Diamonds’s Temperature: 
Unruh effect for bounded trajectories and thermal time hypothesis

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Abstract

We study the Unruh effect for an observer with a finite lifetime, using the thermal time hypothesis. The thermal time hypothesis maintains that: (i) time is the physical quantity determined by the flow defined by a state over an observable algebra, and (ii) when this flow is proportional to a geometric flow in spacetime, temperature is the ratio between flow parameter and proper time. An eternal accelerated Unruh observer has access to the local algebra associated to a Rindler wedge. The flow defined by the Minkowski vacuum of a field theory over this algebra is proportional to a flow in spacetime and the associated temperature is the Unruh temperature. An observer with a finite lifetime has access to the local observable algebra associated to a finite spacetime region called a “diamond”. The flow defined by the Minkowski vacuum of a (four dimensional, conformally invariant) quantum field theory over this algebra is also proportional to a flow in spacetime. The associated temperature generalizes the Unruh temperature to finite lifetime observers.

Furthermore, this temperature does not vanish even in the limit in which the acceleration is zero. The temperature associated to an inertial observer with lifetime \(T\), which we denote as “diamond’s temperature”, is \(T_D = \frac{2\hbar}{\pi k_bT}\). This temperature is related to the fact that a finite lifetime observer does not have access to all the degrees of freedom of the quantum field theory. However, we do not attempt to provide any physical interpretation of our proposed assignment of a temperature.

1 Introduction

This work has two motivations. First, to study the Unruh effect\(^{1}\) for an observer with a finite lifetime. Second, to probe the thermal time hypothesis\(^{2}\). The Unruh effect is the theoretical observation that the vacuum state \(\Omega\) of a quantum field theory on Minkowski spacetime looks like a thermal equilibrium state for an uniformly accelerated observer with acceleration \(a\). The observed temperature is the Davis-Unruh temperature \(T_U = \frac{\hbar a}{2\pi k_b c}\), and is closely related to Hawking’s black hole temperature (\(c\) is the speed of light, \(\hbar\) and \(k_b\) are the Planck and Boltzmann constants.)

There exist many derivations of the Unruh effect\(^{3-4}\). One generally assumes that the observer moves over along an infinite worldline of constant acceleration. A model detector moving on this world line and interacting with the vacuum state of the field becomes excited and reaches a thermal state. For a free field, this can be computed integrating the two-point functions along the infinite trajectory. Alternatively, one can consider the region causally connected to the world line. This
region is the Rindler wedge $W$. A quantization scheme for the field restricted to $W$ leads to
the definition of particle states. The conventional vacuum $\Omega$ is a thermal distribution state of
these particle. Another approach is based on Bisognano-Wichman theorem: World lines of
uniformly accelerated observers in $W$, parametrized by proper time, are orbits of the action of a
one-parameter group, which can be interpreted as the time flow of the accelerated observers. The
vacuum state $\Omega$ is a KMS state with temperature $T_U$, with respect to this time flow, over the
algebra of the local observables in $W$. Intuitively, the reason the accelerated observer does not see
a pure state can be viewed as related to the fact that he has no access to part of $\Omega$. Each observer
instantaneity surface is cut into two parts by the edge of the wedge. The degrees of freedom of the
field on the other side of the edge are inaccessible to the observer. Since $\Omega$ has vacuum correlations
across the edge, the two sets of degrees of freedom on the two sides of the edge are entangled. The
restriction of $\Omega$ to the edge interior, therefore, fails to be pure a state.

Doubts have been raised on the universality of the Unruh effect, for instance in Ref. [8]. One of
the sources of the doubts is that most derivations of the effect consider infinite trajectories, while
the trajectories of realistic detectors have a beginning and an end. In our opinion, the theoretical
support for the Unruh effect is rather robust, and these doubts do not have much consistency.
Nonetheless, the issue of observers with finite lifetime is interesting. The region causally connected
to a finite trajectory is far smaller than $W$. Typically, it is the intersection of the future of the
beginning point of the trajectory with the past of its end point. Such a region is called a “diamond”.
In suitable Lorentz coordinates $x^\mu = (x^0, \vec{x})$, any diamond can be represented as the region $D_L$, with $L > 0$, defined by
\[ |x^0| + |\vec{x}| < L. \tag{1} \]
$T = 2L$ is the lifetime of the observer if he moves inertially. In a sense, only the finite region $D_L$
can take part in the thermalization process. More precisely, the observer has access only to the
degrees of freedom of the field that lie within $|\vec{x}| < L$ on the $x^0 = 0$ simultaneity surface. Does this
observer see a thermal state? Is there a sense in which there is an associated temperature? These
questions can be addressed in a variety of ways. Here we consider the possibility of adapting the
Bisognano-Wichman approach to this case.

Let us now come to the thermal time hypothesis. Consider a system with a large number of
degrees of freedom. We describe our experimental knowledge of its state as a statistical state. A
generic state determines a flow: In quantum theory, a state over an observable algebra determines
a flow called the modular flow, or the Tomita flow. In the classical case, the relation between a
state and a its modular flow reduces to the relation between the distribution $\rho$ on phase space
representing the state and the Hamiltonian flow of $\ln \rho$. The thermal time hypothesis demands
that the flow determined by the statistical state coincides with what we perceive as the physical
flow of time. A second part of the hypothesis refers to the special cases in which a geometrical
background provides an independent notion of temporal flow: if the geometric flow and the modular
are related, the ratio of the rates of the two flows is identified as the temperature. The hypothesis
was initially motivated by the problem of time in quantum gravity.

The Unruh effect can be interpreted as an example in which the thermal time hypothesis is
realized. The relation between the Unruh effect and the thermal time hypothesis is the following.
The vacuum state $\Omega$, restricted to the algebra of the observables in the wedge $W$ generates a
modular flow. This modular flow is precisely proportional to the proper time flow of uniformly
accelerated observers. For each observer, the ratio of the two flows is $T_U$. According to the thermal
time hypothesis, this means that the natural physical time generated by $\Omega$ on the algebra of the
observables in the wedge is the proper time of an accelerated observer, and the corresponding
temperature is the Davis-Unruh temperature.

A natural question, then, is whether there are other cases in which the thermal time associated
to a spacetime region has a physical meaning. Local algebras associated to diamonds are very natural objects in algebraic quantum field theory. As mentioned, a diamond is the region causally connected to an observer with finite lifetime. It is natural to wonder whether the modular flow generated by the vacuum restricted to the local algebra associated to a diamond—the “diamonds’ flow”, or the “diamonds’ time”—may have any significance at all.

In general, there is no reason to expect that a modular flow should have a geometric interpretation and lead to a notion of temperature. Remarkably, however, the modular flow of a diamond is associated to a geometric action in the case of a conformally invariant quantum field theory in four dimensions. Does this lead to a notion of temperature? The modular parameter $s$ cannot be directly proportional to the proper time along the geometric flow, because the modular parameter runs all along $\mathbb{R}$ while the proper time $\tau$ of an observer in $D_L$ is bounded. However, we can nevertheless define a local temperature as the local ratio of the two flows. This allows us to define a temperature which is a natural generalization of the Unruh-Davies temperature to observers with a finite lifetime. In this paper we explicitly compute this temperature. For fixed acceleration, this temperature reduces to the Unruh temperature when $L$ is large.

Furthermore, this temperature does not vanish even for non-accelerating observers. The (minimal) temperature associated an inertial observer with lifetime $T$ is the temperature

$$T_D = \frac{2\hbar}{\pi k_b T}. \quad (2)$$

This temperature is related to the fact that a finite lifetime observer does not have access to all the degrees of freedom of the field. But we leave any physical interpretation of this temperature for future investigations. In particular, it would be interesting to study whether it is related to the response of model detectors.

Section 2 briefly presents our main tools (definition of local observable algebra, modular automorphism and KMS condition) and the thermal time hypothesis. In Section 3, we recall the discussion of the Unruh effect in the Bisognano-Wichman approach. Section 4 presents our new results, namely the discussion of the generalization of the Unruh effect to diamonds $D_L$ and its physical interpretation. Section 5 briefly discusses the case of the future cone. Minkowski metric has signature $(+, -, -, -)$. From now on, we take $c = \hbar = k_b = 1$.

## 2 Tools

### 2.1 Local observable algebra

Consider a quantum field theory on four dimensional Minkowski space. We restrict our attention to conformally invariant quantum field theories. The conformal group is the group of transformations $x \mapsto x'$ that preserve the causal structure: $(x'^\mu - y'^\mu)(x'_\mu - y'_\mu) = 0$ iff $(x^\mu - y^\mu)(x_\mu - y_\mu) = 0$. In four dimensions, this is the fifteen-parameters group generated by Poincaré transformations, dilatations and proper conformal transformations. An infinitesimal transformation is given by

$$\delta x^\mu = \omega^{\mu \nu} v_\nu + a^\mu + \lambda x^\mu + |x|^2 k^\mu - 2x^\mu k_\nu x^\nu, \quad (3)$$

where $(\omega_{\mu \nu} = -\omega_{\nu \mu}, a^\mu, \lambda, k^\mu)$ are infinitesimal parameters. We assume that both the dynamics and the vacuum of our quantum field are invariant under these conformal transformations. Since mass terms break conformal invariance, this means that we are dealing with a massless field. Equivalently, we are considering a regime in which masses can be neglected.

We recall a few elements on local observable algebras\[16\]. We have a Hilbert space $H$ carrying a unitary representation $U$ of the Poincaré group. If the theory is conformally invariant, $U$ extends
to a representation of the entire conformal group. There is a ray $\Omega$ in $H$, called the physical vacuum, invariant under the action of the Poincaré group (or the conformal group). A field is an operator valued distribution over Minkowski space $M$. There is no operator corresponding to the value $\phi(x)$ of the field $\phi$ at a given point $x$; there is an operator $\phi(f)$ corresponding to the quantity obtained smearing out the field with a smooth function $f$

$$\phi(f) = \int \phi(x) f(x) \, d^4x.$$  \hspace{1cm} (4)

Consider a finitely extended open subset $O$ of $M$. The algebra generated by all operators $\phi(f)$, where $f$ has support in $O$, is interpreted as representing physical operations that can be performed within $O$. By completing this algebra in the norm topology, we obtain a $\mathcal{C}^*$-algebra $\mathcal{R}(O)$. This is called the local observable algebra associated to $O$. We have an algebra $\mathcal{R}(O)$ associated to each open subset $O$. The Reeh-Schlieder theorem states that the vacuum $\Omega$ is a cyclic and separating vector for $\mathcal{R}(O)$ as soon as the causal complement of $O$ (the set of all points which lie space-like to all points of $O$) contains a non-void open set.$^a$

### 2.2 Modular automorphisms

The Tomita-Takesaki theorem associates a preferred flow $\sigma_s$ to any $\mathcal{C}^*$-algebra $\mathcal{A}$ acting on a Hilbert space with a preferred cyclic and separating vector $\Omega$. The flow is built as follows. Let $S$ be the conjugate linear operator from $\mathcal{A}\Omega$ to $\mathcal{A}\Omega$

$$Sa\Omega \equiv a^*\Omega$$ \hspace{1cm} (5)

for any $a \in \mathcal{A}$. $S$ is closeable with respect to the strong topology and its closure - still denoted by $S$- has a unique polar decomposition

$$S = J\Delta^{1/2}$$ \hspace{1cm} (6)

where $\Delta$ is a selfadjoint positive operator (unbounded in general) called modular operator and $J$ is antiunitary. This allows us to define a one-parameter group of automorphisms of $\mathcal{A}$: the modular group $\sigma_s$ of $\mathcal{A}$ associated to $\Omega$

$$\sigma_s a \equiv \Delta^{is}a\Delta^{-is}$$ \hspace{1cm} (7)

(see [15] for details on the definition of $\Delta^{is}$) with $s \in \mathbb{R}$. If $\mathcal{A}$ is the algebra of local observables on an open set $O$ of Minkowski space, with non-void causal complement, we call modular group of $O$ the modular group of $\mathcal{R}(O)$ associated to the vacuum $\Omega$. We denote it as $\sigma_s(O)$. We write $\Delta(O)$ for the corresponding modular operator.

### 2.3 KMS condition

Let us now recall the relation between modular flow and thermodynamics. Consider a physical system with a finite number of degrees of freedom, say $N$ particles in a finite box with volume $V$. In statistical physics, an equilibrium state is a state of maximal entropy. If $H$ is the hamiltonian describing $S$, the mean value of any observable $a$ when $S$ is in an equilibrium state at inverse temperature $\beta$ is

$$\omega(a) = \frac{1}{Z} \text{Tr} \left( e^{-\beta H} a \right)$$ \hspace{1cm} (8)

where the partition function $Z$ is given by

$$Z \equiv \text{Tr}(e^{-\beta H}).$$ \hspace{1cm} (9)

$^a\Omega$ is cyclic but not separating for the algebra associated to the entire Minkowski space $M$. Intuitively, a separating state can be thought as a density matrix with nonvanishing components on all basis states.

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Consider now the thermodynamical limit of this system: send the volume of the box \( V \) and the number of particles \( N \) to infinity, keeping the density \( N/V \) fixed. In this limit the total energy \( E \) goes to infinity and therefore the partition function \( \Theta \) is no longer defined. One therefore needs a way to characterize the equilibrium states alternative to equations (8) and (9). This can be obtained as follows. An equilibrium state \( \omega \) of inverse temperature \( \beta \) can be directly characterized as a state over the algebra satisfying the KMS condition. This requires that for any two observables \( a \) and \( b \)

\[
\omega((\alpha_t a)b) = \omega(b(\alpha_{t+i\beta} a))
\]

where \( \alpha_t \) denotes the time translation

\[
\alpha_t a = e^{iHt}a e^{-iHt}
\]

extended to a complex argument, and that the function

\[
z \mapsto \omega(b(\alpha_z a))
\]

is analytic in the strip

\[
0 < \text{Im} \, z < \beta.
\]

A state \( \omega \) satisfying these conditions is called KMS with respect to the time translation \( \alpha_t \). One can shows that both conditions (10) and (12) reduce to (8) in the case of a finite dimensional system. These conditions represent the correct extension of the definition of an equilibrium state to infinite-dimensional systems.

Now, the remarkable fact is that, given a \( C^\ast \)-algebra \( \mathcal{A} \) acting on a Hilbert space \( H \) with a cyclic and separating vector \( \Omega \), the faithful state \( \omega \) over \( \mathcal{A} \) defined by

\[
\omega(a) = \langle \Omega, a\Omega \rangle
\]

for any \( a \in \mathcal{A} \) (we use Dirac notations) is KMS with respect to the modular group \( \sigma_s \) defined by \( \Omega \). More precisely one has

\[
\omega((\sigma_s a)b) = \omega(b(\sigma_{s^{-1}} a)).
\]

Define \( \alpha_{\beta s} \equiv \sigma_s \). Then

\[
\omega((\alpha_{-\beta s} a)b) = \omega(b(\alpha_{-\beta s+i\beta} a)).
\]

Therefore, if we define

\[
t = -\beta s
\]

we obtain condition (10) (as far as (12) is concerned, see ref. [7]). In other words: an equilibrium state at inverse temperature \( \beta \) can be characterized as a faithful state over the observable algebra whose modular automorphism group \( \sigma_s \) is the time translation group. The modular parameter \( s \) is proportional to the time \( t \). The proportionality factor is minus the inverse temperature. (The minus sign is there only for historical reasons: the mathematicians have defined the modular flow with sign opposite to that given by the physicists).

### 2.4 Thermal time hypothesis

The modular group equips a system in a generic state with a dynamics. The existence of such an intrinsic dynamics suggests a solution to a longstanding open puzzle in the physics of general covariant systems. The dynamical laws of a generally covariant system determine correlations between observables, but they do not single out any of these observables as time. This “timeless” structure is sufficient to provide all dynamical predictions. However, this timeless structure leaves us without any understanding of the physical basis for the flow of time, a flow that has evident
The physical time depends on the state. When the system is in a state $\omega$, the physical time flow is given by the modular flow of $\omega$.

The second part of the hypothesis demands then that if the modular flow is proportional to a flow in spacetime, parametrized with the proper time $\tau$, then the ratio of the two flows is the temperature

$$\beta = \frac{1}{T} \equiv -\frac{\tau}{s}.$$  \hfill (18)

In other words, when $t$ is independently defined, (17) becomes a definition of the temperature. In previous applications of this idea, only cases in which $\frac{\tau}{s}$ is exactly constant along the flow were considered. In this paper, we extend the framework slightly, by considering a case in which the modular flow is proportional to a flow in spacetime, but the constant of proportionality between $\frac{d}{ds}$ and $\frac{d}{d\tau}$ varies along the flow. In such a situation, it is natural to consider a local notion of temperature, defined at each point by

$$\beta(s) = \frac{1}{T(s)} = -\frac{d\tau(s)}{ds}. \hfill (19)$$

The thermal time hypothesis is based on the idea that the origin of time is thermodynamical, and that the time flow is state-dependent. In a sense, the thermal time hypothesis is an inversion and a generalization of KMS theory. Given a time flow $\sigma$, KMS theory identifies the thermal states as the KMS states of the time flow. But given the KMS state, the time flow $\sigma$ is precisely its modular flow. Since any generic state defines a modular flow, the idea here is that, generically, this modular flow is in fact the physical time that governs macroscopic thermodynamics.

As far as conventional systems are concerned, notice the following remarkable fact: if we measure the statistical distribution of an equilibrium state, we can infer the hamiltonian from the result. We can then define time as the parameter of the flow generated by the hamiltonian. This provide an operational characterization of time. On the other hand, it is well known that it is difficult to provide a purely dynamical (as opposed to statistical/thermodynamical) characterization of what is time. In a generally covariant context, where definitely no physical time is determined by the dynamics, the thermal time hypothesis identifies the physical time as the modular flow for generic states.

The thermal time hypothesis has been checked in several examples. The most striking one concerns the cosmic microwave background (CMB). The statistical state distribution describing the CMB determines a flow that coincides with the time coordinate of usual Friedman-Robertson-Walker metric. The associated temperature is the CMB temperature. The Rindler wedge provides another example in which the hypothesis is realized.

For a detailed discussion of this issue and the motivations of the thermal time hypothesis, see refs. [20]. On the large literature on problem of time in general relativity see for instance ref. [10] and references therein.

3 Time in the Rindler Wedge

Consider a uniformly accelerated observer. For simplicity, let us assume that the motion is in a plane with constant $x^2$ and constant $x^3$, and that the acceleration four-vector writes $(0,a,0,0)$ in
the local observer frame. Here \( a \in \mathbb{R}^+ \). The world line of this observer can be written as

\[
x^1 = \sqrt{\frac{1}{a^2} + (x^0 + C')^2 + C}
\]

(20)

where \( C \) and \( C' \) are constants. With a Poincaré transformation, we can take \( C \) and \( C' \) to zero

\[
x^1 = \sqrt{a^{-2} + (x^0)^2}.
\]

(21)

A Lorentz boost in the \( x_1 \) direction reads

\[
\Lambda(\rho) = \begin{pmatrix}
  \cosh \rho & \sinh \rho & 0 & 0 \\
  \sinh \rho & \cosh \rho & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix}
\]

(22)

(\( \rho \in \mathbb{R} \)). The trajectory (21) is an orbit of this group. The proper time \( \tau \) along the orbit is proportional to the boost parameter \( \rho \), and the proportionality factor is given by \( a \)

\[
\rho = a \tau.
\]

(23)

We can therefore parametrize the trajectory (21) with the proper time \( \tau \) and write it as

\[
x^\mu(\tau) = (a^{-1}\sinh a\tau, a^{-1}\cosh a\tau, 0, 0)
\]

(24)

so that

\[
\Lambda(a\tau)x(\tau_0) = x(\tau_0 + \tau).
\]

(25)

That is, \( \Lambda(a\tau) \) generates a proper-time translation of proper time \( \tau \) along the accelerated world line.

Consider a quantum field theory defined on \( M \) and let \( K \) denote the representation of the generator of this boost, so that

\[
U(\Lambda(\rho)) = e^{i\rho K}.
\]

(26)

Then the operator

\[
U(\tau) = e^{i a \tau K}
\]

(27)

can be viewed as an operator that generates the evolution in proper-time seen by the accelerated observer.

The region causally connected to the accelerated observer, namely the set of the points that can exchange signals with a uniformly accelerated observer is the Rindler wedge \( W \), defined by

\[
x^1 > |x^0|.
\]

(28)

The wedge \( W \) has a non void causal complement thus the modular group \( \sigma_s(W) \) is defined. The modular operator \( \Delta(W) \) is known. It corresponds precisely to a Lorentz boost

\[
\Delta(W) = e^{-2\pi K}.
\]

(29)

therefore the modular flow \( \sigma_s(W) \) (defined in (21)) is given by the boost

\[
U(s) = e^{-2\pi s K}.
\]

(30)

This flow is precisely proportional to the geometric flow (27). The relation between the modular parameter \( s \) and the proper time \( \tau \) is obtained comparing the two operators: a translation by \( s \) generates a shift in proper time

\[
\tau = -\frac{2\pi}{a} s.
\]

(31)
From (18), the temperature is determined by
\[ \beta = -\frac{\tau}{s} = \frac{2\pi}{a}. \] (32)

The result is therefore that the state \( \Omega \), restricted to the Rindler wedge generates a time flow which is the one of an accelerated observer, with an associated temperature which is the Unruh-Davis temperature.
\[ T = \frac{1}{\beta} = \frac{a}{2\pi} = T_U. \] (33)

### 4 The time of the diamonds

#### 4.1 Diamond’s modular group

Consider now an observer that lives for a finite time. Let \( x_i \) be the event of his birth and \( x_f \) the event of his death, which is in the past of \( x_i \). To begin with, assume that the life of the observer is the straight segment from \( x_i \) to \( x_f \). Let
\[ T = 2L = |(x_f - x_i)| \] (34)
be the observer lifetime, namely the proper time between \( x_i \) and \( x_f \). Let \( D \) be the region which is in the past of \( x_f \) and in the future of \( x_i \). This region is called a Diamond. It is the region with which the observer can exchange signals during his lifetime: he can send a signal, and receive a response, to each point of \( D \). Imagine that the observer sends out measuring apparatus on spacecrafts to make local measurements on the quantum field, and receives back the signals from his apparatus. Then he will be capable of observing, at most, the local observables in \( D \). The local algebra to which he has access is therefore \( \mathcal{R}(D) \). Assuming that the quantum field theory is in its vacuum state \( \Omega \), we can ask what kind of state will this observer observe. The answer is the restriction of \( \Omega \) to \( \mathcal{R}(D) \). Let us therefore study the properties of this restriction.

With a Poincaré transformation, we can always choose Minkowski coordinates such that
\[ x^\mu_i = (-L,0,0,0), \quad x^\mu_f = (L,0,0,0). \] (35)
The region \( D \) is then the region defined by
\[ |x^0| + |\vec{x}| < L. \] (36)

Now, the key observation is that there is a conformal transformation \( K : M \mapsto M \) that sends the diamond \( D \) to the wedge \( W \)
\[ K(D) = W. \] (37)

Explicitly, if \( L = 1 \), \( K \) is given by
\[ x'^\mu = \frac{2x^\mu + \delta_1^\mu (1 + |x|^2 - 2x^1)}{1 - |x|^2 - 2x^1}. \] (38)

Using this map, it is immediate to find out the modular flow \( \sigma_s(D) \) of the diamond local algebra \( \mathcal{R}(D) \): this is simply obtained by mapping the modular flow of the wedge to the diamond \( \mathcal{R}(D) \). More precisely, we consider the one parameter group of transformations \( \Lambda_D(\rho) \) of the diamond into itself given by
\[ \Lambda_D(\rho) \equiv K^{-1} \Lambda(\rho) K. \] (39)
Since this is a composition of conformal transformations, $\Lambda_D$ is a one-parameter subgroup of conformal transformations. A straightforward calculation (see ref. [7]) gives $\Lambda_D(\rho) : x_\mu \mapsto x_\mu(\rho)$ where

$$x^\mu(\rho) = \frac{2x^\mu + \delta^\mu_0 \left( 2x^0 \text{ch} \rho + (1 + x^\mu x_\mu) \text{sh} \rho - 2x^0 \right)}{2x^0 \text{sh} \rho + (1 + x^\mu x_\mu) \text{ch} \rho + (1 - x^\mu x_\mu)}.$$  \hspace{1cm} (40)

As in the case of the wedge, the modular operator $\Delta(D)^i_\sigma$ is the representation of the conformal transformation $\Lambda_D(\rho)$ for $\rho = -2\pi s$.

This generalizes easily to a diamond $D_L$ of arbitrary dimension $L > 0$. The modular operator $\Delta(D_L)^i_\sigma$ is the representation of the conformal transformation $\Lambda_{D_L}(\rho) = L\Lambda_D(\rho)L^{-1}$ where $L$ is the dilatation by a factor $L$. Explicitly

$$x^\mu(\rho) = L \frac{2Lx^\mu + \delta^\mu_0 \left( 2Lx^0 \text{ch} \rho + (L^2 + x^\mu x_\mu) \text{sh} \rho - 2Lx^0 \right)}{2Lx^0 \text{sh} \rho + (L^2 + x^\mu x_\mu) \text{ch} \rho + (L^2 - x^\mu x_\mu)}.$$  \hspace{1cm} (41)

### 4.2 Temperature

The orbits of the modular group of the diamond are uniformly accelerated trajectories. Indeed, consider a uniformly accelerated observer moving from $x_i$ to $x_f$ with acceleration $a$. For simplicity, assume that the motion is in the plane $x^2 = x^3 = 0$. Its worldline is given by (20) with $C' = 0$ and

$$C = -a^{-1}\sqrt{1 + a^2L^2}.$$  \hspace{1cm} (42)

We can parametrize this orbit using the proper time along the orbit. This gives

$$x^\mu(\tau) = (a^{-1}\text{sh} a\tau, a^{-1}\text{ch} a\tau + C, 0, 0)$$  \hspace{1cm} (43)

where $\tau$ runs from $-\tau_0$ to $\tau_0$ with

$$\tau_0 \equiv a^{-1} \text{arcsh} aL.$$  \hspace{1cm} (44)

The line of universe [43] is precisely an orbit of the diamond’s modular group $\sigma_s(D_L)$. Let us fix the origin of the modular parameter $s$ so that $s = 0$ when $\tau = 0$. This gives $x^\mu = x^\mu(0) = (0, a^{-1} + C, 0, 0)$. Inserting this value of $x^\mu$ in (41) gives, with simple algebra,

$$x^0(\rho) = \frac{L^{-1} \text{sh} \rho}{a^{-1} \text{ch} \rho - C},$$  \hspace{1cm} (45)

$$x^1(\rho) = \frac{L^2}{C - a^{-1} \text{ch} \rho}$$

and $x^2(\rho) = x^3(\rho) = 0$. Indeed, it is not difficult to check that for any $\rho \in \mathbb{R}$, $x_1(\rho) - C$ is positive and satisfies $(x_1(\rho) - C)^2 = a^{-2} + x_0(\rho)^2$ so [45] is solution of (21). Moreover

$$\lim_{\rho \to -\infty} x(\rho) = x_i, \quad \lim_{\rho \to \infty} x(\rho) = x_f$$  \hspace{1cm} (46)

thus the orbit of $x(\rho)$ under the action of $\sigma_s(D_L)$ coincides with the worldline of the uniformly accelerated observer.

We have therefore two parameterizations of the same worldline. One [43] given by the proper time $\tau \in [-\tau_0, \tau_0]$, the other [45] given by the modular parameter $\rho = -2\pi s \in [-\infty, +\infty]$. To find the relation between these two parameterizations, we can compare $x^\mu(\tau)$ with $x^\mu(\rho)$. This gives

$$x^0 = a^{-1} \text{sh} a\tau = \frac{L^{-1} \text{sh} \rho}{a^{-1} \text{ch} \rho - C},$$  \hspace{1cm} (47)

$$x^1 = a^{-1} \text{ch} a\tau + C = \frac{L^2}{C - a^{-1} \text{ch} \rho}.$$  \hspace{1cm} (48)
It is convenient to write $\lambda \equiv \text{arcsh} \, aL$, so that $aC = - \text{ch} \, \lambda$. Using this, we have

$$\text{sh} \, a\tau = \frac{\text{sh} \, \rho \, \text{sh} \, \lambda}{\text{ch} \, \rho + \text{ch} \, \lambda}, \quad \text{ch} \, a\tau = \frac{1 + \text{ch} \, \lambda \, \text{ch} \, \rho}{\text{ch} \, \rho + \text{ch} \, \lambda}.$$  

(49)

This gives

$$\tau(\rho) = \frac{1}{a} \text{arcth} \left( \frac{\text{sh} \, \rho \, \text{sh} \, \lambda}{1 + \text{ch} \, \lambda \, \text{ch} \, \rho} \right).$$  

(50)

Differentiating, we get

$$\frac{d\tau(\rho)}{d\rho} = \frac{L}{\text{ch} \, \rho + \text{ch} \, \lambda}. $$  

(51)

Recalling that $\rho = -2\pi s$, and the definition of $\lambda$ we conclude

$$\frac{d\tau(s)}{ds} = -2\pi \frac{d\tau(\rho)}{d\rho} = -\frac{2\pi L}{\text{ch} \, \rho + \text{ch} \, \lambda} = -\frac{2\pi L}{\sqrt{1 + a^2L^2} + \text{ch} (2\pi s)}.$$  

(52)

The thermal time hypothesis (19) defines the temperature

$$\beta(s) = -\frac{d\tau(s)}{ds} = \frac{2\pi L}{\sqrt{1 + a^2L^2} + \text{ch} (2\pi s)}.$$  

(53)

Noticing from (45) that $\beta = -\frac{2\pi}{La} x^1$, we can use (43) to rewrite this inverse temperature as

$$\beta(\tau) = \frac{2\pi}{La^2} \left( \sqrt{1 + a^2L^2} - \text{ch} \, a\tau \right)$$  

(54)

In particular, at $x^0 = 0$, that is $s = 0$, we have

$$\beta_0 \equiv \beta(0) = \frac{2\pi L}{\sqrt{a^2L^2 + 1} + 1}. $$  

(55)

We analyze the meaning of this result in the following two sections.

### 4.3 Unruh effect for the observer with a finite lifetime

First, consider the case in which $a$ is large. This corresponds to orbits that stay near the boundary of the diamond. In particular, at $x^0 = 0$ the observer is near the edge $|\vec{x}| = L$ of the diamond. In the limit of large $a$, we have

$$\beta_0 = \frac{2\pi}{a}, $$  

(56)

which is the Davis-Unruh temperature. Therefore, the thermal time hypothesis associates precisely the Davis-Unruh temperature to a uniformly accelerated observer with a finite lifetime that passes near the edge. For fixed acceleration $a$, an observer with finite lifetime, in the middle of his life, observes the Davis-Unruh temperature if his lifetime $2L$ is large enough. The first order correction in $1/L$ gives

$$\beta_0 = \frac{2\pi}{a} - \frac{2\pi}{a^2L}, $$  

(57)

while the exact formula for every $L$ is given by (53). This result provides a complete generalization of the Unruh effect to observers with finite lifetime. In order to better appreciate this result, consider the dependence of the temperature from the proper time. In the limit of large $L$, this is given by

$$\beta(\tau) = \frac{2\pi}{a} \left( 1 - \frac{\text{ch} \, (a\tau)}{aL} \right). $$  

(58)

In Figure 1 we have plotted this function in the finite region of the observer proper time. Notice that the function goes rapidly to zero at the boundaries of the interval of the life of the observer, but it is nearly flat over most of the region. That is, for the observer measuring his proper time, $\beta(\tau)$ is constant and $\sim \beta_0$ for most of his lifetime.
4.4 Diamond’s temperature

Next consider the case in which, for a given lifetime $T$, the acceleration is small. In particular, consider the case in which $a = 0$. This is the observer moving along a straight line from $x_i$ to $x_f$. Interestingly, the temperature associated to this observer does not vanish. The inverse temperature $\beta$ reaches a maximum value at $x^0 = 0$, which is

$$\beta_D = \pi L.$$  

Therefore the thermal time hypothesis associates the finite temperature

$$T_D = \frac{1}{\beta_D} = \frac{2}{\pi T}$$

(60)

to an observer with a finite lifetime $T$. Inserting back the dimensional constants $s, \hbar$ and $k_B$, we get equation (2). What is the meaning of this temperature?

It is tempting to conjecture that this temperature is due to the fact that the observer cannot observe all the degrees of freedom of the quantum field. In particular, he can only observe the degrees of freedom that at time zero are inside the sphere $|\vec{x}| < L$. Because of the quantum field theoretical correlations between the inside and the outside of this sphere, the vacuum state restricted to this sphere is not a pure state. If this is correct, there should be a physical temperature naturally associated to observers with a finite lifetime, whether they are accelerating or not. We leave the study of this idea for future investigations (comparison with entanglement entropy of [17] may be interesting).

5 Future cone

For completeness, we describe here also an observer that is born at $x_i$ and lives forever. The region causally connected to this observer is the future cone $V^+$ of $x_i$. Since this has a non void causal complement, the vacuum restricted to $V^+$ is a separating state and we can search the action of its modular group $\sigma_s(V^+)$. In fact this action is also geometrical. Taking $x_i$ in the origin, it corresponds to the dilatation

$$x^\mu \mapsto x^\mu(s) \equiv e^{-2\pi s} x^\mu.$$  

(61)
As for the wedge and the diamond, we can interpret the orbits of this action as motions of a physical observer. An inertial observer with constant speed $\vec{v} = (v^1, v^2, v^3)$ with respect to the origin, which leaves the origin at $\tau = 0$ follows the world line

$$x^\mu(\tau) = K^\mu \tau$$

(62)

where $K^0 \equiv c\gamma$, $K^i = v^i\gamma$ with $\gamma \equiv (1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$. The (closure of the) line of universe (61) for $s \in \mathbb{R}$ coincides with the line of universe (62) where $\tau$ runs from 0 to $+\infty$. The relation between the modular parameter $s$ and the proper time $\tau$ is easily obtained as

$$\tau(s) = e^{-2\pi s}.$$  

(63)

Therefore the temperature is

$$\beta(s) = 2\pi e^{-2\pi s}.$$  

(64)

The temperature is finite and non zero at the birth of the observer and then it goes exponentially to zero, as it should for a long living observer moving with constant speed.

### 6 Conclusion

Starting from the thermal time hypothesis, we have obtained the result that the Unruh effect exists also for observers with a finite lifetime. Restauring dimensional constants, the thermal time hypothesis associates the temperature (54)

$$T(\tau) = \frac{\hbar L a^2}{2\pi k_b c^3\left(\sqrt{1 + \frac{a^2 L^2}{c^2}} - \frac{a\tau}{c}\right)}$$

(65)

to an observer living for a finite time and moving with constant finite acceleration $a$ between two spacetime points at proper temporal distance $T = 2L/c$ from each other. The temperature depends on the observer proper time $\tau$ lapsed from the middle of his lifetime. In the limit of large $L$, we recover the Davis-Unruh temperature. For large $L$, an observer with a large acceleration sees the Davis-Unruh temperature for most of his lifetime.

Furthermore, it follows from the thermal time hypothesis that also an inertial observer with finite lifetime has an associated temperature. The minimum value of this temperature is the diamonds temperature $T_D$ (2)

$$T_D = \frac{2\hbar}{\pi k_b T}. $$

(66)

We have speculated that this temperature reflects the fact that the observer has no access to distant degrees of freedom. It would be interesting to understand whether this temperature is related to the response of model thermometers with finite lifetime.

Altogether, the thermal time hypothesis seems to play an interesting role in these physical situations. Much more work is needed to study the validity of this hypothesis, and its relation with thermodynamical and statistical temporal phenomena.

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