UNIFIED TREATMENT OF MIXED VECTOR-SCALAR SCREENED COULOMB POTENTIALS FOR FERMIONS

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Abstract

The problem of a fermion subject to a general mixing of vector and scalar screened Coulomb potentials in a two-dimensional world is analyzed and quantization conditions are found.
1 Introduction

Although the vector Coulomb potential does not hold relativistic bound-state solutions, its screened version \( \sim e^{-|x|/\lambda} \) is a genuine binding potential and its solutions have been found for fermions.\(^1\) The problem has also been analyzed for scalar\(^2\) and pseudoscalar\(^3\) couplings. The Klein-Gordon equation with vector,\(^4\) scalar\(^5\) and arbitrarily mixed vector-scalar\(^6\) couplings has not been exempted. As has been emphasized in Refs.\(^2\) and \(^4\), the solution of relativistic equations with this sort of potential may be relevant in the study of pionic atoms, doped Mott insulators, doped semiconductors, interaction between ions, quantum dots surrounded by a dielectric or a conducting medium, protein structures, etc.

In the present paper it is shown that the problem of a fermion under the influence of a mixed vector-scalar screened Coulomb potential, except for possible isolated energies, can be mapped into a Sturm-Liouville problem for the upper component of the Dirac spinor with an effective symmetric Morse-like potential, or an effective screened Coulomb potential in particular circumstances. In all of those circumstances, the quantization conditions are obtained. Beyond its potential physical applications, this sort of mixing shows to be a powerful tool to obtain a deeper insight about the nature of the Dirac equation and its solutions.

2 The Dirac equation with mixed vector-scalar potentials in a 1+1 dimension

In the presence of time-independent vector and scalar potentials the 1+1 dimensional time-independent Dirac equation for a fermion of rest mass \( m \) reads

\[
\mathcal{H}\Psi = E\Psi, \quad \mathcal{H} = c\sigma_1 p + \sigma_3 \left(mc^2 + V_s\right) + V_v,
\]

where \( E \) is the energy of the fermion, \( c \) is the velocity of light and \( p \) is the momentum operator. \( \sigma_1 \) and \( \sigma_3 \) are 2×2 Pauli matrices and the vector and scalar potentials are given by \( V_v \) and \( V_s \), respectively. Introducing the unitary operator \( U(\delta) = \exp \left[-i (\delta - \pi/2) \sigma_1/2\right] \), with \(-\pi/2 \leq \delta \leq \pi/2\), the transform of the Hamiltonian\(^1\), \( H = U\mathcal{H}U^{-1} \), takes the form

1
\[ H = \sigma_1 cp + \sigma_2 \cos \delta \left( mc^2 + V_s \right) - \sigma_3 \sin \delta \left( mc^2 + V_s \right) + V_v. \]  

(2)

In terms of the upper (\( \phi \)) and lower (\( \chi \)) components of the transform of the spinor \( \Psi \) under the action of the operator \( U \), \( \psi = U \Psi \), the Dirac equation, choosing \( V_v = V_s \sin \delta \), i.e., \(|V_s| \geq |V_v|\), becomes

\[
\begin{align*}
\hbar c \phi' - \cos \delta \left( mc^2 + V_s \right) \phi &= i \left[ E + \sin \delta mc^2 \right] \chi \\
\hbar c \chi' + \cos \delta \left( mc^2 + V_s \right) \chi &= i \left[ E - \sin \delta \left( mc^2 + 2V_s \right) \right] \phi.
\end{align*}
\]

(3)

where the prime denotes differentiation with respect to \( x \). Note that the charge conjugation is put into practice by the simultaneous changes \( E \to -E \) and \( \delta \to -\delta \) while changing the sign of any one of the components of the spinor \( \psi \). Taking advantage of this symmetry we can restrict our attention to nonnegative values of \( \delta \). Using the expression for \( \chi \) obtained from the first line of (3), and inserting it into the second line, one arrives at the following second-order differential equation for \( \phi \):

\[ -\frac{\hbar^2}{2} \phi'' + \left( \frac{\cos^2 \delta}{2c^2} V_s^2 + \frac{mc^2 + E \sin \delta}{c^2} V_s + \frac{\hbar \cos \delta}{2c} V_s' - \frac{E^2 - m^2 c^4}{2c^2} \right) \phi = 0. \]

(4)

Therefore, the solution of the relativistic problem is mapped into a Sturm-Liouville problem for the upper component of the Dirac spinor. In this way one can solve the Dirac problem by recurring to the solution of a Schrödinger-like problem. The solutions for \( E = -\sin \delta mc^2 \), excluded from the Sturm-Liouville problem, can be obtained directly from the Dirac equation (3).

3  The mixed vector-scalar screened Coulomb potential

Now let us focus our attention on a scalar potentials in the form

\[ V_s = -\frac{\hbar cg}{2\lambda} \exp \left( -\frac{|x|}{\lambda} \right), \]

(5)

where the coupling constant, \( g \), is a dimensionless real parameter and \( \lambda \), related to the range of the interaction, is a positive parameter. The solution
for \( E = -\sin \delta mc^2 \) is not continuous at \( x = 0 \) and should be discarded. For \( E \neq -\sin \delta mc^2 \) the Sturm-Liouville problem transmutes into

\[
-\frac{\hbar^2}{2m} \phi''_{\varepsilon} + V_{\text{eff}}^{(e)} \phi_{\varepsilon} = E_{\text{eff}} \phi_{\varepsilon},
\]

(6)

where \( E_{\text{eff}} = (E^2 - m^2c^4) / (2mc^2) \) and

\[
V_{\text{eff}}^{(e)} = V_1^{(e)} \exp \left(-\frac{|x|}{\lambda}\right) + V_2 \exp \left(-2\frac{|x|}{\lambda}\right)
\]

(7)

with

\[
V_1^{(e)} = -\frac{mc^2\lambda c g}{2\lambda} \left(1 + \frac{E}{mc^2} \sin \delta - \frac{\lambda c^2}{2\lambda} \varepsilon \cos \delta\right), \quad V_2 = \frac{mc^2\lambda^2 c^2 g^2}{8\lambda^2} \cos^2 \delta
\]

(8)

where \( \varepsilon \) stands for the sign function (\( \varepsilon = x/|x| \) for \( x \neq 0 \)).

### 3.1 The effective screened Coulomb potential \((V_v = \pm V_s)\)

For this class of effective potential, the discrete spectrum arises when \( V_{1}^{(e)} < 0 \) and \( V_2 = 0 \), corresponding to \( V_v = \pm V_s \). Bound-state solutions are feasible only if \( g > 0 \). Defining the dimensionless quantities

\[
y = y_0 \exp \left(-\frac{|x|}{2\lambda}\right), \quad y_0 = 2 \sqrt{\frac{\lambda g}{\lambda c} \left(1 \pm \frac{E}{mc^2}\right)}, \quad \mu = \frac{2\lambda}{\lambda c} \sqrt{1 - \left(\frac{E}{mc^2}\right)^2}
\]

(9)

and using (6)-(8) one obtains the differential Bessel equation \( y^2 \phi'' + y\phi' + (y^2 - \mu^2) \phi = 0 \), where the prime denotes differentiation with respect to \( y \). The solution finite at \( y = 0 \) (\( |x| = \infty \)) is given by the Bessel function of the first kind and order \( \mu \):\[7\] \( \phi(y) = N_\mu J_\mu(y) \), where \( N_\mu \) is a normalization constant. In fact, the normalizability of \( \phi \) demands that the integral \( \int_0^\infty y^{-1} |J_\mu(y)|^2 dy \) must be convergent. Since \( J_\mu(y) \) behaves as \( y^\mu \) at the lower limit, one can see that \( \mu \geq 1/2 \) so that square-integrable Dirac eigenfunctions are allowed only if \( \lambda \geq \lambda_c / 4 \). The boundary conditions at \( x = 0 \) (\( y = y_0 \)) imply that \( dJ_\mu(y)/dy|_{y=y_0} = 0 \) for even states, and \( J_\mu(y_0) = 0 \) for odd states. Since the Dirac eigenenergies are dependent on \( \mu \) and \( y_0 \), it follows that those last equations are quantization conditions.
3.2 The effective Morse-like potential \((V_v \neq \pm V_s)\)

Let us define \(z = z_0 \exp \left(-\frac{|x|}{\lambda}\right), \ z_0 = g \cos \delta, \text{ and} \)

\[
\rho_\varepsilon = \frac{\lambda}{\lambda_c \cos \delta} \left(1 + \frac{E_m c^2}{mc} \sin \delta - \frac{\lambda_c}{2\lambda} \varepsilon \cos \delta \right), \quad \nu = \frac{\lambda}{\lambda_c} \sqrt{1 - \left(\frac{E}{mc}\right)^2}, \quad (10)
\]

so that \(z\phi''_\varepsilon + \phi'_\varepsilon + \left(-\frac{1}{4} + \frac{\rho_\varepsilon}{z} + \frac{1/4 - \nu^2}{z^2} \right) \phi_\varepsilon = 0. \) Now the prime denotes differentiation with respect to \(z. \) Following the steps of Refs. [4] and [5], we make the transformation \(\phi_\varepsilon = z^{-1/2} \Phi_\varepsilon \) to obtain the Whittaker equation:\[7\]

\[
\Phi''_\varepsilon + \left(-\frac{1}{4} + \frac{\rho_\varepsilon}{z} + \frac{1/4 - \nu^2}{z^2} \right) \Phi_\varepsilon = 0, \quad (11)
\]

whose solution vanishing at the infinity is written as \(\Phi_\varepsilon = N_\varepsilon z^{\nu+1/2} e^{-z/2} M(a_\varepsilon, b, z), \) where \(N_\varepsilon \) is a normalization constant and \(M \) is a regular solution of the confluent hypergeometric equation (Kummer’s equation):[7]

\[
\xi M'' + (b - \xi)M' - a_\varepsilon M = 0, \text{ with } a_\varepsilon = \nu + \frac{1}{2} - \rho_\varepsilon \text{ and } b = 2\nu + 1. \quad (12)
\]

Now we are ready to write the physically acceptable solutions on both sides of the \(x-\)axis by recurring to the symmetry \(\phi_\varepsilon(-x) \sim \phi_{-\varepsilon}(x). \) They are

\[
\phi = z^{\nu} e^{-z/2} \left[\theta(-x) C^{(-)}(a_-, b, z) + \theta(+x) C^{(+)}(a_+, b, z) \right]
\]

\[
\chi = z^{\nu} e^{-z/2} \left[\theta(-x) D^{(-)}(a_+, b, z) + \theta(+x) D^{(+)}(a_-, b, z) \right], \quad (13)
\]

where \(C^{(\pm)} \) and \(D^{(\pm)} \) are normalization constants and \(\theta(x) \) is the Heaviside function. The continuity of the wavefunctions at \(x = 0 \) plus the substitution of (13) into the Dirac equation (4), and making use of a pair of recurrence formulas involving the solution of Kummers’ equation,[7] lead to the quantization condition

\[
\frac{M(a_+ + 1, b, z_0)}{M(a_+, b, z_0)} = \sqrt{\frac{a_+ - 2\nu}{a_+}}. \quad (14)
\]
4 Concluding remarks

The quantization conditions for a general mixing of vector and scalar screened Coulomb potentials in a two-dimensional world have been put forward in a unified way. Of course, we have to analyze the nature of the spectra as a function of the potential parameters. This task, including the complete set of eigenvectors, will be reported elsewhere.

Acknowledgements

This work was supported by CAPES, CNPq and FAPESP.

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