THE CONTACT INVARIANT IN SUTURED FLOER HOMOLOGY

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Abstract. We describe an invariant of a contact 3-manifold with convex boundary as an element of Juhász’s sutured Floer homology. Our invariant generalizes the contact invariant in Heegaard Floer homology in the closed case, due to Ozsváth and Szabó.

The goal of this paper is to present an invariant of a contact 3-manifold with convex boundary. The contact class is an element of sutured Floer homology, defined by András Juhász, and generalizes the contact class in Heegaard Floer homology in the closed case, as defined by Ozsváth and Szabó [OS3] and reformulated by the authors in [HKM2]. In this paper we assume familiarity with convex surface theory (cf. [Gi1, H1]), Heegaard Floer homology (cf. [OS1, OS2]), and sutured manifold theory (cf. [Ga]).

Sutured manifold theory was introduced by Gabai [Ga] to study and construct taut foliations. A sutured manifold \((M, \Gamma)\) is a compact oriented 3-manifold (not necessarily connected) with boundary, together with a compact subsurface \(\Gamma = A(\Gamma) \sqcup T(\Gamma) \subset \partial M\), where \(A(\Gamma)\) is a union of pairwise disjoint annuli and \(T(\Gamma)\) is a union of tori. We orient each component of \(\partial M - \Gamma\), subject to the condition that the orientation changes every time we nontrivially cross \(A(\Gamma)\). Let \(R_+(\Gamma)\) (resp. \(R_-(\Gamma)\)) be the open subsurface of \(\partial M - \Gamma\) on which the orientation agrees with (resp. is the opposite of) the boundary orientation on \(\partial M\). Moreover, \(\Gamma\) is oriented so that the orientation agrees with the orientation of \(\partial R_+(\Gamma)\) and is the opposite that of \(\partial R_-(\Gamma)\). A sutured manifold \((M, \Gamma)\) is balanced if \(M\) has no closed components, \(\pi_0(A(\Gamma)) \to \pi_0(\partial M)\) is surjective, and \(\chi(R_+(\Gamma)) = \chi(R_-(\Gamma))\) on every component of \(M\). In particular, \(\Gamma = A(\Gamma)\) and every boundary component of \(\partial M\) nontrivially intersects the suture \(\Gamma\). In this paper, all our sutured manifolds are assumed to be balanced.

In what follows, we view the suture \(\Gamma\) as the union of cores of the annuli, i.e., as a disjoint union of simple closed curves.

In the setting of contact structures, a natural condition to impose on a contact 3-manifold \((M, \xi)\) with boundary is to require that \(\partial M\) be convex, i.e., there is a contact vector field transverse to \(\partial M\). To a convex surface \(F\) one can associate its dividing set \(\Gamma_F\) — it is defined as the isotopy class of multicurves \(\{x \in F | X(x) \in \xi(x)\}\), where \(X\) is a transverse contact vector field. It was explained in [HKM3] that dividing sets and sutures on balanced sutured manifolds can be viewed as equivalent objects. As usual, our contact structures are assumed to be cooriented.

Date: This version: September 10, 2007. (The pictures are in color.)
1991 Mathematics Subject Classification. Primary 57M50; Secondary 53C15.
Key words and phrases. tight, contact structure, open book decomposition, fibered link, Dehn twists, Heegaard Floer homology, sutured manifolds.

KH supported by an NSF CAREER Award (DMS-0237386); GM supported by NSF grant DMS-0410066; WHK supported by NSF grant DMS-0406158.
In a pair of important papers [Ju1, Ju2], András Juhász generalized (the hat versions of) Ozsváth and Szabó’s Heegaard Floer homology [OS1, OS2] and link Floer homology [OS4] theories, and assigned a Floer homology group $SFH(M, \Gamma)$ to a balanced sutured manifold $(M, \Gamma)$. (A similar theory was also worked out by Lipshitz [Li2].) An important property of this sutured Floer homology is the following: if $(M, \Gamma) \xrightarrow{T} (M', \Gamma')$ is a sutured manifold decomposition along a cutting surface $T$, then $SFH(M', \Gamma')$ is a direct summand of $SFH(M, \Gamma)$.

The main result of this paper is the following:

**Theorem 0.1.** Let $(M, \Gamma)$ be a balanced sutured manifold, and let $\xi$ be a contact structure on $M$ with convex boundary, whose dividing set on $\partial M$ is $\Gamma$. Then there exists an invariant $EH(M, \Gamma, \xi)$ of the contact structure which lives in $SFH(-M, -\Gamma)/\{\pm 1\}$.

Here we are using $\mathbb{Z}$-coefficients. Note that there is currently a $\pm 1$ ambiguity when $\mathbb{Z}$-coefficients are used.

The paper is organized as follows. Section 1 is devoted to discussing the contact-topological preliminaries for obtaining a partial open book decomposition of a contact 3-manifold with convex boundary. We define the contact invariant in Section 2 and prove that it is independent of the choices made in Section 3. We discuss some basic properties of the contact class in Section 4 and compute some examples in Section 5. Finally, we explain the relationship to sutured manifold decompositions in Section 6.

### 1. Contact structure preliminaries

Let $(M, \Gamma)$ be a sutured manifold. Let $\xi$ be a contact structure on $M$ with convex boundary so that the dividing set $\Gamma_{\partial M}$ on $\partial M$ is isotopic to $\Gamma$. Such a contact manifold will be denoted $(M, \Gamma, \xi)$.

The following theorem is the key to obtaining a partial open book decomposition, slightly generalizing the work of Giroux [Gi2] to the relative case. For more detailed expositions of Giroux’s work, see [Co2, Et].

**Theorem 1.1.** There exists a Legendrian graph $K \subset M$ whose endpoints, i.e., univalent vertices, lie on $\Gamma \subset \partial M$ and which satisfies the following:

1. There is a neighborhood $N(K) \subset M$ of $K$ so that (i) $\partial N(K) = T \cup (\cup_i D_i)$, (ii) $T$ is a convex surface with Legendrian boundary, (iii) $D_i \subset \partial M$ is a convex disk with Legendrian boundary, (iv) $T \cap \partial M = \cup_i \partial D_i$, (v) $\#(\partial D_1 \cap \Gamma_{\partial M}) = 2$, and (vi) there is a system of pairwise disjoint compressing disks $D'_j$ for $N(K)$ so that $\partial D'_j \subset T$, $|\partial D'_j \cap \Gamma_T| = 2$, and each component of $N(K) - \cup_j D'_j$ is a standard contact 3-ball, after rounding the corners.

2. Each component $H$ of the complement $M - N(K)$ is a handlebody with convex boundary. There is a system of pairwise disjoint compressing disks $D^*_k$ for $H$ so that $|\partial D^*_k \cap \Gamma_H| = 2$ and $H - \cup_k D^*_k$ is a standard contact 3-ball, after rounding the corners.

Here $|\cdot|$ denotes the geometric intersection number and $\#(\cdot)$ denotes the number of connected components. A standard contact 3-ball is a tight contact 3-ball $B^3$ with convex
boundary and \( \#\Gamma_{\partial B^3} = 1 \). We say that a handlebody \( H \) with convex boundary admits a product disk decomposition if condition (2) of Theorem 11 holds.

**Proof.** Since \( F = \partial M \) is convex, there is an \( I = [0,1] \)-invariant contact neighborhood \( F \times I \subset M \) so that \( F \times \{1\} = \partial M \). First we take a polyhedral decomposition of \( F_0 = F \times \{0\} \) so that the 1-skeleton is Legendrian and the boundary of each 2-cell intersects \( \Gamma_{F_0} \) in two points. During this process we need to use the Legendrian realization principle and slightly isotop \( F_0 \). Next, extend the polyhedral decomposition on \( F_0 \) to \( M - (F \times I) \) so that the 1-skeleton \( K' \) is Legendrian and the boundary of each 2-cell has Thurston-Bennequin invariant \( tb = -1 \). Finally, we need to connect \( K' \) to \( \partial M \). To achieve this, for each intersection point \( p \) of \( K'|F_0 \) with \( \Gamma_{F_0} \), we add an edge \( \{p\} \times [0,1] \subset F \times [0,1] \) to \( K' \). The resulting graph is our desired \( K \).

We now prove that \( K \) satisfies the properties of the theorem. There is a collection of compressing disks in \( (M - N(K')) - (F \times I) \) which intersect the dividing set \( \Gamma_{\partial N(K')} \) at exactly two points, by the \( tb = -1 \) condition. Without loss of generality, these compressing disks can be made convex with Legendrian boundary. Then the compressing disks cut up \( M - N(K') \) into a disjoint union of standard contact 3-balls and \( F \times I \). After removing standard neighborhoods of \( \{p\} \times I \) from \( F \times I \), the remaining contact 3-manifold admits a product disk decomposition. This implies that \( M - N(K) \) admits a product disk decomposition. \( \square \)

The next theorem is the relative version of the subdivision theorem of Giroux [Gi2] for contact cellular decompositions, which implies that on a closed manifold \( M \) any two open books corresponding to a fixed \((M, \xi)\) become isotopic after a sequence of positive stabilizations to each.

**Theorem 1.2.** Let \( K \) and \( K' \) be Legendrian graphs on \((M, \Gamma, \xi)\) which satisfy the conditions of Theorem 11 and \( K \cap K' = \emptyset \). Then there exists a common Legendrian extension \( L \) of \( K \) and \( K' \) so that \( L = K_n \) is obtained inductively from \( K = K_0 \) by attaching Legendrian arcs \( c_i, i = 1, \ldots, n - 1 \), to the standard neighborhood \( N(K_i) \) of \( K_i \) so that the following hold:

1. The arc \( c_i \) has both endpoints on the dividing set of \( \partial(M - N(K_i)) \), and \( \text{int}(c_i) \subset \text{int}(M - N(K_i)) \).
2. \( N(K_{i+1}) = N(c_i) \cup N(K_i) \), and \( K_{i+1} \) is a Legendrian graph so that \( N(K_{i+1}) \) is its standard neighborhood.
3. There is a Legendrian arc \( d_i \) on \( \partial(M - N(K_i)) \) with the same endpoints as \( c_i \), after possible application of the Legendrian realization principle. The arc \( d_i \) intersects \( \Gamma_{\partial(M - N(K_i))} \) only at its endpoints.
4. The Legendrian knot \( \gamma_i = c_i \cup d_i \) bounds a disk in \( M - N(K_i) \) and has \( tb(\gamma_i) = -1 \) with respect to this disk. This implies that \( c_i \) and \( d_i \) are Legendrian isotopic relative to their endpoints inside the closure of \( M - N(K_i) \).

The graph \( L = K'_m \) is similarly obtained from \( K' = K'_0 \) by inductively attaching Legendrian arcs \( c'_i \) to \( N(K'_i) \) to obtain \( K'_{i+1} \).

**Proof.** Consider the invariant neighborhood \( F \times [0,1] \) with \( F = F_1 = \partial M \) as in the first paragraph of Theorem 11 and take a copy \( F_{1-\delta} \) near \( F \). The procedure for finding a common refinement \( L \) of \( K \) and \( K' \) is as follows:
(1) First add Legendrian arcs to $K$ so that $K_i, i \gg 0$, contains a Legendrian 1-skeleton of the protective layer $F_{1-\delta}$ (as in the first paragraph of Theorem 1.1).

(2) Subdivide the Legendrian 1-skeleton of $F_{1-\delta}$ sufficiently, so that every arc of $K'$ near $\partial M$ (we assume they are all of the form $\{p\} \times [1-\delta, 1]$) intersects the 1-skeleton.

(3) Next add enough Legendrian arcs of the type $\{p\} \times [1-\delta, 1]$, where $p \in \Gamma_F$, so that $K_i \supset (K \cup K') \cap (F \times [1-\delta, 1]), i \gg 0$.

(4) Finally, apply the contact subdivision procedure of [Gi2] away from $F \times [1-\delta, 1]$.

Steps (2) and (4) involve the same procedure used in [Gi2], namely, given a face $\Delta$ of the contact cellular decomposition with Legendrian boundary $\partial \Delta \subset K$, take a Legendrian arc $c_i \subset \Delta$ with endpoints on $\partial \Delta$, so that $c_i$ cuts $\Delta$ into $\Delta_1, \Delta_2$, each of which has $tb(\partial \Delta_i) = -1$.

Next we discuss what happens in Steps (1) and (3). There are two types of arcs to attach in Step (1). The first type is an arc $c_i \subset F_{1-\delta}$ from $(p, 1-\delta)$ to $(q, 1-\delta)$, $p, q \in \Gamma_F$, which does not intersect $\Gamma_F \times \{1-\delta\}$ in its interior. Figure 1 depicts this situation. In this case there is an arc $d_i$ which passes through $R_\pm(\Gamma)$ and is isotopic to $c_i$ rel endpoints inside $M - N(K_i)$, so that $\gamma_i = c_i \cup d_i$ bounds a disk in $M - N(K_i)$ and has $tb(\gamma_i) = -1$ with respect to this disk. Another way of thinking about the attachment of $c_i$ is to slide both endpoints of $c_i$ towards $F_1 = \partial M$, along $\Gamma_{\partial (M-N(K_i))}$, so that both endpoints of $c_i$ are then placed on $\Gamma_{\partial M}$. Figure 2 depicts the neighborhood of $c_i$ after the isotopy. The disk $D$ with $tb(\partial D) = -1$ is easy to see in this diagram. Once the arcs of the first type are attached, we need to attach arcs $c_i \subset F_{1-\delta}$ of the second type in order to complete the Legendrian 1-skeleton of $F_{1-\delta}$. These $c_i$ connect interior points of arcs of the first type and do not intersect $\Gamma_F \times \{1-\delta\}$. This is given in Figure 3. The top endpoint of $c_i$ in Figure 3 can be moved to the left and the bottom endpoint can be moved to the right, both along $\Gamma_{\partial (M-N(K_i))}$, so that both endpoints now lie on $\Gamma_{\partial M}$. The result is the same situation as given in Figure 2.

Next we consider Step (3), which is depicted in Figure 4. In the figure, the face $F_1 = \partial M$ is in the back, and the thickening of the Legendrian 1-skeleton of $F_{1-\delta}$ is to the front. The thickening of the arc $c_i$ is the cylinder to the left emanating from $\Gamma_{\partial M}$.
2. Definition of the contact class

We briefly recall Juhász’ sutured Floer homology theory \cite{Ju1, Ju2}.

A compatible Heegaard splitting for a sutured manifold \((M, \Gamma)\) consists of a Heegaard surface \(\Sigma\) (not necessarily connected) with nonempty boundary, together with two sets of pairwise disjoint simple closed curves that do not intersect \(\partial \Sigma\), the \(\alpha\)-curves \(\alpha_1, \ldots, \alpha_r\) and the \(\beta\)-curves \(\beta_1, \ldots, \beta_r\). Then \(M\) is obtained from \(\Sigma \times [-1, 1]\) by gluing compressing disks along \(\alpha_i \times \{-1\}\) and along \(\beta_i \times \{1\}\), and thickening. We take the suture \(\Gamma\) to be \(\partial \Sigma \times \{0\}\).

Let \(T_\alpha = \alpha_1 \times \cdots \times \alpha_r\) and \(T_\beta = \beta_1 \times \cdots \times \beta_r\), viewed in \(\text{Sym}^r(\Sigma)\). Then let \(CF(\Sigma, \alpha, \beta)\) be the free \(\mathbb{Z}\)-module generated by the points \(x = (x_1, \ldots, x_r)\) in \(T_\alpha \cap T_\beta\). The suture \(\Gamma\) plays the role of the basepoint in sutured Floer homology. Denote by \(M_{x,y}\) the 0-dimensional (after quotienting by the natural \(\mathbb{R}\)-action) moduli space of holomorphic maps \(u\) from the unit disk \(D^2 \subset \mathbb{C}\) to \(\text{Sym}^r(\Sigma)\) that (i) send \(1 \mapsto x, -1 \mapsto y, S^1 \cap \{\text{Im } z \geq 0\}\) to \(T_\alpha\) and
Figure 4. Step (3) of the subdivision process. The blue arc is $c_i$ and the green arc is $d_i$.

$S^1 \cap \{\text{Im } z \leq 0\}$ to $\mathbb{T}_\beta$, and (ii) avoid $\partial \Sigma \times \text{Sym}^{-1}(\Sigma) \subset \text{Sym}(\Sigma)$. Then define

$$\partial x = \sum_{\mu(x,y) = 1} \#(M_{x,y}) y,$$

where $\mu(x,y)$ is the relative Maslov index of the pair and $\#(M_{x,y})$ is a signed count of points in $M_{x,y}$. The homology $SFH(M,\Gamma)$ of this complex is shown to be independent of the various choices made in the definition. In particular, it is independent of the choice of a “weakly admissible” Heegaard decomposition.

**Interlude on orientation conventions.** The convention for $(M,\Gamma)$ (consistent with that of Ozsváth-Szabó and Juhász) is as follows: The Heegaard surface $\Sigma$ is an oriented surface whose oriented boundary is $\Gamma$. The suture/dividing set $\Gamma$ is also defined as the boundary of $R_+(\Gamma)$ ($=\text{minus the boundary of } R_-(\Gamma)$). If $\Sigma$ splits $M$ into two compression bodies $H_1$ and $H_2$, and $\partial H_1 = \Sigma, \partial H_2 = -\Sigma$, then the boundaries of the compressing disks for $H_1$ are the $\alpha_i$ and the boundaries of the compressing disks for $H_2$ are the $\beta_i$. Then $SFH(M,\Gamma)$ is the Floer homology $SFH(\Sigma,\alpha,\beta)$, in that order. We will now describe $SFH(M,-\Gamma)$, $SFH(-M,\Gamma)$, and $SFH(-M,-\Gamma)$, using the same data $(\Sigma,\alpha,\beta)$. The reader is warned that the $\Sigma,\alpha,\beta$ that appear in this interlude will be different from the $\Sigma,\alpha,\beta$ that appear subsequently.

- $SFH(M,-\Gamma)$: If we keep the same orientation for $M$ and switch the orientation of $\Sigma$, then we must switch $\alpha$ and $\beta$. Hence $SFH(M,-\Gamma) = SFH(-\Sigma,\beta,\alpha)$. Also $R_+(\Gamma) = R_+(-\Gamma)$.
- $SFH(-M,\Gamma)$: If we switch the orientation of $M$, then $\partial H_1 = -\Sigma$ and $\partial H_2 = \Sigma$. Since the orientation of $\Gamma$ is unchanged, the orientation of $\Sigma$ is unchanged. Hence $SFH(-M,\Gamma) = SFH(\Sigma,\beta,\alpha)$. Observe that $R_\pm(M,\Gamma) = R_\pm(-M,\Gamma)$.
- $SFH(-M,-\Gamma) = SFH(-\Sigma,\alpha,\beta)$, from the above considerations.

Given $(M,\Gamma,\xi)$, the decomposition of $M$ into $M - N(K)$ and $N(K)$ from Theorem 1.1 gives us a partial open book decomposition $(S,R_+/(\Gamma),h)$, which we now describe. (The terminology partial open book decomposition is used because $M$ can be constructed from a page $S$ and a partially-defined monodromy map $h : P = S - R_+/(\Gamma) \rightarrow S$.) The tubular
portion $T$ of $-\partial N(K)$ is split by the dividing set into positive and negative regions, with respect to the orientation of $-\partial N(K)$ or $\partial (M - N(K))$. Let $P$ be the positive region. Next, if $D_i \subset \partial N(K)$ are the attaching disks of $N(K)$, then consider $R_+ (\Gamma) - \cup_i D_i$; from now on this subsurface of $\partial M$ will be called $R_+ (\Gamma)$. Then the page $S$ is obtained from the closure $\overline{R_+ (\Gamma)}$ of $R_+ (\Gamma)$ by attaching the positive region $P$. Let $\sim$ be an equivalence relation on $S \times [-1, 1]$ given by $(x, t) \sim (x, t')$, where $x \in \partial S$ and $t, t' \in [-1, 1]$. Then $S \times [-1, 1]/\sim$ can be identified with $M - N(K)$, where $\Gamma_{\partial (M - N(K))} = \partial S \times \{0\}$. The manifold $M$ is then obtained from $S \times [-1, 1]/\sim$ by attaching thickenings of the compressing disks $D_i^\beta$ corresponding to the meridians of $N(K)$. The suture $\Gamma$ on $\partial M$ is $\partial (\overline{R_+ (\Gamma)}) \times \{1\}$ — we emphasize that this $\Gamma$ is no longer quite the same as the dividing set $\Gamma_{\partial M}$. Also observe that $\partial D_i^\beta \cap (\partial (\overline{R_+ (\Gamma)}) \times \{1\}) = \emptyset$. The Heegaard surface is $\Sigma = \partial (S \times [-1, 1]/\sim) - (R_+ (\Gamma) \times \{1\})$. One set of compressing disks is $\{D_i^\beta\}$ and the other set $\{D_i^\alpha\}$ gives a disk decomposition of $M - N(K)$. Let $h$ be the monodromy map — it is obtained by first pushing $P$ across $N(K)$ to $T - P \subset \partial (M - N(K))$, and then following it with an identification of $M - N(K)$ with $S \times [-1, 1]/\sim$.

Suppose that $S$ is obtained by successively attaching $r$ 1-handles to the union of $\overline{R_+ (\Gamma)}$ and the previously attached 1-handles. Let $a_i, i = 1, \ldots, r$, be properly embedded arcs in $P = S - \overline{R_+ (\Gamma)}$ with endpoints on $A = \partial P - \Gamma$, so that $S - \cup_i a_i$ deformation retracts onto $\overline{R_+ (\Gamma)}$. A collection $\{a_1, \ldots, a_r\}$ of such arcs is called a basis for $(S, R_+ (\Gamma))$. In fact, $\{a_1, \ldots, a_r\}$ is a basis for $H_1 (P, A)$. Next let $b_i$ be an arc which is isotopic to $a_i$ by a small isotopy so that the following hold:

1. The endpoints of $a_i$ are isotoped along $\partial S$, in the direction given by the boundary orientation of $S$.
2. The arcs $a_i$ and $b_i$ intersect transversely in one point in the interior of $S$.
3. If we orient $a_i$, and $b_i$ is given the induced orientation from the isotopy, then the sign of the intersection $a_i \cap b_i$ is $+1$.

Then the $\alpha$-curves are $\partial (a_i \times [-1, 1])$ and the $\beta$-curves are $(b_i \times \{1\}) \cup (h(b_i) \times \{-1\})$, viewed on $S \times [-1, 1]/\sim$. The $\alpha$-curves and $\beta$-curves avoid the suture $\Gamma$ and hence determine a Heegaard splitting $(\Sigma, \beta, \alpha)$, which is easily seen to be weakly admissible. Now the union of $\Sigma$ and $R_+ (\Gamma) = R_+ (\Gamma) \times \{1\}$ bound $S \times [0, 1]/\sim$. Define the orientation on $\Sigma$ to be the outward orientation inherited from $S \times [0, 1]/\sim$. Then $\partial \Sigma$ is oriented oppositely from $\partial (R_+ (\Gamma))$, and we have $\partial \Sigma = -\Gamma$. Hence $SFH (\Sigma, \beta, \alpha) = SFH (-M, -\Gamma)$.

The contact class is basically the $EH$ class which was defined in [HKM2]. The only difference is that we are not using a full basis for $S$ and that the contact class sits in $SFH (-M, -\Gamma)/\{\pm1\}$. Let $T_\alpha = \alpha_1 \times \cdots \times \alpha_r$ and $T_\beta = \beta_1 \times \cdots \times \beta_r$, viewed in $\text{Sym}^r (\Sigma)$. Let $CF (\Sigma, \beta, \alpha)$ be the chain group generated by the points in $T_\beta \cap T_\alpha$. Let $x_i$ be the intersection point $(a_i \cap b_i) \times \{1\}$ lying in $S \times \{1\}$. Then $x = (x_1, \ldots, x_r)$ is a cycle in $CF (\Sigma, \beta, \alpha)$ due to the placement of $\Gamma$, and its class in $SFH (-M, -\Gamma)$ will be written as $EH (S, h, \{a_1, \ldots, a_r\})$. Figures 5 and 6 depict the situation described above.
Figure 5. The first figure shows a small portion of $M$ near an intersection point of $\Gamma$ and $K$ and is drawn from the point of view of the interior of $M - N(K)$. The second figure shows the page $S \times \{1\}$ and a basis arc $a$. Notice that in this figure $R_+(\Gamma)$ denotes a proper subsurface of the previous $R_+(\Gamma)$. The boundary of the new $R_+(\Gamma)$ is the new $\Gamma$. We have also applied edge-rounding to connect the dividing curves on $\partial M - \bigcup_i D_i$ to those on the tubular portion $T$. The third figure shows the Heegaard surface $\Sigma$. Notice that the arc $\beta - b$ is obtained by pushing $b$ through $N(K)$, while $\alpha - a$ is obtained by isotoping $a$ through $M - N(K)$. The subsurface of $\Sigma$ indicated in the fourth figure is $S \times \{-1\}$. Notice that the single point of intersection $a \cap b$ is replaced by two copies of itself in this figure. In doing sutured Floer homology computations, we only consider holomorphic disks which miss complementary regions of $\alpha \cup \beta$ containing $\Gamma$. This ensures that the computations may be done in the subsurface $S \times \{-1\}$ of $\Sigma$.

3. WELL-DEFINITION OF THE CONTACT CLASS

Theorem 3.1. $EH(S, h, \{a_1, \ldots, a_r\}) \in SFH(-M, -\Gamma)/\{\pm 1\}$ is an invariant of the contact structure.

Once the invariance is established, the contact class will be written as $EH(M, \Gamma, \xi)$. As usual, there is a $\pm 1$ indeterminacy, which we sometimes suppress.

Outline of Proof. We need to prove that $EH(S, h, \{a_1, \ldots, a_r\})$ is (i) independent of the choice of basis $\{a_1, \ldots, a_r\}$, (ii) invariant under stabilization, and (iii) only depends on the isotopy class of $h$. The dependence only on the isotopy class of $h$ is identical to the proof
Figure 6. The partial open book decomposition. The curve $\Gamma$ is the suture on $\partial M$. (It has been pushed into $R_+(\Gamma)$ for better viewing.) The top face is $S \times \{1\}$ and contains $R_+(\Gamma)$. In this figure, a single 1-handle is attached to $R_+(\Gamma)$. The point $x$ is then the contact class.

given in [HKM2], and will be omitted. The proof of the independence of choice of basis is similar to that of [HKM2], and we highlight only the differences in Section 3.1.

We define a positive stabilization in the relative case as follows: Let $c$ be a properly embedded arc in $S$; in particular, it can pass through $R_+(\Gamma)$. Attach a 1-handle to $S$ at the endpoints of $c$ to obtain $S'$, and let $\gamma$ be a closed curve obtained by gluing $c$ and a core of the 1-handle. Then a positive stabilization of $(S, R_+(\Gamma), h)$ is $(S', R_+(\Gamma), R_\gamma \circ h)$, where $R_\gamma$ is a positive Dehn twist about $\gamma$. We emphasize that the order of composing $R_\gamma$ and $h$ is important, since we are not allowed a global conjugation of $S$ when we only have a partial open book.

Every two partial open book decompositions representing the same contact structure $\xi$ become isotopic after performing a sequence of positive stabilizations to each. The proof follows from Theorem 1.2. Let us consider the case when $c_0$ is attached to $K_0$. Since $M - N(K) \simeq S \times [-1, 1]/\sim$, Condition 4 of Theorem 1.2 implies that $c_0$ can be viewed as a Legendrian arc on $S \times \{0\} \subset S \times [-1, 1]/\sim$. Attaching a neighborhood of $c_0$ to $N(K_0)$ is equivalent to drilling out a standard neighborhood of the Legendrian arc $c_0 \subset S \times \{0\}$. Recall that the monodromy map $h$ sends $P \times \{1\}$ to $S \times \{-1\}$. Since the drilling takes place in the region $S \times [-1, 1]$, after $h$ is applied, the new monodromy map is $R_\gamma \circ h$.

The invariance under stabilization in the closed case can be argued as follows, once we establish the independence of choice of basis: Suppose the stabilization occurs along the properly embedded arc $c$ in $S$. Take a basis $\{a_1, \ldots, a_r\}$ for $S$ so that all the $a_i$ are disjoint from $c$. The arc $c$ may be separating or nonseparating (even boundary-parallel), but there is a suitable basis in either case. Then take the basis $\{a_0, a_1, \ldots, a_r\}$ for $S'$, where $a_0$ is the cocore of the 1-handle attached onto $S$. Then $\beta_0$ and $\alpha_0$ intersect exactly once, and $\beta_0$ is disjoint from the other $\alpha_i$ by construction. Although there is a natural chain isomorphism between $CF(\beta, \alpha)$ and $CF(\beta \cup \{\beta_0\}, \alpha \cup \{\alpha_0\})$, where $\alpha = \{\alpha_1, \ldots, \alpha_r\}$ and $\beta = \{\beta_1, \ldots, \beta_r\}$, it is not clear that the identification is consistent with the stabilization and handleslide maps in [OS1]. In Lemma 3.5 we prove that $EH(S, h, \{a_1, \ldots, a_r\})$ is indeed mapped to
efficiently
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Same as that of Lemma 3.4 of [HKM2].

A of intersections in its isotopy class, where the endpoints of the arcs are allowed to move in

Lemma 3.2.

Let $\{a_1, a_2, \ldots, a_r\}$ be a basis for $(S, R_+(\Gamma))$. After possibly reordering the $a_i$’s, suppose $a_1$ and $a_2$ are adjacent arcs on $A = \partial P - \Gamma$, i.e., there is an arc $\tau \subset A$ with endpoints on $a_1$ and $a_2$ such that $\tau$ does not intersect any $a_i$ in $\text{int}(\tau)$. Define $a_1 + a_2$ as the isotopy class of $a_1 \cup \tau \cup a_2$, relative to the endpoints. Then the modification $\{a_1, a_2, \ldots, a_r\} \mapsto \{a_1 + a_2, a_2, \ldots, a_r\}$ is called an arc slide.

**Lemma 3.2.** $EH(S, h)$ is invariant under an arc slide $\{a_1, a_2, \ldots, a_r\} \mapsto \{a_1+a_2, a_2, \ldots, a_r\}$.

**Proof.** Same as that of Lemma 3.4 of [HKM2].

Let $\{a_1, \ldots, a_r\}$ and $\{b_1, \ldots, b_r\}$ be two bases for $(S, R_+(\Gamma))$. Assume that the two bases intersect transversely and efficiently, i.e., each pair of arcs $a_i, b_j$ realizes the minimum number of intersections in its isotopy class, where the endpoints of the arcs are allowed to move in $A$. In particular, there are no bigons consisting of a subarc of $a_i$ and a subarc of $b_j$, and no triangles consisting of a subarc of $a_i$, a subarc of $b_j$, and a subarc of $A$.

**Lemma 3.3.** Suppose that each component of $\bar{P}$ intersects $\Gamma$ along at least two arcs. Then there is a sequence of arc slides which takes $\{a_1, \ldots, a_r\}$ to $\{b_1, \ldots, b_r\}$.

Here $\bar{P}$ is the closure of $P$. The condition of the lemma is easily satisfied by performing a trivial stabilization along a boundary-parallel arc. Moreover, a trivial stabilization is easily seen to preserve the $EH$ class.

**Proof.** Consider a connected component $Q$ of $S - \bigcup_{i=1}^r a_i - \bar{R_+(\Gamma)}$. Then $Q$ is a (partially open) polygon whose boundary $\partial Q$ consists of $2k$ arcs, $k - 1$ of which are $a_i$ or $a_i^{-1}$, $k$ of which are subarcs $\tau_1, \ldots, \tau_k$ of $\partial P - \Gamma$, and one which is a subarc $\gamma$ of $\Gamma$. (For the moment we have oriented the $a_i$, and the notation $a_i^{-1}$ means that the orientation of $a_i$ and the orientation of $\partial Q$ are opposite.) Since each component of $\bar{P}$ intersects $\Gamma$ along at least two arcs, there is at least one arc $c \subset \partial Q$ of type $a_i$ or $a_i^{-1}$ so that $c^{-1}$ does not appear on $\partial Q$. Otherwise, $Q$ glues up to a component of $P$ whose closure intersects $\Gamma$ along one arc.

Suppose $(\bigcup_{i=1}^r a_i) \cap (\bigcup_{i=1}^r b_i) \neq 0$. We will apply a sequence of arc slides to $\{a_i\}_{i=1}^r$ to obtain $\{a_i'\}_{i=1}^r$ which is disjoint from $\{b_i\}_{i=1}^r$. After possibly reordering the arcs, there is a subarc $b_i^0 \subset b_1$ in $Q$ with endpoints on $a_1$ and $\tau_1$ on $\partial Q$. The subarc $b_i^0$ separates $Q$ into two regions $Q_1$ and $Q_2$, only one of which (say $Q_1$) has $\gamma$ as a boundary arc. If $c = a_1$, then we can slide $c$ around the boundary of $Q_2$ to obtain $a_1'$ which has fewer intersections with $b_1$. If $a_1$ and $a_1^{-1}$ both occur on $\partial Q$, then we have a problem if $Q_1$ has $\gamma$ as a boundary arc and $Q_2$ has $a_1^{-1}$ as a boundary arc. To maneuver around this problem, move $c$ via a sequence of arc slides so that the resulting $c'$ is parallel to $\gamma$ (and protects it). Then we can slide $a_1$ around the boundary of $\partial Q_1$ as in [HKM2].
Finally, suppose that \( \{a_i\}_{i=1}^r \) and \( \{b_i\}_{i=1}^r \) are disjoint. Let \( Q \) be a component of \( S - \bigcup_i a_i - R_+(\Gamma) \), as before. Let \( b_1 \) be an arc in \( Q \) which is not parallel to any \( a_i \) or \( a_i^{-1} \). The arc \( b_1 \) cuts \( Q \) into two components \( Q_1 \) and \( Q_2 \), where \( Q_1 \) has \( \gamma \) as a boundary arc. We claim that there is some \( a_i \) on \( \partial Q_2 \) so that \( a_i^{-1} \) is not on \( \partial Q_2 \). Otherwise, the arcs on \( \partial Q_2 \) will be paired up and \( b_1 \) will cut off a subsurface of \( S \) whose closure does not intersect \( \Gamma \). Moreover, we may also assume that \( a_i \) is not parallel to any \( b_j \). Now, apply a sequence of arc slides to \( a_i \) so that it becomes parallel to \( b_1 \).

\[ \square \]

3.2. Stabilization. A properly embedded arc \( c \) in \( S \) that intersects \( \Gamma \) efficiently is said to have \textit{complexity} \( n \) if there are \( n \) subarcs of \( c \) in \( \overline{P} \), both of whose endpoints are on \( \Gamma \). Let \( (S', h') \) be a positive stabilization of a partial open book decomposition \( (S, h) \) along a properly embedded arc \( c \subset S \) that intersects \( \Gamma \) efficiently. Then \( (S', h') \) is a stabilization of \( (S, h) \) of \textit{complexity} \( n \) if \( c \) has complexity \( n \).

The following lemma gives the fundamental property of arcs of complexity zero:

\textbf{Lemma 3.4.} An arc \( c \) has complexity zero if and only if there is a basis for \( (S, R_+(\Gamma)) \) which is disjoint from \( c \).

\textit{Proof.} If an arc \( c \) has complexity \( > 0 \), then there is a subarc \( c_0 \subset c \) in \( \overline{P} \), both of whose endpoints are on \( \Gamma \). It is impossible to find a basis that does not intersect \( c_0 \), since the pair \( (c_0, \partial c_0) \) is homotopically nontrivial in \( (\overline{P}, \Gamma) \).

On the other hand, suppose the complexity is zero. Then there are at most two subarcs of \( c \cap \overline{P} \). If \( c \) lies in \( R_+(\Gamma) \), then it does not intersect any basis of \( (S, R_+(\Gamma)) \). If \( c \) is entirely contained in \( P \), then it is easy to see that there is a basis which avoids \( c \). (There are two cases, depending on whether \( c \) cuts off a subsurface of \( P \) whose boundary does not intersect \( \Gamma \).)

Let \( c_1 \) be a subarc of \( c \cap \overline{P} \). Depending on \( c \), there may be two such, i.e., \( i = 1, 2 \), or only one such, i.e., \( i = 1 \). In either case, for each \( c_i \), one of its endpoints is on \( \Gamma \) and the other on \( \partial P - \Gamma \). If \( c_i \) is a trivial subarc, i.e., forms a triangle in \( P \), together with an arc of \( \Gamma \) and an arc of \( \partial P - \Gamma \), then \( c_i \) does not obstruct the formation of a basis and can in fact be isotoped out of \( P \). Therefore assume that \( c_i \) is nontrivial. Let \( \tau_1, \gamma, \tau_2 \) be subarcs of \( \partial P \) in counterclockwise order, where \( \gamma \subset \Gamma \) contains an endpoint of \( c_1 \), and \( \tau_i \) are subarcs of \( \partial P - \Gamma \). Also let \( \tau \) be the component of \( \partial P - \Gamma \) with the other endpoint of \( c_1 \). Then take the first arc \( a \) of the basis to be parallel to \( c_1 \), with endpoints on either \( \tau \) and \( \tau_1 \) or \( \tau \) and \( \tau_2 \). We have a choice of either of these unless \( c_2 \) also has an endpoint on \( \gamma \), in which case only one of these will work (and not intersect \( c_2 \)). The nontriviality of \( c_1 \) implies that \( a \) is neither boundary-parallel, i.e., cuts off a disk in \( P \), nor parallel to \( \Gamma \). If the arc \( a \) cuts off a subsurface of \( P \) whose boundary does not intersect \( \Gamma \), then we must replace it with disjoint, nonseparating arcs \( a_1, \ldots, a_{2g} \), where \( g \) is the genus of the cut-off surface. Once we cut \( P \) along \( a \) or \( a_1, \ldots, a_{2g} \), then \( c_1 \) becomes trivial in the cut-open surface, and can be ignored. We apply the same technique to \( c_2 \), if necessary, and the rest of the basis can be found without difficulty. \[ \square \]

\textbf{Lemma 3.5.} Let \( (S', h') \) be a stabilization of \( (S, h) \) along an arc \( c \) of complexity zero. Suppose \( \{a_1, \ldots, a_r\} \) is a basis for \( S \) which is disjoint from \( c \) and \( \{a_0, a_1, \ldots, a_r\} \) is a basis for \( S' \), where \( a_0 \) is the cocore of the 1-handle attached to obtain \( S' \). Then \( EH(S, h, \{a_1, \ldots, a_r\}) \) is
mapped to $EH(S', h', \{a_0, \ldots, a_r\})$ under the stabilization and handleslide maps defined in [OS1].

**Proof.** If no $\beta_i$, $i > 0$, nontrivially intersects $\alpha_0$, we have a standard stabilization. Suppose there is some $\beta_i$, $i > 0$, which nontrivially intersects $\alpha_0$. This means $h(b_i)$ nontrivially intersected $c$ before stabilization. Let $\eta$ be a closed curve on $S'$, obtained from $c$ by attaching the core of the 1-handle. In the left-hand diagram of Figure 7, a portion of $h(b_i)$ intersecting $c$ is drawn; $h'(b_i)$ is obtained from $h(b_i)$ by applying $R_{\eta}$. In order to remove its intersection with $\alpha_0$, we handleslide $\beta_i$ across $\beta_0$ to obtain $\gamma_i$, as indicated on the right-hand side of Figure 7. In Figures 7 and 8, we place a dot in a region of $\partial(S \times [-1, 1]/\sim) - \bigcup_{i=0}^{r} \alpha_i - \bigcup_{i=0}^{r} \beta_i$ to indicate that any point in the region has a path in the region that connects to $\Gamma = \partial \Sigma$.

![Figure 7](image)

**Figure 7.** The left-hand diagram shows the closed curve $\eta$ and an intersection point of $h(b_i)$ and $\eta$, on a page $S$. The middle diagram is the Heegaard diagram before the handleslide, and right-hand diagram gives the Heegaard diagram after the handleslide. The dotted curves are on the page $S \times \{-1\}$ and the solid curves are on $S \times \{1\}$. The short blue arc is on $S \times \{-1\}$ and is the handleslide arc.

Let $\gamma_j$, $j \neq i$, be small pushoffs of $\beta_j$. Figure 8 shows the corresponding arcs in $P$. The left-hand side of Figure 8 corresponds to $i = 0$ (and Figure 7), and the right-hand side corresponds to $i \neq 0$. We claim that the triple-diagram $(\Sigma, \gamma, \beta, \alpha, \Gamma)$ is weakly admissible. Recall that a triple-diagram is weakly admissible if each nontrivial triply-periodic domain which can be written as a sum of doubly-periodic domains has both positive and negative coefficients. Consider the situation on the right-hand side of Figure 8. The connected components of $\Sigma - \bigcup_i \alpha_i - \bigcup_i \beta_i - \bigcup_i \gamma_i$ are numbered as in the diagram. Due to the placement of $\Gamma$, the only potential doubly-periodic domain involving $\beta$, $\alpha$ in the vicinity of the right-hand diagram is $\Sigma$. Similarly, the only domain involving $\gamma$, $\beta$ is $\Sigma$. The only one involving $\gamma$, $\alpha$ is $\Sigma$. Taking linear combinations, we have

$$a(D_1 + D_2 - D_4 - D_5) + b(D_1 + D_6 - D_3 - D_4) + c(D_2 + D_3 - D_5 - D_6)$$

$$= (a + b)D_1 + (a + c)D_2 + (-b + c)D_3 - (a + b)D_4 - (a + c)D_5 + (b - c)D_6.$$
Since the coefficients come in pairs, e.g., \( a + b \) and \( -(a + b) \), if any of \( a + b, b + c, a - c \) does not vanish, then the triply-periodic domain has both positive and negative coefficients. Hence, if any of \( a + b, b + c, a - c \) does not vanish, then the triply-periodic domain has both positive and negative coefficients. Hence, if any of \( \alpha_i, \beta_i, \gamma_i \) is used for some \( i > 0 \), then we are done. Otherwise, we may assume that none of \( \alpha_i, \beta_i, \gamma_i \) is used in the periodic domain, for any \( i > 0 \). This allows us to assume that only \( \alpha_0, \beta_0, \gamma_0 \) are used for the boundary of the periodic domain.

Once all the \( \alpha_i, \beta_i, \gamma_i, i > 0 \), are ignored, the \( D_4 \) region of the left-hand diagram becomes connected to \( \Gamma \), and we can apply the same procedure.

We claim that we obtain \( x' = (x'_0, \ldots, x'_r) \in CF(\beta, \alpha) \) with the top generator \( \Theta \in CF(\gamma, \beta) \). We obtain \( x' = (x'_0, \ldots, x'_r) \), as depicted in Figure 8. As in the proof of the weak admissibility, we start with the vicinity of right-hand side, i.e., \( i > 0 \). There are no other holomorphic triangles besides the obvious small triangle involving \( \Theta, x_i, \) and \( x'_i \), due to the placement of the dots. Once the \( \alpha_i, \beta_i, \gamma_i, i > 0 \), are exhausted, \( D_4 \) becomes connected to \( \Gamma \), and we only have the small triangle with vertices \( \Theta_0, x_0, x'_0 \).

Let the new \( \beta \) be the old \( \gamma \) and let the new \( x \) be the old \( x' \). After a sequence of such handleslides (and renamings), we obtain \( \alpha \) and \( \beta \) so that \( \alpha_0, \beta_0 \) intersect once and do not intersect any other \( \alpha_i \) and \( \beta_i \). The standard stabilization map simply forgets \( x_0 \). □

**Figure 8.** The blue arc is \( a_i \), the green arc is \( b_i \), and the red arc is a pushoff of \( b_i \).

**Proposition 3.6.** Let \((S', h')\) be a stabilization of \((S, h)\) along an arc \( c \) of complexity \( n \). Then there exists a stabilization of complexity at most \( n - 1 \) taking \((S', h')\) to \((S'', h'')\), and two stabilizations of complexity at most \( n - 1 \) which take \((S, h)\) to \((S'', h'')\).

**Proof.** A stabilization corresponds to drilling out a neighborhood of a properly embedded Legendrian arc \( c = c \times \{0\} \subset S \times \{0\} \) inside \( S \times [-1, 1] \). We may assume that \( \Gamma_{\partial(S \times [-1, 1])} = \partial S \times \{0\} \) and that \( S \times \{0\} \) is a convex surface with dividing set \( \Gamma_{S \times \{0\}} = \partial S \times \{0\} \). Then \( c \) can be realized as a Legendrian arc on \( S \times \{0\} \) after applying the Legendrian realization principle (see e.g. [HI]). The nonisolating condition, required in the Legendrian realization principle, is easily met. The arc \( c \) may be thought of having twisting number \(-\frac{1}{2}\) relative to \( S \times \{0\} \), since it begins and ends on the dividing set and intersects no other dividing curves. Our strategy is to attach a Legendrian arc \( d \subset P \times \{0\} \) with one endpoint on \( (\partial P - \Gamma) \times \{0\} \) and the other endpoint on \( c \) to obtain a Legendrian graph \( L = c \cup d \) on \( S \times \{0\} \). (Again, the nonisolating condition is satisfied.) The endpoint of \( d \) on
c must lie on a component of \( c \cap \overline{P} \) with both endpoints on \( \Gamma \). Then \( S'' \times [-1, 1]/\sim \) will be \( (S \times [-1, 1]/\sim) - N(L) \). We will exploit two ways of obtaining the trivalent graph \( L \). One way is to stabilize along \( c \) and then along \( d \). For the other way, subdivide \( c = c_1 \cup c_2 \) at the trivalent vertex of \( L \). We label the arcs \( c_1, c_2 \) so that \( d, c_1, c_2 \) are in counterclockwise order about the trivalent vertex. Then first stabilize along \( c_1 \cup d \) and then along \( c_2 \). See Figure 9.

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{figure9.png}
\end{array}
\]

**Figure 9.** The red arcs are \( \Gamma \) and the region drawn is \( P \).

Next refer to Figure 10. We claim that if we stabilize along the arc \( c_1 \cup d \) (its thickening is drawn in the diagram to the left), and then stabilize along \( c_2 \), given as the green arc, then the two stabilizations are equivalent to the stabilizations on the right-hand side. To see this, we isotop \( c_2 \) inside \( B = (S \times [-1, 1]/\sim) - N(c_1 \cup d) \), while constraining its endpoints to lie on \( \Gamma_{\partial B} \). The key point to observe is that \( c_2 \) can be slid “to the right” so that it lies above the neighborhood of \( c_1 \cup d \).

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{figure10.png}
\end{array}
\]

**Figure 10.**

Another way to see this, without the use of contact topology, is the following. Let \( c = c_1 \cup c_2 \), \( c' = c_1 \cup d \), and \( c'' = c_2 \cup d \). Also let \( \gamma, \gamma', \) and \( \gamma'' \) be closed curves as depicted in
Figure 11. which extend c, c', c''. Then $R_{c'} \circ R_{\gamma} = R_{c''} \circ R_{c'}$ (on a punctured torus). (Here $R_{\gamma}$ is the positive Dehn twist about $\gamma$.) □

Proposition 3.7. Let $(S', h')$ be a stabilization of $(S, h)$ along an arc $c$ of complexity $n$. Then $EH(S, h)$ is mapped to $EH(S', h')$ under the stabilization and handleslide maps defined in [OS1].

Proof. Let $(\Sigma, \beta, \alpha)$ be a compatible Heegaard splitting for $(M, \Gamma)$ corresponding to some basis for $(S, h)$. (Similarly define $(\Sigma, \beta^{(i)}, \alpha^{(i)})$, for $(S^{(i)}, h^{(i)})$, defined subsequently.) Suppose $(S', h')$ is a stabilization of $(S, h)$ of complexity zero. Then the map $\Phi : SFH(\beta, \alpha)/\{\pm 1\} \to SFH(\beta', \alpha')/\{\pm 1\}$, defined as a composition of handleslide maps and stabilization maps, sends $EH(S, h) \mapsto EH(S', h')$ by Lemma 3.5 and the invariance under basis change. Assume inductively that stabilizations of complexity at most $n-1$ take $EH$ classes to $EH$ classes.

Then, by Proposition 3.6 (we are using the same notation as the proposition), the map $\Phi_{12} : SFH(\beta', \alpha')/\{\pm 1\} \to SFH(\beta'', \alpha'')/\{\pm 1\}$ sends $EH(S', h') \mapsto EH(S'', h'')$ and the map $\Phi_{02} : SFH(\beta, \alpha)/\{\pm 1\} \to SFH(\beta'', \alpha'')/\{\pm 1\}$ sends $EH(S, h) \mapsto EH(S'', h'')$. Now, $\Phi_{01} : SFH(\beta, \alpha)/\{\pm 1\} \to SFH(\beta', \alpha')/\{\pm 1\}$ satisfies $\Phi_{02} = \Phi_{12} \circ \Phi_{01}$ by Theorem 2.1 of [OS5], which states that the maps $\Phi_{ij}$ do not depend on the particular sequence of handleslides, stabilizations, and isotopies chosen. This implies that $\Phi_{01}$ maps $EH(S, h) \mapsto EH(S', h')$. □

4. Properties of the contact class

In this section we collect some basic properties of the contact class $EH(M, \Gamma, \xi)$. Most of the properties are analogs of properties of the contact class that are well-known in the case when $M$ is closed. The theorem which does not have an analog in the closed case (for obvious reasons) is the restriction theorem (Theorem 4.5).

Consider a partial open book decomposition $(S, h)$ for $(M, \Gamma, \xi)$. The notion of a right-veering $(S, h)$ can be defined in the same way as in [HKM]: If for every $x \in \partial P - \Gamma$ and every properly embedded arc $a \subset P$ which begins at $x$ and has both endpoints on $\partial P - \Gamma$, $h(a)$ is to the right of $a$, then we say $(S, h)$ is right-veering.

Proposition 4.1. If $(S, h)$ is not right-veering, then $(M, \Gamma, \xi)$ is overtwisted. Any overtwisted $(M, \Gamma, \xi)$ admits a partial open book decomposition $(S, h)$ which is not right-veering.
Proof. The first assertion is proved in the same way as the analogous statement in [HKM1].

For the second assertion, note that Example 1 of Section 5 gives a partial open book decomposition of a neighborhood of an overtwisted disk with a left-veering arc. Attach the neighborhood of a Legendrian arc \(a\) which connects the boundary of the overtwisted disk to \(\Gamma \subset \partial M\), and then complete it to a partial open book for \((M, \Gamma)\). The left-veering arc from Example 1 survives to give a left-veering arc. \(\square\)

**Proposition 4.2.** If \((M, \Gamma, \xi)\) admits a partial open book decomposition \((S, h)\) which is not right-veering, then \(EH(M, \Gamma, \xi) = 0\).

Proof. Same as that of [HKM2]. \(\square\)

By combining Propositions 4.1 and 4.2, we obtain the following:

**Corollary 4.3.** If \((M, \Gamma, \xi)\) is overtwisted, then \(EH(M, \Gamma, \xi) = 0\).

The next proposition describes the effect of Legendrian surgery on the contact invariant.

**Proposition 4.4.** If \(EH(M, \Gamma, \xi) \neq 0\) and \((M', \Gamma', \xi')\) is obtained from \((M, \Gamma, \xi)\) by a Legendrian \((-1)\)-surgery along a closed Legendrian curve \(L\), then \(EH(M', \Gamma', \xi') \neq 0\).

Proof. We can easily extend \(L\) to a Legendrian skeleton \(K\) as given in Theorem 1.1. Hence there is a partial open book decomposition \((S, R_+(\Gamma), h)\) so that \(L \subset P\). Take a basis \(\{a_1, \ldots, a_r\}\) so that \(a_1\) intersects \(L\) once and \(a_i \cap L = \emptyset\) for \(i > 1\). Push the \(a_i\) off to obtain \(b_i\) and \(c_i\), as drawn in Figure 12. Let \(\alpha_i = (a_i \times \{1\}) \cup (a_i \times \{-1\})\), \(\beta_i = (b_i \times \{1\}) \cup (h(b_i) \times \{-1\})\), and \(\gamma_i = (c_i \times \{1\}) \cup (R_+ h(c_i) \times \{-1\})\). Here \(\gamma\) is the curve \(h(L)\) on the page \(S\).

As Baldwin observed in [Ba], there is a comultiplication \(CF(\gamma, \alpha) \rightarrow CF(\gamma, \beta) \otimes CF(\beta, \alpha)\) which takes \(x''\) to \(x' \otimes x\). We are assuming that \([x'] \neq 0\). Also, \([x'] \neq 0\) is immediate from the fact that \(\beta_1\) and \(\gamma_1\) intersect only at \(x'_1\), due to our choice of basis. Therefore, we have \([x''] \neq 0\). \(\square\)

In [H2], the first author exhibited a tight contact structure on a handlebody which became overtwisted after Legendrian surgery. By Proposition 4.4, its contact invariant must vanish.

---

**Figure 12.**
Theorem 4.5. Let $(M, \xi)$ be a closed contact 3-manifold and $N \subset M$ be a compact submanifold (without any closed components) with convex boundary and dividing set $\Gamma$. If $EH(M, \xi) \neq 0$, then $EH(N, \Gamma, \xi|_N) \neq 0$.

Proof. Consider the partial open book decomposition $(S, R_+(\Gamma), h)$ for $(N, \Gamma, \xi|_N)$, obtained by decomposing $N$ into $N(K)$ and $N - N(K)$. Now, the complement $N' = M - N$ can be similarly decomposed into $N(K')$ and $N' - N(K')$, where $K'$ is a Legendrian graph in $N'$ with univalent vertices on $\Gamma$. We may assume that the univalent vertices of $K'$ and $K$ do not intersect. Then an open book decomposition for $(M, \xi)$ can be obtained from the Heegaard decomposition $(N - N(K)) \cup N(K')$ and $(N' - N(K')) \cup N(K)$. Indeed, both handlebodies are easily seen to be product disk decomposable. The page $T$ for $(M, \xi)$ can be obtained from $(S, R_+(\Gamma), h)$ by successively attaching 1-handles, subject to the condition that none of the handles be attached along $\partial P$, where $P = S - R_+(\Gamma)$. The monodromy map $g : T \to T$ extends $h : P \to S$.

Let $\{a_1, \ldots, a_r, a'_1, \ldots, a'_s\}$ be a basis for $T$ which extends a basis $\{a_1, \ldots, a_r\}$ for $(S, R_+(\Gamma))$. For such an extension to exist, $T - P$ must be connected. This is possible if $M - N$ is connected — simply take suitable stabilizations to connect up the components of $T - P$. If $M - N$ is disconnected, we apply a standard contact connected sum inside $M - N$ to connect up disjoint components of $M - N$. This has the effect of attaching 1-handles to $T$ away from $P$ and extending the monodromy map by the identity. The contact manifold $(M, \xi)$ has been modified, but it is easy to see that the contact class of the connected sum is nonzero if and only if the original contact class is nonzero.

Let $x$ be the generator of $EH(N, \Gamma, \xi|_N)$ with respect to $\{a_1, \ldots, a_r\}$ and $(x, x')$ be the generator of $EH(M, \xi)$ with respect to $\{a_1, \ldots, a_r, a'_1, \ldots, a'_s\}$. If $\partial(\sum_i c_i y_i) = x$, then we claim that $\partial(\sum_i c_i (y_i, x')) = (x, x')$. Indeed, each of the intersection points of $x'$ must map to itself via the constant map — this uses up all the intersection points of $x'$. We then erase all the $\alpha_i$ and $\beta_i$ corresponding to $x'$, and are left with the Heegaard diagram for $(S, R_+(\Gamma))$. \qed

Comparison with other invariants. We now make some remarks on the relationship to the contact invariant in the closed case and to Legendrian knot invariants.

1. If we start with a closed $(M, \xi)$, then we can remove a standard contact 3-ball $B^3$ with $\Gamma_{\partial B^3} = S^1$. The sutured manifold is called $M(1)$ in Juhász [Ju1]. In this case, $SFH(-M(1)) = \widehat{HF}(\bar{M})$, and the contact element in $SFH(-M(1))$ coincides with the Ozsváth-Szabó contact class in $\widehat{HF}(\bar{M})$. (Think of the disk $R_+$ being squashed to a point to give the basepoint $z$.)

2. A Legendrian knot $L$ has a standard neighborhood $N(L)$ which has convex boundary. The dividing set $\Gamma_{\partial N(L)}$ satisfies $\# \Gamma_{\partial N(L)} = 2$ and $|\Gamma_{\partial N(L)} \cap \partial D^2| = 2$, where $D^2$ is the meridian of $N(L)$. Moreover, the framing for $L$ induced by $\xi$ agrees with the framing induced by the ribbon in $N(L)$ which contains $L$ and has boundary on $\Gamma_{\partial N(L)}$.

If $(M, \xi)$ is closed, then $EH(M - N(L), \Gamma_{\partial N(L)}, \xi)$ is an invariant of the Legendrian knot $L$ which sits in $SFH(-(M - N(L)), -\Gamma_{\partial N(L)})$. On the other hand, if we choose the suture $\Gamma$ on $\partial N(L)$ to consist of two parallel meridian curves, then $SFH(-(M - N(L)), -\Gamma) =$
\(\widehat{HFK}(−M, L)\). Hence we are slightly off from the Legendrian knot invariants of Ozsváth-Szabó-Thurston [OST] and Lisca-Ozsváth-Stipsicz-Szabó [LOSS] — the Legendrian knot invariant currently does not sit in \(\widehat{HFK}(−M, L)\). We also only have one invariant, rather than two.

5. Examples

In this section we calculate a few basic examples.

**Example 1:** Standard neighborhood of an overtwisted disk. Let \(D\) be a convex surface which is a slight outward extension of an overtwisted disk. Then consider its \([0, 1]\)-invariant contact neighborhood \(M = D \times [0, 1]\). After rounding the edges, we obtain \(\Gamma_{\partial M}\) which consists of three closed curves, two which are \(\gamma \times \{0, 1\}\), where \(\gamma\) is a closed curve in the interior of \(D^2\), and one which is \(\partial D^2 \times \{\frac{1}{2}\}\). Our Legendrian graph \(K\) is an arc \(\{p\} \times [0, 1]\), where \(p \in \gamma\). Then \(M − N(K)\), after edge-rounding, is a solid torus whose dividing set consists of two parallel homotopically nontrivial curves, each of which intersects the meridian once. See the left-hand diagram of Figure 13 for \(M − N(K)\). The meridian of \(M − N(K)\) gives a product disk decomposition, and the decomposition into \(M − N(K)\) and \(N(K)\) gives rise to a partial open book decomposition. The left-hand diagram of Figure 13 shows the arc \(a\) on \(P\). The arc \(a\) is isotoped through the fibration \(N(K)\) (rel endpoints) and then through the fibration \(M − N(K)\) (rel endpoints). The resulting arc on \(S \times \{1\}\) is \(h(a)\).

Next consider the right-hand diagram of Figure 13 which shows a page \(S\) of the partial open book decomposition. The region \(R_+(\Gamma) \subset S\) has two connected components — an annulus and a disk — the red curves denote their boundary \(\Gamma\). The region \(P\) is the 1-handle which connects the annulus and the disk. The blue arc is the arc \(a \times \{-1\}\) and the green arc represents \(h(b) \times \{-1\}\), both viewed as sitting on \(S \times \{-1\}\). The unique intersection point \(x\) between \(a \times \{1\}\) and \(b \times \{1\}\) is now viewed as two points on \(S \times \{−1\}\), both labeled \(x\). (This way, all of the holomorphic disk counting can be done on \(S \times \{-1\}\).)

Now, \(CF(β, a)\) is generated by two points \(x, y\). (This is because \(h(a)\) is to the left of \(a\) at one of its endpoints.) It is easy to see that \(\partial y = x\), so \(SFH(−M, −Γ) = 0\) and \(EH(M, Γ, ξ) = 0\).

**Example 2:** A \([0, 1]\)-invariant neighborhood \(M\) of a convex surface \(F\), none of whose components \(F − Γ_F\) are disks. Figure 14 depicts the case when \(F\) is a torus with two parallel dividing curves. Let \(γ_1\) and \(γ_2\) be the two components of \(Γ_F\). Then take \(K = \{p_1, p_2\} \times [0, 1]\), where \(p_i \in γ_i\), \(i = 1, 2\). It is not hard to see that \(M − N(K)\) is product disk decomposable. The page \(S\) is obtained from \(R_+(Γ)\), which is a disjoint union of two annuli, by attaching two bands as in Figure 13. Hence \(P\) consists of two connected components, one with a cocore \(a_1\) and the other with a cocore \(a_2\). The black arcs on the left-hand diagram of Figure 14 are obtained from \(a_i\) by isotopy through \(N(K)\) rel endpoints, and the green arcs \(h(a_i)\) by further isotoping through \(M − N(K)\). The arc \(h(a_i)\) is obtained from \(a_i\) via what looks like a “half positive Dehn twist” about a boundary component. The arc \(h(a_1)\) only intersects \(a_1\) and the arc \(h(a_2)\) only intersects \(a_2\); hence \(h(a_1), a_1\) and \(h(a_2), a_2\) are independent. To determine \(SFH(−M, −Γ)\) and verify that \(EH(M, Γ, ξ) \neq 0\), the only nontrivial regions that need to be considered are \(D_1\) and \(D_2\), which are annuli with corners. (All other domains nontrivially
intersect the suture $\Gamma$.) We now apply Lipshitz' formula (Corollary 4.10 of [Li1])

$$\mu(D_i) = n_x(D_i) + n_y(D_i) + e(D_i),$$

for computing the index of $D_i$. Here $e(D_i)$ is the Euler measure of $D_i$ and $n_x(D_i)$ is the (weighted) intersection number of the $r$-tuple $x$ with $D_i$. We compute that $\mu(D_i) = 2(\frac{1}{4}) + 2(\frac{1}{4}) - 1 = 0$, which has index $\neq 1$. Hence it follows that $SFH(-M,-\Gamma) = \mathbb{Z}^4 = \mathbb{Z}^2 \otimes \mathbb{Z}^2$ and $EH(M,\Gamma, \xi) \neq 0$. (The fact that we can view $SFH(-M,-\Gamma)$ as $\mathbb{Z}^2 \otimes \mathbb{Z}^2$ follows from the observation that the two intersection points of $\alpha_1$ and $\beta$ and the two intersection points of $\alpha_2$ and $\beta$ are independent. We can also observe that $(M,\Gamma)$ admits an annulus decomposition into two solid tori of the type considered in the next example with $n = 2$, and each sutured solid torus contributes $\mathbb{Z}^2$. Then we can use the tensor product formula, proved by Juhász [Ju2, Proposition 8.10].) The general situation is similar.

**Example 3:** Solid torus $M = S^1 \times D^2$ with slope $\infty$ and $\#\Gamma = 2n > 2$. According to the classification of tight contact structures on the solid torus [Gi3, Hi], there is a 1-1
correspondence between isotopy classes of tight contact structures on \((M, \Gamma)\) rel boundary and isotopy classes of dividing sets on the meridian disk \(D^2 = \{pt\} \times D^2\). Figure 15 depicts the case \(n = 4\). There are two tight contact structures with \(\partial\)-parallel \(\Gamma\). (A convex surface \(T\) has \(\partial\)-parallel dividing set \(\Gamma_T\) if each component of \(\Gamma_T\) cuts off a half-disk which intersects no other component of \(\Gamma_T\).) The contact structures with \(\partial\)-parallel set are distinguished by their relative Euler class. The monodromy map \(h\) in the \(\partial\)-parallel case is calculated in Figure 15.

On the right-hand side of Figure 15, \(a_1, a_2, a_3\) is a basis for \((S, R_+(\Gamma), h)\), in counterclockwise order. (The \(\alpha_i\) are \(\partial\)(\(a_i \times [-1, 1]\)).) Label the intersections \(\alpha_i \cap \beta\) by \(x_i, y_j, z_k\), where \(i, j, k = 1, 2\). There are regions of \(\Sigma - \cup_i \alpha_i - \cup_i \beta_j\) which do not intersect \(\Gamma\) — all such regions are quadrilaterals, but use the same \(\alpha_i\) or \(\beta_j\) twice, so are not valid. Hence there are no holomorphic disks, and the boundary map is the zero map. Therefore, \(SFH(-M, -\Gamma) = \mathbb{Z}^8 = \mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^2\) and \(EH(M, \Gamma, \xi) \neq 0\). Moreover, if we split \(SFH(-M, -\Gamma)\) according to relative \(\text{Spin}^c\)-structures, then we have a direct sum \(\mathbb{Z} \oplus \mathbb{Z}^3 \oplus \mathbb{Z}^3 \oplus \mathbb{Z}\). The two \(\partial\)-parallel tight contact structures use up the first and last \(\mathbb{Z}\) summands, and the others have \(EH\) classes that live in the remaining \(\mathbb{Z}^3 \oplus \mathbb{Z}^3\). We leave it as an exercise to verify that the \(EH\) classes of the remaining tight contact structures are nonzero. There are \(4 + 2\) tight contact structures \(\xi_i, i = 1, \ldots, 6\), corresponding to each \(\mathbb{Z}^3\) summand, but we expect the \(EH\) classes to distinguish them — this means that we believe the \(EH(\xi_i)\) are linearly dependent.

**Example 4:** A basic slice \(M = T^2 \times [0, 1]\) (see [H1]). Write \(T_i = T^2 \times \{i\}, i = 0, 1, \) and take an oriented identification \(T^2 \simeq \mathbb{R}^2 / \mathbb{Z}^2\). Normalize so that \(#\Gamma_{T_i} = 2, \Gamma_{T_i}\) are linear, and slope(\(\Gamma_{T_0}\)) = 0, slope(\(\Gamma_{T_1}\)) = \(\infty\). A basic slice can be obtained from the convex torus \(T_0\) by attaching a single bypass along a linear arc of slope \(-1 < s < 0\) and thickening. Pick a point \(p\) on the connected component of \(\Gamma_{T_0}\) which contains the endpoints of the bypass arc of attachment, as indicated in Figure 16. Then we let \(K\) be \(p\) times an interval. It is not hard to see that \(M - N(K)\) is product disk decomposable.
Figure 16. The left-hand diagram gives $T_0$ and the arc of attachment for the bypass, which is attached from the front. The right-hand diagram gives $T_1$. Here the sides are identified and the top and the bottom are identified to give $T^2$. The other basic slice is obtained by switching the $+$ and $-$ signs.

The page $S$ is a thrice-punctured sphere obtained from $R_+ (\Gamma)$, a disjoint union of two annuli, by adding a 1-handle to connect the two annuli. If $a \subset P$ is the cocore of the 1-handle, then we compute that $h$ is a positive Dehn twist about the connected component of $\partial S$ which contains the endpoints of $a$. See Figure 17. One easily computes that $SFH(-M, -\Gamma) = \mathbb{Z}^4$ and $EH(M, \Gamma, \xi) \neq 0$. There are two basic slices, which are distinguished by the relative Euler class. They account for $\mathbb{Z} \oplus \mathbb{Z} \subset SFH(-M, -\Gamma)$. The remaining $\mathbb{Z} \oplus \mathbb{Z}$ come from tight contact structures with an extra $\pi$-rotation, and are also distinguished by their relative Euler classes. (See Example 6.)

Example 5: Attaching a bypass. Consider the contact structure $(M, \Gamma, \xi)$. Let $D$ be a bypass which is attached to $M$ from the exterior, along a Legendrian arc of attachment $c \subset \partial M$. If $M' = M \cup N(D)$ and $(M', \Gamma', \xi')$ is the resulting contact 3-manifold with convex boundary, then we express the monodromy map $(S', R_+ (\Gamma'), h')$ for $(M', \Gamma', \xi')$ in terms of...
$(S, R_+(\Gamma), h)$ for $(M, \Gamma, \xi)$. Assume without loss of generality that $c$ does not intersect the neighborhood $N(K)$ of the Legendrian skeleton $K$ for $M$. Let $p_1, p_2$ be the endpoints of $c$ on $\Gamma$ and let $c_{\pm} = c \cap R_{\pm}(\Gamma)$. Next let $\partial D = c \cup d$, where $c$ and $d$ intersect only at their endpoints $p_1, p_2$. The key observation is that a bypass attachment corresponds to two handle attachments: a 1-handle $N(d)$ attached at $p_1$ and $p_2$, followed by a canceling 2-handle. Then $(M - N(K)) \cup N(d)$ is product disk decomposable and gives a fibration structure $S' \times [-1, 1]$. Let $a$ be the arc in $P'$ given in Figure 18. Then, after isotoping it through the 2-handle, $a$ is isotopic rel endpoints to $a' \cup a''$, where $a'$ lies in the positive region and $a''$ lies in the negative region of $\partial((M - N(K)) \cup N(d))$. Next, isotop $a''$ through the fibration $M - N(K)$ to obtain $a'''$ which lies in the positive region of $\partial(M - N(K))$. Then $h'(a) = a' \cup a'''$. All the other arcs of $P'$, i.e., those that were in $P$, are unaffected. Summarizing, $S'$ is obtained from $S$ by attaching a 1-handle from $p_1$ to $p_2$ as in Figure 19, the new $P'$ is the union of $P$ and a neighborhood of the arc $a$, $h'(a) = a' \cup a'''$, and $h' = h$ on $P$.

**Figure 18.**

**Figure 19.** The left-hand diagram shows a portion of $S$ before bypass attachment and the right-hand diagram is $S'$, after modification. The green arc is $a'$ and the blue arc is $a$.

**Example 6:** Tight contact structures on $T^2 \times I$. Using Examples 4 and 5, we analyze partial open book decompositions obtained from a basic slice by successively attaching bypasses.
These calculations were motivated by computations of open books for tight contact structures on $T^3$ by Van Horn-Morris [VH], and echo the results obtained there. In particular, we will build up to an open book for the following tight contact structure on $T^2 \times [0,4]$: If $(x, y, t)$ are coordinates on $T^2 \times [0,4]$, then consider $\ker \alpha$, where 
\[
\alpha = \cos(\frac{\pi}{2}t) dx - \sin(\frac{\pi}{2}t) dy.
\]
Perturb $\ker \alpha$ along $\partial(T^2 \times [0,4])$ so that it becomes convex with \#$\Gamma_{T^0} = \#\Gamma_{T^4} = 2$, and slope($\Gamma_{T^0}$) = slope($\Gamma_{T^4}$) = $\infty$.

(a) Start with the basic slice $M = T^2 \times [0,1]$ from Example 4, depicted in Figure 16. Take a linear Legendrian arc $c$ of slope $1 < s < \infty$ so that the component of $\Gamma_{T^1}$ that contains the endpoints of $c$ is different from that containing $p$. (If the endpoints of $c$ are on the same component as $p$, then the resulting contact structure is overtwisted.) See the left-hand diagram of Figure 21. Attaching a bypass along $c$ and thickening yields $M' = T^2 \times [0,2]$ with \#$\Gamma_{T^i} = 2$, slope($\Gamma_{T^i}$) = 0, $i = 0, 2$. Moreover, the contact structure is obtained from the restriction of $\alpha$ to $T^2 \times [0,2]$, by perturbing the boundary. Hence it has “torsion” $\approx \pi$, where we indicate that the term is used informally by the use of quotations. If $c = c_+ \cup c_-$, where $c_{\pm} = c \cap R_{\pm}(\Gamma_{T^i})$, then we obtain $a$ (as in Example 5) by pushing $c_+$ slightly to the right. We isotop $c_-$ to $c'_+$ in the positive region of $\partial(M - N(p))$ — this is readily done by referring to Figure 17. The result is $h'(a) = c_+ c'_+$ as shown in Figure 20. The monodromy map $h'$ is the restriction of $R_{\gamma_1} R_{\gamma_2} R_{\delta_1}^2$ to $P$, where $\gamma_1$ and $\gamma_2$ are parallel to the two boundary components in the middle and $\delta_1$ is the core curve of the middle vertical annulus. (Refer to the right-hand diagram.) Since the only regions of $\Sigma - \cup_i \alpha_i - \cup_i \beta_i$ which do not intersect the suture $\Gamma'$ for $M'$ are quadrilaterals, it is easy to compute that $EH \neq 0$. The contact class lives in $SFH(-M', -\Gamma') = \mathbb{Z}^2 \otimes \mathbb{Z}^2$ from Example 2 (but is in a different $\mathbb{Z}$ summand).

(b) Next attach another bypass along $T_2$ to obtain $M'' = T^2 \times [0,3]$, where \#$\Gamma_{T_0} = \#\Gamma_{T_3} = 2$, slope($\Gamma_{T_0}$) = 0, and slope($\Gamma_{T_3}$) = $\infty$. The contact structure has “torsion” $\approx \frac{3\pi}{2}$. See Figure 21. Doing a similar calculation as before, we obtain Figure 22. The monodromy $h''$ is

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig20}
\caption{The monodromy map. The right-hand diagram is the same as the left-hand one, but drawn more symmetrically. The top and bottom of each annulus is identified.}
\end{figure}
the product \( R_{\gamma_1} R_{\gamma_2} R_{\gamma_3} R_{\delta_1} R_{\delta_2} \), where \( \gamma_1, \gamma_2, \gamma_3 \) are parallel to the two boundary components in the middle and \( \delta_1, \delta_2 \) are the core curves of the middle vertical annuli. Let \{a_1, a_2, a_3\} be the basis, ordered from left to right in Figure 22. (The arcs are on \( S'' \times \{-1\} \), where \( S'' \) is the page. The \( \alpha \)-curves are \( \alpha_i = \partial(a_i \times [-1, 1]) \).) To determine whether \( EH \neq 0 \), we apply the Sarkar-Wang algorithm [SW] (also see [Pl]) to isotop \( \beta \) so that the only regions of \( \Sigma - \cup_i \alpha_i - \cup_i \beta_i \) which do not intersect the suture are quadrilaterals. (Such a diagram we call combinatorial.) Let \( G \) (in black) be the trivalent graph in the bottom diagram of Figure 22. By isotoping \( \beta_3 \) across \( G \), the Heegaard diagram becomes combinatorial. Now label the intersections \( \alpha_1 \cap \beta \) as \( x_1, \ldots, x_{14}, x_{15} = x_1 \) from top to bottom, \( \alpha_2 \cap \beta \) as \( y_1, \ldots, y_9, y_{10} = y_1 \) from left to right, and \( \alpha_3 \cap \beta \) as \( z_1, \ldots, z_4 = z_1 \) from left to right. Observe that each intersection of the graph with \( \alpha_i \) corresponds to two consecutive intersections with the new \( \beta_3 \).

We now directly prove that \( EH \neq 0 \). If \( c_1 c_2 c_3 c_4 \) is a quadrilateral which avoids the suture, and \( \partial(c_1, c_3, y) = (c_2, c_4, y) + \ldots \) (for some \( y \)), then we say that \( c_1, c_3 \) are the “from” corners of the quadrilateral and \( c_2, c_4 \) are the “to” corners of the quadrilateral. One readily calculates that the only quadrilateral where the “to” corners are in \( \{x_1, y_1, z_1\} \) is \( y_1 y_2 z_1 z_2 \). Now,

\[
\partial(x_1, y_2, z_2) = (x_1, y_1, z_1) + (x_1, y_3, z_2),
\]

where the second term comes from a bigon. Next, the only quadrilateral where the “to” corners are in \( \{x_1, y_3, z_2\} \) is \( x_1 x_2 y_3 y_4 \). We compute that

\[
\partial(x_2, y_4, z_2) = (x_1, y_3, z_2) + (x_3, y_4, z_2).
\]

Continuing, the only quadrilateral with “to” corners in \( \{x_3, y_4, z_2\} \) is \( x_3 x_4 z_3 z_2 \). We have

\[
\partial(x_4, y_4, z_3) = (x_3, y_4, z_2) + (x_7, y_1, z_3).
\]

There is nothing else with “to” corners in \( \{x_7, y_1, z_3\} \). Hence \( EH \neq 0 \).

(c) Now consider \( M''' = T^2 \times [0, 4] \), where \#\( \Gamma_{T_0} = \#\Gamma_{T_1} = 2 \), slope(\( \Gamma_{T_0} \)) = slope(\( \Gamma_{T_1} \)) = 0, and the contact structure \( \xi''' \) has “torsion” \( \approx 2\pi \). (This is the contact structure described in the first paragraph of Example 6.) The page \( S''' \) is obtained from the page \( S'' \) for \( M'' \) by attaching an annulus along the right-hand boundary in Figure 22. (We can draw the
annulus to be vertical and parallel to the previous annuli which were successively attached, starting with $S$.) Then the monodromy $h''' = R_{\gamma_1} R_{\gamma_2} R_{\gamma_3} R_{\gamma_4} R_{\delta_1}^{-2} R_{\delta_2}^{-2} R_{\delta_3}^{-2}$, where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$
are parallel to the three boundary components in the middle and \( \delta_1, \delta_2, \delta_3 \) are the core curves of the middle vertical annuli. One can probably directly show that \( \text{EH}(M'', \Gamma'', \xi'') \neq 0 \) with some patience, but instead we invoke Theorem 4.5 and observe that \( (M'', \xi'') \) can be embedded in the unique Stein fillable contact structure on \( T^3 \), which has nonvanishing contact invariant.

(d) Recall that a contact manifold \((M, \xi)\) has \( n\pi \)-torsion with \( n \) a positive integer if it admits an embedding \((T^2 \times [0, 1], \eta_{n\pi}) \hookrightarrow (M, \xi)\), where \((x, y, t)\) are coordinates on \( T^2 \times [0, 1] \simeq \mathbb{R}^2/\mathbb{Z}^2 \times [0, 1] \) and \( \eta_{n\pi} = \ker(\cos(n\pi t)dx - \sin(n\pi t)dy) \). In a subsequent paper \cite{GHV}, Ghiggini, the first author, and Van Horn-Morris prove that if \((M, \xi)\) has torsion \( \geq 2\pi \) with \( M \) closed, then \( \text{EH}(M, \xi) = 0 \) when \( \mathbb{Z} \)-coefficients are used.

6. Relationship with sutured manifold decompositions

A sutured manifold decomposition \((M, \Gamma) \xrightarrow{T} (M', \Gamma')\) is well-groomed if for every component \( R \) of \( \partial M - \Gamma \), \( T \cap R \) is a union of parallel oriented nonseparating simple closed curves if \( R \) is nonplanar and arcs if \( R \) is planar. According to \cite{HKM3}, to each well-groomed sutured manifold decomposition there is a corresponding convex decomposition \((M, \Gamma) \xrightarrow{(T, \Gamma_T)} (M', \Gamma')\), where \( T \) is a convex surface with Legendrian boundary and \( \Gamma_T \) is a dividing set which is \( \partial \)-parallel, i.e., each component of \( \Gamma_T \) cuts off a half-disk which intersects no other component of \( \Gamma_T \).

In \cite{Ju2}, Juhász proved the following theorem:

**Theorem 6.1** (Juhász). Let \((M, \Gamma) \xrightarrow{T} (M', \Gamma')\) be a well-groomed sutured manifold decomposition. Then \( \text{SFH}(-M', -\Gamma') \) is a direct summand of \( \text{SFH}(-M, -\Gamma) \).

More precisely, \( \text{SFH}(-M', -\Gamma') \) is the direct sum \( \oplus_S \text{SFH}(-M, -\Gamma, s) \), where \( s \) ranges over all relative Spin\(^c\)-structures which evaluate maximally on \( T \). (The are called “outer” in \cite{Ju2}.)

In this section we give an alternate proof from the contact-topological perspective. We remark that our proof does not require the use of the Sarkar-Wang algorithm. In particular, our proof will imply the following:

**Theorem 6.2.** Let \((M, \Gamma, \xi)\) be the contact structure obtained from \((M', \Gamma', \xi')\) by gluing along a \( \partial \)-parallel \((T, \Gamma_T)\). Under the inclusion of \( \text{SFH}(-M', -\Gamma') \) in \( \text{SFH}(-M, -\Gamma) \) as a direct summand, \( \text{EH}(M', \Gamma', \xi') \) is mapped to \( \text{EH}(M, \Gamma, \xi) \).

**Corollary 6.3.** \( \text{EH}(M', \Gamma', \xi') \neq 0 \) if and only if \( \text{EH}(M, \Gamma, \xi) \neq 0 \).

**Proof.** The “if” direction is given by Theorem 4.5. The “only if” direction follows from Theorem 6.2. \( \square \)

The corollary is a gluing theorem for tight contact structures which are glued along a \( \partial \)-parallel dividing set, and does not require any universally tight condition which was needed for its predecessors, e.g., Colin’s gluing theorem \cite{Co1}.

**Proof.** Let \( T \) be a convex surface with Legendrian boundary and \( \Gamma_T \) be its dividing set, which we assume is \( \partial \)-parallel. Let \((M, \Gamma, \xi)\) be a contact structure obtained from \((M', \Gamma', \xi')\) by
gluing along \((T, \Gamma_T)\). Suppose without loss of generality that \(T\) is oriented so that the disks cut off by the \(\partial\)-parallel arcs are negative regions of \(T - \Gamma_T\). Let \(L \subset T\) be a Legendrian skeleton of the positive region, with endpoints on \(\Gamma\). We observe that there exist compressing disks \(D_i\) in \(M - N(L)\) with \(\#(\Gamma_{\partial(M-N(L))} \cap \partial D_i) = 2\) so that splitting \(M - N(L)\) along the \(D_i\) gives \((M', \Gamma')\). (The disks are basically the disks in \(T\) cut off by \(L\).) See the left-hand side of Figure 23.

Next we extend \(L\) to a Legendrian skeleton \(K\) for \((M, \Gamma, \xi)\), which satisfies the conditions of Theorem 1.1. Before doing this, it would be convenient to “protect” \(T\) and \(L\) as follows. Since \(T\) is convex, there exists a \([-1, 1]\)-invariant neighborhood \(T = T_0\). Let \(T_{-\varepsilon}, T_{0}\) and \(T_{\varepsilon}\) be parallel copies of \(T_0\), where \(T_t = T \times \{t\}\). Then take parallel Legendrian graphs \(L_{i\varepsilon}, i = -1, 0, 1\), on \(T_{\varepsilon}\), and thicken to obtain \(N(L_{i\varepsilon})\). Here \(L = L_0\). Now, apply the technique of Theorem 1.1 to extend \(L_{-\varepsilon} \cup L_0 \cup L_{\varepsilon}\) to \(K\). We may assume that the extension does not intersect the region between \(T_{-\varepsilon}\) and \(T_{\varepsilon}\) and is disjoint from \(L_{-\varepsilon} \cup L_0 \cup L_{\varepsilon}\). (This is because we can use the compressing disks \(D_i\), together with their translates corresponding to \(T_{\pm \varepsilon}\).)

Let \((S, R_+(\Gamma), h)\) be the partial open book decomposition of \((M, \Gamma, \xi)\) corresponding to the decomposition into \(N(K)\) and \(M - N(K)\). The three connected components of \(S - R_+(\Gamma)\) corresponding to \(L_{i\varepsilon}\) will be called \(Q_{i\varepsilon}\). (Of course there may be other connected components.) If \(a\) is a properly embedded arc on \(Q = Q_0\) (with endpoints away from \(\Gamma\)), let \(a_{i\varepsilon}\) be the corresponding copy on \(Q_{i\varepsilon}\). Then \(h(a)\) is a concatenation \(a'a''a''\), where \(a'\) and \(a''\) are arcs which switch levels \(Q \leftrightarrow Q_{\varepsilon}\) as given in Figure 24. The proof that \(h(a)\) is indeed as described is similar to the computation of Example 3 in Section 5.

Next, a page of the partial open book decomposition \((S', R_+(\Gamma'), h')\) of \((M', \Gamma', \xi')\) is given in Figure 25. In particular, all of \(Q\) becomes part of the new \(R_+(\Gamma')\) and we cut \(R_+(\Gamma) \subset S\) along the arcs \(d_i\) that correspond to the \(D_i\). (If the \(d_i\) are the dotted blue arcs in Figure 24, then \(D_i = d_i \times [-1, 1]\).)
Figure 24.

Figure 25.

Case 1: \( Q \) has no genus. Suppose \( Q \) has no genus and hence is a \( 2k \)-gon, where \( k \) of the edges are subarcs \( \gamma_0, \ldots, \gamma_{k-1} \) of \( \Gamma \) and \( k \) of the edges are subarcs of \( \partial S \). Assume the \( \gamma_i \) are in counterclockwise order about \( \partial Q \). Consider a basis for \( Q \) consisting of \( k - 1 \) arcs \( a_1, \ldots, a_{k-1} \), where \( a_i \) is parallel to \( \gamma_i \). Let \( b_i \) be the pushoffs of \( a_i \) satisfying (1), (2) and (3) of Section 2 and let \( x_i \) be the intersection of \( a_i \) and \( b_i \) on \( Q \). Also let \( \alpha_i = \partial (a_i \times [-1, 1]) \) and \( \beta_i = (b_i \times \{1\}) \cup (h(b_i) \times \{-1\}) \), viewed on \( S \times [-1, 1] / \sim \). (This \([-1, 1]\)-coordinate is different from the one used previously for the invariant neighborhood of \( T \).) Complete the \( \alpha_i \) and \( \beta_i \) into a compatible Heegaard decomposition for \((M, \Gamma)\) by adding \( \alpha'_i \) and \( \beta'_i \). Hence \( \alpha = (\alpha_1, \ldots, \alpha_{k-1}, \alpha'_1, \ldots, \alpha'_l) \) and \( \beta = (\beta_1, \ldots, \beta_{k-1}, \beta'_1, \ldots, \beta'_l) \).

Claim. The only \((k + l - 1)\)-tuples \( y \) of \( T_\alpha \cap T_\beta \) whose corresponding first Chern class \( c_1(\mathfrak{s}_y) \) evaluates maximally on \( T \) have the form \((y_1, \ldots, y_{k-1}, y'_1, \ldots, y'_l)\), where \( y_i \) is an intersection of some \( \alpha_{j_1} \) and \( \beta_{j_2} \), and \( y'_i \) is an intersection of some \( \alpha'_{j_1} \) and \( \beta'_{j_2} \).

Proof. We describe \( T \) as it sits inside the partial open book \((S, R_+(\Gamma), h)\). First isotop the \( d_i \) above to \( d'_i \) as given in Figure 26. Let \( T' \) be the union of the \( d'_i \times [-1, 1] \) and the region \( Q' \times \{1\} \subset Q \times \{1\} \) bounded by the \( d'_i \times \{1\} \) and the \( \gamma_i \). Then isotop \( T' \) to obtain \( T \) so
that \( T \cap \Sigma \) is the union of (the shaded region) \( \times \{ -1 \} \) and \( Q \times \{ 1 \} \). Here \( \Sigma \) is the Heegaard surface for \((S,R_\ast(\Gamma),h)\). Observe that the gradient flow line corresponding to \( x_i \in \alpha_i \cap \beta_i \) intersects \( T \). The same also holds for any \( y_i \in \alpha_j' \cap \beta_j' \), since \( h(b_{j_2}) \cap Q \) is contained in the shaded region. On the other hand, the gradient flow line for any intersection point of \( \alpha_j' \) and \( \beta_j' \) does not intersect \( T \), as \( \beta_j' \) does not enter \( Q \times \{ -1 \} \) or the shaded region due to the “protection” afforded by the \( Q \) region. (The only \( \beta_j' \) that enter \( Q \times \{ -1 \} \) come from \( Q_{-\varepsilon} \).) In that case we see in Figure 26 that they are represented by the green arcs outside the shaded region.) Therefore, in order to maximize \( \langle c_1(s_y), T \rangle \), the intersection point on \( \alpha_i \) must lie on \( \beta_j \). This forces \( \alpha_i \) to be paired with \( \beta_j \) and \( \alpha_i' \) to be paired with \( \beta_j' \).

Still assuming that \( Q \) has no genus, consider \( y \) as in the above claim. The only \( \beta_i \) which intersects \( \alpha_1 \) is \( \beta_1 \), and their sole intersection is \( x_1 \). Hence \( x_1 \) occurs in \( y \) — this uses up \( \alpha_1 \) and \( \beta_1 \). Next, the only \( \beta_i \) besides \( \beta_1 \) which intersects \( \alpha_2 \) is \( \beta_2 \). Continuing in this manner, \( \{x_1, \ldots, x_{k-1}\} \subset y \). Hence, the inclusion

\[
\Phi : CF(\{\beta_1', \ldots, \beta_l'\}, \{\alpha_1', \ldots, \alpha_l'\}) \to CF(\beta, \alpha),
\]

\[
(y_1', \ldots, y_l') \mapsto (x_1, \ldots, x_{k-1}, y_1', \ldots, y_l'),
\]

is as a direct summand of \( CF(\beta, \alpha) \). Moreover, when counting holomorphic disks from \( y' = (x_1, \ldots, x_{k-1}, y_1', \ldots, y_l') \) to \( y'' = (x_1, \ldots, x_{k-1}, y_1'', \ldots, y_l'') \), the positioning of \( \Gamma \) and the \( x_i \) implies that no subarcs of \( \alpha_i \) and \( \beta_i \) can be used as the boundary of a nontrivial holomorphic map. Hence holomorphic disks from \( y' \) to \( y'' \) are in 1-1 correspondence with holomorphic disks from \( (y_1', \ldots, y_l') \) to \( (y_1'', \ldots, y_l'') \). This implies that \( \Phi \) is a chain map. The homology of \( CF(\{\beta_1', \ldots, \beta_l'\}, \{\alpha_1', \ldots, \alpha_l'\}) \) is \( SFH(-M', -\Gamma') \), and image under the injection \( \Phi \) is the part of \( SFH(-M, -\Gamma) \) which is outer. It is also easy to see that \( EH(M', \Gamma', \xi') \) is mapped to \( EH(M, \Gamma, \xi) \) under \( \Phi \).

**Case 2: \( Q \) has genus.** In general, \( Q \) has genus \( g \) and there are \( k \) attaching arcs along \( \Gamma \). For a general picture, we can think of Figure 24 with added handles. The basis for \( Q \) can be chosen to consist of the same arcs \( a_1, \ldots, a_{k-1} \) parallel to \( \gamma_1, \ldots, \gamma_{k-1} \) as in Figure 24 plus additional arcs \( a_{k+2j-2}, a_{k+2j-1} \), with one pair for each handle \( j = 1, \ldots, g \). These additional
arcs can be taken to end on the subarc $\tau$ of $\partial S$ between $\gamma_{k-1}$ and $\gamma_0$, where $\gamma_{k-1}, \tau, \gamma_0$ are in counterclockwise order about $\partial Q$. Figure 27 depicts the case when $k = 1, g = 1$.

Let $a_1, \ldots, a_m$, $m = k+2g-1$, be a basis for $Q$, and $b_i, x_i, \alpha_i, \beta_i, i = 1, \ldots, m$, and $\alpha'_j, \beta'_j$, $j = 1, \ldots, l$, be as before. The curves that nontrivially intersect $Q \times \{-1\}$ are $\alpha_i$, $\beta_i$, and curves $\beta'_j$ of the form $((b_i)_{-\varepsilon} \times \{1\}) \cup (h((b_i)_{-\varepsilon}) \times \{-1\})$. Here $(b_i)_{-\varepsilon}$ is an arc on $Q_{-\varepsilon}$ which is parallel to $b_i \subset Q$. Moreover, the curves $\alpha'_j$ and $\beta'_j$ which pass through $Q_{\varepsilon} \times \{-1\}$ can be taken to be $\partial((a_i)_{\varepsilon} \times [-1, 1])$ and $((b_i)_{\varepsilon} \times \{1\}) \cup (h((b_i)_{\varepsilon}) \times \{-1\})$, respectively. This is due to the protective layer between $T_{-\varepsilon}$ and $T_\varepsilon$. See Figure 27. We can represent $T$ as before, as the union of (the shaded region) $\times \{-1\}$ and $Q \times \{1\}$, and prove the analogous Claim by using this description of $T$ to show that the $\alpha_{j1}$ and $\beta_{j2}$ must be paired to maximize the first Chern class on $T$. Hence we only consider $y = (y_1, \ldots, y_m, y'_1, \ldots, y'_l)$, where $y_i$ is an intersection of some $\alpha_{j1}$ and $\beta_{j2}$, and $y'_i$ is an intersection of some $\alpha'_{j1}$ and $\beta'_{j2}$. As before, if $k > 1$, then we argue that the only $\beta_i$ which intersects $\alpha_1$ is $\beta_1$, the only $\beta_i$ besides $\beta_1$ which intersects $\alpha_2$ is $\beta_2$, etc., and that $\{x_1, \ldots, x_{k-1}\} \subset y$.

Since the subset $\{x_1, \ldots, x_{k-1}\}$ has no effect on what follows, we assume that $k = 1$. Moreover, for simplicity, we assume that $g = 1$. (The higher genus case is not much more difficult.) Consider the pair of curves $\alpha_1, \alpha_2$ (corresponding to the handle) and the corresponding $\beta$ curves. Let $x_1, v_1$ be the intersection points of $\alpha_1 \cap \beta_1$, $v_2$ be the intersection point of $\alpha_1 \cap \beta_2$, $w_1, w_3$ be the intersection points of $\alpha_2 \cap \beta_1$, and $x_2, w_2$ be the intersection points of $\alpha_2 \cap \beta_2$. See Figure 28. Hence the summand $W \subset CF(\beta, \alpha)$ corresponding to the generators which evaluate maximally on $T$ is generated by $(y_1, y_2, y')$, where $(y_1, y_2)$ is one of

$$(x_1, x_2), (x_1, w_2), (v_1, x_2), (v_1, w_2), (v_2, w_1), (v_2, w_3),$$

and $y'$ is of the type $(y'_1, \ldots, y'_l)$.
Now we examine the holomorphic disks in \( Sym^{m+l}(\Sigma) \) that avoid \( \partial \Sigma \times Sym^{m+l-1}(\Sigma) \), in order to compute the boundary maps. The realization of the holomorphic disk as a holomorphic map \( \pi \) to \( \Sigma \) cannot have “mixed” boundary components, i.e., each boundary component must consist solely of subarcs of \( \alpha_i \) and \( \beta_j \), or consist solely of subarcs of \( \alpha'_i \) and \( \beta'_j \), since all generators of \( W \) consist of intersections of \( \alpha_j \) and \( \beta_k \), and intersections of \( \alpha'_j \) and \( \beta'_k \) only. Consider the regions of \( \Sigma - \bigcup_i \alpha_i - \bigcup_i \beta_i - \bigcup_j \alpha'_j - \bigcup_j \beta'_j \) which nontrivially intersect \( \Gamma \) — they are indicated by red dots in Figure 27. As a consequence, all the regions in \( Q \times \{-1\} \) besides the regions with black dots and the regions 1, 2, and 3, have multiplicity zero. Now, regions 1, 2, and 3 have \( \alpha'_j \) curves on the boundary. Hence, if the multiplicity is nonzero, the image of \( \pi \) must extend across those curves, in the end engulfing the red-dotted regions in \( Q_\varepsilon \times \{-1\} \). Therefore, the multiplicities of regions 1, 2, and 3 are forced to be zero.

The following are the only possible holomorphic disks:

1. quadrilaterals in \( Q \times \{-1\} \) (the regions with black dots in Figure 27);
2. holomorphic disks from \( y' \) to \( y'' \) that do not enter \( Q \times \{-1\} \).

The quadrilaterals give rise to boundary maps:

\[
\begin{align*}
\partial(u_1, v_2) &= (x_2, v_1), \\
\partial(v_1, w_2) &= (v_2, w_1), \\
\partial(v_2, w_3) &= (x_1, w_2), \\
\partial(x_1, x_2) &= \partial(x_2, v_1) = \partial(v_2, w_1) = \partial(x_1, w_2) = 0,
\end{align*}
\]

where we are referring to the Heegaard diagram with the \( \alpha'_i, \beta'_i \) erased. If \( y_1 \) denotes a linear combination of tuples of type \( (y'_1, \ldots, y'_l) \) (by slight abuse of notation), then

\[
\partial((x_1, x_2, y_1) + (u_1, v_2, y_2) + (x_2, v_1, y_3) + \ldots) = 0
\]

implies

\[
(x_1, x_2, \partial y_1) + (x_2, v_1, y_2) + (u_1, v_2, \partial y_2) + (x_2, v_1, \partial y_3) + \ldots = 0.
\]

Here \( \partial y_1 \) refers to the boundary map for the Heegaard diagram with \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) erased. This implies that \( \partial y_1 = \partial y_2 = 0 \), and \( \partial y_3 = y_2 \). Hence,

\[
(x_1, x_2, y_1) + (u_1, v_2, y_2) + (x_2, v_1, y_3) + \cdots = (x_1, x_2, y_1) + \partial(u_1, v_2, y_3) + \cdots,
\]
and the inclusion
\[ \Phi : CF(\{\beta'_1, \ldots, \beta'_l\}, \{\alpha'_1, \ldots, \alpha'_l\}) \to CF(\beta, \alpha), \]
\[ (y'_1, \ldots, y'_l) \mapsto (x_1, x_2, y'_1, \ldots, y'_l), \]
induces an isomorphism of \( SFH(-M', -\Gamma') \) onto a direct summand of \( SFH(-M, -\Gamma) \), as before.

Acknowledgements. We thank András Juhász for patiently explaining sutured Floer homology and András Stipsicz for helpful discussions. We also thank John Etnyre for the discussions which led to the reformulation of the contact invariant in [HKM2].

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