CONTROLLABILITY OF NEUTRAL STOCHASTIC FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION WITH HURST PARAMETER LESSER THAN 1/2

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ABSTRACT. This article investigates the controllability for neutral stochastic delay functional integro-differential equations driven by a fractional Brownian motion, with Hurst parameter lesser than 1/2. We employ the theory of resolvent operators developed by [10] combined with the Banach fixed point theorem to establish sufficient conditions to prove the desired result.

1. Introduction. The controllability plays a crucial role in both deterministic and stochastic control systems. It has various applications in industry, biology, and physics. There are diverse notions of controllability in the literature, both for linear and nonlinear dynamical systems. The controllability of systems represented by differential equations in infinite-dimensional spaces is well-developed using different approaches. The most common method consists of transforming the controllability problem into a fixed point strategy in an appropriate functional space. For instance, by employing the measures of noncompactness and the Mönch fixed point theorem, [15] derived some sufficient conditions for controllability of impulsive mixed-type integro-differential systems with finite delay, [22] considered the controllability result for fractional neutral integro-differential equations with nonlocal conditions, and [11] investigated the controllability of fractional integro-differential equations with nondense domain. Moreover, with the aid of Krasnoselskii’s fixed point and Leray-Schauder theorems, [18] discussed the exact controllability of neutral fractional integro-differential systems with state-dependent delay. For further details on the theory of fractional differential equations, we refer to [2] and the references therein.

Generally, the deterministic models fluctuate due to noises or stochastic perturbations, see e.g. [1, 3, 5, 7, 14, 19, 23] just to mention a few. The models perturbed by a fractional Brownian motion (abbreviated FBM) are of particular interest. This is due to the crucial properties provided by this stochastic process.

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like the short/long-range dependence. The equations with FBM are often used to model phenomena with long memory in fields as hydrology, seismology, economics, and telecommunications. There is a vast literature related to the study of the controllability problems for different kinds of systems described by differential equations driven by FBM in Hilbert space, see for instance, [1, 4, 8, 21]. However, few works consider the controllability of stochastic integro-differential equations driven by FBM. As examples, [12] discussed the controllability problem for neutral stochastic integro-differential equations governed by FBM, [20] studied the complete and approximate controllability of nonlinear fractional neutral stochastic integro-differential inclusions with FBM and [13] obtained the controllability of impulsive neutral stochastic integro-differential systems with infinite delay driven by FBM. Notice that all these previous works consider equations driven by FBM with Hurst parameter \( H \in (\frac{1}{2}, 1) \). Through this paper, we aim to show the controllability of neutral stochastic delay integro-differential equations perturbed by a FBM with Hurst parameter lesser than \( \frac{1}{2} \). To the best of our knowledge, this is the first work that deals with the case \( H < 1/2 \).

After this brief outline of the literature, we will now describe precisely the problem investigated in this paper. We consider the following neutral stochastic functional integro-differential equation with finite delay:

\[
\begin{align*}
\frac{d}{dt}[x(t) + g(t, x(t - r(t)))] &= [A(x(t) + g(t, x(t - r(t)))] + Lu(t)dt \\
+ \left[ \int_0^t B(t - s) [x(s) + g(s, x(s - r(s)))] ds + f(t, x(t - \rho(t))] \right] dt \\
+ \sigma(t)dB^H(t), \quad 0 \leq t \leq T, \\
x(t) &= \varphi(t), \quad -\tau \leq t \leq 0,
\end{align*}
\]

where \( A : D(A) \subset X \to X \) is a closed linear operator, for all \( t \geq 0 \), \( B(t) \) is a closed linear operator with domain \( D(B(t)) \supset D(A) \). The control function \( u(\cdot) \) takes values in \( L^2([0, T], U) \), the Hilbert space of admissible control functions for a separable Hilbert space \( U \). \( L \) is a bounded linear operator form \( U \) into \( X \). \( B^H \) is a fractional Brownian motion with Hurst parameter \( H < 1/2 \) on a real and separable Hilbert space \( Y \). \( r, \rho : [0, +\infty) \to [0, \tau] \) (\( \tau > 0 \)) are continuous and \( f, g : [0, +\infty) \times X \to X \), \( \sigma : [0, +\infty] \to L^2_2(Y, X) \) are appropriate functions. Here \( L^2_2(Y, X) \) denotes the space of all \( Q \)-Hilbert-Schmidt operators from \( Y \) into \( X \) (see Section 2 below). We mention that a variant of this equation without the term involving the operator \( B(t) \) has been studied in [5] by using the theory of analytic semi-groups and fractional powers associated to its generator.

The outline of this paper is as follows. In Section 2, we introduce some notations, concepts, and preliminary results about fractional Brownian motion, Wiener integral over Hilbert spaces, and we recall some preliminary results about resolvent operators. Section 3 investigates the controllability of the system (1) by using Banach fixed point theorem. In the last section, we give an illustrative example.

2. Preliminaries. This section collects some basic notions on Wiener integrals with respect to an infinite-dimensional fractional Brownian motion. Furthermore, we recall some preliminary results about resolvent operators, which will be used throughout this paper.

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space. Consider a time interval \([0, T]\) with arbitrary fixed horizon \( T \) and let \( \{\beta^H(t), t \in [0, T]\} \) the one-dimensional fractional Brownian motion with Hurst parameter \( H \in (0, 1/2) \). This means by definition...
that \( \beta^H \) is a centered Gaussian process with covariance function:

\[
R_H(s, t) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}).
\]

Moreover \( \beta^H \) has the following Wiener integral representation:

\[
\beta^H(t) = \int_0^t K_H(t, s) d\beta(s),
\]

where \( \beta = \{ \beta(t) : t \in [0, T] \} \) is a Wiener process, and \( K_H(t, s) \) is a square integrable kernel given by (see \cite{16})

\[
K_H(t, s) = c_H \left[ \left( \frac{t}{s} \right)^{H - 1/2} (t-s)^{H - 1/2} - (H - \frac{1}{2}) \frac{t^{3/2-H}}{s^{3/2-H}} \int_s^t u^{-\frac{3/2-H}{2}} (u-s)^{-\frac{1}{2}} du \right]
\]

for \( t > s \), where \( c_H = \sqrt{\frac{(1-2H)}{2(1-2H, H+\frac{1}{2})}} \) and \( \beta(.) \) is the Beta function. We put \( K_H(t, s) = 0 \) if \( t \leq s \). And from (3) it follows that:

\[
|K(t, s)| \leq 2c_H \left( (t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}} \right).
\]

In the sequel, we will use the following inequality:

\[
\| \frac{\partial K}{\partial t^H} (t, s) \| \leq c_H \left( \frac{1}{2} - H \right) (t-s)^{H-\frac{3}{2}}.
\]

We denote by \( \mathcal{H} \) the closure of set of indicator functions \( \{ 1_{[0, t]}, t \in [0, T] \} \) with respect to the scalar product \( \langle 1_{[0, t]}, 1_{[0, s]} \rangle_{\mathcal{H}} = \langle R_H(t, s) \rangle \).

The mapping \( 1_{[0, t]} \rightarrow \beta^H(t) \) can be extended to an isometry between \( \mathcal{H} \) and the first Wiener chaos and we will denote by \( \beta^H(\varphi) \) the image of \( \varphi \) by the previous isometry.

It’s known that \( \mathcal{H} = I^1_T(?) \) and \( C^\gamma([0, T]) \subseteq \mathcal{H} \) if \( \gamma > 1/2 - H \) where \( C^\gamma([0, T]) \) is the space of \( \gamma \)-Hölder continuous functions and \( I^\alpha_T(L^2) \) is the image of \( L^2([0, T]) \) by the operator \( I^\alpha_T \) defined by:

\[
I^\alpha_T f(x) = \frac{1}{\Gamma(\alpha)} \int_0^T (y - x)^{\alpha-1} f(y) dy.
\]

Let us consider the operator \( K^\alpha_{H, T} \) from \( \mathcal{H} \) to \( L^2([0, T]) \) defined by

\[
(K^\alpha_{H, T} \varphi)(s) = K(T, s) \varphi(s) + \int_s^T (\varphi(r) - \varphi(s)) \frac{\partial K}{\partial r}(r, s) dr.
\]

We refer to \cite{16} for the proof of the fact that \( K^\alpha_{H, T} \) is an isometry between \( \mathcal{H} \) and \( L^2([0, T]) \). Moreover for any \( \varphi \in \mathcal{H} \), we have

\[
\int_0^T \varphi(s) d\beta^H(s) := \beta^H(\varphi) = \int_0^T (K^*_{H, T} \varphi)(t) d\beta^H(t).
\]

We also have for \( 0 \leq t \leq T \)

\[
\int_0^t \varphi(s) d\beta^H(s) = \int_0^t (K^*_{H, T} \varphi 1_{[0, t]})(s) d\beta^H(s) = \int_0^t (K^*_{H, T} \varphi)(s) d\beta^H(s),
\]

where \( K^*_{H, T} \) is defined in the same way as in (6) with \( t \) instead of \( T \). In the next we will use the notation \( K^*_{H, t} \) without specifying the parameter \( t \) in \( [0, T] \).

Let \( X \) and \( Y \) be two real, separable Hilbert spaces and let \( \mathcal{L}(Y, X) \) be the space of bounded linear operator from \( Y \) to \( X \). For the sake of convenience, we shall use the same notation to denote the norms in \( X, Y \) and \( \mathcal{L}(Y, X) \). Let \( Q \in \mathcal{L}(Y, Y) \) be an operator defined by \( Qe_n = \lambda_n e_n \) with finite trace \( trQ = \sum_{n=1}^{\infty} \lambda_n < \infty \).
where \( \lambda_n \geq 0 \) \((n = 1, 2, \ldots)\) are non-negative real numbers and \( \{e_n\} \) \((n = 1, 2, \ldots)\) is a complete orthonormal basis in \( Y \). We define the infinite dimensional FBM on \( Y \) with covariance \( Q \) as

\[
B^H(t) = B^H_Q(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta^H_n(t),
\]

where \( \beta^H_n \) are real independent FBM. This process is a \( Y \)-valued Gaussian, it starts from 0, has zero mean and covariance:

\[
\mathbb{E} \langle B^H(t), x \rangle \langle B^H(s), y \rangle = R(s, t) \langle Q(x), y \rangle \quad \text{for all} \ x, y \in Y \ \text{and} \ t, s \in [0, T].
\]

In order to define Wiener integrals with respect to the \( Q \)-FBM, we introduce the space \( \mathcal{L}_2^Q(Y, X) \) of all \( Q \)-Hilbert-Schmidt operators \( \psi : Y \to X \). We recall that \( \psi \in \mathcal{L}(Y, X) \) is called a \( Q \)-Hilbert-Schmidt operator, if

\[
\|\psi\|_{\mathcal{L}_2^Q}^2 := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \psi e_n\|^2 < \infty,
\]

and that the space \( \mathcal{L}_2^Q \) is a separable Hilbert space.

Now, let \( \{\phi(s); s \in [0, T]\} \) be a function with values in \( \mathcal{L}_2^Q(Y, X) \). The Wiener integral of \( \phi \) with respect to \( B^H \) is defined by

\[
\int_0^t \phi(s) dB^H(s) := \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \phi(s) e_n d\beta^H_n(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \langle K^\ast_H(\phi e_n)(s), \beta^H_n(s) \rangle ds,
\]

where \( \beta_n \) is the standard Brownian motion used to present \( \beta^H_n \) as in (2), and the above sum is finite when \( \sum_{n=1}^{\infty} \lambda_n \|K^\ast_H(\phi e_n)\|^2 < \infty \).

Now we turn to state some notations and basic facts about the theory of resolvent operators needed in the sequel. For additional details on resolvent operators, we refer to \cite{10, 17}.

Let \( A : D(A) \subset X \to X \) be a closed linear operator and for all \( t \geq 0 \), \( B(t) \) a closed linear operator with domain \( D(B(t)) \supset D(A) \). Let us denote by \( Z \) the Banach space \( D(A) \), the domain of operator \( A \), equipped with the graph norm

\[
\|y\|_Z := \|Ay\| + \|y\| \quad \text{for} \ y \in Z.
\]

Let us consider the following Cauchy problem

\[
\begin{aligned}
\quad v'(t) &= Av(t) + \int_0^t B(t-s)v(s)ds \quad \text{for} \ t \geq 0, \\
\quad v(0) &= v_0 \in X.
\end{aligned}
\]

**Definition 2.1.** \cite{10} A resolvent operator of equation (8) is a bounded linear operator valued function \( R(t) \in \mathcal{L}(X) \) for \( t \geq 0 \), satisfying the following properties:

(i) \( R(0) = I \) and \( \|R(t)\| \leq Ne^{\beta t} \) for some constants \( N \) and \( \beta \).

(ii) For each \( x \in X \), \( R(t)x \) is strongly continuous for \( t \geq 0 \).

(iii) For \( x \in Z \), \( R(\cdot)x \in \mathcal{C}^1([0, +\infty); X) \cap \mathcal{C}([0, +\infty); Z) \) and

\[
R'(t)x = AR(t)x + \int_0^t B(t-s)R(s)x ds = R(t)Ax + \int_0^t R(t-s)B(s)x ds, \quad \text{for} \ t \geq 0.
\]

The resolvent operator plays a vital role in studying the existence of solutions and establishing a variation of constants formula for non-linear systems. For this
reason, to assure the existence of the resolvent operator, we make the following hypotheses:

(\(H.1\)) A is the infinitesimal generator of a \(C_0\)-semigroup \((T(t))_{t \geq 0}\) on \(X\).

(\(H.2\)) For all \(t \geq 0\), \(B(t)\) is a continuous linear operator from \((Z, \| \cdot \|_Z)\) into \((X, \| \cdot \|_X)\). Moreover, there is a locally integrable function \(c : \mathbb{R}^+ \to \mathbb{R}^+\) such that for any \(y \in Z\), \(t \mapsto B(t)y\) belongs to \(W^{1,1}([0, +\infty), X)\) and

\[
\frac{d}{dt} B(t)y \leq c(t)\|y\|_Z \quad \text{for} \quad y \in Z \quad \text{and} \quad t \geq 0.
\]

**Theorem 2.2.** ([9]) Assume that hypotheses (\(H.1\)) and (\(H.2\)) hold. Then the Cauchy problem (8) admits a unique resolvent operator \((R(t))_{t \geq 0}\).

The following lemma proves that the resolvent operator \((R(t))_{t \geq 0}\) satisfies a Lipschitz condition:

**Lemma 2.3.** Under conditions (\(H.1\)) and (\(H.2\)), we have:

\[
\|R(t)x - R(s)x\| \leq M |t - s| \|x\|_Z \quad \text{for all} \quad t, s \in [0, T] \quad \text{and} \quad x \in Z,
\]

where \(M = \left(1 + T\|B(0)\| + T \int_0^T c(s)ds\right)\sup_{t \in [0, T]} \|R(t)\|\).

**Proof.** Let \(t, s \in [0, T]\) and \(x \in Z\). By assumption (\(H.2\)), we have

\[
\|B(t)x\|_X \leq \|B(0)x\| + \int_0^t c(u)du \|x\|_Z
\]

and

\[
\|R'(t)x\|_X \leq \sup_{u \in [0, T]} \|R(u)\| \|Ax\| + \sup_{u \in [0, T]} \|R(u)\| \int_0^T \|B(u)x\|_X du
\]

\[
\leq \sup_{u \in [0, T]} \|R(u)\| \left(1 + T\|B(0)\| + T \int_0^T c(u)du\right) \|x\|_Z,
\]

which entails that \(\|R(t)x - R(s)x\| \leq M |t - s| \|x\|_Z\).

\(\Box\)

3. **Main result.** The following part of this paper moves on to prove the controllability of the stochastic system (1). For this task, we assume that the following conditions are in force.

(\(H.3\)) The function \(f : [0, +\infty) \times X \to X\) satisfies the following Lipschitz conditions:

that is, there exist positive constants \(C_i := C_i(T), i = 1, 2\) such that, for all \(t \in [0, T]\) and \(x, y \in X\)

\[
\|f(t, x) - f(t, y)\| \leq C_1\|x - y\|, \quad \|f(t, x)\|^2 \leq C_2(1 + \|x\|^2).
\]

(\(H.4\)) The function \(g : [0, +\infty) \times X \to X\) satisfies the following conditions:

(i) There exist positive constants \(C_i := C_i(T), i = 3, 4\) such that for all \(t \in [0, T]\) and \(x, y \in X\)

\[
\|g(t, x) - g(t, y)\| \leq C_3\|x - y\|, \quad \|g(t, x)\|^2 \leq C_4(1 + \|x\|^2).
\]

(ii) The function \(g\) is continuous in the quadratic mean sense:

\[
\forall x \in C([0, T], L^2(\Omega, X)), \quad \lim_{t \to s} \mathbb{E}\|g(t, x(t)) - g(s, x(s))\|^2 = 0.
\]

(\(H.5\)) The function \(\sigma : [0, +\infty) \to L^0_2(Y, X)\) satisfies the following conditions:
Proof.

(i) There exists a constant $C_5 > 0$ such that, for all $t, s \in [0, T]$
\[ \| \sigma(t) - \sigma(s) \|_{L_2^2} \leq C_5 | t - s |^\gamma, \quad \gamma > 1/2 - H. \]

(ii) $\forall t \in [0, T]; \forall y \in D(A), \sigma(t)y \in D(A)$.

(iii) There exists a constant $C_6 > 0$ such that $\int_0^T \| A\sigma(t) \|_{L_2^2} dt \leq C_6$.

(H.6) The linear operator $W$ from $L^2([0, T], U)$ into $X$ defined by:
\[ Wu = \int_0^T R(T - s)Lu(s)ds \]
has an inverse operator $W^{-1}$ that takes values in $L^2([0, T], U) \setminus KerW$, where
\[ kerW = \{ x \in L^2([0, T], U), \text{ } Wx = 0 \}, \]
and there exist finite positive constants $M_L$ and $M_W$ such that $\| L \| \leq M_L$ and $\| W^{-1} \| \leq M_W$.

Moreover, we assume that $\varphi \in C([-\tau, 0], L^2(\Omega, X))$.

Similar to the deterministic situation we give the following definition of mild solutions for equation (1).

Definition 3.1. An $X$-valued process $\{ x(t), \text{ } t \in [-\tau, T] \}$, is called a mild solution of equation (1) if

i) $x(.) \in C([-\tau, T], L^2(\Omega, X))$,

ii) $x(t) = \varphi(t), \text{ } -\tau \leq t \leq 0$,

iii) For arbitrary $t \in [0, T]$, we have
\[
\begin{align*}
x(t) &= R(t)(\varphi(0) + g(0, \varphi(-r(0)))) - g(t, x(t - r(t))) \\
&\quad + \int_0^t R(t - s)[Lu(s) + f(s, x(s - \rho(s)))]ds + \int_0^t R(t - s)\sigma(s)dB^H(s) \text{ } \mathbb{P} - a.s.
\end{align*}
\]

The concept of controllability of neutral integro-differential stochastic functional differential equation is the following:

Definition 3.2. The system (1) is said to be controllable on the interval $[-\tau, T]$, if for every initial stochastic process $\varphi$ defined on $[-\tau, 0]$ and $x_1 \in X$, there exists a stochastic control $u \in L^2([0, T], U)$ such that the mild solution $x(.)$ of equation (1) satisfies $x(T) = x_1$.

The main result of this work is given in the next theorem.

Theorem 3.3. Suppose that (H.1)–(H.6) hold. Then, the system (1) is controllable on $[-\tau, T]$ provided that
\[
C_3^2 + D^2C_2^2T^2 + D^2M_2^2M_3^2C_3^2T + D^4M_2^4M_3^2C_3^2T^3 < \frac{1}{4}.
\]

Proof. Throughout the proof, we will use the following notations:
\[
D := \sup_{t \in [0, T]} \| R(t) \|, \quad \bar{\sigma} := \sup_{t \in [0, T]} \| \sigma(t) \|_{L_2^2}.
\]

Fix $T > 0$ and let $B_T := C([-\tau, T], L^2(\Omega, X))$ be the Banach space of all continuous functions from $[-\tau, T]$ into $L^2(\Omega, X)$, equipped with the supremum norm $\| \xi \|_{B_T} = \sup_{u \in [-\tau, T]} (\mathbb{E} \| \xi(u) \|^2)^{1/2}$ and let us consider the set
\[
S_T = \{ x \in B_T : x(s) = \varphi(s), \text{ for } s \in [-\tau, 0] \}.
\]
$S_T$ is a closed subset of $B_T$ endowed with the norm $\| \cdot \|_{B_T}$. Thanks to the hypothesis (H.6), we can define the following control:

$$u(t) = W^{-1}(x_1 - R(T)(\varphi(0) + g(0, \varphi(-r(0)))) + g(T, x(T - r(T))) - \int_0^T R(T - s)f(s, x(s - \rho(s)))ds - \int_0^T R(T - s)\sigma(s)dB^H(s))(t). \quad (9)$$

We define the operator $\psi$ on $S_T$ by:

$$\psi(x)(t) = \varphi(t), \ \forall t \in [-\tau, 0],$$

and for all $t \in [0, T]$

$$\psi(x)(t) = R(t)(\varphi(0) + g(0, \varphi(-r(0)))) - g(t, x(t - r(t))) + \int_0^t R(t - s)[Lu(s) + f(s, x(s - \rho(s)))]ds + \int_0^t R(t - s)\sigma(s)dB^H(s).$$

Then, the controllability of the system (1) is equivalent to find a fixed point for the operator $\psi$. Next, we will show by using Banach fixed point theorem that $\psi$ has a unique fixed point. We divide the subsequent proof into two steps.

**Step 1.** For arbitrary $x \in S_T$, let us prove that $t \mapsto \psi(x)(t)$ is continuous on the interval $[0, T]$ in the $L^2(\Omega, X)$-sense.

Let us consider $0 < t < T$ and $h > 0$ small enough. Then for any fixed $x \in S_T$, we have

$$\mathbb{E}\|\psi(x)(t + h) - \psi(x)(t)\|^2 \leq 5\mathbb{E}\|\psi(x)(t + h) - \psi(x)(t)\|^2 + 5\mathbb{E}\|g(t + h, x(t + h - r(t + h))) - g(t, x(t - r(t)))\|^2
+ 5\mathbb{E}\|\int_0^{t+h} R(t - s)f(s, x(s - \rho(s)))ds - \int_0^t R(t - s)f(s, x(s - \rho(s)))ds\|^2
+ 5\mathbb{E}\|\int_0^{t+h} R(t - s)\sigma(s)dB^H(s) - \int_0^t R(t - s)\sigma(s)dB^H(s)\|^2
+ 5\mathbb{E}\|\int_0^{t+h} R(t + h - s)Lu(s)ds - \int_0^t R(t - s)Lu(s)ds\|^2
= \sum_{1 \leq i \leq 5} 5J_i(h).$$

The continuity of the terms $J_1$, $J_2$, and $J_3$ can be proved by similar arguments as those used to prove Theorem 3.3 in [6]. Then, it suffices to show that $J_4$ and $J_5$ possess the desired regularity. For the sake of clarity of the paper, we restrict ourselves to the continuity of $J_4$. Thanks to the boundedness of the operators $L$ and $W^{-1}$, the same calculus provides the regularity of the term $J_5$.

$$J_4 = \mathbb{E}\|\int_0^{t+h} R(t + h - s)\sigma(s)dB^H(s) - \int_0^t R(t - s)\sigma(s)dB^H(s)\|^2
\leq 2\mathbb{E}\|\int_0^t (R(t + h - s) - R(t - s))\sigma(s)dB^H(s)\|^2
+ 2\mathbb{E}\|\int_{t}^{t+h} R(t + h - s)\sigma(s)dB^H(s)\|^2
\leq J_{41}(h) + J_{42}(h).$$
By (7), we get that

\[
J_{41}(h) = 2 \sum_{n=1}^{\infty} \lambda_n \int_0^t \| K_n^*(R(t+h-s) - R(t-s))\| \sigma(s)e_n^2 ds \\
\leq 4 \sum_{n=1}^{\infty} \lambda_n \int_0^t K^2(t,s)\| (R(t+h-s) - R(t-s))\| \sigma(s)e_n^2 ds \\
+ 8 \sum_{n=1}^{\infty} \lambda_n \int_0^t \left\| \int_s^t (R(t+h-r) - R(t+h-s) \\
+ R(t-s) - R(t-r)) \sigma(r)e_n^2 \frac{\partial K}{\partial r}(r,s)dr \right\|^2 ds \\
\leq I_1 + I_2 + I_3. \tag{10}
\]

We estimate the various terms of the right-hand side of (10) separately. For the first term, we have:

\[
I_1 = \sum_{n=1}^{\infty} f_n(h) \quad \text{where} \\
f_n(h) = 4 \lambda_n \int_0^t K^2(t,s)\| (R(t+h-s) - R(t-s))\| \sigma(s)e_n^2 ds.
\]

By using the strong continuity of $R(t)x$, we get:

\[
\lim_{h \to 0} K^2(t,s)\| (R(t+h-s) - R(t-s))\| = 0,
\]

and since

\[
\lambda_n K^2(t,s)\| (R(t+h-s) - R(t-s))\| e_n^2 \\
\leq 4D^2\sigma^2 K^2(t,s) \in L^1((0,t),ds),
\]

then, we conclude by the Lebesgue dominated theorem that $\lim_{h \to 0} f_n(h) = 0$. Besides, we have:

\[
|f_n(h)| \leq 16D^2 \lambda_n \int_0^t K^2(t,s)\| \sigma(s)e_n^2 ds,
\]

and since

\[
\sum_{n=1}^{\infty} 16D^2 \lambda_n \int_0^t K^2(t,s)\| \sigma(s)e_n^2 ds \leq 16D^2\sigma^2 \int_0^t K^2(t,s)ds < \infty.
\]

Then, we conclude by the double limit theorem that

\[
\lim_{h \to 0} I_1 = \lim_{h \to 0} \sum_{n=1}^{\infty} f_n(h) = \sum_{n=1}^{\infty} \lim_{h \to 0} f_n(h) = 0. \tag{11}
\]

For the second term, we have: $I_2 = \sum_{n=1}^{\infty} g_n(h)$ where

\[
g_n(h) = 8 \lambda_n \int_0^t \left( \int_s^t \kappa_n(t,s,r,h)\frac{\partial K}{\partial r}(r,s)dr \right)^2 ds.
\]

In the above

\[
\kappa_n(t,s,r,h) := \| (R(t+h-r) - R(t+h-s) + R(t-s) - R(t-r)) \sigma(r)e_n^2 \|
\]
The strong continuity of $R(t)x$ provides:

$$\lim_{h \to 0} \| (R(t + h - r) - R(t + h - s) + R(t - s) - R(t - r)) \sigma(r) e_n \| \frac{\partial K}{\partial r}(r, s) = 0.$$ 

Using Lemma 2.3 together with inequality (5), we get

$$\| (R(t + h - r) - R(t + h - s) + R(t - s) - R(t - r)) \sigma(r) e_n \| \frac{\partial K}{\partial r}(r, s) \leq 2MC_H(1/2 - H)\| \sigma(r) e_n \|_{L^2(H^{-1/2} \in L^2(\mathbb{R}, ds)).}$$

then, we conclude again by the dominated convergence theorem that

$$\lim_{h \to 0} \int_s^t \| (R(t + h - r) - R(t + h - s) + R(t - s) - R(t - r)) \sigma(r) e_n \| \frac{\partial K}{\partial r}(r, s) dr = 0.$$ 

Furthermore, Lemma 2.3 and inequality (5) entail

$$\left( \int_s^t \| (R(t + h - r) - R(t + h - s) + R(t - s) - R(t - r)) \sigma(r) e_n \| \frac{\partial K}{\partial r}(r, s) dr \right)^2 \leq 2M^2C_H^2(1/2 - H)^2(t - s)^{2H} \int_s^t \| \sigma(r) e_n \|_2^2 dr \in L^1(0, t, ds).$$

Then we conclude by the Lebesgue dominated theorem that $\lim_{h \to 0} g_n(h) = 0$. On account of:

$$g_n(h) \leq \frac{16M^2C_H^2(1/2 - H)^2}{H(2H + 1)} t^{2H + 1} \int_0^t \lambda_n \| \sigma(r) e_n \|_2^2 dr,$$

and

$$\sum_{n=1}^{\infty} \int_0^t \lambda_n \| \sigma(r) e_n \|_2^2 dr \leq 2T \tilde{\sigma}^2 + 2 \int_0^t \| A \sigma(r) \|_{L^2}^2 dr < \infty,$$

we conclude by the double limit theorem that

$$\lim_{h \to 0} I_2 = \lim_{h \to 0} \sum_{n=1}^{\infty} g_n(h) = \sum_{n=1}^{\infty} \lim_{h \to 0} g_n(h) = 0. \quad (12)$$

Similar computations can be used to estimate the term $I_3$, indeed, we have: $I_3 = \sum_{n=1}^{\infty} l_n(h)$, where

$$l_n(h) = 8\lambda_n \int_0^t \left( \int_s^t \| (R(t + h - s) - R(t - s))(\sigma(s) e_n - \sigma(r) e_n) \| \frac{\partial K}{\partial r}(r, s) dr \right)^2 ds.$$ 

Again, the strong continuity of $R(t)x$ gives us:

$$\lim_{h \to 0} \| (R(t + h - s) - R(t - s))(\sigma(s) e_n - \sigma(r) e_n) \| \frac{\partial K}{\partial r}(r, s) = 0.$$ 

By assumption (H.5) and inequality (5), we have

$$\| (R(t + h - s) - R(t - s))(\sigma(s) e_n - \sigma(r) e_n) \| \frac{\partial K}{\partial r}(r, s) \leq \frac{2DC_H^2C_H(1/2 - H)}{\sqrt{\lambda_n}} (r - s)^{\gamma + H - 3/2} \in L^1((s, t), dr)$$

Once more, we conclude by the Lebesgue dominated theorem that:

$$\lim_{h \to 0} \int_s^t \| (R(t + h - s) - R(t - s))(\sigma(s) e_n - \sigma(r) e_n) \| \frac{\partial K}{\partial r}(r, s) dr = 0.$$
On the other hand, we have
\[
\left( \int_s^t \left\| (R(t + h - s) - R(t - s)) \sigma(s)e_n - \sigma(r)e_n \right\| \frac{\partial K}{\partial r}(r, s) \, dr \right)^2 \leq \frac{4D^2C^2_H(1/2 - H)^2}{\lambda_n(\gamma + H - 1/2)^2} (t - s)^{2\gamma + 2H - 1} \in L^1((0, t), ds).
\]

One more time, the Lebesgue dominated theorem gives:
\[
\lim_{h \to 0} l_n(h) = 0. \tag{13}
\]

In view of (5) we have
\[
l_n(h) \leq 32\lambda_n D^2C^2_H(1/2 - H) \int_0^t \left( \int_s^t \left\| \sigma(s)e_n - \sigma(r)e_n \right\| (r - s)^{H - 3/2} \, dr \right)^2 ds. \tag{14}
\]

Now, let \( \alpha \in (1, \gamma + H + 1/2) \). By Hölder’s inequality and assumption (H.5), we get
\[
\sum_{n=1}^{\infty} \lambda_n \int_0^t \left( \int_s^t (r - s)^{-3 + 2\alpha} \, dr \right)^2 \left( \int_s^t (r - s)^{2H - 2\alpha} \, dr \right)^2 \, ds < \infty \tag{15}
\]

Combining inequalities (13), (14), (15) and the double limit theorem, we get that
\[
\lim_{h \to 0} I_3 = \lim_{h \to 0} \sum_{n=1}^{\infty} l_n(h) = \sum_{n=1}^{\infty} \lim_{h \to 0} l_n(h) = 0. \tag{16}
\]

Inequalities (11), (12) and (16) imply that \( \lim_{h \to 0} J_{41}(h) = 0 \). By the same arguments, we have
\[
J_{42}(h) = 2 \sum_{n=1}^{\infty} \lambda_n \int_t^{t+h} \left\| K_{t+h}(R(t + h - s)\sigma(s)e_n) \right\|^2 ds \leq 4 \sum_{n=1}^{\infty} \lambda_n \int_t^{t+h} K^2(t + h, s) \left\| R(t + h - s)\sigma(s)e_n \right\|^2 ds + 8 \sum_{n=1}^{\infty} \lambda_n \int_t^{t+h} \left\| \int_s^{t+h} (R(t + h - r) - R(t + h - s)) \sigma(r)e_n \frac{\partial K}{\partial r}(r, s) \, dr \right\|^2 ds + 8 \sum_{n=1}^{\infty} \lambda_n \int_t^{t+h} \left\| \int_s^{t+h} R(t + h - s) \sigma(r)e_n - \sigma(s)e_n \frac{\partial K}{\partial r}(r, s) \, dr \right\|^2 ds \leq I'_1 + I'_2 + I'_3.
\]

By means of (4), we get
\[
I'_1 \leq 16D^2c^2_H \sum_{n=1}^{\infty} \lambda_n \int_t^{t+h} \left( (t + h - s)^{2H - 1} + s^{2H - 1} \right) \left\| \sigma(s)e_n \right\|^2 ds \leq \frac{16c^2_H D^2\alpha^2}{H} \left( h^{2H - 1} + (t + h)^{2H - 1} - t^{2H} \right). \tag{17}
\]
Using Hölder’s inequality, Lemma 2.3 together with inequality (5), we get
\[ I_2 \]
\[
\leq 8M^2c_H^2(1/2 - H)^2 \sum_{n=1}^{\infty} \lambda_n \int_t^{t+h} \left( \int_s^{t+h} (r-s)^{-1/2} \| \sigma(r)e_n \|_2 \right)^2 dr \]
\[
\leq 8M^2c_H^2(1/2 - H)^2 \sum_{n=1}^{\infty} \lambda_n \int_t^{t+h} \left( \int_s^{t+h} (r-s)^{2H-1} dr \int_s^{t+h} \| \sigma(r)e_n \|_2^2 \right) ds \]
\[
\leq 16M^2c_H^2(1/2 - H)^2 \int_t^{t+h} \left( \frac{1}{2H}(t+h-s)^{2H} \int_s^{t+h} \left( \| A\sigma(r) \|_2^2 + \| \sigma(r) \|_2^2 \right) dr \right) ds \]
\[
\leq \frac{8M^2c_H^2(1/2 - H)^2h^{2H+1}}{H(2H+1)} \int_0^T \left( \| A\sigma(r) \|_2^2 + \| \sigma(r) \|_2^2 \right) dr. \tag{18} \]
Inequality (5), condition (H.5) and Hölder’s inequality give
\[ I_3 \]
\[
\leq \delta \sum_{n=1}^{\infty} \lambda_n \int_t^{t+h} \left( \int_s^{t+h} \| \sigma(r)e_n - \sigma(s)e_n \| (r-s)^{-3/2} dr \right)^2 ds \]
\[
\leq \delta \sum_{n=1}^{\infty} \lambda_n \int_t^{t+h} \left( \int_s^{t+h} (r-s)^{-3/2+\gamma} dr \int_s^{t+h} \| \sigma(r)e_n - \sigma(s)e_n \|^2 (r-s)^{-3/2-\gamma} dr \right) ds \]
\[
\leq \delta C_5^2 \int_t^{t+h} \left( \int_s^{t+h} (r-s)^{-3/2+\gamma} dr \right)^2 ds \]
\[
\leq \frac{\delta C_5^2}{2(H+\gamma)(H+\gamma-1/2)^2} h^{2(H+\gamma)}. \tag{19} \]
where \( \delta = 8D^2c_H^2(1/2 - H)^2 \).

Inequalities (17), (18) and (19) imply that \( \lim_{h \to 0} J_{42}(h) = 0 \).

Thus, we conclude that the function \( t \to \psi(x)(t) \) is continuous on \([0, T]\) in the \( L^2 \)-sense.

**Step 2.** Now, we are going to show that \( \psi \) is a contraction mapping in \( S_T \). Let \( x, y \in S_T \), we obtain for any fixed \( t \in [0, T] \)
\[
\| \psi(x)(t) - \psi(y)(t) \|^2 \leq 4 \| g(t, x(t-r(t))) - g(t, y(t-r(t))) \|^2 \\
+ 4 \| \int_0^t R(t-s)(f(s, x(s-\rho(s))) - f(s, y(s-\rho(s))) ds \|^2 \\
+ 4 \| \int_0^t R(t-v)LW^{-1} \{ g(T, x(T-r(T))) - g(T, y(T-r(T))) \} (v)dv \|^2 \\
+ 4 \| \int_0^t R(t-v)LW^{-1}q_T(v)dv \|^2 \]
where,
\[
q_T := \int_0^T R(T-s)[f(s, x(s-\rho(s))) - f(s, y(s-\rho(s)))] ds. \]
By virtue of the boundedness of the operators \( L \) and \( W^{-1} \), and Lipschitz property of \( g \) and \( f \) combined with Hölder’s inequality, we obtain for all \( t \in [0, T] \):

\[
\mathbb{E}\|\psi(x)(t) - \psi(y)(t)\|^2 \leq 4C_3^2 \mathbb{E}\|x(t - r(t)) - y(t - r(t))\|^2
\]

\[
+ 4tD^2C_3^2 \int_0^t \mathbb{E}\|x(s - \rho(s)) - y(s - \rho(s))\|^2 ds
\]

\[
+ 4tD^2M_2^2M_1^2C_3^2 \mathbb{E}\|x(T - r(T)) - y(T - r(T))\|^2
\]

\[
+ 4tD^4M_2^2M_1^2C_1^2T \int_0^T \mathbb{E}\|x(s - \rho(s)) - y(s - \rho(s))\|^2 ds.
\]

Consequently,

\[
\sup_{s \in [-T,T]} \mathbb{E}\|\psi(x)(t) - \psi(y)(t)\|^2 \leq K \sup_{s \in [-T,T]} \mathbb{E}\|x(s) - y(s)\|^2,
\]

where

\[
K = 4[C_3^2 + D^2C_3^2T^2 + D^2M_1^2M_1^2C_3^2T + D^4M_1^2M_1^2C_1^2T^3].
\]

Hence \( \psi \) is a contraction mapping on \( S_T \) and therefore has a unique fixed point, which is a mild solution of equation (1) on \([−T, T]\). Clearly, \( \psi(x)(T) = x_1 \) which implies that the system (1) is controllable on \([−T, T]\). This completes the proof. \( \square \)

4. Example. By way of illustration, we consider the following stochastic integro-differential equation with finite delays \( \tau_1 \) and \( \tau_2 \), \( 0 \leq \tau_1, \tau_2 \leq \tau < \infty \), of the form:

\[
\begin{aligned}
\frac{\partial}{\partial t}[x(t, \xi) + \hat{g}(t, x(t - \tau_1, \xi))] &= \frac{\partial^2}{\partial^2 \xi}[x(t, \xi) + \hat{g}(t, x(t - \tau_1, \xi))] \\
+ \int_0^t b(t-s) \frac{\partial^2}{\partial^2 \xi}[x(s, \xi) + \hat{g}(s, x(s - \tau_1, \xi))]ds \\
+ \hat{f}(t, x(t - \tau_2, \xi)) + \mu(t, \xi) + \sigma(t) \frac{dB^H}{dt}(t), \quad t \geq 0,
\end{aligned}
\]

\[x(t, 0) + g(t, x(t - \tau_1, 0)) = 0, \quad t \geq 0,
\]

\[x(t, \pi) + g(t, x(t - \tau_1, \pi)) = 0, \quad t \geq 0,
\]

\[x(s, \xi) = \varphi(s, \xi), \quad -\tau \leq s \leq 0 \text{ a.s.}
\]

where \( B^H \) denotes a fractional Brownian motion, \( \hat{f}, \hat{g} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \) are continuous functions and \( b : \mathbb{R}_+ \to \mathbb{R} \) is continuous function and \( \varphi : [-\tau, 0] \times [0, \pi] \to \mathbb{R} \) is a given continuous function such that \( \varphi(s, \cdot) \in L^2([0, \pi]) \) is measurable and satisfies \( \mathbb{E}\|\varphi\|^2 < \infty \).

Let \( X = Y = L^2([0, \pi]) \). Define the operator \( A : D(A) \subset X \to X \) given by \( A = \frac{\partial^2}{\partial^2 \xi} \) with domain:

\[D(A) = \{ x \in X : x'' \in X, \ x(0) = x(\pi) = 0 \},\]

and,

\[Ax = \sum_{n=1}^{\infty} n^2 < x, e_n > X e_n, \quad x \in D(A)\]

where \( e_n := \sqrt{2 \sin nx, n = 1, 2, \ldots} \) is an orthogonal set of eigenvector of \(-A\).

It is known that \( A \) is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators \( (T(t))_{t \geq 0} \) on \( X \), which is given by

\[T(t)x = \sum_{n=1}^{\infty} e^{-n^2t} < x, e_n > X e_n.\]
Furthermore, \( \|T(t)\| \leq e^{-\tau^2 t} \) for every \( t \geq 0 \).

Let \( B : D(A) \subset X \to X \) be the operator given by \( B(t)x = b(t)Ax, \) for \( t \geq 0 \) and \( x \in D(A) \). Define the operator \( W : L^2([0,T],U) \to X \) by:
\[
Wu(\xi) = \int_0^T R(T-s)\mu(t,\xi)ds, \quad 0 \leq \xi \leq \pi,
\]
\( W \) is a bounded linear operator but not necessarily one-to-one. Let \( KerW = \{ x \in L^2([0,T],U), \ Wx = 0 \} \) be the null space of \( W \) and \([KerW]^\perp\) be its orthogonal complement in \( L^2([0,T],U) \). Let \( \tilde{W} : [KerW]^\perp \to Range(W) \) be the restriction of \( W \) to \([KerW]^\perp\), \( \tilde{W} \) is one-to-one operator. The inverse mapping theorem says that \( \tilde{W}^{-1} \) is bounded since \([KerW]^\perp\) and \( Range(W) \) are Banach spaces. So that \( \tilde{W}^{-1} \) is bounded and takes values in \( L^2([0,T],U) \setminus KerW \), hence assumption (H.6) is satisfied. We suppose that:

(i) The operator \( Lu : [0,T] \to X \), defined by:
\[
Lu(t)\xi = \mu(t,\xi), \quad \xi \in [0,\pi], \quad u \in L^2([0,T],U).
\]

(ii) For \( t \in [0,T] \), \( \tilde{f}(t,0) = \hat{g}(t,0) = 0 \).

(iii) There exist positive constants \( C_1 \) and \( C_2 \), such that
\[
|\tilde{f}(t,\xi_1) - \tilde{f}(t,\xi_2)| \leq C_1|\xi_1 - \xi_2|, \quad \text{for } t \in [0,T] \text{ and } \xi_1, \xi_2 \in \mathbb{R},
\]
\[
|\hat{g}(t,\xi_1) - \hat{g}(t,\xi_2)| \leq C_2|\xi_1 - \xi_2|, \quad \text{for } t \in [0,T] \text{ and } \xi_1, \xi_2 \in \mathbb{R}.
\]

(iv) There exist positive constants \( C_3 \) and \( C_4 \), such that
\[
|\tilde{f}(t,\xi)| \leq C_3(1+|\xi|^2), \quad \text{for } t \in [0,T] \text{ and } \xi \in \mathbb{R},
\]
\[
|\hat{g}(t,\xi)| \leq C_4(1+|\xi|^2), \quad \text{for } t \in [0,T] \text{ and } \xi \in \mathbb{R}.
\]

(v) The function \( \sigma : [0,\infty) \to L^2([0,\pi],L^2([0,\pi])) \) satisfies assumptions (H.6).

Define the operators \( f, g : \mathbb{R}_+ \times L^2([0,\pi]) \to L^2([0,\pi]) \) by
\[
f(t,\phi)(\xi) = \tilde{f}(t,\phi(-\tau_2)(\xi)) \quad \text{for } \xi \in [0,\pi] \text{ and } \phi \in L^2([0,\pi]),
\]
and
\[
g(t,\phi)(\xi) = \hat{g}(t,\phi(-\tau_1)(\xi)) \quad \text{for } \xi \in [0,\pi] \text{ and } \phi \in L^2([0,\pi]).
\]

If we put:
\[
\begin{align*}
\{ \quad x(t)(\xi) &= x(t,\xi), \quad \text{for } t \in [0,T] \text{ and } \xi \in [0,\pi], \\
\{ \quad x(t)(\xi) &= \varphi(t,\xi), \quad \text{for } t \in [-\tau,0] \text{ and } \xi \in [0,\pi].
\end{align*}
\]

Then, equation (20) takes the following abstract form:
\[
\begin{align*}
\begin{cases}
\begin{aligned}
d[x(t) + g(t,x(t-r(t)))] &= [A[x(t) + g(t,x(t-r(t)))] + Lu(t)]dt \\
&+ \left[ \int_0^t B(t-s) [x(s) + g(s,x(s-r(s)))] ds + f(t,x(t-r(t))) \right] dt \\
&+ \sigma(t)dB^H(t), \quad 0 \leq t \leq T,
\end{aligned}
\end{cases}
\end{align*}
\]
\[
x(t) = \varphi(t), \quad -\tau \leq t \leq 0,
\]

Moreover, if \( b \) is bounded and \( C^1 \) such that \( b' \) is bounded and uniformly continuous, then (H.2) is satisfied, hence equation (20) has a resolvent operator \( (R(t))_{t \geq 0} \) on \( X \). Besides, the continuity of \( f \) and \( \hat{g} \) and assumption (ii) it ensues that \( f \) and \( g \) are continuous. In accordance with assumption (iii) we obtain
\[
\|f(t,\phi_1) - f(t,\phi_2)\|_{L^2([0,\pi])} \leq C_1\|\phi_1 - \phi_2\|_{L^2([0,\pi])},
\]
Furthermore, by assumption (14), it follows that
\[ \|g(t, \phi_1) - g(t, \phi_2)\|_{L^2([0,\pi])} \leq C_3\|\phi_1 - \phi_2\|_{L^2([0,\pi])}. \]

Moreover, it is possible to choose the constants in such a way that:
\[ -\mathcal{L} is controllable on \([-\tau, T]\).

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