Higher order WKB corrections to black hole entropy in brick wall formalism

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Abstract We calculate the statistical entropy of a quantum field with an arbitrary spin propagating on the spherical symmetric black hole background by using the brick wall formalism at higher orders in the WKB approximation. For general spins, we find that the correction to the standard Bekenstein–Hawking entropy depends logarithmically on the area of the horizon. Furthermore, we apply this analysis to the Schwarzschild and Schwarzschild–AdS(dS) black holes and discuss our results.

1 Introduction

Thermodynamics and statistical mechanics of black holes are the most exciting and rapidly developing areas of black hole physics. The analogy between thermodynamics and black holes was originally introduced by Bekenstein \cite{1,2} by assigning the area $A_h$ of the black hole to the entropy $S_{BH}$. This correspondence was placed on the formal footing by Hawking's discovery that black holes radiate thermally with the characteristic temperature ($T_H$) \cite{3,4},

$$T_H = \frac{\hbar \kappa}{2\pi} \left( \frac{c}{k_B} \right),$$

where $\kappa$ is the surface gravity of the black hole and $\hbar, c, k_B$ denote the Planck constant, the speed of light and the Boltzmann constant, respectively. Upon using the above expression for the temperature, the Bekenstein–Hawking entropy is given by

$$S_{BH} = \frac{k_B A_h}{4\ell_p^2},$$

with the Planck length, $\ell_p = \sqrt{G_N \hbar / c^3}$.

There are several approaches to derive the black hole entropy. In the Euclidean approach \cite{5–9}, using the analytic continuation to the Euclidean sector and imposing the Matsubara period $\beta = T_H^{-1}$, the free energy and hence entropy have been obtained for the regular Euclidean solution of the vacuum Einstein equations. In the context of the string theory, the statistical entropy for extremal \cite{10–14} and near extremal \cite{15,16} black holes is determined by explicit counting of the microstates associated with the $D$-branes. Apart from these, the calculation of the black hole entropy and the corresponding quantum corrections have been studied using various methods like-spin networks \cite{17,18}, entanglement between the degrees of freedom separated by Killing horizons \cite{19–21}, the conformal anomaly methods \cite{22–24} and the quantum tunneling approach \cite{25,26}. Very recently, the Euclidean approach mentioned above, has been implemented to derive the logarithmic corrections to Bekenstein–Hawking entropy for extremal as well as non-extremal black holes in the various dimensions by incorporating the zero modes \cite{27}.

There is another efficient method, proposed by 't Hooft \cite{28}, of computing the black hole entropy. This formalism, commonly known as \textit{brick wall model}, is the semi-classical approach wherein the gravitational field (metric) of the black hole is treated classically, while the remaining degrees of freedom, leaving outside the horizon, are handled quantum mechanically. The entropy of these quantum degrees of freedom, calculated via statistical mechanics, is identified with the entropy of the black hole. An important ingredient in the brick wall model is the boundary condition on the probe fields near the horizon. Any quantum fields near the black hole have a crucial property. Namely, the density of states of the quantum mechanical Hamiltonian of the fields blows out in the vicinity of the horizon. This results in the divergence of the statistical entropy \cite{29–32}. In the context of brick wall approach, this divergence can be controlled by putting a static spherically symmetric mirror near the horizon, at
which the fields are required to satisfy Dirichlet or Neumann boundary conditions [28]. In other words, the distance between the horizon and the mirror (brick wall) behaves as an ultra violate cut-off. The canonical (statistical) entropy obtained by using this prescription is finite and agrees with Bekenstein–Hawking formula (2). This procedure also holds for any spacetime endowed with the horizon [33]. They found that the expectation values of energy-momentum tensors for quantum fields in the ground state in the brick wall model matches exactly with the difference between Hartle–Hawking and Boulware states.

All the methods of computing the statistical entropy of the black hole, in spite of their differences in the underlying assumptions and methodology, correctly reproduce the same leading order result. However, the quantum corrections are generally different for different approaches. For instance, the coefficient of the logarithmic corrections to the entropy obtained from fluctuations around the stable canonical ensemble [35] is different from the one obtained using tunneling formalism [26, 36]. It is then natural to seek for the extension of the brick wall model beyond the leading order. Such an extension of the brick wall approach up to sixth order in the WKB approximation has been reported in Ref. [37]. They showed that for spherically symmetric black holes, in four dimensions as well as in six dimensions, the corrections to the brick wall entropy can be expressed as $F(A_h)\ln(A_h/c_p^2)$ where $A_h$ is area of the horizon and the form of $F(A_h)$ depends on the specific details of the black hole.

Although the brick wall approach is implemented to obtain the entropy of a quantum scalar field in various background geometries and in diverse dimensions [38–43], relatively less attention has been given for its generalization to other type of fields, e.g., fermions, photons or gravitons [44, 45]. Any computation of the entropy for a quantum field with generic spin degrees of freedom, even at the leading order, seems to be difficult [46–48]. It is therefore worthwhile to analyze the higher order WKB corrections to the canonical entropy for arbitrary spins. In this work, by using the formalism given in Ref. [37] we calculate, up to second order in the WKB approximation, the statistical entropy of a massless minimally coupled field $\Phi_p$ with an arbitrary spin $|p|$ ($|p| = 0, 1/2, 1, \ldots$), propagating in the background gravitational field (3). The equations of motion for $\Phi_p$ are given by [49, 50]

$$p \square \Phi_p = 0$$

where $p \square$ is the generalized d’Alembertian operator

$$p \square = -\frac{r^2}{g} \partial_r^2 - 2pr \left(1 - \frac{r \partial_r g}{2g}\right) \partial_t + (r^2g)^{-p} \partial_r \left[(r^2 g)^{p+1} \partial_r\right] + \frac{1}{\sin^2 \theta} \partial_\theta (\sin \theta \partial_\theta) + 2ip \frac{\cos \theta}{\sin^2 \theta} \partial_\phi + \frac{1}{\sin^2 \theta} \partial_\phi^2 - \left(p^2 \cos^2 \theta - p\right).$$

For the static and spherically symmetric background, we can consider the ansatz

$$\Phi_p(t, r, \theta, \phi) = e^{-iEt/h} \tilde{R}_p(r) S^p_{lm}(\theta, \phi).$$

The paper is organized in the following manner. In the next section, we briefly outline the method for extending the brick wall entropy to higher order in the WKB approximation and generalize it for arbitrary spins. Section 3 is devoted to explicit calculations of the leading order and second order brick wall entropies. Applications of this approach to specific black holes are exhibited in Sect. 4. We summarize our results in Sect. 5.

### 2 Brick Wall approach for arbitrary spin

We consider a $3 + 1$ dimensional spherically symmetric black hole metric,\(^1\)

$$ds^2 = -g(r) dt^2 + \frac{dr^2}{g(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

where the form of the metric coefficient $g(r)$ depends upon the specific black hole. The event horizon $r_h$ is defined by the condition $g(r) = 0$. For a spherically symmetric black hole, the Hawking temperature is given by

$$T_H = \frac{\hbar \kappa}{2\pi} = \frac{\hbar g'(r_h)}{4\pi}$$

where $\kappa$ is the surface gravity and the prime denotes differentiation with respect to the radial coordinate. We consider the massless minimally coupled field $\Phi_p$ with an arbitrary spin $|p|$ ($|p| = 0, 1/2, 1, \ldots$), propagating in the background gravitational field (3). The equations of motion for $\Phi_p$ are given by [49, 50]

$$p \square \Phi_p = 0$$

where $p \square$ is the generalized d’Alembertian operator

$$p \square = -\frac{r^2}{g} \partial_r^2 - 2pr \left(1 - \frac{r \partial_r g}{2g}\right) \partial_t + (r^2g)^{-p} \partial_r \left[(r^2 g)^{p+1} \partial_r\right] + \frac{1}{\sin^2 \theta} \partial_\theta (\sin \theta \partial_\theta) + 2ip \frac{\cos \theta}{\sin^2 \theta} \partial_\phi + \frac{1}{\sin^2 \theta} \partial_\phi^2 - \left(p^2 \cos^2 \theta - p\right).$$

For the static and spherically symmetric background, we can consider the ansatz

$$\Phi_p(t, r, \theta, \phi) = e^{-iEt/h} \tilde{R}_p(r) S^p_{lm}(\theta, \phi).$$

\(^1\)Throughout the paper we shall work with the signature $(-, +, +, +)$, and $c = G = k_B = 1$. 
Substituting this in Eq. (5), we get the radial equation

\[
(r^2 g) - p \frac{d}{dr} \left[ (r^2 g)^{p+1} \frac{d \widetilde{R}_p(r)}{dr} \right] + \left[ \frac{r^2 \omega^2}{g} - \lambda \right] \widetilde{R}_p(r) = 0,
\]

while the spin weighted spherical harmonics \( S^p_{\ell m}(\theta, \phi) \) satisfy

\[
\left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + 2i p \frac{\cos \theta}{\sin^2 \theta} \frac{\partial}{\partial \phi} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right.
\]

\[- \left( p^2 \cot^2 \theta - p \right) S^p_{\ell m}(\theta, \phi) = -\lambda S^p_{\ell m}(\theta, \phi) \tag{9}\]

with

\[
\lambda = \ell (\ell + 1) - p(p + 1); \quad \ell \geq |p|
\]

and \( E, l, \) and \( m \) are the energy, orbital angular momentum and azimuthal angular momentum associated with the given normal mode, respectively. We can recast (8) into a more appropriate form by transforming the dependent variable \( \widetilde{R}_p(r) \) as

\[
\widetilde{R}_p(r) = R_p(r) - \frac{1}{(r^2 g)^{p+1}}. \tag{11}\]

After performing some algebra, we get

\[
R_p''(r) + \frac{1}{g^2} \left[ \frac{E^2}{r^2} - \frac{g \lambda}{r^2} \right] R_p(r) + \frac{2 i p E}{h} \left[ \frac{1}{g} - \frac{r g'}{2 g^2} \right] R_p(r)
\]

\[- (p + 1) \left[ \frac{g'}{r g}(p + 1) + \frac{g^2}{4 g^2} (p - 1) \right.
\]

\[+ \frac{g''}{2 g} + \frac{p}{r^2} \right] R_p(r) = 0. \tag{12}\]

Our task is to solve Eq. (12) for each component\(^2\) of the spin \(|p|\) and determine the number of radial modes of \( \Gamma^p(E) \) of the field \( \Phi_p \) with the energy less than \( E \). The total number of modes \( \Gamma(E) \) is obtained by summing over all possible values of \( p \). When we perform the sum, the terms proportional to \( i p E \) in Eq. (12) vanish. Therefore, we can write Eq. (12) as

\[
R_p''(r) + \left[ \frac{V_1'(r)}{h^2} - V_2(r) \right] R(r) = 0 \tag{13}\]

with the definitions

\[
V_1(r) = \frac{1}{g} \left( E^2 - \frac{g \lambda h^2}{r^2} \right)^{1/2},
\]

\[
V_2(r) = (p + 1) \left[ \frac{g'}{r g}(p + 1) + \frac{g^2}{4 g^2} (p - 1) + \frac{g''}{2 g} + \frac{p}{r^2} \right]. \tag{14}\]

In almost all cases (except for \( 1 + 1 \) dimensions) it is impossible to get an exact analytic solution of Eq. (13) and the WKB approximation (at the leading order) is frequently used in the literature. We here compute the higher order WKB correction for a quantum field with a general spin \(|p|\).

We start by considering the WKB ansatz for \( R_p(r) \) in Eq. (13),

\[
R_p(r) = \frac{1}{\sqrt{Q(r)}} e^{\frac{i}{\hbar} \int d'r' Q(r')} \tag{15}\]

where \( Q(r) \) is an unknown phase. In order to analyze the higher order WKB approximation, we expand \( Q(r) \) in powers of \( \hbar^2 \) [37, 52] as

\[
Q(r) = \sum_{i=0}^{\infty} Q_{2i}(r) \hbar^{2i}. \tag{16}\]

After substituting Eqs. (15) and (16) into Eq. (13) and collecting the equal powers of \( \hbar \), we get

\[
Q_0(r) = \pm V_1(r) = \pm \frac{1}{g} \left( E^2 - \frac{g \lambda h^2}{r^2} \right)^{1/2},
\]

\[
Q_2(r) = \frac{3 Q_2^0(r)}{8 Q_0^2(r)} - \frac{Q_0'(r)}{4 Q_0^2(r)} - \frac{V_2(r)}{2 Q_0(r)},
\]

\[
Q_4(r) = -\frac{5 Q_2^2(r)}{2 Q_0(r)} - \frac{1}{4 Q_0^2(r)} (4 Q_2(r) V_2(r) + Q_2'(r))
\]

\[+ \left( \frac{3 Q_2^2(r) V_1'(r) - Q_2(r) V_1''(r)}{4 V_2^3(r)} \right) \tag{17}\]

for \( \hbar^0, \hbar^2, \) and \( \hbar^4 \) orders, respectively. In fact, all the higher order functions can be expressed in terms of potential \( V_1(r) \) and \( V_2(r) \). Upon using the semi-classical quantization scheme [28] together with the series expansion of \( Q(r) \), we write the expression for the density of states \( \Gamma(E) \) with energy less than \( E \) as

\[
\Gamma(E) = \sum_{i=0}^{\infty} \Gamma_{2i}(E). \tag{18}\]

\(^2\)For a given value of spin \(|p|\), there are \( 2p + 1 \) equivalent spin states. (E.g. for fermions we have \(|p| = 1/2 \) and \( p = 1/2, -1/2, \).)
Here $\Gamma_2^i(E)$ denotes the number of states at $i$th order and it is given by

$$\Gamma_2^i(E) = \frac{\hbar^2(2i+1)^2}{\pi} \sum_{\ell=-i}^{\infty} \int_{\rho_h+\epsilon}^{L} dr 
\times \int_{|p|}^{\epsilon_m} d(\ell 2\ell + 1) Q_2^i(r, \ell, p). \tag{19}$$

We have regularized our computations by introducing the ultra violate and infrared cut-off [28] such that the quantum field $\Phi_p$ satisfy: $\Phi_p(r \leq \rho_h + \epsilon) = \Phi_p(r \geq L) = 0$ with $\epsilon \ll 1$ and $L \gg \rho_h$. There are a few points which we would like to emphasize at this stage. Normal modes of the quantum field in the region close to horizon undergoes an infinite blue shift and eventually gives divergent contribution to the density of states. In fact, this feature is always present when we study the quantum field theory in any spacetime with horizon [53–56]. The ultra violate cut-off introduced above precisely regularize this divergence. On the other hand, the density of states for the massless field at the large distance (asymptotic) gives divergent contribution to free energy. This divergence is cured by imposing the infrared cut-off. For the massive case the asymptotic density of states behave smoothly.

Another point which we would like to mention is that in Eq. (19) we have replaced the sum over $\ell$ values by the corresponding integral. The maximum of the $\ell$ integral is chosen such that $Q_0(r, \ell, p)$ is real. Such an approximation of replacing sum by integration is valid for the leading as well as higher order WKB analysis. For the general spin, situation is slightly different. In that case, there exist a lower limit on the $\ell$ integration and it is related to the consistency of the eigenspectra of the spin weighted spherical harmonics [49, 57] (for more details, see [58]). This modification complicates the higher order of the WKB analysis.

Using the density of states (19), we can compute the free energy for bosons (−) and fermions (+) from their standard expressions,

$$F_{2i}^\pm = -\int_0^\infty dE \frac{\Gamma_2^i(E)}{e^{\beta E} \pm 1} \tag{20}$$

where $\beta$ is the inverse of the Hawking temperature. The statistical entropy associated with the free energy $F_{2i}^\pm$ is given by

$$S_{2i}^\pm = \beta^2 \left( \frac{\partial F_{2i}^\pm}{\partial \beta} \right). \tag{21}$$

### 3 Leading and second order computations

In this section, we shall compute the statistical entropy of the quantum field with spin $|p|$ propagating on the spherically symmetric black hole background at the leading and second order in the WKB approximation. The total entropy is obtain by combining the leading and second order results for the corresponding species.

#### 3.1 Leading order

From Eq. (19), we have the following leading order expression:

$$\Gamma_0(E) = \frac{1}{\hbar \pi} \sum_p \int_{\rho_h+\epsilon}^{L} dr \int_{|p|}^{\epsilon_m} d(2\ell + 1) \frac{1}{\sqrt{g(r)}} \left( E^2 - g(r)\lambda^2 \right)^{1/2} \tag{22}$$

where $\lambda$ is given by Eq. (10). It is convenient to work with variables $\tilde{\lambda}(r)$, $E$ and $G(\tilde{\lambda}, E)$, which are defined as

$$\tilde{\lambda} = g\lambda \frac{h^2}{r^2} = g\lambda \frac{h^2}{r^2} \left( \ell (\ell + 1) - p(p + 1) \right), \tag{23}$$

$$\tilde{E} = E^2, \tag{24}$$

$$G(\tilde{\lambda}, \tilde{E}) = (\tilde{E} - \tilde{\lambda})^{1/2}. \tag{25}$$

Using these, we rewrite the density of states as

$$\Gamma_0(E) = \frac{1}{\pi \hbar^3} \int_{\rho_h+\epsilon}^{L} dr \int_{|p|}^{\epsilon_m} d\tilde{\lambda} G(\tilde{\lambda}, \tilde{E}) \left[ \sum_p \int_{\lambda_0}^{\tilde{\lambda}_{\text{max}}} d\lambda G(\tilde{\lambda}, \tilde{E}) \right]. \tag{26}$$

The maximum of $\tilde{\lambda}$ is located where the function $G(\tilde{\lambda}, \tilde{E})$ vanishes and the lower limit is dictated by the spin $|p|$. Using Eqs. (10) and (25), we find

$$\tilde{\lambda}_{\text{max}} = \tilde{E}; \quad \lambda_0 = \frac{g(r)\lambda^2}{r^2} \left[ |p|(|p|+1) - p(p + 1) \right]. \tag{27}$$

After performing the $\tilde{\lambda}$ integral and summing over $p$, we obtain

$$\Gamma_0(E) = \frac{2}{3\pi \hbar^3} \left( 2|p| + 1 \right) \tilde{E}^{3/2} \int_{\rho_h+\epsilon}^{L} r^2 g^2(r) \tag{28}$$

Substituting this in Eq. (20) and integrating over $E$ yields

$$F_0^- = -\frac{2}{45} \left( \frac{\pi^3 (2|p| + 1)}{h^3 \beta^4} \right) \int_{\rho_h+\epsilon}^{L} r^2 g^2(r), \tag{29}$$

$$F_0^+ = -\frac{14}{360} \left( \frac{\pi^3 (2|p| + 1)}{h^3 \beta^4} \right) \int_{\rho_h+\epsilon}^{L} r^2 g^2(r) \tag{30}$$

for bosons and fermions, respectively.
Now, the radial integrations are evaluated by expanding the metric near the horizon,
\[ g(r) \approx g'(r_h)(r - r_h) + \frac{g''(r_h)}{2}(r - r_h)^2 + \mathcal{O}((r - r_h)^3). \]
(31)

At the quadratic order in the metric expansion, we get
\[ F_0^- = \frac{1}{45} \left( \frac{r_h^2}{h^3} \right) \left[ \frac{r_h^2 g''(r_h)}{2} - \frac{r_h^2}{4k^3} \ln \left( \frac{r_h^2}{k} \right) \right], \]
\[ F_0^+ = \frac{7}{8} \frac{1}{45} \left( \frac{\pi^3}{2} \right) \left[ \frac{r_h^2 g''(r_h)}{2} - \frac{r_h^2}{4k^3} \ln \left( \frac{r_h^2}{k} \right) \right] \]
(32)
(33)

where we have used the coordinate invariant cut-off (proper distance of the brick wall from the horizon)
\[ \tilde{\epsilon} = \sqrt{\frac{4\epsilon}{g'(r_h)}}, \]
(34)

and \( \kappa \) is the surface gravity given by Eq. (4). The canonical entropy computed from Eq. (21) becomes
\[ S_0^- = f^-(p) \frac{r_h^2}{90 \tilde{\epsilon}^2} \]
\[ - f^-(p) \left[ \frac{r_h^2 g''(r_h)}{360} - \frac{r_h^2}{90} \ln \left( \frac{r_h^2}{\tilde{\epsilon}^2} \right) \right], \]
\[ S_0^+ = f^+(p) \frac{7}{8} \frac{r_h^2}{90 \tilde{\epsilon}^2} \]
\[ - f^+(p) \left[ \frac{7}{8} \frac{r_h^2 g''(r_h)}{360} - \frac{r_h^2}{90} \ln \left( \frac{r_h^2}{\tilde{\epsilon}^2} \right) \right]. \]
(35)
(36)

Here \( f^\pm(p) = (2|p| + 1) \) is the spin degeneracy factor for fermions and bosons, respectively. Our leading order results are in agreement with that of given in the literature [44–48].

3.2 Second order analysis

We now calculate the density of states of the quantum field with energy less than \( \tilde{E} \) at the second order \( (i = 1) \) in the WKB approximation. Upon using Eqs. (23), (24), and (25) we can write Eq. (19) as
\[ I(\tilde{E}, r) = \sum_{p=1}^{2|p|+1} \int_{\tilde{\epsilon}_0}^{\tilde{\epsilon}_m} d\tilde{\lambda} \left( \frac{Q_2^{(0)}(r)}{G(\tilde{\lambda}, \tilde{E})} + \tilde{\kappa} \frac{Q_2^{(1)}(r)}{G^3(\tilde{\lambda}, \tilde{E})} \right) + \frac{\tilde{\lambda}^2}{G^5(\tilde{\lambda}, \tilde{E})} Q_2^{(2)}(r) \]
(38)

where
\[ I(\tilde{E}, r) = \sum_{p=1}^{2|p|+1} \int_{\tilde{\epsilon}_0}^{\tilde{\epsilon}_m} d\tilde{\lambda} \left( \frac{Q_2^{(0)}(r)}{G(\tilde{\lambda}, \tilde{E})} + \tilde{\kappa} \frac{Q_2^{(1)}(r)}{G^3(\tilde{\lambda}, \tilde{E})} \right) + \frac{\tilde{\lambda}^2}{G^5(\tilde{\lambda}, \tilde{E})} Q_2^{(2)}(r) \]
(39)

and we have defined
\[ Q_2^{(0)}(r) = -\frac{g'(r)}{2r}, \]
\[ Q_2^{(1)}(r) = \frac{g''(r)}{8g(r)} - \frac{3g'(r)}{4r} + \frac{3g(r)}{4r^2} + \frac{g''(r)}{8r}, \]
\[ Q_2^{(2)}(r) = \frac{5}{32} \frac{g^2(r)}{g(r)} - \frac{5}{8} \frac{g'(r)}{r} + \frac{5}{8} \frac{g(r)}{r^2}, \]
(40)
(41)
(42)

It is worthwhile to point out that Eqs. (40), (41), and (42) have structures similar to those given in Ref. [37]. While the additional piece \( Q_2^{(0)}(r) \) depends explicitly on the spin orientation. This term would vanish for \( p = 0 \) (scalar). Therefore, we expect that the \( \tilde{\lambda} \) integration over the terms \( Q_2^{(0)}, Q_2^{(1)}, \) and \( Q_2^{(2)} \) should yield the similar forms given in Ref. [37]. However, the corresponding integral over \( Q_2^{(0)} \) gives a completely different structure. We shall evaluate these integrals one by one.

The definitions (24) and (25) allow us to write inverse differentials with respect to \( \tilde{E} \). By successive applications of the Leibniz rule of integral calculus and by interchanging the order of \( \tilde{\lambda} \) integration with \( \tilde{E} \) differentiation, we obtain
\[ I(\tilde{E}, r) = \sum_{p=1}^{2|p|+1} \left( \tilde{E}^{(2|p|+1)} \right)^{1/2} \]
\[ \times \left[ 2Q_2^{(0)}(r) - 4Q_2^{(1)}(r) + \frac{16}{3} Q_2^{(2)}(r) \right] \]
\[ + \left[ \frac{4\tilde{E}}{G(\tilde{\lambda}, \tilde{E})} - \frac{8}{3} \frac{\partial}{\partial \tilde{E}} \left( \frac{\tilde{E}^2}{2G(\tilde{\lambda}, \tilde{E})} \right) \right] \left( \tilde{E} - \tilde{\lambda}_0 \right)^{3/2} \]
(44)

In getting the above expression, we have used
\[ \sum_{p=1}^{2|p|+1} \left( \tilde{E} - \tilde{\lambda}_0 \right)^{3/2} \left( 2\tilde{E} + 3\tilde{\lambda}_0 \right) = 2\tilde{E}^{5/2} (2|p| + 1). \]
\[\sum_{p=1}^{2|p|+1} (\tilde{E} - \tilde{\lambda}_0)^{3/2} (8\tilde{E}^2 + 12\tilde{E}\tilde{\lambda}_0 + 15\tilde{\lambda}_0^2) = 8\tilde{E}^{7/2} (2|p| + 1) \]

where \(\tilde{\lambda}_0\) is given by Eq. (27).

Next, we substitute \(\tilde{Q}^{(0)}_2\) in Eq. (39) and evaluate the integral to get

\[J(\tilde{E}, r) = -2 \sum_{p=1}^{2|p|+1} (\tilde{E} - \tilde{\lambda}_0)^{1/2} p \left[ \frac{(p + 2) g'(r)}{2r} + p \frac{g^2(r)}{8g(r)} + (p + 1) \frac{g(r)}{2r^2} + \frac{g''(r)}{4} \right]. \quad (45)\]

As mentioned before, this term vanishes for \(|p| = 0\). For \(|p| > 0\), we can execute the sum by noting that

\[\sum_{p=1}^{2|p|+1} p (\tilde{E} - \tilde{\lambda}_0)^{1/2} = \tilde{E}^{1/2} (1 + |p|)(1 + 2|p|), \]
\[\sum_{p=1}^{2|p|+1} p^2 (\tilde{E} - \tilde{\lambda}_0)^{1/2} = \frac{1}{3} \tilde{E}^{1/2} (1 + |p|)(1 + 2|p|)(3 + 4|p|). \]

Using the above formulas and introducing the theta function, we write

\[J(\tilde{E}, r) = -\tilde{E}^{1/2} \tilde{J}(r) \Theta(|p|) \quad (46)\]

where

\[\tilde{J}(r) = (1 + |p|)(1 + 2|p|) \left[ \frac{(9 + 4|p|) g'(r)}{3r} + (3 + 4|p|) \frac{g^2(r)}{12g(r)} + (3 + 2|p|) \frac{2g(r)}{3r^2} + \frac{g''(r)}{2} \right]. \quad (47)\]

Note that \(\Theta(|p|)\) for \(|p| = 0\) (scalar) vanishes, while it is unity for \(|p| \neq 0\).

We should emphasize that when \(\tilde{\lambda} = \tilde{\lambda}_{max} = \tilde{E}\) (turning point) the function \(G(\tilde{E}, \tilde{\lambda})\) vanishes and all the derivatives with respect to \(\tilde{E}\) diverge. Consequently, integration of the terms like \(G^{-3}\tilde{\lambda}\) and \(G^{-5}\tilde{\lambda}^2\) in Eq. (38) leads to the finite as well as diverging contributions. The divergent part is given by the last term in Eq. (44). This divergence is due to the fact that the WKB approximation is not valid as we move closer to the turning point. On the other hand, we observe that the result for \(J(\tilde{E}, r)\) contains only finite term. Thus, the structure of the divergent terms appearing for scalar particle [37] remain unchanged even for the generic spin.

Then, the density of states \(F_2(E)\) can be obtained by collecting the finite part of Eqs. (44) and (46) and substituting them into Eq. (37).

\[F_2(E) = \frac{E(2|p| + 1)}{\pi \hbar} \int_{r_{\chi+\epsilon}}^{L} dr \frac{r^2}{g(r)} \left\{ \left(2Q_2^{(0)}(r) - 4Q_2^{(1)}(r) \right) \right. \]
\[- 4Q_2^{(1)}(r) + \frac{16}{3} Q_2^{(2)}(r) \]
\[- \Theta(|p|) (1 + |p|) \left( \frac{(9 + 4|p|) g'(r)}{3r} + \frac{(3 + 4|p|) g^2(r)}{12g(r)} + \frac{2(3 + 2|p|) g(r)}{3r^2} + \frac{g''(r)}{2} \right) \}
\[\left. + \left(2Q_2^{(0)}(r) - 4Q_2^{(1)}(r) \right) \right\} \quad (48)\]

Substituting the above expression in Eq. (20) and integrating over \(E\), yields

\[F_2^- = -\frac{(2|p| + 1)\pi}{6\beta^2 \hbar} \int_{r_{\chi+\epsilon}}^{L} dr \frac{r^2}{g(r)} \left\{ \left(2Q_2^{(0)}(r) - 4Q_2^{(1)}(r) \right) \right. \]
\[- \Theta(|p|) (1 + |p|) \left( \frac{(9 + 4|p|) g'(r)}{3r} + \frac{(3 + 4|p|) g^2(r)}{12g(r)} + \frac{2(3 + 2|p|) g(r)}{3r^2} + \frac{g''(r)}{2} \right) \}
\[\left. + \left(2Q_2^{(0)}(r) - 4Q_2^{(1)}(r) \right) \right\} \quad (49)\]

\[F_2^+ = -\frac{(2|p| + 1)\pi}{12\beta^2 \hbar} \int_{r_{\chi+\epsilon}}^{L} dr \frac{r^2}{g(r)} \]
\[- \Theta(|p|) (1 + |p|) \left( \frac{(9 + 4|p|) g'(r)}{3r} + \frac{(3 + 4|p|) g^2(r)}{12g(r)} + \frac{2(3 + 2|p|) g(r)}{3r^2} + \frac{g''(r)}{2} \right) \}
\[\left. + \left(2Q_2^{(0)}(r) - 4Q_2^{(1)}(r) \right) \right\} \quad (50)\]

Finally, by using the metric expansion (31) and performing the radial integration, we get

\[F_2^- = -\frac{(2|p| + 1)\pi}{6\beta^2 \hbar} \left( \frac{4r_h^2}{3\varepsilon^2 g'(r_h)} - \left( \frac{2r_h}{3} + \frac{g''(r_h)}{6g'(r_h)} \right) \right) \]
\[\times \ln \left( \frac{r_h^2}{\varepsilon^2} \right) \]
\[+ \Theta(|p|) (1 + |p|) \left( \frac{(3 + 4|p|)}{6} \left( \frac{r_h^2}{\varepsilon^2 g'(r_h)} \right) \right. \]
\[- \left( \frac{r_h}{4} (7 + 4|p|) + \frac{2r_h^2 g''(r_h)}{24g'(r_h)} (9 + 4|p|) \right) \]
\[\times \ln \left( \frac{r_h^2}{\varepsilon^2} \right) \quad (51)\]
\[ F^+_2 = -\frac{(2|p| + 1)\pi}{12\beta^2\hbar} \left\{ \left[ \frac{4r_h^2}{3\epsilon^2 g'(r_h)} \right] \ln \left( \frac{r_h^2}{\epsilon^2} \right) \right\} \]

\[ - \left( \frac{2r_h^2}{3} + \frac{g''(r_h)r_h^2}{6g'(r_h)} \right) \ln \left( \frac{r_h^2}{\epsilon^2} \right) \]

\[ + \Theta(|p|)(1 + |p|) \left[ \left( \frac{3 + 4|p|}{6} \right) \left( \frac{r_h^2}{\epsilon^2} \right) \right] \]

\[ - \left( \frac{r_h^2}{4} \left( 7 + 4|p| \right) + \frac{r_h^2 g''(r_h)}{288} (9 + 4|p|) \right) \]

\[ \times \ln \left( \frac{r_h^2}{\epsilon^2} \right) \right\}. \quad (52) \]

Thus, the canonical entropy at the second order in the WKB approximation for the quantum field of spin \(|p|\) is given by

\[ S^- = f^-(p) \left\{ \left[ \frac{r_h^2}{9\epsilon^2} - \frac{\kappa r_h g''(r_h)}{18} \right] \ln \left( \frac{r_h^2}{\epsilon^2} \right) \right\} \]

\[ + \Theta(|p|)(1 + |p|) \left[ \left( \frac{3 + 4|p|}{14} \right) \left( \frac{r_h^2}{\epsilon^2} \right) \right] \]

\[ - \left( \frac{r_h^2}{48} \left( 7 + 4|p| \right) + \frac{r_h^2 g''(r_h)}{576} (9 + 4|p|) \right) \]

\[ \times \ln \left( \frac{r_h^2}{\epsilon^2} \right) \right\} \]. \quad (53) \]

\[ S^+ = f^+(p) \left\{ \left[ \frac{r_h^2}{18\epsilon^2} - \frac{\kappa r_h g''(r_h)}{18} \right] \ln \left( \frac{r_h^2}{\epsilon^2} \right) \right\} \]

\[ + \Theta(|p|)(1 + |p|) \left[ \left( \frac{3 + 4|p|}{14} \right) \left( \frac{r_h^2}{\epsilon^2} \right) \right] \]

\[ - \left( \frac{r_h^2}{48} \left( 7 + 4|p| \right) + \frac{r_h^2 g''(r_h)}{576} (9 + 4|p|) \right) \]

\[ \times \ln \left( \frac{r_h^2}{\epsilon^2} \right) \right\} \]. \quad (54) \]

Equations (53) and (54) constitute the main result. For the scalar field, our result is consistent with that given in Ref. [37]. The canonical entropies (53) and (54) contain the quadratic as well as logarithmic divergent terms in the ultra-violate regime of \(\epsilon \rightarrow 0\). However, these divergences are even present at the leading order of the WKB approximations (35) and (36). Various approaches have been suggested in the literature for regularizing these divergences [29, 31, 32], and it was shown that they can be absorbed into the renormalization of the coupling constants appearing in the one-loop effective action [59–63].

We now write the expression for the total canonical entropy up to the second order in the WKB approximation by adding the contributions from the leading (35), (36) and second (53), (54) order expressions,

\[ S^\pm = S_0^\pm + S_2^\pm. \quad (55) \]

It is worthwhile to point out that the invariant cut-off \(\bar{\epsilon}\) can be adjusted to match the leading order term with the standard Bekenstein–Hawking entropy (2). For the scalar field, the degeneracy factor \(f^-(p = 0) = 1\) is unity. Thus, by setting \(\bar{\epsilon}^2_{sc} = (11\beta^2/90\pi)\) the standard expression for \(S_{BH}\), at leading order, is obtained [37]. In general, the value of the invariant cut-off depends upon the type of the field. The massless spin-1/2 (Weyl fermions) we have \(f^+(p = 1/2) = 2\) and in this case we set \(\bar{\epsilon}^2_{wF} = (169\beta^2/1440\pi)\). For the electromagnetic field \((p = 1)\) we have only two polarizations corresponding to two helicities while the remaining one is unphysical gauge degree of freedom. Hence, the effective spin degeneracy factor for this case is \(f^-(p = 1) = 1 = 2\). The appropriate invariant cut-off length for electromagnetic field (photon) is \(\bar{\epsilon}^2_{ph} = (57\beta^2/90\pi)\). Thus, we write the corresponding expressions for the canonical entropy,

\[ S_{sc} = \frac{A_h}{4\ell_p^2} - \left[ \frac{\kappa}{10r_h} + \frac{g''(r_h)}{60} \frac{A_h}{4\pi} \ln \left( \frac{A_h}{\ell_p^2} \right) \right] \]

\[ S_{wF}^+ = \frac{A_h}{4\ell_p^2} - \left[ \frac{471}{360} \left( \frac{\kappa}{r_h} + \frac{657}{4320} g''(r_h) \right) \frac{A_h}{16\pi} \right] \]

\[ \times \ln \left( \frac{A_h}{\ell_p^2} \right), \quad \text{(56)} \]

\[ S_{ph}^+ = \frac{A_h}{4\ell_p^2} - \left[ \frac{122}{15} \left( \frac{\kappa}{r_h} + \frac{77}{90} g''(r_h) \right) \frac{A_h}{16\pi} \right] \]

\[ \times \ln \left( \frac{A_h}{\ell_p^2} \right), \quad \text{(57)} \]

for the massless scalar, the fermion and the electromagnetic field, respectively.

Equations (53) and (54) represent the second order WKB correction to the leading (zeroth) order WKB approximation (see Eqs. (35), (36)) for bosons and fermions, respectively. It is possible to compute the higher order WKB corrections to the canonical entropies. Indeed, for scalar field, these corrections were presented in [37]. At the fourth order \((i = 2\) in Eq. (19)), the density of states is \(\Gamma_4(E) \propto E^{-1}\) and therefore free energy becomes independent of \(\beta\) (i.e., scale invariant). Hence, the fourth order contribution to the canonical entropy vanishes. For the sixth order \((i = 3\) density of states is \(\Gamma_6(E) \propto E^{-3}\) and the corresponding expression for canonical entropy is given by

\[ S_6 = -\frac{13892\pi^2 r_h^2}{45045} + \left[ \frac{9\pi^2}{77} g''(r_h) r_h^2 + \frac{30\pi^2}{77} \kappa r_h \right] \]

\[ \times \ln \left( \frac{r_h^2}{\epsilon^2} \right) \right\} \]. \quad (59)
[37]. Also, note that the UV divergent structure arising from sixth order has the similar form as compare to the corresponding leading and second order terms. From the above analysis, it is natural to expect that the brick wall entropy does indeed provide corrections to the Bekenstein–Hawking entropy at all orders in WKB approximation. We hope that the generic structures for fourth and sixth order WKB corrections to the canonical entropy for the scalar field continue to be valid for the field with arbitrary spins. There are some important points which we would like to emphasize. As mentioned earlier, the WKB approximation arises when we try to solve the radial differential equation (12) for the modes of the quantum field under consideration. In the standard brick wall formalism, existed in the literature, the canonical expressions for free energy and entropies were computed at the leading order in the WKB approximation. In our work, we obtain the possible modifications to the standard (leading order) result by taking into account the higher order WKB terms. This is different from considering the higher order loop corrections to the gravitational partition function. For instance, at the tree level the gravitational partition function reproduces the standard Bekenstein–Hawking entropy [5]. While, at one-loop level, the Bekenstein–Hawking entropy receives logarithmic corrections [21, 27, 59]. The situation is slightly different in the brick wall formalism. Here, even at the leading order in WKB approximation, one can get the logarithmic corrections to the entropy and canonical expressions for free energy and entropies. Comparing (66) for scalar field and obtain

\[ S_{(0)sc} = \frac{r_h^2}{90 \epsilon^2} - \left[ \frac{r_h^2}{360} - \frac{2}{r_h^2} - \frac{r_h}{90} \left( \frac{1}{2r_h} \right) \right] \ln \left( \frac{r_h^2}{\epsilon^2} \right) \]

(61)

where \( r_h = 2M \). Now we first substitute these expressions in the leading order expression (35) for the brick wall entropy for scalar field and obtain

\[ S_{(0)sc} = \frac{r_h^2}{90 \epsilon^2} - \left[ \frac{r_h^2}{360} - \frac{2}{r_h^2} - \frac{r_h}{90} \left( \frac{1}{2r_h} \right) \right] \ln \left( \frac{r_h^2}{\epsilon^2} \right) \]

This expression is in complete agreement with the corresponding result in [21]. The expression for second order canonical entropy can be obtain from Eq. (53) and it is given by

\[ S_{(2)sc} = \frac{r_h^2}{9 \epsilon^2} - \frac{1}{18} \ln \left( \frac{r_h}{\epsilon} \right). \]

We now use Eq. (56) and write the expression for total entropy \( S_{sc} = S_{(0)sc} + S_{(2)sc} \) in terms of the horizon area \( A_h \), as

\[ S_{sc} = \frac{A_h}{4 \ell_p^2} - \frac{1}{60} \ln \left( \frac{A_h}{\ell_p^2} \right). \]

(63)

Similarly, the total entropy for fermions and the Maxwell fields is given by

\[ S_{af} = \frac{A_h}{4 \ell_p^2} - \frac{23}{96} \ln \left( \frac{A_h}{\ell_p^2} \right), \]

(65)

\[ S_{ph} = \frac{A_h}{4 \ell_p^2} - \frac{131}{144} \ln \left( \frac{A_h}{\ell_p^2} \right). \]

5 Schwarzschild–AdS(dS) black hole

In this case the metric is given by

\[ ds^2 = \left( 1 - \frac{2M}{r} - \frac{\delta \Lambda}{3} r^2 \right) dt^2 + \left( 1 - \frac{2M}{r} - \frac{k \Lambda}{3} r^2 \right)^{-1} dr^2 + r^2 d\Omega^2 \]

(66)

where \( \Lambda > 0 \) is the cosmological constant and \( \delta = -1(1) \) for AdS (dS) cases, respectively. For S–AdS, there is only one horizon, while for S–dS black hole we have event horizon (at \( r = r_h \)) as well as the cosmological horizon (at \( r = r_c \)).

First we consider the S–AdS case and compute the canonical entropies. Comparing (66) for \( \delta = -1 \) with Eq. (3), the metric coefficient and the surface gravity are
given by

\[ g(r) = \left(1 - \frac{r_h}{r}\right) \left[1 + \frac{A}{3} (r^2 + rr_h + r_h^2)\right]; \quad (67) \]

\[ \kappa = \frac{1}{2r_h} \left(1 + A r_h^2\right). \]

where \( r_h \) satisfies

\[ 2 M = r_h \left(1 + \frac{r_h^2}{3}\right). \quad (68) \]

Substituting the above equations into Eqs. (56)–(58), we obtain

\[ S_{sc}^+ = \frac{A_h}{4 \epsilon_p^2} - \left[\frac{1}{60} + \frac{A}{80 \pi} A_h\right] \ln \left(\frac{A_h}{\epsilon_p}\right); \]

\[ S_{wm}^+ = \frac{A_h}{4 \epsilon_p^2} - \left[\frac{23}{96} + \frac{471}{500} \left(\frac{A A_h}{32 \pi}\right)\right] \ln \left(\frac{A_h}{\epsilon_p}\right). \quad (69) \]

\[ S_{ph}^+ = \frac{A_h}{4 \epsilon_p^2} - \left[\frac{131}{144} + \frac{61}{240} \left(\frac{A A_h}{\pi}\right)\right] \ln \left(\frac{A_h}{\epsilon_p}\right). \]

Note that, unlike the Schwarzschild case, here the coefficients of logarithmic terms depend on the area.

Now consider the Schwarzschild–de Sitter black hole. We first compute the entropies at the event horizon which is located at \( r_h \). The metric coefficient and the surface gravity for the event horizon are given by

\[ g(r) = \left(1 - \frac{r_h}{r}\right) \left[1 - \frac{A}{3} (r^2 + rr_h + r_h^2)\right]; \quad (70) \]

\[ \kappa = \frac{1}{2r_h} \left(1 - A r_h^2\right), \]

and \( r_h \) satisfy

\[ 2 M = r_h \left(1 - \frac{A}{3} r_h^2\right). \quad (71) \]

Substituting the above equations into Eqs. (56)–(58), we obtain

\[ S_{sc}^{-} \big|_{r_h} = \frac{A_h}{4 \epsilon_p^2} - \left[\frac{1}{60} + \frac{A}{80 \pi} A_h\right] \ln \left(\frac{A_h}{\epsilon_p}\right), \]

\[ S_{wm}^{-} \big|_{r_h} = \frac{A_h}{4 \epsilon_p^2} - \left[\frac{23}{96} + \frac{471}{500} \left(\frac{A A_h}{32 \pi}\right)\right] \ln \left(\frac{A_h}{\epsilon_p}\right). \quad (72) \]

\[ S_{ph}^{-} \big|_{r_h} = \frac{A_h}{4 \epsilon_p^2} - \left[\frac{131}{144} + \frac{61}{240} \left(\frac{A A_h}{\pi}\right)\right] \ln \left(\frac{A_h}{\epsilon_p}\right). \]

Next, we calculate the canonical entropies near the cosmological horizon which is located at \( r_h \ll r = r_c = \sqrt{\frac{3}{\Lambda}} \). We first consider the brick wall near the cosmological horizon with the proper distance \( \epsilon_c \). Note this proper distance is in general different from the one used to construct the brick wall near the event horizon.

In this case the surface gravity is given by \( \kappa_c = -\frac{A}{3} r_c \). Using this and following the similar procedure given above we arrive at

\[ S_{sc}^+ \big|_{c} = \frac{A_c}{4 \epsilon_p^2} - \frac{A A_c}{90 \pi} \ln \left(\frac{A_c}{\epsilon_p}\right); \]

\[ S_{wm}^+ \big|_{c} = \frac{A_c}{4 \epsilon_p^2} + \frac{223}{1800} \left(\frac{A A_h}{16 \pi}\right) \ln \left(\frac{A_c}{\epsilon_p}\right). \quad (73) \]

This gives us the entropies due to scalar, Weyl fermions and photons at the cosmological horizon. The total entropy for the S–dS black hole is then sum of the entropies of the corresponding species at the event horizon and cosmological horizon. At the leading order our result is in agreement with the one given in [46–48].

6 Summary

In this work, we have computed the canonical entropy for a massless quantum field with arbitrary spin propagating in \( 3 + 1 \) dimensional spherically symmetric black hole background up to the second order in the WKB approximation. The generic structure of the leading and second order expressions for the free energy as well as entropy remains the same. However, the second order term contributes significantly to the entropy than the leading order. The total entropy up to the second order was obtained by combining the leading and second order results. The total entropy obtained in this manner contains quadratic as well as logarithmic divergent parts. These divergences can be cured by renormalizing the gravitational coupling constant [31, ...
However, the coefficient for the renormalized gravitational constant is found to be different for different species of matter fields [62]. Consequently, the proper cut-off distance of the brick wall from the horizon depends upon the type of field. Thus, by choosing the invariant cut-off appropriately, we were able to express the total entropy as a combination of the standard Bekenstein–Hawking entropy ($S_{BH}$) and the logarithmic correction. The logarithmic contribution to the black hole entropy have been discovered earlier in several different approaches such as [17, 18, 22, 26, 27, 59]. However, the coefficient of the logarithmic term is generally found to be different for different methods. In our case, the prefactor of the logarithmic term depends on the type of the field. For the scalar field, our result matches with that of given in Ref. [37]. It is important to note that these ultra violate divergences appearing in our analysis are independent of the order of the WKB approximation, as can be easily verified by comparing the leading and the second order expressions. We have also encountered the divergence in the evaluation of $\lambda$ integrals. This divergence occurs near the turning point of the WKB potential. However, the structure of these additional divergent terms still prevails irrespective of the additional spin degrees of freedom of the quantum field. Finally, we have applied our analysis to the Schwarzschild and Schwarzschild–AdS (dS) black holes and obtained the expression for the massless scalar, fermion and electromagnetic field, respectively. For Schwarzschild black hole, the brick wall entropy at the leading order in the WKB approximation (which also contain the logarithmic term) matches with known result [21].

In the present work, we have restricted our computations up to the second order in the WKB approximation. The WKB analysis for the scalar field presented in [37] shows that the brick wall entropy receives no contribution at fourth order while, at the sixth order, it receive significant contribution. We hope that these findings also hold for the quantum field with an arbitrary spin. As mentioned earlier these higher order WKB corrections are generally different from the loop corrections to the gravitational partition function. In this sense the WKB corrections which we are getting have different origin. These WKB corrections are need to be addressed in a sophisticated manner and are subject to future investigations. Another important aspect that we would like to explore is the thermodynamical stability of the black holes. In the tunneling mechanism, it was shown that the inclusion of the quantum corrections makes the black holes stable via phase transition Refs. [69–71]. It will be worthwhile to study the phase transitions and thermodynamic stability of the black holes by using the brick wall approach at higher orders in the WKB approximation. We would like to address these issues in near future.

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