CONNECTEDNESS PROPERTIES OF THE SPACE OF COMPLETE NONNEGATIVELY CURVED PLANES

IGOR BELEGRADEK AND JING HU

Abstract. We prove that the set of complete Riemannian metrics of nonnegative curvature on the plane equipped with the compact-open topology cannot be made disconnected by removing a finite dimensional subspace. A similar result holds for the associated moduli space. The proof combines properties of subharmonic functions with results of infinite dimensional topology and dimension theory. A key step is a characterization of the conformal factors that make the standard Euclidean metric on the plane into a complete metric of nonnegative sectional curvature.

1. Introduction

Spaces of constant curvature metrics on surfaces is the subject of Teichmüller theory. The spaces of Riemannian metrics (and the associated moduli spaces) have been studied under various geometric assumptions such as positive scalar [KS93, Ros07, Mar12], negative sectional [FO09, FO10a, FO10b, FO10c], positive Ricci [Wra11], nonnegative sectional [KPT05, BKS11, BKS, Ott], while curvature-free results about spaces of metrics can be found in [Ebi70, NW00].

This paper studies connectedness properties of the set of complete nonnegatively curved metrics on $\mathbb{R}^2$ equipped with the compact-open topology. The main theme is deciding when two metrics can be deformed to each other through complete nonnegatively curved metrics outside a given subset, and how large the space of such deformations is.

The starting point is a result of Blanc-Fiala [BF42] that any complete nonnegatively curved metric on $\mathbb{R}^2$ is conformally equivalent to the standard Euclidean metric $g_0$, i.e. isometric to $e^{-2u}g_0$ for some smooth function $u$, see [Hub57, Theorem 15] and [Gri99, Corollary 7.4] for generalizations.

The sectional curvature of $e^{-2u}g_0$ equals $e^{2u}\Delta u$, where $\Delta$ is the Euclidean Laplacian. Thus $e^{-2u}g_0$ has nonnegative curvature if and only if $u$ is subharmonic. Characterizing subharmonic functions that correspond to complete

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metrics is not straightforward, and doing so is the main objective of this paper. A basic property [HK76, Theorem 2.14] of a subharmonic function $u$ on $\mathbb{R}^2$ is that the limit

$$\alpha(u) := \lim_{r \to \infty} \frac{M(r, u)}{\log r}$$

exists in $[0, \infty]$, where $M(r, u) := \sup \{u(z) : |z| = r\}$. For example, by Liouville’s theorem $\alpha(u) = 0$ if and only if $u$ is constant, while any nonconstant harmonic function $u$ satisfies $\alpha(u) = \infty$ (see Proposition 2.2).

In the separable Fréchet space $C^\infty(R^2)$ equipped with the compact-open $C^\infty$ topology consider the subset $S_\alpha$ consisting of smooth subharmonic functions with $\alpha(u) \leq \alpha$. Since $S_0$ consists of constants, it is homeomorphic to $\mathbb{R}$. We note in Lemmas 2.4–2.5 that the subset $S_\alpha$ is closed convex, and not locally compact when $\alpha \neq 0$.

Completeness of $e^{-2u}g_0$ is immediate when $\alpha(u) < 1$. Appealing to more delicate properties of subharmonic functions due to Huber [Hub57] and Hayman [Hay60] we prove:

**Theorem 1.1.** The metric $e^{-2u}g_0$ is complete if and only if $\alpha(u) \leq 1$.

Recall that complete Riemannian metrics on any manifold form a dense subset in the space of all Riemannian metrics, e.g. $e^{-2u}g_0$ is the endpoint of the curve $g_s := (s + e^{-2u})g_0$ where the metric $g_s$ is complete for $s > 0$ [FM78]. This led the first author to speculate in [Bel12] that the set of $u$’s for which $e^{-2u}g_0$ is complete and nonnegatively curved is “probably neither closed nor convex”, hence Theorem 1.1 came as a surprise.

A Riemannian metric on $\mathbb{R}^2$ can be thought of as a smooth (i.e. $C^\infty$) map from $\mathbb{R}^2$ to the space of symmetric, positive definite bilinear forms. Let $\mathcal{R}^k(\mathbb{R}^2)$ be the set of all Riemannian metrics on $\mathbb{R}^2$ equipped with the compact-open $C^k$ topology (i.e. topology of $C^k$ uniform convergence on compact sets), where $0 \leq k \leq \infty$. With this topology $\mathcal{R}^k(\mathbb{R}^2)$ becomes metrizable and separable (see Lemma 3.3). Let $\mathcal{R}_{\geq 0}^k(\mathbb{R}^2) \subset \mathcal{R}^k(\mathbb{R}^2)$ denote the subspace of complete metrics of nonnegative sectional curvature. Like any subset of a separable metric space, $\mathcal{R}_{\geq 0}^k(\mathbb{R}^2)$ is separable and metrizable.

As we note in Lemma 3.1 any metric conformally equivalent to $\mathbb{R}^2$ can be written uniquely as $\phi^* e^{-2u}g_0$ where $g_0$ is the standard Euclidean metric, $u$ is a smooth function, and $\phi \in \text{Diff}^{+}_{0,1}(\mathbb{R}^2)$, the group of self-diffeomorphisms of the plane fixing the complex numbers 0, 1 and isotopic to the identity. Hence by Theorem 1.1 the map $(u, \phi) \to \phi^* e^{-2u}g_0$ defines a continuous bijection

$$\Pi_k : S_1 \times \text{Diff}^{+}_{0,1}(\mathbb{R}^2) \to \mathcal{R}_{\geq 0}^k(\mathbb{R}^2),$$

(1.2)
where $\text{Diff}^+_0(\mathbb{R}^2)$ is given the compact-open $C^\infty$ topology. The parametrization $\Pi_k$ is not a homeomorphism, in fact $\Pi_k^{-1}$ is discontinuous at each point of $\mathcal{R}^{k,c}_{\geq 0}(\mathbb{R}^2)$, see Proposition 3.3. (That $\Pi_k$ is not a homeomorphism for $k < \infty$ is immediate because $\Pi_k$ factors as $\Pi_\infty$ followed by $\text{id}: \mathcal{R}^{\infty,c}_0(\mathbb{R}^2) \to \mathcal{R}^{k,c}_{\geq 0}(\mathbb{R}^2)$ and $\text{id}$ is clearly not a homeomorphism).

Yet the existence of the parametrization (1.2) has nontrivial consequences because the parameter space $S_1 \times \text{Diff}^+_0(\mathbb{R}^2)$ is homeomorphic to $\ell^2$ by the following results of infinite dimensional topology:

- any closed convex non-locally-compact subset of a separable Fréchet space is homeomorphic to $\ell^2$, the separable Hilbert space [DT81, Theorem 2]; this applies to $S_\alpha \subset C^\infty(\mathbb{R}^2)$ with $\alpha \neq 0$.
- $\text{Diff}^+_0(\mathbb{R}^2)$ is homeomorphic to $\ell^2$ [Yag, Theorem 1.1].
- $\ell^2$ is homeomorphic to $(-1,1)^\mathbb{N}$, the countably infinite product of open intervals [And66].

Our first application demonstrates that any two metrics can be deformed to each other in a variety of ways, while bypassing a given countable set:

**Theorem 1.3.** If $K$ is a countable subset of $\mathcal{R}^{k,c}_{\geq 0}(\mathbb{R}^2)$ and $X$ is a separable metrizable space, then for any distinct points $x_1, x_2 \in X$ and any distinct metrics $g_1, g_2$ in $\mathcal{R}^{k,c}_{\geq 0}(\mathbb{R}^2) \setminus K$ there is an embedding of $X$ into $\mathcal{R}^{k,c}_{\geq 0}(\mathbb{R}^2) \setminus K$ that takes $x_1, x_2$ to $g_1, g_2$, respectively.

The above theorem hinges on the following facts:

- Like any continuous one-to-one map to a Hausdorff space, the map $\Pi_k$ restricts to a homeomorphism on every compact subset, e.g. the Hilbert cube.
- Every separable metrizable space embeds into the Hilbert cube [Dug66, Theorem IX.9.2].
- The complement in $\ell^2$ of the countable union of compact sets is homeomorphic to $\ell^2$ [And67, cf. BP75, Theorem V.6.4]), and hence contains an embedded Hilbert cube.

A space is *finite dimensional* if it is homeomorphic to a subset of a Euclidean space. A space is *continuum-connected* if every two points lie in a continuum (a compact connected space); thus a continuum-connected space is connected but not necessarily path-connected.

By Theorem 1.3 any two metrics lie in an embedded copy of $\mathbb{R}^n$ for any $n$. Since $\mathbb{R}^n$ cannot be separated by subspace of codimension $\geq 2$ we conclude:
Theorem 1.4. The complement of every finite dimensional subspace of $\mathcal{R}_{\geq 0}^{k,c}(\mathbb{R}^2)$ is continuum-connected. The complement of every closed finite dimensional subspace of $\mathcal{R}_{\geq 0}^{k,c}(\mathbb{R}^2)$ is path-connected.

Let $\mathcal{M}_{k,c}^{\geq 0}(\mathbb{R}^2)$ denote the moduli space of complete nonnegatively curved metrics, i.e. the quotient space of $\mathcal{R}_{\geq 0}^{k,c}(\mathbb{R}^2)$ by the $\text{Diff}(\mathbb{R}^2)$-action via pullback.

The moduli space $\mathcal{M}_{k,c}^{\geq 0}(\mathbb{R}^2)$ is rather pathological, e.g. it is not a $T_1$ space (in the proof of Proposition 3.4 we exhibit a non-flat metric $g \in \mathcal{R}_{\geq 0}^{k,c}(\mathbb{R}^2)$ whose isometry lies in every neighborhood of the isometry class of $g_0$). Consider the map $S_1 \to \mathcal{M}_{k,c}^{\geq 0}(\mathbb{R}^2)$ sending $u$ to the isometry class of $e^{-2u}g_0$. Its fibers lie in the orbits of a $\text{Conf}(\mathbb{R}^2)$-action of $C^\infty(\mathbb{R}^2)$, so each fiber is the union of countably many finite dimensional compact sets, which by dimension theory arguments easily implies:

Theorem 1.5. The complement of a subset $S$ of $\mathcal{M}_{\geq 0}^{k,c}(\mathbb{R}^2)$ is path-connected if $S$ is countable, or if $S$ is closed and finite dimensional.

Theorems 1.3–1.5 yield a deformation between any two given metric $g_1, g_2$ that runs in a separable metrizable space, a continuum, or a path. As we explain in Remark 4.1 this deformation can be arranged to bypass any given set of complete flat metrics.

The proofs of Theorems 1.3–1.5 remain true if $\mathcal{R}_{\geq 0}^{k,c}(\mathbb{R}^2)$ is substituted with the subspace of $\mathcal{R}_{\geq 0}^k(\mathbb{R}^2)$ of nonnegatively curved metrics that are conformally equivalent to $g_0$. The only change is to replace $S_1$ with $S_\infty$.

That $\mathcal{R}_{\geq 0}^{k,c}(\mathbb{R}^2)$ and $\mathcal{M}_{\geq 0}^{k,c}(\mathbb{R}^2)$ cannot be separated by a countable set was announced by the first author in [Bel12] who at that time only knew that $e^{-2u}g_0$ is complete when $\alpha(u) < 1$ and incomplete for $\alpha(u) > 1$. Completeness of $e^{-2u}g_0$ when $\alpha(u) = 1$ was established in collaboration with the second author, which led to stronger applications.

Theorem 1.1 is proved in Section 2. Some auxiliary results are collected in Section 3 while proofs of Theorems 1.3–1.5 are given in Section 4.

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2. Subharmonic functions and complete metrics

We adopt notation of the introduction: $g_0$ is the standard Euclidean metric, $u$ is a smooth subharmonic function on $\mathbb{R}^2$, and $M(r,u) := \max\{u(z) : |z| = r\}$. Set $\mu(t,u) := M(u,e^t)$; when $u$ is understood we simply write $M$, $\mu$. Subharmonicity of $u$ implies that $\mu$ is a convex function [HK76, Theorem 2.13]. Hence $\mu$ has left and right derivatives everywhere, and they are equal outside
a countable subset, and so the same holds for \( M \). By the maximum principle \[HK76\] Theorem 2.3, \( M \) is strictly increasing (except when \( u \) is constant), and hence the same is true for \( \mu \). As mentioned in the introduction, the limit

\[ \alpha(u) := \lim_{r \to \infty} \frac{M(u,r)}{\log r} \]

exists in \([0, \infty]\), and \( \alpha(u) = 0 \) if and only if \( u \) is constant \[HK76\] Theorem 2.14.

Recall that a Riemannian manifold is incomplete if and only if it contains a locally rectifiable (or equivalently, a smooth) path that eventually leaves every compact set and has finite length. In the manifold \((\mathbb{R}^2, e^{-2u} g_0)\) the length of a path \( \gamma \) equals \( \int_{\gamma} e^{-u} ds \).

**Lemma 2.1.** The metric \( g = e^{-2u} g_0 \) is complete if \( a(u) < 1 \) and incomplete if \( a(u) > 1 \) or \( \alpha(u) = \infty \).

**Proof.** If \( \alpha(u) = \infty \), then incompleteness of \( g \) can be extracted from \[Hub57\], see \[LRW84\] for a stronger result, who constructed a finite length locally rectifiable path that goes to infinity in \( \mathbb{R}^2 \). If \( \alpha(u) = 0 \), then \( u \) is constant, so \( g \) is complete.

So we can assume that \( \alpha(u) \) is positive and finite in which case Hayman \[Hay60\] Theorem 2 and Remark (i) on page 75] proved that there is a constant \( c \) and a measure zero subset \( Z \) of the unit circle such that \( 0 \leq M(r) - u(re^{i\theta}) \leq c \) for every \( \theta \notin Z \) and all \( r > r(\theta) \).

Suppose \( \alpha(u) > 1 \), and fix \( \theta \notin Z \), and the corresponding ray \( \gamma(r) = re^{i\theta} \), \( r > r(\theta) \) on which \( 0 \leq M(r) - u(re^{i\theta}) \leq c \). Then \( \int_{\gamma} e^{-u} \) is bounded above and below by positive multiples of \( \int_{\gamma} e^{-M} \). As \( \frac{M(r)}{\log r} \to \alpha \), for any \( \alpha_0 \in (1, \alpha(u)) \) there is \( r_0 \) with \( M(r) > \alpha_0 \log r \) for all \( r > r_0 \). Shortening \( \gamma \) to \( r > r_0 \), we get \( \int_{\gamma} e^{-B} \leq \int_{r_0}^{\infty} r^{-\alpha_0} < \infty \) proving incompleteness of \( g \).

Suppose \( \alpha(u) \in (0,1) \). Fix \( \alpha_1 \in (\alpha(u),1) \) and any smooth path \( \sigma \) going to infinity. Find \( r_1 \) with \( M(r) < \alpha_1 \log r \) for all \( r > r_1 \). Now \( u(re^{i\theta}) \leq M(r) \) implies

\[ \int_{\sigma} e^{-u} ds \geq \int_{r_1}^{\infty} r^{-\alpha_1} dr = \infty \]

so \( g \) is complete. \( \Box \)

**Proposition 2.2.** If \( u \) is harmonic and nonconstant, then \( e^{-2u} g_0 \) is not complete and \( \alpha(u) = \infty \).

**Proof.** If \( \alpha(u) \) is finite, then by rescaling we may assume that \( \alpha(u) < 1 \) so that \( e^{-2u} g_0 \) is complete by Lemma 2.1. Since \( u \) is harmonic, \( e^{-2u} g_0 \) has zero
curvature, and hence \( e^{-2u}g_0 = \psi^*g_0 \) for some \( \psi \in \text{Diff}(\mathbb{R}^2) \). It follows that \( \psi \) is conformal, and hence \( \psi \) or its composition with the complex conjugation is affine. Therefore \( \psi^*g_0 \) is a constant multiple of \( g_0 \), hence \( u \) is constant. \( \square \)

\textbf{Remark 2.3.} There is a purely analytic proof of incompleteness: since \( u \) is harmonic it is the real part of an entire function \( f \). Thus \( e^u \) is \( |e^f| \) where \( e^f \) is an entire function with no zeros, and hence so is \( e^{-f} \). Huber \cite{Hub57} Theorem 7 proves that there is a path going to infinity such that the integral of \( |e^{-f}|^{-1} = e^u \) over the path is finite. (Actually, Huber’s result applies to any non-polynomial entire function in place of \( e^{-f} \), and his proof is much simplified when the entire function has finitely many zeros, as happens for \( e^{-f} \); the case of finitely many zeros is explained on \cite{Kap60} page 71).

\textbf{Lemma 2.4.} \( S_\alpha \) is a closed convex subset in the Fréchet space \( C^\infty(\mathbb{R}^2) \).

\textit{Proof.} Convexity is immediate: If \( u = su_j + (1 - s)u_0 \) with \( s \in [0, 1] \) and \( u_j \in S_\alpha \), then \( u \) is subharmonic, and \( M(r, u) \leq sM(r, u_j) + (1 - s)M(r, u_0) \), so dividing by \( \log r \) and taking \( r \to \infty \) yields \( \alpha(u) \leq s\alpha(u_j) + (1 - s)\alpha(u_0) \leq \alpha \).

Fix \( u_j \in S_\alpha \) and a smooth function \( u \) such that \( u_j \to u \) in \( C^\infty(\mathbb{R}^2) \). Clearly \( u \) is subharmonic, and also \( M(r, u_j) \to M(r, u) \) for each \( r \), so \( \mu(\cdot, u_j) \to \mu(\cdot, u) \) pointwise. Set \( \mu_j := \mu(\cdot, u_j) \), \( \mu := \mu(\cdot, u) \). Since \( \mu_j \), \( \mu \) are convex, \( \mu_j' \), \( \mu' \) exist outside a countable subset \( \Sigma \) \cite{RV73} page 7, Theorem C], and the convergence \( \mu_j \to \mu \) is uniform on compact sets \cite{RV73} page 17, Theorem E], which easily implies that \( \mu_j' \to \mu' \) outside \( \Sigma \) \cite{RV73} Exercise C(9), page 20].

By convexity \( \mu_j' \), \( \mu' \) are non-decreasing outside \( \Sigma \) \cite{RV73} page 5, Theorem B]. It follows that \( \mu_j' \leq \alpha \) outside \( \Sigma \) for if \( \mu_j'(t_1) = \alpha_1 > \alpha \), then \( \mu_j' \geq \alpha_1 \) for \( t \geq t_1 \), so integrating we get \( \mu_j(t) - \mu_j(t_1) \geq \alpha_1(t - t_1) \) which contradicts

\[ \lim_{t \to \infty} \frac{\mu_j(t)}{t} \leq \alpha. \]

Since \( \mu_j' \to \mu' \) we get \( \mu' \leq \alpha \) outside \( \Sigma \). Integrating gives \( \mu(t) \leq \alpha(t - t_0) + \mu(t_0) \) everywhere, so \( u \in S_\alpha \). \( \square \)

\textbf{Lemma 2.5.} If \( \alpha > 0 \), then \( S_\alpha \) is not locally compact.

\textit{Proof.} A closed subset of a locally compact space is locally compact, so since \( S_\alpha \) is closed in \( S_\infty \), we may assume \( \alpha \) is finite. Fix any nonconstant \( u \in S_\alpha \) and show that it has no compact neighborhood. Since \( 0 < \alpha < \infty \) we know that \( u \) is not harmonic, so there is a disk closed \( D \) where \( \Delta u \) is \( > 0 \). Now any sufficiently small \( C^\infty \) variation \( \tilde{u} \) of \( u \) is in \( S_\alpha \) provided \( \tilde{u} - u \) is supported in \( D \), and by a standard argument \( \tilde{u} \)’s cannot all lie in a compact neighborhood of \( u \). \( \square \)
Theorem 2.6. \( S_1 \) equals the set of smooth subharmonic functions \( u \) such that the metric \( e^{-2u}g_0 \) is complete.

Proof. If \( u \notin S_1 \), then \( e^{-2u}g_0 \) is incomplete by Lemma \([2.1]\). Suppose \( u \in S_1 \) while \( e^{-2u}g_0 \) is incomplete, and aim for a contradiction. By incompleteness of \( g \) there is a smooth path \( \gamma \) in \( \mathbb{R}^2 \) going to infinity such that \( \int_{\gamma} e^{-u} ds < \infty \). Now \( u \leq M \) implies \( \int_{\gamma} e^{-M} ds < \infty \). It is convenient to replace \( M \), \( \mu \) with nearby smooth functions with similar properties which is possible by a result of Azagra \([Aza13]\) that there is a smooth convex function \( \nu \) defined on \( \mathbb{R} \) such that \( \mu - 1 \leq \nu \leq \mu \). Note that \( \frac{\nu(t)}{t} \to 1 \) as \( t \to \infty \).

For \( r > 0 \) set \( N(r) := \nu(\log r) \); the function \( (x,y) \to N(r) \) is subharmonic:

\[
\Delta N = \nu''(t_x^2 + t_y^2) + \nu' \Delta t = \nu'' r^{-2}.
\]

Here \( t_x, t_y \) are partial derivatives of \( t = \log r \); note that \( \Delta t = 0 \) while \( t_x = \frac{x}{r} \) and \( t_y = \frac{y}{r} \).

Set \( d(r) := \frac{N(r)}{\log r} - 1 \) so that \( e^{-N(r)} = r^{-1-d(r)} \). Since \( u \in S_1 \setminus C \), we get \( u \notin S_\alpha \) for \( \alpha < 1 \) so that \( d(r) \to 0 \) as \( r \to \infty \). Also \( M - 1 \leq N \leq M \) so that \( \int_{\gamma} r^{-1-d} ds = \int_{\gamma} e^{-N} ds < \infty \).

In deriving a contradiction it helps consider the following cases.

If \( d' \geq 0 \) for all large \( r \), then since \( d \) converges to zero as \( r \to \infty \), we must have \( d \leq 0 \) for large \( r \), so after shortening \( \gamma \) we get

\[
\int_{\gamma} r^{-1-d} ds \geq \int_{\gamma} r^{-1} ds = \infty
\]

which is a contradiction.

If the sign of \( d' \) changes \( \pm 1 \) as \( r \to \infty \), then there is a point where \( d' \) and \( d'' \) are both negative, which contradicts subharmonicity of \( N \) for \( r > 1 \) as

\[
0 \leq \Delta N = N'' + \frac{N'}{r} = d'' \log r + d' \left( \frac{\log r}{r} + \frac{2}{r} \right).
\]

It remains to deal with the case when \( d' \leq 0 \) for large \( r \). Multiply (2.7) by \( r \log r \) to get

\[
0 \leq r \log^2 r \left( d'' + d' \left( \frac{1}{r} + \frac{2}{r \log r} \right) \right) = (r \log r)^2 d' \]

which integrates over \([\rho, r]\) to \( d'(\rho) \rho \log^2 \rho \leq d'(r) r \log^2 r \). Since \( d' \leq 0 \) for all large \( r \), we conclude that \( c := d'(\rho) \rho \log^2 \rho \) is a nonpositive constant. Set \( f(r) := -\frac{c}{r \log^2 r} \) so that \( f' = \frac{c}{r \log^2 r} \leq d' \). Integrating \( d' - f' \geq 0 \) over \([r, R]\) gives \( d(R) - f(R) \geq d(r) - f(r) \) and since \( d(R), f(R) \) tend to zero as \( R \to \infty \), we get \( d(r) \leq f(r) \) for all large \( r \). Hence \( \int_{\gamma} r^{-1-d} ds \geq \int_{\gamma} r^{-1} ds = e^c \int_{\gamma} r^{-1} ds = \infty \) which again is a contradiction. \( \square \)
3. Loose ends

In this section we justify various claims made in the introduction.

Lemma 3.1. If $g$ is conformal to $g_0$, then there are unique $\phi \in \Diff_{0,1}^+(\mathbb{R}^2)$ and $v \in C^\infty(\mathbb{R}^2)$ such that $g$ equals $\phi^*e^vg_0$.

Proof. Any metric $g$ conformal to $g_0$ can be written as $\psi^*e^fg_0$ where $\psi \in \Diff(\mathbb{R}^2)$ and $f \in C^\infty(\mathbb{R}^2)$. Note that $\psi^*e^fg_0 = e^{f \circ \psi} \psi^*g_0$. To see uniqueness suppose $\phi_1^*e^{v_1}g_0 = \phi_2^*e^{v_2}g_0$ and rewrite it as

$$ (\phi_1 \circ \phi_2^{-1})^*g_0 = e^{v_2 - v_1 \circ \phi_1 \circ \phi_2^{-1}}g_0 $$

so that $\phi_1 \circ \phi_2^{-1}$ is a conformal automorphism of $\mathbb{R}^2$ that preserves orientation and fixes 0, 1. It follows that $\phi_1 \circ \phi_2^{-1}$ is the identity, hence (3.2) implies $v_1 = v_2$.

To prove existence recall that any diffeomorphism of $\mathbb{R}^2$ is isotopic either to the identity or to the reflection $z \rightarrow \bar{z}$. The metric $e^{f \circ \psi} \psi^*g_0$ does not change when we compose $\psi$ with an isometry of $g_0$, and composing $\psi$ with an affine map results in rescaling which can be subsumed into $f \circ \psi$ changing it by an additive constant. Thus composing with $z \rightarrow \bar{z}$ if needed, and with an affine map we can arrange $\psi$ to lie in $\Diff_{0,1}^+(\mathbb{R}^2)$.

The properties of $C^\infty(\mathbb{R}^2)$ and $\mathcal{R}^k(\mathbb{R}^2)$ stated in the introduction follow from

Lemma 3.3. Given $0 \leq k \leq \infty$ equip $C^\infty(M,N)$ with the topology of uniform $C^k$ convergence on compact sets. Then $C^\infty(M,N)$ is

(1) separable and metrizable,
(2) completely metrizable if $k = \infty$,
(3) a Fréchet space if $N$ is a Euclidean space and $k = \infty$.

Proof. The space $C^\infty(M,N)$ sits in $C^k(M,N)$ which embeds as a closed subset into $C^0(M,J^k(M,N))$ where $J^k(M,N)$ is the space of $k$-jets which is a $C^0$ manifold, see e.g. [Hir94, Section 2.4]. Separability is implied by having a countable basis, and since the latter property is inherited by subspaces it suffices to show that $C^0(M,J^k(M,N))$ has a countable basis, but in general if the spaces $X$, $Y$ have a countable basis and if $X$ is locally compact, then $C^0(X,Y)$ with the compact-open topology has a countable basis [Dug66, Theorem XII.5.2]. Similarly, metrizability is inherited by subspaces, and complete metrizability is inherited by closed subspaces, while $C^0(X,Y)$ with the compact-open topology is completely metrizable whenever $X$ is locally compact and $Y$ is completely metrizable [Hir94 Theorem 2.4.1]. A proof of (3) can be found in [Tré67, Example 10.I].

\[\square\]
Proposition 3.4. If $\Pi_k$ is the bijection (1), then $\Pi_k^{-1}$ is discontinuous at every point of $\mathcal{R}_{\geq 0}^{k,c}(\mathbb{R}^2)$.

Proof. Fix a positive $\epsilon \ll 1$. Let $f: [0, \infty) \to [0, \infty)$ be a convex smooth function with $f(x) = 2\epsilon$ for $x \in [0, \epsilon]$ and $f(x) = x$ if $x \geq 3\epsilon$. The surface of revolution in $\mathbb{R}^3$ obtained by rotating the curve $x \to (x, 0, f(x))$ about the $z$-axis has a complete metric of nonnegative curvature. Identify the surface with the $xy$-plane by projecting along the $z$-axis, and let $g$ denote the corresponding metric on $\mathbb{R}^2$. Write $g = \Pi_k(u, \phi)$; clearly $g$ is not flat, so $u$ is non-constant. The metric balls $B^g(j, j)$, $B^{g_0}(0, j)$ of radius $j$ are isometric. So if $\psi_j$ is a self-diffeomorphism of $\mathbb{R}^2$ that isometrically takes $B^{g_0}(0, j)$ to $B^g(0, j)$, then $\psi_j^*g$ converges to $g_0$ in $\mathcal{R}_{\geq 0}^{k,c}(\mathbb{R}^2)$. One can arrange $\psi_j$ to preserve orientation and isometrically map the $g_0$-segment $[-d^g(0, j), 0]$ onto the $g$-segment $[0, j]$. Then $\psi_j^{-1}(1) = d^g(0, 1) - d^g(0, j)$, so if $l_j(z) := z d^g(0, 1) - d^g(0, j)$, then the composition $\phi_j = \psi_j \circ l_j$ lies in $\text{Diff}^+_0(\mathbb{R}^2)$. For $w := -\log d^g(0, 1)$, we get

$$\psi_j^* g = \phi_j^* e^{-2w}g = \phi_j^* e^{-2w} \phi^* e^{-2w} g_0 = (\phi \circ \phi_j)^* e^{-2(w+u)} g_0.$$ 

Now $(w+u, \phi \circ \phi_j)$ does not converge to $(0, \text{id})$ as $j \to 0$ because $w$ is constant, while $u$ is not, so $w+u \neq 0$. Thus $\Pi_k^{-1}$ is discontinuous at $g_0$.

To prove discontinuity at any $\varphi^* e^{-2w}g_0$ with $(v, \varphi) \in S_1 \times \text{Diff}^+_0(\mathbb{R}^2)$ note that $\varphi^* e^{-2v} \psi_j^* g$ converges to $\varphi^* e^{-2v} g_0$. Write

$$\varphi^* e^{-2v} \psi_j^* g = \varphi^* e^{-2v} (\phi \circ \phi_j)^* e^{-2(w+u)} g_0 = (\phi \circ \phi_j \circ \varphi)^* e^{-2(w+u+v)} g_0$$

where $v_j = v \circ (\phi \circ \phi_j)^{-1}$. If $(w+u+v_j, \phi \circ \phi_j \circ \varphi) \to (v, \varphi)$, then $\phi \circ \phi_j \to \text{id}$, so $v_j \to v$, and we again get a contradiction since $w+u \neq 0$. 

4. Metric deformation with obstacles

Proof of Theorem 1.3. As was explained in the introduction, $S_1 \times \text{Diff}^+_0(\mathbb{R}^2)$ is homeomorphic to $\ell^2$, and since $\Pi_k^{-1}(K)$ is a countable union of compact sets, its complement is homeomorphic to $\ell^2$, which contains the Hilbert cube and hence an embedded copy of $X$. Applying an affine self-homeomorphism of $\ell^2$ one can ensure that the embedding maps $x_1, x_2$ to $\Pi_k^{-1}(g_1), \Pi_k^{-1}(g_2)$, respectively. Since $X$ sits in an embedded copy of the Hilbert cube (a compact set), and since $\mathcal{R}_{\geq 0}^{k,c}(\mathbb{R}^2)$ is Hausdorff, the restriction of $\Pi_k$ to the embedded copy of $X$ is a homeomorphism onto its image, which has desired properties. 

Proof of Theorem 1.4. Let $S$ be a finite dimensional subspace of $\mathcal{R}_{\geq 0}^{k,c}(\mathbb{R}^2)$. Fix two points $g_1, g_2$ in the complement of $S$. Theorem 1.3 implies $g_1, g_2$ lie in a subspace $X$ of $\mathcal{R}_{\geq 0}^{k,c}(\mathbb{R}^2)$ that is homeomorphic to $\mathbb{R}^n$ with $n \geq \dim(S) + 2$. 

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Since $S \cap X$ has dimension $\leq \dim(S)$ \cite[Theorem 1.1.2]{Eng78}, its codimension in $X \cong \mathbb{R}^n$ is $\geq 2$, hence the points $g_1, g_2$ lie is a continuum in $X$ that is disjoint from $S$ \cite[Theorem 1.8.19]{Eng78}.

If $S$ is closed one can say more, namely, $S \cap X$ is a closed subset of $X$ of dimension $\leq \dim(S) \leq n - 2$, so by Alexander duality $X \setminus S$ is path-connected, giving a path in $X \setminus S$ joining $g_1, g_2$. \hfill $\square$

Proof of Theorem 1.5. Let $q: S_1 \to \mathcal{M}_{k,c}^\leq(S^2)$ denote the continuous surjection sending $u$ to the isometry class of $e^{-2u}g_0$.

If $S$ is countable, it suffices to show that every fiber of $q$ is the union of countably many compact sets because then the complement of a countable subset in $\mathcal{M}_{k,c}^\leq(S^2)$ is the image of $S_1$ with a countable collection of compact subsets removed, which is homeomorphic to $\ell^2$ \cite[Theorem V.6.4]{BP75}, and of course the continuous image of $\ell^2$ is path-connected.

A function $v \in S_1$ lies in the fiber over the isometry class of $e^{-2u}g_0$ if and only if $e^{-2u}g_0 = \psi^* e^{-2u}g_0 = e^{-2u\psi^*} \psi^* g_0$ for some $\psi \in \text{Diff}(\mathbb{R}^2)$. Note that $\psi$ necessarily lies in $\text{Conf}(\mathbb{R}^2)$, the group of conformal automorphism of $\mathbb{R}^2$, i.e. either $\psi$ or $r\psi$ equals $z \mapsto az+b$ for some $a, b \in \mathbb{C}$, where $a \neq 0$ and $r(z) = \bar{z}$. Since $\psi^* g_0 = |a|^2g_0$, we conclude that $v = u \circ \psi - \log |a|$. In summary, $v, u \in S_1$ lie in the same fiber if and only if $v = u \circ \psi - \log |a|$ where either $\psi$ or $r\psi$ equals $z \mapsto az+b$ with $a \neq 0$. Thus the fiber through $u$ is the intersection of $S_1$ and the image of the continuous map $o: \text{Conf}(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ sending $\psi$ to $u \circ \psi - \log |a|$. Since $\text{Conf}(\mathbb{R}^2)$ is a Lie group, it is the union of countably many compact sets, and hence so is its image under any continuous map. Since $S_1$ is closed, every fiber is the union of countably many compact sets.

Now suppose that $S$ is closed and finite dimensional. Let $\hat{S}$ be the $q$-preimage of $S$. Fix two points $g_1, g_2$ in the complement of $S$, which are $q$-images of $u_1, u_2 \in S_1$, respectively. By Theorem 1.3 we may assume that $u_1, u_2$ lie in an embedded copy $\hat{Q}$ of the Hilbert cube. It is enough to show that $\hat{S}$ is finite dimensional in which case $\hat{Q} \setminus \hat{S}$ is path-connected by the proof of Theorem 1.4 in fact, much more is true: the complement to any finite dimensional closed subset of the Hilbert cube is acyclic \cite[Lemma 2.1]{Kro74}. Since $S$ is closed, $\hat{Q} \setminus \hat{S}$ is compact, so the restriction of $q$ to $\hat{Q} \setminus \hat{S}$ is a continuous surjection $\hat{q}: \hat{Q} \setminus \hat{S} \to q(\hat{Q}) \cap S$ of compact separable metrizable spaces, and in particular a closed map, which is essential for what follows.

Since the target of $\hat{q}$ lies in $S$, it is finite dimensional. Finite-dimensionality of the domain of $\hat{q}$ would follow from a uniform upper bound on the dimension of the fibers of $\hat{q}$ \cite[Theorem 1.12.4]{Eng78}. Each fiber lies in the image of the orbit map $o: \text{Conf}(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ described above. Since $\text{Conf}(\mathbb{R}^2)$ is a Lie group, it is the union of countably many $l$-dimensional compact domains (here $l = 4$).
Restricting $o$ to each such domain is a continuous map from a compact space to the Hausdorff space, which is therefore a closed map, and hence the dimension of the target is $< 2l$, which is the sum of the dimension of the domain and the largest dimension of the fiber [Eng78, Theorem 1.12.4]. By the sum theorem for the dimension of a countable union of closed subsets [Eng78, Theorem 1.5.4] the image of $o$ has dimension $< 2l$. □

Remark 4.1. Theorems 1.3–1.5 yield a deformation between any two given metric $g_1, g_2$ that runs in a separable metrizable space, a continuum, or a path, and we now show that this deformation can be arranged to bypass any given set of complete flat metrics $F$. We do so in the setting of Theorem 1.3; the other two proofs are similar. Set $P_0 := S_0 \times \text{Diff}_{0,1}^+(\mathbb{R}^2)$, which we identify with $l^2$. Flat metrics are parametrized by $P_0$ which is a closed linear subset of infinite infinite codimension in $P_1$. If $P'_0 := P_0 \setminus \{\Pi^{-1}_k(g_1), \Pi^{-1}_k(g_2)\}$, then $P'_0$ has property $Z$ in $P'_1$, see [AHW69, Lemma 1], so that [AHW69, Theorem 3] implies that $P_1 \setminus P'_0$ is homeomorphic to $P_1$, after which the proof is finished as in Theorem 1.3.

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Igor Belegradek, School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160

*E-mail address: ib@math.gatech.edu*

Jing Hu, School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160

*E-mail address: jhu61@math.gatech.edu*