Correlators of heavy–light quark currents in HQET: OPE at three loops

K.G. Chetyrkin a,b,c, A.G. Grozin d,e,∗

a Institut für Theoretische Teilchenphysik, Karlsruher Institut für Technologie, Karlsruhe, Germany
b Institut für Theoretische Physik, Universität Regensburg, 93040 Regensburg, Germany
c II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany
d Budker Institute of Nuclear Physics, Novosibirsk, Russia
e Novosibirsk State University, Novosibirsk, Russia

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Abstract

Coefficient functions of the operator product expansion of correlators of HQET heavy–light quark currents are calculated up to operators of dimension 4 up to 3 loops.
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1. Heavy–light currents and their correlators in HQET

QCD problems with a single heavy quark $Q$ having momentum $P = Mv + p$ ($M$ is its on-shell mass, $v^2 = 1$) can be described by heavy quark effective theory [1] (HQET, see, e.g., [2–4]) if its characteristic residual momentum is small ($p \ll M$), and characteristic momenta of light quarks and gluons are also small. QCD operators are expanded in $1/M$, the coefficients are HQET operators of corresponding dimensionalities. For example, QCD heavy–light quark currents at the leading order in $1/M$ are equal to the matching coefficients times the HQET heavy–light currents. These matching coefficients are known at 2 [5,6] and 3 loops [7]. Anomalous dimensions of all HQET heavy-light currents are the same and known at 2 [8–10] and 3 loops [11]. Cor-

* Corresponding author.
E-mail addresses: konstantin.chetyrkin@partner.kit.edu (K.G. Chetyrkin), A.G.Grozin@inp.nsk.su (A.G. Grozin).

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relators of such currents at small distances can be calculated using operator product expansion (OPE); coefficient functions of operators up to dimension 3 are known up to 2 loops [12–14]. The \( G^2 \) contribution vanishes at 1 loop; the \( \langle G^2 \rangle \) one is known at 1 loop [12]. Contributions of quark condensates up to dimension 8 are known at tree level [12,15]. Here we calculate the perturbative contribution expanded up to \( m^4 \) (\( m \) is the light-quark mass) and condensate contributions up to dimension 4 at 3 loops. The perturbative spectral densities of correlators of some QCD heavy–light currents with \( m = 0 \) in the threshold region at 3 loops were calculated [16]; they are related to the HQET spectral density by the corresponding matching coefficients.

If our heavy quark is \( b \), there are 2 different HQETs: with \( c \) quark and without it. The heavy–light HQET currents in these 2 theories are related by the decoupling coefficient, which is known up to 3 loops [17]. The HQET current in HQET with \( c \) is related to the QCD currents by the matching coefficients. In this paper we shall work in HQET without \( c \) quarks. There are \( n_i = 3 \) dynamic flavors \((u, d, s)\) and the static \( b \) quark.

At the leading order in \( 1/M \) the heavy-quark spin does not interact with gluon field. We may rotate it at will without affecting physics (heavy-quark spin symmetry [18]). We may even switch it off (superflavor symmetry [19]). We shall use the effective theory of a scalar static antiquark. This particle has no antiparticle; its field \( \phi^* \) contains only annihilation operators. Its coordinate-space free propagator in the \( v \) rest frame is \( \delta(\vec{x})S_0(x^0) \) where \( S_0(t) = -i\theta(t) \). The momentum-space propagator \( S_0(p) = 1/(p^0 + i0) \) does not depend on \( \vec{p} \). Static-quark lines cannot form loops.

We consider the current

\[
j_0 = \phi_0^* q_0 = Z_j(\alpha_s(\mu)) j(\mu)
\]

where \( q_0 \) is the bare light-quark field. The correlator of 2 currents in the \( v \) rest frame

\[
\langle T j_0(x) \bar{j}_0(0) \rangle = \delta(\vec{x}) \Pi_0(x^0)
\]

is non-zero only for \( x^0 \geq 0 \) (the symbol \( T \) is superfluous: the product \( \bar{j}_0(0) j_0(x) = 0 \)). The momentum-space correlator

\[
\int d^4x \langle T j_0(x) \bar{j}_0(0) \rangle e^{ip \cdot x} = \Pi_0(p^0)
\]

does not depend on \( \vec{p} \). They are related by the 1-dimensional Fourier transform

\[
\Pi_0(\omega) = \int_0^\infty dt \Pi_0(t)e^{i\omega t}, \quad \Pi_0(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \Pi_0(\omega)e^{-i\omega t}.
\]

The correlator \( \Pi_0(\omega) \) has a cut from 0 to \( +\infty \), the discontinuity gives the spectral density

\[
\rho_0(\omega) = \frac{1}{2\pi} \left[ \Pi_0(\omega + i0) - \Pi_0(\omega - i0) \right].
\]

The correlator is expressed via the spectral density by the dispersion representation:

\[
\Pi_0(\omega) = i \int_0^\infty \frac{dv \rho_0(v)}{\omega - v + i0}, \quad \Pi_0(\omega) = \theta(\omega) \int_0^{+\infty} d\omega \rho_0(\omega)e^{-i\omega t}.
\]

We can analytically continue \( \Pi_0(t) \) from \( t > 0 \) to \( t = -i\tau, \tau > 0 \) and obtain the Euclidean correlator

\[\text{...}\]
The spectral density can be reconstructed from it by the inverse Mellin transform
\[ \rho_0(\omega) = \frac{1}{2\pi i} \int_{-i\infty}^{a+i\infty} d\tau \, \Gamma(\tau) e^{\omega \tau}, \tag{8} \]
where \( a \) is to the right from all singularities of \( \rho_0(\tau) \).

Borel transform of the correlator \( \Pi(\omega) \) is often used in sum rules. In HQET it is defined by
\[ \hat{\mathcal{B}}_E F(\omega) = \lim_{k \to \infty} \left( \frac{\omega}{k} \right)^{k+1} \frac{k!}{d\omega} \left. F(\omega) \right|_{\omega = -E_k}. \tag{9} \]
It is equivalent to the correlator in imaginary time \( \Pi(\tau) \). For example, for the function
\[ F(\omega) = \frac{1}{(\nu - \omega - i0)^n}, \]
it is
\[ \hat{\mathcal{B}}_E F(\omega) = \frac{e^{-v/E}}{\Gamma(n) E^{n-1}}. \]

The Fourier transform (4) of \( F(\omega) \) is
\[ F(t) = i\theta(t) \left( \frac{it}{\Gamma(n)} \right)^n e^{-i(\nu - i0)t}; \]
its analytical continuation from the half-axis \( t > 0 \) to the half-axis \( t = -i\tau, \tau > 0 \) is
\[ F(\tau) = i \frac{\tau^{n-1}}{\Gamma(n)} e^{-\nu \tau}. \]

Therefore,
\[ \hat{\mathcal{B}}_E \Pi(\omega) = -i \Pi(\tau = 1/E). \tag{10} \]

The static-antiquark propagator in a gluon field is
\[ \delta(\vec{x}_1 - \vec{x}_0) S_0(x_1^0 - x_0^0)[x_1, x_0], \]
where
\[ [x_1, x_0] = P \exp \left[ -ig_0 \int_{x_0}^{x_1} dx_\mu A_\mu(x) \right]. \tag{11} \]
is the Wilson line in the antiquark representation, the integral is taken along the straight line from \( x_0 \) to \( x_1 \). Therefore, the correlator can be written as
\[ \Pi_0(t) = \langle q_0(\nu t)[\nu t, 0]\bar{q}_0(0) \rangle. \tag{12} \]

We can consider a more general object [20]
\[ F_0(x) = \langle q_0(x)[x, 0]\bar{q}_0(0) \rangle, \tag{13} \]
where \( x \) is not necessarily timelike. The bilocal vacuum average can be expressed via \( F_0(x) \):
\[ \langle \bar{q}_0(0)[0, x]\Gamma q(x) \rangle = -\text{Tr} \Gamma F_0(x), \tag{14} \]
where \( [0, x] \) is the Wilson line in the quark (fundamental) representation, and \( \Gamma \) is a Dirac matrix.
The correlator of the \( \overline{\text{MS}} \) renormalized currents \( j(\mu) \) still contains ultraviolet (UV) divergences when \( t = 0 \). Subtracting these divergences we obtain the renormalized correlator \( \Pi(t; \mu) \).

The dispersion representation should contain 3 subtractions:

\[
\Pi(\omega; \mu) = -i\omega^3 \int_0^\infty \frac{d\epsilon \rho(\epsilon; \mu)}{\epsilon^3(\epsilon - \omega - i0)} + \sum_{n=0}^2 c_n \omega^n ,
\]

\[
\Pi(t; \mu) = \theta(t) \int_0^\infty d\omega \rho(\omega; \mu) e^{-i\omega t} + \sum_{n=0}^2 c_n t^n \delta(n)(t) .
\]

Divergences of the correlator of renormalized currents are in subtraction terms in (15): in coordinate space they are at \( t = 0 \), in momentum space they are polynomial in \( \omega \). More exactly, in dimensional regularization only \( c_2 \) contains \( 1/\epsilon^2 \) divergences, whereas power divergences in \( c_{0,1} \) are not seen in this scheme. The renormalized spectral density is simply given by

\[
\rho_0(\omega) = Z_j^2(\alpha_s(\mu)) \rho(\omega; \mu) .
\]

The correlator has 2 Dirac structures

\[
\Pi = A + B \gamma .
\]

It is convenient to introduce the currents with definite parities \( P = \pm 1 \):

\[
j_P = \frac{1 + P \gamma}{2} j .
\]

Their correlators are

\[
P \Pi_P \frac{1 + P \gamma}{2} \quad \text{where} \quad \Pi_P = P(A + PB) = \frac{P}{4} \text{Tr}(1 + P \gamma) \Pi
\]

(we shall see soon why it is convenient to introduce the factor \( P \) here).

For sufficiently large \(-\omega\) the operator product expansion (OPE) is valid

\[
\Pi(\omega; \mu) = \sum_i C_i(\omega; \mu) \langle O_i(\mu) \rangle ,
\]

where \( O_i \) are all possible operators. If the \( q \) mass is small, we can include operators with powers of \( m(\mu) \) in the set \( O_i \) and calculate the Wilson coefficients \( C_i \) treating \( q \) as massless. Then the terms with even-dimensional \( O_i \) have Dirac structure \( \gamma \), and those with odd-dimensional \( O_i \) have the structure 1.

Currently we are in the world where the antiquark \( \bar{Q} \) has quantum numbers \( 0^+ \). Then S-wave \( \bar{Q}q \) mesons have \( j_P = 1^+ \), and P-wave ones \( 1^- \) and \( 3^- \). The currents \( j_{\pm} \) have quantum numbers of \( \frac{1}{2}^\pm \) mesons (currents with quantum numbers of mesons with \( j > \frac{1}{2} \) necessarily involve derivatives, we don’t consider them). The matrix elements of our currents are

\[
\langle 0 | j_P(\mu) | M \rangle = F(\mu) u ,
\]

where the meson states are normalized as

\[
\langle M, \bar{\rho}^i | M', \bar{\rho} \rangle = (2\pi)^3 \delta(\bar{\rho}^i - \bar{\rho})
\]

in the \( u \) rest frame, and \( u \) is the Dirac wave function of the \( \frac{1}{2}^P \) meson \( M \) satisfying \( \gamma u = P u \) and normalized as \( u^+u = 1 \). The contribution of the meson \( M \) to the correlator \( \Pi_P \) and its spectral density \( \rho_P \) is
Table 1

| $P$ | $\Gamma_1$ | $\Gamma_2$ | $\Pi_{12}$ |
|-----|-----|-----|-----|
| +1  | $\gamma_5$ | $\gamma_5$ | $2\pi_+$ |
|     | $\gamma_i$ | $\gamma_i$ | $2\pi_+\delta^{ij}$ |
| −1  | $1$ | $1$ | $2\pi_-$ |
|     | $\gamma_5 R_i$ | $\gamma_5 R_j$ | $2\pi_-\delta^{ij}$ |

$\Pi_M(t) = |F|^2 e^{-i\vec{\Lambda}t} \theta(t)$,  
$\Pi_M(\tau) = |F|^2 e^{-\vec{\Lambda}\tau}$,  
$\Pi_M(\omega) = \frac{i|F|^2}{\omega - \vec{\Lambda} + i0}$,  
$\rho_M(\omega) = |F|^2 \delta(\omega - \vec{\Lambda})$,  

(22)

where $\vec{\Lambda}$ is the residual energy of this meson. If there are several mesons with given quantum numbers, we get sums of contributions (22); sums become integrals in the continuum spectrum.

Now let’s switch on the spin (and parity) $\frac{1}{2}^-$ of the static antiquark $\bar{Q}$ (still at $M = \infty$). The static antiquark is now described by the field $\tilde{h}$ satisfying $\tilde{h}\tilde{\gamma} = -\tilde{h}$. The free propagator of this field contains the extra factor $(1 - \bar{\gamma})/2$ as compared to the scalar case. The currents are

$$j_{\bar{\gamma}0} = \tilde{h}_0 \Gamma q_0,$$

(23)
in the $\nu$ rest frame the set of independent Dirac structures $\Gamma$ is $1, \gamma_i, \gamma^i \gamma_j, \gamma^i \gamma^j \gamma^k$, where square brackets mean antisymmetrization. Instead of this set, we can use $1, \gamma_5^{HV}, \gamma^i, \gamma_5^{HV} \gamma^i$, where $\gamma_5^{HV}$ is the ’t Hooft–Veltman $\gamma_5$. In HQET renormalized currents with the anticommuting $\gamma_5^{AC}$ coincide with the corresponding currents with $\gamma_5^{HV}$, because their anomalous dimensions are the same [5] (contrary to the QCD case). Therefore, in the following we shall just use $\gamma_5$. The correlator of $j_1^+$ and $j_2$ is

$$\Pi_{12} = -\text{Tr} \bar{\Gamma}_1 \frac{1 - \gamma_0}{2} \Gamma_2 \Pi,$$

(24)

where $\Pi$ is the correlator with the scalar static antiquark, and minus comes from the fermion loop. The Dirac matrices $\Gamma$ either commute or anticommute with $\gamma_0$: $\Gamma\gamma^0 = -P\gamma_0 \Gamma$. Both $\Gamma_{1,2}$ must have the same $P$ (otherwise the correlator vanishes), and

$$\Pi_{12} = -\Pi P \frac{1}{2} \text{Tr} \bar{\Gamma}_1 \Gamma_2.$$

(25)

The same formula works for the spectral densities.

$S$-wave mesons with light-fields quantum numbers $j^P = \frac{1 +}{2}$ become degenerate doublets $0^-, 1^-$; $P$-wave ones with $j^P = \frac{1 -}{2}, \frac{3 -}{2}$ form degenerate doublets $0^+, 1^+$ and $1^- + 2^+$. The currents with $\Gamma$ anticommuting with $\gamma^0 (\gamma_5, \gamma^i; P = +1)$ have quantum numbers of the $S$-wave $0^-, 1^-$ mesons ($j^P = \frac{1 +}{2}$); those with $\Gamma$ commuting with $\gamma^0 (1, \gamma_5 \gamma^i; P = -1)$ have quantum numbers of the $P$-wave $0^+, 1^+$ mesons ($j^P = \frac{1 -}{2}$). The spectral density of correlator of the currents with quantum numbers of $0^0$ mesons is $2\rho_{\pm}$, and for $1^+$ mesons it is $2\rho_{\pm} \delta^{ij}$ (Table 1, eq. (25)); this is the reason why we introduced the factor $P$ in (18). A $0^-$ meson contribution to the spectral density is $|F_0|^2 \delta(\omega - \vec{\Lambda})$; for $1^{-}$ one it is $|F_1|^2 \delta^{ij} \delta(\omega - \vec{\Lambda})$, where

$$\langle 0 | \tilde{h} \gamma_5 q | 0^- \rangle = F_0^{-} (\mu), \quad \langle 0 | \tilde{h} \gamma_5 q | 1^- \rangle = F_1^{-} (\mu) \bar{e}$$

(26)
(\vec{e} \text{ is the polarization vector of the } 1^- \text{ meson}). Therefore

$$F_{0^-}(\mu) = F_{1^-}(\mu) = \sqrt{2} F(\mu)$$

(27)

(this is an example of the heavy-quark spin symmetry). The case of \(0^+\), \(1^+\) mesons is similar.

Usually the relativistic normalization of one-particle states is used:

$$\langle M, P^0|M, P \rangle = (2\pi)^3 2 P^0 \delta(\vec{P} - \vec{P})$$

(28)

(it becomes meaningless when the meson mass \(M \to \infty\), and thus is not usable in HQET; in the meson rest frame \(|M\rangle_r = \sqrt{2M}|M\rangle\)). The spin-symmetry result (27) can be written in a completely Lorentz-invariant way:

$$\langle 0|\vec{h}\Gamma q|1^-\rangle_r = \sqrt{M} F(\mu) \text{Tr} \Gamma \mathcal{M}, \quad \mathcal{M} = \frac{1 + \gamma_5}{2} \times \left\{ \begin{array}{ll} \gamma_5 & \text{for } 0^- , \\ \gamma_\tau & \text{for } 1^- . \end{array} \right.$$  

(29)

For example \((P^\mu = M v^\mu)\) is the meson momentum

$$\langle 0|\vec{h}\gamma_5\gamma^\mu q|0^-\rangle_r = \frac{2 F(\mu)}{\sqrt{M}} P^\mu, \quad \langle 0|\vec{h}\gamma_5 q|0^-\rangle_r = \frac{2 F(\mu)}{\sqrt{M}} M ,$$

$$\langle 0|\vec{h}\gamma^\mu q|1^-\rangle_r = \frac{2 F(\mu)}{\sqrt{M}} M e^\mu, \quad \langle 0|\vec{h}\frac{1}{2}[\gamma^\mu, \gamma^\nu] q|1^-\rangle_r = \frac{2 F(\mu)}{\sqrt{M}} (e^\mu P^\nu - e^\nu P^\mu).$$

Similarly, for \(0^+\), \(1^+\) mesons \((j^P = \frac{1}{2}^-)\)

$$\mathcal{M} = \frac{1 - \gamma_5}{2} \times \left\{ \begin{array}{ll} 1 & \text{for } 0^+ , \\ \gamma_5 & \text{for } 1^+ . \end{array} \right. \quad \bar{\psi}\mathcal{M} = \mathcal{M}\bar{\psi} = -\mathcal{M}.$$  

(30)

Of course, phases of \(|M\rangle\) states (and hence of \(\mathcal{M}\)) can be redefined.

The vacuum average (13) for a general (timelike or spacelike) \(x\) has 2 Dirac structures

$$F_0(x) = -\frac{1}{4} \left[ F_S(x^2) - i \gamma_5 F_V(x^2) \right],$$

$$F_S(x^2) = \langle \bar{q}(0)[0, x] q(x) \rangle, \quad F_V(x^2) = \frac{i}{x^2} \langle \bar{q}(0)[0, x] \gamma_5 q(x) \rangle.$$  

(31)

In HQET \(x = v t\), and the scalar functions \(F_{S, V}\) have positive argument \(x^2 = t^2\). If \(x\) is spacelike, the argument \(x^2\) is negative. When we analytically continue HQET results to \(t = -i \tau\), we obtain \(F_{S, V}\) of negative argument \(-\tau^2\).  

2. Perturbative contribution

Perturbative contributions to the correlator are shown in Fig. 1: the one-loop diagram; one of three two-loop ones; and two examples of three-loop diagrams.

We use integration by parts (IBP) to reduce three-loop diagrams to master integrals with the C++ program\(^1\) FIRE6 [24]. Generation of Feynman diagram was done with QGRAF [25] and evaluation of color factors with the FORM [26] package COLOR [27].

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\(^1\) We have also used the Mathematica program LiteRed 1.4 [21,22] and the REDUCE package Grinder [23] for testing purposes and the identification of the master integrals.
There are three non-trivial master integrals. Two of them are known exactly as hypergeometric functions with $\varepsilon$: \cite{28} and \cite{23,29}; for the last one, only a few terms of the $\varepsilon$ expansion are known \cite{16}, but this is sufficient for our purpose. This IBP procedure and the master integrals are reviewed in \cite{30}.

The renormalized perturbative correlator is

$$
\Pi_p(\tau; \mu) = \frac{N_c}{\pi^2} \left\{ \frac{1}{2\tau^3} \left[ 1 + C_F \frac{\alpha_s}{4\pi} \left[ 6L_\tau + 4 \left( \frac{\pi^2}{3} + 2 \right) \right] \right.ight.
+ C_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[ 18L_\tau^2 + \left( \frac{40}{3} \pi^2 + 43 \right) L_\tau - 8\xi_3 + \frac{8}{45} \pi^4 + \frac{52}{3} \pi^2 + 153 \right] \right.
+ C_A \left[ 22L_\tau^2 + \left( \frac{76}{9} \pi^2 + 75 \right) L_\tau - 104\xi_3 - \frac{8}{45} \pi^4 - \frac{5}{27} \pi^2 + \frac{6413}{72} \right]
- T_{F nl} \left[ 8L_\tau^2 + 4 \left( \frac{8}{9} \pi^2 + 7 \right) L_\tau - 32\xi_3 - \frac{16}{27} \pi^2 + \frac{589}{18} \right] \right\}
+ P \frac{m}{4\tau^2} \left[ 1 + C_F \frac{\alpha_s}{4\pi} \left[ 12L_\tau + 4 \left( \frac{\pi^2}{3} + 3 \right) \right] \right.
+ C_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[ 72L_\tau^2 + 2 \left( \frac{32}{3} \pi^2 + 71 \right) L_\tau - 20\xi_3 + \frac{8}{45} \pi^4 + \frac{52}{3} \pi^2 + \frac{233}{4} \right] \right.
+ C_A \left[ 44L_\tau^2 + \frac{2}{3} \left( \frac{38}{3} \pi^2 + 205 \right) L_\tau - 116\xi_3 - \frac{8}{45} \pi^4 + \frac{31}{27} \pi^2 + \frac{4981}{36} \right]
- T_{F nl} \left[ 16L_\tau^2 + \frac{8}{3} \left( \frac{4}{3} \pi^2 + 17 \right) L_\tau - 32\xi_3 - \frac{16}{27} \pi^2 + \frac{401}{9} \right] \right\}
- \frac{m^2}{8\pi} \left[ 1 + 6C_F \frac{\alpha_s}{4\pi} \left( 3L_\tau + 1 \right) \right.
+ C_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[ 162L_\tau^2 + \left( \frac{16}{3} \pi^2 + 109 \right) L_\tau + 40\xi_3 - \frac{32}{45} \pi^4 - 12\pi^2 + \frac{507}{8} \right] \right.
+ C_A \left[ 66L_\tau^2 - \left( \frac{4}{3} \pi^2 - 125 \right) L_\tau - 22\xi_3 - \frac{4}{15} \pi^4 - \pi^2 + \frac{2789}{24} \right]
- T_{F nl} \left[ 24L_\tau^2 + 36L_\tau + \frac{229}{6} \right] \right\}
- 2 \sum \frac{m_i^2}{\tau} C_F T_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left( \frac{\pi^2}{3} - 2 \right)
$$
\[ + P \frac{m^3}{8} \left[ L_\tau + C_F \frac{\alpha_s}{4\pi} \left[ 12 L_{\tau}^2 + 10 L_\tau - 2 \left( \frac{2}{3} \pi^2 - 3 \right) \right] \right. \\
+ C_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[ 96 L_{\tau}^3 + 2 \left( \frac{8}{3} \pi^2 + 59 \right) L_{\tau}^2 + \left( 16 \zeta_3 - \frac{76}{3} \pi^2 + \frac{249}{2} \right) L_\tau \right. \\
- 140 \zeta_5 + \frac{64}{3} \pi^2 \zeta_3 + 134 \zeta_3 - \frac{26}{15} \pi^4 - \frac{40}{3} \pi^2 + \frac{91}{24} \right] \\
\left. + C_A \left[ \frac{88}{3} L_{\tau}^3 - \frac{2}{3} (2 \pi^2 - 161) L_{\tau}^2 - \left( 16 \zeta_3 + \frac{97}{9} \pi^2 - \frac{889}{6} \right) L_\tau \right. \\
+ 60 \zeta_5 - \frac{20}{3} \pi^2 \zeta_3 + 89 \zeta_3 - \frac{\pi^4}{45} - \frac{277}{27} \pi^2 + \frac{4357}{72} \right] \\
- T_F n_l \left[ \frac{32}{3} L_{\tau}^3 + \frac{104}{3} L_{\tau}^2 - \frac{2}{3} \left( \frac{16}{3} \pi^2 - 71 \right) L_\tau + 32 \zeta_3 - \frac{80}{27} \pi^2 + \frac{245}{18} \right] \Bigg] \right. \\
- P m \left( \sum m_i^2 \right) C_F T_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left( 6 L_\tau + \frac{\pi^2}{3} - \frac{2}{3} \right) \\
- \frac{m^4}{32} \left[ L_\tau - \frac{1}{4} + C_F \frac{\alpha_s}{4\pi} \left[ 18 L_{\tau}^2 - \frac{17}{2} L_\tau - \frac{8}{3} \pi^2 + \frac{45}{2} \right] \right. \\
\left. + C_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[ 186 L_{\tau}^3 + \left( \frac{16}{3} \pi^2 - \frac{259}{2} \right) L_{\tau}^2 + \left( 16 \zeta_3 - \frac{212}{3} \pi^2 + \frac{4155}{8} \right) L_\tau \right. \\
- 420 \zeta_5 + 64 \pi^2 \zeta_3 + 351 \zeta_3 - \frac{20}{9} \pi^4 - \frac{113}{9} \pi^2 - \frac{10609}{32} \right] \\
\left. + C_A \left[ \frac{154}{3} L_{\tau}^3 - \left( \frac{4}{3} \pi^2 - \frac{63}{2} \right) L_{\tau}^2 - \left( 16 \zeta_3 + \frac{200}{9} \pi^2 - \frac{2603}{8} \right) L_\tau \right. \\
+ 180 \zeta_5 - 20 \pi^2 \zeta_3 + 179 \zeta_3 - \frac{\pi^4}{15} + \frac{835}{108} \pi^2 - \frac{64801}{288} \right] \\
- T_F n_l \left[ \frac{56}{3} L_{\tau}^3 + 2 L_{\tau}^2 - \left( \frac{64}{9} \pi^2 - \frac{235}{2} \right) L + 64 \zeta_3 + \frac{32}{27} \pi^2 - \frac{6137}{72} \right] \Bigg] \right. \\
+ m^2 \sum m_i^2 \frac{1}{4} \tau C_F T_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left( 6 L_\tau + \frac{17}{6} \right) \\
- \frac{\sum m_i^4}{4} \tau C_F T_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left[ \left( \frac{\pi^2}{3} - \frac{11}{4} \right) L_\tau - 3 \zeta_3 - \frac{\pi^2}{2} + \frac{65}{8} \right] \\
+ O(m^5, \alpha_s^3) \right\}, \quad (32) \]

where \( \alpha_s = \alpha_s^{(n)}(\mu), m = m^{(n)}(\mu) \) is the mass of the quark \( q \) in our current (1), \( m_i = m_i^{(n)}(\mu) \) are all light-flavor masses (see the last diagram in Fig. 1), and

\[ L_\tau = \log \frac{\mu \tau e^{\gamma_E}}{2}. \quad (33) \]

The coefficient functions \( C_m^{(n)}(\mu) \) with \( n = 0, 1, 2 \) satisfy simple renormalization group (RG) equations (while \( m^3 \) mixes with \( \bar{q}q \), and \( m^4 \) mixes with \( \bar{m}q \) and \( G^2 \), Sect. 3). Its solution is
\[ C_{mn} \sim \exp \left\{ \frac{\alpha_s}{4\pi} (-2\gamma_0 L_\tau + c_1) + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[ -2\beta_0 \gamma_0 L_\tau^2 - 2(\gamma_1 - 6c_1) L_\tau + c_2 \right] \right\} + \mathcal{O}(\alpha_s^3) \}
\]
\[ \gamma_k = 2\gamma_{jk} - n\gamma_{mk}, \quad \gamma_0 = -6C_F(n + 1). \] (34)

Here

\[ \gamma_a(\alpha_s) = \frac{d}{d\log \mu} \left( \frac{Z}{\alpha_s} \right) = \sum_{n=0}^{\infty} \gamma_{an} \left( \frac{\alpha_s}{4\pi} \right)^{n+1} \quad (a = j, m), \]
\[ \beta(\alpha_s) = \frac{1}{2} \frac{d}{d\log \mu} \left( \frac{Z}{\alpha_s} \right) = \sum_{n=0}^{\infty} \beta_n \left( \frac{\alpha_s}{4\pi} \right)^{n+1}. \]

Our results (32) for \( n = 0, 1, 2 \) satisfy this condition.

The renormalized spectral density of the OPE terms having dimensionalities \( \leq 2 \) is

\[ \rho^{d\leq 2}_F(\omega; \mu) = \frac{N_c}{\pi^2} \left\{ \frac{\alpha_s}{4\pi} \left[ 1 - C_F \frac{\alpha_s}{4\pi} \left( 6L_\omega - \frac{4}{3}\pi^2 - 17 \right) \right] + C_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[ 18L_\omega^2 - \left( \frac{40}{3}\pi^2 + 97 \right) L_\omega - 8\xi_3 + \frac{8}{45}\pi^4 + \frac{103}{3}\pi^2 + \frac{1173}{8} \right] + C_A \left[ 22L_\omega^2 - \left( \frac{76}{9}\pi^2 + 141 \right) L_\omega - 104\xi_3 - \frac{8}{45}\pi^4 + \frac{238}{27}\pi^2 + \frac{20057}{72} \right] - T_F m_l \left[ 8L_\omega^2 - 4 \left( \frac{8}{9}\pi^2 + 13 \right) L_\omega - 32\xi_3 + \frac{92}{27}\pi^2 + \frac{1849}{18} \right] \right\} \right\} + \frac{P m_\omega}{4} \left[ 1 - 4C_F \frac{\alpha_s}{4\pi} \left( 3L_\omega - \frac{\pi^2}{3} - 6 \right) \right] + 4C_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[ 18L_\omega^2 - \left( \frac{16}{3}\pi^2 + \frac{143}{2} \right) L_\omega - 5\xi_3 + \frac{2}{45}\pi^4 + \frac{20}{3}\pi^2 + \frac{1377}{16} \right] + C_A \left[ 11L_\omega^2 - \frac{1}{3} \left( \frac{19}{3}\pi^2 + \frac{337}{2} \right) L_\omega - 29\xi_3 - \frac{2}{45}\pi^4 + \frac{61}{108}\pi^2 + \frac{13069}{144} \right] - T_F m_l \left[ 4L_\omega^2 - 2 \left( \frac{4}{3}\pi^2 + 29 \right) L_\omega - 8\xi_3 + \frac{2}{27}\pi^2 + \frac{1097}{36} \right] \right\} \right\} + \frac{m^2}{8} \left[ 1 - 6C_F \frac{\alpha_s}{4\pi} (3L_\omega - 1) \right] + 2C_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[ 8L_\omega^2 - \left( \frac{8}{3}\pi^2 + \frac{109}{2} \right) L_\omega + 20\xi_3 - \frac{16}{45}\pi^4 - \frac{39}{2}\pi^2 + \frac{507}{16} \right] + C_A \left[ 33L_\omega^2 + \left( \frac{2}{3}\pi^2 - \frac{125}{2} \right) L_\omega - 11\xi_3 - \frac{2}{15}\pi^4 - 6\pi^2 + \frac{2789}{48} \right] - T_F m_l \left[ 12L_\omega^2 - 18L_\omega - 2\pi^2 + \frac{229}{12} \right] \right\} \right\} + \frac{2}{3} \left( \sum m_l^2 \right) C_F T_F \left( \frac{\alpha_s}{4\pi} \right)^2 (\pi^2 - 6) + \mathcal{O}(\alpha_s^3) \} , \] (35)

where
Fig. 2. Quark-condensate contributions to the correlator.

\[ L_\omega = \log \frac{2\omega}{\mu}. \]  

(36)

Terms up to two loops agree with [12]; the remaining ones are new. Multiplying the leading \( m^0 \) term in the HQET spectral density (35) by the corresponding matching coefficients [5,6], we reproduce the leading \( \delta^0 \) terms in the 3-loop QCD spectral densities (10), (14) in [16].

3. Quark and gluon condensates (dimensions 3 and 4)

Some 0-, 1-, and 2-loop diagrams for the quark condensate contribution are shown in Fig. 2. Starting from 2 loops (the last diagram in the figure) contributions proportional to the singlet sum \( \sum m_i \langle \bar{q}_i q_i \rangle \) appear. Our result for the coordinate-space correlator is

\[
\begin{align*}
\Pi_F^Q(\tau; \mu) &= -P \left( \frac{\langle \bar{q} q \rangle}{4} \right) \left[ 1 + 6C_F \frac{\alpha_s}{4\pi} \right. \\
&\quad + C_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ 2C_F \left[ 4 \left( \frac{2}{3} \pi^2 - 1 \right) L_\tau - 16\zeta_3 + \frac{10}{3} \pi^2 + 11 \right] \\
&\quad - C_A \left[ 4 \left( \frac{\pi^2}{3} - 7 \right) L_\tau - 8\zeta_3 + \pi^2 - \frac{149}{3} \right] - 16T_F n_l \left( L_\tau + \frac{4}{3} \right) \right\} \\
&\quad + C_F \left( \frac{\alpha_s}{4\pi} \right)^3 \left\{ C_F^2 \left[ 4 \left( 18\zeta_3 + \frac{4}{9} \pi^4 + \frac{8}{3} \pi^2 - 35 \right) L_\tau \\
&\quad + \frac{1600}{3} \zeta_5 - \frac{928}{9} \pi^2 \zeta_3 - \frac{140}{3} \zeta_3 + \frac{479}{135} \pi^4 - \frac{8}{9} \pi^2 + 157 \right] \\
&\quad + C_F C_A \left[ \frac{176}{3} \left( \frac{2}{3} \pi^2 - 1 \right) L_\tau^2 - 4 \left( 141\zeta_3 - \frac{4}{45} \pi^4 - \frac{902}{27} \pi^2 - \frac{737}{9} \right) L_\tau \\
&\quad - \frac{1}{3} \left( 1216\zeta_5 - \frac{424}{3} \pi^2 \zeta_3 + \frac{23654}{270} \zeta_3 - \frac{3799}{81} \pi^4 - \frac{27122}{81} \pi^2 - \frac{23669}{27} \right) \right\} \\
&\quad - C_A^2 \left[ \frac{88}{3} \left( \frac{\pi^2}{3} - 7 \right) L_\tau^2 - 4 \left( 33\zeta_3 + \frac{2}{15} \pi^4 - \frac{164}{27} \pi^2 - \frac{1409}{9} \right) L_\tau \\
&\quad - 72\zeta_5 + 12\pi^2 \zeta_3 - \frac{4856}{27} \zeta_3 + \frac{199}{810} \pi^4 + \frac{4094}{243} \pi^2 - \frac{69583}{81} \right] \right. \\
&\left. \right] \\
\end{align*}
\]

\[ 2 \text{ We have used the well-known method of projectors [31,32] for computation of various condensate contributions (at 1 loop a similar method was used in [33,34]).} \]
\[ + \left( \sum m_i \langle \bar{q} i, q i \rangle \right)^{\gamma_j} + \left( \frac{\alpha_s}{4\pi} \right)^{\gamma_j} \left\{ \frac{1}{2} \left( \frac{4}{3} \pi^2 - 11 \right) \right\} \]

\[ + \alpha_s \frac{C_F}{4\pi} \left[ C_F \left( \frac{4}{3} \pi^2 - 11 \right) L_\tau + 12 \zeta_3 + \frac{8}{45} \pi^4 + \frac{44}{9} \pi^2 + \frac{76}{3} \right] \]

\[ + C_A \left[ \left( \frac{100}{3} \pi^2 - 311 \right) L_\tau - 256 \zeta_3 - \frac{7}{15} \pi^4 - \frac{211}{9} \pi^2 + \frac{3559}{6} \right] \]

\[ - 2 T_F n_1 \left[ \frac{4}{3} \left( \frac{4}{3} \pi^2 - 11 \right) L_\tau - 16 \zeta_3 - \frac{44}{27} \pi^2 + \frac{313}{9} \right] \right\}, \tag{37} \]

where \( \langle \bar{q} q \rangle \) is renormalized at \( \mu \). The terms up to 2 loops in the dimension-3 contribution agree with [12].

Finiteness of the renormalized coefficient function \( C_{\bar{q} q} \) provides an independent confirmation of \( 2 \gamma_j - \gamma_{\bar{q} q} \) at 3 loops [11]. This anomalous dimension vanishes at 1 loop; \( \gamma_{\bar{q} q} = -\gamma_m \), hence \( C_{\bar{q} q} \) has the structure (34) with \( n = -1 \). We need one more term:

\[ C_{\bar{q} q} \sim \exp \left\{ c_1 \frac{\alpha_s}{4\pi} + \left( \frac{\alpha_s}{4\pi} \right)^2 \left[ -2(\gamma_1 - \beta_0 c_1) L_\tau + c_2 \right] \right\} \]

\[ + \left( \frac{\alpha_s}{4\pi} \right)^3 \left[ -4\beta_0 (\gamma_1 - \beta_0 c_1) L_\tau^2 - 2(\gamma_2 - 2\beta_0 c_2 - \beta_1 c_1) L_\tau + c_3 \right] + \mathcal{O}(\alpha_s^4) \right\}, \tag{38} \]

where \( \gamma_k = 2\gamma_{jk} + \gamma_{mnk}, \gamma_0 = 0 \). Hence the \( \alpha_s \) term contains no \( L_\tau \), the \( \alpha_s^2 \) one contains \( L_\tau^2 \), etc.

The dimension-3 operators \( O_3 = (m^3, m \sum m_i^2, \bar{q} q)^T \) satisfy the renormalization group equation [35–38]

\[ \frac{d O_3}{d \log \mu} + \gamma_3 O_3 = 0, \quad \gamma_3 = \begin{pmatrix} 3\gamma_m & 0 & 0 \\ 0 & 3\gamma_m & 0 \\ \gamma & \gamma' & -\gamma_m \end{pmatrix} \tag{39} \]

\[ \gamma = -\frac{N_c}{4\pi^2} \left\{ 2 + 8 C_F \frac{\alpha_s}{4\pi} \right\} \]

\[ + C_F \left[ C_F (96 \zeta_3 - 131) - C_A (48 \zeta_3 - 109) - 20 T_F n_1 \right] \left( \frac{\alpha_s}{4\pi} \right)^2 + \mathcal{O}(\alpha_s^3) \right\}, \]

\[ \gamma' = 24 \frac{N_c}{\pi^2} C_F T_F \left( \frac{\alpha_s}{4\pi} \right)^2 + \mathcal{O}(\alpha_s^3). \]

Therefore, the coefficient functions \( C_3 = (C_{m^3}, C_m \sum m_i^2, C_{\bar{q} q})^T \) satisfy the renormalization group equation

\[ \frac{d C_3}{d \log \mu} = \frac{\partial C_3}{\partial L_\tau} - 2\beta \frac{\partial C_3}{\partial \log \alpha_s} = (\gamma_3^T - 2 \gamma_j) C_3. \tag{40} \]

The dimension-4 operators \( m O_3 \) satisfy the renormalization group equation similar to (39) but with the anomalous dimension \( \gamma_3 + \gamma_m \). Hence we obtain

\[ \left[ \frac{\partial}{\partial L_\tau} - 2\beta \frac{\partial}{\partial \log \alpha_s} + 2\gamma_j - 3 \gamma_m \right] C_{m^3} = \gamma C_{\bar{q} q}. \]
Our results (32), (37) satisfy these equations.

It is well known that the gluon condensate contribution vanishes at 1 loop. In the fixed-point gauge the static quark does not interact with gluons, and the only remaining diagram is shown in Fig. 3. But the $G^2$ correction to the massless quark propagator $S(x, 0)$ vanishes after vacuum averaging [39]. The 2- and 3-loop contributions are

\[
\Pi_G^B(\tau; \mu) = \frac{(G^2)^\tau}{48} T_F \left(\frac{\alpha_s}{4\pi}\right)^2 \left\{ \left[ C_F \left(\frac{4}{3}\pi^2 - 11\right) + \frac{C_A}{2} \left(\frac{4}{3}\pi^2 - 23\right) \right] \right.
\]
\[
+ \frac{3\alpha_s}{4\pi} \left[ C_F \left(\frac{4}{3}\pi^2 - 11\right) \right. L_\tau + \frac{124}{3\pi^2 - 449} 56\zeta_3 + \frac{136}{135}\pi^4 - \frac{140}{9}\pi^2 - \frac{23}{2} \right] \]
\[
+ \frac{C_A}{2} \left[ \left(\frac{4}{3}\pi^2 - 11\right) \right. L_\tau + \frac{124}{3\pi^2 - 449} 56\zeta_3 - \frac{20}{3}\pi^4 + 100\pi^2 + 167 \right] \]
\[
+ \frac{2}{3} C_F T_F n_f \left[ \left(\frac{4}{3}\pi^2 - 11\right) \right. L_\tau - \frac{44}{27}\pi^2 + \frac{259}{9} \right] \]
\[
- \frac{1}{3} C_A T_F n_f \left[ \left(\frac{4}{3}\pi^2 - 23\right) \right. L_\tau - \frac{44}{27}\pi^2 + \frac{253}{9} \right] \right\} ,
\]

(41)

where $G^2 = G_{\mu\nu}^a G^{a\mu\nu}$. The anomalous dimension of this operator is [40–42]

\[
\gamma G^2 = -2 \frac{d\beta}{d\log\alpha_s} ,
\]

(42)

and hence the coefficient function must have the structure

\[
C_{G^2} \sim \left(\frac{\alpha_s}{4\pi}\right)^2 \left\{ 1 + \frac{\alpha_s}{4\pi} [2(\beta_0 - \gamma_{j0})L_\tau + c] + O(\alpha_s^2) \right\} .
\]

(43)

Our result (41) satisfies this condition.

The flavor-singlet dimension-4 operators $O_4 = (\sum m_i^4, (\sum m_i^2)^2, \sum m_i \bar{q}_i q_i, G^2)^T$ satisfy the renormalization group equation [35–38]
\[ \frac{dO_4}{d\log \mu} + \gamma_4 O_4 = 0, \tag{44} \]

\[ \gamma_4 = \begin{pmatrix} 4\gamma_m & 0 & 0 & 0 \\ 0 & 4\gamma_m & 0 & 0 \\ \gamma & \gamma' & 0 & 0 \\ -\frac{d\gamma}{d\log \alpha_s} & -\frac{d\gamma'}{d\log \alpha_s} & 4\frac{d\gamma_m}{d\log \alpha_s} & -2\frac{d\beta}{d\log \alpha_s} \end{pmatrix} \]

and the corresponding coefficient functions \( C_4 = (C_{\sum m_i^4}, C_{(\sum m_i^2)^2}, C_{\sum m_i\bar{q}_i q_i}, C_{G^2})^T \) — the equation

\[ \frac{dC_4}{d\log \mu} = (\gamma_4^T - 2\gamma_J)C_4; \tag{45} \]

hence,

\[ \left[ \frac{\partial}{\partial L_f} - 2\beta \frac{\partial}{\partial \log \alpha_s} + 2\gamma_J - 4\gamma_m \right] C_{\sum m_i^4} = \gamma C_{\sum m_i\bar{q}_i q_i} - \frac{d\gamma}{d\log \alpha_s} C_{G^2}, \]

\[ \left[ \frac{\partial}{\partial L_f} - 2\beta \frac{\partial}{\partial \log \alpha_s} + 2\gamma_J - 4\gamma_m \right] C_{(\sum m_i^2)^2} = \gamma' C_{\sum m_i\bar{q}_i q_i} - \frac{d\gamma'}{d\log \alpha_s} C_{G^2}, \]

\[ \left[ \frac{\partial}{\partial L_f} - 2\beta \frac{\partial}{\partial \log \alpha_s} + 2\gamma_J \right] C_{\sum m_i\bar{q}_i q_i} = 4\frac{d\gamma_m}{d\log \alpha_s} C_{G^2}. \]

The second equation here is satisfied trivially, because \( C_{(\sum m_i^2)^2} = O(\alpha_s^3). \) Our results (32), (37), (41) satisfy these equations.

4. Higher-dimensional condensates

The tree diagram in Fig. 2 can be written exactly in \( x = vt: \)

\[ \Pi^q(t) = i\theta(t)\langle q(vt)[vt, 0]\bar{q}(0) \rangle. \tag{46} \]

It is expressed via the bilocal quark condensate [43] which has 2 Dirac structures:

\[ \langle q(x)[x, 0]\bar{q}(0) \rangle = -\frac{\langle \bar{q}q \rangle}{4}\left[ f_S(x^2) - \frac{i\gamma}{d} f_V(x^2) \right]. \tag{47} \]

Its expansion in \( x \) via local quark condensates is known up to dimension 8 [15]. We use the bases of local condensates [44]

\[ Q^3 = \langle \bar{q}q \rangle, \quad Q^5 = i\langle \bar{q}G_{\mu\nu}\sigma^{\mu\nu}q \rangle, \quad Q^6 = \langle \bar{q}f q \rangle, \]

\[ Q^1 = \langle \bar{q}G_{\mu\nu}G^{\mu\nu}q \rangle, \quad Q^2 = i\langle \bar{q}G_{\mu\nu}\tilde{G}^{\mu\nu}\gamma_5 q \rangle, \]

\[ Q^3 = \langle \bar{q}G_{\mu\lambda}\gamma_\lambda\sigma^{\mu\nu}q \rangle, \quad Q^4 = i\langle \bar{q}D_\mu J_\nu\sigma^{\mu\nu}q \rangle, \quad A = i\langle \bar{q}D_\alpha D_\beta D_\gamma D_\delta \bar{q}_{\epsilon\kappa\lambda\mu}^\gamma q \rangle, \]

\[ Q^8 = i\langle \bar{q}[[G_{\mu\lambda}, \tilde{G}_{\nu\lambda}]_+, D^\mu]_+\gamma_\gamma q \rangle, \quad \bar{Q}^8 = -\langle \bar{q}[[G_{\mu\lambda}, \tilde{G}_{\nu\lambda}]_+, D^\mu]_+\gamma_\gamma q \rangle, \]

\[ Q^5 = i\langle \bar{q}[\Phi G_{\mu\nu}, G^{\mu\nu}]q \rangle, \quad \bar{Q}^5 = \langle \bar{q}D^2 f q \rangle, \quad Q^6 = \langle \bar{q}G_{\mu\nu}, J_\mu^\gamma q \rangle, \quad \bar{Q}^6 = \langle \bar{q}[\tilde{G}_{\mu\nu}, J_\mu^\gamma]_+\gamma_\gamma q \rangle, \tag{48} \]

where \( G_{\mu\nu} = gG_{\mu\nu}^{\alpha\beta}J_\alpha J_\beta , \quad J_\mu^\alpha = gJ_\mu^{\alpha\beta}J_\beta , \quad J_\mu^\alpha = D^\nu G_{\mu\nu}^{\alpha\beta} = g \sum\bar{q}_i\gamma_\mu t^\alpha_i q_i \); \( \sigma_{\mu\nu} = \gamma_{[\mu}\gamma_{\nu]} \); operators containing \( \tilde{G}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu
u\rho\sigma}G^{\rho\sigma} \) and \( \gamma_5 = \frac{i}{2}\varepsilon_{\alpha\beta\gamma\delta}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta \) are understood as short notations.
for the expressions from which both $\varepsilon$ tensors are eliminated using $\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\delta} = -4! \delta^\mu_{[\alpha} \delta^\nu_{\beta} \delta^\rho_{\gamma} \delta^\sigma_{\delta]}$.

The anomalous condensate $A$ does not vanish in the $\overline{\text{MS}}$ scheme; it is a finite combination of dimension-8 gluon condensates [44].

We obtain the contribution of bilinear quark condensates up to dimension 8 to the correlator at the tree level

\[
\Pi_P^0(\tau) = \frac{-1}{4} \left\{ P \, Q^3 - \frac{\tau}{d} m Q^3 - \frac{\tau^2}{21d} \left[ \frac{1}{2} Q^5 - m^2 Q^3 \right] \right. \\
+ \frac{\tau^3}{3! d(d + 2)} \left[ \frac{1}{2} Q^6 + \frac{3}{2} m Q^5 - 3m^3 Q^3 \right] \\
+ \frac{P}{4! d(d + 2)} \left[ \frac{\tau^4}{d(d + 2)} \left[ 5A - \frac{5}{2} Q^8 + \frac{1}{4} Q^5 - \frac{1}{2} Q^4 - 3Q^3_5 + 5Q^8 \right] \right. \\
\left. \left. + 5m(3Q^7_1 - Q^2_7 - 3Q^3 + Q^4_4) - 15m^2(Q^6 + m Q^5 - m^3 Q^3) \right] \right. \\
\left. + \mathcal{O}(\tau^5) \right\}. \tag{49}
\]

The terms up to dimension 7 at $m = 0$ agree with [12].

5. Conclusion

The results obtained here can be used for extracting numerical values of $F_P$ (and hence $f_B = f_{B^*}$, $f_{B_s}/f_B$ and similar quantities for $0^+, 1^+$ mesons) and $\Lambda_P$ (and hence $m_{B_s} - m_B$, $m_{B(0^+)} - m_B$, $m_{B_s(0^+)} - m_{B(0^+)}$, $m_B$) from HQET sum rules (1/$m_b$ corrections should be calculated separately).

For sufficiently small $\tau$ the correlator $\Pi_P(\tau; \mu)$ is given by the truncated OPE series

\[
\Pi_P(\tau; \mu) = \int_0^\infty d\omega \, \rho_P^{d \leq 2}(\omega; \mu) e^{-\omega \tau} + \Pi_P^{d \geq 3}(\tau; \mu), \tag{50}
\]

where the coefficient functions are known as truncated series in $\alpha_s$. On the other hand, we can represent it as

\[
\Pi_P(\tau; \mu) = \int_0^\infty d\omega \, \rho_P(\omega; \mu) e^{-\omega \tau}, \tag{51}
\]

where the spectral density is given by the ground-state meson contribution (22) plus the continuum of excited states. We can use the rough model of the continuum contribution [45]

\[
\rho_P(\omega; \mu) = \left| F_P(\mu) \right|^2 \delta(\omega - \Lambda_P) + \rho_P^{d \leq 2}(\omega; \mu) \theta(\omega - \omega_{c,P}), \tag{52}
\]

where $\omega_{c,P}$ is the effective continuum threshold. Equating these two expressions, we obtain the sum rule
\[ |F_P(\mu)|^2 e^{-\hat{\Lambda}_P \tau} = \int_0^{\omega_{eP}} d\omega \rho_p^{d_{\leq 2}}(\omega; \mu) e^{-\omega \tau} + \Gamma_p^{d_{\geq 3}}(\tau; \mu). \quad (53) \]

It is approximately valid at sufficiently large \( \tau \), where the continuum contribution is small, and the uncertainty introduced by its rough model is not essential. If there is a window of \( \tau \) where both conditions are satisfied, we can use this sum rule to extract an approximate value of \( F_P(\mu) \).

Differentiating (53) in \( \tau \) and dividing by (53) we obtain the sum rule for the ground-state residual energy
\[
\hat{\Lambda}_P = \int_0^{\omega_{eP}} d\omega \rho_p^{d_{\leq 2}}(\omega; \mu) \omega e^{-\omega \tau} - d\Gamma_p^{d_{\geq 3}}(\tau; \mu)/d\tau \int_0^{\omega_{eP}} d\omega \rho_p^{d_{\leq 2}}(\omega; \mu) e^{-\omega \tau} + \Gamma_p^{d_{\geq 3}}(\tau; \mu). \quad (54) 
\]

The continuum thresholds \( \omega_{eP} \) are tuned in such a way that the resulting \( \hat{\Lambda}_P \) do not depend on \( \tau \) in the region of applicability.

**CRediT authorship contribution statement**

**K.G. Chetyrkin:** Investigation, Methodology, Software, Validation, Writing – review & editing. **A.G. Grozin:** Investigation, Validation, Writing – original draft.

**Declaration of competing interest**

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

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**References**

[1] E. Eichten, B.R. Hill, An effective field theory for the calculation of matrix elements involving heavy quarks, Phys. Lett. B 234 (1990) 511–516, [https://doi.org/10.1016/0370-2693(90)92049-O](https://doi.org/10.1016/0370-2693(90)92049-O).

[2] M. Neubert, Heavy quark symmetry, Phys. Rep. 245 (1994) 259–396, [https://doi.org/10.1016/0370-1573(94)90091-4](https://doi.org/10.1016/0370-1573(94)90091-4), arXiv:hep-ph/9306320.

[3] A.V. Manohar, M.B. Wise, Heavy Quark Physics, Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol., vol. 10, Cambridge University Press, Cambridge, 2000.

[4] A.G. Grozin, Heavy Quark Effective Theory, Springer Tracts Mod. Phys., vol. 201, Springer, Berlin, 2004.

[5] D.J. Broadhurst, A.G. Grozin, Matching QCD and HQET heavy-light currents at two loops and beyond, Phys. Rev. D 52 (1995) 4082–4098, [https://doi.org/10.1103/PhysRevD.52.4082](https://doi.org/10.1103/PhysRevD.52.4082), arXiv:hep-ph/9410240.

[6] A.G. Grozin, Decoupling of heavy quark loops in light-light and heavy-light quark currents, Phys. Lett. B 445 (1998) 165–167, [https://doi.org/10.1016/S0370-2693(98)01439-7](https://doi.org/10.1016/S0370-2693(98)01439-7), arXiv:hep-ph/9810358.

[7] S. Bekavac, A.G. Grozin, P. Marquard, J.H. Piclum, D. Seidel, M. Steinhauser, Matching QCD and HQET heavy-light currents at three loops, Nucl. Phys. B 833 (2010) 46–63, [https://doi.org/10.1016/j.nuclphysb.2010.02.025](https://doi.org/10.1016/j.nuclphysb.2010.02.025), arXiv:0911.3356.

[8] X.-D. Ji, M. Musolf, Subleading logarithmic mass dependence in heavy meson form-factors, Phys. Lett. B 257 (1991) 409–413, [https://doi.org/10.1016/0370-2693(91)91916-J](https://doi.org/10.1016/0370-2693(91)91916-J).
[9] D.J. Broadhurst, A.G. Grozin, Two loop renormalization of the effective field theory of a static quark, Phys. Lett. B 267 (1991) 105–110, https://doi.org/10.1016/0370-2693(91)90532-U, arXiv:hep-ph/9908362.

[10] V. Giménez, Two loop calculation of the anomalous dimension of the axial current with static heavy quarks, Nucl. Phys. B 375 (1992) 582–622, https://doi.org/10.1016/0550-3213(92)90112-O.

[11] K.G. Chetyrkin, A.G. Grozin, Three loop anomalous dimension of the heavy-light quark current in HQET, Nucl. Phys. B 666 (2003) 289–302, https://doi.org/10.1016/S0550-3213(03)00490-5, arXiv:hep-ph/0303113.

[12] D.J. Broadhurst, A.G. Grozin, Operator product expansion in static quark effective field theory: large perturbative corrections, Phys. Lett. B 274 (1992) 421–427, https://doi.org/10.1016/0370-2693(92)900363.

[13] E. Bagan, P. Ball, V.M. Braun, H.G. Dosch, QCD sum rules in the effective heavy quark theory, Phys. Lett. B 278 (1992) 457–464, https://doi.org/10.1016/0370-2693(92)90585-R.

[14] M. Neubert, Heavy meson form-factors from QCD sum rules, Phys. Rev. D 45 (1992) 2451–2466, https://doi.org/10.1103/PhysRevD.45.2451.

[15] A.G. Grozin, Methods of calculation of higher power corrections in QCD, Int. J. Mod. Phys. A 10 (1995) 3497–3529, https://doi.org/10.1142/S0217751X95010674, arXiv:hep-ph/9412238.

[16] A. Czarnecki, K. Melnikov, Threshold expansion for heavy light systems and flavor off diagonal current-current correlators, Phys. Rev. D 66 (2002) 011502, https://doi.org/10.1103/PhysRevD.66.011502, arXiv:hep-ph/0110028.

[17] A.G. Grozin, A.V. Smirnov, V.A. Smirnov, Decoupling of heavy quarks in HQET, J. High Energy Phys. 11 (2006) 022, https://doi.org/10.1088/1126-6708/2006/11/022, arXiv:hep-ph/0609280.

[18] N. Isgur, M.B. Wise, Weak transition form-factors between heavy mesons, Phys. Lett. B 237 (1990) 527–530, https://doi.org/10.1016/0370-2693(90)91219-2.

[19] H. Georgi, M.B. Wise, Superflavor symmetry for heavy particles, Phys. Lett. B 243 (1990) 279–283, https://doi.org/10.1016/0370-2693(90)90851-V.

[20] V.M. Braun, K.G. Chetyrkin, B.A. Kniehl, Renormalization of parton quasi-distributions beyond the leading order: spacelike vs. timelike, J. High Energy Phys. 07 (2020) 161, https://doi.org/10.1007/JHEP07(2020)161, arXiv:2004.01043.

[21] R.N. Lee, Presenting LiteRed: a tool for the Loop InTEgrals REDuction, arXiv:1212.2685, December 2012.

[22] R.N. Lee, LiteRed 1.4: a powerful tool for reduction of multiloop integrals, J. Phys. Conf. Ser. 523 (2014) 012059, https://doi.org/10.1088/1742-6596/523/1/012059, arXiv:1310.1145.

[23] A.G. Grozin, Calculating three loop diagrams in heavy quark effective theory with integration by parts recurrence relations, J. High Energy Phys. 03 (2000) 013, https://doi.org/10.1088/1126-6708/2000/03/013, arXiv:hep-ph/0002266.

[24] A.V. Smirnov, F.S. Chuharev, FIRE6: Feynman Integral REDuction with modular arithmetic, arXiv:1901.07808, 2019.

[25] P. Nogueira, Automatic Feynman graph generation, J. Comput. Phys. 105 (1993) 279–289, https://doi.org/10.1006/jcph.1993.1074.

[26] J.A.M. Vermaseren, New features of FORM, arXiv:math-ph/0010025, 2000.

[27] T. Van Ritbergen, A.N. Schellekens, J.A.M. Vermaseren, Group theory factors for Feynman diagrams, Int. J. Mod. Phys. A 14 (1) (1999) 41–96, https://doi.org/10.1142/S0217751X99000038, arXiv:hep-ph/9802376.

[28] M. Beneke, V.M. Braun, Heavy quark effective theory beyond perturbation theory: renormalons, the pole mass and the residual mass term, Nucl. Phys. B 426 (1994) 301–343, https://doi.org/10.1016/0550-3213(94)90314-X, arXiv:hep-ph/9402364.

[29] A.G. Grozin, Lectures on multiloop calculations, Int. J. Mod. Phys. A 19 (2004) 473–520, https://doi.org/10.1142/S0217751X04016775, arXiv:hep-ph/0307297.

[30] A.G. Grozin, Higher radiative corrections in HQET: in: A. Ali, M. Ivanov (Eds.), Helmholtz International Summer School on Heavy Quark Physics, Verlag Deutsches Elektronen-Synchrotron, 2008, pp. 55–88, DESY-PROC-2009-07, http://www-library.desy.de/preparach/desy/proc/proc09-07.pdf, arXiv:0809.4540.

[31] S.G. Gorishnii, S.A. Larin, F.V. Tkachov, The algorithm for OPE coefficient functions in the MS scheme, Phys. Lett. B 124 (1983) 217–220, https://doi.org/10.1016/0370-2693(83)91439-9.

[32] S.G. Gorishnii, S.A. Larin, Coefficient functions of asymptotic operator expansions in minimal subtraction scheme, Nucl. Phys. B 283 (1987) 452–476, https://doi.org/10.1016/0550-3213(87)90283-5.

[33] D.J. Broadhurst, S.C. Generalis, Can mass singularities be minimally subtracted?, Phys. Lett. B 142 (1984) 75–79, https://doi.org/10.1016/0370-2693(84)91139-0.

[34] D.J. Broadhurst, S.C. Generalis, Dimension eight contributions to light quark QCD sum rules, Phys. Lett. B 165 (1985) 175–180, https://doi.org/10.1016/0370-2693(85)90715-4.

[35] V.P. Spiridonov, K.G. Chetyrkin, Nonleading mass corrections and renormalization of the operators $m \bar{\psi} \psi$ and $G_{\mu \nu}^2$, Sov. J. Nucl. Phys. 47 (1988) 522–527, Yad. Fiz. 47 (1988) 818–826.
[36] K.G. Chetyrkin, J.H. Kühn, Quartic mass corrections to $R_{\text{had}}$, Nucl. Phys. B 432 (1994) 337–350, https://doi.org/10.1016/0550-3213(94)90605-X, arXiv:hep-ph/9406299.

[37] K.G. Chetyrkin, M.F. Zoller, Leading QCD-induced four-loop contributions to the $\beta$-function of the Higgs self-coupling in the SM and vacuum stability, J. High Energy Phys. 06 (2016) 175, https://doi.org/10.1007/JHEP06(2016)175, arXiv:1604.00853.

[38] P.A. Baikov, K.G. Chetyrkin, QCD vacuum energy in 5 loops, in: PoS RADCOR2017, 2018, 025.

[39] V.A. Novikov, M.A. Shifman, A.I. Vainshtein, V. I. Zakharov, Calculations in external fields in quantum chromodynamics. Technical review, Fortschr. Phys. 32 (1984) 585–622.

[40] J.C. Collins, A. Duncan, S.D. Joglekar, Trace and dilatation anomalies in gauge theories, Phys. Rev. D 16 (1977) 438–449, https://doi.org/10.1103/PhysRevD.16.438.

[41] N.K. Nielsen, The energy momentum tensor in a nonabelian quark gluon theory, Nucl. Phys. B 120 (1977) 212–220, https://doi.org/10.1016/0550-3213(77)90040-2.

[42] V.P. Spiridonov, Anomalous dimension of $G_{\mu \nu}^2$ and $\beta$ function, Tech. Rep. P-0378, IYaI https://lib-extopc.kek.jp/preprints/PDF/1986/8601/8601315.pdf, 1984.

[43] S.V. Mikhailov, A.V. Radyushkin, Nonlocal condensates and QCD sum rules for pion wave function, JETP Lett. 43 (1986) 712, Pis’ma Zh. Eksp. Teor. Fiz. 43 (1986) 551.

[44] A.G. Grozin, Y.F. Pinelis, Contribution of higher gluon condensates to the light quark vacuum polarization, Z. Phys. C 33 (1987) 419–425, https://doi.org/10.1007/BF01552548.

[45] M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, QCD and resonance physics. Theoretical foundations, Nucl. Phys. B 147 (1979) 385–447, https://doi.org/10.1016/0550-3213(79)90022-1.