Boundary conditions for Einstein’s field equations: Analytical and numerical analysis

Olivier Sarbach\textsuperscript{1,2,3}, and Manuel Tiglio\textsuperscript{2,4,5}.
1 Department of Mathematics, Louisiana State University, Lockett Hall, Baton Rouge, Louisiana 70803-4918
2 Department of Physics and Astronomy, Louisiana State University, 202 Nicholson Hall, Baton Rouge, LA 70803-4001
3 Theoretical Astrophysics 130-33 California Institute of Technology, Pasadena, CA 91125
4 Center for Computation and Technology, 302 Johnston Hall, Louisiana State University, Baton Rouge, LA 70803-4001
5 Center for Radiophysics and Space Research, Cornell University, Ithaca, NY 14853

Outer boundary conditions for strongly and symmetric hyperbolic formulations of 3D Einstein’s field equations with a live gauge condition are discussed. The boundary conditions have the property that they ensure constraint propagation and control in a sense made precise in this article the physical degrees of freedom at the boundary. We use Fourier-Laplace transformation techniques to find necessary conditions for the well posedness of the resulting initial-boundary value problem and integrate the resulting three-dimensional nonlinear equations using a finite-differencing code. We obtain a set of constraint-preserving boundary conditions which pass a robust numerical stability test. We explicitly compare these new boundary conditions to standard, maximally dissipative ones through Brill wave evolutions. Our numerical results explicitly show that in the latter case the constraint variables, describing the violation of the constraints, do not converge to zero when resolution is increased while for the new boundary conditions, the constraint variables do decrease as resolution is increased. As an application, we inject pulses of “gravitational radiation” through the boundaries of an initially flat spacetime domain, with enough amplitude to generate strong fields and induce large curvature scalars, showing that our boundary conditions are robust enough to handle nonlinear dynamics.

We expect our boundary conditions to be useful for improving the accuracy and stability of current binary black hole and binary neutron star simulations, for a successful implementation of characteristic or perturbative matching techniques, and other applications. We also discuss limitations of our approach and possible future directions.

I. INTRODUCTION

In many numerical simulations in General Relativity one integrates Einstein’s field equations on a spatially compact domain with artificial timelike boundaries, effectively truncating the computational domain. This raises the question of how to specify boundary conditions. In this article we address this question within the context of the Cauchy formulation, in which the field equations split into evolution and constraint equations. We adopt a free evolution approach, in which the constraints are solved on the initial time slice only and the future of the initial slice is computed by integrating the evolution equations. Boundary conditions within this approach should ideally satisfy the following three requirements: (i) be compatible with the constraints in the sense of guaranteeing that initial data which solves the constraint equations yields constraint-satisfying solutions, (ii) permit to control, in some sense, the gravitational degrees of freedom at the boundary, and (iii) be stable in the sense of yielding a well posed initial-boundary value problem (IBVP).

There are several motivations for the construction of boundary conditions satisfying the above properties. First, recent detailed analysis of binary neutron star evolutions\textsuperscript{1} showed that the presence of artificial boundaries in current state of the art evolutions, which use rather ad hoc boundary conditions, dramatically affects the dynamics in the strong field region, near the stars. While it can be argued that this effect should disappear when placing the boundaries further and further away from the strong field region, until ideally, the region of interest in the computational domain is causally disconnected from the boundaries, in practice, this would require huge computer resources, especially in the three-dimensional case. Even though some kind of mesh refinement should help in placing the boundaries far away, there is in any case a minimum resolution needed in the far region in order to reasonably represent wave propagation, thus constraining the size of the computational domain for a given amount of memory. Next, an understanding of boundary conditions is important if the evolution equations include elliptic equations, since in this case effects from the boundaries can propagate with infinite speed, and so have an immediate effect on the fields being evolved. Examples of cases in which elliptic equations arise include elliptic gauge conditions (such as maximal slicing\textsuperscript{2} or minimal distortion conditions\textsuperscript{3}) and constraint projection\textsuperscript{4}\textsuperscript{,}\textsuperscript{5}\textsuperscript{,}\textsuperscript{6} methods. Finally, isolating the physical degrees of freedom at the boundaries should be important in view of Cauchy-characteristic (see\textsuperscript{7} for a review) or Cauchy-perturbative\textsuperscript{8}\textsuperscript{,}\textsuperscript{9}\textsuperscript{,}\textsuperscript{10} matching techniques, where a Cauchy code is coupled to a characteristic
Cauchy code and the outer module might be based on completely different formulations. It is important to communicate only the physical degrees of freedom at the boundary between the two codes, since the Cauchy code and the outer module might be based on completely different formulations.

Boundary conditions satisfying requirement (i) can be constructed by analyzing the constraint propagation system, which constitutes an evolution system for the constraint variables and is a consequence of Bianchi’s identities and the evolution equations. If it can be cast into first order symmetric hyperbolic form, the imposition of homogeneous maximally dissipative boundary conditions \([11, 12]\) for the constraint propagation system guarantees that a smooth enough solution of the evolution system (if it exists) which satisfies the constraints initially automatically satisfies the constraints at later times. Homogeneous maximally dissipative boundary conditions consist in a linear relation between the in- and outgoing characteristic fields of the system, which is chosen such that an energy estimate can be derived. This energy estimate implies that the unique solution of the constraint propagation system with zero initial data is zero, i.e. that the constraints are preserved during evolution. Maximally dissipative boundary conditions for the constraint variables usually translate into differential conditions for the fields satisfying the main evolution equations, since the constraint variables depend on derivatives of the main variables. This means that the resulting boundary conditions for the main system usually are not of maximally dissipative type and, as discussed below, analyzing well posedness of the corresponding IBVP becomes more difficult.

Requirement (ii), controlling the physical degrees of freedom, is a difficult one since there are no known local expressions for the energy or the energy flux density in General Relativity. Nevertheless, one should be able to control the physical degrees of freedom in some approximate sense, as for example in the weak field regime approximation, in which one linearizes the equations around flat spacetime. In this approximation, it might be a good idea to specify conditions through the Weyl tensor, since it is invariant with respect to infinitesimal coordinate transformations of Minkowski spacetime. More precisely, since there are two gravitational degrees of freedom, we should specify two linearly independent combinations of the components of the Weyl tensor. Below we will discuss boundary conditions which involve the Newman-Penrose complex scalars \(\Psi_0\) and \(\Psi_4\) (see, for instance, \([13]\)) with respect to a null tetrada adapted to the time-evolution vector field and the normal to the boundary. If the boundary is at null infinity these scalars represent the in- and outgoing gravitational radiation, respectively. Furthermore, it turns out that these scalars are invariant with respect to infinitesimal coordinate transformations and tetrada rotations for linear fluctuations of any Petrov type D solution represented in an adapted background tetrada \([14]\). This class of solutions not only comprises flat spacetime but also the family of Kerr solutions describing stationary rotating black holes. Since in many physical situations one is interested in modeling asymptotically flat spacetimes, such that if the outer boundary is placed sufficiently far away from the strong field region spacetime can be described by a perturbed Kerr black hole, we expect the boundary condition \(\Psi_0 = 0\) to be a good approximation for a “non-reflecting” wave condition. These boundary conditions are actually part of the family of conditions imposed in the formulation of Ref. \([15]\), to date the only known well posed initial-boundary value formulation of the vacuum Einstein equations, and were also considered in \([16]\).

Requirement (iii), the well posedness of the resulting IBVP, turns out to be a difficult problem as well: For quasilinear symmetric hyperbolic systems with maximally dissipative boundary conditions there are well-known well posedness theorems \([17, 18, 19]\) which state that a (local in time) solution exists in some appropriate Hilbert space, that the solution is unique, and that it depends continuously on the initial and boundary data. The proof of Ref. \([17]\) is based on these techniques. There, using a formulation based on tetrada fields, the authors manage to obtain a symmetric hyperbolic system by adding suitable combinations of the constraints to the evolution equations in such a way that the constraints propagate tangentially to the boundary. In this way, the issue of preserving the constraints becomes, in some sense, trivial. (See \([20]\) for a treatment in spherical symmetry in which the constraints propagate tangentially to the boundary as well.) However, for the more commonly used metric formulations it seems difficult to achieve tangential propagation for the constraints with a symmetric or strongly hyperbolic system, and therefore, one has to deal with either constraint propagation across the boundary or systems that are not strongly or symmetric hyperbolic. Here we choose to deal with constraint propagation across the boundary and strongly or symmetric hyperbolic systems. There has been a lot of effort in understanding such systems, both at the analytical \([21, 22, 23, 24, 25, 26, 27, 28, 29]\) and numerical \([2, 13, 22, 23, 26, 31, 32, 33]\) level. Although partial proofs of well posedness have been obtained using symmetric hyperbolic systems with maximally dissipative boundary conditions \([22, 24, 25, 26]\), it seems that these kind of boundary conditions are not flexible enough since constraint-preserving boundary conditions usually yield differential conditions.

In this article we construct constraint-preserving boundary conditions (CPBC) for a family of first order strongly and symmetric hyperbolic evolution systems for Einstein’s equations. For definiteness, we focus on the formulation presented in Ref. \([34]\), which is a generalization of the Einstein-Christoffel type formulations \([35, 36, 37, 38]\) with a Bona-Masso type of gauge condition \([39, 40]\) for the lapse. However, our approach is quite general and should also be applicable to other hyperbolic formulations of Einstein’s equations. In Section II we briefly review the family of formulations considered here and recall under which conditions the main evolution equations are strongly or symmetric...
We first analyze evolutions of these four systems through a robust stability test \[45, 46\]. In this test, random initial and boundary data is specified at different resolutions, and the growth rate in the time evolved fields is observed. A growth rate that becomes \textit{larger} as resolution is increased is a strong indication of a numerical instability, while for numerical stability the growth rate should be \textit{bounded} by a constant that is independent of resolution. We find that systems that violate the determinant condition fail to pass the robust stability test, as expected. However, we also find that at least some systems with CPBC with Weyl control which satisfy the determinant condition are numerically unstable as well, although in a somewhat weaker sense (explained in the text) that reminds the numerical evolution of weakly hyperbolic systems \[17\]. In contrast to this, the systems with CPBC without Weyl control which satisfy the determinant condition that we have evolved pass the robust stability test for the length of our simulations (usually between 100 and 1,000 crossing times).

Next, we concentrate on evolutions of Brill waves \[48\] and confirm the expectations drawn from the robust stability test in what concerns numerical stability. Using these waves we further concentrate on a detailed comparison between the results using maximally dissipative boundary conditions for the main evolution system and stable CPBC. Our convergence tests strongly suggest that in the former case the constraint variables \textit{do not} converge to zero in the limit of infinite resolution, implying that one \textit{does not} obtain a solution to Einstein’s field equations, while in the latter case for the same resolutions the constraint variables do converge to zero.

Next we concentrate on the stable CPBC case and evolve pure gauge solutions, using high order accurate finite difference operators which satisfy the summation by parts property. The operators used are eighth order accurate in the interior points and fourth order accurate at and near the boundary points.

As a final numerical experiment, we also concentrate on the stable CPBC case and inject pulses of gravitational radiation through the boundaries of an initially flat spacetime. We inject pulses of large enough amplitude to create very large curvature in the interior (as measured by curvature invariants), showing that our CPBC are strong enough
to handle very non-linear dynamics. The order of magnitude achieved by the curvature invariant measured corresponds to being at roughly \( r = 0.7 \) from the singularity, in a Schwarzschild spacetime of mass one. In these simulations this curvature is produced solely by the injected pulses.

A summary of the results and conclusions are presented in Sect. IX. Technical details, like the derivation of the constraint propagation system and of the characteristic fields, and a special family of solutions to the linearized IBVP with Weyl control are found in [A] [B] and [C]

II. STRONGLY HYPERBOLIC FORMULATIONS WITH A LIVE GAUGE

In this section we review the family of hyperbolic formulations of Einstein’s field equations constructed in [34], which is an extension of the Einstein-Christoffel type [35, 36] of formulations which incorporates a generalization of the Bona-Masso [39, 40] slicing conditions. It consists of 34 evolution equations for the variables \( \{N, g_{ij}, K_{ij}, d_{kij}, A_i\} \), where \( N \) is the lapse function, \( g_{ij} \) the three-metric, \( K_{ij} \) the extrinsic curvature, and where the extra variables \( d_{kij} \) and \( A_i \) represent the first order spatial derivatives of the three-metric and of the logarithm of the lapse, respectively.

The evolution equations are obtained from the ADM evolution equations in vacuum by adding suitable constraints to the right-hand side of the equations (see [A] for more details). Following the notation of Ref. [34], we have

\[
\begin{align*}
\partial_0 N &= - F(N, K, x^\mu), \quad (1) \\
\partial_0 g_{ij} &= - 2K_{ij}, \quad (2) \\
\partial_0 K_{ij} &= R_{ij} - \partial_i(A_j) + \Gamma_{i}^{k} K_{kl} - A_i A_j - 2K_{ij} (1 + K_{ij}) \\
&+ \gamma g_{ij} C + \zeta g^{kl} C_{k(i)l}, \quad (3) \\
\partial_0 d_{kij} &= -2\partial_k K_{ij} - 2A_k K_{ij} + \eta g_{k(i} C_{j)} + \chi g_{ij} C_k + \frac{2}{N} g_{(i} \partial_j) \partial_k \beta^l, \quad (4) \\
\partial_0 A_i &= - \frac{\partial F}{\partial N} A_i - \frac{1}{N} \frac{\partial F}{\partial K} (g^{kl} \partial_i K_{kl} - d_{kij} K^{kli}) - \frac{1}{N} \frac{\partial F}{\partial x^i} + \xi C_i. \quad (5)
\end{align*}
\]

Here, \( \partial_0 = (\partial_t - \langle L_\beta \rangle)/N \), and \( F(N, K, x^\mu) \) is a function that is smooth in all its arguments and that satisfies \( \sigma = (\partial_k F)/(2N) > 0 \). For the simulations below we shall choose \( F = N \cdot K \) which corresponds to time-harmonic slicing, but for our analytical results we shall leave \( F \) unspecified for generality. We assume that the shift vector \( \beta^i \) is a fixed, a priori specified vector field. The parameters \( \beta^i, \gamma, \zeta, \eta, \chi, \xi \) control the dynamics off the constraint hypersurface, defined by the vanishing of the following expressions

\[
\begin{align*}
C &= \frac{1}{2} (g^{kl} R_{kl} - K^{kl} K_{kl} + K^2), \quad (6) \\
C_i &= g^{kl} (\partial_k K_{li} - \partial_l K_{ki}) + \frac{1}{2} (d^k - 2d^k) K_{ki} + \frac{1}{2} d^k K_{ki}, \quad (7) \\
C_{kij} &= d_{kij} - \partial_k g_{ij}, \quad (8) \\
C_{l(kij)} &= \partial_i (d_{kij}), \quad (9) \\
C_i^{(A)} &= A_i - \partial_i (N A_j), \quad (10) \\
C_{ij}^{(A)} &= \frac{1}{N} \partial_i (N A_{j}) \quad (11)
\end{align*}
\]

Here, \( C = 0 \) is the Hamiltonian constraint, \( C_i = 0 \) the momentum one, and \( C_{kij} = 0, C_{l(kij)} = 0, C_i^{(A)} = 0 \) and \( C_{ij}^{(A)} = 0 \) are artificial constraints that arise as a consequence of the introduction of the extra variables \( d_{kij}, A_i \) [33].

The Ricci tensor \( R_{ij} \) belonging to the three-metric is written as

\[
R_{ij} = \frac{1}{2} g^{kl} (\partial_k d_{lij} + \partial_k d_{(ij)l}) + \partial_i (d_{klij}) - \partial_i (d_{lkij})
\]

\[
+ \frac{1}{2} d^{kl} d_{jkl} + \frac{1}{2} (d_k - 2b_k) \Gamma_{ij}^{k} - \Gamma_{ij}^{k} \Gamma_{ik}^{l}, \quad (12)
\]

where \( b_j \equiv d_{kij} g^{ki}, d_k \equiv d_{kij} g^{ij} \) and

\[
\Gamma_{ij}^{k} = \frac{1}{2} g^{kl} (2d_{(ij)k} - d_{kij}).
\]
The evolution equations (12,13,15) have the form of a quasilinear first order system,
\[ \partial_t u = A(u)^j \partial_j u + F(u), \]
where \( u = (N, g_{ij}, K_{ij}, d_{kij}, A_i) \) and the matrix-valued functions \( A^j, \ j = 1, 2, 3, \) and the vector-valued function \( F \) are smooth. We are looking for solutions with given initial data on some three-dimensional manifold which satisfies the constraints equations. In order to guarantee the existence of solutions we restrict the freedom in choosing the parameters \( \sigma, \gamma, \zeta, \eta, \chi, \xi \) by demanding that the corresponding initial-value formulation is well posed. This means that given smooth initial data, a (local in time) solution should exist in some appropriate Hilbert space, be unique, and depend continuously on the initial data. Although in this article we are interested in the numerical evolution of spacetimes on a spatially compact region with boundaries, well posedness of the problem in the absence of boundaries is a necessary condition for obtaining a numerical stable and consistent evolution inside the domain of dependence of the initial slice. An easy and intuitive way of finding necessary conditions for well posedness is to look at high frequency perturbations of smooth solutions: Let \( p = (t, x^i) \) be a fixed point and \( u_0 \) a smooth solution in a neighborhood of \( p \). Perturb \( u_0 \) according to \( u(t, x^i) = u_0(t, x^i) + \varepsilon \tilde{u}(t) \exp(i \omega n_i x^i), \) with \( \varepsilon, \omega \) real, and \( n_i \) a constant one-form on \( \mathbb{R}^3 \) which is normalized such that \( g(p)^{ij} n_i n_j = 1 \). If we evaluate the evolution equations at a point near \( p \), divide by \( \omega \varepsilon \), and take first the limit \( \omega \to \infty \) and then the limit \( \varepsilon \to 0 \), the evolution equations reduce to

\[ \partial_t \tilde{u} = [N(p)A(p, n) + \beta(p)^j n_j] \tilde{u}, \]

where the matrix \( A(p, n) \) is given by

\[
A(p, n) = \begin{pmatrix}
0 & 0 & 0 \\
-\partial_n \delta_{ij} + (1 + \zeta) \delta_{ij} - \zeta n_i n_j - 2n_i A_j + \gamma g_{ij} n_j
\end{pmatrix},
\]

where we have set \( \Delta_j = b_j - d_j \) and where the index \( n \) refers to the contraction with \( n^j \). Notice that by rescaling and rotating the coordinates \( x' \) one can always achieve that \( g(p)^{ij} = \delta_{ij} \), and by rescaling the coordinate \( t \) one can achieve that \( N(p) = 1 \). For this reason, we drop the entry \( p \) in the following. We call the system (13) the associated frozen coefficient problem. A necessary condition for the well posedness of the initial-value formulation defined by the (non-linear) Eqs. (12,13,15) is the well posedness of the associated frozen coefficient problem. If some extra smoothness properties are satisfied (see [3]) this condition is also a sufficient one. The frozen coefficient problem is well posed if the matrix \( N A(n) + \beta n \) is diagonalizable and has only real eigenvalues for each \( n \). Clearly, this is true if and only if the matrix \( A(n) \) is diagonalizable and has only real eigenvalues. As we have shown in [3], this can be easily analyzed by taking advantage of the block structure of \( A(n) \): Suppose \( u \) is an eigenvector of \( A(n) \) with eigenvalue \( \mu \). Then

\[
\mu \begin{pmatrix}
u^{(0)} \\
u^{(1)} \\
u^{(2)}
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 \\
0 & A & 0 \\
0 & B & 0
\end{pmatrix} \begin{pmatrix}
u^{(0)} \\
u^{(1)} \\
u^{(2)}
\end{pmatrix},
\]

where \( u^{(0)} = (N, g_{ij}), u^{(1)} = (K_{ij}), u^{(2)} = (d_{kij}, A_i) \), and where the matrices \( A \) and \( B \) are read off from Eq. (15). A sufficient condition for \( A(n) \) to be diagonalizable and possesses only real eigenvalues can be obtained by considering the equation

\[ \mu^2 K_{ij} = ABK_{ij}. \]

Explicitly, we have

\[ \mu^2 K_{ij} = K_{ij} + A n_i K_j n_j + B n_i n_j K + C g_{ij} (K_{nn} - K), \]

where the coefficients \( A, B \) and \( C \) are

\[
A = -2 - \frac{\chi}{2} - \frac{1}{4} (3\zeta - 1)\eta - \xi, \tag{17}
\]

\[
B = \frac{\chi}{2} + \frac{1}{4} (3\zeta - 1)\eta + \xi + 1 + 2\sigma, \tag{18}
\]

\[
-2C = \chi - \frac{1}{2} (\zeta + 1)\eta + (2 - \eta + 2\chi). \tag{19}
\]
We now demand that $AB$ is diagonalizable and has only positive eigenvalues. As we have shown in [34] this guarantees that $A(n)$ is diagonalizable and has only real eigenvalues. Representing $AB$ in an orthonormal basis $e_1^i$, $e_2^i$, $e_3^i$ such that $e_1^i = n_i$, we find

$$
\mu^2 K_{nn} = (1 + A + B)K_{nn} + (B - C)K_A^A, \quad (20)
$$
$$
\mu^2 K_{nA} = (1 + A/2)K_{nA}, \quad (21)
$$
$$
\mu^2 K_A^A = (1 - 2C)K_A^A, \quad (22)
$$
$$
\mu^2 \hat{K}_{AB} = \hat{K}_{AB}, \quad (23)
$$

where $A = 2, 3$ and $\hat{K}_{AB} = K_{AB} - \delta_{AB}d^{CD}K_{CD}/2$. From this we immediately see that $AB$ is diagonalizable with only positive eigenvalues if and only if

$$
\lambda_1 = 2\sigma, \quad (24)
$$
$$
\lambda_2 = \frac{1}{2}(2\gamma + 1)(2 + 2\chi - \eta) - \frac{1}{2} \zeta \eta, \quad (25)
$$
$$
\lambda_3 = -\frac{1}{8}(2\chi - \eta + 4\xi) - \frac{3}{8} \zeta \eta, \quad (26)
$$

are positive and if $B - C = \lambda_1 + \frac{1}{2}(\lambda_2 + 1) - 2\lambda_3 = 0$ whenever $\lambda_1 = \lambda_2$. In [34] we derive the characteristic fields, which are given by the projections of $u$ onto the eigenspaces of $A(n)$. These fields play an important role in the construction of boundary conditions. Using these fields, we also derive in [34] sufficient conditions for the non-linear evolution system to be strongly hyperbolic and thus yield a well posed initial value formulation.

### III. CONSTRAINT PROPAGATION SYSTEM

In order to obtain a solution to Einstein’s equations, not only do we have to solve the evolution equations but also the constraints. We want to follow a free evolution scheme, in which the constraints are solved initially only. For this scheme to be valid, we have to guarantee that any solution for such initial data automatically satisfies the constraints in the computational domain everywhere and at every time. At the numerical level, since the constraints are already violated initially due to truncation or roundoff errors, we have to guarantee that the numerical solution to the evolution equations converge to a constraint-satisfying solution to the continuum equations in the limit of infinite resolution. In order to show this, the constraint propagation system, which gives the change of the constraint variables under the flux of the main evolution system, plays an important role. In this section we show that for a suitable range of the parameters $\gamma, \zeta, \eta, \chi, \xi$ this system can be cast into first order symmetric hyperbolic form. We then specify boundary conditions that guarantee that zero initial data for this system leads to zero constraint variables at later times.

The constraint propagation system is derived in [34] up to lower order terms which are linear algebraic expressions in the constraint variables and whose precise form are not needed for the purpose of this article. In order to analyze under which conditions the system is symmetric hyperbolic, it is convenient to decompose $C_{kij}$ into its trace and trace-less parts,

$$
C_{kij} = E_{kij} + \frac{1}{2} (g_{l(i}B_{j)k} - g_{k(i}B_{j)l}) + \frac{1}{3} g_{ij}W_{lk},
$$

where $E_{kij} = E_{[k][ij]}$ is trace-less with respect to all pair of indices and where $B_{ij}$ and $W_{ij}$ are determined by the traces $S_{ki} = C^{\alpha}_{(k)is}$, $A_{ki} = C^{\alpha}_{[ki]s}$ and $V_{lk} = C^{\alpha}_{iks}$,

$$
B_{ki} = \frac{4}{3} S_{ki} - \frac{12}{5} A_{ki} - \frac{4}{5} V_{ki}, \quad W_{lk} = \frac{9}{5} V_{lk} + \frac{12}{5} A_{lk}.
$$

(Notice that $S_{ij}$ is symmetric trace-less while $A_{ij}$ and $V_{ij}$ are antisymmetric.) In terms of the variables $U = (C_i^{(A)}, C_{kij}, E_{kij}, C_i, s_{ij}A_{ij}, V_{ij}, C_{ij}^{(A)})$, where $A_{ij} = A_{ij} + V_{ij}/2 + C_{ij}^{(A)}$, the non-linear constraint propagation system has the form

$$
\partial_t U = A_C(u)^2 \partial_j U + B(u)U, \quad (27)
$$

where $A_C(u)^2 = A^{(A)} + B^{(A)} + C^{(A)}$.
where the principal symbol $A_C(u, n) = A_C(u)^j n_j$ is given by

$$A_C(u, n)U = \begin{pmatrix}
0 \\
P \\
\frac{1}{2}(2 + 2\chi - \eta)C_n \\
-(2\gamma + 1)n_i C + \zeta S_{ni} - \hat{A}_{ni} \\
-\frac{3}{4} (n_i \hat{C}_j - \frac{1}{4} g_{ij} C_n) \\
\frac{1}{4} (2\chi - \eta) + \xi n_j \hat{C}_j \\
(\eta + 3\chi) n_j \hat{C}_j \\
\xi n_i \hat{C}_j
\end{pmatrix},$$

and where the matrix $B(u)$ depends on the main fields $u$ and their spatial derivatives, but not on $U$. The system Eq. (27) is called symmetric hyperbolic if there exists a symmetric positive definite matrix $H$ which may depend on $u$ but not on $n$ such that $HA_C(u, n)$ is symmetric for all $u$ and $n$. From the above representation of the principal symbol it is not difficult to see that the system is symmetric hyperbolic if the following conditions hold

$$(2\gamma + 1)(2 + 2\chi - \eta) > 0,$$

$$\zeta \eta < 0,$$

$$2\chi - \eta + 4\xi < 0.$$  \hspace{1cm} (28), (29), (30)

Notice that these conditions automatically imply that $\lambda_2 > 0$ and $\lambda_3 > 0$, which are necessary conditions for the main evolution system to be strongly hyperbolic. If the conditions (28), (29), (30) are satisfied, a symmetrizer is given by the quadratic form

$$UTHU = g^{ij}C_i^{(A)}C_j^{(A)} + C^{kij}C_{kij} + E^{klij}E_{klij} + \hat{V}^{ij}\hat{V}_{ij} + \hat{C}^{ij}\hat{C}_{ij}$$

$$+ \frac{2(2\gamma + 1)}{2 + 2\chi - \eta} C^2 + C^i C_i + \frac{4}{3} \frac{\zeta}{\eta} S^{ij} S_{ij} + \frac{4}{|2\chi - \eta + 4\xi|} \tilde{A}^{ij} \tilde{A}_{ij},$$

where

$$\tilde{A}_{ij} = A_{ij} + \frac{1}{2} V_{ij} + C_{ij}^{(A)},$$

$$\tilde{V}_{ij} = V_{ij} - \frac{4(\eta + 3\chi)}{(2\chi - \eta) + 4\xi} \hat{A}_{ij},$$

$$\tilde{C}_{ij} = C_{ij}^{(A)} - \frac{4\xi}{(2\chi - \eta) + 4\xi} \hat{A}_{ij}. $$

The symmetrizer allows us to obtain an energy estimate for solutions to Eq. (27) on a domain $O$ of $\mathbb{R}^3$ with smooth boundary $\partial O$ and suitable boundary conditions on $\partial O$. Defining the energy norm

$$E_{\text{constraints}} = \int_O U^T H U \, d^3x,$$

differentiating with respect to $t$ and using the constraint propagation system, Eq. (27), we obtain

$$\frac{d}{dt} E_{\text{constraints}} = 2 \int_O U^T H [N(A_{ij}^t \partial_t U + B U) + \beta^j \partial_t U] \, d^3x$$

$$= \int_{\partial O} U^T H N A_C(n) U \, d^2x$$

$$+ \int_O U^T [NHB + NB^T H - \partial_t (NHA_C^t + H\beta^t)] U \, d^3x,$$

where here $n_i$ denotes the unit outward one-form to the boundary. In the last step we have used the fact that $A_C(n)$ is symmetric with respect to the scalar product defined by (34) and assumed that the shift is tangential to the boundary at $\partial O$. As one can easily verify,

$$U^T H A_C(n) U = \frac{1}{2} g^{ij} \left( V_i^{(+)} V_j^{(+)} - V_i^{(-)} V_j^{(-)} \right),$$

$$\text{(34)}.$$
where $V^i_{\pm} = -C_i \pm (A_{ni} - \zeta S_{ni}) \pm (2\gamma + 1)n_i C$ are the in (+) and out (−) going characteristic constraint fields. Therefore, if we impose the boundary conditions

$$V^i_{\pm} = S^j_i V^j_{\pm},$$

(35)

where the matrix $(S^j_i)$ satisfies $g^{ij} S^k_i S^l_j v_{kl} \leq g^{ij} v_{ik} v_{lj}$ for all one-forms $v_{kl}$, we obtain the energy estimate

$$\frac{d}{dt} E_{\text{constraints}} \leq \text{const} \cdot E_{\text{constraints}},$$

(36)

where the constant only depends on bounds for $NB$ and $H^{-1} \partial_t (NH A^l_C + H^3)$. Since $E_{\text{constraints}} \geq 0$ this proves that $E_{\text{constraints}}(t) = 0$ for all $t > 0$ provided that $E_{\text{constraints}}(t = 0) = 0$. For this reason, we call the three conditions constraint-preserving boundary conditions.

**IV. BOUNDARY CONDITIONS FOR THE MAIN EVOLUTION SYSTEM**

In this section we consider the main evolution system, Eqs. (1,2,3,4,5), on a open domain $O$ of $\mathbb{R}^3$ with smooth boundary $\partial O$. We also assume that the shift vector is chosen such that it is tangential to $\partial O$ at the boundary. This means that at the boundary we have six ingoing characteristic fields, denoted by $v^i_{nn}^{(+)}$, $\hat{v}^i_{AB}^{(+)}$, $v^i_{nB}^{(+)}$ (see [12] for their definition; here and in the following, $A, B = 2, 3$, and quantities with a hat denote trace-free two by two matrices), and thus we have to provide six independent boundary conditions. Following the classification scheme of Ref. [24] one can show that the first field is a gauge field, the second ones are physical fields, and the last are constraint-violating fields. We stress that this classification scheme does only make precise sense in the linearized regime for plane waves propagating in the normal direction to the boundary (see [24] for a more detailed discussion about this).

If we forgot about the constraints, we could give data to the six ingoing fields. The simplest possibility would be to freeze the ingoing fields to their values given by the initial data. Provided the evolution system is symmetric hyperbolic this would yield a well posed IBVP. However, in the presence of constraints, the boundary conditions have to ensure that no constraint-violating modes enter the boundary, i.e. the boundary conditions have to be compatible with the constraints. In fact, the numerical results of section VII show explicitly that freezing of the ingoing fields to their initial values does not, in general, provide a solution to Einstein’s equations: The constraints are violated.

Instead of the freezing non-constraint preserving boundary conditions just mentioned, we impose the three constraint-preserving boundary conditions [35]. This fixes three of the six conditions we are allowed to specify at $\partial O$. We complete these three conditions in the following two ways:

1. **CPBC without Weyl control**
   Here we adopt the simplest possibility and impose algebraic boundary conditions on the “gauge” and “physical” fields,

   $$v^i_{nn}^{(+)} = a v^i_{nn}^{(-)} + h,$$
   (37)

   $$\hat{v}^i_{AB}^{(+)} = b \hat{v}^i_{AB}^{(-)} + \hat{h}_{AB},$$
   (38)

   where $a$ and $b$ are constants satisfying $|a| \leq 1$, $|b| \leq 1$, and $h$ and $\hat{h}_{AB}$ are functions on $\partial O$ describing the boundary data. The justification for the bounds on $a$ and $b$ will become clear in the next section. Notice that the choice $a = b = -1$ results in Dirichlet conditions in the sense that data is imposed on some components of the extrinsic curvature while the choice $a = b = 1$ yields boundary conditions that impose data on combinations of spatial derivatives of the three metric (see the definitions of $v^i_{nn}$ and $\hat{v}^i_{AB}$ in [12]). In our simulations below, we choose $a = b = 0$ which yields Sommerfeld-like conditions.

2. **CPBC with Weyl control**
   Here we replace Eq. (35) by a similar condition for the Weyl tensor. We impose the boundary conditions

   $$v^i_{nn} = a v^i_{nn}^{(-)} + s,$$
   (39)

   $$\hat{w}^i_{AB} = c \hat{w}^i_{AB}^{(-)} + \hat{H}_{AB},$$
   (40)

   where $|c| \leq 1$ and where $\hat{w}^i_{AB}$ are defined in terms of the electric ($E_{ij}$) and magnetic ($B_{ij}$) parts of the Weyl tensor in the following way: Let $e_1$, $e_2$, $e_3$ be an orthonormal triad with respect to the three metric $g_{ij}$ such that $e_1$ coincides with the unit outward normal $n^i$ to the boundary. Then, $\hat{w}^i_{AB}^{(±)} =$
\((e_A^i e_B^j - \delta_{AB} \delta^{CD} e_C^i e_D^j)/2\) \(\left( E_{ij} \pm n^k \varepsilon_{kij} B_j^l \right)\), where \(\varepsilon_{kij}\) is the volume element associated to \(g_{ij}\). If the vacuum equations hold, the electric and magnetic components can be determined by

\[
E_{ij} = \partial_h K_{ij} + K_{ia} K^a_j + \nabla_{(i} A_{j)} + A_i A_j, \\
B_{ij} = \varepsilon_{rsi}(\nabla^r K^s_j),
\]

where one uses the evolution equation \((3)\) in order to re-express \(\partial_h K_{ij}\) in terms of spatial derivatives of the main variables. The boundary conditions \((40)\) correspond to the conditions imposed by Friedrich and Nagy \((12)\). In the symmetric hyperbolic system considered in \((12)\), where the components of the Weyl tensor are evolved as independent fields, these boundary conditions arise naturally when analyzing the structure of the equations since they give rise to maximally dissipative boundary conditions. In particular, a well-posed initial-boundary value problem incorporating the condition \((40)\) is derived in \((15)\). In contrast to this, the boundary conditions \((40)\) are not maximally dissipative for our symmetric hyperbolic system since they are not even algebraic.

The conditions \((40)\) can also be expressed in terms of the Newman-Penrose scalars \(\Psi_0\) and \(\Psi_4\) with respect to a null tetrad which is adapted to the boundary in the following sense: Let \(e_0\) denote the future-directed unit normal to the \(t = \text{const}\) slices. Together with the above vectors \(e_1, e_2, e_3\) it forms a tetrad. From this, we construct the following Newman-Penrose null tetrad:

\[
l^\mu = \frac{1}{\sqrt{2}} (e_0^\mu + e_1^\mu), \quad k^\mu = \frac{1}{\sqrt{2}} (e_0^\mu - e_1^\mu), \quad m^\mu = \frac{1}{\sqrt{2}} (e_2^\mu + ie_3^\mu).
\]

Then, we find

\[
\Psi_0 = \hat{w}_i^{(-)} - i\hat{w}_i^{(+)}, \quad \Psi_4 = \hat{w}_i^{(-)} + i\hat{w}_i^{(+)},
\]

and the boundary condition \((40)\) is simply

\[
\Psi_0 = c\Psi_4,
\]

where the star denotes complex conjugation (we could generalize this boundary condition by allowing for complex values of \(c\)). Notice that \(\Psi_0\) and \(\Psi_4\) are not uniquely determined by the unit normals \(e_0\) and \(n\): with respect to a rotation of \(e_2, e_3\) about the angle \(\phi\), these quantities transform through \(\Psi_0 \mapsto e^{2i\phi} \Psi_0, \Psi_4 \mapsto e^{-2i\phi} \Psi_4\), the factor 2 reflecting the spin of the graviton. However, the boundary condition \((40)\) is indeed invariant with respect to such transformations. In our simulations below, we shall choose \(c = 0\) corresponding to an outgoing radiation condition.

Finally, as discussed in the introduction, \(\Psi_0\) and \(\Psi_4\) represent, respectively, in- and outgoing radiation when evaluated at null infinity and are gauge-invariant quantities for linearizations about a Kerr background.

V. FOURIER-LAPLACE ANALYSIS

We now analyze the well posedness of the IBVP defined by the evolution equations \((1)\) \((2)\) \((3)\) \((4)\) \((5)\), the CPBC \((35)\) and the boundary conditions \((37)\) \((38)\) or \((37)\) \((40)\). We derive necessary conditions for well posedness by verifying a determinant condition in the high frequency limit. We assume that the parameters are such that the evolution equations are strongly hyperbolic, since otherwise the problem is ill posed even in the absence of boundaries, and such that the constraint propagation system is symmetric hyperbolic.

Let \(p \in \partial O\) be a point on the boundary. By taking the high frequency limit we obtain the associated frozen coefficient problem at \(p\). After rescaling and rotating the coordinates if necessary, we can achieve that \(N(p) = 1, g(p)_{ij} = \delta_{ij}\) and that the domain of integration is \(x^+ > 0\). In this way, we obtain a constant coefficient problem on the half space. Introducing the operator \(\hat{\partial}_h = \partial_t - \beta^l \partial_i\), it is given by

\[
\hat{\partial}_h K_{ij} = \frac{1}{2} \left( -\partial^k d_{kij} + (1 + \zeta) \partial^k d_{(ij)k} + (1 - \zeta) \partial_{(i} b_{j)} - \partial_{(i} d_{j)} - 2 \partial_{(i} A_{j)} \right) + \frac{\gamma}{2} \delta_{ij} \partial^k (b_k - d_k),
\]

\[
\hat{\partial}_h d_{kij} = -2 \partial_k K_{ij} + \eta g_{ik} \left( \partial^l K_{jl} - \partial_j K^l \right) + \chi g_{ij} \left( \partial^l K_{kl} - \partial_k K \right),
\]

\[
\hat{\partial}_h A_i = -2 \sigma \partial_i K + \xi \left( \partial^l K_{lj} - \partial_l K \right).
\]
Notice that this system is equivalent to the one that one would obtain by linearizing the evolution equations around flat spacetime in a slicing with respect to which the three metric is flat, the lapse is one and the shift is constant and tangential to the boundary, but not necessarily zero.

Since we have a linear constant coefficient problem on the half plane, we can solve these equations by means of a Laplace transformation in time and a Fourier transformation in the $x^2$ and $x^3$ directions. That is, we write the solution as a superposition of solutions of the form $u(t,x^1,x^2,x^3) = \tilde{u}(\omega x^1) \exp(\omega(z t + i \omega_A(x^A + \beta^A t)))$, where $z \in \mathbb{C}$ with Re$(z) > 0$, $\delta^{AB}\omega_A\omega_B = 1$, $A = 2,3$ and $\tilde{u} \in L^2(\mathbb{R}_+)$. (Notice that for such solutions, $\hat{\partial}_0 u = \omega z u$.) Substituting this into Eqs. (44-46) one obtains a system of ordinary differential equations coupled to algebraic conditions. Since there are six in- and six outgoing modes, there are twelve independent differential equations. The remaining equations which are algebraic can be used in order to eliminate the characteristic fields which have zero speeds, and one ends up with a closed system of twelve linear ordinary differential equations. Because the system is strongly hyperbolic we expect exactly six linearly independent solutions that decay as $x^1 \to \infty$, and six solutions that blow up as $x^1 \to -\infty$. Since we require the solution to lie in $L^2$ we only consider the six decaying solutions. The determinant condition follows in verifying that the boundary conditions with homogeneous boundary data annihilate these six solutions. If the determinant condition is violated, the problem admits solutions of the form $u(t,x^1,x^2,x^3) = \tilde{u}(\omega x^1) \exp(\omega(z t + i \omega_A(x^A + \beta^A t)))$ for some Re$(z) > 0$, where $\omega$ can be arbitrarily large, and the system is ill posed since $|u(t,x^1,x^2,x^3)|/|u(0,x^1,x^2,x^3)| = \exp(\omega(Re(z)t)) \to \infty$ as $\omega \to \infty$ for each fixed $t$. Thus, the determinant condition is necessary for the well posedness of the IBVP and, as we will see, will yield nontrivial conditions.

A convenient way for finding the six decaying solutions is to look at the second order equation for $K_{ij}$, which is a consequence of Eqs. (44-46),

$$\partial_0^2 K_{ij} = \partial^k \partial_k K_{ij} + A \partial_i \partial_0 K_{jkl} + B \partial_i \partial_j + C \delta_{ij} \left( \partial^l \partial_l K_{kl} - \partial^i \partial_i K \right),$$

(47)

where the coefficients $A$, $B$ and $C$ are given in Eqs. (44-46). We show below that there are exactly six solutions to this second order system which have the form $K_{ij}(t,x^1,x^2,x^3) = \tilde{K}_{ij}(\omega x^1) \exp(\omega(z t + i \omega_A(x^A + \beta^A t)))$, with Re$(z) > 0$ and such that $\tilde{K}_{ij}$ decays as $x^1 \to \infty$. Since in Fourier-Laplace space the operator $\hat{\partial}_0$ is just multiplication with the nonzero factor $\omega z$, corresponding solutions to the original system can be obtained by determining $d_{klj}$ and $A_j$ from Eqs. (45) and (46) respectively.

In order to find the decaying solutions of Eq. (47) we make the ansatz $\tilde{K}_{ij} = k_{ij} \exp(\omega \nu x)$, where here and in the following we set $x = x^1$ for simplicity, and where $\nu$ is a complex number with negative real part to be determined. It is also convenient to introduce a unit two-vector $\tilde{\eta}^A$ which is orthogonal to $\tilde{\omega}_A$, and to decompose $k_{ij}$ in the components $k_{xx}, k_{x\eta} = k_{x\eta} \tilde{\eta}^B, k_{\eta\eta} = k_{\eta\eta} \tilde{\eta}^B, k_{A\eta} = \delta^{AB} k_{AB}, k_{\omega\omega} = k_{\omega\omega} \tilde{\omega}^B$ and $k_{\omega\eta} = (\tilde{\omega}^A \tilde{\omega}^B - \tilde{\eta}^A \tilde{\eta}^B) k_{AB}/2$. Using this ansatz, Eq. (47) splits into the following two decoupled systems,

$$z^2 \begin{pmatrix} k_{xx} \\ k_{x\eta} \\ k_{\eta\eta} \end{pmatrix} = M_1(\nu) \begin{pmatrix} k_{xx} \\ k_{x\eta} \\ k_{\eta\eta} \end{pmatrix}, \quad z^2 \begin{pmatrix} k_{xx} \\ k_{A\eta} \\ k_{\omega\omega} \end{pmatrix} = M_2(\nu) \begin{pmatrix} k_{xx} \\ k_{A\eta} \\ k_{\omega\omega} \end{pmatrix},$$

(48)

where the matrices $M_1(\nu)$ and $M_2(\nu)$ are given by

$$M_1(\nu) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

$$M_2(\nu) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

and $a_{ij}$ are defined in Eqs. (47-49).

The matrix $M_1(\nu)$ has the eigenvalue-eigenvector pair

$$z^2 = \nu^2 - 1, \quad \begin{pmatrix} k_{xx} \\ k_{x\eta} \\ k_{\eta\eta} \end{pmatrix} = \begin{pmatrix} 1 \\ i\nu \\ 0 \end{pmatrix},$$

$$z^2 = \lambda_3 (\nu^2 - 1), \quad \begin{pmatrix} k_{xx} \\ k_{x\eta} \\ k_{\eta\eta} \end{pmatrix} = \begin{pmatrix} -i\nu \\ 0 \\ 1 \end{pmatrix}.$$
The two vectors are always linearly independent from each other since \( \text{Re}(z) > 0 \). This yields the solution
\[
\begin{pmatrix}
\tilde{K}_{x\eta} \\
\tilde{K}_{\omega\eta}
\end{pmatrix} = \sigma_1 \begin{pmatrix} 1 \\ -i\nu_3 \end{pmatrix} e^{-\omega_3 x} + \sigma_2 \begin{pmatrix} i\nu_3 \\ 1 \end{pmatrix} e^{-\omega_2 x},
\]
(49)
where \( \sigma_1, \sigma_2 \) are two constants and where here and in the following \( \nu_l = \sqrt{\lambda_l^{-1} z^2 + 1}, l = 1, 2, 3, 4 \) (\( \lambda_4 = 1 \)) where the branch of the square root for which \( \text{Re}(\nu) > 0 \) for \( \text{Re}(z) > 0 \) is chosen. Similarly, after obtaining the eigenvalues and eigenvectors of \( M_2(\nu) \) one obtains the solution
\[
\begin{pmatrix}
\tilde{K}_{xx} \\
\tilde{K}_A \\
\tilde{K}_{x\omega} \\
\tilde{K}_{\omega\omega}
\end{pmatrix} = \sigma_3 v_3 e^{-\omega_4 x} + \sigma_4 v_4 e^{-\omega_5 x} + \sigma_5 v_5 e^{-\omega_2 x} + \sigma_6 v_6 e^{-\omega_1 x},
\]
(50)
where
\[
v_3 = \begin{pmatrix} 1 \\ -1 \\ -i\nu_4 \\ \frac{1}{2} - \nu_4^2 \end{pmatrix}, \quad v_4 = \begin{pmatrix} -i\nu_3 \\ i\nu_3 \\ -\frac{1}{2}(\nu_3^2 + 1) \end{pmatrix}, \quad v_6 = \begin{pmatrix} \nu_1^2 \\ -1 \\ -1 \\ -\frac{1}{2} \end{pmatrix},
\]
\[
v_5 = \begin{pmatrix} 2(B-C)\nu_2^2 + A + B + 2C \\ -2(A+B+2C)\nu_2^2 + A - B + 4C \\ -i\nu_2(A+3B) \\ -\frac{1}{2}(A+3B) \end{pmatrix}
\]
if \( \lambda_1 \neq \lambda_2 \), \( v_5 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \) if \( \lambda_1 = \lambda_2 \).

Therefore, we have obtained six linearly independent solutions which decay exponentially as \( x \to \infty \) and thus lie in \( L^2(O) \). They are parameterized by the constants (which depend on \( \omega \) and \( z \)) \( \sigma_1, \ldots, \sigma_6 \). A necessary condition for the IBVP to be well posed is that these constants are uniquely determined by the boundary data. Before checking this condition, it is instructive to have a closer look at the six-parameter family of solutions given by Eqs. (49) and (50). Let us first compute the momentum constraint variable \( C_i = \partial^j K_{ij} - \partial_i K \): It has the form
\[
C_i(t, x^1, x^2, x^3) = \mathcal{C}_i(\omega x^1) \exp(\omega(t + i\bar{\omega}_A(x^4 + \beta A t)))
\]
where
\[
\mathcal{C}_\eta = i\omega(1-\nu_2^2)\sigma_2 e^{-\omega_3 x},
\]
(51)
\[
\mathcal{C}_x = -\frac{i}{2}\omega(1-\nu_2^2)\sigma_4 e^{-\omega_3 x} + 2\omega\nu_2\bar{\sigma}_5 e^{-\omega_2 x},
\]
(52)
\[
\mathcal{C}_\omega = -\frac{1}{2}\nu_3(1-\nu_3^2)\sigma_4 e^{-\omega_3 x} - 2i\bar{\sigma}_5 e^{-\omega_2 x},
\]
(53)
where \( \bar{\sigma}_5 = (1-\nu_2^2)(\lambda_1 - \lambda_2)\sigma_5 \) if \( \lambda_1 \neq \lambda_2 \) and \( \bar{\sigma}_5 = \sigma_5 \) otherwise. Thus, \( C_i = 0 \) for this family of solutions if and only if \( \sigma_2 = \sigma_4 = \sigma_5 = 0 \). In other words, the three-parameter subfamily of solutions parameterized by \( \sigma_2, \sigma_4 \) and \( \sigma_5 \) are constraint-violating modes. Next, consider an infinitesimal coordinate transformation parametrized by a vector field \( (X^\mu) = (f, X^4) \), and assume zero shift for simplicity. With respect to such a transformation, the linearized lapse and extrinsic curvature change according to
\[
N \mapsto N + \partial_i f,
\]
(54)
\[
K_{ij} \mapsto K_{ij} - \partial_i \partial_j f.
\]
(55)
On the other hand, the linearization of Eq. (11) around a Minkowski background \( \mathcal{M} \) yields
\[
\partial_i N = -2\sigma K.
\]
(56)
We see that the choice \( f(t, x^1, x^2, x^3) = -2\exp(\omega(zt - \nu_1 x + i\bar{\omega}_A x^4)) \) leaves Eq. (50) invariant and induces the transformation
\[
\begin{pmatrix}
\tilde{K}_{xx} \\
\tilde{K}_A \\
\tilde{K}_{x\omega} \\
\tilde{K}_{\omega\omega}
\end{pmatrix} \mapsto \begin{pmatrix}
\tilde{K}_{xx} \\
\tilde{K}_A \\
\tilde{K}_{x\omega} \\
\tilde{K}_{\omega\omega}
\end{pmatrix} - \begin{pmatrix} \nu_1^2 \\ -1 \\ -i\nu_1 \\ -\frac{1}{2} \end{pmatrix} e^{-\omega_1 x},
\]
(57)
while $\tilde{K}_{\eta}$ and $\tilde{K}_{\omega}$ remain invariant. Therefore, it is possible to gauge away the solution parametrized by $\sigma_6$ and we call this solution a gauge mode from hereon. The remaining family of solutions parametrized by $\sigma_1$ and $\sigma_3$ are physical modes: They satisfy the constraints and $\sigma_1$ and $\sigma_3$ are gauge-invariant.

Next, we verify that the integration constants $\sigma_1,...,\sigma_6$ are uniquely determined by the boundary conditions. First, we notice that the expressions (51, 52, 53) for the Fourier-Laplace transformation of the momentum constraint yield a three-parameter family of solutions for the constraint propagation system, Eq. (2). Since this system is symmetric hyperbolic and since we specify homogeneous maximally dissipative boundary conditions for it (see Eq. (35)), the corresponding IBVP is well posed. In particular, zero is the only solution with trivial initial data. This implies that $\sigma_2 = \sigma_4 = \sigma_5 = 0$. We stress that such a conclusion cannot be drawn if the constraint propagation is strongly but not symmetric hyperbolic, see Ref. [2] for a counterexample.

Next, using Eqs. (45) and (46), we find the following expressions for the relevant characteristic fields at the boundary

$$v_{xx} = K_{xx} + \Omega K_A^T \pm \frac{1}{\sqrt{\lambda_1}} \rho \zeta (1 - \Omega) \partial^A K_{ax} + \lambda_1 (1 - \Omega) C_{ax},$$

$$\hat{v}_{AB} = \hat{K}_{AB} \pm \frac{1}{\omega z} \left[ \partial_x \hat{K}_{AB} - (1 + \rho) \partial (A \hat{K}_B) x \right]^{TF},$$

$$\hat{w}_{AB} = \omega \hat{K}_{AB} \pm \left[ \partial_x \hat{K}_{AB} - \partial (A \hat{K}_B) x \right]^{TF} + \frac{1}{\omega z} \left[ -2 \rho \partial_A \hat{B}_K + \rho \partial (A \hat{B}_B) \right]^{TF},$$

where

$$\Omega = \frac{1 + 2 \rho_1 + \rho_2 - 4 \rho_3}{2(\rho_1 - \rho_2)}, \text{ if } \rho_1 \neq \rho_2 \text{ and } \Omega \text{ arbitrary otherwise},$$

and where $[...]^{TF}$ denotes the trace-free part with respect to the metric $\delta_{AB}$. Plugging into this the six-parameter family of solutions given in Eqs. (10) and (50), taking into account the vanishing of the constraint-violating modes, $\sigma_2 = \sigma_4 = \sigma_5 = 0$, and evaluating at $x = 0$, we obtain

$$\hat{v}_{xx}^{(\pm)} = \frac{z}{\sqrt{\lambda_1}} \left[ z \pm \sqrt{2} \left( 1 + \Omega \right) \rho \zeta (1 - \Omega) \left[ \sigma_3 + \sigma_6 \mp \frac{\zeta}{\sqrt{\lambda_1}} (\nu_3 \sigma_3 + \nu_1 \sigma_6) \right],$$

$$\hat{v}_{\omega \omega}^{(\pm)} = - \left[ \sqrt{\lambda_1} \rho \zeta (1 - \Omega) \left[ \sigma_3 - \frac{\zeta}{\sqrt{\lambda_1}} \right] \right],$$

$$\hat{v}_{\omega \eta}^{(\pm)} = - \frac{i}{\sqrt{\lambda_1}} \left[ z \pm \sqrt{2} \left( 1 + \Omega \right) \rho \zeta (1 - \Omega) \right] \sigma_1,$$

$$\hat{v}_{\omega \eta}^{(\pm)} = - \left[ z \pm \sqrt{2} \left( 1 + \Omega \right) \rho \zeta (1 - \Omega) \right] \sigma_1.$$
Since the function \( z \mapsto (z + \sqrt{z^2 + 1})^2 \) maps \( \text{Re}(z) > 0 \) onto the outside of the unit disk minus the negative real axis, while the function \( z \mapsto (z - \sqrt{z^2 + 1})^2 \) maps \( \text{Re}(z) > 0 \) onto the inside of the unit disk minus the negative real axis, and since \( |c| \leq 1 \), it follows that the gauge-invariant constants \( \sigma_1 \) and \( \sigma_3 \) vanish. Now it also becomes clear why we restricted the range of the parameter \( c \): If \( |c| > 1 \), there are nontrivial exponential growing solutions with either \( \sigma_1 \neq 0 \) or \( \sigma_3 \neq 0 \), and the system is ill posed.

In order to analyze the consequences of the boundary condition \( \tilde{v}_{xx}^{(+)} = a \tilde{v}_{xx}^{(-)} \), we assume that either \( \Omega = 1 \) (in which case \( \tilde{v}_{xx}^{(\pm)} \) are pure gauge fields) or \( \zeta = -\lambda_1 \). In both cases we obtain, using \( \sigma_3 = 0 \) and Eq. (64),

\[
[z^2 + \lambda_1(1 - \Omega)] \left[ (1 - a)z + (1 + a)\sqrt{z^2 + \lambda_1} \right] \sigma_6 = 0.
\]

The term in the second bracket is non-vanishing for \( \text{Re}(z) > 0 \) if and only if \( |a| \leq 1 \) (see Lemma 1 of Ref. [30] for a proof). The term in the first bracket never vanishes provided that \( \Omega \leq 1 \).

Summarizing, the CPBC with Weyl control and a choice of parameters that allows for \( \Omega = 1 \) or such that \( \zeta = -\lambda_1 \) and \( \Omega \leq 1 \) does always yield an initial-boundary value formulation that satisfies the determinant condition, as long as the main evolution system is strongly hyperbolic and the constraint propagation system is symmetric hyperbolic (see sections III and IV).

### B. CPBC without the Weyl control

Next, we analyze the family of boundary conditions [37, 38], where we set the boundary data \( h \) and \( \hat{h}_{AB} \) to zero. In this case, the result is less robust and depends more strongly on the choice of the parameters. For example, assume again that \( \Omega = 1 \) which implies that \( \sigma_6 = 0 \). Now Eq. (64) has to be replaced by

\[
\left[ (1 + b)(2z^2 - \zeta)\sqrt{z^2 + 1} + (1 - b)z(2z^2 + 1) \right] \sigma_3 = 0.
\]

By taking \( z \) real and positive and by taking the limits \( z \to 0 \) and \( z \to \infty \) one sees that the expression inside the square bracket changes its sign if \( -1 < b \leq 1 \) and \( \zeta > 0 \). In this case it does not follow that \( \sigma_3 \) is zero and the system suffers from the presence of ill posed modes.

For simplicity, let us assume that \( \lambda_1 = \lambda_2 = \lambda_3 = 1 \) and that \( \zeta = -1 \). All of the simulations below will satisfy these conditions. In those cases, we obtain

\[
\begin{align*}
\tilde{f}(z)\sqrt{z^2 + 1} \sigma_1 & = 0, \\
f_b(z) \left[ 2z^2 + 1 \right] \sigma_3 + \sigma_6 & = 0, \\
f_a(z) \left[ (1 - \Omega)(\sigma_3 + \sigma_6) + z^2\sigma_6 \right] & = 0,
\end{align*}
\]

where \( f_a(z) = (1 - a)z + (1 + a)\sqrt{z^2 + 1} \). Since \( |a| \leq 1 \), \( |b| \leq 1 \), \( f_a \) and \( f_b \) never vanish for \( \text{Re}(z) > 0 \) and the only condition we obtain is \( \Omega \leq 3/2 \).

### VI. NUMERICAL IMPLEMENTATION

In this section we discuss how to numerically implement the IBVP with or without Weyl control. For quasilinear first order symmetric hyperbolic systems with maximally dissipative boundary conditions there are well known methods for discretizing the problem such that numerical stability is guaranteed at the linearized level (see [50, 57, 58, 59] for a recent application in the context of numerical relativity, and references therein for the original work). Unfortunately, in our case, the boundary conditions are more complicated since the CPBC [37] and the conditions [40] which control the Weyl tensor are not even algebraic, and so we cannot apply this methods for discretizing the boundary conditions. So we first describe our method for implementing the constraint-preserving boundary conditions, and then explain how the differential equations are discretized.

For simplicity, we focus on the case where the coupling coefficients \( S^j_i, a, b \) and \( c \) vanish, which corresponds to Sommerfeld-like conditions, as discussed in Section IV. The boundary conditions [43] and [44] are implemented in the following way: For all points that lie on the boundary we add to the right-hand side of the evolution equations terms that are linear in the quantities \( V^{(+)j} \) and \( \omega^{(+)}_{AB} - \hat{H}_{AB} \) (which are zero if the boundary conditions are satisfied). More precisely, we replace the right-hand side of equations [44, 45] by

\[
\partial_b K_{ij} = \text{(as before)} + p^a_{ij} V^{(+)a} + q_{ij}^{AB}(\omega^{(+)}_{AB} - \hat{H}_{AB}),
\]

(73)
\begin{align}
\partial_0 d_{ki} &= \text{(as before)} + p^0_{ki} V^{(+)}_a + q_{ki}^{AB} (\dot{w}^{(+)}_{AB} - \dot{H}_{AB}), \\
\partial_0 A_i &= \text{(as before)} + p^0_i V^{(+)}_a + q_{i}^{AB} (\dot{w}^{(+)}_{AB} - \dot{H}_{AB}),
\end{align}

where the matrix coefficients \( p^0_{ij}, q_{ij}^{AB} \), etc., are allowed to depend on the three metric and the unit outward normal one-form \( n_i \) to the boundary. (The \( q \)'s are set to zero for the case without Weyl control.) It is clear that these extra terms change the principal part of the equations. The idea is to choose the \( p \)'s and \( q \)'s in such a way that with respect to the unit outward normal \( n_i \) to the boundary the ingoing characteristic fields \( v^{(+)}_{AA}, v^{(+)}_{nA} \) and \( \dot{w}^{(+)}_{AB} \) become zero speed fields while the speed of the other fields remains unchanged. This has the effect of eliminating the normal derivatives in the evolution equations for \( v^{(+)}_{AA}, v^{(+)}_{nA} \) and \( \dot{w}^{(+)}_{AB} \), and hence these equations are \textit{intrinsic} to the boundary. We will see that this requirement uniquely determines the \( p \)'s and \( q \)'s. In order to see this we first notice that the ingoing characteristic constraint fields have the form

\begin{align}
V^{(+)}_n &= \frac{1}{2} \frac{\partial}{\partial n} \left[ (1 + \sqrt{\lambda_2}) v^{(+)}_{AA} + (1 - \sqrt{\lambda_2}) v^{(-)}_{AA} \right] + Q(\partial_1 u, u), \\
V^{(+)}_A &= -\frac{1}{2} \frac{\partial}{\partial n} \left[ (1 + \sqrt{\lambda_3}) v^{(+)}_{nA} + (1 - \sqrt{\lambda_3}) v^{(-)}_{nA} \right] + Q_A(\partial_1 u, u), \\
\dot{w}^{(+)}_{AB} &= \frac{\partial}{\partial n} \dot{v}^{(+)}_{AB} + Q_{AB}(\partial_1 u, u),
\end{align}

where \( \partial / \partial n \) denotes the normal derivative to the boundary and where the expressions \( Q, Q_A \) and \( Q_{AB} \) only depend on the variables \( u \) and their derivatives \( \partial_1 u \) tangential to the boundary. Therefore, the principal symbol corresponding to the modified system (73,74,75) and the direction of the unit outward normal \( n_i \) is given by

\begin{align}
\mu v^{(+)}_{nn} &= \pm \sqrt{\lambda_1} v^{(+)}_{nn} + p^{(\pm)}_{nn} \dot{v}^{(+)}_a + q_{nn}^{CD} v^{(+)}_{CD}, \\
\mu v^{(+)}_{AA} &= \pm \sqrt{\lambda_2} v^{(+)}_{AA} + p^{(\pm)}_{AA} \dot{v}^{(+)}_a + q_{AA}^{CD} v^{(+)}_{CD}, \\
\mu v^{(+)}_{nA} &= \pm \sqrt{\lambda_3} v^{(+)}_{nA} + p^{(\pm)}_{nA} \dot{v}^{(+)}_a + q_{nA}^{CD} v^{(+)}_{CD}, \\
\mu v^{(+)}_{AB} &= \pm \sqrt{\lambda_2} v^{(+)}_{AB} + p^{(\pm)}_{AB} \dot{v}^{(+)}_a + q^{CD}_{AB} v^{(+)}_{CD}, \\
\mu v^{(0)}_{Ann} &= 0 + p^{(0)}_{Ann} \dot{v}^{(0)}_a + q^{CD}_{Ann} v^{(0)}_{CD}, \\
\end{align}

where \( p^{(\pm)}_{nn}, \; p^{(0)}_{Ann} \) and \( q^{(\pm)}_{nn}, \; q^{(0)}_{Ann} \) are defined in terms of \((p^0_{ij}, p^0_{ikj}, p^0_{i})\) and \((q^0_{ij}, q^0_{ikj}, q^0_{i})\) in the same way as \( v^{(\pm)}_{nn}, \; v^{(0)}_{Ann} \) in terms of \((K_{ij}, d_{ki}, A_i)\), and where

\begin{align}
\dot{V}^{(+)}_n &= \frac{1}{2} \left[ (1 + \sqrt{\lambda_2}) v^{(+)}_{AA} + (1 - \sqrt{\lambda_2}) v^{(-)}_{AA} \right], \\
\dot{V}^{(+)}_A &= -\frac{1}{2} \left[ (1 + \sqrt{\lambda_3}) v^{(+)}_{nA} + (1 - \sqrt{\lambda_3}) v^{(-)}_{nA} \right].
\end{align}

Assume first that \( \lambda_2 = \lambda_3 = 1 \). In this case, we see immediately that \( v^{(\pm)}_{nn}, \; v^{(0)}_{Ann} \) can only remain characteristic fields (i.e., fields with respect to which the principal symbol is diagonal) if all \( p \)'s and \( q \)'s are zero except for \( p^{(\pm)}_{AA}, \; p^{(\pm)}_{nA} \) and \( q^{(\pm)}_{CD} \). Then, the choice

\begin{align}
p^{(\pm)}_{AA} V^{(+)}_n &= -\dot{V}^{(+)}_n, \\
p^{(\pm)}_{nA} \dot{V}^{(+)}_a &= \dot{V}^{(+)}_a, \\
q^{(\pm)}_{AB} \dot{w}^{(+)}_{CD} &= -\dot{w}^{(+)}_{AB}
\end{align}

yields zero speeds for the variables \( v^{(+)}_{AA}, v^{(+)}_{nA} \) and \( \dot{w}^{(+)}_{AB} \). The matrix coefficients \( p^{0}_{ij}, \ldots q^{AB} \) are easily obtained from this by applying the inverse transformation to the one that defines the characteristic fields in terms of the main variables (see 43). If \( \lambda_2 \neq 1 \) or \( \lambda_3 \neq 1 \) it is not possible to retain the fields \( v^{(+)}_{AA}, v^{(+)}_{nA} \) as characteristic fields. In this case, we replace \( v^{(+)}_{AA}, v^{(+)}_{nA} \) by \( \bar{v}^{(+)}_{AA} \equiv \lambda_2 K^A_A + D^A_A \) and \( \bar{v}^{(+)}_{nA} \equiv \lambda_3 K^A_n + D^A_n \), respectively, where \( D^A_A \) and \( D^A_n \) are given in 43 and set

\begin{align}
p^{(\pm)}_{AA} V^{(+)}_n &= \frac{-2 \sqrt{\lambda_2}}{\sqrt{\lambda_2 + 1}} V^{(+)}_n,
\end{align}
equations with a free (nonnegative) multiplicative parameter \( \sigma \)

\[ p_{nA} v_B^{(+)} = \frac{2 \sqrt{\lambda_3}}{\sqrt{\lambda_3} + 1} v_A^{(+)} , \]

\[ q_{AB} w_{CD}^{(+)} = -w_{AB}^{(+)} , \]

and all other \( p \)'s and \( q \)'s to zero. This implies \( \mu \bar{v}_{nA}^{(+)} = 0, \mu \bar{w}_{nA}^{(+)} = 0, \) so \( \bar{v}_{nA}^{(+)} \) and \( \bar{w}_{nA}^{(+)} \) are zero speed fields while the speeds of the remaining characteristic fields are unchanged.

Next, we discretize the evolution system (1, 2, 73, 74, 75) using the method of lines. The matrices \( p \) and \( q \) are set to zero at interior points and are chosen as described above at boundary points. The spatial discretization uses difference operators satisfying summation by parts (SBP) (see, for example, [43]). In this paper we use two of these difference operators, which we call D2-1 and D8-4. These operators satisfy SBP with respect to diagonal norms; it can be seen [51] that the use of these kind of norms implies that the order of the operator at and near boundary points is half that one in the interior.

D8-4 is one of Strand’s three-parametric difference operators of order eight in the interior and four at and close to boundaries, chosen in Ref. [60] to minimize its spectral radius and therefore its associated Courant limit. We have not been able to extend the associated Kreiss-Oliger dissipation for this operator in a straightforward way such as to include the presence of boundaries (see [60] for more details). The refore, here, we simply set the dissipative operator to zero near boundaries. However, as we will discuss later, the resulting dissipation operator does not seem to yield a stable numerical scheme in the presence of CPBC (as opposed to maximally dissipative boundary conditions) and a better dissipation operator is needed.

The following initial data is used in the 3D evolutions of the next section. In each case, the spatial domain is the cube \( \{(x, y, z) \in [-1, 1]^3\} \) of side length 2.

VII. SIMULATIONS: SPACETIMES EVOLVED AND FORMULATIONS OF THE IBVP USED

A. Spacetimes evolved
1. Random data or robust stability test

Here we consider initial data corresponding to flat space and add a random perturbation to it. Therefore, initially, the fields are chosen to be

\[ g_{ij} = \delta_{ij} + \epsilon \mathcal{R}^g_{ij} \]  
\[ K_{ij} = \epsilon \mathcal{R}^k_{ij} \]  
\[ d_{kij} = \epsilon \mathcal{R}^d_{kij} \]  
\[ N = 1 + \epsilon \mathcal{R}^n \]  
\[ A_i = \epsilon \mathcal{R}^a_i \]  

where the different \( \mathcal{R} \) quantities are random numbers which are uniformly distributed in \([-1, 1]\). Similarly, at each timestep the boundary data \( h, \hat{h}_{AB} \) and \( \hat{H}_{AB} \) (see Eqs. (37), (38) and (40), respectively) for the “gauge” and “physical” degrees of freedom is set to a random number of the same order, \( \epsilon \mathcal{R} \). In the simulations shown below we choose \( \epsilon = 10^{-5} \).

This robust stability test \([45, 46]\) is designed to test the numerical stability of the scheme by finding out whether there is, at fixed time, unbounded growth as resolution is increased. One of the advantages of the test is that it excites all frequency modes allowed by a given resolution and it is therefore useful in spotting instabilities (if present) that could otherwise remain hidden in some convergence tests. However, this test is not a convergence test, since the constraints are not satisfied and since different random data is used when resolution is changed.

2. Brill waves

Axisymmetric Brill waves \([48]\) are evolved in the next section. The corresponding initial data for the three-metric is given in Cartesian Coordinates \( x, y, z \) in the form

\[ ds^2 = g_{ij}dx^i dx^j = \Psi^4 \left[ e^{2q} \left( \frac{x dx + y dy}{\rho} \right)^2 + e^{2q}dz^2 + \left( \frac{x dy - y dx}{\rho} \right)^2 \right], \]  

where \( \rho = \sqrt{x^2 + y^2} \), the function \( q \) has the form

\[ q = A_0 \rho^2 \exp \left( -\frac{x^2 + y^2 + z^2}{\sigma_r^2} \right), \]  

and the conformal factor \( \Psi \) is obtained by solving the Hamiltonian constraint for time-symmetric initial data. In order to do so, we use the numerical elliptic solver BAM and the IDBrill Thorn, both publicly available from the CACTUS distribution \([54]\). The initial data for \( K_{ij} \), \( N \), \( A_i \) and \( d_{kij} \) is given by

\[ K_{ij} = 0, \]  
\[ N = 1, \]  
\[ A_i = 0, \]  
\[ d_{kij} = D_k g_{ij}, \]  

where \( D_k \) is the D2-1 finite difference operator in the \( k \) direction. For these Brill wave simulations we choose the D2-1 operator not only for computing the initial data for \( d_{kij} \) but also for the discretization of the right-hand side of the evolution equations. There is no advantage in using a higher order accurate difference operator because the elliptic solver is only second order accurate. In the evolutions shown below, \( \sigma_r = 1/5 \). For weak enough waves the solution bounces at the origin and disperses to infinity, while for strong waves an apparent horizon is typically found \([55]\). In the evolutions below we use very weak waves, corresponding to an amplitude \( A = 10^{-2} \) (while the critical solution is believed to correspond to \( A \approx 4.8 \) \([55]\)). As we shall see, even in these very weak field evolutions there is a clear violations of the constraints when non constraint-preserving boundary conditions are used. Even though these waves are axisymmetric, we have evolved them in full 3D.
3. Gauge solutions

Initial data for a static solution corresponding to a gauge transformation of flat spacetime can be obtained by setting $K_{ij}$, $N$ and $A_i$ to the same expressions as above [Eqs. (101,102,103)], while the three-metric is obtained by starting from the flat metric in spherical coordinates,

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2),$$

performing a coordinates transformation

$$r = \tilde{r} \left(1 - a e^{-r^2/\sigma^2}\right)$$

and transforming the resulting metric to the Cartesian coordinates $(x, y, z) = (\tilde{r} \sin \vartheta \cos \phi, \tilde{r} \sin \vartheta \sin \phi, \tilde{r} \cos \vartheta)$. Initial data for $d_{kij}$ is obtained by computing analytically the gradient of the resulting Cartesian components of the three-metric, $d_{kij} = \partial_k g_{ij}$. This gauge solution has been used to compare the stability properties of different formulations of Einstein’s vacuum equations [57]. In our simulations, we choose $a = 0.1$ and $\sigma_0 = 0.2$.

4. Injecting pulses of gravitational radiation through the boundaries

Finally, we consider an example where we start with flat initial data, i.e. $N = 1$, $g_{ij} = \delta_{ij}$, $K_{ij} = 0$, $d_{kij} = 0$, $A_k = 0$, but inject a pulse of gravitational radiation by choosing as boundary data

$$h = 0,$$

$$\dot{h}_{22} = -\dot{h}_{33} = \dot{h}_{23} = -2\alpha(t - t_0)\sigma_t^{-2}e^{-(t-t_0)^2/\sigma_t^2-\rho^2/\sigma^2}$$

where $\rho$ is a “polar coordinate at each face”. For example, at the $x = \pm 1$ faces $\rho = \sqrt{y^2 + z^2}$ and similarly for the other faces. In the simulations shown below we choose $t_0 = 1.2$, $\sigma_t = \sigma_r = 0.2$ and amplitudes $\alpha = 0.01$ and $2$.

B. Formulations of the IBVP used

In the simulations below, we choose $F = N K$ corresponding to time harmonic slicing, set the shift to zero everywhere and at all times and restrict ourselves to the case in which the parameter $\zeta$ in Eq. (3) is set to $\zeta = -1$. Furthermore, we only consider choices of parameters for which $\lambda_2 = 1$ and $\lambda_3 = 1$, and set the parameter $\Omega$ defined in Eq. (62) to zero. This implies that the characteristic directions lie along the light cone or the normal to the $t = \text{const}$ hypersurfaces.

There are two subsets of parameter space which fulfill these requirements:

1. Mono-parametric family (parametrized by the nonzero value of $\chi$):

   $$\gamma = -\frac{1}{2}, \quad \zeta = -1, \quad \eta = 2, \quad \xi = -\frac{\chi}{2}, \quad \chi \neq 0.$$  

2. Bi-parametric family (parametrized by the parameters $\eta$ and $\gamma \neq -1/2$):

   $$\zeta = -1, \quad \chi = -\frac{\gamma(2 - \eta)}{1 + 2\gamma}, \quad \xi = -\frac{\chi}{2} + \eta - 2, \quad \gamma \neq -\frac{1}{2}, \quad \eta.$$  

One can show that for these families the evolution system [12 3 13] is symmetric hyperbolic. However, notice that the mono-parametric family violates the condition [28] which means that the constraint propagation system is not symmetric hyperbolic. The bi-parametric family satisfies the conditions [28, 24, 31] if and only if $0 < \eta < 2$. According to the analysis in the previous section the determinant condition is satisfied for this subfamily. In this article, we consider the following four cases:

- **CPBC without Weyl control: a completely ill posed case**

  The bi-parametric family is used in this case, with parameters $(\eta, \gamma) = (3, 0)$, and CPBC without Weyl control. As mentioned above, the resulting parameters violate the condition for the constraint propagation system to be symmetric hyperbolic. Therefore, there is no guarantee that the IBVP is well posed. As a matter of fact, it turns out that the determinant condition is violated in this case and thus the problem admits ill posed modes. All the simulations performed below confirm this fact.
• **CBPC with Weyl control**
  Here we choose the bi-parametric family and CPBC with Weyl control. The parameters \( \eta \) and \( \gamma \) are chosen to be either \((\eta, \gamma) = (1, 0)\) or \((\eta, \gamma) = (7/4, -2/3)\); both choices satisfy the determinant condition. Although the growth rate is smaller for the second choice, in both cases we find that the system is numerically unstable. However, we also notice that the instability seems to be milder than in the completely ill posed case: In the weak Brill wave runs, for instance, one needs to run for several crossing times or use very high resolution before noticing the lack of convergence. A similar situation arises in the numerical evolution of weakly hyperbolic systems \[47\] with periodic boundary conditions.

• **CPBC without Weyl control**
  Again, we choose the bi-parametric family, but now we consider CPBC without Weyl control, and choose \((\eta, \gamma) = (1, 0)\). The numerical simulations presented below suggest that the system is numerically stable.

• **Maximally dissipative**
  The mono-parametric family with \( \chi = -1 \) is used with maximally dissipative boundary conditions. This system should be stable, since the evolution equations are symmetric hyperbolic and so the IBVP is well posed. The simulations below confirm this. But more importantly, they provide an explicit demonstration that evolutions of a system with constraint-preserving boundary conditions, when numerically stable, are more accurate than standard, maximally dissipative ones, as in the latter case the boundary conditions introduce constraint violations that do not converge to zero.

**VIII. SIMULATIONS**

For each of the four IBVP described above we run the robust stability test and evolve weak Brill waves. For the case of CPBC without Weyl control, we also evolve the gauge solution and inject pulses of gravitational radiation from the boundary of an initially flat spacetime.

**A. Random, or robust stability test**

For these runs, the Courant factor is \( \lambda = 0.5 \) and the resolution varies from \( 21^3 \), \( 41^3 \) to \( 81^3 \) gridpoints. The spatial derivatives are discretized using the D2-1 operator, and a dissipation parameter of \( \sigma_{\text{diss}} = 0.05 \) is chosen. Recall that random initial and boundary data is given here. In Figure 1 we show the energy of the constraints versus light crossing time. The energy of the constraints is defined to be the sum over all gridpoints of the sum of the square of the components of the constraint variables divided by the number of gridpoints.

As expected, in the completely ill posed case, the energy of the main variables grows at fixed time as resolution is increased, which makes the code crash in the timescale of less than a crossing time (there is also an increase in the energy at fixed resolution as a function of time, though this does not necessarily represent a numerical instability). Figure 1 shows that there is also a similar kind of growth in the energy for the constraints. The timescale and explosive kind of the numerical instability are similar to those found in initial value problems that are completely ill posed due to the presence of complex eigenvalues in the principal part \[47, 61\]. This similarity is, indeed, expected, as at the analytical level the problem admits exponentially in time growing modes, where the exponential factor increases with the frequency as shown in section \[17\]. Recall, however, that the main evolution system is symmetric hyperbolic; thus the boundary conditions are responsible for the instabilities.

In the CPBC case with Weyl control the runs also show evidence of a numerical instability both in the main and constraint variables, though the growth rate is somehow smaller than before. What is somehow unexpected is the resolution dependent growth in the constraint variables. After all, at the continuum, the constraints’ evolution is governed by a symmetric hyperbolic system with maximally dissipative boundary conditions which constitutes a well posed problem by itself. However, it seems that the main system is unstable and that at the discrete level, the instabilities do have an effect on the constraint variables. In order to gain a better insight into this problem we have analyzed the semi-discrete problem, where only the space derivatives are discretized. Even in the simpler case of linearizations about Minkowski spacetime we found that while the discrete constraints do obey a symmetric hyperbolic system it is far from obvious that the way in which we implement the constraint-preserving boundary conditions allow for a semi-discrete energy estimate. It also seems difficult to represent the boundary conditions in a different way such that a semi-discrete energy estimate can be shown for the constraint propagation system. We will not attempt to address this question further in this article. Notice that even if we found such a discretization, the system could still suffer from the presence of more weakly ill posed gauge or physical modes that are undetected by the determinant condition. An explicit example of such weakly ill posed modes is given in \[17\].
FIG. 1: Random data runs for the D2-1 derivative, with Courant factor $\lambda = 0.5$ and dissipation $\sigma_{diss} = 0.05$. The energy of the constraints is defined to be the sum over all gridpoints of the sum of the square of the components of the constraint variables divided by the number of gridpoints.

As opposed to the previous cases, in the CPBC case without Weyl control the runs strongly suggest that the resulting systems is numerically stable and that the continuum problem might be well posed. At early times the constraints’ energy decrease while at around 100 crossing times the constraint variables start to slowly grow in time. However, the growth rate does not become larger with increasing resolution, as opposed to the previous cases.

Finally, the random runs with maximally dissipative boundary conditions also strongly suggest that the system is numerically stable, though this is expected because in this case we know that the IBVP is well posed.
B. Brill wave evolutions

In these simulations we use the smaller Courant factor of $\lambda = 0.25$. Figure 2 shows weak Brill wave runs for the four formulations of the IBVP described in Section VII B. The expectations based on the random data runs of the previous section are confirmed. Namely, the completely ill posed CPBC case is manifestly numerically unstable. The CPBC with Weyl control case that satisfies the determinant condition seems to be unstable as well, though in a “weaker” sense while the two remaining cases (the maximally dissipative one and the CPBC without Weyl control case that satisfies the determinant condition) seem to be numerically stable.

![Brill wave plots](image)

FIG. 2: Weak Brill runs for the four formulations of the IBVP used in the simulations of Fig. 1. The expectations based on that figure regarding numerical stability (or its lack) are here confirmed.

Figure 3 shows the last two cases on a shorter timescale and with a higher resolution run added. Notice how in the CPBC case the constraint variables seem to converge to zero, while in the maximally dissipative case with the same initial data and resolution the lack of convergence to zero is evident.

C. Gauge solutions

Here, we concentrate on the CPBC without Weyl control case. Since the initial data is given in analytic form we use the high order accurate finite differencing operator $D_8\!-\!4$. The results are shown in Figure 4. Notice that the high order accurate scheme results in a much faster convergence of the constraints to zero as resolution is increased. Here, the Courant factor is $\lambda = 0.25$ and the dissipation parameter $\sigma_{\text{diss}} = 10^{-4}$.
FIG. 3: Same runs as previous figure for the two numerically stable cases, but on a shorter timescale and with more resolution. Notice the lack of convergence to zero in the maximally dissipative case, as opposed to the constraint-preserving boundary conditions case.

FIG. 4: Gauge solution simulations with a high order scheme, Courant factor $\lambda = 0.25$, and dissipation $\sigma_{diss} = 10^{-4}$

D. Injecting pulses of gravitational radiation through the boundaries

As in the previous gauge simulations, we use the CPBC formulation without Weyl control. However, we have found that in this case for the same resolutions used in the previous simulations a numerical instability shows up after some time. This is probably due to the fact that the dissipative operator is zero near the boundary and is not negative semi-definite with respect to the scalar product for which SBP holds. Therefore for this case we have used the D2-1 operator, with Courant factor $\lambda = 0.25$, and dissipation parameter $\sigma_{diss} = 0.05$. In Figure 4 we show the energy of the constraints as a function of time for two resolutions and the two amplitudes $\alpha = 0.01$ and $\alpha = 2$. The energy starts from zero, as the initial data consists of Minkowski spacetime. After a short time the energy quickly grows as the pulses are injected into the domain through the six boundaries, until the pulses have been completely injected at roughly one crossing time, time at which the energy stays approximately constant (at fixed resolution).

Figure 5 shows the maximum (in the computational domain) of the curvature invariant $J := R_{abcd}R^{abcd} = 8(E^2 - B^2)$. This curvature invariant starts at zero as well and remains small (compared to one) for amplitude $\alpha = 0.01$ while for amplitude $\alpha = 2$ it increases to very large values as the pulses are injected. For example, at $t = 1.2$ we have $J \approx 3 \times 10^2$. To have an idea of how strong the curvature is, recall that for the Schwarzschild solution $J = 48m^2/r^6$, where $m$ is the mass of the black hole and $r$ is the area radial coordinate. Therefore, a curvature of $J \approx 3 \times 10^2$ would correspond to being inside a black hole of mass $m = 1$ at $r \approx 0.7$. This curvature is being caused solely by the injection of pulses through the boundaries, showing that the latter are able to handle very non-linear dynamics.
IX. CONCLUSIONS

We derived a family of outer boundary conditions for first order hyperbolic formulations of Einstein’s field equations with live gauges. These boundary conditions have the property of being constraint-preserving in the sense that they guarantee that any smooth solution to the evolution equations subject to these boundary conditions automatically satisfy the constraints if so initially. Furthermore, we have discussed different possibilities for using constraint-preserving boundary conditions in order to control the gauge and physical degrees of freedom at the boundary. One of these possibilities (which we called CPBC with Weyl control) is attractive from a physical point of view since it provides boundary data to the Weyl scalars $\Psi_0$ and $\Psi_4$, which at null infinity represent the in- and outgoing radiation. Furthermore, these scalars are gauge-invariant for linearizations about the Kerr metric, an approximation that should be good in many simulations provided the boundaries can be moved sufficiently far from the strong field region. We also discussed simpler ways (which we called CPBC without Weyl control) of controlling the physical degrees of freedom; however, their physical interpretation is less clear.

Next, we analyzed the stability of the resulting IBVP by analytical and numerical means. By considering high-
frequency perturbations we obtained the corresponding frozen coefficient problem which is linear and can be analyzed using Fourier-Laplace transformations. In this way, one obtains a determinant condition which is a necessary condition for the well-posedness of the problem. The satisfaction of this condition restricts the freedom in the choice of parameters in the formulation and the boundary conditions. In particular, we found that the violation of the determinant condition leads to the presence of ill-posed constraint-violating or gauge modes. These modes are ill posed in the sense that they grow exponentially in time with an exponential factor that is unbounded. One could say that the boundary conditions are responsible for the presence of these modes since, in the absence of boundaries, the initial value problem is well posed because the evolution equations are strongly hyperbolic. A further example of boundary conditions leading to an instability of an otherwise stable initial-value problem is given in C.

Next, we performed three-dimensional numerical simulations. In order to do so, we extended an earlier finite-differencing code and implemented the constraint-preserving boundary conditions. We first performed a robust stability test which consists in specifying random initial and boundary data for different resolutions and checking that the time evolution of the fields does not exhibit unbounded resolution-dependent growth. We found that the set of CPBC without Weyl control considered in this article is robustly stable in this sense. We also compared the results from these boundary conditions with results obtained by simply freezing the ingoing fields of the main evolution system to their initial values. While both systems are robustly stable, we found very strong evidence for the freezing boundary condition to yield constraint-violating solutions: In contrast to the CPBC, the constraint variables seem to converge to a nonzero value for freezing boundary conditions. We further tested the CPBC without Weyl control by evolving a gauge solution and weak Brill waves, and found that in both cases the constraint variables seem to converge to zero. Finally, as an application, we considered a situation in which one starts with flat initial data and injects “pulses of gravitational radiation” through the boundaries, of enough amplitude to create very large curvature in the interior.

We expect the CPBC without Weyl control to be useful for many applications, like improving the accuracy and stability of current binary black hole and binary neutron star simulations or for a successful implementation of characteristic or perturbative matching techniques [7, 8, 9, 10].

Unfortunately, our numerical results for the CPBC with Weyl control do not pass the robust stability test, although the instability in this case is much weaker than for the case of CPBC that violate the determinant condition. There are several possible causes for this instability. First, it is not clear if this instability is caused by a “bad” discretization of the problem — the continuum IBVP being well posed — or if the continuum problem is actually ill posed and thus the cause for the instability observed. Another possible source of problem is the cubical domain used in our simulations, which has edges and corners. It might be that we violate some compatibility conditions at such points and that this causes an instability [66]. Whatever the cause of the instability might be, it would be nice to have a better analytic insight into the problem. Although the determinant condition used here is necessary for the well posedness of the IBVP it is not sufficient, not even for the frozen coefficient problem. When making such a statement we have to specify in what sense we expect the IBVP to be well posed. In the linear regime, we expect the problem to be well posed with respect to a Hilbert space that controls the $L^2$ norm of the fields and the $L^2$ norm of the constraint variables. In this case, there are explicit examples [50] that show that the satisfaction of the determinant condition is not sufficient for well posedness. A different examples which applies to the IBVP considered in this article is given in C. Clearly, it would be much more satisfactory to derive CPBC with and without Weyl control for which well posedness can be guaranteed.

X. ACKNOWLEDGMENTS

This research was supported in part by the NSF under Grants No: PHY0244335, PHY0326311, INT0204937 to Louisiana State University, the Horace Hearne Jr. Institute for Theoretical Physics, NSF Grant No. PHY-0099568 to Caltech, and NSF Grants No. PHY0354631 and PHY0312072 to Cornell University. This research used the resources of the Center for Computation and Technology at Louisiana State University, which is supported by funding from the Louisiana legislature’s Information Technology Initiative.

We are especially indebted to Mark Miller for many discussions, suggestions and help throughout this project. We also thank L. Buchman, C. Gundlach, L. Kidder, L. Lehner, L. Lindblom, G. Nagy C. Palenzuela, O. Reula, M. Scheel, S. Teukolsky and J. York for many helpful discussions and comments.

APPENDIX A: A DERIVATION OF THE MAIN EVOLUTION AND CONSTRAINT PROPAGATION SYSTEM

In this appendix we re-derive the main evolution system, carefully keeping track of the constraints that are being used, and find the constraint propagation system with the help of the (twice contracted) Bianchi identities. We start
with the $3 + 1$ split of the four-dimensional Ricci tensor,

$$(4) R_{ij} = (3) R_{ij} - \frac{1}{N} (3) \nabla_i (3) \nabla_j N - 2K_i d_{ij}^l + KK_{ij} - \partial_0 K_{ij}, \quad (A1)$$

where $(3) R_{ij}$ and $(3) \nabla_i$ are the Ricci tensor and the covariant derivative, respectively, belonging to the three metric $g_{ij}$, $N$ is the lapse, $K_{ij}$ the extrinsic curvature, and $\partial_0$ denotes the operator $N^{-1}(\partial_t - \mathcal{E}_\beta)$. In order to obtain first order equations we rewrite the spatial derivatives of the three-metric as

$$\partial_k g_{ij} = d_{kij} - C_{kij}, \quad (A2)$$

where $d_{kij}$ are new fields and $C_{kij} = 0$ are constraints. Correspondingly, we can split the Christoffel symbols, $(3) \Gamma^k_{ij}$, belonging to the three metric as

$$(3) \Gamma^k_{ij} = \Gamma^k_{ij} - \hat{\Gamma}^k_{ij}, \quad (A3)$$

where

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} (2d_{(ij)l} - d_{ijkl}), \quad (A4)$$

are Christoffel symbols belonging to a new connection $\nabla$ which is torsion-free but not necessarily metric (we have $\nabla_k g_{ij} = - C_{kij}$, so $\nabla$ is only metric if the constraints $C_{kij} = 0$ are satisfied), and where

$$\hat{\Gamma}^k_{ij} = \frac{1}{2} g^{kl} (2C_{(ij)l} - C_{lij}). \quad (A5)$$

Since the symbols $\hat{\Gamma}^k_{ij}$ correspond to the difference between the connections $(3) \nabla$ and $\nabla$, they are actually the components of tensor field. Substituting Eq. (A2) into the expression for the Ricci tensor, we obtain

$$(3) R_{ij} = R_{ij} - \hat{R}_{ij}, \quad (A6)$$

where $R_{ij}$ is given by Eq. (12) and where

$$\hat{R}_{ij} = \nabla_k \hat{\Gamma}^k_{ij} - \nabla_i \hat{\Gamma}^k_{kj} - \hat{\Gamma}^k_{lk} \hat{\Gamma}^l_{ij} + \hat{\Gamma}^k_{lj} \hat{\Gamma}^l_{ik} - \Gamma^l_{ij} C^{kl} + \frac{1}{2} C_{(i}^{kl} d_{ijkl} - g^{kl} C_{k(j)l}. \quad (A7)$$

Here, we have defined $C_{kij} = \partial_t C_{kij}$. Note that the hatted quantities vanish if $C_{kij} = 0$. Next, we replace the gradient of the logarithm of the lapse by a new field $A_i$ minus a corresponding constraint variable,

$$\frac{\partial_i N}{N} = A_i - C_i^{(A)}, \quad (A8)$$

and rewrite

$$\frac{1}{N} (3) \nabla_i (3) \nabla_j N = \nabla_i A_j + A_i A_j - \nabla_i C_j^{(A)}$$

$$- 2A_i C_j^{(A)} + C_j^{(A)} C_j^{(A)} + \hat{\Gamma}^k_{ij} (A_k - C_k^{(A)}). \quad (A9)$$

Using Eqs. (A6) and (A9), we rewrite Eq. (11) as

$$\partial_0 K_{ij} = R_{ij} - \nabla_i C_j^{(A)} - A_i A_j - 2K_i^l K_{lj} + K K_{ij} - \Lambda_{ij}, \quad (A10)$$

where

$$\Lambda_{ij} \equiv (4) R_{ij} + \hat{R}_{ij} - \nabla_i C_j^{(A)} - 2A_i C_j^{(A)} + C_j^{(A)} C_j^{(A)} + \hat{\Gamma}^k_{ij} (A_k - C_k^{(A)}) \quad (A11)$$

groups together the four Ricci tensor (which vanishes in vacuum) and the constraint variables. An evolution equation for $K_{ij}$ in vacuum can be obtained from the identity (A10) by setting $\Lambda_{ij}$ to zero. However, in order to obtain a strongly hyperbolic evolution system, one needs to set $\Lambda_{ij}$ equal to suitable combinations of the constraint variables.
and Eqs. (A11,A15,A16), a lengthy but straightforward calculation yields

\[ [\partial_0, \partial_k]T_{i_1i_2...i_r} = \frac{\partial N}{\partial x^r} \partial_0 T_{i_1i_2...i_r} + \frac{1}{N} (T_{i_2...i_r} \delta_k \partial_0 \beta^s + ... + T_{i_1i_2...i_r-1} \delta_k \partial_0 \beta^s), \]  

(A12)

for any \( r \)-rank symbol \( T_{i_1i_2...i_r} \), where the Lie derivative \( L_\beta \) of \( T_{i_1i_2...i_r} \) is formally defined by

\[ L_\beta T_{i_1i_2...i_r} = \beta^k \partial_k T_{i_1i_2...i_r} + T_{k_1...i_r} \partial_i \beta^k + ... + T_{i_1i_2...i_r-1} \delta_k \partial_0 \beta^k. \]

Using the evolution equations \( \partial_0 g_{ij} = -2K_{ij} \) and \( \partial_0 N = -F(N, K, x^\mu) \) for the three metric and the lapse, respectively, we obtain

\[ \partial_0 d_{ki} = -2\partial_0 K_{ij} - 2A_k K_{ij} + \frac{2}{N} g_{i\ell} \partial_\ell \partial_k \beta^j + \Lambda_{ki} , \]  

(A13)

\[ \partial_0 A_i = -\frac{\partial F}{\partial N} A_i - \frac{\partial F}{\partial K} \left( g^{kl} \partial_l K_{ki} - d_k^{kl} K_{kl} \right) - \frac{1}{N} \frac{\partial F}{\partial x^i} + \Lambda_i , \]  

(A14)

where

\[ \Lambda_{ki} \equiv \partial_0 c_{kj} + 2c^{(A)}_{kj} K_{ij} , \]  

(A15)

\[ \Lambda_i \equiv \partial_0 c^{(A)}_i + \frac{\partial F}{\partial N} c^{(A)}_i - \frac{\partial F}{\partial K} c_{ikl} K_{kl} . \]  

(A16)

Finally, we rewrite the Hamiltonian and momentum constraint. Let \( (4)G_{\mu\nu} \) be the Einstein tensor, and let \( \mu = 0 \) denote the contraction with the vector field \( \partial_0 \). Then, we have

\[ (4)G_{00} = C - \dot{C}, \quad (4)G_{0j} = -(C_j - \dot{C}_j), \]

where the expressions for \( C \) and \( C_j \) are given by Eqs. and where

\[ \dot{C} = \frac{1}{2} g^{kl} \dot{R}_{kl} , \quad \dot{C}_j = \frac{1}{2} g^{kl} (C_{kli} - 2C_{klij}) K_{ij} + \frac{1}{2} K^{kl} C_{jkl} . \]  

(A17)

The main evolution equations are \( \partial_0 g_{ij} = -2K_{ij}, \partial_0 N = -F(N, K, x^\mu) \), and Eqs. where one sets the quantities

\[ \Lambda_{ij}(\gamma, \zeta) \equiv \Lambda_{ij} + \gamma g_{ij} C + \zeta g^{kl} C_{k(ij)l} , \]  

(A18)

\[ \Lambda_{kj}(\eta, \chi) \equiv \Lambda_{kj} - \eta g_{kj} C_j + \chi g_{ij} C_i , \]  

(A19)

\[ \Lambda_i(\xi) \equiv \Lambda_i - \xi C_i , \]  

(A20)

to zero.

With this information it is not very difficult to find the constraint propagation system using the commutation relation and the twice contracted Bianchi identities (written in 3+1 form)

\[ \partial_0 (4)G_{00} = \frac{1}{N^2} (3) \nabla^i \left( N^2 (4)G_{0i} \right) + 2K^{ij} (4)G_{00} + K^{ij} (4)R_{ij} - K g^{ij} (4)R_{ij} , \]  

(A21)

\[ \partial_0 (4)G_{0j} = \frac{1}{N^2} (3) \nabla^i \left( N^2 (4)G_{00} \right) + K^{ij} (4)G_{0j} \]

\[ + \frac{1}{N} \nabla^i \left( N (4)R_{ij} \right) - \frac{1}{N^3} (3) \nabla^i \left( N g^{rs} (4)R_{rs} \right) . \]  

(A22)

Substituting \( G_{00} = C - \dot{C}, \ G_{0j} = -(C_j - \dot{C}_j) \) and using the equations \( \Lambda_{ij}(\gamma, \zeta) = 0, \ \Lambda_{kj}(\eta, \chi) = 0 \) and \( \Lambda_i(\xi) = 0 \) and Eqs. , a lengthy but straightforward calculation yields

\[ \partial_0 C = - \left( 1 + \chi - \frac{\eta}{2} \right) g^{kl} \partial_0 C_k + l.o., \]  

(A23)

\[ \partial_0 C_j = -(1 + 2\gamma) \partial_0 C_j - g^{kl} g^{ij} \partial_k \left( C_{k[ijl]} + \frac{1}{2} C_{ijkl} - \xi C_{k(ij)l} \right) \]
in the constraint variables, with coefficients that depend on the main variables and their spatial derivatives. The direction where we can express the characteristic fields as

\[ n = \frac{1}{2} \left( g_{i[k} \partial_l] C_j + g_{j[k} \partial_l] C_i \right) + \chi g_{ij} \partial_l] C_k] + l.o., \]

(A25)

\[ \partial_0 C_{kij} = l.o., \]  

(A26)

\[ \partial_0 C_{k(A)} = l.o., \]  

(A27)

\[ \partial_0 C_{ij}^{(A)} = \xi \partial_l] C_k] + l.o., \]  

(A28)

where we have defined \( C^{(A)} = N^{-1} \partial_0 (NC_i]^{(A)} \) and where the lower order terms, l.o., are linear algebraic expressions in the constraint variables, with coefficients that depend on the main variables and their spatial derivatives. The hard part of the calculation is to show that no spatial derivatives of \( C_{kij} \) and \( C_{ij}^{(A)} \) other than the antisymmetric ones (which can be re-expressed in terms of the constraint variables \( C_{lkij} \) and \( C_{ij}^{(A)} \), respectively) enter the lower order terms.

**APPENDIX B: CHARACTERISTIC FIELDS AND STRONG HYPERBOLICITY**

Given a first order evolution system with principal symbol \( A(n) \), the characteristic fields with respect to a fixed direction \( n_i \) are defined to be the projections of the main fields onto the eigenspaces of \( A(n) \). In order to find the characteristic fields for the symbol defined by Eq. (15), it is convenient to choose an orthonormal basis of three-vectors \( e_1, e_2, e_3 \), such that \( e_1 = n^i n_j \) and express the main variables \( K_{ij}, d_{kij} \) and \( A_i \) with respect to this basis:

\[ K_{ab} = K_{ij} e_i^a e_j^b, \quad d_{cab} = d_{kij} e_i^c e_j^b e_k^a, \quad A_a = A_i e_i^a. \]  

(B1)

Here and in the following, we assume that \( n_i \) is normalized such that \( g^{ij} n_i n_j = 1 \). Assuming that \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) are positive (which is a necessary condition for strong hyperbolicity) and defining

\[ \Omega = \frac{1 + 2 \lambda_1 + \lambda_2 - 4 \lambda_3}{2(\lambda_1 - \lambda_2)}, \]

if \( \lambda_1 \neq \lambda_2 \) and \( \Omega \) arbitrary otherwise, we can express the characteristic fields as (\( n \) refers to the the index \( a = 1 \) and \( A, B, C, \ldots \) to the indices \( a = 2, 3 \))

\[ v_{(\pm) nn} = K_{nn} + \Omega K_A^A \pm \frac{1}{\sqrt{\lambda_2}} (D_{nn} + \Omega D_A^A), \]

(B3)

\[ v_{(\pm) AA} = \pm \frac{1}{\sqrt{\lambda_2}} D_A^A, \]

(B4)

\[ v_{(\pm) A} = K_{nA} \pm \frac{1}{\sqrt{\lambda_3}} D_{nA}, \]

(B5)

\[ \hat{v}_{AB}^{(\pm)} = \hat{K}_{AB} \pm \hat{D}_{AB}, \]

(B6)

where

\[ D_{nn} = \frac{\zeta}{2} d_{nn} + \frac{1}{2} (1 - \zeta) b_n - \frac{1}{2} d_n - A_n + \frac{\gamma}{2} (b_n - d_n), \]

\[ D_A^A = \frac{1}{2} k^{AB} [-d_{nAB} + (1 + \zeta) d_{AB} n] + \gamma (b_n - d_n), \]

\[ D_{nA} = -\frac{1}{4} (1 - \zeta) (d_{nn} A - b_A) + \frac{1}{4} (1 + \zeta) d_{An} - \frac{1}{4} d_A - \frac{1}{2} A_A, \]

\[ \hat{D}_{AB} = \frac{1}{2} [-d_{nAB} + (1 + \zeta) d_{AB} n] - \frac{1}{4} d_{CD} \delta^{CD} [-d_{nCD} + (1 + \zeta) d_{CD} n]. \]

Here, \( d_k = g^{ij} d_{kij}, b_j = g^{kj} d_{kij} \) and \( \hat{K}_{AB} = K_{AB} - \delta_{AB} \delta^{CD} K_{CD}/2 \). \( v_{nn}^{(\pm)}, v_{AA}^{(\pm)}, v_{A}^{(\pm)}, \hat{v}_{AB}^{(\pm)} \) have the speeds \( \mu = \pm \sqrt{\lambda_1}, \mu = \pm \sqrt{\lambda_2}, \mu = \pm \sqrt{\lambda_3}, \) and \( \mu = \pm 1 \), respectively. (These are the speeds with respect to the normal derivative operator. The speeds with respect to the time evolution vector field are obtained from these after the transformation \( \mu \to \eta_{\mu} + \beta^i n_i \).) The remaining characteristic fields have speeds \( \mu = 0 \). For the following representation, we assume that the conditions [28, 29], which are necessary for the constraint propagation system to be symmetrizable, hold.
Defining \( \omega = (\eta - 2\chi - 2)^{-1} \), the zero speed characteristic fields are

\[
N_i, \quad b_{ij},
\]
\[
v_{An}^{(0)} = d_{Ann} - \chi\omega(b_A - d_A), \quad v_{A \delta B}^{(0)} = d_{A n \delta} - \frac{1}{2} \eta\omega\delta_{AB}(b_n - d_n),
\]
\[
v_{AB}^{(0)} = d_{ABC} - \eta\omega\delta_{A(BC)} - \chi\omega\delta_{BC}(b_A - d_A), \quad v_n^{(0)} = A_n + \sigma\omega[(2 + \eta + 3\chi)b_n - (2\eta + \chi)d_n] - (2\sigma + \xi\omega)(b_n - d_n),
\]
\[
v_A^{(0)} = A_n - \xi\omega(b_A - d_A).
\]

Having obtained the characteristic fields explicitly, it is not difficult to verify the additional smoothness requirements for the system to be strongly hyperbolic: Namely, we have to construct a symmetric positive definite matrix \( H = H(p, n, u) \) which symmetrizes the principal symbol \( A(p, n, u) \) defined in Eq. (13). The system is called strongly hyperbolic if \( H \) depends smoothly on \( p, n, u \) and is such that the matrix \( HA \) is symmetric for all \( p, n, u \). The smoothness requirements is needed for the pseudo-differential calculus while the symmetry condition allows for appropriate energy estimates. The matrix \( H \) can be obtained from the quadratic form which is built by summing over the square of the eigenfields:

\[
u^T H \nu = N^2 + \delta^{ij} \delta^{kl} g_{ik}g_{jl} + \delta^{AB} \delta^{(0)} v_{An} v_{Bn} + \delta^{AB} \delta^{CD} v_{AnC} v_{BnD} + \delta^{AB} \delta^{CD} \delta^{EF} v_{ABC} v_{DEF} + \nu_n^2 + \frac{\delta^{AB} \nu_B^2}{2} + \nu_A^2 \delta^{AB} \delta^{(0)} v_A v_B^2 + \frac{\delta^{AB} \delta^{CD} \delta^{EF} \nu_A^2 \nu_B^2}{2}.
\]

Clearly, this quadratic form depends smoothly on \( n_1 \), since it can be written in terms of contractions with \( g^{ij} \) and \( n_i \). \( H \) is also smooth in the other variables provided that the parameters are smooth functions, and provided that the function \( \Omega \) can be chosen smoothly. The latter condition is nontrivial if there are crossing points in the values swept by \( \lambda_1 \) and \( \lambda_2 \). In all our simulations, we choose smooth parameters such that \( \lambda_1 = \lambda_2 = \lambda_3 = 1 \) and set \( \Omega = 0 \), so the system is strongly hyperbolic.

Finally, we give the inverse transformation which allows to recover the main variables from the characteristic ones. First, compute

\[
K_{AB} = \frac{1}{2}(v_{AB}^{(+)} - v_{AB}^{(-)}), \quad K_{nA} = \frac{1}{2}(v_{nA}^{(+)} + v_{nA}^{(-)}), \quad K^A = \frac{1}{2}(v_{AA}^{(+)} + v_{AA}^{(-)}), \quad K_{nn} = \frac{1}{2}(v_{nn}^{(+)} - v_{nn}^{(-)}) - \Omega K^A,
\]

from which \( K_{ab} \) is obtained. Next, compute

\[
D_{AB} = \frac{1}{2}(v_{AB}^{(+) - v_{AB}^{(-)}}, \quad D_{nA} = \frac{\sqrt{A}}{2} (v_{nA}^{(+)} - v_{nA}^{(+) - v_{AA}^{(-)}}, \quad D^A = \frac{\sqrt{A}}{2} (v_{AA}^{(+)} - v_{AA}^{(-)}), \quad D_{nn} = \frac{\sqrt{A}}{2} (v_{nn}^{(+) - v_{nn}^{(-)}) - \Omega D^A.
\]

Next, set

\[
\begin{pmatrix} b_n \\ d_n \end{pmatrix} = M_1 \begin{pmatrix} 2D_{AA} - \zeta \delta_{AB} v_{AB}^{(0)} \\ D_{nn} + D_{AA} + v_n^{(0)} \end{pmatrix}, \quad \begin{pmatrix} b_A \\ d_A \end{pmatrix} = M_2 \begin{pmatrix} 2D_{nA} + v_A^{(0)} - \frac{1}{2}(1 - \zeta \delta_{BC} v_{BC}^{(0)} + \frac{1}{2} \delta_{BC} v_{BC}^{(0)} - \frac{\zeta}{2} v_{Ann}^{(0)} \end{pmatrix},
\]

where

\[
M_1 = \begin{pmatrix} 2D_{AA} - \zeta \delta_{AB} v_{AB}^{(0)} \\ D_{nn} + D_{AA} + v_n^{(0)} \end{pmatrix}, \quad M_2 = \begin{pmatrix} 2D_{nA} + v_A^{(0)} - \frac{1}{2}(1 - \zeta \delta_{BC} v_{BC}^{(0)} + \frac{1}{2} \delta_{BC} v_{BC}^{(0)} - \frac{\zeta}{2} v_{Ann}^{(0)} \end{pmatrix}.
\]
where

\[
M_1 = \frac{1}{\sigma} \left( \frac{2 + 3\gamma - 2\xi + 2\omega(2\chi - \eta) - \eta\zeta}{2(1 + 2\gamma + \eta\omega)} - 1 \right),
\]

\[
M_2 = \left( \frac{1}{1 + \frac{2\omega(\eta + 3\chi) + 1}{2(\eta + 3\chi)}} \right).
\]

Notice that as a consequence of the conditions \(1 + 2\gamma + \eta\zeta - \omega[(1 + 2\gamma)(1 + 2\chi - \eta) - \eta\zeta] < 0\) and the matrix \(M_1\) is well defined. Using this, compute

\[
d_{Ann} = v^{(0)}_{Ann} + \chi\omega(b_A - d_A),
\]

\[
d_{Abb} = v^{(0)}_{Abb} + \frac{1}{2} \eta\omega\delta_{AB}(b_n - d_n),
\]

\[
d_{ABC} = v^{(0)}_{ABC} + \eta\omega\delta_{A(B}(b_{C}) - d_{C})) + \chi\omega\delta_{BC}(b_A - d_A),
\]

\[
A_n = v^{(0)}_n - \sigma\omega[(2 + \eta + 3\chi)b_n - (2\eta + \chi)d_n] + (2\sigma + \xi)\omega(b_n - d_n),
\]

\[
A_A = v^{(0)}_A + \xi\omega(b_A - d_A),
\]

\[
d_{nAB} = -2\bar{D}_{AB} + (1 + \zeta)d_{(AB)n} + \delta_{AB} \left[ \gamma(b_n - d_n) - D^C_{C} \right],
\]

\[
d_{mnA} = \zeta^{-1} \left[ 2D_{mA} - \frac{1}{2}(1 + \zeta)\delta^{BC}(b_{BCA} - d_{ABC}) + A_A \right] + b_A - \frac{1}{2} d_A,
\]

\[
d_{mnn} = 2\zeta^{-1} \left[ D_n - \frac{1}{2}(1 - \zeta)b_n + \frac{1}{2} d_n + A_n - \frac{\gamma}{2} (b_n - d_n) \right].
\]

from which one can compute the components of \(d_{kij}\) and \(A_i\) with respect to the orthonormal basis.

Finally, one obtains the coordinate components of the main variables by contracting with the dual basis \(\theta^a_i\), which is defined by \(\theta^a_i(e^i_j) = \delta^a_b\), \(a, b = 1, 2, 3\):

\[
K_{ij} = K_{aib} \theta^a_i \theta^b_j, \quad d_{kij} = d_{acab} \theta^c_i \theta^a_j, \quad A_i = A_a \theta^a_i.
\]

### APPENDIX C: A SPECIAL FAMILY OF SOLUTIONS

In this appendix we show explicitly that the linearized IBVP with Weyl control cannot be well posed in \(L^2\) if the parameter \(\Omega\) defined in Eq. (22) is one; independent on whether or not the determinant condition is satisfied. In order to see this, we consider the following family of solutions:

Let \(w_j\) be a one-form that satisfies \(\partial^i w_j = 0, \partial^i \partial_i w_j = 0,\) and \(\partial_k w_j = 0,\) and let

\[
K_{ij} = \partial_i w_j,
\]

\[
d_{kij} = -2t \partial_k \partial_i w_j,
\]

\[
A_i = 0.
\]

It is not difficult to check that these expressions satisfy the evolution equations \(44, 45, 46\) and the constraints \(\partial^i(d_k - b_k) = 0, \partial_k d_{kij} = 0, \partial^i K_{ij} - \partial_j K = 0.\) For systems without boundaries, these solutions are trivial if appropriate fall off conditions on the fields are demanded since then the harmonic condition on \(w_j\) implies that it must vanish. However, if boundaries are present, \(w_j\) may be nontrivial. The electric and magnetic components of the linearized Weyl tensor corresponding to the solutions \(w_j\) are

\[
E_{ij} = 0, \quad B_{ij} = \frac{1}{2} \partial_i \varepsilon_j \varepsilon_{rs} \partial^r w^s,
\]

and \(B_{ij}\) vanishes if the one-form \(w_j\) is closed. In particular, this is true if \(w_j\) is exact, i.e., \(w_j = \partial_j f\) for some time-independent harmonic function \(f.\) In this case, we also have

\[
v^{(0)}_{nn} = (1 - \Omega) \left( \frac{\partial^2 f - \frac{\xi}{\sqrt{\lambda_1}} \partial_n^3 f}{\sqrt{\lambda_1}} \right).
\]
Therefore, if $\Omega = 1$, the family of solutions with $w_j = \partial_j f$ and $f$ harmonic and time-independent shows that the linearized IBVP with the boundary conditions is not well posed in $L^2$ since then the boundary conditions are satisfied with homogeneous data and since the initial data depends only on second derivatives of $f$ whereas $d_{kij}$ depends on third derivatives of $f$ for $t > 0$. This results in frequency dependent growth of the solution of the form $|k|t$, where $k$ is a characteristic wave number of the initial data.

[1] M. Miller, P. Gressman and W. M. Suen, Towards a Realistic Neutron Star Binary Inspiral: Initial Data and Multiple Orbit Evolution in Full General Relativity, Phys. Rev. D 69 (2004) 064026.
[2] A. Lichnerowicz, L’intégration des équations de la gravitation relativiste et le problème des n corps, J. Math. Pure Appl. 23 (1944) 37–63.
[3] L. Smarr and J. W., Kinematical Conditions In The Construction Of Space-Time, Phys. Rev. D 17 (1978) 2529–2551.
[4] M. Anderson and R. A. Matzner, Extended Lifetime in Computational Evolution of Isolated Black Holes, arXiv:gr-qc/0307055.
[5] E. Schnetter, Ph.D. thesis, Universität Tübingen (2003), [http://w210.ub.uni-tuebingen.de/dbt/volltexte/2003/819/].
[6] M. Holst, L. Lindblom, R. Owen, H. P. Pfeiffer, M. A. Scheel and L. E. Kidder, Optimal Constraint Projection for Hyperbolic Evolution Systems, Phys. Rev. D 70 (2004) 084017.
[7] J. Winicour, Characteristic Evolution and Matching, Living Rev. Relativity 4, (2001), 3.
[8] L. Rezzolla, A. M. Abrahams, R. A. Matzner, M. E. Rupright and S. L. Shapiro, Cauchy-perturbative matching and outer boundary conditions: Computational studies, Phys. Rev. D 59 (1999) 064001.
[9] M. E. Rupright, A. M. Abrahams and L. Rezzolla, Cauchy-perturbative matching and outer boundary conditions. I: Methods and tests, Phys. Rev. D 58 (1998) 044005.
[10] A. M. Abrahams et al. [Binary Black Hole Grand Challenge Alliance Collaboration], Gravitational wave extraction and outer boundary conditions by perturbative matching, Phys. Rev. Lett. 80 (1998) 1812–1815.
[11] P.D. Lax, and R.S. Phillips, Local Boundary Conditions for Dissipative Symmetric Linear Differential Operators,, Commun. Pure Appl. Math. 13 (1960) 427–455.
[12] K.O. Friedrichs, Symmetric Positive Linear Differential Equations, Commun. Pure Appl. Math. 11 (1958) 333–418.
[13] J. M. Stewart, Advanced general relativity, Cambridge University Press (1996).
[14] S. A. Teukolsky, Perturbations Of A Rotating Black Hole. 1. Fundamental Equations For Gravitational Electromagnetic, And Neutrino Field Perturbations, Astrophys. J. 185 (1973) 635–647.
[15] H. Friedrich and G. Nagy, The initial boundary value problem for Einstein’s vacuum field equations, Comm. Math. Phys. 201 (1999) 619–655.
[16] J. M. Bardeen and L. T. Buchman, Numerical tests of evolution systems, gauge conditions, and boundary conditions for 1D colliding gravitational plane waves, Phys. Rev. D 65 (2002) 064037.
[17] J. Rauch, Symmetric positive systems with boundary characteristics of constant multiplicity, Trans. Am. Math. Soc. 291 (1985) 167–187.
[18] P. Secchi, The initial boundary value problem for linear symmetric hyperbolic systems with characteristic boundary of constant multiplicity, Diff. Int. Eq. 9 (1996) 671–700.
[19] P. Secchi, Well-Posedness of Characteristic Symmetric Hyperbolic Systems, Arch. Rat. Mech. Anal. 134 (1996) 155–197.
[20] M. S. Iriondo and O. A. Reula, On free evolution of selfgravitating, spherically symmetric waves, Phys. Rev. D 65 (2002) 044024.
[21] J. M. Stewart, The Cauchy problem and the initial boundary value problem in numerical relativity Class. Quantum Grav. 15 (1998) 2865–2889.
[22] B. Szilágyi, B. G. Schmidt and J. Winicour, Boundary conditions in linearized harmonic gravity, Phys. Rev. D 65 (2002) 064015.
[23] B. Szilágyi and J. Winicour, Well-Posed Initial-Boundary Evolution in General Relativity, Phys. Rev. D 68 (2003) 041501.
[24] G. Calabrese, J. Pullin, O. Reula, O. Sarbach and M. Tiglio, Well posed constraint-preserving boundary conditions for the linearized Einstein equations, Commun. Math. Phys. 240 (2003) 377–395.
[25] G. Calabrese and O. Sarbach, Detecting ill posed boundary conditions in General Relativity, J. Math. Phys. 44 (2003) 3888–3899.
[26] G. Calabrese, Ph.D. thesis (unpublished).
[27] S. Frittelli and R. Gomez, Boundary conditions for hyperbolic formulations of the Einstein equations, Class. Quant. Grav. 20 (2003) 2379–2392, Einstein boundary conditions for the 3+1 Einstein equations, Phys. Rev. D 68 (2003) 044014, Einstein boundary conditions in relation to constraint propagation for the initial-boundary value problem of the Einstein equations, Phys. Rev. D 69 (2004) 124020, Einstein boundary conditions for the Einstein equations in the conformal-traceless decomposition, Phys. Rev. D 70 (2004) 064008.
[28] C. Gundlach and J. M. Martín-García, Symmetric hyperbolic form of systems of second-order evolution equations subject to constraints, Phys. Rev. D 70 (2004) 044031.
[29] C. Gundlach and J. M. Martín-García, Symmetric hyperbolicity and consistent boundary conditions for second-order Einstein equations, Phys. Rev. D 70 (2004) 044032.
[30] O. Reula and O. Sarbach, A model problem for the initial-boundary value formulation of Einstein’s field equations, Journal
of Hyperbolic Differential Equations, 2 (2005) 397–435.

[31] G. Calabrese, L. Lehner and M. Tiglio, Constraint-preserving boundary conditions in numerical relativity, Phys. Rev. D 65 (2002) 104031.

[32] L. Lindblom, M. A. Scheel, L. E. Kidder, H. P. Pfeiffer, D. Shoemaker and S. A. Teukolsky, Controlling the Growth of Constraints in Hyperbolic Evolution Systems, Phys. Rev. D 69 (2004) 124025.

[33] C. Bona, T. Ledvinka, C. Palenzuela-Luque and M. Zacek, Constraint-preserving boundary conditions in the Z4 Numerical Relativity formalism, [arXiv:gr-qc/0411110]

[34] O. Sarbach and M. Tiglio, Exploiting gauge and constraint freedom in hyperbolic formulations of Einstein’s equations, Phys. Rev. D 66 (2002) 064023.

[35] S. Fritelli and O. A. Reula, First-order symmetric-hyperbolic Einstein equations with arbitrary fixed gauge, Phys. Rev. Lett. 76, (1996) 4667–4670.

[36] A. Anderson and J. W. York, Fixing Einstein’s equations, Phys. Rev. Lett. 82 (1999) 4384–4387.

[37] S. D. Hern, Ph.D. thesis, University of Cambridge, 1999, [arXiv:gr-qc/0004036]

[38] L. E. Kidder, M. A. Scheel and S. A. Teukolsky, Extending the lifetime of 3D black hole computations with a new hyperbolic system of evolution equations, Phys. Rev. D 64 (2001) 064017.

[39] C. Bona, J. Masso, E. Seidel and J. Stela, A New formalism for numerical relativity, Phys. Rev. Lett. 75 (1995) 600–603.

[40] C. Bona, J. Masso, E. Seidel and J. Stela, First order hyperbolic formalism for numerical relativity, Phys. Rev. D 56 (1997) 3405–3415.

[41] H. O. Kreiss, Initial boundary value problems for hyperbolic systems, Commun. Pure Appl. Math. 23 (1970) 277–298.

[42] H.O. Kreiss, J. Lorenz, “Initial-Boundary Value Problems and the Navier-Stokes Equations,” Academic Press, (1978).

[43] B. Gustafsson, H. Kreiss, and J. Oliger, “Time-dependent problems and difference methods,” John Wiley & Sons, New York (1995).

[44] A. Majda and S. Osher, Initial-Boundary Value Problems for Hyperbolic Equations with Uniformly Characteristic Boundary, Commun. Pure Appl. Math. 28 (1975) 607–675.

[45] B. Szilagyi, R. Gomez, N. T. Bishop and J. Winicour, Cauchy boundaries in linearized gravitational theory, Phys. Rev. D 62 (2000) 104006.

[46] M. Alcubierre et al., Toward standard testsbeds for numerical relativity, Class. Quant. Grav. 21 (2004) 589–613.

[47] G. Calabrese, J. Pullin, O. Sarbach and M. Tiglio, Convergence and stability in numerical relativity, Phys. Rev. D 66 (2002) 041501.

[48] D. Brill, On the Positive Definite Mass of the Bondi-Weber-Wheeler Time-Symmetric Gravitational Waves, Ann. Phys. 7 (1959) 466–483.

[49] M. Tiglio, Dynamical control of the constraints growth in free evolutions of Einstein’s equations, [arXiv:gr-qc/0304062]

[50] M. Tiglio, L. Lehner and D. Neilsen, 3D simulations of Einstein’s equations: symmetric hyperbolicity, live gauges and dynamic control of the constraints, Phys. Rev. D 70 (2004) 104018.

[51] H. Kreiss and G. Scherer, “On the existence of energy estimates for difference approximations for hyperbolic systems”, Tech. Report, Dept. of Scientific Computing, Uppsala University, 1977.

[52] H. Kreiss and J. Oliger, “Methods for the approximate solution of time dependent problems” (Geneva: GARP Publication Series, 1973).

[53] B. Strand, Summation by Parts for Finite Difference Approximations for d/dx, Journal of Computational Physics 110 (1994) 47–67.

[54] http://www.cactuscode.org

[55] M. Alcubierre, G. Allen, B. Brugmann, G. Lanfermann, E. Seidel, W. M. Suen and M. Tobias, Gravitational collapse of gravitational waves in 3D numerical relativity, Phys. Rev. D 61 (2000) 041501.

[56] M. Alcubierre, G. Allen, B. Brugmann, E. Seidel and W. M. Suen, Towards an understanding of the stability properties of the 3+1 evolution equations in general relativity, Phys. Rev. D 62 (2000) 124011.

[57] G. Calabrese, L. Lehner, D. Neilsen, J. Pullin, O. Reula, O. Sarbach and M. Tiglio, Novel finite-differencing techniques for numerical relativity: application to black hole excision, Class. Quant. Grav. 20 (2003) L245–L252.

[58] G. Calabrese, L. Lehner, O. Reula, O. Sarbach and M. Tiglio, Summation by parts and dissipation for domains with excised regions, Class. Quant. Grav. 21 (2004) 5735–5758.

[59] L. Lehner, D. Neilsen, O. Reula and M. Tiglio, The discrete energy method in numerical relativity: Towards long-term stability, Class. Quant. Grav. 21 (2004) 5819–5848.

[60] L. Lehner, O. Reula and M. Tiglio, in preparation.

[61] G. Calabrese, J. Pullin, O. Sarbach and M. Tiglio, Stability properties of a formulation of Einstein’s equations, Phys. Rev. D 66 (2002) 064011.

[62] These parameters can be constants or a priori specified functions on spacetime. See Ref. [46, 50] where time-dependent parameters are used in order to control growth of constraint violations.

[63] Notice that the constraint variables $C_{kij}$ and $C_{ij}^{(A)}$ are automatically zero if the constraints $C_{kij} = 0$, $C_{ij}^{(A)} = 0$ are satisfied, and so they are redundant. However, we will need these variables in the next section in order to cast the constraint propagation system into first order form.

[64] Here we also assume that $F(N, K = 0, \mu^\alpha) = 0$ for all $N$, which is satisfied by the time-harmonic slicing condition adopted in our simulations.

[65] This can also be verified directly by introducing the expressions $C_{kij}$ into the Fourier-Laplace transformed of the CPBC.
[66] See, for instance, [24, 26] for a detailed study of compatibility conditions in domains with corners.