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ORIGINAL ARTICLE

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Abstract
We study the unique existence of weak solutions for initial boundary value problems associated with different class of fractional diffusion equations including variable order, distributed order, and multiterm fractional diffusion equations. So far, different definitions of weak solutions have been considered for these class of problems. This includes definition of solutions in a variational sense and definition of solutions from properties of their Laplace transform in time. The goal of this article is to unify these two approaches by showing the equivalence of these two definitions. Such a property allows also to show that the weak solutions under consideration combine the advantages of these two classes of solutions, which include representation of solutions by a Duhamel type of formula, suitable properties of Laplace transform of solutions, resolution of the equation in the sense of distributions, and explicit link with the initial condition.

Keywords
distributed order fractional diffusion equations, multiterm fractional diffusion equations, representation of solutions, variable order fractional diffusion equations, well-posedness

MSC (2020)
35R11, 35B30, 35R05

1 INTRODUCTION

1.1 Settings

Let $\Omega$ be a bounded and connected open subset of $\mathbb{R}^d$, $d \geq 2$, with Lipschitz boundary $\partial \Omega$. Let $a := (a_{i,j})_{1 \leq i,j \leq d} \in L^\infty(\Omega; \mathbb{R}^{d^2}) \cap H^1(\Omega; \mathbb{R}^{d^2})$ be symmetric, that is,

$$a_{i,j}(x) = a_{j,i}(x), \quad x \in \Omega, \quad i, j = 1, \ldots, d,$$

and fulfill the ellipticity condition: There exists a constant $c > 0$ such that

$$\sum_{i,j=1}^{d} a_{i,j}(x)\xi_i \xi_j \geq c|\xi|^2, \quad \text{for a.e.} x \in \Omega, \quad \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d. \quad (1.1)$$
Assume that \( q \in L^2(\Omega) \) is nonnegative and define the operator \( A \) by

\[
A u(x) := -\sum_{i,j=1}^{d} \partial_{x_i} \left( a_{i,j}(x) \partial_{x_j} u(x) \right) + q(x) u(x), \ x \in \Omega.
\]

I set also \( \rho \in L^\infty(\Omega) \) obeying

\[
0 < c_0 \leq \rho(x) \leq C_0 < +\infty, \ x \in \Omega. \tag{1.2}
\]

From now on, I set \( \mathbb{R}_+ = (0, \infty) \) and, for any Banach space \( X \) and all \( p \in [1, +\infty] \), we denote by \( L^p_{up-loc}(\mathbb{R}_+; X) \) the set of measurable functions \( f \) on \( \mathbb{R}_+ \) taking values in \( X \) and satisfying

\[
\forall T > 0, \ f \big|_{(0,T)} \in L^p(0,T; X).
\]

In the same way, I denote by \( W^{1,p}_{up-loc}(\mathbb{R}_+; X) \) the set of functions \( f \in L^p_{up-loc}(\mathbb{R}_+; X) \) satisfying

\[
\forall T > 0, \ f \big|_{(0,T)} \in W^{1,p}(0,T; X),
\]

and I introduce the function \( K \in L^1_{up-loc}(\mathbb{R}_+; L^\infty(\Omega)) \cap C^\infty(\mathbb{R}_+; L^\infty(\Omega)) \) satisfying the following condition:

\[
\inf\{ \tau > 0 : e^{-\tau} K \in L^1(\mathbb{R}_+; L^\infty(\Omega)) \} = 0. \tag{1.3}
\]

Then, I define the operator \( I_K \) by

\[
I_K g(t,x) = \int_0^t K(t-s,x)g(s,x)ds, \ g \in L^1_{up-loc}(\mathbb{R}_+; L^2(\Omega)), \ x \in \Omega, \ t \in \mathbb{R}_+.
\]

I introduce the Caputo and Riemann–Liouville fractional derivative with kernel \( K \) as follows:

\[
\partial^K_\gamma g(t,x) = I_K \partial_\gamma g(t,x), \quad D^K_\gamma g(t,x) = \partial_\gamma I_K g(t,x), \ g \in W^{1,1}_{up-loc}(\mathbb{R}_+; L^2(\Omega)), \ x \in \Omega, \ t \in \mathbb{R}_+.
\]

In this article, I consider the following initial boundary value problem (IBVP in short):

\[
\begin{aligned}
(\phi(x) \partial^K_\gamma + A) u(t,x) &= F(t,x), \quad (t,x) \in \mathbb{R}_+ \times \Omega, \\
u(t,x) &= 0, \quad (t,x) \in \mathbb{R}_+ \times \partial \Omega, \\
u(0,x) &= u_0(x), \quad x \in \Omega.
\end{aligned} \tag{1.4}
\]

Namely, for different values of the kernel \( K \), corresponding to variable order, distributed order, and multiterm fractional diffusion equations, I prove the existence of weak solutions of (1.4) in the sense of a definition involving the Riemann–Liouville fractional derivative \( D^K_\gamma \). In addition, I would like to prove that this unique weak solution is described by a suitable Duhamel type of formula and its Laplace transform in time has the expected properties for such equations. Our goal is to unify the two different main approaches considered so far for defining solutions of (1.4). That is, the definition of solutions of (1.4) in a variational sense involving Riemann–Liouville fractional derivative (see, e.g., [6, 10, 42]) and the definition of solutions in terms of Laplace transform (see, e.g., [19–21, 27]).

### 1.2 | Definitions of weak and Laplace-weak solutions

Before stating our results, I give the definition of solutions under consideration in this article. Inspired by [6, 42], we give the definition of weak solutions of the IBVP (1.4) as follows.
Definition 1.1 (Weak solution). Let the coefficients in (1.4) satisfy (1.1)–(1.2). I say that \( u \in L_{u_p}^1(\mathbb{R}_+, L^2(\Omega)) \) is a weak solution to (1.4) if it satisfies the following conditions.

(i) The following identity,
\[
\rho(x)D^K_t [u - u_0](t, x) + Au(t, x) = F(t, x), \quad t \in \mathbb{R}_+, \ x \in \Omega,
\]
holds true in the sense of distributions in \( \mathbb{R}_+ \times \Omega \).

(ii) We have \( I^*_{K}[u - u_0] \in W^{1,1}_{u_p} (\mathbb{R}_+, D'(\Omega)) \) and the following initial condition,
\[
\langle I^*_{K}[u - u_0](0), \varphi \rangle_{D'(\Omega), C_0^\infty(\Omega)} = 0, \quad \varphi \in C_0^\infty(\Omega),
\]
is fulfilled.

(iii) We have
\[
\tau_0 = \inf \{ \tau > 0 : e^{-\tau t}u \in L^1(\mathbb{R}_+, L^2(\Omega)) \} < \infty
\]
and there exists \( \tau_1 \geq \tau_0 \) such that for all \( p \in \mathbb{C} \) satisfying \( \Re p > \tau_1 \), we have
\[
\hat{u}(p, \cdot) = \int_0^\infty e^{-pt}u(t, \cdot)dt \in H^1_0(\Omega).
\]

Remark 1. The conditions in Definition 1.1 describe the different aspects of the IBVP (1.4). Namely, condition (i) is associated with the equation in (1.4), condition (ii) describes the link with the initial condition of (1.4), and condition (iii) gives the boundary condition of (1.4). Let us also observe that in the spirit of the works [6, 42] (see also [24, 25, 34]), we use in Equation (1.5) the fact that for \( u \in W^{1,1}_{u_p} (\mathbb{R}_+, L^2(\Omega)) \), we have \( \partial^K_t u = D^K_t [u - u(0, \cdot)] \). In that sense, the expression \( \partial^K_t u \) in (1.4) can be defined in a more general context by considering instead the expression \( \partial^K_t u \).

In this article, I study the unique existence of a weak solution of the IBVP (1.4) in the sense of Definition 1.1 in three different contexts:

(1) Variable order fractional diffusion equations where for \( \alpha \in L^\infty(\Omega) \) satisfying
\[ 0 < \alpha_0 \leq \alpha(x) \leq \alpha_M < 1, \quad x \in \Omega, \]
I fix
\[
K(t, x) = \frac{t^{-\alpha(x)}}{\Gamma(1 - \alpha(x))}, \quad t \in \mathbb{R}_+, \ x \in \Omega.
\]  
(1.7)

(2) Distributed order fractional diffusion equations where for a nonnegative weight function \( \mu \in C([0, 1]) \), I define
\[
K(t, x) = \int_0^1 \mu(\alpha) \frac{t^{-\alpha}}{\Gamma(1 - \alpha)} d\alpha, \quad t \in \mathbb{R}_+, \ x \in \Omega.
\]  
(1.8)

(3) Multiterm fractional diffusion equations where, for \( N \in \mathbb{N} \), \( 0 < \alpha_1 < \ldots < \alpha_N < 1 \) and for \( \rho_j \in L^\infty(\Omega), j = 1, \ldots, N \), satisfying (1.2) with \( \rho = \rho_j \), I fix
\[
K(t, x) = \sum_{j=1}^N \rho_j(x) \frac{t^{-\alpha_j}}{\Gamma(1 - \alpha_j)}, \quad t \in \mathbb{R}_+, \ x \in \Omega.
\]  
(1.9)

Let us also recall an alternative definition of weak solutions of (1.4) defined in terms of Laplace transform (see, e.g., [19–21, 27]). In order to distinguish these two definitions of solutions, in the remaining part of this article, this class of weak solutions will be called Laplace-weak solutions. From now on and in all the remaining parts of this article, I denote by \( J \)
the set of functions $F \in L^1_{u_{p-loc}}(\mathbb{R}^+_+; L^2(\Omega))$ for which there exists $J \in \mathbb{N}$ such that $t \mapsto (1 + t)^{-J}F(t, \cdot) \in L^1(\mathbb{R}^+_+; L^2(\Omega))$.

Following [19–21, 27], we give the following definition of Laplace-weak solutions of (1.4).

**Definition 1.2** (Laplace-weak solution). Assume that $K$ is given by either of the three expressions (1.7), (1.8), and (1.9). Let $F \in J$ and let the coefficients and the source term in (1.4) satisfy (1.1)–(1.2). We say that $u \in L^1_{u_{p-loc}}(\mathbb{R}^+_+; L^2(\Omega))$ is a Laplace-weak solution to (1.4) if it satisfies the following conditions.

(i) $\inf\{\tau > 0 : e^{-\tau t}u \in L^1(\mathbb{R}^+_+; L^2(\Omega))\} = 0$.

(ii) There exists $\tau_1 \geq 0$ such that for all $p \in \mathbb{C}$ satisfying $\Re p > \tau_1$, the Laplace transform $\hat{u}(p, \cdot)$ of $u(t, \cdot)$ with respect to $t$ is lying in $H^1_0(\Omega)$ and it solves the following boundary value problem:

\[
\begin{aligned}
(A + p\hat{K}(\rho, \cdot))\hat{u}(p, \cdot) &= \hat{F}(p, \cdot) + p\hat{K}(p, \cdot)u_0 \\
\hat{u}(p, \cdot) &= 0
\end{aligned}
\quad \text{in } \Omega,
\quad \text{on } \partial \Omega.
\]

(1.10)

Note that here $\hat{K}(p, \cdot)$ is well defined for all $p \in \mathbb{C}$ satisfying $\Re p > 0$ thanks to condition (1.3).

**Remark 2.** In Definition 1.2, all the properties of the IBVP (1.4) are described by the boundary value problem (1.10). Indeed, for a solution $u$ of (1.4) satisfying the condition

$$
\tau_2 = \inf\{\tau > 0 : e^{-\tau t}u \in W^{1,1}(\mathbb{R}^+_+; L^2(\Omega))\} < \infty,
$$

we have

$$
\hat{\tau}_2\hat{u}(p, \cdot) = \hat{K}(p, \cdot)[p\hat{u}(p, \cdot) - u(0, \cdot)] = p\hat{K}(p, \cdot)\hat{u}(p, \cdot) - \hat{K}(p, \cdot)u_0, \quad p \in \mathbb{C}, \quad \Re p > \tau_2.
$$

Therefore, applying the Laplace transform in time to Equation of (1.4), we deduce that, for all $p \in \mathbb{C}$ satisfying $\Re p > \tau_1$, $\hat{u}(p, \cdot)$ is the unique solution of (1.10). Combining this with the uniqueness and the analyticity of the Laplace transform, we can conclude that such a solution of (1.4) coincides with the Laplace-weak solution of (1.4). In that sense, this notion of Laplace-weak solutions allows to define solutions of (1.4) in terms of properties of their Laplace transform in time.

The goal of this article is to unify these two definitions by proving the equivalence between Definition 1.1 and Definition 1.2 in some general context with the kernel $K$ given by (1.7), (1.8), (1.9). For this purpose, assuming that $F \in J$ and $u_0 \in L^2(\Omega)$, I will show the unique existence of Laplace-weak solutions of (1.4) in the sense of Definition 1.2. After that, I prove that the Laplace-weak solutions of (1.4) coincides with the unique weak solution of (1.4) in the sense of Definition 1.1. This property shows in particular the equivalence between these definitions. In addition to this equivalence, we give also a Duhamel type of representation of the weak solutions of (1.4) taking the forms (2.5), (3.1), and (4.10).

The analysis of this article is restricted to fractional diffusion equations with elliptic operators that are time independent. The method based on application of Laplace transform in time, under consideration in this article (as well as in the references [19, 27]) for proving the unique existence of Laplace-weak solutions of (1.4), cannot be applied to fractional diffusion equations with time-dependent elliptic operators. For this class of equations, which are common in practice, we cannot establish the equivalence between Definitions 1.1 and 1.2, and a different approach should be considered for proving the unique existence of weak solutions.

### 1.3 Motivations

Recall that anomalous diffusion in complex media has been intensively studied these last decades in different fields with multiple applications in geophysics, environmental, and biological problems. The diffusion properties of homogeneous media are currently modeled, see, for example, [1, 4], by constant order time-fractional diffusion processes where in (1.4), the kernel $K$ takes the form $t^{-\beta} \Gamma(1-\beta)$ with a constant $\beta \in (0, 1)$. However, in some complex media, several physical properties lead to more general model involving variable order, distributed order, and multiterm fractional diffusion equations. For instance, it has been proved that the presence of heterogeneous regions displays space inhomogeneous variations...
and the constant order fractional dynamic models are not robust for long times (see [7]). In this context, the variable order time-fractional model, corresponding to kernel $K$ given by (1.7), is more relevant for describing the space-dependent anomalous diffusion process (see, e.g., [40]). Several variable order diffusion models have been successfully applied in numerous applications in sciences and engineering, including chemistry [5], rheology [38], biology [9], hydrogeology [2], and physics [39, 43]. In the same way, some anomalous diffusion processes such as ultraslow diffusion, where the mean squared variance grows only logarithmically with time, are modeled by fractional diffusion equations with distributed order fractional derivatives with applications in polymer physics and kinetics of particles (see, e.g., [32, 33]). For these different physical models, the goal of this article is to prove existence of weak solutions of (1.4) enjoying several important properties such as resolution of the equation in the sense of distributions, suitable Duhamel representation formula, and expected properties of the Laplace transform in time of the solutions stated in Definition 1.2.

Beside these physical motivations, our analysis is also motivated by applications in other classes of mathematical problems where the Duhamel representation formula, the properties of the Laplace transform in time of solutions, as well as the resolution in the sense of distributions of the equation in (1.4), stated in (1.5)–(1.6), play an important role. This is, for instance, the case for several inverse problems (see, e.g., [11, 13–16, 18, 26]) as well as the study of some dynamical properties (see, e.g., [8, 17, 29]), the derivation of analyticity properties in time of solutions (see, e.g., [30]), and the numerical resolution (see, e.g., [3, 12]) of these equations. In this context, our goal is to exhibit weak solutions that satisfy simultaneously all the above mentioned properties.

### 1.4 Known results

Recall that the well-posedness of the IBVP (1.4) has received a lot of attention these last decades among the mathematical community. For constant order fractional diffusion equations, where in (1.4), the kernel $K$ takes the form $t^{-\beta}/\Gamma(1-\beta)$ with constant values $\beta \in (0, 1)$, several approaches have been considered for defining solutions of (1.4). This includes the definition of solutions of (1.4) in a variational and strong sense considered by [6, 21, 24, 25, 42], the definition of solutions in the mild sense in [36], and the definition of solutions by mean of their Laplace transform in time given by [20, 21]. Such an analysis includes also the study of the IBVP (1.4) with a time-dependent elliptic operator as stated in [24, 25, 42]. Several authors considered also the well-posedness of a more general class of diffusion equations. For instance, the analysis of [22, 42] in some abstract framework can be applied to some class of distributed order and multiterm fractional diffusion equation of the form (1.4) with a kernel $K$ independent of $x$ (for $K$ given by (1.9), the coefficients $\rho_1, \ldots, \rho_N$ are constants). In the same way, we can mention the works of [23, 27] for the study of distributed order fractional diffusion equations and the works of [28, 31] devoted to the study of well-posedness of multiterm fractional diffusion equations with both constant and variable coefficients $\rho_1, \ldots, \rho_N$ in (1.9). To the best of our knowledge, in the article [19], one can find the only result available in the mathematical literature devoted to the study of the well-posedness of variable order fractional diffusion equations (the kernel $K$ given by (1.7)) with a nonvanishing initial condition and general source term. In this last work, the authors give a definition of solutions in terms of Laplace transform comparable to Definition 1.2. We mention also the recent work of [15] where in [15, Theorem 4.1] the unique existence of weak solutions of (1.4) with $K$ given by (1.7), $F$ taking the form of a separated variables function $F(t, x) = \sigma(t)f(x)$, $(t, x) \in \mathbb{R}_+ \times \Omega$, and $u_0 \equiv 0$, is proved. As far as we know, there is no result showing existence of weak solutions of (1.4) satisfying the properties described by Definition 1.1 for variable order fractional diffusion equations with a general source term $F$ and $u_0 \not\equiv 0$.

In all the above-mentioned results, the authors have either considered a variational definition of solutions comparable to Definition 1.1 or a definition of solutions in terms of Laplace transform comparable to the Laplace-weak solutions of Definition 1.2. However, as far as I know, there has been no result so far proving the unification of these two definitions of solutions for variable order, distributed order, or multiterm fractional diffusion equations.

### 1.5 Main results

The main results of this article state the unique existence of Laplace-weak solutions of (1.4) in the sense of Definition 1.2 as well as the equivalence between Definitions 1.1 and 1.2 for the weight function $K$ given by (1.7), (1.8), and (1.9), which correspond to variable, distributed order, and multiterm fractional diffusion equations.

For variable order fractional diffusion equations, our result can be stated as follows.
**Theorem 1.3.** Assume that the conditions (1.1)–(1.2) are fulfilled. Let \( u_0 \in L^2(\Omega) \), \( F \in J \), \( \alpha \in L^\infty(\Omega) \) satisfy
\[
0 < \alpha_0 \leq \alpha(x) \leq \alpha_M < 1, \quad \alpha_M < 2\alpha_0, \ x \in \Omega, \tag{1.11}
\]
and let \( K \) be given by (1.7). Then, there exists a unique Laplace-weak solution \( u \in L^1_{up-loc}(\mathbb{R}^+;L^2(\Omega)) \) of (1.4) in the sense of Definition 1.2. Moreover, the Laplace-weak solution \( u \in L^1_{up-loc}(\mathbb{R}^+;L^2(\Omega)) \) of (1.4) is the unique weak solution of (1.4) in the sense of Definition 1.1. In addition, \( u \) is described by a Duhamel type of formula taking the form (2.5).

For distributed order fractional diffusion equations, our result can be stated as follows.

**Theorem 1.4.** Assume that the conditions (1.1)–(1.2) are fulfilled. Let \( u_0 \in L^2(\Omega) \), \( F \in J \), \( \mu \in C([0, 1]) \) be a nonnegative function satisfying the condition
\[
\exists \alpha_0 \in (0, 1), \exists \epsilon \in (0, \alpha_0), \forall \alpha \in (\alpha_0 - \epsilon, \alpha_0), \mu(\alpha) \geq \frac{\mu(\alpha_0)}{2} > 0, \tag{1.12}
\]
and let \( K \) be given by (1.8). Then, there exists a unique Laplace-weak solution \( u \in L^1_{up-loc}(\mathbb{R}^+;L^2(\Omega)) \) of (1.4) in the sense of Definition 1.2. Moreover, the Laplace-weak solution \( u \in L^1_{up-loc}(\mathbb{R}^+;L^2(\Omega)) \) of (1.4) is the unique weak solution of (1.4) in the sense of Definition 1.1. In addition, \( u \) is described by a Duhamel type of formula taking the form (3.1).

For multiterm fractional diffusion equations, our result can be stated as follows.

**Theorem 1.5.** Assume that the conditions (1.1)–(1.2) are fulfilled with \( \rho \equiv 1 \). Let \( u_0 \in L^2(\Omega) \), \( F \in J \), \( 0 < \alpha_1 < \ldots < \alpha_N < 1 \), and \( \rho_j \in L^\infty(\Omega) \), \( j = 1, \ldots, N \), satisfying (1.2) with \( \rho = \rho_j \), and let \( K \) be given by (1.9). Then there exists a unique Laplace-weak solution \( u \in L^1_{up-loc}(\mathbb{R}^+;L^2(\Omega)) \) of (1.4) in the sense of Definition 1.2. Moreover, the Laplace-weak solution \( u \in L^1_{up-loc}(\mathbb{R}^+;L^2(\Omega)) \) of (1.4) is the unique weak solution of (1.4) in the sense of Definition 1.1. In addition, \( u \) is described by a Duhamel type of formula taking the form (4.10).

### 1.6 Comments about our results

To the best of our knowledge, in Theorem 1.3, we obtain the first result of unique existence of weak solutions of variable order fractional diffusion equations solving in the sense of distributions the equation in (1.4), as stated in (1.5), and with explicit connection to the initial condition \( u_0 \) stated in (1.5)–(1.6). As far as we know, the only other comparable results, when \( u_0 \not\equiv 0 \), can be found in [19] where the authors proved only existence of Laplace-weak solution of (1.4) with \( K \) given by (1.7). In that sense, Theorem 1.3 gives the first general extension of the analysis of [19] by proving that the unique Laplace-weak solution under consideration in [19] is also the unique weak solution in the sense of Definition 1.1.

Let us observe that, in Theorems 1.4 and 1.5 we show, for what seems to be the first time, the unique existence of weak solutions of distributed order and multiterm fractional diffusion equations that enjoy simultaneously the following properties; (1) The weak solution solves in the sense of distributions the equation in (1.4) as stated in (1.5). (2) The weak solution is explicitly connected with the initial condition \( u_0 \) by (1.5)–(1.6). (3) The weak solution is described by a Duhamel type of formula. (4) The weak solution is also a Laplace-weak solution in the sense of Definition 1.1. Indeed, several authors proved unique existence of solutions of distributed order and multiterm fractional diffusion equations enjoying the above properties (1) and (2) (see, e.g., [22, 23, 28]) or the above properties (3) and (4) (see, e.g., [19]). Nevertheless, we are not aware of any results proving existence of weak solutions of distributed order fractional diffusion equations or multiterm fractional diffusion equations with variable coefficients enjoying simultaneously the above properties (1), (2), (3), and (4). In that sense, Theorems 1.4 and 1.5 show that these different properties of solutions of distributed order and multiterm fractional diffusion equations can be unified.

We mention that the assumptions \( \alpha_M < 2\alpha_0 \) and (1.12) are required for technical reason. Namely, in our approach, we need to consider conditions guaranteeing the unique existence of Laplace-weak solutions of (1.4) described by a Duhamel type of formula. Following [19, Remark 1], we show in Proposition 2.3 that this property holds true for \( K \) given by (1.7), with the Duhamel type of formula (2.5), when the condition \( \alpha_M < 2\alpha_0 \) is fulfilled. In the same way, applying [27, Proposition
2.1] and assuming that $K$ is given by (1.8), we show in Proposition 3.2 the unique existence of Laplace-weak solutions of (1.4) described by the Duhamel type of formula (3.1) when (1.12) holds true.

In contrast to Definition 1.2, where the solutions are described by mean of the properties of Laplace transform in time of such class of fractional diffusion equations (see, e.g., [34] for more details), Definition 1.1 gives more explicit properties of solutions of (1.4). Namely, the weak solution of (1.4), in the sense of Definition 1.1, solves the equation in (1.4) in the sense of distribution, as stated in (1.5). Moreover, this class of weak solutions is also explicitly connected with the initial condition $u_0$ by mean of properties (1.5)–(1.6) and condition (iii) gives the boundary condition imposed to weak solutions in the sense of Definition 1.1. By proving the equivalence between Definition 1.1 and Definition 1.2 of weak solutions, we show that the weak solution of (1.4) combine the explicit properties of Definition 1.1 with the properties of Laplace transform of solutions as stated in Definition 1.2.

Let us observe that the results of Theorems 1.3, 1.4, and 1.5 can be applied to the unique existence of solutions of the IBVP (1.4) at finite time (see the IBVP (5.1)). This aspect is discussed in Section 5 of this article with a definition of weak solutions stated in Definition 5.1 by mean of a weak solution at infinite time in the sense of Definition 1.1. Our results for these issue are stated in Theorem 5.2, where we show that the unique solution in the sense of Definition 5.1 is independent of the choice of the final time.

Let us remark that the boundary condition under consideration in (1.4) can be replaced, at the price of some minor modifications, by a more general homogeneous Neumann or Robin boundary condition. In the spirit of the work [21], it is also possible to consider nonhomogeneous boundary conditions. For simplicity, we restrict our analysis to homogeneous Dirichlet boundary conditions.

1.7 Outline

This article is organized as follows. In Section 2, we prove the existence of a Laplace-weak solutions of the IBVP (1.4) in the sense of Definition 1.2 as well as the equivalence between Definitions 1.1 and 1.2, when $K$ is given by (1.7), stated in Theorem 1.3. In the same way, Sections 3 and 4 are, respectively, devoted to the proof of Theorems 1.4 and 1.5. Moreover, in Section 5, we study the same problem at finite time (see the IBVP (5.1)) and we give a definition of solutions in that context stated in Definition 5.1. We prove also in Theorem 1.5 the unique existence of solutions in the sense of Definition 5.1 as well as the independence of the unique solution in the sense of Definition 5.1 with respect to the final time.

2 VARIABLE ORDER FRACTIONAL DIFFUSION EQUATIONS

In this section, we prove the unique existence of a weak solution to the problem (1.4) as well as the equivalence between Definitions 1.1 and 1.2 of weak and Laplace-weak solutions of (1.4) for weight $K$ given by (1.7) with $\alpha \in L^\infty(\Omega)$ satisfying (1.11). For this purpose, let us first recall that the unique existence of solutions close to the Laplace-weak solutions for (1.4) has been proved by [19, Theorem 1.1] in the case of source terms $F \in L^\infty(\mathbb{R}^+; L^2(\Omega))$. We will recall here the representation of Laplace-weak solutions of (1.4) given by [19]. For this purpose, we fix $\delta \in \left(\frac{\pi}{2}, \pi\right)$, $\delta > 0$ and we define the contour in $\mathbb{C}$,

$$\gamma(\delta, \theta) := \gamma_-(\delta, \theta) \cup \gamma_0(\delta, \theta) \cup \gamma_+(\delta, \theta),$$

(2.1)

oriented in the counterclockwise direction with

$$\gamma_0(\delta, \theta) := \{\delta e^{i\beta} : \beta \in [-\theta, \theta]\}, \quad \gamma_\pm(\delta, \theta) := \{s e^{i\theta} : s \in [\delta, \infty)\}.$$

(2.2)

We denote also by $A$ the Dirichlet realization of the operator $\mathcal{A}$ acting on $L^2(\Omega)$ with domain

$$D(A) = \{v \in H^1_0(\Omega) : A v \in L^2(\Omega)\}.$$

Then, following [19], we define the operators
\[ S_0(t)\psi := \frac{1}{2i\pi} \int_{\gamma(\delta, \beta)} e^{ip}(A + \rho p^{\alpha(i)})^{-1}\rho p^{\alpha(i)-1}\psi dp, \quad t > 0, \quad (2.3) \]

\[ S_1(t)\psi := \frac{1}{2i\pi} \int_{\gamma(\delta, \beta)} e^{ip}(A + \rho p^{\alpha(i)})^{-1}\psi dp, \quad t > 0. \quad (2.4) \]

According to [19, Theorem 1.1], the definition of the operator valued functions \( S_0, S_1 \) is independent of the choice of \( \beta \in \left( \frac{\pi}{2}, \pi \right) \), \( \delta > 0 \). I recall the following results borrowed from [19].

**Theorem 2.1** [19], Theorem 1 and Remark 1. Suppose that (1.1) is fulfilled and \( \alpha \in L^\infty(\Omega) \) satisfies the condition

\[ 0 < \alpha_0 \leq \alpha(x) \leq \alpha_M < 1, \quad x \in \Omega. \]

Let \( u_0 \in L^2(\Omega) \) and assume that \( F \in C(\mathbb{R}_+, L^2(\Omega)) \) satisfies \( \langle t \rangle^{-\xi} F \in L^\infty(\mathbb{R}_+; L^2(\Omega)) \) with some \( \xi \in \mathbb{R}_+ \). Then, there exists a unique tempered distribution \( u \) with respect to the time variable \( t \in \mathbb{R} \), supported on \( \mathbb{R}_+ \) and taking values in \( L^2(\Omega) \) whose Laplace transform in time solves (1.10). Moreover, assuming that \( \alpha_M < 2\alpha_0 \), this unique tempered distribution \( u \) takes the form

\[ u(t, \cdot) = S_0(t)u_0 + \int_0^t S_1(t - \tau)F(\tau, \cdot)d\tau, \quad t > 0. \quad (2.5) \]

According to Theorem 2.1, the function \( u \) given by (2.5) will be the unique Laplace-weaksolution of problem (1.4) in the sense of Definition 1.1 provided that \( u \in L^1_{up-loc}(\mathbb{R}_+; L^2(\Omega)) \). I start by proving an extension of this result to the unique existence of a Laplace-weaksolution of problem (1.4) when \( u_0 \in L^2(\Omega) \) and \( F \in \mathcal{F} \). For this purpose, I need the following intermediate result about the operator valued functions \( S_0 \) and \( S_1 \).

**Lemma 2.2.** Let \( \beta \in \left( \frac{\pi}{2}, \pi \right) \). The maps \( t \mapsto S_j(t), \ j = 0, 1 \), defined by (2.3)–(2.4) are lying in \( L^1_{up-loc}(\mathbb{R}_+; B(L^2(\Omega))) \) and there exists a constant \( C > 0 \) depending only on \( A, \rho, \alpha, \beta, \Omega \) such that the estimates

\[ \|S_0(t)\|_{B(L^2(\Omega))} \leq C \max \left( t^{2(\alpha_M - \alpha_0)}, t^2(\alpha_0 - \alpha_M) \right), \quad t > 0, \quad (2.6) \]

\[ \|S_1(t)\|_{B(L^2(\Omega))} \leq C \max \left( t^{2\alpha_M - \alpha_0 - 1}, t^2(\alpha_0 - \alpha_M - 1), 1 \right), \quad t > 0, \quad (2.7) \]

hold true

**Proof.** Throughout this proof, I denote by \( C \) generic positive constants depending only on \( A, \rho, \alpha, \beta, \Omega \), which may change from line to line. I only consider the proof of this lemma for the operator valued function \( S_0 \), for \( S_1 \) one can refer to [15, Lemma 4.2] for the proof of (2.7). In light of [19, Proposition 2.1], for all \( \beta \in (0, \pi) \), we have

\[ \| (A + \rho r^{i\beta})^{\alpha(i)} \|_{B(L^2(\Omega))} \leq C \max \left( r^{\alpha_0 - 2\alpha_M}, r^{2\alpha_M - 2\alpha_0} \right), \quad r > 0, \quad \beta_1 \in (\beta, \beta). \quad (2.8) \]

Using the fact that the operator \( S_0 \) is independent of the choice of \( \delta > 0 \), we can decompose

\[ S_0(t) = H_-(t) + H_0(t) + H_+(t), \quad t > 0, \]

where

\[ H_m(t) = \frac{1}{2i\pi} \int_{\gamma_m(t^{-1}, \beta)} e^{ip}(\rho p^{\alpha(i)} + A)^{-1}\rho p^{\alpha(i)-1} dp, \quad m = 0, \mp, \ t > 0. \]

In order to complete the proof of the lemma, it suffices to prove

\[ \|H_m(t)\|_{B(L^2(\Omega))} \leq C \max \left( t^{2(p_{\alpha_M - \alpha_0})}, t^{2(p_{\alpha_0 - \alpha_M})} \right), \quad t > 0, \quad m = 0, \mp. \quad (2.9) \]
Indeed, these estimates clearly imply (2.6). Moreover, condition (1.11) implies that $2(\alpha_0 - \alpha_M) > -\alpha_M > -1$ and we deduce from (2.6) that $S_0 \in L^1_{up-loc}(\mathbb{R}_+; B(L^2(\Omega)))$. For $m = 0$, using (2.8), we find

$$
\|H_0(t)\|_{B(L^2(\Omega))} \leq C \int_{-\theta}^{\theta} t^{-1} \left\| (A + (t^{-1} e^{i\beta})^{\alpha(\cdot)})^{-1} \right\|_{B(L^2(\Omega))} \left\| t^{-1} e^{i\beta} \alpha^{\alpha(\cdot)} \right\|_{L^\infty(\Omega)} d\beta
\leq C \max \left( t^{2(\alpha_M - \alpha_0)}, t^{2(\alpha_0 - \alpha_M)} \right),
$$

which implies (2.9) for $m = 0$. For $m = \mp$, again we employ (2.8) to estimate

$$
\|H_\mp(t)\|_{B(L^2(\Omega))} \leq C \int_{t^{-1}}^{\infty} e^{r t \cos \theta} \left\| (A + (r e^{i\beta})^{\alpha(\cdot)})^{-1} \right\|_{B(L^2(\Omega))} \left\| r e^{i\beta} \alpha^{\alpha(\cdot)} \right\|_{L^\infty(\Omega)} dr
\leq C \int_{t^{-1}}^{\infty} e^{r t \cos \theta} \max \left( r^{\alpha_0 - 2\alpha_M}, r^{\alpha_M - 2\alpha_0} \right) \left\| r^{\alpha(\cdot)} \right\|_{L^\infty(\Omega)} dr
\leq C \int_{t^{-1}}^{\infty} e^{r t \cos \theta} \left( r^{2(\alpha_0 - \alpha_M)} - 1 \right) \left( r^{2(\alpha_M - \alpha_0)} - 1 \right) dr.
$$

For $t > 1$, we obtain

$$
\|H_\mp(t)\|_{B(L^2(\Omega))} \leq C \int_{1}^{\infty} e^{r t \cos \theta} r^{2(\alpha_0 - \alpha_M)} - 1 dr + C \int_{1}^{1} r^{2(\alpha_0 - \alpha_M)} - 1 dr
\leq C \int_{0}^{\infty} e^{r t \cos \theta} r^{2(\alpha_0 - \alpha_M)} - 1 dr + C (t^{2(\alpha_0 - \alpha_M)} - 1)
\leq C t^{-1} \int_{1}^{\infty} e^{r t \cos \theta} \left( \frac{r}{t} \right)^{2(\alpha_0 - \alpha_M)} - 1 dr + C (t^{2(\alpha_0 - \alpha_M)} - 1)
\leq C \max \left( t^{2(\alpha_0 - \alpha_M)}, t^{2(\alpha_M - \alpha_0)} \right).
$$

In the same way, for $t \in (0, 1]$, we get

$$
\|H_\mp(t)\|_{B(L^2(\Omega))} \leq C \int_{1}^{\infty} e^{r t \cos \theta} r^{2(\alpha_0 - \alpha_M)} - 1 dr \leq C t^{2(\alpha_M - \alpha_0)}.
$$

Combining these two estimates, we obtain

$$
\|H_\mp(t)\|_{B(L^2(\Omega))} \leq C \max \left( t^{2(\alpha_0 - \alpha_M)}, t^{2(\alpha_M - \alpha_0)} \right), \quad t > 0.
$$

This proves that (2.9) also holds true for $m = \mp$. Therefore, estimate (2.6) holds true and we have $S_0 \in L^1_{up-loc}(\mathbb{R}_+; B(L^2(\Omega)))$, which completes the proof of the lemma. \(\square\)

I am now in a position to state the existence of a unique Laplace-weak solution $u \in L^1_{up-loc}(\mathbb{R}_+; B(L^2(\Omega)))$ of (1.4), given by (2.5), for any source term $F \in \mathcal{F}$.

**Proposition 2.3.** Assume that the conditions (1.1)–(1.2) are fulfilled. Let $u_0 \in L^2(\Omega)$, $F \in \mathcal{F}$, $\alpha \in L^\infty(\Omega)$ satisfy (1.11) and let $K$ be given by (1.7). Then, there exists a unique Laplace-weak solution $u \in L^1_{up-loc}(\mathbb{R}_+; B(L^2(\Omega)))$ to (1.4) given by (2.5).

**Proof.** According to Theorem 2.1 and Lemma 2.2, I only need to prove this result for $u_0 \equiv 0$. Using Lemma 2.2, I will complete the proof of Proposition 2.3 by mean of density arguments. Fix

$$
G(t, x) = (1 + t)^{-1} F(t, x), \quad (t, x) \in \mathbb{R}_+ \times \Omega
$$

(2.10)
and recall that $G \in L^1(\mathbb{R}_+; L^2(\Omega))$. Therefore, we can find a sequence $(G_n)_{n \in \mathbb{N}}$ lying in $C_0^\infty(\mathbb{R}_+ \times \Omega)$ such that

$$\lim_{n \to \infty} \|G_n - G\|_{L^1(\mathbb{R}_+; L^2(\Omega))} = 0.$$

Fixing $(F_n)_{n \in \mathbb{N}}$ a sequence of functions defined by

$$F_n(t, x) = (1 + t)^t G_n(t, x), \quad (t, x) \in \mathbb{R}_+ \times \Omega, \quad n \in \mathbb{N},$$

we deduce that the sequence $(F_n)_{n \in \mathbb{N}}$ is lying in $C_0^\infty(\mathbb{R}_+ \times \Omega)$ and we have

$$\lim_{n \to \infty} \|(1 + t)^t (F_n - F)\|_{L^1(\mathbb{R}_+; L^2(\Omega))} = \lim_{n \to \infty} \|G_n - G\|_{L^1(\mathbb{R}_+; L^2(\Omega))} = 0. \quad (2.11)$$

In light of Lemma 2.2, for $t > 0$, we can introduce

$$u_n(t, \cdot) := \int_0^t S_1(t - \tau) F_n(\tau, \cdot) d\tau = \int_0^t S_1(\tau) F_n(t - \tau, \cdot) d\tau, \quad n \in \mathbb{N},$$

$$u(t, \cdot) := \int_0^t S_1(t - \tau) F(\tau, \cdot) d\tau,$$

as elements of $L^1_{u_p-L^\infty}(\mathbb{R}_+; L^2(\Omega))$. We will prove that for all $p \in \mathbb{C}_+ := \{z \in \mathbb{C} : \Re z > 0\}$, the Laplace transform $\hat{u}(p)$ of $u$ is well defined in $L^2(\Omega)$ and we have

$$\lim_{n \to \infty} \|\hat{u}_n(p) - \hat{u}(p)\|_{L^2(\Omega)} = 0. \quad (2.12)$$

Applying estimate (2.7), we obtain

$$\|e^{-pt} u(t, \cdot)\|_{L^2(\Omega)} \leq \int_0^t e^{-\Re pt} \|S_1(\tau)\|_{L^2(\Omega)} e^{-\Re p(t - \tau)} \|F(t - \tau, \cdot)\|_{L^2(\Omega)} d\tau$$

$$\leq C \left( e^{-\Re pt} \max \left( t^{2\alpha_0 - \alpha_1 - 1}, t^{2\alpha_1 - \alpha_0 - 1}, 1 \right) \right) \ast \left( e^{-\Re pt} \|F(\cdot, \cdot)\|_{L^2(\Omega)} \right),$$

for all $t > 0$ and $p \in \mathbb{C}_+$, where $\ast$ denotes the convolution in $\mathbb{R}_+$. Therefore, applying Young’s convolution inequality, we deduce

$$\leq \int_0^\infty \|e^{-pt} u(t, \cdot)\|_{L^2(\Omega)} dt$$

$$\leq C \left( \int_0^\infty e^{-\Re pt} \max \left( t^{2\alpha_0 - \alpha_1 - 1}, t^{2\alpha_1 - \alpha_0 - 1}, 1 \right) dt \right) \left( \int_0^\infty e^{-\Re pt} \|F(\cdot, \cdot)\|_{L^2(\Omega)} dt \right)$$

$$\leq C_p \left( \int_0^\infty e^{-\Re pt} \max \left( t^{2\alpha_0 - \alpha_1 - 1}, t^{2\alpha_1 - \alpha_0 - 1}, 1 \right) dt \right) \|(1 + t)^t (F(\cdot, \cdot))\|_{L^1(\mathbb{R}_+; L^2(\Omega))} < \infty,$$

for all $p \in \mathbb{C}_+$. This proves that $\hat{u}(p)$ is well defined for all $p \in \mathbb{C}_+$ in the sense of $L^2(\Omega)$. In the same way, for all $t > 0$, $p \in \mathbb{C}_+$ and $n \in \mathbb{N}$, we get

$$\|e^{-pt} (u_n - u)(t, \cdot)\|_{L^2(\Omega)}$$

$$\leq \int_0^t e^{-\Re p(t - \tau)} \|S_1(t - \tau)\|_{L^2(\Omega)} e^{-\Re pt} \|F_n(\tau, \cdot) - F(\tau, \cdot)\|_{L^2(\Omega)} d\tau$$

$$\leq C \left( e^{-\Re pt} \max \left( t^{2\alpha_0 - \alpha_1 - 1}, t^{2\alpha_1 - \alpha_0 - 1}, 1 \right) \right) \ast \left( e^{-\Re pt} \|F_n(\cdot, \cdot) - F(\cdot, \cdot)\|_{L^2(\Omega)} \right).$$
Thus, applying Young’s convolution inequality again, we have
\[
\|\hat{u}_n(p) - \hat{u}(p)\|_{L^2(\Omega)} \leq \int_0^\infty \|e^{-pt}(u_n - u)(t, \cdot)\|_{L^2(\Omega)} \, dt \\
\leq C \left( \int_0^\infty e^{-R pt} \max(t^{2\alpha_M - \alpha_M - 1}, t^{2\alpha_0 - \alpha_0 - 1}, 1) \, dt \right) \left( \int_0^\infty e^{-R pt} \|F_n(t, \cdot) - F(t, \cdot)\|_{L^2(\Omega)} \, dt \right) \\
\leq C_p \left( \sup_{t \in \mathbb{R}_+} (1 + t)^{-1} e^{-R pt} \right) \left( \sup_{t \in \mathbb{R}_+} (1 + t)^{-1} e^{-R pt} \right) \|F_n - F\|_{L^1(\mathbb{R}_+; L^2(\Omega))},
\]
for all \( p \in \mathbb{C}_+ \) and \( n \in \mathbb{N} \), and (2.11) implies (2.12). On the other hand, in view of Theorem 2.1, since \( F_n \in L^\infty(\mathbb{R}_+; L^2(\Omega)) \), we have
\[
(\rho p^{\alpha(\cdot)} + A)\hat{u}_n(p, \cdot) = \left( \int_0^\infty e^{-pt} F_n(t, \cdot) \, dt \right), \quad p \in \mathbb{C}_+, \ n \in \mathbb{N}.
\]
In addition, (2.11) implies that, for all \( p \in \mathbb{C}_+ \), we have
\[
\limsup_{n \to \infty} \|\hat{F}_n(p) - \hat{F}(p)\|_{L^2(\Omega)} \leq \limsup_{n \to \infty} \int_0^\infty e^{-R pt} \|F_n(t, \cdot) - F(t, \cdot)\|_{L^2(\Omega)} \, dt \\
\leq \left( \sup_{t \in \mathbb{R}_+} (1 + t)^{-1} e^{-R pt} \right) \limsup_{n \to \infty} \|1 + t\|^{-1} \|F_n - F\|_{L^1(\mathbb{R}_+; L^2(\Omega))} \\
\leq 0.
\]
Therefore, we obtain
\[
\lim_{n \to \infty} \left\|\hat{u}_n(p) - \left( A + \rho p^{\alpha(\cdot)} \right)^{-1} \hat{F}(p, \cdot)\right\|_{L^2(\Omega)} = 0
\]
and (2.12) implies that \( \hat{u}(p) = \left( A + \rho p^{\alpha(\cdot)} \right)^{-1} \hat{F}(p, \cdot), \ p \in \mathbb{C}_+ \). From the definition of the operator \( A \), we deduce that \( \hat{u}(p) \in H^1_0(\Omega) \) solves the boundary value problem (1.10) for all \( p \in \mathbb{C}_+ \). Recalling that the uniqueness of Laplace-weak solutions can be deduced easily from the uniqueness of the solution of (1.10) and the uniqueness of Laplace transform, we conclude that \( u \) is the unique Laplace-weak solutions of (1.4) and the proof is completed. \( \square \)

In view of Proposition 2.3, the first statement of Theorem 1.3 is fulfilled. Let us complete the proof of Theorem 1.3.

**Proof of Theorem 1.3.** Let us first observe that the first statement of Theorem 1.3 is a direct consequence of Proposition 2.3. Therefore, in order to complete the proof of Theorem 1.3, I need to prove that (1.4) admits a unique weak solution \( u \in L^1_{\text{up-loc}}(\mathbb{R}_+; L^2(\Omega)) \) in the sense of Definition 1.1 given by (2.5), which is the unique Laplace-weak solution of (1.4). I divide the proof of our result into three steps. I start by proving the uniqueness of the solution of (1.4) in the sense of Definition 1.2. Then, I prove that (2.5) is a weak solution of (1.4) in the sense of Definition 1.1 for \( u_0 \equiv 0 \). Finally, I show that (2.5) is a weak solution of (1.4) in the sense of Definition 1.1 for \( F \equiv 0 \).

**Step 1.** This step will be devoted to the proof of the uniqueness of weak solutions of (1.4) in the sense of Definition 1.1. For this purpose, let \( u \in L^1_{\text{up-loc}}(\mathbb{R}_+; L^2(\Omega)) \) be a weak solution of (1.4) with \( F \equiv 0 \) and \( u_0 \equiv 0 \). In view of condition (iii) of Definition 1.1, we can fix
\[
\tau_0 = \inf\{\tau > 0 : e^{-\tau t}u \in L^1(\mathbb{R}_+; L^2(\Omega))\}.
\]
Then, using the fact that \( a := (a_{ij})_{1 \leq i, j \leq d} \in H^1(\Omega; \mathbb{R}^d) \), for all \( p \in \mathbb{C} \) satisfying \( \Re p > \tau_0 \), we have \( e^{-pt}Au \in L^1(\mathbb{R}_+; D^K(\Omega)) \) and condition (1.10) implies that
\[
e^{-pt}p^K u = -e^{-pt}Au, \quad t \in \mathbb{R}_+.
\]
It follows that, for all \( p \in \mathbb{C} \) satisfying \( \Re p > \tau_0 \), \( e^{-pt} pD_t^K u \in L^1(\mathbb{R}_+; D'(\Omega)) \). Combining this with condition (ii) of Definition 1.1, we deduce that, for all \( p \in \mathbb{C} \) satisfying \( \Re p > \tau_0 \), \( e^{-pt} pI_K u \in W^{1,1}(\mathbb{R}_+; D'(\Omega)) \). Therefore, multiplying (1.5) by \( e^{-pt} \) with \( p \in \mathbb{C} \) satisfying \( \Re p > \tau_0 \), integrating over \( t \in \mathbb{R}^+ \) and using condition (1.6), we find

\[
0 = \int_0^\infty e^{-pt} (\partial_t [\rho I_K u(t, \cdot)] + A u(t, \cdot)) dt = p\rho I_K u(p, \cdot) + \hat{A}(p, \cdot) = \rho \hat{A}(p, \cdot) + \hat{A}(p, \cdot).
\]

Combining this with condition (iii) of Definition 1.2, we deduce that, for all \( p \in \mathbb{C} \) satisfying \( \Re p > \tau_1 \), \( \hat{u}(p, \cdot) \in H_0^1(\Omega) \) solves the boundary value problem

\[
\begin{cases}
(A + p^{\alpha(\cdot)} \rho) \hat{u}(p, \cdot) = 0 & \text{in } \Omega, \\
\hat{u}(p, \cdot) = 0 & \text{on } \partial \Omega.
\end{cases}
\]

On the other hand, applying [19, Proposition 2.1], we deduce that for all \( p \in \mathbb{C}_+ \) the operator \( A + p^{\alpha(\cdot)} \rho \) is invertible as an operator acting on \( L^2(\Omega) \). Therefore, we get that, for all \( p \in \mathbb{C} \) satisfying \( \Re p > \tau_1 \), \( \hat{u}(p, \cdot) \equiv 0 \) and, combining this with the analyticity and the uniqueness of Laplace transform, we deduce that \( u \equiv 0 \). This completes the proof of the uniqueness of weak solutions of problem (1.4).

**Step 2.** In this step, I will prove that the Laplace-weak solution \( u \in L^1_{\text{up-loc}}(\mathbb{R}_+; L^2(\Omega)) \) of (1.4), given by (2.5), is the weak solution of (1.4) in the sense of Definition 1.1 when \( u_0 \equiv 0 \). Note that the Laplace-weak solution \( u \in L^1_{\text{up-loc}}(\mathbb{R}_+; L^2(\Omega)) \) of (1.4) clearly satisfies condition (iii) of Definition 1.1. Thus, we only need to prove that the Laplace-weak solution \( u \in L^1_{\text{up-loc}}(\mathbb{R}_+; L^2(\Omega)) \) of (1.4) satisfies conditions (i) and (ii) of Definition 1.1. In a similar way to Proposition 2.3, I consider \( G \in L^1(\mathbb{R}_+; L^2(\Omega)) \) given by (2.10) and we fix a sequence \((G_n)_{n \in \mathbb{N}}\) lying in \( C_0^\infty(\mathbb{R}_+ \times \Omega) \) such that

\[
\lim_{n \to \infty} \|G_n - G\|_{L^1(\mathbb{R}_+; L^2(\Omega))} = 0
\]

and \((F_n)_{n \in \mathbb{N}}\) a sequence of functions of \( C_0^\infty(\mathbb{R}_+ \times \Omega) \) defined by

\[
F_n(t, x) = (1 + t)^\gamma G_n(t, x), \quad (t, x) \in \mathbb{R}_+ \times \Omega, \quad n \in \mathbb{N}.
\]

Then, condition (2.11) is fulfilled. According to Proposition 2.3, for all \( n \in \mathbb{N} \), the Laplace-weak solution \( u_n \) of (1.4) with \( F = F_n \) is given by

\[
u_n(t, \cdot) = \int_0^t S_1(t-\tau) F_n(\tau, \cdot) d\tau = \int_0^t S_1(\tau) F_n(t-\tau, \cdot) d\tau, \quad t \in \mathbb{R}_+.
\]

Using the fact that \( F_n \in C_0^\infty(\mathbb{R}_+ \times \Omega), n \in \mathbb{N} \), and applying estimate (2.7), we deduce that \( u_n \in C^1([0, \infty); L^2(\Omega)) \) and \( u_n(0, x) = 0, x \in \Omega \). Moreover, in view of (2.7), applying Young’s convolution inequality, for all \( p \in \mathbb{C}_+ \), we get

\[
\|e^{-pt} u_n\|_{L^1(\mathbb{R}_+; L^2(\Omega))} + \|e^{-pt} \partial_t u_n\|_{L^1(\mathbb{R}_+; L^2(\Omega))} \leq C \left( \|e^{-(\Re p)t\} \|_{L^2(\Omega)} + \|\partial_t F_n(t)\|_{L^2(\Omega)} \right) * \left( e^{-(\Re p)t\} \|S(t)\|_{L^2(\Omega)} \right) \leq C \|F_n\|_{W^{1,1}(\mathbb{R}_+; L^2(\Omega))} \int_0^\infty e^{-(\Re p)t} \max(t^{2\alpha_0-\alpha_M-1}, t^{2\alpha_0-\alpha_M-1}, 1) dt < \infty.
\]

Thus, for all \( n \in \mathbb{N} \), we have \( D_t^K u_n = \partial_t^K u_n \) and we deduce that, for all \( p \in \mathbb{C}_+ \), we have

\[
\hat{D}_t^K u_n(p, \cdot) = p^{\alpha(\cdot)} \hat{u}_n(p, \cdot).
\]

Therefore, using the fact that for all \( p \in \mathbb{C} \) satisfying \( \Re p > \tau_1 \), \( \hat{u}_n(p, \cdot) \) solves (1.10) with \( F = F_n \) and \( u_0 \equiv 0 \), we deduce that

\[
\hat{\rho D}_t^K u_n(p, \cdot) = p^{\alpha(\cdot)} \hat{u}_n(p, \cdot) = -A \hat{u}_n(p, \cdot) + F_n(p, \cdot) = \hat{w}_n(p, \cdot), \quad p \in \mathbb{C}, \Re p > \tau_1,
\]
where \( w_n(t, x) = -Au_n(t, x) + F_n(t, x), (t, x) \in \mathbb{R}_+ \times \Omega \). Combining this with the uniqueness and the analyticity of the Laplace transform in time of \( u_n \), we deduce that the identity

\[
\rho(x)D^K_t u_n(t, x) + Au_n(t, x) = F_n(t, x), \quad (t, x) \in \mathbb{R}_+ \times \Omega
\]

holds true in the sense of distributions on \( \mathbb{R}_+ \times \Omega \). In the same way, using the fact that \( u_n \in C^1([0, \infty); L^2(\Omega)) \) with \( u_n(0, \cdot) \equiv 0 \), we deduce that \( I_K u_n \in C^1([0, \infty); L^2(\Omega)) \) and

\[
I_K u_n(0, x) = 0, \quad x \in \Omega.
\]  

(2.13)

From now on, I will prove that the above properties can be extended by density to \( u \). Fix \( T_1 > 0 \). Applying this last identity, I will show that \( D^K_t u_n \) converges in the sense of \( L^1(0, T_1; D'(\mathbb{R}); L^2(\Omega)) \) to \( -\rho^{-1}u(t, x) + F(t, x) \) as \( n \to \infty \), and then I will complete the proof of the theorem.

Define the space \( L^2(\rho \, dx) \) corresponding to the space of function \( L^2 \) on \( \Omega \) with a measure of density \( \rho \). I define the operator \( A_\rho = \rho^{-1}A \) acting on \( L^2(\rho \, dx) \) with domain \( D(A_\rho) = \{ v \in H^1_0(\Omega) : \rho^{-1}Av \in L^2(\Omega) \} \) and I recall that \( A_\rho \) is a self-adjoint operator with a compact resolvent whose spectrum consists of a nondecreasing unbounded positive eigenvalues. In view of Proposition 2.3, we have \( u \in L^1_u(\mathbb{R}_+; L^2(\Omega)) \) and we deduce that \( \rho^{-1}Au \in L^1(0, T_1; D(A_\rho^{-1})) \). Moreover, there exists \( C > 0 \) depending only on \( A, \rho, \) and \( \Omega \) such that, for all \( n \in \mathbb{N} \), we have

\[
\| \rho^{-1}A(u_n - u) \|_{L^1(0, T_1; D(A_\rho^{-1}))} \leq C \| u_n - u \|_{L^1(0, T_1; L^2(\Omega))}.
\]  

(2.14)

In addition, we have

\[
u_n(t, \cdot) - u(t, \cdot) = \int_0^t S_1(t - \tau)[F_n(\tau) - F(\tau)]d\tau, \quad t \in \mathbb{R}_+, \ n \in \mathbb{N},
\]

and applying Lemma 2.2 and Young’s convolution inequality, we obtain

\[
\| u_n - u \|_{L^1(0, T_1; L^2(\Omega))} \leq C \| t^{2\alpha_0 - \alpha M - 1} \|_{L^1(0, T_1)} \| F_n - F \|_{L^1(0, T_1; L^2(\Omega))}
\]

\[
\leq C \| (1 + t)^{-J}F_n - F \|_{L^1(0, T_1; L^2(\Omega))} \leq C \| (1 + t)^{-J}F_n - F \|_{L^1(\mathbb{R}_+; L^2(\Omega))}
\]

and (2.11) implies that

\[
\lim_{n \to \infty} \| \rho^{-1}Au_n - \rho^{-1}Au \|_{L^1(0, T_1; D(A_\rho^{-1}))} = 0.
\]

In the same way, we have

\[
\lim_{n \to \infty} \| \rho^{-1}F_n - \rho^{-1}F \|_{L^1(0, T_1; D(A_\rho^{-1}))} \leq C \lim_{n \to \infty} \| (1 + t)^{-J}F_n - F \|_{L^1(\mathbb{R}_+; L^2(\Omega))} = 0
\]

and it follows that \( D^K_t u_n \) converges in the sense of \( L^1(0, T_1; D(A_\rho^{-1})) \) to \( -\rho^{-1}Au + F \) as \( n \to \infty \). On the other hand, for all \( \psi \in C_0^\infty(0, T_1) \), we have

\[
\langle D^K_t u_n(t, \cdot), \psi(t) \rangle_{D'(0, T_1), C_0^\infty(0, T_1)} = \langle \delta I_K u_n(t, \cdot), \psi(t) \rangle_{D'(0, T_1), C_0^\infty(0, T_1)}
\]

\[
\quad = -\langle I_K u_n(t, \cdot), \psi(t) \rangle_{D'(0, T_1), C_0^\infty(0, T_1)}.
\]

In addition, repeating the above arguments and applying (2.11), one can check that the sequence \( (u_n)_{n \in \mathbb{N}} \) converges to \( u \) in the sense of \( L^1(0, T_1; L^2(\Omega)) \) as \( n \to \infty \) and, applying again Young’s convolution inequality, we deduce that the sequence
\( I_K u_n \) converges to \( I_K u \) in the sense of \( L^1(0, T_1; L^2(\Omega)) \) as \( n \to \infty \). Thus, sending \( n \to \infty \), we find

\[
\lim_{n \to \infty} \langle D^K_t u_n(t, \cdot), \psi(t) \rangle_{D'(0,T_1), C^0(0,T_1)} = -\lim_{n \to \infty} \langle I_K u_n(t, \cdot), \psi'(t) \rangle_{D'(0,T_1), C^0(0,T_1)} = -\langle I_K u(t, \cdot), \psi'(t) \rangle_{D'(0,T_1), C^0(0,T_1)} = \langle D^K_t u(t, \cdot), \psi(t) \rangle_{D'(0,T_1), C^0(0,T_1)}.
\]

It follows that \( D^K_t u_n \) converges in the sense of \( D'(0, T_1; L^2(\Omega)) \) to \( D^K_t u \) as \( n \to \infty \). Therefore, by the uniqueness of the limit in the sense of \( D'(0, T_1; D(A^{-1}_{-1})) \), we deduce that

\[
\lim_{n \to \infty} \langle D^K_t u_n(t, \cdot), h \rangle_{D'(0,T_1), D(A^{-1}_{+})} = \langle \partial_t I_K u(t, \cdot), h \rangle_{D'(0,T_1), D(A^{-1}_{+})} = \langle D^K_t u(t, \cdot), h \rangle_{D'(0,T_1), D(A_{-1})}.
\]

Step 3. In this step, I will prove that the Laplace-weak solution \( u \in L^1_{up-loc}(\mathbb{R}_+; D(A^{-1}_{-1})) \) of (1.4), given by (2.5), satisfies condition (ii) of Definition 1.1, which implies that \( u \) satisfies all the conditions of Definition 1.1. Thus, (2.5) is a weak solution of (1.4) in the sense of Definition 1.1. This completes the proof of the theorem when \( u_0 \equiv 0 \).

**Lemma 2.4.** Let \( \delta > 0 \) and \( \theta \in (\pi/2, \pi) \). Then we have

\[
\frac{1}{2\pi} \int_{\gamma(\delta, \theta)} \frac{e^{itp}}{p} dp = 1, \quad t \in \mathbb{R}_+.
\] (2.15)
We postpone the proof of this lemma to the end of the present demonstration. Fix \((u_{n,0})_{n \in \mathbb{N}}\) a sequence of \(C_0^\infty(\Omega)\) such that

\[
\lim_{n \to \infty} \|u_{0,n} - u_0\|_{L^2(\Omega)} = 0 \tag{2.16}
\]

and consider

\[u_n(t, \cdot) = S_0(t)u_{0,n}.\]

Fix \(n \in \mathbb{N}\). In view of (2.15) and (2.3), we have

\[
u_n(t, \cdot) - u_{0,n} = \frac{1}{2i\pi} \int_{\gamma(\delta, \vartheta)} \frac{e^{ip}}{p} (A + \rho p \alpha(\cdot))^{-1} \rho p \alpha(\cdot) u_{0,n} - \left( \frac{1}{2i\pi} \int_{\gamma(\delta, \vartheta)} \frac{e^{ip}}{p} dp \right) u_{0,n}
\]

On the other hand, for all \(p \in \mathbb{C} \setminus (-\infty, 0]\), we have

\[
(A + \rho p \alpha(\cdot))^{-1} \rho p \alpha(\cdot) u_{0,n} - u_{0,n} = -(A + \rho p \alpha(\cdot))^{-1} A u_{0,n}
\]

and it follows that

\[
u_n(t, \cdot) - u_{0,n} = -\frac{1}{2i\pi} \int_{\gamma(\delta, \vartheta)} \frac{e^{ip}}{p} (A + \rho p \alpha(\cdot))^{-1} A u_{0,n} \, dp.
\]

On the other hand, in view of [19, Proposition 2.1], there exists a constant \(C > 0\) depending on \(A, \rho, \Omega, \vartheta, \) and \(\alpha\) such that, for all \(p \in \gamma(\delta, \vartheta)\), we have

\[
\left\| (A + \rho p \alpha(\cdot))^{-1} \right\|_{L^2(\Omega)} \leq C \max \left( |p|^{\alpha_0 - 2\alpha_M}, |p|^{\alpha_M - 2\alpha_0} \right).
\]

Therefore, there exists a constant \(C > 0\) depending on \(A, \rho, \Omega, \vartheta, \) and \(\alpha\) such that, for all \(p \in \gamma(\delta, \vartheta)\) and all \(t \in \mathbb{R}_+\), we have

\[
\left\| \frac{e^{ip}}{p} (A + \rho p \alpha(\cdot))^{-1} A u_{0,n} \right\|_{L^2(\Omega)} \leq e^{\Re p} C \max \left( |p|^{\alpha_0 - 2\alpha_M - 1}, |p|^{\alpha_M - 2\alpha_0 - 1} \right),
\]

\[
\left\| \frac{e^{ip}}{p} (A + \rho p \alpha(\cdot))^{-1} A u_{0,n} \right\|_{L^2(\Omega)} \leq e^{\Re p} C \max \left( |p|^{\alpha_0 - 2\alpha_M}, |p|^{\alpha_M - 2\alpha_0} \right).
\]

It follows that

\[
\partial_t \nu_n(t, \cdot) = \partial_t [\nu_n(t, \cdot) - u_{0,n}] = -\frac{1}{2i\pi} \int_{\gamma(\delta, \vartheta)} e^{ip} (A + \rho p \alpha(\cdot))^{-1} A u_{0,n} \, dp = -S_1(t) A u_{0,n}.
\]

Combining this with Lemma 2.2 and the fact that \(A u_{0,n} \in L^2(\Omega)\), we deduce that the map \(u_n\) is lying in \(W^{1,1}_{up-\infty}(\mathbb{R}_+; L^2(\Omega))\). Moreover, we have

\[
u_n(0, \cdot) - u_{0,n} = -\frac{1}{2i\pi} \int_{\gamma(\delta, \vartheta)} p^{-1} (A + \rho p \alpha(\cdot))^{-1} A u_{0,n} \, dp.
\]

Let us prove that

\[
\nu_n(0, \cdot) - u_{0,n} = -\frac{1}{2i\pi} \int_{\gamma(\delta, \vartheta)} p^{-1} (A + \rho p \alpha(\cdot))^{-1} A u_{0,n} \, dp \equiv 0. \tag{2.19}
\]
For this purpose, I fix $\delta < 1, R > 1$ and I consider the contour

$$
\gamma(\delta, R, \theta) := \gamma_-(\delta, R, \theta) \cup \gamma_0(\delta, \theta) \cup \gamma_+(\delta, R, \theta) 
$$

(2.20)

oriented in the counterclockwise direction, where $\gamma_0(\delta, \theta)$ is given by (2.2) and where

$$
\gamma_\pm(\delta, R, \theta) := \{re^{\pm i\theta} : r \in (\delta, R)\}.
$$

Applying the Cauchy formula, for any $R > 1$, we have

$$
\frac{1}{2i\pi} \int_{\gamma(\delta, R, \theta)} p^{-1}(A + \rho p^{\alpha(\cdot)})^{-1}Au_{0,n}dp = \frac{1}{2i\pi} \int_{\gamma_0(R, \theta)} p^{-1}(A + \rho p^{\alpha(\cdot)})^{-1}Au_{0,n}dp,
$$

with $\gamma_0(R, \theta)$ given by (2.2) with $\delta = R$. Sending $R \to +\infty$, we obtain

$$
\frac{1}{2i\pi} \int_{\gamma(\delta, \theta)} p^{-1}(A + \rho p^{\alpha(\cdot)})^{-1}Au_{0,n}dp = \lim_{R \to +\infty} \frac{1}{2i\pi} \int_{\gamma_0(R, \theta)} p^{-1}(A + \rho p^{\alpha(\cdot)})^{-1}Au_{0,n}dp.
$$

On the other hand, applying (2.17), we deduce that

$$
\left\| \frac{1}{2i\pi} \int_{\gamma_0(R, \theta)} p^{-1}(A + \rho p^{\alpha(\cdot)})^{-1}Au_{0,n}dp \right\|_{L^2(\Omega)} \leq C \int_{-\theta}^{\theta} \left\| (A + \rho (Re^{i\beta})^{\alpha(\cdot)})^{-1} \right\|_{B(R^2(\Omega))} \left\| Au_{0,n} \right\|_{L^2(\Omega)} d\beta 
$$

$$
\leq C \max(R^{2\alpha_0-2\alpha_M}, R^{2\alpha_M-2\alpha_0}) \left\| Au_{0,n} \right\|_{L^2(\Omega)}.
$$

In view of (1.11), it follows

$$
\frac{1}{2i\pi} \int_{\gamma(\delta, \theta)} p^{-1}(A + \rho p^{\alpha(\cdot)})^{-1}Au_{0,n}dp = \lim_{R \to +\infty} \frac{1}{2i\pi} \int_{\gamma_0(R, \theta)} p^{-1}(A + \rho p^{\alpha(\cdot)})^{-1}Au_{0,n}dp \equiv 0.
$$

This proves (2.19) and in a similar way to Step 2, we deduce that $I_K[u_n - u_{0,n}] \in W^{1,1}_{uloc}(\mathbb{R}^+; L^2(\Omega))$ satisfies

$$
I_K[u_n - u_{0,n}](0, x) = 0, \quad x \in \Omega.
$$

(2.21)

Therefore, $u_n$ satisfies condition (ii) of Definition 1.1. Now let us show that $u_n$ satisfies condition (i) of Definition 1.1. Applying Lemma 2.2 and (2.18), we deduce that there exists a constant $C > 0$ such that, for all $t > 0$, we have

$$
\|u_n(t, \cdot)\|_{L^2(\Omega)} + \|\partial_t u_n(t, \cdot)\|_{L^2(\Omega)} 
$$

$$
\leq \|S_0(t)u_{0,n}\|_{L^2(\Omega)} + \|S_1(t)Au_{0,n}\|_{L^2(\Omega)} 
$$

$$
\leq C \max(t^{2(\alpha_M-\alpha_0)}, t^{2(\alpha_0-\alpha_M)}, t^{2\alpha_M-\alpha_0-1}, t^{2\alpha_0-\alpha_M-1}) \left\| u_{0,n} \right\|_{D(A)}.
$$
Combining this with (1.11), for all \( p \in \mathbb{C}_+ \), we obtain \( t \mapsto e^{-pt}I_K[u_n(t, \cdot) - u_{0,n}] \in W^{1,1}(\mathbb{R}_+; L^2(\Omega)) \). Thus, for all \( n \in \mathbb{N} \) and all \( p \in \mathbb{C}_+ \), fixing \( v_n(t, \cdot) = u_n(t, \cdot) - u_{0,n}, t > 0 \), and applying (2.21), we get

\[
\hat{D}_K t v_n(p, \cdot) = \int_0^{+\infty} e^{-pt} \partial_t I_K[u_n(t, \cdot) - u_{0,n}] dt \\
= p \int_0^{+\infty} e^{-pt} I_K[u_n(t, \cdot) - u_{0,n}] dt \\
= p \left( \int_0^{+\infty} e^{-pt} I_K u_n(t, \cdot) dt - \int_0^{+\infty} e^{-pt} \frac{t^{1-\alpha(\cdot)}}{\Gamma(2-\alpha(\cdot))} dt \right) \\
= p^{\alpha(\cdot)} \hat{u}_n(p, \cdot) - p^{\alpha(\cdot)-1} u_{0,n}.
\]

Therefore, using the fact that for all \( p \in \mathbb{C} \) satisfying \( \Re p > \tau_1 \), \( \hat{u}_n(p, \cdot) \) solves (1.10) with \( F \equiv 0 \) and \( u_0 = u_{0,n} \), we deduce that

\[
\hat{D}_K t v_n(p, \cdot) = p^{\alpha(\cdot)} \hat{u}_n(p, \cdot) - p^{\alpha(\cdot)-1} u_{0,n} = -\rho^{-1} \hat{u}_n(p, \cdot), \quad p \in \mathbb{C}, \ Re p > \tau_1.
\]

Then in a similar way to Step 2, we find that the identity

\[
\rho(x) D_K t [u_n - u_{0,n}](t, x) + \mathcal{A} u_n(t, x) = 0, \quad (t, x) \in \mathbb{R}_+ \times \Omega
\]

holds true in the sense of distributions on \( \mathbb{R}_+ \times \Omega \), which implies that \( u_n \) satisfies condition (i) of Definition 1.1 with \( u_0 = u_{0,n} \) and \( F \equiv 0 \). I will now extend this result by density to the Laplace-weak solution of (1.4), which is given by (2.5) with \( F \equiv 0 \). For this purpose, fix \( T_1 > 0 \). Let us first observe that applying Lemma 2.2, we obtain

\[
\lim_{n \to +\infty} \| u_n - u \|_{L^1(0, T_1; L^2(\Omega))} \leq C \| t^{2(\alpha_0 - \alpha_M)} \|_{L^1(0, T_1)} \lim_{n \to +\infty} \| u_{0,n} - u_0 \|_{L^2(\Omega)} = 0.
\]

Therefore, repeating the arguments of Step 2, we can prove that \( D_K t u_n \) converges in the sense of \( D'(0, T_1; L^2(\Omega)) \) to \( D_K t u \) and in the sense of \( L^1(0, T_1; D'(\Omega)) \) to \( \rho^{-1} \mathcal{A} u \) as \( n \to \infty \). Then, repeating the arguments used in the last part of Step 2, we deduce that the Laplace-weak solution \( u \) of (1.4) fulfills the conditions (i) and (ii) of Definition 1.1. This completes the proof of Theorem 1.3.

Now that I have completed the proof of Theorem 1.3, let us consider the proof of Lemma 2.4.

**Proof of Lemma 2.4.** Let us first recall that for all \( t > 0 \), the map \( z \mapsto \frac{e^{iz}}{z} \) is meromorphic on \( \mathbb{C} \) with a simple pole at \( z = 0 \). Therefore, the residue theorem implies that for all \( R > \delta \), we have

\[
\frac{1}{2i\pi} \int_{\gamma(\delta, R, \theta)} \frac{e^{ip}}{p} dp = 1 + \frac{1}{2i\pi} \int_{\gamma_1(R, \theta)} \frac{e^{ip}}{p} dp, \quad t \in \mathbb{R}_+,
\]

where I recall that \( \gamma(\delta, R, \theta) \) is given by (2.20) and \( \gamma_1(R, \theta) \) is given by

\[
\gamma_1(R, \theta) := \{ R e^{i\beta} : \beta \in [\theta, 2\pi - \theta] \}.
\]

Sending \( R \to +\infty \), we obtain

\[
\frac{1}{2i\pi} \int_{\gamma(\delta, \theta)} \frac{e^{ip}}{p} dp = \lim_{R \to +\infty} \frac{1}{2i\pi} \int_{\gamma(\delta, R, \theta)} \frac{e^{ip}}{p} dp.
\]

On the other hand, we have

\[
\left| \frac{1}{2i\pi} \int_{\gamma_1(R, \theta)} \frac{e^{ip}}{p} dp \right| \leq \frac{1}{2\pi} \int_0^{2\pi-\theta} e^{iR\cos \beta} d\beta \\
\leq Ce^{iR\cos \theta}.
\]
In view of (2.22) and the fact that $\theta \in (\pi/2, \pi)$, we find
\[
\frac{1}{2i\pi} \int_{\gamma(\delta, \theta)} e^{\lambda \theta} \frac{d\lambda}{\lambda} = 1 + \lim_{R \to +\infty} \frac{1}{2i\pi} \int_{\gamma(R, \delta)} e^{\lambda \theta} \frac{d\lambda}{\lambda}, \quad t \in \mathbb{R}_+.
\]
This proves (2.15) and it completes the proof of Lemma 2.4.

\[\Box\]

3 DISTRIBUTED ORDER FRACTIONAL DIFFUSION EQUATIONS

In this section, I prove the unique existence of a weak solution to the problem (1.4) as well as the equivalence between Definitions 1.1 and 1.2 of weak and Laplace-weak solutions of (1.4) for weight $K$ given by (1.8) with $\mu \in C([0, 1])$ a nonnegative function satisfying (1.12). For this purpose, let us first recall that the unique existence of Laplace-weak solutions for (1.4) has been proved by [27] in the case of source terms $F \in L^\infty(\mathbb{R}_+; L^2(\Omega))$ and extended to source terms $F \in L^1(\mathbb{R}_+; L^2(\Omega))$ by [18, Proposition 5.1]. I will recall here the representation of Laplace-weak solutions of (1.4) given by these works. Like in the previous section, I denote by $A_*$ the operator $\rho^{-1}A$ acting in the space $L^2(\Omega; \rho\, dx)$ with Dirichlet boundary condition. Let $(\varphi_n)_{n \geq 1}$ be an $L^2(\Omega; \rho\, dx)$ orthonormal basis of eigenfunctions of the operator $A_*$ associated with the nondecreasing sequence of eigenvalues $(\lambda_n)_{n \geq 1}$ of $A_*$ repeated with respect to their multiplicity. According to [27, Proposition 2.1], the unique Laplace-weak solution $u$ of (1.4) enjoys the following representation formula,
\[
u(t, \cdot) = S_0,\mu(t) u_0 + \int_0^t S_1,\mu(t - s) F(s, \cdot) ds, \quad t \in \mathbb{R}_+,
\]
where
\[
S_0,\mu(t) \varphi := \sum_{n=1}^{\infty} \left( \frac{1}{2i\pi} \int_{\gamma(\delta, \theta)} e^{\lambda \theta} \frac{\lambda}{\lambda} \varphi(p)^{-1} \delta(p) \varphi_n L^2(\Omega; \rho\, dx) d\lambda \right) \varphi_n, \quad \varphi \in L^2(\Omega),
\]
and
\[
S_1,\mu(t) \varphi := \sum_{n=1}^{\infty} \left( \frac{1}{2i\pi} \int_{\gamma(\delta, \theta)} e^{\lambda \theta} \frac{\lambda}{\lambda} \varphi(p)^{-1} \rho(p) \varphi_n L^2(\Omega; \rho\, dx) d\lambda \right) \varphi_n, \quad \varphi \in L^2(\Omega),
\]
where $\delta(p) := \int_0^\infty p^\alpha \mu(\alpha) d\alpha, \theta \in (\pi/2, \pi), \delta > 0$, and $\gamma(\delta, \theta)$ corresponds to the contour (2.1). According to [27], the map $S_{j,\mu}$ is independent of the choice of $\theta \in (\pi/2, \pi), \delta > 0$. I start by proving an extension of this result to source terms $F \in J$ and by proving that the Laplace-weak solution $u$ given by (3.1) is lying in $L^1_{up-loc}(\mathbb{R}_+; B(\Omega))$. For this purpose, like in the previous section, I need the following intermediate result about the operator valued functions $S_{0,\mu}$ and $S_{1,\mu}$.

Lemma 3.1. Let $\theta \in \left(\frac{\pi}{2}, \pi\right)$. The maps $t \mapsto S_{j,\mu}(t), j = 0, 1$, defined by (3.2)–(3.3) are lying in $L^1_{up-loc}(\mathbb{R}_+; B(\Omega))$ and there exists a constant $C > 0$ depending only on $A, \rho, \theta, \mu, \Omega$ such that the estimates
\[
\|S_{0,\mu}(t)\|_{B(\Omega)} \leq C \max \left( t^{a_0-\varepsilon-1}, t^{a_0}, 1 \right), \quad t > 0,
\]
and
\[
\|S_{1,\mu}(t)\|_{B(\Omega)} \leq C \max \left( t^{a_0-\varepsilon-1}, t^{a_0-1}, 1 \right), \quad t > 0,
\]
hold true.

Proof. For the proof of these results for $S_{1,\mu}$, one can refer to [18, Proposition 5.1] and I only show the above properties for $S_{0,\mu}$. For this purpose, I recall the following estimate from [27, Lemma 2.2],
\[
\frac{1}{|\delta(p) + \lambda_n|} \leq C \max(|p|^{-a_0+\varepsilon}, |p|^{-a_0}), \quad p \in \mathbb{C} \setminus (-\infty, 0), \quad n \in \mathbb{N},
\]
where the positive constant $C$ depends only on $\mu$. We recall also that
\[
|\delta(p)| \leq C \max(|p|, 1), \quad p \in \mathbb{C} \setminus (-\infty, 0),
\]
where the positive constant $C$ depends only on $\mu$. Therefore, we have

$$\frac{|\hat{\vartheta}(p)|}{|p|} \leq C \max(|p|^{-\delta_0+\varepsilon}, |p|^{-1-\delta_0}), \ p \in C \setminus (-\infty,0], \ n \in \mathbb{N},$$

(3.7)

where the positive constant $C$ depends only on $\mu$. For all $t \in (0, +\infty)$ and all $\psi \in L^2(\Omega)$, by taking $\delta = t^{-1}$ in (2.1), (3.7) implies

$$\left\|\sum_{n=1}^{+\infty} \left( \int_{\gamma_0(t-1,\theta)} \frac{\hat{\vartheta}(p)e^{pt}}{p(\hat{\vartheta}(p) + \lambda_n)} dp \right) \langle \psi, \varphi_n \rangle_{L^2(\Omega)} \varphi_{n,k} \right\|_{L^2(\Omega)} \leq C \left( \int_{1}^{+\infty} \max(r^{-\delta_0+\varepsilon}, r^{-\delta_0-1})e^{r \cos \theta} dr \right) \left\| \psi \right\|_{L^2(\Omega)},$$

(3.8)

$$\leq C \max(t^{\alpha_0-\varepsilon-1}, t^{\alpha_0}) \left( \int_{-\theta}^{\theta} e^{\cos \beta} d\beta \right) \left\| \psi \right\|_{L^2(\Omega)},$$

Putting these two estimates together with (2.2), we deduce (3.4) and the fact that $S_{0,\mu} \in L^1_{u_p-loc}(\mathbb{R}_+; B(L^2(\Omega)))$.

Combining Lemma 3.1 with the arguments used in Proposition 2.3, we obtain the following result about the unique existence of Laplace-weak solution for problem (1.4).

**Proposition 3.2.** Assume that the conditions (1.1)–(1.2) are fulfilled. Let $u_0 \in L^2(\Omega)$, $F \in J$, $\mu \in C([0,1])$ be a nonnegative function satisfying (1.12), and let $K$ be given by (1.8). Then, there exists a unique Laplace-weak solution $u \in L^1_{u_p-loc}(\mathbb{R}_+; L^2(\Omega))$ to (1.4) given by (3.1).

Armed with this result, we can now complete the proof of Theorem 1.4.

**Proof of Theorem 1.4.** Let us first observe that the first statement of Theorem 1.4 is a direct consequence of Proposition 3.2. Moreover, the uniqueness of weak solutions in the sense of Definition 1.1 as well as the fact that, for $u_0 \equiv 0$, (1.4) admits a unique weak solution $u \in L^1_{u_p-loc}(\mathbb{R}_+; L^2(\Omega))$ in the sense of Definition 1.1 given by (3.1), can be deduced by mimicking the proof of Theorem 1.3. For this purpose, I only show that $u$ given by (3.1) is a weak solution of (1.4) in the sense of Definition 1.1 when $F \equiv 0$. Since the Laplace-weak solution $u \in L^1_{u_p-loc}(\mathbb{R}_+; L^2(\Omega))$ of (1.4) clearly satisfies condition (iii) of Definition 1.1, I only need to check conditions (i) and (ii). Fix $(u_{0,n})_{n \in \mathbb{N}}$ a sequence of $C^\infty_0(\Omega)$ such that (2.16) is fulfilled and consider

$$u_n(t, \cdot) = S_{0,\mu}(t)u_{0,n}, \ t \in \mathbb{R}_+.$$
Fix \( n \in \mathbb{N} \). In view of (2.15) and (3.2), we have

\[
\begin{align*}
\| u_n(t, \cdot) - u_{0,n} \| & = \sum_{k=1}^{\infty} \left( \frac{1}{2\pi} \int_{\gamma(\delta, \theta)} \frac{\varphi_k}{p(\lambda_k + \vartheta(p))} \, dp \right) \langle u_{0,n}, \varphi_k \rangle_{L^2(\Omega; \rho \, dx)} \varphi_k - \left( \frac{1}{2\pi} \int_{\gamma(\delta, \theta)} \frac{e^{\rho t}}{p} \, dp \right) u_{0,n} \\
& = \sum_{k=1}^{\infty} \left( \frac{1}{2\pi} \int_{\gamma(\delta, \theta)} \frac{e^{\rho t}}{p} \left( \frac{\vartheta(p)}{\lambda_k + \vartheta(p)} - 1 \right) \, dp \right) \langle u_{0,n}, \varphi_k \rangle_{L^2(\Omega; \rho \, dx)} \varphi_k \\
& = -\sum_{k=1}^{\infty} \left( \frac{1}{2\pi} \int_{\gamma(\delta, \theta)} \frac{e^{\rho t}}{p(\lambda_k + \vartheta(p))} \, dp \right) \langle A_n u_{0,n}, \varphi_k \rangle_{L^2(\Omega; \rho \, dx)} \varphi_k.
\end{align*}
\]

On the other hand, applying (3.6), for all \( k \in \mathbb{N} \) and all \( p \in \mathbb{C} \setminus (-\infty, 0] \), we have

\[
\left| \frac{e^{\rho t}}{p(\lambda_k + \vartheta(p))} \langle A_n u_{0,n}, \varphi_k \rangle_{L^2(\Omega; \rho \, dx)} \right| \leq C \max \left( \left| p \right|^{-\alpha_0 + \varepsilon - 1}, \left| p \right|^{-\alpha_0 - 1} \right) \left| \langle A_n u_{0,n}, \varphi_k \rangle_{L^2(\Omega; \rho \, dx)} \right|.
\]

Combining this with the fact that \( u_{0,n} \in D(A_n) \), we deduce that

\[
\partial_t u_n(t, \cdot) = -\sum_{k=1}^{\infty} \left( \frac{1}{2\pi} \int_{\gamma(\delta, \theta)} \frac{e^{\rho t}}{p(\lambda_k + \vartheta(p))} \, dp \right) \langle A_n u_{0,n}, \varphi_k \rangle_{L^2(\Omega; \rho \, dx)} \varphi_k = -S_{1,\rho}(t) A u_{0,n}.
\]

Applying Lemma 3.1 and the fact that \( A u_{0,n} \in L^2(\Omega) \), we deduce that the map \( u_n \) is lying in \( W^{1,1}_{u_p-loc}(\mathbb{R}_+; L^2(\Omega)) \). Moreover, we have

\[
\begin{align*}
u_n(0, \cdot) - u_{0,n} &= -\sum_{k=1}^{\infty} \left( \frac{1}{2\pi} \int_{\gamma(\delta, \theta)} \frac{1}{p(\lambda_k + \vartheta(p))} \, dp \right) \langle A_n u_{0,n}, \varphi_k \rangle_{L^2(\Omega; \rho \, dx)} \varphi_k.
\end{align*}
\]

Applying the arguments used at the end of the proof of [27, Proposition 2.1], we obtain

\[
\frac{1}{2\pi} \int_{\gamma(\delta, \theta)} \frac{1}{p(\lambda_k + \vartheta(p))} \, dp = 0, \quad k \in \mathbb{N}
\]

and it follows that \( I_K[u_n - u_{0,n}] \in W^{1,1}_{u_p-loc}(\mathbb{R}_+; L^2(\Omega)) \) satisfies

\[
I_K[u_n - u_{0,n}](0, x) = 0, \quad x \in \Omega.
\]

This proves that \( u_n \) satisfies condition (ii) of Definition 1.1. Combining Lemma 3.1, Proposition 2.3 with the arguments used in the last step of the proof of Theorem 1.4, we can show that \( u_n \) satisfies also condition (i). In the same way, using Lemma 3.1, Proposition 2.3, and repeating the arguments used in the last step of the proof of Theorem 1.4, we can show that, by density these properties can be extended to \( u \). This proves that the Laplace-weak solution \( u \) of (1.4), given by (3.1), fulfills the conditions (i) and (ii) of Definition 1.1 and it completes the proof of Theorem 1.4.

\[\square\]

## 4  MULTITERM FRACTIONAL DIFFUSION EQUATIONS

In this section, I prove the unique existence of a weak solution to the problem (1.4), with \( \rho \equiv 1 \), as well as the equivalence between Definition 1.1 and 1.2 of weak and Laplace-weak solutions of (1.4) for weight \( K \) given by (1.9) with \( 0 < \alpha_1 < \ldots < \alpha_N < 1 \) and \( \rho_j \in L^\infty(\Omega) \), \( j = 1, \ldots, N \), satisfying (1.2) with \( \rho = \rho_j \). In contrast to variable order and distributed order fractional diffusion equations, I have not found any result in the mathematical literature showing the unique existence
of Laplace-weak solutions for multiterm fractional diffusion equations. For this purpose, I will consider first the proof of this result.

For all \( p \in \mathbb{C} \setminus (-\infty, 0] \), we can consider the following operator:

\[
\left( A + \sum_{k=1}^{N} \rho_k(x) p^\alpha_k \right)^{-1} \in \mathcal{B}(L^2(\Omega)).
\]

For all \( \theta \in (0, \pi) \), I denote by \( D_\theta \) the following set \( D_\theta := \{ r e^{i \beta} : r > 0, \beta \in (-\theta, \theta) \} \). Inspired by [19, Proposition 2.1], I start with the following properties of the above operator.

**Lemma 4.1.** Let \( \theta \in (0, \pi) \). Then, there exists a constant \( C > 0 \) depending only on \( A, \rho_1, \ldots, \rho_N, \alpha_1, \ldots, \alpha_N, \Omega \) and \( \theta \) such that

\[
\left\| \left( A + \sum_{k=1}^{N} \rho_k(x) z^{\alpha_k} \right)^{-1} \right\|_{B(L^2(\Omega))} \leq C |z|^{-\alpha_N}, \quad z \in D_\theta.
\] (4.1)

**Proof.** Let us observe that, since the spectrum of \( A \) is discrete and contained into \( \mathbb{R}^+ \), it is enough to prove (4.1) with \( z \in D_\theta \) satisfying \( |z| > 1 \). For this purpose, from now on I fix \( z = re^{i \beta} \) with \( r \in [1, +\infty), \beta \in (-\theta, \theta) \) and I will show (4.1). In all this proof, \( c_0 \) and \( C_0 \) denote the constants appearing in (1.2). I divide the proof of this result into two steps.

**Step 1:** In this step, I will prove that for all \( \beta \in (-\theta, \theta) \setminus \{0\} \), we have

\[
\left\| \left( A + \sum_{k=1}^{N} \rho_k(x) z^{\alpha_k} \right)^{-1} \right\|_{B(L^2(\Omega))} \leq c_0^{-1} \max\left( |\sin(\alpha_1 \beta)|^{-1}, |\sin(\alpha_N \beta)|^{-1} \right) r^{-\alpha_N}.
\] (4.2)

For this purpose, we assume that \( \beta \in (0, \theta) \), the case of \( \beta \in (-\theta, 0) \) being treated in a similar fashion. Let \( B_\beta \) be the multiplier in \( L^2(\Omega) \), by the function

\[
b_\beta(x) := \left( \sum_{k=1}^{N} \rho_k(x) r^{\alpha_k} \sin(\beta \alpha_k) \right)^{1/2}, \quad x \in \Omega,
\]

in such a way that \( i B_\beta^2 \) is the skew-adjoint part of the operator \( A + \sum_{k=1}^{N} \rho_k(x) r^{\alpha_k} e^{i \beta \alpha_k} \). Applying (1.2), we obtain

\[
0 < c_0^{1/2} \min \left( |\sin(\alpha_1 \beta)|^{1/2}, |\sin(\alpha_N \beta)|^{1/2} \right) r^{\alpha_N/2} \leq b_\beta(x), \quad x \in \Omega,
\]

\[
b_\beta(x) \leq (NC_0)^{1/2} \max \left( |\sin(\alpha_1 \beta)|^{1/2}, \ldots, |\sin(\alpha_N \beta)|^{1/2} \right) r^{\alpha_N/2}, \quad x \in \Omega.
\]

Hence, the self-adjoint operator \( B_\beta \) is bounded and boundedly invertible in \( L^2(\Omega) \), with

\[
\| B_\beta^{-1} \|_{B(L^2(\Omega))} \leq c_0^{-1/2} \max \left( |\sin(\alpha_1 \beta)|^{-1/2}, |\sin(\alpha_N \beta)|^{-1/2} \right) r^{-\alpha_N/2}.
\] (4.3)

Moreover, for each \( z = re^{i \beta} \), it holds true that

\[
A + \sum_{k=1}^{N} \rho_k(x) z^{\alpha_k} = B_\beta \left( B_\beta^{-1} H_\beta B_\beta^{-1} + i \right) B_\beta,
\] (4.4)

where \( H_\beta := A + \sum_{k=1}^{N} \rho_k(x) r^{\alpha_k} \cos(\beta \alpha_k) \). It is clear that the operator \( H_\beta \) is self-adjoint in \( L^2(\Omega) \) with domain \( D(H_\beta) = D(A) \), by the Kato–Rellich theorem. Thus, \( B_\beta^{-1} H_\beta B_\beta^{-1} \) is self-adjoint in \( L^2(\Omega) \) as well, with domain \( B_\beta D(A) \). Therefore, the operator \( B_\beta^{-1} H_\beta B_\beta^{-1} + i \) is invertible in \( L^2(\Omega) \) and satisfies the estimate

\[
\| (B_\beta^{-1} H_\beta B_\beta^{-1} + i)^{-1} \|_{B(L^2(\Omega))} \leq 1.
\]
It follows from this and (3.7) that $A_q + \sum_{k=1}^N \rho_k(x)z^{\alpha_k}$ is invertible in $L^2(\Omega)$, with

$$
(A + \sum_{k=1}^N \rho_k(x)z^{\alpha_k})^{-1} = B^{-1}_\beta(U^{-1}_\beta B^{-1}_\beta + i)^{-1}B^{-1}_\beta,
$$

showing that $(A + \sum_{k=1}^N \rho_k(x)z^{\alpha_k})^{-1}$ maps $L^2(\Omega)$ into $B^{-1}_\beta D(B^{-1}_\beta H z B^{-1}_\beta) = D(A)$. As a consequence, we infer from (4.3) that

$$
\left\| (A + \sum_{k=1}^N \rho_k(x)z^{\alpha_k})^{-1} \right\|_{L^2(\Omega)} \leq \|B^{-1}_\beta H z B^{-1}_\beta + i\|_{L^2(\Omega)} \|B^{-1}_\beta\|^2_{L^2(\Omega)}
\leq c^{-1}_0 \max (|\sin(\alpha_1\beta)|^{-1}, |\sin(\alpha_N\beta)|^{-1}) r^{-\alpha_N}.
$$

From this last estimate, we deduce (4.2).

**Step 2:** I fix $\theta_* \in (0, \min(\theta, \pi/2))$ such that

$$
\frac{|\sin(\alpha_N\theta_*)|}{\cos(\alpha_N\theta_*)} \leq \frac{c_0}{2C_0N}.
$$

In this step, I will prove that for all $\beta \in (-\theta_*, \theta_*)$, we have

$$
\left\| (A + \sum_{k=1}^N \rho_k(x)z^{\alpha_k})^{-1} \right\|_{L^2(\Omega)} \leq 2c^{-1}_0 \cos(\alpha_N\theta_*)^{-1} r^{-\alpha_N},
$$

where I recall that since $\theta_* \in (0, \min(\theta, \pi/2))$, we have $\cos(\alpha_N\theta_*) > 0$. Using the fact that the operator $A$ is positive, for all $v \in D(A)$ and $\beta \in (-\theta_*, \theta_*)$, we get

\[
\begin{align*}
\|H_z v\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} &\geq \langle H_z v, v \rangle_{L^2(\Omega)} \\
&\geq \langle A v, v \rangle_{L^2(\Omega)} + \left\langle \sum_{k=1}^N \rho_k \cos(\beta \alpha_k) v, v \right\rangle_{L^2(\Omega)} \\
&\geq \int_{\Omega} \left( \sum_{k=1}^N \rho_k r^{\alpha_k} \cos(\beta \alpha_k) \right) |v|^2 dx \\
&\geq \int_{\Omega} \rho_N r^{\alpha_N} \cos(\beta \alpha_N) |v|^2 dx \\
&\geq c_0 \cos(\beta \alpha_N) r^{\alpha_N} \|v\|_{L^2(\Omega)}^2.
\end{align*}
\]

Choosing $v = H_z^{-1}u$ in the above inequality, we obtain

$$
\left\| H_z^{-1}u \right\|_{L^2(\Omega)} \leq c^{-1}_0 \cos(\alpha_N\theta_*)^{-1} r^{-\alpha_N} \|u\|_{L^2(\Omega)}
$$

and we deduce that $\left\| H_z^{-1}\right\|_{L^2(\Omega)} \leq c^{-1}_0 \cos(\alpha_N\theta_*)^{-1} r^{-\alpha_N}$. Therefore, recalling that $(-\theta_*, \theta_) \subset (-\pi/2, \pi/2)$, we get

\[
\begin{align*}
\left\| -iH_z^{-1}B_\beta \right\|_{L^2(\Omega)} &\leq \left\| H_z^{-1}\right\|_{L^2(\Omega)} \|B_\beta\|^2_{L^2(\Omega)} \\
&\leq \frac{C_0N r^{\alpha_N} |\sin(\alpha_N\beta)|}{c_0 \cos(\beta \alpha_N)} \leq \frac{C_0N \sin(\alpha_N\theta_*)}{c_0 \cos(\beta \alpha_N)}, \quad \beta \in (-\theta_*, \theta_*)
\end{align*}
\]
and, combining this with (4.5), we find

$$\| -iH_z^{-1}B_\beta^2 \|_{\mathcal{B}(L^2(\Omega))} \leq \frac{1}{2}.$$né

Therefore, for all $\beta \in (-\theta_*, \theta_*)$, the operator $(Id + iH_z^{-1}B_\beta^2)^{-1}$ is invertible and we have

$$\| (Id + iH_z^{-1}B_\beta^2)^{-1} \|_{\mathcal{B}(L^2(\Omega))} \leq \frac{1}{1 - \| -iH_z^{-1}B_\beta^2 \|_{\mathcal{B}(L^2(\Omega))}} \leq 2.$$né

It follows that

$$(A + \sum_{k=1}^{N} \rho_k(x)z^{\alpha_k})^{-1} = (Id + iH_z^{-1}B_\beta^2)^{-1}H_z^{-1}, \quad \beta \in (-\theta_*, \theta_*)$$né

and applying (4.7), we deduce (4.6). Combining (4.2) and (4.6), we deduce (4.1) by choosing

$$C = \min(c_0^{-1} \sin(\alpha_N \theta)^{-1}, c_0^{-1} \sin(\alpha_1 \theta)^{-1}, 2c_0^{-1} \cos(\alpha_N \theta)^{-1}).$$

This completes the proof of the lemma.

I fix $\theta \in \left( \frac{\pi}{2}, \pi \right)$, $\delta \in \mathbb{R}_+$, and applying Lemma 4.1, I consider the operator $R_j(t) = B(\mathcal{L}^2(\Omega))$, $j = 0, 1$ and $t \in \mathbb{R}_+$, given by

$$R_0(t)h = \frac{1}{2i\pi} \int_{\gamma(\delta, \delta)} e^{ip} \left( A + \sum_{k=1}^{N} \rho_k p^{\alpha_k} \right)^{-1} \left( \sum_{k=1}^{N} \rho_k p^{\alpha_k-1} \right) h dp, \quad h \in L^2(\Omega), \quad t \in \mathbb{R}_+.$$ (4.8)

$$R_1(t)h = \frac{1}{2i\pi} \int_{\gamma(\delta, \delta)} e^{ip} \left( A + \sum_{k=1}^{N} \rho_k p^{\alpha_k} \right)^{-1} h dp, \quad h \in L^2(\Omega), \quad t \in \mathbb{R}_+.$$ (4.9)

Note that here since the map $z \mapsto (A + \sum_{k=1}^{N} \rho_k(x)z^{\alpha_k})^{-1}$ is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$ as a map taking values in $B(\mathcal{L}^2(\Omega))$, the definition of $R_j$, $j = 0, 1$, will be independent of the choice of $\delta$ and $\delta \in \left( \frac{\pi}{2}, \pi \right)$. Let us consider

$$u(t, \cdot) = R_0(t)u_0 + \int_{0}^{t} R_1(t-s)F(s, \cdot) ds.$$ (4.10)

Combining the arguments used in Lemma 2.2 with estimate (4.1), I can show the following properties of the maps $R_j : t \mapsto R_j(t)$, $j = 0, 1$.

**Lemma 4.2.** Let $\theta \in \left( \frac{\pi}{2}, \pi \right)$. The maps $t \mapsto S_j(t)$, $j = 0, 1$, defined by (4.8)-(4.9) are lying in $L^1_{up-loc}(\mathbb{R}_+; B(\mathcal{L}^2(\Omega)))$ and there exists a constant $C > 0$ depending only on $A, \rho, \delta, \omega$ such that the estimates

$$\| R_0(t) \|_{\mathcal{B}(L^2(\Omega))} \leq C \max \{ t^{\alpha_N - \alpha_1}, 1 \}, \quad t > 0,$$ (4.11)

$$\| R_1(t) \|_{\mathcal{B}(L^2(\Omega))} \leq C \max \{ t^{\alpha_N - 1}, 1 \}, \quad t > 0,$$ (4.12)

hold true.

This proves that, for $u_0 \in L^2(\Omega)$ and $F \in \mathcal{J}$, $u$ given by (4.10) is lying in $L^1_{up-loc}(\mathbb{R}_+; L^2(\Omega))$. Let us prove that this function $u$ is the unique Laplace-weak solution of (1.4) when $K$ is given by (1.9). For this purpose, I need two intermediate results.

Combining the result of Lemma 4.1 with [19, Theorem 1.1.], we deduce the following.
Lemma 4.3. Let \( u_0 \in L^2(\Omega) \) and \( F \in L^\infty(\mathbb{R}_+; L^2(\Omega)) \). Then, the function

\[
    v(\cdot, t) = R_0(t)u_0 + \int_0^t R_1(t-s)F(s, \cdot)ds + BF(t, \cdot), \quad t \in \mathbb{R}_+, \tag{4.13}
\]

is the unique Laplace-weak solution of (1.4). Here, \( B \) is defined by

\[
    Bh = \frac{1}{2i\pi} \int_{\gamma(\delta, \delta)} p^{-1} \left( A + \sum_{k=1}^N \rho_k p^{\alpha_k} \right)^{-1} hd p. \tag{4.14}
\]

Proof. From now on, for any Banach space \( Y \), I denote by \( S'(\mathbb{R}_+; Y) \) the set of temperate distributions supported in \( [0, +\infty) \) taking values in \( Y \). Since the proof of this result is rather long and similar to [19, Theorem 1.1.], I only give the main idea of its proof when \( u_0 \equiv 0 \).

In the first step of this proof, I introduce the following family of operators acting in \( L^2(\Omega) \):

\[
    \tilde{W}(p) := p^{-1} \left( A + \sum_{k=1}^N \rho_k p^{\alpha_k} \right)^{-1}, \quad p \in \mathbb{C} \setminus \mathbb{R}_-.
\]

Combining Lemma 2.2 with the arguments used in [19, Lemma 2.3.], we can define the map

\[
    \mathcal{R}_2(t) := \frac{1}{2i\pi} \int_{-\infty}^{+\infty} e^{ip\tilde{W}(p+1)}dp = \frac{1}{2i\pi} \int_{-\infty}^{+\infty} e^{ip\tilde{W}(1+i\eta)}d\eta, \quad t \in \mathbb{R} \tag{4.14}
\]

and show that \( \mathcal{R}_2 \in L^\infty(\mathbb{R}; B(L^2(\Omega))) \cap S'(\mathbb{R}_+; B(L^2(\Omega))). \) Moreover, combining Lemma 2.2 with Theorem 19.2 and the following remark in [35], we deduce that \( \mathcal{R}_2(p) = \tilde{W}(p+1) \) for all \( p \in \mathbb{C}_+ \). As a consequence, the operator

\[
    \mathcal{R}_3(t) = e^t \mathcal{R}_2(t) = \frac{1}{2i\pi} \int_{-\infty}^{+\infty} e^{i(p+1)\tilde{W}(p+1)}dp = \frac{1}{2i\pi} \int_{-\infty}^{+\infty} e^{ip\tilde{W}(p)}dp, \quad t \in \mathbb{R}
\]

verifies \( \mathcal{R}_3(p) = \mathcal{R}_2(p-1) = \tilde{W}(p) \) for all \( p \in \{ z \in \mathbb{C}; \Re z \in (1, +\infty) \} \). Following [19, Lemma 2.4.], we can prove that

\[
    \mathcal{R}_3(t) = \frac{1}{2i\pi} \int_{\gamma(\delta, \delta)} e^{ip\tilde{W}(p)}dp, \quad t \in \mathbb{R}_+, \tag{4.15}
\]

and \( \mathcal{R}_3 \in S'(\mathbb{R}_+; B(L^2(\Omega))) \cap L^1_{u_p-loc}(\mathbb{R}_+; B(L^2(\Omega))). \) Using the fact that \( \mathcal{R}_3(p) = \tilde{W}(p) \) for all \( p \in \{ z \in \mathbb{C}; \ Re z \in (1, +\infty) \} \), we deduce that

\[
    \mathcal{R}_3\psi(p) = \tilde{W}(p)\psi, \quad p \in \mathbb{C}_+, \psi \in L^2(\Omega). \tag{4.16}
\]

I denote by \( \mathcal{F} \) the extension of a function \( F \) by 0 on \( (\Omega \times \mathbb{R}) \setminus (\Omega \times \mathbb{R}_+) \). Consider the convolution in time of \( S_2 \) with \( \mathcal{F} \) given by

\[
    (\mathcal{R}_3 \ast \mathcal{F})(x,t) = \int_0^t \mathcal{R}_3(t-s)F(s,x)1_{\mathbb{R}_+}(s)ds, \quad (t,x) \in \mathbb{R} \times \Omega.
\]

I show that \( \mathcal{R}_3 \ast \mathcal{F} \in S'(\mathbb{R}_+; L^2(\Omega)) \) and

\[
    \mathcal{R}_3 \ast \mathcal{F}(p) = \mathcal{R}_3(p)\mathcal{F}(p), \quad p \in \mathbb{C}_+,
\]

with \( \mathcal{R}_3(p) = \int_0^{+\infty} \mathcal{R}_3(t)e^{-pt}dt \) and \( \mathcal{F}(p) = \int_0^{+\infty} F(t)e^{-pt}dt \). Thus, setting \( \partial := \partial_t(\mathcal{R}_3 \ast \mathcal{F}) \in S'(\mathbb{R}_+; L^2(\Omega)) \), we derive from (4.16) that

\[
    \partial(p) = p\mathcal{R}_3 \ast \mathcal{F}(p) = p\mathcal{R}_3(p)\mathcal{F}(p) = \left( A + \sum_{k=1}^N \rho_k p^{\alpha_k} \right)^{-1} \mathcal{F}(p), \quad p \in \mathbb{C}_+.
\]
Therefore, the proof will be completed if we show that \( \bar{v} = v \) with \( v \) given by (4.13). For this purpose, applying (4.11), we deduce that

\[
(R_3 \ast \bar{F})(t) = \frac{1}{2i\pi} \int_{\gamma(\delta, \vartheta)} g(t, p) dp, \; t \in \mathbb{R}_+,
\]

with

\[
g(t, p) := \int_0^t e^{(t-s)p} p^{-1} \left( A + \sum_{k=1}^N \rho_k p^{\alpha_k} \right)^{-1} F(s, \cdot) ds, \; p \in \gamma(\delta, \vartheta).
\]

(4.17)

Thus, for a.e. \( t \in \mathbb{R}_+ \) and all \( p \in \gamma(\delta, \vartheta) \), we have

\[
\partial_t g(t, p) = \int_0^t e^{(t-s)p} p^{-1} \left( A + \sum_{k=1}^N \rho_k p^{\alpha_k} \right)^{-1} F(s, \cdot) ds + p^{-1} \left( A + \sum_{k=1}^N \rho_k p^{\alpha_k} \right)^{-1} F(t, \cdot),
\]

and consequently

\[
\| \partial_t g(t, p) \|_{L^2(\Omega)} \leq \left( \left\| A + \sum_{k=1}^N \rho_k p^{\alpha_k} \right\|_{B(L^2(\Omega))} \right) \left( \int_0^t e^{\|p\|_\infty} ds + |p|^{-1} \right) \|F\|_{L^\infty(\mathbb{R}_+; L^2(\Omega))},
\]

From this and (4.1), it follows that

\[
\| \partial_t g(t, p) \|_{L^2(\Omega)} \leq C |p|^{-(1+\alpha_N)} \|F\|_{L^\infty(\mathbb{R}_+; L^2(\Omega))} = C |p|^{-(1+\alpha_N)} \|F\|_{L^\infty(\mathbb{R}_+; L^2(\Omega))}.\]

As a consequence, the mapping \( p \mapsto \partial_t g(t, p) \in L^1(\gamma(\delta, \vartheta); L^2(\Omega)) \) for any fixed \( t \in \mathbb{R}_+ \) and we have \( \bar{v}(t) = \partial_t [R_3 \ast \bar{F}](t) = \frac{1}{2i\pi} \int_{\gamma(\delta, \vartheta)} \partial_t g(t, p) dp \), or equivalently

\[
\bar{v}(\cdot, t) = \frac{1}{2i\pi} \int_{\gamma(\delta, \vartheta)} \left( \int_0^t e^{(t-s)p} \left( A + \sum_{k=1}^N \rho_k p^{\alpha_k} \right)^{-1} F(s) ds + p^{-1} \left( A + \sum_{k=1}^N \rho_k p^{\alpha_k} \right)^{-1} F(t) \right) dp
\]

in virtue of (4.17). Now, applying the Fubini theorem to the right-hand side of the above identity, we obtain that \( \bar{v} = v \) with \( v \) given by (4.13). Combining this with Lemma 4.2, we deduce that \( v \in L^1_{up-loc}(\mathbb{R}_+; L^2(\Omega)) \) and it is the unique Laplace-weak solution of (1.4) in the sense of Definition 1.2. \( \square \)

I can extend the result of Lemma 4.3 as follows.

**Lemma 4.4.** Let \( u_0 \in L^2(\Omega) \) and \( F \in L^\infty(\mathbb{R}_+; L^2(\Omega)) \). Then, the function \( u \) given by (4.10) is the unique Laplace-weak solution of (1.4).

**Proof.** According to Lemmas 4.2 and 4.3, I only need to show that here the map \( B \) appearing in Lemma 4.2 will be equal to zero. To see this, let us observe that, fixing \( \delta \in (0, 1) \) and applying the Cauchy formula, for any \( R > 1 \) and \( h \in L^2(\Omega) \), we have

\[
\frac{1}{2i\pi} \int_{\gamma(\delta, R, \vartheta)} p^{-1} \left( A + \sum_{k=1}^N \rho_k p^{\alpha_k} \right)^{-1} h dp = \frac{1}{2i\pi} \int_{\gamma(\delta, R, \vartheta)} p^{-1} \left( A + \sum_{k=1}^N \rho_k p^{\alpha_k} \right)^{-1} h dp,
\]

with \( \gamma(\delta, R, \vartheta) \) given by (2.20). Sending \( R \to +\infty \), we obtain

\[
Bh = \lim_{R \to +\infty} \frac{1}{2i\pi} \int_{\gamma(\delta, R, \vartheta)} p^{-1} \left( A + \sum_{k=1}^N \rho_k p^{\alpha_k} \right)^{-1} h dp.
\]
On the other hand, applying Lemma 2.2, we deduce that
\[
\left\| \frac{1}{2i\pi} \int_{\gamma(R)} p^{-1} \left( A + \sum_{k=1}^{N} \rho_k p^{\alpha_k} \right)^{-1} \right\|_{L^2(\Omega)} 
\leq C \int_{-\theta}^{\theta} \left\| \left( A + \sum_{k=1}^{N} \rho_k (Re^{i\beta})^{\alpha_k} \right)^{-1} \right\| \| h \|_{L^2(\Omega)} \, d\beta
\leq CR^{-\alpha N} \| h \|_{L^2(\Omega)}.
\]
Therefore, we have
\[
Bh = \lim_{R \to +\infty} \frac{1}{2i\pi} \int_{\gamma(R)} p^{-1} \left( A + \sum_{k=1}^{N} \rho_k p^{\alpha_k} \right)^{-1} \right\|_{L^2(\Omega)} \equiv 0. \quad \square
\]

Combining Lemmas 4.2 and 4.4 with the density arguments used in Proposition 2.3, we obtain the following results about the unique existence of Laplace-weaksolutions for (1.4).

**Proposition 4.5.** Assume that the conditions (1.1)–(1.2) are fulfilled. Let \( u_0 \in L^2(\Omega) \), \( F \in L^\infty(\Omega) \), \( 0 < \alpha_1 < \ldots < \alpha_N < 1 \), \( \rho_j \in L^\infty(\Omega) \), \( j = 1, \ldots, N \), satisfy (1.2) with \( \rho = \rho_j \), and let \( K \) be given by (1.9). Then, there exists a unique Laplace-weaksolution \( u \in L^1_{u,p-loc}(\mathbb{R}_+; L^2(\Omega)) \) to (1.4) given by (4.10).

Combining Lemma 4.2, Proposition 4.5, and mimicking the proof of Theorem 1.3, we deduce Theorem 1.5.

## 5 WEAK SOLUTION AT FINITE TIME

In a similar way to [19–21, 27], following Definition 1.1 of weak solutions of (1.4), I give the definition of weak solutions of the same problem at finite time. Namely, for \( T > 0 \), let us consider the IBVP

\[
\begin{cases}
(\rho(x)D_t^K + A)v(t, x) = G(t, x), & (t, x) \in (0, T) \times \Omega, \\
v(t, x) = 0, & (t, x) \in (0, T) \times \partial \Omega, \\
v(0, x) = u_0(x), & x \in \Omega.
\end{cases}
\]

I give the following definition of weak solutions of (5.1).

**Definition 5.1.** Let \( F \) be the extension of the function \( G \) by zero to \( \mathbb{R}_+ \times \Omega \). Then, we call weak solution of (5.1) the restriction on \( (0, T) \times \Omega \) of the weak solution \( u \) of the IBVP (1.4) in the sense of Definition 1.1.

Notice that, according to Definition 1.1, any weak solution \( v \) of (5.1) satisfies the following properties:

1) \( v \in L^1(0, T; L^2(\Omega)) \) and the identity

\[
\rho(x)D_t^K[v - u_0](t, x) + Av(t, x) = G(t, x), \quad x \in \Omega, \ t \in (0, T),
\]

holds true in the sense of distributions in \( (0, T) \times \Omega \).

2) We have \( I_K[v - u_0] \in W^{1,1}(0, T; D' (\Omega)) \) and the following initial condition

\[
I_K[v - u_0](0, x) = 0, \quad x \in \Omega,
\]

is fulfilled. Moreover, applying the result of Theorems 1.3, 1.4, and 1.5, I can show the unique existence of weak solutions of (5.1). Let us also observe that the Definition 5.1 of weak solutions depends on the final time \( T \). Nevertheless, I can
show that the unique weak solution of (5.1) in the sense of Definition 5.1 is independent of $T$ and by the same way of the extension of the source term $G$ under consideration in Definition 5.1. All these properties can be summed up as follows.

**Theorem 5.2.** Assume that the condition of Theorems 1.3, 1.4, and 1.5 are fulfilled and assume that the weight $K$ is given by (1.7) or (1.8) or (1.9). Then, for any $G \in L^1(0, T; L^2(\Omega))$ and $u_0 \in L^2(\Omega)$, the IBVP (5.1) admits a unique weak solution $v \in L^1(0, T; L^2(\Omega))$ in the sense of Definition 5.1. Moreover, the unique weak solution of (5.1) has a Duhamel type of representation given by:

1) $v(t, \cdot) = S_0(t)u_0 + \int_0^t S_1(t - s)G(s, \cdot)ds, \quad t \in (0, T),$

when $K$ is given by (1.7). Here, $S_0$ (resp. $S_1$) is defined by (2.3) (resp. (2.4)).

2) $v(t, \cdot) = S_{0, \mu}(t)u_0 + \int_0^t S_{1, \mu}(t - s)G(s, \cdot)ds, \quad t \in (0, T),$

when $K$ is given by (1.8). Here, $S_{0, \mu}$ (resp. $S_{1, \mu}$) is defined by (3.2) (resp. (3.3)).

3) $v(t, \cdot) = R_0(t)u_0 + \int_0^t R_1(t - s)G(s, \cdot)ds, \quad t \in (0, T),$

when $K$ is given by (1.9). Here, $R_0$ (resp. $R_1$) is defined by (4.8) (resp. (4.9)).

Finally, the solution of the IBVP (5.1) in the sense of Definition 5.1 is independent of the choice of the final time $T$.

**Proof.** The proof the first two claims of this theorem are a direct consequence of Theorems 1.3, 1.4, and 1.5 and the discussion in Sections 2, 3, and 4 for the representation of solutions. Therefore, I only need to prove that the unique solution of the IBVP (5.1) in the sense of Definition 5.1 is independent of the choice of the final time $T$. For this purpose, let us consider $T_1 < T_2$ and $G \in L^1(0, T_2; L^2(\Omega))$. For $j = 1, 2$, consider $v_j$ the weak solution of the IBVP (5.1) with $T = T_j$ in the sense of Definition 5.1. In order to prove that the solutions of (5.1) in the sense of Definition 5.1 are independent of $T$, I need to show that the restriction of $v_2$ to $(0, T_1) \times \Omega$ coincides with $v_1$. In view of the first claims of the theorem, one of the following identities holds true:

1) $v_j(t, \cdot) = S_0(t)u_0 + \int_0^t S_1(t - s)G(s, \cdot)ds, \quad t \in (0, T_j),$

2) $v_j(t, \cdot) = S_{0, \mu}(t)u_0 + \int_0^t S_{1, \mu}(t - s)G(s, \cdot)ds, \quad t \in (0, T_j),$

3) $v_j(t, \cdot) = R_0(t)u_0 + \int_0^t R_1(t - s)G(s, \cdot)ds, \quad t \in (0, T_j).$

Thus, in each case, we deduce that

$$v_1(t, x) = v_2(t, x), \quad t \in (0, T_1), \ x \in \Omega.$$  

This shows that the unique solution of the IBVP (5.1) in the sense of Definition 5.1 is independent of the choice of the final time $T$. 

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**CONFLICT OF INTEREST STATEMENT**

The authors declare no potential conflict of interests.
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