General relativistic quantum theories

Foundations. Expanding Universes

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Abstract. Here the space-time is represented by the usual, four-dimensional manifold and at every space-time point is assigned an infinite-dimensional Hilbert space, seat of a (local) quantum description: states, probabilities and expectations. On the space-time manifold is assigned a metric tensor and it is assumed that the quantum fields commutations relations do not only depend on the metric tensor but also on its Ricci tensor: this is a fundamental postulate. This assumption has many relevant consequences: the theory is regularized; the commutators and the propagators are well defined functions and, applying the theory to electroweak interactions, we can obtain a finite and discrete specter of leptons masses.

1. Introduction and summary
We shall discuss (the foundations of) a General Relativistic Quantum Theory: now the Hilbert space of the quantum description is replaced by a fiber bundle (here called quantum bundle) based on a four-dimensional Riemannian manifold (the space-time); the fibers are complex, infinite-dimensional, separable Hilbert spaces; on every fiber (that is at every space-time point) are defined quantum states (vectors or rays), physical quantities (operators), probabilities and expectation-values (Sections 2, 3).

At every space-time point it is also possible to define, in the usual way, creation and annihilation operators; they are assumed to depend on the value of the metric tensor (of the space-time Riemannian manifold) and of the value of its Ricci tensor: this is the main hypothesis. Now the quantum fields and the quantum Lagrangian are, fixed a bundle frame, well defined operator-valued functions; the Pauli-Jordan distributions and the Feynman propagators are true, smooth functions (metric and Ricci tensors dependent): the theory is “physically” regularized, there are no more divergences. The standard quantum field theories are formally recovered for a “null” Ricci tensor, in particular for a “constant”, special relativistic metric tensor; but, in fact, for an isotropic, expanding or contracting metric (Friedmann, Lemaitre), the theory assumes a very different shape. Quite interesting is the case of an “exponentially” expanding space-time metric. (Sections 4, 5).

The metric-dependent modifications of the commutation relations not only regularize the theory but, as a consequence, the values of the leptons masses (and the number of leptons generations) are, by some supplementary assumptions, theoretically predictable. (In principle also the masses of strong interacting particles are predictable). In fact, imposing the equality of the space-time dependent self-mass (logarithmically diverging, when the Ricci tensor goes to zero or when it is simply proportional to the metric tensor) and of a space-time dependent...
counter term, we arrive at a second order partial differential equation; this equation links the d’Alembertian of some classical scalar field to the self-mass term; the finite set of self-values of this equation are related to the leptons masses. The counter term is, eventually a remnant of some quantum field which “breaks the symmetries”; note that here the vacuum expectation-values are space-time dependents (Sections 6, 7).

2. Quantum bundles

A quantum bundle \((\mathcal{H}, X, \pi)\) (or \(\mathcal{H}_X\) ) is a vector bundle (see, for example, Bourbaki [2], Dieudonné [3], Chapters XVI-XX, Lang [8]) where the base space \(X\) is a 4-dimensional Riemannian manifold (the space-time) and the fibers at \(x \in X\), \(\mathcal{H}_x\) are infinite-dimensional, complex, separable Hilbert spaces, all isomorphic to a typical fiber \(\mathcal{H}_0\), consistently to the requirement of space-time homogeneity. Alternatively we can assign on the total space \(\mathcal{H}\) an equivalence relation \(\mathcal{R}\); then \(X := \mathcal{H}/\mathcal{R}\) and \(h \mathcal{R} k\) if and only if \(\pi(h) = \pi(k)\) \((h, k \in \mathcal{H})\); the relation \(\mathcal{R}\) can be thought as “at the same place, at the same time”. A bundle (moving) frame is an isomorphism, \(F_x, \mathcal{H}_x \mapsto \mathcal{H}_0\); if \(s\) is a section of \(\mathcal{H}_X\), \(F_x \circ s\) is an ordinary map from \(X\) to \(\mathcal{H}_0\).

The quantum bundle is here built starting from a (nuclear) vector bundle \(\mathcal{E}_X\) and assigning, for every \(x \in X\), a bilinear, separately continuous, symmetric and positive form \(k_x\) on \(\mathcal{E}_x\) (a metric along the fibers). The map:

\[
(f, g) \in \mathcal{E}_x \times \mathcal{E}_x \mapsto k_x(f, g) = \bar{k}_x(f, \bar{g}) \tag{1}
\]

define, on \(\mathcal{E}_x\), a pre-Hilbert structure \((f \mapsto \bar{f}\) is a conjugate-linear involution); its completion can be identified to \(\mathcal{H}_x\) (see Schwartz [11] or Gelfand [5], Chapter I, for all mathematical details).

If the vectors \(\Phi\) and \(\Psi\) belong to \(\mathcal{H}_x\), we shall denote by \(\langle \Phi, \Psi \rangle_{k,x}\) (or simply by \(\langle \Phi, \Psi \rangle_x\)) their inner product, at \(x \in X\); \(\|\Phi\|_x := \sqrt{\langle \Phi, \Phi \rangle_x}\) is the norm of \(\Phi\). An explicit expression of the inner product, dependent on the space-time metric \(g\) and of its Ricci tensor \(\check{R}^{(g)}\), will be introduced in the following Sections.

The physical states at \(x\) are now represented by (not null) vectors of \(\mathcal{H}_x\). The (bounded) physical quantities are represented by elements of \(\mathcal{A}_x := \text{Lin}(\mathcal{H}_x)\). The average value of the physical quantity represented by \(\xi \in \mathcal{A}_x\), the system being in the state \(\Phi\), at \(x \in X\), can consequently be written as:

\[
\langle \Phi, \xi, \Phi \rangle_x \tag{2}
\]
a generalization of the Born formula.

3. The base space (space-time)

We shall assume that the space-time (the base space of the quantum bundle) is a 4-dimensional, globally- and time-oriented, Riemannian manifold \((X, g, \tau, o)\). \(T_x\) and \(T_x^*\) (its dual) are the tangent and the cotangent vector spaces, at \(x \in X\); \(g_x\) is a \((+1, -1, -1, -1)\) quadratic form on \(T_x\); \(\tau_x\) is a time-orientation function (that is a \(\{+1, -1\}\) valued function, constant on each one of the two connected \(g_x\)–cones of \(T_x\)); \(\alpha_x\) is an orientation of \(T_x\) (and of \(T_x^*\)). Therefore \(g\) is a section of \(T_X^* \otimes T_X\) and \(\tilde{g}\) (the dual of \(g\)) is a section of \(T_X \otimes T_X\); identifying quadratic forms to linear maps, \(g, \tilde{g} = I_T, \tilde{g} \circ g = I_T\).

\(\check{R}^{(g)}\) is the Ricci tensor of \(g\) and \(\check{R}^{(g)}\) is its dual, that is, if \(p \in T^*_x\), \(\check{R}_x(p, p) = R_x(\tilde{g}_x p, \tilde{g}_x p) = \sum_{0 \leq \alpha < \beta \leq 3} R_{\alpha \beta} g^{\alpha \beta} p_\alpha p_\beta\). In the following we shall need the Ricci tensor of the Friedmann-Lemaître metric \(g_x = dt_x^2 - \frac{a^2(t_x)}{b^2(r_x)} dr_x^2\) (expressed in a non-static chart \((t, r_1, r_2, r_3, U) = (t, r, U), U \in X)\):

\[
\check{R}^{(g)} = 3 \left( h'(t) + h^2(t) \right) . dt^2 - \frac{a^2(t)}{b^2(t)} \left( h'(t) + 3 h^2(t) + \frac{2K}{a^2(t)} \right) . dr^2. \tag{3}
\]
Here $dt_x^2 = dt_x \otimes dt_x$, $dx_x^2 = \sum_{1 \leq j \leq 3} dr_j^2 \otimes dr_j^2$; $\alpha$ is a differentiable, positive function and $a(0) = 1$; $h(t) = a'(t)/a(t)$; $r = \sqrt{(r_1^2 + (r_2^2 + (r_3^2)^2}$, $b(r) = 1 + K/r^2$, $K$ ($-\infty < K < +\infty$) is the spatial, constant, sectional curvature. Fixed $s \in \mathbb{R}$, $K/a^2(s)$ is the curvature of the space-like submanifold $\Sigma_s = \{x \in X : t(x) = s\}$. See, for example, Einstein [4], Tolman [12] or Weinberg [13].

For a “generalized” de Sitter metric (the expansion parameter $L$ is $>0$ but, typically, $\neq K$)

\[
    a(t) = \cosh(\sqrt{L}t), \quad h(t) = \sqrt{L}\tanh(\sqrt{L}t), \quad h'(t) = L/\cosh^2(\sqrt{L}t);
\]

in this case $R^{(g)} = 3Lg + 2(K - L) a^2(t) b(r) dr^2$.

4. Creation and annihilation operators

The choice of the typical fiber $\mathcal{H}_0$ (of the quantum bundle) and of the corresponding vector space $E_0$ (Section 2) are quite arbitrary and are done in dependence of what we want to describe. If the main objective is to build a theory of elementary particles and of their interactions, the obvious candidate for the role is a Fock space, direct sum of tensor product (symmetric or antisymmetric) of the one-particle Hilbert spaces, $\mathcal{H}_x^{(ONE)}$, invariant under the action of the (local) $g_x$–orthogonal group (Wigner [17], Schwartz [11], Weinberg [14], Chapter II).

Here $E_x^{(ONE)} \subset \mathcal{H}_x^{(ONE)}$ is chosen to be the nuclear space $S(T_x', \mathbb{C}^N)$ (a Schwartz space of indefinitely differentiable functions of rapid decrease), $1 \leq N < +\infty$; the vectors of $T_x'$ represent the particles energy-impulse. Then, the positive and $\mathbb{C}$–bilinear kernel (which is a tempered distribution of $S(T_x' \times T_x', \mathbb{C}^N \otimes \mathbb{C}^N)$)

\[
    K_x : f, g \mapsto K_x(f, g); \quad f, g \in S(T_x', \mathbb{C}^N), \quad x \in X
\]

assigns a pre-Hilbert structure on $S(T_x', \mathbb{C}^N)$: $(f, g)_x = K_x(\mathcal{T}, g)$; the completion of this pre-Hilbert space can be identified to the one-particle Hilbert space, as in [11] and [17]. Afterwards we can define the local Jordan-Wigner creation and annihilation operators (Fermi statistics),

\[
    a_x^*(f) = \sum_n a_{x,n}^*(f_n), \quad a(f) = (a^*(\mathcal{T}))^*,
\]

such that

\[
    [a_x(f), a_x^* (g)]_+ = K_x(f, g).I_x = \sum_{m,n} K_{x,mn}(f_m, g_n).I_x
\]

and similarly for the local Heisenberg creation and annihilation operators (Bose statistics). Here $m, n = 1, 2, \ldots, N$, $x \in X$, $f, g \in S(T_x', \mathbb{C}^N)$ and $I_x$ is the identity of $\mathcal{H}_x$; the maps $f \mapsto a(f)$, $f \mapsto b(f)$ are $\mathbb{C}$–linear and continuous.

Then let be $\mu_{x,M}^2$ (\(x \in X, M \in \mathbb{R}^+\)) a Lebesgue measure, concentrated on the local mass hyperboloid $\Omega_{x,M} = \{p \in T_x' : 3g_x(p, p) = M^2, \tau_x(p) = \pm 1\}$; $M$ is the effective mass of the particle we want to describe and we shall assume (this is the fundamental postulate) that:

\[
    \langle f, g \rangle_x = K_x(\mathcal{T}, g) = \sum_n \int_{\Omega_{x,M}} f_n(p) g_n(p).d\mu_{x,M}^{(R)}(p)
\]

where $d\mu_{x,M}^{(R)} := \rho_x^{(R)}.d\mu_{x,M}^{+}$ and $\rho_x^{(R)}$ is a real, continuous function of rapid decrease, dependent on the Ricci tensor of $g$, $R^{(g)}$; in fact, for invariance reasons, we really do not have much more choices. Subsequently it is assumed that:

\[
    \rho_x^{(R)}(p) = \mathcal{N}(x, p) \exp \left(-\frac{\lambda}{2M^2} \hat{R}_x(p, p)\right),
\]
$M_P$ is the Planck mass (or some other universal parameter), $\lambda$ is a real parameter, a priori $>0$ or $<0$, $N(x, p)$ is a normalization factor chosen in such a way that on some fixed spatial submanifold $\Sigma_0 = \{ x \in X : t(x) = t_0 \}$ (Section 3) the theory “reduces” to the standard one, that is $\rho^{(R)}_x(p) = 1$, on $\Sigma_0$. In the following we shall always set $t_0 = 0$.

For the de Sitter metric described at the end of Section 3, if $p = \varepsilon dt + \frac{a}{\varepsilon} p \cdot dr$, $p \in \mathbb{R}^3$ (so $M^2 = g(p, p) = \varepsilon^2 - p^2$),

$$
\bar{R}(p, p) = 3M^2 L + 2 \frac{L - K}{\cosh^2(\sqrt{L} t)} p^2,
$$

when $\varepsilon = M^2 + p^2$; therefore, posing $\lambda(t) := \frac{K - L}{M^2} \tanh^2(\sqrt{L} t)$,

$$
\rho^{(R)}_x(p) = \exp\left(-\lambda(t(x)) p^2\right)
$$

which play the role of a “smooth” cut-off, physically generated. Evidently $\lambda(K - L)$ has to be $> 0$.

5. Quantum fields and interactions

If the $w(p)$ are, for every $p \in T_x'$, linear maps $\mathbb{C}^{2N} \mapsto \mathbb{C}^N$ and $F, G$ are functions of $S(T'_x, \mathbb{C}^{2N})$ we can define the bounded, self-adjoint, anti-commuting operators $(n = 1, 2, \ldots, N, i, j = 1, 2, \ldots, 2N)$

$$
\xi_x(F) = \sum_{ij} \xi_{x,j} (F_i) = a_x^*(w.F) + a_x(w.F)
$$

hence

$$
[\xi_x(F), \xi_x(G)] = (\Delta_x(F, G) + \Delta_x(G, F)).I_x.
$$

$\Delta_x$ is a kernel of $S(T'_x, \mathbb{C}^{2N})$, $\Delta_x(F, G) = \sum_{ij} \int (\overline{F}_i m_{ij} G_j)(p).d\mu^{(R)}_x(p); m_{ij}(p) = \sum_n w_{nj}(p) w_{ij}(p) = m_{ij}(p)$; if $\Delta_x(F, F)$ is a unitary. The $\xi$ are essentially a local version of the anti-commuting Majorana “real variables” ([9], see also Weyl [16], Chapter IV); they describe, if $N = 2$, Majorana neutrinos. A “charged” particle, like the electron, can be described by a couple of such operators, $(\xi, \eta)$; usually, the corresponding Dirac fields are defined as $\psi = \frac{1}{\sqrt{2}} (\xi + i\eta)$ so $2 [\psi^*, \psi]_+ = [\xi, \xi]_+ [\eta, \eta]_+$. 

It is now possible to define a metric dependent version of the anti-commuting quantum fields, as $(u \in T_x, \chi_u(p) = e^{-i(p \cdot w)})$:

$$
\xi_j(x, u) := \xi_{x,j}(\chi_u) = \sum_{n} \left( a_{x,n}^* (\chi_u w_{nj}) + a_{x,n} (\chi_u w_{nj}) \right)
$$

which are well defined operator-valued functions, assigned on $T_X$. Note that it is alternatively possible to build the theory starting from the quantum fields, operator-valued functions defined on $T_X$. For a flat metric, the measures $\mu_M$ do not depend on $x$, $\rho^{(R)} = 1$ and the tangent space can be (fixed a space-time origin) identified to $X$: we shall obtain the usual, distributional, quantum fields.

We can also build a metric dependent version of the Pauli-Jordan functions:

$$
\Delta_{ij,M}(x, u) = \int_{\Omega^*_x} e^{ip \cdot u} m_{ij}(p).d\mu^{(R)}_{x,M}(p)
$$

so $[\xi_i(x, u), \xi_j(x, v)]_+ = \Delta_{ij,M} (\Delta_{ij,M} \Delta_{ij,M}) (x, u - v).I_x$. The $u \in T_x \mapsto \Delta_{ij,M}(x, u)$ are now true, differentiable and bounded functions. In fact the functions $m_{ij}$ are tempered, for every
representation of the $g$--orthogonal group, so $|m_{ij}(p)| \leq c \left( \frac{p^2}{M^2} \right)^d$ (from some $c > 0, d \geq 0$) and

$$|\Delta_{ij,M}(x,u)| \leq c \int_{\mathbb{R}^3} \left( \frac{p^2}{M^2} \right)^d \frac{e^{-\lambda_\alpha(t(x)) p^2}}{2\sqrt{M^2 + p^2}} dp$$

$$= c\pi M^2 \int_{\mathbb{R}^+} e^{-\lambda_\alpha(t(x))M^2 q} q^{d+\frac{1}{2}} (1 + q)^{-\frac{1}{2}} dq;$$

the last integral is proportional to a confluent hypergeometric function $\lambda_\alpha \rightarrow \Psi \left( d + \frac{3}{2}, d + 2, \lambda_\alpha M^2 \right)$ (see Gradshteyn-Ryzhik [6], Section 9.2) which is analytic, for $\lambda_\alpha > 0$, but diverge, as $(1/\lambda_\alpha)^{d+1}$, for $\lambda_\alpha \rightarrow 0^+$ (that is for $\lambda \rightarrow 0$ or $M_P \rightarrow +\infty$; or when $t \rightarrow 0$).

The (quantum) Lagrangian densities (the free and the interaction terms) are here built in the usual way (see, for example, [1], [14] and, particularly, Jauch and Rohrlich, [7]) starting from a multilinear, self-adjoint expression, $\mathcal{L}(x,u)$, of the quantum fields; $x \in X, u \in \mathcal{T}_x$. The action integral, at $x \in X$, is obtained formally “summing” over all the vectors of $\mathcal{T}_x$:

$$A_x = \int_{\mathcal{T}_x} \mathcal{L}(x,u).d\nu(u)$$

where $\nu$ is some Lebesque measure assigned on $\mathcal{T}_x$. Note that the measure $\nu$ provides eventually a space-time cut-off and it is physically irrelevant, the relevant ultraviolet divergences (vacuum fluctuations, self masses, self charges) are controlled by the function $\rho(R)$, of rapid decrease.

Hence, when $\lambda_\alpha \approx 0$, we shall almost recover every standard, special relativistic theory. Almost, because some terms, in a perturbation expansion, became very large, when $\lambda_\alpha \approx 0$. For example, when we try to calculate the electron “self-energies”, in quantum electrodynamics, the Weisskopf-Furry term (Weisskopf [15]), depending on $\lambda_\alpha$ and now logarithmic diverging when $\lambda_\alpha \rightarrow 0^+$ (from integrals like to $\int_{\mathbb{R}^+} e^{-\lambda_\alpha(t(x))M^2 q}.q^{\frac{1}{2}} (1 + q)^{-\frac{1}{2}} dq$ is always there; we have to face this problem (Sections 6, 7).

6. Self-masses and counter terms

The Electro-Weak one-loop calculations of the leptons self-energies (self-masses) proceed now in the standard way, the factor $\exp \left( -\lambda_\alpha(t) p^2 \right)$ provides only a “smooth” cutoff. The calculations can be done directly, as in [15] or by the propagators machinery ([1], [7], [14]). In the latter case, for every loop, we have to add to $M$ (the effective mass of the lepton we are considering) a term like $t$ is always the time coordinate):

$$M_L, \alpha_X^* I(t, M_L, M_X);$$

$\alpha_X^*$ is a constant, proportional to the coupling parameter $\alpha_X$; $M_L$ is the mass of the lepton of the loop; $M_X$ is the mass of the particle which mediate the interaction (photons, $Z$--particles or $W$--particles). The functions $I$ can be computed by standard techniques (Feynman parametrization, Wick rotation); for $\lambda_\alpha \rightarrow 0$ and $M_X = 0$

$$I(t, M_L, 0) \approx -\log \left( \lambda_\alpha(t) M_L^2 \right) + \frac{5}{3} - c$$

$$= - \left( \log \tanh^2(\sqrt{t}L) + \log (M_L/M_P)^2 \right) + c$$

while if $M_L << M_X$:

$$I(t, M_L, M_X) \approx -\log \left( \lambda_\alpha(t) M_X^2 \right) + \frac{7}{6} - c$$

$$= - \left( \log \tanh^2(\sqrt{t}L) + \log (M_X/M_P)^2 \right) + c.$$
\( c = (\psi(\frac{3}{2}) + 2\gamma) \), \( c_1 \) and \( c_2 \) are constant, not depending on \( t \) or on \( M_L \) and \( M_X \).

Ignoring the \( W \)-term, proportional to the neutral lepton mass, the charged leptons self-masses, \( M_{SELF} \), are obtained adding the photon term, \( \alpha_s^* A_I(t, M, 0) \) and the \( Z \)-term, \( \alpha_s^* Z_I(t, M, M_Z) \):

\[
M_{SELF}(t)/M = - (\alpha_s^* + \alpha_s^Z) \log \tanh^2(\sqrt{L}t) - \alpha_s^* \log (M_L/M_s)^2, \tag{20}
\]

the constants are absorbed by \( M_s \). \( M_{SELF} \) now depends on \( x \), we are forced to assume that the interaction Lagrangian contains a mass-like counter-term such that we have to add to \( M_{SELF} \) another real function of \( x \), \( M_\Theta \), proportional to \( M \) and such that \( M_{SELF}(x) + M_\Theta(x) = 0 \), for every \( x \in X \).

Alternatively (and perhaps more perspicuously) it is possible to start from a “bare” mass dependent on \( x \), \( M(x) (= M + M_\Theta(x)) \) and to impose that \( M(x) + M_{SELF}(x) = M \) where \( M \) is the effective mass, not dependent on \( x \). In any case we shall assume that the \( M_\Theta \) term is associated to a scalar (“classical”) field \( \Theta \) by a relation like to:

\[
\Delta_x \Theta(x) = - \frac{\theta}{M} M_\Theta(x) \cdot \Theta(x); \tag{21}
\]

here \( \Delta_x \) is the Laplace-Beltrami differential operator of the metric tensor \( g \) and \( \theta (\neq 0) \) is some real parameter.

When \( K > 0 \), \( X \) has the topology of \( \mathbb{R} \times S^3 \); therefore, if \( (t, n) \) is a diffeomorphism \( X \rightarrow \mathbb{R} \times S^3 \) (\( n \in \mathbb{R}^4 \), \( n^2 = 1 \)), posing \( \Theta(x) = T(tx) \cdot S_t(n(x)) \)

\[
T''(t) + 3 h(t) T'(t) + \frac{K \ell (\ell + 2)}{a^2(t)} T(t) = \frac{\theta}{M} M_{SELF}(t) T(t); \tag{22}
\]

the \( S_t \) are the \( S^3 \)-spherical harmonics, \( \ell = 0, 1, 2, \ldots \); note that there are \((\ell + 1)^2 \) independent \( S^3 \)-spherical harmonics, for every \( \ell \). Setting \( a(t)^{3/2} T(t) = U(\sqrt{L} t) \ (\sqrt{L} t = s) \), \( a(t) = \cosh(\sqrt{L} t) \) (Section 3)

\[
U''(s) = \left( w - w_A \log \tanh^2 s - \frac{w_B}{\cosh^2 s} \right) U(s) \tag{23}
\]

where \( w_A = \frac{\theta}{L} (\alpha_s^* + \alpha_s^Z) \), \( w_B = \frac{3}{4} + \frac{K}{4} \ell (\ell + 2) \) and \( w = - \frac{\theta}{L} \alpha_s^* \log (M/M_s)^2 \). For \( w < 0 \), \( U \) oscillates (at infinity) while, for \( w > 0 \) (that is \( M < M_s \)), \( U \) generally diverge (at infinity), except for a finite set of value of \( w \): \( w_n, n = 0, 1, 2, \ldots, n_0 \); in this case \( U \) goes to zero exponentially, at infinity; besides, if \( n \) is even (odd), \( U \) is even (odd).

Approximating \( \log \tanh^2 s \) by \(-1/ \cosh^2 s \), \( U''(s) = \left( w - \frac{w_B - w_A}{\cosh^2 s} \right) U(s) \) and this differential equation can be integrate analytically in term of Jacobi polynomials ([6], Section 8.9), \( z = \tanh s, n = 0, 1, 2, \ldots \)

\[
\mathbf{P}^{(\alpha, \beta)}(n, b) = \frac{1}{2^n} \sum_{a, b \in \mathbb{N}} \left( \begin{array}{c} n + \alpha \\ a \\ \end{array} \right) \left( \begin{array}{c} n + \beta \\ b \\ \end{array} \right) (z + 1)^a (z - 1)^b. \tag{24}
\]

Then \( w_n = \left( \sqrt{w_B - w_A + \frac{1}{4} - \frac{1}{2} - n} \right)^2, n < \sqrt{w_B - w_A + \frac{1}{4} - \frac{1}{2}}, \alpha = \beta = \sqrt{w_B - w_A + \frac{1}{4} - \frac{1}{2} - n} \).
7. Lepton masses

Now we shall assume that the lepton case correspond to $\ell = 1$; if $w_A = 0.40$ and $w_B = 8.32$ (that is $\frac{K}{L} = 2.52$) a numerical integration of the $U$ differential equation leads to $n_0 = 2$ (so there are exactly three generations of leptons) and to

$$w_0 = 4.798, \quad w_1 = 1.708, \quad w_2 = 0.0681. \quad (25)$$

Inserting the experimental value of $\alpha_Z/\alpha_A$ ($\Gamma_Z (\ell, el) \approx \frac{1}{3} M_Z \alpha_Z$, but $\Gamma_Z = 0.084$ $GeV$, $M_Z = 91.2$ $GeV$ (PDG [10]) hence $\alpha_Z = 2.76 \cdot 10^{-3}$ and $\alpha_Z/\alpha_A = 0.378$) we get $\frac{\theta}{L} \alpha_A^* = \frac{\alpha_A}{\alpha_A + \alpha_Z} = 0.29$ (and $M_* = 1.99$ $GeV$). Then

$$\log \frac{M_n}{M_0} = \frac{w_0 - w_n}{0.58}; \quad (26)$$

$n = 0$ for the electron, $n = 1$ for the $\mu$-particle, $n = 2$ for the $\tau$-particle, therefore

$$\log \frac{M_\mu}{M_{el}} = 5.33, \quad \log \frac{M_\tau}{M_{el}} = 8.15 \quad (27)$$

 compatibly to experimental data.

Finally observe that the real function $s \mapsto -w_A \log \tanh^2 s - w_B / \cosh^2 s$ ($w_A, w_B > 0$) diverge positively at the origin, has a negative minimum (if $w_B > w_A$) and goes to zero at infinity; on the contrary if, for example, $a(t) = e^{\frac{t}{2}^2}$ we have to consider a function like $s \mapsto w_A \log (1 - e^{-s^2}) + w_B e^{-s^2} - w_C s^2$; consequently there are typically a denumerable infinity of self values: the metric expansion is, so to speak, too fast.

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