SELFINJECTIVE ALGEBRAS WITHOUT SHORT CYCLES OF INDECOMPOSABLE MODULES

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Dedicated to Zygmunt Pogorzaty on the occasion of his 60th birthday

Abstract. We describe the structure of finite dimensional selfinjective algebras over an arbitrary field without short cycles of indecomposable modules.

Keyword: selfinjective algebra, repetitive algebra, orbit algebra, tilted algebra, algebra of finite representation type, short cycle of modules

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Introduction and the main result.

Throughout the paper, by an algebra we mean a basic indecomposable finite dimensional associative $K$-algebra with identity over a field $K$. For an algebra $A$, we denote by mod $A$ the category of finite-dimensional right $A$-modules, by ind $A$ the full subcategory of mod $A$ formed by the indecomposable modules, by $\Gamma_A$ the Auslander-Reiten quiver of $A$, and by $\tau_A$ and $\tau_A^{-1}$ the Auslander-Reiten translations $D\tau_A$ and $\tau_A D$, respectively. We do not distinguish between a module in ind $A$ and the vertex of $\Gamma_A$ corresponding to it. An algebra $A$ is of finite representation type if the category ind $A$ admits only a finite number of pairwise nonisomorphic modules. It is well known that a hereditary algebra $A$ is of finite representation type if and only if $A$ is of Dynkin type, that is, the valued quiver $Q_A$ of $A$ is a Dynkin quiver of type $A_n$ ($n \geq 1$), $B_n$ ($n \geq 2$), $C_n$ ($n \geq 3$), $D_n$ ($n \geq 4$), $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$ (see [8], [9], [10]). A distinguished class of algebras of finite representation type is formed by the tilted algebras of Dynkin type, that is, the algebras of the form $\text{End}_H(T)$ for a hereditary algebra $H$ of Dynkin type and a (multiplicity-free) tilting module $T$ in mod $H$. An algebra $A$ is called selfinjective if $A_A$ is an injective module, or equivalently, the projective modules in mod $A$ are injective. For a selfinjective algebra $A$, we denote by $\Gamma^*_A$ the stable Auslander-Reiten quiver of $A$, obtained from $\Gamma_A$ by removing the projective modules and the arrows attached to them.

We are concerned with the problem of describing the isomorphism classes of selfinjective algebras of finite representation type. For $K$ algebraically closed, the problem was solved in the 1980’s by C. Riedtmann (see [6], [18], [19], [20]) via the combinatorial
classification of the Auslander-Reiten quivers of selfinjective algebras of finite representation type. Equivalently, Riedtmann's classification can be presented as follows (see [21, Section 3]): a nonsimple selfinjective algebra $A$ over an algebraically closed field $K$ is of finite representation type if and only if $A$ is a socle (geometric) deformation of an orbit algebra $\hat{B}/G$, where $\hat{B}$ is the repetitive category of a tilted algebra $B$ of Dynkin type $A_n$ ($n \geq 1$), $D_n$ ($n \geq 4$), $E_6$, $E_7$, $E_8$, and $G$ is an admissible infinite cyclic group of automorphisms of $\hat{B}$. For an arbitrary field $K$, the problem seems to be difficult (see [3], [4], [13], [26] for some results in this direction and [27, Section 12] for related open problems). An important known result towards solution of this general problem is the description of the stable Auslander-Reiten quiver $\Gamma^*_A$ of a selfinjective algebra $A$ of finite representation type established by C. Riedtmann [18] and G. Todorov [31] (see also [28, Section IV.15]): $\Gamma^*_A$ is isomorphic to the orbit quiver $\mathbb{Z}\Delta/G$, where $\Delta$ is a Dynkin quiver of type $A_n$ ($n \geq 1$), $B_n$ ($n \geq 2$), $C_n$ ($n \geq 3$), $D_n$ ($n \geq 4$), $E_6$, $E_7$, $E_8$, $F_4$ or $G_2$, and $G$ is an admissible infinite cyclic group of automorphisms of the translation quiver $\mathbb{Z}\Delta$. Therefore, we may associate to a selfinjective algebra $A$ of finite representation type a Dynkin graph $\Delta(A)$, called the Dynkin type of $A$, such that $\Gamma^*_A = \mathbb{Z}\Delta/G$ for a quiver $\Delta$ having $\Delta(A)$ as underlying graph. We also mention that, for a tilted algebra $B$ of Dynkin type $\Delta$, the orbit algebras $\hat{B}/G$ are selfinjective algebras of finite representation type whose Dynkin type is the underlying graph of $\Delta$.

Following [17], a short cycle in the module category $\text{mod} A$ of an algebra $A$ is a sequence

$$X \xrightarrow{f} Y \xrightarrow{g} X$$

of two nonzero nonisomorphisms between modules $X$ and $Y$ in $\text{ind} A$. It has been proved in [17, Corollary 2.2] that if $M$ is a module in $\text{ind} A$ which does not lie on a short cycle then $M$ is uniquely determined (up to isomorphism) by its image $[M]$ in the Grothendieck group $K_0(A)$. Moreover, by a result of Happel and Liu [12, Theorem] every algebra $A$ having no short cycles in $\text{mod} A$ is of finite representation type.

The following theorem is the main result of the paper.

**Theorem.** Let $A$ be a selfinjective algebra over a field $K$. The following statements are equivalent.

(i) $\text{mod} A$ has no short cycles.
(ii) $A$ is isomorphic to an orbit algebra $\hat{B}/(\varphi \nu_B^2)$, where $B$ is a tilted algebra of Dynkin type over $K$, $\nu_B$ is the Nakayama automorphism of $B$, and $\varphi$ is a strictly positive automorphism of $\hat{B}$.

The paper is organized as follows. In Section 1 we introduce the orbit algebras of repetitive categories. Section 2 is devoted to presenting basic results from the theory of selfinjective algebras with deforming ideals, playing the fundamental role in the proof of the main theorem. In Section 3 we discuss properties of stable slices of Auslander-Reiten quivers of selfinjective algebras of finite type, essential for further considerations. In Section 4 we describe the selfinjective Nakayama algebras without short cycles of indecomposable modules. In Section 5 we prove the Theorem for the selfinjective algebras of
Dynkin type. In the final Section 6 we complete the proof of the Theorem for arbitrary selfinjective algebras.

For basic background on the representation theory applied in this paper we refer to [1], [2], [28] and [29].

1. Orbit algebras of repetitive categories.

Let $B$ be an algebra and $1_B = e_1 + \cdots + e_n$ a decomposition of the identity of $B$ into a sum of pairwise orthogonal primitive idempotents. We associate to $B$ a selfinjective locally bounded $K$-category $\hat{B}$, called the repetitive category of $B$ (see [14]). The objects of $\hat{B}$ are $e_{m,i}$, for $m \in \mathbb{Z}$, $i \in \{1, \ldots, n\}$, and the morphism spaces are defined as follows

$$\hat{B}(e_{m,i}, e_{r,j}) = \begin{cases} e_jBe_i, & r = m, \\ D(e_iBe_j), & r = m + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $e_jBe_i = \text{Hom}_B(e_iB, e_jB)$, $D(e_iBe_j) = e_jD(B)e_i$ and

$$\bigoplus_{(m,i) \in \mathbb{Z} \times \{1, \ldots, n\}} \hat{B}(e_{m,i}, e_{r,j}) = e_jB \oplus D(Be_j),$$

for any $r \in \mathbb{Z}$ and $j \in \{1, \ldots, n\}$. We denote by $\nu_{\hat{B}}$ the Nakayama automorphism of $\hat{B}$ defined by

$$\nu_{\hat{B}}(e_{m,i}) = e_{m+1,i} \quad \text{for all} \quad (m,i) \in \mathbb{Z} \times \{1, \ldots, n\}.$$

Moreover, an automorphism $\varphi$ of the $K$-category $\hat{B}$ is said to be:

- **positive** if, for each pair $(m,i) \in \mathbb{Z} \times \{1, \ldots, n\}$, we have $\varphi(e_{m,i}) = e_{p,j}$ for some $p \geq m$ and some $j \in \{1, \ldots, n\}$;

- **rigid** if, for each pair $(m,i) \in \mathbb{Z} \times \{1, \ldots, n\}$, there exists $j \in \{1, \ldots, n\}$ such that $\varphi(e_{m,i}) = e_{m,j}$;

- **strictly positive** if it is positive but not rigid.

Thus the automorphisms $\nu_{\hat{B}}^r$, for any $r \geq 1$, are strictly positive automorphisms of $\hat{B}$.

Recall that a group $G$ of automorphisms of $\hat{B}$ is said to be *admissible* if $G$ acts freely on the set of objects of $\hat{B}$ and has finitely many orbits. Then, following P. Gabriel [11], we may consider the orbit category $\hat{B}/G$ of $\hat{B}$ with respect to $G$ whose objects are the $G$-orbits of objects in $\hat{B}$, and the morphism spaces are given by

$$(\hat{B}/G)(a,b) = \left\{(f_{y,x}) \in \prod_{(x,y) \in a \times b} \hat{B}(x,y) \mid g f_{y,x} = f_{g y,g x}, \forall g \in G,(x,y) \in a \times b \right\}$$

for all objects $a, b$ of $\hat{B}/G$. Since $\hat{B}/G$ has finitely many objects and the morphism spaces in $\hat{B}/G$ are finite dimensional, we have the associated finite dimensional selfinjective $K$-algebra $\bigoplus(\hat{B}/G)$ which is the direct sum of all morphism spaces in $\hat{B}/G$, called the orbit algebra of $\hat{B}$ with respect to $G$. We will identify $\hat{B}/G$ with $\bigoplus(\hat{B}/G)$. For example, for each positive integer $r$, the infinite cyclic group $(\nu_{\hat{B}}^r)$ generated by the $r$-th power $\nu_{\hat{B}}^r$ of $\nu_{\hat{B}}$ is an admissible group of automorphisms of $\hat{B}$, and we have the associated selfinjective orbit algebra
called the \( r \)-fold trivial extension algebra of \( B \). In particular, \( T(B)^{(1)} \cong T(B) = B \ltimes D(B) \) is the trivial extension of \( B \) by the injective cogenerator \( D(B) \).

2. Selfinjective algebras with deforming ideals.

In this section we present criteria for selfinjective algebras to be socle equivalent to orbit algebras of the repetitive categories of algebras with respect to infinite cyclic automorphism groups, playing fundamental role in the proof of the Theorem.

Let \( A \) be a selfinjective algebra. For a subset \( X \) of \( A \), we may consider the left annihilator \( l_A(X) = \{ a \in A \mid aX = 0 \} \) of \( X \) in \( A \) and the right annihilator \( r_A(X) = \{ a \in A \mid Xa = 0 \} \) of \( X \) in \( A \). Then by a theorem due to T. Nakayama (see [28, Theorem IV.6.10]) the annihilator operation \( l_A \) induces a Galois correspondence from the lattice of right ideals of \( A \) to the lattice of left ideals of \( A \), and \( r_A \) is the inverse Galois correspondence to \( l_A \). Let \( I \) be an ideal of \( A \), \( B = A/I \), and \( e \) an idempotent of \( A \) such that \( e + I \) is the identity of \( B \). We may assume that \( 1_A = e_1 + \cdots + e_r \) with \( e_1, \ldots, e_r \) pairwise orthogonal primitive idempotents of \( A \), \( e = e_1 + \cdots + e_n \) for some \( n \leq r \), and \( \{ e_i \mid 1 \leq i \leq n \} \) is the set of all idempotents in \( \{ e_i \mid 1 \leq i \leq r \} \) which are not in \( I \). Then such an idempotent \( e \) is uniquely determined by \( I \) up to an inner automorphism of \( A \), and is called a residual identity of \( B = A/I \). Observe also that \( B \cong eAe/eIe \).

We have the following lemma from [25, Lemma 5.1].

**Lemma 2.1.** Let \( A \) be a selfinjective algebra, \( I \) an ideal of \( A \), and \( e \) an idempotent of \( A \) such that \( l_A(I) = Ie \) or \( r_A(I) = eI \). Then \( e \) is a residual identity of \( A/I \).

We recall also the following proposition proved in [22, Proposition 2.3].

**Proposition 2.2.** Let \( A \) be a selfinjective algebra, \( I \) an ideal of \( A \), \( B = A/I \), \( e \) a residual identity of \( B \), and assume that \( IeI = 0 \). The following conditions are equivalent.

(i) \( Ie \) is an injective cogenerator in \( \text{mod } B \).

(ii) \( eI \) is an injective cogenerator in \( \text{mod } B^{\text{op}} \).

(iii) \( l_A(I) = Ie \).

(iv) \( r_A(I) = eI \).

Moreover, under these equivalent conditions, we have \( \text{soc}(A) \subseteq I \) and \( l_{eAe}(I) = eI = r_{eAe}(I) \).

The following theorem proved in [23, Theorem 3.8] (sufficiency part) and [25, Theorem 5.3] (necessity part) will be fundamental for our considerations.

**Theorem 2.3.** Let \( A \) be a selfinjective algebra. The following conditions are equivalent.
(i) $A$ is isomorphic to an orbit algebra $\tilde{B}/(\varphi \nu \tilde{B})$, where $B$ is an algebra and $\varphi$ is a positive automorphism of $\tilde{B}$.

(ii) There is an ideal $I$ of $A$ and an idempotent $e$ of $A$ such that

1. $r_A(I) = eI$;
2. the canonical algebra epimorphism $eAe \to eAe/eIe$ is a retraction.

Moreover, in this case, $B$ is isomorphic to $A/I$.

Let $A$ be a selfinjective algebra, $I$ an ideal of $A$, and $e$ a residual identity of $A/I$. Following [22], $I$ is said to be a deforming ideal of $A$ if the following conditions are satisfied:

(D1) $l_{eAe}(I) = eI = r_{eAe}(I)$;

(D2) the valued quiver $Q_{A/I}$ of $A/I$ is acyclic.

Assume now that $I$ is a deforming ideal of $A$. Then we have a canonical isomorphism of algebras $eAe/eIe \to A/I$ and $I$ can be considered as an $(eAe/eIe)-(eAe/eIe)$-bimodule. Denote by $A[I]$ the direct sum of $K$-vector spaces $(eAe/eIe) \oplus I$ with the multiplication

$$(b, x) \cdot (c, y) = (bc, by + xc + xy)$$

for $b, c \in eA/eIe$ and $x, y \in I$. Then $A[I]$ is a $K$-algebra with the identity $(e + eI, 1_A - e)$, and, by identifying $x \in I$ with $(0, x) \in A[I]$, we may consider $I$ as an ideal of $A[I]$. Observe that $e = (e + eI, 0)$ is a residual identity of $A[I]/I = eA/eIe \cong A/I$, $eA[I]e = (eA/eIe) \oplus eIe$, and the canonical algebra epimorphism $eA[I]e \to eA[I]/eIe$ is a retraction.

The following properties of the algebra $A[I]$ were established in [23, Theorem 4.1].

**Theorem 2.4.** Let $A$ be a selfinjective algebra and $I$ a deforming ideal of $A$. The following statements hold.

(i) $A[I]$ is a selfinjective algebra with the same Nakayama permutation as $A$ and $I$ is a deforming ideal of $A[I]$.

(ii) $A$ and $A[I]$ are socle equivalent.

We note that if $A$ is a selfinjective algebra, $I$ an ideal of $A$, $B = A/I$, $e$ an idempotent of $A$ such that $r_A(I) = eI$, and the valued quiver $Q_B$ of $B$ is acyclic, then by Lemma 2.1 and Proposition 2.2, $I$ is a deforming ideal of $A$ and $e$ is a residual identity of $B$.

The following theorem proved in [23, Theorem 4.1] shows the importance of the algebras $A[I]$.

**Theorem 2.5.** Let $A$ be a selfinjective algebra, $I$ an ideal of $A$, $B = A/I$ and $e$ an idempotent of $A$. Assume that $r_A(I) = eI$ and $Q_B$ is acyclic. Then $A[I]$ is isomorphic to an orbit algebra $\tilde{B}/(\varphi \nu \tilde{B})$ for some positive automorphism $\varphi$ of $\tilde{B}$.

We point out that there are selfinjective algebras $A$ with deforming ideals $I$ such that the algebras $A$ and $A[I]$ are not isomorphic (see [23, Example 4.2]). The following criterion proved in [24, Proposition 3.2] describes a situation when the algebras $A$ and $A[I]$ are isomorphic.

**Theorem 2.6.** Let $A$ be a selfinjective algebra with a deforming ideal $I$, $B = A/I$, $e$ be a residual identity of $B$, and $\nu$ the Nakayama permutation of $A$. Assume that
\[ IeI = 0 \text{ and } e_i \neq e_{\nu(i)}, \text{ for any primitive summand } e_i \text{ of } e. \] Then the algebras \( A \) and \( A[I] \) are isomorphic. In particular, \( A \) is isomorphic to an orbit algebra \( \hat{B}/(\varphi \nu_{\hat{B}}) \) for some positive automorphism \( \varphi \) of \( \hat{B} \).

3. Selfinjective algebras of finite representation type.

Let \( A \) be a selfinjective algebra of finite representation type. We know from the Riedtmann-Todorov theorem that the stable Auslander-Reiten quiver \( \Gamma_A^\nu \) of \( A \) is of the form \( \mathbb{Z}\Delta/G \) for a valued Dynkin quiver \( \Delta \) and an admissible infinite cyclic group \( G \) of automorphisms of the translation quiver \( \mathbb{Z}\Delta \). Following [30], a full valued subquiver \( \Delta \) of \( \Gamma_A \) is said to be a \textit{stable slice} if the following conditions are satisfied:

(1) \( \Delta \) is connected, acyclic and without projective modules.
(2) For any valued arrow \( V \xrightarrow{(d,d')} U \) in \( \Gamma_A \) with \( U \) in \( \Delta \) and \( V \) nonprojective, \( V \) belongs to \( \Delta \) or to \( \tau_A \Delta \).
(3) For any valued arrow \( U \xrightarrow{(e,e')} V \) in \( \Gamma_A \) with \( U \) in \( \Delta \) and \( V \) nonprojective, \( V \) belongs to \( \Delta \) or to \( \tau_A^{-1} \Delta \).

Observe that a stable slice \( \Delta \) of \( \Gamma_A \) is a Dynkin quiver intersecting every \( \tau_A \)-orbit in \( \Gamma_A \) exactly once. A stable slice \( \Delta \) of \( \Gamma_A \) is said to be \textit{semiregular} if \( \Delta \) does not contain both the socle factor \( Q/\text{soc}(Q) \) of an indecomposable projective module \( Q \) and the radical \( \text{rad } P \) of an indecomposable projective module \( P \). Further, following [30], a stable slice \( \Delta \) of \( \Gamma_A \) is said to be \textit{double \( \tau_A \)-rigid} if \( \text{Hom}_A(X, \tau_A Y) = 0 \) and \( \text{Hom}_A(\tau_A^{-1} X, Y) = 0 \) for all indecomposable modules \( X \) and \( Y \) from \( \Delta \).

Recall also that a selfinjective algebra \( A \) is a \textit{Nakayama algebra} if every indecomposable projective module \( P \) in \( \text{mod } A \) is uniserial (its submodule lattice is a chain). We note that then \( A \) is of finite representation type and every indecomposable module in \( \text{mod } A \) is uniserial (see [28, Theorem I.10.5]).

Theorem 3.1. Let \( A \) be a selfinjective algebra of finite representation type. The following statements are equivalent.

(i) \( \Gamma_A \) admits a semiregular stable slice.
(ii) \( A \) is not a Nakayama algebra.

Proof. Let \( A \) be an indecomposable selfinjective Nakayama algebra and \( n \) be the rank of \( K_0(A) \). Note that then each \( \tau_A \)-orbit of \( \Gamma_A \) consists of \( n \) indecomposable modules having the same length (see [11, Corollary V.4.2] or [2, Corollary IV.2.9]). Therefore, a \( \tau_A \)-orbit in \( \Gamma_A \) which contains the radical of an indecomposable projective \( A \)-module consists entirely of the radicals of all indecomposable projective \( A \)-modules. Hence, no stable slice of \( \Gamma_A \) is semiregular, and (i) implies (ii).

Assume \( A \) is not a Nakayama algebra. It implies that there exists a projective module \( P \) in \( \text{ind } A \) such that \( P/\text{soc}(P) \) is not a radical of any indecomposable projective module. Indeed, if \( P_1, \ldots, P_n \) is a complete family of indecomposable projective modules in \( \text{mod } A \) and \( P_k/\text{soc}(P_k) = \text{rad } P_{k+1} \) for \( k \in \{1, \ldots, n\} \), with \( P_{n+1} = P_1 \), then \( P_1, \ldots, P_n \) are uniserial modules, and \( A \) is a Nakayama algebra, a contradiction. We denote by \( \Delta_P \) the full subquiver of \( \Gamma_A \) given by the module \( \tau_A^{-1}(P/\text{soc}(P)) \) and all modules \( X \) in
ind $A$ such that there is a nontrivial sectional path in $\Gamma_A^s$ from $P/soc(P)$ to $X$. Observe that $\Delta_P$ is a stable slice in $\Gamma_A$. We shall show that $\Delta_P$ does not contain $Q/soc(Q)$ for any indecomposable projective module $Q$ in $\text{mod} A$, and hence it is semiregular.

Suppose, on the contrary, that $\Delta_P$ contains $Q/soc(Q)$ for some indecomposable projective module $Q$ in $\text{mod} A$. From the assumption imposed on $P$ we know that $\tau_A^{-1}(P/soc(P)) \neq Q/soc(Q)$. Then $\Gamma_A$ contains a full valued subquiver of the form

\[
\begin{array}{c}
Y_0 \\
X_0 \\
\vdots \\
X_1 \\
X_r \\
Y_r \\
Y_{r-1} \\
\vdots \\
Y_1 \\
Y_0
\end{array}
\]

where $Y_0 = P/soc(P)$, all modules $Y_i$ for $i \in \{1, \ldots, r\}$ belong to $\Delta_P$, $X_r$ is the indecomposable projective module $Q$, $Y_r = Q/soc(Q)$ and $r \geq 1$ is the smallest number with this property. Clearly, $X_0 \to X_1 \to \ldots \to X_r$ form a sectional path of irreducible homomorphisms between modules in $\text{mod} A$ which starts in a direct summand $X_0$ of $rad P/soc(P)$. Then, for $l$ denoting the length function on $\text{mod} A$, we have the inequalities

\[ l(X_{i-1}) + l(Y_i) \geq l(X_i) + l(Y_{i-1}), \]

for all $i \in \{1, \ldots, r\}$, and the equality holds if the number of indecomposable direct summands in the middle term of the Auslander-Reiten sequence ending in $Y_i$ is two. Therefore, $l(X_{i-1}) - l(Y_{i-1}) \geq l(X_i) - l(Y_i)$ for all $i \in \{1, \ldots, r\}$. Since $l(X_r) = l(Q) > l(Q/soc(Q)) = l(Y_r)$ we conclude that $l(X_0) > l(Y_0) = P/soc(P)$, a contradiction because $X_0$ is a proper submodule of $P/soc(P)$. \hfill \Box

For the class of selfinjective algebras of finite representation type without short cycles in the module category we have the following property of stable slices.

**Lemma 3.2.** Let $A$ be a selfinjective algebra which does not admit a short cycle in $\text{mod} A$. Then all stable slices of $\Gamma_A$ are double $\tau_A$-rigid.

**Proof.** Since $\text{mod} A$ has no short cycle, $A$ is of finite representation type [12, Theorem]. Suppose, on the contrary, that there is a stable slice $\Delta$ of $\Gamma_A$ which is not double $\tau_A$-rigid. Without loss of generality we may assume $\text{Hom}_A(X, \tau_A Y) \neq 0$ for some indecomposable modules $X$ and $Y$ from $\Delta$. By $\Delta_Y$, we denote the stable slice of $\Gamma_A$ formed by all modules $M$ in $\text{ind} A$ such that there is a sectional path in $\Gamma_A^s$ from $Y$ to $M$. Since $X, Y$ belong to the same stable slice any nonzero homomorphism $f : X \to \tau_A Y$ factors through a direct sum $\bigoplus_{i=1}^m Z_i$ of modules $Z_i$ from $\Delta_Y$ for some $m \geq 1$. Hence, there is a nonzero homomorphism $g : Z \to \tau_A Y$, where $Z = Z_i$ for some $i \in \{1, \ldots, m\}$. Moreover, $\text{Hom}_A(Y, Z) \neq 0$ because the composition of irreducible homomorphisms corresponding
to the sectional path in $\Gamma_A$ is nonzero \cite{7}. Hence the indecomposable module $Z$ is the middle of a short chain of the form $Y \rightarrow Z \rightarrow \tau_A Y$. Applying \cite{17} Theorem 1.6, we conclude that $Z$ lies on a short cycle in $\mod A$. This contradicts the assumption. \hfill \Box

The following example shows that the converse does not hold in general.

**Example 3.3.** Let $A = KQ/I$ be the bound quiver algebra, where $Q$ is the quiver

$$
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\alpha_1 & \beta_1 & \beta_4 & \alpha_4 \\
\beta_2 & \alpha_3 & \beta_3 & \alpha_3 \\
\end{array}
$$

and the ideal $I = \langle \alpha_1 \alpha_2 - \beta_1 \beta_2, \alpha_3 \alpha_4 - \beta_3 \beta_4, \alpha_2 \beta_3, \beta_2 \alpha_3, \alpha_4 \beta_1, \beta_4 \alpha_1 \rangle$. Then $A$ is an orbit algebra $\tilde{B}/G$, where $B$ is a tilted algebra of type $A_3$ and $G$ is an admissible group of automorphisms of $\tilde{B}$ generated by $\nu^2_B$. The Auslander-Reiten quiver $\Gamma_A$ of $A$ is of the form

$$
\begin{array}{cccc}
P(3) & P(1) & P(5) & P(4) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
P(2) & P(6) & P(2) & P(3) \\
\end{array}
$$

Let $\Delta$ be a stable slice in $\Gamma_A$. We claim that $\Delta$ is double $\tau_A$-rigid. Suppose there are indecomposable modules $X, Y \in \Delta$ such that $\Hom_A(X, \tau_A Y) \neq 0$. Since $A$ is of finite representation type there exists a path of irreducible homomorphisms

$$
X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \ldots \xrightarrow{f_r} X_r = \tau_A Y
$$

between nonprojective indecomposable modules $X_0, \ldots, X_r$ such that $f_r \ldots f_1 \neq 0$. Observe that $r \geq 8$. On the other hand, $X_0, X_1, \ldots, X_r$ are of length at most 3. This contradicts Harada-Sai Lemma (see for example \cite{28} Lemma III.2.1)). Therefore, $\Hom_A(X, \tau_A Y) = 0$ for any indecomposable $X, Y \in \Delta$. Similarly, we prove that $\Hom_A(\tau_A X, Y) = 0$ for any indecomposable $X, Y \in \Delta$. Therefore, $\Delta$ is double $\tau_A$-rigid.

Nevertheless, $P(1) \xrightarrow{f} P(4) \xrightarrow{g} P(1)$, where $\text{Im } f = S(1)$ is the socle of $P(4)$ and $\text{Im } g = S(4)$ is the socle of $P(1)$, forms a short cycle in $\mod A$.

The following consequence of Theorems 2.5, 2.6 and \cite{30} Proposition 3.8 and Theorem 3.9] will be crucial in our proof of the main theorem.
Theorem 3.4. Let $A$ be a selfinjective algebra of finite representation type such that $\Gamma_A$ admits a semiregular double $\tau_A$-rigid stable slice $\Delta$. Moreover, let $M$ be the direct sum of the indecomposable modules lying on $\Delta$, $I = r_A(M)$, and $B = A/I$. Then the following statements hold.

(i) $M$ is a tilting module in $\text{mod } B$.
(ii) $H = \text{End}_B(M)$ is a hereditary algebra of Dynkin type $\Delta$.
(iii) $T = D(M)$ is a tilting module in $\text{mod } H$ with $\text{End}_H(T)$ isomorphic to $B$.
(iv) $I$ is a deforming ideal of $A$ with $r_A(I) = eI$ for an idempotent $e$ of $A$, being a
residual identity of $B = A/I$.
(v) $A[I]$ is isomorphic to an orbit algebra $\widehat{B}/(\psi \nu B)$ for a positive automorphism $\psi$ of
$\widehat{B}$.
(vi) $A$ is socle equivalent to $A[I]$.

4. Selfinjective Nakayama algebras.

The aim of this section is to prove the implication (i)\(\Rightarrow\)(ii) of the Theorem for self-
injective Nakayama algebras.

It is well known that an algebra $B$ is a hereditary Nakayama algebra if and only if $B$
is isomorphic to the algebra

$$T_n(F) = \begin{bmatrix} F & 0 & \cdots & 0 \\ F & F & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F & F & \cdots & F \end{bmatrix}$$

of all lower triangular $n \times n$ matrices over a finite dimensional division $K$-algebra $F$, for some positive natural number $n$.

Proposition 4.1. Let $A$ be a selfinjective Nakayama algebra which does not admit a
short cycle in $\text{mod } A$. Then $A$ is isomorphic to an orbit algebra $\widehat{B}/(\psi \nu B)$, where $B$ is a
hereditary Nakayama algebra and $\varphi$ is a strictly positive automorphism of $\widehat{B}$.

Proof. Let $P$ be an indecomposable projective module in $\text{mod } A$. Since $P$ is a uniserial
module, the radical series

$$P \supset \text{rad } P \supset \text{rad }^2 P \supset \cdots \supset \text{rad }^n P \supset \text{rad }^{n+1} P = 0$$

of $P$ is its unique composition series, and hence $n + 1 = l(P)$ (see [28, Theorem I.10.1]).
Moreover, $n + 1$ is the Loewy length $ll(A)$ of $A$. We define $M_i = \text{rad }^{n-i+1} P$ for
$i \in \{1, \ldots, n\}$. It follows also from [28, Proposition III.8.6 and Theorem III.8.7] that there is in $\Gamma_A$ a sectional path $\Delta$ of the form

$$M_1 \to M_2 \to \cdots \to M_{n-1} \to M_n$$

with $M_1 = \text{soc}(P)$ and $M_n = \text{rad } P$. Moreover, $\Delta$ is a stable slice of $\Gamma_A$. For each
$i \in \{1, \ldots, n\}$, let $P_i$ be the projective cover of $M_i$ in $\text{mod } A$. We note that there is a
sectional path in $\Gamma_A$ of the form

$$P_1 \to P_i/\text{soc}(P_i) \to \cdots \to M_i$$
and $M_i = P_i/\text{rad}^i P_i$, for any $i \in \{1, \ldots, n\}$. We claim that the projective modules $P_1, \ldots, P_n$ are pairwise different. Indeed, suppose that $P_i = P_j$ for some $i < j$ in $\{1, \ldots, n\}$. Clearly, we have in mod $A$ a canonical proper monomorphism $M_i \to M_j$. On the other hand, $M_i = P_i/\text{rad}^i P_i = P_j/\text{rad}^j P_j$ and $M_j = P_j/\text{rad}^j P_j$, and consequently there is in mod $A$ a proper epimorphism $M_j \to M_i$. But then we have in mod $A$ a short cycle $M_i \to M_j \to M_i$, which contradicts the assumption imposed on $A$. Observe now that we have the equalities
\[
P_i/\text{soc}(P_i) = \text{rad} P_{i+1} = \tau_A(P_{i+1}/\text{soc}(P_{i+1}))
\]
for any $i \in \{1, \ldots, n\}$, where $P_{n+1} = P$. We may choose pairwise orthogonal primitive idempotents $e_j$, $j \in \{1, \ldots, m\}$, such that $1_A = e_1 + \ldots + e_m$, $m \geq n$, and $P_i = e_i A$ for any $i \in \{1, \ldots, n\}$. Let $M$ be the direct sum of the modules $M_1, \ldots, M_n$ lying on the stable section $\Delta$, $I = r_A(M)$, and $B = A/I$. Observe that $e = e_1 + \ldots + e_n$ is a residual identity of $B$. Since $A$ is a Nakayama algebra, we conclude that $B$ is also a Nakayama algebra (see [28, Lemma I.10.2]). Moreover, since $M$ is a faithful module in mod $B$, there is in mod $B$ a monomorphism $B \to M'$ for some positive integer $r$ (see [28, Lemma II.5.5]). Then we obtain that $M_1, \ldots, M_n$ form a complete set of pairwise nonisomorphic indecomposable projective modules in mod $B$, $B_1 = M_1 \oplus \ldots \oplus M_n$, and $B$ is a hereditary Nakayama algebra of Loewy length $n$. Clearly, $M_n$ is the unique indecomposable projective-injective module in mod $B$, and the Auslander-Reiten quiver $\Gamma_B$ of $B$ is a full valued translation subquiver of the Auslander-Reiten quiver $\Gamma_A$ of $A$. Consider the indecomposable projective modules $P_{n+1}, \ldots, P_{2n}$ in mod $A$ such that $\text{rad} P_j = P_{j-1}/\text{soc}(P_{j-1})$ for $j \in \{n+1, \ldots, 2n\}$. Further, let $\nu$ be the Nakayama permutation of $A$, which is a permutation of $\{1, \ldots, m\}$. Then it follows from the above discussion that $j = \nu(j - n)$ for any $j \in \{n+1, \ldots, 2n\}$. Clearly, $P_{n+1}, \ldots, P_{2n}$ are pairwise different projective modules in $\Gamma_A$, because the projective modules $P_1, \ldots, P_n$ are pairwise different. We also observe that $P_i \neq P_j$ in $\Gamma_A$ for any $i \in \{1, \ldots, n\}$ and $j \in \{n+1, \ldots, 2n\}$. Indeed, suppose that $P_i = P_j$ for some $i \in \{1, \ldots, n\}$ and $j \in \{n+1, \ldots, 2n\}$. Then there is a short cycle in mod $A$ of the form
\[
P_i \xrightarrow{f} M_n \xrightarrow{g} P_j = P_i
\]
because the factor module $M_i$ of $P_i$ is a submodule of $M_n$ and a factor module of $M_n$ is a submodule of $P_j$, contradicting our assumption on $A$. In particular, we conclude that $m \geq 2n$.

We will prove now that $I$ is a deforming ideal of $A$ with $l_A(I) = Ie$ and $r_A(I) = eI$. We first note that the valued quiver $Q_B$ of $B$ is acyclic, because $B$ is a hereditary Nakayama algebra with $Q_B$ of the form
\[
1 \leftarrow 2 \leftarrow \ldots \leftarrow n - 1 \leftarrow n .
\]
Then, in order to show that $I$ is a deforming ideal of $A$, it is enough to show that $l_A(I) = Ie$, by Proposition [22]. We denote by $J$ the trace ideal of $M$ in $A$, that is, the ideal in $A$ generated by the images of all homomorphisms from $M$ to $A$ in mod $A$. Observe that $J$ is a right $B$-module, and hence $Je = J$ and $JI = 0$. Since mod $M$ is without short cycles and $\Delta$ is a stable slice of $\Gamma_A$, it follows from Lemma [3,2] that $\Delta$ is double $\tau_A$-rigid. Then, applying [30, Lemmas 3.10, 3.11 and 3.12], we obtain that
J ⫋ I, l_A(I) = J, and eIe = eJe. We claim that (1 − e)Ie = (1 − e)Je. Clearly, (1 − e)I = (1 − e)A because 1 − e ∈ I by the definition of e. Further, there is a canonical isomorphism of right eAe-modules (1 − e)Ae ∼= Hom_A(eA, (1 − e)A) in mod A (see [28, Lemma I.8.7]). Let f : eA → (1 − e)A be a nonzero homomorphism in mod A. Since eA = P_1 ⊕ ⋯ ⊕ P_n, it follows from the definition of Δ that there exists a positive integer s and homomorphisms g : eA → M^s, h : M^s → (1 − e)A such that f = hg. But then Im f ⊆ Im h ⊆ (1 − e)J. This shows that (1 − e)Ie = (1 − e)Je ⊆ (1 − e)Je. Then we obtain that Ie = eIe ⊕ (1 − e)Ie ⊆ eJe ⊕ (1 − e)Je = J. Since J ⫋ I we have also J = Je ⊆ Ie. Summing up, we conclude that Ie = J = l_A(I). In particular, we get IeI = 0. Then, applying Proposition 2.2 we obtain that I is a deforming ideal of A with l_A(I) = Ie and r_A(I) = eI.

It follows from Theorem 2.5 that A[I] is isomorphic to an orbit algebra \( \hat{B}/(\psi_\nu_B) \) for some positive automorphism \( \psi \) of \( \hat{B} \). Moreover, by Theorem 2.4 A is socle equivalent to A[I] and both algebras have the same Nakayama permutation. Observe also that, if \( e_i \) is a primitive summand of e, then \( i \in \{1, \ldots, n\} \) and \( e_{\nu(i)} = e_{n+i} \), and consequently \( e_i \neq e_{\nu(i)} \). Then it follows from Theorem 2.6 that the algebras A and A[I] are isomorphic. In particular, we conclude that A is isomorphic to the orbit algebra \( \hat{B}/(\psi_\nu_B) \). Finally, we claim that \( \psi = \varphi_\nu_B \) for a strictly positive automorphism \( \varphi \) of \( \hat{B} \). Since \( P_1, \ldots, P_n, P_{n+1}, \ldots, P_{2n} \) are pairwise different projective modules in \( \Gamma_A \), and \( n+i = \nu(i) \) for any \( i \in \{1, \ldots, n\} \), we conclude that \( \psi = \varphi_\nu_B \) for a positive automorphism \( \varphi \) of \( \hat{B} \). Suppose that \( \varphi \) is a rigid automorphism of \( \hat{B} \). Then \( m = 2n \) and \( \nu^2 \) is the identity permutation of \( \{1, \ldots, m\} \). Then we have in mod A a short cycle of the form

\[
P_1 \xrightarrow{u} P_{n+1} \xrightarrow{v} P_1
\]

with Im u and Im v being simple modules, which contradicts the assumption imposed on A. Therefore, we conclude that A is isomorphic to an orbit algebra \( \hat{B}/(\varphi_\nu^2_B) \) for a strictly positive automorphism \( \varphi \) of \( \hat{B} \).

5. Selfinjective algebras of Dynkin type.

Let B be a triangular algebra (i.e. the quiver \( Q_B \) is acyclic) and \( e_1, \ldots, e_n \) be pairwise orthogonal primitive idempotents of B with \( 1_B = e_1 + \cdots + e_n \). We identify B with the full subcategory \( B_0 \) of the repetitive category \( \hat{B} \) given by the objects \( e_{0,j} \), \( 1 \leq j \leq n \). For a sink \( i \) of \( Q_B \), the reflection \( S_i^+ \) of B at \( i \) is the full subcategory of \( \hat{B} \) given by the objects

\[
e_{0,j}, \quad 1 \leq j \leq n, \quad j \neq i, \quad \text{and} \quad e_{1,i} = \nu_B(e_{0,i}).
\]

Then the quiver \( Q_{S_i^+} \) of \( S_i^+ \) is the reflection \( \sigma_i^+ Q_B \) of \( Q_B \) at \( i \) (see [14]). Observe that \( \hat{B} = \hat{S_i^+} B \). By a reflection sequences of sinks of \( Q_B \) we mean a sequence \( i_1, \ldots, i_t \) of vertices of \( Q_B \) such that \( i_s \) is a sink of \( \sigma_{i_{s-1}}^+ \cdots \sigma_{i_1}^+ Q_B \) for all \( s \in \{1, \ldots, t\} \). Moreover, for a sink \( i \) of \( Q_B \), we denote by \( T_i^+ B \) the full subcategory of \( \hat{B} \) given by the objects

\[
e_{0,j}, \quad 1 \leq j \leq n, \quad \text{and} \quad e_{1,i} = \nu_B(e_{0,i}).
\]
Observe that $T^+_i B$ is the one-point extension $B[I_B(i)]$ of $B$ by the indecomposable injective $B$-module $I_B(i)$ at the vertex $i$. Recall that by a finite dimensional $\hat{B}$-module we mean a contravariant $K$-linear functor $M$ from $\hat{B}$ to the category of $K$-vector spaces such that $\sum_{x \in \text{ob}\hat{B}} \dim_K M(x)$ is finite. We denote by $\text{mod}\hat{B}$ the category of all finite dimensional $\hat{B}$-modules. Finally, for a module $M$ in $\text{mod}\hat{B}$, we denote by $\text{supp}(M)$ the full subcategory of $\hat{B}$ formed by all objects $x$ with $M(x) \neq 0$, and call it the support of $M$.

The following consequences of results proved in [13], [14] describe the supports of finite dimensional indecomposable modules over the repetitive categories $\hat{B}$ of tilted algebras $B$ of Dynkin type.

**Theorem 5.1.** Let $B$ be a tilted algebra of Dynkin type $\Delta$ and $n$ the rank of $K_0(B)$. Then there exists a reflection sequence $i_1, \ldots, i_n$ of sinks of $Q_B$ such that the following statements hold.

(i) $S^+_{i_n} \cdots S^+_{i_1} B = \nu^\hat{B}_{\nu^\hat{B}_B}(B)$.

(ii) For each $r \in \{1, \ldots, n\}$, $S^+_{i_r} \cdots S^+_{i_1} B$ is a tilted algebra of type $\Delta$.

(iii) For every indecomposable nonprojective module $M$ in $\text{mod}\hat{B}$, $\text{supp}(M)$ is contained in one of the full subcategories of $\hat{B}$

$$\nu^m_{\nu^\hat{B}_{\nu^\hat{B}_B}}(S^+_{i_r} \cdots S^+_{i_1} B), \quad r \in \{1, \ldots, n\}, \quad m \in \mathbb{Z}.$$  

(iv) For every indecomposable projective module $P$ in $\text{mod}\hat{B}$, $\text{supp}(P)$ is contained in one of the full subcategories of $\hat{B}$

$$\nu^m_{\nu^\hat{B}_{\nu^\hat{B}_B}}(T^+_{i_r} S^+_{i_{r-1}} \cdots S^+_{i_1} B), \quad r \in \{1, \ldots, n\}, \quad m \in \mathbb{Z}.$$  

Moreover, invoking again [13] and [14], we have the below proposition.

**Proposition 5.2.** Let $B$ be a tilted algebra of Dynkin type and $G$ be an admissible group of automorphisms of $\hat{B}$. Then $G$ is an infinite cyclic group generated by a strictly positive automorphism of $\hat{B}$.

It follows from Theorem 5.1 that the repetitive category $\hat{B}$ of a tilted algebra $B$ of Dynkin type is a locally representation-finite category [11], that is, for any object $x \in \hat{B}$ the number of indecomposable modules $N \in \text{mod}\hat{B}$ satisfying $N(x) \neq 0$ is finite. Then we obtain the following consequence of [11, Theorem 3.6].

**Theorem 5.3.** Let $B$ be a tilted algebra of Dynkin type, $G$ an admissible group of automorphisms of $\hat{B}$, and $A = \hat{B}/G$ the associated selfinjective orbit algebra. Then the following statements hold.

(i) The push-down functor $F_\lambda : \text{mod}\hat{B} \to \text{mod} A$, associated with the Galois covering $F : \hat{B} \to \hat{B}/G = A$, is dense and preserves the almost split sequences.

(ii) The Auslander-Reiten quiver $\Gamma_A$ of $A$ is the orbit quiver $\Gamma_{\hat{B}/G}$ of $\Gamma_{\hat{B}}$ with respect to the induced action of $G$ on $\Gamma_{\hat{B}}$.

(iii) $A$ is of finite representation type.
For a Dynkin quiver $\Delta$ of type $A_n$ ($n \geq 1$), $B_n$ ($n \geq 2$), $C_n$ ($n \geq 3$), $D_n$ ($n \geq 4$), $E_6$, $E_7$, $E_8$, $F_4$ or $G_2$, by a selfinjective algebra of type $\Delta$ we mean an orbit algebra $B/G$, where $B$ is a tilted algebra of type $\Delta$ and $G$ is an admissible group of automorphisms of $B$.

**Proposition 5.4.** Let $A$ be a selfinjective algebra of Dynkin type $\Delta \in \{B_n, C_n, F_4, G_2\}$. Then $A$ is isomorphic to the $r$-fold trivial extension algebra $T(B)^{(r)}$, where $B$ is a tilted algebra of type $\Delta$ and $r$ is a positive integer.

**Proof.** For $\Delta$ of type $B_n$ and $C_n$ this follows from [34, Theorems 1.2, 1.3 and 4.1]. We note that the proofs of these results presented in [34] apply essentially combinatorial characterizations of finite Auslander-Reiten quivers of algebras, established in [15], and the classification of selfinjective configurations of the stable translation quivers $\mathbb{Z}D_{m+1}$ and $\mathbb{Z}A_{2n-1}$ given in [19], [20].

For $\Delta = F_4$ the claim follows by repeating arguments applied in [34] for types $B_n$ and $C_n$ and the classification of selfinjective configurations of the stable translation quiver $\mathbb{Z}E_6$ established in [6]. Similarly, for $\Delta = G_2$, the proof reduces to the classification of selfinjective configurations of the stable translation quiver $\mathbb{Z}D_4$ (see [6 (7.6)]). \qed

Let $B$ be a tilted algebra of Dynkin type $\Delta$ and $n$ the rank of $K_0(B)$. Following [21], $B$ is said to be exceptional if there exists a reflection sequence $i_1, \ldots, i_t$ of sinks in $Q_B$ with $t < n$ and $S_{i_1} \ldots S_{i_t} B$ isomorphic to $B$. Equivalently, $B$ is exceptional if and only if there is an automorphism $\varphi$ of $\hat{B}$ with $\varphi^m = \nu_{\hat{B}}$ for some $m \geq 2$ (see [21, Proposition 3.9]). We have the following consequence of [6, Proposition 1.4 and 1.5], [19, Proposition 3.3], [20, Theorem] and Proposition 5.4.

**Proposition 5.5.** Let $B$ be an exceptional tilted algebra of Dynkin type $\Delta$. Then either $\Delta = A_n$ for some $n \geq 2$ or $\Delta = D_{3m}$ for some $m \geq 2$.

We introduce now exceptional tilted algebras of Dynkin type which play a prominent role in the description of selfinjective algebras of Dynkin types $A_n$ ($n \geq 1$), $D_n$ ($n \geq 4$), $E_6$, $E_7$, $E_8$ (see [21, Section 3] for details). Let $F$ be a division algebra over a field $K$.

Let $T_S^m$ be a Brauer tree with $m$ edges and an exceptional vertex $S$ of multiplicity $m \geq 2$. Then the Brauer tree algebra $A(F, T_S^m)$ of $T_S^m$ over $F$ is a symmetric algebra of Dynkin type $A_n$, isomorphic to the orbit algebra $B(F, T_S^m)/(\varphi)$, for an exceptional tilted algebra $B(F, T_S^m)$ of type $A_n$ and an automorphism $\varphi$ of $B(F, T_S^m)$ with $\varphi^m = \nu_{B(F, T_S^m)}$.

Let $T_S^2$ be a Brauer tree with $m \geq 2$ edges and an extreme exceptional vertex $S$ of multiplicity 2. Then the modified Brauer tree algebra $D(F, T_S^2)$ of $T_S^2$ over $F$ (in the sense of [20, 33]) is a symmetric algebra of Dynkin type $D_{3m}$, isomorphic to the orbit algebra $B^*(F, T_S^2)/(\varphi)$, for an exceptional tilted algebra $B^*(F, T_S^2)$ of type $D_{3m}$ and an automorphism $\varphi$ of $B^*(F, T_S^2)$ with $\varphi^3 = \nu_{B^*(F, T_S^2)}$.

Then we have the following general version of [21, Theorem 3.10].

**Proposition 5.6.** Let $A$ be a selfinjective algebra of Dynkin type $\Delta \in \{A_n, D_n, E_6, E_7, E_8\}$. Then $A$ is isomorphic to an orbit algebra of one of the forms:

1. $B/(\nu_{B}^r)$, $r \geq 1$, $B$ a tilted algebra of type $\Delta \in \{A_n, D_n, E_6, E_7, E_8\}$;
(2) $\tilde{B}/(\sigma \nu^r_B)$, $r \geq 1$, $B$ a tilted algebra of type $\Delta \in \{A_{2p+1}, D_n, E_6\}$, $\sigma$ an automorphism of $\tilde{B}$ of order 2;
(3) $\tilde{B}/(\sigma \nu^r_B)$, $r \geq 1$, $B$ a hereditary algebra with $Q_B$ of the form
$$
\begin{array}{ccc}
2 & 3 & 4 \\
\downarrow & & \\
1 \\
\end{array}
$$
$\sigma$ an automorphism of $\tilde{B}$ of order 3;
(4) $\tilde{B}/(\rho^r)$, $r \geq 1$, $B = B(F, T^m_S)$ the tilted algebra of type $A_n$ (introduced above), $\rho$ an automorphism of $\tilde{B}$ with $\rho^m = \nu_B$;
(5) $\tilde{B}/(\rho^r)$, $r \geq 1$, $B = B^*(F, T^2_S)$ the tilted algebra of type $D_{3m}$ (introduced above), $\rho$ an automorphism of $\tilde{B}$ with $\rho^3 = \nu_B$.

We are now in the position to prove the following proposition, forming an essential step in the proof of the main theorem.

**Theorem 5.7.** Let $B$ be a tilted algebra of Dynkin type $\Delta$, $G$ an admissible group of automorphisms of $\tilde{B}$, and $A = \tilde{B}/G$ the associated orbit algebra. The following statements are equivalent.

(i) $\text{mod } A$ has no short cycles.
(ii) $G = (\varphi \nu^2_B)$ for a strictly positive automorphism $\varphi$ of $\tilde{B}$.

**Proof.** Let $n$ be the rank of $K_0(B)$ and $i_1, \ldots, i_n$ a reflection sequence of sinks in $Q_B$ satisfying the statements of Theorem 5.1. Applying Proposition 5.2 we may choose a strictly positive automorphism $\varphi$ of $\tilde{B}$ generating $G$. Moreover, let $F_\lambda : \text{mod } \tilde{B} \to \text{mod } A$ be the push-down functor associated with the Galois covering $F : \tilde{B} \to \tilde{B}/G = A$. Further, let $e_1, \ldots, e_n$ be pairwise orthogonal primitive idempotents of $B$ such that $1_B = e_1 + \ldots + e_n$. Then $e_{m,i}, (m, i) \in \mathbb{Z} \times \{1, \ldots, n\}$, form the set of objects of $\tilde{B}$. For each $(m, i) \in \mathbb{Z} \times \{1, \ldots, n\}$, we denote by $P(m, i)$ the indecomposable projective module $\text{Hom}_{\tilde{B}}(-, e_{m,i})$ in $\text{mod } \tilde{B}$.

We prove first that (ii) implies (i). Assume that $\varphi = \varphi \nu^2_B$ for a strictly positive automorphism $\varphi$ of $\tilde{B}$. Suppose that there is a short cycle $M \xrightarrow{u} N \xrightarrow{v} M$ in $\text{mod } A$. It follows from Theorem 5.3 that there exist indecomposable modules $X$ and $Y$ in $\text{mod } \tilde{B}$ such that $M = F_\lambda(X)$ and $N = F_\lambda(Y)$. Further, since $F_\lambda : \text{mod } \tilde{B} \to \text{mod } A$ is a Galois covering of module categories (see [11, Theorem 3.6]) the functor $F_\lambda$ induces a $K$-linear isomorphism
$$
\bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\tilde{B}}(X, g^p Y) \cong \text{Hom}_A(F_\lambda(X), F_\lambda(Y)) .
$$

Hence, there is a nonzero nonisomorphism $f : X \to g^p Y$ for some $p \in \mathbb{Z}$. Let $Z = g^p Y$. Clearly, we have $F_\lambda(Z) = F_\lambda(g^p Y) = F_\lambda(Y) = N$. The functor $F_\lambda$ induces also a
K-linear isomorphism
\[ \bigoplus_{s \in \mathbb{Z}} \text{Hom}_B(Z, g^sX) \cong \text{Hom}_A(F_\lambda(Z), F_\lambda(X)) , \]
and hence there is a nonzero nonisomorphism \( h : Z : \to g^sX \) for some \( s \in \mathbb{Z} \). We note that \( \Gamma_B \) is an acyclic quiver whose stable part \( \Gamma_B^s \) is isomorphic to \( \mathbb{Z}\Delta \). Moreover, since \( \hat{B} \) is locally representation-finite, every nonzero nonisomorphism between indecomposable modules in mod \( \hat{B} \) is a sum of compositions of irreducible homomorphisms between indecomposable modules in mod \( \hat{B} \). Then we conclude that \( s \geq 1 \). We have two cases to consider.

Assume first that \( X \) is not projective. It follows from Theorem 5.1 that \( \text{supp}(X) \) is contained in the full subcategory of \( \hat{B} \) of the form \( \nu_B^m(S_{i_1}^+ \ldots S_{i_r}^+ B) \), for some \( r \in \{1, \ldots , n\} \) and \( m \in \mathbb{Z} \). Since \( B' = \nu_B^m(S_{i_1}^+ \ldots S_{i_r}^+ B) \) is a tilted algebra of Dynkin type \( \Delta \) with \( \hat{B}' = \hat{B} \), we may assume that \( \text{supp}(X) \) is contained in \( B \). We note that \( \text{Hom}_B(X, Z) \neq 0 \) implies that the supports \( \text{supp}(X) \) and \( \text{supp}(Z) \) are not disjoint. Then, applying Theorem 5.1 again, we conclude that \( \text{supp}(Z) \) is contained in a full subcategory of \( \hat{B} \) of the form \( T_{i_k}^+ S_{i_{k-1}}^+ \ldots S_{i_1}^+ B \) for some \( k \in \{1, \ldots , n\} \). On the other hand, since \( g = \varphi \nu_B^2 \) with \( \varphi \) strictly positive automorphism of \( \hat{B} \) and \( s \geq 1 \), \( \text{supp}(g^sX) \) is contained in the full subcategory of \( \hat{B} \) given by the objects \( e_{m,i} \) for all \( (m, i) \in \mathbb{Z} \times \{1, \ldots , n\} \) with \( m \geq 2 \). Hence, the supports \( \text{supp}(Z) \) and \( \text{supp}(g^sX) \) are disjoint, and consequently \( \text{Hom}_B(Z, g^sX) = 0 \), a contradiction.

Finally, assume that \( X \) is projective. Then, by Theorem 5.1 \( \text{supp}(X) \) is contained in a full subcategory of \( \hat{B} \) of the form \( \nu_B^m(T_{i_k}^+ S_{i_{k-1}}^+ \ldots S_{i_1}^+ B) \) for some \( r \in \{1, \ldots , n\} \) and \( m \in \mathbb{Z} \). We may assume that \( \text{supp}(X) \) is contained in \( T_{i_1}^+ B \) and hence \( X = P(1, i_1) \). Since there is a nonzero nonisomorphism \( f : X \to Z \), we conclude that \( \text{Hom}_B(X/\text{soc}(X), Z) \neq 0 \). Moreover, since \( \text{supp}(X/\text{soc}(X)) \) is contained in \( S_{i_1}^+ B \), which is the full subcategory of \( \hat{B} \) with the objects \( e_{1,i_1} \) and \( e_{0,i} \), for \( i \in \{1, \ldots , n\} \setminus \{i_1\} \). Then we obtain that \( \text{supp}(Z) \) is contained in the full subcategory of \( \hat{B} \) given by the objects \( \nu_B^s(S_{i_1}^+ B) \), so by the objects \( e_{1,i_1}, e_{2,i_1}, \) and \( e_{0,i}, e_{1,i} \) for \( i \in \{1, \ldots , n\} \setminus \{i_1\} \). Observe also that \( \text{supp}(g^sX) = \text{supp}(g^sP(1, i_1)) \) is contained in the full subcategory of \( \hat{B} \) given by the objects \( g^s(e_{1,i_1}) \) and \( g^s(e_{0,i}) \) with \( i \in \{1, \ldots , n\} \). Clearly, \( g^s(e_{1,i_1}) = e_{k,j} \) for some \( k \geq 3 \) and \( j \in \{1, \ldots , n\} \), because \( g = \varphi \nu_B^2 \) and \( s \geq 1 \). Similarly, for any \( i \in \{1, \ldots , n\} \), we have \( g^s(e_{0,i}) = e_{k,j(i)} \) for some \( k \geq 2 \) and \( j(i) \in \{1, \ldots , n\} \). Since \( \text{Hom}_B(Z, g^sX) \neq 0 \), we conclude that the supports \( \text{supp}(Z) \) and \( \text{supp}(g^sX) \) have a common object. Then \( e_{2,i_1} \) is the unique common object of \( \text{supp}(Z) \) and \( \text{supp}(g^sX) \). But then \( s = 1 \) and \( g(e_{0,i}) = e_{2,i_1} \) for some \( i \in \{1, \ldots , n\} \}. Hence \( \varphi(e_{2,i}) = \varphi(\nu_B^2(e_{0,i})) = g(e_{0,i}) = e_{2,i_1} \). Applying \( \nu_B^{-1} \) to this equality we obtain that \( \varphi(e_{1,i}) = e_{1,i_1} \), and consequently \( \varphi P(1, i) = P(1, i_1) \). Since \( i = i_t \) for some \( t \in \{2, \ldots , n\} \), \( \varphi P(1, i_1) = P(1, j) \) for some \( j \in \{1, \ldots , n\} \), then \( j = i_s \) for some \( s \in \{2, \ldots , n\} \) and \( s \neq t \), because \( \varphi \) induces a strictly positive automorphism of the acyclic Auslander-Reiten quiver \( \Gamma_B \). Then \( gX = gP(1, i_1) = \varphi \nu_B^2 P(1, i_1) = P(3, j) \) and \( \text{supp}(gX) \cap \text{supp}(Z) = \text{supp}(P(3, j)) \cap \text{supp}(Z) \) contains \( e_{2,j} \) because \( \text{Hom}_B(Z, gX) \neq 0 \). Therefore \( e_{2,j} = e_{2,i_1} \), a contradiction. Similarly, if \( \varphi P(1, i_1) = P(k, j) \) for some
$k \geq 2$ and $j \in \{1, \ldots, n\}$, then $gX = gP(1, i_1) = \varphi \nu_B^2 P(1, i_1) = P(k, j)$ for some $k \geq 4$. Hence, the supports $\text{supp}(Z)$ and $\text{supp}(gX)$ are disjoint, and consequently $\text{Hom}_B(Z, gX) = 0$, a contradiction. Summing up, we have proved that (ii) implies (i).

We shall now prove that (i) implies (ii). Assume that $g$ is not of the form $\varphi \nu_B^2$ for a strictly positive automorphism $\varphi$ of $\hat{B}$. We will show that $\text{mod} \ A$ contains a short cycle. Recall that $g$ is a strictly positive automorphism of $\hat{B}$. We have few cases to consider, taking into account Proposition 5.6.

(a) Assume that $g = \sigma \nu_B$ or $g = \sigma \nu_B^2$ for a rigid automorphism $\sigma$ of $\hat{B}$, that is, $A = \hat{B}/G$ is as in Proposition 5.4 with $r \in \{1, 2\}$, or is one of the forms (1), (2), (3) with $r \in \{1, 2\}$, described in Proposition 5.6. Then it follows from the classification of selfinjective configurations of stable quivers $ZA_{2p+1}$, $ZD_n$ and $ZE_6$ given in [6, 18, 19, 20] (see also [16] for a general result) that there is $(m, i) \in \mathbb{Z} \times \{1, \ldots, n\}$ such that $\sigma(\varepsilon_m, i) = \varepsilon_m, i$, and then $\sigma(\varepsilon_k, i) = \varepsilon_k, i$ for all $k \in \mathbb{Z}$. Hence, we obtain that $\sigma P(k, i) = P(k, i)$ for all $k \in \mathbb{Z}$. But then $\sigma \nu_B P(0, i) = \sigma P(1, i) = P(1, i)$ and $\sigma \nu_B^2 P(0, i) = \sigma P(2, i) = P(2, i)$. Observe that we have in mod $\hat{B}$ a sequence of nonzero nonisomorphisms

$$P(0, i) \xrightarrow{f} P(1, i) \xrightarrow{h} P(2, i)$$

with $\text{Im} f = \text{soc}(P(1, i))$ and $\text{Im} h = \text{soc}(P(2, i))$. Then we obtain a short cycle in mod $\ A$ of the form

$$F_\lambda(P(0, i)) \xrightarrow{F_\lambda(f)} F_\lambda(P(1, i)) \xrightarrow{F_\lambda(h)} F_\lambda(P(0, i)),$$

where $F_\lambda(P(2, i)) = F_\lambda(P(0, i))$, because if $g = \sigma \nu_B^2$, then $gP(0, i) = \sigma \nu_B^2 P(0, i) = P(2, i)$. We also note that if $g = \sigma \nu_B$, then $F_\lambda(P(1, i)) = F_\lambda(P(0, i))$.

(b) Assume that $B$ is an exceptional tilted algebra of type $A_n$, and $g$ is an automorphism of $\hat{B}$ with $\sigma^m = \nu_B$ and $m \geq 2$. Then $g = \sigma^p$ for some $p \in \{1, \ldots, 2m\}$, by the assumption imposed on $g$. Applying Proposition 5.6, we conclude that $\hat{B} \cong B(F,T_S^m)$ for the exceptional tilted algebra $B(F,T_S^m)$ associated with a division algebra $F$ and a Brauer tree $T_S^m$ with $n$ edges and an exceptional vertex $S$ of multiplicity $m$. We may assume that $\hat{B} = B(F,T_S^m)$. Then $\hat{B}/(g)$ is the Brauer tree algebra $A(F,T_S^m)$ associated with $F$ and $T_S^m$. Consider the orbit algebra $A_k = \hat{B}/(g^k)$, for $k \in \{1, \ldots, 2m\}$. Then $A_k = FQ_{A_k}/I_{A_k}$, where $Q_{A_k}$ is the quiver of $A_k$ and $I_{A_k}$ is an ideal of the path algebra $FQ_{A_k}$ of $Q_{A_k}$ over $F$. Since $A_1 = A(F,T_S^m)$, the quiver $Q_{A_k}$ contains a cycle $C_1$ of length $l$, with $l$ being the number of edges in $T_S^m$ connected to the exceptional vertex $S$, and the composition of any $lm$ consecutive arrows in $C_1$ does not belong to $I_{A_1}$. Hence, for any $k \in \{1, \ldots, 2m\}$, the quiver $Q_{A_k}$ contains a cycle $C_k$ of length $kl$ such that the composition of any $lm$ consecutive arrows in $C_k$ does not belong to $I_{A_k}$. But then the cycle $C_k$ contains two vertices $x$ and $y$ such that the shortest paths $u$ from $x$ to $y$ and $v$ from $y$ to $x$ in $C_k$ do not belong to $I_{A_k}$, and then the indecomposable projective modules $P_{A_k}(x)$ and $P_{A_k}(y)$ in mod $A_k$ at the vertices $x$ and $y$ lie on a short cycle

$$P_{A_k}(x) \rightarrow P_{A_k}(y) \rightarrow P_{A_k}(x).$$
In particular, since $A = \hat{B}/G = \hat{B}/(\varrho^p)$, mod $A$ admits a short cycle of indecomposable projective modules.

(c) Assume that $B$ is an exceptional tilted algebra of type $D_{3m}$, and $\varrho$ is an automorphism of $\hat{B}$ with $\varrho^3 = \nu_{\hat{B}}$. Then $A = \hat{B}/(\varrho^p)$ for some $p \in \{1, \ldots, 6\}$, by the assumption imposed on $g$. Applying Theorem 5.6, we conclude that $\hat{B} = B^*(F, T^2_S)$ for the exceptional tilted algebra $B^*(F, T^2_S)$ associated with a division algebra $F$ and a Brauer tree $T^2_S$ with $m \geq 2$ edges and an extreme exceptional vertex $S$ of multiplicity 2. We may assume that $B = B^*(F, T^2_S)$. Then $\hat{B}/(\varrho)$ is isomorphic to the modified Brauer algebra $D(F, T^2_S)$ associated with $F$ and $T^2_S$. Consider the orbit algebra $D_k = \hat{B}/(\varrho^k)$, for $k \in \{1, \ldots, 6\}$. Then $D_k = FQ_{D_k}/J_{D_k}$, where $Q_{D_k}$ is the quiver of $D_k$ and $J_{D_k}$ is an ideal in the path algebra $FQ_{D_k}$ of $D_k$ over $F$. Since $D_1 = D(F, T^2_S)$, the quiver $Q_{D_1}$ contains a loop $C_1$ with $C_1^3$ not in $J_{D_1}$. Hence, for any $k \in \{1, \ldots, 6\}$, the quiver $Q_{D_k}$ contains a cycle $C_k$ of length $k$ such that the composition of any 3 consecutive arrows in $C_k$ does not belong to $J_{D_k}$. But then the cycle $C_k$ contains two vertices $x$ and $y$ such that the shortest paths $u$ from $x$ to $y$ and $v$ from $y$ to $x$ in $C_k$ are of length at most 3, and hence do not belong to $J_{D_k}$. Then we conclude that mod $D_k$ contains a short cycle $P_{D_k}(x) \rightarrow P_{D_k}(y) \rightarrow P_{D_k}(x)$, with $P_{D_k}(x)$ and $P_{D_k}(y)$ being the indecomposable projective modules in mod $D_k$ at the vertices $x$ and $y$. In particular, since $A = \hat{B}/G = \hat{B}/(\varrho^p)$, mod $A$ contains a short cycle of indecomposable projective modules.

Summing up, we have proved that (i) implies (ii). \hfill \Box

6. Proof of the Theorem.

Let $A$ be a selfinjective algebra over a field $K$. The implication (ii) \Rightarrow (i) follows from Theorem 5.7. We will show that (i) implies (ii). Assume that mod $A$ has no short cycles. If $A$ is a Nakayama algebra then the statement (ii) follows from Proposition 4.3. Hence, assume that $A$ is not a Nakayama algebra. Applying Theorem 3.3 and Lemma 3.2, we conclude that $\Gamma_A$ admits a semiregular double $\tau_A$-rigid stable slice $\Delta$. We note that $\Delta$ is a Dynkin quiver, by the Riedtmann-Todorov theorem. Let $M$ be the direct sum of the indecomposable modules lying on $\Delta$, $I = r_A(M)$, and $B = A/I$. Then it follows from Theorem 3.1 that $H = \text{End}_B(M)$ is a hereditary algebra of type $\Delta$, $T = D(M)$ is a tilting module in mod $H$ with $\text{End}_H(T)$ isomorphic to $B$, and consequently $B$ is a tilted algebra of Dynkin type $\Delta$. Further, applying Theorem 3.4 again, we obtain that $I$ is a deforming ideal of $A$ with $r_A(I) = eI$ for an idempotent $e$ of $A$, being a residual identity of $B = A/I$, the algebra $A[I]$ is isomorphic to an orbit algebra $\hat{B}/(\psi \nu_{\hat{B}})$ for a positive automorphism $\psi$ of $\hat{B}$, and $A$ is socle equivalent to $A[I]$. We claim that $A$ is isomorphic to $A[I]$. Since $IeI = 0$, it is enough to show, by Proposition 2.6, that $e_i \neq e_{\nu(i)}$ for any primitive summand $e_i$ of $e$, where $\nu$ is the Nakayama permutation of $A$. Suppose that $e_i = e_{\nu(i)}$ for a primitive summand $e_i$ of $e$. Consider the indecomposable projective module $P_i = e_iA$ in mod $A$ given by $e_i$. Then, by the definition of Nakayama permutation, the equality $e_i = e_{\nu(i)}$ implies that $\text{top}(P_i) \cong \text{soc}(P_i)$. But then we have
in mod $A$ a short cycle

$$P_i \overset{f}{\longrightarrow} P_i \overset{f}{\longrightarrow} P_i$$

where $f$ is a homomorphism whose image is $\text{soc}(P_i)$, which contradicts the assumption imposed on $A$. Therefore, $A$ is isomorphic to $A[I]$, and consequently $A$ is isomorphic to the orbit algebra $\hat{B}/(\psi\nu\hat{B})$. Applying now Theorem $5.7$ we conclude that $A$ is isomorphic to an orbit algebra $\hat{B}/(\varphi\nu\hat{B})$ for a strictly positive automorphism $\varphi$ of $\hat{B}$. Summing up, we have proved that (i) implies (ii).

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