The Average-Case Time Complexity of Certifying the Restricted Isometry Property

Yunzi Ding\textsuperscript{*1}, Dmitriy Kunisky\textsuperscript{‡1}, Alexander S. Wein\textsuperscript{§1}, and Afonso S. Bandeira\textsuperscript{¶2}

\textsuperscript{1}Department of Mathematics, Courant Institute of Mathematical Sciences, New York University, USA
\textsuperscript{2}Department of Mathematics, ETH Zurich, Switzerland

Abstract

In compressed sensing, the restricted isometry property (RIP) on $M \times N$ sensing matrices (where $M < N$) guarantees efficient reconstruction of sparse vectors. A matrix has the $(s, \delta)$-RIP property if behaves as a $\delta$-approximate isometry on $s$-sparse vectors. It is well known that an $M \times N$ matrix with i.i.d. $\mathcal{N}(0, 1/M)$ entries is $(s, \delta)$-RIP with high probability as long as $s \lesssim \delta^2 M / \log N$. On the other hand, most prior works aiming to deterministically construct $(s, \delta)$-RIP matrices have failed when $s \gg \sqrt{M}$. An alternative way to find an RIP matrix could be to draw a random gaussian matrix and certify that it is indeed RIP. However, there is evidence that this certification task is computationally hard when $s \gg \sqrt{M}$, both in the worst case and the average case.

In this paper, we investigate the exact average-case time complexity of certifying the RIP property for $M \times N$ matrices with i.i.d. $\mathcal{N}(0, 1/M)$ entries, in the “possible but hard” regime $\sqrt{M} \ll s \lesssim M / \log N$, assuming that $M$ scales proportional to $N$. Based on analysis of the low-degree likelihood ratio, we give rigorous evidence that subexponential runtime $N^{\Omega(s^2/N)}$ is required, demonstrating a smooth tradeoff between the maximum tolerated sparsity and the required computational power. The lower bound is essentially tight, matching the runtime of an existing algorithm due to Koiran and Zouzias [KZ14]. Our hardness result allows $\delta$ to take any constant value in $(0, 1)$, which captures the relevant regime for compressed sensing. This improves upon the existing average-case hardness result of Wang, Berthet, and Plan [WBP16], which is limited to $\delta = o(1)$.

\textsuperscript{*}Email: yding@nyu.edu. Partially supported by NSF grant DMS-1712730.
\textsuperscript{‡}Email: kunisky@cims.nyu.edu. Partially supported by NSF grants DMS-1712730 and DMS-1719545.
\textsuperscript{§}Email: awein@cims.nyu.edu. Partially supported by NSF grant DMS-1712730 and by the Simons Collaboration on Algorithms and Geometry.
\textsuperscript{¶}Email: bandeira@math.ethz.ch. Part of this work was done while ASB was with the Department of Mathematics at the Courant Institute of Mathematical Sciences, and the Center for Data Science, at New York University; and partially supported by NSF grants DMS-1712730 and DMS-1719545, and by a grant from the Sloan Foundation.
1 Introduction

1.1 Restricted Isometry Property

For measuring and reconstructing high-dimensional sparse signals, the compressed sensing technique introduced by Candès and Tao \cite{CT05} and Donoho \cite{Don06} has demonstrated state-of-the-art efficiency and effectiveness in theory and practice. A central property on the sensing matrix, known as the restricted isometry property (RIP) \cite{Can08}, requires that the matrix approximately preserves the norm of sparse vectors.

**Definition 1.1.** A matrix $A \in \mathbb{R}^{M \times N}$ is said to satisfy the $(s, \delta)$-RIP if

$$(1 - \delta)\|x\|^2 \leq \|Ax\|^2 \leq (1 + \delta)\|x\|^2$$

for any $x \in \mathbb{R}^N$ with $\|x\|_0 \leq s$. Here $\| \cdot \|$ denotes the vector $\ell^2$ norm, and $\| \cdot \|_0$ denotes the number of nonzero entries of a vector.

In the context of compressed sensing, in pursuit of reducing the dimension by a constant factor, we are interested in the regime $N/M \to \gamma > 1$. An $(s, \delta)$-RIP sensing matrix with $\delta < \sqrt{2} - 1$ allows for efficient reconstruction of $s/2$-sparse signals in the compressed sensing framework \cite{Can08, CZ13a, CZ13b, FR13}. From a practical point of view, given desired parameters $M, N, s, \delta$, one would like to construct an $(s, \delta)$-RIP matrix suitable for use as a sensing matrix. It appears to be very difficult to deterministically construct RIP matrices for $s \gg \sqrt{M}$, a phenomenon known as the "square bottleneck" \cite{Mix15, BFMM16, BMM17, Gam18}; in fact, the only known success is due to \cite{BDF11}, which constructed $(s, \delta)$-RIP matrices for $s \sim M^{1/2 + \epsilon}$ where $0 < \epsilon \ll 1$. Readers may refer to \cite{BFMW13} for more details.

Meanwhile, randomized algorithms have seen much success in breaking the "square bottleneck" \cite{CT05, BDDW08, FR13}. For example, by simply sampling an $M \times N$ matrix with i.i.d. $\mathcal{N}(0, 1/M)$ entries, one obtains a $(s, \delta)$-RIP matrix with high probability (i.e., probability $1 - o(1)$) if $s \lesssim \delta^2 M/\log N$. While such randomized algorithms generate RIP matrices with more desirable parameters $s, \delta$ than the known deterministic constructions, they suffer from a potential drawback: it is not guaranteed that the output is always $(s, \delta)$-RIP as desired. This motivates the following task known as RIP certification, in which we discard non-RIP samples and confidently keep a sample only when it is indeed RIP.

**Problem 1.2 (RIP certification).** Given a matrix $A \in \mathbb{R}^{M \times N}$, a positive integer $s$, and $\delta \in (0, 1)$, output either “yes” or “no” according to the following rules. If $A$ is not $(s, \delta)$-RIP, the output must always be “no”. If $A$ has i.i.d. $\mathcal{N}(0, 1/M)$ entries, the output must be “yes” with high probability.

Note that this allows false negative errors but not false positive errors, so that such a certifying procedure allows us to reliably obtain an RIP matrix by sampling random matrices until the certifier outputs “yes”.

The worst-case problem of deciding (with certainty) whether or not a given matrix is $(s, \delta)$-RIP is NP-hard \cite{TP13, BDMS13}, even if the input is guaranteed to either be RIP or far from RIP \cite{Wee17}. All known algorithms for this worst-case task \cite{Dev07, AHSC09, FMT12, BFMW13} require time $N^{\tilde{O}(s)}$, which is the time required to enumerate all possible support sets $S \subseteq [N]$ of cardinality $s$.

For the average-case RIP certification problem (Problem 1.2), it is shown in \cite{WBP16} that thresholding $\|A^*A - I_N\|_{\infty}$ gives a polynomial-time certifier of $(s, \delta)$-RIP for i.i.d. sub-gaussian matrices in the regime $s \lesssim \delta \sqrt{M/\log N}$. Note, however, that this does not help surpass the "square
bottleneck”. In fact, the same work [WBP16] shows that when \( s \gg (\delta^2 M / \log N)^{1/(1+\alpha)} \) for some \( \alpha \in [0,1) \), no polynomial-time certifier exists, conditional on an assumption about detecting dense subgraphs, which is a weaker assumption than the planted clique hypothesis (i.e., their assumption is implied by the planted clique hypothesis). A related result by Koiran and Zouzias [KZ14] shows hardness of approximating the RIP parameter \( \delta \), conditional on the planted clique hypothesis. However, both the results [KZ14] and [WBP16] only show hardness in the regime \( \delta = o(1) \).

1.2 Our Contributions

In this paper, we further investigate the average-case hardness of Problem 1.2. For any fixed \( \delta \in (0,1) \) and \( M = \lceil N/\gamma \rceil \) for some fixed \( \gamma > 1 \), in the “possible but hard” regime \( \sqrt{M} \ll s \lesssim M / \log N \), we give evidence that RIP certification requires time \( N^{\tilde{O}(s^2/N)} \) with an analysis of the low-degree likelihood ratio (see Section 1.4). We show that our lower bound is optimal, as it matches the upper bound due to the lazy algorithm proposed in [KZ14]. Here, the notation \( \tilde{O}, \tilde{\Theta} \) and \( \tilde{\Omega} \) hide factors of \( \log N \).

The strategy for our lower bound is to give a reduction to Problem 1.2 from a certain hypothesis testing problem in the negatively-spiked Wishart model. Our arguments will be based on two sequences of distributions over \( \mathbb{R}^{M \times N} \), namely the distribution \( Q_N \) of i.i.d. Gaussian matrices and a distribution \( P_N \) over matrices which have a sparse vector planted in their null-space. We start with the proof that, in the regime \( \sqrt{M} \ll s \lesssim M / \log N \), \( A \sim Q_N \) is RIP with high probability while \( A \sim P_N \) is non-RIP with high probability. We then argue that it is computationally hard to distinguish \( P_N \) from \( Q_N \), assuming the low-degree conjecture (see Section 1.4). From this we infer the hardness of RIP certification.

For the matching upper bound, let us give a brief overview of the lazy algorithm in [KZ14]. The main step of the algorithm involves exhaustive search over subsets of \([N]\) with cardinality \( r \approx s^2/N \), which are interpreted as the possible supports of \( r \)-sparse vectors in \( \mathbb{R}^N \). As \( s \) ranges from \( \sqrt{M} \) up to \( M / \log N \), the runtime of the algorithm \( (N^{\tilde{O}(s^2/N)}) \) smoothly interpolates between polynomial \( (N^{O(1)}) \) and exponential \( (N^{\tilde{O}(N)}) \). Our lower bound suggests that the dependence of the runtime on the sparsity \( s \) achieved by the algorithm is indeed optimal.

Our main contribution in this paper is a precise understanding of the computational power needed for the average-case certification of \((s, \delta)\)-RIP, namely \( N^{\tilde{O}(s^2/N)} \). In contrast, the previous average-case hardness results of [KZ14] [WBP16] only suggest that at least \( N^{\log(N)} \) time is required, since the planted clique and dense subgraph problems (that they give reductions from) can be solved in time \( N^{\log(N)} \). Another strength of our lower bound is that it applies for any fixed \( \delta \in (0,1) \), whereas the hardness results in [KZ14] [WBP16] are restricted to \( \delta = o(1) \). In other words, we are showing that even an easier certification problem is hard. Since any fixed \( \delta < \sqrt{2} - 1 \) is sufficient for compressed sensing applications, our result is the first lower bound to capture the entire regime of relevant \( \delta \) values.

1.3 Spiked Wishart Model and Hardness of RIP Certification

Definition 1.1 tells us that a matrix is not \((s, \delta)\)-RIP for any \( \delta \in (0,1) \) if there exists an \( s \)-sparse vector in its kernel. The computational hardness of RIP certification in this paper is based on the following fact:

Any certifier for \((s, \delta)\)-RIP (a solution to Problem 1.2) can be used to distinguish a random matrix from any matrix that has an \( s \)-sparse vector in its kernel.
Note that the operation on $A \in \mathbb{R}^{M \times N}$ that projects each row of $A$ onto the subspace $\{x\}^\perp$ for some $s$-sparse vector $x \in \mathbb{R}^N$ adds $x$ to the kernel of the resulting matrix, and hence deprives the matrix $A$ of its $(s, \delta)$-RIP property. Therefore, a certifier should be able to tell whether an $(s, \delta)$-RIP matrix $A$ has undergone this operation. We introduce the following spiked Wishart model, which captures the property of the row-wise projection.

**Definition 1.3 (Spiked Wishart model).** Let $\mathcal{X} = (\mathcal{X}_N)$ be a distribution over $\mathbb{R}^N$, $\beta \in [-1, +\infty)$ and $\gamma > 0$. Let $M = \lceil N/\gamma \rceil$. We define two distributions over $\mathbb{R}^{M \times N}$, where $A \in \mathbb{R}^{M \times N}$ is taken as

- Under $\mathcal{Q} = \mathcal{Q}_{N, \gamma}$, draw each row $u_i^\top$ ($i = 1, 2, \ldots, M$) of $A$ i.i.d. from $\mathcal{N}(0, I)$.
- Under $\mathcal{P} = \mathcal{P}_{N, \gamma, \beta, \mathcal{X}}$, draw $x \sim \mathcal{X}$. If $\beta \|x\|^2 \geq -1$, then draw each row $u_i^\top$ ($i = 1, 2, \ldots, M$) of $A$ i.i.d. from $\mathcal{N}(0, I + \beta xx^\top)$; otherwise, draw each row $u_i^\top$ ($i = 1, 2, \ldots, M$) of $A$ i.i.d. from $\mathcal{N}(0, I)$.

We call $\mathcal{P}$ the planted model and $\mathcal{Q}$ the null model. We define the spiked Wishart model as the two distributions taken together: $(\mathcal{P}, \mathcal{Q}) = \text{Wishart}(N, \gamma, \beta, \mathcal{X})$.

We will consider spike priors $\mathcal{X}$ normalized so that $\|x\| \approx 1$. The restriction $\beta \|x\|^2 \geq -1$ in the definition of $\mathcal{P}$ is imposed to ensure that the covariance matrix for each row of $A$ is positive semidefinite. When $\beta < 0$, $\mathcal{P}$ can be viewed as the row-wise partial projection of $\mathcal{Q}$ onto $\{x\}^\perp$, where $x$ is taken from the prior distribution $\mathcal{X}$. In fact, for any $u \sim \mathcal{N}(0, I + \beta xx^\top)$ where $\|x\| = 1$, we have

$$E\langle u, x \rangle^2 = 1 + \beta < 1 = E\langle u, y \rangle^2,$$

for any unit-norm $y \perp x$.

Since we are interested in the signal $x$ being sparse, we will take the spike prior $\mathcal{X}$ to be the following sparse Rademacher prior.

**Definition 1.4 (Sparse Rademacher prior).** Given $\rho \in (0, 1)$, the sparse Rademacher prior $\mathcal{X}_N^\rho$ is the distribution over $\mathbb{R}^N$ where for $x \sim \mathcal{X}_N^\rho$, each entry $x_i$ is distributed independently as

$$x_i = \begin{cases} \frac{1}{\sqrt{\rho N}} & \text{with probability } \frac{\rho}{2}, \\ -\frac{1}{\sqrt{\rho N}} & \text{with probability } \frac{\rho}{2}, \\ 0 & \text{with probability } 1 - \rho. \end{cases}$$

Note that $x \sim \mathcal{X}$ has $E\|x\|^2 = 1$ and $E\|x\|_0 = \rho N$. In order for $x$ to be $s$-sparse, we will take $\rho N < s$.

We now introduce the logic for demonstrating the average-case hardness of certifying RIP for gaussian random matrices. The following proposition serves as an outline of the proof of the lower bound. The informal assertions (1), (2) and (3) are made rigorous in Section 2 in Theorems 2.1, 2.2 and 2.3 respectively.

**Proposition 1.5 (Informal).** Consider the sequence of spiked Wishart models with sparse Rademacher prior $\{\text{Wishart}(N, \gamma, -(1 - \epsilon)\mathcal{X}_N^{s\rho})\}_{\epsilon \in \mathbb{Z}^+}$ where $\gamma > 1$, $\epsilon = \frac{1 - \delta}{\epsilon(1 + \delta)}$ for some constant $\delta \in (0, 1)$, and $\rho N$ is taken as $s/(2N)$. For $A \in \mathbb{R}^{M \times N}$ where $M = \lceil N/\gamma \rceil$, the following three properties hold in the limit $N \to \infty$. 

4
(1) For $A \sim \mathcal{P}$, $\frac{1}{\sqrt{M}} A$ is not $(s, \delta)$-RIP with high probability.

(2) For $A \sim \mathcal{Q}$, $\frac{1}{\sqrt{M}} A$ is $(s, \delta)$-RIP with high probability.

(3) Conditional on the low-degree conjecture (see Section 1.4), any algorithm to distinguish $\mathcal{P}$ from $\mathcal{Q}$ with error probability $o(1)$ requires time $N^{\Omega(s^2/N)}$.

In light of assertions (1) and (2), any certifier for $(s, \delta)$-RIP (a solution to Problem 1.2) can be used to distinguish $\mathcal{P}$ from $\mathcal{Q}$ with error probability $o(1)$. Thus (3) implies that, conditional on the low-degree conjecture, any algorithm to solve Problem 1.2 requires time $N^{\Omega(s^2/N)}$.

The problem of distinguishing two sequences of distributions falls into the setting of hypothesis testing. In the following Section 1.4 we discuss a general method to predict computational hardness of such tasks.

1.4 The Low-Degree Likelihood Ratio

The so-called low-degree method for predicting the amount of computational power required for hypothesis testing tasks has seen fruitful development in the recent years, after its origination from studies on the sum-of-squares (SoS) hierarchy [BHK+19, HS17, HKP+17, Hop18]. This method has proven successful in understanding many classical statistical tasks, including community detection [HS17, Hop18], planted clique [BHK+19, Hop18], PCA and sparse PCA in spiked matrix models [BKW19, KWB19, DKWB19], and tensor PCA [HKP+17, Hop18, KWB19]. Here we give a brief overview of this method; the reader may find more details in [HS17, Hop18] or in the survey article [KWB19].

This method applies to hypothesis testing problems in which we aim to distinguish two sequences of hypotheses $\{\mathcal{P}_N\}$ and $\{\mathcal{Q}_N\}$, where $\mathcal{P}_N$ and $\mathcal{Q}_N$ are probability distributions on $\Omega_N = \mathbb{R}^{d(N)}$ with $d(N) = \text{poly}(N)$. Usually $\mathcal{Q}_N$ is referred to as the “null” distribution (which contains pure noise), and $\mathcal{P}_N$ is referred to as the “planted” distribution (which contains a planted structure). $\mathcal{Q}_N$ naturally induces an inner product on $L^2$ functions $f : \Omega_N \to \mathbb{R}$, given by

$$\langle f, g \rangle_{L^2(\mathcal{Q}_N)} = \mathbb{E}_{Y \sim \mathcal{Q}_N} [f(Y)g(Y)]$$

and the associated norm

$$\|f\|_{L^2(\mathcal{Q}_N)}^2 = \langle f, f \rangle_{L^2(\mathcal{Q}_N)}.$$

For a positive integer $D$, let $\mathbb{R}[Y]_{\leq D}$ denote the polynomials $\Omega_N \to \mathbb{R}$ of degree at most $D$; and for $f : \Omega_N \to \mathbb{R}$, let $f_{\leq D}$ denote the orthogonal projection of $f$ onto $\mathbb{R}[Y]_{\leq D}$ with respect to $\langle \cdot, \cdot \rangle_{L^2(\mathcal{Q}_N)}$. In particular, in the case where $\mathbb{P}_n$ is absolutely continuous with respect to $\mathcal{Q}_N$, the likelihood ratio (which is the Radon-Nikodym derivative) $L_N = d\mathbb{P}_N/d\mathcal{Q}_N$ is well defined, and the behavior of the projection $L_{\leq D}^*_N$ is believed to predict whether or not there exists an $N^\Theta(D)$-time algorithm which succeeds in distinguishing $\mathbb{P}_N$ and $\mathcal{Q}_N$. This is captured by the following informal conjecture based on [HS17, HKP+17, Hop18].

**Conjecture 1.6** (Informal). Let $t : \mathbb{N} \to \mathbb{N}$. For “natural” high-dimensional testing problems specified by $\mathbb{P}_N$ and $\mathcal{Q}_N$, if $\|L_{\leq D}^*\|_{L^2(\mathcal{Q}_N)}$ remains bounded as $N \to \infty$ whenever $D(N) \leq t(N) \cdot \text{polylog}(N)$, then there exists no sequence of functions $f_n : \Omega_N \to \{\mathbb{P}, \mathcal{Q}\}$ with $f_n$ computable in time $N^{O(t(N))}$ that strongly distinguishes $\mathbb{P}_N$ and $\mathcal{Q}_N$, i.e., that satisfies

$$\lim_{N \to \infty} \mathbb{Q}_N[f_N(Y) = q] = \lim_{N \to \infty} \mathbb{P}_N[f_N(Y) = p] = 1.$$  \hfill (1)
In this paper, we will suggest hardness of distinguishing $P_N$ and $Q_N$ in part (3) of Proposition 1.5 based on Conjecture 1.6. This builds on our previous work [DKWB19] which gave a low-degree analysis of sparse PCA (but did not cover the negatively-spiked case that we use here).

**Organization.** The remainder of the paper is organized as follows. In Section 2, we state our lower bound for average-case RIP certification based on the low-degree likelihood ratio, and show that the lower bound is optimal, matching the upper bound due to [KZ14]. In Section 3, we give the proof for the lower bound.

**Notation.** Our standard asymptotic notation $O(\cdot)$, $\Omega(\cdot)$, $\Theta(\cdot)$ always pertains to the limit $N \to \infty$. We also use $\tilde{O}(B)$ to mean $O(B \cdot \text{polylog}(N))$, $\tilde{\Omega}(B)$ to mean $\Omega(B/\text{polylog}(N))$, and $\tilde{\Theta}(B)$ to mean $\tilde{O}(B)$ and $\tilde{\Omega}(B)$. Also recall that $f(N) = o(g(N))$ (or equivalently $f(N) \ll g(N)$) means $f(N)/g(N) \to 0$ as $N \to \infty$ and $f(N) = \omega(g(N))$ (or equivalently $f(N) \gg g(N)$) means $f(N)/g(N) \to \infty$ as $N \to \infty$. We write $A \lesssim B$ to mean $A \leq CB$ for an absolute constant $C$, $A \gtrsim B$ to mean $A \geq CB$ for an absolute constant $C$, and $A \sim B$ to mean $A \lesssim B$ and $A \gtrsim B$. We say that an event happens with high probability if it happens with probability $1 - o(1)$ as $N \to \infty$.

## 2 Main Results

We control the low-degree likelihood ratio (see Section 1.4) to establish an average-case lower bound for the certification of RIP. We follow the proof sketch outlined in Proposition 1.5. Adopting the framework of Proposition 1.5, we consider $s = s(N)$ changing with $N$ but $\delta \in (0, 1)$ held fixed as $N \to \infty$.

**Theorem 2.1.** In the setting of Proposition 1.5, suppose $s(N) \to \infty$ as $N \to \infty$. Under $P$,

$$
\Pr \left[ \frac{1}{\sqrt{M}} A \text{ is } (s, \delta)-\text{RIP} \right] \leq \exp \left( -\frac{\delta^2 M}{12} \right) + 2 \exp \left( -\frac{(1 - \delta)^2 s}{24} \right) = o(1).
$$

(2)

We defer the proof of Theorem 2.1 to Section 3. The following well-known result tells that the random ensemble is RIP with high probability under $Q$.

**Theorem 2.2** (See [WBP16], Proposition 1; also [CT05, BDDW08, FR13]). In the setting of Proposition 1.5 under $Q$,

$$
\Pr \left[ \frac{1}{\sqrt{M}} A \text{ is not } (s, \delta)-\text{RIP} \right] \leq 2 \exp \left[ s \log \left( \frac{9eN}{s} \right) - \frac{\delta^2 M}{256} \right].
$$

(3)

Based on Conjecture 1.6, we are left to show the boundedness of the low-degree likelihood ratio between $P$ and $Q$, which gives evidence for hardness of the certification task. This is demonstrated in the following theorem.

**Theorem 2.3.** Under the setting of Proposition 1.5, let $D = D(N)$ satisfy $D(N) = o(N)$. If one of the following holds for sufficiently large $N$:

(a) $s \geq 2 \max \left( 1, \sqrt{\frac{1}{-6 \log[(1 - \epsilon)/\sqrt{\gamma}]}} \right) \sqrt{DN}$, or

(4)
(b) \((1 - \epsilon)/\sqrt{\gamma} < 1/\sqrt{3}\) and

\[
s \geq 2 \frac{1 - \epsilon}{\sqrt{\gamma}} \sqrt{DN},
\]

then the low-degree likelihood ratio \(\|L^{\leq D}\|\) of \(P\) against \(Q\) remains bounded as \(N \to \infty\).

The proof of Theorem 2.3 is deferred to Section 3. Recall that in Proposition 1.5 we set \(\epsilon = \frac{1 - \delta}{2(1 + \delta)}\).

In Theorem 2.3, case (a) covers all choices of constant \(\delta \in (0, 1)\) (and is thus most relevant for our purposes), while case (b) offers a looser requirement on \(s\) when \(\epsilon\) is close to 1, i.e., \(\delta\) is close to 0.

Thus, for any fixed \(\delta \in (0, 1), \|L^{\leq D}\|\) of \(P\) against \(Q\) remains bounded if \(s \gtrsim \sqrt{DN}\), i.e., \(D \lesssim s^2/N\). Under Conjecture 1.6 this implies that any algorithm which strongly distinguishes \(P_N\) from \(Q_N\) requires time \(N^{\Omega(s^2/N)}\) time. Together, the results in this section verify items (1)-(3) in Proposition 1.5 and thus constitute evidence that RIP certification requires time \(N^{\tilde{\Omega}(s^2/N)}\).

For convenience to reader, we also present below the lazy algorithm proposed in [KZ14], which gives the matching upper bound. In the following context, for \(S \subseteq [N]\) we denote by \(P_S := \sum_{i \in S} e_i e_i^\top\) the projector that zeros out all but the entries indexed by \(S\) of a vector.

**Proposition 2.4.** For a matrix \(X \in \mathbb{R}^{M \times N}\) and \(s \in \mathbb{N}_+\), for any \(\delta \in (0, 1), X\) satisfies the \((s, \delta)\)-RIP if and only if

\[
B_s(X) := \max_{S \subseteq [N]} \|P_S(X^\top X - I_N)P_S\| \leq \delta.
\]

Proposition 2.4 follows directly from the definition of RIP. A naïve way to certify RIP is to compute \(B_s(X)\) by enumerating all \(S \subseteq [N]\) with \(|S| = s\), which would take time \(N^{\Theta(s)}\). Instead, the following algorithm calculates \(B_r(X)\) for \(r = \tilde{\Theta}(s^2/N)\), thereby reducing the total runtime to \(N^{\tilde{\Theta}(s^2/N)}\).

**Algorithm 1** [KZ14] Lazy algorithm: Certification for the \((s, \delta)\)-RIP

**Require:** Input matrix \(X \in \mathbb{R}^{M \times N}\) with unit column vectors; parameters \(r, s \in [N]\) with \(1 < r \leq s\), and a number \(\delta \in (0, 1),\)

1: Compute \(B_r(X)\).
2: if \(\frac{s}{r-1} B_r(X) \leq \delta\) then
3:      return “yes”
4: else
5:      return “no”
6: end if

It is proved in [KZ14] that, with a choice \(r = \tilde{\Theta}(s^2/(\delta^2N))\), Algorithm 1 certifies the \((s, \delta)\)-RIP for \(X = \frac{1}{\sqrt{M}} A\) with high probability, where \(A\) is an \(M \times N\) matrix with i.i.d. Bernoulli random variables as its entries. We remark that with minor changes the same procedure works for \(A\) with i.i.d. standard Gaussian random variables, since each column of \(\frac{1}{\sqrt{M}} A\) is almost a unit vector under the \(L^2\) norm. Based on our lower bound and the effectiveness of Algorithm 1 we conclude that the task of \((s, \delta)\)-RIP certification requires time \(N^{\tilde{\Omega}(s^2/N)}\).

**Remark 2.5.** We remark that the RIP certification problem bears resemblance to sparse PCA, which is the positively-spiked \((\beta > 0)\) case of the spiked Wishart model with a sparse spike prior. In [DKW19], the authors investigated the precise runtime required to solve sparse PCA. This includes an algorithm that improves over the runtime of naïve exhaustive search by enumerating subsets of a particular size smaller than the true sparsity (similar to the lazy algorithm above), as well as a matching lower bound based on the low-degree likelihood ratio.
3 Proof for the Low-Degree Likelihood Ratio Bound

In this section we prove Theorems 2.1 and 2.3. We start with introducing the following two Chernoff-type bounds for Bernoulli and $\chi^2$ sums.

**Lemma 3.1.** Suppose $x$ is taken from the sparse Rademacher prior $X^N_\rho$ per Definition 1.4. For any $\mu \in (0, 1]$, we have

$$\Pr \left[ \|x\|^2 > 1 + \mu \right] \leq \exp \left( -\frac{\mu^2 \rho N}{3} \right), \quad \Pr \left[ \|x\|^2 < 1 - \mu \right] \leq \exp \left( -\frac{\mu^2 \rho N}{2} \right),$$

and therefore

$$\Pr \left[ 1 - \mu \leq \|x\|^2 \leq 1 + \mu \right] \geq 1 - 2 \exp \left( -\frac{\mu^2 \rho N}{3} \right).$$

**Proof.** Note that

$$\Pr \left[ \|x\|^2 > 1 + \mu \right] = \Pr \left[ \|x\|_0 > (1 + \mu)\rho N \right], \quad \Pr \left[ \|x\|^2 < 1 - \mu \right] = \Pr \left[ \|x\|_0 < (1 - \mu)\rho N \right]$$

where $\|x\|_0$ is the sum of $N$ independent Bernoulli($\rho$) random variables, and $\mathbb{E}\|x\|_0 = \rho N$. Therefore (6) and (7) follow from the multiplicative Chernoff bound in [AV79].

**Lemma 3.2** (Chernoff bound for $\chi^2$ distribution). For all $\delta \in (0, 1)$,

$$\Pr \left[ \frac{\chi^2_M}{M} \geq 1 + \delta \right] \leq \exp \left( -\frac{\delta^2 M}{12} \right).$$

**Proof.** By the classical Chernoff bound (see, for example [LM00]),

$$\frac{1}{M} \Pr \left[ \chi^2_M \geq (1 + \delta)M \right] \leq \frac{1}{2} (\delta + \log(1 + \delta)).$$

Now the lemma follows immediately from the observation that for $\delta \in (0, 1)$,

$$\frac{1}{2} (\delta + \log(1 + \delta)) \leq -\frac{\delta^2}{12}.$$

**Proof of Theorem 2.1.** Note that

$$\left\| \frac{1}{\sqrt{M}} Ax \right\|^2 = \frac{1}{M} \sum_{i=1}^{M} (u_i^\top x)^2 \sim \frac{1}{M} \left( \|x\|^2 - (1 - \epsilon)\|x\|^4 \right) \chi^2_M.$$

For $x$ taken from $X^N_\rho$ satisfying $1 - \epsilon \leq \|x\|^2 \leq 1 + \epsilon$, since $-(1 - \epsilon)\|x\|^2 \geq -(1 - \epsilon^2) > -1$, under $\mathbb{P}$ each row of the observation $A$ is taken from $\mathcal{N}(0, I - (1 - \epsilon)xx^\top)$. Furthermore, we deduce from $\|x\|^2 \leq 1 + \epsilon$ that $\|x\|_0 \leq (1 + \epsilon)\rho N \leq s$. Now Lemma 3.2 gives

$$\Pr \left[ \left\| \frac{1}{\sqrt{M}} Ax \right\|^2 \geq (1 - \delta)\|x\|^2 \right] = \Pr \left[ \frac{1}{M} \chi^2_M \geq \frac{1 - \delta}{1 - (1 - \epsilon)\|x\|^2} \right]$$

$$\leq \Pr \left[ \frac{1}{M} \chi^2_M \geq 1 + \delta \right]$$

$$\leq \exp \left( -\frac{\delta^2 M}{12} \right).$$
The first inequality used the fact that
\[
\frac{1 - \delta}{1 - (1-\epsilon)\|x\|^2} \geq \frac{1 - \delta}{1 - (1-\epsilon)^2} \geq \frac{1 - \delta}{2\epsilon} = 1 + \delta.
\]
Therefore we know from Lemma 3.1 that
\[
\Pr \left( \frac{1}{\sqrt{M}} A is (s, \epsilon)-RIP \right) \leq \Pr \left( 1 - \epsilon \leq \|x\|^2 \leq 1 + \epsilon, \left\| \frac{1}{\sqrt{M}} A x \right\|^2 \geq (1 - \delta)\|x\|^2 \right) + \Pr \left( \|x\|^2 < 1 - \epsilon \right) + \Pr \left( \|x\|^2 > 1 + \epsilon \right)
\]
\leq \exp \left( -\frac{\delta^2 M}{12} \right) + 2 \exp \left( -\frac{(1 - \delta)^2 s}{24} \right)
\]
the rightmost sum being \textit{o}(1) given \(s(N) \to \infty\) as \(N \to \infty\).

**Proof of Theorem 2.3.** Let \(L^{\leq D}_{N,M,\beta,X}\) denote the degree-\(D\) likelihood ratio for the spiked Wishart model (see Definition 1.3) with parameters \(N, M, \beta\) and spike prior \(X\). \cite{BKW19} gives the formula
\[
\| L^{\leq D}_{N,M,\beta,X} \|^2 = \mathbb{E}_{v^{(1)},v^{(2)} \sim X_N} \left[ \varphi_{M,D/2} \left( \frac{\beta^2 \langle v^{(1)},v^{(2)} \rangle^2}{4} \right) \right]
\]
\[
= \sum_{d=0}^{D/2} \sum_{d_1,\ldots,d_M \atop \sum d_i=d} \prod_{i=1}^{M} \left( 2 d_i \right) \left( \frac{\beta^2 \langle v^{(1)},v^{(2)} \rangle^2}{4} \right)^d
\]
where \(v^{(1)},v^{(2)}\) are drawn independently from \(X_N\). Here \(\varphi_{N,k}(x)\) is the Taylor series of \(\varphi_N\) around \(x = 0\) truncated to degree \(k\), i.e.
\[
\varphi_M(x) := (1 - 4x)^{-M/2}
\]
\[
\varphi_{M,k}(x) := \sum_{d=0}^{k} x^d \sum_{d_1,\ldots,d_M \atop \sum d_i=d} \prod_{i=1}^{M} \left( 2 d_i \right)
\]
Note that under the setting of Problem 1.5 we are essentially dealing with \(X_N\) a truncated version of the sparse Rademacher prior \(X^{\rho N}_N\): \(x \sim X_N\) is taken as follows.

1. Draw \(x \sim X^{\rho N}_N\).
2. If \(-(1 - \epsilon)\|x\|^2 < -1\), then set \(x = 0\).

Therefore for any non-negative integer \(d\) it holds that
\[
\mathbb{E}_{v^{(1)},v^{(2)} \sim X_N} \langle v^{(1)},v^{(2)} \rangle^{2d} \leq \mathbb{E}_{v^{(1)},v^{(2)} \sim X^{\rho N}_N} \langle v^{(1)},v^{(2)} \rangle^{2d}.
\]
The authors proved in \cite[Theorem 2.14]{DKWB19} that the right-hand side of (9) remains bounded as \(N \to \infty\) for \(X_N\) being the sparse Rademacher prior \(X^{\rho N}_N\), under the conditions which are exactly (4) and (5) upon taking \(\beta = -(1 - \epsilon)\) and \(\rho_N = s/(2N)\). (Note that the first condition in part (a) of Theorem 2.14 in \cite{DKWB19}, which translates to \((1 - \epsilon)/\sqrt{\gamma} < 1\), is automatically satisfied under \(\epsilon \in (0,1)\) and \(\gamma \geq 1\).) Hence we conclude that, under the condition (4) or (5), the LDLR of \(P\) against \(Q\) under the setting of Problem 1.5 also remains bounded as \(N \to \infty\).
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