Most materials of microelectronic devices, including the computer ones, can contain foreign inclusions, which can lead to strong oscillations of the wave field due to the occurrence of re-reflected waves. Rigid inclusions can be found in many different materials, as well as in human biological tissues. In these cases, a complex wave field of re-reflected waves occurs, which leads to oscillations. This problem is modeled here as the problem of wave diffraction on a rigid spherical inclusion near a plane rigid boundary, which generalizes the previously considered problems (Selezov et al., 2018) [1], (Selezov, 1993) [2].

Consider a spherical coordinate system $r, \theta, \varphi$ (the radial, zenithal and azimuthal coordinates), which corresponds to a rectangular Cartesian coordinate system $x, y, z$. The $Oy$ axis is perpendicular to the flat boundary with the origin at the center of an absolutely rigid spherical inclusion (scatterer) and is directed from infinity to the flat boundary.
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When plane waves run from infinity (plane waves propagate along the $Oy$ axis), a diffracted field of repeatedly re-reflected waves appears in the system. A plane wave of displacements propagates from infinity along the $Oy$ axis

$$u_y(0, y, 0, t) = u_0 e^{i(y_0 + \omega t)}. \quad (1)$$

The motion of an elastic medium is described by the equations

$$\left\{ \begin{array}{c}
\nabla^2 - \frac{1}{c^2_e} \frac{\partial^2}{\partial t^2} \psi = 0, \\
\nabla^2 - \frac{1}{c^2_s} \frac{\partial^2}{\partial t^2} \ddot{a} = 0,
\end{array} \right. \quad (2)$$

and the displacement vector is determined by the formula

$$\ddot{a} = \nabla \psi + \nabla \times \ddot{a}, \quad \nabla \cdot \ddot{a} = 0. \quad (3)$$

The boundary conditions on the sphere and on the flat boundary have the form

$$u_r|_{r=a} = 0, \quad u_\theta|_{r=a} = 0, \quad u_y|_{y=-h} = 0, \quad \sigma_{xy}|_{y=-h} = 0. \quad (4)$$

Conditions (4) mean that, on the surface of the sphere $r=a$, the displacement vector is zero. On the plane boundary $y=-h$, its normal component and tangent stress are zero (slippage). The desired functions must also meet the Sommerfeld radiation conditions.

When introducing dimensionless values, the characteristic values are taken as: length [m] — radius of the sphere $a$, time [s] — $1/\omega$, kilogram-mass [kg] — Young modulus.

The equation for the incoming wave determines, in accordance with (2), the potential $\psi$ corresponding to dilation waves

$$\left\{ \begin{array}{c}
\nabla^2 - \frac{1}{c^2_e} \frac{\partial^2}{\partial t^2} \psi (y, t) = 0. \end{array} \right. \quad (5)$$

Equations (2) follow from the equations of elastodynamics

$$G \nabla^2 \ddot{a} + (\lambda + G) \nabla (\nabla \cdot \ddot{a}) = \rho \frac{\partial^2 \ddot{a}}{\partial t^2}$$

with the use of a well-known formula $\nabla \times \nabla \times \ddot{a} = \nabla (\nabla \cdot \ddot{a}) - \nabla^2 \ddot{a}$. As a result, the definition of the operator $\nabla \times \ddot{a}$ in (3) is reduced to the definition of the operator $\nabla^2 \ddot{a}$ (Morse & Feshbach) [3]. In the case of axial symmetry $\frac{\partial}{\partial \phi} = 0$, we get

$$\nabla^2 \ddot{a} = \ddot{e}_r \left[ \nabla^2 a_r - \frac{2}{r^2} a_r - \frac{2}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta a_\theta) \right] +$$

$$+ \ddot{e}_\theta \left[ \nabla^2 a_\theta - \frac{1}{r^2 \sin^2 \theta} + \frac{2}{r^2} \frac{\partial}{\partial \theta} \right] + \ddot{e}_\phi \left[ \nabla^2 a_\phi - \frac{a_\phi}{r^2 \sin^2 \theta} \right]. \quad (6)$$
The third term in expression (6) is also zero, since \( a_\varphi = 0 \). The components \( a_\varphi \) are the projections of the vector \( \vec{a} \) onto the coordinate line \( \varphi \) and are equal to zero in the case of axial symmetry. By analogy with the construction of the equation for \( \psi \) (5) we can introduce a scalar function \( \xi(r, \theta) \) with normalization \( u_0 \), which depends on two arguments, i.e. we obtain the scalar wave equation for the scalar function \( \xi(r, \theta) \)

\[
\left( \nabla^2 - \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} \right) \xi = 0.
\]

From it, after the separation of variables, the Legendre equation and the Bessel equation for spherical functions follow.

Solutions to the problem of elastic wave diffraction by a sphere (Seismic, 2016) [4] in an infinite domain for functions \( \psi \) and \( \xi \) are written as

\[
\psi = \sum_{m=0}^{\infty} \left[ f_m j_m(pr) + a_m h_m^{(2)}(pr) \right] P_m(\cos \theta), \quad \xi = \sum_{m=0}^{\infty} b_m h_m^{(2)}(qr) \frac{\partial}{\partial \theta} P_m(\cos \theta),
\]

(7)

Where \( f_m = -(2m+1) u_0 p^{-1-i(m+1)} \), \( j_m(pr) \) and \( h_m^{(2)}(pr) \) are spherical Bessel and Hankel functions. For example, \( j_m(\xi) = J_{m+1}(\xi)\sqrt{\frac{\pi \xi}{2}} \).

From the first two boundary conditions (4) using (7), we obtain the coefficients \( a_m \) and \( b_m \):

\[
a_m = f_m \Delta_m^{-1} \left\{ m(m+1) j_m(pa) h_m(qa) - pa j'_m(pa) [h_m(qa) + qah'_m(qa)] \right\},
\]

\[
b_m = f_m \Delta_m^{-1} \left\{ m(m+1) j'_m(pa) - h'_m(pa) j_m(pa) \right\},
\]

\[
\Delta_m = pa h'_m(pa) [h_m(qa) + qa h'_m(qa)] - m(m+1) h_m(pa) h_m(qa).
\]

(8)

We find approximate solutions for the field in the far zone \( \frac{r}{a} \gg 1 \) by representing the Hankel functions by their asymptotic expansions at large \( r / a \),

\[
u_r = \sum_{m=0}^{\infty} a_m i^m \frac{1}{r} e^{-ipr} P_m(\cos \theta), \quad u_0 = -\sum_{m=0}^{\infty} b_m i^m \frac{1}{r} e^{-ipr} \frac{\partial}{\partial \theta} P_m(\cos \theta).
\]

(9)

In the Rayleigh approximation, the quantities \( pa \) and \( qa \) satisfy the inequalities \( pa, qa \ll 1 \). In this case, it can be established from (8) and (9) that the dominant coefficients are

\[
a_1 \equiv i \cdot 3a \left[ 1 + 2 \left( \frac{q}{p} \right)^2 \right]^{-1}, \quad b_1 \equiv i \cdot 3a \left[ 1 + 2 \left( \frac{q}{p} \right)^2 \right]^{-1} \left( \frac{q}{p} \right)^2.
\]

(10)

To construct solutions in a semiinfinite region, we use the image method (Jackson, 1962) [5]. A solution satisfying the second two boundary conditions (4) in each \( k \)-th approximation is represented in the form
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\[ \ddot{u}(r, \theta, r^*, \theta^*) = \sum_{k=1}^{\infty} [\ddot{u}_k(r, \theta) + \ddot{u}_k^*(r^*, \theta^*)], \]  

(11)

where the summary components of displacements for the scattered field of multiplicity \( k \) have the form

\[ (\ddot{u}_k + \ddot{u}_k^*)_r = U_{rk} = u_{rk} - u_{rk}^* \cos(\theta + \theta^*) + u_{yk}^* \sin(\theta + \theta^*), \]

\[ U_{yk} = u_{yk} + u_{yk}^* \cos(\theta + \theta^*) - u_{yk}^* \sin(\theta + \theta^*). \]

(12)

Difference in distances from real and imaginary obstacles to a certain point \( r, \theta \) and the time difference of arrivals of \( P \)- and \( S \)-waves in the first approximation are taken into account by the formulas

\[ \exp(-i\alpha_j) = (\cos \eta_j - i \sin \eta_j) \exp(-ipr), \quad \alpha_0 = qr, \quad \alpha_1 = pr^*, \quad \alpha_2 = qr^*, \]

(13)

\[ \eta_0 = pr \left( \frac{q}{p} - 1 \right), \quad \eta_1 = \eta_p = pr(\eta - 1), \quad \eta_2 = \eta_q = pr \left( \frac{q}{p}\eta - 1 \right). \]

Formulas (13) follow from the geometric relations obtained above for the main and mirror obstacles.

By formulas (10) (13), after a series of transformations for a singly scattered field, we find

\[ U_{r1} \equiv -\cos \theta + \left( \frac{2h}{r} \cos \theta - 1 \right) \left( \frac{2h}{r} - \cos \theta \right) (\cos \eta_p - i \sin \eta_p) \eta^{-3} + \]

\[ + \left( \frac{q}{p} \right)^2 (\cos \eta_q - i \sin \eta_q) \frac{2h}{r} \eta^{-3} \sin^2 \theta, \]

(14)

\[ U_{\theta 1} \equiv \left[ - \left( \frac{q}{p} \right)^2 (\cos \eta_q - i \sin \eta_q) - \left( \frac{2h}{r} \cos \theta - 1 \right) (\cos \eta_q - i \sin \eta_q) \eta^{-3} \left( \frac{q}{p} \right)^2 \right. \]

\[ \left. - \left( \frac{2h}{r} - \cos \theta \right) (\cos \eta_p - i \sin \eta_p) \frac{2h}{r} \eta^{-3} \right] \sin \theta. \]

(15)

On the right-hand sides (14), (15), the factor \( \frac{a}{r} \exp(ipr) \) is omitted, the left-hand sides are normalized by a factor \( 3[1+2(q/p)^2]^{-1} \), and displacements are attributed to \( u_0 \). The field of the incident wave and the corresponding field of the wave reflected from the boundary have the form: \( u_y = u^+_y - u^*_y = \exp(ipy) - \exp[i(py - 2ph)] \).

An approximate solution to the scattering problem in the second approximation is represented in the form

\[ u_r \equiv U_{r1} + U_{r2}; \quad u_\theta \equiv U_{\theta 1} + U_{\theta 2}. \]

The addition theorem for spherical wave functions is given in (Friedman & Russek, 1954) [6].
As an example, let us find displacements $u_r$ and $u_\theta$ with the next data: Poisson’s ratio $\nu = 0.25$; $pa = 0.8$; $r/a = 200$. With the selected parameters, the error of the applied formulas does not exceed 10%. Calculations are performed in points with a step of $\pi/36$. The calculation results are shown in Fig. 1, where the oscillations of the re-reflected waves are clearly visible.

Below are the calculations and the scattering diagrams, which show a strongly oscillating wave field.

The calculation results are obtained for the following parameters $pa = 0.10$, $q/p = 3.317$, $h/a = 200$, $r/a = 200$, $pr = 20$, $h/r = 1$, $\nu = 0.45$.

**Fig. 1.** Change in the values of the scattered field $\text{Im} u_r$ and $\text{Re} u_r$ at $h/a = 200$ (solid line) and at $h/r = \infty$ (dash-dotted line) without flat border.

**Fig. 2.** Change in the values of the scattered field $\text{Re} u_r$ and $\text{Im} u_r$. 

As an example, let us find displacements $u_r$ and $u_\theta$ with the next data: Poisson’s ratio $\nu = 0.25$; $pa = 0.8$; $r/a = 200$. With the selected parameters, the error of the applied formulas does not exceed 10%. Calculations are performed in points with a step of $\pi/36$. The calculation results are shown in Fig. 1, where the oscillations of the re-reflected waves are clearly visible.

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О.М. Хіміч 1, І.Т. Селезов 2, В.А. Сидорук 1
1 Інститут кібернетики ім. В.М. Глушкова НАН України, Київ
2 Інститут гідромеханіки НАН України, Київ
E-mail: khimich505@gmail.com, igor.selezov@gmail.com, wolodymyr.sydoruk@gmail.com

ЧИСЕЛЬНЕ МОДЕЛЮВАННЯ ДИФРАКЦІЇ ПРУЖНИХ ХВИЛЬ НА СФЕРІ У НАПІВОБМЕЖЕНОЙ ОБЛАСТІ

Розглянуто проблему розсіювання плоских пружних хвиль твердою сферою, розташованою поблизу плоскої жорсткої межі, що призводить до породження багаторазово відбитих дилатационних та зсувних хвиль. Постановка задачі дається, коли умови ковзання задаються на рівній межі (рівність нулю дотичних напружень). Проблема зводиться до визначення скалярних функцій. Записані загальні розв'язки і побудований приблизний рішення для поля в дальній зоні, які характеризуються тим, що відстань від межі площини до перешкоди набагато більша за радіус кулі. Крім того, наближення Релея використовується, коли хвильове число набагато менше за радіус кулі. Метод зображень використовується для побудови множинно відбитих хвиль. Наведені розрахунки розсіяних хвильових полів, представлені у вигляді діаграм розсіювання, з яких видно сильно коливальне хвильове поле.

Ключові слова: дифракція хвилі, пружні хвилі, сфера, напівобмежена область, метод зображення, коливальне поле, довжина хвилі.