Quark mixing sum rules and the right unitarity triangle

Stefan Antusch,1 Stephen F. King,2 Michal Malinský,3 and Martin Spinrath1

1Max-Planck-Institut für Physik (Werner-Heisenberg-Institut) Föhringer Ring 6, D-80805 München, Germany
2School of Physics and Astronomy, University of Southampton, SO17 1BJ Southampton, United Kingdom
3Department of Theoretical Physics, School of Engineering Sciences, Royal Institute of Technology (KTH) – AlbaNova University Center, Roslagstullsbacken 21, 106 91 Stockholm, Sweden

In analogy with the recently proposed lepton mixing sum rules, we derive quark mixing sum rules for the case of hierarchical quark mass matrices with 1-3 texture zeros, in which the separate up and down type 1-3 mixing angles are approximately zero, and $V_{ub}$ is generated from $V_{cb}$ as a result of 1-2 up type quark mixing. Using the sum rules, we discuss the phenomenological viability of such textures, including up to four texture zeros, and show how the right-angled unitarity triangle, i.e., $\alpha \approx 90^\circ$, can be accounted for by a remarkably simple scheme involving real mass matrices apart from a single element being purely imaginary. In the framework of grand unified theories, we show how the quark and lepton mixing sum rules may combine to yield an accurate prediction for the reactor angle.

1. INTRODUCTION

The origin and nature of quark and lepton masses and mixings remains one of the most intriguing questions left unanswered by the standard model (SM) of particle physics. Within the SM, quark and lepton masses and mixings arise from Yukawa couplings which are essentially free and undetermined. In extensions such as Grand Unified Theories (GUTs), the Yukawa couplings within a particular family may be related, but the mass hierarchy between different families is not explained and supersymmetry (SUSY) does not shed any light on this question either. Indeed, in the SM or GUTs, with or without SUSY, a specific structure of the Yukawa matrices has no intrinsic meaning due to basis transformations in flavour space. For example, one can always work in a basis in which, say, the up quark mass matrix is taken to be diagonal with the quark sector mixing arising entirely from the down quark mass matrix, or vice versa, and analogously in the lepton sector (see e.g. [1]). This is symptomatic of the fact that neither the SM or GUTs are candidates for a theory of flavour.

The situation changes somewhat once these theories are extended to include a family symmetry spontaneously broken by extra Higgs fields called flavons. This approach has recently received a massive impetus due to the discovery of neutrino mass and approximately tri-bimaximal lepton mixing [2] whose simple pattern strongly suggests some kind of a non-Abelian discrete family symmetry might be at work, at least in the lepton sector, and, assuming a GUT-type of structure relating quarks and leptons at a certain high energy scale, within the quark sector too. The observed neutrino flavour symmetry may arise either directly or indirectly from a range of discrete symmetry groups [3]. Examples of the direct approach, in which one or more generators of the discrete family symmetry appears in the neutrino flavour group, are typically based on $S_4$ or a related group such as $A_4$ or $PSL(2,7)$ [4]. Models of the indirect kind, in which the neutrino flavour symmetry arises accidentally, include also $A_4$ as well as $\Delta_27$ and the continuous flavour symmetries like, e.g., $SO(3)$ [5] or $SU(3)$ [6] which accommodate the discrete groups above as subgroups [7].

Theories of flavour based on a spontaneously broken family symmetry are constructed in a particular basis in which the vacuum alignment of the flavons is particularly simple. This then defines a preferred basis for that particular model, which we shall refer to as the “flavour basis.” In such frameworks, the resulting low energy effective Yukawa matrices are expected to have a correspondingly simple form in the flavour basis associated with the high energy simple flavon vacuum alignment. This suggests that it may be useful to look for simple Yukawa matrix structures in a particular basis, since such patterns may provide a bottom-up route towards a theory of flavour based on a spontaneously broken family symmetry.

Unfortunately, experiment does not tell us directly the structure of the Yukawa matrices, and the complexity of the problem, in particular, the basis ambiguity from the bottom-up perspective, generally hinders the prospects of deducing even the basic features of the underlying flavour theory from the experimental data. We are left with little alternative but to follow an ad hoc approach pioneered some time ago by Fritzsch [8, 9] and currently represented by the myriads of proposed effective Yukawa textures (see e.g. [10–18]) whose starting assumption is that (in some basis) the Yukawa matrices exhibit certain nice features such as symmetries or zeros in specific elements which have become known as “texture zeros.” For example, in his classic paper, Fritzsch pioneered the idea of having six texture zeros in the 1-1, 2-2, 1-3 entries of the Hermitian up and down quark Yukawa (or mass)
matrices. Unfortunately, these six-zero textures are no longer consistent with experiment, since they imply the bad prediction $|V_{ub}| \sim \sqrt{m_s/m_b}$, so texture zeroists have been forced to retreat to the (at most) four-zero schemes discussed, for example, in [16–18] which give up on the 2-2 texture zeros allowing the good prediction $|V_{ub}| \sim m_s/m_b$.

However, four-zero textures featuring zeros in the 1-1 and 1-3 entries of both up and down Hermitian mass matrices may also lead to the bad prediction $|V_{ub}|/|V_{cb}| \sim \sqrt{m_u/m_c}$ unless $|V_{cb}|$ results from the cancellation of quite sizeable up- and down-type quark 2-3 mixing angles, leading to non-negligible induced 1-3 up- and down-type quark mixing. Another possibility is to give up on the 1-3 texture zeros, as well as the 2-2 texture zeros, retaining only two texture zeros in the 1-1 entries of the up and down quark matrices. Here we reject both of these options, and instead choose to maintain up to four texture zeros without invoking cancellations, for example by making the 1-1 element of the up (but not down) quark mass matrix nonzero, while retaining 1-3 texture zeros in both the up and down quark Hermitian matrices, as suggested in [17].

In this paper we discuss phenomenologically viable textures for hierarchical quark mass matrices which have both 1-3 texture zeros and negligible 1-3 mixing in both the up and down quark mass matrices. Such textures clearly differ from the textures discussed in [16] and [18], but include some cases discussed in [17], as remarked above. Our main contribution in this paper is to derive quark mixing sum rules applicable to textures of this type, in which $V_{ub}$ is generated from $V_{cb}$ as a result of 1-2 up-type mixing, in direct analogy to the lepton sum rules derived in [14, 20]. Another important result of our study is to use the sum rules to show how the right-angled unitarity triangle, i.e., $\alpha \approx 90^\circ$, can be accounted for by a remarkably simple scheme involving real mass matrices apart from a single element of either the up or down quark mass matrix being purely imaginary. Fritzsch and Xing have previously emphasized how their four-zero scheme with 1-1 and 1-3 texture zeros in the Hermitian up and down mass matrices can be used to accommodate right unitarity triangles, but since their scheme involves large 2-3 and non-negligible 1-3 up and down qu mixing, our sum rules are not applicable to their case. Therefore, the textures in Refs. [16] and [18] do not allow us to explain $\alpha \approx 90^\circ$ by simple structures with a combination of purely real and purely imaginary matrix elements. Recently, it has become increasingly clear that current data is indeed consistent with the hypothesis of a right unitarity triangle, with the best fits giving $(\alpha = 90.7^{\pm 4.5}_{\pm 2.9})^\circ$ [21], and this provides additional impetus for our scheme. The phenomenological observation that $\alpha \approx \pi/2$ has also motivated other approaches (see e.g. [22, 23]) which are complementary to the approach developed in this paper.

The layout of the rest of the paper is as follows. In Section 2, we derive the quark mixing sum rules, assuming zero up and down quark 1-3 mixing angles. In Section 3, using the sum rules, we discuss the phenomenological viability of quark mass matrix textures with 1-3 texture zeros, show how modifications in the up sector can achieve viability, and show how $\alpha \approx 90^\circ$ allows each matrix element to be either real or purely imaginary. In Section 4, in the framework of GUTs, we discuss the implications of zero 1-3 mixing for the charged lepton and neutrino sectors, and show that the quark mixing sum rules may be used to yield an accurate prediction for the reactor angle. Finally, Section 5 concludes the paper. Appendix A shows that textures with nonzero 1-3 elements in the up sector are disfavoured.

2. QUARK MIXING SUM RULES FROM NEGLIGIBLE 1-3 UP AND DOWN MIXING

2.1. Conventions

The mixing matrix in the quark sector, the Cabibbo-Kobayashi-Maskawa (CKM) matrix $U_{CKM}$, is defined as the unitary matrix occurring in the charged current part of the SM interaction Lagrangian expressed in terms of the quark mass eigenstates. These mass eigenstates can be determined from the mass matrices in the Yukawa sector, namely

$$L_Y = -\bar{u}_L^i (m_u)_{ij} u_R^j - \bar{d}_L^i (m_d)_{ij} d_R^j + \text{H.c.},$$

where $M_u$ and $M_d$ are the mass matrices of the up-type and down-type quarks, respectively. The change from the flavour into the mass basis is achieved via bi-unitary transformations

$$V_{ur} M_u V_{ur}^\dagger = \text{diag}(m_u, m_c, m_t),$$

$$V_{dr} M_d V_{dr}^\dagger = \text{diag}(m_d, m_s, m_b),$$

where $V_{ur}$, $V_{ur}$, $V_{dl}$ and $V_{dr}$ are unitary $3 \times 3$ matrices. The CKM matrix $U_{CKM}$ (in the “raw” form, i.e. before the “unphysical” phases were absorbed into redefinitions of the quark mass eigenstate field operators) is then given by

$$U_{CKM} = V_{ur} V_{dr}^\dagger.$$

In this paper we shall use the standard (or so-called Particle Data Group (PDG) [22]) parameterisation for the CKM matrix (after eliminating the “unphysical” phases) with the structure

$$U_{CKM} = R_{23} U_{13} R_{12},$$

where $R_{23}, R_{12}$ denote real (i.e. orthogonal) matrices, and the unitary matrix $U_{13}$ contains the observable phase $\delta_{CKM}$. For more details, see e.g. [22]. Other alternative parametrisations, motivated by the observation of $\alpha \approx 90^\circ$ (see e.g. [24]) have been suggested, but we prefer to stick to the standard one here.
However, in order to construct the “physical” CKM matrix $U_{CKM}$ in any given theory of flavour one should begin with the “raw” CKM matrix $U'_{CKM}$ defined in Eq. (4), where $V_{uL}$ and $V_{dL}$ on the right-hand side are general unitary matrices. Recall that a generic $3 \times 3$ unitary matrix $V^\dagger$ can be always written in terms of three angles $\theta_{ij}$, three phases $\delta_{ij}$ (in all cases $i < j$) and three phases $\gamma_i$ in the form

$$V^\dagger = U_\mu V_{uL} U_{13} U_{12} \text{diag}(e^{i\gamma_1}, e^{i\gamma_2}, e^{i\gamma_3}),$$

(6)

where the three unitary transformations $U_{23}, U_{13}, U_{12}$ are defined as

$$U_{12} = \begin{pmatrix} c_{12} & s_{12} e^{-i\delta_{12}} & 0 \\ -s_{12} e^{i\delta_{12}} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(7)

(and analogously for $U_{13}, U_{23}$). As usual, $s_{ij}$ and $c_{ij}$ are abbreviations for $\sin \theta_{ij}$ and $\cos \theta_{ij}$, and the $\theta_{ij}$ angles can be always made positive by a suitable choice of the $\delta_{ij}$'s. It is convenient to use this parameterisation for both $V_{uL}^\dagger$ and $V_{dL}^\dagger$, where the phases $\gamma_i$ can immediately be absorbed into the qurk mass eigenstates. Thus, they can be dropped and one is effectively left only with

$$V_{uL}^\dagger = U_{23} U_{uL} U_{13} U_{12}^\dagger \quad \text{and} \quad V_{dL}^\dagger = U_{23} U_{dL} U_{13} U_{12}^\dagger,$$

(8)

where $V_{uL}^\dagger$ involves the angles $\theta_{12}^u$ and phases $\delta_{12}^u$, while $V_{dL}^\dagger$ involves the angles $\theta_{12}^d$ and phases $\delta_{12}^d$. Using Eqs. (4) and (5) $U'_{CKM}$ can be written as

$$U'_{CKM} = U_{12}^\dagger U_{13}^\dagger U_{23}^\dagger U_{12}$$

(9)

On the other hand, $U'_{CKM}$ can be also parametrised along the lines of Eq. (4).

$$U_{CKM} = U_{23} U_{13} U_{12} \text{diag}(e^{i\gamma_1}, e^{i\gamma_2}, e^{i\gamma_3}).$$

(10)

By comparing Eq. (10) to Eq. (5), we see that the angles $\theta_{ij}$ are the standard PDG ones in $U_{CKM}$, and five of the six phases of $U'_{CKM}$ in Eq. (10) may be removed leaving the standard PDG phase in $U_{CKM}$ identified as

$$\delta_{CKM} = \delta_{13} - \delta_{23} - \delta_{12}.$$  

(11)

2.2. Mixing angle sum rules

Let us now suppose that $\theta_{ij}^d = \theta_{ij}^u = 0$. From the SM point of view, this corresponds to just a convenient choice of basis but, as discussed in the Introduction, it becomes a nontrivial assumption at the level of a specific underlying model of flavour. For models where zero 1-3 mixing is realised with flavour symmetries, see e.g. [28]. For $\theta_{ij}^d = \theta_{ij}^u = 0$, Eq. (9) simplifies to

$$U'_{CKM} = U_{12}^\dagger U_{23}^\dagger U_{13} U_{12}^\dagger.$$

(12)

Then, by equating the right-hand sides of Eqs. (10) and (12) and expanding to leading order in the small mixing angles, we obtain the following relations (up to cubic terms in the physical quark mixing angles):

$$\theta_{23} e^{-i\delta_{23}} = \theta_{23}^d e^{-i\delta_{23}^d} - \theta_{23}^u e^{-i\delta_{23}^u},$$  

(13)

$$\theta_{13} e^{-i\delta_{13}} = -\theta_{12}^d e^{-i\delta_{12}^d} (\theta_{23}^d e^{-i\delta_{23}^d} - \theta_{23}^u e^{-i\delta_{23}^u}),$$  

(14)

$$\theta_{12} e^{-i\delta_{12}} = \theta_{12}^d e^{-i\delta_{12}^d} - \theta_{12}^u e^{-i\delta_{12}^u}.$$  

(15)

Let us first consider Eq. (14), which can be transformed into

$$\theta_{13} e^{-i\delta_{13}} = -\theta_{12}^d e^{-i(\delta_{13}^d + \delta_{23})},$$

(16)

where $\theta_{13}$ and $\theta_{23}$ stand for the measurable 1-3 and 2-3 mixing angles in the quark sector, respectively. Taking the modulus of Eq. (16), the 1-2 angle entering the up-sector rotation $(V_{uL})$ in the flavour basis obeys

$$\theta_{12}^u = \frac{\theta_{13}}{\theta_{23}} = \left(4.96 \pm 0.30\right)^\circ.$$  

(17)

where the 1σ errors are displayed [22].

Similarly, combining Eq. (15) with Eq. (16) one receives

$$\theta_{12} = \frac{\theta_{13}}{\theta_{23}} e^{-i(\delta_{13} - \delta_{23}^d - \delta_{12})} = \theta_{12}^d e^{-i(\delta_{12}^d - \delta_{12})}.$$  

(18)

This, together with the identification Eq. (11) gives rise to the quark sector sum rule

$$\theta_{12}^d = \left| \theta_{12} - \frac{\theta_{13}}{\theta_{23}} e^{-i\delta_{CKM}} \right| = \left(12.0^{+0.39}_{-0.22}\right)^\circ.$$  

(19)

which is valid up to higher order corrections. The present best-fit value and the 1σ errors are also displayed. Needless to say, the relations (17) and (19) apply at the scale at which the flavour structure emerges, often close to the scale of Grand Unification. Thus, in principle, the renormalisation group (RG) effects should be taken into account. However, due to the smallness of the mixing angles in the quark sector and the hierarchy of the quark masses, the RG corrections to the above relations are very small and can be neglected to a very good approximation.

2.3. Phase sum rule

It is interesting that, with the 1-2 mixing angles in the up and down sector derived from the physical parameters, the 1-2 phase difference in the up and down
sectors can also be determined. Indeed, combining all three equations (13) and (15), one obtains
\[ \frac{\theta_{13} \theta_{12}}{\theta_{23}} e^{i \delta_{\text{PKM}}} = -\theta_{12}^{u} (\theta_{12}^{d} e^{-i (\delta_{12}^{d} - \delta_{12}^{u})} - \theta_{12}^{u}) \] . (20)

Using Eqs. (17) and (19) we can solve Eq. (20) for \( \delta_{12}^{d} - \delta_{12}^{u} \) and obtain (at 1σ level)
\[ \delta_{12}^{d} - \delta_{12}^{u} = (91.5^{+5.5}_{-4.0})^{\circ} , \] (21)
which is remarkably close to π/2. We emphasise that this is a consequence of zero 1-3 mixing in the up and down sectors, \( \theta_{13}^{u} = \theta_{13}^{d} = 0 \).

We now show that, assuming quark textures with negligible 1-3 up and down quark mixing, corresponding to 1-3 texture zeros for hierarchical quark mass matrices, \( \delta_{12}^{d} - \delta_{12}^{u} \) is approximately equal to \( \alpha \). This comes from the definition of the unitarity triangle angle \( \alpha \):
\[ \alpha = \arg \left( \frac{V_{td} V_{tb}^{*}}{V_{ud} V_{ub}^{*}} \right) = \arg \left( \frac{(s_{12} s_{23} c_{13}) e^{i d_{12}^{u}}}{c_{12} c_{23} s_{13} e^{i d_{12}^{d}}} \right) = \arg \left( 1 - \theta_{12}^{u} \theta_{23}^{d} e^{-i \delta_{12}^{u}} \right) . \] (22)

For the second term in the argument, we can use Eqs. (13), (14) and (15)
\[ \alpha \approx \arg \left( 1 + \frac{\theta_{23} e^{i \delta_{23}} (\theta_{12}^{d} e^{i \delta_{12}^{d} - \theta_{12}^{u} e^{i \delta_{12}^{u}}})}{\theta_{12}^{u} e^{i \delta_{12}^{u}} \theta_{23} e^{i \delta_{23}}} \right) = \arg \left( \theta_{12}^{d} e^{i (\delta_{12}^{d} - \delta_{12}^{u})} \right) = \delta_{12}^{d} - \delta_{12}^{u} . \] (23)

Thus, one can see that the angle \( \alpha \) is nothing but the phase difference \( \delta_{12}^{d} - \delta_{12}^{u} \), corresponding to a very simple phase sum rule
\[ \alpha \approx \delta_{12}^{d} - \delta_{12}^{u} . \] (24)

3. QUARK MASS MATRICES WITH 1-3 TEXTURE ZEROS

3.1. Real/imaginary matrix elements for \( \alpha = 90^{\circ} \)

According to the phase sum rule in Eq. (24), the experimental observation that \( \alpha \approx 90^{\circ} \), or the equivalent determination in Eq. (21), suggests looking at quark mass matrices with 1-3 texture zeros and with \( \delta_{13}^{u} \) or \( \delta_{13}^{d} \) at the special values ±π/2. This would correspond to a set of rather specific textures of the quark mass matrices with, for example, purely imaginary 1-2 elements in either \( M_{u} \) or \( M_{d} \) while the 2-2 elements remain real. For example, in the relation between the phases of the mixing angles and the phases of the matrix elements is discussed.

For instance, the following patterns naturally emerge:
\[ M_{u} = \begin{pmatrix} a_{u} & -i b_{u} & 0 \\ * & c_{u} & d_{u} \\ * & * & e_{u} \end{pmatrix}, \quad M_{d} = \begin{pmatrix} a_{d} & b_{d} & 0 \\ * & c_{d} & d_{d} \\ * & * & e_{d} \end{pmatrix} \] (25)

and/or
\[ M_{u} = \begin{pmatrix} a_{u} & b_{u} & 0 \\ * & c_{u} & d_{u} \\ * & * & e_{u} \end{pmatrix}, \quad M_{d} = \begin{pmatrix} a_{d} & i b_{d} & 0 \\ * & c_{d} & d_{d} \\ * & * & e_{d} \end{pmatrix} . \] (26)

where \( a_{u}, b_{u}, c_{u}, d_{u}, e_{u} \) and \( a_{d}, b_{d}, c_{d}, d_{d}, e_{d} \) are real parameters, and the elements marked by \( \ast \) are irrelevant as long as the hierarchy of the mass matrix is large enough, or, equivalently, as long as the mixing angles in \( V_{uR} \) and \( V_{dR} \) are small. These textures are all phenomenologically viable, and consistent with \( \alpha = 90^{\circ} \), and their simple phase structure provides a post justification of our assumption of 1-3 texture zeros and negligible 1-3 up- and down-type quark mixing. However, the above textures are clearly not the most predictive ones and, for example, do not relate the up and down quark 1-2 mixing angles to masses. This requires additional assumptions, such as additional texture zeros and Hermitian or symmetric matrices, as we now discuss.

3.2. Four-zero textures confront the sum rules

Under the additional assumptions of symmetric or Hermitian mass matrices in the 1-2 block and zero textures in the 1-1 positions of the quark mass matrices, i.e.,
\[ M_{u} = \begin{pmatrix} 0 & b_{u} & 0 \\ b_{u} & c_{u} & d_{u} \\ * & * & e_{u} \end{pmatrix}, \quad M_{d} = \begin{pmatrix} 0 & i b_{d} & 0 \\ \pm i b_{d} & c_{d} & d_{d} \\ * & * & e_{d} \end{pmatrix} \] (27)

we obtain as additional predictions the Gatto-Sartori-Tonin (GST) relations [23] with 1σ errors displayed,
\[ \theta_{12}^{u} = \sqrt{\frac{m_{u}}{m_{c}}} = (2.61^{+0.54}_{-0.46})^{\circ} , \] (28)
\[ \theta_{12}^{d} = \sqrt{\frac{m_{d}}{m_{s}}} = (13.2^{+3.4}_{-3.3})^{\circ} . \] (29)

Here we already see a conflict in the up sector. The prediction for \( \theta_{12}^{u} \) from the sum rule in Eq. (17) is quite different (several σ away) from the GST relation above. That suggests that the texture in the up sector should be modified to be in good agreement with experiment. By contrast the prediction from the sum rule in Eq. (19) for \( \theta_{12}^{d} \) is in very good agreement (within the errors) with the GST result in Eq. (23), and therefore it is quite plausible to keep the simple texture ansatz for the down sector.

Combining Eqs. (25) and (26) with the sum rules in Eqs. (17) and (19), the two relations
\[ \left| \theta_{12} - \frac{\theta_{13}}{\theta_{23}} e^{-i \delta_{\text{PKM}}} \right| = \sqrt{\frac{m_{d}}{m_{s}}} \] (30)
and

\[ \frac{\theta_{13}}{\theta_{23}} = \sqrt{\frac{m_u}{m_c}} \]  

emerge. We emphasize that these results do not hold for the textures in \[18\] where the 2-3 up and down quark mixings are large and the 1-3 up and down quark mixings are non-negligible.

The compatibility of Eq. \[30\] with the experimental results for the down-type quark masses and mixing parameters \[25\] is illustrated in Fig. 1. We note that RG running for the quark masses, as well as their potential SUSY threshold corrections, are very similar for the first two generations and thus cancel out in their ratio. For our estimates, we have considered the running quark masses at the top mass scale \(m_t(m_t)\) \[30\]. \(\delta_{\text{CKM}}\) is extracted for given \(\sqrt{m_d/m_s}\). The solid blue line shows the relation for best-fit values of the parameters while the dashed blue lines indicate the range with 1σ errors included. The dashed horizontal and vertical black lines (and solid black lines) show the 1σ errors (and best-fit values) for \(\delta_{\text{CKM}}\) and \(\sqrt{m_d/m_s}\), respectively. The relation of Eq. \[30\] is well compatible with the present data. Future more precise experimental measurements (for instance at LHCb or \(B\) factories) and, in particular, an improved knowledge on \(m_d\) (e.g. from lattice QCD) are required to test it more accurately.

In the following, we consider some examples of possible modifications to the textures in the up sector which are phenomenologically acceptable, while leaving the successful down sector texture unchanged, and retaining the successful real and imaginary scheme which leads to the right unitarity triangle. As discussed in Appendix A, the idea of relaxing the up quark 1-3 texture zero is disfavoured, so we restrict ourselves to either relaxing the up quark 1-1 texture zero, or relaxing symmetry in the 1-2 up quark sector, as discussed below.

### 3.3. Relating the up quark 1-1 texture zero

One possible modification is to introduce a nonzero element in the 1-1 position of the up quark mass matrix, i.e.

\[ M_u = \begin{pmatrix} a_u & b_u & 0 \\ b_u & c_u & d_u \\ 0 & * & * \end{pmatrix}, \quad M_d = \begin{pmatrix} 0 & ib_d & 0 \\ \pm ib_d & c_d & d_d \\ * & * & c_d \end{pmatrix}. \]  

(32)

As a result, we obtain the up sector relation

\[ m_u = a_u - \frac{b_u^2}{c_u} \]  

(33)

which allows to adjust \(a_u\), which is of the order of the up quark mass, while \(b_u/c_u \approx \theta_{12}^u\) has to be equal to the value obtained in Eq. \[17\] using the sum rule. For the down sector, there is still the successful prediction from Eq. \[29\] leading to the successful sum rule relation of Eq. \[30\]. Furthermore, as discussed in section \[5\] the Dirac phase of the CKM matrix is correct. We note that there exist several variants of the texture. For example, we can choose the 1-2 element of \(M_d\) real, the 1-2 element of \(M_u\) purely imaginary and all the other elements also real. These variants are valid as long as \(b_u/c_u\) is real and \(b_d/c_d\) is purely imaginary, or vice versa.

We emphasize that the elements marked by "*" are irrelevant as long as the hierarchy of the mass matrix is large enough, so they may be replaced by zeros or, if the matrices are Hermitian, the 3-1 elements may be zero while the 3-2 elements are determined by Hermiticity. Since the sum rule in Eq. \[13\] shows that \(V_{cb}\) is determined only by the difference in 2-3 mixing angles in the up and down sectors, it is also possible to set either \(d_u\) or \(d_d\) equal to zero without changing the physical predictions. In this way it is possible to arrive at some of the four-zero textures discussed, for example, in \[17\]. However, we emphasize that here we are additionally assuming the real and imaginary structures consistent with the right unitarity triangle and this was not discussed in \[17\].

### 3.4. Relating the up quark 1-2 symmetry

A second option for a texture consistent with experimental data consists in relaxing the symmetry of the 1-2 block in the up sector, while keeping the texture zero in
more precise by using the sum rule which relates $1$, $2-2$ and $1-3$ texture zeros, this prediction can be made on textures, which are based on Hermitian matrices with $1-$ down-type quark masses. Thus, applying Eq. (19) at low energies, and taking the present experimental data for the quark mixing angles, and for $\theta_{23}^{MNS}$ (taken from [32]), one can make the rather precise prediction

\[ \theta_{13}^{MNS} = (2.84^{+0.22}_{-0.18})^\circ \]  

which gives $\sin^2 \theta_{13}^{MNS} = 0.0025^{+0.0004}_{-0.0003}$ and holds under the assumption of texture zeros in the $1-3$ elements of the mass matrices (or more precisely $\theta_{13}^{u} = \theta_{13}^{d} = \theta_{13}^{\nu} = 0$) and $\theta_{12}^{d} = 3 \theta_{12}^{\nu}$. Of course, Eq. (37) is only a single example out of a larger variety of predictions which may arise in unified flavour models (see e.g. [33]). We emphasise that the main use of Eq. (19) in this context is that it allows to “determine” the down quark mixing $\theta_{12}^{d}$, which is generically involved in relations between quark and lepton mixing angles, from measurable quantities.

We note that in the lepton sector, the RG corrections (see e.g. [34]) can be significant, depending on the absolute neutrino mass scale (and on $\tan \beta$ in a SUSY framework) and other effects such as canonical normalisation on the mixing angles can also be sizeable [20]. Furthermore, relaxing the $1$-$1$ texture zero in the up quark sector may switch on a nonzero $1$-$3$ mixing angle in the neutrino sector via partially constrained sequential dominance [33].

5. CONCLUSIONS

In this paper we have discussed phenomenologically viable textures for hierarchical quark mass matrices which have both $1$-$3$ texture zeros and negligible $1$-$3$ mixing in both the up and down quark mass matrices. Such textures differ from the textures discussed in [16] and [18]. Our main contribution in this paper has been to derive quark mixing sum rules applicable to textures of this type, in which $V_{ub}$ is generated from $V_{cb}$ as a result of $1$-$2$ up-type mixing, in direct analogy to the lepton sum rules derived in [14, 20]. An important result of our study is to show how the right-angled unitarity triangle, i.e., $\alpha \approx 90^\circ$, can be accounted for by a remarkably simple scheme involving real mass matrices apart from a single element of either the up or down quark mass matrix being purely imaginary. The experimental result that $\alpha \approx 90^\circ$ therefore provides an impetus for having hierarchical textures compatible with negligible $1$-$3$ mixing in both the up and down quark mass matrices. This is probably the most important take-home message of this paper.

The quark mixing sum rules in Eqs. (17) and (19) relate the up and down quark $1$-$2$ mixing angles to observable parameters in the CKM matrix. Using these sum rules the four-zero texture with $1$-$1$ and $1$-$3$ texture zeros and a $2$-$1$ symmetric or Hermitian structure, is shown to be viable for the down quark sector but not for the up quark sector. However, it is possible to have four-zero textures compatible with our sum rules by, for example, filling in the up quark $1$-$1$ texture zero, then having Hermitian matrices with either of the $2$-$3$ elements in the up or down sector set equal to zero as in [17]. However, we emphasize that here we are additionally assuming the real and imaginary structures consistent with the right unitarity triangle and this was not discussed in [17].

In the framework of GUTs, it is natural to have $1$-$3$ texture zeros for both the quark and charged lepton sectors, and in such a case we have shown that the quark mixing sum rules may be used to yield an accurate prediction for the reactor angle; see Eq. (37). However, we caution that this prediction is subject to considerable theoretical uncertainty due to the model dependence of the quark-lepton mixing angle relations, RG and canonical normalisation effects, as well as the assumption that the underlying neutrino $1$-$3$ mixing angle is zero. Indeed, relaxing the $1$-$1$ texture zero in the up quark sector will
shown that energy theory of flavour from experimental data. We have provide the only bottom-up way of deducing a high energy theory of flavour via rather simple Yukawa matrix structures are expected. Indeed, the study of simple Yukawa textures may provide the only bottom-up way of deducing a high energy theory of flavour from experimental data. We have shown that $\alpha \approx 90^\circ$ may provide a clue towards such a high energy theory of flavour via rather simple Yukawa matrices involving 1-3 texture zeros whose nonzero elements are either real or purely imaginary. Such patterns could be achieved, in principle, by appropriate alignment of the vacuum expectation values of flavour symmetry breaking flavon fields, and in particular their phases. It would be interesting to build a theory of flavour along these lines.

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Appendix A: Textures with nonzero 1-3 elements are disfavoured

With nonzero 1-3 elements, $\delta_{\text{CKM}}$ depends not only on $\delta_{12}^u - \delta_{12}^d$ but also on other parameters (in particular $\delta_{13}^u$ and $\delta_{13}^d$) and the simple quark mixing sum rules in Eqs. (17) and (19) are no longer valid. Examples of this type of texture include (with real parameter $f_u$)

$$M_u = \begin{pmatrix} 0 & b_u & f_u \\ b_u & c_u & d_u \\ * & * & e_u \end{pmatrix}, \quad M_d = \begin{pmatrix} 0 & ib_d & 0 \\ ib_d & c_d & d_d \\ * & * & e_d \end{pmatrix} \quad (A1)$$

and

$$M_u = \begin{pmatrix} 0 & b_u & if_u \\ b_u & c_u & d_u \\ * & * & e_u \end{pmatrix}, \quad M_d = \begin{pmatrix} 0 & ib_d & 0 \\ ib_d & c_d & d_d \\ * & * & e_d \end{pmatrix} \quad (A2)$$

but also variations with different elements chosen either purely imaginary or real.

We will demonstrate our approach for this case by means of the texture in Eq. (A1). The starting point is here (similar to Eqs. (13)-15)

$$\theta_{23} e^{-i\delta_{23}} = \theta_{23}^{d} e^{-i\delta_{23}^{d}} - \theta_{23}^{u} e^{-i\delta_{23}^{u}}, \quad (A3)$$

$$\theta_{13} e^{-i\delta_{13}} = -\theta_{13}^{d} e^{-i\delta_{13}^{d}} - \theta_{13}^{u} \theta_{23} e^{-i(\delta_{13}^{u}+\delta_{23})}, \quad (A4)$$

$$\theta_{12} e^{-i\delta_{12}} = \theta_{12}^{d} e^{-i\delta_{12}^{d}} - \theta_{12}^{u} e^{-i\delta_{12}^{u}}, \quad (A5)$$

where we have also neglected terms of order $O(\theta_{13} \theta_{12})$.

From our texture ansatz, we know the phases $\delta_{12}^{u/d}$, $\delta_{23}^{u/d}$ and $\delta_{13}$. For the values of $\theta_{12}^{u/d}$, we take the values from the GST relations, i.e. Eq. (20), which hold here because of the zeros in the 1-1 position and the symmetric structure for the first two generations. Then we can calculate $\theta_{23}^{u/d}$ and $\delta_{\text{CKM}}$ in terms of the known quantities and obtain

$$\delta_{\text{CKM}} = \left(78.83 \pm 3.30 \right)^\circ.$$  

This result is several standard deviations away from the measurements.

Beyond the particular example discussed above, we found that the inconsistency of the prediction for $\delta_{\text{CKM}}$ also appears in all other cases with $\delta_{23}$, $\delta_{12}^{u/d}$ and $\delta_{13}^{u/d} \in \{0, \pm \pi/2\}$. Furthermore, the same happens for textures with $f_u = 0$ and $f_d \neq 0$, where $f_d$ denotes the 1-3 element of $M_d$. We conclude that under these conditions textures with nonzero 1-3 elements are disfavoured.

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