Decay of periodic entropy solutions to degenerate nonlinear parabolic equations

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Abstract

Under a precise nonlinearity-diffusivity condition we establish the decay of space-periodic entropy solutions of a multidimensional degenerate nonlinear parabolic equation.

1 Introduction

In the half-space $\Pi = \mathbb{R}_+ \times \mathbb{R}^n$, $\mathbb{R}_+ = (0, +\infty)$, we consider the nonlinear parabolic equation

$$u_t + \text{div}_x(\varphi(u) - a(u)\nabla_x u) = 0,$$

(1.1)

where the flux vector $\varphi(u) = (\varphi_1(u), \ldots, \varphi_n(u))$ is merely continuous: $\varphi_i(u) \in C(\mathbb{R})$, $i = 1, \ldots, n$, and the diffusion matrix $a(u) = (a_{ij}(u))_{i,j=1}^n$ is Lebesgue measurable and bounded: $a_{ij}(u) \in L^\infty(\mathbb{R})$, $i, j = 1, \ldots, n$. We also assume that the matrix $a(u) \geq 0$ (nonnegative definite). This matrix may have nontrivial kernel. Hence (1.1) is a degenerate (hyperbolic-parabolic) equation. In particular case $a \equiv 0$ it reduces to a first order conservation law

$$u_t + \text{div}_x \varphi(u) = 0.$$

(1.2)

Equation (1.1) is endowed with the initial condition

$$u(0, x) = u_0(x).$$

(1.3)

Let $g(u) \in BV_{loc}(\mathbb{R})$ be a function of bounded variation on any segment in $\mathbb{R}$. We will need the bounded linear operator $T_g : C(\mathbb{R})/C \to C(\mathbb{R})/C$, where $C$ is the space of constants. This operator is defined up to an additive constant by the relation

$$T_g(f)(u) = g(u-)f(u) - \int_0^u f(s)dg(s),$$

(1.4)
where \(g(u-) = \lim_{v \to u^-} g(v)\) is the left limit of \(g\) at the point \(u\), and the integral in (1.4) is understood in accordance with the formula

\[
\int_0^u f(s) dg(s) = \text{sign } u \int_{J(u)} f(s) dg(s),
\]

where \(\text{sign } u = 1\), \(J(u)\) is the interval \([0, u]\) if \(u > 0\), and \(\text{sign } u = -1\), \(J(u) = [u, 0]\) if \(u \leq 0\). Observe that \(T_g(f)(u)\) is continuous even in the case of discontinuous \(g(u)\). For instance, if \(g(u) = \text{sign}(u - k)\) then \(T_g(f)(u) = \text{sign}(u - k)(f(u) - f(k))\). Notice also that for \(f \in C^1(\mathbb{R})\) the operator \(T_g\) is uniquely determined by the identity \(T_g(f)'(u) = g(u)f'(u)\) (in \(\mathcal{D}'(\mathbb{R})\)). In the case \(f'(u), g(u) \in L^2_{\text{loc}}(\mathbb{R})\) the function \(T_g(f) \in C(\mathbb{R})/C\) can be defined by the identity \(T_g(f)'(u) = g(u)f'(u) \in L^1_{\text{loc}}(\mathbb{R})\). As is easy to see, in this case the correspondence \(g \to T_g(f)\) is a linear continuous map from \(L^2_{\text{loc}}(\mathbb{R})\) into \(C(\mathbb{R})/C\).

We fix some representation of the diffusion matrix \(a(u)\) in the form \(a(u) = b^\top(u)b(u)\), where \(b(u) = (b_{ij}(u))_{i,j=1}^n\) is matrix-valued function with measurable and bounded entries, \(b_{ij}(u) \in L^\infty(\mathbb{R})\). We recall the notion of entropy solution of the Cauchy problem (1.1), (1.3) introduced in [7].

**Definition 1.1.** A function \(u = u(t, x) \in L^\infty(\Pi)\) is called an entropy solution (e.s. for short) of (1.1), (1.3) if the following conditions hold:

(i) for each \(r = 1, \ldots, n\) the distributions

\[
\text{div}_x B_r(u(t, x)) \in L^2_{\text{loc}}(\Pi),
\]

where vectors \(B_r(u) = (B_{r1}(u), \ldots, B_{rn}(u)) \in C(\mathbb{R}, \mathbb{R}^n)\), and \(B'_{ri}(u) = b_{ri}(u), r, i = 1, \ldots, n;\)

(ii) for every \(g(u) \in C^1(\mathbb{R}), r = 1, \ldots, n\)

\[
\text{div}_x T_g(B_r)(u(t, x)) = g(u(t, x)) \text{div}_x B_r(u(t, x)) \text{ in } \mathcal{D}'(\Pi); \quad (1.6)
\]

(iii) for any convex function \(\eta(u) \in C^2(\mathbb{R})\)

\[
\eta(u) + \text{div}_x(T_{\eta'}(\varphi)(u)) - D^2 \cdot T_{\eta'}(A(u)) + \eta''(u) \sum_{r=1}^n (\text{div}_x B_r(u))^2 \leq 0 \text{ in } \mathcal{D}'(\Pi); \quad (1.7)
\]

(iv) \(\text{ess lim}_{t \to 0} u(t, \cdot) = u_0\) in \(L^1_{\text{loc}}(\mathbb{R}^n).\)
In (1.7) the operator

\[ D^2 \cdot T_{\eta'}(A(u)) \doteq \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} T_{\eta'}(A_{ij}(u)). \]

Relation (1.7) means that for any non-negative test function \( f = f(t,x) \in C_0^\infty(\Pi) \)

\[ \int_{\Pi} [\eta(u)f_t + T_{\eta'}(\varphi)(u) \cdot \nabla_x f + T_{\eta'}(A(u)) \cdot D^2 f] dt dx \geq 0, \]

where \( D^2 f \) is the symmetric matrix of second order derivatives of \( f \), and " \cdot " denotes the standard scalar multiplications of vectors or matrices.

**Remark 1.1.** The chain rule postulated in (ii) actually holds for arbitrary locally bounded Borel function \( g(u) \in L^\infty_{loc}(\mathbb{R}) \). First of all, observe that since \( B'(u) \in L^\infty(\mathbb{R}, \mathbb{R}^n) \), then \( T_{g}(B_r)(u) = \int_0^u g(v)B'_r(v)dv \). (1.8)

For \( R > 0 \) we consider the set \( F_R \) consisting of bounded Borel functions \( g(u) \) such that \( \sup|g(u)| \leq R \) and the chain rule (1.6) holds. By (ii) \( F_R \) contains all bounded functions \( g(u) \in C^1(\mathbb{R}) \) such that \( \sup|g(u)| \leq R \). Let \( g_l(u) \in F_R, l \in \mathbb{N}, \) be a sequence which converges pointwise to a function \( g(u) \) as \( l \to \infty \). It is clear that \( g(u) \) is a Borel function and \( \sup|g(u)| \leq R \). Observe that the functions \( g_l(u(t,x)) \) are bounded and measurable (because \( g_l \) are Borel functions), the sequence \( g_l(u(t,x)) \to g(u(t,x)) \) pointwise as \( l \to \infty \), and \( \|g_l(u(t,x))\|_\infty \leq R \). This implies that

\[ g_l(u(t,x)) \operatorname{div}_x B_r(u(t,x)) \to g(u(t,x)) \operatorname{div}_x B_r(u(t,x)) \text{ in } L^2_{loc}(\Pi). \]

From the other hand, it follows from (1.8) that \( T_{g_l}(B_r)(u(t,x)) \) converges to \( T_{g}(B_r)(u(t,x)) \) as \( l \to \infty \) uniformly on \( \Pi \). Therefore, we can pass to the limit as \( l \to \infty \) in relation (1.6) with \( g = g_l \) and derive that (1.6) holds for our limit function \( g(u) \). Thus, \( g(u) \in F_R \). We see that \( F_R \) is closed under pointwise convergence and by Lebesgue theorem \( F_R \) contains all Borel functions \( g \) such that \( \sup|g| \leq R \). Since \( R > 0 \) is arbitrary, we find that (1.6) holds for all bounded Borel functions. It only remains to notice that the behavior of \( g(u) \) out of the segment \([-M,M]\), where \( M = \|u\|_\infty \), does not matter. Therefore, (1.6) holds for any Borel function bounded on \([-M,M]\), in particular, for each locally bounded Borel function.

Remark also that for correctness of (1.6) we have to choose the Borel representative for \( g(u) \) (recall that \( g(u) \) is defined up to equality on a set of full measure and such the
representative exists). Notice that \( T_g(B_r)(u) \) does not depend on the choice of a Borel representative of \( g \). Hence, the right-hand side of (1.6) does not depend on this choice either.

If to be precise, in [7] the representation \( a = b^\top b \) was used with \( b = a(u)^{1/2} \). In order to make the definition invariant under linear changes of the independent variables, we have to extend the class of admissible representations. For instance, let us introduce the change \( y = y(t,x) = qx - tc \), where \( q : \mathbb{R}^n \to \mathbb{R}^n \) is a nondegenerate linear map and \( c \in \mathbb{R}^n \). This can be written in the coordinate form as

\[
y_i = \sum_{j=1}^n q_{ij}x_j - c_i t, \quad i = 1, \ldots, n.
\]  

As is easy to verify, the function \( u = v(t,y(t,x)) \) is an e.s. of (1.1), (1.3) with initial data \( u_0 = v_0(y(0,x)) \) if and only if \( v(t,y) \) is an e.s. of the problem

\[
v_t + \text{div}_y(\tilde{\varphi}(v) - \tilde{a}(v)\nabla_y v) = 0, \quad v(0,y) = v_0(y),
\]

where

\[
\tilde{\varphi}(v) = q\varphi(v) - cv, \quad \tilde{a}(v) = qa(v)q^\top,
\]

corresponding to the representation \( \tilde{a}(v) = (b(v)q^\top)^\top(b(v)q^\top) \), \( b(v)q^\top \in L^\infty(\mathbb{R}) \), of the diffusion matrix \( \tilde{a}(v) \). We remark that

\[
\text{div}_y(B(v)q^\top)_r \big|_{y=y(t,x)} = \sum_{l,j=1}^n \partial_{y_l}B_{rj}(v)q_{lj} = \sum_{j=1}^n \partial_{x_j}B_{rj}(u) = \text{div}_x B_r(u).
\]

Recall that \( B_r'(u) = b_r(u) = (b_{r1}(u), \ldots, b_{rn}(u)), r = 1, \ldots, n. \)

We underline that the matrix \( bq^\top \) is not necessarily symmetric and therefore differs from \( a(v)_{1/2} \).

Now, we suppose that the initial function \( u_0 \) is periodic with a lattice of periods \( L: u_0(x + e) = u_0(x) \) a.e. in \( \mathbb{R}^n \) for all \( e \in L \). Denote \( \mathbb{T}^n = \mathbb{R}^n/L \) the corresponding torus (which can be identified with a fundamental parallelepiped), \( dx \) is the Lebesgue measure on \( \mathbb{T}^n \), normalized by the condition \( dx(\mathbb{T}^n) = V, V \) being volume of a fundamental parallelepiped (observe that \( V \) does not depend on its choice). Then there exists a space-periodic e.s. \( u = u(t,x) \) of the problem (1.1), (1.3), \( u(t,x+e) = u(t,x) \) a.e. in \( \Pi \) for all \( e \in L \). This can be written as \( u \in L^\infty((0, +\infty) \times \mathbb{T}^n) \). This solution can be constructed as a limit of the sequence \( u_k \) of solutions to the regularized problem

\[
u_t + \text{div}_x(\varphi_k(u) - a_k(u)\nabla_x u) = 0,
\]
with smooth flux vectors $\varphi_k(u)$, and smooth and strictly positive definite diffusion matrices $a_k(u)$, which approximate $\varphi(u)$ and $a(u)$, respectively. For details, we refer to [7, 8]. It is known [7, Theorem 1.2] that e.s. $u(t, x)$ satisfies the maximum principle: $|u(t, x)| \leq \|u_0\|_{\infty}$ for a.e. $(t, x) \in \Pi$.

Our main result is the long time decay property of entropy solutions. Let

$$L' = \{ \xi \in \mathbb{R}^n | \xi \cdot e \in \mathbb{Z} \ \forall e \in L \}$$

be a dual lattice to the lattice of periods $L$,

$$I = \frac{1}{V} \int_{T^n} u_0(x) dx$$

be the mean value of initial data.

**Theorem 1.1.** Assume that the following nonlinearity-diffusivity condition holds: for all $\xi \in L'$, $\xi \neq 0$ there is no vicinity of $I$, where simultaneously the function $\xi \cdot \varphi(u)$ is affine and the function $a(u)\xi \cdot \xi \equiv 0$. Then

$$\text{ess lim}_{t \to +\infty} u(t, x) = I \ \text{in} \ L^1(\mathbb{T}^n). \quad (1.11)$$

The decay property was established in [9] under the more restrictive version of the nonlinearity-diffusivity condition, in the case of locally Lipschitz flux vector when $\varphi'(u) \in L^\infty_{\text{loc}}(\mathbb{R}, \mathbb{R}^n)$. This conditions reads: for all $\tau \in \mathbb{R}$, $\xi \in L'$, $\xi \neq 0$

$$\text{meas}\{ u \in \mathbb{R}, |u| \leq \|u_0\|_{\infty} \mid \tau u + \xi \cdot \varphi'(u) = a(u)\xi \cdot \xi = 0 \} = 0. \quad (1.12)$$

In the hyperbolic case $a \equiv 0$ decay property (1.11) was proved in [18], see also the previous papers [6, 10, 17].

Let us show that our nonlinearity-diffusivity condition is exact. In fact, if it fails, we can find an interval $(I - \delta, I + \delta)$, a nonzero vector $\xi \in L'$, and a constant $c \in \mathbb{R}$ such that $\xi \cdot \varphi(u) - cu \equiv \text{const}$, $a(u)\xi \cdot \xi \equiv 0$ on this interval. Then, as is easily verified, the function $u(t, x) = I + \delta \sin(2\pi(\xi \cdot x - ct))$ is an $x$-periodic (with the lattice of periods $L$) e.s. of $(1.1)$, $(1.3)$ with the periodic initial data $u_0(x) = I + \delta \sin(2\pi(\xi \cdot x))$, which has the mean value $I$. It is clear that $u(t, x)$ does not converges as $t \to +\infty$ in $L^1(\mathbb{T}^n)$, and the decay property fails.

## 2 Some properties of periodic entropy solutions

**Lemma 2.1.** For each convex $\eta(u)$ there exists a locally finite nonnegative $x$-periodic Borel measure $\mu_\eta$ on $\Pi$ such that

$$\eta(u)_t + \text{div}_x(T_\eta(\varphi)(u)) - D^2 \cdot T_\eta(A(u)) = -\mu_\eta \ \text{in} \ \mathcal{D}'(\Pi). \quad (2.1)$$
This measure can be identified with a finite measure on \( \mathbb{R}_+ \times \mathbb{T}^n \). Moreover, for almost each (a.e.) \( t_1, t_2, 0 < t_1 < t_2 \),

\[
\mu((t_1, t_2) \times \mathbb{T}^n) \leq \int_{\mathbb{T}^n} \eta(u(t_1, x)) \, dx - \int_{\mathbb{T}^n} \eta(u(t_2, x)) \, dx. \tag{2.2}
\]

**Proof.** It follows from (1.7) that for every convex \( \eta(u) \) the distribution

\[
\eta(u)_t + \text{div}_x (T_\eta(\varphi)(u)) - D^2 \cdot T_\eta(A(u)) \leq 0 \text{ in } \mathcal{D}'(\Pi).
\]

By the Schwartz theorem on the representation of nonnegative distributions we conclude that (2.1) holds for some locally finite nonnegative Borel measure \( \mu_\eta \) on \( \Pi \). It is clear that \( \mu_\eta \) is periodic with respect to \( x \) and can be treated as a measure on \( \mathbb{R}_+ \times \mathbb{T}^n \). Let \( \alpha(t) \in C^1_0(\mathbb{R}_+), \beta(y) \in C^2_0(\mathbb{R}^n), \alpha(t), \beta(y) \geq 0, \int_{\mathbb{R}^n} \beta(y) \, dy = 1 \). Applying (2.1) to the test function \( \alpha(t)\beta(x/k) \), with \( k \in \mathbb{N} \), we arrive at the relation

\[
\langle \mu_\eta, \alpha(t)\beta(x/k) \rangle = \int_\Pi \alpha(t)\beta(x/k) \, d\mu_\eta(t, x) = \int_\Pi \eta(u(t, x)) \alpha'(t)\beta(x/k) \, dt \, dx + \frac{1}{k} \int_\Pi T_\eta(\varphi)(u) \cdot \nabla_y \beta(x/k) \alpha(t) \, dt \, dx + \frac{1}{k^2} \int_\Pi T_\eta(A(u)) \cdot D^2_\eta \beta(x/k) \alpha(t) \, dt \, dx.
\]

Multiplying this equality by \( k^{-n} \) and passing to the limit as \( k \to \infty \), we obtain

\[
\int_{\mathbb{R}_+ \times \mathbb{T}^n} \alpha(t) \, d\mu_\eta(t, x) = \int_{\mathbb{R}_+ \times \mathbb{T}^n} \eta(u(t, x)) \alpha'(t) \, dt \, dx, \tag{2.3}
\]

where we use the known property

\[
\lim_{k \to \infty} k^{-n} \langle \mu, \alpha(t)\beta(x/k) \rangle = \int_{\mathbb{R}_+ \times \mathbb{T}^n} \alpha(t) \, d\mu(t, x) \int_{\mathbb{R}^n} \beta(y) \, dy
\]

for an arbitrary \( x \)-periodic locally finite measure on \( \Pi \). Identity (2.3) means that

\[
\frac{d}{dt} \int_{\mathbb{T}^n} \eta(u(t, x)) \, dx = - \int_{\mathbb{T}^n} d\mu_\eta(t, x) \text{ in } \mathcal{D}'(\mathbb{R}_+),
\]

where \( \int_{\mathbb{T}^n} d\mu_\eta(t, x) \) is treated as the measure \( \mu_\eta^t \) on \( \mathbb{R}_+ \) defined by the relation

\[
\langle \mu_\eta^t, \alpha(t) \rangle = \int_{\mathbb{R}_+ \times \mathbb{T}^n} \alpha(t) \, d\mu_\eta(t, x),
\]
that is, \( \mu^t \eta \) is the projection of \( \mu \eta \) on the \( t \)-axis. It follows from \( (2.3) \) that for all \( t_1, t_2 \in \mathbb{R}_+ \), \( t_1 < t_2 \), being Lebesgue points of the function \( t \to \int_{\mathbb{T}^n} \eta(u(t,x))dx \) relation \( (2.2) \) holds.

**Corollary 2.1.** Let \( \eta(u) \) be a convex function. Then

(i) The function \( I(t) = \int_{\mathbb{T}^n} \eta(u(t,x))dx \) decreases: \( I(t_2) \leq I(t_1) \) for a.e. \( t_1, t_2 \geq 0 \), \( t_2 > t_1 \);

(ii) for a.e. \( t > 0 \)

\[
\frac{1}{V} \int_{\mathbb{T}^n} u(t,x)dx = I \overset{\text{def}}{=} \frac{1}{V} \int_{\mathbb{T}^n} u_0(x)dx; \tag{2.4}
\]

(iii) the measure \( \mu \eta \) is finite on \( \mathbb{R}_+ \times \mathbb{T}^n \). Moreover,

\[
\mu \eta(\mathbb{R}_+ \times \mathbb{T}^n) = \int_{\mathbb{T}^n} \eta(u_0(x))dx - \text{ess lim}_{t \to \infty} \int_{\mathbb{T}^n} \eta(u(t,x))dx \leq 2V \max_{|r| \leq M} |\eta(r)|,
\]

where \( M = \|u_0\|_\infty \).

**Proof.** Assertion (i) readily follows from \( (2.2) \). Applying (i) to the entropies \( \eta(u) = \pm u \), we obtain that for a.e. \( t, \tau > 0 \)

\[
\int_{\mathbb{T}^n} u(t,x)dx = \int_{\mathbb{T}^n} u(\tau,x)dx,
\]

and \( (2.4) \) follows in the limit as \( \tau \to 0 \) with the help of initial condition (iv) of Definition 1.1.

To prove (iii), we pass to the limits in \( (2.2) \) as \( t_1 \to 0 \), \( t_2 \to \infty \), taking again into account the initial condition. \( \square \)

**Lemma 2.2.** In the notations of Definition 1.1 the distributions

\[
\text{div}_x B_r(u(t,x)) \in L^2(\mathbb{R}_+ \times \mathbb{T}^n) \quad \forall r = 1, \ldots, n,
\]

and for almost all \( \tau > 0 \)

\[
\int_{(\tau, +\infty) \times \mathbb{T}^n} \sum_{r=1}^n (\text{div}_x B_r(u(t,x)))^2 dt dx \leq \frac{1}{2} \int_{\mathbb{T}^n} (u(\tau,x))^2 dx \leq \frac{1}{2} \int_{\mathbb{T}^n} (u_0(x))^2 dx. \tag{2.5}
\]
Proof. In view of relation (1.7) with $\eta(u) = u^2/2$

$$\sum_{r=1}^{n} (\text{div}_x B_r(u))^2 \leq -\{(u^2/2)_t + \text{div}_x (T_u(\varphi)(u)) - D^2 \cdot T_u(A(u))\} \text{ in } \mathcal{D}'(\Pi).$$

Applying this relation to the nonnegative test function $k^{-n} \alpha(t) \beta(x/k)$, with $\alpha(t) \in C^1_0(\mathbb{R}^+)$, $\beta(y) \in C^2_0(\mathbb{R}^n)$, $k \in \mathbb{N}$, and passing to the limit as $k \to \infty$, we obtain, like in the proof of Lemma 2.1, that

$$\int_{\mathbb{R}^+ \times \mathbb{T}^n} \sum_{r=1}^{n} (\text{div}_x B_r(u))^2 \alpha(t) dt dx \leq \int_{\mathbb{R}^+ \times \mathbb{T}^n} u^2 \alpha'(t) dt dx.$$

This relation implies that for almost all $\tau, T > 0$, $\tau < T$

$$\int_{(\tau,T) \times \mathbb{T}^n} \sum_{r=1}^{n} (\text{div}_x B_r(u))^2 dt dx \leq \int_{\mathbb{T}^n} (u(\tau, x))^2 dx - \frac{1}{2} \int_{\mathbb{T}^n} (u(T, x))^2 dx \leq \frac{1}{2} \int_{\mathbb{T}^n} (u(\tau, x))^2 dx,$$

and (2.5) follows in the limit as $T \to +\infty$. We take also into account that for a.e. $t_0 \in (0, \tau)$

$$\int_{\mathbb{T}^n} (u(\tau, x))^2 dx \leq \int_{\mathbb{T}^n} (u(t_0, x))^2 dx$$

by Corollary 2.1(i), which implies in the limit as $t_0 \to 0$ that

$$\int_{\mathbb{T}^n} (u(\tau, x))^2 dx \leq \int_{\mathbb{T}^n} (u_0(x))^2 dx.$$

Lemma 2.3. Let $u_1 = u_1(t, x)$, $u_2 = u_2(t, x)$ be $x$-periodic e.s. of (1.1), (1.3) with initial functions $u_{10}(x), u_{20}(x) \in L^\infty(\mathbb{T}^n)$, respectively. Then for a.e. $t, \tau > 0$ such that $t > \tau$

$$\int_{\mathbb{T}^n} |u_1(t, x) - u_2(t, x)| dx \leq \int_{\mathbb{T}^n} |u_1(\tau, x) - u_2(\tau, x)| dx \leq \int_{\mathbb{T}^n} |u_{10}(x) - u_{20}(x)| dx. \quad (2.6)$$

Proof. As was established in [7, 11] (for the isotropic case see earlier paper [3]) by application of a variant of Kruzhkov’s doubling variables method,

$$(|u_1 - u_2|)_t + \text{div}_x [\text{sign}(u_1 - u_2)(\varphi(u_1) - \varphi(u_2))] - D^2 \cdot [\text{sign}(u_1 - u_2)(A(u_1) - A(u_2))] \leq 0 \quad \text{in } \mathcal{D}'(\Pi).$$
Applying this relation to a nonnegative test function $k^{-n}\alpha(t)\beta(x/k)$ like in the proofs of Lemmas 2.1-2.2 and passing to the limit as $k \to \infty$, we arrive at the relation

$$
\int_{\mathbb{R}_+ \times \mathbb{T}^n} |u_1(t, x) - u_2(t, x)|\alpha'(t)dt\,dx \geq 0 \quad \forall \alpha(t) \in C^1_0(\mathbb{R}_+), \alpha(t) \geq 0.
$$

This means that

$$
\frac{d}{dt} \int_{\mathbb{T}^n} |u_1(t, x) - u_2(t, x)|\,dx \leq 0 \quad \text{in } \mathcal{D}'(\mathbb{R}_+).
$$

Therefore, for a.e. $t, \tau, \delta \in \mathbb{R}_+$ such that $t > \tau > \delta$

$$
\int_{\mathbb{T}^n} |u_1(t, x) - u_2(t, x)|\,dx \leq \int_{\mathbb{T}^n} |u_1(\tau, x) - u_2(\tau, x)|\,dx \leq \int_{\mathbb{T}^n} |u_1(\delta, x) - u_2(\delta, x)|\,dx,
$$

and (2.6) follows in the limit as $\delta \to 0$. □

**Corollary 2.2.** Let $u(t, x)$ be a periodic e.s. of (1.1), (1.3). Then the family $u(t, \cdot)$, $t > 0$, is pre-compact in $L^2(\mathbb{T}^n)$ (after possible correction on a set of null measure of values $t$).

**Proof.** Let

$$
E = \{ \ t > 0 \mid (t, x) \text{ is a Lebesgue point of } u \text{ for a.e. } x \in \mathbb{R}^n \}\}.
$$

It is known (see, for example, [19, Lemma 1.2]) that $E$ is a set of full measure and $t \in E$ is a common Lebesgue point of the functions $t \to \int u(t, x)g(x)dx$ for all $g(x) \in L^1(\mathbb{R}^n)$. It is clear that for $t \in E$ and $\Delta x \in \mathbb{R}^n (t, x)$ is a Lebesgue point of $|u(t, x + \Delta x) - u(t, x)|$ for a.e. $x \in \mathbb{R}^n$. By [19, Lemma 1.2] again, $t \in E$ is a Lebesgue point of the functions $t \to \int_{\mathbb{T}^n} |u(t, x + \Delta x) - u(t, x)|\,dx$ for all $\Delta x \in \mathbb{R}^n$.

Applying Lemma 2.3 to e.s. $u(t, x), u(t, x + \Delta x)$, we find that for each $t \in E$

$$
\int_{\mathbb{T}^n} |u(t, x + \Delta x) - u(t, x)|\,dx \leq \int_{\mathbb{T}^n} |u_0(x + \Delta x) - u_0(x)|\,dx.
$$

and $|u(t, x)| \leq M = \|u_0\|_{\infty}$. Therefore, for all $t \in E$ and all $\Delta x \in \mathbb{R}^n$

$$
\int_{\mathbb{T}^n} (u(t, x + \Delta x) - u(t, x))^2\,dx \leq 2M \int_{\mathbb{T}^n} |u(t, x + \Delta x) - u(t, x)|\,dx \leq 2M \int_{\mathbb{T}^n} |u_0(x + \Delta x) - u_0(x)|\,dx.
$$

This means that the family $u(t, \cdot)$, $t \in E$, is uniformly bounded and equicontinuous in $L^2(\mathbb{T}^n)$. By the Kolmogorov-Riesz criterion this implies the pre-compactness of the family $u(t, \cdot)$ in $L^2(\mathbb{T}^n)$. □
3 Reduction of the problem

We will need the following algebraic statement.

Lemma 3.1. Let $A$ be a lattice in $X = \mathbb{R}^n$, $A_0$ be a subgroup of $A$, $X_0$ be a linear span of $A_0$. Assume that $A_0 = A \cap X_0$. Then any basis of $A_0$ can be completed to a basis of $A$.

Proof. Let $\xi_1, \ldots, \xi_k$ be a basis of $A_0$, that is, any element $\xi \in A_0$ is uniquely represented as $\xi = \sum_{i=1}^{k} n_i \xi_i$ with integer coefficients $n_i$. It is clear that $A_0$ is a lattice and therefore $\xi_1, \ldots, \xi_k$ is a basis of the vector space $X_0$. We consider the natural projection $\text{pr} : X \to X/X_0$. Then $B = \text{pr}(A)$ is an additive subgroup of $X/X_0$. We will show that $B$ is a lattice, i.e., a discrete subgroup of $X/X_0$. It is sufficient to show that any ball $B_R = \{ x \in X/X_0 \mid p(x) \leq R \}$ contains only finite set of points of $B$. Here

$$p(x) = \min\{ |\xi - y| \mid y \in X_0 \}, \quad x = \text{pr}(\xi),$$

is the factor-norm. Here, and in the sequel, $|z|$ denotes the Euclidean norm of a finite-dimensional vector $z$. We assume that $x = \text{pr}(\xi) \in B$, where $\xi \in A$, and that $p(x) \leq R$. We can choose $y \in X_0$ such that $|\xi - y| = p(x) \leq R$. Recall that $\xi_i, i = 1, \ldots, k$ is a basis of $X_0$. Therefore, we can represent $y = \sum_{i=1}^{k} s_i \xi_i, s_i \in \mathbb{R}$. Let $n_i = [s_i] \in \mathbb{Z}$ be integer parts of $s_i$, so that $\alpha_i = s_i - n_i \in [0, 1)$. Then

$$|\xi - \sum_{i=1}^{k} n_i \xi_i| = |\xi - y + \sum_{i=1}^{k} \alpha_i \xi_i| \leq |\xi - y| + \sum_{i=1}^{k} |\alpha_i| |\xi_i| \leq R + C, \quad C = \sum_{i=1}^{k} |\xi_i| = \text{const.}$$

Thus,

$$\xi - \sum_{i=1}^{k} n_i \xi_i \in A_{R+C} = \{ \zeta \in A \mid |\zeta| \leq R + C \}. $$

Since $A$ is a lattice, the set $A_{R+C}$ is finite. Taking into account that $\sum_{i=1}^{k} n_i \xi_i \in A_0 \subset X_0$, we find that

$$x = \text{pr}(\xi) = \text{pr} \left( \xi - \sum_{i=1}^{k} n_i \xi_i \right) \in \text{pr}(A_{R+C}).$$
Hence, $B \cap B_R \subset \text{pr}(A_{R+C})$ is a finite set for each $R > 0$. We conclude that $B$ is a lattice. Therefore, there exists a basis $x_i$, $i = k + 1, \ldots, m$, of the free abelian group $B$. We can find such $\xi_i \in A$ that $x_i = \text{pr}(\xi_i)$. Then for every $\xi \in A$ there exist unique $n_i \in \mathbb{Z}$, $i = k + 1, \ldots, m$, such that $\eta = \xi - \sum_{i=k+1}^{m} n_i \xi_i \in X_0$. We also observe that $\eta \in A$. Thus, $\eta \in A \cap X_0 = A_0$. Since $\xi_i$, $i = 1, \ldots, k$ is a basis of $A_0$, then there exist unique $n_i \in \mathbb{Z}$, $i = 1, \ldots, k$, such that $\eta = \sum_{i=1}^{k} n_i \xi_i$. We conclude that

$$\xi = \sum_{i=1}^{m} n_i \xi_i$$

and this representation is unique. This means that $\xi_i$, $i = 1, \ldots, m$ is a basis of $A$. The proof is complete.

**Remark 3.1.** The requirement $A \cap X_0 = A_0$ is exact: if a basis of $A_0$ can be completed to a basis of $A$ then $A \cap X_0 = A_0$. In fact, let $\xi_1, \ldots, \xi_k, \xi_{k+1}, \ldots, \xi_m$ be a basis of $A$ completing a basis $\xi_1, \ldots, \xi_k$ of $A_0$ and $\xi \in A \cap X_0$. Since $\xi_1, \ldots, \xi_k$ is also a basis of the linear space $X_0$, we have two representation

$$\xi = \sum_{i=1}^{m} n_i \xi_i = \sum_{i=1}^{k} s_i \xi_i, \quad n_i \in \mathbb{Z}, \quad s_i \in \mathbb{R}.$$

Taking into account that the vectors $\xi_1, \ldots, \xi_m$ are linearly independent, we conclude that these representations must coincide. In particular, $n_i = 0$ for $i > k$ and $\xi = \sum_{i=1}^{k} n_i \xi_i \in A_0$.

Hence, $A \cap X_0 \subset A_0$. Since the inverse inclusion $A_0 \subset A \cap X_0$ is evident, we conclude that $A \cap X_0 = A_0$.

We define

$$L'_0 = \{ \xi \in L' \mid \text{the function } \xi \cdot \varphi(u) \text{ is affine in some vicinity of } I \}.$$ 

It is clear that for any vector $\xi \in X_0$, where $X_0$ is the linear span of $L'_0$, the function $\xi \cdot \varphi(u)$ is affine in some vicinity of $I$, this means that $\xi \cdot \varphi(u) = a(\xi)u + \text{const}$ in this vicinity. Since the map $\xi \rightarrow a(\xi)$ is linear, there exist a unique vector $\bar{c} \in X_0$ such that $a(\xi) = \bar{c} \cdot \xi$. 

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Obviously, $L' \cap X_0 \subset L_0'$ and $L_0'$ is a subgroup of the lattice $L'$ satisfying the condition of Lemma 3.1. Let $m$ be a rank of $L_0'$, $d = n - m$, $\zeta_i$, $i = d + 1, \ldots, n$, be a basis of $L_0'$. By Lemma 3.1 this basis can be completed to a basis $\zeta_i$, $i = 1, \ldots, n$, of the lattice $L'$. Then the vectors $\zeta_i$, $i = 1, \ldots, n$ forms a basis of $\mathbb{R}^n$ as well. We introduce the linear change $y = y(t, x)$, $y_i = \zeta_i \cdot x - c_i t$, $i = 1, \ldots, n$, where $c_i = \bar{c} \cdot \zeta_i = a(\zeta_i)$. After this change, we obtain entropy solution $v(t, y)$ of the problem

$$v_t + \text{div}_y (\tilde{\varphi}(v) - \tilde{a}(v) \nabla_y v) = 0, \quad v(0, y) = v_0(y) = u_0(x(0, y)), $$

where, in view of (1.10), $\tilde{\varphi}_i(v) = \zeta_i \cdot \varphi(v) - c_i v$, $\tilde{a}_{ij}(v) = a(v)\zeta_i \cdot \zeta_j$. By the construction the last $m$ flux component $\tilde{\varphi}_i(v)$, $i = d + 1, \ldots, n$ are constant on some interval $(\alpha, \beta) \ni I$. We notice that $v(t, y)$ is periodic with respect to $y$ with the standard lattice of periods $\mathbb{Z}^n$, this follows from the fact that $y(0, x)$ is an isomorphism from $L$ into $\mathbb{Z}^n$. We underline that the mean value $I$ remains the same under the described change. The nonlinearity-diffusivity condition of Theorem 1.1 converts to the requirement: for all $\kappa \in \mathbb{Z}^n$, $\kappa \neq 0$, there is no vicinity of $I$, where simultaneously the function $\kappa \cdot \tilde{\varphi}(u)$ is affine and the function $\tilde{a}(u)\kappa \cdot \kappa \equiv 0$. This directly follows from the relations

$$\kappa \cdot \tilde{\varphi}(u) = \xi \cdot \varphi(u), \quad \tilde{a}(u)\kappa \cdot \kappa = a(u)\xi \cdot \xi,$$

where $\xi = \sum_{i = 1}^{n} \kappa_i \zeta_i \in L'$. The condition that the function $\xi \cdot \varphi(u)$ is affine in some vicinity of $I$ is equivalent to the inclusion $\xi \in L_0' \Leftrightarrow \bar{\kappa} = 0$, where we denote by $\bar{\kappa}$, $\bar{\kappa}$ the orthogonal projections of $\kappa$ on the spaces

$$\mathbb{R}^d = \{ \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \mid \xi_i = 0, \forall i = d + 1, \ldots, n \},$$

$$(\mathbb{R}^d)^\perp = \{ \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \mid \xi_i = 0, \forall i = 1, \ldots, d \},$$

respectively. Therefore, the nonlinearity-diffusivity condition reduces to the requirement: $\forall \kappa = \bar{\kappa} \in \mathbb{Z}^n, \kappa \neq 0$, the function $\tilde{a}(u)\kappa \cdot \kappa \neq 0$ in any vicinity of $I$.

Hence, going back to the original notations of equation (1.11), we can suppose that

(R1) $L = \mathbb{Z}^n$ (and in particular the volume $V = 1$);

(R2) the function $\xi \cdot \varphi(u)$, $\xi \in L' = \mathbb{Z}^n$, is affine if and only if $\xi \in (\mathbb{R}^d)^\perp$;

(R3) the function $\xi \cdot \varphi(u) \equiv c(\xi) = \text{const}$ on an interval $(\alpha, \beta) \ni I$ for all $\xi \in \mathbb{Z}^n \cap (\mathbb{R}^d)^\perp$;

(R4) the function $a(u)\xi \cdot \xi \neq 0$ in any vicinity of $I$ for all $\xi \in \mathbb{Z}^n \cap (\mathbb{R}^d)^\perp$.

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4 Preliminaries

4.1 Measure valued functions and H-measures

Recall (see [4, 20]) that a measure-valued function on a domain \( \Omega \subset \mathbb{R}^N \) is a weakly measurable map \( x \mapsto \nu_x \) of \( \Omega \) into the space \( \text{Prob}_0(\mathbb{R}) \) of probability Borel measures with compact support in \( \mathbb{R} \).

The weak measurability of \( \nu_x \) means that for each continuous function \( g(\lambda) \) the function \( x \to \langle \nu_x, g(\lambda) \rangle = \int g(\lambda) d\nu_x(\lambda) \) is measurable on \( \Omega \).

A measure-valued function \( \nu_x \) is said to be bounded if there exists \( M > 0 \) such that \( \text{supp} \nu_x \subset [-M, M] \) for almost all \( x \in \Omega \).

Measure-valued functions of the kind \( \nu_x(\lambda) = \delta(\lambda - u(x)) \), where \( u(x) \in L^\infty(\Omega) \) and \( \delta(\lambda - u^*) \) is the Dirac measure at \( u^* \in \mathbb{R} \), are called regular. We identify these measure-valued functions and the corresponding functions \( u(x) \), so that there is a natural embedding of \( L^\infty(\Omega) \) into the set \( \text{MV}(\Omega) \) of bounded measure-valued functions on \( \Omega \).

Measure-valued functions naturally arise as weak limits of bounded sequences in \( L^\infty(\Omega) \) in the sense of the following theorem by L. Tartar [20].

**Theorem 4.1.** Let \( u_k(x) \in L^\infty(\Omega) \), \( k \in \mathbb{N} \), be a bounded sequence. Then there exist a subsequence (we keep the notation \( u_k(x) \) for this subsequence) and a bounded measure valued function \( \nu_x \in \text{MV}(\Omega) \) such that

\[
\forall g(\lambda) \in C(\mathbb{R}) \quad g(u_k) \to_{k \to \infty} \langle \nu_x, g(\lambda) \rangle \quad \text{weakly-}* \text{ in } L^\infty(\Omega). \tag{4.1}
\]

Besides, \( \nu_x \) is regular, i.e., \( \nu_x(\lambda) = \delta(\lambda - u(x)) \) if and only if \( u_k(x) \to u(x) \) in \( L^1_{\text{loc}}(\Omega) \) (strongly).

Another useful tool for evaluation of weak convergence is Tartar’s H-measures and their variants. H-measures were introduced by L. Tartar [21] and independently by P. Gérard in [5]. We recall the notion of H-measure in the simple case of scalar sequences.

Let \( S = \{ \xi \in \mathbb{R}^N \mid |\xi| = 1 \} \) be the unit sphere in \( \mathbb{R}^N \), \( \pi : \mathbb{R}^N \setminus \{0\} \to S, \pi(\xi) = \xi/|\xi| \), be the orthogonal projection on the sphere. Let

\[
F(u)(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i \xi \cdot x} u(x) dx, \quad \xi \in \mathbb{R}^N,
\]

be the Fourier transformation extended as a unitary operator on the space \( L^2(\mathbb{R}^N) \), \( u \to \overline{u}, u \in \mathbb{C} \), be the complex conjugation.

Now, we assume that \( u_k = u_k(x) \) is a bounded sequence in \( L^\infty(\Omega) \) (more generally, in \( L^2_{\text{loc}}(\Omega) \)) weakly convergent to 0,
Proposition 4.1 (see [24, 5]). There exists a nonnegative Borel measure \( \mu \) in \( \Omega \times S \) and a subsequence \( u_r(x) = u_k(x), k = k_r, \) such that

\[
\langle \mu, \Phi_1(x) \Phi_2(x) \psi(\xi) \rangle = \lim_{r \to \infty} \int_{\mathbb{R}^n} F(u_r \Phi_1)(\xi) \overline{F(u_r \Phi_2)(\xi)} \psi(\pi(\xi)) d\xi
\]

for all \( \Phi_1(x), \Phi_2(x) \in C_0(\Omega) \) and \( \psi(\xi) \in C(S) \).

Remark 4.1. It follows from (4.2) and the Plancherel identity that \( pr_\Omega \) \( \mu \leq C \) meas, and that (4.2) remains valid for all \( \Phi_1(x), \Phi_2(x) \in L^2(\Omega) \), cf. [15, Remark 2(a)]. Here we denote by meas the Lebesgue measure on \( \Omega \).

The measure \( \mu \) is called an H-measure corresponding to \( u_r(x) \). It is clear that \( \mu = 0 \) if and only if the sequence \( u_r \to 0 \) as \( r \to \infty \) in \( L^2_{pr}(\Omega) \). In some sense, the H-measure \( \mu = \mu(x, \xi) \) indicates the strength of oscillations of the sequence \( u_r \) at the point \( x \) and in the direction \( \xi \).

In the sequel we will use the special sequences obtained from a given function by rescaling of independent variables. Namely, let \( u(t, x) \in L^\infty(\Pi) \) be a function periodic over the space variables \( x \) (with the standard lattice of periods \( \mathbb{Z}^n \)), and \( s_k \in \mathbb{R}_+, p_k, q_k \in \mathbb{N} \) be sequences converging to infinity as \( k \to \infty \). We consider the sequence

\[
u_k = u_k(t, x) = u(s_k t, p_k x + q_k \bar{x}), \]

where \( \bar{x}, \bar{x} \) are the orthogonal projection of \( x \in \mathbb{R}^n \) into the spaces \( \mathbb{R}^d = \{ x = (x_1, \ldots, x_n) \mid x_i = 0, i = d + 1, \ldots, n \} \) and \( (\mathbb{R}^d)^\perp = \{ x = (x_1, \ldots, x_n) \mid x_i = 0, i = 1, \ldots, d \} \), respectively. By the periodicity, the function \( u \) and the functions \( u_k \) can be treated as elements of \( L^\infty(\mathbb{R}_+ \times \mathbb{T}^n) \), where \( \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \) is the standard torus.

Lemma 4.1. Suppose that the sequence \( I_0(s_k t) = \int_{\mathbb{T}^n} u(s_k t, x) dx \to v(t) \) weakly-* in \( L^\infty(\mathbb{R}_+) \) as \( k \to \infty \). Then \( u_k \to v(t) \) weakly-* in \( L^\infty(\Pi) \) as \( k \to \infty \).

Proof. For given \( r \in \mathbb{Z}^n, r \neq 0 \), we choose \( \bar{k} \in \mathbb{N} \) so large that \( \min(p_k, q_k) > |r|^2 \) for all \( k \geq \bar{k} \). Observe that the projections \( \tilde{r}, \tilde{r} \in \mathbb{Z}^n \) and, by the periodicity of \( u(t, \cdot) \),

\[
\int_{\mathbb{T}^n} e^{2\pi i r \cdot x} u_k(t, x) dx = \int_{\mathbb{T}^n} e^{2\pi i r \cdot x + \bar{r} / p_k} u(s_k t, p_k \bar{x} + \bar{r} + q_k \bar{x}) dx =
\]

\[
\int_{\mathbb{T}^n} e^{2\pi i r \cdot x + \bar{r} / p_k} u(s_k t, p_k \bar{x} + q_k \bar{x}) dx = e^{2\pi i |\bar{r}|^2 / p_k} \int_{\mathbb{T}^n} e^{2\pi i r \cdot x} u_k(t, x) dx, \tag{4.3}
\]

\[
\int_{\mathbb{T}^n} e^{2\pi i \tilde{r} \cdot x} u_k(t, x) dx = \int_{\mathbb{T}^n} e^{2\pi i \tilde{r} \cdot x + q_k \bar{x}} u(s_k t, p_k \bar{x} + q_k \bar{x} + \bar{r}) dx =
\]

\[
\int_{\mathbb{T}^n} e^{2\pi i \tilde{r} \cdot x + q_k \bar{x}} u(s_k t, p_k \bar{x} + q_k \bar{x}) dx = e^{2\pi i |\tilde{r}|^2 / q_k} \int_{\mathbb{T}^n} e^{2\pi i \tilde{r} \cdot x} u_k(t, x) dx. \tag{4.4}
\]
Since either \( \tilde{r} \neq 0 \) or \( \bar{r} \neq 0 \) while \( p_k > |r|^2 \geq |\tilde{r}|^2, q_k > |r|^2 \geq |\bar{r}|^2 \), one of the factors \( e^{2\pi i \tilde{r}/p_k}, e^{2\pi i \bar{r}/q_k} \) is different from 1. Therefore, it follows from (4.3), (4.4) that for \( k \geq \bar{k} \)

\[
\int_{\mathbb{T}^n} e^{2\pi i r \cdot x} u_k(t, x) dx = 0 \quad \forall t > 0.
\]

This implies that for each \( a(t) \in L^1(\mathbb{R}_+) \)

\[
\int_{\mathbb{R}_+ \times \mathbb{T}^n} a(t) e^{2\pi i r \cdot x} u_k(t, x) dtdx \to 0 = \int_{\mathbb{R}_+ \times \mathbb{T}^n} v(t) a(t) e^{2\pi i r \cdot x} dtdx. \tag{4.5}
\]

If \( r = 0 \) we have

\[
\int_{\mathbb{T}^n} u_k(t, x) dx = \int_{\mathbb{T}^n} u(s_k t, p_k \tilde{x} + q_k \bar{x}) dx = \int_{\mathbb{T}^n} u(s_k t, x) dx = I_0(s_k t)
\]

because the map \( x \to p_k \tilde{x} + q_k \bar{x} \) keeps the Lebesgue measure on \( \mathbb{T}^n \). Since \( I_0(s_k t) \) weakly-* converges to the function \( v(t) \), we conclude that for each \( a(t) \in L^1(\mathbb{R}_+) \)

\[
\int_{\mathbb{R}_+ \times \mathbb{T}^n} a(t) u_k(t, x) dtdx \to \int_{\mathbb{R}_+} v(t) a(t) dt = \int_{\mathbb{R}_+ \times \mathbb{T}^n} v(t) a(t) dtdx. \tag{4.6}
\]

From relations (4.5), (4.6) it follows that

\[
\int_{\mathbb{R}_+ \times \mathbb{T}^n} f(t, x) u_k(t, x) dtdx \to \int_{\mathbb{R}_+ \times \mathbb{T}^n} v(t) f(t, x) dtdx \tag{4.7}
\]

for every function \( f(t, x) = \sum_{r \in I} a_r(t) e^{2\pi i r \cdot x} \), where the parameter \( r \) runs over a finite set \( I \subset \mathbb{Z}^n \) while \( a_r(t) \in L^1(\mathbb{R}_+) \). Since the space of such functions is dense in \( L^1(\mathbb{R}_+ \times \mathbb{T}^n) \), relation (4.7) implies that \( u_k \rightharpoonup v(t) \) as \( k \to \infty \) weakly-* in \( L^\infty(\mathbb{R}_+ \times \mathbb{T}^n) \). By the periodicity, the above limit relation holds also weakly-* in the space \( L^\infty(\Pi) \). The proof is complete.

**Corollary 4.1.** Assume in addition that for a dense set of functions \( p(u) \in C(\mathbb{R}) \)

\[
\int_{\mathbb{T}^n} p(u(s_k t, x)) dx \to c_p = \text{const weakly-* in } L^\infty(\mathbb{R}_+). \tag{4.8}
\]

Then the sequence \( u_k \) converges to a constant measure valued function \( \nu_{t, x} \equiv \nu \) in the sense of relation (4.1).
Proof. Since the set of functions \( p(\lambda) \), for which relation (4.8) holds, is dense in \( C([-M, M]) \), where \( M = \|u\|\infty \), this relation remains valid for all \( p(\lambda) \in C([-M, M]) \). It is clear that the functional \( p \to c_p \) is linear and continuous. By the Riesz-Markov representation theorem \( c_p = \langle \nu, p(\lambda) \rangle = \int p(\lambda)d\nu(\lambda) \) with some Borel measure \( \nu \) on \([-M, M]\). Evidently, \( c_p \geq 0 \) for \( p(\lambda) \geq 0 \) and \( c_p = 1 \) for \( p \equiv 1 \), which implies that \( \nu \) is a probability measure. By Lemma 4.1 we conclude that for each \( p(\lambda) \in C(\mathbb{R}) \)

\[
\lim_{k \to \infty} c_p = \langle \nu, p(\lambda) \rangle \quad \text{weakly-}* \quad \text{in} \quad L^\infty(\Pi),
\]

which is exactly (4.1) with \( \nu_{t,x} \equiv \nu \). \( \square \)

Remark 4.2. Condition (4.8) is always satisfied in the case when \( u(t, x) \) is a periodic e.s. of (1.1), (1.3). In fact, any function \( p(u) \in C^2(\mathbb{R}) \) is a difference of two convex functions. By Corollary 2.1(i) the function \( I_p(t) = \int_{\mathbb{T}^n} p(u(t, x))dx \) is a difference of two decreasing functions (after possible extraction of a set of null measure). Hence \( I_p(t) \) is a function of bounded variation and there exists a limit \( c_p \) of this function as \( t \to +\infty \). This implies that the sequence \( I_p(s_k t) \) converges in \( L^1_{\text{loc}}(\mathbb{R}^+) \) to the constant \( c_p \). Hence, limit relation (4.8) holds, even in the stronger topology of \( L^1_{\text{loc}} \).

Let \( u_k = u(k^2 t, k^2 \bar{x} + k\bar{x}) \) be a sequence of the considered above kind (with \( s_k = p_k = k^2 \), \( q_k = k \)). Passing to a subsequence, we may assume that this sequence converges weakly-* in \( L^\infty(\Pi) \). Then the sequence

\[
I_0(k^2 t) = \int_{\mathbb{T}^n} u(k^2 t, x)dx = \int_{\mathbb{T}^n} u_k(t, x)dx
\]

converges weakly-* in \( L^\infty(\mathbb{R}^+) \) to some function \( v(t) \). By Lemma 4.1 the function \( v(t) \) is a weak-* limit of the sequence \( u_k \).

We are going to investigate the H-measure corresponding to the sequence \( u_k - v(t) \). We suppose that the family \( u(t, \cdot), t > 0, \) is pre-compact in \( L^2(\mathbb{T}^n) \). As was shown in [16] Lemma 3.1, this requirement implies that the Fourier series

\[
u(t, x) = \sum_{\kappa \in \mathbb{Z}^n} a_{\kappa}(t)e^{2\pi i \kappa \cdot x}, \quad a_{\kappa}(t) = \int_{\mathbb{T}^n} e^{-2\pi i \kappa \cdot x} u(t, x)dx,
\]

converge in \( L^2(\mathbb{T}^n) \) uniformly in \( t \geq 0 \).

By Proposition 4.1 there exists \( \mu = \mu(t, x, \tau, \xi) \), an H-measure corresponding to some subsequence of \( u_k - v \). Denote by

\[
S_0 = \left\{ \pi(\hat{\xi}) \in S \mid \hat{\xi} = (\tau, \xi) \neq 0, \ \tau \in \mathbb{R}, \xi \in \mathbb{Z}^n, |\hat{\xi}| \cdot |\xi| = 0 \right\}.
\]

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where

\[ \pi(\tau, \xi) = \frac{(\tau, \tilde{\xi})}{(\tau^2 + |\xi|^2)^{1/2}} \]

is the orthogonal projection on the sphere

\[ S = \{ (\tau, \xi) \in \mathbb{R}^{n+1} \mid \tau^2 + |\xi|^2 = 1 \}. \]

**Proposition 4.2.** The support of H-measure \( \mu \) is contained in \( \Pi \times S_0 \), that is, \( \mu(\Pi \times (S \setminus S_0)) = 0 \).

**Proof.** For \( m \in \mathbb{N} \) we introduce the sets

\[ S_m = \{ \pi(\hat{\xi}) \in S \mid \hat{\xi} = (\tau, \xi) \neq 0, \tau \in \mathbb{R}, \xi \in \mathbb{Z}^n, |\tilde{\xi}| \cdot |\bar{\xi}| = 0, |\xi| \leq m \}. \]

It is clear that \( S_m \) is a closed subset of \( S \) (it is the union of the finite set of circles \{ \( (p, q|\xi|^{-1}\xi) \mid p^2 + q^2 = 1 \} \), where \( \xi \in \mathbb{Z}^n, |\tilde{\xi}| \cdot |\bar{\xi}| = 0, 0 < |\xi| \leq m \), and \( S_0 = \bigcup_{m=1}^{\infty} S_m \).

Let

\[ u(t, x) = \sum_{\kappa \in \mathbb{Z}^n} a_\kappa(t) e^{2\pi i \kappa \cdot x} \]

be the Fourier series for \( u(t, \cdot) \) in \( L^2(T^n) \). Then

\[ u_k(t, x) = u(k^2 t, k^2 \tilde{x} + k\bar{x}) = \sum_{\kappa \in \mathbb{Z}^n} a_\kappa(k^2 t) e^{2\pi i (k^2 \tilde{\kappa} + k\bar{\kappa}) \cdot x}. \tag{4.10} \]

We denote \( b_{0,k} = a_0(k^2 t) - v(t) \); \( b_{\kappa,k} = a_\kappa(k^2 t) \), where \( \kappa \in \mathbb{Z}^n, \kappa \neq 0 \). Let \( \alpha(t) \in C_0(\mathbb{R}_+) \), and \( \beta(x) \in L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n) \) be such function that its Fourier transform is a continuous compactly supported function:

\[ \tilde{\beta}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} \beta(x) dx \in C_0(\mathbb{R}^n). \tag{4.11} \]

We take \( R = \max_{\xi \in \text{supp} \tilde{\beta}} |\xi| \). Let \( \Phi(t, x) = \alpha(t) \beta(x) \). By (4.10) we find that

\[ (u_k(t, x) - v(t))\Phi(t, x) = \sum_{\kappa \in \mathbb{Z}^n} b_{\kappa,k}(t) \alpha(t) e^{2\pi i (k^2 \tilde{\kappa} + k\bar{\kappa}) \cdot x} \beta(x). \tag{4.12} \]

Observe that the Fourier transform of \( e^{2\pi i (k^2 \tilde{\kappa} + k\bar{\kappa}) \cdot x} \beta(x) \) in \( \mathbb{R}^n \) coincides with \( \hat{\beta}(\xi - (k^2 \tilde{\kappa} + k\bar{\kappa})) \). Since for \( k > 2R + 1 \) supports of these functions do not intersect, then for such \( k \) the series

\[ \sum_{\kappa \in \mathbb{Z}^n} b_{\kappa,k}(t) \alpha(t) \tilde{\beta}(\xi - (k^2 \tilde{\kappa} + k\bar{\kappa})) \tag{4.13} \]

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is orthogonal in $L^2(\mathbb{R}^n)$ for each $t > 0$. Besides, by the Plancherel equality
\[
\|\tilde{\beta}(\xi - (k^2 \tilde{\kappa} + k\bar{\kappa}))\|_{L^2(\mathbb{R}^n)} = \|\tilde{\beta}\|_2 = \|\beta\|_2,
\]
and
\[
\sum_{\kappa \in \mathbb{Z}^n} |b_{\kappa,k}(t)\alpha(t)|^2 \|\tilde{\beta}(\xi - (k^2 \tilde{\kappa} + k\bar{\kappa}))\|_{L^2(\mathbb{R}^n)}^2 = |\alpha(t)|^2 \|\beta\|_2^2 \cdot \|u(k^2 t, \cdot) - v(t)\|_{L^2(\mathbb{T}^n)}^2 < +\infty.
\]
Therefore, orthogonal series (4.13) converges in $L^2(\mathbb{R}^n)$ for each $t > 0$. Moreover, by the uniform converges of the Fourier series (4.9)
\[
\sum_{\kappa \in \mathbb{Z}^n, |\kappa| > N} |b_{\kappa,k}(t)|^2 = \sum_{\kappa \in \mathbb{Z}^n, |\kappa| > N} |a_\kappa(k^2 t)|^2 \to 0 \quad (N \to \infty) \tag{4.14}
\]
uniformly with respect to $t > 0$. Hence, series (4.13) converges in $L^2(\mathbb{R}^n)$ uniformly with respect to $t$. Since the Fourier transformation is an isomorphism on $L^2(\mathbb{R}^n)$, we conclude that series (4.12) also converges in $L^2(\mathbb{R}^n)$ uniformly with respect to $t$. Since $\alpha(t) \in C_0(\mathbb{R})$, this implies that (4.12) converges in $L^2(\Pi)$, and
\[
F((u_k - v)\Phi)(\hat{\xi}) = \sum_{\kappa \in \mathbb{Z}^n} F^t(\alpha b_{\kappa,k})(\tau)\tilde{\beta}(\xi - (k^2 \tilde{\kappa} + k\bar{\kappa})), \quad \hat{\xi} = (\tau, \xi), \tag{4.15}
\]
where $F^t(h)(\tau) = \int_{\mathbb{R}} e^{-2\pi i \tau t} h(t) dt$ denotes the Fourier transform over the time variable (we extend functions $h(t) \in L^2(\mathbb{R}_+)$ on the whole line $\mathbb{R}$, setting $h(t) = 0$ for $t < 0$). It follows from (4.15) that for $k > 2R + 1$
\[
\int_{\mathbb{R}^{n+1}} |F(\Phi(u_k - v))(\hat{\xi})|^2 \psi(\pi(\hat{\xi})) d\hat{\xi} = 
\]
\[
\sum_{\kappa \in \mathbb{Z}^n} \int_{\mathbb{R}^{n+1}} |F^t(\alpha b_{\kappa,k})(\tau)|^2 \tilde{\beta}(\xi - (k^2 \tilde{\kappa} + k\bar{\kappa}))^2 \psi(\pi(\hat{\xi})) d\hat{\xi}, \tag{4.16}
\]
where $\psi(\hat{\xi}) \in C(\mathbb{S})$ is arbitrary. Now we fix $\varepsilon > 0$. Recall that $b_{\kappa,k} = a_\kappa(k^2 t)$ for $\kappa \neq 0$, and by (4.14) there exists $m \in \mathbb{N}$ such that
\[
\sup_{t > 0} \sum_{\kappa \in \mathbb{Z}^n, |\kappa| > m} |a_\kappa(t)|^2 < \varepsilon.
\]
Then
\[
\sum_{\kappa \in \mathbb{Z}^n, |\kappa| > m} \int_{\mathbb{R}^{n+1}} |F'(\alpha b_{\kappa,k})(\tau)|^2 |\tilde{\beta}(\xi - (k^2 \bar{k} + k\bar{k}))|^2 d\xi =
\sum_{\kappa \in \mathbb{Z}^n, |\kappa| > m} \int_{\Pi} |\alpha(t)a_{\kappa}(k^2 t)|^2 |\beta(x)|^2 dt dx \leq
\|\Phi\|_2^2 \cdot \sup_{t > 0} \sum_{\kappa \in \mathbb{Z}^n, |\kappa| > m} |a_{\kappa}(t)|^2 \leq \varepsilon \|\Phi\|_2^2. \tag{4.17}
\]

Now we suppose that \(\|\psi\|_\infty \leq 1\) and \(\psi(\hat{\xi}) = 0\) on the set \(S_m\). By (4.17)
\[
\sum_{\kappa \in \mathbb{Z}^n, |\kappa| > m} \int_{\mathbb{R}^{n+1}} |F'(\alpha b_{\kappa,k})(\tau)|^2 |\tilde{\beta}(\xi - (k^2 \bar{k} + k\bar{k}))|^2 |\psi(\pi(\hat{\xi}))| d\hat{\xi} < \varepsilon \|\Phi\|_2^2. \tag{4.18}
\]

Since continuous function \(\psi(\hat{\xi})\) is uniformly continuous on the compact \(S\) then we can find such \(\delta > 0\) that \(|\psi(\hat{\xi}_1) - \psi(\hat{\xi}_2)| < \varepsilon\) whenever \(\hat{\xi}_1, \hat{\xi}_2 \in S\), \(|\hat{\xi}_1 - \hat{\xi}_2| < \delta\). Suppose that \(\kappa \neq 0\), \(\tilde{\beta}(\xi - (k^2 \bar{k} + k\bar{k})) \neq 0\). Then \(|\xi - (k^2 \bar{k} + k\bar{k})| \leq R\). For a fixed \(\tau \in \mathbb{R}\) we denote \(\hat{\xi} = (\tau, \xi), \hat{\eta} = (\tau, k^2 \bar{k} + k\bar{k}); \hat{\eta}_1 = (\tau, k^2 \bar{k})\) if \(\bar{k} \neq 0\), \(\hat{\eta}_1 = (\tau, k\bar{k})\) if \(\bar{k} = 0\). By the evident estimate \(|\pi(x) - \pi(y)| \leq \frac{2|x - y|}{|y|}\), for each \(\kappa \in \mathbb{Z}^n, \kappa \neq 0\),
\[
|\pi(\hat{\xi}) - \pi(\hat{\eta})| \leq \frac{2|\xi - (k^2 \bar{k} + k\bar{k})|}{|\eta|} \leq \frac{2R}{k|\kappa|} \leq \frac{2R}{k}, \tag{4.19}
\]
\[
|\pi(\hat{\eta}) - \pi(\hat{\eta}_1)| \leq \frac{2k|\bar{k}|}{k^2 |k\bar{k}|} \leq \frac{2|\bar{k}|}{k} \quad \text{if} \quad \bar{k} \neq 0; \quad \hat{\eta} = \hat{\eta}_1 \quad \text{if} \quad \bar{k} = 0. \tag{4.20}
\]

In view of (4.19), (4.20), we have
\[
|\pi(\hat{\xi}) - \pi(\hat{\eta}_1)| \leq \frac{2(R + |\bar{k}|)}{k} \tag{4.21}
\]

By (4.21) for all \(k > k_0 = 2(R + m)/\delta\) and all \(\kappa \in \mathbb{Z}^n\) such that \(0 < |\kappa| \leq m\) and \(\tilde{\beta}(\xi - (k^2 \bar{k} + k\bar{k})) \neq 0\) the inequality \(|\pi(\hat{\xi}) - \pi(\hat{\eta}_1)| < \delta\) holds, which implies the estimate
\[
|\psi(\pi(\hat{\xi}))| = |\psi(\pi(\hat{\xi})) - \psi(\pi(\hat{\eta}_1))| < \varepsilon. \tag{4.22}
\]

We use here that \(\pi(\hat{\eta}_1) \in S_m\) and, therefore, \(\psi(\pi(\hat{\eta}_1)) = 0\). In view of (4.22), for all
\[ k > k_0 \]

\[
\sum_{\kappa \in \mathbb{Z}^n, 0 < |\kappa| \leq m} \int_{\mathbb{R}^{n+1}} |F^t(\alpha b_{\kappa,k})(\tau)|^2 |\tilde{\beta}(\xi - (k^2 \hat{\kappa} + k \tilde{\kappa}))|^2 |\psi(\pi(\hat{\xi}))| d\hat{\xi} \\
\leq \varepsilon \sum_{\kappa \in \mathbb{Z}^n, 0 < |\kappa| \leq m} \int_{\mathbb{R}^{n+1}} |F^t(\alpha b_{\kappa,k})(\tau)|^2 |\tilde{\beta}(\xi - (k^2 \hat{\kappa} + k \tilde{\kappa}))|^2 d\hat{\xi} \leq \\
\varepsilon \|\beta\|_2^2 \sum_{\kappa \in \mathbb{Z}^n} \int_{\mathbb{R}} |\alpha(t)b_{\kappa,k}(t)|^2 dt \leq \varepsilon \|\Phi\|_2^2 \sup_{t > 0} \sum_{\kappa \in \mathbb{Z}^n} |b_{\kappa,k}(t)|^2 = \\
\varepsilon \|\Phi\|_2^2 \sup_{t > 0} \|u(k^2 t, \cdot) - v(t)\|_\mathcal{L}_2(\mathbb{R}^n) \leq C\varepsilon \|\Phi\|_2^2, \quad (4.23)
\]

where \( C = 4\|u\|_\infty^2 \). Further, in the case \( \kappa = 0 \) when \( \hat{\eta}_1 = \hat{\eta} = (\tau, 0) \in S_m \), we have the estimate

\[
|\pi(\hat{\xi}) - \pi(\hat{\eta})| \leq \frac{2|\xi|}{|\tau|} \leq \frac{2R}{|\tau|}
\]

for \( |\xi| \leq R \). Taking \( |\tau| > T \equiv \frac{2R}{\delta} \), we find

\[
|\psi(\pi(\hat{\xi}))| = |\psi(\pi(\hat{\xi})) - \psi(\pi(\hat{\eta}))| < \varepsilon.
\]

Therefore,

\[
\int_{\mathbb{R}^{n+1}} \theta(|\tau| - T)|F^t(\alpha b_{0,k})(\tau)|^2 |\tilde{\beta}(\xi)|^2 |\psi(\pi(\hat{\xi}))| d\hat{\xi} \leq C\varepsilon \|\Phi\|_2^2. \quad (4.24)
\]

Here \( \theta(r) = \begin{cases} 1 & , \quad r > 0, \\ 0 & , \quad r \leq 0 \end{cases} \) is the Heaviside function.

For \( |\tau| \leq T \) we are reasoning in the following way. Since \( \alpha(t)b_{0,k}(t) = \alpha(t)(a_{0,k}(t) - v(t)) \to 0 \) as \( k \to \infty \), and \( \|\alpha b_{0,k}\|_1 \leq C_1 = 2\|u\|_\infty\|\alpha\|_1 \), the Fourier transform \( F^t(\alpha b_{0,k})(\tau) \to 0 \) for all \( \tau \in \mathbb{R} \) and uniformly bounded: \( |F^t(\alpha b_{0,k})(\tau)| \leq C_1 \). By Lebesgue dominated convergence theorem

\[
\int_{\mathbb{R}} \theta(T - |\tau|)|F^t(\alpha b_{0,k})(\tau)|^2 d\tau \to 0.
\]

Therefore (recall that \( \|\psi\|_\infty \leq 1 \),

\[
\int_{\mathbb{R}^{n+1}} \theta(T - |\tau|)|F^t(\alpha b_{0,k})(\tau)|^2 |\tilde{\beta}(\xi)|^2 |\psi(\pi(\hat{\xi}))| d\hat{\xi} \leq \\
\|\beta\|_2 \int_{\mathbb{R}} \theta(T - |\tau|)|F^t(\alpha b_{0,k})(\tau)|^2 d\tau \to 0. \quad (4.25)
\]
In view of (4.24), (4.25) we find
\[
\limsup_{k \to \infty} \int_{\mathbb{R}^{n+1}} |F^t(\alpha b_{0,k})(\tau)|^2 |\tilde{\beta}(\xi)|^2 |\psi(\pi(\hat{\xi}))| d\hat{\xi} \leq C \varepsilon \|\Phi\|_2^2.
\] (4.26)

Using (4.16), (4.18), (4.23) and (4.26), we arrive at the relation
\[
\limsup_{k \to \infty} \int_{\mathbb{R}^{n+1}} |F(\Phi(u_k - v))(\hat{\xi})|^2 |\psi(\pi(\hat{\xi}))| d\hat{\xi} \leq C_2 \varepsilon \|\Phi\|_2^2, \tag{4.27}
\]
where \(C_2\) is a constant. By the definition of H-measure and Remark 4.1
\[
\lim_{k \to \infty} \int_{\mathbb{R}^{n+1}} |F(\Phi(u_k - v))(\hat{\xi})|^2 |\psi(\pi(\hat{\xi}))| d\hat{\xi} = \langle \mu, |\Phi(t,x)|^2 \psi(\hat{\xi}) \rangle = \int_{\Pi \times (S \setminus S_m)} |\Phi(t,x)|^2 \psi(\hat{\xi}) |d\mu(t,x,\hat{\xi})|
\]
and (4.27) implies that
\[
\int_{\Pi \times (S \setminus S_m)} |\Phi(t,x)|^2 \psi(\hat{\xi}) d\mu(t,x,\hat{\xi}) \leq C_2 \varepsilon \|\Phi\|_2^2
\]
for all \(\psi(\hat{\xi}) \in C_0(S \setminus S_m)\) such that \(0 \leq \psi(\hat{\xi}) \leq 1\). Therefore, we can claim that
\[
\int_{\Pi \times (S \setminus S_m)} |\Phi(t,x)|^2 d\mu(t,x,\hat{\xi}) \leq C_2 \varepsilon \|\Phi\|_2^2,
\]
and since \(S \setminus S_0 \subset S \setminus S_m\), we obtain the relation
\[
\int_{\Pi \times (S \setminus S_0)} |\Phi(t,x)|^2 d\mu(t,x,\hat{\xi}) \leq C_2 \varepsilon \|\Phi\|_2^2,
\]
which holds for arbitrary positive \(\varepsilon\). Therefore,
\[
\int_{\Pi \times (S \setminus S_0)} |\Phi(t,x)|^2 d\mu(t,x,\hat{\xi}) = 0. \tag{4.28}
\]

Since for every point \((t_0, x_0) \in \Pi\) one can find functions \(\alpha(t), \beta(x)\) with the prescribed above properties in such a way that \(\Phi(t,x) = \alpha(t)\beta(x) \neq 0\) in a neighborhood of \((t_0, x_0)\), we derive from (4.28) the desired relation \(\mu(\Pi \times (S \setminus S_0)) = 0\). 

\(\square\)
4.2 Variants of H-measures

We will need the variant of H-measures with “continuous indexes” introduced in [12], see also subsequent papers [13,14,15,18]. Let \( u_k(x) \) be a bounded sequence in \( L^\infty(\Omega) \). Passing to a subsequence if necessary, we can suppose that this sequence converges to a bounded measure valued function \( \nu_x \in \text{MV}(\Omega) \) in the sense of relation (4.1). We introduce the measures \( \gamma^k(\lambda) = \delta(\lambda - u_k(x)) - \nu_x(\lambda) \) and the corresponding distribution functions \( U_k(x,p) = \gamma^k((p, +\infty)) \), \( u_0(x,p) = \nu_x((p, +\infty)) \) on \( \Omega \times \mathbb{R} \). Observe that \( U_k(x,p), u_0(x,p) \in L^\infty(\Omega) \) for all \( p \in \mathbb{R} \), see [12, Lemma 2]. We define the set

\[
E = E(\nu_x) = \left\{ p_0 \in \mathbb{R} \mid u_0(x,p) \xrightarrow{p \to p_0} u_0(x,p_0) \text{ in } L^1_{\text{loc}}(\Omega) \right\}.
\]

As was shown in [12, Lemma 4], the complement \( \mathbb{R} \setminus E \) is at most countable and if \( p \in E \) then \( U_k(x,p) \xrightarrow{k \to \infty} 0 \) weakly-* in \( L^\infty(\Omega) \).

The following result was established in [12] (see also [13,14,15]).

**Proposition 4.3.** There exist a family of locally finite complex Borel measures \( \{\mu_{pq}\}_{p,q \in E} \) on \( \Omega \times S \) and a subsequence \( U_r(x) = \{U_{k_r}(x,p)\}_{p \in E} \) such that for all \( \Phi_1(x), \Phi_2(x) \in C_0(\Omega) \) and \( \psi(\xi) \in C(S) \)

\[
\langle \mu_{pq}, \Phi_1(x)\Phi_2(x)\psi(\xi) \rangle = \lim_{r \to \infty} \int_{\mathbb{R}^n} F(\Phi_1 U_r(\cdot, p))(\xi) \overline{F(\Phi_2 U_r(\cdot, q))(\xi)} \psi(\pi(\xi)) d\xi. \tag{4.29}
\]

Moreover for every \( p_1, \ldots, p_l \in E, \xi_1, \ldots, \xi_l \in C \) the measure \( \sum_{i,j=1}^{l} \xi_i \xi_j \mu_{pq} \geq 0 \).

Since \( |U_r(x,p)| \leq 1 \), the projection \( \text{pr}_\Omega \mu_{pq} \leq \text{meas} \) and the measures \( \mu_{pq} \) admits disintegration: \( \mu_{pq}(x,\xi) = \mu^p_x(\xi) d\alpha \), where \( \mu^p_x(\xi) \in M(S) \), \( p, q \in E \), \( x \in \Omega \), is a family of finite Borel measures on \( S \). This means that for every \( f(x,\xi) \in C_0(\Omega \times S) \) the function

\[
x \to \langle \mu^p_x(\xi), f(x,\xi) \rangle = \int_S f(x,\xi) d\mu^p_x(\xi)
\]

is Lebesgue-measurable, bounded and

\[
\int_{\Omega \times S} f(x,\xi) d\mu^p_x(x,\xi) = \int_\Omega \langle \mu^p_x(\xi), f(x,\xi) \rangle dx. \tag{4.30}
\]

Let \( D \subset E \) be a countable dense subset of \( E \). As was demonstrated in [18], for almost all \( x \in \Omega \) and all \( p_1, q_1, p_2, q_2 \in D \)

\[
\text{Var}(\mu^p_{x q_2} - \mu^p_{x q_1}) \leq 2(\nu_x((p_1, p_2)))^{1/2} + 2(\nu_x((q_1, q_2)))^{1/2}.
\]
and these estimates implies that the maps \( D^2 \ni (p,q) \to \mu^{pq}_x \) admit right and left continuous in \( M(S) \) extensions \( \mu^{pq+}_x \) and \( \mu^{pq-}_x \), respectively, on the space of all pairs \((p,q) \in \mathbb{R}^2\), where \( x \) runs over some set of full measure in \( \Omega \). Actually, these extensions do not depend on the choice of a set \( D \). Namely, for different sets \( D \) the corresponding measures \( \mu^{pq\pm}_x \) coincide for all \( x \in \Omega' \), where \( \Omega' \subset \Omega \) is a set of full Lebesgue measure, independent of \( p,q \). The measures \( \mu^{pq+}_x \) inherits the nonnegative definiteness property: for every \( p_1,\ldots,p_l \in \mathbb{R}, \zeta_1,\ldots,\zeta_l \in \mathbb{C} \) the measures \( \sum_{i,j=1}^{l} \zeta_i \overline{\zeta_j} \mu^{pq\pm}_{x,i,j} \geq 0 \). As was shown in [18 Corollary 1], this property implies that for each \( p,q \in \mathbb{R} \) for every Borel set \( A \subset S \)

\[
|\mu^{pq\pm}_x|(A) \leq (\mu^{pq\pm}_x(A) \mu^{pq\pm}_x(A))^{1/2}, \tag{4.31}
\]

where \(|\mu|\) denotes the variation of a measure \( \mu \), i.e., the smallest nonnegative measure \( m \) with the property \(|\mu(A)| \leq m(A) \) for every Borel set \( A \).

The following property connecting the \( H \)-measure \( \mu^{pq\pm}_x \) and the measure valued function \( \nu_x \), corresponding to the same subsequence \( \psi_r \), was proved in [18 Corollary 2].

**Lemma 4.2.** For almost all \( x \in \Omega \) (independent of \( p \)) for all \( p \) belonging to the smallest segment \([a,b]\) containing \( \text{supp}\nu_x \) the following holds: If \( S_+, S_- \subset S \) are Borel sets such that

\[
\mu^{pq+}_x(S \setminus S_+) = \mu^{pq-}_x(S \setminus S_-) = 0
\]

then \( S_+ \cap S_- \neq \emptyset \). In particular, \( \text{supp}\mu^{pq+}_x \cap \text{supp}\mu^{pq-}_x \neq \emptyset \).

Denote by \( s_{\alpha,\beta}(u) = \max(\alpha, \min(\beta, u)) \) the cut-off function, where \( \alpha, \beta \in \mathbb{R}, \alpha < \beta \). Let \( \psi(u) = (\psi_1(u), \ldots, \psi_N(u)) \in C(\mathbb{R}, \mathbb{R}^N) \) be a continuous vector-function. Suppose that for every \( \alpha, \beta \in \mathbb{R}, \alpha < \beta \) the sequences of distributions

\[
\text{div} \psi(s_{\alpha,\beta}(u_k(x))) \quad \text{are pre-compact in} \quad H^{-1}_\text{loc}(\Omega),
\]

where the Sobolev space \( H^{-1}_\text{loc}(\Omega) \) is a locally convex space, consisting of distributions \( l \in \mathcal{D}'(\Omega) \) such that for all \( f \in C^\infty_0(\Omega) \) the distribution \( fl \in H^{-1} = H^{-1}(\mathbb{R}^N) \) (the latter space is dual to the Sobolev space \( H^1 = W^1_2(\mathbb{R}^N) \)), the topology in \( H^{-1}_\text{loc}(\Omega) \) is generated by seminorms \( ||fl||_{H^{-1}} \). Extracting a subsequence if necessary, we can assume that H-measures \( \mu^{pq\pm}_x \) are well-defined. Let \( H_+(x), H_-(x) \) be the minimal linear subspaces of \( \mathbb{R}^N \) containing \( \text{supp}\mu^{pq_+p_0}_x, \text{supp}\mu^{pq_-p_0}_x \), respectively, where \( p_0 \in \mathbb{R} \) is fixed.

**Proposition 4.4.** For a.e. \( x \in \Omega \) for all \( \xi \in H_+(x) \cap H_-(x) \) the function \( \xi \cdot \psi(u) \) is constant in some vicinity \(|u - p_0| < \delta\) of \( p_0 \).
The statement of Proposition 4.5 follows from a general result of [18, Theorem 3.2].

In the case of the sequence \( u_k = u(k^2t, k^2 \vec{x} + k \vec{x}) \) it follows from Proposition 4.2 that \( \mu_{t,x}^{pp}(S \setminus S_0) = 0 \). More precisely, we assume that \( u(t, x) \in L^\infty(\Pi) \) is an \( x \)-periodic function with the standard lattice of periods \( \mathbb{Z}^n \). We consider the sequence \( u_k = u(k^2t, k^2 \vec{x} + k \vec{x}) \). Passing to a subsequence (and keeping the notation \( u_k \) for this subsequence), we may assume that \( u_k \to \nu_{t,x} \) in the sense of relation (4.1), and that the H-measures \( \mu_{t,x}^{pq} \) are defined.

**Proposition 4.5.** Assume that the family \( u(t, \cdot), t > 0 \), is pre-compact in \( L^2(\mathbb{T}^n) \) (possibly, after correction on a set of full measure). Then there exists a set \( \Pi' \subset \Pi \) of full Lebesgue measure such that for all \( (t, x) \in \Pi' \) and all \( p, q \in \mathbb{R} \) \( \mu_{t,x}^{pq}(S \setminus S_0) = 0 \).

**Proof.** Let \( M = \| u \|_\infty \). Observe that by Lemma 4.1 for each \( g(\lambda) \in C(\mathbb{R}) \) the weak-* limit of \( g(u_k) \) does not depend on \( x \):

\[
g(u_k) \underset{k \to \infty}{\rightharpoonup} v_g(t) \text{ weakly-* in } L^\infty(\Pi). \tag{4.32}
\]

Let \( G \subset C(\mathbb{R}) \) be a countable dense set. Then the set \( A \subset \mathbb{R}_+ \) of common Lebesgue points of the functions \( v_g(t), g \in G \) has full measure. Obviously, for each \( t \in A \) the functional \( g \to v_g(t) \) is uniquely extended to a linear continuous functional on the whole \( C(\mathbb{R}) \). Therefore, for some compactly supported measure \( \nu_t \)

\[
v_g(t) = \langle \nu_t, g(\lambda) \rangle \quad \forall g(\lambda) \in C(\mathbb{R}).
\]

Since \( v_g(t) = 0 \) whenever \( g \equiv 0 \) on \([-M, M]\), we see that \( \text{supp } \nu_t \subset [-M, M] \). By (4.32) we conclude that \( \nu_t \) is a limit measure valued function of the sequence \( u_k \). Hence, the measure valued function \( \nu_{t,x} = \nu_t \) does not depend on \( x \). By (4.29), (4.2) we find that the measure \( \mu_{t,x}^{pp} = \mu_{t,x}^{pp}(\hat{\xi})dtdx \) coincides with the H-measure \( \mu = \mu(t, x, \hat{\xi}) \) corresponding to the sequence \( U_k(t, x, p) = U(k^2t, k^2 \vec{x} + k \vec{x}, p) - v(t, p) \), where \( U(t, x, p) = \theta(u(t, x) - p) \) (\( \theta(r) \) being the Heaviside function), \( v(t, p) = \nu_t((p, +\infty)) \) is a weak-* limit of the sequence \( U_k(t, x, p) \). By Proposition 4.2 we claim that \( \mu(\Pi \times (S \setminus S_0)) = 0 \). From (4.30) it follows that for all \( f(t, x) \in C_0(\Pi), g(\hat{\xi}) \in C(S) \)

\[
\int_{\Pi} \langle \mu_{t,x}^{pp}(\hat{\xi}), g(\hat{\xi}) \rangle f(t, x)dtdx = \int_{\Pi \times S} f(t, x)g(\hat{\xi})d\mu(t, x, \hat{\xi}). \tag{4.33}
\]

Obviously, (4.33) remains valid for Borel functions \( g(\hat{\xi}) \). Taking \( g(\hat{\xi}) \) being the indicator function of the set \( S \setminus S_0 \), we arrive at the relation

\[
\int_{\Pi} \langle \mu_{t,x}^{pp}(\hat{\xi}), g(\hat{\xi}) \rangle f(t, x)dtdx = 0 \quad \forall f(t, x) \in C_0(\Pi).
\]

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Therefore, \( \langle \mu_{t,x}^{pp}(\hat{\xi}), g(\hat{\xi}) \rangle = 0 \) for almost all \((t, x) \in \Pi\). Now we fix some countable dense set \(D \subset E\) and choose a set of full Lebesgue measure \(\Pi_0 \subset \Pi\) such that \(\langle \mu_{t,x}^{pp}(\hat{\xi}), g(\hat{\xi}) \rangle = 0\) for all \((t, x) \in \Pi_0\) and \(p \in D\). By the definition of measures \(\mu_{t,x}^{pp^+}, \mu_{t,x}^{pp^-}\), they are, respectively, right- and left-continuous in \(M(S)\) with respect to \(p \in \mathbb{R}\), for some set \(\Pi' \subset \Pi_0\) of full measure of values \((t, x)\). Therefore, \(\langle \mu_{t,x}^{pp^+}(\hat{\xi}), g(\hat{\xi}) \rangle = 0\), that is, \(\mu_{t,x}^{pp^+}(S \setminus S_0) = 0\) for all \(p \in \mathbb{R}\), \((t, x) \in \Pi'\). To complete the proof, it only remains to apply relation \((4.31)\).

\[ \square \]

5 The proof of the decay property

We assume that \(u(t, x)\) is an \(x\)-periodic e.s. of \((1.1), (1.3)\). As was demonstrated in section 3 we may suppose that conditions \((R1)-(R4)\) hold. In view of \((1.10)\) for every \(k \in \mathbb{N}\) the function \(u_k = u(k^2t, k^2\tilde{x} + k\tilde{x})\) is an e.s. of the equation

\[ u_t + \text{div}_x(\varphi^k(u) - a^k(u)\nabla_x u) = 0, \quad (5.1) \]

where \(\varphi^k(u)\) is the vector with components \(\varphi^k_i(u) = \varphi_i(u), i = 1, \ldots, d, \varphi^k_i(u) = k\varphi_i(u), i = d+1, \ldots, n\), while the symmetric matrix \(a^k(u)\) has the entries \((a^k(u))_{ij} = \epsilon_{ki}\epsilon_{kj}a_{ij}(u), 1 \leq i, j \leq n\), where \(\epsilon_{ki} = k^{-1}\) for \(1 \leq i \leq d\), \(\epsilon_{ki} = 1\) for \(d < i \leq n\).

If \(a(u) = b(u)^\top b(u)\) is an admissible representation then \(a^k(u) = b_k(u)^\top b_k(u), (b_k)_{rj} = \epsilon_{kj}b_{rj}\), is an admissible representation for the matrix \(a^k(u)\). Let \(\alpha, \beta \in \mathbb{R}, \alpha < \beta, 0 < (\alpha, \beta) = \max(\alpha, \min(\beta, u))\) be the corresponding cut-off function, \(A^k(u)\) be a primitive of the matrix \(a^k(u)\). By Definition \((1.1)(i, ii)\) and Remark \((1.1)\)

\[ \sum_{j=1}^{n}[(A^k(s_{\alpha,\beta}(u_k)))_{ij}]_{x_j} = \sum_{j=1}^{n} \sum_{r=1}^{n} (T(b_k)_{rj}(u))_{x_j} = \sum_{r=1}^{n} (b_k)_{rj}(u_k) \delta_{\alpha,\beta}(u_k) \text{div}_x(B_k)_{rj}(u_k), \quad (5.2) \]

where the matrix \(B_k\) is a primitive of \(b_k\), \((B_k)'(u) = b_k(u)\), and \((B_k)_{rj}\) is the vector with components \((B_k)_{rj}(u), j = 1, \ldots, n\). By \(\chi_{\alpha,\beta}(u)\) we denote the indicator function of the interval \((\alpha, \beta)\). By Lemma \((2.2)\) \(\text{div}_x(B_k)_{rj}(u_k) \in L^2_{\text{loc}}(\Pi)\), and

\[ \int_{(\delta, +\infty) \times \mathbb{T}^n} \sum_{r=1}^{n} |\text{div}_x(B_k)_{rj}(u_k)|^2 dt dx \leq \frac{1}{2} \int_{\mathbb{T}^n} |u_k(\delta, x)|^2 dx - \text{ess lim} \frac{1}{2} \int_{\mathbb{T}^n} |u_k(T, x)|^2 dx, \quad (5.3) \]
where \( \delta > 0 \) is a common Lebesgue point of the functions \( t \to \int_{\mathbb{T}^n} |u_k(t, x)|^2 dx, \ k \in \mathbb{N} \).

Since the maps \( x \to y = k^2 \bar{x} + k \bar{x} \) keep the Lebesgue measure on the torus \( \mathbb{T}^n \), then for each \( t > 0 \)
\[
\int_{\mathbb{T}^n} |u_k(t, x)|^2 dx = \int_{\mathbb{T}^n} |u(k^2 t, k^2 \bar{x} + k \bar{x})|^2 dx = \int_{\mathbb{T}^n} |u(k^2 t, y)|^2 dy,
\]
and it follows from (5.3) that for almost each \( \delta > 0 \)
\[
\int_{(\delta, +\infty) \times \mathbb{T}^n} \sum_{r=1}^n |\text{div}_x(B_k)_r(u_k)|^2 dtdx \leq \frac{1}{2} \int_{\mathbb{T}^n} |u(k^2 \delta, y)|^2 dy - \text{ess lim}_{T \to +\infty} \frac{1}{2} \int_{\mathbb{T}^n} |u(T, y)|^2 dy \xrightarrow{k \to \infty} 0,
\]
which implies that for all \( r = 1, \ldots, n \)
\[
\text{div}_x(B_k)_r(u_k) \to 0 \text{ in } L^2_{\text{loc}}(\Pi) \tag{5.4}
\]
as \( k \to \infty \). It follows from (5.2) that for \( 1 \leq i \leq n \) the distributions
\[
\sum_{j=1}^n [(A_k(s_{\alpha,\beta}(u_k)))_{ij}]_{x_j} \to 0 \text{ in } L^2_{\text{loc}}(\Pi)
\]
This implies that
\[
\sum_{i=1}^n \sum_{j=1}^n [(A^k(s_{\alpha,\beta}(u_k)))_{ij}]_{x_i x_j} \to 0 \tag{5.5}
\]
as \( k \to \infty \) in the Sobolev space \( H^{-1}_{\text{loc}}(\Pi) \).

Observe that for every \( g(u) \in C(\mathbb{R}) \)
\[
g(s_{\alpha,\beta}(u)) = \text{sign}^+(u - \alpha)(g(u) - g(\alpha)) - \text{sign}^+(u - \beta)(g(u) - g(\beta)) + g(\alpha) = T_{\text{sign}^+(u-\alpha)}(g)(u) - T_{\text{sign}^+(u-\beta)}(g)(u) + \text{const},
\]
where \( \text{sign}^+ u = (\max(u, 0))^' \) is the Heaviside function. Using this relation and Lemma 2.1 we obtain
\[
(s_{\alpha,\beta}(u_k))_t + \text{div}_x \phi^k(s_{\alpha,\beta}(u_k)) - D^2 \cdot A^k(s_{\alpha,\beta}(u_k)) = \mu_{\beta}^k - \mu_{\alpha}^k \tag{5.6}
\]
in \( \mathcal{D}'(\Pi) \), where \( \mu^k_r, r \in \mathbb{R} \) are nonnegative measures defined by (2.1) with \( \varphi(u) = \varphi^k(u) \), \( A(u) = A^k(u) \), \( u = u_k \), and \( \eta(u) = (u - r)^+ \max(u - r, 0) \). By Corollary 2.1(iii)
\[
\mu^k_r(\mathbb{R}_+ \times \mathbb{T}^n) \leq \int_{\mathbb{T}^n} \left( u_0\left(k^2 \tilde{x} + k\tilde{r}\right) - r \right)^+ dx = \int_{\mathbb{T}^n} \left( u_0(x) - r \right)^+ dx,
\]
and the sequence \( \mu^k_r - \mu^k_r \) is bounded in the space of measures \( M_{\text{loc}}(\Pi) \) equipped with standard locally convex topology generated by semi-norms \( p_K(\mu) = |\mu|(K) \), where \( K \) is an arbitrary compact subset of \( \Pi \). By Murat’s interpolation lemma [11], it follows from (5.6) and (5.5) that the sequence of distributions
\[
(s^k_{\alpha,\beta}(u_k))_t + \text{div}_x \varphi^k(s^k_{\alpha,\beta}(u_k)) \quad (5.7)
\]
is pre-compact in \( H_{\text{loc}}^{-1}(\Pi) \).

By requirement (ii) of Definition 1.1
\[
\text{div}_x(B_k)_r(s^k_{\alpha,\beta}(u_k)) = \chi_{\alpha,\beta}(u_k) \text{div}_x(B_k)_r(u_k) \to 0
\]
as \( k \to \infty \) in \( L^2_{\text{loc}}(\Pi) \) and therefore in \( H_{\text{loc}}^{-1}(\Pi) \) as well. Since
\[
\sum_{j=d+1}^n B_{rj}(s^k_{\alpha,\beta}(u_k))_{x_j} = \text{div}_x(B_k)_r(s^k_{\alpha,\beta}(u_k)) - \frac{1}{k} \sum_{j=1}^d B_{rj}(s^k_{\alpha,\beta}(u_k))_{x_j},
\]
we claim that the sequences of distributions
\[
\sum_{j=d+1}^n B_{rj}(s^k_{\alpha,\beta}(u_k))_{x_j} \to 0 \quad (5.8)
\]
in \( H_{\text{loc}}^{-1}(\Pi) \) for all \( r = 1, \ldots, n \). In particular, these sequences are pre-compact in \( H_{\text{loc}}^{-1}(\Pi) \).

Taking into account Remark 4.2 and Corollary 4.1, we conclude that the sequence \( u_k \) converges as \( k \to \infty \) to a constant measure valued function \( \nu_{t,x} \equiv \nu \) (in the sense of relation (4.1)). Let \([a_0, b_0]\) be the smallest segment containing supp \( \nu \). We suppose that \( a_0 < b_0 \) and are going to get a contradiction.

Notice that by Corollary 2.1(ii) for a.e. \( t > 0 \int_{\mathbb{T}^n} u(t, x)dx = \int_{\mathbb{T}^n} u_0(x)dx = I \). By Lemma 4.1 we find that \( u_k \to I \) as \( k \to \infty \) weakly-\( * \) in \( L^\infty(\Pi) \). In view of relation (4.1)
\[
I = \int \lambda d\nu(\lambda) \in (a_0, b_0).
\]
By our assumption (R3) the functions \( \varphi_i(u) \), \( i = d + 1, \ldots, n \), are constant on some interval \((a_1, b_1) \supset I \). Without loss of generality we can assume that \((a_1, b_1) \subset (a_0, b_0)\).
Lemma 5.1. There exists an interval \((a_2, b_2) \subset (a_1, b_1)\) such that
\[
\int s_{a_2, b_2}(\lambda) d\nu(\lambda) = I. \tag{5.9}
\]

**Proof.** Denote \(I_1 = \int s_{a_1, b_1}(\lambda) d\nu(\lambda)\). If \(I_1 = I\), there is nothing to prove, we set \((a_2, b_2) = (a_1, b_1)\). If \(I_1 < I\), then the continuous function \(f(a) = \int s_{a, b_1}(\lambda) d\nu(\lambda)\) takes values \(f(a_1) = I_1 < I, f(I) > I\). Therefore, there exists \(a_2 \in (a_1, I)\) such that \(f(a_2) = I\). Taking \(b_2 = b_1\), we conclude that (5.9) holds. In the case \(I_1 > I\) we consider the continuous function \(g(b) = \int s_{a_1, b}(\lambda) d\nu(\lambda)\) and observe that \(g(I) < I, g(b_1) = I_1 > I\). Therefore, there is a value \(b_2 \in (I, b_1)\) such that \(g(b_2) = I\) and (5.9) follows with \(a_2 = a_1\). \(\square\)

Let \((a_2, b_2)\) be an interval indicated in Lemma 5.1. We consider the sequence \(v_k = s_{a_2, b_2}(u_k)\). In correspondence with (4.1)
\[
v_k \rightharpoonup \int s_{a_2, b_2}(\lambda) d\nu(\lambda) = I.
\]

Observe that in the case when \((\alpha, \beta) \cap (a_2, b_2) = (\alpha_1, \beta_1) \neq \emptyset\)
\[
s_{\alpha, \beta}(v_k) = s_{\alpha_1, \beta_1}(u_k).
\]
Otherwise, \(s_{\alpha, \beta}(v_k) \equiv c = \text{const}\). Since the flux functions \(\varphi_i(u), i = d + 1, \ldots, n\), are constant on the segment \([a_2, b_2]\), we conclude that \(\varphi_i(s_{\alpha, \beta}(v_k))_{x_i} = 0\) and in view of (5.7) for all \(\alpha, \beta \in \mathbb{R}, \alpha < \beta\) the sequences of distributions
\[
(s_{\alpha, \beta}(v_k))_{t} + \sum_{i=1}^{d} \varphi_i(s_{\alpha, \beta}(v_k))_{x_i} \tag{5.10}
\]
are pre-compact in \(H_{-1}^{loc}(\Pi)\). Similarly, relation (5.8) implies that the sequences
\[
\sum_{j=d+1}^{n} B_{r_j}(s_{\alpha, \beta}(v_k))_{x_j} \tag{5.11}
\]
are pre-compact in \(H_{-1}^{loc}(\Pi)\) as well.

As follows from relation (4.1), the limit measure valued function for the sequence \(v_k\) is a constant measure valued function \(\nu_{t,x} \equiv \nu_1 = s_{a_2, b_2}^*\nu\) is push-forward measure of \(\nu\) under the map \(s_{a_2, b_2}(\lambda)\). Since \([a_2, b_2] \subset [a_0, b_0]\), \([a_2, b_2]\) is the minimal segment containing \(\text{supp} \nu_1\). Passing to a subsequence if necessary, we can consider H-measures \(\mu_{t,x}^{pq}\) corresponding to the sequence \(v_k\). Let \(p \in (a_2, b_2), H_+(t, x), H_-(t, x)\) be the linear
spans of supports $\text{supp } \mu_{t,x}^{pp+}$ and $\text{supp } \mu_{t,x}^{pp-}$, respectively. As follows from Proposition 4.4 and pre-compactness of sequences (5.10), (5.11), for a.e. $(t,x) \in \Pi$ for all $(\tau, \xi) \in H_+(t,x) \cap H_-(t,x)$ the functions

$$u \to \tau u + \varphi(u) \cdot \hat{\xi} = \tau u + \sum_{j=1}^d \varphi_j(u) \xi_j = \text{const}; \quad (5.12)$$

$$u \to B_r(u) \cdot \hat{\xi} = \sum_{j=d+1}^n B_{r_j}(u) \xi_j = \text{const}, \quad r = 1, \ldots, n, \quad (5.13)$$

in some vicinity of $p$. For fixed $(t,x)$ we denote $H_\pm = H_\pm(t,x)$ and introduce the sets $S_\pm = H_\pm \cap S_0$. Since $S \setminus S_\pm \subset (S \setminus H_+) \cap (S \setminus S_0)$ then

$$\mu_{t,x}^{pp\pm}(S \setminus S_\pm) \leq \mu_{t,x}^{pp\pm}(S \setminus H_+) + \mu_{t,x}^{pp\pm}(S \setminus S_0) = 0,$$

where we use Proposition 4.5. We remark that by Corollary 2.2 the family $u(t, \cdot), t > 0$ is pre-compact in $L^2(\mathbb{T}^n)$, and the conditions of Proposition 4.5 are actually satisfied.

Applying Lemma 4.2, we obtain that for a.e. $(t,x) \in \Pi$ the set $H_+(t,x) \cap H_-(t,x) \cap S_0$ is not empty and for all $(\tau, \xi) \in H_+(t,x) \cap H_-(t,x) \cap S_0$ identities (5.12), (5.13) hold. In particular, we can take $p = I \in (a_0, b_0)$. Then it follows from (5.12) and assumption (R2) that $\hat{\xi} = 0$, $\tau = 0$. Further, in view of (5.13) $b(u) \hat{\xi} = \frac{d}{du} B(u) \hat{\xi} = 0$ in some vicinity of $I$. This implies that $a(u) \hat{\xi} = b(u)^+ b(u) \hat{\xi} = 0$ a.e. in this vicinity. By assumption (R4), we claim that $\hat{\xi} = 0$. Hence, $\hat{\xi} = (\tau, \xi) = 0$, which contradicts to the condition $\xi \in S$. We conclude that $a_0 = b_0 = I$, that is, $\nu = \delta(\lambda - I)$. This means that the limit measure valued function $\nu_{t,x}(\lambda) \equiv \nu(\lambda) = \delta(\lambda - I)$ of the sequence $u_k$ is regular, and by Theorem 4.1 $u_k \to I$ as $k \to \infty$ in $L^1_{\text{loc}}(\Pi)$. This implies that for a.e. $t > 0$ $u_k(t, \cdot) \to I$ as $k \to \infty$ in $L^1(\mathbb{T}^n)$. We fix such $t = t_0$. Then

$$\int_{\mathbb{T}^n} |u(k^2 t_0, x) - I| dx = \int_{\mathbb{T}^n} |u(k^2 t_0, k^2 \bar{x} + k \bar{x}) - I| dx = \int_{\mathbb{T}^n} |u_k(t_0, x) - I| dx \to 0. \quad (5.14)$$

For $t > k^2 t_0$ we use Lemma 2.3 for e.s. $u_1 = u$, $u_2 \equiv I$ and find that for almost each such $t$

$$\int_{\mathbb{T}^n} |u(t, x) - I| dx \leq \int_{\mathbb{T}^n} |u(k^2 t_0, x) - I| dx,$$

which, together with (5.14), implies the desired decay relation (1.11).
Remark 5.1. Using methods of paper [19], we can extend our results to the case of almost periodic (in the Besicovitch sense) initial function $u_0(x)$. Let

$$C_R = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid |x| = \max_{i=1,\ldots,n} |x_i| \leq R/2 \}, \quad R > 0,$$

be the mean $L^1$-norm of a function $u(x) \in L^1_{\text{loc}}(\mathbb{R}^n)$. Recall, (see [2]) that Besicovitch space $B^1(\mathbb{R}^n)$ is the closure of trigonometric polynomials, i.e., finite sums $\sum a_\lambda e^{2\pi i \lambda \cdot x}$, with $i^2 = -1$, $\lambda \in \mathbb{R}^n$, in the quotient space $B^1(\mathbb{R}^n)/B^1_0(\mathbb{R}^n)$, where

$$B^1(\mathbb{R}^n) = \{ u \in L^1_{\text{loc}}(\mathbb{R}^n) \mid N_1(u) < +\infty \}, \quad B^1_0(\mathbb{R}^n) = \{ u \in L^1_{\text{loc}}(\mathbb{R}^n) \mid N_1(u) = 0 \}.$$

The space $B^1(\mathbb{R}^n)$ is a Banach space equipped with the norm $\|u\|_1 = N_1(u)$.

It is known [2] that for each $u \in B^1(\mathbb{R}^n)$ there exist the mean value

$$\int_{\mathbb{R}^n} u(x)dx = \lim_{R \to +\infty} R^{-n} \int_{C_R} u(x)dx$$

and, more generally, the Bohr-Fourier coefficients

$$a_\lambda = \int_{\mathbb{R}^n} u(x)e^{-2\pi i \lambda \cdot x}dx, \quad \lambda \in \mathbb{R}^n.$$

The set

$$Sp(u) = \{ \lambda \in \mathbb{R}^n \mid a_\lambda \neq 0 \}$$

is at most countable and is called the spectrum of an almost periodic function $u$. We denote by $M(u)$ the smallest additive subgroup of $\mathbb{R}^n$ containing $Sp(u)$.

Suppose that $u_0 \in B^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $I = \int_{\mathbb{R}^n} u_0(x)dx$, $M_0 = M(u_0)$, and that $u = u(t, x)$ is an e.s. of (1.1), (1.3). Arguing as in [19], we may conclude that $u(t, \cdot) \in B^1(\mathbb{R}^n)$ and that $M(u(t, \cdot)) \subset M_0$ for a.e. $t > 0$. The decay property is modified as follows.

Theorem 5.1. Assume that for all $\xi \in M_0$, $\xi \neq 0$ there is no vicinity of $I$, where simultaneously the function $\xi \cdot \varphi(u)$ is affine and the function $a(u)\xi \cdot \xi \equiv 0$. Then

$$\text{ess lim}_{t \to +\infty} u(t, x) = I \quad \text{in } B^1(\mathbb{R}^n).$$
6 Acknowledgements

This work was supported by the Ministry of Education and Science of the Russian Federation (project no. 1.445.2016/1.4) and by the Russian Foundation for Basic Research (grant 18-01-00258-a.)

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