A SURVEY OF RESULTS ON CONSERVATION LAWS WITH DETERMINISTIC AND RANDOM INITIAL DATA

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Abstract. This expository paper examines key results on the dynamics of nonlinear conservation laws with random initial data and applies some theorems to physically important situations. Conservation laws with some nonlinearity, e.g. Burgers’ equation, exhibit discontinuous behavior, or shocks, even for smooth initial data. The introduction of randomness in any of several forms into the initial condition renders the analysis extremely complex. Standard methods for tracking a multitude of shock collisions are difficult to implement, suggesting other methods may be needed. We review several perspectives into obtaining the statistics of resulting states and shocks. We present a spectrum of results from a number of works, both deterministic and random. Some of the deep theorems are applied to important discrete examples where the results can be understood in a clearer, more physical context.

1. Introduction.

1.1. Background. In the effort to create a mathematical description of turbulence, an important building block is the study of shocks and rarefactions together with random initial conditions. Although the model is a somewhat coarse description of turbulence in practice, Burgers’ equation is extensively studied [21, 26, 27, 5, 2, 24, 10] as a test case for new methods and types of randomness. It also possesses the surprising feature of producing discontinuous solutions, even from smooth initial data. From there it is then reasonable to seek broader classes of equations to which these properties can be extended.

The link between many-particle systems and fluid mechanics, shock waves, and PDEs poses important problems in understanding the continuum limit. In this paper, we provide a summary of various results in the field and potential directions for open problems in the future. Our aim is to tie together a number of vastly different approaches across the scope of kinetic theory involving conservation laws (not limited simply to the widely studied Burgers’ equation) in a comprehensive note that serves both as a general introduction and as a starting point for readers who may wish to delve into the more technical aspects in the references herein. We also highlight the role of various types of randomness in the initial conditions, and the

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preservation (or lack thereof) of certain prescribed structure in the solution as time advances. We illuminate the theory with a series of discrete examples. Randomness in these initial conditions, together with formation and interaction of resulting shocks, forms a basic model that is a first step toward a mathematical description of turbulence. This is subsequently useful in a wide array of applications in fields such as engineering, one such application being attempts to control turbulent flows.

For the sticky particle model, one can consider an aggregation process with an arbitrary number of conserved quantities using serveral approaches, for example via a mean-field model with origins in statistical mechanics [25], or by representing the particles by approximating them by a piecewise polynomial reconstruction over an auxiliary Cartesian mesh [9]. An excellent treatment of the large scale limit can be found in [39]. Another important question is how well the microscopic dynamics of the problem can be understood, for example by approximating non-Markovian stochastic processes by a Markov process [40]. One important case of this model involves putting together a system of conservation laws along with pressureless Euler and pressureless attractive/repulsive Euler-Poisson equations [32]. One can also consider a probabilistic interpretation of the sticky particle model, constructing a solution for a nonlinear stochastic differential equation [11, 12].

When random initial conditions are introduced, an important concept is that of the notion of a stochastic entropy solution, i.e. the methods used by Kruzkov, as well as related existence and uniqueness theory [14]. The regularity of these random entropy solutions to scalar hyperbolic conservation laws has also been studied with various kinds of initial data [38, 17]. Fractional stochastic perturbation on nonlinear hyperbolic equations can also be studied, along with existence and uniqueness of the solution via a Lax-Oleinik formula [36]. For systems of hyperbolic conservation laws, multi-level Monte Carlo methods can be used to analyze their solutions and quantify the resulting uncertainty [31]. Other applications include that of a large particle moving through a sea of small particles, with micro and macro behaviors connected by fluctuation and dissipation [18] and interfaces moving in a disordered medium, i.e. fluid invasion [33].

1.2. Burgers’ equation derived from pressureless limit. In the continuum, one represents gas dynamics in one dimension by density and velocity fields, $\rho(t, x)$ and $u(t, x)$, respectively. The Euler equations, given by

$$
\rho_t + (\rho v)_x = 0 \text{ (conservation of mass)},
$$

$$
(\rho v)_t + (\rho v^2 + p)_x = 0 \text{ (conservation of momentum)},
$$

$$
(\rho E)_t + (\rho v + pv)_x = 0 \text{ (conservation of energy)},
$$

(1)

provide a starting point for the approach of Brenier and Grenier [7], which we detail in Section 2. One then considers inelastic collisions (under which kinetic energy is not conserved) and take the pressureless limit, formally obtaining the system

$$
\partial_t \rho + \partial_x (\rho u) = 0
$$

$$
\partial_t (\rho u) + \partial_x (\rho u^2) = 0
$$

(2)

In working with this system, Radon measures (see [34], p. 455) provide a key tool in making the interpretation of the equations precise in the most general case. The appropriate system of conservation laws provides conservation of mass and momentum, and the unknowns include these quantities for specific particles, along
with velocity. Under some basic assumptions, the mathematical tool of the Radon-Nikodym derivative, which is essentially the derivative of a measure \([34]\) (p. 385), then allows them to define in a rigorous sense the quotient of these measures (momentum and mass), a mathematical analog to velocity being the quotient of momentum over mass.

1.3. **Analyzing Burgers’ equation from another perspective: Flow maps.**

In Section 3, we analyze a similar problem with a very different approach. The same system from \([7]\) is presented in E, Rykov, and Sinai \([35]\), and is equivalent, under smooth solutions, to Burger’s equation and a transport equation

\[
\begin{align*}
  u_t + \left( \frac{u^2}{2} \right)_x &= 0 \\
  \rho_t + (\rho u)_x &= 0
\end{align*}
\]  

(3)

From here one can define

\[ x = \varphi_t (y) = y + tu_0 (y). \]  

(4)

Formal inversion of this flow map yields

\[ u (x, t) = u_0 \left( \varphi_t^{-1} (x) \right), \rho (x, t) = \rho_0 \left( \varphi_t^{-1} (x) \right) \frac{\partial x}{\partial y}^{-1}. \]  

(5)

The issue is that this flow map is only invertible up to some time \(t\), at which point \(\varphi_t (x)\) is no longer one-to-one and the inverse not well-defined. Specifically, we have a whole interval mapped into one point, forming a shock. However, we note that the map \(\varphi_t\) still defines a partition of the real line (mapping some points into points and sometimes intervals into a single point). It is from this observation that E, Rykov, and Sinai \([35]\) construct solution formulas (in the appropriate sense, to be defined later) given by

\[
\begin{align*}
  \varphi_t (y) &= \int_{C_t (y)} \left( \eta + tu_0 (\eta) \right) dP_0 (\eta) \int_{C_t (y)} dP_0 (\eta), \\
  u (x, t) &= \int_{D_t (x)} u_0 (\eta) dP_0 (\eta) \\
  \rho (x, t) &= \int_{D_t (x)} \frac{\partial x}{\partial y}^{-1} dP_0 (\eta)
\end{align*}
\]  

(6)

where \(D_t (x), C_t (x)\) have to do with the aforementioned partition.

1.4. **Entropy solution and variational approach using Stieltjes integral.**

A similar variational approach is studied in Huang and Wang \([22]\), where an entropy solution is constructed from a generalised potential

\[
F (y; x, t) = \int_{x}^{y} t u_0 (\eta) + \eta - x d\mu_0 (\eta).
\]  

(7)

They construct a set \(S (x, t)\) to serve the role of determining what point or interval is mapped into a point \((x, t)\). Here the notation denotes a special kind of integral known as the Stieltjes integral, whereby integration is from the right limit of 0 to the left limit of \(y\). Unlike the traditional Riemann or Lebesgue integrals, this may result in different values for the expression, as shown in Section 4.

1.5. **Introduction of random initial conditions for more general conservation Laws.**

In the work of Menon and Srinivasan \([28]\) and Kaspar and Rezakhanlou \([23]\), further analysis on these equations was performed to gain deeper understanding of particle dynamics and the evolution of the system starting with random initial data. The analysis went beyond Burgers’ equation to the more general case of a \(C^1\), convex flux, and considered initial data that was random rather than
deterministic. Here, the initial conditions were restricted to processes that were *spectrally negative*, that is, stochastic processes with jumps but only in one direction. In particular, only downward jumps were permitted, as upward jumps lead to an immediate breakdown of the statistics, in that other behavior such as rarefaction is observed. Furthermore, this initial stochastic process was also assumed to be Markov. Using the Levy-Khinchine representation for the Laplace exponent and other methods, formal calculations were used to establish a number of results. One remarkable assertion proven states that if one starts with strong Markov, spectrally negative initial data, then this Markov property persists in the entropy solution for any positive time \( t > 0 \). This closure result can also hold for a non-stationary process and can obtain an equation for a generator in the case of specific kinds of stochastic processes. These results are described in Section 5.

In a few special cases, such as Burgers’ equation under white noise initial conditions in [16], one has the remarkable achievement of a closed form solution up to the level of special functions. By starting from the full Burgers’ equation and taking the vanishing viscosity limit, one is lead to the variational blueprint for this set of exact results, along with a crisp geometric visualisation. We will examine this case in further detail in Section 6.

1.6. **Extension of results to flux functions with lesser regularity.** Subsequently, we extend these results to a further class of nonlinear flux functions, providing results and derivations of two different hierarchies. These were checked rigourously against various examples involving Riemann initial data, a prototypical but important base case that is the building block to more complicated or even random initial conditions. In particular, the second hierarchy derived shows consistency through shock interactions without any sort of extraneous resetting or additional conditions imposed, a feature absent in many classical methods. This is presented in Section 7.

1.7. **Open problems.** Finally, in Section 8, we discuss future work in these areas, including the possibility of proving a rigourous closure theorem for these hierarchies. Other applications such as more computational testing of these equations under various forms of random initial data are also open problems.

2. **Burgers’ equation and the sticky particle model.** Brenier and Grenier [7] consider Burgers’ equation applied to a pressureless gas, described at a discrete level by a large collection of sticky particles. The “sticky” part of the description corresponds to the particles remaining together after inelastic collisions in accordance with conservation of mass and momentum (but notably, *not* conservation of energy). At a continuum level, the model is described by density and velocity fields \( \rho (t,x) \) and \( u(t,x) \), respectively, that must satisfy

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0, \\
\partial_t \rho u + \partial_x (\rho u^2) &= 0.
\end{align*}
\]  
(8)

The system (8) follows from formally letting the pressure go to zero in (1) or letting the temperature approach zero in the Boltzmann equation. In particular, the system (8) can be shown to be equivalent to the inviscid Burgers’ equation under the assumptions of smooth solutions and positive densities.

However, for the considerations of the sticky particle model, such a reduction is not as immediate. Application of (8) to this problem presents several difficulties: (i) under discrete particles, the fields are no longer functions but must be considered...
as measures, (ii) the velocity field needs to be well-defined almost everywhere with respect to the measure prescribed by \( \rho \), and (iii) the system must be supplemented by some entropy conditions. In addition, an obvious choice such as the condition

\[
\partial_t (\rho U(u)) + \partial_x (\rho u U(u)) \leq 0
\]

for any smooth, convex \( U \) is shown to be insufficient to guarantee uniqueness.

In this section we aim to illustrate how the flux function is constructed by considering a discrete example with a finite number of particles. We also show detail how the solution of the conservation law is linked to the Hamilton-Jacobi equation in a viscosity sense.

2.1. **Discrete example for Burgers’ equation.** For definiteness, take \( n = 4 \) particles with masses \( \{ m_i \}_{i=1}^4 \) at positions \( \{ x_i \}_{i=1}^4 \) and with initial velocities \( \{ v_i \}_{i=1}^4 \), respectively, given by

\[
\begin{align*}
m_0^1 &= \frac{1}{4}, & m_0^2 &= \frac{1}{4}, & m_0^3 &= \frac{1}{3}, & m_0^4 &= \frac{1}{6}, \\
v_0^1 &= 2, & v_0^2 &= 1, & v_0^3 &= -\frac{1}{2}, & v_0^4 &= 1, \\
x_0^1 &= -3, & x_0^2 &= -2, & x_0^3 &= 1, & x_0^4 &= 3.
\end{align*}
\]

These are plotted in Figure 1(a) along with characteristics as the dynamics evolve in time. Our goal is to show how we build a PDE to model the dynamics of this system, and that the solution of this PDE matches our intuition about what should occur with sticky particles. Define for notational convenience \( M_i = \sum_{j=1}^i m_j, \; M_0 = 0. \) The initial distribution is given by \( M^0(x) = \rho^0((-\infty, x]) \) where \( \rho^0 \) jumps by \( \Delta M_i \) at the points \( x_i \) (so clearly \( M^0(x_i^-) = M_{i-1}, \; M^0(x_i^+) = M_i \)). Now we define the function \( a(m) = u_0(x) \) for \( M^0(x^-) \leq m \leq M^0(x^+) \). In this case, this simplifies to

\[
a(m) := v_i, \; M_{i-1} \leq m \leq M_i
\]

and is plotted in Fig. 1(b).

Now, we construct the flux function:

\[
A(m) = \int_0^m a(m') \, dm'
\]

and plot it in Figure 1(c).

We want to construct a weak solution to the differential equation

\[
\partial_t M + \partial_x (A(M)) = 0
\]

and show that it works for the flux function proposed above. A weak solution will satisfy

\[
\int_{-\infty}^{\infty} \int_0^{\infty} \varphi_t M + \varphi_x A(M) = 0
\]

for every smooth function \( \varphi \in C^\infty_c(\mathbb{R} \times (0, \infty)) \). We want our flux function to satisfy the Rankine-Hugoniot condition at each shock. This means if we have \( M = M_l \) as the left-hand limit and \( M = M_r \) as the right-hand limit, \( A(M) \) should satisfy

\[
\frac{A(M_r) - A(M_l)}{M_r - M_l} = \sigma
\]
where $\sigma$ is the slope of the parametrised curve describing the shock. In particular, choosing a test function with compact support in the region $R$ sketched in Fig. 1(a) yields

$$A(M_r) - A(M_l) = \frac{v_1 M_1 + v_2 (M_2 - M_1) - v_1 M_1}{M_r - M_l} = \frac{v_2 m_2}{m_2} = v_2.$$  

(16)

Hence, the Rankine-Hugoniot condition is satisfied. Similarly, one verifies it holds at the other discontinuities.

2.2. Verifying that the potential $\Psi$ is a viscosity solution. Another result of Brenier and Grenier involves linking this problem with the Hamilton-Jacobi equation. To this end, they define a potential

$$\Psi(x,t) = \int_{-\infty}^{x} M(t, y) dy.$$  

(17)

This is a viscosity solution in the sense of Crandall-Lions of the following Hamilton-Jacobi equation:

$$\partial_t \Psi + A(\partial_x \Psi) = 0$$  

(18)

and is derived from the second Hopf formula. Indeed, in Bardi and Evans [3], the following is proven.

**Theorem 2.1.** Assume $A : \mathbb{R}^n \to \mathbb{R}$ is continuous and $\Psi_0 : \mathbb{R}^n \to \mathbb{R}$ is uniformly Lipschitz and convex. Then

$$\Psi(x,t) = \sup_{y\in \mathbb{R}^n} \inf_{z\in \mathbb{R}^n} \{u_0(z) + y \cdot (x - z) - tA(y)\} \quad \text{(for } t > 0, \ x \in \mathbb{R})$$  

(19)

is the unique uniformly continuous viscosity solution of

$$\Psi_t + A(\partial_x \Psi) = 0 \text{ in } \mathbb{R}^{n+1}$$  

$$\Psi(\cdot, 0) = \Psi_0(\cdot)$$  

(20)
This is valuable because it provides a solution to the Hamilton-Jacobi equation without the usual convexity assumptions on the flux function $A$. Performing some rearrangements, we have that

$$
\Psi(x,t) = \sup_m \left\{ \inf_z \left( u_0(z) - m \cdot z + m \cdot x - tA(y) \right) \right\} = \sup_m \left\{ -\Psi^*_0(m) + m \cdot x - tA(m) \right\},
$$

where $\Psi^*_0$ is the Legendre-Fenchel transform given by

$$
\Psi^*_0(m) = \inf \left\{ u_0(z) - y \cdot z \right\}
$$

i.e., in the notation of Brenier and Grenier [7], equation (25)

$$
\Psi(t,x) = \sup_{0 \leq m \leq 1} \left\{ xm - \Phi_0(m) - tA(m) \right\}.
$$

Here $\Phi_0(m)$ is the Legendre-Fenchel transform of $\Psi(t,x)$, evaluated at the initial time $t = 0$. The geometric interpretation is that $\Phi(t,m)$ forms the convex hull of $\Phi_0(t,m)$ on the interval $m \in [0,1]$.

To illustrate this construction, consider a discrete example with mass density function $M$ along and corresponding function $\Psi$ as graphed in Figure 2.

This corresponds to our example of discrete point masses $m_i$ at positions $x_i$. Clearly the function $\Psi$ is continuous but not $C^2$ or even $C^1$. To show that $\Psi$ is a viscosity solution, we need to show that, given $\phi \in C^2(\mathbb{R} \times [0,\infty])$ such that $\phi(x_0,t) = \Psi(x_0,t)$, the following hold:

(i) $\Psi$ is continuous (this is trivial).
(ii) $\phi \geq \Psi$ in a neighborhood of $x_0$ implies $\partial_t \phi + A(\partial_x \phi) \leq 0$.
(iii) $\phi \leq \Psi$ in a neighborhood of $x_0$ implies $\partial_t \phi + A(\partial_x \phi) \geq 0$.

Notationally, we will use a prime to denote the $x$ derivative of $\phi$, i.e. $\phi'(x_0,0) = \frac{d}{dx} \phi(x_0,0)$, for the remainder of this section.

To satisfy (ii), we let $\phi$ be a $C^2$ function which lies above $\Psi$. However, we argue that choosing such a function is impossible. Observe that for $\phi \geq \Psi$ to be satisfied in a neighborhood of the point $x_2$, we need $\phi' \leq M_1$ in a one-sided neighborhood of $x_2$ ("left" of $x_2$). For if not, then $\phi' > M_1$ in $(x_2 - \varepsilon, x_2)$, some $\varepsilon > 0$, and clearly

$$
\phi(x_2 - \varepsilon, 0) < \phi(x_2, 0) - \varepsilon M_1 = \Psi(x_2 - \varepsilon, 0),
$$

violating our assumption. Thus, $\phi' \leq M_1$ in a neighborhood $(x_2 - \varepsilon, x_2)$. 

**Figure 2.** (a) Cumulative distribution function of mass as a function of position; (b) Construction of $\Psi(0,x)$. 


Similarly, one can show that \( \phi' \geq M_2 \) in a neighborhood \( (x_2, x_2 + \epsilon) \), \( \epsilon > 0 \). But since \( M_1 < M_2 \), this implies \( \phi' \) is discontinuous at \( x_2 \), so that \( \phi''(x_2, 0) \) does not exist. Hence it is impossible to find a \( C^2 \) function \( \phi \) satisfying \( \phi \geq \Psi \) in a neighborhood of \( x_2 \).

In the case of (iii), it is indeed plausible that one can find such a \( \phi \). Then one wishes to verify that
\[
[\partial_t \phi + A(\partial_x \phi)](x_0, 0) \geq 0.
\] (25)

Let \( \phi \) be given such that \( \phi \in C^2(\mathbb{R} \times [0, \infty)) \), \( \phi(x_0, 0) = \Psi(x_0, 0) \), and \( \phi \leq \Psi \) in a neighborhood of \( x_0 \). By considering the cases (i) \( x_0 \notin \{x_i\}_{i=1}^4 \) and (ii) \( x_0 \) taking one of the values \( \{x_i\}_{i=1}^4 \), one can show condition (iii) holds. The algebraic details are not particularly enlightening and are hence omitted.

Similar results relating to ballistic aggregation of particles are studied in [41].

2.3. Returning to the discrete example. Let us return to our discrete example with four point masses. Recall in this setting that \( M_i := \sum_{j=1}^i m_i \). We sketch \( M(0, x) \) in Fig. 2(a) and \( \Psi_0(x) = \Psi(0, \cdot) = \int_{-\infty}^x M(t, y) \, dy \) in Fig. 2(b). Clearly, \( \Psi_0(m) \) is given by

\[
\Psi_0(x) = \begin{cases} 
0 & x < x_1 \\
M_1(x-x_1) & x_1 < x < x_2 \\
M_1(x-x_1) + M_2(x-x_2) & x_2 < x < x_3 \\
M_1(x-x_1) + M_2(x-x_2) + M_3(x-x_3) & x_3 < x < x_4 \\
M_1(x-x_1) + M_2(x-x_2) + M_3(x-x_3) + M_4x & x_4 < x 
\end{cases}
\] (26)

It is convenient to express \( \Psi_0(x) \) in terms of positive parts of functions rather than the piecewise construction in (26), i.e. we write

\[
\Psi_0(x) = \sum_{i=1}^4 M_1(x-x_i)_+ = m_1(x-x_1)_+ + m_2(x-x_2)_+ \\
+ m_3(x-x_3)_+ + m_4(x-x_4)_+ 
\] (27)

This is easy to see intuitively (for example, the \( M_1(x-x_1) \) term becomes \( m_1(x-x_1) \) for \( x > x_2 \), and the \( M_2(x-x_2) \) contributes the \( m_1(x-x_2) \) part). Now, we compute \( \Phi_0(m) = \Psi^*(0, m) = \sup_{x \in \mathbb{R}} \{xm - \Psi(0, x)\} \). We differentiate the expression and find:

\[
Q_x := (xm - \Psi(0, x))_x = m - m_1H(x-x_1) - m_2H(x-x_2) \\
- m_3H(x-x_3) - m_4H(x-x_4).
\] (28)

Note: (i) For \( m \neq m_1 \), it has no critical points, (ii) for any value of \( m \), \( Q_x \) is a decreasing function. For example, for \( m < m_1 \), clearly the maximum is attained at \( x = x_1 \), and hence

\[
\Phi_0(m) = x_1m - \Psi(0, x_1) = x_1m - m_1(x_1-x_1)_+ - m_2(x_1-x_2)_+ \\
- m_3(x_1-x_3)_+ - m_4(x_1-x_4)_+ = x_1m
\] (29)

After performing similar computations for other cases and combining the results, one is lead to

\[
\Phi_0(m) = x_1m + \sum_{i=1}^{n-1} \left( m - \sum_{j=1}^i m_j \right)(x_{i+1}-x_i).
\] (30)
To justify (29) in the paper, we consider the discrete case as in Fig. 2(b) and observe

\[ A(m) = \begin{cases} 
    v_1 m & m < M_1 \\
    v_1 m_1 + v_2 (m - m_1) & M_1 < m < M_2 \\
    v_1 m_1 + v_2 m_2 + v_3 (m - M_2) & M_2 < m < M_3 \\
    v_1 m_1 + v_2 m_2 + v_3 m_3 + v_4 (m - M_3) & M_3 < m < M_4 
\end{cases} \]

\[ = v_1 m + \sum_{i=1}^{3} \left( m - \sum_{j=1}^{i} m_i \right) (v_{i+1} - v_i) \quad (31) \]

using the same arguments as before for rearranging the function. Both arguments carry over easily using induction for the algebra.

Then the expression

\[ \Phi_n(t,m) = x_1(t) m + \sum_{i=1}^{n-1} \left( m - \sum_{j=1}^{i} m_j \right) (x_{i+1}(t) - x_i(t)) \quad (32) \]

is just given by the convex hull of

\[ \Phi^0(m) + t A(m) = (x_1 + tv_1) m + \sum_{i=1}^{n-1} \left( m - \sum_{i=1}^{j} m_i \right) (x_{i+1} + tv_{i+1} - x_i - tv_i) \quad (33) \]

To illustrate this, we return to our example. Recalling (10), note \( \sum m_i = 1 \). One can readily compute the quantity (33) by breaking it into cases for the values of \( m \), and has

\[ \text{LHS of (33)} = \begin{cases} 
    (-3 + 2t) m & m < \frac{1}{4} \\
    (1 - t) m - \frac{1}{2} (1 - t) & \frac{1}{4} < m < \frac{1}{2} \\
    (1 - \frac{t}{2}) m - \frac{1}{4} (7 - 4t) & \frac{1}{2} < m < \frac{3}{4} \\
    (3 + t) m - \frac{1}{12} (41 + 3t) & \frac{3}{4} < m < 1 
\end{cases} \quad (34) \]

Now we want to calculate (34) for different values of \( t \) and illustrate what the convex hull looks like. By simple geometry, one observes that particles 1 and 2 will collide and stick at time \( t = 1 \), and subsequently that will collide with 3 at some time \( 1 < t < 2 \) (see Figure 3(a), where the double-dashed line represents \( t = 2 \)). Note that although this is defined piecewise, it is continuous since the boundary values match at \( \frac{1}{4}, \frac{1}{2}, \) and \( \frac{5}{6} \). We consider (34) for three different values of \( t \).

For the initial state, \( t = 0 \), one has

\[ \text{LHS of (33)} \big|_{t=0} = \begin{cases} 
    -3m & m < \frac{1}{4} \\
    -2m - \frac{1}{2} & \frac{1}{4} < m < \frac{1}{2} \\
    m - \frac{7}{12} & \frac{1}{2} < m < \frac{3}{4} \\
    3m - \frac{3}{12} & \frac{3}{4} < m < 1 
\end{cases} \quad (35) \]

Similarly, for \( t = 1, 2 \), we have, respectively

\[ \text{LHS of (33)} \big|_{t=1} = \begin{cases} 
    -m & m < \frac{1}{4} \\
    -m & \frac{1}{4} < m < \frac{1}{2} \\
    m - \frac{3}{12} & \frac{1}{2} < m < \frac{3}{4} \\
    4m - \frac{3}{4} & \frac{3}{4} < m < 1 
\end{cases} \quad (36) \]

\[ \text{LHS of (33)} \big|_{t=2} = \begin{cases} 
    m & m < \frac{1}{4} \\
    \frac{1}{4} & \frac{1}{4} < m < \frac{1}{2} \\
    \frac{1}{4} & \frac{1}{2} < m < \frac{3}{4} \\
    5m - \frac{47}{12} & \frac{3}{4} < m < 1 
\end{cases} \quad (37) \]
which is once again continuous.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{(a) Representation of mass in cumulative distribution form up to a point \( x \) in the \( xt \) plane; (b)-(d) Plot of the expression \( \Phi^0(m) + tA(m) \) for times \( t = 0, 1, 2 \) respectively in solid lines; following the dashed lines forms the convex hull, yielding the Legendre transform \( \Phi_n(t, m) \). Note that for (b) and (c), the expression and its convex hull are identical, and in (d) there is a distinction, with the convex hull indicated by the dashed blue line.}
\end{figure}

These are graphed in Figure 3(b), (c), and (d), respectively. Recall that we wanted to consider the convex hull of the expression above. In the first two cases, the resulting function is already convex. In the third case (after shocks have occurred), the function is not convex and the convex hull is formed by taking the piecewise linear function given by the double-dashed line for \( 0 < m < \frac{5}{6} \) and the piece \( 5m - \frac{11}{12} \) for \( \frac{5}{6} < m < 1 \).

3. Analysis of Burgers’ equation using flow maps.

3.1. Definition of weak solution for conservation laws. Another approach to this problem is to use a Generalised Variational Principle (GVP) as in E, Rykov, and Sinai [35]. The first task is to formulate the equations in a weak form. Specifically, consider the system of conservation laws

\begin{equation}
\begin{aligned}
\rho_t + (\rho u)_x &= 0 \\
(\rho u)_t + (\rho u^2)_x &= 0.
\end{aligned}
\end{equation}

We want to define a weak solution by having (38) hold when we multiply by a test function and integrate. Since \( \rho \) is a purely singular measure for any time \( t > 0 \), it is incorrect to write

\begin{align*}
\int \rho \phi_t + \rho u \phi_x &= 0 + \text{boundary terms} \\
\int \rho u \phi_t + \rho u^2 \phi_x &= 0 + \text{boundary terms}
\end{align*}

\begin{equation}
\end{equation}
Instead, we take a family \((P_t, I_t)\) of Borel measures that are weakly continuous with respect to \(t\) such that \(I_t\) is absolutely continuous with respect to \(P_t\) for each fixed \(t\). Then we can define the Radon-Nikodym derivative as

\[
 u(x, t) = \frac{dI_t}{dP_t}(x),
\]

i.e.

\[
 \int_A u(x, t) \, dP_t(x) = \int_A dI_t(x),
\]

for any measurable set \(A\) in the appropriate set of functions. More specifically,

\[
 \int u(x, t) f(x) \, dP_t(x) = \int f(x) \, dI_t(x)
\]

for any measurable function \(f\). We can now integrate and call \((P_t, I_t, u)_{t \geq 0}\) a weak solution of \((38)\) if it satisfies the resulting equality. More precisely, for any \(f, g \in C^1_0(\mathbb{R})\) and \(0 < t_1 < t_2\), we need

\[
 \int_{t_1}^{t_2} \{ \rho_t (\eta) f(\eta) + (\rho_t u) x f \} \, d\tau d\eta
\]

\[
 = \int g(\eta) \, dP_{t_2}(\eta) - \int g(\eta) \, dP_{t_1}(\eta) - \int_{t_1}^{t_2} \int f'(\eta) u(\eta, \tau) \, dP_\tau(\eta)
\]

\[
 = \int f(\eta) \, dP_{t_2}(\eta) - \int f(\eta) \, dP_{t_1}(\eta) - \int_{t_1}^{t_2} \int f'(\eta) \, dI_\tau(\eta) = 0,
\]

where the boundary terms from the integration by parts in the second term drop out since \(f\) has compact support. Note that there is no integration by parts in evaluating the first term \((f\) has no dependence on \(t\) so we just integrate \(\rho_t\)). To write the second equation of \((38)\) in weak form, we proceed similarly:

\[
 \int_{t_1}^{t_2} \{ \rho(\eta) u(\eta), x g(\eta) + (\rho(\eta) u^2) x g(\eta) \} \, d\tau d\eta
\]

\[
 = \int g(\eta) \, dI_{t_2}(\eta) - \int g(\eta) \, dI_{t_1}(\eta) - \int_{t_1}^{t_2} \int g'(\eta) u(\eta, \tau) \, dI_\tau(\eta) = 0
\]

Rewriting, our definition of a weak solution under the above assumptions becomes:

\[
 \int f(\eta) \, dP_{t_2}(\eta) - \int f(\eta) \, dP_{t_1}(\eta) = \int_{t_1}^{t_2} \int f'(\eta) \, dI_\tau(\eta)
\]

\[
 \int g(\eta) \, dI_{t_2}(\eta) - \int g(\eta) \, dI_{t_1}(\eta) = \int_{t_1}^{t_2} \int g'(\eta) u(\eta, \tau) \, dI_\tau(\eta)
\]

which matches Definition 1 in [35].
3.2. Discrete example. We again consider our discrete example with four particles, with initial conditions specified by (10). We first want to compute the flow map under the following initial velocity field:

\[
u_0(x) = \begin{cases} 
0 & x < -3 \\
2 & -3 \leq x < -2 \\
1 & -2 \leq x < 1 \\
-\frac{1}{2} & 1 \leq x < 3 \\
1 & 3 \leq x
\end{cases}
\] (46)

An equivalent method would be to consider a velocity field with \(\delta\)-functions at each point mass, but this is omitted for brevity. For time \(t\) before the first collision at \((x_{12}, t^*_1) = (-1, 1)\), the flow map is as follows:

\[
\phi_t(x) = \begin{cases} 
x & x < -3 \\
-3 + 2t & x = -3 \\
x + 2t & -3 < x < -2 \\
-2 + t & x = -2 \\
x + t & -2 < x < 1 \\
x - \frac{1}{2} & 1 \leq x < 3 \\
3 + t & x = 3 \\
x + t & 3 < x
\end{cases}
\] (47)

Therefore, the inverse flow map is given by

\[
\phi_t^{-1}(x) = \begin{cases} 
x & x < -3 \\
\{0\} & -3 \leq x < -3 + 2t \\
[-3, -3 + 2t] & x = -3 + 2t \\
x & -3 + 2t < x < -2 \\
\{0\} & -2 \leq x < -2 + t \\
[-2, -2 + t] & x = -2 + t \\
x & -2 + t < x < 1 \\
[1 - \frac{t}{2}, 1] & x = 1 - \frac{t}{2} \\
\{0\} & 1 - \frac{t}{2} < x \leq 1 \\
x & 1 < x < 3 \\
\{0\} & 3 \leq x < 3 + t \\
[3, 3 + t] & x = 3 + t \\
x & 3 + t < x
\end{cases}
\] (48)

The elements of the partition \(\xi_t\) are then given by real numbers in the intervals \((-\infty, -2 + t), (-2 + 2t, 1 - \frac{t}{2})\), and \((1 + t, \infty)\) along with the intervals \([-2 + 2t, 1 - \frac{t}{2}]\) and \([1 - \frac{t}{2}, 1 + t]\). As we increase time, these intervals will grow and eventually merge when we have collisions between particles 1 and 2, and then merge again when the resulting particle collides with 3.

From here we can identify the element \(C_t(y)\) corresponding to the element of the partition \(\xi_t\) containing \(y\), and reconstruct the solution using equation (1.10) in the paper:

\[
\phi_t(y) = \frac{\int_{C_t(y)} (\eta + tu_0(\eta)) dP_0(\eta)}{\int_{C_t(y)} dP_0(\eta)}, \quad u(x, t) = \frac{\int_{D_t(x)} u_0(\eta) dP_0(\eta)}{\int_{D_t(x)} dP_0(\eta)}.
\] (49)

Construction of the inverse flow map is shown in Figure 4. Note that our calculations are formal, but can be made rigorous through application of the lemmas and
Theorem in [35]. In a similar vein, one can also apply front tracking methods as described in [20].

![Figure 4. Evolution of the discrete example and mapping back using the flow map. Highlighted in blue (long-short dash lines) are intervals unchanged under the flow map. In red (long dashed line) are intervals for which the flow map inverse is undefined. The points in green correspond to single points for which an entire interval is mapped back onto, which occurs in notably many cases. For example, $\varphi_{t_2}^{-1}(I_2^*) = \{0\}$ and $\varphi_{t_1}^{-1}(-2) = \{0\}$.

3.3. GVP and the continuous case. We give a brief idea of generalisation to the continuous case. To do this, we will need several assumptions. We provide both the technical definition and an explanation of the physical meaning of each. We first let $P_0, I_0 \in M$, the space of Radon measures on $\mathbb{R}^1$, $P_0 \geq 0$. A Radon measure is defined as a measure that is inner regular (for all Borel sets $B$, $m(B) = \sup \{m(K) | K \subset B \text{ compact}\}$) measure defined on the $\sigma$-algebra of a
Hausdorff topological space $X$ that is locally finite (for every point of $X$, there exists a neighborhood $U$ such that $m(U) < \infty$).

**Assumption 1.** For any compact $\Lambda \subset \mathbb{R}^1$, one has $P_0(\Lambda) < \infty$ and $P_0$ is either discrete or absolutely continuous with respect to Lebesgue measure. If $P_0$ is absolutely continuous, then we assume $\rho_0(x) > 0$ for all points $x$ in the support of $\rho_0$.

If $\text{Supp}(\rho_0)$ is unbounded, then we assume

$$
\int_0^x sdP_0(s) \to +\infty \text{ as } |x| \to \infty. 
$$

The first statement corresponds to the physical requirement that we do not have an infinite amount of mass on any finite interval (intervals are precompact in $\mathbb{R}$), and that the distribution of masses must be either (i) fully discrete, with point masses $\{m_i\}$ at locations $\{x_i\}$, or (ii) there are not any such point masses anywhere on $\mathbb{R}$. In this second case, we then assume it has a density, with either a finite cutoff, or density out to infinity that does not fall off too sharply (less sharply than $\frac{1}{s^{2+\varepsilon}}$, for example).

**Assumption 2.** The initial distribution of momentum $I_0$ is absolutely continuous with respect to $P_0$. We can define a Radon-Nikodym derivative $u(\cdot, 0) = \frac{dI_0}{dP_0}$ and this is the initial velocity. When $P_0$ is absolutely continuous, we assume further that $u(\cdot, 0)$ is continuous as well. This corresponds simply to the requirement that momentum is zero on intervals where there is no mass.

**Assumption 3.** For all $z > 0$, we have

$$
\sup_{|x| \leq z} |u_0(x)| \leq b_0(z) \text{ where } \lim_{|z| \to \infty} \frac{1}{z} b_0(z) = 0.
$$

Physically this means that the initial velocities of the particles can not increase as $O(|z|)$ or more as position goes to infinity.

Under these assumptions, we have a way of constructing the partition $\xi_t$ using the initial data, using a Generalised Variational Principle. We have that $y \in \mathbb{R}^1$ is a left endpoint of an element of $\xi_t$ if and only for every $y^-, y^+ \in \mathbb{R}$ such that $y^- < y < y^+$, we have the following:

$$
\frac{\int_{[y^-, y^+]} (\eta + tu_0(\eta)) dP_0(\eta)}{\int_{[y^-, y^+]} dP_0(\eta)} < \frac{\int_{[y^-, y^+]} (\eta + tu(\eta)) dP_0(\eta)}{\int_{[y^-, y^+]} dP_0(\eta)}.
$$

**4. Entropy solution and variational approach.** We now consider the work of Huang and Wang [22] on the system of one-dimensional pressureless gas equations given by

$$
\rho_t + (\rho u)_x = 0 \quad (53)
$$

$$
(\rho u)_t + (\rho u^2)_x = 0.
$$

A basis of their approach entails generalising characteristics when the flow map breaks down (is no longer one-to-one). In particular, they consider the generalised potential given by

$$
F(y; x, t) = \int^y_0 (tu_0(\eta) + \eta - x) \, dm_0(\eta) 
$$

and first note that if $\rho, u$ are bounded and measurable functions, then $m(x, t) = \int_{(0, \infty)} p\, dx - \rho u\, dt$ is independent of path since $\rho_t + (\rho u)_x$ is conserved. Indeed,
\[ \rho_t = - (\rho u)_x = (-\rho u)_x. \] Further, we have \( m_x = \rho, m_t = -\rho u \). One then readily verifies that (53) is then equivalent to

\[ m_t + um_x = 0 \]

\[ (m_x u)_x + (m_x u^2)_x = 0 \] (55)

A weak solution to the system (55) is defined as follows.

**Definition 4.1.** Let \( m(x,t) \) be of bounded variation locally in \( x \), and \( u(x,t) \) be bounded and \( m_x \)-measurable. Assume \( m_x, um_x \) are weakly continuous in \( t \). We call \((\rho, u) = (m_x, u)\) a weak solution of (55) given that

\[ \int \int \varphi_t m dx dt - \int \int \varphi u dm dt = 0 \]

\[ \int \int \psi_t u + \psi u^2 dm dt = 0 \] (56)

is satisfied for all \( \varphi, \psi \in C_0^\infty (\mathbb{R}_+^2) \), where the integrals are the Lebesgue-Stieltjes integrals.

We understand the initial value in the sense that as we take \( \eta \) to the lower limit, the measures \( \rho, \rho u \) converge weakly ([6], p. 57), where \( u_0 \) is bounded and is measurable with respect to \( \rho_0 \). We also have the following entropy condition: we call \((\rho, u)\) an entropy condition if

\[ \frac{u(x_2, t) - u(x_1, t)}{x_2 - x_1} \leq \frac{1}{t} \] (57)

holds for any \( x_1 < x_2 \), a.e. in \( t > 0 \), and \( pu^2 \) converges weakly to \( \rho_0 u_0^2 \) as \( t \to 0 \).

Their main result is the following:

**Theorem 4.2.** (Existence). Let \( \rho_0 \geq 0 \in M_{loc}(\mathbb{R}) \), the space of Radon measures defined on \( \mathbb{R} \) (or \( m_0(x) = \rho([0, x]) \) increasing) and let \( u_0 \) be bounded and measurable with respect to \( \rho_0 \), then the system (55) admits at least one entropy solution.

In proving the result, a number of technical lemmas are required. By using the generalised potential, one uses them to construct an entropy solution for an increasing function \( m_0(x) = \rho_0([0, x]) \). The trivial case \( \rho_0 = 0 \) is excluded. We state several of the lemmas without proof; more details can be found in [22].

**Lemma 4.3.** For any point \((x, t)\), the function \( F(y; x, t) \) considered as a function of \( y \) has a finite lower bound.

Now we define

\[ v(x, t) = \min_y F(y; x, t), \]

\[ S(x, t) = \{ y | \text{there exists } y_n \to y \text{ s.t. } F(y_n; x, t) \to v(x, t) \}, \] (58)

i.e. \( S(x, t) \) as the set of points \( y \) for which we can find a sequence \( y_n \) approaching this limit. Using the fact that \( F \) is left continuous in \( y \), for every \( y_0 \in S(x, t) \), one has

\[ v(x, t) = \begin{cases} F(y_0; x, t) & \text{if } F(y; x, t) \text{ achieves its minimum at } y_0, \\ F(y_0 + 0; x, t) & \text{otherwise} \end{cases} \] (59)

leading to the following lemma.
Lemma 4.4. Assume \( y_0 \in S(x,t) \), \([m_0(y_0)] = m_0(y_0 + 0) - m_0(y_0 - 0) > 0\). Then
\[
v(x,t) = \min_y F(y;x,t) = \begin{cases} 
F(y_0;0;x,t) & \text{if } x \leq y_0 + tu_0(y_0) \\
F(y_0 + 0;x,t) & \text{if } x > y_0 + tu_0(y_0)
\end{cases}
\] (60)

In other words, in the first case, \( F \) achieves its minimum, and in the second it does not.

Lemma 4.5. Let \((x_n,t_n)\) and \(y_n \in S(x_n,t_n)\) converge to \((x,t)\) and \(y_0\), respectively. Then \(y_0 \in S(x,t)\).

From Lemma 4.3, we have that \(\inf \{y \in \text{spt}\{\rho_0\}\} \) is finite if \(y_m(x,t) = -\infty\). Similarly, \(\sup \{y \in \text{spt}\{\rho_0\}\} \) is finite if \(y^m(x,t) = +\infty\). For each point \((x_0,t_0)\), introduce left and right backward generalised characteristics \(L_1, L_2\):
\[
L_1: x = x_0 + \frac{x_0 - y_0(x_0,t_0)}{t_0} (t - t_0)
\]
\[
L_2: x = x_0 + \frac{x_0 - y^*(x_0,t_0)}{t_0} (t - t_0)
\] (61)
and claim that there is only one minimum point of \(F(y;x,t)\) for each \((x,t)\) along backward lines \(L_1, L_2\).

Lemma 4.6. For any \(y_0 \in S(x_0,t_0)\), \(y_*(x,t) = y^*(x,t)\) holds along the lines
\[
L: x = x_0 + \frac{x_0 - y_0}{t_0} (t - t_0).
\] (62)

Furthermore, \(y_*(x,t) = y^*(x,t) \leq y_*(x_0,t_0)\) along the line \(L_1\), and \(y_*(x,t) = y^*(x,t) = y^*(x_0,t_0)\) along the line \(L_2\).

Lemma 4.7. \(y_*, y^*\) are increasingly monotonic in \(x\). In particular, \(y^*(x_1,t) \leq y_*(x_2,t)\) holds for any \(x_1 < x_2\).

4.1. Constructing the generalised potential for a discrete example. In conveying the ideas of the construction of the solution and application of the lemmas, it is useful to return to our discrete example given by four point masses on the real line (see (10)). We have \(m_0(x) = \rho_0([0,x])\). For negative \(x\) this is interpreted as \(m_0(x) = -\rho_0((x,0])\). Hence
\[
m_0(x) = \rho_0([0,x]) = \begin{cases} 
-m_1^0 - m_2^0 & x < x_1^0 \\
-m_2^0 & x_1^0 \leq x < x_2^0 \\
0 & x_2^0 \leq x \leq x_3^0 \\
m_3^0 & x_3^0 < x < x_4^0 \\
m_3^0 + m_4^0 & x_4^0 < x
\end{cases}
\] (63)

Evaluating the Stieltjes integral for the generalised potential (54) for \(x = 0\) and \(t = 1\), one has
Figure 5. Graphs of (a) $\int_{y=0}^{y=0} tu_0(\eta) \, d\eta_0(\eta)$ (with $t = 1$), (b) $\int_{y=0}^{y=0} (\eta - x) \, d\eta_0(\eta)$, (c) $F(y; 0, 1)$

Some of the intermediary steps are illustrated in Figure 5.

Lemma 4.3 says that for a given $(x, t)$, $F(y; x, t)$ has a finite lower bound. This is trivial: for a given point $(x, t)$, as the generalised potential takes on only (at most) five distinct nonzero values. Next we want to define $v(x, t)$ and the set $S(x, t)$.

We have

$$v(x, t) = \min_{y} F(y; x, t) = \min \{ \alpha_i \}_{i=1}^{5}$$

(65)

where $\alpha_i$ are the values that the piecewise function $F$ may take on, and define

$$S(x, t) = \{ y \mid \text{there exists } y_n \to y \text{ s.t. } F(y_n; x, t) \to v(x, t) \}.$$  

(66)

In this example, $v(0, 1) = -\frac{1}{2}$ and for any $y \leq x_1^0 = -3$, we can clearly find $y_n \to y$ from the left so that this condition holds. In fact, $F(y; 0, 1)$ achieves its minimum at any such $y$, so that

$$v(0, 1) = F(y; 0, 1) = -\frac{1}{2}.$$  

(67)
Lemma 4.4 indicates that for a point $y_0 \in S(x,t)$ ($y_0 \leq -3$ in our example) and $[m_0(y_0)] = m(y_0 + 0) - m(y_0 - 0) > 0$ (which forces $y_0 = -3$), then

$$v(x,t) = \min_y F(y;x,t) = \begin{cases} F(y_0;x,t) & \text{if } x \leq y_0 + tu_0(y_0) \\ F(y_0 + 0;x,t) & \text{if } x > y_0 + tu_0(y_0) \end{cases}$$

$$v(0,1) = \min_y F(y;0,1) = F(y_0 + 0;0,1) = -\frac{1}{4}$$  \hspace{1cm} (68)

for $(x,t) = (0,1)$, since $0 > -3 + 2$.

Lemma 4.5 says that for $(x_n,t_n)$ and $y_n \in S(x_n,t_n)$ that converge to $(x,t)$ and $y_0$, respectively, then $y_0 \in S(x,t)$. In our example, this corresponds to choosing $(x_n,t_n) \to (0,1)$ and $y_n \in S(x_n,t_n)$. Recall

$$F(y;x_n,t_n) = \begin{cases} \frac{3}{4} t_n + \frac{1}{4} \left( x_0^2 - x_n \right) + \frac{1}{4} \left( x_0^2 - x_n \right) & y \leq -3 \\ \frac{1}{4} t_n + \frac{1}{4} \left( x_0^2 - x_n \right) & -3 < y \leq -2 \\ 0 & -2 < y \leq 1 \\ -\frac{1}{6} t_n + \frac{1}{6} \left( x_0^2 - x_n \right) & 1 < y \leq 3 \\ \frac{1}{6} \left( x_0^2 - x_n \right) & 3 < y \end{cases}$$ \hspace{1cm} (69)

Here we will have $S(x_n,t_n) = \{ y \leq -3 \} = S(x,t)$ for $(x_n,t_n)$ sufficiently close to $(x,t) = (0,1)$. Indeed,

$$F(y;\varepsilon,1+\delta) = \begin{cases} -\frac{1}{4} + \frac{3}{4} \delta - \frac{1}{4} \varepsilon & y \leq -3 \\ -\frac{1}{4} + \frac{3}{4} \delta - \frac{1}{4} \varepsilon & -3 < y \leq -2 \\ 0 & -2 < y \leq 1 \\ \frac{1}{12} - \frac{1}{6} \delta - \frac{1}{3} \varepsilon & 1 < y \leq 3 \\ \frac{5}{12} - \frac{1}{6} \delta - \frac{1}{3} \varepsilon & 3 < y \end{cases}$$ \hspace{1cm} (70)

So clearly if $\{y_n\}$ converges, its limit is also in $S(x,t)$.

5. Introduction of random initial conditions and generalisation to conservation laws. Thus far, we have seen several examples of the evolution of conservation laws under purely deterministic initial conditions. Other questions of interest concern the behavior of the solutions under random initial conditions. It is typical to assign an initial condition in the form of a stochastic process with some structure, for example a Markov ([1], p. 144) or Feller process. A Feller process ([1], p. 150) is essentially a special kind of Markov process whose transition kernel is built from a semigroup with the contraction property. A natural first question arising from this is whether there are universality classes for which the structure of the initial condition is preserved under the conservation law as time evolves [29]. It is also of interest whether one can expect certain properties to persist in time. For example, the Markov property in the continuum states conditioning the value of the solution at a point $x$ based on information in the past or future offers no additional information about the solution at $x$. This is a very desirable condition to work with in the context of probability theory. However, even if the Markov property is imposed on the initial conditions, it may not persist for even an infinitesimal amount of time. Another direction of interest is the extension of these results to more general $C^1$, convex flux functions, beyond the special case of Burgers’ equation, where $f(u) = u^2/2$. Among many excellent references for a clear explanation of several kinds of these stochastic processes are [1, 4, 37].

Several key results along these lines are developed in [30, 28]. First, results are obtained involving the Lax equation from multiple different perspectives. They
note that a stationary, spectrally negative Feller process can be characterised by a generator $A(t)$ acting on smooth test functions $\varphi \in C_1^c(\mathbb{R})$:

$$A\varphi(y) = b(y, t) \varphi'(y) + \int_{-\infty}^{y} (\varphi(z) - \varphi(y)) n(y, dz, t)$$  \hspace{1cm} (71)

where $b(y, t)$ is the drift term and $n(y, dz, t)$ describes the jump density and satisfies

$$\int 1 \wedge |y - z|^2 n(y, dz, t) < \infty \text{ for all } y \in \mathbb{R}. \hspace{1cm} (72)$$

By introducing a second operator which involves the flux function $f$, defined by

$$B\varphi(y) = -f'(y) b(y, t) \varphi'(y) - \int_{-\infty}^{y} \frac{f(y) - f(z)}{y - z} (\varphi(z) - \varphi(y)) n(y, dz, t), \hspace{1cm} (73)$$

they show that evolution of the process is governed by the Lax equation

$$\partial_t A = [A, B] = AB - BA. \hspace{1cm} (74)$$

These equations hold not only for the special case of Burgers’ equation, as first shown in [8], but for more general fluxes. The structure of the result is shown to depend only on the convexity assumption on the flux function $f$ rather than Burgers’ equation being a special case.

Another important result expands on kinetic theory. By expanding out the commutator in (74), equations that describe shock clustering are obtained. Namely, the drift satisfies

$$\partial_t b(y, t) = -f''(y) b^2(y, t), \hspace{1cm} (75)$$

and the jump density

$$\partial_t n(x, y, t) + \partial_y (n V_y(y, z, t)) + \partial_z (n V_z(y, z, t)) = Q(n, n) + n \left( \left( \frac{f(y) - f(z)}{y - z} - f'(y) \right) \partial_y b - b f''(y) \right) \hspace{1cm} (76)$$

where the velocities $V_y$ and $V_z$ are prescribed by

$$V_y(y, z, t) = \left( \frac{f(y) - f(z)}{y - z} - f'(y) \right) b(y, t), \hspace{1cm} V_z(y, z, t) = \left( \frac{f(y) - f(z)}{y - z} - f'(z) \right) b(z, t) \hspace{1cm} (77)$$

and $Q$ is a collision kernel describing various interactions between shocks.

It is proven that the equation (76) and the Lax equation (74) are in fact equivalent. They also prove a closure property, stating that if initial conditions are strong Markov (that is, have the Markov property with respect to stopping times, [1], p. 97) and satisfy some other additional assumptions, then a Markov property persists in time. More precisely, one has (Thm 2, [28]):

**Theorem 5.1.** Define the inverse Lagrangian process $a(x, t)$ by the following:

$$I(s; x, t) = \int_0^{s} \left( u_0(r) - (f')^{-1} \left( \frac{x - r}{t} \right) \right) dr,$$

$$a(x, t) = \arg^+ \min_{s \in \mathbb{R}} I(s; x, t), \hspace{1cm} (78)$$

and let $u_0$ be a spectrally negative strong Markov process such that the growth condition

$$\lim_{|s| \to \infty} I(s; x, t) = +\infty \hspace{1cm} (79)$$
holds a.s. Under the law $\mu_0$ and for any fixed $t > 0$, the inverse Lagrangian process $a(x, t)$ is Markov.

6. **Exact solution for a special case of random initial conditions.** With the introduction of random initial conditions into Burgers’ equation and more general conservation laws, in general, the most one can hope to obtain for a solution in terms of complicated generators, as we have seen in Section 5. However, for a few specialised kinds of initial data, one can obtain exact, closed-form solutions up to the level of special functions. One such case is that of Frachebourgh and Martin [16], for which white noise initial data is considered, and expressions for the one- and two-point functions obtained, building off a key result of [19]. This is also closely related to a problem of ballistic aggregation they addressed in [15]. The closed analytical forms are expressed in terms of Airy functions. The starting point involves the inviscid Burgers’ equation

$$\frac{\partial}{\partial t} u(x, t) + u(x, t) \frac{\partial}{\partial x} u(x, t) = \nu \frac{\partial^2}{\partial x^2} u(x, t)$$

(80)

and using variational methods in tandem with taking the limit of viscosity parameter $\nu \downarrow 0$. To this end, one may introduce the potential $\frac{\partial \Psi(x, t)}{\partial x} = u(x, t)$ and use the Cole-Hopf transformation $\Psi(x, t) = -2\nu \ln \theta(x, t)$ along with other methods ([13], p. 207) to show that $\theta(x, t)$ satisfies the heat equation. Consequently, the solution to (80) is then given by

$$u(x, t) = \frac{\int_{-\infty}^{\infty} dy \frac{x-y}{t} \exp \left( -\frac{1}{2\nu} F(x, y, t) \right)}{\int_{-\infty}^{\infty} dy \exp \left( -\frac{1}{2\nu} F(x, y, t) \right)}$$

(81)

where

$$F(x, y, t) = \frac{(x-y)^2}{2t} - \psi(y)$$

$$\psi(y) = -\Psi(y, 0) = -\int_0^y dy' u(y', 0),$$

(82)

the latter of which depends, obviously, on the given initial conditions. In the limit of interest $\nu \downarrow 0$, the only contributions from (81) arise from where $F$ has a minimum, i.e. at point(s) prescribed by

$$\xi(x, t) = \min_y F(x, y, t),$$

(83)

and write, formally

$$u(x, t) = \frac{x - \xi(x, t)}{t}.$$

(84)

Due to the scaling properties of the solution, it is sufficient to set $t = 1$ herein. To find this minimum, $\xi(x, 1)$, consider the following geometric illustration of the solution. Picture $\psi(x)$ as a sample Brownian motion path, and a parabola $P(x) = (x-y)^2/2 + C$. One then adjusts $P(x) (= P_y(x))$ by sliding it down, decreasing $y$ so that it touches the Brownian path but does not cross it. This may occur at multiple points, or at just one point. If there are two such points for a value $x^*$, labelled $\xi_-$ and $\xi_+$, then the function $F(x, y, t)$ has a discontinuity at $x^*$, $\mu$ which is called a shock. We then characterise this shock by two parameters:

$$\mu = \xi_+ - \xi_-, \nu = x^* - \xi_-, $$

(85)

named strength and wavelength.
For a shock at a point \( x^* \), we slide the parabola \((x - x^*)^2/2\) down until we have (at least) two contact points with the Brownian path, but in such a way that the parabola does not cross the Brownian path. If there are more than two, we consider only the first and last contact points. These points are given by \((\xi, (\xi - x^*)^2/2)\) and \((\xi +, (\xi^+ - x^*)^2/2)\). The shock is then described by the parameters \( \mu = \xi^+ - \xi^- \) and \( \nu = x^* - \xi^- \). This figure is based off Figure 1, [16].

These notions are illustrated in Figure 6. This forms the basis of relating the one-point functions \( p_1(x, u) \), the probability that the velocity field at a point \( x \) takes values between \( u \) and \( u + du \), with the object \( \rho_1(\mu, \eta) \). In particular, one can adjust the coordinate system and has

\[
\rho_1(\mu, \eta) = \mathbb{E}\left\{ \psi(x) \leq P_\nu(x), x \in \mathbb{R}, \text{ first contact with } P_\nu(x) \text{ at } (0, 0) ; \text{ last contact with } P_\nu(x) \text{ at } (\mu, \eta) \right\}
\]  

Similarly, one can consider two such parabolas and build the two-point functions \( p_2(x_1, u_1, x_2, u_2) \), giving the probability that the velocity field at points \( x_1, x_2 \) take values between \( u_1 + du \), \( u_2 + du \), in terms of \( \rho_2(0, \mu_1, \eta_1, x_1, \mu_2, \eta_2) \).

Integrating up the shock strength distribution \( \rho_1(\mu) = \int_{-\infty}^{\infty} \rho_1(\mu, \eta) d\eta \), they obtain that the shock distribution as a function of strength is given by

\[
\rho_1(\mu) = 2\alpha^3 \mu \sum_{k \geq 1} e^{-\alpha \mu} \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\zeta_1 \int_{-\infty}^{\infty} d\zeta_2 \frac{e^{-i\mu(\zeta_1 + \zeta_2)/2}}{Ai(i\zeta_1)Ai(i\zeta_2)} \int_{-\infty}^{\infty} d\eta' e^{i\eta'(\zeta_1 - \zeta_2)}
\]

In the same vein, they obtain results for \( \rho_2 \), thus giving an exact, closed-form expression for the characterisation of the system dynamics in this case of Burgers’ equation with white noise initial data.
7. Further extension of results to Flux functions with less regularity. As we have seen in the previous sections, conservation laws such as Burgers’ equation can admit an exact solution in some special cases, even with random initial data, as in [16]. More general expressions entailing generators and semigroup theory can be extended to the case of an arbitrary $C^1$ flux function in [28] and provide more insight into the solution. Using a specific set of test functions and probability theory, we show that one may consider a nonlinear flux function that is only continuous and obtain meaningful results for the hierarchy of equations describing the formation and interaction of shocks. The results obtained are in terms of various $n$-point functions, representations of probabilities (or probability densities) that the solution field takes certain values at a given time and a number of different positions.

7.1. Density of states approach. We build the solution to a piecewise linear flux function defined by piecewise linear interpolation between the points

$$f(u_i) = f_i, \ 1 \leq i \leq M$$

(88)

with slopes consequently given by

$$c_k = \frac{f_{k+1} - f_k}{u_{k+1} - u_k}.$$ 

(89)

The flux function is illustrated in Figure 7. Burgers’ equation

$$u_t + uu_x = 0$$

(90)

can be written in its entropy-entropy flux pair form in terms of smooth test functions $\varphi$ and $\psi$ as

$$\partial_t \varphi (u(x,t)) = -\partial_x \psi (u(x,t)).$$

(91)

Further technical details of the equivalence between the entropy-entropy flux pair form (91) and (90) can be found in [42].
Taking expectations of (91) and formally passing derivatives inside yields
\[
E \{ \partial_t \varphi (u (x, t)) \} = \partial_t E \{ \varphi (u (x, t)) \} = -E \{ \partial_x \psi (u (x, t)) \} .
\] (92)

As this is a piecewise linear function with \( M - 1 \) distinct slopes, it makes sense to consider a test function \( \varphi \) with discrete values on the set of points \( x \in \{ x_i \}_{i=1}^M \), extended for other \( x \) to make it left continuous. One then has
\[
\partial_t \sum_{i=1}^M \varphi (u_i) p_1 (x, t; u_l) = - \sum_{i=1}^M \sum_{m=1}^M (\psi (u_m) - \psi (u_l)) p_2 (x, x^+, t; u_l, u_m) .
\] (93)

We now choose the derivative of the test function \( \varphi \) to be a discretised version of a \( \delta \)-function as \( u_k \) for a given \( k \), i.e.
\[
\varphi'_k (u) = \frac{1_{[u_k, u_{k+1})} (u)}{u_{k+1} - u_k} ,
\] (94)
\[
\varphi_k (u) = 1_{[u_k, \infty)} (u) , \quad \psi'_k (u) = \frac{c_k 1_{[u_k, u_{k+1})} (u)}{u_{k+1} - u_k} , \quad \psi (u) = c_k 1_{[u_k, \infty)} (u) .
\] (95)

These are further illustrated in Figure 8(a-b). Substituting the expressions (95) into (93), we obtain
\[
\sum_{l=k+1}^M \partial_l p_1 (x, t; u_l) = \sum_{l=k+1}^M \sum_{m=1}^k c_k p_2 (x, x^+, t; u_l, u_m) - \sum_{l=1}^k \sum_{m=k+1}^M c_k p_2 (x, x^+, t; u_l, u_m)
\] (96)

This is significant as it provides a relation for the change (in time) of the value of the one-point distribution at position \( x \) as a function of the difference of two terms involving the two-point distribution at the same point \( x \) and the value of the solution at its right limit \( x^+ \). This difference of terms can be interpreted as a discrete derivative or a finite difference. By generalising the procedure outlined above, we obtain similar results for the hierarchy at the \( n \)-point level. Thus, the expression for the \( n \)-point function at points \( x_1, ..., x_n \) depends on analogous pairs of terms with interplay between value of the \( n + 1 \) point function at \( x_i^+ \) and \( x_i \). The relation (96) applies uniformly before shock interactions, but the meaningful case one should visualise is when a shock is present at position \( x \). This is shown in Figure 8(c).

### 7.2. Density of shocks approach.

Although the first approach outlined above accurately describes the initial formation and propagation of shocks, it requires additional techniques to persist through shock collisions. By considering a second method and changing the interpretation of the \( n \)-point function to be inclusive of both the value of the velocity field \( u \) at \( x \) and \( x^+ \), one is able to derive a second hierarchy that accurately described the system dynamics even through these interactions. Specifically, the new definition linked the information of the value of the solution at both limits, making \( x_i \) and \( x_i^+ \) always appear together, replacing the old expressions as follows:
\[
p_1 (x, t; u) \rightarrow f_1 (x_-, x, t; u, v)
\]
\[
p_2 (x_1, x_2, t; u_1, u_2) \rightarrow f_2 (x_1^-, x_1, x_2^-, x_2, t; u_1, v_1, u_2, v_2) .
\] (97)
Figure 8. (a)-(b) Construction of the test functions $\varphi_k(u)$ and $\psi_k(u)$; (c) Illustration of a shock, with positive contribution from $\partial_t p_1(x, t; u_i)$ (upward arrow, blue), and negative contribution from $p_2(x, x^*, t; u_l, u_r)$ (right arrow, red).

In deriving this hierarchy, it clearly makes sense to have a transport term of the form

$$\partial_t f_1(x_-, x, t; u, v)$$  \hspace{1cm} (98)

along with a free-streaming term

$$c_{uv} \partial_x f_1(x_-, x, t; u, v).$$  \hspace{1cm} (99)

When considering the hierarchy at the nth level, (99) is generalised by having a sum of $n$ terms, one for each shock between $u_i$ and $v_i$. On the other side of equation, terms for various interactions causing creation or destruction of the shock $u$ and $v$ are added. Using Taylor series, one picks up appropriate rate constants and obtains

$$\partial_t f_1(x, t; u, u + 1) + c_{u,u+1} \partial_x f_1(x, t; u, u + 1)$$

$$= - \sum_{w < u} (c_{w,u+1} - c_{u+1,w}) \partial_2 f_2(x, x, t; u, u + 1, w)$$

$$- \sum_{w > u+1} (c_{w,u} - c_{u,u+1}) \partial_1 f_2(x, x, t; w, u, u + 1)$$  \hspace{1cm} (100)

for $u < v$ (for which $v = u + 1$ yields the only nontrivial case) and

$$\partial_t f_1(x, t; u, v) + c_{uv} \partial_x f_1(x, t; u, v)$$

$$= \sum_{w \leq u + 1} (c_{uw} - c_{w}) \partial_2 f_2(x, x, t; u, w, w, v)$$

$$- \sum_{w \leq v + 1} (c_{uv} - c_{w}) \partial_2 f_2(x, x, t; u, v, v, w)$$

$$- \sum_{w \geq u - 1} (c_{uw} - c_{uv}) \partial_1 f_2(x, x, t; w, u, u, v).$$  \hspace{1cm} (101)
for \( v < u \). We use the notation \( u + 1 \) to mean the nearest neighbor state, with the states in ascending order. When tested against examples, it is confirmed that these equations accurately model the dynamics after shocks collide without modifications, unlike the first approach or in many classical results.

A related problem is studied in [23], where an expression for the statistics is obtained in the problem with an initial condition bounded to the case \( 0 \leq x \leq L \). In an approach reminiscent of statistical mechanics methods, limits of a large number of particles are taken to obtain these results.

8. **Open problems and ideas for future research.** There remain a number of open problems, particularly with the introduction of randomness into the initial conditions for conservation laws. Additional problems include the more detailed analysis and perhaps direct computation with broader classes of initial conditions. Although the Markov property imposed on random initial conditions does not persist under the evolution of time for Burgers’ equation, the strong Markov property with only downward jumps do persist for Burgers’ equation. Other classes of random initial data and/or more general conservation laws can be considered and explored further.

A closure theorem for the \( n \)-point function is another possible direction for future work. Such a result would allow the time derivative of the \( n \)-point function to be expressed explicitly in terms of the spatial derivatives of \( n \)-point functions of different arguments. It is possible, for example, that the \( n+1 \)-point functions may factor into lower point functions for larger values of \( n \). Such a reduction, should it exist, would serve to greatly enhance the understanding of conservation laws from yet another perspective.

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