Product and anti-Hermitian structures on the tangent space

E. Peyghan, A. Razavi and A. Heydari
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Abstract

Noting that the complete lift of a Riemannian metric $g$ defined on a differentiable manifold $M$ is not 0-homogeneous on the fibers of the tangent bundle $TM$. In this paper we introduce a new lift $\tilde{g}_2$ which is 0-homogeneous. It determines on slit tangent bundle a pseudo-Riemannian metric, which depends only on the metric $g$. We study some of the geometrical properties of this pseudo-Riemannian space and define the natural almost complex structure and natural almost product structure which preserve the property of homogeneity and find some new results.

Keywords: Almost complex structure, almost anti-Hermitian structure, almost product structure, complete lift metric, 0-homogeneous lift.

1. Introduction.

The importance of the complete lift $g_2$ of a Riemannian metric $g$ is well known in Riemannian geometry, Finsler geometry and Physics, and has many applications in Biology too (see [1]). The tensor field $g_2$ determines a pseudo-Riemannian structure on slit tangent bundle $\tilde{TM} = TM \setminus \{0\}$, but $g_2$ is not 0-homogeneous on the fibers of the tangent bundle $TM$. Therefore, we cannot study some global properties of the pseudo-Riemannian space $(\tilde{TM}, g_2)$. For instance we can not prove a theorem of Gauss-Bonnet type for this space (see [4]).

In this paper, we define a new kind of lift $\tilde{g}_2$ to $TM$ of the Riemannian metric $g$. Thus $\tilde{g}_2$ determines on $\tilde{TM}$ a pseudo-Riemannian structure, which is 0-homogeneous on the fibers of $TM$ and depends only on $g$. Some geometrical properties of $\tilde{g}_2$ such as the Levi-Civita connection are studied.

Almost complex and almost product structures are among the most important geometrical structures which can be considered on a manifold. Geometric properties of these structures have been studied in (see [2] to [7], [11], [12], [15], [16]). We introduce the natural almost complex and product structures $\tilde{J}$ and $\tilde{Q}$ which depend only on $g$ and preserve the property of homogeneity. Then we get almost anti-Hermitian structure $(\tilde{g}_2, \tilde{J})$ and almost product structure $(\tilde{g}_2, \tilde{Q})$. By considering twin tensor of $\tilde{g}_2$, we construct almost para-Hermitian and Hermitian structures on $\tilde{TM}$.

Let $M$ be a smooth manifold, $TM$ its tangent bundle and $\chi(M)$ the algebra of vector fields on $M$. A $K$-structure on $M$ is a family of endomorphisms $K$ on $TM$ such that $K^2 = \varepsilon I$, where $\varepsilon = \pm 1$. Thus $\varepsilon = 1$ corresponds to an almost product structure, while $\varepsilon = -1$ provides an almost complex structure.
A $K$-structure is \textit{integrable} if and only if there exists an a linear torsionless connection on $M$ such that $\nabla K = 0$, or equivalently the Nijenhuis tensor $N_K$ vanishes. In this case, $\nabla$ is called \textit{almost complex (product) connection} if $K$ be an almost complex (product) structure.

If $g$ is a metric on $M$ such that $g(KX, KY) = \sigma g(X, Y)$, $\sigma = \pm 1$, for arbitrary vector fields $X$ and $Y$ on $M$, then we shall say that the metric $g$ is $K$-metric.

The definition above unifies the following four cases:

The case $\varepsilon = 1, \sigma = 1$ corresponds to the (pseudo-) Riemannian \textit{almost product manifold} $(M, g, K)$, the case $\varepsilon = 1, \sigma = -1$ provides the \textit{almost para-Hermitian manifold} $(M, g, K)$, the case $\varepsilon = -1, \sigma = 1$ is known as the \textit{almost Hermitian manifold} $(M, g, K)$, and finally the case $\varepsilon = -1, \sigma = -1$ corresponds to the \textit{almost anti-Hermitian manifold} $(M, g, K)$.

Let us introduce a $(0, 2)$ tensor field $h$, the twin of $g$, by $h(X, Y) = g(KX, Y)$.

Then

$$h(X, Y) = \varepsilon \sigma h(Y, X), h(KX, KY) = \sigma h(X, Y).$$

Notice that for $\varepsilon \sigma = 1$, the twin tensor is a metric, while for $\varepsilon \sigma = -1$ the twin tensor is a 2-form.

Let $\psi$ be a $(0, 3)$ tensor fields defined by the formula

$$\psi(X, Y, Z) = g((\nabla_X K)Y, Z) \equiv (\nabla_X h)(Y, Z) \quad (1.1)$$

Obviously, if the tensor fields $\psi$ vanishes then $\nabla K = 0$ for a torsionless (Levi-Civita) connection and the Nijenhuis tensor $N_K$ is forced to vanish, too ($[2]$).

2. \textbf{The Complete Lift}

Let $\Gamma^k_{ij}$ be the coefficients of the Riemannian connection of $M$, then $N_j^h = \Gamma^h_{0j} = y^a \Gamma_{aj}^h(x)$ can be regarded as coefficients of the canonical nonlinear connection $N$ of $TM$, where $(x^a, y^b)$ are the induced coordinates in $TM$.

$N$ determines a horizontal distribution on $\overline{TM}$, which is supplementary to the vertical distribution $V$, such that, we have:

$$T_u \overline{TM} = N_u \oplus V_u, \quad \forall u \in \overline{TM}. \quad (2.1)$$

The adapted basis to $N$ and $V$ is given by $\{X_h, X^?_i\}$ where

$$X_h = \frac{\partial}{\partial x^h} - y^a \Gamma_{ah}^m \frac{\partial}{\partial y^m}, \quad X^?_i = \frac{\partial}{\partial y^i} \quad (2.2)$$

and its dual basis is $\{dx^i, dy^i\}$ where

$$\delta y^i = dy^i + y^a \Gamma_{ai}^j dx^j. \quad (2.3)$$

The indices $a, b, \ldots, a, \overline{a}, \overline{b}, \ldots$, run over the range $\{1, 2, \ldots, n\}$. The summation convention will be used in relation to this system of indices. By straightforward calculations, we have the following lemma.

\textbf{Lemma 1}. The Lie bracket of the adapted frame of $TM$ satisfies the following:

1) $[X_i, X_j] = y^a K_{jia}^m X^m_i$,
2) $[X_i, X^?_j] = \Gamma^?_{ij}^m$,
3) $[X^?_i, X^?_j] = 0$.
where $K_{ij}^m$ denote the components of the curvature tensor of $M$.

Let $(M, g)$ be a Riemannian space, $M$ being a real $n$-dimensional manifold and $(TM, \pi, M)$ its tangent bundle. On a domain $U \subset M$ of a local chart, $g$ has the components $g_{ij}(x), (i, j, ... = 1, ..., n)$. Then on the domain of chart $\pi^{-1}(U) \subset TM$ we consider the functions $g_{ij}(x, y) = g_{ij}(x), \forall (x, y) \in \pi^{-1}(U)$ and put

$$\|y\| = \sqrt{g_{ij}(x)y^iy^i}.$$  

Then, $\|y\|$ is globally defined on $TM$, differentiable on $\overline{TM}$ and continuous on the null section.

The complete lift of $g$ to $TM$ is defined by

$$g_2(x, y) = 2g_{ij}(x)dx^i\delta y^i, \forall (x, y) \in \overline{TM}.$$  

Then, $g_2$ is not $0$-homogeneous on the fibers of $TM$.

Namely, for the homothety $h_t : (x, y) \rightarrow (x, ty)$ for all $t \in R^+$ we get

$$(g_2 \circ h_t)(x, y) = 2tg_{ij}(x)dx^i\delta y^i = tg_2(x, y) \neq g_2(x, y).$$

On $\overline{TM}$ we define an almost complex structure $J$ by

$$J(X_i) = -X_\tau, \quad J(X_\tau) = X_i, \quad i = 1, ..., n.$$  

It is known that $(\overline{TM}, J, g_2)$ is an almost anti-Hermitian manifold. Moreover, the integrability of the almost complex structure $J$ implies that $(M, g)$ is locally flat. (see [7])

Also, we define almost product structure $Q$ on $\overline{TM}$ by

$$Q(X_i) = X_\tau, \quad Q(X_\tau) = X_i, \quad i = 1, ..., n.$$  

Then, $(\overline{TM}, Q, g_2)$ is an almost product manifold. Also, the integrability of the almost product structure $Q$ implies that $(M, g)$ is locally flat.

The previous space, called "the geometrical model on $TM$ of the Riemannian space $(M, g)$", is important in the study of the geometry of initial Riemannian space $(M, g)$ ([6], [7]).

3. The 0-homogeneous lift of the Riemannian metric $g$

We can eliminate the inconvenience of the complete lift, introducing a new kind of lift to $TM$ of the Riemannian metric $g$. Then we obtain the Levi-Civita connection for this metric.

**Definition**. Let $\tilde{g}_2$ be a the tensor field on $\overline{TM}$ defined by

$$\tilde{g}_2(x, y) = \frac{2}{\|y\|}g_{ij}(x)dx^i\delta y^i.$$  

where $\|y\|$ was defined in (2.4). Then $\tilde{g}_2$ is called the 0-homogeneous lift of the Riemannian metric $g$ to $\overline{TM}$.

We get, evidently:

**Theorem 2.** The following properties hold:

1. The pair $(\overline{TM}, \tilde{g}_2)$ is a pseudo-Riemannian space, depending only on the metric $g$.
2. $\check{g}_2$ is 0-homogeneous on the fibers of the tangent bundle $TM$.

In order to study the geometry of the pseudo-Riemannian space $(\widetilde{T}\mathcal{M}, \check{g}_2)$ we can apply the theory of the $(h, \nu)$-Riemannian metric on $TM$ given in the books [6], [7] and [9]. Looking at the relation (2.5) and (3.1) we can assert:

**Theorem 3.** The lifts $g_2$ and $\check{g}_2$ coincide on the hyper unit tangent sphere $g_0(x_0)y^i y^j = 1$, for every point $x_0 \in M$.

Let $\check{\nabla}$ be the Riemannian connection of $TM$ with coefficient $\Gamma_{BC}^A$, that is:

$$
\check{\nabla}_{x_t}X_j = \check{\Gamma}_{ij}^m X_m + \check{\Gamma}_{im}^j X_m,
\check{\nabla}_{x_t}X_j = \check{\Gamma}_{ij}^m X_m + \check{\Gamma}_{jm}^i X_m,
\check{\nabla}_{x^e}X_j = \check{\Gamma}_{ij}^m X_m + \check{\Gamma}_{jm}^i X_m
$$

(3.2)

Then, we have

$$
\check{\nabla}_{x_t} dx^h = -\check{\Gamma}_{m^i}^h dx^m - \check{\Gamma}_{m^i}^h dy^m,
\check{\nabla}_{x_t} dy^h = -\check{\Gamma}_{m^i}^h dx^m - \check{\Gamma}_{m^i}^h dy^m,
\check{\nabla}_{x^e} dx^h = -\check{\Gamma}_{m^i}^h dx^m - \check{\Gamma}_{m^i}^h dy^m,
\check{\nabla}_{x^e} dy^h = -\check{\Gamma}_{m^i}^h dx^m - \check{\Gamma}_{m^i}^h dy^m
$$

(3.3)

Since the torsion tensor $T(X,Y)$ of $\check{\nabla}$ defined by $T(X,Y) = \check{\nabla}_X Y - \check{\nabla}_Y X - [X,Y]$ vanishes, we have the following relations by means of Lemma 1 and (3.2).

(1) $\check{\Gamma}_{ij}^h = \Gamma_{ij}^h$
(2) $\check{\Gamma}_{ji}^h = \Gamma_{ji}^h + y^a K_{ja}$
(3) $\check{\Gamma}_{ij}^h = \Gamma_{ij}^h$
(4) $\check{\Gamma}_{ij}^h = \Gamma_{ij}^h + \Gamma_{ji}^h$
(5) $\check{\Gamma}_{ij}^h = \Gamma_{ij}^h$
(6) $\check{\Gamma}_{ij}^h = \Gamma_{ij}^h$

(3.4)

Furthermore, we have the following lemma.

**Lemma 4.** The connection coefficients $\Gamma_{BC}^A$ of $\check{\nabla}$ of the complete metric $\check{g}_2$ satisfy the following relations:

(1) $\Gamma_{ji}^h = \Gamma_{ij}^h$
(2) $\Gamma_{ji}^h = y^a K_{aji}$
(3) $\check{\Gamma}_{ij}^h = \check{\Gamma}_{ij}^h$
(4) $\check{\Gamma}_{ij}^h = \check{\Gamma}_{ij}^h + \check{\Gamma}_{ji}^h$
(5) $\check{\Gamma}_{ij}^h = \check{\Gamma}_{ij}^h$
(6) $\check{\Gamma}_{ij}^h = \check{\Gamma}_{ij}^h$
(7) $\check{\Gamma}_{ij}^h = \check{\Gamma}_{ij}^h$
(8) $\check{\Gamma}_{ij}^h = \check{\Gamma}_{ij}^h$

(3.5)

**Proof.** The condition compatibility $\check{\nabla}$ is equivalent with following equations:

$$
g_{ij} \check{\Gamma}_{jm}^r + g_{jr} \check{\Gamma}_{im}^r = 0
\check{\Gamma}_{jm}^r - \check{\Gamma}_{jm}^r + g_{jr} (\check{\Gamma}_{im}^r - \check{\Gamma}_{im}^r) = 0
\check{\Gamma}_{jm}^r + g_{jr} \check{\Gamma}_{jm}^r = 0
$$

(3.6)

(3.7)
From (3.10) we have $\bar{\Gamma}_{\tau \tau} = 0$, thus we get (7). From (3.4), (3.9) and (3.7), we have

$$g_{ij} \bar{\Gamma}_{\tau m}' - g_{j}^{\rho} \bar{\Gamma}_{\tau \rho} + \frac{1}{\| y \|^2} g_{ij} y_m = 0$$

Thus we get (3). From (3) and (3.4), we have (4).

From (3.9), (4) and (3.4), we have

$$g_{ij} \bar{\Gamma}_{\tau m}' + \frac{1}{2 \| y \|^2} g_{ij} y_m + \frac{1}{2 \| y \|^2} g_{ij} y_m + \frac{1}{\| y \|^2} g_{ij} y_m = 0,$$

then we obtain (8).

From (3.4) and (5) we have

$$g_{ij} \bar{\Gamma}_{jm} - g_{j}^{\rho} \bar{\Gamma}_{mn} = g_{ij} (\bar{\Gamma}_{mn} - \bar{\Gamma}_{mj}) = g_{ij} (\bar{\Gamma}_{jm} + \bar{\Gamma}_{mj}) + y_a (K_{jiam} - K_{imaj}),$$

thus we get (2).

From (3.4) and (8.1) and (8.2), we have

$$g_{ij} \bar{\Gamma}_{jm} = -g_{j}^{\rho} \bar{\Gamma}_{mn} = g_{ij} (\bar{\Gamma}_{mn} - \bar{\Gamma}_{mj}) = g_{ij} (\bar{\Gamma}_{jm} - \bar{\Gamma}_{mj})$$

thus we obtain (5) and (6). From (9.1) and (9.2), we have (1).

4. The almost anti-Hermitian structure $(\tilde{g}, \tilde{J})$

The almost complex structure $J$ defined in (2.6) has not the property of homogeneity. The $F(\tilde{T}M)$-linear mapping $J : \chi(\tilde{T}M) \rightarrow \chi(\tilde{T}M)$, applies the 1-homogeneous vector fields $X_i$ into 0-homogeneous vector fields $X_i$, $i = 1, \ldots, n$. Therefore, we consider the $F(\tilde{T}M)$-linear mapping $\tilde{J} : \chi(\tilde{T}M) \rightarrow \chi(\tilde{T}M)$, given on the adapted basis by

$$\tilde{J}(X_i) = -\| y \| X_i, \quad \tilde{J}(X_i) = \frac{1}{\| y \|} X_i, \quad i = 1, \ldots, n.$$
**Theorem 5.** \((\widetilde{TM}, \tilde{g}_2, \tilde{J})\) is an almost anti-Hermitian manifold.

**Proof.** It follows easily that
\[
\tilde{g}_2(JX_1, JX_1) = -\tilde{g}_2(X_1, X_1), \quad \tilde{g}_2(JX_\tau, JX_\tau) = -\tilde{g}_2(X_\tau, X_\tau).
\]

Hence
\[
\tilde{g}_2(JX, JY) = -\tilde{g}_2(X, Y), \quad \forall X, Y \in \chi(\widetilde{TM}).
\]

**Proposition 6.** The Nijenhuis tensor field of the almost complex structure \(\tilde{J}\) on \(\widetilde{TM}\) is given by
\[
N_j(X, Y) = (y_j \delta^a_i - y_i \delta^a_j - y^a K_{jia}^a)X_\tau,
\]
\[
N_j(X_\tau, X_\tau) = \frac{1}{\|y\|^2} (y_j \delta^a_i - y_i \delta^a_j - y^a K_{jia}^a)X_\tau,
\]
\[
N_j(X_\tau, X_\tau) = \frac{1}{\|y\|^2} (y_j \delta^a_i - y_i \delta^a_j + y^a K_{jia}^a)X_\tau.
\]

**Proof.** Recall that the Nijenhuis tensor field \(N_j\) defined by \(\tilde{J}\) is given by
\[
N_j(X, Y) = \tilde{J}[X, \tilde{J}Y] - \tilde{J}[\tilde{J}X, Y] - \tilde{J}[X, Y] - [X, Y], \quad \forall X, Y \in \chi(\widetilde{TM}).
\]
Replacing the basis \((X_1, X_\tau)\) in the above formula and using following relation:
\[
X_i(\|y\|) = 0, \quad X_\tau(\|y\|) = \frac{y_i}{\|y\|}
\]
We get the proof.

**Theorem 7.** The almost complex structure \(\tilde{J}\) is a complex structure on \(\widetilde{TM}\) if and only if the Riemannian space \((M, g)\) is of constant sectional curvature 1.

**Proof.** From the condition \(N_j = 0\), one obtains:
\[
\{K_{jia}^a - (g_{ia} \delta^a_j - g_{ja} \delta^a_i)\} y^a = 0.
\]
Differentiating with respect to \(y^b\), taking \(y^c = 0\) \(\forall a \in \{1, \ldots, n\}\), it follows that the curvature tensor field of \(\nabla\) has the expression
\[
K_{jik}^a = g_{ia} \delta^a_j - g_{ja} \delta^a_i = (4.2)
\]
Using by the Schur theorem (in the case where \(M\) is connected and \(\dim M \geq 3\)) it follows that \((M, g)\) has the constant sectional curvature 1.

**Corollary 8.** \((\widetilde{TM}, \tilde{g}_2, \tilde{J})\) is an anti-Hermitian manifold if and only if the space \((M, g)\) is of constant sectional curvature 1.

From (4.2) we have
\[
R_y = (n-1)g_y, \quad (n > 1)
\]
where \(R_y\) is the Ricci tensor and
\[
S = n(n-1).
\]
where \(S\) is the scalar tensor.
Corollary 9. If the structure $(\tilde{g}_2, \tilde{J})$ is a Hermitian structure on $\tilde{T}M$ then $(M, g)$ is an Einstein space with positive scalar curvature.

Since $R_{ij} = R_{ji}$ then from (4.3) we get:

**Corollary 10.** If the almost complex structure $\tilde{J}$ is a complex structure then $(M, R_{\tilde{g}}(x))$ is a Riemannian space.

5. The almost product structure $(\tilde{g}_2, \tilde{Q})$

The almost product structure $Q$ defined in (2.7) has not the property of homogeneity. The $F(\tilde{T}M)$-linear mapping $Q : \chi(\tilde{T}M) \to \chi(\tilde{T}M)$, applies the 1-homogeneous vector fields $X_i$ into 0-homogeneous vector fields $X_j(i = 1, ..., n)$. Therefore, we consider the $F(\tilde{T}M)$-linear mapping $\tilde{Q} : \chi(\tilde{T}M) \to \chi(\tilde{T}M)$, given on the adapted basis by

$$\tilde{Q}(X_i) = ||y|| X_j, \quad \tilde{Q}(X_j) = \frac{1}{||y||} X_i, \; (i = 1, ..., n). \; \text{(5.1)}$$

Obviously, $\tilde{Q}$ is a tensor field of type (1,1) on $\tilde{T}M$, that is homogeneous on the fibers of $TM$. It is not difficult to prove:

**Theorem 11.** $(\tilde{T}M, \tilde{g}_2, \tilde{Q})$ is an almost product manifold.

In order to find conditions that $\tilde{Q}$ be a product structure, we have to put zero for the Nijenhuis tensor field of $\tilde{Q}$, $N_{\tilde{g}}(X, Y) = \tilde{Q}(\tilde{Q}X, Y) - \tilde{Q}(\tilde{Q}Y, X) - [X, \tilde{Q}Y] + [Y, \tilde{Q}X], \; \forall X, Y \in \chi(\tilde{T}M)$.

**Theorem 12.** $(\tilde{T}M, \tilde{g}_2, \tilde{Q})$ is a product manifold if and only if the space $(M, g)$ is of constant sectional curvature -1.

**Proof.** Similar to proposition 6 and theorem 7, by putting $N_{\tilde{g}} = 0$ we get

$$K_{jii} = -(g_{ii} \tilde{S}_j^i - g_{ji} \tilde{S}_i^j). \; \text{(5.2)}$$

Therefore, using by the Schur theorem, it follows that $(M, g)$ has the constant sectional curvature -1.

**Theorem 13.** If the structure $(\tilde{g}_2, \tilde{Q})$ is a product structure on $\tilde{T}M$ then $(M, g)$ is an Einstein space with negative scalar curvature.

**Proof.** From (5.2) we have $R_{ij} = (1 - n)g_{ij}, S = n(1 - n)$ for $n > 1$.

Since $R_{ij} = R_{ji}$ then we get:

**Corollary 14.** If the almost product structure $\tilde{Q}$ is a product structure then $(M, R_{\tilde{g}}(x))$ is a Riemannian space.
6. Almost Hermitian and para-Hermitian structures on $\overline{TM}$

In this section, we get twin tensor of metric $\tilde{g}_2$ and by using it introduce Almost Hermitian and para-Hermitian structures on $\overline{TM}$. Then we show that these structures are not Kahlerian or para-Kahlerian.

**Lemma 15.** The twin tensor of structure $(\tilde{g}_2, \tilde{J})$ is a metric that is given by

$$ h_j = -2g_{ij}dx^i dx^j + \frac{2g_{ij}}{\|y\|^2} \delta y^i \delta y^j $$

**Proof.** From relation $h_j(X, Y) = \tilde{g}_2(JX, Y)$ we have:

$$ h_j(X_i, X_j) = -\|y\| \tilde{g}_2(X_\tau, X_\tau) = -2g_{ij}, h_j(X_\tau, X_\tau) = \frac{1}{\|y\|^2} \tilde{g}_2(X_i, X_\tau) = \frac{2}{\|y\|^2} g_{ij} $$

$$ h_j(X_i, X_\tau) = -\|y\| \tilde{g}_2(X_\tau, X_\tau) = 0. $$

**Theorem 16.** $(\overline{TM}, h_j, \tilde{Q})$ is an almost para-Hermitian manifold.

**Proof.** Straightforward computations, we obtain

$$ h_j(\tilde{Q}X_i, \tilde{Q}X_j) = 2g_{ij} = -h_j(X_i, X_j), h_j(\tilde{Q}X_\tau, \tilde{Q}X_\tau) = -\frac{2}{\|y\|^2} g_{ij} = -h_j(X_\tau, X_\tau) $$

$$ h_j(\tilde{Q}X_\tau, \tilde{Q}X_\tau) = 0 = -h_j(X_\tau, X_\tau) $$

Therefore

$$ h_j(\tilde{Q}X_i, \tilde{Q}X_j) = -h_j(X_i, Y) $$

By definition $\Omega^{h_j}_\tilde{Q}(X, Y) = h_j(\tilde{Q}X, Y)$, the associated almost simplectic structure $\Omega^{h_j}_\tilde{Q}$ is given in adapted basis by

$$ \Omega^{h_j}_\tilde{Q} = \frac{4}{\|y\|^2} g_{ij}(x)dx^i \wedge \delta y^j. $$

**Theorem 17.** The space $(\overline{TM}, h_j, \tilde{Q})$ cannot be an almost para-Kählerian manifold.

**Proof.** since, $d(\frac{1}{\|y\|^2}) = -\frac{1}{\|y\|^2} d\|y\|$ and $d(g_{ij}dx^i \wedge \delta y^j) = 0$ then, the exterior differential of $\Omega^{h_j}_\tilde{Q}$ satisfies the equation:

$$ d\Omega^{h_j}_\tilde{Q} = -\frac{4}{\|y\|^2} d\|y\| \wedge \Omega^{h_j}_\tilde{Q}. $$

It follows, easily that $d\Omega^{h_j}_\tilde{Q} \neq 0$ on $\overline{TM}$, i.e., $\Omega^{h_j}_j$ is not closed.

From theorem 12,16, we have:

**Theorem 18.** $(\overline{TM}, h_j, \tilde{Q})$ is a para-Hermitian manifold if and only if the space $(M, g)$ is of constant sectional curvature -1.

**Lemma 19.** The Levi-Civita connection coefficients $\tilde{\nabla}^{h_j}$ of $h_j$ satisfy the following relations:
Theorem 20. $\nabla^h_j$ is an almost complex connection.

**Proof.** From (1.1) we have

$$\tilde{g}_2((\nabla^h_j J)Y, Z) = (\nabla^h_j h_j)(Y, Z)$$

Since $\nabla^h_j$ is Levi-Civita connection for $h_j$ then

$$\tilde{g}_2((\nabla^h_j J)Y, Z) = 0$$

i.e. $\nabla^h_j J = 0$.

Similarly previous case, the twin tensor of structure $(\tilde{g}_2, \tilde{Q})$ is a metric that is

$$h_Q = 2g_\delta dx^i dy^i + \frac{2g_\delta}{\|y\|^2} \delta y^i \delta y^j$$

Obviously, $h_Q$ is 0-homogeneous on the fibers of $TM$.

**Theorem 21.**

1. $(TM, h_Q, J)$ is an almost Hermitian structure on $TM$.
2. The associated almost simplectic structure $\Omega^h_Q$ is given in adapted basis by

$$\Omega^h_Q = \frac{4}{\|y\|^2} g_\delta(x) \delta y^i \wedge dx^i$$

**Theorem 22.** The space $(TM, h_Q, J)$ cannot be an almost Kählerian manifold.

From corollary 8 and theorem 21 we obtain following theorem.

**Theorem 23.** $(TM, h_Q, J)$ is a Hermitian manifold if and only if the space $(M, g)$ is of constant sectional curvature 1.

**Lemma 24.** The Levi-Civita connection coefficients $\nabla^h_Q$ of $h_Q$ satisfy the following relations:

1. $\Gamma^h_{ji} = \Gamma^h_{ji}$,
2. $\Gamma^h_{ji} = \frac{1}{2} y^a K^h_{jia}$,
3. $\Gamma^h_{ji} = -\frac{1}{2\|y\|^2} y^a K^h_{aji}$,
4. $\Gamma^h_{ji} = -\frac{1}{2\|y\|^2} y^a K^h_{ajj}$,
5. $\Gamma^h_{ji} = \Gamma^h_{ji}$,
6. $\Gamma^h_{ji} = 0$,
7. $\Gamma^h_{ji} = 0$,
8. $\Gamma^h_{ji} = \frac{1}{\|y\|^2} (g_{ji} y^h - \delta^h y_j - \delta^h y_j)$.

**Theorem 25.** $\nabla^h_Q$ is an almost product connection.
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Department of Mathematics and Computer Science
AmirKabir University.
Tehran.Iran.
E-mail address: e_peyghan@aut.ac.ir
E-mail address: arazavi@aut.ac.ir

Faculty of Science of Tarbiatmodares University.
Tehran.Iran.
E-mail address: abasheydari@gmail.com