Mean-field type Quadratic BSDEs

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January 29, 2022

Abstract

In this paper, we give several new results on solvability of a quadratic BSDE whose generator depends also on the mean of both variables. First, we consider such a BSDE using John-Nirenberg’s inequality for BMO martingales to estimate its contribution to the evolution of the first unknown variable. Then we consider the BSDE having an additive expected value of a quadratic generator in addition to the usual quadratic one. In this case, we use a deterministic shift transformation to the first unknown variable, when the usual quadratic generator depends neither on the first variable nor its mean, the general case can be treated by a fixed point argument.

1 Introduction

Let \( \{W_t := (W^1_t, \ldots, W^d_t)^*, 0 \leq t \leq T\} \) be a \( d \)-dimensional standard Brownian motion defined on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Denote by \( \{\mathcal{F}_t, 0 \leq t \leq T\} \) the augmented natural filtration of the standard Brownian motion \( W \).

In this paper, we study the existence and uniqueness of an adapted solution of the following BSDE:

\[
Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s]) \, ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T].
\] (1.1)

When \( f \) does not depend on \((\bar{y}, \bar{z})\), BSDE (1.1) is the classical one, and it is extensively studied in the literature, see the pioneer work of Bismut \([1, 2]\) as well as Pardoux and Peng \([10]\). When \( f(t, y, \bar{y}, z, \bar{z}) \) is scalar valued and quadratic in \( z \) while it does not depend on \((\bar{y}, \bar{z})\), BSDE (1.1) is the so-called quadratic BSDE and has been studied

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by Kobylanski [13], Briand and Hu [4, 5]. BSDE (1.1) (called mean-field type BSDE) arises naturally when studying mean-field games, etc. We refer to [6] for the motivation of its study. When the generator $f$ is uniformly Lipschitz in the last four arguments, BSDE (1.1) is shown in a straightforward manner to have a unique adapted solution, and the reader is referred to Buckdahn et al. [6] for more details. For the general generator $f(s, y, \bar{y}, z, \bar{z})$ depending quadratically on $z$, BSDE (1.1) is a quadratic one involving both $E[Y]$ and $E[Z]$. The comparison principle (see [13]) is well known to play a crucial role in the study of quadratic BSDEs (see [13]). Unfortunately, the comparison principle fails to hold for BSDE (1.1) (see, e.g. [6] for a counter-example for comparison with Lipschitz generators), the derivation of its solvability is not straightforward. Up to our best knowledge, no study on quadratic mean-field type BSDEs is available. To tackle the difficulty of lack of comparison principle, we use the John-Nirenberg inequality for BMO martingales to address the solvability.

Furthermore, we study the following alternative of mean-field type BSDE, which admits a quadratic growth in the mean of the second unknown variable $E[Z]$:

$$Y_t = \xi + \int_t^T \left[ f_1(s, Y_s, E[Y_s], Z_s, E[Z_s]) + E[f_2(s, Y_s, E[Y_s], Z_s, E[Z_s])] \right] ds - \int_t^T Z_s \cdot dW_s,$$

where $f_2$ is allowed to grow quadratically in both $Z$ and $E[Z]$, the function $f_1$ also admits a quadratic growth in the second unknown variable for the scalar case.

To deal with the additive expected value of $f_2$, Cheridito and Nam [8] introduced Krasnoselskii fixed point theorem to conclude the existence and uniqueness, by observing that the range of the expected value of $f_2$ is (locally) compact. Here we observe the following fact: the expected value of $f_2$ has no contribution to the second unknown variable $Z$ if $f_1$ depends neither on the first variable $Z$ nor on its mean. Hence we use the shift transformation to remove the expectation of $f_2$. In the general case, we apply the same kind of technique and the contraction mapping principle.

Let us close this section by introducing some notations. Denote by $\mathcal{S}_\infty(\mathbb{R}^n)$ the totality of $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted essentially bounded continuous processes, and by $||Y||_{\infty}$ the essential supremum norm of $Y \in \mathcal{S}_\infty(\mathbb{R}^n)$. It can be verified that $(\mathcal{S}_\infty(\mathbb{R}^n), ||\cdot||_{\infty})$ is a Banach space. Let $M = (M_t, \mathcal{F}_t)$ be a uniformly integrable martingale with $M_0 = 0$, and for $p \in [1, \infty)$ we set

$$\|M\|_{BMO_p(\mathbb{P})} := \sup_\tau \left[ \mathbb{E}_\tau \left[ \left( \langle M \rangle_\tau^\frac{p}{2} \right)^{\frac{2}{p}} \right] \right],$$

where the supremum is taken over all stopping times $\tau$. The class $\{M : \|M\|_{BMO_p} < \infty\}$ is denoted by $BMO_p$, which is written as $BMO_p(\mathbb{P})$ whenever it is necessary to indicate the underlying probability, and observe that $\|\cdot\|_{BMO_p}$ is a norm on this space and $BMO_p(\mathbb{P})$ is a Banach space.

Denote by $\mathcal{E}(M)$ the stochastic exponential of a one-dimensional local martingale $M$ and by $\mathcal{E}(M)_s^t$ that of $M_t - M_s$. Denote by $\beta \cdot M$ the stochastic integral of a scalar-valued adapted process $\beta$ with respect to a local continuous martingale $M$. 

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For any real $p \geq 1$, $S^p(\mathbb{R}^n)$ denotes the set of $\mathbb{R}^n$-valued adapted and càdlàg processes $(Y_t)_{t \in [0,T]}$ such that

$$||Y||_{S^p(\mathbb{R}^n)} := \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t|^p \right]^{1/p} < +\infty,$$

and $\mathcal{M}^p(\mathbb{R}^{d \times n})$ denotes the set of adapted processes $(Z_t)_{t \in [0,T]}$ with values in $\mathbb{R}^{d \times n}$ such that

$$||Z||_{\mathcal{M}^p(\mathbb{R}^{d \times n})} := \mathbb{E} \left[ \left( \int_0^T |Z_s|^2 ds \right)^{p/2} \right]^{1/p} < +\infty.$$

The rest of our paper is organized as follows. In Section 2, we study BSDE (1.1) when $f(t, y, \bar{y}, z, \bar{z})$ is scalar valued and quadratic in $z$, and uniformly Lipschitz in $(y, \bar{y}, z)$, and prove by the contraction mapping principle that BSDE (1.1) has a unique solution. In Section 3, we study scalar-valued BSDE (1.2) when $f_2(t, y, \bar{y}, z, \bar{z})$ is both quadratic in $z$ and $\bar{z}$, and $f_1$ is quadratic in $z$. Finally, in Section 4, we study BSDE (1.2) in the multi-dimensional case, where we suppose that $f_2(t, y, \bar{y}, z, \bar{z})$ is both quadratic in $z$ and $\bar{z}$, and $f_1$ is Lipschitz in $z$ and $\bar{z}$.

2 Quadratic BSDEs with a mean term involving the second unknown variable

In this section we consider the following BSDE:

$$Y_t = \xi + \int_t^T f(s, Y_s, E[Y_s], Z_s, E[Z_s]) ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0,T].$$

(2.1)

We first recall the following existence and uniqueness, a priori estimate for one-dimensional BSDEs.

**Lemma 2.1.** Assume that (i) the function $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ has the following growth and locally Lipschitz continuity in the last two variables:

$$|f(s, y, z)| \leq g_s + \beta |y| + \frac{\gamma}{2} |z|^2, \quad y \in \mathbb{R}, z \in \mathbb{R}^d;$$

$$|f(s, y, z_1) - f(s, y, z_2)| \leq C|y_1 - y_2| + C(1 + |z_1| + |z_2|)|z_1 - z_2|, \quad y_1, y_2 \in \mathbb{R}, z_1, z_2 \in \mathbb{R}^d;$$

(2.2)

(ii) the process $f(\cdot, y, z)$ is $\mathcal{F}_t$-adapted for each $y \in \mathbb{R}$, $z \in \mathbb{R}^d$; and (iii) $g \in L^1(0, T)$.

Then for bounded $\xi$, the following BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T]$$

(2.3)

has a unique solution $(Y, Z)$ such that $Y$ is (essentially) bounded and $Z \cdot W$ is a BMO martingale. Furthermore, we have

$$e^{\gamma |Y_t|} \leq \mathbb{E}_t \left[ e^{\gamma e^{\beta(T-t)} \xi + \gamma \int_t^T |g(s)| e^{\beta(s-t)} ds} \right].$$
The following lemma plays an important role in our subsequent arguments. It indicates that following the proof of [12, Theorem 3.3, page 57] can give a more precise dependence of the two constants $c_1, c_2$ on $\beta \cdot M$.

**Lemma 2.2.** For $K > 0$, there are constants $c_1 > 0$ and $c_2 > 0$ depending only on $K$ such that for any BMO martingale $M$, we have for any one-dimensional BMO martingale $N$ such that $\|N\|_{BMO_2(\mathbb{P})} \leq K$,

$$c_1 \|M\|_{BMO_2(\mathbb{P})} \leq \|\tilde{M}\|_{BMO_2(\mathbb{P})} \leq c_2 \|M\|_{BMO_2(\mathbb{P})}$$

(2.4)

where $\tilde{M} := M - \langle M, N \rangle$ and $d\tilde{\mathbb{P}} := \mathcal{E}(N)^\infty d\mathbb{P}$.

### 2.1 Main Results

We make the following three assumptions. Let $C$ be a positive constant.

\textbf{(2.1)} Assume that there are positive constants $C$ and $\gamma$ and $\alpha \in [0, 1)$ such that the function $f : \Omega \times [0, T] \times \mathbb{R}^2 \times (\mathbb{R}^d)^2 \to \mathbb{R}$ has the following linear-quadratic growth and globally-locally Lipschitz continuity: for all $(\omega, s, y_i, z_i, \bar{z}_i) \in \Omega \times [0, T] \times \mathbb{R}^2 \times (\mathbb{R}^d)^2$ with $i = 1, 2$,

$$|f(\omega, s, y, \bar{y}, z, \bar{z})| \leq C(2 + |y| + |\bar{y}| + |\bar{z}|^{1+\alpha}) + \frac{1}{2} \gamma |z|^2;$$

$$|f(s, y_1, \bar{y}_1, z_1, \bar{z}_1) - f(s, y_2, \bar{y}_2, z_2, \bar{z}_2)| \leq C \left\{ |y_1 - y_2| + |\bar{y}_1 - \bar{y}_2| + (1 + |\bar{z}_1|^\alpha + |\bar{z}_2|^\alpha)|\bar{z}_1 - \bar{z}_2| 

+ (1 + |z_1| + |z_2|)|z_1 - z_2| \right\};$$

(2.5)

The process $f(\cdot, y, \bar{y}, z, \bar{z})$ is $\mathcal{F}_t$-adapted for each $(y, \bar{y}, z, \bar{z})$.

\textbf{(2.2)} The terminal condition $\xi$ is uniformly bounded by $C$.

We have the following two theorems.

The first one is a result concerning local solutions. For this, let us introduce some notations. For $\varepsilon > 0$, and $r_{\varepsilon} > 0$, we define the ball $\mathbb{B}_\varepsilon$ by

$$\mathbb{B}_\varepsilon := \left\{ (Y, Z) : Y \in S^\infty, \ Z \cdot W \in BMO_2(\mathbb{P}), \ ||Y||_{\infty, [T-\varepsilon, T]} + ||Z \cdot W||_{BMO_2, [T-\varepsilon, T]} \leq r_\varepsilon \right\}.$$

**Theorem 2.3.** Let assumptions \textbf{(2.1)} and \textbf{(2.2)} be satisfied with $\alpha \in [0, 1)$. Then, for any bounded $\xi$, there exist $\varepsilon > 0$ and $r_{\varepsilon} > 0$ such that the following BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s]) \, ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T]$$

(2.6)

has a unique local solution $(Y, Z)$ in the time interval $[T - \varepsilon, T]$ with $(Y, Z) \in \mathbb{B}_\varepsilon$.
Example 2.4. The condition on $f$ means that $f$ is of linear growth with respect to $(y, \bar{y})$, and of $|\bar{z}|^{1+\alpha}$ growth. For example, for $\alpha \in (0, 1)$,

$$f(s, y, \bar{y}, z, \bar{z}) = 1 + |y| + |\bar{y}| + \frac{1}{2}|z|^2 + |\bar{z}|^{1+\alpha}.$$  

The second theorem is a result about global solutions.

Theorem 2.5. Let assumption (A2) be satisfied. Moreover, assume that there is a positive constant $C$ such that the function $f : \Omega \times [0, T] \times \mathbb{R}^2 \times (\mathbb{R}^d)^2 \to \mathbb{R}$ has the following linear-quadratic growth and globally-locally Lipschitz continuity: for $\forall (\omega, s, y_i, \bar{y}_i, z_i, \bar{z}_i) \in \Omega \times [0, T] \times \mathbb{R}^2 \times (\mathbb{R}^d)^2$ with $i = 1, 2$,

$$|f(s, 0, 0, 0, 0) + h(\omega, s, y, \bar{y}, z, \bar{z})| \leq C(1 + |y| + |\bar{y}|),$$
$$|f(\omega, s, 0, 0, z_1, 0) - f(\omega, s, 0, 0, z_2, 0)| \leq C(1 + |z_1| + |z_2|)|z_1 - z_2|,$$
$$|h(s, y_1, \bar{y}_1, z_1, \bar{z}_1) - h(s, y_2, \bar{y}_2, z_2, \bar{z}_2)| \leq C(|y_1 - y_2| + |\bar{y}_1 - \bar{y}_2| + |z_1 - z_2| + |\bar{z}_1 - \bar{z}_2|)$$

where for $(\omega, s, y, \bar{y}, z, \bar{z}) \in \Omega \times [0, T] \times \mathbb{R}^2 \times (\mathbb{R}^d)^2$,

$$h(s, y, \bar{y}, z, \bar{z}) := f(s, y, \bar{y}, z, \bar{z}) - f(s, 0, 0, z, 0).$$

The process $f(\cdot, y, \bar{y}, z, \bar{z})$ is $\mathcal{F}_t$-adapted for each $(y, \bar{y}, z, \bar{z}) \in \mathbb{R}^2 \times (\mathbb{R}^d)^2$.

Then, the following BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s]) \, ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T]$$

has a unique adapted solution $(Y, Z)$ on $[0, T]$ such that $Y$ is bounded. Furthermore, $Z \cdot W$ is a $\text{BMO}(\mathbb{P})$ martingale.

Example 2.6. The inequality (2.7) requires that $f$ is bounded with respect to the last variable $\bar{z}$. The following function

$$f(s, y, \bar{y}, z, \bar{z}) = 1 + s + |y| + |\bar{y}| + \frac{1}{2}|z|^2 + |\sin(\bar{z})|, \quad (s, y, \bar{y}, z, \bar{z}) \in [0, T] \times \mathbb{R}^2 \times (\mathbb{R}^d)^2,$$

satisfies such an inequality.

2.2 Local solution: the proof of Theorem 2.3

We prove Theorem 2.3 (using the contraction mapping principle) in the following three subsections: in Subsection 2.2.1, we construct a map (which we call quadratic solution map) in a Banach space; in Subsection 2.2.2, we show that this map is stable in a small ball; and in Subsection 2.2.3, we prove that this map is a contraction.
2.2.1 Construction of the map

For a pair of bounded adapted process $U$ and BMO martingale $V \cdot W$, we consider the following quadratic BSDE:

$$Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[V_s]) \, ds - \int_t^T Z_s \cdot dW_s, \; t \in [0, T].$$

(2.10)

As

$$|f(s, y, \mathbb{E}[U_s], Z_s, \mathbb{E}[V_s])| \leq C(2 + |\mathbb{E}[U_s]| + |\mathbb{E}[V_s]|^{1+\alpha}) + C|y| + \frac{1}{2} \gamma |z|^2,$$

in view of Lemma 2.1, it has a unique adapted solution $(Y, Z)$ such that $Y$ is bounded and $Z \cdot W$ is a BMO martingale. Define the quadratic solution map $\Gamma : (U, V) \mapsto \Gamma(U, V)$ as follows:

$$\Gamma(U, V) := (Y, Z), \; \forall (U, V \cdot W) \in \mathcal{S}^\infty \times \text{BMO}_2(\mathbb{P}).$$

It is a transformation in the Banach space $\mathcal{S}^\infty \times \text{BMO}_2(\mathbb{P})$.

Let us introduce here some constants and a quadratic (algebraic) equation which will be used in the next two subsections.

Define

$$C_\delta := e^{\frac{6}{1-\alpha} \gamma CT e^{CT} + \frac{1-\alpha}{2} (\frac{3}{1-\alpha} \gamma C e^{CT}) \frac{1}{\delta}} (1 + \frac{1-\alpha}{2}) \frac{1+\alpha}{1-\alpha};$$

(2.11)

$$\beta := \frac{1}{2} (1-\alpha) C T^{-\alpha} (2(1+\alpha)) \frac{1+\alpha}{1-\alpha};$$

(2.12)

$$\mu_1 := (1-\alpha) \left(1 + \frac{1-\alpha}{(1+\alpha)\gamma}\right) = 1 - \alpha + \frac{(1-\alpha)^2}{(1+\alpha)\gamma};$$

(2.13)

$$\mu_2 := \frac{1}{2} (1+\alpha) \left(1 + \frac{1-\alpha}{(1+\alpha)\gamma}\right) = \frac{1}{2} (1+\alpha) + \frac{1-\alpha}{2\gamma};$$

(2.14)

$$\mu := (\beta + C\mu_1) \frac{\gamma^2}{\alpha - 1} + 2C\mu_2.$$

Consider the following standard quadratic equation of $A$:

$$\delta A^2 - \left(1 + 4\gamma^{-2} e^{\gamma|\xi|\delta} \right) A + 4\gamma^{-2} e^{\gamma|\xi|\delta} + 4\mu C_\delta e^{\frac{3\gamma e^{CT}}{1-\alpha} \gamma|\xi|\varepsilon} = 0.$$

The discriminant of the quadratic equation reads

$$\Delta := \left(1 + 4\gamma^{-2} e^{\gamma|\xi|\delta}\right)^2 - 4\delta \left[4\gamma^{-2} e^{\gamma|\xi|\delta} + 4\mu C_\delta e^{\frac{3\gamma e^{CT}}{1-\alpha} \gamma|\xi|\varepsilon}\right]$$

(2.15)

$$= \left(1 - 4\gamma^{-2} e^{\gamma|\xi|\delta}\right)^2 - 16\mu \delta C_\delta e^{\frac{3\gamma e^{CT}}{1-\alpha} \gamma|\xi|\varepsilon}.$$

Take

$$\delta := \frac{1}{8} \gamma^2 e^{-\gamma|\xi|\delta}, \; \varepsilon \leq \min \left\{ \frac{1}{3C} e^{-CT}, \frac{1}{8\mu C_\delta} \gamma^{-2} e^{\gamma (1 + \frac{3\gamma e^{CT}}{1-\alpha})|\xi|\varepsilon} \right\},$$

(2.16)

$$A := \frac{1 + 4\gamma^{-2} e^{\gamma|\xi|\delta} - \sqrt{\Delta}}{2\delta} = \frac{3 - 2\sqrt{\Delta}}{4\delta} \leq \frac{3}{4\delta} = 6\gamma^{-2} e^{\gamma|\xi|\delta},$$

$$= \frac{3}{4\delta} = 6\gamma^{-2} e^{\gamma|\xi|\delta},$$
and we have

\[ \Delta \geq 0, \quad 1 - \delta A = \frac{1 + 2\sqrt{\Delta}}{4} > 0, \]

\[ \gamma^{-2} e^{\gamma |\xi|_{\infty}} + \mu C_\delta \frac{e^{\frac{nCT}{\tau}} |\xi|_{\infty}}{1 - \delta A} + \frac{1}{4} A = \frac{1}{2} A. \]  

(2.17)

Throughout this section, we base our discussion on the time interval \([T - \varepsilon, T]\).

We shall prove Theorem 2.3 by showing that the quadratic solution map \(\Gamma\) is a contraction on the closed convex set \(B_\varepsilon\) defined by

\[ B_\varepsilon := \left\{ (U, V) : \begin{array}{l}
U \in S^\infty, \\
V \cdot W \in BMO_2(\mathbb{P}), \\
|V \cdot W|_{BMO_2}^2 \leq A, \\
e_1^{2 - \alpha} |U|_{\infty} \leq C_\delta e^{\frac{2\gamma e^{CT}}{1 - \delta A}} |\xi|_{\infty} \end{array} \right\} \]

(2.18)

(where \((U, V)\) is defined on \(\Omega \times [T - \varepsilon, T]\)) for a positive constant \(\varepsilon\) (to be determined later).

2.2.2 Estimation of the quadratic solution map

We shall show the following assertion: \(\Gamma(B_\varepsilon) \subset B_\varepsilon\), that is,

\[ \Gamma(U, V) \in B_\varepsilon, \quad \forall (U, V) \in B_\varepsilon. \]

(2.19)

*Step 1. Exponential transformation.*

Define

\[ \phi(y) := \gamma^{-2} [\exp (\gamma |y|) - \gamma |y| - 1], \quad y \in \mathbb{R}. \]

(2.20)

Then, we have for \(y \in \mathbb{R},\)

\[ \phi'(y) = \gamma^{-1} [\exp (\gamma |y|) - 1] \text{sgn}(y), \quad \phi''(y) = \exp (\gamma |y|), \quad \phi''(y) - \gamma |\phi'(y)| = 1. \]

(2.21)

Using Itô’s formula, we have for \(t \in [T - \varepsilon, T],\)

\[ \phi(Y_t) + \frac{1}{2} \mathbb{E}_t \left[ \int_t^T |Z_s|^2 \, ds \right] \leq \phi(|\xi|_{\infty}) + C \mathbb{E}_t \left[ \int_t^T |\phi'(Y_s)| \left( 2 + |Y_s| + |\mathbb{E}[U_s]| + |\mathbb{E}[V_s]|^{1+\alpha} \right) \, ds \right]. \]

(2.22)

Since (in view of the definition of notation \(\beta\) in (2.12))

\[ C |\phi'(Y_s)| |\mathbb{E}[V_s]|^{1+\alpha} \leq \frac{1}{4} |\mathbb{E}[V_s]|^2 + \beta |\phi'(Y_s)|^{\gamma - \alpha}, \]

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we have
\[\phi(Y_t) + \frac{1}{2} E_t \left[ \int_t^T |Z_s|^2 \, ds \right] \]
\[\leq \phi(|\xi|_\infty) + CE_t \left[ \int_t^T |\phi'(Y_s)| (2 + |Y_s| + |E[U_s]|) \, ds \right] \]
\[+ \beta E_t \left[ \int_t^T |\phi'(Y_s)|^{\frac{2}{\gamma}} \, ds \right] + \frac{1}{4} E_t \left[ \int_t^T |E[V_s]|^2 \, ds \right] \]
\[\leq \phi(|\xi|_\infty) + CE_t \left[ \int_t^T |\phi'(Y_s)| ((1 + |Y_s|) + (1 + |E[U_s]|)) \, ds \right] \]
\[+ \beta E_t \left[ \int_t^T |\phi'(Y_s)|^{\frac{2}{\gamma}} \, ds \right] + \frac{1}{4} E_t \left[ \int_t^T |E[V_s]|^2 \, ds \right]. \tag{2.23} \]

In view of the inequality for \( x > 0, \)
\[1 + x \leq \left( 1 + \frac{1 - \alpha}{\gamma(1 + \alpha)} \right) e^{\frac{\gamma(1+\alpha) x}{4}}, \]
we have
\[CE_t \left[ \int_t^T |\phi'(Y_s)| ((1 + |Y_s|) + (1 + |E[U_s]|)) \, ds \right] \]
\[\leq CE_t \left[ \int_t^T |\phi'(Y_s)| \left( 1 + \frac{1 - \alpha}{\gamma(1 + \alpha)} \right) \left( e^{\frac{\gamma(1+\alpha) |Y_s|}{4}} + e^{\frac{\gamma(1+\alpha) |E[U_s]|}{4}} \right) \, ds \right]. \tag{2.24} \]

Since (by Young’s inequality)
\[|\phi'(Y_s)| \left( e^{\frac{\gamma(1+\alpha) |Y_s|}{4}} + e^{\frac{\gamma(1+\alpha) |E[U_s]|}{4}} \right) \leq (1 - \alpha) |\phi'(Y_s)|^{\frac{2}{\gamma}} + \frac{1 + \alpha}{2} \left( e^{\frac{2 \alpha \gamma |Y_s|}{4}} + e^{\frac{2 \alpha \gamma |E[U_s]|}{4}} \right), \]
in view of the definition of the notations \( \mu_1 \) and \( \mu_2 \) in (2.13), we have
\[CE_t \left[ \int_t^T |\phi(Y_s)| ((1 + |Y_s|) + (1 + |E[U_s]|)) \, ds \right] \]
\[\leq C \mu_1 E_t \left[ \int_t^T |\phi'(Y_s)|^{\frac{2}{\gamma}} \, ds \right] + C \mu_2 E_t \left[ \int_t^T \left( e^{\frac{2 \alpha \gamma |Y_s|}{4}} + e^{\frac{2 \alpha \gamma |E[U_s]|}{4}} \right) \, ds \right]. \tag{2.25} \]

In view of inequality (2.23), we have
\[\phi(Y_t) + \frac{1}{2} E_t \left[ \int_t^T |Z_s|^2 \, ds \right] \]
\[\leq \phi(|\xi|_\infty) + (\beta + C \mu_1) E_t \left[ \int_t^T |\phi'(Y_s)|^{\frac{2}{\gamma}} \, ds \right] \]
\[+ C \mu_2 E_t \left[ \int_t^T \left( e^{\frac{2 \alpha \gamma |Y_s|}{4}} + e^{\frac{2 \alpha \gamma |E[U_s]|}{4}} \right) \, ds \right] + \frac{1}{4} E_t \left[ \int_t^T |E[V_s]|^2 \, ds \right] \]
\[\leq \phi(|\xi|_\infty) + C \mu_2 + \gamma^{\frac{2}{\alpha \gamma}} (\beta + C \mu_1) E_t \left[ \int_t^T e^{\frac{2 \alpha \gamma |Y_s|}{4}} \, ds \right] \]
\[+ C \mu_2 E_t \left[ \int_t^T e^{\frac{2 \alpha \gamma |U_s|}{4}} \, ds \right] + \frac{1}{4} E \left[ \int_t^T |V_s|^2 \, ds \right]. \tag{2.26} \]
Step 2. Estimate of $e^{\|Y\|_\infty}$.

In view of the last inequality of Lemma 2.1, we have
\begin{equation}
E_t \left[ e^{\frac{3}{1-\alpha} \gamma |Y_t|} \right] \leq E_t \left[ e^{\frac{3}{1-\alpha} \gamma e^{CT}(|\xi|_\infty + C\int_t^T (2+|E[U_s]|+|E[V_s]|)^{1+\alpha} ds)} \right].
\end{equation}

(2.27)

Since (by Young’s inequality)
\begin{equation}
\frac{3e^{CT}}{1-\alpha} \gamma C |E[V_s]|^{1+\alpha} \leq \frac{1-\alpha}{2} \left( \frac{3e^{CT}}{1-\alpha} \gamma C \left( \frac{1+\alpha}{2\delta} \right)^{\frac{1+\alpha}{2}} \right)^{\frac{2}{1-\alpha}} + \delta |E[V_s]|^2,
\end{equation}
in view of the definition of notation $C_\delta$ in (2.11), we have
\begin{equation}
e^{\frac{\gamma}{1-\alpha} |Y_t|} \leq C_\delta \left[ e^{\left( \frac{\gamma}{1-\alpha} e^{CT} (|\xi|_\infty + C\gamma |U|_\infty) \right)} E \left[ e^{\delta \int_t^T |E[V_s]|^2 ds} \right] \right].
\end{equation}

(2.29)

and therefore using Jensen’s inequality,
\begin{equation}
e^{\frac{\gamma}{1-\alpha} |Y_t|} \leq C_\delta \left[ e^{\left( \frac{\gamma}{1-\alpha} e^{CT} (|\xi|_\infty + C\gamma |U|_\infty) \right) E} \right] E \left[ e^{\delta \int_t^T |E[V_s]|^2 ds} \right].
\end{equation}

(2.30)

It follows from (2.16) and the definition of $B_\varepsilon$ that $\|\sqrt{\delta} V \cdot W\|_{BMO^2(\mathbb{F})}^2 \leq \delta A < 1$, applying the John-Nirenberg inequality to the BMO martingale $\sqrt{\delta} V \cdot W$, we have
\begin{equation}
e^{\frac{\gamma}{1-\alpha} |Y_t|} \leq C_\delta \left[ e^{\left( \frac{\gamma}{1-\alpha} e^{CT} (|\xi|_\infty + C\gamma |U|_\infty) \right) E} \right] \left( \frac{C_\delta e^{\left( \frac{\gamma}{1-\alpha} e^{CT} (|\xi|_\infty + C\gamma |U|_\infty) \right) E}}{1-\delta A} \right).
\end{equation}

(2.31)

Since $3e^{CT} C \varepsilon \leq 1$ (see the choice of $\varepsilon$ in (2.16)) and $(U, V) \in B_\varepsilon$, we have
\begin{equation}
e^{\left( \frac{\gamma}{1-\alpha} e^{CT} (|\xi|_\infty + C\gamma |U|_\infty) \right)} \leq C_\delta \left[ e^{\left( \frac{\gamma}{1-\alpha} e^{CT} (|\xi|_\infty + C\gamma |U|_\infty) \right) E} \right] \frac{1}{1-\delta A} \left( C_\delta \left[ e^{\left( \frac{\gamma}{1-\alpha} e^{CT} (|\xi|_\infty + C\gamma |U|_\infty) \right) E} \right] \right)^\frac{1}{2} \left( C_\delta \left[ e^{\left( \frac{\gamma}{1-\alpha} e^{CT} (|\xi|_\infty + C\gamma |U|_\infty) \right) E} \right] \right)^\frac{1}{4} \left( \frac{1}{1-\delta A} \right)^\frac{1}{4}.
\end{equation}

(2.32)

which gives a half of the desired result (2.19).

Step 3. Estimate of $\|Z \cdot W\|_{BMO^2}^2$.

From inequality (2.26) and the definition of notation $\mu$ in (2.14), we have
\begin{equation}
\frac{1}{2} E_t \left[ \int_t^T |Z_s|^2 ds \right] \leq \gamma^{-2} e^{\gamma |\xi|_\infty} + \mu C_\delta \frac{e^{\gamma CT} e^{\gamma |\xi|_\infty}}{1-\delta A} \varepsilon + \frac{1}{4} A.
\end{equation}

(2.33)

In view of (2.17), we have
\begin{equation}
\frac{1}{2} \|Z \cdot W\|_{BMO^2}^2 \leq \gamma^{-2} e^{\gamma |\xi|_\infty} + \mu C_\delta \frac{e^{\gamma CT} e^{\gamma |\xi|_\infty}}{1-\delta A} \varepsilon + \frac{1}{4} A = \frac{1}{2} A.
\end{equation}

(2.34)

The other half of the desired result (2.19) is then proved.
2.2.3 Contraction of the quadratic solution map

For \((U, V) \in \mathcal{B}_\varepsilon\) and \((\tilde{U}, \tilde{V}) \in \mathcal{B}_\varepsilon\), set

\[
(Y, Z) := \Gamma(U, V), \quad (\tilde{Y}, \tilde{Z}) := \Gamma(\tilde{U}, \tilde{V}).
\]

That is,

\[
Y_t = \xi + \int_t^T f(s, Y_s, E[U_s], Z_s, E[V_s]) \, ds - \int_t^T Z_s \, dW_s, \\
\tilde{Y}_t = \xi + \int_t^T f(s, \tilde{Y}_s, E[\tilde{U}_s], \tilde{Z}_s, E[\tilde{V}_s]) \, ds - \int_t^T \tilde{Z}_s \, dW_s.
\]

(2.35)

We can define the vector process \(\beta\) in an obvious way such that

\[
|\beta_s| \leq C(1 + |Z_s| + |\tilde{Z}_s|), \quad f(s, Y_s, E[U_s], Z_s, E[V_s]) - f(s, Y_s, E[U_s], \tilde{Z}_s, E[\tilde{V}_s]) = (Z_s - \tilde{Z}_s)\beta_s.
\]

(2.36)

Then \(\tilde{W}_t := W_t - \int_0^t \beta_s \, ds\) is a Brownian motion under the equivalent probability measure \(\bar{P}\) defined by

\[
d\bar{P} := \mathcal{E}(\beta \cdot W)_0^T \, dP,
\]

and from the above-established a priori estimate, there is \(K > 0\) such that \(\|\beta \cdot W\|_{BMO}^2 \leq K^2 := 3C^2T + 6C^2A\).

In view of the following equation

\[
Y_t - \tilde{Y}_t + \int_t^T (Z_s - \tilde{Z}_s) \, dW_s = \int_t^T \left[ f(s, Y_s, E[U_s], \tilde{Z}_s, E[V_s]) - f(s, \tilde{Y}_s, E[\tilde{U}_s], \tilde{Z}_s, E[\tilde{V}_s]) \right] \, ds,
\]

(2.37)

taking square and then the conditional expectation with respect to \(\bar{P}\) (denoted by \(\bar{E}_t\)) on both sides of the last equation, we have the following standard estimates:

\[
|Y_t - \tilde{Y}_t|^2 + \bar{E}_t \left[ \int_t^T |Z_s - \tilde{Z}_s|^2 \, ds \right]
\]

\[
= \bar{E}_t \left[ \left( \int_t^T \left( f(s, Y_s, E[U_s], \tilde{Z}_s, E[V_s]) - f(s, \tilde{Y}_s, E[\tilde{U}_s], \tilde{Z}_s, E[\tilde{V}_s]) \right) \, ds \right)^2 \right]
\]

\[
\leq C^2 \bar{E}_t \left[ \left( \int_t^T \left( |Y_s - \tilde{Y}_s| + |E[U_s - \tilde{U}_s]| + (1 + |E[V_s]|^\alpha + |E[\tilde{V}_s]|^\alpha) |E[V_s - \tilde{V}_s]| \right) \, ds \right)^2 \right]
\]

\[
\leq 3C^2(T - t)^2(\|U - \tilde{U}\|_\infty^2 + |Y - \tilde{Y}|_\infty^2)
\]

\[
+ 3C^2 \int_t^T (1 + |E[V_s]|^\alpha + |E[\tilde{V}_s]|^\alpha)^2 \, ds \int_t^T |E[V_s - \tilde{V}_s]|^2 \, ds
\]

\[
\leq 3C^2(T - t)^2(\|U - \tilde{U}\|_\infty^2 + |Y - \tilde{Y}|_\infty^2)
\]

\[
+ 9C^2 \int_t^T (1 + |E[V_s]|^{2\alpha} + |E[\tilde{V}_s]|^{2\alpha}) \, ds \int_t^T |E[V_s - \tilde{V}_s]|^2 \, ds
\]

(2.38)
We have for \( t \in [T - \varepsilon, T] \),
\[
\int_t^T (1 + |E[V_s]|^{2\alpha} + |E[\tilde{V}_s]|^{2\alpha}) \, ds \\
\leq \int_t^T (1 + E[|V_s|^2]^{\alpha} + E[|\tilde{V}_s|^2]^{\alpha}) \, ds \\
\leq \varepsilon + \varepsilon^{1-\alpha} E\left[ \int_t^T |V_s|^2 \, ds \right]^{\alpha} + \varepsilon^{1-\alpha} E\left[ \int_t^T |\tilde{V}_s|^2 \, ds \right]^{\alpha}
\]
(2.39)

Concluding the above estimates, we have for \( t \in [T - \varepsilon, T] \),
\[
|Y_t - \tilde{Y}_t|^2 + E_t\left[ \int_t^T |Z_s - \tilde{Z}_s|^2 \, ds \right] \\
\leq 3C^2\varepsilon^2(|U - \tilde{U}|^2_\infty + |Y - \tilde{Y}|^2_\infty) \\
+ 9C^2 (T^\alpha + 2 + 2\alpha A) \varepsilon^{1-\alpha} \| (V - \tilde{V}) \cdot W \|_{BMO_2(P)}^2.
\]
(2.40)

In view of estimates (2.4), noting that \( 1 - 3C^2\varepsilon^2 \geq \frac{2}{3} \), we have for \( t \in [T - \varepsilon, T] \),
\[
|Y - \tilde{Y}|^2_\infty + 3\varepsilon^2 (|Z - \tilde{Z}| \cdot W)\|_{BMO_2(P)}^2 \\
\leq 9C^2 \varepsilon^2 |U - \tilde{U}|^2_\infty + 27C^2 (T^\alpha + 2 + 2\alpha A) \varepsilon^{1-\alpha} \| (V - \tilde{V}) \cdot W \|_{BMO_2(P)}^2.
\]
(2.41)

It is then standard to show that there is a very small positive number \( \varepsilon \) such that the quadratic solution map \( \Gamma \) is a contraction on the previously given set \( \mathcal{B}_\varepsilon \), by noting that \( A \leq 6\gamma^{-2}e^{\gamma|\xi|_\infty} \) from (2.10). The proof is completed by choosing a sufficiently small \( r_\varepsilon > 0 \) such that \( \mathcal{B}_\varepsilon \subset \mathcal{B}_r \).

### 2.3 Global solution: the proof of Theorem 2.5

Let us first note that there exists a constant \( \tilde{C} > 0 \) such that \( |\xi|^2 \leq \tilde{C} \) and
\[
|2xh(s, y, \bar{y}, z, \bar{z}) + 2xf(s, 0, 0, 0, 0)| \leq \tilde{C} + \tilde{C}|x|^2 + \tilde{C}(|y|^2 + |\bar{y}|^2)
\]
for any \((x, y, \bar{y}, z, \bar{z}) \in R \times R^2 \times (R^2)^2\). Let \( \alpha(\cdot) \) be the unique solution of the following ordinary differential equation:
\[
\alpha(t) = \tilde{C} + \int_t^T \tilde{C} \, ds + \int_t^T (2\tilde{C} + \tilde{C})\alpha(s) \, ds, \quad t \in [0, T].
\]

It is easy to see that \( \alpha(\cdot) \) is a continuous decreasing function and we have
\[
\alpha(t) = \tilde{C} + \int_t^T \tilde{C}[1 + 2\alpha(s)] \, ds + \tilde{C} \int_t^T \alpha(s) \, ds, \quad t \in [0, T].
\]
Define

\[ \lambda := \sup_{t\in[0,T]} \alpha(t) = \alpha(0). \]

As \(|\xi|^2 \leq \tilde{C} \leq \lambda\), Theorem 2.3 shows that there exists \(\eta_\lambda > 0\) which only depends on \(\lambda\), such that BSDE has a local solution \((Y, Z)\) on \([T - \eta_\lambda, T]\) and it can be constructed through the Picard iteration.

Consider the Picard iteration:

\[
Y_t^0 = \xi + \int_t^T Z_s^0 dW_s; \quad \text{and for } j \geq 0,
\]

\[
Y_t^{j+1} = \xi + \int_t^T [f(s, 0, 0, Z_s^{j+1}, 0) - f(s, 0, 0, 0)] ds
+ \int_t^T [f(s, 0, 0, 0) + f(s, Y_s^j, E[Y_s^j], Z_s^j, E[Z_s^j]) - f(s, 0, 0, Z_s^j, 0)] ds
- \int_t^T Z_s^{j+1} dW_s
= \xi + \int_t^T [f(s, 0, 0, 0) + h(s, Y_s^j, E[Y_s^j], Z_s^j, E[Z_s^j])] ds
- \int_t^T Z_s^{j+1} d\tilde{W}_s^{j+1}, \quad t \in [T - \eta_\lambda, T],
\]

where

\[ f(s, 0, 0, Z_s^{j+1}, 0) - f(s, 0, 0, 0) = Z_s^{j+1}\beta_s^{j+1} \]

and the process

\[ \tilde{W}_t^{j+1} = W_t - \int_0^t \beta_s^{j+1} 1_{[T - \eta_\lambda, T]}(s) ds \]

is a Brownian motion under an equivalent probability measure \(P^{j+1}\) which we denote by \(\tilde{P}\) with loss of generality, and under which the expectation is denoted by \(\tilde{E}\). Using Itô’s formula, it is straightforward to deduce the following estimate for \(r \in [T - \eta_\lambda, T]\),

\[
\tilde{E}_r[|Y_t^{j+1}|^2] = \tilde{E}_r[\int_t^T |Z_s^{j+1}|^2 ds]
= \tilde{E}_r[|\xi|^2] + \tilde{C} \tilde{E}_r \int_t^T 2Y_s^{j+1} \left[ f(s, 0, 0, 0) + h(s, Y_s^j, E[Y_s^j], Z_s^j, E[Z_s^j]) \right] ds
\leq \tilde{C} \int_t^T \tilde{E}_r[|Y_s^j|^2] ds + \tilde{C} \int_t^T \{\tilde{E}_r[|Y_s^j|^2] + |E[Y_s^j]|^2 + 1\} ds. \tag{2.42}
\]

In what follows, we show by induction the following inequality:

\[ |Y_t^j|^2 \leq \alpha(t), \quad t \in [T - \eta_\lambda, T]. \tag{2.43} \]

In fact, it is trivial to see that \(|Y_t^0|^2 \leq \alpha(t)|, and let us suppose \(|Y_t^j|^2 \leq \alpha(t)| for \(t \in [T - \eta_\lambda, T]\). Then, from (2.42),

\[ \tilde{E}_r[|Y_t^{j+1}|^2] \leq \tilde{C} + \tilde{C} \int_t^T [1 + 2\alpha(s)] ds + \tilde{C} \int_t^T \tilde{E}_r[|Y_s^{j+1}|^2] ds, \quad t \in [T - \eta_\lambda, T]. \]
From the comparison theorem, we have
\[ \tilde{E}_r[|Y_t^{j+1}|^2] \leq \alpha(t), \quad t \in [T - \eta, T]. \]

Setting \( r = t \), we have
\[ |Y_t^{j+1}|^2 \leq \alpha(t), \quad t \in [T - \eta, T]. \]

Therefore, inequality (2.43) holds.

As \( Y_t = \lim_j Y^j_t \), our constructed local solution \((Y, Z)\) in \([T - \eta, T]\) satisfies the following estimate:
\[ |Y_t|^2 \leq \alpha(t), \quad t \in [T - \eta, T]. \]

In particular, \( |Y_{T-\eta}|^2 \leq \alpha(T - \eta) \leq \lambda. \)

Taking \( T - \eta \) as the terminal time and \( Y_{T-\eta} \) as terminal value, Theorem 2.3 shows that BSDE has a local solution \((Y, Z)\) on \([T - 2\eta, T]\) through the Picard iteration. Once again, using the Picard iteration and the fact that \( |Y_t|^2 \leq \alpha(t) \), for \( t \in [T - 2\eta, T - \eta] \), we deduce that \( |Y_T|^2 \leq \alpha(T - \eta) \leq \lambda. \) We now show that \( Z \cdot W \) is a \( BMO(P) \) martingale.

Identical to the proof of inequality (2.22), we have
\[
\phi(Y_t) + \frac{1}{2} E_t \left[ \int_T^t |Z_s|^2 \, ds \right] \\
\leq \phi(|\xi|_\infty) + C E_t \left[ \int_T^t |\phi'(Y_s)| (1 + |Y_s| + E[|Y_s|]) \, ds \right] \\
\leq \phi(|\xi|_\infty) + C\phi'(\lambda) E_t \left[ \int_T^t (1 + \lambda + \lambda) \, ds \right].
\]

Consequently, we have
\[
\|Z \cdot W\|_{BMO_2(P)}^2 = \sup_\tau E_\tau \left[ \int_\tau^T |Z_s|^2 \, ds \right] \\
\leq 2\phi(|\xi|_\infty) + 4C\phi'(\lambda)(1 + \lambda)T.
\]

Finally, we prove the uniqueness. Let \((Y, Z)\) and \((\tilde{Y}, \tilde{Z})\) be two adapted solutions. Then, we have (recall that \( \beta \) is defined by (??))
\[
Y_t - \tilde{Y}_t = \int_t^T \left[ h(s, Y_s, E[Y_s], \tilde{Z}_s, E[\tilde{Z}_s]) - h(s, \tilde{Y}_s, E[\tilde{Y}_s], \tilde{Z}_s, E[\tilde{Z}_s]) \right] \, ds \\
- \int_t^T (Z_s - \tilde{Z}_s) \, d\tilde{W}_s, \quad t \in [0, T].
\]

Similar to the first two inequalities in (2.38), for any stopping time \( \tau \) which takes values
in $[T - \varepsilon, T]$, we have
\[
|Y_t - \tilde{Y}_t|^2 + \tilde{E}_r \left[ \int_{\tau}^{T} |Z_s - \tilde{Z}_s|^2 \, ds \right]
= \tilde{E}_r \left[ \left( \int_{\tau}^{T} \left[ h(s, Y_s, E[Y_s], \tilde{Z}_s, E[Z_s]) - h(s, \tilde{Y}_s, E[\tilde{Y}_s], \tilde{Z}_s, E[\tilde{Z}_s]) \right] \, ds \right)^2 \right]
\leq C^2 \tilde{E}_r \left[ \left( \int_{\tau}^{T} \left[ |Y_s - \tilde{Y}_s| + |E[Y_s - \tilde{Y}_s]| + |E[Z_s - \tilde{Z}_s]| \right] \, ds \right)^2 \right]
\leq 6C^2\varepsilon^2|Y - \tilde{Y}|^2_{\infty} + 3C^2\varepsilon \tilde{E}_r \left[ \int_{\tau}^{T} |Z_s - \tilde{Z}_s|^2 \, ds \right] + 3C^2\varepsilon \tilde{E}_r \left[ \int_{T-\varepsilon}^{T} |Z_s - \tilde{Z}_s|^2 \, ds \right]
\leq 6C^2\varepsilon^2|Y - \tilde{Y}|^2_{\infty} + 3C^2\varepsilon \|Z - \tilde{Z}\cdot W\|_{BMO_2(\bar{P})}^2 + 3C^2\varepsilon \|\tilde{Z} \cdot W\|_{BMO_2(\bar{P})}^2.
\]

Therefore, we have (on the interval $[T - \varepsilon, T]$)
\[
\left| Y - \tilde{Y} \right|^2_{\infty} + c_1^2 \left\| (Z - \tilde{Z}) \cdot W \right\|_{BMO_2(\bar{P})}^2
\leq 6C^2\varepsilon^2\left| Y - \tilde{Y} \right|^2_{\infty} + 3C^2(1 + c_2^2)\varepsilon \left\| (Z - \tilde{Z}) \cdot W \right\|_{BMO_2(\bar{P})}^2.
\] (2.48)

Note that since
\[
|\beta| \leq C(1 + |Z| + |\tilde{Z}|),
\]
the two generic constants $c_1$ and $c_2$ only depend on the sum
\[
\left\| Z \cdot W \right\|_{BMO_2(\bar{P})}^2 + \left\| \tilde{Z} \cdot W \right\|_{BMO_2(\bar{P})}^2.
\]

Then when $\varepsilon$ is sufficiently small, we conclude that $Y = \tilde{Y}$ and $Z = \tilde{Z}$ on $[T - \varepsilon, T]$. Repeating iteratively with a finite of times, we have the uniqueness on the given interval $[0, T]$.

3 The expected term is additive and has a quadratic growth in the second unknown variable

Let us first consider the following quadratic BSDE with mean term:
\[
Y_t = \xi + \int_{t}^{T} \left[ f_1(s, Z_s) + E[f_2(s, Z_s, E[Z_s])] \right] \, ds - \int_{t}^{T} Z_s \, dB_s, \quad t \in [0, T] \quad (3.1)
\]
where $f_1 : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfies: $f_1(\cdot, z)$ is an adapted process for any $z$, and
\[
|f_1(t, 0)| \leq C, \quad |f_1(t, z) - f_1(t, z')| \leq C(1 + |z| + |z'|)|z - z'|, \quad (3.2)
\]
and $f_2 : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfies: $f_2(\cdot, z, \tilde{z})$ is an adapted process for any $z$ and $\tilde{z}$, and
\[
|f_2(t, z, \tilde{z})| \leq C(1 + |z|^2 + |\tilde{z}'|^2). \quad (3.3)
\]
**Proposition 3.1.** Let us suppose that $f_1$ and $f_2$ be two generators satisfying the above conditions and $\xi$ be a bounded random variable. Then (3.1) admits a unique solution $(Y, Z)$ such that $Y$ is bounded and $Z \cdot W$ is a BMO martingale.

**Proof.** Let us first prove the existence. We solve this equation in two steps:

Step one. First solve the following BSDE:

$$ \tilde{Y}_t = \xi + \int_t^T f_1(s, Z_s) \, ds - \int_t^T Z_s \cdot dW_s. $$

(3.4)

It is well known that this BSDE admits a unique solution $(\tilde{Y}, Z)$ such that $\tilde{Y}$ is bounded and $Z \cdot W$ is a BMO martingale.

Step two. Define

$$ Y_t = \tilde{Y}_t + \int_t^T E[f_2(s, Z_s, \mathbb{E}[Z_s])] \, ds. $$

(3.5)

Then

$$ Y_t = \xi + \int_t^T [f_1(s, Z_s) + E[f_2(s, Z_s, \mathbb{E}[Z_s])] \, ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T]. $$

(3.6)

The uniqueness can be proved in a similar way: Let $(Y^1, Z^1)$ and $(Y^2, Z^2)$ be two solutions. Then set

$$ \tilde{Y}_t^1 = Y_t^1 - \int_t^T E[f_2(s, Z_s^1, \mathbb{E}[Z_s^1])] \, ds, \quad \tilde{Y}_t^2 = Y_t^2 - \int_t^T E[f_2(s, Z_s^2, \mathbb{E}[Z_s^2])] \, ds. $$

$(\tilde{Y}^1, Z^1)$ and $(\tilde{Y}^2, Z^2)$ being solution of the same BSDE (3.4), from the uniqueness of solution to this BSDE,

$$ \tilde{Y}^1 = \tilde{Y}^2, \quad Z^1 = Z^2, $$

Hence $Y^1 = Y^2$, and $Z^1 = Z^2$.

Now we consider a more general form of BSDE with a mean term:

$$ Y_t = \xi + \int_t^T [f_1(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s]) + E[f_2(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s])] \, ds $$

$$ - \int_t^T Z_s \cdot dW_s. $$

(3.7)

Here for $i = 1, 2$, $f_i : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfies: for any $(y, \bar{y}, z, \bar{z})$, $f_i(t, y, \bar{y}, z, \bar{z})$ is an adapted process, and

$$ |f_1(t, y, \bar{y}, 0, \bar{z})| + |f_2(t, 0, 0, 0, 0)| \leq C, $$$$|f_1(t, y, z, \bar{y}, \bar{z}) - f_1(t, y', z', \bar{y}', \bar{z}')| \leq C(|y - y'| + |\bar{y} - \bar{y}'| + |z - z'| + (1 + |z| + |z'|)|z - z'|), $$

$$ |f_2(t, y, z, \bar{y}, \bar{z}) - f_2(t, y', z', \bar{y}', \bar{z}')| \leq C(|y - y'| + |\bar{y} - \bar{y}'| + (1 + |z| + |z'| + |\bar{z}| + |\bar{z}'|)|z - z'| + |\bar{z} - \bar{z}'|). $$

We have the following result.
Theorem 3.2. Assume that \( f_1 \) and \( f_2 \) satisfy the above conditions, and \( \xi \) is a bounded random variable. BSDE \((3.7)\) has a unique solution \((Y, Z)\) such that \( Y \in \mathcal{S}^\infty \) and \( Z \cdot W \in BMO_2(\mathbb{P}) \).

Example 3.3. The condition on \( f_1 \) means that this function should be bounded with respect to \((y, \bar{y}, \bar{z})\). For example,

\[
 f_1(s, y, \bar{y}, z, \bar{z}) = 1 + |\sin(y)| + |\sin(\bar{y})| + \frac{1}{2}|z|^2 + |\sin(\bar{z})|,
\]

\[
 f_2(s, y, \bar{y}, z, \bar{z}) = 1 + |y| + |y'| + \frac{1}{2}(|z| + |\bar{z}|)^2.
\]

Proof. We prove the theorem by a fixed point argument. Let \( U, V \) and \( \phi \), we define the map \( \Gamma : (\bar{U}, \bar{V}) \rightarrow \Gamma(U, V) = (Y, Z) \) on \( \mathcal{S}^\infty \times BMO_2(\mathbb{P}) \) as the unique solution to BSDE with mean:

\[
 Y_t = \xi + \int_t^T \left[ f_1(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[V_s]) + E[f_2(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[Z_s]) \right] ds
 - \int_t^T Z_s \cdot dW_s. \tag{3.8}
\]

And we define the map \( \tilde{\Gamma} : (U, V) \rightarrow (\tilde{Y}, \tilde{Z}) \) on \( \mathcal{S}^\infty \times BMO_2(\mathbb{P}) \). Set

\[
 \tilde{Y}_t = Y_t - \int_t^T E[f_2(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[Z_s]) ds,
\]

then \((\tilde{Y}, \tilde{Z})\) is the solution to

\[
 \tilde{Y}_t = \xi + \int_t^T f_1(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[V_s]) ds - \int_t^T Z_s dB_s.
\]

As \( |f_1(t, y, \bar{y}, 0, \bar{z})| \leq C \), we have

\[
 |f_1(t, y, \bar{y}, z, \bar{z})| \leq C + \frac{C}{2}|z|^2.
\]

Applying Ito’s formula to \( \phi_C(|\bar{Y}|) \),

\[
 \phi_C(|\bar{Y}|) + \int_t^T \frac{1}{2} \phi''_C(|\bar{Y}|)|Z_s|^2 ds \leq \phi(\xi) + \int_t^T |\phi''_C(|\bar{Y}|)|(C + \frac{C}{2}|Z_s|^2) ds - \int_0^T \phi'(\bar{Y}_s) Z_s dW_s, \tag{3.9}
\]

we can prove that there exists a constant \( K > 0 \) such that

\[
 ||Z||_{BMO} \leq K.
\]

Let \((U^1, V^1)\) and \((U^2, V^2)\), and \((Y^1, Z^1)\) and \((Y^2, Z^2)\) be the corresponding solution, and

\[
 \tilde{Y}^1_t = \xi + \int_t^T f_1(s, U^1_s, \mathbb{E}[U^1_s], Z^1_s, \mathbb{E}[V^1_s]) ds - \int_t^T Z^1_s dB_s,
\]

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\[ \tilde{Y}_t^2 = \xi + \int_t^T f_1(s, U_s^2, E[U_s^2], Z_s^2, E[V_s^2]) ds - \int_t^T Z_s^2 dB_s. \]

There exists a bounded adapted process \( \beta \) such that
\[ f_1(s, U_s^1, E[U_s^1], Z_s^1, E[V_s^1]) - f_1(s, U_s^1, E[U_s^1], Z_s^2, E[V_s^1]) = \beta(s)(Z_s^1 - Z_s^2). \]

Then \( \tilde{W}_t := W_t - \int_0^T \beta_s ds \) is a Brown motion under the equivalent probability measure \( \tilde{\mathbb{P}} \) defined by
\[ d\tilde{\mathbb{P}} = \mathbb{E}_0^T(\beta \cdot W_t^\tau) d\mathbb{P}. \]

We have
\[ \begin{align*}
\Delta \tilde{Y}_t + \int_t^T \Delta Z_s d\tilde{W}_s \\
= \int_t^T \left[ f_1(s, U_s^1, E[U_s^1], Z_s^2, E[V_s^1]) - f_1(s, U_s^2, E[U_s^2], Z_s^2, E[V_s^2]) \right] ds, \quad t \in [0, T].
\end{align*} \]

For any stopping time \( \tau \) which takes values in \( [T - \varepsilon, T] \), taking square and then the conditional expectation with respect to \( \tilde{\mathbb{P}} \) (denoted by \( \tilde{\mathbb{E}} \)), we have
\[ \begin{align*}
|\Delta \tilde{Y}_t|^2 + \tilde{\mathbb{E}}_\tau \int_\tau^T |\Delta Z|^2 ds \\
= \tilde{\mathbb{E}}_\tau \left( \int_t^T \left[ f_1(s, U_s^1, E[U_s^1], Z_s^2, E[V_s^1]) - f_1(s, U_s^2, E[U_s^2], Z_s^2, E[V_s^2]) \right] ds \right)^2 \\
\leq C^2 \tilde{\mathbb{E}}_\tau \left( \int_\tau^T (|\Delta U_s| + |E[\Delta U_s]| + |E[\Delta V_s]|) ds \right)^2 \\
\leq C^2 \varepsilon (|\Delta U|_\infty^2 + |\Delta V \cdot W|_{BMO_2(\mathbb{P})}^2).
\end{align*} \]

Therefore, we have (on the interval \( [T - \varepsilon, T] \))
\[ |\Delta \tilde{Y}|_\infty^2 + c_1^2 \|W\|_{BMO_2(\mathbb{P})}^2 \leq C^2 \varepsilon (|\Delta U|_\infty^2 + |\Delta V \cdot W|_{BMO_2(\mathbb{P})}^2). \]

Note that since
\[ |\beta| \leq C(1 + |Z_1| + |Z_2|), \]
the generic constant \( c_1 \) and \( c_2 \) only depend on \( C \) and \( K \).

As
\[ \Delta Y_t = \Delta \tilde{Y}_t + \int_t^T E \left[ f_2(s, U_s^1, E[U_s^1], Z_s^1, E[Z_s^1]) - f_2(s, U_s^2, E[U_s^2], Z_s^2, E[Z_s^2]) \right] ds, \]
then
\[ |\Delta Y_t| \leq |\Delta \tilde{Y}_t| + C E \left[ \int_t^T (|\Delta U_s| + |E[\Delta U_s]|) ds \right] \\
+ C E \left[ \int_t^T (1 + |Z_s^1| + |Z_s^2| + |E[\Delta Z_s]| + |E\Delta Z_s|) \right] (|\Delta Z_s| + |E[\Delta Z_s]|) ds \]

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we deduce that

$$|\Delta Y_t|^2 \leq C^2 \left( |\Delta \bar{Y}_t|^2 + E \left[ \int_t^T \left( |\Delta U_s| + |\mathbb{E} \Delta U_s| \right) ds \right]^2 \right)$$

$$E \left[ \int_t^T \left( |Z_s^1| + |Z_s^2| + |\mathbb{E}[Z_s^1]| + |\mathbb{E}[Z_s^2]| \right) \left( |\Delta Z_s| + |\mathbb{E} \Delta Z_s| \right) ds \right]^2,$$

which implies that

$$|\Delta Y|^2 \leq C^2 \left( |\Delta \bar{Y}|^2 + \varepsilon |\Delta U|^2 + |(\Delta Z) \cdot W|^2_{BMO_{2}(\mathbb{P})} \right)$$

$$\leq C^2 \varepsilon \left( |\Delta U|^2 + ||V \cdot W||^2_{BMP_2(\mathbb{P})} \right).$$

Then when $\varepsilon$ is sufficiently small, we conclude that the application is contracting on $[T - \varepsilon, T]$. Repeating iteratively with a finite of times, we have the existence and uniqueness on the given interval $[0, T]$. \hfill \Box

4 Multi-dimensional Case

In this section, we will study the multi-dimensional case of $(3.7)$, where $f$ is Lipschitz. Our result generalizes the corresponding one of Cheridito and Nam [8].

We first consider the following BSDE with mean term:

$$Y_t = \xi + \int_t^T \left[ f_1(s, Z_s, \mathbb{E}[Z_s]) + E[f_2(s, Z_s, \mathbb{E}[Z_s])] \right] ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T]$$

where $f_1 : \Omega \times [0, T] \times \mathbb{R}^{d \times n} \times \mathbb{R}^{d \times n} \to \mathbb{R}^n$ satisfies: for any $z$ and $\bar{z}$, $f_1(t, z, \bar{z})$ is an adapted process, and

$$|f_i(t, 0, 0)| \leq C, \quad |f_i(t, z, \bar{z}) - f_i(t, z', \bar{z}')| \leq C(|z - z'| + |\bar{z} - \bar{z}'|), \quad (4.2)$$

and $f_2 : \Omega \times [0, T] \times \mathbb{R}^{d \times n} \times \mathbb{R}^{d \times n} \to \mathbb{R}^n$ satisfies: for any $z, \bar{z}$, $f_2(t, z, \bar{z})$ is an adapted process, and

$$|f_2(t, 0, 0)| \leq C, \quad |f_2(t, z, \bar{z}) - f_2(t, z', \bar{z}')| \leq C(1 + |z| + |z'| + |\bar{z}| + |\bar{z}'|)(|z - z'| + |\bar{z} - \bar{z}'|). \quad (4.3)$$

**Proposition 4.1.** Let us suppose that $f_1$ and $f_2$ be two generators satisfying the above conditions and $\xi \in L^2(\mathcal{F}_T)$. Then $(4.1)$ admits a unique solution $(Y, Z)$ such that $Y \in \mathcal{S}^2$ and $Z \in \mathcal{M}^2$.

**Proof.** Let us first prove the existence. Our proof is divided into the following two steps.

Step one. First consider the following BSDE

$$\bar{Y}_t = \xi + \int_t^T f_1(s, Z_s, \mathbb{E}[Z_s]) ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T]. \quad (4.4)$$

It admits a unique solution $(\bar{Y}, Z)$ such that $\bar{Y} \in \mathcal{S}^2$ and $Z \in \mathcal{M}^2$, see Buckdahn et al. [6].
Step two. Define

\[ Y_t = \bar{Y}_t + \int_t^T E[f_2(s, Z_s, \mathbb{E}[Z_s])] \, ds, \quad t \in [0, T]. \] (4.5)

Then, we have for \( t \in [0, T] \),

\[ Y_t = \xi + \int_t^T \left[ f_1(s, Z_s, \mathbb{E}[Z_s]) + E[f_2(s, Z_s, \mathbb{E}[Z_s])] \right] \, ds - \int_t^T Z_s \cdot dW_s. \] (4.6)

The uniqueness can be proved in a similar way: Let \((Y^1, Z^1)\) and \((Y^2, Z^2)\) be two solutions. Then set

\[ \bar{Y}_t^1 = Y_t^1 - \int_t^T E[f_2(s, Z^1_s, \mathbb{E}[Z^1_s])] \, ds, \quad \bar{Y}_t^2 = Y_t^2 - \int_t^T E[f_2(s, Z^2_s, \mathbb{E}[Z^2_s])] \, ds. \]

\((\bar{Y}^1, Z^1)\) and \((\bar{Y}^2, Z^2)\) being solution of the same BSDE (4.4), from the uniqueness of solution to this BSDE,

\[ \bar{Y}^1 = \bar{Y}^2, \quad Z^1 = Z^2, \]

Hence \(Y^1 = Y^2\), and \(Z^1 = Z^2\).

Now we consider the following more general form of BSDE with a mean term:

\[ Y_t = \xi + \int_t^T \left[ f_1(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s]) + E[f_2(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s])] \right] \, ds \]
\[ - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T]. \] (4.7)

Here for \( i = 1, 2 \), \( f_i : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{d \times n} \times \mathbb{R}^{d \times n} \to \mathbb{R}^n \) satisfies: for any \((y, \bar{y}, z, \bar{z})\), \( f_i(t, y, \bar{y}, z, \bar{z}) \) is an adapted process, and

\[ |f_1(t, y, \bar{y}, 0, 0)| + |f_2(t, 0, 0, 0, 0)| \leq C, \]

(4.8)

\[ |f_1(t, y, z, \bar{y}, \bar{z}) - f_1(t, y', z', \bar{y}', \bar{z}')| \leq C(|y - y'| + |\bar{y} - \bar{y}'| + |z - z'| + |\bar{z} - \bar{z}'|), \]

(4.9)

\[ |f_2(t, y, z, \bar{y}, \bar{z}) - f_2(t, y', z', \bar{y}', \bar{z}')| \leq C(|y - y'| + |\bar{y} - \bar{y}'| + (1 + |z| + |z'| + |\bar{z}| + |\bar{z}'|)(|z - z'| + |\bar{z} - \bar{z}'|)). \]

(4.10)

We have the following result.

**Theorem 4.2.** Assume that \( f_1 \) and \( f_2 \) satisfy the inequalities (4.8)-(4.10), and \( \xi \in L^2(F_T) \). Then BSDE (4.7) has a unique solution \((Y, Z)\) such that \( Y \in \mathcal{S}^2 \) and \( Z \in \mathcal{M}^2 \).

**Example 4.3.** The condition (4.8) requires that both functions \( f_1 \) and \( f_2 \) should be bounded with respect to \((y, \bar{y})\). For example,

\[ f_1(s, y, \bar{y}, z, \bar{z}) = 1 + |\sin(y)| + |\sin(\bar{y})| + |z| + |\bar{z}|, \]

\[ f_2(s, y, \bar{y}, z, \bar{z}) = 1 + |y| + |y'| + \frac{1}{2}(|z| + |\bar{z}|)^2. \]
Proof. We prove the theorem by a fixed point argument. Let \( U \in \mathcal{S}^2 \), and \( V \in \mathcal{M}^2 \), we define \((Y, Z \cdot W) \in \mathcal{S}^2 \times \mathcal{M}^2 \) as the unique solution to BSDE with mean:

\[
Y_t = \xi + \int_t^T \left[ f_1(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[Z_s]) + E[f_2(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[Z_s])] \right] ds - \int_t^T Z_s \cdot dW_s. \tag{4.11}
\]

We define the map \( \Gamma : (U, V) \mapsto \Gamma(U, V) = (Y, Z) \) on \( \mathcal{S}^2 \times \mathcal{M}^2 \). Set

\[
\tilde{Y}_t = Y_t - \int_t^T E \left[ f_2(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[Z_s]) \right] ds, \quad t \in [0, T].
\]

Then \((\tilde{Y}, Z)\) is the solution to

\[
\tilde{Y}_t = \xi + \int_t^T f_1(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[Z_s]) ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T].
\]

As \(|f_1(t, y, \bar{y}, z, \bar{z})| \leq C\), we have

\[
|f_1(t, y, \bar{y}, z, \bar{z})| \leq C(1 + |z| + |\bar{z}|).
\]

Applying Ito’s formula to \(|\tilde{Y}|^2\), we have

\[
|\tilde{Y}_t|^2 + \int_t^T |Z_s|^2 ds \leq |\xi|^2 + 2C \int_t^T |\tilde{Y}_s|(C + |Z_s| + |\mathbb{E}[Z]||) ds - 2 \int_t^T (Y_s, Z_s \cdot dW_s). \tag{4.12}
\]

Further, using standard techniques, we can prove that there exists a constant \( K > 0 \) such that

\[
||Z||_{\mathcal{M}^2} \leq K.
\]

For \((U^i, V^i) \in \mathcal{S}^2 \times \mathcal{M}^2\), define \((Y^i, Z^i) = \Gamma(U^i, V^i)\) with \( i = 1, 2 \). Further, set for \( t \in [0, T]\),

\[
\tilde{Y}_t^1 := \xi + \int_t^T f_1(s, U_s^1, \mathbb{E}[U_s^1], Z_s^1, \mathbb{E}[Z_s^1]) ds - \int_t^T Z_s^1 \cdot dW_s,
\]

\[
\tilde{Y}_t^2 := \xi + \int_t^T f_1(s, U_s^2, \mathbb{E}[U_s^2], Z_s^2, \mathbb{E}[Z_s^2]) ds - \int_t^T Z_s^2 \cdot dW_s.
\]

We have

\[
\Delta \tilde{Y}_t + \int_t^T \Delta Z_s dW_s
\]

\[
= \int_t^T \left[ f_1(s, U_s^1, \mathbb{E}[U_s^1], Z_s^1, \mathbb{E}[Z_s^1]) - f_1(s, U_s^2, \mathbb{E}[U_s^2], Z_s^2, \mathbb{E}[Z_s^2]) \right] ds, \quad t \in [0, T].
\tag{4.13}
\]
For any $t \in [T - \varepsilon, T]$, taking square and then expectations on both sides of the last equality, we have

\[
\mathbb{E}(|\Delta \tilde{Y}_t|^2) + \mathbb{E}\left[\int_t^T |\Delta Z_s|^2 ds\right]
\]

\[
= \mathbb{E}\left(\int_t^T \left| f_1(s, U^1_s, \mathbb{E}[U^1_s], Z^1_s, \mathbb{E}[Z^1_s]) - f_1(s, U^2_s, \mathbb{E}[U^2_s], Z^2_s, \mathbb{E}[Z^2_s])\right| ds\right)^2
\]

\[
\leq C^2 \mathbb{E}\left(\left[\int_T^T (|\Delta U_s| + |\mathbb{E}[\Delta U_s]| + |\Delta Z_s| + |\mathbb{E}[\Delta Z_s]|) ds\right]^2\right)
\]

\[
\leq C^2 \varepsilon (|\Delta U|^2_s + |\Delta Z|^2_M).
\]

Therefore, we have (on the interval $[T - \varepsilon, T]$) for a sufficiently small $\varepsilon > 0$,

\[
\left|\Delta \tilde{Y}_t\right|^2 + \|\Delta Z\|^2_M \leq C^2 \varepsilon (|\Delta U|^2_s).
\]  

(4.15)

As

\[
\Delta Y_t = \Delta \tilde{Y}_t + \int_t^T E \left[ f_2(s, U^1_s, \mathbb{E}[U^1_s], Z^1_s, \mathbb{E}[Z^1_s]) - f_2(s, U^2_s, \mathbb{E}[U^2_s], Z^2_s, \mathbb{E}[Z^2_s])\right] ds,
\]

we have

\[
|\Delta Y_t| \leq |\Delta \tilde{Y}_t| + C E \left[\int_t^T (|\Delta U_s| + |\mathbb{E}[\Delta U_s]|) ds\right]
\]

\[
+ CE \left[\int_t^T (1 + |Z^1_s| + |Z^2_s| + |\mathbb{E}[Z^1_s]| + |\mathbb{E}[Z^2_s]|) (|\Delta Z_s| + |\mathbb{E}[\Delta Z_s]|) ds\right].
\]

Moreover, we deduce that

\[
|\Delta Y_t|^2 \leq C^2 \left|\Delta \tilde{Y}_t\right|^2 + E \left[\int_t^T (|\Delta U_s| + |\mathbb{E}[\Delta U_s]|) ds\right]^2
\]

\[
+ E \left[\int_t^T (1 + |Z^1_s| + |Z^2_s| + |\mathbb{E}[Z^1_s]| + |\mathbb{E}[Z^2_s]|) (|\Delta Z_s| + |\mathbb{E}[\Delta Z_s]|) ds\right]^2
\]

In view of (4.15)

\[
\left|\Delta \tilde{Y}_t\right|^2 + \|\Delta Z\|^2_M \leq C^2 \varepsilon (|\Delta U|^2_s).
\]

Then when $\varepsilon$ is sufficiently small, we conclude that the application is contracting on $[T - \varepsilon, T]$. Repeating iteratively with a finite of times, we have the existence and uniqueness on the given interval $[0, T]$.

\[\square\]
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