On the Exel crossed product of topological covering maps

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Abstract

For dynamical systems defined by a covering map of a compact Hausdorff space and the corresponding transfer operator, the associated crossed product $C^*$-algebras $C(X) \rtimes_{\alpha,\mathbb{N}}$ introduced by Exel and Vershik are considered. An important property for homeomorphism dynamical systems is topological freeness. It can be extended in a natural way to in general non-invertible dynamical systems generated by covering maps. In this article, it is shown that the following four properties are equivalent: the dynamical system generated by a covering map is topologically free; the canonical imbedding of $C(X)$ into $C(X) \rtimes_{\alpha,\mathbb{N}}$ is a maximal abelian $C^*$-subalgebra of $C(X) \rtimes_{\alpha,\mathbb{N}}$; any nontrivial two sided ideal of $C(X) \rtimes_{\alpha,\mathbb{N}}$ has non-zero intersection with the imbedded copy of $C(X)$; a certain natural representation of $C(X) \rtimes_{\alpha,\mathbb{N}}$ is faithful. This result is a generalization to non-invertible dynamics of the corresponding results for crossed product $C^*$-algebras of homeomorphism dynamical systems.

1 Introduction

A dynamical system generated by a homeomorphism of a compact Hausdorff topological space leads to crossed product $C^*$-algebras of continuous functions on the space by the action of the additive group of integers via composition of continuous functions with the iterations of the generating homeomorphism. The interplay between topological properties of the dynamical system (or more general actions of groups) such as minimality, transitivity, freeness and others on the one hand, and properties of ideals, subalgebras and representations of the corresponding crossed product $C^*$-algebra on the other hand have been a subject of intensive investigations at least since the 1960’s. This interplay and its implications for operator

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representations of the corresponding crossed product algebras, spectral and harmonic analysis and non-commutative analysis and non-commutative geometry are fundamental for the mathematical foundations of quantum mechanics, quantum field theory, string theory, integrable systems, lattice models, quantization, symmetry analysis and, as it has become clear recently, in wavelet analysis and its applications in signal and image processing (see [7, 11, 24, 32, 33, 34, 36, 37, 51] and references therein).

In the works of Zeller-Meier [52], Effros, Hahn [17], Elliott [19], Archbold, Quigg, Spielberg [2, 38, 44], Kishimoto, Kawamura, Tomiyama [30, 31, 49] it was observed that the property of topological freeness of the dynamics for a homeomorphism, or for more general actions of groups, is closely linked with the structure of the ideals in the corresponding crossed product $C^*$-algebra and in particular with the existence of non-zero intersections between ideals and the algebra of continuous functions imbedded as a $C^*$-subalgebra into the crossed product $C^*$-algebra. The property of topological freeness has also been observed to be equivalent or closely linked to the position of the algebra of continuous functions inside the crossed product, namely whether it is a maximal abelian subalgebra or not. (For recent developments in this direction for reversible dynamical systems see also [45, 46, 47, 48].) This interplay has been considered both for the universal crossed product $C^*$-algebra and for the reduced crossed product $C^*$-algebra, the latter providing one of the important insights into the significance of those properties for representations of the crossed product.

One of the basic motivating points for this paper is the following pivotal theorem summarizing results established in [2, 19, 30, 31, 49, 52]. This theorem concerns the dynamical systems generated by a homeomorphism of a compact Hausdorff topological space, and establishes for such dynamical systems equivalences between topological freeness and properties of ideals and of the canonical subalgebra of continuous functions inside the crossed product $C^*$-algebra. It can be found for example in the book by Tomiyama [49, Theorem 5.4] in the following convenient formulation.

**Theorem 1.** The following three properties are equivalent for a compact Hausdorff space $X$ and a homeomorphism $\sigma$ of $X$:

(i) The non-periodic points of $(X, \sigma)$ are dense in $X$;

(ii) Any non-zero closed ideal $I$ in the crossed product $C^*$-algebra $C(X) \rtimes_{\alpha} \mathbb{Z}$ satisfies $I \cap C(X) \neq \{0\}$;

(iii) $C(X)$ is a maximal abelian $C^*$-subalgebra of $C(X) \rtimes_{\alpha} \mathbb{Z}$. 

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In [45, 46, 47, 48], extensions and modifications of this result and the interplay between dynamics and properties of the canonical subalgebra and ideals have been investigated for dynamical systems that are not topologically free on more general spaces than Hausdorff topological spaces, both in the context of algebraic crossed product by \( \mathbb{Z} \) and for the corresponding Banach algebra and \( C^* \)-algebra crossed products in the case of homeomorphism dynamical systems or more general dynamical systems generated by invertible map. Also in these works, this interplay has been considered on the side of representations as well as with respect to duality in the crossed product algebras.

In the recent years, substantial efforts are made in establishing broad interplay between \( C^* \)-algebras and non-invertible dynamical systems, actions of semigroups, equivalence relations, (semi-)groupoids, correspondences (see for example works by Exel, Arzumanian, Vershik [20, 21, 22, 3, 4], Deaconu [12], Renault [39, 40, 41], Adji, an Huef, Laca, Nielsen, Raeburn [1, 23], Bratteli, Dutkay, Jorgensen, Evans [5, 6, 7, 13, 14, 15, 16, 25], Dai, Larson [10], Kawamura, Kajiwara, Watatani [26, 29, 50], Ostrovskyi, Samoilenko [36], Cuntz, Krieger [9], Matsumoto [35], Eilers [18], Carlsen, Silvestrov [8] and references therein).

The notion of topological freeness for dynamical systems generated by a homeomorphism can in a natural way be extended to such in general non-invertible dynamical systems. The main result of this paper is Theorem 6, extending Theorem 1 to non-invertible dynamical systems generated by covering maps on compact Hausdorff spaces and to the corresponding crossed product \( C^* \)-algebras \( C(X) \rtimes_{\alpha, L} \mathbb{N} \) introduced by Exel and Vershik in [22]. We also add a fourth equivalent condition of faithfulness of a certain specified explicitly representation of \( C(X) \rtimes_{\alpha, L} \mathbb{N} \). More precisely, in Theorem 6, we show that the following four properties are equivalent: the dynamical system generated by a covering map is topologically free; the canonical embedding of \( C(X) \) into \( C(X) \rtimes_{\alpha, L} \mathbb{N} \) is a maximal abelian \( C^* \)-subalgebra of \( C(X) \rtimes_{\alpha, L} \mathbb{N} \); any nontrivial two sided ideal of \( C(X) \rtimes_{\alpha, L} \mathbb{N} \) has a non-zero intersection with the imbedded copy of \( C(X) \); a certain natural representation of \( C(X) \rtimes_{\alpha, L} \mathbb{N} \) is faithful. It should be notices that since \( C(X) \rtimes_{\alpha, L} \mathbb{N} \) can be constructed as a singly generated dynamical system (cf. [12], [40] and [22]), it follows from Katsura’s work in [27] and [28] (which applies to a much more general class of dynamical systems) that the first and the third condition above are equivalent. Theorem 6 also extends a theorem in [22] where it was shown that for covering maps, topological freeness of the dynamical system implies that every non-zero ideal in the crossed product \( C(X) \rtimes_{\alpha, L} \mathbb{N} \) has a nonzero intersection with the imbedded copy of \( C(X) \).
(we use this result in our proof of Theorem 6). In the course of our investigation preceding Theorem 6 we construct two representations of $C(X) times_{\alpha, \mathcal{L}} \mathbb{N}$, one having zero intersection of the kernel with the imbedded copy of $C(X)$ and the other being faithful. The condition of faithfulness of the first representation is then shown to be one of the four equivalent conditions in Theorem 6.

2 Crossed product $C^*$-algebra for dynamics of covering maps

Let $X$ be a compact Hausdorff space and let $T : X \to X$ be a covering map, i.e., $T$ is continuous and surjective and there exists for every $x \in X$ an open neighborhood $V$ of $x$ such that $T^{-1}(V)$ is a disjoint union of open sets $(U_\alpha)_{\alpha \in I}$ satisfying that $T$ restricted to each $U_\alpha$ is a homeomorphism from $U_\alpha$ onto $V$.

If we let $\alpha$, $\mathcal{L}$ and $\mathcal{L}$ be the maps from $C(X)$ to $C(X)$ given by

\[ \alpha(f) = f \circ T, \]
\[ \mathcal{L}(f)(x) = \sum_{y \in T^{-1}(x)} f(y), \]

and

\[ \mathcal{L}(f) = \mathcal{L}(1_X)^{-1}\mathcal{L}(f), \]

(that these maps are well defined and maps $C(X)$ into $C(X)$ is showed in [22]), then $\mathcal{L}$ is a transfer operator for $\alpha$.

We will as in [22] denote $\alpha(\mathcal{L}(1_X))$ by ind$(E)$, and we will for every $k \geq 1$ let

\[ I_k = \text{ind}(E)\alpha(\text{ind}(E)) \cdots \alpha^{k-1}(\text{ind}(E)). \]

Since $\mathcal{L}$ is a transfer operator for $\alpha$, one can associate the $C^*$-algebra $C(X) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ to the dynamical system $(X, T)$ (here $C(X) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ is the crossed-product associated to the triple $(C(X), \alpha, \mathcal{L})$ by Exel in [21]). Exel and Vershik have in [22] studied this $C^*$-algebra. Since $T$ is a covering map there exists a finite open covering $\{V_i\}_{i=1}^t$ of $X$ such that the restriction of $T$ to each $V_i$ is injective. Let $\{v_i\}_{i=1}^t$ be a partition of unit subordinate to $\{V_i\}_{i=1}^t$ and let

\[ u_i = (\alpha(\mathcal{L}(1_X))v_i)^{1/2}. \]

Exel and Vershik have characterized $C(X) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ by the following theorem:
Theorem 2 ([22, Theorem 9.2]). The $C^*$-algebra $C(X) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ is the universal $C^*$-algebra generated by a copy of $C(X)$ and an isometry $s$ subject to the relations

1. $sf = \alpha(f)s$,
2. $s^*fs = \mathcal{L}(f)$,
3. $1 = \sum_{i=1}^t u_isss^*u_i$,

for all $f \in C(X)$.

3 Two representations of $C(X) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$

Let $X$ be a compact Hausdorff space and let $T : X \rightarrow X$ be a covering map. Let $H$ be a Hilbert space with an orthonormal basis $(e_x)_{x \in X}$ indexed by $X$. For $f \in C(X)$, let $M_f$ be the bounded operator on $H$ defined by

$$M_f(e_x) = f(x)e_x, \quad x \in X,$$

and let $S$ be the bounded operator on $H$ defined by

$$S(e_x) = (\mathcal{L}(1_X)(x))^{-1/2} \sum_{y \in T^{-1}(\{x\})} e_y, \quad x \in X.$$ 

It is easy to check that we for all $k \geq 1$ and all $x \in X$ have

$$S^k(e_x) = \sum_{y \in (T^k)^{-1}(\{x\})} (I_k(y))^{-1/2} e_y,$$

and

$$(S^*)^k(e_x) = (I_k(x))^{-1/2} e_{T^k(x)}.$$ 

Proposition 3. Let $X$ be a compact Hausdorff space, let $T : X \rightarrow X$ be a covering map and let $H$, $M_f$ and $S$ be as above. Then there exists a representation $\psi$ of $C(X) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ on $H$ such that $\psi(f) = M_f$ for every $f \in C(X)$ and $\psi(s) = S$.

We furthermore have that $\ker(\psi) \cap C(X) = \{0\}$.

Proof. Let $f \in C(X)$. It is straightforward to check that $SM_f = M_{\alpha(f)}S$, that $S^*M_fS = M_{\mathcal{L}(f)}$, and that $\sum_{i=1}^t M_uSS^*M_u = M_{1_X}$, so the existence of $\psi$ follows from Theorem 2.

Let $f \in C(X)$, and assume that there is an $x \in X$ such that $f(x) \neq 0$. Then $M_f e_x \neq 0$, so $f \notin \ker(\psi)$, which proves that $\ker(\psi) \cap C(X) = \{0\}$. □
Let \( \widetilde{H} \) be a Hilbert space with an orthonormal basis \((e_{(x,n)})_{(x,n)\in X\times\mathbb{Z}}\) indexed by \( X \times \mathbb{Z} \). For \( f \in C(X) \), let \( \widetilde{M}_f \) be the bounded operator on \( \widetilde{H} \) defined by
\[
\widetilde{M}_f(e_{(x,n)}) = f(x)e_{(x,n)}, \quad (x,n) \in X \times \mathbb{Z},
\]
and let \( \widetilde{S} \) be the bounded operator on \( \widetilde{H} \) defined by
\[
\widetilde{S}(e_{(x,n)}) = (\mathcal{L}(1_X)(x))^{-1/2} \sum_{y \in T^{-1}(x)} e_{(y,n+1)}, \quad (x,n) \in X \times \mathbb{Z}.
\]
It is easy to check that we for all \( k \geq 1 \), all \( n \in \mathbb{Z} \) and all \( x \in X \) have
\[
\widetilde{S}^k(e_{(x,n)}) = \sum_{y \in (T^k)^{-1}(x)} (I_k(y))^{-1/2} e_{(y,n+k)},
\]
and
\[
(\widetilde{S}^*)^k(e_{(x,n)}) = (I_k(x))^{-1/2} e_{(T^k(x),n-k)}.
\]

**Proposition 4.** Let \( X \) be a compact Hausdorff space, let \( T : X \rightarrow X \) be a covering map and let \( \widetilde{H} \), \( \widetilde{M}_f \) and \( \widetilde{S} \) be as above. Then there exists a faithful representation \( \widetilde{\psi} \) of \( C(X) \rtimes_{\alpha,\mathbb{L}} \mathbb{N} \) on \( \widetilde{H} \) such that \( \widetilde{\psi}(f) = \widetilde{M}_f \) for every \( f \in C(X) \) and \( \widetilde{\psi}(s) = \widetilde{S} \).

**Proof.** Let \( f \in C(X) \). It is straightforward to check that \( \widetilde{S}\widetilde{M}_f = \widetilde{M}_{\alpha(f)}\widetilde{S} \), that \( \widetilde{S}^*\widetilde{M}_f\widetilde{S} = \widetilde{M}_{\mathbb{L}(f)} \), and that \( \sum_{i=1}^t \widetilde{M}_u_i \widetilde{S}\widetilde{S}^*\widetilde{M}_u_i = \widetilde{M}_{1_X} \), so the existence of \( \widetilde{\psi} \) follows from Theorem 2.

Let \( z \in \mathbb{T} \). Let \( U_z \) be the bounded operator on \( \widetilde{H} \) defined by
\[
U_z e_{(x,n)} = z^n e_{(x,n)}, \quad (x,n) \in X \times \mathbb{Z}.
\]
We then have that \( U_z\widetilde{M}_f U_z^* = \widetilde{M}_f \) for every \( f \in C(X) \), and that \( U_z\widetilde{S} U_z^* = z\widetilde{S} \). Thus
\[
(D, z) \mapsto U_z DU_z^*, \quad D \in B(\widetilde{H}), \ z \in \mathbb{T}
\]
is an action of the circle group on \( B(\widetilde{H}) \) such that \( \widetilde{\psi} \) is covariant relative to this action and the gauge action on \( C(X) \rtimes_{\alpha,\mathbb{L}} \mathbb{N} \).

Let \( f \in C(X) \), and assume that there is an \( x \in X \) such that \( f(x) \neq 0 \). Then \( \widetilde{M}_f e_{(x,0)} \neq 0 \), so \( f \notin \ker(\widetilde{\psi}) \), which proves that \( \widetilde{\psi} \) is faithful on \( C(X) \). Thus it follows from [22, Theorem 4.2] that \( \widetilde{\psi} \) is faithful. \( \square \)

Recall from [22, Theorem 8.9] that there exists a conditional expectation \( G : C(X) \rtimes_{\alpha,\mathbb{L}} \mathbb{N} \rightarrow C(X) \) such that \( G(f s^k(s^*)^l g) = \delta_{k,l} f I_n^{-1} g \) for \( f, g \in C(X) \) and \( k, l \in \mathbb{N} \), where \( \delta \) is the Kronecker symbol.
Lemma 5. Let $X$ be a compact Hausdorff space, let $T : X \to X$ be a covering map and let $\tilde{\psi}$ be the representation of Proposition 4. Then we have for all $b \in C(X) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$:

1. $\langle \tilde{\psi}(b)e_{(x,n)}, e_{(x,n)} \rangle = G(b)(x)$ for all $(x, n) \in X \times \mathbb{Z}$.

2. $b \in C(X)$ if and only if $\langle \tilde{\psi}(b)e_{(x,m)}, e_{(y,n)} \rangle = 0$ for all $(x, m), (y, n) \in X \times \mathbb{Z}$ with $(x, m) \neq (y, n)$.

3. If $(x, m), (y, n) \in X \times \mathbb{Z}$ and $\langle \tilde{\psi}(b)e_{(x,m)}, e_{(y,n)} \rangle \neq 0$, then there exist $k, l \in \mathbb{N}$ such that $k + m = l + n$ and such that $T^l(x) = T^k(y)$, and there exist an open neighbourhood $U$ of $x$ and an open neighbourhood $V$ of $y$ such that if $x' \in U$, $y' \in V$ and $T^l(x') = T^k(y')$, then $\langle \tilde{\psi}(b)e_{(x',m)}, e_{(y',n)} \rangle \neq 0$.

Proof. (1): Fix $(x, n) \in X \times \mathbb{Z}$. It follows from [22, Proposition 2.3] that the set of elements of $C(X) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ which can be written as a finite sum of elements of the form $fs^k(s^*)^lg$ where $f, g \in C(X)$ and $k, l \in \mathbb{N}$, is dense in $C(X) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$. Since both $b \mapsto \langle \tilde{\psi}(b)e_{(x,n)}, e_{(x,n)} \rangle$ and $b \mapsto G(b)(x)$ are linear and continuous maps, it follows that it is enough to show that if $f, g \in C(X)$ and $k, l \in \mathbb{N}$, then $\langle \tilde{\psi}(fs^k(s^*)^lg)e_{(x,m)}, e_{(x,n)} \rangle = \delta_{k,l}f(x)I_k^{-1}(x)g(x)$, and it is straightforward to check that this is the case.

(2): It is clear that if $f \in C(X)$, then $\langle \tilde{\psi}(f)e_{(x,m)}, e_{(y,n)} \rangle = 0$ for all $(x, m), (y, n) \in X \times \mathbb{Z}$ with $(x, m) \neq (y, n)$.

Let $b \in C(X) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ and assume that $\langle \tilde{\psi}(b)e_{(x,m)}, e_{(y,n)} \rangle = 0$ for all $(x, m), (y, n) \in X \times \mathbb{Z}$ with $(x, m) \neq (y, n)$. It then follows from what we have just shown that $\langle \tilde{\psi}(b)e_{(x,m)}, e_{(y,n)} \rangle = \langle \tilde{\psi}(G(b))e_{(x,m)}, e_{(y,n)} \rangle$ for all $(x, m), (y, n) \in X \times \mathbb{Z}$, and thus that $\tilde{\psi}(b) = \tilde{\psi}(G(b))$. Since $\tilde{\psi}$ is faithful, it follows that $b = G(b) \in C(X)$.

(3): Let $(x, m), (y, n) \in X \times \mathbb{Z}$ and assume that $\varepsilon := \langle \tilde{\psi}(b)e_{(x,m)}, e_{(y,n)} \rangle \neq 0$. It follows from [22, Proposition 2.3] that there exists a finite subset $F$ of $C(X) \times \mathbb{N} \times \mathbb{N} \times C(X)$ such that

$$\left\| b - \sum_{(f,k,l,g) \in F} fs^k(s^*)^lg \right\| < \varepsilon/3.$$ 

If $(f, k, l, g) \in F$, then we have

$$\tilde{\psi}(b)e_{(x,m)} = \sum_{y' \in (T^k)^{-1}(T^l((x)))} f(y')(I_k(y')I_l(y'))^{-1/2}g(x)e_{(y',m-l+k)},$$

where $\tilde{\psi}$ is the representation of Proposition 2.3.
so either do we have that $y \in (T^k)^{-1}(T^l(\{x\}))$ and $n = m - l + k$, and thus that $k + m = l + n$ and $T^l(x) = T^k(y)$, or we have $\langle \tilde{\psi}(s^k(s^*)^l \varphi) e_{(x,m)}, e_{(y,n)} \rangle = 0$.

Let $F' := \{(f,k,l,g) \in F \mid k + m = l + n$ and $T^l(x) = T^k(y)\}$. Since $\sum_{(f,k,l,g) \in F} \langle \tilde{\psi}(s^k(s^*)^l \varphi) e_{(x,m)}, e_{(y,n)} \rangle \neq 0$, it follows that $F' \neq \emptyset$. Let $r$ be the number of elements of $F'$, let $k := \max\{k' \mid (f',k',l',g') \in F'\}$ and let $l := k + m - n$. Choose for each $(f',k',l',g') \in F'$ an open neighbourhood $U_{(f',k',l',g')}$ of $x$ and an open neighbourhood $V_{(f',k',l',g')}$ of $y$ such that for each $x' \in U_{(f',k',l',g')}$ exists a unique $y' \in V_{(f',k',l',g')}$ such that $T^l(x') = T^k(y')$, and such that

$$|f(y)(I_k(y)I_l(y))^{-1/2}g(x) - f(y')(I_k(y')I_l(y'))^{-1/2}g(x')| < \varepsilon/(3r)$$

for $x' \in U_{(f',k',l',g')}$ and $y' \in V_{(f',k',l',g')}$ with $T^l(x') = T^k(y')$.

Let $U = \bigcap_{(f',k',l',g') \in F'} U_{(f',k',l',g')}$ and $V = \bigcap_{(f',k',l',g') \in F'} V_{(f',k',l',g')}$. Then $U$ is an open neighbourhood of $x$, $V$ is an open neighbourhood of $y$, and if $x' \in U$, $y' \in V$ and $T^l(x') = T^k(y')$, then we have

$$|\langle \tilde{\psi}(b) e_{(x',m)}, e_{(y',n)} \rangle| \geq \varepsilon - |\langle \tilde{\psi}(b) e_{(x,m)}, e_{(y,n)} \rangle - \langle \tilde{\psi}(b) e_{(x,m)}, e_{(y,n)} \rangle|$$

$$\geq \varepsilon - \left| \langle \tilde{\psi}(b) e_{(x,m)}, e_{(y,n)} \rangle - \left\langle \sum_{(f,k,l,g) \in F} \tilde{M}_f S^k \tilde{S}^* \tilde{M}_g e_{(x,m)}, e_{(y,n)} \right\rangle \right|$$

$$- \left| \left\langle \sum_{(f,k,l,g) \in F} \tilde{M}_f S^k \tilde{S}^* \tilde{M}_g e_{(x',m)}, e_{(y,n)} \right\rangle \right|$$

$$- \left| \left\langle \sum_{(f,k,l,g) \in F} \tilde{M}_f S^k \tilde{S}^* \tilde{M}_g e_{(x',m)}, e_{(y',n)} \right\rangle - \langle \tilde{\psi}(b) e_{(x',m)}, e_{(y',n)} \rangle \right|$$

$$\geq \varepsilon - \varepsilon/3 - \sum_{(f,k,l,g) \in F'} \left| f(y)(I_k(y)I_l(y))^{-1/2}g(x) \right|$$

$$- f(y')(I_k(y')I_l(y'))^{-1/2}g(x') \right| - \varepsilon/3 > 0.$$

\[ \square \]

4 The main theorem

Let $X$ be a compact Hausdorff space and let $T : X \to X$ be a covering map. As in [22], we say that $(X,T)$ is topological free if for every pair of
nonnegative integers \((k, l)\) with \(k \neq l\), the set \(\{x \in X \mid T^k(x) = T^l(x)\}\) has empty interior. If the space \(X\) is infinite, and we consider dynamical systems generated by covering maps, then the class of topologically free systems contains the subclass of irreducible dynamical systems, defined as follows (see [22, Proposition 11.1]). Two points \(x, y \in X\) are said to be trajectory-equivalent (see e.g. [3]) when there are \(n, m \in \mathbb{N}\) such that \(T^n(x) = T^m(y)\). We will denote this by \(x \sim y\). A subset \(Y \subseteq X\) is said to be invariant if \(x \sim y \in Y\) implies that \(x \in Y\). It is easy to see that \(Y\) is invariant if and only if \(T^{-1}(Y) = Y\). The covering map \(T\) and the dynamical system it generates is said to be irreducible when there is no closed (equivalently open) invariant set other than \(\emptyset\) and \(X\) (see e.g. [3]). Notice that irreducibility is weaker than the condition of minimality defined in [12]. In [22], it was shown that, for dynamical systems generated by covering maps of infinite spaces, irreducibility of the system is equivalent to simplicity of \(C(X) \rtimes \mathbb{N}\).

**Theorem 6.** Let \(X\) be a compact Hausdorff space, and let \(T : X \to X\) be a covering map. Then the following are equivalent:

1. \((X, T)\) is topological free.
2. Every nontrivial ideal of \(C(X) \rtimes \mathbb{N}\) has a nontrivial intersection with \(C(X)\).
3. The representation of Proposition 3 is faithful.
4. \(C(X)\) is a maximal abelian C*-subalgebra of \(C(X) \rtimes \mathbb{N}\).

**Proof.** (1) \(\Rightarrow\) (2): That (1) implies (2) is proven in [22, Theorem 10.3].

(2) \(\Rightarrow\) (3): Assume that (2) holds and let \(\psi\) be the representation of Proposition 3. It follows from Proposition 3 that \(\ker(\psi) \cap C(X) = \{0\}\), so \(\ker(\psi) = \{0\}\). Thus (3) holds.

(1) \(\Rightarrow\) (4): Assume that \((X, T)\) is topological free. Let \(b \in C(X) \rtimes \mathbb{N}\) and assume that \(bf = fb\) for all \(f \in C(X)\). Let \(\tilde{\psi}\) be the representation of Proposition 4. We want to show that \(b \in C(X)\). It follows from Lemma 5(2) that it is enough to show that \(\langle \tilde{\psi}(b) e_{(x,m)}, e_{(y,n)} \rangle = 0\) for all \((x, m), (y, n) \in X \times \mathbb{Z}\) with \((x, m) \neq (y, n)\).

Fix \((x, m), (y, n) \in X \times \mathbb{Z}\) with \((x, m) \neq (y, n)\). Assume first that \(x \neq y\). Choose \(f \in C(X)\) such that \(f(x) \neq 0\) and \(f(y) = 0\). We then have that

\[
f(x) \langle \tilde{\psi}(b) e_{(x,m)}, e_{(y,n)} \rangle = \langle \tilde{\psi}(bf) e_{(x,m)}, e_{(y,n)} \rangle = \langle \tilde{\psi}(fb) e_{(x,m)}, e_{(y,n)} \rangle = \langle \tilde{\psi}(b) e_{(x,m)}, \tilde{M} e_{(y,n)} \rangle = 0,
\]

where \(\tilde{M}\) is the shift operator.
so \( \langle \tilde{\psi}(b)e_{(x,m)}, e_{(y,n)} \rangle = 0 \) as wanted.

Assume then that \( x = y \) and \( m \neq n \). Assume that \( \langle \tilde{\psi}(b)e_{(x,m)}, e_{(y,n)} \rangle \neq 0 \). It follows from Lemma 5(3) that there exist \( k, l \in \mathbb{N} \) such that \( T^l(x) = T^k(x) \) and \( k + m = l + n \), and two open neighbourhoods \( U \) and \( V \) of \( x \) such that if \( x' \in U \), \( y' \in V \) and \( T^l(x') = T^k(y') \), then \( \langle \tilde{\psi}(b)e_{(x',m)}, e_{(y',n)} \rangle \neq 0 \). We must have that \( k \neq l \), and since we are assuming that \( (X,T) \) is topological free, it follows that there exist \( x' \in U \) and \( y' \in V \) such that \( T^l(x') = T^k(y') \) and \( x' \neq y' \). But it then follows from what we have just proved that \( \langle \tilde{\psi}(b)e_{(x',m)}, e_{(y',n)} \rangle \) is both zero and non-zero, so we have a contradiction.

Thus \( \langle \tilde{\psi}(b)e_{(x',m)}, e_{(y',n)} \rangle = 0 \) for all \( (x,m), (y,n) \in X \times \mathbb{Z} \) with \( (x,m) \neq (y,n) \). Hence \( b \in C(X) \), which proves that (4) holds.

(4) \( \implies \) (1) and (3) \( \implies \) (1): Assume that \( (X,T) \) is not topological free. We will then show that the representation of Proposition 3 is not faithful, and that \( C(X) \) is not maximal abelian. Since \( (X,T) \) is not topological free, there exist \( k \neq l \) such that \( \{ x \in X \mid T^k(x) = T^l(x) \} \) has non-empty interior. Choose a non-empty open subset \( U \) of \( \{ x \in X \mid T^k(x) = T^l(x) \} \) such that \( T^k \) and \( T^l \) are injective on \( U \), and let \( x_0 \in U \). Choose an \( f \in C(X) \) with \( \text{supp } f \subseteq U \) and \( f(x_0) \neq 0 \).

Let \( \{ V_i \}_{i=1}^t \) be a finite open covering of \( X \) such that the restriction of \( T \) to each \( V_i \) is injective, let \( \{ v_i \}_{i=1}^t \) be a partition of unit subordinate to \( \{ V_i \}_{i=1}^t \), let \( Z := \{ 1, 2, \ldots, t \} \), let \( u_i := (\alpha(L_i(1)))v_i \) for \( i \in Z \), and let

\[
\begin{align*}
\psi_j := u_{j_0} & \alpha(u_{j_1}) \alpha^2(u_{j_2}) \cdots \alpha^{i-1}(u_{j_{i-1}})
\end{align*}
\]

for \( j = (j_0, j_1, j_2, \ldots, j_{i-1}) \in Z^i \). It then follows from Proposition 7.4 and Proposition 8.2 of [22] that \( \sum_{j \in Z^i} u_j \alpha(s^j) = 1 \).

Let \( h \in C(X) \). If \( j \in Z^i \), then we have \( f \alpha^k(L^i(hfu_j)) = hf \alpha^k(L^i(fu_j)) \).

It follows that we have

\[
\begin{align*}
fs^k(s^j)^i & fh = fs^k(s^j)^i fh = fs^k(s^j)^i fh \sum_{j \in Z^i} u_j \alpha(s^j)^{i-1}u_j^*
\end{align*}
\]

\[
= fs^k \sum_{j \in Z^i} L^i(hfu_j)(s^j)^iu_j^* = \sum_{j \in Z^i} f \alpha^k(L^i(hfu_j))s^k(s^j)^iu_j^*
\]

\[
= \sum_{j \in Z^i} hfu_j(s^j)^iu_j^* = hf \alpha^k(L^i(fu_j))s^k(s^j)^iu_j^* = hf s^k(s^j)^i fh \sum_{j \in Z^i} u_j \alpha(s^j)^{i-1}u_j^*
\]

\[
= hf s^k(s^j)^i fh.
\]

Thus \( fs^k(s^j)^i f \in C(X)' \).

Let \( \tilde{\psi} \) be the representation of Proposition 4. It is easy to check that

\[
\langle \tilde{\psi}(fs^k(s^j)^i f)e_{(x_0,l)}, e_{(x_0,k)} \rangle = f(x_0) \langle I_k(x_0)I_l(x_0) \rangle^{-1/2} f(x_0) \neq 0,
\]

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and that $\langle \tilde{\psi}(h) e_{(x,0)}, e_{(x,0)} \rangle = 0$ for all $h \in C(X)$. It follows that $f s^k(s^*)^l f \notin C(X)$. This shows that $C(X)' \setminus C(X)$ is non-empty, and thus that $C(X)$ is not a maximal abelian $C^*$-subalgebra of $C(X) \rtimes_{\alpha,\mathbb{C}} \mathbb{N}$. Hence $\neg(1) \implies \neg(4)$ or equivalently $(4) \implies (1)$.

Let $\psi$ be the representation of Proposition 3. It is easy to check that we for all $x, y \in X$ have $\langle \psi(f s^k(s^*)^l f) e_x, e_y \rangle = \delta_{x,y} f(x)(I_k(x)I_l(x))^{-1/2} f(x)$, and thus that $\psi(f s^k(s^*)^l f) = \psi(f(I_k I_l)^{-1/2} f)$. We have already seen that $f s^k(s^*)^l f \notin C(X)$, and since $f(I_k I_l)^{-1/2} f \in C(X)$, it follows that $\psi$ is not faithful. Hence $\neg(1) \implies \neg(3)$ or equivalently $(3) \implies (1)$. \hfill $\square$

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