The kinematics of particles moving in rainbow spacetime

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The kinematics of particles moving in rainbow spacetime is studied in this paper. In particular the geodesics of a massive particle in rainbow flat spacetime is obtained when the semi-classical effect of its own energy on the background is taken into account. We show that in general the trajectory of a freely falling particle remains unchanged which is still a straight line as in the flat spacetime. The implication to the Unruh effect in rainbow flat spacetime is also discussed.

I. INTRODUCTION

Recently the formalism of rainbow gravity has been proposed as a generalization of doubly special relativity when incorporating with curved spacetime\textsuperscript{1}. It can be viewed as a phenomenological model at the semi-classical level of quantum gravity where the quantum effect of moving particles on the background is taken into account. One key ingredient in this formalism is that there is no single fixed background for all observers, but dependent on the energy $E$ of probes. Corresponding to a modified dispersion relation as proposed in doubly special relativity\textsuperscript{2, 3, 4, 5, 6, 7, 8}, the dual or position space is defined by requiring that the contraction between momentum and infinitesimal displacement be a linear invariant. Specifically, given a modified dispersion relation as

$$E^2 f_1^2(E, \eta) - p^2 f_2^2(E, \eta) = m_0^2, \quad (1)$$

the dual space $dx^a$ is endowed with an energy dependent invariant

$$ds^2 = -\frac{1}{f_1^2(E, \eta)} dt^2 + \frac{1}{f_2^2(E, \eta)} dx^2, \quad (2)$$

where $f_1$ and $f_2$ are two general functions of energy $E$ and $\eta$ which is a dimensionless parameter labeling the magnitude of correction terms. To go back to special relativity as $E/E_p \ll 1$ where $E_p$ is the Planck energy, one requires that $f_1(E, \eta)$ and $f_2(E, \eta)$ approach to one at this limit\textsuperscript{13}.

The invariant in equation (2) is usually named as a rainbow flat metric. It can be extended to incorporate with curvature when the usual effect of classical gravity is taken into account. In this general case the background spacetime is described by a rainbow metric which is given in terms of a one-parameter family of orthonormal frame fields,

$$g(E) = \eta^{ab} e_a(E) \otimes e_b(E). \quad (3)$$

Then through the standard process, the corresponding one-parameter family of connections $\nabla(E)_{\mu}$ and curvature tensors $R(E)_{\mu\nu\lambda}$ can be constructed, leading to modified Einstein’s field equations

$$G_{\mu\nu}(E) = 8\pi G(E) T_{\mu\nu}(E) + g_{\mu\nu}(E) \Lambda(E), \quad (4)$$

where Newton’s constant as well as the cosmological constant is conjectured to be energy dependent as one expects from the viewpoint of renormalization group theory, for instance $G(E) = g^2(E) G$ and $\Lambda(E) = h^2(E) \Lambda$.

This proposal has received considerable attention recently and other stimulated work on this formalism can be found in \textsuperscript{9, 10, 11, 12, 13, 14, 15}. In this paper we intend to study the kinematics of massive particles moving in such a one-parameter spacetime described by rainbow metric. In particular the semi-classical effect on the background due to its own energy of the moving particle is concerned. We firstly present a general discussion on the equation of geodesics of particles moving in rainbow spacetime, then consider solutions to this equation in the special case that the background is described by the rainbow flat metric. Interestingly, our results show that in this case the geodesics is the same as that one in usual flat spacetime. We will briefly discuss the possible implications to the Unruh effect as well.

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II. PARTICLES MOVING IN RAINBOW SPACETIME

For dust in rainbow spacetime, its energy momentum tensor can be written as
\[ T_{\mu\nu}(E) = \rho T_{\mu}(E) T_{\nu}(E), \] (5)

where \( T_{\mu}(E) \) is the four-velocity of the particle, satisfying the ordinary normalized condition
\[ T^{\mu}(E) T_{\mu}(E) = -1. \] (6)

Now using the Bianchi identity or the conservation of the energy momentum tensor, we have
\[ \rho \nabla(E) \mu T^{\mu}(E) T^{\nu}(E) + \rho T^{\mu}(E) \nabla(E) \nu T^{\nu}(E) + T^{\mu}(E) T^{\nu}(E) \nabla(E) \mu \rho = 0. \] (7)

Furthermore, contracting both sides with \( \delta^{\lambda}_{\nu} + T_{\nu} T^{\lambda} \) and employing the normalized condition \( \delta \), the above equation yields
\[ T^{\nu}(E) \nabla(E) \mu T^{\lambda}(E) = 0, \] (8)

which shows that a free particle also goes along the geodesics in rainbow spacetime.

III. GEODESICS IN RAINBOW FLAT SPACETIME

In this section, we shall investigate the geodesics of a single particle with energy \( E \) moving in rainbow flat spacetime which is described by the metric \( \eta \). Particularly we intend to answer the question of how its trajectory may be changed when the semi-classical effect due to its own energy on the background is taken into account. In this case the energy appearing in the rainbow metric \( \eta \) is identified with the energy of the particle itself. Thus recalling the general definition of the energy of a massive particle in curved spacetime, we have
\[ E = -g_{00} p^0 = \frac{m_0}{f_1(E)} \frac{dt}{ds}, \] (9)

where \( s \) is understood as the proper time of this particle, and for convenience the parameter \( \eta \) in the function \( f_1 \) has been set as unit since it does not play an important role in our present discussion. On the other hand from \( \delta \), we may write the equation of geodesics as
\[ \frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\rho\sigma}(E) \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds} = 0. \] (10)

Obviously if \( f_2(E) = f_2(E) = 1 \), we have \( \Gamma^\mu_{\rho\sigma}(E) = 0 \) and the corresponding geodesics is a straight line in Minkowski spacetime. But in general, it is unclear whether the connection still vanishes since it manifestly depends on the energy of the moving particle, which should not be thought of as a constant for granted. As a matter of fact, given a specific form of the function \( f_1 \) we may obtain the energy of the particle in terms of the zeroth component of the four-velocity, namely \( \frac{dt}{ds} \), from \( \delta \), while whether the quantity of \( \frac{dt}{ds} \) is constant or not is completely determined by the equation of geodesics. Therefore to obtain the trajectory of a freely falling particle moving in rainbow spacetime we need solve for both the geodesics equation \( \delta \) and the energy equation \( \delta \) altogether.

Next we will show that the energy of the moving particle remains a constant indeed by solving these two equations. For simplicity but without loss of generality, we consider the case in two dimensional spacetime, where the components of connection are:
\[ \Gamma^0_{00} = -\frac{1}{f_1} \frac{df_1}{dt}, \quad \Gamma^0_{11} = -\frac{f_2^2}{f_1} \frac{df_2}{dt}, \]
\[ \Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{f_1} \frac{df_1}{dx}, \]
\[ \Gamma^1_{00} = -\frac{f_2}{f_1} \frac{df_2}{dx}, \quad \Gamma^1_{11} = -\frac{1}{f_2} \frac{df_2}{dx}, \]
\[ \Gamma^1_{01} = \Gamma^1_{10} = -\frac{1}{f_2} \frac{df_2}{dt}. \] (11)

Thus the equations of geodesics reads
\[ \frac{d^2 t}{ds^2} - \frac{1}{f_1} \frac{df_1}{dt} \left( \frac{dt}{ds} \right)^2 - \frac{f_2^2}{f_1} \frac{df_2}{dt} \left( \frac{dx}{ds} \right)^2 = 0, \]
\[ -2 \frac{1}{f_1} \frac{df_1}{dx} \frac{dt}{dx} \frac{dt}{ds} \frac{ds}{dx} = 0, \]
\[ \frac{d^2 x}{ds^2} - \frac{f_2^2}{f_1} \frac{df_1}{dx} \left( \frac{dt}{ds} \right)^2 - \frac{1}{f_2} \frac{df_2}{dx} \left( \frac{dx}{ds} \right)^2 = 0, \]
\[ -2 \frac{1}{f_2} \frac{df_2}{dt} \frac{dt}{dx} \frac{ds}{dx} = 0. \] (12)

From now on we denote \( \frac{dt}{ds} \) as \( \lambda \), then from \( \delta \) we may express \( E \) as a function of \( \lambda \) such that \( f_1(E) \) as well as \( f_2(E) \) can be rewritten as
\[ f_1 = G_1(\lambda), \quad f_2 = G_2(\lambda). \] (13)

Denoting \( \{ t, x \} \) as \( \{ x_0, x_1 \} \), respectively, we obtain
\[ \frac{\partial f_1}{\partial x_j} = \frac{dG_1}{dx} \frac{\partial x_j}{ds} \frac{ds}{dx} \frac{dt}{dx}, \quad \frac{\partial f_2}{\partial x_j} = \frac{dG_2}{dx} \frac{\partial x_j}{ds} \frac{dt}{dx}, \] (14)

where \( i = 1, 2 \) and \( j = 0, 1 \). Thus the first equation in \( \delta \) reduces into
\[ A \frac{d^2 t}{ds^2} = 0, \] (15)

where \( A \) is some complicated coefficient function depending on the specific form of functions \( f_1(E) \) and \( f_2(E) \). Usually \( A \) does not vanish and we may see this from the example given below. Hence we finally obtain
\[ \frac{d^2 t}{ds^2} = 0. \] (16)
Plugging it into (9) we show that the energy $E$ of the particle is indeed a constant along the geodesics in rainbow flat spacetime. As a consequence we also have

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} = 0,$$

such that all connection components vanish and the second equation in (12) reduces to

$$\frac{d^2 x}{ds^2} = 0. \quad (18)$$

Therefore the geodesics is still a straight line in energy-independent coordinate system. Above considerations can be extended to the four dimensional case straightforwardly as the equation (14) plays a crucial role in the proof which is independent of the dimension of spacetime.

For explicitness we would like to provide an example by specifying

$$f_1^2(E) = f_2^2 = 1 - \frac{l_p}{E}, \quad f_2^2(E) = 1,$$

where $l_p \sim 1/E_p$ is the Planck length. Then the non-vanishing components read as

$$\Gamma^0_{00} = -\frac{1}{f} \frac{\partial f}{\partial t}, \quad \Gamma^1_{00} = -\frac{1}{f} \frac{\partial f}{\partial x},$$

$$\Gamma^0_{01} = \Gamma^0_{10} = -\frac{1}{f} \frac{\partial f}{\partial x} \quad (20)$$

and the equation of geodesics can be written as

$$\frac{d^2 t}{ds^2} - \frac{1}{f} \frac{\partial f}{\partial t} \left( \frac{dt}{ds} \right)^2 - 2 \frac{1}{f} \frac{\partial f}{\partial x} \frac{dt}{ds} \frac{dx}{ds} = 0,$$

$$\frac{d^2 x}{ds^2} - \frac{1}{f} \frac{\partial f}{\partial x} \left( \frac{dt}{ds} \right)^2 = 0. \quad (21)$$

On the other hand, from (9) we obtain

$$f^4(E) - f^2(E) + l_p m \lambda = 0, \quad (22)$$

which yields a non-perturbative solution to $f$ as

$$f^2(E) = \frac{1 + \sqrt{1 - 4l_p m \lambda}}{2} \equiv 1 + \frac{k}{2}. \quad (23)$$

This requires

$$\lambda \leq 1 - 4l_p m. \quad (24)$$

Thus using (23) the connection components can be computed and plugging them into the first equation of geodesics, we find it becomes

$$\left( k + \frac{3}{4k} \right) \frac{d^2 t}{ds^2} = 0. \quad (25)$$

Because of the constraint (24), we know $k + \frac{3}{4k} \neq 0$, thus the geodesics is the same as the one in flat spacetime.

$$\frac{d^2 t}{ds^2} = \frac{d^2 x}{ds^2} = 0. \quad \text{Thus } \lambda \text{ is a constant, and the energy of the particle has a form}$$

$$E = \frac{2m \lambda}{1 + \sqrt{1 - 4l_p m \lambda}}. \quad (26)$$

At the classical limit $l_p \to 0$, comparing with the standard result in special relativity, we can easily fix it as

$$\lambda = \frac{1}{\sqrt{1 - v^2}}. \quad (27)$$

The energy has a cutoff at $\lambda = \frac{1}{4m l_p}$,

$$E_{\text{max}} = \frac{1}{2l_p}. \quad (28)$$

as one expects from the viewpoint of doubly special relativity.

### IV. IMPLICATIONS TO UNRUH EFFECT

In this section we present a brief discussion on Unruh effect in rainbow flat spacetime. It is well known that an accelerating observer in Minkowski spacetime will detect a thermal bath surrounding him and the Unruh temperature is supposed to be proportional to the magnitude of the proper acceleration

$$T = 2\pi a. \quad (29)$$

This identification has also been justified in de-Sitter and Anti de-Sitter spacetimes [14, 17]. Here we assume this identification is still valid in rainbow flat spacetime. Consider a detector is moving in a spacetime with a rainbow metric as

$$ds^2 = - \frac{dt^2}{f^2} + \frac{dx^2}{g^2}, \quad (30)$$

where for simplicity we take

$$f^2 = g^2 = 1 - (l_p E)^2. \quad (31)$$

As usual, we consider the case that the trajectory of the detector in the energy-independent coordinates is hyperbolic, namely

$$x^2 - t^2 = \frac{1}{a^2}, \quad (32)$$

such that the proper time $\tau$ can be defined as

$$d\tau = \frac{1}{af} d\eta, \quad (33)$$

where

$$x = \frac{1}{a} \cosh \eta, \quad t = \frac{1}{a} \sinh \eta. \quad (34)$$
It is easy to check that the four-velocity satisfies the normalized condition

\[ g_{ab}U^aU^b = g_{ab}\left(\frac{\partial}{\partial \tau}\right)^a\left(\frac{\partial}{\partial \tau}\right)^b = -1. \]  

(35)

Furthermore, we may obtain

\[ P^0 = m \frac{dx^0}{d\tau} = mf(E) \cosh \eta, \]

\[ P^1 = m \frac{dx^1}{d\tau} = mf(E) \sinh \eta, \] 

(36)

such that the energy and the momentum read

\[ E = -g_{00}P^0 = \frac{m}{f} \cosh \eta, \]

\[ P^1 = g_{11}P^1 = \frac{m}{f} \sinh \eta. \]  

(37)

Then from (31) and (37) we have

\[ E^2 - l_p^2 E^4 = m^2 \cosh^2 \eta, \]  

(38)

such that

\[ \frac{dE}{d\eta} = \frac{m^2 \cosh \eta \sinh \eta}{E(1 - 2l_p^2 E^2)}. \]  

(39)

Obviously along the trajectory the energy of the detector is not a constant but has a form

\[ E^2 = \frac{1}{2} \left(1 - \sqrt{1 - 4l_p^2 m^2 \cosh^2 \eta} \right). \]  

(40)

It goes back to the usual form \( E = m \cosh \eta \) as \( l_p \to 0 \). From (40) we also notice that the parameter \( \eta \) is constrained by \( \cosh \eta \leq \frac{1}{2l_p} \), implying that our classical picture is not valid any more as the energy of the detector approaches the Planck scale.

Finally, the proper acceleration for the moving particle can be computed by

\[ a^\mu = \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}(E) \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau}, \]  

(41)

where

\[ \Gamma^0_{00} = \Gamma^0_{11} = \Gamma^1_{01} = \Gamma^1_{10} = -\frac{1}{f} \frac{\partial f}{\partial \tau} \]

\[ = \frac{am^2 l_p^2 \sinh \eta}{f^2(1 - 2l_p^2 E^2)}. \]

\[ \Gamma^0_{00} = \Gamma^0_{11} = \Gamma^1_{01} = \Gamma^1_{10} = -\frac{1}{f} \frac{\partial f}{\partial x} \]

\[ = \frac{am^2 l_p^2 \cosh \eta}{f^2(1 - 2l_p^2 E^2)}. \]  

(42)

Then after a long calculation, it turns out that the magnitude of the acceleration is

\[ |a|^2 = a^2 f^2 \left\{ \left[ 1 + \frac{l_p^2 E^2}{1 - 2l_p^2 E^2} \right] \left(\frac{2}{E^2 f^2}\right)^2 \right. \]

\[ - \frac{l_p^4 E^4}{(1 - 2l_p^2 E^2)^2} \left(1 - \frac{m^2}{E^2 f^2}\right) \}. \]  

(43)

Obviously this formula can be trusted only when \( E \ll 1/l_p \). Nevertheless, with the assumption that the thermal effect is proportional to the proper acceleration of the detector in flat spacetime, this result implies that the usual Unruh temperature might be modified due to the semi-classical effect of the detector energy.

V. CONCLUDING REMARKS

In this paper we have attempted to study the kinematics of massive particles moving in rainbow spacetime. The equation of geodesics has been given. Especially we considered a single particle moving in flat rainbow spacetime and proved that its trajectory will not change due to the semi-classical effect of its own energy. Whether this conclusion can be extended to other rainbow curved spacetimes awaits further investigations.

We also considered the possible implications to Unruh effect in rainbow flat spacetime. Fixing the trajectory of the moving detector as a hyperbola in energy-independent coordinate system, we find the proper acceleration receives modifications due to the rainbow effect of its own energy, implying the Unruh temperature might also be corrected. It is noteworthy that our discussion presented here is preliminary and the detailed investigation calls for the quantum field theory over the curved spacetime endowed with rainbow metrics.

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In [1] the coordinates $t, x$ are also named as energy independent ones. Alternatively some energy dependent coordinates can be defined. The relations between these two sorts of coordinates and their physical interpretations can be found in [1] as well.