Dark solitons revealed by particle losses in the 1D Bose gas

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The un-obvious correspondence between the dark solitons from the Gross-Pitaevskii equation and the lowest excitations of the Lieb-Liniger model has been in scientific discussion for years. Recently, few different approaches based on alternative symmetry breaking mechanism were proposed to extract the mean-field solitons directly from the underlying many-body problem. Here, we access much larger systems of hundreds of atoms with computational basis built by Bogoliubov states and in parallel with a simple many-body wave function Ansatz. We elucidate that particle losses, inevitable in all experiments with ultracold gases, serve as a physical mechanism of symmetry breaking that unifies previous proposals. We observe that the losses can push the system initiated in the appropriate eigenstates towards a product state with macroscopically occupied mean-field soliton. In such cases, losses increase the coherence of the system accompanied by robust structures moving with constant speed, as the mean-field solitons do. We expect that with well-set losses places a new possibility of conditional states engineering opens.

The famous Lieb-Liniger model \[1, 2\] describes particles that move along a circle and interact via delta interatomic potentials. Such a simple interaction turns out to be a well-suited approximation for realistic interactions between neutral slow atoms. Thus there has been great interest in this model for many years. The model can predict phenomena such as fermionization or pre-thermalization \[3\] to name a few, and is still in the active field of theoretical and experimental physics and mathematics \[4, 5\].

The same system is often treated within a simple mean-field approximation, based on the non-linear Schrödinger equation (NLSE):

\[
\partial_t \phi(x,t) = \left( -\frac{1}{2} \partial_x^2 + gN|\phi(x,t)|^2 \right) \phi(x,t), \tag{1}
\]

where the wave function \(\phi(x,t)\) is interpreted as an orbital occupied by a macroscopic number of atoms \(N\). The latter equation \(1\) is useful in many disciplines of physics ranging from quantum optics to hydrodynamics \[6, 7\] and has in particular attracted the attention of mathematicians. It is a rare example of a practical model that supports solitonic solutions \[8\]. When atoms repel each other, i.e. \(g > 0\), solitons are rarefactions in the gas density, which move with a constant speed and are unusually robust thanks to a balance between the dispersion and the nonlinearity \[9\]. Such solitons, called dark solitons are engineered in cold bosonic clouds using a phase imprinting method \[10, 11\].

There is a puzzling link between the mean-field solitons and the solutions of the underlying many-body Lieb-Liniger model. More than a decade after the seminal paper by E. Lieb \[11\], a coincidence between the dispersion relations of dark solitons and certain many-body eigenstates, the so called type-II elementary excitations, was observed \[12, 13\]. These type-II excitations are simply the many-body eigenstates that minimize the energy for a fixed total momentum, sometimes called yrast states \[14\]. Further relations between the yrast states and solitons were presented in \[15\].

There have been efforts to show how the mean-field solitons can be extracted directly from the yrast state. The groups in Japan \[18–20\] and New Zealand \[21\] showed in a series of works that the single-particle density calculated for the appropriate superposition of yrast states indeed resembles solitonic shape. In turn, \[22\] argues that solitons can be found in properly high correlation functions. In all cases, the emerging solitonic profiles were, unlike mean-field solitons, blurred during dynamics \[18, 20, 21\].

Figure 1. (color online) Yrast states are understood as a superposition of mean-field solitons with unknown centers (left). The system wave-function is conditioned to particle losses and collapses to a superposition of more localized groups of solitons (right). Counter-intuitively, a dip in the density emerges far from where the losses occurred.

Here, we unify different approaches by employing a numerical analysis to the many-body system and a simple but powerful Ansatz for the yrast state. The physical mechanism used to break the translational symmetry and reveal the solitons are the particle losses. These are of practical importance in cold gases where the typical loss rate is around 0.1 Hz, while experiments using ultra-cold gases could last on the order of seconds. We show that for lossy dynamics, a system that initiated in an yrast state evolves spontaneously towards a product state with...
a macroscopically-occupied mean-field soliton. The first-order correlation increases as a result of the dissipative dynamics. Finally, the structures revealed by losses are robust and move with a constant speed, which is the same as the mean-field solitons.

The particle losses are incorporated in the standard approach of using a master equation [24–26]:

\[ \partial_t \hat{\rho} = -i \left[ \hat{H}, \hat{\rho} \right] + \gamma \int dx \left( \hat{\Psi}(x) \hat{\rho} \hat{\Psi}^\dagger(x) - \frac{1}{2} \left[ \hat{N}(x), \hat{\rho} \right]_+ \right), \]

where \( \hat{\rho} \) is the density matrix of the system, \( \hat{\Psi}(x) \) is the bosonic field operator, \( \hat{N}(x) = \hat{\Psi}^\dagger(x) \hat{\Psi}(x) \) is the operator of density and \( \gamma \) is the rate of one-body losses. The symbol \( \left[ \hat{N}(x), \hat{\rho} \right]_+ \) stands for \( \hat{N}(x) \hat{\rho} + \hat{\rho} \hat{N}(x) \).

It is noted that many authors have investigated particle losses in a boson gas, but most often by utilizing single-mode approximations [25–28]. Such a simplification cannot be justified in our case; therefore, we study the full multi-mode dynamics.

The unitary part of the system dynamics for bosons is governed by the Lieb-Liniger Hamiltonian:

\[ H = -\frac{1}{2} \int \hat{\Psi}^\dagger(x) \partial_x^2 \hat{\Psi}(x) + \frac{g}{2} \int \hat{\Psi}^\dagger(x) \hat{\Psi}(x)^2 \]

The exact solutions for the eigenstates of the Hamiltonian [3] for particles moving in a box with periodic boundary conditions were previously identified in 1963. Among the eigenstates, there are special ones that are called elementary excitations [1], which can be divided into two families. Until Ref. [1], only one family was expected, known as the so-called Bogoliubov excitations. The second unexpected family consists of the yrast states [1], called elementary excitations [1], which can be divided beyond zero momentum mode. In practice it means, that they are spanned by dozens of the Bogoliubov states.

Let’s us remind essentials of the Bogoliubov approximation. It is based on the assumption that almost all atoms occupy a single orbital, here the plane wave with momentum \( \hbar k = 0 \). Typically one keeps in the Hamiltonian [3] only terms up to \( O(N^2) \), which can be written in the form:

\[ \hat{H}_B = \sum_{k \neq 0} \omega_k \hat{b}_k^\dagger \hat{b}_k + \text{const}, \]

where \( \omega_k = \sqrt{\epsilon_k + 2g(N-K)/L} \) and \( \epsilon_k = 2\pi^2 k^2/L^2 \), with the operators \( \hat{b}_k \) and \( \hat{b}_k^\dagger \) obeying the bosonic commutation rules. The eigenstates of the Hamiltonian \( H_B \) are Fock states \( |n\rangle_B := |n_1, n_2, ..., n_{K-1}, 1\rangle \), which are eigenstates of \( \hat{b}_k^\dagger \hat{b}_k \) with eigenvalues \( n_k \). The states with one quasi-particle with momentum \( 2\pi K/L \), i.e. \( |n_{K-1}, 1\rangle \), build the Bogoliubov branch of excitation, i.e. phonons and quasiparticles. It has been proposed in [17] that the yrast states can be approximated by \( K \) quasi-particles in momentum \( 2\pi/L \), i.e. \( |n_{K-1}, 1\rangle \), build the Bogoliubov branch of excitation, i.e. phonons and quasiparticles. Using exact diagonalization we find numerically the yrast states \( |K\rangle \) – it turns out that they are spanned by dozens of the Bogoliubov states.

Within such method we can access regimes of large number of atoms – examples will be given for \( N = 1000 \). The price we have to pay comes from the assumptions of the Bogoliubov approximation: we restrict our study to regimes in which only small fraction of atoms is beyond zero momentum mode. In practice it means, that we focus on yrast states with small momentum, \( K \ll N \), corresponding to grey solitons.

The usage of Bogoliubov basis significantly reduces the complexity of numerical representation of the Lieb-Liniger eigenstates, but a remaining obstacle is to solve the dynamical master equation [2]. The standard method of solving the master equation is the stochastic wave approach [32–34]. In this method one always deals with pure states \( |\psi(t)\rangle \), which are generated in appropriate stochastic evolution. The stochastic wave approach is equivalent to dynamics given by Eq. (2) – a mixture of all possible conditional states obeys the master equation. The stochastic evolution is generated in the following way: given a state \( |\psi(t)\rangle \) at time \( t \) one draws \( |\psi(t + \Delta t)\rangle \) as one of the two possibilities:

- a) \( \hat{\Psi}(x_1)|\psi(t)\rangle \) with probability \( p = \gamma \langle \hat{N} \rangle \Delta t \) or,
b) \( e^{-(i\hat{H}+\gamma N/2)\Delta t} \tilde{\psi}(t) \) with probability \( 1-p \).

In the former case, the position \( x_1 \) at which the particle is annihilated is drawn from the probability distribution \( \langle \psi(t)|\hat{\Psi}^\dagger(x)|\psi(t) \rangle \). After each time-step the wavefunction is normalized.

The stochastic waves are interpreted as single realization of the system – strictly speaking they represent a state under condition, that some particles were lost at certain random positions. Here, we search for solitons within such stochastic states. In what follows, the system is initially always in an yrast state \(|\psi(t=0)\rangle = |K\rangle\).

First we study a case of such fast losses and short timescales that the free evolution between losses can be omitted. Then the conditional wave function for \( m \) lost particles is

\[
|\tilde{\psi}\rangle \propto \hat{\Psi}(x_m)\hat{\Psi}(x_{m-1})\ldots \hat{\Psi}(x_1)|\psi(0)\rangle ,
\]

where \( x_i \) is the position at which \( i \)-th particle loss happened. Actually, the average density in the conditional state \(|\tilde{\rho}(x)\rangle := \langle \hat{\Psi}(x)|\hat{\Psi}(x)\rangle \) is proportional to the \((m+1)\)-order correlation function \( \langle \psi(0)|\hat{\Psi}(x_1)\ldots \hat{\Psi}(x_m)|\psi(0)\rangle \), being in fact the objects investigated in [22].

In Fig. 2 we show the average densities evaluated in states \((5)\) after different number \( m \) of lost particles (thin dashed lines) and compare them with densities \(|\phi_{MF}(x)|^2\) of the solitonic solutions of Eq. (1) (thick gray lines). The mean-field solutions are known analytically [33], but they involve rather complicated formulas. The details of the exact solutions are nicely described in [19]. The left panel of Fig. 2 corresponds to results for an ideal gas case, which should be also close to that of the interacting systems with healing length long enough, \( \xi > L \) (see [36]). In this limit we obtain very good agreement between the density emergent in the many-body calculation and the mean-field soliton. The interacting case (the right panel of Fig. 2) turned out to be much more involved, as the size of the basis grows rapidly after each lost atoms.

In the conditional states for ideal gas \( |\psi(0)\rangle \) density emerges in the many-body calculation and the mean-field soliton. The interacting case (the right panel of Fig. 2) turned out to be much more involved, as the size of the basis grows rapidly after each lost atoms. In the conditional densities for up to \( m = 10 \) losses, there emerge density dips, which are shallow, broad and far from the mean-field soliton. The origin of this discrepancy requires a bit deeper understanding of the quantum dark solitons, as explained below.

The insight is gained via an Ansatz for the yrast state. The Ansatz is built from a product states of \( N \) particles occupying a single orbital \(|\phi\rangle\) represented by \(|\phi\rangle^{\otimes N}\). First let us remark, that the continuous superposition of product states shifted by the translation operator \( e^{-i\hat{P}_y} \) over all possible shifts \( y \):

\[
|\psi_{\text{Ansatz}}\rangle \propto \int_0^L dy e^{i\pi Ky}e^{-i\hat{P}_y}|\phi\rangle^{\otimes N}
\]

is, irrespectively of the orbital \(|\phi\rangle\), an eigenstate of the total momentum operator \( \hat{P} \) with the eigenvalue \( 2\pi K \). Therefore, the state \(|\psi\rangle\) belongs to the same momentum subspace as the yrast state \(|K\rangle\). One may think about the many-body state \(|\psi\rangle\) as a variational Ansatz for the yrast state, with an orbital \(|\phi\rangle\), chosen to minimize the total energy. On the other hand it is known that energies of the yrast states \(|K\rangle\) and the mean-field solitons agrees with each other [12]. Therefore, as an Ansatz for yrast \(|K\rangle\) we choose the state \(|\psi\rangle\) with \( \phi = \phi_{MF}(x) \) being the solitonic solution of the NLSE with the average single particle momentum \((-i\partial_x)\) equal to \( 2\pi K/NL \). Then also the energy of the state \(|\psi\rangle\) is very close to the exact yrast state (Suppl. Mat.). The Ansatz is based on the intuition, partially spread in the community, that the quantum solitons should be somehow related to the product states of the mean-field solitons but smeared over the whole circle. In the ideal gas limit the Ansatz \(|\psi\rangle\) gives correct exact form of the yrast states \(|\psi_{\text{Ansatz}}\rangle \equiv \lim_{g \to 0} |K\rangle\) for any \( K \neq 0 \).

Ansatz \(|\psi\rangle\) is suitable to compute all important quantities. For instance, the conditional state reads:

\[
\hat{\Psi}(x_m)\hat{\Psi}(x_{m-1})\ldots \hat{\Psi}(x_1)|\psi_{\text{Ansatz}}\rangle \propto \int dy e^{i\pi Ky} \left( \prod_{i=1}^m \phi_{MF}(x_i-y) \right) e^{-i\hat{P}_y}\phi_{MF})^{\otimes (N-m)}.
\]

Due to the factors \( \phi_{MF}(x_i-y) \) the solitons centered close to places \( x_i \), where losses occurred, enter to the conditional state with lower weights, as compared to solitons with density dip far from \( x_i \). Physically, losses are happening more often at places where the gas is denser, therefore far from the solitonic density notch. The intuition behind the Ansatz and the lossy dynamics is visualized in Fig. 1.

Once we set the same parameters \( x_i \) in the Ansatz and in the many-body numerical simulation, then both methods lead to almost the same single-particle densities, as shown in Fig. 2. This agreement encourages us to look also at phase relations, which are very special for
solitons. A phase $\arg \{\phi_{\text{MF}}(x)\}$ of the soliton changes quickly within density notch, but it is linear far from it (see thick gray line in the left panel of Fig. 3). Therefore if the system is in the soliton state then majority of atoms surrounds the circle with constant velocity, apart from the place of rarefaction, where is a flow of fast particles moving in the opposite direction to the rest. We will extract the phase from the many-body wave functions using the normalized first order correlation function:

$$g_1(x) := \frac{\langle \hat{\Psi}(x) \hat{\Psi}(0) \rangle}{\sqrt{\langle \hat{N}(x) \rangle \langle \hat{N}(0) \rangle}}. \quad (8)$$

As shown in Fig. 3 the many-body calculation for the $g_1$ function are matching those obtained via the Ansatz (9). In the limit of many losses, the first order correlation function evaluated in the conditional state (8) converges to $e^{-i \arg \{\phi_{\text{MF}}(x)\}}$, which is $g_1$ function of the product state $|\phi_{\text{MF}}\rangle \otimes |N\rangle$. This result is actually quite surprising in the context of condensation. In the thermodynamic limit the divergence of the length scale at which $|g_1|$ decays, called the correlation length, is a fingerprint of Bose-Einstein condensation, probed experimentally for instance in [37]. As shown in Fig. 3 the correlation increases due to losses. We understand it in the following way: initial state is the superposition of macroscopically occupied modes. The local phase of such superposition is scrambled, resulting in reduced value of $|g_1|$ function. Particle losses select solitons, as sketched in Fig. 4 and push the state towards randomly chosen product state $|\phi_N\rangle \otimes |N\rangle$. The latter is a perfect condensate, with $|g_1| = 1$.

Having established correspondence between the yrast state and the mean-field solitons for the static properties, we turn now to dynamical ones. The real mean-field solitons should move with a constant speed and maintain their shape. On the other hand, different many-body constructions of solitons [18, 21, 23] led to density dips smeared out in dynamics on the time-scale $t_s = 1/g$.

Figure 3. (color online) Complex phase (left) and modulus (right) of the first order correlation function (8) computed in the conditional states with different number of lost atoms (5) using many-body calculation (dashed lines) and the Ansatz (9) (solid lines) contrasted with the mean-field result (thick gray line). Number of lost atoms, color-code are the same as in Fig. 2. Parameters ($N, K, g$) = (1000, 100, 0.05).

For our test-bed parameters, which are within experimentally studied regime [38], it would be impossible to track numerically the exact dynamics. On the other hand we checked that the spectrum of our yrast states is in the phononic regime, such that the average of Bogoliubov Hamiltonian in our yrast state is close to the exact value of the energy. Therefore as the generator of time-evolution we use (28), neglecting the higher order dynamical effects (like the Landau and Beliaev dampings). The snapshots of such dynamics are shown in Fig. 4. The changes in shape results from the losses happening during dynamics, but the density dips do not wash out, even after time $t = t_s$. We verified that, the dips move with practically constant speed, close to the one of the corresponding mean-field soliton (up to 7%).

In conclusion, we studied a system of many interacting particles in the presence of one-body losses to better understand how the mean-field solitons emerge and evolve. We performed many-body calculations for a large number of atoms using Bogoliubov states as the computational basis. The proper interpretation of the results required an appeal to the Ansatz (9), which is a many-body wave function that is built from product states with an unknown position and appropriately chosen relative phases.

The perfect agreement between the Ansatz (9) and the many body calculations is shown in Figs. 2 and 3, which indicate that the Ansatz is a fair approximation to the yrast state. In addition, one can easily decompose the conditional state after $m$ losses (8) in the states (9), which leaves $N - m$ atoms but with different momenta. In this way, one recovers the observations presented in [19, 21], that a conditional state is close to a wave packet of yrast states (see Supplementary Material). On the other hand, the conditional state after $m$-losses is equal to the high-order correlation function, as studied in [22]. Therefore, the physical phenomenon of particle losses unifies the previously studied symmetry breaking mechanisms. It appears that only finite-size effects make a soliton dynamically unstable in such works.

The results of this work show the importance of the spatial effects that are induced by losses. Many lost
atoms would produce phonons, shallow spontaneous solitons [39], and, in 2D and 3D, maybe even vortices. This, apart from the fundamental interest in the underlying physics, would lead to quick changes in spatial correlations, affecting the performance of quantum devices, such as interferometers.

Another important route to follow is to study what conditional state one can engineer by forcing losses to occur at chosen locations. This could be implemented using the light assisted loss technique [40] or with the electron-gun method [41]. With the well-scheduled loss locations, one may posit pushing the system towards the superposition of two macroscopic solitons, analogous to Schrödinger' cat-like state.

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SUPPLEMENTARY MATERIAL:
DARK SOLITONS REVEALED BY PARTICLE LOSSES IN THE 1D BOSE GAS

1. Ansatz for yrast state

In the main text we use the following Ansatz for an yrast state $|K\rangle$:

$$|\psi_{\text{Ansatz}}\rangle = C \int_0^L dy e^{\frac{i2\pi}{N} Ky} e^{-i\hat{P} y} |\phi_{\text{MF}}\rangle \otimes N, \quad (9)$$

where $\phi_{\text{MF}}(x)$ is the solitonic solution of the nonlinear Schrödinger equation which has the average momentum equal to $\frac{2\pi L}{N}$,

$$\langle \hat{\rho} \rangle := -i \int_0^L dx \, \phi_{\text{MF}}^*(x) \partial_x \phi_{\text{MF}}(x) = \frac{2\pi}{L} K/N. \quad (10)$$

The normalization factor of the wavefunction Eq. (9) is evaluated numerically, according to:

$$1 = \langle \psi_{\text{Ansatz}} | \psi_{\text{Ansatz}} \rangle = C^2 \int_0^L dy' \int_0^L dy \, e^{\frac{i2\pi}{N} Ky} \left( |\phi_{\text{MF}}\rangle \otimes N e^{-i\hat{P} (y-y')} |\phi_{\text{MF}}\rangle \otimes N \right) \quad (11)$$

$$= C^2 \int_0^L dy' \int_0^L dy \, e^{\frac{i2\pi}{N} Ky} \left( \int_0^L dx \, \phi_{\text{MF}}^*(x) \phi_{\text{MF}}(x-y+y') \right)^N. \quad (12)$$

The overlap $\int_0^L dx \, \phi_{\text{MF}}^*(x) \phi_{\text{MF}}(x-y+y')$ in the expression above is computed numerically.

To simulate particle losses during evolution one needs to act on the Ansatz (9) with the field operator. To some extent it can be performed analytically. If particle loss happened at position $x_1$, then the conditional wave-function $|\psi_1\rangle$ is given by:

$$|\tilde{\psi}_1\rangle \propto \hat{\Psi}(x_1)|\psi_{\text{Ansatz}}\rangle = C \int_0^L dy e^{\frac{i2\pi}{N} Ky} \hat{\Psi}(x_1) e^{-i\hat{P} y} |\phi_{\text{MF}}\rangle \otimes N \quad (13)$$

$$= C \int_0^L dy e^{\frac{i2\pi}{N} Ky} \hat{\Psi}(x_1) |\phi_{\text{MF}}(x-y)\rangle \otimes N \quad (14)$$

$$= C \int_0^L dy e^{\frac{i2\pi}{N} Ky} \left(\sqrt{N} \phi_{\text{MF}}(x_1-y) |\phi_{\text{MF}}(x-y)\rangle \otimes (N-1) \right), \quad (15)$$

where on uses the fact $\hat{\Psi}(x) |f\rangle \otimes N = f(x) |f\rangle \otimes (N-1)$. The state $\hat{\Psi}(x_1)|\psi_{\text{Ansatz}}\rangle$ is not normalized, that is why we used proportionality symbol $\propto$ in the equation above. Writing explicit formula wouldn’t have much sense here, as anyhow in the quantum trajectories method [42] the conditional wave-function has to be normalized "by hand" after each time step. Moreover, for brevity we omit in the notation the parameter $x_1$, although $|\tilde{\psi}_1\rangle$ depends on it.

Iterating Eq. (15), one can write down a conditional state after $m$ subsequent particle losses which occurred at positions $x_1$, $x_2$, ..., $x_m$:

$$|\tilde{\psi}_m\rangle \propto \left( \prod_{j=1}^{m} \hat{\Psi}(x_j) \right) |\psi_{\text{Ansatz}}\rangle \propto \int_0^L dy e^{\frac{i2\pi}{N} Ky} \left( \prod_{j=1}^{m} \phi_{\text{MF}}(x_j-y) \right) |\phi_{\text{MF}}(x-y)\rangle \otimes (N-m). \quad (16)$$
Having the conditional wave-function, as defined in (16), one can write down its single particle density and the first

Figure 5. From top to bottom: conditional density, modulus of the normalized first order correlation function $g_1$ and argument of $g_1$ (phase) obtained from the Ansatz. The corresponding mean-field results are marked with thick gray line. Parameters: $(N, K, g) = (100, 10, 0.5)$ (on the left) and $(N, K, g) = (1000, 100, 0.05)$ (on the right). In the left and the right column the healing length, the relative total momentum $K/N$ in the initial state and the fraction of lost particles are the same. In the case of $N = 100$ atoms initially (the left panels) the number of lost particles equals $m = 0$ (blue), $m = 5$ (orange), $m = 10$ (green), $m = 15$ (red) and $m = 20$ (brown), whereas for $N = 1000$ atoms initially (the right panels) the number of lost atoms equals to $m = 0$ (blue), $m = 50$ (orange), $m = 100$ (green), $m = 150$ (red) and $m = 199$ (brown).
order correlation function:

\[ \rho_m(x) := \left\langle \tilde{\Psi}(x) \tilde{\Psi}(x) \right\rangle = \left\| \tilde{\Psi}(x) \tilde{\psi}_m \right\|^2 \propto \int_0^L dy \int_0^L dy' \left[ e^{i \pi K (y-y')} \left( \prod_{j=1}^m \phi_{MF}^*(x_j-y') \phi_{MF}(x_j-y) \right) \right] \tag{17} \]

\[ \phi_{MF}^*(x-y') \phi_{MF}(x-y) \left( \phi_{MF}^*(u-y') \phi_{MF}(u-y) \right)^{N-m-1}. \tag{18} \]

\[ G_1(x) := \left\langle \tilde{\Psi}(x) \tilde{\Psi}(0) \right\rangle \propto \int_0^L dy \int_0^L dy' \left[ e^{i \pi K (y-y')} \left( \prod_{j=1}^m \phi_{MF}^*(x_j-y') \phi_{MF}(x_j-y) \right) \right] \tag{19} \]

\[ \phi_{MF}^*(x-y') \phi_{MF}(-y) \left( \phi_{MF}^*(u-y') \phi_{MF}(u-y) \right)^{N-m-1}. \tag{20} \]

In fact after first jump, we compute \( \rho_1 \), which is then used to draw the position of the second loss, i.e. \( x_2 \). Repeating this procedure we also obtain the places at which the subsequent particles were lost.

Having the representation of an yrast state with the Ansatz (9) one can compute conditional states after many, even dozens, of lost atoms. The results are shown in Fig. 5. In the case of \( N = 1000 \) atoms, the results in the case of 20% of lost atoms agree with the mean-field predictions. In the smaller sample, with \( N = 100 \) atoms, also the results for the conditional states fall eventually on the mean-field, but one needs as many as 80% of atoms to be lost. One has to remember that at some point the losses would completely change the state by introducing large fluctuations of number of atoms (and therefore of healing length) and momentum. Clearly – the larger the system the closer it is to the mean-field result.

Figure 6. Distributions of the conditional state \( |\tilde{\psi}\rangle_m \) in the yrast states approximated by the Ansatz (9) with \( (N-m) \) atoms and momentum. The conditional state \( |\tilde{\psi}\rangle_m \) defined in Eq. (16) is obtained in the stochastic process of measuring \( m \) atoms. The two results for \( m = 10 \) lost atoms corresponds to different realization of the stochastic process. Parameters \( (N, K, g) = (100, 48, 0.5) \) (left) and \( (N, K, g) = (1000, 400, 0.05) \) (right).

Finally one can decompose the conditional state after \( m \) lost atoms in the yrast states, approximated with Ansatzes (9), but with \( N-m \) atoms and different total momenta \( 2\pi K'/L \). We show examples of such decomposition in Fig. 6 for systems with initially \( N = 100 \) atoms (left) and \( N = 1000 \) (right). In both cases, the initial state was close to the yrast state with \( |N/2\rangle \). Clearly, the initial yrast state is transformed by losses into wave-packet (compare with LS [21]). For \( m = 1 \) the conditional state is covered by the yrast states in 98.8% in the case \( N = 100 \) and in 99.9% in the case \( N = 1000 \). For \( m = 10 \) the conditional state is covered by the yrast states only in 63.1% in the case \( N = 100 \), but in 96.6% in the case \( N = 1000 \).

2. Bogoliubov approximation

As written in the main text we use the Bogoliubov basis to perform numerical calculations. Here we write details of the Bogoliubov approximation stressing the unusual points which we made to improve the many-body computations.
The starting point is the Lieb-Liniger Hamiltonian written in the second quantization:

\[ \hat{H} = \frac{2\pi^2}{L^2} \sum_k k^2 \hat{a}_k^\dagger \hat{a}_k + \frac{g}{2L} \sum_{k_1, k_2 \neq k} \hat{a}_{k_1}^\dagger \hat{a}_{k_2} \hat{a}_{k_1} \hat{a}_{k_2}, \]  

(21)

with \( \hat{a}_k \) (\( \hat{a}_k^\dagger \)) annihilating (creating) a boson with momentum \( 2\pi k/L \) (with plane waves \( \phi_k(x) = e^{i2\pi k x/L} / \sqrt{L} \) as orbitals).

Our goal is to understand structure and dynamics initiated in the lowest energy state in the subspace with the total momentum \( 2\pi K/L \) with \( K \ll N \). In this case, provided that interaction is weak, we expect the average occupation of orbital with momentum \( k = 0 \) will be macroscopic, namely \( \langle \hat{a}_{k=0}^\dagger \hat{a}_{k=0} \rangle = O(N) \) when averaged in our states of interest. This is necessary condition to perform the Bogoliubov approximation. The Hamiltonian \( \hat{H} \) may be rewritten to the form:

\[ \hat{H} = \frac{2\pi^2}{L^2} \sum_k k^2 \hat{a}_k^\dagger \hat{a}_k + \frac{g}{2L} \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 + \frac{g}{2L} \sum_{k \neq 0} \left( 4\hat{a}_0^\dagger \hat{a}_k^\dagger \hat{a}_0 \hat{a}_k + \hat{a}_k^\dagger \hat{a}_{-k}^\dagger \hat{a}_0 \hat{a}_0 \right) + \hat{H}_R, \]

(22)

where \( \hat{H}_R \) contains all terms of the Hamiltonian \( \hat{H} \) which have less than two operators from the set \( \{ \hat{a}_0, \hat{a}_0^\dagger \} \). In the next step we would like to approximate mean values of operators \( \langle \hat{a}_0^\dagger \hat{a}_0 \rangle, \langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 \rangle, \langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \rangle \) by a constant. Note that while it is reasonable for the terms with two operators from the set \( \{ \hat{a}_0, \hat{a}_k^\dagger \} \), in case of \( \langle \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 \hat{a}_0 \rangle \) we need to be more subtle – here even small correction to \( \langle \hat{a}_0^\dagger \hat{a}_0 \rangle \), of the order of \( O(1) \), eventually leads to an error of the order of \( O(N) \). To avoid this problem, we use the fact that total number of particles is fixed \( N = \hat{a}_0^\dagger \hat{a}_0 + \sum_{k \neq 0} \hat{a}_k^\dagger \hat{a}_k \), therefore:

\[ \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 \hat{a}_0 = (\hat{a}_0^\dagger \hat{a}_0)^2 - \hat{a}_0^\dagger \hat{a}_0 = N^2 - 2N \sum_{k \neq 0} \hat{a}_k^\dagger \hat{a}_k + \left( \sum_{k \neq 0} \hat{a}_k^\dagger \hat{a}_k \right)^2 - \hat{a}_0^\dagger \hat{a}_0. \]

(23)

Typically, in Bogoliubov approximation one assumes \( \langle \hat{a}_k^\dagger \hat{a}_k \rangle = O(N) \). Here, given that \( K \gg 1 \), we use \( \langle \hat{a}_k^\dagger \hat{a}_k \rangle = N - K \). After substituting \( \hat{H} \) to \( \hat{H}_R \) and applying this approximation we get:

\[ \hat{H}_{\text{approx}} = \frac{2\pi^2}{L^2} \sum_k k^2 \hat{a}_k^\dagger \hat{a}_k + \frac{g}{2L} \left( N(N-1) - 2((N-K) + K) \sum_{k \neq 0} \hat{a}_k^\dagger \hat{a}_k + \left( \sum_{k \neq 0} \hat{a}_k^\dagger \hat{a}_k \right)^2 + K \right) + \frac{g}{2L} \sum_{k \neq 0} \left( 4(N-K) \hat{a}_k^\dagger \hat{a}_{-k}^\dagger + (N-K) \hat{a}_k^\dagger \hat{a}_{-k}^\dagger + (N-K) \hat{a}_k \hat{a}_{-k} + (N-K) \hat{a}_{-k}^\dagger \hat{a}_{-k} \right) + \hat{H}_R. \]

(24)

In the next step we only rearrange terms to obtain the expression:

\[ \hat{H}_{\text{approx}} = \frac{g}{2L} N(N-1) + \frac{2\pi^2}{L^2} \sum_k k^2 \hat{a}_k^\dagger \hat{a}_k + \frac{g}{2L} \sum_{k \neq 0} \left( (N-K) \hat{a}_k^\dagger \hat{a}_k + (N-K) \hat{a}_k^\dagger \hat{a}_{-k} + (N-K) \hat{a}_{-k}^\dagger \hat{a}_{-k} + \left( \sum_{k \neq 0} \hat{a}_k^\dagger \hat{a}_k \right)^2 + K \right) = \hat{H}_B + \hat{H}_R. \]

(25)

The last term \( \hat{R} \) is assumed to be small, as it contains only terms with at most one operator \( \hat{a}_0^\dagger \) or \( \hat{a}_0 \). \( \hat{H}_B \) is well known Bogoliubov Hamiltonian calculated for \( N - K \) particles and it may be easily diagonalized by applying Bogoliubov transformation:

\[ \hat{a}_k := u_k \hat{b}_k - v_k \hat{b}_k^\dagger, \]

(26)

\[ \hat{a}_k^\dagger := u_k \hat{b}_k^\dagger - v_k \hat{b}_k, \]

(27)

where \( u_k, v_k = \left( \frac{\omega_k / \epsilon_k - \sqrt{\epsilon_k / \omega_k}}{2} \right) / \sqrt{\omega_k / \epsilon_k + 2g(N-K)/L} \), \( \epsilon_k = \sqrt{\omega_k / \epsilon_k + 2g(N-K)/L} \) and \( \omega_k = \sqrt{\epsilon_k \epsilon_k + 2g(N-K)/L} \). Operators \( \hat{b}_k \) and \( \hat{b}_k^\dagger \) obey the bosonic commutation relation, i.e. \( \{ \hat{b}_k, \hat{b}_k^\dagger \} = 1 \). In these new operators \( \hat{H}_B \) is indeed diagonal:

\[ \hat{H}_B = \sum_{k \neq 0} \omega_k \hat{b}_k^\dagger \hat{b}_k. \]

(28)
It is worth to mention that in the limit of weak interaction $g \to 0$ such approach (i.e. approximating $\hat{a}_0^\dagger$, $\hat{a}_0$ operators by $\sqrt{N-K}$ instead of $\sqrt{N}$) works pretty well even for $K$ of the order $N$ (in contrast to usual approach, where error arises with $K$ even for $g \to 0$). More precisely, for the state $|n_0 = N - K, n_1 = K\rangle$ expectation value of the total energy calculated on Bogoliubov Hamiltonian $\left(\frac{g}{2L}N(N-1) + \langle H_B \rangle\right)$ gives exactly the same value as the one calculated on full Hamiltonian $\left(\langle \hat{H} \rangle = \frac{K}{2} + \frac{g}{2L}(N(N-1) + 2K(N-K))\right)$.

Finally also in $\hat{R}$ we replace the creation and annihilation operators $\hat{a}_k^\dagger$ and $\hat{a}_k$ with the Bogoliubov operators $\hat{b}_k$ and $\hat{b}_k^\dagger$ using the transformation given by Eqs. (26) and (27) so our approximated Hamiltonian:

$$\hat{H}_{\text{approx}} = \frac{g}{2L}N(N-1) + \hat{H}_B + \hat{R},$$

is written only with the Bogoliubov operators $\hat{b}_k$ and $\hat{b}_k^\dagger$. The only approximation we have made to get this form was replacing $\hat{a}_0^\dagger \to \sqrt{N-K}$, $\hat{a}_0 \to \sqrt{N-K}$; all other steps were made rigorously. We use the whole approximated Hamiltonian, including also the third and fourth order terms in $\hat{b}_k$, i.e. $\hat{R}$, to find the approximated yrast states. To generate dynamics we used only $\hat{H}_B$ part.

The computational basis is formed from the eigenstates of the Hamiltonian $\hat{H}_B$:

$$|n\rangle_B := |n_1, n_{-1}, n_2 \ldots \rangle_B = \frac{(b_k^\dagger)^{n_1} (b_{k-1}^\dagger)^{n_{-1}} \ldots (b_{k_{\text{max}}}^\dagger)^{n_{k_{\text{max}}}}}{\sqrt{n_1! n_{-1}! \ldots n_{k_{\text{max}}}!}} |0\rangle_B,$$

where $|0\rangle_B$ is the Bogoliubov vacuum. The states (30) are eigenstates of $\hat{b}_k^\dagger \hat{b}_k$, with eigenvalues $n_k$.

We do not write down the explicit forms of operator $\hat{R}$. This would be lengthy and complicated. In numerical codes, when we build the matrix of the Hamiltonian in the Bogoliubov basis (30), we identify in flow which terms from the part $\hat{R}$ will have non-zero contribution.

Also the field operator, which is the jump operator in the stochastic evolution, is represented in the Bogoliubov basis:

$$\hat{\Psi}(x) \approx \sqrt{(N-K)/L} + \sum_{k\neq 0} e^{i 2\pi k x/L} \left( u_k \hat{b}_k - v_k \hat{b}_k^\dagger \right) / \sqrt{L}.$$  

3. Numerical methods

As described in the main text we use as the basis Bogoliubov states $|n_1, n_{-1}, \ldots \rangle_B$. To perform numerical calculation we introduce two cut-offs. First one, $k_{\text{max}}$, is for the maximal single particle momenta, i.e. our basis is formed from states $|n_{-k_{\text{max}}}, n_{-k_{\text{max}}+1}, \ldots , n_{k_{\text{max}}}\rangle_B$. Then, to find an approximation for the yrast state $|K\rangle$ we include in the computational basis only the Bogoliubov states which are energetically close to the state $|n_1 = K\rangle_B$ and have the
Figure 8. Energy as a function of momentum for different interaction strengths $g$, from bottom to top $g = 0.5, 1, 2$. Solid lines refer to exact solutions of the Lieb-Liniger model, whereas the corresponding points refer to the mean-field spectrum and Ansatz \textsuperscript{9} (results from Ansatz are slightly below mean-field results). Number of atoms is $N = 31$.

The energy of the Bogoliubov state is estimated within Bogoliubov approximation:

$$E_n = \sum_{k=-k_{\text{max}}}^{k_{\text{max}}} n_k \omega_k,$$

where $\omega_k = \sqrt{\epsilon_k + 2g(N - K)/L}$ and $\epsilon_k = 2\pi^2 k^2/L^2$. The estimate $E_n$ is chosen only to help in selecting the candidates for basis. The number of states in the basis is denoted with $D$.

Once the basis is constructed for the given $D$ and $k_{\text{max}}$, then we compute the matrix of the Lieb-Liniger Hamiltonian

$$h_{nn'} := \langle n_{-k_{\text{max}}}', \ldots, n_{k_{\text{max}}}' | \hat{H} | n_{-k_{\text{max}}}, \ldots, n_{k_{\text{max}}} \rangle_B.$$

In computing, we do an approximation, as described in the previous section: we keep even terms with four Bogoliubov operators $\hat{b}_k$ and $\hat{b}_k^\dagger$, but we do the replacements $\hat{a}_0, \hat{a}_0^\dagger \rightarrow \sqrt{N - K}$. Finally the matrix $h_{nn'}$ is diagonalized. The lowest energy state is our approximation for the yrast state $|K\rangle$.

To test if the constructed basis is sufficient we look how the energy of our numerical candidates for the yrast state converges when we increase $D$ and $k_{\text{max}}$. Tests for $|K = 100\rangle$, with $N = 1000$ atoms and coupling strength $g = 0.05$ are shown in Fig. 7. Relying on this test we choose in this case $D = 200$ and $k_{\text{max}} = 3$.

We also benchmark our approximation with the exact formula for the energy of yrast state of the Lieb-Liniger model \textsuperscript{1}, by solving numerically the Bethe equations to find the Lieb’s quasimomenta and computing energy from them. In Fig. 8 we show the comparison between the exact results, mean-field model and Ansatz. The interaction strength are much smaller than the ones used in the main text (which will be $g = 0.05$), although with similar healing lengths.

We also compared the energy calculated with the many-body calculations using Bogoliubov basis and with the Ansatz, and the mean-field. The comparison is given in Fig. 9. In this Figure we also present the result of the many-body calculation but just for a single state $|n_1 = K\rangle_B$ in the basis, namely with the cut-off $D = 1$.

4. Ideal gas case

We use the extreme case of the ideal gas, $g \rightarrow 0$ as the test-bed for our numerics. In the ideal gas case the yrast states are just Fock states

$$|K\rangle = |n_0 = N - K, n_1 = K\rangle,$$
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Figure 9. Energy as a function of momentum for the target interaction strength \( g = 0.05 \). The solid line is the result from mean-field. Results marked by the points which are close to the solid line where obtained within the Ansatz. There are two data series from the many-body calculations using Bogoliubov basis – with just a single state in the basis, i.e. \( |n_1 = K\rangle_B \), and with \( D = 200 \) states.

with \( N - K \) atoms occupying the orbital \( 1/\sqrt{L} \) and \( K \) in the plane wave \( e^{i2\pi x/L}/\sqrt{L} \). Therefore the momentum of this state is \( 2\pi K L \). In the position representation this Fock state reads:

\[
\psi_S(x) := \langle x_1, x_2, \ldots x_N | n_0 = N - K, n_1 = K \rangle = \frac{1}{L^{N/2}} \frac{1}{\sqrt{(N \choose K)}} \sum_{\sigma \in S} e^{\frac{2\pi i}{N} (x_{\sigma(1)} + x_{\sigma(2)} + \ldots + x_{\sigma(K)})},
\]

where the final sum is over all combinations of choosing \( K \) elements from the set with \( N \) elements. On the other hand the solution of the NLSE corresponding to average momentum per particle \( p = \frac{1}{N} \left( \frac{2\pi K}{L} \right) \) equals to:

\[
\phi_{MF}(x, t) = \sqrt{\frac{N - K}{N}} + \sqrt{\frac{K}{N}} e^{i \frac{2\pi}{L} (x - \frac{\pi}{L} t)}.
\]

The state is a wave moving with velocity \( \frac{\pi}{L} \). It should be clear that all grey solitons in the limit \( g \rightarrow 0 \) have velocity \( \frac{\pi}{L} \), although they can have any average momentum.

The single-particle "soliton" (37) is connected with the yrast state (36) via identity:

\[
\psi_S(x) = \frac{1}{L^{\sqrt{(N \choose K)}}} \left( \frac{N - K}{K} \right)^{-K/2} \left( \frac{N}{N - K} \right)^{N/2} \int_0^L dy e^{i \frac{2\pi}{L} K y} \prod_{i=1}^N \phi_{MF}(x_i - y, 0).
\]

a. Density and the first order correlation function \( g_1 \).

Before any loss happened, the density is uniform \( \langle \hat{\Psi}^\dagger(x) \hat{\Psi}(x) \rangle = \frac{N}{L} \). The normalized first order correlation function \( g_1 \) is equal to

\[
g_1(x) = \frac{\langle \hat{\Psi}^\dagger(x) \hat{\Psi}(0) \rangle}{\sqrt{\langle \hat{\Psi}^\dagger(x) \hat{\Psi}(x) \rangle \langle \hat{\Psi}^\dagger(0) \hat{\Psi}(0) \rangle}} = \frac{N - K}{N} + \frac{K}{N} e^{i \frac{2\pi}{L} x}.
\]

It is also easy to write down density and the first order correlation function after loss of a single particle from position \( x_1 \):

\[
\hat{\rho}(x) = A \left( (N - K)(N - K - 1) + K(K - 1) + 2(N - K)K \left( 1 + \cos \left( \frac{2\pi}{L} (x - x_1) \right) \right) \right),
\]

\[
G_1(x) = A \left( (N - K)(N - K - 1) + K(K - 1)e^{-i \frac{2\pi}{L} x} + (N - K)K \left( 1 + e^{-i \frac{2\pi}{L} x} + e^{-i \frac{2\pi}{L} x_1} + e^{-i \frac{2\pi}{L}(x-x_1)} \right) \right),
\]

where \( x_1 \) is the position where the particle was lost and \( A \) is the normalization factor.
Figure 10. Snapshots of the single-particle density at 4 instants of time: $t/t_g = 0, 0.25, 0.5, 0.75, 1$, where $t_g = 1/g = 20$. The upper panel is the one, which was shown in the main text.

5. Dynamics

In the main text we presented snapshots of randomly chosen two stochastic wave dynamics. For completeness we present snapshots of more realizations. The number of lost particles is different in different realization being however always less than 8. The other snapshots are given in Fig. 10.

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