New and More Solitary Wave Solutions for the Klein-Gordon-Schrödinger Model Arising in Nucleon-Meson Interaction

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This paper considers methods to extract exact, explicit, and new single soliton solutions related to the nonlinear Klein-Gordon-Schrödinger model that is utilized in the study of neutral scalar mesons associated with conserved scalar nucleons coupled through the Yukawa interaction. Three state of the art integration schemes, namely, the \( e^{-\Phi(\xi)} \)-expansion method, Kudryashov’s method, and the tanh-coth expansion method are employed to extract bright soliton, dark soliton, periodic soliton, combo soliton, kink soliton, and singular soliton solutions. All the constructed solutions satisfy their existence criteria. It is shown that these methods are concise, straightforward, promising, and reliable mathematical tools to untangle the physical features of mathematical physics equations.

Keywords: traveling wave solution, tanh-coth method, \( e^{-\Phi(\xi)} \)-expansion method, Kudryashov's method, Klein-Gordon-Schrödinger equation

1. INTRODUCTION

Many of the problems arising in mathematically-oriented scientific fields such as physics and engineering are described by partial differential equations (PDEs). PDEs are used to depict an ample variety of phenomena such as dislocations in crystals, superconductivity, laser pulses in two-phase [1, 2], waves in ferromagnetic materials, and many more [3–6]. Many theories such as electromagnetism, diffusion, fluid flow, etc. are presented to understand the dynamics of PDEs [7, 8]. Therefore, exploring exact solutions for PDEs plays an important role in such fields. These solutions might be essential and important for exploring some physical phenomena. The majority of PDEs are not exactly solvable with existing mathematical techniques. Especially for higher order nonlinear PDEs, existing methods are not able to find exact solutions. However, due to the invention of algebraic system solvers such as Mathematica and Maple, many integrating schemes have been proposed, such as the \( e^{-\Phi(\xi)} \)-expansion method, Hirota’s bilinear method, the homogeneous balance reduction of the PDE to a quadrature problem, the truncated Painlevé expansion, etc. [9–13].

Solitons are formed because of an interplay between nonlinear and dispersive effects. The importance of such waves lies in their roles in telecommunication systems as well as other physical
This coupled system describes the interplay of a meson field with a nucleon field and is significant in modern physics. Here \( N = N(x, t) \) is a meson field, \( W = W(x, t) \) is a complex scalar nucleon field, and \( c \) is a real constant.

### 3. SOLITON SOLUTIONS OF (1+1)-DIMENSIONAL KGS EQUATIONS

In this section, three state of the art integration schemes, Raza et al. [34], Asokan and Vinodh [35], and Ullah et al. [36] are employed to extract bright soliton, dark soliton, dipole and combo soliton, kink soliton, and singular soliton solutions.

The following wave transformation

\[
W(x, t) = w(\xi)e^{-(kx+\omega t)}, \quad N(x, t) = n(\xi),
\]

is applied, where \( \xi = x - \alpha t \), to obtain traveling wave solutions for the proposed model given by Equations (1) and (2). In the above transformation the wave number is \( \omega \), \( k \) is the frequency, and \( \alpha \) is the velocity of the soliton.

Plugging Equation (3) into Equations (1) and (2), then equating the real parts gives

\[
(w(\xi))^2 + w(\xi)n(\xi) - (k^2 + \omega)w(\xi) = 0, \quad (\alpha^2 - c^2)(n(\xi)) + n(\xi) + (w(\xi))^2 = 0.
\]

The imaginary part of Equation (1) gives the velocity of soliton, i.e., \( \alpha = -2k \).

Solving Equation (4) for \( n(\xi) \), we get

\[
n(\xi) = \frac{(k^2 + \omega)w(\xi) - (w(\xi))^\prime}{w(\xi)}. \quad (6)
\]

After plugging the value of \( n(\xi) \) in Equation (5), we obtain the following ODE as

\[
(\alpha^2 - c^2)\left((k^2 + \omega)w - \frac{w^\prime}{w}\right)^\prime + \frac{w^\prime}{w} + w^2 = 0.
\]

In accordance with the \( e^{-\Phi(\xi)} \) expansion scheme [34], the solution of Equation (7) has the form

\[
w(\xi) = \sum_{i=0}^{N} a_i(e^{-\Phi(\xi)})^i. \quad (8)
\]

The homogenous balance method gives \( N = 1 \). For \( N = 1 \), Equation (8) becomes

\[
w(\xi) = a_0 + a_1e^{-\Phi(\xi)}, \quad (9)
\]

here \( \Phi(\xi) \) is the solution of the following ODE

\[
\Phi'(\xi) = e^{-\Phi(\xi)} + me^{-\Phi(\xi)} + l. \quad (10)
\]
By substituting Equation (9) into Equation (7) a system of equations for $a_0$ and $a_1$ is retrieved by comparing the coefficients of $e^{-\Phi(x)}$ equal to zero. By finding unknowns $a_0$ and $a_1$ from the obtained system and inserting them into Equation (9), solutions of the coupled KGS Equations (1) and (2) are obtained. The obtained results are summarized in the following sets.

**SET 1**

\[
\begin{align*}
a_0 &= -\frac{l}{\sqrt{2}}, \\
\omega &= -k^2 - \frac{l^2}{2} + 2m, \\
c &= -\alpha.
\end{align*}
\]

**SET 2**

\[
\begin{align*}
a_0 &= 0, \\
a_1 &= \pm \sqrt{2}, \\
l &= 0, \\
\omega &= -k^2 + 2m, \\
c &= -\alpha.
\end{align*}
\]

Soliton solutions for **SET 1** are calculated.

When $s > 0$ and $m \neq 0$, then

\[
W(x, t) = -e^{i(-kx + \omega t)} \left[ \frac{l}{\sqrt{2}} + \frac{2\sqrt{2} m}{-l - \sqrt{s} \tanh \left[ \frac{1}{2} \sqrt{s}(C + \xi) \right]} \right]
\]

and

\[
N(x, t) = -\frac{s}{2} + \frac{2sm}{(l \cos \left[ \frac{1}{2} \sqrt{-s}(C + \xi) \right] - \sqrt{-s} \sin \left[ \frac{1}{2} \sqrt{-s}(C + \xi) \right])^2}
\]

The **Figure 1**, depicts the graphical representation of the absolute values of the obtained soliton solutions given in Equation (11) and Equation (12). In **Figure 1A** represents the kink soliton and **Figure 1B** represents the bright soliton.

When $s < 0$ and $m \neq 0$, then

\[
W(x, t) = \pm e^{i(-kx + \omega t)} \left[ \frac{l}{\sqrt{2}} \left. + \frac{2\sqrt{2} m}{-l - \sqrt{-s} \tan \left[ \frac{1}{2} \sqrt{-s}(C + \xi) \right]} \right] \right]
\]

and

\[
N(x, t) = 2m - \frac{1}{2} \left( \frac{k^2 - 2m}{l} \right) \coth \left[ \frac{1}{2} \sqrt{2} \left( C + \xi \right) \right]
\]

The **Figure 2**, depicts the graphical representation of the absolute values of the obtained soliton solutions given in Equation (13) and Equation (14). In **Figure 2A** represents the periodic soliton and **Figure 2B** represents the singular soliton.

When $s > 0$ and $m = 0$ and $l \neq 0$ then

\[
W(x, t) = -e^{i(-kx + \omega t)} \left[ \frac{l}{\sqrt{2}} \left. + \frac{2\sqrt{2} m}{-l - \sqrt{2} \tanh \left[ \frac{1}{2} \sqrt{2}(C + \xi) \right]} \right] \right]
\]

and

\[
N(x, t) = 2m - \frac{1}{2} \left( \frac{k^2}{l} \right) \coth \left[ \frac{1}{2} \sqrt{2} \left( C + \xi \right) \right]
\]
When $s = 0$ and $m \neq 0$ and $l \neq 0$, then

$$W(x, t) = \pm e^{i(-kx + \alpha t)} \frac{l}{\sqrt{2}(-1 + l(C + \xi))}$$

and

$$N(x, t) = 2m + l^2 \left( -\frac{1}{2} - \frac{2}{(1 + l(C + \xi))^2} \right)$$

When $s = 0$ and $m = 0$ and $l = 0$, then

$$W(x, t) = \pm e^{i(-kx + \alpha t)} \frac{\sqrt{2}}{C + \xi}$$

and

$$N(x, t) = -\frac{j^2}{2} + 2\mu - \frac{2}{(C + \xi)^2}$$

where $C$ is the constant of integration and $s = l^2 - 4m$

Soliton solutions for Set 2 are calculated.

When $s > 0$ and $m \neq 0$, then

$$W(x, t) = \mp e^{i(-kx + \alpha t)} \sqrt{2m \cosh[\sqrt{-m}(C + \xi)]}$$

provided that $m < 0$.

$$N(x, t) = 2m(\coth^2[\sqrt{-m}(C + \xi)])$$

provided that $m < 0$.

When $s < 0$ and $m \neq 0$, then

$$W(x, t) = \pm e^{i(-kx + \alpha t)} \sqrt{2m \cosh[\sqrt{-m}(C + \xi)]}$$

provided that $m > 0$.

$$N(x, t) = -2m(\cot^2[\sqrt{m}(C + \xi)])$$

provided that $m > 0$, where $C$ is the constant of integration and $s = l^2 - 4m$

According to the tanh-coth method [35], the estimated solution of Equation (7) has the form

$$w(\xi) = \sum_{i=0}^{N} a_i \tanh(m\xi) + \sum_{i=1}^{N} b_i \tanh^{-1}(m\xi).$$

The homogenous balance method, gives $N = 1$. For $N = 1$, the above equation takes the form

$$w(\xi) = a_0 + a_1 \tanh(m\xi) + b_1 \coth(m\xi).$$

Substituting Equation (26) into Equation (7), a system of nonlinear equations for $a_0$, $a_1$, and $b_1$ is obtained by the comparison of the coefficients of $\tanh(m\xi)$ to zero. Upon solving the obtained system for $a_0$, $a_1$, and $b_1$ and plugging them in Equation (26), the solutions of the coupled KGS Equations (1) and (2) are obtained.

The obtained results are summarized in the following sets.

**SET 1**

$$a_0 = 0 = a_1, \quad b_1 = \pm \sqrt{2}m,$$

$$\omega = -k^2 - 2m^2, \quad \alpha = -c$$

**SET 2**

$$a_0 = 0, \quad a_1 = \pm \sqrt{2}m, \quad b_1 = \pm \sqrt{2}m,$$

$$\omega = -k^2 - 8m^2, \quad \alpha = -c$$
FIGURE 3 | 3D plots of $|W(x, t)|$ and $|N(x, t)|$ given in Equations (32) and (33) with values of parameters as $m = 1$, $k = 1$ and $\xi = x + 2t$.

FIGURE 4 | 3D plots of $|W(x, t)|$ and $|N(x, t)|$ given in Equations (38) and (39) with values of parameters as $m = 1$, $d = 1$, $k = 1$ and $\xi = x + 2t$.

SET 3

$$a_0 = 0 = b_1, \quad a_1 = \pm \sqrt{2}m,$$
$$\omega = -k^2 - 2m^2, \quad \alpha = -c$$

(29)

Singular soliton solutions relative to SET 1 are obtained as

$$W(x, t) = \pm e^{(\xi - kx + \omega t)} \sqrt{2}m \coth[m\xi],$$

(30)

and

$$N(x, t) = -2m^2 \coth^2[m\xi]$$

(31)

A dark-singular combo soliton solution relative to SET 2 is attained as

$$W(x, t) = \pm e^{(\xi - kx + \omega t)} \sqrt{2}m \left( \coth[m\xi] + \tanh[m\xi] \right),$$

(32)

and a singular soliton is calculated as

$$N(x, t) = -8m^2 \coth^2[2m\xi]$$

(33)

The Figure 3, depicts the graphical representation of the absolute values of the obtained soliton solutions given in Equation (32) and Equation (33). In Figure 3A represents the dark-singular combo soliton and Figure 3B represents the singular soliton.

A dark soliton solution relative to SET 3 is attained as

$$W(x, t) = \pm e^{(\xi - kx + \omega t)} \sqrt{2}m \tanh[m\xi],$$

(34)
and bright soliton is calculated as

$$N(x, t) = -2m^2 (1 - \text{sech}^2[m\xi])$$  \hspace{1cm} (35)$$

According to Kudryashov’s method [36], the predicted solution of Equation (7) has the following form

$$w(\xi) = \sum_{i=0}^{N} a_i \left( \frac{1}{1 + d(\cosh(\xi) + \sinh(\xi))} \right)^i.$$  \hspace{1cm} (36)$$

The homogenous balance method gives \( N = 1 \). For \( N = 1 \), Equation (36) becomes

$$w(\xi) = a_0 + \frac{a_1}{1 + d(\sinh(\xi) + \cosh(\xi))}. \hspace{1cm} (37)$$

Inserting Equation (37) into ODE Equation (7), an algebraic system of equations for \( a_0 \) and \( a_1 \) is obtained by equating every coefficient of different powers of \( \frac{1}{1 + d(\cosh(\xi) + \sinh(\xi))} \) to zero. The obtained system is solved for \( a_0 \) and \( a_1 \), and replacing these values in Equation (37) gives solutions of the coupled KGS Equations (1) and (2). The following solution set arises

**SET 1**

$$a_0 = \pm \frac{1}{\sqrt{2}}, \quad a_1 = \mp \sqrt{2}, \quad c = -\alpha,$$

$$\omega = \frac{1}{2}(-1 - 2k^2)$$

A kink soliton solution is given as

$$W(x, t) = e^{(kx + \omega t)\sqrt{2m}} \left( \pm \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{1 + d(\cosh(\xi) + \sinh(\xi))} \right)$$  \hspace{1cm} (38)$$

and a bright soliton solution is obtained as

$$N(x, t) = -\frac{(-1 + d(\sinh(\xi) + \cosh(\xi)))^2}{2(1 + d(\sinh(\xi) + \cosh(\xi)))^2}. \hspace{1cm} (39)$$

The Figure 4, depicts the graphical representation of the absolute values of the obtained soliton solutions given in Equation (38) and Equation (39). In Figure 4A represents the kink soliton and Figure 4B represents the bright soliton.

3.1. Novelty of the Results

It is worth mentioning here that the proposed model has been solved for the first time by the \( e^{-\Phi(\xi)} \)-expansion method, tanh-coth expansion technique, and Kudryashov’s method to extract solitonic structures. The results presented in this piece of research could be very useful in discussing the physical properties of the different nonlinear evolution equations emerging in quantum mechanics, fluid dynamics, and plasma physics. The solitonic structures obtained in this study could attract the attention of researchers working in the field of optical fiber communication systems. The comparison of our results, with the outcomes of [20, 21], show that bright and dark solitons as well as dipole soliton, singular soliton, and kink soliton solutions have been found for the first time in this article.

4. RESULTS AND DISCUSSION

It is important to clarify that the analytical methods utilized in this article are truly state of the art techniques for extracting the soliton solution of the non-linear Klein-Gordon-Schrödinger model. It is important to note here that each integration method has its own benefits and disadvantages compared to other accessible strategies. For example, the inverse scattering method is not useful for log-law, power law, and dual-power law nonlinearities. Only bright solitons are recovered by the semi-inverse variational algorithm. Likewise, here, the \( e^{-\Phi(\xi)} \)-expansion technique gives bright soliton, kink soliton, periodic soliton, and singular soliton solutions. The second method applied in this research extracts dark soliton, dark-singular combo soliton, and singular soliton solutions. The third method employed here obtains bright soliton and kink soliton solutions.

5. CONCLUSION

In this article, new soliton solutions have been obtained by utilizing three well-known integration architectures namely, the tanh-coth expansion strategy, Kudryashov’s strategy, and the \( e^{-\Phi(\xi)} \)-expansion strategy. To the best of our knowledge, these fresh examples of soliton solutions have been obtained for the first time for the KGS model. Since the invention of symbolic computation tools, the solution procedures have been simplified, and therefore the described methods are becoming more efficient in solving many physical problems. The outcomes of this paper consist of dispersive solitons incorporating CQS and cubic nonlinearities. Kudryashov’s method along with the generalized tanh method extract singular, bright, singular periodic, and a combo type of solitons for the given model. The advantage of these techniques is quite evident as they have no limitations in finding such wave profiles.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article.Supplementary material, further inquiries can be directed to the corresponding author/s.

AUTHOR CONTRIBUTIONS

All authors contributed equally to the writing of this paper and read and approved the final version of the manuscript.
REFERENCES

1. Hiki Y, Kogure Y, Itoh A. Effect of crystal dislocation on superconductivity of aluminum. Jpn J Appl Phys. (1987) 26:3. doi: 10.1143/JJAP.26.392

2. Mustafa H, Matthews DTA, Ramer GRBE. Investigation of the ultrashort pulsed laser processing of zinc at 515 nm, morphology, crystallography and ablation threshold. Mater. Design. (2019) 169:103922. doi: 10.1016/j.matdes.2019.107675

3. Raza N, Jhangeer A, Rezazadeh H, Bekir A. Explicit solutions of the (2+ 1)-dimensional Hirota–Maccari system arising in nonlinear optics. Int J Modern Phys B. (2019) 33:1950360. doi: 10.1142/S021797921950360

4. Raza N, Javid A. Optical dark and singular solitons to the Biswas–Milovic equation in nonlinear optics with spatio-temporal dispersion. Optik. (2018) 158:1049–57. doi: 10.1016/j.ijleo.2017.12.186

5. Ali KK, Wazwaz AM, Osman MS. Optical soliton solutions to the generalized nonautonomous nonlinear Schrödinger equations in optical fibers via the sine-Gordon expansion method. Optik. (2020) 208:164132. doi: 10.1016/j.ijleo.2019.164132

6. Raza N, Sial S, Kaplan M. Exact periodic and explicit solutions of higher dimensional equations with fractional temporal evolution. Optik. (2018) 156:628–34. doi: 10.1016/j.ijleo.2017.11.107

7. Wang YP, Xia DF. Generalized solitary wave solutions for the Klein-Gordon-Schrödinger equations. Comput Math Appl. (2009) 58:2300–6. doi: 10.1016/j.camwa.2009.03.012

8. Kumar D, Singh J, Kumar S, Sushila. Numerical computation of Klein-Gordon equations arising in quantum field theory by using homotopy analysis transform method. Alexandria Eng J. (2014) 53:469–74. doi: 10.1016/j.aej.2014.02.001

9. Petrovic NZ, Bohra M. General Jacobi elliptic function expansion method applied to the generalized (3 + 1)-dimensional nonlinear Schrödinger equation. Opt Quantum Electron. (2016) 48:4. doi: 10.1007/s11082-016-0522-1

10. Abdou MA. The extended F-expansion method and its application for a class of nonlinear evolution equations. Chaos Solitons Fract. (2007) 31:95–104. doi: 10.1016/j.chaos.2005.09.030

11. Hsu CS, Chiu HM. A cell mapping method for nonlinear deterministic and stochastic systems-Part II. J Appl Mech. (1986) 53:702–10. doi: 10.1115/1.3171834

12. Ray SS. An application of the modified decomposition method for the solution of the coupled Klein-Gordon-Schrödinger equation. Commun Nonlinear Sci Numer Simul. (2008) 13:1311–7. doi: 10.1016/j.cnsns.2006.12.010

13. Biswas A, Triki H. 1-Soliton solution of the Klein-Gordon–Schrödinger’s equation with power law nonlinearity. Appl Math Comput. (2010) 217:3869–74. doi: 10.1016/j.amc.2010.09.046

14. Dehghani M, Talei A. Numerical solution of the Yukawa-coupled Klein-Gordon-Schrödinger equations via a Chebyshev pseudo spectral multi domain method. Appl Math Model. (2012) 36:2340–9. doi: 10.1016/j.apm.2011.08.030

15. Yumak A, Boubaker K, Petkova P. An attempt to give exact solitary and periodic wave polynomial solutions to the nonlinear Klein-Gordon-Schrödinger equations. Chaos Solitons Fract. (2015) 81:299–302. doi: 10.1016/j.chaos.2015.09.031

16. Wang J, Liang D, Wang Y. Analysis of a conservative high-order compact finite difference scheme for the Klein-Gordon-Schrödinger equation. J Comput Appl Math. (2019) 358:84–96. doi: 10.1016/j.cam.2019.02.018

17. Kong L, Chen M, Yin X. A novel kind of efficient symplectic scheme for Klein-Gordon–Schrödinger equation. Appl Numer Math. (2019) 135:481–86. doi: 10.1016/j.apnum.2019.05.005

18. Liang H. Linearly implicit conservative schemes for long-term numerical simulation of the Klein-Gordon–Schrödinger equations. Appl Math Comput. (2014) 238:475–84. doi: 10.1016/j.amc.2014.04.032

19. Fukuda M, Tsutsumi M. On the Yukawa-coupled Klein-Gordon-Schrödinger equations in three space dimensions. Proc Jpn Acad. (1975) 51:402–5. doi: 10.3792/pjpa/1195518563

20. Fukuda M, Tsutsumi M. On coupled Klein-Gordon-Schrödinger equations. J Math Anal Appl. (1978) 66:358–78. doi: 10.1016/0022-247X(78)90239-1

21. Raza N, Arshed S, Sial S. Optical solitons for coupled Fokas-Lenells equation in birefringence fibers. Modern Phys Lett B. (2019) 33:1950317. doi: 10.1142/S0217984919503172

22. Asokan R, Vinodh D. The tanh-coth Method for Soliton and Exact Solutions of the Sawada-Kotera Equation. Int J Pure Appl Math. (2017) 117:19–27. Available online at: https://acadpubl.eu/jsi/2017-117-11-14/articles/13/3.pdf

23. Ullah N, Rehman H, Imran MA, Abdeljawad T. Highly dispersive optical solitons with cubic law and cubic-quintic-septic law nonlinearities. Results Phys. (2020) 17:103021. doi: 10.1016/j.rinp.2020.103021

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