BTZ Black Hole with Gravitational Chern-Simons: Thermodynamics and Statistical Entropy

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ABSTRACT

Recently, the BTZ black hole in the presence of the gravitational Chern-Simons term has been studied and it is found that the usual thermodynamic quantities, like as the black hole mass, angular momentum, and entropy, are modified. But, for large values of the gravitational Chern-Simons coupling where the modification terms dominate the original terms some exotic behaviors occur, like as the roles of the mass and angular momentum are interchanged and the entropy depends more on the inner-horizon area than the outer one. A basic physical problem of this system is that the form of entropy does not guarantee the second law of thermodynamics, in contrast to the Bekenstein-Hawking entropy. Moreover, this entropy does not agree with the statistical entropy, in contrast to a good agreement for small values of the gravitational Chern-Simons coupling. Here I find that there is another entropy formula where the usual Bekenstein-Hawking form dominates the inner-horizon term again, as in the small gravitational Chern-Simons coupling case, such as the second law of thermodynamics can be guaranteed. I also find that the new entropy formula agrees with the statistical entropy based on the holographic anomalies for the whole range of the gravitational Chern-Simons coupling. This reproduces, in the limit of vanishing Einstein-Hilbert term, the recent result about the exotic BTZ black holes, where their masses and angular momenta are completely interchanged and the entropies depend only on the area of the inner horizon. I compare the result of the holographic approach with the classical-symmetry-algebra-based approach, and I find exact agreements even with the higher-derivative term of gravitational Chern-Simons. This provides a non-trivial check of the AdS/CFT-correspondence, in the presence of higher-derivative terms in the gravity action.

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I. Introduction

The gravitational Chern-Simons term in Einstein gravity, with a \textit{vanishing} cosmological constant $\Lambda$, produces a propagating, massive, spin-2 mode, although the separate actions do not \cite{1,2}. (This system is known as the “Topologically Massive Gravity” (TMG) in the literatures.) So, a massive object in this theory has the gravitational Chern-Simons \textit{dressing} whose size is governed by the inverse of the graviton’s mass, which is proportional to the coupling constant of the gravitational Chern-Simons term.

Recently, the BTZ black hole system as a \textit{trivial} solution of the gravitational-Chern-Simons-corrected gravity in three-dimensional anti-de Sitter space (AdS) with a negative cosmological constant $\Lambda = -1/l^2$ (I call this the gravitational-Chern-Simons-corrected/dressed BTZ (GCS-BTZ) black hole) has been studied in the context of the higher-derivative/curvature gravities \cite{3,4,5,6,7,8,9}. And, it is found that the usual thermodynamic quantities of the BTZ black hole, like as the black hole’s mass, angular momentum, and entropy are modified as

\begin{align}
M &= m + xj/l, \quad J = j + xlm, \quad (1.1) \\
S &= \frac{2\pi r_+}{4G\hbar} + x\frac{2\pi r_-}{4G\hbar}, \quad (1.2)
\end{align}

which shows some \textit{mixings} between the original BTZ black hole’s mass $m$ and angular momentum $j$, and also some deviation, proportional to the \textit{inner}-horizon’s area, from the usual Bekenstein-Hawking form \cite{10,11} in the entropy \cite{5,7,8,9}. Here, the parameter $x$ is proportional to the gravitational Chern-Simons coupling constant. These modifications would be the results of the gravitational Chern-Simons dressing in the $AdS$ space, which have been absent in the usual topologically massive gravity with $\Lambda = 0$.

But, that does not change much about the physical contents of the usual BTZ black hole when the parameter $x$ is not large enough, more exactly when it is smaller than a critical value of the coupling constant. In fact, there is a good agreement in the entropy (1.2) with the statistical entropy, based on the conformal field theory (CFT) for the Virasoro algebras at the spatially infinite boundary \cite{7,8}, as in the usual BTZ black hole systems \cite{12,13,14}.

However, for large values of the gravitational Chern-Simons coupling where the modification terms dominate the original terms some exotic behaviors occur, like as the roles of mass and angular momentum are interchanged and its black hole entropy depends more on the \textit{inner}-horizon area than the outer one. Actually, similar phenomena have been also known for some time in several other contexts \cite{15} where the masses and angular momenta are \textit{completely} interchanged and the black hole entropies depend \textit{only} on the areas of the inner horizon (I have called these kinds of black holes as the \textit{exotic black holes} \cite{16}), in completely contrast to the Bekenstein-Hawking entropy formula \cite{10,11}. This looks similar to the suggestion in Ref.
But a basic physical problem of those approaches is that the second law of thermodynamics is not guaranteed with their entropy formulae, in contrast to the Bekenstein-Hawking form [10]; actually, without the guarantee of the second law, there is no justification for identifying the entropies, even though they satisfy the first law, with the inner-horizon areas [11]. Moreover, those entropies do not agree with the statistical entropies, in contrast to a good agreement for small values of the gravitational Chern-Simons coupling, though this has not been well known in the literatures.

In the usual system of black holes, the first law of thermodynamics uniquely determines (up to an arbitrary constant) the black hole entropy with a given Hawking temperature $T_+$ and chemical potential for the outer (event) horizon $r_+$. In this context, there is no choice in the entropy of the GCS-BTZ black hole, other than (1.2), which is problematic for large values of $x$.

But recently, I have proposed a new entropy for the case of the exotic black holes [16], which corresponds to the $|x| \to \infty$ limit, such as the entropy has the usual Bekenstein-Hawking form, which is proportional to the area of the outer horizon. And I have found that the new entropy formula agree well with the statistical entropy, based on the CFT at the spatial infinity.

In this paper I argue that the new approach can be generalized to large but finite values of $x$ also: By considering the characteristic angular velocity and temperature as those of the inner horizon, a new entropy formula is found from the first law of thermodynamics. This new entropy agrees well with the statistical entropy. But, for small values of $x$, the system behaves like as an ordinary BTZ black hole with the characteristic angular velocity and temperature as those of the outer horizon with the known entropy formula (1.2), which agrees with the statistical entropy as well. So, I argue that there are two different phases of the GCS-BTZ black hole, depending on its gravitational Chern-Simons coupling constant. For each phase, the second law of the thermodynamics is guaranteed and there are good agreements with the statistical entropies.

The plan of this paper is as follows.

In Sec. II, I consider the thermodynamics of the GCS-BTZ black hole and find the new entropy formula for the large gravitational Chern-Simons coupling $|\tilde{\beta}| > 1$, as well as the usual entropy formula for the small coupling $|\tilde{\beta}| < 1$, from a new re-organization of the first law of thermodynamics. I study also the Smarr formula and find the same form as in the usual BTZ black holes without the gravitational Chern-Simons term.

In Sec. III, the statistical entropy, based on the holographic anomalies, is considered, and I find perfect agreements with the thermodynamic entropies that have been studied in Sec. II,
for the whole range of the gravitational Chern-Simons coupling. The new entropy formula, as well as the ordinary one, is supported by the CFT approach, which is robust in the context of the AdS/CFT correspondence.

In Sec. IV, the classical symmetry algebra approach, based on the Chern-Simons formulation of three-dimensional gravity, is considered for comparison with the holographic anomaly approach of Sec. III, and I find exact agreements between them. This provides a non-trivial check of the AdS/CFT-correspondence, in the presence of higher-derivative terms in the gravity action. In order to ensure that the exact factor matching even with the gravitational Chern-Simons term is a solid result, by carefully fixing the subtleties involving the normalization differences between the different bases and conventions in the literatures, I include some details of the computations and useful formulae in Appendix A.

In Sec. V, I conclude with several discussions.

In Appendix B, I briefly review on the derivation of the Cardy formula and its higher-order corrections for completeness.

I shall omit the speed of light $c$ and the Boltzman’s constant $k_B$ in this paper for convenience, by adopting the units of $c \equiv 1$, $k_B \equiv 1$. But, I shall keep the Newton’s constant $G$ and the Planck’s constant $\hbar$ in order to clearly distinguish the quantum (gravity) effects with the classical ones.

II. Thermodynamics of the GCS-BTZ black hole

A. The BTZ black hole in gravitational-Chern-Simons-corrected gravity

The (2+1)-dimensional gravity with a gravitational Chern-Simons term and a negative cosmological constant $\Lambda = -1/l^2$ is described by the action on a manifold $\mathcal{M}$ [1, 2] [ omitting some possible boundary terms ]

$$I_g = \frac{1}{16\pi G} \int_\mathcal{M} d^3x \sqrt{-g} \left( R + \frac{2}{l^2} \right) + I_{GCS},$$  \hspace{1cm} (2.1)

where the gravitational Chern-Simons term is given by $^3$ [ the Greek letters $(\mu, \nu, \alpha, \cdots)$ denote the space-time indices and Latin $(a, b, c, \cdots)$ denote the internal Lorentz indices; I take the metric convention $\eta_{ab} = \text{diag}(-1, 1, 1)$ for the internal Lorentz indices, and the indices are raised and lowered by the metric $\eta_{ab}$ ]

$$I_{GCS} = \frac{\hat{\beta} l}{64\pi G} \int_\mathcal{M} d^3x \ e^{\alpha\mu} \left( R_{ab\mu\nu} \omega^{ab} + \frac{2}{3} \omega_{c\mu} \omega^{c\nu} \omega^{a\beta} \right).$$  \hspace{1cm} (2.2)

$^3$Note that the dimensionless coupling constant $\hat{\beta} = x$ is related to the one used in Refs. [1, 2] as $\hat{\beta} = -1/(\mu l)$, in Ref. [8] as $\hat{\beta} = -\hat{\beta}_S/l$, and in Ref. [7] as $\hat{\beta} = -32\pi G\beta_{KL}/l$. 

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Here, the spin-connection 1-form \( \omega^a_b = \omega^a_{b\mu} dx^\mu \), \( \omega_{ab\mu} = -\omega_{ba\mu} \) is determined by the torsion-free condition
\[
d e^a + \omega^a_b \wedge e^b = 0 \tag{2.3}
\]
with the dreibeins 1-form \( e^a = e^a_\mu dx^\mu \), and the curvature is then \( R_{ab\mu\nu} = \partial_\mu \omega_{ab\nu} + \omega_{a\rho}^c \omega_{\rho b\nu} - (\mu \leftrightarrow \nu) \). [I take the same definitions as in Ref. [7] for the curvature 2-form \( R_{ab} = (1/2) R_{ab\mu\nu} dx^\mu \wedge dx^\nu \) and the spin-connection 1-form \( \omega_{ab} \). Some useful formulae are summarized in Appendix A.] Note that \( I_{GCS} \) is of the third-derivative order, rather than the second as in the Einstein-Hilbert term, so this is the first higher-derivative correction in three-dimensional spacetimes.

The resulting equations of motion, by varying \( I_g \) of (2.1) with respect to the metric\( ^4 \), are
\[
R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - \frac{1}{l^2} g^{\mu\nu} = \hat{\beta} l C^{\mu\nu}, \tag{2.4}
\]
where the Cotton tensor \( C^{\mu\nu} \) is defined by
\[
C^{\mu\nu} = \frac{1}{\sqrt{-g}} \epsilon^{\mu\rho\sigma} \nabla_\rho (R^\nu_\sigma - \frac{1}{4} \delta^\nu_\sigma R), \tag{2.5}
\]
which is traceless and covariantly conserved \( [1] \). From the fact that the Einstein equation (2.4) gives a constant curvature scalar \( R = -6/l^2 \), the equation (2.4) can be further reduced to
\[
R^{\mu\nu} = \frac{2}{l^2} g^{\mu\nu} + \hat{\beta} l C^{\mu\nu} = \frac{2}{l^2} g^{\mu\nu} + \frac{2}{l^2} \epsilon^{\mu\rho\sigma} \nabla_\rho R^\nu_\sigma. \tag{2.6}
\]

It would be a non-trivial task to find the general black hole solutions for the third-derivative-order equations\( ^5 \). However, there is a trivial solution, e.g., the BTZ solution because it satisfies the equation (2.6) trivially with \( C^{\mu\nu} = \epsilon^{\mu\rho\sigma} \nabla_\rho (R^\nu_\sigma - \frac{1}{4} \delta^\nu_\sigma R)/\sqrt{-g} = 0 \) \( [3] \). This looks like a too-trivial situation which does not have any higher-derivative effect of the gravitational Chern-Simons term. But actually this is not the case, as we will see, since there are some non-trivial shifts in the physical parameters of the black hole \( [5, 7, 8] \); actually, the BTZ solution is rich enough to show some genuine effect of the gravitational Chern-Simons term. So, I concentrate hereafter only the BTZ solution, which is given by the metric \( [19] \)
\[
ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (d\phi + N^\phi dt)^2 \tag{2.7}
\]
\(^4\)The variations of \( I_{GCS} \) depends only the metric, though it does not look clear at first sight, as \( \delta I_{GCS} = (l^3/8\pi G) \int_M d^4x \sqrt{-g} C^{\mu\nu}\delta g_{\mu\nu} \) \( [1, 7] \).
\(^5\)Recently, a non-trivial two-parameter family of black hole solutions have been found \( [18] \), but it does not seem that its properties have been fully elucidated yet.
with

\[ N^2 = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l^2 r^2}, \quad N^\phi = -\frac{r_+ r_-}{l r^2}. \]  

Here, \( r_+ \) and \( r_- \) denote the outer and inner horizons, respectively.

In the absence of the gravitational Chern-Simons term, the conserved mass and angular momentum of the black hole are given by

\[ m = \frac{r_+^2 + r_-^2}{8Gl^2}, \quad j = \frac{2r_+ r_-}{8Gl}, \]  

respectively. Note that these parameters satisfy the usual mass/angular momentum inequality \( m^2 \geq j^2/l^2 \), in order that the horizon exists or the conical singularity is not naked and the equality holds for the extremal black hole, where the inner and outer horizons overlap.

But, in the presence of the gravitational Chern-Simons term, it has been found that these “bare” parameters are shifted as (1.1), i.e.,

\[ M = m + \hat{\beta} j/l, \quad J = j + \hat{\beta} m, \]  

respectively and these modifications would be the results of gravitational Chern-Simons term in \( AdS \) space. One remarkable result of these modifications is that the usual mass/angular momentum inequality is not valid generally

\[ M^2 - J^2/l^2 = (1 - \hat{\beta}^2)(m^2 - j^2/l^2), \]  

but it depends on the values of the gravitational Chern-Simons coupling constant \( \hat{\beta} \): For small values of coupling \( |\hat{\beta}| < 1 \), the usual inequality is preserved, i.e., \( M^2 \geq J^2/l^2 \); however, for the large values of coupling \( |\hat{\beta}| > 1 \), one has an anomalous inequality with an exchanged role of the mass and angular momentum as \( J^2/l^2 \geq M^2 \); on the other hand, at the critical value \( |\hat{\beta}| = 1 \), the modified mass and angular momentum are “always” saturated, i.e., \( M^2 = J^2/l^2 \), regardless of inequality of the bare parameters \( m \) and \( j \).

**B. Black hole thermodynamics**

Since the solution (2.7) has the same form of the metric as the usual BTZ solution, it has the same form of the Hawking temperature and angular velocity of the outer (event) horizon \( r_+ \) as the BTZ also

\[ T_+ = \left. \frac{\hbar \kappa}{2\pi} \right|_{r_+} = \frac{\hbar (r_+^2 - r_-^2)}{2\pi l r_+^2}, \quad \Omega_+ = \left. -N^\phi \right|_{r_+} = \frac{r_-}{l r_+} \]  

\(^6\text{This has been computed in several different approaches, e.g., the super-angular momentum’s in Ref. [18], the quasi-local method’s in Ref. [5], the ADM’s in Ref. [20], the holography’s in Refs. [7, 8]. But, they all give the same result.}\)
with the surface gravity function $\kappa = \partial N^2 / (2 \partial r)$.

Now, by considering the first law of thermodynamics as

$$\delta M = \Omega_+ \delta J + T_+ \delta S$$

with $T_+$ and $\Omega_+$ as the characteristic temperature and angular velocity of the system, one can easily determine the black hole entropy as

$$S = \frac{2\pi r_+}{4G\hbar} + \hat{\beta} \frac{2\pi r_-}{4G\hbar}.$$  \hspace{1cm} (2.14)

There is no other choice in the entropy in this usual context [15, 8, 9]. In fact, this has been computed also in rather formal contexts, like as the Euclidean method of conical singularity [8] and Wald’s formalism [9], but the same entropy has been obtained.

However, an inherent problem of all those approaches is that there is no general proof about the second law of thermodynamics when higher-derivative/curvature terms are included in general [21]. In our case of (2.14), there are two contributions: One is the usual Bekenstein-Hawking term

$$S_{BH} = \frac{2\pi r_+}{4G\hbar};$$

which guarantees the second law from Hawking’s area theorem [10, 11], which saying the increase of the area of the outer horizon $A_+ = 2\pi r_+$. Another term is proportional to the inner-horizon area $A_- = 2\pi r_-$ and this comes from the gravitational Chern-Simons term. But, in this second part, the second law would be questionable since some of the basic assumptions for the Hawking’s area theorem, i.e., cosmic censorship conjecture might not be valid for the inner horizon, in general. Moreover, the usual instability of the inner horizon makes it difficult to apply the Raychaudhuri’s equation to get the area theorem, even without worrying about other assumptions for the theorem; actually, this seems to be the situation that really occurs in our GCS-BTZ black holes also [22, 23].

But, there is a novel situation where the total entropy (2.14) still satisfies the second law, though all its constituents do not. This is the case where the usual Bekenstein-Hawking term dominates the exotic term proportional to $A_-$. Since $r_+ \geq r_-$ is always satisfied, this condition is equivalent to $|\hat{\beta}| < 1$. Actually, this is the case where the usual mass/angular momentum inequality holds, as I have shown in the previous sub-section II.A, the system behaves as an ordinary BTZ black hole, though there are some shifts in the mass, angular momentum, and entropy.

On the other hand, for large values of coupling $|\hat{\beta}| > 1$, where the exotic term dominates the Bekenstein-Hawking term, the above argument does not guarantee the second law of thermodynamics generally. Then, without the guarantee of the second law of thermodynamics,
there is no justification for identifying entropy (2.14), even though it satisfies the first law of thermodynamics (2.13), and its characteristic temperature and angular velocity have the usual identifications [11].

So, in order to avoid the problem for the large couplings, we need another form of the entropy which is dominated by a term linearly proportional to the outer horizon area \( A_+ \), following the Bekenstein’s general argument [11], which should be valid in my case also. Recently, I have studied the extreme limit \( |\hat{\beta}| \to \infty \) of the system and found that the new entropy formula can be determined by a new re-organization of the first law of thermodynamics; here, I consider its generalization to my case. A crucial fact for the new formulation is by observing the following identities in the BTZ system

\[
\delta m = \Omega_+ \delta j + \frac{(r_+^2 - r_-^2)}{2\pi l^2 r_+} \left( \frac{2\pi \delta r_+}{4G} \right),
\]

\[
= \Omega_- \delta j + \frac{(r_-^2 - r_+^2)}{2\pi l^2 r_-} \left( \frac{2\pi \delta r_-}{4G} \right),
\]

where

\[
\Omega_- = -N^\phi \bigg|_{r_-} = \frac{r_+}{l r_-}
\]

is the angular velocity for the inner horizon; these identities show a symmetry between \( r_+ \) and \( r_- \), which would reflect the symmetry in the metric (2.7), (2.8) and the bare parameters (2.9).

Then, the first identity (2.16) produces the usual first law of thermodynamics with the Hawking temperature \( T_+ \), angular velocity \( \Omega_+ \) for the outer horizon, and Bekenstein-Hawking entropy \( S_{BH} \):

\[
\delta m = \Omega_+ \delta j + T_+ \delta S_{BH}.
\]

The second identity (2.17) is an interesting re-arrangement of the first identity by replacing \( r_+ \) with \( r_- \); this would be remarkable since the first law does not uniquely determine (up to a constant) the black hole entropy, as well as the characteristic temperature and angular velocity, in contrast to usual belief; actually, the second identity (2.17) implies that the system can be also considered as a black hole with the entropy

\[
S_- = \frac{2\pi r_-}{4G\hbar},
\]

which is proportional to the inner-horizon area \( A_- \), and the characteristic temperature\(^7\)

\[
T_- = \frac{\hbar \kappa}{2\pi \bigg|_{r_-}} = \frac{\hbar (r_-^2 - r_+^2)}{2\pi l^2 r_-}
\]

\(^7\)I have used the definition of \( \kappa \) as \( \nabla^\nu (\chi^\mu \chi_\mu) = -\kappa \chi^\nu \) for the horizon Killing vector \( \chi^\mu \) in order to determine its sign, as well as its magnitude.
and angular velocity $\Omega_-$ for the inner horizon:

$$\delta m = \Omega_\delta j + T_\delta S_-. \quad (2.22)$$

Here, the physical relevances of the parameters $T_-$ and $\Omega_-$ are not clear. But, here and below, I use $T_-$, $\Omega_-$ just for convenience in identifying the new entropy, from the “assumed” first law of thermodynamics (2.22).\(^8\)

Now, let me consider, from (2.10),

$$\delta M - \Omega_\delta J = \delta m - \Omega_\delta j + \hat{\beta}(\delta j/l - \Omega_- l \delta m), \quad (2.23)$$

instead of $\delta M - \Omega_+ \delta J$ in (2.13). Then, it is easy to see that the first two terms in the right hand side become $T_- \delta S_-$, by using the second identity (2.17) or from (2.22). And also, the final two terms in the bracket become $T_- \delta S_{BH}$, by using the first identity (2.16) and another identity

$$\Omega_- = \Omega_-^1 l^{-2}. \quad (2.24)$$

So, finally I find that (2.23) becomes a new re-arrangement of the first law as

$$\delta M = \Omega_\delta J + T_- \delta S_{\text{new}}, \quad (2.25)$$

with the new black hole entropy

$$S_{\text{new}} = \frac{2\pi r_-}{4\sqrt{h}} + \frac{\hat{\beta} 2\pi r_+}{4\sqrt{h}}. \quad (2.26)$$

With the above new entropy formula, it is easy to see that the previous argument for the second law of thermodynamics of (2.14) in the small values of coupling $|\hat{\beta}| < 1$ can now be applied to that of (2.26) in the large values of coupling $\hat{\beta} > 1$.

On the other hand, for the large but “negative” values of coupling $\hat{\beta} < -1$, the entropy formula (2.26) would not guarantee the second law of thermodynamics nor the positiveness of the entropy: The entropy would “decrease” indefinitely, with the negative values, as the outer horizon $r_+$ be increased, following the area theorem. But, there is a simple way of resolution from the new form of the first law (2.25). It is to consider

$$S_{\text{new}} \equiv -S_{\text{new}} = -\frac{2\pi r_-}{4\sqrt{h}} - \frac{\hat{\beta} 2\pi r_+}{4\sqrt{h}}, \quad (2.27)$$

$$T_- \equiv -T_- = \frac{\hbar(r_+^2 - r_-^2)}{2\pi l^2 r_-}, \quad (2.28)$$

\(^8\)The positive-valued surface gravity and temperature with $T = |\kappa_-/(2\pi)|$ (as in Ref. [23]) produces an incorrect sign in front of the $TdS$ term in (2.25).
instead of $S_{\text{new}}, T_-$, and actually this seems to be the unique choice: One might consider $S_{\text{new}}'' = \frac{2\pi r_-}{4G\hbar} - \hat{\beta}^2\frac{2\pi r_+}{4G\hbar}$, but then the first law (2.25) is not satisfied.

C. Smarr formula and its universality

So far, I have argued that there are two different phases of the GCS-BTZ black hole, depending on its gravitational Chern-Simons coupling. The physics is quite different in the two phases, having different thermodynamic functions, $T_+, \Omega_+, S$ for $|\hat{\beta}| < 1$ and $T_-, \Omega_-, S_{\text{new}}$ for $|\hat{\beta}| > 1$. But, for each phase, the second law, as well as the (assumed) first law of thermodynamics, is guaranteed.

On the other hand, it is known that the bare BTZ black hole satisfies the three-dimensional Smarr formula [24]

$$m = \frac{1}{2}T_+ S_{BH} + \Omega_+ j.$$  \hspace{1cm} (2.29)

So, an interesting question would be whether this formula is deformed in the presence of the higher-derivative/curvature terms in the action; also, the study of this relation would be important in that it could show some universal characteristics of the system, in connection with other thermodynamic systems which look completely different.

This would be a non-trivial question in the general asymptotically-AdS space [25]. But, in our GCS-BTZ case, the Smarr formula (2.29) is unchanged from some magic of the system. The magic comes, first, from the following identity, in addition to (2.29),

$$m = \frac{1}{2}T_- S_- + \Omega_- j.$$  \hspace{1cm} (2.30)

and this can be considered as another re-arrangement of the three-dimensional Smarr formula (2.29), which has never been considered in the literatures. And also, by considering (2.24) and $T_-\Omega_- = -T_+/l$, one has the identities

$$j/l = \frac{1}{2}T_- S_+ + l\Omega_- m$$  \hspace{1cm} (2.31)

$$= \frac{1}{2}T_+ S_- + l\Omega_+ m.$$  \hspace{1cm} (2.32)

The first and second identities come from (2.29) and (2.30), respectively.

Then, from all these magical identities, one can easily find the following Smarr formulae for the gravitational-Chern-Simons-corrected mass and angular momentum, $M$ and $J$, respectively

$$M = \frac{1}{2}T_+ S + \Omega_+ J,$$  \hspace{1cm} (2.33)

$$M = \frac{1}{2}T_- S_{\text{new}} + \Omega_- J$$  \hspace{1cm} (2.34)

$$= \frac{1}{2}T_- S_{\text{new}}' + \Omega_- J,$$  \hspace{1cm} (2.35)
by considering (2.29) and (2.32), and (2.30) and (2.31), respectively. Here, (2.29) and (2.33) describe the black holes with $|\hat{\beta}| < 1$ since $T_+^+$ and $\Omega_+^+$ are considered as the characteristic parameters of the system. Similarly, (2.30) and (2.34) describe those with $|\hat{\beta}| > 1$.

So, I have found that the two Smarr formulae (2.29) and (2.30) extend to the GCS-BTZ black hole with the corrected $M$, $J$, and the entropies $S$, $S_{\text{new}}$, or $S_{\text{new'}}$. However, it is not clear whether the covariance of Smarr formula is just a result of the speciality of the gravitational Chern-Simons term or there are other deep reasons.

III. Statistical entropy: The holographic anomaly approach

In the usual context of the AdS/CFT correspondence [26], the central charges for the CFT on the asymptotic AdS boundary are identified by evaluating the anomalies of the CFT effective action, from the regularized bulk gravity action [27, 28, 29].

Recently, the approach has been applied to the action (2.1), and it is found that there are anomalies in the expectation values of the boundary stress tensor $T_{ij} = 2\delta I_0[\gamma_{ij}]/\sqrt{-\gamma}\delta\gamma_{ij}$, for the boundary metric $ds^2 = \gamma_{ij}dx^i dx^j \simeq -r^2 dx^+ dx^-$ with $r$ taken to infinity,

$$\langle T_{++}(x^+) \rangle = -\frac{\hbar \hat{c}^+}{24\pi}, \quad \langle T_{--}(x^+) \rangle = -\frac{\hbar \hat{c}^-}{24\pi},$$  \hspace{1cm} (3.1)

with the central charges [ I follow the conventions of [29] ]

$$\hat{c}^\pm = \gamma^\pm \frac{3l}{2G\hbar}$$  \hspace{1cm} (3.2)

with $\gamma^\pm = 1 \pm \hat{\beta}$ for the right/left-moving sectors with the superscripts + and −, respectively. Here, I have defined $\hat{c}^\pm \sim O(\hbar^{-1})$ as the quantum-mechanical central charges of the boundary CFT, due to (quantum) anomaly $\langle 2\gamma_{ij} \delta \log Z/\sqrt{-\gamma}\delta\gamma_{ij} \rangle = -\hat{c} R^{(2)}/(24\pi) \sim O(\hbar^{-1})$ such as they contain the Planck’s constant $\hbar$, intrinsically. However, the bulk computation of the holographic anomaly has no $\hbar$ because it just uses the classical action and equations of motion, e.g., $T_{\pm\pm} = -c^\pm/(24\pi) \sim O(1)$, with classical numbers $c^\pm \sim O(1)$. Note that the quantum-mechanical central charges $\hat{c}^\pm$ defined in this way have the correct “1/$\hbar$”-factor for the semiclassical black hole entropy, like as the Bekenstein-Hawking entropy (2.15), via the Cardy formula [30, 31].

It seems that this 1/$\hbar$-factor is closely related to that of the black hole entropy in Euclidean action approach [32], where 1/$\hbar$-factor occurs from the zero-loop approximation of the effective action $\log Z \approx -I_{\text{eff}}\hbar^{-1}$.

By considering (3.1) as the anomalous transformations of the boundary stress tensors under the diffeomorphism $\delta x^\pm = -\xi^\pm(x^\pm)$,

$$\delta_{\xi^+} T_{++} = 2\partial_+ \xi^+ T_{++} + \xi^+ \partial_+ T_{++} - \frac{\hbar \hat{c}^+}{24\pi} \partial_+^2 \xi^+$$
\[ \delta_{\xi^{-}}T_{--} = 2\partial_{-}\xi^{-}T_{--} + \xi^{-}\partial_{-}T_{--} - \frac{\hbar\hat{c}^{-}}{24\pi}\partial^{3}\xi^{-} = \frac{1}{i}[T_{--}, \hat{L}^{-}[\xi^{-}]] \]  

(3.3)

with the generators

\[ \hat{L}^{\pm}[\xi^{\pm}] = \frac{1}{\hbar}\int dx^{\pm}T_{\pm\pm}\xi^{\pm}(x^{\pm}) + \frac{\hat{c}^{\pm}}{24}, \]  

(3.4)

one can obtain a pair of quantum Virasoro algebras

\[ [\hat{L}^{\pm}_{m}, \hat{L}^{\pm}_{n}] = (m - n)\hat{L}^{\pm}_{m+n} + \frac{\hat{c}^{\pm}}{12}m(m^2 - 1)\delta_{m+n,0} \]  

(3.5)

for a monochromatic basis \( \xi^{\pm} = e^{imx^{\pm}} \) with the integer numbers \( m, n \). Here I note that this reduces to the usual result for the holographic conformal anomaly in the \( \hat{\beta} \to 0 \) limit [27, 28, 29], whereas \( \hat{\beta} \)-dependent terms come from the holographic gravitational anomaly due to the gravitational Chern-Simons term [7, 8].

Now, let me consider the ground state Virasoro generators, expressed in terms of the black hole’s mass and angular momentum:

\[ \hat{L}^{\pm}_{0} = \frac{lM \pm J}{2\hbar} + \frac{\hat{c}^{\pm}}{24} = \frac{\gamma^{\pm}(lm \pm j)}{2\hbar} + \frac{\hat{c}^{\pm}}{24}. \]  

(3.6)

With the Virasoro algebras of \( \hat{L}^{\pm}_{m} \) in the standard form, which are defined on the plane, one can use the Cardy formula for the asymptotic states [30, 33, 31, 34, 35]

\[ \log \rho(\hat{\Delta}^{\pm}) \simeq 2\pi \sqrt{\frac{1}{6} \left( \hat{c}^{\pm} - 24\hat{\Delta}_{\text{min}}^{\pm} \right) \left( \hat{\Delta}^{\pm} - \frac{\hat{c}^{\pm}}{24} \right)}, \]  

(3.7)

where \( \hat{\Delta}^{\pm} \) are the eigenvalues, called conformal weights, of the operator \( \hat{L}_{0} \) for black-hole quantum states \( |\hat{\Delta}^{\pm}\rangle \), and \( \hat{\Delta}_{\text{min}}^{\pm} \) are their minimum values. Here, I note that the above Cardy formula, which comes from the saddle-point approximation of the CFT partition function on a torus, is valid only if the following two conditions are satisfied:

\[ \frac{24\hat{\Delta}_{\text{eff}}^{\pm}}{\hat{c}_{\text{eff}}^{\pm}} \gg 1, \]  

(3.8)

\[ \frac{\hat{c}_{\text{eff}}^{\pm}}{\hat{\Delta}_{\text{eff}}^{\pm}} \gg 1, \]  

(3.9)
where \( \hat{\Delta}^\pm_{\text{eff}} = \Delta^\pm - \hat{c}^\pm / 24 \) and \( \hat{c}^\pm_{\text{eff}} = \hat{c}^\pm - 24\hat{\Delta}^\pm_{\text{min}} \) are the effective conformal weights and central charges, respectively; from the first condition, the higher-order correction terms are exponentially suppressed as \( e^{-2\pi \epsilon^\pm (\hat{\Delta}^\pm \pm \hat{\Delta}^\pm_{\text{min}})} \) with \( \epsilon^\pm \equiv 24 \hat{\Delta}^\pm_{\text{eff}} / \hat{c}^\pm_{\text{eff}} \); from the second condition, the usual saddle-point approximation is reliable, i.e., \( \rho(\hat{\Delta}^\pm) \) dominates in the partition function (see Appendix B for the details).

Then, the statistical entropy for the asymptotic states becomes

\[
S_{\text{stat}} = \log \rho(\hat{\Delta}^+) + \log \rho(\hat{\Delta}^-) = \frac{\pi}{4G\hbar} |\gamma^+(r_+ - r_-)| + \frac{\pi}{4G\hbar} |\gamma^-(r_+ - r_-)|
\]

\[
= \frac{\pi}{4G\hbar} (|\gamma^+| + |\gamma^-|)r_+ + \frac{\pi}{4G\hbar} (|\gamma^+| - |\gamma^-|)r_-, \quad (3.10)
\]

where I have chosen \( \hat{\Delta}^\pm_{\text{(min)}} = 0 \), as usual \([12, 13, 14]\); from (3.6), this corresponds to the \( \text{AdS}_3 \) vacuum solution with \( m = -1/(8G) \) and \( j = 0 \), in the usual context, but it has a permanent rotation as well, in the new context \([7]\),

\[
M = -\frac{1}{8G}, \quad J = -\frac{l\hat{\beta}}{8G}. \quad (3.11)
\]

Note that the correct “1/\( \hbar \)”-factor for the semiclassical black hole entropy comes from the appropriate recovering of \( \hbar \) in (3.2) and (3.6). According to the conditions of validity (3.8), (3.9), this entropy formula is valid only when both of the two conditions

\[
(r_+ \pm r_-) \gg l, \quad (3.12)
\]

\[
(r_+ \pm r_-) \gg \hbar G \quad (3.13)
\]

are satisfied. The usual semiclassical limit of large black hole (area), in which the back-reaction of the emitted radiation from the black hole is neglected \([36]\) and so the thermodynamic entropy formula (2.14) and (2.26) from the first law can be reliable, agrees with the condition (3.13). So, there would be no obstacles to compare the statistical entropy (3.10) with the thermodynamical one. Note that from another condition (3.12) we are considering a more restricted class of black hole systems\(^9\), though this does not seem to be needed, in general.

Now, let me consider the following four cases, depending on the values of \( \hat{\beta} \): (a). \( |\hat{\beta}| < 1 \), (b). \( \hat{\beta} > 1 \), (c). \( \hat{\beta} < -1 \), and (d). \( |\hat{\beta}| = 1 \).

(a). \( |\hat{\beta}| < 1 \): In this case, I have \( |\gamma^\pm| = \gamma^\pm \), and the statistical entropy (3.10) becomes

\[
S_{\text{stat}} = \frac{2\pi r_+}{4G\hbar} + \hat{\beta} \frac{2\pi r_-}{4G\hbar} \quad (3.14)
\]

\(^9\)At this state, the condition of large central charges \( \hat{c}^\pm \gg 1 \), i.e., \( l \gg \hbar G \) \([12]\), which would be related to the leading supergravity approximation of AdS/CFT correspondence \([26]\), is not needed yet.
from $\gamma^+ + \gamma^- = 2$, $\gamma^+ - \gamma^- = 2\hat{\beta}$. This agrees exactly with the usual entropy formula (2.14).

And, this is the case where $\hat{c}^\pm$ and $\hat{\Delta}^\pm - \hat{c}^\pm/24$ are positive definite such as the Cardy formula (3.7) has a well-defined meaning. In the gravity side also, it shows the usual behavior with the “positive” mass and angular momentum, satisfying the normal inequality $M^2 \geq J^2/l^2$.

(b). $\hat{\beta} > 1$: In this case, I have $|\gamma^+| = \gamma^+$, $|\gamma^-| = -\gamma^-$, and so the statistical entropy (3.10) becomes

$$S_{\text{stat}} = \frac{2\pi r_-}{4G\hbar} + \hat{\beta} \frac{2\pi r_+}{4G\hbar}. \quad (3.15)$$

This agrees exactly with the new entropy formula (2.26), which guarantees the second law of thermodynamics even in this case. And, this is the case where there is some abnormal change of the role of the mass and angular momentum due to $J^2/l^2 \geq M^2$, even though $M$ and $J$ both are positive definite, as usual. Moreover, in the CFT side also, this is not the usual system because $\hat{c}^- = \gamma^- 3l/(2G\hbar)$ and $\hat{\Delta}^- - \hat{c}^-/24 = \gamma^- (m l - j)/2\hbar$ are negative valued, though their self-compensations of the negative signs produce the real and positive statistical entropy. The application of the Cardy formula to the case of negative $\hat{c}^-$ and $\hat{\Delta}^- - \hat{c}^-/24$ might be questioned, due to the existence of negatives-norm states with the usual condition $\hat{L}^+_n |\hat{\Delta}^-\rangle = 0$ ($n > 0$) for the highest-weight state $|\hat{\Delta}^-\rangle$. However, this problem can be easily cured by considering another representation of the Virasoro algebra with $\hat{L}^-_n \equiv -\hat{L}^-_{-n}$, $\hat{c}^- \equiv -\hat{c}^-$, and $\hat{L}^-_n |\hat{\Delta}^-\rangle = 0$ ($n > 0$) for the new highest-weight state $|\hat{\Delta}^-\rangle$ [37]; this implies that the Hilbert space need to be “twisted” in which the whole states vectors be constructed from the twisted highest-weight state $|\hat{\Delta}^+\rangle \otimes |\hat{\Delta}^-\rangle$. The formula (3.10), which is invariant under this substitution, should be understood in this context. On the other hand, if I take the limit $\hat{\beta} \to \infty$, in which there is only the gravitational Chern-Simons term, this becomes the “exotic” black hole system that occur in several different contexts [15, 16]; however, note that this can not be obtained from (3.14).

(c). $\hat{\beta} < -1$: In this case, I have $|\gamma^+| = -\gamma^+$, $|\gamma^-| = \gamma^-$, and the statistical entropy (3.10) becomes

$$S_{\text{stat}} = -\frac{2\pi r_-}{4G\hbar} - \hat{\beta} \frac{2\pi r_+}{4G\hbar}. \quad (3.16)$$

Note that this is positive definite, and this should be the case from its definition $S_{\text{stat}} = \log(\rho(\hat{\Delta}^+_n)\rho(\hat{\Delta}^-_n)) \geq 0$ for the number of possible states $\rho(\hat{\Delta}^+_n) \geq 1$. This agrees exactly with the new entropy formula (2.27), which guarantees the second law. And, this is the case where $M$ can be negative and $J$ has the opposite direction to the bare one $j$, in contrast to the positive definite $M$ and $J$ in the cases of (a) and (b), as well as the anomalous inequality $J^2/l^2 \geq M^2$. In
the CFT side, $\hat{c}^+$ and $\hat{\Delta}^+ - \hat{c}^+/24$ become negative-valued, now, and I need to twist this right-moving sector, rather than the left-moving one as in the case of (b), $\hat{L}_n^+ \equiv -\hat{L}_{-n}^+$, $\hat{c}^+ \equiv -\hat{c}^+$, and $\hat{L}_{n}^+ |\hat{\Delta}^+\rangle = 0 \, (n > 0)$ for the twisted highest-weight state $|\hat{\Delta}^+\rangle \otimes |\hat{\Delta}^-\rangle$.

(d). $|\hat{\beta}| = 1$: In this case, one of $\gamma^\pm$ vanishes, i.e., $\gamma^+ = 0$, $\gamma^- = 2$ for $\hat{\beta} = -1$, and $\gamma^+ = 2$, $\gamma^- = 0$ for $\hat{\beta} = 1$. The statistical entropy becomes

$$S_{stat} = \frac{2\pi}{4G\hbar}(r_+ - r_-) \, \quad (\hat{\beta} = -1),$$

(3.17)

$$S_{stat} = \frac{2\pi}{4G\hbar}(r_+ + r_-) \, \quad (\hat{\beta} = +1).$$

(3.18)

Note that (3.18) can be reproduced from (3.14) and (3.15), but (3.17) from (3.14) and (3.16). So, statistical entropies (3.17) and (3.18) agree exactly with the usual entropy formula (3.14)~(3.16). As I have remarked previously in Sec. II. (b), this is the case where the mass/angular momentum inequality saturates $M^2 = J^2/l^2$, regardless of $m$ and $j$. In fact, they satisfy

$$M = \pm J/l = \frac{(r_+ \pm r_-)^2}{8G\ell^2} \geq 0$$

(3.19)

for $\hat{\beta} = \pm 1$, respectively. So, for non-extremal bare black holes with $r_+ > r_-$, the mass $M$ is positive definite, but $J$ changes its direction for $\hat{\beta} = -1$. For extremal bare black holes with $r_+ = r_-$, one has $M = J = 0$, as well as $S_{stat} = S = 0$ satisfying the Nernst formulation of the third law of thermodynamics [38] for $\hat{\beta} = -1$, whence $M = J = (G\hbar/(2\pi\ell^2))S_{stat} = r_+^2/(2G\ell^2) > 0$ without satisfying the third law for $\hat{\beta} = 1$, as in all other cases of (a)~(c) and in the usual Kerr black hole [39]. But, there are some subtleties about this in the fully corrected entropies; see Sec. V about this issue.

In summary, I have found exact agreements between the thermodynamical black hole entropies which have been evaluated in the bulk (AdS) gravity side and the CFT entropies in the asymptotic boundary, for the whole range of the coupling constant $\hat{\beta}$. So, the new entropy formula for the strong coupling $|\hat{\beta}| > 1$ seems to be supported by the CFT approach also. This reveals the AdS/CFT correspondence in the sub-leading order with the higher-derivative term of gravitational Chern-Simons, as well as in the leading order with the Einstein-Hilbert action.

IV. Comparison with a classical symmetry algebra approach: Exact agreements

There is an alternative approach to compute the Virasoro algebras with central charges. This is based on the classical symmetry algebras of the asymptotic isometry of $AdS_3$ [40, 41,
\[\{L^\pm_m, L^\pm_n\} \approx i(m-n)L^\pm_{m+n} + \frac{ie^\pm}{12}m(m^2 - 1)\delta_{m+n,0}, \quad (4.1)\]

with “classical” central charges \(c^\pm\) and the Dirac bracket \(\{\ , \\}^\ast\ [45]\).

It is well known that there is an exact agreement with the anomaly based approaches of Sec. III by the mapping \[31\], with the appropriate recovering of \(\hat{h}, \hat{c}^\pm, \hat{L}^\pm = c^\pm, L^\pm\ [31\).]

\[\hat{c}^\pm = \frac{c^\pm}{\hbar}, \quad \hat{L}^\pm_m = \frac{L^\pm_m}{\hbar}, \quad (4.2)\]

in the absence of the gravitational Chern-Simons term \[27, 28, 29\] \(10\). So, the statistical entropy agrees with the Bekenstein-Hawking entropy also. But, this is a quite non-trivial fact, and actually this provides an explicit check of the AdS/CFT correspondence by comparing the classical data \((c^\pm, L^\pm)\), which can be directly computed, with the quantum data \((\hat{c}^\pm, \hat{L}^\pm)\) in the anomaly approach, which can be identified only indirectly through the (conjectured) AdS/CFT-correspondence.

So, it would be interesting to consider the classical approach in the presence of the gravitational Chern-Simons term also and compare with the results from the anomaly approach of Sec. III in order to see whether they both agree or not. This would provide a non-trivial check of the AdS/CFT-correspondence beyond the Einstein-Hilbert action; there are some works already in this direction \[4, 49, 50\], but there are several aspects which should be clarified.

There are two “classically” equivalent approaches for this purpose. These are the purely gravity approach of Brown-Henneaux \[40\] and Chern-Simons (CS) approach. Here, let me consider the latter approach since it is easier and provides some explicit computations of the symmetry generators and their Dirac brackets of \((4.1)\) even far from asymptotic boundary, which are not available in the former approach. Moreover, it can reveal the holographic phenomena explicitly and the novel boundary effects to the derivative of Dirac delta function, which are the mathematical origin of the classical central terms \[43\].

### A. Chern-Simons gauge theory with boundaries

It is well known that CS (gauge) theory with boundaries produces central terms in Virasoro algebras, as well as in Kac-Moody algebras, even at “classical” level; this has been first spelled \(10\)The classical algebra with the higher curvature terms was computed in Ref. \[46\] by transforming the gravity action with the higher curvature terms into the usual Einstein-Hilbert action with some auxiliary tensor matter fields. The same central charges and Virasoro generators have been obtained in the anomaly approach also recently \[6\]. But the validity of Ref. \[46\] is unclear since there would be non-trivial contributions in the generators \(L^\pm_m\) and central charges from the matter fields in general \[14, 47\], though the agreement seems to be plausible in the context of AdS/CFT \[48, 6\].
out in [41], but rigorously computed later in [43, 44]. This is a general field theoretic result only if some appropriate boundary conditions are satisfied, regardless of the physical contents of the CS theory. Moreover, this is not an artifact of a “classical” theory, but persists even in quantum theory because it can not be removed from some quantum effects due to normal orderings [44].

So, if a theory can be expressed as the CS theory with the appropriate boundary conditions, one can quickly identify the Kac-Moody and Virasoro algebras with the classical central terms. This is actually the case of three-dimensional Einstein gravity with a cosmological constant $\Lambda$, where the usual BTZ black hole or the three-dimensional Kerr-de Sitter solutions ($KdS_3$) are admitted, depending on the sign of $\Lambda$ [41, 43].

The generalization of this approach to some more general class of gravity systems, i.e., with matter couplings [14] or with higher curvature terms [46, 6, 9] would not be possible, in general. But, the three-dimensional gravity with a gravitational Chern-Simons term is an exceptional case since the gravitational Chern-Simons term itself can also be expressed as the CS theory for another choice of the invariant quadratic forms of the Lie algebra, for a non-vanishing $\Lambda$ [51]; on the other hand, for the case of $\Lambda = 0$, the quadratic forms are not well defined since they are degenerate. So, for the most general form of the invariant quadratic forms which admit the new choice for the gravitational Chern-Simons action as well, one can express the Einstein gravity with the gravitational Chern-Simons term and non-vanishing $\Lambda$ as a CS gauge theory [51, 4].

Moreover, in the GCS-BTZ black holes, there is no difference in the metric form, though there are some shifts in the ADM mass and angular momentum, and so there is no difference in the boundary conditions for the corresponding CS theory; however, this would not be valid generally for other non-trivial solutions in which there are some important deformations of the metric itself.

Hence, all the previous results about the bare BTZ black hole can be applied to the GCS-BTZ case also, from the general results of the Kac-Moody and Virasoro algebras for the CS theory.

**B. $SO(2,2)$ Chern-Simons gravity with the gravitational Chern-Simons term**

For the (2+1)-dimensional space with a negative cosmological constant $\Lambda = -1/l^2$, symmetry of the space is $SO(2,2)$, which has the following commutation relations among the generators of the Lie group

$$[J_a, J_b] = \epsilon_{ab}^\ c J_c, \quad [J_a, P_b] = \epsilon_{ab}^\ c P_c, \quad [P_a, P_b] = \frac{1}{l^2} \epsilon_{ab}^\ c J_c. \quad (4.3)$$
The most general form of the invariant quadratic forms are [51, 4]

\[
\langle J_a, J_b \rangle = \alpha \eta_{ab}, \quad \langle J_a, P_b \rangle = \beta \eta_{ab}, \quad \langle P_a, P_b \rangle = \frac{\alpha}{l^2} \eta_{ab}.
\] (4.4)

Here, \(\alpha\) and \(\beta\) are some arbitrary constants, but the ratio of \(\langle J_a, J_b \rangle\) and \(\langle P_a, P_b \rangle\) are completely fixed by the algebras (4.3).

The algebras (4.3) and the quadratic forms (4.4) look unusual. But, if I introduce

\[
J^\pm_a = \frac{1}{2} (J_a \pm l P_a),
\] (4.5)

(4.3) and (4.4) become

\[
[J^\pm_a, J^\pm_b] = \epsilon_{ab}^c J^\pm_c, \quad [J^\pm_a, J^\mp_b] = 0,
\] (4.6)

\[
\langle J^\pm_a, J^\pm_b \rangle = \frac{1}{2} (\alpha \pm \beta l) \eta_{ab}, \quad \langle J^\pm_a, J^\mp_b \rangle = 0.
\] (4.7)

This is the usual form of the \(SL(2, \mathbb{R}) \times SL(2, \mathbb{R})\) Lie algebra but with different values of the quadratic forms of the two sectors.

Now, by considering the Lie-algebra-valued one-form

\[
A = \omega^a J_a + e^a P_a = A^+ + A^-,
\]

\[
A^\pm = (\omega^a \pm \frac{e^a}{l}) J^\pm_a
\] (4.8)

with the triads \(e^a = e^a_\mu dx^\mu\) and the spin connections \(\omega^a = (1/2) \epsilon^{abc} \omega_{\mu bc} dx^\mu\) \(^{11}\), the CS action becomes \(\langle A \wedge B \rangle\) is understood as \(\langle A \wedge B \rangle\), up to some boundary terms,

\[
I_{CS}[A] = \frac{k}{4\pi} \int_M \left\langle A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\rangle
\]

\[
= \frac{k}{4\pi} \Omega^+ \int_M Tr \left( A^+ \wedge dA^+ + \frac{2}{3} A^+ \wedge A^+ \wedge A^+ \right) - (+ \leftrightarrow -)
\]

\[
= \frac{k\beta}{\pi} \int_M Tr \left( e \wedge R + \frac{1}{3} e \wedge e \wedge e \right)
\]

\[
+ \frac{k\alpha}{2\pi} \int_M Tr \left( \omega \wedge \left( d\omega + \frac{2}{3} \omega \wedge \omega \right) + \frac{e}{l^2} \wedge T \right),
\] (4.9)

where \(\Omega^\pm = \beta l \pm \alpha\), \(Tr(J^\pm_a J^\pm_b) = (1/2) \eta_{ab}\) and

\[
R = d\omega + \omega \wedge \omega
\]

\[
= \frac{1}{2} F^a_{\mu \nu} dx^\mu \wedge dx^\nu
\]

\(^{11}\)The definition depends on the signature of the internal metric \(\eta_{ab}\). Our formulae are the case where the number of negative signatures is odd. For more details about my conventions, see Appendix A.
\begin{align}
  \frac{1}{2} e^a_{\alpha} e^b_{\beta} R^\alpha_{\beta \nu \mu} dx^\nu \wedge dx^\mu, \\
  T &= de + 2 \omega \wedge e \\
  &= \frac{1}{2} T^\alpha_{\nu \mu} dx^\nu \wedge dx^\mu \\
  &= \frac{1}{2} e^a_{\alpha} T^\alpha_{\nu \mu} dx^\nu \wedge dx^\mu 
\end{align}

(4.10)

are the curvature and torsion 2-forms, respectively.

The equations of motion of the CS gravity, by treating $A^+$ and $A^-$ “independently”, become the usual forms

\begin{align}
  F^\pm &= dA^\pm + A^\pm \wedge A^\pm \\
  &= R + \frac{1}{l^2} e \pm \frac{1}{l} T \wedge e = 0 
\end{align}

(4.11)

or

\begin{align}
  T &= 0, \\
  R + \frac{1}{l^2} e \wedge e = 0, 
\end{align}

(4.12)

where I have chosen the boundary conditions [41, 42, 44, 43], for each time slice,

\begin{align}
  A_0|_{\partial M} \propto A_\varphi|_{\partial M}, \\
  \oint_{\partial M} dtd\varphi \left< A_\varphi, A_\varphi \right> = \text{fixed},
\end{align}

(4.14)

(4.15)

with the boundary action

\begin{align}
  I_S = -\frac{k}{4\pi} \oint_{\partial M} dtd\varphi \left< A_\varphi, A_0 \right>.
\end{align}

(4.16)

Here, I note that the equivalence of the equations (4.11) or (4.12, 4.13) and the Einstein equations (2.4) can be achieved only after solving the torsion-free condition (4.12) first. This should be the case since the spin-connections $\omega$ are not independent variables but are determined by the torsion-free condition (2.3) already. Actually, by plugging (4.12) into the action (4.9), it is a standard computation to show that (4.9) is equivalent to the gravity action (2.1), up to some boundary terms, with the couplings (see Appendix A for details)

\begin{align}
  k\beta = -\frac{1}{4G}, \quad \frac{\alpha}{l^2\beta} = \hat{\beta}.
\end{align}

(4.17)

But at this point, there is one subtlety here: The whole CS equations of motion are not available when one of $\Omega^\pm$’s vanishes and this occurs with $\beta l = \alpha$ or $\hat{\beta} = 2$. In this critical case I have only one sector of the solutions in (4.11), such as the torsion-free condition (4.12)
is not “necessarily” required. So, the equivalence of CS gravity (4.9) with the gravitational-Chern-Simons-corrected gravity (2.1) can not be achieved in this case, generally. However, if I restrict the solution space to the torsion-free ones only, the equivalence is admitted still. This is actually the situation that I consider in this paper since the BTZ solution (2.7) satisfies (4.12) and (4.13), which do not depend on the choice of \( \omega \) or \( e \).

Now, in order to study the black hole solution (2.7), in the context of the CS gravity, it is convenient to introduce a proper radial coordinate \( \rho \), such as (2.7) can be written as

\[
 ds^2 = -\sinh^2 \rho \left( \frac{r_+ dt}{l} - r_- d\phi \right)^2 + l^2 d\rho^2 + \cosh^2 \rho \left( \frac{r_- dt}{l} - r_+ d\phi \right)^2 
\]  

(4.18)

with

\[
 r^2 = r_+^2 \cosh^2 \rho - r_-^2 \sinh^2 \rho. 
\]  

(4.19)

In these coordinates, the (outer) event horizon is at \( \rho = 0 \) and hence this metric describes the exterior of the horizon for real values of \( \rho \), but the interior for imaginary values of \( \rho \). Then, it is easily checked that the 1-form gauge connections are given, in the proper coordinates, by

\[
 A^{\pm 0} = \pm \frac{r_+ + r_-}{l} \left( \frac{dt}{l} \mp d\phi \right) \sinh \rho, 
\]

\[
 A^{\pm 1} = \pm d\rho, 
\]

\[
 A^{\pm 2} = \frac{r_+ + r_-}{l} \left( \frac{dt}{l} \mp d\phi \right) \cosh \rho. 
\]  

(4.20)

[ The superscript indices denote the group indices \( a = 0, 1, 2 \).] In matrix form \(^{13}\), this becomes

\[
 A^\pm = \frac{1}{2} \begin{pmatrix} 
 \pm d\rho & z_\pm e^{\mp \rho} dx^\pm \\
 z_\pm e^{\pm \rho} dx^\mp & \mp d\rho 
\end{pmatrix}, 
\]  

(4.21)

where \( z_\pm \equiv (r_+ \mp r_-)/l \) and \( x^\pm = t/l \pm \phi \). From this, the polar components \(^{14}\) in the proper coordinates can be obtained as

\[
 A^\pm_\rho = \pm J_1, \quad A^\pm_\varphi = \mp z_\pm (U^{-1} J_2 U), \quad A^\pm_t = \mp l A^\pm_\varphi 
\]  

(4.22)

\(^{12}\)Note that the sign convention of \( \phi \) differs from Ref. [41], such as it agrees with the original BTZ metric (2.7) [19]. This agrees also with Refs. [7, 8, 9, 29].

\(^{13}\)I take \( J_0 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), \( J_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), \( J_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), and \( \epsilon_{012} = 1 \) as in Ref. [41]. The final results about the Virasoro algebras, however, do not depend on the choice of the representation.

\(^{14}\)Here, \( A_\rho = \hat{\rho}^i A_i \), \( A_\varphi = \hat{\varphi}^i A_i \), for the orthogonal unit vectors \( \hat{\rho}, \hat{\varphi} \) on the spatial boundary \( \partial \Sigma \) with \( M = \Sigma \times \bar{R} \); \( \Sigma \) is a 2-dimensional disc of space, and \( \bar{R} \) is a 1-dimensional infinite, real manifold of time.
with
\[ U = \begin{pmatrix} e^{\mp \rho/2} & 0 \\ 0 & e^{\mp \rho/2} \end{pmatrix}. \] (4.23)

Here, I note that this solution satisfies the boundary conditions (4.14) and (4.15) for any radius \( \rho \), such as the solution can be implemented even at the boundary whose radius may be arbitrary, from 0 (at \( r_+ \)) to \( \infty \). And, the condition (4.15) implies the micro – canonical ensemble, from the relation \( \langle A^\pm, A^\pm \rangle \) \( \sim (m \pm j/l) \).

C. Symmetry algebras with classical central terms and statistical entropy

The CS action has diffeomorphism (Diff) symmetries. If there are boundaries, central terms appear in the symmetry algebras, even at the classical level. For a spatial and time-independent Diff:

\[
\begin{align*}
\delta f_x^\mu &= -\delta_k^\mu f_x^{\pm k}, \\
\delta f A_i^{\pm a} &= f^{\pm k} \partial_k A_i^{\pm a} + (\partial_i f^{\pm k}) A_k^{\pm a}, \\
\delta f A_0^{\pm a} &= f^{\pm k} \partial_k A_0^{\pm a},
\end{align*}
\] (4.24)

the Lagrangian of (4.9) transforms as \( \delta f L_{CS} = dX_f^+/dt \) with \( X_f^\pm = (k\Omega^\pm)/(4\pi) \int_\Sigma \int d^2x Tr(f^{\pm \rho} A_\rho^{\pm \varphi} A_\varphi^{\pm \rho}) \) when the boundary conditions “\( A^{\pm a}_{\rho|\partial\Sigma=\text{constant}} \)” is imposed, which is a quite natural choice according to the explicit BTZ solution (4.22).

Then, the conserved Noether charges become

\[
\begin{align*}
Q^\pm (f) &= \frac{k\Omega^\pm}{4\pi} \int_\Sigma d^2x Tr(f^{\pm k} A_k^i e^{ij} F_i^{\pm j}) \\
&\quad - \frac{k\Omega^\pm}{4\pi} \int_{\partial\Sigma} d\varphi Tr(2f^{\pm \rho} A_\rho^{\pm \varphi} A_\varphi^{\pm \rho} + f^{\pm \varphi} A_\varphi^{\pm \rho} A_\rho^{\pm \varphi} + f^{\pm \varphi} A_\rho^{\pm \varphi} A_\varphi^{\pm \rho}) \\
&= Q_B^\pm (f) + Q_S^\pm (f)
\end{align*}
\] (4.25)

with the bulk and boundary parts \( Q_B^\pm (f) \) and \( Q_S^\pm (f) \), respectively; the last constant term, proportional to \( Tr(A_\rho A_\varphi) \), in (4.25) was included to obtain the standard Virasoro central term, with the help of the ambiguities in the definition of Noether charge. These satisfy the Virasoro algebras with classical central terms in Dirac bracket algebras

\[
\{Q^\pm (f), Q^\pm (g)\}^* \approx \{Q_S^\pm (f), Q_S^\pm (g)\}^*
\]

\[
= Q_S([f, g]) - \frac{k\Omega^\pm}{2\pi} Tr(A_\rho^+ A_\rho^-) \int_{\partial\Sigma} d\varphi (f^{\pm \rho} \partial_\varphi g^{\pm \rho} - f^{\pm \rho} \partial_\varphi g^{\pm \rho}),
\] (4.26)

where \([f, g]^k = f^\varphi \partial_\varphi g^k - g^\varphi \partial_\varphi f^k\) is Lie bracket on the boundary circle (\( \partial\Sigma \)).
Under the $Diff$ generated by the Noether charges $Q^\pm(f)$, the gauge fields of (4.22), representing the BTZ black hole, have the transformations

$$
\delta_f A^\pm_\varphi = \frac{1}{2} \left( z^\pm \ e^{\pm \rho} (f^{\pm \rho} \pm \partial_\varphi f^{\pm \varphi}) \right) \ ,
$$

$$
\delta_f A^\pm_\rho = 0.
$$

(4.27)

This implies that the black hole solution (4.22) admits the isometries, i.e., $\delta_f A_i^\pm = 0$ as $\rho \to \infty$ when

$$
f^{\pm \rho}|_{\partial \Sigma} = - \partial_\varphi f^{\pm \varphi}|_{\partial \Sigma}
$$

(4.28)

is satisfied, though not necessarily for arbitrary $\rho$. This exactly agrees, to the leading order, with the asymptotic isometries found by Brown-Henneaux [40]. Contrary to the existence of the central term itself, this result is a purely non-Abelian effect which comes from the off-diagonal parts.

Now, by substituting (4.28) with the insertion of $Tr(A^\pm_\rho A^\pm_\rho) = 1/2$ for the black hole solution (4.22), the algebras (4.26) become the standard Virasoro algebras, in the coordinate space, with classical central charges

$$
c^\pm = -12k\Omega^\pm Tr(A^\pm_\rho A^\pm_\rho) = \gamma^\pm \frac{3l}{2G}
$$

(4.29)

with $\gamma^\pm = 1 \pm \hat{\beta}$. In the $\hat{\beta} \to 0$ limit, these classical central charges reduce to the usual result of Brown-Henneaux [40] for the asymptotic isometry of $AdS_3$ and also agrees exactly with that of conformal anomaly computation [27, 28, 29]. But interestingly, the $\hat{\beta}$-dependent central charges give an exact agreement also with the semi–classical central charges

$$
\hat{c}^\pm = \frac{c^\pm}{h}.
$$

(4.30)

as in (4.2), that have been obtained from gravitational anomaly computation; this seems to be a quite non-trivial result since I don’t see any general proof about the equivalence of the two central charges even without the gravitational Chern-Simons term though it seems to be quite plausible in the context of AdS/CFT correspondence, which identifies the “classical” asymptotic CFT of AdS space on the one hand with the “quantum-mechanical” CFT on the boundary on the other hand.

\[15\] There are several other ways to implement the $Diff$ even for the finite values of $\rho$ [52], where there are some $RG$–flows of the central charges and conformal weights without changing the statistical entropies. So, there remains the question on the very place where the black hole’s degrees of freedom live.
The more familiar momentum-space Virasoro algebras (4.1) can be obtained by defining the boundary parts of the Noether charges in (4.25) as

\[ Q^\pm_S(f) \equiv \frac{1}{2\pi} \oint_{\partial \Sigma} d\varphi f^\pm \varphi \left( \sum_n L_n^\pm e^{in\varphi} \right) \]  

(4.31)

and the central charges are given by (4.29). The ground state generators, from the definition, become

\[ L_0^\pm = -\frac{k\Omega^\pm}{4\pi} \oint_{\partial \Sigma} d\varphi \, Tr(A_\varphi^+ A_\varphi^+ + A_\rho^+ A_\rho^+) = \gamma^\pm \frac{1}{2} (lm \pm j) + \frac{c^\pm}{24}. \]  

(4.32)

Note that the \( \hat{\beta} \)-dependent terms, as well as \( \hat{\beta} \)-independent terms, agree exactly with \( \hat{L}_0^\pm = L^\pm / \hbar \) of (3.6). So, if I define the black hole’s mass and angular momentum \textit{canonically} as in (3.6), from the general consideration of CFT on the torus [53],

\[ L_0^\pm = \frac{lM \pm J}{2} + \frac{c^\pm}{24} \]  

(4.33)

one obtains the same mass and angular momentum as in the anomaly approach [7, 8], which agree with the usual ADM quantities of (2.10) [5, 20] also. It does not seem that this is not just a coincidence but there be some deep reasons involving the \textit{holographic} principle; however, our CFT computation of the statistical entropy does not depend on the manners of identifications of \( M \) and \( J \) but only on the geometrical quantities of \( r_+ \) and \( r_- \), such as the CFT computation provides a quite independent estimation of the \textit{would-be} black hole entropy.

Now with the Virasoro algebras with “classical” data of the central charges (4.29) and the ground state generator \( L_0^\pm \) in (4.32), it is straightforward to obtain the corresponding quantum Virasoro algebras [31]: If I consider the canonical quantization

\[ \{L_m^\pm, L_n^\pm\} = i\hbar \{L_m^\pm, L_n^\pm\}^* \]  

(4.34)

for the quantum operators \( L_m^\pm \) and a rescaling transformation

\[ L_m^\pm \rightarrow \hbar (\hat{L}_m^\pm : +h a^\pm \delta_{m,0}) \]  

(4.35)

for the normal ordered operators \( :\hat{L}_m^\pm : \) with some possible normal ordering constants \( a^\pm \), one can easily find the corresponding quantum Virasoro algebras

\[ [:\hat{L}_m^\pm : :\hat{L}_m^n :] = (m - n) :\hat{L}_{m+n}^\pm : + \frac{\hat{c}_\text{tot}^\pm}{12} m(m^2 - 1) \delta_{m,-n} \]  

(4.36)

with

\[ \hat{c}_\text{tot}^\pm = \frac{c^\pm}{\hbar} + c^\pm_{\text{quant}}. \]  

(4.37)
Here, the quantum correction $c_{\text{quant}}^{\pm}$ is due to some operator re-orderings and it is order of $O(1)$.

With the Virasoro algebras of $\hat{L}_m$ in the standard form, which is defined on the plane, one can use the Cardy formula for the asymptotic states [31, 34, 35] as in (3.7)

$$\log\rho(\hat{\Delta}^{\pm}) \simeq 2\pi \left[ \frac{1}{6} \left( \frac{c^{\pm}_{\text{tot}}}{24} - 24 \hat{\Delta}^{\pm}_{\text{min}} \right) \left( \hat{\Delta}^{\pm} - \frac{c^{\pm}_{\text{tot}}}{24} \right) \right],$$

where $\hat{\Delta}^{\pm}$ are the eigenvalues of $\hat{L}_0^{\pm}$: for the black-hole quantum states $\mid \hat{\Delta}^{\pm} \rangle$, and $\hat{\Delta}^{\pm}_{\text{min}}$ are their minimum values. When expressed in terms of the classical generators $L_0^{\pm}$ and the central charges $c^{\pm}$ through

$$\hat{\Delta}^{\pm} = \frac{L_0^{\pm}}{\hbar} - \hbar a^{\pm},$$

one obtains

$$\log\rho(L_0^{\pm}) \simeq 2\pi \frac{1}{\hbar} \left[ \frac{1}{6} \left( c^{\pm} - 24L_0^{\pm}_{\text{min}} + \hbar c_{\text{quant}}^{\pm} + 24\hbar^2 a^{\pm} \right) \left( L_0^{\pm} - c^{\pm}_{24} - \frac{\hbar c_{\text{quant}}^{\pm}_{24}}{24} - \hbar^2 a^{\pm} \right) \right].$$

This approach shows explicitly how the classical Virasoro generators $L_0^{\pm}$ and central charges $c^{\pm}$ can give the correct order of the semiclassical Bekenstein-Hawking entropy (2.15),

$$S_{\text{BH}} \simeq \frac{A_+}{4\hbar G},$$

with $\sqrt{c^{\pm}L_0^{\pm}} \sim A_+/G$; the quantum corrections due to reordering give the negligible order of $O(1)$ effect to the entropy when one considers the large black holes with $A_+/(\hbar G) \gg 1$.

Then, the statistical entropy for the asymptotic states becomes [omitting the small quantum corrections of the order of $O(1)$]

$$S_{\text{stat}} = \log\rho(L_0^{+}) + \log\rho(L_0^{-}) = \frac{\pi}{4\hbar G}(|\gamma^+| + |\gamma^-|)r_+ + \frac{\pi}{4\hbar G}(|\gamma^+| - |\gamma^-|)r_-, \quad (4.42)$$

where I have chosen $L_0^{\pm}_{\text{(min)}} = 0$, which corresponds to the $AdS_3$ vacuum solution with $m = -1/(8G)$ and $j = 0$, in agreement with (3.10). This has exact matchings with (3.10) in the $\beta$-dependent correction terms, as well as $\beta$-independent terms. I note also that the “$1/\hbar$”-factor in the black hole entropy (4.42) was generated in the process of canonical quantization of the classical Virasoro algebras.

So, the statistical entropy, based on the classical symmetry algebras, agrees with the thermodynamic black hole entropy even in the correction terms due to the gravitational Chern-Simons term, as well as the usual one for the Einstein-Hilbert action. This might a subtle issue because
of some normalization differences between the different bases and conventions in the literatures. Actually, there are ubiquitous factor “2” differences between different bases. So, I have included some details about the transformations of the formulae between the different bases and conventions in the Appendix A in order to ensure that this exact factor matching is a solid result, actually.

V. Summary and discussions

I have studied the thermodynamics of BTZ black hole in the presence of the higher-derivative corrections of the gravitational Chern-Simons term and its solid connection with the statistical approaches, based on the holographic anomalies and the classical symmetry algebras.

The main results are as follows:

First, for the case of large coupling $|\hat{\beta}| > 1$ the new entropy formula is proposed from the purely thermodynamic point of view such as the second law of thermodynamics be guaranteed.

Second, I have found supports of the proposal from the CFT based approaches which reproduce the new entropy formulae for $|\hat{\beta}| > 1$, as well as the usual entropy formula for the small coupling case of $|\hat{\beta}| \leq 1$.

Third, I have found the exact “factor” matchings between the holographic anomaly approach and the classical symmetry algebra approach from the Chern-Simons formulation of the three-dimensional gravity. This would provide a non-trivial check of the AdS/CFT-correspondence in the presence of higher-derivative terms in the gravity action.

Now, several comments are in order.  

1. On the general validity of Cardy formula with higher-derivative/curvature corrections: It is interesting to note that the statistical entropy (3.10) from the Cardy formula (3.7) has basically the same form for both the Einstein-Hilbert action and the gravitational-Chern-Simons-corrected action; the only changes are some correction terms in the central charges and conformal weights themselves, rather than considering the higher-order corrections to the Cardy formula as in Ref. [35]. This seems to be true even in the presence of higher-“curvature” terms [46, 6, 9] and also in the supersymmetric black holes [55]. So, there should exist some explanations about this and actually this is the case. This comes from the fact that the higher-derivative/curvature actions do not necessarily imply the quantum corrections though the converse can be true [56]. So, if the higher-derivative/curvature gravities are treated semi-classically by neglecting the back-reaction effects, which are quantum effects, such as (3.9) or (3.13) is satisfied, the saddle- point approximation for the Cardy formula (3.7) and so the entropy formula (3.10) are good approximations, even with the higher-derivative/curvature terms.

\footnote{This has been proved more rigorously and extensively in Ref. [54].}
in the gravity action [35]. There is another factor whose departure from unity is order of \( O[\exp\{-2\pi \hat{\Delta}^\pm (\hat{\Delta}^\pm - \hat{\Delta}^\pm_{\text{min}}) / \hat{c}_{\text{eff}} \}] \), but this correction, if there is, is not comparable with the leading term (3.10) and other higher-order corrections, by departing the semi-classical limit of (3.13); in our case of the GCS-BTZ black holes there is already the corrections of order of \( O(r_-/r_+) \) in the leading entropy (3.10), but this dominates the exponentially suppressed corrections. Hence, the leading Cardy formula (3.7) would have quite general validity for any kinds of semiclassical black holes in the higher-derivative/curvature gravities unless the condition (3.8) or (3.12) is violated.

2. Higher-order corrections to the saddle-point approximation: By relaxing the semiclassical condition of (3.9) or (3.13) but keeping only the condition (3.8) or (3.12), the higher order corrections in the Cardy formula (3.7) can be evaluated by the steepest descent method, known as the Rademacher expansion [57, 58]. The statistical entropy then becomes, up to fourth order, (see Appendix B for the details) from (B.12)

\[
S_{\text{stat}}^{(4)} = (S_0^+ + S_0^-) - \frac{3}{2} \log(S_0^+ S_0^-) + \log(\hat{c}_{\text{eff}}) + \log(\pi^3/18) - \frac{3}{8} \left( \frac{1}{S_0^+} + \frac{1}{S_0^-} \right) + O((S^\pm)^{-2})
\]

\[
= S_{\text{stat}} - \frac{3}{2} \log \left( \frac{\pi}{G\hbar} \right)^2 |\gamma^+ \gamma^-|(r_+^2 - r_-^2) + \log \left( \gamma^+ \gamma^- \left( \frac{3l}{2G\hbar} \right)^2 \right) + \log(\pi^3/18)
\]

\[
- \frac{3}{8} \left( \frac{G\hbar}{\pi} \right)^2 \frac{S_{\text{stat}}}{|\gamma^+ \gamma^-|(r_+^2 - r_-^2)} + O((G\hbar)^2/r_+^2, (G\hbar)^2/r_-^2),
\]

(5.1)

where \( S_0^\pm \) denote the right/left-moving parts of the leading entropy formula (3.10), i.e., \( S_0^+ = \log \rho(\hat{\Delta}^\pm) \) with \( S_0^+ + S_0^- = S_{\text{stat}} \) and this is the expansion about the Planck constant \( \hbar \). It would be a challenging problem to compute the loop-corrected black hole entropies in the gravity side also and compare with the above CFT result (5.1). Actually, the loop corrections in the gravity side would not be trivial in this case since there would be now some propagating mode(s) with the gravitational Chern-Simons term [1, 2, 59], in contrast to the usual BTZ black hole [35, 60].

2\( \frac{1}{2} \). Subtleties of extremal and near-extremal black holes: If I consider extremal bare black holes with \( r_+ = r_- \), i.e., \( \hat{\Delta}^-_{\text{eff}} = 0 \), which saturates the mass bound \( m = j/l \) and has vanishing temperatures, there seem to exist some subtleties in the above manipulations. Namely, the condition (3.13) does not apply and the back-reaction effect would not be negligible anymore in this case, such as I would need to consider “infinite” higher-order corrections in the steepest-descent approximations, which seems to be highly divergent from (5.1); other infinite series of exponential correction terms are actually of the form \( O((\hat{\Delta}^\pm - \hat{\Delta}^\pm_{\text{min}})^m (\hat{c}_{\text{eff}})^n \exp\{-2\pi \hat{\Delta}^\pm (\hat{\Delta}^\pm - \hat{\Delta}^\pm_{\text{min}}) / \hat{c}_{\text{eff}} \}) \) with some positive integers \( m \) and \( n \) [35] such as the problematic part does not contribute further. But, actually this is not quite correct as can be seen easily in the original partition function (B.1). In the case of extremal bare black holes, the left-moving sector is
absent in the partition function because of $\hat{L}_0 - \hat{c}^- / 24 = 0$ such as total partition function is given by, from (B.12),

$$S_{\text{stat}(4): \text{extreme}} = \frac{2\pi r_+}{4Gh} |\gamma^+| - \frac{3}{2} \log \left( \frac{2\pi r_+}{4Gh} |\gamma^+| \right) + \log \left( \frac{\gamma^+ 3l}{2Gh} \right) + \frac{1}{2} \log(\pi^3/18)$$

$$- \frac{3}{8} \left( \frac{4Gh}{2\pi r_+} \right) \frac{1}{|\gamma^+|} + O((Gh)^2/r_+^2).$$

(5.2)

This gives the correct Bekenstein-Hawking entropy for the leading term as can be also read from (5.1) and there is no divergence in each order\(^\dagger\). This implies that, in the “near-extremal” case, the naive divergence in each term of (5.1) would cancel each other and one would have only some finite entropy. Actually, this seems to be supported also by the exact Raedmacher expansion which shows that the exact entropy with all higher-order corrections is bounded by, up to some exponentially suppressed terms, the Bekenstein-Hawking entropy, i.e., $0 \leq S_{\text{exact}} < S_{\text{BH}}$ [61]; if there are no cancelations, the exact entropy $S_{\text{exact}}$ would easily violate the above Birmingham-Sen’s bound. On the other hand, it is important to note that the condition for the right-moving sector only can be satisfied, though not possible for the left-moving sector, such as the extremal bare black hole with a vanishing temperature does not always imply the necessity of the higher-order corrections; however, its relevance to the back-reaction effect is not clear [36].

On the other hand, the case of critical coupling $|\hat{\beta}| = 1$, which has the extremal bound $M^2 = J^2/l^2$ but a non-vanishing temperature, has similar subtleties. In this case, one of $\gamma^\pm$ vanishes such as the condition (3.8) would be ambiguous, even though overall $\gamma^\pm$ factor can be canceled for a non-vanishing $\gamma^\pm$. And, the condition (3.9) can not be satisfied either such as its entropy has similar divergence problem from (5.1), as in the bare extremal black holes. The resolution is similar to the bare-extremal black hole, and the appropriate statistical entropies are given by, from (B.12),

$$S_{\text{stat}(4): \hat{\beta} = \pm 1} = \frac{2\pi (r_+ \pm r_-)}{4Gh} - \frac{3}{2} \log \left( \frac{2\pi (r_+ + r_-)}{4Gh} \right) + \log \left( \frac{2 \cdot 3l}{2Gh} \right) + \frac{1}{2} \log(\pi^3/18)$$

$$- \frac{3}{8} \left( \frac{4Gh}{2\pi (r_+ + r_-)} \right) + O((Gh)^2/(r_+^2 r_-^2))$$

(5.3)

for $\hat{\beta} = \pm 1$ and these agree with the entropies (3.17, 3.18) in the leading order. But, if I consider the extremal bare black holes further with $r_+ = r_-$, the entropy for $\hat{\beta} = 1$ case reduces to (5.2), whereas that for $\hat{\beta} = -1$ case has divergent higher order terms with the vanishing

\(^\dagger\)Interestingly, the factor “3/2” in the logarithmic term agrees with the corresponding corrections in the induced WZW model at the horizon within the context of CS gravity, in contrast to the factor “2” mismatches in the non-extremal black holes [35]. But, it is subtle to compare this with the purely gravity manipulation since there is no clear way to resolve a similar divergence problem.
entropy in the leading term. This subtleties can be resolved again in the original partition function language; there, the right-moving sector is absent, i.e., $L_0^+ - c^+ / 24 = 0$ due to $\gamma^+ = 1$, whereas the left-moving sector is also absent, i.e., $L_0^- - c^- / 24 = 0$ due to $r_+ = r_-$ such as one has only a single ground state with $\rho(\hat{\Delta}^+, \hat{\Delta}^-) = 1$; this system satisfies the Nernst formulation of the third law of thermodynamics [38], i.e., $S_{\text{stat}} = \log \rho = 0$, to all orders!

3. **Probing inside the outer horizon by the gravitational Chern-Simons action**: Although there are some solid supports from the second law of thermodynamics and the CFT approaches, the inner horizon’s data, which are required in the complete formulae, look strange still; of course, the necessity of the inner horizon’s data seems to be a quite general feature with quantum corrections from the result of (5.1), but the problem is that it occurs even at the leading, classical level. Actually, this would be much strange in the Euclidean method of conical singularity [8] or in the Wald’s approach to compute the black hole entropy, which gives the same entropy formula with the inner-horizon term even though it is given by some integrals over the outer horizon [8, 62]. So, understanding the roles of the inner horizon’s data appearing in the thermodynamics relations would be a challenging problem; some possible probing, in the context of the AdS/CFT, beyond the event horizon have been considered recently [22, 23, 63], but this need further studies.

4. **Classical (in)stability of the $|\hat{\beta}| > 1$ black holes**: For the large coupling of $|\hat{\beta}| > 1$, the black-hole angular momentum is greater than its mass $J^2/l^2 \geq M^2$, and there are three known cases which show this “exotic” property, including the gravitational Chern-Simons case, in $D = 3$ and 5 [15, 16]. There are no similar black hole solutions in $D = 2$ and 4, as far as I know. In $D \geq 6$, the “ultra-spinning” black holes are possible in Einstein gravity [64], but it seems that there is a classical instability under small perturbations [65]. So, it would be interesting to investigate this classical (in)stability in our exotic cases also; there might exist some topological reasons for this, but it is not clear in our case since there are propagating modes also [59], in contrast to the ordinary BTZ black hole and $KdS_3$ solution [43].

5. **The first law of thermodynamics vs. Hawking radiation for $|\hat{\beta}| > 1$**: A difficult problem of the new entropy formula for the case of large coupling $|\hat{\beta}| > 1$ is that it requires rather unusual characteristic temperature $T_- = \kappa/(2\pi)|_{r_-}$, which is negative-valued (for $\hat{\beta} > 1$), or $T_-' = -T_-$ (for $\hat{\beta} < -1$), and angular velocity $\Omega_-$, which is the inner-horizon angular velocity in BTZ if I “assume” the first law of thermodynamics. The negative-valued temperature might be understood from the existence of the upper bound of mass in (2.11), i.e., $M \leq J/l$ with positive $M$ and $J$, as in the spin-systems [66]. However, the very meanings of $T_-, T_-'$ and $\Omega_-$ in the Hawking radiation are not clear since the radiation spectrum is determined by the metric alone, in the standard analysis initiated by Hawking [67]. So, *does this work imply that two black holes with identical BTZ metrics will emit radiation with different spectra, one a black*
body spectrum corresponding to a positive temperature $T_+$ for the weak coupling of $|\hat{\beta}| \leq 1$ and one a very non-black-body spectrum corresponding to a negative temperature $T_-$ for the strong coupling of $|\hat{\beta}| > 1$? Or, does this imply that the first law of thermodynamics is “not” satisfied for $|\hat{\beta}| > 1$?

This would be a difficult question whose complete answer is still missing. But here, I would like to only mention the possible limitation of the standard approach in this system and how this might be circumvented. An important point for this is that dynamical geometry responds differently under the emission of Hawking radiation. For example, the emission of energy $\omega$ would reduce the black holes’s mass $M$ from the conservation of energy, but this corresponds to the change of the angular momentum $j$, as well as $m$, in the ordinary BTZ black hole context, due to the mixing of the mass and angular momentum as in (2.10). This is in sharp contrast to the case of ordinary black hole. This seems to be a key point to understand the Hawking radiation in our system, and in this argument the conservations of energy and angular momentum, which are not well enforced in the standard computation, would have a crucial role. So, in this respect, the Parikh and Wilczek’s approach [68], which provides a direct derivation of Hawking radiation as a quantum tunneling by considering the global conservation law naturally, would be an appropriate framework to study the problem. Currently this is under study.

6. Green’s function method and thermal equilibrium: The Green’s function methods for determining the temperature of a black hole require an equilibrium with matter at the corresponding temperature [69]. This work now implies that the analysis, for the strong coupling of $\hat{\beta} > 1$, assumes such an equilibrium with “some exotic surrounding matter” which has a negative temperature$^{18}$, with an upper bound of energy levels as in spin systems: Otherwise, i.e., with the ordinary surrounding matter, the negative temperature black hole can not be at equilibrium with positive temperature surroundings since an object with a negative temperature is hotter than one with any positive temperature.

I suspect that this would be rather easy to understand in our case from the fact that in the AdS space the artificial container is not needed in order to study the canonical (or grand-canonical) ensemble [72, 73]. But, in the context without the explicit container, there is a critical angular velocity [73] at which the action of the black hole or the partition function of its corresponding CFT diverges. However, I note that the critical value is the same as the lower bound of $\Omega_-$, such as we are beyond the critical point with our angular velocity $\Omega_-$. So, from this fact, it seems clear that the ensemble, if there is, in this strong coupling regime would be quite different from that of the usual BTZ black hole such as one can not simply apply the

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$^{18}$The determination of the equilibrium temperature from the “fundamental period” in the thermal Green’s function, known as the KMS (Kugo-Martin-Schwinger) condition [70], can be defined without the implicit assumption of a positive temperature, though not quite well-known in the gravity community (see Ref. [71], for example).
usual result to the strong coupling case. It seems that we need an independent analysis for this case.

Appendix A. Conventions and some useful formulae in differential forms

In this appendix, I summarize the conventions and some useful formulae in differential forms used in this paper. I have also included some details about the computations in order to ensure that the exact factor matching, which is directly related to the relation in (4.17) is a quite solid result, regardless of some normalization differences between different bases. I have used the Lorentzian metric for the internal Lorentz indices $\eta_{ab} = \text{diag}(-1, 1, 1)$ and $\epsilon_{012} = -\epsilon^{012} = 1$. [ For the s-negative signatures in the metric generally, a number of formulae will contain the factor of $(-1)^s$ [74, 59, 75]. ]

The invariant quadratic forms for the $\text{SL}(2, \mathbb{R})$ generators are (4.7), i.e.,

$$\langle J^\pm_a, J^\pm_b \rangle = \frac{1}{2} (\alpha \pm \beta l) \eta_{ab}, \quad \langle J^\pm_a, J^\mp_b \rangle = 0,$$

(A.1)

and the Lorentz indices are raised and lowered by the metric $\eta_{ab}$. [ One can consider conveniently the invariant form as $\langle J^\pm_a, J^\pm_b \rangle = (1/2) \eta_{ab}$ as in the Sec. IV, but the final results do not depend on the representations; thus, I will keep (A.1) in this appendix. ]

Now let me prove (4.9), (4.11), and the relations in (4.17). To this end, I first note that the CS action in (4.9) can be written, in the component form for the internal space, as

$$\frac{4\pi}{k} I_{CS}[A] = \int_M \frac{1}{2} \Omega^+ \left( \eta_{ab} A^{+a} \wedge dA^{+b} + \frac{1}{3} \epsilon_{abc} A^{+a} \wedge A^{+b} \wedge A^{+c} \right) - (+ \leftrightarrow -),$$

(A.2)

where I have used

$$\langle A^+ \wedge [A^+ + A^+] \rangle = \left( A^+ \wedge \frac{1}{2} [A^+, A^+] \right) + (+ \leftrightarrow -)$$

$$= \frac{1}{2} A^{+a} \wedge A^{+b} \wedge A^{+d} \epsilon_{abc} \langle J^+_d, J^+_c \rangle + (+ \leftrightarrow -)$$

$$= \frac{1}{2} A^{+a} \wedge A^{+b} \wedge A^{+c} \cdot \frac{1}{2} \epsilon_{abc}(\alpha + \beta l) + \frac{1}{2} A^{-a} \wedge A^{-b} \wedge A^{-c} \cdot \frac{1}{2} \epsilon_{abc}(\alpha - \beta l)$$

$$= \frac{1}{4} \Omega^+ \epsilon_{abc} A^{+a} \wedge A^{+b} \wedge A^{+c} - \frac{1}{4} \Omega^- \epsilon_{abc} A^{-a} \wedge A^{-b} \wedge A^{-c}$$

(A.3)

and

$$\langle A^+ \wedge dA \rangle = \langle A^+ \wedge dA^+ \rangle + (+ \leftrightarrow -)$$

$$= A^{+a} \wedge dA^+ \langle J^+_a, J^+_b \rangle + (+ \leftrightarrow -)$$
\[ A^+a \wedge dA^+b + \frac{1}{2}(\alpha + \beta)\eta_{ab} + A^-a \wedge dA^-b + \frac{1}{2}(\alpha - \beta)\eta_{ab} \]

\[ = \frac{1}{2}\Omega^+\eta_{ab}A^+a \wedge dA^+b - \frac{1}{2}\Omega^-\eta_{ab}A^-a \wedge dA^-b, \quad (A.4) \]

with the one-form gauge fields \( A = A^+a^+J_a + A^-a^-J_a \) and \( \Omega^\pm = \beta \lambda \pm \eta \).

By considering \( A^\pm a = \omega^a \pm e^a/l \) with the spin connections \( \omega^a \) and the triads \( e^a \), one can find that the CS action \((A.2)\) becomes, after some manipulations,

\[
\frac{4\pi}{k} I_{CS} = \int_M \left[ \alpha \omega^a \wedge \left( d\omega_a + \frac{1}{3} \epsilon_{abc} \omega^b \wedge \omega^c \right) + \frac{\alpha}{l^2} e^a \wedge (d\omega_a + \epsilon_{abc} \omega^b \wedge e^c) 
+ \beta e^a \wedge \left( 2d\omega_a + \epsilon_{abc} \omega^b \wedge \omega^c + \frac{1}{3l^2} \epsilon_{abc} e^b \wedge e^c \right) - \beta d(\omega^a \wedge e_a) \right]
\[
= \alpha \int_M \left[ \omega^a \wedge \left( d\omega_a + \frac{1}{3} \epsilon_{abc} \omega^b \wedge \omega^c \right) + \frac{1}{l^2} e^a \wedge T_a \right] 
+ \beta \int_M \left( 2e^a \wedge R_a + \frac{1}{3l^2} \epsilon_{abc} e^a \wedge e^b \wedge e^c \right) - \beta \oint_{\partial M} \omega^a \wedge e_a, \quad (A.5) \]

where I have defined the curvature two-form, in \textit{vector} form basis,

\[
R^a = \frac{1}{2} \epsilon_{bac} R^{bc} 
= d\omega_a + \frac{1}{2} \epsilon_{abc} \omega^b \wedge \omega^c \quad (A.6) \]

from \( R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega^{cb} \) and \( \omega^{ab} = -\epsilon_{abc} \omega^c, \omega^a = (1/2)\epsilon^{abc} \omega_{bc} \) [note the difference in the numerical factors of the quadratic terms in \((A.6)\) and the bracket of the first term in the final result of \((A.5)\) such as the latter can not be expressed as \( R^a \) only]. The negative sign comes from \((-1)^s\) factor when we consider \( \epsilon_{abc} e^{ade} = (-1)^s (\delta^d_b \delta^e_c - \delta^d_c \delta^e_b) \) for \( s \) negative signatures in the metric \( \eta_{ab} \). This becomes \((4.9)\) in the compact form notation with the trace \( Tr \), up to the boundary term–actually this becomes a “half” of the Gibbons-Hawking’s boundary term \( 2\oint_M K \), for the extrinsic curvature scalar \( K \) of the boundary, in the gravity action \([76]\). Note also that there are factor “2” difference in the triple wedge products of \( \omega \)’s between \((4.9)\) and \((A.5)\).

Now, in order to determine the coefficients \( \alpha, \beta \), I need to compare the result \((A.5)\) in the \textit{vector} basis with that of the usual \textit{tensor} form basis. To this end, I first note that

\[
I_1 \equiv \int 2e^a \wedge R_a = \int \epsilon_{abc} e^a \wedge R^{bc} 
= \int \epsilon_{abc} e_{\mu} \cdot \frac{1}{2} R^{bc}_{\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho 
= \frac{1}{2} \int d^3 x \epsilon^{\mu\rho} \epsilon_{abc} e^a_{\mu} R^{bc}_{\nu\rho} 
= \frac{1}{2} \int d^3 x \epsilon^{\mu\rho} \epsilon_{abc} e^a_{\mu} \epsilon_\beta e^\beta R^{a\beta}_{\nu\rho} 
\]

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\[
\frac{1}{2} \int d^3 x \sqrt{-g} \, e^{\mu \nu} \epsilon_{\alpha \beta \mu} R^{\alpha \beta \nu} = - \int d^3 x \sqrt{-g} \, R, \tag{A.7}
\]

where I have denoted \( R_{bc\nu\rho} = \partial_\nu \omega_{bc} + \omega^b_{\mu \nu} \omega^{\mu \rho} - (\nu \leftrightarrow \rho) \) in the second line and I have used \( dx^\mu \wedge dx^\nu \wedge dx^\rho = e^{\mu \nu} \, d^3 \mathbf{x} \) in the third line; \( \epsilon^{\alpha \beta \mu} e^\alpha_{\mu} e^\beta_{\nu} = e \epsilon_{\mu \nu} \) with \( e = \sqrt{-g} \) [ \( e \) is the determinant of \( e^a_{\mu} \)] due to \( g_{\mu \nu} = e^a_{\mu} \eta_{ab} e^b_{\nu} \) in the fourth line; the negative sign in the final line comes from \((-1)^s\) factor with \( s = 1 \). This is the usual Einstein-Hilbert action, up to the sign.

Similarly, one can show that
\[
I_2 \equiv \int \frac{1}{3l^2} \epsilon_{abc} e^a \wedge e^b \wedge e^c = \int d^3 x \frac{1}{3l^2} e^{\mu \nu} \epsilon_{abc} e^a_{\mu} \wedge e^b_{\nu} \wedge e^c_{\rho} = - \int d^3 x \sqrt{-g} \frac{2}{l^2}, \tag{A.8}
\]

where I have used \( \epsilon^{abc} \epsilon^{\mu \nu \rho} = (-1)^s 3! \) in the final line. This is the cosmological constant action.

Next, I note that
\[
I_3 \equiv \int \omega^a \wedge (d \omega_a + \frac{1}{3} \epsilon_{abc} \omega^b \wedge \omega^c) = \int \frac{1}{2} \epsilon^{abc} \omega_{bc} \left[ d \left( \frac{1}{2} \epsilon_{ade} \omega^{de} \right) + \frac{1}{3} \epsilon_{abc} \left( \frac{1}{2} \epsilon^{bde} \omega_{de} \right) \wedge \left( \frac{1}{2} \epsilon^{efg} \omega_{fg} \right) \right] = \int \frac{1}{2} \left( \omega_{bc} \wedge d \omega^{cb} + \frac{2}{3} \omega^b_{c} \wedge \omega^c_{d} \wedge \omega^d_{b} \right), \tag{A.9}
\]

The final line is the gravitational Chern-Simons 3-form in the tensor basis appeared in Refs. [5, 7, 9] and the first line is in the vector form basis that appeared in Refs. [51, 74, 4, 59, 50, 49], up to overall coefficients. The relation to the component (tensor) form for the spacetime indices is given by
\[
I_3 = \int \frac{1}{2} \left( \omega_{bc} \wedge R^{cb} - \frac{1}{3} \omega^b_{c} \wedge \omega^c_{d} \wedge \omega^d_{b} \right) = \int \frac{1}{2} \left( \omega_{bc} \cdot \frac{1}{2} R^{cb}_{\nu \rho} - \frac{1}{3} \omega^b_{c \nu} \omega^c_{\rho} \omega^d_{b \rho} \right) dx^\mu \wedge dx^\nu \wedge dx^\rho = - \int d^3 x \frac{1}{4} e^{\mu \nu} \left( \omega_{bc} R^{bc}_{\nu \rho} + \frac{2}{3} \omega^b_{c \nu} \omega^c_{\rho} \omega^d_{b \rho} \right). \tag{A.10}
\]

This expression is what appeared in Refs. [1, 2, 8].

Finally, I note that
\[
I_4 \equiv \int \frac{1}{l^2} e^a \wedge T_a = \int \frac{1}{l^2} e^a_{\mu} \cdot \frac{1}{2} T_{\rho \nu} dx^\mu \wedge dx^\nu \wedge dx^\rho = \int d^3 x \frac{1}{l^2} e^{\mu \rho} e^a_{\mu} T_{\rho \nu} dx^\mu \wedge dx^\nu \wedge dx^\rho, \tag{A.11}
\]

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where $T_{a
u\rho} = \partial_{\nu} e^{a}_{\rho} + e^{b}_{bc} \omega^{b}_{\nu\rho} - (\nu \leftrightarrow \rho)$ is the torsion tensor. This action is what appeared in Refs. [51, 4, 50, 49].

Collecting all formulae together, I arrive at the following action for the generalized CS gravity, up to the boundary term in (A.5),

$$I_{CS} = \frac{k}{4\pi} \alpha (I_{3} + I_{4}) + \frac{k}{4\pi} \beta (I_{1} + I_{2})$$

$$= -\frac{k \beta}{4\pi} \int_{M} d^{3}x \sqrt{-g} \left( R + \frac{2}{l^{2}} \right) - \frac{k \alpha}{16\pi} \int_{M} d^{3}x \epsilon^{\mu\nu\rho} \left( \omega_{bc\mu} R^{bc}_{\nu\rho} + \frac{2}{3} \omega^{b}_{\epsilon\mu} \omega^{c}_{\delta\nu} \omega^{d}_{\rho} \right)$$

$$+ \frac{k \alpha}{8\pi} \int_{M} d^{3}x \frac{1}{l^{2}} \epsilon^{\mu\nu\rho} e_{\alpha\mu} T_{a\nu\rho}. \quad (A.12)$$

This is the expression that appeared in Refs. [1, 2, 8], but it is easy to compare with other expressions in Refs. [7, 9, 50, 49] from the above formulae. Now, in order that the first term becomes the ordinary Einstein-Hilbert action $I_{EH} = (1/16\pi G) \int_{M} (R + 2/l^{2})$ with a negative cosmological constant $\Lambda = -1/l^{2}$ in (2.1) I choose $k\beta = -1/4G$, as in (4.17). Then, the gravitational Chern-Simons term becomes, in several equivalent expressions,

$$I_{GCS} \equiv \frac{k}{4\pi} \alpha I_{3}$$

$$= \frac{1}{64\pi G} \frac{\alpha}{\beta} \int_{M} d^{3}x \epsilon^{\mu\nu\rho} \left( \omega_{bc\mu} R^{bc}_{\nu\rho} + \frac{2}{3} \omega^{b}_{\epsilon\mu} \omega^{c}_{\delta\nu} \omega^{d}_{\rho} \right)$$

$$= -\frac{1}{32\pi G} \frac{\alpha}{\beta} \int_{M} \left( \omega_{bc} \wedge d\omega^{cb} + \frac{2}{3} \omega^{b}_{\epsilon\mu} \omega^{c}_{\delta\nu} \omega^{d}_{\rho} \right)$$

$$= -\frac{1}{16\pi G} \frac{\alpha}{\beta} \int_{M} \omega^{a} \wedge \left( d\omega_{a} + \frac{1}{3} \epsilon_{abc} \omega^{b} \wedge \omega^{c} \right). \quad (A.13)$$

By comparing the first line with (2.2) and Refs. [1, 2], [8] (the published version), I find $\hat{\beta} = \alpha/\beta = -1/\mu l = -\beta_{S}/l$ for the coefficient $\mu$ in Refs. [1, 2] and $\beta_{S}$ in Ref. [8], as I have claimed in (4.17); by comparing the second line with Ref. [7], I find $\hat{\beta} = \alpha/\beta = -32\pi G \beta_{KL}/l$ for the coefficient $\beta_{KL}$ in Ref. [7]; by comparing the third line with Refs. [50, 49], I find $\hat{\beta} = \alpha/\beta = -16\pi G \alpha_{3}/l$. From these relations one can ensure that the central charges between the anomaly approaches of Refs. [7, 8] and the classical symmetry approaches of Refs. [50, 16] agree exactly, even in the presence of gravitational Chern-Simons term,

$$\hat{c}^{+\pm}_{\text{tot}} = \frac{1}{\hbar} \left( 1 \mp 16\pi G \alpha_{3}/l \right) \frac{3l}{2G} \quad (A.14)$$
Appendix B. Cardy formula and its higher-order corrections

In this appendix, I briefly review the physicist’s derivation of Cardy formula and its higher-order corrections, for completeness of my discussions in this paper.

To this end, let me begin with the partition function of the CFT on a torus, with the modular parameters $\tau, \bar{\tau}$ [30, 35]

$$Z[\tau, \bar{\tau}] = Tr e^{2\pi i (\hat{L}_0 - \frac{c}{12})} e^{-2\pi i (\hat{\bar{L}}_0 - \frac{\hat{\bar{c}}}{12})}. \quad (B.1)$$

This is invariant under the modular transformations $\tau \rightarrow (a\tau + b)/(c\tau + d)$ (similarly for $\bar{\tau}$), with the some integers $a, b, c, d$ satisfying $ad - bc = 1$, and the Virasoro generators $\hat{L}_m, \hat{\bar{L}}_m$ are defined on the “plane” with central charges $\hat{c}, \hat{\bar{c}}$, with the algebras in the standard form,

$$[\hat{L}_m, \hat{L}_n] = (m - n)\hat{L}_{m+n} + \frac{\hat{c}}{12} m(m^2 - 1)\delta_{m+n,0},$$

$$[\hat{L}_m, \hat{\bar{L}}_n] = (m - n)\hat{\bar{L}}_{m+n} + \frac{\hat{\bar{c}}}{12} m(m^2 - 1)\delta_{m+n,0},$$

$$[\hat{L}_m, \hat{\bar{L}}_n] = 0. \quad (B.2)$$

The density of states $\rho(\hat{\Delta}, \hat{\bar{\Delta}})$ for the eigenvalues $\hat{L}_0 = \hat{\Delta}, \hat{\bar{L}}_0 = \hat{\bar{\Delta}}$ is given as a contour integral (I suppress the $\bar{\tau}$-dependence for simplicity, but the computation is similar to the $\tau$-part)

$$\rho(\hat{\Delta}) = \int_C d\tau e^{-2\pi i (\hat{\Delta} - \frac{\hat{c}}{12})} Z[\tau], \quad (B.3)$$

where the contour $C$ encircles the origin in the complex $q = e^{2\pi i \tau}$ plane. The general evaluation of this integral would be impossible unless $Z[\tau]$ is known completely. But, due to the modular invariance of (B.1), one can easily compute its asymptotic formula through the steepest-descent approximation. In particular, (B.1) is invariant under $\tau \rightarrow -1/\tau$ [30] such that

$$Z[\tau] = Z[-1/\tau] = e^{-2\pi i (\hat{\Delta}_{\min} - \frac{\hat{c}}{12})} \tilde{Z}[-1/\tau], \quad (B.4)$$

where $\tilde{Z}[-1/\tau] = Tr e^{-2\pi i (\hat{L}_0 - \hat{\Delta}_{\min})/\tau}$ approaches a constant value $\rho(\hat{\Delta}_{\min})$ as $\tau \rightarrow i0_+$, which defines the steepest-descent path for a “real” value of $\hat{\Delta} \geq \hat{\Delta}_{\min}$. With the help of this property, (B.3) is evaluated as, by expanding the integrand around the steepest-descent path $\tau_*$,

$$\rho(\hat{\Delta}) = \int_C d\tau e^{\eta(\tau)} \tilde{Z}[-1/\tau] \quad (B.5)$$

$$= e^{\eta(\tau_*)} \tilde{Z}[-1/\tau_*] \times \int_C d\tau \exp \left\{ \frac{1}{2} \eta^{(2)}(\tau - \tau_*)^2 + \sum_{n \geq 3} \frac{1}{n!} \eta^{(n)}(\tau - \tau_*)^n \right\} \times \left[ 1 + \sum_{m \geq 1} \frac{1}{m!} \tilde{Z}^{(m)}(\tau - \tau_*)^m \right]. \quad (B.6)$$
Here, \( \eta(\tau) = -2\pi i \hat{\Delta}_{\text{eff}} \tau + 2\pi i \hat{c}_{\text{eff}}/(24\tau) \), which dominates \( \tilde{Z}[1/\tau] \) in the region of interest, gets the maximum
\[
\eta(\tau_*) = 2\pi \sqrt{\frac{\hat{c}_{\text{eff}} \hat{\Delta}_{\text{eff}}}{6}},
\] (B.7)
with \( \tau_* = i \hat{c}_{\text{eff}}/24 \hat{\Delta}_{\text{eff}} \), when
\[
\frac{24 \hat{\Delta}_{\text{eff}}}{\hat{c}_{\text{eff}}} \gg 1
\] (B.8)
is satisfied. Here, \( \eta^{(n)} = (d^n \eta/d\tau^n)|_{\tau=\tau_*} \), \( \tilde{Z}^{(m)} = (d^n \tilde{Z}/d\tau^n)|_{\tau=\tau_*} \), and \( \hat{c}_{\text{eff}} = \hat{c} - 24 \hat{\Delta}_{\text{min}} \), \( \hat{\Delta}_{\text{eff}} = \Delta - \hat{c}/24 \); \( \hat{\Delta}_{\text{min}} \) is the minimum of \( \hat{\Delta} \). Here, I am assuming “\( \hat{c}_{\text{eff}}, \hat{\Delta}_{\text{eff}} > 0 \)” since, otherwise, the saddle-point approximation is not valid for real valued \( \hat{c}_{\text{eff}}, \hat{\Delta}_{\text{eff}} \).

Then, in the limit of \( \epsilon \to \infty \) with \( \tau_* = i/\epsilon \), the higher-order correction terms in the bracket \([\ ]\) of (B.6) are exponentially suppressed as \( e^{-2\pi\epsilon(\hat{\Delta}-\hat{\Delta}_{\text{min}})} \), hence (B.6) is simplified as, up to the exponentially suppressing terms,
\[
\rho(\hat{\Delta}) = e^{2\pi \sqrt{\hat{c}_{\text{eff}} \hat{\Delta}_{\text{eff}}/6}} \times \int_C \! d\tau \exp \left\{ \frac{1}{2} \eta^{(2)}(\tau - \tau_*)^2 + \sum_{n \geq 3} \frac{1}{n!} \eta^{(n)}(\tau - \tau_*)^n \right\},
\] (B.9)
where I have used \( \tilde{Z}[i\infty] = 1 \). This is known as the Cardy formula [30]. Note that here I need
\[
\hat{c}_{\text{eff}} \hat{\Delta}_{\text{eff}} \gg 1
\] (B.10)
in order that the approximation is reliable, i.e., \( e^{\eta(\tau_*)} \) dominates in the integral of (B.5), as well as the condition (B.8), such as \( \tilde{Z}[-1/\tau] \) is slowly varying near \( \tau_* \).

The integrals above could be evaluated by the steepest-descent method but the direct computation would be quite involved if one wants to go beyond the Gaussian integral. But fortunately there exits an exact, closed expression, due to Raedmacher [57], with the result [58, 61]
\[
\rho(\hat{\Delta}) = e^{2\pi \sqrt{\hat{c}_{\text{eff}} \hat{\Delta}_{\text{eff}}/6}} \times \left( \frac{\hat{c}_{\text{eff}}}{96 \hat{\Delta}_{\text{eff}}^3} \right)^{1/4} I_1(2\pi \sqrt{\hat{c}_{\text{eff}} \hat{\Delta}_{\text{eff}}/6}),
\] (B.11)
up to the exponentially suppressed terms. So, its corresponding entropy \( S_{\text{stat}} = \log \rho(\hat{\Delta}) \) becomes, with \( S_0 = 2\pi \sqrt{\hat{c}_{\text{eff}} \hat{\Delta}_{\text{eff}}/6} \),
\[
S_{\text{stat}} = S_0 + \ln \left( \frac{\hat{c}_{\text{eff}}}{96 \hat{\Delta}_{\text{eff}}^3} \right)^{1/4} I_1(S_0)
= S_0 + \ln \left( \frac{\hat{c}_{\text{eff}}}{96 \hat{\Delta}_{\text{eff}}^3} \right)^{1/4} - \frac{3}{8} S_0^{-1} + O((S_0)^{-2}),
\] (B.12)
where $I_n(x)$ is the modified Bessel function of the first kind, and I have used its asymptotic series expansion for large $x$:

\[
I_1(x) = \frac{1}{\sqrt{2\pi x}} e^x \left[ 1 - \frac{3}{8} x^{-1} + O(x^{-2}) \right].
\]  

\[\text{(B.13)}\]

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