Determination of optimal insurance company reimbursement by using exponential utility functions

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Abstract. In an insurance company, it is very possible that there is an imbalance between the amount of the premium and the amount of claims submitted by the insured. In managing the risk of such an imbalance, the insurance company must be able to make a decision to determine the appropriate amount of premium based on a number of claims by the insured. To make these decisions, insurance companies can do so by using certain utility functions. This paper examines the use of exponential utility functions in transferring risk between insurance companies, optimal Pareto exchanges, equilibrium, Bowley solutions, and optimal reinsurance purchases. Furthermore, using the exponential utility function is calculated the optimal amount of reimbursement that will be carried out by the insurer. The results of the discussion obtained the factors that influence the calculation and the optimal amount of reimbursement value, from a property policy or group policy that needs to be carried out by the insurer. So that based on the results of the optimal reimbursement calculation, it can be used to determine the balance between the amount of the premium and the amount of claims that may be submitted by the insured.

1. Introduction
Insurance companies are engaged in business activities that focus on risk management issues. This happens because the insurance company is willing to provide compensation or compensation to the insured or insured owner for loss, damage, or legal liability to third parties that may be suffered by insured because of an uncertain event, or give compensation for death or life someone insured [1; 2; 3].

The existence of insurance companies is because insured are willing to pay premiums on the expectation of their claims. As a result, insurance companies (insurers) collect premiums greater than the expected size of the claim [4; 5]. In the management of insurance companies, it is very possible that there is an imbalance between the amount of the premium and the amount of the claim. This can threaten the stability of insurance companies [6; 7]. Therefore, insurance companies must be careful in managing the risks that may occur against insured, and risks that may occur within the insurance company itself. In managing this risk, insurance companies must be able to make decisions in determining the amount of premium for a number of claims submitted by insured [8; 9].

The decision to determine the size of the premium made by the insurance company is based on preference as a particular consequence. One thing that also needs to be considered is the determination.
of the premium is the occurrence of inflation [10]. Therefore, insurance companies must be able to
choose an action that will lead to an uncertain event [11; 12]. A theory that provides knowledge about
decision making to deal with uncertain events in insurance problems, and which explains why insured
is willing to pay a premium that is greater than net premium is a mathematical expectation or known
as utility theory [13; 14]. Therefore, this paper examines optimal reimbursement calculations in
insurance companies. In the calculation of the reimbursement is done using the exponential utility
function. The objective of the optimal reimbursement calculation is to find a balance between the
amount of premium and the amount of claims that may be submitted by the insured. This research
is useful for insurance companies in determining optimal reimbursement, especially when utility
functions are exponential.

2. Mathematical models
This section discusses mathematical models including: utility functions, transfer of risk between
insurance companies, transfer of optimal Pareto risk, equilibrium, Bowley solutions, and purchase of
optimal reinsurance.

2.1. Utility function
A utility function is based on the choice of the decision maker for various outcome distributions,
which illustrates how the utility of a particular value or circumstance is for decision makers. In general
the utility scale is expressed between 0 and 1, where 1 represents the most preferred value or state and
0 represents the least preferred value or situation. An insurer who acts as a decision maker does not
have to be individually, but may also be an alliance, legal entity, or government. Suppose \( u(x) \) stating
the utility function, the determination of the form of utility functions for insurer is probably an easy
process [13; 14].

There are several types of functions, but this paper only examines the exponential utility function.
In general, the exponential utility function is expressed as:

\[
\begin{align*}
    u(x) & = \frac{1}{a} (1 - e^{-ax}) ; \quad a > 0 \\
\end{align*}
\]

(1)

with \( a \) stated the parameters, and \( x \) value or state of benefits. The properties of the utility function are
\( u'(x) > 0 \) and \( u''(x) < 0 \). Based on these properties indicate that the utility function is an ascending
concave. Decision makers with an elevated concave utility function are called risk avoiders. Risk
avoidance coefficient of utility function \( u(x) \) is given by:

\[
    r(x) = \frac{-u'(x)}{u''(x)} = -\frac{d}{dx} \ln u'(x),
\]

(2)

assuming that the risk avoidance function \( r(x) > 0 \) [13; 14].

To determine the amount of an appropriate premium (equitable), it is assumed that insurer has a
utility function \( u \), and premiums are viewed in terms of their utility. If \( u(x) \) a concave function and \( S \)
random variable, then Jensen’s inequality applies \( E[u(S)] \geq u(E[S]) \). Suppose \( \pi \) the amount of
premium, based on Jensen’s inequality can be determined that \( E[u(x + \pi - S)] \leq u(E[x + \pi - S]) = u(x) \).
Premium \( \pi \) obtained as a solution of the equation [2; 11]:

\[
    E[u(x + \pi - S)] = u(x).
\]

(3)

Equation (3) generally cannot be solved analytically to search \( \pi \). If distribution \( S \) have a central
moment around the average, then \( \pi \) can be predicted using the following equation [2; 3]:

\[
    \pi \approx E[S] + \frac{1}{2} r(x) Var(S).
\]

(4)

For the exponential utility function in equation (1), the amount of the premium \( \pi \) can be determined
that:
\[ \pi = \frac{1}{a} \ln E[e^{aS}] . \]  

(5)

2.2. Risk transfer between insurance companies

Suppose \( n \) the number of companies, and \( X_i \) (\( i = 1, \ldots, n \)) is a company surplus \( i \) at the end of the year. The amount of surplus at the end of the year \( X_i \) is a surplus at the beginning of the year, plus premiums received, and deducted by claims paid. Suppose \( x \) surplus early in the year \( (t = 0) \), \( c \) premium received per unit of time, and \( S(t) \) aggregate claims paid up to time \( t \). The surplus function can be expressed as \([1; 5; 8]\):

\[ X(t) = x + ct - S(t), \ t \geq 0 . \]  

(6)

Because \( X_i \), random variables, it is assumed that the distribution is shared \((X_1, \ldots, X_n)\) is known. So, with the transfer of risk the company can try to improve the situation in several ways. The result of this risk transfer, for example a random vector \((Y_1, \ldots, Y_n)\) where \( Y_i \) \((i = 1, \ldots, n)\) change in surplus from the company \( i \). The combination of surplus after transfer of risk with a surplus before the transfer of risk must be equal, meaning:

\[ Y_1 + \ldots + Y_n = X_1 + \ldots + X_n . \]  

(7)

Furthermore, suppose \( X_1 + \ldots + X_n = X \), one simple form of risk transfer formula is:

\[ Y_i = q_i X + c_i , \]  

(8)

where \( q_i \) proportion of company parts \( i \) \((q_1 + \ldots + q_n = 1)\), and \( c_i \) deterministic part of payment by the company \( i \) \((c_1 + \ldots + c_n = 0)\) \([2; 5]\).

2.3. Pareto optimal risk transfer

Suppose the company \( i \) use the utility function \( u_i \) to evaluate risk transfer. So value \((Y_1, \ldots, Y_n)\) for companies \( i \) is \( E[u_i(Y_i)]\), \( i = 1, \ldots, n \). A company is said to achieve optimal Pareto, if there is no other risk transfer that is better for all companies. Geometrically, the Pareto optimal exchange member is identical to the solution of the selection of positive constants \( k_1, \ldots, k_n \) which maximizes the equation

\[ k_1 E[u_1(Y_1)] + \ldots + k_n E[u_n(Y_n)] . \]

Using variations in calculus and the utility of the utility function, equivalence conditions are obtained \( k_i u_i'(Y_i) \) that does not depend on each other \( i \). Based on equation \((7)\) can be determined Pareto optimal risk transfer \([7; 8]\).

If it is assumed that the utility function is an exponential form like equation \((1)\), from parameter \((8)\) the parameters can be obtained:

\[ q_i = \frac{a}{a_i} , \text{ with } \frac{1}{a} = \frac{1}{a_1} + \ldots + \frac{1}{a_n} , \text{ and } \]

\[ c_i = \frac{1}{a} \left( \ln k_i - \sum_{j=1}^{n} \frac{a}{a_j} \ln k_j \right) . \]  

(10)

In general there will be as many \((n - 1)\) Member parameters for Pareto optimal risk transfer.

2.4. Equilibrium

Suppose there is \( n \) the company formed a group. Company \( i \) \((i = 1, \ldots, n)\) can buy a payment (reimbursement) \( R_i \) from the group for a premium \( \pi_j \) \((i = 1, \ldots, n)\), where \( R_i \) is a random variable. So that the transfer of risk can be formulated with the equation \( Y_i = X_i - \pi_j + R_i \), with equilibrium that:

\[ R_1 + \ldots + R_n = \pi_1 + \ldots + \pi_n . \]  

(11)

Suppose \( P \) intensity premium which is a random variable with \( P > 0 \) and \( E[P] = 1 \) unit, the appropriate premium is determined by the equation:

\[ \pi_j = E[PR_j] = E[R_j] + \text{Cov}(P, R_j) . \]  

(12)
Random variable $P$ is a function of $X_1, \ldots, X_n$.\[[7; 11]\].

Company $i$ want to choose $R_i$ which maximizes $E[u_i(X_i - \pi_i + R_i)]$, where $\pi_i = E[PR_i]$. Value of $R_i$ the optimal can be generated based on conditions:

$$\frac{u'_i(X_i - \pi_i + R_i)}{E[u'_i(X_i - \pi_i + R_i)]} = P.$$  \hspace{1cm} (13)

It is said that $(P, R_1, \ldots, R_n)$ an equilibrium when (11) and (13) is met for $i = 1, \ldots, n$. In the case of an exponential utility function (1), based on conditions (13) will give the equation:

$$R_i = -X_i - \frac{1}{a_i} \ln P + d_i,$$  \hspace{1cm} (14)

where $d_i$ an arbitrary constant. So with equation (11) and conditions that $E[P]$=1 unit, it was found that [2; 7]:

$$P = \frac{e^{-ax}}{E[e^{-ax}]}.$$  \hspace{1cm} (15)

If selected $d_i$ such that $E[R_i] = 0$, using equation (15) the equation will be obtained

$$R_i = -(X_i - \mu_i) + q_i(X - \mu), \text{ where } \mu_i = E[X_i] \text{ and } \mu = E[X],$$

then:

$$\pi_i = -\text{Cov}(P, X_i) + q_i \text{Cov}(P, X), \text{ and } Y_i = q_i X - \pi_i + \mu_i - q_i \mu.$$  \hspace{1cm} (16)

Equation (16) shows that risk transfer is truly Pareto optimal. If it is assumed that $a_1, \ldots, a_n$ small, then the estimation of the first order is obtained by obtaining the equation [13; 15]:

$$P \approx 1 - a(X - \mu), \text{ and } \pi_i \approx a \text{Cov}(X, X_i) - a q_i \text{Var}(X).$$  \hspace{1cm} (17)

2.5. Bowley solution

Suppose there are two insurance companies, namely insurer and reinsurer. Insurer can buy a payment $R = R_1$ for premiums $\pi = \pi_1$, which is calculated based on conditions $\pi = E[PR]$. Because given an intensity premium $P$, Insurer chooses $R$ to maximize their utility expectations. As in equation (13), $R$ generated based on conditions that:

$$\frac{u'_1(X_1 - \pi + R)}{E[u'_1(X_1 - \pi + R)]} = P.$$  \hspace{1cm} (18)

It is assumed that the reinsurer knows the utility function and has this monopoly. So insurer will choose $P$ to maximize utility expectations, that is $E[u_2(X_2 + \pi - R)]$. In this equation $R$ depend on $P$ as shown in equation (18). Results of completion of $(P, R)$ called the Bowley solution [7; 8].

Suppose there are two utility functions known for insurer and reinsurer as follows:

$$u_1(x) = \frac{1}{\alpha}(1 - e^{-ax}) \text{ and } u_2(x) = x.$$  \hspace{1cm} (19)

where insurer has a risk avoidance coefficient $\alpha > 0$, and risk-neutral reinsurer. So the insurer request function is:

$$R = -X - \frac{1}{\alpha} \ln P + d,$$  \hspace{1cm} (20)

where $d$ constant changes, and reinsurer chooses $P$ in order to maximize expectations of profits:

$$\pi - E[R] = -E(\Pi X_1) - \frac{1}{\alpha} E[P \ln P] + E[X_1] + \frac{1}{\alpha} E[\ln P].$$  \hspace{1cm} (21)

Using calculus variations will be generated $P$ determined from the equation:

$$-X_1 - \frac{1}{\alpha} \ln P + \frac{1}{\alpha P} = c.$$  \hspace{1cm} (22)
where the constant $c$ must be chosen so that $E[P] = 1$. Equations (21) and (22) are Bowley solutions [8; 15].

2.6. Optimal Reinsurance Purchases

Suppose that a reinsurance premium is determined based on general principles $\pi = E[PR]$. Suppose also $\pi = H(R)$ is a premium that the insurer must pay for reimbursement $R$. Mathematically $H$ is a function of a set of random variables. The problem is choosing $R$ to maximize the equation:

$$E[u_1(X_1 - H(R) + R)].$$

If the function $H$ concave, then $R$ optimal can be determined as a solution to the equation:

$$u_1'(X_1 - H(R) + R) = H'(R).$$

Where $H'(R)$ as a gradient from $H$ on $R$ which is a random variable, as follows:

$$\frac{d}{dt} H(R + tQ)|_{t=0} = E[QH'(R)], \text{ for all } Q.$$

If $H(R) = E[PR]$, then $H'(R) = P$ [4; 15].

If for example $H(R) = \frac{1}{\beta} \ln E[Pe^{\beta R}]$, where is the parameter $\beta > 0$ and $P$ random variable with $P > 0$ and $E[P] = 1$, then:

$$H'(R) = \frac{Pe^{\beta R}}{E[Pe^{\beta R}]}.$$

Furthermore, suppose $u_1$ exponential utility function like equation (1). Using (24) optimal reimbursement is:

$$R = -\frac{\alpha}{\alpha + \beta} X_1 - \frac{1}{\alpha + \beta} \ln P + d$$

where $d$ changeable constants. Please note, that this result is only for special cases $P = 1$, that is $H$ exponentially shaped with parameters $\beta$ [8; 15].

3. Numerical illustration

This section of the discussion carried out includes: data and information, as well as optimal reimbursement calculations.

3.1. Data and information

The data used in this numerical illustration is simulation data. For example, from an insurance company, property insurance data is obtained as given in Table-1. It is assumed that there are only two insurance companies, namely one insurer and one reinsurer. It is also assumed that the insurer is risk avoidance with utility functions $u_1(x)$ exponent, and risk-neutral reinsurer with utility functions $u_2(x)$ linear.
3.2. Optimal reimbursement calculation

Based on the data in Table-1, using equation (6) obtained surplus data at the end of the year \((X_1)\) for each property, as in Table-2 of the Initial Surplus column. Next, it is assumed that the parameters \(\alpha = 0.5\) utility function \((19)\) become \(u_1(x) = (1/0.5)(1-e^{-0.5x})\) and \(u_2(x) = x\). The premium that will be paid by the insurer to the reinsurer is calculated based on the exponential principle. If assumed parameters \(\beta = 0.005\), then the equation becomes \(H(R) = (1/0.005)\ln[E(PEe^{0.005R})]\). Because of the utility function \(u_2(x) = x\) earned value \(P = 1\). The amount \(d_1 = E[X_1] = \sum x u_1(x) = \sum x (1/0.5)(1-e^{-0.5x})\), by entering parameter values \(\alpha = 0.5\) and \(\beta = 0.005\) into equation (27), can be used to calculate the amount of reimbursement for each property. The results are given in Table-2 column \(R\).

Based on the calculation results presented in Table-2 above it can be interpreted that the optimal reimbursement value for each property is less than the expected surplus amount. The greater the surplus obtained from a property insurance, the reimbursement that must be purchased will be smaller, and vice versa. In other words, if a property can generate a large enough surplus, then the reimbursement that must be paid by the insurer to the reinsurer will be smaller.
### Table 2: Reimbursement of Each Property

| Property | Initial Surplus $X_1$ | $E[X_1]$ | $R$ |
|----------|------------------------|---------|----|
| 1        | 5,500                  | 11,000  | 463,600.0 |
| 2        | 7,522                  | 15,044  | 461,761.8 |
| 3        | 10,873                 | 21,746  | 458,715.5 |
| 4        | 5,585                  | 11,170  | 463,522.7 |
| 5        | 2,455                  | 4,910   | 466,368.2 |
| 6        | 20,629                 | 41,258  | 449,846.4 |
| 7        | 30,728                 | 61,456  | 440,665.5 |
| 8        | 24,369                 | 48,738  | 446,446.4 |
| 9        | 10,903                 | 21,806  | 458,688.2 |
| 10       | 10,913                 | 21,826  | 458,679.1 |
| 11       | 11,033                 | 22,066  | 458,570.0 |
| 12       | 8,036                  | 16,072  | 461,294.5 |
| 13       | 16,720                 | 33,440  | 453,400.0 |
| 14       | 8,650                  | 17,300  | 460,736.4 |
| 15       | 21,181                 | 42,362  | 449,344.5 |
| 16       | 1,219                  | 2,438   | 467,491.8 |
| 17       | 2,300                  | 4,600   | 466,509.1 |
| 18       | 21,430                 | 42,860  | 449,118.2 |
| 19       | 6,397                  | 12,794  | 462,784.5 |
| 20       | 7,857                  | 15,714  | 461,457.3 |
| **Total**| **468,600**            |         |     |

### 4. Conclusion

In this paper, an exponential utility function has been applied in calculating the transfer of risk between an insurance company, Pareto optimal risk transfer, equilibrium, and Bowley solutions, and an optimal reimbursement purchased by an insurer and reinsurer. The optimal reimbursement value for each property of a group of policies that is reinsured is less than the expectation of the group's surplus policy. The greater the surplus obtained, the reimbursement to be purchased is smaller, and vice versa. In other words, if a property can generate a large enough surpluses, then the reimbursement that must be paid by the insurer to the reinsurer will experience a smaller decrease. The limitation of the results of this study is that the optimal reimbursement calculation here only applies to exponential utility functions.

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