Separation of variables and integral relations for special functions

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To the memory of Felix M. Arscott

Abstract

We show that the method of separation of variables gives a natural generalisation of integral relations for classical special functions of one variable. The approach is illustrated by giving a new proof of the “quadratic” integral relations for the continuous $q$-ultraspherical polynomials. The separating integral operator $M$ expressed in terms of the Askey-Wilson operator is studied in detail: apart from writing down the characteristic (“separation”) equations it satisfies, we find its spectrum, eigenfunctions, inversion, invariants (invariant $q$-difference operators), and give its interpretation as a fractional $q$-integration operator. We also give expansions of the $A_1$ Macdonald polynomials into the eigenfunctions of the separating operator $M$ and vice versa.
1. Introduction

One of the most powerful methods for solving the spectral problem of a quantum integrable system is that of Separation of Variables (SoV). Given a quantum integrable system with \( n \) degrees of freedom defined by a complete set of \( n \) commuting Hermitian operators (integrals of motion) \( H_j \) acting in some Hilbert space \( \mathcal{H} \),

\[
[H_j, H_k] = 0, \quad j, k = 1, 2, \ldots, n, \quad (1.1)
\]

one tries to find a linear operator \( M \) sending any common eigenvector \( P_\lambda \) of the \( H_j \)'s,

\[
H_j P_\lambda = h_{\lambda,j} P_\lambda, \quad (1.2)
\]

labelled by the quantum numbers \( \lambda = \{\lambda_1, \ldots, \lambda_n\} \), into the product \( F_\lambda \)

\[
M : P_\lambda \rightarrow F_\lambda = \prod_{j=1}^{n} f_{\lambda,j}(y_j) \quad (1.3)
\]

of functions \( f_{\lambda,j}(y_j) \) of one variable each referred to as partial, or separated, or factorized functions. Notice that the operator \( M \) does not depend on the spectrum \( \{h_{\lambda,j}\}_{j=1}^{n} \) (or the quantum numbers \( \lambda \)) and is thought of as an intertwiner between two representations of the eigenfunctions of the integrals \( H_j \), namely: the initial representation and the separated (or factorized) \( y \)-representation. The essence of the method is that the original multivariable spectral problem (1.2) is transformed by such a transformation into a set of (simpler) multiparameter spectral problems, each being in one variable, given by the separation equations of the form

\[
\mathcal{D}_j \left( y_j, \frac{\partial}{\partial y_j}; h_{\lambda,1}, \ldots, h_{\lambda,n} \right) f_{\lambda,j}(y_j) = 0, \quad j = 1, \ldots, n, \quad (1.4)
\]

where \( \mathcal{D}_j \) are usually some differential or finite-difference operators in variable \( y_j \) depending on the spectral parameters \( h_{\lambda,k} \). In the context of the classical Hamiltonian mechanics the above construction corresponds precisely to the standard definition of SoV in the Hamilton-Jacobi equation.

For a recent review of the SoV method see [39]. See also [28, 29] as for applications of SoV to the 3-particle Calogero-Sutherland model and its relativistic (or \( q \))-analog, respectively. We would like to stress here that our definition of SoV is not restricted by purely coordinate changes of variables, as it is usually (historically) supposed (see, for instance, the book [24] and references therein). Instead, we allow the separating transform to be a generic canonical transformation in classical mechanics or a linear (integral) operator in quantum case.

The general multiparameter spectral theory for systems of equations like (1.4) including, in particular, questions related to completeness of eigenfunctions, the Parseval equality and the like, has been studied extensively [1, 11, 2, 11, 4, 42, 45]. The book [42] (see also survey article [11]) gives a comprehensive treatment of multiparameter spectral theory in finite dimensional cases. Various results concerning unbounded (e.g. differential) operators are brought together in the monograph [11]
where the author provides a study of the infinite dimensional case via the theory of several commuting operators in Hilbert space \([H_j(y)]\).

In the present paper we will not concentrate on the multiparameter spectral problem because for the particular system we study we do know explicitly the spectrum and the solution of the separation equation. Although, when doing SoV for more complicated integrable systems, like Toda lattice or elliptic Calogero-Moser problem, we expect the above-mentioned theory will play an important role.

The main problem of SoV (apart from the study of the separation equation and its solution) is to try to describe the separating operator \(M\) (and its inversion) in the most explicit terms. The eigenvectors \(P_\lambda\) in a particular representation become special functions or orthogonal polynomials in many variables (for instance, Jack or Macdonald polynomials) while \(f_{\lambda,j}\) in (1.3) become special functions in one variable. In this interpretation the corresponding kernel (nucleus) \(M\) of the integral operator \(M\) is some special function in \(2n\) variables and the method of SoV provides us with two fundamental integral relations between special functions in one and many variables:

\[
\prod_{i=1}^{n} f_{\lambda,i}(y_i) = \int \cdots \int dx_1 \cdots dx_n M(y|x) P_\lambda(x_1, \ldots, x_n) \quad \text{(1.5a)}
\]

\[
P_\lambda(x_1, \ldots, x_n) = \int \cdots \int dy_1 \cdots dy_n \tilde{M}(x|y) \prod_{i=1}^{n} f_{\lambda,i}(y_i). \quad \text{(1.5b)}
\]

Here \(\tilde{M}(x|y)\) represents the kernel of the inverse \(M^{-1}\) and, we repeat, both kernels do not depend on \(\lambda\).

The first relation in (1.3) looks like a “product formula” for the one-variable functions \(f_{\lambda,j}(y_j)\) while the second one gives an integral representation for the special function \(P_\lambda(x_1, \ldots, x_n)\) of many variables in terms of the special functions, \(f_{\lambda,j}(y_j)\), of one variable. These two types of integral relations are not entirely new in the theory of special functions. In particular, when \(n = 2\) both integral relations have appeared before in one-variable theory. Below we give a brief review of the literature on integral relations/equations (1.3) for the special functions and orthogonal polynomials in one variable. Let us start with the relations of 2-nd type (1.5b).

In 1846 Liouville [30] demonstrated that Lamé polynomials satisfy certain non-linear integral equations of the form

\[
f(x) = \int \int dy_1 dy_2 \tilde{M}(x|y) f(y_1) f(y_2). \quad \text{(1.6)}
\]

However, he failed to specify for which of the eight types of Lamé polynomials these equations were valid. Later on Sleeman [40] showed, using the results obtained by Arscott in [3] (see also the work [1] and the book [2]), that integral equations and relations may be constructed which are satisfied by all eight types of Lamé polynomials and Lamé functions of the second type. The kernels (nuclei) appearing in all these equations are the “potential” Green’s functions. These results were further generalized in [40] to construction of integral equations and relations satisfied by ellipsoidal wave functions of the first and third kinds, the kernels now being the “free
space” Green’s functions. The partial differential equations for the kernels \( \tilde{\mathcal{M}}(x|y) \) as well as the corresponding equality of the form (1.6) for these cases were first written down in [7] (see also the book [5]) under the names “Second integral theorem” and “Second integral equation”. Actually, the very existence of the relation (1.6) is closely connected to the corresponding 2-parameter spectral problem for the Lamé polynomials/functions and for the ellipsoidal wave functions as was first realized by Arscott [5]. This idea was generalized further to the case of \( n \)-parameter spectral problem where one gets the following equation

\[
f(x) = \int \cdots \int dy_1 \cdots dy_n \tilde{\mathcal{M}}(x|y) \prod_{j=1}^{n} f(y_j)
\]

(1.7)

as well as equation of the form

\[
\prod_{i=1}^{n} f(x_i) = \int \cdots \int dy_1 \cdots dy_n \tilde{\mathcal{M}}(x|y) \prod_{j=1}^{n} f(y_j).
\]

(1.8)

An abstract formulation (in Hilbert space) of both above-mentioned equations (and also the theory of solvability of some operator equations behind it) and the characteristic (determinantal) equations for the corresponding kernels \( \tilde{\mathcal{M}}(x|y) \) can be found in the monograph [41] (Chapter 6 titled “An abstract relation”), see also the works [23, 13]. For the case when \( f(y) \) is the ellipsoidal wave function the relations (1.7) and (1.8) (with \( n = 2 \)) were derived for the first time by Malurkar [32] and Möglich [34], respectively.

Let us proceed now to the 1-st type integral relations (1.3a). The particular form of these relations

\[
f(y_1) f(y_2) = \int dx \, \mathcal{M}(y|x) f(x)
\]

(1.9)

is known in the literature as product formulas for the orthogonal polynomials and special functions in one variable. The first example was provided by Gegenbauer [21] for his polynomials. This approach was developed further for several other classical special functions and orthogonal polynomials [14, 38, 14, 27, 1, 32, 33, 20, 14] and culminated in [16] where the product formulas for angular and radial spheroidal wave functions were presented. These formulas generalize the results for the Gegenbauer (ultraspherical) polynomials and functions as well as for the Mathieu functions. The modern approach to the construction of some of the corresponding kernels \( \mathcal{M}(y|x) \) employs a PDE technique based on Riemann’s integration method to solve a Cauchy problem for an associated hyperbolic differential equation (cf. [13, 33, 16]).

Product formulas are very useful identities associated with a discrete or continuous function system. Generally, such a formula represents the product of any two values of an orthogonal polynomial in terms of a Stieltjes integral which depends linearly on the polynomial itself. It is a very important tool in the harmonic analysis of orthogonal expansions where it usually defines a convolution product on a function space (this space usually becomes a Banach algebra under this operation and the convolution product plays the same role as ordinary convolution in Fourier
analysis). Notice that the right hand side of a product formula usually defines a generalized translation operator and that this in turn can be used to establish an appropriate convolution structure on the space and often a hypergroup structure on the corresponding interval (cf. [15] for details). The derivation of new product formulas yields new convolution structures and hypergroups; moreover it may be used to obtain more information about the special functions $f$ themselves, since they are usually not given in closed form as soon as they lie in the “land beyond Bessel” [8].

In 1914 Whittaker [46] conjectured that the Heun differential equation [18, 22] is the simplest equation of Fuchsian type whose solution cannot be represented by a contour integral (of a simpler, elementary function); instead the nearest approach to such a solution is to find an integral equation satisfied by a solution of the differential equation. This is indeed a general situation for the higher special functions (cf. [8]) where one could not hope to get as many explicit formulas and representations as there are for special functions of the hypergeometric type. For instance, the technique of series solutions leads to a higher order recursion relation involving at least three successive coefficients (for functions of the Heun type). This observation applies not only to power series but to series of other forms. Although there is no proof that a two-term recursion cannot be obtained. Usually, no integral expressions for higher special functions are available; instead they satisfy certain integral equations like the ones mentioned above.

In the present paper we aim to demonstrate that the integral equations similar to those just discussed appear very naturally in the unified approach of SoV. One (among the many) way to introduce a multivariable special function generalizing a given one-variable special function is to associate it to a quantum integrable family, i.e. to obtain it as a common eigenfunction of a commutative ring of $n$ quantum integrals of motion. In such a manner many known functions can be described and characterized. Assuming the hypothesis that any quantum integrable system can be separated we arrive at an interesting problem of studying various integral operators, in an attempt to factorize a particular function or polynomial. That was our leading motivation for studying 3-variable Jack polynomials in [28] and 3-variable Macdonald polynomials in [29] which are symmetric orthogonal polynomials associated to the root system $A_2$ characterized by being common eigenfunctions of the corresponding quantum Calogero-Sutherland [14, 43] and Ruijsenaars [37] models of the trigonometric type. It appears that there are explicit separating operators $M$ for both problems which factorize polynomials and which can be explicitly inverted giving new integral representations for the related polynomials.

Here we apply the SoV technique to even simpler 2-variable case of the $A_1$ Macdonald polynomials. In particular, we give a new proof of the “quadratic” integral equations for the continuous $q$-ultraspherical polynomials. The separating integral operator $M$ expressed in terms of the Askey-Wilson operator is studied in detail: apart from writing down the characteristic (“separation”) equations it satisfies, we find its spectrum, eigenfunctions, inversion, invariants (invariant $q$-difference operators), and give its interpretation as a fractional $q$-integration operator. We also give expansions of the $A_1$ Macdonald polynomials into the eigenfunctions of the separating operator $M$ and vice versa. The structure of the operators $M$ for $A_1$ and $A_2$
cases turns out to be quite similar which indicates an intimate relation between the two SoV problems.

The structure of the paper is as follows. In Section 2 we collect basic information about continuous $q$-ultraspherical polynomials including the product formula in integral form. In Section 3 we give the main properties of the Askey-Wilson type integral operator $M_{\alpha\beta}$ which provides us with an important tool for construction and description of the separating operator $M$ in the key Section, 4. In the next Section, 5, we derive expansions for the $A_1$ Macdonald polynomials in terms of the eigenfunctions of the separating operator $M$, as well as the dual expansions. As a bi-product we obtain an interesting interpretation of $M$ as a quantum integrable map (discrete-time system) and find its invariants (integrals of motion): the $q$-difference operators $N_1$ and $N_2$. We provide some concluding remarks in the final Section, 6, and give an Appendix where the classical SoV for the trigonometric $A_1$ type Ruijsenaars model is performed and a crucial (for the present paper) inter-relation between the $A_2$ and $A_1$ problems is explained in detail.

2. Continuous $q$-ultraspherical polynomials

The main references for the formulas and notations in this Section are [21, 26].

Continuous $q$-ultraspherical polynomials $C_n(\xi;\beta|q)$ is an important class of orthogonal polynomials of basic hypergeometric type within the Askey scheme. They can be obtained from the general Askey-Wilson polynomials [10] through a specification of their four parameters, so that $C_n(\xi;\beta|q)$ depend only on one parameter $\beta$, apart from the degree $n$ and basic parameter $q$. They are defined through the (terminated) $2\phi_1$ basic hypergeometric series

$$C_n(\xi;\beta|q) = \frac{(\beta;q)_n}{(q;q)_n} e^{i\theta} \ _2\phi_1^q \left[ q^{-n}, \beta \begin{array}{c} q^{-1}, q^{-1} e^{-2i\theta} \end{array} \right] = \sum_{k=0}^{n} (\beta;q)_k (\beta;q)_{n-k} (q;q)_{n-k} e^{i(n-2k)\theta}, \quad \xi = \cos \theta. \quad (2.1)$$

Notice that in the limit $q \uparrow 1$ we get usual Gegenbauer (or ultraspherical) polynomials $C_n^\lambda(\xi)$ (see [18, 26])

$$\lim_{q \uparrow 1} C_n(\xi;\beta|q) = \sum_{k=0}^{n} \frac{(\lambda)_k (\lambda)_{n-k}}{k!(n-k)!} e^{i(n-2k)\theta} = C_n^\lambda(\xi). \quad (2.2)$$

Polynomials $C_n(\xi;\beta|q)$ are orthogonal polynomials on $[-1,1]$ with the following measure ($|q| < 1, |\beta| < 1$):

$$\frac{1}{2\pi} \int_{-1}^{1} C_m(\xi;\beta|q) C_n(\xi;\beta|q) \left( \frac{e^{2i\theta};q)_\infty}{(\beta e^{2i\theta};q)_\infty} \right)^2 \frac{d\xi}{\sqrt{1-\xi^2}} = \frac{(\beta,\beta q;q)_\infty (\beta^2;q)_n}{(\beta^2,q;q)_\infty (q;q)_n} \frac{(1-\beta)}{(1-\beta q^n)} \delta_{mn}. \quad (2.3)$$
They satisfy the recurrence relation
\[ 2(1 - \beta q^n) \xi C_n(\xi; \beta|q) = (1 - q^{n+1}) C_{n+1}(\xi; \beta|q) + (1 - \beta^2 q^{n-1}) C_{n-1}(\xi; \beta|q) \] (2.4)
and \( q \)-difference equation of the form
\[ (1 - q)^2 D_q[w(\xi; q^{\frac{1}{2}} \beta|q) D_q y(\xi)] + \lambda_n w(\xi; \beta|q) y(\xi) = 0, \quad y(\xi) = C_n(\xi; \beta|q), \] (2.5)
where
\[ w(\xi; \beta|q) := \left| \frac{(e^{2i\theta}; q)_\infty}{(\beta e^{2i\theta}; q)_\infty} \right|^2, \quad \lambda_n = 4q^{-n+1}(1 - q^n)(1 - \beta^2 q^n) \] (2.6)
and \( D_q f(\xi) := \frac{\delta_q f(\xi)}{\delta_q \xi} \) with \( \delta_q f(e^{i\theta}) = f(q^{\frac{1}{2}} e^{i\theta}) - f(q^{-\frac{1}{2}} e^{i\theta}), \ \xi = \cos \theta. \)

There is a simple generating function for these polynomials:
\[ \frac{(\beta e^{\theta} z, \beta e^{-\theta} z; q)_\infty}{(e^{\theta} z, e^{-\theta} z; q)_\infty} = \sum_{n=0}^{\infty} C_n(\xi; \beta|q) z^n, \quad \xi = \cos \theta. \] (2.7)

Finally, let us mention the product formula in integral form (see (8.4.1)-(8.4.2) in \[20\] and work \[30\])
\[ C_n(\xi; \beta|q) C_n(\eta; \beta|q) = \frac{(\beta^2; q)_n}{(q; q)_n} \beta^{-n/2} \int_{-1}^{1} K(\xi, \eta, \zeta; \beta|q) C_n(\zeta; \beta|q) \, d\zeta \] (2.8)
where \( (\xi = \cos \theta, \eta = \cos \phi, \zeta = \cos \psi) \)
\[ K(\xi, \eta, \zeta; \beta|q) = \frac{(q, \beta, \beta; q)_\infty}{2\pi(\beta^2; q)_\infty \sqrt{1 - \zeta^2}} \times w(e^{i\psi}; \beta^2 e^{i\theta + i\phi}, \beta^2 e^{-i\theta - i\phi}, \beta \zeta e^{i\phi - i\theta}). \] (2.9)

Function \( w(x; a, b, c, d) \) in \[2.9\] is defined as follows:
\[ w(x; a, b, c, d) := \frac{(x^2, x^{-2}; q)_\infty}{(ax, ax^{-1}, bx, bx^{-1}, cx, cx^{-1}, dx, dx^{-1}; q)_\infty}. \] (2.10)

In the Section 4 we will give a new proof of this product formula.

**Remark 1** It is always possible to give a general expression for the kernel \( K \) as an infinite series. Indeed, consider a product formula in integral form for some polynomials \( p_n(x) \):
\[ p_n(x) p_n(y) = \int_a^b dz K(x, y, z) p_n(z) \] (2.11)
and let the orthogonality relation for the polynomials be given by
\[ \int_a^b p_n(x) p_m(x) w(x) \, dx = b_n \delta_{nm}. \] (2.12)
Then, the left hand side $p_n(x)p_n(y)$ of (2.11) can be thought of as the expression for the Fourier coefficients in the expansion of the function $K(x, y, z)$ into the basis of orthogonal polynomials $p_n(z)$. Hence, we always have the formula for the kernel $K(x, y, z)$ in terms of the product of 3 polynomials $p_n(x)p_n(y)p_n(z)$ as a consequence of (2.11) and (2.12):

$$K(x, y, z) = w(z) \sum_{n=0}^{\infty} \frac{p_n(x)p_n(y)p_n(z)}{b_n}.$$ 

Applying this to the kernel (2.9) and using (2.3) we obtain the expansion

$$K(\xi, \eta, \zeta; |q|) = \frac{(q, \beta^2; q_\infty)(q_\infty)}{2\pi(\beta, \beta; q_\infty)\sqrt{1-\zeta^2}} \left[ \frac{(e^{2i\psi}; q_\infty)}{(\beta e^{2i\psi}; q_\infty)} \right]^2 \times$$

$$\times \sum_{n=0}^{\infty} \beta^{n/2} (1 - \beta q^n) \left( \frac{(q; q_n)_\infty}{(\beta^2; q_n)_\infty} \right)^2 C_n(\xi; \beta|q) C_n(\eta; \beta|q) C_n(\zeta; \beta|q). \quad (2.13)$$

**Remark 2** It was shown in [36] that in the limit $q \uparrow 1$ the product formula (2.8) goes into the product formula due to Gegenbauer [21]

$$\frac{C_n^\lambda(x) C_n^\lambda(y)}{C_n^\lambda(1)} = \int_{-1}^{1} K(x, y, z) \frac{C_n^\lambda(z)}{C_n^\lambda(1)} \, dz, \quad (2.14)$$

where

$$K(x, y, z) = \left\{ \begin{array}{ll}
\frac{(1 - x^2 - y^2 - z^2 + 2xyz)^{\lambda-1}}{B(\lambda, \frac{1}{2})[(1 - x^2)(1 - y^2)]^{\lambda-1} \beta^2}, & 1 - x^2 - y^2 - z^2 + 2xyz > 0, \\
0, & 1 - x^2 - y^2 - z^2 + 2xyz \leq 0.
\end{array} \right.$$

Now we describe the relation between the $q$-ultraspherical polynomials and $A_1$ Macdonald polynomials. Let $\text{Sym}(x_1, x_2)$ be the space of symmetric Laurent polynomials in two variables $x_1, x_2$. Let us introduce the polynomials $P_\lambda \in \text{Sym}(x_1, x_2)$,

$$P_\lambda(x_1, x_2) = x_+^{\frac{1}{|\lambda|}} \frac{(q; q)_\lambda}{(t; q)_{\lambda_21}} C_{\lambda_21}(\frac{x_+ + x_-}{2}; t|q), \quad x_\pm := (x_1 x_2^{\pm1})^{\frac{1}{2}}, \quad (2.15)$$

depending on real parameters $q, t \in (0, 1)$ and indexed by the pairs of integers $\lambda \equiv \{\lambda_1, \lambda_2\} \in \mathbb{Z}^2$ such that $\lambda_1 \leq \lambda_2$. We use the notation $|\lambda| \equiv \lambda_1 + \lambda_2, \lambda_{21} \equiv \lambda_2 - \lambda_1$, $t = q^g, \ g > 0$.

**Proposition 1** The polynomials $P_\lambda(x_1, x_2)$ (2.14) coincide with the standard Macdonald polynomials for the root system $A_1$.

**Proof.** From (2.3)–(2.6) it follows that $P_\lambda$ are the eigenfunctions

$$H_j P_\lambda = h_{\lambda,j} P_\lambda \quad (2.16)$$
of two finite-difference operators in the space \( \text{Sym}(x_1, x_2) \)

\[
H_1 = v_{12} T_{q,x_1} + v_{21} T_{q,x_2}, \quad H_2 = T_{q,x_1} T_{q,x_2}
\]  

(2.17)

where

\[
v_{jk} = t^\frac{j}{2} x_j - t^{-\frac{j}{2}} x_k \quad x_j - x_k
\]

(2.18)

and \( T_{q,x} \) is the \( q \)-shift operator in the variable \( x \)

\[
T_{q,x} f(x) = f(q x).
\]

(2.19)

The corresponding eigenvalues \( h_{\lambda, j} \) are

\[
h_{\lambda, 1} = t^{-\frac{j}{2}} q^{\lambda_1} + t^\frac{j}{2} q^{\lambda_2}, \quad h_{\lambda, 2} = q |\lambda|.
\]

(2.20)

Introducing the basis of monomial symmetric functions \( m_\lambda \) in \( \text{Sym}(x_1, x_2) \)

\[
m_\lambda(x_1, x_2) = \begin{cases} 
  x_{1_1} x_{2_2} + x_{1_2} x_{2_1}, & \lambda_1 < \lambda_2 \\
  x_{1_1} x_{2_2}^{\lambda_2_1}, & \lambda_1 = \lambda_2
\end{cases}
\]

(2.21)

and using the definition (2.1) we observe that \( P_\lambda \) expands in terms of \( m_\nu \) as

\[
P_\lambda = \sum_{\nu < \lambda, |\nu| = |\lambda|} u_{\lambda \nu} m_\nu,
\]

(2.22)

\[
u < \lambda \iff \lambda_1 \leq \nu_1 \leq \nu_2 \leq \lambda_2.
\]

(2.23)

From (2.1) there follows also the normalization condition \( u_{\lambda \lambda} = 1 \). The enlisted properties of \( P_\lambda \) constitute precisely the definition of \( A_1 \) Macdonald polynomials (see [31], Chapter VI).

Let us introduce also the separated polynomials \( f_\lambda(y) \)

\[
f_\lambda(y) = y^{\lambda_1} \frac{\phi_1\left( t, q^{-\lambda_{21}} ; t^{-1} q^{1-\lambda_{21}} ; q, q^{-2} q y \right)}{t^{-\lambda_{21}} (t; q)_{\lambda_{21}} (q; q)_{\nu_1-\lambda_{11}} (q; q)_{\nu_2-\lambda_{11}}}
\]

(2.24)

(2.25)

It follows from (2.3) and (2.4) that \( f_\lambda(y) \) satisfies the \( q \)-difference equation

\[
\left[ t (1 - q y) T_{q^2,y} - t^{\frac{1}{2}} (t - q y) h_{\lambda, 1} T_{q,y} + (t^2 - q y) h_{\lambda, 2} \right] f_\lambda(y) = 0
\]

(2.26)

and expands in \( y \) as

\[
f_\lambda(y) = \sum_{k=\lambda_1}^{\lambda_2} \chi_k y^k, \quad \chi_k = (t^{-2} q)^{k-\lambda_1} \frac{(t, q^{-\lambda_{21}} ; q)_{k-\lambda_1}}{(t, q^{-\lambda_{21}} ; q)_{k}}.
\]

(2.27)
In particular,
\[ \chi_{\lambda_1} = 1, \quad \chi_{\lambda_2} = t^{-\lambda_{21}}. \quad (2.28) \]

**Remark 3** In [29] a general formula is given for the separated polynomials for any root system \( A_{n-1} \). Putting \( n = 2 \) in the formula (5.1) from [29] one obtains another expression for \( f_\lambda(y) \):
\[ f_\lambda(y) = y^{\lambda_1}(y; q)_{1-2g} 2\phi_1 \left[ t^{-2}q^{1-\lambda_{21}}, t^{-1}q^{1-\lambda_{21}}; q, y \right]. \quad (2.29) \]
The equivalence of (2.25) and (2.29) follows from the \( q \)-analog of Euler’s transformation formula [20], (1.4.3).

The formula (2.8) can now be interpreted as the one expressing SoV for the pair of integrals of motion (2.17) for the quantum integrable system with 2 degrees of freedom (which is the \( A_1 \) type trigonometric Ruijsenaars’ system [37]).

**Remark 4** As one can see, the \( A_1 \) Macdonald polynomials \( P_\lambda(x_1, x_2) \) (2.15) are already in the factorized form when written in terms of the variables \( x_\pm \). This corresponds to another (trivial) SoV which is purely coordinate one (local transform): \((x_1, x_2) \mapsto (x_+, x_-)\), as opposed to the integral separating transform (2.8). This is a simple demonstration of the fact that SoV for a given integrable system (a given multi-variable special function) might be not unique in general.

### 3. Operator \( M_{\alpha\beta} \) and its properties

In this and the next sections we introduce and study the integral operator \( M_{\xi} \) performing the separation of variables in the \( A_1 \) Macdonald polynomials (2.15). The operator is most conveniently described in terms of a slightly more general integral operator \( M_{\alpha\beta} \) which, in turn, is closely related to the fractional \( q \)-integration operator \( I^\alpha \).

Our main technical tool is the famous Askey-Wilson integral identity [10, 20]
\[ \frac{1}{2\pi i} \int_{\Gamma_{abcd}} \frac{dx}{x} w(x; a, b, c, d) = \frac{2(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty} \quad (3.1) \]
where \( w \) is given by (2.10). The cycle \( \Gamma_{abcd} \) depends on complex parameters \( a, b, c, d \) and is defined as follows. Let \( C_{z,r} \) be the counter-clockwise oriented circle with the center \( z \) and radius \( r \). If \( \max(|a|, |b|, |c|, |d|, |q|) < 1 \) then \( \Gamma_{abcd} = C_{0,1} \). The identity (3.1) can be continued analytically for the values of parameters \( a, b, c, d \) outside the unit circle provided the cycle \( \Gamma_{abcd} \) is deformed appropriately. In general case
\[ \Gamma_{abcd} = C_{0,1} + \sum_{z=a, b, c, d} \sum_{k \geq 0 \atop |z|q^k > 1} (C_{zq^k, \varepsilon} - C_{z^{-1}q^{-k}, \varepsilon}), \quad (3.2) \]
\( \varepsilon \) being small enough for \( C_{z \pm 1q \pm k, \varepsilon} \) to encircle only one pole of the denominator.
Now put
\[ a = y q^\frac{\alpha}{2}, \quad b = y^{-1} q^\frac{\alpha}{2}, \quad c = r q^\frac{\beta}{2}, \quad d = r^{-1} q^\frac{\beta}{2}, \] (3.3)
and introduce the notation
\[ L_q(\nu; x, y) := (\nu x y, \nu x y^{-1}, \nu x^{-1} y, \nu x^{-1} y^{-1}; q)_\infty. \] (3.4)
The kernel
\[ M_{\alpha\beta}(r, y|x) = \frac{(1 - q)(q; q^2)_\infty (x, x^{-2}; q)_\infty L_q(q^{\frac{\alpha + \beta}{2}}; r, y)}{2B_q(\alpha, \beta) L_q(q^{\frac{\alpha}{2}}; y, x) L_q(q^{\frac{\beta}{2}}; r, x)} \] (3.5)
defines the integral operator
\[ (M^r_{\alpha\beta} f)(y) := \frac{1}{2\pi i} \int_{r_{\alpha\beta}} \frac{dx}{x} M_{\alpha\beta}(r, y|x) f(x), \] (3.6)
the contour \( r_{\alpha\beta} \) being obtained from \( r_{abcd} \) by the substitutions (3.3). Notice that in (3.5)–(3.6) we consider \( x \) and \( y \) as arguments and \( r \) as a parameter.

The following properties of the operator \( M_{\alpha\beta} \) are proved in [29], Appendix B.

**Proposition 2**

1. Let \( \text{Refl}(x) \) be the space of reflexive (invariant w.r.t. \( x \to x^{-1} \)) Laurent polynomials in \( x \). Then \( M_{\alpha\beta} : \text{Refl}(x) \to \text{Refl}(y) \).

2. In particular, from the Askey-Wilson identity (3.1) it follows immediately that \( M : 1 \to 1 \).

3. More generally, consider a Laurent polynomial \( R_{j_1 j_2 k_1 k_2}^{\alpha\beta}(r, x) \in \text{Refl}(x) \), where \( j_{1,2}, k_{1,2} \in \mathbb{Z}_{\geq 0} \)
\[ R_{j_1 j_2 k_1 k_2}^{\alpha\beta}(r, x) := (q^{\frac{\alpha}{2}} y x, q^{\frac{\alpha}{2}} y x^{-1}; q)_{j_1} (q^{\frac{\alpha}{2}} y^{-1} x, q^{\frac{\alpha}{2}} y^{-1} x^{-1}; q)_{j_2} \times (q^{\frac{\beta}{2}} r x, q^{\frac{\beta}{2}} r x^{-1}; q)_{k_1} (q^{\frac{\beta}{2}} r^{-1} x, q^{\frac{\beta}{2}} r^{-1} x^{-1}; q)_{k_2}. \] (3.7)

Then
\[ M_{\alpha\beta} : R_{j_1 j_2 k_1 k_2}^{\alpha\beta}(r, x) \to \frac{(q^{\alpha}; q)_{j_1 + j_2} (q^{\beta}; q)_{k_1 + k_2}}{(q^{\alpha+\beta}; q)_{j_1 + j_2 + k_1 + k_2}} \times (q^{\frac{\alpha+\beta}{2}} r y; q)_{j_1 + k_1} (q^{\frac{\alpha+\beta}{2}} r y^{-1}; q)_{j_2 + k_2} \times (q^{\frac{\alpha+\beta}{2}} r^{-1} y; q)_{j_1 + k_2} (q^{\frac{\alpha+\beta}{2}} r^{-1} y^{-1}; q)_{j_2 + k_2}. \] (3.8)
The last formula allows to calculate effectively the action of \( M_{\alpha\beta} \) on any polynomial from \( \text{Refl}(x) \).
4. The inversion of $M_{\alpha\beta}^r$ is given by the formula

$$\left(M_{\alpha\beta}^r\right)^{-1} = M_{-\alpha,\alpha+\beta}^r,$$

the corresponding kernel being

$$\tilde{M}_{\alpha\beta}^r(x|y) = \frac{(1-q)(q; q)_\infty^2 (y^2; y^{-2}; q)_\infty \mathcal{L}_q(q^{\frac{\beta}{2}}; r, x)}{2B_q(-\alpha, \alpha + \beta) \mathcal{L}_q(q^{-\frac{\beta}{2}}; y, x) \mathcal{L}_q(q^{\frac{\alpha+\beta}{2}}; r, y)}. \quad (3.10)$$

5. The operator $M_{\alpha\beta}^r$ simplifies drastically when either of the parameters $\alpha$, $\beta$ takes negative integer values. Let $\alpha = -g$, $g \in \mathbb{Z}_{\geq 0}$. Then $M_{\alpha\beta}^r$ turns into the $q$-difference operator of order $g$:

$$M_{-g,\beta}^r : f(x) \to \sum_{k=0}^{g} \xi_k(r, y) f(q^{k-\frac{\beta}{2}}y) \quad (3.11)$$

where

$$\xi_k(r, y) = (-1)^k q^{-\frac{k(k-1)}{2}} \left[ \frac{g}{k} \right] q^{-2k} (1 - q^{g-2k}y^{-2}) \times \left( q^{\frac{\beta-g}{2}}ry, q^{\frac{\beta-g}{2}}r^{-1}y; q \right)_k (q^{\frac{\beta-g}{2}}ry^{-1}, q^{\frac{\beta-g}{2}}r^{-1}y^{-1}; q)_{g-k} \left( q^{\beta-g}; q \right)_g (q^{-k}y^{2}; q)_{g+1}. \quad (3.12)$$

It was mentioned briefly in [29] that, similarly to the nonrelativistic case [28], the operator $M_{\alpha\beta}^r$ has a natural interpretation in terms of a fractional $q$-integration operator. To clarify this remark, take the formula (3.5) for the kernel $M_{\alpha\beta}^r(r, y|x)$ and rewrite it in the form

$$\mathcal{M}_{\alpha\beta}(r, y|x) = \frac{\psi_{\alpha}^{\beta-1}(x)}{\psi_{\alpha+\beta-1}^{\beta}(y) \mathcal{I}^\alpha(r, y|x)} \quad (3.13)$$

with

$$\psi_{\alpha}^{\nu}(x) := \frac{\mathcal{L}_q(q^{\frac{\nu}{2}}; r, x)}{\Gamma_q(\nu + 1) \mathcal{L}_q(q^{\frac{\nu+1}{2}}; r, x)}, \quad (3.14)$$

where the kernel $\mathcal{I}^\alpha(r, y|x)$ is defined as follows:

$$\mathcal{I}^\alpha(r, y|x) = \frac{(1-q)(q; q)_\infty^2 (x^2; x^{-2}; q)_\infty \mathcal{L}_q(q^{\frac{1}{2}}; r, y)}{2\Gamma_q(\alpha) \mathcal{L}_q(q^{\frac{3}{2}}; y, x) \mathcal{L}_q(q^{\frac{1}{2}}; r, x)}. \quad (3.15)$$

In the operator form we have

$$M_{\alpha\beta}^r = \frac{1}{\psi_{\alpha+\beta-1}^{\beta-1}} \circ \mathcal{I}^\alpha \circ \psi_{\alpha}^{\beta-1} \quad (3.16)$$
where $\psi^\nu_r$ are thought of as multiplication operators and the operator $I^\alpha$ corresponds to the kernel $I^\alpha(r, y|x)$ (3.15)

$$I^\alpha : f(x) \to \frac{1}{2\pi i} \int_{\Gamma_{\alpha,1}} \frac{dx}{x} I^\alpha(r, y|x) f(x)$$  \hspace{1cm} (3.17)

with $r$ considered as a parameter.

The following properties of the operator $I^\alpha$ can be obtained mostly from those of $M^\alpha_{\alpha\beta}$ enlisted in the Proposition 2 using the correspondence (3.16).

**Proposition 3**

1. The operator $I^\alpha$ possesses the group property with respect to the parameter $\alpha$

$$I^{\alpha+\beta} = I^\alpha \circ I^\beta.$$  \hspace{1cm} (3.18)

2. Define an analog $\psi^\nu_r$ of the power function by the formula (3.14). Then

$$I^\alpha : \psi^\nu_r(x) \to \psi^{\nu+\alpha}_r(y).$$  \hspace{1cm} (3.19)

3. For $\alpha = -g$, $g \in \mathbb{Z}_{\geq 0}$ the operator $I^\alpha$ turns into the finite-difference operator

$$I^{-g} : f(x) \to \sum_{k=0}^g \zeta_{g,k}(r, y) f(q^k - \frac{2}{q}y),$$  \hspace{1cm} (3.20)

$$\zeta_{g,k}(r, y) = (-1)^k q^{\frac{k(k-1)}{2}} \binom{g}{k} \frac{y^{-2k} (1 - q^{g-2k}y^{-2})}{(1 - q)^g (q^{-k}y^{-2}; q)_{g+1} \mathcal{L}_q(\frac{1}{q^{k/2}}; r, y)} \mathcal{L}_q(\frac{1}{q^{k/2}}; r, q^{k/2}y).$$  \hspace{1cm} (3.21)

Moreover

$$I^{-g} = (I^{-1})^g,$$  \hspace{1cm} (3.22)

where

$$I^{-1} : f(x) \to \frac{\mathcal{L}_q(\frac{1}{q^{1/2}}; r, y)}{(1 - q)(1 - y^2)} \left( \frac{f(q^{1/2}y)}{\mathcal{L}_q(\frac{1}{q^{1/2}}; r, q^{1/2}y)} - \frac{y^2 f(q^{-1/2}y)}{\mathcal{L}_q(\frac{1}{q^{1/2}}; r, q^{-1/2}y)} \right).$$  \hspace{1cm} (3.23)

4. In the limit $q \uparrow 1$ we have

$$I^{-1} \to -\frac{y^2}{1 - y^2} \frac{d}{dy},$$  \hspace{1cm} (3.24)

$$\psi^\nu_r(y) \to \frac{(y + y^{-1} - r - r^{-1})^\nu}{\Gamma(\nu + 1)},$$  \hspace{1cm} (3.25)

so, for instance, equality (3.19) for $\alpha = -1$ gives the formula

$$I^{-1} \psi^\nu_r = \psi^{\nu-1}_r.$$  \hspace{1cm} (3.26)
The identity (3.18), like the equivalent identity $M_{\alpha\beta} : 1 \to 1$, is a disguised form of the Askey-Wilson (AW) integral (3.1). The formula (3.19) is also proved quite directly using AW integral. The formulas (3.20) and (3.21) are equivalent, respectively, to (3.11) and (3.12). The equality (3.22) is easily proved by induction.

The properties of the operator $I^\alpha$ described in the Proposition 3 allow to consider it as a fractional power of the first order difference operator $I^{-1}$ (3.23). In the limit $q \uparrow 1$ the operator $I^{-1}$ (3.24) is reduced to the pure differentiation operator by the change of variable $\xi = y + y^{-1} - 2$. The operator $I^\alpha$, respectively, is reduced to the well-known Riemann-Liouville-Weyl fractional integral

$$I^\alpha : f(x) \to \int_x \frac{dy}{\Gamma(\alpha)} \frac{(y - x)^{\alpha-1}}{\Gamma(\alpha)} f(y).$$

(3.27)

In the general case $q \neq 1$, however, there is no change of variable reducing $I^{-1}$ (3.23) to the pure $q$-derivative $(D_q f)(x) = (f(x) - f(qx))/((1 - q)x)$, and our operator $I^\alpha$ (3.17) cannot be reduced, respectively, to the fractional $q$-integration operator studied in [2, 3, 4]. It is not our task to develop here the complete theory of the operator $I^\alpha$ (3.17) although we believe that this new $q$-fractional integration operator deserves more comprehensive study.

4. Separating operator $M_\xi$

Let us specify the values of the arguments and parameters in the kernel $M_{\alpha\beta}(r, y|x)$ (3.3) to be

$$\alpha = \beta = g, \quad y = y_-, \quad x = x_-, \quad r = t^{-1}y_+, \quad y_\pm = (y_1 y_2^{\pm 1})^{\pm 1},$$

(4.1)

(in contrast to $\alpha = g, \beta = 2g$, as for $A_2$ case [29]), and denote the resulting kernel as $M(y_+, y_-|x_-)$:

$$M(y_+, y_-|x_-) = \frac{(1 - q)(q; q)_\infty^2 (x_+^2, x_-^2; q)_\infty}{2B_q(g, g) L_q(t^{\frac{1}{2}}; y_-, x_-) L_q(t^{\frac{1}{2}}; x_-, t^{-1}y_+)}.$$

(4.2)

Assuming $\xi$ to be an arbitrary complex parameter, we introduce the integral operator $M_\xi$ acting on functions $f(x_1, x_2)$ by the formula

$$(M_\xi f)(y_1, y_2) \equiv \frac{1}{2\pi i} \int_{t^{-1}y_+, y_-} \frac{dz}{z} \frac{d}{dz} M(y_+, y_-|z) f(t^{-\frac{1}{2}}\xi y_+ z, t^{-\frac{1}{2}}\xi y_+ z^{-1}).$$

(4.3)

It is apparent from (4.3) that $M_\xi$ acts on $x_+$ as a simple scaling:

$$M_\xi : \varphi(x_1 x_2) \to \varphi(t^{-1}\xi^2 y_1 y_2).$$

(4.4)

Now we can formulate the main result of the paper.
Proposition 4

Theorem 1

The operator $M_\xi$ \([4.3]\) performs the factorisation of (or, in other words, the SoV for) the $A_1$ Macdonald polynomials \([2.13]\):

$$M_\xi : P_\lambda(x_1, x_2) \rightarrow F_\lambda(y_1, y_2) \equiv c_{\lambda, \xi} f_\lambda(y_1) f_\lambda(y_2),$$

(4.5)

where the factorized (or separated) polynomial $f_\lambda(y)$ is given in \([2.23]\) and the normalization coefficient $c_{\lambda, \xi}$ is

$$c_{\lambda, \xi} = t^{-2\lambda_1 + \lambda_2 \xi |\lambda|} \frac{(t; q)^{\lambda_2}}{(t^2; q)^{\lambda_2}}.$$  

(4.6)

Note that the product formula \([2.8]\) for the $q$-ultraspherical polynomials is equivalent to the formula \([4.3]\). To observe it, it is sufficient to make substitutions $t = \beta$, $\lambda_2 = n$, $z = e^{i\psi}$, $y_1 = te^{-2i\theta}$, $y_2 = te^{-2i\phi}$ and to replace the integral over the unit circle by the integral from $-1$ to $1$.

Taking into account the inversion formulas \([3.9]\) and \([3.10]\) for $M_\alpha^r$ we notice that the kernel of the inverse operator $M_\xi^{-1}$ is obtained from $M_{\alpha, \beta}(r, y|x)$ \([3.5]\) by the substitution $\alpha = -g$, $\beta = 2g$, $x := y_-$, $y := x_-$, $r := t^{-\frac{1}{2}} \xi - 1$:

$$\tilde{M}(x_+, x_- | y_-) = \frac{(1 - q)(q; q)_\infty^2 (y^2, y^{-2}; q)_\infty}{2B_q(-g, 2g)} L_q\left( t^{-\frac{1}{2}}; x_-, t^{-\frac{1}{2}} \xi^{-1} x_+ \right).$$

(4.7)

As the immediate corollary the above remark and the Theorem 1 we obtain the following result.

Theorem 2

The inversion of the operator $M_\xi$ \([4.3]\) is given by the formula

$$(M_\xi^{-1} f)(x_1, x_2) = \frac{1}{2\pi i} \int \frac{dz}{z} \tilde{M}(x_+, x_- | z) f(t^{\frac{1}{2}} \xi^{-1} x_+ z, t^{\frac{1}{2}} \xi^{-1} x_+ z^{-1}).$$

(4.8)

The operator $M_\xi^{-1}$ provides an integral representation for the $A_1$ Macdonald polynomials in terms of the factorized polynomials $f_\lambda(y)$

$$M_\xi^{-1} : c_{\lambda, \xi} f_\lambda(y_1) f_\lambda(y_2) \rightarrow P_\lambda(x_1, x_2).$$

(4.9)

In contrast to the formula \([4.3]\) which paraphrases an already known result, the formula \([4.9]\) leads to a new integral relation for the $q$-ultraspherical polynomials. Note that for positive integer $g$ the operator $M_\xi^{-1}$ becomes a $q$-difference operator (cf. Proposition 2.5).

The proof of the Theorem 1 mimics that in \([29]\) for the $A_2$ case and consists of the following steps.

Proposition 4

The operator $M_\xi$ maps bijectively $\text{Sym}(x_1, x_2)$ onto $\text{Sym}(y_1, y_2)$.  

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Proposition 5 The following operator identity ("quantum characteristic equation", cf. [29]) is true
\[(1 - qy_j) T_{q^2, y_j} M_{\xi} - t^\frac{1}{2}(1 - qt^{-1}y_j) T_{q, y_j} M_{\xi} H_1 + t(1 - qt^{-2}y_j) M_{\xi} H_2 = 0, \quad (4.10)\]
where $H_1$ and $H_2$ are quantum integrals of motion given in (2.17).

Proposition 6 The separated polynomial $f_\lambda(y)$ (2.24) is the only Laurent polynomial solution to the separated equation (2.24).

Proposition 7 The normalization coefficient $c_{\lambda, \xi}$ in (4.5) is given by the formula (4.6).

The main statement of the Theorem 1 follows then by a standard argument (cf. [29]). The Propositions 4 and 5 imply that $M_{\xi} P_{\lambda}$ is a Laurent polynomial in $y_1$ and $y_2$ satisfying the separated equation (2.24) in both variables $y_1, y_2$. Then, by virtue of the Proposition 6, it is factorized into $f_{\lambda}(y_1) f_{\lambda}(y_2)$. It remains only to calculate the normalization coefficient $c_{\lambda, \xi}$ which is done by the Proposition 7.

To prove the Proposition 7 we shall calculate explicitly the matrix of $M_{\xi}$ in special bases in $\text{Sym}(x_1, x_2)$ and $\text{Sym}(y_1, y_2)$. Define the bases $p_\nu$ and $r_\nu$ in $\text{Sym}(x_1, x_2)$ and, respectively, $\tilde{p}_\nu$ and $\tilde{r}_\nu$ in $\text{Sym}(y_1, y_2)$ for $\nu = \{\nu_1 \leq \nu_2\} \in \mathbb{Z}^2$:

\[p_\nu = (x_1 x_2)^{\nu_1} (\xi^{-1} x_1, \xi^{-1} x_2; q)_{\nu_2}, \quad r_\nu = (x_1 x_2)^{\nu_2} (t \xi x_1^{-1}, t \xi x_2^{-1}; q)_{\nu_2}, \quad (4.11)\]

\[\tilde{p}_\nu = (y_1 y_2)^{\nu_1} (y_1; q)_{\nu_2}, \quad \tilde{r}_\nu = (y_1 y_2)^{\nu_2} (t^2 y_1^{-1}, t^2 y_2^{-1}; q)_{\nu_2}. \quad (4.12)\]

Note that

\[p_\nu = (x_1 x_2)^{\nu_1} R_{000; 010}^{a\lambda g} (t^{-1/2} \xi^{-1} x_+, x_-), \quad r_\nu = (x_1 x_2)^{\nu_2} R_{000; 010}^{a\lambda g} (t^{-1/2} \xi^{-1} x_+, x_-) \quad (4.13)\]

where $R_{j_1 j_2 k_1 k_2}^{a\lambda g}$ is defined by (3.7). Then, using (3.8), we obtain explicit formulas for the action of $M_{\xi}$ on the bases $p_\nu$, $r_\nu$:

\[M_{\xi} : p_\nu \rightarrow \mu_{\nu}^{(p)} \tilde{p}_\nu, \quad M_{\xi} : r_\nu \rightarrow \mu_{\nu}^{(r)} \tilde{r}_\nu \quad (4.14)\]

where

\[\mu_{\nu}^{(p)} = t^{-\nu_1} \xi^{2 \nu_1} \frac{(t; q)_{\nu_2}}{(t^2; q)_{\nu_2}}, \quad \mu_{\nu}^{(r)} = t^{-\nu_2} \xi^{2 \nu_2} \frac{(t; q)_{\nu_1}}{(t^2; q)_{\nu_1}}. \quad (4.15)\]

The invertibility of $M_{\xi}$ being obvious from (4.14), the Proposition 7 is thus proved. Actually, we have obtained much stronger result, having found an effective way of calculating the action of $M_{\xi}$ on any polynomial.

Remark 5 There are involutions $U_{\xi}$ and $V$ in $\text{Sym}(x_1, x_2)$ and $\text{Sym}(y_1, y_2)$, respectively,

\[U_{\xi} : \varphi(x_1, x_2) \rightarrow \varphi(t \xi^2 x_1^{-1}, t \xi^2 x_2^{-1}), \quad V : \varphi(y_1, y_2) \rightarrow \varphi(t^2 y_1^{-1}, t^2 y_2^{-1}) \quad (4.16)\]

\[16\]
\[ U_ξ p_ν = t^{2ν_1} ξ^{4ν_1} r_ν \, , \quad V \bar{p}_ν = t^{4ν_1} \bar{r}_ν \, , \quad \bar{ν} \equiv \{-ν_2, -ν_1\} \, , \] (4.17)

which are intertwined by the operator \( M_ξ: \)

\[ M_ξ U_ξ = V M_ξ \, . \] (4.18)

**Remark 6** One can identify Sym\((x_1, x_2)\) and Sym\((y_1, y_2)\) setting \( y_j \equiv ξ^{-1} x_j \). Then \( p_ν \equiv \bar{p}_ν \) becomes the eigenbasis for \( M_ξ \). Similarly, setting \( y_j \equiv t ξ^{-1} x_j \) one obtains \( r_ν \equiv \bar{r}_ν \) as the eigenbasis for \( M_ξ \).

For a more detailed discussion of the bases \( p_ν, r_ν \) see the next Section.

Now we shall prove the Proposition 6.

**Proof.** In [29] the proof of the identity analogous to (4.10) was based on shift relations for the kernel \( M \). Here we choose another strategy and prove (4.10) purely algebraically.

Note first that the Hamiltonians \( H_j \) (2.17) are described by Jacobian matrices in the basis \( r_ν \) (not true for \( p_ν \!))

\[ H_j r_ν = a_j^ν r_ν + b_j^ν r_{ν+1, ν+2} + c_j^ν r_{ν, ν-1} \, , \quad j = 1, 2 \, , \] (4.19)

where

\[
\begin{align*}
a_1^ν & = t^{-\frac{1}{2}} q^{ν_1} + t^{\frac{1}{2}} q^{ν_2} \\
b_1^ν & = -t^{-\frac{1}{2}} q^{ν_1} (1 - q^{ν_2}) \\
c_1^ν & = t^{\frac{1}{2}} ξ^2 q^{-ν_1+2ν_2-2} (1 - q^{ν_2}) \\
a_2^ν & = q^{ν_1} \\
b_2^ν & = -q^{ν_2} (1 - q^{ν_2}) \\
c_2^ν & = t^2 ξ^2 q^{2ν_2-2} (1 - q^{ν_2}).
\end{align*}
\] (4.20)

Now apply the left-hand-side of (4.10) to the basis \( r_ν \). Using the relations

\[
\begin{align*}
\bar{r}_{ν+1, ν+2} & = \bar{r}_ν (1 - q^{ν_2-1} t^2 y_1^{-1})^{-1} (1 - q^{ν_2-1} t^2 y_2^{-1})^{-1}, \quad (4.21a) \\
\bar{r}_{ν, ν-1} & = \bar{r}_ν y_1^{-1} y_2^{-1} (1 - q^{ν_2-1} t^2 y_1^{-1})^{-1} (1 - q^{ν_2-1} t^2 y_2^{-1})^{-1}, \quad (4.21b) \\
T_{q,y_j} \bar{r}_ν & = \frac{q^{y_j} (1 - q^{-1} t^2 y_j^{-1})}{1 - q^{ν_2-1} t^2 y_j^{-1}} \bar{r}_ν \quad (4.22)
\end{align*}
\]

one can express all the \( r \)'s in terms of \( r_ν \). Discarding the common multiplier \( r_ν \) we obtain a rather clumsy rational expression in \( y_1, y_2, t, ξ, q \) and \( q^{ν_2} \) which, nevertheless, can be shown to be identically zero by a direct computation.

The Proposition 6 is a direct corollary of the Proposition 12 from [29].

To complete the proof of the Theorem 4 it remains to prove the Proposition 7. The proof repeats, with appropriate adjustments, the proof of the Proposition 7 from [29].

**Proof.** Using the variables \( x_± \) and comparing the asymptotics at \( x_- \to \infty \) of the monomial symmetric function \( m_λ \) (2.27)

\[ m_λ \sim x_+^{λ_1} x_-^{λ_2} \, , \quad x_- \to \infty \] (4.23)
and of the polynomial $r_{\nu}$ \cite{1.11}

$$r_{\nu} \equiv x_{+}^{2\nu_{21}} \left( t\xi x_{-}^{-1} x_{+}^{1}, t\xi x_{-}^{-1} x_{+}^{1}; q \right)_{\nu_{21}}$$

$$\sim (-1)^{\nu_{21}} q^{\frac{1}{2}\nu_{21}(\nu_{21}-1)\nu_{21}^{2} \xi^{2} x_{+}^{1} x_{-}^{-1}, x_{-} \to \infty}$$ \hspace{1cm} \quad (4.24)

we conclude that the expansion of $m_{\lambda}$ in $r_{\nu}$ is triangular

$$m_{\lambda} = (-1)^{\lambda_{21}} q^{-\frac{1}{2}\lambda_{21}(\lambda_{21}-1)} t^{-\lambda_{21}} \xi^{-\lambda_{21}} r_{\lambda} + \sum_{\nu < \lambda \atop \nu \neq \lambda} \cdots r_{\nu}, \hspace{1cm} (4.25)$$

and, consequently, the expansion of $P_{\lambda}$ in $r_{\nu}$ has the same structure. Then using (4.14)–(4.15) and the asymptotics of $\tilde{r}_{\nu}$

$$\tilde{r}_{\nu} \sim (-1)^{\nu_{21}} q^{\frac{1}{2}\nu_{21}(\nu_{21}-1)\nu_{21}^{2} \xi^{2} x_{+}^{1} x_{-}^{-1}, x_{-} \to \infty}$$ \hspace{1cm} \quad (4.26)

we obtain

$$M_{\xi} P_{\lambda} \sim t^{-\lambda_{21}} \xi^{1} \left( t; q \right)^{\lambda_{21}} y_{+}^{\nu_{21}} y_{-}^{\lambda_{21}}, \hspace{1cm} y_{-} \to \infty \quad (4.27)$$

On the other hand, the right hand side of (4.5) has the asymptotics

$$c_{\lambda, \xi} f_{\lambda}(y_{1}) f_{\lambda}(y_{2}) \sim c_{\lambda, \xi} \xi^{1} \chi_{\lambda_{1}} \chi_{\lambda_{2}} y_{+}^{\nu_{21}} y_{-}^{\lambda_{21}}, \hspace{1cm} y_{-} \to \infty \quad (4.28)$$

Comparing (4.27) and (4.28), and using formula (2.28) we obtain the desired expression (4.6) for $c_{\lambda, \xi}$. \hfill \blacksquare

5. Bases $p_{\nu}$ and $r_{\nu}$

In this section we establish some more properties of the polynomial bases $p_{\nu}$, $r_{\nu}$, $\tilde{p}_{\nu}$, $\tilde{r}_{\nu}$ introduced by (1.11), (1.12) in the previous Section. Being the eigenfunctions of the separating integral operator $M_{\xi}$ (see Remark 3), these bases must play an important role in SoV.

Our first observation is that the bases can be interpreted also as eigenfunctions of certain quantum integrable systems. The following statement is easily verified by a direct calculation.

**Proposition 8** The Laurent polynomials $p_{\nu}$, $\tilde{p}_{\nu}$, $r_{\nu}$, $\tilde{r}_{\nu}$ are the eigenfunctions, respectively, of the commuting pairs of $q$-difference operators $N_{j}$, $\tilde{N}_{j}$, $Q_{j}$, $\tilde{Q}_{j}$ ($j=1,2$)

$$N_{j} p_{\nu} = q^{\nu_{j}} p_{\nu}, \quad \tilde{N}_{j} \tilde{p}_{\nu} = q^{\nu_{j}} \tilde{p}_{\nu}, \quad Q_{j} r_{\nu} = q^{-\nu_{j}} r_{\nu}, \quad \tilde{Q}_{j} \tilde{r}_{\nu} = q^{-\nu_{j}} \tilde{r}_{\nu}, \quad (5.1)$$

$$N_{1} = -\frac{x_{2}(\xi-x_{1})}{\xi(x_{1}-x_{2})} T_{q,x_{1}} + \frac{x_{1}(\xi-x_{2})}{\xi(x_{1}-x_{2})} T_{q,x_{2}}, \quad N_{2} = -\frac{\xi-x_{1}}{x_{1}-x_{2}} T_{q,x_{1}} + \frac{\xi-x_{2}}{x_{1}-x_{2}} T_{q,x_{2}}, \quad (5.2a)$$
\[ \widetilde{N}_1 = -\frac{y_2(1-y_1)}{y_1-y_2} T_{q,y_1} + \frac{y_1(1-y_2)}{y_1-y_2} T_{q,y_2}, \quad \widetilde{N}_2 = -\frac{1-y_1}{y_1-y_2} T_{q,y_1} + \frac{1-y_2}{y_1-y_2} T_{q,y_2}, \]  

(5.2b)

\[ Q_1 = -\frac{x_1(t\xi-x_1)}{t\xi(x_1-x_2)} T^{-1}_{q,x_1} + \frac{x_1(t\xi-x_2)}{t\xi(x_1-x_2)} T^{-1}_{q,x_2}, \quad Q_2 = -\frac{t\xi-x_1}{x_1-x_2} T^{-1}_{q,x_1} + \frac{t\xi-x_2}{x_1-x_2} T^{-1}_{q,x_2}, \]  

(5.2c)

\[ \tilde{Q}_1 = -\frac{y_2(t^2-y_1)}{t^2(y_1-y_2)} T^{-1}_{q,y_1} + \frac{y_1(t^2-y_2)}{t^2(y_1-y_2)} T^{-1}_{q,y_2}, \quad \tilde{Q}_2 = -\frac{t^2-y_1}{y_1-y_2} T^{-1}_{q,y_1} + \frac{t^2-y_2}{y_1-y_2} T^{-1}_{q,y_2}. \]  

(5.2d)

Notice that the above spectral problems are already separated in the variables \( x_j \) (\( y_j \)). The integral operator \( M_\xi \) obviously intertwines these pairs of first order difference operators

\[ M_\xi N_j = \widetilde{N}_j M_\xi, \quad M_\xi Q_j = \widetilde{Q}_j M_\xi. \]  

(5.3)

Note also that, by virtue of (1.10),

\[ U_\xi N_j U_\xi^{-1} = Q_{3-j}, \quad V \widetilde{N}_j V^{-1} = \widetilde{Q}_{3-j}. \]  

(5.4)

**Remark 7** Choosing the identification \( y_j \equiv \xi^{-1}x_j \) of the spaces \( \text{Sym}(x_1, x_2) \) and \( \text{Sym}(y_1, y_2) \), see Remark 3, one can identify \( N_j \equiv \widetilde{N}_j \). By virtue of (5.3) the operator \( M_\xi \) commutes then with \( N_j \equiv \widetilde{N}_j \) and can be considered as the time-shift operator for the *discrete-time quantum integrable system* with integrals of motion \( N_1 \) and \( N_2 \). Same is true for another identification \( y_j \equiv t\xi^{-1}x_j \) and the operators \( Q_j \equiv \widetilde{Q}_j \).

Having defined several polynomial bases in \( \text{Sym}(x_1, x_2) \), such as \( P_\lambda, p_\nu, r_\nu \), it is natural to calculate the transition matrices between them. The solution is given by the following Proposition.

**Proposition 9** The expansion of \( P_\lambda \) (2.13) in the bases \( p_\nu \) (\( r_\nu \)) (4.11) is given by

\[ P_\lambda = \sum_{\nu<\lambda} \pi^\nu_\lambda p_\nu = \sum_{\nu<\lambda} \rho^\nu_\lambda r_\nu, \]  

(5.5)

where

\[ \pi^\nu_\lambda = (-1)^{\nu_2} \xi |\lambda| - 2\nu_1 q^{2\nu_1 - 2\lambda_2 + 1} (t; q)_{\lambda_2 - \nu_1} (t; q)_{\nu_2 - \lambda_1} (q; q)_{\lambda_2}, \]  

(5.6a)

\[ \rho^\nu_\lambda = (-1)^{\nu_2} (t \xi)^{-2
 \nu_2 + |\lambda|} q^{2\nu_2 - 2\lambda_1 - 1 - |\nu|} (t; q)_{\nu_2 - \lambda_1} (t; q)_{\lambda_2 - \nu_1} (q; q)_{\lambda_2}, \]  

(5.6b)

The dual expansions of the bases \( p_\lambda \) and \( r_\lambda \) (4.11) in terms of the \( A_1 \) Macdonald polynomials \( P_\nu \) (2.13) are as follows:

\[ p_\lambda = \sum_{\nu<\lambda} Q^\nu_\lambda P_\nu, \quad r_\lambda = \sum_{\nu<\lambda} R^\nu_\lambda P_\nu, \]  

(5.7)

where

\[ Q^\nu_\lambda = (-1)^{\nu_2} \xi^{\lambda_1 - |\nu|} q^{\lambda_2 - (|\nu| - 1)\lambda_1 - \frac{|\nu|}{2} + \frac{\nu_2^2}{2}} (tq; q)_{\lambda_2} (tg; q)_{\nu_2 - \lambda_1} (tq; q)_{\nu_2 - \lambda_1} (tg; q)_{\nu_2 - \lambda_1} (q; q)_{\lambda_2}, \]  

(5.8a)

\[ R^\nu_\lambda = (-1)^{\nu_2} (t \xi)^{2\lambda_2 - |\nu|} q^{\lambda_2 - (|\nu| + 1)\lambda_2 + \frac{|\nu|}{2} + \frac{\nu_2^2}{2}} (tq; q)_{\lambda_2} (tg; q)_{\nu_2 - \lambda_1} (tq; q)_{\nu_2 - \lambda_1} (tg; q)_{\nu_2 - \lambda_1} (q; q)_{\lambda_2}. \]  

(5.8b)
which give rise to the recurrence relations directly from the expansion (4.25):
\[ \rho^\nu_\lambda (a_j^\nu - h_{\lambda j}) + \rho^{\nu_1-1,\nu_2}_\lambda y_j^{\nu_1-1,\nu_2} + \rho^{\nu_1,\nu_2+1}_\lambda c_j^{\nu_1,\nu_2+1} = 0, \quad j = 1, 2. \] (5.9)

Solving the above equations with respect to \( \rho^{\nu_1-1,\nu_2}_\lambda \) and \( \rho^{\nu_1,\nu_2+1}_\lambda \) one obtains the recurrence relations for the coefficients \( \rho^\nu_\lambda \)

\[ \rho^{\nu_1-1,\nu_2}_\lambda = - \frac{(1 - q^{\nu_1-\lambda_1})(1 - tq^{\nu_2-\nu_1})}{q^{\nu_2-\lambda_1}(1 - q^{\nu_2-\nu_1})} \rho^\nu_\lambda, \] (5.10a)

\[ \rho^{\nu_1,\nu_2+1}_\lambda = - \frac{(1 - q^{\nu_2-\lambda_2})(1 - tq^{\nu_2-\lambda_1})}{q^{\nu_2-\lambda_1}2\xi^2(1 - q^{\nu_2+1})} \rho^\nu_\lambda. \] (5.10b)

As the initial value for the recurrence relations we can use \( \rho^\lambda_\lambda \) which is extracted directly from the expansion (4.25):

\[ \rho^\lambda_\lambda = (-1)^{\lambda_2} q^{-\frac{1}{2}\lambda_2(\lambda_2-1)} t^{-\lambda_2} \xi^{-\lambda_2}. \] (5.11)

It is easy to see then that the recurrence relations (5.10) together with the initial condition (5.11) possess the unique solution (5.7b) for \( \nu < \lambda \). Note that the solution is compatible with the triangularity condition \( \nu < \lambda \) since from (5.10) it follows that \( \rho^\nu_\lambda = 0 \) for \( \nu_1 < \lambda_1 \) or \( \nu_2 > \lambda_2 \).

Similarly, for the \( R \)-coefficients we obtain from \( r_\lambda = \sum_{\nu < \lambda} R^\nu_\lambda P_\nu \) the equations

\[ (a_j^\lambda - h_{\lambda j}) R^\nu_\lambda + b_j^\lambda R^\nu_{\lambda_1+1,\lambda_2} + c_j^\lambda R^\nu_{\lambda_1,\lambda_2-1} = 0 \] (5.12)

which give rise to the recurrence relations

\[ R^\nu_{\lambda_1+1,\lambda_2} = \frac{(1 - q^{\nu_1-\lambda_1})(1 - tq^{\nu_2-\nu_1})}{(1 - q^{\lambda_2})(1 - tq^{\lambda_2})} R^\nu_\lambda, \] (5.13a)

\[ R^\nu_{\lambda_1,\lambda_2-1} = \frac{(1 - q^{\nu_2-\lambda_2})(1 - tq^{\nu_2-\nu_1})}{q^{2\lambda_2-\lambda_2^2}2\xi^2(1 - q^{\lambda_2+1})} R^\nu_\lambda \] (5.13b)

which, in turn, being combined with the initial condition

\[ R^\lambda_\lambda = (-1)^{\lambda_2} q^{-\frac{1}{2}\lambda_2(\lambda_2-1)} t^{\lambda_2} \xi^{\lambda_2} \] (5.14)

produce the formula (5.8b).

The easiest way to obtain the corresponding formulas for \( \pi^\nu_\lambda \) and \( Q^\nu_\lambda \) is to use the involution \( U_\xi \) (4.16). Noting that

\[ U_\xi P_\lambda = t^{\lambda_2} \xi^{2|\lambda|} P_\lambda \] (5.15)

and using the equality \( U_\xi r_\nu = t^{2\nu_1} \xi^{4\nu_1} r_\nu \) (4.17) one obtains the formulas

\[ \pi^\nu_\lambda = (t\xi^2)^{|\lambda| - 2\nu_1} \rho^\nu_\lambda \] (5.16)
and

\[ Q^\nu_\lambda = (t\xi^2)^{2\lambda_1-|\nu|} R^\nu_\lambda \]  \hspace{1cm} (5.17)

which lead directly to the expressions (5.6a) and (5.8a).

**Corollary.** The matrices \( \rho^\nu_\lambda \) and \( R^\nu_\lambda \), resp. \( \pi^\nu_\lambda \) and \( Q^\nu_\lambda \), being mutually inverse, we obtain the following algebraic identities:

\[
\sum_{\mu<\nu<\lambda} R^\nu_\mu \rho^\mu_\nu = 0, \quad \sum_{\mu<\nu<\lambda} \rho^\nu_\mu R^\mu_\nu = 0, \quad \sum_{\mu<\nu<\lambda} Q^\nu_\mu \pi^\mu_\nu = 0, \quad \sum_{\mu<\nu<\lambda} \pi^\nu_\mu Q^\mu_\nu = 0,
\]

if \( \mu<\lambda \) and \( \mu\neq \lambda \), otherwise \( R^\lambda_\mu \rho^\mu_\lambda = 1, \ Q^\lambda_\mu \pi^\mu_\lambda = 1 \).

The identities can also be verified directly with the help of the \( q \)-Vandermonde formula (20, formula (1.5.3)):

\[ 2\phi_1 \left[ \frac{q^{-n}, b}{c}; q, q \right] = \frac{(c/b; q)_n}{(c; q)_n} b^n. \]

Using the relation (1.15) together with (1.14) it is easy to transform the expansions (5.3) and (5.7) into, respectively,

\[ F_\lambda = \sum_{\nu<\lambda} \tilde{\pi}^\nu_\lambda \tilde{p}_\nu = \sum_{\nu<\lambda} \tilde{\rho}^\nu_\lambda \tilde{r}_\nu \]  \hspace{1cm} (5.18)

and

\[ \tilde{p}_\lambda = \sum_{\nu<\lambda} \tilde{Q}^\nu_\lambda F_\nu, \quad \tilde{r}_\lambda = \sum_{\nu<\lambda} \tilde{R}^\nu_\lambda F_\nu \]  \hspace{1cm} (5.19)

where

\[ \tilde{\pi}^\nu_\lambda = \pi^\nu_\lambda \mu^\nu_\mu, \quad \tilde{\rho}^\nu_\lambda = \rho^\nu_\lambda \mu^\nu_\mu \]  \hspace{1cm} (5.20)

and

\[ \tilde{Q}^\nu_\lambda = Q^\nu_\lambda (\mu^\nu_\lambda)^{-1}, \quad \tilde{R}^\nu_\lambda = R^\nu_\lambda (\mu^\nu_\lambda)^{-1} \]  \hspace{1cm} (5.21)

(for the definition of \( \mu^\nu_\lambda \) and \( \mu^\nu_\mu \) see (1.13)).

The second expansion in (5.18) can also be found in the book [2]. Indeed, consider the formula (i) in the Exercise 8.1 from [20], i.e. the expansion of the product of two little \( q \)-Jacobi polynomials in the form

\[ p_n(x_1; a, b; q) p_n(x_2; a, b; q) = (-aq)^n q^{\binom{n}{2}} \frac{(aq; q)_n}{(a; q)_n} \sum_{m=0}^{n} \frac{(q^{-n}, abq^{-1-m}, x_1^{-1}, x_2^{-1}; q)_m}{(q, bq; q)_m} \times \]

\[ \times \left( \frac{x_1 x_2 q^{1-m}}{a} \right)^m \sum_{k=0}^{m} \frac{(q^{-m}, b^{-1} q^{-m}; q)_k}{(q, aq, x_1 q^{1-m}, x_2 q^{1-m}; q)_k} \frac{(ab x_1 x_2)^k q^{k^2+2k}}{k!}. \]

If we substitute now \( n = \lambda_2 \), \( x = t^{-2} y \), \( a = t^{-1} q^{-\lambda_2} \), \( b = t^2 q^{-1} \) and use the formula (2.27) for the separated polynomials \( f_\lambda(y) \) then we arrive to the expansion (5.18) of the polynomial \( F_\lambda \) in terms of the polynomials \( \tilde{r}_\nu \).
6. Concluding remarks

We have demonstrated that even in the simplest case of the $A_1$ type Macdonald polynomials the method of separation of variables is capable to provide a new interpretation of the product formula (2.8) for classical orthogonal polynomials of one variable (Theorem 1) and even to produce a new result: the integral relation (4.8) for the same polynomials (Theorem 2).

The SoV approach proposed in [39, 28, 29] and in the present paper seems to be quite general and could provide thus a useful tool for studying the integral relations between various one- and multidimensional special functions. The most interesting next application of the method would be the case of the $A_n$ Macdonald polynomials, $n > 2$. The work in this direction is in progress. We hope also to apply the same method to the pair $(A_2, A_1)$ in the cases of periodic Toda lattice and elliptic Calogero-Moser and Ruijsenaars models elsewhere, thereby getting some (new) integral relations for the Mathieu and Lamé types of functions and for their multivariable (and $q$-) analogs.

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Appendix A. Classical Ruijsenaars system

An integrable Hamiltonian system with $n$ degrees of freedom is determined by a $2n$-dimensional symplectic manifold (phase space) and $n$ independent functions (Hamiltonians) $H_j$ commuting with respect to the Poisson bracket

$$\{H_j, H_k\} = 0, \quad j, k = 1, \ldots, n. \quad (A.1)$$

For our purposes it is more convenient to use Weil canonical variables $(T_{x_j}, x_j)$

$$\{T_{x_j}, T_{x_k}\} = \{x_j, x_k\} = 0, \quad \{T_{x_j}, x_k\} = -iT_{x_j} x_k \delta_{jk}, \quad j, k = 1, \ldots, n \quad (A.2)$$

rather than the traditional coordinates–momenta.

To find a SoV means then to find a canonical transformation $M : (x, T_x) \mapsto (y, T_y), \quad M : H_j^{(x)} \mapsto H_j^{(y)}$ such that there exist $n$ relations

$$\Phi(y_j, T_{y_j}; H_1^{(y)}, \ldots, H_n^{(y)}) = 0, \quad j = 1, \ldots, n, \quad (A.3)$$

separating the variables $y_j$. The most common way to describe a canonical transformation is the one in terms of its generating function $F(y|x)$. 

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Presently, no algorithm is known of constructing a SoV for any given integrable system. Nevertheless, there exists a fairly effective practical recipe based on the classical inverse scattering method. A detailed description of the procedure with many examples can be found in the review paper [39]. Here we describe very briefly its main steps and apply it afterwards to the classical trigonometric 2-particle Ruijsenaars system.

A Lax matrix for a given integrable system is a matrix \( L(u) \) dependent on a “spectral parameter” \( u \in \mathbb{C} \) such that its characteristic polynomial obeys two conditions

\[(i) \quad \text{Poisson involutivity:} \quad \{ \det(L(u) - v \cdot 1), \det(L(\tilde{u}) - \tilde{v} \cdot 1) \} = 0, \quad \forall u, \tilde{u}, v, \tilde{v} \in \mathbb{C}; \]

\[(ii) \quad \det(L(u) - v \cdot 1) \quad \text{generates all integrals of motion} \quad H_i. \]

A Baker-Akhiezer (BA) function is the eigenvector

\[ L(u) f(u) = v(u) f(u) \quad \text{(A.4)} \]

of the Lax matrix \( L(u) \), provided that a normalization of the eigenvectors \( f(u) \) is fixed

\[
\sum_{i=1}^{n} \alpha_i(u) f_i(u) = 1, \quad (f(u) \equiv (f_1(u), \ldots, f_n(u))^t).
\]

(A.5)

Supposing \( L(u) \) to be a rational function in \( u \), the pair \((u, v)\) can be thought of as a point of the algebraic spectral curve

\[ \det(L(u) - v \cdot 1) = 0. \]

(A.6)

The BA function \( f(u) \) is then a meromorphic function on the spectral curve.

The recipe for finding an SoV is simple:

\[ \text{The separation variables} \quad y_j \quad \text{are poles of the Baker-Akhiezer function, provided it is properly normalized. The corresponding eigenvalues} \quad T_{y_j} \quad \text{of} \quad L(y_j), \quad \text{or some functions of them, serve as the canonically conjugated variables.} \]

It is easy to see that the pairs \((y_j, T_{y_j})\) thus defined satisfy the separation equations (A.3) for \( \Phi_j \equiv \det(L(y_j) - T_{y_j} \cdot 1) \). The canonicity of the variables \((y_j, T_{y_j})\) should be verified independently. No general recipe is known how to guess the proper (that is producing canonical variables) normalization for the BA function. In many cases the simplest standard normalization \( \vec{\alpha} = \vec{\alpha}_0 \equiv (0, \ldots, 0, 1) \) works. In other cases the vector \( \vec{\alpha} \) may depend on the spectral parameter \( u \) and the dynamical variables \((T_{x_j}, x_j)\). We shall refer to such normalization as a dynamical one.

To be more concrete, let \( f_i^{(j)} = \text{res}_{u=y_j} f_i(u) \) and \( T_{y_j} = v(y_j) \). Then from (A.4)–(A.3) we have the overdetermined system of \( n+1 \) linear homogeneous equations for \( n \) components \( f_i^{(j)} \) of the vector \( f^{(j)} \):

\[
\left\{ \begin{array}{l}
L(y_j) f^{(j)} = T_{y_j} f^{(j)}; \\
\sum_{i=1}^{n} \alpha_i(y_j) f_i^{(j)} = 0.
\end{array} \right. \quad \text{(A.7)}
\]
The pair \((u,v) \equiv (y_j,T_{y_j})\) is thus determined from the condition
\[
\text{rank} \left( \frac{\alpha(u)}{L(u) - v \cdot 1} \right) = n - 1
\]  
(A.8)
where \(\alpha\) is understood as a row-vector. Finally, the condition (A.8) can be rewritten as the following vector equation:
\[
\bar{\alpha} \cdot (L(u) - v \cdot 1)^{\wedge} = 0,
\]  
(A.9)
where the wedge denotes the matrix of cofactors. Eliminating \(v\) from (A.9) we get an algebraic equation for \(y_j\)
\[
B(u) = \det \left( \bar{\alpha} \cdot L(u) \right) = 0.
\]  
(A.10)
Also, from equations (A.9) we can get formulas for \(v\) in the form
\[
v = A(u)
\]  
with \(A(u)\) being some rational functions of the entries of \(L(u)\).

To validate the choice of normalization \(\alpha(u)\) it remains to verify (somehow) the canonicity of brackets between the whole set of separation variables, namely: between zeros \(y_j\) of \(B(u)\) and their conjugated variables \(T_{y_j} \equiv v(y_j) = A(y_j)\). To do this final calculation one needs information about Poisson brackets between entries of the Lax matrix \(L(u)\).

Now we apply the above scheme to the 2-particle Ruijsenaars system. But, first, let us recall the definition of the \(n\)-particle system.

The \(n\)-particle (\(A_{n-1}\) type) trigonometric Ruijsenaars model [37] is formulated in terms of the Weyl canonical system of coordinates \((T_{x_j},x_j), |x_j| = 1, T_{x_j} \in \mathbb{R}\) \((j = 1, 2, \ldots, n)\) with the Poisson brackets (A.2). The mutually Poisson commuting integrals of motion \(H_i\) are defined as follows
\[
H_i = \sum_{J \subset \{1, \ldots, n\}} \left( \prod_{j \in J} v_{jk} \right) \left( \prod_{j \in J} T_{x_j} \right), \quad i = 1, \ldots, n,
\]  
(A.11)
where
\[
v_{jk} = \frac{t^{\frac{1}{2}}x_j - t^{\frac{1}{2}}x_k}{x_j - x_k}, \quad 0 \leq t \leq 1,
\]  
(A.12)
t being a parameter of the model. The \(n \times n\) Lax matrix (or the \(L\)-operator) is given by the formula
\[
L(u) = D(u)E(u)
\]  
(A.13)
where
\[
D_{jk} = \frac{(1 - t)(t^n - u)}{2t^{\frac{1}{2}}} \frac{\left( \prod_{i \neq j} v_{ji} \right) T_{x_j} \delta_{jk}}{t^n - u - t x_j + x_k}, \quad E_{jk} = \frac{t^n + u}{t^n - u} \frac{t x_j + x_k}{t x_j - x_k}.
\]  
(A.14)
The characteristic polynomial of the matrix $L(u)$ (A.13) generates the integrals (A.11)

$$(-1)^n t^{n(n-1)/2} (t^n - u)(1 - u)^n \det(z \cdot \mathbb{1} - L(u))$$

$$= \sum_{k=0}^{n} (-1)^k t^{n-k} (t^k - u)(1 - u)^k (t^n - u)^{n-k} H_{n-k} z^k$$  \hspace{1cm} (A.15)

where it is assumed $H_0 \equiv 1$.

In [29] we considered the case $n = 3$ and constructed a classical (as well as quantum) SoV for it, giving explicitly the generating function of the separating canonical transform in terms of Euler’s dilogarithm function [25]

$$\text{Li}_2(z) := - \int_0^z \frac{dt}{t} \ln(1 - t) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}.$$  \hspace{1cm} (A.16)

In the present paper we are concerned with the 2-particle (or the $A_1$) case. Putting $n = 2$ we have, respectively:

$$D(u) = \frac{(1 - t)(t^2 - u)}{2t^2(1 - u)} \begin{pmatrix} v_{12} T_{x_1} & 0 \\ 0 & v_{21} T_{x_2} \end{pmatrix}, \quad E_{jk} = \frac{t^2 + u}{t^2 - u} - \frac{tx_j + x_k}{tx_j - x_k}.$$  \hspace{1cm} (A.17)

The characteristic polynomial takes the form

$$t(1 - u) \det(z \cdot \mathbb{1} - L(u)) = t(1 - u) z^2 - t^2(t - u) H_1 z + (t^2 - u) H_2$$  \hspace{1cm} (A.18)

where the integrals of motion $H_j$ are given by the formulas

$$H_1 = \frac{t \frac{3}{2} x_1 - t \frac{3}{2} x_2}{x_1 - x_2} T_{x_1} + \frac{t \frac{3}{2} x_2 - t \frac{3}{2} x_1}{x_2 - x_1} T_{x_2}, \quad H_2 = T_{x_1} T_{x_2}.$$  \hspace{1cm} (A.19)

The standard choice of the normalization $\vec{\alpha} = \vec{\alpha}_0 \equiv (0, 1)$ leads to a trivial SoV in the coordinates $x_\pm = (x_1 x_2^{\pm1})^{1/2}$ (separating the center-of-mass motion). There exists, however, another, more complicated SoV. Let us choose the following normalization $\vec{\alpha}$ of the Baker-Akhiezer function

$$\vec{\alpha} \equiv (\alpha_1, \alpha_2) = \begin{pmatrix} \frac{t^2 + u}{t^2 - u} - \frac{t \xi + x_1}{t \xi - x_1} \\ \frac{t^2 + u}{t^2 - u} - \frac{t \xi + x_2}{t \xi - x_2} \end{pmatrix},$$  \hspace{1cm} (A.20)

where $\xi \in \mathbb{C}$ is a free parameter. Notice that the chosen normalization is dynamical and dependent on the spectral parameter $u$.

Now introduce the functions $A_j(u)$

$$A_1(u) = L_{11}(u) - \frac{\alpha_1}{\alpha_2} L_{12}(u), \quad A_2(u) = L_{22}(u) - \frac{\alpha_2}{\alpha_1} L_{21}(u),$$  \hspace{1cm} (A.21)

$$A_j(u) = T_{x_j} a_j(u),$$  \hspace{1cm} (A.22)

$$a_j(u) = \frac{(t^2 - u)(\xi - x_j)(\xi u - x_{3-j})}{(1 - u)(t \xi - x_j)(\xi u - t x_{3-j})}, \quad j = 1, 2.$$  \hspace{1cm} (A.23)

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The separation variables \((T_{y_1}, y_j), \ j = 1, 2\), are defined then by the equations
\[
T_{y_j} = A_1(y_j) = A_2(y_j), \quad j = 1, 2;
\]  
(A.24)
which can be put, alternatively, as four equations for four variables \(y_1, y_2, T_{y_1}, T_{y_2}\) in the form
\[
T_{y_j} = T_{x_k} a_k(y_j), \quad j, k = 1, 2.
\]  
(A.25)
From the invariance of \(a_1(u)/a_2(u)\) with respect to the change \(u \to tx_1x_2/u\xi^2\) we can deduce the relation
\[
y_1y_2\xi^2 = tx_1x_2.
\]  
(A.26)
Having checked by straightforward calculations the relation
\[
(1 - y) a_1(y) a_2(y) - t^2(t - y) [v_{12}a_2(y) + v_{21}a_1(y)] + (t^2 - y) = 0
\]  
(A.27)
for \(a_j\)'s, we obtain at once the separation equation for each pair of \((T_y, y)\) variables
\[
(1 - y) T_y^2 - t^2(t - y) H_1 T_y + (t^2 - y) H_2 = 0. 
\]  
(A.28)
The (Weyl type) canonical Poisson brackets \([A.2]\) for the variables \(T_{y_1}, y_j\) are easily checked on a computer. The other way to establish that the separated transformation from \((T_{x_1}, x_j)\) to \((T_{y_1}, y_j)\) is canonical is to give an explicit generating function for it which has the simplest form in terms of new (canonical) \(\pm\)-variables:
\[
x_\pm = x_1x_2^{\pm \frac{1}{2}}, \quad T^\pm_x = T_{x_1} T_{x_2}^{\pm 1},
\]  
(A.29)
\[
y_\pm = y_1y_2^{\pm \frac{1}{2}}, \quad T^\pm_y = T_{y_1} T_{y_2}^{\pm 1},
\]  
(A.30)
\[
x_\mp = t - \frac{1}{2}\xi y_\mp.
\]  
(A.31)
The generating function \(F(T^+_{y_1}, y_- | x_+, x_-)\) has the form
\[
F(T^+_{y_1}, y_- | x_+, x_-) = \ln T^+_{y_1} \ln(\xi^{-1} t^{\frac{1}{2}} x_+) + \tilde{F}(y_+, y_- | x_+, x_-),
\]  
(A.32)
\[
\tilde{F} = i \left(\mathcal{L}(t^{\frac{1}{2}}; y_-, x_-) + \mathcal{L}(t^{\frac{1}{2}}; t^{-\frac{1}{2}}\xi^{-1} x_+, x_-) - \mathcal{L}(t; t^{-\frac{1}{2}}\xi^{-1} x_+, y_-)\right) - i \text{Li}_2(x^2) - i \text{Li}_2(x^{-2}),
\]  
(A.33)
where we have introduced the notation
\[
\mathcal{L}(\nu; x, y) := \text{Li}_2(\nu x y) + \text{Li}_2(\nu x y^{-1}) + \text{Li}_2(\nu x y^{-1}) + \text{Li}_2(\nu x^{-1} y^{-1}).
\]  
(A.34)
Function \(\tilde{F}(y_+, y_- | x_+, x_-)\) \([A.33]\) satisfies the equations
\[
x_+ \partial_{x_+} \tilde{F} = i \ln \frac{T^+_x}{T^+_y}, \quad x_- \partial_{x_-} \tilde{F} = i \ln T^-_x, \quad y_- \partial_{y_-} \tilde{F} = -i \ln Y^-_y, \quad y_+ \partial_{y_+} \tilde{F} = 0.
\]  
(A.35)
There is striking similarity between the classical SoV performed above for the case \( n = 2 \) and the one for \( n = 3 \) constructed in [29]. Indeed, we recall that there we had the generating function \( F(T^+_y, y_-, x_+, x_-) \) ((2.31) of [29]) of the form

\[
F := i \ln T^+_y \ln(t^2 x_+) + \tilde{F}
\]

and an extra condition which replaced (A.31) was as follows:

\[
x_+ = t^{-\frac{3}{2}} y_+,
\]

while the definition of the \( x_\pm \)-variables was a bit different:

\[
x_+ = x_1^{1/2} x_2^{1/2} x_3^{-1}, \quad x_- = x_1^{1/2} x_2^{-1/2}.
\]

Actually, there is a simple and an elegant explanation of this similarity; moreover, in the general case of \( n \) degrees of freedom we could say that a SoV for the \( A_{n-1} \) problem with the standard normalization vector \( \vec{\alpha}_0 \equiv (0, 0, \ldots, 0, 1) \) implies a SoV for the \( A_{n-2} \) problem with the non-standard normalization vector \( \tilde{\alpha} = (\alpha_1, \ldots, \alpha_{n-1}) \) (cf. (A.20) for \( n = 3 \)),

\[
\tilde{\alpha} = \left( \frac{t^{n-1} + u - t \xi + x_1}{t^{n-1} - u + t \xi - x_1}, \ldots, \frac{t^{n-1} + u - t \xi + x_{n-1}}{t^{n-1} - u + t \xi - x_{n-1}} \right),
\]

if we choose \( \xi = x_n \). Let us demonstrate this explicitly.

If we remove the last (\( n \)th) row and the last column from the Lax matrix \( L^{(n)}(u) \) then, as one can check easily, we get \((n-1) \times (n-1)\) Lax matrix \( \tilde{L}(u) \) for an integrable system of \( n-1 \) degrees of freedom which is equivalent through a simple canonical transformation to the standard \((n-1)\)-particle Ruijsenaars model. This 1-degree-of-freedom-less system with the Lax matrix \( \tilde{L}^{(n-1)}(u) \) directly inherits the non-standard SoV with the normalization (A.37) from the standard one (with the \( \tilde{\alpha}_0 \)) for the system with \( n \) degrees of freedom. Indeed, to see this, it is sufficient to note that separation variables \( (T_y, y) \) for both systems (one with \( L^{(n)}(u) \) and one with \( \tilde{L}(u) \)) are defined from the intersection of two spectral curves:

\[
\left\{ \begin{array}{l}
\det(L^{(n)}(y) - T_y \cdot \mathbb{1}) = 0, \\
\det(\tilde{L}(y) - T_y \cdot \mathbb{1}) = 0.
\end{array} \right.
\]

In other words, the condition of the standard SoV for the first problem,

\[
\text{rank} \left( L^{(n)}(y) - T_y \cdot \mathbb{1} \right) = n - 1,
\]

implies the following condition of SoV for the second problem:

\[
\text{rank} \left( \tilde{L}(y) - T_y \cdot \mathbb{1} \right) = n - 2.
\]
where 
\[
\tilde{\alpha}_1 = \left( t^n + u - \frac{tx_n + x_1}{t - u}, \ldots, t^n + u - \frac{tx_n + x_{n-1}}{t - u} \right).
\]

Applying this to our \((A_2, A_1)\) pair, we have proved in [23] the standard SoV for the
\(A_2\) problem \((L^{(3)}(u) = L(u), n = 3)\), hence, the same separation variables can work
also for the related \(\tilde{L}(u)\) problem which, in turn, is related to the initial 2-degrees-of-freedom \((A_1)\) problem (with the Lax matrix \(L^{(2)}(u) = L(u), n = 2\)) through the
following canonical transformation:

\[
y = t^{-1} \tilde{y}, \quad T_x = t^{\frac{3}{2}} v_3 \tilde{T}_x, \quad \xi = x_3, \quad T_y = t^{\frac{3}{2}} \frac{1 - \tilde{y}}{t^{\frac{3}{2}} - t^{-\frac{3}{2}} y} \tilde{T}_y. \quad (A.41)
\]

The analogous transformation connects the corresponding quantum \(A_2\) and \(A_1\) systems. Indeed, let us establish an equivalence of the separating kernel \(\mathcal{M}^{(2)}\) constructed in the Section 4 for the case of the \(A_1\) Macdonald polynomials and the \(\mathcal{M}^{(3)}\) from [23] for the 3-variable case. Let us write down both kernels: for the 2-variable case
\[
\mathcal{M}^{(2)} \equiv \mathcal{M}_{g, q}(r, y|x) = \frac{(1 - q) (q, q, x^2, x^{-2}; q)_{\infty} \mathcal{L}_q(t; y, r)}{2B_q(g, g) \mathcal{L}_q(t^{\frac{3}{2}}; x, y) \mathcal{L}_q(t^{\frac{3}{2}}; x, r)} \quad (A.42)
\]
and for the 3-variable case
\[
\mathcal{M}^{(3)} \equiv \mathcal{M}_{g, 2g}(\tilde{r}, y|x) = \frac{(1 - q) (q, q, x^2, x^{-2}; q)_{\infty} \mathcal{L}_q(t^{\frac{3}{2}}; y, \tilde{r})}{2B_q(g, 2g) \mathcal{L}_q(t^{\frac{3}{2}}; x, y) \mathcal{L}_q(t; x, \tilde{r})}. \quad (A.43)
\]

Substituting
\[
y = y_-, \quad x = x_-, \quad r = t^{-1} y_+ = t^{-\frac{1}{2}} \xi^{-1} x_+ \quad (A.44)
\]
and, respectively,
\[
y = \tilde{y}_-, \quad x = x_-, \quad \tilde{r} = t^{-\frac{3}{2}} \tilde{y}_+ = x_+ x_3^{-1}, \quad (A.45)
\]
and also
\[
B_q(a, b) = (1 - q)_{(q, q^{a+b}; q)_{\infty}} (q^a, q^b, q)_{\infty}, \quad (A.46)
\]
we get
\[
\mathcal{M}^{(2)} = \frac{\mathcal{L}_q(t; y_-, t^{-1} y_+)}{(t^{2}; q)_{\infty}} \frac{(q, t, x^2_-, x^{-2}_-; q)_{\infty}}{2 \mathcal{L}_q(t^{\frac{3}{2}}; x_-, y_-) \mathcal{L}_q(t^{\frac{3}{2}}; x_-, t^{-\frac{3}{2}} \xi^{-1} x_+)} \quad (A.47)
\]
\[
\mathcal{M}^{(3)} = \frac{\mathcal{L}_q(t^{\frac{3}{2}}; y_-, t^{-\frac{3}{2}} \tilde{y}_+)}{(t^3; q)_{\infty}} \frac{(q, t, x^2_-, x^{-2}_-; q)_{\infty}}{2 \mathcal{L}_q(t^{\frac{3}{2}}; x_-, y_-) \mathcal{L}_q(t; x_-, x_+ x_3^{-1})}. \quad (A.48)
\]
Identifying the variables
\[
\xi \equiv x_3, \quad \tilde{y}_j \equiv t y_j, \quad j = 1, 2, \quad (A.49)
\]
we obtain the equivalence of two operators:

\[ M^{(2)} = W_y^{-1} \circ M^{(3)} \circ W_x \]  \hspace{1cm} (A.50)

where

\[
W_x = \frac{(t; q)_{\infty}}{(t^2; q)_{\infty}} \frac{\mathcal{L}_q(t; x, x^{-1} x_+)}{\mathcal{L}_q(t^2; x, t^{-\frac{1}{2}} \xi^{-1} x_+)} = (t, tx_1 x^{-1}, tx_2 x^{-1}; q)_{\infty},
\]

\[
W_y = \frac{(t^2; q)_{\infty}}{(t^3; q)_{\infty}} \frac{\mathcal{L}_q(t^3; y, t^{-\frac{3}{2}} \bar{y}^+)}{\mathcal{L}_q(t; y, t^{-2} \bar{y}^+)}, (t^2, t^{-1} \bar{y}_1, t^{-1} \bar{y}_2; q)_{\infty}.
\]

In particular, the relations between \( q \)-shift operators are as follows (cf. (A.41)):

\[
W_x \circ T_{q,x} \circ W_x^{-1} = \frac{1 - tx_j x_3^{-1}}{1 - x_j x_3^{-1}} \bar{T}_{q,x} = t^j v_j \bar{T}_{q,x},
\]

\[
W_y \circ T_{q,y} \circ W_y^{-1} = \frac{1 - \bar{y}_j}{1 - t^{-1} \bar{y}_j} \bar{T}_{q,y} = t^j \bar{T}_{q,y}.
\]

(A.53)  \hspace{1cm} (A.54)
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