Divides with cusps and Kirby diagrams for line arrangements

Sakumi Sugawara,∗ Masahiko Yoshinaga†

Dedicated to the memory of Prof. Stefan Papadima

July 6, 2021

Abstract

The complement of a complexified real line arrangement is an affine surface. It is classically known that such a space has a handle decomposition up to 2-handles. We will describe the handle decomposition induced from Lefschetz hyperplane section theorem for such a space. To describe the Kirby diagram, we introduce the notion of the divide with cusps which is a generalization of the divide introduced by A’Campo.

Keywords: line arrangements, divides, Kirby diagrams

MSC classes: 32S50, 32Q55, 52C35

1 Introduction

Let A = {H1, . . . , Hn} be a set of hyperplanes in a complex vector space Cℓ. The complement M(A) = Cℓ \bigcup_{H \in A} H is an important topological space that has been studied from the topological and combinatorial perspective ([OT92]). One of the most peculiar properties of M(A) is the so-called minimality, that is, M(A) is homotopy equivalent to a finite CW complex such that the number of k-cells is equal to the k-th Betti number bk(M(A)) for all k ≥ 0 ([R02, DP03] (see also [F93] for the case ℓ = 2). Note that the original proof did not give a precise description of attaching maps of minimal CW complexes. Later several approaches to this problem were developed for complexified real arrangements. For example, descriptions via Lefschetz hyperplane section theorem ([Y07] and via discrete Morse theory ([SS07] are known.

When ℓ = 2, the complement M(A) is a complex smooth affine surface. It has been well known that such a space is a 4-manifold which has a handle decomposition by 0-, 1-, and 2-handles. Such handle decompositions are completely described by Kirby diagrams ([Ak16, KS99, GS99]). However, as far as the authors know, there are no results available in the literature concerning explicit handle decompositions and Kirby diagrams for M(A). (Recently, the handle decompositions for the complement of real 1-dimensional subspaces in Rℓ were obtained [IO20]..)

The purpose of the present paper is to give a description of handle decompositions and Kirby diagrams of M(A) for complexified real line arrangements.

Our approach consists of two steps. The first step is describing the handle decomposition explicitly ([1]). This part is a refinement of a previous description ([Y07, Y12]) of the attaching

∗Department of Mathematics, Graduate School of Science, Hokkaido University, North 10, West 8, Kita-ku, Sapporo 060-0810, JAPAN E-mail: sugawara.sakumi.f5@elms.hokudai.ac.jp
†Department of Mathematics, Faculty of Science, Hokkaido University, North 10, West 8, Kita-ku, Sapporo 060-0810, JAPAN E-mail: yoshinaga@math.sci.hokudai.ac.jp
maps of 2-cells on 1-skeletons. We will describe the attaching of 2-handles using piecewise linear maps.

The second step is describing the Kirby diagram explicitly. For this purpose, we introduce the notion of the \textit{divide with cusps}.

Simply speaking, a \textit{divide} \( C \) is a union of immersed curves in the 2-dimensional unit disk \( D^2 \). Although the divide was originally introduced in the context of complex plane curve singularities, it is of interest in low dimensional topology since one can associate a link \( L(C) \) in \( S^3 \) to a divide \( C \) ([AC98]). The notion of the divide can be considered as a “2-dimensional way” of describing certain links. There are also several generalizations of the notion of divide, e.g., oriented divides and free divides ([GI02a, GI02b]), etc. (In particular, the notion of cusp in the present paper may be considered as “divisors” (in the sense of [GI02b]) equipped with half-integer signs \( \pm \frac{1}{2} \).

The \textit{divide with cusps} is, roughly speaking, a generalization that allows cusps in the union of curves \( C \) in the \( D^2 \). Since the tangent space is well-defined also at a cusp, one can associate a link as in the case of divides. Our main result asserts that there exists a divide with cusps such that its link represents the Kirby diagram of \( M(\mathcal{A}) \).

The paper is organized as follows. In §2 we prepare piecewise linear descriptions of the 3-sphere \( S^3 \) and certain circles in it which will be used later. In §3 we define divides with cusps and associated links. In §4 we give a piecewise linear description of handle decompositions for the complements to complexified real line arrangements. Here Kirby diagrams (the framed links consisting of boundaries of carved disks and attaching circles of 2-handles) are described in terms of piecewise linear curves introduced in the previous section. In §5 we describe the Kirby diagram using divide with cusps.

## 2 Standard and PL models for \( S^3 \)

Let \( D^2 \subset \mathbb{R}^2 \) be the 2-disk. The starting point of the theory of divides ([AC98]) is to describe the 3-sphere \( S^3 \) in terms of tangent vectors on \( D^2 \) as follows.

\[
\{(x, y) \mid x \in D^2, y \in T_x \mathbb{R}^2, |x|^2 + |y|^2 = 1\},
\]

where \( x = (x_1, x_2), y = (y_1, y_2) \), and \( |x| = \sqrt{x_1^2 + x_2^2} \). We will also identify \( T \mathbb{R}^2 \) with \( \mathbb{C}^2 \) by \((x, y) \mapsto x + \sqrt{-1} y\). Therefore, we sometimes call \( x \) (resp. \( y \)) the real part (resp. imaginary part) of \((x, y)\).

In this paper, we will use the following piecewise linear (rectangular) model for \( S^3 \). For \( x = (x_1, x_2) \), let us denote the maximum norm by \( ||x||_\infty := \max\{|x_1|, |x_2|\} \). Let \( R > 0 \) be a positive real number. The rectangle \( \text{Rect}(R, 1) \) is defined as

\[
\text{Rect}(R, 1) := [-R, R] \times [-1, 1] = \{(x = (x_1, x_2) \in \mathbb{R}^2 \mid -R \leq x_1 \leq R, -1 \leq x_2 \leq 1\}.
\]

For \( x \in \text{Rect}(R, 1) \), let \( \delta(x) \) be the distance between \( x \) and the boundary \( \partial \text{Rect}(R, 1) \) with respect to the norm \( || \cdot ||_\infty \). In other words, \( \delta(x) = \min\{x_1 + R, R - x_1, x_2 + 1, 1 - x_2\} \). We will use the following piecewise linear presentation \( S^3(R, 1) \) of the sphere (see Figure 1).

\[
S^3 \cong S^3(R, 1) := \{(x, y) \mid x \in \text{Rect}(R, 1), y \in T_x \mathbb{R}^2, ||y||_\infty = \delta(x)\}. \quad (1)
\]

Next, we define a special family of circles which are defined as images of piecewise linear maps from the boundary of the square \( \partial \text{Rect}(1, 1) \). Let \( a_1 = (a_1, b), a_2 = (a_2, b) \in \text{Rect}(R, 1) \) be two points with the same height \( b \). For simplicity, we assume \( a_1 \leq a_2, 0 \leq |a_1|, |a_2| \ll R \), and \( 0 \leq b < 1 \). We decompose the boundary \( \partial \text{Rect}(1, 1) \) into four edges \( R_1, R_2, R_3, R_4 \) as shown in Figure 2.
Figure 1: Rect(5, 1) and a circle $S(\mathbf{a}, \mathbf{a})$ with $\mathbf{a} = (2, 0.3)$

Figure 2: decomposition into $R_1, \ldots, R_4$

Let $\mathbf{v} = (v_1, v_2) \in \partial \text{Rect}(1, 1) = R_1 \cup R_2 \cup R_3 \cup R_4$. Define the map $F : \partial \text{Rect}(1, 1) \rightarrow S^3(R, 1)$ as follows.

$$F(\mathbf{v}) = \begin{cases} 
    \left( \left( \frac{a_2-a_1}{2}v_1 + \frac{a_1+a_2}{2}, b \right), (1-b)\mathbf{v} \right), & v_2 = -1 (R_1), \\
    \left( \left( \frac{a_2-a_1}{2}v_2 + \frac{a_1+a_2}{2}, b \right), (1-b)\mathbf{v} \right), & v_1 = 1 (R_2), \\
    \left( \left( \frac{a_1-a_2}{2}v_1 + \frac{a_1+a_2}{2}, b \right), (1-b)\mathbf{v} \right), & v_2 = 1 (R_3), \\
    \left( \left( \frac{a_2-a_1}{2}v_2 + \frac{a_1+a_2}{2}, b \right), (1-b)\mathbf{v} \right), & v_1 = -1 (R_4),
\end{cases}$$

(See also Figure 3). We denote the image of this map by $S(\mathbf{a}_1, \mathbf{a}_2) = F(R_1) \cup F(R_2) \cup F(R_3) \cup F(R_4)$.

Since the argument of the tangent vector is preserved, $S(\mathbf{a}_1, \mathbf{a}_2)$ is an embedded (piecewise linear) circle. When $\mathbf{a}_1 = \mathbf{a}_2$, $S(\mathbf{a}, \mathbf{a})$ is a circle (square) as in Figure 1.

Figure 3: $F(R_1), F(R_2), F(R_3)$, and $F(R_4)$

For simplicity, we draw $S(\mathbf{a}_1, \mathbf{a}_2)$ as in Figure 4.

**Proposition 2.1.** $S(\mathbf{a}_1, \mathbf{a}_2)$ is isotopic to $S(\mathbf{a}_1, \mathbf{a}_1)$. Furthermore, $S(\mathbf{a}_1, \mathbf{a}_2)$ is a trivial knot.
In this section, we introduce the notion of the divide with cusps and associated links.

**3 Divides with cusps**

In this section, we introduce the notion of the divide with cusps and associated links.

Let $X$ be a compact 1-dimensional manifold. Then, $X$ is expressed as a disjoint union of finitely many intervals and circles $\bigcup_j I_j \sqcup \bigcup_k S_k$, where $I_j$ and $S_k$ are intervals and circles, respectively. We have $\partial X = \bigcup_j \partial I_j$.

**Definition 3.1.** A divide with cusps $C$ is the image of a continuous map $\alpha : (X, \partial X) \to (D^2, \partial D^2)$ satisfying the following conditions.

(i) $\alpha^{-1}(\partial D) = \partial X$, $\alpha|_{\partial X}$ is injective, and $C$ is transversal to $\partial D$.

(ii) There are finitely many points $p_1, \ldots, p_s$ in the interior of $X$ such that

- $\alpha$ is an immersion on $X \setminus \{p_1, \ldots, p_s\}$ and $\alpha(p_i)$ is a cusp of $C$.
- $\alpha^{-1}(\alpha(p_i)) = \{p_i\}$.

(iii) Except for cusps and double points, there are no singularities.
Let $C$ be a divide with cusps. We consider tangent space at $x = (x_1, x_2) \in C$. If $x$ is neither double points nor cusps, then $T_x C$ is as usual. Note that there is a natural tangent space $T_x C$ at a cusp (Figure 7). At a double point, $T_x C$ is a union of two tangent spaces.

To a divide with cusps $C$, the link $L(C)$ is defined as

$$L(C) = \{(x, y), |x| \in C, y \in T_x(C), |x|^2 + |y|^2 = 1\} \subset S^3.$$  

By definition, when $x \in C$ is in the interior of $D^2$, there are exactly two tangent vectors $\pm y \in T_x C$ which is contained in $L(C)$. The vector $y$ is determined by the argument. Hence (over the interior) the link $L(C)$ can be drawn in $C \times S^1$. Note that two kinds of cusps are corresponding to positive and negative half twists (Figure 8).

**Remark 3.2.** The moves of divides with cusps in Figure 9 do not change the isotopy type of the corresponding links. The proof is similar to that of [CP00 Lemma 1.3].
Example 3.3. Let $C_1$, $C_2$, and $C_3$ be, respectively, a smooth circle, a circle with an outward cusp, and a circle with an inward cusp. Then $L(C_1)$, $L(C_2)$, and $L(C_3)$ are, respectively, the Hopf link, the trivial knot, and the trefoil. (Figure 10.)

4 Handle decomposition of $M(A)$

4.1 Setting

A real line arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ is a finite set of affine lines in the affine plane $\mathbb{R}^2$. Each line is defined by some affine linear form

$$\alpha_H(x_1, x_2) = ax_1 + bx_2 + c = 0,$$

with $a, b, c \in \mathbb{R}$ and $(a, b) \neq (0, 0)$. A connected component of $\mathbb{R}^2 \setminus \bigcup_{H \in \mathcal{A}} H$ is called a chamber. The set of all chambers is denoted by $\text{ch}(\mathcal{A})$. The affine linear equation (2) defines a complex
Proposition 4.1. Based on this description, we have the following.

Proof. (1) Assume \((x, y) \in M(A)\). Then for any \(t \in \mathbb{R}\), \((x + ty, y) \in M(A)\).

(2) Suppose \(y\) is not parallel to any of \(H \in A\). Then for any \(x \in \mathbb{R}^2\), \((x, y) \in M(A)\).

Now let us fix a generic line \(F\) such that \(F\) does not separate intersections of \(A\). We can choose coordinates \(x_1, x_2\) so that \(F\) is given by \(x_2 = 0\) and all intersections of \(A\) are contained in the upper half-plane \(\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}\). Let

\[
\text{ch}_F(A) := \{C \in \text{ch}(A) \mid C \cap F = \emptyset\}.
\]

We set \(H_1 \cap F\) has coordinates \((a_i, 0)\). By changing the numbering of lines and signs of the defining equation \(\alpha_i\) of \(H_i \in A\) we may assume that

- \(a_1 < a_2 < \cdots < a_n\), and
- the half-plane \(H_i^- = \{\alpha_i < 0\}\) contains the negative direction (leftward) of \(F\) (equivalently, \(H_i^+ = \{\alpha_i > 0\}\) contains the positive direction).

Let us recall the construction in [Y07] briefly. There exists a (after suitable compactification near the boundary) Morse function \(\varphi : M(A) \longrightarrow \mathbb{R}_{\geq 0}\) such that

- \(\varphi^{-1}(0) = M(A) \cap F_C\),
- for each \(C \in \text{ch}_F(A)\) there is a unique critical point \(p_C \in C\),
- the chamber \(C\) is the stable manifold of \(p_C\),
- there are no other critical points.

Based on these facts, it was proved that \(M(A) \setminus \bigcup_{C \in \text{ch}_F(A)} C\) is homeomorphic to an open neighborhood of \(F_C \cap M(A)\). The homeomorphism/retraction was originally constructed by using the gradient flow ([Y07, Corollary 5.1.5]). Below we construct a piecewise linear retraction.

After appropriate changes of coordinates, we may further assume the following. (See Figure [Y11])

- There exists a positive real number \(R_0 > 0\) such that all the intersection points of \(A\) are lying in \(\{(x_1, x_2) \mid 1 < x_2 < R_0\}\).
- Denote by \(\theta_i\) the angle between the \(x_1\)-axis (from the positive side) and \(H_i\). Then \(\frac{\pi}{4} \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n \leq \frac{3\pi}{4}\).
- Take a sufficiently large real number \(R > 0\) \((R \gg R_0)\) so that all intersections \(H_i \cap \{x_2 = R_0\}, i = 1, \ldots, n\), are contained in \(\{-R + R_0 < x_1 < R - R_0\}\).
Let $\tilde{H}_C$ be a small tubular neighborhood of $H_C$. Define $M_1$ as

$$M_1 = \left( \mathbb{C}^2 \setminus \bigcup_{H \in A} \tilde{H}_C \right) \cap \{(x, y) \mid x \in [-R, R] \times [-1, R_0], ||y||_\infty \leq R\},$$

which is a compact 4-dimensional manifold with boundary whose interior is homeomorphic to $M(A)$ (see [Dur83]).

Next, we remove tubular neighborhoods of each chamber $C$. Let $\tilde{C}$ be the tubular neighborhood of $C$ in $M(A)$. Define $M_2$ as

$$M_2 = \left( M_1 \setminus \bigcup_{C \in \text{ch}(A)} \tilde{C} \right) \cup \{(x, y) \mid x \in \text{Rect}(R, 1), ||y||_\infty \leq R\} \setminus \bigcup_{H \in A} \tilde{H}_C.$$

Note that if $C \cap F \neq \emptyset$, then the addition of $\tilde{C}$ does not change the topology. On the other hand, $\tilde{C}$ for $C \in \text{ch}_F(A)$ corresponds to a 2-handle. Hence $M(A)$ is obtained from $M_2$ by attaching 2-handles $\tilde{C}$ for each $C \in \text{ch}_F(A)$.

Let $\rho : [1, R_0] \to [0, 1]$ be the function defined by $\rho(h) = \frac{h - 1}{R_0 - 1}$. Next, we define

$$M_3 = \left\{ (x, y) \in M_2 \mid \begin{array}{l} \text{If } x_2 \geq 1, \text{ then } \\
\text{x_2 \leq x_1 + R, x_2 \leq -x_1 + R, } \\
\text{and } \rho(x_2) \leq ||y||_\infty \leq 1 \end{array} \right\}.$$

Note that the inequalities $x_2 \leq x_1 + R, x_2 \leq -x_1 + R$ mean that the real part of the point is below the blue dashed lines in Figure 11. Since $M_3$ is obtained from $M_2$ just by deforming near the boundary, $M_2$ and $M_3$ are homeomorphic. Let us define $M_4$ as

$$M_4 := \{(x, y) \mid x \in \text{Rect}(R, 1), y \in T_x \mathbb{R}^2, ||y||_\infty \leq \delta(x)\} \setminus \bigcup_{H \in A} \tilde{H}_C.$$

(c. f. Figure 12)

Now we will construct an explicit contraction from $M_3$ to $M_4$. 

![Figure 11: Setting](image-url)
4.2 Explicit contraction

As we noted, $M_2, M_3$, and $M_4$ are homeomorphic. Define the map $\sigma : M_3 \to M_4$ by

$$\sigma(x, y) := \begin{cases} 
(x, y), & \text{if } x_2 \leq 1 - ||y||_{\infty}, \\
\left( x_1 - \frac{y_1}{y_2}(x_2 - 1 + ||y||_{\infty}), 1 - ||y||_{\infty} - \frac{y_1}{y_2}, y \right), & \text{if } x_2 \geq 1 - ||y||_{\infty} \text{ and } |y_1| \leq |y_2|,
\end{cases}$$

(8)

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ (Figure 13). Note that the first case is for the edges $R_1$ and $R_3$, the second case is for the edges $R_2$ and $R_1$, and the third case is for $(x, y) \in M_4$.

Let $\sigma_t(x, y) = (1 - t) \cdot (x, y) + t \cdot \sigma(x, y)$. It is straightforward that if $(x, y) \in M_3$, then $\sigma_t(x, y) \in M_3$ for $0 \leq t \leq 1$. Since $\sigma_0$ is the identity on $M_3$ and $\sigma_1 = \sigma$, $\sigma_t (0 \leq t \leq 1)$ gives a contraction of $M_3$ to $M_4$.

4.3 Attaching maps of 2-handles

In this section, we describe the Kirby diagrams (the framed links of attaching circles and the boundaries of carved disks) associated with the handle decompositions obtained in the previous section. Our 0-handle, namely $D^4$, is the piecewise linear disk that we introduced in §2

$$D^4(R, 1) \approx \{(x, y) \mid x \in R, y \in T_x \mathbb{R}^2, ||y||_{\infty} \leq \delta(x)\}. \quad (9)$$

Note that $\partial D^4(R, 1) = S^3(R, 1)$ and $D^4(R, 1) \cap M_1 = M_4$.

Next, we consider 1-handlebody. Note that the intersection $D^4(R, 1) \cap M_1$ is equal to $D^4(R, 1) \setminus \bigcup_{i=1}^{n} H_i$. The intersection $D^4(R, 1) \cap H_i$ is homeomorphic to a disk whose boundary $\partial(D^4(R, 1) \cap H_i) = S^3(R, 1) \cap H_i$ is a circle. This circle is presented as “dotted circle” in the Kirby diagram (AK16, KB10, GS99). We note that $S^3(R, 1) \cap H_i$ is the knot corresponding to the divide $\overline{H}_i := H_i \cap R$. Thus the Kirby diagram of the 1-handlebody is the link associated with the divide consisting of $n$ intervals $\{\overline{T}_1, \ldots, \overline{T}_n \}$ in $R$ (Figure 14).

From the description in the previous section, to each chamber $C \in \mathcal{A}$, the union of the neighborhood of $C$

$$\{(x, y) \mid x \in C, ||y||_{\infty} \leq \rho(x_2)\} \quad (10)$$

and the trajectories of the contraction

$$\{\sigma_t(x, y) \mid x \in C, ||y||_{\infty} = \rho(x_2), 0 \leq t \leq 1\} \quad (11)$$

gives an explicit 2-handle $h^2(C)$ attached to $M_4$ (Figure 12).
Figure 13: $\sigma : M_3 \to M_4$

Figure 14: Divide whose link (with dots) represents the 1-handlebody
Let \( P = (x_1, x_2) \in C \). The real part of \( \sigma(x_1, x_2, -\rho(x_2), -\rho(x_2)) \) is \( (x_1 - x_2 + 1 - \rho(x_2), 1 - \rho(x_2)) \) which we denote by \( a_1(P) \). Similarly, the real part of \( \sigma(x_1, x_2, \rho(x_2), -\rho(x_2)) \) is \( (x_1 + x_2 - 1 + \rho(x_2), 1 - \rho(x_2)) \) which we denote by \( a_2(P) \).

**Theorem 4.2.** The attaching circle of the 2-handle \( h^2(C) \) for \( C \in \text{ch}_F(A) \) is equal to \( S(a_1, a_2) \) with zero framing. (Figure 15)

**Proof.** The first assertion is clear from the definition of \( \sigma \). The framing is computed as the linking number of \( S(a_1(P), a_2(P)) \) and \( S(a_1(P'), a_2(P')) \), where \( P' = P + (0, \varepsilon) \). By Proposition 2.3 the linking number is zero.

\[
\delta_i(C) = \begin{cases} 
+1, & \text{if } \alpha_i(C) > 0, \\
-1, & \text{if } \alpha_i(C) < 0. 
\end{cases} \tag{12}
\]

We note that \( \delta_i(C) = +1 \) (resp. \( \delta_i(C) = -1 \)) is equivalent to that \( C \) is sitting in the right (resp. left) side of \( H_i \). To each \( C = C_i \in \text{ch}_F(A) \), define a curve \( \gamma(C) \) with cusps by joining the following parts (these parts are drawn in the strip \( h_i' - \varepsilon < x_2 < h_i' + \varepsilon \) in \( \text{Rect}(R, 1) \)).

(i) In the left side of \( H_1 \), if \( \delta_1(C) = +1 \), then draw a leftward cusp, if \( \delta_1(C) = -1 \), then draw a round curve as in Figure 16.

(ii) According to \( (\delta_i(C), \delta_{i+1}(C)) = (+1, -1), (-1, +1), \) or \( \delta_i = \delta_{i+1} \), we draw a (two) curves between \( H_i \) and \( H_{i+1} \) as in Figure 17.

(iii) In the right side of \( H_n \), if \( \delta_n(C) = +1 \), then draw a round curve, if \( \delta_n(C) = -1 \), then draw a rightward cusp as in Figure 18.

In the next section, we describe \( S(a_1(P), a_2(P)) \) as a link associated with a divide with cusps.

## 5 Kirby diagrams using divides with cusps

### 5.1 Main result

Let \( b = |\text{ch}_F(A)| \) and \( \text{ch}_F(A) = \{C_1, \ldots, C_b\} \). For each \( 1 \leq s \leq b \), choose a point \( P_s = (k_s, h_s) \in C_s \). We may assume that \( 1 < h_1 < \cdots < h_b < R_0 \). We also choose \( 0 < h_i' < 1 \) and \( \varepsilon > 0 \) (sufficiently small) such that \( 1 > h_i' > \cdots > h_b' > 0 \), \( |h_i' - h_{i+1}'| > 2\varepsilon \) and \( \varepsilon < h_b' < h_1' < 1 - \varepsilon \). (E.g., \( h_i' = 1 - \rho(h_i) \))

We define the sign \( \delta_i(C) \) of \( C \in \text{ch}_F(A) \) with respect to \( H_i \) as follows.

\[
\delta_i(C) = \begin{cases} 
+1, & \text{if } \alpha_i(C) > 0, \\
-1, & \text{if } \alpha_i(C) < 0. 
\end{cases} \tag{12}
\]

We note that \( \delta_i(C) = +1 \) (resp. \( \delta_i(C) = -1 \)) is equivalent to that \( C \) is sitting in the right (resp. left) side of \( H_i \). To each \( C = C_i \in \text{ch}_F(A) \), define a curve \( \gamma(C) \) with cusps by joining the following parts (these parts are drawn in the strip \( h_i' - \varepsilon < x_2 < h_i' + \varepsilon \) in \( \text{Rect}(R, 1) \)).

(i) In the left side of \( H_1 \), if \( \delta_1(C) = +1 \), then draw a leftward cusp, if \( \delta_1(C) = -1 \), then draw a round curve as in Figure 16.

(ii) According to \( (\delta_i(C), \delta_{i+1}(C)) = (+1, -1), (-1, +1), \) or \( \delta_i = \delta_{i+1} \), we draw a (two) curves between \( H_i \) and \( H_{i+1} \) as in Figure 17.

(iii) In the right side of \( H_n \), if \( \delta_n(C) = +1 \), then draw a round curve, if \( \delta_n(C) = -1 \), then draw a rightward cusp as in Figure 18.

11
\[ \delta_i(C) > 0 \quad \text{or} \quad \delta_i(C) < 0 \]

Figure 16: The curve \( \gamma(C) \) at the left side of \( H_1 \).

\[ \delta_n(C) < 0 \quad \text{or} \quad \delta_n(C) > 0 \]

Figure 18: The curve \( \gamma(C) \) at the right side of \( H_n \).
**Theorem 5.1.** The link corresponding to \(\{\gamma(C_1), \ldots, \gamma(C_b)\}\) (with zero framing) present the attaching circles of the 2-handles of \(M(A)\).

We will prove Theorem 5.1 in §5.2.

The curves \(\gamma(C_1), \ldots, \gamma(C_b)\) sometimes contain redundant cusps. Therefore, we define reduced curves \(\gamma'(C_1), \ldots, \gamma'(C_b)\) which are obtained from the previous curves by the moves formulated in Remark 3.2.

**Definition 5.2.** For a chamber \(C \in \text{ch}_F(A)\), let \(m_1 := \min \{i \mid \delta_i(C) = -1\}\) and \(m_2 := \max \{i \mid \delta_i(C) = +1\}\). Let us define the curve \(\gamma'(C)\) by joining above (i), (ii), (iii) for the lines \(H_{m_1}, H_{m_1+1}, \ldots, H_{m_2}\) instead of \(H_1, \ldots, H_n\). We call \(\gamma'(C)\) the reduced curve of \(\gamma(C)\). (Figure 20)

**Proposition 5.3.** The divides \((\gamma'(C), \overline{H}_1, \ldots, \overline{H}_n)\) and \((\gamma(C), \overline{H}_1, \ldots, \overline{H}_n)\) give isotopic links in \(S^3(R, 1)\).

**Proof.** If \(m_1 = 1, m_2 = n\), then \(\gamma'(C) = \gamma(C)\). Suppose \(m_1 > 1\) (or \(m_2 < n\)). Then \(\delta_1(C) = +1\). Hence at the left side of \(\overline{H}_1, \gamma(C)\) has a leftward cusp. Using the moves in Remark 3.2 (Figure 9), \(\gamma(C)\) is deformed to \(\gamma'(C)\). \(\square\)

**Example 5.4.** Let us consider 4-lines \(A = \{H_1, H_2, H_3, H_4\}\) as in Figure 19. Let \(\text{ch}_F(A) = \{C_1, \ldots, C_6\}\) and fix \(P_i \in C_i\) as shown in the figure. Then the curves \(\gamma_i := \gamma(C_i)\) \((i = 1, \ldots, 6)\) are as in Figure 19 (below). The curves \(\gamma_i, i \geq 3,\) are reduced. However, \(\gamma_1\) and \(\gamma_2\) have reductions.

![Figure 19: Generic four lines and its divide with cusps](image)

The reduced ones \(\gamma'_1, \gamma'_2\) are as in Figure 20.
5.2 Proof of Theorem 5.1

In this section, we prove Theorem 5.1. Let \( P \in C \). Here we will show that the link \( \mathbb{S}(a_1(P),a_2(P)) \) is isotopic to the link determined by the curve constructed as in §5.1 (for \( a_1(P) \) and \( a_2(P) \)). The latter can be deformed to \( \gamma(C) \) (and to \( \gamma(C) \)) by the moves in Remark 5.2 (Figure 21).

First, we consider the image of \( R_2 \) and \( R_4 \) in \( \mathbb{S}(a_1(P),a_2(P)) \). The argument \( \theta \) of tangent vectors satisfies \( -\frac{\pi}{4} \leq \theta < \frac{\pi}{4} \) (for \( R_2 \)) or \( \frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4} \) (for \( R_4 \)). Hence by the assumptions on \( A \) (§4.1), the image of \( R_2 \) is isotopic to the constant vectors with argument \( \theta = 0 \), and that of \( R_4 \) is isotopic to \( \theta = \pi \). Therefore, the image of \( R_2 \cup R_4 \) is isotopic to a part of a simple closed curve corresponding to the tangent vectors of the upper line of \( \gamma(C) \).

Next, we consider the marginal case (the case corresponding to Figure 22). Suppose that \( a_1(P) \) is in between \( H_{i-1} \) and \( H_i \). Let us consider \( \sigma(R_4) \) and \( \sigma(R_4) \). We shift \( \sigma(R_4) \) upward by \( \varepsilon \), and \( \sigma(R_4) \) downward by \( \varepsilon \). By connecting two left most points by a segment with tangent vectors of argument \( \frac{3\pi}{4} \), we obtain a piecewise linear curve \( c : [0,1] \rightarrow \text{Rect}(R,1) \cap H_1^- \) (red segments in Figure 16) and tangent vectors \( \psi(t) \in T_{c(t)}\mathbb{R}^2 \). Note that \( \{(c(t),\psi(t)) \mid 0 \leq t \leq 1\} \) forms the (\( \varepsilon \)-shifted) image \( \sigma(R_4) \cup \sigma(R_1) \). Now suppose that \( \delta(C) = -1 \). Then \( \psi(0) \) is directing the negative half space \( H_i^- \) and \( \psi(1) \) is directing the positive half space \( H_i^+ \).

There exist a curve \( \tilde{c}(t) \) (the blue curve in Figure 22) and tangent vector \( \tilde{\psi}(t) \in T_{\tilde{c}(t)}\mathbb{R}^2 \) such that

- The curve \( \tilde{C} = \{\tilde{c}(t) \mid t \in [0,1]\} \) is smooth.
- \( \tilde{c}(0) = c(0), \tilde{c}(1) = c(1) \).
- \( \tilde{\psi}(t) \in T_{\tilde{c}(t)}\tilde{C}, \tilde{\psi}(t) \neq 0 \).
- Let \( \tilde{\psi}(t) = (\tilde{v}_1(t),\tilde{v}_2(t)) \). Then \( \tilde{v}_2(0) = \tilde{v}_2(1) = 0, v_1(0) < 0, v_1(1) > 0, \) and \( \tilde{v}_2(t) > 0 \) for \( 0 < t < 1 \).

Then we have
• both $\mathbf{v}(0)$ and $\tilde{\varphi}(0)$ are directing negative side of $H_i$,
• both $\mathbf{v}(1)$ and $\tilde{\varphi}(1)$ are directing the positive side of $H_i$, and
• for $0 < t < 1$, $v_2(t) < 0$ and $\tilde{v}_2(t) < 0$ hold.

Therefore, $s \cdot \tilde{\varphi}(t) + (1 - s) \cdot \mathbf{v}(t)$ is nonzero for all $0 \leq s \leq 1$, and $s \cdot (\tilde{\mathbf{c}}(t), \tilde{\varphi}(t)) + (1 - s) \cdot (\mathbf{c}(t), \mathbf{v}(t))$ $(0 \leq s \leq 1)$ gives an isotopy between $(\tilde{\mathbf{c}}(t), \tilde{\varphi}(t))$ and $(\mathbf{c}(t), \mathbf{v}(t))$. The other case $(\delta_i(C) = +1)$ is similar.

![Isotopy between \((\tilde{\mathbf{c}}(t), \tilde{\varphi}(t))\) and \((\mathbf{c}(t), \mathbf{v}(t))\)](image)

Figure 22: Isotopy between $(\tilde{c}(t), \tilde{\varphi}(t))$ and $(c(t), v(t))$

Next, we consider the case $\delta_i(C) = +1, \delta_{i+1}(C) = -1$. The intersection $[a_1(P), a_2(P)] \cap \{\alpha_i \geq 0, \alpha_{i+1} \leq 0\}$ is an interval. Fix a linear parametrization $c : [0, 1] \rightarrow [a_1(P), a_2(P)] \cap \{\alpha_i \geq 0, \alpha_{i+1} \leq 0\}$. Then the image of $\sigma(R_3)$ is expressed as $\{(c(t), v(t)) | t \in [0, 1]\}$, where $v(t) \in T_{c(t)}\mathbb{R}^2$ which points to the fixed point $P \in C$ (i.e., $v(t)$ is a positive scalar multiple of the vector $c(t) \cdot \vec{P}$). In particular, since $\delta_i(C) = +1$, $v(0) \in T_{c(0)}\mathbb{R}^2$ is directing to the positive side of $H_i$. Similarly, $v(1) \in T_{c(1)}\mathbb{R}^2$ is directing to the negative side of $C_{i+1}$. Let $v(t) = (v_1(t), v_2(t))$. Then we also have $v_2(t) > 0$ for all $t \in [0, 1]$. There exists a curve $\tilde{c}(t)$ and tangent vector $\tilde{\varphi}(t) \in T_{c(t)}\mathbb{R}^2$ such that (Figure 23)

• The curve $\tilde{C} = \{\tilde{c}(t) | t \in [0, 1]\}$ has a unique upward cusp.
• $\tilde{c}(0) = c(0), \tilde{c}(1) = c(1)$.
• $\tilde{\varphi}(t) \in T_{\tilde{c}(t)}\tilde{C}$.
• Let $\tilde{\varphi}(t) = (\tilde{v}_1(t), \tilde{v}_2(t))$. Then $\tilde{v}_2(0) = \tilde{v}_2(1) = 0$, and $\tilde{v}_2(t) > 0$ for $0 < t < 1$.

Then we have

• both $\mathbf{v}(0)$ and $\tilde{\varphi}(0)$ are directing the positive side of $H_i$,
• both $\mathbf{v}(1)$ and $\tilde{\varphi}(1)$ are directing negative side of $H_{i+1}$, and
• for $0 < t < 1$, $v_2(t) > 0$ and $\tilde{v}_2(t) > 0$ hold.

Therefore, $s \cdot \tilde{\varphi}(t) + (1 - s) \cdot \mathbf{v}(t)$ is nonzero for all $0 \leq s \leq 1$, and $s \cdot (\tilde{c}(t), \tilde{\varphi}(t)) + (1 - s) \cdot (c(t), v(t))$ $(0 \leq s \leq 1)$ gives an isotopy between $(\tilde{c}(t), \tilde{\varphi}(1))$ and $(c(t), v(t))$. Other cases $(\delta_i(C), \delta_{i+1}(C)) = (1, -1), (1, 1), (-1, -1)$ are similar. This completes the proof of Theorem 5.1.
5.3 Further examples

Let $\Sigma_{g,n}$ be the oriented surface of genus $g$ with $n$ boundary components. Since $\Sigma_{0,2}$ is the annulus, the interior of $\Sigma_{0,2} \times \Sigma_{0,2}$ is homeomorphic to $(\mathbb{C}^*)^2$ which is the complement of the arrangement of two lines intersecting at one point. Hence, the minimal handle decomposition of $\Sigma_{0,2} \times \Sigma_{0,2}$ can be expressed in terms of divide with cusps (Figure 24). The below are related examples.

**Example 5.5.** The space $\Sigma_{0,4} \times \Sigma_{0,2}$ is realized as the complexified complement of different arrangements $A_1 = \{x_1 = 0\}, \{x_2 = 0\}, \{x_1 = x_2\}, \{x_1 = -x_2\}$, and $A_2 = \{x_1 = 0\}, \{x_2 = 0\}, \{x_2 = 1\}, \{x_2 = 2\}$. (Figure 25)

**Example 5.6.** The space $\Sigma_{0,4} \times \Sigma_{0,3}$ is realized as the complexified complement of the arrangement $A = \{x_1 = 0\}, \{x_1 = 1\}, \{x_2 = 0\}, \{x_2 = 1\}, \{x_2 = 2\}$. (Figure 26)

**Acknowledgements.** Masahiko Yoshinaga was partially supported by JSPS KAKENHI Grant Numbers JP19K21826, JP18H01115. The authors thank Prof. Masaharu Ishikawa for helpful
Figure 26: A divide with cusps for $\Sigma_{0,4} \times \Sigma_{0,3}$

comments on the first draft. They also thank the referee for careful reading and comments to improve the paper.

References

[AC98] N. A’Campo, Generic immersions of curves, knots, monodromy and Gordian number. *Inst. Hautes Études Sci. Publ. Math.* No. 88 (1998), 151-169.

[Ak16] S. Akbulut, 4-manifolds. Oxford Graduate Texts in Mathematics, 25. *Oxford University Press, Oxford*, 2016. xii+262 pp.

[CM98] J. A. Calvo, K. C. Millett, Minimal edge piecewise linear knots. *Ideal knots*, 107–128, Ser. Knots Everything, 19, *World Sci. Publ., River Edge, NJ*, 1998.

[CP00] O. Couture, B. Perron, Representative braids for links associated to plane immersed curves. *J. Knot Theory Ramifications* 9 (2000), no. 1, 1–30.

[DP03] A. Dimca, S. Papadima, Hypersurface complements, Milnor fibers and higher homotopy groups of arrangements. *Ann. of Math. (2)* 158 (2003), no. 2, 473–507.

[Dur83] A. H. Durfee, Neighborhoods of algebraic sets. *Trans. Amer. Math. Soc.* 276 (1983), no. 2, 517-530.

[F93] M. Falk, Homotopy types of line arrangements. *Invent. Math.* 111 (1993), no. 1, 139–150.

[GI02a] W. Gibson, M. Ishikawa, Links and Gordian numbers associated with generic immersions of intervals. *Topology Appl.* 123 (2002), no. 3, 609-636.

[GI02b] W. Gibson, M. Ishikawa, Links of oriented divides and fibrations in link exteriors. *Osaka J. Math.* 39 (2002), no. 3, 681-703.
[GS99] R. E. Gompf, A. I. Stipsicz, 4-manifolds and Kirby calculus. Graduate Studies in Mathematics, 20. *American Mathematical Society, Providence, RI*, 1999. xvi+558 pp.

[IO20] G. Ishikawa, M. Oyama, Topology of complements to real affine space line arrangements. *J. Singul.* **22** (2020), 373-384.

[K89] R. Kirby, The topology of 4-manifolds. Lecture Notes in Mathematics, 1374. *Springer-Verlag, Berlin*, 1989. vi+108 pp.

[OT92] P. Orlik, H. Terao, Arrangements of Hyperplanes. Grundlehren Math. Wiss. **300**, Springer-Verlag, New York, 1992.

[R02] R. Randell, Morse theory, Milnor fibers and minimality of hyperplane arrangements. *Proc. Amer. Math. Soc.* **130** (2002), no. 9, 2737–2743.

[SS07] M. Salvetti, S. Settepanella, Combinatorial Morse theory and minimality of hyperplane arrangements. *Geom. Topol.* **11** (2007), 1733–1766.

[Y07] M. Yoshinaga, Hyperplane arrangements and Lefschetz’s hyperplane section theorem. *Kodai Math. J.* **30** (2007) no. 2, 157–194.

[Y12] M. Yoshinaga, Minimal stratifications for line arrangements and positive homogeneous presentations for fundamental groups. *Configuration Spaces: Geometry, Combinatorics and Topology*, 503-533, CRM Series, 14, Ed. Norm., Pisa, 2012.