QUALITATIVE ANALYSIS OF A LOTKA-VOLTERRA COMPETITION-DIFFUSION-ADVECTION SYSTEM

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Abstract. This paper performs an in-depth qualitative analysis of the dynamic behavior of a diffusive Lotka-Volterra type competition system with advection terms under the homogeneous Dirichlet boundary condition. First, we obtain the existence, multiplicity and explicit structure of the spatially nonhomogeneous steady-state solutions by using implicit function theorem and Lyapunov-Schmidt reduction method. Secondly, by analyzing the distribution of eigenvalues of infinitesimal generators, the stability of spatially nonhomogeneous positive steady-state solutions and the non-existence of Hopf bifurcations at spatially nonhomogeneous positive steady-state solutions are given. Finally, two concrete examples are provided to support our previous theoretical results. It should be noticed that an elliptic operator with advection term is not self-adjoint, which causes some trouble in the spatial decomposition, explicit expressions of steady-state solutions and some deductive processes related to infinitesimal generators. Moreover, unlike other work, the advection rate here depends on the spatial position, which increases some difficulties in the investigation of the principal eigenvalue.

1. Introduction. The classical Lotka-Volterra model of two populations with diffusion can be expressed as follows:

\[
\begin{align*}
    u_t &= D_1 \Delta u + u(1-u-cv), \quad t > 0, x \in \Omega, \\
    v_t &= D_2 \Delta v + v(1-v-bu), \quad t > 0, x \in \Omega,
\end{align*}
\]

where \( \Omega \) is a connected bounded open domain in \( \mathbb{R}^n \), with a smooth boundary \( \partial \Omega \); \( u(x, t) \) and \( v(x, t) \) represent the population densities of two competing species at time \( t \) and position \( x \); positive constants \( D_1 \) and \( D_2 \) are random diffusion rates of species \( u \) and \( v \), respectively; constants \( c \) and \( b \) represent the interaction between the two populations. According to the population interaction relationship, model (1) can be divided into three categories: (i) if \( b > 0, c > 0 \), it is a competitionleta...
model; (ii) if \( b < 0 \), it is a predator-prey model; (iii) if \( b < 0, c < 0 \), it is a reciprocal model. There have been many classical examples of model (1), which address problems related to spatial ecology and evolution, one can refer to [3, 11, 15, 17, 19, 21, 24, 25, 35] and the references therein. Of course, in order to reflect the effects of the past time on the population, many researchers have added reasonable time delays to the existing models and obtained interesting results, one can refer to [2, 5, 8, 10, 12, 14, 22] and the references therein.

For ecosystems such as rivers, oceans and deserts, the movement of species is not just random diffusion, but also some directed movement incurred by the water flow, ocean currents and shifting sands. So naturally we need to add advection terms to the original reaction-diffusion model to better describe species changes. However, the dynamic behavior of reaction-diffusion equations with advection terms has not been fully and extensively studied. The main reason is that the linear operator with only the diffusion term is self-adjoint, and becomes no longer self-adjoint when the advection term is added. Therefore, the existing research method for the reaction-diffusion Lotka-Volterra model may not be applicable. The most typical example is that there are some different properties of the principal eigenvalue between the operator \( \Delta \) and the operator \( D\Delta - \beta \nabla \) [6]. In other words, what we are doing can also be regarded as the further study and application of the spectral theory of non-self-adjoint operators to some extent. In this respect, Lou et al. [20], Zhou and his collaborators [29, 33, 34] investigated the global dynamics of a Lotka-Volterra competition-diffusion-advection system by using the prior estimate and the theory of monotone dynamical system when the diffusion rates and the advection rates are constants and there are some proportion relationships between them. These works are of great significance in the study of Lotka-Volterra model with advection terms added, which is worthy of deep investigation. So it’s natural to ask, what if the advection rates are related to the spatial position, or the advection rates are not related to the diffusion coefficients?

In this paper, we investigate the following reaction-diffusion-advection Lotka-Volterra type competition system (2) with Dirichlet boundary conditions and initial conditions:

\[
\begin{aligned}
  u_t &= D_1 u_{xx} - \beta_1(x) u_x + r u(1 - u - cv), & t > 0, L_1 < x < L_2, \\
  v_t &= D_2 v_{xx} - \beta_2(x) v_x + r v(1 - v - bu), & t > 0, L_1 < x < L_2, \\
  u(t, L_1) &= u(t, L_2) = 0, & t > 0, \\
  v(t, L_1) &= v(t, L_2) = 0, & t > 0, \\
  u(0, x) = u_0 \geq 0, & v(0, x) = v_0 \geq 0, & L_1 < x < L_2,
\end{aligned}
\]

where the initial data \( u_0, v_0 \in C^2([L_1, L_2]) \); \( u(x,t) \) and \( v(x,t) \) represent the population densities of two competing species at time \( t \) and position \( x \); \( D_1 \) and \( D_2 \) are random diffusion rates of species \( u \) and \( v \), respectively; \( \beta_1(x), \beta_2(x) \in C^2([L_1, L_2]) \) account for the advection coefficients that illustrate the tendency of the biased movement of species \( u \) and \( v \), respectively; \( r \) is the intrinsic growth of species \( u \) and \( v \); \( c \) and \( b \) are interspecific competition coefficients of species \( u \) and \( v \), respectively. Moreover, \( \beta_1(x) > 0 \), \( \beta_2(x) > 0 \), and all the parameters in model (2) are assumed to be positive. The Dirichlet boundary condition implies that the exterior environment is hostile and the species cannot move across the boundary of environment.

As we know, non-zero steady-state solutions and periodic solutions in a Dirichlet boundary value problem are all spatially nonhomogeneous. The study of homogeneous steady-state solution has been extensively studied, while the discussion about spatially nonhomogeneous steady-state solutions is much less because the relevant
investigation is complicated. For example, the existence of a spatially nonhomogeneous steady-state solution can be obtained by variational and topological methods [4, 7, 31], but its explicit algebraic form is not clear. In fact, its explicit algebraic form largely determines the specific form of the characteristic equation of the linearized system at the spatially nonhomogeneous steady-state solution, and so plays a key role in the analysis of stability and bifurcation phenomena. As early as in 1996, Busenberg and Huang [2] used the implicit function theorem and ingenious construction methods to study the existence, stability and Hopf bifurcation of a spatially nonhomogeneous steady-state solution in a logistic equation with delay and diffusion term on one-dimensional space domain. This groundbreaking work about the dynamics near a spatially nonhomogeneous steady-state solution provides new ideas for researchers, typically such as [5, 16, 18, 28, 32]. However, the work of Busenberg and Huang [2] hardly involves the multiplicity of spatially nonhomogeneous steady-state solutions, which naturally becomes one of our interests and the focus of this paper. In this paper, different from [2], we use Lyapunov-Schmidt reduction method [13] to investigate the existence, multiplicity, and asymptotical forms of the spatially nonhomogeneous steady-state solution for model (2). Few people considered the effect of changing of the intrinsic growth rate on the nonhomogeneous steady-state solution. According to [26], there are many determinants of the intrinsic growth rate, among which the influence of temperature is the most important. Further, Birch [1] discovered the connection between temperature changes and the generations of species reproduce and intrinsic growth rates. Therefore, in order to better describe the dependence of the presence of spatially nonhomogeneous steady-state solutions near the trivial steady-state solution of (2) on the intrinsic growth rate, it is reasonable to regard the intrinsic growth rate as a bifurcation parameter. Throughout this paper, we assume that two competing species have the same intrinsic growth rate $r$, and distinguish three cases to study the different effects of simultaneous changes in intrinsic growth rates on two competing species (see Theorems 2.3-2.6,4.1). Moreover, the existence and uniqueness of spatially nonhomogeneous steady-state solutions are established in each case (see Theorems 2.3-2.6), which indicates that our work on system (2) is more integrated and in-depth than others. In addition, as mentioned earlier, it is a major breakthrough to obtain an explicit form of the solution that plays an important role in studying the stability and Hopf bifurcation of the spatially nonhomogeneous steady-state solution. At the same time, we need to emphasize that although this paper concentrates on the exploration of the reaction-diffusions-advection equation (2), our research method is also applicable to other Dirichlet problems and even Neumann problems. Furthermore, the elliptic operator with advection term is not self-adjoint, which causes some troubles in spatial decomposition, explicit expressions of steady-state solutions and some deductive processes related to an infinitesimal generator. Most importantly, unlike other work, the advection rates here depend on the spatial position, which adds some difficulties to the investigation of the principal eigenvalue.

The rest of the paper is organized as follows. In Section 2, for the three different cases of intrinsic growth of species, we obtain the existence and multiplicity of spatially nonhomogeneous steady state solutions by using Lyapunov-Schmidt reduction method. In Section 3, we analyze the eigenvalue distribution of the infinitesimal generator of the linearized system (2) at a steady-state solution. In Section 4, based on the discussion in Section 3, we further obtain the asymptotic behavior
and stability of the steady-state solution of system (2), and conclude that there is no Hopf bifurcation in (2). Finally, we provide two concrete examples and numerical simulation to prove the validity of our study in Section 5.

Throughout this paper, let $H^k(\Omega)(k \geq 0)$ be the Sobolev space of $L^2$-functions $f(x)$ defined on $\Omega = (L_1, L_2)$ whose derivatives $\frac{\partial^k f}{\partial x^n}$ $(n = 1, 2, ..., k)$ belong to $L^2(\Omega)$. Also, we denote the spaces $X = H^2(\Omega, \mathbb{R})$ and $Y = L^2(\Omega, \mathbb{R})$, where $H^2(\Omega, \mathbb{R}) = \{ u \in H^2(\Omega, \mathbb{R}) | u(x) = 0 \text{ for all } x \in \partial \Omega \}$. For any subspace $Z$ of $X$ or $Y$, we also define the complexification of $Z$ to be $Z_\mathbb{C} \triangleq Z \oplus iZ = \{ x_1 + ix_2 | x_1, x_2 \in Z \}$. Denote by $C^k = C^k(\mathbb{X}, \mathbb{Y})$ the Banach space of $k$-times continuously differentiable mappings from $\mathbb{X}$ into $\mathbb{Y}$ equipped with the supremum norm. For a linear operator $L : \mathbb{Z}_1 \rightarrow \mathbb{Z}_2$, we denote the domain of $L$ by $\mathcal{D}(L)$, the range of $L$ by $\text{Ran}(L)$. For the complex-valued Hilbert space $\mathbb{Y}_\mathbb{C}$, we use the standard inner product $\langle u, v \rangle = \int_\Omega \bar{u}^T(x)v(x)dx$.

2. Preliminaries. In this section, we study the existence and multiplicity of the spatially nonhomogeneous steady state solution of system (2), which satisfy the following boundary value problem:

$$
\begin{align*}
D_1 u_{xx} - \beta_1(x)u_x + ru(1 - u - cv) &= 0, & t > 0, L_1 < x < L_2, \\
D_2 v_{xx} - \beta_2(x)v_x + rv(1 - v - bu) &= 0, & t > 0, L_1 < x < L_2, \\
u(t, L_1) &= u(t, L_2) = 0, & t > 0, \\
v(t, L_1) &= v(t, L_2) = 0, & t > 0.
\end{align*}
$$

(3)

To find solutions of (3), define $F : \mathbb{X}^2 \times \mathbb{R}^2 \rightarrow \mathbb{Y}^2$ by

$$
F(U, r) = \begin{pmatrix} F_1(U, r) \\ F_2(U, r) \end{pmatrix} = \begin{pmatrix} D_1 u_{xx} - \beta_1(x)u_x + ru(1 - u - cv) \\ D_2 v_{xx} - \beta_2(x)v_x + rv(1 - v - bu) \end{pmatrix}
$$

for all $U = (u, v)^T \in \mathbb{X}^2$. We shall try to solve $F(U, r) = 0$ for $U \in \mathbb{X}^2$ and $r \in \mathbb{R}$. It is easy to see that, for every fixed parameter value $r \in \mathbb{R}$, $F(U, r) = 0$ always has a trivial solution $(0, 0)^T$. Namely, $F(0, r) = 0$ for all values of the parameter $r$. If we want to prove the uniqueness of these solutions by the implicit function theorem, we need to compute the Fréchet derivative of $F$ with respect to $U$ evaluated at $(0, r)$. Thus, we have

$$
\mathcal{L}_r U = \begin{pmatrix} D_1 u_{xx} - \beta_1(x)u_x + ru & 0 \\ 0 & D_2 v_{xx} - \beta_2(x)v_x + rv \end{pmatrix}
$$

For the Hilbert space $\mathbb{X}^2$, we define the standard inner product

$$
\langle U, V \rangle = \int_\Omega \bar{U}^T(x)V(x)dx
$$

and the adjoint operation of $\mathcal{L}_r$ by $\mathcal{L}_r^* \mathcal{L}_r = \langle \mathcal{L}_r^* U, V \rangle$, where $U, V \in \mathbb{X}^2$. Since we shall discuss variables on one-dimensional space, then by simple calculation we can get the adjoint operator of $\mathcal{L}_r$ as follows

$$
\mathcal{L}_r^* U = \begin{pmatrix} D_1 u_{xx} + \beta_1(x)u_x + \beta_{1x}(x)u + ru & 0 \\ 0 & D_2 v_{xx} + \beta_2(x)v_x + \beta_{2x}(x)v + rv \end{pmatrix}
$$

Before discussing the kernel of $\mathcal{L}_r$, we first introduce the following lemma:

**Lemma 2.1 ([30]).** The following eigenvalue problem:

$$
\begin{align*}
-\varphi_{xx}(x) + q(x)\varphi_x(x) &= \lambda \varphi(x), & t > 0, L_1 < x < L_2, \\
\varphi(t, L_1) &= \varphi(t, L_2) = 0, & t > 0,
\end{align*}
$$

(4)
has a sequence of eigen-pairs \( \{ (\lambda_n(q), \varphi_n(q)) \}_{n=1}^{\infty} \). Among them, \( \lambda_1(q) \) denotes the unique positive principal eigenvalue, which is simple and real, and \( \varphi_1(q) \) denotes its associated eigenvector, which is positive for all \( x \in (L_1, L_2) \). Moreover, \( \{ \varphi_n(q) \}_{n=1}^{\infty} \) is a complete orthogonal system in the Lebesgue space \( L^2([L_1, L_2]) \) of integrable functions defined on \([L_1, L_2]\). For convenience, let \( \Omega = (L_1, L_2) \).

Obviously, the adjoint equation of (4) is as follows:

\[
\begin{aligned}
-\varphi_{xx}(x) - g(x)\varphi_x(x) - q(x)\varphi(x) &= \lambda \varphi(x), & t > 0, L_1 < x < L_2, \\
\varphi(t, L_1) &= \varphi(t, L_2) = 0, & t > 0.
\end{aligned}
\]  

(5)

According to the Theorem 2.3.20 in [30], there are similar conclusions to Lemma 2.1, that is, (5) also has a sequence of eigen-pairs \( \{ (\lambda_n(q), \varphi_n(q)) \}_{n=1}^{\infty} \), and has a unique positive principal eigenvalue \( \lambda_1(q) \), which is simple and real. Similarly, \( \{ \varphi_n(q) \}_{n=1}^{\infty} \) is a complete orthonormal system in the Lebesgue space \( L^2(\Omega) \) of integrable functions defined on \( \Omega \), where \( \varphi_1(q) > 0 \) is the principal eigenvector associated with \( \lambda_1(q) \). In this article we can choose a suitable \( \varphi_1(q) > 0 \) such that \( \int_\Omega \varphi_1(q) \varphi_1(q) dx = 1 \) for all \( x \in \Omega \). If not, we can normalize it to get the desired results. In particular, we have \( \lambda_1(q) = \lambda_1(q) \) because

\[
\lambda_1(\varphi_1, \varphi_1) = \langle \lambda_1 \varphi_1, \varphi_1 \rangle = \langle \mathfrak{L} \varphi_1, \varphi_1 \rangle = \langle \varphi_1, \mathfrak{L}^* \varphi_1 \rangle = \langle \varphi_1, \lambda_1 \varphi_1 \rangle = \lambda_1 \langle \varphi_1, \varphi_1 \rangle,
\]

where \( \mathfrak{L} \varphi = -\varphi_{xx} + g(x)\varphi_x \) and \( \mathfrak{L}^* \varphi = -\varphi_{xx} - g(x)\varphi_x - q(x)\varphi \).

For convenience, let \( \mu_1 = D_1 \lambda_1(\partial_1^2), \varphi_1 = \varphi_1(\partial_1^2) \) and \( \varphi_i = \varphi_i(\partial_1^2) \) \((i = 1, 2)\) for all \( x \in \Omega \). Naturally, we will consider whether \( \mu_1 \) and \( \mu_2 \) are equal. This is why we make the following three different assumptions for the intrinsic growth rate \( r \):

(H1): \( \mu_1 \neq \mu_2 \), \( |r - \mu_1| \ll 1 \) and \( r \neq D_2 \lambda_1(\partial_1^2) \) for each \( n \in \mathbb{N}_+ \triangleq \{1, 2, 3, \cdots\} \);

(H2): \( \mu_1 \neq \mu_2 \), \( |r - \mu_2| \ll 1 \) and \( r \neq D_1 \lambda_1(\partial_1^2) \) for each \( n \in \mathbb{N}_+ \);

(H3): \( \mu_1 = \mu_2 = \mu_*, |r - \mu_*| \ll 1 \).

Based on the above three assumptions, we have the following results.

**Lemma 2.2.** (i): Under the assumption (H1), the kernel of \( \mathcal{L}_{\mu_1} \) is given by

\[
\mathcal{K}_1 = \text{span}\{q_1\}, \quad \text{and the kernel of } \mathcal{L}^*_{\mu_1} \text{ is given by } \mathcal{K}^*_1 = \text{span}\{p_1\}, \quad \text{where } q_1 = (\varphi_1, 0)^T, p_1 = (\varphi_1, 0)^T;
\]

(ii): Under the assumption (H2), the kernel of \( \mathcal{L}_{\mu_2} \) is given by \( \mathcal{K}_2 = \text{span}\{q_2\}, \quad \text{and the kernel of } \mathcal{L}^*_{\mu_2} \text{ is given by } \mathcal{K}^*_2 = \text{span}\{p_2\}, \quad \text{where } q_2 = (0, \varphi_2)^T, p_2 = (0, \varphi_2)^T;
\]

(iii): Under the assumption (H3), the kernel of \( \mathcal{L}_{\mu_*} \) is given by \( \mathcal{K}_3 = \text{span}\{q_1, q_2\}, \quad \text{and the kernel of } \mathcal{L}^*_{\mu_*} \text{ is given by } \mathcal{K}^*_3 = \text{span}\{p_1, p_2\}, \quad \text{where } q_1, q_2, p_1, p_2 \) are given in (i) and (ii).

2.1. \( \mu_1 \neq \mu_2 \), \( |r - \mu_1| \ll 1 \) and \( r \neq D_2 \lambda_1(\partial_1^2) \) for each \( n \in \mathbb{N}_+ \). As mentioned earlier, we consider \( r \) as a bifurcation parameter. Our purpose is to find the non-zero solutions of the nonlinear functional equation \( F(U, r) = 0 \) with \( U \) close to \( \mathbf{0} \), and \( r \) close to \( \mu_1 \) in \( \mathbb{R} \setminus \{ D_2 \lambda_1(\partial_1^2) \} n \in \mathbb{N}_+ \). We now perform a Lyapunov-Schmidt reduction to obtain finite-dimensional bifurcation. Firstly, we have the following spatial decompositions:

\[
X^2 = \mathcal{K}_1 \oplus X_1, \quad \mathcal{Y}^2 = \text{Ran} \mathcal{L}_{\mu_1} \oplus \mathcal{K}^*_1,
\]

where \( X_1 = \{ y \in X^2 | (p_1, y) = 0 \} \). Obviously, the operator \( \mathcal{L}_{\mu_1} : X^2 \to \mathcal{Y}^2 \) is Fredholm with index zero, and \( \mathcal{L}_{\mu_1}|_{X_1} : X_1 \to \text{Ran} \mathcal{L}_{\mu_1} \) is invertible and has a bounded inverse.
Let $\mathcal{P}_1$ and $\mathcal{I} - \mathcal{P}_1$ denote the projection operators from $\mathbb{Y}^2$ to $\text{Ran}\mathcal{L}_{\mu_1}$ and $\mathbb{K}_1^*$, respectively. Then, $\mathcal{P}_1 F = F - p_1(p_1, F)$ for all $F = (F_1, F_2)^T \in \mathbb{Y}^2$, that is,

\[
\mathcal{P}_1 F = \begin{pmatrix} F_1 - \tilde{\phi}_1 \int_{\Omega} \tilde{\phi}_1 F_1 \, dx \\ F_2 \end{pmatrix}.
\]

According to the direct sum decomposition of the space $\mathbb{X}^2$, we have $U = zq_1 + w$ for each $U \in \mathbb{X}^2$, where $z = \langle p_1, U \rangle$ and $w = U - zq_1$. Thus, $F(U, r) = 0$ is equivalent to the following system:

\[
\mathcal{P}_1 F(zq_1 + w, r) = 0, \quad (\mathcal{I} - \mathcal{P}_1) F(zq_1 + w, r) = 0,
\]

where $z \in \mathbb{R}$ and $w = (w_1, w_2)^T \in \mathbb{X}_1$. Thus, the first equation of (6) can be rewritten as

\[
T(z, w, r) \triangleq \mathcal{P}_1 F(zq_1 + w, r) = 0
\]

Clearly,

\[
T(z, w, r) = \begin{bmatrix} T_1(z, w_1, w_2, r) \\ T_2(z, w_1, w_2, r) \end{bmatrix}
\]

satisfies

\[
T_1(0, 0, w_2, r) = 0, \quad T_2(z, w_1, 0, r) = 0,
\]

for all $z \in \mathbb{R}, w = (w_1, w_2)^T \in \mathbb{X}_1, r \in \mathbb{R}$, where

\[
T_1(z, w_1, w_2, r) = F_1(zq_1 + w, r) - \phi_1 \int_{\Omega} \tilde{\phi}_1 F_1(zq_1 + w, r) \, dx,
\]

\[
T_2(z, w_1, w_2, r) = F_2(zq_1 + w, r).
\]

And the Jacobian matrix of $T_1$ (respectively, $T_2$) with respect to $w_1$ (respectively, $w_2$) evaluated at $(0, 0, 0, \mu_1)$ is non-invertible (respectively, invertible). Applying the implicit function theorem, we obtain an open neighborhood $N$ of $0$ in $\mathbb{R}$, an open neighborhood $\Lambda$ of $\mu_1$ in $\mathbb{R}$ and a continuously differentiable mapping $w(z, r) = (w_1(z, r), w_2(z, r))^T : N \times \Lambda \rightarrow \mathbb{X}_1$ such that $w_1(0, \mu_1) = 0$, $w_2(z, r) \equiv 0$, and

\[
\mathcal{P}_1 F(zq_1 + w(z, r), r) = 0
\]

for all $(z, r) \in N \times \Lambda$. Substituting $w = w(z, r)$ into the second equation of (6), we have

\[
\mathcal{B}_1(z, r) \triangleq (\mathcal{I} - \mathcal{P}_1) F(zq_1 + w(z, r), r) = 0
\]

(7)

Thus, we reduce the original bifurcation problem to the problem of finding zeros of the mapping $\mathcal{B}_1 : N \times \Lambda \rightarrow \mathbb{K}_1^*$. We refer to $\mathcal{B}_1$ as the bifurcation map of system (3). Furthermore, $\mathcal{B}_1(0, r) = 0$ and $\frac{\partial \mathcal{B}_1}{\partial z}(0, \mu_1) = 0$. Therefore, each solution to $\mathcal{B}_1(z, r) = 0$ in $N \times \Lambda$ one-to-one corresponds to some solution to $F(U, r) = 0$.

Next, let us solve and study $\mathcal{B}_1(z, r) = 0$. Calculating the inner product of (7) with $p_1 = (\tilde{\phi}_1, 0)^T$, we have

\[
f(z, r) \triangleq < p_1, \mathcal{B}_1(z, r) >
\]

\[
= < p_1, (\mathcal{I} - \mathcal{P}_1) F(zq_1 + w(z, r), r) >
\]

\[
= < p_1, F(zq_1 + w(z, r), r) > - < p_1, \mathcal{P}_1 F(zq_1 + w(z, r), r) >
\]

\[
= < p_1, F(zq_1 + w(z, r), r) >
\]

\[
= \int_{\Omega} \tilde{\phi}_1(x) F_1(zq_1 + w(z, r), r) \, dx.
\]
Obviously, \( f(0, r) = 0, f_z(0, r) = r - \mu_1 \) for all \( r \in \mathbb{R} \). It follows that \( f : \mathbb{R}^2 \to \mathbb{R} \) takes the form
\[
f(z, r) = zg(z, r)
\]
where
\[
g(z, r) = r - \mu_1 - r\kappa_1z - \frac{1}{2}r^2\kappa_2z^2 + o(z^2)
\]
and
\[
\kappa_1 = \int_{\Omega} \tilde{\phi}_1\phi_1^2(x)dx > 0,
\]
\[
\kappa_2 = \int_{\Omega} \tilde{\phi}_1\phi_1(x)(2\frac{\partial^2 w_1}{\partial z^2} + c\frac{\partial^2 w_2}{\partial z^2})dx.
\]
Next, we need to give the concrete algorithm of \( \partial^2_w \) takes the form
\[
\Phi = (\frac{\partial^2 w_1}{\partial z^2}, \frac{\partial^2 w_2}{\partial z^2})^T.
\]
For each \( u = (u_1, u_2)^T \) and \( v = (v_1, v_2)^T \in K_1 \), we denote
\[
F_{\mu_1}(u, v) = \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} F(t_1u + t_2v, \mu_1)|_{t_1 = t_2 = 0}
\]
\[
= -\mu_1 \left( \begin{array}{c} u_1(v_1 + cv_2) + v_1(u_1 + cu_2) \\ u_2(v_2 + bv_1) + v_2(u_2 + bu_1) \end{array} \right). \]

It follows from (6) that
\[
\mathcal{P}_1F_{\mu_1}^2(q_1, q_1) + \mathcal{L}_{\mu_1} \frac{\partial^2 w}{\partial z^2} = 0,
\]
and hence that
\[
\frac{\partial^2 w}{\partial z^2} = -\mathcal{L}_{-1} \mathcal{P}_1F_{\mu_1}^2(q_1, q_1).
\]
Clearly, \( g(0, \mu_1) = 0 \) and \( g_z(0, \mu_1) = -\mu_1 \kappa_1 \neq 0 \). By using the implicit function theorem we know that there exist a constant \( \delta_1 > 0 \) satisfying \( \delta_1 < \min\{|\mu_1 - D_2\lambda_n(\frac{\partial^2}{\partial z^2})|n \in \mathbb{N}_+\} \) and a continuously differentiable mapping \( z : \Lambda \rightarrow (\mu_1 - \delta_1, \mu_1 + \delta_1) \) \( \to \mathbb{R} \) such that \( g(z_r, r) \equiv 0 \) for all \( r \in \Lambda \). In fact, due to \( \kappa_1 \neq 0 \) we have
\[
z_r = \frac{r - \mu_1}{\kappa_1} + o(|r - \mu_1|).
\]
Thus, we have the following result:

**Theorem 2.3.** Under the assumption \((H1)\), there exist a constant \( \delta_1 > 0 \) satisfying \( \delta_1 < \min\{|\mu_1 - D_2\lambda_n(\frac{\partial^2}{\partial z^2})|n \in \mathbb{N}_+\} \) and a continuously differentiable mapping \( r \to z_r \) from \( (\mu_1 - \delta_1, \mu_1 + \delta_1) \) \( \to \mathbb{R} \) such that Eq.(3) has a semi-trivial steady-state solution of the form
\[
U_r = (u_r, v_r)^T = z_rq_1 + w(z_r, r),
\]
with
\[
u_r = z_r\phi_1 + w_1(z_r, r), \ v_r = 0
\]
existing for \( r \in \Lambda \triangleq (\mu_1 - \delta_1, \mu_1) \cup (\mu_1, \mu_1 + \delta_1) \) and satisfying
\[
\lim_{r \to \mu_1} U_r = 0.
\]

Note that \( \frac{\partial}{\partial r} \) is the principal eigenvalue of the one-dimensional eigenvalue problem (4) with \( q(x) = \frac{\delta_1(x)}{\delta_1} \), and the associated eigenfunction is \( \phi_1 > 0 \) on \( \Omega \), then \( u_r = z_r\phi_1 + w_1(z_r, r) \) given by Theorem 2.3 is positive (respectively, negative) if \( r \in (\mu_1, \mu_1 + \delta_1) \) (respectively, \( r \in (\mu_1 - \delta_1, \mu_1) \)).
2.2. $\mu_1 \neq \mu_2$, $|r - \mu_2| \ll 1$ and $r \neq D_1\lambda_n(\frac{\beta_1}{D_1})$ for each $n \in \mathbb{N}_+$. In this section, our purpose is to find the non-zero solutions of the nonlinear functional equation $F(U, r) = 0$ with $U$ close to $0$, and $r$ close to $\mu_2$ in $\mathbb{R}\setminus\{D_1\lambda_n(\frac{\beta_1}{D_1}) | n \in \mathbb{N}_+\}$. We now perform a Lyapunov-Schmidt reduction to obtain finite-dimensional bifurcation. Firstly, we have the following decompositions:

$$\mathbb{X}^2 = \mathbb{K}_2 \oplus \mathbb{X}_2, \quad \mathbb{Y}^2 = \text{Ran} \mathcal{L}_{\mu_2} \oplus \mathbb{K}_2^\ast,$$

where $\mathbb{X}_2 = \{y \in \mathbb{X}^2 | (p_2, y) = 0\}$. Obviously, the operator $\mathcal{L}_{\mu_2} : \mathbb{X}^2 \to \mathbb{Y}^2$ is Fredholm with index zero. $\mathcal{L}_{\mu_2}|_{\mathbb{X}_2} : \mathbb{X}_2 \to \text{Ran} \mathcal{L}_{\mu_2}$ is invertible and has a bounded inverse.

Let $\mathcal{P}_2$ and $\mathcal{I} - \mathcal{P}_2$ denote the projection operators $\mathbb{Y}^2$ to $\text{Ran} \mathcal{L}_{\mu_2}$ and $\mathbb{K}_2^\ast$, respectively. Using a similar argument to the previous subsection, we can obtain an open neighborhood $N$ of 0 in $\mathbb{R}$, an open neighborhood $\Lambda$ of $\mu_2$ in $\mathbb{R}$ and a continuously differentiable mapping $w(z, r) = (w_1(z, r), w_2(z, r))^T : N \times \Lambda \to \mathbb{X}_2$ such that $w_1(z, r) \equiv 0$, $w_2(0, \mu_2) = 0$, and

$$\mathcal{P}_2 F(zq_2 + w(z, r), r) = 0$$

for all $(z, r) \in N \times \Lambda$. Thus, we reduce the original bifurcation problem to the problem of finding zeros of the mapping $\mathfrak{B}_2 : N \times \Lambda \to \mathbb{K}_2^\ast$ defined by

$$\mathfrak{B}_2(z, r) = (\mathcal{I} - \mathcal{P}_2) F(zq_2 + w(z, r), r) \quad (16)$$

Calculating the inner product of (16) with $p_2 = (0, \tilde{\phi}_2)^T$, we have

$$f(z, r) \triangleq < p_2, \mathfrak{B}_2(z, r) > = < p_2, (\mathcal{I} - \mathcal{P}_2) F(zq_2 + w(z, r), r) > = < p_2, F(zq_2 + w(z, r), r) > - < p_2, \mathcal{P}_2 F(zq_2 + w(z, r), r) >$$

$$= < p_2, F(zq_2 + w(z, r), r) > = \int_\Omega \tilde{\phi}_2(x) F(zq_2 + w(z, r), r) dx$$

Obviously, $f(0, r) = 0, f_z(0, r) = r - \mu_2$ for all $r \in \mathbb{R}$. It follows that $f : \mathbb{R}^2 \to \mathbb{R}$ takes the form

$$f(z, r) = zg(z, r) \quad (17)$$

where

$$g(z, r) = r - \mu_2 - r\hat{\kappa}_1 z - \frac{1}{2} r^2 \hat{\kappa}_2 z^2 + o(z^2) \quad (18)$$

and

$$\hat{\kappa}_1 = \int_\Omega \tilde{\phi}_2 \phi_2^2(x) dx > 0, \quad \hat{\kappa}_2 = \int_\Omega \tilde{\phi}_2 \phi_2(x) (2 \frac{\partial^2 w_2}{\partial z^2} + b \frac{\partial^2 w_1}{\partial z^2}) dx \quad (19)$$

where

$$\frac{\partial^2 w}{\partial z^2} = -\mathcal{L}_{\mu_2}^{-1} \mathcal{P}_2 F_{\mu_2}^\ast (q_2, q_2). \quad (20)$$

Clearly, $g(0, \mu_2) = 0$ and $g_z(0, \mu_2) = -r\hat{\kappa}_1 \neq 0$. By using the implicit function theorem we know that there exist a constant $\delta_2 > 0$ satisfying $\delta_2 < \min\{|\mu_2 - D_1\lambda_n(\frac{\beta_1}{D_1})| | n \in \mathbb{N}_+\}$ and a continuously differential mapping $z : \Lambda \triangleq (\mu_2 - \delta_2, \mu_2 + \delta_2) \to \mathbb{R}$ such that $g(z, r) \equiv 0$ for $r \in \Lambda$. In fact, due to $\hat{\kappa}_1 \neq 0$ we have

$$z_r = \frac{r - \mu_2}{r \hat{\kappa}_1} + o(|r - \mu_2|). \quad (21)$$
Thus, we have the following result:

**Theorem 2.4.** Under the assumption (H2), then there exist a constant $\delta_2 > 0$ satisfying $\delta_2 < \min\{|\mu_2 - D_1 \lambda_n(\frac{\beta}{D_1})| | n \in \mathbb{N}_+\}$ and a continuously differentiable mapping $r \rightarrow z_r$ from $(\mu_2 - \delta_2, \mu_2 + \delta_2)$ to $\mathbb{R}$ such that Eq.(3) has a semi-trivial steady-state solution of the form

$$U_r = (u_r, v_r)^T = z_r q_2 + w(z_r, r),$$

with

$$u_r = 0, \quad v_r = z_r \phi_2 + w_2(z_r, r)$$

existing for $r \in \Lambda \triangleq (\mu_2 - \delta_2, \mu_2) \cup (\mu_2, \mu_2 + \delta_2)$ and satisfying

$$\lim_{r \to \mu_2} U_r = 0$$

Note that $\frac{\partial x}{\partial z}$ is the principal eigenvalue of the one-dimensional eigenvalue problem (4) with $q(x) = \frac{\beta(x)}{D_2}$, and the associated eigenfunction is $\phi_2 > 0$ on $\Omega$, then $v_r = z_r \phi_2 + w_2(z_r, r)$ given by Theorem 2.4 is positive (respectively, negative) if $r \in (\mu_2, \mu_2 + \delta_2)$ (respectively, $r \in (\mu_2 - \delta_2, \mu_2)$).

2.3. $\mu_1 = \mu_2 = \mu_*$, $|r - \mu_*| \ll 1$. In this case, we also have the following decompositions:

$$X^2 = K_3 \oplus X_3, \quad Y^2 = \text{Ran}\mathcal{L}_{\mu_*} \oplus X_3,$$

where $X_3 = \{y \in X^2 | \langle p_i, y \rangle = 0, \ i = 1, 2\}$. Obviously, the operator $\mathcal{L}_{\mu_*} : X^2 \to Y^2$ is Fredholm with index zero. $\mathcal{L}_{\mu_*}, \text{Ran}\mathcal{L}_{\mu_*}$ are invertible and has a bounded inverse.

Let $\mathcal{P}_3$ and $\mathcal{I} - \mathcal{P}_3$ the projection operators $Y^2$ to $\text{Ran}\mathcal{L}_{\mu_*}$ and $X_3$, respectively. Then, $\mathcal{P}_3 F = F - p_1(p_1, F) - p_2(p_2, F)$ for $F = (F_1, F_2)^T \in Y^2$, that is,

$$\mathcal{P}_3 F = \begin{pmatrix} F_1 - \tilde{\phi}_1 \int_{\Omega} \tilde{\phi}_1 F_1 dx \\ F_2 - \tilde{\phi}_2 \int_{\Omega} \tilde{\phi}_2 F_2 dx \end{pmatrix}.$$  

According to the direct sum decomposition of the space $X^2$, we have $U = z_1 q_1 + z_2 q_2 + w$ for $U \in X^2$, where $z_1 = \langle p_1, U \rangle$, $z_2 = \langle p_2, U \rangle$ and $w = U - z_1 q_1 - z_2 q_2$. Thus, $F(U, r) = 0$ is equivalent to the following system:

$$\mathcal{P}_3 F(U, r) = 0, \quad (\mathcal{I} - \mathcal{P}_3) F(U, r) = 0,$$

(23)

Then, the first equation of (23) can be rewritten as

$$T(z_1, z_2, w_1, w_2, r) \triangleq \mathcal{P}_3 F(z_1 q_1 + z_2 q_2 + w, r) = 0,$$

and

$$T(z_1, z_2, w_1, w_2, r) = \begin{bmatrix} T_1(z_1, z_2, w_1, w_2, r) \\ T_2(z_1, z_2, w_1, w_2, r) \end{bmatrix} = \begin{bmatrix} F_1 - \tilde{\phi}_1 \int_{\Omega} \tilde{\phi}_1 F_1 dx \\ F_2 - \tilde{\phi}_2 \int_{\Omega} \tilde{\phi}_2 F_2 dx \end{bmatrix}$$

satisfies $T_1(0, z_2, 0, w_1, w_2, r) = 0, T_2(z_1, 0, w_1, 0, r) = 0$ for all $(z_1, z_2)^T \in \mathbb{R}^2, (w_1, w_2)^T \in X_3$ and $T_2(w(0, 0, \mu_*), \Lambda) = \mathcal{P}_3 \mathcal{L}_{\mu_*} = \mathcal{L}_{\mu_*}$. Applying the implicit function theorem, we obtain an open neighborhood $\tilde{N}$ of $(z_1, z_2)^T = (0, 0)^T$ in $\mathbb{R}^2$, an open neighborhood $\Lambda$ of $\mu_*$ in $\mathbb{R}$ and a continuously differentiable mapping:

$$w(z_1, z_2, r) = (w_1(z_1, z_2, r), w_2(z_1, z_2, r))^T : \tilde{N} \times \Lambda \to X_3$$

such that $w_1(0, z_2, r) = 0$, and $w_2(z_1, 0, r) = 0$ for all $z_1, z_2 \in \mathbb{R}$, and

$$\mathcal{P}_3 F(z_1 q_1 + z_2 q_2 + w(z_1, z_2, r), r) = 0.$$
for all \((z_1, z_2, r) \in \hat{N} \times \Lambda\). Substituting \(w = w(z_1, z_2, r)\) into the second equation of (23), we have

\[
\mathcal{B}_3(z_1, z_2, r) \overset{\equiv}{=} (I - \mathcal{P}_3)F(z_1q_1 + z_2q_2 + w(z_1, z_2, r), r) = 0
\]  
(24)

Thus, we reduce the original bifurcation problem to the problem of finding zeros of the mapping \(\mathcal{B}_3 : \hat{N} \times \Lambda \rightarrow \mathbb{K}_3\). Obviously, \(\mathcal{B}_3(0, 0, r) = 0\) and \(\frac{\partial \mathcal{B}_3}{\partial z_1}(0, 0, \mu_*) = 0\), and \(\frac{\partial \mathcal{B}_3}{\partial z_2}(0, 0, \mu_*) = 0\).

Next, let us solve and study \(\mathcal{B}_3(z_1, z_2, r) = 0\). Calculating the inner product of (24) with \(p_1\) and \(p_2\), we have \(f(z_1, z_2, r) = 0\), where \(f = (f_1, f_2)^T\) is given by

\[
f_1(z_1, z_2, r) = \int_{\Omega} \tilde{\phi}_1 F_1(z_1q_1(x) + z_2q_2(x) + w(z_1, z_2, r), r)dx
\]

and

\[
f_2(z_1, z_2, r) = \int_{\Omega} \tilde{\phi}_2 F_2(z_1q_1(x) + z_2q_2(x) + w(z_1, z_2, r), r)dx
\]

Clearly, \(f_1(0, z_2, r) = f_2(z_1, 0, r) = 0\) for all \(z_1, z_2 \in \mathbb{R}\), and

\[
\left. \frac{\partial (f_1, f_2)}{\partial (z_1, z_2)} \right|_{(z_1, z_2) = (0, 0)} = \begin{pmatrix} r - \mu_* & 0 \\ 0 & r - \mu_* \end{pmatrix}
\]

Thus, we can rewrite \(f_j(z_1, z_2, r)\) as

\[f_j(z_1, z_2, r) = z_j g_j(z_1, z_2, r), \quad j = 1, 2,\]

where \(g_j(z_1, z_2, r)\) are polynomials in their arguments, and take the form

\[
g_1(z_1, z_2, r) = r - \mu_* - r(\kappa_{11}z_1 + \kappa_{12}z_2) - \frac{r}{2} [l_{11}^1 z_1^2 + l_{11}^2 z_2^2 + l_{12}^2 z_1 z_2 + \cdots],
\]

\[
g_2(z_1, z_2, r) = r - \mu_* - r(\kappa_{21}z_1 + \kappa_{22}z_2) - \frac{r}{2} [l_{21}^2 z_1^2 + l_{22}^2 z_2^2 + l_{12}^2 z_1 z_2 + \cdots],
\]

where

\[
\kappa_{11} = \int_{\Omega} \tilde{\phi}_1 \phi_1^2 dx > 0,
\]

\[
\kappa_{12} = c \int_{\Omega} \tilde{\phi}_1 \phi_1 \phi_2 dx > 0,
\]

\[
\kappa_{21} = b \int_{\Omega} \tilde{\phi}_2 \phi_1 \phi_2 dx > 0,
\]

\[
\kappa_{22} = \int_{\Omega} \tilde{\phi}_2 \phi_2^2 dx > 0,
\]

and

\[
l_{20}^1 = 2 \int_{\Omega} \tilde{\phi}_1 \phi_1 \frac{\partial^2 w_1}{\partial z_1^2} dx,
\]

\[
l_{11}^1 = c \int_{\Omega} \tilde{\phi}_1 \phi_1 \frac{\partial^2 w_1}{\partial z_1^2} dx + \int_{\Omega} \tilde{\phi}_1 \phi_1 [4 \frac{\partial^2 w_1}{\partial z_1 \partial z_2} + 2c \frac{\partial^2 w_2}{\partial z_1 \partial z_2}] dx,
\]

\[
l_{10}^2 = 2c \int_{\Omega} \tilde{\phi}_1 \phi_2 \frac{\partial^2 w_1}{\partial z_1 \partial z_2} dx + c \int_{\Omega} \tilde{\phi}_1 \phi_1 \frac{\partial^2 w_2}{\partial z_2^2} dx,
\]

\[
l_{20}^2 = 2b \int_{\Omega} \tilde{\phi}_2 \phi_1 \frac{\partial^2 w_2}{\partial z_1 \partial z_2} dx + b \int_{\Omega} \tilde{\phi}_2 \phi_2 \frac{\partial^2 w_1}{\partial z_1^2} dx,
\]

\[
l_{11}^1 = b \int_{\Omega} \tilde{\phi}_2 \phi_1 \frac{\partial^2 w_2}{\partial z_2^2} + \int_{\Omega} \tilde{\phi}_2 \phi_2 [4 \frac{\partial^2 w_2}{\partial z_1 \partial z_2} + 2b \frac{\partial^2 w_1}{\partial z_1 \partial z_2}] dx,
\]
\[
\frac{d^2}{dx^2} = 2 \int_\Omega \frac{\partial^2 w_2}{\partial z_2^2} dx.
\]

Next, we need to give the concrete algorithm of \(\frac{\partial^2 w}{\partial z_1^2}, \frac{\partial^2 w}{\partial z_2^2}\) and \(\frac{\partial^2 w}{\partial z_2^2}\).

For \(u = (u_1, u_2)^T\) and \(v = (v_1, v_2)^T \in \mathbb{K}_3\), let
\[
\mathcal{F}_{\mu_*}(u, v) = -\mu_* \left( u_1 (v_1 + c v_2) + v_1 (u_1 + c u_2) \right),
\]

It follows from (23) that
\[
\mathcal{P}_3 \mathcal{F}_{\mu_*}(q_1, q_1) + \mathcal{L}_{\mu_*} \frac{\partial^2 w}{\partial z_1^2} = 0,
\]
\[
\mathcal{P}_3 \mathcal{F}_{\mu_*}(q_1, q_2) + \mathcal{L}_{\mu_*} \frac{\partial^2 w}{\partial z_1 \partial z_2} = 0,
\]
\[
\mathcal{P}_3 \mathcal{F}_{\mu_*}(q_2, q_2) + \mathcal{L}_{\mu_*} \frac{\partial^2 w}{\partial z_2^2} = 0,
\]

and hence that
\[
\frac{\partial^2 w}{\partial z_1^2} = -\mathcal{L}_{\mu_*}^{-1} \mathcal{P}_3 \mathcal{F}_{\mu_*}(q_1, q_1),
\]
\[
\frac{\partial^2 w}{\partial z_1 \partial z_2} = -\mathcal{L}_{\mu_*}^{-1} \mathcal{P}_3 \mathcal{F}_{\mu_*}(q_1, q_2),
\]
\[
\frac{\partial^2 w}{\partial z_2^2} = -\mathcal{L}_{\mu_*}^{-1} \mathcal{P}_3 \mathcal{F}_{\mu_*}(q_2, q_2).
\]

Our purpose is to find zero points \((z_1, z_2)\) of \(f(\cdot, \cdot, r)\) in the neighborhood of \((z_1, z_2, r) = (0, 0, \mu_*).\) Firstly, we discuss the existence of non-boundary zero points \((z_1, z_2)\) of \(f(\cdot, \cdot, r),\) i.e., \(z_1 z_2 \neq 0\) satisfying \(g_j(z_1, z_2, r) = 0, j = 1, 2.\) It is easy to see that \(g_j(z_1, z_2, \mu_*) = -\mu_* \kappa,\) where
\[
\kappa = \begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix}.
\]

Clearly, \(g_j(z_1, z_2, \mu_*) = -\mu_* \kappa\) invertible if and only if \(\kappa_{11} \kappa_{22} \neq \kappa_{12} \kappa_{21}.\) In what follows, we distinguish two cases to discuss the zeros of \(g(\cdot, \cdot, r).\)

2.3.1. \(\kappa_{11} \kappa_{22} \neq \kappa_{12} \kappa_{21}.\) If \(\kappa_{11} \kappa_{22} \neq \kappa_{12} \kappa_{21}\) then applying the implicit function theorem, there exist a positive constant \(\delta_3 > 0\) and a continuously differentiable mapping \(z_r = (z_1r, z_2r)^T\) from \((\mu_* - \delta_3, \mu_*) + \delta_3)\) to \(\mathbb{R}^2\) such that
\[
g_j(z_1r, z_2r, r) = 0
\]
for \(r \in (\mu_* - \delta_3, \mu_*) + \delta_3.\) In addition, we have
\[
z_{1r} = \frac{(r - \mu_*) d_1}{r(\kappa_{11} \kappa_{22} - \kappa_{12} \kappa_{21})} + o(|r - \mu_*|),
\]
\[
z_{2r} = \frac{(r - \mu_*) d_2}{r(\kappa_{11} \kappa_{22} - \kappa_{12} \kappa_{21})} + o(|r - \mu_*|),
\]
where
\[
d_1 = \kappa_{22} - \kappa_{12}, d_2 = \kappa_{11} - \kappa_{21}.
\]

Hence, we have the following result:
Theorem 2.5. Under the assumption (H3) if $\kappa_{11}\kappa_{22} \neq \kappa_{12}\kappa_{21}$ then there exist a constant $\delta_4 > 0$ and a continuously differentiable mapping $r \rightarrow (z_{1r}, z_{2r})^T$ from $(\mu_* - \delta_3, \mu_* + \delta_3)$ to $\mathbb{R}^2$ such that Eq. (3) has a nontrivial steady-state solution of the form

$$U_r = (u_r, v_r)^T,$$

with

$$\begin{cases} u_r = z_{1r}\phi_1 + w_1(z_{1r}, z_{2r}, r), \\ v_r = z_{2r}\phi_2 + w_2(z_{1r}, z_{2r}, r), \end{cases}$$

which exists for $r \in \Lambda \triangleq (\mu_* - \delta_3, \mu_*) \cup (\mu_* + \delta_3)$ and satisfies

$$\lim_{r \rightarrow \mu_*} U_r = 0$$

Furthermore, this nontrivial steady-state solution $U_r = (u_r, v_r)^T$ is positive if and only if $d_1d_2 > 0$ and $r \in (\mu_*, \mu_* + \delta_3)$.

2.3.2. $\kappa_{11}\kappa_{22} = \kappa_{12}\kappa_{21}$. If $\kappa_{11}\kappa_{22} = \kappa_{12}\kappa_{21}$, i.e., $\kappa$ is a singular matrix, then there exist non-zero vectors $\theta_j = (\theta_{j1}, \theta_{j2})^T \in \mathbb{R}^2$ and $\theta_j^* = (\theta_{j1}^*, \theta_{j2}^*) \in \mathbb{R}^2, j = 0, 1, 1$ such that $\kappa\theta_0 = 0, \theta_0^* = 0$, and $\theta_j^*\theta_s = \delta_{js} \neq 0, j, s = 0, 1$, where $\mathbb{R}^2$ is the Euclidean space of 2-dimensional row vector. Apparently, we have

$$\theta_j^*g(a_1\theta_0 + a_2\theta_1, r) = 0, \quad \theta_j^*g(a_1\theta_0 + a_2\theta_1, r) = 0$$

where $a_1, a_2 \in \mathbb{R}, z = a_1\theta_0 + a_2\theta_1$. Notice that $\theta_1^*g(0, \mu_*) = 0$ and $\theta_1^*g_2(0, \mu_*) = -\mu_1\theta_1^*\theta_1 \neq 0$. From the implicit function theorem, there are a positive constant $\delta_4 > 0$ and a continuously differentiable mapping $a_2 : (-\delta_4, \delta_4) \times (\mu_* - \delta_4, \mu_* + \delta_4) \rightarrow \mathbb{R}$ such that $a_2(0, \mu_*) = 0$ and $\theta_1^*g(a_1\theta_0 + a_2(a_1, r)\theta_1, r) = 0$ for all $(a_1, r) \in (-\delta_4, \delta_4) \times (\mu_* - \delta_4, \mu_* + \delta_4)$. Substituting $a_2 = a_2(a_1, r)$ into the first equation of (32), we obtain $G(a_1, r) = 0$, where

$$G(a_1, r) = \theta_0^*g(a_1\theta_0 + a_2(a_1, r)\theta_1, r)$$

$$(\theta_0^* + \theta_2^*)(r - \mu_*) - \frac{1}{2} r(a_1\theta_0 + a_2(a_1, r)\theta_1)^T M(a_1\theta_0 + a_2(a_1, r)\theta_1)$$

and

$$M = \theta_0^* \left( \begin{array}{cc} \frac{1}{2} \ell_1^2 & \frac{1}{2} \ell_1^1 \\ \ell_1^1 & \ell_1^2 \end{array} \right) + \theta_2^* \left( \begin{array}{cc} \frac{1}{2} \ell_2^2 & \frac{1}{2} \ell_2^1 \\ \ell_2^1 & \ell_2^2 \end{array} \right).$$

Obviously, in the neighborhood of $(a_1, r) = (0, \mu_*), G(a_1, r)$ has no zero if

$$(\theta_0^* + \theta_2^*)(r - \mu_*)\theta_0^T M\theta_0 < 0.$$
Note that \( \kappa \theta_0 = 0 \) and \( r, \kappa_{ij} (i, j = 1, 2) \) are all positive, then \( \theta_0, \theta_0^2 < 0 \) and the two components of \( z^+ \) (and \( z^- \)) have different signs, i.e., the two components of \( U^r_+ \) (and \( U^r_- \)) have different signs for sufficiently small \( |r - \mu| \). These steady states are meaningless in biology, and so this situation will not be discussed further.

Finally, \( f(\cdot, \cdot, r) \) may have boundary (also called semi-trivial) zero points \((z_1, 0)^T\) or \((0, z_2)^T\) with \( z_1 z_2 \neq 0 \) satisfying \( f_1(z_1, 0, r) = 0 \) and \( f_2(0, z_2, r) = 0 \), respectively, that is, \( g_1(z_1, 0, r) = 0 \) and \( g_2(0, z_2, r) = 0 \). Due to \(-\kappa \mu_1 < 0\), applying the implicit function theorem, \( g_1(\cdot, 0, r) = 0 \) has a zero \( z_1 \) depending on \( r \), which tends to \( 0 \) as \( r \to \mu_1 \) and is positive (respectively, negative) if \( r > \mu_1 \) (respectively, \( r < \mu_1 \)). Similarly, due to \(-\kappa \mu_2 < 0\), applying the implicit function theorem, \( g_2(\cdot, 0, r) = 0 \) has a zero \( z_2 \) depending on \( r \), which tends to \( 0 \) as \( r \to \mu_2 \) and is positive (respectively, negative) if \( r > \mu_2 \) (respectively, \( r < \mu_2 \)).

Nontrivial zero points \((z_{1r}, 0)^T\) and \((0, z_{2r})^T\) of \( f(\cdot, \cdot, r) \) correspond to boundary (also called semi-trivial) solutions \((u, 0)^T\) and \((0, v)^T\) of system (3), respectively, where \( u = z_{1r} \phi_1 + w_1(z_{1r}, 0, r) \) and \( v = z_{2r} \phi_2 + w_2(0, z_{2r}, r) \) satisfy

\[
\begin{align*}
D_1 u_{xx} - \beta_1(x) u_x + ru(1 - u) &= 0, & t > 0, L_1 < x < L_2, \\
u(t, L_1) &= u(t, L_2) = 0, & t > 0.
\end{align*}
\]

and

\[
\begin{align*}
D_2 v_{xx} - \beta_2(x) v_x + rv(1 - v) &= 0, & t > 0, L_1 < x < L_2, \\
v(t, L_1) &= v(t, L_2) = 0, & t > 0,
\end{align*}
\]

respectively.

**Theorem 2.6.** (i): There exist a constant \( \delta_5 > 0 \) and a continuously differentiable mapping \( r \to z_{1r} \) from \((\mu_1 - \delta_5, \mu_1 + \delta_5)\) to \( \mathbb{R} \) such that (2) with \( 0 < |r - \mu| < \delta_5 \) has a non-constant semi-trivial steady-state solution \((u_r, 0)^T\), with \( u_r \) given by \( u_r = z_{1r} \phi_1 + w_1(z_{1r}, 0, r) \), which exists for \( r \in \Lambda \triangleq (\mu_1 - \delta_5, \mu_1 + \delta_5) \), and satisfies

\[
\lim_{r \to \mu_1} u_r = 0.
\]

In addition, \( u_r \) is positive (respectively, negative) if \( r > \mu_1 \) (respectively, \( r < \mu_1 \)).

(ii): There exist a constant \( \delta_6 > 0 \) and a continuously differentiable mapping \( r \to z_{2r} \) from \((\mu_2 - \delta_6, \mu_2 + \delta_6)\) to \( \mathbb{R} \) such that (2) with \( 0 < |r - \mu_2| < \delta_6 \) has a non-constant semi-trivial steady-state solution \((0, v_r)^T\), with \( v_r \) given by \( v_r = z_{2r} \phi_2 + w_2(0, z_{2r}, r) \), which exists for \( r \in \Lambda \triangleq (\mu_2 - \delta_6, \mu_2) \cup (\mu_2, \mu_2 + \delta_6) \), and satisfies

\[
\lim_{r \to \mu_2} v_r = 0.
\]

In addition, \( v_r \) is positive (respectively, negative) if \( r > \mu_2 \) (respectively, \( r < \mu_2 \)).

**Remark 1.** From the previous discussion, we know that when we discuss the semi-trivial steady-state solutions \((u_r, 0)^T\) or \((0, v_r)^T\), we only need to consider (33) or (34) separately, so we do not need to consider whether \( \mu_1 \) and \( \mu_2 \) are equal to each other for a single equation.

3. **Eigenvalue problem.** From Theorems 2.3-2.6, there exists an open set \( \Lambda \) in \( \mathbb{R} \) with the bifurcation value on its boundary such that model (3) with \( r \in \Lambda \) has a spatially nonhomogeneous steady-state solution. The details are shown in Table 1.
Consider a small perturbation $\hat{U}(x,t) = U(x,t) - U_r(x)$. Substituting it into model (2) and dropping the hats for simplicity of notations, we obtain the following linearization of model (2) at $U_r(x)$:

$$
\begin{align*}
&u_t = D_1 u_{xx} - \beta_1(x) u_x + ru(1 - u_r - cv_r) - ru_r(u + cv), \quad t > 0, \ L_1 < x < L_2, \\
v_t = D_2 v_{xx} - \beta_2(x) v_x + rv(1 - v_r - bu_r) - rv_r(v + bu), \quad t > 0, \ L_1 < x < L_2, \\
u(t, L_1) = u(t, L_2) = 0, \quad t > 0, \\
v(t, L_1) = v(t, L_2) = 0, \quad t > 0.
\end{align*}
$$

(35)

Rewrite system (35) as the following abstract Cauchy problem:

$$
\frac{dU}{dt} = A_r U
$$

(36)

where $U = (u, v)^T \in \mathbb{X}^2$, and the linear operator $A_r : \mathbb{X}^2 \to \mathbb{Y}^2$ is defined as

$$
A_r \psi = \begin{pmatrix}
D_1 \psi_{xx}^1 - \beta_1(x) \psi_x^1 + r(1 - 2u_r - cv_r)\psi^1 - cru_r\psi^2 \\
- brv_r\psi^1 + D_2 \psi_{xx}^2 - \beta_2(x) \psi_x^2 + r(1 - 2v_r - bu_r)\psi^2
\end{pmatrix}
$$

for all $\psi = (\psi^1, \psi^2)^T \in \mathbb{X}^2$. Then $A_r$ is the infinitesimal generator of the semigroup induced by the solutions of (35), where the domain $\mathcal{D}(A_r) = \mathbb{X}^2_C$. And the spectral set of $A_r$ is

$$
\sigma(A_r) = \{ \lambda \in \mathbb{C} : \Delta(r, \lambda) \psi = 0 \text{ for some } \psi \in \mathbb{X}^2_C \backslash \{0\} \},
$$

where

$$
\Delta(r, \lambda) \psi = A_r \psi - \lambda \psi.
$$

(37)

For later applications, it is also useful to consider the adjoint operator $A_r^* \psi$ of $A_r$. The adjoint operator $A_r^*$ is defined by

$$
A_r^* \psi = \begin{pmatrix}
D_1 \psi_{xx}^1 + \beta_1(x) \psi_x^1 + \beta_1(x) \psi_x^1 + r(1 - 2u_r - cv_r)\psi^1 - brv_r\psi^2 \\
- cru_r\psi^1 + D_2 \psi_{xx}^2 + \beta_2(x) \psi_x^2 + \beta_2(x) \psi_x^2 + r(1 - 2v_r - bu_r)\psi^2
\end{pmatrix}
$$

The domain of $A_r^*$ is $\mathbb{X}^2_C$ and the spectral set of $A_r^*$ is

$$
\sigma(A_r^*) = \{ \lambda \in \mathbb{C} : \Delta^*(r, \lambda) \psi = 0 \text{ for some } \psi \in \mathbb{X}^2_C \backslash \{0\} \},
$$

where

$$
\Delta^*(r, \lambda) \psi = A_r^* \psi - \lambda \psi.
$$

(38)

Noticing that $\mathcal{D}(\Delta(r, \lambda)) = \mathbb{X}^2_C$, and $\mathbb{X}^2_C$ is dense in $\mathbb{Y}^2_C$, we have

$$
\langle \tilde{\psi}, \Delta(r, \lambda) \psi \rangle = \langle \Delta^*(r, \lambda) \tilde{\psi}, \psi \rangle
$$

(39)
for \( \psi, \tilde{\psi} \in \mathbb{C}_2 \). We know that \( \Delta^*(r, \lambda) \) is the adjoint operator of \( \Delta(r, \lambda) \), and its point spectrum is the same as that of \( \Delta(r, \lambda) \):
\[
\sigma(\Delta(r, \lambda)) = \sigma(\Delta^*(r, \lambda)).
\]
Namely, \( \lambda \in \sigma(A_r) \) if and only if \( \bar{\lambda} \in \sigma(A_*^r) \).

In view of Section 2, we will investigate the eigenvalues of \( A_r \) under each of the three assumptions (H1), (H2) and (H3). For this purpose, we first give the following lemma when the bifurcation parameter approaches the bifurcation point.

**Lemma 3.1.** (i): Under the assumption (H1),
\[
\langle U_r, p_1 \rangle = \int_{\Omega} u_r \tilde{\phi}_1 dx = \frac{\rho_{1r}}{\kappa_1} \quad \text{and} \quad \lim_{r \to \mu_1} \frac{U_r}{\rho_{1r}} = \frac{q_1}{\kappa_1},
\]
where \( \rho_{1r} = \frac{r - \mu_1}{r} + o(|r - \mu_1|) \).

(ii): Under the assumption (H2),
\[
\langle U_r, p_2 \rangle = \int_{\Omega} v_r \phi_2 dx = \frac{\rho_{2r}}{\kappa_1} \quad \text{and} \quad \lim_{r \to \mu_2} \frac{U_r}{\rho_{2r}} = \frac{q_2}{\kappa_1},
\]
where \( \rho_{2r} = \frac{r - \mu_2}{r} + o(|r - \mu_2|) \).

(iii): Under the assumption (H3),
\[
\langle U_r, p_1 \rangle = \int_{\Omega} u_r \tilde{\phi}_1 dx = \rho_r d_1, \quad \langle U_r, p_2 \rangle = \int_{\Omega} v_r \phi_2 dx = \rho_r d_2
\]
and
\[
\lim_{r \to \mu_*} \frac{U_r}{\rho_r} = d_1 q_1 + d_2 q_2
\]
where
\[
\rho_r = \frac{r - \mu_*}{r(\kappa_{11} \kappa_{22} - \kappa_{12} \kappa_{21})} + o(|r - \mu_*|).
\]

It is well known that eigenvalues of \( A_r \) with zero real parts play a key role in the analysis of stability and bifurcation of the steady-state solution. In what follows, we will investigate the existence of zero eigenvalue and purely imaginary eigenvalues of \( A_r \) associated with the nonhomogeneous steady-state solutions \( U_r \) established in Section 2 under each of the three assumptions (H1), (H2) and (H3).

3.1. Eigenvalues of \( A_r \) under the assumption (H1).

**Lemma 3.2.** Under the assumption (H1), for each \( r \in \Lambda, \, 0 \notin \sigma(A_r) \).

Proof. If \( 0 \in \sigma(A_r) \), then there exists some \( \psi \in \mathbb{X}_2 \setminus \{0\} \) such that
\[
\Delta(r, 0) \psi = 0 \quad (40)
\]
Note that \( U_r = z_r q_1 + w(z_r, r), \Delta(\mu_1, 0) = \mathcal{L}_{\mu_1} \), and \( \ker \mathcal{L}_{\mu_1} = \mathbb{K}_1 = \text{span}\{q_1\}, q_1 = (\phi_1, 0)^T \), then \( \psi \) can be decomposed as follows
\[
\psi(x) = a(r) q_1 + \rho_{1r} b(r), \quad (41)
\]
where \( a(r) \in \mathbb{R}, \, b(r) \in \mathbb{X}_1, \, a(\mu_1) = a_0 \neq 0, \, b(\mu_1) = b_0 \). Substituting (41) into (40), we have
\[
\Delta(r, 0) a(r) q_1 + \Delta(r, 0) \rho_{1r} b(r) = 0 \quad (42)
\]
for all \( r \in \Lambda \). Note that
\[
\lim_{r \to \mu_1} \frac{\Delta(r, 0) a(r) q_1}{\rho_{1r}} = a_0 \mu_1 \left( \phi_1 - \frac{2}{\kappa_1} \phi_1^0 \right),
\]
\[ \lim_{r \to \mu_1} \Delta(r,0)b(r) = \mathcal{L}_{\mu_1}b_0. \]

Hence,
\[ a_0\mu_1 \left( \phi_1 - \frac{\zeta}{\kappa_1} \phi_1^2 \right) + \mathcal{L}_{\mu_1}b_0 = 0. \] (43)

Calculating the inner product of the above equation with \( p_1 \), we have
\[ \left\langle a_0\mu_1 \left( \phi_1 - \frac{\zeta}{\kappa_1} \phi_1^2 \right) + \mathcal{L}_{\mu_1,r}b_0(x), p_1 \right\rangle = -a_0\mu_1 = 0, \]
i.e., \( a_0 = 0 \), which contradicts \( a(\mu_1) = a_0 \neq 0 \). That is to say, \( 0 \notin \sigma(\mathcal{A}_r) \). \hfill \square

Based on the above discussion, we have the following conclusions.

**Theorem 3.3.** Under the assumption (H1), there exists a constant \( \delta_1 > 0 \) such that for each \( r \in \Lambda \triangleq (\mu_1 - \delta_1, \mu_1) \cup (\mu_1, \mu_1 + \delta_1) \), model (3) has exactly one spatially nonhomogeneous steady state solution, whose associated infinitesimal generator \( \mathcal{A}_r \) has no zero eigenvalue.

**Lemma 3.4.** Under the assumption (H1), for each \( r \in \Lambda \), \( \mathcal{A}_r \) has no purely imaginary eigenvalue.

**Proof.** If \( \mathcal{A}_r \) has a purely imaginary eigenvalue \( \lambda = i\omega(\omega > 0) \), then there exists some \( \psi = (\psi^1, \psi^2)^T \in \mathbb{X}_C^2 \setminus \{0\} \) such that \( \tilde{\Delta}(r,\omega)\psi = 0 \) is solvable for some \( \omega > 0 \), where
\[
\tilde{\Delta}(r,\omega)\psi = \begin{pmatrix} \frac{D_1\psi^1_{xx} - \beta_1(x)\psi^1_x + [r(1 - 2u_r - cv_r) - i\omega]\psi^1 - cru_r\psi^2}{brv_r\psi^1 + D_2\psi^2_{xx} - \beta_2(x)\psi^2_x + [r(1 - 2v_r - bu_r) - i\omega]\psi^2} \end{pmatrix}.
\] (44)

Next, for \( r \in \Lambda \) suppose that \( (\omega, \psi) \in (0, \infty) \times (\mathbb{X}_C^2 \setminus \{0\}) \) is a solution of \( \tilde{\Delta}(r,\omega)\psi = 0 \). Note that \( \lim_{r \to \mu_1} \tilde{\Delta}(r,\omega) = \mathcal{L}_{\mu_1} \), and \( \ker \mathcal{L}_{\mu_1} = \mathbb{K}_1 = \text{span}\{q_1\}, q_1 = (\phi_1, 0)^T \), then \( \psi \) can be decomposed into
\[ \psi = a(r)q_1 + \rho_1(r), \] (45)
where \( a(r) \in \mathbb{C} \), \( b(r) \in \mathbb{X}_C \), \( a(\mu_1) = a_0 \neq 0 \), \( b(\mu_1) = b_0 \). Substituting (45) into (44) and then calculating the inner product with \( p_1 \), we have
\[
0 = \int_{11} \tilde{\phi}_1(D_1\psi^1_{xx} - \beta_1(x)\psi^1_x + [r(1 - 2u_r - cv_r) - i\omega]\psi^1 - cru_r\psi^2)dx
= a(r)(r - \mu_1 - i\omega - 2\rho_1) + o(\rho_1^2).
\]

It follows that
\[ a(r) \left( \frac{r - \mu_1 - i\omega}{\rho_1} - 2r \right) + o(\rho_1) = 0. \]

If \( \eta = \lim_{r \to \mu_1} \frac{\omega}{\rho_1} \neq 0 \), then \( a_0(-\mu_1 - i\eta) = 0 \). Due to the fact that \( \mu_1 > 0 \), the above formula holds if and only if \( a_0 = 0 \). However, this is a contradiction. Therefore, \( \mathcal{A}_r \) has no purely imaginary eigenvalue. \hfill \square

Hence, we have the following conclusions.

**Theorem 3.5.** Under the assumption (H1), there exist a constant \( \delta_1 > 0 \) such that for each \( r \in \Lambda \triangleq (\mu_1 - \delta_1, \mu_1) \cup (\mu_1, \mu_1 + \delta_1) \), model (3) has exactly one spatially nonhomogeneous steady state solution, whose associated infinitesimal generator \( \mathcal{A}_r \) has no purely imaginary eigenvalue.

Using similar theoretical analysis as above, we can draw the following conclusions.
Theorem 3.6. Under the assumption (H2), there exist a constant $\delta_2 > 0$ such that for each $r \in \Lambda \ni (\mu_2 - \delta_2, \mu_2) \cup (\mu_2, \mu_2 + \delta_2)$, model (3) has exactly one spatially nonhomogeneous steady-state solution, whose associated infinitesimal generator $A_r$ has neither zero eigenvalue nor purely imaginary eigenvalue.

3.2. Eigenvalues of $A_r$ under the assumption (H3). Different from the previous subsection, here we assume $\mu_1 = \mu_2 = \mu_*$ and study the eigenvalue problem of $A_r$ under this condition.

Lemma 3.7. Under the assumption (H3), for each $r \in \Lambda$, $0 \in \sigma(A_r)$ if and only if $\det D = d_1 d_2 = 0$.

Proof. Under the assumption (H3), if $0 \in \sigma(A_r)$, then there exists some $\psi = (\psi^1, \psi^2)^T \in X_\mathbb{C}^2 \setminus \{0\}$ such that

$$\Delta(r,0) \psi = 0.$$  \hspace{1cm} (46)

Note that $\Delta(\mu_*,0) = L_{\mu_*}$ and $\ker L_{\mu_*} = \mathbb{K}_3 = \text{span}\{q_1, q_2\}$, $q_1 = (\phi_1, 0)^T$, $q_2 = (0, \phi_2)^T$ then $\psi$ can be decomposed into

$$\psi(x) = a_1(r)q_1 + a_2(r)q_2 + \rho_r b(r),$$  \hspace{1cm} (47)

where $a(r) = (a_1(r), a_2(r))^T \in \mathbb{R}^2$, $b(r) \in \mathbb{K}_3$, $a(\mu_*) = a_0 = (a_{10}, a_{20})^T \neq 0$, $b(\mu_*) = b_0$. Substituting (47) into (46), we have

$$\Delta(r,0)[a_1(r)q_1 + a_2(r)q_2] + \Delta(r,0)\rho_r b(r) = 0,$$  \hspace{1cm} (48)

for all $r \in \Lambda$. Note that

$$\lim_{r \to \mu_*} \frac{\Delta(r,0)[a_1(r)q_1 + a_2(r)q_2]}{\rho_r} = \mu_*|\kappa| a_0 - \mu_*\kappa M a_0 - \mu_* D K a_0,$$

where

$$M = \begin{pmatrix} d_1\phi_1 + cd_2\phi_2 & 0 \\ 0 & bd_1\phi_1 + d_2\phi_2 \end{pmatrix},$$

$$K = \begin{pmatrix} \phi_1^2 & c\phi_1\phi_2 \\ b\phi_1\phi_2 & \phi_2^2 \end{pmatrix},$$

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},$$

and

$$|\kappa| = \det \kappa = \kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21}, \quad \kappa = \text{diag}(\phi_1, \phi_2).$$

Hence,

$$\mu_*|\kappa| a_0 - \mu_*\kappa M a_0 - \mu_* D K a_0 + L_{\mu_*} b_0 = 0.$$  \hspace{1cm} (49)

Calculating the inner product of the above equation with $p_1$ and $p_2$, respectively, we have

$$\left\{ \begin{array}{l} -\mu_*(a_{10}\kappa_{11} + a_{20}\kappa_{12}) = 0, \\
-\mu_*(a_{10}\kappa_{21} + a_{20}\kappa_{22}) = 0,
\end{array} \right.$$  

i.e., $-\mu_* D K a_0 = 0$. If $\det D = d_1 d_2 \neq 0$, then $a_0 = 0$, which contradicts $a(\mu_*) \neq 0$. Hence, $\det D = d_1 d_2 = 0$.

Next, we assume that $\det D = d_1 d_2 = 0$. In order to prove that $0 \in \sigma(A_r)$, it suffices to find a solution $\psi \neq 0$ to

$$\Delta(r,0) \psi = 0.$$  

Substituting $\psi(x) = a_1 q_1 + a_2 q_2 + \rho_r b$ into $\Delta(r,0) \psi = 0$ with $a = (a_1, a_2)^T \in \mathbb{R}^2$ and $b \in \mathbb{K}_3$, we have $H(a,b,r) = 0$, where

$$H(a,b,r) = \frac{\Delta(r,0)[a_1 q_1 + a_2 q_2]}{\rho_r} + \Delta(r,0)b.$$  \hspace{1cm} (50)
Clearly,
\[ H(a, b, \mu_\ast) = \mu_\ast |\kappa| x a - \mu_\ast x M a - \mu_\ast DK a + L_{b, b}. \]
It follows from \( \det \mathcal{D} = d_1 d_2 = 0 \) that we can choose \( a_0 \neq 0 \) such that \( \mu_\ast DK a_0 \in \text{Ran} \mathcal{L}_{\mu_\ast} \). Hence, there exists \( b_0 \in X_3 \backslash \{ 0 \} \) such that \( H(a_0, b_0, \mu_\ast) = 0 \). Moreover, we have
\[ H_a(a_0, b_0, \mu_\ast) = \mu_\ast |\kappa| x - \mu_\ast x M a - \mu_\ast DK, \quad H_b(a_0, b_0, \mu_\ast) = L_{\mu_\ast}. \]
Note that \( H_b(a_0, b_0, \mu_\ast) \) is always invertible. Firstly, suppose \( H_a(a_0, b_0, \mu_\ast) \) is invertible, then from the implicit function theorem, we obtain a continuously differentiable mapping: \( r \to (a(r), b(r)) \) from \( \Lambda \) to \( \mathbb{R}^2 \times X_3 \) such that \( a(\mu_\ast) = a_0 \neq 0, b(\mu_\ast) = b_0 \neq 0 \), and \( H(a, b, r) \equiv 0 \). This implies that \( \Delta(r, 0) \psi = 0 \) has a nontrivial solution \( \psi(x) = a_1(r)q_1 + a_2(r)q_2 + \rho_b b(r, x) \). Hence, \( 0 \in \sigma(A_r) \).

If \( H_a(a_0, b_0, \mu_\ast) \) is non-invertible, since \( H_b(a_0, b_0, \mu_\ast) \) is always invertible, then by the implicit function theorem, there exists a continuously differentiable mapping: \( (r, a) \to b(r, a) \) from \( \Lambda \times \mathbb{R}^2 \) to \( X_3 \) such that \( b(\mu_\ast, a_0) = b_0 \neq 0 \) and \( H(a, b(r, a), r) \equiv 0 \), which implies that \( \Delta(r, 0) \psi = 0 \) has a nontrivial solution \( \psi(x) = a_1 q_1 + a_2 q_2 + \rho_b b(r, a) \) for \( (r, a) \in \Lambda \times \mathbb{R}^2 \). In other words, \( 0 \in \sigma(A_r) \). The proof is completed. \( \square \)

In summary, we have the following conclusion.

**Theorem 3.8.** Under the assumption \((H3)\), if \( \kappa_{11} \kappa_{22} \neq \kappa_{12} \kappa_{21} \), and \( d_1 d_2 > 0 \) then there exists a constant \( \delta_3 > 0 \) such that for each \( r \in (\mu_\ast, \mu_\ast + \delta_3) \), model \((3)\) has exactly one spatially nonhomogeneous positive steady-state solution \( U_r \), whose associated infinitesimal generator \( A_r \) has no zero eigenvalue.

Next, we will investigate the existence of purely imaginary eigenvalues of the infinitesimal generator \( A_r \) associated with the spatially nonhomogeneous positive steady-state solution \( U_r \) under the assumption of Theorem 3.8.

**Lemma 3.9.** Under the assumption \((H3)\), for each \( r \in (\mu_\ast, \mu_\ast + \delta_3) \), \( A_r \) has no purely imaginary eigenvalue.

**Proof.** If \( A_r \) has a purely imaginary eigenvalue \( \lambda = i \omega(\omega > 0) \), then there exists some \( \psi = (\psi^1, \psi^2)^T \in X_3^2 \backslash \{ 0 \} \) such that
\[ \Delta(r, \omega) \psi = 0 \tag{51} \]
is solvable for some \( \omega > 0 \). For \( r \in (\mu_\ast, \mu_\ast + \delta_3) \) suppose that \( (\omega, \psi) \in (0, \infty) \times (X_3^2 \backslash \{ 0 \}) \) is a solution of \((51)\). Note that
\[ \lim_{r \to \mu_\ast} \Delta(r, \omega) = \mathcal{L}_{\mu_\ast} \]
and \( \ker \mathcal{L}_{\mu_\ast} = \mathbb{K}_3 = \text{span}\{ q_1, q_2 \}, q_1 = (\phi_1, 0)^T, q_2 = (0, \phi_2)^T \) then \( \psi \in X_3^2 \backslash \{ 0 \} \) can be decomposed into
\[ \psi = a_1(r)q_1 + a_2(r)q_2 + \rho_b(r) \]
with \( a(r) = (a_1(r), a_2(r))^T \in \mathbb{C}^2 \backslash \{ 0 \} \) and \( b(r) = (b_1(r), b_2(r))^T \in X_3 \mathbb{C} \). Calculating the inner product of \((51)\) with \( p_1 \) and \( p_2 \), respectively, we have
\[ 0 = \langle p_1, \Delta(r, \omega) \psi \rangle \]
\[ = \int_{\Omega} \hat{\psi}_1(D_1 \psi_{xx}^1 - \beta_1(x) \psi_x^1 + [r(1 - 2u_r - cu_r) - i\omega] \psi^1 - cru_r \psi^2)dx \]
Due to the fact that equation (53) is non-singular, and it follows that
\[0 = \langle p_2, \tilde{\Delta}(r, \omega)\psi \rangle\]
\[= \int_{\Omega} \tilde{\phi}_2(D_2\psi_{zz}^2 - \beta_2(x)\psi_z^2)dx + \int_{\Omega} r(1 - 2v_r - bu_r - i\omega)\psi^2 - br_v \psi^3 dx\]
\[= a_2(r - \mu_* - i\omega)\]
\[-2ra_1d_1d_2\kappa_{11}\rho_r - 2rd_2\rho_r^2 \int_{\Omega} \tilde{\phi}_1\phi_1b_1dx - 2ra_1 \int_{\Omega} \tilde{\phi}_1\phi_1w_1dx - 2r \int_{\Omega} \tilde{\phi}_1w_1b_1dx\]
\[-ra_1d_2d_2\kappa_{12}\rho_r - crd_2\rho_r^2 \int_{\Omega} \tilde{\phi}_1\phi_2b_1dx - cra_1 \int_{\Omega} \tilde{\phi}_1\phi_2w_1dx - cr \int_{\Omega} \tilde{\phi}_1w_2b_1dx\]
\[-ra_2d_1d_2\kappa_{12}\rho_r - crd_1\rho_r^2 \int_{\Omega} \tilde{\phi}_2\phi_1b_1dx - cra_2 \int_{\Omega} \tilde{\phi}_2\phi_1w_1dx - cr \int_{\Omega} \tilde{\phi}_2w_1b_2dx\]
\[= a_1(r - \mu_* - i\omega) - 2ra_1d_1d_2\kappa_{11}\rho_r - ra_1d_2d_2\kappa_{12}\rho_r - ra_2d_1d_2\kappa_{12}\rho_r + o(\rho_r^2),\]
and
\[0 = \langle p_2, \tilde{\Delta}(r, \omega)\psi \rangle\]
\[= \int_{\Omega} \tilde{\phi}_2(D_2\psi_{zz}^2 - \beta_2(x)\psi_z^2)dx + \int_{\Omega} r(1 - 2v_r - bu_r - i\omega)\psi^2 - br_v \psi^3 dx\]
\[= a_2(r - \mu_* - i\omega)\]
\[-2ra_2d_2d_2\kappa_{22}\rho_r - 2rd_2\rho_r^2 \int_{\Omega} \tilde{\phi}_2\phi_2b_2dx - 2ra_2 \int_{\Omega} \tilde{\phi}_2\phi_2w_2dx - 2r \int_{\Omega} \tilde{\phi}_2w_2b_2dx\]
\[-ra_2d_1d_{21}\kappa_{21}\rho_r - brd_1\rho_r^2 \int_{\Omega} \tilde{\phi}_2\phi_1b_2dx - bra_2 \int_{\Omega} \tilde{\phi}_2\phi_1w_2dx - br \int_{\Omega} \tilde{\phi}_2w_1b_2dx\]
\[-ra_1d_2d_{21}\kappa_{21}\rho_r - brd_2\rho_r^2 \int_{\Omega} \tilde{\phi}_2\phi_2b_1dx - bra_1 \int_{\Omega} \tilde{\phi}_2\phi_2w_1dx - br \int_{\Omega} \tilde{\phi}_2w_2b_1dx\]
\[= a_2(r - \mu_* - i\omega) - 2ra_2d_2d_{22}\kappa_{22}\rho_r - ra_2d_1d_{21}\kappa_{21}\rho_r - ra_1d_2d_{21}\kappa_{21}\rho_r + o(\rho_r^2).\]
It follows that
\[
\begin{cases}
(\frac{r-\mu_* - i\omega}{r}) - 2rd_1\kappa_{11} - rd_2\kappa_{12})a_1 - rd_1\kappa_{12}a_2 + o(\rho_r) = 0,
(\frac{r-\mu_* - i\omega}{r}) - 2rd_2\kappa_{22} - rd_1\kappa_{21})a_2 - rd_2\kappa_{21}a_1 + o(\rho_r) = 0.
\end{cases}
\tag{52}
\]
Let \(r \to \mu_*\), then we have
\[
\begin{cases}
(\mu_* d_1\kappa_{11} + i\eta)a_{10} + \mu_* d_1\kappa_{12}a_{20} = 0,
(\mu_* d_2\kappa_{22} + i\eta)a_{20} + \mu_* d_2\kappa_{21}a_{10} = 0,
\end{cases}
\tag{53}
\]
where \(a_{10} = a_i(\mu_*) \neq 0(i=1,2)\), \(\eta = \lim_{r \to \mu_*} \frac{\rho_r}{r} \neq 0.\)

For the above equation (53), there exists a non-zero solution \(a_0 = (a_{10}, a_{20})^T \neq 0\) if and only if the coefficient matrix is singular, i.e.,
\[
\mu_*^2 d_1 d_2 (\kappa_{11} \kappa_{22} - \kappa_{12} \kappa_{21}) - \eta^2 + i\eta \mu_* (\kappa_{11} d_1 + \kappa_{22} d_2) = 0.
\]
Separating the real and imaginary parts, we obtain
\[
\begin{cases}
\mu_*^2 d_1 d_2 (\kappa_{11} \kappa_{22} - \kappa_{12} \kappa_{21}) - \eta^2 = 0,
\eta \mu_* (\kappa_{11} d_1 + \kappa_{22} d_2) = 0.
\end{cases}
\tag{54}
\]
Due to the fact that \(\mu_* > 0, d_1 d_2 > 0, \kappa_{ij} > 0 (i,j = 1,2)\) and \(\kappa_{11} \kappa_{22} \neq \kappa_{12} \kappa_{21}\), there is no solution to the above equation. That is to say, the coefficient matrix of equation (53) is non-singular, and \(a_0 = 0\) is the unique solution, which contradicts that \(a(\mu_*) = a_0 \neq 0.\) Therefore, \(\mathcal{A}_r\) has no purely imaginary eigenvalue.

Based on the above discussion, we come to the following conclusion.

**Theorem 3.10.** Under the assumption (H3), if \(\kappa_{11} \kappa_{22} \neq \kappa_{12} \kappa_{21} \) and \(d_1 d_2 > 0\) then there exists a constant \(\delta_3 > 0\) such that for each \(r \in (\mu_*, \mu_* + \delta_3)\), model (3) has exactly one spatially nonhomogeneous positive steady-state solution \(U_r,\) whose associated infinitesimal generator \(\mathcal{A}_r\) has no purely imaginary eigenvalue.
4. Local stability. In this section, we shall investigate the stability of the spatially nonhomogeneous steady-state solution by analyzing the real parts of the eigenvalues of the infinitesimal generator $A_r$. It should be noted that here the steady-state solution to be studied is positive.

**Theorem 4.1.** (i): Under the assumption (H1), there exist a constant $0 < \delta_1 < \min\{\mu_1 - D_2\lambda_n(\frac{\partial}{\partial x}) ||n \in \mathbb{N}_+\}$ such that for each $r \in (\mu_1, \mu_1 + \delta_1)$ system (3) has exactly one spatially nonhomogeneous positive steady-state solution, whose associated infinitesimal generator $A_r$ has only eigenvalues with negative real parts. In other words, this spatially nonhomogeneous positive steady-state solution is locally asymptotically stable;

(ii): Under the assumption (H2), there exist a constant $0 < \delta_2 < \min\{\mu_2 - D_2\lambda_n(\frac{\partial}{\partial x}) ||n \in \mathbb{N}_+\}$ such that for each $r \in (\mu_2, \mu_2 + \delta_2)$ system (3) has exactly one spatially nonhomogeneous positive steady-state solution, whose associated infinitesimal generator $A_r$ has only eigenvalues with negative real parts. In other words, this spatially nonhomogeneous positive steady-state solution is locally asymptotically stable;

(iii): Under the assumption (H3) if $\kappa_{11r} \neq \kappa_{12r}, d_1 > 0$ and $d_2 > 0$ (respectively, $d_1 < 0$ and $d_2 < 0$) then there exist a constant $\delta_3 > 0$ such that for each $r \in (\mu_\ast, \mu_\ast + \delta_3)$ system (3) has exactly one spatially nonhomogeneous positive steady-state solution, whose associated infinitesimal generator $A_r$ has only eigenvalues with negative real parts (respectively, has at least one eigenvalue with positive real parts). In other words, this spatially nonhomogeneous positive steady-state solution is locally asymptotically stable (respectively, unstable).

**Proof.** We just prove conclusions (i) and (iii) because the proof of conclusion (ii) is similar. First, let’s prove the conclusion (i).

Within the $\delta_1$-neighborhood of the bifurcation value $\mu_1$, we consider a sequence $\{r_n\}_{n=1}^\infty \subseteq (\mu_1 - \delta_1, \mu_1 + \delta_1)$ that satisfies the conditions $\lim_{n \to \infty} r_n = \mu_1$, $0 < |\mu_n - \mu_1| \ll 1 (n \geq 1)$, and that $A_{r_n}$ has an eigenvalue $\lambda_n$ with an associated eigenfunction $\psi_n$. Namely,

$$A_{r_n}\psi_n = \lambda_n \psi_n.$$  \hfill (55)

Similar to the previous section, rewrite $\psi_n$ as

$$\psi_n = a(r_n)q_1 + p_{1r_n}b(r_n, x),$$

where $a(r_n) \in \mathbb{R}\{0\}$, $b(r_n) \in X_{1C}$, $a(\mu_1) = a_0 \neq 0$, $b(\mu_1) = b_0$. Calculating the inner product of $p_1$ with $\Delta(r_n, \lambda_n)$, we have

$$a(r_n)(r_n - \mu_1 - \lambda_n - 2r_n\rho_{1r_n}) + o(\rho_{1r_n}^2) = 0.$$  

It follows that

$$a(r_n)(\frac{r_n - \mu_1 - \lambda_n}{\rho_{1r_n}} - 2r_n) + o(\rho_{1r_n}) = 0.$$  

Letting $n \to \infty$, we have

$$a_0(-\mu_1 - \lim_{n \to \infty} \frac{\lambda_n}{\rho_{1r_n}}) = 0,$$

i.e.,

$$\lim_{n \to \infty} \frac{\lambda_n}{\rho_{1r_n}} = -\mu_1 < 0.$$
Hence, by the arbitrariness of \( \{r_n\}_{n=1}^{\infty} \), the all eigenvalues of \( \mathcal{A}_{r_n} \) have negative real parts. This completes the proof of conclusion (i).

Next, we only prove conclusion (iii). Similarly, within the \( \delta_3 \)-neighborhood of the bifurcation value \( \mu_* \), we consider a sequence \( \{r_n\}_{n=1}^{\infty} \subseteq (\mu_* - \delta_3, \mu_* + \delta_3) \) that satisfies the conditions \( \lim_{n \to \infty} r_n = \mu_* \), \( 0 < |r_n - \mu_*| \ll 1 \ (n \geq 1) \), and that \( \mathcal{A}_{r_n} \) has an eigenvalue \( \lambda_n \) with an associated eigenfunction \( \psi_n \). Namely,

\[
\mathcal{A}_{r_n} \psi_n = \lambda_n \psi_n. \tag{56}
\]

For (H3), since the parameters and solution we obtained earlier are more complicated than the previous two cases, the above method is no longer applicable. Next, we have another kind of decomposition of the eigenfunction \( \psi_n \):

\[
\psi_n = c_n U_{r_n} + \rho_{r_n} V_n, \quad n \geq 1,
\]

where \( U_{r_n} = (u_{r_n}, v_{r_n})^T \) is given as (31), \( c_n \in \mathbb{C}^2 \) and \( V_n \in X_3^2 \) satisfy that

\[
c_n = \| U_{r_n} \|_{Y_C}^2, \quad (U_{r_n}, V_n) = 0.
\]

Clearly, we have

\[
\lim_{n \to \infty} c_n \rho_{r_n}^2 = \frac{1}{d_1^2 \int_\Omega \phi_1^2 dx + d_2^3 \int_\Omega \phi_2^3 dx} > 0.
\]

Calculating the inner product of the two sides of the equation (56) with \( U_{r_n} \), we obtain

\[
\lambda_n = (U_{r_n}, \mathcal{A}_{r_n} \psi_n) = (U_{r_n}, \mathcal{A}_{r_n} (c_n U_{r_n} + \rho_{r_n} V_n)) = c_n (U_{r_n}, \mathcal{A}_{r_n} U_{r_n}).
\]

Furthermore,

\[
\lim_{n \to \infty} \frac{\lambda_n}{\rho_{r_n}} = \lim_{n \to \infty} \frac{c_n (U_{r_n}, \mathcal{A}_{r_n} U_{r_n})}{\rho_{r_n}} = \lim_{n \to \infty} \frac{c_n \rho_{r_n}^2 (U_{r_n}, \mathcal{A}_{r_n} U_{r_n})}{\rho_{r_n}^3}
\]

\[
= -\mu_*(d_1^3 \int_\Omega \phi_1^3 dx + d_2^3 \int_\Omega \phi_2^3 dx + d_3^3 \int_\Omega \phi_3^3 dx + d_1 d_2 \delta \int_\Omega \phi_1 \phi_2 dx)
\]

\[
\times \left( \frac{d_1^2 \int_\Omega \phi_1^2 dx + d_2^2 \int_\Omega \phi_2^2 dx + d_2^3 \int_\Omega \phi_2^3 dx + d_1 d_2 \delta \int_\Omega \phi_1 \phi_2 dx} \right), \tag{57}
\]

and

\[
\sgn \left( \lim_{n \to \infty} \frac{\lambda_n}{\rho_{r_n}} \right) = -\sgn \left\{ d_1^3 \int_\Omega \phi_1^3 dx + d_2^3 \int_\Omega \phi_2^3 dx + d_3^3 \int_\Omega \phi_3^3 dx + d_1 d_2 \delta \int_\Omega \phi_1 \phi_2 dx \right\}.
\]

Thus, if \( d_1 > 0 \) (respectively, \( d_1 < 0 \)) and \( d_2 > 0 \) (respectively, \( d_2 < 0 \)), then \( \sgn \left( \lim_{n \to \infty} \frac{\lambda_n}{\rho_{r_n}} \right) > 0 \) (respectively, \( \sgn \left( \lim_{n \to \infty} \frac{\lambda_n}{\rho_{r_n}} \right) < 0 \)) and \( \kappa_{11} \kappa_{22} - \kappa_{12} \kappa_{21} > 0 \) (respectively, \( \kappa_{11} \kappa_{22} - \kappa_{12} \kappa_{21} < 0 \)), that is, \( \mathcal{A}_{r_n} \) has only eigenvalues with negative real parts (respectively, has at least one eigenvalue with positive real parts). In addition, this proof method can be applied to (H1) and (H2) to obtain results consistent with the foregoing. The proof is complete. \( \square \)
5. Example and numerical simulation. In this section, we consider the following system to check the validity of the main results obtained in Sections 2-4:

\[
\begin{align*}
    u_t &= u_{xx} - \beta_1 u_x + ru(1 - \frac{1}{3}u), & t > 0, x \in (0, \pi), \\
    v_t &= D_2 v_{xx} - \beta_2 v_x + rv(1 - \frac{1}{6}u), & t > 0, x \in (0, \pi), \\
    u(t,0) &= u(t,\pi) = 0, & t > 0, \\
    v(t,0) &= v(t,\pi) = 0, & t > 0, \\
    u(0,x) &= \bar{u}_0 \geq 0, v(0,x) = v_0 \geq 0, & x \in (0, \pi).
\end{align*}
\] (58)

From the Corollary 1 of Murray and Sperb [23] and Lemma 3.2 of [9] we have \(\mu_1 = \frac{4+\beta_1^2}{4} , \mu_2 = \frac{4D_2^2 + \beta_1^2}{4D_2^2} \), \(\phi_1 = \exp\{\frac{\beta_1 x}{2}\} \sin x > 0\), \(\phi_2 = \exp\{\frac{\beta_2 x}{2D_2}\} \sin x > 0\) and \(\tilde{\phi}_1 = \exp\{-\frac{\beta_1 x}{2}\} \sin x > 0\), \(\tilde{\phi}_2 = \exp\{-\frac{\beta_2 x}{2D_2}\} \sin x > 0\) for all \(x \in (0, \pi)\). In what follows, we consider the following two sets of parameters:

\[
\beta_1 = 2, \quad \beta_2 = 1, \quad D_2 = \frac{1}{2},
\] (59)

and

\[
\beta_1 = \beta_2 = 2, \quad D_2 = 1.
\] (60)

Obviously, parameters (59) satisfies assumption (H1) or (H2), while parameters (60) satisfies assumption (H3).

5.1. \(\beta_1 = 2, \beta_2 = 1, D_2 = \frac{1}{2}\). In this case, we have \(\dot{\kappa}_1 = \int_0^\pi \dot{\phi}_2 \phi_2^2 dx = \frac{\pi}{10}(e^\pi + 1) > 0\). Thus, in view of Theorem 2.4, we have the following result.

**Corollary 1.** There exist a positive constant \(0 < \delta \ll 1\) and a continuously differentiable mapping \(r \rightarrow z_r\), from \((1-\delta, 1+\delta)\) to \(\mathbb{R}\) such that (58) with parameters (59) has a spatially nonhomogeneous semi-trivial steady-state solution \((0, v_r)^T\) satisfying that

\[
\lim_{r \rightarrow 0^+} v_r = 0,
\]

and \(v_r\) is positive (respectively, negative) when \(1 < r < 2\) (respectively, \(0 < r < 1\)).

It is well known that negative steady-states are meaningless in biology, and we are concerned with the stability and Hopf bifurcation of the positive steady-state solution. Therefore, we only consider the case where \(1 < r < 2\). In view of Theorem 4.1(ii), we have the following conclusion.

**Corollary 2.** For each fixed \(1 < r < 2\), the spatially nonhomogeneous boundary steady-state solution obtained in Corollary 1 is locally asymptotically stable.

Next, we present a numerical simulation to illustrate our analytical results. It follows from Corollary 2 that system (58) with parameters (59) and \(r = 1.1\) has a spatially nonhomogeneous boundary steady-state solution \(U_r\), which is locally asymptotically stable. The initial conditions \(u_0(x) = \frac{\sin x}{10}\) and \(v_0(x) = \frac{\sin x}{20}\) for all \(x \in (0, \pi)\) are selected to obtain the simulation results in Figure 1, which confirmed the correctness of our previous work.

5.2. \(\beta_1 = \beta_2 = 2, D_2 = 1\). In this case, we can get the following quantities by simple calculation.

\[
\kappa_{11} = \frac{3}{10}(e^\pi + 1), \quad \kappa_{12} = \frac{1}{10}(e^\pi + 1), \quad \kappa_{21} = \frac{1}{20}(e^\pi + 1), \quad \kappa_{22} = \frac{3}{10}(e^\pi + 1)
\]

\[
d_1 = \frac{1}{5}(e^\pi + 1) > 0, \quad d_2 = \frac{1}{4}(e^\pi + 1) > 0, \quad \kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21} = \frac{17}{200}(e^\pi + 1)^2 > 0.
\]

Hence, in view of Theorem 2.5, we have the following result.
Corollary 3. There exist a positive constant $\delta > 0$ and a continuously differentiable mapping $r \to z_r$ from $(2 - \delta, 2 + \delta)$ to $\mathbb{R}$ such that (58) with parameters (60) has a spatially nonhomogeneous nontrivial steady-state solution $U_r$ satisfying

$$\lim_{r \to 2} U_r = 0.$$  

More precisely, we can get the specific expression of the steady state solution as follows

$$U_r = \begin{pmatrix} 40(r - 2) \frac{e^{\pi} \sin x}{17(\pi + 1)r} \\ 50(r - 2) \frac{e^{\pi} \sin x}{17(\pi + 1)r} \end{pmatrix}^T$$

for all $r \in (0, 2) \cup (2, \infty)$ and $x \in (0, \pi)$, and $U_r$ is positive (respectively, negative) when $r > 2$ (respectively, $0 < r < 2$).

Similar to the subsection 5.1, based on biological significance, we only consider the positive steady-state solution here. That is to say, we only consider the case where $r > 2$. In view of Theorem 4.1(iii), we have the following conclusion.

Corollary 4. For each fixed $r > 2$, the infinitesimal generator $A_r$ has only eigenvalues with negative real parts. In other words, this spatially nonhomogeneous positive steady-state solution obtained in Corollary 3 is locally asymptotically stable.

Next, we present some numerical simulations to illustrate our analytical results. It follows from Corollary 4 that system (58) with parameters (59) and $r = 2.1$ has a spatially nonhomogeneous positive steady-state solution $U_r$, which is locally asymptotically stable. The initial conditions $u_0(x) = \frac{\sin x}{17}$ and $v_0(x) = \frac{\cos x}{20}$ for all $x \in (0, \pi)$ are selected to obtain the simulation results as shown in Figure 2, which confirms the correctness of our previous work.

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Figure 2. Solutions of (58) with parameters (60) and \( r = 2.1 \) tend to a spatially nonhomogeneous positive steady-state.

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