Abstract. We consider a class of restless bandit problems that finds a broad application area in stochastic optimization, reinforcement learning and operations research. In our model, there are $N$ independent 2-state Markov processes that may be observed and accessed for accruing rewards. The observation is error-prone, i.e., both false alarm and miss detection may happen. Furthermore, the user can only choose a subset of $M$ ($M < N$) processes to observe at each discrete time. If a process in state 1 is correctly observed, then it will offer some reward. Due to the partial and imperfect observation model, the system is formulated as a restless multi-armed bandit problem with an information state space of uncountable cardinality. Restless bandit problems with finite state spaces are PSPACE-HARD in general. In this paper, we establish a low-complexity algorithm that achieves a strong performance for this class of restless bandits. Under certain conditions, we theoretically prove the existence (indexability) of Whittle index and its equivalence to our algorithm. When those conditions do not hold, we show by numerical experiments the near-optimal performance of our algorithm in general.

Key words. restless bandits, continuous state space, observation errors, Whittle index

AMS subject classifications. 65K05, 90C15, 90C40

1. Introduction. The exploration-exploitation (EE) dilemma is well posed in optimization-over-time problems and mathematically modeled in various forms for reinforcement learning, to which a major category, multi-armed bandits (MAB), belongs. In the classical MAB model, a player chooses 1 out of $N$ statistically independent arms to pull and possibly accrues reward determined by the state of the chosen arm which transits to a new state according to a known Markovian rule [2]. The states of other arms remain frozen. The objective is to maximize the expected total discounted reward over an infinite time horizon:

$$\max_{\pi \in \Pi} E_\pi \sum_{t=1}^{\infty} [\beta^{t-1} R(t)|S(t), A(t)],$$

where $\Pi$ is the set of all feasible policies, $\beta \in (0,1)$ the discount factor, $A(t) \in \{1,2,\cdots,N\}$ the arm chosen at time $t$ under a policy $\pi \in \Pi$, $S(t)$ the joint state of all arms at time $t$ and $R(t)$ the obtained reward determined by the state of the chosen arm. For the infinite-horizon problem, there must exist an optimal stationary policy $\pi^*$, independent of time $t$, that maps the joint arm state space to an arm to choose. If each arm’s state space has cardinality of $K$, then the joint state space has a size of $K^N$, leading to the geometric growth of the algorithmic complexity with the number of arms. Gittins in 1970s solved the problem by showing the optimality of an index policy, i.e., for each state of each arm we can compute an index solely dependent on the parameters of the arm and choosing the arm with a state of the highest index at each time is optimal [1]. Gittins index policy clearly reduces the complexity of the problem to a linear growth with the number of arms as they are decoupled when computing the index function (of the states of each arm). Whittle in 1988 generalized Gittins index to the restless MAB model where arms not chosen may also change states or offer reward [9]. Whittle’s generalization, referred to as
Whittle index, has been demonstrated with a strong performance through numerical studies [5]. However, establishing the existence of Whittle index (indexability) and the computation of Whittle index if it exists are both difficult in general. Papadimitriou and Tsitsiklis in 1999 have shown that the restless MAB with a finite state space is PSPACE-HARD [7].

In this paper, we extend the work in [5] (for a perfect observation model) and [6] (for the myopic policy on stochastically identical arms) to build a near-optimal algorithm with low complexity for a class of restless bandits with an infinite state space and an imperfect observation model. Our model also belongs to the general framework of partially observable Markov decision processes (POMDP) [8]. Consider $N$ processes each of which evolves on a 2-state Markov chain whose state is observed if and only if the process is chosen. Furthermore, the observation is error-prone: state 1 may be observed as 0 and vice versa. At time $t$, the player obtains reward of amount $B_n$ if and only if channel $n$ is currently chosen and accurately observed in state 1. This observation model finds a broad application area in communication networks [6]. Under resource constraints, the player's goal is to select $M$ ($M < N$) channels at each time for maximizing the long-term reward. By formulating the belief vector as the system state for decision making, we show that the indexability is satisfied under certain conditions. Furthermore, we propose an efficient algorithm to compute the approximated Whittle index that achieves a near-optimal performance in general, even if the conditions for indexability do not hold.

The paper is organized as follows. Our main results are in section 2, our new algorithm is in section 3, experimental results are in section 4, and the conclusions follow in section 5.

2. Main results. Consider a restless MAB with $N$ internal 2-state Markov chains (arms) of potentially different transition probabilities. At each time $t$, the player chooses a subset consisting of $M$ ($M < N$) arms to observe their states. Let $\delta = \Pr(O = 1|S = 0)$ and $\epsilon = \Pr(O = 0|S = 1)$ denote, respectively, the miss detection and false alarm probabilities in the observation model, where $S \in \{0 \text{ (bad)}, 1 \text{ (good)}\}$ denotes the state of an arm and $O$ the observation (detection outcome) of the state.

Under the optimal detector, the error probabilities $\delta$ and $\epsilon$ follow the curve of receiver operating characteristics (ROC) [6]. Since one cannot minimize both $\delta$ and $\epsilon$ simultaneously, we fix the constraint on miss detection $\delta \leq \delta_0$ for some $\delta_0 \in (0, 1)$ which yields the optimal false arm probability $\epsilon_0$, simply denoted by $\epsilon$. If arm $n$ ($1 \leq n \leq N$) in state $S = 1$ is observed in state 1 (i.e., $S = 1$ and $O = 1$), then the player accrues $B_n$ units of reward from this arm. One application example of this observation model is cognitive radios, where a secondary user aims to utilize a frequency band (channel/arm) currently unused by the primary users. Due to energy and policy constraints on the sensor of the secondary user, only a subset of channels can be sensed at each time and if any of them is sensed idle ($O = 1$), the user can send certain packets over it to its receiver and obtain an ACK in the end of the time slot if the channel is indeed idle ($S = 1$); otherwise no ACK from this channel would be received. Then the reward $B_n$ is just the bandwidth of channel $n$. Clearly, the hard constraint here should be on the miss detection probability $\delta$ to guarantee the satisfaction of the primary users, i.e., the disturbance (when a secondary user senses a busy channel as idle and subsequently sends data over it) to the primary users should be capped.

2.1. System Model and Belief Vector. At each discrete time $t$, the internal state $(0/1)$ of an arm cannot be observed before deciding whether or not to observe the arm. Therefore, we cannot use the states of the Markov chains as the system
state for decision making. Applying the general POMDP theory to our model the belief state vector consisting of probabilities that arms are in state 1 given all past observations is a sufficient statistics for making future decisions [8]:

\[ \Omega(t) = (\omega_1(t), \omega_2(t), \ldots, \omega_N(t)) \]

where \(\omega_n(t)\) is the belief state of arm \(n\) at time \(t\) and \(S_n(t)\) its internal state. According to the Bayes’ rule, the belief state (of any arm) itself evolves as a Markov chain with an infinite state space:

\[
\begin{align*}
\omega_n(t+1) &= \begin{cases} 
 p_{11}^{(n)}, & n \in A(t), ACK_n(t) \ (S_n(t) = 1, O_n(t) = 1) \\
 T_n(\frac{\epsilon_{\omega_n(t)}}{\omega_n(t)+1-\omega_n(t)}), & n \in A(t), No \ ACC_n(t) \\
 \omega_n(t), & n \notin A(t)
\end{cases}
\end{align*}
\]

where \(A(t) \subset \{1, 2, \ldots, N\}\) is the set of chosen arms at time \(t\) with \(|A(t)| = M, S_n(t)\) and \(O_n(t)\) respectively the state and observation from arm \(n\) at time \(t\) if \(n \in A(t)\), \(ACK_n(t)\) the acknowledgement of successful utilization of arm \(n\) for slot \(t\), \(T_n(\cdot)\) the one-step belief update operator without observation, and \(P^{(n)} = \{p_{ij}^{(n)}\}_{i,j \in \{0,1\}}\) the transition matrix of the internal Markov chain of arm \(n\). Furthermore, the \(k\)-step belief update of an unobserved arm for \(k\) consecutive slots starting from any belief state \(\omega\) is

\[ T_n^k(\omega) = \frac{p_{01}^{(n)} - (p_{11}^{(n)} - p_{01}^{(n)}) k (p_{01}^{(n)} - (1 + p_{01}^{(n)} - p_{11}^{(n)}) \omega)}{1 + p_{01}^{(n)} - p_{11}^{(n)}}. \]

For simplicity of notations, we denote \(T_n^1(\cdot)\) by \(T_n(\cdot)\). At time \(t = 1\), the initial belief state \(\omega_n(1)\) of arm \(n\) can be set as the stationary distribution \(\omega_{n,o}\) of the internal Markov chain:

\[ \omega_n(1) = \omega_{n,o} = \lim_{k \to \infty} T_n^k(\omega') = \frac{p_{01}^{(n)}}{p_{01}^{(n)} + p_{10}^{(n)}}, \]

where \(\omega_{n,o}\) is the unique solution to \(T_n(\omega) = \omega\) and \(\omega' \in [0, 1]\) an arbitrary probability. Given the initial belief vector \(\Omega(1) = (\omega_1(1), \omega_2(1), \ldots, \omega_N(1))\), we arrive at the following constrained optimization problem:

\[ \max_{\pi: \Omega(t) \to A(t)} E_{\pi} \sum_{t=1}^{\infty} \beta^{t-1} R(t) |\Omega(1)|, \]

subject to \(|A(t)| = M, \ t \geq 1\).

It is clear that fixing \(\Omega(1)\), the action-dependent belief vector \(\Omega(t)\) takes possible values geometrically growing with time \(t\), leading to a high-complexity in solving the problem, i.e., curse of dimensionality. In the following, we adopt Whittle’s original idea of Lagrangian relaxation to decouple arms for an index policy and show some crucial properties of the value functions of a single arm.

---

This manuscript is for review purposes only.

---

\(^1\)Here we assume the internal Markov chain with transition matrix \(P^{(n)}\) is irreducible and aperiodic.
2.2. Arm Decoupling by Lagrangian Relaxation. Consider the following relaxed version of the restless MAB:

\[
\begin{align*}
(2.9) & \quad \max_{\pi: \Omega(t) \to A(t)} \mathbb{E}_\pi \left[ \sum_{t=1}^{\infty} \sum_{n=1}^{N} \beta^{t-1} \mathbb{I}(n \in A(t)) \cdot S_n(t) \cdot O_n(t) \cdot B_n|\Omega(1) \right] \\
(2.10) & \quad \text{subject to } \mathbb{E}_\pi \left[ \sum_{t=1}^{\infty} \sum_{n=1}^{N} \beta^{t-1} \mathbb{I}(n \notin A(t))|\Omega(1) \right] = \frac{N-M}{1-\beta}.
\end{align*}
\]

Clearly constraint (2.10) is a relaxation on the player’s action \(A(t)\) from (2.8). Applying the Lagrangian multiplier \(\lambda\) to constraint (2.10), we arrive at the following *unconstrained* optimization problem:

\[
(2.11) \quad \max_{\pi: \Omega(t) \to A(t)} \mathbb{E}_\pi \left[ \sum_{t=1}^{\infty} \sum_{n=1}^{N} \beta^{t-1} \mathbb{I}(n \in A(t)) S_n(t) O_n(t) B_n + \lambda \mathbb{I}(n \notin A(t))|\Omega(1) \right].
\]

Fixing \(\lambda\), the above optimization is equivalent to \(N\) independent unconstrained optimization problem as shown below: for each \(n \in \{1, 2, \cdots, N\}\),

\[
(2.12) \quad \max_{\pi: \omega_n(t) \to \{0, 1\}} \mathbb{E}_\pi \left[ \sum_{t=1}^{\infty} \beta^{t-1} \mathbb{I}(n \in A(t)) S_n(t) O_n(t) B_n + \lambda \mathbb{I}(n \notin A(t))|\omega_n(1) \right].
\]

Here \(\pi\) is a single-arm policy that maps the belief state of the arm to the binary action \(u = 1\) (chosen/activated) or \(u = 0\) (unchosen/made passive). It is thus sufficient to consider a single arm for solving problem (2.11). For simplicity, we will drop the subscript \(n\) in consideration of a single-armed bandit problem without loss of generality. Let \(V_{\beta,m}(\omega)\) denote the value of (2.12) with \(\lambda = m\) and \(\omega_n(1) = \omega\), it is straightforward to write out the dynamic equation of the single-armed bandit problem as follows:

\[
(2.13) \quad V_{\beta,m}(\omega) = \max\{V_{\beta,m}(\omega; u = 1); V_{\beta,m}(\omega; u = 0)\},
\]

where \(V_{\beta,m}(\omega; u = 1)\) and \(V_{\beta,m}(\omega; u = 0)\) denote, respectively, the maximum expected total discounted reward that can be obtained if the arm is activated or made passive at the current belief state \(\omega\), followed by an optimal policy in subsequent slots. Since we consider the infinite-horizon problem, a stationary optimal policy can be chosen and the time index \(t\) is not needed in (2.13). Define the nonlinear operator \(\phi(\cdot)\) as

\[
(2.14) \quad \phi(\omega) = \frac{\epsilon \omega}{\epsilon \omega + 1 - \omega}.
\]

It is easy to see that \(T \circ \phi(\cdot)\) is Lipschitz continuous on \([0, 1]\):

\[
(2.15) \quad |T \left( \frac{\epsilon \omega}{\epsilon \omega + 1 - \omega} \right) - T \left( \frac{\epsilon \omega'}{\epsilon \omega' + 1 - \omega'} \right)| = \left| \frac{p_{11} \epsilon \omega + (1 - \omega) p_{01}}{\epsilon \omega + 1 - \omega} - \frac{p_{11} \epsilon \omega' + (1 - \omega') p_{01}}{\epsilon \omega' + 1 - \omega'} \right| \leq \frac{|p_{11} - p_{01}|}{\epsilon} |\omega - \omega'|.
\]

We assume that \(\epsilon \neq 0\) (otherwise the problem is reduced to that considered in [5]) and \(p_{11} \neq p_{01}\) (otherwise the belief update is independent of observations or actions).
and the problem becomes trivial). Without loss of generality, set $B = 1$. We have

$$
V_{\beta,m}(\omega; u = 1) = (1 - \epsilon)\omega + \beta[(1 - \epsilon)\omega V_{\beta,m}(p_{n1})
+ (1 - (1 - \epsilon)\omega)V_{\beta,m}(T(\phi(\omega)))],
$$

$$
V_{\beta,m}(\omega; u = 0) = m + \beta V_{\beta,m}(T(\omega)).
$$

Define passive set $P(m)$ as the set of all belief states at which taking the passive action $u = 0$ is optimal:

$$
P(m) \overset{\Delta}{=} \{ \omega : V_{\beta,m}(\omega; u = 1) \leq V_{\beta,m}(\omega; u = 0) \}.
$$

It is clear that $P(m)$ changes from the empty set to the closed interval $[0, 1]$ as $m$ increases from $-\infty$ to $\infty$. However, such change may not be monotonic as $m$ increases. If the passive set $P(m)$ increases monotonically with $m$, then for each value $\omega$ of the belief state, one can define the unique $m$ that makes it join $P(m)$ and stay in the set forever. Intuitively, such $m$ measures how attractive it is to activate the arm at the belief state $\omega$ compared to other belief states in a well-ordered manner: the larger $m$ required for it to be passive, the more incentives to activate at the belief state $\omega$ without $m$. This Lagrangian multiplier $m$ is thus called 'subsidy for passivity' by Whittle who formalized the following definition of indexability and Whittle index [9].

**Definition 2.1.** A restless multi-armed bandit is indexable if for each single-armed bandit with subsidy, the passive set of arm states increases monotonically from $\emptyset$ to the whole state space as $m$ increases from $-\infty$ to $+\infty$. Under indexability, the Whittle index of an arm state is defined as the infimum subsidy $m$ such that the state remains in the passive set.

For our model in which the arm state is given by the belief vector, the indexability is equivalent to the following:

$$
\text{If } V_{\beta,m}(\omega; u = 1) \leq V_{\beta,m}(\omega; u = 0),
\text{ then } \forall m' > m, V_{\beta,m'}(\omega; u = 1) \leq V_{\beta,m'}(\omega; u = 0).
$$

Under indexability, the Whittle index $W(\omega)$ of arm state $\omega$ is defined as

$$
W(\omega) \overset{\Delta}{=} \inf \{ m : V_{\beta,m}(\omega; u = 1) \leq V_{\beta,m}(\omega; u = 0) \}.
$$

In the following we give some useful properties of the value functions $V_{\beta,m}(\omega; u = 1)$, $V_{\beta,m}(\omega; u = 0)$ and $V_{\beta,m}(\omega)$. Our strategy is to first establish those properties for finite horizons and then extend them to the infinite horizon by the uniform convergence of the value functions of the former to the latter. Define the $T$-horizon value function $V_{1,T,\beta,m}(\omega)$ as the maximum expected total discounted reward achievable over the next $T$ time slots starting from the initial belief state $\omega$. Then

$$
V_{1,T,\beta,m}(\omega) = \max\{V_{1,T,\beta,m}(\omega; u = 1); V_{1,T,\beta,m}(\omega; u = 0)\},
$$

where $V_{1,T,\beta,m}(\omega; u = 1)$ and $V_{1,T,\beta,m}(\omega; u = 0)$ denote, respectively, the maximum expected total discounted reward achievable given the initial active and passive actions over the next $T$ time slots starting from the initial belief state $\omega$:

$$
V_{1,T,\beta,m}(\omega; u = 1) = (1 - \epsilon)\omega + (1 - \epsilon)\omega\beta V_{1,T-1,\beta,m}(p_{n1})
+ (1 - (1 - \epsilon)\omega)\beta V_{1,T-1,\beta,m}(T(\frac{\epsilon\omega}{1 - (1 - \epsilon)\omega})),
$$

$$
V_{1,T,\beta,m}(\omega; u = 0) = m + \beta V_{1,T-1,\beta,m}(T(\omega)),
$$

$$
V_{1,0,\beta,m}(\cdot) \equiv 0.
$$
From the above recursive equations, we can analyze $V_{1,T,\beta,m}(\omega)$ by backward induction on $T$. It is easy to see that for any $\omega$,

$$V_{1,1,\beta,m}(\omega; u = 1) = (1 - \epsilon)\omega, \quad V_{1,1,\beta,m}(\omega; u = 0) = m. \quad (2.24)$$

Therefore $V_{1,T,\beta,m}(\omega)$ is the maximum of two linear equations and thus piecewise linear and convex for $T = 1$ (in both $\omega$ and $m$). Exploiting the Bayes’ rule that leads to the following term

$$V_{1,T,\beta,m}(\omega) = (1 - (1 - \epsilon)\omega)\beta V_{1,T-1,\beta,m}(\mathcal{T}(\frac{\epsilon\omega}{1 - (1 - \epsilon)\omega})), \quad (2.25)$$

which has a coefficient $(1 - (1 - \epsilon)\omega)$ also appeared as the denominator of the expression inside the linear operator $\mathcal{T}$, the recursive equation set (2.21) and (2.22) shows that $V_{1,T,\beta,m}(\omega)$ is the maximum of two convex and piecewise linear functions and thus piecewise linear and convex for any $T > 1$ (in both $\omega$ and $m$). Motivated by the Lipschitz continuity of $\mathcal{T} \circ \phi$, we show in Lemma 2.3 that $V_{1,T,\beta,m}(\omega)$ is also Lipschitz continuous under certain conditions. In the following, we first establish a monotonic property of $V_{1,T,\beta,m}(\omega)$ in the case of $p_{11} > p_{01}$ (positively correlated Markov chain).

**Lemma 2.2.** If $p_{11} > p_{01}$, then $V_{1,T,\beta,m}(\omega)$ is monotonically increasing with $\omega \in [0, 1]$ for any $T \geq 1$.

**Proof.** Since $V_{1,T,\beta,m}(\omega)$ is piecewise linear, it is differentiable almost everywhere except on a null set (under the Lebesgue measure on $\mathbb{R}$) consisting of finite points among which both the left and right derivatives at any point exist but not equal. To prove that the continuous function $V_{1,T,\beta,m}(\omega)$ is monotonically increasing with $\omega$, we only need to show

$$V'_{1,T,\beta,m}(\omega) \geq 0, \quad \forall \omega \in (0, 1), \quad (2.26)$$

where $V'_{1,T,\beta,m}(\omega)$ denotes the right derivative of $V_{1,T,\beta,m}(\cdot)$ as a function of the belief state with $m$ fixed. From (2.24), the value function $V_{1,1,\beta,m}(\omega) = \max\{(1 - \epsilon)\omega, m\}$ is monotonically increasing with nonnegative right derivative $1 - \epsilon$ or $0$. Assume (2.26) is true for $T \geq 1$, then for $T + 1$ we have $V_{1,T+1,\beta,m}(\omega) = \max\{f_T(\omega), g_T(\omega)\}$ with:

$$\begin{cases}
  f_T(\omega) = (1 - \epsilon)\omega + (1 - \epsilon)\omega V_{1,T,\beta,m}(p_{11}) \\
  + (1 - (1 - \epsilon)\omega)\beta V_{1,T,\beta,m}(\mathcal{T} \circ \phi(\omega)), \\
  g_T(\omega) = m + \beta V_{1,T,\beta,m}(\mathcal{T}(\omega)).
\end{cases} \quad (2.27)$$

From the above, we have

$$\begin{cases}
  f'_T(\omega) = (1 - \epsilon) + (1 - \epsilon)\beta V_{1,T,\beta,m}(p_{11}) - (1 - \epsilon)\beta V_{1,T,\beta,m}(\mathcal{T} \circ \phi(\omega)) \\
  + V'_{1,T,\beta,m}(\mathcal{T} \circ \phi(\omega)) \frac{\epsilon \beta (p_{11} - p_{01})}{1 - (1 - \epsilon)\omega}, \\
  g'_T(\omega) = \beta (p_{11} - p_{01}) V'_{1,T,\beta,m}(\mathcal{T}(\omega)),
\end{cases} \quad (2.28)$$

where $f'_T(\cdot)$, $g'_T$ and $V'_{1,T,\beta,m}(\cdot)$ denote the right derivatives of the corresponding functions. We have used the fact that $\phi(\cdot)$ is monotonically increasing and when $p_{11} > p_{01}$, $\mathcal{T}(\cdot)$ is also monotonically increasing and that

$$\phi'(\omega) = \frac{\epsilon}{(1 - (1 - \epsilon)\omega)^2}. \quad (2.29)$$

This manuscript is for review purposes only.
By the induction hypothesis and (2.28), if \( p_{11} > p_{01} \) then \( g_T(\omega) \) is monotonically increasing (since \( g_T'(\omega) \geq 0 \)) and

\[
f_T'(\omega) = (1 - \epsilon) + (1 - \epsilon)\beta V_{1,T,\beta,m}(p_{11}) - (1 - \epsilon)\beta V_{1,T,\beta,m}(T(\frac{\epsilon \omega}{\epsilon \omega + 1 - \omega}))
\]

\[
+ (1 - \epsilon)\beta V_{1,T,\beta,m}(T(\frac{\epsilon \omega}{\epsilon \omega + 1 - \omega}))(T \circ \phi)'(\omega)
\]

\[
\geq (1 - \epsilon) + \frac{\epsilon \beta(p_{11} - p_{01})}{1 - (1 - \epsilon)\omega} V_{1,T,\beta,m}(T(\frac{\epsilon \omega}{\epsilon \omega + 1 - \omega}))
\]

> 0,

where both the first and second inequalities are due to the monotonically increasing property of \( V_{1,T,\beta,m}(\cdot) \) under the assumption that \( p_{11} > p_{01} \) by our induction hypothesis and

\[
p_{01} \leq T(\omega) \leq p_{11}, \quad 0 \leq \phi(\omega) \leq 1, \quad \forall \omega \in [0,1].
\]

This proves the monotonically increasing property of \( f_T(\omega) \). Hence \( V_{1,T+1,\beta,m}(\omega) = \max\{f_T(\omega), g_T(\omega)\} \) is also monotonically increasing and the proof by induction is finished. \( \Box \)

Now we show that under a constraint on the discount factor \( \beta \in (0,1) \), the value function \( V_{1,T,\beta,m}(\omega) \) is a Lipschitz function:

**Lemma 2.3.** Suppose that the discount factor \( \beta \in (0,1) \) satisfies

\[
\beta < \frac{1}{(2 - \epsilon)p_{11} - p_{01}}.
\]

Then \( \forall T \geq 1 \) and \( \forall \omega, \omega' \in [0,1] \),

\[
|V_{1,T,\beta,m}(\omega) - V_{1,T,\beta,m}(\omega')| \leq C|\omega - \omega'|, \quad \text{where}
\]

\[
C = \frac{1 - \epsilon}{1 - (2 - \epsilon)\beta(p_{11} - p_{01})}.
\]

**Proof.** We prove this lemma by induction. Without loss of generality, assume \( \omega < \omega' \). For the case of \( T = 1 \),

\[
|V_{1,1,\beta,m}(\omega) - V_{1,1,\beta,m}(\omega')| = \begin{cases} 
0, & \text{if } m \leq (1 - \epsilon)\omega'; \\
(1 - \epsilon)\omega' - m, & \text{if } (1 - \epsilon)\omega \leq m < (1 - \epsilon)\omega'; \\
(1 - \epsilon)\omega - \omega', & \text{if } m < (1 - \epsilon)\omega.
\end{cases}
\]

Thus \( |V_{1,1,\beta,m}(\omega) - V_{1,1,\beta,m}(\omega')| \leq (1 - \epsilon)|\omega - \omega'| \leq C|\omega - \omega'| \).

Assume that for \( T \geq 1 \), \( |V_{1,T,\beta,m}(\omega) - V_{1,T,\beta,m}(\omega')| \leq C|\omega - \omega'| \) holds, i.e., neither the left nor the right derivative of \( V_{1,T,\beta,m}(\cdot) \) can exceed \( C \). To prove \( |V_{1,T+1,\beta,m}(\omega) - V_{1,T+1,\beta,m}(\omega')| \leq C|\omega - \omega'| \), recall the definitions of \( f_T(\omega) \) and \( g_T(\omega) \) in (2.27) and their right derivatives \( f_T'(\omega) \) and \( g_T'(\omega) \) in (2.28) and observe the following inequalities:

\[
|V_{1,T,\beta,m}(p_{11}) - V_{1,T,\beta,m}(T(\frac{\epsilon \omega}{\epsilon \omega + 1 - \omega}))| \leq C|p_{11} - T(\frac{\epsilon \omega}{\epsilon \omega + 1 - \omega})| 
\]

\[
\leq C|p_{11} - p_{01}|,
\]

\[
\frac{\epsilon}{1 - (1 - \epsilon)\omega}|V_{1,T,\beta,m}(T(\frac{\epsilon \omega}{\epsilon \omega + 1 - \omega}))| \leq C.
\]
Thus we have the following lower and upper bounds on \( f_T'(\omega) \) and \( g_T'(\omega) \):

\[
\tag{2.36}
\begin{align*}
(1 - \epsilon) - (2 - \epsilon)\beta C|p_{11} - p_{01}| & \leq f_T'(\omega) \leq (1 - \epsilon) + (2 - \epsilon)\beta C|p_{11} - p_{01}|, \\
-C\beta|p_{11} - p_{01}| & \leq g_T'(\omega) \leq C\beta|p_{11} - p_{01}|.
\end{align*}
\]

Note that for the case of \( p_{11} < p_{01} \), the left derivative of \( V_{1,T,\beta,m}(\cdot) \) has been used due to the monotonically decreasing property of \( T(\cdot) \). From (2.28) and (2.36), we have

\[
|V'_{1,T+1,\beta,m}(\omega)| \leq (1 - \epsilon) + (2 - \epsilon)\beta C|p_{11} - p_{01}|
\]

\[
= (1 - \epsilon) + (2 - \epsilon)\beta|p_{11} - p_{01}| \frac{1 - \epsilon}{1 - (2 - \epsilon)\beta|p_{11} - p_{01}|}
\]

\[
= \frac{1 - \epsilon}{1 - (2 - \epsilon)\beta|p_{11} - p_{01}|}
\]

\[
= C.
\]

Since \( V_{1,T+1,\beta,m}(\omega) \) is absolutely continuous, the above implies that

\[
|V_{1,T+1,\beta,m}(\omega) - V_{1,T+1,\beta,m}(\omega')| \leq C|\omega - \omega'|.
\]

The proof is thus finished by the induction process. \( \blacksquare \)

Last, we give a lemma establishing the order of \( V'_{1,T,\beta,m}(\cdot; u = 1) \) and \( V'_{1,T,\beta,m}(\cdot; u = 0) \) under certain conditions which further leads to a threshold structure of the optimal single-arm policy as detailed in Sec 2.3.

**Lemma 2.4.** Suppose that \( p_{11} > p_{01} \) and \( \beta \leq \frac{1}{(3-\epsilon)(p_{11} - p_{01})} \), we have

\[
\tag{2.37}
V'_{1,T,\beta,m}(\omega; u = 1) \geq V'_{1,T,\beta,m}(\omega; u = 0),
\]

where \( V'_i_{1,T,\beta,m}(\omega; u = i) \) denotes the right derivative of \( V_{1,T,\beta,m}(\cdot; u = i) \) at \( \omega \) for \( i \in \{0, 1\} \). The above inequality is also true if \( p_{01} > p_{11} \) and \( \beta \leq \frac{1}{(5-2\epsilon)(p_{01} - p_{11})} \).

**Proof.** Again, we prove by induction on the time horizon \( T \). When \( T = 1 \), it is clear that \( V_{1,1,\beta,m}(\omega; u = 1) = (1 - \epsilon)\omega \) and \( V_{1,1,\beta,m}(\omega; u = 0) = m \):

\[
\tag{2.38} V'_{1,1,\beta,m}(\omega; u = 1) = 1 - \epsilon > V'_{1,1,\beta,m}(\omega; u = 0) = 0.
\]

Assume that \( V'_{1,1,\beta,m}(\omega; u = 1) \geq V'_{1,T,\beta,m}(\omega; u = 0) \) for \( T \geq 1 \). From (2.36), we have, in case of \( p_{01} > p_{11} \) and \( \beta \leq \frac{1}{(5-2\epsilon)(p_{01} - p_{11})} \),

\[
C\beta(p_{01} - p_{11}) \leq (1 - \epsilon) - (2 - \epsilon)\beta C(p_{01} - p_{11}),
\]

which shows that \( f_T'(\omega) \geq g_T'(\omega) \).

When \( p_{11} > p_{01} \), \( V_{1,T,\beta,m}(\omega) \) is increasing with \( \omega \) with nonnegative right derivatives by Lemma 2.2. We can thus obtain tighter bounds on \( f_T'(\omega) \) and \( g_T'(\omega) \):

\[
\tag{2.39}
\begin{align*}
(1 - \epsilon) \leq f_T'(\omega) \leq (1 - \epsilon) + (2 - \epsilon)\beta C(p_{11} - p_{01}), \\
0 \leq g_T'(\omega) \leq C\beta(p_{11} - p_{01}).
\end{align*}
\]

If \( \beta \leq \frac{1}{(3-\epsilon)(p_{11} - p_{01})} \), we have

\[
C\beta(p_{11} - p_{01}) \leq (1 - \epsilon),
\]

which shows that \( f_T'(\omega) \geq g_T'(\omega) \). The proof is thus complete. \( \blacksquare \)
2.3. Threshold Policy and Indexability. In this section, we show that the optimal single-arm policy is a threshold policy under the constraints on the discount factor \( \beta \) specified in Sec. 2.2 and analyze the conditions for indexability. First, for a finite-horizon single-armed bandit, a threshold policy \( \pi \) is defined by a time-dependent real number \( \omega_{T,\beta}(m) \) such that

\[
\omega_{T,\beta}(m) = \begin{cases} 
1, & \text{if } \omega > \omega_{T,\beta}(m); \\
0, & \text{if } \omega \leq \omega_{T,\beta}(m).
\end{cases}
\]

In the above \( u_{T,\beta}(\omega) \in \{0, 1\} \) is the action taken under \( \pi \) at the current state \( \omega \) with \( T \) slots remaining. Intuitively, the larger \( \omega \) is, the larger expected immediate reward to accrue and thus more attractive to activate the arm. We formally prove this statement under certain conditions in the following theorem.

**Theorem 2.5.** Suppose that \( p_{11} > p_{01} \) and \( \beta \leq \frac{1}{(3-\epsilon)(p_{11}-p_{01})} \). For any \( T \geq 1 \), the optimal single-arm policy \( \pi^* \) is a threshold policy, i.e., there exists \( \omega_{T,\beta}^*(m) \in \mathbb{R} \) such that under \( \pi^* \), the optimal action is

\[
u_{T,\beta}^*(\omega) = \begin{cases} 
1, & \text{if } \omega > \omega_{T,\beta}^*(m); \\
0, & \text{if } \omega \leq \omega_{T,\beta}^*(m).
\end{cases}
\]

Furthermore, at the threshold \( \omega_{T,\beta}^*(m) \),

\[
V_{1,T,\beta,m}(\omega; u = 0) = V_{1,T,\beta,m}(\omega_{T,\beta}^*(m); u = 1).
\]

The conclusion is also true for the case of \( p_{01} > p_{11} \) and \( \beta \leq \frac{1}{(5-2\epsilon)(p_{01}-p_{11})} \).

**Proof.** At \( T = 1 \), \( V_{1,1,\beta,m}(\omega; u = 1) = (1-\epsilon)\omega, V_{1,1,\beta,m}(\omega; u = 0) = m \). Thus we can choose \( \omega_{T,\beta}^*(m) \) as follows:

\[
\omega_{T,\beta}^*(m) = \begin{cases} 
c, & \text{if } m \geq 1 - \epsilon; \\
\frac{m}{1-\epsilon}, & \text{if } 0 \leq m < 1 - \epsilon; \\
b, & \text{if } m < 0,
\end{cases}
\]

where \( b < 0, c > 1 \) are arbitrary constants.

For \( T \geq 1 \), when the condition on \( \beta \) is satisfied, Lemma 2.4 shows that

\[
h_T(\omega) := V_{1,T,\beta,m}(1; u = 1) - V_{1,T,\beta,m}(1; u = 0),
\]

\[
h_T(\omega) \geq 0, \quad \forall \omega \in (0, 1).
\]

This shows that \( h_T(\cdot) \) is monotonically increasing and either has no zeros in the interval \([0, 1]\) or intersects with it over a closed interval (which can be a single point) only. Specially,

\[
\begin{align*}
V_{1,T,\beta,m}(0; u = 1) &= \beta V_{1,T-1,\beta,m}(p_{01}), \\
V_{1,T,\beta,m}(0; u = 0) &= m + \beta V_{1,T-1,\beta,m}(p_{01}), \\
V_{1,T,\beta,m}(1; u = 1) &= (1-\epsilon) + \beta V_{1,T-1,\beta,m}(p_{11}), \\
V_{1,T,\beta,m}(1; u = 0) &= m + \beta V_{1,T-1,\beta,m}(p_{11}).
\end{align*}
\]

Consider the following three regions of \( m \).

i) \( 0 \leq m < 1 - \epsilon \). In this case, \( V_{1,T,\beta,m}(0; u = 1) \leq V_{1,T,\beta,m}(0; u = 0) \) and
respectively can be shown similarly.

ii) \( m < 0 \). In this case, \( V_{1,T,\beta,m}(0; u = 1) > V_{1,T,\beta,m}(0; u = 0) \) and \( V_{1,T,\beta,m}(1; u = 1) > V_{1,T,\beta,m}(1; u = 0) \). So \( h_T(\cdot) \) is strictly positive over \([0,1]\) and we can choose \( \omega^*_T(\beta,m) = b \) with any \( b < 0 \).

iii) \( m \geq (1-\epsilon) \). In this case, always choosing the passive action is clearly optimal as the expected immediate reward is uniformly upper-bounded by \( m \) over the whole belief state space. We can thus choose \( \omega^*_T(\beta,m) = c \) with any \( c > 1 \).

In conclusion, when the conditions in the theorem are satisfied, the optimal finite-horizon single-arm policy is a threshold policy for any horizon length \( T \geq 1 \).

In the next theorem, we show that the optimal single-arm policy over the infinite horizon is also a threshold policy under the same conditions.

**Theorem 2.6.** Fix the subsidy \( m \). The finite-horizon value functions \( V_{1,T,\beta,m}(\cdot) \), \( V_{1,T,\beta,m}(\cdot; u = 1) \) and \( V_{1,T,\beta,m}(\cdot; u = 0) \) uniformly converge to the infinite-horizon value functions \( V_{\beta,m}(\cdot) \), \( V_{\beta,m}(\cdot; u = 1) \) and \( V_{\beta,m}(\cdot; u = 0) \) which are consequently obedient to the same properties established in Lemmas 2.2 and 2.3 and Theorem 2.5.

**Proof.** Consider the single-armed bandit with subsidy. We will apply two different scenarios to analyze the relation between \( V_{1,T,\beta,m}(\omega) \) and \( V_{\beta,m}(\omega) \). First, we apply a \( T \)-horizon optimal policy to the first \( T \) time slots and the infinite-horizon optimal policy to the future slots. Suppose that the initial belief state is \( \omega \) and after \( T \) slots the belief state becomes to a policy-dependent random variable \( \omega(T+1) \). Then the maximum expected total discounted reward under this policy is \( V_{1,T,\beta,m}(\omega) + \beta T \mathbb{E}[V_{\beta,m}(\omega(T+1))] \) where the expectation is taken w.r.t. \( \omega(T+1) \).

By the definition of \( V_{\beta,m}(\omega) \), we have

\[
V_{1,T,\beta,m}(\omega) + \beta T \mathbb{E}[V_{\beta,m}(\omega(T+1))] \leq V_{\beta,m}(\omega).
\]

In the second scenario, we apply the infinite-horizon optimal policy to the RMAB and the expected total discounted reward over the first \( T \) slots is \( V_{\beta,m}(\omega) - \beta T \mathbb{E}[V_{\beta,m}(\omega(T+1))] \). By the definition of \( V_{1,T,\beta,m}(\omega) \), we have

\[
V_{\beta,m}(\omega) - \beta T \mathbb{E}[V_{\beta,m}(\omega(T+1))] \leq V_{1,T,\beta,m}(\omega).
\]

Combining the two inequalities above, we obtain

\[
0 \leq V_{\beta,m}(\omega) - V_{1,T,\beta,m}(\omega) \leq \beta T \max\{1,m\} \frac{1}{1-\beta} \to 0 \quad \text{as} \quad T \to \infty.
\]

Thus \( V_{1,T,\beta,m}(\cdot) \) uniformly converges to \( V_{\beta,m}(\cdot) \) as \( T \to \infty \). The uniform convergence of \( V_{1,T,\beta,m}(\cdot; u = 1) \) and \( V_{1,T,\beta,m}(\cdot; u = 0) \) to \( V_{\beta,m}(\cdot; u = 1) \) and \( V_{\beta,m}(\cdot; u = 0) \) respectively can be shown similarly.

For any \( \epsilon > 0 \), there exists a \( T_0 \geq 1 \) such that \( |V_{\beta,m}(\omega) - V_{1,T,\beta,m}(\omega)| < \epsilon \) for any \( T \geq T_0 \) and any \( \omega \in [0,1] \). Therefore the properties of monotonicity and Lipschitz-continuity of \( V_{1,T,\beta,m}(\omega) \) established under certain conditions in Lemmas 2.2 and 2.3 and the convexity of the finite-horizon value functions also hold for \( V_{\beta,m}(\omega) \). Recall the definition of \( h_T(\omega) \) in (2.41) and define the continuous function (as the difference of two convex functions \( V_{\beta,m}(\cdot; u = 1) \) and \( V_{\beta,m}(\cdot; u = 0) \))

\[
(2.43) \quad h(\cdot) = V_{\beta,m}(\cdot; u = 1) - V_{\beta,m}(\cdot; u = 0) = \lim_{T \to \infty} h_T(\cdot).
\]

The above limit is well defined due to the uniform convergence of \( V_{1,T,\beta,m}(\cdot; u = i) \) to \( V_{\beta,m}(\cdot; u = i) \) as \( T \to \infty \) (for \( i \in \{0,1\} \)). Since \( h_T(\omega) \) is monotonically increasing
with \( \omega \) for any \( T \geq 1 \) according to Lemma 2.4, \( h(\omega) \) is also monotonically increasing
which shows that the infinite-horizon optimal policy is a threshold policy.

By far we have established the threshold structure of the optimal single-arm policy
with subsidy based on the analysis of \( V_{\beta,m}(\omega) \) as a function of the belief state \( \omega \) with \( m \)
fixed. To study the indexability condition, we now analyze the properties of \( V_{\beta,m}(\omega) \)
as a function of the subsidy \( m \) with the starting belief \( \omega \) fixed. From Definition 2.1
and the threshold structure of the optimal policy, the indexability of our model is
reduced to requiring that the threshold \( \omega^*_m(m) \) is monotonically increasing with \( m \) (if
the threshold is a closed interval then the right end is selected). Note that for the
infinite-horizon problem, the threshold \( \omega^*_m(m) \) is independent of time. Furthermore,
\( V_{\beta,m}(\omega) \) is also convex in \( m \) as for any \( m_1, m_2 \in \mathbb{R} \) and \( \lambda \in (0, 1) \) the optimal policy
\( \pi^*_m(\lambda m_1 + (1-\lambda)m_2) \) achieving \( V_{\beta,\lambda m_1 + (1-\lambda)m_2}(\omega) \) applied respectively on the problem
with subsidies \( m_1 \) and \( m_2 \) cannot outperform those achieving \( V_{\beta,m_1}(\omega) \) and \( V_{\beta,m_2}(\omega) \).
Specifically, let \( r_a \) be the expected total discounted reward from the active action and
\( r_p(m) \) that from the passive action under \( \pi^*_m(\lambda m_1 + (1-\lambda)m_2) \) applied to the problem
with subsidy \( m \), then

\[
\begin{align*}
(2.44) \quad & \lambda V_{\beta,m_1}(\omega) + (1-\lambda)V_{\beta,m_2}(\omega) \geq r_a + \lambda r_p(m_1) + (1-\lambda)r_p(m_2) \\
(2.45) \quad & = r_a + r_p(\lambda m_1 + (1-\lambda)m_2) \\
(2.46) \quad & = V_{\beta,\lambda m_1 + (1-\lambda)m_2}(\omega).
\end{align*}
\]

Since \( V_{\beta,m}(\omega) \) is convex in \( m \), its left and right derivatives with \( m \) exist at every point
\( m_0 \in \mathbb{R} \). Furthermore, consider two policies \( \pi^*_m(m_1) \) and \( \pi^*_m(m_2) \) achieving \( V_{\beta,m_1}(\omega) \)
and \( V_{\beta,m_2}(\omega) \) for any \( m_1, m_2 \in \mathbb{R} \), respectively. With a similar interchange argument
of \( \pi^*_m(m_1) \) and \( \pi^*_m(m_2) \) as above, we have

\[
\begin{align*}
(2.47) \quad & V_{\beta,m_1}(\omega) \geq V_{\beta,m_2}(\omega) - \frac{|m_1 - m_2|}{1-\beta}, \\
(2.48) \quad & V_{\beta,m_2}(\omega) \geq V_{\beta,m_1}(\omega) - \frac{|m_1 - m_2|}{1-\beta}.
\end{align*}
\]

Therefore \( |V_{\beta,m_1}(\omega) - V_{\beta,m_2}(\omega)| \leq \frac{1}{1-\beta}|m_1 - m_2| \) and \( V_{\beta,m}(\omega) \) is Lipschitz continuous
in \( m \). By the Rademacher theorem, \( V_{\beta,m}(\omega) \) is differentiable almost everywhere in \( m \).
For a small increase of \( m \), the rate at which \( V_{\beta,m}(\omega) \) increases is at least the expected
total discounted passive time under any optimal policy for the problem with subsidy
\( m \) starting from the belief state \( \omega \). In the following theorem, we formalize this relation
between the value function and the passive time as well as a sufficient condition for
the indexability of our model.

**Theorem 2.7.** Let \( \Pi^*_m(m) \) denote the set of all optimal single-arm policies achieving \( V_{\beta,m}(\omega) \) with initial belief state \( \omega \). Define the passive time

\[
(2.49) \quad D_{\beta,m}(\omega) \triangleq \max_{\pi^*_m(m) \in \Pi^*_m(m)} \mathbb{E} \sum_{t=1}^{\infty} \beta^{t-1} I(u(t) = 0)|\omega(1) = \omega|.
\]

The right derivative of the value function \( V_{\beta,m}(\omega) \) with \( m \), denoted by \( \frac{dV_{\beta,m}(\omega)}{(dm)^+} \), exists
at every value of \( m \) and

\[
(2.50) \quad \frac{dV_{\beta,m}(\omega)}{(dm)^+} \bigg|_{m=m_0} = D_{\beta,m_0}(\omega).
\]
Furthermore, the single-armed bandit is indexable if at least one of the following conditions is satisfied:

1. for any \( m_0 \in [0, 1 - \epsilon) \) the optimal policy is a threshold policy with threshold \( \omega^*_m(m_0) \in [0, 1) \) (if the threshold is a closed interval then the right end is selected) and

\[
(2.51) \quad \frac{dV_{\beta,m}(\omega^*_m(m_0); u = 0)}{(dm)^+} \bigg|_{m=m_0} > \frac{dV_{\beta,m}(\omega^*_m(m_0); u = 1)}{(dm)^+} \bigg|_{m=m_0}.
\]

2. for any \( m_0 \in \mathbb{R} \) and \( \omega \in P(m_0) \), we have

\[
(2.52) \quad \frac{dV_{\beta,m}(\omega; u = 0)}{(dm)^+} \bigg|_{m=m_0} \geq \frac{dV_{\beta,m}(\omega; u = 1)}{(dm)^+} \bigg|_{m=m_0}.
\]

**Proof.** The proof of (2.50) follows directly from the argument in Theorem 1 in [3] and is omitted here. To prove the sufficiency of (2.51), we note that if it is true then there exists a \( \Delta m > 0 \) such that

\[
V_{\beta,m}(\omega^*_m(m_0); u = 0) - V_{\beta,m}(\omega^*_m(m_0); u = 0) > V_{\beta,m}(\omega^*_m(m_0); u = 1) - V_{\beta,m}(\omega^*_m(m_0); u = 1), \forall m \in (m_0, m_0 + \Delta m).
\]

Since \( V_{\beta,m}(\omega^*_m(m_0); u = 0) = V_{\beta,m}(\omega^*_m(m_0); u = 0) > V_{\beta,m}(\omega^*_m(m_0); u = 1) \) which implies that the threshold \( \omega^*_m(m_0) \) remains in the passive set as \( m \) continuously increases so \( P(m) \) is monotonically increasing with \( m \). This conclusion is clearly true for the trivial case of \( m < 0 \) or \( m \geq 1 - \epsilon \). The sufficiency of (2.52) is obvious as any \( \omega \in P(m) \) is impossible to escape from \( P(m) \) as \( m \) increases due to the nondereasing property of \( V_{\beta,m}(\omega; u = 0) - V_{\beta,m}(\omega; u = 1) \) enforced by (2.52).

Theorem 2.7 essentially provides a way for checking the indexability condition in terms of the passive times. For example, equation (2.51) is equivalent to for any \( m \in [0, 1 - \epsilon) \),

\[
\beta[(1 - \epsilon)\omega^*_\beta(m)D_{\beta,m}(p_{11}) + (1 - (1 - \epsilon)\omega^*_\beta(m))]D_{\beta,m}(\mathcal{T}(\frac{\epsilon\omega^*_\beta(m)}{\epsilon\omega^*_\beta(m) + 1 - \omega^*_\beta(m)})) < 1 + \beta D_{\beta,m}(\mathcal{T}(\omega^*_\beta(m))).
\]

The above strict inequality clearly holds if \( \beta < 0.5 \) since \( D_{\beta,m}(\cdot) \in [0, \frac{1}{1-\beta}] \) for any \( m \in \mathbb{R} \). When \( \beta = 0.5 \), we prove by contradiction that the strict inequality (2.54) must hold under the threshold structure of the optimal policy. If \( \omega^*_\beta(m) = 0 \) then (2.54) is clearly true. Assume that the left and right sides of (2.54) are equal and \( \omega^*_\beta(m) \neq 0 \).

In this case, we have

\[
(2.55) \quad D_{\beta,m}(p_{11}) = \frac{1}{1 - \beta},
\]

\[
(2.56) \quad D_{\beta,m}(\mathcal{T}(\omega^*_\beta(m))) = 0.
\]

Equation (2.56) implies that starting from \( \mathcal{T}(\omega^*_\beta(m)) \), always activating the arm is strictly optimal. This means that the threshold \( \omega^*_\beta(m) \) is strictly below \( p_{11} \) and we have a contradiction to (2.55). Another easier way to see that the bandit is indexable
if $\beta \leq 0.5$ is that (2.52) would be satisfied since no strict inequality is required. However, condition (2.54) provides a convenient way for approximately computing the passive times as well as the value functions which leads to an efficient algorithm for evaluating the indexability and solving for the Whittle index function for any $\beta \in (0, 1)$, as detailed in the next section.

**Corollary 2.8.** The restless bandit is indexable if $\beta \leq 0.5$.

### 2.4. The Approximated Whittle Index

The threshold structure of the optimal single-arm policy under certain conditions yields the following iterative nature of the dynamic equations for both $D_{\beta,m}(\omega)$ and $V_{\beta,m}(\omega)$. Define the first crossing time

\begin{equation}
L(\omega, \omega') = \min_{0 \leq k < \infty} \{k : T^k(\omega) > \omega'\}.
\end{equation}

In the above $T^0(\omega) = \omega$ and we set $L(\omega, \omega') = +\infty$ if $T^k(\omega) \leq \omega$ for all $k \geq 0$.

Clearly $L(\omega, \omega')$ is the minimum time slots required for a belief state $\omega$ to stay in the passive set $P(m)$ before the arm is activated given a threshold $\omega' \in [0, 1)$. Consider the nontrivial case where $p_{01}, p_{11} \in (0, 1)$ and $p_{01} \neq p_{11}$ such that the Markov chain of the internal arm states is aperiodic and irreducible and that the belief update is action-dependent. From (2.5), if $p_{11} > p_{01}$ then

\begin{equation}
L(\omega, \omega') = \begin{cases} 
0, & \omega > \omega' \\
\frac{p_{01} - \omega' (1 - p_{11} + p_{01})}{1 - p_{11} + p_{01}}, & \omega \leq \omega' < \omega_0 \\
\log \frac{p_{11} - \omega' (1 - p_{01} + p_{11})}{p_{01} - \omega' (1 - p_{01} + p_{11})}, & \omega \leq \omega', \omega' \geq \omega_0 \\
\infty, & \omega \leq \omega', \omega' \leq \omega_0 
\end{cases}
\end{equation}

or if $p_{11} < p_{01}$ then

\begin{equation}
L(\omega, \omega') = \begin{cases} 
0, & \omega > \omega' \\
1, & \omega \leq \omega', T(\omega) > \omega' \\
\infty, & \omega \leq \omega', T(\omega) \leq \omega'
\end{cases}
\end{equation}

Suppose that the following conditions are satisfied such that the optimal single-arm policy is a threshold policy and the indexability holds:

\begin{equation}
\beta \leq \min\{\frac{1}{(3 - \epsilon)(p_{11} - p_{01})}, 0.5\}, \quad \text{if } p_{11} > p_{01} \\
\frac{1}{(5 - 2\epsilon)(p_{01} - p_{11})}, 0.5\}, \quad \text{if } p_{11} < p_{01}.
\end{equation}

To solve for the Whittle index function $W(\omega)$, given the current arm state $\omega$, we aim to find out the minimum subsidy $m$ that makes it as a threshold:

\begin{align}
V_{\beta,m}(\omega) &= V_{\beta,m}(\omega; u = 1) = V_{\beta,m}(\omega; u = 0), \\
V_{\beta,m}(\omega; u = 1) &= (1 - \epsilon)\omega + \beta[\epsilon V_{\beta,m}(p_{11})] \\
&\quad + (1 - (1 - \epsilon)\omega)\epsilon \phi(\omega)], \\
V_{\beta,m}(\omega; u = 0) &= m + \beta V_{\beta,m}(T(\omega)).
\end{align}

Given a threshold $\omega_0^\beta(m) \in [0, 1)$ and any $\omega \in [0, 1]$, the value function $V_{\beta,m}(\omega)$ can
be expanded by the first crossing time as
\[ V_{\beta,m}(\omega) = \frac{1 - \beta^{L(\omega,\omega^*_m(m))}}{1 - \beta} m + \beta^{L(\omega,\omega^*_m(m))} V_{\beta,m}(T^{L(\omega,\omega^*_m(m))}(\omega); u = 1) \]
\[ = \frac{1 - \beta^{L(\omega,\omega^*_m(m))}}{1 - \beta} m + \beta^{L(\omega,\omega^*_m(m))} \left\{ (1 - \epsilon) T^{L(\omega,\omega^*_m(m))}(\omega) \right\} \]
\[ + \beta \left[ (1 - \epsilon) T^{L(\omega,\omega^*_m(m))}(\omega) V_{\beta,m}(p_{11}) + (1 - (1 - \epsilon)) T^{L(\omega,\omega^*_m(m))}(\omega) \right] \\
\cdot V_{\beta,m} \left( T \left( \frac{\epsilon T^{L(\omega,\omega^*_m(m))}(\omega)}{\epsilon T^{L(\omega,\omega^*_m(m))}(\omega) + 1 - T^{L(\omega,\omega^*_m(m))}(\omega)} \right) \right) \right\}. \]
\[ \tag{2.64} \]

There is no doubt that the last item of the above equation has caused us trouble in solving for \( V_{\beta,m}(\omega) \). However, if we let
\[ f(\omega, \omega^*_m(m)) = T \left( \frac{\epsilon T^{L(\omega,\omega^*_m(m))}(\omega)}{\epsilon T^{L(\omega,\omega^*_m(m))}(\omega) + 1 - T^{L(\omega,\omega^*_m(m))}(\omega)} \right) \]
\[ = p_{11} \frac{\epsilon T^{L(\omega,\omega^*_m(m))}(\omega) + p_{01} (1 - T^{L(\omega,\omega^*_m(m))}(\omega))}{\epsilon T^{L(\omega,\omega^*_m(m))}(\omega) + 1 - T^{L(\omega,\omega^*_m(m))}(\omega)} \]
and construct iteratively the sequence \( \{k_n\} \) as \( k_{n+1} = f(k_n, \omega^*_m(m)) \) with \( k_0 = \omega \). We then get the following sequence of equations:
\[ V_{\beta,m}(k_0) = \frac{1 - \beta^{L(k_0,\omega^*_m(m))}}{1 - \beta} m + \beta^{L(k_0,\omega^*_m(m))} \left\{ (1 - \epsilon) T^{L(k_0,\omega^*_m(m))}(k_0) + \beta \left[ (1 - \epsilon) \right. \right. \]
\[ \left. \left. T^{L(k_0,\omega^*_m(m))}(k_0) \right] V_{\beta,m}(p_{11}) + (1 - (1 - \epsilon)) T^{L(k_0,\omega^*_m(m))}(k_0) \right] V_{\beta,m}(k_1) \right\} \]
\[ V_{\beta,m}(k_1) = \frac{1 - \beta^{L(k_1,\omega^*_m(m))}}{1 - \beta} m + \beta^{L(k_1,\omega^*_m(m))} \left\{ (1 - \epsilon) T^{L(k_1,\omega^*_m(m))}(k_1) + \beta \left[ (1 - \epsilon) \right. \right. \]
\[ \left. \left. T^{L(k_1,\omega^*_m(m))}(k_1) \right] V_{\beta,m}(p_{11}) + (1 - (1 - \epsilon)) T^{L(k_1,\omega^*_m(m))}(k_1) \right] V_{\beta,m}(k_2) \right\} \]
\[ \ldots \]
\[ V_{\beta,m}(k_n) = \frac{1 - \beta^{L(k_n,\omega^*_m(m))}}{1 - \beta} m + \beta^{L(k_n,\omega^*_m(m))} \left\{ (1 - \epsilon) T^{L(k_n,\omega^*_m(m))}(k_n) + \beta \left[ (1 - \epsilon) \right. \right. \]
\[ \left. \left. T^{L(k_n,\omega^*_m(m))}(k_n) \right] V_{\beta,m}(p_{11}) + (1 - (1 - \epsilon)) T^{L(k_n,\omega^*_m(m))}(k_n) \right] V_{\beta,m}(k_{n+1}) \right\} \]
\[ \ldots \]

For sufficiently large \( n \), we can get an estimation of \( V_{\beta,m}(\omega) = V_{\beta,m}(k_0) \) with an arbitrarily small error by setting \( V_{\beta,m}(k_{n+1}) = 0 \) whose error is discounted by \( \beta \) in computing \( V_{\beta,m}(k_n) \) thus causing a geometrically decreasing error propagation in the backward computation process for \( V_{\beta,m}(k_0) \). Note that we first compute \( V_{\beta,m}(p_{11}) \) in the same way by setting \( k_0 = p_{11} \) in the above equation set. Therefore we can have an estimation of \( V_{\beta,m}(\omega) \) with arbitrarily high precision for any \( \omega \in [0, 1] \). Interestingly, extensive numerical results found that \( \{k_n\} \) quickly converges to a limit belief state \( k \) (independent of \( k_0 \)) or oscillates within a small neighborhood of \( k \). Specifically, after 4 iterations, the difference \( |k_4 - k| \) becomes quite small so we can set \( V_{\beta,m}(k_5) = \ldots \)
the solved based on the following equations:

\[
\begin{align*}
V_{\beta,m}(p_{11}) &= \frac{1 - \beta^{L(p_{11}, \omega^*_m(m))}}{1 - \beta} m + \beta^{L(p_{11}, \omega^*_m(m))} \{(1 - \epsilon)T^{L(p_{11}, \omega^*_m(m))}(p_{11}) \\
&\quad + \beta \left[(1 - \epsilon)T^{L(p_{11}, \omega^*_m(m))}(p_{11})V_{\beta,m}(p_{11}) + (1 - (1 - \epsilon)T^{L(p_{11}, \omega^*_m(m))}(p_{11}))V_{\beta,m}(k_1)\right]\}, \\
V_{\beta,m}(k_1) &= \frac{1 - \beta^{L(k_1, \omega^*_m(m))}}{1 - \beta} m + \beta^{L(k_1, \omega^*_m(m))} \{(1 - \epsilon)T^{L(k_1, \omega^*_m(m))}(k_1) + \beta \left[(1 - \epsilon)T^{L(k_1, \omega^*_m(m))}(k_1)\right] \}
\end{align*}
\]

\[
\begin{align*}
V_{\beta,m}(k_n) &= \frac{1 - \beta^{L(k_n, \omega^*_m(m))}}{1 - \beta} m + \beta^{L(k_n, \omega^*_m(m))} \{(1 - \epsilon)T^{L(k_n, \omega^*_m(m))}(k_n) \\
&\quad + \beta \left[(1 - \epsilon)T^{L(k_n, \omega^*_m(m))}(k_n)\right] \}
\end{align*}
\]

According to Theorem 2.7, the passive time \(D_{\beta,m}(\omega)\) can also be approximately solved based on the following equations:

\[
\begin{align*}
D_{\beta,m}(p_{11}) &= \frac{1 - \beta^{L(p_{11}, \omega^*_m(m))}}{1 - \beta} + \beta^{L(p_{11}, \omega^*_m(m))} + 1 \{(1 - \epsilon)T^{L(p_{11}, \omega^*_m(m))}(p_{11}) \}
\end{align*}
\]

\[
\begin{align*}
D_{\beta,m}(k_1) &= \frac{1 - \beta^{L(k_1, \omega^*_m(m))}}{1 - \beta} + \beta^{L(k_1, \omega^*_m(m))} + 1 \{(1 - \epsilon)T^{L(k_1, \omega^*_m(m))}(k_1) \}
\end{align*}
\]

\[
\begin{align*}
D_{\beta,m}(k_n) &= \frac{1 - \beta^{L(k_n, \omega^*_m(m))}}{1 - \beta} + \beta^{L(k_n, \omega^*_m(m))} + 1 \{(1 - \epsilon)T^{L(k_n, \omega^*_m(m))}(k_n) \}
\end{align*}
\]

Substituting \(\omega\) for \(\omega^*_m(m)\) in the above \(n + 1\) linear equations with \(n + 1\) unknowns (first solving for \(\omega = p_{11}\)), we can obtain \(V_{\beta,m}(\omega')\) and \(D_{\beta,m}(\omega')\) for any \(\omega' \in [0, 1]\) according to the linear equation sets. The indexability condition (2.51) in Theorem 2.7 can be checked online: for the original multi-armed bandit problem and for each arm at state \(\omega(t)\) at time \(t\), we compute its approximated Whittle index \(W(\omega(t))\) by solving a set of linear equations, which has a polynomial complexity of the iteration number \(n\), independent of the decision time \(t\). At time \(t\), for each arm, if \(W(\cdot)\) is found to be nondecreasing with the arm states \((\omega(1), \omega(2), \cdots, \omega(t))\) appeared so far starting from the initial belief vector \(\Omega(1)\) defined in (2.1), then the indexability has not been violated. Interestingly, extensive numerical studies have shown that the indexability is always satisfied as illustrated in Sec. 4.

For large \(\beta \in (0, 1]\) where the threshold structure of the optimal policy or the indexability may not hold \((i.e., \text{condition (2.60) is not satisfied})\), we can still use the above process to solve for the subsidy \(m\) that makes (2.61) true if it exists. Note that after computing the value functions appeared in (2.61) in terms of \(m\), both
\[ V_{\beta,m}(\omega; u = 1) \text{ and } V_{\beta,m}(\omega; u = 0) \text{ are linear (affine) in } m \text{ and their equality gives a unique solution of } m \text{ if their linear coefficients are not equal. This } m, \text{ if exists, can thus be used as the approximated Whittle index } W(\omega) \text{ without requiring indexability or threshold-based optimal policy. If it does not exist, we can simply set } W(\omega) = \omega. \]

The existence of such an \( m \) is defined as the relaxed indexability in [3]. Note that extensive numerical studies have shown that the relaxed indexability of our model with imperfect state observations is always satisfied as well. Before summarizing our general algorithm for all \( \beta \in (0, 1) \) in Sec. 3, we solve for the approximated Whittle index function in closed-form for the simplest case of 0-iteration, which is referred to as the imperfect Whittle index. Note that if \( \epsilon \to 0 \) then \( \mathcal{T}(\frac{\epsilon}{\epsilon^2 + 1}) \to p_0 \). Thus when \( \epsilon \) is sufficiently small, we can approximate \( V_{\beta,m}(\mathcal{T}(\frac{\epsilon}{\epsilon^2 + 1})) \) by \( V_{\beta,m}(p_0) \). Under this approximation, we have, for any \( \omega \in [0, 1] \),

\[
\begin{align*}
V_{\beta,m}(\omega; u = 1) &= (1 - \epsilon)\omega + \beta (1 - \epsilon)\omega V_{\beta,m}(p_{11}) + (1 - (1 - \epsilon)\omega) V_{\beta,m}(p_{01}) \\
V_{\beta,m}(\omega; u = 0) &= m + \beta V_{\beta,m}(\mathcal{T}(\omega)) \\
V_{\beta,m}(\omega) &= \frac{1 - \beta L(\omega, \omega^*(m))}{1 - \beta} m + \beta L(\omega, \omega^*(m)) \{(1 - \epsilon)\mathcal{T}(\omega, \omega^*(m)) + (1 - (1 - \epsilon)\omega) V_{\beta,m}(p_{11}) + (1 - (1 - \epsilon)\omega) V_{\beta,m}(p_{01})\}.
\end{align*}
\]

By using the above three equations, we can directly solve for \( V_{\beta,m}(p_{01}) \) and \( V_{\beta,m}(p_{11}) \) in closed-form.

When \( p_{11} > p_{01} \), \( V_{\beta,m}(p_{01}) = \)

\[
\begin{align*}
\frac{(1 - \epsilon)p_{11} + (1 - (1 - \epsilon)p_{11}) V_{\beta,m}(p_{01})}{1 - (1 - \epsilon)p_{11}}, & \quad \text{if } \omega^*(m) < p_{01} \\
\frac{(1 - \beta)(1 - (1 - \epsilon)p_{11} + (1 - \epsilon)p_{11})}{(1 - (1 - \epsilon)p_{11})(1 - \beta L(p_{01}, \omega^*(m)))}, & \quad \text{if } p_{01} \leq \omega^*(m) < \omega_0 \\
\frac{m}{1 - \beta}, & \quad \text{if } \omega^*(m) \geq \omega_0
\end{align*}
\]

The approximate Whittle index is given by

\[
W(\omega) = \begin{cases} 
\frac{\omega(1 - \epsilon)(1 - \beta p_{11} + \beta(1 - \epsilon)p_{11})}{1 - (1 - \epsilon)p_{11}}, & \text{if } \omega \leq p_{01} \\
\frac{(1 - \beta)(1 - (1 - \epsilon)p_{11} + (1 - \epsilon)p_{11})}{(1 - (1 - \epsilon)p_{11})}, & \text{if } p_{01} < \omega < \omega_0 \\
\frac{(1 - \beta)(1 - (1 - \epsilon)p_{11} + (1 - \epsilon)p_{11})}{(1 - \epsilon)\omega}, & \text{if } \omega_0 \leq \omega < p_{11} \\
(1 - \epsilon)\omega, & \text{if } \omega \geq p_{11}
\end{cases}
\]

where

\[
\begin{align*}
C_1 &= \frac{(1 - \beta)(1 - \epsilon)p_{11})}{(1 - \beta)(1 - \epsilon)p_{11})(1 - \beta L(p_{01}, \omega)) + (1 - \epsilon)(1 - \beta)L(p_{01}, \omega) + 1 - \beta L(p_{01}, \omega)} \mathcal{T}(p_{01}, \omega) \mathcal{T}(p_{01}, \omega) \\
C_2 &= \frac{(1 - \beta)(1 - \epsilon)p_{11})(1 - \beta L(p_{01}, \omega) + 1 - \beta L(p_{01}, \omega))}{(1 - \beta)(1 - \epsilon)p_{11})(1 - \beta L(p_{01}, \omega) + 1 - \beta L(p_{01}, \omega))}.
\end{align*}
\]

This manuscript is for review purposes only.
Similarly, when $p_{01} > p_{11}$, we have

$$V_{\beta,m}(p_{11}) = \begin{cases} 
(1-\beta)(1-\epsilon)p_{01} + \beta(1-\epsilon)p_{01}, & \text{if } \omega_{\beta}^*(m) < p_{11} \\
(1-\beta)(1-\epsilon)p_{01} + \beta(1-\epsilon)p_{01} + \omega - \beta T(\omega), & \text{if } p_{11} \leq \omega_{\beta}^*(m) < T(p_{11}) \\
1 + \beta(1-\epsilon)p_{01} - \beta^2(1-\epsilon)T(p_{11}) + \omega - \beta T(\omega), & \text{if } \omega_{\beta}^*(m) \geq T(p_{11})
\end{cases}$$

The approximate Whittle index is given by

$$W(\omega) = \begin{cases} 
\frac{\omega(1-\epsilon)(1-\beta)p_{01} + \beta^2(1-\epsilon)p_{01}}{1-\beta(1-\beta)p_{01} + \beta(1-\epsilon)p_{01}}, & \text{if } \omega \leq p_{11} \\
1 - \beta(1-\epsilon)p_{01} + \beta(1-\epsilon)p_{01} + \omega - \beta T(\omega), & \text{if } p_{11} < \omega < \omega_{0} \\
1 + \beta(1-\epsilon)p_{01} - \beta^2(1-\epsilon)T(p_{11}) + \omega - \beta T(\omega), & \text{if } \omega_{0} \leq \omega < T(p_{11}) \\
1 + \beta(1-\epsilon)p_{01} - \beta^2(1-\epsilon)T(p_{11}) + \omega - \beta T(\omega), & \text{if } T(p_{11}) \leq \omega < p_{01} \\
(1-\epsilon)\omega, & \text{if } \omega \geq p_{01}
\end{cases}$$

where

$$C_3 = \frac{1 - \beta(1 - (1-\epsilon)p_{01})}{1 + \beta(1+\beta)(1-\epsilon)p_{01} - \beta^2(1-\epsilon)T(p_{11})},$$

$$C_4 = \frac{\beta(1-\epsilon)T(p_{11})(1-\beta) + \beta^2(1-\epsilon)p_{01}}{1 + \beta(1+\beta)(1-\epsilon)p_{01} - \beta^2(1-\epsilon)T(p_{11})}.$$
**Algorithm 3.1 Whittle Index Policy**

**Input:** \( \beta \in (0, 1) \), \( T \geq 1, N \geq 2, 1 \leq M < N \), iteration number \( k \)

**Input:** initial belief state \( \omega_n(1) \), \( \mathbf{P}^{(n)}, B_n, n = 1, \ldots, N \)

1: for \( t = 1, 2, \ldots, T \) do
2:   for \( n = 1, \ldots, N \) do
3:     Set the threshold \( \omega_n^*(m) = \omega_n(t) \) in (2.64)
4:     Compute \( L(p_{11}, \omega_n(t)) \) and set \( \omega = p_{11} \) in (2.64)
5:     Compute \( L(T \circ \phi(\omega), \omega_n(t)) \) and set \( \omega = T \circ \phi(\omega) \) in (2.64)
7:     Expand (2.64) to the \( k \)th step and solve for \( V_{\beta,m}(p_{11}) \)
6:     Compute \( L(T \circ \phi(\omega), \omega_n(t)) \) and set \( \omega = T \circ \phi(\omega) \) in (2.64)
8:     Expand (2.64) to the \( k \)th step and solve for \( V_{\beta,m}(T \circ \phi(\omega)) \) from \( V_{\beta,m}(p_{11}) \)
9:     Solve for \( V_{\beta,m}(\omega; u = 1) \) by \( V_{\beta,m}(p_{11}) \) and \( V_{\beta,m}(T \circ \phi(\omega)) \) as in (2.62)
10: Solve for \( V_{\beta,m}(\omega; u = 0) \) by \( V_{\beta,m}(T(\omega)) \) as in (2.63)
11: Evaluate the solvability of the linear equation of \( m: \ V_{\beta,m}(\omega; u = 1) = V_{\beta,m}(\omega; u = 0) \)
13: Set \( W(\omega_n(t)) = \omega_n(t)B_n \) and skip Step 14 if the above is unsolvable
14: Compute \( W(\omega_n(t)) \) as the solution to \( V_{\beta,m}(\omega; u = 1) = V_{\beta,m}(\omega; u = 0) \)
15: end for
16: Choose the top \( M \) arms with the largest Whittle Indices \( W(\omega_n(t)) \)
17: Observe the selected \( K \) arms and accrue reward \( O_n(t)S_n(t)B_n \) from each observed arm
18: for \( n = 1, \ldots, N \) do
19:   Update the belief state \( \omega_n(t) \) according to (2.3)
20: end for
21: end for

---

**Fig. 1.** Approximated \( W(\omega) \) \((p_{11} < p_{01})\)  
**Fig. 2.** Approximated \( W(\omega) \) \((p_{11} > p_{01})\)
Table 1

**Experiment Setting**

| System     | $\{p^{(1)}_{11}\}_{i=1}^7$ | $\{p^{(1)}_{01}\}_{i=1}^7$ | $\{B_i\}_{i=1}^7$ |
|------------|----------------------------|----------------------------|-------------------|
| System-1   | $\{0.3, 0.6, 0.4, 0.7, 0.2, 0.6, 0.8\}$ | $\{0.1, 0.4, 0.3, 0.4, 0.1, 0.3, 0.5\}$ | $\{0.8800, 0.2200, 0.3300, 0.1930, 1.0000, 0.2558, 0.1549\}$ |
| System-2   | $\{0.6, 0.4, 0.2, 0.2, 0.4, 0.1, 0.3\}$ | $\{0.8, 0.6, 0.4, 0.9, 0.8, 0.6, 0.7\}$ | $\{0.5150, 0.6666, 1.0000, 0.6296, 0.5833, 0.8100, 0.6700\}$ |
| System-3   | $\{0.1, 0.4, 0.3, 0.4, 0.1, 0.3, 0.5\}$ | $\{0.3, 0.6, 0.4, 0.7, 0.2, 0.6, 0.8\}$ | $\{0.7273, 0.3636, 0.5000, 0.3377, 1.0000, 0.3939, 0.2955\}$ |
| System-4   | $\{0.6, 0.7, 0.2, 0.6, 0.4, 0.5, 0.3\}$ | $\{0.8, 0.4, 0.9, 0.5, 0.7, 0.2, 0.6\}$ | $\{0.4286, 0.5000, 0.5397, 0.5143, 0.5306, 1.0000, 0.6190\}$ |

Table 2

**Experiment Setting (continued)**

| System | Example | $\epsilon$ | $\beta$ | Meet threshold conditions? | Meet indexability conditions? |
|--------|---------|------------|---------|----------------------------|-------------------------------|
| System-1 | 1 | 0.3 | 0.999 | yes | no |
|         | 2 | 0.1 | 0.999 | yes | no |
| System-2 | 3 | 0.3 | 0.29 | yes | yes |
|         | 4 | 0.1 | 0.29 | yes | yes |
|         | 5 | 0.3 | 0.48 | no | yes |
|         | 6 | 0.1 | 0.48 | no | yes |
| System-3 | 7 | 0.3 | 0.69 | yes | no |
|         | 8 | 0.1 | 0.69 | yes | no |
|         | 9 | 0.3 | 0.48 | yes | yes |
|         | 10 | 0.1 | 0.48 | yes | yes |
| System-4 | 11 | 0.3 | 0.29 | yes | yes |
|         | 12 | 0.1 | 0.29 | yes | yes |
|         | 13 | 0.3 | 0.48 | no | yes |
|         | 14 | 0.1 | 0.48 | no | yes |
|         | 15 | 0.3 | 0.999 | no | no |
|         | 16 | 0.1 | 0.999 | no | no |
Fig. 3. Example-1

Fig. 4. Example-2

Fig. 5. Example-3

Fig. 6. Example-4

Fig. 7. Example-5

Fig. 8. Example-6

This manuscript is for review purposes only.
Fig. 9. Example-7

Fig. 10. Example-8

Fig. 11. Example-9

Fig. 12. Example-10

Fig. 13. Example-11

Fig. 14. Example-12
REFERENCES

[1] J. C. Gittins, *Bandit processes and dynamic allocation indices*, Journal of the Royal Statistical Society: Series B, 41 (1979), pp. 148–177.

[2] J. C. Gittins, K. D. Glazebrook, and R. R. Weber, *Multi-Armed Bandit Allocation Indices*, Wiley, Chichester, 2nd ed., 2011.

[3] K. Liu, *Index policy for a class of partially observable markov decision processes*, (2021), https://arxiv.org/abs/2107.11939.

[4] K. Liu, R. R. Weber, and Q. Zhao, *Indexability and whittle index for restless bandit problems involving reset processes*, Proc. of the 50th IEEE Conference on Decision and Control, (2011), pp. 7690–7696.

[5] K. Liu and Q. Zhao, *Indexability of restless bandit problems and optimality of whittle index for dynamic multichannel access*, IEEE Transactions on Information Theory, 56 (2010), pp. 5547–5567.

[6] K. Liu, Q. Zhao, and B. Krishnamachari, *Dynamic multichannel access with imperfect channel state detection*, IEEE Transactions on Signal Processing, 58 (2010), pp. 2795–2808.

[7] C. H. Papadimitriou and J. N. Tsitsiklis, *The complexity of optimal queueing network control*, Mathematics of Operations Research, 24 (1999), pp. 293–305.

[8] E. J. Sondik, *The optimal control of partially observable markov processes over the infinite horizon: discounted costs*, Operations Research, 26 (1978), pp. 282–304.

[9] P. Whittle, *Restless bandits: Activity allocation in a changing world*, Journal of Applied Probability, 25 (1988), pp. 287–298.

This manuscript is for review purposes only.