ON THE MONOTONICITY OF $q$-SCHURER-STANCU TYPE POLYNOMIALS

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Received 05 October, 2015

Abstract. Some properties of monotonicity and convexity of the $q$-Schurer-Stancu operators are considered. The paper contains also numerical examples based on Matlab algorithms, which verify these properties.

2010 Mathematics Subject Classification: 41A10; 41A36

Keywords: generalized Schurer-Stancu operators, $q$-integers, monotonicity, convexity

1. Preliminaries

In the last decades, the application of $q$-calculus represents one of the most interesting areas of research in approximation theory. Lupas [12] introduced in 1987 a $q$-type of the Bernstein operators and in 1997 another generalization of these operators based on $q$-integers was introduced by Phillips [16]. Their approximation properties were studied by Videnški [18], N. Mahmudov [13], T. Acar and A. Aral [1] and O. Dalmanoglu [9, 10]. In time, many authors have been studied new classes of $q$-generalized operators ([2–4, 6, 7, 17]).

Before proceeding further, we mention some basic definitions and notations from $q$-calculus. For any fixed real number $q > 0$, the $q$-integer $[k]_q$, for $k \in \mathbb{N}$ is defined as

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q}, & q \neq 1, \\ k, & q = 1. \end{cases}$$

The $q$-factorial integer and the $q$-binomial coefficients are:

$$[k]_q! = \begin{cases} [k]_q[k-1]_q \ldots [1]_q, & k = 1, 2, \ldots \\ 1, & k = 0. \end{cases}$$

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, (n \geq k \geq 0).$$
The q-analogue of $(x-a)^n_q$ is the polynomial

$$(x-a)^n_q = \begin{cases} 1, & \text{if } n = 0, \\ (x-a)(x-q)(x-q^2)\ldots(x-q^{n-1}a), & \text{if } n \geq 1. \end{cases}$$

Let $p$ be a non-negative integer and let $\alpha, \beta$ be some real parameters satisfying the conditions $0 \leq \alpha \leq \beta$. In 2003, D. Bărboşu [8] introduced for any $f \in C[0,1+p]$ and $x \in [0,1]$ the Schurer-Stancu operators as follows

$$S_{m,p}^{(\alpha,\beta)}(f,q,x) = \sum_{k=0}^{m+p} p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right),$$

where $p_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k}$.

Recently, P.N. Agrawal, V. Gupta and A.S. Kumar [5] introduced the class of q-Schurer-Stancu operators. For any $m \in \mathbb{N}$, $p$ a fixed non-negative integer number and $\alpha, \beta$ some real parameters satisfying the conditions $0 \leq \alpha \leq \beta$, they constructed the class of generalized q-Schurer-Stancu operators

$$\tilde{S}_{m,p}^{(\alpha,\beta)} : C[0,1+p] \to C[0,1],$$

as follows

$$\tilde{S}_{m,p}^{(\alpha,\beta)}(f,q,x) = \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) f\left(\frac{[k]_q + \alpha}{[m]_q + \beta}\right), \quad x \in [0,1], \quad (1.1)$$

where $\tilde{p}_{m,k}(x) = \left[ \frac{m+p}{k} \right]_q x^k (1-x)^{m+p-k}$.

If $\alpha = \beta = 0$ the above operators reduce to the Bernstein-Schurer operators introduced by Muraru in [14].

**Lemma 1 ([5]).** For the operators defined in (1.1) the following properties hold

1. $\tilde{S}_{m,p}^{(\alpha,\beta)}(e_0,q,x) = 1$,
2. $\tilde{S}_{m,p}^{(\alpha,\beta)}(e_1,q,x) = \frac{\alpha}{[m]_q + \beta} + \frac{[m+p]_q}{[m]_q + \beta} x$,
3. $\tilde{S}_{m,p}^{(\alpha,\beta)}(e_2,q,x) = \frac{\alpha^2}{([m]_q + \beta)^2} + \frac{[m+p]_q^2}{([m]_q + \beta)^2} x^2 + \frac{2\alpha[m+p]_q}{([m]_q + \beta)^2} x + \frac{[m+p]_q x(1-x)}{([m]_q + \beta)^2}$.

The next result is based on Popoviciu’s technique and it is expressed in terms of the first order modulus of continuity.
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Theorem 1 ([5]). If \( f \in C[0, 1 + p] \) and \( q \in (0, 1) \) then
\[
\left\| \tilde{S}_{m,p}^{(\alpha, \beta)}(f, q, x) - f(x) \right\| \leq \frac{5}{4} \omega_f(\delta_m)
\]
holds, where
\[
\delta_m = \frac{1}{[m]_q + \beta} \sqrt{[m + p]_q + 4(q^m[p]_q + \alpha - \beta)^2}.
\]

2. MONOTONICITY OF THE \( q \)-SCHURER-STANCU OPERATORS

Oruc and Philips [15] showed that for a convex function \( f \) on \([0,1]\), the \( q \)-Bernstein polynomials are monotonic decreasing. In this section we will prove a similar result for \( q \)-Schurer-Stancu operators.

Theorem 2. Let \( f \) be a convex and increasing function on \([0,p + 1]\). Then, for \( 0 < q \leq 1 \) and \( \beta \leq \frac{[p]_q}{q^p} \),
\[
\tilde{S}_{m-1,p}^{(\alpha, \beta)}(f, q, x) \geq \tilde{S}_{m,p}^{(\alpha, \beta)}(f, q, x),
\]
for \( 0 \leq x \leq 1 \) and \( m \geq 2 \).

Proof. For \( 0 < q < 1 \) we have
\[
\sum_{k=0}^{m+p-1} \left[ \begin{array}{c} m + p - 1 \\ k \end{array} \right]_q x^k \prod_{s=m+p-k}^{m+p-1} (1 - q^s x)^{-1} \left( [k]_q + \frac{\alpha}{[m-1]_q + \beta} \right)
\]
and using the following relation
\[
x^k \prod_{s=m+p-k}^{m+p-1} (1 - q^s x)^{-1} = \psi_k(x) + q^{m+p-k-1} \psi_{k+1}(x)
\]
we find
\[
\sum_{k=0}^{m+p-1} \left[ \begin{array}{c} m + p - 1 \\ k \end{array} \right]_q x^k \prod_{s=m+p-k}^{m+p-1} (1 - q^s x)^{-1} \left( \tilde{S}_{m-1,p}^{(\alpha, \beta)}(f, q, x) - \tilde{S}_{m,p}^{(\alpha, \beta)}(f, q, x) \right)
\]
\[
\begin{align*}
&= \sum_{k=0}^{m+p-1} f \left( \frac{[k]_q + \alpha}{[m-1]_q + \beta} \right) \binom{m+p-1}{k} \left[ \psi_k(x) + q^{m+p-k-1} \psi_{k+1}(x) \right] \\
&- \sum_{k=0}^{m+p} f \left( \frac{[k]_q + \alpha}{[m]_q + \beta} \right) \binom{m+p}{k} \psi_k(x) + \sum_{k=0}^{m+p-1} f \left( \frac{[k]_q + \alpha}{[m-1]_q + \beta} \right) \binom{m+p-1}{k} \psi_k(x) \\
&+ \sum_{k=1}^{m+p} q^{m+p-k} f \left( \frac{[k-1]_q + \alpha}{[m-1]_q + \beta} \right) \binom{m+p-1}{k-1} \psi_k(x) \\
&+ q^{m+p-k} f \left( \frac{[k-1]_q + \alpha}{[m]_q + \beta} \right) \binom{m+p-1}{k-1} - f \left( \frac{[m+p]_q + \alpha}{[m]_q + \beta} \right) \binom{m+p}{k} \psi_k(x) \\
&+ \sum_{k=1}^{m+p} \binom{m+p}{k} a_k \psi_k(x) + \left\{ f \left( \frac{[m+p-1]_q + \alpha}{[m-1]_q + \beta} \right) - f \left( \frac{[m+p]_q + \alpha}{[m]_q + \beta} \right) \right\} \psi_{m+p}(x) \\
&+ \left\{ f \left( \frac{\alpha}{[m-1]_q + \beta} \right) - f \left( \frac{\alpha}{[m]_q + \beta} \right) \right\} \psi_0(x),
\end{align*}
\]

where

\[
a_k = f \left( \frac{[k]_q + \alpha}{[m-1]_q + \beta} \right) \frac{[m+p]_q}{[m+p]_q} q^{m+p-k} f \left( \frac{[k-1]_q + \alpha}{[m-1]_q + \beta} \right) \frac{[k]_q}{[m+p]_q}
\]

From (2.2) it is clear that each \( \psi_k(x) \) is non-negative on \([0, 1]\) for \(0 \leq q \leq 1\) and thus, it suffices to show that each \( a_k \) is non-negative.

Since \( f \) is convex on \([0, p+1]\), for any \( t_0, t_1 \) such that \( 0 \leq t_0 < t_1 \leq p+1 \) and any \( \lambda, 0 < \lambda < 1 \), we have

\[
f(\lambda t_0 + (1-\lambda)t_1) \leq \lambda f(t_0) + (1-\lambda) f(t_1).
\]

Let \( t_0 = \frac{[k-1]_q + \alpha}{[m-1]_q + \beta}, \ t_1 = \frac{[k]_q + \alpha}{[m-1]_q + \beta} \) and \( \lambda = q^{m+p-k} \frac{[k]_q}{[m+p]_q} \). Then \( 0 \leq t_0 < t_1 \leq p+1 \) and \( 0 < \lambda < 1 \) for \( 1 \leq k \leq m + p - 1 \). If we replace them in the relation (2.3), it follows

\[
q^{m+p-k} \frac{[k]_q}{[m+p]_q} f \left( \frac{[k-1]_q + \alpha}{[m-1]_q + \beta} \right) + \frac{[m+p-k]_q}{[m+p]_q} f \left( \frac{[k]_q + \alpha}{[m-1]_q + \beta} \right)
\]
Using the inequality \([k]_q ([k]_q + \alpha) \geq ([k]_q + \alpha)[k]_q\) and \(f\) increasing function, it follows

\[
q^{m+p-k} \frac{[k]_q}{[m]_q} \cdot \frac{[k]_q + \alpha}{[m]_q} + \frac{[m+p-k]_q}{[m]_q} \cdot \frac{[k]_q + \alpha}{[m]_q}.
\]

From the inequality (2.4) we obtain \(a_k \geq 0, k = 1, m + p - 1\).

Therefore \(\tilde{S}_{m-1,p}(f,q,x) \geq \tilde{S}_{m,p}(f,q,x)\).

For \(q = 1\) and \(0 < x < 1\) in a similar way the property (2.1) is verified.

For \(q = 1\) and \(x = 1\) we have

\[
\tilde{S}_{m-1,p}(f,1,1) - \tilde{S}_{m,p}(f,1,1) = f \left( \frac{m+p-1+\alpha}{m+1+\beta} \right) - f \left( \frac{m+p+\alpha}{m+\beta} \right) \geq 0.
\]

Theorem 3. If \(f\) is convex, then for all \(m \geq 1\) and \(0 < q \leq 1\) it follows

i) \(\tilde{S}_{m,p}(f,q,x) \geq f(x)\), for \(x \in [0,1]\). \(f\) increasing on \([0,1]\) and \(\beta = \alpha + \varepsilon, \varepsilon \in [0,q^m[p]_q];\)

ii) \(\tilde{S}_{m,p}(f,q,x) \geq f(x)\), for \(x \in \left[0, \frac{\alpha}{\beta - q^m[p]_q}\right]\). \(f\) increasing on \([0,1]\) and \(\beta > \alpha + q^m[p]_q;\)

iii) \(\tilde{S}_{m,p}(f,q,x) \geq f(x)\), for \(x \in \left(\frac{\alpha}{\beta - q^m[p]_q}, 1\right]\). \(f\) decreasing on \([0,1]\) and \(\beta > \alpha + q^m[p]_q.\)

Proof. We consider the knots \(x_k = \frac{[k]_q + \alpha}{[m]_q + \beta}, 0 \leq k \leq m + p\). From Lemma 1 it follows

\[
\sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) = 1, \quad \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) x_k = \frac{\alpha}{[m]_q + \beta} + \frac{[m+p]_q}{[m]_q + \beta}.
\]
Using the convexity of function \( f \) we have

\[
\tilde{S}_{m,p}(\alpha, \beta, q) = \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) f(x_k) \geq f \left( \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x)x_k \right)
\]

\[
= f \left( \frac{\alpha}{[m]_q + \beta} + \frac{[m+p]_q}{[m]_q + \beta} x \right).
\]

The following inequalities hold

\begin{align*}
\text{a)} & \quad \frac{\alpha}{[m]_q + \beta} + \frac{[m+p]_q}{[m]_q + \beta} x \geq x \text{ for } \beta = \alpha + \epsilon, \, \epsilon \in [0, q^m[p]], \, x \in [0, 1]; \\
\text{b)} & \quad \frac{\alpha}{[m]_q + \beta} + \frac{[m+p]_q}{[m]_q + \beta} x \geq x \text{ for } \beta > \alpha + q^m[p], \, x \in \left[ 0, \frac{\alpha}{\beta - q^m[p]} \right]; \\
\text{c)} & \quad \frac{\alpha}{[m]_q + \beta} + \frac{[m+p]_q}{[m]_q + \beta} x \leq x \text{ for } \beta > \alpha + q^m[p], \, x \in \left[ \frac{\alpha}{\beta - q^m[p]}, 1 \right].
\end{align*}

The theorem is proved using the monotony of function \( f \) and the inequalities a)-c).

\[\square\]

3. NUMERICAL EXAMPLE

Davis [11] proved that for any convex function \( f \), the classical Bernstein polynomial is convex and the sequence of Bernstein polynomials is monotonic decreasing. Oruc and Philips [15] extend these results for the Bernstein operators in \( q \)-calculus for \( 0 < q \leq 1 \). In this section we will verify numerically these properties for the \( q \)-Schurer-Stancu operators.

**Table 1.** The \( q \)-Schurer-Stancu operators

| \( x \) | \( S_{30,6}^{(3,5)}(f,q,x) \) | \( S_{90,6}^{(3,5)}(f,q,x) \) |
|---|---|---|
| 0 | 0.029115231597413 | 0.02656252311249 |
| 0.1 | 0.089811163243826 | 0.08388394253406 |
| 0.2 | 0.191687642176340 | 0.179475155093384 |
| 0.3 | 0.347058215579779 | 0.326749960898829 |
| 0.4 | 0.57005753689721 | 0.538887425606676 |
| 0.5 | 0.876786656660820 | 0.831510780091276 |
| 0.6 | 1.285573442130916 | 1.222376845885256 |
| 0.7 | 1.817111513724159 | 1.731579581814080 |
| 0.8 | 2.494715950321931 | 2.381768592612226 |
| 0.9 | 3.34454197167635 | 3.198383491124964 |
| 1 | 4.395900582819147 | 4.209905050209471 |
In Table 1 are calculated the values of the \(q\)-Schurer-Stancu operators \(\mathcal{S}^{(3,5)}_{30.6}(f;q,x)\) and \(\mathcal{S}^{(3,5)}_{90.6}(f;q,x)\) for \(f(x) = x^3 e^x + 1\) and \(q = 0.9\). Also, in the Figure 1 are given the graphics of these operators.

**Figure 1.** The monotonicity of the \(q\)-Schurer-Stancu operators

In the next part of this section we will give some numerical examples which verify the inequalities proved in Theorem 3.

**Example 1.** If \(n = 50\), \(p = 5\), \(\alpha = 3\), \(\beta = 3.0211\), \(q = 0.9\), \(f(x) = x^3 e^x + 1\), it follows \(\mathcal{S}^{(\alpha,\beta)}_{m,p}(f;q,x) \geq f(x)\) for all \(x \in [0,1]\).

**Figure 2.** The \(q\)-Schurer-Stancu operators for increasing function and \(\beta = \alpha + \epsilon\)
Example 2. If \( n = 50, p = 5, \alpha = 3, q = 0.9, \beta = 8.0211, f(x) = x^3 e^{x+1} \), it follows \( \tilde{S}_{m,p}^{(a,b)}(f,q,x) \geq f(x) \) for all \( x \in [0, 0.375] \).

![Figure 3](image3.png)

**Figure 3.** The \( q \)-Schurer-Stancu operators for increasing function and \( \beta > \alpha + q^m[p]_q \)

Example 3. If \( n = 50, p = 5, \alpha = 3, q = 0.9, \beta = 8.0211, f(x) = e^{-x^2} \), it follows \( \tilde{S}_{m,p}^{(a,b)}(f,q,x) \geq f(x) \) for all \( x \in [0.375, 1] \).

![Figure 4](image4.png)

**Figure 4.** The \( q \)-Schurer-Stancu operators for decreasing function

ACKNOWLEDGMENT.

Project financed from Lucian Blaga University of Sibiu research grants LBUS-IRG-2015-01, No.2032/7.
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