Holes in the Infrastructure of Global Hyperelliptic Function Fields

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Abstract

We prove that the number of “hole elements” \( H(K) \) in the infrastructure of a hyperelliptic function field \( K \) of genus \( g \) with finite constant field \( \mathbb{F}_q \) with \( n+1 \) places at infinity, of whom \( n'+1 \) are of degree one, satisfies

\[
\left\lfloor \frac{H(K)}{\text{Pic}^0(K)} \right\rfloor - \frac{n'}{q} = O(16^g n q^{-3/2}).
\]

We obtain an explicit formula for the number of holes using only information on the infinite places and the coefficients of the \( L \)-polynomial of the hyperelliptic function field. This proves a special case of a conjecture by E. Landquist and the author on the number of holes of an infrastructure of a global function field.

Moreover, we investigate the size of a hole in case \( n = n' \), and show that asymptotically for \( n \to \infty \), the size of a hole next to a reduced divisor \( D \) behaves like the function \( \frac{n^{g-\deg D}}{(g-\deg D)!} \).

1 Introduction

When considering the infrastructure of a global function field \( K/\mathbb{F}_q(x) \), it turns out that its set of \( f \)-representations represents the divisor class group \( \text{Pic}^0(K) \) (see [Font09]). Some elements of \( \text{Pic}^0(K) \) are directly represented by reduced ideals of \( \mathcal{O} \), the integral closure of \( \mathbb{F}_q[x] \) in \( K \), while others need additional information for the infinite places. We say that an element of \( \text{Pic}^0(K) \) is a hole element if it does not correspond to a reduced ideal. (Note that we will make this more precise in Section 3.) Two natural questions is: how many hole elements are there? And if some of them cluster together, what is the size of this cluster?

These questions are related to applications. When implementing arithmetic in \( \text{Pic}^0(K) \) using the infrastructure, it is most efficient if one avoids hole elements. Hence, if the number of hole elements is small, the chance that one avoids

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hole elements is big: it makes sense to optimize the algorithms for arithmetic for this case.

In practical experiments made by E. Landquist [Lan09] and the author [Fon09a] with a small number of infinite places, it turns out that the chance that a random element of Pic⁰(K) is a hole element is about \( \frac{1}{q} \). There is also a heuristic explanation of this phenomenon: hole elements only occur near to reduced ideals of degree \(< g \). Assuming that the norms of reduced ideals are uniformly distributed in \((\mathbb{F}_q)_{\leq g}[x]\), one obtains that the chance that a random reduced ideal has degree \(< g \) should be around \( \frac{1}{q} \) [Fon09a, p. 132].

In this paper, we will show that this conjecture is (almost) true for hyperelliptic \(^1\) function fields \( K \): the probability is not \( \frac{1}{q} \) but \( \frac{n}{q} \), if \( n' + 1 \) is the number of infinite places of \( K/\mathbb{F}_q(x) \) which have degree one. The error term is indeed dominated by \( q^{-3/2} \) in this case, with an additional factor of \( 16^g n \), where \( g \) is the genus of \( K \) and \( n + 1 \) the total number of infinite places of \( K/\mathbb{F}_q(x) \).

Another question related to holes of infrastructures is their “size”: hole elements group “around” reduced ideals of degree \(< g \). When we define a hole next to a reduced ideal \( a \) as the set of hole elements whose reduced ideal is \( a \), one can give upper and lower bounds for the size of a hole in case \( n = n' \), i.e. all infinite places of \( K/\mathbb{F}_q(x) \) are of degree one. It turns out that asymptotically for \( n \to \infty \), assuming that \( a \) avoids certain places, our bounds imply that the size behaves like the function \( \frac{n}{(g - \deg a)^{g - \deg a}} \).

We first investigate the arithmetic of hyperelliptic function fields in Section 2 to get an explicit description of the set of reduced divisors. In Section 3 we sketch how the infrastructure looks like and make more precise what the holes in it are. Moreover, we consider the elliptic function field case and give a (mostly) “local” criterion whether a divisor is reduced. By generalizing this definition of being reduced, we describe generating functions of these sets of reduced divisors vanishing at a set \( S \) of place\(^2\)s in Section 4. We investigate how these functions change when \( S \) changes and obtain an explicit description of the set of holes. Next, we consider the case of \( |S| = 1 \) in Section 5, show that the generating function is rational and give estimations for certain coefficients of the generating function. In Section 6, we use the results to show our first main result, namely a bound on the number of holes and an explicit formula, and in Section 7 we show the result on the size of holes. Finally, we conclude in Section 8 with a few conjectures for the case of arbitrary global function fields.

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\(^1\)We consider elliptic function fields as a special case of hyperelliptic function fields.

\(^2\)In the above notation, \( |S| = n + 1 \).
2 Arithmetic of Hyperelliptic Function Fields

We begin with reviewing the arithmetic of a hyperelliptic function field $K$. We want to work with a representation of $K/R$, where $R$ is a quadratic rational subfield of $K$, without having the notion of an infinite place of $R$. We begin with very general results on the arithmetic of a function field.

Let $k$ be a perfect field and $K$ a hyperelliptic function field with exact field of constants $k$. Let $g$ denote the genus of $K$. If $g > 1$, let $R$ be the unique rational subfield of $K$ with $[K : R] = 2$ and let $\text{Con}_{K/R}$ denote the conorm map $\text{Div}(R) \to \text{Div}(K)$.

Let $p$ be a place of $K$ of degree one. The first few results are true for arbitrary function fields as well: the only requirement is that $p$ is a place of degree one. For more information, see [Hes02, Fon09b].

**Definition 2.1.** Let $D \in \text{Div}(K)$ be a divisor. We say that $D$ is reduced with respect to $p$ if $L(D) = k$ and $\nu_p(D) = 0$. Denote the set of reduced divisors by $\text{Red}_p(K)$.

Note that we always have $0 \in \text{Red}_p(K)$. Reduced divisors allow to describe the divisor class group $\text{Pic}^0(K) = \text{Div}^0(K) / \text{Princ}(K)$:

**Proposition 2.2.** The map

$$\text{Red}_p(K) \to \text{Pic}^0(K), \quad D \mapsto D - (\deg D)p + \text{Princ}(K)$$

is a bijection, mapping $0 \in \text{Red}_p(K)$ to the neutral element of $\text{Pic}^0(K)$.

One can state a few properties on reduced divisors; we will see that in the hyperelliptic case, some of these properties already characterize reduced divisors in hyperelliptic function fields:

**Lemma 2.3.** Let $D \in \text{Red}_p(K)$. Then $D \geq 0$ and $\deg D \leq g$. In case $R$ is any rational subfield of $K$, and we have $D' \in \text{Div}(R)$ with $\text{Con}_{K/R}(D') \leq D$, then $D' \leq 0$.

In particular, if $K$ is a rational function field, this shows that $\text{Red}_p(R) = \{0\}$, i.e. $\text{Pic}^0(K) = 0$.

**Proof.** First, as $1 \in L(D)$, we must have $D \geq 0$. Next, if $\deg D > g$, we have $\dim L(D) \geq \deg D + 1 - g > 1$ by Riemann’s Inequality, contradicting $L(D) = k$.

Now assume that $R$ is any rational subfield of $K$. Let $q = p \cap R$ and let $D' \in \text{Div}(R)$ with $\text{Con}_{K/R}(D') \leq D$. Without loss of generality, we can assume $D' \geq 0$. Now $\deg q = 1$, whence $D'' := D' - (\deg D')q$ is principal; i.e. there exists some $x \in R^*$ with $(x)_R = D''$. Now

$$\text{Con}_{K/R}(D') = \text{Con}_{K/R}(D'') + (\deg D')q$$

$$= \text{Con}_{K/R}(D'') + (\deg D')\text{Con}_{K/R}(q)$$

$$= (x)_K + (\deg D')\text{Con}_{K/R}(q)$$
Note that \( \nu_p(\text{Con}_{K/R}(D')) = 0 \), whence \( \text{Con}_{K/R}(D') \leq D \) and \( D' \geq 0 \) imply \( \nu_q(D') = 0 \). Therefore, \( \nu_q(D'') = -\deg D' \leq 0 \), whence \((x)_K \leq D \). But this implies \( x^{-1} \in L(D) = k \), whence \((x) = 0\), forcing \( D' = 0 \).

Even though we restrict to hyperelliptic function fields, the previous results hold as well for general function fields. But from now on, we need that \( K \) is hyperelliptic. Let us first handle the case \( g = 1 \).

**Proposition 2.4.** If \( g = 1 \), let \( E \) denote the set of places of degree one. Then

\[
E \to \text{Red}_p(K), \quad q \mapsto \begin{cases} 
0 & \text{if } q = p, \\
q & \text{if } q \neq p
\end{cases}
\]

is a bijection.

It turns out that in case \( g > 1 \), the converse of the lemma holds as well. Let \( R \) be the unique rational subfield of index 2.

**Theorem 2.5.** [CF+A 06, p. 306, Section 14.1.2] Assume that \( g > 1 \). Let \( D \in \text{Div}(K) \) be a divisor with \( D \geq 0 \), \( \deg D \leq g \), \( \nu_p(D) = 0 \) such that, if \( D' \in \text{Div}(R) \) satisfies \( 0 \leq \text{Con}_{K/R}(D') \leq D \), then \( D' \leq 0 \). Then \( D \in \text{Red}_p(K) \).

Therefore, we have the explicit description of \( \text{Red}_p(K) \) as the set

\[
\left\{ D \in \text{Div}(K) \left| \begin{array}{l} 
D \geq 0, \\ \nu_p(D) = 0, \\ \deg D \leq g, \\ \forall D' \in \text{Div}(R) : \text{Con}_{K/R}(D') \leq D \Rightarrow D' \leq 0 
\end{array} \right. \right\},
\]

together with a bijection \( \text{Red}_p(K) \to \text{Pic}^0(K) \), \( D \mapsto D - (\deg D)p + \text{Princ}(K) \).

We will show in the next section how this leads to a combinatorial approach to describe the hole elements in \( \text{Pic}^0(K) \).

### 3 Relating Certain Reduced Divisors to the Infrastructure

Assume that \( K \) is hyperelliptic of genus \( g \) with exact constant field \( k \). Let \( R \) be a rational subfield \( R \) of index \([K : R] = 2\); in case \( g > 1 \), \( R \) is unique. Let \( p \) be a place of \( K \) of degree one. We have seen that we have a bijection \( \text{Red}_p(K) \to \text{Pic}^0(K) \) and an explicit description of \( \text{Red}_p(K) \). Let \( S \) be a set of places of \( K \) containing \( p \); for convenience, let \( S_1 := \{ p \in S \mid \deg p = 1 \} \). We are interested in the set

\[
\text{Red}_S(K) := \{ D \in \text{Red}_p(K) \mid \nu_p(D) = 0 \text{ for all } p' \in S \};
\]

more precisely, we are interested how its size compares to \( \text{Red}_p(K) = \text{Red}_{\{p\}}(K) \).

The reason why we are interested in this set is that it appears in studying infrastructures. Let \( x \in K^\times \) be an element whose poles are precisely the elements in \( S \). Let \( O_S \) be the integral closure of \( k[x] \) in \( K \); then the “infinite places” of
the extension $K/k(x)$, i.e. the places of $K$ lying over the infinite place of $k(x)$, are exactly $S$. We have a surjection from $\text{Div}(K)$ onto the (nonzero fractional) ideal group $\text{Id}(\mathcal{O}_S)$ of $\mathcal{O}_S$, given by

$$\text{ideal}_S : \sum_p n_p p \mapsto \prod_{p \notin S'} (m_p \cap \mathcal{O}_S)^{-n_p},$$

where $m_p$ is the maximal ideal in the valuation ring $\mathcal{O}_p$ of $p$. Note that $\text{ideal}_S(\text{Princ}(K)) = \text{PId}(\mathcal{O}_S)$, i.e. the principal divisors map onto the group of non-zero fractional principal ideals of $\mathcal{O}_S$. Now consider the map

$$\Psi : \mathbb{A}^{S \setminus \{p\}}_{\mathbb{A}}, \quad f \mapsto (-\nu_p(f))_{p \in S}.$$

Fix a divisor $D$ and the corresponding ideal $a = \text{ideal}_S(D)$; we consider the set of reduced divisors in $\text{Red}_p(K)$ resp. $\text{Red}_S(K)$ mapping onto ideals equivalent to $a$, i.e. we consider

$$\text{Red}_p(K, D) := \{ D' \in \text{Red}_p(K) \mid \text{ideal}_S(D') \text{PId}(\mathcal{O}_S) = \text{ideal}_S(D) \text{PId}(\mathcal{O}_S) \}$$

and

$$\text{Red}_S(K, D) := \{ D' \in \text{Red}_S(K) \mid \text{ideal}_S(D') \text{PId}(\mathcal{O}_S) = \text{ideal}_S(D) \text{PId}(\mathcal{O}_S) \}.$$

Define the map

$$\text{gen}_D : \text{Red}_p(K, D) \to \mathbb{A}^{S \setminus \{p\}}_{\mathbb{A}}, \quad D' \to \mu \mathcal{O}_S^* \text{ if } (\frac{1}{p}) = \text{ideal}_S(D') \text{ideal}_S(D)^{-1};$$

then the combination

$$\Phi : \text{Red}_p(K, D) \to \mathbb{A}^{S \setminus \{p\}}_{\mathbb{A}}/\Psi(\mathcal{O}_S^*), \quad D' \mapsto \Psi(\text{gen}_D(D')) + (\nu_p(D'))_{p \in S \setminus \{p\}} + \Psi(\mathcal{O}_S^*)$$

turns out to be a bijection; the subset $\text{Red}_S(K, D)$ of $\text{Red}_p(K, D)$ maps onto a subset of the torus $\mathbb{A}^{S \setminus \{p\}}_{\mathbb{A}}/\Psi(\mathcal{O}_S^*)$. Now the elements of $\text{Red}_S(K, D)$ correspond to the reduced ideals in the ideal class of $a = \text{ideal}_S(D)$, and the image of $\text{Red}_S(K, D)$ under $\Phi$ equals the image of the infrastructure in the ideal class of $a$ under the distance map. Hence, we see that the hole elements in the infrastructure – namely, elements of $\mathbb{A}^{S \setminus \{p\}}_{\mathbb{A}}/\Psi(\mathcal{O}_S^*)$ which lie not in the image of $\Phi$ – are the elements in $\text{Red}_p(K, D) \setminus \text{Red}_S(K, D)$; looking at all ideal classes at once, they correspond to the elements in $\text{Red}_p(K) \setminus \text{Red}_S(K)$.

For that reason, the comparison of $\text{Red}_p(K)$ and $\text{Red}_S(K)$ is related to the problem of counting the number of holes in the infrastructure.

In case $g = 1$, i.e. $K$ is elliptic, the question can be answered easily:

**Proposition 3.1.** Assume that $g = 1$. Then

$$\text{Red}_p(K) \setminus \text{Red}_S(K) = S_1 \setminus \{p\}.$$ 

**Proof.** The elements of $\text{Red}_p(K)$ are $D = 0$ and $D = q$, where $q$ ranges over all rational places of $K$ except $p$. 

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Corollary 3.2. Let \( g = 1 \) and \( k \) be a finite field of \( q \) elements. Then

\[
\frac{|\text{Red}_S(K)|}{|\text{Red}_p(K)|} = 1 - \frac{|S_1| - 1}{q} + O(q^{-3/2})
\]

for \( q \to \infty \). Hence, the probability that a random reduced divisor \( D \in \text{Red}_p(K) \) is not in \( \text{Red}_S(K) \) is approximately \( \frac{|S_1| - 1}{q} \).

Proof. By the proposition,

\[
\frac{|\text{Red}_S(K)|}{|\text{Red}_p(K)|} = \frac{|\text{Red}_p(K)| - (|S_1| - 1)}{|\text{Red}_p(K)|} = 1 - \frac{|S_1| - 1}{|\text{Pic}^0(K)|}.
\]

Now Hasse-Weil gives \( |\text{Pic}^0(K)| = q + 1 - t \) with \( |t| \leq 2\sqrt{q} \), whence, for \( q \geq 7 \),

\[
\frac{1}{q-t+1} \leq q^{-3/2} \frac{2 + q^{-1/2}}{1 - 2q^{-1/2} - q^{-1}},
\]

which implies the claim. \( \square \)

In case \( g > 1 \), the problem is harder. We begin with another classification of reduced divisors. Note that if \( q \) is a place of \( R \) which is inert in \( K \), and if \( p \) is a place of \( K \) lying above \( q \), then \( \deg p = 2 \deg q \geq 2 \); in particular, no place of degree one of \( K \) lies above a place of \( R \) inert in \( K \). Again, let \( \sigma \) denote the unique non-trivial \( R \)-automorphism of \( K \). One directly obtains the following classification:

Lemma 3.3. Let \( p \) be a place of \( K \) of degree one.

(a) If \( \sigma(p) = p \), then \( p \cap R \) ramifies in \( K \) and \( \text{Con}_{R/K}(p \cap R) = 2p \).

(b) If \( \sigma(p) \neq p \), then \( p \cap R \) splits in \( K \) and \( \text{Con}_{R/K}(p \cap R) = p + \sigma(p) \).

In case \( \deg p > 1 \), one can also have that \( p \cap R \) is inert in \( K \) in case \( \sigma(p) = p \). \( \square \)

We can now reformulate our explicit description of the reduced divisors using signatures:

Proposition 3.4. Let \( D \in \text{Div}(K) \). Then \( D \in \text{Red}_S(K) \) if, and only if,

(i) \( \deg D \leq g \);

(ii) \( \nu_q(D) = 0 \) for all \( q \in S \) and

(iii) for all places \( q \) of \( R \),

\[
\{\nu_p(D) \mid p \cap R = q\} \in \begin{cases} 
\{\{0\}\} & \text{if } q \text{ is inert in } K, \\
\{\{0\}, \{1\}\} & \text{if } q \text{ ramifies in } K, \\
\{\{0, 0\}, \{0, 1\}, \ldots, \{0, g\}\} & \text{if } q \text{ splits in } K.
\end{cases}
\]
Proof. Clearly, the condition that $D' \in \text{Div}(R)$ with $\text{Con}_{K/R}(D') \leq D$ implies $D' = 0$ is local, i.e. it suffices to check it for $D' = q$ for all places $q$ of $R$.

Let $q$ be a place of $R$ and let $p_1, p_2$ be all places of $K$ lying above $q$ (with $p_1 = p_2$ possible). If $\text{sig}(q) = (1, 2)$, then $\text{Con}_{K/R}(q) = p_1 = p_2$. If $\text{sig}(q) = (2, 1)$, then $\text{Con}_{K/R}(q) = 2p_1 = 2p_2$. If $\text{sig}(q) = (1, 1, 1)$, then $\text{Con}_{K/R}(q) = p_1 + p_2$. This, together with $\deg D \leq g$, shows that the above listed possibilities for $\{\nu_{p_1}(D), \nu_{p_2}(D)\}$ correspond to the cases where $\text{Con}_{K/R}(q) \not\leq D$. \hfill $\square$

We have seen how the hole elements of an infrastructure correspond to the set $\text{Red}_S(K) \setminus \text{Red}_p(K)$, and we obtained an explicit combinatorial description of $\text{Red}_S(K)$. This will be used in the next sections to obtain information on $|\text{Red}_S(K) \setminus \text{Red}_p(K)|$. Finally, we have investigated the case of $g = 1$ and shown that in this case our main result is true.

4 Generating Functions for $|\text{Red}_S(K)|$

Let $S$ be an arbitrary set of places of $K$. We want to consider the subset of divisors of $\text{Red}_S(K)$ of a fixed degree, and describe their quantity using a generating function. Since this does not make that much sense for finite sequences, we extend the definition of $\text{Red}_S(K)$ to include divisors of higher degree; these additional elements are not relevant for computational reasons, but allow to relate the so obtained generating functions of $\text{Red}_S(K)$ with the zeta function of $K$.

We begin with defining a filtration $\text{Red}_S(K) = \bigcup_{d=0}^g \text{Red}_d^S(K)$, where $\text{Red}_d^S(K)$ contains the reduced divisors in $\text{Red}_S(K)$ of degree $d$, and extending $\text{Red}_d^S(K)$ for $d > g$. For that, define

$$\text{Red}_d^S(K) := \left\{ D \in \text{Div}(K) \mid D \geq 0, \deg D = d, \forall q \in S : \nu_q(D) = 0 \forall D' \in \text{Div}(R) : (\text{Con}_{K/R}(D') \leq D \Rightarrow D' \leq 0) \right\}$$

for any $d \in \mathbb{N}$. Then, the classification of Proposition 3.4 also holds:

**Proposition 4.1.** Let $D \in \text{Div}(K)$. Then $D \in \text{Red}_d^S(K)$ if, and only if,

(i) $\deg D = d$;

(ii) $\nu_q(D) = 0$ for all $q \in S$ and

(iii) for all places $q$ of $R$,

$$\{\nu_p(D) \mid p \cap R = q\} \in \begin{cases} \{0\} & \text{if } q \text{ is inert in } K, \\ \{0, 1\} & \text{if } q \text{ ramifies in } K, \\ \{0, 0, 0, 1, \ldots, 0, g\} & \text{if } q \text{ splits in } K. \end{cases}$$

$\square$
Moreover, in the case that $p \in S$, we have the disjoint union
\[
\text{Red}_S(K) = \bigcup_{d=0}^{g} \text{Red}_S^d(K).
\]
Consider $C_n(S) := |\text{Red}_S^d(K)|$; we are interested in the generating function
\[
h_S(t) := \sum_{n=0}^{\infty} C_n(S)t^n \in \mathbb{Q}[[t]]
\]
and its relation to $h_\emptyset(t)$. We begin with a statement on the relation of $\text{Red}_S^d(K)$ if we modify $S$ in certain ways.

**Proposition 4.2.** Let $S' = S \cup \{p_1, p_2\}$ with $p_i \not\in S$, and let $d \in \mathbb{N}$.

(a) Assume that $p_1 \neq p_2 = \sigma(p_1)$. Then we have the disjoint union
\[
\text{Red}_S^d(K) = \bigcup_{i=0}^{\infty} \{D + ip_1, D + ip_2 \mid D \in \text{Red}_{S'}^{d-i \deg(p_1)}(K)\}.
\]

(b) Assume that $p_1 = p_2 = \sigma(p_1)$, and that $p_1 \cap R$ is not inert in $K$. Then we have the disjoint union
\[
\text{Red}_S^d(K) = \bigcup_{i=0}^{1} \{D + ip_1 \mid D \in \text{Red}_{S'}^{d-i \deg(p_1)}(K)\}.
\]

(c) Assume that $p_1 = p_2 = \sigma(p_1)$, and that $p_1 \cap R$ is inert in $K$. Then we have
\[
\text{Red}_S^d(K) = \text{Red}_{S'}^d(K).
\]

(d) Assume that $p_1 = p_2$ and $\sigma(p_1) \in S$. Then we have the disjoint union
\[
\text{Red}_S^d(K) = \bigcup_{i=0}^{\infty} \{D + ip_1 \mid D \in \text{Red}_{S'}^{d-i \deg(p_1)}(K)\}.
\]

**Proof.** This is clear from the generalization of Proposition 3.4.

This allows us to state how to obtain $h_S(t)$ from $h_{S'}(t)$ in these cases:

**Theorem 4.3.** Let $S' = S \cup \{p_1, p_2\}$ with $p_i \not\in S$.

(a) Assume that $p_1 \neq p_2 = \sigma(p_1)$. Then
\[
h_{S'}(t) = \frac{1 - t^\deg p_1}{1 + t^\deg p_1} h_S(t).
\]
(b) Assume that \( p_1 = p_2 = \sigma(p_1) \), and that \( p_1 \cap R \) is not inert in \( K \). Then

\[
h_S'(t) = \frac{1}{1 + t^\deg p_1} h_S(t).
\]

(c) Assume that \( p_1 = p_2 = \sigma(p_1) \), and that \( p_1 \cap R \) is inert in \( K \). Then

\[
h_S'(t) = h_S(t).
\]

(d) Assume that \( p_1 = p_2 \) and \( \sigma(p_1) \in S \). Then

\[
h_S'(t) = (1 - t^\deg p_1) h_S(t).
\]

Proof. Using the proposition, we obtain

\[
\begin{align*}
(a) \quad & C_d(S) = C_d(S') + 2 \sum_{i=1}^{+\infty} C_{d-i \deg p_1}(S') , \\
(b) \quad & C_d(S) = C_d(S') + C_{d-\deg p_1}(S'), \\
(c) \quad & C_d(S) = C_d(S') \quad \text{and} \\
(d) \quad & C_d(S) = \sum_{i=0}^{+\infty} C_{d-i \deg p_1}(S').
\end{align*}
\]

Therefore,

\[
\begin{align*}
(a) \quad & h_S(t) = h_S'(t) \left( 1 + 2 \sum_{i=1}^{+\infty} t^i \deg p_1 \right), \\
(b) \quad & h_S(t) = h_S'(t)(1 + t^\deg p_1) , \\
(c) \quad & h_S(t) = h_S'(t) \quad \text{and} \\
(d) \quad & h_S(t) = h_S'(t) \sum_{i=0}^{+\infty} t^i \deg p_1.
\end{align*}
\]

Now, the geometric series gives

\[
\sum_{i=0}^{+\infty} t^i \deg p_1 = \frac{1}{1 - t^\deg p_1},
\]

and multiplying with \( 1 + t^\deg p_1 \) gives

\[
1 + 2 \sum_{i=1}^{+\infty} t^i \deg p_1 = \frac{1 + t^\deg p_1}{1 - t^\deg p_1}.
\]

Plugging this in and solving for \( h_S'(t) \), we obtain the claim. \( \blacksquare \)

With this, we can explicitly describe \( h_S(t) \) in terms of \( h_\emptyset(t) \) when \( S \) is a finite set not containing places lying above inert places of \( R \).
Corollary 4.4. Let $S$ be a finite set of places of $K$, containing no places lying above inert places of $R$. For $i \in \mathbb{N}$, let $S_i = \{ p \in S \mid \deg p = i \}$, and let

- $n_i = \{ p \in S_i \mid \sigma(p) = p \}$,
- $\ell_i = \frac{1}{2} \{ p \in S_i \mid p \neq \sigma(p) \in S_i \}$,
- $m_i = \{ p \in S_i \mid \sigma(p) \not\in S \}$.

Then

$$h_S(t) = h_\emptyset(t) \cdot \prod_{i=1}^{\infty} \left( (1 - t^i)(1 + t^i)^{-\ell_i - n_i - m_i} \right).$$

Proof. Let $S' = S \cup \sigma(S)$. If $S'_i$ and $n'_i, \ell'_i, m'_i$ are defined in a similar manner, than $n'_i = n_i, \ell'_i = \ell_i + m_i, m'_i = 0$. Using the Theorem, we obtain

$$h_{S'}(t) = h_\emptyset(t) \cdot \prod_{i=1}^{\infty} \left( \frac{1 - t^i}{1 + t^i} \right)^{\ell'_i} \cdot \prod_{i=1}^{\infty} \left( \frac{1}{1 + t^i} \right)^{n'_i} = h_\emptyset(t) \cdot \prod_{i=1}^{\infty} \left( (1 - t^i)(1 + t^i)^{-\ell'_i - n'_i} \right) = h_\emptyset(t) \cdot \prod_{i=1}^{\infty} \left( (1 - t^i)^{\ell_i + m_i}(1 + t^i)^{-\ell_i - m_i - n_i} \right).$$

Using the Theorem a second time and the fact that

$$S' \setminus S = \{ \sigma(p) \in S \mid \sigma(p) \not\in S \},$$

we get

$$h_{S'}(t) = h_S(t) \cdot \prod_{i=1}^{\infty} (1 - t^i)^{m_i}.$$

Putting everything together, we have

$$h_S(t) = h_{S'}(t) \prod_{i=1}^{\infty} (1 - t_i)^{-m_i} = h_\emptyset(t) \cdot \prod_{i=1}^{\infty} \left( (1 - t^i)^{\ell_i}(1 + t^i)^{-\ell_i - m_i - n_i} \right),$$

what we had to show.

Next, we want to find bounds for the coefficients of the Taylor expansion of the rational functions involved in describing the relation of $h_S(t)$ to $h_\emptyset(t)$. For that, we need a small lemma on formal power series.

Lemma 4.5. Let $f = \sum_{n=0}^{\infty} a_n t^n, g = \sum_{n=0}^{\infty} b_n t^n \in \mathbb{C}[t[t]$ and $a, b \in \mathbb{R}_{\geq 0}$ such that $|a_n| \leq a^n, |b_n| \leq b^n$. If $fg = \sum_{n=0}^{\infty} c_n$, then $|c_n| \leq (a+b)^n$ for all $n \in \mathbb{N}$. In case $a_0 = b_0 = 1$, we have $c_0 = 1$ and $c_1 = a_1 + b_1$. 

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Proof. We have \(c_n = \sum_{i=0}^{n} a_i b_{n-i}\), whence

\[|c_n| \leq \sum_{i=0}^{n} |a_i||b_{n-i}| \leq \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i} = (a + b)^n.\]

Finally, \(a_0 = b_0 = 1\) clearly implies \(c_0 = 1\) and \(c_1 = a_1 + b_1\).

Using this, we obtain how two generating functions \(h_S(t)\) and \(h_{S'}(t)\) differ in case \(S' \subset S\):

**Corollary 4.6.** Assume that \(S\) is finite and contains no places lying above inert places of \(R\), and let \(S' \subset S\) be a subset. Let

\[h_S(t) = h_{S'}(t) \sum_{n=0}^{\infty} a_n t^n.\]

Then \(a_0 = 1\), \(a_1 = -(|S| - |S'|)\) and \(|a_n| \leq (|S| - |S'|)^n\) for \(n \in \mathbb{N}\).

**Proof.** Define \(S_i, S_i', n_i, n_i', m_i, m_i'\) and \(\ell_i, \ell_i'\) as before. Then

\[h_S(t) = h_{S'}(t) \prod_{i=1}^{\infty} \left(\frac{1-t_i}{1+t_i} \right)^{\ell_i - \ell_i'} \left(\frac{1}{1+t_i} \right)^{n_i + m_i - n_i' - m_i'}.\]

Note that \(\ell_i' \leq \ell_i\) and \(n_i' \leq n_i\), whence \(\ell_i - \ell_i' \geq 0\) and \(n_i' - n_i \geq 0\). Moreover, \(\tilde{M}_i := \max\{m_i - m_i', 0\} \leq \ell_i - \ell_i'\) and \(m_i' - m_i = \tilde{M}_i - \hat{M}_i\) with \(\hat{M}_i := \max\{m_i' - m_i, 0\}\).

Write

\[\sum_{n=0}^{\infty} c_n t^n = \left(\frac{1-t_i}{1+t_i} \right)^{\ell_i - \ell_i'} \left(\frac{1}{1+t_i} \right)^{n_i + m_i - n_i' - m_i'}.\]

Hence, by applying the lemma repeatedly, we obtain

\[c_1 = \begin{cases} -(|S| - |S'|) & \text{if } i = 1, \\ 0 & \text{otherwise} \end{cases}\]

and \(|c_n| \leq 2(\ell_i - \ell_i' - \tilde{M}_i) + (n_i - n_i') + \hat{M}_i + \tilde{M}_i)^n = (|S_i| - |S_i'|)^n\).

□

We can now make an explicit statement on the number of hole elements \(|\text{Red}_p(K) \setminus \text{Red}_S(K)|\) for all hyperelliptic curves.

**Corollary 4.7.** Assume that \(g \geq 1\), let \(S\) be finite and let \(p \in S\) be a place of degree one. There exists efficiently computable \(c_1, \ldots, c_g \in \mathbb{Z}\) such that

\[|\text{Red}_p(K) \setminus \text{Red}_S(K)| = (|S_1| - 1)C_{g-1}(\{p\}) - \sum_{j=0}^{g-2} \sum_{i=1}^{g-j} c_i C_j(\{p\}).\]
and
\[
||\text{Red}_p(K) \setminus \text{Red}_S(K)| - (|S_1| - 1)C_{g-1}(\{p\})| \\
\leq \sum_{j=0}^{g-2}(g - j)(|S| - 1)^{g-j}C_j(\{p\}).
\]

Proof. By Corollary 4.6, we have that 
\[h_S(t) = h_{\{p\}}(t) \cdot \sum_{n=0}^{\infty} a_n t^n \text{ with } a_0 = 1,\]
\[a_1 = -(|S_1| - 1) \text{ and } |a_n| \leq (|S| - 1)^n.\]

Therefore,
\[h_{\{p\}}(t) - h_S(t) = \left(\sum_{n=0}^{\infty} C_n(\{p\}) t^n\right) \cdot \left(|S_1| - 1 - \sum_{n=2}^{\infty} a_n t^n\right).
\]

Now the coefficient of \(t^d\) equals \(|\text{Red}_p^d(K)| - |\text{Red}_S^d(K)|\), whence
\[
|\text{Red}_p(K) \setminus \text{Red}_S(K)| = -\sum_{d=1}^{g} \sum_{i=0}^{d-1} C_i(\{p\}) a_{d-i}
\]
\[
= -\sum_{i=0}^{g-2} \sum_{d=i+1}^{g} a_{d-i} C_i(\{p\}) = -a_1 C_{g-1}(\{p\}) - \sum_{i=0}^{g-2} a_i C_i(\{p\}),
\]
which implies the first equality. By using \(|a_n| \leq (|S| - 1)^n\), we get
\[
||\text{Red}_p(K) \setminus \text{Red}_S(K)| - (|S_1| - 1)C_{g-1}(\{p\})| \\
\leq \sum_{i=0}^{g-2} \sum_{d=1}^{g} (|S| - 1)^d C_i(\{p\}) \leq \sum_{i=0}^{g-2} (g - i)(|S| - 1)^{g-i} C_i(\{p\}).
\]

In this section we found a description of the generating function \(h_S(t)\) of \(\text{Red}_S(K)\) in terms of \(h_\emptyset(t)\) and a rational factor, of which we have information on its coefficients in the Taylor expansion. This allowed us to give a bound on \(||\text{Red}_p(K) \setminus \text{Red}_S(K)| - (|S_1| - 1)C_{g-1}(\{p\})||\) in terms of \(|S| - 1\) and \(C_i(\{p\})\), \(0 \leq i \leq g\).

Our next goal is to obtain information on \(h_{\{p\}}(t)\), i.e. on the \(C_i(\{p\})\)'s; then, we can combine this information with the above result to obtain our main result on \(|\text{Red}_p(K) \setminus \text{Red}_S(K)|\).

5 Counting Reduced Divisors of Certain Degrees

In this section, we want to obtain information on \(h_{\{p\}}(t)\). In particular, we show that all \(h_S(t)\) are rational as long as \(S\) is finite and relate \(h_{\{p\}}(t)\) to the \(L\)-polynomial of \(K\).

Let \(k = \mathbb{F}_q\), the field of \(q\) elements. We begin considering \(C_d(\emptyset) = |\text{Red}_\emptyset^d(K)|\)
for all \(d \in \mathbb{N}\). Later, we will relate the \(C_d(\emptyset)\)'s with the \(C_d(\{p\})\)'s.
For \( d \in \mathbb{N} \), we also consider the sets
\[
\text{Div}^d_+(K) := \{ D \in \text{Div}(K) \mid D \geq 0, \deg D = d \}
\]
and
\[
\text{Div}^d_+(R) := \{ D \in \text{Div}(R) \mid D \geq 0, \deg D = d \}.
\]

Set \( A_n(K) := |\text{Div}^d_+(K)| \) and \( A_n(R) := |\text{Div}^d_+(R)| \).

**Proposition 5.1.** Let \( d \geq 0 \). For every \( D \in \text{Div}^d_+(K) \), there exists a unique integer \( r \) with \( 0 \leq r \leq d/2 \) and two unique divisors \( D_R \in \text{Div}^r_+(R) \), \( D_K \in \text{Red}^{d-2r}_0(K) \) such that \( D = \text{Con}_{K/R}(D_R) + D_K \).

**Proof.** For \( D \in \text{Div}^d_+(K) \), consider
\[
A(D) := \{ D_R \in \text{Div}^r_+(R) \mid D_R \geq 0, \text{Con}_{K/R}(D_R) \leq D \}.
\]
This set turns out to be a finite lattice when ordered with \( \leq \), whence it has a maximal element, say \( D_R \). Then \( D_K := D - \text{Con}_{K/R}(D_R) \geq 0 \) and, if \( r := \deg D_R \), by the maximality of \( D_R \), \( D_K \in \text{Red}^{d-2r}_0(K) \). The uniqueness is clear from the lattice structure of \( A(D) \).

**Corollary 5.2.** For \( d \geq 0 \), we have
\[
A_d(K) = \sum_{r=0}^{[d/2]} A_r(R)C_{d-2r}(\emptyset).
\]

The zeta function of \( K \) is given by
\[
Z_K(t) := \sum_{n=0}^{\infty} A_n(K)t^n,
\]
and the zeta function of \( R \) is given by
\[
Z_R(t) := \sum_{n=0}^{\infty} A_n(R)t^n = \frac{1}{(1-t)(1-qt)}.
\]
(See [Sti93, Chapter 5].)

Consider the formal power series \( h_\emptyset(t) = \sum_{d=0}^{\infty} C_d(\emptyset)t^d \in \mathbb{Q}[[t]] \). The following result shows its relation to the zeta function of \( K \) and the zeta function of \( R \):

**Lemma 5.3.** Let \( f(t) = Z_R(t^2) \cdot h_\emptyset(t) \in \mathbb{Q}[[t]] \) as a formal power series; write \( f(t) = \sum_{n=0}^{\infty} a_nt^n \). Then, for \( d \geq 0 \), \( a_n = A_n(K) \), i.e. \( f(t) = Z_K(t) \).
**Proof.** We have

\[
f = \left( \sum_{n=0}^{\infty} A_n(R)t^{2n} \right) \cdot \left( \sum_{m=0}^{\infty} C_m(\emptyset)t^m \right).
\]

Hence, the coefficient of \( t^d \) in the product is given by \( \sum_{2n+m=d} A_n(R)C_m(\emptyset) \). But this means \( n \leq \lfloor d/2 \rfloor \), i.e. we can write this sum as \( \sum_{n=0}^{\lfloor d/2 \rfloor} A_n(R)C_{d-2n}(\emptyset) \), which equals \( A_d(K) \) by the previous corollary.

Therefore, we see that

\[
h_{\emptyset}(t) = \frac{Z_K(t)}{Z_R(t^2)} = (1 - t^2)(1 - qt^2)Z(t).
\]

Now \( Z_K(t) \) is a rational function as well: by [Sti93, p. 193, Theorem V.1.15], \( Z_K(R) = \frac{L_K(t)}{(1-t)(1-qt)} \), where \( L_K \in \mathbb{Z}[t] \) with \( \deg L_K = 2g \); the polynomial \( L_K \) is called the \( L \)-polynomial of \( K \). Hence,

\[
h_{\emptyset}(t) = \frac{(1 - t^2)(1 - qt^2)L_K(t)}{(1-t)(1-qt)} = \frac{(1 + t)(1 - qt^2)L_K(t)}{1 - qt}
\]

is a rational function with a simple pole in \( t = q^{-1} \); in particular, \( h_{\emptyset}(t) \) is a convergent power series with radius of convergence \( q^{-1} \). Therefore, we obtain:

**Theorem 5.4.** For any finite set \( S \) of places of \( K \), \( h_S(t) \) is a convergent power series with radius of convergence \( q^{-1} \). In particular,

\[
h_{\emptyset}(t) = \frac{(1 + t)(1 - qt^2)L_K(t)}{1 - qt},
\]

where \( L_K \) is the \( L \)-polynomial of \( K \).

Important as well is the fact that \((1 - qt)h_{\emptyset}(t)\) is a polynomial of degree \( 2g+3 \), namely \((1 + t)(1 - qt^2)L_K(t)\). Write

\[
L_K(t) = \sum_{i=0}^{2g} a_it^i.
\]

Then \( a_0 = 1 \), \( a_{2g} = q^g \) and \( a_{2g-i} = q^{g-i}a_i \) for \( 0 \leq i \leq g \). Moreover, \( a_1 = N - (q + 1) \), where \( N = A_t(K) = |\text{Div}_+(K)| \) is the number of rational places of degree one.

**Lemma 5.5.** We have

\[
(1 - qt)h_{\emptyset}(t) = 1 + (C_1(\emptyset) - q)t + \sum_{d=2}^{\infty} (C_d(\emptyset) - qC_{d-1}(\emptyset))t^d
\]
and

\begin{align*}
(1 + t)(1 - qt^2)L_K(t) &= 1 + (a_1 + 1)t + (a_1 + a_2 - q)t^2 \\
&+ \sum_{i=3}^{2g} (a_i + a_{i-1} - qa_{i-2} - qa_{i-3})t^i \\
&+ (q^g - qa_{2g-1} - qa_{2g-2})t^{2g+1} \\
&- q(q^g + a_{2g-1})t^{2g+2} - q^{g+1}t^{2g+3}.
\end{align*}

Proof. This is an easy and direct computation. \hfill \Box

Using \((1 - qt)h_0(t) = (1 + t)(1 - qt^2)L_K(t)\) and comparing coefficients, we obtain:

**Corollary 5.6.** For \(d \in \{4, \ldots, 2g\}\), we have

\begin{align*}
C_0(\emptyset) &= 1, \\
C_1(\emptyset) &= q + a_1 + 1, \\
C_2(\emptyset) &= q^2 + a_1q(1 + q^{-1}) + a_2, \\
C_3(\emptyset) &= q^3 + a_1q^2 + a_2q(1 + q^{-1}) + a_3 - q \quad \text{and} \\
C_d(\emptyset) &= q^d + \sum_{i=1}^{d-3} a_iq^{d-i}(1 - q^{-2}) + a_{d-2}q^2 + a_{d-1}q(1 + q^{-1}) + a_d - q^{d-2}. 
\end{align*}

Proof. The equalities for \(d \leq 2\) follow directly from the lemma. For \(d \geq 3\), we have

\(C_d(\emptyset) - q^d = q(C_{d-1}(\emptyset) - q^{d-1}) + a_d + a_{d-1} - qa_{d-2} - qa_{d-3}\).

Plugging in \(d = 3\) and the formula for \(C_2(\emptyset)\), we obtain

\(C_3(\emptyset) - q^3 = q^2a_1 + q(1 + q^{-1})a_2 + a_3 - q\).

Now, for \(d = 4\), we similarly obtain

\(C_4(\emptyset) - q^4 = q^3(1 - q^{-2})a_1 + q^2a_2 + q(1 + q^{-1})a_3 + a_4 - q^2\).

Now let \(d \geq 4\); then, using induction,

\begin{align*}
C_{d+1}(\emptyset) - q^{d+1} &= q(C_d(\emptyset) - q^d) + a_{d+1} + a_d - qa_{d-1} - qa_{d-2} \\
&= \sum_{i=1}^{d-3} q^{d+1-i}(1 - q^{-2})a_i + q^3a_{d-2} + q^2(1 + q^{-1})a_{d-1} \\
&+ qa_d - q^{d-1} + a_{d+1} + a_d - qa_{d-1} - qa_{d-2} \\
&= \sum_{i=1}^{d-3} q^{d+1-i}(1 - q^{-2})a_i + q^3(1 - q^{-2})a_{d-2} + q^2a_{d-1} \\
&+ q(1 + q^{-1})a_d + a_{d+1} - q^{d-1},
\end{align*}

what we had to show. \hfill \Box

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We have further information on the integers \(a_i\). The result we need in the following are the Hasse-Weil bounds:

**Proposition 5.7 (Hasse-Weil Bounds).** For \(i = 0, \ldots, 2g\), we have \(|a_i| \leq \left(\frac{2g}{i}\right)q^{i/2}\).

**Proof.** By Hasse-Weil [Sti93, p. 193, Theorem V.1.15 and p. 197, Theorem V.2.1],

\[ L_K(t) = \prod_{i=1}^{2g}(1 - \alpha_i t) \text{ with } |\alpha_i| = q^{1/2}. \]

Therefore,

\[ a_i = (-1)^i \sum_{1 \leq j_1 < \cdots < j_i \leq 2g} \prod_{t=1}^{i} \alpha_{j_t}. \]

The sum has \(\binom{2g}{i}\) terms, whence we obtain the specified bound. \(\square\)

Using them, we can make explicit statements on the cardinality of \(C_d(\{p\})\) for the case \(\deg p = 1\):

**Theorem 5.8.** Let \(S = \{p\}\) with \(\deg p = 1\). Then, for \(d \in \{1, \ldots, g\}\),

\[ C_d(S) \leq \sum_{i=0}^{d} \binom{2g}{i} q^{d-i/2} \]

and \(\left| C_d(S) - q^d \right| \leq 2q^{d-1} + \sum_{i=1}^{d} \binom{2g}{i} q^{d-i/2} \).

For the proof, we need a rather technical lemma.

**Lemma 5.9.** Let \(S = \{p\}\) with \(\deg p = 1\).

(a) If \(\sigma(p) = p\),

\[ C_0(S) = 1, \]

\[ C_1(S) = q + a_1 \]

and \(C_d(S) = \sum_{i=0}^{d-2} a_i q^{d-i}(1 - q^{-1}) + a_{d-1}q + a_d\)

for \(d \in \{2, \ldots, 2g\}\).

(b) If \(\sigma(p) \neq p\),

\[ C_0(S) = 1, \]

\[ C_1(S) = q + a_1, \]

\[ C_2(S) = q^2(1 - q^{-1} + q^{-2}) + a_1q + a_2, \]

\[ C_3(S) = q^3(1 - q^{-1} - q^{-2}) + a_1q^2(1 - q^{-1} - q^{-2}) + a_2q + a_3 \]

and \(C_d(S) = \sum_{i=0}^{d-4} a_i q^{d-i}(1 - q^{-1} - q^{-2} + q^{-3}) + a_{d-3}q^3(1 - q^{-1} - q^{-2}) + a_{d-2}q^2(1 - q^{-1} - q^{-2}) + a_{d-1}q + a_d\)

for \(d \in \{4, \ldots, 2g\}\).
Proof.

(a) By Proposition 4.2 (b), \( C_d(\emptyset) = C_d(S) + C_{d-1}(S) \), whence \( C_0(S) = C_0(\emptyset) \) and, for \( d > 0 \), \( C_d(S) = C_d(\emptyset) - C_{d-1}(S) \). Hence, with Corollary 5.6 one obtains

\[
\begin{align*}
C_0(S) &= 1, \\
C_1(S) &= q + a_1, \\
C_2(S) &= q^2 - q + a_1 q + a_2, \\
\text{and} \quad C_3(S) &= q^3 - q^2 + a_1 q^2 (1 - q^{-1}) + a_2 q + a_3.
\end{align*}
\]

For \( d > 3 \), using induction and Corollary 5.6, we see that

\[
\begin{align*}
C_d(S) &= C_d(\emptyset) - C_{d-1}(S) \\
&= \left[ q^d + \sum_{i=1}^{d-3} a_i q^{d-i}(1 - q^{-2}) + a_{d-2} q^2 + a_{d-1} q (1 + q^{-1}) + a_d - q^{d-2} \right] \\
- \left[ \sum_{i=0}^{d-3} a_i q^{d-1-i} (1 - q^{-1}) + a_{d-2} q + a_{d-1} \right] \\
&= \sum_{i=0}^{d-2} a_i q^{d-i} (1 - q^{-1}) + a_{d-1} q + a_d,
\end{align*}
\]

what we had to show.

(b) By Proposition 4.2 (a), \( C_d(\emptyset) = \sum_{i=0}^{d} C_i(S) \), whence \( C_0(S) = C_0(\emptyset) \) and \( C_d(S) = C_d(\emptyset) - C_{d-1}(\emptyset) \) for \( d \in \{1, \ldots, g\} \). Hence, with Corollary 5.6

\[
\begin{align*}
C_0(S) &= 1, \\
C_1(S) &= q + a_1, \\
C_2(S) &= q^2 (1 - q^{-1} + q^{-2}) + a_1 q + a_2, \\
C_3(S) &= q^3 (1 - q^{-1} - q^{-2}) + a_1 q^2 (1 - q^{-1} - q^{-2}) + a_2 q + a_3, \\
C_4(S) &= q^4 (1 - q^{-1} + q^{-3}) + a_1 q^3 (1 - q^{-1} + q^{-3}) \\
&\quad + a_2 q^2 (1 - q^{-1}) + a_3 q + a_4.
\end{align*}
\]

For \( d > 4 \),

\[
\begin{align*}
C_d(S) &= C_d(\emptyset) - C_{d-1}(\emptyset) \\
&= \sum_{i=0}^{d-4} a_i q^{d-i} (1 - q^{-1} - q^{-2} + q^{-3}) + a_{d-3} q^3 (1 - q^{-1} - q^{-2}) \\
&\quad + a_{d-2} q^2 (1 - q^{-1} - q^{-2}) + a_{d-1} q + a_d.
\end{align*}
\]

\( \Box \)
Proof of Theorem 6.8. Note that both for \( \sigma(p) = p \) and \( \sigma(p) \neq p \), one quickly obtains from the Lemma that

\[
|C_d(S)| \leq \sum_{i=0}^{d} q^{d-i}|a_i|
\]

and

\[
|C_d(S) - q^d| \leq 2q^{d-1} + \sum_{i=1}^{d} q^{d-i}|a_i|
\]

using \( 0 < 1 - q^{-1} < 1, 0 < 1 - q^{-1} + q^{-2} < 1, 0 < 1 - q^{-1} - q^{-2} < 1, 0 < 1 - q^{-1} - q^{-2} + q^{-3} < 1, 0 < 1 + q^{-1} < 2 \) and \( 0 < 1 + q^{-1} - q^{-2} < 2 \). Now \( q^{d-i}|a_i| \leq (\frac{2g}{i})q^{d-i/2} \) by the Hasse-Weil bounds, whence we can conclude. □

In this section, we have shown that \( h_S(t) \) is rational for every finite set of places \( S \). Moreover, we have given bounds on the coefficients of \( h_p(t) \) for some place \( p \) of degree one. In the next section, we will combine these bounds with Corollary 4.7 to obtain our first two main results.

6 Counting the Number of Hole Elements

The first main result gives an explicit bound on how much the number of hole elements deviates from \( (|S_1| - 1)q^{g-1} \). The dominant part of the error term turns out to be \( 2g(|S| - 1)q^{g-3/2} \), which reminds of the Hasse-Weil bound on the divisor class group: namely, we have \( |\text{Pic}^0(K)| \in [(\sqrt{q} - 1)^{2g}, (\sqrt{q} + 1)^{2g}] \) and \( (\sqrt{q} \pm 1)^{2g} = q^g \pm 2gq^{g-1/2} + \ldots \).

Theorem 6.1. Let \( S \) be a finite set of places of \( K \), containing a place of degree one and no places lying over inert places of \( R \). Assume that \( g \geq 1 \) and \( q^{1/2} > |S| + g \). We have

\[
|\text{Red}_p(K) \setminus \text{Red}_S(K)| - (|S_1| - 1)q^{g-1}\n\]

\[
\leq 2g(|S_1| - 1)q^{g-3/2} + 2^{2g}g^{g-1}(|S| - 1)^2q^{g-2}.
\]

For the proof, we need two technical lemmata. We also use the abbreviation \( C_d := C_d(\{p\}) \).

Lemma 6.2. Assume that \( g \geq 2 \) and that \( q^{1/2} \geq \max\{|S| - 1, 2\} \). We then have

\[
|\text{Red}_p(K) \setminus \text{Red}_S(K)| - (|S_1| - 1)C_{g-1}| \leq 2^{2g-2}g^{g-1}(|S| - 1)^2q^{g-2}.
\]

Proof. For \( |S| = 1 \) there is nothing to show; hence, assume that \( |S| > 1 \). Note that \( |S| - 1 \leq q^{1/2} \leq \frac{1}{2}q \) (as \( q \geq 4 \)) gives \( q - (|S| - 1) \geq \frac{1}{2}q \) and \( q^{1/2} + (|S| - 1) \leq 2q^{1/2} \). We show the result in three steps.
(i) Clearly, for $0 \leq i \leq j \leq g - 2$,
\[
\binom{2g}{i} = \frac{(2g)!}{i!(2g-i)!} \leq \frac{(2g)!}{j!(2g-j)!} \leq \frac{(2g)!}{(g+2)!}.
\]
We have
\[
\sum_{i=0}^{j} \binom{2g}{i} q^{i-j/2} \leq \frac{(2g)!}{(g+2)!} q^{j} \sum_{i=0}^{j} \binom{2g}{i} q^{-i/2} = \frac{(2g)!}{(g+2)!} (q + q^{1/2})^{j}.
\]

(ii) We have
\[
\begin{aligned}
&\sum_{j=0}^{g-2} (g-j)(|S|-1)^{g-j} \sum_{i=0}^{j} \binom{2g}{i} q^{i-j/2} \\
&\leq g \frac{(2g)!}{(g+2)!} (|S|-1)^{g} \sum_{j=0}^{g-2} \left( \frac{q + q^{1/2}}{|S|-1} \right)^{j} \\
&= g \frac{(2g)!}{(g+2)!} (|S|-1)^{g} \left( \frac{q + q^{1/2}}{|S|-1} \right)^{g-1} - 1 \\
&\leq g(2g)^{g-2} (|S|-1)^{2g-1} \frac{q^{g-1} - (1 + q^{-1/2})^{g-1}}{q + q^{1/2} + 1 - |S|} \\
&\leq 2^{g-2} g^{g-1} (|S|-1)^{2g-2} q^{g-2} = 2^{g-3} g^{g-1} (|S|-1)^{2} q^{g-2}.
\end{aligned}
\]

(iii) By Theorem 5.8, we have
\[
||\text{Red}_p(K) \setminus \text{Red}_S(K)|| - (|S_1|-1)C_{g-1} \leq \sum_{j=0}^{g-2} (g-j)(|S|-1)^{g-j} \sum_{i=0}^{j} \binom{2g}{i} q^{i-j/2} \leq 2^{g-3} g^{g-1} q^{g-2} (|S|-1)^{2}.
\]

\[\square\]

**Lemma 6.3.** Assume that $g \geq 2$ and that $q^{1/2} > |S| + g$. We then have
\[
|C_{g-1} - q^{g-1}| \leq 2gq^{g-3/2} + 10g^{3} - 32g^{2} - 2q^{g-2}.
\]

**Proof.** Note that $q^{1/2} > |S| + g \geq 3$, i.e. $q > 9$.

First, assume that $g \geq 3$. Note that for $2 \leq i \leq g - 1$,
\[
\binom{2g}{i} = \binom{g-1}{i} \binom{g-1-i}{(2g)!} = \frac{(2g)!}{(g-1)(g-2)(g-1)!} \binom{g-1}{i},
\]

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whence
\[
\sum_{i=1}^{g-1} \binom{2g}{i} q^{g-1-i/2} \leq \frac{(2g)!}{(g-1)(g-2)(g+1)!} q^{g-1} \sum_{i=2}^{g-1} \binom{g-1}{i} q^{-i/2} \leq \frac{9(2g)! \cdot 2^{g-1}}{g^2(g+1)!} q^{g-2}.
\]

In case \( g = 2 \), this term is non-negative. Thus, using Theorem 5.8
\[
|C_{g-1} - q^{g-1}| \leq 2gq^{g-3/2} + 2q^{g-2} + \sum_{i=2}^{g-1} \binom{2g}{i} q^{g-1-i/2} \
\leq 2gq^{g-3/2} + \left[ 2 + \frac{9(2g)! \cdot 2^{g-1}}{g^2(g+1)!} \right] q^{g-2}.
\]
Using \( \frac{(2g)!}{g^2(g+1)!} \leq g^{g-1}2g^{-1} \), we obtain the bound \( 2gq^{g-3/2} + 10q^{g-3}2^{2g-2}q^{g-2} \). \( \Box \)

**Proof of Theorem 6.1.** First, for \( g = 1 \) or \( |S| = 1 \)
\[
||\text{Red}_p(K) \ \text{Red}_S(K)\| - (|S| - 1)| = 0
\]
by Proposition 3.1 in case \( g = 1 \) respectively the definition of \( \text{Red}_S(K) \) if \( |S| = 1 \).
Hence, assume that \( g > 1 \) and \( |S| \geq 2 \). We have
\[
||\text{Red}_p(K) \ \text{Red}_S(K)\| - (|S| - 1)q^{g-1} | \
\leq ||\text{Red}_p(K) \ \text{Red}_S(K)\| - (|S| - 1)C_{g-1}| + (|S| - 1)|C_{g-1} - q^{g-1}|.
\]
Now, with the two lemmata, this can be bounded by
\[
2g(|S| - 1)q^{g-3/2} + 2g^{g-3}g^{g-1}(|S| - 1)^2q^{g-2} + 10g^{g-3}2^{2g-2}(|S| - 1)q^{g-2} \
\leq 2g(|S| - 1)q^{g-3/2} + 2g^{g-1}(|S| - 1)^2q^{g-2}
\]
as \( 2^{-1} + 10g^{-2}(|S| - 1)^{-1} \leq 2 \). \( \Box \)

Using this theorem, we can also prove our second main result which states that the probability of “stepping into a hole”, i.e. that a random element of \( \text{Red}_p(K) \) lies in \( \text{Red}_S(K) \), equals \( \frac{|S| - 1}{q} \), with an error of \( \mathcal{O}(16^g(|S| - 1)q^{-3/2}) \):

**Corollary 6.4.** Assume that \( g \geq 1 \) and let \( S \) be as in Theorem 6.1. For \( q \to \infty \), we have
\[
\left| \frac{|\text{Red}_p(K) \ \text{Red}_S(K)\| - |S| - 1}{|\text{Red}_p(K)|} \right| = \mathcal{O}(16^g(|S| - 1)q^{-3/2}).
\]

**Proof.** Assume that \( q^{1/2} > |S| + g \). Note that \( |\text{Red}_p(K)| \in [(\sqrt{q} - 1)2^g, (\sqrt{q} + 1)2^g] \) by the Hasse-Weil bounds \( \text{[Lor96 p. 287, Corollary 6.3 and Remark 6.4]} \).
Now
\[
\frac{|\text{Red}_p(K) \setminus \text{Red}_S(K)|}{|\text{Red}_p(K)|} - \frac{|S_1| - 1}{q} \\
\leq \frac{|\text{Red}_p(K) \setminus \text{Red}_S(K)|}{|\text{Red}_p(K)|} - \frac{(|S_1| - 1)q^{g-1}}{|\text{Red}_p(K)|} \\
+ \frac{(|S_1| - 1)q^{g-1}}{|\text{Red}_p(K)|} - \frac{(|S_1| - 1)q^{g-1}}{q^g} \\
\leq \frac{q^g}{(\sqrt{q} - 1)^{2g}} 2g(|S_1| - 1)q^{-3/2} + 2^{2g}q^{-2}(|S| - 1)^2q^{-2} \\
+ \frac{q^g}{(\sqrt{q} - 1)^{2g}} |q^g - |\text{Red}_p(K)|| \\
\leq 2^{2g+1}g(|S_1| - 1)q^{-3/2} + 2^{4g}(|S| - 1)^2q^{-2} \\
+ 2^{2g}(|S_1| - 1)\max\{(1 + q^{-1/2})^{2g} - 1, 1 - (1 - q^{-1/2})^{2g}\}
\]
using $\sqrt{q} - 1 < 2$ as $\sqrt{q} > 2$. Now
\[
1 - (1 - q^{-1/2})^{2g} = \sum_{i=1}^{2g} \binom{2g}{i}(-1)^{i+1}q^{i/2} \leq \sum_{i=1}^{g} \binom{2g}{2i-1} q^{-i/2} < 4^g q^{-1/2}
\]
and, analogously, $(1 + q^{-1/2})^{2g} - 1 < 4^g q^{-1/2}$, whence we obtain
\[
\frac{|\text{Red}_p(K) \setminus \text{Red}_S(K)|}{|\text{Red}_p(K)|} - \frac{|S_1| - 1}{q} \\
\leq 2^{2g+1}g(|S_1| - 1)q^{-3/2} + 2^{4g}(|S| - 1)^2q^{-2} + 2^{4g}(|S_1| - 1)q^{-3/2} \\
\leq 2^{4g}(1 - 2g^2)q^{-1/2}(|S| - 1) + 1)(|S| - 1)q^{-3/2} \\
< 2^{4g+2}(|S| - 1)^{-3/2}
\]
as $|S_1| \leq |S|$, $2^{1-2g}q < 1$ and $q^{-1/2}(|S| - 1) < 1$. \hfill \Box

Finally, we will give an explicit formula for $C_d(S)$ in the case that all places of $S$ have degree one, i.e. $S = S_1$, and that $S \neq \emptyset$. For that, we compute the Taylor expansion of $h_S(t)$. For convenience, we set $(-1)_0 := 1$.

**Theorem 6.5.** Let $S = S_1$ be a finite, non-empty set of places of degree one. Set

- $n = |\{p \in S \mid \sigma(p) = p\}|$,
- $\ell = \frac{1}{2}|\{p \in S \mid p \neq \sigma(p) \in S\}|$,
- $m = |\{p \in S \mid \sigma(p) \notin S\}|$,
and let $L_K(t) = \sum_{i=0}^{\infty} a_i t^i$. Then
\[
C_0(S) = a_0 = 1, \\
C_1(S) = q - 2\ell - m - n + 1 + a_1, \quad \text{and for } i > 1, \\
C_i(S) = \sum_{k=0}^{i} \sum_{j=0}^{k} \sum_{p=0}^{i} \left( \ell + m + n - 2 + i - k \right) \binom{k}{j} (-1)^{i-j} \binom{\ell}{k-j} \left( -1 \right)^i \binom{i}{j} \binom{\ell}{k-j} q^p a_{j-p} \\
- \sum_{k=0}^{i-2} \sum_{j=0}^{k} \sum_{p=0}^{i} \left( \ell + m + n - 4 + i - k \right) \binom{k}{j} (-1)^{i-j} \binom{\ell}{k-j} \left( -1 \right)^i \binom{i}{j} \binom{\ell}{k-j} q^{p+1} a_{j-p}.
\]

We begin with a small lemma on the Taylor expansion on $(1 + \lambda t)^n$ with $n \in \mathbb{Z}$.

**Lemma 6.6.** Let $\lambda \in \mathbb{C}^*$ and $n \in \mathbb{N}$.

(a) We have
\[
(1 + \lambda t)^n = \sum_{i=0}^{\infty} \binom{n}{i} \lambda^i t^i.
\]

(b) We have
\[
\left( \frac{1}{1 + \lambda t} \right)^n = \sum_{i=0}^{\infty} \binom{i + n - 1}{i} (-\lambda)^i t^i.
\]

(In case $n = 0$, we need $\binom{-1}{0} = 1$.)

**Proof.** Part (a) is clear since $\binom{n}{i} = 0$ for $i > n$. For part (b), the case $n = 0$ is clear since $\binom{k-1}{0} = 1$ and $\binom{k-1}{k} = 0$ for $k > 0$. For $n > 0$, we have
\[
\left( \frac{1}{1 + \lambda t} \right)^n = (\lambda)^{1-n} \sum_{i=0}^{\infty} \binom{i+n-1}{i} \frac{\lambda^i}{t^i} = (\lambda)^{1-n} \sum_{i=0}^{\infty} \binom{i}{i-n} (-\lambda)^i t^i.
\]

finally, note that $\binom{i+n-1}{n-1} = \binom{i+n-1}{i}$.

**Proof of Theorem 6.5.** Note that
\[
\sum_{i=0}^{\infty} C_i(S) t^i = \frac{(1 - t)^\ell (1 - qt^2) L_K(t)}{(1 - qt)(1 + t)^{\ell + m + n - 1}}.
\]

Using the lemma, it suffices to compute the Taylor expansion of
\[
(1 - qt^2) \sum_{i=0}^{\infty} \binom{\ell}{i} (-1)^i t^i \sum_{j=0}^{\infty} q^j t^j \sum_{k=0}^{\infty} \binom{k + \ell + m + n - 2}{k} (-1)^k t^k \sum_{p=0}^{\infty} a_p t^p,
\]

\[=:A\]
which can be obtained by multiplying out. First,
\[
\sum_{i=0}^{\infty} \binom{\ell}{i} (-1)^i t^i \cdot \sum_{j=0}^{\infty} q^j t^j \cdot \sum_{p=0}^{\infty} a_p t^p = \sum_{i=0}^{\infty} \binom{\ell}{i} (-1)^i t^i \cdot \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} q^p a_{j-p} t^j
\]
\[
= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} \binom{\ell}{i-j} (-1)^{i-j} q^p a_{j-p} t^i.
\]
Using this, we obtain that \( A \) equals
\[
\sum_{k=0}^{\infty} \binom{k + \ell + m + n - 2}{k} (-1)^k t^k \cdot \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} \binom{\ell}{i-j} (-1)^{i-j} q^p a_{j-p} t^i
\]
\[
= \sum_{k=0}^{\infty} \sum_{i=0}^{k} \sum_{j=0}^{i} \binom{\ell + m + n - 2 + k - i}{k-i} (-1)^{k-i} \binom{\ell}{i-j} q^p a_{j-p} t^k.
\]

Note that
\[
b_0 = a_0 = 1 \quad \text{and} \quad b_1 = (q - \ell - (\ell + m + n - 1)) a_0 + a_1 = q - 2\ell - m - n + 1 + a_1.
\]

If we then multiply \( A \) by \( 1 - qt^2 \), and use these relations, we obtain the claim. \( \Box \)

Note that one can also compute the Taylor expansion by working in \( \mathbb{Z}[[x]]/(x^{g+1}) \): multiplying two elements requires \( \mathcal{O}(g^2) \) multiplications and additions in \( \mathbb{Z} \), whence one can compute \( h_S(t) \) from \( L_K \) using a square-and-multiply method in \( \mathcal{O}(\log |S| \cdot g^2) \) multiplications and additions in \( \mathbb{Z} \). Also note that the coefficients can be effectively bounded, using Corollary 4.6 and Theorem 5.7.

Hence, we obtained a bound on the number of hole elements as well an exact formula, as well as a strategy how to quickly evaluate the formula. The error terms in the bounds are by no means optimal, but they suffice for our needs.

7 On the Size of Holes

Using the methods from Section 3 and Section 4, we can state some results on the size of holes; holes can be thought of as clusters of hole elements. First, we want to make this informal definition more precise.

On Red_p(K), define the equivalence relation
\[
D \sim_S D' \iff \text{ideal}_S(D) = \text{ideal}_S(D').
\]
It turns out that every equivalence class contains exactly one element of Red_S(K). For \( D \in \text{Red}_S(K) \), we call \( h(D) := [D]_{\sim_S} \setminus \{D\} \) the hole associated to \( D \). Every element of \( h(D) \) is a hole element, as well as any hole element is contained in some \( h(D) \).
Proposition 7.1. Let $D$ be an element of degree one and assume that $|S| = 2$.

Then the following statements hold:

(a) $|S| = 2$ implies $D = g - \deg D$.

For all $D' \in h(D)$ we have $D \leq D'$; hence, in case $\deg D = g$, $h(D) = 0$. Assuming that $S \setminus \{p\}$ contains a place $q$ of degree one with $\sigma(q) \in S$, $\deg D < g$ implies $D + q \in h(D)$ by Proposition 3.4, whence $h(D) \neq 0$. We need the assumption that $\sigma(q) \in S$, as otherwise it could happen that $\nu_{\sigma(q)}(D) > 0$, whence $D + q \geq \Con_{K/R}(q \cap R)$.

For the rest of this section, we assume that all places in $S$ are of degree one.

**Proposition 7.1.** Let $D \in \Div_S(K)$ and set

- $n = |\{p' \in S \setminus \{p\} \mid \sigma(p') = p'\}|$,
- $\ell = \frac{1}{2}|\{p' \in S \setminus \{p\} \mid p' \neq \sigma(p') \in S \setminus \{p\}\}|$,
- $r = |\{p' \in S \setminus \{p\} \mid \sigma(p') = p\}|$,
- $m = |\{p' \in S \setminus \{p\} \mid \sigma(p') \notin S\}|$ and
- $m_D' = |\{p' \in S \setminus \{p\} \mid \sigma(p') \notin S, \nu_{\sigma(p')}(D) = 0\}|$.

Then the following statements hold:

(a) We have $r, n, m, \ell, m_D' \in \mathbb{N}$ with $m_D' \leq m, r \leq 1$ and $|S| = n + m + r + 2\ell + 1$.

(b) In case $m_D' < |S| - 1$, we have $h(D) = \emptyset$ if, and only if, $\deg D = g$. The “if” part also holds if $m_D' = |S| - 1$.

(c) In case $|S| = 1$, we have $h(D) = \emptyset$.

(d) There is a bijection between $h(D)$ and the set

$$A_{g - \deg D}^{n, \ell, m_D'} := \left\{(a, b, c) \mid a = (a_i)_i \in \{0, 1\}^n, b = (b_i)_i \in \mathbb{Z}^\ell, c = (c_i)_i \in \mathbb{N}^{m_D'} + r \mid 1 \leq \sum_{i=1}^n a_i + \sum_{i=1}^\ell |b_i| + \sum_{i=1}^{m_D'} c_i \leq g - \deg D \right\}.$$  

**Proof.** Part (a) is clear, and part (b) follows from the above discussion. Part (c) is clear as in that case, $\Red_S(K) = \Red_p(K)$. For part (d), let

- $\{p' \in S \setminus \{p\} \mid \sigma(p') = p'\} = \{p_{1,1}, \ldots, p_{1,n}\}$
- $\{p' \in S \setminus \{p\} \mid p' \neq \sigma(p') \in S \setminus \{p\}\} = \{p_{2,1}, \ldots, p_{2,\ell}, \sigma(p_{2,1}), \ldots, \sigma(p_{2,\ell})\}$
- $\{p' \in S \setminus \{p\} \mid \sigma(p') = p\} = \{p_{3,1}, \ldots, p_{3,r}\}$ and
- $\{p' \in S \setminus \{p\} \mid \sigma(p') \notin S, \nu_{\sigma(p)}(D) = 0\} = \{p_{4,1}, \ldots, p_{4,m_D'}\}$.
Define the map
\[ \Psi : A_{g - \deg D}^{n, \ell, m_D^{-r}} \to \text{Div}(K), \]
\[ ((a_i)_i, (b_i)_i, (c_i)_i) \rightarrow D + \sum_{i=1}^{n} a_i p_{1,i} + \sum_{i=1}^{\ell} \max\{b_i, 0\} p_{2,i} \]
\[ + \sum_{i=1}^{\ell} \max\{-b_i, 0\} \sigma(p_{2,i}) + \sum_{i=m_D^{-r}+1}^{m_D^{-r}+r} c_i p_{3,i}. \]

Clearly, \( \deg \Psi((a, b, c)) \leq g \), ideal \(_S(\Psi((a, b, c))) = \text{ideal}_S(D) \) and \( \Psi((a, b, c)) \neq D \) for all \( (a, b, c) \in A_{g - \deg D}^{n, \ell, m_D^{-r}} \). Therefore, it suffices to show that the image of \( \Psi \) lies in \( \text{Red}_p(K) \). But this follows directly from the definition of \( \Psi \), \( A_{g - \deg D}^{n, \ell, m_D^{-r}} \) and Proposition 3.4.

In the following we assume that a divisor \( D \in \text{Red}_S(K) \) is fixed, and we use \( n, m, \ell, m_D' \) as in the proposition.

Hence, to estimate the size of \( h(D) \), we have to estimate the size of \( A_{g - \deg D}^{n, \ell, m_D'^{-r}} \).

For that, define \( A_s^{n, \ell, m_D'^{-r}} \) as the set
\[ \left\{ (a, b, c) \mid a = (a_i)_i \in \{0, 1\}^n, \ b = (b_i)_i \in \mathbb{Z}^\ell, \ c = (c_i)_i \in \mathbb{N}^{m_D'^{-r}} \right\}, \]
for \( s \in \mathbb{N} \); then \( A_{\deg D - d}^{n, \ell, m_D'^{-r}} = \bigcup_{s=1}^{\deg D - d} A_s^{n, \ell, m_D'^{-r}} \) is a disjoint union. Set
\[ f_{n, \ell, m_D'^{-r}}(t) := \sum_{s=0}^{\infty} |A_s^{n, \ell, m_D'^{-r}}| t^s; \]
then, one quickly obtains
\[ f_{n, \ell, m_D'^{-r}}(t) = \sum_{s=0}^{\infty} \left( 1 + 2 \sum_{s=0}^{\infty} t^s \right)^\ell \left( \sum_{s=0}^{\infty} t^s \right)^{m_D'^{-r}} \]
\[ = \left( 1 + t \right)^n \left( 1 + 2 \frac{1}{1 - t} \right)^\ell (1 - t)^{-m_D'^{-r}} \]
\[ = (1 + t)^{n+\ell} (1 - t)^{-m_D'^{-r} - \ell}. \]

Using Lemma 6.6 this equals (with \( \binom{-1}{0} = 1 \))
\[ \sum_{s=0}^{\infty} \sum_{i=0}^{s} \binom{n + \ell}{s - i} \left( \binom{i + m_D' + r + \ell - 1}{i} \right) i^s, \]
whence \( |A_s^{n, \ell, m_D'^{-r}}| = a_{n, \ell, m_D'^{-r} + \ell} \). Combining this with \( h(D) = \sum_{s=1}^{g - \deg D} a_{s}^{n, \ell, m_D'^{-r} + \ell} \),
we obtain:
Proposition 7.2. We have that
\[ |h(D)| = \sum_{s=1}^{g-\deg D} \sum_{i=0}^{s} \binom{n+\ell}{s-i} \binom{i+m'_D+r+\ell-1}{i}. \]

This allows us to give an upper and lower bound for $|h(D)|$. We begin with the upper bound.

Proposition 7.3. Assume that $|S| \geq 2$. We then have
\[ |h(D)| + 1 \leq \binom{|S| - (m - m'_D) - 1 + g - \deg D}{g - \deg D} \leq \binom{|S| - 1 + g - \deg D}{g - \deg D}. \]

Proof. Set $s := g - \deg D$. Clearly, one can embed $A^s_n, \ell, m'_D + r$ into $A^s_{n,0,0,n+m'_D+r+2\ell}$ by $((a_i), (b_i), (c_i)) \mapsto ((), (a_1, \ldots, a_n, b_1^+, \ldots, b_{\ell}^+, b_1^-, \ldots, b_{\ell}^-, c_1, \ldots, c_{w'_D+r}))$ with $b_i^+ := \max\{b_i, 0\}$ and $b_i^- := \max\{-b_i, 0\}$. Hence, we get
\[ |A^s_n, \ell, m'_D + r| \leq |A^s_{0,0,n+m'_D+r+2\ell}| = \sum_{i=1}^{s} \binom{i+n+m'_D+r+2\ell-1}{n+m'_D+r+2\ell-1}. \quad (\ast) \]

In case $n + m'_D + r + 2\ell = 0$, we get $|A^s_n, \ell, m'_D + r| = 0$. Hence, assume that $t := n + m'_D + r + 2\ell - 1 \geq 0$, and $t \leq |S| - 2$; in that case, we get
\[ |A^s_n, \ell, m'_D + r| \leq \sum_{i=0}^{s} \binom{i+\ell}{i} - 1 = \binom{s+t+1}{s} - 1 \leq \binom{s+|S|-1}{s} - 1 \leq \frac{(|S|-1+s)}{s!} - 1, \]
which results in the claim.

Proposition 7.4. Assume that $|S| \geq 2$. We then have
\[ |h(D)| + 1 \geq \sum_{i=0}^{g-\deg D} \binom{|S| - (m - m'_D) - 2}{i}. \]

Proof. Set $s := g - \deg D$. First, assume that $m'_D + r + \ell = 0$. Then $n + 1 = |S|$, and
\[ |h(D)| + 1 = \sum_{i=0}^{s} \binom{|S| - 1}{i} \geq \sum_{i=0}^{s} \binom{|S| - 2}{i}. \]

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Next, assume that \( m'_D + r + \ell > 0 \), which implies \( |S| \geq 2 \). We have that

\[
|h(D)| + 1 = \sum_{i=0}^{s} \sum_{j=0}^{i} \binom{n+\ell}{i-j} \left( j + m'_D + r + \ell - 1 \right)
\geq \sum_{i=0}^{s} \sum_{j=0}^{i} \binom{n+\ell}{i-j} \left( |S| - (m - m'_D) - (n + \ell) - 2 \right)
= \sum_{i=0}^{s} \left( |S| - (m - m'_D) - 2 \right)
\]

by Vandermonde’s Identity.

Finally, we want to analyze the situation in the case \( |S| \to \infty \) when \( m = m'_D \). In this case, the above bounds give

\[
\frac{|S| - (m - m'_D) - 1 + (g - \deg D))^{g - \deg D}}{(g - \deg D)!}
\leq \frac{|S| - (m - m'_D) - 2}{g - \deg D} - 1 \leq \sum_{i=0}^{g - \deg D} \left( \frac{|S| - 2}{i} \right) - 1
\leq |h(D)| \leq \frac{|S| - (m - m'_D) - 1 + (g - \deg D))^{g - \deg D}}{(g - \deg D)!}.
\]

Hence, we obtain:

**Corollary 7.5.** For \( |S| \to \infty \),

\[
|h(D)| \sim \frac{|S| - (m - m'_D))^{g - \deg D}}{(g - \deg D)!}.
\]

*Note that in case \((S \cup \sigma(S)) \cap \text{supp} D = \emptyset\), \( m - m'_D = 0 \).*

Note that the set of places of \( K \) of degree one is finite; hence, \( |S| \to \infty \) does not make sense if \( K \) is fixed. With \( |S| \to \infty \) in the corollary, we mean that for any sequence of hyperelliptic function fields \( K_i \), all of genus \( g \), with sets of places \( S^{(i)} \) of degree one of \( K_i \) with \( |S^{(i)}| \to \infty \), and any \( D_i \in \text{Red}_{S^{(i)}}(K_i) \) with \( \deg D_i \) independent of \( i \), we have

\[
\lim_{i \to \infty} \frac{h(D_i) \cdot (g - \deg D_i)!}{(|S^{(i)}| - (m_{S^{(i)}} - m'_{S^{(i)}, D}))^{g - \deg D_i}} = 1.
\]

Finally, for the special case \( D = 0 \), the hole gets as big as it can:
Proposition 7.6. Assume that $|S| \geq 2$ and $S \setminus \{p\}$ contains a place lying above an unramified place of $R$. For $D \in \text{Red}_S(K)$, we have that $|h(D)|$ is maximal if, and only if, $D = 0$; in that case, 

\[
|h(0)| = \sum_{s=1}^{g} \sum_{i=0}^{s} \binom{n + \ell}{i + m + r + \ell - 1}.
\]

In particular, for $|S| \to \infty$, $|h(0)| \sim \frac{|S|^g}{g^\ell}$. 

Proof. Clearly, $D = 0$ is the only reduced divisor for which $g - \deg D = g$. For any divisor $D \in \text{Red}_S(K)$, the map $h(D) \to h(0)$, $D' \mapsto D' - D$ is injective. As there exists a place $p' \in S \setminus \{p\}$ which does not lie over a ramified place of $R$, $gp' \in h(0)$ lies not in the image of the above map $h(D) \to h(0)$ if $D \neq 0$. 

We have seen that the size of $h(D)$ only depends on $\deg D$, $S$, and $m_D$, and were able to give a precise formula for $|h(D)|$. Moreover, we were able to give lower and upper bounds for $|h(D)|$ which shows the behavior for $|S| \to \infty$.

8 Conclusion

We have shown that at least in the case of hyperelliptic function fields, the number of hole elements in the infrastructure of a global function fields behaves as expected, namely the number of hole elements is $nq^{g-1}$, where $n + 1$ is the number of infinite places, up to an error term of $O(q^{g-3/2})$. Moreover, we obtained an explicit formula for the number of hole elements involving only certain information on $S$ and the $L$-polynomial of $K$.

A natural question is whether this holds as well for all global function fields, and if not, how one has to adjust the bounds. So far, the author is not aware of any answer to this question. Based on the results in this paper and on the experiments in [Lan09, Fon09a], we conjecture:

Conjecture 8.1. Let $K$ be a function field of genus $g$ with exact constant field $\mathbb{F}_q$, and assume that $S$ is a finite set of places of $K$ containing at least one place of degree one. Then, for $q \to \infty$, 

\[
||\text{Red}_p(K) \setminus \text{Red}_S(K)|| - (|S_1| - 1) = 2g(|S_1| - 1)q^{g-3/2} + O(q^{g-2})
\]

and 

\[
\frac{||\text{Red}_p(K) \setminus \text{Red}_S(K)|| - |S_1| - 1}{q} = O(q^{-3/2}),
\]

where the $O$-constants only depend on $|S|$ and $g$.

Moreover, we have shown that the size of a hole next to a reduced divisor $D \in \text{Red}_S(K)$ depends highly on $g - \deg D \in \{0, \ldots, g\}$: in case $g = \deg D$, the size is zero. Otherwise, if $S = S_1$, there usually exists at least one hole element next to $D$. Assuming that $\sigma(S)$ meets supp $D$ in $m_S$ places, the size of the hole next to $D$ behaves like $\frac{(S_{g-m_S})^{g-\deg D}}{(g-\deg D)}$ for $|S| \to \infty$.

We conjecture that this holds in a similar way for all function fields.
Conjecture 8.2. Let $K$ be a function field of genus $g$ with exact constant field $\mathbb{F}_q$, and assume that $S$ is a non-empty finite set of places of $K$, all of degree one. Then there exists another finite set $S'$ with $S \subseteq S'$ of size $|S'| = O(|S|)$ such that for all $D \in \text{Red}_{S'}(K)$, we asymptotically have

$$|h(D)| \sim \frac{|S|^{g - \deg D}}{(g - \deg D)!}$$

as $|S| \to \infty$ with $S' \cap \text{supp} D = \emptyset$. Moreover, $|h(D)|$ only depends on $\deg D$ (and of course $S$ and $K$) as long as $S' \cap \text{supp} D = \emptyset$.

In the case of hyperelliptic function fields with quadratic rational subfield $R$ and the unique non-trivial $R$-automorphism $\sigma$ of $K$, we can set $S' = S \cup \sigma(S)$ and obtain $|S'| \leq 2|S|$. We moreover assume that a result similar to Proposition 7.6 holds in general:

Conjecture 8.3. Let $K$ be a function field of genus $g$ with exact constant field $\mathbb{F}_q$, and assume that $S$ is a non-empty finite set of places of $K$, all of degree one, with at least one place $\neq p$ which lies outside a finite set only dependent of $K$. Then $|h(D)|$ is maximal for $D \in \text{Red}_S(K)$ if, and only if, $D = 0$.

In the hyperelliptic case, the finite set of places of $K$ which have to be avoided are the places lying above places of $R$ which ramify in $K$.

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