(2+1)-dimensional Chern-Simons bi-gravity with AdS Lie bialgebra as an interacting theory of two massless spin-2 fields

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We introduce a new Lie bialgebra structure for the anti de Sitter (AdS) Lie algebra in (2+1)-dimensional spacetime. By gauging the resulting AdS Lie bialgebra, we write a Chern-Simons gauge theory of bi-gravity involving two dreibeins rather than two metrics, which describes two interacting massless spin-2 fields. Our ghost-free bi-gravity model which has no any local degrees of freedom, has a suitable free field limit. By solving its equations of motion, we obtain a new black hole solution which has two curvature singularities and two horizons.

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I. INTRODUCTION

There are different theories of gravity in three-dimensional spacetime and each of them has own advantages and has been widely studied. General relativity is a classical theory which describes interactions of a single massless spin-2 particle (graviton) [1–5]. Three-dimensional general relativity, without cosmological constant, is equivalent to a Chern-Simons gauge theory with the Poincaré gauge group ISO(2,1) [1]. But, the Chern-Simons gauge theories with gauge groups SO(2,2) or SO(3,1) are equivalent to adding negative or positive cosmological constants to three-dimensional general relativity [1].

It has been shown that does not exist any consistent theory (with at most two derivatives of the fields) involving interactions of many massless spin-2 fields in spacetime dimensions $d > 3$, because of their ghost problems or their discontinuity in the number of degrees of freedom at their free field limits [6]. Although, in (2+1)-dimensional spacetime an exotic consistent interacting theory of many massless spin-2 fields has been constructed in [7], but the physical consequences of such interacting model has not been studied in detail. Theories which describe massless spin-2 fields in (2+1)-dimensional spacetime, have no local degrees of freedom, hence the Chern-Simons theory which is a topological model, is a suitable candidate to construct a (2+1)-dimensional interacting theory of massless spin-2 fields.

On the other hand, in past years, “massive gravity” theories which have local degrees of freedom and describe the interactions of the massive spin-2 fields (gravitons), have been developed. Massive gravity theories have been greatly studied after resolution of their theoretical difficulties (see for a review [8, 9]). Topologically massive gravity [10–12], new massive gravity (NMG) [14–17] and general massive gravity [17] are three higher derivative theories of massive gravity involving auxiliary fields. dRGT massive gravity [18–21] is a bi-metric theory of massive gravity, and describes a massive together with a massless spin-2 particles. The non-dynamical reference metric of the dRGT model is promoted to a dynamical metric by introducing a kinetic term for it, resulting in the zwei-dreibein gravity (ZDG) [22, 23] (see also [24, 25]). The ZDG model has been generalized to obtain a parity-violating model which is called General Zwei-Dreibein Gravity (GZDG) [21, 26]. These massive gravity models have not Chern-Simons formulations, but they are Chern-Simons-like theories of gravity (see for a review [21]). In ref. [27], the GZDG$^+$ model has been introduced by adding a constraint term to the GZDG model for fixing torsion. Moreover, during the past few years two different extensions of the Poincaré algebra, i.e. the Maxwell algebra [28–35] and the semi-simple extension of the Poincaré algebra (AdS-Lorentz algebra) [35–38] have been applied to construct some group-theoretical gravity theories in four and three spacetime dimensions. Recently, we have studied a (2+1)-dimensional interacting model of two massless spin-2 fields by gauging a new Lie algebra [39]. Now, in this paper, we are interested in the study of an interacting theory of two massless spin-2 fields which obtains by gauging a new Lie bialgebra. The resulting bi-gravity model, just like the ZDG model, has been formulated in terms of two dreibeins rather than two metrics. But, unlike the ZDG model, it is a massless zwei-dreibein gravity theory.

The outline of the paper is as follows: In section two, we construct a new Lie bialgebra using the AdS Lie algebra so(2,2) in (2+1)-dimensional spacetime. In Section three, using the obtained Lie bialgebra (and the corresponding Manin triple), we propose a Chern-Simons gauge invariant bi-metric theory of gravity involving two different dreibein fields which describes two interacting massless spin-2 fields. We compare our bi-gravity model with the “massive gravity” theories such as NMG and ZDG. We also solve the equations of motion using the BTZ black hole metric for one of the metrics, and obtain a new black hole in the other metric solution. Some concluding remarks and discussions are given at the end.
II. ADS LIE BIALGEBRA

In this section, we introduce a new bialgebra \cite{10, 11} which is obtained by use of the AdS Lie algebra in (2+1)-dimensional spacetime. In (2+1)-dimensional spacetime, the commutation relations of the six-dimensional AdS Lie algebra are as follows \cite{3}:

\[ [J_a, J_b] = \epsilon_{abc} J_c, \quad [J_a, P_b] = \epsilon_{abc} P_c, \quad [P_a, P_b] = \frac{1}{\ell^2} \epsilon_{abc} J^c, \]

where \(\ell^{-2} = -\Lambda\) is a constant, \(\epsilon_{012} = -1\), and \(J_a\) and \(P_a\) \((a = 0, 1, 2)\) are the Lorentz and translation generators, respectively. By letting the basis of the AdS algebra as \(\{X_1, \ldots, X_6\} = \{J_0, J_1, J_2, P_0, P_1, P_2\}\), and using the structure constants \(f_{ab}^c\) of the AdS Lie algebra \cite{11} and the following Jacobi and mixed-Jacobi identities \cite{41}:

\[ f_{ab}^c f_{cm}^d + f_{ca}^m f_{mb}^d + f_{bc}^m f_{ma}^d = 0, \]

\[ f_{mc} f_{nb}^m n f_{ab}^m n f_{mb}^m = 0, \]

\[ f_{mc} f_{nb}^m n f_{mb}^m = 0, \]

we respectively obtain the following structure constants \(f_{ab}^c\) of the dual Lie algebra:

\[ f_{35}^a = -a, \quad f_{35}^a = a, \quad f_{21}^a = -\Lambda a, \quad f_{35}^a = -a, \quad f_{21}^a = -\Lambda a, \]

where \(a\) is an arbitrary constant. By letting the basis of the dual Lie algebra as \(\{\tilde{X}^1, \ldots, \tilde{X}^6\} = \{\tilde{P}_0, \tilde{P}_1, \tilde{P}_2, \tilde{J}_0, \tilde{J}_1, \tilde{J}_2\}\), the commutation relations of the dual Lie algebra can be written in the following form:

\[ [\tilde{J}_1, \tilde{J}_0] = -\epsilon_{1bc} \tilde{J}_c, \quad [\tilde{J}_1, \tilde{P}_0] = -\epsilon_{1bc} \tilde{P}_c, \quad [\tilde{J}_1, \tilde{P}_1] = \epsilon^{-\epsilon_{1bc} \tilde{J}_c}, \]

where \(\tilde{J}_a\) and \(\tilde{P}_a\) \((a = 0, 1, 2)\) are the generators of spacetime rotation and translation related to the dual geometric structures such as metric and spin connection (see below). The dual Lie algebra \cite{11} is very similar to the AdS Lie algebra \cite{11}, but in \cite{3} we have the commutation relations between generators with indefinite "1" \((J_1, P_1)\) and generators with indefinite "0" \(J_2, P_2\), only. In other words, generators with indefinite "1" \(j = 0, 2^n\) \((J_j, P_j)\) commute with each other. The commutation relations \cite{11} together with \cite{3} describe AdS Lie bialgebra. Now, using \([X_a, X_b] = f_{abc} X_c\) and \(f_{abc} \tilde{X}^c = f_{abc} X_c\) \cite{10, 11}, one can obtain the commutation relations between the generators of the AdS Lie algebra \(J_a, P_a\) and the generators of the dual Lie algebra \(\tilde{J}_a, \tilde{P}_a\) as follows:

\[ [J_a, \tilde{P}_b] = \epsilon_{abc} (a P^c - \tilde{P}_c), \quad [P_a, \tilde{J}_b] = -\epsilon_{abc} (a P^c + \tilde{P}_c), \quad [J_a, \tilde{J}_b] = -\epsilon_{abc} (a J^c + \tilde{J}_c), \quad [P_a, \tilde{P}_b] = \epsilon^{-\epsilon_{abc} J^c}, \]

where the indices \(i\) and \(j\) \((i, j = 0, 2)\) are the algebra indices. The commutation relations \cite{11} together with \cite{11} and \cite{3}, describe AdS Manin triple which is a 12-dimensional Lie algebra.

III. (2+1)-DIMENSIONAL CHERN-SIMONS GRAVITY WITH ADS LIE BIALGEBRA

In this section, we use the AdS Lie bialgebra, which is discussed in the previous section, to construct a new \((2+1)\)-dimensional Chern-Simons bi-gravity. Using the relation \(f_{AB} C \Omega_{CD} + f_{AD} C \Omega_{CB} = 0 \) \cite{42, 47} an ad-invariant metric \(\Omega_{AB} = \langle X_A, X_B \rangle\) for the AdS Manin triple is obtained as follows:

\[ \langle J_a, P_b \rangle = -\alpha \eta_{ab}, \quad \langle J_a, \tilde{P}_b \rangle = \beta \delta_{ab} + \alpha \delta_{a1} \delta_{b1}, \quad \langle P_a, \tilde{P}_b \rangle = \alpha^2 \delta_{a1} \delta_{b1}, \quad \langle P_a, \tilde{J}_b \rangle = \beta \delta_{ab} - \alpha \delta_{a1} \delta_{b1}, \quad \langle J_a, \tilde{J}_b \rangle = \langle P_a, \tilde{J}_b \rangle = \langle \tilde{J}_a, \tilde{J}_b \rangle = 0, \quad \langle P_a, \tilde{J}_b \rangle = \langle \tilde{J}_a, \tilde{J}_b \rangle = 0, \]

where \(\delta_{ab} = diag(1, 1, 1)\) is the Kronecker delta function, and \(\alpha\) and \(\beta\) are arbitrary constants. The ad-invariant metric should be non-degenerate, and then, we have \(\beta \neq 0\). Now, we use the AdS Manin triple to construct a gauge symmetric Chern-Simons action, \(I_{cs} = \frac{i}{4\pi} f_{M} (\{h \wedge dh\} + \frac{1}{3} \{h \wedge [h, h]\})\), where \(h = h_{\mu} dx^\mu\) is an AdS Manin triple valued Murer-Cartan one-form gauge field as follows:

\[ h_{\mu} = h_{\mu} B X_B = e_{\mu}^a P_a + \omega_{\mu}^a J_a + \tilde{e}_a^\mu \tilde{P}_a + \tilde{\omega}_a^\mu \tilde{J}_a, \]

where the Greek indices \(\mu = 0, 1, 2\) are the spacetime indices, \(e_{\mu}^a\) and \(\omega_{\mu}^a\) are the ordinary dreibein and spin connection, and \(\tilde{e}_a^\mu\) and \(\tilde{\omega}_a^\mu\) are dreibein and spin connection corresponding to the generators of the dual Lie algebra, respectively. In this point of view, we obtain a gauge theory which has two metric tensors \(\eta_{\mu\nu} = e_{\mu}^a e_{\nu}^b \eta_{ab}\) and \(f_{\mu\nu\rho} = \tilde{e}_a^\mu \tilde{e}_b^\nu \eta_{ab}\). We use the infinitesimal gauge parameter \(u = p^\mu P_\mu + \tau^a J_a + P_\mu \tilde{P}_a + \tilde{P}_\mu \tilde{J}_a\) together with the commutation relations \cite{11, 3, 11} and the gauge transformations \(h_{\mu} \rightarrow h_{\mu} = U^{-1} h_{\mu} U + \tilde{\eta}_{\mu} \partial U\), with \(U = e^{-u} \approx 1 - u\) and \(U^{-1} = e^u \approx 1 + u\), to obtain the following transformations for the gauge fields:

\[ \delta e^\mu_{\nu} = -\partial_{\mu} e^c + \epsilon_{abc} (\rho_a \omega_{b\mu} + \tau_\alpha \epsilon_{\mu c}), \]

\[ \delta \omega_{\mu}^a = -\partial_{\mu} \tau^c + \epsilon_{abc} (\epsilon^{-2} \rho_a \epsilon_{\mu b} + \tau_a \omega_{b\mu}), \]

\[ \delta \tilde{e}_a^\mu = -\partial_{\mu} \tilde{P}_a + \epsilon_{abc} (\rho_a \tilde{\omega}_{b\mu} - \tilde{\tau}_a \epsilon_{\mu b} + \tilde{\lambda}_a \omega_{b\mu}), \]

\[ \delta \tilde{\omega}_a^\mu = -\partial_{\mu} \tilde{J}_a + \epsilon_{abc} (\epsilon^{-2} (\rho_a \tilde{\epsilon}_{b\mu} + \tilde{\tau}_a \omega_{b\mu} + \tilde{\lambda}_a \omega_{b\mu})), \]

\[ + \epsilon_{abc} (\tilde{\tau}_a \omega_{b\mu} + \epsilon^{-2} (\rho_a \tilde{\epsilon}_{b\mu} + \tilde{\lambda}_a \omega_{b\mu})). \]
The Ricci curvature two-form $R = R_{\mu\nu} dx^\mu \wedge dx^\nu$ can be written as:

$$R_{\mu\nu} = \partial_{[\mu} h_{\nu]} + [h_{\mu}, h_{\nu}] = R_{\mu\nu}^a X_A,$$

$$= T_{\mu\nu}^a \, P_a + R_{\mu\nu}^a \, J_a + \tilde{T}_{\mu\nu}^a \, \tilde{P}_a + \tilde{R}_{\mu\nu}^a \, \tilde{J}_a,$$

such that the torsion $T_{\mu\nu}^a$ and the standard Riemannian curvature $R_{\mu\nu}^a$ are as follows:

$$T_{\mu\nu}^a = \partial_{[\mu} e^a_{\nu]} + \epsilon_{\nu b c e^a_{\mu]} \, \epsilon_{\mu}^{\partial} \left( e_{\nu}^{b, \nu} - e_{\nu}^{b, \nu} \right),$$

$$T_{\mu\nu}^1 = \partial_{[\mu} e^1_{\nu]} + \epsilon_{\nu b c e^1_{\mu]} \, \epsilon_{\mu}^{\partial} \left( e_{\nu}^{b, \nu} + e_{\nu}^{b, \nu} \right),$$

$$R_{\mu\nu}^1 = \partial_{[\mu} \omega_{\nu]}^1 + \frac{1}{2} \epsilon_{\nu e c} \left( \omega_{[\mu}^{b e c} \, \epsilon_{\nu a, e}^{\omega_{\nu]} + \epsilon_{\nu a, e}^{\omega_{\nu]} \right),$$

$$+ \epsilon_{\nu e c} \left( \epsilon_{\nu e c}^{\omega_{\nu]} + \omega_{[\mu}^{b e c} \, \epsilon_{\nu a, e}^{\omega_{\nu]} \right),$$

$$R_{\mu\nu}^1 = \partial_{[\mu} \omega_{\nu]}^1 + \frac{1}{2} \epsilon_{\nu e c} \left( \omega_{[\mu}^{b e c} \, \epsilon_{\nu a, e}^{\omega_{\nu]} + \epsilon_{\nu a, e}^{\omega_{\nu]} \right),$$

and in the same way, the field strengths $\tilde{T}_{\mu\nu}^a$ and $\tilde{R}_{\mu\nu}^a$ which can be interpreted as dual torsion and dual Riemannian curvature respectively, have the following forms:

$$\tilde{T}_{\mu\nu}^j = \partial_{[\mu} \tilde{e}^j_{\nu]} - \epsilon_{\nu e c} \left( \omega^1_{[\mu}^{b e c} + \omega_{[\mu}^{b e c} \right),$$

$$\tilde{T}_{\mu\nu}^1 = \partial_{[\mu} \tilde{e}^1_{\nu]} + \epsilon_{\nu e c} \left( \omega^1_{[\mu}^{b e c} + \omega_{[\mu}^{b e c} \right),$$

$$\tilde{R}_{\mu\nu}^j = \partial_{[\mu} \tilde{\omega}_{\nu]}^j - \epsilon_{\nu e c} \left( \omega^1_{[\mu}^{b e c} + \omega_{[\mu}^{b e c} \right),$$

$$\tilde{R}_{\mu\nu}^1 = \partial_{[\mu} \tilde{\omega}_{\nu]}^1 + \epsilon_{\nu e c} \left( \omega^1_{[\mu}^{b e c} + \omega_{[\mu}^{b e c} \right),$$

Using (11) as well as (33-36), one obtains the following Chern-Simons bi-gravity model with the AdS Manin triple as a gauge symmetry:

$$I = I_{\text{CS}}(e, \omega) + I_{\text{CS}}(\tilde{e}, \tilde{\omega}) + I_{\text{int}}(e, \omega, \tilde{e}, \tilde{\omega}),$$

(11)

where the first term is

$$I_{\text{CS}} = -4 a \tilde{a} G_{\text{EC}}(e, \omega, \Lambda),$$

and the Einstein-Cartan action $I_{\text{EC}}$ is

$$I_{\text{EC}}(e, \omega, \Lambda) = -\frac{1}{16 \pi G M} \int d^4 x \epsilon^{\mu\nu\rho\sigma} \epsilon_{\mu} \left( D_{\lambda} \omega_{\nu} - \frac{\Lambda}{3} \epsilon_{abc} e^{a}_{\nu} e^{b}_{\rho} \right).$$

The second term in (11) is the Einstein-Cartan action with $\tilde{e}^j_\mu = 0$, $\tilde{\omega}^{i\mu}_\nu \neq 0$, $\tilde{e}^{1}_\mu \neq 0$, as follows:

$$I_{\text{CS}} = -\frac{\tilde{a} a}{4 \pi} \int d^4 x \epsilon^{\mu\nu\rho\sigma} \epsilon_{\mu} \left( D_{\lambda} \tilde{\omega}_{\nu} - \frac{\Lambda}{3} \epsilon_{abc} e^{a}_{\nu} e^{b}_{\rho} \right).$$

The third term in (11) includes some interaction terms between the fields $\{e^a_\mu, \omega^i_\mu\}$ and the fields $\{\tilde{e}^a_\mu, \tilde{\omega}^i_\mu\}$ as follows:

$$I_{\text{int}} = \frac{1}{4 \pi} \int d^4 x \epsilon^{\mu\nu\rho\sigma} \left\{ a \tilde{a} \left( D_{\lambda} e^{a}_{\mu} - \frac{\Lambda}{3} \epsilon_{abc} e^{a}_{\nu} e^{b}_{\rho} \right) + \left( \tilde{e}^{1}_\mu D_{\lambda} e^{a}_{\nu} - \frac{\Lambda}{3} \epsilon_{abc} e^{a}_{\nu} e^{b}_{\rho} \right) + \epsilon_{\nu e c} \left( D_{\lambda} e^{a}_{\nu} + \epsilon_{abc} e^{a}_{\nu} e^{b}_{\rho} \right) \right\},$$

(17)

The Chern-Simons bi-gravity model (11) which is invariant under the gauge transformations (17), has no any local degrees of freedom, and is a ghost-free model which describes two interacting massless spin-2 fields in (2+1)-dimensional spacetime. Two dreibeins in the action (11) are related to their corresponding metric tensors as follows:

$$g_{\mu\nu} = e^{a}_\mu e^{b}_\nu \eta_{ab}, \quad f_{\mu\nu} = \tilde{e}^{a}_\mu \tilde{e}^{b}_\nu \eta_{ab}.$$  

(13)

In the absence of the interaction terms $I_{\text{int}}$, the free field limit of (11),

$$I_{\text{free}} = I_{\text{CS}}(e, \omega) + I_{\text{CS}}(\tilde{e}, \tilde{\omega}),$$

(14)

similar to (11) has no any local degrees of freedom, and is invariant under the following gauge transformations:

$$\delta e^{a}_\mu = -\partial_{\lambda} e^{a}_\mu + \epsilon_{abc} (\rho_a e^{b}_\mu + \tau_a e^{c}_\mu),$$

$$\delta \tilde{e}^{a}_\mu = -\partial_{\lambda} \tilde{e}^{a}_\mu + \epsilon_{abc} (\ell^2 \rho_a e^{b}_\nu + \tau_a e^{c}_\mu),$$

$$\delta \omega^{i\mu}_\nu = -\partial_{\lambda} \omega^{i\mu}_\nu + \epsilon_{abc} (\ell^2 \rho_a e^{b}_\nu + \tau_a \omega^{c}_\mu),$$

$$\delta \tilde{\omega}^{i\mu}_\nu = -\partial_{\lambda} \tilde{\omega}^{i\mu}_\nu + \epsilon_{abc} (\ell^2 \rho_a \tilde{e}^{b}_\nu + \tau_a \omega^{c}_\mu).$$

(15)

Now, by assuming the following relations among the fields and constants:

$$\beta \tilde{e}^{a}_\mu = -\frac{1}{m^2} f_{\mu a}, \quad \beta \tilde{\omega}^{a}_\mu = h_{\mu a},$$

$$\alpha = -\tau, \quad \alpha \ell^{-2} = \Lambda_0, \quad \alpha \ell^{-2} = 2m^2 \beta,$$

(16)

the Chern-Simons action (11) can be rewritten in the following form:

$$I_{\text{CS}} = \frac{1}{2 \pi} I_{\text{NMG}} + \frac{1}{4 \pi} \int d^4 x \epsilon^{\mu\nu\rho\sigma} \left\{ -a^2 \alpha^2 \partial^2 \omega^{\mu}\nu - a a \left( D_{\lambda} e^{a}_{\mu} - \frac{\Lambda}{3} \epsilon_{abc} e^{a}_{\nu} e^{b}_{\rho} \right) + \alpha a \left( D_{\lambda} e^{a}_{\mu} - \frac{\Lambda}{3} \epsilon_{abc} e^{a}_{\nu} e^{b}_{\rho} \right) + \beta \epsilon_{abc} e^{a}_{\nu} e^{b}_{\rho} \epsilon^{c}_{\sigma} + \alpha \epsilon_{abc} e^{a}_{\nu} e^{b}_{\rho} \epsilon^{c}_{\sigma} \right\},$$

(18)
Again, using the following redefinition of the fields and constants, 
\[ e_1^a = e^a, \quad \omega_1^a = \omega^a, \quad e_2^a = e^a, \quad \omega_2^a = \omega^a, \]
\[ a_2 = 1, \quad \alpha = M_p, \quad \epsilon^2 = -\alpha_1 m^2, \quad \beta = M_p (a - \frac{\beta_3}{\alpha_1}), \]
the Chern-Simons model (11) can be rewritten in another form as follows:
\[ I = \frac{1}{2\pi} I_{ZDG}(\sigma = -1, \tilde{\epsilon}_{\mu} j = \tilde{\omega}_{\mu} = 0) \]
\[ + \frac{1}{4\pi M_p} \int d^4x \epsilon^{\mu
u\rho\sigma} \left\{ \alpha_1 \left( \tilde{e}_{\mu} \tilde{D}_\nu \epsilon_{\rho} - \tilde{e}_{\mu} \tilde{D}_\nu \epsilon_{\rho} \right) \right\} \]
\[ + \beta \left( \tilde{\omega}_{\mu} \tilde{D}_\nu \epsilon_{\rho} + \tilde{e}_{\mu} \tilde{D}_\nu \epsilon_{\rho} + \epsilon^2 \epsilon_{ab} \epsilon_{\mu} \epsilon_{cb} \epsilon_{\rho} \right) \]
\[ + 2\alpha_1 \left( \tilde{e}_{\mu} \left( \epsilon_{\nu} \tilde{D}_\rho \epsilon_{\mu} - \epsilon^2 \epsilon_{\nu} \epsilon_{\mu} \right) - \tilde{e}_{\mu} \left( \epsilon_{\nu} \tilde{D}_\rho \epsilon_{\mu} - \epsilon^2 \epsilon_{\nu} \epsilon_{\mu} \right) \right), \]
where \( I_{ZDG}(\sigma = -1, \tilde{\epsilon}_{\mu} j = \tilde{\omega}_{\mu} = 0) \) is the ZDG action [22]:
\[ I_{ZDG} = -\frac{1}{2} M_p \int d^4x \epsilon^{\mu\nu\rho\sigma} \left\{ \sigma \epsilon_{\mu\nu} \tilde{D}_{\rho\sigma} + \epsilon_{\mu} \tilde{D}_{\nu\rho} \right\} \]
\[ + \frac{1}{4\alpha_1 m^2} \epsilon_{abc} \epsilon_{\mu a} \epsilon_{\nu b} \epsilon_{\rho c} + \frac{1}{3\alpha_2 m^2} \epsilon_{abc} \epsilon_{\mu a} \epsilon_{\nu b} \epsilon_{\rho c} \]
\[ - \beta_3 m^2 \epsilon_{abc} \epsilon_{\mu a} \epsilon_{\nu b} \epsilon_{\rho c} - \frac{1}{2} \left( \epsilon_{\mu} \tilde{D}_{\nu} \epsilon_{\rho} - \epsilon^2 \epsilon_{\mu} \epsilon_{\rho} \right), \quad (19) \]
with the sign parameter \( \sigma = -1 \) and the fields \( \tilde{e}_{\mu} j = \tilde{\omega}_{\mu} = 0, \) \( \tilde{\sigma}_{\mu} = \tilde{\omega}_{\mu} = 0, \) \( (j = 0, 2) \), where \( M_p \) is the Planck mass, \( \alpha_1 \) and \( \alpha_2 \) are cosmological constants, \( \beta_1 \) and \( \beta_3 \) are coupling constants, and \( \epsilon_{\mu a}^I \) and \( \alpha_{\mu a}^I \) \( (I = 1, 2) \) are pairs of the dreibein and spin connection one-forms, respectively. Note that the zero values of the fields \( \tilde{e}_{\mu} j = \tilde{\omega}_{\mu} = 0 \), in ZDG action is imposed by the dual Lie algebra [3]. Variations of the action [11] with respect to the fields \( e_{\mu a}, \tilde{e}_{\mu a}, \omega_{\mu}, \) and \( \omega_{\mu} \) give the corresponding equations of motion, respectively:
\[ T_{\mu\nu} = \tilde{T}_{\mu\nu} = R_{\mu\nu} = \tilde{R}_{\mu\nu} = 0, \quad (20) \]
where \( T_{\mu\nu}, \tilde{T}_{\mu\nu}, R_{\mu\nu}, \) and \( \tilde{R}_{\mu\nu} \) are defined in [9]-[10].

A. Black hole solution

Now, we use the BTZ black hole metric \( g_{\mu\nu} \) [43] to obtain the following solution for the equations of motion [20]:
\[ ds^2 = -N^2(r) dt^2 + \frac{dr^2}{N^2(r)} + r^2 (N^2(r) dt^2 + d\phi^2), \quad (21) \]
\[ df^2 = -\frac{4\Lambda N^4}{a^2 N^2} dt^2 + \frac{4\Lambda N^2}{a^2 N^4} dr^2 + r^2 (N^2 dt^2 + d\phi^2), \quad (22) \]
where \( ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \) \( df^2 = f_{\mu\nu} dx^\mu dx^\nu, \) and \( N^2(r) = -M + \frac{r^2}{\ell_f^2}, \quad N^2(r) = -\frac{J^2}{2r^2}, \)
\[ N_f^2(r) = -M_f + \frac{r^2}{\ell_f^2}, \quad N_f^2(r) = -\frac{J + 2 D}{ar^2}, \]
\( \{x^0, x^1, x^2\} = \{t, r, \phi\} \) are the coordinates of the spacetime, \( M, J, D, M_f \) and \( \ell_f \) are arbitrary constants, and the spin connections \( \omega^\mu_a(r) \) and \( \tilde{\omega}^\mu_a(r) \) are obtained as follows:
\[ \omega^0 = 2DN dt + (1 - a) N d\varphi, \]
\[ \omega^1 = \left( \frac{\partial J^2}{r} \left( -\frac{\ell_f^2}{\ell_f^2} \right) + (e^2 + 2 \ell_f^2) r \right) dt \]
\[ + \frac{J}{2r} \left( 2(1 - a) \left( -\frac{\ell_f^2}{\ell_f^2} \right) \right) d\varphi, \]
\[ \omega^2 = \frac{2D}{a N} (2 N^2 + N^2 (1 - 2 \ell_f^2)) dt \]
\[ + N \left( \frac{2(1 - a)}{a} \left(-\frac{\ell_f^2}{\ell_f^2} \right) \right) d\varphi, \]
\[ \tilde{\omega}^0 = \frac{2D}{a N} (2 N^2 + N^2 (1 - 2 \ell_f^2)) dt \]
\[ + \frac{\ell_f^2}{r} \left( -\frac{2r}{a} \right) dt + \frac{J}{2r} \left( \frac{2(1 - a)}{a} \left( -\frac{\ell_f^2}{\ell_f^2} \right) \right) d\varphi, \]
\[ \tilde{\omega}^2 = \frac{2D}{a N^2} (N^2 - N_f^2) dt \]
\[ \text{The Kretschmann scalar } K = R_{\mu\nu\rho\sigma} R^\mu\nu\rho\sigma \text{ for the metric } f_{\mu\nu}, \text{ and then } f_{\mu\nu} \text{ has two curvature singularities at} \]
\[ r_s = \sqrt{\frac{\ell_f^2}{2} (\ell_f M_f + \sqrt{\ell_f^2 M_f^2 - J^2})}, \quad |\ell_f M_f > |J|, \quad (24) \]
\[ r_s = -\sqrt{\frac{\ell_f^2}{2} (\ell_f M_f - \sqrt{\ell_f^2 M_f^2 - J^2})}, \quad |\ell_f M_f > |J|, \quad (25) \]
where \( N_f(r) \) vanishes. \( f_{\mu\nu} \) has also two horizons at
\[ r_s = \sqrt{\frac{\ell_f^2}{2} (\ell_f M_f + \sqrt{\ell_f^2 M_f^2 - J^2})}, \quad |\ell_f M_f > |J|, \]
where \( N_f(r) \) vanishes. We use suitable values of the arbitrary constants \( M, M_f, \ell \) and \( \ell_f \) to have \( r_s > r_s^+, \) such that \( r_s \) is the event horizon of the black hole. Then, depending on the values of these constants, we have three different situations: \( r_s > r_s^+, \quad r_s > r_s^-, \text{ and } r_s < r_s^- \). To investigate the asymptotic behavior of this solution, we keep only the dominant terms. For very large values of \( r \), \( ds^2 \) has the following form:
\[ ds^2 \sim -\frac{r^2}{\ell_f^2} dt^2 + \frac{\ell_f^2}{r^2} d\phi^2 + r^2 d\varphi^2, \]
which is the AdS spacetime. But the metric \( df^2 \), for large values of \( r \), approaches to the following one:
\[ df^2 \sim -\frac{4\ell_f^2}{a^2 r_f^2} dt^2 + \frac{4\ell_f^2}{a^2 r_f^2} dr^2 + r^2 d\varphi^2, \]
which is clearly different from the AdS spacetime. This new black hole is different from the black hole solutions of the three dimensional \( f-g \) theory, which are asymptotically AdS and have coordinate singularities [44, 45].
IV. CONCLUSIONS

We have obtained a Lie bialgebra for the AdS Lie algebra in (2+1)-dimensional spacetime. Applying a Manin triple corresponding to the AdS Lie bialgebra as a gauge symmetry algebra of the Chern-Simons theory, we have introduced a new (2+1)-dimensional bi-metric gravity model. Our ghost-free Chern-Simons bi-gravity action is an exactly soluble model without any local degrees of freedom which describes two interacting massless spin-2 fields. Its free field limit has also no degrees of freedom, and is a ghost-free action. Our model is different from known three dimensional bi-metric massive gravity theories such as dRGT and ZDG models. The black hole solution (21)-(23) of our model is different from the previously obtained three dimensional bi-gravity black hole solutions [44, 45]. In our solution, one of the metrics is the BTZ black hole metric, and the other metric is a new black hole metric with two curve singularities and two horizons unlike two coordinate singularities of the previously obtained solutions of the three dimensional \( f-g \) theory. Our solution is also different from the other solutions in its asymptotic behaviour, and unlike other solutions, it has not asymptotically AdS form.

It is also interesting to study details of the new black hole metric (22) as well as gravity/CFT correspondence at the boundary of the bi-gravity model [11], which we leave them to later. Chern-Simons formulation of our interacting model simplifies its quantization, which may be interesting in the context of quantum gravity. Study of (3+1)-dimensional version of the AdS Lie bialgebra, and resultant (3+1)-dimensional gauge invariant interacting model is also a useful task which may have interesting features.

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$[47]$ $f_{AB}^C$ is the structure constant of the AdS Manin triple.