K-RING OF THE CLASSIFYING SPACE OF THE SYMMETRIC GROUP ON FOUR LETTERS

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ABSTRACT. We describe $K(BS_4)$ and make a connection of the order of the bundle induced from the standard representation over the four dimensional skeleton of $BS_4$ with the stable homotopy group $\pi_3^s = \mathbb{Z}_{24}$ explaining the reasons of this connection by pulling this bundle over lens spaces.

1. INTRODUCTION

$K$-rings of the classifying spaces of cyclic groups, so-called lens spaces, are well-known, [2]. In [3] and [4], we described the $K$-ring of the classifying space of the dihedral and the generalized quaternion group in a naive way.

A natural continuation of the problem seems to be to study the $K$-rings of the classifying spaces of symmetric groups. Since, the representation theory of symmetric groups is very complicated, this problem turns out to be interesting even for the first non-trivial case which is the classifying space of the symmetric group on 4 letters. In this note, we will describe $K(BS_4)$ by making connections with the integral cohomology of $S_4$, which is described in [6] via the Atiyah-Hirzebruch Spectral Sequence (AHSS) of the $K$-ring.

We will also make a fancy connection of the $\tilde{K}$-order of the bundle induced from the standard representation over the four dimensional skeleton of $BS_4$ with the stable homotopy group $\pi_3^s = \mathbb{Z}_{24}$.

2. REPRESENTATIONS

There are 5 conjugacy classes of $S_4$: $C_1 = \{1\}, C_2 = (12), C_3 = (123), C_4 = (1234), C_5 = (12)(34)$. And thus, there are 5 irreducible representations: The trivial one dimensional representation $d_1$, the sign representation $d_1$ which is also one dimensional, the two dimensional representation $d_2$, the three dimensional representation $d_3$ which takes value +1 on the conjugacy class $C_2$, and the three dimensional representation $d_1d_3$. You can see the character table of $S_4$ in [6].

Date: 28 October, 2013.
2000 Mathematics Subject Classification. Primary 55R50; Secondary 20C10.
Key words and phrases. Topological K-Theory, Representations of Symmetric Groups.
The relations of the representation ring $R(S_4)$ are:

$$d_1^2 = 1, \quad d_2^2 = 1 + d_1 + d_2, \quad d_3^2 = 1 + d_2 + d_3 + d_1d_3, \quad d_2d_3 = d_3 + d_1d_3, \quad d_1d_2 = d_2.$$ 

Hence, any representation of $S_4$ can be written as an integral linear combination of the 5 irreducible representations described above.

From the above relations it is clear that the representation ring $R(S_n)$ is generated by $d_1$ and $d_3$ since $d_2 = d_3^2 - 1 - d_3 - d_1d_3$. In fact, this observation is true for any symmetric group: Hooks generate the representation ring of any symmetric group, $[5]$.

Let $res$ denote the restriction operator for a representation to a subgroup. We have the following observations for the restriction of the representations of $S_4$ which will be useful to understand the filtrations of the corresponding vector bundles on the AHSS of the $K$-ring.

Let $Z_2$ be the cyclic subgroup generated by (12), then $res(d_1) = \eta$, $res(d_2) = 2$ and $res(d_3) = 1 + 2\eta$ where $\eta$ denotes the tautological one dimensional complex representation of $Z_2$.

Let $Z_3$ be the cyclic subgroup of $S_4$ generated by (123), then $res(d_1) = 1$, $res(d_2) = \eta + \eta^2$ and $res(d_3) = 1 + \eta + \eta^2$ where $\eta$ denotes the tautological one dimensional complex representation of $Z_3$.

Let $Z_4$ be the cyclic subgroup of $S_4$ generated by (1234), then $res(d_1) = \eta^2$, $res(d_2) = 1 + \eta^2$ and $res(d_3) = \eta + \eta^2 + \eta^3$ where $\eta$ denotes the tautological one dimensional complex representation of $Z_4$.

3. Cohomology

The integral cohomology of $S_4$ is generated by 4 elements $a_2, a_3, a_4, b_4$ with dimensions $\dim a_2 = 2, \dim a_3 = 3, \dim a_4 = 4$ and $\dim b_4 = 4$ indicated by subscripts. And the cohomology ring is described in [6] in the following way:

$$H^*(S_4) = \mathbb{Z}[a_2, a_3, a_4, b_4]/I$$

where $I$ is the ideal generated by the elements $2a_2, 2a_3, 4a_4, 3b_4$ and, something weird, by the elements $a_2a_3^{2j} - a_2^{j+1}(a_4 + a_3^2)^j$ for all $j \geq 1$.

The relations $a_2a_3^{2j} = a_2^{j+1}(a_4 + a_3^2)^j$ for all $j \geq 1$ and also the whole information about the 2-primary part of the cohomology are obtained by means of the cohomology of the dihedral subgroup $D_8$ mentioned in the previous section through the group inclusion homomorphism $i : D_8 \to S_4$ by Thomas and for details we advise the reader to take a look at [6].

Note also that 2-primary part and 3-primary part of the cohomology are totally separated as it is immediate to deduce the relations $a_2b_4 = 0, a_3b_4 = 0, a_4b_4 = 0$ from the relations $2a_2 = 0, 2a_3 = 0, 4a_4 = 0, 3b_4 = 0$. 


Since the $K$-ring of a topological space is only related to the even dimensional part of the integral cohomology through the Atiyah-Hirzebruch spectral sequence, we will extract only the even dimensional cohomology groups and they can be tabulated like $H^{2i}(BS_4) =$

\[
Z
\]
\[
Z_2(a_2)
\]
\[
Z_2(a_2^2) \oplus Z_4(a_4) \oplus Z_3(d_4)
\]
\[
Z_2(a_2^3) \oplus Z_2(a_2a_4) \oplus Z_3(a_2^2)
\]
\[
Z_2(a_2^4) \oplus Z_2(a_2^2a_4) \oplus Z_4(a_2^3) \oplus Z_3(d_4^2)
\]
\[
Z_2(a_2^5) \oplus Z_2(a_2^3a_4) \oplus Z_2(a_2a_4^2) \oplus Z_2(a_2^2a_4)
\]
\[
\oplus_{i=0}^{3k-1} Z_2(a_2^{4k-2i}a_4^i) \oplus_{i=0}^{k-1} Z_2(a_3^{4k-4i}a_4^{3i}) \oplus Z_4(a_2^{3k}) \oplus Z_3(d_4^{3k})
\]
\[
\oplus_{i=0}^{3k} Z_2(a_2^{6k+1-2i}a_4^i) \oplus_{i=0}^{k-1} Z_2(a_3^{4k-4i}a_4^{3i+2})
\]
\[
\oplus_{i=0}^{3k} Z_2(a_2^{6k+2-2i}a_4^i) \oplus_{i=0}^{k-1} Z_2(a_3^{4k-4i}a_4^{3i+4}) \oplus Z_4(a_4^{3k+1}) \oplus Z_3(d_4^{3k+1})
\]
\[
\oplus_{i=0}^{3k+1} Z_2(a_2^{6k+3-2i}a_4^i) \oplus_{i=0}^{k} Z_2(a_3^{4k+2-4i}a_4^{3i})
\]
\[
\oplus_{i=0}^{3k+1} Z_2(a_2^{6k+4-2i}a_4^i) \oplus_{i=0}^{k-1} Z_2(a_3^{4k-4i}a_4^{3i+2}) \oplus Z_4(a_4^{3k+2}) \oplus Z_3(a_4^{3k+2})
\]
\[
\oplus_{i=0}^{3k+2} Z_2(a_2^{6k+5-2i}a_4^i) \oplus_{i=0}^{k} Z_2(a_3^{4k+2-4i}a_4^{3i+1})
\]

where $k \geq 1$ and $Z_n(g)$ denotes the cyclic group of order $n$ with the generator $g$.

Thomas, in [6], indicates that the even cohomology ring $H^{even}(S_n)$ is generated by Chern characters of the standard representation for any $n$. We now expect that the standard representation will play a central role in $K$-ring as well.

4. K-Ring

Let $d_1$, $d_2$, $d_3$ denote the induced vector bundles over the classifying space $BS_4$. Since the corresponding representations generate the representation ring, these vector bundles generate $K(BS_4)$ due to Atiyah-Segal Completion Theorem. We set reduced elements $v = d_1 - 1$, $\delta = d_2 - 2$, $\phi = d_3 - 3$ in $\tilde{K}(BS_4)$. Now, the relations in the representation ring transform to the set of relations:

\[
2v = -v^2
\]
\[
3\delta + \delta^2 = v
\]
\[
4\phi + \phi^2 = 3v + \delta + v\phi
\]
\[
\delta\phi = 3v + v\phi - 3\delta
\]
\[
v\delta = v^2
\]

There is an AHSS which looks like

\[
E^{p-1}_2 = H^p(S_4; \tilde{K}(S^n)) \Rightarrow K(BS_4)
\]
on the main diagonal of the second page and which converges to $K(BS_4)$. The groups $E^{p-1}_2$ on the limit page are called filtrations and they are generated by the reduced vector bundles which are trivial on $(p - 1)$-th skeleton of $BS_4$, but non-trivial on
the $p$-th skeleton. In particular, the generators of the filtrations are some particular elements in $\tilde{K}(BS_4)$. Note that these elements are related to the cohomology of $BS_4$ since they are limits of the elements on the second page. These classes probably corresponds to Chern characters of these bundles, but we don’t go on this direction further in this note.

Since $\tilde{K}(S^p) = \mathbb{Z}$ when $p$ is even and zero otherwise, we have only even dimensional filtrations $E_\infty^{2j-2j}$ on the second page of the AHSS. Because of that, only even dimensional cohomology ring $H^{even}(S_4)$ plays a role for $K(BS_4)$.

Firstly, let us explain the filtration $E_\infty^{2,-2}$. We claim that AHSS collapses on second page at $(2, -2)$ and $E_\infty^{2,-2}$ is $Z_2$ and is generated by $v$. Let us pick a subgroup $Z_2$ in $S_4$, say generated by (12). Then the subgroup inclusion $Z_2 \to S_4$ induces ring homomorphism $\tilde{K}(BS_4) \to \tilde{K}(BZ_2)$ and also group homomorphisms $E_\infty^{p,-p}(BS_4) \to E_\infty^{p,-p}(BZ_2)$ on AHSSs by naturality. Under the former map, since it corresponds to restriction of the representation, $v$ maps to $v$ where the last same letter denotes the tautological generator of $\tilde{K}(BZ_2)$. We recall that $v \in \tilde{K}(BZ_2)$ generates $E_\infty^{2,-2}(BZ_2)$ of the AHSS of $\tilde{K}(BZ_2)$. Therefore, $v$ generates $E_\infty^{2,-2}(BS_4)$. We also note that $v$ corresponds to the first Chern character of the standard representation although it is the reduction of the sign representation.

The same kind of arguments show that $v^j = (-2)^{j-1}v$ generates a $Z_2$ summand on the filtration $E_\infty^{2j-2j}$ for all $j \geq 1$ and these summands correspond to the summands $Z_2(a_2^j)$ on $E_\infty^{2j-2j}$.

Next, we will deal with $E_\infty^{4,-4}$. As we said the $Z_2$ summand of $E_\infty^{4,-4}$ is generated by $v^2$. Now, we should explain its remaining $Z_{12}$ summand for which we claim that it survives.

Let us pick a subgroup $Z_4$ in $S_4$, say generated by (1234). Then the subgroup inclusion $Z_4 \to S_4$ induces ring homomorphism $\tilde{K}(BS_4) \to \tilde{K}(BZ_4)$ and also group homomorphisms $E_\infty^{p,-p}(BS_4) \to E_\infty^{p,-p}(BZ_4)$ on AHSSs by naturality. At this point we observe that the pull-back of $\phi$ to $BZ_4$ lives on $E_\infty^{2,-2}(BZ_4)$ so that it can not be a generator of $E_\infty^{4,-4}$ of the AHSS of our ring.

Because of that, we set $x = \phi + v$ and now $x$ should be a generator of $E_\infty^{4,-4}$, since it creates no conflict with the restrictions to cyclic subgroups $Z_4$ and $Z_3$ as well: $x$ pulls back to a bundle in the form $\mu^2 + \ldots$ over both $BZ_4$ and $BZ_3$ where $\mu = \eta - 1$ is the reduction of the tautological line bundle. Thus it must have order 12 in $E_\infty^{4,-4}$. Therefore, the AHSS collapses on second page at the coordinate $(4, -4)$ too. But, it will not be the case on the dimension 6.

Really, by the similar elementary explanations, the higher filtrations are observed as below:

\[
E_\infty^{2,-2} = Z_2(v) \\
E_\infty^{4,-4} = Z_2(v^2) \oplus Z_{12}(x)
\]
\[ E_6^{5,-6} = Z_2(v^3) \oplus Z_2(v\phi) \] (AHSS is not collapsing from hereafter!)
\[ E_8^{8,-8} = Z_2(v^4) \oplus Z_2(v^2\phi) \oplus Z_12(x^2) \]
\[ E_{10}^{10,-10} = Z_2(v^5) \oplus Z_2(v^3\phi) \oplus Z_2(v\phi^2) \]
\[ E_{12}^{12,-12} = Z_2(v^6) \oplus Z_2(v^4\phi) \oplus Z_2(v^2\phi^2) \oplus Z_12(x^3) \]

We observe that only the parts of the even cohomology related to the class \( a_3 \in H^3(S_4) \) are not surviving on the AHSS; they vanish. The other parts of the even cohomology stay as they are. Hence, we have

**Theorem 1:**

\[ K(\text{BS}_4) = Z[v, \phi] / \left( \begin{array}{c}
2v + v^2 \\
12\phi + 7\phi^2 + \phi^3 - 4v - v\phi \\
24\phi + 26\phi^2 + 9\phi^3 + \phi^4 \\
2v\phi - 8v - 24\phi^2 - 14\phi^3 - 2\phi^4 + v\phi^2
\end{array} \right) \]

As we expected, the \( K \)-ring is just a change of variables in the \( R \)-ring.

5. A Little Homotopy

Third stable homotopy group of spheres is \( \pi_3^s = Z_{24} \) and it is the image of the \( J \)-homomorphism

\[ J : \widetilde{KO}(S^4) \to \pi_3^s. \]

Its generator is

\[ S^4 \xrightarrow{1} BO \xrightarrow{B} B(Z \times \text{BS}_4^+) \]

Now, it follows from the previous section, that the order of the map

\[ \phi : \text{BS}_4^{(4)} \to BU \]

is 24 where \( \text{BS}_4^{(4)} \) is the four dimensional skeleton of \( \text{BS}_4 \). We believe that it is not a coincidence that the order of \( \phi \) in \( \widetilde{K}(\text{BS}_4^{(4)}) \) is the same as the order of \( \pi_3^s \).

We can explain the reasons a little bit behind our claim by pulling the map \( \phi \) over some lens spaces which is our fundamental trick. As we said before, we have natural subgroups which induce projections \( BZ_3 \to \text{BS}_4 \) and \( BZ_4 \to \text{BS}_4 \) and the pull-backs of \( \phi \) over these lens spaces by these projections are equal to \( \mu^2 \) and \( 2\mu + h.o.t. \) respectively where \( \mu \) is the reduction of the corresponding Hopf bundle. On the other hand, \( r(\mu^2) = 2w + w^2 \) where \( r \) is the relaification map and \( r(2\mu) = 2w \). Now, the \( KO \)-order, and the \( J \)-order as well, of \( 2w \) over the skeletons \( BZ_3^{(4)} \) and \( BZ_4^{(4)} \) are 3 and 4 respectively. These should correspond to the local \( J \)-order of the generator of
the group $\tilde{KO}(S^4)$ at prime 3 and to the half of the local $J$-order of that generator at prime 2, respectively. We guess, we should write down $KO$-ring of $BS_4$ to understand that 2-local behaviour since realification seems to change the order at prime 2.

Having this connection, we will make the following expected conjectures.

For all $n$ and in the limit $n \to \infty$, let $\phi$ denote the family of the standard generators of $\tilde{K}(BS_n)$ which are induced from the standard representations. We claim that $\phi$ detect the $\text{Im} J$ part of $\pi^*_s$. This conjecture seems to be quite possibly true due to Hegenbarth’s work, [1], on the $k$-theory of the classifying space of the infinite symmetric group where his mod $p$ generators suggest that connection.

A stronger and much more fancy conjecture would be that

**Conjecture 2:** $\phi$ detect $\pi^*_s$.

In other words, we claim that representations of symmetric groups are saying quite a lot about the homotopy groups of spheres. We note that it becomes incredibly complicated even to write the polynomial relations in $K(BS_n)$ when $n \to \infty$, yet alone the unimaginable geometric work behind. $n = 5$ stays as a messy exercise.

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