A FAMILY OF SOBOLEV ORTHOGONAL POLYNOMIALS ON THE UNIT BALL

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Abstract. A family of orthonormal polynomials on the unit ball $B^d$ of $\mathbb{R}^d$ with respect to the inner product
\[
\langle f, g \rangle = \int_{B^d} \Delta [(1 - \|x\|^2) f(x)] \Delta [(1 - \|x\|) g(x)] dx,
\]
where $\Delta$ is the Laplace operator, is constructed explicitly.

1. Introduction

In a recent study [1] on the numerical solution of the nonlinear Poisson equation $-\Delta u = f(\cdot, u)$ on the unit disk with zero boundary conditions, Atkinson [1] asked the question of finding an explicit orthogonal basis for the inner product defined by
\[
\langle f, g \rangle := \frac{1}{\pi} \int_{B^2} \Delta [(1 - x^2) f(x, y)] \Delta [(1 - x^2 - y^2) g(x, y)] dx dy
\]
on the unit disk $B^2$ of the Euclidean plane, where $\Delta$ is the usual Laplace operator. The purpose of this note is to provide an answer for this question.

We shall consider more generally the analogous inner product on the unit ball $B^d$ in $\mathbb{R}^d$. We call orthogonal polynomials with respect to such an inner product Sobolev orthogonal polynomials. In the theory of orthogonal polynomials of one variable, the name Sobolev is associated with polynomials that are orthogonal with respect to an inner product defined using both functions and their derivatives; see, for example, [4] and the references therein. As far as we know, Sobolev orthogonal polynomials have not been studied in the case of several variables.

Our main result, given in Section 2, is a family of orthonormal polynomials with respect to $(\cdot, \cdot)_\Delta$ on $B^d$ that are constructed using spherical harmonics and Jacobi polynomials in Section 2. For $d = 1$, orthogonal polynomials with respect to this inner product has been studied recently in [3]. The explicit formula can be used to study further properties of the orthogonal basis. In particular, it turns out that the orthogonal expansion of a function $f$ in this basis can be computed without involving the derivatives of $f$. This will be discussed in Section 3.

2. Sobolev Orthogonal Polynomials

For $x \in \mathbb{R}^d$, let $\|x\|$ denote the usual Euclidean norm of $x$. The unit ball in $\mathbb{R}^d$ is $B^d := \{ x : \|x\| \leq 1 \}$. Its surface is $S^{d-1} := \{ x : \|x\| = 1 \}$. The volume of $B^d$
and the surface area of $S^{d-1}$ are denoted by $\text{vol}(B^d)$ and $\omega_{d-1}$, respectively,

$$\text{vol}(B^d) = \omega_{d-1}/d \quad \text{and} \quad \omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2).$$

Let $\Pi^d = \mathbb{R}[x_1, \ldots, x_d]$ be the ring of polynomials in $d$ variables and let $\Pi^d_n$ denote the subspace of polynomials of total degree at most $n$. We consider the inner product defined on the polynomial space by

$$\langle f, g \rangle_\Delta := \frac{1}{4d^2 \text{vol}(B^d)} \int_{B^d} \Delta[(1-\|x\|^2)f(x)]\Delta[(1-\|x\|^2)g(x)]dx.$$

The constants are chosen so that $\langle 1, 1 \rangle_\Delta = 1$. As pointed out in [1], the inner product is well defined and positive definite on $\Pi^d$. Let $V^d_n(\Delta)$ denote the space of orthogonal polynomials with respect to $\langle f, g \rangle_\Delta$. It follows from the general theory of orthogonal polynomials in several variables ([2]) that the dimension of $V^d_n(\Delta)$ is $\binom{n+d-1}{d-1}$. A polynomial $P$ is in $V^d_n(\Delta)$ if it is orthogonal to all polynomials of lower degrees with respect to $\langle f, g \rangle_\Delta$. If $\{P_\alpha\}$ is a basis of $V^d_n(\Delta)$ and $\langle P_\alpha, P_\beta \rangle_\Delta = 0$ whenever $\alpha \neq \beta$, it is called a mutually orthogonal basis. If, in addition, $P_\alpha$ is normalized so that $\langle P_\alpha, P_\alpha \rangle_\Delta = 1$, the basis is called orthonormal. Our objective in this section is to find an explicit orthonormal basis for $V^d_n(\Delta)$.

The presence of the Laplace operator suggests that we make use of harmonic polynomials, which are homogeneous polynomials that satisfy the equation $\Delta P = 0$. Let $\mathcal{H}^d_n$ denote the space of harmonic polynomials of degree $n$. It is well known that

$$\dim \mathcal{H}^d_n = \binom{n+d-1}{d-1} - \binom{n+d-3}{d-1} := \sigma_n.$$

The restriction of $Y \in \mathcal{H}^d_n$ on $S^{d-1}$ are called spherical harmonics. They are orthogonal on $S^{d-1}$. We will use the spherical polar coordinates $x = rx'$ for $x \in \mathbb{R}^d$, $r \geq 0$, and $x' \in S^{d-1}$. For $Y \in \mathcal{H}^d_n$, we use the notation $Y(x)$ to denote the harmonic polynomials and use $Y(x')$ to denote the spherical harmonics. This agrees with $x = rx'$ since $Y$ is a homogeneous polynomial, $Y(x) = r^\nu Y(x')$. Throughout this paper, we use the notation $\{Y^\nu_\mu : 1 \leq \nu \leq \sigma_n\}$ to denote an orthonormal basis for $\mathcal{H}^d_n$, that is,

$$\sum_{\mu=1}^{\nu} \frac{1}{\omega_{d-1}} \int_{S^{d-1}} Y^\mu_\mu(x')Y^\nu_{\nu'}(x')d\omega(x') = \delta_{\mu,\nu}\delta_{\nu,m}, \quad 1 \leq \mu, \nu \leq \sigma_n,$$

where $d\omega$ stands for the surface measure on $S^{d-1}$. In terms of the spherical polar coordinates, $x = rx'$, $r > 0$ and $x' \in S^{d-1}$, the Laplace operator can be written as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_0,$$

where $\Delta_0$ is the spherical Laplacian on $S^{d-1}$. It is well-known that

$$\Delta_0 Y(x') = -(n+d-2)Y(x'), \quad Y \in \mathcal{H}^d_n, \quad x' \in S^{d-1}.$$

The spherical harmonics have been used to construct orthogonal polynomials on the unit ball. For later use, let us mention an orthogonal basis with respect to the inner product

$$\langle f, g \rangle_\mu := c_\mu \int_{B^d} f(x)g(x)W_\mu(x)dx, \quad W_\mu(x) = (1-\|x\|^2)^\mu,$$
where \( \mu > -1 \) and \( c_\mu \) is the normalization constant of \( W_\mu \). Let \( \mathcal{V}_n^d(W_\mu) \) denote the space of orthogonal polynomials of degree \( n \). A mutually orthogonal basis for \( \mathcal{V}_n(W_\mu) \) is given by (9)

\[
(2.4) \quad P_{j,\nu}^n(W_\mu; x) = I_j^{(\nu,n-2j+\frac{d-2}{2})}(2\|x\|^2 - 1)Y_{\nu}^{n-2j}(x), \quad 0 \leq j \leq n/2,
\]

where \( P_{j,\nu}^n \) denotes the Jacobi polynomial of degree \( j \), which is orthogonal with respect to \( (1-x)^\alpha (1+x)^\beta \) on \([-1, 1] \), and \( \{Y_{\nu}^{n-2j} : 1 \leq j \leq \sigma_{n-2j}\} \) is a basis for \( \mathcal{H}_{n-2j}^d \).

In view of (2.4) we will look for a basis with respect to \( \langle f, g \rangle \Delta \) in the form of

\[
(2.5) \quad Q_{j,\nu}^n(x) = q_j(2\|x\|^2 - 1)Y_{\nu}^{n-2j}(x), \quad 0 \leq 2j \leq n, \quad Y_{\nu}^{n-2j} \in \mathcal{H}_{n-2j}^d,
\]

where \( q_j \) is a polynomial of degree \( j \) in one variable.

**Lemma 2.1.** Let \( Q_{j,\nu}^n \) be defined as above. Then

\[
\Delta \left[ (1 - \|x\|^2)Q_{j,\nu}^n(x) \right] = 4(J_{\beta}q_j)(2r^2 - 1)Y_{\nu}^{n-2j}(x),
\]

where \( \beta = n - 2j + \frac{d-2}{2} \) and

\[
\langle J_{\beta}q_j \rangle(s) = (1 - s^2)q_j''(s) + (\beta - 1 - (\beta + 3)s)q_j'(s) - (\beta + 1)q_j(s).
\]

**Proof.** Using spherical-polar coordinates, (2.2) and (2.3) show that

\[
\Delta \left[ (1 - \|x\|^2)Q_{j,\nu}^n(x) \right] = \Delta \left[ (1 - r^2)q_j(2r^2 - 1)Y_{\nu}^{n-2j}(x) \right]
\]

\[
= 4r^{n-2j} \left[ 4r^2(1 - r^2)q_j''(2r^2 - 1) + 2((\beta + 1) - (\beta + 3)r^2)q_j'(2r^2 - 1) \right.
\]

\[
- (\beta + 1)q_j(2r^2 - 1) \right] Y_{\nu}^{n-2j}.
\]

Setting \( s \mapsto 2r^2 - 1 \) gives the stated result. \( \square \)

**Lemma 2.2.** Let \( p_k^\beta \in \Pi_k := \Pi_k^1 \) be orthogonal with respect to the inner product

\[
\langle f, g \rangle \beta := \int_{-1}^1 (J_{\beta}f)(s)(J_{\beta}g)(s)(1 + s)^\beta ds, \quad \beta > -1.
\]

Then the polynomials \( Q_{j,\nu}^n \) in (2.3) with \( q_j = p_k^\beta \), where \( \beta_k = k + (d - 2)/2 \), form a mutually orthogonal basis for \( \mathcal{V}_n^d(\Delta) \).

**Proof.** It is easy to see that \( \langle f, g \rangle \beta \) is indeed a positive definite inner product on the space of polynomials of one variables, so that the orthogonal polynomials with respect to \( \langle f, g \rangle \beta \) exist (see Lemma 2.3 below). Using the formula

\[
\int_{B^d} f(x)dx = \int_0^1 r^{d-1} \int_{S^{d-1}} f(rx')d\omega(x')dr,
\]

the definition of \( Q_{j,\nu}^n \) and (2.3) shows immediately that

\[
\langle Q_{j,\nu}^n, Q_{j',\nu'}^n \rangle \Delta := \delta_{\nu,\nu'}\delta_{n-2j,n'-2j'}
\]

\[
\times \frac{1}{4d} \int_0^1 r^{d+2(n-2j)-1}4^2(J_{\beta_{n-2j}}q_j)(2r^2 - 1)(J_{\beta_{n'-2j'}}q_{j'})(2r^2 - 1)dr.
\]

In the nonzero case we have \( \beta_{n-2j} = \beta_{n'-2j'} \). Thus, a change of variable \( r \mapsto \sqrt{(1 + s)/2} \) shows that

\[
(2.6) \quad \langle Q_{j,\nu}^n, Q_{j',\nu'}^n \rangle \Delta = \delta_{\nu,\nu'}\delta_{n-2j,n'-2j'}\frac{1}{d^2\beta_{n-2j}}(q_j, q_{j'})(\beta_{n-2j})
\]

which proves the stated result. \( \square \)
We note that \( q_j^{\beta_n-2j} \) should be understood as one member (of degree \( j \)) in the orthogonal family \( \{q_k^{\beta_n-2j}\} \).

**Lemma 2.3.** The polynomials \( p_j^\beta \) defined by

\[
p_0^\beta(s) = 1, \quad p_j^\beta(s) = (1 - s)P_{j-1}^{(2, \beta)}(s), \quad j \geq 1,
\]

are orthogonal with respect to the inner product \( \langle f, g \rangle_\beta \).

**Proof.** We need the following property of the Jacobi polynomials [2, p. 782, (22.7.8)],

\[
(2.8) \quad (1 - s)P_{j-1}^{(2, \beta)}(s) = \frac{2}{2j + \beta + 1} \left[ (j + 1)P_{j-1}^{(1, \beta)}(s) - jP_j^{(1, \beta)}(s) \right].
\]

The Jacobi polynomial \( P_{j-1}^{(1, \beta)} \) satisfies a differential equation

\[
(1 - s^2)y'' + (-1 + \beta - (3 + \beta)s)y' + (j - 1)(j + \beta + 1)y = 0.
\]

Using these two facts, we easily deduce that

\[
\frac{2j + \beta + 1}{2} \mathcal{J}_\beta \left[ (1 - s)P_{j-1}^{(2, \beta)}(s) \right] = (j + 1)\mathcal{J}_\beta P_{j-1}^{(1, \beta)}(s) - j\mathcal{J}_\beta P_j^{(1, \beta)}(s)
\]

\[
= (j + 1) \left[ (-j - 1)(j + \beta + 1) - (\beta + 1) \right] P_{j-1}^{(1, \beta)}(s)
\]

\[
- j(j + \beta + 2) - (\beta + 1) P_j^{(1, \beta)}(s)
\]

\[
= -j(j + 1) \left[ (j + \beta)P_{j-1}^{(1, \beta)}(s) - (j + \beta + 1)P_j^{(1, \beta)}(s) \right].
\]

We need yet another formula of Jacobi polynomials [2, p. 782, (22.7.8)],

\[
(2.9) \quad (2j + \beta + 1)P_j^{(0, \beta)}(s) = (j + \beta + 1)P_j^{(1, \beta)}(s) - (j + \beta)P_{j-1}^{(1, \beta)}(s)
\]

which implies immediately that

\[
\mathcal{J}_\beta \left[ (1 - s)P_{j-1}^{(2, \beta)}(s) \right] = 2j(j + 1)P_j^{(0, \beta)}(s).
\]

Hence, for \( j, j' \geq 1 \), we conclude that

\[
(2.10) \quad (p_j^\beta, p_{j'}^\beta) = \int_{-1}^{1} \mathcal{J}_\beta \left[ (1 - s)P_{j-1}^{(2, \beta)}(s) \right] \mathcal{J}_\beta \left[ (1 - s)P_{j'-1}^{(2, \beta)}(s) \right] (1 + s)^\beta ds
\]

\[
= 4j(j + 1)j'(j' + 1) \int_{-1}^{1} P_j^{(0, \beta)}(s)P_{j'}^{(0, \beta)}(s)(1 + s)^\beta ds = 0
\]

whenever \( j \neq j' \). Furthermore, for \( j \geq 1 \), we have

\[
(p_0^\beta, p_j^\beta) = -2j(j + 1)(\beta + 1) \int_{-1}^{1} P_j^{(0, \beta)}(s)(1 + s)^\beta ds = 0
\]

since \( \langle \mathcal{J}_\beta p_0^\beta(s) \rangle = \langle \mathcal{J}_\beta 1 \rangle = -(\beta + 1) \).

As a consequence of the above lemmas, we have found a mutually orthogonal basis with respect to \( \langle \cdot, \cdot \rangle_\Delta \).

**Theorem 2.4.** A mutually orthogonal basis for \( \mathcal{V}_n^\beta(\Delta) \) is given by

\[
Q_{0,0}^\beta(x) = Y_0^\beta(x),
\]

\[
Q_{j,\nu}^\beta(x) = (1 - \|x\|^{2})P_{j-1}^{(2, n-2j+\frac{\nu}{2}-1)}Y_{\nu}^{n-2j}(x), \quad 1 \leq j \leq \frac{n}{2},
\]
where \( \{Y_{\nu}^{n-2j} : 1 \leq \nu \leq \sigma_{n-2j}\} \) is an orthonormal basis of \( \mathcal{H}_{n-2j}^d \). Furthermore,

\[
\langle Q_{0,\nu}^n, Q_{0,\nu}^n \rangle_{\Delta} = \frac{2n + d}{d}, \\
\langle Q_{j,\nu}^n, Q_{j,\nu}^n \rangle_{\Delta} = \frac{8j^2(j + 1)^2}{d(n + d/2)}.
\]

**Proof.** The fact that \( Q_{j,\nu}^n \in \mathcal{V}_{n}^d(\Delta) \) follows from Lemma 2.2 and Lemma 2.3. To compute the norm of \( Q_{0,\nu}^n \) we use the fact that

\[
\Delta [(1 - ||x||^2)Y_{\nu}^{n-2j}(x)] = -2dY_{\nu}^{n}(x) - 4(x, \nabla)Y_{\nu}^{n} = -2(d + 2n)Y_{\nu}^{n}(x)
\]

by Euler’s formula on homogeneous polynomials, which shows that

\[
\langle Q_{0,\nu}^n, Q_{0,\nu}^n \rangle_{\Delta} = \frac{(2n + d)^2}{d} \int_0^1 r^{d-1+2n}dr \frac{1}{\omega_{d-1}} \int_{S^{d-1}} [Y_{\nu}^n(x)]^2 dx = \frac{2n + d}{d}.
\]

Furthermore, using the equation (2.6) and (2.10), we have

\[
\langle Q_{j,\nu}^n, Q_{j,\nu}^n \rangle_{\Delta} = \frac{1}{d^{2\beta_j}} (p_j, p_j)_{\beta_j} = \frac{4j^2(j + 1)^2}{d^{2\beta_j}} \int_{-1}^1 \left[ P_j^{(0,\beta_j)}(s) \right]^2 (1 + s)^{\beta_j} ds
\]

\[
= \frac{8j^2(j + 1)^2}{d(\beta_j + 2j + 1)} = \frac{8j^2(j + 1)^2}{d(n + d/2)}
\]

where we have used the well known formula for the norm of the Jacobi polynomial (see, for example, [6] p. 68)). \( \square \)

The explicit formula of the basis (2.11) leads to the following interesting result, which relates \( \mathcal{V}_{n}^d(\Delta) \) to orthogonal polynomials with respect to \( W_2(x) = (1 - ||x||)^2 \).

**Corollary 2.5.** For \( n \geq 1 \),

\[
\mathcal{V}_{n}^d(\Delta) = \mathcal{H}_{n}^d \oplus (1 - ||x||^2)\mathcal{V}_{n-1}^d(W_2).
\]

**Proof.** Using the basis (2.21) for \( \mathcal{V}_{n-1}^d(W_2) \), it follows that we actually have

\[
Q_{j,\nu}^n(x) = (1 - ||x||^2)P_{j-1,\nu}^{n-2}(W_2; x)
\]

for \( j \geq 1 \), from which the stated result follows. \( \square \)

In the case of \( d = 2 \), an orthonormal basis for the space \( \mathcal{H}_{n}^2 \) is given by

\[
Y_1^n(x,y) = \sqrt{\frac{2}{\pi}} r^n \cos n\theta \quad \text{and} \quad Y_2^n(x,y) = \sqrt{\frac{2}{\pi}} r^n \sin n\theta
\]

in polar coordinates \( x = r \cos \theta, y = r \sin \theta \). Hence, a mutually orthogonal basis for \( \mathcal{V}_{n}^2(\Delta) \) is given by

\[
Q_{0,1}^n(x,y) = Y_1^n(x,y), \quad Q_{0,2}^n(x,y) = Y_2^n(x,y), \\
Q_{1,1}^n(x,y) = (1 - x^2 - y^2)P_{j-1,\nu}^{2,n-2}(2x^2 + 2y^2 - 1)Y_1^{n-2j}(x,y), \quad 1 \leq j \leq \frac{n}{2} \\
Q_{1,2}^n(x,y) = (1 - x^2 - y^2)P_{j-1,\nu}^{2,n-2}(2x^2 + 2y^2 - 1)Y_2^{n-2j}(x,y), \quad 1 \leq j \leq \frac{n-1}{2}
\]

which becomes an orthonormal basis upon dividing by the square root of the norm given by (2.13). Without normalization, this gives

\[
\mathcal{V}_1^2(\Delta) = \text{span}\{x, y\}, \quad \mathcal{V}_2^2(\Delta) = \text{span}\{x^2 - y^2, xy, 1 - x^2 - y^2\}, \\
\mathcal{V}_3^2(\Delta) = \text{span}\{x^3 - 3xy^2, 3y^2 - x^2y, x(1 - x^2 - y^2), y(1 - x^2 - y^2)\},
\]

for example.
3. Expansions in Sobolev Orthogonal Polynomials

Let $H^2(B^d)$ denote the space of functions for which $\langle f, f \rangle_{\Delta}$ is finite. This is not the $L^2$ space on $B^d$ since the definition of $\langle \cdot, \cdot \rangle_{\Delta}$ require that $f$ has second order derivatives. Nevertheless the standard Hilbert space theory shows that every $f \in H^2(B^d)$ can be expanded into a series in Sobolev orthogonal polynomials. In other words,

$$H^2(B^d) = \bigoplus_{n=0}^{\infty} \mathcal{V}^d_n(\Delta) : \quad f = \sum_{n=0}^{\infty} \text{proj}_n f,$$

where $\text{proj}_n : H^2(B^d) \to \mathcal{V}^d_n(\Delta)$ is the projection operator, which can be written in terms of the orthonormal basis \((2.11)\) as

$$\text{proj}_n f(x) = \sum_{0 \leq j \leq n/2} H_j^{-1} \sum_{\nu=0}^{\sigma_{n-2j}} \hat{f}^n_{j,\nu} Q^n_{j,\nu}(x), \quad \hat{f}^n_{j,\nu} = \langle f, Q^n_{j,\nu} \rangle_{\Delta}.$$

where $H_j = (Q^n_{j,\nu}, Q^n_{j,\nu})_\Delta$ are independent of $\nu$ as shown in \((2.12)\). Let $P^n_\Delta(x, y)$ denote the reproducing kernel of $\mathcal{V}^d_n(\Delta)$.

In terms of the orthonormal basis \((2.11)\) in the previous section, the reproducing kernel can be written as

$$P^n_\Delta(x, y) = \sum_{0 \leq j \leq n/2} H_j^{-1} \sum_{\nu} Q^n_{j,\nu}(x)Q^n_{j,\nu}(y).$$

The projection operator can be written as an integral operator with $P^n_\Delta$ as its kernel, which means that

$$\text{proj}_n f(x) = \langle f, P^n_\Delta(x, \cdot) \rangle_{\Delta} = \frac{1}{4d^2 \text{vol}(B^d)} \int_{B^d} \Delta[(1-\|y\|^2)f(y)]\Delta[(1-\|y\|)]P^n_\Delta(x, y)dy$$

where $\Delta$ is applied on $y$ variable.

It turns out that the orthogonal expansion can be computed without involving derivatives of $f$.

**Proposition 3.1.** For $j \geq 1$, let $\beta_j = n - 2j + (d - 2)/2$; then

$$\hat{f}^n_{j,\nu} = \frac{8j(j+1)}{d^2 \text{vol}(B^d)} \left[ (\beta_j + j)(\beta_j + j + 1) \int_{B^d} f(x)Q^n_{j,\nu}(x)dx ight. \\
- \frac{1}{2j} \int_{S^{d-1}} f(y')Y^{n-2j}_{\nu}(y')d\omega(y') \right];$$

Furthermore, for $j = 0$,

$$\hat{f}^n_{0,\nu} = \frac{d + 2n}{d} \frac{1}{\omega_d} \int_{S^{d-1}} Y^{n-2j}_{\nu}(y')f(y')d\omega(y').$$

**Proof.** By \((2.20)\), $P^n_{j,\nu}(W_0; x) = P^{(0,\beta_j)}_j(2\|x\|^2 - 1)Y^{n-2j}_{\nu}(x)$. Let $j \geq 1$. By Lemma \((2.21)\) and \((2.24)\),

$$\Delta \left[(1-\|x\|^2)Q^n_{j,\nu}(x)\right] = 8j(j+1)P^n_{j,\nu}(W_0; x).$$

Applying Green’s identity

$$ \int_{B^d} (u\Delta v - v\Delta u)dx = \int_{S^{d-1}} \left( \frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} \right) dv$$
with \( v(x) = (1 - \|x\|^2)f(x) \) and \( u = Q_{j\beta}^n \) shows then

\[
\hat{f}_{j,\nu}^n = \frac{8j(j+1)}{4d^2 \text{vol}(B^d)} \int_{B^d} \Delta \left[ (1 - \|x\|^2) f(x) \right] P_{j,\nu}(W_0; x) dx
\]

\[
= \frac{2j(j+1)}{d^2 \text{vol}(B^d)} \left[ \int_{B^d} (1 - \|x\|^2) f(x) \Delta P_{j,\nu}(W_0; x) dx - 2 \int_{S_{d-1}} Y_{\nu}^{n-2j}(x') f(x') d\omega \right]
\]

where we have used the fact that \( P_{j,\beta}((1,\beta)) = 1 \). Let \( \partial P_{(0,\beta)} \) denote the derivative of \( P_{(0,\beta)} \). Using \( 2.8 \) and \( 2.7 \) it is easy to see that

\[
\Delta [P_{j,\beta}^n(W_0; x)] = \frac{8j(j+1)}{4d^2 \text{vol}(B^d)} \int_{B^d} \Delta \left[ (1 - \|x\|^2) f(x) \right] P_{j,\nu}(W_0; x) dx
\]

\[
= \frac{2j(j+1)}{d^2 \text{vol}(B^d)} \left[ \int_{B^d} (1 - \|x\|^2) f(x) \Delta P_{j,\nu}(W_0; x) dx - 2 \int_{S_{d-1}} Y_{\nu}^{n-2j}(x') f(x') d\omega \right]
\]

Let us denote the expression in the square bracket by \( M_j \). The Jacobi polynomial \( P_{j,\beta}^n(s) \) satisfies the differential equation

\[
(1 - s^2)y'' - (\beta + (\beta + 2)s)y' + j(j + \beta + 1)y = 0.
\]

Hence, changing variable \( 2r^2 - 1 \rightarrow s \), we conclude that

\[
2(1 - r^2)M_j = -j(j + \beta + 1)P_{j,\beta}^n(s) + \frac{1}{2}(j + \beta + 1)(1 + s)P_{j-1,\beta}^{(1,\beta+1)}(s).
\]

On the other hand, using \( 2.8 \), \( 2.7 \) and the fact that \( 2.7 \) p. 782

\[
(2j + \beta + 1)(1 + s)P_{j-1,\beta}^{(1,\beta+1)}(s) = 2(j + \beta)P_{j,\beta}^{(1,\beta)}(s) + 2jP_{j-1}^{(1,\beta)}(s)
\]

we conclude that

\[
2(1 - r^2)M_j = \frac{(\beta_j + j + 1)(\beta_j + j)}{2j + \beta + 1} \left[ -jP_{j,\beta}^{(1,\beta)}(s) + (j + 1)P_{j-1,\beta}^{(1,\beta+1)}(s) \right]
\]

\[
= \frac{1}{2}(\beta_j + j)(\beta_j + j)(1 - s)P_{j-1}^{(2,\beta+1)}(s)
\]

\[
= (\beta_j + j + 1)(\beta_j + j)(1 - r^2)P_{j-1}^{(2,\beta+1)}(2r^2 - 1).
\]

Consequently, we have proved that

\[
(1 - \|x\|^2)\Delta [P_{j,\mu}^n(W_0; x)] = 4(\beta_j + j + 1)(\beta_j + j)(1 - r^2)P_{j-1,\mu}^{(2,\beta+1)}(2r^2 - 1)Y_{\nu}^{n-2j}(x)
\]

which leads to the stated result for \( j \geq 1 \) by \( 3.3 \). The proof of \( j = 0 \) is similar but easier, in which we need to use \( 2.4 \).

Let us denote by \( P_n(W_\mu; x, y) \) the reproducing kernel of \( \mathcal{V}_n^d(W_\mu) \), which can be written as

\[
P_n(W_\mu; x, y) = \sum_{|\alpha|=n} A_{\alpha,\mu}^{-1}P_\alpha(W_\mu; x)P_\alpha(W_\mu; y),
\]

where \( A_{\alpha,\mu} = c_\mu \int_{B^d} [P_\alpha(W_\mu; y)]^2 W_\mu(y) dy \) in which \( c_\mu \) is the normalization of \( W_\mu \). Let us also denote by \( C_n^\lambda(t) \) the Gegenbauer polynomial of degree \( n \), and by \( x \cdot y \) the usual dot product of \( x, y \) in \( \mathbb{R}^d \).
Corollary 3.2. For \( f \in H^2(B^d) \) and \( x \in B^d \),

\[
\text{proj}_n f(x) = Y_n f(x) + (1 - \|x\|^2) \frac{4}{\binom{d}{2} \text{vol}(B^d)} \int_{B^d} f(y) P_{n-2}(W_2; x, y)(1 - \|y\|^2) dy 
- \frac{(n + d/2)}{4} (1 - \|x\|^2) \sum_{1 \leq j \leq n/2} \frac{P_j^{(2n-2j+d/2)}(2\|x\|^2 - 1)}{P_j^{(2n-2j+d/2)}(1)} Y_n-2j f(x),
\]

where with \( x' = x/\|x\| \in S^{d-1} \),

\[
Y_m f(x) = \|x\|^m \int_{S^{d-1}} f(y') \frac{m + (d - 2)/2}{(d - 2)/2} C_m^{d-2}(x \cdot y') d\omega(y').
\]

Proof. The values of \( H_j = \langle Q^n_{j,\nu}, Q^n_{j,\nu} \rangle_\Delta \) are given in (2.12). It follows immediately that

\[
\sum_{\nu=1}^{\sigma_n} H_{j,\nu}^{-1} \tilde{f}_{0,\nu} Q^n_{j,\nu}(x) = \frac{1}{\omega_{n-1}} \int_{S^{d-1}} f(y') \sum_{\nu=1}^{\sigma_n} Y_{\nu}^{n}(y') Y_{\nu}^{n}(x) d\omega(y') = Y_n f(x),
\]

where the last step follows from the summation formula of spherical harmonics,

\[
\sum_{\nu=1}^{\sigma_n} Y_{\nu}^{n}(x) Y_{\nu}^{n}(y) = \|x\|^n \sum_{\nu=1}^{\sigma_n} Y_{\nu}^{n}(x') Y_{\nu}^{n}(y) = \|x\|^n \frac{n + (d - 2)/2}{(d - 2)/2} C_n^{d-2}(x' \cdot y)
\]

for \( x', y \in S^{d-1} \). Furthermore, setting \( f = Q^n_{j,\nu} \) with \( j \geq 1 \) in (2.12) also shows

\[
H_j = \frac{8j(j+1)}{d^2 \text{vol}(B^d)} (\beta_j + j)(\beta_j + j + 1) \int_{B^d} |Q^n_{j,\nu}(x)|^2 dx.
\]

Hence, it follows from (3.2) and (2.12) that

\[
(3.4)
H_{j,\nu}^{-1} \tilde{f}_{j,\nu} = \frac{\int_{B^d} f(y) Q^n_{j,\nu}(y) dy}{\int_{B^d} |Q^n_{j,\nu}(y)|^2 dy} - \frac{n + d/2}{2j(j+1)} \frac{1}{\omega_{d-1}} \int_{S^{d-1}} f(y') Y_{\nu}^{n-2j}(y') d\omega(y').
\]

The relation (2.14) readily shows that

\[
(3.5)
\int_{B^d} |Q^n_{j,\nu}(y)|^2 dy = \frac{1}{4} \binom{d}{2} \text{vol}(B^d) c_2 \int_{B^d} |P^n_{j,\nu}(W_2; y)|^2 W_2(y) dy.
\]

We multiply (3.3) by \( Q^n_{j,\nu}(x) \) and sum over \( \nu \) and \( j \). Using (3.5) and the fact that \( P_j^{(2n-2j+d/2)}(1) = j(j+1)/2 \), the stated result follows from (3.2) and (3.1). \( \square \)

It follows from this corollary that the orthogonal expansion of \( f \) with respect to \( \langle \cdot, \cdot \rangle_\Delta \) coincides with the spherical harmonic expansion of \( f \) when restricted on \( S^{d-1} \).

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References

[1] K. Atkinson, Solving the nonlinear Poisson equation on the unit disk, Reports on Computational Mathematics #160, Department of Mathematics, University of Iowa, 2005. http://www.math.uiowa.edu/~atkinson/papers-misc.htm

[2] M. Abramowitz and I. Stegun, Handbook of Mathematical Functions, 9th print., Dover Publ., New York, 1970.

[3] C. F. Dunkl and Yuan Xu, Orthogonal polynomials of several variables, Cambridge Univ. Press, 2001.

[4] W. Gautschi, Orthogonal Polynomials: Computation and Approximation, Oxford Univ. Press, 2004.

[5] O. Hansen, Orthogonal polynomials for the solution of semilinear two-point boundary value problems, preprint, 2005.

[6] G. Szegő, Orthogonal Polynomials, Amer. Math. Soc. Colloq. Publ. Vol.23, Providence, 4th edition, 1975.

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