Geometric-Algebra Adaptive Filters

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Abstract

This document introduces a new class of adaptive filters, namely Geometric-Algebra Adaptive Filters (GAAFs). Those are generated by formulating the underlying minimization problem (a least-squares cost function) from the perspective of Geometric Algebra (GA), a comprehensive mathematical language well-suited for the description of geometric transformations. Also, differently from the usual linear-algebra approach, Geometric Calculus (the extension of Geometric Algebra to differential calculus) allows to apply the same derivation techniques regardless of the type (subalgebra) of the data, i.e., real, complex-numbers, quaternions etc. Exploiting those characteristics (among others), a general least-squares cost function is posed, from which the GAAFs are designed. They provide a generalization of regular adaptive filters for any subalgebra of GA. From the obtained update rule, it is shown how to recover the following least-mean squares (LMS) adaptive filter variants: real-entries LMS, complex LMS, and quaternions LMS. Mean-square analysis and simulations in a system identification scenario are provided, showing almost perfect agreement for different levels of measurement noise.

Index Terms

Adaptive filtering, geometric algebra, quaternions.

I. INTRODUCTION

Since many decades ago, linear algebra (LA) has been the mathematical lingua franca across many scientific disciplines. Engineering sciences have resorted to the analytical tools of LA to understand and document their theoretical and experimental results. This is particularly true in signal processing, where basically all the theory can be described by matrices, vectors, and a norm induced by an inner product.

Adaptive filtering, which inherited the mathematical mindset of its parent disciplines (signal processing and control theory), has been successful in expanding its results based on LA. In the design of adaptive filters (AFs) for estimating vectors with real entries, there is no doubt about the efficiency of LA and standard vector calculus. Even if the number field is changed, e.g., from real numbers to complex numbers, requiring new calculus rules to be adopted (Cauchy-Riemann conditions) [1], LA is still a reliable tool.

However, the history of mathematics is richer than what is usually covered in engineering courses [2]. One may ask: how LA constructed its reputation along the years? Why is it largely adopted? And more importantly: is it the only way to describe and understand linear transformations? No, it is not. As a matter of fact, it can be shown that the tools of LA are only a subset of something larger. This work takes advantage of this more comprehensive theory, namely geometric algebra (GA), which encompasses not only LA but a number of other algebraic systems [3], [4].

One may interpret LA-based AFs as instances for geometric estimation, since the vectors to be estimated represent directed lines in an underlying vector space. However, to estimate areas, volumes, and hypersurfaces, a regular adaptive filter designed in light of LA might not be very helpful. As it turns out, LA has limitations regarding the representation of geometric structures [4]. Take for instance the inner product between two vectors: it always results in a scalar. Thus, one may wonder if it is possible to construct a new kind of product that takes two vectors (directed lines) and returns an area (or hypersurface, for a vector space with dimension $n > 3$). Or even a product that takes a vector and an area and returns a volume (or hypervolume). Similar ideas have been present since the advent of algebra, in an attempt to establish a deep connection with geometry.

The “new” product aforementioned is the geometric product, which is the product operation of GA (as defined in the next sections). In fact, GA and its product are anything but new. They have been available since the second half of the 19th century, about the same time LA started its ascension to be largely adopted. The geometric product allows one to actually map a set of vectors not only onto scalars, but also onto hypersurfaces, hypervolumes, and so on.
on. Thus, the use of GA increases the portfolio of geometric shapes and transformations one can represent. Also, its extension to calculus, geometric calculus (GC), allows for a clear and compact way to perform calculus with hypercomplex quantities, i.e., elements that generalize the complex numbers for higher dimensions (Section III-D).

It can be shown that multivectors (the fundamental hypercomplex elements of GA) are originated by operating elements of an orthonormal basis for an n-dimensional vector space over \( \mathbb{R} \) (real numbers) via the geometric product [3]–[5]. It means that hypercomplex quantities, e.g., complex numbers, quaternions etc., can be originated without resorting to a number field greater than the reals. This is an interesting feature of geometric algebras. It greatly simplifies the task of performing calculus with hypercomplex-valued elements, avoiding the need to adopt specific calculus rules for each type of multivector – GC inherently implements that.

Taking advantage of that, this work uses GA and GC to expand concepts of adaptive filtering theory and introduce new elements into it. The GAAFs devised herein are able to naturally estimate hypersurfaces, hypervolumes, and elements of greater dimensions (multivectors). In this sense, this research exploits the tools of GA and GC to generate a new class of AFs capable of encompassing the regular ones. For instance, filters like the regular least-mean squares (LMS) – with real entries, the Complex LMS (CLMS) – with complex entries, and the Quaternion LMS (QLMS) – with quaternion entries, are recovered as special cases of the more comprehensive GA-LMS introduced by this work.

The text is organized as follows. Section II presents the notation and explains the system identification (system ID) scenario for testing the GAAFs. Section III covers the fundamentals of GA, providing a brief history on the subject, several important definitions and relations, and the very basics of GC. Hopefully, only very specific points will require consulting extra literature. Whenever this happens, proper references are cited.

Section IV recasts standard linear estimation results into GA. Definitions like random multivectors, array of multivectors and array product are provided. More importantly, it is shown that the cost function of the system ID problem is a particular case of a general cost function that can only be written using geometric multiplication.

Section V introduces GAAFs for the system ID application. The problem is posed in light of GA, the gradient of the cost function is calculated and the GA-LMS is devised. Also, mean-square analysis (steady state) is provided with the support of the energy conservation relations [1].

Experiments\(^1\) for a system ID scenario are shown in Section VI to assess the performance of the GAAFs. Simulations depicting several learning curves corroborate theoretical predictions almost perfectly. This study is performed for four different subalgebras of GA and highlights the unique features originated from the combination of GA and adaptive filtering theories.

Finally, discussion, conclusion, and topics for future research are presented in Section VII.

II. PRELIMINARIES

This section introduces preliminary material to support the exposition made in the following sections.

A. Notation

Boldface letters are used for random quantities and normal font letters for deterministic quantities. For instance, \( z \) is a realization (deterministic) of the random variable \( Z \).

Capital letters are used for general multivectors (see Definition 6 further ahead) and matrices. For instance, \( A \) is a general multivector, while \( a \) is a random multivector. There are only two matrices in this text: rotation matrix \( R \) and identity \( I_d \).

Small letters represent arrays of multivectors, vectors and scalars. For example, \( z \) is an array of multivectors, a vector or scalar. The type of the variable will be clear from the context. Also, it is important to notice the following exceptions (properly justified in the body of the text): the small letter \( r \) represents a rotor (a kind of multivector); the small letter \( d \) represents a general multivector; the small letter \( v \) represents a general multivector.

The time-dependency of a scalar or multivector quantity is denoted by parentheses, while subscripts are employed to denote the time-dependency of arrays and vectors. For instance, \( u(i) \) is a time-varying scalar and \( u_i \) is a time-varying array or vector, while \( U(i) \) is a time-varying general multivector.

\(^1\)All source codes and scripts are available on openga.org, a companion website to this text.
Fig. 1. The system identification scenario.

B. The System Identification Problem

A system identification scenario is adopted in Section VI to evaluate the performance of the standard GAAFs devised in Section V.

Consider the schematic depicted in Fig. 1. The goal is to estimate the entries (coefficients) of an unknown plant (system) modeled by an $M \times 1$ vector $w^o$, which relates the system input-output via

$$d(i) = u_i^H w^o + v(i),$$

where $u_i$ is an $M \times 1$ vector (sometimes known as the regressor, which collects input samples as $u_i = [u(i) u(i - 1) \cdots u(i - M + 1)]$, $H$ denotes Hermitian conjugate, and $v(i)$ represents measurement noise typically modeled as a white Gaussian process with variance $\sigma_v^2$ [1], [6].

At each iteration, the unknown plant and the adaptive system $w_i$ are fed with the same regressor $u_i$. The output $d(i)$ of the unknown system is contaminated by measurement noise $v(i)$ and the adaptive system output is subtracted from $d(i)$. This generates the output estimation error $e(i)$ which is fed back into the estimator in order to update its coefficients $w_i$. That iterative process continues until the adaptive system has converged (steady-state), minimizing $e(i)$, usually in the mean-square sense. At that stage ($i \to \infty$), $w_i$ is the best estimate of the unknown plant $w^o$.

III. Fundamentals of Geometric Algebra

Geometric Algebra (also called Clifford Algebra after the British mathematician William Kingdon Clifford) was first developed as a mathematical language to unify all the different algebraic systems trying to express geometric relations/transformations, e.g., rotation and translation [3]–[5], [7]. All the following geometric systems are particular cases (subalgebras) of GA: vector and matrix algebras, complex numbers, and quaternions (see Section III-B). Depending on the application, one system is more appropriate than the others, and sometimes it is necessary to employ two or more of those algebras in order to precisely describe the geometric relations. Before the advent of GA, this eventually resulted in lack of clarity due to the extensive translations from one system to the other.

A complete coverage of GA theory is not in the scope of this text. For an in-depth discussion about GA theory & history, and its importance to Physics, please refer to [3]–[5], [7]–[9]. For applications of GA in engineering and computer science, check [10]–[15]. Finally, to contextualize the development of GA with the advent of abstract algebra, the reading of the historical report in [2] is recommended.

A. Constructing the Geometric Algebra of a Vector Space

In this Section, the Geometric algebra of a vector space is gradually constructed. Along the process, a series of definitions are presented. The explanation starts with the definition of algebra.

**Definition 1** (Definition of Algebra). A vector space $V$ over the reals $\mathbb{R}$, equipped with a bilinear product $V \times V \to V$ denoted by $\circ$, is said to be an algebra over $\mathbb{R}$ if the following relations hold $\forall \{a, b, c \} \in V$ and $\{\alpha, \beta\} \in \mathbb{R}$ [5], [7]:

$$
(a + b) \circ c = a \circ c + b \circ c \quad \text{(Left distributivity)}
$$

$$
c \circ (a + b) = c \circ a + c \circ b \quad \text{(Right distributivity)}
$$

$$
(\alpha a) \circ (\beta b) = (\alpha \beta)(a \circ b) \quad \text{(Compatibility with scalars)}.
$$

(2)
The associative property, i.e., \((a \circ b) \circ c = a \circ (b \circ c)\), does not necessarily hold for the product \(\circ\).

In a nutshell, the GA of a vector space \(\mathbb{V}\) over the reals \(\mathbb{R}\), namely \(\mathcal{G}(\mathbb{V})\), is a geometric extension of \(\mathbb{V}\) which enables algebraic representation of orientation and magnitude. Vectors in \(\mathbb{V}\) are also vectors in \(\mathcal{G}(\mathbb{V})\). The properties of \(\mathcal{G}(\mathbb{V})\) are defined by the signature of \(\mathbb{V}\):

**Definition 2** (Signature of a Vector Space/Algebra). Let \(\mathbb{V} = \mathbb{R}^n = \mathbb{R}^{p,q,r}\), with \(n = p + q + r\). The signature of a vector space (and by extension of the algebra constructed from it) is expressed in terms of the values \(\{p, q, r\}\), i.e., \(\mathbb{R}^n = \mathbb{R}^{p,q,r}\) has signature \(\{p, q, r\}\). An orthonormal basis of \(\mathbb{R}^n = \mathbb{R}^{p,q,r}\) has \(p\) vectors that square to 1, \(q\) vectors that square to \(-1\), and \(r\) vectors that square to 0.

In the signal processing literature, which is built on top of the theory of linear algebra (LA), one usually considers only vector spaces for which the basis elements square to \(1\), i.e., \(q = r = 0 \Rightarrow \mathbb{R}^{p,0,0} = \mathbb{R}^{n,0,0} = \mathbb{R}^n\). Thus, one can say that \(\mathbb{R}^{p,0,0}\) has Euclidean signature (see [3, p.42 and p.102]). GA allows for a more comprehensive approach to vector spaces. It naturally takes into account the so-called pseudo-Euclidean spaces, where \(q\) and \(r\) can be different than zero. Such feature allows to build algebras with pseudo-Euclidean signatures. From here on, the derivations require only Euclidean signatures, except when otherwise noted.

The main product of the algebra \(\mathcal{G}(\mathbb{V})\) is the so-called geometric product. Before defining it, it is first necessary to define the *inner* and *outer* products. Those are approached by considering vectors \(a\) and \(b\) in the vector space \(\mathbb{R}^n\).

**Definition 3** (Inner Product of Vectors). The inner product \(a \cdot b\), \(\{a, b\} \in \mathbb{R}^n\), is the usual vector product of linear algebra, defining the (linear) algebra generated by the vector space \(\mathbb{R}^n\). This way, \(a \cdot b\) results in a scalar,

\[
a \cdot b = |a||b|\cos\theta,
\]

in which \(\theta\) is the angle between \(a\) and \(b\). Additionally, the inner product is commutative, i.e., \(a \cdot b = b \cdot a\). See Fig. 2.

**Definition 4** (Outer Product of Vectors). The outer product \(a \wedge b\), \(\{a, b\} \in \mathbb{R}^n\), is the usual product in the exterior algebra of Grassman [7]. The multiplication \(a \wedge b\) results in an oriented area or bivector. Such an area can be interpreted as the parallellogram (hyperplane) generated when vector \(a\) is swept on the direction determined by vector \(b\) (See Fig. 2). The resulting bivector (oriented area) is uniquely determined by this geometric construction. That is the reason it may be considered as a kind of product of the vectors \(a\) and \(b\). This way, \(a \wedge b = C\), where \(C\) is the oriented area (bivector). Alternatively, the outer product can be defined as a function of the angle \(\theta\) between \(a\) and \(b\)

\[
a \wedge b = C = I_{a,b}|a||b|\sin\theta,
\]

where \(I_{a,b}\) is the unit bivector\(^2\) that defines the orientation of the hyperplane \(a \wedge b\) [4, p.66].

The outer product is noncommutative, i.e., \(a \wedge b = -b \wedge a\). This can be concluded from Fig. 2: the orientation of the area generated by sweeping \(a\) along \(b\) \((a \wedge b)\) is opposite to the orientation of the area generated by sweeping \(b\) along \(a\) \((b \wedge a)\).

For a detailed exposition on the nature of the outer product, please refer to [4, p.20] and [7, p.32].

**Definition 5** (Geometric Product of Vectors). The geometric product is defined as

\[
ab \triangleq a \cdot b + a \wedge b,
\]

in terms of the inner \((\cdot)\) and outer \((\wedge)\) products ([5], Sec. 2.2).

**Remark 1.** Note that in general the geometric product is noncommutative because \(a \wedge b = -(b \wedge a)\), resulting in \(ab = -ba\). Also, it is associative, \(a(bc) = (ab)c\), \(\{a, b, c\} \in \mathbb{R}^n\).

In this text, from now on, all products are geometric products, unless otherwise noted.

Next, the general element of a geometric algebra, the so-called multivector, is defined.

\(^2\)An unit bivector is the result of the outer product between two unit vectors, i.e., vector with unitary norm.
Definition 6 (Multivector (Clifford number)). A is a multivector (Clifford number), the basic element of a Geometric Algebra $\mathcal{G}$,

$$A = \langle A \rangle_0 + \langle A \rangle_1 + \langle A \rangle_2 + \cdots = \sum_g \langle A \rangle_g,$$

(6)

which is comprised of its g-grades (or g-vectors) $\langle \cdot \rangle_g$, e.g., $g = 0$ (scalars), $g = 1$ (vectors), $g = 2$ (bivectors, generated via the geometric multiplication of two vectors), $g = 3$ (trivectors, generated via the geometric multiplication of three vectors), and so on. The ability to group together scalars, vectors, and hyperplanes in a unique element (the multivector $A$) is the foundation on top of which GA theory is built on.

Remark 2. Recall Section II-A: except where otherwise noted, scalars ($g = 0$) and vectors ($g = 1$) are represented by lower-case letters, e.g., $a$ and $b$, and general multivectors by upper-case letters, e.g., $A$ and $B$. Also, in $\mathbb{R}^3$, $\langle A \rangle_g = 0$, $g > 3$, i.e., there are no grades greater than three [4, p.42].

Definition 7 (Grade operator). To retrieve the grade $p$ of a multivector,

$$\langle A \rangle_p \triangleq A_p; \quad p = 0 \Rightarrow \langle A \rangle_0 \equiv \langle A \rangle.$$

(7)

This way, multivectors are the elements that populate the geometric algebra of a given vector space. Moreover, the concept of a multivector, which is central in GA theory, allows for “summing apples and oranges” in a well-defined fashion. Vectors can be added to (multiplied by) scalars, which can then be added to (multiplied by) bivectors, and so on, without having to adopt special rules: the same algebraic tools can be applied to any of those quantities (subalgebras). This represents an amazing analytic advantage when compared to linear algebra, where scalars and vectors belong to separated realms. It also gives support to the idea presented in Section I: the field of real numbers, combined with a sophisticated algebra like GA, is enough to perform analysis with hypercomplex quantities (there might be no need for a number field more comprehensive than $\mathbb{R}$, e.g., the complex numbers field $\mathbb{C}$).

Now let the set of vectors $\{\gamma_k\} \in \mathbb{R}^n, k = 1, 2, \cdots, n,$

$$\{\gamma_1, \gamma_2, \cdots, \gamma_p, \gamma_{p+1}, \cdots, \gamma_{p+q}, \gamma_{p+q+1}, \cdots, \gamma_n\},$$

(8)

with $n = p + q + r$ (recall Definition 2), for which the following relations hold

$$\gamma_k^2 = \begin{cases} 1, & k = 1, \cdots, p \text{ (square to 1)} \\ -1, & k = p + 1, \cdots, p + q \text{ (square to -1)} \\ 0, & k = p + q + 1, \cdots, n \text{ (square to 0)}, \end{cases}$$

(9)

be an orthonormal basis of $\mathbb{R}^n$. Using that, the Geometric (Clifford) algebra can be formally defined:

Definition 8 (Clifford Algebra). Given an orthonormal basis of $\mathbb{R}^n$, its elements form a Geometric (Clifford) algebra $\mathcal{G}(\mathbb{R}^n)$ via the geometric product according to the rule [5], [7]

$$\gamma_k \gamma_j + \gamma_j \gamma_k = 2\gamma_k^2 \delta_{k,j}, \quad k, j = 1, \cdots, n,$$

(10)
where $\delta_{k,j} = 1$ for $k = j$, and $\delta_{k,j} = 0$ for $k \neq j$, which emphasizes the noncommutativity of the geometric product.

Thus, a basis for the geometric algebra $\mathcal{G}(\mathbb{R}^n)$ is obtained by multiplying the $n$ vectors in (8) (plus the scalar 1) according to (10). This procedure generates $2^n$ members (multivectors), defining an algebra and its dimension.

**Definition 9 (Subspaces and dimensions).** Consider a vector space $\mathcal{V}$, whose basis has dimension $n$, which generates the complete Geometric Algebra of $\mathcal{V}$ (or $\mathcal{G}(\mathcal{V})$). Adding and multiplying $g$ linearly-independent vectors ($g \leq n$) in $\mathcal{V}$ generates a linear subspace $\mathcal{G}^g(\mathcal{V})$ (closed under the geometric product) of $\mathcal{G}(\mathcal{V})$. The dimension of each subspace $\mathcal{G}^g(\mathcal{V})$ is $\binom{n}{g}$. Thus, the dimension of the complete algebra $\mathcal{G}(\mathcal{V})$ is (3), p.19

\[ \dim\{\mathcal{G}(\mathcal{V})\} = \sum_{g=0}^{n} \dim\{\mathcal{G}^g(\mathcal{V})\} = \sum_{g=0}^{n} \binom{n}{g} = 2^n \tag{11} \]

When $n = 3 \Rightarrow \mathcal{V} = \mathbb{R}^3$, which is the main case studied in this work, (8) becomes

\[ \{\gamma_1, \gamma_2, \gamma_3\}. \tag{12} \]

This way, according to (11), $\mathcal{G}(\mathbb{R}^3)$ has dimension $2^3 = 8$, with basis

\[ \{1, \gamma_1, \gamma_2, \gamma_3, \gamma_{12}, \gamma_{23}, \gamma_{31}, I\}, \tag{13} \]

which, as aforementioned, is obtained by multiplying the elements of (12) (plus the scalar 1) via the geometric product. Note that (13) has one scalar, three orthonormal vectors $\gamma_i$ (basis for $\mathbb{R}^3$), three bivectors (oriented areas) $\gamma_{ij} \triangleq \gamma_i \wedge \gamma_j, i \neq j$ ($\gamma_i \cdot \gamma_j = 0, i \neq j$), and one trivector (pseudoscalar$^3$) $I \triangleq \gamma_1 \gamma_2 \gamma_3 = \gamma_{123}$ (Fig. 3).

To illustrate the geometric multiplication between elements of $\mathcal{G}(\mathbb{R}^3)$, take two multivectors $A = \gamma_1$ and $B = 2\gamma_1 + 4\gamma_3$. Then, $AB = \gamma_1(2\gamma_1 + 4\gamma_3) = \gamma_1 \cdot (2\gamma_1 + 4\gamma_3) + \gamma_1 \wedge (2\gamma_1 + 4\gamma_3) = 2 + 4(\gamma_1 \wedge \gamma_3) = 2 + 4\gamma_{13}$ (a scalar plus a bivector).

In the sequel, it is shown how the geometric algebra $\mathcal{G}(\mathbb{R}^n)$ encompasses subalgebras of interest, e.g., rotor algebra. In particular, some well-known algebras like complex numbers and quaternion algebras are retrieved from the complete $\mathcal{G}(\mathbb{R}^n)$ via isomorphism.

### B. Subalgebras and Isomorphisms

As pointed out in Definition 9, adding and multiplying $g$ linearly-independent vectors in a given set $\mathcal{V}$ generates a subalgebra $\mathcal{G}^g(\mathcal{V})$ (closed under the geometric product) of $\mathcal{G}(\mathcal{V})$. This endows the GA of $\mathcal{V}$ with the capability of encompassing previously known algebras, like the ones originated by real, complex, and quaternion numbers.

In abstract algebra, two structures are said to be *isomorphic* if they have equivalent algebraic properties, enabling the use of one or the other interchangeably [4], [5]. In other words, the algebras are mutually identified, with well-defined correspondences (bijective relationship) between their elements.

$^3$The proper definition of pseudoscalar is given further ahead in (29).
Thus, it is clear that $G$ is the visual representation of the pseudovector of $\gamma_j$. This way, $G$ is also a subalgebra of $\gamma_j$. This helps to visualize any subalgebra of $\gamma_j$. A special group of subalgebras, the so-called even-grade subalgebras, will be necessary during the development of the GAAFs.

**Definition 10** (Even-Grade Algebra). A (sub)algebra is said to be even-grade (or simply even), and denoted $\gamma^+$, if it is composed only by even-grade elements, i.e., scalars ($g = 0$), bivectors ($g = 2$), 4-vectors ($g = 4$), and so on. For instance, a multivector $A$ in the even subalgebra $\gamma^+(\mathbb{R}^3)$ has the general form

$$A = \langle A \rangle_0 + \langle A \rangle_2,$$

where $\langle A \rangle_1 = \langle A \rangle_3 = 0$. (15)

The even subalgebra $\gamma^+(\mathbb{R}^n)$ is known as the algebra of rotors, i.e., its elements are able to apply $n$-dimensional rotations to vectors in $\mathbb{R}^n$. □

**Remark 3.** Similarly, the odd-grade part of an algebra is composed only by odd-grade elements and denoted $\gamma^-$. For $A$ in $\gamma^-$, $A = \langle A \rangle_0 + \langle A \rangle_3$, where $\langle A \rangle_0 = \langle A \rangle_2 = 0$. This way, $\gamma^\pm(\mathbb{R}^3) = \gamma^+(\mathbb{R}^3) + \gamma^-(\mathbb{R}^3)$. Note that, differently from $\gamma^+$, $\gamma^-$ is not a subalgebra since it is not closed under the geometric product – it is only a subspace. □

2) **Rotor Algebra of $\mathbb{R}^2$ (Complex Numbers):** The complex-numbers algebra is isomorphic to the even subalgebra $\gamma^+(\mathbb{R}^2)$, which has basis

$$\{1, \gamma_{12}\}.$$ (16)

Thus, it is clear that $\gamma^+(\mathbb{R}^2)$ is also a subalgebra of $\gamma^+(\mathbb{R}^3)$ (with basis given by (13)).

Fig. 4 shows the oriented area (bivector) created by the geometric multiplication between $\gamma_1$ and $\gamma_2$. That area is the visual representation of the pseudovector of $\gamma^+(\mathbb{R}^2)$, namely $\gamma_{12}$. The isomorphism to the complex algebra is established by identifying the imaginary unit $j$ with the pseudovector, $j = \gamma_{12} = \gamma_1 \gamma_2 = \gamma_1 \wedge \gamma_2$. From Table I it is known that $(\gamma_{12})^2 = -1$. Then, due to the isomorphism, $j^2 = -1$.

Section VI-C resorts to this isomorphism to test the performance of a GA-based AF which is equivalent to the Complex LMS (CLMS) [23].
3) Rotor Algebra of \( \mathbb{R}^3 \) (Quaternions): The even subalgebra \( \mathbb{G}^+(\mathbb{R}^3) \) has basis
\[
\{1, \gamma_{12}, \gamma_{23}, \gamma_{31}\}.
\] (17)

By adopting the following correspondences, \( \mathbb{G}^+(\mathbb{R}^3) \) is shown to be isomorphic to quaternion algebra \([5], [24]\):
\[
i \leftrightarrow -\gamma_{12} \quad j \leftrightarrow -\gamma_{23} \quad k \leftrightarrow -\gamma_{31},
\] (18)
where \( \{i, j, k\} \) are the three imaginary unities of quaternion algebra. The minus signs are necessary to make the product between two bivectors equal to the third one and not minus the third, e.g.
\[
(-\gamma_{12})(-\gamma_{23}) = \gamma_{13} = -\gamma_{31},
\]
just like in quaternion algebra, i.e. \( ij = k, \) \( jk = i, \) and \( ki = j \) \([5]\). Again, from Table I it is known that \( (\gamma_{12})^2 = -1 = i^2, \) \( (\gamma_{23})^2 = -1 = j^2, \) and \( (\gamma_{31})^2 = -1 = k^2. \)

A rotor \( r \in \mathbb{G}^+(\mathbb{R}^n) \) can be generated from the geometric multiplication of two unit vectors in \( \mathbb{R}^n. \) Given \( \{a, b\} \in \mathbb{R}^n, |a| = |b| = 1, \) with an angle \( \theta \) between them, and using the geometric product (Definition 5), a rotor can be defined as \([3, p. 107]\)
\[
r = ab = a \cdot b + a \wedge b
= |a||b| \cos \theta + I_{a,b}|a||b| \sin \theta
= \cos \theta + I_{a,b} \sin \theta
= e^{I_{a,b} \theta},
\] (19)
where the definitions of inner product (Definition 3) and outer product (Definition 4) of vectors was used. The result is the exponential form of a rotor.

Using (19), the rotation operator can be defined:

**Definition 11** (Rotation operator). Given the vector \( x \in \mathbb{R}^n, \) a rotated version can be obtained by applying the GA rotation operator \( r(\cdot)\tilde{r} \) to it,
\[
x \rightarrow r(\cdot)\tilde{r},
\] (20)
where \( r \in \mathbb{G}^+(\mathbb{R}^n), \) \( \tilde{r} \) is its reverse\(^4\), and \( r\tilde{r} = 1 \), i.e., \( r \) is a unit rotor.

The unity constraint is necessary to avoid the rotation operator to scale the vector \( x, \) i.e., to avoid changing its norm. A lower case letter was adopted to represent the rotor \( r \) (see to Section II-A)

It is possible to show (see Fig. 5) that \( x \) is rotated by an angle of \( 2\theta \) about the normal of the oriented area \( I_{a,b} \) (rotation axis) \([4]\). This way, the structure of a rotor highlights the rotation angle and axis. Similarly, quaternions can be represented in an exponential shape \([24], [25]\).

The rotor \( r \) can also be expressed in terms of its coefficients. For the 3D case, \( r \in \mathbb{G}^+(\mathbb{R}^3), \) and
\[
r = \langle r \rangle + \langle r \rangle_2 = r_0 + r_1\gamma_{12} + r_2\gamma_{23} + r_3\gamma_{31},
\] (21)
in which \( \{r_0, r_1, r_2, r_3\} \) are the coefficients of \( r. \) Note that quaternions, which can also represent rotations in three-dimensional space, have four coefficients as well.

\(^4\)The proper definition of reverse of a (rotor) multivector is given further ahead in (25).
Fig. 5. (a) A rotor can be generated from the geometric multiplication of two unit vectors in $\mathbb{R}^n$. (b) Applying the rotation operator: the vector $x$ is rotated by an angle of $2\theta$ about the normal $n$ of the oriented area $I_{a,b}$.

C. Useful Definitions and Properties

This section lists a number of extra definitions and properties of GA which are used throughout the text. They are provided as a consulting list (which is referred to when necessary), and can be skipped at a first reading.

The GA scalar product $\ast$ between two multivectors $A$ and $B$ is defined as $A \ast B = \langle AB \rangle$, in which $\langle \cdot \rangle \equiv \langle \cdot \rangle_0$.

From that, the magnitude of a multivector is defined as $|A|^2 = A \ast \tilde{A} = \sum_g |A|^2_g$, in terms of the magnitude $|A|^g$ of the grades.

**Definition 12** (Inner Product of $p$-vectors). The inner product of a $p$-vector $A_p = \langle A \rangle_p$ with a $q$-vector $B_q = \langle B \rangle_q$ is

$$A_p \cdot B_q = B_q \cdot A_p \triangleq \langle A_p B_q \rangle_{|p-q|}. \quad (22)$$

For example, the inner product between a $p$-vector $B$ and a vector $a$ is $B \cdot a = \langle Ba \rangle_{|p-1|}$. Thus, multiplying a multivector $B$ by a vector reduces its grade by 1.

**Definition 13** (Outer Product of $p$-vectors). The outer product of a $p$-vector $A_p = \langle A \rangle_p$ with a $q$-vector $B_q = \langle B \rangle_q$ is

$$A_p \wedge B_q \triangleq \langle A_p B_q \rangle_{p+q}. \quad (23)$$

For example, the outer product between a $p$-vector $B$ and a vector $a$ is $B \wedge a = \langle Ba \rangle_{|p+1|}$. Thus, multiplying a multivector $B$ by a vector increases its grade by 1. Note that $A_p \wedge B_q \neq B_q \wedge A_p$, i.e., the outer product is non-commutative.

**Remark 4.** The outer product of a vector $a$ with itself is $a \wedge a \triangleq 0$. Thus, $aa \equiv a \cdot a$.

**Definition 14** (Properties).

Addition is commutative,

$$A + B = B + A.$$  

Multiplication is non-commutative for general multivectors,

$$AB \neq BA.$$  

Addition and multiplication are associative,

$$(A + B) + C = A + (B + C),$$  

$$(AB)C = A(BC).$$  

There exist unique additive and multiplicative identities 0 and 1,

$$A + 0 = A,$$

$$1A = A.$$  

Every multivector has a unique additive inverse $-A$,

$$A + (-A) = 0.$$  

(24)
The reversion operation of GA is the extension of the complex conjugate in linear algebra.

**Definition 15 (Reversion).** The reverse of a multivector $A$ is defined as

$$\tilde{A} \triangleq \sum_{g=0}^{n} (-1)^{g(g-1)/2} \langle A \rangle_{g}. \quad (25)$$

For example, the reverse of a 2-vector $A = \langle A \rangle_0 + \langle A \rangle_1 + \langle A \rangle_2$ is $\tilde{A} = \langle A \rangle_0 + \langle A \rangle_1 + \langle A \rangle_2 = \langle A \rangle_0 + \langle A \rangle_1 - \langle A \rangle_2$. The reversion operation of GA is the extension of the complex conjugate in linear algebra.

**Remark 5.** Note that since the 0-grade of a multivector is not affected by reversion, mutually reverse multivectors, say $A$ and $\tilde{A}$, have the same 0-grade, $\langle A \rangle_0 = \langle \tilde{A} \rangle_0$.

**Definition 16 (Scalar Product).** The scalar product between two multivectors is

$$A \ast B \triangleq \langle AB \rangle, \quad (26)$$

i.e., it is the scalar part (0-grade) of the geometric multiplication between $A$ and $B$. For the special case of vectors, $a \ast b = \langle ab \rangle = a \cdot b$.

**Definition 17 (Magnitude).**

$$|A| \triangleq \sqrt{A \ast A} = \sqrt{\sum_{g} |\langle A \rangle_{g}|^2}. \quad (27)$$

**Definition 18 (Cyclic reordering).** The scalar part of a product of two multivectors is order invariant. This way,

$$\langle AB \rangle = \langle BA \rangle \Rightarrow \langle AB \cdots C \rangle = \langle B \cdots CA \rangle. \quad (28)$$

**Remark 6.** From that it follows that the scalar product is commutative, $A \ast B = \langle AB \rangle = \langle BA \rangle = B \ast A$.

**Definition 19 (Pseudoscalar).** The pseudoscalar $I$ is the highest grade of an algebra $\mathcal{G}$. In 3D Euclidean space,

$$I \triangleq a \wedge b \wedge c, \quad (29)$$

in which $\{a, b, c\}$ are linearly-independent vectors in $\mathcal{G}$. $I$ commutes with any multivector in $\mathcal{G}$, hence the name pseudoscalar.

**Definition 20 (Inversion).** Every nonzero vector $a$ has a multiplicative inverse defined as [4],

$$a^{-1} \triangleq \frac{a}{a^2}, \quad a^2 \neq 0 \Rightarrow aa^{-1} = \frac{a}{a^2} = 1. \quad (30)$$

**Definition 21 (Versor).** A multivector $A$ that can be factored into a product of $n$ vectors

$$A = a_1a_2\cdots a_n, \quad (31)$$

is called versor. Moreover, if the vectors $\{a_1, a_2, \cdots, a_n\}$ are invertible it is possible to show that $A$ has a multiplicative inverse $A^{-1} \triangleq a_n^{-1}\cdots a_2^{-1}a_1^{-1}$. For a detailed explanation, please refer to [5, Eq.(25)] and [3, pp.103].

**Definition 22 (Frame and Reciprocal Frame).** A set of vectors $\{a_1, a_2, \cdots, a_n\}$ defining a geometric algebra $\mathcal{G}$ is said to be a frame if and only if $A_n = a_1 \wedge a_2 \wedge \cdots \wedge a_n \neq 0$. This is equivalent to saying $\{a_1, a_2, \cdots, a_n\}$ are linearly independent.

Given a frame $\{a_1, a_2, \cdots, a_n\}$, it is possible to obtain a reciprocal frame [3] $\{a^1, a^2, \cdots, a^n\}$ via the equations $a_ka^j = \delta_{k,j}$, $j, k = 1, 2, \cdots, n$, where $\delta_{k,j} = 1$, for $j = k$, is the Kronecker delta.

**Definition 23 (Decomposition into Grades).** According to Definition 8, given the frame $\{a_1, a_2, \cdots, a_n\}$, a basis for the geometric algebra $\mathcal{G}(I)^5$, in which $I = a_1 \wedge a_2 \wedge \cdots \wedge a_n$ is its pseudovector, can be constructed by geometrically

$^5\mathcal{G}(I)$ is adopted in the literature to denote the geometric algebra whose pseudovector is $I$. In fact, since $I$ results from the geometric multiplication of the elements in the basis of the underlying vector space $\mathcal{V}$, the forms $\mathcal{G}(\mathcal{V})$ and $\mathcal{G}(I)$ are equivalent. See [3, p.19].
multiplying the elements of the frame. The resulting \(2^n\) members (multivectors) of the basis are grouped like \(\{\alpha_1, \alpha_2, \cdots, \alpha_{2^n}\}\), where \(\alpha_{2^n} = I\). The same procedure can be adopted for a reciprocal frame \(\{a^1, a^2, \cdots, a^n\}\), originating a reciprocal basis for \(\mathcal{G}(I)\), \(\{\alpha^1, \alpha^2, \cdots, \alpha^{2^n}\}\), where \(\alpha^{2^n} = I\). From that, any multivector \(B \in \mathcal{G}(I)\) can be decomposed into its grades like [3]

\[
B = \sum_K a^K(\alpha_K \ast B) = \sum_K a^K(\alpha_K B), \quad K = 1, \cdots, 2^n. \tag{32}
\]

This procedure will be very useful when performing geometric calculus operations.

\section*{D. Geometric Calculus}

The Geometric Calculus generalizes the standard concepts of calculus to encompass the GA theory. In the sequel, some basic relations are defined in order to be promptly used in the design of the AFs. For a detailed discussion on the subject, please refer to [3], [26].

\begin{definition}[Differential operator] The differential operator \(\partial\) (also used throughout this work in the form \(\nabla\)) has the algebraic properties of any other multivector in \(\mathcal{G}(I)\) [3]. Thus, it can be decomposed into its grades by applying Definition 23

\[
\partial = \sum_K a^K(a_K \ast \partial). \tag{33}
\]

Whenever necessary, the differential operator will present a subscript indicating the variable (multivector) with respect to the derivation is performed. For instance, \(\partial_X\) is a derivative with respect to the multivector \(X\).

\begin{definition}[Differential and overdot notation] Let \(F = F(X) \in \mathcal{G}(I)\). Given the product of two general multivectors \(AX\), in which \(A = F(X)\), the following notation

\[
\partial_X(AX) \tag{34}
\]

indicates that only \(X\) is to be differentiated [5], [26]. This is particularly useful to circumvent the limitations imposed by the noncommutativity of GA: note that since the differential operator has the algebraic properties of a multivector in \(\mathcal{G}(I)\), one cannot simply assume \(\partial_X AX = A \partial_X X\). Recall that, in general, \(\partial_X A \neq A \partial_X X\) (Definition 14). Thus, the overdot notation provides a way to comply with the noncommutativity of multivectors with respect to the geometric product.

\begin{proposition}[Basic Multivector Differential] Given two multivectors \(X\) and \(A\), it holds [11]

\[
(A \ast \partial_X)X = \dot{\partial}_X(\dot{X} \ast A) = \dot{\partial}_X(\dot{X}A) = A. \tag{35}
\]

\end{proposition}

\begin{remark} By similar means, it can be shown that the following relation holds: \((A \ast \partial_X)\dot{X} = \dot{\partial}_X(\dot{X}A) = \dot{A}\).
\end{remark}

\section*{IV. Linear Estimation in GA}

This section shows how one can use GA to address linear estimation. Key differences between GA-based and LA-based formulations are highlighted. In particular, the concepts of random multivectors and array of multivectors are introduced in order to support the derivation and performance analysis of the GA-AFs.

The following LA minimization (least-squares) problem will be utilized to motivate the transition from LA to GA,

\[
\min \left\| d - \hat{d} \right\|^2, \tag{36}
\]

in which \(\{d, \hat{d}\} \in \mathbb{R}^n, n = \{1, 2, \cdots\}\) and \(\hat{d}\) is the estimate for \(d\).

To formulate (36) in the GA framework, the concepts of multivectors (Definition 26) and arrays of multivectors (Definition 27) are used. This way, as shown further ahead, the GA version of (36) offers a way to extend that minimization problem for hypercomplex quantities.

This paper focuses on an special case of (36). Define \(d\) according to (1) and \(\hat{d} = u^*w\), in which \(u\) and \(w\) are \(M \times 1\) arrays of multivectors, the regressor and the weight arrays, respectively, and \(*\) denotes the reverse array
(see ahead Definition 28). The estimate for $d$ is obtained from a collection of $M$ input samples (regressor). Such a way of defining $d$ is widely employed across adaptive filtering literature [1], [6];

From that, a GA-based minimization cost function (CF) is generated to estimate the coefficients of $w$, which generates an estimate for $d$. This CF is connected to the system identification problem (see Section VI).

A. Useful Definitions

Some definitions are necessary before stating the general GA cost function. In the first one, the concept of random variable is simply extrapolated to allow for hypercomplex random quantities,

**Definition 26** (Random Multivectors). A random multivector is defined as a multivector whose grade values are random variables. Take for instance the following random multivector in $\mathfrak{G}(\mathbb{R}^3)$ (the GA formed by the vector space $\mathbb{R}^3$)

$$A = \langle A \rangle_0 + \langle A \rangle_1 + \langle A \rangle_2 + \langle A \rangle_3 = a_0 + a_1 \gamma_1 + a_2 \gamma_2 + a_3 \gamma_3 + a_4 \gamma_{12} + a_5 \gamma_{23} + a_6 \gamma_{31} + a_7 I.$$  

(37)

The terms $a_0, \cdots, a_7$ are real-valued random variables, i.e., they are drawn from a stochastic process described by a certain probability density function with a mean and a variance [1, Chapter A]. Note that random multivectors/variables are denoted in boldface letters throughout the whole text.

The next definition introduces the concept of arrays of multivectors,

**Definition 27** (Arrays of Multivectors). An array of multivectors is a collection of general multivectors. Given $M$ multivectors $\{U_1, U_2, \cdots, U_M\}$ in $\mathfrak{G}(\mathbb{R}^3)$, the $M \times 1$ array collects them as follows

$$u = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_M \end{bmatrix} = \begin{bmatrix} u_{10} + u_{11} \gamma_1 + u_{12} \gamma_2 + u_{13} \gamma_3 + u_{14} \gamma_{12} + u_{15} \gamma_{23} + u_{16} \gamma_{31} + u_{17} I \\ u_{20} + u_{21} \gamma_1 + u_{22} \gamma_2 + u_{23} \gamma_3 + u_{24} \gamma_{12} + u_{25} \gamma_{23} + u_{26} \gamma_{31} + u_{27} I \\ \vdots \\ u_{M0} + u_{M1} \gamma_1 + u_{M2} \gamma_2 + u_{M3} \gamma_3 + u_{M4} \gamma_{12} + u_{M5} \gamma_{23} + u_{M6} \gamma_{31} + u_{M7} I \end{bmatrix}.$$  

(38)

The array is denoted using lower case letters, the same as scalars and vectors (1-vectors). However, the meaning of the symbol will be evident from the context. Also, the name array was chosen to avoid confusion with vectors (1-vectors) in $\mathbb{R}^n$, which in this text have the usual meaning of collection of real numbers. In this sense, an array of multivectors can be interpreted as a “vector” that allows for hypercomplex entries.

Array $u$ in (38) can be rewritten to highlight its grades,

$$u = \begin{bmatrix} u_{10} \\ u_{20} \\ \vdots \\ u_{M0} \end{bmatrix} + \begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ u_{M1} \end{bmatrix} \gamma_1 + \cdots + \begin{bmatrix} u_{17} \\ u_{27} \\ \vdots \\ u_{M7} \end{bmatrix} I.$$  

(39)

Finally, there are also arrays of random multivectors,

$$u = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_M \end{bmatrix},$$  

(40)

which of course are denoted using boldface type.

Next, the reverse array is defined,

**Definition 28** (Reverse Array). The reverse array is the extension of the reverse operation of multivectors to include arrays of multivectors. Given the array $u$ in (38), its reverse version, denoted by the symbol $^*$, is

$$u^* = \begin{bmatrix} \bar{U}_1 \\ \bar{U}_2 \\ \cdots \bar{U}_M \end{bmatrix}.$$  

(41)

Note that the entries in $u^*$ are the reverse counterparts of the entries in $u$.
Definition 29 (Array Product). Given two $M \times 1$ arrays of multivectors, $u$ and $w$, the product between them is defined as

$$u^T w = U_1 W_1 + U_2 W_2 + \cdots + U_M W_M,$$

in which $^T$ represents the transpose array. The underlying product in each of the terms $U_j W_j, j = \{1, \cdots, M\}$, is the geometric product. Thus, the array product $u^T w$ results in the general multivector $\sum_{j=1}^{M} U_j W_j$. In a similar fashion,

$$u^* w = \sum_{j=1}^{M} \tilde{U}_j W_j,$$

(43)

where $^*$ represents the reverse array.

Observe that due to the noncommutativity of the geometric product, $u^T w \neq w^T u$ in general.

Remark 8. This text adopts the following notation to represent a product between an array and itself: given the array $u$, $\|u\|^2 \triangleq u^* u$. Note this is the same notation employed to denote the squared norm of a vector in $\mathbb{R}^n$ in linear algebra. However, here $\|u\|^2$ is a general multivector, i.e., it is not a pure scalar value which in linear algebra provides a measure of distance. In GA, the distance metric is given by the magnitude of a multivector (see Definition 17), which is indeed a scalar value. Thus, for an array $u$ and a multivector $U$,

$$\|u\|^2 = u^* u : \text{is a multivector}$$

$$|U|^2 = U^* \tilde{U} = \tilde{U}^* U = \sum_g |\langle U \rangle_g|^2 : \text{is a scalar.}$$

(44)

Finally, note that $\|u\|^2 = (u^* u) = u^* u = \|u\|^2$, i.e., $\|u\|^2$ is equal to its own reverse.

Definition 30 (Product Between Multivector and Array). Here the multivector $U$ is simply geometrically multiplied with each entry of the array $w$. Due to the noncommutativity of the geometric product, two cases have to be considered. The first is $Uw$,

$$Uw = U \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_M \end{bmatrix} = \begin{bmatrix} UW_1 \\ UW_2 \\ \vdots \\ UW_M \end{bmatrix},$$

(45)

and the second is $wU$,

$$wU = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_M \end{bmatrix} U = \begin{bmatrix} W_1 U \\ W_2 U \\ \vdots \\ W_M U \end{bmatrix}.$$ 

(46)

With the previous definitions, the general GA cost function can be formulated.

B. General Cost Function in GA

Following the guidelines in [3, p.64 and p.121], one can formulate a minimization problem by defining a general CF in GA. The following CF is a “mother” cost function,

$$J(D, A_k, X, B_k) = \left| D - \sum_{k=1}^{M} A_k X B_k \right|^2,$$

(47)

where $D, X, A_k, B_k$ are general multivectors. The term $\sum_{k=1}^{M} A_k X B_k$ represents the canonical form of a linear transformation applied to the multivector $X$ ([3, p.64 and p.121]).

In the sequel, it will be shown how to retrieve the standard CF from (47).
1) The Standard Shape: The standard cost function (least-squares) \( J_s \) is obtained from (47) by making \( D = d \) (a general multivector), \( X = 1 \), \( A_k = \tilde{U}_k \), \( B_k = W_k \),

\[
J_s(w) = \left| d - \sum_{k=1}^{M} \tilde{U}_k W_k \right|^2 = |d - u^* w|^2 ,
\]

(48)

where \( M \) is the system order (the number of taps in the filter), and the definition of array product (43) was employed to make \( \sum_{k=1}^{M} \tilde{U}_k W_k = u^* w \). Note that a lower case letter was adopted to represent the general multivector \( d \) (an exception to the convention used in this text). This is done to emphasize the shape similarity to the usual cost function \( \| d - u^H w \|^2 \) used in system identification applications in terms of a scalar \( d \) and vectors \( u \) and \( w \), with \( H \) denoting the Hermitian conjugate [1], [6]. Similarly to its linear-algebra counterpart, \( d \) is estimated as a linear combination of the entries of the regressor \( u \), which are random multivectors. Thus, the error quantity to be minimized is defined as

\[
e = d - u^* w.
\]

(49)

The performance analysis of the GAAFs (Section V-C) requires the least-mean-squares counterpart of (48),

\[
J_s(w) = E |e|^2 = E |d - u^* w|^2
\]

(50)

in which \( e \) and \( d \) are random multivectors, \( u^* \) is an \( M \times 1 \) array of multivectors, and \( E \) is the expectation operator. Notice that (50) has the exact same shape as the least-mean-squares cost function used in linear algebra-based adaptive filtering [1], [6].

It will be shown in Section V how to devise the standard GAAFs, which are able to minimize (50). In fact, the GAAFs are derived from the steepest-descent recursion, which iteratively minimizes (50) providing the instantaneous cost

\[
J_s(i) = E |d - u^* w_{i-1}|^2 .
\]

(51)

\( J_s(i) \triangleq J_s(w_{i-1}) \) is the learning curve associated to the cost function \( J_s(w) \) (see [1, Chapter 9]).

V. GEOMETRIC-ALGEBRA ADAPTIVE FILTERS (STANDARD)

In this section, the GAAFs are motivated following a least-squares approach, deriving the GA-LMS to minimize the cost function (48) in an adaptive manner. In the sequel, by modeling the observed data \( d(i) \) and \( u_i \) as stochastic processes, a mean-square analysis (steady-state) is performed.

The GAAFs to be designed must provide an estimate for the array of multivectors \( w \) via a recursive rule of the form,

\[
w_i = w_{i-1} + \mu G,
\]

(52)

where \( i \) is the (time) iteration, \( \mu \) is the AF step size, and \( G \) is a multivector valued quantity related to the estimation error (49).

A proper selection of \( G \) is required to make \( J(w_i) < J(w_{i-1}) \) at each iteration. This section adopts the steepest-descent rule [1], [6] and the analytical guidelines of [27], in which the AF is designed to follow the opposite direction of the gradient of the cost function, namely \( \partial_w J(w_{i-1}) \). This way, \( G \) is proportional to \( \partial_w J(w_{i-1}) \),

\[
G \triangleq - B \partial_w J(w_{i-1}),
\]

(53)

what yields the general form of an AF,

\[
w_i = w_{i-1} - \mu B \partial_w J(w_{i-1}),
\]

(54)

in which \( B \) is a general multivector, in contrast with the standard case in which \( B \) would be a matrix [1]. Choosing \( B \) appropriately is a requirement to define the type of adaptive algorithm, what is detailed in the following subsections.
A. GA Least-Mean Squares (GA-LMS)

The GA-LMS is supposed to adaptively minimize the cost function (48), reproduced below for ease of reference

$$J_s(w_{i-1}) = |d(i) - u_i^* w_{i-1}|^2 = |e(i)|^2.$$  \hfill (55)

Writing (55) in terms of its grades allows for applying GC in order to derive the GAAFs further ahead. This way

$$J(w_{i-1}) = |e(i)|^2 = e(i) \star e(i) = \sum_{A=1}^{2^n} \gamma_A e_A \star \sum_{A=1}^{2^n} e_A \tilde{\gamma}_A$$

$$= \sum_{A=1}^{2^n} e_A^2,$$

where

$$e_A = d_A - \hat{d}_A.$$  \hfill (57)

To move on to the calculation of the gradient of $J(w_{i-1})$ (required to obtain the GA-LMS AF), it is necessary to find an expression for $\hat{d}_A$ as a function of the grades of $u_i^* w_{i-1}$. Defining $\hat{d}(i) \triangleq u_i^* w_{i-1}$ (a multivector resultant from an array product) and using (32), $\hat{d}(i)$ can be written as

$$\hat{d}(i) = u_i^* w_{i-1} = \sum_{A=1}^{2^n} \gamma_A (\tilde{\gamma}_A u_i^* w_{i-1}).$$  \hfill (58)

Since $u_i$ and $w_{i-1}$ are arrays with $M$ multivector (Clifford numbers) entries, they can be written in terms of $2^n$ grades of $M$-dimensional arrays with real entries

$$u_i^* = \sum_{A=1}^{2^n} (u_i^* \gamma_A) \tilde{\gamma}_A = \sum_{A=1}^{2^n} u_{i,A}^T \tilde{\gamma}_A,$$  \hfill (59)

and

$$w_{i-1} = \sum_{A=1}^{2^n} \gamma_A (\tilde{\gamma}_A w_{i-1}) = \sum_{A=1}^{2^n} \gamma_A w_{i-1,A},$$  \hfill (60)

where $u_{i,A}^T$ and $w_{i-1,A}$ are respectively $1 \times M$ and $M \times 1$ arrays with real entries. Also, (32) was utilized once more. Plugging (59) and (60) back into (58)\(^6\)

$$\hat{d}(i) = u_i^* w_{i-1} = \sum_{A=1}^{2^n} \gamma_A (\tilde{\gamma}_A u_i^* w_{i-1})$$

$$= \sum_{A=1}^{2^n} \gamma_A (\sum_{B=1}^{2^n} u_{i,B}^T \tilde{\gamma}_B \sum_{C=1}^{2^n} \gamma_C w_C)$$

$$= \sum_{A=1}^{2^n} \gamma_A \sum_{B,C=1}^{2^n} (\tilde{\gamma}_A u_{i,B}^T \gamma_C w_C)$$

$$= \sum_{A=1}^{2^n} \gamma_A \sum_{B,C=1}^{2^n} (\tilde{\gamma}_A \gamma_C w_B) u_{i,B}^T \cdot w_C$$

$$= \sum_{A=1}^{2^n} \gamma_A \hat{d}_A,$$

in which

$$\hat{d}_A = \sum_{B,C=1}^{2^n} (\tilde{\gamma}_A \gamma_B \gamma_C) (u_{i,B}^T \cdot w_C), \ A = 1, \ldots, 2^n$$  \hfill (61)

is the expression of $\hat{d}_A$ as a function of the grades of $u_i^* w_{i-1}$.

The last step before performing the actual gradient calculation is to define the multivector derivative with respect to $w$ in terms of its grades (see Definition 24

$$\partial_w \triangleq \sum_{A=1}^{2^n} \gamma_A (\tilde{\gamma}_A \partial_w) = \sum_{A=1}^{2^n} \gamma_A \partial_w,A.$$  \hfill (63)

This is the case since the differential operator has the algebraic properties of a multivector in $\mathfrak{G}(\mathbb{R}^n)$ ([3, p.45]).

\(^6\)From now on, the iteration subscripts $i$ and $i - 1$ are omitted from $u_{i,A}$ and $w_{i-1,A}$ for clarity purposes.
With all the previous quantities (multivectors and arrays) described in terms of their GA grades, the gradient calculation is performed as follows

$$\partial_w J(w_{i-1}) = \left( \sum_{D=1}^{2^n} \gamma_D \partial_w d_A \right) \left( \sum_{A=1}^{2^n} e_A^2 \right) = \sum_{A,D=1}^{2^n} \gamma_D \partial_w, D e_A^2,$$

(64)
in which

$$\partial_w, D e_A^2 = 2e_A(\partial_w, D e_A) = 2e_A(\partial_w, D(d_A - \hat{d}_A)) = -2e_A(\partial_w, Dd_A),$$

(65)
where \( \partial_w, Dd_A = 0 \) since \( d_A \) does not depend on the weight vector \( w \). Plugging (65) into (64) results in

$$\partial_w, J(w_{i-1}) = -2\sum_{A,D=1}^{2^n} \gamma_D e_A(\partial_w, D\hat{d}_A).$$

(66)

Using (62) to rewrite \( \partial_w, D\hat{d}_A \) yields

$$\partial_w, D\hat{d}_A = \partial_w, D \left[ \sum_{B,C=1}^{2^n} (\gamma_A\gamma_B\gamma_C)(u_B^T \cdot w_C) \right] = \sum_{B,C=1}^{2^n} (\gamma_A\gamma_B\gamma_C)\partial_w, D(u_B^T \cdot w_C).$$

(67)
Now it is important to notice that the term \( \partial_w, D(u_B^T \cdot w_C) \) will be different from zero only when \( D = C \), i.e., when \( \partial_w, D \) and \( w_C \) are of same grade – recall that \( \partial_w, D \) has the same algebraic properties as a multivector in \( \mathbb{G}(R^n) \). This way, \( \partial_w, D(u_B^T \cdot w_C) = u_B \) for \( D = C \), or adopting the Kronecker delta function \( \delta_{BC} \) [27]

$$\partial_w, D(u_B^T \cdot w_C) = \delta_{CD}u_B^T.$$  

(68)
Plugging it back into (67) results in

$$\partial_w, D\hat{d}_A = \sum_{B,C=1}^{2^n} (\gamma_A\gamma_B\gamma_C)\delta_{CD}u_B^T.$$  

(69)
Finally, substituting (69) into (66), the gradient is obtained

$$\partial_w, J(w_{i-1}) = -2\sum_{A,D=1}^{2^n} \gamma_D e_A \sum_{B,C=1}^{2^n} (\gamma_A\gamma_B\gamma_C)\delta_{CD}u_B^T$$

$$= -2\sum_{A,D=1}^{2^n} e_A \sum_{B=1}^{2^n} \gamma_D (\gamma_A\gamma_B\gamma_D)u_B^T$$

$$= -2\sum_{A,D=1}^{2^n} e_A (\gamma_A\gamma_D)$$

$$= -2\sum_{A,D=1}^{2^n} e_A (\gamma_D u_A^T)$$

$$= -2\sum_{A=1}^{2^n} e_A u_A^T = -2u_i e(i).$$

(70)
In the AF literature, setting \( B \) equal to the identity matrix in (54) (the general form of an AF) results in the steepest-descent update rule ([1, Eq.8-19]). In GA though, the multiplicative identity is the multivector (scalar) 1 (see Definition 14). This way, substituting (70) into (54) and setting \( B = 1 \) yields the GA-LMS update rule

$$w_i = w_{i-1} + \mu u_i e(i).$$

(71)
where the 2 in (70) was absorbed by the step size \( \mu \).

Note that the GA-LMS (71) has the same shape of the regular LMS AFs [1], [6], namely the real-valued LMS (\( u \) and \( w \) have real-valued entries) and the complex-valued LMS (\( u \) and \( w \) have complex-valued entries). However, in the previous derivation, no constraints were put on the entries of the arrays \( u \) and \( w \) – they can be any kind of multivector. This way, the update rule (71) is valid for any \( u \) and \( w \) whose entries are general multivectors in \( \mathbb{G}(R^n) \). In other words, the update rule (71) generalizes the standard LMS AF for several types of \( u \) and \( w \) entries: general multivectors, rotors, quaternions, complex numbers, real numbers – any subalgebra of \( \mathbb{G}(R^n) \).

This is a very interesting result, accomplished due to the comprehensive analytic tools provided by Geometric Calculus. Recall that, in adaptive filtering theory, the transition from real-valued AFs to complex-valued AFs requires one to abide by the rules of differentiation with respect to a complex variable, represented by the Cauchy-Riemann
conditions (see [1, Chapter C, p.25]). Similarly, quaternion-valued AFs require further differentiation rules that are captured by the Hamilton-real (HR) calculus [17], [18], [20] and its generalized version (GHR) [22]. Although those approaches are successful, note that each time the underlying algebra is changed, the analytic tools need an update as well. This is not the case if one resorts to GA and GC to address the minimization problem. In this sense, GC proves itself as an extremely versatile analytic tool, providing a simple and unique way to perform calculus in any subalgebra of $\mathbb{G}(\mathbb{R}^n)$.

B. Data Model in GA

In order to carry on performance analyses of the GAAFs, this work adopts an specific data model. The goal of the analysis is to derive an expression for the mean-square error (MSE) in steady-state of standard GAAFs via energy conservation relations (ECR) [1].

**Definition 31** (Steady-State MSE in GA). As in linear algebra, the steady-state MSE in GA must be scalar-valued. To this end, the MSE is defined as

$$\text{MSE} = \xi \triangleq \lim_{i \to \infty} \mathbb{E} \left( \| e(i) \|^2 \right) = \lim_{i \to \infty} \left( \mathbb{E} \| e(i) \|^2 \right),$$

(72)
i.e., it involves the calculation of the scalar part (0-grade) of the multivector $\| e(i) \|^2 = \bar{e}(i)e(i)$.

The ECR technique is an energy balance in terms of the following error quantities

$$\begin{cases}
\Delta w_{i-1} \triangleq (w^o - w_{i-1}) \text{ weight-error array} \\
 e_a(i) = u_i^* \Delta w_{i-1} \text{ a priori estimation error} \\
 e_p(i) = u_i^* \Delta w_i \text{ a posteriori estimation error}
\end{cases}$$

(73)
together with the AF’s recursion.

This way, since adaptive filters are non-linear, time-varying, and stochastic, it is necessary to adopt a set of assumptions (stationary data model) ([1, p.231]),

**Definition 32** (Stationary Data Model).

1. There exists an array of multivectors $w^o$ such that $d(i) = u_i^* w^o + v(i)$ ;
2. The noise sequence $\{v(i)\}$ is independent and identically distributed (i.i.d.) with constant variance $\mathbb{E} \bar{v}(i)v(i) = \mathbb{E} \| v(i) \|^2$ ;
3. The noise sequence $\{v(i)\}$ is independent of $u_j$ for all $i, j,$ and all other data;
4. The initial condition $w_{-1}$ is independent of all $\{d(i), u_i, v(i)\}$ ;
5. The expectation of $u_i^* u_i$ is denoted by $\mathbb{E} u_i^* u_i > 0$ ;
6. The random quantities $\{d(i), u_i, v(i)\}$ are zero mean.

Similarly to the definition of $d$, a lower case letter was adopted to represent the general multivector $v$ (another exception to the convention used in this text).

The steady-state excess mean-square error (EMSE) is defined from the a priori estimation error $e_a(i)$,

**Definition 33** (Steady-State EMSE).

$$\text{EMSE} = \zeta \triangleq \lim_{i \to \infty} \mathbb{E} \left( \| e_a(i) \|^2 \right) = \lim_{i \to \infty} \left( \mathbb{E} \| e_a(i) \|^2 \right).$$

(74)
Similar to (72), it involves the calculation of the scalar part (0-grade) of the multivector $\| e_a(i) \|^2 = \bar{e}_a(i)e_a(i)$.

As will be seen ahead, the analysis procedure requires the expectation of $\bar{v}(i)v(i)$ to be calculated.

**Definition 34** (Expectation of $\bar{v}(i)v(i)$). Given a random multivector $v \in \mathbb{G}(\mathbb{R}^3)$ (see (37)),

$$v = (v)_0 + (v)_1 + (v)_2 + (v)_3 \quad \begin{align} v &= v_0 + v_1 \gamma_1 + v_2 \gamma_2 + v_3 \gamma_3 + v_4 \gamma_{12} + v_5 \gamma_{23} + v_6 \gamma_{31} + v_7 I, \end{align}$$

(75)
where each coefficient $v_k$, with $k = 0, \cdots, 7$ is an i.i.d. random variable.

The geometric product $\tilde{v}v$ is
\[
\tilde{v}v = v_0^2 + v_0v_1\gamma_1 + v_0v_2\gamma_2 + v_0v_3\gamma_3 + v_0v_4\gamma_{12} + v_0v_5\gamma_{23} - v_0v_6\gamma_{31} - v_0v_7I +
\]
\[
v_1^2 + v_1v_0\gamma_1 - v_1v_2\gamma_2 + v_1v_3\gamma_3 + v_1v_4\gamma_{12} - v_1v_5\gamma_{23} - v_1v_6\gamma_{31} + v_1v_7I +
\]
\[
\vdots
\]
\[
v_7^2 + v_7v_5\gamma_1 + v_7v_6\gamma_2 + v_7v_4\gamma_3 + v_7v_3\gamma_{12} + v_7v_1\gamma_{23} - v_7v_2\gamma_{31} + v_7v_0I.
\]

(76)

Thus, applying the expectation operator to (76) results in
\[
E\tilde{v}v = Ev_0^2 + Ev_1^2 + Ev_2^2 + Ev_3^2 + Ev_4^2 + Ev_5^2 + Ev_6^2 + Ev_7^2,
\]
(77)

since the expectations of the cross-terms are zero. Each term $Ev_k^2$, $k = 0, \cdots, 7$, is said to be the variance of $v_k$ and denoted $Ev_k^2 \equiv \sigma_k^2$. This way, (77) becomes
\[
E\tilde{v}v = 8\sigma_v^2
\]
(78)

Remark 9. Note that in general $E\tilde{v}v = \text{dim}\{\mathcal{G}^g(\mathbb{R}^n)\} \sigma_v^2$ for $v \in \mathcal{G}^g(\mathbb{R}^n)$, in which $\mathcal{G}^g(\mathbb{R}^n)$ can be any subspace of $\mathcal{G}(\mathbb{R}^n)$ (see Definition 9). When the complete geometric algebra is used, i.e., $v \in \mathcal{G}(\mathbb{R}^n)$, $\text{dim}\{\mathcal{G}(\mathbb{R}^n)\}$ is given by (11), and thus $E\tilde{v}v = 2^n\sigma_v^2$. The expectation (78) is the particular case when $n = 3$, i.e., when the complete geometric algebra of $\mathcal{G}^3$, namely $\mathcal{G}(\mathbb{R}^3)$, is adopted (Section III-B1).

Similarly, the expectation of $u^*u_i$ is necessary during the analysis.

Definition 35 (Expectation of $u^*u$). Using (43),
\[
Eu^*u = E\tilde{U}_1U_1 + E\tilde{U}_2U_2 + \cdots + E\tilde{U}_MU_M,
\]
(79)

where $U_j$, $j = 1, \cdots, M$ is a general multivector.

The terms $\tilde{U}_jU_j$ are geometric products. For the case $U_j \in \mathcal{G}(\mathbb{R}^3)$,
\[
\tilde{U}_jU_j = u_{j0}^2 + u_{j0}u_{j1}\gamma_1 + u_{j0}u_{j2}\gamma_2 + \cdots + u_{j0}u_{j6}\gamma_{31} - u_{j0}u_{j7}I +
\]
\[
u_{j1}^2 + u_{j0}u_{j1}\gamma_1 - u_{j1}u_{j2}\gamma_2 + \cdots + u_{j1}u_{j3}\gamma_{31} - u_{j1}u_{j5}I +
\]
\[
\vdots
\]
\[
u_{j7}^2 + u_{j7}u_{j5}\gamma_1 + u_{j7}u_{j6}\gamma_2 + \cdots + u_{j7}u_{j2}\gamma_{31} + u_{j7}u_{j0}I.
\]

(80)

where each coefficient $u_{jk}$, with $k = 0, \cdots, 7$ is drawn from a white Gaussian noise process. Thus,
\[
E\tilde{U}_jU_j = Eu_{j0}^2 + Eu_{j1}^2 + Eu_{j2}^2 + Eu_{j3}^2 + Eu_{j4}^2 + Eu_{j5}^2 + Eu_{j6}^2 + Eu_{j7}^2,
\]
(81)

since the expectations of the cross-terms are zero. Each term $Eu_{jk}^2$, with $j = 1, \cdots, M$ and $k = 0, \cdots, 7$, is said to be the variance of $u_{jk}$ and denoted $Eu_{jk}^2 \equiv \sigma_{uj}^2$. Note that this result is also obtained if it is assumed $U_j$, $j = 1, \cdots, M$, is a circular Gaussian random multivector. This assumption \footnote{Here the circularity condition – explained in [1, p. 8] for complex-valued random variables – is extended to encompass random multivectors.} considers that the grades of a random multivector are independent Gaussian random variables.

This way, (81) becomes
\[
E\tilde{U}_jU_j = 8\sigma_u^2, \text{ for } j = 1, \cdots, M
\]
(82)

which substituted in (79) yields
\[
Eu^*u = M(8\sigma_u^2)
\]
(83)

where $M$ is the number of taps in the filter, as pointed out in Definition 27).
is given by (11), and thus $\mathbf{E}u^*u = M(2^n\sigma_u^2)$. The expectation (83) is the particular case when $n = 3$, i.e., when the complete geometric algebra of $\mathbb{R}^3$, namely $\mathfrak{g}(\mathbb{R}^3)$, is adopted (Section III-B1).

From the stationary linear data model (Definition 32),
\[
e(i) = d(i) - u_i^*w_{i-1} \\
\quad = u_i^*(w^o - w_{i-1}) + v(i) \\
\quad = e_a(i) + v(i).
\]
Thus,
\[
E\tilde{e}(i)e(i) = E\tilde{e}_a(i)e_a(i) + E\tilde{v}(i)v(i),
\]
and using (72) and (74),
\[
\text{MSE} = \text{EMSE} + E\tilde{v}(i)v(i).
\]

The resulting energy equation leads to a variance relation from which the MSE and the EMSE can be derived.

C. Steady-State Analysis

In this section, the ECR technique [1] is applied step-by-step in order to obtain an expression for the EMSE of any GAAF whose update rule has the following general shape
\[
w_i = w_{i-1} + \mu u_i f(e(i)),
\]
where $f(\cdot)$ is a multivector-valued function of the estimation error $e(i)$. Depending on the type of the GAAF (LMS, NLMS etc), $f(\cdot)$ assumes a specific value.

The ECR technique performs an interplay between the energies of the weight array $w$ and the error $e$ at two successive time instants, say $i - 1$ and $i$. Quantities related to time instant $i - 1$ are labeled a priori, and those related to $i$ are named a posteriori. Equating a priori and a posteriori quantities allows for studying the mean-square performance of adaptive filters – transient, steady-state, and tracking (this work focuses on steady-state only). As a result, an expression for the variance relation is obtained, which is then particularized for each AF of interest.

For details on the ECR procedure, please refer to [1, p.228].

Subtracting (87) from the optimal weight vector $w^o$ yields
\[
\Delta w_i = \Delta w_{i-1} - \mu u_i f(e(i)),
\]
in which $\Delta w_i = w^o - w_i$. Multiplying from the left by $u_i^*$ (array product),
\[
u_i^*\Delta w_i = u_i^* [\Delta w_{i-1} - \mu u_i f(e(i))]
\]
\[
e_p(i) = e_a(i) - \mu \|u_i\|^2 f(e(i)),
\]
where $e_p(i) = u_i^*\Delta w_i$ is the a posteriori error, $e_a(i) = u_i^*\Delta w_{i-1}$ is the a priori error (See (73)), and in the last equation $\|u_i\|^2 = u_i^*u_i$ (See (44)).

Assuming that the multivector $\|u_i\|^2$ is a versor composed by the product of invertible vectors (see Definition 21), and $u_i \neq 0$, then it has a multiplicative inverse $\Gamma(i) = (\|u_i\|^2)^{-1}$. This allows for solving (89) for $f(e(i))$
\[
f(e(i)) = \frac{1}{\mu} \Gamma(i) [e_a(i) - e_p(i)],
\]
and plugging back into (88) results in
\[
\Delta w_i = \Delta w_{i-1} - u_i \Gamma(i) [e_a(i) - e_p(i)],
\]
which can be rearranged as
\[
\Delta w_i + u_i \Gamma(i) e_a(i) = \Delta w_{i-1} + u_i \Gamma(i) e_p(i).
\]
The left-hand side (LHS) is expanded as
\[
\text{LHS} = \left( \Delta w_i + u_i \Gamma(i) e_a(i) \right) \ast \left( \Delta w_i + u_i \Gamma(i) e_a(i) \right)
\]
in which \( \ast \) is the GA scalar product and \( \sim \) is the reverse. Further expansion gives
\[
\text{LHS} = |\Delta w_i|^2 + |u_i \Gamma(i) e_a(i)|^2 + 2 \Delta w_i \ast (\bar{e}_a(i) \Gamma(i) u_i^*) + (u_i \Gamma(i) e_a(i)) \ast \bar{\Delta} w_i,
\]
(95)
in which \( \Gamma(i) = \bar{\Gamma}(i) \) since \( \| \bar{u}_i \|^2 = \| u_i \|^2 \) holds (See Remark 8). Applying the definition of GA scalar product and observing that the third and fourth terms of (95) are each other’s reverse\(^8\), their sum can be written as
\[
2 \left( \Delta w_i \bar{e}_a(i) \Gamma(i) u_i^* \right) = 2 \left( \bar{e}_a(i) \Gamma(i) u_i^* \Delta w_i \right),
\]
(96)
where the cyclic reordering property (28) for the 0-grade operator was used. Note that the term \( u_i^* \Delta w_i \) is the definition of the \textit{a posteriori} error \( e_p(i) \) (refer to (89)). This way, (94) assumes the form
\[
|\Delta w_i|^2 + 2 \left( \bar{e}_a(i) \Gamma(i) e_p(i) \right) + |u_i \Gamma(i) e_a(i)|^2.
\]
(97)
Similar procedures allow to expand the right-hand side (RHS) of (93) as
\[
|\Delta w_{i-1}|^2 + 2 \left( e_p(i) \Gamma(i) e_a(i) \right) + |u_i \Gamma(i) e_p(i)|^2.
\]
(98)
Substituting (97) (LHS) and (98) (RHS) into (93) yields the following energy relation
\[
|\Delta w_i|^2 + |u_i \Gamma(i) e_a(i)|^2 = |\Delta w_{i-1}|^2 + |u_i \Gamma(i) e_p(i)|^2.
\]
(99)
which balances out \textit{a priori} and \textit{a posteriori} terms. Note that the terms enclosed by the 0-grade operator (in (97) and (98)) are each others reverse. Thus, since their 0-grade are exactly the same (see Remark 5), they are mutually cancelled.

Taking the expectation of the terms of (99) with respect to the random quantities \( d(i) \) and \( u_i \) results in
\[
E |\Delta w_i|^2 + E |u_i \Gamma(i) e_a(i)|^2 = E |\Delta w_{i-1}|^2 + E |u_i \Gamma(i) e_p(i)|^2.
\]
(100)
Calculating the limit of (100) when \( i \to \infty \) gives
\[
E |u_i \Gamma(i) e_a(i)|^2 = E |u_i \Gamma(i) e_p(i)|^2, i \to \infty,
\]
(101)
in which the steady-state condition \( E |\Delta w_i|^2 = E |\Delta w_{i-1}|^2 \) = \textit{constant} as \( i \to \infty \) was employed [1, p.237].

Plugging (89) into (101) results in
\[
E |u_i \Gamma(i) e_a(i)|^2 = E |u_i \Gamma(i)(e_a(i) - \mu \| u_i \|^2 f)|^2, i \to \infty.
\]
(102)
The right-hand side of (102) is expanded as
\[
E |u_i \Gamma(i) e_a(i)|^2 = 2 \mu E \left( u_i \Gamma(i) e_a(i) \bar{f} u_i^* \right) + \mu^2 E |u_i f|^2.
\]
(103)
Plugging (103) back into (102) and cancelling out the term \( E |u_i \Gamma(i) e_a(i)|^2 \) on both sides results in
\[
2 \mu E \left( u_i \Gamma(i) e_a(i) \bar{f} u_i^* \right) = \mu^2 E |u_i f|^2.
\]
(104)
Using the cyclic reordering property on the left-hand side of (104) to make \( u_i^* u_i \Gamma(i) = 1 \), the so-called \textit{variance relation} is obtained
\[
2 E \left( e_a(i) \bar{f} \right) = \mu E |u_i f|^2.
\]
(105)
\(^8\)Given mutually reverse multivectors \( A \) and \( \bar{A} \), the following relation holds \( \langle A \rangle = \langle \bar{A} \rangle \) (See Remark 5). Thus, \( \langle A \rangle + \langle \bar{A} \rangle = 2 \langle A \rangle \).
1) GA-LMS: For the GA-LMS, function $f$ is given by $f(e(i)) = e(i) = e_a(i) + v(i)$ (see (84)). Substituting into (105)

$$2E\langle e_a(i)\left(\bar{e}_a(i) + \bar{v}(i)\right)\rangle = \mu E\left[u_i\left(e_a(i) + v(i)\right)\right]^2.$$  \hspace{1cm} (106)

The left-hand side of (106) becomes

LHS (106) \hspace{1cm} = 2E\langle e_a(i)\bar{e}_a(i)\rangle + 2E\langle e_a(i)\bar{v}(i)\rangle \\
\hspace{1cm} = 2E\langle e_a(i)^2\rangle + 2\left(E\langle e_a(i)\rangle*E\langle \bar{v}(i)\rangle\right) \\
\hspace{1cm} = 2E\langle e_a(i)^2\rangle, \hspace{1cm} (107)

where the fact that $v(i)$ is independent of any other random quantity was used. Additionally, it is assumed here that the entries of $v(i)$ (and $\bar{v}(i)$) are drawn from a zero-mean white Gaussian process, thus $E\langle \bar{v}(i)\rangle = 0$.

The right-hand side of (106) is expanded as

RHS (106) \hspace{1cm} = \mu E\langle u_i(e_a(i) + v(i)) - (\bar{e}_a(i) + \bar{v}(i))u_i^*\rangle \\
\hspace{1cm} = \mu E\langle \|u_i\|^2\|e_a(i)\|^2\rangle + 2\mu E\langle \|u_i\|^2\|e_a(i)\|\bar{v}(i)\rangle \\
\hspace{1cm} \hspace{1cm} + \mu E\langle \|u_i\|^2\|v(i)\|^2\rangle. \hspace{1cm} (108)

Since $v(i)$ is statistically independent from other quantities and $E\bar{v}(i) = E\langle v(i)\rangle = 0$ (Definition 32), the term $2E\langle \|u_i\|^2\|e_a(i)\|\rangle = 2E\langle \|u_i\|^2\|e_a(i)\||\rangle*E\langle \bar{v}(i)\rangle = 0$. This way,

RHS (106) \hspace{1cm} = \mu E\langle \|u_i\|^2\|e_a(i)\|^2\rangle + \mu E\langle \|u_i\|^2\|v(i)\|^2\rangle. \hspace{1cm} (109)

Substituting (107) and (109) into (106) gives

$$2E\langle e_a(i)^2\rangle = \mu E\langle \|u_i\|^2\|e_a(i)\|^2\rangle + \mu E\langle \|u_i\|^2\|v(i)\|^2\rangle. \hspace{1cm} (110)$$

Observing that $2E\langle e_a(i)^2\rangle = 2E\langle \|e_a(i)\|^2\rangle$, (110) can be rewritten as

$$E\langle (2 - \mu \|u_i\|^2)\|e_a(i)\|^2\rangle = \mu E\langle \|u_i\|^2\|v(i)\|^2\rangle. \hspace{1cm} (111)$$

Adopting the separation principle (see [1, p.245]), i.e., in steady state $\|u_i\|^2$ is independent of $e(i)$ (and consequently of $e_a(i)$), (111) becomes

$$\langle (2 - \mu \|u_i\|^2)E\langle \|e_a(i)\|^2\rangle = \mu \langle \|u_i\|^2\|v(i)\|^2\rangle. \hspace{1cm} (112)$$

In the most general case, i.e., when all the multivectors belong to the complete algebra $\mathcal{G}(\mathbb{R}^n)$, the terms $E\langle \|v(i)\|^2\rangle$ and $E\langle \|u_i\|^2\rangle$ are calculated as described in Remarks 9 and 10, namely, $E\langle \|v(i)\|^2\rangle = 2^n\sigma_v^2$ and $E\langle \|u_i\|^2\rangle = M(2^n\sigma_u^2)$. Substituting into (112) yields

$$2 - \mu M(2^n\sigma_u^2)\langle E\langle \|e_a(i)\|^2\rangle = \mu M(2^n\sigma^2_u)(2^n\sigma_v^2). \hspace{1cm} (113)$$

It is important to notice that since Remark 10 is obtained considering inputs (regressor entries) drawn from a circular Gaussian process (see [1, p. 8]), the present analysis holds only for that kind of input.

Finally, the expression for the GA-LMS steady-state EMSE using the complete algebra $\mathcal{G}(\mathbb{R}^n)$ is given by

$$\zeta_{\text{LMS}} = \frac{\mu M4^n\sigma_u^2\sigma_v^2}{2 - \mu M2^n\sigma_u^2}, \hspace{1cm} i \to \infty. \hspace{1cm} (114)$$

For the special case $n = 3$, (114) becomes

$$\zeta_{\text{LMS}} \hspace{1cm} (n = 3) = \frac{32\mu M\sigma_u^2\sigma_v^2}{1 - 4\mu M\sigma_u^2}, \hspace{1cm} i \to \infty. \hspace{1cm} (115)$$

Table II summarizes the EMSE theoretical values for several algebras. Those are useful in Section VI, where GAAFs with entries in $\mathcal{G}(\mathbb{R}^3)$, $\mathcal{G}^+ (\mathbb{R}^3)$, $\mathcal{G}^+ (\mathbb{R}^2)$, and $\mathcal{G} (\mathbb{R})$ have their steady-state performance assessed. Notice that for $\mathcal{G} (\mathbb{R})$ the EMSE for the LMS with real-valued entries is recovered (compare to Equation 16.10 in [1, p.246] for white Gaussian inputs).

To obtain the respective MSE, one should add $E\langle \|v(i)\|^2\rangle$ (Definition 34 and Remark 9) to the EMSE expression, as pointed out in (86).
the core idea of this work: the GAAFs can estimate hypercomplex quantities which generalize real, complex, and real entries) adopts a different value for $W$ assume. However, setting $n$ by basis (13). The derivation of the GAAFs puts no restriction on the values the vector space dimension $A$. Multivector Entries

As aforementioned, the measurement noise multivector $v$ has each of its coefficients drawn from a white Gaussian stationary process with variance $\sigma_v^2$.

### A. Multivector Entries

The underlying geometric algebra in this case is $G(\mathbb{R}^n)$, with $n = 3$, i.e., the one whose multivectors are described by basis (13). The derivation of the GAAFs puts no restriction on the values the vector space dimension $n$ can assume. However, setting $n = 3$ (generating a GA with dimension 8) provides a didactic example that captures the core idea of this work: the GAAFs can estimate hypercomplex quantities which generalize real, complex, and quaternion entries.

Here the optimal weight array is,

$$w^o = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_M \end{bmatrix} = \begin{bmatrix} 0.55 + 0\gamma_1 + 1\gamma_2 + 2\gamma_3 + 0.71\gamma_{12} + 1.3\gamma_{23} + 4.5\gamma_{31} + 3I \\ 0.55 + 0\gamma_1 + 1\gamma_2 + 2\gamma_3 + 0.71\gamma_{12} + 1.3\gamma_{23} + 4.5\gamma_{31} + 3I \\ \vdots \\ 0.55 + 0\gamma_1 + 1\gamma_2 + 2\gamma_3 + 0.71\gamma_{12} + 1.3\gamma_{23} + 4.5\gamma_{31} + 3I \end{bmatrix},$$

where all multivector entries are the same, namely \(\{0.55 + 0\gamma_1 + 1\gamma_2 + 2\gamma_3 + 0.71\gamma_{12} + 1.3\gamma_{23} + 4.5\gamma_{31} + 3I\}\). Those values were selected in an aleatory manner. Note that the coefficient of $\gamma_1$ is zero. However, it was kept in (116) to emphasize the structure of the $G(\mathbb{R}^3)$ basis.

VI. SIMULATIONS

This Section shows the performance of the computational implementation of GAAFs in a system identification task. The optimal weight array $w^o$ to be estimated has $M$ multivector-valued entries (number of taps), namely $W_j$, $j = 1, \cdots, M$, $w^o = [W_1 W_2 \cdots W_M]^T$. Each case studied in the sequel (multivector, rotor, complex, and real entries) adopts a different value for $W_j$ (which will be highlighted in a timely manner).

As aforementioned, the measurement noise multivector $v$ has each of its coefficients drawn from a white Gaussian stationary process with variance $\sigma_v^2$.

### TABLE II

| Complete GA of $\mathbb{R}^n$ | $\frac{\mu M \sigma_n^2 \sigma_v^2}{2 - \mu M 2^n \sigma_v^2}$ |
|-------------------------------|-------------------------------------------------------------|
| Any subspace $g^o(\mathbb{R}^n)$ | $\frac{\mu M \gamma^o}{2 - \mu M \gamma^o}$ |
| Even Algebras | $\frac{\mu M \left( \sum_k \binom{n}{2k} \right) \sigma_v}{2 - \mu M \sigma_v}^2$, for $k = 0, 1, 2, 3, \cdots$ |
| Complete GA of $\mathbb{R}^3$ | $\frac{32 \mu M \sigma_3^2 \sigma_v^2}{1 - 4 \mu M \sigma_3^2}$ |
| Rotor GA of $\mathbb{R}^3$ (Quaternions) | $\frac{\mu M \left( \binom{3}{0} + \binom{3}{1} \right) \sigma_v^2}{2 - \mu M \sigma_v^2}^2$ |
| Rotor GA of $\mathbb{R}^2$ (Complex) | $\frac{\mu M \left( \binom{2}{0} + \binom{2}{1} \right) \sigma_v^2}{2 - \mu M \sigma_v^2}^2$ |
| Rotor GA of $\mathbb{R}$ (Real) | $\frac{\mu M \sigma_1^2 \sigma_v^2}{2 - \mu M \sigma_1^2}$ |

All the AFs were implemented using the Geometric Algebra Algorithms Expression Templates (GAALET) [28], a C++ library for evaluation of GA expressions. For implementation details, please refer to openga.org.
Fig. 6 shows several learning curves (MSE and EMSE) for the GA-LMS estimating the weight array (116) with $M = 10$. The step size value is $\mu = 0.005$ for all simulated curves. Notice the perfect agreement between the theoretical error levels (obtained with (115)) and the simulated steady-state error. Those experiments show that the GA-LMS is indeed capable of estimating multivector-valued quantities, supporting what was previously devised in Section V.

Fig. 7 depicts the steady-state error as a function of the system order (number of taps) $M$. Multiple curves are provided, each for a specific value of measurement noise $\sigma_v^2$. Theory and experimental values agree throughout the entire tested range $M = [1, 40]$. 
Fig. 6. GA-LMS: MSE and EMSE learning curves for $M = 10$, $\mu = 0.005$, and $\sigma_v^2 = \{10^{-2}, 10^{-3}, 10^{-5}\}$. The curves are averaged over 100 experiments.
Fig. 7. GA-LMS: steady-state MSE and EMSE as functions of the system order (number of taps) $M$ for $\sigma_v^2 = \{10^{-2}, 10^{-3}, 10^{-5}\}$. The simulated steady-state value is obtained by averaging the last 200 points of the ensemble-average learning curve for each $M$. Notice how the simulated curves agree with the model.

B. Rotor Entries

In this case, the underlying geometric algebra is $\mathcal{G}^+(\mathbb{R}^3)$ (isomorphic to quaternions – see Section III-B3), and the optimal weight array is

$$w^o = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_M \end{bmatrix} = \begin{bmatrix} 0.55 + 0.71\gamma_{12} + 1.3\gamma_{23} + 4.5\gamma_{31} \\ 0.55 + 0.71\gamma_{12} + 1.3\gamma_{23} + 4.5\gamma_{31} \\ \vdots \\ 0.55 + 0.71\gamma_{12} + 1.3\gamma_{23} + 4.5\gamma_{31} \end{bmatrix},$$

(117)

where all rotor entries are the same, namely \{0.55 + 0.71\gamma_{12} + 1.3\gamma_{23} + 4.5\gamma_{31}\}.

Differently from Section VI-A, it was chosen to show only one EMSE learning curve to avoid an overwhelming amount of similar figures\textsuperscript{10}. The EMSE learning curve is depicted in Fig. 8 together with steady-state MSE and EMSE for several values of $M$. Note how the theoretical and experimental values agree.

All in all, the AF is shown to be capable of estimating a weight array with rotor-valued quantities. Thus, resorting to the isomorphism between $\mathcal{G}^+(\mathbb{R}^3)$ and quaternion algebra (see Section III-B3), that filter is naturally suited for estimating weight arrays whose entries are quaternions. This way, the GA-LMS becomes an alternative to the quaternion-LMS (QLMS) [17], [18], [20], [22].

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\textsuperscript{10}All the C++ source codes, MATLAB\textsuperscript{®} scripts, and instructions necessary to generate the learning curves are available on openga.org. The reader is encouraged to explore the online material in order to see the GAAFs performance in several different scenarios.
C. Complex Entries

The underlying geometric algebra in this section is $\mathbb{G}^+(\mathbb{R}^2)$ (isomorphic to complex numbers – see Section III-B2), and the optimal weight array is

$$w^o = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_M \end{bmatrix} = \begin{bmatrix} 0.55 + 0.71 \gamma_{12} \\ 0.55 + 0.71 \gamma_{12} \\ \vdots \\ 0.55 + 0.71 \gamma_{12} \end{bmatrix},$$

(118)

where all complex entries are the same, namely $\{0.55 + 0.71 \gamma_{12}\}$.

Fig. 9 shows an EMSE learning curve for $M = 10$ and steady-state errors for several values of $M$. The GA-LMS is shown to be capable of estimating a weight array with complex-valued quantities and the experimental values corroborate the theory. Thus, the GA-LMS becomes an alternative to the complex-LMS (CLMS) [23].

Fig. 8. Rotor entries. (a) EMSE learning curve for $M = 10$, $\mu = 0.005$, and $\sigma_v^2 = 10^{-3}$ (100 experiments). (b) Steady-state MSE and EMSE versus the number of taps for $\mu = 0.005$ and $\sigma_v^2 = 10^{-3}$. 
Fig. 9. Complex entries. (a) EMSE learning curve for $M = 10$, $\mu = 0.005$, and $\sigma_v^2 = 10^{-3}$ (100 experiments). (b) Steady-state MSE and EMSE versus the number of taps for $\mu = 0.005$ and $\sigma_v^2 = 10^{-3}$.

D. Real Entries

Finally, the most basic type of LMS, i.e., the one that estimates arrays with real-valued entries, is recovered via isomorphism with the geometric algebra $\mathbb{S}^+(\mathbb{R})$. Thus, the optimal weight array is

$$w^o = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_M \end{bmatrix} = \begin{bmatrix} 0.55 \\ 0.55 \\ \vdots \\ 0.55 \end{bmatrix},$$

where all real entries are the same, namely 0.55 (recall that $\gamma_0 = 1$). The array $w^o$ in (119) has the same shape of the so-called weight vector from regular adaptive filtering theory [1], [6].

Fig. 10 shows an EMSE learning curve for $M = 10$ and steady-state errors for several values of $M$. Once again, theoretical and experimental values agree.
VII. CONCLUSION

The formulation of GA-based adaptive techniques is still in its infancy. The majority of AF algorithms available in the literature resorts to specific subalgebras of GA (real, complex numbers and quaternions). Each of them requires an specific set of tools in order to pose the problem and perform calculus. In this sense, the development of the GAAFs is an attempt to unify those different adaptive-filtering approaches under the same mathematical language. Additionally, as shown throughout the text, GAAFs have improved estimation capabilities since they are not limited to 1-vector estimation (like LA-based AFs). Instead, they can naturally estimate any kind of multivector. Also, given a type of GAAF, say GA-LMS, the shape of its update rule is invariant with respect to the multivector subalgebra. This is only possible due to the use of GA and GC.

On top of the theoretical contributions, the experimental validation provided in Section VI shows that the GAAFs are ready for use in a system identification scenario. Nevertheless, it must be said that there might exist a number of different applications in need of a tool like GAAFs. It is expected that any estimation problem posed in terms of hypercomplex quantities could benefit from this work. For instance, GAAFs may be useful in data fusion, where different signals are supposed to be integrated in the same “package” and then processed. The multivector (and by extension the array of multivectors) can be interpreted as a fundamental information package that aggregates scalar, vector, bivector, and so on, quantities.

Besides storing all the codes and scripts to perform the experiments, the website openga.org is an interface between engineers and researchers interested in contributing for the development of GA-based algorithms (not necessarily AFs). Thus, the possibilities of new GAAF applications can be amplified by the network of contributors that will (hopefully) expand as time goes by. The reader is encouraged to download the source code, develop his own ideas, and communicate the results to openga.org. Also, feedback about the codes and scripts, as well as application ideas for GAAFs are welcomed and appreciated.

Finally, the combination of GA and AF theories shows that the use of more comprehensive mathematics can indeed circumvent limitations of already-known tools, leading the way to pose new questions and formulate new
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