Four-spin-exchange- and magnetic-field-induced chiral order in two-leg spin ladders

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(Dated: October 14, 2018)

We propose a mechanism of a vector chiral long-range order in two-leg spin-1/2 and spin-1 antiferromagnetic ladders with four-spin exchanges and a Zeeman term. It is known that for one-dimensional quantum systems, spontaneous breakdown of continuous symmetries is generally forbidden. Any vector chiral order hence does not appear in spin-rotationally [SU(2)]-symmetric spin ladders. However, if a magnetic field is added along the $S^z$ axis of ladders and the SU(2) symmetry is reduced to the U(1) one, the $z$ component of a vector chiral order can emerge with the remaining U(1) symmetry unbroken. Making use of Abelian bosonization techniques, we actually show that a certain type of four-spin exchange can yield a vector chiral long-range order in spin-1/2 and spin-1 ladders under a magnetic field. In the chiral-ordered phase, the $Z_2$ interchain-parity (i.e., chain-exchange) symmetry is spontaneously broken. We also consider effects of perturbations breaking the parity symmetry.

PACS numbers: 75.10.Jm,75.30.Kz,75.40.Cx,75.50.Ee

I. INTRODUCTION

Quantum spin ladder systems have attracted much attention for more than a decade. There are numerous compounds with a ladder structure. It is well known now that ladders contain considerably rich physics: for example, they provide fascinating ordered states, string order parameters detecting different topological sectors, interesting exactly solvable points, etc. In particular, physics of two-leg ladders has been greatly developed. Low-energy properties of nonfrustrated two-leg ladders have been well understood: as soon as an infinitesimal antiferromagnetic (AF) type spin-liquid state occurs, even while even if an interchain coupling with an arbitrary strength is introduced between two-leg AF Haldane-gapped chains, the gap always survives. Based on these properties of standard ladders, many physicists recently have been focusing on more realistic or frustrated ladder systems. Among them, effects of four-spin exchanges have been studied intensively now. Although it is generally difficult to understand the effects of four-spin exchanges compared with conventional two-spin ones, thanks to the discovery of a duality relation in two-leg spin-1/2 ladders, powerful analytical tools (conformal field theory, Bethe ansatz, etc.) and accurate numerical methods in (1+1) dimensions, physical properties of four-spin exchanges in ladders have been gradually elucidated. For instance, it has been shown that in spin-rotationally [SU(2)]-symmetric spin-1/2 ladder, four-spin exchanges can cause novel phases: a staggered dimer order, a scalar chiral one, and a spin liquid with a strong (but short-range) vector chirality correlation. Moreover, in SU(2)-symmetric spin-1 ladders with a biquadratic term, a certain parameter region where a spin-nematic correlation is dominant is predicted.

In this paper, motivated by the above studies of spin ladders, we investigate two-leg spin-1/2 and spin-1 AF ladders with four-spin exchanges in the presence of a magnetic field, which reduces the SU(2) symmetry to a U(1) one. For these ladders, we propose a mechanism of a vector chiral long-range order which spontaneously breaks the parity symmetry for the interchain (rung) direction. The paper is organized as follows. Throughout the paper, we apply the effective field-theory approach and focus on the weak two- and four-spin rung-coupling regime. In Sec. II, we first review simple spin-1/2 ladders without four-spin exchanges under a magnetic field using Abelian bosonization techniques and discuss the physical meaning of locking the bosonic fields. Then, based on the discussion, we prove the mechanism of a vector chiral order in the next two sections: Secs. III and IV are devoted to investigating spin-1/2 and spin-1 cases, respectively. In Sec. V, we shortly consider a few perturbation terms breaking the rung-parity symmetry: celebrated Dzyaloshinsky-Moriya (DM) interactions and a parity-breaking three-spin term. Finally, we summarize our results and then discuss them in Sec. VI.

II. BOSONIZATION FOR SPIN-1/2 LADDERS

We begin with a fundamental spin-1/2 AF ladder, whose Hamiltonian is

$$
\hat{H}_{\text{lad}} = J \sum_{n,j} S_{n,j} \cdot S_{n,j+1} + J_{\perp} \sum_j S_{1,j} \cdot S_{2,j} - H \sum_{n,j} S_{n,j}^z, \tag{1}
$$

where $S_{n,j}$ is the spin-1/2 operator on site $j$ of the $n$th chain ($n = 1, 2$), $J > 0$ ($J_{\perp}$) is the intrachain (rung)
coupling, and $H$ is a magnetic field. At the decoupled point of $J_{\perp} = 0$, the low-energy effective Hamiltonian for the $n$th chain is a Gaussian model

$$
\mathcal{H}_n^{\text{eff}} = \int dx \left[ \frac{v}{2} (\partial_x \phi_n)^2 + K (\partial_x \theta_n)^2 \right],
$$

where $x = j a_0$ ($a_0$ is the lattice constant), $\phi_n$ is the scalar boson field, and $\theta_n$ is the dual to $\phi_n$: these two fields satisfy a canonical commutation relation $[\phi_n(x), \partial_y \theta_n(y)] = i \delta(x - y)$. Quantities $v$ and $K$ denote the spin-wave velocity and the Tomonaga-Luttinger liquid (TLL) parameter, respectively. The value of $K$ ($v$) runs from 1/2 ($\pi J_0 a_0 / 2$) to 1 (0) with $H$ increasing from 0 to the upper critical field $2J$. Spin operators are also bosonized as

$$
S_{n,j}^\pm \approx M + \frac{a_0}{\sqrt{\pi}} \partial_x \phi_n + (-1)^j A_1 \sin(4\pi \phi_n + 2\pi M j) + \cdots, \quad (2a)
$$

$$
S_{n,j}^z \approx e^{i\sqrt{\pi} \theta_n} \{(-1)^j B_0 + B_1 \sin(4\pi \phi_n + 2\pi M j) + \cdots\}, \quad (2b)
$$

where $M(H) = \langle S_{n,j}^z \rangle$, and $A_0$ and $B_0$ are nonuniversal constants.\textsuperscript{16} From this bosonization framework, the low-energy theory of the ladder (1) with $|J_{\perp}| \ll J$ is written as

$$
\mathcal{H}_\text{eff} = \int dx \sum_{j=\pm} v_j \left[ K_j^{-1} (\partial_x \phi_j)^2 + K_j (\partial_x \theta_j)^2 \right] + \frac{J_{\perp}}{a_0} \left[ B_0 \cos(2\pi \theta_j) + \frac{A_1^2}{2} \cos(8\pi \phi_j) \right] - \frac{A_1^2}{2} \cos(8\pi \phi_j + 4\pi M j) + \sqrt{\frac{2}{\pi}} M J_{\perp} \partial_x \phi_j + \cdots, \quad (3)
$$

where we introduced new fields $\phi_{\pm} = (\phi_1 \pm \phi_2) / \sqrt{2}$ and $\theta_{\pm} = (\theta_1 \pm \theta_2) / \sqrt{2}$ that obey $[\phi_{\pm}(x), \partial_y \theta_{\pm}(y)] = i \delta(x - y)$. New TLL parameters $K_{\pm}$ and velocities $v_{\pm}$ are estimated as

$$
K_{\pm} \approx K \left( 1 \mp \frac{J_{\perp} a_0}{2 \pi v} \right), \quad v_{\pm} \approx v \left( 1 \pm \frac{J_{\perp} a_0}{2 \pi v} \right). \quad (4)
$$

The boson linear term $\partial_x \phi_+$ can be absorbed into the Gaussian (boson quadratic) part by the shift $\phi_+ = \phi_+ + \sqrt{2\pi} K_j M \theta_j / v_j$ which accompanies the correction of the magnetization, $\langle S_{n,j} \rangle = (1 - J_{\perp} a_0 / \pi K_j) M = \hat{M}$. Hereafter, we will again use symbols $\phi_+$ and $M$ for $\phi_+$ and $\hat{M}$, respectively.

The $U(1)$ spin rotation around the $S^z$ axis, the translation by one site for the chain direction, the site-parity operation for the same direction and the rung-parity transformation (i.e., exchange between two chains) correspond to $\theta_+ \to \theta_+ + \text{const}$, $(\phi_+, \theta_+) \to (\phi_+ + \sqrt{2\pi} M + 1/2, \theta_+ + \sqrt{2\pi})$, $(\phi_-(x), \phi_-(x), \theta_-(x)) \to (-\phi_-(x) + \sqrt{2\pi} / 2, -\phi_-(x), \theta_-(x))$ and $(\phi_-, \theta_-) \to -(\phi_-, \theta_-)$, respectively. The Hamiltonian (3) hence does not contain $\theta_+$ vertex terms, $\phi_+$ ones without spatially oscillating factors $e^{\pm i n 4\pi M_j}$, and sine terms with $\phi_\pm$ and $\theta_\pm$.

The form of Eq. (3) tells us that the low-energy physics of the ladder (1) can be described by two parts, $(\phi_+, \theta_+)$ and $(\phi_-, \theta_-)$ sectors. In the present notation, scaling dimensions of $e^{in\sqrt{2}\pi \phi_+}$ and $e^{im\sqrt{2}\pi \phi_+}$ are, respectively, $n^2/(2K_{\pm})$ and $2m^2K_{\pm}$. The $(\phi_-, \theta_-)$ sector always takes a massive spectrum due to relevant cosine terms, irrespective of the value of $M$. Since $\cos(2\pi \theta_-)$ is usually more relevant than $\cos(\sqrt{2}\pi \phi_-)$ in the ladder, namely $K_- > 1/2$,\textsuperscript{17} the dual field $\theta_-$ is pinned. For the AF-rung case, $(\theta_-) = \pm \sqrt{\pi}/2$, whereas for the FM-rung case, $(\theta_-) = 0$. On the other hand, physics of the $(\phi_+, \theta_+)$ sector depends on $M$. When $M = 0$, i.e., $H$ is smaller than the lower critical field $H_\perp$ of the ladder,\textsuperscript{18} $\cos(\sqrt{2}\pi \phi_+)$ is relevant and a spin gap thus emerges. This state is nothing but the Haldane or rung-singlet spin liquids. It is known that pinning the field $\phi_+$ corresponds to the emergence of nonlocal string orders.\textsuperscript{6} Conversely, if $M \neq 0$ ($H > H_\perp$), $\cos(\sqrt{8}\pi \phi_+ + 4\pi M j)$ is irrelevant due to the factor $e^{\pm i 4\pi M j}$ and the gap vanishes. Therefore in the case of $M = 0$, $(\phi_+, \theta_+)$ sector is described by a Gaussian model, which indeed corresponds to the field-induced TLL phase. These scenarios in the ladder (1) would be robust against small perturbations conserving symmetries of the lattice (e.g., XXZ anisotropy, further-neighbor exchanges, etc.).

As typical order parameters for two-leg ladder systems, one can find three quantities: the $z$ component of spin vector chirality $V_j = \langle S_{1,j}^z \times S_{2,j}^z \rangle^2$ and two magnetic moments $\mathcal{N}_{\pm,j} = S_{1,j}^z \pm S_{2,j}^z$. It is important to note that $V_j$ and $\mathcal{N}_{-,j}$ are odd for the rung-parity operation, while $\mathcal{N}_{+,j}$ is even. In the SU(2)-symmetric case with $H = 0$, which corresponds to the gapped spin-liquid state, expectation values of these order parameters must be zero because the presence of them means the spontaneous breakdown of the continuous SU(2) symmetry and such a symmetry breaking is generally forbidden by Mermin-Wagner-Hohenberg argument.\textsuperscript{19} The gapped spin-liquid state still remains if $H < H_\perp$ and $M = 0$. Meanwhile, in the U(1)-symmetric case with $M \neq 0$ their finite expectation values can be allowed since they do not violate the U(1) symmetry. Formula (2) enables us to represent these order parameters as

$$
V_j \approx -B_0^2 \sin(\sqrt{2}\pi \theta_j) + \cdots, \quad (5a)
$$

$$
\mathcal{N}_{-,j} \approx - (1 - 2J_{\perp} a_0 / \pi K_j) \sin(\sqrt{2}\pi \phi_+) + a_0 \sqrt{\frac{2}{\pi}} \partial_x \phi_+ + \cdots, \quad (5b)
$$

$$
\mathcal{N}_{+,j} \approx (1 - 2J_{\perp} a_0 / \pi K_j) \sin(\sqrt{2}\pi \phi_+) + 2M + a_0 \sqrt{\frac{2}{\pi}} \partial_x \phi_+ + \cdots. \quad (5c)
$$

We see that $V_j$ and $\mathcal{N}_{-,j}$ are actually odd for the rung-
parity transformation $(\phi_-, \theta_-) \rightarrow -(\phi_-, \theta_-)$, whereas $N_{+}^{-1}$ is even. The leading part of $N_{+}^{-1}$ consists of two fields $\phi_\perp$. As we mentioned above, since $\phi_\perp$ is not locked in the case of $M \neq 0$, a magnetic order with $\langle N_{+}^{-1} \rangle \neq 0$ is shown to disappear in the case. On the other hand, remarkably, the leading term of $V_j$ contains only the field $\theta_-$ that can be locked even in the U(1)-symmetric case with $M \neq 0$.

From the above discussion on order parameters, it is inferred that the vector chiral order emerges in a certain class of U(1)-symmetric ladders under a magnetic field. For the case of the simple ladder (1), we find $\langle V_j \rangle = 0$ because of $(\theta_-) = 0 \pm \sqrt{\pi}/2$. However, in other words, it is expected that if the $\theta_-$-locked position is moved even slightly by additional perturbations, the chiral order would appear. We show below that some types of four-spin exchanges are indeed such a desirable perturbation.

Here, we again remark on Eq. (5). As one saw above, for the simple spin $1/2$ ladder (1) with $M = 0$, boson fields in both $(\phi_+, \theta_\perp)$ and $(\phi_-, \theta_-)$ sectors are locked. The physical meanings of locking $\phi_\perp$ or $\theta_-$, however, have not been discussed well so far. Equation (5) clearly shows that locking $\theta_-$ can induce the vector chirality, while locking $\phi_\perp$ can do magnetic orders. Moreover, one finds that Eq. (5a) is analogous to the celebrated supercurrent formula in a Josephson junction.\textsuperscript{20}

### III. FOUR-SPIN EXCHANGES IN U(1)-SYMMETRIC SPIN-$1/2$ LADDERS

As additional four-spin exchanges to the U(1)-symmetric spin ladder (1), let us consider the following terms:

\begin{equation}
H_{||} = V_{\parallel} \sum_j \langle S_{1,j} \cdot S_{1,j+1} \rangle (S_{2,j} \cdot S_{2,j+1}),
\end{equation}

\begin{equation}
H_{rr} = V_{rr} \sum_j \langle S_{1,j} \cdot S_{2,j} \rangle (S_{1,j+1} \cdot S_{2,j+1}),
\end{equation}

\begin{equation}
H_{\times} = V_{\times} \sum_j \langle S_{1,j} \cdot S_{2,j+1} \rangle (S_{2,j} \cdot S_{1,j+1}).
\end{equation}

The well known four-spin cyclic term satisfies $V_{\parallel} = V_{rr} = -V_{\times}$.\textsuperscript{3} The bosonized spin operators $(2)$ and symmetry arguments provide the following bosonized expressions of four-spin exchanges:

\begin{equation}
H_{||} \approx V_{\parallel} \int \frac{dx}{a_0} C_{11} \cos(\sqrt{8\pi} \phi_-) + \cdots,
\end{equation}

\begin{equation}
H_{rr} \approx V_{rr} \int \frac{dx}{a_0} M^2 C_{rr,1} \cos(\sqrt{2\pi} \theta_-) + C_{rr,2} \cos(2\sqrt{2\pi} \theta_-) + C_{rr,3} \cos(\sqrt{8\pi} \phi_-) + C_{rr,4} \cos(2\sqrt{8\pi} \phi_-) + \cdots,
\end{equation}

\begin{equation}
H_{\times} \approx V_{\times} \int \frac{dx}{a_0} - M^2 C_{\times,1} \cos(\sqrt{2\pi} \theta_-) + C_{\times,2} \cos(2\sqrt{2\pi} \theta_-) + C_{\times,3} \cos(\sqrt{8\pi} \phi_-) + C_{\times,4} \cos(2\sqrt{8\pi} \phi_-) + \cdots,
\end{equation}

where $C_{11}, C_{rr},$ and $C_{\times,2}$ are nonuniversal constants, and we have written down only important terms. We note that (i) $C_{11}, C_{rr},$ and $C_{\times,2}$ with $q = 1, 2,$ and $4$ are all positive, (ii) $C_{rr}$ is the same order as $C_{\times,2}$ when $q = 1, 2,$ and $4$, and (iii) $C_{rr}$ and $C_{\times,3}$ depend strongly on the magnetization $M$. In Eq. (7a), we have used the bosonized dimerization operator $-\langle -1 \rangle^j S_{n,j} S_{n,j+1} \sim \cos(\sqrt{4\pi} a_n + 2Mq_j) + \cdots$. The $\theta_-$ vertex operators and $\phi_\perp$ ones without oscillating factors are absent in Eq. (7) because the four-spin exchanges (6) conserve all the symmetries of the ladder (1). It is remarkable that $H_{rr}$ and $H_{\times}$ include a new $\theta_-$-term, cos$(2\sqrt{2\pi} \theta_-)$, which can make the locking position of $\theta_-$ shift. From the calculation in Eq. (7), we find that a four-spin exchange term with the form

\begin{equation}
(S_{1,i} \cdot S_{2,j})(S_{1,k} \cdot S_{2,l})
\end{equation}

always contains cos$(2\sqrt{2\pi} \theta_-)$.\textsuperscript{21} The scaling dimension of cos$(2\sqrt{2\pi} \theta_-)$ is $2/K_- (\approx 2/K)$, and thus it becomes relevant when $K_- > 1$. Equation (4) and the property of $K$ suggest that the condition $K_- > 1$ is realized when the system is sufficiently close to the saturated state $(K \rightarrow 1)$ and the rung coupling is AF $(J_1 > 0)$. If we introduce perturbations such as further-neighbor exchanges for the intrachain direction, the parameter regime with $K_- > 1$ could expand. In addition, four-spin exchanges would vary the value of $K_-$ slightly.

Let us consider a spin ladder with four-spin terms $V_{\parallel}$ and $V_{\times}$ under the condition $K_- > 1$. In its bosonized Hamiltonian, the TLL of the $(\phi_+, \theta_\perp)$ sector is still stabilized by symmetries of the ladder. On the other hand, the $(\phi_-, \theta_-)$ sector is described as

\begin{equation}
H_{\phi_-, \theta_-} = \text{Gaussian part}
\end{equation}

\begin{equation}
+ v_- \int \frac{dx}{a_0^2} \left( g_1 \frac{g_1}{a_1} \cos(\sqrt{2\pi} \phi_-) + \frac{g_2}{a_2^2} \cos(2\sqrt{2\pi} \theta_-), \quad \begin{cases} g_1 \propto B_{\parallel}^2 J_\perp + M^2 (C_{rr,1} V_{rr} - C_{\times,1} V_{\times}), \\ g_2 \propto C_{rr,2} V_{rr} + C_{\times,2} V_{\times}, \end{cases} \right)
\end{equation}

where $\alpha \sim a_0$ is the short-distance cut off, $g_{1,2}$ are dimensionless coupling constants, $d_1 = 2 - 1/(2K_-)$ and $d_2 = 2 - 2/K_- = 1/(2K_-)$ and $2/K_- = 2$ are the scaling dimensions of cos$(\sqrt{2\pi} \theta_-)$ and cos$(2\sqrt{2\pi} \theta_-)$, respectively. Here we have omitted irrelevant $\phi_\perp$-vertex terms. Equation (9a) is a well-known double sine-Gordon (DSG) model.\textsuperscript{22} It is widely believed that the DSG model exhibits an Ising-type quantum phase transition. In fact, one easily finds that when $g_2$ is positively increased enough compared with $|g_1|$, the potential form of the DSG model changes from a single-well to a double-well type. The Ising transition takes place just between the single-well (disordered) phase and the double-well (ordered) one. For the ordered phase, the locking position of $\theta_-$ deviates from 0 and $\pm \sqrt{\pi}/2$. It indeed means the emergence of a finite vector chirality $\langle V_j \rangle$ and the spontaneous breaking of the $Z_2$ rung-parity symmetry. Note that even in the chiral-ordered phase, the U(1) symmetry around the $S^z$
axis ($\theta_+ \to \theta_+ + \text{const}$) is conserved. It is expected that even via the Ising transition, there is no singular behavior in the magnetization curve since the DSG model in the $(\phi_-, \theta_-)$ sector does not directly couple to the field $H$. Equation (9b) indicates that the system favors the vector chiral long-range order, for example, under the following conditions:

\begin{align}
(1) \quad V_{rr} &\gtrsim J_\perp > 0 \quad \text{and} \quad V_{\perp} \sim 0, \\
(2) \quad V_{rr} &\gtrsim J_\perp > 0 \quad \text{and} \quad V_{\perp} \sim 0, \\
(3) \quad V_{rr} &\sim V_{\perp} \gtrsim J_\perp > 0,
\end{align}

etc. It also tells us that the cyclic exchange term with $V_{rr} = -V_{\perp}$ possesses only a low possibility of causing the chiral order. From these arguments based on the bosonization, we conclude that the vector chiral long-range order emerges in the field-induced TLL, when a sufficiently strong four-spin exchange (8) is introduced in the ladder with $K_- > 1$. Furthermore, if $K_- > 1$ and $M = 0$ ($H < H_1$) simultaneously hold in a U(1)-symmetric ladder, i.e., if $K_- > 1$ holds when the $(\phi_+, \theta_+)$ sector has a gapped spectrum (although this situation is hard to occur within a realistic ladder system), it is also possible that a four-spin exchange (8) yields a gapped chiral-ordered state.

In order to see the global phase structure of the DSG model around the trivial Gaussian fixed point $(g_1, g_2) = (0, 0)$, the renormalization-group (RG) analysis is very useful. Using the operator product expansion technique,\textsuperscript{23} we obtain the following one-loop RG equation for the DSG model:\textsuperscript{24,25}

\begin{align}
\frac{d g_1}{d L} &= \left( 2 - \frac{1}{2K_-} \right) g_1 - \pi g_1 g_2, \\
\frac{d g_2}{d L} &= \left( 2 - \frac{2}{K_-} \right) g_2 - \pi g_1^2,
\end{align}

where $L$ is the scaling parameter: $\alpha \to \alpha e^L$. The RG equations have two nontrivial fixed points $(\pm \sqrt{\pi g_1^2 / \pi, d_1 / \pi}) = (\pm g_1^*, g_2^*)$. These must correspond to the above-mentioned Ising transition fixed point although their values $g_1^*, g_2^*$ would not be reliable. Analyzing the RG equations around $(g_1, g_2) = (0, 0)$ and $(\pm g_1^*, g_2^*)$, we can draw the RG flow and the ground-state phase diagram as in Fig. 1. As expected, it shows that if $g_2$ sufficiently grows, the vector chiral order appears.

The vector chirality $V_j$ must play the role of the order parameter near the Ising transition. Therefore applying known results of the two-dimensional Ising model\textsuperscript{9,26} we can predict several properties of the chirality $V_j$. For instance, the chirality increases as $(V_j) \sim (g_2 - g_2^*)^{1/8}$ in the vicinity of the critical point $(g_1^*, g_2^*)$. In addition, the chirality correlation function near the critical point is predicted to behave as follows:

\[
\lim_{j \to \infty} \langle V_j V_0 \rangle \approx \begin{cases} 
C_1 e^{-|j|/4} / \sqrt{|j|} & (g_2 < g_2^*) \\
C_0 |j|^{1/4} & (g_2 = g_2^*) \\
C_3 (g_2 - g_2^*)^{1/4} + C_2 e^{-|j|/\xi_2} / \sqrt{|j|} & (g_2 > g_2^*)
\end{cases},
\]

where $C_n$ are nonuniversal constants, and $\xi_{1(2)}$ is the correlation length in the disordered (ordered) phase of the DSG model. The result (12) would be useful in numerically detecting the chiral-ordered phase. Furthermore, if the chiral-ordered phase is realized in a ladder compound, these features of the vector chirality could be observed experimentally, in principle.

As we already mentioned in the Introduction, in spin-$\frac{1}{2}$ ladder systems, there is a duality relation that has been studied in Ref. 3. Let us here apply the duality to our spin-$\frac{1}{2}$ AF ladders with four-spin exchanges. The duality transformation is defined as $S_{1,j} = \frac{1}{2}(T_1,j + T_2,j) + T_1,j × T_2,j$ and $S_{2,j} = \frac{1}{2}(T_1,j + T_2,j) - T_1,j × T_2,j$. New operators $T_{n,j}$ obey the same algebra as spin-$\frac{1}{2}$ operators. This mapping therefore makes an arbitrary spin-$\frac{1}{2}$ ladder with $\{S_{n,j}\}$ change into a new dual ladder with $\{T_{n,j}\}$. It is remarkable that the duality transformation conserves the total spin in each rung $(S_{1,j} + S_{2,j} = T_{1,j} + T_{2,j})$ and changes the vector chirality in each rung to a Néel-type moment $-2S_{1,j} × S_{2,j} = T_{1,j} - T_{2,j}$. On the other hand, recall that a spin-$\frac{1}{2}$ ladder $\mathcal{H}_s = H_{\text{ad}}[J, J_\perp] + H_{\text{xx}}[V_\perp]$ is shown to possess the vector chiral long-range order near the saturation if the condition $K_- > 1$ is satisfied. Combining this prediction and the above properties of the duality, we can find that the dual model for $\mathcal{H}_s$ has a rung Néel order $(T_{1,j} - T_{2,j}) \neq 0$ in the vicinity of the saturation $(T_{1,j} + T_{2,j} \to 1)$. The Hamiltonian of the
dual ladder is represented as

\[ \mathcal{H}_{\text{dual}} = \left( \frac{J}{2} - \frac{V_z}{8} \right) \sum_{n,j} T_{n,j} \cdot T_{n,j+1} + J_1 \sum_{j} T_{1,j} \cdot T_{2,j} \]

\[ + \left( \frac{J}{2} + \frac{V_z}{8} \right) \sum_{j} T_{1,j} \cdot T_{2,j+1} + T_{2,j} \cdot T_{1,j+1} \]

\[ + V_\tau \sum_{j} (T_{1,j} \cdot T_{2,j})(T_{1,j+1} \cdot T_{2,j+1}) \]

\[ + (2J + \frac{V_z}{2}) \sum_{j} (T_{1,j} \cdot T_{1,j+1})(T_{2,j} \cdot T_{2,j+1}) \]

\[ + (-2J + \frac{V_z}{2}) \sum_{j} (T_{1,j} \cdot T_{2,j+1})(T_{2,j} \cdot T_{1,j+1}) \]

\[ - H \sum_{n,j} T_{n,j}^z. \] (13)

Since this model has a strong rung-coupling term and strong four-spin ones, it cannot be analyzed within the weak-rung-coupling approach in this paper.

**IV. SPIN-1 LADDERS**

Let us now turn to the two-leg spin-1 ladder (1) where spin-\(\frac{1}{2}\) operators are replaced with spin-1 ones. One will encounter a scenario similar to the spin-\(\frac{1}{2}\) case below.

In the decoupled case with zero field, each AF chain has a Haldane gap, \(\simeq 0.41J\). When the magnetic field \(H\) exceeds it, a field-induced TLL appears due to the \(S^z = 1\) magnon condensation. For this TLL phase, the effective field theory has been established.\(^{27-30}\) The effective Hamiltonian consists of the Gaussian part, which corresponds to the TLL, plus the part of \(S^z = 0\) and \(-1\) massive magnons. In contrast to the spin-\(\frac{1}{2}\) case, the TLL parameter \(K\) is larger than 1.\(^{31}\) The effective theory enables us to write the field theory form of spin operators as follows:

\[ S_{n,j}^z \sim M - a_0 \partial_x \phi_n/\sqrt{\pi} + D_1 \cos(\sqrt{4\pi} \phi_n - 2\pi M j) \]

\[ + (-1)^j[D_2 \sigma_n \cos(\sqrt{\pi} \phi_n - \pi M j)] \]

\[ + D_3 (\eta_n e^{i\sqrt{\pi} \theta_n} + \text{h.c.})] + \cdots, \] (14a)

\[ S_{n,j}^x \sim (-1)^j e^{-i\sqrt{\pi} \theta_n} \mu_n [E_2 + E_3 \cos(\sqrt{4\pi} \phi_n - 2\pi M j)] \]

\[ + E_1 e^{-i\sqrt{\pi} \theta_n} \cos(\sqrt{\pi} \phi_n - \pi M j) (\xi_{L,n} + i \xi_{R,n}) \]

\[ + \cdots, \] (14b)

where \(D_q\) and \(E_q\) are nonuniversal constants, \((\phi_n, \theta_n)\) are the massless boson fields for the \(S^z = 1\) condensed magnon, the set \((\xi_{L,n}, \sigma_n, \mu_n)\) describes the \(S^z = 0\) magnon sector, and \(\eta_n\) is the \(S^z = -1\) magnon field. For the massive \(S^z = 0\) magnon sector, \(\langle \sigma_n \rangle = 0\) and \(\langle \mu_n \rangle \neq 0\) hold. [For a more detailed explanation of Eq. (14), see Ref. 30.] If we consider only part of the TLL sector, the symmetries of the spin-1 ladder can be expressed by fields \(\phi_n\) and \(\theta_n\): a U(1) rotation around the \(S^z\) axis and the one-site translation respectively correspond to \(\theta_n \rightarrow \theta_n + \text{const}\) and \((\phi_n, \theta_n) \rightarrow (\phi_n - M \sqrt{\pi}, \theta_n + \sqrt{\pi})\).\(^{30}\) These symmetry operations are very similar to those of the spin-\(\frac{1}{2}\) case.

Substituting the formula (14) into the rung coupling of the spin-1 ladder (1) and integrating out the massive-magnon part in its effective Hamiltonian,\(^{25}\) we obtain a two-component field theory that is the same type as Eq. (3). Namely, the \((\phi_+, \theta_+\) sector again brings a TLL, and the \((\phi_-, \theta_-\) sector takes a massive spectrum with pinning \(\theta_-\) to 0 or \(\pm \sqrt{\pi}/2\).\(^{33}\) The TLL state is strongly protected by symmetries of the ladder like the case of the spin-\(\frac{1}{2}\) ladder. Formula (14) further tells us that the vector chirality is written as

\[ \mathcal{V}_j \sim \mu_1 \mu_2 \sin(\sqrt{2\pi} \theta_-) + \cdots, \] (15)

and leading parts of \(N_{\pm,j}\) are written as a function of \(\phi_\pm\). These field theory results are indeed analogous to those of the spin-\(\frac{1}{2}\) case, Eq. (5). The formula (15) shows that \(\langle \mathcal{V}_j \rangle = 0\) holds in the standard spin-1 ladder (1) with \(M \neq 0\).

For the TLL phase of the spin-1 ladder, let us add a four-spin interaction term. In this paper, we focus on a so-called biquadratic term\(^{11,12}\)

\[ \mathcal{H}_b = V \sum_j (S_{1,j} \cdot S_{2,j})^2, \] (16)

that is one of realistic, familiar four-spin terms in spin-1 systems. Employing the field theory formula (14) and symmetry arguments, and then tracing out massive magnons, we obtain the bosonized form of \(\mathcal{H}_b\), which contains \(\cos(2\sqrt{2\pi} \theta_-)\). As a result, the \((\phi_-, \theta_-\) sector including effects of \(\mathcal{H}_b\) could be described by a DSG model that has the same form as Eq. (9a). For the present case, the coupling constants \(g_{1,2}\) are evaluated as

\[ g_1 \propto \tilde{C}_1 J_\perp + M^2 \tilde{C}_2 V + \cdots, \]

\[ g_2 \propto \tilde{C}_3 V + \mathcal{O}_1 \left( \frac{V}{j} \right) V + \mathcal{O}_2 \left( \frac{J}{j} \right) J_\perp, \] (17)

where \(\tilde{C}_q\) are nonuniversal positive constants, and the second and third terms of \(g_2\) originate from the trace-out procedure of massive magnons. As we already mentioned, since the TLL parameter \(K\) of the spin-1 AF chain is larger than 1, \(K_0 > 1\) would hold in a wide parameter regime where \(\cos(2\sqrt{2\pi} \theta_-)\) is relevant. From Eq. (17) and Fig. 1, we can conclude that if \(g_2 \gg |g_1|\), namely, \(V \gg |J_\perp|\), under the condition \(K_0 > 1\), a vector chiral long-range order emerges like the spin-\(\frac{1}{2}\) case. In addition, Eq. (15) indicates that around the Ising transition, the chirality correlation function in spin-1 ladders behaves as Eq. (12). As one easily expects from Eqs. (8) and (14), besides the biquadratic term, other four-spin exchanges with the form (8) can also cause the vector chiral order.
V. PARITY-BREAKING PERTURBATIONS

From Sec. II to Sec. IV, we have studied two-leg spin ladders with the $Z_2$ rung-parity symmetry. In this section, we shortly discuss effects of rung-parity-breaking perturbations. Let us focus on the following three kinds of realistic perturbations for spin-$\frac{1}{2}$ ladders:

$$H_{\text{ADM}} = \sum_j D_u (S_{1,j} \times S_{2,j})^2,$$

(18a)

$$H_{\text{SDM}} = \sum_j (-)^j D_s (S_{1,j} \times S_{2,j})^2,$$

(18b)

$$H_{\text{chiral}} = \sum_{p,q,r} A_{pqr} S_p \cdot (S_q \times S_r),$$

(18c)

where $p, q, r$ and $r$ in Eq. (18c) represent a site index on a spin ladder, $\{n, j\}$. The uniform DM term (18a) and the staggered one (18b) often emerge in sufficiently low-crystal-symmetric magnets, and their origin is spin-orbit coupling. The third three-spin term (18c) might be unfamiliar. It originates from virtual electron-hopping processes on a triangle plaquette $\{p, q, r\}$ in a Mott-insulating ladder under an external magnetic field. The coupling constant $A_{pqr} \propto \sin(e\Phi_{pqr})$, where $e$ is the electron charge and $\Phi_{pqr}$ is the magnetic flux enclosed by the plaquette $\{p, q, r\}$. It therefore depends heavily on the direction and the strength of the magnetic field, and vanishes when the plaquette plane and the direction of the magnetic field is parallel or the magnetic field is absent. (For a more detailed discussion on the three-spin term, see Refs. 14 and 15.)

Formula (2) allows us to bosonize the DM terms as follows:

$$H_{\text{ADM}} \approx -D_u \int \frac{dx}{a_0} B_2^2 \sin(\sqrt{2\pi} \theta) + \cdots,$$

(19a)

$$H_{\text{SDM}} \approx -D_s \int \frac{dx}{a_0} (-)^j B_2^2 \sin(\sqrt{2\pi} \theta) + \cdots.$$  

(19b)

This result is obvious from the bosonized expression of vector chirality, Eq. (5a). As we already discussed, the phase field $\theta_{\text{ladd}}$ is locked at 0 or $\pm \sqrt{\pi}/2$ in the spin ladder $H_{\text{ladd}}$ due to the potential $\cos(\sqrt{2\pi} \theta)$. Equation (19a) clearly shows that as soon as a uniform DM term is added to $H_{\text{ladd}}$, the locking position of $\theta_{\text{ladd}}$ immediately varies and the vector chirality takes a finite expectation value. In contrast, since a small staggered DM term is irrelevant due to the factor $(-1)^j$, it hardly affects the low-energy physics of the ladder $H_{\text{ladd}}$.

Similarly, one can bosonize the three-spin term (18c).

VI. CONCLUSIONS AND DISCUSSIONS

We have studied effects of four-spin exchanges in two-leg spin-$\frac{1}{2}$ and spin-1 AF ladders under a magnetic field, utilizing bosonization techniques. The magnetic field reduces the spin SU(2) symmetry to the U(1) one, and as a result, the emergence of vector chirality $\langle V \rangle$ becomes possible. In Secs. III and IV, certain types of four-spin exchanges with the form (8) are indeed shown to yield the vector chiral long-range order $\langle V \rangle \neq 0$ with the $Z_2$ rung-parity symmetry spontaneously broken and the U(1) one unbroken, if the four-spin exchanges are large enough and the TLL parameter $K$ is larger than 1. Spin-1 ladders more easily satisfy the condition $K > 1$ rather than spin-$\frac{1}{2}$ ones: for the spin-$\frac{1}{2}$ case, $K > 1$ is predicted to hold only near the saturated state, while for the spin-1 case, it would hold in a wide region from the lower critical magnetic field to the upper one. An Ising-type phase transition takes place between the vector-chiral-ordered state and the standard spin liquid. Around the transition, the vector chirality correlation function follows Eq. (12). The Ising transition would not accompany any singularity of the magnetization curve, because the $\langle \phi \rangle$ sector, which brings the transition, does not directly couple to the magnetic field $H$. It is noteworthy that for spin-$\frac{1}{2}$ ladders with a four-spin cyclic exchange, the chiral order is hard to generate. For spin-$\frac{1}{2}$ ladders with four-spin exchanges, we have also applied the duality transformation. The dual spin ladder (13) is predicted to possess a Néel order for the rung direction. In Sec. V, we have briefly considered effects of rung-parity-breaking perturbations on spin ladders. In particular, we clearly find that a uniform DM interaction (18a) yields a finite expectation value of vector chirality in the ladder $H_{\text{ladd}}$ even if it is small, while a staggered DM term (18b) is irrelevant in the same ladder. Moreover, it is expected that some types of these perturbations including Eq. (18a) could play the role of the trigger of the vector chiral order generated by four-spin exchanges and a magnetic field.
Our predictions in the spin-$\frac{1}{2}$ and spin-1 cases suggest that for general spin-$S$ AF ladders, a four-spin exchange such as Eq. (8) and a magnetic field can also induce a vector chiral order. Within the bosonization framework, it is difficult to quantitatively calculate critical values of the four-spin exchanges: $V_{rr}^c$, $V_{xx}^c$, and $V^c$. Determining them accurately is an interesting future problem.

Unfortunately, the predicted vector chiral long-range order has never been detected in experiments. However, since it is suggested that effects of four-spin exchange cannot be negligible in some magnets and a magnetic field is one of the few things that we can control, there is a possibility that an experimental evidence of the chiral order is found. It is shown that a polarized neutron scattering experiment can detect a vector chiral order. In addition, if four-point spin correlation functions such as Eq. (8) and a magnetic field can also induce a hopping processes in Mott insulators.

In optical lattice have been greatly developed and they make it possible to construct several insulating states. It is also known that spin-phonon couplings can induce four-spin terms. Therefore magnets with such a strong coupling might be another candidate for a chiral-ordered state.

It is shown in a previous work that in spin-$\frac{1}{2}$ ladders, a four-spin cyclic exchange can yield a $M = 1/4$ plateau state with a Néel-type order when $J_{\perp} \gg J$. In our weak-rung-coupling approach, such a plateau phase could also appear if a vertex operator $\cos(2\sqrt{8\pi}\phi_+ + 8\pi M_j)$ in the $(\phi_+, \theta_+)$ sector becomes relevant, i.e., $K_+ < 1/4$ simultaneously hold in the weak-rung-coupling regime, we can expect that the vector chiral order, caused by four-spin exchanges, still survives in the plateau state. In this plateau regime, there is the possibility that a subleading term of $V_{\perp}$, $\sin(2\sqrt{8\pi}\theta_+ + 4\pi M_j)$, causes a staggered vector chiral order. We, however, note that the conditions $K_+ < 1/4$ and $K_- > 1$ are difficult to occur in real ladders.

Spin-$\frac{1}{2}$ ladders with four-spin exchanges possess an exactly solvable SU(4)-symmetric point, around which we can construct its effective field theory. The relationship between the SU(4)-point field theory and the bosonized one in the present paper has not been established well. Making it clear is an interesting open problem, and must contribute to a more sophisticated understanding of spin-$\frac{1}{2}$ ladders.

We finally note that besides a magnetic field, other U(1)-symmetric terms such as XXZ anisotropy would help the emergence of the chiral order in AF spin ladders with four-spin exchanges.

Acknowledgments

The author thanks Masahisa Tsuchiizu for useful comments on the DSG model. This work was supported by a Grant-in-Aid for Scientific Research (B) (No. 17340100) from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

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