A NOTE ON ASYMPTOTICALLY EXACT A POSTERIORI ERROR ESTIMATES FOR MIXED LAPLACE EIGENVALUE PROBLEMS

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ABSTRACT. We derive optimal and asymptotically exact a posteriori error estimates for the approximation of the Laplace eigenvalue problem. To do so, we combine two results from the literature. First, we use the hypercircle techniques developed for mixed eigenvalue approximations with Raviart-Thomas Finite elements. In addition, we use the post-processings introduced for the eigenvalue and eigenfunction based on mixed approximations with the Brezzi-Douglas-Marini Finite element. To combine these approaches, we define a novel additional local post-processing for the fluxes that appropriately modifies the divergence. Consequently, the new flux can be used to derive upper bounds and still shows good approximation properties. Numerical examples validate the theory and motivate the use of an adaptive mesh refinement.

1. INTRODUCTION

In many examples from physics to industrial applications, the solution of eigenvalue problems plays an essential role. Similar as for standard source problems, the Finite element method seems to be a very promising method to discretize these problems due to its flexibility and good approximation properties. Numerous works deal with the analysis in general frameworks where issues such as stability, convergence properties and a priori error estimates are discussed, see [3, 9].

Since in general one can not assume high regularity of the eigenfunctions on arbitrary domains [24], the requirement for an adaptive mesh refinement strategy is obvious. Central to this approach is the derivation of an efficient and reliable a posteriori error estimator, as already developed for Finite element methods in general [1, 32], and for eigenvalue problems in particular in [17].

In this work we consider the Laplace eigenvalue problem and approximate it using a mixed method, see [26, 16, 10]. By that we get access to the hypercircle theory, see [29, 25], eventually leading to asymptotically exact upper bounds and local efficiency. However, unlike for standard source problems, see [23, 13, 33, 18, 22], a more profound approach is needed since the orthogonality of the corresponding errors is no longer exactly satisfied.

For eigenvalue problems this was first introduced in the work [6], by means of the Raviart-Thomas Finite element. To discuss details, note that we have

\[ \|\sigma_h - \sigma\|^2_0 + \|\nabla(u - u_h^{**})\|^2_0 = \|\sigma_h - \nabla u_h^{**}\|^2_0 - 2(\sigma_h - \sigma, \nabla(u - u_h^{**})) , \]

where \( \lambda, u, \sigma \) are the eigenvalue, eigenfunction and its gradient, \( \lambda_h, u_h, \sigma_h \) are the corresponding approximations and \( u_h^{**} \) denotes some \( H^1 \)-conforming post-processed...
function of \( u_h \). The first term on the right-hand side of (1) is computable and can therefore be used to define an a posteriori estimator \( \eta \). The astonishing observation in [6] was then that in the case of an approximation using the Raviart-Thomas Finite element, the second term \( 2(\sigma_h - \sigma, \nabla(u - u_h^*)) \) converges of higher order. Consequently, \( \eta \) is an asymptotically exact upper bound for the errors on the left-hand side of (1).

Unfortunately, the method of [6] has the drawback of a reduced accuracy of the eigenvalue and the eigenfunction since the Raviart-Thomas space does not allow an optimal approximation. In [4] (using ideas from [20]) the same authors (and collaborators) were able to achieve an optimal approximation by using the Brezzi-Douglas-Marini (BDM) Finite element instead. However, this was only possible by paying the price of unknown constants in the a posteriori estimates since the additional term in (1) is not of higher order any more as was also observed in [5].

The goal of this work is to combine the advantages from both works. More precisely, we use the BDM Finite Element and the post-processing techniques for the eigenvalue and the eigenfunction as in [4], and consider modifications of the approaches from [6] to derive asymptotically exact upper bounds. For the latter, we introduce an additional (local) post-processing for the flux variable \( \sigma_h \), where we correct its divergence to fit the additional term in (1), which consequently converges again with higher order.

The rest of the paper is organized as follows. Section 2 discusses the problem setting and its approximation. In section 3 we present the local post-processing technique for the eigenfunction and the eigenvalue. The main results are then discussed in section 4. While we first recapture the standard a posteriori error analysis based on (1) and reveal its breakdown due to a slow convergence of the additional terms, we then introduce the novel post-processing of the flux and derive the asymptotically exact upper bound. In the last section 5 we present two numerical examples to validate our findings. The appendix, see section 6, considers some additional results needed in the analysis.

2. Problem setting

Let \( \Omega \subset \mathbb{R}^d \) be a polygon or polyhedron for \( d = 2, 3 \), respectively. We consider the mixed formulation of the Laplace eigenvalue problem with homogeneous Dirichlet boundary conditions, i.e. we want to find a \( \lambda \in \mathbb{R} \), \( u \in L^2(\Omega) \) and \( \sigma \in H(\mathrm{div}, \Omega) \) such that

\begin{align}
-(\sigma, \tau) - (\mathrm{div} \tau, u) &= 0 \quad \forall \tau \in H(\mathrm{div}, \Omega), \\
-(\mathrm{div} \sigma, v) &= \lambda (u, v) \quad \forall v \in L^2(\Omega).
\end{align}

We approximate (2) by a mixed method using the BDM Finite element for the approximation of \( \sigma \) and a piece-wise polynomial approximation of \( u \). To this end let \( \mathcal{C}_h \) be a regular triangulation of \( \Omega \) into triangles and tetrahedrons in two and three dimensions respectively. Let \( k \geq 1 \) be a fixed integer (see Remark 1 for a comment regarding the lowest order case). We introduce the spaces

\[ U_h := \{ v_h \in L^2(\Omega) : v_h|_K \in \mathbb{P}^k(K) \ \forall K \in \mathcal{C}_h \}, \]
\[ \Sigma_h := \{ \tau_h \in H(\mathrm{div}, \Omega) : \tau_h|_K \in \mathbb{P}^{k+1}(K, \mathbb{R}^d) \ \forall K \in \mathcal{C}_h \}, \]

where \( \mathbb{P}^l(K) \) denotes the space of polynomials of order \( l \geq 0 \) on \( K \), and \( \mathbb{P}^l(K, \mathbb{R}^d) \) denotes the corresponding vector-valued version. An approximation of (2) then
seeks \( \lambda_h \in \mathbb{R}, u_h \in U_h \) and \( \sigma_h \in \Sigma_h \) such that

\[
\begin{align*}
(3a) & \quad -(\sigma_h, \tau_h) - (\text{div} \tau_h, u_h) = 0 \quad \forall \tau_h \in \Sigma_h, \\
(3b) & \quad -(\text{div} \sigma_h, v_h) = \lambda_h(u_h, v_h) \quad \forall v_h \in U_h.
\end{align*}
\]

Review article \[9\] (for example) states that problem (3) defines a well-approximation of the continuous eigenvalue problem (2) in the sense that it does not produce any spurious modes and that eigenfunctions are approximated with the proper multiplicity. The approximation results are summarized in the following. To this end let \( s > 1/2 \) and let \((\lambda, u, \sigma)\) be a solution of the eigenvalue problem (2) with the regularity \( u \in H^{1+s}(\Omega) \) and \( \sigma \in \text{H(div,} \Omega) \cap H^s(\Omega, \mathbb{R}^d) \) (for the regularity results see \[10\] \[19\] \[21\]). Then there exists a discrete solution of (3) such that

\[
\begin{align*}
(4a) & \quad \|u - u_h\|_0 \lesssim h^r |u|_{r+1}, \\
(4b) & \quad |\sigma - \sigma_h|_0 \lesssim h^{r'} |u|_{r'+1}, \\
(4c) & \quad \|\text{div}(\sigma - \sigma_h)\|_0 \lesssim h^{r'}(|u|_{r+1} + |u|_{r'+1}),
\end{align*}
\]

where \( r = \min\{s, k + 1\} \) and \( r' = \min\{s, k + 2\} \). If \( s \) is big enough we have \( k' = \min\{s, k + 2\} \). Above estimates follow from the abstract theory from \[9\] \[16\] and \[26\], and the approximation results of the source problem, see \[10\]. In addition we have

\[
\|u - u_h\|_{1,h} \lesssim h^{r-1} |u|_{r+1},
\]

where

\[
\|u - u_h\|_{1,h}^2 := \sum_{K \in \mathcal{C}_h} \|\nabla(u - u_h)\|_{0,K}^2 + \sum_{F \in \mathcal{F}_h} \frac{1}{h_F} \|[u_h]\|_{0,F}^2.
\]

Here \([\cdot]\) denotes the standard jump operator, \(\mathcal{F}_h\) the set of facets of the triangulation \(\mathcal{C}_h\), and \(h_F\) the diameter of a facet \(F \in \mathcal{F}_h\). Note that above results demand a careful choice of the approximated eigenfunction \(u_h\) and the approximated gradient \(\sigma_h\). An example, well established in the literature, is given by a normalization such that \(\|u_h\|_0 = \|u\|_0 = 1\) and by choosing the sign \((u, u_h) > 0\). Note that this also fixes \(\sigma\) and \(\sigma_h\) by \[2a\] and \[3a\], respectively. The case of eigenvalues with a higher multiplicity demands more carefullness, particularly if an a posteriori analysis is considered, and we refer to \[9\] \[11\] for more details. For simplicity, we assume for the rest of this work that \(\lambda\) is a simple eigenvalue and that the above choice of sign and scaling of the continuous and the discrete eigenfunctions is applied. Further, for simplicity, we will call \((\lambda, u, \sigma)\) the solution of (2), keeping in mind that a different scaling and sign can be chosen.

**Remark 1.** Although the schemes proposed in this work are computable also for the lowest order case \(k = 0\), one does not observe any high-order convergence of the post processed variables defined later in the work. The reason for this is that the Aubin-Nitsche technique, needed in the analysis, can not be applied for this case.

3. Local post-processing for \(u_h\) and \(\lambda_h\)

For a sufficiently smooth solution, estimates \(4\) and \(5\) show that there is a gap of two between the order of convergence of \(\|\sigma - \sigma_h\|_0\) and \(\|u - u_h\|_{1,h}\). In \[16\] the following identity is proven

\[
\lambda - \lambda_h = \|\sigma - \sigma_h\|_0^2 - \lambda_h \|u - u_h\|_0^2.
\]
which, due to (4), gives the estimate (using $r \leq r'$)

\begin{equation}
|\lambda - \lambda_h| \lesssim h^{2r}|u|_{r+1} + h^{2r'}|u|_{r'+1} \lesssim h^{2r}(|u|_{r+1} + |u|_{r'+1}).
\end{equation}

We see that the order of convergence of $|\lambda - \lambda_h|$ is dominated by the order of the $L^2$-error of the eigenfunction. The reduced convergence of $u_h$ (compared to the $L^2$-error of $\sigma$) is well known for mixed methods and can be improved by means of a local post-processing, see [2] [31], and particularly for eigenfunctions in [14]. Consequently, using the ideas from [20], we can then also get an improved eigenvalue.

For a given integer $l \geq 0$ let $\Pi'$ denote the $L^2$-projection onto element-wise polynomials of order $l$. Consider the spaces

\begin{align}
U_h^* := \{v_h \in L^2(\Omega) : v_h|_K \in \mathbb{P}^{k+2}(K) \forall K \in \mathcal{C}_h\}, \quad \text{and} \quad U_h^{**} := U_h^* \cap H^1_0(\Omega),
\end{align}

then we define $u_h^* \in U_h^*$ by

\begin{align}
(\nabla u_h^*, \nabla v_h^*)_K = (\sigma_h, \nabla v_h^*)_K \quad \forall v_h^* \in (\text{id} - \Pi')_h \mathbb{P}^{k+2}(K), \forall K \in \mathcal{C}_h,
\end{align}

\begin{equation}
\Pi^k u_h^* = u_h.
\end{equation}

For the discretization of the standard source problem (i.e. the Poisson problem), it is known that the kernel inclusion property $\nabla \Sigma_h \subseteq U_h$ (see [10]) and commuting interpolation operators yield a super convergence property of the projected error $\|\Pi^k u - u_h\|_0$ given by $\rho(h)O(h^r)$. Here $\rho(h)$ is a function that depends on the regularity of the problem and for which we have $\rho(h) \to 0$ as $h \to 0$. For convex domains we have $\rho(h) = O(h)$. This super convergence of the projected error is the fundamental ingredient to derive the enhanced approximation properties of $u_h^*$.

Unfortunately the same technique, i.e. the one from the standard source problem, does not work for the eigenvalue problem and an improved convergence estimate of $\|\Pi^k u - u_h\|_0$ is more involved. This has been discussed for the lowest order case in [19], for a more general setting including eigenvalue clusters in [11], for Maxwell eigenvalue problems in [12] and for the Stokes problem for example in [20]. Unfortunately, these results are only presented using the full $\|\cdot\|_{div}$-norm (or the corresponding mixed norm) for $\Sigma$ and $\Sigma_h$. While such an estimate is applicable for an approximation of (3) using Raviart-Thomas Finite elements, the BDM case is not covered since (4c) and (4b) show different convergence orders which would spoil the estimate. As the author is not aware of a detailed proof that can be found in the literature, it will be given in the appendix in section 6. Note however, that these results are already used for example in [4] (without proof). The resulting super convergence reads as

\begin{equation}
\|\Pi^k u - u_h\|_0 \lesssim \rho(h)(\|u - u_h\|_0 + ||\sigma - \sigma_h||_0),
\end{equation}

which in combination with the techniques from [31], then yield the approximation properties (see also [28])

\begin{align}
\|u - u_h^*\|_0 &\lesssim \rho(h)h^r(|u|_{r+1} + |u|_{r'+1}),
\end{align}

\begin{align}
\|u - u_h^*\|_{1,h} &\lesssim h^r(|u|_{r+1} + |u|_{r'+1}).
\end{align}

Since $u_h^*$ is not $H^1$-conforming the final post-processing step consists of the application of an averaging operator $I^a : U_h^* \to U_h^{**}$ often also called Oswald operator, see [27] and [15] for details on the approximation properties. We set $u_h^{**} := I^a(u_h^*)$.
for which we have by (10)

\[ \|u - u_h^*\|_0 \lesssim \rho(h)h^r(|u|_{r+1} + |u|_{r'+1}), \quad (11a) \]

\[ \|\nabla(u - u_h^*)\|_0 \lesssim h^r(|u|_{r+1} + |u|_{r'+1}). \quad (11b) \]

We conclude this section by introducing a post-processing of the eigenvalue. As in [20] 4 we define

\[ \lambda_h^* := \frac{\text{div} \sigma_h, u_h^*}{(u_h^*, u_h^*)}. \tag{12} \]

The following lemma was given in [20]. Since we need some intermediate steps in the sequel, we include the proof.

**Lemma 1.** Let \( s > 1/2 \) and let \((\lambda, u, \sigma)\) be the solution of (2) with the regularity \( u \in H^{1+s}(\Omega) \) and \( \sigma \in H(\text{div}, \Omega) \cap H^s(\Omega, \mathbb{R}^d) \). Further let \( \|u_h^*\|_0 \neq 0 \). There holds

\[ |\lambda - \lambda_h^*| \lesssim (\rho(h)h^{r+r} + h^{2r'})(|u|_{r+1} + |u|_{r'+1}), \]

where \( r = \min\{s, k + 1\} \) and \( r' = \min\{s, k + 2\} \).

**Proof.** Since \( \|u\|_0 = 1 \) we have using that \( \text{div} \Sigma_h \subseteq U_h \) and (8b)

\[ \lambda_h^* - \lambda^* = -\text{div} \sigma, u = \lambda(u, u) = \lambda, \]

\[ (\sigma_h, \sigma_h) = (\text{div} \sigma_h, u_h) = (\Pi h^k \text{div} \sigma_h, u_h) = (\text{div} \sigma_h, u_h^*), \]

\[ (\sigma - \sigma_h) = (\sigma - \sigma_h, \sigma - \sigma_h) = (\sigma, \sigma) + (\sigma, \sigma_h) - 2(\sigma, \sigma_h) \]

\[ = \lambda + \lambda_h^*(u_h^*, u_h^*) + 2(\text{div} \sigma_h, u). \]

Using \( \lambda_h^* - \lambda^* = \lambda_h^*(u, u) + \lambda_h^*(u_h^*, u_h^*) - 2\lambda_h^*(u, u_h^*) \) we have in total

\[ \lambda - \lambda_h^* \]

\[ = \|\sigma - \sigma_h\|_0^2 - \lambda_h^*(u_h^*, u_h^*) - 2(\text{div} \sigma_h, u) - \lambda_h^*, \]

\[ = \|\sigma - \sigma_h\|_0^2 + \lambda_h^*(u, u) - 2\lambda_h^*(u, u_h^*) - \lambda_h^*\|u - u_h^*\|_0^2 - 2(\text{div} \sigma_h, u) - \lambda_h^*, \]

and thus again with \( \|u\|_0 = 1 \) this gives

\[ \lambda - \lambda_h^* = \|\sigma - \sigma_h\|_0^2 - \lambda_h^*\|u - u_h^*\|_0^2 - 2(\text{div} \sigma_h + \lambda_h^* u_h^*, u). \quad (13) \]

Since \( (\text{div} \sigma_h + \lambda_h^* u_h^*, u_h^*) = 0 \) (according to the definition of \( \lambda_h^* \)), the last term can be written as

\[ (\text{div} \sigma_h + \lambda_h^* u_h^*, u) \]

\[ = (\text{div} \sigma_h + \lambda_h^* u_h^*, u - u_h^*), \]

\[ = (\text{div}(\sigma_h - \sigma), u - u_h^*) + (\text{div} + \lambda_h^* u_h^*, u - u_h^*), \]

\[ = (\text{div}(\sigma_h - \sigma), u - u_h^*) + (-\lambda u + \lambda_h^* u_h^*, u - u_h^*) \]

\[ = (\text{div}(\sigma_h - \sigma), u - u_h^*) + \lambda_h(u_h^* - u, u - u_h^*) - (\lambda - \lambda_h^*)(u, u - u_h^*). \]

By the Cauchy-Schwarz inequality we finally get

\[ |\lambda - \lambda_h^*| \leq \|\sigma - \sigma_h\|_0^2 + \lambda_h^*\|u - u_h^*\|_0^2 + 2\|\text{div}(\sigma_h - \sigma)\|_0\|u - u_h^*\|_0 + |\lambda - \lambda_h^*|\|u - u_h^*\|_0 \]

\[ \lesssim \|\sigma - \sigma_h\|_0^2 + \|u - u_h^*\|_0^2 + \|\text{div}(\sigma_h - \sigma)\|_0\|u - u_h^*\|_0 + |\lambda - \lambda_h^*|^2. \]

Thus, for \( h \) small enough, the last term can be moved to the left hand side, and we can conclude the proof using (10) and (1).
4. A posteriori analysis

In this section we provide an a posteriori error analysis and define an appropriate error estimator. We follow [6] where the authors derived an error estimator using the variables $\sigma_h$ and $u_h^*$. While this works for a mixed approximation of (4) using the Raviart-Thomas Finite element of order $k$ (as was done in [6]), this does not work for our setting. To discuss the problematic terms and to motivate our modification, we present more details in the following. Since $\sigma = \nabla u$ we have

$$
\|\sigma_h - \nabla u_h^*\|_0^2 = \|\sigma_h - \sigma + \nabla u_h^*\|_0^2 \\
= \|\sigma_h - \sigma\|_0^2 + \|\nabla(u - u_h^*)\|_0^2 + 2(\sigma_h - \sigma, \nabla(u - u_h^*)).
$$

Using integration by parts, $u_h^* \in H_0^1(\Omega)$ and $-\div \sigma_h = \lambda_h u_h$, the last term can be written as

$$(\sigma_h - \sigma, \nabla(u - u_h^*)) = -(\div(\sigma_h - \sigma), u - u_h^*) \\
= - (\lambda_h u_h - \lambda u, u - u_h^*) \\
= - (\lambda_h u_h + \lambda u_h - \lambda u, u - u_h^*) \\
= - (\lambda_h - \lambda)(u_h, u - u_h^*) - \lambda(u_h - u, u - u_h^*).$$

In total this gives the guaranteed upper bound

$$
\|\sigma_h - \sigma\|_0^2 + \|\nabla(u - u_h^*)\|_0^2 \leq \\
\|\sigma_h - \nabla u_h^*\|_0^2 + 2\|\lambda_h - \lambda\|_{\infty} \|u - u_h^*\|_0 + 2\lambda \|u_h - u\|_0 \|u - u_h^*\|_0.
$$

In [6] the first term on the right hand side is the (computable) proposed error estimator, whereas the second and third are high-order terms. Compared to our setting we can see the problem since

$$
\|\sigma_h - \nabla u_h^*\|_0^2 \lesssim h^{2k+4}, \\
\|\lambda_h - \lambda\|_{\infty} \|u - u_h^*\|_0 \lesssim h^{3k+4}, \\
\|u_h - u\|_0 \|u - u_h^*\|_0 \lesssim h^{2k+4},
$$

where for simplicity, i.e. to allow a simpler comparison, we assumed a smooth solution. Whereas the second term converges with an increased rate (compared to $2k + 4$), the bad convergence order of $\|u - u_h\|_0$, see equation (7), spoils the estimate of the last term. As we can see in the proof above, the problem can be traced back to the identity $-\div \sigma_h = \lambda_h u_h$, since this is the point in the proof where $u_h$ appears first.

To fix this problem we propose another post-processing. Whereas the first two post-processing routines where used to increase the convergence rate of the error of the eigenfunction and eigenvalue i.e. $u_h^*$ (and $u_h^*$) and $\lambda_h$, respectively, we now aim to construct a $\sigma_h^*$ with a fixed divergence constraint rather than improving its approximation properties measured in the $L^2$-norm. To this end we define the space

$$
\Sigma_h^* := \{\tau_h \in H(\div, \Omega) : \tau_h|_K \in \mathbb{P}^{k+3}(K, \mathbb{R}^d) \forall K \in \mathcal{C}_h, \\
\tau_h \cdot n|_F \in \mathbb{P}^{k+1}(F) \forall F \in \mathcal{F}_h\}.
$$

The space $\Sigma_h^*$ reads as a BDM space of order $k + 3$ with a reduced polynomial order of the normal traces. Note that other choices of $\Sigma_h^*$ are possible, see Remark 2.

The basic idea now is to find a $\sigma_h^* \in \Sigma_h^*$ being as "close" as possible to $\sigma_h$ (i.e. 
being a good approximation) such that the divergence is modified appropriately using the additional high-order normal-bubbles (i.e. functions with a zero normal component). Since these bubbles are defined locally, this can be done in an element-wise procedure. Now let $\xi_h \in \Sigma_h^*$ be arbitrary. Proposition 2.3.1 in \cite{10} shows that the following degrees of freedom (here applied to $\xi_h$)

(14a) facet moments: $\int_F \xi_h \cdot nr_h \, ds \quad \forall r_h \in \mathbb{P}^{k+1}(F) \ \forall F \in \mathcal{F}_h$,  

(14b) div moments: $\int_K \text{div} \xi_h q_h \, dx \quad \forall q_h \in \mathbb{P}^{k+2}(K)/\mathbb{P}^0(K) \ \forall K \in \mathcal{C}_h$,  

(14c) vol moments: $\int_K \xi_h \cdot l_h \, dx \quad \forall l_h \in \mathbb{H}^{k+3}(K) \ \forall K \in \mathcal{C}_h$,  

where $\mathbb{H}^{k+3}(K) := \{ l_h \in \mathbb{P}^{k+3}(K, \mathbb{R}^d) : \text{div} \ l_h = 0, l_h \cdot n|_{\partial K} = 0 \}$, are unisolvent. By that we can define the post processed flux $\sigma_h^* \in \Sigma_h^*$ by

(15a) $\int_F (\sigma_h^* - \sigma_h) \cdot nr_h \, ds \quad \forall r_h \in \mathbb{P}^{k+1}(F) \ \forall F \in \mathcal{F}_h$,  

(15b) $\int_K (\text{div} \sigma_h^* + \lambda_h u_h^*) q_h \, dx \quad \forall q_h \in \mathbb{P}^{k+2}(K)/\mathbb{P}^0(K) \ \forall K \in \mathcal{C}_h$,  

(15c) $\int_K (\sigma_h^* - \sigma_h) \cdot l_h \, dx \quad \forall l_h \in \mathbb{H}^{k+3}(K) \ \forall K \in \mathcal{C}_h$.  

Note that since $\sigma_h$ is normal continuous, i.e. the normal trace coincides on a common facet of two neighboring elements, the boundary constraints (15a) of $\sigma_h^*$ can be set locally on each element (boundary) separately. Further, since $\sigma_h \cdot n$ and $\sigma_h^* \cdot n$ have the same polynomial degree $k + 1$, the moments from (15a) result in $\sigma_h \cdot n = \sigma_h^* \cdot n$. In Remark 3 we also make a comment on the choice of (15b).

**Theorem 1.** Let $\sigma_h^* \in \Sigma_h^*$ be the function defined by (15), then there holds  

$$ - \text{div} \sigma_h^* = \lambda_h u_h^*.$$

Let $s > 1/2$ be the solution of the eigenvalue problem (2) with the regularity $\sigma \in H(\text{div}, \Omega) \cap H^s(\Omega, \mathbb{R}^d)$. There holds the a priori error estimate

$$ ||\sigma - \sigma_h^*||_0 \lesssim h^{r'}(|u|_{r+1} + |u|_{r'+1}),$$

where $r' = \min\{s, k + 2\}$ and $r = \min\{s, k + 1\}$.

**Proof.** We start with the proof of the divergence identity. Let $K \in \mathcal{C}_h$ and $q_h \in \mathbb{P}^{k+2}(K)$ be arbitrary, then we have

$$ - \int_K \text{div} \sigma_h^* q_h \, dx = - \int_K \text{div} \sigma_h^* (q_h - \Pi^0 q_h) \, dx - \int_K \text{div} \sigma_h^* \Pi^0 p_h \, dx =$$

$$ = \int_K \lambda_h u_h^* (q_h - \Pi^0 q_h) \, dx - \Pi^0 q_h \int_{\partial K} \sigma_h^* \cdot n \, ds,$$

where the second step followed due to (15b) and the Gauss theorem. Using (15a) and (3b), the last integral can be written as

(16a) $- \Pi^0 q_h \int_{\partial K} \sigma_h^* \cdot n \, ds = - \Pi^0 q_h \int_{\partial K} \sigma_h \cdot n \, ds = - \int_K \Pi^0 q_h \text{div} \sigma_h \, dx$

(16b) $= \int_K \Pi^0 q_h \lambda_h u_h \, dx = \int_K \Pi^0 q_h \lambda_h u_h^* \, dx,$
where we used (8b) in the last step. All together this gives
\[- \int_K \text{div} \sigma_h^* q_h \, dx = \int_K \lambda_h u_h^* q_h \, dx,\]
from which we conclude the proof as
\[\int_K \lambda_h u_h^* q_h \, dx,\]
and let
\[\text{instead. First, the triangle inequality gives}\]
\[\|I_h^* \lambda - \sigma_h^*\|_0 \leq \|\lambda - \sigma_h^*\|_0 + \|I_h^* \lambda - \sigma_h^*\|_0.\]
Since the first term can be bounded by the properties of \(I_h^*\), we continue with the latter which can be written as
\[\|I_h^* \lambda - \sigma_h^*\|_0 = \|I_h^* (\lambda - \sigma_h^*)\|_0 \leq \|I_h^* (\lambda - \sigma_h^*)\|_0 + \|I_h^* (\sigma - \sigma_h^*)\|_0.\]
By the definition of the interpolation operators and similar steps as above we have
\[I_h^* (\sigma - \sigma_h^*) = \sigma_h,\]
and thus the term most to the right simplifies to
\[\|I_h^* (\lambda - \sigma_h^*)\|_0 = \|I_h^* (\lambda - \sigma)\|_0 \leq \|I_h^* (\sigma - \sigma_h^*)\|_0 + \|\sigma - \sigma_h\|_0.\]
We continue with the other term. For this let \(\psi_i^{\text{div}}\) be the hierarchical dual basis functions of the highest order divergence moments from (14b) given by \(\int_K \text{div}(\cdot) q_i \, dx\) with \(q_i \in \mathbb{P}^{k+2}(K)/\mathbb{P}^0(K)\). Similarly let \(\psi_i^{\text{div}}\) be the hierarchical dual basis functions of the highest order vol moments from (14c) given by \(\int_K (\cdot) \cdot l_i \, dx\) with \(l_i \in \mathbb{H}^{k+3}(K)/\mathbb{H}^{k+1}(K)\). An explicit construction of these basis functions can be found for example in [8, 34]. Also let \(N_{\text{div}}\) and \(N_E\) be the corresponding index sets. Using (2b), (15b) and (15c), this then gives
\[(I_h^* - I_h)(\sigma - \sigma_h^*) = \sum_{i \in N_{\text{div}}} \int_K (\lambda u - \lambda_h u_h^*) q_i \, dx \psi_i^{\text{div}} + \sum_{i \in N_E} \int_K (\sigma - \sigma_h) l_i \, dx \psi_i^{\text{div}},\]
which implies that (using that the norms of the \(q_i, l_i\) and \(\psi_i^{\text{div}}, \psi_i^{\text{div}}\) is bounded)
\[\|(I_h^* - I_h)(\sigma - \sigma_h^*)\|_0 \leq \|\lambda u - \lambda_h u_h^*\|_0 + \|\sigma - \sigma_h\|_0 \leq |\lambda|\|u - u_h^*\|_0 + |\lambda - \lambda_h|\|u_h^*\|_0 + \|\sigma - \sigma_h\|_0.\]
Since \(\|u_h^*\|_0 \leq \|u_h^* - u\|_0 + \|u\|_0\), we can conclude the proof by the approximation properties of \(I_h^*\) and \(I_h^*\) (see Proposition 2.5.1 in [10]), estimates (7) and (10) and by \(\rho(h)h^{r'} \leq h^{r'}\) and \(h^{2r} \leq h^{r'}\).

**Remark 2.** Instead of choosing \(\Sigma_h^*\) as above, one can for example also use the standard Raviart-Thomas space of order \(k+2\) denoted by \(RT^{k+2}\). Since \(\text{div} RT^{k+2} = U_h^*\) it is again possible to set \(\text{div} \sigma_h^* = \lambda_h u_h^*\) (using the appropriate degrees of freedom). However, since the normal trace of \(\sigma_h^*\) is now in \(\mathbb{P}^{k+2}(F)\) on each facet \(F \in \mathcal{F}_h\), one has to be more careful defining the edge moments. Precisely, we would now set
\[\Pi^{k+1}(\sigma_h^* \cdot n) = \sigma_h \cdot n, \quad \text{and} \quad (\text{id} - \Pi^{k+1})(\sigma_h^* \cdot n) = 0,\]
where the projection has to be understood as the \(L^2\)-projection on the facets.
Remark 3. One might be curious why we do not use $\lambda_h^*$ instead of $\lambda_h$ in the definition of $\text{div}\,\sigma_h^*$ in (15). Indeed, as can be seen in the proof this is a crucial choice since we used in (16) that the mean value of the divergence is fixed by the constant normal moments (first equal sign) and thus coincides with $\Pi^0(\lambda_h u_h)$ (third equal sign). Choosing $\lambda_h^*$ in (15a) would then lead to a mismatch of the low-order and high-order parts of the divergence.

We are now in the position of defining the local error estimator on each element $K \in C_h$ by

$$\eta(K) := ||\nabla u_h^* - \sigma_h^*||_K,$$

and the corresponding global estimator by

$$\eta := \left( \sum_{K \in C_h} \eta(K)^2 \right)^{1/2} = ||\nabla u_h^* - \sigma_h^*||_0.$$

Theorem 2. Let $(\lambda, u, \sigma)$ be the solution of (2). Let $(\lambda_h, u_h, \sigma_h)$ the the solution of (4) and let $u_h^*$ and $\sigma_h^*$ be the post-processed solutions. There holds the reliability estimate

$$||\nabla u - \nabla u_h^*||_0^2 + ||\sigma - \sigma_h^*||_0^2 \leq \eta^2 + \text{hot}(h),$$

where $\text{hot}(h) := 2(|(\sigma_h^* - \sigma, \nabla (u - u_h^*))|)$ with

$$\text{hot}(h) \leq \rho(h) (h^{2r+2r'} + \rho(h) h^{2r'})(|u|_{r+1} + |u|_{r'+1}),$$

is a high-order term compared to $O(h^{2r'})$ as $h \to 0$. Further, there holds the efficiency

$$\eta \leq ||\nabla u - \nabla u_h^*||_0 + ||\sigma - \sigma_h^*||_0.$$

Proof. Following the same steps as at the beginning of this section we arrive at

$$||\nabla u - \nabla u_h^*||_0^2 + ||\sigma - \sigma_h^*||_0^2 = ||\nabla u_h^* - \sigma_h^*||_0^2 + 2(\sigma_h^* - \sigma, \nabla (u - u_h^*))|.$$ 

For the last term we now have

$$(\sigma_h^* - \sigma, \nabla (u - u_h^*)) = -(\text{div}(\sigma_h^* - \sigma), u - u_h^*)$$

$$= -(\lambda_h u_h^* - \lambda u, u - u_h^*)$$

$$= -(\lambda_h - \lambda)(u_h^* - u) - \lambda(u_h^* - u, u - u_h^*).$$

Whereas the first term converges of order

$$|\lambda_h - \lambda|(u_h^* - u, u - u_h^*)| \leq |\lambda_h - \lambda||u_h^*||_0||u - u_h^*||_0$$

$$\lesssim \rho(h) h^{2r+2r'}(|u|_{r+1} + |u|_{r'+1}),$$

we have for the second term

$$|\lambda||(u_h^* - u, u - u_h^*)| \leq |\lambda||u_h^* - u||_0||u - u_h^*||_0$$

$$\lesssim \rho(h)^2 h^{2r'}(|u|_{r+1} + |u|_{r'+1}).$$

It remains to show that $\text{hot}(h) \leq \rho(h) (h^{2r+2r'} + \rho(h) h^{2r'})$ is of higher order compared to $h^{2r'}$. Due to the additional $\rho(h)$ in the upper bound of $\text{hot}(h)$, we only have to show that $2r' \leq 2r + r'$. For the low regularity case, i.e. $s = r = r'$, and the case
of full regularity, i.e. \( r = k + 1 \) and \( r' = k + 2 \), this follows immediately. For the case where \( r = k + 1 \) and \( r' = s \) with \( k + 1 < s < k + 2 \), we also have

\[
2r' = 2s < k + 2 + s < 2(k + 1) + s = 2r + r',
\]
from which we conclude the proof of the reliability.

The efficiency estimate follows by the triangle inequality and \( \sigma = \nabla u \). \( \square \)

Using the estimator from above we are now also able to derive an upper bound for \( \lambda_h^* \). To this end let

\[
\eta_{\lambda} := \eta^2 + \|\sigma_h - \sigma_h^*\|_0^2 + (\lambda_h^* u_h^* - \lambda_h u_h, u_h^*).
\]

The last two terms from the estimator \( \eta_{\lambda} \) are needed to measure the difference between the quantities used in \( \sigma_h \) and the functions used in the definition of \( \lambda_h^* \). Unfortunately the authors do not see how the definition of \( \lambda_h^* \) can be changed such that only \( \sigma^*_h \) and \( u_h^* \) are used, which would allow a direct estimate by \( \eta \).

**Theorem 3.** Let \((\lambda, u, \sigma)\) be the solution of \((2)\). Let \((\lambda_h, u_h, \sigma_h)\) be the the solution of \((1)\) and let \( u_h^* \) and \( \sigma^*_h \) be the post-processed solutions. There holds the estimate

\[
|\lambda - \lambda_h^*| \leq \eta_h + \text{hot}(h) + \tilde{\text{hot}}(h),
\]
where \( \text{hot}(h) := \|u_h^* - u_0\|_0^2 + \|u - u_h^*\|_0^2 \) with

\[
\text{hot}(h) \leq \rho(h)^2 2^{2r'}(|u|_{r+1} + |u|_{r'+1}),
\]
and \( \text{hot}(h) \) are higher order terms compared to \( \mathcal{O}(\rho(h)h^{r+r'} + h^{2r'}) \) as \( h \to 0 \).

**Proof.** According to \((13)\) we have the equation

\[
(\lambda - \lambda_h^*, u - u_h) = (\text{div} \sigma_h + \lambda_h^* u_h^*, u - u_h^*) - 2(\text{div} \sigma_h + \lambda_h^* u_h^*, u).
\]

Note that the second term on the right side is already of higher order, thus we only consider the rest. The idea is to modify the terms including \( \sigma_h \) such that we can use the results from the previous theorem. By the triangle inequality we have

\[
\|\sigma - \sigma_h\|_0 \leq \|\sigma - \sigma_h^*\|_0 + \|\sigma_h^* - \sigma_h\|_0.
\]

Since the error \( \|\sigma_h^* - \sigma_h\|_0 \) is computable and \( \|\sigma - \sigma_h\|_0 \) can be bounded by the estimator from the previous theorem, we are left with an estimate for the last term on the right hand side of \((17)\).

In contrast to the the proof of Lemma \((7)\) we now add and subtract \( u_h^* \) (and not \( u_h \)) which gives

\[
(\text{div} \sigma_h + \lambda_h^* u_h^*, u - u_h^*) = (\text{div} \sigma_h + \lambda_h^* u_h^*, u - u_h^*) + (\text{div} \sigma_h + \lambda_h^* u_h^*, u - u_h^*)
\]

\[
= (\text{div} \sigma_h + \lambda_h^* u_h^*, u - u_h^*) + (\lambda_h^* u_h^* - \lambda_h u_h, u_h^*).
\]

The last term is computable and will be used in the estimator. For the first one we have using that \( u_h^* \in H_0^1(\Omega) \) and integration by parts

\[
(\text{div} \sigma_h + \lambda_h^* u_h^*, u - u_h^*)
\]

\[
= (\text{div}(\sigma_h - \sigma), u - u_h^*) + (\text{div} \sigma + \lambda_h^* u_h^*, u - u_h^*),
\]

\[
= - (\sigma_h - \sigma, \nabla(u - u_h^*)) + (-\lambda u + \lambda_h^* u_h^*, u - u_h^*),
\]

\[
\leq \|\sigma_h - \sigma\|_0^2 + \|\nabla(u - u_h^*)\|_0^2
\]

\[
+ \lambda_h^* \|u_h^* - u\|_0 \|u - u_h^*\|_0 + |\lambda - \lambda_h^*| \|u - u_h^*\|_0,
\]

\[
\leq |\sigma_h - \sigma|_0^2 + \|\nabla(u - u_h^*)\|_0^2
\]

\[
+ \lambda_h^* \|u_h^* - u\|_0 \|u - u_h^*\|_0 + |\lambda - \lambda_h^*|^2 + \|u - u_h^*\|_0^2.
\]


The first term can be estimated as before, thus for \( h \) small enough we have
\[
|\lambda - \lambda_h| \lesssim \|\sigma - \sigma_h\|_0^2 + \|\nabla (u - u_h)\|_0^2 + \|\sigma_h - \sigma_h^*\|_0^2
+ |(\lambda_h u_h^* - \lambda_h u_h, u_h^* + \tilde{\eta}(h),
\lesssim \eta^2 + \|\sigma_h - \sigma_h^*\|_0^2 + |(\lambda_h u_h^* - \lambda_h u_h, u_h^*)| + \tilde{\eta}(h) + \tilde{\eta}(h).
\]
To show that \( \tilde{\eta}(h) \) and \( \tilde{\eta}(h) \) are of higher order compared to \( O(\rho(h) h^{r+r'} + h^{2r'}) \),
one follows the same steps as in the proof of Theorem 2.

5. Numerical examples

In this section we discuss some numerical examples to validate our theoretical findings. All methods were implemented in the Finite element Library Netgen/NGSolve, see [www.ngsolve.org](http://www.ngsolve.org) and [30].

5.1. Convergence on a unit square. The first example considers the unit square domain \( \Omega = (0,1)^2 \). The eigenfunction and the smallest eigenvalue of (2) is given by \( u = 2 \sin(2\pi x) \sin(2\pi y) \) and \( \lambda = 2\pi^2 \), respectively. We start with an initial mesh with \( |\mathcal{C}_h| = 32 \) elements and use a uniform refinement. Note that for simplicity we used a structured mesh for this example, thus we have
\[
\eta := \frac{\eta^2}{\|\nabla u - \nabla u_h^*\|_0^2 + \|\sigma - \sigma_h^*\|_0^2}, \quad \text{and} \quad \eta^2 := \frac{\eta^2}{\|\nabla u - \nabla u_h^*\|_0^2 + \|\sigma - \sigma_h^*\|_0^2}.
\]
Since \( \Omega \) is convex we have for this example that \( \rho(h) \sim h \), thus we expect the following convergence orders (for simplicity recalled here)
\[
\|u - u_h^*\|_0 \lesssim h^{k+3}, \quad \|\nabla (u - u_h^*)\|_0 \lesssim h^{k+2}, \quad \|\sigma - \sigma_h^*\|_0 \lesssim h^{k+2}, \quad \|\lambda - \lambda_h\|_0 \lesssim h^{2(k+2)}.
\]
In accordance to the theory all errors converge with the optimal orders. Further the high-order term \( \tilde{\eta}(h) \) converges faster than the estimator \( \eta \) as predicted by Theorem 2. Note that this results in an efficiency \( \eta \) converging to one, i.e. the error estimator is asymptotically exact. Also the estimator for the error of the eigenvalue converges appropriately and shows a good efficiency \( \eta^2 \). The same conclusions can be made for \( k = 2 \), however, the error of the eigenvalues \( \lambda_h \) and \( \lambda_h^* \) converge so fast that rounding errors dominate on the finest meshes. For the same reason we also do not present any numbers for \( \tilde{\eta}(h) \) since this term converges even faster resulting in very small numbers already on coarse meshes.

5.2. Adaptive refinement on the L-shape. For the second example we choose the L-shape domain \( \Omega = (-1,1)^2 \setminus ((0,1) \times [-1,0]) \) where the first eigenvalue reads as \( \lambda \approx 9.6397238440219 \). Note that all digits except the last two have been proven to be correct, see [7]. In this example the corresponding eigenfunction is singular, thus we expect a suboptimal convergence of order \( O(N^{-2/3}) \) on a uniform refined mesh, where \( N \) denotes the number of degrees of freedom. To this end we solve
the problem using an adaptive mesh refinement. The refinement loop is defined as usual by

SOLVE $\rightarrow$ ESTIMATE $\rightarrow$ MARK $\rightarrow$ REFINE $\rightarrow$ SOLVE $\rightarrow \ldots$
and is based on the local contributions $\eta(K)$ as element-wise refinement indicators. In the marking step we mark an element if $\eta(K) \geq \frac{1}{4} \max_{K \in C_h} \eta(K)$. The refinement routine then refines all marked elements plus further elements in a closure step to guarantee a regular triangulation. In Figure 1 we see the error history of the post processed eigenvalue $\lambda^*_h$, its estimator $\eta_\lambda$ and the estimator for the eigenfunction error $\eta$ for polynomial order $k = 2, 3$. We can observe an optimal convergence $O(N^{-2(k+2)})$, $O(N^{-2(k+2)})$ and $O(N^{-(k+2)})$, for $|\lambda - \lambda^*_h|$, $\eta_\lambda$ and $\eta$, respectively. Further $\eta_\lambda$ shows a good efficiency.

![Figure 1](image.png)

**Figure 1.** Convergence history of the L-shape example using an adaptive refinement for $k = 2, 3$.

6. **Appendix**

In this section we present a proof of the super convergence estimate
\[
\| \Pi^k u - u_h \|_0 \lesssim \rho(h)(\|u\|_0 + \|u_h\|_0 + \|\sigma - \sigma_h\|_0).
\]

For this we will follow very similar steps as in [12] with several changes in order to get the proper $h$ scaling. We define the auxillary problem: find $\hat{u}_h \in U_h$ and $\hat{\sigma}_h \in \Sigma_h$ such that

\begin{align}
-(\hat{\sigma}_h, \tau_h) - (\text{div} \tau_h, \hat{u}_h) &= 0 \quad \forall \tau_h \in \Sigma_h, \\
-(\text{div} \hat{\sigma}_h, v_h) &= \lambda(u, v_h) \quad \forall v_h \in U_h.
\end{align}

Note that above solution provides the property
\[
\lambda^*(\hat{u}_h, u_h) = -(\text{div} \hat{\sigma}_h, \hat{u}_h) = (\sigma_h, \hat{\sigma}_h) = -(\text{div} \hat{\sigma}_h, u_h) = \lambda(u, u_h).
\]

**Lemma 2.** Let $(\lambda, u, \sigma)$ be the solution of (2), and let $(\hat{u}_h, \hat{\sigma}_h)$ be the solution of (18). There holds the estimate
\[
\| \Pi^k u - \hat{u}_h \|_0 \lesssim \rho(h)(\|\sigma - \hat{\sigma}_h\|_0 + h\|\text{div}(\sigma - \hat{\sigma}_h)\|_0).
\]
Proof. We solve the continuous problem: Find $\Theta \in H(\text{div}, \Omega)$ and $\Psi \in L^2(\Omega)$ such that

\begin{align}
(20a) \quad -& (\Theta, \tau) - (\text{div} \tau, \Psi) = 0 \quad \forall \tau \in H(\text{div}, \Omega),
(20b) \quad -(\text{div} \Theta, v) = \lambda (\Pi^k u - \widehat{u}_h, v) \quad \forall v \in L^2(\Omega).
\end{align}

Note that we have the regularity $\Theta \in H^s(\Omega, \mathbb{R}^d)$ and $\Psi \in H^{1+s}(\Omega)$ with $s > 1/2$, and there holds the stability estimate (see for example $[19]$)

\begin{equation}
\|\Theta\|_s + \|\Psi\|_{1+s} \lesssim \|\Pi^k u - \widehat{u}_h\|_0.
\end{equation}

This then gives

\[\|\Pi^k u - \widehat{u}_h\|_0^2 = -(\text{div} \Theta, \Pi^k u - \widehat{u}_h) = -(\Pi^k \text{div} \Theta, \Pi^k u - \widehat{u}_h),\]

\[= -(\text{div} I_h \Theta, \Pi^k u - \widehat{u}_h) = -(\text{div} I_h \Theta, u - \widehat{u}_h),\]

where we used the commuting diagram property of the BDM-interpolation operator $I_h$ and the $L^2$ projection $\Pi^k$, see $[10]$. By problems $(18)$ and $(20)$ we then have

\[-(\text{div} I_h \Theta, u - \widehat{u}_h) = (I_h \Theta, \sigma - \widehat{\sigma}_h) = (I_h \Theta, \sigma - \sigma_h) + (\Theta, \sigma - \widehat{\sigma}_h) = (I_h \Theta, \sigma - \sigma_h) - (\text{div}(\sigma - \widehat{\sigma}_h), \Psi) = (I_h \Theta - \Theta, \sigma - \widehat{\sigma}_h) + (\text{div}(\sigma - \widehat{\sigma}_h), \Pi^k \Psi - \Psi),\]

where the last step followed by $(\text{div}(\sigma - \widehat{\sigma}_h), \Pi^k \Psi) = 0$. By the interpolation properties of $\Pi^k$ and $I_h$ and the stability $[21]$ we conclude

\[\|\Pi^k u - \widehat{u}_h\|_0^2 \lesssim |\sigma - \widehat{\sigma}_h||h^s|\|\Theta\|_s + \|\text{div}(\sigma - \widehat{\sigma}_h)||h^{1+s}|\|\Psi\|_{1+s}\]

\[\lesssim h^s (|\sigma - \widehat{\sigma}_h|_0 + h|\text{div}(\sigma - \widehat{\sigma}_h)|_0)\|\Pi^k u - \widehat{u}_h\|_0.\]

\[\square\]

Lemma 3. Let $(\lambda, u, \sigma)$ be the solution of $(2)$, $(\lambda_h, u_h, \sigma_h)$ be the solution of $(3)$ and let $(\widehat{u}_h, \widehat{\sigma}_h)$ be the solution of $(18)$. There holds the estimate

\[|\sigma - \widehat{\sigma}_h|_0 + h|\text{div}(\sigma - \widehat{\sigma}_h)|_0 \lesssim h|u_h - \widehat{u}_h|_0 + h|u - u_h|_0 + |\sigma - \sigma_h|_0.\]

Proof. We start with the estimate of the divergence term. By the triangle inequality we have

\[\|\text{div}(\sigma - \widehat{\sigma}_h)\|_0 \leq \|\text{div}(\sigma - \sigma_h)\|_0 + \|\text{div}(\sigma_h - \widehat{\sigma}_h)\|_0,\]

Using $\text{div} \Sigma_h = U_h$ gives

\[\|\text{div}(\sigma_h - \widehat{\sigma}_h)\|_0 = \sup_{v_h \in U_h} \frac{(\text{div}(\sigma_h - \widehat{\sigma}_h), v_h)}{\|v_h\|_0} \leq \sup_{v_h \in U_h} \frac{(\lambda_h u_h - \lambda u, v_h)}{\|v_h\|_0} \leq \|\lambda_h u_h - \lambda u\|_0,\]

thus since also $\|\text{div}(\sigma - \sigma_h)\|_0 = \|\lambda u - \lambda_h u_h\|_0$ we have with $[6]$ and a small enough mesh size $h$ that

\[\|\text{div}(\sigma_h - \widehat{\sigma}_h)\|_0 \lesssim \|\lambda_h u_h - \lambda u\|_0 \lesssim |u - u_h|_0 + |\lambda - \lambda_h| \lesssim |u - u_h|_0 + |\sigma - \sigma_h|_0.\]
For the second term we proceed similarly. The triangle inequality gives \( \| \sigma - \hat{\sigma} \|_0 \leq \| \sigma - \sigma_h \|_0 + \| \sigma_h - \hat{\sigma}_h \|_0 \). For the latter we then have
\[
\| \sigma_h - \hat{\sigma}_h \|_0 = \sup_{\tau_h \in \Sigma_h} \frac{(\sigma_h - \hat{\sigma}_h, \tau_h)}{\| \tau_h \|_0} = \sup_{\tau_h \in \Sigma_h} \frac{(u_h - \hat{u}_h, \text{div} \tau_h)}{\| \text{div} \tau_h \|_0} \lesssim h \| u_h - \hat{u}_h \|_0
\]
where we used that \( \| \text{div} \tau_h \|_0 \lesssim h \| \tau_h \|_0 \) which follows from standard scaling arguments. □

Lemma 4. Let \((\lambda, u, \sigma)\) be the solution of (2), \((\lambda_h, u_h, \sigma_h)\) be the solution of (3) and let \((\hat{u}_h, \hat{\sigma}_h)\) be the solution of (18). There holds the estimate
\[
\| u_h - \hat{u}_h \|_0 \lesssim \| \Pi^k u - \hat{u}_h \|_0.
\]

Proof. Using equation (19) the proof follows with exactly the same steps as in the proof of Lemma 11 in [12] or Lemma 6.3 in [11]. □

Combining above results we have the super convergence property.

Corollary 1. Let \((\lambda, u, \sigma)\) be the solution of (2), \((\lambda_h, u_h, \sigma_h)\) be the solution of (3) and let \((\hat{u}_h, \hat{\sigma}_h)\) be the solution of (18). For \( h \) small enough there holds the super convergence property
\[
\| \Pi^k u - u_h \|_0 \lesssim \rho(h) \left( h \| u - u_h \|_0 + \| \sigma - \sigma_h \|_0 \right).
\]

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DATA AVAILABILITY

All datasets generated during the current study are available in the repository https://doi.org/10.5281/zenodo.6417423

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