DISCONJUGACY CONDITIONS AND SPECTRUM STRUCTURE OF CLAMPED BEAM EQUATIONS WITH TWO PARAMETERS

YANQIONG LU* AND RUYUN MA

Department of Mathematics, Northwest Normal University,
Lanzhou 730070, China

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ABSTRACT. In this work, we apply the ‘disconjugacy theory’ and Elias’s spectrum theory to study the disconjugacy $u^{(4)} + \beta u'' - \alpha u = 0$ with two parameters $\alpha, \beta \in \mathbb{R}$ and the spectrum structure of the linear operator $u^{(4)} + \beta u'' - \alpha u$ coupled with the clamped beam conditions $u(0) = u'(0) = u(1) = u'(1) = 0$. As the application of our results, we obtain the global structure of nodal solutions of the corresponding nonlinear analogue based on the bifurcation theory.

1. Introduction. Fourth-order equations appear as model equations of elastic beams, see Gupta[9]. The deformation of an elastic beam whose both ends clamped may be described by

\begin{align}
\frac{d^4}{dt^4} u(t) + \beta \frac{d^2}{dt^2} u(t) - \alpha u &= f(t, u(t)), \quad t \in (0, 1), \\
u(0) &= u'(0) = u(1) = u'(1) = 0.
\end{align}

Existence of solutions and positive solutions of the boundary value problem (1)-(2), or its generalizations, has been studied by several authors, see for examples, Agarwal and Chow[1], Cabada and Enguiça [3], Ma et al.[13], Rynne [17], Webb et al. [19] and Xu and Han [21].

Recently, Cabada and Enguiça[3] made an exhaustive study of the fourth order linear operator $u^{(4)} + Mu$ coupled with the clamped beam conditions (2). They obtained the exact values on the real parameter $M$ for which this operator satisfies an anti-maximum principle. Such a property is equivalent to the fact that the related Green’s function is nonnegative in $[0, 1] \times [0, 1]$. When $M < 0$ they obtained the best estimate by means of the spectral theory and for $M > 0$ they attained the optimal value by studying the oscillation properties of the solutions of the homogeneous equation $u^{(4)} + Mu = 0$. By using the method of lower and upper solutions they deduced the existence of solutions for nonlinear problems coupled with (2). More precisely, they proved the following

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* Corresponding author.
Theorem 1.1 ([3], Theorem 2.3). Given $h \in C[0,1]$ a nontrivial function such that $h(t) \geq 0$ for all $t \in [0,1]$, the boundary value problem
\[ u^{(4)}(t) - m^4u(t) = h(t), \quad t \in (0,1), \quad u(0) = u'(0) = u(1) = u'(1) = 0 \tag{3} \]
has a unique positive solution $u$ if $0 \leq m < m_1$ and no positive solution if $m > m_1$, where $m_1 \approx 4.73004$ is the first positive solution of the equation
\[ \cos(m) \cosh(m) = 1. \tag{4} \]

Theorem 1.2 ([3], Theorem 3.7). If $0 < m \leq m_0$, then
\[ u^{(4)}(t) + m^4u(t) \geq 0, \quad t \in (0,1), \quad u(0) = u'(0) = u(1) = u'(1) = 0 \tag{5} \]
implies $u \geq 0$, here $m_0 \approx 5.553$ is the smallest positive solution of the equation
\[ \tanh \frac{m}{\sqrt{2}} = \tan \frac{m}{\sqrt{2}}. \tag{6} \]

Ma et al. [13] also obtained the exact values on the real parameter $M \in (-m_1^4, m_1^4)$ for which confirm the positivity of linear equation $u^{(4)} + Mu = 0$ by using the 'disconjugacy theory' [6] and the spectrum theory Elias [7, 8] and Rynne [18]. In addition, they obtained the spectrum structure of the linear operator $u^{(4)} + Mu$ coupled with the clamped beam conditions (2).

A natural question is whether or not exist the optimal condition of the positivity and spectrum structure for the general equation $u^{(4)} + \beta u'' - \alpha u = 0$. Note that [4] give the description of the interval of parameters for the general linear nth-order ordinary differential equation is disconjugate on a given bounded interval $I$. For example, let
\[ T_4[M]u(t) = u^{(4)} + a_1(t)u'' + a_2(t)u'' + (a_n(t) + M)u = 0, \quad t \in [0,1]. \]
If there exists $M \in \mathbb{R}$ be such that $T_4[M]u(t) = 0$ is disconjugate on $[0,1]$, then $T_4[M]u(t) = 0$ is disconjugate on $[0,1]$ if and only if $M \in (M - \lambda_1, M + \lambda_2)$, where $\lambda_1 > 0$ is the least positive eigenvalues of $T_4[M]u(t) = 0, u(0) = u'(0) = u(1) = u'(1) = 0$ and $\lambda_2 < 0$ is the maximum of the biggest negative eigenvalues of $T_4[M]u(t) = 0$ with $u(0) = u'(0) = u''(0) = u(1) = 0$ and $u(0) = u'(0) = u''(0) = u(1) = 0$. This is an important result of disconjugacy theory, however, the choice of $M$ limits the widespread application. It is the purpose of this paper to apply the 'disconjugacy theory' [4, 6] and the spectrum theory Elias [7, 8] and Rynne [18] to study the choice of $\alpha$ to provide the disconjugacy of the equation $u^{(4)} + \beta u'' - \alpha u = 0$ and the spectrum structure of the linear operator $u^{(4)} + \beta u'' - \alpha u$ coupled with the clamped beam conditions (2). As the applications of our results, we show the global structure of nodal solutions of the corresponding nonlinear problem (1), (2). For other results on the existence of solutions for fourth-order boundary value problems, see [2, 10, 12, 14, 15, 20] and the reference therein.

The rest of the paper is arranged as follows: in Section 2, we state some preliminary results about disconjugacy and the spectrum of some general linear operators. Section 3 is devoted to showing that the sufficient conditions and necessary and sufficient conditions for the equation $u^{(4)} + \beta u'' - \alpha u = 0$ is disconjugate on $[0,1]$. And we also obtain the spectrum structure of the corresponding fourth-order linear operator. In Section 4, we apply our results on linear problems to show the global structure of nodal solutions of the nonlinear problem (1), (2).
2. Preliminaries.

**Definition 2.1** ([6], Page 1). Let \( p_k \in C[a, b] \) for \( k = 1, \ldots, n \). A linear differential equation of order \( n \)

\[
Ly \equiv y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0 \tag{7}
\]

is said to be disconjugate on an interval \([a, b]\) if every nontrivial solution has less than \( n \) zeros on \([a, b]\), multiple zeros being accounted according to their multiplicity.

**Definition 2.2** ([6]). The functions \( y_1, \ldots, y_n \in C^n[a, b] \) are said to form a Markov system if the \( n \) Wronskians

\[
W_k := W[y_1, \ldots, y_k] = \begin{vmatrix}
  y_1 & \cdots & y_k \\
  \cdots & \cdots & \cdots \\
  y_1^{(k-1)} & \cdots & y_k^{(k-1)}
\end{vmatrix}, \quad (k = 1, \ldots, n) \tag{8}
\]

are positive throughout on \([a, b]\).

**Lemma 2.3** ([6], Theorem 3 in Page 94). The equation (7) has a Markov fundamental system of solutions on \([a, b]\) if and only if it is disconjugate on \([a, b]\).

**Lemma 2.4** ([6], Theorem 2 in Page 91). The equation (7) has a Markov fundamental system of solutions if and only if \( L \) has a representation

\[
Ly \equiv v_1v_2 \cdots v_n D^{-1}D \cdots D^{-1}D^{-1}v_1 y,
\]

where \( D = d/dt \), and

\[
1 = W_0, \; v_1 = W_1, \; v_k = W_{k}W_{k-2}/W_{k-1}^2, \quad k = 2, \ldots, n.
\]

Suppose the equation (7) is disconjugate on \([a, b]\). Let \( f \) be a continuous function on \([a, b]\). Let \( k \) be a positive integer, the \((k, n-k)\) two-point boundary value problem

\[
\begin{align*}
Ly = f(t), & \quad t \in (a, b), \\
\{y^{(i)}(a) = 0, & \quad i = 0, 1, \ldots, k-1, \\
y^{(j)}(b) = 0, & \quad j = 0, 1, \ldots, n-k-1 \}
\end{align*} \tag{10}
\]

has a unique solution \( y \), since the corresponding homogeneous problem has no nontrivial solution. The solution can be represented in the form

\[
y(t) = \int_a^b G(t, s)f(s)ds,
\]

where the Green’s function \( G(t, s) \) is defined by the following properties

(1) As a function of \( t \), \( G(t, s) \) is a solution of (7) on \([a, s]\) and on \((s, b]\) and satisfies the \( n \) boundary conditions (10);

(2) As a function of \( t \), \( G(t, s) \) and its first \( n-2 \) derivatives are continuous at \( t = s \), while

\[
G^{(n-1)}(s + 0, s) - G^{(n-1)}(s - 0, s) = 1.
\]

**Lemma 2.5** ([6], Theorem 11 in Page 106; [8], Theorem 0.13; [5], Theorem 3.1). Suppose the equation (7) is disconjugate on \([a, b]\). Then the Green’s function of \((k, n-k)\) two-point boundary value problem satisfies

\[
(-1)^{n-k}G(t, s) > 0, \quad a < s < b, \quad a < t < b.
\]
Note that Cabada and Saavedra [5, Theorem 3.1] also obtain the sign of the Green’s function of \((k,n − k)\) two-point boundary value problem.

For \(2m\)th order \((m ≥ 1)\), self-adjoint, disconjugate differential operator, Rynne [18] showed a spectrum theorem which will be the foundation of this paper.

Suppose that for each \(i = 1,· · ·,2m\), we have a function \(ρ_i ∈ C^{2m−1}[a,b]\) with \(ρ_i > 0\) on \([a,b]\), and any \(u ∈ C^{2m}[a,b]\). Let

\[
L_0u = ρ_0u, \quad L_iu = ρ_i(L_{i−1}u)', \quad i = 1,· · ·,2m.
\]

The functions \(L_iu, \ i = 0,· · ·,2m\) will be called the \textit{quasi-derivatives} of \(u\). We consider the Banach spaces

\[
X = \{u ∈ C^{2m}[a,b] \mid u \text{ satisfies (10)}\} \quad \text{and} \quad Y = C^0[a,b].
\]

Denote

\[
E = \{u ∈ C^{2m−1}[a,b] \mid u \text{ satisfies (10)}\}
\]

with the norm \(||·||_{2m−1}\) which, for convenience, we will write as \(||·||\). Define an operator \(L : X → Y\)

\[
Lu = (-1)^m L_{2m}u, \quad u ∈ X.
\]

For each integer \(k ≥ 1\) and \(ν ∈ \{+,-\}\), let \(S_{κ,ν}\) denote the set of functions \(u ∈ E\) such that:

(1) \(u\) has only simple zeros in \((a,b)\) and no quasi-derivative of \(u\) is zero at \(a\) or \(b\), other than those specified in (10);

(2) \(u\) has exactly \(k − 1\) zeros in \((a,b)\);

(3) \(νu > 0\) in a deleted neighborhood of \(t = 0\).

\textbf{Lemma 2.6 ([18], Theorem 2.4).} Assume that \(Lu = 0\) is disconjugate on \([a,b]\), and the boundary conditions (10) are such that \(Lu\) is formally self-adjoint, that is

\[
⟨Lu,v⟩ = ⟨u,Lv⟩, \quad u,v ∈ X,
\]

where \(⟨·,·⟩\) denotes the standard \(L^2(a,b)\) inner product. Assume that

\(H0\) \(p ∈ C^0[a,b]\), and \(p ≥ 0\) on \([a,b]\), while \(p ≠ 0\) on any interval of \([a,b]\).

Then for each \(k ≥ 1\) and each \(ν ∈ \{+,-\}\), problem

\[
Lu = μp(t)u, \quad t ∈ (a,b),
\]

\[y^{(i)}(a) = 0, \quad i = 0,1,· · ·,k − 1;\]

\[y^{(j)}(b) = 0, \quad j = 0,1,· · ·,n − k − 1\]

has a unique solution \((μ_k,ψ_k) ∈ \mathbb{R}_+ × S_{κ,ν}\) with \(||ψ_k|| = 1\). In addition:

(1) \(σ(L,p) = \{μ_k : k ≥ 1\}\);

(2) If \(k' > k ≥ 1\) then \(μ_{k'} > μ_k > 0\);

(3) \(\lim_{k → ∞} μ_k = ∞\).

\textbf{Definition 2.7 ([6], Page 99).} Suppose that the equation (7) is not disconjugate on \([a,b]\). Let \(η(a)\) be the supremum of all \(c > a\) such that (7) is disconjugate on \([a,c]\). We call \(η(a)\) the first right conjugate of \(a\).

Let \(y_k(t,a)\) be the solution of (7) which satisfies at \(t = a\) for \(k = 1,· · ·,n − 1\) the initial conditions

\[y_k^{(n−k)}(a) = 1, \quad y_k^{(n−j)}(a) = 0 \quad (j = 1,· · ·,n; j ≠ k).\]
Denote
\[ W_k(t,a) = \begin{vmatrix} y_1(t,a) & \ldots & y_k(t,a) \\ \vdots & \ddots & \vdots \\ y_{1}^{(k-1)}(t,a) & \ldots & y_{k}^{(k-1)}(t,a) \end{vmatrix} \]
the Wronskian of \( y_1(t,a), \ldots, y_k(t,a) \).

**Definition 2.8** ([6], Page 99). Let \( w(a) \) be the least \( s > a \) in \([a, b] \), if one exists, at which one of the Wronskian \( W_i(s,a), \ldots, W_{n-1}(s,a) \) vanishes.

**Remark 1.** It may be note that \( W_n(s,a), \ s \in [a, b] \) never vanishes because the solutions \( y_1, \ldots, y_n \) are linearly independent.

**Lemma 2.9** ([6], Page 99). \( \eta(a) = w(a) \).

**Lemma 2.10** ([6], Page 94; [4], Theorem 1.2). Let \( L_n \) and \( L_m \) be two general linear differential operators of order \( n \) and \( m \) respectively. If the equations \( L_n u(t) = 0 \) and \( L_m u(t) = 0 \) are disconjugate on the interval \( I \), then the composite \( n + m \)th order equation \( L_n(L_m u(t)) = 0 \) is also disconjugate on \( I \).

Let \( T_n[M]y(t) \equiv Ly(t) + My(t) = 0 \), \( t \in [a, b] \) with \( M \in \mathbb{R} \) is a parameter, \( E_k = \{ y \in C^n[a, b] \mid u(a) = u'(a) = \cdots = u^{(k-1)}(a) = u(b) = \cdots = u^{(n-k-1)}(b) = 0 \} \), \( 1 \leq k \leq n - 1 \).

**Lemma 2.11** ([4], Theorem 2.1). Let \( \bar{M} \in \mathbb{R} \) and \( n \geq 2 \) be such that \( T_n[M]y(t) = 0 \) is a disconjugate equation on \( I \). Then, \( T_n[M]u(t) = 0 \) is a disconjugate equation on \([a, b] \) if, and only if, \( M \in (\bar{M} - \lambda_1, \bar{M} - \lambda_2) \), where
1. \( \lambda_1 = +\infty \) if \( n = 2 \) and for \( n > 2 \), \( \lambda_1 > 0 \) is the least positive eigenvalues on \( T_n[M] \) in \( E_k \), with \( n - k \) even.
2. \( \lambda_2 < 0 \) is the maximum of the biggest negative eigenvalues on \( T_n[M] \) in \( E_k \), with \( n - k \) odd.

3. **Disconjugacy for the equation** \( u^{(4)} + \beta u'' - \alpha u = 0 \). Let us consider the linear boundary value problem
\[
\begin{align*}
  u^{(4)}(t) + \beta u''(t) - \alpha u(t) &= 0, \quad t \in (0, 1), \\
  u(0) &= u'(0) = u(1) = u'(1) = 0.
\end{align*}
\]  
(11)

where \( \alpha, \beta \in \mathbb{R} \) are two parameters.

First, we consider the case \( \alpha = 0 \). In this case, (11) reduces to
\[
\begin{align*}
  u^{(4)}(t) + \beta u''(t) &= 0, \quad t \in (0, 1), \\
  u(0) &= u'(0) = u(1) = u'(1) = 0.
\end{align*}
\]  
(12)

The problem (12) has nontrivial solution if and only if \( \beta \) solves the equation
\[
\sqrt{\beta} \sin \sqrt{\beta} = 2(1 - \cos \sqrt{\beta}).
\]  
(13)

Moreover, the first positive solution of the equation (13) is \( \beta_1 = 4\pi^2 \).

**Theorem 3.1.** For every \( \beta \in (-\infty, \beta_1) \), the equation \( u^{(4)} + \beta u'' = 0 \) is disconjugate on \([0, 1] \). Moreover, the choice of \( \beta \) is optimal.

**Proof.** We divide the proof into two cases.

Case 1 \( \beta \in (0, \beta_1) \). In the virtue of Lemma 2.3, we only need to find a Markov fundamental system of solutions of \( u^{(4)} + \beta u'' = 0 \).
For each fixed $k \in \{1, 2, 3, 4\}$, let $u_k$ be the unique solution of the initial value problem
\[
\begin{align*}
  u^{(4)}(t) + \beta u''(t) &= 0, \quad t \in (0, 1), \\
  u^{(j)}(0) &= 0, \quad j = 0, 1, 2, 3, \quad j \neq 4 - k, \\
  u^{(4-k)}(0) &= (\sqrt{\beta})^{4-k}.
\end{align*}
\]
Then
\[
\begin{align*}
  u_1(t) &= \sqrt{\beta}t - \sin \sqrt{\beta}t, \quad t \in [0, 1], \\
  u_2(t) &= 1 - \cos \sqrt{\beta}t, \quad t \in [0, 1], \\
  u_3(t) &= \sqrt{\beta}t, \quad t \in [0, 1], \\
  u_4(t) &= 1, \quad t \in [0, 1].
\end{align*}
\]
Let $z_1(t) = u_1(t + \sigma)$, $z_2(t) = -u_2(t + \sigma)$, $z_3(t) = u_3(t + \sigma)$, $z_4(t) = -u_4(t + \sigma)$. Then
\[
\begin{align*}
  W_1[z_1](t) &= \sqrt{\beta}(t + \sigma) - \sin \sqrt{\beta}(t + \sigma) > 0; \\
  W_2[z_1, z_2](t) &= 2\sqrt{\beta}(1 - \cos \sqrt{\beta}(t + \sigma)) - \beta(t + \sigma) \sin \sqrt{\beta}(t + \sigma) > 0; \\
  W_3[z_1, z_2, z_3](t) &= (\sqrt{\beta})^3[\sqrt{\beta}(t + \sigma) - \sin \sqrt{\beta}(t + \sigma)] > 0; \\
  W_4[z_1, z_2, z_3, z_4](t) &= \beta^3 > 0.
\end{align*}
\]
Hence, it follows from (14) yields that the functions $\{z_1, z_2, z_3, z_4\}$ form a Markov fundamental system of solutions of $u^{(4)} + \beta u'' = 0$ on $[0, 1]$ if $\beta \in (0, \beta_1)$ and $\sigma \in (0, 1)$ is small enough. It follows from Lemma 2.3 that $u^{(4)} + \beta u'' = 0$ is disconjugate on $[0, 1]$ if $\beta \in (0, \beta_1)$.

Case 2 $\beta \in (-\infty, 0]$. In this case, since $u^{(4)} + \beta u'' = (u'' + \beta u)' = 0$, this together with the fact $u'' = 0$ is disconjugate on $[0, 1]$ and $u'' + \beta u = 0$ is disconjugate on $[0, 1]$ and Lemma 2.10 implies that $u^{(4)} + \beta u'' = 0$ is disconjugate on $[0, 1]$ with $\beta \leq 0$.

Therefore, $u^{(4)} + \beta u'' = 0$ is disconjugate on $[0, 1]$ if $\beta \in (-\infty, \beta_1)$. Moreover, from the Lemma 2.9, the result is optimal. \hfill \Box

Remark 2. Note that in the case $\beta > 0$, the disconjugacy of $u^{(4)} + \beta u'' = 0$ also follows from Lemma 2.10. However, $u'' + \beta u$ is disconjugate on $[0, 1]$ with $\beta \in (0, \pi^2)$, it yields that $u^{(4)} + \beta u'' = (u'' + \beta u)' = 0$ is also disconjugate on $[0, 1]$ with $\beta \in (0, \pi^2) \subset (0, 4\pi^2)$. Therefore, in the case of $\beta > 0$, we obtain the optimal range of $\beta$ by using Lemma 2.3 and Lemma 2.9.

Obviously, Theorem 3.1 together with Lemma 2.5 and Lemma 2.6 implies the following

Corollary 1. Let $\beta \in (-\infty, \beta_1)$ and (H0) hold. Then
(1) The Green’s function $G_1(t, s)$ of (12) satisfies
\[
G_1(t, s) > 0, \quad 0 < s < 1, \quad 0 < t < 1;
\]
(2) For each $k \geq 1$ and each $\nu \in \{+, -\}$, the problem
\[
u u^{(4)} + \beta u'' = \mu p(t)u, \quad u(0) = u'(0) = u(1) = u'(1) = 0
\]
has a unique solution $(\mu_k, \psi_k) \in \mathbb{R}_+ \times S_{k, \nu}$ with $||\psi_k|| = 1$. In addition:
(i) $\sigma(\mathcal{L}_1, p) = \{\mu_k : \ k \geq 1\}$. Here $\mathcal{L}_1u := u^{(4)} + \beta u''$, $u \in X_2$ and $X_2 := \{u \in C^4[0, 1] \mid u \text{ satisfies (2)}\}$;
(ii) If $k' > k \geq 1$ then $\mu_{k'} > \mu_k > 0$;
(iii) \( \lim_{k \to \infty} \mu_k = \infty. \)

Denote \( \mu_1 > 0 \) is the least positive eigenvalues on the operator \( u^{(4)} + \beta u'' - \alpha_0 u \) coupled with \( u(0) = u'(0) = u(1) = u'(1) = 0 \) and \( \lambda_1 := \max\{\rho_1, \rho_2\} < 0, \) here \( \rho_1 < 0 \) is the biggest negative eigenvalues on the operator \( u^{(4)} + \beta u'' - \alpha_0 u \) coupled with \( u(0) = u'(0) = u''(0) = u(1) = 0 \) and \( \rho_2 < 0 \) is the biggest negative eigenvalues on the operator \( u^{(4)} + \beta u'' - \alpha_0 u \) coupled with \( u(0) = u'(0) = u''(0) = u'(1) = 0. \)

It follows from Theorem 3.1 and Lemma 2.11 yields that a necessary and sufficient condition for disconjugacy of \( u^{(4)} + \beta u'' - \alpha u = 0 \) on \([0, 1]\).

**Corollary 2.** Let \( \beta \in (\infty, \beta_1). \) Then the equation \( u^{(4)} + \beta u'' - \alpha u = 0 \) is disconjugate on \([0, 1]\) if, and only if \( \alpha \in (\lambda_1, \mu_1). \)

Next, we consider the case \( \alpha > 0. \) In this case, by computing, the problem (11) has a nontrivial solution if and only if \((\alpha, \beta)\) solve the equation

\[
2\sqrt{\alpha}(\cos \mu \cosh \lambda - 1) + \beta(\sinh \lambda \sin \mu) = 0, \tag{15}
\]

where

\[
\lambda = \sqrt{\frac{\beta^2 + 4\alpha - \beta}{2}}, \quad \mu = \sqrt{\frac{\beta^2 + 4\alpha + \beta}{2}}. \tag{16}
\]

Especially, if \( \beta = 0, \) then (11) degenerate the boundary value problem

\[
u^{(4)} - \alpha u = 0, \quad u(0) = u'(0) = u(1) = u'(1) = 0.
\]

This problem has nontrivial solution if and only if

\[
\cos \sqrt{\alpha} \cosh \sqrt{\alpha} = 1
\]

and \( \sqrt{\alpha} \approx 4.73004 \) is the smallest positive solution of the above equation, see Cabada and Enguiça [3, Page 3114].

To this end, in the virtue of Lemma 2.3, we only need to find a Markov fundamental system of solutions of \( u^{(4)} + \beta u'' - \alpha u = 0 \) on \([0, 1]\).

Suppose equation \( u^{(4)} + \beta u'' - \alpha u = 0 \) has a general solution

\[
u(t) = C_1 \cos \mu t + C_2 \sin \mu t + C_3 \sinh \lambda t + C_4 \cosh \lambda t, \quad t \in [0, 1],
\]

where \( C_i, \ i = 1, 2, 3, 4 \) are arbitrary constants.

Let \( u_1 \) be the unique solution of the initial value problem

\[
u^{(4)}(t) + \beta \nu''(t) - \alpha \nu(t) = 0, \quad t \in (0, 1),
\]

\[
u(0) = \nu'(0) = \nu''(0) = 0, \quad \nu'''(0) = \sqrt{\alpha} \sqrt{\beta^2 + 4\alpha}.
\]

Then \( u_1(t) = \mu \sinh \lambda t - \lambda \sin \mu t, \ t \in [0, 1]. \)

Let \( u_2 \) be the unique solution of the initial value problem

\[
u^{(4)}(t) + \beta \nu''(t) - \alpha \nu(t) = 0, \quad t \in (0, 1),
\]

\[
u(0) = \nu'(0) = \nu'''(0) = 0, \quad \nu''(0) = \sqrt{\beta^2 + 4\alpha}.
\]

Then \( u_2(t) = \cosh \lambda t - \cos \mu t, \ t \in [0, 1]. \)

Let \( u_3 \) be the unique solution of the initial value problem

\[
u^{(4)}(t) + \beta \nu''(t) - \alpha \nu(t) = 0, \quad t \in (0, 1),
\]

\[
u(0) = \nu''(0) = \nu'''(0) = 0, \quad \nu'(0) = \sqrt{\alpha} \sqrt{\beta^2 + 4\alpha}.
\]

Then \( u_3(t) = \mu^3 \sinh \lambda t + \lambda^3 \sin \mu t, \ t \in [0, 1]. \)
Let $u_4$ be the unique solution of the initial value problem
\begin{align*}
  u^{(4)}(t) + \beta u''(t) - \alpha u(t) &= 0, \quad t \in (0, 1), \\
  u'(0) = u''(0) = u'''(0) = 0, \quad u(0) = \sqrt{\beta^2 - 4\alpha}.
\end{align*}
Then $u_4(t) = \mu^2 \cosh \lambda t - \lambda^2 \cos \mu t$, $t \in [0, 1]$.

Let
\begin{align*}
  z_1(t) &= u_1(t + \sigma), \quad z_2(t) = -u_2(t + \sigma), \quad z_3(t) = u_3(t + \sigma), \quad z_4(t) = -u_4(t + \sigma).
\end{align*}
Then
\begin{align*}
  W_1[z_1](t) &= \mu \sinh \lambda(t + \sigma) - \lambda \sin \mu(t + \sigma); \\
  W_2[z_1, z_2](t) &= -[2\sqrt{\alpha}[\cosh \lambda(t + \sigma) \cos \mu(t + \sigma) - 1] \\
  &\quad + \beta(\sinh \lambda(t + \sigma) \sin \mu(t + \sigma))]; \\
  W_3[z_1, z_2, z_3](t) &= \begin{vmatrix}
  u_1 & u_2 & u_3 \\
  u_1' & u_2' & u_3' \\
  u_1'' & u_2'' & u_3''
\end{vmatrix} \\
  &= u_3' \begin{vmatrix}
  u_1 & u_2 & u_3 \\
  u_1' & u_2' & u_3'
\end{vmatrix} - u_2' \begin{vmatrix}
  u_1 & u_2 & u_3 \\
  u_1' & u_2' & u_3'
\end{vmatrix} + u_1' \begin{vmatrix}
  u_1 & u_2 & u_3 \\
  u_1' & u_2' & u_3'
\end{vmatrix} \\
  &= (2\alpha \sqrt{\alpha} \mu + \alpha \lambda^3 + \sqrt{\alpha} \mu^3) \sinh \lambda(t + \sigma) \\
  &\quad - (2\alpha \sqrt{\alpha} \lambda + \alpha \mu^3 + \sqrt{\alpha} \lambda^5) \sin \mu(t + \sigma); \\
  W_4[z_1, z_2, z_3, z_4](t) &= \alpha(\beta^2 + 4\alpha)^2 > 0,
\end{align*}
where $\sigma \in (0, 1)$ is small enough.

To show that the functions $\{z_1, z_2, z_3, z_4\}$ form a Markov fundamental system of solutions of $u^{(4)} + \beta u'' - \alpha u = 0$ on $[0, 1]$, we only need to provide the value range of $\alpha, \beta$ to ensure $W_1[z_1] > 0, W_2[z_1, z_2] > 0, W_3[z_1, z_2, z_3] > 0, W_4[z_1, z_2, z_3, z_4] > 0$.

For convenience, in the sequel of the proof we replace $W_1[z_1], W_2[z_1, z_2], W_3[z_1, z_2, z_3], W_4[z_1, z_2, z_3, z_4]$ with $W_i, i = 1, 2, 3, 4$.

For any fixed $\sigma \in (0, 1)$ is small enough, by the sign of $W_1'(t)$ and $W_1''(t)$, it is not difficult to verify that $W_1(t)$, $W_3(t)$ is positive on $[0, 1]$, so we only discuss the positivity of $W_2(t)$. Clearly, we can get that the first derivative and second derivative of $W_2$ as follows:
\begin{align*}
  W_2'(t) &= (\lambda^2 + \mu^2)[\lambda \cosh \lambda(t + \sigma) \sin \mu(t + \sigma) - \mu \sinh \lambda(t + \sigma) \cos \mu(t + \sigma)], \\
  W_2''(t) &= (\lambda^2 + \mu^2)^2 \sinh \lambda(t + \sigma) \sin \mu(t + \sigma).
\end{align*}
Thus, we only need to show that $W_2''(t)$ has at most one zero point on $[0, 1]$, which is equivalent to solve the equation
\[
  \lambda \tan \mu(t + \sigma) = \mu \tanh \lambda(t + \sigma), \quad t \in [0, 1].
\]

It is very difficult to solve the above equation since the parameter $\lambda, \mu$ is not given. Hence, we give some sufficient conditions to ensure the disconjugacy of $u^{(4)} + \beta u'' - \alpha u = 0$ on $[0, 1]$.

**Theorem 3.2.** If $\alpha_0 > 0, \beta \in \mathbb{R}$ satisfy
\[
  \frac{\alpha_0}{(2\pi)^4} + \frac{\beta}{(2\pi)^2} < 1, \quad \text{and} \quad 2\sqrt{\alpha_0}(1 - \cosh \lambda \cos \mu) - \beta \sinh \lambda \sin \mu > 0.
\]
where \( \lambda, \mu \) depend on \( \alpha_0, \beta, \) which is defined by (16). Then the equation \( u^{(4)} + \beta u'' - \alpha_0 u = 0 \) is disconjugate on \( [0, 1] \). Moreover, the equation \( u^{(4)} + \beta u'' - \alpha u = 0 \) is disconjugate on \( [0, 1] \) if, and only if \( \alpha \in (\lambda_1 + \alpha_0, \mu_1 + \alpha_0) \).

**Proof.** From (16) and (17), we imply that \( \mu \in (0, 2\pi) \), which means that \( W''_2(t) \) at most changes one sign in \([0, 1]\). Hence, there exists \( t_0 \in \left(\frac{1}{2}, 1\right) \), such that

\[
W''_2(t) \geq 0, \quad t \in (0, t_0), \quad W''_2(t) \leq 0, \quad t \in (t_0, 1].
\]

This concludes that

\[
W''_2(t) \geq W'_2(0) \geq 0, \quad t \in (0, t_0), \quad W'_2(t) \geq W'_2(1), \quad t \in (t_0, 1].
\]

Moreover, if \( W'_2(1) \geq 0 \), then \( W_2(t) \geq W_2(0) > 0, \quad t \in [0, 1] \); if \( W'_2(1) \leq 0 \), then there exists \( t_1 \in (t_0, 1) \), such that \( W'_2(t_1) = 0, \quad t \in (t_0, 1), \quad W'_2(t) < 0, \quad t \in (t_1, 1] \). That is to say, \( W_2(t) \geq W_2(0) > 0, \quad t \in (0, t_1), \quad W_2(t) \geq W_2(1) = 2\sqrt{\alpha_0} (1 - \cosh \lambda \cos \mu - \beta \sinh \lambda \sin \mu), \quad t \in (t_1, 1] \). Therefore, it follows from \( 2\sqrt{\alpha_0} (1 - \cosh \lambda \cos \mu - \beta \sinh \lambda \sin \mu) > 0 \) yields that \( W_2(1) > 0 \) for any fixed \( \sigma \in (0, 1) \) is small enough. Subsequently, \( W_2(t) > 0, \quad t \in [0, 1] \).

This together with the fact \( W_2(t) > 0, \quad W'_2(t) > 0, \quad W'_3(t) > 0, \quad t \in [0, 1] \) and Lemma 2.3 conclude that the equation \( u^{(4)} + \beta u'' - \alpha_0 u = 0 \) is disconjugate on \([0, 1]\). Moreover, from Lemma 2.11, the equation \( u^{(4)} + \beta u'' - \alpha u = 0 \) is disconjugate on \([0, 1]\) if, and only if \( \alpha \in (\lambda_1 + \alpha_0, \mu_1 + \alpha_0) \).

**Remark 3.** Theorem 3.2 generalize the Theorem 1.1 ([3, Theorem 2.3]) and [13, Theorem 3.1]. Note that if \( \alpha > 0 \), then the basic solutions of equation in (11) is \( \{ e^{\lambda t}, e^{-\lambda t}, \cos \mu t, \sin \mu t \} \), however, if \( \alpha = 0, \beta > 0 \), its basic solutions is \( \{1, t, \cos \mu t, \sin \mu t \} \), this reveals (15) cannot be directly degenerated to (13).

If \( \beta = 0 \), then (17) is equivalent to \( \alpha < 16\pi^4 \) and \( \alpha < \alpha_* \), which yields that \( \alpha < \alpha_* \). From [13, Theorem 3.1], \( \alpha < \alpha_* \) is sharp condition for the disconjugacy of \( u^{(4)} - \alpha u = 0 \) on \([0, 1]\).

Last, we consider the case \( \alpha < 0 \). We divide three subcases to discuss: (a) \( -\frac{\beta^2}{4} < \alpha < 0 \), (b) \( \alpha = -\frac{\beta^2}{4} \), and (c) \( \alpha < -\frac{\beta^2}{4} \).

**Subcase (a)** \( -\frac{\beta^2}{4} < \alpha < 0 \). In this case, we also divide two cases: \( \beta > 0 \) and \( \beta < 0 \).

1. For any fixed \( -\frac{\beta^2}{4} < \alpha < 0, \beta > 0 \). Let us consider the following equation with \( \alpha, \beta \)

\[
\beta \sin \lambda \sin \mu = 2\sqrt{-\alpha}(1 - \cos \mu \cos \lambda),
\]

where \( \lambda = \sqrt{\frac{\beta - \sqrt{\beta^2 + 4\alpha}}{2}}, \mu = \sqrt{\frac{\beta^2 + 4\alpha + \beta}{2}} \) and \( 0 < \lambda < \sqrt{\frac{\beta^2}{2}} < \mu < \sqrt{\beta} < 2\pi \).

Clearly, the equation \( u^{(4)} + \beta u'' - \alpha u = 0 \) has a general solution

\[
u(t) = C_1 \cos \mu t + C_2 \sin \mu t + C_3 \sin \lambda t + C_4 \cos \lambda t, \quad t \in [0, 1],
\]

where \( C_i, i = 1, 2, 3, 4 \) are arbitrary constants. The initial value problem

\[
\begin{align*}
u^{(4)}(t) + \beta \nu''(t) - \alpha \nu(t) &= 0, & t & \in (0, 1), \\
\nu(0) &= \nu'(0) = \nu''(0) = 0, & \nu'''(0) &= \sqrt{-\alpha} \sqrt{\beta^2 + 4\alpha}
\end{align*}
\]

has a unique solution \( \nu(t) = \mu \sin \lambda t - \lambda \sin \mu t, \quad t \in [0, 1] \).

The initial value problem

\[
\begin{align*}
u^{(4)}(t) + \beta \nu''(t) - \alpha \nu(t) &= 0, & t & \in (0, 1), \\
\nu(0) &= \nu'(0) = \nu''(0) = 0, & \nu'''(0) &= \sqrt{\beta^2 + 4\alpha}
\end{align*}
\]
has a unique solution $u_2(t) = \cos \lambda t - \cos \mu t$, $t \in [0, 1]$. 

The initial value problem

$$u^{(4)}(t) + \beta u''(t) - \alpha u(t) = 0, \quad t \in (0, 1),$$

$$u(0) = u''(0) = u'''(0) = 0, \quad u'(0) = \sqrt{-\alpha \beta^2 + 4\alpha}$$

has a unique solution $u_3(t) = \mu^3 \sin \lambda t - \lambda^3 \sin \mu t$, $t \in [0, 1]$. 

The initial value problem

$$u^{(4)}(t) + \beta u''(t) - \alpha u(t) = 0, \quad t \in (0, 1),$$

$$u'(0) = u''(0) = u'''(0) = 0, \quad u(0) = \sqrt{\beta^2 + 4\alpha}$$

has a unique solution $u_4(t) = \mu^2 \cos \lambda t - \lambda^2 \cos \mu t$, $t \in [0, 1]$. 

Let

$$y_1(t) = u_1(t + \sigma), \ y_2(t) = -u_2(t + \sigma), \ y_3(t) = u_3(t + \sigma), \ y_4(t) = -u_4(t + \sigma). \quad (19)$$

Then

$$W_1[y_1](t) = \mu \sin \lambda(t + \sigma) - \lambda \sin \mu(t + \sigma);$$

$$W_2[y_1, y_2](t) = 2\sqrt{-\alpha}(1 - \cos \mu(t + \sigma) \cos \lambda(t + \sigma))$$

$$- \beta \sin \lambda(t + \sigma) \sin \mu(t + \sigma);$$

$$W_3[y_1, y_2, y_3](t) = \begin{vmatrix}
 u_1 & -u_2 & u_3 \\
 u_1' & -u_2' & u_3' \\
 u_1'' & -u_2'' & u_3''
\end{vmatrix}$$

$$= \lambda \mu (\lambda^2 - \mu^2)^2 [\mu \sin \lambda(t + \sigma) - \lambda \sin \mu(t + \sigma)];$$

$$W_4[y_1, y_2, y_3, y_4](t) = -\alpha(\beta^2 + 4\alpha)^2 > 0, \quad \sigma \in (0, 1) \text{ small enough}. \quad (2)$$

For any fixed $-\frac{\beta^2}{\alpha} < \alpha < 0, \ \beta < 0$. Let us consider the equation with $\alpha, \ \beta$

$$2\sqrt{-\alpha} (\cosh \mu \cosh \lambda - 1) + \beta \sinh \lambda \sinh \mu = 0,$$

where $\lambda = \sqrt{-\beta + \sqrt{\beta^2 + 4\alpha}}$, $\mu = \sqrt{-\beta - \sqrt{\beta^2 + 4\alpha}}$ and $\lambda > \mu > 0$.

It is easy to see that the equation $u^{(4)} + \beta u'' - \alpha u = 0$ has a general solution

$$u(t) = C_1 \cosh \mu t + C_2 \sinh \mu t + C_3 \sinh \lambda t + C_4 \cosh \lambda t, \quad t \in [0, 1],$$

here $C_i, i = 1, 2, 3, 4$ are arbitrary constants. The initial value problem

$$u^{(4)}(t) + \beta u''(t) - \alpha u(t) = 0, \quad t \in (0, 1),$$

$$u(0) = u'(0) = u''(0) = 0, \quad u'''(0) = \sqrt{-\alpha \beta^2 + 4\alpha}$$

has a unique solution $u_1(t) = \mu \sinh \lambda t - \lambda \sinh \mu t$, $t \in [0, 1]$. 

The initial value problem

$$u^{(4)}(t) + \beta u''(t) - \alpha u(t) = 0, \quad t \in (0, 1),$$

$$u(0) = u'(0) = u''(0) = 0, \quad u'''(0) = \sqrt{\beta^2 + 4\alpha}$$

has a unique solution $u_2(t) = \cosh \lambda t - \cosh \mu t$, $t \in [0, 1]$. 

The initial value problem

$$u^{(4)}(t) + \beta u''(t) - \alpha u(t) = 0, \quad t \in (0, 1),$$

$$u(0) = u''(0) = u'''(0) = 0, \quad u'(0) = \sqrt{-\alpha \beta^2 + 4\alpha}$$

has a unique solution $u_3(t) = \lambda^3 \sinh \mu t - \mu^3 \sinh \lambda t$, $t \in [0, 1]$. 


The initial value problem

\[ u^{(4)}(t) + \beta u''(t) - \alpha u(t) = 0, \quad t \in (0,1), \]

\[ u'(0) = u''(0) = u'''(0) = 0, \quad u(0) = \sqrt{\beta^2 + 4 \alpha} \]

has a unique solution \( u_4(t) = \lambda^2 \cosh \mu t - \mu^2 \cosh \lambda t, \quad t \in [0,1]. \)

Let

\[ z_1(t) = u_1(t + \sigma), \quad z_2(t) = -u_2(t + \sigma), \quad z_3(t) = u_3(t + \sigma), \quad z_4(t) = -u_4(t + \sigma). \quad (20) \]

Then

\[ W_1[z_1](t) = \mu \sinh \lambda(t + \sigma) - \lambda \sinh \mu(t + \sigma); \]
\[ W_2[z_2, z_3](t) = -2 \sqrt{-\alpha} (\cosh \mu(t + \sigma) \cosh \lambda(t + \sigma) - 1) \]
\[ - \beta \sinh \lambda(t + \sigma) \sinh \mu(t + \sigma); \]
\[ W_3[z_1, z_2, z_3](t) = \begin{vmatrix} u_1 & u_2 & -u_3 \\ u'_1 & u'_2 & -u'_3 \\ u''_1 & u''_2 & -u''_3 \end{vmatrix} \]
\[ = \lambda \mu (\lambda^2 - \mu^2)^2 [\mu \sinh \lambda(t + \sigma) - \lambda \sinh \mu(t + \sigma)]; \]
\[ W_4[z_1, z_2, z_3, z_4](t) = -\alpha (\beta^2 + 4 \alpha)^2 > 0, \quad \sigma \in (0,1) \text{ small enough.} \]

**Theorem 3.3.** Assume that \( \alpha_0 \in (-\frac{\beta^2}{4}, \sqrt{2}), \beta < 4 \pi^2. \)

1. If \( 0 < \beta < 4 \pi^2 \) and \( \alpha, \beta \) satisfy the inequality

\[ 2 \sqrt{-\alpha_0} (1 - \cos \mu \cos \lambda) - \beta \sin \lambda \sin \mu > 0, \quad (21) \]

here \( \lambda, \mu \) depend on \( \alpha_0, \beta \) defined as follows

\[ \lambda = \sqrt{-\beta + \sqrt{\beta^2 + 4 \alpha_0}}, \quad \mu = \sqrt{-\beta - \sqrt{\beta^2 + 4 \alpha_0}}. \quad (22) \]

Then the equation \( u^{(4)} + \beta u'' - \alpha_0 u = 0 \) is disconjugate on \( [0,1]. \)

2. If \( \beta \leq 0, \) then the equation \( u^{(4)} + \beta u'' - \alpha_0 u = 0 \) is disconjugate on \( [0,1]. \)

Moreover, the equation \( u^{(4)} + \beta u'' - \alpha u = 0 \) is disconjugate on \( [0,1] \) if, and only if \( \alpha \in (\lambda_1 + \alpha_0, \mu_1 + \alpha_0). \)

**Proof.** To this end, in the virtue of Lemma 2.3, we only need to find a Markov fundamental system of solutions of \( u^{(4)} + \beta u'' - \alpha_0 u = 0. \)

1. If \( 0 < \beta < 4 \pi^2. \) In this case, we only prove \( \{y_1, y_2, y_3, y_4\} \) is a Markov fundamental system of solutions of \( u^{(4)} + \beta u'' - \alpha_0 u = 0, \) where \( y_1, y_2, y_3, y_4 \) defined by \( (19). \)

Obviously, from (22), \( 0 < \lambda < \sqrt{2} \pi < \mu < \sqrt{\beta} < 2 \pi. \) This together with the property of \( W'_1[y_1](t) = \lambda \mu [\cos \lambda(t + \sigma) - \cos \mu(t + \sigma)] \) implies that

\[ W_1[y_1](t) = \mu \sin \lambda(t + \sigma) - \lambda \sin \mu(t + \sigma) > 0, \quad t \in [0,1]. \]

Similarly, we get \( W_3[y_1, y_2, y_3](t) > 0, \quad t \in [0,1]. \)

By a simple computation, we have

\[ W'_2[y_1, y_2](t) = (\mu^2 - \lambda^2) [\lambda \cos \lambda(t + \sigma) \sin \mu(t + \sigma) - \mu \sin \lambda(t + \sigma) \cos \mu(t + \sigma)], \]

and

\[ W'_2[y_1, y_2](t) = (\mu^2 - \lambda^2)^2 \sin \lambda(t + \sigma) \sin \mu(t + \sigma), \quad t \in [0,1]. \]
The value ranges of $\lambda, \mu$ yields that $W''_2[y_1, y_2](t)$ is positive near $0$ and at most once change its sign, which means that there exist $t_0 \in (0, 1)$, such that $W''_2[y_1, y_2](t)$ is increasing on $[0, t_0]$ and decreasing on $[t_0, 1]$. It follows that

$$W''_2[y_1, y_2](t) > W''_2[y_1, y_2](0) > 0, \quad t \in [0, t_0],$$

and

$$W''_2[y_1, y_2](t) > W''_2[y_1, y_2](1), \quad t \in [t_0, 1]$$

for $\sigma \in (0, 1)$ small enough. If

$$W''_2[y_1, y_2](1) = (\mu^2 - \lambda^2)|\lambda\cos(1 + \sigma)\sin\mu(1 + \sigma) - \mu\sin\lambda(1 + \sigma)\cos\mu(1 + \sigma)| \geq 0,$$

then

$$W_2[y_1, y_2](t) \geq W_2[y_1, y_2](0) = 2\sqrt{-\alpha_0}(1 - \cos\mu\cos\lambda\sigma) - \beta\sin\mu\sigma\sin\lambda\sigma > 0.$$

If $W''_2[y_1, y_2](1) < 0$, then

$$W_2[y_1, y_2](t) > \min\{W_2[y_1, y_2](0), \ W_2[y_1, y_2](1)\}, t \in [0, 1].$$

Therefore, $W_2[y_1, y_2](t) > 0, t \in [0, 1]$ is equivalent to the inequality $2\sqrt{-\alpha_0}(1 - \cos\mu\cos\lambda) - \beta\sin\lambda\sin\mu > 0$ holds. Moreover, if $\alpha_0 \in (-\frac{\beta^2}{4}, 0)$ and $\beta \in (0, 4\pi^2)$ satisfy (21), then $W_i[y_1, \ldots, y_i](t) > 0, i = 1, 2, 3, 4, t \in [0, 1]$, which implies that the basic solutions $\{y_1, y_2, y_3, y_4\}$ is a Markov fundamental system of solutions of $u^{(4)} + \beta u'' - \alpha_0 u = 0$ on $[0, 1]$. This together with Lemma 2.3 yields that $u^{(4)} + \beta u'' - \alpha_0 u = 0$ is disconjugate on $[0, 1].$

(2) $\beta \leq 0$. Notice that if $\beta = 0$, then $\alpha_0 = 0$ and the equation $u^{(4)} + \beta u'' - \alpha_0 u = 0$ is disconjugate on $[0, 1]$ by Theorem 3.1. Thus we only discuss the case $\beta < 0$. We claim that $\{z_1, z_2, z_3, z_4\}$ is a Markov fundamental system of solutions of $u^{(4)} + \beta u'' - \alpha_0 u = 0$, here $z_1, z_2, z_3, z_4$ defined by (20).

It follows from $\lambda > \mu > 0$ and $W_1[z_1](t) = \lambda\mu[\cos\lambda(t + \sigma) - \cos\mu(t + \sigma)] > 0$ that

$$W_1[z_1](t) = \mu\sinh\lambda(t + \sigma) - \lambda\sinh\mu(t + \sigma) > 0, t \in [0, 1],$$

here $\sigma > 0$ is small enough. By a similar way, we have $W_3[z_1, z_2, z_3](t) > 0, t \in [0, 1].$

Since

$$W_2[z_1, z_2](t) = (\lambda^2 - \mu^2)|\lambda\sinh\mu(t + \sigma)\cosh\lambda(t + \sigma) - \mu\cosh\mu(t + \sigma)\sinh\lambda(t + \sigma)|,$$

and

$$W_2'[z_1, z_2](t) = (\lambda^2 - \mu^2)^2\sinh\mu(t + \sigma)\sinh\lambda(t + \sigma) > 0,$$

it follows that

$$W_2'[z_1, z_2](t) \geq W_2'[z_1, z_2](0) = (\lambda^2 - \mu^2)(\lambda\sinh\mu\sigma\cosh\lambda\sigma - \mu\cosh\mu\sigma\sinh\lambda\sigma) > 0, t \in [0, 1]$$

for $\sigma \in (0, 1)$ small enough. Therefore, $W_2[z_1, z_2](t) \geq W_2[z_1, z_2](0) > 0, t \in [0, 1]$. Moreover, if $\alpha_0 \in (-\frac{\beta^2}{4}, 0)$ and $\beta \in (-\infty, 0)$, then $W_i[z_1, \ldots, z_i](t) > 0, i = 1, 2, 3, 4, t \in [0, 1]$, which implies that $\{z_1, z_2, z_3, z_4\}$ is a Markov fundamental system of solutions of $u^{(4)} + \beta u'' - \alpha_0 u = 0$ on $[0, 1]$. This together with Lemma 2.3 yields that $u^{(4)} + \beta u'' - \alpha_0 u = 0$ is disconjugate on $[0, 1]$. Based on it and Lemma 2.11, the equation $u^{(4)} + \beta u'' - \alpha u = 0$ is disconjugate on $[0, 1]$ if, and only if $\alpha \in (\lambda_1 + \alpha_0, \mu_1 + \alpha_0)$.
Remark 4. Note that Theorem 3.3 gives the value range of $\alpha$ for the disconjugacy of $u^{(4)} + \beta u'' - \alpha_0 u = 0$ on $[0, 1]$ in the subcase (a), which is very useful to verify the disconjugacy. For example, let $\alpha_0 = -\frac{\pi^4}{16}, \beta = \pi^2$, then we can verify that
\[ \lambda = \frac{\sqrt{2 - \sqrt{2}}}{2}, \quad \mu = \frac{\sqrt{2 + \sqrt{2}}}{2} \]
and
\[ 2\sqrt{-\alpha_0}(1 - \cos \mu \cos \lambda) - \beta \sin \lambda \sin \mu = \pi^2 \left[\frac{\sqrt{2}}{2} (1 - \cos \mu \cos \lambda) - \sin \lambda \sin \mu\right] \approx 0.7332 > 0. \]
So, from Theorem 3.3, the equation $u^{(4)} + \pi^2 u'' + \frac{\pi^4}{16} u = 0$ is disconjugate on $[0, 1]$. This together with Lemma 2.9 and [4, Proposition 1.5] implies that $\lambda_1 \approx -2.4674, \mu_1 \approx 21.8849$. Moreover, $u^{(4)} + \pi^2 u'' - \alpha u = 0$ is disconjugate on $[0, 1]$ if, and only if $\alpha \in (-14.6189, 9.7334)$.

Let $\alpha_0 = -\frac{\beta}{4}, \beta = -\pi^2$, then we can compute that $\lambda = \frac{\sqrt{3}}{2}, \mu = \frac{1}{2} \pi$ and verify that $W_{\lambda}(z_1, \ldots, z_4) > 0, i = 1, 2, 3, 4, t \in [0, 1]$. Hence, from Theorem 3.3, the equation $u^{(4)} - \pi^2 u'' + \frac{\beta^2}{16} u = 0$ is disconjugate on $[0, 1]$. Moreover, $u^{(4)} - \pi^2 u'' - \alpha u = 0$ is disconjugate on $[0, 1]$ if, and only if $\alpha \in (\lambda_1 - \frac{3\pi^4}{16}, \mu_1 - \frac{3\pi^4}{16})$.

Subcase (b) $\alpha = -\frac{\beta^2}{4}$. Let $\alpha = -\frac{\beta^2}{4}$ and $\beta \in \mathbb{R}$. The the problem (11) can be rewritten as
\[ u^{(4)}(t) + \beta u''(t) + \frac{\beta^2}{4} u(t) = 0, \quad t \in (0, 1), \]
\[ u(0) = u(1) = u'(0) = u'(1) = 0. \]
It is not difficult to compute that if $\beta \leq 0$, then (23) has only trivial solution $u \equiv 0$; if $\beta > 0$, then (23) has nontrivial solutions if and only if $\beta$ is a solution of the equation
\[ \sin \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{2}} \cos \frac{\sqrt{3}}{\sqrt{2}}. \]
Moreover, $\beta_2 \approx 40.3815$ is the first positive solution of (24).

Theorem 3.4. For any fixed $\beta < \beta_2$, the equation $u^{(4)} + \beta u'' + \frac{\beta^2}{4} u = 0$ is disconjugate on $[0, 1]$. Moreover, the value range of $\beta$ is optimal and $u^{(4)} + \beta u'' - \alpha u = 0$ is disconjugate on $[0, 1]$ if, and only if $\alpha \in (\lambda_1 - \frac{\beta^2}{4}, \mu_1 - \frac{\beta^2}{4})$.

Proof. To this end, in the virtue of Lemma 2.3, we only need to find a Markov fundamental system of solutions of $u^{(4)} + \beta u'' - \alpha u = 0$. Notice that if $\beta = 0$, then $\alpha = 0$ and the equation $u^{(4)} + \beta u'' - \alpha u = 0$ is disconjugate on $[0, 1]$ by Theorem 3.1.

Case 1 $0 < \beta < \beta_2$. For fixed $k \in \{1, 2, 3, 4\}$, let $u_k$ be the unique solution of initial value problem
\[ u^{(4)}(t) + \beta u''(t) + \frac{\beta^2}{4} u(t) = 0, \quad t \in (0, 1), \]
\[ u^{(j)}(0) = 0, \quad j \neq 4 - k, \quad u^{(4-k)}(0) = 2\lambda^{4-k}, \]
where $\lambda = \frac{\sqrt{2}}{2}$. Then
\[ u_1(t) = -\lambda t \cos \lambda t + \sin \lambda t, \quad t \in [0, 1], \]
\[ u_2(t) = \lambda t \sin \lambda t, \quad t \in [0, 1], \]
\[ u_3(t) = -\lambda t \cos \lambda t + 3 \sin \lambda t, \quad t \in [0, 1], \]
\[ u_4(t) = 2 \cos \lambda t + \lambda t \sin \lambda t, \quad t \in [0, 1]. \]

Set \( y_1(t) = u_1(t + \sigma), \ y_2(t) = -u_2(t + \sigma), \ y_3(t) = u_3(t + \sigma), \ y_4(t) = -u_4(t + \sigma). \) It is not difficult to obtain that
\[
W_1[y_1](t) = \sin \frac{\sqrt{\beta}}{\sqrt{2}}(t + \sigma) - \sqrt{\beta} \frac{\sqrt{\beta}}{\sqrt{2}}(t + \sigma) \cos \frac{\sqrt{\beta}}{\sqrt{2}}(t + \sigma) > 0;
\]
\[
W_2[y_1, y_2](t) = \frac{\beta \sqrt{\beta}}{2 \sqrt{2}}(t + \sigma)^2 - \frac{\beta \sqrt{\beta}}{2 \sqrt{2}}(t + \sigma) > 0;
\]
\[
W_3[y_1, y_2, y_3](t) = \sqrt{2}\beta \frac{\sqrt{\beta}}{\sqrt{2}}(t + \sigma) - \frac{\sqrt{\beta}}{\sqrt{2}}(t + \sigma) \cos \frac{\sqrt{\beta}}{\sqrt{2}}(t + \sigma) > 0;
\]
\[
W_4[y_1, y_2, y_3, y_4](t) = 2\beta^3 > 0.
\]

Therefore, for any fixed \( \sigma \in (0, 1) \) small enough, if \( 0 < \beta < \beta_2 \), then \( \{y_1, y_2, y_3, y_4\} \) is a Markov fundamental system of solutions of \( u^{(4)} + \beta u'' + \frac{\beta^2}{4} u = 0 \) on \( [0, 1] \).

Case 2 \( \beta < 0 \). For fixed \( k \in \{1, 2, 3, 4\} \), let \( u_k \) be the unique solution of initial value problem
\[
u^{(4)}(t) + \beta u''(t) + \frac{\beta^2}{4} u(t) = 0, \quad t \in (0, 1),
\]
\[
u^{(j)}(0) = 0, \quad j \neq 4 - k, \quad u^{(4-k)}(0) = 2\lambda^4 - k,
\]
where \( \lambda = \sqrt{-\frac{\beta}{2}} \). Then
\[
u_1(t) = \lambda t \cosh \lambda t - \sinh \lambda t, \quad t \in [0, 1];
\]
\[
u_2(t) = \lambda t \sinh \lambda t, \quad t \in [0, 1];
\]
\[
u_3(t) = -\lambda t \cosh \lambda t + 3 \sinh \lambda t, \quad t \in [0, 1];
\]
\[
u_4(t) = 2 \cosh \lambda t - \lambda t \sinh \lambda t, \quad t \in [0, 1].
\]

Set \( z_1(t) = u_1(t + \sigma), \ z_2(t) = -u_2(t + \sigma), \ z_3(t) = u_3(t + \sigma), \ z_4(t) = -u_4(t + \sigma). \) Then
\[
W_1[z_1](t) = \frac{-\beta}{\sqrt{2}}(t + \sigma) \cosh \frac{-\beta}{\sqrt{2}}(t + \sigma) - \frac{-\beta}{\sqrt{2}}(t + \sigma) > 0;
\]
\[
W_2[z_1, z_2](t) = \frac{-\beta}{\sqrt{2}} \sinh^2 \frac{-\beta}{\sqrt{2}} + \frac{\beta}{2 \sqrt{2}}(t + \sigma)^2(t + \sigma) > 0;
\]
\[
W_3[z_1, z_2, z_3](t) = \beta^2(t + \sigma) \cosh \frac{-\beta}{\sqrt{2}}(t + \sigma) + \beta \frac{-\beta}{\sqrt{2}} \sinh \frac{-\beta}{\sqrt{2}}(t + \sigma) > 0;
\]
\[
W_4[z_1, z_2, z_3, z_4](t) = 16\lambda^6 = -2\beta^3 > 0.
\]

Subsequently, if \( \beta < 0 \) and \( \sigma \in (0, 1) \) small enough, then \( \{z_1, z_2, z_3, z_4\} \) is a Markov fundamental system of solutions of \( u^{(4)} + \beta u'' + \frac{\beta^2}{4} u = 0 \) on \( [0, 1] \).

Therefore, \( u^{(4)} + \beta u'' + \frac{\beta^2}{4} u = 0 \) is disconjugate on \([0, 1]\) if \( \beta \in (-\infty, \beta_2) \). Moreover, from the Lemma 2.9, the result is optimal.

In addition, from Lemma 2.11, \( u^{(4)} + \beta u'' - \alpha u = 0 \) is disconjugate on \([0, 1]\) if, and only if \( \alpha \in (\lambda_1 - \frac{\beta^2}{4}, \mu_1 - \frac{\beta^2}{4}) \). \( \square \)
Moreover, \(u\) has a unique solution \(u\) here

Then

\[
a^2 - b^2 = -\beta, \quad 2ab = \sqrt{-\beta^2 - 4\alpha}, \quad a^2 + b^2 = 2\sqrt{-\alpha}.
\]

Let us consider the following equation with \(u\), \(\beta\)

\[
\begin{align*}
b \tanh \frac{a}{\sqrt{2}} &= \frac{a}{\sqrt{2}} \tan \frac{b}{\sqrt{2}}, \quad (25) \\
a &= \frac{\sqrt{2\sqrt{-\alpha} + \sqrt{-\beta^2 - 4\alpha} + \sqrt{2\alpha\sqrt{-\alpha} - \sqrt{-\beta^2 - 4\alpha}}}}{2}, \\
&= \frac{\sqrt{2\sqrt{-\alpha} + \sqrt{-\beta^2 - 4\alpha} - \sqrt{2\alpha\sqrt{-\alpha} - \sqrt{-\beta^2 - 4\alpha}}}}{2}. \quad (26)
\end{align*}
\]

It is worthy to point out that if \(\beta = 0\), then (25) is degenerated (6). So the equation \(u^{(4)} + \alpha u = 0\) has nontrivial solution if and only if \(\alpha < 0\) solve the equation (6), i.e.

\[
\tanh \frac{\sqrt{-\alpha}}{\sqrt{2}} = \tan \frac{\sqrt{-\alpha}}{\sqrt{2}}.
\]

Moreover, \(\sqrt{-\alpha^2} \approx 5.553\) is the first positive solution of (6), see [3, 13].

In this case, the equation \(u^{(4)} + \beta u'' - \alpha u = 0\) has the general solution

\[
\begin{align*}
u(t) &= e^{\frac{t}{2}}(C_1 \cos \frac{b}{\sqrt{2}}t + C_2 \sin \frac{b}{\sqrt{2}}t) + e^{-\frac{t}{2}}(C_3 \cos \frac{b}{\sqrt{2}}t + C_4 \sin \frac{b}{\sqrt{2}}t), \quad t \in [0, 1],
\end{align*}
\]

here \(C_i, \ i = 1, 2, 3, 4\) are arbitrary constants. Then the initial value problem

\[
\begin{align*}
u^{(4)}(t) + \beta u''(t) - \alpha u(t) &= 0, \quad t \in (0, 1), \\
u(0) = u'(0) = u''(0) = u'''(0) = 0, \quad \frac{1}{2}\sqrt{-\alpha}\sqrt{-\beta^2 - 4\alpha}
\end{align*}
\]

has a unique solution \(u_1(t) = \frac{a}{\sqrt{2}} \cosh \frac{a}{\sqrt{2}}t \sin \frac{b}{\sqrt{2}}t - \frac{b}{\sqrt{2}} \cosh \frac{b}{\sqrt{2}}t \sinh \frac{a}{\sqrt{2}}t, \quad t \in [0, 1].\)

The initial value problem

\[
\begin{align*}
u^{(4)}(t) + \beta u''(t) - \alpha u(t) &= 0, \quad t \in (0, 1), \\
u(0) = u'(0) = u''(0) = u''''(0) = 0, \quad \frac{1}{2}\sqrt{-\alpha}\sqrt{-\beta^2 - 4\alpha}
\end{align*}
\]

has a unique solution \(u_2(t) = \sinh \frac{a}{\sqrt{2}}t \sin \frac{b}{\sqrt{2}}t, \quad t \in [0, 1].\) The initial value problem

\[
\begin{align*}
u^{(4)}(t) + \beta u''(t) - \alpha u(t) &= 0, \quad t \in (0, 1), \\
u(0) = u'(0) = u''(0) = u''''(0) = 0, \quad \frac{1}{2}\sqrt{-\alpha}\sqrt{-\beta^2 - 4\alpha}
\end{align*}
\]

has a unique solution

\[
\begin{align*}
u_3(t) &= \frac{a(3b^2 - a^2)}{2\sqrt{2}} \sin \frac{b}{\sqrt{2}}t \cosh \frac{a}{\sqrt{2}}t + \frac{b(3a^2 - b^2)}{2\sqrt{2}} \cos \frac{b}{\sqrt{2}}t \sinh \frac{a}{\sqrt{2}}t, \quad t \in [0, 1].
\end{align*}
\]

The initial value problem

\[
\begin{align*}
u^{(4)}(t) + \beta u''(t) - \alpha u(t) &= 0, \quad t \in (0, 1),
\end{align*}
\]
\[ u'(0) = u''(0) = u'''(0) = 0, \quad u(0) = \frac{1}{2} \sqrt{-\beta^2 - 4\alpha} \]

has a unique solution \( u_4(t) = ab \cosh \frac{\alpha}{\sqrt{2}} t \cos \frac{b}{\sqrt{2}} t + \frac{\beta}{2} \sinh \frac{\alpha}{\sqrt{2}} t \sin \frac{b}{\sqrt{2}} t, \quad t \in [0, 1]. \]

Set
\[ y_1(t) = u_1(t + \sigma), \quad y_2(t) = -u_2(t + \sigma), \quad y_3(t) = u_3(t + \sigma), \quad y_4(t) = -u_4(t + \sigma). \quad (27) \]

Then
\[ W_1[y_1](t) = \frac{a}{\sqrt{2}} \cosh \frac{\alpha}{\sqrt{2}} (t + \sigma) \sin \frac{b}{\sqrt{2}} (t + \sigma) \]
\[ - \frac{b}{\sqrt{2}} \cos \frac{b}{\sqrt{2}} (t + \sigma) \sinh \frac{\alpha}{\sqrt{2}} (t + \sigma); \]
\[ W_2[y_1, y_2](t) = \frac{b^2}{2} \sin^2 \frac{a}{\sqrt{2}} (t + \sigma) - \frac{a^2}{2} \sin^2 \frac{b}{\sqrt{2}} (t + \sigma); \]
\[ W_3[y_1, y_2, y_3](t) = \begin{vmatrix}
    y_1 & y_2 & y_3 \\
    y_1' & y_2' & y_3' \\
    y_1'' & y_2'' & y_3''
\end{vmatrix}
\]
\[ = y_3'' - \frac{(-\beta^2 - 4\alpha)}{4} \left( \frac{a}{\sqrt{2}} \cosh \frac{\alpha}{\sqrt{2}} (t + \sigma) \sin \frac{b}{\sqrt{2}} (t + \sigma) \right. \]
\[ - \left. \frac{b}{\sqrt{2}} \sin \frac{b}{\sqrt{2}} (t + \sigma) \cos \frac{\alpha}{\sqrt{2}} (t + \sigma); \right) \]
\[ W_4[y_1, y_2, y_3, y_4](t) = - \frac{\alpha(\beta^2 + 4\alpha)^2}{8} > 0, \quad \sigma \in (0, 1) \text{ is small enough}. \]

**Theorem 3.5.** Assume that \( \alpha, \beta \) satisfy the following inequality
\[ a > 0, \quad 0 < b < 2\sqrt{2\pi}, \quad \frac{b}{\sqrt{2}} \tanh \frac{a}{\sqrt{2}} - \frac{a}{\sqrt{2}} \tan \frac{b}{\sqrt{2}} < 0, \quad (28) \]

here \( a = a(\alpha, \beta), b = b(\alpha, \beta) \) is defined by (26). Then the equation \( u^{(4)} + \beta u'' - \alpha_0 u = 0 \) is disconjugate on \([0, 1]\). Moreover, \( u^{(4)} + \beta u'' - \alpha u = 0 \) is disconjugate on \([0, 1]\) if, and only if \( \alpha \in (\lambda_1 + \alpha_0, \mu_1 + \alpha_0) \).

**Proof.** To this end, in the virtue of Lemma 2.3, we only show \( \{y_1, y_2, y_3, y_4\} \) is a Markov fundamental system of solutions of \( u^{(4)} + \beta u'' - \alpha u = 0 \) on \([0, 1]\), here \( y_1, y_2, y_3, y_4 \) defined by (27).

By a simple derivative computation,
\[ W_2''[y_1, y_2](t) = \frac{a^2 b^2}{2} \left[ \cosh(\sqrt{2}a(t + \sigma)) - \cos(\sqrt{2}b(t + \sigma)) \right] \geq 0, \quad t \in [0, 1], \]
it follows that \( W_2'[y_1, y_2](t) \geq W_2''[y_1, y_2](0) > 0, \) so \( W_2[y_1, y_2](t) > W_2[y_1, y_2](0) > 0, \quad t \in [0, 1]. \) It is easy to see that the positivity of \( W_1[y_1] \) is equivalent to the positivity of \( W_3[y_1, y_2, y_3] \) and
\[ W_4[y_1, y_2, y_3, y_4](t) = - \frac{\alpha(\beta^2 + 4\alpha)^2}{8} > 0, \quad \sigma \in (0, 1) \text{ is small enough}. \]

Since \( a > 0, \quad 0 < b < 2\sqrt{2\pi}, \) the function \( \frac{a^2 + b^2}{2} \sin(\sqrt{2}a(t + \sigma)) \sin(\sqrt{2}b(t + \sigma)) \) is positive near 0 and at most once change its sign on \([0, 1]\). This concludes that \( W_1[y_1](t) \) increasing and decreasing again on \([0, 1]\). This together with \( \frac{b}{\sqrt{2}} \tanh \frac{a}{\sqrt{2}} - \frac{a}{\sqrt{2}} \tan \frac{b}{\sqrt{2}} < 0 \)
implies $W_1[y_1](t) > 0$, $t \in [0, 1]$. Similarly, we have $W_3[y_1, y_2, y_3](t) > 0$, $t \in [0, 1]$. Therefore, from Lemma 2.3, the equation $u^{(4)} + \beta u'' - \alpha u = 0$ is disconjugate on $[0, 1]$.

**Remark 5.** Theorem 3.5 generalize the Theorem 1.2([3, Theorem 3.7]) and [13, Theorem 3.2]. Note that if $\beta = 0$, then $a = b = \sqrt{-\alpha}$ and (28) is equivalent to

$$0 < a < 2\sqrt{2}\pi, \quad \tan \frac{a}{\sqrt{2}} > \tanh \frac{a}{\sqrt{2}}.$$  

This imply that $\alpha^* < \alpha < 0$, which is a sharp condition for the disconjugacy of $u^{(4)} - \alpha u = 0$ on $[0, 1]$.

Let $\beta = 64, \alpha = -1296$. Then $\alpha < -\frac{\beta^2}{4} = -1026$. By computing,

$$a = \sqrt{18 + 2\sqrt{17} + \sqrt{18 - 2\sqrt{17}}} \approx 8.2462,$$

and

$$b = \sqrt{18 + 2\sqrt{17} - \sqrt{18 - 2\sqrt{17}}} \approx 2$$

and

$$\frac{b}{\sqrt{2}} \tanh \frac{a}{\sqrt{2}} - \frac{a}{\sqrt{2}} \tan \frac{b}{\sqrt{2}} \approx 2 \tanh(5.8310) - 5.8310 \tan(\sqrt{2}) \approx -35.5197.$$

From Theorem 3.5, the equation $u^{(4)} + 64u'' - 1296u = 0$ is disconjugate on $[0, 1]$ and the equation $u^{(4)} + 64u'' - \alpha u = 0$ is disconjugate on $[0, 1]$ with $\alpha \in (\lambda_1 - 1296, \mu_1 + 1296)$.

Note that Theorem 3.2-3.5 together with Lemma 2.5 and Lemma 2.6 imply the following

**Corollary 3.** Let $\alpha, \beta \in \mathbb{R}$ such that the equation $u^{(4)} + \beta u'' - \alpha u = 0$ is disconjugate on $[0, 1]$, and (H0) hold. Then

1. The Green’s function $G_2(t, s)$ of (11) satisfies

$$G_2(t, s) > 0, \quad 0 < s < 1, \quad 0 < t < 1;$$

2. For each $k \geq 1$ and each $\nu \in \{+, -\}$, the problem

$$u^{(4)} + \beta u'' - \alpha u = \mu p(t)u, \quad u(0) = u'(0) = u(1) = u'(1) = 0$$

has a unique solution $(\mu_k, \psi_k) \in \mathbb{R}_+ \times S_{k, \nu}$ with $\|\psi_k\| = 1$. In addition:

(i) $\sigma(L_2, p) = \{ \mu_k : k \geq 1 \}$.

(ii) If $k' > k \geq 1$ then $\mu_{k'} > \mu_k > 0$;

(iii) $\lim_{k \to \infty} \mu_k = \infty$.

**Remark 6.** The expressibility of the Green function $G_2(t, s)$ continuously depend on parameters $\alpha$ and $\beta$ (see [6, Page106]). This means that $G_2(t, s)$ has five different expressions, however, it always satisfies

$$G_2(t, s) > 0, \quad (t, s) \in (0, 1) \times (0, 1).$$

**Remark 7.** From Corollary 3, the disconjugacy of the equation $u^{(4)} + \beta u'' - \alpha u = 0$ is an important property to obtain the spectrum structure of the linear operator $u^{(4)} + \beta u'' - \alpha u = 0$ coupled with the clamped beam conditions $u(0) = u(1) = u'(0) = u'(1) = 0$, which provides an important theoretical basis for studying the steady-state solution of elastic beam equations in engineering and mechanics.
Notice that [4, Theorem 2.1] give an important characterization of disconjugacy, however, the choice of \( M \) limits its widespread application to verify disconjugacy of linear differential equations, Theorem 3.1-Theorem 3.5 provide just one approach to obtain the suitable \( M \). Based on our results and [4, Theorem 2.1], our results also give some optimal interval of \( \alpha \) to provide the disconjugacy of equation \( u^{(4)} + \beta u'' + \alpha u = 0 \) on \([0, 1]\). In addition, the Markov system together with [4, Proposition 1.5] also provide the computation formula of eigenvalues \( \lambda_1 \) and \( \mu_1 \).

4. Some applications. As application of the main results in previous sections, let us consider the existence of nodal solutions of the nonlinear problem

\[
\begin{align*}
\tag{29}
\frac{d^4 u}{dt^4} + \beta \frac{d^2 u}{dt^2} - \alpha u &= rf(t, u(t)), & 0 < t < 1, \\
u(0) = u(1) = u'(0) = u'(1) = 0,
\end{align*}
\]

where \( \alpha, \beta \in \mathbb{R} \) such that the equation \( u^{(4)}(t) + \beta u''(t) - \alpha u(t) = 0 \) is disconjugate on \([0, 1]\), \( r \) is a parameter and \( f \) satisfies the following assumption:

(H1) \( f \in C(\mathbb{R}, \mathbb{R}) \) satisfies \( f(\cdot, u) \geq 0 \) uniformly for \( t \in [0, 1] \) and for all \( u \neq 0 \).

(H2) There exist \( f_0, f_\infty \in (0, \infty) \), such that

\[
\lim_{s \to +\infty} \frac{f(t, s)}{s} = f_\infty, \quad \lim_{s \to 0} \frac{f(t, s)}{s} = f_0 \quad \text{uniformly for } t \in [0, 1].
\]

(H3) There exist \( f_0, f_\infty \in (0, \infty) \), such that

\[
\lim_{s \to -\infty} \frac{f(t, s)}{s} = 0, \quad \lim_{s \to +\infty} \frac{f(t, s)}{s} = f_\infty, \quad \lim_{s \to 0} \frac{f(t, s)}{s} = f_0, \quad \text{uniformly for } t \in [0, 1].
\]

From Corollary 3, the eigenvalue problem

\[
\begin{align*}
\tag{30}
\frac{d^4 u}{dt^4} + \beta \frac{d^2 u}{dt^2} - \alpha u &= \eta u, & u(0) = u'(0) = u(1) = u'(1) = 0
\end{align*}
\]

has a sequence simple and real eigenvalues \( \eta_k, k = 1, 2, \cdots, \) and for each \( \eta_k \), the corresponding eigenfunction \( \psi_k \) exactly changes \( k - 1 \) signs on \((0, 1)\).

By applying the bifurcation theorem [11, 16] and a similar argument of[13, Theorem 6.1-6.2] and [21, Theorem 3.1], we can directly establish the following conclusions:

**Theorem 4.1.** Let (H1) and (H2) hold. Assume that for some \( k \in \mathbb{N}^+ \), either

\[
\frac{\eta_k}{f_0} < r < \frac{\eta_k}{f_\infty} \quad \text{or} \quad \frac{\eta_k}{f_\infty} < r < \frac{\eta_k}{f_0}.
\]

Then problem (29) has two solutions \( u^+_k \) and \( u^-_k \) such that \( u^+_k \) changes exactly \( k - 1 \) signs in \((0, 1)\) and is positive near 0, and \( u^-_k \) changes exactly \( k - 1 \) signs in \((0, 1)\) and is negative near 0.

**Theorem 4.2.** Let (H1), (H3) hold and \( \{\eta_k\}^\infty_1 \) be the eigenvalues of (30). If

\[
\lambda > \frac{\eta_k}{f_0},
\]

then problem (29) has at least \( 2k - 1 \) nontrivial solutions. That is to say, there exist solutions \( w_1, \ldots, w_k \), such that for \( 1 \leq j \leq k \), \( w_j \) has exactly \( j - 1 \) simple zeros on the open interval \((0, 1)\) and \( w'_j(0) < 0 \) and there exist solutions \( z_2, \ldots, z_k \), such that for \( 2 \leq j \leq k \), \( z_j \) has exactly \( j - 1 \) simple zeros on the open interval \((0, 1)\) and \( z'_j(0) > 0 \).

**Remark 8.** Theorem 4.1-4.2 generalizes the results of [13, Theorem 6.1-6.2] and [21, Theorem 3.1].
Remark 9. Let \( \alpha = -\frac{1}{16} \pi^4 \in (-\frac{\pi^4}{4}, 0) \), \( \beta = \pi^2 \) and for fixed constants \( a \in (0, \infty) \) and \( b \in (0, \infty) \) with \( a < b \), we consider the nonlinear problem

\[
\begin{align*}
\quad u^{(4)} + \pi^2 u'' + \frac{1}{16} \pi^4 u &= \lambda f(u), \quad t \in (0, 1), \\
\quad u(0) = u'(0) = u(1) = u'(1) = 0,
\end{align*}
\]

where \( r > 0 \) is a parameter and

\[
f(s) = \begin{cases} 
  g(s), & s \in [0, \infty), \\
  -g(-s), & s \in (-\infty, 0);
\end{cases} \quad \text{and} \quad g(s) = \begin{cases} 
  as, & s \in [0, 3], \\
  3a + (b - a)(s - 3), & s \in [3, 5], \\
  b(s - 3) + a, & s \in [5, \infty).
\end{cases}
\]

Let \( \eta \) is the eigenvalue of the problem

\[
\begin{align*}
\quad u^{(4)} + \pi^2 u'' + \frac{1}{16} \pi^4 u &= \eta u, \\
\quad u(0) = u'(0) = u(1) = u'(1) = 0.
\end{align*}
\]

Then set \( \lambda = \alpha + \frac{\sqrt{\pi^2 + \pi^4 + 4a}}{2} \) and

\[
\begin{align*}
\quad u_1(t) &= \mu \sin \lambda t - \lambda \sin \mu t, \\
\quad u_2(t) &= \cos \lambda t - \cos \mu t, \\
\quad u_3(t) &= \mu^3 \sin \lambda t - \lambda^3 \sin \mu t, \\
\quad u_4(t) &= \mu^2 \cos \lambda t - \lambda^2 \cos \mu t.
\end{align*}
\]

Clearly, \( 0 < \lambda < \frac{\sqrt{\pi^2}}{2} < \mu < \pi \). Moreover, we can implies that \( 1 - \cos \mu \cos \lambda > 0 \) and \( \eta \) is the eigenvalue of the problem (30) if and only if \( \alpha \in (-\frac{\pi^4}{4}, 0) \) and \( \eta = \alpha + \frac{\sqrt{\pi^2 + \pi^4 + 4a}}{2} \) satisfies

\[
\begin{vmatrix}
\quad u_1(0) & u_1'(0) & u_1(1) & u_1'(1) \\
\quad u_2(0) & u_2'(0) & u_2(1) & u_2'(1) \\
\quad u_3(0) & u_3'(0) & u_3(1) & u_3'(1) \\
\quad u_4(0) & u_4'(0) & u_4(1) & u_4'(1)
\end{vmatrix} = \sqrt{-\alpha(\pi^2 + 4a)} \left[ 2 \sqrt{-\alpha(1 - \cos \lambda \cos \mu)} - \pi^2 \sin \lambda \sin \mu \right] = 0.
\]

By using Mathlab, it is not difficult to find that the first and second negative zero points of

\[
2 \sqrt{-\alpha(1 - \cos \lambda \cos \mu)} - \pi^2 \sin \lambda \sin \mu
\]

are \( \alpha_1 \doteq -4.41762 \times 10^{-28} \) and \( \alpha_2 \doteq -9.26862 \times 10^{-24} \), respectively, and

\[
6 < \eta_1 \doteq \frac{1}{16} \pi^4 + 4.41762 \times 10^{-28} < \eta_2 \doteq \frac{1}{16} \pi^4 + 9.26862 \times 10^{-24}.
\]

Obviously, \( f_0 = a \) and \( f_\infty = b \). This together Theorem 4.1 yields that for any \( r \in (\frac{\pi^2}{4}, \frac{\eta_2}{a}) \) problem (31) has two solutions \( u_1^+ \) and \( u_1^- \), such that \( u_1^+ > 0 \) in \( (0, 1) \) and \( u_1^- < 0 \) in \( (0, 1) \). And for any \( r \in (\frac{\pi^2}{4}, \frac{\eta_2}{a}) \) problem (31) has two solutions \( u_2^+ \) and \( u_2^- \), such that \( u_2^+ \) changes exactly one sign in \( (0, 1) \) and is positive near 0, and \( u_2^- \) changes exactly one sign in \( (0, 1) \) and is negative near 0. Especially, if \( a = 2, b = 12, \) and \( r \in (\frac{\pi^2}{12}, \frac{\eta_2}{2}) \), then Theorem 4.1 also holds for the nonlinear problem

\[
\begin{align*}
\quad u^{(4)} + \pi^2 u'' + \frac{1}{16} \pi^4 u &= f(u), \quad t \in (0, 1), \\
\quad u(0) = u'(0) = u(1) = u'(1) = 0.
\end{align*}
\]
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E-mail address: linmu8610@163.com
E-mail address: mary@nwnu.edu.cn