Diagrammatic proof of the large $N$ melonic dominance in the SYK model

V. Bonzom · V. Nador · A. Tanasa

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Abstract
A crucial result on the celebrated Sachdev–Ye–Kitaev model is that its large $N$ limit is dominated by melonic graphs. In this letter, we offer a rigorous, diagrammatic proof of that result by direct, combinatorial analysis of its Feynman graphs.

Keywords Sachdev–Ye–Kitaev model · Melonic graphs · Tensor models

Mathematics Subject Classification 81Q30 · 05C22

Introduction
The Sachdev–Ye model [24] was initially introduced within a condensed matter perspective in the early nineties. Kitaev [17], in a series of talks, exposed its connection...
to the celebrated AdS/CFT correspondence. The model is now known as the Sachdev–Ye–Kitaev (SYK) model.

It has attracted a huge deal of interest from both condensed matter, see for example [22], and high-energy physics communities, see for example [2,3,6,12,13,20,23] and references within.

A crucial property of the SYK model is that the large $N$ limit, where $N$ is the number of fermions, is dominated by the set of melonic graphs. Remarkably, those graphs had been known to dominate the large $N$ limit of random tensor models [14], $N$ being here the size of the tensor. This is true for the Gurau’s colored tensor model [5], Tanasa’s multiorientable model [10,25], Tanasa and Carrozza’s model [9]. Even for tensor models whose large $N$ limit does not consist of melonic graphs, the universality class of melonic graphs is easily stumbled upon [4].

The large $N$ melonic dominance is mentioned in Kitaev’s original talk or in the seminal article [20], to name only a few of the very well-known references mentioning it. It is a direct cause of interesting properties of the model, in particular the form of the Schwinger–Dyson equation for the two-point function, which leads to conformal invariance in the infrared regime, or the low-energy regime.

The large $N$ dominance of melonic graphs in both the SYK model and tensor models triggered interesting developments, starting from the Gurau–Witten model [26] and the CTKT model [19]. This has motivated $1/N$ expansions for new tensorial models [1,2,8,16] and the new field of tensor quantum mechanics [7,11,18,21].

A proposal which is very close to both the SYK model and unitary-invariant tensor models is the so-called colored SYK model, initially introduced in [12,15], where the fermions come in several flavors, or colors, and only different colors can interact. In this model and in the Gurau–Witten model, combinatorial methods originating from the study of tensor models can be used, for instance to identify the Feynman graphs which contribute at a given order in the $1/N$ expansion [6,15].

However, combinatorial proofs have been of limited use so far in the original SYK model. One reason is that colors (or flavors) have been crucial to most combinatorial results in tensor models but the Feynman graphs of the SYK model have no colors. (Only recently new methods have been found to deal with tensor models without colors [1,2,8,16], but have not been applied to the original SYK model to the best of our knowledge.)

In this letter, we give a mathematically rigorous proof of the dominance of melonic graphs of the SYK model, see Theorem 1. One might expect it to be difficult without the notion of colors. As it turns out, even for graphs without colors, the set of melonic graphs is simple enough that our direct, combinatorial approach remains quite elementary. It revolves around the fact that ultimately, melonic graphs are characterized by their two cuts, so we can study the large $N$ behavior of Feynman graphs under two cuts.

We present the set of Feynman graphs to be studied in Sect. 1, then we introduce melonic graphs and prove Theorem 1 in Sect. 2.
1 Diagrammatic expansion of the SYK model

1.1 Definition of the SYK model

The SYK model has \( N \) Majorana fermions coupled via a \( q \)-body random interaction

\[
H_{\text{SYK}} = i^{q/2} \sum_{i_1, \ldots, i_q} J_{i_1 \ldots i_q} \psi_{i_1}(t) \cdots \psi_{i_q}(t),
\]

where \( J_{i_1 \ldots i_q} \) is the coupling constant. Furthermore, the model is quenched, which by definition means that the coupling \( J \) is a random tensor with a Gaussian distribution such that

\[
\langle J_{i_1 \ldots i_q} \rangle = 0 \quad \text{and} \quad \langle J_{i_1 \ldots i_q} J_{j_1 \ldots j_q} \rangle = 6 J^2 N^{-(q-1)} \prod_{m=1}^{q} \delta_{i_m,j_m}.
\]

The fields \( \psi_i(t) \) satisfy fermionic anticommutation relations \( \{ \psi_i(t), \psi_j(t) \} = \delta_{i,j} \). This anticommutation property excludes graphs with tadpoles (also known as loops, in a graph theoretical language); the model being \((0+1)\)-dimensional, the Feynman amplitude of such a graph is zero.

1.2 Stranded structure of the Feynman graphs

When doing perturbation theory, the interaction term is represented by a vertex with \( q \) incident fermionic lines. Each fermionic line \( m = 1, \ldots, q \) carries an index \( i_m = 1, \ldots, N \) which is contracted at the vertex with a coupling constant \( J_{i_1 \ldots i_q} \). The free energy expands onto those connected, \( q \)-regular and tadpoleless (or loopless, in a graph theoretical language) graphs.

Averaging over the disorder is performed using standard QFT Wick contractions between pairs of \( J \)s, with covariance (2). Each vertex thus receives an additional line, which we represent as a dashed line and call a disorder line. An example of such a Feynman graph of the SYK model is given in Fig. 1.
The above description of the Feynman graphs is however not enough to describe the $1/N$ expansion as it ignores the indices of the random couplings. Indeed a disorder line propagates in fact $q$ field indices, where the field index of fermionic line incident on a vertex is identified with the index of a fermionic line at another vertex. We thus have to represent a disorder line as a line made of $q$ strands where each strand connects fermionic lines as follows

$$\langle J_{i_1} \cdots J_{i_q} J_{j_1} \cdots J_{j_q} \rangle = \sum_{i_1, \ldots, i_q, j_1, \ldots, j_q} \prod_{j=1}^{q} \delta_{i_j j_j} \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \delta_{i_q j_q}$$

Here, the gray disks represent the Feynman vertices.

We denote $\mathbb{G}$ the set of Feynman graphs of the SYK model with stranded disorder lines. For $G \in \mathbb{G}$, we further denote $G_0 \subset G$ the $q$-regular graph obtained by removing the strands of the disorder lines, see Fig. 2. Due to the quenching averaging, the fermionic free energy over the disorder $G_0$ is connected. Moreover, due to the Wick contractions, each vertex has exactly one incident disorder line. This implies that $G$ and $G_0$ have an even number of vertices.

Let us consider graphs with two vertices and $G_{0,\text{min}} = \begin{array}{c} \vdots \end{array}$. There are $q!$ such graphs corresponding to permutations of the strands of the disorder line connecting these two vertices. However, among these $q!$ graphs there is only one which maximizes the number of vertices. This single graph is:

$$G_{\text{min}} = \begin{array}{c} \vdots \end{array}$$
The field index is preserved along each fermionic line and along each strand of disorder lines. This means that there is a free sum for each cycle made of those lines, thereby contributing to a factor $N$.

**Definition 1 (Faces)** A cycle made of alternating fermionic lines and strands of disorder lines is called a face. We denote $F(G)$ the number of faces of $G \in \mathcal{G}$.

$G \in \mathcal{G}$ thus receives a factor $N$ per face. It also receives a factor $N^{-(q-1)}$ for each disorder line. The weight of a graph is thus

$$W(G) = N^{\delta(G)}$$

with

$$\delta(G) = F(G) - (q - 1)V(G)/2$$

where $V(G)$ is the number of vertices. To find the large $N$ limit, we can thus find the graphs which maximize the number of faces at fixed number of vertices.

**Definition 2 (Melonic graphs)** We call dipole the following two-point graph,

$$D = \quad \ldots \quad \overline{\ldots}$$

A melonic move is the insertion of a dipole on a fermionic line

$$- \quad \rightarrow \quad -$$

A melonic graph is a graph obtained from $G_{\text{min}}$ by repeated melonic moves.

An example is provided in Fig. 3.

A dipole is made of two vertices connected by $(q - 1)$ fermionic lines and a disorder line. A priori, there are $q!$ ways of connecting the strands of the disorder line. Among these $q!$ possibilities, we chose for $D$ the one which creates the maximal number of faces.

The main theorem of this paper states:

**Theorem 1** The weight of $G \in \mathcal{G}$ is bounded as

$$\delta(G) \leq 1$$

Moreover, the graphs such that $\delta(G) = 1$ are the melonic graphs.
2 Proof of the melonic dominance in the large $N$ limit

2.1 Some properties of melonic graphs

The number of faces of melonic graphs is easily found.

**Proposition 1** A melonic move adds two vertices and $q - 1$ faces. The number of faces of melonic graphs is

$$F(G) = q + (q - 1) \frac{V(G) - 2}{2}$$

(9)

hence, $\delta(G) = 1$ for melonic graphs.

**Proof** The first statement is trivial from the definition of the melonic move. The number of faces is then obtained by induction. Indeed, $F(G_{\text{min}}) = q$ at $V(G) = 2$ for $G_{\text{min}}$ the only melonic graph with two vertices. The induction is completed by using the first statement. $\delta(G) = 1$ follows from the expression of $\delta(G)$ in (5).

Obviously, one can always find a dipole in a melonic graph, by definition. It will be convenient to notice that there are always more than one.

**Proposition 2** A melonic graph with at least four vertices has at least two dipoles.

**Proof** We proceed by induction on the number of vertices. There is a single melonic graph with four vertices,
which indeed has two dipoles.

Assume that the proposition holds for graphs with at most $V - 2 \geq 4$ vertices and let $G$ be a melonic graph with $V$ vertices. By construction, it can be obtained by a melonic move on the fermionic line $e$ of $G'$, a melonic graph with $V - 2$ vertices. From the induction hypothesis, $G'$ has at least two dipoles. If $e$ is not an edge connecting the two vertices of a dipole, then the melonic move $G' \rightarrow G$ increases the number of dipoles. If $e$ connects two vertices of a dipole in $G'$, then the total number is unchanged from $G'$ to $G$ since cutting $e$ destroys one but the melonic move itself adds one. $\square$

Melonic graphs satisfy a gluing rule which generalizes the melonic move. Let $G_1, G_2 \in \mathbb{G}$ be two melonic graphs and $e_1$ in $G_1, e_2$ in $G_2$ two fermionic lines. If one cuts open $e_1$ in $G_1$ and $e_2$ in $G_2$, then there are two ways to glue the half-edges of $e_1$ with those of $e_2$. To avoid this ambiguity, we use orientations.

**Definition 3** If $(G, e)$ is a graph $G$ with an oriented fermionic line $e$, denote $G^{(e)}$ the two-point graph obtained by cutting $e$ into two half-edges with their induced orientations. For two such graphs $(G_1, e_1)$ and $(G_2, e_2)$, denote $G_1^{(e_1)} \star G_2^{(e_2)}$ the unique connected graph obtained by gluing $G_1^{(e_1)}$ with $G_2^{(e_2)}$ in the only way which respects the orientations of the half-edges,

$$G_1^{(e_1)} \star G_2^{(e_2)} = H_1 H_2$$

**(11)**

**Proposition 3** Let $G_1, G_2 \in \mathbb{G}$ be two melonic graphs and $e_1$ in $G_1, e_2$ in $G_2$ two oriented fermionic lines. Then, $G_1^{(e_1)} \star G_2^{(e_2)}$ is melonic.

**Proof** This is proved by induction on the number of vertices of $G_1$. If $G_1$ is melonic and has two vertices, then $G_1 = G_{\text{min}}$ and the insertion of $G_1^{(e_1)}$ is the melonic move on $e_2$ (for any orientations of $e_1$ and $e_2$).

Assume the proposition holds for graphs $G_1$ with $V - 2$ vertices and consider a new melonic $G_1$ with $V$ vertices. It is obtained from a melonic move performed on a
fermionic line $e'_1$ of the melonic graph $G'_1$. The idea is then to find the line $e_1$ in $G'_1$, form $G'_1(e_1) \ast G'_2(e_2)$ which is melonic from the induction hypothesis and then perform the melonic move on $e'_1$ to get $G'_1(e_1) \ast G'_2(e_2)$ which will thus be melonic too. This is summarized in the following commutative diagram

$$
\begin{align*}
\text{Cutting } e_1 \text{ and} \\
\text{inserting } G'_2(e_2) & \\
\text{Melonic} \\
\text{move on } e'_1 & \\
G'_1 & \longrightarrow G'_1(e_1) \ast G'_2(e_2) \\
G_1 & \longrightarrow G_1(e_1) \ast G_2(e_2) \\
\text{Cutting } e_1 \text{ and} \\
\text{inserting } G'_2(e_2) & \\
\end{align*}
$$

(12)

We thus want to use the path from $G'_1$ to $G'_1(e_1) \ast G'_2(e_2)$ which goes right then down. When $e_1$ and $e'_1$ are distinct in $G'_1$, this is straightforward:

$$
G'_1(e_1) \ast G'_2(e_2) = H'_1 \bigcirc e'_1 \bigcirc H_2
$$

(13)

By the induction hypothesis, this graph is melonic. Then, it remains after the melonic insertion on $e'_1$ (by definition).

However, if $e_1$ is incident to or part of the dipole which is inserted from $G'_1$ to $G_1$, it means it does not exist in $G'_1$, as it is created by the melonic move. We distinguish two cases.

\begin{itemize}
  \item $e_1$ is a fermionic line connecting the two vertices of the dipole. Then from Proposition 2, we know that $G_1$ has at least one other dipole. Therefore, one can redefine $G'_1$ has the melonic graph obtained from $G_1$ by removing the latter. Then, the fermionic line $e_1$ can be identified without issues in $G'_1$ and the reasoning above applies.

  \item $e_1$ is incident to the dipole, i.e., $(G_1, e_1) = H_1 \bigcirc e_1 \bigcirc H_1$. Then, $G_1(e_1) \ast G_2(e_2)$ has the form

$$
G_1(e_1) \ast G_2(e_2) = H_1 \bigcirc e_1 \bigcirc H_2
$$

(14)
\end{itemize}
The graph $G'_1$ is $G'_1 = G_1 \overset{\hat{e}}{\to} G_1'$ and is melonic. From the induction hypothesis, $G'_1 \ast G'_2 = G_1 \overset{\hat{e}}{\to} G_2$ is melonic. Then, so is $G'_1 \ast G'_2$ since it is obtained by a melonic move on $G'_1 \ast G'_2$.

$\square$

### 2.2 2-cuts

We recall that following (5), we are looking for the graphs which maximize the number of faces at fixed number of vertices. Let us denote the maximal number of faces on $V$ vertices

$$F_{\text{max}}(V) = \max_{\{G \in \mathbb{G}, V(G) = V\}} F(G)$$

and the set of graphs maximizing $F(G)$ at fixed $V$

$$\mathbb{G}_{\text{max}}(V) = \{G \in \mathbb{G} \text{ s.t. } V(G) = V \text{ and } F(G) = F_{\text{max}}(V)\}.$$  

A two-cut is a pair of edges in a graph whose removal (or equivalently cutting) disconnects the graph.

**Proposition 4** Let $G \in \mathbb{G}$ and $e_1, e_2$ two fermionic lines in $G$ which belong to the same face. If $\{e_1, e_2\}$ is not a 2-cut in $G$, then $G \notin \mathbb{G}_{\text{max}}(V(G))$.

In other words, if there exist two lines in the same face which do not form a two-cut, the graph is not dominant at large $N$.

**Proof** There are two cases to distinguish: whether $\{e_1, e_2\}$ is a two-cut or not in $G_0$, $\{e_1, e_2\}$ is not a two-cut in $G_0$. We draw $G$ as

$$G = e_1 \overset{\text{dotted line}}{\to} e_2$$

where the dotted line represents the paths alternating fermionic lines and strands of disorder lines which constitute the face of $e_1$ and $e_2$. Now, consider $G'$ obtained by cutting $e_1$ and $e_2$ and regluing the half lines in the unique way which creates one additional face,
$G'$ is connected since $\{e_1, e_2\}$ is not a two-cut in $G_0$, and hence $G' \in \mathcal{G}$. No other faces of $G$ are affected. Therefore, $F(G') = F(G) + 1$, and thus $G \not\in G_{\max}(V(G))$.

$\{e_1, e_2\}$ is a 2-cut in $G_0$. An example of this situation is when $G_0$ is melonic but $G$ is not, the disorder lines are added in a way which does not respect melonicity. In this case, $G$ looks like

\begin{equation}
G = \begin{array}{c}
\includegraphics{diagram1.png}
\end{array}
\end{equation}

(19)

i.e., $H_L$ and $H_R$ are both connected, and the only lines between them are $e_1, e_2$ and some disorder lines. Consider $G'$ obtained by cutting $e_1$ and $e_2$ and regluing the half lines as follows

\begin{equation}
G' = \begin{array}{c}
\includegraphics{diagram2.png}
\end{array}
\end{equation}

(20)

Notice that $G' \not\in \mathcal{G}$ since $G'_0$ consists of two connected components $G'_0L$ and $G'_0R$.

Consider a disorder line $e_0$ between them. It joins two vertices $v_L$ in $G'_0L$ and $v_R$ in $G'_0R$. We perform the contraction of the disorder line $e_0$ as follows
Diagrammatic proof of the large $N$ melonic dominance in the…

\[ G' = H_L \quad \quad \quad \quad H_R \]

\[ \rightarrow G'' = H_L \quad \quad \quad \quad H_R \]

It removes $v_L$, $v_R$ and $e_0$ and joins the pending fermionic lines which were connected by the strands of $e_0$. The key point is that $G'' \in \mathbb{G}$ now since the contraction of $e_0$ connects the two disjoint components of $G'_0$ by $q$ fermionic lines.

Let us now analyze the variations of the number of faces from $G$ to $G''$. First from $G$ to $G'$: in $G$, the lines $e_1$, $e_2$ belong to the same face, while $e_L$ and $e_R$ may or may not belong to the same face in $G'$, hence

\[ F(G) \leq F(G'). \tag{22} \]

Then, the contraction of $e_0$ does not change the number of faces. Indeed, the faces of $G'$ which do not go along $e_0$ are not affected. As for those which go along $e_0$, they follow paths

\[ v_{Li} \rightarrow v_L \rightarrow v_R \rightarrow v_{Ri} \tag{23} \]
for $i = 1, \ldots, q$ (some of those $q$ paths may belong to common faces). In $G''$, they become paths going directly from $v_{Li}$ to $v_{Ri}$. There is thus a one-to-one correspondence between the faces of $G'$ and those of $G''$. Therefore, $F(G) \leq F(G'')$.

To conclude the proof, notice that $G''$ has two vertices less than $G$. Therefore, we can perform a melonic insertion on any fermionic line of $G''$ to get a graph $\tilde{G} \in G$ with $V(G) = V(\tilde{G})$ and

$$F(\tilde{G}) = F(G'') + q - 1$$

as in Proposition 1. For $q > 1$, it comes that $F(G) < F(\tilde{G})$ and thus $G \notin G_{\text{max}}(V(G))$.

Using the notation of Proposition 4, Theorem 1 rewrites as

$$\bigcup_{V \text{ even}} G_{\text{max}}(V) = \{G \in G \text{ s.t. } \delta(G) = 1\} = \{\text{Melonic graphs}\}.$$ (25)

### 2.3 Large $N$ limit

We now prove Theorem 1.

**Proof** We proceed by induction.

First, $G_{\text{min}}$ is melonic by definition. It has $F(G_{\text{min}}) = q$ and $V(G_{\text{min}}) = 2$ hence satisfies $\delta(G_{\text{min}}) = 1$. Since it is the only graph on two vertices, the theorem indeed holds on two vertices.

Let $V \geq 4$ even. We assume the theorem is true up to $V - 2$ vertices and consider $G \in G$ with $V(G) = V$ vertices.

We need to investigate pairs $\{e_1, e_2\}$ with $e_1, e_2$ two fermionic lines belonging in a common face. Notice that such a pair exists. If it was not the case, then all faces would be of length 2 (i.e., one fermionic line and one disorder line) which implies $G = G_{\text{min}}$, which is impossible since $G$ has $V \geq 4$ vertices.

Let $\{e_1, e_2\}$ be a pair of fermionic lines in the same face. Due to Proposition 4, we know that it is a two-cut in $G$. The graph therefore takes the form

$$G = \begin{array}{c}
H_L
\end{array}
\begin{array}{c}
\bigcirc
\end{array}
\begin{array}{c}
H_R
\end{array}
\begin{array}{c}
e_1
\end{array}
\begin{array}{c}
e_2
\end{array}$$

where $H_L, H_R$ are connected, two-point graphs (in the sense that $e_1$ and $e_2$ are hanging out). We cut $e_1$ and $e_2$ and glue the resulting half lines to close $H_L$ and $H_R$ into $G_L, G_R$, and use “reverse” orientations as follows

$$G_L = \begin{array}{c}
H_L
\end{array}
\begin{array}{c}
\bigcirc
\end{array}
\begin{array}{c}
e_L
\end{array}
\begin{array}{c}
G_R = \begin{array}{c}
e_R
\end{array}
\begin{array}{c}
\bigcirc
\end{array}
\begin{array}{c}
H_R
\end{array}$$

(27)
In the notation of Proposition 3, we thus have

\[ G = G_{L}^{(e_{L})} \star G_{R}^{(e_{R})}. \]  

(28)

Since \( e_{1} \) and \( e_{2} \) are in the same face, we have

\[ F(G) = F(G_{L}) + F(G_{R}) - 1, \]  

(29)

and \( F(G) \) is maximal if and only if \( F(G_{L}) \) and \( F(G_{R}) \) are. From the induction hypothesis, this requires \( G_{L} \) and \( G_{R} \) to be melonic. Then, \( G = G_{L}^{(e_{L})} \star G_{R}^{(e_{R})} \) is melonic too according to Proposition 3.

We get as a corollary of Theorem 1 and Proposition 4 a characterization of melonic graphs.

**Corollary 1** \( G \in \mathbb{G} \) is melonic if and only if all pairs \( \{e_{1}, e_{2}\} \) of fermionic lines which belong in a common face are two cuts.

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