Constraining the Size of the Corona with Fully Relativistic Calculations of Spectra of Extended Coronae. I. The Monte Carlo Radiative Transfer Code

Wenda Zhang, Michal Dovčiak, and Michal Bursa
Astronomical Institute, Czech Academy of Sciences, Boční 11, 1401 CZ-14100 Prague, Czech Republic; zhang@asu.cas.cz
Received 2019 January 7; revised 2019 March 15; accepted 2019 March 20; published 2019 April 24

Abstract

The size and geometry of the X-ray-emitting corona in an active galactic nucleus (AGN) are still not well constrained. Dovčiak & Done proposed a method based on calculations assuming a point-like lamp-post corona. To perform more self-consistent calculations of energy spectra of extended coronae, we develop MONK, a Monte Carlo radiative transfer code dedicated to calculations of Comptonized spectra in the Kerr spacetime. In MONK we assume a Klein–Nishina scattering cross section and include all general relativistic effects. We find that for a corona located above the disk, the spectrum is not isotropic, but has harder and less luminous spectra toward observers at lower inclinations, owing to anisotropic illumination of the seed photons. This anisotropy also leads to an underestimated size for the corona if we assume it to be a point-like, isotropic source located on the rotation axis of the black hole, demonstrating the necessity for more self-consistent calculations. We also study the effect of motion and geometry of the corona on the emergent spectrum. Finally, we discuss the implication of anisotropic coronal emission for the reflection spectrum in AGNs as well as black hole X-ray binaries (BHXRBs). We find that by assuming the coronal emission to be isotropic, one may underestimate the soft excess in AGNs, and the reflection continuum and iron K fluorescent line flux in BHXRBs.

Key words: galaxies: active – methods: numerical – radiative transfer – relativistic processes

1. Introduction

The hard X-ray (≥2 keV) spectrum of active galactic nuclei (AGNs) is usually dominated by a non-thermal component with a high-energy cutoff at tens to hundreds of keV (e.g., Fabian et al. 2017; Tortosa et al. 2018). The hard X-rays are generally believed to be dominated by radiation from the corona, which contains hot (with a temperature of tens to hundreds of keV), optically thin plasma that Comptonizes UV/optical disk photons (e.g., Haardt & Maraschi 1991). The coronal emission also irradiates the cold disk to produce the reflection emission (see Fabian & Ross 2010, and references therein).

For optically thin thermal plasma Comptonizing low-frequency thermal radiation, above ∼3kBT_e (where k_B is the Boltzmann constant and T_e is the temperature of the thermal radiation) the spectrum is expected to be a power law with a high-energy cut-off (e.g., Sunyaev & Titarchuk 1980). As shown by analytical studies and Monte Carlo simulations, the slope of the power law is a function of the electron temperature and the optical depth (e.g., Sunyaev & Titarchuk 1980; Pozdnyakov et al. 1983) while the cut-off energy depends sensitively on the electron temperature. Hence the electron temperature and optical depth of the corona can be constrained by analyzing the hard X-ray spectrum. In contrast, the size and geometry of the corona are still not clear.

A constraint on the size of the corona may offer us the opportunity to discriminate models for the formation of AGN coronae, because different mechanisms predict distinct sizes of the corona. For example, we would expect a compact corona if its formation is due to the “aborted” jet (e.g., Ghisellini et al. 2004). In contrast in the “two-phase” corona model an extended corona is expected (Haardt & Maraschi 1991). A constraint on the size of the corona will also tell us whether some physical processes, such as pair production, are important (e.g., Guilbert et al. 1983; Zdziarski & Lightman 1985; Fabian et al. 2015).

To date, the most promising constraint on the size of the corona in AGNs comes from analysis of strongly lensed quasars, where the microlensing by stellar components in the lensing galaxy results in a complex magnification pattern (see the review of Wambsganss 2006). The relative motion of the lensed quasar, the galaxy and its stellar components, and the observer leads to uncorrelated variability. The variability amplitude depends sensitively on the size of the emitting region, with a larger amplitude from a smaller emitting region. Thus the size of the source region can be estimated by modeling the light curve (e.g., Kochanek 2004; Kochanek et al. 2007). Reis & Miller (2013) compiled measurements of the X-ray-emitting region in lensed quasars and found the sizes to be in the range from ~1 to tens of gravitational radii. However, the paucity of strongly lensed quasars limits the application of this method.

Dovčiak & Done (2016, hereafter DD16) proposed a method to estimate the size of the corona in AGNs with simultaneous UV/X-ray observations. In this method one first obtains the mass accretion rate with UV luminosity, and then calculates the seed photon spectrum received by a lamp-post corona at an assumed height. They found that the spectrum is close to blackbody, such that NTHCOMP1 (Zdziarski et al. 1996; Życki et al. 1999) can be utilized to calculate the Comptonized spectrum. Given that Comptonization conserves the number of photons (neglecting the double Compton effect), one can calculate the X-ray flux density. Together with the observed X-ray luminosity, a constraint can be put on the size of the corona.

The DD16 method uses NTHCOMP, which assumes that the seed photons illuminate the corona isotropically. Apparently

1 https://heasarc.gsfc.nasa.gov/xanadu/xspec/manual/node200.html
this is not the case for AGN coronae, and as we will show later, anisotropic illumination of the seed photons would lead to anisotropic emission from AGN coronae. Another limitation in the DD16 method is that the calculation of the Comptonized flux is made assuming a lamp-post corona, thus implicitly assuming that the spectra do not change much across the extended corona. This assumption may not hold for a large corona. To perform a more realistic, self-consistent modeling of coronal emission, we develop MONK, a general relativistic Monte Carlo code dedicated to calculations of Comptonized spectra in the Kerr spacetime. We assume a Klein–Nishina cross section and take all general relativistic effects into account. In general the polarization of emission is sensitive to the geometry of the source. For AGN coronae, X-ray polarimetry may play an important role in breaking the degeneracy of coronal shape (e.g., Schnittman & Krolik 2010; Beheshtipour et al. 2017; Tamborra et al. 2018). In MONK we also have the option of performing polarized radiative transfer. If this option is switched on, we also propagate the polarization vector along the null geodesics and take into account the change in photon polarization angle and degree due to scattering.

### 2. Procedure

We follow Dolence et al. (2009) to account for disk emissivity with the “superphoton” scheme. One “superphoton” is a package of several identical photons. Each superphoton is parameterized by: its energy at infinity $E_\infty$, weight $w$, and polarization degree $\delta$, which are constants of motion; and four-position $x^\mu$, wavevector $k^\mu$, and polarization vector $f^\mu$, which need to be propagated along the geodesic. The weight $w$ has the physical meaning of photon generation rate per unit time in a distant observer’s frame.

We first generate superphotons according to disk emissivity, and then ray-trace the superphotons along null geodesics. While the superphoton is inside the corona, we set the step size of ray-tracing to be much less than the scattering mean free path, and for each step we evaluate the scattering optical depth covariantly. If the superphoton is scattered, we sample the differential cross section and calculate the energy, momentum, and polarization properties after scattering. The propagation terminates if the superphoton enters the event horizon, arrives at infinity, or hits the disk.

The energy and polarization spectra can be reconstructed by counting superphotons arriving at infinity. For an observer at inclination $i$, the luminosity density (in counts per unit time per energy interval) at energy $E$ is

$$L_E = \frac{4\pi \sum w_k \delta_k \cos 2\psi_k}{\Delta E \Delta \Omega},$$

where the sum is performed over all superphotons with $\theta_\infty \in (i - \Delta i/2, i + \Delta i/2]$ and $E_\infty \in (E - \Delta E/2, E + \Delta E/2]$, $\Delta E$ is the width of the energy bin, $\Delta \Omega = 2\pi[\cos(i - \Delta i/2) - \cos(i + \Delta i/2)]$ is the solid angle subtended by the inclination bin, and $E_\infty$, $\theta_\infty$ are the energy and polar angle of the superphotons at infinity, respectively. As Compton scattering only induces linear polarization, we always let the Stokes parameter $V = 0$. For the other two Stokes parameters:

$$Q_E = \frac{4\pi \sum w_k \delta_k \cos 2\psi_k}{\Delta E \Delta \Omega},$$

$$U_E = \frac{4\pi \sum w_k \delta_k \sin 2\psi_k}{\Delta E \Delta \Omega},$$

where $\psi$ is the polarization angle measured at infinity in a comoving frame attached to the observer, while $x$- and $y$-axes are identified as $\partial/\partial \theta$ and $\partial/\partial \phi$, respectively.

#### 2.1. The Kerr Metric

The covariant Kerr metric in the Boyer–Lindquist coordinates is

$$g_{\mu\nu} = \begin{pmatrix}
(1 - 2r) & 0 & 2a \sin^2 \theta \\
0 & -\Delta & 0 \\
0 & 0 & \rho^2
\end{pmatrix},$$

where $\Delta \equiv r^2 - 2r + a^2$, and $A \equiv (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta$. Here we let $G = M = c = 1$.

As the Kerr metric is stationary and axially symmetric, we immediately obtain two constants of motion: the photon energy at infinity $E_\infty \equiv -p_r$, and the photon angular momentum about the axis of symmetry $L_z \equiv p_\phi$. Carter (1968) found another constant of motion: $Q \equiv p_\rho^2 - a^2 \cos^2 \theta + L_z^2 \cot^2 \theta$. We denote $l \equiv L_z/E_\infty$, $Q \equiv Q/E_\infty$.

#### 2.2. Ray-tracing in the Kerr Spacetime

In MONK we implement two independent methods of ray-tracing photons along null geodesics: the first one is to integrate the geodesic equation directly, and the second is to make use of the separability of the Hamilton–Jacobi equation and integrate the equations of motion of $r$ and $\theta$. We find these two methods to give consistent results.

#### 2.2.1. Integration of the Geodesic Equation

The propagation of a photon is described by

$$\frac{dx^\mu}{d\lambda} = k^\mu,$$

where $\lambda$ is the affine parameter. We normalize $\lambda$ in such a way that $k_\lambda = -1$. The propagation of $k^\mu$ is governed by the geodesic equation:

$$\frac{dk^\mu}{d\lambda} = -\Gamma^\mu_{\nu\rho} k^\nu k^\rho,$$

where $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols. We utilize the SIMS package (Bursa 2017) to perform the ray-tracing; this uses the Verlet algorithm to evaluate the integral, following Dolence et al. (2009).
2.2.2. Equations of Motion after Separation of the Hamilton–Jacobi Equation

The Hamilton–Jacobi equation of the Kerr spacetime is separable (Carter 1968). As a result, we can relate the following two integrals of \( r \) and \( \mu \equiv \cos \theta \), respectively, where \( r \) and \( \theta \) are two Boyer–Lindquist coordinates (Chandrasekhar 1983):

\[
I = s_r \int \frac{dr}{\sqrt{R(r)}} = s_{\mu} \int \frac{d\mu}{\sqrt{M(\mu)}},
\]

where \( s_r, s_{\mu} = \pm 1 \). We set \( s_r \) and \( s_{\mu} \) in such a way that the integrals are always positive. For massless particles

\[
R(r) = r^4 + (a^2 - l^2 - Q)r^2 + 2[Q + (l - a)^2]r
- a^2 Q,
\]

\[
M(\mu) = Q + (a^2 - l^2 - Q)\mu^2 - a^2 \mu^4.
\]

Starting with a superphoton at \( x_0^\mu \) with wavevector \( k_0^\mu \) and setting \( I = 0 \), we can ray-trace the photon’s trajectory in the following way: for each step we give \( I \) a small increment such that \( I \to I + dI \), and then invert the integrations in Equation (7) to solve for \( r \) and \( \mu \). We also change the signs of \( k^r \) and \( k^\theta \) upon encountering the respective turning points. At any step one can solve for \( k^\mu \) given \( I, Q \), and the signs of \( k^r \) and \( k^\theta \) following Carter (1968). The affine parameter can be obtained by evaluating the following equation (Carter 1968):

\[
\lambda = \int \frac{r^2 dr}{\sqrt{R(r)}} + a^2 \int \frac{\mu^2 d\mu}{\sqrt{M(\mu)}}.
\]

In this paper we are dealing with axially symmetric problems and we do not study variability properties, hence evaluation of \( \lambda \) is orthonormal to the photon wavevector \( k^\mu \):

\[
f^\mu f_\mu = 1, \quad k^\mu f_\mu = 0,
\]

and is parallel-transported along the null geodesic:

\[
k^\mu \nabla_\mu f^\nu = 0.
\]

For vectors of this kind one can find a complex constant: the Walker–Penrose constant \( \kappa_{wp} \), which is conserved along the null geodesic (Walker & Penrose 1970; Chandrasekhar 1983). So instead of propagating \( f^\mu \) using Equation (13), we just keep a record of \( \kappa_{wp} \). Only when the photon interacts with an electron do we solve for \( f^\mu \), calculate \( f'^\mu \) and \( k'^\mu \) after the interaction, and subsequently re-evaluate \( \kappa_{wp} \).

2.4. Orthonormal Tetrad

One can construct an orthonormal tetrad \( e_\alpha^\mu \) for an observer that is comoving with the fluid. To transform a vector \( f \) from the tetrad frame to the coordinate frame,

\[
f^\mu = e_\alpha^\mu f^{(\alpha)},
\]

and to transform back,

\[
f^{(\alpha)} = e_\alpha^{(\alpha)} f^\mu.
\]

We illustrate the method to obtain the orthonormal tetrad in the Appendix.

2.5. Superphoton Generation

To find the generation rate of superphotons emitted by an optically thick accretion disk on the equatorial plane, we first notice that the flux density defined as follows is a relativistic invariant (Kulkarni et al. 2011; Schnittman & Krolik 2013):

\[
F_\nu = I_\nu \frac{1}{u'} \cos \theta \mathrm{em} dS d\Omega,
\]

where \( I_\nu \) is the specific intensity, \( 1/u' \) is the factor that transforms time from the rest frame of the disk fluid into the distant observer’s frame, \( \theta_{\mathrm{em}} \) is the polar emission angle with respect to disk norm in the rest frame of the disk fluid, \( dS \) is the proper area, and \( d\Omega \) is the solid angle. It is easy to see that the number of photons emitted per unit time in the distant observer’s frame

\[
N = \int \frac{F_\nu d\nu}{h \nu}
\]

is also an invariant. For photons emitted from an annulus on the equatorial plane that is centered at Boyer–Lindquist radius \( r \) and has a width of \( dr \),

\[
\frac{1}{u'} = \sqrt{1 - 2/r + 4\Omega a/r - \Omega^2 (r^2 + a^2 + 2a^2/r)}
= \sqrt{\frac{\Delta}{\rho^2}}
\]

where \( \Omega \) is the angular velocity of the disk fluid; the proper area (Wilkins & Fabian 2012) is

\[
dS = \frac{2\pi \Omega}{\sqrt{\Delta}} \sqrt{r^2 + a^2 + \frac{2a^2r}{\rho^2}} dr.
\]
where $\Gamma$ is the Lorentzian factor of the disk fluid as measured by a zero-angular-momentum observer (ZAMO). For Keplerian disks,
\begin{equation}
\Omega = \frac{1}{r^{7/2} + a}.
\end{equation}

The velocity in the $\phi$ direction as measured by a stationary observer is
\begin{equation}
v_0 = \frac{(\Omega - \omega)A}{r^2\sqrt{\Delta}},
\end{equation}

where $\omega$ is the frame-dragging angular velocity; hence the Lorentzian factor is
\begin{equation}
\Gamma = \frac{1}{\sqrt{1 - v_0^2}}.
\end{equation}

If the local spectrum is that of a color-corrected blackbody with a color correction factor $f_{\text{col}}$ and an effective temperature $T_{\text{eff}}$, then
\begin{equation}
I_\nu = f_{\text{limb}} \frac{1}{f_{\text{col}}} \frac{2\hbar v^3}{c^2} \frac{1}{e^{\hbar k_{\mu} T_{\text{eff}}} - 1},
\end{equation}

where $f_{\text{limb}}$ is the limb-darkening factor; we have
\begin{equation}
N = \frac{4\zeta(3)k_B^2 f_{\text{limb}} \cos \theta_{\text{em}} dS d\Omega T_{\text{eff}}^3}{c^2 h^3 f_{\text{col}} u^2},
\end{equation}

where $\zeta$ is the Riemann zeta function. Therefore from the annulus, we sample $N_e$ photons each with weight $w = \dot{N}/N_e$ into the solid angle $d\Omega$. The energies of the sampled photons at infinity follow the Planckian distribution with a temperature of $g_{\text{col}} T_{\text{eff}}$, where $g = -1/k_{\mu} U^\mu$ is the redshift factor and $U^\mu$ is the four-velocity of a disk particle. We use a rejection method as illustrated in detail in Section 9.4 of Pozdnyakov et al. (1983) to sample the Planckian distribution.

We assume a Novikov–Thorne emissivity profile with zero torque at the inner edge of the disk. The local spectrum is that of a color-corrected blackbody. If the polarization option is switched off, we assume that the disk emission is isotropic in the local frame, and the seed superphotons emitted by the disk are unpolarized. Otherwise we model the disk atmosphere as a semi-infinite planar atmosphere (Chandrasekhar 1960), as in Dovčiak et al. (2008) and Li et al. (2009), to calculate the polarization properties and the angular distribution of the seed superphotons. In this case the polarization angle is perpendicular to the meridian plane, i.e., the plane containing both the axis of symmetry and the line of sight to the observer, and the polarization degree increases monotonically with the polar emission angle. Given the emission angle, the intensity and polarization degree of the radiation can be calculated using Chandrasekhar’s $H$-functions (Chandrasekhar 1960). We utilize the method described in Bosma & de Rooij (1983) to evaluate the $H$-functions.

2.6. Comptonization

While the superphoton is traveling inside the corona, for each step we evaluate the scattering probability assuming a Klein–Nishina cross section. If the superphoton is scattered, first we sample the momentum of the scattering electron and boost $k^\mu$ and $j^\mu$ into the electron rest frame. Then we sample the momentum of the scattered photon and calculate the polarization vector after scattering. Finally we boost $k^\mu$ and $j^\mu$ back into the Boyer–Lindquist frame and update $E,$ $\delta,$ and $k_{wp}$ of the superphoton. If the polarization option is switched off, we always assume that the incoming superphoton is unpolarized.

2.6.1. Covariant Evaluation of the Scattering Optical Depth

While we propagate the superphoton inside the corona, for each step we evaluate the covariant scattering optical depth following Younsi et al. (2012):
\begin{equation}
\tau_\nu = -\int_{\lambda_0}^{\lambda_{\text{lim}}} \alpha_{0,\nu}(\lambda) k_\nu U^\nu |_{\lambda} d\lambda,
\end{equation}

where $\lambda_0$ and $\lambda_1$ are the affine parameters of the superphoton at the beginning and end of the step, and $\alpha_{0,\nu}$ is the scattering coefficient in the fluid rest frame.

The superphoton energy measured in the fluid frame is $E_0 = -k_\nu U^\nu |_{\lambda_0}$. Denoting $x_0 \equiv E_0/\left(m_e c^2 \right)$ where $m_e$ is the electron rest mass, the scattering coefficient of the superphoton with respect to a population of electrons can be evaluated via the following integral (e.g., Pozdnyakov et al. 1983):
\begin{equation}
\alpha_{0,\nu} = \int \frac{dN_e}{d\Omega d^3p} (1 - \mu_e \beta_e) \sigma(x) d^3p,
\end{equation}

where $dN_e/d\Omega d^3p$ is the electron velocity distribution, $\mu_e = \cos \theta_e,$ $\theta_e$ is the angle between the momentum of the photon and that of the electron, $\beta_e = v/e/c$ is the electron velocity, $\gamma_e$ is the Lorentz factor of the electron, $\sigma(x)$ is the scattering cross section, and $x = \gamma_e x_0 (1 - \mu_e \beta_e)$ is the dimensionless photon energy in the electron rest frame.

In the electron rest frame, the differential scattering cross section is (Berestetskii et al. 1971; Connors et al. 1980)
\begin{equation}
\frac{d\sigma}{d\Omega} = - \frac{r_0^2}{2} \left( \frac{x'}{x} \right)^2 \left( \frac{x}{x'} + \frac{x'}{x} - \sin^2 \theta' \right) - \frac{r_0^2}{2} \left( \frac{x}{x'} + \frac{x'}{x} - \sin^2 \theta' \right) [X_\nu \cos 2\psi' + Y_\nu \sin 2\psi'],
\end{equation}

where $r_0$ is the classical electron radius, $x'$ is the photon energy after scattering, and $\theta'$ and $\phi'$ are the polar and azimuthal angles of the photon wavevector after scattering. The coordinate system is defined in such a way that the $z$-axis is aligned with the photon wavevector, while the $x$- and $y$-axes are defined by two orthonormal unit vectors in the plane perpendicular to the photon wavevector. For a light beam with polarization degree $\delta$ and polarization angle $\psi,$
\begin{equation}
X_\nu = \delta \cos 2\psi, \quad Y_\nu = \delta \sin 2\psi.
\end{equation}

By integrating Equation (29), we obtain the total cross section:
\begin{equation}
\sigma(x) = \pi r_0^2 \left[ \left( 1 - \frac{2}{x} - \frac{2}{x^2} \right) \ln(1 + 2x) + \frac{1}{2} + \frac{4}{x} \right],
\end{equation}

which is the Klein–Nishina formula $\sigma_{KN}$ and is independent of the polarization degree.
For thermal electrons with temperature $T$, their velocities follow the Maxwell–Jüttner distribution:

$$\frac{dN_e}{d\gamma_e} \propto \frac{\gamma_e^2 \beta_e}{\theta_T K_2(1/\theta_T)} e^{-\gamma_e \theta_T},$$  \hspace{1cm} (32)

where $\theta_T \equiv k_B T/m_e c^2$ is the dimensionless electron temperature and $K_2$ is the modified Bessel function of order 2.

With Equations (28), (31), and (32) we can calculate the optical depth $\tau$ in the case of a thermal electron. With optical depth $\tau$, the scattering probability is $P = 1 - e^{-\tau}$. For optically thin coronae, we introduce a bias factor $b \gg 1$ following the practice of Pozdnyakov et al. (1983) and Dolence et al. (2009) to enhance the statistics at high energy, such that the scattering probability becomes $P = 1 - \frac{1}{b \tau}$, and after scattering the superphoton “splits” into two superphotons with appropriate weights. We generate a random number $\epsilon$ by sampling a uniform distribution between 0 and 1, and the condition for scattering is $\epsilon \leq P$.

### 2.6.2. Sampling Electron Momentum

If the photon is scattered, we first sample the momentum of the scattering electron. The probability density of the scattering electron having momentum $p_e = [\gamma_e m_e, \gamma_e m_e \mathbf{v}]$ is

$$P(p_e) \propto \frac{dN_e}{d^3p} (1 - \mu_e \beta_e) \sigma_{KN}.$$  \hspace{1cm} (33)

We sample the probability density distribution with a rejection method, following Pozdnyakov et al. (1983) and Canfield et al. (1987).

### 2.7. Scattering in the Electron Rest Frame

Once we sample the four-velocity of the scattering electron, we can obtain the energy, the wavevector, and the polarization vector of the photon in the electron rest frame with a generic Lorentz transformation. For the polarization vector in the electron rest frame $f_e$, we make an additional transformation according to Equation (15), in which we set $\alpha$ in such a way that $f'_e = 0$, as required by the gauge we choose.

For convenience we set up such a 3D Cartesian coordinate system in which the $z$- and $x$-axes are aligned with the wavevector and polarization vector of the incoming photons, respectively. The differential cross section for the photon to be scattered into a solid angle $d\Omega'$ centered on polar angle $\theta'$ and azimuthal angle $\phi'$ is

$$\frac{d\sigma}{d\Omega'} = \frac{r_0^2}{2} \left( \frac{x'}{x} \right)^2 \left[ 1 + \frac{x'}{x} \left( \frac{x'}{x} - \sin^2 \theta' - \delta \sin^2 \theta' \cos 2\phi' \right) \right],$$  \hspace{1cm} (34)

where $x'$, the dimensionless photon energy after scattering, is related to $\theta'$ by the Compton recoil relation:

$$x' = \frac{x}{1 + x(1 - \cos \theta')}.$$  \hspace{1cm} (35)

Hence the joint probability density of $x'$ and $\phi'$ is

$$p(x', \phi'; \delta) = \frac{1}{\sigma_{KN}} \frac{dx'}{d\Omega'} \frac{d\cos \theta'}{dx'}. \hspace{1cm} (36)$$

By integrating Equation (36) over $\phi'$, we obtain the probability density of $x'$:

$$p(x') = \frac{1}{\sigma_{KN}} \frac{\pi \delta_0^2}{x'^2} \left[ \frac{x}{x'} + \frac{x'}{x} - \sin^2 \theta' \right]. \hspace{1cm} (37)$$

which is independent of $\delta$. We sample $x'$ following Kahn (1954), and subsequently solve for the scattering polar angle $\theta'$ with the Compton recoil relation. The probability density distribution of the scattering azimuthal angle $\phi'$ is

$$p(\phi'|x'; \delta) = \frac{p(x', \phi'; \delta)}{p(x')} = \frac{1}{2\pi} \frac{1}{2\pi} \frac{\delta \sin^2 \theta' \cos 2\phi'}{2\pi \left( \frac{x'}{x} + \frac{x'}{x} - \sin^2 \theta' \right)}. \hspace{1cm} (38)$$

It is trivial to calculate the cumulative distribution of $\phi'$:

$$\text{cdf}(\phi') = \frac{\phi'}{2\pi} - \frac{\delta \sin^2 \theta' \sin 2\phi'}{4\pi \left( \frac{x'}{x} + \frac{x'}{x} - \sin^2 \theta' \right)}. \hspace{1cm} (39)$$

With $\theta'$ and $\phi'$ we can derive the photon wavevector in the electron rest frame after scattering, $k'_e$. We set up a new coordinate system, with the $z$-axis aligned with $k'_e$ and

$$e'_z = \frac{k_e \times k'_e}{|k_e \times k'_e|},$$  \hspace{1cm} (40)

$$e'_z = k'_e \times e'_z,$$  \hspace{1cm} (41)

such that $e'_z$ is the unit vector in the scattering plane and perpendicular to $k'_e$ while $e'_z$ is the unit vector perpendicular to the scattering plane. The two normalized Stokes parameters of the scattered photon in the coordinate system $\{e'_z, e'_z\}$ are (Connors et al. 1980)

$$Q' = \frac{1}{N} \sin^2 \theta' - \delta \sin^2 \theta' \cos 2\phi',$$  \hspace{1cm} (42)

$$U' = \frac{2}{N} \delta \cos \theta' \sin 2\phi',$$  \hspace{1cm} (43)

where

$$N = \frac{x'}{x} + \frac{x}{x'} - \sin^2 \theta' - \delta \sin^2 \theta' \cos 2\phi'.$$  \hspace{1cm} (44)

The polarization degree and angle after scattering are

$$\delta = \sqrt{Q'^2 + U'^2},$$  \hspace{1cm} (45)

$$\psi' = \frac{1}{2} \arctan \left( \frac{Q'}{U'} \right).$$  \hspace{1cm} (46)

The polarization vector is then $f'_e = \cos \psi' e'_z + \sin \psi' e'_z$. The energy, wavevector, and polarization vector after scattering in the Boyer–Lindquist frame can be obtained from $x'$, $k'_e$, and $f'_e$ by a series of transformations including rotation, generic Lorentz boost, and tetrad transform. Finally we can evaluate the Walker–Penrose constant of the superphoton after scattering following Equation (14).
3. Comparison with Previous Codes

3.1. Spectrum and Polarization from a Novikov–Thorne Disk

In this section we compare the energy and polarization spectra of a Novikov–Thorne disk around a stellar-mass black hole with Dovčiak et al. (2008). The parameters are $a = 1$, $M = 14 \, M_\odot$, $M = 1.4 \times 10^{18} \, \text{g s}^{-1}$, and $f_{\text{col}} = 1.7$. The observer’s inclination is $60^\circ$. In Figure 1 we compare the results calculated by MONK with that by Dovčiak et al. (2008), and find them to agree well.

3.2. Energy Spectra of Spherical Plasma Clouds Comptonizing Low-frequency Radiation

In this section we compare the energy spectra of spherical plasma clouds Comptonizing low-frequency radiation calculated by MONK with that calculated by GRMONTY (Dolence et al. 2009). The isotropic, low-frequency primary radiation is located in the center of the cloud. Both the plasma and the primary radiation are thermal, and their temperatures are $4 \, m_e c^2$ and $10^{-8} \, m_e c^2$, respectively. For comparison we take three different optical depths: $\tau_T = 10^{-3}$, 0.1, and 3, corresponding to Figures 7–9 of Dolence et al. (2009). Here $\tau_T \equiv n_e \sigma_T R$, where $n_e$ is the electron number density and $R$ is the radius of the cloud.

In Figures 2–4 we present the spectra calculated with MONK and GRMONTY, and find them to be in good agreement. The spectra by GRMONTY are read from their paper using WEBPLOTDIGITIZER. The polarization option is turned off.

3.3. Polarization Degree of Radiation from a Scattering Disk Atmosphere

We calculate the angular dependence of the polarization degree of the radiation emerging from the surface of a pure scattering disk atmosphere and compare the result with STOKES (Goosmann & Gaskell 2007). The optical depth is defined as $\tau_T = n_e \sigma_T h$, where $h$ is the half-thickness of the disk. The primary photons are isotropic and uniformly located on the disk midplane. To compare with STOKES, we assume a Thomson cross section and stationary electrons.

Figure 1. Energy and polarization spectra from an optically thin disk around a stellar-mass black hole. Parameters: $a = 1$, $M = 14 \, M_\odot$, $M = 1.4 \times 10^{18} \, \text{g s}^{-1}$, and $f_{\text{col}} = 1.7$. The results from MONK and Dovčiak et al. (2008) are plotted in cyan and magenta, respectively. From top to bottom: the energy spectrum, the polarization degree, and the polarization angle.

Figure 2. Spectra of a spherical plasma cloud ($T_r = 4 \, m_e c^2$) Comptonizing a central photon source ($T_{\text{ph}} = 10^{-3} \, m_e c^2$). The Thomson optical depth $\tau_T = n_e \sigma_T R = 10^{-3}$, where $n_e$ is the electron number density, $\sigma_T$ is the Thomson scattering cross section, and $R$ is the radius of the cloud. The cyan solid and magenta dotted lines represent spectra by MONK and GRMONTY, respectively.

Figure 3. Same as Figure 2, but for $\tau_T = 0.1$.

Figure 4. Same as Figure 2, but for $\tau_T = 3$.

https://automeris.io/WebPlotDigitizer/
Equation (42), we utilize the Rayleigh matrix method to calculate the Stokes parameters of the scattered photons following Schnittman & Krolik (2013). The results are presented in Figure 5, in which good agreement is seen for various optical depths.

4. Results

In this section we showcase a few spectra of disk–corona systems calculated by MONK, leaving a systematic study to a future work. Note that the polarization option is switched off for all calculations included in this section.

4.1. Stationary Spherical Coronae

In this section we investigate the spectral properties of disk–corona systems in which the coronae are spherical and stationary (i.e., the angular velocity of the fluid is the same with a ZAMO observer).

4.1.1. Dependence on the Optical Depth

In Figure 6 we present the spectra of disk–corona systems consisting coronae of different optical depths. The disk surrounds a Kerr black hole with spin $a = 0.998$ and mass $M = 10^7 M_\odot$. We assume that the disk extends down to the innermost stable orbit and there is zero torque at the inner edge of the disk. The radiative efficiency for a standard disk around a spin 0.998 black hole is expected to be $\sim 0.32$. The outer boundary of the disk is taken to be $1000 GM/c^2$. The mass accretion rate $\dot{M} = 4.32 \times 10^{23}$ g s$^{-1}$, corresponding to a bolometric luminosity of $1.24 \times 10^{44}$ erg s$^{-1}$, $\sim 10\%$ of the Eddington luminosity. We take the color correction $f_{\text{col}} = 2.4$ that is expected for AGN disks (Ross et al. 1992; Done et al. 2012). The spherical corona is located $10 GM/c^2$ above the disk, has a radius of $R_c = 4 GM/c^2$, and has an electron temperature of 100 keV. The observer is located at an inclination of $30^\circ$. Results for different optical depths are plotted in different colors and line styles, as indicated in the plot. The spherical corona is located $10 GM/c^2$ above the disk, has a radius of $4 GM/c^2$, and has an electron temperature of 100 keV. The other parameters are $a = 0.998$, $M = 10^7 M_\odot$, and $M = 4.32 \times 10^{23}$ g s$^{-1}$.

4.1.2. Dependence on the Observer’s Inclination

In Figure 7 we present the spectra of a disk–corona system as seen by observers at various inclinations. The parameters are the same as in Section 4.1.1, except that $\tau_T = 0.2$ and various inclinations are assumed. To highlight the distinction we enlarge the spectra between 0.5 and 20 keV in the inset plot, in which the spectra for inclinations of $60^\circ$ and $80^\circ$ are multiplied by factors of 2 and 4, respectively, for clarity. The spectra depend sensitively on the observer’s inclination. The spectrum at the inclination of $10^\circ$ flattens toward low energy, and seems to be harder than that observed at larger inclinations below $\sim 5$ keV.

We measure the photon indices between 2 and 10 keV by fitting the spectra with a power-law model using the least-squares method. The indices are 2.69, 2.70, and 2.73 for inclinations of $10^\circ$, $60^\circ$, and $80^\circ$, respectively. For comparison
we calculate the spectrum of a spherical plasma cloud Comptonizing central isotropic radiation, the same geometry as in Section 3.2. In this case we take $\tau_T = 0.2$ and $T_e = 100$ keV. We also take the temperature of the thermal radiation to be $0.026$ keV, the temperature of the thermal radiation received by a lamp-post corona with a height of $10 \, GM/c^2$. We measure the photon index in the isotropic case and find it to be 2.88 between 2 and 10 keV, softer than all spectra presented in Figure 7.

The angular dependence is more clearly seen in Figure 8, where we present the Comptonized spectra as well as the contribution of photons with different numbers of scatterings. Between $\sim 0.1$ and 2 keV the spectrum observed at higher inclination is softer and more luminous. By inspecting the contributions of different scattering orders (middle and bottom panels), we find that this is mainly caused by the difference in the scattering spectrum of the first order.

The angular dependence is due to anisotropic illumination of the seed photons. For isotropic electrons scattering anisotropic photons, the first scattering spectrum would be highly anisotropic, with more scattering power backward in the direction of the seed photons than forward, while the high-order spectra are expected to be isotropic (e.g., Ghisellini et al. 1991; Haardt & Maraschi 1991; Haardt 1993). Therefore, for the geometry we assume, where the seed photons are emitted by the thin disk below the corona, a more powerful first scattering spectrum at larger inclination is expected, consistent with what we see in Figure 8. This indicates that in the frame of the corona the radiation illuminating the underlying disk is more luminous than the radiation that arrives at an observer at infinity. This is contrary to the assumption of an isotropic corona usually taken in modeling the reflection spectrum from the disk illuminated by a lamp-post corona. We will discuss this topic further in Section 5.

4.1.3. Dependence on the Size of the Corona

In Figure 9 we present the spectra of disk–corona systems with different coronal radii. The parameters are the same as in Section 4.1.1, but with various sizes of the corona and $\tau_T = 0.2$. The most prominent effect is that the non-thermal luminosity becomes brighter as $R_c$ increases. In the lower panel of Figure 9 we show the spectra normalized by $R_c^2$. For coronal emission the normalized spectra agree with each other quite well, indicating that the Comptonized spectra scale with $R_c^2$.

For each spectrum in Figure 9 we estimate the size of the corona using the DD16 method with the measured photon index and luminosity. For coronal sizes of 1, 2, 4, and 8 $GM/c^2$, the DD16 method gives estimates of 0.63, 1.30, 2.55, and 5.00 $GM/c^2$ for the coronal radius, respectively, about half the input sizes. The reason is that in NTHCOMP the emergent spectrum is obtained under the assumption that the seed photons are uniformly distributed and isotropic. As shown in Section 4.1.2, with the same optical depth the spectrum in the isotropic case is softer than the spectrum observed at low inclination if the seed photons are illuminated anisotropically. Hence the DD16 method, which assumes isotropic seed photons, requires a larger than assumed optical depth to explain the spectrum. This subsequently leads to an...
overestimate of the luminosity and an underestimate of the size of the corona.

We also calculate the photon index and luminosity as measured by observers at inclinations of $10^\circ$ and $60^\circ$ and estimate the size of the corona using the DD16 method. For an inclination of $10^\circ$, the DD16 method gives estimates of 0.64, 1.31, 2.57, and 5.08 \( GM/c^2 \) for the coronal radius for input corona sizes of 1, 2, 4, and 8 \( GM/c^2 \), respectively; whereas for an inclination of $60^\circ$, the estimated sizes are 0.64, 1.26, 2.55, and 5.14 \( GM/c^2 \), respectively. For both inclinations the estimated sizes are close to those estimated when the observer’s inclination is $30^\circ$.

4.2. Rotating Spherical Coronae

In this section we investigate the effect of coronal rotation on the emergent spectrum. The set-up is the same as in Section 4.1.2, except that the corona is not stationary, but rotating about the black hole. To assess the effect of rotation, we make a simple assumption on coronal motion that the emergent spectrum. The set-up is the same as in Section 4.1.2, except that the corona is not stationary, but rotating about the black hole. To assess the effect of rotation, we make a simple assumption on coronal motion that

4.3. Co-rotating Slab Coronae

In this section we present the spectra for different disk–corona systems in which the coronae are of slab geometry above the disk. We assume that the coronae are co-rotating with the underlying Keplerian thin disks.

4.3.1. Dependence on the Observer’s Inclination

In Figure 11 we present the spectra of a disk–corona system as seen by observers at various inclinations. The co-rotating slab corona has a height of 10 \( GM/c^2 \), a thickness of 2 \( GM/c^2 \), and a radius of 4 \( GM/c^2 \). Its temperature is 100 keV, and its optical depth in the vertical direction is \( \tau_T = n_e \sigma_T h = 0.2 \), where \( h = 1 \ GM/c^2 \) is the half-thickness of the disk. The other parameters are the same as in Section 4.1.1. The observers at lower inclinations see less luminous X-ray emission, the same as for a rotating spherical corona (Section 4.2).

There is also a difference in the spectral shape below \( \sim 5 \) keV. To highlight the distinction we plot the enlarged spectra between 0.5 and 20 keV in the inset plot, in which the spectra for inclinations of $60^\circ$ and $80^\circ$ are multiplied by a factor of 2 and 4, respectively, for clarity. Similarly with spherical coronae, the low-inclination spectrum below \( \sim 5 \) keV seems to be harder than that observed at larger inclinations, and flattens toward low energy.

4.3.2. Dependence on the Size of the Corona

In Figure 12 we present the energy spectra of disk–corona systems consisting of slab coronae of different sizes. The parameters are the same as in Section 4.3.1, but with different

Figure 9. Upper panel: same as Figure 6, but for different coronal radii as indicated in the plot and $\tau_T = 0.2$. The observer’s inclination is $30^\circ$. Lower panel: spectra normalized by $R_c^2$, where $R_c$ is the coronal radius in units of $GM/c^2$.

Figure 10. Spectra of a disk–corona system that contains a rotating spherical corona as seen by observers at different inclinations, in dashed lines. Everywhere in the corona the fluid is rotating with a linear velocity of \( c/2 \) as measured by a ZAMO observer. The other parameters are the same as in Figure 7. For comparison we also plot the spectra for a stationary corona in solid lines.
Similarly to Figure 9, in the lower panel of Figure 12 we plot the spectra normalized by $R_c^2$. Unlike spherical coronae, the spectra of slab coronae do not simply scale with $R_c^2$, but become harder and more luminous as the radius increases, due to increased optical depth in the horizontal direction.

5. Discussion

As seen in Figure 7, the coronal emission is not isotropic even for a stationary spherical corona. Owing to anisotropic illumination of the seed photons, the Comptonized spectrum is more powerful in the direction of the seed photons, indicating that for the coronae in AGNs above the disk, in the rest frame of the corona the Comptonized radiation striking the underlying disk would be more powerful than that emitted toward an observer at infinity. This is contrary to the assumption of an isotropic corona, which is usually taken in modeling and interpreting the reflection spectrum, with a few exceptions (e.g., Henri & Petrucci 1997; Petrucci & Henri 1997; Malzac et al. 1998).

To investigate the difference between the illuminating and direct emission, we calculate the angle-dependent spectra for spherical plasma Comptonizing highly anisotropic seed photons that are moving in a single direction. We take the electron temperature $T_e = 100$ keV and the optical depth $\tau_T = 0.2$. To assess the effect in both AGNs and black hole X-ray binaries (BHXRBs), we perform calculations for two different seed photon temperatures of 0.026 and 0.58 keV, the temperatures of seed photons as received by lamp-post coronae with a height of 10 $GM/c^2$ above black holes accreting at 10% of the Eddington rate with masses of $10^9 M_\odot$ and $10 M_\odot$, respectively. In Figure 13 we present the spectra for observers at various inclinations, where the inclination is defined to be the angle between the line of sight and the direction of the seed photons. Hence, inclinations of $10^\circ$, $30^\circ$, and $60^\circ$ correspond to distant observers; whereas inclinations of $120^\circ$, $150^\circ$, and $170^\circ$ represent observers on the underlying disk.

In the upper panel of Figure 13 we present the results for the AGN case. Above ~3 keV the coronal emission is more or less isotropic. As the illuminating radiation interacts with the cold disk mainly via Compton down-scattering and photoionization, the estimate of the flux density of the reflection emission above ~3 keV will not be affected much by assuming an isotropic corona. However, there is large contrast between the spectra of the radiation received by observers on the disk (with inclinations of $120^\circ$, $150^\circ$, and $170^\circ$) and those seen by distant observers (with inclinations of $10^\circ$, $30^\circ$, and $60^\circ$) in the range of ~0.1–3 keV. A common practice in modeling the reflection spectrum is to take the photon index of the observed X-ray continuum to be the photon index of the illuminating radiation. In the range of 2–10 keV, the photon indices are 2.68, 2.64, and 2.65 for inclinations of $10^\circ$, $30^\circ$, and $60^\circ$, respectively; while the observers on the disk see softer spectra, with indices of 2.86, 3.12, and 3.27 for inclinations of $120^\circ$, $150^\circ$, and $170^\circ$, respectively. Therefore by assuming the coronal radiation to be isotropic in the rest frame of the corona, one may underestimate the photon index of the illuminating radiation. This can be eased if the photon index is measured in a harder energy band. For example, in the energy band of 10–79 keV, the photon indices differ by no more than 0.2 among different inclinations.

Many AGN spectra contain “soft excess” below 1 keV compared with the 2–10 keV continuum (e.g., Turner & Pounds 1988). One popular scenario for the origin of the soft excess in AGNs is that it is due to hard coronal emission reflecting off ionized disk material (e.g., Crummy et al. 2006). Unlike the reflection continuum, the flux of the soft excess
would be underestimated due to anisotropy of the coronal emission below 3 keV.

The spectra for the BHXRB case are presented in the lower panel of Figure 13. Similarly to the AGN case, the spectra seen by distant observers are harder and less luminous, and flatten toward low energy. The photon index of the illuminating emission would be underestimated if one assumes that the coronal emission is isotropic in the rest frame of the corona. Even if the photon index is measured in a harder band, e.g., 10–79 keV, the photon indices are 2.10, 2.15, and 2.36 for observers at inclinations of 10°, 30°, and 60°, respectively, while much softer spectra are seen by observers on the disk, with photon indices of 3.24, 3.23, and 3.21 for observers at inclinations of 120°, 150°, and 170°, respectively.

For an isotropic lamp-post corona, the reflection fraction is expected to be close to unity in the Newtonian case (Dauser et al. 2014). In the case of fast spinning black hole, the gravitational field focuses the radiation toward the disk and reduces the intensity received by an observer at infinity, thus a larger reflection fraction is expected, especially for a compact corona close to the black hole (e.g., Miniutti & Fabian 2004; Dauser et al. 2014). However, for the BHXRB case, the coronal emission is anisotropic in the range of ∼3–60 keV, a harder band than AGNs. In this case the estimate of the flux density of the reflection continuum will also be affected. We calculate the luminosity above 1 keV, and find that the luminosity measured by observers on the disk is in general larger than the luminosity measured by distant observers. For instance, at 170° the luminosity is ∼3 times that at 10°. In this case a reflection fraction larger than unity can be observed even without a fast spinning black hole.

For the BHXRB case the largest contrast in the spectra is around 10 keV. As the photoionization cross section decreases rapidly with energy above the absorption edge, photons with energy just above the iron K edge are most important for the production of fluorescent iron K photons (George & Fabian 1991). Hence the anisotropy of the coronal emission has an even larger effect in estimating the flux of the fluorescent iron K line than the reflection continuum.

While the analysis above hints qualitatively at the consequence of assuming an isotropic corona, to quantitatively assess the effect more issues have to be taken into account. The angular distribution of the seed photons is more complicated. The gravitational field of the black hole will play an important role in modifying the energies and trajectories of the illuminating photons, especially when the black hole has large spin. For an observer at infinity the direct emission is contaminated by the reflection emission, and to assess the effect one needs to take into account the reflected photons arriving at infinity as well. Such calculations can be done with MONK with a reflection model, by calculating the reflection spectrum for photons arriving at the disk and propagating the reflected photons in the same fashion as the thermal seed photons. However, this is beyond the scope of this paper and will be carried out in a future work.

We also investigate the angle-dependent spectra for slab plasma clouds and present the results in Figure 14. It is obvious that spectra depend on the observer’s inclination in a similar fashion to the spherical case, therefore the conclusion we draw based on the case of a spherical cloud should apply to the slab geometry as well.

6. Summary

The size of the X-ray corona in AGNs is still not well constrained. To date the most promising constraint comes from analysis of strongly lensed quasars, but their paucity limits the application of this method. DD16 developed a method to measure the size of the corona with simultaneous X-ray and UV observations, while the coronal spectrum was calculated assuming a lamp-post geometry. To perform more self-consistent calculations of the coronal spectra, we develop MONK, a Monte Carlo radiative transfer code that is dedicated to the Comptonization process in the Kerr spacetime. We include all general relativistic effects and assume a Klein–Nishina scattering cross section. We compare the results from MONK with those from previous codes and find them to be consistent.

We calculate spectra of disk–corona systems in AGNs that consist of Novikov–Thorne disks on the equatorial plane and optically thin coronae above the geometrically thin disks. For stationary spherical coronae, the coronal emission is dependent on inclination, owing to anisotropic illumination of the seed photons, with observers at lower inclinations seeing harder and less luminous coronal emission. The non-thermal emission

---

**Figure 13.** Upper panel: the spectra of a spherical plasma cloud Comptonizing thermal seed photons, as seen by observers at different inclinations. The seed photons are moving in a single direction and the inclination is defined as the angle made by the direction of the seed photons and the line of sight to the observer. Spectra observed at different inclinations are plotted in different colors and line styles, as indicated in the plot. The plasma has a temperature of 100 keV and a Thomson optical depth of 0.2. The temperature of the seed photons is 0.026 keV. The injection rate of seed photons is 1 count s⁻¹. Lower panel: same as the upper panel, but for a seed photon temperature of 0.58 keV. We also indicate the energy of the neutral iron K edge at 7.1 keV with a vertical dashed line.

---

**Figure 14.** Summary of the comparison between MONK and other codes for different inclination angles. The size of the X-ray corona in AGNs is still not well constrained. To date the most promising constraint comes from analysis of strongly lensed quasars, but their paucity limits the application of this method. DD16 developed a method to measure the size of the corona with simultaneous X-ray and UV observations, while the coronal spectrum was calculated assuming a lamp-post geometry.
scales with the coronal radius squared. We find that the sizes estimated with the DD16 method are around half the assumed size. The reason for this discrepancy is that DD16 assumes isotropic coronal emission that is softer than the spectrum observed at low inclinations at the same optical depth. For a spherical corona that is rotating about the axis of symmetry, at a large inclination the spectrum is more or less the same as that for a stationary spherical corona, while at low inclination the observed spectrum is less luminous than in the stationary case due to the beaming effect. For co-rotating slab coronae, the spectra depend on the observer’s inclination in a similar fashion to spherical coronae. However, unlike spherical coronae, as the size increases the spectrum also hardens. A thorough study of coronal emission will be carried out in a future work.

We discuss the implication of anisotropic coronal emission for modeling and interpreting the illuminating spectrum. For AGNs this would lead to an inaccurate estimate of the spectral shape of the illuminating coronal emission as well as an underestimate of the flux of the illuminating radiation that is accounting for the soft excess. For BHXRBs, this would also lead to an underestimated reflection fraction and iron K flux.

We thank the anonymous referee for his/her careful reading of the manuscript and useful comments and suggestions. The authors thank Giorgio Matt for valuable comments, and Iossif Papadakis and Jason Dexter for useful discussion. The authors acknowledge financial support provided by Czech Science Foundation grant 17-02430S. This work is also supported by the project RVO:67985815. This research makes use of MATPLOTLIB (Hunter 2007), a Python 2D plotting library that produces publication quality figures.

### Appendix

#### Tetrads

The method for obtaining an orthonormal tetrad attached to an observer with arbitrary four-velocity can be found in Sadowski et al. (2011). In the following we give the expressions for the tetrads in two specific cases: (1) the tetrad attached to a ZAMO observer; (2) the tetrad attached to an observer moving in the azimuthal direction.

##### A.1. ZAMO Tetrad

The four-velocity of a ZAMO observer is

\[ U = \left[ U^t, 0, 0, -\frac{g_{\phi \phi}}{g_{\theta \theta}} U^\phi \right], \]

where

\[ U^t = \sqrt{\frac{g_{\phi \phi}}{g_{\theta \theta}} - g_{\theta \theta} g_{\phi \phi}}. \]

The orthonormal tetrad attached to a ZAMO observer is

\[ e_{(t)} = U, \]

\[ e_{(r)} = \left[ 0, 1, 0, 0 \right], \]

\[ e_{(\theta)} = \left[ 0, 0, -\frac{1}{\sqrt{g_{\theta \theta}}}, 0 \right], \]

\[ e_{(\phi)} = \left[ 0, 0, 0, \frac{1}{\sqrt{g_{\phi \phi}}} \right]. \]

##### A.2. Azimuthal Tetrad

For an observer moving in the azimuthal direction with an angular velocity \( \Omega \equiv d\phi/dt \), its four-velocity is

\[ U = [U^t, 0, 0, \Omega U^\phi], \]

where

\[ U^t = \sqrt{\frac{1}{g_{\theta \theta} + 2\Omega g_{\phi \phi} + \Omega^2 g_{\phi \phi}}}. \]

Denoting

\[ k_1 = g_{\phi \phi} U^t + g_{\phi \phi} U^\phi, \]

\[ k_2 = g_{\phi \phi} U^t + g_{\phi \phi} U^\phi, \]

then the time component of \( e_{(\phi)} \) can be written as

\[ e_{(t)} = \frac{-\text{sign}(k_1)k_2}{\sqrt{(g_{\theta \theta} g_{\phi \phi} - g_{\theta \theta} g_{\phi \phi})^2 + g_{\theta \theta} g_{\phi \phi} (U^\phi)^2 + 2g_{\theta \theta} U^t U^\phi}}, \]

and the tetrad is

\[ e_{(t)} = U, \]
\[ e_{(r)} = \begin{bmatrix} 0, -\frac{1}{\sqrt{6_\theta}}, 0, 0 \end{bmatrix}, \quad (59) \]

\[ e_{(\theta)} = \begin{bmatrix} 0, 0, -\frac{1}{\sqrt{6_\theta}}, 0 \end{bmatrix}, \quad (60) \]

\[ e_{(\phi)} = \begin{bmatrix} e'_{(\phi)}, 0, 0, -\frac{k_1}{k_2} e'_{(\phi)} \end{bmatrix}. \quad (61) \]

**References**

Beheshtipour, B., Krawczynski, H., & Malzac, J. 2017, *ApJ*, 850, 14

Berestetskii, V. B., Lifshitz, E. M., & Pitaevskii, V. B. 1971, Relativistic Quantum Theory (Oxford: Pergamon)

Bosma, P. B., & de Rooij, W. A. 1983, *A&A*, 126, 283

Bursa, M. 2017, in Proc. RAGtime 17–19: Workshops on Black Holes and Neutron Stars, ed. Z. Stuchlík, G. Török, & V. Karas (Opava: Silesian University in Opava), 7

Canfield, E., Howard, W. M., & Liang, E. P. 1987, *ApJ*, 323, 565

Carter, B. 1968, *PhRv*, 174, 1559

Chandrasekhar, S. 1960, *Radiative Transfer* (New York: Dover)

Chandrasekhar, S. 1983, The Mathematical Theory of Black Holes (New York: Oxford Univ. Press)

Connors, P. A., Stark, R. F., & Piran, T. 1980, *ApJ*, 235, 224

Crummy, J., Fabian, A. C., Gallo, L., & Ross, R. R. 2006, *MNRAS*, 365, 1067

Dauser, T., García, J., Parker, M. L., Fabian, A. C., & Wilms, J. 2014, *MNRAS*, 444, L100

Dexter, J., & Agol, E. 2009, *ApJ*, 696, 1616

Dolence, J. C., Gammie, C. F., Moscibrodzka, M., & Leung, P. K. 2009, *ApJS*, 184, 387

Done, C., Davis, S. W., Jin, C., Blaes, O., & Ward, M. 2012, *MNRAS*, 420, 1848

Dovčiak, M., & Done, C. 2016, *AN*, 337, 441

Dovčiak, M., Muleri, F., Goosmann, R. W., Karas, V., & Matt, G. 2008, *MNRAS*, 391, 32

Fabian, A. C., Lohfink, A., Belmont, R., Malzac, J., & Coppi, P. 2017, *MNRAS*, 467, 2566

Fabian, A. C., Lohfink, A., Kara, E., et al. 2015, *MNRAS*, 451, 4375

Fabian, A. C., & Ross, R. R. 2010, *SSRv*, 157, 167

George, I. M., & Fabian, A. C. 1991, *MNRAS*, 249, 352

Ghisellini, G., George, I. M., Fabian, A. C., & Done, C. 1991, *MNRAS*, 248, 14

Ghisellini, G., Haardt, F., & Matt, G. 2004, *A&A*, 413, 535

Goosmann, R. W., & Gaskell, C. M. 2007, *A&A*, 465, 129

Guilbert, P. W., Fabian, A. C., & Rees, M. J. 1983, *MNRAS*, 205, 593

Haardt, F. 1993, *ApJ*, 413, 680

Haardt, F., & Maraschi, L. 1991, *ApJL*, 380, L51

Henri, G., & Petrucci, P. O. 1997, *A&A*, 326, 87

Hunter, J. D. 2007, *CSE*, 9, 90

Kahn, H. 1954, Applications of Monte Carlo AECU-3259 (Santa Monica, CA: RAND Corporation)

Kochanek, C. S. 2004, *ApJ*, 605, 58

Kochanek, C. S., Dai, X., Morgan, C., Morgan, N., & Pounds, G. S. C. 2007, Statistical Challenges in Modern Astronomy IV, Vol. 371 (San Francisco, CA: ASP), 43

Kulkarni, A. K., Penna, R. F., Shcherbakov, R. V., et al. 2011, *MNRAS*, 414, 1183

Li, L.-X., Narayan, R., & McClintock, J. E. 2009, *ApJ*, 691, 847

Li, L.-X., Zimmerman, E. R., Narayan, R., & McClintock, J. E. 2005, *ApJS*, 157, 335

Malzac, J., Jourdain, E., Petrucci, P. O., & Henri, G. 1998, *A&A*, 336, 807

Miniutti, G., & Fabian, A. C. 2004, *MNRAS*, 349, 1435

Misner, C. W., Thorne, K. S., & Wheeler, J. A. 1973, Gravitation (San Francisco: Freeman)

Petrucci, P. O., & Henri, G. 1997, *A&A*, 326, 99

Pozdnyakov, L. A., Sobol, I. M., & Sunyaev, R. A. 1983, *ASPv*, 2, 189

Rauch, K. P., & Blandford, R. D. 1994, *ApJ*, 421, 46

Reis, R. C., & Miller, J. M. 2013, *ApJL*, 769, L7

Ross, R. R., Fabian, A. C., & Mineshige, S. 1992, *MNRAS*, 258, 189

Schnittman, J. D., & Krolik, J. H. 2010, *ApJ*, 712, 908

Schnittman, J. D., & Krolik, J. H. 2013, *ApJ*, 777, 11

Sadowski, A., Bursa, M., Abramowicz, M., et al. 2011, *A&A*, 532, A41

Sunyaev, R. A., & Titarchuk, L. G. 1980, *A&A*, 86, 121

Tamborra, F., Matt, G., Bianchi, S., & Dovčiak, M. 2018, *A&A*, 619, A105

Tortosa, A., Bianchi, S., Marinucci, A., Matt, G., & Petrucci, P. O. 2018, *A&A*, 614, A37

Turner, T. J., & Pounds, K. A. 1988, *MNRAS*, 232, 463

Walker, M., & Penrose, R. 1970, *CMaPh*, 18, 265

Wambsganss, J. 2006, Saas-Fee Advanced Course 33: Gravitational Lensing: Strong, Weak and Micro (Berlin: Springer), 453

Wilkins, D. R., & Fabian, A. C. 2012, *MNRAS*, 424, 1284

Younsi, Z., Wu, K., & Fuerst, S. V. 2012, *A&A*, 545, A13

Zdziarski, A. A., Johnson, W. N., & Magdziarz, P. 1996, *MNRAS*, 283, 193

Zdziarski, A. A., & Lightman, A. P. 1985, *ApJL*, 294, L79

Życki, P. T., Done, C., & Smith, D. A. 1999, *MNRAS*, 300, 561

The Astrophysical Journal, 875:148 (13pp), 2019 April 20

Zhang, Dovčiak, & Bursa

\[ e_{(r)} = \begin{bmatrix} 0, -\frac{1}{\sqrt{6_\theta}}, 0, 0 \end{bmatrix}, \quad (59) \]

\[ e_{(\theta)} = \begin{bmatrix} 0, 0, -\frac{1}{\sqrt{6_\theta}}, 0 \end{bmatrix}, \quad (60) \]

\[ e_{(\phi)} = \begin{bmatrix} e'_{(\phi)}, 0, 0, -\frac{k_1}{k_2} e'_{(\phi)} \end{bmatrix}. \quad (61) \]