The boundary of the Milnor fiber
for some non-isolated germs of complex surfaces.

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Abstract.

We study the boundary $L_t$ of the Milnor fiber for the non-isolated singularities in $\mathbb{C}^3$ with equation $z^m - g(x, y) = 0$ where $g(x, y)$ is a non-reduced plane curve germ. We give a complete proof that $L_t$ is a Waldhausen graph manifold and we provide the tools to construct its plumbing graph. As an example, we give the plumbing graph associated to the germs $z^2 - (x^2 - y^3)y^l = 0$ with $l \geq 2$. We prove that the boundary of the Milnor fiber is a Waldhausen manifold new in complex geometry, as it cannot be the boundary of a normal surface singularity.

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1. Introduction.

In [M-P] the authors state with a sketch of proof that the boundary $L_t$ of the Milnor fiber of a non-isolated surface singularity in $\mathbb{C}^3$ is a Waldhausen graph manifold (non-necessarily "reduziert"). These manifolds are conveniently described by a plumbing graph. In [M-P-W] we determine the plumbing graph for the boundary of the Milnor fiber of Hirzebruch singularities $z^m - x^k y^l = 0$. The present paper is devoted to the study of germs with equation $z^m - g(x, y) = 0$ where $g(x, y)$ is a non-reduced plane curve germ. For them:

1) We prove in details that $L_t$ is indeed a Waldhausen manifold (Section 4). The Waldhausen decomposition for $L_t$ is obtained by gluing two specific Waldhausen sub-manifolds along boundary torii: the trunk and the (non-necessarily connected) vanishing zone.
2) We prove that the vanishing zone is in fact a Seifert manifold and we elucidate its structure (Section 5).
3) We show how to obtain the trunk (Section 2) and how to determine the gluing between the two sub-manifolds (Section 4).

Necessary results about Seifert and Waldhausen manifolds are recalled in section 3. The dictionary which translates Waldhausen decompositions into plumbing graphs provided by [N] can then be used to obtain the canonical plumbing graph for $L_t$.

In section 7, the plumbing graph is given for the singularities $z^2 - (x^2 - y^3)y^l = 0$ with $l \geq 2$. We prove that the boundary of their Milnor fiber are Waldhausen manifolds new in complex geometry, as they cannot be the boundary of a normal surface singularity. This fact does not depend on the orientation on $L_t$.

Information about the homology of $L_t$ is given in section 8. For Hirzebruch singularities we obtain the following result.

**Theorem 8.1.** Let $f(x, y, z) = z^m - x^k y^l = 0$ be the equation of a Hirzebruch singularity. Assume that $gcd(m, k, l) = 1$, that $1 \leq k < l$ and that $m \geq 2$. Let $d = gcd(k, l)$ and write $k = k/d$ and $l = l/d$. Then $H_1(L_t, \mathbb{Z})$ is equal to the direct sum of a free abelian group of rank $2(m - 1)(d - 1)$ and a torsion group. The torsion subgroup is the direct sum of $(m - 1)$ cyclic factors. One of them is of order $m k l$ and the other $(m - 2)$ factors are of order $k l$.

In section 6, we expound when $L_t$ is a lens space for the germs under consideration in this paper. The reason why lens spaces come up is explained at the end of section 2.

2. Definitions and main results.

We consider germs $f(x, y, z) \in \mathbb{C}\{x, y, z\}$ such that $f(0, 0, 0) = 0$. We deal with germs
We denote by $B^{2n}_r$ the 2n-ball with radius $r > 0$ centered at the origin of $\mathbb{C}^n$ and by $S^{2n-1}_r$ the boundary of $B^{2n}_r$. We set $F_0 = B^6_\varepsilon \cap f^{-1}(0)$ and $L_0 = S^5_\varepsilon \cap f^{-1}(0)$. According to the theory of Milnor [Mi], extended by Burghelea and Verona [B-V]) in the non-isolated case, the homeomorphism classes of the pairs $(B^6_\varepsilon, F_0)$ and $(S^5_\varepsilon, L_0)$ do not depend on $\varepsilon > 0$ if it is sufficiently small. As a consequence, we shall usually remove ”$\varepsilon$” from our notations.

The restriction $f|B^6_\varepsilon \cap f^{-1}(B^2_\eta - \{0\}) \rightarrow (B^2_\eta - \{0\})$ is a locally trivial differentiable fibration whose isomorphism class does not depend on $\eta > 0$ provided that $\eta$ is sufficiently small ($0 < \eta << \varepsilon$). See Milnor [Mi] and also Hamm-Lê [H-L]. Therefore, the diffeomorphism classes of the manifolds $F_t = B^6_\varepsilon \cap f^{-1}(t)$ and $L_t = S^5_\varepsilon \cap f^{-1}(t)$ do not depend on $t$ if $0 < |t| \leq \eta$. We say that $F_t$ is the Milnor fiber of $f$ and that $L_t$ is the boundary of the Milnor fiber. $F_t$ is oriented by its complex structure and $L_t$ is oriented as the boundary of $F_t$.

We denote by $n : \tilde{F}_0 \rightarrow F_0$ the normalisation. It follows from the arguments in Durfee [D] that the boundary $\bar{L}_0$ of an algebraic neighborhood of $n^{-1}(0)$ is well defined. We shall call $\bar{L}_0$ the boundary of the normalisation.

The strategy used to obtain the boundary of the Milnor fiber for non-isolated singularities is the following. Let $\Sigma(f)$ be the singular locus of $f$. By hypothesis $\Sigma(f)$ is a curve. Let $K_0 = L_0 \cap \Sigma(f)$ be the link of the singular locus in $L_0$. Let $\tilde{K}_0 = n^{-1}(K_0)$ be the pull-back of $K_0$ in $\tilde{L}_0$. A good resolution of $\tilde{F}_0$ provides a Waldhausen decomposition for $\tilde{L}_0$ as a union of Seifert leaves such that $\tilde{K}_0$ is a union of Seifert leaves. Let $M_0$ be a tubular neighborhood of $\tilde{K}_0$ in $\tilde{L}_0$. The closure $\tilde{N}_0$ of $(\tilde{L}_0 - M_0)$ is called the trunk of $L_t$. In 4.6 we define a submanifold $M_t$ of $L_t$ called the vanishing zone around $K_0$. A slightly less general version of theorem 4.7 can be easily stated as follows.

**Theorem.**

1) The closure $N_t$ of $L_t \setminus M_t$ is diffeomorphic to the trunk $\tilde{N}_0$.
2) The manifold $M_t$ is a Seifert manifold.

The construction (see 4.6) of the vanishing zone is so precise that it gives rise to a very explicit description of $M_t$. To each irreducible component $\sigma_i$ of the singular locus of $f$ corresponds a connected component $M_t(i)$ of $M_t$. An hyperplane section argument provides a plane curve germ $(z^m - y^{n_i})$ and an integer $k$. Let $d = gcd(n_i, k)$. In section 5 we prove the following result.

**Theorem 5.4.** The vanishing zone $M_t(i)$ is the mapping torus of a diffeomorphism $h : G_t \rightarrow G_t$ such that :

1) $G_t$ is diffeomorphic to the Milnor fiber of the plane curve germ $z^m - y^{n_i}$.
2) The diffeomorphism $h$ is finite of order $n_i/d$.
3) If $d < n_i$, $h$ has exactly $m$ fixed points and the action of $h$ has order $n_i/d$ on all other points.
4) Around a fixed point $h$ is a rotation of angle $-2\pi k/n_i$.

**Remark.** When $f$ is not analytically equivalent to $z^m - g(x, y)$ one can have vanishing
zones which are Waldhausen but not Seifert, or Seifert manifolds of a more complicated nature.

It is stated in [M-P] that $L_t$ is never homeomorphic to $\tilde{L}_0$. But the particular case when $L_t$ is a lens space is not treated in [M-P] and is rather delicate. To produce a complete proof of this statement in a forthcoming paper, the first two authors need a characterization of the germs $z^m - g(x, y)$ for which $L_t$ is a lens space. Theorem 6.5 solves the problem.

**Theorem 6.5.** The boundary of the Milnor fiber of a irreducible germ $f(x, y, z) = z^m - g(x, y)$ is a lens space iff $f$ is analytically equivalent to $z^2 - xy^l$.

**Remark.** For our purpose lens spaces are defined as graph manifolds which are obtained from a plumbing graph which is a "bamboo" with genus zero vertices.

For technical reasons, we use in this paper a polydisc $B(\alpha) = B^2_\alpha \times B^2_\beta \times B^2_\gamma$ where $0 < \alpha \leq \beta \leq \gamma \leq \epsilon$ in place of a standard ball $B^6_\epsilon$.

**Definition.** The polydisc $B(\alpha)$ is a Milnor polydisc for $f$ if:

i) For each $\alpha'$ with $0 < \alpha' \leq \alpha$ the pair $(B(\alpha'), f^{-1}(0) \cap B(\alpha'))$ is homeomorphic to the pair $(B^6_\epsilon, f^{-1}(0) \cap B^6_\epsilon)$.

ii) For each $\alpha'$ with $0 < \alpha' \leq \alpha$ there exists $\eta$ with $0 < \eta << \alpha'$ such that:

1) the restriction of $f$ to $W(\alpha', \eta) = B(\alpha') \cap f^{-1}(B^2_\eta - \{0\})$ is a locally trivial differentiable fibration on $(B^2_\eta - \{0\})$

2) this fibration does not depend on $\alpha'$ (when $0 < \alpha' \leq \alpha$) up to isomorphism.

3. Three-dimensional manifolds.

In this section, we recall some facts pertaining to 3-dimensional manifolds in a setting appropriate to our needs.

We consider differentiable, compact (usually connected) 3-manifolds $M$ possibly with boundary. When the boundary $\partial M$ is non-empty, we assume that it is a disjoint union of torii. Manifolds are oriented. Classifications are done up to orientation preserving diffeomorphism. In the situations we meet, $M$ is quite often the boundary of a complex surface $V$. The complex structure gives rise to an orientation of $V$ and $M = \partial V$ receives an orientation via the boundary homomorphism $\partial: H_4(V \text{mod}\partial V; \mathbb{Z}) \to H_3(\partial V; \mathbb{Z})$.

3.1 Seifert foliations.

In this paper, we only need to consider orientable Seifert fibrations (to be called Seifert
foliations, since we have too many fibrations present). As our manifolds are oriented and compact, we may define a Seifert foliation on \( M \) as an orientable foliation by circles. Thanks to a theorem of Epstein [E], this is equivalent to requiring that there exists a fixed point free \( S^1 \)-action on \( M \) such that the leaves coincide with the orbits.

An exceptional orbit (leaf) is one such that the isotropy subgroup is non-trivial. It is a finite cyclic subgroup of order \( \alpha \geq 2 \). The slice theorem (and orientability of \( M \)) imply that for each exceptional leaf there exist:

i) a tubular neighborhood which is a union of leaves

ii) an orientation preserving diffeomorphism of this neighborhood with the mapping torus of a rotation of order \( \alpha \) on an oriented 2-disc, sending leaves to leaves.

A Seifert invariant for an exceptional leaf is defined as follows. Suppose that the rotation angle on the 2-disc is equal to \( \frac{2\pi \beta^*}{\alpha} \). We need the orientation of the 2-disc to get the correct sign for the angle. We have \( \gcd(\alpha, \beta^*) = 1 \) and we choose \( \beta^* \) such that \( 0 < \beta^* < \alpha \). Now let \( \beta \) be any integer such that \( \beta \beta^* \equiv 1 \pmod{\alpha} \). The pair \((\alpha, \beta)\) is a Seifert invariant of the exceptional leaf. See [Mo] pages 135 to 140. The choice of a \( \beta \) in its residue class \( \pmod{\alpha} \) is related to the choice of a section of the foliation near the exceptional leaf.

Let \( r \geq 0 \) be the number of boundary components of \( M \). The space of leaves is a compact connected orientable surface of genus \( g \geq 0 \) with \( r \) boundary components.

Suppose now that sections of the foliation are chosen on each boundary component of \( M \) and that they are kept fixed during the following discussion. We then choose a \( \beta \) for each exceptional leaf. Once these choices have been made, the Euler number \( e \in \mathbb{Z} \) is defined. See [Mo] for details. Essentially it is the obstruction to extend the section already defined on some part of the orbit space. The integer \( e \) depends on the choice of the \( \beta \)'s, but the rational number \( e_0 = e - \sum \frac{\beta_i}{\alpha_i} \) does not. Of course, if \( r > 0 \) the numbers \( e \) and \( e_0 \) still depend on the choice of a section on the boundary of \( M \).

### 3.2 Waldhausen manifolds and plumbings graphs.

The manifolds \( \tilde{L}_0 \) and \( L_t \) we study in this paper are graphed manifolds in Waldhausen’s sense [W]. They will appear in the following dress.

A finite decomposition \( M = \bigcup M_i \) of a 3-manifold \( M \) is Waldhausen if:

1) Each \( M_i \) is a Seifert manifold

2) If \( i \neq j \) the intersection \( M_i \cap M_j \) is either empty or equal to a union of common boundary components.

A manifold is Waldhausen if it admits a Waldhausen decomposition. It is best described by a plumbing graph. To begin with, we consider oriented 3-manifolds which are circle bundles over a closed surface (we only need here to consider these). Such a bundle is characterised by its Euler number and the genus of the base space. Two bundles may be glued together by an operation called plumbing. See [N] for details.
A 3-manifold constructed by plumbing is represented by a graph. The vertices represent the bundles. They carry two integral weights: the genus $g$ of the base space and the Euler number $e$. An edge represents a plumbing operation. The dual graph of a good resolution for a normal surface singularity is also weighted like this. If understood as a plumbing graph, it describes the boundary of a semi-algebraic neighbourhood of the exceptional locus. See [N] for details. In [N] Neumann assigns a canonical plumbing graph to each Waldhausen manifold. Particularly useful are the bamboo $o\ldots o$ with genus zero vertices and Euler numbers $e \leq -2$ for a lens space (see p.327 thm. 6.1) and the star-shaped tree for the other Seifert spaces (see p.327 cor.5.7).

3.3 Mapping torii.

**Definition.** Let $G$ be an oriented differentiable surface and let $h : G \to G$ be an orientation preserving diffeomorphism. The mapping torus $T(h)$ of $h$ is the quotient of the product $G \times \mathbb{R}$ by the equivalence relation $(x, t + 1) \sim (h(x), t)$. The manifold $G \times \mathbb{R}$ is oriented by the product orientation, $\mathbb{R}$ being equipped with the usual one. This orientation projects down to $T(h)$.

The mapping torus $T(h)$ fibers over the circle $S^1$ with fiber $G$ and $h$ is "the" monodromy (well defined up to isotopy) of this fibration. Suppose now that $h$ is of finite order. Then the lines $\{x\} \times \mathbb{R}$ in $G \times \mathbb{R}$ project onto circles in $T(h)$. The mapping torus thus receives a foliation in circles. The following important property of $e_0$ is useful for computations. For a proof see [P].

**Theorem.** Let $h$ be an orientation preserving diffeomorphism acting on a closed surface. Then the rational Euler number $e_0$ of the Seifert foliation on the mapping torus $T(h)$ vanishes.

3.4 Comments.

i) The plumbing graph for $L_t$ can be obtained as follows. The plumbing graph for the trunk is part of the plumbing graph for the normalised surface. From the mapping torus of the vertical monodromy, we obtain the Seifert-Waldhausen invariants of the vanishing zone by the dictionary given in [P]. Then [N] gives the plumbing graph for the vanishing zone. The pasting of two Seifert pieces along a common boundary component is represented in the plumbing graph by a bamboo having vertices with $g = 0$.

ii) Neumann proves in [N] that the boundary of a normal surface singularity cannot be a non-trivial connected sum. However, the boundary $L_t$ of the Milnor fiber can be a non-trivial connected sum, as proved in [M-P]. In this case, the canonical plumbing graph is non-connected.

iii) Usually when lens spaces $L(n, q)$ are considered it is implicitly assumed that $n \geq 2$. In this paper we shall call generalised lens space an oriented 3-manifold which is
orientation preserving diffeomorphic to $L(n,q)$ or $S^3$ or $S^1 \times S^2$. They are exactly the 3-manifolds which admit a genus one Heegaard decomposition. A beautiful result of F. Bonahon [B] says that the Heegaard decomposition is unique up to isotopy.

iv) A manifold which has two Seifert structures (one of them being non-orientable) is a frequent pebble in the shoe. Let $h$ be ”the” orientation-preserving involution of the annulus $S^1 \times [0,1]$. The mapping torus of $h$ is a Seifert manifold which has two exceptional leaves with $\alpha = 2$. This is the Seifert structure that Waldhausen calls $Q$. See [W]. We shall not meet the other Seifert structure.

4. From the boundary of the normalisation to the boundary of the Milnor fiber.

Let $g \in \mathbb{C}\{x,y\}$ be non-reduced and such that $g(0,0) = 0$. Let $\prod_{i=1}^lg_i^{n_i}$ be the factorisation of $g$ into a product of irreducible factors with $g_i$ prime to $g_j$ if $i \neq j$. We choose the indices in such a way that $n_i > 1$ iff $i \leq i_0$ for some $i_0$ with $1 \leq i_0 \leq l$. We choose the coordinate axis such that $x$ is prime to $\partial g/\partial y$.

Now let $f(x,y,z) = z^m - g(x,y)$ and let $\Gamma = \{\partial g/\partial y = 0\} \cap \{f = 0\}$. The singular locus $\Sigma(f)$ of $f$ is the intersection of $\{z = 0\}$ with $\{g'(x,y) = 0\}$ where $g'(x,y) = \prod_{i=1}^l g_i^{n_i}$.

(4.1) Now let $B(\alpha)$ be a Milnor polyball as defined at the end of section 2. Let $S$ be the boundary of $B(\alpha)$ and let $S(\alpha) = S^1_\alpha \times IntB^3_\beta \times IntB^2_\gamma$. We choose $0 < \alpha \leq \beta \leq \gamma \leq \epsilon$ such that:

1. $\Gamma \cap S \subset S(\alpha)$ and $(\{g = 0\} \cap \{z = 0\} \cap S) \subset S(\alpha)$
2. $L_0 = (\{f = 0\} \cap S) \subset \{|z| < \gamma\}$
3. In $B(\alpha)$ the curve $\Gamma$ intersects transversally the hyperplanes $H_a = \{x = a\}$ for all $a$ with $0 < |a| \leq \alpha$.

Let $F_0 = f^{-1}(0) \cap B(\alpha)$. Then $L_0 = S \cap F_0$ is the boundary of $F_0$. The link $K_0$ of the singular locus $\Sigma(f)$ of $f$ is by definition $K_0 = \Sigma(f) \cap L_0$.

Now let $n : \tilde{F}_0 \rightarrow F_0$ be the normalisation of $F_0$. We have seen in section 2 that $\tilde{L}_0 = n^{-1}(L_0)$ can be identified with the boundary of the normalisation. Finally let $\tilde{K}_0 = n^{-1}(K_0)$ be the pull-back of $K_0$ by the normalisation.

Remark 4.2 The resolution theory implies that there exists a decomposition of $\tilde{L}_0$ as a union of Seifert manifolds such that $\tilde{K}_0$ is a union of Seifert leaves.

Let $\varphi : \mathbb{C}^3 \rightarrow \mathbb{C}^2$ be the projection defined by $\varphi(x,y,z) = (x,z)$. For a small $\theta$ with $0 < \theta << \alpha$ we denote by $M_0$ the union of the connected components of $\varphi^{-1}(S^1_\alpha \times B^2_\beta)$.
which meet $K_0$.

**Proposition 4.3** There exists a sufficiently small $\theta$ such that:
1) $M_0 \subset S(\alpha)$
2) $M_0 \cap \{z = 0\} = K_0$
3) $n^{-1}(M_0) = M_0$ is a tubular neighborhood of $\tilde{K}_0$ in $\tilde{L}_0$. Moreover $\tilde{K}_0$ is the ramification locus of $\varphi \circ n$ restricted to $\tilde{M}_0$.

**Corollary 4.4** The closure $N_0$ of $(L_0 - M_0)$ in $L_0$ is a Waldhausen manifold.

**Proof of the Corollary.** The restriction of the normalisation $n$ to the closure $\tilde{N}_0$ of $(\tilde{L}_0 - \tilde{M}_0)$ in $\tilde{L}_0$ is a diffeomorphism onto $N_0$. But $\tilde{N}_0$ is a Waldhausen manifold by remark 4.2.

**Proof of Proposition 4.3.** From (4.1) fact 1. we have $K_0 \subset S(\alpha)$. Then there exists $\theta$ such that $M_0 \subset S(\alpha)$. We can choose $\theta$ small enough such that $L_0 \cap \{|z| \leq \theta\}$ is a tubular neighborhood of $(g = 0) \cap L_0$ in $L_0$. This proves 2). The singular locus of $\varphi$ restricted to $F_0$ is the curve $\Gamma$. Let $\Delta = \varphi(\Gamma)$. We can choose $\theta$ still smaller in order that $\Delta \cap (S^{1}_\alpha \times B^{2}_\beta) = S^{1}_\alpha \times \{0\}$. As $\Sigma(f) = \{z = 0\} \cap \Gamma$ this proves 3).

(4.5) From the definition of $B(\alpha)$ given at the end of section 1, there exists a very small $\eta$ with $0 < \eta << \theta < \alpha$ such that $f$ restricted to $W(\alpha, \eta) = B(\alpha) \cap f^{-1}(B^{2}_\eta - \{0\})$ is a locally trivial fibration on $(B^{2}_\eta - \{0\})$. When $0 < |t| \leq \eta$ we say that $F_t = W(\alpha, \eta) \cap f^{-1}(t)$ is “the” Milnor fiber of $f$ and that $L_t = F_t \cap S$ is the boundary of the Milnor fiber of $f$.

In $S$ we consider $S(\alpha) = S^{1}_\alpha \times B^{2}_\beta \times \text{Int}B^{2}_\gamma$ and $S(\beta) = B^{2}_\alpha \times S^{1}_\beta \times \text{Int}B^{2}_{\gamma}$. As $\alpha, \beta, \gamma$ have been chosen such that $L_0 = (f^{-1}(0) \cap S) \subset (\bar{S}(\alpha) \cup \bar{S}(\beta))$ (see (4.1) fact 2) there exists $\eta$ with $0 < \eta << \alpha$ such that $L_t \subset (\bar{S}(\alpha) \cup \bar{S}(\beta))$ for all $t$ with $0 \leq |t| \leq \eta$.

(4.6) Let $M(\eta)$ be the union of the connected components of $S \cap \{|f| \leq \eta\} \cap \{|z| \leq \theta\}$ which meet $K_0$. Let $N(\eta)$ be the closure of $(W(\alpha, \eta) \cap S) - M(\eta)$ in $S$. For any $t$ with $0 \leq |t| \leq \eta$ let $M_t = L_t \cap M(\eta)$ and let $N_t = L_t \cap N(\eta)$ be the closure of $(L_t - M_t)$ in $L_t$.

**Theorem 4.7** There exists a sufficiently small $\eta$ such that for any $t$ with $0 < |t| \leq \eta$ we have
1) $M_t \subset S(\alpha)$
2) $f$ restricted to $N(\eta)$ is a fibration on $B^{2}_\eta$ with fiber $N_t$ for $0 \leq |t| \leq \eta$
3) $M_t$ has a Seifert structure such that the restriction of $z$ on any Seifert leaf is constant.

**Remark 4.8.** Theorem 4.7 enables us to describe $L_t$ as the union of the Seifert manifold $M_t$ with the manifold $N_t$ which is diffeomorphic to the Waldhausen submanifold $\tilde{N}_0$ of $\tilde{L}_0$ defined in proposition 4.3. Moreover, the intersection $M_t \cap N_t$ is equal to $\partial M_t = \partial N_t$ which is a disjoint union of torii. Hence we have:

**Corollary 4.9.** $L_t$ is a Waldhausen manifold.

As $f$ induces a deformation retraction of $M(\eta)$ onto the link $K_0$ we say that $M_t$ is the vanishing zone around $K_0$. Up to a diffeomorphism, $N_t$ is a common Waldhausen submanifold of $L_t$, $L_0$ and $\tilde{L}_0$. This is why we say that $N_t$ (resp $\tilde{N}_0$) is the trunk of $L_t$.
L_t can be constructed as the union of \( \tilde{N}_0 \) with \( M_t \) and small collars attached on the boundaries. These small collars are defined with the help of the restriction of \( f \) on the boundary of \( N(\eta) \).

**Proof of 4.7.** Proposition 4.3 implies that \( M_0 = M(\eta) \cap f^{-1}(0) \) is included in \( S(\alpha) \). As \( S(\alpha) \) is open, we may choose \( \eta \) sufficiently small in order that \( M(\eta) \subset S(\alpha) \). Thus point 1) is proved.

As noticed in 4.5, for a sufficiently small and for \( t \) such that \( 0 \leq |t| \leq \eta \) we have \( L_t \subset \bar{S}(\alpha) \cup \bar{S}(\beta) \). Let \( L(\eta) = N(\eta) \cup M(\eta) \). We restrict \( \eta \) to have

\[
\left( L(\eta) \cap \{ z = 0 \} \cap \left\{ \frac{\partial g}{\partial y} = 0 \right\} \cap \{|x| = \alpha\} \right) \subset K_0
\]

\[
\left( L(\eta) \cap \{ z = 0 \} \cap \left\{ \frac{\partial g}{\partial x} = 0 \right\} \cap \{|y| = \beta\} \right) \subset K_0
\]

The above inclusions imply that the restriction of \( f \) to \( L(\eta) - K_0 \) is a fibration. The boundary of \( N(\eta) \) (which is equal to the boundary of \( M(\eta) \)) is included in \( S(\alpha) \) and in \( \{|z| = \theta\} \). In proposition 4.3 we have chosen \( \theta \) such that \( \partial N_0 = \partial M_0 \) does not meet \( \{ \frac{\partial g}{\partial y} = 0 \} \). Hence, for a sufficiently small \( \eta \) the boundary of \( N(\eta) \) does not meet \( \{ \frac{\partial g}{\partial y} = 0 \} \) either. This proves 2).

We consider the projection \( \varphi \) defined in 4.3. For \( 0 < |t| \leq \eta \) let us denote by \( \varphi_t \) the restriction of \( \varphi \) to \( M_t \). The singular locus of \( \varphi_t \) is \( M_t \cap \{ g' = 0 \} = M_t \cap \{ z^m = t \} \). For each \( c \) with \( 0 \leq |c| \leq \theta \) we have

\[
\varphi^{-1}(S^1_\alpha \times \{ c \}) = M_t \cap \{ z = c \}
\]

This gives a foliation in circles on \( M_t \) with leaves defined by \( M_t \cap \{ z = c \} \). This ends the proof of theorem 4.7.

5. The vertical monodromy.

With the notations of 4.1, the link \( K_0 \) of the singular locus of \( f \) has \( i_0 \) connected components. We choose \( i \) with \( 1 \leq i \leq i_0 \) and we denote by \( K_i \) the component of \( K_0 \) which corresponds to the irreducible factor \( g_i \) of \( g \). More precisely:

\[
K_i = (S \cap \{ z = 0 \} \cap \{ g_i(x,y) = 0 \})
\]

Let \( M(i) \) be the connected component of the vanishing zone \( M(\eta) \) (see 4.6) which contains \( K_i \). Let \( \pi : M(\eta) \to S^1_\alpha \) be the projection on the \( x \)-axis. Let \( M_t(i) = M_t \cap M(i) \). Let \( \pi_t \) be \( \pi \) restricted to \( M_t(i) \) with \( 0 < |t| \leq \eta \).
Lemma 5.1. The projection $\pi_t$ is a fibration. Moreover the Seifert leaves constructed in 4.7 are transverse to the fibers of $\pi_t$.

Proof of lemma 5.1. The equation of the singular locus of $\pi_t$ is $\{z = 0\} \cap \{\frac{\partial a}{\partial y} = 0\}$. This curve does not meet $M_t(i)$ when $t \neq 0$.

We now choose $a$ with $|a| = \alpha$ and $P \in K_i \cap \{x = a\}$. Let $U(P)$ be the connected component of $\pi^{-1}(a) \cap M(i)$ which contains the point $P$. Let $f_P$ denote $f$ restricted to $U(P)$. Then $f_P$ is a plane curve germ with an isolated singular point at $P$ and $G_t = U(P) \cap M_t(i)$ is its Milnor fiber.

Definition 5.2. The vertical monodromy around $K_i$ is the first return diffeomorphism $h : G_t \to G_t$ along the Seifert leaves of $M_t(i)$.

Remark 5.3. Let $(s^r, w(s))$ be a Puiseux expansion of the branch $g_i(x, y) = 0$. Then $G'_t = M_t(i) \cap \pi^{-1}(a)$ has $r$ connected components. There exists a monodromy $h' : G'_t \to G'_t$ for the fibration $\pi_t$ such that $(h')^r$ is the vertical monodromy $h$.

Consider the following decomposition $g = g_i^{n_i} \cdot g''$ in $\mathbb{C}\{x, y\}$ with $g''$ prime to $g_i$. Let $k$ be the intersection multiplicity at the origin between $g_i$ and $g''$. Let $d = gcd(n_i, k)$.

Theorem 5.4. The vanishing zone $M_t(i)$ around $K_i$ is the mapping torus of $h : G_t \to G_t$ and we have:
1) $G_t$ is diffeomorphic to the Milnor fiber of the plane curve germ $z^m - y^{n_i}$.
2) The vertical monodromy $h$ is finite of order $n_i/d$.
3) If $d < n_i$ the vertical monodromy $h$ has exactly $m$ fixed points and the action of $h$ has order $n_i/d$ on all other points.
4) Around a fixed point $h$ is a rotation of angle $-2\pi k/n_i$.

Proof of theorem 5.4. The fact that the vanishing zone is the mapping torus of $h$ is an immediate consequence of lemma 5.1 and definition 5.2.

We first prove statements 1) to 4) when $g_i(x, y) = y$. In this case, $G_t$ is the Milnor fiber of $f(a, y, z) = z^m - y^{n_i} g''(a, y)$ with $g''$ prime to $y$. Hence $f(a, y, z)$ has at $P = (a, 0, 0)$ the topological type of $z^m - y^{n_i}$. Thus point 1) is proved. A Seifert leaf of $M_t(i)$ is in the hyperplane $\{z = c\}$ with $0 \leq |c| \leq \theta$. It is parametrised by $x = ae^{iv}$ with $v \in [0, 2\pi]$. Moreover, there exists a unity $u(a)$ in $\mathbb{C}\{a\}$ such that $g''(a, y) = a^k u(a) + y(\ldots)$. Hence, the intersection points $(a, y, c)$ of $G_t$ with this Seifert leaf satisfy an equation of the following type:

$$ y^{n_i} = (a^k u(a) + y(\ldots))^{-1}(c^m - t) $$

If $y \neq 0$ the order of $h$ is equal to the order of a rotation of angle $-2\pi k/n_i$ on the parametrised leaf. This order is equal to $n_i/d$. Moreover $y = 0$ if and only if $c^m = t$. Hence, we have exactly $m$ fixed points for $h$ when $z$ is equal to each $m$-th root of $t$. The equation $(\ast)$ gives directly the angle of rotation around the $m$ fixed points.

In the general case, we consider the Puiseux expansion $(s^r, w(s))$ of $g_i(x, y)$. If we
make the substitution of variables \( x = s^r, y' = y - w(s) \) and \( f'(s, y', z) = f(s^r, y' + w(s), z) \)
we are back to the preceding case with \( f \) replaced by \( f' \).

6. When is the boundary of the Milnor fiber a lens space?

In this Section, we assume that \( f \) is irreducible. In 4.8 we have described the boundary \( L_t \) of the Milnor fiber by gluing the vanishing zone \( M_t \) to the trunk \( N_0 = \tilde{N}_0 \).

**Proposition 6.1.** 1) A connected component of \( M_t \) is never a solid torus.
2) When \( m > 2 \) a connected component of \( M_t \) has \( m \) exceptional leaves or has a basis with non-zero genus or both.

**Proof of proposition 6.1.** In theorem 5.4 we have described a connected component \( M_t(i) \) of \( M_t \) as the mapping torus of the vertical monodromy \( h \) acting on a differentiable surface \( G_t \) which is diffeomorphic to the Milnor fiber of the plane curve germ \( z^{n_i} - y^{n_i} \) with \( n_i \geq 2 \). Hence \( G_t \) is always connected and never diffeomorphic to a disc. As a consequence \( M_t(i) \) is never a solid torus.

When \( m > 2 \) the surface \( G_t \) has non-zero genus. Then:

i) If \( h \) is the identity, the basis of \( M_t(i) \) is \( G_t \) itself which has non-zero genus.

ii) If \( h \) is not the identity, we have proved in 5.4 that \( h \) has exactly \( m \) fixed points and hence \( M_t(i) \) has \( m \) exceptional leaves.

**Proposition 6.2.** If \( L_t \) is a lens space, then the trunk \( N_0 \) is a solid torus and \( M_t \) is connected with a connected boundary.

Proof of proposition 6.2. The boundary components of a Seifert manifold which is not a solid torus are incompressible. If the trunk were not a solid torus, \( L_t \) would contain incompressible torii.

**Remark 6.3.** By construction, the number of connected components of \( M_t \) is equal to the number of irreducible components of the singular locus \( \Sigma(f) \) of \( f \).

**Corollary 6.4.** If \( L_t \) is a lens space, then \( \Sigma(f) \) is an irreducible germ of curve at the origin of \( \mathbb{C}^3 \).

**Theorem 6.5.** The boundary of the Milnor fiber of a irreducible \( f(x, y, z) = z^m - g(x, y) \) is a lens space iff \( f \) is analytically equivalent to \( z^2 - x y^l \).

**Proof of theorem 6.5.** In [M-P-W] section 4 it is proved that the lens space \( L(2l, 1) \) is indeed the boundary of the Milnor fiber of \( z^2 - x y^l \).

Conversely, when \( L_t \) is a lens space, propositions 6.1 and 6.2 and corollary 6.4 imply that \( m = 2 \), that \( N_0 \) is a solid torus and that \( \Sigma(f) \) is irreducible. Hence we can write \( g(x, y) = g_1(x, y)^l \cdot g''(x, y) \) with \( g_1 \) irreducible , \( l = n_1 \geq 2, g'' \) being either reduced and prime to \( g_1 \) or a unity.
Let $\psi : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$ be the projection defined by $\psi(x, y, z) = (x, y)$. Let $S_1$ be the boundary of the polydisc $B_1 = B_2^{x} \times B_2^{y}$ with $0 < \alpha \leq \beta$ such that $B_1$ is a Milnor polydisc for $g$. Let $K_1 = S_1 \cap \{g_1 = 0\}$. By construction $\psi(M_0)$ is a tubular neighborhood of $K_1$ in $S_1$ and the closure $W$ of its complement in $S_1$ is $\psi(N_0)$.

Let us consider the Milnor fibration $\rho = g_1/|g_1| : W \to S^1$ for the plane curve germ $g_1$. Let $G_1$ be the Milnor fiber of this fibration. Then $\rho \circ \psi : N_0 \to S^1$ is a fibration with fiber $G_1'$ which is a ramified covering of $G_1$ induced by $\psi$. The ramification values of this covering are $G_1 \cap \{g'' = 0\}$. Hence the cardinality of the set of ramification values is equal to the intersection multiplicity $m_0(g_1, g'')$ of $g_1$ and $g''$ at the origin of $\mathbb{C}^2$.

As $m = 2$ this covering has degree 2. Hence

$$\chi(G_1) = 1 - \mu(g_1)$$

$$\chi(G_1') = 2(1 - \mu(g_1)) - m_0(g_1, g'')$$

As $N_0$ is a solid torus, $G_1'$ is a disjoint union of discs. The only solution for the second equation just above is $\mu(g_1) = 0$ and $m_0(g_1, g'')$ either equals 1 or 0.

When $g''$ is not a unity, i.e. $m_0(g_1, g'') = 1$, then we can choose the axis in such a way that $g_1(x, y) = y$ and $g''(x, y) = x$. As a consequence we obtain that $f(x, y, z) = z^2 - xy^l$.

Otherwise, we can choose the second axis in such a way that $g_1(x, y) = y$. Then, $f(x, y, z) = z^2 - y^l$ with $l \geq 3$ as $f$ is irreducible. Then the vertical monodromy on the identity on a surface which has non-zero genus. Then the Vanishing-zone is a Seifert manifold whose basis has non-zero genus. In this case, we never get a lens space.

End of proof of theorem 6.5.

Remark. The reducible case $z^2 - y^2$ is treated in [M-P]. It is prove that $L_t$ is then diffeomorphic to $S^2 \times S^1$.

7. Examples.

In this section we apply the method presented above to the singularities with equation $z^2 - (x^2 - y^3)y^l = 0$ ($l \geq 2$). The ingredients necessary to get the Waldhausen structure are stated in proposition 7.1 for $l$ odd and proposition 7.2 for $l$ even. The proof of these propositions is immediate from the theorems proved in section 4 and 5. The invariants of the Waldhausen structure can then be computed using the classical results recalled in section 3. From these and from [N] we can get the canonical plumbing graph.

**Proposition 7.1.** Suppose that $l$ is odd and write $l = 2\bar{l} + 1$ ($\bar{l} \geq 1$). Then:
1. The trunk is the Waldhausen manifold $Q$.
2. The vanishing zone is connected with one boundary component. More precisely, it is the mapping torus of an orientation preserving diffeomorphism $h$ of order $l$ acting on
the Milnor fiber of the plane curve singularity \( z^2 - y^l = 0 \). It has two fixed points. The rotation angle at the fixed points is equal to \( \frac{-2}{l} \pi \). On the complement of the fixed points the diffeomorphism \( h \) induces a free action of a cyclic group of order \( l \).

3. The Waldhausen \((\alpha, \beta)\) for the gluing between the trunk and the vanishing zone is equal to \((l + 3, 1)\).

**Proposition 7.2.** Suppose that \( l \) is even and write \( l = 2\bar{l} \) \((\bar{l} \geq 1)\). Then:
1. The trunk is a thickened torus \( S^1 \times S^1 \times [0, 1] \).
2. The vanishing zone is connected and has two boundary components. More precisely, it is the mapping torus of an orientation preserving diffeomorphism \( h \) of order \( \bar{l} \) acting on the Milnor fiber of the plane curve singularity \( z^2 - y^l = 0 \). Each boundary component of the fiber is invariant under \( h \). The diffeomorphism \( h \) has two fixed points. The rotation angle at each fixed point is \( \frac{-2}{\bar{l}} \pi \). On the complement of the fixed points the diffeomorphism \( h \) induces a free action of a cyclic group of order \( l \).
3. The Waldhausen \((\alpha, \beta)\) for the gluing of the two boundary components of the vanishing zone through the thickened torus is equal to \((l + 3, 1)\).

We now describe the plumbing graphs. We call "bamboo" a graph with the following shape \( o-o-...-o \). The length of a bamboo is its number of vertices. All vertices in the plumbing graphs have genus equal to zero. Most of them have Euler number equal to \(-2\). As a consequence, we only point out Euler numbers which are different from \(-2\).

To construct the plumbing graph when \( l \) is odd, we start with a bamboo of length \((l + 4)\). At one extremity, we glue two bamboos of length one. At the other extremity, we glue two bamboos of length two. The extremity of these glued bamboos has Euler number equal to \(-\bar{l}\).

To construct the plumbing graph when \( l \) is even, we start with a circuit with \((l + 3)\) vertices. At one vertex of the circuit, we glue two bamboos of length one and Euler number \(-l - 1\).

**Theorem 7.3.** The boundary \( L_t \) of the Milnor fiber of the non-isolated singularity with equation \( z^2 - (x^2 - y^3)y^l \) \((l \geq 2)\) is not diffeomorphic to the boundary of a normal surface singularity, whatever the orientation on \( L_t \) may be.

**Proof of theorem 7.3.** If it were, the quadratic form associated to the plumbing graph would be definite (negative definite if \( L_t \) is oriented as the boundary of a resolution. See [H-N-K]). But the graph contains a full subgraph which has an indefinite quadratic form: the circuit when \( l \) is even and the maximal full subgraph with \(-2\) Euler numbers when \( l \) is odd. See [H].

8. The homology of the boundary of the Milnor fiber.

**Theorem 8.1.** Let \( f(x, y, z) = z^m - x^k y^l = 0 \) be the equation of a Hirzebruch singularity.
Assume that $\gcd(m, k, l) = 1$, that $1 \leq k < l$ and that $m \geq 2$. Let $d = \gcd(k, l)$ and write $\bar{k} = k/d$ and $\bar{l} = l/d$. Then $H_1(L_t, \mathbb{Z})$ is equal to the direct sum of a free abelian group of rank $2(m-1)(d-1)$ and a torsion group. The torsion subgroup is the direct sum of $(m-1)$ cyclic factors. One of them is of order $mk\bar{l}$ and the other $(m-2)$ factors are of order $\bar{k}\bar{l}$.

The proof is a consequence of the description we give for $L_t$ in [M-P-W]. The main ingredient is the determination of the monodromy $\mathbb{Z}[t, t^{-1}]$ module associated to the vanishing zone. As we proved in [M-P-W] that $L_t$ is in fact a Seifert manifold, one can check that the result fits with [B-L-P-Z].

**Theorem 8.2.** When $l$ is odd, the group $H_1(L_t, \mathbb{Z})$ for the singularity $z^2 - (x^2 - y^3)y^l$ is cyclic of order $4l$. When $l$ is even it is the direct sum of the integers $\mathbb{Z}$ and a torsion group of order $l(l + 3)$.

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