Large deviations for conditional guesswork

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Abstract

The guesswork problem was originally studied by Massey to quantify the number of
guesses needed to ascertain a discrete random variable. It has been shown that for a
large class of random processes the rescaled logarithm of the guesswork satisfies the large
deviation principle and this has been extended to the case where \( k \) out \( m \) sequences are
guessed. The study of conditional guesswork, where guessing of a sequence is aided by
the observation of another one, was initiated by Arıkan in his simple derivation of the
upper bound of the cutoff rate for sequential decoding. In this note, we extend these
large deviation results to the setting of conditional guesswork.

1 Introduction

Let \((X, Y)\) be a pair of random variables with \(X\) and \(Y\) taking values in a finite alphabet
set \(\mathcal{X}\) and a countable alphabet set \(\mathcal{Y}\), respectively. Here, \(X\) is the random variable to be
guessed by a series of truthfully answered questions of the form “Is \(X = x\)?”, while \(Y\) is a
correlated random variable that is directly observed. For example, in sequential decoding, one
can think of \(X\) as channel input and \(Y\) as channel output. We call \(G(X)\) a guessing function
of \(X\) if \(G : \mathcal{X} \mapsto \{1, 2, \cdots, |\mathcal{X}|\}\) is a one-to-one function. A guessing function determines the
order in which guesses are made; that is, \(G(x)\) is number of queries needed when \(X = x\).
This guesswork problem, originally proposed by Massey [11], arises for instance when a
cryptanalyst must try out possible secret keys one at a time after narrowing the possibilities
by some cryptanalysis. We call \(G(X|Y)\) a guessing function of \(X\) given \(Y\), if \(G(X|y)\) is
a guessing function of \(X\) for any given value \(Y = y\). The study of conditional guesswork
was initiated by Arıkan [1] in his simple derivation of the upper bound of the cutoff rate
for sequential decoding. This was motivated by Jacobs-Berlekamp’s observation [9] on the
relationship between sequential decoding and guessing.

It is not hard to see that the average number of guesses \(\mathbb{E}G(X)\) is minimized when one
makes guesses of values of \(X\) from the most likely to the least likely. Such a guessing function
is called optimal. Massey [11] lower bounded \(\mathbb{E}G(X)\) in terms of Shannon entropy \(H(X)\).
It is observed by Arıkan [1] that \(1/(1 + \alpha)\)-Rényi entropy \(H_{1/(1+\alpha)}(X)\) is the appropriate
metric to measure the logarithm of the \(\alpha\)-th moment \(\mathbb{E}G(X)\alpha\) for \(\alpha > 0\). Arıkan’s bound
is asymptotically sharp when one considers a long sequence of independent and identically
distributed (i.e., i.i.d.) random variables. This result was subsequently extended by Malone-
Sullivan [10] to Markov processes with finite state spaces, and by Pfister-Sullivan [12] to more
general processes for \(\alpha > -1\). This asymptotic behavior inspires the recent study of large
deviations for guesswork by Christiansen-Duffy [3] and for the guesswork of guessing \(k\) out
\(m\) mutually independent sequences by Christiansen-Duffy-du Pin Calmon-Médard [4]. Re-
cently, Duffy-Li-Médard [6] employed the large deviation results of guesswork to give a simple
derivation of Shannon’s channel coding theorem [13] for additive noise channels. Motivated by potential applications in coding-decoding of concatenated codes, we extend these large deviation results to the setting of conditional guesswork.

2 Negative moments of guesses

We derive bounds on negative moments of guessing functions, which complement Arıkan’s [1] results on positive moments. Our lower bounds for negative moments resemble the upper bounds of Arıkan for positive moments, while our upper bound takes the form of Arıkan’s lower bound for positive moments. Our proof for the lower bound mirrors Arıkan’s approach, while our upper bounds proof uses a somewhat different technique, based on the reverse Hölder’s inequality, inspired by a remark by Arıkan [1].

**Theorem 2.1.** Let $G(X)$ and $G(X|Y)$ be arbitrary guessing functions. For $\alpha \in (-1, 0)$, we have
\[
\mathbb{E}G(X)^\alpha \leq (1 + \log |\mathcal{X}|)^{-\alpha} \left( \sum_x p_X(x)^{1+\alpha} \right)^{1+\alpha},
\]
\[
\mathbb{E}G(X|Y)^\alpha \leq (1 + \log |\mathcal{X}|)^{-\alpha} \sum_y \left( \sum_x p_{X,Y}(x,y)^{1+\alpha} \right)^{1+\alpha},
\]
where $|\mathcal{X}|$ is the cardinality of $\mathcal{X}$, and $p_X(x)$ and $p_{X,Y}(x,y)$ are the probability mass functions of $X$ and $(X,Y)$, respectively.

**Proof.** Recall the following reverse Hölder inequality: for any $p, q > 0$ such that $1/p - 1/q = 1$, and any $a_i, b_i > 0$, we have
\[
\sum_i a_i b_i \geq \left( \sum_i a_i^p \right)^{1/p} \left( \sum_i b_i^{-q} \right)^{-1/q}.
\]
The upper bound in unconditional case follows by taking $p = 1 + \alpha$ and $q = -(1 + 1/\alpha)$, and
\[
a_i = (i^\alpha \mathbb{P}(G(X) = i))^{1/(1+\alpha)}, \quad b_i = i^{-\alpha/(1+\alpha)}, \quad i = 1, \ldots, |\mathcal{X}|,
\]
as well noting that
\[
\sum_{i=1}^{\lfloor X \rfloor} i^{-1} \leq 1 + \log |\mathcal{X}|.
\]
Then the upper bound in conditional case follows readily from the definition.
\[
\mathbb{E}G(X|Y)^\alpha = \sum_y p_Y(y) \mathbb{E}G(X|Y = y)^\alpha
\]
\[
\leq (1 + \log |\mathcal{X}|)^{-\alpha} \sum_y p_Y(y) \left( \sum_x p_{X,Y}(x,y)^{1\alpha} \right)^{1+\alpha}
\]
\[
= (1 + \log |\mathcal{X}|)^{-\alpha} \sum_y \left( \sum_x p_{X,Y}(x,y)^{1+\alpha} \right)^{1+\alpha}.
\]
\[\square\]
**Theorem 2.2.** Let $G(X|Y)$ be an optimal conditional guessing function. For $\alpha \in (-1, 0)$, we have

$$\mathbb{E}G(X|Y)^\alpha \geq \sum_y \left( \sum_x p_{X,Y}(x,y) \right)^{\frac{1}{1+\alpha}}.$$ 

**Proof.** The statement can be proved in the same manner as that of Proposition 4 in [1]. We include the proof for completeness. For any optimal guessing function $G(X|Y)$, we have

$$G(x|y) = \sum_{x': G(x'|y) \leq G(x|y)} 1 \leq \sum_{x': G(x'|y) \leq G(x|y)} \left( \frac{p_{X|Y}(x'|y)}{p_{X|Y}(x|y)} \right)^{\frac{1}{1+\alpha}} \leq \sum_x \left( \frac{p_{X|Y}(x'|y)}{p_{X|Y}(x|y)} \right)^{\frac{1}{1+\alpha}}.$$ 

Then the proof readily follows from

$$\mathbb{E}G(X|Y)^\alpha = \sum_{x,y} p_{X,Y}(x,y) G(x|y)^\alpha \geq \sum_{x,y} p_{X,Y}(x,y) \left( \sum_{x'} \left( \frac{p_{X|Y}(x'|y)}{p_{X|Y}(x|y)} \right)^{\frac{1}{1+\alpha}} \right)^{\alpha} = \sum_y p_Y(y) \left( \sum_x p_{X|Y}(x|y) \right)^{\frac{1}{1+\alpha}} \left( \sum_x \left( \frac{p_{X,Y}(x,y)^{\frac{1}{1+\alpha}}} \right) \right)^{1+\alpha}.$$

Recall that the Rényi entropy of order $\alpha > 0$ (or simply $\alpha$-Rényi entropy) of $X$ is defined as

$$H_\alpha(X) = \frac{\alpha}{1-\alpha} \log \left( \sum_x p_X(x)^\alpha \right)^{1/\alpha}.$$ 

Unlike conditional Shannon entropy, there is no commonly accepted notation of conditional Rényi entropy. We refer to [7] for discussions of different definitions. We follow Arimoto’s notation [2] and define the conditional $\alpha$-Rényi entropy of $X$ given $Y$ as

$$H_\alpha(X|Y) = \frac{\alpha}{1-\alpha} \log \sum_y \left( \sum_x p_{X,Y}(x,y)^\alpha \right)^{1/\alpha}.$$ 

(1)

Rewrite our bounds in Theorem 2.1 and Theorem 2.2 in terms of Rényi entropies. We will see the following operational characterization of Rényi entropies. This connection was first identified in [1] for $\alpha > 0$.

**Corollary 2.3.** Let $X_{1,n} = (X_1, \cdots, X_n)$ and $Y_{1,n} = (Y_1, \cdots, Y_n)$ be two random sequences. Let $G(X_{1,n})$ and $G(X_{1,n}|Y_{1,n})$ be optimal guessing functions. Suppose the pairs $(X_i, Y_i)$ are jointly independent and have identical distribution. Let $-1 < \alpha < 0$. Then we have

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}G(X_{1,n}|Y_{1,n})^\alpha = \alpha H_{1/(1+\alpha)}(X_1|Y_1).$$
In particular, we have
\[ \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E} G(X_{1,n})^\alpha = \alpha H_{1/(1+\alpha)}(X_1). \]

Proof. Since \((X_i, Y_i)\) are i.i.d., we have
\[
\sum_{y_{1,n}} \left( \sum_{x_{1,n}} p_{X_{1,n}, Y_{1,n}}(x_{1,n}, y_{1,n}) \right)^{1+\alpha} = \left( \sum_{y_1} \left( \sum_{x_1} p_{X_1, Y_1}(x_1, y_1) \right)^{1+\alpha} \right)^n.
\]
Then the statements readily follow from Theorem 2.1 and Theorem 2.2. \(\square\)

3 Large deviations for optimal guessing functions

Let \(\mathbb{X} = (X_1, \ldots, X_n, \ldots)\) and \(\mathbb{Y} = (Y_1, \ldots, Y_n, \ldots)\) be a pair of random sequences. We denote by \(X_{1,n} = (X_1, \ldots, X_n)\) and \(Y_{1,n} = (Y_1, \ldots, Y_n)\) the truncated sequences of length \(n\). The scaled cumulant generating function of the sequence \(\{n^{-1} \log G(X_{1,n} | Y_{1,n})\}_{n \in \mathbb{N}}\) is defined as
\[
\Lambda(\alpha) = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E} e^{\alpha \log G(X_{1,n} | Y_{1,n})} = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E} G(X_{1,n} | Y_{1,n})^\alpha, \tag{2}
\]
provided the limit exists. The conditional \(\alpha\)-Rényi entropy of \(\mathbb{X}\) given \(\mathbb{Y}\) is defined as
\[
H_\alpha(\mathbb{X}|\mathbb{Y}) = \lim_{n \to \infty} \frac{1}{n} H_\alpha(X_{1,n} | Y_{1,n}), \tag{3}
\]
provided the limit exists, and the definition of \(H_\alpha(X_{1,n} | Y_{1,n})\) is given in (1). As \(\alpha \to 1\), we have the classical conditional Shannon entropy. By taking limits, we have
\[
H_0(\mathbb{X}|\mathbb{Y}) = \log |\mathbb{X}|,
\]
and
\[
H_\infty(\mathbb{X}|\mathbb{Y}) = -\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(G(X_{1,n} | Y_{1,n}) = 1), \tag{4}
\]
whenever the limit exists.

Corollary 2.3 and its counterpart Proposition 5 in [1] shows that there is a close connection between the scaled cumulant generating function of conditional guesswork and the conditional Rényi entropy of the corresponding random sequences. The following regularity assumption, which trivially holds for i.i.d. sequences, is analogous to Assumption 1 in [3]. It will be the base of our study of large deviations for optimal guessing functions.

Assumption 3.1. Suppose the scaled cumulant generating function \(\Lambda(\alpha)\) exists for \(\alpha > -1\), and it has a continuous derivative. Furthermore,
\[
\Lambda(\alpha) = \alpha H_{1/(1+\alpha)}(\mathbb{X}|\mathbb{Y}).
\]

Proposition 3.2. Under the above assumption, for all \(\alpha \leq -1\), we have
\[
\Lambda(\alpha) = -H_\infty(\mathbb{X}|\mathbb{Y}).
\]
Proof. Notice that $\Lambda(\alpha)$ is the limit of a sequence of bounded convex functions. By Assumption 3.1, the limit $H_{\infty}(X|Y) = \lim_{\alpha \downarrow -1} \Lambda(\alpha)$ exists. By definition, we have

$$
\mathbb{E}(G(X_{1,n}|Y_{1,n}))^\alpha = \sum_{i=1}^{[X]^n} i^\alpha \mathbb{P}(G(X_{1,n}|Y_{1,n}) = i).
$$

For $\alpha \leq -1$, we have

$$
\mathbb{E}(G(X_{1,n}|Y_{1,n}))^\alpha \geq \mathbb{P}(G(X_{1,n}|Y_{1,n}) = 1),
$$

and

$$
\mathbb{E}(G(X_{1,n}|Y_{1,n}))^\alpha \leq \mathbb{P}(G(X_{1,n}|Y_{1,n}) = 1) \sum_{i=1}^{[X]^n} i^{-1}.
$$

Then the result follows from the existence of $H_{\infty}(X|Y)$ and the simple fact that

$$
\lim_{n \to \infty} \frac{1}{n} \log \sum_{i=2}^{[X]^n} i^{-1} = 0.
$$

The Legendre transform of $\Lambda(\alpha)$ is defined as

$$
\Lambda^*(x) = \sup_{\alpha \in \mathbb{R}} (x\alpha - \Lambda(\alpha)).
$$

(5)

Define $\gamma = \lim_{\alpha \downarrow -1} \Lambda'(\alpha)$. One can check that

$$
\Lambda^*(x) = H_{\infty}(X|Y) - x, \ x \in [0, \gamma],
$$

(6)

and

$$
\Lambda^*(x) = \infty, \ x > \log |X|.
$$

Recall that $x \in \mathbb{R}$ is called an exposed point of $\Lambda^*$ if for some $\alpha \in \mathbb{R}$ and all $x \neq y$,

$$
\alpha x - \Lambda^*(x) > \alpha y - \Lambda^*(y),
$$

and $\alpha$ is called an exposing hyperplane.

**Theorem 3.3.** Under Assumption 3.1, the sequence $\{n^{-1} \log G(X_{1,n}|Y_{1,n})\}_{n \in \mathbb{N}}$ satisfies the large deviation principle with the rate function $\Lambda^*(x)$, i.e., for any closed set $F \subset \mathbb{R}$,

$$
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(n^{-1} \log G(X_{1,n}|Y_{1,n}) \in F) \leq - \inf_{x \in F} \Lambda^*(x),
$$

(7)

and for any open set $J \subset \mathbb{R}$,

$$
\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(n^{-1} \log G(X_{1,n}|Y_{1,n}) \in J) \geq - \inf_{x \in J} \Lambda^*(x).
$$

(8)

**Proof.** It suffices to consider $F, J \subset [0, \log |X'|]$, since the sequence $\{n^{-1} \log G(X_{1,n}|Y_{1,n})\}_{n \in \mathbb{N}}$ is supported on this range. The upper bound (7) readily follows from Gärtner-Ellis' Theorem (Theorem 2.3.6 in [5]), which assumes the existence of $\Lambda(\alpha)$ and that 0 is in the interior of $\{\alpha \in \mathbb{R} : \Lambda(\alpha) < \infty\}$. These assumptions are satisfied by Assumption 3.1 and Proposition
3.2. Regarding the lower bound (8), we split $[0, \log |\mathcal{X}|]$ into $[0, \gamma]$ and $(\gamma, \log |\mathcal{X}|]$. For any open set $J \subset (\gamma, \log |\mathcal{X}|]$, Gärtner-Ellis’ Theorem says that
\[
\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(n^{-1} \log G(X_{1,n}|Y_{1,n}) \in J) \geq - \inf_{x \in J \cap \mathcal{F}} \Lambda^*(x),
\]
where $\mathcal{F}$ is the set of exposed points of $\Lambda^*$. Assumption 3.1 implies that $\Lambda^*(x)$ has at most a finite number of points in $(\gamma, \log |\mathcal{X}|]$ without exposing hyperplanes. The continuity of $\Lambda^*$ implies that
\[
\inf_{x \in J \cap \mathcal{F}} \Lambda^*(x) = \inf_{x \in J} \Lambda^*(x).
\]
Therefore, the lower bound (8) holds when $J \subset (\gamma, \log |\mathcal{X}|]$. Owing to the representation (6), $\Lambda^*$ has no exposed points in $[0, \gamma]$. We need a different argument for the case $J \subset [0, \gamma]$. Without loss of generality, we can assume that $\gamma > 0$. For any $x \in J \subset [0, \gamma]$ and $\epsilon > 0$ small enough, we have $B(x, \epsilon) := (x - \epsilon, x + \epsilon) \subset J$. One can verify that
\[
\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(n^{-1} \log G(X_{1,n}|Y_{1,n}) \in J) \geq \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(n^{-1} \log G(X_{1,n}|Y_{1,n}) \in B(x, \epsilon)).
\]
Since $x \in J$ is arbitrary and $\epsilon$ can be arbitrarily small, using the representation (6), the lower bound (8) will hold if we can show that
\[
\lim \liminf_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(n^{-1} \log G(X_{1,n}|Y_{1,n}) \in B(x, \epsilon)) = x - H_\infty(X|Y). \tag{9}
\]
The proof proceeds in two cases.

Case 1. There is some $x^* > \gamma$ such that $\Lambda^*(x^*)$ is finite. We select $\epsilon$ such that $x^* - \epsilon > \gamma$.

Since lower bound (8) holds for $B(x^*, \epsilon)$, we have
\[
\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(n^{-1} \log G(X_{1,n}|Y_{1,n}) \in B(x^*, \epsilon)) \geq - \inf_{y \in B(x^*, \epsilon)} \Lambda^*(y). \tag{10}
\]

We define the set
\[
\mathcal{X}^n(y_{1,n}, x^*, \epsilon) = \{x_{1,n} \in \mathcal{X}^n : n^{-1} \log G(x_{1,n}|y_{1,n}) \in B(x^*, \epsilon)\}. \tag{11}
\]

One can verify that
\[
\lim \liminf_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log |\mathcal{X}^n(y_{1,n}, x^*, \epsilon)| = x^*. \tag{12}
\]

Notice that
\[
\mathbb{P}(n^{-1} \log G(X_{1,n}|Y_{1,n}) \in B(x^*, \epsilon)) \leq \sum_{y_{1,n} \in Y^n} |\mathcal{X}^n(y_{1,n}, x^*, \epsilon)| p_{Y_{1,n}}(y_{1,n}) \sup_{x_{1,n} \in \mathcal{X}^n(y_{1,n}, x^*, \epsilon)} p_{X_{1,n}|Y_{1,n}}(x_{1,n}|y_{1,n}) \leq \sum_{y_{1,n} \in Y^n} |\mathcal{X}^n(y_{1,n}, x^*, \epsilon)| p_{Y_{1,n}}(y_{1,n}) \inf_{x_{1,n} \in \mathcal{X}^n(y_{1,n}, x^*, \epsilon)} p_{X_{1,n}|Y_{1,n}}(x_{1,n}|y_{1,n}).
\]

In the second inequality, we use the monotonicity of conditional guesswork and the fact that $x + \epsilon < x^* - \epsilon$. Notice that the cardinality $|\mathcal{X}^n(y_{1,n}, x^*, \epsilon)|$ is independent of the choice of $y_{1,n}$. Combine the above upper bound with (10) and (12), we have
\[
-\Lambda^*(x^*) \leq x^* + \lim \liminf_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \sum_{y_{1,n} \in Y^n} p_{Y_{1,n}}(y_{1,n}) \inf_{x_{1,n} \in \mathcal{X}^n(y_{1,n}, x^*, \epsilon)} p_{X_{1,n}|Y_{1,n}}(x_{1,n}|y_{1,n}).
\]
Let $x^* \to \gamma$ in the above inequality. Using the representation (6), we have

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \sum_{y_1,n \in \mathcal{Y}^n} p_{Y_1,n}(y_1,n) \inf_{x_1,n \in \mathcal{X}^n(y_1,n,x,\epsilon)} p_{X_1,n|Y_1,n}(x_1,n|y_1,n) \geq -H_\infty(\mathcal{X}|\mathcal{Y}).$$

Notice that

$$\mathbb{P}(n^{-1} \log G(X_1,n|Y_1,n) \in B(x,\epsilon)) \geq \sum_{y_1,n \in \mathcal{Y}^n} |\mathcal{X}^n(y_1,n,x,\epsilon)|p_{Y_1,n}(y_1,n) \inf_{x_1,n \in \mathcal{X}^n(y_1,n,x,\epsilon)} p_{X_1,n|Y_1,n}(x_1,n|y_1,n),$$

where $\mathcal{X}^n(y_1,n,x,\epsilon)$ is defined as in (11) with $x^*$ replaced by $x$. The lower bound (9) follows from (13) and the facts that $|\mathcal{X}^n(y_1,n,x,\epsilon)|$ is independent of $y_1,n$ and that as in (12)

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log |\mathcal{X}^n(y_1,n,x,\epsilon)| = x.$$

This结论s the first case that there is some $x^* > \gamma$ such that $\Lambda^*(x^*)$ is finite.

**Case 2.** We have that $\Lambda'(x) = \infty$ for all $x > \gamma$, which implies that $\Lambda(\alpha)$ is a linear function with slope $\gamma$ for $\alpha > -1$. To see this, using the definition of $\Lambda^*(x)$ in (5), we can see that $x\alpha - \Lambda(\alpha)$ is monotonically increasing for $\alpha > -1$. Using the differentiability of $\Lambda(\alpha)$ in Assumption 3.1, we have $\Lambda'(\alpha) < x$ for any $x > \gamma$. Recall that $\gamma = \lim_{\alpha \to -1} \Lambda'(\alpha)$. We have $\Lambda'(\alpha) = \gamma$ for $\alpha > -1$. The fact $\Lambda(0) = 0$ and the representation (6) imply that $\gamma = H_\infty(\mathcal{X}|\mathcal{Y})$ and $\Lambda^*(\gamma) = 0$. Since $\Lambda'(0) = H(\mathcal{X}|\mathcal{Y})$ is the only zero of $\Lambda^*(x)$, we must have $\gamma = H_\infty(\mathcal{X}|\mathcal{Y}) = H(\mathcal{X}|\mathcal{Y}) = H_0(\mathcal{X}|\mathcal{Y}) = \log |\mathcal{X}|$. The proof in the second case proceeds by contradiction. Assume that the lower bound (9) does not hold. Then, there is some $x \in J$ such that

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(n^{-1} \log G(X_1,n|Y_1,n) \in B(x,\epsilon)) < x - H_\infty(\mathcal{X}|\mathcal{Y}).$$

For any fixed $\delta > 0$ and $\epsilon > 0$ small enough, we have

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(n^{-1} \log G(X_1,n|Y_1,n) \in B(x,\epsilon)) < x - H_\infty(\mathcal{X}|\mathcal{Y}) - \delta.$$

For $n$ large enough, we have

$$\mathbb{P}(n^{-1} \log G(X_1,n|Y_1,n) \in B(x,\epsilon)) < e^{n(x - H_\infty(\mathcal{X}|\mathcal{Y}) - \delta)}.$$ (14)

Since $(e^{n(x-\epsilon)}, e^{n\gamma})$ can be covered by at most $\frac{e^{n\gamma} - e^{n(x-\epsilon)}}{e^{n(x+\epsilon)} - e^{n(x-\epsilon)}}$ translations of $(e^{n(x-\epsilon)}, e^{n(x+\epsilon)})$, the monotonicity of conditional guesswork and (14) imply that for $n$ large enough

$$\mathbb{P}(n^{-1} \log G(X_1,n|Y_1,n) \in (x - \epsilon, \gamma]) \leq \frac{e^{n\gamma} - e^{n(x-\epsilon)}}{e^{n(x+\epsilon)} - e^{n(x-\epsilon)}} \cdot e^{n(x - H_\infty(\mathcal{X}|\mathcal{Y}) - \delta)} \leq \frac{e^{n(-\delta+\epsilon)}}{e^{2n\epsilon} - 1},$$

which approaches 0 as $n \to \infty$. Using the definition of $H_\infty(\mathcal{X}|\mathcal{Y})$ given in (4), we have that for any fixed $\delta' > 0$ and $n$ large enough

$$\mathbb{P}(G(X_1,n|Y_1,n) = 1) < e^{n(\delta' - H_\infty(\mathcal{X}|\mathcal{Y}))}.$$ (15)

The monotonicity of conditional guesswork and the above upper bound imply that

$$\mathbb{P}(n^{-1} \log G(X_1,n|Y_1,n) \in [1, x - \epsilon]) \leq e^{n(x-\epsilon)} e^{n(\delta' - H_\infty(\mathcal{X}|\mathcal{Y}))}.$$ (16)
Since \( x < \gamma = H_\infty(\mathcal{X}|\mathcal{Y}) \), we can select \( \delta' > 0 \) small enough such that the above probability approaches 0 as \( n \to \infty \). The upper bound (15) together with the upper bound (16) contradict the fact that
\[
\mathbb{P}(n^{-1} \log G(X_{1:n}|Y_{1:n}) \in [1, \gamma]) = 1.
\]
Hence, the lower bound (9) must hold. \qed

**Remark 1.** Under Assumption 3.1, we have
\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E} G(X_{1:n}|Y_{1:n}) = \Lambda(1) = H_{1/2}(\mathcal{X}|\mathcal{Y}), \tag{17}
\]
whereas
\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E} \log G(X_{1:n}|Y_{1:n}) = \Lambda'(0) = H(\mathcal{X}|\mathcal{Y}), \tag{18}
\]
which is the zero of the rate function \( \Lambda^*(x) \).

## 4 Discussion of parallel guesswork

The multi-user guesswork problem was studied by Christiansen-Duffy-du Pin Calmon-Médard [4]. Suppose \( m \) users independently select strings from a finite, but potentially large, list. An inquisitor who knows the selection probabilities of each user is equipped with a method that enables the testing of each (user, string) pair, one at a time, for whether that string had been selected by that user. The inquisitor wishes to identify any \( k \leq m \) of the strings with the smallest number of total guesses. Therefore, a multi-user guesswork strategy is a querying order of (user, string) pairs. Unlike the single-user guesswork, there is no stochastically dominant strategy in the multi-user case if \( k < m \) (Lemma 1, [4]). The following round-robin strategy, constructed in [4], satisfies the large deviation principle and asymptotically meets the bound of the multi-user guesswork. For the round-robin strategy, each guess allows up to \( m \) parallel queries, that is, to query the most likely string of one user followed by the most likely string of a second user and so forth, for each user in a round-robin fashion, before moving to the second most likely string of each user. This is equivalent to that \( m \) guessers work independently on \( m \) parallel strings. We call such a strategy parallel guesswork and extend the large deviation results for single-user conditional guesswork to parallel conditional guesswork.

Define \([m] = \{1, \ldots, m\}\). For \( i \in [m] \), let \( X_{1:n}^i = (X_1^i, \ldots, X_n^i) \) and \( Y_{1:n}^i = (Y_1^i, \ldots, Y_n^i) \) be \( m \) pairs of random sequences of length \( n \). Guesses for the \( m \) sequences \( X_{1:n}^i \) are made simultaneously, and we assume the outcome for \( X_{1:n}^i \) only depends on \( Y_{1:n}^i \). Let \( G(X_{1:n}^i|Y_{1:n}^i) \) be an optimal single-user conditional guessing strategy for \( (X_{1:n}^i, Y_{1:n}^i) \). For any \( 1 \leq k \leq m \), we define
\[
G_{k,m}(\{(X_{1:n}^i, Y_{1:n}^i)\}_{i \in [m]}) = k\text{-min}(G(X_{1:n}^1|Y_{1:n}^1), \ldots, G(X_{1:n}^m|Y_{1:n}^m)), \tag{19}
\]
where \( k\text{-min}(v) \) gives the \( k \)-th smallest component of the vector \( v \). The unconditional analogue of (19) is studied in [4]. We know that \( \{n^{-1} \log G(X_{1:n}^i|Y_{1:n}^i)\}_{n \in \mathbb{N}} \) satisfies the large deviation principle under the regularity Assumption 3.1. Suppose the \( m \) pairs \( (X_{1:n}^i, Y_{1:n}^i) \) are independent. Then we can apply the contraction principle (Theorem 4.2.1 in [5]) to show that \( \{n^{-1} \log G_{k,m}(\{(X_{1:n}^i, Y_{1:n}^i)\}_{i \in [m]})\}_{n \in \mathbb{N}} \) also satisfies the large deviation principle.
Let $\Lambda_i(\alpha)$ be the scaled cumulant generating function of \(\{n^{-1} \log G(X^i_{1:n},Y^i_{1:n})\}_{n \in \mathbb{N}}\) (see definition (2)). We denote by $\Lambda^*_i(x)$ the Legendre transform of $\Lambda_i(\alpha)$. Let $H_\alpha(X^i|Y^i)$ be the conditional $\alpha$-Rényi entropy of $X^i = (X^i_1,X^i_2,\ldots)$ given by $Y^i = (Y^i_1,Y^i_2,\ldots)$ (see definition (3)).

**Theorem 4.1.** Suppose the $m$ pairs $(X^i_{1:n},Y^i_{1:n})$ are jointly independent. Suppose that $\Lambda_i(\alpha)$ satisfies Assumption 3.1. Then $\{n^{-1} \log G_{k,m}((X^i_{1:n},Y^i_{1:n}))\}_{n \in \mathbb{N}}$ satisfies the large deviation principle with the rate function

\[
I_{k,m}(x) = \max_{i_1,\ldots,i_m} \left\{ \Lambda^*_i(x) + \sum_{l=2}^{k} \delta_i(x) + \sum_{l=k+1}^{m} \gamma_i(x) \right\},
\]

where

\[
\delta_i(x) = \begin{cases} 
\Lambda^*_i(x) & \text{if } x \leq H(X^i|Y^i) \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
\gamma_i(x) = \begin{cases} 
\Lambda^*_i(x) & \text{if } x \geq H(X^i|Y^i) \\
0 & \text{otherwise}
\end{cases}
\]

The scaled cumulant generating function of $\{n^{-1} \log G_{k,m}((X^i_{1:n},Y^i_{1:n}))\}_{n \in \mathbb{N}}$ is

\[
\Lambda_{k,m}(\alpha) = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E} e^{\alpha \log G_{k,m}((X^i_{1:n},Y^i_{1:n}))_{i \in [m]}} = \sup_{x \in [0,\log |\mathcal{X}|]} (\alpha x - I_{k,m}(x)).
\]

We omit the proof of this statement since it can be proved in the same manner as Theorem 5 in [4] with a slight change of notations. As observed in [4], the rate function is not necessarily convex. Convexity of the rate function is ensured if all users select strings using the same stochastic property, whereupon the result in Theorem 4.1 simplifies greatly.

**Corollary 4.2.** Under assumptions in Theorem 4.1, we also assume the $m$ pairs $(X^i_{1:n},Y^i_{1:n})$ have identical distribution. Let $\Lambda(\alpha)$ be the common scaled cumulant generating function with the Legendre transform $\Lambda^*(x)$. Define $H_\alpha(X|Y) = H_\alpha(X^i|Y^i)$ to be the common conditional $\alpha$-Rényi entropy. Then the rate function in (20) simplifies to

\[
I(k,m,x) = \begin{cases} 
k \Lambda^*(x), & x \in [0,H(X|Y)], \\
(m-k+1) \Lambda^*(x), & x \in (H(X|Y),\log |\mathcal{X}|].
\end{cases}
\]

The scaled cumulant generating function in (21) is

\[
\Lambda_{k,m}(\alpha) = \begin{cases} 
k \Lambda(\alpha), & \alpha \leq 0, \\
(m-k+1) \Lambda(\alpha/m-k+1), & \alpha > 0.
\end{cases}
\]

**Remark 2.** Since $H(X|Y)$ is zero of $\Lambda^*(x)$, it is also the zero of $\Lambda^*_m(x)$. Similar to (17), we have

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E} G_{k,m}((X^i_{1:n},Y^i_{1:n}))_{i \in [m]} = \Lambda_{k,m}(1) = H_{m-k+1}(X|Y),
\]

where the second identity follows from (23) and Assumption 3.1. Analogous to (18), we have

\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E} \log G_{k,m}((X^i_{1:n},Y^i_{1:n}))_{i \in [m]} = \Lambda'_{k,m}(0) = H(X|Y).
\]


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