Tomographic reconstruction of quantum states in $N$ spatial dimensions

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Most quantum tomographic methods can only be used for one-dimensional problems. We show how to infer the quantum state of a non-relativistic $N$-dimensional harmonic oscillator system by simple inverse Radon transforms. The procedure is equally applicable to finding the joint quantum state of several distinguishable particles in different harmonic oscillator potentials. A requirement of the procedure is that the angular frequencies of the $N$ harmonic potentials are incommensurable.

We discuss what kind of information can be found if the requirement of incommensurability is not fulfilled and also under what conditions the state can be reconstructed from finite time measurements. As a further example of quantum state reconstruction in $N$ dimensions we consider the two related cases of an $N$-dimensional free particle with periodic boundary conditions and a particle in an $N$-dimensional box, where we find a similar condition of incommensurability and finite recurrence time for the one-dimensional system.

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I. INTRODUCTION

"What do we know about the state of a physical system given a certain set of measurements?" While this question permeates all of physics it is particularly slippery in quantum physics. Here, measurements disturb a system and potentially alter the outcome of subsequent measurements. This obstacle is overcome by using a vast ensemble of uncorrelated and identical quantum systems, where one measurement is performed on each ensemble member whereafter this member is discarded. The ensemble must be chosen large enough to permit measurements of all the quantities of interest and to obtain statistically significant data about these quantities. Introducing ensembles, this way furthermore cements the role of the quantum state as the complete statistical information of the quantum system. Following this ensemble approach, we shall in the present paper consider the quantum state as being described by the density operator $\hat{\rho}$, which can describe both pure and mixed states. Our goal shall be to find this operator.

In principle, since the quantum state is completely characterized by its density operator’s matrix elements in a complete basis $\{|\lambda\rangle\}$, one could just measure (the real and imaginary values of) all these matrix elements $\langle \lambda | \hat{\rho} | \lambda' \rangle$ - i.e. the density matrix. However, such a general set of observations may be very difficult to perform experimentally, and instead it will be our aim to find the quantum state from experimentally realizable measurements. Specifically, we shall adopt the quantum tomographic approach where only measurements of the spatial distribution is made at different points of time.

From these diagonal elements of the density operator in the position representation, $\Pr(x,t) = \langle x' | \hat{\rho} | x \rangle$, and the known time evolution due to the Hamiltonian we obtain the full density matrix $\langle x' | \hat{\rho} | x \rangle$ or, equivalently, the phase space distribution $W(x,p)$.

Sofar, most methods in quantum state tomography have been concerned with systems with only one spatial dimension. We shall in this paper present the extension of two well-known methods of quantum state tomography to $N$ dimensions. In section II we shall consider the harmonic oscillator by a treatment similar to that in [1]. This is not a trivial extension as revealed by a simple consideration of the dimensionality of the sets of measurements and the quantum state: $\Pr(x,t)$ and $\langle x' | \hat{\rho} | x \rangle$ are both of dimensionality two in the spatial one-dimensional case, whereas in the $N$-dimensional case $\Pr(x_1, \ldots, x_N,t)$ is of dimension $N+1$, but the density matrix $\langle x'_1, \ldots, x'_N | \hat{\rho} | x_1, \ldots, x_N \rangle$ is of dimensionality $2N$.

In section III we will treat the case of free particles considered in one dimension in [3], but with periodic boundary conditions and in a box potential. We finally give a summary of the paper in section IV.

II. THE HARMONIC OSCILLATOR

It is shown in [2] that there is a $1:1$ correspondence between the density operator and the quantum characteristic function $\tilde{W}(\xi)$, where $\xi$ is a complex variable. This means that instead of directly finding $\hat{\rho}$ we may just as well find the quantum characteristic function. This is traditionally the main trick used in the quantum state reconstruction of the harmonic oscillator. Before proceeding to the multidimensional case we will briefly recapitulate this procedure in one dimension. The quantum characteristic function can be found from $\hat{\rho}$ by:

$$\tilde{W}(\xi) = \text{Tr} \left( e^{\xi \hat{a}^\dagger - \xi^* \hat{a}} \hat{\rho} \right)$$

(1)

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Under the harmonic oscillator Hamiltonian \( \hat{H} = \hbar \omega (\hat{a}^{\dagger} \hat{a} + 1/2) \) with angular frequency \( \omega \), the ladder operators in the Heisenberg picture evolve according to \( \hat{a}(t) = \hat{a} e^{-i \omega t} \), \( \hat{a}^{\dagger}(t) = \hat{a}^{\dagger} e^{i \omega t} \). Letting \( \theta = \omega t \), the position operator \( \hat{x} \) evolves according to:

\[
\hat{x}(\theta) = \cos(\theta) \hat{x} + \sin(\theta) \hat{p} = \frac{1}{\sqrt{2}} (\hat{a}^{\dagger} e^{i \theta} + \hat{a} e^{-i \theta}).
\]

Please note that we use dimensionless coordinates \( x \) and \( p \). We now make a change of variables in \( \xi \) from the complex number \( \xi = \frac{1}{\sqrt{2}} i \eta e^{i \theta} \):

\[
\tilde{w}(\eta, \theta) = \sqrt{2} \tilde{W}
\]

It will prove convenient to choose the variables to be in the intervals \(-\infty < \eta < \infty \) and \( 0 \leq \theta < \pi \). Note that the position distribution \( \text{Pr}(x, \theta) \) at the time \( \theta/\omega \) is a Fourier transform of the quantum characteristic function \( \tilde{w}(\eta, \theta) \):

\[
\text{Pr}(x, \theta) = \text{Tr} \left\{ \hat{\rho} \delta[\hat{x}(\theta) - x] \right\} = \text{Tr} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} d\eta \tilde{w}(\eta, \theta) e^{-i\eta x} \right\}
\]

By inverting this Fourier transformation, we can find the quantum characteristic function \( \tilde{w}(\eta, \theta) \) from the measured position distributions:

\[
\tilde{w}(\eta, \theta) = \int_{-\infty}^{\infty} dx \text{Pr}(x, \theta) e^{i\eta x}.
\]

This is the main equation of quantum tomography in one dimension: We can find the quantum state (through the quantum characteristic function) by observing the position distribution for \( 0 \leq \theta < \pi \) corresponding to one half period of the oscillator.

For completeness, we write the result in terms of the Wigner-function; the complex Fourier transform of the characteristic function \( \tilde{w}(\eta, \theta) \). The Wigner function is a quasi phase-space distribution whose marginals along rotated lines are the measured position distributions:

\[
W(x, p) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\eta \int_{0}^{\pi} dx' |\eta| \tilde{w}(\eta, \theta) e^{-i\eta[\cos(\theta)x' + \sin(\theta)p]}
\]

\[
= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\eta \int_{0}^{\pi} dx' \int_{-\infty}^{\infty} dx'' |\eta| \times \text{Pr}(x', \theta) e^{i\eta[x - \cos(\theta)x' + \sin(\theta)p]}.
\]

A. The multidimensional oscillator

We will now proceed to show that it is possible to reconstruct the joint quantum state of a multidimensional harmonic oscillator under certain conditions.

The Hamiltonian is now \( \hat{H} = \sum_{j=1}^{N} \hbar \omega_j (\hat{a}_j^{\dagger} \hat{a}_j + 1/2) \) and we measure the set of \( N \) mutually commuting position operators \( \hat{x}_j, \ j = 1 \ldots N \). For notational simplicity, we arrange these operators in a vector \( \hat{x} = (\hat{x}_1, \ldots, \hat{x}_N) \). The \( N \)-dimensional quantum characteristic function is now \( \tilde{W}(\xi) \), where \( \xi \) is a vector of \( N \) complex variables \( \xi_j \). As before, we let \( \xi_j = \frac{1}{\sqrt{2}} i \eta_j e^{i \theta_j} \) with \(-\infty < \eta_j < \infty \) and \( 0 \leq \theta_j < \pi \). This yields the \( N \)-dimensional equivalent of \( \tilde{w}(\eta, \theta) \):

\[
\tilde{w}(\eta, \theta) = \tilde{W}(\xi) = \text{Tr} \left\{ \exp \left\{ i \sum_{j=1}^{N} \eta_j \hat{x}_j(\theta_j) \right\} \hat{\rho} \right\} = \int_{-\infty}^{\infty} d^N x \text{Pr}(x, \theta) e^{i\eta x}.
\]

To gain full knowledge of the function \( \tilde{w}(\eta, \theta) \) we must be able to vary the \( N \) variables \( \theta_j \) independently of each other on the interval \( 0 \leq \theta_j < \pi \). This is naturally not possible in general, since we can only vary the \( N \) variables comprising \( \theta \) through variation of the one parameter \( t \). A way to clearly see this restriction is by noticing that the present reconstruction scheme relies on a Fourier-transformation, which preserves dimensionality. While we measure the joint spatial distribution in \( N \) dimensions for different \( t \), this equals \( N + 1 \) dimensional measurements, while the quantum state (e.g. the characteristic function or Wigner function) is a \( 2N \)-dimensional object.

It is important to realize what kind of limitations are implied by the inability to vary the \( \theta_j \)'s independently. It is always possible to find the quantum states of a single degree of freedom, corresponding to tracing out all other degrees of freedom. The limitation comes about when trying to find the joint quantum state of the \( N \)-dimensional system, in particular the correlations between the different degrees of freedom and entanglement. A simple illustration of this is offered by a two-dimensional harmonic oscillator with \( \omega_1 = \omega_2 \). Let us consider measuring the observable \( \langle \hat{x}_1(\omega t)\hat{x}_2(\omega t) \rangle \). By using \( \tilde{w}(\eta, \theta) \) we find:

\[
\langle \hat{x}_1(\omega t)\hat{x}_2(\omega t) \rangle = \left[ \langle \cos(\omega t)\hat{x}_1 + \sin(\omega t)\hat{p}_1 \rangle \times \langle \cos(\omega t)\hat{x}_2 + \sin(\omega t)\hat{p}_2 \rangle \right] = \cos^2(\omega t) \langle \hat{x}_1\hat{x}_2 \rangle + \sin^2(\omega t) \langle \hat{p}_1\hat{p}_2 \rangle + \cos(\omega t) \sin(\omega t) \langle \hat{x}_1\hat{p}_2 + \hat{p}_1\hat{x}_2 \rangle.
\]
We return now to the problem with (7): The $\theta_j$’s are all varied through the one parameter $t$. The obvious solution to this problem is to devise some means to vary the $\theta_j$’s independently. There are important situations where this is indeed possible, e.g., the case of several entangled light fields. In this case a full reconstruction may be done, regardless of the values of the $\omega_j$’s, as can be seen in Fig. 1. The independent variation of $\theta_j$ can here be achieved simply by delaying the measurement on the subsystems by introduction of, for example, a variable delay line.

Another possibility for varying the $\theta_j$’s independently would be to vary the times for the subsystems independently. For this purpose one might use a method closely analogous to the so-called “twin-paradox” from special relativity. For instance, imagine two spin-0 particles in each their one-dimensional harmonic oscillator. One may then leave the one subsystem undisturbed while independently vary the elapsed time for the two subsystems compared to the first, effectively giving a means to time dilation will hereby delay the second subsystem relative to its direction of mechanical oscillation. To avoid direct disturbance of the second oscillator it should also be noticed that this procedure is equally applicable to a quantum system comprised of several distinguishable non-interacting particles in separate harmonic potentials.

Returning to the general problem of full state reconstruction by (7), we shall discuss under what circumstances this is indeed possible. By considering all times $t \geq 0$ and choosing the $\omega_j$’s mutually incommensurable, i.e., their ratios are irrational numbers, we can find a unique $t$ to reach any $\theta$ as long as $\theta_j/\theta_k$ with $j \neq k$ is an irrational number. To see this, remember that $\theta_j = [\omega_j t]_\pi$, with $[\cdot]_\pi$ being the modulus function with respect to $\pi$. The whole scheme can be pictured as letting $N$ initially coinciding points move around a circle with mutually incommensurable angular frequencies: If the points coincide at one angle (which we have chosen to be $\theta = 0$), then no pair will ever again coincide at this angle. The situation is illustrated for $N = 2$ in figure 1. A small technical detail in this respect is that since we have chosen the intervals of $0 \leq \theta_j < \pi$ and $-\infty < \eta_j, x_j < \infty$, then each time a $\theta_j$ surpasses an integer multiple of $\pi$ we must let $\eta_j \to -\eta_j$ in (7).

We have found the function $\tilde{w}(\eta, \theta)$ except on the values of $\theta$ where two or more $\theta_j/\theta_k$, $j \neq k$ is a rational number. Fortunately, this non-available set of $\theta$ values has measure zero, and since $\tilde{w}(\eta, \theta)$ is uniformly continuous (and thereby non-singular), the inability to find $\tilde{w}(\eta, \theta)$ on a set of measure zero is of no consequence. The uniform continuity of $\tilde{w}(\eta, \theta)$ is a consequence of $\hat{\rho}$ belonging to the trace class $\mathcal{B}$. The price we pay to gain knowledge of the $N$-dimensional state as compared to the one-dimensional case is that we must measure the joint position distribution of all coordinates for all times instead of just half of the oscillators’ period. In actual applications, where infinite measurement times are not available, one would presumably use frequencies of the $N$ oscillators whose ratios are rational numbers and measure for the recurrence time of the joint system. The frequencies should then be chosen so that the $\theta$-space is sufficiently closely sampled for a reliable reconstruction. The exact amount of information obtained in such an experiment will be quantified in subsection 4.D.

It should also be noticed that this procedure is equally applicable to a quantum system comprised of several distinguishable non-interacting particles in separate harmonic potentials.

For completeness we give the formula for reconstruction of the $N$-dimensional Wigner function:

$$W(x, p) = \lim_{T \to \infty} \frac{1}{2 \pi (2\pi)^N} \int_0^{T'} dt \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d^N x' \times$$

$$\left| \eta \right| \left| (-1)^{\sum_{j=1}^{N} F_j(t)} \text{Pr}(x', [\omega j]_{\pi}) \times \exp \left\{ i \sum_{j=1}^{N} (-1)^{F_j(t)} \eta_j \left[ x_j - x_j' \cos(\omega j t_{\pi}) - p_j \sin(\omega j t_{\pi}) \right] \right\} \right|,$$

where $F_j(t) = \text{Floor}(\omega_j t/\pi)$, and the Floor-function rounds downwards to the nearest integer.
C. An example of realizing incommensurable frequencies

In the above discussion, it was demonstrated that the quantum state of an $N$-dimensional harmonic oscillator could be exactly reconstructed if the $N$ frequencies were incommensurable. It may be noticed that the reconstruction of the joint quantum state did not require interactions between the $N$ degrees of freedom. We will now give a brief example of how it is possible, by introducing interactions between the oscillators, to reconstruct the full quantum state when all the frequencies are identical and equal to $\omega$. We imagine the $N$ oscillators arranged in a line, and introduce nearest-neighbor interaction terms in the Hamiltonian. We let $\kappa \leq \omega$ be a real coupling constant:

$$\hat{H} = \sum_{j=1}^{N} \hbar \omega \left( \hat{a}^+_j \hat{a}_j + \frac{1}{2} \right) + \hat{H}_{\text{int}}$$

(9)

$$\hat{H}_{\text{int}} = \sum_{j=1}^{N} \hbar \kappa \left( \hat{a}^+_j \hat{a}_{j+1} + \hat{a}^+_j \hat{a}_{j+1} \hat{a}_j \right).$$

(10)

Arranging now the $N$ annihilation operators in a column vector $\mathbf{a} = (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_N)^T$, we find from the Heisenberg equation of motion:

$$\frac{d}{dt} \begin{pmatrix} \hat{a}_1(t) \\ \hat{a}_2(t) \\ \vdots \\ \hat{a}_N(t) \end{pmatrix} = -i \begin{pmatrix} \omega & \kappa & 0 & \cdots & 0 \\ \kappa & \omega & \kappa & \cdots & 0 \\ 0 & \kappa & \omega & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega \end{pmatrix} \begin{pmatrix} \hat{a}_1(t) \\ \hat{a}_2(t) \\ \vdots \\ \hat{a}_N(t) \end{pmatrix},$$

where the matrix $\mathcal{D}$ is tri-diagonal and real. Since there are no terms containing $\hat{a}^+_j$, we can perform a usual orthogonal diagonalization of this matrix to yield $N$ new modes, characterized by new annihilation operators $\hat{a}^+_j$. The eigenvalues $\omega_j$ of $\mathcal{D}$ are well-known from e.g. Hückel molecular orbital theory and solid-state physics:

$$\omega_j = \omega + 2\kappa \cos \left( \frac{2j\pi}{N} \right), \quad k \in \{1, 2, \ldots, N\}.$$  

(11)

In this way, the new ladder operators have simple time evolutions $\hat{a}^+_j(t) = \hat{a}^+_j(0) \exp(-i\omega_j t)$, and we can again find position operators $\hat{x}^+_j(t) = \frac{1}{\sqrt{2}} \hat{a}^+_j(t) + \hat{a}^+_j(t)$. So if we choose $\kappa$ so that the $\lambda_j$’s are incommensurable, we can use the reconstruction method from subsection 11B. Experimentally, one still has only to measure the $\hat{x}^+_j$’s since the $\hat{x}^+_j$’s are merely linear combinations of these.

D. Commensurable frequencies and partial information

We have shown above that a complete tomographic reconstruction of the state of an $N$-dimensional quantum oscillator system is possible if all the oscillator angular frequencies are mutually incommensurable. This naturally leads to the question of which aspects of the quantum state can, and which cannot, be obtained from such tomographic measurements if some of the angular frequencies are commensurable. We will seek to quantify this degree of information through the moments of the ladder operators $\hat{a}_j$ and $\hat{a}^+_j$. For this to be meaningful, we must assume that these moments are finite. For convenience we shall be considering the Weyl-ordered (i.e. symmetrically ordered) products. Letting $\mathbf{r}$ and $\mathbf{s}$ be $N$-vectors with non-negative integer components, these moments are generally of the form:

$$S(\mathbf{r}, \mathbf{s}) = \left\langle \prod_{j=1}^{N} r_j! s_j! (\hat{a}_j)^{s_j} (\hat{a}^+_j)^{r_j-s_j} \right\rangle_w,$$

where $r_j \in \mathbb{N}_0$ and $s_j \in \{0, 1, \ldots, r_j\}$.  

(12)

That is, all factors in the product $S(\mathbf{r}, \mathbf{s})$ is the sum of all symmetric permutations of a number $s_j$ of the operator $\hat{a}_j$ and a number $(r_j - s_j)$ of the operator $\hat{a}^+_j$. Here we let $\{\}$$_w$ stand for the Weyl ordering, which is the same as the sum of all permutations divided by the number of terms. For example:

$$\{\hat{a}^+_1 \hat{a} \hat{a}^+_1\} _w = \{\hat{a}^+_1 \hat{a} \hat{a}^+_1\} _w = \frac{1}{2} (\hat{a}^+_1 \hat{a} \hat{a}^+_1 + \hat{a} \hat{a}^+_1 \hat{a}^+_1).$$

A quantum state is completely characterized if all moments of the form in $\{\}$$_w$ are specified. In this way we are recasting the question of to what extent the quantum state can be reconstructed into the question of how many of the moments $S(\mathbf{r}, \mathbf{s})$ can be found.

To keep things transparent, we shall initially consider only two oscillators with commensurable frequencies and only later generalize to the case of $N$ oscillators. We shall need the recurrence time for the system, $T = 2\pi/\omega$ so that $\theta_j = \omega_j t = \alpha_j \omega \tau = \alpha_j \theta$, making $\alpha_j$ a positive integer. Since we are measuring the joint $\mathbf{x}$-distributions at different times, it is natural to consider moments of these distributions $\left\langle [\hat{x}_1(\theta_1)]^r [\hat{x}_2(\theta_2)]^s \right\rangle$. Recalling $\{\}$$_w$:

$$2^{\frac{\alpha_1 + \alpha_2}{2}} \left\langle [\hat{x}_1(\theta_1)]^r [\hat{x}_2(\theta_2)]^s \right\rangle$$

$$= 2^{\frac{\alpha_1 + \alpha_2}{2}} \left\langle [\hat{x}_1(\alpha_1 \theta)]^r [\hat{x}_2(\alpha_2 \theta)]^s \right\rangle$$

$$= \left\langle (\hat{a}_1 e^{-\alpha_1 \theta} + \hat{a}^+_1 e^{\alpha_1 \theta})^r (\hat{a}_2 e^{-\alpha_2 \theta} + \hat{a}^+_2 e^{\alpha_2 \theta})^s \right\rangle$$

$$= \sum_{s_1=0}^{r_1} \sum_{s_2=0}^{r_2} \sum_{s_1=0}^{s_1} \sum_{s_2=0}^{s_2} S(r_1, r_2, s_1, s_2) e^{i\theta(\alpha_1(r_1-2s_1)+\alpha_2(r_2-2s_2))}.$$  

(13)

Since the set of functions $\{e^{in\theta}, n \in \mathbb{N}\}$ is linearly independent on the interval $[0:2\pi]$ we can find all the moments $S(r_1, r_2, s_1, s_2)$ if there are no two of the exponential functions in the sum $\{\}$$_w$ that have the same period in $\theta$. Indeed, to find all moments (and not only the symmetric ones) of order $(r_1, r_2)$, we must know all
symmetric moments of this and lower order. A precise way to state this is that there must be no recurrences in the following lists, where we keep the \( r_j' \)'s fixed in each list and let the \( s_j' \)'s assume all possible values: 
\[
(\alpha_1 [r_1' - 2s_1'] + \alpha_2 [r_2' - 2s_2'])_{r_j'} \quad \text{with} \quad r_j' \leq r_j \quad \text{and} \quad s_j' \leq r_j'.
\]
Finding these moments can then be done, for instance, by Fourier transformation since the aforementioned exponential functions are orthogonal.

Furthermore, one should notice that reconstructing the quantum state through the moments of the ladder operators, one in principle needs knowledge only of a small but finite interval of the angle \( \theta \), and not the whole interval \([0; \pi]\). A similar result is found in [10]. The fundamental assumption that allows for reconstruction from any small finite \( \theta \)-interval is that of finiteness of the moments of the ladder operators and position operators in (13). The earlier discussed method of state reconstruction via Fourier transformation, [14], does not suffer from this limitation. On the other hand not all these moments need be finite, only the ones we use in the reconstruction.

Finally, one should notice that some of the moments can always be found, regardless of the value of \( \alpha_1/\alpha_2 \). This trivially includes the moments \( \langle \hat{x}_1^{j_1} \hat{x}_2^{j_2} \rangle \), but also the moments \( \langle \hat{a}_1^{j_1} \hat{a}_2^{j_2} \rangle \) and their complex conjugates for any \((r_1, r_2)\) since these moments evolve with the unique largest numerical frequency.

We now proceed to discuss the practical usefulness of this approach to reconstruction. It is easy to show that it is possible to find all moments with \( r_1 < \alpha_2 \) and/or \( r_2 < \alpha_1 \), so one may indeed settle for reconstructing moments of only low order, e.g. \( r_1 + r_2 < \max(\alpha_1, \alpha_2) \). The reason for this is both the difficulty in precisely measuring higher moments of the joint position distribution and that finding higher moments of the ladder operators in general requires the ability to measure very rapid variations in the joint position distribution (see [13]). In addition, it is not necessary to measure a continuum of angles if one is only interested in moments up to a certain order, but only a number of angles equaling this number, which makes the procedure practically feasible. In this way one only has to solve a number of equations with an equal number of unknowns.

Generalizing the above results to arbitrary \( N \), we let \( T = 2\pi/\omega \) be the recurrence time of the system and \( \theta_j = \alpha_j \theta \). The positive integers \( \alpha_j \) are arranged in an \( N \)-vector \( \alpha \), and we find that to reconstruct the moments up to \( S(r, s) \) one needs to measure \( \left\langle \prod_{j=1}^{N} \hat{x}_j^{j_j} (\theta_j) \right\rangle \) and that there can be no recurring numbers in each of the lists (once more the \( r_j' \)'s are fixed in each list while \( s_j' \) assumes all possible values): 
\[
(\alpha \cdot [r' - 2s'])_{r_j'} \quad \text{with} \quad r_j' \leq r_j \quad \text{and} \quad s_j' \leq r_j'.
\]
Like before, we can in particular always find all moments \( \left\langle \prod_{j=1}^{N} \hat{x}_j^{r_j} \right\rangle \), since they are directly measured, and also \( \left\langle \prod_{j=1}^{N} \hat{a}_j^{r_j} \right\rangle \) and its complex conjugate for all \( r \).

It is amusing to note that if all the moments of the ladder operators are finite and the angular frequencies are incommensurable, it is in principle possible to reconstruct the full quantum state from measurements made in a small but finite time interval regardless of the dimension \( N \) - an impossible task if just two angular frequencies are commensurable.

One may also remark that an \( N \)-dimensional Gaussian state can always be completely reconstructed if no two \( \omega_j \)'s are equal: The required measurements are the joint position distribution for either 4 points of time or any finite continuous interval of time.

Lastly, we make a short comment on a possible strategy for guessing the quantum state in the case of commensurable frequencies. Even though we do not know the moments of the ladder operators individually, we still find the sum of two or more - this number determined by the how many occurrences there are of a particular number in the aforementioned lists. One method of guessing the state from an incomplete set of data is the Maximum Entropy Principle, due to Jaynes [11], which has been used in several reconstruction schemes [12]-[17]. The Maximum Entropy principle says that in case one has a set of data which could have come about due to several different quantum states, one should choose the state with the largest entropy. In our present scenario, this means that if we only know the sum of, say, \( n_{r,s} \) different symmetric moments, the Maximum Entropy principle would ascribe equal Lagrange multipliers to each observable in the Maximum Entropy density operator.

### III. THE FREE PARTICLE

The tomographic reconstruction of the completely free particle has been considered in [18], but as stated herein, cannot be used beyond the one-dimensional case. Here, we instead study the semi-continuous case of the free particle with different boundary conditions. In the first two subsections [11A] and [11B] we shall study the free particle with periodic boundary conditions, also valid for the planar rotor [20]. In the last subsection [11C] we shall give a brief treatment of the particle in a box where it will be seen that the different boundary condition has an important effect on the available information.

#### A. The one-dimensional case

As in the case of the oscillator, we shall first treat the one-dimensional case and later extend this to \( N \) dimensions. In the present case the reconstruction of the quantum state will be done through finding the matrix elements of \( \hat{p} \) by simple inversion of Fourier transforms used in [18]. Let us consider a particle with mass \( m \) on the spatial interval \( 0 \leq x < L \). The Hamiltonian is 
\[
\hat{H} = \hat{p}^2/2m,
\]
yielding the eigenstates \( |n\rangle \) with energy \( E(n) = \hbar \Omega n^2 \). Here \( \Omega = \pi \hbar /mL^2 \) and \( \hbar \) is Planck's
constant. The eigenstate $|n\rangle$ in the $x$-representation is:

$$\langle x|n \rangle_t = \frac{1}{\sqrt{L}} e^{2\pi inx/L} e^{-itn^2t}. \quad (14)$$

We can use this to find the position distribution at any time:

$$\Pr(x, t) = \langle x|\hat{\rho}|x \rangle_t = \sum_{n,n'= -\infty}^{\infty} \rho(n, n') \frac{1}{L} e^{2\pi i (n-n')x/L} e^{-i\Omega(n^2-n'^2)t}. \quad (15)$$

It will now be convenient to change variables from $n$ and $n'$ to $\bar{n} = n + n'$, $\Delta n = n - n'$. Note that $\bar{n}$ and $\Delta n$ are both either even or odd. Changing summation variables in this manner yields:

$$\Pr(x, t) = \frac{1}{L} \left( \sum_{\bar{n}= -\infty}^{\infty} \sum_{\Delta n = -\infty}^{\infty} + \sum_{\bar{n}= -\infty}^{\infty} \sum_{\Delta n = -\infty}^{\infty} \right) \times$$

$$\rho \left( \frac{\bar{n} + \Delta n}{2}, \frac{\bar{n} - \Delta n}{2} \right) \times$$

$$e^{2\pi i \Delta nx/L} e^{-i\Omega\bar{n}\Delta n t}. \quad (16)$$

Our task is to invert this equation to find the matrix elements of $\hat{\rho}$. Fortunately, this is quite easy. Notice that there are two exponential functions in $\Omega^2$ and both can be used with a Fourier transformation to pick out certain values of $\bar{n}$ and $\Delta n$: The first exponential function in $x$ contains only $\bar{n}$ and once $\bar{n}$ is fixed, the other exponential function can be used to select $\Delta n$.

We let $N_T \in \mathbb{N}$ and $2T = 2\pi/\Omega$ be the minimum required measurement time, which corresponds to twice the time a classical particle with the lowest non-zero energy ($n = 1$) would take to traverse the length $L$. Also, we shall choose $\beta \neq 0$ and obtain:

$$\int_0^L dx \ e^{-2\pi i \beta x/L} \frac{1}{2N_T} \int_{-N_T}^{N_T} dt \ e^{2\pi i \nu/\Omega t} \Pr(x, t) =$$

$$\frac{1}{2N_T} \int_{-N_T}^{N_T} dt \ \sum_{\bar{n}= -\infty}^{\infty} \sum_{\Delta n = -\infty}^{\infty} \rho \left( \frac{\bar{n} + \beta}{2}, \frac{\bar{n} - \beta}{2} \right) e^{i\Omega(\bar{n}-\bar{n})t} =$$

$$\sum_{\bar{n}= -\infty}^{\infty} \rho \left( \frac{\bar{n} + \beta}{2}, \frac{\bar{n} - \beta}{2} \right) \delta_{\nu, \bar{n}} =$$

$$\rho \left( \frac{\nu + \beta}{2}, \frac{\nu - \beta}{2} \right) \beta \text{ and } \nu \text{ of same parity.} \quad (17)$$

We have found all elements of the density matrix except those for which $\beta = 0$, i.e. the diagonal in the (momentum) $n$-representation. If we were to choose $\beta = 0$, we would always obtain the result of unity in $\Omega^2$, as this is the same as taking the trace of $\hat{\rho}$ in the $x$-basis. This inability to find the diagonal was also pointed out for finite observation times for the completely free particle in $\mathbb{R}^3$, and will unfortunately carry over to the multidimensional case. This limitation arises because all probability densities of the eigenstates are identical: $| \langle x|n \rangle^2 | = 1/L$ whereby, for example, a thermal state and any pure eigenstate $|n\rangle\langle n|$ have the same probability distribution at all times. Even though finding the diagonal of a density matrix is impossible given the rest of the matrix, one can use the Schwartz inequality to constrain the size of the diagonal elements through $|\rho(n, n')|^2 \leq \rho(n, n)\rho(n', n')$.

### B. $N$-dimensional free particle with periodic boundary conditions

We will now move on to the $N$-dimensional case. We shall work in the $N$-dimensional interval $x_j \in [0, L_j]$. The energies of the eigenstates are $E(n) = \hbar \sum_{j=1}^N \Omega_j n_j^2$ and in extension of (10) we introduce the $N$-vectors of intergers $\bar{n}$ and $\Delta n$. This yields:

$$\Pr(x, t) = \prod_{j=1}^N \left( \sum_{\bar{n}_j = \text{even}} \delta_{\Omega_j, \bar{n}_j} \sum_{\Delta n_j \text{ even}} \sum_{\Delta n_j \text{ odd}} \sum \sum \sum \right) \times$$

$$\frac{1}{L_j} e^{2\pi i \Delta n_j x_j/L_j} e^{-i\Omega_j \bar{n}_j \Delta n_j t} \times$$

$$\rho \left( \frac{\bar{n} + \Delta n}{2}, \frac{\bar{n} - \Delta n}{2} \right). \quad (18)$$

We shall try to invert this equation to find the matrix elements of $\hat{\rho}$. Of course, we cannot approach this completely as in the one-dimensional case and select a particular vector $\bar{n}$ through Fourier-transforms, having only the parameter $t$ to vary. We can, however, expand the time interval of integration to incorporate all points of time:

$$\lim_{T'' \to \infty} \frac{1}{2T''} \int_{-T''}^{T''} dt \ e^{i\sum_{j=1}^N \Omega_j (\bar{n}_j - \bar{n}_j) \beta_j t} =$$

$$\delta_{\sum_{j=1}^N \Omega_j (\bar{n}_j - \bar{n}_j) \beta_j, 0}. \quad (19)$$

If the $\Omega_j$'s are mutually incommensurable, the only possibility for this Kronecker delta-function to give a non-zero result is for $\nu_j = \bar{n}_j \forall j$. Remembering that $\Omega_j = \frac{\pi h}{m_j L_j^2}$, the condition of incommensurability of the $\Omega_j$'s is equivalent to demanding the elements of the list:

$$(m_j L_j^2) \text{ be incommensurable}.$$
we only obtain useful results for \( \beta_j \neq 0 \). With these limitations we can exactly reconstruct the rest of the density matrix. Additionally, \( \beta_j \) and \( \bar{n}_j \) have the same parity for all \( j \):

\[
\rho \left( \frac{\nu + \beta}{2}, \frac{\nu - \beta}{2} \right) = \lim_{T' \to \infty} \frac{1}{2T'} \int_{-T'}^{T'} \int_0^L d^N x \times e^{-2\pi i \sum_{j=1}^N \beta_j x_j / L_1} e^{i \sum_{j=1}^N \Omega_j \nu_j \beta_j t} \times \Pr(x, t),
\]

(20)

The special feature of this system, which allows the treatment above, is that all its eigenenergies are rational numbers (actually a integers) times some minimum energy. This can in fortunate circumstances, i.e. if the products of the energy eigenstates in the position representation are reasonably placid, allow state reconstruction equations like that in (17). These reconstruction equations will make use of a finite measurement time, and by using the trick in (19), can be used in multiple dimensions. Another example of this kind of system is the quantum mechanical rotor with fixed angular momentum projection, which will be the topic of a forthcoming article [10].

C. Particle in a box

In the last two subsections we found ourselves unable to determine the momentum distribution for the free particle with periodic boundary conditions. The reason for this was that all the energy eigenstates of the system had the same spatial distribution. In this subsection we shall show how imposing other boundary conditions can completely alter this situation. In particular, we shall study the particle in a box, where the spatial density is zero at the boundaries of the box. As usual, we shall first treat the one-dimensional case and later move on to the \( N \)-dimensional case.

We choose \( x \in [0, L] \) whereby \( E(n) = \hbar \Omega' n^2 \), now with \( n \in \mathbb{N} \). We use \( \Omega' = \Omega / 4 = \frac{\pi^2}{4mL^2} \), yielding the energy eigenstates:

\[
\langle x | n \rangle = \sqrt{\frac{2}{L}} \sin \left( \frac{n \pi x}{L} \right) e^{-i\Omega' t}.
\]

These eigenstates all have different spatial distributions, and we already suspect that we shall be able to reconstruct the momentum distribution. Following section III B \( \bar{16} \) we find the joint position distribution, similar to \( \text{IIIB } \bar{16} \):

\[
\Pr(x, t) = \epsilon(x | \beta | x) t
\]

\[
= \sum_{n, n'=1}^\infty \rho(n, n') \frac{2}{L} \sin \left( \frac{n \pi x}{L} \right) \sin \left( \frac{n' \pi x}{L} \right) e^{-i\Omega'(n^2-n'^2)t}
\]

\[
= \sum_{n, n'=1}^\infty \rho(n, n’) \frac{1}{L} \left\{ \cos \left( \frac{(n - n') \pi x}{L} \right) - \cos \left( \frac{(n + n') \pi x}{L} \right) \right\} e^{-i\Omega(n^2-n'^2)t}.
\]

Making once more the substitution \( \tilde{n} = n + n’ \) and \( \Delta n = n - n’ \) we arrive at the box equivalent of \( \text{\text{IIIB } \bar{16}} \):

\[
\Pr(x, t) = \frac{1}{L} \left( \sum_{\tilde{n}=0}^{\infty} \sum_{\Delta n = -\infty}^{\infty} \rho(\tilde{n} + \Delta n, -\Delta n) \times \Pr(x, t) \times \right)
\]

\[
\sum_{\tilde{n}=0}^{\infty} \sum_{\Delta n = -\infty}^{\infty} \left[ \cos \left( \frac{\Delta n \pi x}{L} \right) - \cos \left( \frac{\tilde{n} \pi x}{L} \right) \right] e^{-i\Omega \Delta n t}.
\]

(21)

Recalling that the set of functions \( \{ \cos(k \pi x / L) \} \), \( k \in \mathbb{N} \), is orthogonal on \( x \in [0, L] \) we can use the cosine functions in (21) and the exponential function in time to select a certain term in the sum. Selecting \( \nu \in \mathbb{N} \) and \( \beta \in \mathbb{Z} \) with \( \nu > |\beta| \) and letting \( T' = 2\pi / \Omega' \) we obtain the equivalent of (17) (21):

\[
\rho \left( \frac{\nu + \beta}{2}, \frac{\nu - \beta}{2} \right) = 2 \int_0^L dx \cos \left( \beta \frac{\pi x}{L} \right) \times
\]

\[
\frac{1}{2N \Omega T'} \int_{-N \Omega T'}^{N \Omega T'} dt e^{i\Omega' \nu t} \Pr(x, t), \quad \nu > |\beta|.
\]

(22)

Here it is apparent that we can reconstruct the full density matrix, including the momentum distribution. Thus, the choice of boundary conditions has allowed us to overcome the limitation encountered in the case of periodic boundary conditions.

Moving on to the multidimensional case where \( E(n) = \hbar \sum_{j=1}^N \Omega_j' n_j^2 \) we find the equivalent of \( \text{\text{IIIB } \bar{16}} \):

\[
\Pr(x, t) = \left\{ \prod_{j=1}^N \sum_{\tilde{n}_j=0}^{\infty} \sum_{\Delta n_j = -\infty}^{\infty} \rho(\tilde{n}_j + \Delta n_j, -\Delta n_j) \times
\]

\[
\cos \left( \frac{\tilde{n}_j \pi x_j}{L_j} \right) - \cos \left( \frac{\tilde{n}_j \pi x_j}{L_j} \right) \right] e^{-i\Omega_j \tilde{n}_j \Delta n_j t} \right\} \times
\]

\[
\frac{1}{L_j} \left[ \cos \left( \frac{\Delta n_j \pi x_j}{L_j} \right) - \cos \left( \frac{\tilde{n}_j \pi x_j}{L_j} \right) \right] e^{-i\tilde{n}_j \Delta n_j t} \right\} \times
\]

\[
\rho \left( \frac{\tilde{n} + \Delta n}{2}, \frac{\tilde{n} - \Delta n}{2} \right).
\]

(23)

Selecting the two \( N \)-vectors \( \nu \) and \( \beta \) with \( \nu_j > |\beta_j| \) and demanding incommensurability of the elements in the list \( (m_j L_j^2) \), we find the reconstruction formula corresponding to (20):

\[
\rho \left( \frac{\nu + \beta}{2}, \frac{\nu - \beta}{2} \right) = \lim_{T' \to \infty} \frac{1}{2T'} \int_{-T'}^{T'} \int_0^L d^N x \times
\]

\[
\prod_{j=1}^N \cos \left( \beta_j \frac{\pi x_j}{L_j} \right) e^{i \sum_{j=1}^N \Omega_j' \nu_j \beta_j t} \times
\]

\[
\Pr(x, t), \quad \nu_j > |\beta_j|.
\]

(24)
As in the one-dimensional particle in a box, it is in the multidimensional case possible to completely reconstruct the density matrix.

IV. SUMMARY

We have shown how to extend two common schemes of quantum state tomography from one to \( N \) dimensions: The harmonic oscillator and the free particle on a finite interval. In both cases a complete reconstruction required extension of the time interval of observation to all times and required incommensurability of the eigenenergy differences for the different \( N \) spatial dimensions.

For the harmonic oscillator, we quantified the information that can be reconstructed if some of the \( N \) frequencies are commensurable. This was done through reconstructing the moments of the ladder operators, and also constitutes a reconstruction method for these moments if they are finite. This partial reconstruction only required measurements at a finite number of points of time.

For the free particle on a finite interval we showed that all off-diagonal elements of the density matrix in the energy-representation can be reconstructed in the case of periodic boundary conditions, and that a full reconstruction is possible for the box potential.

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[17] For simplicity, we choose the ordering parameter \( s = 0 \). This choice corresponds to the choice of the Wigner distribution as the complex Fourier transform of \( \rho \).
[18] For the massive harmonic oscillator, this means measuring \( x \) in units of \( \sqrt{\hbar/m\omega} \) and \( p \) in units of \( \sqrt{\hbar m\omega} \)
[19] Actually, no further information on the quantum state can be gained by considering other functions of the position distribution. This can be seen by expanding the function in its moments of the position operators, and realizing that the resulting equation is merely a linear combination of equations obtained in [13].
[20] For the rotor, one must substitute \( ma^2 \rightarrow I \), where \( I \) is the moment of inertia.
[21] Performing the necessary sums, it is easiest to let the sums extend over all positive and negative \( \bar{n} \) and \( \Delta n \) (each sum containing only even or odd indices) and then in the end set all \( \rho(n, n') = 0 \) if \( n \leq 0 \) or \( n' \leq 0 \).