Integrable semi-discretizations of the reduced Ostrovsky equation

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Received 17 July 2014, revised 18 January 2015
Accepted for publication 11 February 2015
Published 12 March 2015

Abstract
Based on our previous work on the reduced Ostrovsky equation (J. Phys. A: Math. Theor. 45 355203), we construct its integrable semi-discretizations. Since the reduced Ostrovsky equation admits two alternative representations, one being its original form, the other the differentiated form (the short wave limit of the Degasperis–Procesi equation) two semi-discrete analogues of the reduced Ostrovsky equation are constructed possessing the same N-loop soliton solution. The relationship between these two versions of semi-discretizations is also clarified.

Keywords: integrable discretization, 3-reduction of the BKP/CKP hierarchy, reduced Ostrovsky equation, short wave limit of the Degasperis–Procesi equation

1. Introduction
In this paper, we consider integrable discretizations of the reduced Ostrovsky equation

\[ \partial_x \left( \partial_t + u \partial_x \right) u - 3u = 0, \] (1.1)

which is a special case (\( \beta = 0 \)) of the Ostrovsky equation

\[ \partial_x \left( \partial_t + u \partial_x + \beta \partial_x^3 \right) u - \gamma u = 0. \] (1.2)

The Ostrovsky equation was originally derived as a model for weakly nonlinear surface and internal waves in a rotating ocean [1, 2]. Later on, the same equation was derived for different physical situations by several authors [4, 5]. Equation (1.1) especially appears as a model for...
high-frequency waves in a relaxing medium [5, 6]. Note that the reduced Ostrovsky equation (1.1) is sometimes called the Vakhnenko equation [7–9], the Ostrovsky–Hunter equation [10], or the Ostrovsky–Vakhnenko equation [11, 12]. Travelling wave solutions were investigated in [2, 3, 13]. Vakhnenko et al constructed the N (loop) soliton solution of the reduced Ostrovsky equation using a hodograph (reciprocal) transformation and Hirota’s bilinear method [7, 8]. The same problem was approached from the point of view of the inverse scattering method [9].

Differentiating the reduced Ostrovsky equation (1.1), we obtain

\[ u_{xxx} + 3u_xu_{xx} + uu_{xxx} - 3u_x = 0, \]  

which is known as the short wave limit of the Degasperis–Procesi (DP) equation [14, 15]. The following equation is derived from the DP equation [16]

\[ U_T + 3\epsilon^3 U_x - U_{TXX} + 4UU_X = 3U_XU_{XX} + UUU_{XXX}, \]  

by taking a short wave limit \( \epsilon \to 0 \) with \( U = \epsilon^2 (\mu + cU_t + \cdots), T = \epsilon t, X = \epsilon^{-1}x \) and \( \epsilon = 1 \).

It is noted that the short wave limit of the DP equation can also be rewritten in an alternative form

\[ (\partial_t + \partial_x) m = -3 \, mu_x, \quad m = 1 - u_{xx}. \]  

Based on this connection, Matsuno [15] constructed an N-soliton solution of the short wave model of the DP equation from the N-soliton solution of the DP equation [17, 18]. This N-soliton formula is equivalent to the one obtained by Vakhnenko et al [7, 8].

As already mentioned, the reduced Ostrovsky equation (1.1), as well as its differentiated form (1.3), has attracted much attention in the past. Hone and Wang constructed the Lax pairs for both of equations [14]. The bi-Hamiltonian structure for the reduced Ostrovsky equation (1.1) was found by Brunelli and Sakovich [11]; its integrability and wave-breaking was studied in [19]. Interestingly, the short wave limit of the DP equation (1.3) also serves as an asymptotic model for propagation of surface waves in deep water under the condition of small aspect ratio [20]. Most recently, the inverse scattering transform problem for the short wave limit of the DP equation (1.3) was solved by a Riemann–Hilbert approach [12].

The reduced Ostrovsky equation (1.1) is known to be related to the Tzitzeica equation [21, 23, 24], and also the so-called Dodd–Bullough–Mikhailov equation [25–27], by a reciprocal transformation. Based on this reciprocal link between the reduced Ostrovsky equation and 3-reduction of the B-type or C-type two-dimensional Toda lattice, i.e. the \( A_2^{(2)} \) 2D-Toda lattice, multi-soliton solutions to both the reduced Ostrovsky equation (1.1) and its differentiation version were constructed by the authors [28].

How to construct its integrable discrete analogue for a soliton equation has been an important topic since the discovery of soliton theory. Although several approaches have been developed starting from the mid-1970s, it remains a challenging and mysterious problem and has to be dealt with on a case-by-case base. Ablowitz and Ladik originated a method of integrable discretization based on the Lax pair of a soliton equation [29, 30]. Almost at the same time, Hirota proposed an intriguing and universal approach based on the bilinear form of a soliton equation [31–33]. Another successful way to discretize soliton equations was proposed by Date et al [34–39] via the transformation group theory, which gives a large number of integrable discretizations. One of the most interesting examples is the discrete KP equation, or the so-called Hirota–Miwa equation [39, 40]. This can be viewed as the master equation of discrete systems due to the fact that integrable discretization of many soliton equations, such as the discrete KdV equation and discrete sine-Gordon equation, can be
obtained from the Hirota–Miwa equation by reduction. Suris also developed a general Hamiltonian approach for integrable discretizations of integrable systems, see [41].

The aim of this work is to construct integrable semi-discretizations of the reduced Ostrovsky equation (1.1) and its differentiated form (1.3) by virtue of Hirota’s bilinear method. The remainder of this paper is organized as follows. In section 2, by constructing a semi-discrete analogue of a set of bilinear equations reduced from the period 3-reduction of the $B_\infty$ or $C_\infty$ two-dimensional Toda system, we derive a semi-discrete reduced Ostrovsky equation based on equation (1.3) and provide its N-loop soliton solution in terms of the Pfafian. Then, an alternative semi-discrete reduced Ostrovsky equation is constructed based on equation (1.1) which shares the same N-loop soliton solution. It is interesting that a connection between two semi-discrete versions exists in analogue to a link between their continuous counterparts. We conclude our paper with some comments and further topics in section 4.

2. Integrable semi-discretization of the short wave limit of the DP equation (1.3)

It is shown in [28] that bilinear equations for the reduced Ostrovsky equation (1.3) are

$$\left( \frac{1}{2} D_y D_s - 1 \right) f \cdot f = fg, \quad \left( \frac{1}{2} D_y D_s - 1 \right) g \cdot g = f^2,$$

which originate from a period three-reduction of BKP (CKP) hierarchy [24]. Here $D_y D_s$ is the Hirota $D$-operator as defined by

$$D^m D^n y f(y, s) \cdot g(y, s) = \left( \frac{\partial}{\partial y} - \sum_{i} \frac{1}{m} \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_i} \right)^m \left( \frac{\partial}{\partial s} - \sum_{i} \frac{1}{n} \frac{\partial}{\partial s_i} \frac{\partial}{\partial s_i} \right)^n f(y, s) g(y', s') \bigg|_{y=y', s=s'}.$$

For the sake of convenience, we set $y=x_1$, $s=x_{-1}$. Under this reduction, one of the tau-functions $f$ turns out to be a square of a Pfafian [42]

$$\tau^2 = cf,$$  

where $\tau = \text{Pf}(1, 2, \cdots, 2N)$ is a Pfafian whose elements are given by

$$\text{Pf}(i, j) = c_{i,j} + \frac{p_i - p_j}{p_i + p_j} \exp(z_i + z_j),$$

with

$$c_{i,j} = -c_{j,i}, \quad \xi_i = p_i y + p_i^{-1} s + \xi_0, \quad c = \prod_{i=1}^{2N} 2p_i.$$

It was shown in [28] that bilinear equations (2.1) and (2.2), together with (2.3), yield the reduced Ostrovsky equation (1.3) through a hodograph transformation

$$x = y - 2(\ln r)_s, \quad t = s,$$

and a dependent transformation

$$u = -2(\ln r)_{ss} = -(\ln f)_{ss}.$$
Remark 2.1. In accordance with the integrable discretizations, which will be constructed hereafter, we choose an alternative hodograph transformation mentioned in remark 2.15 of [28].

2.1. Semi-discrete analogues of equations (2.1)–(2.3)

Based on the results briefly mentioned above, we attempt to construct an integrable semi-discrete analogue of the reduced Ostrovsky equation (1.3). The key point is how to discretize the bilinear equations (2.1) and (2.3). To this end, we start with Gram-type determinants

$$g_l = \det_{1 \leq i, j \leq 2N} \left( m_{ij}(l) \right), \quad f_l = \det_{1 \leq i, j \leq 2N} \left( m_{ij}'(l) \right),$$

where

$$m_{ij}(l) = C_{ij} + \frac{1}{p_i + p_j} \phi_i^{(0)}(l) \phi_j^{(0)}(l),$$

$$m_{ij}'(l) = C_{ij} + \frac{1}{p_i + p_j} \left( -\frac{p_j}{p_i} \right) \frac{1 + bp_i}{1 - bp_j} \phi_i^{(0)}(l) \phi_j^{(0)}(l),$$

with

$$C_{ij} = C_{ji}, \quad \phi_i^{(n)}(l) = p_i^l \left( \frac{1 + bp_i}{1 - bp_j} \right) e^{\xi_i}, \quad \xi_i = p_i^{-1}s + \xi_{i0}.$$ 

Here $2b$ (not $b$) is the mesh size in the $y$-direction. A relation between $f_l$ and $g_l$ is shown by the following lemma.

Lemma 2.2. The above determinants of $f_l$ and $g_l$ satisfy

$$(D_l - 2b) g_{l+1}, \quad g_l = -2bf_l^2. \quad (2.7)$$

Proof. It can be easily verified that

$$\partial_\phi m_{ij}(l) = \phi_i^{(-1)}(l) \phi_j^{(-1)}(l),$$

$$m_{ij}(l + 1) = m_{ij}(l) + \frac{2b}{(1 - bp_i)(1 - bp_j)} \phi_i^{(0)}(l) \phi_j^{(0)}(l),$$

and

$$m_{ij}'(l) = m_{ij}(l) - \frac{1}{1 - bp_j} \phi_i^{(-1)}(l) \phi_j^{(0)}(l).$$

Then by using the following formulas for a $N \times N$ determinant $M$ with $M_{ij}$ denoting the cofactor of the element $m_{ij}$
we have

\[
\frac{\partial}{\partial s}|M| = \sum_{i,j=1}^{N} \frac{\partial m_{ij}}{\partial s} M_{ij}, \quad \left| \begin{array}{cc} m_{ij} & a_i \\ b_j & d \end{array} \right| = d |M| - \sum_{i,j=1}^{N} a_i b_j M_{ij},
\]

\[
\partial g_i = \begin{vmatrix} m_{ij}(l) & \phi_i^{(-1)}(l) \\ -\phi_j^{(-1)}(l) & 0 \end{vmatrix},
\]

\[
g_{i+1} = \begin{vmatrix} m_{ij}(l) & \frac{2b}{1-bp_j} \phi_i^{(0)}(l) \\ -\frac{1}{1-bp_j} \phi_j^{(0)}(l) & 1 \end{vmatrix},
\]

\[
f_i = \begin{vmatrix} m_{ij}(l) & \phi_i^{(-1)}(l) \\ \frac{1}{1-bp_j} \phi_j^{(0)}(l) & 1 \end{vmatrix} = \begin{vmatrix} m_{ij}(l) & \frac{1}{1-bp_i} \phi_i^{(0)}(l) \\ \phi_j^{(-1)}(l) & 1 \end{vmatrix}.
\]

Furthermore, we can show

\[
(\partial_i - 2b) g_{i+1} = \begin{vmatrix} m_{ij}(l) & \phi_i^{(-1)}(l) & \frac{2b}{1-bp_i} \phi_i^{(0)}(l) \\ -\phi_j^{(-1)}(l) & 0 & 0 \\ -\frac{1}{1-bp_j} \phi_j^{(0)}(l) & 0 & 1 \end{vmatrix} + \begin{vmatrix} m_{ij}(l) & \frac{2b(\partial_i - b)}{1-bp_i} \phi_i^{(0)}(l) \\ -\frac{1}{1-bp_j} \phi_j^{(0)}(l) & -b \end{vmatrix} + \begin{vmatrix} m_{ij}(l) & \frac{2b}{1-bp_i} \phi_i^{(0)}(l) \\ (\partial_i - b) \frac{1}{1-bp_j} \phi_j^{(0)}(l) & -b \end{vmatrix} + \begin{vmatrix} m_{ij}(l) & \phi_i^{(-1)}(l) & \frac{2b}{1-bp_i} \phi_i^{(0)}(l) \\ -\phi_j^{(-1)}(l) & 0 & 0 \\ -\frac{1}{1-bp_j} \phi_j^{(0)}(l) & 0 & 1 \end{vmatrix}.
\]
By using the Jacobi identity of determinant and the relations (2.8)–(2.11), we obtain
\[ \partial_{ij} = \partial_{ij} - \partial_{ij} \times \partial_{ij} \times \partial_{ij} \times \partial_{ij} \]
which is nothing but equation (2.7). □

**Remark 2.3.** Equation (2.7) is an integrable discretization of the bilinear equation (2.2) in the y-direction. Note that \( b \) is the mesh size. In the limit of \( b \to 0 \), we have
\[ f_i \to f, \quad g_i \to g, \quad g_{i+1} \to g + 2bg_y, \]
then it follows that
\[ \frac{1}{2b} D_{ij} g_{i+1} \cdot g_i = \frac{1}{2} D_{ij} g \cdot g. \]
Therefore, equation (2.7) converges to equation (2.2) as \( b \to 0 \).

Next, we perform reduction in order to mimic the period 3-reduction of CKP/BKP hierarchy in the continuous case. To this end, we let \( C_{ij} \) take a special value as follows
\[ C_{ij} = \delta_{j,2N+1-i}c_i, \quad c_{2N+1-i} = c_i, \]  
and further assume
\[ c_{ij} = -C_{ij} \frac{2p_i^2}{p_j} \frac{1 - bp_j}{1 + bp_j}. \]

By imposing a reduction condition
\[ p_i^3 \left( 1 - b^2p_{2N+1-i}^2 \right) = -p_{2N+1-i}^3 \left( 1 - b^2p_i^2 \right), \]
which can be written as
\[ \frac{p_i^3 \left( 1 - bp_{2N+1-i} \right)}{p_{2N+1-i} \left( 1 + bp_i \right)} = -\frac{p_{2N+1-i}^3 \left( 1 - bp_i \right)}{p_i \left( 1 + bp_{2N+1-i} \right)}, \]
it then follows that
\[
c_{i,j} = -\delta_{j,2N+1-i}c_i \frac{2p_j^2}{p_{2N+1-i}} \frac{1 - bp_{2N+1-i}}{1 + bp_i} = \delta_{i,2N+1-j}c_{2N+1-i} \frac{2p_{2N+1-i}^2}{p_i} \frac{1 - bp_i}{1 + bp_{2N+1-i}} = -c_{j,i}.
\]

Thus, we can define a Pfaffian
\[
\tau_i = \text{Pf}(1,2,\cdots, 2N)_i
\]
whose elements are
\[
(i,j)_i = c_{i,j} + \frac{p_i - p_j}{p_i + p_j} \phi_i^{(0)}(l)\phi_j^{(0)}(l).
\]

The relations between the Pfaffian \(\tau_i\) and the Gram-type determinants \(f_i\), \(g_i\) are stated by the following lemma.

**Lemma 2.4.** The above Pfaffian \(\tau_i\) and determinants \(f_i\), \(g_i\) satisfy
\[
(D_i - b)\tau_{i+1} \cdot \tau_i = -bcg_{i+1}, \quad (2.15)
\]
\[
\tau_i^2 = cf_i, \quad (2.16)
\]
where
\[
c' = \prod_{i=1}^{2N} \frac{2p_i^2}{1 + bp_i}.
\]

**Proof.** We first list two Pfaffian identities which will be used in the proof
\[
\text{Pf}_{1 \leq i < j \leq 2N} \left( \begin{array}{cc}
\delta a_{ij} - a_i b_j + a_j b_i \\
a_i & b_j \\
\end{array} \right) = \text{Pf} \left( \begin{array}{c}
a_{ij} \\
a_i \\
\end{array} \right), \quad (2.17)
\]
\[
\text{det}_{1 \leq i < j \leq 2N} \left( \begin{array}{ccc}
a_{ij} & a_i & b_j \\
\alpha & a & \beta \\
\gamma & d_j & \delta \\
\end{array} \right) = \text{Pf} \left( \begin{array}{c}
\alpha a_{ij} - a_i c_j + a_j c_i \\
\beta a_{ij} - b_i d_j + b_j d_i \\
\gamma a_{ij} - a_i d_j + a_j d_i \\
\end{array} \right) \text{Pf} \left( \begin{array}{c}
\delta a_{ij} - b_i c_j + b_j c_i \\
\end{array} \right). \quad (2.18)
\]

Since
\[
\delta_{i,i} = \phi_i^{(0)}(l)\phi_j^{(-1)}(l) - \phi_i^{(-1)}(l)\phi_j^{(0)}(l),
\]
\[
(i,j)_{i+1} = (i,j)_i + \phi_i^{(0)}(l+1)\phi_j^{(0)}(l) - \phi_i^{(0)}(l)\phi_j^{(0)}(l+1),
\]
\[
(D_i - b)(i,j)_{i+1} = -b(i,j)_i + \phi_i^{(0)}(l+1)\phi_j^{(-1)}(l) - \phi_i^{(-1)}(l)\phi_j^{(0)}(l+1),
\]

\[\text{J. Phys. A: Math. Theor. 48 (2015) 135203} \quad \text{B-F Feng et al} \]

\[\]
we have

$$\tau_l = \text{Pf}\left( (i, j)_l \begin{array}{c} \varphi_i^{(0)}(l) \\ \varphi_i^{(0)}(l+1) \end{array} \right),$$  \hspace{1cm} (2.19)

$$\partial_s \tau_l = \text{Pf}\left( (i, j)_l \begin{array}{c} \varphi_i^{(-1)}(l) \\ \varphi_i^{(0)}(l) \end{array} \right),$$  \hspace{1cm} (2.20)

$$\tau_{l+1} = \text{Pf}\left( (i, j)_l \begin{array}{c} \varphi_i^{(0)}(l) \\ \varphi_i^{(0)}(l+1) \end{array} \right),$$  \hspace{1cm} (2.21)

$$(\partial_s - b)\tau_{l+1} = \text{Pf}\left( (i, j)_l \begin{array}{c} \varphi_i^{(-1)}(l) \\ \varphi_i^{(0)}(l+1) \end{array} \right),$$  \hspace{1cm} (2.22)

by referring to the identity (2.17). Furthermore, by using the Pfaffian identify (2.18)

$$(D_s - b)\tau_{l+1} \cdot \tau_l = \tau_l (\partial_s - b)\tau_{l+1} - \tau_{l+1}\partial_s \tau_l$$

$$= \text{Pf}\left( (i, j)_l \begin{array}{c} \varphi_i^{(-1)}(l) \\ \varphi_i^{(0)}(l+1) \end{array} \right) \text{Pf}\left( (i, j)_l \begin{array}{c} \varphi_i^{(0)}(l) \\ \varphi_i^{(0)}(l+1) \end{array} \right)$$

$$- \text{Pf}\left( (i, j)_l \begin{array}{c} \varphi_i^{(0)}(l) \\ \varphi_i^{(0)}(l) \end{array} \right) \text{Pf}\left( (i, j)_l \begin{array}{c} \varphi_i^{(-1)}(l) \\ \varphi_i^{(0)}(l) \end{array} \right)$$

$$= \begin{vmatrix} m_{ij}(l) & \varphi_i^{(-1)}(l) & \varphi_i^{(0)}(l) \\ \varphi_j^{(0)}(l+1) & -b & 1 \\ \varphi_j^{(0)}(l) & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} m_{ij}(l) - \varphi_i^{(0)}(l)\varphi_j^{(0)}(l) & \varphi_i^{(-1)}(l) & \varphi_i^{(0)}(l) \\ \varphi_j^{(0)}(l+1) & -b & 1 \\ \varphi_j^{(0)}(l) & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} m_{ij}(l) - \varphi_i^{(1)}(l)\varphi_j^{(1)}(l) & \varphi_i^{(-1)}(l) \\ b\left(\varphi_j^{(0)}(l+1) + \varphi_j^{(0)}(l)\right) & -b \end{vmatrix}$$

$$= -b\text{det}\left( m_{ij}(l) - \varphi_i^{(0)}(l)\varphi_j^{(0)}(l) + \varphi_i^{(-1)}(l)(\varphi_j^{(1)}(l+1) + \varphi_j^{(1)}(l)) \right)$$

$$= -b\text{det}\left( c_{i,j} + \frac{2p_i^2(1+bp_i)}{p_i(p_i+bp_i)}\varphi_i^{(0)}(l)\varphi_j^{(0)}(l) \right)$$

$$= -bc'\text{det}\left( \frac{p_i(1-bp_i)}{2p_i^2(1+bp_i)}c_{i,j} + \frac{1}{p_i+p_j}\varphi_i^{(0)}(l)\varphi_j^{(0)}(l) \right)$$

$$= -bc'\text{det}\left( C_{i,j} + \frac{1}{p_i+p_j}\varphi_i^{(0)}(l)\varphi_j^{(0)}(l) \right)$$

$$= -bc'g_{l+1}. \hspace{1cm} (2.23)$$
Thus equation (2.15) is proved. Next, we prove the relation (2.16).

\[ f_i = \det \left( \delta_{j,2N+1-i}c_i - \frac{p_j}{p_i} \frac{1 + b p_i}{1 - b p_j} \phi_i^{(0)}(l) \phi_j^{(0)}(l) \right) \]

\[ = \prod_{i=1}^{2N} \left( \frac{1 + b p_i}{2p_i(1 - b p_i)} \det \left( -\delta_{j,2N+1-i}c_i \frac{2p_j^2 - 1 - b p_j}{p_j} \frac{1 + b p_i}{1 + b p_j} + \frac{2p_i}{p_i + p_j} \phi_i^{(0)}(l) \phi_j^{(0)}(l) \right) \right) \]

\[ = \frac{1}{c'} \left( d_{i,j} + \left( \frac{p_i - p_j}{p_i + p_j} - 1 \right) \phi_i^{(0)}(l) \phi_j^{(0)}(l) \right)^2. \]  

(2.24)

Therefore equation (2.16) holds. □

**Remark 2.5.** Obviously, equation (2.16) converges to (2.3) as \( b \to 0 \) since \( \tau_l \to \tau, f_i \to f \) and \( c' \to c \) under this limit.

**Remark 2.6.** Multiplying both sides of equation (2.15) by \( 2\tau_l \tau_{l+1} \), we have

\[ (D_s - 2b) \tau_{l+1} \tau_l = -2bc' g_{l+1} \tau_{l+1} \tau_l \]

using a bilinear identify \( D_s f^2 \cdot g^2 = 2fgD_s f \cdot g \). Furthermore, by referring to the relation (2.3), we have

\[ \left( \frac{1}{2b} D_s - 1 \right) f_i^2 \cdot f_j^2 = -\frac{1}{c'} g_{l+1} \tau_{l+1} \tau_l \]

which converges to (2.1) as \( b \to 0 \) since \( g_{l+1} \to g \) and \( \tau_{l+1} \tau_l/c' \to \tau^2/c = f \) under this limit.

### 2.2. Integrable semi-discretization of the short wave limit of the DP equation (1.3)

Summarizing what we have discussed in the previous subsection, the following three relations

\[ (D_s - 2b) g_{l+1} \cdot g_l = -2bf_i^2, \]  

(2.25)

\[ (D_s - b) \tau_{l+1} \cdot \tau_l = -bc' g_{l+1}, \]  

(2.26)

\[ \tau_l^2 = c'f_i, \]  

(2.27)

constitute the semi-discrete analogue of bilinear equations (2.1)–(2.3). Let us construct an integrable semi-discretization of the reduced Ostrovsky equation based on bilinear equations (2.25)–(2.27). First, we rewrite equations (2.25) and (2.26) into

\[ \left( \ln \frac{g_{l+1}}{g_l} \right)_l - 2b = -2bf_i^2 \frac{f_i^2}{g_{l+1}g_l}, \]  

(2.28)
and

$$\left( \ln \frac{\tau_{i+1}}{\tau_i} \right) - b = -bc \frac{\Delta s_{i+1}}{\tau_{i+1} \tau_i}$$

respectively. Introducing a discrete hodograph transformation

$$x_i = 2lb - 2(\ln \tau_i)_s, \quad t = s,$$

and a dependent variable transformation

$$u_t = -2(\ln \tau_i)_s = -\left( \ln f_i \right)_s,$$

it then follows that the nonuniform mesh, which is defined by \( \delta_i = x_{i+1} - x_i \), can be expressed as

$$\delta_i = 2b - 2 \left( \ln \frac{\tau_{i+1}}{\tau_i} \right)_s = 2bc \frac{\Delta s_{i+1}}{\tau_{i+1} \tau_i}$$

with the use of (2.29). Differentiating equation (2.32) with respect to \( s \), one obtains

$$\frac{d\delta_i}{ds} = -2 \left( \ln \frac{\tau_{i+1}}{\tau_i} \right)_s = u_{i+1} - u_i.$$  

Introducing an auxiliary variable \( \eta = f_i / g_i \), we then have

$$\frac{4}{\delta_i} = \frac{1}{b^2} \frac{g_i}{g_{i+1}} \eta \eta_{i+1},$$

where (2.27) is used. Further, one obtains

$$\left( \ln \frac{\eta_{i+1}}{\eta_i} \right)_s + \delta_i = 2b \frac{f_i^2}{g_{i+1} g_i}$$

by referring to (2.29) and (2.28). Taking the logarithmic derivative of (2.34) with respect to \( s \) leads to

$$\left( \ln \frac{\eta_{i+1}}{\eta_i} \right)_s - \left( \ln \frac{g_i}{g_{i+1}} \right)_s = -\frac{2}{\delta_i} \frac{d\delta_i}{ds}.$$  

Substituting equation (2.33) into equation (2.36) and referring to equations (2.28) and (2.35), one obtains

$$\frac{u_{i+1} - u_i}{\delta_i} = \frac{1}{2} \left( \ln \frac{\eta_{i+1}}{\eta_i} \right)_s + \frac{1}{2} \left( \ln \frac{g_{i+1}}{g_i} \right)_s \quad \text{or} \quad \frac{u_{i+1} - u_i}{\delta_i} = -\frac{1}{2} (\ln \eta_{i+1})_s + b - \frac{f_i^2}{g_i g_{i+1}}$$

$$= -\frac{1}{2} (\ln \eta_{i+1})_s + b - \frac{1}{2} \left( \ln \frac{\eta_{i+1}}{\eta} \right)_s - \frac{1}{2} \delta_i$$

$$= -(\ln \eta_i)_s + b - \frac{1}{2} \delta_i.$$
which can be recast into
\[
(\ln \eta_{i+1})_t = -\frac{u_{i+1} - u_i}{\delta_i} + b - \frac{1}{2} \delta_i. \tag{2.38}
\]

A substitution of (2.38) back into (2.35) leads to
\[
-\frac{u_{i+1} - u_i}{\delta_i} + u_i - u_{i-1} + \frac{1}{2} \delta_i + \frac{1}{2} \delta_{i-1} = \frac{8b^3}{\delta_i^2} \eta_{i+1}. \tag{2.39}
\]

Defining
\[
m_i = \frac{2}{\delta_i + \delta_{i-1}} \left( - \frac{u_{i+1} - u_i}{\delta_i} + \frac{u_i - u_{i-1}}{\delta_{i-1}} \right) + 1,
\]
and taking the logarithmic derivative on both sides of (2.39), we have
\[
\frac{d \ln m_i}{ds} = (\ln \eta)_t - (\ln \eta_{i+1})_t - \frac{2}{\delta_i} \frac{d \delta_i}{ds} - \frac{d}{ds} \left( \delta_i + \delta_{i-1} \right)
= -\frac{u_{i+1} - u_{i-1}}{\delta_{i-1}} + \frac{1}{2} \delta_{i-1} + \frac{u_{i+1} - u_i}{\delta_i} + \frac{1}{2} \delta_i - \frac{2(u_{i+1} - u_i)}{\delta_i} = \frac{u_{i+1} - u_{i-1}}{\delta_i + \delta_{i-1}}
= -\frac{u_i - u_{i-1}}{\delta_{i-1}} - \frac{u_{i+1} - u_i}{\delta_i} - \frac{u_{i+1} - u_{i-1}}{\delta_i + \delta_{i-1}} - \frac{1}{2} \left( \delta_i - \delta_{i-1} \right). \tag{2.40}
\]

As a result, by defining forward difference and average operators
\[
\Delta u_i = \frac{u_{i+1} - u_i}{\delta_i}, \quad M u_i = \frac{u_i + u_{i-1}}{2},
\]
we can summarize what we have deduced into the following theorem.

**Theorem 2.7.** The semi-discrete analogue of the short wave limit of the DP equation
\[
\begin{align*}
\frac{d m_i}{ds} &= m_i \left( -2 \Delta u_i - \frac{M(\delta_i \Delta u_i)}{M \delta_i} - \frac{1}{2}(\delta_i - \delta_{i-1}) \right), \\
\frac{d \delta_i}{ds} &= u_{i+1} - u_i, \\
m_i &= \frac{-2 \Delta u_i}{M \delta_i} + 1,
\end{align*}
\]
is determined from the following equations
\[
\begin{align*}
(D_s - 2b)\dot{g}_{i+1} \cdot \dot{g}_i &= -2b \dot{f}_i^2, \\
(D_s - b)\dot{f}_{i+1} \cdot \dot{f}_i &= -bc \dot{g}_{i+1}, \\
\tau_i^2 &= c' \dot{f}_i,
\end{align*}
\]
through discrete hodograph transformation \( x_t = 2tb - 2(\ln \eta)_t \), \( \delta_i = x_{i+1} - x_i \), \( t = s \) and dependent variable transformation \( u_i = -2(\ln \eta)_t = -(\ln f)_t \).

Let us consider the continuous limit when \( b \to 0 \). The dependent variable \( u \) is a function of \( l \) and \( s \). Meanwhile, we regard it as a function of \( x \) and \( t \), where \( x \) is the space coordinate at the \( l \)th lattice point and \( t \) is the time, defined by
Then in the continuous limit, $b \to 0$ ($\delta_i \to 0$), we have

$$2M \Delta u_t = \frac{u_{t+1} - u_t}{\delta_t} + \frac{u_t - u_{t-1}}{\delta_{t-1}} \to 2u_x,$$

$$M \left( \frac{\delta_t \Delta u_t}{\delta_t} \right) = \frac{u_{t+1} - u_{t-1}}{\delta_t + \delta_{t-1}} \to u_x.$$ 

Moreover, since

$$\frac{\partial x}{\partial s} = \frac{\partial x_0}{\partial s} + \sum_{j=0}^{l-1} \partial \delta_j = \frac{\partial x_0}{\partial s} + \sum_{j=0}^{l-1} (u_{j+1} - u_j) \to u,$$

we then have

$$\partial_x = \partial_t + \frac{\partial x}{\partial s} \partial_x \to \partial_t + u \partial_x.$$ 

Consequently, the third equation in (2.41) converges to $m = 1 - u_{xx}$, whereas the first equation in (2.41) converges to

$$(\partial_t + u \partial_x)m = -3mu_x,$$  

which is exactly the short wave limit of the DP equation (1.3).

Based on the results in the previous section, we can provide an $N$-soliton solution to the semi-discrete reduced Ostrovsky equation

**Theorem 2.8.** The $N$-soliton solution to the semi-discrete analogue for the short wave limit of the DP equation (2.41) takes the following parametric form

$$u_t = -2(1\ln \tau_i)_{xx}, \quad x_t = 2lb - 2(\ln \tau_i)_s$$

where $\tau_i$ is a Pfaffian

$$\tau_i = \text{Pf}(1,2,\cdots,2N)_i,$$  

whose elements are

$$(i,j)_i = c_{ij} + \frac{p_i - p_j}{p_i + p_j} \phi_i^{(0)}(l) \phi_j^{(0)}(l), \quad \phi_i^{(0)}(l) = p_i^n \left( \frac{1 + bp_i}{1 - bp_i} \right)^{i-1} e^{p_i^3 l + p_i^2 s},$$

under the reduction condition

$$p_i^3 \left( 1 - b^2 p_{2N+1-i}^2 \right) = -p_i^3 \left( 1 - b^2 p_{2N+1-i}^2 \right), \quad i = 1, 2, \ldots, N.$$  

3. Integrable semi-discretization of the reduced Ostrovsky equation (1.1)

3.1. Bilinear equation for the reduced Ostrovsky equation (1.1)

In this section, we will deduce an integrable semi-discrete analogue to the reduced Ostrovsky equation (1.1). It was pointed out by the authors that a single bilinear equation (2.53) in [28]
yields the reduced Ostrovsky equation (1.1). In order to be consistent with the N-soliton solution given in the previous section, we start with

$$\left( D_{x,1} - D_{x,0}^3 \right) \tau + 3D_{x,2}^2 \tau = 0,$$

(3.1)

which is a dual bilinear equation of (2.47) in [28] for the extended BKP hierarchy. Imposing the same period 3-reduction by requesting $D_{x,1} = D_{x,3} = 0$ and assuming $y = x_1$, $s = x_{-1}$, equation (3.1) is reduced to

$$\left( D_s D_{x}^3 - 3D_{x}^2 \right) \tau \cdot \tau = 0. \quad (3.2)$$

Prior to proceeding to the semi-discretization of equation (1.1), let us briefly show how the reduced Ostrovsky equation (1.1) is derived from equation (3.2) through the same hodograph transformation (2.5) and dependent variable transformation (2.6) defined in the previous section. By defining $\rho^{-1} = 1 - 2(\ln r)_y$, a conversion formula

$$\left\{ \begin{array}{l}
\frac{\partial}{\partial y} = \frac{1}{\rho} \frac{\partial}{\partial x}, \\
\frac{\partial}{\partial x} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x},
\end{array} \right. \quad (3.3)$$

can be easily obtained from the hodograph transformation (2.5). By using the relations

$$\frac{D_s D_{x}^3 \tau \cdot \tau}{\tau^2} = 2(\ln r)_{xys} + 12(\ln r)_{xs}(\ln r)_{ys},$$

$$\frac{D_x^2 \tau \cdot \tau}{\tau^2} = 2(\ln r)_{xs},$$

equation (3.2) is converted to

$$2(\ln r)_{xys} = 6(\ln r)_x \left( 1 - 2(\ln r)_y \right), \quad (3.4)$$

and is further reduced to

$$\rho u_{ys} = 3u. \quad (3.5)$$

With the use of the conversion formulas (3.3), we finally arrive at

$$\partial_1 (\partial_1 + u \partial_1) u = 3u, \quad (3.6)$$

which is exactly the reduced Ostrovsky equation (1.1).

3.2. Semi-discrete analogue of the reduced Ostrovsky equation (1.1)

In order to obtain a discrete analogue for the bilinear equation (3.2), we first prove a bilinear equation associated with the modified BKP.

Lemma 3.1. Assume a Pfaffian $\tau_l = Pf(1,2, \cdots, 2N)_l$ with an element determined by

$$c_{ij} = \frac{p_i - p_j}{p_i + p_j} \phi^{(0)}_i (l) \phi^{(0)}_j (l), \quad (3.7)$$

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where

\[
\varphi_{i}^{(n)}(l) = p_i^n \left( \frac{1 + b p_i}{1 - b p_i} \right)^l e^{\xi_l}, \quad \xi_l = p_i^{-1} s + p_i^{-3} r + \xi_{0,l}.
\]

Then the Pfaffian \( \tau_i \) satisfies the following bilinear equation

\[
\left( (D_r - b)^3 - (D_r - b^3) \right) \tau_{i+1} \cdot \tau_i = 0. \quad (3.8)
\]

**Proof.** First, we define the Pfaffian elements in addition to (3.7):

\[
\text{Pf}(i, d_m) = \varphi_{i}^{(n)}(l), \quad \text{Pf}(d_m, d_n) = 0,
\]

\[
\text{Pf}(i, d_i') = \varphi_{i}^{(0)}(l + 1), \quad (d_m, d_i')_{kl} = (-b)^{m}.
\]

Then the following differential and difference formulas are obtained previously or can be easily verified

\[
\partial_j \tau_{l} = \text{Pf} \left( (i, j) \mid \varphi_{i}^{(-1)}(l) \varphi_{i}^{(0)}(l) \right) = \text{Pf} \left( 1, 2, \ldots, 2N, d_{-1}, d_0 \right),
\]

\[
\partial_j^2 \tau_{l} = \text{Pf} \left( (i, j) \mid \varphi_{i}^{(-2)}(l) \varphi_{i}^{(0)}(l) \right) = \text{Pf} \left( 1, 2, \ldots, 2N, d_{-2}, d_0 \right),
\]

\[
\tau_{l+1} = \text{Pf} \left( (i, j) \mid \varphi_{i}^{(0)}(l) \varphi_{i}^{(0)}(l + 1) \right) = \text{Pf} \left( 1, 2, \ldots, 2N, d_0, d_i' \right),
\]

\[
(\partial_j - b) \tau_{l+1} = \text{Pf} \left( (i, j) \mid \varphi_{i}^{(-1)}(l) \varphi_{i}^{(0)}(l + 1) \right) = \text{Pf} \left( 1, 2, \ldots, 2N, d_{-1}, d_i' \right),
\]

\[
(\partial_j - b)^2 \tau_{l+1} = \text{Pf} \left( (i, j) \mid \varphi_{i}^{(-2)}(l) \varphi_{i}^{(0)}(l + 1) \right) = \text{Pf} \left( 1, 2, \ldots, 2N, d_{-2}, d_i' \right),
\]

\[
\frac{1}{3} (\partial_j^3 - \partial_j) \tau_{l+1} = \text{Pf} \left( (i, j) \mid \varphi_{i}^{(-2)}(l) \varphi_{i}^{(-1)}(l) \right) = \text{Pf} \left( 1, 2, \ldots, 2N, d_{-2}, d_{-1} \right).
\]

Moreover, the following relations can be further verified

\[
(\partial_j - b^3) \tau_{l+1} = \text{Pf} \left( (i, j) \mid \varphi_{i}^{(-3)}(l) \varphi_{i}^{(0)}(l + 1) \right) = \text{Pf} \left( 1, 2, \ldots, 2N, d_{-1}, d_i' \right),
\]

\[
-2 \text{Pf} \left( (i, j) \mid \varphi_{i}^{(2)}(l) \varphi_{i}^{(-1)}(l) \varphi_{i}^{(0)}(l) \varphi_{i}^{(0)}(l + 1) \right) = \text{Pf} \left( 1, 2, \ldots, 2N, d_{-1}, d_0, d_i' \right),
\]

\[
-2 \text{Pf} \left( (i, j) \mid \varphi_{i}^{(-2)}(l) \varphi_{i}^{(-1)}(l) \varphi_{i}^{(0)}(l) \varphi_{i}^{(0)}(l + 1) \right) = \text{Pf} \left( 1, 2, \ldots, 2N, d_{-1}, d_{-1}, d_0, d_i' \right).
\]
\[(\partial_x - b)^3 \tau_{l+1} = \text{Pf} \left( (i, j), \begin{pmatrix} \phi_i^{(3)}(l) & \phi_i^{(0)}(l + 1) \\ -b^3 & \partial \end{pmatrix} \right) \]

\[
+ \text{Pf} \left( (i, j), \begin{pmatrix} \phi_i^{(2)}(l) & \phi_i^{(1)}(l) & \phi_i^{(0)}(l + 1) \\ 0 & 0 & b^2 \\ 0 & -b & 1 \end{pmatrix} \right) \]

\[
= \text{Pf} \left( 1, 2, \ldots, 2N, d_{-1}, d_{l} \right) + \text{Pf} \left( 1, 2, \ldots, 2N, d_{-2}, d_{-1}, d_{0}, d_{l} \right),
\]

thus, we get
\[
\frac{1}{3} \left( (\partial_x - b)^3 - (\partial_y - b^3) \right) \tau_{l+1} = \text{Pf} \left( 1, 2, \ldots, 2N, d_{-2}, d_{-1}, d_{0}, d_{l} \right).
\]

Then an algebraic identity of the Pfaffian [42]
\[
\text{Pf} \left( \ldots, d_{-2}, d_{-1}, d_{0}, d_{l} \right) \text{Pf}(\ldots) = \text{Pf}(\ldots, d_{-2}, d_{-1}) \text{Pf}(\ldots, d_{0}, d_{l})
- \text{Pf}(\ldots, d_{-2}, d_{0}) \text{Pf}(\ldots, d_{-1}, d_{l})
+ \text{Pf}(\ldots, d_{-2}, d_{l}) \text{Pf}(\ldots, d_{0}, d_{l}),
\]

derives
\[
\frac{1}{3} \left( (\partial_x - b)^3 - (\partial_y - b^3) \right) \tau_{l+1} \times \tau_l = \frac{1}{3} \left( \partial_x^3 - \partial_y \right) \tau_l \times \tau_{l+1} - \partial_x \tau_l
\]
\[
\times (\partial_x - b) \tau_{l+1} + (\partial_y - b)^2 \tau_{l+1} \times \partial_y \tau_l,
\]

which is equivalent to
\[
\left( (D_x - b)^3 - (D_y - b^3) \right) \tau_{l+1} \cdot \tau_l = 0.
\]

Next, we preform a reduction in parallel to period three-reduction for the continuous case. Imposing the same reduction condition (2.45), which is also of the form
\[
\frac{1}{p_i^3} + \frac{1}{p_{2N+1-i}^3} = b^2 \left( \frac{1}{p_i} + \frac{1}{p_{2N+1-i}} \right),
\]

and note that \(\tau_l\) is rewritten as
\[
\tau_l = \prod_{i=1}^{2N} \phi_i^{(0)}(l) \text{Pf} \left( \frac{\delta_{j,2N+1-i}c_{i,j}}{\phi_i^{(0)}(l)\phi_{2N+1-i}^{(0)}(l)} + \frac{p_i - p_j}{p_i + p_j} \right),
\]

it can easily shown that the Pfaffian \(\tau_l\) satisfies
\[
\partial_x \tau_l = b^2 \partial_x \tau_l,
\]

therefore we have
\[
\left( D_x^3 - 3bD_x^2 + 2b^2D_x \right) \tau_{l+1} \cdot \tau_l = 0.
\]
In what follows, we construct a semi-discrete reduced Ostrovsky equation based on equation (3.10). First, using the following relations

\[
\begin{align*}
\frac{D_t \tau_{i+1} \cdot \tau_i}{\tau_{i+1} \tau_i} &= \left( \ln \frac{\tau_{i+1}}{\tau_i} \right)_s, \\
\frac{D_t^2 \tau_{i+1} \cdot \tau_i}{\tau_{i+1} \tau_i} &= \left( \ln \left( \frac{\tau_{i+1}}{\tau_i} \right) \right)_{ss} + \left( \left( \ln \frac{\tau_{i+1}}{\tau_i} \right)_s \right)^2, \\
\frac{D_t^3 \tau_{i+1} \cdot \tau_i}{\tau_{i+1} \tau_i} &= \left( \ln \frac{\tau_{i+1}}{\tau_i} \right)_{ss} + 3 \left( \ln \frac{\tau_{i+1}}{\tau_i} \right)_s \left( \ln \left( \frac{\tau_{i+1}}{\tau_i} \right)_s \right) + \left( \left( \ln \frac{\tau_{i+1}}{\tau_i} \right)_s \right)^3,
\end{align*}
\]

one obtains

\[
\left( \ln \frac{\tau_{i+1}}{\tau_i} \right)_{sss} = \left( b - \ln \frac{\tau_{i+1}}{\tau_i} \right)_s \times \left[ 3 \left( \ln \left( \frac{\tau_{i+1}}{\tau_i} \right) \right)_{ss} - \left( \ln \frac{\tau_{i+1}}{\tau_i} \right)_s \left( 2b - \ln \frac{\tau_{i+1}}{\tau_i} \right)_s \right],
\]

from equation (3.10). Next, using the discrete hodograph transformation (2.30) and dependent variable transformation (2.31), equation (3.11) reads

\[
\frac{d}{ds} (u_{i+1} - u_i) = \frac{3}{2} \delta_l (u_t + u_{t+1}) - \frac{1}{4} \delta_l \left( \delta_l^2 - 4b^2 \right). \tag{3.12}
\]

Obviously, the evolution equation for the nonuniform mesh \( \delta_l \) remains the same as equation (2.33). In summary, we have the following theorem:

**Theorem 3.2.** The bilinear equation

\[
\left( \frac{1}{b} D_s^3 - 3D_s^2 + 2bD_s \right) \tau_{i+1} \cdot \tau_i = 0
\]

determines a semi-discrete analogue of the reduced Ostrovsky equation (1.1)

\[
\begin{align*}
\frac{d}{ds} (u_{i+1} - u_i) &= \frac{3}{2} \delta_l (u_t + u_{t+1}) - \frac{1}{4} \delta_l \left( \delta_l^2 - 4b^2 \right), \\
\frac{d}{ds} \delta_l &= u_{i+1} - u_i.
\end{align*}
\]

through the dependent variable transformation \( u_t = -2(\ln \tau_i)_s \) and the discrete hodograph transformation \( x_t = 2lb - 2\left( \ln \tau_i \right)_s, \delta_l = x_{i+1} - x_i \).

Now we turn to check if equation (3.13) converges to equation (1.1) in the continuous limit. To this end, by dividing \( \delta_l \) on both sides of equation (3.12), we have
\[
\frac{1}{\delta_l} \frac{d}{ds}(u_{l+1} - u_l) = \frac{3}{2} (u_l + u_{l+1}) - \frac{1}{4} \delta_l^2 + b^2, \\
(3.14)
\]

which converges to exactly the reduced Ostrovsky equation (1.1)

\[
\partial_x (\partial_t + u \partial_x) u = 3u,
\]
as \(b \to 0\) (\(\delta_l \to 0\)).

Regarding the \(N\)-soliton solution, it is obvious that equation (3.13) admits the same solution as the semi-discrete reduced Ostrovsky equation (2.33) proposed previously.

So far, we have constructed semi-discrete analogues of the reduced Ostrovsky equation (1.1) and its differentiated form (1.3). In light of the link between (1.3) and (1.1), let us find a connection between (2.41) and (3.13). First, by taking a backward difference of equation (3.14), we obtain

\[
\frac{1}{\delta_l} \frac{d}{ds}(u_{l+1} - u_l) - \frac{1}{\delta_{l-1}} \frac{d}{ds}(u_l - u_{l-1}) = \frac{3}{2} (u_{l+1} - u_{l-1}) - \frac{1}{4} (\delta_l^2 - \delta_{l-1}^2).
\]

(3.15)

On the other hand, by substituting the third equation into the first equation in (2.41) and eliminating \(m_l\), one arrives at exactly the same equation (3.15).

**Remark 3.3.** Although we have derived semi-discrete analogues of the reduced Ostrovsky equation (1.1) and its differentiated form (1.3) from totally different bilinear equations, the connection between them is clear here. In other words, the semi-discrete analogue for the short wave limit of the DP equation is simply a backward difference of the semi-discrete reduced Ostrovsky equation. This finding corresponds to the fact that a differentiation of the reduced Ostrovsky equation (1.1) with respect to spatial variable \(x\) gives rise to the short wave limit of the DP equation (1.3) in the continuous case. Forward difference and differentiation are two typical operators corresponding to discrete and continuous systems, respectively. In the world of integrable systems, we observe a perfect correspondence between these two operators and discrete and continuous systems.

Lastly, for the sake of convenience, we list the \(\tau\)-functions for one- and two-soliton solutions.

**One-soliton** For \(N = 1\), we have

\[
\tau_l = \text{Pf}(1, 2) = c_1 + \frac{p_1 - p_2}{p_1 + p_2} e^{\eta_l(i) + \eta_{l+1}(i)}, \\
(3.16)
\]

where \(c_1\) is a nonzero constant,

\[
e^{\eta_{l}(i)} = \left( \frac{1 + bp_1}{1 - bp_1} \right)^i e^{\frac{1}{p_1} + \frac{1}{p_2}},
\]

and \(p_1, p_2\) are related by a constraint

\[
\frac{1}{p_1^3} + \frac{1}{p_2^3} = b^2 \left( \frac{1}{p_1} + \frac{1}{p_2} \right). \\
(3.17)
\]
Two-soliton For \( N = 2 \), we have
\[
\tau_1 = \text{Pf}(1, 2, 3, 4) = \text{Pf}(1, 2)\text{Pf}(3, 4) - \text{Pf}(1, 3)\text{Pf}(2, 4) + \text{Pf}(1, 4)\text{Pf}(2, 3)
\]
\[
= \frac{p_1 - p_2}{p_1 + p_2} e^{\eta_1(l)}e^{\eta_2(l)}e^{\eta_3(l)}e^{\eta_4(l)} + \frac{p_3 - p_4}{p_3 + p_4} e^{\eta_1(l)}e^{\eta_2(l)}e^{\eta_3(l)}e^{\eta_4(l)}
\]
\[
+ \left( c_1 + \frac{p_1 - p_4}{p_1 + p_4} e^{\eta_1(l)}e^{\eta_2(l)}e^{\eta_3(l)}e^{\eta_4(l)} \right) \left( c_2 + \frac{p_2 - p_3}{p_2 + p_3} e^{\eta_1(l)}e^{\eta_2(l)}e^{\eta_3(l)}e^{\eta_4(l)} \right),
\]
under the condition
\[
\frac{1}{p_1^3} + \frac{1}{p_4^3} = b^2 \left( \frac{1}{p_1} + \frac{1}{p_4} \right), \quad \frac{1}{p_2^3} + \frac{1}{p_3^3} = b^2 \left( \frac{1}{p_2} + \frac{1}{p_3} \right).
\]
Letting \( c_1 = c_2 = 1 \) and \( e^{\eta_1} = \frac{p_3 - p_4}{p_3 + p_4} \) and \( e^{\eta_2} = \frac{p_1 - p_4}{p_1 + p_4} \), the above \( \tau \)-function can be rewritten as
\[
\tau_1 = 1 + e^{\eta_1(l)}e^{\eta_2(l)} + e^{\eta_3(l)}e^{\eta_4(l)} + b_{12} e^{\eta_2(l)}e^{\eta_3(l)}e^{\eta_4(l)} + \tau_1 + \tau_2,
\]
where
\[
b_{12} = \frac{(p_1 - p_2)(p_1 - p_3)(p_4 - p_2)(p_4 - p_3)}{(p_1 + p_2)(p_1 + p_3)(p_4 + p_2)(p_4 + p_3)}.\]
In the continuous limit \( b \to 0 \), it is obvious that above one- and two-soliton solutions for the semi-discrete reduced Ostrovsky equation converge to the one- and two-soliton solutions for the reduced Ostrovsky equation listed in [28].

4. Conclusion and further topics

There are two versions of the reduced Ostrovsky equation one is the original form (1.1), the other is its differentiated form, or is also called the short wave limit of the DP equation (1.3). In this paper, we have constructed their integrable semi-discretizations separately based on their different bilinear forms. Two versions of integrable semi-discretizations of the reduced Ostrovsky equation share the same \( N \)-soliton solution in terms of Pfaffians, which converges to the \( N \)-soliton solution of the continuous Ostrovsky equation (1.1), as well as its differentiated form (1.3). The connection between the two versions of integrable discretizations is made clear. In the continuous case, the short wave limit of the DP equation (1.3) is the differentiated form of the reduced Ostrovsky equation, whereas in the discrete case, the semi-discrete short wave limit of the DP equation is the forward difference for the semi-discrete reduced Ostrovsky equation.

Similar to our previous results [43–45], the semi-discrete reduced Ostrovsky equation proposed here can serve as an integrable numerical scheme, the so-called self-adaptive moving mesh method, for numerical simulation. It seems that the semi-discrete reduced Ostrovsky equation (3.13) has more advantages than the semi-discrete analogue of the short wave limit of the DP equation in serving as a self-adaptive moving mesh method. We would like to report our results on this topic in a forthcoming paper. Finally, we have not succeeded in constructing an integrable fully discrete reduced Ostrovsky equation. If we could have done so, then a newly integrable discrete Tzitzeica equation might be constructed due to a direct link between these two equations. This is a further topic to be explored in the future. Another problem to be solved is the integrable discretization of the DP equation which is a more challenging problem.
compared with those of the Camassa–Holm equation and the reduced Ostrovsky equation. We are tackling this problem based on our previous work on the DP equation [46].

Acknowledgments

This work of BF is partially supported by the National Natural Science Foundation of China (No. 11428102).

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