Extremal Myers-Perry black holes coupled to Born-Infeld electrodynamics in odd dimensions

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Abstract

Employing higher order perturbation theory, we find a new class of perturbative extremal rotating black hole solutions with Born-Infeld electric charge in odd $D$-dimensional spacetime. The seed solution is an odd dimensional extremal Myers-Perry black hole with equal angular momenta to which a perturbative nonlinear electric Born-Infeld field charge $q$ is added maintaining the extremality condition. The perturbations are performed up to third-order. We also study some physical properties of these black holes. In particular, it is shown that the values of the gyromagnetic ratio of the black holes are modified by the perturbative parameter $q$ and the Born-Infeld parameter $\beta$.

I. INTRODUCTION

Finding rotating black holes solutions in higher dimensions is a difficult task due to the size and complexity of the equations. For this reason the number of analytic solutions in closed form is very limited. An exception is pure general relativity, where there are exact $D$-dimensional rotating black holes. Indeed, the generalization of the Kerr metric to higher $D$ dimensions was performed by Myers and Perry \cite{1}. These Myers-Perry black holes in general possess $N = [(D-1)/2]$ independent
angular momenta as stated in [1], where \([X]\) denotes the integer part of \(X\). This, in turn, implies, that \(D\)-dimensional rotating black hole solutions fall into two classes, namely, even-\(D\) and odd-\(D\). The insertion of a cosmological constant also yields an exact rotating solution in \(D\)-dimensions [2].

The inclusion of any other charge into the solutions, be it electric, magnetic, or dilatonic, to name a few, obliges the introduction of alternative techniques to study rotating black holes in higher dimensions. There are then two opposed situations that can be dealt with some ease: slow rotation and rotation in the extremal regime. For slow rotation, charged black holes with a single rotation parameter in higher dimensions have been studied perturbatively in [3–7], and numerically in [8] for asymptotically flat solutions and in [9] for asymptotically anti-de Sitter solutions. The incorporation of a dilaton coupling in this slow rotation regime in four dimensions has been done in [10] and it is also possible to find the corresponding higher-dimensional solutions.

For the rotating solutions in the extremal regime there are perturbative methods that work in odd \(D\)-dimensions when one includes some type of charge. The correctness of these perturbative solutions can be performed by making use of a Smarr-type formula for black holes in \(D\) dimensions [11]. Employing perturbation theory with the electric charge as the perturbation parameter, charged rotating Einstein-Maxwell black holes have been constructed in five dimensions [12]. This perturbative method was also applied to obtain extremal Einstein-Maxwell black holes with equal magnitude angular momenta in odd dimensions [13]. These solutions have then been generalized by including a scalar dilaton field [14, 15]. One can also deal with Born-Infeld electric charge, indeed, extremal rotating Einstein-Born-Infeld black holes in five-dimensional spacetime have been studied [16].

In this paper, we want to generalize the five-dimensional studied performed in [16] to any odd \(D\) dimension, \(5 \leq D < \infty\). The odd-dimensional case, in opposition to the even-dimensional case, can be treated explicitly perturbatively when the \(N\) angular momenta of the black hole have all equal magnitude, as the resulting system of field equations simplifies remarkably. Using a prescribed perturbative method we are able to find extremal rotating Einstein-Born-Infeld black holes. We start from the extremal Myers-Perry black holes with equal \(N = [(D - 1)/2]\) angular momenta [1], then we evaluate the perturbative series up to third order in the electric charge parameter \(q\), and finally we study the physical properties of these black holes. In particular, we analyze the effects of the perturbative parameter \(q\) and the Born-Infeld parameter \(\beta\) on the gyromagnetic ratio of these fast rotating black holes. The structure of this paper is the following: In section II the field equations of the nonlinear Born-Infeld theory in Einstein gravity are displayed and a new class of perturbative charged rotating solutions in odd dimensions is obtained. In section III the
physical quantities of the solutions are calculated and their properties discussed. In Sec. IV the mass formula for these black holes is presented. In section V we draw some conclusions. The formulas for the metric and the gauge potential in $D$ dimensions are given in the Appendix.

II. METRIC AND BASIC EQUATIONS

Our departure point is the Einstein-Hilbert action coupled to the Born-Infeld nonlinear gauge field in $D$ dimensions

$$S = \int dx^D \sqrt{-g} \left( \frac{R}{16\pi G_D} + L(F) \right),$$

where $G_D$ is the Newton constant in $D$ dimensions, $g$ is the determinant of the $D$-dimensional metric $g_{\mu\nu}$, $R$ is the Ricci curvature scalar. $L(F)$ is the Lagrangian of the nonlinear Born-Infeld gauge field given by

$$L(F) = 4\beta^2 \left( 1 - \sqrt{1 + \frac{F^2}{2\beta^2}} \right),$$

where $F = F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor, $A_\mu$ is the electromagnetic vector potential, and $\beta$ is the Born-Infeld parameter. In the limit $\beta \to \infty$, $L(F)$ reduces to the Lagrangian of the standard Maxwell field, $L(F) = F$. By varying the action with respect to the gravitational field $g_{\mu\nu}$ and the gauge field $A_\mu$ one obtains the equations of motion for these fields. For the gravitational field the equations are

$$G_{\mu\nu} = \frac{1}{2}g_{\mu\nu}L(F) + \frac{2F_{\mu\eta}F_{\nu}^{\eta}}{\sqrt{1 + \frac{F^2}{2\beta^2}},$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor formed out from the Ricci tensor $R_{\mu\nu}$ and scalar $R$. For the Born-Infeld electromagnetic field one finds the following equations

$$\partial_\mu \left( \sqrt{-g}F^{\mu\nu} \right) = 0.$$
\[
\begin{align*}
\frac{ds^2}{W} = & g_{tt} dt^2 + \frac{dr^2}{W} + r^2 \left[ \sum_{i=1}^{N-1} \left( \prod_{j=0}^{i-1} \cos^2 \theta_j \right) d\theta_i^2 + \sum_{k=1}^{N} \left( \prod_{l=0}^{k-1} \cos^2 \theta_l \right) \sin^2 \theta_k d\varphi_k^2 \right] \\
+ & V \left[ \sum_{k=1}^{N} \left( \prod_{l=0}^{k-1} \cos^2 \theta_l \right) \sin^2 \theta_k \varepsilon_k d\varphi_k \right]^2 - 2B \sum_{k=1}^{N} \left( \prod_{l=0}^{k-1} \cos^2 \theta_l \right) \sin^2 \theta_k \varepsilon_k d\varphi_k dt,
\end{align*}
\]

where \( \theta_0 \equiv 0, \theta_i \in [0, \pi/2] \) for \( i = 1, \ldots, N - 1 \), \( \theta_N \equiv \pi/2 \) for \( k = 1, \ldots, N \), and \( \varepsilon_k = \pm 1 \) denotes the sense of rotation in the \( k \)-th orthogonal plane of rotation, and the metric functions \( g_{tt} \), \( W \), \( V \), and \( B \) depend only on the radial coordinate \( r \). An adequate parametrization for the gauge potential is given by

\[
A_{\mu} dx^\mu = a_t dt + a_\varphi \sum_{k=1}^{N} \left( \prod_{l=0}^{k-1} \cos^2 \theta_l \right) \sin^2 \theta_k \varepsilon_k d\varphi_k,
\]

where the gauge potential functions \( a_t \) and \( a_\varphi \) depend only on the radial coordinate \( r \).

We now consider perturbations around the Myers-Perry solution, with a Born-Infeld electric charge \( q \) as the perturbative parameter. Taking into account the seed solution and the symmetry with respect to charge reversal, the functions for the metric and gauge potential take the form

\[
g_{tt} = -1 + \frac{2\hat{M}}{r^{D-3}} + q^2 g_{tt}^{(2)} + O(q^4),
\]

\[
W = 1 - \frac{2\hat{M}}{r^{D-3}} + \frac{2\hat{J}^2}{Mr^{D-1}} + q^2 W^{(2)} + O(q^4),
\]

\[
V = \frac{2\hat{J}^2}{Mr^{D-3}} + q^2 V^{(2)} + O(q^4),
\]

\[
B = \frac{2\hat{J}}{r^{D-3}} + q^2 B^{(2)} + O(q^4),
\]

\[
a_t = qa_t^{(1)} + q^3 a_t^{(3)} + O(q^5),
\]

\[
a_\varphi = qa_\varphi^{(1)} + q^3 a_\varphi^{(3)} + O(q^5),
\]

Here \( g_{tt}^{(2)} \) is a second-order perturbative term with the other perturbative terms defined similarly. The quantities \( \hat{M} \) and \( \hat{J} \) are integration constants related to the mass and angular momentum, respectively.
It is important to be able to fix the integration constants. First one fixes the angular momenta at any perturbative order and then imposes the extremality condition in all orders. We also assume asymptotic flatness and regularity at the horizon. With these assumptions we are able to fix the constants of integration.

We now introduce a parameter $\nu$ for the extremal Myers-Perry solutions in $D$ dimensions by

$$\hat{M} = \frac{(D-1)^{D-1}}{4(D-3)^{D-3}} \nu^{D-3}, \quad \hat{J} = \frac{(D-1)^{D-1}}{4(D-3)^{D-3}} \nu^{D-2}. \quad (13)$$

Inserting the metric, Eq. (5), and the gauge potential, Eq. (6), together with the perturbation expansions, Eqs. (7)-(12), into the field equations Eqs. (3)-(4), we can solve these equations order by order. The perturbative expansions for the metric and the gauge potential functions are exhibited in Appendix A for generic values of the Born-Infeld parameter $\beta$. One may note that in the Maxwell limit, $\beta \to \infty$, these perturbative solutions reduce to the odd dimensional perturbative charged rotating black holes in Einstein-Maxwell theory presented in [13] and the accuracy of these solutions can be checked by using Smarr’s formula [11]. Also, for the case $D = 5$ one finds the equations found in [16].

III. PHYSICAL QUANTITIES

Writing the asymptotic behavior of the metric and the gauge potential one can extract the mass $M$, the equal-magnitude angular momenta $|J_i| = J$, the electric charge $Q$, and the magnetic moments $\mu_{\text{mag}}$. Indeed, one can set the asymptotic behavior as [13–16]

$$g_{tt} = -1 + \frac{\hat{M}}{r^{D-3}} + \ldots, \quad B = \frac{2\hat{J}}{r^{D-3}} + \ldots, \quad a_t = \frac{\hat{Q}}{r^{D-3}} + \ldots, \quad a_{\varphi} = \frac{\hat{\mu}_{\text{mag}}}{r^{D-3}} + \ldots, \quad (14)$$

where $\hat{M}$, $\hat{J}$, $\hat{Q}$, and $\hat{\mu}_{\text{mag}}$ are parameters with dimensions of mass, angular momentum, charge, and magnetic momentum, respectively, and we have put $\hat{Q} = q$. These parameters are related to the mass $M$, angular momentum $J$, electric charge $Q$, and magnetic momentum $\mu_{\text{mag}}$ through

$$\hat{M} = \frac{16\pi G_D}{(D-2)A(S^{D-2})} M, \quad \hat{J} = \frac{4\pi G_D}{A(S^{D-2})} J, \quad \hat{Q} = \frac{4\pi G_D}{(D-3)A(S^{D-2})} Q, \quad \hat{\mu}_{\text{mag}} = \frac{4\pi G_D}{(D-3)A(S^{D-2})} \mu_{\text{mag}}, \quad (15)$$
and where $A(S^{D-2})$ is the area of the unit $(D - 2)$-sphere. Comparing the above expansions to the asymptotic behavior of the solution, Eqs. (A1)-(A6), we obtain

$$M = \frac{A(S^{D-2})}{64\pi G_D} \left[ \frac{2\nu^{2(D-3)}(D - 2)(D - 1)^{D-1} + 16q^2(D - 3)^{D-2}}{\nu^{D-3}(D - 1)^{D-2} - (D - 3)^{D-3}} \right] + O(q^4), \quad (16)$$

$$Q = \frac{A(S^{D-2})(D - 3)}{4\pi G_D} q, \quad (17)$$

$$J = \frac{A(S^{D-2})\nu^{D-2}(D - 1)^{D-2}}{16\pi G_D(D - 3)^{D-3}} , \quad (18)$$

$$\mu_{\text{mag}} = \frac{A(S^{D-2})(D - 3)}{4\pi G_D} \left\{ q\nu - \frac{4q^3(D - 3)^{D-3}(D^3 - 5D^2 + 7D - 3)}{\nu^{2D-7}(D - 2)^2(D - 1)^D}\right. \right.
- \frac{4q^3(D - 3)^{D+1}(D - 2)}{3(3D - 5)(3D - 7)(D - 1)^{D-1}\beta^2\nu^{2D-4}} \left. + O(q^5) \right\}. \quad (19)$$

The gyromagnetic ratio $g$ is given by

$$g = \frac{2M\mu_{\text{mag}}}{QJ} = (D - 2) - \frac{4q^2[(D^2 - 4D + 3)(D - 3)^{D-3} - 2(D - 2)(D - 3)^{D-2}]}{(D - 2)(D - 1)^{D-1}\nu^{2(D-3)}}
- \frac{4q^2(D - 3)^{D+1}(D - 2)}{3(3D - 5)(3D - 7)(D - 1)^{D-1}\beta^2\nu^{2D-4}} + O(q^4). \quad (20)$$

We show the behavior of the magnetic moment $\mu_{\text{mag}}$ and the gyromagnetic ratio $g$ of the perturbative extremal charged rotating black holes in the Einstein-Born-Infeld theory versus $\beta$ in Fig. 1 and Fig. 2 respectively. From these figures we find out that the magnetic moment $\mu_{\text{mag}}$ and the gyromagnetic ratio $g$ increase with increasing $\beta$ in any odd dimension. One can also see that for some small value of $\beta$ the gyromagnetic ratio and magnetic moment $\mu_{\text{mag}}$ are zero and then become negative. We speculate that this change of sign comes from significantly different distributions of charge for high and low values of the Born-Infeld parameter $\beta$. The interested reader can see [15] for a detailed discussion about a possible interpretation of this sign reversal. In the Maxwell’s limit, $\beta \rightarrow \infty$, the magnetic moment and the gyromagnetic ratio reduce to

$$\mu_{\text{mag}} = \frac{A(S^{D-2})(D - 3)}{4\pi G_D} \left\{ q\nu - \frac{4q^3(D - 3)^{D-3}(D^3 - 5D^2 + 7D - 3)}{\nu^{2D-7}(D - 2)^2(D - 1)^D}\right. \right.
- \frac{4q^3(D - 3)^{D+1}(D - 2)}{3(3D - 5)(3D - 7)(D - 1)^{D-1}\beta^2\nu^{2D-4}} \left. + O(q^5) \right\}. \quad (21)$$

$$g = (D - 2) - \frac{4q^2[(D^2 - 4D + 3)(D - 3)^{D-3} - 2(D - 2)(D - 3)^{D-2}]}{(D - 2)(D - 1)^{D-1}\nu^{2(D-3)}} + O(q^4). \quad (22)$$
FIG. 1: The third-order perturbative values for the magnetic moment $\mu_{\text{mag}}$ versus Born-Infeld parameter $\beta$ for the perturbative extremal charged rotating black holes in the Einstein-Born-Infeld theory in 5, 7 and 9 dimensions with $\nu = 1.16$ and $q = 0.09$.

FIG. 2: The third-order perturbative values for the gyromagnetic ratio $g$ versus Born-Infeld parameter $\beta$ for the perturbative extremal charged rotating black holes in the Einstein-Born-Infeld theory in 5, 7 and 9 dimensions with $\nu = 1.16$ and $q = 0.09$.

which are exactly the expressions obtained for the odd dimensional perturbative charged rotating black holes in the Einstein-Maxwell theory [13]. The event horizon of these rotating Born-Infeld black holes is located at

$$r_H = \sqrt{\frac{D-1}{D-3} \nu + \frac{4(D-3)^{D-\frac{5}{2}}}{(D-2)(D-1)^{D-\frac{3}{2}} \nu^{2D-7}}} q^2 + O(q^4).$$

(23)

Its value does not depend on $\beta$ up to this order.

IV. THE MASS FORMULA

Now, the constant horizon angular velocities $|\Omega_i| = \Omega$ can be defined by imposing that the Killing vector field

$$\chi = \xi + \Omega \sum_{k=1}^{N} \epsilon_k \eta_k,$$

(24)
is null on and orthogonal to the horizon, with \( \chi \) defined as \( \chi = \partial_t \) and \( \eta_k = \partial_{\phi_k} \). The horizon electrostatic potential \( \Psi_H \) of these black holes is then given by

\[
\Psi_H = (a_t + \Omega a_\varphi)|_{r=r_H},
\]

and the surface gravity \( \kappa \) is defined by

\[
\kappa^2 = -\frac{1}{2} \left( \nabla_\mu \chi_\nu \right) \left( \nabla^\mu \chi^\nu \right) |_{r=r_H}.
\]

Finally, the horizon angular velocities \( |\Omega_i| = \Omega \) and the horizon area \( A_H \) are then given by

\[
\begin{align*}
\Omega &= \frac{D - 3}{\nu(D-1)} - \frac{8q^2(D-3)^{D-1}}{\nu^{2D-5}(D-2)(D-1)^D} + O(q^4), \\
A_H &= \frac{\sqrt{2} \, A(S^{D-2})(D-1)^{\frac{(D-1)}{2}}}{2(D-3)^{\frac{(D-2)}{2}}} + O(q^4),
\end{align*}
\]

respectively. Taking into account the quantities previously defined, it is straightforward to see that these black holes satisfy the Smarr mass formula up to third-order \[11\], namely

\[
\frac{D-3}{D-2}M = N \Omega J + \frac{D-3}{D-2} \Psi_H Q.
\]

Noting that the surface gravity \( \kappa \) vanishes for extremal solutions, one has in our odd-dimensional case

\[
\frac{D-3}{D-2}M = N \Omega J + \frac{D-3}{D-2} \Psi_H Q.
\]

For \( D = 5 \) the result in \[16\] is recovered.

**V. CONCLUSIONS**

We have constructed a new class of perturbative charged rotating black hole solutions in higher odd dimensions in the presence of a nonlinear Born-Infeld gauge field. We have restricted to the case of extremal black holes with equal angular momenta. These solutions are asymptotically flat and their horizons have spherical topology. Our strategy for obtaining these solutions was through a perturbative method up to the third-order for the perturbative charge parameter \( q \). We have started from rotating Myers-Perry black hole solutions in odd dimensions, and then investigated the effects of adding a charge parameter \( q \) and the Born-Infeld parameter \( \beta \) to the solutions. We have calculated the mass, angular momentum, electric charge, magnetic moment, gyromagnetic ratio, and horizon radius of these Born-Infeld black holes. For large \( \beta \) the solutions reduce to the perturbative rotating Einstein-Maxwell solutions in odd dimensions \[13\], as expected. Recently, it was shown that in five dimensions the Born-Infeld parameter \( \beta \) may modify the value of the gyromagnetic ratio relative to the corresponding Einstein-Maxwell rotating black holes \[16\]. Here, we obtain a similar result for odd-dimensional black holes.
Appendix A

We give the perturbative expressions for the metric and the gauge potential in the Einstein-Born-Infeld theory for general odd $D$. The solutions up to third-order read

\[
g_{tt} = -1 + \frac{(D - 1) \frac{D-1}{2} \nu^{D-3}}{2(D-3) \frac{D-3}{2} r^{D-3}} - q^2 \left[ \frac{2(D - 3)}{(D - 2)r^{2(D-3)}} - \frac{4(D - 3) \frac{D-1}{2}}{(D - 2)(r^{(D-3)}(D-1) \frac{D-1}{2})} \right] + O(q^4), \tag{A1}
\]

\[
W = 1 - \frac{(D - 1) \frac{D-1}{2} \nu^{D-3}}{2(D-3) \frac{D-3}{2} r^{D-3}} + \frac{(D - 1) \frac{D-1}{2} \nu^{D-1}}{2(D-3) \frac{D-3}{2} r^{D-1}} - \frac{2q^2}{(D - 2)} \left\{ \frac{(D - 3) \frac{D-1}{2}}{(r^{(D-3)}(D-1) \frac{D-1}{2})} \right\} + O(q^4), \tag{A2}
\]

\[
V = \frac{(D - 1) \frac{D-1}{2} \nu^{D-1}}{2(D-3) \frac{D-3}{2} r^{D-3}} + \frac{4(D - 3) \frac{D-1}{2}}{(D - 2)(D - 1) \nu^{D-5} r^{D-3}} \]

\[
B = \frac{(D - 1) \frac{D-1}{2} \nu^{D-2}}{2(D-3) \frac{D-3}{2} r^{D-3}} - \frac{2\nu(D - 3)q^2}{(D - 2)r^{2(D-3)}} + O(q^4), \tag{A4}
\]

\[
a_t = \frac{q}{r^{D-3}} + q^3 \left\{ \int \int S_1 \left( \frac{D - 3}{2} r^2 - \left( \frac{D - 1}{2} \right) \nu^2 \right)^2 S_1 \int \frac{r^{D-2}}{S_1 \left( \frac{(D - 3)}{2} r^2 - \left( \frac{D - 1}{2} \right) \nu^2 \right)^2} dr \right\} + O(q^5), \tag{A5}
\]

\[
a_\varphi = -\frac{\nu q}{r^{D-3}} + q^3 r^2 \left\{ \int \int - \left( \frac{D - 1}{2} \right) \left( \frac{D - 3}{2} \right) \frac{D-5}{2} S_5 r^{3D-3} \nu^{D-3} \int S_1 S_5 \int \frac{r^{D-2}}{S_1 \left( \frac{(D - 3)}{2} r^2 - \left( \frac{D - 1}{2} \right) \nu^2 \right)^4} dr \right\} + O(q^5), \tag{A6}
\]
where in the above equations $S_1$, $S_2$, $S_3$, $S_4$, and $S_5$ are

$$S_1 = \left( -(D - 2) (D - 3) \frac{D+1}{2} 2^{3-D} \frac{D+1}{D-2} \nu^{D-3} - (D - 3) \frac{D+1}{2} 2^{3-D} \beta^2 (D - 1) \nu^{D-1} \right) r^{D+1} + (D - 1)^3 (D - 3) \frac{D+1}{2} 2^{3-D} r^{D-1} \nu^{D-1} + 3 (D - 2) (D - 3)^4 (D - 1) \frac{D+1}{2} 2^{3-D} r^4 \nu^2 D^{-6} - \left( (D - 2)^2 + \frac{D - 1}{2} \right) (3 D - 7) (D - 3)^2 (D - 1) \frac{D+1}{2} 2^{3-D} \nu^2 D^{-4} + (3 D - 5) (D - 3)^2 (D - 1) \frac{D+1}{2} 2^{3-D} \nu^2 D^{-2}, \quad (A7)$$

$$S_2 = \left( \frac{(D - 3) \frac{3D-7}{2} \nu^2 \beta^2}{(D - 1) \frac{D+2}{2} 2^{D+2} (D - 2)^2} - \frac{16}{3} \frac{(D - 3) \frac{1+3D}{2}}{(3 D - 7) (D - 1) \frac{D+1}{2} (3 D - 5)} \right) r^{3D-3} - 16 (D - 3)^{D-2} 2^{D-2} \left( \frac{1}{8} (D - 3)^{2} (D - 2) \nu^{D-3} + \frac{\beta^2 (D - 1) \nu^{D-1}}{D - 2} \right) r^{2D} + 8 (D - 1) \frac{D+1}{2} 2^{1-D} (D - 3) \frac{D+1}{2} \left( \frac{1}{2} (D - 2) (D - 3)^2 \nu^2 D^{-6} + \frac{\beta^2 (D - 1) \nu^2 D^{-4}}{D - 2} \right) r^{D+3} - 32 (D - 1) \frac{D+1}{2} 2^{D-1} (D - 3) \frac{D+1}{2} \left( \frac{17}{4} - \frac{15}{4} (D + D^2) \right) \nu^2 D^{-4} + \frac{\beta^2 (D - 1) \nu^2 D^{-2}}{(D - 2)^2} r^{D+1} + \frac{8}{2^{D+1}} (2 D - 3) (D - 3) \frac{D+1}{2} (D - 1) \frac{D+1}{2} r^{D-1} \nu^2 D^{-2} - (D - 2) (D - 3)^3 \left( \frac{D - 1}{2} \right)^{D-1} \nu^2 D^{-9} + 4 (D - 3)^2 \left( \frac{D - 1}{2} \right)^{D-1} \left( \frac{103}{4} + \frac{133}{4} (D - 5) + \frac{9}{4} (D^3) \right) r^4 \nu^3 D^{-7} (3 D - 7)^{-1} + 8 \left( \frac{27}{8} - \frac{7}{2} (D + \frac{9}{8} D^2) \right) (D - 3)^2 (D - 2) (D - 1)^{D-1} 2^{1-D} r^2 \nu^3 D^{-5} (3 D - 5)^{-1} + \frac{2}{3} (D - 3)^2 (D - 5) (D - 1)^{D-2} \nu^3 D^{-3} + 4 (D - 1)^{D-2} (D - 3)^{D-1} 2^{1-D} r^2 \nu^2 D^{-1}, \quad (A8)$$

$$S_3 = -r^{D+1} \nu^2 D^{-6} (D - 2) (D - 1) \frac{D+1}{2} (D - 3)^{D-1} \frac{D+1}{2} \nu^2 D^7 \frac{3}{3 D - 5} + \frac{(D - 2) (D - 1) \frac{D+1}{2} (D - 3)^{D-1} \frac{D+1}{2} \nu^2 D^9}{3 D - 7} + r^{3D-3} (D - 3) \frac{D+1}{2} (D - 1) \frac{D+1}{2} \nu^2 D^{-9} + (D - 3) (D - 1) r^{D-1} \nu^2 D^{-4} - 9 r^{D+1} \nu^2 D^{-6} + 6 r^{D+1} \nu^2 D^{-6} (D) - 4 \frac{r^{D+1} \nu^2 D^{-4} D^2}{(D - 2)^2 (D - 3)} + 16 \frac{r^{D+1} \nu^2 D^{-4} (D)}{(D - 2)^2 (D - 3)} - 8 \frac{r^{3D-3} (D - 3) \nu^2 D^{-3} \beta^2}{(D - 1)^{D-1} (D - 2)} + 4 \frac{r^{2D-2} (D - 3)^{D-1} \nu^2 D^{-1}}{(D - 1)^{D-5} (D - 2)} - 4 \frac{r^{3D-3} (D - 3) \nu^2 D^{-3} \beta^2 (D - 1)^{-D}}{(D - 2)^2} - \frac{27}{2} r^2 D (D - 1)^{D-1} \nu^4 D^{-12} \beta^2 (D - 3)^{-D} (D) + \frac{28}{3} \frac{r^{3D-3} (D - 3) \nu^2 D^{-12} \beta^2 (D - 3)^{-D}}{(D - 1)^D (D - 2)} + \frac{27}{2} r^2 D (D - 1)^{D-1} \nu^4 D^{-12} \beta^2 (D - 3)^{-D} + \frac{9}{2} r^2 D (D - 1)^{D-1} \nu^4 D^{-12} \beta^2 (D - 3)^{-D} - 24 \frac{r^{3D-3} (D - 3) \nu^2 D^{-3} \beta^2 D^2}{(D - 1)^D (D - 2)}.$
\[ + \frac{27}{2} r^{2D-2} (D-1)^{D-1} \nu^{A-D-10} \beta^2 (D-3)^{-D} (D) + 16 \frac{r^{3D-3} (D-3)^{D-3} \beta^2 D^3}{(D-1)^D (D-2)^2} \]
\[ + \frac{1}{3} \frac{(D-1)^{D-1} \nu^{A-D-5}}{(D-3)^{D-5}} - \frac{12 \nu^{D+1} \beta^2 \nu^{D-4}}{(D-2)^2 (D-3)} - \frac{9 \nu^{2D-2} (D-1)^{D-1} \nu^{A-D-10} \beta^2 D^2}{(D-3)^D} \]
\[ + \frac{1}{2} r^{2D-2} (D-1)^{D-1} \nu^{A-D-10} \beta^2 (D-3)^{-D} D^3 - \frac{4 \nu^{2D} (D-1)^{D-3} \beta^2 \nu^{D-3} D^2}{(D-1)^D (D-2)^2} \]
\[ - \frac{1}{2} r^{2D} (D-1)^{D-1} \nu^{A-D-12} \beta^2 (D-3)^{-D} D^3 + \frac{16 \nu^{3D-3} (D-3)^{D-3} \beta^2 (D)}{(D-1)^D (D-2)^2} \]
\[ + 24 \frac{r^{3D-3} (D-3)^{D-3} \beta^2 (D-1)^{-D} (D)}{(D-2)^2} + \frac{4 \nu^{3D-3} (D-3)^{D} (D-1)^{-D} 2D D^3}{(D-2)^2} \]
\[ - 12 \frac{r^{2D} (D-3)^{D-3} (D-1)^{-D} \nu^{D-3} (D)}{(D-2)^2} + \frac{16 \nu^{2D} (D-3)^{D-3} \beta^2 \nu^{D-3} (D)}{(D-1)^D (D-2)^2} \]
\[ - \frac{4}{(D-1)^D} \nu^2 (3 D - 7) (3 D - 5) \]
\[ + \frac{8 \nu^{3D-3} (D-3)^{D-3} \beta^2 D^3}{(D-1)^D (D-2)} - \frac{27}{2} r^{2D-2} (D-1)^{D-1} \nu^{A-D-10} \beta^2 (D-3)^{-D}, \]
\[ S_4 = \sum_{i=0}^{D-5} (i + 1) \nu^{2i} \left( \frac{D-1}{2} \right)^i \left( \frac{D-3}{2} \right)^{D-2i-5} \]
\[ S_5 = r^{D-1} - \frac{1}{2} \frac{(D-1)^{D-1} \nu^{D-3} r^2}{(D-3)^{D-3}} + \frac{1}{2} \frac{(D-1)^{D-1} \nu^{2D-4}}{(D-3)^{D-3}}, \]

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