $m \geq 2n$. Let $M$ be a closed and connected manifold of dimension $m$. Then the following two are equivalent.

\begin{enumerate}
\item A round fold map $f : M \to \mathbb{R}^n$ satisfying the followings exist.
  \begin{enumerate}
  \item Regular fibers of $f$ are disjoint unions of standard spheres and the number of connected components of the fiber of a point in a proper core of $f$ and the number of connected components of $S(f)$ are $l > 1$.
  \item For any integer $1 \leq j \leq l - 1$, we can take $(j, j + 1)$-part bundles of $f$ which are trivial smooth bundles with fibers diffeomorphic to disjoint unions of manifolds diffeomorphic to the cylinder $S^{m-n} \times [-1,1]$ with the interior of a standard closed disc $D^{m-n+1}$ smoothly embedded in the interior of the manifold removed.
  \end{enumerate}
\item $M$ is represented as a connected sum of $l-1$ manifolds having the structures of smooth $S^{m-n}$-bundles over $S^n$ and a manifold in a class of $\Theta_{(m,n)}$.
\end{enumerate}
\end{theorem}
Proof. Assume that a round fold map \( f : M \to \mathbb{R}^n \) satisfying the condition (1) exists.

By Theorems 8 and 10, \( M \) is represented as a connected sum of a finite number of closed manifolds admitting round fold maps with singular sets consisting of 2 connected components and each manifold is represented as a connected sum of a manifold having the structure of a smooth \( S^{m-n} \)-bundle over \( S^n \) and a manifold in a class of \( \Theta_{(m,n)} \).

Conversely, assume that the condition (2) holds. By Theorem 1 (1) or Example 2, each manifold having the structure of a smooth \( S^{m-n} \)-bundle over \( S^n \) appearing as an ingredient of the connected sum admits a \( C^\infty \) trivial round fold map with the singular set consisting of 2 connected components such that the inverse image of an axis of the map is diffeomorphic to the cylinder \( S^{m-n} \times [-1,1] \). By Proposition 2, Example 3 or Theorem 5, a manifold \( M \) represented as a connected sum of a finite number of manifolds having the structures of smooth \( S^{m-n} \)-bundles over \( S^n \) and a manifold in a class of \( \Theta_{(m,n)} \) admits a round fold map \( f : M \to \mathbb{R}^n \) satisfying the condition (1).

This completes the proof. \( \square \)

Last, we show the following theorem.

**Theorem 12.** Let \( m, n \in \mathbb{N}, n \geq 2 \) and \( m \geq 2n \). Let \( M \) be a closed and connected manifold of dimension \( m \). Let \( F \) be a closed and \((n-1)\)-connected manifold of dimension \( m-n \). Then the following two are equivalent.

1. A round fold map \( f : M \to \mathbb{R}^n \) satisfying the followings exist.
   a. The number of connected component of the singular set \( S(f) \) is larger than 2.
   b. \( f \) is almost \( C^\infty \) trivial about a sequence \( \{k_j\}_{j=1}^3 \) and we can take \((k_1,k_2)\)-part bundles and \((k_2,k_3)\)-part bundles so that the corresponding fibers of the bundles are diffeomorphic to a manifold represented as a disjoint union of a manifold diffeomorphic to \( F \times [-1,1] \) with an \((m-n+1)\)-dimensional standard closed disc smoothly embedded in the interior removed and the cylinder \( S^{m-n} \times [-1,1] \) and the cylinder \( S^{m-n} \times [-1,1] \) with the interior of an \((m-n+1)\)-dimensional standard closed disc smoothly embedded in the interior removed, respectively.
   c. For any integer \( 1 \leq j \leq l-1 \), we can take \((j,j+1)\)-part bundles of \( f \) which are trivial smooth bundles with fibers diffeomorphic to disjoint unions of manifolds diffeomorphic to the cylinder \( S^{m-n} \times [-1,1] \) with the interior of a standard closed disc \( D^{m-n+1} \) embedded in the interior of the manifold removed.

2. \( M \) is represented as a connected sum of a manifold having the structures of smooth \( S^{m-n} \)-bundles over \( S^n \), a manifold having the structure of a smooth \( F \)-bundle over \( S^n \) and a manifold in a class of \( \Theta_{(m,n)} \).

**Proof.** Assume that a round fold map \( f : M \to \mathbb{R}^n \) satisfying the condition (1) exists. These assumptions enable us to apply Proposition 4 and by a canonical decomposing operation, we obtain a pair of round fold maps. More precisely, one of the round fold maps is a \( C^\infty \)-trivial round fold map as in Theorem 1 such that the fiber of a point in a proper core of \( f \) is a disjoint union of \( F \), the other is a round fold map as discussed in Theorem 9 (let \( E \) be a manifold diffeomorphic to...
the cylinder $S^{m-n} \times [-1, 1]$ in the situation of Theorem 9) and $M$ is a manifold as in the condition (2).

Conversely, from Theorem 1 (1) and Proposition 2, a manifold as in the condition (2) always admits a round fold map satisfying the condition (2).

This completes the proof. $\square$

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