A model for conservative chaos constructed from multi-component Bose-Einstein condensates with a trap in 2 dimensions

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To show a mechanism leading to the breakdown of a particle picture for the multi-component Bose-Einstein condensates (BECs) with a harmonic trap in high dimensions, we investigate the corresponding 2-d nonlinear Schrödinger equation (Gross-Pitaevskii equation) with use of a modified variational principle. A molecule of two identical Gaussian wavepackets has two degrees of freedom (DFs), the separation of centers-of-masses and the wavepacket width. Without the inter-component interaction (ICI) these DFs show independent regular oscillations with the degenerate eigen-frequencies. The inclusion of ICI strongly mixes these DFs, generating a fat mode that breaks a particle picture, which however can be recovered by introducing a time-periodic ICI with zero average. In case of the molecule of three wavepackets for a three-component BEC, the increase of amplitude of ICI yields a transition from regular to chaotic oscillations in the wavepacket breathing.

Recently a great number of theoretical and experimental efforts have been devoted to Bose-Einstein condensates (BECs) [1, 2]. As well as single-component BECs, the trapping techniques can create multi-component condensates which involve inter-component nonlinear interactions. The multi-component BEC, far from being a trivial extension of the single-component one, presents novel and fundamentally different scenarios for its ground state and excitations. In particular, it has been observed that BEC can reach an equilibrium state characterized by separation of the components in different domains [3].

BEC has a dual aspect of waves and particles. The wave nature is high-lighted in the phenomenon of interference leading to fringe patterns [1, 2]. On the other hand, the particle nature of BECs can be seen in typical localized states like vortices and solitons. In fact solitons were observed in the quasi-one dimensional BEC [4, 5].

Among the works that emphasize a role of the particle picture for BEC with a harmonic trap in high-dimensions, those of Pérez-García et al. are the most noteworthy [6, 7, 8, 9]. We focus on their two important assertions. The first one made for a single-component BEC is as follows [8]: If the phase of BEC wave function will be suitably corrected, the center-of-mass for a wavepacket displaced from the origin obeys Newtonian dynamics and the Ehrenfest theorem is valid even for the nonlinear Schrödinger equation (NSE). Furthermore the center-of-mass is decoupled from dynamics of the shape of a wavepacket. The second assertion is concerned with the multi-component BEC [9]: If the distance between wavepackets with each linked to the individual component is much larger than their typical widths, the picture of soliton molecules composed of solitonic atoms (wavepackets) holds for the multi-component BEC with a harmonic trap in high-dimensions. The large distance as above is guaranteed by a sufficiently large centrifugal force due to non-vanishing angular momentum.

Pérez-García et al.'s second assertion is quite interesting because the solitonic structures like wavepackets are believed to be dynamically unstable in two and higher spatial dimensions. However, we have several criticisms against their assertion: (1) Inter-component interaction has a tendency to swell out individual wavepackets and breaks their picture of interacting particles, as evidenced in Fig.1(a); (2) Collective coordinates for the width and phase of wave packets, which should be coupled with the center-of-mass, are not taken into consideration, although being studied intensively in other works of their own [6, 11].

In this Letter, choosing the 2-d multi-component BEC with a harmonic trap, we develop a refined variational principle to derive an effective dynamics for interacting wave packets and pinpoint the weak points of the idea of Pérez-García et al. [9]. Then we shall propose a new model to restore the picture of soliton molecules in two dimensions. In particular, by controlling a parameter we shall see a transition of the wavepacket breathing from regular to chaotic oscillations. We numerically analyze the subject on the basis of a three-component BEC corresponding to the “three-body problem”.

The multi-component BEC at zero temperature is described by NSE (or Gross-Pitaevskii(GP) equation). We shall consider a system of $n$ complex fields
ψ_j(t, r) = \sqrt{\frac{1}{\pi w_j^2}} \exp \left[ -\frac{(r - R_j)^2}{2w_j^2} \right] \exp(i\Theta_j(t, r)),
\Theta_j(t, r) = \alpha_j(t, r) - \frac{1}{2}(r - R_j)^T\beta_j(t, r) - R_j. \quad (2)

(One may also choose excited states responsible for vortices.) \( R = (X, Y) \) is the center-of-mass and \( w \) is width of the circular WP.

\[ \alpha = (\alpha^X, \alpha^Y) \quad (3) \]

\[ \beta = \begin{pmatrix} \beta_{XX} & \beta_{XY} \\ \beta_{YX} & \beta_{YY} \end{pmatrix}, \quad \beta_{XY} = \beta_{YX} = 0 \quad (4) \]

are respectively the first- and second-order coefficients of Taylor-expansion of the phase \( \Theta \) w.r.t. \( r - R \) (a trivial constant phase is suppressed). \( R, w, \alpha \) and \( \beta \) constitute collective coordinates below.

The expansion of \( \Theta \) in Eq. (2) is the most natural, although the existing works\[ 4, 6, 11, 13 \] employ an expansion w.r.t. \( r \) rather than \( r - R \). The advantage of our expansion is that \( \alpha \) and \( \beta \) turn out to have a transparent correspondence with the velocity of center-of-mass and the WP breathing, respectively (see \( 5 \) and \( 9 \) below).

According to the variational principle, the action function deriving Eq. (1) is obtained from Lagrangian density for field variables:

\[ \mathcal{L} = \sum_{j=1}^{n} \left[ \frac{i}{2}(\psi_j^*\dot{\psi}_j - \psi_j\dot{\psi}_j^*) + \frac{1}{2}|\nabla\psi_j|^2 + \frac{\Lambda g}{2} |\psi_j|^4 \right] + \sum_{k>j}^n \Lambda g |\psi_j|^2 |\psi_k|^2. \quad (5) \]

In fact, the multi-component GP equation is obtained from Lagrange equation,

\[ \begin{align*}
\frac{\partial}{\partial t}\psi_j &= \left[ \Delta + \sum_{k \neq j} (\psi_j^* \psi_k + \psi_k^* \psi_j) \right] \psi_j,
\frac{\partial}{\partial x_j} \alpha_j &= \frac{1}{2} \lambda_j^2 (2 + \frac{\Lambda g}{2})
\end{align*} \]

with

\[ U(\{R_j\}, \{w_j\}) = \sum_{j<k} \frac{\Lambda g}{\pi (w_j^2 + w_k^2)} e^{-\frac{(r_j - r_k)^2}{w_j^2 + w_k^2}}. \quad (7) \]

Lagrange equations of motion for phase variables \( \alpha \) and \( \beta \) lead to

\[ \alpha_j = \dot{R}_j \quad (8) \]

and

\[ \beta_j^{XX} = \beta_j^{YY} = \frac{\dot{w}_j}{w_j}, \quad \beta_j^{XY} = \beta_j^{YX} = 0. \quad (9) \]

Noting that \( \alpha \) and \( \beta \) in \( 5 \) and \( 9 \) (and also all higher-order phase coefficients) adiabatically follow \( R \)
and \( w \), we can rewrite Lagrange equations for the latter by eliminating the former: Equation of motion for \( \mathbf{R} \),
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{R}_j} \right) - \frac{\partial L}{\partial R_j} = 0, \tag{10}
\]
is reduced to
\[
\ddot{R}_j + \mathbf{R} + \frac{\partial}{\partial R_j} U(\{R_j\}, \{w_j\}) = 0.
\]
Similarly for \( w \) we have \( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{w}_j} \right) - \frac{\partial L}{\partial w_j} = 0 \), which is reduced to
\[
\ddot{w}_j + w_j - \frac{1}{2} \left( 1 + \frac{g}{2\pi} \right) + \frac{\partial}{\partial w_j} U(\{R_j\}, \{w_j\}) = 0. \tag{11}
\]

A couple of Eqs. \( \text{(10)} \) and \( \text{(11)} \) indicate: Contrary to Pérez-García et al.’s theory that ignored the role of WP width, the width and center-of-mass are strongly correlated, leading to a breakdown of the picture of a solitonic molecule based on the center-of-masses alone. To examine the problem in detail, we consider the two-component BEC in two dimensions and explore the fate of a WP.

Let us define the center-of-mass of two components \( R_0 = \frac{R_1 + R_2}{\sqrt{2}} \) and the relative displacement \( Q = \frac{R_1 - R_2}{\sqrt{2}} \), and suppress the global translational degree of freedom \( (R_0 = \dot{R}_0 = 0) \). Owing to rotational symmetry the angular momentum \( M \) is a constant of motion: \( M = \frac{\partial L_{\text{eff}}}{\partial \dot{\theta}} = Q^2 \dot{\theta} \) with \( \theta \) as a polar angle for \( Q \). Choosing a synchronous width dynamics \( w_1 = w_2 = w \) with the canonical momentum \( p_w = \frac{\partial}{\partial w} (L_{\text{eff}} - M \dot{\theta}) \), the effective Hamiltonian is given by
\[
H_{\text{eff}} = \frac{1}{2} P_Q^2 + \frac{1}{4} P_w^2 + V(Q, w)
\]
with
\[
V(Q, w) = \frac{1}{2} Q^2 + \frac{M^2}{2 Q^2} + w^2 + \frac{1}{w} \left( 1 + \frac{g}{2\pi} \right)
\]
\[
+ \frac{\Lambda g}{2\pi w^2} e^{-\frac{Q^2}{2 w^2}}. \tag{13}
\]

Equation \( \text{(13)} \) indicates: In the absence of ICI (\( \Lambda = 0 \)), \( w \) and \( Q \) show independent regular oscillations around the minima of individual potentials, \( w = w_0 = (1 + g/2\pi)^{1/4} \) and \( Q = Q_0 = \sqrt{M} \), respectively. Interestingly, however, for any set of values \( M \) and \( g \) the typical oscillation frequencies \( \omega_0^Q \) for \( Q \) and \( \omega_0^w \) for \( w \) are always degenerate: \( \omega_0^Q = \omega_0^w = 2 \) (in unit of frequency of the harmonic potential, \( \omega \)). Therefore, by switching on a small but non-vanishing ICI, dynamics of \( w \) and that of \( Q \) will be strongly mixed to remove the degeneracy and normal modes, fat (optical) and skinny (acoustic), are formed. In the fat mode, \( Q \) and \( w \) change in anti-phase, namely the decrease of the relative displacement is accompanied by the blowup of WPs, while in the skinny mode, \( Q \) and \( w \) change in phase (see Fig. \( \text{II(b)} \)). The emergence of the fat mode brings about the wave interference between adjacent WPs as shown in Fig. \( \text{II(a)} \).

This provides a mechanism leading to the breakdown of Pérez-García et al.’s picture of interacting particles.

Let us introduce a time-periodic ICI with zero time-average and the amplitude \( \Lambda g \),
\[
g_{ij} = \Lambda g \cos(\Omega t) \quad (i \neq j) \tag{14}
\]
rather than sticking to the static ICI. Here the frequency scale \( \omega_0^w(= 2) \) is essential. If \( \Omega \gg \omega_0^w \), the width will feel actually no ICI, showing a stable and regular breathing proper to a single component BEC with a harmonic trap. For \( \Omega \) near but larger than \( \omega_0^w \), one can expect a sufficiently long stable breathing until the time \( t = 100[\omega_0^w]^{-1} \) of initial three Gaussian WPs in 3-component BEC with \( g = 2 \pi, \Lambda g = 3, \Omega = 100 \) and \( M = 9 \). See the persistence of a particle picture.

To suppress the wave interference between adjacent WPs and recover the particle picture, we proceed to introduce a time-periodic ICI with zero time-average and the amplitude \( \Lambda g \),
mentum) and
\[ j \gtrsim \omega \]
\[ \gtrsim \Omega \]
for \( \Omega = 0 \) and a recovery of stable oscillatory structure in Fig. 3(a') to broad ones in Fig. 3(b') and corresponding power spectra show a transition from a distinguished chaotic breathing oscillations when the amplitude \( \Lambda_g \)

increased. In fact, in the case of \( \Omega = 5 \), the oscillation is regular for \( \Lambda_g \)

irregular for \( \Lambda_g \)

increased. For a fixed value of \( \Omega \) in the range \( \Omega \approx \omega_w^0 \), we see a transition from regular to chaotic breathing oscillations when the amplitude \( \Lambda_g \)

Note: even for a single-component NSE, the initial circular symmetry of WP is slightly broken during the time evolution, due to the nonlinear interaction; Each of the expectations with non-zero ICI removes the degeneracy and induces the fat circular symmetry of WP.

FIG. 3: Time evolution of WP width \((w_j)\) (upper panel) and corresponding power spectrum (lower panel). \( g = 2\pi, \Omega = 5 \) and \( M = 9 \). \( \Lambda_g \) is a tunable parameter. (a)(a') \( \Lambda_g = 0 \), (b)' \( \Lambda_g = 2 \), (c)(c') \( \Lambda_g = 4 \).

The regular and chaotic oscillations without blowup are observed even for \( \Omega < \omega_w^0 \) and the blowup occurs at resonances, \( \Omega = \omega_w^0 \) and \( 2\omega_w^0 \), even for a small amplitude \( \Lambda_g \). However, Fig 4 clearly conveys that the three-component BEC with the oscillating ICI restores a picture of interacting particles and can exhibit the transition from regular to chaotic motions without the blowup. Thus we have obtained a model of conservative chaos constructed from the three-component BEC with a harmonic trap in 2 dimensions.

To develop Pérez-García et al.’s idea, we have examined the multi-component repulsive BEC in a 2-d harmonic trap. In the absence of the inter-component interaction (ICI), the wavepacket (WP) breathing and the motion of relative distance between WPs have oscillation frequencies always degenerate for any set of the angular momentum and intra-component interaction. The non-zero ICI removes the degeneracy and induces the fat mode that breaks a picture of soliton molecules. We have therefore proposed a new model with a time-periodic ICI with zero time-average, which can elongate the time interval without any blowup of WPs. Provided that the frequency is near the characteristic breathing frequency \( \omega_w^0 \), a transition from regular to chaotic oscillations occurs as the amplitude of ICI is increased. For attractive BECs without a trap or in free space, the oscillating nonlinearity with non-zero time-average is known to stabilize high-dimensional WPs \([13, 14]\). In a similar way, in the case of the multi-component repulsive BEC with a trap,
the analogous picture holds for high-dimensional WPs mutually interacting through time-periodic ICI with zero average. The amplitude of ICI is found to control the chaoticity of breathing motions of WPs. The analysis of chaos in space coordinates and the comparison of the present results based on NSE with the effective dynamics with a few degrees-of-freedom in Eqs. [10] and [11] will constitute subjects which we intend to study in future.

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