COUPLED PROBLEMS ON STATIONARY FLOW OF ELECTRORHEOLOGICAL FLUIDS UNDER THE CONDITIONS OF NONHOMOGENEOUS TEMPERATURE DISTRIBUTION.

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Abstract. We set up and study a coupled problem on stationary non-isothermal flow of electrorheological fluids. The problem consist in finding functions of velocity, pressure and temperature which satisfy the motion equations, the condition of incompressibility, the equation of the balance of thermal energy and boundary conditions.

We introduce the notions of a $P$-generalized solution and generalized solution of the coupled problem. In case of the $P$-generalized solution the dissipation of energy is defined by the regularized velocity field, which leads to a nonlocal model.

Under weak conditions, we prove the existence of the $P$-generalized solution of the coupled problem. The existence of the generalized solution is proved under the conditions on smoothness of the boundary and on smallness of the data of the problem.

1. Introduction

Electrorheological fluids are smart materials which are concentrated suspensions of polarizable particles in a nonconducting dielectric liquid. In moderately large electric fields, the particles form chains along the field lines, and these chains then aggregate to form columns [11]. These chainlike and columnar structures cause dramatic changes in the rheological properties of the suspensions. The fluids become anisotropic, the apparent viscosity (the resistance to flow) in the direction orthogonal to the direction of electric field abruptly increases, while the apparent viscosity in the direction of the electric field changes not so drastically.

The chainlike and columnar structures are destroyed under the action of large stresses, and then the apparent viscosity of the fluid decreases and the fluid becomes less anisotropic.

On the basis of experimental results the following constitutive equation was developed in [5]:

$$\sigma_{ij}(p, u, E) = -p\delta_{ij} + 2\varphi(I(u), |E|, \mu(u, E))\varepsilon_{ij}(u), \quad i, j = 1, \ldots, n, \quad n = 2 \text{ or } 3. \quad (1.1)$$

Here, $\sigma_{ij}(p, u, E)$ are the components of the stress tensor which depend on the pressure $p$, the velocity vector $u = (u_1, \ldots, u_n)$ and the electric field strength $E = (E_1, \ldots, E_n)$, $\delta_{ij}$ are the components of the unit tensor (the Kronecker delta), and $\varepsilon_{ij}(u)$ are the components of the rate of strain tensor

$$\varepsilon_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad (1.2)$$
Moreover, \( I(u) \) is the second invariant of the rate of strain tensor

\[
I(u) = \sum_{i,j=1}^{n} (\varepsilon_{ij}(u))^2, \tag{1.3}
\]

and \( \phi \) the viscosity function depending on \( I(u), |E| \) and \( \mu(u,E) \), where

\[
(\mu(u,E))(x) = \left( \frac{u(x)}{|u(x)|}, \frac{E(x)}{|E(x)|} \right)^2 = \frac{(\sum_{i=1}^{n} u_i(x)E_i(x))^2}{(|u(x)|^2)(\sum_{i=1}^{n} (E_i(x))^2)}. \tag{1.4}
\]

So \( \mu(u,E) \) is the square of the scalar product of the unit vectors \( \frac{u}{|u|} \) and \( \frac{E}{|E|} \). The function \( \mu \) is defined by (1.4) in the case of an immovable frame of reference. If the frame of reference moves uniformly with a constant velocity \( \vec{u} = (\vec{u}_1, \ldots, \vec{u}_n) \), then we set:

\[
\mu(u,E)(x) = \left( \frac{u(x) + \vec{u}}{|u(x) + \vec{u}|}, \frac{E(x)}{|E(x)|} \right)^2. \tag{1.5}
\]

As the scalar product of two vectors is independent of the frame of reference, the constitutive equation (1.1) is invariant with respect to the group of Galilei transformations of the frame of reference that are represented as a product of time-independent translations, rotations and uniform motions.

It is obvious that \( \mu(u,E)(x) \in [0,1] \), and for fixed \( y_1, y_2 \in \mathbb{R}_+ \), where \( \mathbb{R}_+ = \{ z \in \mathbb{R}, \ z \geq 0 \} \), the function \( y_3 \rightarrow \varphi(y_1, y_2, y_3) \) reaches its maximum at \( y_3 = 0 \) and its minimum at \( y_3 = 1 \) when the vectors \( u(x) + \vec{u} \) and \( E \) are correspondingly orthogonal and parallel.

The function \( \mu \) defined by (1.4), (1.5) is not specified at \( E = 0 \) and at \( u = 0 \), and there does not exist an extension of \( \mu \) by continuity to the values of \( u = 0 \) and \( E = 0 \). However, at \( E = 0 \) there is no influence of the electric field. Therefore,

\[
\varphi(y_1, 0, y_3) = \varphi(y_1), \quad y_3 \in [0,1], \tag{1.6}
\]

and the function \( \mu(u,E) \) need not be specified at \( E = 0 \). Likewise, in case that the measure of the set of points \( x \) at which \( u(x) = 0 \) is zero, the function \( \mu \) need not also be specified at \( u = 0 \). But in the general the function \( \mu \) is defined as follows:

\[
\mu(u,E)(x) = \left( \frac{\alpha \vec{I} + u(x) + \vec{u}}{\alpha \sqrt{n} + |u(x) + \vec{u}|}, \frac{E(x)}{|E(x)|} \right)^2, \tag{1.7}
\]

where \( \vec{I} \) denotes a vector with components equal to one, and \( \alpha \) is a small positive constant. If \( u(x) \neq 0 \) almost everywhere in \( \Omega \), we may choose \( \alpha = 0 \).

The viscosity function \( \varphi \) is identified by approximation of flow curves, see [5], and it was shown in [5] that it can be represented as follows:

\[
\varphi(I(u), |E|, \mu(u,E)) = b(|E|, \mu(u,E))(\lambda + I(u))^{-\frac{1}{2}} + \psi(I(u), |E|, \mu(u,E)), \tag{1.8}
\]

where \( \lambda \) is a small parameter, \( \lambda \geq 0 \).

In the special case that the direction of the velocity vector \( u(x) \) at each point \( x \) at which \( E(x) \neq 0 \) is known, the function \( x \rightarrow \mu(u,E)(x) \) becomes well-known, and the viscosity functions (1.8) takes the form

\[
\varphi(I(u), |E|, x) = e(|E|, x)(\lambda + I(u))^{-\frac{1}{2}} + \psi_1(I(u), |E|, x), \tag{1.9}
\]
where
\[ e(|E|, x) = b(|E|, \mu(u, E)(x)), \]
\[ \psi_1(I(u), |E|, x) = \psi(I(u), |E|, \mu(u, E)(x)). \] (1.10)

Here and thereafter the function of electric field \( E \) is assumed to be known. The equations for the functions \( E \) and \( (p, v) \) are separated, see [5].

In actual practice, the temperature fields in electrorheological fluids are nonuniform, and in many cases this non-homogeneity drastically affects on the flow of electrorheological fluids. Because of this, we reckon below that the functions \( b, \psi, e, \psi_1 \) in (1.8) and (1.9) depend also on the temperature \( \tau \). Therefore, we consider the following viscosity functions
\[ \varphi_1 = b(|E|, \mu(u, E), \tau)(\lambda + I(u))^{-\frac{1}{2}} + \psi(I(u), |E|, \mu(u, E), \tau), \] (1.11)
\[ \varphi_2 = e(|E|, \tau, x)(\lambda + I(u))^{-\frac{1}{2}} + \psi_1(I(u), |E|, \tau, x), \] (1.12)
and the stress tensor is defined by
\[ \sigma_{ij} = -p\delta_{ij} + 2\varphi_k\varepsilon_{ij}(u), \quad k = 1 \text{ or } 2, \quad i, j = 1, \ldots, n. \] (1.13)

Further we study coupled problems on non isothermal flow of electrorheological fluids with the constitutive equation (1.13) for \( k \) equals one and two.

In Sections 2 and 3 we introduce equations governing the process of non isothermal flows of electrorheological fluids. The coupled problem on non-isothermal flow consists in finding functions of velocity, pressure and temperature which satisfy the motion equations, the condition of incompressibility, the equation of the balance of thermal energy and boundary conditions.

We introduce the notions of a \( P \)-generalized and generalized solution of the coupled problem. In case of the \( P \)-generalized solution the dissipation of energy is defined by the regularized velocity field, which leads to a nonlocal model.

Some auxiliary results are set in Section 4, and the existence of a \( P \)-generalized solution of the coupled problem for the viscosity function \( \varphi_2 \) is proved in Section 5.

Assuming that the data of the problem are small and the boundary of the domain under consideration is smooth, we prove in Section 6 the existence of a generalized solution of the coupled problem for the viscosity function \( \varphi_2 \).

In Section 7 we consider coupled problem for the viscosity function \( \varphi_1 \) and formulate a result on the existence of the \( P \)-generalized solution for \( \varphi_1 \). A proof of this result is contained in Sections 8 and 9.

2. Equations of flow and thermal balance, boundary conditions.

We consider stationary problem on non-isothermal flow of the electrorheological fluid. The inertia forces are assumed to be small as compared with the internal forces caused by the viscous stresses. Then the equations of motion take the following form:
\[ \frac{\partial p}{\partial x_i} - 2\frac{\partial}{\partial x_j}[\varphi_k\varepsilon_{ij}(u)] = K_i \quad \text{in } \Omega, \quad i = 1, \ldots, n, \quad k = 1 \text{ or } 2, \] (2.1)
where \( K_i \) are the components of the volume force vector \( K \). In (2.1) and below the Einstein convention on summation over repeated index is applied.

The velocity function \( u \) meets the incompressibility condition
\[ \text{div } u = \sum_{i=1}^n \frac{\partial u_i}{\partial x_i} = 0 \quad \text{in } \Omega. \] (2.2)
The equation of the balance of thermal energy is the following:

\[
\chi \sum_{i=1}^{n} \frac{\partial^2 \tau}{\partial x_i^2} + 2\varepsilon \varphi_k I(u) - u_i \frac{\partial \tau}{\partial x_i} = 0 \quad \text{in } \Omega, \quad k = 1 \text{ or } 2. \tag{2.3}
\]

Here \(\chi\) is the thermal diffusivity, \(\varepsilon = (\rho c_p)^{-1}\), where \(\rho\) is the density, \(c_p\) specific heat. We reckon that \(\chi\) and \(\varepsilon\) are positive constants.

The first, second and third terms in the left-hand side of (2.3) determine the increments of the temperature in the unit of time produced respectively by heat conduction, dissipation of energy and thermal convection.

We assume that \(\Omega\) is a bounded domain in \(\mathbb{R}^n\), \(n = 2\) or \(3\) with a Lipschitz continuous boundary \(S\). Suppose that \(S_1\) and \(S_2\) are open subsets of \(S\) such that \(S_1\) is non-empty, \(S_1 \cap S_2 = \emptyset\), and \(\overline{S}_1 \cup \overline{S}_2 = S\). We consider mixed boundary conditions for the functions \(u, p\), wherein velocities are specified on \(S_1\) and surface forces are given on \(S_2\), i.e.

\[
\left| u \right|_{S_1} = \hat{u}, \tag{2.4}
\]

\[
\left. -p\delta_{ij} + \varphi_k \varepsilon_{ij}(u) \nu_j \right| \bigg|_{S_2} = F_i, \quad i = 1, \ldots, n, \quad k = 1 \text{ or } 2. \tag{2.5}
\]

Here \(F_i\) and \(\nu_j\) are components of the vectors of surface force \(F = (F_1, \ldots, F_n)\) and the unit outward normal \(\nu = (\nu_1, \ldots, \nu_n)\) to \(S\), respectively.

Temperature field on the boundary \(S\) is considered to be given

\[
\left| \tau \right|_{S} = \hat{\tau}. \tag{2.6}
\]

We assume that \(\hat{u} \in H^{\frac{1}{2}}(S_1), \hat{\tau} \in H^{\frac{1}{2}}(S)\). Then there exist functions \(\tilde{u}\) and \(\tilde{\tau}\) such that

\[
\tilde{u} \in H^1(\Omega)^n, \quad \left| \tilde{u} \right|_{S_1} = \hat{u}, \quad \text{div } \tilde{u} = 0, \tag{2.7}
\]

\[
\tilde{\tau} \in H^1(\Omega), \quad \left| \tilde{\tau} \right|_S = \hat{\tau}. \tag{2.8}
\]

Suppose also

\[
K = (K_1, \ldots, K_n) \in L_2(\Omega)^n, \quad F \in L_2(S_2)^n. \tag{2.9}
\]

3. **P-generalized solution of the coupled problem for the function \(\varphi_2\), and the existence theorem.**

We consider coupled problem for the function \(\varphi_2\) defined by (1.12). We assume that the functions \(e\) and \(\psi_1\) satisfy the following conditions:

**\(C1\):** \(e : (y_1, y_2, x) \rightarrow e(y_1, y_2, x)\) is a function continuous in \(\mathbb{R}_+ \times \mathbb{R} \times \overline{\Omega}\) and in addition

\[
0 \leq e(y_1, y_2, x) \leq a_0, \quad (y_1, y_2, x) \in \mathbb{R}_+ \times \mathbb{R} \times \overline{\Omega}, \tag{3.1}
\]

\(a_0\) being a positive constant.

**\(C2\):** \(\psi_1 : y_1, y_2, y_3, x \rightarrow \psi_1(y_1, y_2, y_3, x)\) is a function continuous in \(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \overline{\Omega}\), and for an arbitrary fixed \((y_2, y_3, x) \in \mathbb{R}_+ \times \mathbb{R} \times \overline{\Omega}\) the function \(\psi_1(\cdot, y_2, y_3, x) : y_1 \rightarrow \psi_1(y_1, y_2, y_3, x)\) is continuously differentiable in \(\mathbb{R}_+\), and the following inequalities
hold:

\begin{align}
\alpha_2 & \geq \psi_1(y_1, y_2, y_3, x) \geq \alpha_1, \quad (3.2) \\
\psi_1(y_1, y_2, y_3, x) + 2 \frac{\partial \psi_1}{\partial y_1} (y_1, y_2, y_3, x) y_1 & \geq \alpha_3, \quad (3.3) \\
\left| \frac{\partial \psi_1}{\partial y_1}(y_1, y_2, y_3, x) \right| y_1 & \leq \alpha_4, \quad (3.4)
\end{align}

where \( \alpha_i, 1 \leq i \leq 4, \) are positive constants.

The inequalities (3.1) and (3.2) indicate that the viscosity function is bounded from below and above by positive constants. (3.3) implies that for fixed values of \( |E|, \tau, x \) the derivative of the function \( I(u) \to G(u) \) is positive, where \( G(u) \) is the second invariant of the stress deviator \( G(u) = 4\varphi_2 I(u) \). This means that in the case of simple shear flow the shear stress increases with increasing shear rate. (3.4) is a restriction on \( \frac{\partial \psi_1}{\partial y_1} \) for large values of \( y_1 \). The inequalities (3.2)–(3.4) are natural from the physical point of view.

We consider the following spaces:

\[
X = \{ u | u \in H^1(\Omega)^n, u|_{S_1} = 0 \}, \\
V = \{ u | u \in X, \text{div}\ u = 0 \}.
\]

By means of Korn's inequality, the expression

\[
\| u \|_X = \left( \int_{\Omega} I(u) \, dx \right)^{\frac{1}{2}}
\]

defines a norm on \( X \) and \( V \) being equivalent to the norm of \( H^1(\Omega)^n \).

Everywhere below we use the following notations: if \( Y \) is a normed space, we denote by \( Y^* \) the dual of \( Y \), and by \( (f, h) \) the duality between \( Y^* \) and \( Y \), where \( f \in Y^*, h \in Y \). In particular, if \( f \in L_2(\Omega) \) or \( f \in L_2(\Omega)^n \), then \( (f, h) \) is the scalar product in \( L_2(\Omega) \) or in \( L_2(\Omega)^n \), respectively. The sign \( \rightharpoonup \) denotes weak convergence in a Banach space.

Define a bilinear form \( \pi \) in \( H^1(\Omega) \times H^1(\Omega) \) as follows:

\[
\pi(\theta_1, \theta_2) = \int_{\Omega} \sum_{i=1}^{n} \frac{\partial \theta_1}{\partial x_i} \frac{\partial \theta_2}{\partial x_i} \, dx, \quad \theta_1, \theta_2 \in H^1(\Omega).
\]

The expression

\[
\| \theta \|_1 = (\pi(\theta, \theta))^{\frac{1}{2}}
\]

defines a norm in \( H^1_0(\Omega) \) that is equivalent to the norm of \( H^1(\Omega) \).

We introduce a mapping \( N : X \times H^1_0(\Omega) \to X^* \) by

\[
(N(v, \zeta, h) = 2 \int_{\Omega} [e(|E|, \tilde{\tau} + \zeta, x)(\lambda + I(\tilde{u} + v))^{-\frac{1}{2}}
+ \psi_1(I(\tilde{u} + v), |E|, \tilde{\tau} + \zeta, x)] \varepsilon_{ij}(\tilde{u} + v) \varepsilon_{ij}(h) \, dx, \quad v, h \in X, \quad \zeta \in H^1_0(\Omega),
\]

and we assume that \( \lambda > 0 \) in (3.10).
Determine an operator \( A : X \times H^1_0(\Omega) \to H^{-1}(\Omega) \), where \( H^{-1}(\Omega) \) is the dual space of \( H^1_0(\Omega) \), as follows:

\[
\begin{align*}
(A(v, \zeta), \xi) &= \int_{\Omega} (\tilde{\tau} + \zeta)(\tilde{u} + v_i) \frac{\partial \xi}{\partial x_i} \, dx + 2\varepsilon \int_{\Omega} |e(\nabla(\tilde{\tau} + \zeta, x))(\lambda + I(\tilde{u} + v))^{-\frac{1}{2}} \\
&\quad + \psi_1(I(\tilde{u} + v), |E|, \tilde{\tau} + \zeta, x)] \cdot \epsilon(\tilde{u} + v)\xi dx - \chi \pi(\tilde{\tau}, \xi), \quad v \in X, \quad (\zeta, \xi) \in H^1_0(\Omega)^2.
\end{align*}
\]

(3.11)

Here \( P \) is an operator of regularization given by

\[
Pu(x) = \int_{\mathbb{R}^n} \omega(|x - x'|)u(x') \, dx', \quad x \in \overline{\Omega},
\]

(3.12)

where

\[
\omega \in C^\infty(\mathbb{R}_+), \quad \text{supp } \omega = [0, a], \quad \omega(z) \geq 0 \quad z \in \mathbb{R}_+,
\]

\[
\int_{\mathbb{R}^n} \omega(|x|) \, dx = 1, \quad a \text{ is a small positive constant.}
\]

(3.13)

In (3.12) we assume that the function \( u \) is extended to \( \mathbb{R}^n \).

We denote by \( B \) the operator \( \text{div} \), i.e.

\[
Bu = \text{div} \, u.
\]

(3.14)

It is obvious that \( B \in \mathcal{L}(X, L^2(\Omega)) \), and we denote by \( B^* \) the adjoint to \( B \) operator.

Consider the following problem: find a triple \((v, p, \zeta)\) such that

\[
(v, p, \zeta) \in X \times L^2(\Omega) \times H^1_0(\Omega),
\]

(3.15)

\[
\begin{align*}
(N(v, \zeta), h) - (B^*p, h) &= (K + F, h), \quad h \in X, \\
(Bv, q) &= 0, \quad q \in L^2(\Omega), \\
\pi(\zeta, \xi) - \frac{1}{\chi} (A(v, \zeta), \xi) &= 0, \quad \xi \in H^1_0(\Omega).
\end{align*}
\]

(3.16)

(3.17)

(3.18)

Here we use the notations

\[
(K, h) = \int_{\Omega} K_i h_i \, dx, \quad (F, h) = \int_{S_2} F_i h_i \, ds, \quad h \in X.
\]

(3.19)

The triple \((u = \tilde{u} + v, p, \tau = \tilde{\tau} + \zeta)\), where \( v, p, \zeta \) is a solution of the problem (3.15)–(3.18), will be called the \( P \)-generalized solution of the problem (2.1)–(2.6) for the viscosity function \( \varphi_2 \); in the special case that \( P \) is a unit operator the triple \((u = \tilde{u} + v, p, \tau = \tilde{\tau} + \zeta)\) is a generalized solution of the problem (2.1)–(2.6).

Indeed, by use of Green’s formula it can be seen that smooth generalized solution of the problem (2.1)–(2.6) is a classical solution of this problem. On the contrary, if \((u, p, \theta)\) is a classical solution of the problem (2.1)–(2.6), then the triple \((v = u - \tilde{u}, p, \zeta = \tau - \tilde{\tau})\) is a solution of the problem (3.15)–(3.18), wherein \( P \) is the unit operator.

From the physical point of view the presence of the operator \( P \) in (3.18) means that the value of dissipation of energy at a point \( x \) depends on the values of the rate of strain tensor at points belonging to some small vicinity of the point \( x \). Therefore, the presence of the operator \( P \) in (3.18) is natural from the physical point of view and it leads to a nonlocal model.
Theorem 3.1. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n = 2$ or 3 with a Lipschitz continuous boundary $S$, and suppose that the conditions $(C1)$, $(C2)$, (2.7)–(2.9) are satisfied. Then there exists a solution of the problem (3.15)–(3.18).

4. Auxiliary results.

Lemma 4.1. Assume that the conditions $(C1)$, $(C2)$, (2.7), (2.8) are satisfied. Then for an arbitrary fixed $\zeta \in H^1_0(\Omega)$ the partial function $N(\cdot, \zeta): v \to N(v, \zeta)$, where $N(v, \zeta)$ is given by (3.10), is a Lipschitz-continuous strictly monotone and coercive mapping of $X$ into $X^*$, i.e.

$$
\|N(v, \zeta) - N(w, \zeta)\|_{X^*} \leq \mu_1 \|v - w\|_X, \quad v, w \in X, 
$$

(4.1)

$$
(N(v, \zeta) - N(w, \zeta), v - w) \geq \mu_2 \|v - w\|_X^2, \quad v, w \in X, 
$$

(4.2)

$$
(N(v, \zeta), v) \geq \mu_3 \|v\|_{X^*}^2 - \mu_4 \|v\|_X, \quad v \in X, 
$$

(4.3)

where

$$
\mu_1 = 2a_2 + 4(a_4 + a_0 \lambda^{-\frac{1}{2}}), \quad \mu_2 = \min(2a_1, 2a_3),
$$

$$
\mu_3 = 2a_1, \quad \mu_4 = 2(a_0 \lambda^{-\frac{1}{2}} + a_2)\left(\int_{\Omega} I(\tilde{u}) \, dx\right)^\frac{1}{2}. 
$$

(4.4)

Lemma 4.2. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n = 2$ or 3 with a Lipschitz continuous boundary $S$, and let the operator $B \in L(X, L^2(\Omega))$ be defined by (3.14) Then, the inf-sup condition

$$
\inf_{\mu \in L^2(\Omega)} \sup_{v \in X} \frac{(Bv, \mu)}{\|v\|_X \|\mu\|_{L^2(\Omega)}} \geq \beta_1 > 0 
$$

(4.5)

holds true. The operator $B$ is an isomorphism from $V^\perp$ onto $L^2(\Omega)$, where $V^\perp$ is orthogonal complement of $V$ in $X$, and the operator $B^*$ that is adjoint to $B$, is an isomorphism from $L^2(\Omega)$ onto the polar set

$$
V^0 = \{f \in X^*, \quad (f, u) = 0, \quad u \in V\}. 
$$

(4.6)

Moreover,

$$
\|B^{-1}\|_{L(L^2(\Omega), V^\perp)} \leq \frac{1}{\beta_1}, 
$$

(4.7)

$$
\|(B^*)^{-1}\|_{L(V^0, L^2(\Omega))} \leq \frac{1}{\beta_1}. 
$$

(4.8)

For a proof see [1]. Lemma 4.2 is a generalization of the inf-sup condition in case that the operator $\text{div}$ acts in the subspace $H^1_0(\Omega)^n$ (see [3]). This result was first established in an equivalent form by Ladyzhenskaya and Solonnikov in [6].

Theorem 4.1. Suppose the conditions $(C1)$, $(C2)$, (2.7)–(2.9) are satisfied. Then for an arbitrary $\zeta \in H^1_0(\Omega)$ there exists a unique pair $(v(\zeta), p(\zeta))$ such that

$$
v(\zeta) \in X, \quad p(\zeta) \in L^2(\Omega),
$$

(4.9)

$$
(N(v(\zeta), \zeta), h) - (B^* p(\zeta), h) = (K + F, h), \quad h \in X, 
$$

(4.10)

$$
(Bv(\zeta), q) = 0, \quad q \in L^2(\Omega). 
$$

(4.11)
In addition
\[ \|v(\zeta)\|_X \leq \frac{1}{\mu_3} (\|K + F\|_{V^*} + \mu_4), \quad \zeta \in H^1_0(\Omega), \] (4.12)
and the function \( \zeta \to (v(\zeta), p(\zeta)) \) is a compact mapping of \( H^1_0(\Omega) \) into \( X \times L^2(\Omega) \).

The condition
\[ \zeta_k \rightharpoonup \zeta_0 \quad \text{in} \quad H^1_0(\Omega) \]
implies
\[ v(\zeta_k) \to v(\zeta_0) \quad \text{in} \quad X, \]
\[ p(\zeta_k) \to p(\zeta_0) \quad \text{in} \quad L^2(\Omega). \]

**Proof.** It follows from (4.10), (4.11) and (3.6) that the function \( v(\zeta) \) is a solution of the following problem:
\[ v(\zeta) \in V, \quad (N(v(\zeta), \zeta), h) = (K + F, h), \quad h \in V. \] (4.13)
Lemma 4.1 and the results on solvability of equations with monotone operators, see e.g. [7] imply that there exists a unique solution of the problem (4.13). Equality (4.13) yields
\[ N(v(\zeta), \zeta) - K - F \in V^0. \] (4.14)
By virtue of Lemma 4.2 and (4.14) there exists a unique function \( p(\zeta) \in L^2(\Omega) \) such that the pair \((v(\zeta), p(\zeta))\) is a solution of the problem (4.9)--(4.11).

Taking \( h = v(\zeta) \) in (4.10), we obtain
\[ (N(v(\zeta), \zeta), v) = (K + F, v(\zeta)) \leq \|K + F\|_{V^*} \|v(\zeta)\|_X, \] (4.15)
and (4.12) follows from (4.3) and (4.15). Inequality (4.12) implies
\[ \|p(\zeta)\|_{L^2(\Omega)} \leq c, \quad \zeta \in H^1_0(\Omega). \] (4.16)
Let now
\[ \zeta_k \rightharpoonup \zeta_0 \quad \text{in} \quad H^1_0(\Omega). \] (4.17)
We take the following notations:
\[ v^m = v(\zeta_m), \quad I_m = I(\tilde{u} + v^m), \quad \varepsilon^m_{ij} = \varepsilon_{ij}(\tilde{u} + v^m), \quad m = 0, 1, 2, \ldots, \]
\[ \varphi_{mk} = e(|E|, \tilde{\tau} + \zeta_k, x)(\lambda + I(\tilde{u} + v^m))^{-\frac{1}{2}} \]
\[ + \psi_1(I(\tilde{u} + v^m), |E|, \tilde{\tau} + \zeta_k, x), \quad k, m = 0, 1, 2, \ldots. \] (4.18)
It follows from (4.13) that
\[ (N(v^k, \zeta_k), h) = (K + F, h), \quad h \in V, \quad k = 1, 2, \ldots \]
\[ (N(v^0, \zeta_0), h) = (K + F, h), \quad h \in V. \]
Taking \( h = v^k - v^0 \), we obtain from here by (3.10) that
\[ \int_{\Omega} (\varphi_{kk} \varepsilon^{k}_{ij} - \varphi_{00} \varepsilon^{0}_{ij}) (\varepsilon^{k}_{ij} - \varepsilon^{0}_{ij}) \, dx = 0. \] (4.19)
(4.19) implies
\[ M^1_k + M^2_k = 0, \] (4.20)
where

\[ M_k^1 = \int_\Omega (\varphi_{kk} \varepsilon_{ij}^k - \varphi_{0k} \varepsilon_{ij}^0)(\varepsilon_{ij}^k - \varepsilon_{ij}^0) \, dx, \quad (4.21) \]

\[ M_k^2 = \int_\Omega (\varphi_{0k} - \varphi_{00}) \varepsilon_{ij}^0(\varepsilon_{ij}^k - \varepsilon_{ij}^0) \, dx. \quad (4.22) \]

The Cauchy inequality and the notation (4.18) imply

\[ |M_k^2| \leq \left( \int_\Omega (\varphi_{0k} - \varphi_{00})^2 I_0 \, dx \right)^{\frac{1}{2}} \left( \int_\Omega I(v^k - v^0) \, dx \right)^{\frac{1}{2}}. \quad (4.23) \]

By virtue of (4.17), we can extract a subsequence \( \{\zeta_\eta\} \) from the sequence \( \{\zeta_k\} \) such that

\[ \zeta_\eta \rightarrow \zeta_0 \text{ a.e. in } \Omega. \quad (4.24) \]

Now the continuity of the functions \( e \) and \( \psi_1 \) (see (C1), (C2)) yields

\[ \varphi_{0\eta} \rightarrow \varphi_{00} \text{ a.e. in } \Omega. \]

The functions \( \varphi_{0\eta} \) are bounded by a constant \( a_0 \lambda^{-\frac{1}{2}} + a_2 \). Therefore, the Lebesgue theorem and (4.23) imply that

\[ \lim M_\eta^2 = 0. \quad (4.25) \]

By analogy with the stated above, we can see that from any subsequence \( \{M_\eta^m\} \) extracted from the sequence \( \{M_k^2\} \), one can extract a subsequence \( \{M_\eta^m\} \) such that \( \lim M_\eta^m = 0 \). Therefore, \( \lim M_k^2 = 0 \), and by (4.20), we get

\[ \lim M_k^1 = 0. \quad (4.26) \]

It follows from (3.10), (4.2), (4.18) and (4.21) that

\[ M_k^1 = \frac{1}{2} \left[ (N(v^k, \zeta_k) - N(v^0, \zeta_k), v^k - v^0) \right] \geq \frac{1}{2} \mu_2 \|v^k - v^0\|^2_X. \]

This inequality together with (4.26) yields

\[ v(\zeta_k) \rightarrow v(\zeta_0) \text{ in } X. \quad (4.27) \]

We have

\[ \|N(v(\zeta_k), \zeta_k) - N(v(\zeta_0), \zeta_0)\|_{X^*} \leq \|N(v(\zeta_k), \zeta_k) - N(v(\zeta_0), \zeta_k)\|_{X^*} + \|N(v(\zeta_0), \zeta_k) - N(v(\zeta_0), \zeta_0)\|_{X^*}. \]

By virtue of (4.1) and (4.27) the first term on the right-hand side of this inequality tends to zero, and (4.24) implies that the second term also tends to zero.

Therefore,

\[ N(v(\zeta_k), \zeta_k) \rightarrow N(v(\zeta_0), \zeta_0) \text{ in } X^*. \quad (4.28) \]

It follows from (4.10) that

\[ B^* p(\zeta_k) - B^* p(\zeta_0) = N(v(\zeta_k), \zeta_k) - N(v(\zeta_0), \zeta_0) \text{ in } X^*. \quad (4.29) \]

Since the operator \( B^* \) is an isomorphism from \( L_2(\Omega) \) onto \( V^0 \), we obtain from (4.28) and (4.29) that

\[ p(\zeta_k) \rightarrow p(\zeta_0) \text{ in } L_2(\Omega), \]

and the theorem is proved. ■
We assign a mapping \( G : H^1_0(\Omega) \rightarrow H^{-1}(\Omega) \) as follows:

\[
H^1_0(\Omega) \ni \zeta \mapsto G(\zeta) = A(v(\zeta), \zeta) \in H^{-1}(\Omega).
\]

Here \( A : X \times H^1_0(\Omega) \rightarrow H^{-1}(\Omega) \) is given by (3.11) and \( v(\zeta) \) is the solution of the problem (4.9)–(4.11).

**Lemma 4.3.** Suppose the conditions (C1), (C2), (2.7)–(2.9) are satisfied. Then \( G \) is a compact mapping of \( H^1_0(\Omega) \) into \( H^{-1}(\Omega) \), the condition \( \zeta_k \rightarrow \zeta_0 \) in \( H^1_0(\Omega) \) implies \( G(\zeta_k) \rightarrow G(\zeta_0) \) in \( H^{-1}(\Omega) \).

**Proof.** Let \( \zeta_k \rightarrow \zeta_0 \) in \( H^1_0(\Omega) \). Then Theorem 4.1 and the embedding theorem yield

\[
\begin{align*}
    v(\zeta_k) &\rightarrow v(\zeta_0) \quad \text{in } L^2(\Omega)^n, \\
    \varepsilon_{ij}(v(\zeta_k)) &\rightarrow \varepsilon_{ij}(v(\zeta_0)) \quad \text{in } L^2(\Omega), \quad i, j = 1, \ldots, n, \\
    \zeta_k &\rightarrow \zeta_0 \quad \text{in } L^2(\Omega).
\end{align*}
\]

In view of (4.31), (4.32) a subsequence \( \{\zeta_{k_m}\} \) can be extracted from the sequence \( \{\zeta_k\} \) such that

\[
\begin{align*}
    \zeta_{k_m} &\rightarrow \zeta_0 \quad \text{a.e. in } \Omega, \\
    I(v(\zeta_{k_m})) &\rightarrow I(v(\zeta_0)) \quad \text{a.e. in } \Omega.
\end{align*}
\]

We represent the mapping \( G \) as a sum of two mappings \( G = G_1 + G_2 \), which we define as follows:

\[
\begin{align*}
    (G_1(\zeta), \xi) &= \int_\Omega (\mathbf{\tau} + \zeta)(\bar{u}_i + v(\zeta)_i) \frac{\partial \xi}{\partial x_i} \, dx - \chi \pi(\mathbf{\tau}, \xi), \quad \zeta, \xi \in H^1_0(\Omega), \\
    (G_2(\zeta), \xi) &= 2\varepsilon \int_\Omega |e(v(\zeta), \mathbf{\tau} + \zeta, x)(\lambda + I(\bar{u} + v(\zeta))^{-\frac{1}{2}} \\
    &+ \psi_3(I(\bar{u} + v(\zeta)), |v(\zeta)|, |\mathbf{\tau} + \zeta, x) I(P(\bar{u} + v(\zeta))) \xi \, dx, \quad \zeta, \xi \in H^1_0(\Omega).
\end{align*}
\]

If we will argue that

\[
\begin{align*}
    G_1(\zeta_k) &\rightarrow G_1(\zeta_0) \quad \text{in } H^{-1}(\Omega), \\
    G_2(\zeta_k) &\rightarrow G_2(\zeta_0) \quad \text{in } H^{-1}(\Omega),
\end{align*}
\]

then the lemma will be proved.

We denote

\[
f_{ki} = (\mathbf{\tau} + \zeta)(\bar{u}_i + v(\zeta)_i) - (\mathbf{\tau} + \zeta_0)(\bar{u}_i + v(\zeta_0)_i).
\]

It follows from (4.35) that

\[
\left| (G_1(\zeta_k) - G_1(\zeta_0), \xi) \right| = \left| \int_\Omega f_{ki} \frac{\partial \xi}{\partial x_i} \, dx \right| \leq \left( \sum_{i=1}^n \int_\Omega f_{ki}^2 \, dx \right)^{\frac{1}{2}} \|\xi\|_1,
\]

\( \|\cdot\|_1 \) being the norm defined by (3.9).

(4.30) and (4.32) yield

\[
f_{ki} \rightarrow 0 \quad \text{in } L^2(\Omega) \quad \text{as } k \rightarrow \infty, \quad i = 1, \ldots, n.
\]

By (4.40) and (4.41), we obtain (4.37).
By applying (3.1), (3.2), the Hölder inequality, and the notations (4.18), we obtain
\[
\left| \frac{1}{2\varepsilon} (G_2(\zeta_k) - G_2(\zeta_0), \xi) \right| = \left| \int_{\Omega} [\varphi_{kk} I(P(\bar{u} + v^k)) 
- \varphi_{00} I(P(\bar{u} + v^0))] \xi \, dx \right|
\leq (\delta \alpha_{1k} + \alpha_{2k}) \|\xi\|_{L_6(\Omega)} \leq c(\delta \alpha_{1k} + \alpha_{2k}) \|\xi\|_1, \\
\xi \in H_0^1(\Omega).
\] (4.42)

Here
\[
\delta = a_0 \lambda^{-\frac{1}{2}} + a_2, \\
\alpha_{1k} = \sum_{i,j=1}^{n} \left( \|P(\varepsilon_{ij}^k - \varepsilon_{ij}^0)\|_{L_{12}(\Omega)} \|P(\varepsilon_{ij}^k + \varepsilon_{ij}^0)\|_{L_{12}(\Omega)} \right), \\
\alpha_{2k} = \left( \int_{\Omega} (\varphi_{kk} - \varphi_{00}) \frac{6}{5} (I(P(\bar{u} + v^0))) \frac{6}{5} \, dx \right)^{\frac{5}{6}}.
\] (4.43) (4.44) (4.45)

(4.31) yields
\[
\lim \alpha_{1k} = 0. \\
\] (4.46)

By (4.33), (4.34) and the Lebesgue theorem, we obtain that \(\lim \alpha_{2k} = 0\). By analogy with the stated above, we can see that from any subsequence \(\{\zeta_m\}\) extracted from the sequence \(\{\zeta_k\}\), one can extract a subsequence \(\{\zeta_i\}\) such that \(\lim \alpha_{2i} = 0\). Therefore,
\[
\lim \alpha_{2k} = 0. \\
\] (4.47)

(4.42), (4.46) and (4.47) imply (4.38), and the lemma is proved. ■

5. PROOF OF THE THEOREM 3.1.

Consider the problem: find a function \(\zeta\) such that
\[
\zeta \in H_0^1(\Omega), \\
\pi(\zeta, \xi) - \frac{1}{\chi} (A(v(\zeta), \zeta, \xi) = 0, \quad \xi \in H_0^1(\Omega),
\] (5.1) (5.2)

where \(v(\zeta)\) is the solution of the problem (4.13). In this case there exists a unique function \(p(\zeta) \in L_2(\Omega)\) such that the pair \((v(\zeta), p(\zeta))\) is the solution of the problem (4.9)–(4.11).

It is obvious that, if \(\zeta\) is a solution of the problem (5.1), (5.2), then the triple \((v = v(\zeta), p = p(\zeta), \zeta)\) is a solution of the problem (3.15)–(3.18).

Let \(\{Z_k\}\) be a sequence of finite dimensional subspaces in \(H_0^1(\Omega)\), such that
\[
Z_k \subset Z_{k+1}, \quad \lim_{k \to \infty} \inf_{h \in Z_k} \|w - h\|_1 = 0, \quad w \in H_0^1(\Omega).
\] (5.3)

We seek an approximate solution of the problem (5.1), (5.2) of the form
\[
\zeta_k \in Z_k, \quad \pi(\zeta_k, \xi) - \frac{1}{\chi} (A(v(\zeta_k), \zeta_k, \xi) = 0, \quad \xi \in Z_k,
\] (5.4)

where \(v(\zeta_k)\) is the solution of the problem (4.13) for \(\zeta = \zeta_k\).
Applying the integration by parts, we obtain
\[
\int_{\Omega} \zeta(\tilde{u}_i + v_i) \frac{\partial \zeta}{\partial x_i} \, dx = - \int_{\Omega} \frac{\partial \zeta}{\partial x_i} (\tilde{u}_i + v_i) \zeta \, dx = 0, 
\]
\(v \in V, \quad \zeta \in H^1_0(\Omega).\) \hfill (5.5)

Taking into account (2.7), (2.8), (3.1), (3.2), (4.12) and (5.5), we reduce from (3.11) that
\[
\left| \frac{1}{\lambda} \left( A(v(\zeta), \zeta), \zeta \right) \right| \leq c \|\zeta\|_1, \quad \zeta \in H^1_0(\Omega). 
\] \hfill (5.6)

Therefore,
\[ y(\zeta) = \pi(\zeta, \zeta) - \frac{1}{\lambda} (A(v(\zeta), \zeta), \zeta) \geq \|\zeta\|_2^2 - c \|\zeta\|_1, \] \hfill (5.7)
and \(y(\zeta) \geq 0\) for \(\|\zeta\|_1 \geq c.\)

From the corollary of Brouwer’s fixed point theorem \cite{2} it follows that there exists a solution of the problem (5.4) and \(\|\zeta_k\|_1 \leq c.\) Consequently, we can extract a subsequence \(\{\zeta_{\eta}\}\) from the sequence \(\{\zeta_k\}\) such that
\[
\zeta_{\eta} \rightharpoonup \zeta_0 \text{ in } H^1_0(\Omega). 
\] \hfill (5.8)

Let \(\eta_0\) be a fixed positive number and \(\xi \in Z_{\eta_0}.\) By (5.8) and Lemma 4.3, we pass to the limit in (5.4) with \(k\) replaced by \(\eta.\) Then, we get
\[
\zeta_0 \in H^1_0(\Omega), \quad \pi(\zeta_0, \xi) - \frac{1}{\lambda} (A(v(\zeta_0), \zeta_0), \xi) = 0, \quad \xi \in Z_{\eta_0}, 
\] \hfill (5.9)
(5.3) and (5.9) imply that the function \(\zeta = \zeta_0\) is a solution of the problem (5.1), (5.2). The theorem is proved. \(\blacksquare\)

6. Generalized solution of the coupled problem for the function \(\varphi_2.\)

We denote
\[ Q = H^1_0(\Omega) \cap L_\infty(\Omega), \]
and determine a mapping \(A_1 : X \times W^{1,\frac{6}{5}}_0(\Omega) \to Q^*\) as follows:
\[
(A_1(v, \zeta), \xi) = - \int_{\Omega} \frac{\partial (\tilde{\tau} + \zeta)}{\partial x_i} (\tilde{u}_i + v_i) \xi \, dx 
+ 2\varepsilon \int_{\Omega} \left[ e(|E|, \tilde{\tau} + \zeta, x) (\lambda + I(\tilde{u} + v))^{-\frac{1}{2}} 
+ \psi_1(I(\tilde{u} + v), |E|, \tilde{\tau} + \zeta, x) I(\tilde{u} + v) \right] \xi \, dx - \chi \pi(\tilde{\tau}, \xi), 
\]
\(v \in X, \quad \zeta \in W^{1,\frac{6}{5}}_0(\Omega), \quad \xi \in Q.\) \hfill (6.1)

Here \(W^{1,\frac{6}{5}}_0(\Omega)\) is the closure of \(D(\Omega)\) for the norm of the Sobolev space \(W^{1,\frac{6}{5}}(\Omega).\)

The expression
\[
\|\zeta\|_2 = \left( \int_{\Omega} \left( \sum_{i=1}^{n} \left| \frac{\partial \zeta}{\partial x_i} \right|^{\frac{6}{5}} \right)^{\frac{5}{6}} \, dx \right)^{\frac{6}{5}}, 
\] \hfill (6.2)
defines a norm in \(W^{1,\frac{6}{5}}_0(\Omega)\) which is equivalent to the norm of \(W^{1,\frac{6}{5}}(\Omega).\)
Consider the problem: find a triple \((v, p, \zeta)\) such that
\[
(v, p, \zeta) \in X \times L_2(\Omega) \times W^{1, \frac{8}{3}}(\Omega), \tag{6.3}
\]
\[
(N(v, \zeta), h) - (B^* p, h) = (K + F, h), \quad h \in X, \tag{6.4}
\]
\[
(Bv, q) = 0, \quad q \in L_2(\Omega), \tag{6.5}
\]
\[
\pi(\zeta, \xi) - \frac{1}{\chi} (A_1, (v, \zeta), \xi) = 0, \quad \xi \in Q. \tag{6.6}
\]
The triple \((u = \bar{u} + v, p, \tau = \bar{\tau} + \zeta)\), where \((v, p, \zeta)\) is a solution of the problem (6.3)–(6.6), is a generalized solution of the problem (2.1)–(2.6).

Indeed, by use of the Green formula, one can verify that, if \((v, p, \zeta)\) is a solution of the problem (6.3)–(6.6), then \(u, p, \tau\) with \(u = \bar{u} + v\), and \(\tau = \bar{\tau} + \zeta\) is a solution of the problem (2.1)–(2.6) in the distribution sense. On the contrary, if \(u, p, \tau\) is a solution of the problem (2.1)–(2.6) such that (6.3) holds with \(v = u - \bar{u}, \zeta = \tau - \bar{\tau}\), then \((v, p, \zeta)\) is a solution of the problem (6.3)–(6.6).

**Theorem 6.1.** Let \(\Omega\) be a bounded domain in \(\mathbb{R}^n\), \(n = 2 \text{ or } 3\) with a boundary of the class \(C^4\). Suppose the conditions \((C1), (C2), (2.7) - (2.9)\) are satisfied.

Assume also that
\[
\inf_{w \in V} \frac{\tilde{c}}{\chi} \left[ \|\bar{u} + w\|_{H^1(\Omega)^n} + \frac{1}{2a_1} \left( \|K + F\|_{V^*} + 2(a_0 \lambda^{-\frac{1}{2}} + a_2) \left( \int_{\Omega} I(\bar{u} + w) dx \right)^{\frac{1}{2}} \right) \right] < 1, \tag{6.7}
\]
where \(\tilde{c}\) is a constant depending on the domain \(\Omega\). Then, there exists a solution of the problem (6.3)–(6.6)

**Proof.**

1) Let \(\{P_k\}_{k=1}^\infty\) be a sequence of regularizing operators assigned by
\[
P_k u(x) = \int_{\mathbb{R}^n} \omega_k(|x - x'|) u(x') dx', \quad x \in \overline{\Omega}, \tag{6.8}
\]
where
\[
\omega_k \in C^\infty(\mathbb{R}_+), \quad \text{supp } \omega_k = [0, b_k], \quad \omega_k(z) \geq 0, \quad z \in \mathbb{R}_+, \quad \int_{\mathbb{R}^n} \omega_k(|x|) dx = 1, \tag{6.9}
\]
and also \(\lim b_k = 0\). In this case we have
\[
\lim ||P_k u - u||_{H^1(\Omega)^n} = 0, \quad u \in H^1(\Omega)^n, \tag{6.10}
\]
\[
\lim ||P_k u - u||_{L_2(\Omega)^n} = 0, \quad u \in L_2(\Omega)^n,
\]
and it is assumed here that the function \(u\) is prolonged in \(\Omega_1, \Omega_1 \supset \overline{\Omega}\), such that \(u\) belongs to \(H^1(\Omega_1)^n \text{ or } L_2(\Omega_1)^n\), respectively.

By virtue of the Theorem 3.1 for each operator \(P = P_k\) there exists a solution of the problem (3.15)–(3.18), which we denote by \(v^k, p_k, \zeta_k\). In this case the function \(\zeta_k\) is a solution of the following problem:
\[
\zeta_k \in H^1_0(\Omega), \quad \sum_{i=1}^n \frac{\partial^2 \zeta_k}{\partial x_i^2} = \beta_k - \sum_{i=1}^n \frac{\partial^2 \varphi_i}{\partial x_i^2} \quad \text{in } D^*(\Omega), \tag{6.11}
\]
where
\[
\beta_k = \frac{1}{\chi} (\bar{u}_i + v^k_i) \frac{\partial (\bar{\tau} + \zeta_k)}{\partial x_i} - \frac{2\pi}{\chi} \varphi_{kk} I(P_k (\bar{u} + v^k)), \tag{6.12}
\]
and \( \varphi_{kk} \) is defined in (4.18).

We have

\[
\left\| (\bar{u}^i + v^k_i) \frac{\partial (\bar{\tau} + \zeta_k)}{\partial x_i} \right\|_{L^1(\Omega)} \leq M_{1k} + M_{2k},
\]

where

\[
M_{1k} = \left\| (\bar{u}^i + v^k_i) \frac{\partial \bar{\tau}}{\partial x_i} \right\|_{L^1(\Omega)}, \quad M_{2k} = \left\| (\bar{u}^i + v^k_i) \frac{\partial \zeta_k}{\partial x_i} \right\|_{L^1(\Omega)}.
\]

(6.13)

It follows from (2.7)–(2.9), (4.4) and (4.12) that

\[
M_{1k} = \sum_{i=1}^n \left\| \bar{u}^i + v^k_i \right\|_{L^6(\Omega)} \left\| \frac{\partial \bar{\tau}}{\partial x_i} \right\|_{L^6(\Omega)} \leq c_1 \| \bar{u} + v^k \|_{H^1(\Omega)} \| \bar{\tau} \|_2 \leq c_2.
\]

(6.15)

(2.7), (4.4) and (4.12) imply

\[
M_{2k} \leq \sum_{i=1}^n \left\| \bar{u}^i + v^k_i \right\|_{L^6(\Omega)} \left\| \frac{\partial \zeta_k}{\partial x_i} \right\|_{L^6(\Omega)} \leq c_3 (\| \bar{u} \|_{H^1(\Omega)} + \| v^k \|_X) \| \zeta_k \|_2 \leq c_3 \dot{c} \| \zeta_k \|_2,
\]

where

\[
\dot{c} = \| \bar{u} \|_{H^1(\Omega)} + \frac{1}{2a_1} \left( \| K + F \|_{V^*} + 2(a_0 \lambda^{-\frac{1}{2}} + a_2) \left( \int_{\Omega} (\bar{u})dx \right)^\frac{1}{2} \right).
\]

(6.16)

Taking note of (6.13), (6.15) and (6.16), we obtain

\[
\left\| (\bar{u}^i + v^k_i) \frac{\partial (\bar{\tau} + \zeta_k)}{\partial x_i} \right\|_{L^1(\Omega)} \leq c_3 \dot{c} \| \zeta_k \|_2 + c_2.
\]

(6.18)

(2.7), (3.1), (3.2) and (4.12) yield

\[
\| \varphi_{kk} \mathbf{I}(P_k(\bar{u} + v^k)) \|_{L^1(\Omega)} \leq c_4.
\]

(6.19)

It follows from (6.12), (6.18) and (6.19) that

\[
\| \beta_k \|_{L^1(\Omega)} \leq \frac{c_3 \dot{c}}{\dot{\lambda}} \| \zeta_k \|_2 + c_5.
\]

(6.20)

Since the boundary \( S \) of the class \( C^4 \) and

\[
\frac{\partial^2 \bar{\tau}}{\partial x_i^2} \in H^{-1}(\Omega) \subset W^{-1, \frac{2}{3}}(\Omega), \quad i = 1, \ldots, n,
\]

where \( W^{-1, \frac{2}{3}}(\Omega) \) is the space dual to \( W^{1, 6}(\Omega) \), we obtain from (6.11) and known results [13], [14], Section 1.1 that

\[
\| \zeta_k \|_2 \leq c_6 (\| \beta_k \|_{W^{-1, \frac{2}{3}}(\Omega)} + \| \bar{\tau} \|_{W^{1, \frac{2}{3}}(\Omega)}).
\]

(6.21)

The embedding of \( L^1(\Omega) \) into \( W^{-1, \frac{2}{3}}(\Omega) \) is continuous because \( W^{1, 6}(\Omega) \subset C(\overline{\Omega}) \) for \( n = 2 \) and 3. So that by (6.20) and (6.21), we obtain

\[
\| \zeta_k \|_2 \leq c_7 \| \beta_k \|_{L^1(\Omega)} + c_8 \leq \frac{c_9 \dot{c}}{\dot{\lambda}} \| \zeta_k \|_2 + c_{10}.
\]

(6.22)

That is

\[
\left( 1 - \frac{c_9 \dot{c}}{\dot{\lambda}} \right) \| \zeta_k \|_2 \leq c_{10}
\]

(6.23)

It is evident that, if a function \( \bar{u} \) satisfies the conditions (2.7), then the function \( \bar{u} + w \), where \( w \in V \), satisfies (2.7) also. Therefore, we can consider that, in the operators \( N, A \) and
Taking into account that (6.27) and the Lebesgue theorem, we deduce

\[ \gamma(w) = \frac{c_0}{X} \left( \|u + w\|_{H^1(\Omega)} + \frac{1}{2a_1} \left( \|K + F\|_{X^*} + 2(a_0\lambda^{-\frac{1}{3}} + a_2) \left( \int_\Omega (\bar{u} + w) dx \right)^{\frac{1}{2}} \right) \right). \]  

(6.25)

If the condition (6.7) with \( \tilde{c} = c_0 \) is satisfied, then there exists a function \( \tilde{w} \in V \) such that \( \gamma(\tilde{w}) < 1 \), and by virtue of (6.24) the sequence \( \{\zeta_k\} \) is bounded in \( W^{1,\frac{4}{3}}_0(\Omega) \).

We consider that the function \( \tilde{u} \) is replaced by the function \( \bar{u} + \tilde{w} \) in the operators \( N, A \) and \( A_1 \), and we still denote by \( \tilde{u} \) the function \( \bar{u} + \tilde{w} \). Then for the new function \( \tilde{u} \), we have in (6.23) \( c_0\tilde{c}\chi^{-1} < 1 \). Therefore, a subsequence \( \{v^m, p_m, \zeta_m\} \) can be extracted from the sequence \( \{\tilde{v}^k, p_k, \zeta_k\} \) such that

\[ \zeta_m \to \zeta_0 \quad \text{in} \quad W^{1,\frac{4}{3}}_0(\Omega), \]  

(6.26)

\[ \zeta_m \to \zeta_0 \quad \text{in} \quad L^{\frac{4}{3}}(\Omega) \quad \text{and a.e. in} \quad \Omega, \]  

(6.27)

\[ v^m \to v^0 \quad \text{in} \quad X, \]  

(6.28)

\[ N(v^m, \zeta_m) \to \alpha \quad \text{in} \quad X^*, \]  

(6.29)

\[ p_m \to p_0 \quad \text{in} \quad L^2(\Omega), \]  

(6.30)

2). Now we are concerned with the passage to the limit. It follows from (6.4) and (6.5) that

\[ (N(v^m, \zeta_m), h) - (B^* p_m, h) = (K + F, h), \quad h \in X, \]  

(6.31)

\[ (Bv^m, q) = 0, \quad q \in L^2(\Omega). \]  

(6.32)

Observing (6.28)–(6.30), we pass to the limit in (6.31), (6.32), and obtain

\[ \alpha - B^* p_0 = K + F \quad \text{in} \quad X^*, \]  

(6.33)

\[ \text{div} \, v^0 = 0. \]  

(6.34)

Lemma 4.1 implies

\[ (N(v^m, \zeta_m) - N(g, \zeta_m), v^m - g) \geq 0, \quad g \in X. \]  

(6.35)

Taking into account that \( (p_m, Bv_m) = 0 \), by (6.28) and (6.31), we obtain

\[ (N(v^m, \zeta_m), v^m) = (K + F, v^m) \to (K + F, v^0), \]  

(6.36)

and

\[ \lim (N(v^m, \zeta_m), g) - (B^* p_0, g) = (K + F, g), \quad g \in X. \]  

(6.37)

By using (C1), (C2), (6.27) and the Lebesgue theorem, we deduce

\[ [c(|E|, \tilde{\tau} + \zeta_m, x) (\lambda + I(\bar{u} + g))^{-\frac{1}{2}} + \psi_1(I(\bar{u} + g), |E|, \tilde{\tau} + \zeta_m, x)] \]  

\[ \times \varepsilon_{ij}(\bar{u} + g) \to [c(|E|, \tilde{\tau} + \zeta_0, x) (\lambda + I(\bar{u} + g))^{-\frac{1}{2}} \]  

\[ + \psi_1(I(\bar{u} + g), |E|, \tilde{\tau} + \zeta_0, x)] \varepsilon_{ij}(\bar{u} + g) \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad m \to \infty. \]  

(6.38)
(6.28) yields
\[ \varepsilon_{ij}(v^m - g) \to \varepsilon_{ij}(v^0 - g) \text{ in } L_2(\Omega). \] (6.39)

It follows from (3.10), (6.38) and (6.39) that
\[ (N(g, \zeta_m), v^m - g) \to (N(g, \zeta_0), v^0 - g). \] (6.40)

Observing (6.36), (6.37) and (6.40), we pass to the limit in (6.35), this gives
\[ (K + F - N(g, \zeta_0) + B^* p_0, v^0 - g) \geq 0, \quad g \in X. \] (6.41)

We choose \( g = v^0 - \gamma h, \ h \in X, \gamma > 0, \) and consider \( \gamma \to 0. \) Then, (4.1) implies
\[ (K + F - N(v^0, \zeta_0) + B^* p_0, h) \geq 0. \] (6.42)

This inequality holds for any \( h \in X. \) Replacing \( h \) by \(-h\) shows that here the equality holds true. Consequently, the triple \((v = v^0, p = p_0, \zeta = \zeta_0)\) meets the equations (6.4) and (6.5).

It follows from (4.2) that
\[ (N(v^m, \zeta_m) - N(v^0, \zeta_0), v^m - v^0) \geq \mu_2 \|v^m - v^0\|^2_X. \] (6.43)

Granting (6.36), (6.37), (6.40) one can recognize that the left-hand side of (6.43) tends to zero. Therefore,
\[ v^m \to v^0 \text{ in } X, \] (6.44)

and we can regard that
\[ I(v^m) \to I(v^0) \text{ a.e. in } \Omega. \] (6.45)

Otherwise, we can extract a subsequence, still denoted by \( \{v^m\} \), such that (6.45) holds true.

3). The pair \((v^m, \zeta_m)\) meets the following equation (see (3.11) and (3.18))
\[ \pi(\zeta_m, \xi) - \frac{1}{\chi}(U_m, \xi) = 0, \quad \xi \in H^1_0(\Omega) \cap L_\infty(\Omega) = Q, \] (6.46)

where
\[ (U_m, \xi) = -\int_\Omega \frac{\partial (\bar{\tau} + \zeta_m)}{\partial x_i}(\bar{u}_i + v^m_i)\xi \ dx + 2 \varepsilon \int_\Omega \varphi_{mm} I(P_m(\bar{u} + v^m))\xi \ dx - \chi \pi(\bar{\tau}, \xi), \] (6.47)

\( \varphi_{mm} \) being defined in (4.18).

By virtue of (6.44) \( v^m \to v^0 \) in \( L_0(\Omega)^n \), so that by (6.26), we obtain
\[ \lim_{m \to \infty} \int_\Omega \frac{\partial (\bar{\tau} + \zeta_m)}{\partial x_i}(\bar{u}_i + v^m_i)\xi \ dx = \int_\Omega \frac{\partial (\bar{\tau} + \zeta_0)}{\partial x_i}(\bar{u}_i + v^0_i)\xi \ dx, \quad \xi \in Q. \] (6.48)

We have
\[ \left| \int_\Omega [\varphi_{mm} I(P_m(\bar{u} + v^m)) - \varphi_{00} I(\bar{u} + v^0)]\xi \ dx \right| \leq \|\xi\|_{L_\infty(\Omega)} (\mathcal{M}_1^m + \mathcal{M}_2^m), \] (6.49)

where
\[ \mathcal{M}_1^m = \left| \int_\Omega (\varphi_{mm} - \varphi_{00}) I(\bar{u} + v^0) dx \right|, \] (6.50)
\[ \mathcal{M}_2^m = \left| \int_\Omega \sum_{i,j=1}^n \varphi_{mm}(P_m \varepsilon_{ij}^m - \varepsilon_{ij}^0)(P_m \varepsilon_{ij}^m + \varepsilon_{ij}^0) dx \right|, \] (6.51)

\( \varepsilon_{ij}^m \) being defined in (4.18).
Taking into consideration (C1), (C2), (6.27) and (6.45), we deduce from the Lebesgue theorem that

\[ \lim M^n_m = 0. \]  

(6.52)

(C1) and (C2) imply

\[
M^n_m \leq (a_0 \lambda^{-\frac{1}{2}} + a_2) \sum_{i,j=1}^{n} \| P_m \varepsilon^n_{ij} - \varepsilon^0_{ij} \|_{L^2(\Omega)} \\
\times \| P_m \varepsilon^n_{ij} + \varepsilon^0_{ij} \|_{L^2(\Omega)} \leq c \sum_{i,j=1}^{n} \| P_m \varepsilon^n_{ij} - \varepsilon^0_{ij} \|_{L^2(\Omega)} \\
\leq c \sum_{i,j=1}^{n} (\| P_m (\varepsilon^n_{ij} - \varepsilon^0_{ij}) \|_{L^2(\Omega)} + \| P_m \varepsilon^0_{ij} - \varepsilon^0_{ij} \|_{L^2(\Omega)}) 
\]  

(6.53)

It follows from (6.10) and the Banach-Steinhaus theorem, that the norms of the operators \( P_m \) are uniformly bounded in \( H^1(\Omega)^n \) and \( L^2(\Omega) \). So that (6.10), (6.44) and (6.53) yield \( \lim M^n_m = 0 \). Therefore, the pair \( (v = v^0, \zeta = \zeta_0) \) satisfies the equation (6.6), and the theorem is proved. ■

Theorem 6.1 is a generalization of the result obtained in [17] for a model of non-linear viscous fluid.

7. Coupled problem for the viscosity function \( \varphi_1 \).

For the viscosity function \( \varphi_1 \) defined by (1.11), we take the equation of the balance of thermal energy in the following form:

\[
\chi \sum_{i=1}^{n} \frac{\partial^2 \tau}{\partial x_i^2} + 2\varepsilon \varphi_1(|E|, \mu(u, E), I(Pu), \tau) I(Pu) - u_i \frac{\partial \tau}{\partial x_i} = 0 \quad \text{in} \; \Omega, 
\]  

(7.1)

where

\[
\varphi_1(|E|, \mu(u, E), I(Pu), \tau) = b(|E|, \mu(u, E), \tau)(\lambda + I(Pu))^{-\frac{1}{2}} + \psi(I(Pu), |E|, \mu(u, E), \tau), 
\]  

(7.2)

and \( P \) is an operator of regularization determined in (3.12), (3.13).

The presence of the operator \( P \) in (7.1) means that the model is not local and it is natural from the physical point of view (see Section 3).

The motion equations, the condition of incompressibility and boundary conditions are given by (2.1) under \( k = 1 \), (2.2), and (2.4)–(2.6).

We assume that (2.7)–(2.9) hold and the functions \( b \) and \( \psi \) satisfy the following conditions which are similar to the conditions (C1) and (C2).

\textbf{(C1a):} \( b : (y_1, y_2, y_3) \rightarrow b(y_1, y_2, y_3) \) is a function continuous in \( \mathbb{R}_+ \times [0,1] \times \mathbb{R} \), and in addition

\[
0 \leq b(y_1, y_2, y_3) \leq a_0, \quad (y_1, y_2, y_3) \in \mathbb{R}_+ \times [0,1] \times \mathbb{R}. 
\]  

(7.3)

\textbf{(C2a):} \( \psi : (y_1, y_2, y_3, y_4) \rightarrow \psi(y_1, y_2, y_3, y_4) \) is a function continuous in \( \mathbb{R}_+ \times \mathbb{R}_+ \times [0,1] \times \mathbb{R} \), and for an arbitrary fixed \( (y_2, y_3, y_4) \in \mathbb{R}_+ \times [0,1] \times \mathbb{R} \) the function
\[ \psi(., y_2, y_3, y_4) : y_1 \to \psi(y_1, y_2, y_3, y_4) \] is continuously differentiable in \( \mathbb{R}_+ \), and the following inequalities hold

\[ a_2 \geq \psi(y_1, y_2, y_3, y_4) \geq a_1, \tag{7.4} \]

\[ \psi(y_1, y_2, y_3, y_4) + 2 \frac{\partial \psi}{\partial y_1}(y_1, y_2, y_3, y_4)y_1 \geq a_3, \tag{7.5} \]

\[ \left| \frac{\partial \psi}{\partial y_1}(y_1, y_2, y_3, y_4) \right| y_1 \leq a_4 \tag{7.6} \]

We introduce mappings \( N_1 : X \times H^1_0(\Omega) \to X^* \) and \( A_1 : X \times H^1_0(\Omega) \to H^{-1}(\Omega) \) as follows:

\[
(N_1(v, \zeta), h) = 2 \int_{\Omega} \left[ b(|E|, \mu(\bar{u} + v, E), \bar{\tau} + \zeta)(\lambda + I(\bar{u} + v))^{-\frac{1}{2}} \right. \\
+ \left. \psi(I(\bar{u} + v), |E|, \mu(\bar{u} + v, E), \bar{\tau} + \zeta)\varepsilon_{ij}(\bar{u} + v)\varepsilon_{ij}(h) \right] dx, \\
v, h \in X, \quad \zeta \in H^1_0(\Omega), \\
A_1(v, \zeta)(\xi) = -\int_{\Omega} (\bar{u}_i + v_i) \frac{\partial (\bar{\tau} + \zeta)}{\partial x_i} \xi \, dx \\
+ 2\varepsilon \int_{\Omega} \varphi_1(|E|, \mu(\bar{u} + v, E), I(P(\bar{u} + v)), \bar{\tau} + \zeta)I(P(\bar{u} + v))\xi \, dx - \chi \pi(\bar{\tau}, \xi), \\
v \in X, \quad \zeta, \xi \in H^1_0(\Omega). \tag{7.7} \]

Consider the following problem: find a triple \((v, p, \zeta)\) such that

\[
(v, p, \zeta) \in X \times L^2(\Omega) \times H^1_0(\Omega), \tag{7.9} \\
(N_1(v, \zeta), h) - (B^* p, h) = (K + F, h), \quad h \in X, \tag{7.10} \\
(Bv, q) = 0, \quad q \in L^2(\Omega), \tag{7.11} \\
\pi(\zeta, \xi) - \frac{1}{\chi} \left(A_1(v, \zeta), \xi\right) = 0, \quad \xi \in H^1_0(\Omega). \tag{7.12} \]

Here we use the notations (3.19).

If \((v, p, \zeta)\) is a solution of the problem (7.9)–(7.12), then \((\bar{u} + v, p, \bar{\tau} + \zeta)\) is a generalized solution of the problem (2.1) for \( k = 1 \), (2.2), (2.4)–(2.6) and (7.1).

**Theorem 7.1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( n = 2 \) or \( 3 \), with a boundary \( S \) of the class \( C^1 \), and suppose the conditions (C1a), (C2a) and (2.7), (2.9) are satisfied.

Assume also that there exists a function \( \bar{\tau} \) such that (2.8) holds and in addition

\[
\bar{\tau} \in L^\infty(\Omega). \tag{7.13} \]

Then exists a solution of the problem (7.9)–(7.12)

8. Lemmas and estimation of the term generated by the convection.

Below we suppose that \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), \( n = 2 \) or \( 3 \) with a boundary \( S \) of the class \( C^1 \).
Lemma 8.1. Let $\rho(x)$ be the distance from $x$ to $S$. Then for an arbitrary $\gamma > 0$ there exists a function $\alpha_{\gamma} \in C^1(\Omega)$ such that

\begin{align*}
\alpha_{\gamma} &= 1 \quad \text{in some vicinity of } S \text{ depending on } \gamma, \\
\alpha_{\gamma}(x) &= 0 \quad \text{at } \rho(x) \geq 2\delta(\gamma), \quad \delta(\gamma) = \exp\left(-\frac{1}{\gamma}\right), \\
\left| \frac{\partial}{\partial x_k} \alpha_{\gamma}(x) \right| &\leq \frac{\gamma}{\rho(x)} \quad \text{at } \rho(x) \leq 2\delta(\gamma), \quad k = 1, \ldots, n. \quad (8.3)
\end{align*}

The function $\alpha_{\gamma}$ satisfying the above conditions was constructed by Hopf [4], see also [7].

Lemma 8.2. There exists a constant $c$ depending only on $\Omega$ such that

\begin{equation}
\left\| \frac{1}{\rho} w \right\|_{L^2(\Omega)} \leq c \|w\|_1, \quad w \in H^1_0(\Omega). \tag{8.4}
\end{equation}

This lemma is proved by a partition of unity and local maps followed by the application of the Hardy inequality, see e.g. [18], Lemma 1.10, Chapter 2.

Lemma 8.3. Suppose a function $\tilde{\tau}$ satisfies (2.8) and (7.13). Then for an arbitrary $\beta > 0$ one can construct a function $G$ such that

\begin{equation}
G \in H^1(\Omega) \cap L_{\infty}(\Omega), \quad G|_S = \tilde{\tau}|_S, \tag{8.5}
\end{equation}

and in addition

\begin{equation}
\left\| \zeta \frac{\partial G}{\partial x_i} \right\|_{L^p(\Omega)} \leq \beta \|\zeta\|_1, \quad \zeta \in H^1_0(\Omega), \quad i = 1, \ldots, n. \tag{8.6}
\end{equation}

Proof. Consider the function

\begin{equation}
G = \alpha_{\gamma} \tilde{\tau}. \tag{8.7}
\end{equation}

This function satisfies (8.5). It follows from (7.13), (8.7) and Lemma 8.1 that

\begin{equation}
\left| \frac{\partial G}{\partial x_i}(x) \right| = \left| \left( \frac{\partial}{\partial x_i} (\alpha_{\gamma} \tilde{\tau}) \right)(x) \right| \leq \frac{\gamma}{\rho(x)}|\tilde{\tau}(x)| + c \left| \frac{\partial \tilde{\tau}}{\partial x_i}(x) \right| \leq c_1 \left( \frac{\gamma}{\rho(x)} + \left| \frac{\partial \tilde{\tau}}{\partial x_i}(x) \right| \right) \quad \text{at } \rho(x) \leq 2\delta(\gamma), \quad i = 1, \ldots, n. \tag{8.8}
\end{equation}

(8.8) implies

\begin{equation}
\left\| \zeta \frac{\partial G}{\partial x_i} \right\|_{L^p(\Omega)} \leq c_1 \left( \gamma \|\zeta\|_{L^p(\Omega)} + \left( \int_{\rho \leq 2\delta(\gamma)} \left| \frac{\partial \tilde{\tau}}{\partial x_i} \right| \frac{\rho}{\rho(x)} dx \right)^{\frac{\rho}{\rho}} \right) \leq c_2 \left( \frac{\gamma}{\rho} \|\zeta\|_{L^2(\Omega)} + \|\zeta\|_{L^p(\Omega)} \sigma_i(\gamma) \right), \tag{8.9}
\end{equation}

where

\begin{equation}
\sigma_i(\gamma) = \left( \int_{\rho \leq 2\delta(\gamma)} \left( \frac{\partial \tilde{\tau}}{\partial x_i} \right)^2 dx \right)^{\frac{1}{2}}, \quad i = 1, \ldots, n. \tag{8.10}
\end{equation}

Here we have applied the H"{o}lder inequality with the indexes $\frac{2}{\rho}$ and $\frac{3}{\rho}$ to the integral on the right-hand side of (8.9).

(8.9) and Lemma 8.2 yield

\begin{equation}
\left\| \zeta \frac{\partial G}{\partial x_i} \right\|_{L^p(\Omega)} \leq c_3 \|\zeta\|_1 (\gamma + \sigma_i(\gamma)), \quad i = 1, \ldots, n. \tag{8.11}
\end{equation}
Since $\frac{\partial \eta}{\partial x_i} \in L_2(\Omega)$, we obtain from (8.10) that $\sigma_i(\gamma) \to 0$ as $\gamma \to 0$, and the Lemma is proved.

We consider a mapping $A_2 : V \times H_0^1(\Omega) \to H^{-1}(\Omega)$ that is defined as follows:

$$
(A_2(v, \zeta), \xi) = -\int_{\Omega} (\tilde{u}_i + v_i) \frac{\partial (G + \zeta)}{\partial x_i} \xi \, dx, \quad v \in X, \quad \zeta, \xi \in H_0^1(\Omega).
$$

(8.12)

**Lemma 8.4.** Suppose the conditions (2.7), (2.8) and (7.13) are satisfied. Then for an arbitrary $\eta > 0$ one can determine the function $G$ so that (8.5) is satisfied, and in addition

$$
|(A_2(v, \zeta), \xi)| \leq c|\zeta|_1 + \eta(|v|_X^2 + |\zeta|_1^2), \quad v \in V, \quad \zeta \in H_0^1(\Omega).
$$

(8.13)

**Proof.** It follows from (8.12) that

$$
(A_2(v, \zeta), \xi) = M_1 + M_2 + M_3,
$$

(8.14)

where

$$
M_1 = -\int_{\Omega} \tilde{u}_i \frac{\partial G}{\partial x_i} \zeta \, dx, \quad M_2 = -\int_{\Omega} v_i \frac{\partial G}{\partial x_i} \zeta \, dx, \quad M_3 = -\int_{\Omega} (\tilde{u}_i + v_i) \frac{\partial \zeta}{\partial x_i} \zeta \, dx.
$$

(8.15)

(2.7) and (8.6) yield

$$
|M_1| \leq \sum_{i=1}^n \|	ilde{u}_i\|_{L_6(\Omega)} \left\| \frac{\partial G}{\partial x_i} \right\|_{L_\infty(\Omega)} \leq \beta c_1 \|	ilde{u}\|_{H^1(\Omega)} \|\zeta\|_1 \leq c_2 |\zeta|_1,
$$

(8.16)

$$
|M_2| \leq \sum_{i=1}^n \|v_i\|_{L_6(\Omega)} \left\| \frac{\partial G}{\partial x_i} \right\|_{L_\infty(\Omega)} \leq \beta c_3 \|v\|_X \|\zeta\|_1 \leq \frac{1}{2} \beta c_3 (|v|_X^2 + |\zeta|_1^2).
$$

(8.17)

Since $\text{div}(\tilde{u} + v) = 0$, we obtain by integration by parts that

$$
M_3 = -\int_{\Omega} (\tilde{u}_i + v_i) \frac{\partial \zeta}{\partial x_i} \zeta \, dx = \int_{\Omega} (\tilde{u}_i + v_i) \zeta \frac{\partial \zeta}{\partial x_i} \, dx = 0.
$$

(8.18)

Now (8.13) follows from (8.16)–(8.18) and Lemma 8.3.

9. **Proof of Theorem 7.1.**

1) It follows from the proof of Theorem 4.1, that if $(v, p, \zeta)$ is a solution of the problem (7.9)–(7.12), then

$$
|v|_X \leq \frac{1}{\mu_3} (|K + F|^0 + \mu_4).
$$

(9.1)

By (3.12), (3.13) and (9.1), we conclude that there exists a constant $b_1$ such that

$$
|I(P(\tilde{u} + v))|_{C(\overline{\Omega})} \leq b_1.
$$

(9.2)

Consider a function $\varphi_3$ such that $\varphi_3$ is a function continuous and bounded in $\mathbb{R}_+ \times [0, 1] \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ and in addition

$$
\varphi_3(z) = \begin{cases}
\varphi_1(z) & \text{at } z = (z_1, z_2, z_3, z_4) \in \mathbb{R}_+ \times [0, 1] \times [0, b_1] \times \mathbb{R},
0 & \text{at } z = (z_1, z_2, z_3, z_4) \in \mathbb{R}_+ \times [0, 1] \times (b_2, \infty) \times \mathbb{R},
\end{cases}
$$

(9.3)
where \( b_2 > b_1 \) and \( \varphi_1 \) is the function defined by (7.2), i.e.

\[
\varphi_1(z) = b(z_1, z_2, z_4)(\lambda + z_3)^{-\frac{1}{2}} + \psi(z_3, z_1, z_2, z_4).
\]

(9.4)

Define operators \( N_2 : X \times H^1_0(\Omega) \to X^* \) and \( A_3 : X \times H^1_0(\Omega) \to H^{-1}(\Omega) \) by

\[
(N_2(v, \zeta), h) = 2 \int_{\Omega} [b(|E|, \mu(\bar{u} + v, E), G + \zeta)(\lambda + I(\bar{u} + v))^{-\frac{1}{2}} + \psi(I(\bar{u} + v), |E|, \mu(\bar{u} + v, E), G + \zeta)] \epsilon_{ij}(\bar{u} + v) \epsilon_{ij}(h) \, dx,
\]

\[
v, h \in X, \quad \zeta \in H^1_0(\Omega),
\]

(9.5)

\[
(A_3(v, \zeta), \xi) = 2\varepsilon \int_{\Omega} \varphi_3(|E|, \mu(\bar{u} + v, E), I(P(\bar{u} + v)), G + \zeta) I(P(\bar{u} + v)) \xi \, dx - \chi \pi(G, \xi),
\]

\[
v \in X, \quad \zeta, \xi \in H^1_0(\Omega),
\]

(9.6)

where \( G \) is the function defined in Lemma 8.3.

We consider the problem: find a triple of functions \((v, p, \theta)\) satisfying

\[
(v, p, \theta) \in X \times L_2(\Omega) \times H^1_0(\Omega),
\]

(9.7)

\[
(N_2(v, \theta), h) - (B^* p, h) = (K + F, h), \quad h \in X,
\]

(9.8)

\[
(Bv, q) = 0, \quad q \in L_2(\Omega),
\]

(9.9)

\[
\pi(\theta, \xi) - \frac{1}{\chi} [\Phi_2(v, \theta, \xi) + 3(A_3(v, \theta, \xi)) = 0, \quad \xi \in H^1_0(\Omega).
\]

(9.10)

It follows from the proof of theorem 4.1 that the inequality (9.1) is also fulfilled for the solution of the problem (9.7)–(9.10). Therefore, (9.2) holds, and if \((v, p, \theta)\) is a solution of the problem (9.7)–(9.10), then the triple \((v, p, \xi = G - \bar{\tau} + \theta)\) is a solution of the problem (7.9)–(7.12).

The problem (9.7)–(9.10) is equivalent to the following problem: find a pair \((v, \theta)\) such that

\[
(v, \theta) \in V \times H^1_0(\Omega),
\]

(9.11)

\[
(N_2(v, \theta), h) = (K + F, h), \quad h \in V,
\]

(9.12)

\[
\pi(\theta, \xi) - \frac{1}{\chi} [(A_2(v, \theta, \xi) + A_3(v, \theta, \xi)) = 0, \quad \xi \in H^1_0(\Omega).
\]

(9.13)

Indeed, if \((v, p, \theta)\) is a solution of the problem (9.7)–(9.10), then the pair \((v, \theta)\) is a solution of the problem (9.11)–(9.13). Conversely, let \((v, \theta)\) be a solution of the problem (9.11)–(9.13). By virtue of (9.12) and Lemma 4.2 there exists a unique function \( p \in L_2(\Omega) \) such that the triple \((v, p, \theta)\) is a solution of the problem (9.7)–(9.10).

2) Let us prove the existence of a solution of the problem (9.11)–(9.13).

We define a mapping \( U : V \times H^1_0(\Omega) \to V^* \times H^{-1}(\Omega) \) by

\[
(U(v, \theta), (h, \xi)) = (N_2(v, \theta), h) + \pi(\theta, \xi) - \frac{1}{\chi} [(A_2(v, \theta, \xi) + A_3(v, \theta, \xi)] - (K + F, h),
\]

\[
v, h \in V, \quad \theta, \xi \in H^1_0(\Omega).
\]

(9.14)

It is easy to verify that the problem (9.11)–(9.13) is equivalent to the following one: find a pair \((v, \theta)\) such that

\[
(v, \theta) \in V \times H^1_0(\Omega),
\]

(9.15)

\[
(U(v, \theta), (h, \xi)) = 0, \quad (h, \xi) \in V \times H^1_0(\Omega).
\]

(9.16)
Indeed, taking the pair \((h, 0)\) in (9.16), we obtain (9.12), and taking the pair \((0, \xi)\), we get (9.13). Conversely, by adding (9.12) and (9.13), we get (9.16).

Let \(\{V_m\}\) and \(\{Z_m\}\) be sequences of finite-dimensional subspaces in \(V\) and \(H^1_0(\Omega)\), respectively, such that

\[
\lim_{m \to \infty} \inf_{g \in V_m} \|u - g\|_X = 0, \quad u \in V; \tag{9.17}
\]

\[
\lim_{m \to \infty} \inf_{h \in Z_m} \|w - h\|_1 = 0, \quad w \in H^1_0(\Omega), \tag{9.18}
\]

\[V_m \subset V_{m+1}, \quad Z_m \in Z_{m+1}. \tag{9.19}\]

We search for an approximate solution \(v_m, \theta_m\) of the problem (9.11)–(9.13) in the form

\[
(v_m, \theta_m) \in V_m \times Z_m, \tag{9.20}
\]

\[
(N_2(v_m, \theta_m), h) = (K + F, h), \quad h \in V_m, \tag{9.21}
\]

\[
\pi(\theta_m, \xi) - \frac{1}{\chi}[(A_2(v_m, \theta_m), \xi) + (A_3(v_m, \theta_m), \xi)] = 0, \quad \xi \in Z_m. \tag{9.22}
\]

(9.3) and (9.6) imply

\[
|A_3(v, \theta), \theta| \leq c_1 \int_{\Omega} |\theta| dx + \chi(\pi(G, G))^{\frac{1}{2}} \|\theta\|_1 \leq c_2 \|\theta\|_1. \tag{9.23}
\]

Inequality (4.3) is also realized for the operator \(N_2\). So that by using (8.13), (9.14) and (9.23), we obtain

\[
(U(v, \theta), (v, \theta)) \geq \mu_3 \|v\|_X^2 + \|\theta\|_1^2 - c_3 \|v\|_X - c_4 \|\theta\|_1 - \frac{\eta}{\chi} (\|v\|_X^2 + \|\theta\|_1^2), \quad (v, \theta) \in V \times H^1_0(\Omega). \tag{9.24}
\]

By Lemma 8.4 we can reckon that

\[
\eta = \frac{1}{2} \chi \min(\mu_3, 1). \tag{9.25}
\]

Then there exists a constants \(r > 0\) such that

\[
(U(v, \theta), (v, \theta)) \geq 0, \quad \text{if} \quad \|v\|_X + \|\theta\|_1 \geq r. \tag{9.26}
\]

Therefore, for an arbitrary \(m \in \mathbb{N}\) there exists a pair \((v_m, \theta_m)\) satisfying

\[
(v_m, \theta_m) \in V_m \times Z_m, \quad (U(v_m, \theta_m), (h, \xi)) = 0, \quad (h, \xi) \in V_m \times Z_m. \tag{9.27}
\]

This pair is the solution of the problem (9.20)–(9.22), and moreover

\[
\|v_m\|_X + \|\theta_m\|_1 \leq r. \tag{9.28}
\]

It follows from the proof of Theorem 4.1 that

\[
\|v_m\|_X \leq \frac{1}{\mu_3}(\|K + F\|_{V^*} + \mu_4). \tag{9.29}
\]
3) By virtue of (9.27), we can extract a subsequence \( \{v_k, \theta_k\} \) from the sequence \( \{v_m, \theta_m\} \) such that

\[
\begin{align*}
v_k & \rightharpoonup v_0 \quad \text{in} \quad V, \quad (9.29) \\
v_k & \rightarrow v_0 \quad \text{in} \quad L_4(\Omega)^n \quad \text{and a.e. in} \quad \Omega, \quad (9.30) \\
\theta_k & \rightharpoonup \theta_0 \quad \text{in} \quad H_0^1(\Omega), \quad (9.31) \\
\theta_k & \rightarrow \theta_0 \quad \text{in} \quad L_2(\Omega) \quad \text{and a.e. in} \quad \Omega, \quad (9.32) \\
N_2(v_k, \theta_k) & \rightarrow F \quad \text{in} \quad X^*. \quad (9.33)
\end{align*}
\]

Let \( k_0 \) be a fixed positive number, and \( h \in V_{k_0} \). Observing (9.33), we pass to the limit in (9.21) with \( m \) replaced by \( k \), and obtain

\[
(F, h) = (K + F, h), \quad h \in V_{k_0}. \quad (9.34)
\]

Since \( k_0 \) is an arbitrary positive integer, we get by (9.17) and (9.34) that

\[
F = K + F \quad \text{in} \quad V^*. \quad (9.35)
\]

We present the operator \( N_2 \) in the form

\[
N_2(v, \zeta) = \tilde{N}(v, v, \zeta), \quad (9.36)
\]

where the operator \((v, w, \zeta) \rightarrow \tilde{N}(v, w, \zeta)\) is considered as a mapping of \( V \times V \times H_0^1(\Omega) \) into \( V^* \) according to

\[
(\tilde{N}(v, w, \zeta), h) = 2 \int_{\Omega} [b(|E|, \mu(\tilde{u} + w, E), G + \zeta)(\lambda + I(\tilde{u} + v))^{-\frac{1}{2}}
+ \psi(I(\tilde{u} + v), E, \mu(\tilde{u} + w, E), G + \zeta)] \varepsilon_{ij}(\tilde{u} + v) \varepsilon_{ij}(h) \, dx,
\]

\[
v, w \in V, \quad \zeta \in H_0^1(\Omega). \quad (9.37)
\]

For an arbitrary fixed \( w \in V \) and \( \zeta \in H_0^1(\Omega) \) the operator \( \tilde{N}(., w, z) : v \rightarrow \tilde{N}(v, w, z) \) is monotone (see (4.2)). Therefore,

\[
(\tilde{N}(v_k, v_k, \theta_k) - \tilde{N}(z, v_k, \theta_k), v_k - z) \geq 0, \quad z \in V, \quad k \in \mathbb{N}. \quad (9.38)
\]

(9.30), (9.32) and Lebesgue theorem give

\[
\lim \| \tilde{N}(z, v_k, \theta_k) - \tilde{N}(z, v_0, \theta_0) \|_{X^*} = 0, \quad z \in V. \quad (9.39)
\]

By (9.21), (9.29) and (9.36), we obtain

\[
(\tilde{N}(v_k, v_k, \theta_k), v_k) = (K + F, v_k) \rightarrow (K + F, v_0), \quad (9.40)
\]

and (9.17) yields

\[
\lim (\tilde{N}(v_k, v_k, \theta_k), z) = (K + F, z), \quad z \in V. \quad (9.41)
\]

Observing (9.29), (9.39)–(9.41), we pass to the limit in (9.38).

Then we get

\[
(K + F - \tilde{N}(z, v_0, \theta_0), v_0 - z) \geq 0, \quad z \in V. \quad (9.42)
\]

We choose \( z = v_0 - \gamma h, \gamma > 0, h \in V \), and consider \( \gamma \rightarrow 0 \). Then, taking into account that \( \tilde{N}(., v_0, \theta_0) : z \rightarrow \tilde{N}(z, v_0, \theta_0) \) is a Lipschitz continuous mapping of \( V \) into \( V^* \) (see (4.1)), we obtain

\[
(K + F - \tilde{N}(v_0, v_0, \theta_0), h) \geq 0. \quad (9.43)
\]
This inequality holds for any \( h \in V \). Therefore, replacing \( h \) by \(-h\) shows that equality holds true in (9.43). From here by (9.29), (9.31) and (9.36), we deduce
\[
(v_0, \theta_0) \in V \times H^1_0(\Omega), \quad (N_2(v_0, \theta_0), h) = (K + F, h), \quad h \in V. \tag{9.44}
\]
(8.12), (9.6), (9.30)–(9.32) imply
\[
\begin{align*}
\lim (A_2(v_k, \theta_k), \xi) &= (A_2(v_0, \theta_0), \xi), \\
\lim (A_3(v_k, \theta_k), \xi) &= (A_3(v_0, \theta_0), \xi), \quad \xi \in H^1_0(\Omega). \tag{9.45}
\end{align*}
\]
\(^{(9.44)}\)

Bearing in mind (9.31) and (9.45), we pass to the limit in (9.22) for fixed \( \xi \in Z_{k_0} \). Then by (9.18), we deduce
\[
\pi(\theta_0, \xi) - \frac{1}{\chi} [ (A_2(v_0, \theta_0), \xi) + (A_3(v_0, \theta_0), \xi) ] = 0, \quad \xi \in H^1_0(\Omega). \tag{9.46}
\]
\(^{(9.46)}\)

It follows from (9.44) and (9.46) that the pair \((v = v_0, \theta = \theta_0)\) is a solution of the problem (9.11)–(9.13). Hence, there exists a unique function \( p \) such that the triple \((v = v_0, p, \theta = \theta_0)\) is a solution of the problem (9.7)–(9.10).

(9.28) and (9.29) yield
\[
\|v_0\|_X \leq \frac{1}{\mu_3} (\|K + F\|_{V^*} + \mu_4). \tag{9.47}
\]
\(^{(9.47)}\)

Since (9.1) implies (9.2), we get \( \|I(P(\tilde{u} + v_0))\|_{C(\bar{\Omega})} \leq b_1 \). From here by (7.8), (9.3) and (9.6), we obtain that the triple \((v = v_0, p, \zeta = G - \tilde{\tau} + \theta_0)\) is a solution of the problem (7.9)–(7.12). The theorem is proved. \( \blacksquare \)

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