ON THE CONJECTURE $\mathcal{O}$ OF GGI FOR $G/P$

DAEWOONG CHEONG

Abstract. In this paper, we show that general homogeneous manifolds $G/P$ satisfy Conjecture $\mathcal{O}$ of Galkin, Golyshev and Iritani which 'underlies' Gamma conjectures I and II of them. Our main tools are the quantum Chevalley formula for $G/P$ and a theory on nonnegative matrices including Perron-Frobenius theory.

1. Introduction

Let $X$ be a Fano manifold, i.e., a smooth projective variety whose anti-canonical line bundle is ample. The quantum cohomology ring $H^*(X, \mathbb{C})$ of $X$ is a certain deformation of the classical cohomology ring $H^*(X, \mathbb{C})$ (§2.4 below). For $\sigma \in H^*(X, \mathbb{C})$, define the quantum multiplication operator $[\sigma]$ on $H^*(X, \mathbb{C})$ by $[\sigma](\tau) = \sigma \star \tau$ for $\tau \in H^*(X, \mathbb{C})$, where $\star$ denotes the quantum product in $H^*(X, \mathbb{C})$. Let $\delta_0$ be the absolute value of a maximal modulus eigenvalue of the operator $[c_1(X)]$, where $c_1(X)$ denotes the first Chern class of the tangent bundle of $X$. In [7], Galkin, Golyshev and Iritani say that $X$ satisfies Conjecture $\mathcal{O}$ if

1. $\delta_0$ is an eigenvalue of $[c_1(X)]$.
2. The multiplicity of the eigenvalue $\delta_0$ is one.
3. If $\delta$ is an eigenvalue of $[c_1(TX)]$ such that $|\delta| = \delta_0$, then $\delta = \delta_0 \xi$ for some $r$-th root of unity, where $r$ is the Fano index of $X$.

In fact, in addition to Conjecture $\mathcal{O}$, Galkin, Golyshev and Iritani proposed two more conjectures called Gamma conjectures I, II, which can be stated under the Conjecture $\mathcal{O}$. Let us briefly introduce Gamma conjectures I, II in order to explain how it underlies them. Consider the quantum connection of Dubrovin

$$\nabla z_{\partial z} = z \frac{\partial}{\partial z} - \frac{1}{z}(c_1(X)\star) + \mu,$$

acting on $H^*(X, \mathbb{C}) \times \mathbb{C}[z, z^{-1}]$, where $\mu$ is the grading operator on $H^*(X)$ defined by $\mu(\tau) = (k - \dim X)\tau$ for $\tau \in H^{2k}(X, \mathbb{C})$. This has a regular singularity at $z = \infty$ and an irregular singularity at $z = 0$. Flat sections near $z = \infty$ can be constructed through flat sections near $z = 0$ classified by their exponential growth order, and they are put into correspondence with cohomology classes. To be precise, if $X$ satisfies Conjecture $\mathcal{O}$, we can take a flat section $s_0(z)$ with the smallest asymptotics $\sim e^{-\delta_0/z}$ as $z \to +0$ along $\mathbb{R}_{>0}$. We transport $s_0(z)$ to $z = \infty$ and identify the corresponding class $A_X$ called the principal asymptotic class of $X$. Then Gamma conjecture I states that the cohomology class $A_X$ is equal to the Gamma class $\Gamma_X$. Here $\Gamma_X := \prod_{i=1}^n \Gamma(1 + \theta_i) \in H^*(X)$, where $\theta_i$ are the Chern roots of the tangent bundle $TX$ for $i = 1, \ldots, n$. Under further assumption of semisimplicity of the ring $H^*(X)$, we can identify cohomology classes $A_\delta$ corresponding to each eigenvalue $\delta$ in similar way. The classes $A_\delta$ form a basis of $H^*(X, \mathbb{C})$. Then Gamma conjecture II,

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\textsuperscript{1}We use this notation for the quantum cohomology ring with the multiplication $\star$, and without quantum variables.
A refinement of a part of Dubrovin’s conjecture \([3]\), states that there is an exceptional collection \(\{E_\delta \mid \delta\text{ eigenvalues of } [c_1(X)]\}\) of the derived category \(D^{ab}_{coh}(X)\) such that for each \(\delta\),

\[
A_\delta = \hat{\Gamma}_X \text{Ch}(E_\delta),
\]

where \(\text{Ch}(E_\delta) := \sum_{k=0}^{\dim X} (2\pi i)^k \text{ch}_k(E_\delta)\) is the modified Chern character. In this situation, Gamma conjecture I says that the exceptional object \(O\) corresponds to \(\delta_0\). See \([7]\) and \([3]\) for details on these materials.

As far as we know, the Conjecture \(O\) has thus far been proved for the ordinary, Lagrangian and orthogonal Grassmannians. For the ordinary Grassmannian, Galkin, Golyshev and Iritani \([7]\) proved Conjecture \(O\) together with Gamma conjectures I, II by using the quantum Satake of Golyshev and Manivel \([8]\). In fact we noticed that there were already two earlier papers proving Conjecture \(O\) for the ordinary Grassmannian. In 2006, Galkin and Golyshhev \([6]\) gave a very short proof of Conjecture \(O\) using a theorem of Seibert and Tian \([13]\) and some elementary considerations. In 2003, Rietsch \([11]\) gave a full description of eigenvalues and corresponding (simultaneous) eigenvectors of quantum multiplication operators for the Grassmannian, which actually proves Conjecture \(O\), by using a Peterson’s result and some combinatorics. Very recently we proved the Conjecture \(O\) for Lagrangian and orthogonal Grassmannian \([2]\), following Rietsch \([11]\).

As for toric Fano manifold, Galkin, Golyshev and Iritani \([7]\) proved Gamma conjectures I, II modulo Conjecture \(O\), and then Galkin \([5]\) has made some progress on Conjecture \(O\) by showing that the quantum cohomology ring of a toric Fano manifold contains a field as a direct summand.

It is natural to consider general homogeneous space \(X = G/P\) as next targets for Gamma conjectures. Indeed, here we prove Conjecture \(O\) for homogeneous spaces as a first step into this project. A scheme of proof of Conjecture \(O\) is to use the so-called quantum Chevalley formula which computes the multiplication \(\sigma_1 \star \sigma_2\) of two basis elements with \(\sigma_1\) or \(\sigma_2\) in \(H^2(X, \mathbb{Z})\), and a theory on nonnegative matrices including Perron-Frobenius theory.

To be precise, first note that since structure constants of the quantum product in \(H^*(X, \mathbb{C})\) in a certain basis are three point genus zero Gromov-Witten invariants, they are all nonnegative and hence the matrix \(M(X)\) of \([c_1(X)]\) with respect to this basis is a nonnegative matrix. Therefore once we prove that \(M(X)\) is irreducible (§3.1 below), then by the celebrated Perron-Frobenius theorem (§3.1 below), the conditions (1) and (2) are automatically satisfied. We remark that the use of Perron-Frobenius theorem in the proof of (1) and (2) is due to Kaoru Ono (p. 21 of \([7]\)). However, the Perron-Frobenius theorem does not assert that the Fano index \(r\) of \(X\) is equal to the number \(h\) of eigenvalues of maximal modulus, which is to be shown for the condition (3). For the condition (3), we first find a new ordered basis for \(H^*(X)\), proper to apply Proposition \(3.4\) below, to show that \(r\) divides \(h\). Then to show that conversely \(h\) divides \(r\), we bring a theory on directed graphs, a disguise of nonnegative matrices, into our situation and construct a certain number of cycles at a fixed vertex in the directed graph in question. The lengths of these cycles are used to show that \(h\), in turn, divides \(r\), and hence \(r = h\). This fact together with Proposition \(3.3\) proves the condition (3).

Lastly, we point out that one of advantages of our approach is that if one know one of eigenvalues (of not necessarily maximal modulus) of \([c_1(X)]\), then one can recover other eigenvalues of the same modulus from the known eigenvalue by rotating it by a fixed angle depending on the Fano index of \(X\) with the aid of Proposition \(3.3\).

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2. Quantum Cohomology of $G/P$

2.1. Notations. Throughout this paper, $G$ denotes a complex, connected, semisimple, algebraic group, $B$ a fixed Borel subgroup, and $T$ the maximal torus in $B$. As usual, $\mathfrak{g}$, $\mathfrak{b}$, and $\mathfrak{t}$ denote their Lie algebra of $G$, $B$ and $T$, respectively. Denote the set of all roots by $R$, the set of positive respectively negative roots relative to $B$ by $R^+$ respectively $R^-$. Then we have decomposition of root spaces $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\alpha}$. Let $\Delta$ be the set of simple roots and $I$ be an indexing set for $\Delta$. The parabolic subgroups $P$ of $G$ containing $B$ correspond to subsets $\Delta_P$ of $\Delta$. Let $I_P \subset I$ be an indexing set for $\Delta_P$, and $I^P = I \setminus I_P$. Let $R^+_P$ be the subset of positive roots which can be written as sums of roots in $\Delta_P$ and let $R_P = R^+ \cup (-R^+_P)$. Then the Lie algebra $\mathfrak{p}$ has a decomposition of root spaces $\mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R_P} \mathfrak{g}_{\alpha}$.

Let $W$ be the Weyl group of $G$, i.e., $W = N_G(T)/T$. Then $W$ is generated by simple reflections $s_i$, $i \in I$, where $s_i$ is a simple reflection corresponding to $\alpha_i$. For $u \in W$, the length of $u$, denoted by $l(u)$, is defined to be the minimum number of simple reflections whose product is $u$. The element in $W$ of the longest length is denoted by $w_0$. Then the opposite Borel subgroup $B^-$ is written as $B^- = w_0 B w_0$.

Let $W_P$ be a subgroup of $W$ generated by the generators $s_i$ with $i \in I_P$. Note that the generators $s_i$ with $i \in I_P$ are precisely ones such that $s_i \subset P$. We will write $[u]$ for cosets $uW_P$ in $W/W_P$. It is well-known that each coset $[u]$ has a unique representative of minimal length. Let $W^P$ be the subset of $W$ consisting of such representatives in the cosets. Let $w^P_0$ be the minimal length representative in $[w_0]$, and hence $w^P_0$ is an element of the longest length in $W^P$. The length of $[u]$, denoted $l([u])$, is defined to be the length of a minimal length representative in the coset $[u]$. The dual of $u$ in $W^P$, denoted $u^\vee$, is defined to be the minimal length representative of the coset $[w_0 u]$. Note that $l(w^P_0) = \dim G/P$ and $l(u^\vee) = l(w^P_0) - l(u)$. Let $0^P$ be the element of $W^P$ of minimum length. In fact, $w^P_0$ is the identity of $W$, and so $l(w^P_0) = 0$.

2.2. Cohomology. For convenience, throughout we will identify the element $u \in W^P$ with the element $[v] \in W/W_P$ if $u$ is a minimal length representative in $[v]$. For $u \in W^P$, let $X(u) = B_u P$ be the Schubert variety corresponding to $u$ and $Y(u) = B^- u P$ the opposite Schubert variety corresponding to $u$. Then $X(u)$ is a subvariety of $G/P$ of dimension $l(u)$ and $Y(u)$ is a subvariety of $G/P$ of codimension $l(u)$. Let $\sigma(u)$ respectively $\sigma_v$ be the cohomology class $[X(u)]$ respectively $[Y(u)]$. Then we have the following classical results.

1. For $u \in W^P$, $\sigma_u \in H^{2l(u)}(X)$, $\sigma(u) \in H^{2n-2l(u)}(X)$, and $\sigma_u = \sigma(u^\vee)$.
2. $H^\bullet(X) = \bigoplus_{u \in W^P} \mathbb{Z} \sigma_u = \bigoplus_{u \in W^P} \mathbb{Z} \sigma(u)$. In particular, $H^2(X) = \bigoplus_{i \in I^P} \mathbb{Z} \sigma_{s_i}$, and $H^{2n-2}(X) = \bigoplus_{i \in I^P} \mathbb{Z} \sigma(s_i)$.
3. $\int_X \sigma_u \sigma_v = 1$ if $v = u^\vee$, and 0 otherwise.

2.3. Degrees. Since the cohomology group $H^{2n-2}(X)$ can be canonically identified with the homology groups $H_2(X)$ by Poincaré duality, elements of $H^{2n-2}(X)$ may be referred to as curve classes. By a degree $d$, we mean an effective class in $H^{2n-2}(X)$, i.e., a nonnegative integral linear combination of the Schubert generators $\sigma(s_i)$ with $i \in I^P$. A degree $d = \sum_{i \in I^P} d_i \sigma(s_i)$ may be identified with $(d_i)_{i \in I^P}$.

For $\alpha \in R^+$, write $\alpha = \sum_{i \in I} m_{\alpha, \alpha_i} \alpha_i$ for some $m_{\alpha, \alpha_i} \in \mathbb{Z}_{\geq 0}$. Then we define the degree of $\alpha$ as

$$d(\alpha) = \sum_{i \in I^P} m_{\alpha, \alpha_i} \frac{\alpha_i}{\langle \alpha, \alpha \rangle} \sigma(s_i).$$

Note that $d(\alpha_i) = \sigma(s_i)$ if $i \in I^P$, and $d(\alpha_i) = 0$ otherwise, since $m_{\alpha_i, \alpha_j} = \delta_{i,j}$ for $i, j \in I$. Let $h_\alpha = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$ and $\omega_\alpha$ be the fundamental weight corresponding to $\alpha_i$, so that $h_\alpha$ and $\omega_\alpha$ are dual
bases for $i \in I$. Then $h_\alpha(\omega_\alpha) = m_{\alpha,\alpha}(\alpha, \alpha)/(\alpha, \alpha)$, and hence we have
\[
    d(\alpha) = \sum_{i \in I^p} h_\alpha(\omega_\alpha) \sigma(s_i).
\]

Set
\[
    \rho_p = \frac{1}{2} \sum_{\alpha \in R^+ \setminus R^+_P} \alpha \quad \text{and} \quad n_\alpha = 4 \frac{\langle \rho_p, \alpha \rangle}{\langle \alpha, \alpha \rangle}.
\]

**Lemma 2.1.** The first Chern class of $X = G/P$ can be written as
\[
    c_1(X) = 4 \sum_{i \in I^p} n_i \sigma_{s_i} = 2 \sum_{i \in I^p} h_\alpha(\rho_p) \sigma_{s_i},
\]
where $n_i := n_{s_i}$.

**Remark 2.2.** Note that in the Schubert basis $\{\sigma_{s_i} \mid i \in I^p\}$ of $H^2(X)$, the first Chern class $c_1(X)$ has a positive integral coefficients $n_i$, and $n_i$ can be written as $n_i = \int_{X(u)} c_1(X)$. The positivity of $n_i$ plays important role in our proof together with nonnegativity of Gromov-Witten invariants below (§2.4). It is obvious that $2 \leq n_i \leq \dim X + 1$ for all $i \in I^p$. We recall that if we let $r := \gcd\{n_i \mid i \in I^p\}$, then $r$ is equal to the Fano index of $X$.

### 2.4 Quantum cohomology of $G/P$. To define the quantum cohomology ring of $X$, we begin by defining Gromov-Witten invariants. Given $u, v, w \in W^p$ and $d = \sum d_i \sigma(s_i)$ with $l(u) + l(v) + l(w) = \dim F + \int_X c_1(X) \cdot d$, the three pointed, genus zero Gromov-Witten invariant associated with $u, v, w$ and $d$, denoted $c_{u,v,w}^{d}$, is defined as the number of morphisms $f : \mathbb{P}^1 \to X$ of degree $d$ such that the three fixed points $f(0), f(1)$ and $f(\infty)$ on $\mathbb{P}^1$ pass through general translates of $X(u), X(v)$ and $X(w)$, respectively.

For each $i \in I^p$, take a variable $q_i$, and let $\mathbb{Z}[q]$ be the polynomial ring with indeterminates $q_i, i \in I^p$. We will regard $\mathbb{Z}[q]$ as a graded $\mathbb{Z}$-algebra by assigning to $q_i$ the degree $2n_i$. For a degree $d = \sum d_i \sigma(s_i)$, let $q^d$ stand for $\prod q_i^{d_i}$.

The quantum cohomology ring of $X$, denoted $qH^*(X)$, is defined to be, as a $\mathbb{Z}[q]$-module,
\[
    qH^*(X) = H^*(X) \otimes \mathbb{Z}[q].
\]
The Schubert classes $\sigma_u$ with $u \in W^p$ form a $\mathbb{Z}[q]$-basis for $qH^*(X)$. The multiplication is defined as
\[
   \sigma_u \ast \sigma_v = \sum_d \sum_w c_{u,v,w}^{d} \sigma_w,
\]
where the sums are taken over all $w \in W^p$ and degrees $d$ such that $l(u) + l(v) = l(w) + \int_X c_1(X) \cdot d$.

The quantum product of two general Schubert classes $\sigma_u$ and $\sigma_v$ is far from completely understood, whereas if one of them is of degree two, then the so-called quantum Chevalley formula, due to Fulton and Woodward(II), gives an explicit description of the coefficients in (2.1).

**Proposition 2.3.** (Quantum Chevalley formula) For fixed $i \in I^p$, and $u \in W^p$, the quantum product of $\sigma_{s_i}$ and $\sigma_u$ is given by
\[
   \sigma_{s_i} \ast \sigma_u = \sum_{\alpha} h_\alpha(\omega_\alpha) \sigma_v + \sum_{\alpha} q^{d(\alpha)} h_\alpha(\omega_\alpha) \sigma_w,
\]
where the first sum is taken over roots $\alpha \in R^+ \setminus R^+_P$ for which $l(v) = l(u) + 1$ and $v$ is the minimal length representative in $[v_{s_\alpha}]$, and the second sum is taken over roots $\alpha \in R^+ \setminus R^+_P$ for which $l(w) = l(u) + 1 - n_\alpha$. 

Remark 2.4. Since $h_\alpha$ and $\omega_\alpha$ are dual bases for $i \in I$, by the very definition of $R^+_I$, if $\alpha \in R^+_I$, then $h_\alpha(\omega_\alpha) = 0$ for all $i \in I^P$, and if $\alpha \in R^+ \setminus R^+_I$, then there is an $i \in I^P$ such that $h_\alpha(\omega_\alpha) \neq 0$.

Some authors often define the quantum cohomology ring of $X$ without using the quantum variables $q_i$, $i \in I^P$. We denote this ring by $H^*(X, \mathbb{C})$. In our language, the ring $H^*(X, \mathbb{C})$ is identified with the specialization of $qH^*(X, \mathbb{C})$ at $q_i = 1$ for all $i \in I^P$, i.e.,

$$H^*(X, \mathbb{C}) = qH^*(X, \mathbb{C}) \otimes \mathbb{C}[q]/ < q_i - 1 \mid i \in I^P > .$$

Note that $H^*(X, \mathbb{C})$ is a finite dimensional vector space over $\mathbb{C}$, while $qH^*(X, \mathbb{C})$ is not over $\mathbb{C}$, but over $\mathbb{C}[q]$.

3. Nonnegative matrices

In this section, we review Perron-Frobenius theory on nonnegative matrices and some related results which will be used later. Details on these materials can be found in [10] and [11].

3.1. Irreducible matrices.

Definition. A nonnegative matrix $M$ is said to be cogredient to a matrix $M'$ if there is a permutation matrix $P$ such that $M = P^T M' P$.

A nonnegative matrix $M$ is called reducible if it is cogredient to a matrix in the form

$$M' = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix},$$

where $A, D$ are square submatrices. If it is not reducible, then $M$ is called irreducible.

Remark 3.1. (1) Note that if $V$ is a vector space with an ordered basis $B = \{v_1, \ldots, v_m\}$ and $T$ is an operator on $V$, then the matrix $[T]_B$ of $T$ with respect to the basis $B$ is reducible if and only if there is a nontrivial proper coordinate subspace under $T$, equivalently there is a ordered basis $B'$ with respect to which $[T]_{B'}$ is in the form $(3.2)$, where $B'$ is obtained from $B$ by reordering elements of $B$.

(2) Suppose $[T]_B$ is reducible. Let $V_0$ denote a nontrivial proper coordinate subspace of $V$ invariant under $T$. We point out that if $V_0$ contains a basis element $v_i \in B$, then $V_0$ contain all basis elements $v_j \in B$ such that the coefficient $b_{ji}$ of $v_j$ is nonzero in $T(v_i) = \sum_{k=1}^m b_{ki} v_k$. More generally, suppose $T = \sum_{i=1}^l c_i T_i$ for some operators $T_i$ and positive numbers $c_i$. Then the coefficient $b_{ji}$ is nonzero if and only if there exists an $1 \leq r \leq l$ such that the coefficient $b'_{ji}$ of $v_j$ is nonzero in $T_r(v_i) = \sum_{k=1}^m b'_{ki} v_k$.

Proposition 3.2. (Perron-Frobenius Theorem) Let $M$ be an irreducible matrix. Then $M$ has a real positive eigenvalue $\delta_0$ such that

$$\delta_0 \geq |\delta|$$

for any eigenvalue $\delta$ of $M$. Furthermore, there is an positive eigenvector corresponding to $\delta_0$.

Definition. For an irreducible matrix $M$, we define the index of imprimitivity of $M$, denoted $h(M)$, to be the number of eigenvalues of maximal modulus. If $h(M) = 1$, then $M$ is said to be primitive; otherwise, it is imprimitive.

If $M$ is an irreducible matrix, then eigenvalues of the same modulus are completely determined by one of them.

Proposition 3.3. Let $M$ be an irreducible matrix with $h(M) = h$. Then the set of eigenvalues of $M$ is invariant under rotation by $\frac{2\pi}{h}$, but not by smaller angles than $\frac{2\pi}{h}$. 
Definition. A matrix in the form
\[
\begin{bmatrix}
0 & A_{12} & 0 & \cdots & 0 & 0 \\
0 & 0 & A_{2,3} & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & A_{k-1,k} \\
A_{k1} & 0 & \cdots & 0 & \cdots & 0
\end{bmatrix}
\]
(3.3)
is said to be in the superdiagonal \((m_1, m_2, \ldots, m_k)\)-block form if the block \(A_{i,i+1}\) is a \((m_i \times m_{i+1})\) matrix for \(i = 1, \ldots, k-1\), and \(A_{k1}\) is a \((m_k \times m_1)\) matrix.

If an irreducible matrix \(M\) is cogredient to a matrix \(M'\) in the form (3.3), much spectral information of \(M\) can be read off from \(M'\). Among them, first comes the index of imprimitivity.

Proposition 3.4. (Minc) Let \(M\) be an irreducible matrix with \(h(M) = h\). Then \(M\) is cogredient to a matrix in the form (3.3) such that all \(k\) blocks are nonzero if and only if \(k\) divides \(h\).

3.2. Directed graphs. When we deal with spectral properties of nonnegative matrices, mostly we are interested in only zero pattern of their entries. One of ways of encoding this pattern is through the so-called directed graph. We list a multiple of definitions related with directed graphs.

Definition. (1) A directed graph \(D\) consists of data (Ver, Arc), where Ver is a set and Arc is a binary relation on Ver, i.e., a subset of \(\text{Ver} \times \text{Ver}\). Elements of Ver are called vertices and elements of Arc are called arcs. For convenience, we assume that \(\text{Ver} = \{v_1, \ldots, v_m\}\).

(2) A sequence of arcs \((v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), (v_{i_3}, v_{i_4}), \ldots, (v_{i_{k-1}}, v_{i_k})\) in \(D\) is called a path from \(v_{i_1}\) to \(v_{i_k}\) which we will denote by \(\text{PATH}(v_{i_1}: v_{i_k})\). The length of a path is the number of arcs in the sequence. A path of length \(k\) from a vertex to itself is called a cycle of length \(k\).

(3) The adjacency matrix \(A\) of \(D\), denoted \(A = A(D) = (a_{i,j})\), is a \(m \times m\) square matrix with entries 0 or 1 defined by
\[
a_{i,j} = \begin{cases} 
1 & \text{if } (v_j, v_i) \in \text{Arc}, \\
0 & \text{otherwise}.
\end{cases}
\]

(4) A directed graph \(D\) is said to be associated with an nonnegative matrix \(M\) if the adjacency matrix \(A(D)\) has the same zero pattern as \(M\).

(5) A directed graph is strongly connected if for any ordered pair \((v_i, v_j)\) with \(i \neq j\), there is a path from \(v_i\) to \(v_j\).

(6) Let \(D\) be a strongly connected directed graph. The index of imprimitivity of \(D\), denoted \(h(D)\), is defined to be the g.c.d of lengths of all cycles in \(D\).

Remark 3.5. Let us make a few remarks on these definitions.

(1) We note that a directed graph \(D\) can be visualized by a diagram in which an arc \((v_i, v_j)\) is represented by a directed line going from \(v_i\) to \(v_j\). We will refer to the diagram associated to \(D\) as a directed graph.

(2) Let \(V\) be a vector space with an ordered basis \(B = \{v_1, \ldots, v_m\}\), and, for \(i = 1, \ldots, l\), let \(T_i\) an operator on \(V\) with \([T_i]_B\) nonnegative. For positive real numbers \(c_1, c_2, \ldots, c_l\), let \(T := \sum_{i=1}^l c_i T_i\). Then we can associate to \(T\) a directed graph
\[
D(T : B) = (\text{Ver}(T : B), \text{Arc}(T : B)).
\]
Take $\text{Ver}(T : B) = \{v_1, ..., v_m\}$, and define a relation $\text{Arc}(T : B)$ on $\text{Ver}(T : B)$ by

$$(v_i, v_j) \in \text{Arc}(T : B) \iff b_{ji} \neq 0 \text{ in } T(v_i) = \sum_{k=1}^{m} b_{ki}v_k,$$

$$\iff \exists \, r, 1 \leq r \leq l, \text{ such that } b'_{ji} \neq 0 \text{ in } T_r(v_i) = \sum_{k=1}^{m} b'_{ki}v_k.$$ 

Note that the matrix $[T]_B$ has the same zero pattern as the adjacency matrix $A(D(T : B))$, and hence the directed graph $D(T : B)$ is associated with the matrix $[T]_B$.

Now we compare properties of nonnegative matrices and their associated directed graphs.

**Proposition 3.6.** (1) A nonnegative matrix is irreducible if and only if the associated directed graph is strongly connected.

(2) The index of imprimitivity of an irreducible matrix is equal to the index of imprimitivity of the associated directed graph.

**4. Main result**

The quantum cohomology ring $H^*(X, \mathbb{C})$ is a finite dimensional complex vector space with the Schubert basis $S$ consisting of Schubert classes $\sigma_u$ for $u \in W^P$. Arrange elements of $S$ linearly once and for all to make $S$ into an ordered basis. We will denote this ordered basis by $S$, too.

**Proposition 4.1.** Let $M(X)$ be a matrix of the operator $[c_1(X)]$ on $H^*(X, \mathbb{C})$ with respect to $S$. Then $M(X)$ is irreducible.

**Proof.** Since $[c_1(X)] = \sum_{i \in I^P} n_i[\sigma_s]$ for positive $n_i$ and each matrix $[\sigma_s]_S$ is nonnegative, the matrix $M(X) = [c_1(X)]_S$ is nonnegative. To show that $M$ is irreducible, suppose that $M(X)$ is reducible. Then there is a nontrivial proper coordinate subspace $V$ of $H^*(X)$ which is invariant under the operator $[c_1(X)]$. Let $W^p_0$ be the index set for Schubert classes which span $V$, so that $V = \bigoplus_{u \in W^p_0} \mathbb{C}[\sigma_u]$. Let $W^p_1 = W^P \setminus W^p_0$. We first claim that the element $w^p_0$ belongs to $W^p_0$, and the element $0^p$ belongs to $W^p_1$. Indeed, if $w^p_0$ does not lie in $W^p_0$, choose an element $v_1 \in W^p_0$, and a reduced expression of $v_1 = s_i \cdots s_s$. Next, we fix $j \in I^P$ with $j \neq i_1$. Note that $v_2 := v_1 s_j$ has a length $l(v_2) = l(v_1) + 1$, since $s_i \cdots s_s s_j$ is a reduced expression for $v_2$ (Exchange condition in p. 14 of [F]). Now consider the quantum Chevalley formula for $\sigma_u \ast \sigma_j$. We can easily see that in the classical part of $\sigma_u \ast \sigma_j$, there is a term $h_\alpha(\omega_\alpha)\sigma_\omega$, which is nonzero since $h_\alpha(\omega_\alpha) = 1$. Thus $\sigma_v \in V$ by Remark 3.1(2), and so $v_2 \in W^p_0$. Repeating this process, we get elements $v_1, v_2, ..., v_m, ...$ of $W^p_0$ such that $l(v_m) = l(v_{m-1}) + 1 = l(v_1) + m - 1$. Since $w^p_0$ is a unique element of the longest length in $W^p$, we reach the situation where $v_k = w^p_0$ for some $k$.

Now let us show that $0^p$ lies in $W^p_1$. Since $W^p_1$ is nonempty by the assumption, we can take an element $w \in W^p_1$. Then there always exist $\alpha \in R^+ \setminus R^+_P$ and $u \in W^P$ such that $w = us_\alpha$ and $l(u) + 1 = l(w)$. By Remark 2.7 there is an $k \in I^P$ such that $h_\alpha(\omega_{\alpha k}) \neq 0$. So since the classical part in the quantum Chevalley formula for $\sigma_u \ast \sigma_k$ has a nonzero term $h_\alpha(\omega_{\alpha k})\sigma_\omega$, there is a basis element $\sigma_\omega$ with a nonzero coefficient in the expansion of $[c_1(X)](\sigma_u)$ in the basis $S$. But if $u \in W^p_0$, then, by the same reason as above, we would have $w \in W^p_0$, which contradicts the assumption that $w \in W^p_1$. Thus $u \in W^p_1$. Continuing this process, we will end up with $0^p \in W^p_1$.

To derive a contradiction, we take an element $w(m) \in W^p_0$ whose length is minimal among elements in $W^p_0$. There are two possible cases to consider.

Case I: $l(w(m)) \leq n_j - 1$ for some $j \in I^P$.

Case II: $l(w(m)) > n_j - 1$ for all $i \in I^P$. In Case I, we choose an element $u \in W^p_0$ with $l(u) = n_j - 1$. Existence of such an element $u \in W^p_0$ can be shown in the same way as we showed that $w^p_0 \in W^p_0$.
Now we consider the quantum Chevelley formula for $\sigma_u \ast \sigma_s$. It reads that there is a nonzero term $h_{\alpha_i}(\omega_{\alpha_i}) q^{d(\alpha_i)} \sigma_0 \sigma_P \equiv q \sigma_P \equiv q \sigma_0 \sigma_P$ in the expansion of $\sigma_u \ast \sigma_s$, in the basis $S$, and so there is a basis element $\sigma_0 \sigma_P$ with a nonzero coefficient in the expansion of $[c_1(X)](\sigma_u)$. Then by Remark 3.1 (2), $\sigma_0 \sigma_P \in V$, i.e., $0^P \in W_0^P$, since $u \in W_0^P$. But this violates the fact that $0^P \in W_1^P$.

In Case II, for any $k \in I^P$, in the expansion of the product $\sigma_v \sigma_a \sigma_s$, there is a term $q^{d(\alpha)} \sigma_a \sigma_v$ with a nonzero coefficient for some $\alpha \in R^+ \setminus R^+_P$ and $v \in W^P$, e.g., $q^{d(\alpha_k)} \sigma_v$. As before, this implies $v \in W_1^P$. Note that $l(w(m)) + 1 - n_k \geq l(v)$, and $n_k \geq 2$ (Remark 2.2). Thus $l(w(m)) > l(v)$. This contradicts the minimality of the length of $w(m)$. Therefore for any cases, $M(X)$ is not reducible. □

Remark 4.2. (1) We note that the proof of Proposition 3.1 works for any quantum multiplication operators of the form $[\sigma] = \sum_{i \in I^P} a_i [\sigma_{s_i}]$ for positive real numbers $a_i$.

(2) We mention that a general idea of proving the irreducibility of $M(X)$ was taken from Lemma 9.3 (p. 384) in [12], where Riettsch used some Peterson result to show the irreducibility of the matrix $[\sigma] S$ for a flag manifold $X$ of type $A$, where $\sigma = \sum_{w \in W^P} \sigma_w \in H^*(X, \mathbb{C})$.

Recall that $r = \text{g.c.d.} \{ n_i \mid i \in I^P \}$ is the Fano index of $X$, and $S$ is an ordered basis of $H^*(X)$. The order on $S$ induces a linear order on the index set $W^P$, denoted $\prec$. It is necessary for us to give a new linear order on $W^P$. First, we make a partition on the set $W^P$ into $r$ subsets. For each $0 \leq a \leq r - 1$, let $W^P(a)$ be the subset of $W^P$ consisting of elements $u$ with $l(u) \equiv a \mod r$, and we assign to each $W^P(a)$ the weight $a$. Now we define a linear order $\prec_q$ on $W^P$ as

$$u \prec_q v \iff \begin{cases} u \in W^P(a), & v \in W^P(b), \text{ and } a > b, \\ u, v \in W^P(a), & \text{ and } u < v. \end{cases}$$

The order $\prec$ on $W^P$ naturally makes the Schubert basis for $H^*(X)$ into an ordered basis, denoted $S_q$, in the way that the basis elements $\sigma_u$ for $u \in W^P(r - 1)$ come first, $\sigma_u$ for $u \in W^P(r - 2)$ second, and so on.

Lemma 4.3. The matrix $M(X)_q$ of the operator $[c_1(X)]$ on $H^*(X)$ with respect to the ordered basis $S_q$ is in the superdiagonal $(m_1, m_2, ..., m_r)$-block form (3.3) with $k = r$, where $m_i := |W^P(r - i - 1)|$ for $i = 1, ..., r$, where $W^P(-1) := W^P(r - 1)$.

Proof. It is obvious that the blocks $A_{i,i+1}$ for $i = 1, ..., r - 1$ and $A_{r,1}$ are nonzero. Now note that if the basis element $\sigma_u$ appears with a nonzero coefficient in the expansion of $[c_1(X)](\sigma_u)$ in the basis $S_q$, then by the degree condition of quantum multiplication we have

$$l(v) \equiv l(u) + 1 \mod r.$$

This proves the lemma. □

Corollary 4.4. The Fano index $r$ of $X$ divides the index of imprimitivity $h(M(X))$ of $M(X)$.

Proof. Since $M(X)$ is cogredient to $M(X)_q$, the corollary immediately follows from Lemma 4.3 and Proposition 3.4. □

Definition. For a homogeneous space $X = G/P$, define the directed graph $D(X)$ as

$$D(X) = D([c_1(X)], S),$$

where $D([c_1(X)], S)$ was defined in Remark 3.5 (2).

Since $M(X)$ is irreducible, its associated directed graph $D(X)$ is strongly connected by Proposition 3.6, and so it makes sense to consider the index of imprimitivity of $D(X)$. We will show that $h(D(X))$ divides $r$ by following lemma.

Lemma 4.5. For each $i \in I^P$, there is a cycle of length $n_i$ in $D(X)$ through the fixed vertex $\sigma_0^P$. 

Proof. First note that \( l(w^P_0) = \dim G/P \geq n_i - 1 \) for all \( i \in I^P \), and if \( v \in W^P \), then there is a path from the vertex \( \sigma_0^P \) to the vertex \( \sigma_v \) of the length \( l(v) \). Indeed, given \( v \in W^P \), there is a \( u \in W^P \) such that \( l(u) = l(v) - 1 \) and the basis element \( \sigma_u \) appears, with a nonzero coefficient, in the expansion of \( [c_1(X)](\sigma_u) \) in the basis \( S_q \). Remark (3.3) (2) implies that there is a path \( \text{PATH}(u : v) \) from \( \sigma_u \) to \( \sigma_v \) of length 1. By the induction hypothesis, there is the shortest path \( \text{PATH}(0^P : u) \) from \( \sigma_0^P \) to \( \sigma_u \) of length \( l(u) \). Now join two paths \( \text{PATH}(0^P : u) := \text{PATH}(0^P : u) \sqcup_u \text{PATH}(u : v) \). Then \( \text{PATH}(0^P : v) \) is the shortest path from \( \sigma_0^P \) to \( \sigma_v \) of length \( l(v) \). Now fix \( i \in I^P \), and choose \( u \) with \( l(u) = n_i - 1 \). Consider the quantum multiplication \( \sigma_u \star \sigma_s \). By the quantum Chevalley formula, there is a basis element \( q_i \sigma_0^P \) with a nonzero coefficient in the expansion of \( \sigma_u \star \sigma_s \) in the basis \( \{q^d w \mid d \text{ degrees and } w \in W^P \} \). By Remark (3.3) this implies that there is a cycle of length \( n_i \) at \( \sigma_0^P \) in \( D(X) \).

Let \( h_0 \) be the g.c.d of the lengths of all cycles of \( D(X) \) through \( \sigma_0^P \). Note that \( h(D(X)) \) divides \( h_0 \) and \( h(D(X)) \) and Lemma (4.5). Thus \( h(M(X)) \) divides \( r \), since \( h(D(X)) = h(M(X)_q) = h(M(X)) \). This fact together with Corollary (4.4) implies

**Corollary 4.6.** The imprimitivity index of the irreducible matrix \( M(X) \) is equal to the Fano index \( r \).

**Theorem 4.7.** Homogeneous spaces \( X = G/P \) satisfy Conjecture \( \mathcal{O} \).

**Proof.** Since \( M(X) \) is irreducible, Condition (1) and (2) are satisfied by Proposition (3.2). Condition (2) follows from Corollary (4.6) and Proposition (3.3). □

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KOREA INSTITUTE FOR ADVANCED STUDY, 85 Hoegiro, Dongdaemun-gu, Seoul, 130-722, Korea

E-mail address: daevoongc@kias.re.kr