Bayes Estimation of Traffic Intensity in M/M/1 Queue under a Precautionary Loss Function

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Abstract

Bayesian estimators of the traffic intensity in an M/M/1 queue are derived under a new weighted square error loss function. These Bayes estimators are obtained based on two different priors of the traffic intensity, and compared with the corresponding Maximum likelihood (ML) and Bayesian estimators under a precautionary loss function. Finally, a numerical example is given to illustrate the results.

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1. Introduction

Queueing theory has many applications in communication theory, computer designs, manufacturing process and transportation systems. Statistical inference in stochastic process and in particular inference in queueing process has drawn the attention of researchers in the past few years. The problem of estimation and hypothesis testing concerning the parameters of the queueing process, such as the arrival rate, service rate, traffic intensity is important from the point of view of design of queues and comparison of the performances of queueing systems.

The earliest work on parameter estimation in queues is by Clarke (1957), who derived the MLE estimators of the arrival rate and service rate in the single server M/M/1 queue. Armero and Bayarri (1994) obtained the Bayes estimator of the traffic intensity $\rho$ in M/M/1 queue. Sharma and Kumar (1999)
discussed classical and Bayesian estimators of various characteristics of a M/M/1 queue under squared error loss function. Mukherjee and Chowdhury (2005) obtained the Bayes estimator of the traffic intensity $\rho$ in a M/M/1 queue under squared error loss function and LINEX loss function. Dey (2006) obtained Bayes estimators of traffic intensity and various queue characteristics under the assumptions of different priors and the usual squared error loss function.

The object of the present paper is to obtain Bayes estimators for an M/M/1 queue under entropy loss function. The rest of this paper is organized as follows. In Second Section, some useful preliminaries are introduced. The Bayes estimators of the traffic intensity $\rho$ under the different priors are obtained under in Section 3. In Section 4, a numerical example is given to illustrate our results and a conclusion is also given in the finally section, Section 5.

2. Preliminaries

Let us consider a M/M/1 queuing system with the mean arrival rate $\lambda$ and mean service time $1/\mu$. The analysis for such a queue is now folk lore in the queuing literature, and we know that the random variable $X$ representing the number of customers in the system under steady state has the distribution specified by the probability mass function (pmf) (see for Kleinrock, 1975, p96):

$$P(x \mid \rho) = (1 - \rho)^x \rho^x, \quad x = 0,1,2,\cdots$$

where $\rho = \lambda / \mu$, $0 < \rho < 1$ represents the traffic intensity for the given queuing system M/M/1.

The likelihood function corresponding to the pmf (1) is given by

$$L(\rho) = (1 - \rho)^n \rho^t$$

where $t = \sum_{i=0}^{n} x_i$ is the observation value of $T = \sum_{i=0}^{n} X_i$.

In the Bayesian approach, we further assume some prior knowledge about the queuing parameter $\rho$ is available to the investigator from past experiences with the underlying queuing system. The prior knowledge can often be summarized in terms of the so-called prior densities on the parameter space of $\rho$.

In the following discussion, we assume the following priors:

(i) The quasi-prior: For the situation where the experimenter has no prior information about the parameter $\rho$, one may use the quasi density as given by

$$\pi_1(\rho; c) \propto \frac{1}{\rho^c}, \quad \rho > 0, c > 0$$

Hence, $c = 0$ leads to a diffuse prior and $c = 1$ to a non-informative prior.

(ii) The Beta prior: The most widely used prior distribution of $\rho$ is the Beta prior distribution with parameters $a$ and $b$ ($>0$) with pdf given by

$$\pi_2(\rho; a, b) = \frac{1}{B(a,b)} \rho^{a-1}(1 - \rho)^{b-1}, \quad 0 < \rho < 1, a, b > 0$$
3. Bayes Estimation

In Bayesian analysis, a commonly used loss function is the squared error loss function (SELF) \( L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2 \), which is a symmetrical loss function that assigns equal losses to overestimation and underestimation. However, in many practical problems, overestimation and underestimation will make different consequents. Thus using of the symmetric loss function may be inappropriate, Varian (1975) and Zellner (1986) proposed an asymmetric loss function known as the LINEX function, and Norstrom (1996) proposed an asymmetric loss function known as precautionary loss function with the following form:

\[
L(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\hat{\theta}}
\]  

(5)

Where \( \hat{\theta} \) is an estimator of \( \theta \). The loss function (5) infinitely near to the origin to prevent underestimation, thus giving conservative estimators, especially when low failure rates are being estimated. It is very useful when underestimation may lead to serious consequences.

The Bayes estimator under precautionary (5) is denoted by \( \hat{\theta}_B \) and is given by

\[
\hat{\theta}_B = [E(\theta^2 | X)]^{1/2}
\]  

(6)

In this section, we shall discuss the estimation of the traffic intensity \( \rho \) of the M/M/1 queue (1) under precautionary loss function

\[
L(\hat{\rho}, \rho) = \frac{(\hat{\rho} - \rho)^2}{\hat{\rho}}
\]  

(7)

3.1. Bayes Estimation under Quasi-Prior

We consider the case when the prior density of \( \rho \) is quasi-prior (3). The likelihood function (2) is combined with the prior (3) by using the Bayes theorem to obtain the posterior density:

\[
h_1(\rho | x) = \frac{(1 - \rho)^n \rho^{i-c}}{\int_0^1 (1 - \rho)^n \rho^{i-c} d\rho}
\]  

(8)

It is obvious that \( \rho | X \) is distributed with Beta distribution \( \text{Beta}(T - c + 1, n + 1) \). Then, under the precautionary loss (7), the Bayes estimator of \( \rho \) is

\[
\hat{\rho}_1 = [E(\rho^2 | X)]^{1/2} = [\int_0^1 \rho^2 h_1(\rho | X) d\rho]^{1/2}
\]
\[ \int_0^1 \rho^{T-c+2} (1-\rho)^n d\rho = \left[ \frac{(T-c+2)(T-c+1)}{(T-c+n+3)(T-c+n+2)} \right]^{1/2} \]  

(9)

3.2. Bayes Estimation under Beta Prior

In this subsection, we consider the case when the prior density of \( \rho \) is Beta distribution with parameters \( a \) and \( b (>0) \). The posterior density of \( \rho \) can be obtained by using Bayes theorem as:

\[ h_2(\rho | x) = \frac{(1-\rho)^{a+b-1} \rho^{t+a-1}}{\int_0^1 (1-\rho)^{a+b-1} \rho^{t+a-1} d\rho} \]  

(10)

It is obvious that \( \rho | X \) is distributed with Beta distribution \( \text{Beta}(T - c + 1, n + 1) \). Then, under the precautionary loss (7), the Bayes estimator of \( \rho \) is

\[ \hat{\rho}_2 = [E(\rho^2 | X)]^{1/2} = [\int_0^1 \rho^2 h_2(\rho | X) d\rho]^{1/2} \]

\[ = \int_0^1 \rho^{T+a+1} (1-\rho)^{n+b-1} d\rho \]

\[ = \left[ \frac{(T+a+1)(T+a)}{(T+a+b+n+1)(T+a+b+n)} \right]^{1/2} \]  

(11)

4. Numerical Results

We generated 2000 samples of size \( n=50 \) from the geometric distribution as mentioned in section 2. Table 1 and Table 2 respectively show Bayesian estimators of the parameter for different values of the parameter \( \rho \) under the prior distributions (i) quasi-prior density (ii) Beta prior density.

Table 1. Estimators and squared error of estimators with \( \rho = 0.5, n = 50 \)

| \( c \) | \( \hat{\rho}_1 \) | ER(\( \hat{\rho}_1 \)) | a | b | \( \hat{\rho}_2 \) | ER(\( \hat{\rho}_2 \)) |
|---|---|---|---|---|---|---|
| 0.0 | 0.4967 | 0.0023 | 1.0 | 1.0 | 0.4967 | 0.0023 |
| 1.0 | 0.4917 | 0.0025 | 1.0 | 1.5 | 0.4943 | 0.0023 |
| 1.5 | 0.4891 | 0.0026 | 1.0 | 2.0 | 0.4919 | 0.0024 |
| 2.0 | 0.4865 | 0.0027 | 1.5 | 1.0 | 0.4992 | 0.0022 |
| 2.5 | 0.4839 | 0.0028 | 2.0 | 1.5 | 0.4993 | 0.0022 |
| 3.0 | 0.4812 | 0.0029 | 2.0 | 3.0 | 0.4922 | 0.0023 |
Table2. Estimators and squared error of estimators with $\rho = 0.8, n = 50$

| c   | $\hat{\rho}_1$ | $ER(\hat{\rho}_1)$ | a   | b   | $\hat{\rho}_2$ | $ER(\hat{\rho}_2)$ |
|-----|----------------|---------------------|-----|-----|----------------|---------------------|
| 0   | 0.7956         | 6.8898e-004         | 1.0 | 1.0 | 0.7956         | 6.8898e-004         |
| 1.0 | 0.7948         | 7.0810e-004         | 1.0 | 1.5 | 0.7940         | 7.1308e-004         |
| 1.5 | 0.7943         | 7.1830e-004         | 1.0 | 2.0 | 0.7924         | 7.4208e-004         |
| 2.0 | 0.7939         | 7.2894e-004         | 1.5 | 1.0 | 0.7960         | 6.08004e-004        |
| 2.5 | 0.7935         | 7.3080e-004         | 2.0 | 1.5 | 0.7948         | 6.9286e-004         |
| 3.0 | 0.7931         | 7.5153e-004         | 2.0 | 3.0 | 0.7901         | 7.8587e-004         |

5. Conclusion

A Bayesian approach to estimation in M/M/1 queue has been presented. Unlike the frequentist approach, which has been shown to possess a number of undesirable properties, Bayesian technique lends itself to lucid analysis. It has been observed from Table1 and Table2 that there is very little change in values of the Bayesian estimators of the traffic intensity $\rho$. We have considered two priors with the sole intention of providing alternatives to the practitioner. We believe these are quite flexible and capable of modeling a wide variety of prior information.

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