PROOFS OF SOME CONJECTURES OF KEITH AND ZANELLO ON $t$-REGULAR PARTITION

AJIT SINGH AND RUPAM BARMAN

Abstract. For a positive integer $t$, let $b_t(n)$ denote the number of $t$-regular partitions of a nonnegative integer $n$. In a recent paper, Keith and Zanello established infinite families of congruences and self-similarity results modulo 2 for $b_t(n)$ for certain values of $t$. Further, they proposed some conjectures on self-similarities of $b_t(n)$ modulo 2 for certain values of $t$. In this paper, we prove their conjectures on $b_3(n)$ and $b_{25}(n)$. We also prove a self-similarity result for $b_{21}(n)$ modulo 2.

1. Introduction and statement of results

A partition of a positive integer $n$ is any non-increasing sequence of positive integers whose sum is $n$. The number of such partitions of $n$ is denoted by $p(n)$. Let $t$ be a fixed positive integer. A $t$-regular partition of a positive integer $n$ is a partition of $n$ such that none of its part is divisible by $t$. Let $b_t(n)$ denote the number of $t$-regular partitions of $n$. The generating function of $b_t(n)$ is given by

$$\sum_{n=0}^{\infty} b_t(n)q^n = \frac{f_t}{f_1},$$

(1.1)

where $f_k := (q^k; q^k)^{\infty} = \prod_{j=1}^{\infty} (1 - q^{jk})$ and $k$ is a positive integer.

In a very recent paper [2], Keith and Zanello studied $t$-regular partition for certain values of $t$. They proved various congruences for $b_t(n)$ modulo 2 for certain values of $t \leq 28$, and posed several open questions. One of the congruences they proved for $b_3(n)$ is the following:

$$\sum_{n=0}^{\infty} b_3(26n + 14)q^n \equiv \sum_{n=0}^{\infty} b_3(2n)q^{13n} \pmod{2}.$$  

(1.2)

More generally, they conjectured that:

Conjecture 1.1. [2 Conjecture 6] For any prime $p > 3$, let $\alpha \equiv -24^{-1} \pmod{p^2}$, $0 < \alpha < p^2$. It holds for a positive proportion of primes $p$ that

$$\sum_{n=0}^{\infty} b_3(2(pm + \alpha))q^n \equiv \sum_{n=0}^{\infty} b_3(2n)q^{pn} \pmod{2}.$$  

(1.3)

The congruence (1.2) is a specific case of (1.3) corresponding to $p = 13$. In [5], we proved a specific case of (1.3) corresponding to $p = 17$. The aim of this...
article is to prove two conjectures of Keith and Zanello. Our first theorem confirms Conjecture 1.1.

**Theorem 1.2.** Conjecture 1.1 is true.

Keith and Zanello also studied 2-divisibility of $b_{25}(n)$ and proved several congruences for primes $p \equiv 11, 13, 17, 19 \pmod{20}$ and $p \equiv 31, 39 \pmod{40}$. To be specific, if $p \equiv 11, 13, 17, 19 \pmod{20}$ is prime, then they proved that

$$b_{25}(8(p^2n + kp - 3 \cdot 4^{-1}) + 5) \equiv 0 \pmod{2}$$

for all $1 \leq k < p$, where $3 \cdot 4^{-1}$ is taken modulo $p^2$. Further, they conjectured the following:

**Conjecture 1.3.** [2, Conjecture 28] For a positive proportion of primes $p$, it holds that

$$\sum_{n=0}^{\infty} b_{25}(2pn + \alpha)q^n \equiv q^\beta \sum_{n=0}^{\infty} b_{25}(2n + 1)q^n \pmod{2},$$

for some $\alpha$ and $\beta$ depending on $p$.

Our second theorem confirms Conjecture 1.3.

**Theorem 1.4.** Conjecture 1.3 is true.

Next, we prove a self-similarity result for $b_{21}(n)$ modulo 2. More precisely, we prove the following theorem:

**Theorem 1.5.** For a positive proportion of primes $p$, it holds that

$$\sum_{n=0}^{\infty} b_{21}(p(n + 11\gamma + 1))q^n \equiv \sum_{n=0}^{\infty} b_{21}(n + 1)q^n \pmod{2},$$

for some $\gamma$ depending on $p$.

2. Preliminaries

We recall some definitions and basic facts on modular forms. For more details, see for example [3, 4]. We first define the matrix groups

$$\text{SL}_2(\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \pmod{N} \right\},$$

and

$$\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N}, \text{ and } b \equiv c \equiv 0 \pmod{N} \right\},$$

where $N$ is a positive integer. A subgroup $\Gamma$ of $\text{SL}_2(\mathbb{Z})$ is called a congruence subgroup if $\Gamma(N) \subseteq \Gamma$ for some $N$. The smallest $N$ such that $\Gamma(N) \subseteq \Gamma$ is called the level of $\Gamma$. For example, $\Gamma_0(N)$ and $\Gamma_1(N)$ are congruence subgroups of level $N$. 
Let \( \mathbb{H} := \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \) be the upper half of the complex plane. The group
\[
GL_2^+(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^4 : a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0 \right\}
\]
acts on \( \mathbb{H} \) by
\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az + b}{cz + d} \quad \text{for all } z \in \mathbb{H}.
\]
We identify \( \infty \) with \( 1/0 \) and define \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} r/s = \frac{ar + bs}{cr + ds} \) where \( r/s \in \mathbb{Q} \cup \{ \infty \} \). This gives an action of \( GL_2^+(\mathbb{R}) \) on the extended upper half-plane \( \mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{ \infty \} \). Suppose that \( \Gamma \) is a congruence subgroup of \( SL_2(\mathbb{Z}) \). A cusp of \( \Gamma \) is an equivalence class in \( P^1 = \mathbb{P}^1 = \mathbb{Q} \cup \{ \infty \} \) under the action of \( \Gamma \).

The group \( GL_2^+(\mathbb{R}) \) also acts on functions \( f : \mathbb{H} \to \mathbb{C} \). In particular, suppose that \( \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2^+(\mathbb{R}) \). If \( f(z) \) is a meromorphic function on \( \mathbb{H} \) and \( \ell \) is an integer, then define the slash operator \( |\ell\gamma \) by
\[
(f|\ell\gamma)(z) := (\det \gamma)^{\ell/2}(cz + d)^{-\ell}f(\gamma z).
\]

**Definition 2.1.** Let \( \Gamma \) be a congruence subgroup of level \( N \). A holomorphic function \( f : \mathbb{H} \to \mathbb{C} \) is called a modular form with integer weight \( \ell \) on \( \Gamma \) if the following hold:

1. We have
\[
f(\begin{bmatrix} a & b \\ c & d \end{bmatrix} z) = (cz + d)^\ell f(z)
\]
for all \( z \in \mathbb{H} \) and all \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma \).

2. If \( \gamma \in SL_2(\mathbb{Z}) \), then \( (f|\ell\gamma)(z) \) has a Fourier expansion of the form
\[
(f|\ell\gamma)(z) = \sum_{n \geq 0} a_n(\gamma) q^n,
\]
where \( q_N := e^{2\pi iz/N} \).

In addition, if \( a_{\gamma}(0) = 0 \) for all \( \gamma \in SL_2(\mathbb{Z}) \), then \( f \) is called a cusp form.

For a positive integer \( \ell \), the complex vector space of modular forms (resp. cusp forms) of weight \( \ell \) with respect to a congruence subgroup \( \Gamma \) is denoted by \( M_\ell(\Gamma) \) (resp. \( S_\ell(\Gamma) \)).

**Definition 2.2.** [4, Definition 1.15] If \( \chi \) is a Dirichlet character modulo \( N \), then we say that a modular form \( f \in M_\ell(\Gamma_1(N)) \) (resp. \( S_\ell(\Gamma_1(N)) \)) has Nebentypus character \( \chi \) if
\[
f(\begin{bmatrix} a & b \\ c & d \end{bmatrix} z) = \chi(d)(cz + d)^\ell f(z)
\]
for all \( z \in \mathbb{H} \) and all \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) \). The space of such modular forms (resp. cusp forms) is denoted by \( M_\ell(\Gamma_0(N), \chi) \) (resp. \( S_\ell(\Gamma_0(N), \chi) \)).

In this paper, the relevant modular forms are those that arise from eta-quotients. Recall that the Dedekind eta-function \( \eta(z) \) is defined by
\[
\eta(z) := q^{1/24}(q; q)_\infty = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),
\]
where \( q := e^{2\pi i z} \) and \( z \in \mathbb{H} \). A function \( f(z) \) is called an eta-quotient if it is of the form

\[
 f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_{\delta}},
\]

where \( N \) is a positive integer and \( r_{\delta} \) is an integer. We now recall two theorems from [4, p. 18] which are very useful in checking modularity of eta-quotients.

**Theorem 2.3.** [4, Theorem 1.64] If \( f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_{\delta}} \) is an eta-quotient such that

\[
 \ell = \frac{1}{2} \sum_{\delta \mid N} r_{\delta} \equiv 0 \pmod{24}
\]

and

\[
 \sum_{\delta \mid N} \delta r_{\delta} \equiv 0 \pmod{24},
\]

then \( f(z) \) satisfies

\[
 f \left( \frac{az + b}{cz + d} \right) = \chi(d)(cz + d)^{\ell} f(z)
\]

for every \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) \). Here the character \( \chi \) is defined by \( \chi(d) := \left( \frac{-1}{d} \right)^{\ell s/d} \), where \( s := \prod_{\delta \mid N} \delta^{r_{\delta}} \).

Suppose that \( f \) is an eta-quotient satisfying the conditions of Theorem 2.3 and that the associated weight \( \ell \) is a positive integer. If \( f(z) \) is holomorphic (resp. vanishes) at all of the cusps of \( \Gamma_0(N) \), then \( f(z) \in M_{\ell}(\Gamma_0(N), \chi) \) (resp. \( S_{\ell}(\Gamma_0(N), \chi) \)). The following theorem gives the necessary criterion for determining orders of an eta-quotient at cusps.

**Theorem 2.4.** [4, Theorem 1.65] Let \( c, d \) and \( N \) be positive integers with \( d \mid N \) and \( \gcd(c, d) = 1 \). If \( f \) is an eta-quotient satisfying the conditions of Theorem 2.3 for \( N \), then the order of vanishing of \( f(z) \) at the cusp \( \frac{c}{d} \) is

\[
 \frac{N}{24} \sum_{\delta \mid N} \frac{\gcd(d, \delta)^2 r_{\delta}}{\gcd(d, \frac{d}{\delta})} \delta^s.
\]

We next recall the definition of Hecke operators. Let \( m \) be a positive integer and

\[
 f(z) = \sum_{n=0}^{\infty} a(n) q^n \in M_{\ell}(\Gamma_0(N), \chi).
\]

Then the action of Hecke operator \( T_m \) on \( f(z) \) is defined by

\[
 f(z)|T_m := \sum_{n=0}^{\infty} \sum_{d \mid \gcd(n, m)} \chi(d) d^{\ell - 1} a \left( \frac{nm}{d^2} \right) q^n.
\]

In particular, if \( m = p \) is prime, we have

\[
 f(z)|T_p := \sum_{n=0}^{\infty} \left( a(np) + \chi(p) p^{\ell - 1} a \left( \frac{n}{p} \right) \right) q^n.
\]

We adopt the convention that \( a(n/p) = 0 \) when \( p \nmid n \).

We finally recall a result of Serre [7] (also see [6, Proposition 4.2]) about the action of Hecke operator on cusp forms. For a number field \( K \), let \( \mathcal{O}_K \) denote its ring of integers.
Theorem 2.5. [7, Exercise 6.4] Suppose that
\[ f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k(\Gamma_0(N), \chi) \]
has coefficients in \( O_K \), and \( M \) is a positive integer. Furthermore, suppose that \( k > 1 \). Then a positive proportion of the primes \( p \equiv -1 \pmod{MN} \) have the property that
\[ f(z) \mid T_p \equiv 0 \pmod{M}. \]

3. Proof of Theorem 1.2

Proof. We first recall the following even-odd dissection of the 3-regular partitions [2, (6)]:
\[ \sum_{n=0}^{\infty} b_3(n)q^n = \frac{f_1}{f_3} + \frac{q f_3}{f_1} \pmod{2}. \]
Extracting the terms with even powers of \( q \), we obtain
\[ \sum_{n=0}^{\infty} b_3(2n)q^n \equiv \frac{f_4}{f_3} \pmod{2}. \]  \((3.1)\)

Let
\[ A(z) := \prod_{n=1}^{\infty} \frac{(1 - q^{24n})^2}{(1 - q^{48n})} = \frac{\eta^2(24z)}{\eta(48z)}. \]
Then using the binomial theorem we have
\[ A(z) = \frac{\eta^2(24z)}{\eta(48z)} \equiv 1 \pmod{2}. \]

Define \( B(z) \) by
\[ B(z) := \left( \frac{\eta^4(24z)}{\eta(72z)} \right) A(z) = \frac{\eta^6(24z)}{\eta(72z)\eta(48z)}. \]
Modulo 2, we have
\[ B(z) \equiv \frac{\eta^4(24z)}{\eta(72z)} = q^{24} \frac{q^{24}q^{144}}{q^{72}q^{72}}. \]  \((3.2)\)
Combining \((3.1)\) and \((3.2)\), we obtain
\[ B(z) \equiv \sum_{n=0}^{\infty} b_3(2n)q^{24n+1} \pmod{2}. \]  \((3.3)\)
Now, \( B(z) \) is an eta-quotient with \( N = 3456 \). We next prove that \( B(z) \) is a modular form. We know that the cusps of \( \Gamma_0(3456) \) are represented by fractions \( \frac{c}{d} \), where \( d \mid 3456 \) and gcd\((c,d) = 1 \). By Theorem 2.5, we find that \( B(z) \) vanishes at a cusp \( \frac{c}{d} \) if and only if
\[ L := 12 \frac{\gcd(d, 24)^2}{\gcd(d, 48)^2} - 2 \frac{\gcd(d, 72)^2}{3 \gcd(d, 48)^2} - 1 > 0. \]
We now consider the following four cases according to the divisors of 3456 and find the values of \( G_1 := \frac{\gcd(d, 24)^2}{\gcd(d, 48)^2} \) and \( G_2 := \frac{\gcd(d, 72)^2}{\gcd(d, 48)^2} \). Let \( d \) be a divisor of \( N = 3456 \).
Case (i). For \( d = 2^{r_1}3^{r_2} \), where \( 0 \leq r_1 \leq 3 \) and \( 0 \leq r_2 \leq 1 \), we find that
\[ G_1 = G_2 = 1. \text{ Hence, } L > 0. \]

Case (ii). For \( d = 2^i 3^r \), where \( 0 \leq r_1 \leq 3 \) and \( 2 \leq r_2 \leq 3 \), we find that \( G_1 = 1 \) and \( G_2 = 9 \). Hence, \( L > 0. \)

Case (iii). For \( d = 2^i 3^r \), where \( 4 \leq r_1 \leq 7 \) and \( 0 \leq r_2 \leq 1 \), we find that \( G_1 = G_2 = 1/4 \). Hence, \( L > 0. \)

Case (iv). For \( d = 2^i 3^r \), where \( 4 \leq r_1 \leq 7 \) and \( 2 \leq r_2 \leq 3 \), we find that \( G_1 = 1/4 \) and \( G_2 = 9/4 \). Hence, \( L > 0. \)

Thus, \( B(z) \) vanishes at every cusp \( \frac{a}{b} \). Using Theorem 2.3, we find that the weight of \( B(z) \) is equal to 2. Also, the associated character for \( B(z) \) is given by \( \chi_1 = (\frac{2}{13^3}). \) This proves that \( B(z) \in S_2(\Gamma_0(3456), \chi_1) \). Also, the Fourier coefficients of \( B(z) \) are all integers. Hence by Theorem 2.5, a positive proportion of the primes \( p \equiv -1 \pmod{6912} \) have the property that

\[ B(z) \mid T_p \equiv 0 \pmod{2}. \quad (3.4) \]

Let \( B(z) = \sum_{n=1}^{\infty} a(n)q^n \). Then, (3.3) yields

\[ \sum_{n=1}^{\infty} b_3 \left( \frac{2(n-1)}{24} \right) q^n = \sum_{n=1}^{\infty} a(n)q^n \pmod{2}. \quad (3.5) \]

Now, from (3.4) we obtain

\[ B(z) \mid T_p = \sum_{n=1}^{\infty} (a(pm) + p\chi_1(p)a(n/p))q^n \equiv 0 \pmod{2} \]

which yields

\[ \sum_{n=1}^{\infty} a(pm)q^n \equiv \sum_{n=1}^{\infty} a(n/p)q^n \pmod{2}. \quad (3.6) \]

Combining (3.5) and (3.6), we find that

\[ \sum_{n=1}^{\infty} b_3 \left( \frac{2(pm-1)}{24} \right) q^n = \sum_{n=1}^{\infty} b_3 \left( \frac{2(n/p-1)}{24} \right) q^n \pmod{2} \]

\[ = \sum_{n=1}^{\infty} b_3 \left( \frac{2(n-1)}{24} \right) q^n \pmod{2} \]

\[ = \sum_{n=0}^{\infty} b_3 \left( \frac{2n}{24} \right) q^{pn+p} \pmod{2}. \]

Multiplying both sides by \( q^{-p} \) we obtain

\[ \sum_{n=p}^{\infty} b_3 \left( \frac{2(pm-1)}{24} \right) q^{n-p} \equiv \sum_{n=0}^{\infty} b_3 \left( \frac{2n}{24} \right) q^n \pmod{2} \]

which yields

\[ \sum_{n=0}^{\infty} b_3 \left( \frac{2(pm + p^2 - 1)}{24} \right) q^n \equiv \sum_{n=0}^{\infty} b_3 \left( \frac{2n}{24} \right) q^n \pmod{2} \]
Let \( \alpha = \frac{p^2 - 1}{24} \). Since, \( p \equiv -1 \pmod{6912} \), so \( \alpha \) is a positive integer, and \( \alpha \equiv -24^{-1} \pmod{p^2} \), \( 0 < \alpha < p^2 \). Replacing \( n \) by \( 24n \) and then substituting \( q^{24} \) by \( q \) we get
\[
\sum_{n=0}^{\infty} b_3(2(n + \alpha))q^n \equiv \sum_{n=0}^{\infty} b_3(2n)q^{pn} \pmod{2}.
\]
This completes the proof of the theorem. \( \square \)

4. PROOF OF THEOREM 1.4

Proof. Putting \( t = 25 \) in (1.1) we have
\[
\sum_{n=0}^{\infty} b_{25}(n)q^n = \frac{f_{25}}{f_1}.
\]
(4.1)

We use identity [1, (4)], namely
\[
f_1f_5 \equiv f_1^6 + qf_5^6 \pmod{2}.
\]
Dividing both sides by \( f_1^3 \) we obtain
\[
\frac{f_5}{f_1} \equiv f_1^4 + q\frac{f_5}{f_1} \pmod{2}.
\]
(4.2)

Therefore, by (4.1) and (4.2) we have
\[
\sum_{n=0}^{\infty} b_{25}(n)q^n = \frac{f_{25}}{f_1} = \frac{f_{25}f_5}{f_1 f_1}
\]
\[
\equiv f_1^4 f_5^4 + q^6 f_5^4 f_{25}^6 f_1 + q^5 f_5^{10} f_1 + q^5 f_5 f_{25}^6 f_1^4 \pmod{2}.
\]
Extracting the terms involving \( q^{2n+1} \), and then dividing by \( q \) and replacing \( q^2 \) by \( q \), we find that
\[
\sum_{n=0}^{\infty} b_{25}(2n + 1)q^n \equiv \frac{f_5^5}{f_1} + q^2 \frac{f_1^2 f_{25}^2}{f_5} \pmod{2}
\]
\[
\equiv f_1^4 f_5^4 + q^6 f_5^4 f_{25}^6 f_1 + q^5 f_5^{10} f_1 + q^7 f_1^2 f_{25}^2 \pmod{2}.
\]

Extracting the terms involving \( q^{2n} \), we obtain
\[
\sum_{n=0}^{\infty} b_{25}(4n + 1)q^{2n} \equiv f_2^2 f_{10}^2 + q^2 f_2 f_{10} f_{50} \pmod{2}.
\]
(4.3)

Define \( F(z) \) by
\[
F(z) := \eta^2(2z)\eta^2(10z) + \eta(2z)\eta^2(10z)\eta(50z).
\]
(4.4)

Combining (4.3) and (4.4), we obtain
\[
F(z) \equiv \sum_{n=0}^{\infty} b_{25}(4n + 1)q^{2n+1} \pmod{2}.
\]
(4.5)

Now using Theorems 2.3 and 4.1, we find that \( \eta^2(2z)\eta^2(10z) \in S_2(\Gamma_0(100), \chi_3) \) and \( \eta(2z)\eta^2(10z)\eta(50z) \in S_2(\Gamma_0(100), \chi_3) \) for some Nebentypus character \( \chi_3 \) and hence \( F(z) \in S_2(\Gamma_0(100), \chi_3) \). Also, the Fourier coefficients of \( F(z) \) are all integers.
Hence by Theorem 2.5 a positive proportion of the primes \( p \equiv -1 \) (mod 200) have the property that

\[
F(z) \mid T_p \equiv 0 \pmod{2}. \tag{4.6}
\]

Let \( F(z) = \sum_{n=1}^{\infty} d(n)q^n \). Then, (4.5) yields

\[
\sum_{n=1}^{\infty} b_{25}(2(n - 1) + 1)q^n = \sum_{n=1}^{\infty} d(n)q^n \quad \text{(mod 2).} \tag{4.7}
\]

Now, from (4.6) we obtain

\[
F(z) \mid T_p = \sum_{n=1}^{\infty} (d(pm) + p\chi_3(p)d(n/p))q^n \equiv 0 \pmod{2}
\]

which yields

\[
\sum_{n=1}^{\infty} d(pm)q^n \equiv \sum_{n=1}^{\infty} d(n/p)q^n \quad \text{(mod 2).} \tag{4.8}
\]

Combining (4.7) and (4.8) we find that

\[
\sum_{n=1}^{\infty} b_{25}(2n - 1)q^n \equiv \sum_{n=1}^{\infty} b_{25}(2(n/p - 1) + 1)q^n \quad \text{(mod 2)}
\]

\[
\equiv \sum_{n=1}^{\infty} b_{25}(2(n - 1) + 1)q^{pn} \quad \text{(mod 2)}
\]

\[
\equiv \sum_{n=0}^{\infty} b_{25}(2n + 1)q^{pn+p} \quad \text{(mod 2).}
\]

Replacing \( n \) by \( n + 1 \) on the left side and then dividing both sides by \( q \) we obtain

\[
\sum_{n=0}^{\infty} b_{25}(2pn + \alpha)q^n \equiv q^\beta \sum_{n=0}^{\infty} b_{25}(2n + 1)q^{pn} \pmod{2},
\]

where \( \alpha = 2p - 1 \) and \( \beta = p - 1 \). This completes the proof of the theorem. \( \square \)

5. Proof of Theorem 1.5

Proof. We begin with the identity \( [2, \text{Section 7}] \), namely

\[
\sum_{n=0}^{\infty} b_{21}(4n + 1)q^n \equiv \frac{f_4^4}{f_4} \pmod{2}. \tag{5.1}
\]

Let

\[
G(z) := \prod_{n=1}^{\infty} \frac{(1 - q^{24n})^2}{(1 - q^{88n})} = \frac{\eta^2(24z)}{\eta(48z)}.
\]

Then using the binomial theorem we have

\[
G(z) = \frac{\eta^2(24z)}{\eta(48z)} \equiv 1 \pmod{2}.
\]

Define \( H(z) \) by

\[
H(z) := \left( \frac{\eta^4(72z)}{\eta(24z)} \right) G(z) = \frac{\eta^4(72z)\eta(24z)}{\eta(48z)}.
\]
Modulo 2, we have
\[ H(z) \equiv \frac{\eta^4(72z)}{\eta(24z)} = q^{11} \frac{(q^{72}; q^{72})^4}{(q^{24}; q^{24})^\infty}, \]  
(5.2)

Combining (5.1) and (5.2), we obtain
\[ H(z) \equiv \sum_{n=0}^{\infty} b_{21}(4n + 1)q^{24n+11} \pmod{2}. \]  
(5.3)

Now using Theorems 2.3 and 2.4, we find that
\[ H(z) \in S_2(\Gamma_0(3456), \chi_2) \]  
for some Nebentypus character \( \chi_2 \). Also, the Fourier coefficients of \( H(z) \) are all integers. Hence by Theorem 2.5, a positive proportion of the primes \( p \equiv -1 \pmod{6912} \) have the property that
\[ H(z) \mid T_p \equiv 0 \pmod{2}. \]  
(5.4)

Let \( H(z) := \sum_{n=1}^{\infty} c(n)q^n \). Then, (5.3) yields
\[ \sum_{n=1}^{\infty} b_{21} \left( \frac{4(n - 11)}{24} + 1 \right) q^n \equiv \sum_{n=1}^{\infty} c(n)q^n \pmod{2}. \]  
(5.5)

Now, from (5.4) we obtain
\[ H(z) \mid T_p = \sum_{n=1}^{\infty} (c(pn) + p\chi_2(p)c(n/p))q^n \equiv 0 \pmod{2} \]
which yields
\[ \sum_{n=1}^{\infty} c(pn)q^n \equiv \sum_{n=1}^{\infty} c(n/p)q^n \pmod{2}. \]  
(5.6)

Combining (5.5) and (5.6) we find that
\[ \sum_{n=1}^{\infty} b_{21} \left( \frac{4(pm - 11)}{24} + 1 \right) q^n \equiv \sum_{n=1}^{\infty} b_{21} \left( \frac{4(n/p - 11)}{24} + 1 \right) q^n \pmod{2} \]
\[ \equiv \sum_{n=1}^{\infty} b_{21} \left( \frac{4(n - 11)}{24} + 1 \right) q^{pn} \pmod{2} \]
\[ \equiv \sum_{n=0}^{\infty} b_{21} \left( \frac{4n}{24} + 1 \right) q^{pm+11p} \pmod{2}. \]

Multiplying both sides by \( q^{-11p} \) we obtain
\[ \sum_{n=11p}^{\infty} b_{21} \left( \frac{pm - 11}{6} + 1 \right) q^{n-11p} \equiv \sum_{n=0}^{\infty} b_{21} \left( \frac{n}{6} + 1 \right) q^{pn} \pmod{2} \]
which yields
\[ \sum_{n=0}^{\infty} b_{21} \left( \frac{pm + 11(p^2 - 1)}{6} + 1 \right) q^n \equiv \sum_{n=0}^{\infty} b_{21} \left( \frac{n}{6} + 1 \right) q^{pn} \pmod{2}. \]
Let $\gamma = \frac{p^2 - 1}{6}$. Since $p \equiv -1 \pmod{6912}$, so $\gamma$ is a positive integer, and $\gamma \equiv -6^{-1} \pmod{p^2}$, $0 < \gamma < p^2$. Replacing $n$ by $6n$ and then substituting $q^6$ by $q$ we get

$$\sum_{n=0}^{\infty} b_{21}(pn + 11\gamma + 1)q^n \equiv \sum_{n=0}^{\infty} b_{21}(n + 1)q^{pn} \pmod{2}.$$ 

This completes the proof of the theorem. \hfill \Box

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Department of Mathematics, Indian Institute of Technology Guwahati, Assam, India, PIN- 781039

Email address: ajit18@iitg.ac.in

Department of Mathematics, Indian Institute of Technology Guwahati, Assam, India, PIN- 781039

Email address: rupam@iitg.ac.in