Symplectic and orthogonal Lie algebra technology for bosonic and fermionic oscillator models of integrable systems

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Abstract

To provide tools, especially $L$-operators, for use in studies of rational Yang-Baxter algebras and quantum integrable models when the Lie algebras $so(N)$ ($b_n$, $d_n$) or $sp(2n)$ ($c_n$) are the invariance algebras of their $R$ matrices, this paper develops a presentation of these Lie algebras convenient for the context, and derives many properties of the matrices of their defining representations and of the ad-invariant tensors that enter their multiplication laws. Metaplectic-type representations of $sp(2n)$ and $so(N)$ on bosonic and on fermionic Fock spaces respectively are constructed. Concise general expressions (see (5.2) and (5.5) below) for their $L$-operators are obtained, and used to derive simple formulas for the $T$ operators of the rational $RTT$ algebra of the associated integral systems, thereby enabling their efficient treatment by means of the algebraic Bethe ansatz.

1 Introduction

Let the compact real simple Lie algebra $\mathfrak{g}$ with generators $X_i$ and totally antisymmetric structure constants be defined by

$$[X_i, X_j] = \iota c_{ijk} X_k.$$  \hfill (1.1)

Let $V$ denote the defining representation of $\mathfrak{g}$ with matrices $x_i$ given by $X_i \mapsto \gamma x_i$, where $\gamma = 1$ for all series of simple Lie algebras except $a_n = su(n+1)$, for which $\gamma = \frac{1}{2}$. We chose the basis so that

$$x_i^\dagger = x_i, \quad \text{Tr} x_i = 0, \quad \text{Tr} x_i x_j = 2\delta_{ij}. \hfill (1.2)$$

We define also the $L$-matrix by

$$L = x_i \otimes X_i \equiv x_i X_i \hfill (1.3)$$
acting on $V \otimes H$ where $V$ is the defining representation of $g$ and $H$ any other representation. This matrix is an object of major importance for work on integrable systems defined with the aid of the rational Yang Baxter algebra on $V \otimes V \otimes H$

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v) \quad , \quad (1.4)$$

for which $g$ is the Lie algebra under which the corresponding rational $R$ matrix is invariant. Here the subscripts 1 and 2 refer to the auxiliary vector spaces $V$, and the $T_i(u)$ act on $V \otimes H$ where $H$ is the quantum space. The $T_i(u)$ are defined in terms of the $X_i$, and may be used to provide the observables of the integrable system under study. In fact, one uses $L$ in the construction of one-site solutions of $T(u)$ of $[12]$, and then follows the procedure laid down by Sklyanin $[1]$, to build $N$-site integrable models of spin chain type based on the Lie algebra $g$. In the study of integrable quantum mechanical models, it is widely accepted that much deeper insight into their structures − why things work as they do − is attained by generalising from some simple basic model, in which usually $g = su(2)$, to its counterpart for general $g$. Generalisation from $a_1 = su(2)$ to $a_{n-1} = su(n)$ is normally the easiest to achieve, and in part this is due to the fact that much more detailed and systematic knowledge is available in this case than is conveniently at hand for $c_n = sp(2n), b_n = so(2n+1)$ and $d_n = so(2n)$. The present paper originated in the study $[2]$ of integrable systems in which the quantum space is the representation space of the metaplectic representation $[3,4]$ of $c_n$ for which there is an elegant realisation in terms of $n$ pairs of (bosonic) oscillator variables. The system in question derives its interest in large part from the fact that it is one for which the determination of its physical eigenstates can be carried out, for $g$ ‘larger’ than $a_1$, by an unnested application of the algebraic Bethe ansatz procedure $[3,4]$. (Cf. the treatment of $[5,6]$ with that of $[5]$ for $g = su(3)$, or of $[3]$ and $[4]$, which give rise to nested Bethe equations for the determination of the physical eigenstates of the models studied.) Now the metaplectic realisation of $c_n$ in terms of bosonic oscillators has a close analogue at a fairly deep structural level in which the spinor representations of $d_n = so(2n)$ are realised in terms of $n$ pairs of Dirac fermionic operators. Just as the Hilbert space of the metaplectic representation of $c_n$ consists of two irreducible subspaces of even and odd states, so also does the fermionic construction give rise to the two spinor representations of $d_n$ of opposite chirality, carried by the subspaces of even and odd total fermion number. A deep treatment of this situation is contained in $[13]$. In general what can be done for one of the two types of theories admits a parallel treatment in the other. It is perhaps worth remarking that we here use the word metaplectic in the way that is usual in theoretical physics, primarily in relation to the $c_n$ context. It is used more generally in the mathematical literature, as can be traced from $[11]$. Having discovered the relevance of the fermionic realisation of $d_n$ in the context of integrable systems, it was natural to consider the extension of our results to $b_n$. This is easily achieved by adjoining a Majorana fermion to the variables of the realisation of $d_n$. However the need to take account of this renders awkward the Fock space treatment of the spinor representation of $b_n$ that has been realised, making related models less favourable candidates for integrable system study. In this context, we observe that in the $c_n$ and $d_n$ examples, there is an obvious relationship between the models we build and the Lie algebra decompositions associated with the hermitian symmetric spaces $[12]$ $Sp(2n, \mathbb{R})/U(n)$ and $SO(2n)/U(n)$. There is no such relationship in the case of our $b_n = so(2n+1)$ examples.
To account for our interest in the two ‘metaplectic’ type theories, we remark that they are very favourably placed amongst the larger family of algebraic systems (see [13] for a review and [14] for an interesting example) that have nice descriptions in terms of oscillators. At the simplest level of construction, the ansatz

$$X_i = \gamma A^\dagger x_i A,$$  \hspace{1cm} (1.5)

produces an operator realisation of \(\mathfrak{g}\) whenever the column vector \(A\) has either bosonic or fermionic creation operators as its set of \(\text{dim} \mathcal{V}\) components. Proof in each case is elementary.

Roughly speaking, the two types of model that we are focusing our attention upon use only half as many pairs as bosonic or fermionic variables as eq. (1.5) does. To see this it is simplest to look ahead to eq. (1.2) for \(c_n\) and (4.8) for \(d_n\) where a single vector corresponding to \(A\) contains all creation and annihilation operators of the system. The advantages are not only that we have smaller Fock spaces to deal with and thus a gain in tractability, but also that specially simple properties of the operators \(L\) and \(T(u)\) come into play, which allow considerable simplifications. In particular, we obtain strikingly concise general formulas, valid for all \(n\), for the \(L\)-operators, see (5.2) for \(c_n\) and (5.3) for \(d_n\) below. These expressions make it very easy to show that \(L\) then obeys a quadratic equation. It is this property which makes the solution \(T(u)\), displayed below as eq. (5.16) for a one-site solution of eq. (1.4), take on its particularly simple form. Another construction which, when it is compared to (1.5), in some sense shares the economy of the models we are talking about now, uses a vector of hermitian Majorana fermions. For one example of this, see [15]; for a more recent one, see [16]. Indeed, if the fermionic models of Sec. 4.2 are formulated in terms of Majorana fermions, their relationship to models like those of [15] and [16] can be seen to be quite close.

Our continuing studies of these problems have called for the systematic assembly of information about \(c_n = sp(2n)\) and \(so(N)\) \((b_n\) and \(d_n)\). One reason why the work described here is claimed to be novel concerns the description we give of the matrices \(x_i\) of the defining representations \(\mathcal{V}\) of these algebras. We have found it best to use a set labelled by a single index, rather than the more common alternative in which one uses, for example for \(so(N)\), a basis in \(\mathcal{V}\) given by \(x_{ij} = -x_{ji} = (E_{ij} - E_{ji})/2\), where \(E_{ij}\) has as its only non-zero element a one where row \(i\) meets column \(j\). Our choice arises mainly because the physical description of the models we are interested in favours a Cartan-Weyl basis for \(\mathfrak{g}\), and the two-index labelled basis for \(\mathfrak{g}\) does not relate very well to this. One consequence of our choice of basis is that it brings into the formalism ad-invariant analogues of the \(f\) and \(d\) tensors of \(a_{n-1} = \mathfrak{su}(n)\). One of the benefits is seen in the ease with which one accumulates all the necessary data about the Lie algebras, representations of them including the ones important to our programme, and the associated \(L\) matrices.

The paper is organised as follows. Sec. 2.1 contains all the necessary information about the description of the Lie algebras \(c_n\), \(b_n\) and \(d_n\) in Cartan-Weyl form. It is presented so that as much of the data as possible is encapsulated in the \(L\) matrix of the Lie algebra. It turns out that we can present the matrix \(L\) in an essentially standard form whose main features are determined by the way the positive roots are related to the simple ones: Sec. 2.1 aims at explaining this carefully. One says ‘essentially’ because the pattern observed for \(c_n\) and \(b_n\), and available (and very simply) also for \(a_n\), and even \(g_2\), is not visible immediately in the corresponding \(L\) for \(d_n\). The features of \(d_n\) which obscure the simple pattern seen in other cases are associated with the fishtail extremity of its Dynkin diagram. However, by viewing the two equivalent fishtail-roots of the diagram suitably, the \(d_n\) pattern can be made to conform satisfactorily to that enjoyed by the other Lie algebras studied here.
Sec. 3 derives a compendium of results involving the matrices \( x_i \) of \( \mathfrak{g} \) for \( c_n \) and, in unified form, for \( so(N) \), certain associated matrices \( y_{\alpha} \), and the various ad-invariant tensors that enter naturally into the product laws of these matrices. The matrices \( y_{\alpha} \) enter necessarily alongside the \( x_i \) to complete the basis of the space of all traceless \( 2n \times 2n \) hermitian matrices for \( c_n \) and \( N \times N \) for \( so(N) \). The treatment perhaps goes further than the needs of this paper make apparent. Even so it is certainly not comprehensive, although it should indicate viable methods of approaching further identities.

Sec. 4 describes the bosonic oscillator description of the metaplectic representation of \( c_n \), and compares and contrasts this with the fermionic analogues for \( d_n \) and \( b_n \), giving also some discussion of all the corresponding Fock spaces.

Sec. 5 discusses various matters associated with the fact that the \( L \)-operators dealt with in this paper obey quadratic equations. It begins by displaying concise and tractable formulas for the \( L \) operators of each of the oscillator systems of Sec. 4, and thence derives the important quadratic equations satisfied by these \( L \)-operators, mentioning also some \( a_n \) examples based on (1.3) for the purposes of comparison. Also it is shown how the equations for \( L = x_iX_i \) can be used to provide product laws for the matrices/operators \( X_i \) involved in \( L \). There are new results here. They are simple because \( L \) obeys a quadratic equation. But one could in principle seek similar results – for \( X_iX_jX_k \) – whenever \( L \) obeys a cubic equation, and beyond. Also the practical relevance to the solution of the corresponding \( RTT \) equations (1.4) of the fact that \( L \)-operators obey quadratic equations is indicated briefly. Further, the correlation between that property of such an \( L \)-operator acting in a space \( V \otimes \mathcal{H} \) (cf. (1.3) ), and the fact that this space reduces, under the action of the relevant invariance algebra \( \mathfrak{g} \), to a direct sum of exactly two subspaces, in each of which the quadratic Casimir \( C_{V \otimes \mathcal{H}}^{(2)} \) of \( \mathfrak{g} \) takes on a fixed value, is noted, and a survey is given of the situations in which such a reduction takes place.

2 Cartan-Weyl form of the defining representations

We present the Lie algebra \( \mathfrak{g} \) of rank \( n \) and \( \text{dim} \ \mathfrak{g} \) in Cartan-Weyl form with generators \( H = (H_1, \ldots, H_n) \) of its Cartan subalgebra, positive roots \( r_{\alpha} \), and raising and lowering operators \( E_{\pm\alpha}, \alpha \in \{1, 2, \ldots, \frac{1}{2}(\text{dim} \ \mathfrak{g} - n)\} \). Then we have

\[
[H, E_{\pm\alpha}] = \pm r_{\alpha} E_{\pm\alpha}, \quad [E_{\alpha}, E_{-\alpha}] = r_{\alpha} \cdot H,
\]

together with well-known non-trivial expressions for \( [E_{\pm\alpha}, E_{\pm\beta}] \) whenever \( r_{\alpha} \pm r_{\beta} \) is a non-zero root of \( \mathfrak{g} \). The \( X_i \) of (1.3) are related to the Cartan-Weyl generators according to

\[
\{X_i; i \in \{1, \ldots, \text{dim} \ \mathfrak{g}\}\} = \{H_r; r \in \{1, \ldots, n\}\} \cup \{U_{\alpha}, V_{\alpha}; \alpha \in \{1, \ldots, \frac{1}{2}(\text{dim} \ \mathfrak{g} - n)\}\},
\]

where \( \sqrt{2}E_{\pm\alpha} = U_{\alpha} \pm iV_{\alpha} \). For the defining representation \( V \) of \( \mathfrak{g} \), we have \( X_i \mapsto \gamma x_i \) with \( \gamma = 1 \), (except for \( a_n \) when \( \gamma = 1/2 \)) so that we also employ the notation

\[
H \mapsto h, \quad E_{\pm\alpha} \mapsto e_{\pm\alpha}, \quad \sqrt{2}e_{\pm\alpha} = u_{\alpha} \pm iv_{\alpha}.
\]

Further, for compact real \( \mathfrak{g} \), we have \( x_i^\dagger = x_i \) for all \( i \), and hence \( h^\dagger = h \), \( e_{+\alpha}^\dagger = e_{-\alpha} \), but no such result will hold for a non-compact representation of \( \mathfrak{g} \). We now turn case by case to the
definition of $L$-matrices for the Lie algebras $\mathfrak{g}$ of immediate interest, intending to comment below on the nearly standard form of the results displayed.

General references for background on Lie algebras and their representations are [17–19]. Another source of valuable information is [20].

2.1 The Lie algebras $c_n$

We begin with the case of $c_2 = sp(4)$, but everything generalises naturally for $c_n = sp(2n)$. For $c_2$, the simple roots are $\mathbf{r}_1 = (1, -1)$ and $\mathbf{r}_2 = (0, 2)$ (see [17]: p64). The remaining positive roots are $\mathbf{r}_3 = \mathbf{r}_1 + \mathbf{r}_2 \equiv \mathbf{r}_{12} = (1, 1)$ and $\mathbf{r}_4 = \mathbf{r}_1 + \mathbf{r}_3 = 2\mathbf{r}_1 + \mathbf{r}_2 \equiv \mathbf{r}_{112} = (2, 0)$. Hence, in addition to (2.1), we have

$$[E_1, E_2] = \sqrt{2}E_3 \quad , \quad [E_1, E_3] = \sqrt{2}E_4 \quad ,$$

(2.4)

together with results obtained from these in a well-known manner. In this basis, $L$ from eq. (1.3) is of the form

$$L = h_1 H_1 + h_2 H_2 + \sum_{\alpha=1}^{4} (e_{\alpha} E_{-\alpha} + e_{-\alpha} E_{\alpha}) .$$

(2.5)

If we write the left tensor factor as matrices and the right one as operators, so that $L$ reads

$$
\begin{pmatrix}
H_1 & E_{-1} & E_{-3} & \sqrt{2}E_{-4} \\
E_{-1} & H_2 & \sqrt{2}E_{-2} & E_{-3} \\
E_{-3} & \sqrt{2}E_{-2} & -H_2 & -E_{-1} \\
\sqrt{2}E_{-4} & E_{-3} & -E_{-1} & -H_1
\end{pmatrix},
$$

(2.6)

then one can read the explicit forms of the matrices $h_1$, $h_2$ and $e_{\pm \alpha}$ off (2.6) and verify that they do indeed provide a representation of the Lie algebra relations of $c_2$, obtained from (2.1) and (2.4). Furthermore, and in part because of the $\sqrt{2}$ factors (which multiply the raising generators which belong to the long roots of $c_2$), the $x_i$ clearly obey (1.2), and also

$$x_i^T = -Jx_iJ^{-1} ,$$

(2.7)

where $J$ is the standard symplectic form, which in our representation (and in [21]) is given by

$$
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix} .
$$

(2.8)

We turn next to $c_3$. Here the symplectic form $J$ has, as its only non-zero entries, $(1, 1, 1, -1, -1, -1)$ down the main antidiagonal. The simple roots of $c_3$, as given $\in \mathbb{R}^3$ ([17]: p64) are

$$\mathbf{r}_1 = (1, -1, 0) , \quad \mathbf{r}_2 = (0, 1, -1) , \quad \mathbf{r}_3 = (0, 0, 2) ,$$

(2.9)

and other positive roots are given by

$$\mathbf{r}_4 = \mathbf{r}_1 + \mathbf{r}_2 \equiv \mathbf{r}_{12} , \quad \mathbf{r}_5 = \mathbf{r}_2 + \mathbf{r}_3 \equiv \mathbf{r}_{23} , \quad \mathbf{r}_6 = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 \equiv \mathbf{r}_{123} ,$$

$$\mathbf{r}_7 = 2\mathbf{r}_2 + \mathbf{r}_3 \equiv \mathbf{r}_{223} , \quad \mathbf{r}_8 = \mathbf{r}_1 + 2\mathbf{r}_2 + \mathbf{r}_3 \equiv \mathbf{r}_{1223} , \quad \mathbf{r}_9 = 2\mathbf{r}_1 + 2\mathbf{r}_2 + \mathbf{r}_3 \equiv \mathbf{r}_{1123} .$$

(2.10)
Given these the requisite form of $L$ for $c_3$ can directly be inferred to be

\[
\begin{pmatrix}
  H_1 & & & & & \\
  E_1 & H_2 & & & & \\
  E_{12} & E_2 & H_3 & & & \\
  E_{123} & E_{23} & \sqrt{2}E_3 & -H_3 & & \\
  \sqrt{2}E_{1223} & E_{1223} & E_{123} & -E_{12} & -E_1 & -H_1 \\
\end{pmatrix}
\]  \quad (2.11)

For clarity, we left out the entries involving the $E_{-\alpha}$ that should occupy the places above the main diagonal of (2.11). We have moreover employed a notation that at first glance might seem unduly clumsy in order to make evident some features of our construction of $L$ that we return to below. We can however immediately read off (2.11) explicit expressions for the $6 \times 6$ matrices of the defining representation $V$ of $c_3$, namely $h_1, h_2, h_3$ and $e_{\pm \alpha}$ for $\alpha \in \{1, \ldots, 9\}$, or, assigned in accordance with (2.9) and (2.10), for

\[
\alpha \in \{1, 2, 3, 12, 23, 123, 223, 1223, 11223\} \quad (2.12)
\]

Then the Lie algebra of $c_3$ can be calculated from the matrices of $V$, and seen to be of Cartan-Weyl form with roots given correctly by (2.9) and (2.10). In other words $L$ as given by (2.11) is the Cartan-Weyl form of $L = x_i X_i$, wherein the matrices $x_i$ of $V$ obey the same algebra (1.1) as the abstract generators $X_i$.

We wish next to state in full all the steps of the recipe that enables the above construction to be applied to $c_n$ of arbitrary rank $n$, and beyond. Consider first the placement of the $E_{\pm \alpha}$ in (2.11) in relation to the root system of $c_3$. It can be seen to follow a systematic pattern, that will allow the exact form of $L$ to be written down directly for arbitrary $c_n$. The form of $L$ displayed above for $c_2$ in (2.6) of course conforms to the same pattern although we did not stop there to point this out. The patterns are easiest to see if one writes (2.6) schematically, displaying, at the site occupied by a given raising generator, only the corresponding root label

\[
\begin{pmatrix}
  x \\
  1 \\
  12 \\
  x \\
\end{pmatrix},
\]  \quad (2.13)

and similarly (cf. (2.11)) for $c_3$

\[
\begin{pmatrix}
  x \\
  1 \\
  12 \\
  23 \\
\end{pmatrix}.
\]  \quad (2.14)

Since inspection of these displays surely makes plain their salient features, we now state our recipe in full. First a natural choice of Cartan subalgebra generators is made and they are placed down the main diagonal. Second the raising generators corresponding to the simple roots are placed down the first sub-diagonal. The symmetry present after performing the first step dictates how to perform the second one. Third, the positive root sums determine the rest.
of the assignments according to the uniform pattern implicit in (2.13) and (2.14). Explicitly then consider e.g. the entry to (2.11) that sits in the place $L_{51}$, and look at the portion of the first sub-diagonal subtended by $L_{51}$. We find there, occupying the places $L_{54}, L_{43}, L_{32}, L_{21}$, the entries $-E_2, \sqrt{2}E_3, E_2, E_1$. Disregarding the minus sign and the root-two, these entries correspond to simple roots whose sum is $r_{1223}$, and we therefore associate $E_{1223}$ with the place $L_{51}$, so that the stage summarised in (2.14) is reached. Fourth, while we have disregarded the signs of elements $L_{ij}$ for the identification of what raising operators to assign to places in $L$ below the first sub-diagonal, these signs have to be supplied. This is simple to do because the pattern of signs in $L$ possesses an evident and uniform pattern, trivial here and similarly for the case of $b_n$ treated below. These patterns stem directly from our use of Racah metric forms, like (2.8) here and (2.17) below for $b_n$, in determining transposition properties, (2.7) here and (2.16) below for $b_n$ and $d_n$, of the matrices $x_i$ of $V$. Fifth, factors $\sqrt{2}$ are inserted as needed in order to normalise the matrices $x_i$ so that $\text{Tr} x_i x_j = 2 \delta_{ij}$. It is easy to check the trace and transposition properties of the matrices of $V$ read off displays like (2.13) by considering the matrix $A = a_i x_i$ for $a_i \in \mathbb{R}$ for which $\text{Tr} A^2 = a_i a_i$ and $A^T = -JAJ^{-1}$.

A state of affairs analogous to what has just been described is observed, below straightforwardly for $b_n$ and (in a considerably more subtle form) for $d_n$, and found elsewhere [22] also for $g_2$.

Our purposes here do not require us to address questions regarding the $a_n$ family, but it is easy to present $L$ for this family in a way that embodies very simply the features in focus just now. See the book [23] which uses matrices like $L$ in a fashion that is quite similar to what is done here. Similarly see [11]. We turn next to the case of $b_n$.

### 2.2 The Lie algebras $b_n$

In this case, $b_n = so(2n + 1)$, it is convenient to present results for $n = 2$. Despite the fact that $b_2 \cong c_2$, the results for $n = 2$ do make completely evident the full generalisation for all $n$. The simple and positive roots are in this case chosen to be $r_1 = (1, -1)$, $r_2 = (0, 1)$ [17], $r_3 = r_1 + r_2 \equiv r_{12} = (1, 0)$ and $r_4 = r_1 + 2r_2 \equiv r_{122} = (1, 1)$, and we have

$$\begin{pmatrix}
H_1 & E_{-1} & E_{-3} & E_{-4} & 0 \\
E_{1} & H_2 & E_{-2} & 0 & -E_{-4} \\
E_3 & E_2 & 0 & -E_{-2} & -E_{-3} \\
E_4 & 0 & -E_2 & -H_2 & -E_{-1} \\
0 & -E_4 & -E_3 & -E_1 & -H_1
\end{pmatrix} \quad . \quad (2.15)$$

Again all of $h_1$, $h_2$ and the $e_{\pm\alpha}$ can be read off (2.15) and shown to obey the correct Lie algebra relations. Here the corresponding $5 \times 5$ matrices $x_i$ satisfy (2.2) and

$$x_i^T = -M x_i M^{-1} \quad , \quad (2.16)$$

where $M$ is the Racah [21] form for $b_2$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix} \quad . \quad (2.17)$$
The evident generalisation to \( b_n \) (for which the short roots are \( E_3, E_{23} \) and \( E_{123} \)) gives for \( n = 3 \)

\[
\begin{pmatrix}
H_1 & H_2 \\
E_1 & H_3 \\
E_{12} & E_2 \\
E_{123} & E_{23} \\
E_{123} & 0 \\
0 & E_{1233}
\end{pmatrix}
\quad . \quad (2.18)
\]

Here the simple roots are \( \mathbf{17} \)

\[
\mathbf{r}_1 = (1, -1, 0) \ , \ \mathbf{r}_2 = (0, 1, -1) \ , \ \mathbf{r}_3 = (0, 0, 1)
\quad , \quad (2.19)
\]

and the remaining positive roots can be inferred from (2.18) to be given by

\[
\mathbf{r}_{12} = \mathbf{r}_1 + \mathbf{r}_2 = (1, 0, -1) \ , \quad (2.18)
\]

\[
\mathbf{r}_{23} = \mathbf{r}_2 + \mathbf{r}_3 = (0, 1, 0) \ , \quad (2.18)
\]

\[
\mathbf{r}_{123} = \mathbf{r}_{12} + \mathbf{r}_3 = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 = (1, 0, 0) \ , \quad (2.18)
\]

\[
\mathbf{r}_{233} = \mathbf{r}_{23} + \mathbf{r}_3 = \mathbf{r}_2 + 2\mathbf{r}_3 = (0, 1, 1) \ , \quad (2.18)
\]

\[
\mathbf{r}_{1233} = \mathbf{r}_{123} + \mathbf{r}_3 = \mathbf{r}_1 + \mathbf{r}_2 + 2\mathbf{r}_3 = (1, 0, 1) \ , \quad (2.18)
\]

\[
\mathbf{r}_{1233} = \mathbf{r}_{123} + \mathbf{r}_2 = \mathbf{r}_1 + 2\mathbf{r}_2 + 2\mathbf{r}_3 = (1, 1, 0) \ . \quad (2.18)
\]

One can also read from (2.18), whose upper triangular part involving the lowering generators has been suppressed, explicit matrices \( h_1, h_2, h_3 \) and \( e_{\pm \alpha} \) where \( \alpha \) runs through the set of values

\[
1, 2, 3, 12, 23, 123, 233, 1223, 12233.
\quad (2.21)
\]

Then one can check that they satisfy the correct Lie algebra relations, with the same expressions for the roots as given in (2.19) and (2.20). We could also have presented the \( b_2 \) results in such a fashion using notation for \( b_2 = E_3 = E_{12} \) since for \( b_2 \) we have \( \mathbf{r}_3 = \mathbf{r}_1 + \mathbf{r}_2 = \mathbf{r}_{12} \), etc. The display (2.18) should make apparent that its nature and the properties conform, apart from minor variation of detail, to the description in Sec. 2.1 of the standard method of construction of \( L \)-operators.

### 2.3 The Lie algebras \( d_n \)

We begin by treating the case of \( d_4 = so(8) \). The positive roots of \( d_4 \) are given \( \mathbf{17} \) by \( l_i \pm l_j \), for \( 1 \leq i < j \leq 4 \), where the \( l_i \) denote the standard basis vectors of \( \mathbb{R}^4 \) (and similarly for \( d_n \)).

For the simple roots we take

\[
\mathbf{r}_1 = (1, -1, 0, 0) \ , \ \mathbf{r}_2 = (0, 1, -1, 0) \ , \ \mathbf{r}_3 = (0, 0, 1, -1) \ , \ \mathbf{r}_4 = (0, 0, 1, 1)
\quad . \quad (2.22)
\]

Then for the remaining positive roots we have

\[
\mathbf{r}_{12} = (1, 0, -1, 0) \ , \ \mathbf{r}_{23} = (0, 1, 0, -1) \ , \ \mathbf{r}_{123} = (1, 0, 0, -1) \ , \ \mathbf{r}_{24} = (0, 1, 0, 1)
\quad , \quad (2.23)
\]

\[
\mathbf{r}_{124} = (1, 0, 0, 1) \ , \ \mathbf{r}_{234} = (0, 1, 1, 0) \ , \ \mathbf{r}_{1234} = (1, 0, 1, 0) \ , \ \mathbf{r}_{12234} = (1, 1, 0, 0)
\quad . \quad (2.24)
\]
The standard form of $L$ for $d_4$ is

$$
\begin{pmatrix}
H_1 & H_2 \\
E_1 & H_2 \\
E_{12} & E_2 & H_3 \\
E_{123} & E_{23} & E_3 & H_4 \\
E_{124} & E_{24} & E_4 & 0 & -H_4 \\
E_{1234} & E_{234} & 0 & -E_4 & -E_3 & -H_3 \\
0 & -E_{1234} & -E_{1234} & -E_{124} & -E_{123} & -E_1 & -H_1 \\
0 & -E_{12234} & -E_{1234} & -E_{124} & -E_{123} & -E_1 & -H_1
\end{pmatrix}.
$$

(2.25)

As before $h_r$, $r \in \{1, 2, 3, 4\}$ and the $e_{\pm \alpha}$ can be read off, their Lie algebra relations checked and seen to be correct. So also can the $x_i$ be obtained. They satisfy (1.3) and (2.16), where, for $M$, we use the $8 \times 8$ analogue of (2.17). But we see that the nice pattern of assignments of generators to places in $L$ has been modified for $d_4$, not only because $E_4$ appears below the first main sub-diagonal, but also in other respects. Much the same applies to general $d_n$.

To see exactly that and how (2.25) and its $2n \times 2n$ generalisation for $L$ for $d_n$ conform to the pattern observed above for $c_n$ and $b_n$, we recall the Dynkin diagram of $d_n$ has a line of linked points labelled $1, 2, \ldots (n - 3)$ leading to the node $(n - 2)$ from which separate the two lines of the fishtail to the points $(n - 1)$ and $n$. For $d_4$, above, it is the fishtail roots $r_3$ and $r_4$ are somehow responsible for obscuring the standard nature of the display (2.25) for $L$, and thereby the correct method of generalising it. One way to find a suitable analogue of previous patterns locked within (2.25) is simply to leave out row 5 and column 5, which reveals a reduced display given schematically by

$$
\begin{pmatrix}
x & x \\
1 & x \\
12 & 2 & x \\
123 & 23 & 3 & x \\
1234 & 234 & 0 & 4 & x \\
12234 & 0 & 234 & 24 & 2 & x \\
0 & 12234 & 1234 & 124 & 12 & 1 & x
\end{pmatrix}.
$$

(2.26)

One could alternatively have left out row and column 4 from (2.25). All the features seen in previous cases are apparent here. The zero arises here because $r_3 + r_4$ is not a root, just as zeros in (2.15), (2.18) and (2.25) reflect the fact that if $r$ is a root then $2r$ is not. The schematic nature of (2.26) should be emphasised. It does not, in the present case, even formally coincide with an $L$-operator, but it does unambiguously indicate how the required $L$-operator is to be written down: it tells which $E_\alpha$ must go in which place, leaving only the easy insertion of signs to complete the construction.

For the case of $d_3 \cong a_3 = su(4)$, the standard form for $L$ is

$$
\begin{pmatrix}
H_1 & H_2 \\
E_1 & H_2 \\
E_{12} & E_2 & H_3 \\
E_{13} & E_3 & 0 & -H_3 \\
E_{123} & 0 & -E_3 & -E_2 & -H_2 \\
0 & -E_{123} & -E_{13} & -E_{12} & -E_1 & -H_1
\end{pmatrix}.
$$

(2.27)
which corresponds, as the above discussion requires, to the ‘reduced display’

\[
\begin{pmatrix}
  x & x \\
  1 & x \\
  12 & 2 & x \\
  123 & 0 & 3 & x \\
  0 & 123 & 13 & 1 & x \\
\end{pmatrix}
\] .

(2.28)

2.4 Mention of \( g_2 \)

Finally we refer to a construction [22] of an \( L \)-operator for \( g_2 \) that is based on the fact that \( g_2 \) is a non-symmetric subalgebra of \( b_3 \).

We here exhibit the lower triangular form of \( L \) in schematic form,

\[
\begin{pmatrix}
  x & x \\
  1 & x \\
  12 & 2 & x \\
  112 & 12 & 1 & x \\
  1112 & 112 & 0 & 1 & x \\
  0 & 11122 & 1112 & 112 & 12 & 1 & x \\
\end{pmatrix}
\] .

(2.29)

The pattern of the places of the non-simple roots relative to the simple ones here, despite its non-trivial nature, conforms very closely to that found for the \( c_n \) and \( b_n \) families. We refer to [22] for explanation and full detail.

3 Explicit results for the defining representations

3.1 Results for \( c_n \)

We begin with the important fact that, in virtue of (1.2) and (2.7), the matrices \( x_i \) possess the completeness relation

\[
x_{iab} x_{icd} = (x_i \otimes x_i)_{ac,bd} = \delta_{ad} \delta_{cb} - J_{ac} J_{bd} = P_{ac,bd} - K_{ac,bd} .
\]

(3.1)

The indices \( i, a \) here vary respectively through ranges from 1 to \( n(2n + 1) = \dim c_n \), \( 2n = \dim V \), which for \( c_2 \) is equal to 10 and 4. Also \( P \) is the permutation operator, and \( K \), to within a constant factor, projects onto the trivial representation in the decomposition of \( V \otimes V \).

However in many contexts to which the algebras \( c_n \) relate, it is necessary to introduce the number \( n(2n - 1) - 1 = (2n + 1)(n - 1) \) (equal to 5 for \( c_2 \)) of matrices \( y_\alpha \) that complete the basis – an \( su(2n) \) basis in fact – of all traceless hermitian \( 2n \times 2n \) matrices. We define them so that

\[
y_\alpha^\dagger = y_\alpha, \quad \text{Tr } y_\alpha = 0, \quad \text{Tr } y_\alpha y_\beta = 2\delta_{\alpha\beta}, \quad \text{Tr } x_i y_\alpha = 0 ,
\]

and

\[
J y_\alpha J^{-1} = y_\alpha^T .
\]

(3.2)

(3.3)

To account for the number of the \( y_\alpha \), we note that there are \( n(2n + 1) \) symmetric matrices \( Jx_i \) and \( n(2n - 1) \) antisymmetric ones: \( Jy_\alpha \) and \( J \) itself.
Since \( x_i x_j + x_j x_i \) is converted to its transpose under conjugation by \( J \), it follows that we can write
\[
x_i x_j + x_j x_i = \frac{2}{n} \delta_{ij} + d_{ij \alpha} y_\alpha ~.
\] (3.4)

Given an explicit choice of the \( y_\alpha \), (3.4) defines the ad-invariant tensor \( d_{ij \alpha} \) so that \( d_{ii \alpha} = 0 \), as well as \( d_{ji \alpha} = d_{ij \alpha} \). Thus we have the product law
\[
x_i x_j = \frac{1}{n} \delta_{ij} + \frac{1}{2} i c_{ijk} x_k + \frac{1}{2} d_{ij \alpha} y_\alpha ~,
\] (3.5)
so that
\[
ic_{ijk} = \text{Tr} x_i x_j x_k, \quad d_{ij \alpha} = \text{Tr} x_i x_j y_\alpha ~.
\] (3.6)

Since the \( x_i \) and \( y_\alpha \) have the same trace properties \((1.2)\) as a standard set of \( 2n \times 2n \) Gell-Mann matrices \( \lambda_A \) of \( a_{2n-1} = su(2n) \), their well-known \cite{24} completeness relation yields, upon use of (3.1),
\[
y_{\alpha \beta} y_{\alpha \gamma} = \delta_{\alpha \beta} \delta_{\alpha \gamma} + J_{\alpha \gamma} J_{\beta \gamma} - \frac{1}{n} \delta_{\alpha \beta} \delta_{\alpha \gamma} = (P + K - \frac{1}{n} I)_{\alpha \beta \gamma} ~,
\] (3.7)
where \( P \) and \( K \) are as in (3.1). It is obvious that alongside (3.3), we should write also
\[
x_i y_\alpha = \frac{1}{2} i h_{\alpha \beta \gamma} y_\beta \quad \text{and} \quad y_\alpha y_\beta = \frac{1}{n} \delta_{\alpha \beta} \delta_{\alpha \gamma} + \frac{1}{2} i h_{\alpha \beta \gamma} x_i + \frac{1}{2} d_{\alpha \beta \gamma} y_\gamma ~.
\] (3.8)

These are justified by consideration of behaviours under conjugation with \( J \), the requirement that \( d_{\alpha \alpha \gamma} = 0 \), and by noting that the taking of traces explains why certain tensors appear twice in the three product laws.

One of the roles of the tensor \( d_{ij \alpha} \) arises in the need to define Casimir operators of degree four and higher \cite{25, 26}. For \( c_2 \) for example, alongside \( C_2 = X_i X_i = \frac{1}{2} \text{Tr} L^2 \), we have
\[
C_4 = \frac{1}{4} \text{Tr} L^4 = t_{ijkl} X_i X_j X_k X_l ~,
\] (3.9)
where \( t_{ijkl} \) is a totally symmetric ad-invariant fourth rank tensor. If one wishes to deal only with quantities which carry \( c_2 \) vector indices, then the argument of \cite{27} is used to show that \( x_i x_j x_k \), where the round brackets denote symmetrisation with unit weight over all indices enclosed by them, is converted into minus its transpose by conjugation with \( J \). Thus one can write
\[
x_i x_j x_k = v_{ijkl} x_k ~,
\] (3.10)
thereby defining a tensor which is totally symmetric
\[
v_{ijkl} = \text{Tr} x_i x_j x_k x_l = \text{Tr} x_i x_j x_k x_l ~,
\] (3.11)
and which can serve in the role of \( t_{ijkl} \) in (3.9).

But \( v_{ijkl} \) can be expressed in terms of the \( d_{ij \alpha} \). Thus insert \((3.2)\) into
\[
x_i x_j x_k = x_i \{ x_j , x_k \} + x_j \{ x_k , x_i \} + x_k \{ x_i , x_j \} ~,
\] (3.12)
and use (3.5) and (3.8). This produces a set of unwanted terms involving the \( y_\alpha \), which vanish by use of a suitable and obvious Jacobi identity, allowing the explicit identification
\[
v_{ijkl} = \frac{1}{n} \delta_{(ij} \delta_{k)l} + \frac{1}{4} d_{\alpha (ij} d_{k)l \alpha} ~,
\] (3.13)
which too is totally symmetric.
To go further for example for $c_3$ to build a sextic Casimir, one can repeat the argument of [27] for a totally symmetrised five-fold product of matrices $x_i$, or use the ad-invariant totally symmetric sixth rank tensor

$$d_{\alpha\beta\gamma}d_{(ij}^\alpha d_{kl}^\beta d_{pq)}^\gamma .$$

Here we raised certain indices to exempt them from the symmetrisation effect of the round brackets, which is trivial from a metric point of view.

One use of the $h_{\alpha\beta}$ is to define a set $R$ of matrices $(R_i)_\alpha^\beta = -ih_{\alpha\beta}$. These define a representation $R$ of $c_n$ of dimension $(2n+1)(n-1)$. To prove this one uses a simple rearrangement of the ordinary Jacobi identity of $xyy$ type, to obtain

$$[R_i, R_j] = ic_{ijk}R_k .$$

For $c_2$, dim $R_i = 5$, so that $R$ is the vector representation of $b_2 = so(5) \cong c_2$.

Turning next to identities of various sorts, we use completeness relations to get

$$x_i x_i = (2n+1) , \quad y_\alpha y_\alpha = \frac{1}{n}(2n+1)(n-1) ,$$

and

$$x_i x_j x_i = -x_j , \quad y_\alpha x_i y_\alpha = (1 - \frac{1}{n}) x_i ,$$

$$x_i y_\alpha x_i = y_\alpha , \quad y_\alpha y_\beta y_\alpha = -(1 + \frac{1}{n}) y_\beta$$

Another important class of identities includes the following

$$c_{ijk} c_{ijl} = 4(n+1)\delta_{kl} ,$$

$$d_{i\alpha\alpha} d_{i\beta\beta} = 4(n+1)\delta_{\alpha\beta} ,$$

$$d_{i\alpha\alpha} d_{i\beta\gamma} = \frac{4}{n}(n^2-1)\delta_{jk} ,$$

$$h_{\alpha\beta} h_{\delta\beta} = 4(n-1)\delta_{ij} ,$$

$$h_{\alpha\beta} h_{\alpha\gamma} = 4n\delta_{\beta\gamma} ,$$

$$d_{\alpha\gamma\mu} d_{\beta\gamma\mu} = \frac{4}{n}(n-2)(n+1)\delta_{\alpha\beta} .$$

All these identities can be proved directly by the same method (although it is easier to get the third and the fifth from their predecessors). For the second, one uses the definition of $d_{ij\gamma}$ as a trace, followed by (3.1) and elementary properties of the $y_\alpha$. One consequence of (3.18) and (3.5) is the formula giving the $y_\alpha$ in terms of the $x_i$

$$2(n+1)y_\alpha = d_{ij\alpha}x_ix_j .$$

A similar approach (in the proof of which (3.17) is very useful) works also for identities like

$$c_{pq\alpha} c_{qj\beta} c_{r\beta\gamma} = -2(n+1)c_{ijk}$$

$$c_{pq\alpha} d_{ip\alpha} d_{jq\alpha} = \frac{2}{n}(n+2)(n-1)c_{ijk} .$$

There are many like this of evident structure all proved the same way.

To finish the discussion of identities, one recalls the set of Jacobi-type identities listed as first class identities in [24] and repeated in [26]. There are many possibilities: $xxx, xxy, xyy$
and $yyy$ versions for each type. Two of these have been used, as needed, above. Another one gives rise to the useful result

$$c_{ijl} c_{kml} = \frac{4}{n} (\delta_{ik} \delta_{jm} - \delta_{im} \delta_{jk}) + d_{ika} d_{jma} - d_{ima} d_{jka} \quad . \tag{3.21}$$

And for second class identities – those that depend on the use of characteristic equations – see [28].

Finally, we give the eigenvalues of the Casimir operator $C_2$ for $c_n$ for the representations $V = (1,0,\ldots,0)$, $R = (0,1,0,\ldots,0)$ and $Ad = (2,0,\ldots,0)$, the adjoint representation with matrices $(F_i)_{jk} = -i_{ij}$. We need $x_i x_i$, $(R_i R_i)_{\beta\gamma} = h_{i\alpha}\beta h_{i\alpha}\gamma$ and $(F_i F_i)_{kl} = c_{ijk} c_{jkl}$. Using (3.16) and (3.18), it follows that $C_2$ has eigenvalues $(2n+1), 4n$ and $4(n+1)$ for $V$, $R$ and $Ad$. These results agree with general formulas for the eigenvalues of Casimir operators given in [28].

### 3.2 Results for so($N$)

We may use for $D$ matrices $x_i$ which obey (1.2) and $M x_i M^{-1} = -x_i^T$ for all $N$, unifying the discussion and properties of tensors for $b_n = so(2n+1)$ and $d_n = so(2n)$. So $\dim x_i = N$. Since $(M x_i)^T = -M x_i$, we can introduce also matrices $y_{\alpha}$ such that $M y_{\alpha}$, like $M$ itself, is symmetric. One needs a number $\frac{1}{2}(N-1)(N+2)$ of them. In fact, it is simpler to pass from the representation of these used so far to an equivalent one with antisymmetric matrices $\hat{x}_i$ and symmetric matrices $\hat{y}_\alpha$. We do this here without supplying any hats. The change of representation does not affect the values or the properties of the tensors which enter the product laws below, but simplifies the derivation of the properties. These laws are

$$x_i x_j = \frac{2}{N} \delta_{ij} + \frac{1}{2} i c_{ijk} x_k + \frac{1}{2} d_{i\alpha j\alpha} y_\alpha \quad ,$$

$$x_i y_\alpha = \frac{1}{2} i h_{i\alpha \beta} y_\beta + \frac{1}{2} d_{i\alpha j\alpha} x_j \quad ,$$

$$y_\alpha y_\beta = \frac{2}{N} \delta_{\alpha\beta} + \frac{1}{2} i h_{i\alpha \beta} x_i + \frac{1}{2} d_{o\beta \gamma} y_\gamma \quad . \tag{3.22}$$

Since the discussion follows the pattern of Sec. 3.1 closely, and the notation conforms to that used there, we simply list results. Completeness relations:

$$x_i x_{i cd} = (x_i \otimes x_i)_{ac, bd} = \delta_{ad} \delta_{cb} - \delta_{ac} \delta_{bd} = (P - Q)_{ac, bd} \quad , \tag{3.23}$$

$$y_\alpha y_{\alpha cd} = \delta_{ad} \delta_{cb} - \delta_{ac} \delta_{bd} - \frac{2}{N} \delta_{ab} \delta_{cd} = (P + Q - \frac{2}{N} I)_{ac, bd} \quad , \tag{3.24}$$

where, as before, $P$ is the permutation operator, and $Q$ is the analogue for $so(N)$ of the operator $K$ used above, (3.1, for $c_n$. Analogues of (3.16):

$$x_i x_i = (N-1) \quad , \quad y_\alpha y_\alpha = \frac{1}{N} (N + 2)(N - 1) \quad . \tag{3.25}$$

Analogues of (3.17):

$$x_i x_j x_i = x_j \quad , \quad y_\alpha x_i y_\alpha = -(1 + \frac{2}{N}) x_i \quad ,$$

$$x_i y_\alpha x_i = -y_\alpha \quad , \quad y_\alpha y_\beta y_\alpha = (1 - \frac{2}{N}) y_\beta \quad . \tag{3.26}$$

Analogues of (3.18):

$$c_{ijk} c_{jkl} = 2(N - 2) \delta_{kl} \quad ,$$
\[ d_{ij\alpha} d_{ij\beta} = 2(N - 2)\delta_{\alpha\beta} \]
\[ d_{ij\alpha} d_{ik\alpha} = \frac{2}{N} (N^2 - 4)\delta_{jk} \]
\[ h_{ij\alpha\beta} h_{ij\alpha\beta} = 2(N + 2)\delta_{ij} \]
\[ h_{ij\alpha\beta} h_{ij\alpha\gamma} = 2N\delta_{\beta\gamma} \]
\[ d_{ij\gamma\mu} d_{ij\gamma\mu} = \frac{2}{N} (N - 2)(N + 4)\delta_{\alpha\beta} \].

(3.27)

The discussion of fourth rank totally symmetric tensors applies here with obvious minor changes. The matrices of the representation \((2,0,\ldots,0)\) of dimension \(\frac{1}{2}(N+2)(N-1)\) can be defined here too using \((S_i)_{\alpha\beta} = -i\hbar_{i\alpha\beta}\). We note here that \((2,0,\ldots,0)\) is not the adjoint representation of \(so(N)\) but that \((0,1,0,\ldots,0)\) is. This is in contrast to the situation for \(c_n\) for which \((2,0,\ldots,0)\) is adjoint, and for which the coefficients \(h_{i\alpha\beta}\) of \((3.8)\) define the representation \((0,1,0,\ldots,0)\), denoted \(R\) above.

Finally one observes that results for \(c_n\) translate into those for \(so(N)\) by means of the substitution \(2n = N \mapsto -N\) accompanied, often, by a change of sign. This is of course not a new observation – see \([29]\) – although the sets of identities given are. Another matter discussed in \([21]\) ‘explains’ the last remarks of the previous paragraph. For further discussion of such matters and many other related ones, see also the recent paper \([30]\).

4 Special oscillator representations

4.1 The metaplectic representation \(M_n\) of \(c_n\)

Let \(a_\mu, \mu \in \{1,2,\ldots,n\}\), denote a set of bosonic annihilation operators such that

\[ [a_\mu, a_\nu^\dagger] = \delta_{\mu\nu} \] (4.1)

Then for \(c_n\) we form the vector \(v\) with components \(v_\alpha, \alpha \in \{1,2,\ldots,2n\}\) given by

\[ v^T = (a_1^\dagger, \ldots, a_n^\dagger, a_{n+1}, \ldots, a_{2n}) \] (4.2)

and define the operators

\[ X_i = \frac{1}{2} v^T x_i (Jv) \] (4.3)

where \(J\) is the symplectic form defined in \((2.7)\). It is now easy to use \((4.3)\) and \((4.2)\), as well as \((2.7)\), to show (what the notation \(X_i\) in \((1.3)\) implies), that these \(X_i\) obey the same commutation relations as do the \(x_i\). Since the \(x_i\) by construction give a representation of \(c_n\), so also do the operators \(X_i\) of \((4.3)\). It acts in the Fock space of \(n\) sets of harmonic oscillator variables and constitutes the (reducible) metaplectic representation \(M_n\) \([1,17]\) of \(c_n\) or of \(sp(2n,\mathbb{R})\). By use of the explicit forms used in section two for the matrices \(x_i\) of \(c_2\) as well as \((2.3)\), we find the results for \(c_2\)

\[ H_1 = \frac{1}{2} \{a_1^\dagger, a_1\} \] , \[ H_2 = \frac{1}{2} \{a_2^\dagger, a_2\} \] , \[ E_1 = a_1^\dagger a_2 \] , \[ E_2 = -\frac{1}{\sqrt{2}} a_1^{\dagger 2} \] , \[ E_3 = a_1^\dagger a_2^\dagger \] , \[ E_4 = -\frac{1}{\sqrt{2}} a_2^{\dagger 2} \] , \[ E_{-1}^\dagger = E_1 \] , \[ E_{-\alpha}^\dagger = -E_\alpha , \alpha = 2,3,4 \] .

(4.4)

The hermiticity properties seen here reflect the fact that \((1.4)\) defines the generators of an infinite dimensional unitary representation of the non-compact group \(Sp(4,\mathbb{R})\). The operators

\[ J_z = \frac{1}{2} (H_1 - H_2) , J_+ = a_1^\dagger a_2 , J_- = J_+^\dagger \] ,

(4.5)
and $\frac{1}{2}(H_1 + H_2)$ span the maximum compact subalgebra of $c_2$. Also the non-compact raising generators of $c_2$, namely $E_4$, $E_3$, $E_2$ are the components of a standard set of spherical components $K_n$ of a vector operator (in the precise sense of Racah) with respect to the $su(2)$ algebra generated by the operators $J$ in (4.3). The same applies to their adjoints $E_{-2}$, $-E_{-3}$, $E_{-4}$. Furthermore, and notably, the components of each of these vectors constitute a set of commuting operators, which is crucial for building up the basis states of $M_2$, and for the solution of Gaudin models $[32]$, and in algebraic Bethe ansatz work $[4]$.

As all the operators $X_i$ of (4.4) are bilinear in creation and annihilation operators, it follows that the Fock space $B_2$ of $M_2$ breaks into two subspaces $B_{2\pm}$ of even and odd states in which the operators of $c_2$ act irreducibly. The ground state of $B_{2+}$ is the Fock vacuum $|0\rangle$ such that $a_1|0\rangle = a_2|0\rangle = 0$, and $B_{2+}$ involves states of integral $j$, $J^2 = j(j+1)$. These are constructed as $su(2)$ multiplets with respect to $J$ by action of the (commutative) $K_\mu$ on the Fock vacuum. Indeed, states involving $p$ applications of $K_\mu$ components have $j = p$. The situation with regard to $B_{2-}$ is similar; its ground states are a $j = \frac{1}{2}$ doublet of states $a_1^\dagger|0\rangle$, $a_2^\dagger|0\rangle$, i.e. the states $|\frac{1}{2} \pm \frac{1}{2}\rangle$ in standard, $|jm\rangle$ notation, and the action of the $K_\mu$ produces $su(2)$ multiplets with $j$-values equal to one-half of an odd integer. Since $p$ applications of $K_\mu$ components to the states of the ground state multiplet produces a state of $j = p + \frac{1}{2}$, it follows that $M_2$ contains exactly one multiplet of states of each of the allowed $j$-values.

Most of the previous discussion generalises easily and obviously from the case of $c_2$ to $c_n$.

The maximal compact sub-algebra of the algebra of $c_n$ is of type $u(n)$. The non-compact ‘creation’ generators transform according to the representation of its $su(n)$ sub-algebra with highest weight twice that of the $n$-dimensional defining representation, $V = (1,0,\ldots,0)$ in highest weight notation, of $su(n)$. This does agree with the fact that the non-compact ‘creation’ generators of $M_2$ belong to the vector representation of $su(2)$. For $B_{2n+}$, we see that the Fock vacuum is the ground state of the irreducible subspace of even states. These are arranged into $su(n)$ multiplets of the type $(2p,0,\ldots,0)$, which correspond to totally symmetric tensors. Turning to $B_{2n-}$, it is clear that its ground states (lowest weight states) constitute an $su(n)$ multiplet of states which transform according to the representation $D$ of $su(n)$. Application of non-compact ‘creation’ generators to these produces $su(n)$ multiplets of states $(2p+1,0,\ldots,0)$. Thus $M_2$ involves one multiplet of states for each $su(n)$ representation that corresponds to a totally symmetric tensor, and no others.

One may use the completeness relation (3.3), and $v^T Jv = -n$, to show that the eigenvalue of $C_2$ for $M_n$ is

$$-\frac{n}{2}(2n+1) \quad (4.6)$$

We can see that the work of section 4.1 involves the splitting of $c_n$, $c_n = u(n) \oplus n_+ \oplus n_-$, where $u(n)$ is the maximal compact subalgebra of our non-compact realisation of $c_n$, and where $n_+$ contains the non-compact creation generators of $c_n$ and, notably, like $n_-$ is commutative. This is further exactly the same splitting as that of the Lie algebra underlying $[12]$ the non-compact hermitian symmetric space $Sp(2n,\mathbb{R})/U(n)$. The non-compactness of $M_\mu$ follows from the hermiticity relations seen in (4.4). In general for $c_n$, the generators of $n_-$ are minus the adjoints of those in $n_+$.

### 4.2 Spinor representations of $d_n$

Let $c_\mu$, $\mu \in \{1,\ldots,n\}$, and $\pi_\mu = c_\mu^\dagger$ denote a set of fermionic creation and annihilation operators

$$\{c_\mu, \pi_\nu\} = \delta_{\mu\nu} \quad (4.7)$$
Now we form the vector \( v \)

\[
v^T = (c_1, \ldots, c_n, \pi_n, \ldots, \pi_1)
\]

and define

\[
X_i = \frac{1}{2} v^T x_i (M v)
\]

where \( M \) is as in (2.17). It can be shown that, using (4.7) and (2.16) that the \( X_i \) of (4.9) obey the same commutation relations as do the matrices \( x_i \). In this case of course we use those constructed for \( d_n \) in section 2.3.

As before, we may translate (4.9), e.g. in the case of \( d_3 \), into the explicit operator form

\[
H_1 = N_1 - \frac{1}{2}, \quad H_2 = N_2 - \frac{1}{2}, \quad H_3 = N_3 - \frac{1}{2},
\]

\[
E_1 = c_1 \pi_2, \quad E_2 = c_2 \pi_3, \quad E_{12} = c_1 \pi_3,
\]

\[
E_3 = c_2 c_3, \quad E_{13} = c_1 c_3, \quad E_{123} = c_1 c_2,
\]

\[
E_{-\alpha} = E^\dagger_{\alpha}, \quad \text{for all } \alpha.
\]

Here \( N_1 = c_1 \pi_1 \) is a fermion number operator, etc. Again use of fermionic anticommutation relations to do a check of any Lie algebra relation here works correctly. The representation of \( d_n \) just constructed acts in the Fock space \( F_n \) of \( n \) Dirac fermions of dimension \( 2^n \). The situation resembles closely that described in section 4.1. The space \( F_n \) breaks up into two subspaces \( F_{n\pm} \) of states of even and odd total fermion number in which (4.9) acts irreducibly because all the \( X_i \) are bilinear in fermionic variables. This also agrees with the fact that \( d_n \) has two inequivalent irreducible spinor representations of dimension \( 2^{n-1} \). See [33], which contains a detailed description of the Fock space of dimension \( 2^{8} = 256 \) in the case \( n = 8 \).

We note here too the splitting of the Lie algebra \( d_n = u(n) \oplus n_+ \oplus n_- \), where the \( n_\pm \) are Abelian. This corresponds to that of the compact hermitian symmetric space \( SO(2n)/U(n) \). Again the fact that \( n_\pm \) is Abelian is important for the construction of the basis of the \( F_{n\pm} \). In fact each of the fundamental representations of \( su(n) \) occurs once in \( F_n \), so that e.g. for \( n = 4 \) the basis of \( F_{n+} \) contains two singlets (the states of fermion number zero and four) and a 6 = (0, 1, 0), while \( F_{n-} \) contains 4 = (1, 0, 0) and \( \bar{4} = (0, 0, 1) \).

### 4.3 The spinor representation of \( b_n \)

In this case we have to augment the \( c_\mu \) and \( \pi_\mu \) used in section 4.2 by a single Majorana fermion \( c \) in order to realise a representation of \( b_n = so(2n + 1) \) like that of (4.9). This is a theme that has been developed considerably in previous work [15]. For a striking application of such thinking, see also [34] for Majorana parafermions.

The operator \( c \) obeys

\[
c^\dagger = c, \quad c^2 = \frac{1}{2}, \quad \{c, c_\mu\} = 0, \quad \{c, \pi_\mu\} = 0.
\]

Forming

\[
v^T = (c_1, \ldots, c_n, c, \pi_n, \ldots, \pi_1)
\]

we employ once more the definition (4.9). This time it must be checked by an independent calculation, one which uses (4.12) and (4.11), that the \( X_i \) obey the same commutation relations as do the \( x_i \), in this case those that belong to the defining representation \( V \) of \( b_n \).
The explicit operators $X_i$ here consist of the first two lines of (4.10) and in addition

\[ E_3 = c_3 c \quad , \quad E_{23} = c_2 c \quad , \quad E_{123} = c_1 c \quad , \quad E_{1233} = c_1 c_3 \quad , \quad E_{233} = c_2 c_3 \quad , \quad E_{12233} = c_1 c_3 \quad , \]

(4.13)

together with hermitian conjugation relations. Checks of their commutation relations using (4.7) and (4.11) do work.

In contrast to what was seen in section 4.1 and 4.2, in the splitting $b_n = u(n) \oplus n_+ \oplus n_-$, the $n_\pm$ are not commutative, making use of the elements of $n_+$ less convenient. This should be seen alongside the fact that there is no hermitian space with $G = SO(2N+1)$ and $H = U(N)$.

5 Quadratic equations for $L$

5.1 Case of $c_n$

We apply the completeness relation (3.1) of the matrices $x_i$ of $c_n$ to $L = x_i X_i$, where $X_i$ is given by (4.3). This gives

\[ L_{ab} = \frac{1}{2} v_b (J v)_a + \frac{1}{2} (J v)_a v_b \quad , \]

(5.1)

or

\[ L = (J v) v^T - \frac{1}{2} \quad . \]

(5.2)

Hence

\[ L^2 = \frac{1}{4} - (J v) v^T + (J v) (v^T J v) v^T \quad . \]

(5.3)

Since $(v^T J v) = -n$ follows from (3.1) and (3.2), we can eliminate $(J v) v^T$ to obtain a quadratic relation for $L$. It is

\[ L^2 + (n + 1)L + \frac{1}{4}(2n + 1) = 0 \quad . \]

(5.4)

5.2 Cases of $b_n$ and $d_n$

The completeness relations for the matrices $x_i$ of $b_n$ and $d_n$ take the same form (3.1), and as above, we get

\[ L = \frac{1}{2} - (M v) v^T \quad . \]

(5.5)

Since we find $v^T M v = n$ for $so(2n)$ and $n + \frac{1}{2}$ for $so(2n + 1)$, and hence $\frac{1}{2} N$ for all $so(N)$, it follows that for all $so(N)$

\[ L^2 + \frac{1}{2} (N - 2)L - \frac{1}{4}(N - 1) = 0 \quad . \]

(5.6)

The result $\text{Tr} L = 0$ is compatible with (5.6), which implies that the eigenvalues of $L$ are $\frac{1}{2}$ and $-\frac{1}{2}(N - 1)$, because the former occurs with multiplicity $(N - 1)$. 17
5.3 Some other examples

We here compare the examples of the two previous sub-sections with some \( a_{n-1} = su(n) \) examples some of which employ the definition \((1.5)\).

We look first at the case of \( su(2) \), setting \( \gamma = \frac{1}{2} \) in \((1.5)\) and using \( x_i = \sigma_i \), where the \( \sigma_i \) are Pauli matrices.

By considering the angular momentum addition problem for \( J = J_1 + J_2 \) where \( j_1 = \frac{1}{2} \) and \( j_2 = j \) is arbitrary, it is seen that

\[
L = x_i X_i = \sigma_i (J_2)_i ,
\]

has eigenvalues \( j \) and \(- (j + 1)\) corresponding to the total angular momentum values \( j \pm \frac{1}{2} \). It follows that \( L \) obeys the quadratic equation

\[
(L + (j + 1)) (L - j) = 0.
\]

Equation \((5.8)\) is compatible with \( \text{Tr} L = 0 \), in view of the multiplicities of its two eigenvalues. Equation \((5.8)\) could also have been obtained by the method of Secs. 5.1 and 5.2.

The present example differs from those of these sub-sections, in that \( L \) refers to an arbitrary representation of \( su(2) \), as opposed to specific ones albeit given in operator form. Indeed, for \( su(2) \), each tensor product of the fundamental with another representation decomposes into exactly two irreducible components, so that in this case the quadratic relation is a property of the Lie algebra rather than of the representations involved. Therefore we have a quadratic relation \((5.8)\) for each finite dimensional irreducible representation of \( su(2) \), the vector space on which it acts being the quantum space \( \mathcal{H} \) of some model.

We turn next to the case of \( a_{n-1} = su(n) \). Setting \( \gamma = \frac{1}{2} \) and \( x_i = \lambda_i \), where these are the usual Gell-Mann \( \lambda \)-matrices of \( su(n) \), we use \((1.5)\). It is to be noted that the latter definition uses twice as many oscillator pairs as does our \( c_n \) construction, \((4.2)\).

Then the method of Secs. 5.1 and 5.2 shows that

\[
L = x_i X_i = \lambda_i X_i ,
\]

satisfies the equation

\[
(L + 1 + \frac{N}{n}) (L + \frac{N}{n} - N) = 0.
\]

The operator \( N = a_j^\dagger a_j \) here is the total bosonic number operator, which enters the calculations via \( A^\dagger A = N \) in the notation of \((1.5)\).

Putting \( n = 2 \) and replacing the operator \( N \) by its eigenvalue \( 2j \) leads back to \((5.8)\). On the other hand, comparing this \( su(2) \) example with the \( c_1 \cong su(2) \) representation \( \mathcal{M}_1 \), we note the former used two while the latter needed only one oscillator pair of variables.

Since the representation operators \( X_i \) in eq. \((1.5)\) involve exactly one creator and one annihilator each, the bosonic number operator \( N \) forms an invariant of this representation, \( i.e. [N, X_i] = 0 \). Thus the oscillator representation decomposes into invariant subspaces, one for each eigenvalue \( \lambda \) of \( N \). Each subspace corresponds to the subspace of the bosonic Fock space generated by all states of a given number \( \lambda \) of quanta. Whereas for \( n = 2 \), each finite dimensional irreducible representation of \( su(2) \) occurs exactly once in this decomposition, for higher \( n \) the oscillator representation contains due to the bosonic operators only representations which are totally symmetric tensor products of the fundamental representation. For \( su(n) \), these are all representations whose highest weight is of the form \((\lambda, 0, \ldots, 0)\) in terms...
of the fundamental weights or which correspond to a Young diagram with just one row of length $\lambda$.

In fact the $n = 3$ example deals with the direct product $(1, 0) \otimes (\lambda, 0)$ where $\lambda$ is the eigenvalue of the operator $N$. Now

$$ (1, 0) \otimes (\lambda, 0) = (\lambda + 1, 0) \oplus (\lambda - 1, 1) \quad . $$

(5.11)

For the latter two, the quadratic Casimir operator of $su(3)$, which in general has eigenvalues (see e.g. [35], where a factor $\frac{1}{9}$ occurs instead of $\frac{1}{3}$ here),

$$ C_2(\lambda, \mu) = \frac{1}{3} (\lambda^2 + \lambda \mu + \mu^2 + 3\lambda + 3\mu) \quad , $$

(5.12)

has eigenvalues $\frac{1}{3}(\lambda + 1)(\lambda + 4)$ and $\frac{1}{3}(\lambda + 1)^2$. Thus $L = x_iX_i$ with $X_i$ given by (1.5) has eigenvalues $-\frac{1}{3}\lambda$ and $-\frac{1}{6}(\lambda + 3)$. Of course, (5.10) agrees with this when restricted to $n = 3$.

5.4 Relevance of the quadratic equation

If we consider rational solutions $T(u)$ and $R(u)$ of eq. (1.4) that are symmetric under the Lie algebra $g$, we expect $T(u)$ to be of the form

$$ T(u) = \sum_{j=1}^{k} f_j(u)P^{(j)} \quad , $$

(5.13)

where $P^{(j)}$, $j \in \{1, \ldots, k\}$, are the invariant projectors decomposing

$$ \mathcal{V} \otimes \mathcal{H} = V_1 \oplus \cdots \oplus V_k \quad (5.14) $$

into irreducible components. The $f_j(u)$ in eq. (5.13) denote rational functions which are to be determined from the condition (1.4).

In Sec. 5 we have shown that the operators $L$ in our representations given by eq. (4.3) and (4.9) satisfy quadratic relations. This reflects the fact that $\mathcal{V} \otimes \mathcal{H} = V_1 \oplus V_2$ decomposes under the action of the invariance algebra $g$ into two components which are the eigenspaces of the quadratic Casimir operator $C^{(2)}_{\mathcal{V} \otimes \mathcal{H}}$. The $L$-operator has the same invariant subspaces, since

$$ C^{(2)}_{\mathcal{V} \otimes \mathcal{H}} = \sum_j (x_j \otimes 1 + 1 \otimes X_j)^2 $$

$$ = \mu_1P^{(1)} + \mu_2P^{(2)} $$

$$ = C^{(2)}_\mathcal{V} \otimes 1 + 1 \otimes C^{(2)}_\mathcal{H} + 2L \quad . $$

(5.15)

In such cases, we can use, for insertion into the RTT equation (1.4), the simple ansatz

$$ T(u) = u \mathbb{1} + \eta L \quad (5.16) $$

with a constant $\eta$, since this incorporates the generic linear combination of all (i.e. both) invariant projectors as demanded by (5.13).

The advantage of (5.16) is furthermore, that this is precisely the same form of $T(u)$ as it appears for rational $su(n)$ symmetric models where an algebraic Bethe ansatz is not only available, but also tractable.
In fact, for $c_n = sp(2n)$, 
\[ R(u) = u \mathbb{1} + \eta P - \frac{w\eta}{\eta(n+1) + u} K \] (5.17)

together with $T(u)$ in (5.16) provide a solution of (1.4). Here $P_{ac,bd} = \delta_{ac} \delta_{bd}$ flips the two tensor factors, and $K_{ac,bd} = J_{ac} J_{bd}$ projects (up to a factor) onto the trivial representation $\mathcal{V} \otimes \mathcal{V}$, where $\mathcal{V}$ denotes the defining representation of $c_n$.

The analogue for $g = so(N)$ ($N = 2n + 1$ for $b_n$ and $N = 2n$ for $d_n$) is
\[ R(u) = u \mathbb{1} + \eta P - \frac{w\eta}{2\eta(N-2) + u} Q, \] (5.18)

where $Q_{ac,bd} = \delta_{ac} \delta_{bd}$ projects (again up to a factor) onto the trivial component of $\mathcal{V} \otimes \mathcal{V}$, where $\mathcal{V}$ here refers to the defining representation of $so(N)$. The $R$-matrices (5.17) and (5.18) are given in [8]. They were found in the study of models in which the quantum space $\mathcal{H}$ is a particular finite-dimensional irreducible representation.

5.5 Quadratic relations for finite dimensional quantum spaces

Of course, the arguments of Sec 5.4 are also available for finite dimensional irreducible representations $\mathcal{H}$ of $g$ for which $\mathcal{V} \otimes \mathcal{H}$ decomposes into exactly two irreducible components.

For $a_n$ we can select all representations with highest weight $(\lambda, 0, \ldots, 0)$ due to eq. (5.10). But besides this, we have for the conjugate representations
\[ (1, 0, \ldots, 0) \otimes (0, \ldots, 0, \lambda) = (1, 0, \ldots, 0, \lambda) \oplus (0, \ldots, 0, \lambda - 1), \] (5.19)

and similarly for the other representations whose highest weight is an integer multiple of a single fundamental weight.

Since tensor products of $b_n$, $c_n$ and $d_n$ representations tend to split into smaller components than their $a_n$ counterparts, one expects fewer quadratic relations in these cases. In fact, for $b_n$, there is only one pair of representations
\[ (1, 0, \ldots, 0) \otimes (0, \ldots, 0, 1) = (1, 0, \ldots, 0, 1) \oplus (0, \ldots, 0, 1). \] (5.20)

A similar decomposition is available for $c_n$, where
\[ (1, 0, \ldots, 0) \otimes (0, \ldots, 0, 1) = (1, 0, \ldots, 0, 1) \oplus (0, \ldots, 0, 1, 0) \] (5.21)

except for $c_2$, where all
\[ (1, 0) \otimes (0, \lambda) = (1, \lambda) \oplus (1, \lambda - 1) \] (5.22)

due to the isomorphism $c_2 \cong b_2$.

Finally, for $d_n$ we find quadratic relations for the decompositions
\[ (1, 0, \ldots, 0) \otimes (0, \ldots, 0, \lambda) = (1, 0, \ldots, 0, \lambda) \oplus (0, \ldots, 0, 1, \lambda - 1) \] (5.23)

and
\[ (1, 0, \ldots, 0) \otimes (0, \ldots, 0, \lambda, 0) = (1, 0, \ldots, 0, \lambda, 0) \oplus (0, \ldots, 0, \lambda - 1, 0) \] (5.24)
for both spinor representations of $d_n = so(2n)$. The algebra $d_3$ is an exception to these rules and admits many more quadratic relations due to $a_3 \cong d_3$. We remark that the $R$-matrices (5.17) and (5.18) are found [8] in the study of these pairs of finite-dimensional representations where the corresponding $L$ matrix satisfies a quadratic relation.

If one could relax the requirement that the auxiliary space $\mathcal{V}$ is chosen as the defining representation (on which much of our discussion in the previous sections relied), there were a few more examples of quadratic relations. For $b_n$, they are

\[(0, \ldots, 0, 1) \otimes (\lambda, 0, \ldots, 0) = (\lambda, 0, \ldots, 0, 1) \oplus (\lambda - 1, 0, \ldots, 0, 1).\]  
(5.25)

For $c_n$, there are no more results, but for $d_n$,

\[(0, \ldots, 0, 1) \otimes (\lambda, 0, \ldots, 0) = (\lambda, 0, \ldots, 0, 1) \oplus (\lambda - 1, 0, \ldots, 0, 1, 0)\]  
(5.26)

and

\[(0, \ldots, 0, 1, 0) \otimes (\lambda, 0, \ldots, 0) = (\lambda, 0, \ldots, 0, 1, 0) \oplus (\lambda - 1, 0, \ldots, 0, 1).\]  
(5.27)

For $d_4$ there are in addition

\[(0, 0, 0, 1) \otimes (0, 0, \lambda, 0) = (0, 0, \lambda - 1, 1) \oplus (1, 0, \lambda - 1, 0)\]  
(5.28)

and

\[(0, 0, 0, \lambda) \otimes (0, 0, 1, 0) = (0, 0, 1, \lambda - 1) \oplus (1, 0, 0, \lambda - 1).\]  
(5.29)

### 5.6 Consequences for $a_n$

One can usefully proceed further and convert the quadratic equations for $L$ into product laws for the operators $X_i$ involved in $L$.

Look briefly first at (5.34), which refers to $a_1$, although no new result emerges in this case. Using the product formula for $\sigma_i \sigma_j$ in (5.33) gives $J^2 = j(j+1)$ and $-i \epsilon_{ijk} J_i J_j = J_k$. The latter equation, and likewise its counterparts in other cases below, contains no new information:

\[-i \epsilon_{ijk} J_i J_j = -\frac{i}{2} \epsilon_{ijk} [J_i, J_j] = \frac{1}{2} \epsilon_{ijk} \epsilon_{ijl} J_l = J_k.\]  
(5.30)

In the case of (5.10), when the $X_i$ are the generators of the irreducible representation $(\lambda, 0, \ldots, 0)$ of $su(n)$, the Kronecker delta term of the product law [24] for $\lambda_i \lambda_j$ leads to

\[C_2 = \frac{i}{2} \text{Tr} L^2 = X_i X_i = \frac{1}{2} (n - 1) \lambda (1 + \frac{\lambda}{n}) .\]  
(5.31)

The part containing $f_{ijk}$ is treated as in the previous example. Thus we are left with the useful result

\[d_{ijk} X_i X_j = \frac{n + 2 \lambda}{2n} (n - 2) X_k .\]  
(5.32)

which vanishes at $n = 2$.

We note first that (5.32) enables the calculation of the eigenvalues for $(\lambda, 0, \ldots, 0)$ of higher order Casimir operators. For example,

\[C_3 = d_{ijk} X_i X_j X_k = \frac{1}{7} (n - 1) (n - 2) \lambda (1 + \frac{\lambda}{n}) (1 + \frac{2 \lambda}{n}) .\]  
(5.33)

which for $n = 3$ reads

\[C_3(\lambda, 0) = \frac{1}{15} \lambda (\lambda + 3)(2\lambda + 3) .\]  
(5.34)

This agrees to within a factor due to normalisations with the result in [25]

\[C_3(\lambda, \mu) = \frac{1}{18} (\lambda - \mu)(\lambda + 2\mu + 3)(2\lambda + \mu + 3) .\]  
(5.35)
5.7 Consequences for $c_n$

Here, when the operators $X_i$ are those of $M_n$ as displayed in (1.4), use of (3.5) in (5.4) gives

\[ X_i X_i = -\frac{1}{4} n(2n + 1), \]
\[ i c_{ijk} X_i X_j = -2(n + 1) X_k, \]
\[ d_{ij\alpha} X_i X_j = 0. \]  

(5.36)

Of these, the first agrees with (4.6), the second gives nothing new, but the latter does and is useful. It is worth remarking also that the operators $X_i$ in (5.36) are represented by infinite dimensional matrices in the Fock space of $M_n$. As above, one use of (5.36) is to calculate the eigenvalues for $M_n$ of the higher order Casimir operators of $c_n$. For example, for

\[ d_{\alpha(ij) d_{kl}\alpha} X_i X_j X_k X_l = 0, \]  

(5.37)

only one term under the symmetrisation brackets contributes, doing so according to

\[ \frac{1}{3} d_{i\kappa\alpha} d_{j\lambda\alpha} X_i X_j X_k X_l = \frac{1}{3} d_{i\kappa\alpha} d_{j\lambda\alpha} [X_i, X_j] X_k X_l, \]  

(5.38)

upon use of (5.36) again. Now use of (3.20) and (3.18) give an expression for the eigenvalue

\[ \frac{2}{3} (n^2 - 1)(n + 2)(2n + 1). \]  

(5.39)

One could pursue further the question of what is the ‘best’ definition of $C_4$, and go on to $C_r$ for higher even $r$.

Another use of (5.36) arose in the search [2] for solutions $T(u)$ of the Yang-Baxter algebra relations when, for the quantum space, we have taken the Hilbert space of $M_n$. We may use (3.1) and (3.7) to write the $c_n$ invariant solution matrix $R(u)$ (5.17) [2,36] of the Yang-Baxter equation in the form

\[ R_{13}(u) = a(u) I \otimes I + b(u) x_i \otimes x_i + c(u) y_\alpha \otimes y_\alpha. \]  

(5.40)

Then a natural strategy for passing to the required $T(u)$ is at hand. This replaces the auxiliary space 3 of $R_{13}$ in (5.40) by the quantum space $H$, and passes from (5.40) to $T(u)$, via $x_i \mapsto X_i$ and $y_\alpha \propto d_{ij\alpha} x_i x_j \mapsto Y_\alpha$, with $Y_\alpha \propto d_{ij\alpha} X_i X_j$. Since in the present case, the latter is seen from (5.36) to be a vanishing operator, solution of (1.4) turns out, as might be anticipated, to be particularly simple [2].

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