Boundedness of intrinsic square functions and commutators on generalized central Morrey spaces

Wenna Lu and Jiang Zhou

College of Mathematics and System Sciences, Xinjiang University, Ürümqi, People’s Republic of China

ABSTRACT
In this paper, the authors establish the boundedness for a large class of intrinsic square functions \(G_\alpha, g_\alpha, g_{\tilde{\lambda}, \alpha}\) and their commutators \([b, G_\alpha], [b, g_\alpha]\) and \([b, g_{\tilde{\lambda}, \alpha}]\) generated with \(\lambda\)-central \(BMO\) functions \(b \in CBMO_{\tilde{\lambda}}(\mathbb{R}^n)\) on generalized central Morrey spaces \(B^{q, \varphi}(\mathbb{R}^n)\) for \(1 < q < \infty, 0 < \alpha \leq 1\), respectively. All of the results are new even on the central Morrey spaces \(B^{q, \lambda}(\mathbb{R}^n)\).

1. Introduction
Morrey [1], to study the local behavior of solutions to second-order elliptic partial differential equations, introduced the Morrey space in 1938. It is well known that the space plays a significant role in studying the regularity of solutions to partial differential equations. Since then, many scholars have also considered the mapping properties of some classical operators in harmonic analysis on Morrey space (see [2,3]). The study on the intrinsic square function characterizations of function spaces has attracted a lot of attentions in recent years. To be precise, Wilson [4] first introduced intrinsic square functions to settle a conjecture proposed by Fefferman and Stein [5] on the boundedness of the Lusin-area function on the weighted Lebesgue space. Meanwhile, Wilson proved such operators are bounded from \(L^p(\mathbb{R}^n)\) to itself and also extended the weighted case in [4]. In 2012, Wang [6] obtained the boundedness of intrinsic square functions and their commutators generated by \(BMO(\mathbb{R}^n)\) functions on weighted Morrey spaces. Later on, Wu and Zheng [7] generalized these consequences to the generalized Morrey spaces. For richer achievements and further developments in this subject, we refer the readers to [8,9], etc.

Now, let us first recall the definitions of the intrinsic square functions (see [4]).
For $0 < \alpha \leq 1$, let $C_\alpha$ be the family of functions $\phi : \mathbb{R}^n \to \mathbb{R}$ such that $\phi$ has support contained in $\{ x \in \mathbb{R}^n : |x| \leq 1 \}$, $\int_{\mathbb{R}^n} \phi(x) dx = 0$, and for all $x, x' \in \mathbb{R}^n$

$$|\phi(x) - \phi(x')| \leq |x - x'|^\alpha.$$ 

For $(y, t) \in \mathbb{R}^{n+1}_+$ and $f \in L^1_{loc}(\mathbb{R}^n)$, set

$$A_\alpha f(t, y) = \sup_{\phi \in C_\alpha} |f * \phi_t(y)|,$$

where $\phi_t(x) = \frac{1}{t^n} \phi(\frac{x}{t})$.

The varying-aperture intrinsic square (intrinsic Lusin) function is defined by

$$G_{\alpha, \beta}(f)(x) = \left( \int \int_{\Gamma_{\beta(x)}} (A_\alpha f(t, y))^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

where

$$\Gamma_{\beta(x)} = \{ (y, t) \in \mathbb{R}^{n+1}_+ : |x - y| < \beta t \}.$$ 

If $\beta \equiv 1$, we denote $G_{\alpha, 1}(f)$ by $G_\alpha(f)$. Moreover, for any $0 < \alpha \leq 1$ and $\beta \geq 1$, there is a pointwise relation between the function $G_{\alpha, \beta}(f)(x)$ and $G_\alpha(f)(x)$ as:

$$G_{\alpha, \beta}(f)(x) \leq \beta^{3n+2\alpha} G_\alpha(f)(x).$$

A classical example is the kernel function $\phi_t(x) = P_t(x)$ (Poisson kernel), We know that the intrinsic square functions are independent of any particular kernels, and it dominates pointwise the classical Lusin area integral and some of its corresponding real-variable generalizations. On the other hand, we should pay attention to the fact that the function $G_{\alpha, \beta}(f)(x)$ depends on the kernels with uniform compact support.

The intrinsic Littlewood-Paley $g$-function and the intrinsic $g^*_\lambda$-function are defined respectively by

$$g_\alpha f(x) = \left( \int_0^\infty (A_\alpha f(t, x))^2 \frac{dt}{t} \right)^{1/2},$$

and

$$g^*_\lambda \alpha f(x) = \left( \int \int_{\mathbb{R}^{n+1}_+} \left( \frac{t}{t + |x - y|} \right)^{n\lambda} (A_\alpha f(t, y))^2 \frac{dydt}{t^{n+1}} \right)^{1/2}.$$

Let $b$ be a locally integrable function on $\mathbb{R}^n$, in this paper, we also consider the commutators generated by the function $b$ and the above intrinsic square functions, which are defined respectively by the expressions (see [6]):

$$A_{\alpha, \lambda} f(t, y) = \sup_{\phi \in C_\alpha} \left| \int_{\mathbb{R}^n} (b(y) - b(z)) \phi_t(y - z) f(z) dz \right|,$$

$$[b, \mathcal{G}_\alpha](f)(x) = \left( \int \int_{\Gamma_{\beta(x)}} (A_{\alpha, \lambda} f(t, y))^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$
\[ [b, g_{\alpha}] f(x) = \left( \int_0^\infty \left( A_{\alpha,b} f(t,y) \right)^2 \frac{dt}{t} \right)^{1/2}, \]

and

\[ [b, g_{\alpha}^*] f(x) = \left( \int \int_{\mathbb{R}^{n+1}} \left( \frac{t}{t + |x - y|} \right)^{n\lambda} \left( A_{\alpha,b} f(t,y) \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2}. \]

Wiener [10,11] explored a way to describe the behavior of a function at infinity by considering the appropriate weighted \( L^q(\mathbb{R}^n) \) spaces, and then, Beurling [12] employed this idea and obtained a pair of dual Banach spaces \( A^q \) and \( B^q \), where \( 1/q + 1/q' = 1 \). Subsequently, Lu and Yang [13,14] introduced some new homogeneous Hardy space \( \dot{H}A^q \) related to the homogeneous spaces \( \dot{A}^q \), and obtained that the dual space of \( \dot{H}A^q \) was the central bounded mean oscillation space \( CBMO^q(\mathbb{R}^n) \), which satisfies the condition:

\[ \| f \|_{CBMO^q(\mathbb{R}^n)} = \sup_{r > 0} \left( \frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^q dx \right)^{1/q} < \infty, \]

here and in the sequel, for \( r > 0 \), \( B(0,r) \) denotes the open ball centered at 0 of radius \( r \), \( |B(0,r)| \) the Lebesgue measure of the ball \( B(0,r) \) and

\[ f_{B(0,r)} = \frac{1}{|B(0,r)|} \int_{B(0,r)} f(x) dx. \]

In fact, the space \( CBMO^q(\mathbb{R}^n) \) can be regarded as a local version of the space of bounded mean oscillation space \( BMO(\mathbb{R}^n) \). However, their properties are quite different, for example, the famous John-Nirenberg inequality for \( BMO(\mathbb{R}^n) \) space no longer holds in the \( CBMO^q(\mathbb{R}^n) \) space. In addition, Alvarez, Guzmán-Partida and Lakey [15] pointed out that \( BMO(\mathbb{R}^n) \) is strictly included in \( \cap_{q>1} CBMO^q \). Furthermore, to study the relationship between central \( BMO(\mathbb{R}^n) \) spaces and Morrey spaces, they introduced \( \lambda \)-central bounded mean oscillation spaces and \( \lambda \)-central Morrey spaces, respectively.

**Definition 1.1** ([15]): Let \( \lambda < 1/n \) and \( 1 < q < \infty \). The \( \lambda \)-central bounded mean oscillation spaces \( CBMO^{q,\lambda}(\mathbb{R}^n) \) is defined as

\[ CBMO^{q,\lambda}(\mathbb{R}^n) = \{ f \in L^q_{\text{loc}}(\mathbb{R}^n) : \| f \|_{CBMO^{q,\lambda}(\mathbb{R}^n)} < \infty \}, \]

where

\[ \| f \|_{CBMO^{q,\lambda}(\mathbb{R}^n)} = \sup_{r > 0} \left( \frac{1}{|B(0,r)|^{1+\lambda q}} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^q dx \right)^{1/q} < \infty. \]

**Remark 1.1:** If two functions which differ by a constant are regarded as a function in the space \( CBMO^{q,\lambda}(\mathbb{R}^n) \), then \( CBMO^{q,\lambda}(\mathbb{R}^n) \) becomes a Banach space. when \( \lambda = 0 \), the space \( CBMO^{q,\lambda}(\mathbb{R}^n) \) reduces to the space \( CBMO^q(\mathbb{R}^n) \).
Definition 1.2 ([15]): Let $\lambda \in \mathbb{R}$ and $1 < q < \infty$. The $\lambda$-central Morrey space $B^{q,\lambda}(\mathbb{R}^n)$ is defined by

$$B^{q,\lambda}(\mathbb{R}^n) = \{ f \in \mathcal{L}^q_{\text{loc}}(\mathbb{R}^n) : \| f \|_{B^{q,\lambda}(\mathbb{R}^n)} < \infty \},$$

where

$$\| f \|_{B^{q,\lambda}(\mathbb{R}^n)} = \sup_{r > 0} \frac{1}{|B(0, r)|^{1 + \lambda q}} \left( \int_{B(0, r)} |f(x)|^q \, dx \right)^{1/q}.$$

Remark 1.2: $B^{q,\lambda}(\mathbb{R}^n)$ space is a Banach space continuously included in $CBMO^{q,\lambda}(\mathbb{R}^n)$ space. When $\lambda < -1/q$, $B^{q,\lambda}(\mathbb{R}^n)$ space reduces to $[0]$, and $B^{q,-1/q}(\mathbb{R}^n) = \mathcal{L}^q(\mathbb{R}^n)$.

Remark 1.3:

(i) If $\lambda_1 < \lambda_2$, it follows from the property of monotone functions that $CBMO^{q,\lambda_1}(\mathbb{R}^n) \subset CBMO^{q,\lambda_2}(\mathbb{R}^n)$ and $B^{q,\lambda_1}(\mathbb{R}^n) \subset B^{q,\lambda_2}(\mathbb{R}^n)$ for $1 < q < \infty$.

(ii) If $1 < q_1 < q_2 < \infty$, $CBMO^{q_2,\lambda}(\mathbb{R}^n) \subset CBMO^{q_1,\lambda}(\mathbb{R}^n)$ for $\lambda < 1/n$, and $B^{q_2,\lambda}(\mathbb{R}^n) \subset B^{q_1,\lambda}(\mathbb{R}^n)$ for $\lambda \in \mathbb{R}$.

In [16], Mizuhara introduced the generalized Morrey spaces, and established the boundedness of some classical operators. After that, many authors have considered the mapping properties of various variant operators and their commutators on the generalized Morrey spaces. Naturally, the generalized central Morrey spaces, as special local Morrey spaces, are also very significant function spaces in the study of boundedness of related operators, see for example [7,17–20], and references therein.

Next, we recall the definition of the generalized central Morrey space.

Definition 1.3 ([17]): Let $\varphi(r)$ be a positive measurable function on $\mathbb{R}^+$, and $1 < q < \infty$. The generalized central Morrey space $B^{q,\varphi}(\mathbb{R}^n)$ is defined by

$$B^{q,\varphi}(\mathbb{R}^n) = \{ f \in \mathcal{L}^q_{\text{loc}}(\mathbb{R}^n) : \| f \|_{B^{q,\varphi}(\mathbb{R}^n)} < \infty \},$$

where

$$\| f \|_{B^{q,\varphi}(\mathbb{R}^n)} = \sup_{r > 0} \frac{1}{\varphi(r)} \left( \frac{1}{|B(0, r)|} \int_{B(0, r)} |f(x)|^q \, dx \right)^{1/q}.$$

Remark 1.4: If $1 < q_1 < q_2 < \infty$, $B^{q_2,\varphi}(\mathbb{R}^n) \subset B^{q_1,\varphi}(\mathbb{R}^n)$. Note that if we take $\varphi(r) = r^{n\lambda}$, then $B^{q,\varphi}(\mathbb{R}^n) = B^{q,\lambda}(\mathbb{R}^n)$.

The main purpose in this paper is to establish the boundedness of the operators $\mathcal{G}_\alpha$, $g^*_\alpha$, $g_\alpha$ and their commutators generated with the $\lambda$-central bounded mean oscillation function $b \in CBMO^{q,\lambda}(\mathbb{R}^n)$ on the generalized central Morrey spaces, respectively.

Our main results can be formulated as follows.

Theorem 1.1: Let $1 < q < \infty$, $0 < \alpha \leq 1$ and the pair $(\varphi_1, \varphi_2)$ satisfy the condition

$$\int_{r}^{\infty} \text{ess inf}_{t < \tau < \infty} \frac{\varphi_1(\tau) \tau^{n/q}}{t^{n/q+1}} \, dt \lesssim \varphi_2(r).$$

Then the operator $\mathcal{G}_\alpha$ is bounded from $B^{q,\varphi_1}(\mathbb{R}^n)$ to $B^{q,\varphi_2}(\mathbb{R}^n)$. 
Theorem 1.2: Let $1 < q < \infty$, $0 < \alpha \leq 1$ and the pair $(\varphi_1, \varphi_2)$ satisfy the condition
\[
\int_r^\infty \frac{\text{ess inf}_{t < \tau < \infty} \varphi_1(\tau) \tau^{n/q}}{t^{n/q+1}} \, dt \lesssim \varphi_2(r).
\]
Then we have $\bar{\lambda} > 3 + \frac{2\alpha}{n}$, the operator $g_{\bar{\lambda}, \alpha}^*$ is bounded from $B^{q, \varphi_1}(\mathbb{R}^n)$ to $B^{q, \varphi_2}(\mathbb{R}^n)$.

Theorem 1.3: Let $1 < q < \infty$, $0 < \alpha \leq 1$, $b \in \text{CBMO}^{p, \lambda}(\mathbb{R}^n)$, $0 < \lambda < 1/n$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{p}$ and the pair $(\varphi_1, \varphi_2)$ satisfy the condition
\[
\int_r^\infty t^{n\lambda} \left(1 + \ln \frac{t}{r}\right) \frac{\text{ess inf}_{t < \tau < \infty} \varphi_1(\tau) \tau^{n/q_1}}{t^{n/q_1+1}} \, dt \lesssim \varphi_2(r).
\]
Then we have
\[
\| [b, G_\alpha] (f) \|_{B^{q, \varphi_2}(\mathbb{R}^n)} \lesssim \| b \|_{\text{CBMO}^{p, \lambda}(\mathbb{R}^n)} \| f \|_{B^{q, \varphi_1}(\mathbb{R}^n)}.
\]

Theorem 1.4: Let $1 < q < \infty$, $0 < \alpha \leq 1$, $b \in \text{CBMO}^{p, \lambda}(\mathbb{R}^n)$, $0 < \lambda < 1/n$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{p}$ and the pair $(\varphi_1, \varphi_2)$ satisfy the condition
\[
\int_r^\infty t^{n\lambda} \left(1 + \ln \frac{t}{r}\right) \frac{\text{ess inf}_{t < \tau < \infty} \varphi_1(\tau) \tau^{n/q_1}}{t^{n/q_1+1}} \, dt \lesssim \varphi_2(r).
\]
Then for $\bar{\lambda} > 3 + \frac{2\alpha}{n}$, we have
\[
\| [b, g_{\bar{\lambda}, \alpha}^*] (f) \|_{B^{q, \varphi_2}(\mathbb{R}^n)} \lesssim \| b \|_{\text{CBMO}^{p, \lambda}(\mathbb{R}^n)} \| f \|_{B^{q, \varphi_1}(\mathbb{R}^n)}.
\]

In addition, the author in [4] proved the operators $G_\alpha$ and $g_\alpha$ are pointwise comparable. Therefore, as applications of Theorems 1.1 and 1.3, we have the following conclusions.

Corollary 1.1: Let $1 < q < \infty$, $0 < \alpha \leq 1$ and the pair $(\varphi_1, \varphi_2)$ satisfy the condition
\[
\int_r^\infty \frac{\text{ess inf}_{t < \tau < \infty} \varphi_1(\tau) \tau^{n/q}}{t^{n/q+1}} \, dt \lesssim \varphi_2(r).
\]
Then the operator $g_\alpha$ is bounded from $B^{q, \varphi_1}(\mathbb{R}^n)$ to $B^{q, \varphi_2}(\mathbb{R}^n)$.

Corollary 1.2: Let $1 < q < \infty$, $0 < \alpha \leq 1$, $b \in \text{CBMO}^{p, \lambda}(\mathbb{R}^n)$, $0 < \lambda < 1/n$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{p}$ and the pair $(\varphi_1, \varphi_2)$ satisfy the condition
\[
\int_r^\infty t^{n\lambda} \left(1 + \ln \frac{t}{r}\right) \frac{\text{ess inf}_{t < \tau < \infty} \varphi_1(\tau) \tau^{n/q_1}}{t^{n/q_1+1}} \, dt \lesssim \varphi_2(r).
\]
Then we have
\[
\| [b, g_\alpha] (f) \|_{B^{q, \varphi_2}(\mathbb{R}^n)} \lesssim \| b \|_{\text{CBMO}^{p, \lambda}(\mathbb{R}^n)} \| f \|_{B^{q, \varphi_1}(\mathbb{R}^n)}.
\]

Remark 1.5: According to Remark 1.4, one can see that all of the results are new on the central Morrey spaces $B^{q, \lambda}(\mathbb{R}^n)$. 

-W. LU AND J. ZHOU-
The rest of this paper is organized as follows. In Section 2 we establish several auxiliary lemmas, which are the important ingredients of this paper. The detailed proofs of Theorems 1.1–1.4 are given in Section 3. We would like to remark that some arguments are taken from [7,17].

As a rule, we use the symbol $f \lesssim g$ to denote there exists a positive constant $C$ such that $f \leq Cg$, and the notation $f \approx g$ means that there exist positive constants $C_1, C_2$ such that $C_1 g \leq f \leq C_2 g$. For any set $E \subset \mathbb{R}^n$, $\chi_E$ denotes its characteristic function and $E^c$ denotes its complementary set.

2. Some auxiliary lemmas

We present some important lemmas, which will play significant roles in proving the theorems.

Lemma 2.1 ([4]): Let $1 < q < \infty$, $0 < \alpha \leq 1$. Then $G_\alpha$ is bounded from $L^q(\mathbb{R}^n)$ to itself.

Lemma 2.2 ([20]): Suppose that $f \in CBMO^{q,\lambda}$, $1 \leq q < \infty$, $\lambda < \frac{1}{n}$ and $r_1, r_2 \in \mathbb{R}^+$. Then

$$\left( \frac{1}{|B(0, r_1)|^{1 + \lambda q}} \int_{B(0, r_1)} |f(x) - f_{B(0, r_2)}|^q dx \right)^{1/q} \leq C \left( 1 + \left| \ln \frac{r_1}{r_2} \right| \right) \| f \|_{CBMO^{q,\lambda}}.$$

Lemma 2.3: Let $1 < q < \infty$ and $0 < \alpha \leq 1$. Then the inequality

$$\| G_\alpha(f) \|_{L^q(B(0, r))} \lesssim r^{n/q} \int_{2r}^\infty t^{-n/q - 1} \| f \|_{L^q(B(0, t))} dt$$

holds for any ball $B(0, r)$ and for all $f \in L^q_{\text{loc}}(\mathbb{R}^n)$.

Proof: For any $r > 0$, set $B = B(0, r)$ and $2B = B(0, 2r)$, we write

$$f(x) = f(x) \chi_{2B}(x) + f(x) \chi_{(2B)^c}(x) := f_1(x) + f_2(x)$$

and have

$$\| G_\alpha(f) \|_{L^q(B)} \leq \| G_\alpha(f_1) \|_{L^q(B)} + \| G_\alpha(f_2) \|_{L^q(B)}$$

$$:= I_1 + I_2.$$ 

For $I_1$, since $f_1 \in L^q(\mathbb{R}^n)$, $G_\alpha(f_1) \in L^q(\mathbb{R}^n)$, by Lemma 2.1, we get

$$I_1 = \| G_\alpha(f_1) \|_{L^q(B)} \leq \| G_\alpha(f_1) \|_{L^q(\mathbb{R}^n)} \lesssim \| f_1 \|_{L^q(\mathbb{R}^n)}$$

$$\lesssim r^{n/q} \int_{2r}^\infty t^{-n/q - 1} \| f \|_{L^q(B(0, t))} dt.$$
We now estimate $I_2$, note that the fact $\|\phi\|_{L^\infty} \lesssim 1$, we obtain that

$$|f_2 \ast \phi_t(y)| = \left| \frac{1}{t^n} \int_{|y-z| \leq t} \phi\left(\frac{y-z}{t}\right)f_2(z) \, dz \right| \lesssim \frac{1}{t^n} \int_{|y-z| \leq t} |f_2(z)| \, dz.$$  

It is clear that $x \in B$, $(y, t) \in \Gamma(x)$, and $z \in (2B)^c$, which deduces that

$$r \leq |z| - |x| \leq |x - z| \leq |x - y| + |y - z| \leq 2t.$$  

From this, we get

$$G_\alpha(f_2)(x) \lesssim \left( \int_{\Gamma(x)} \left( \frac{1}{t^n} \int_{|y-z| \leq t} |f_2(z)| \, dz \right)^2 \, dy \, dt \frac{1}{t^{n+1}} \right)^{1/2}$$

$$\leq \left( \int_{t > \frac{2}{2}} \int_{|x-y| < t} \left( \int_{|z-x| \leq 2t} |f_2(z)| \, dz \right)^2 \, dy \, dt \frac{1}{t^{3n+1}} \right)^{1/2}$$

$$\lesssim \left( \int_{t > \frac{2}{2}} \left( \int_{|z-x| \leq 2t} |f_2(z)| \, dz \right)^2 \, dt \frac{1}{t^{2n+1}} \right)^{1/2}.$$  

The Fubini theorem and the fact that $|z - x| \geq |z| - |x| \geq \frac{1}{2} |z|$ lead to the following result

$$G_\alpha(f_2)(x) \lesssim \left( \int_{\mathbb{R}^n} \left( \int_{t > \frac{|z-x|}{2}} \frac{dt}{t^{2n+1}} \right)^2 |f_2(z)| \, dz \right)^{1/2}$$

$$\lesssim \int_{|z| > 2r} \frac{|f(z)|}{|x - z|^n} \, dz$$

$$\lesssim \int_{|z| > 2r} \frac{|f(z)|}{|z|^n} \, dz \quad (|z| \approx |x - z|)$$

$$\lesssim \int_{|z| > 2r} |f(z)| \int_{|z|}^\infty \frac{1}{t^{n+1}} \, dt \, dz$$

$$\lesssim \int_{2r}^\infty \int_{2r < |z| < t} |f(z)| \frac{1}{t^{n+1}} \, dz \, dt$$

$$\lesssim \int_{2r}^\infty t^{-n/q-1} \|f\|_{L^q(B(0,t))} \, dt.$$  

Thus, we have

$$I_2 = \|G_\alpha(f_2)\|_{L^q(B)} \lesssim r^{n/q} \int_{2r}^\infty t^{-n/q-1} \|f\|_{L^q(B(0,t))} \, dt.$$  

Combining the estimates $I_1$ and $I_2$, the proof of Lemma 2.3 is completed.  ■
Lemma 2.4: Let $1 < q < \infty$ and $0 < \alpha \leq 1$. Then for any ball $B(0, r)$ and for all $f \in L^q_{\text{loc}}(\mathbb{R}^n)$, the operator
\[
G_{\alpha, 2}(f)(x) = \left( \int_0^\infty \int_{|x-y| \leq 2t} \left( A_{\alpha f}(t, y) \right)^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}, \quad j \in \mathbb{Z}^+
\]
satisfies the following inequality
\[
\|G_{\alpha, 2}(f)\|_{L^q(B(0, r))} \lesssim 2^{j(3+2\alpha/n)} r^{n/q} \int_{2r}^\infty t^{-n/q-1} \|f\|_{L^q(B(0, t))} dt.
\]

Proof: For $0 < \alpha \leq 1$ and $\beta \geq 1$, we know that
\[
G_{\alpha, \beta}(f)(x) \leq \beta^{3+2\alpha/n} G_{\alpha}(f)(x).
\]
Now, we set $\beta = 2^j > 1$, which, together with Lemma 2.3, gives the desired result. 

Lemma 2.5: Let $1 < q < \infty$, $0 < \alpha \leq 1$, $b \in \text{CBMO}^{p, \lambda}(\mathbb{R}^n)$, $0 < \lambda < 1/n$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{p}$. Then the inequality
\[
\|[b, G_{\alpha}](f)\|_{L^q(B(0, r))} \lesssim r^{n/q} \|b\|_{\text{CBMO}^{p, \lambda}(\mathbb{R}^n)} \int_{2r}^\infty t^{n\lambda} \left( 1 + \ln \frac{t}{r} \right) \|f\|_{L^{q_1}(B(0, t))} \frac{dt}{t^{n/q_1+1}}
\]
holds for any ball $B(0, r)$ and for all $f \in L^q_{\text{loc}}(\mathbb{R}^n)$.

Proof: Let $1 < q < \infty$, $0 < \alpha \leq 1$, $b \in \text{CBMO}^{p, \lambda}(\mathbb{R}^n)$, $0 < \lambda < 1/n$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{p}$. For any $r > 0$, set $B = B(0, r)$ and $2B = B(0, 2r)$, we write
\[
f(x) = f(x) \chi_{2B}(x) + f(x) \chi_{(2B)^c}(x) := f_1(x) + f_2(x),
\]
and
\[
[b, G_{\alpha}](f)(x) = [b - b_B, G_{\alpha}](f)(x)
\]
\[
\leq (b(x) - b_B)G_{\alpha}(f_1)(x) + G_{\alpha}((b - b_B)f_1)(x)
\]
\[
+ (b(x) - b_B)G_{\alpha}(f_2)(x) + G_{\alpha}((b - b_B)f_2)(x).
\]
Hence, we have
\[
\|[b, G_{\alpha}](f)\|_{L^q(B(0, r))}
\]
\[
\leq \|(b(x) - b_B)G_{\alpha}(f_1)\|_{L^q(B(0, r))} + \|G_{\alpha}((b - b_B)f_1)\|_{L^q(B(0, r))}
\]
\[
+ \|(b(x) - b_B)G_{\alpha}(f_2)\|_{L^q(B(0, r))} + \|G_{\alpha}((b - b_B)f_2)\|_{L^q(B(0, r))}
\]
\[
:= J_1 + J_2 + J_3 + J_4.
\]
For $J_1$, by Hölder’s inequality and the boundedness of $G_{\alpha}$, we get
\[
J_1 = \left( \int_B |b(x) - b_B|^q |G_{\alpha}(f_1)(x)|^q dx \right)^{1/q}
\]
\[
J_1 \lesssim \left( \int_B |b(x) - b_B|^p \, dx \right)^{1/p} \left( \int_B |G_\alpha(f_1)(x)|^{q_1} \, dx \right)^{1/q_1}
\]

\[
\lesssim t^{n/p + n\lambda} \| b \|_{\text{CBMO}^{p,\lambda}((\mathbb{R}^n))} \| f \|_{L^{q_1}(2B)}
\]

\[
\lesssim t^{n/p + n\lambda + n/q_1} \| b \|_{\text{CBMO}^{p,\lambda}((\mathbb{R}^n))} \int_{2r}^{\infty} \| f \|_{L^{q_1}(B(0,t))} \frac{dt}{t^{n/q_1 + 1}}
\]

\[
\lesssim t^{n/q} \| b \|_{\text{CBMO}^{p,\lambda}((\mathbb{R}^n))} \int_{2r}^{\infty} t^{n\lambda} \| f \|_{L^{q_1}(B(0,t))} \frac{dt}{t^{n/q_1 + 1}}
\]

Similarly, for \( J_2 \), we also have

\[
J_2 = \left( \int_B |G_\alpha((b - b_B)f_1)(x)|^{q_1} \, dx \right)^{1/q_1}
\]

\[
\lesssim \left( \int_{2B} |b(x) - b_B|^p \, dx \right)^{1/p} \left( \int_{2B} |f(x)|^{q_1} \, dx \right)^{1/q_1}
\]

\[
\lesssim \left( \int_{2B} |b(x) - b_{2B}|^p \, dx \right)^{1/p} \left( \int_{2B} |f(x)|^{q_1} \, dx \right)^{1/q_1}
\]

\[
+ |2B|^{1/p} |b_B - b_{2B}| \left( \int_{2B} |f(x)|^{q_1} \, dx \right)^{1/q_1}
\]

\[
\lesssim \left( \int_{2B} |b(x) - b_{2B}|^p \, dx \right)^{1/p} \| f \|_{L^{q_1}(2B)}
\]

\[
\lesssim t^{n/p + n\lambda} \| b \|_{\text{CBMO}^{p,\lambda}((\mathbb{R}^n))} \| f \|_{L^{q_1}(2B)}
\]

\[
\lesssim t^{n/q} \| b \|_{\text{CBMO}^{p,\lambda}((\mathbb{R}^n))} \int_{2r}^{\infty} t^{n\lambda} \left( 1 + \ln \frac{t}{r} \right) \| f \|_{L^{q_1}(B(0,t))} \frac{dt}{t^{n/q_1 + 1}}
\]

For \( J_3 \), by Lemma 2.3, we know that

\[
G_\alpha(f_2)(x) \lesssim \int_{2r}^{\infty} t^{-n/q_1 - 1} \| f \|_{L^{q_1}(B(0,t))} \, dt,
\]

which, together with Hölder’s inequality, implies that

\[
J_3 \lesssim \left( \int_B |b(x) - b_B|^q \, dx \right)^{1/q} \int_{2r}^{\infty} \| f \|_{L^{q_1}(B(0,t))} \frac{dt}{t^{n/q_1 + 1}}
\]

\[
\lesssim t^{n/p + n\lambda + n/q_1} \| b \|_{\text{CBMO}^{p,\lambda}((\mathbb{R}^n))} \int_{2r}^{\infty} \| f \|_{L^{q_1}(B(0,t))} \frac{dt}{t^{n/q_1 + 1}}
\]

\[
\lesssim t^{n/q} \| b \|_{\text{CBMO}^{p,\lambda}((\mathbb{R}^n))} \int_{2r}^{\infty} t^{n\lambda} \left( 1 + \ln \frac{t}{r} \right) \| f \|_{L^{q_1}(B(0,t))} \frac{dt}{t^{n/q_1 + 1}}.
\]
For \( J_4 \), since \(|y - x| < t\), it follows that \(|x - z| < 2t\). By the Minkowski inequality, we have

\[
G_a((b - b_B)f_2)(x) \leq \left( \int \int \int_{|x-z|<2t} |b(z) - b_B||f_2(z)|dz \right)^2 \left( \frac{dydt}{t^{3n+1}} \right)^{1/2}
\]

\[
\leq \left( \int_0^\infty \left( \int_{|x-z|<2t} |b(z) - b_B||f_2(z)|dz \right)^2 \frac{dt}{t^{2n+1}} \right)^{1/2}
\]

\[
\leq \int_{|z|>2r} |b(z) - b_B||f_2(z)| \frac{1}{|x-z|^n} dz.
\]

Note that \(|z - x| \geq \frac{1}{2}|z|\), by applying the Fubini theorem, we get

\[
J_4 \leq \int_{B(0,r)} \left( \int_{|z|>2r} |b(z) - b_B||f_2(z)| \frac{1}{|x-z|^n} dz \right)^{1/q} \frac{dx}{r^{n/q}}
\]

\[
\leq r^{n/q} \int_{|z|>2r} |b(z) - b_B||f_2(z)| \frac{1}{|z|^n} dz
\]

\[
\leq r^{n/q} \int_{|z|>2r} |b(z) - b_B||f(z)| \int_0^\infty \frac{1}{t^{n+1}} dt dz
\]

\[
\leq r^{n/q} \int_2^\infty \int_{B(0,t)} |b(z) - b_B||f(z)| dz \frac{1}{t^{n+1}} dt
\]

\[
+ r^{n/q} \int_2^\infty \int_{B(0,t)} |b(z) - b||f(z)| dz \frac{1}{t^{n+1}} dt
\]

\[= J_{41} + J_{42}.\]

For \( J_{41} \), by the Hölder inequality, we have

\[
\int_{B(0,t)} |b(z) - b_B||f(z)| dz
\]

\[
\leq t^{n(1-1/q)} \left( \int_{B(0,t)} |b(z) - b_B||f(z)|^q dz \right)^{1/q}
\]

\[
\leq t^{n(1-1/q)+n/p+n\lambda} \|b\|_{CBMO^{p,\lambda}(\mathbb{R}^n)} \|f\|_{L^{q1}(B(0,t))}.
\]

Thus,

\[
J_{41} \leq r^{n/q} \|b\|_{CBMO^{p,\lambda}(\mathbb{R}^n)} \int_2^\infty t^{n(1-1/q)+n/p+n\lambda} \|f\|_{L^{q1}(B(0,t))} \frac{1}{t^{n+1}} dt
\]

\[
\leq r^{n/q} \|b\|_{CBMO^{p,\lambda}(\mathbb{R}^n)} \int_2^\infty t^{n\lambda} \left( 1 + \ln \frac{t}{r} \right) \|f\|_{L^{q1}(B(0,t))} \frac{dt}{t^{n/q1+1}}.
\]

For \( J_{42} \), by Lemma 2.2, we have

\[
J_{42} = r^{n/q} \int_2^\infty |b_B(0,t) - b_B| \int_{B(0,t)} |f(z)| dz \frac{1}{t^{n+1}} dt
\]
\[
\begin{align*}
& \leq \frac{1}{r} \int_{B(0, t)} \left( \frac{1}{|B(0, t)|} \int_{B(0, t)} |b(x) - b_{B(0, r)}|^p \, dx \right)^{1/p} \int_{B(0, t)} |f(z)| \, dz \left( \frac{1}{t^{n+1}} + 1 \right) \\
& \leq r^{n/q} \|b\|_{CBMO^{p, \lambda}(\mathbb{R}^n)} \int_{2r}^{\infty} t^{n\lambda} \left( 1 + \ln \frac{t}{r} \right) \left\| f \right\|_{L^{q1}(B(0, t))} \frac{dt}{t^{n+1}} \\
& \lesssim r^{n/q} \|b\|_{CBMO^{p, \lambda}(\mathbb{R}^n)} \int_{2r}^{\infty} t^{n\lambda} \left( 1 + \ln \frac{t}{r} \right) \left\| f \right\|_{L^{q1}(B(0, t))} \frac{dt}{t^{n/q + 1}}.
\end{align*}
\]

Combining all of the above estimates, we finish the proof of Lemma 2.5.

3. Proofs of Theorems 1.1–1.4

Now we are in a position to prove Theorems 1.1–1.4.

**Proof of Theorem 1.1:** The method of the proof is standard, by Lemma 2.3 and a change of variables \( t = s - \frac{q}{n} \), we obtain that

\[
\left\| \mathcal{G}_\alpha(f) \right\|_{B^{q, q2}(\mathbb{R}^n)} \lesssim \sup_{r > 0} \frac{1}{\varphi_2(r)} \left( \frac{1}{|B(0, r)|} \int_{B(0, r)} \frac{1}{|B(0, t)|} \int_{2r}^{\infty} \left\| f \right\|_{L^{q1}(B(0, t))} \frac{dt}{t^{n+1}} \right)^{1/q} \\
\lesssim \sup_{r > 0} \frac{1}{\varphi_2(r)} \int_0^{r - \frac{q}{n}} \left\| f \right\|_{L^{q1}(B(0, s - \frac{q}{n}))} \, ds \\
\lesssim \sup_{r > 0} \frac{1}{\varphi_2(r - \frac{q}{n})} \int_0^{r} \left\| f \right\|_{L^{q1}(B(0, s - \frac{q}{n}))} \, ds \\
= \sup_{r > 0} \frac{r}{\varphi_2(r - \frac{q}{n})} \frac{1}{r} \int_0^{r} \left\| f \right\|_{L^{q1}(B(0, s - \frac{q}{n}))} \, ds.
\]

If we set

\[
\omega(t) = \varphi_2(t - \frac{q}{n})^{-1} t, \quad v(t) = \varphi_1(t - \frac{q}{n})^{-1} t,
\]

since the pair \((\varphi_1, \varphi_2)\) satisfies the following condition

\[
\int_r^{\infty} \frac{\inf_{t < \tau < \infty} \varphi_1(\tau) \, \tau^{n/q}}{t^{n/q + 1}} \, d\tau \lesssim \varphi_2(r),
\]

it follows that

\[
\sup_{t > 0} \frac{\omega(t)}{t} \int_0^t \frac{dr}{\inf_{0 < s < r} v(s)} < \infty.
\]

This leads to the following inequality (see [21])

\[
\ess \sup_{t > 0} \omega(t) \mathcal{H} g(t) \lesssim \ess \sup_{t > 0} v(t) g(t)
\]

holds for all non-negative and non-increasing functions \( g \) on \((0, \infty)\), where \( \mathcal{H} \) is the classical Hardy operator, that is,

\[
\mathcal{H} g(t) = \frac{1}{t} \int_0^t g(r) \, dr.
\]
Therefore, let \( g(t) = \|f\|_{L^q(B(0, r^{1/2})^n)} \), we have
\[
\|G_\alpha(f)\|_{B^{q\psi_2}(\mathbb{R}^n)} \lesssim \sup_{r > 0} \frac{r}{\bar{\varphi}_1(r^{-\frac{q}{n}})} \|f\|_{L^q(B(0, r^{-\frac{q}{n}}))} \lesssim \|f\|_{B^{q\psi_1}(\mathbb{R}^n)}.
\]

The proof of Theorem 1.1 is completed. ■

**Proof of Theorem 1.2:** It is easy to see that the following fact
\[
\left( \frac{g^*_{\bar{\lambda}, \alpha}(f)(x)}{A} \right)^2
= \int_0^\infty \int_{|x-y| < t} \left( \frac{t}{t + |x-y|} \right)^{n\bar{\lambda}} \left( A_\alpha f(t, y) \right)^2 \frac{dy dt}{t^{n+1}}
+ \int_0^\infty \int_{|x-y| \geq t} \left( \frac{t}{t + |x-y|} \right)^{n\bar{\lambda}} \left( A_\alpha f(t, y) \right)^2 \frac{dy dt}{t^{n+1}}
\leq (G_\alpha(f)(x))^2 + \sum_{j=1}^\infty \int_0^\infty \int_{|x-y| < 2^jt} \left( \frac{t}{t + |x-y|} \right)^{n\bar{\lambda}} \left( A_\alpha f(t, y) \right)^2 \frac{dy dt}{t^{n+1}}
\leq (G_\alpha(f)(x))^2 + \sum_{j=1}^\infty 2^{-jn\bar{\lambda}} \left( A_\alpha f(t, y) \right)^2 \frac{dy dt}{t^{n+1}}
\leq (G_\alpha(f)(x))^2 + \sum_{j=1}^\infty 2^{-jn\bar{\lambda}} \left( G_{\alpha, 2j}(f)(x) \right)^2.
\]

Thus,
\[
\|g^*_{\bar{\lambda}, \alpha}(f)\|_{B^{q\psi_2}(\mathbb{R}^n)} \lesssim \|G_\alpha(f)\|_{B^{q\psi_2}(\mathbb{R}^n)} + \sum_{j=1}^\infty 2^{-jn\bar{\lambda}} \|G_{\alpha, 2j}(f)\|_{B^{q\psi_2}(\mathbb{R}^n)}.
\]

Theorem 1.1 tells us the fact that
\[
\|G_\alpha(f)\|_{B^{q\psi_2}(\mathbb{R}^n)} \lesssim \|f\|_{B^{q\psi_1}(\mathbb{R}^n)}.
\]

Therefore, to obtain the proof of Theorem 1.2, it suffices to show that
\[
\sum_{j=1}^\infty 2^{-jn\bar{\lambda}} \|G_{\alpha, 2j}(f)\|_{B^{q\psi_2}(\mathbb{R}^n)} \lesssim \|f\|_{B^{q\psi_1}(\mathbb{R}^n)}, \quad \text{for } \bar{\lambda} > 3 + \frac{2\alpha}{n}.
\]

In fact, by Lemma 2.4 and similar to the proof of Theorem 1.1, for any \( \bar{\lambda} > 3 + \frac{2\alpha}{n} \), we get that
\[
\sum_{j=1}^\infty 2^{-jn\bar{\lambda}} \|G_{\alpha, 2j}(f)\|_{B^{q\psi_2}(\mathbb{R}^n)} \lesssim \sum_{j=1}^\infty 2^{-jn\bar{\lambda}} 2^{-3 + \frac{2\alpha}{n}}
\]
\[
\times \sup_{r>0} \frac{1}{\varphi_2(r^{-\frac{q}{n}})} \int_{0}^{r} \|f\|_{L^q(B(0,s^{-\frac{q}{n}}))} ds \\
\lesssim \|f\|_{B^{q,\varphi_1}(\mathbb{R}^n)}.
\]

The proof of Theorem 1.2 is finished. ■

**Proof of Theorem 1.3:** By Lemma 2.5, the proof of Theorem 1.3 is only a repetition of Theorem 1.1. Therefore we omit the details here. ■

**Proof of Theorem 1.4:** Similar to the proof of Theorem 1.2, together with Theorem 1.3, we derive that

\[
\| [b, g^{\alpha}_{\lambda, \alpha}] (f) \|_{B^{q,\varphi_2}(\mathbb{R}^n)} \\
\leq \| [b, \mathcal{G}_\alpha] (f) \|_{B^{q,\varphi_2}(\mathbb{R}^n)} + \sum_{j=1}^{\infty} 2^{-jn\lambda^2} \| [b, \mathcal{G}_\alpha, 2^j] (f) \|_{B^{q,\varphi_2}(\mathbb{R}^n)} \\
\lesssim \| b \|_{CBMOp^{\lambda, \alpha}(\mathbb{R}^n)} \| f \|_{B^{q,\varphi_1}(\mathbb{R}^n)}.
\]

The proof of Theorem 1.4 is completed. ■

**Acknowledgements**

The authors would like to express their gratitude to the referee for his very valuable comments.

**Disclosure statement**

No potential conflict of interest was reported by the author(s).

**Funding**

The research was supported by the NNSF of China [grant number 12061069].

**References**

[1] Morrey CB. On the solutions of quasi-linear elliptic partial differential equations. Trans Amer Math Soc. 1938;43:126–166.
[2] Chiarenza J, Frasca M. Morrey spaces and Hardy-Littlewood maximal functions. Rend Mat Appl. 1987;7(3):273–279.
[3] Satorn S. Necessary and sufficient conditions for boundedness of commutators of fractional integral operators on classical Morrey spaces. Hokkaido Math J. 2006;35:683–696.
[4] Wilson M. The intrinsic square function. Rev Mat Iberoam. 2007;23:771–791.
[5] Fefferman C, Stein EM. $H^p$ spaces of several variables. Acta Math. 1972;129:137–193.
[6] Wang H. Intrinsic square functions on the weighted Morrey spaces. J Math Anal Appl. 2012;396:302–314.
[7] Wu X, Zheng T. Commutators of intrinsic square functions on generalized Morrey spaces. J Inequal Appl. 2014;2014:128.
[8] Wang H. Boundedness of intrinsic square functions on the weighted weak Hardy spaces. Integr Equ Oper Theory. 2013;75:135–149.
[9] Wang H. Weak type estimates for intrinsic square functions on weighted Morrey spaces. Anal Theory Appl. 2013;29(2):104–119.
[10] Wiener N. Generalized harmonic analysis. Acta Math. 1930;29:117–258.
[11] Wiener N. Tauberian theorems. Ann Math. 1932;33:1–100.
[12] Beurling A. Construction and analysis of some convolution algebras. Ann Inst Fourier (Grenoble). 1964;14:1–32.
[13] Lu S, Yang D. The Littlewood-Paley function and φ-transform characterization of a new Hardy space $H_{K_2}$ associated with Herz space. Studia Math. 1992;101:285–298.
[14] Lu S, Yang D. The central BMO spaces and Littlewood-Paley operators. Approx Theory Appl. 1995;11:72–94.
[15] Alvarez J, Guzmán-Partida M, Lakey J. Spaces of bounded $\lambda$-central mean oscillation, Morrey spaces, and $\lambda$-central Carleson measures. Collect Math. 2000;51:1–47.
[16] Mizuhara T. Boundedness of some classical operators on generalized Morrey spaces. Conf Proc. 1990;90:183–189.
[17] Fan Y. Boundedness of sublinear operators and their commutators on generalized central Morrey spaces. J Inequal Appl. 2013;2013:411.
[18] Fu Z, Lin Y, Lu S. $\lambda$-central BMO estimates for commutators of singular integral operators with rough kernels. Acta Math Sin. 2008;24(3):373–386.
[19] Si Z, Xue Q. $\lambda$-central BMO estimates for commutators of multilinear maximal operators. Acta Math Sin. 2013;29(4):729–742.
[20] Yu X, Tao X. Boundedness for a class of generalized commutators on $\lambda$-central Morrey spaces. Acta Math Sin. 2013;29(10):1917–1926.
[21] Carro M, Pick L, Soria J, et al. On embeddings between classical Lorentz spaces. Math Inequal Appl. 2001;4(3):397–428.