Deformed quantum statistics in two-dimensions

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It is known from the early work of May in 1964 that ideal Bose gas do not exhibit condensation phenomenon in two dimensions. On the other hand, it is also known that the thermostatics arising from $q$-deformed oscillator algebra has no connection with the spatial dimensions of the system. Our recent work concerns the study of important thermodynamic functions such as the entropy, occupation number, internal energy and specific heat in ordinary three spatial dimensions, where we established that such thermostatics is developed by consistently replacing the ordinary thermodynamic derivatives by the Jackson derivatives. The thermostatics of $q$-deformed bosons and fermions in two spatial dimensions is an unresolved question and that is the subject of this investigation. We study the principal thermodynamic functions of both bosons and fermions in the two dimensional $q$-deformed formalism and we find that, different from the standard case, the specific heat of $q$-boson and $q$-fermion ideal gas, at fixed temperature and number of particle, are no longer identical.

I. INTRODUCTION

The phenomenon of Bose-Einstein condensation of the standard Bose gas is well established. The condensation signifies a macroscopically large occupation number of the zero energy state corresponding to a temperature of 3.14 K. The experimentally observed critical temperature corresponds to 2.2 K for liquid He$^4$ when it becomes a superfluid. Indeed the phenomenon of Bose condensation involves the macroscopically large occupation number $N_0$ corresponding to zero energy while the average occupation number $N_p$ corresponding to non-zero energies remains finite. The thermostatics of an ideal, standard, Bose gas is well established and the standard behavior of all the thermodynamic functions below and above the critical temperature is well described [1]. In particular, the specific heat shows a discontinuous behavior as a function of temperature, the point of discontinuity generally known as the lambda point.

It is well-known from the work of May [2] that the ideal gas obeying the standard Bose-Einstein statistics does not condense in two dimensional space i.e., two space dimensions and one time dimension. It is indeed known that the specific heat for an ideal gas of Fermi particles is identically the same as that for an ideal Bose gas for all $T$ and $N$ in two dimensions, despite the great difference in the distribution functions of the two systems at low temperatures. The early work of May also includes an investigation of the extreme relativistic limit of such systems.

The question of what is implied by the absence of Bose condensation in two dimensions has been discussed in Ref.[3] and [4]. This last work raises the question of what is special about the dimensionality two. Let us note that the above properties are rigorously true for ideal uniform two dimensional Fermi and Bose gases in the thermodynamic limit [5]. It is relevant to note that in Ref.[3] anomalous thermodynamics of Coulomb interacting massless Dirac fermions in two-spatial dimensions has been outlined and in Ref.[7], has been studied the thermostatics of the interacting Bose gas in two and three dimensions.

On the other hand, because of complicated topology of the configuration space for indistinguishable particles in two dimensions, Feynman’s path-integral formulation allows exotic quantum statistics which interpolates between fermions and bosons [8, 9]. Analogously, quon algebra introduces $q$-deformed commutation relation with violations of Pauli exclusion principle and Bose statistics not related to the spatial dimension of the system [10].

In describing complex systems, quantum algebra and quantum groups have been the subject of intensive research in several physical fields such as cosmic strings and black holes [11], conformal quantum mechanics [12], nuclear and high energy physics [13, 14, 15], fractional quantum Hall effect and high-$T_c$ superconductors [16]. From the seminal work of Biedenharn [17] and Macfarlane [18], it was clear that the $q$-calculus, originally introduced by Heine [19] and by Jackson [20] in the study of the basic hypergeometric series [21], plays a central role in the representation of the quantum groups with a deep physical meaning [22, 23, 24]. Furthermore, it is remarkable to observe that the $q$-calculus is very well suited for to describe fractal and multifractal systems. As soon as the system exhibits a discrete-scale invariance, the natural tool is provided by Jackson $q$-derivative and $q$-integral, which constitute the natural generalization of the regular derivative and integral for discretely self-similar systems [25].

In this framework, the thermostatics of such deformed bosons and fermions and the properties of $q$-deformed quantum mechanics have been studied in Ref.[26, 27, 28]. It was found that there are consequences of the deformation in the thermostatics of $q$-bosons and $q$-fermions and the theory is based on the introduction of basic numbers and it was further shown that the thermostatics involving the various thermodynamic functions can be fully described if the ordinary thermodynamic
derivatives are replaced by the Jackson derivatives \[^{20}\] in a systematic manner. On general grounds, one might remark that while the ideal gases are described by the standard Bose-Einstein and Fermi-Dirac statistics, the statistics of real, complex gases can thus be described by the thermostatistics based on the q-deformed algebra.

In view of the above facts, it is worthwhile to ask whether the specific heat of spinless q-bosons and q-fermions, at fixed \(T\) and \(N\), are identical in two space dimensions? One may ask the further question whether the specific heat of Einstein condensation of a relativistic gas has been studied in Ref.\[^{29}\] within the standard thermostatistics based on the Bose-Einstein and F ermi-Dirac statistics, the undeformed statistics of real, complex gases can thus be described by the thermodynamics derivatives and q-fermions. Furthermore, in Ref.\[^{30}\] thermodynamics of ideal and statistically interacting quantum gas in \(D\) dimensions has been studied in the framework of fractional statistics \[^{31}\].

We begin with a review of the standard quantum thermostatistics of bosons and fermions, i.e., the undeformed gas in Section 2. The basis of the q-deformed quantum thermostatistics in the framework of q-calculus is presented in Section 3. The specific heat of q-bosons and q-fermions is investigated in Section 4. The last section contains a summary and conclusions.

II. UNDEFORMED QUANTUM STATISTICS, ANALYTICAL AND NUMERICAL APPROACH

Let us briefly review some basic properties of undeformed \((q = 1)\) non-interacting bosons and fermions in two dimensions. Such results will be very useful in the q-deformation extension described in the next sections.

The grand canonical partition function for an ideal Bose gas \((\Omega_+, \kappa = +1)\) or an ideal Fermi gas \((\Omega_-, \kappa = -1)\) at the temperature \(T\) is given by

\[
\Omega_\kappa = -\kappa T \ln \mathcal{Z}_\kappa ,
\]

where we have set the Boltzmann constant equal to unity and the logarithm of the grand partition function \(\mathcal{Z}_\kappa\) is given by

\[
\log \mathcal{Z}_\kappa = -\kappa \sum_i \ln(1 - \kappa z e^{-\beta \epsilon_i}) ,
\]

and \(\beta = 1/T\).

The average occupation number of bosons or fermions can be derived from the relation

\[
N_\kappa = z_\kappa \frac{\partial}{\partial z_\kappa} \ln \mathcal{Z}_\kappa = \sum_i \frac{1}{z_\kappa} \exp(u_i) - \kappa ,
\]

where \(z_\kappa = e^{\beta \mu_\kappa}\) is the fugacity, \(u_i = \beta \epsilon_i\) and \(\mu_\kappa\) is the chemical potential associated with the boson \((\kappa = +1)\) or fermion \((\kappa = -1)\). For Bose statistics, the fugacity must satisfy \(z_+ < 1\) \((\mu_+ < 0, \text{ negative chemical potential})\) in order to assure the non-negativity of the occupation numbers.

For a large (two dimensional) volume \(V_2\) and a large number of particles, the sum over all single particle energy states can be transformed to an integral over the energy, according to

\[
\sum_i f(u_i) \implies \frac{V_2}{\lambda^2} \int_0^\infty f(u) du ,
\]

where \(u = \beta \epsilon, \epsilon = \hbar^2 k^2/2m\) is the kinetic energy and \(\lambda = \hbar/(2\pi m T)^{1/2}\) is the thermal wavelength.

In the thermodynamic limit, when both \(N\) and \(V\) tend to infinity but the ratio \(N/V\) remains finite, the average number of particles in Eq.(3) can thus be written as

\[
N_\kappa = \frac{V_2}{\lambda^2} \int_0^\infty \frac{1}{z_\kappa} \exp(u) - \kappa du .
\]

The above integral can be evaluated analytically and we obtain

\[
N_\kappa = -\kappa \frac{V_2}{\lambda^2} \ln(1 - \kappa z_\kappa) .
\]

In the case of bosons \((\kappa = 1)\), the right hand side of the above equation has no upper bound and diverges logarithmically as \(z_+ \rightarrow 1\), there is no temperature below which the ground state can be said to be macroscopically occupied in comparison to the excited states. Therefore, as it is well-known, no Bose condensation occurs in two dimensional non-interacting Bose systems \[^{2}\]. Furthermore, as first established by May \[^{2}\], the internal energies of two systems of spinless bosons and fermions at the same fixed temperature \(T\) and number of particle \(N\) differ only by a quantity proportional to \(N\) and do not depend on \(T\), therefore, the two systems have the same specific heat \(C_v(T, N)\).

Since we shall be exploring the same properties for q-deformed bosons and fermions, let us briefly review the crucial points of the demonstration of the above property.

The internal energy can be derived from the grand partition function by means of the following thermodynamic derivative

\[
U_\kappa(T, z_\kappa) = -\frac{\partial}{\partial \beta} \log Z = \frac{V_2}{\lambda^2} \int_0^\infty \frac{u}{z_\kappa} \exp(u) - \kappa du .
\]

By introducing the Bose-Einstein and Fermi-Dirac functions

\[
h_\kappa(n; z_\kappa) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{u^{n-1}}{z_\kappa} \exp(u) - \kappa du = \sum_{i=1}^{\infty} (\kappa z_\kappa)^i / i^n ,
\]

the form for the internal energy can be cast into the more compact expression

\[
U_\kappa(T, z_\kappa) = \frac{V_2}{\lambda^2} \frac{h_2^\kappa(z_\kappa)}{z_\kappa} .
\]
Let us observe that the above internal energy is calculated at a fixed temperature and fugacity or, equivalently, at a fixed number of particles. In fact, by inverting Eq. (6), we have the fugacities $z_\kappa$ as a function of $N_\kappa$

$$z_\kappa = \kappa \left[ 1 - \exp \left( -\kappa \frac{\lambda^2}{V_2} N_\kappa \right) \right]. \quad (10)$$

In order to compare the internal energy at the same $T$ and $N$, the fugacities of bosons ($\kappa = +1$) and fermions ($\kappa = -1$) must be related by the following relations

$$z_+ = 1 - \sigma_N, \quad (11)$$

$$z_- = \frac{1}{\sigma_N} - 1, \quad (12)$$

where we have defined $\sigma_N = \exp(-N\lambda^2/V)$. The above equations can be equivalently set as

$$z_- = \frac{z_+}{1 - z_+}. \quad (13)$$

By using the property of the dilogarithmic functions

$$h_2^\kappa(z_-) - h_2^\kappa(z_+) = \frac{1}{2} (\ln \sigma_N)^2, \quad (14)$$

it follows that

$$U_-(T, N) - U_+(T, N) = \frac{1}{2} N \rho_N, \quad (15)$$

where we have set

$$\rho_N = \frac{N}{V_2} \frac{h^2}{2\pi \hbar}. \quad (16)$$

Therefore, the right hand side of Eq. (15) does not explicitly depend on $T$ and the specific heat $C_v = \frac{\partial U}{\partial T} \bigg|_{V,N}$, at the same temperature and number of particle, are identical for fermions and bosons [2].

As we will see in the next Section, in the $q$-deformed theory of bosons and fermions it is not possible to find an analytical expression analogous to Eq. (10). Eq. (10) is no longer correct, consequently, it is crucial to test, for the following developments, that the above properties (14) and (15) can be easily obtained numerically. For further developments, it is useful to introduce the variable $y$, defined as

$$y = \frac{N}{V_2} \lambda^2. \quad (18)$$

With the above definition, the corresponding Eq. (19), for bosons/fermions systems at the same $T$, $N$ and $V$, can be derived as

$$h_2^\kappa(z_\kappa) = y \implies z_\kappa = [h_2^\kappa(y)]^{-1}, \quad (19)$$

where the inverse function introduced above refers to the symbolic form. We can evaluate the internal energy as

$$U_\kappa(T, N) = \frac{T N}{y} h_2^\kappa[z_\kappa(y)]. \quad (20)$$

In order to show that the difference of internal energy of fermions-bosons, at the same $T$ and $N$, does not depend on $T$, it is sufficient to show that the following equation holds

$$\Delta h_2^\kappa(y) = h_2^\kappa[z_-(y)] - h_2^\kappa[z_+(y)] = \alpha_1 y^2, \quad (21)$$

where $z_-(y)$ and $z_+(y)$ are obtained from Eq. (19) and $\alpha_1$ is a dimensionless constant. In fact, if the last equivalence of Eq. (21) is verified, we have

$$\Delta U(N) = U^- - U^+ = \alpha_1 N \rho_N. \quad (22)$$

Performing the numerical evaluation of the function $\Delta h_2^\kappa(y)$, we can see that it has a parabolic behavior on the variable $y$ with the dimensionless coefficient $\alpha_1 = 1/2$ (as we know analytically from Eq. (14)). The statistical variable $\chi = \sum_i (\Delta h_2^\kappa(y_i) - \alpha_1 y_i^2) \approx 10^{-8}$, therefore, this numerical approach gives a very reliable test and it will be applied in the next Section on the framework of $q$-deformed bosons and fermions.

### III. $q$-DEFORMED QUANTUM STATISTICS

Let us briefly review the basic properties of $q$-oscillator algebra and the generalized thermodynamic properties of $q$-deformed bosons and fermions [26, 27].

The symmetric $q$-oscillator algebra is defined, in terms of the creation and annihilation operators $c, c^\dagger$ and the $q$-number operator $N$, by [28, 34, 35, 36]

$$[c, c^\dagger]_\kappa = [c^\dagger, c]_\kappa = 0, \quad cc^\dagger - \kappa q^\kappa c c^\dagger = q^{-N}, \quad (23)$$

$$[N, c^\dagger] = c^\dagger, \quad [N, c] = -c, \quad (24)$$

where the deformation parameter $q$ is real and $[x, y]_\kappa = xy - \kappa qx$ where, as before, $\kappa = 1$ for $q$-bosons with commutators and $\kappa = -1$ for $q$-fermions with anticommutators.

Furthermore, the operators obey the relations

$$c^\dagger c = [N], \quad cc^\dagger = [1 + \kappa N], \quad (25)$$

where the $q$-basic number is defined as

$$[x] = \frac{q^{2x} - q^{-x}}{q - q^{-1}}. \quad (26)$$

The transformation from Fock space to the configuration space (Bargmann holomorphic representation) may be accomplished by means the Jackson derivative (JD) [20]

$$D^{(q)}_x f(x) = \frac{f(qx) - f(q^{-1}x)}{x(q - q^{-1})}, \quad (27)$$
which reduces to the ordinary derivative when \( q \) goes to unity. Therefore, the JD occurs naturally in \( q \)-deformed structures and we will see that it plays a crucial role in the \( q \)-generalization of the thermodynamics relations.

Thermal average of an observable can be computed by following the usual prescription of quantum mechanics. Accordingly, the Hamiltonian of the non-interacting \( q \)-deformed oscillators (fermions or bosons) expected to have the form

\[
H = \sum_i (\epsilon_i - \mu) N_i .
\]  

(28)

Let us note that the Hamiltonian is deformed and depends on \( q \) since the number operator is deformed by means Eq.(26) and it is not linear in \( c^\dagger c \). Therefore, although the logarithm of the grand partition function has the same functional expression as in the undeformed case, Eq.(2), the standard thermodynamic relations in the usual form are ruled out (for instance, it is verified that \( N \neq z \cfrac{\partial}{\partial z} \log Z \) \[26, 27\].

In Ref.\[26\], we have shown that the entire structure of thermodynamics is preserved if ordinary derivatives are replaced by the use of an appropriate Jackson derivative

\[
\cfrac{\partial}{\partial z} \rightarrow D_z^{(q)} .
\]  

(29)

Consequently, the number of particles in the \( q \)-deformed theory can be derived from the relation

\[
N = z D_z^{(q)} \log Z = \sum_i n_i ,
\]  

(30)

where \( n_i \) is the mean occupation number expressed as

\[
n_i = \cfrac{1}{q-q^{-1}} \log \left( \cfrac{z^{-1} e^{\beta \epsilon_i} - \kappa q^{-\kappa}}{z^{-1} e^{\beta \epsilon_i} - \kappa q^{\kappa}} \right) .
\]  

(31)

In this context, it is relevant to observe that the statistical origin of such \( q \)-deformation lies in the modification, relative to the standard case, of number of states \( W \) of the system corresponding to the set of occupational number \( n_i \) \[26\]. In literature, other statistical generalization are present, such as the so-called nonextensive thermostatistics or superstatistics with a completely different origin \[34, 35\].

The usual Leibniz chain rule is ruled out for the JD and therefore derivatives encountered in thermodynamics must be modified as follows. First we observe that the JD applies only with respect to the variable in the exponential form such as \( z = e^{\beta \mu} \) or \( y_i = e^{-\beta \epsilon_i} \). Therefore for the \( q \)-deformed case, any thermodynamic derivative of functions which depend on \( z \) or \( y_i \) must be transformed to derivatives in one of these variables by using the ordinary chain rule and then evaluating the JD with respect to the exponential variable. For instance, in the case of the internal energy in the \( q \)-deformed case, we can write this prescription explicitly as

\[
U = \left. -\cfrac{\partial}{\partial \beta} \log Z \right|_z = \kappa \sum_i \cfrac{\partial y_i}{\partial \beta} D_z^{(q)} \log(1 - \kappa z y_i) .
\]  

(32)

In this case we obtain the correct form of the internal energy

\[
U = \sum_i \epsilon_i n_i ,
\]  

where \( n_i \) is the mean occupation number expressed in Eq.(31).

In the thermodynamic limit, for a large (two dimensional) volume \( V_2 \) and a large number of particles, the sum over states can be replaced by the integral, similar to the correspondence in Eq.(4). However, in a \( q \)-deformed theory the standard integral must be consistently generalized to the \( q \)-integral, inverse operator of the JD, defined, for \( 0 < q < 1 \) in the interval \([0,1]\), as \[23, 37\]

\[
\int_0^a f(x) \, dq_x = a (q^{-1} - q) \sum_{n=0}^\infty q^{2n+1} f(q^{2n+1} a) ,
\]  

(34)

while in the interval \([0,\infty)\)

\[
\int_0^\infty f(x) \, dq_x = (q^{-1} - q) \sum_{n=-\infty}^\infty q^{2n+1} f(q^{2n+1}) .
\]  

(35)

Following the above prescriptions (see, for example, Ref.\[37\] for a detailed description of the \( q \)-integral properties), we gain the \( q \)-analogue of Eq.(4) as follows \[10\]

\[
\sum_i f(u_i) = I_q = \cfrac{V_2}{\pi^2} \cfrac{2}{q+q^{-1}} \cfrac{1}{z^2} \int_0^\infty f(u(k)) \, dq_x \, dq_y ,
\]  

(36)

where \( u(k) = \beta \hbar^2 k^2 / 2m \) and holds the constraint: \( k^2 = k_x^2 + k_y^2 \). By taking into account the rules related to changing the variable of \( q \)-integration \[37\], we have verified that for \( 0.6 < q < 1.4 \) the above integration can be well approximately expressed as

\[
I_q \approx \cfrac{V_2}{\lambda^2} \left( \cfrac{1}{q+q^{-1}} \cfrac{1}{z^2} \int_0^\infty f(u) \, dQ u \right) ,
\]  

(37)

where \( Q = q^2 \) (a change of variable \( u = \beta \hbar^2 k^2 / 2m \) also involves a corresponding change of base). Therefore, in the thermodynamic limit, Eq.(31) and Eq.(32), respectively, becomes

\[
N_n(T, \kappa) = \cfrac{V_2}{\lambda^2} h_1^\kappa (z_\kappa, q) ,
\]  

(38)

\[
U_n(T, \kappa) = \cfrac{V_2}{\beta \lambda^2} h_2^\kappa (z_\kappa, q) ,
\]  

(39)

where we have defined the \( q \)-deformed \( h_n^\kappa (z_\kappa, q) \) as

\[
h_n^\kappa (z_\kappa, q) = \cfrac{1}{\Gamma(n)} \int_0^\infty \cfrac{u^{n-1}}{q-q^{-1}} \log \left( \cfrac{z^{-1} e^{\beta u} - \kappa q^{-\kappa}}{z^{-1} e^{\beta u} - \kappa q^{\kappa}} \right) \, dQ u .
\]  

(40)

It must be stressed that the above equation is quite different from the definition of the generalized function introduced in Eq.(21) in our earlier work \[21\]. This is an important notion in our present work. It should also
be noted that, to the best of our knowledge, this is the first time that $q$-integrals are numerically employed in thermostatistics calculations. In the limit $q \to 1$, the deformed $h^*_n(z_n, q)$ functions reduce to the standard $h^*_n(z_n)$ for bosons and fermions, defined in Eq. (5).

As in the undeformed boson case, we need to set the range of the $q$-boson fugacity $z_B = z_q$ which will correspond to non-negative occupation number. In the case of $q$-bosons we see that the condition is $z_B < 1/q$ for $q > 1$ and $z_B < 1$ for $q < 1$. Also in this case the number of particle, expressed by Eq. (38), diverges logarithmically as $z_B \to 1/q$ (if $q > 1$) and $z_B \to q$ (if $q < 1$). Therefore, no Bose condensation occurs in two dimensional $q$-boson gas.

Moreover, it should be pointed out that we also have to require the existence of the JD of the mean occupation number which is encountered in the calculation of thermodynamic quantities such as the specific heat and this changes the upper bound of the fugacity $z_B$. In the following, we thus will require the condition $z_B < z_q$, where we have defined

$$z_q = \begin{cases} q^{-2} & \text{if } q > 1; \\ q^2 & \text{if } q < 1. \end{cases} \quad (41)$$

At this point, we are able to see if the difference of the internal energy of $N$ $q$-fermions and $q$-bosons at fixed $T$ does not depend on $T$ and the specific heats are equal, as in undeformed case. These properties must be verified numerically because of we are unable to get the analytic expression of Eq. (38), therefore, we follow the numerical procedure tested for $q = 1$ in the second part of Section II.

In the $q$-deformed theory, the previous definition of the variable $y$ of Eq. (38) must be changed with

$$y_q = \frac{q + q^{-1}}{2} \frac{N}{V_2} \lambda^2, \quad (42)$$

consequently, we can obtain the fugacities $z_\kappa = z_\kappa(y_q, q)$ as

$$z_\kappa = [h^*_1(y_q, q)]^{-1}, \quad (43)$$

and the internal energy as

$$U_\kappa(T, N) = \frac{T N}{y_q} h^*_2[z_\kappa(y_q, q)]. \quad (44)$$

As before, the difference between the internal energy of fermions and bosons at the same $N$ and $T$, does not depend on $T$ if the following relation holds

$$\Delta h_2(y_q, q) = h_2[\zeta - (y_q, q)] - h_2[\zeta + (y_q, q)] = \alpha_q y_q^2, \quad (45)$$

where $\zeta - (y_q, q)$ and $\zeta + (y_q, q)$ are obtained from Eq. (43) and $\alpha_q$ is a dimensionless constant. It may be noted that $\alpha_q \to 1/2$ in the limit $q \to 1$.

In Fig. 1, we plot the coefficient $\alpha_q$ for different values of $q$, while in Fig. 2 it is possible to check the reliability of the quadratic approximation of Eq. (45), $\chi^2 = \sum_i (\Delta h_2(y_q, q) - \alpha_q y_q^2)^2$, related to the difference of the internal energy of fermions and bosons system at fixed $T$ and $N$. As we can see from Fig. 2, the quadratic behavior of $h_2(y_q, q)$ holds only for small $q$-deformation effect ($q \approx 1$) or at small value of the variable $y$ (or the fugacity $z$), therefore, the difference of the internal energy is not rigorously independent of the temperature and the specific heats of bosons and fermions, at fixed $T$ and $N$ are not exactly equal. In the next Section, we will give an explicit evaluation of the specific heat for different values of the deformation parameter $q$.

![FIG. 1: Plot of the dimensionless coefficient $\alpha_q$ of Eq. (45) as a function of $q$.](image1)

![FIG. 2: Behavior of the $\chi^2$ related to quadratic fit in Eq. (45) as a function of $q$.](image2)

### IV. Specific Heat of Boson and Fermion Systems

We are now able to calculate the specific heat of the $q$-boson and $q$-fermion gas, starting from the thermodynamic definition of Eq. (17).

Carrying out the JD prescription, described earlier in Sec. III, Eq. (17) in the $q$-deformed theory can be written as

$$C_v = -\beta^2 \sum_i \frac{\partial^2 \epsilon_i}{\partial \beta} \frac{1}{q - q^{-1}} D^{(q)} \log \left( 1 - \kappa q^{-\kappa} \alpha_i \right),$$

(46)
where \( \alpha_i = z e^{-\beta \epsilon_i} \) and

\[
\frac{\partial \alpha_i}{\partial \beta} = \left( \frac{1}{z} \frac{\partial z}{\partial \beta} - \epsilon_i \right) \alpha_i. \tag{47}
\]

For this purpose we first need, therefore, the derivative of the fugacity with respect to \( T \) (or \( \beta \)), keeping \( V \) and \( N \) constant. Accordingly, we observe that the following identity holds (since the number of particles is kept constant)

\[
\frac{\partial}{\partial \beta} \sum_i \log \left( \frac{1 - \kappa q^{-\kappa} \alpha_i}{1 - \kappa q^{\kappa} \alpha_i} \right) = 0. \tag{48}
\]

In accordance with the JD recipe about the thermodynamical relations, the above equation can be written as

\[
\sum_i \frac{\partial \alpha_i}{\partial \beta} D^{(q)}_{\alpha_i} \log \left( \frac{1 - \kappa q^{-\kappa} \alpha_i}{1 - \kappa q^{\kappa} \alpha_i} \right) = 0. \tag{49}
\]

Evaluating in the thermodynamical limit (\( V \to \infty \)) and by using the definition in Eq. (40), we obtain

\[
\left. \frac{1}{z} \frac{\partial z}{\partial \beta} \right|_{V,N} = \frac{1}{\beta} \frac{D^{(q)}_{\alpha_i} h_3^{(z, q)}}{D^{(q)}_{\alpha_i} h_1^{(z, q)}}. \tag{50}
\]

By using the above relation in Eq. (46), we obtain the specific heat for a system of bosons and fermions at fixed \( T \) and \( N \)

\[
\frac{C_v \lambda^2}{V_2} \equiv \frac{C_v}{N} y = 2 z \kappa D^{(q)}_{\alpha_i} h_3^{(z, q)} - z_n \frac{(D^{(q)}_{\alpha_i} h_3^{(z, q)})^2}{D^{(q)}_{\alpha_i} h_1^{(z, q)}}. \tag{51}
\]

In Figs. 3 and 4, we plot the behavior of the specific heat \( C_v \lambda^2/V_2 \) for a boson and fermion system at fixed temperature and number of particles (let us remember that we have not taken into account the degeneracy factor due to the spin quantum number) for two different values of \( q \). As we can see from the figures, the specific heats of boson and fermion are no longer equal for \( q \neq 1 \) and this difference becomes more relevant by increasing the value of the deformation parameter \( q \).

It must be emphasized here that if we want to have a correct comparison, we must plot the specific heat as a function of the variable \( y \) and not as a function of the variable \( z \kappa(y, q) \). Same \( T \) and \( N \) does not imply the same fugacity (which is a different function of \( N \) and \( T \) for bosons and fermions) but the same variable \( y \). To better clarify this aspect we report in Fig. 5 and 6 the specific heat \( C_v \lambda^2/V_2 \) as a function of the fugacity for bosons and fermions, respectively (remember that the range of meaningful fugacities \( z_B \), for boson gas, is limited by the condition \( q \leq 1 \)). Let us observe that the modification of the specific heat increasing with the value of the deformation parameter \( q \) becomes very remarkable in the fermion case.

V. CONCLUSION

Understanding properties of quantum matter confined to two spatial dimensions has been at the forefront of theoretical and experimental physics. In the last years there was a growing importance high energy physics, low-dimensional systems computers, superfluid and superconducting films, quantum Hall and related two-dimensional

![FIG. 3: The specific heat \( C_v \lambda^2/V_2 \) for bosons (B) and fermions (F), at fixed \( T \) and \( N \), as a function of the variable \( y \) for \( q = 0.7 \).](image)

![FIG. 4: The specific heat as in Fig. 3 for \( q = 0.9 \).](image)

![FIG. 5: The specific heat \( C_v \lambda^2/V_2 \) for bosons as a function of fugacity \( z_B \) for different values of \( q \).](image)
FIG. 6: The specific heat $C_0 \lambda^2/V_2$ for a fermion gas as a function of fugacity $z_F$.

electron gases and low-dimensional trapped Bose gases. On the other hand $q$-deformed quantum and statistical theory, inspired by the quantum groups formulation, arise as the underlying mathematical structure in several physical complex systems.

In this paper we have investigated the structure of the $q$-deformed quantum statistics in two-dimensions by working consistently in the framework of the $q$-calculus with the use of the Jackson derivatives and the $q$-integration. In this context, we have shown that, as in the undeformed case, ideal $q$-Bose gas does not exhibit condensation, the specific heat is a continuous function on the relevant thermodynamical variable. However, we have shown that the difference of the internal energy of fermions and bosons, at fixed $N$ and $T$, depends on $T$. This, as a matter of fact, has no counterpart in the standard case and implies that the specific heats, at fixed $N$ and $T$, of bosons and fermions are no longer equal.

This different behavior from the undeformed quantum theory can be dealt with in the statistical behavior of a complex systems, intrinsically contained in $q$-deformation, whose underlying dynamics is spanned in many-body interactions and/or long-time memory effects. This aspect has just outlined in several papers. For example in Ref. [14] it has been shown that $q$-deformation plays a significant role in understanding higher-order effects in many-body nuclear interactions. Moreover, the strong effects on the deformation, that we have found especially in the $q$-fermion specific heat, could be connected to an intrinsically presence of complex many-body effective interactions on $q$-deformation theory. In this context, it appears relevant to observe that nonanalytic temperature behavior of the specific heat of Fermi liquid can be explained within two dimensional interactions beyond the weak-coupling limit [41].

Let us now address a different perspective to the formulation in two dimensions, by adding the following remarks. The subject of anyons has been well investigated in the recent past [42]. Planar physical systems, in two space and one time dimensions, display many peculiar and interesting quantum properties owing to the unusual structure of rotation, Lorentz and Poincaré groups in two spatial dimensions and thus lead to a theory of intermediate statistics, interpolating between Bose statistics at one end and Fermi statistics at the other. Such anyons are described by a theory based on the permutation group which is the braid group. Since the real world is described in 3+1 dimensions, anyons may not be real particles. On one hand, the theory based only on a deformation of the oscillator algebra which is a generalization of the ordinary boson or fermion oscillator algebra may not have the features of a full-fledged theory of anyons since it does not have the advantage of the braid group characteristic of two dimensions. On the other hand, a theory formulated on the basis of detailed balancing can describe intermediate statistics purely on the basis of thermostatistics and this formulation leads to a description in terms of the basic numbers characteristic of the $q$-deformed oscillator algebra. Conventional wisdom might indicate that such generalization can have no relation to the algebra of deformed harmonic oscillators since oscillators exist in any dimensions. Accordingly, the connection between $q$-deformation and two dimensions is an open question which has not been dealt with satisfactorily in the literature.

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