Analysis of some basic approaches to finite strain elasto-plasticity in view of reference change

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Abstract

There is a large variety of concepts used to generalize the classical Prandtl-Reuss relations of infinitesimal elasto-plasticity to finite strains. In this work, some basic approaches are compared in a qualitative way with respect to a certain invariance property. These basic approaches include the additive hypoelasto-plasticity with corotational stress rates, additive plasticity in the logarithmic strain space, and multiplicative hyperelasto-plasticity.

The notion of weak invariance is introduced in this study. Roughly speaking, a material model is weakly invariant under a certain transformation of the local reference configuration if this reference change can be neutralized by a suitable transformation of initial conditions, leaving the remaining constitutive relations intact. We analyse the basic models in order to find out if they are weakly invariant under arbitrary volume-preserving transformations of the reference configuration.

It is shown that the weak invariance property corresponds to a generalized symmetry which provides insights into underlying constitutive assumptions. This property can be used for a systematic study of different frameworks of finite strain elasto-plasticity. In particular, it can be used as a classification criterion.

Keywords: finite strain elasto-plasticity, reference change, weak invariance, hypoelasto-plasticity, logarithmic elasto-plasticity, multiplicative plasticity

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Nomenclature

F deformation gradient
C right Cauchy-Green tensor
B left Cauchy-Green tensor
L velocity gradient tensor
D strain rate tensor (stretching tensor)
W continuum spin tensor (vorticity tensor)
1. Introduction

The main purpose of the mathematical material theory is to provide some general concepts useful in construction and analysis of appropriate constitutive relations (Truesdell and Noll, 1965; Haupt, 1996). In particular, any systematic study of competitive approaches to finite strain elasto-plasticity should be based on a set of sound principles. In this context, the following popular concepts and criteria have proved to be especially useful:

- the principle of material objectivity or material frame-indifference (Noll, 1958; Truesdell and Noll, 1965; Truesdell, 1969; Bressan, 1972; Svendsen and Bertram, 1999; Bertram and Svendsen, 2001; Muschik and Restuccia, 2008; Korobeynikov, 2008),
requirement of thermodynamic consistency (Coleman and Noll, 1963; Coleman, 1964; Truesdell, 1963; Halphen and Nguyen Quoc Son, 1975), prevention of the shear oscillatory response (Lehmann, 1972; Dienes, 1979; Nagtegaal and de Jong, 1982), prevention of the energy dissipation within the elastic range (Simo and Pister, 1984; Bruhns et al., 1999).

In this study we introduce the notion of weak invariance of constitutive relations under arbitrary isochoric change of the local reference configuration. This weak invariance cannot be attributed to general principles of material modeling. Instead, we will demonstrate that it is a powerful tool of analysis and classification, especially useful within finite strain elasto-plasticity.

We start by recalling that the local reference configuration is, in general, a fictitious configuration, set at will to designate material points in a unique manner. Hence, any arbitrary volume-preserving change of the local reference configuration is always possible since the transformed configuration can act as a new reference. The main idea of the paper is as follows. The constitutive models of elasto-plasticity can be subdivided into two major groups: those models which are weakly invariant under arbitrary volume-preserving reference changes and those which do not fulfill this invariance.

In order to get an intuitive insight into the weak invariance, we start with a simple example. First, let us consider the well-known constitutive relations governing a elastic-perfectly-plastic material behaviour in the geometrically linear setting. These relations include the classical Prandtl-Reuss kinematic equations

\[ \varepsilon = \varepsilon_e + \varepsilon_p, \]  

where symmetric tensors \( \varepsilon, \varepsilon_e, \varepsilon_p \) stand for the total, elastic, and plastic strains, respectively. The isotropic Hooke’s law implies that the stress tensor \( \sigma \) is a linear function of \( \varepsilon_e \)

\[ \sigma = k \text{tr}(\varepsilon_e) \mathbf{1} + 2\mu \varepsilon^D_e. \]  

Here, \( k > 0, \mu > 0 \) are material parameters and \( (\cdot)^D \) represents the deviatoric part. The Mises-Huber yield function with a constant yield stress \( K > 0 \) takes the form

\[ f := \|\sigma^D\| - \sqrt{2/3}K, \]  

where \( \|X\| := \sqrt{\text{tr}(XX^T)} \). A six-dimensional flow rule is introduced in combination with the Kuhn-Tucker constraints:

\[ \dot{\varepsilon}_p = \lambda_p \frac{\sigma^D}{\|\sigma^D\|}, \]  

\[ f \leq 0, \lambda_p \geq 0, f\lambda_p = 0. \]
Here, \( \lambda_p \) stands for the plastic multiplier. Within the local approach, the history of the total strain tensor \( \varepsilon(t) \) is given. The system of equations is closed by appropriate initial conditions imposed on plastic strains

\[
\varepsilon_p|_{t=t_0} = \varepsilon_p^0, \quad \text{tr}(\varepsilon_p^0) = 0.
\]

(6)

Note that the flow rule (4) is incompressible, which is a typical assumption in the context of metal plasticity. Thus, \( \text{tr}(\varepsilon_p) \equiv 0 \).

Let us consider an arbitrary second-rank tensor \( \varepsilon_0 = \text{const} \) such that \( \text{tr}(\varepsilon_0) = 0 \). The system (1)–(6) has the following remarkable property. If the prescribed loading program is shifted according to

\[
\varepsilon_{\text{new}}(t) := \varepsilon(t) - \varepsilon_0,
\]

(7)

and the new initial conditions are obtained by the same shift

\[
\varepsilon_p^{\text{new}}|_{t=t_0} = \varepsilon_p^0 - \varepsilon_0,
\]

(8)

then the same stress response \( \sigma(t) \) is predicted by the constitutive equations.

**Proof:** In order to check this invariance property, we consider the following candidate function for the new plastic strain

\[
\varepsilon_p^{\text{new}}(t) := \varepsilon_p(t) - \varepsilon_0.
\]

(9)

Let us show that this candidate function corresponds to the new solution. For this candidate, it follows immediately from (7) and (9) that the elastic strains remain invariant

\[
\varepsilon_0^{\text{new}}(t) \equiv \varepsilon_0^{\text{new}}(t) - \varepsilon_0^{\text{new}}(t) \equiv \varepsilon_0^{\text{new}}(t) - \varepsilon_0^{\text{new}}(t) \equiv \varepsilon_0(t).
\]

(10)

Therefore, according to the Hooke’s law (2), the same stresses are predicted:

\[
\sigma^{\text{new}}(t) = \sigma(t), \quad f^{\text{new}}(t) = f(t).
\]

(11)

Thus, the Kuhn-Tucker conditions (5) are identically satisfied if we put

\[
\lambda_p^{\text{new}}(t) : = \lambda_p(t).
\]

(12)

Moreover, the candidate function \( \varepsilon_p^{\text{new}}(t) \) satisfies the initial condition (8). By differentiating (9) with respect to time, it becomes obvious that \( \varepsilon_p^{\text{new}}(t) \) satisfies the evolution equation:

\[
\dot{\varepsilon}_p^{\text{new}} = \dot{\varepsilon}_p + \lambda_p \frac{\sigma^D}{\|\sigma^D\|} \equiv \lambda_p^{\text{new}} \frac{(\sigma^{\text{new}})^D}{\| (\sigma^{\text{new}})^D \|}.
\]

(13)

Therefore, the candidate \( \varepsilon_p^{\text{new}}(t) \) is indeed a solution for the new local loading \( \varepsilon^{\text{new}}(t) \). It remains to recall that \( \sigma^{\text{new}}(t) = \sigma(t) \). The assertion is proved.  

In the remainder of this paper, the aforementioned invariance of the stress response under the transformation (7) will be seen as a weak invariance. Here, the adjective “weak” points to the fact that an additional freedom in transformation of the initial conditions (8) weakens the restrictions imposed on the material model. In the next section, the notion of the weak invariance will be reformulated in the finite strain context. As an extension to finite strains, a weak invariance under isochoric change of the reference configuration will be introduced, and a proper transformation of the initial conditions (similar to (8)) will be a crucial part of this concept.

Considering the generalization of small strain relations to finite strains, the following aspect should be taken into account. If the small strain relations are weakly invariant under the shift (7), one may expect that this invariance property will be inherited by the finite strain counterpart.

In the modern literature on elasto-plasticity there exists a large variety of frameworks used to generalize the small strain relations (1)–(6) to finite strains (Xiao et al., 2006). In this paper, some of the most popular approaches will be analysed from the viewpoint of the reference change. These approaches include the additive hypoelasto-plasticity with corotational stress rates (cf. Section 3), additive plasticity in the logarithmic strain space (cf. Section 4), and multiplicative hyperelasto-plasticity (cf. Section 5). In case of small strains and rotations, these approaches are reduced to the geometrically linear theory (1)–(6). It is shown that some of these models are weakly invariant under the reference change, but some of them not. Thus, different models can be categorized using the weak invariance as a criterion. Moreover, the weak invariance corresponds to a certain generalized material symmetry. Just like any other symmetry property, it brings new insights into our understanding of the underlying constitutive assumptions.

2. Weak invariance under the reference change

Let \( T \) be the Cauchy stress tensor (also known as true stresses) at a certain material point and \( F \) be the deformation gradient from the local reference configuration \( \tilde{K} \) to the current configuration \( K \). The history of the deformation gradient from the initial state at \( t_0 \) up to a certain time instance \( t \) is captured by \( F(t'), t' \in [t_0, t] \). Suppose that the state of the material at \( t_0 \) is uniquely determined by some initial data \( Z_0 \). Thus, for simple materials in the sense of Noll (1972) we obtain the following:

\[
T(t) = T_{t_0 \leq t' \leq t} (F(t'), Z_0). \tag{14}
\]

\(^1\)Interestingly, the transformation (8) was previously considered in combination with the shift (7) by Rubin (2001) to question the status of the total strain as a state variable.
Figure 1: Commutative diagram: change of the reference configuration \( \tilde{K} \) to the new reference \( \tilde{K}^{\text{new}} \).

Here, the right-hand side represents a response functional of the material. In certain applications, the data \( Z_0 \) may be given by the set of initial values of internal variables. In general, \( Z_0 \) can encapsulate both scalar and tensor-valued quantities.

Now let us consider a mapping \( F_0 = \text{const} \), such that \( \det(F_0) = 1 \). Let \( \tilde{K}^{\text{new}} := F_0 \tilde{K} \) be a new reference configuration (cf. Fig. 1). We define the new deformation gradient (relative deformation gradient) by the following push-forward operation

\[
F^{\text{new}}(t) := F(t) F_0^{-1}. \tag{15}
\]

We say that the material model (14) is weakly invariant under the transformation (15) when there is a new set of initial data

\[
Z_0^{\text{new}} = Z_0^{\text{new}}(Z_0, F_0) \tag{16}
\]

such that the same mechanical response is predicted:

\[
T_{t_0 \leq t' \leq t}(F(t'), Z_0) = T_{t_0 \leq t' \leq t}(F^{\text{new}}(t'), Z_0^{\text{new}}). \tag{17}
\]

If a certain model is invariant under arbitrary volume-preserving reference changes, we may be free in our use of the terminology and say that the model is weakly invariant, or even \( w \)-invariant.

**Remark 1.** Let us recall the classical “strong” invariance requirement, which operates with a semi-infinite deformation history (Noll, 1958; Truesdell and Noll, 1965)

\[
T_{-\infty < t' \leq t}(F(t')) = T_{-\infty < t' \leq t}(F^{\text{new}}(t')). \tag{18}
\]

The strong invariance (18) under arbitrary isochoric transformations of the reference would immediately imply a fluid-like material behavior. Thus, the strong invariance under arbitrary

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Note that a *semi-infinite* deformation history was originally introduced by Noll (1958) without any initial conditions. Nevertheless, we follow Noll (1972) and consider a *limited* deformation history in (14), since it provides the correct framework for the classical plasticity models.
transformations is too restrictive for elasto-plasticity. In contrast to (18), the invariance relations (17) are “weak” in the sense that the new initial conditions can counteract the reference change. Owing to this “weakness”, a reasonable analysis of elasto-plasticity is possible with this concept.

In order to prevent any misconception, a clear distinction should be made between the weak invariance (16), (17) and the strong invariance (18). The difference between both concepts may disappear in case of constitutive relations without initial conditions.

Remark 2. Since the Kirchhoff stress is given by $S = \det(F)T$ and $\det(F) = \det(F^{\text{new}})$, the same Kirchhoff stress is predicted upon the reference change by any w-invariant model.

Remark 3. Note that the velocity gradient $L$, the strain rate $D$ and the continuum spin $W$ are reference independent (in that sense they represent truly Eulerian quantities)

$$L^{\text{new}} = F^{\text{new}}(F^{\text{new}})^{-1} = \ddot{F}F^{-1} = L, \quad (19)$$

$$D^{\text{new}} = \text{sym}(L^{\text{new}}) = \text{sym}(L) = D, \quad (20)$$

$$W^{\text{new}} = \text{skew}(L^{\text{new}}) = \text{skew}(L) = W. \quad (21)$$

All material models in the form

$$\mathbf{T}(t) = \mathbf{T}_{t_0 \leq t' \leq t} (\mathbf{L}(t'), \det(\mathbf{F}(t)), Z_0) \quad (22)$$

can be represented in a weakly invariant form, if $Z_0$ is not affected by the reference change ($Z_0^{\text{new}} = Z_0$). A non-trivial example will be considered in Section 3.2.

The notion of w-invariance can be formulated on the reference configuration as well. Indeed, suppose that the principle of material frame-indifference holds. Following Truesdell and Noll (1965), the material model (14) takes the reduced form with respect to the 2nd Piola-Kirchhoff stress tensor $\tilde{T}$ and the right Cauchy-Green tensor $C$:

$$\tilde{T}(t) = \tilde{T}_{t_0 \leq t' \leq t} (C(t'), Z_0), \quad \text{where } C(t') := F^T(t') F(t'). \quad (23)$$

The model (23) is weakly invariant under the reference change if there is a new set $Z_0^{\text{new}} = Z_0^{\text{new}}(Z_0, F_0)$ such that

$$\tilde{T}_{t_0 \leq t' \leq t} (C(t'), Z_0) = F_0^{-1} \tilde{T}_{t_0 \leq t' \leq t} (C^{\text{new}}(t'), Z_0^{\text{new}}) F_0^{-T}. \quad (24)$$

3Another example is given by the new model of finite strain elasto-plasticity, which was proposed recently by Volokh (2013).

4Observe that the reduced form (23) corresponds to the active interpretation of the principle of frame-indifference. Thus, indifference with respect to superimposed rigid-body motions is postulated as a constitutive assumption (Svendsen and Bertram, 1999; Bertram and Svendsen, 2001).
where the new right Cauchy-Green tensor is obtained using the push forward operation

$$C_{\text{new}}(t) := F_0^{-T}C(t)F_0^{-1}. \quad (25)$$

The 2nd Piola-Kirchhoff stress must obey (24) in order to ensure that the true stresses remain invariant. An example of such model will be given in Section 5.

Dealing with constitutive equations enjoying the w-invariance, the reference configuration can be chosen in a rather arbitrary manner. The only restriction is imposed by the incompressibility condition $\det(F_0) = 1$.

Two physical observations are helpful for a better understanding of the w-invariance:

(p) Whereas the strong invariance under arbitrary isochoric reference changes is observed only for fluids, the plastic flow of solids may also exhibit some fluid-like features, which are formalized here using w-invariance.

(e) The current deformation of the crystal lattice and not the overall deformation of the metallic material determine the Cauchy stresses [Anand et al., 1987; Bertram, 1998; Rubin, 2001].

Observations (p) and (e) impose some restrictions on the plastic and elastic constitutive relations, respectively. These observations are complimentary to each other and they motivate the current study from a physical standpoint.

3. Additive hypoelasto-plasticity

3.1. General framework of the additive hypoelasto-plasticity

As a first example, let us consider the classical approach to elasto-plasticity (cf. Neale (1981); Nemat-Nasser (1982)) which is based on the additive decomposition of the strain rate tensor $D$ into the elastic part $D_e$ and the plastic part $D_p$:

$$D = D_e + D_p. \quad (26)$$

The elastic part $D_e$ is connected to a certain objective rate of the Kirchhoff stress tensor via hypoelastic relations

$$D_e = \mathbb{H} : \dot{S}, \quad (27)$$

where $\mathbb{H}$ is a fourth rank compliance tensor, $S$ is the Kirchhoff stress. For simplicity, we suppose here that $\mathbb{H}$ is constant. Next, the Mises-Huber yield function is considered

$$f := \|S^D\| - \sqrt{2/3}K, \quad (28)$$

$^5$Equation (27) corresponds to the so-called grade-zero hypoelasticity. It is a special case of relations of Eulerian rate type (cf. Eshraghi et al. [2013a]).
where \( K > 0 \) stands for the yield stress. The flow rule is given by

\[
D_p = \lambda_p \frac{S^D}{\|S^D\|}.
\]  
(29)

Here, \( \lambda_p \geq 0 \) stands for the plastic multiplier. The Kuhn-Tucker conditions take the same form as in the small strain case

\[
f \leq 0, \quad \lambda_p \geq 0, \quad f \lambda_p = 0.
\]  
(30)

The initial conditions are imposed on the Kirchhoff stresses

\[
S|_{t=t_0} = S_0, \quad Z_0 := \{S_0\}.
\]  
(31)

Note that the objective stress rate which appears in (27) has to be specified. In this work we imply the consistency criterion proposed by Prager (1960) in order to restrict the class of possible stress rates. The consistency criterion states that \( \ddot{S} = 0 \) implies \( f = \text{const} \), where the yield function \( f \) is given by (28). In order to satisfy this condition, the stress rate \( \ddot{S} \) must be a corotational derivative (cf. Xiao et al. (2000a); Bruhns et al. (2004))

\[
\ddot{S} := \dot{S} + S\Omega - \Omega S, \quad \Omega \in \text{Skew}.
\]  
(32)

Here, Skew stands for the set of skew-symmetric tensors, the skew-symmetric operator \( \Omega \) is referred to as a spin tensor, superimposed dot stands for the material time derivative. There are infinitely many ways of defining the spin tensors \( \Omega \) (Xiao et al., 1998; Korobeynikov, 2011; Hashiguchi and Yamakawa, 2012). Obviously, the properties of the resulting constitutive equations are strongly dependent on the specific choice of \( \Omega \). In particular, we have the following theorem.

**Theorem 1.** Constitutive relations \( (26) - (32) \) are weakly invariant under the isochoric reference change if and only if the spin tensor \( \Omega \) remains invariant under such a change.

**Proof:** After some algebraic computations, the constitutive equations can be rewritten in the compact form (cf. equation (45) in Bruhns et al. (1999)):

\[
\ddot{S} = \ddot{\mathcal{S}}(S,D),
\]  
(33)

where the right-hand side depends solely on \( S \) and \( D \). Consider a local loading process \( F(t'), t' \in [t_0, t] \) and let \( F_0 = \text{const} \) such that \( \det(F_0) = 1 \). As already mentioned in Section 2, the material

\footnote{Note that the yield function is formulated in terms of the Kirchhoff stress, which is believed to be a natural choice for metals. For other groups of materials, like certain geomaterials, the true stresses can be used (cf. Yamakawa et al., 2013).}

\footnote{Equation (33) is a remarkable result since the plastic part \( D_p \) is excluded.}
model \((33)\) is w-invariant if and only if it predicts the same Kirchhoff stress \(S\) for the new process \(F(t')F_0^{-1}, t' \in [t_0, t]\). According to \((31)\), the set of initial data is given by \(Z_0 = \{S_0\}\). Thus, the transformation rule \((16)\) must take the following trivial form

\[Z_0^{new} = Z_0 = \{S_0\}\]  

which is independent of \(F_0\).

Assume that the model \((33)\) is w-invariant. We need to show that the spin tensor \(\Omega\) is invariant (reference independent). First, recall that the strain rate tensor \(D\) fulfils the invariance. Thus, the right-hand side of \((33)\) represents an invariant quantity. Therefore, \((33)\) implies that \(\dot{S}\) is invariant as well. Taking \((32)\) into account, we conclude that

\[S\Omega^{new} - \Omega^{new}S = S\Omega - \Omega S.\]  

Here, the spin tensors \(\Omega\) and \(\Omega^{new}\) correspond to loading histories \(F(t')\) and \(F(t')F_0^{-1}\) respectively. Observe that arbitrary initial conditions \((31)\) can be imposed on the Kirchhoff stresses.\(^8\) Therefore, \((35)\) must hold for all symmetric \(S\). For the difference \(\Omega^{\text{diff}} := \Omega^{\text{new}} - \Omega \in \text{Skew}\) we obtain from \((35)\)

\[S\Omega^{\text{diff}} = \Omega^{\text{diff}}S \quad \text{for all } S \in \text{Sym}. \]  

Thus, for all vectors \(x\) we have

\[(x \otimes x)\Omega^{\text{diff}} = \Omega^{\text{diff}}(x \otimes x).\]  

Therefore, obviously,

\[x \otimes (x\Omega^{\text{diff}}) = (\Omega^{\text{diff}}x) \otimes x.\]  

But, on the other hand, \(\Omega^{\text{diff}} \in \text{Skew}\) yields

\[x\Omega^{\text{diff}} = (\Omega^{\text{diff}})^{\top}x = -\Omega^{\text{diff}}x.\]  

Substituting this into \((38)\), we arrive at the following for all \(x\)

\[-x \otimes (\Omega^{\text{diff}}x) = (\Omega^{\text{diff}}x) \otimes x.\]  

Thus, \(\alpha_x \in \mathbb{R}\) exists such that \(\Omega^{\text{diff}}x = \alpha_x x\) and from \((40)\) we come to

\[-\alpha_x x \otimes x = \alpha_x x \otimes x.\]  

\(^8\)The only restriction is imposed by \(\|S_0^D\| \leq \sqrt{2/3K}\).
For all \( x \neq 0 \), we have \( \alpha_x = 0 \) and
\[
\Omega^\text{diff} x = 0, \quad \Omega^\text{diff} = 0.
\] (42)

In other words, relation (35) implies \( \Omega^\text{diff} = \Omega^\text{new} - \Omega = 0 \). Hence, (35) holds only if \( \Omega^\text{new} = \Omega \).

For the proof in the opposite direction, assume that the spin \( \Omega \) fulfills the invariance. In which case, obviously, the model (33) is w-invariant. The theorem is proved \( \blacksquare \).

Different models of elasto-plasticity can be constructed by an appropriate choice of the spin tensor \( \Omega \). Let us consider some of the most popular options.

### 3.2. Zaremba-Jaumann and Green-Naghdi rates

Probably, the most simple choice for the spin tensor \( \Omega \) is the continuum spin \( W \), which yields the Zaremba-Jaumann rate, also known as Zaremba-Jaumann-Noll rate
\[
\Omega^ZJ := W = \text{skew}(L), \quad \dot{S}^ZJ := \dot{S} + S\Omega^ZJ - \Omega^ZJ S.
\] (43)

Since the continuum spin \( W \) is invariant (recall Remark 3), the corresponding system of equations is w-invariant as well. One important drawback of such an approach is that the stress response exhibits non-physical oscillations under the simple shear (Lehmann (1972); Dienes (1979)). In order to prevent the oscillations of stresses, the so-called Green-Naghdi rate\(^9\) can be implemented. Its definition makes use of the rotation tensor \( R \) which results from the polar decomposition of the deformation gradient:
\[
\Omega^\text{GN} := \dot{R}R^T \in \text{Skew}, \quad \dot{S}^\text{GN} := \dot{S} + S\Omega^\text{GN} - \Omega^\text{GN} S.
\] (44)

It can be easily shown that this spin tensor depends on the choice of the reference configuration (cf. Korobeynikov (2008)). Indeed, in general we have
\[
\Omega^\text{GN} \neq W.
\] (45)

Therefore, a loading history and a time instance \( t' \) exist such that \( \Omega^\text{GN}(t') \neq W(t') \). By choosing \( F_0 := F(t') \), where \( X := (\det X)^{-1/3}X \), we obtain
\[
F^\text{new}(t') = F(t')(F(t'))^{-1} = (\det F(t'))^{1/3}1.
\] (46)

In this special case, the new GN-spin coincides with the continuum spin:
\[
(\Omega^\text{GN})^\text{new}(t') = W(t') \neq \Omega^\text{GN}(t').
\] (47)

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\(^9\)The reader interested in the numerical treatment of constitutive equations for a variety of different spin tensors is referred to Abbasi and Parsa (2006).

\(^{10}\)It is also known as Green-Naghdi-Dienes rate, Green-McInnis rate or polar rate.
Thus, in general,

\[(\Omega^{GN})^{new} \neq \Omega^{GN}.\]  (48)

In other words, \(\Omega^{GN} = \dot{R} \dot{R}^T\) is not a purely Eulerian quantity, since it depends on the choice of the reference configuration. Hence, according to Theorem 1, the constitutive equations based on (44) are not w-invariant.

**Remark 4.** The spin \(\Omega^{GN}\) fulfills invariance under the rigid rotation of the reference configuration. Indeed, for \(F_0 = Q_0 \in \text{Orth}\), we get \(\dot{R}^{new} = R Q_0^{-1}\). Therefore, \(\dot{R}^{new}(R^{new})^{-1} = R Q_0^{-1} Q_0 R^{-1} = \dot{R} R^{-1}\) for \(F_0 = Q_0 \in \text{Orth}\). Thus, initially isotropic materials can be modeled using this stress rate.

Now let us illustrate the implications of w-invariance using a simple numerical example. To be definite, the compliance tensor \(H\) which appears in (27) is defined here through its inverse \(H^{-1}\):

\[H^{-1} : X = \frac{E}{3(1-2\nu)} \text{tr}(X) \mathbf{1} + \frac{E}{1+\nu} X^D, \text{ for all } X \in \text{Sym}.\]  (49)

We insert the values \(E = 500\) and \(\nu = 0.3\) (all quantities are non-dimensional). The yield stress is given by \(K = \sqrt{3} \cdot 100\). First, we simulate a simple shear with an abrupt load path change according to the following loading program:

\[F(t) = \begin{cases} 
\mathbf{1} + t \mathbf{e}_1 \otimes \mathbf{e}_3 & \text{if } t \in [0, 2) \\
(1 + (t - 2) \mathbf{e}_2 \otimes \mathbf{e}_3)(\mathbf{1} + 2 \mathbf{e}_1 \otimes \mathbf{e}_3) & \text{if } t \in [2, 4]
\end{cases},\]  (50)

where \(\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}\) is a fixed orthonormal basis. Since \(F(0) = \mathbf{1}\), the initial configuration coincides with the reference configuration here. As an initial condition at \(t = 0\) we assume zero stresses. The simulated stress response is shown by solid lines in Fig. 2 (left) for the Zaremba-Jaumann model and in Fig. 2 (right) for the Green-Naghdi model.

Now let us restart the simulation at the time instance \(t = 2\) using a new reference configuration \(\tilde{K}^{new} = F_0 \tilde{K}\) where \(F_0 = F(2) = \mathbf{1} + 2 \mathbf{e}_1 \otimes \mathbf{e}_3\). In other words, the new reference configuration corresponds to the configuration occupied by the body at \(t = 2\). The corresponding initial conditions (at \(t = t_0 = 2\)) are formulated with respect to the Kirchhoff stresses. The initial values are taken from the simulation results obtained in the previous simulation. Within the time interval \(t \in [2, 4]\), the material is subjected to the simple shear

\[F^{new}(t) = F(t) F_0^{-1} = \mathbf{1} + (t - 2) \mathbf{e}_2 \otimes \mathbf{e}_3.\]  (51)

The simulation results are shown in Figure 2 by dotted lines. Observe that exactly the same stress response is predicted by the Zaremba-Jaumann model after restart, since this model is w-invariant under arbitrary isochoric changes of the reference configuration. For the Green-Naghdi model, in contrast, a divergent response is obtained.
The set of initial data at $t = t_0 = 2$ is given by the initial value of the Kirchhoff stress tensor, but these data are not sufficient to describe the material state in a unique manner if the model is not w-invariant. For such models, additional information concerning the concrete choice of the reference is required.

3.3. Models based on the logarithmic rate

A common drawback of the models relying on the Zaremba-Jaumann and Green-Naghdi rates is that the constitutive equations are not integrable within the elastic range. In other words, even for $\mathbf{D} = \mathbf{D}_e$ the stress state depends on the loading path, which is inconsistent with the classical notion of elasticity (Simo and Pister, 1984). Moreover, it was proved by Bruhns et al. (1999) that, among all possible models of the corotational rate type with constant elastic moduli, the model based on the logarithmic rate is the only choice which allows for obtaining integrable stress-strain relations and avoiding the non-physical energy-dissipation in the elastic range. For that reason, the logarithmic rate deserves serious consideration.

The corotational logarithmic rate is given by

$$\dot{\mathbf{S}}^{\log} := \dot{\mathbf{S}} + \mathbf{S}\Omega^{\log} - \Omega^{\log}\mathbf{S},$$

(52)

where $\Omega^{\log}$ stands for the so-called log-spin (Xiao et al., 1997). This logarithmic rate has the following remarkable property: The strain rate $\mathbf{D}$ equals the logarithmic rate of the Eulerian logarithmic strain measure $\ln \mathbf{V}$

$$\mathbf{D} = (\ln \mathbf{V})^{\log}, \quad \mathbf{V} := \sqrt{\mathbf{F}^T \mathbf{F}}.$$ 

(53)

The logarithmic rate is uniquely determined by the kinematic requirement (53). Property (53) has become the basis for several advanced models of plasticity, including models
with plastic anisotropy and models for shape memory alloys (cf. Müller and Bruhns (2006); Arghavani et al. (2011); Naghdabadi et al. (2013); Teeriaho (2013); Xiao (2013); Zhu et al. (2014)). For the discussion of this seminal idea in historical perspective the reader is referred to Xiao et al. (2006) and Zhilin et al. (2013). The closed form coordinate-free expression for the logarithmic spin tensor $\Omega^{\log}$ is given in Appendix A. The log-spin $\Omega^{\log}$ is a smooth function of the strain rate $D$, the continuum spin $W$ and the left Cauchy-Green tensor $B := FF^T$

$$\Omega^{\log} = \Omega^{\log}(D, W, B) \in \text{Skew}. \quad (54)$$

All arguments are essential in (54). Observe that the first two arguments are invariant under the reference change, but the left Cauchy-Green tensor $B$ is not. Let us show that the log-spin tensor depends on the choice of the reference configuration. In general, $\Omega^{\log}(D, W, B) \neq W$, but in the special case where $B$ is isotropic, the log-spin coincides with the continuum spin (cf. Appendix A):

$$\Omega^{\log}(D, W, B) = W, \text{ if } B = \beta 1, \beta > 0. \quad (55)$$

Let us consider a time instance $t'$ and a corresponding set $\{D, W, B\}$ such that $\Omega^{\log}(D, W, B) \neq W$. By choosing $F_0 := F(t')$, we arrive at $t = t'$

$$F^{\text{new}}(t') = F(t')(F(t'))^{-1} = (\det F(t'))^{1/3} 1, \quad B^{\text{new}}(t') = (\det F(t'))^{2/3} 1. \quad (56)$$

Thus, at $t = t'$, we have

$$(\Omega^{\log})^{\text{new}}(t') = W(t') \neq \Omega^{\log}(t'). \quad (57)$$

Therefore, the log-spin $\Omega^{\log}$ depends on the choice of the reference configuration. It follows from Theorem 1 that the model based on the logarithmic rate is not $w$-invariant.

Remark 5. The logarithmic spin meets invariance with respect to rigid rotation of the reference configuration. Indeed, for $F_0 = Q_0 \in \text{Orth}$, we get $B^{\text{new}} = B$. Therefore, $\Omega^{\log} \equiv \Omega^{\log}$ for $F_0 = Q_0 \in \text{Orth}$. Hence, initially isotropic materials can be modeled using this stress rate. Moreover, constitutive modeling of initially anisotropic materials is also possible, as discussed in Xiao et al. (2007).

4. Additive elasto-plasticity in the logarithmic strain space

Now we proceed to another popular class of constitutive equations. Following Papadopoulos and Lu (1998); Miehe et al. (2002); Schröder et al. (2002), we consider a framework which adopts the

\[\text{Remark 5.} \quad \text{Observe that } \ln V \text{ and } \Omega^{\log} \text{ are reference dependent. As it was pointed out by an anonymous reviewer, the reference dependencies counteract each other so that } (\ln V)^{\log} \text{ becomes reference independent, which is a noticeable property of the log-spin.} \]
structure of the the geometrically linear relations (1) – (6). In these relations, the linearized strain tensor \( \varepsilon(t) \) is formally replaced by the Lagrangian logarithmic strain (Hencky strain)

\[
H(t) := \frac{1}{2} \ln(C(t)).
\]

(58)

The resulting stress response \( \sigma \) is now understood as a certain Lagrangian stress measure which is power conjugate to the logarithmic strain \( H \)

\[
\sigma : \dot{H} = \tilde{T} : \left( \frac{1}{2} \dot{C} \right) \text{ for all } \dot{C} \in Sym.
\]

(59)

Thus, \( \sigma \) and \( H \) are dual variables. In order to obtain the relation between \( \sigma \) and the conventional stress measures we rewrite (59) in the form

\[
\sigma : \left( \frac{1}{2} \frac{\partial \ln(C)}{\partial C} : \dot{C} \right) = \tilde{T} : \dot{C} \text{ for all } \dot{C} \in Sym.
\]

(60)

Next, taking into account that the derivative \( \frac{\partial \ln(C)}{\partial C} \) is super-symmetric (cf. Itskov (2007)), we obtain

\[
\left( \frac{\partial \ln(C)}{\partial C} : \sigma \right) : \dot{C} = \tilde{T} : \dot{C} \text{ for all } \dot{C} \in Sym.
\]

(61)

Therefore,

\[
\tilde{T} = \frac{\partial \ln(C)}{\partial C} : \sigma, \quad T = \frac{1}{\det F} F \left( \frac{\partial \ln(C)}{\partial C} : \sigma \right) F^T.
\]

(62)

Following the terminology of Miehe et al. (2002), the system of constitutive equations is subdivided into three modules:

- equation (58) as a geometric preprocessor (\( C \mapsto H \))
- equations (1) – (6) as a constitutive model (\( H \equiv \varepsilon \mapsto \sigma \))
- equation (62) as a geometric postprocessor (\( \sigma \mapsto \tilde{T} \))

As an alternative to (58), a series of other generalized strain measures can be implemented. Here we concentrate on logarithmic strain for the following reason. As deduced by Itskov (2004), the logarithmic strain yields the most appropriate results in case of rigid-plastic material subjected to simple shear.

\[\text{Numerical procedures for the computation of conventional stresses basing on } \sigma \text{ were discussed, among others, by Plešek and Kruisová (2006) and Itskov (2007).}\]
Let us show that the corresponding constitutive relations are not w-invariant under the reference change. Toward that end, consider an example of simple shear given by
\[ F(t) = 1 + t \, e_2 \otimes e_3. \]  
(63)
For the model under consideration, the initial state is described by \( Z_0 = \{ \varepsilon^0_p \} \) (recall the initial condition (6)). To be definite, we assume zero stress at \( t = 0 \)
\[ T|_{t=0} = 0. \]  
(64)
Due to (1), (2), and (62), the initial condition (64) implies
\[ \sigma|_{t=0} = 0, \quad \varepsilon_p|_{t=0} = H|_{t=0} = 0. \]  
(65)
Next, consider a reference change which is described by \( F_0 = 1 + 2 \, e_1 \otimes e_3 \). The new load path is then given by
\[ F^{\text{new}}(t) = F(t) F_0^{-1}, \quad C^{\text{new}}(t) = F_0^{-T} C(t) F_0^{-1}, \quad H^{\text{new}}(t) = \frac{1}{2} \ln(C^{\text{new}}(t)). \]  
(66)
In order to obtain zero stress at \( t = 0 \), the following initial condition must be imposed on \( \varepsilon_p^{\text{new}} \)
\[ \varepsilon_p^{\text{new}}|_{t=0} = H^{\text{new}}|_{t=0}. \]  
(67)
Let the yield functions \( f \) and \( f^{\text{new}} \) correspond to processes (63), (65) and (66), (67) respectively. Synchronized yielding is a necessary condition of w-invariance, which is possible only if \( f(t) = f^{\text{new}}(t) \). Thus, the following condition is necessary for the w-invariance
\[ \| \sigma^D \| = \|(\sigma^{\text{new}})^D\|. \]  
(68)
The lack of the w-invariance becomes evident upon examination of the elasticity relations. Indeed, assuming that for small \( t \) the material remains in the elastic range, we have
\[ \varepsilon_p(t) = H|_{t=0} = 0, \quad \varepsilon_p^{\text{new}}(t) = H^{\text{new}}|_{t=0}. \]  
(69)
Combining this with (2), and taking into account that \( \text{tr}H = \text{tr}H^{\text{new}} = 0 \), the condition (68) is reduced to
\[ \| H(t) \| = \| H^{\text{new}}(t) - H^{\text{new}}(0) \|. \]  
(70)
As shown in Fig. 3, this necessary condition is violated. Since (68) does not hold true, the time instance of plastification depends on the choice of the reference configuration. Thus, the constitutive equations are not w-invariant under the reference change. Roughly speaking, the transformation of the reference cannot be neutralized by any transformation of the initial conditions.
5. Multiplicative hyperelasto-plasticity

Finally, let us analyse a model of hyperelastic-perfectly-plastic response, which is covered as a special case by the classical model of Simo (1992). The model is based on the multiplicative decomposition of the deformation gradient into the elastic part \( \hat{F}_e \) and the plastic part \( F_p \)

\[
\mathbf{F} = \hat{F}_e F_p. \tag{71}
\]

This decomposition was originally considered by Bilby et al. (1957) and Kröner (1958) for metallic materials\(^{13}\). It can be postulated as a phenomenological assumption or it can be derived from the concept of material isomorphism under some general assumptions (Bertram, 1998, 2003). The decomposition (71) gives rise to the weighted stress tensor operating on the intermediate configuration (also known as stress-free configuration)

\[
\hat{\mathbf{S}} := \hat{F}_e^{-1} \mathbf{S} \hat{F}_e^{-T}. \tag{72}
\]

Implementing this stress measure, hyperelastic relations are assumed in the form of

\[
\dot{\mathbf{S}} = 2 \rho_p \frac{\partial \psi(\hat{\mathbf{C}}_e)}{\partial \hat{\mathbf{C}}_e}, \tag{73}
\]

where \( \hat{\mathbf{C}}_e := \hat{F}_e^T \hat{F}_e \) stands for the elastic right Cauchy-Green tensor, \( \psi(\hat{\mathbf{C}}_e) \) is the free energy per unit mass and \( \rho_p \) denotes the mass density in the reference configuration. In what follows we suppose that the elastic potential \( \psi(\hat{\mathbf{C}}_e) \) is isotropic. Next, the plastic strain rate \( \dot{\mathbf{D}}_p \) is introduced:

\[
\dot{\mathbf{D}}_p := \text{sym}(\hat{F}_p F_p^{-1}). \tag{74}
\]

\(^{13}\) As discussed in Clayton et al. (2014); Le and Günther (2014), two different mechanical interpretations of terms “elastic” and “plastic” were suggested by Bilby and Kröner.
The yield function is formulated using the Kirchhoff stress tensor $\mathbf{S}$

$$f := \|\mathbf{S}^D\| - \sqrt{2/3}K,$$

just as was done within the additive hypoelasto-plasticity (cf. Section 3). Note that this yield function can be formulated in terms of the Mandel tensor, operating on the reference configuration:

$$f = \sqrt{\text{tr}[(\mathbf{C}\tilde{T})^D]^2} - \sqrt{2/3}K. \quad (76)$$

A six-dimensional flow rule is postulated on the intermediate configuration adopting the symmetric Mandel tensor $\mathbf{\hat{C}}^e\mathbf{\hat{S}}$

$$\mathbf{\dot{D}}_p = \lambda_p (\mathbf{\hat{C}}^e\mathbf{\hat{S}})^D, \quad (77)$$

in combination with the Kuhn-Tucker constraints

$$f \leq 0, \quad \lambda_p \geq 0, \quad f\lambda_p = 0. \quad (78)$$

In order to simplify the numerical treatment, the constitutive equations can be transformed into the reference configuration (cf. Lion (1997)). Introducing $\mathbf{C}_p := \mathbf{F}_p^T \mathbf{F}_p$, the strain energy function takes the form

$$\psi(\mathbf{\hat{C}}^e) = \psi(\mathbf{C}_p^{-1}). \quad (79)$$

The system of constitutive equations and initial conditions is now written on the reference configuration as

$$\mathbf{\tilde{T}} = 2\rho_R \frac{\partial \psi(\mathbf{C}_p^{-1})}{\partial \mathbf{C}}|_{\mathbf{C}_p=\text{const}}, \quad \mathbf{\dot{C}}_p = 2\lambda_p (\mathbf{C}\tilde{T})^D \mathbf{C}_p, \quad \mathbf{C}_p|_{t=t_0} = \mathbf{C}_0^p. \quad (80)$$

The constitutive relations (76), (78), (80) are w-invariant under arbitrary isochoric reference change.

**Proof:** In order to prove the w-invariance, we modify the proof presented in Section 1 (cf. equations (9)–(13)). Let $\mathbf{C}(t)$ be a given local loading process. Consider arbitrary $\mathbf{F}_0 = \text{const}$, such that $\det(\mathbf{F}_0) = 1$. Since the reference change (15) is volume preserving, the referential mass density remains invariant

$$\rho_{R_{\text{new}}} = \rho_{R}. \quad (81)$$

For the analysed material model, the set of initial data which appears in (23) is given by $\mathcal{Z}_0 = \{\mathbf{C}_p^0\}$. The initial conditions (80) are transformed according to

$$\mathbf{C}_p^{\text{new}}|_{t=t_0} = \mathbf{C}_p^{\text{new} 0}, \quad \mathbf{C}_p^{\text{new} 0} := \mathbf{F}_0^{-T} \mathbf{C}_p^0 \mathbf{F}_0^{-1}. \quad (82)$$

14In the particular case of Neo-Hookean strain energy function, an efficient and robust numerical procedure can be developed using an explicit update formula presented by Shutov et al. (2013).
We need to check that the weak invariance relation (24) holds true, where the new loading is now given by
\[ C_{\text{new}}(t) := F_0^{-T}C(t)F_0^{-1}. \] (83)

First, we introduce the candidate function
\[ C_p^{\text{new}}(t) := F_0^{-T}C_p(t)F_0^{-1}. \] (84)

Obviously, the initial conditions (82) are satisfied by this candidate. In the following we will show that this candidate function represents the solution for the new loading (83). Toward that end, we note that the potential relation (80) implies the following representation of the Mandel stress tensor on the reference configuration
\[ C\tilde{T} = g(CC_p^{-1}), \] (85)

where \( g \) is a suitable isotropic function. Relying on functions \( C_{\text{new}}(t) \) and \( C_p^{\text{new}}(t) \) we introduce formally the quantity \( \tilde{T}_{\text{new}}(t) \)
\[ \tilde{T}_{\text{new}}^{\text{new}} = 2\rho \frac{\partial \psi(C_{\text{new}}^{\text{new}} - 1)}{\partial C_{\text{new}}^{\text{new}}} \bigg|_{C_p^{\text{new}}=\text{const}}. \] (86)

In analogy to (85), we have now
\[ C_{\text{new}}^{\text{new}}\tilde{T}_{\text{new}}^{\text{new}} = g(C_{\text{new}}^{\text{new}}(C_p^{\text{new}})^{-1}). \] (87)

Since \( g \) is isotropic, \( F_0^{-T}g(CC_p^{-1})F_0^T = g(F_0^{-T}CC_p^{-1}F_0^T) \). Combining this with (83), (84), and (87), for the deviatoric part of the Mandel stress we arrive at
\[ F_0^{-T}(C\tilde{T})^D F_0^T = (C_{\text{new}}^{\text{new}}\tilde{T}_{\text{new}})^D. \] (88)

Combining this with (76) it can be shown that the yield function remains invariant
\[ f^{\text{new}} := \sqrt{\text{tr}[(C_{\text{new}}^{\text{new}}\tilde{T}_{\text{new}})^D]^2} - \sqrt{2/3K}, \quad f^{\text{new}}(t) \equiv f(t). \] (89)

Thus, the Kuhn-Tucker conditions (78) are identically satisfied if we put
\[ \lambda_p^{\text{new}}(t) := \lambda_p(t). \] (90)

Now, differentiating (84) with respect to \( t \), we have:
\[ \dot{C}_p^{\text{new}}(t) = F_0^{-T}\dot{C}_p(t)F_0^{-1} \equiv 2\lambda_p F_0^{-T}(C\tilde{T})^D F_0^T F_0^{-T}C_p F_0^{-1}. \] (91)
Substituting (88), (84), and (90) into this relation, we obtain

\[
\dot{C}_{p}^{\text{new}}(t) = 2\lambda_{p}^{\text{new}} \left( C_{p}^{\text{new}} \tilde{T}_{\text{new}} \right)^{D} C_{p}^{\text{new}}.
\] (92)

Thus, the candidate function \(C_{p}^{\text{new}}(t)\) is indeed a solution of the problem (76), (78), (80) where all “old” quantities are formally replaced by their new counterparts. It follows from (88) that

\[
F_{0}^{-T}(C\tilde{T})F_{0}^{T} = C_{\text{new}}\tilde{T}_{\text{new}}.
\] (93)

After some simple computations, one gets from (93) the desired weak invariance relation (cf. (24))

\[
\tilde{T} = F_{0}^{-1}\tilde{T}_{\text{new}}F_{0}^{-T}.
\] (94)

Thus, the system (76), (78), (80) is w-invariant under arbitrary isochoric reference change. ■

Due to the w-invariance, there is no preferred reference configuration for this model: any reference can be used if the mass density is preserved (cf. (81)). An arbitrary volume-preserving change of the reference implies merely a certain transformation of initial conditions according to (82). Thus, at the stage of constitutive modeling, there is no need to specify the reference explicitly.

Remark 6. Let us consider an isotropic hyperelastic material in the popular form

\[
\tilde{T} = 2\rho_{k} \frac{\partial \psi(C)}{\partial C}.
\] (95)

Roughly speaking, this type of material is covered by the model of Simo (1992) as a special case. Indeed, for the large yield stress (as \(K \to \infty\)) the plastic flow is frozen (\(C_{p} = \text{const}\)) and (80) is reduced to (95) by eliminating \(C_{p}\). Observe that the hyperelastic relation (95) is w-invariant under arbitrary isochoric reference change only if it describes an elastic fluid, namely

\[
\psi(C) = \psi_{\text{vol}}(\det C).
\] (96)

On the other hand, the model of Simo (1992) is w-invariant even for general (isotropic) free energy \(\psi\). This seeming contradiction is resolved by recognizing that a w-invariant material model of hyperelasticity is given by (80) and not by (95). In contrast to (95), the w-invariant relation (80) includes a tensor-valued internal constant \(C_{p}\).

A different performance of relations (80) and (95) regarding the w-invariance reveals the following fundamental difference between these two types of hyperelasticity. In the case of relation (95), the function \(\psi\) contains information about the stress-free configuration and this configuration must be known a priori. No information about the stress-free configuration is contained in \(\psi\) which appears in (80). Since the stress-free configuration does not have to
be known during construction of the free energy function, the relations of \((80)\) are useful for applications dealing with pre-stressed hyperelastic structures.

**Remark 7.** For the material model under consideration, it follows from \((83)\) and \((84)\) that

\[
\psi(C C_p^{-1}) = \psi(C^{\text{new}} C_p^{\text{new}^{-1}}). \tag{97}
\]

If \(C_p\) is understood as a certain referential metric tensor (which is not necessary related to plastic strain), then the invariance property \((97)\) corresponds to the so-called material covariance of the strain energy (cf. Section 3.4 in Marsden and Hughes (1983) or equation (3.9) in Lu and Papadopoulos (2004)).

**Remark 8.** Note that the simplifying assumptions of isotropic elasticity \((73)\) and isotropic yield function \((75)\) made in this section are not crucial for the \(w\)-invariance. The \(w\)-invariance can be shown even for a broad class of anisotropic material models, based on a double split of the deformation gradient (Lion, 2000; Helm, 2001; Tsakmakis and Willuweit, 2004; Dettmer and Reese, 2004; Henann and Anand, 2009). In particular, dealing with a model of multiplicative viscoplasticity with nonlinear kinematic hardening (Shutov and Kreißig, 2008), the \(w\)-invariance was proved by Shutov et al. (2012).

It would be wrong to assume that all plasticity models with the multiplicative split \((71)\) are automatically \(w\)-invariant. In particular, some models of additive hypoelasto-plasticity with the logarithmic rate can be rendered in the multiplicative format (Xiao et al., 2000b). As we already know from Section 3, such models are not \(w\)-invariant.

### 6. Discussion and conclusion

Material properties such as “isotropic”, “fluid”, “solid” etc. are defined by certain (strong) invariance requirements imposed upon the constitutive relations (Truesdell and Noll, 1965). In this work, a new notion of weak invariance under the reference change is introduced, which involves a proper transformation of the initial conditions and leaves the remaining constitutive equations unchanged. Just like the material symmetries are characterized by the strong invariance under a certain group of transformations of the reference, the notion of \(w\)-invariance yields new material symmetries in a broader sense. Especial usefulness of the \(w\)-invariance lies in its “weakness” since it can be imposed on solids even under general isochoric transformations. A meaningful classification of elasto-plastic models is obtained using this notion. In particular, the \(w\)-invariance can be implemented as a tool for a systematic study of different approaches to finite strain elasto-plasticity.

Obviously, similar ideas can be implemented within viscoplasticity and viscoelasticity. The concept of \(w\)-invariance should be compatible with any internal constraints imposed on the material. In the current study, the incompressibility condition \(\det(F_0) = 1\) stems naturally from the incompressibility of the plastic flow. In that way, the considerations are restricted to
the group of unimodular transformations. For materials with additional internal constraints, a proper subgroup of the unimodular group can be considered.

No preferred reference configuration exists for weakly invariant material models. If a certain model does not fulfill the weak invariance, the concrete choice of the reference configuration should be explicitly specified at the stage of constitutive modeling in order to avoid ambiguity.

Apart from these theoretical aspects, there is a clear impact of the w-invariance on the practical applicability of material models. The w-invariance is useful for the simulation of multi-stage forming processes, since each intermediate state can be considered as a new reference configuration (cf. Shutov et al. (2012)). Another important aspect is that the deformation prehistory of the as-received metallic material is often unknown. As a result, the as-received state is typically associated with a new reference. Such lax treatment of the reference configuration is justified for w-invariant models.

The notion of the w-invariance can be used to simplify the mathematical analysis of constitutive equations (cf. Shutov and Kreißig (2010); Shutov et al. (2011)). Furthermore, any time discretization technique should preserve the material symmetries of the constitutive relations. For the same reasons, the numerical methods should retain the w-invariance since it represents a generalized material symmetry. Some advantages of the numerical algorithms which exactly preserve the w-invariance were discussed by Shutov et al. (2013) in the specific context of multiplicative inelasticity.

The w-invariance of constitutive equations should not be confused with the concept of material isomorphism (Bertram, 1998, 2003). The later represents an axiomatic approach to material modeling, based on the assumption that the elastic properties remain constant. The w-invariance is a generalized symmetry restriction which does not necessarily imply constant elastic properties.

Note that only some basic approaches to elasto-plasticity were tested in this study using the simple case of elastic-perfectly-plastic material. This test is sufficient as it includes the relevant effects of the geometric nonlinearity.

In the field of additive hypoelasto-plasticity, two basic restrictions are commonly adopted: the consistency criterion of Prager (1st consistency condition) and the need to avoid any energy dissipation in the elastic range (2nd consistency condition). Recall that the constitutive relations based on the logarithmic stress rate are the only option if both consistency conditions have to be satisfied (the case of constant elastic moduli is considered here). Unfortunately, as it was shown in Section 3, models based on the logarithmic rate are not w-invariant. Thus, within the additive hypoelasto-plasticity, it is impossible to combine the mentioned consistency conditions and the w-invariance under arbitrary isochoric reference change. Another modeling framework

\[15\] On the other hand, for some applications, the incompressibility assumption can be dropped. For instance, a purely volumetric reference change with isotropic \( \mathbf{F}_0 \) was considered by Landgraf et al. (2014).
relies on geometrically linear equations of elasto-plasticity in combination with the Lagrangian Hencky strain. An essential advantage of this framework lies in its simplicity. This setting is free from the non-physical energy dissipation in the case of frozen plastic flow. Unfortunately, as shown in Section 4, the material model is not w-invariant. Finally, it was shown in Section 5 that the framework of multiplicative hyperelasto-plasticity can be formulated in a consistent manner such that any isochoric change of the reference configuration would merely imply a transformation of the initial conditions. Table 1 is convenient to summarize the main results of the current study.

| Type of model                          | Energy dissipation in elastic range | Shear oscillation | w-invariance under reference change |
|----------------------------------------|-------------------------------------|-------------------|-------------------------------------|
| Additive hypoelasto-plasticity:        |                                     |                   |                                     |
| Zaremba-Jaumann rate                   | dissipation                         | oscillation       | w-invariant                         |
| Green-Naghdi rate                      | dissipation                         | no oscillation    | not w-invariant                     |
| Logarithmic rate                       | no dissipation (for constant elastic moduli) | no oscillation | not w-invariant                     |
| Add. plast. with log. strain           | no dissipation                      | no oscillation    | not w-invariant                     |
| Multiplicative plasticity             | no dissipation                      | no oscillation    | w-invariant                         |

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**Appendix A**

Let $b_1$, $b_2$, $b_3$ be the eigenvalues of $B$. We recall the closed form coordinate-free expression for the log-spin $\Omega^\log$ (see, for instance, Xiao et al. (1997))

$$\Omega^\log = W + N^\log,$$

$$N^\log = \begin{cases} 0 & \text{if } b_1 = b_2 = b_3 \\ 2 \text{skew}(\nu \ B \ D) & \text{if } b_1 \ne b_2 = b_3 \\ 2 \text{skew}(\nu_1 \ B \ D + \nu_2 \ B^2 \ D + \nu_3 \ B^3 \ D B) & \text{if } b_1 \ne b_2 \ne b_3 = b_1 \end{cases},$$

$$\nu = \frac{1}{b_1 - b_2} \left( \frac{1 + b_1/b_2}{1 - b_1/b_2} + \frac{2}{\ln(b_1/b_2)} \right),$$
\[
\nu_j = \frac{1}{\Delta} \sum_{i=1}^{3} (-b_i)^{3-j} \left( \frac{1+\varepsilon_i}{1-\varepsilon_i} + \frac{2}{\ln(\varepsilon_i)} \right), \quad j = 1, 2, 3, \\
\Delta = (b_1 - b_2)(b_2 - b_3)(b_3 - b_1), \quad \varepsilon_1 = b_2/b_3, \quad \varepsilon_2 = b_3/b_1, \quad \varepsilon_3 = b_1/b_2.
\]

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