Comparison Of Tests For Isomorphism In Planetary Gear Trains

I Rajasri Reddy, Ch VInay Kumar Reddy, VVD Rao, A Chandra Shekar
1 Sumathi Reddy Institute of Technology for Women, India
2 SR Engineering College, India
3 BITS-Pilani, Hyderabad, India
4 NallaMalla Reddy Engineering College, Hyderabad, India

e-mail: principalsritw@gmail.com

Abstract: There are plenty of ways available for synthesis and analysis of Planetary Gear Trains (PGTs) of one DOF. However, every method has its own shortcomings. In this paper a comparison is made between Characteristic polynomial, Eigenvalues and Eigenvectors, Hamming Number Method and Modified Path Matrix (MPM) method. There are many methods available to test isomorphism in PGTs, out of which these four methods was analyzed in this paper. For a given PGT with a number of links and a single Degree of Freedom (DOF), adjacency matrix is enough to find out the Eigenvalues and Eigenvectors. Isomorphism of PGT is determined using the Eigenvalues. If Eigenvalues are similar then the PGTs are isomorphic. Similarly, if the characteristic polynomials of two PGTs are same then it represents the isomorphic PGTs. Characteristic polynomials are determined from the Adjacency matrix. Hamming method also uses adjacency matrix to generate Hamming matrix. Hamming strings are developed from Hamming Matrix. Uniform hamming strings of two PGTs indicates isomorphism in PGTs. Along with isomorphism, Symmetry also known from the Hamming method. Modified path matrix method uses a connectivity matrix to generate MPM. If train values of two PGTs are same then they will be isomorphic otherwise non isomorphic. As per literature as the number of links increases the results may not be accurate with Characteristic coefficients and Eigenvalues methods, though all these methods are used to detect isomorphism among a group of PGTs. Whereas with the Hamming number approach, one can detect isomorphism, symmetry and number of possible level combinations of an PGT with a single hamming matrix.

1. Introduction
Researchers using Graph theory for synthesis and analysis of PGTs from the decades. A kinematic chain is converted as a graph for easy analysis. Buchsbaum and Freudenstein are pioneers [1] developed conditions to be satisfied by a graph of a PGT. The correspondence between the graph and displacement equations was developed by Freudenstein [2]. In a labeled graph lower pairs or turning pairs are represented by thin lines and geared pairs by double line or Thick line. Levels are denoted on lower pairs which trace the location of the rotational axis. In a ‘n’ link PGT there must be (n-1) turning pairs and (n-2) gear pairs. Transfer vertex exists in each fundamental circuit [3]. Characteristic polynomial method was developed by Uicker and Raicu [4] which is the fundamental research on kinematic chains to determine isomorphism. This concept is adapted to geared kinematic chains up to 4 links by Lung-Win...
Tsai [5]. A computerized analysis of geared kinematic chains using characteristic polynomials was done by Ravi Shankar and Mruthyunjaya [6]. Canonical graph representation using Pseudo isomorphic graphs was developed by Chatterjee and Tsai [7]. Canonical graph is another way to represent the geared kinematic chains in which each PGT is divided into a number of basic entities. Zongyu Chang [8] proposed the Eigenvectors and Eigenvalues method to detect isomorphism in PGTs. Complicated computations required if the Eigenvalues of PGTs are identical to confirm the isomorphism. Efficiency of this method is very low. Fuzzy adjacency matrix is used in fuzzy logic method [9]. Nomographs concept was introduced by E. L. Ismail to determine the velocity of an PGT[10]. Rajasri et. all used the Hamming number approach to detect isomorphism of an PGT [11,12]. Ali Hasan et al.[13] used Modified Path Matrix (MPM) to identify the isomorphism inPGTs. Identifying isomorphic PGTs is very important to reduce the duplication of PGTs in the generation process [14]. From a group of PGTs, one best isomorphic PGT is selected for further process of generation. All the isomorphic PGTs are having different structural properties i.e. they may not generate equal number of next level graphs, they may vary in velocity ratios, they may vary in structural arrangement etc. This should be identified by the structural aspects like symmetry of an PGT, which is only possible with the hamming matrix.

2. Methodology

2.1 Characteristic Polynomial Method
This method was introduced by Uicker and Raicu [4] and is further developed by Tsai [5]. The connectivity of the links will be known from the Distance matrix and it is same as adjacency matrix. Using matrix algebra, characteristic polynomial coefficient is determined. If the coefficient of characteristic polynomial is the same for two PGTs then the two PGTs are said to be isomorphic. The condition
\[|x^1 - A_1| = |x^1 - A_2|\]
must be satisfied in case if the PGTs are isomorphic. One need to calculate the determinant of the matrix \((X^I - A)\), to determine the linkage characteristic Polynomial of a PGT. Where \(X\) represents the variable, \(I\) represent Identity Matrix and \(A\) is Adjacency Matrix. Characteristic coefficient of a PGT is calculated using the MAT lab program. For finding isomorphism in a PGT the characteristic polynomial test is sufficient. Coefficient must be unique for a given topology.

2.2. Eigenvalues and Eigenvectors Method
Eigenvalues and Eigenvectors of an PGT are computed using a MATLAB Program. These are derived through adjacency matrix. Isomorphism is confirmed if the two PGTs have unique Eigenvalues otherwise not. To confirm the isomorphism after identical PGTs, one needs to find Eigenvectors which requires a lot of computational efforts. Rows should be interchanged in a sequential manner to calculate the eigenvectors. A row transformation matrix \((R)\) is generated after interchanging the rows.

2.3. Hamming Number Method:
Information and communication theory first used hamming distances to know the planar [10] Kinematic chains (KCs) and PGTshaving ‘n’ links with single DOF. Adjacency matrix is generated first based on the link connectivity then hamming matrix is calculated using adjacency matrix. Hamming strings are generated from hamming matrices. Unique hamming string for the two PGTs is the sign for isomorphism. Symmetry is identified with hamming strings of a PGT [15, 16].

2.4. Modified Path Matrix Method
MPM method is used to identify the isomorphism in PGTs and also to compare the geared kinematic chains by its train values. If the train values of two PGTs are identical then the two PGTs are said to be isomorphic otherwise non isomorphic. MPM is a square symmetric matrix \([a_{ij}]\) calculated based on the connectivity of the links[13]. It is calculated as follows
$$a_{ij} = \begin{cases} \min \sum_{i}^{j=k} \frac{1}{(LkC)_i} \text{ i.e. Minimum summation of joint value:} \\
\text{If vertex connected to vertex j by lower pair or ig erpair w ere} \text{Lk - Nodal value of lower pair:} \\
\text{Hk - Nodal value of ig erpair:} \\
0 \text{ otherwise (including i = f)} \end{cases}$$

In this method joint value is identified to each joint. Then the least path value is calculated based on the shortest path of the link using joint values. Elements in the Modified Path Matrix are generated from least path value. Summation of elements in each row generates a pair value. The summation of all pair values results in a train value of a PGT. If train values are similar then the PGTs are isomorphic otherwise non isomorphic.

3. Illustrative Examples

3.1. Characteristic Polynomial Method

Four Six link PGTs with one DOF are shown in Figs. 1, 2, 3 and 4. These PGTs are considered to compare the above four methods to identify isomorphism. Table: 1 consists of four PGTs along with their Adjacency matrices:

| Table 1: Six link PGTs and their Adjacency matrices |
|---------------------------------------------------|
| 6 Link PGTs | Adjacency Matrix |
| Figure 1    | \( A1 = \begin{bmatrix} 0 & 1 & 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 1 & 2 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 \end{bmatrix} \) |
| Figure 2    | \( A2 = \begin{bmatrix} 0 & 1 & 1 & 2 & 0 & 2 \\ 1 & 0 & 2 & 1 & 2 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \) |
| Figure 3    | \( A3 = \begin{bmatrix} 0 & 1 & 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 1 & 2 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{bmatrix} \) |
The characteristic coefficient of an PGT is shown in Figure 1 & 2 are,

\[ CP_1 = [1.0 \ -0.0 \ -21.0 \ -16.0 \ 70.0 \ 24.0 \ -33.0] \]

\[ CP_2 = [1.0 \ 0.0 \ -21.0 \ -16.0 \ 70.0 \ 24.0 \ -33.0] \]

The characteristic coefficient values of both the PGTs are identical, so the given two PGTs shown in Figs. 1 and 2 are isomorphic to each other.

Consider two PGTs shown in Figure 3 and Figure 4. The characteristic polynomials of PGTs shown in Figs. 3 and 4 are \( CP_3 = CP_4 = [1.0 \ -0.0 \ -21.0 \ -16.0 \ 73.0 \ 36.0 \ -48.0] \).

Characteristic Polynomials of PGTs shown in Figure 3 & 4 are also identical hence the two PGTs are isomorphic but the PGTs shown in Figure 1 and Figure 3 are non-isomorphic. Similarly, PGT shown in Figure 2 & 4 are also non-isomorphic.

### 3.2. Eigenvalues and Eigenvectors Method

Consider the Adjacency matrix of the PGT in Figure 1. Eigenvalues and Eigenvectors are given as

\[
\text{Eigenvalues for Figure 1} = \begin{bmatrix} -3.1819 \\ -2.6868 \\ -0.8938 \\ 0.5936 \\ 1.5884 \\ 4.5806 \end{bmatrix}
\]

\[
\text{Eigenvectors} = \begin{bmatrix} -0.2858 & 0.4760 & 0.3428 & 0.6376 & -0.1083 & 0.3948 \\ -0.5033 & -0.4885 & 0.1012 & -0.2936 & -0.4006 & 0.5011 \\ 0.4062 & 0.1864 & -0.6101 & 0.0849 & -0.5726 & 0.3050 \\ 0.5033 & -0.4885 & 0.1012 & 0.2936 & 0.4006 & 0.5011 \\ 0.2858 & 0.4760 & 0.3428 & -0.6376 & 0.1083 & 0.3948 \\ -0.4062 & 0.1864 & -0.6101 & -0.0849 & 0.5726 & 0.3050 \end{bmatrix}
\]

Similarly, the Eigenvalues and Eigenvectors of the PGTs shown in Figure 2 are given as

\[
\text{Eigenvalue for Figure 2} = \begin{bmatrix} -3.1819 \\ -2.6868 \\ -0.8938 \\ 0.5936 \\ 1.5884 \\ 4.5806 \end{bmatrix}
\]
Eigenvectors =
\[
\begin{bmatrix}
0.5033 & -0.4885 & 0.1012 & 0.2936 & 0.4006 & 0.5011 \\
-0.5033 & -0.4885 & 0.1012 & -0.2936 & -0.4006 & 0.5011 \\
0.2858 & 0.4760 & 0.3428 & -0.6376 & 0.1083 & 0.3948 \\
-0.2858 & 0.4760 & 0.3428 & 0.6376 & -0.1083 & 0.3948 \\
0.4062 & 0.1864 & -0.6101 & 0.0849 & -0.5726 & 0.3050 \\
-0.4062 & 0.1864 & -0.6101 & -0.0849 & 0.5726 & 0.3050 \\
\end{bmatrix}
\]

Eigenvalues of both the PGTs are identical; hence both the PGTs are isomorphic. As a second case, Figure 3 and Figure 4 are two PGTs to be considered for further evaluation, Eigenvalues and Eigenvectors are determined from the adjacency matrix. From the results we can conclude that the Eigenvalues of both the PGTs are same.

Eigenvalue for $\mathbf{g}_1$ =
\[
\begin{bmatrix}
-3.2855 \\
-2.4176 \\
-1.1871 \\
0.6707 \\
1.6673 \\
4.5522 \\
\end{bmatrix}
\]

From the above evaluation results, it is clear that when the Eigenvalues of the PGTs are uniform then those PGTs are isomorphic, and it is also clear that one need not go for the calculation of Eigenvectors to test the isomorphism. It has been proved with testing of several PGTs. One can quickly calculate Eigenvalues. With this one can avoid robust calculation which is required to calculate the Eigenvectors.

3.3. Hamming Number Method:
To test isomorphism in PGTs using hamming approach, consider two 6-link PGTs with single DOF shown in Fig: 1 and Fig: 2. Hamming Matrix for Fig: 1 and Fig: 2 are given below

Hamming Matrix for $\mathbf{g}_1$ =
\[
\begin{bmatrix}
00 & 10 & 07 & 08 & 08 & 03 \\
10 & 00 & 07 & 12 & 08 & 09 \\
07 & 07 & 00 & 09 & 03 & 06 \\
08 & 12 & 09 & 00 & 10 & 07 \\
08 & 08 & 03 & 10 & 00 & 07 \\
03 & 09 & 06 & 07 & 07 & 00 \\
\end{bmatrix}
\]

Hamming string for Fig: 1 = 228 [46  46  36 36  32  32]

Hamming Matrix for $\mathbf{g}_2$ =
\[
\begin{bmatrix}
00 & 12 & 10 & 08 & 09 & 07 \\
12 & 00 & 08 & 10 & 07 & 09 \\
10 & 08 & 00 & 08 & 03 & 07 \\
08 & 10 & 08 & 00 & 07 & 03 \\
09 & 07 & 03 & 07 & 00 & 06 \\
07 & 09 & 07 & 03 & 06 & 00 \\
\end{bmatrix}
\]

Hamming string for Figure 2 =228 [46 46 36 36 32 32]

From the results of above two PGTs, it is observed that the Hamming string is same for both PGTs. Hence both the PGTs are isomorphic. Another example, consider two PGTs with four gear pairs with one DOF shown in Figure 3and Figure 4, the corresponding Hamming matrices and Hamming strings are given below,
Hamming matrix for Figure 3:
\[
\begin{bmatrix}
00 & 10 & 07 & 07 & 09 & 07 \\
10 & 00 & 07 & 11 & 09 & 03 \\
07 & 07 & 00 & 08 & 04 & 06 \\
07 & 11 & 08 & 00 & 10 & 08 \\
09 & 09 & 04 & 10 & 00 & 06 \\
07 & 03 & 06 & 08 & 06 & 00
\end{bmatrix}
\]

Hamming string for Figure 3 = 224 [44 40 40 38 32 30]

Hamming matrix for Figure 4:
\[
\begin{bmatrix}
00 & 11 & 07 & 08 & 08 \\
11 & 00 & 09 & 10 & 07 & 03 \\
10 & 09 & 00 & 08 & 04 & 06 \\
07 & 10 & 09 & 00 & 10 & 08 \\
08 & 07 & 04 & 07 & 00 & 06 \\
08 & 03 & 06 & 07 & 06 & 00
\end{bmatrix}
\]

Hamming string for Figure 4 = 224 [44 40 40 38 32 30]

Hamming strings of PGTs shown in Figure 3 and Figure 4 are same. It means both the PGTs are isomorphic. Along with isomorphism, symmetry is also known from Hamming matrix.

3.4 Modified Path Matrix Method

As an example consider two 6-link PGTs with single DOF shown in Figure 1 and Figure 2. Modified Path Matrix for Figure 1 and Figure 2 are given below

Modified Path Matrix for Figure 1:
\[
\begin{bmatrix}
00 & 04 & 09 & 04 & 07 & 10 \\
04 & 00 & 07 & 02 & 04 & 08 \\
09 & 07 & 00 & 08 & 10 & 16 \\
04 & 02 & 08 & 00 & 04 & 07 \\
07 & 04 & 10 & 04 & 00 & 09 \\
10 & 08 & 16 & 07 & 09 & 00
\end{bmatrix} = \frac{1}{36}
\]

Train values of above two PGTs are identical; hence both the PGTs are isomorphic. As a second example consider two PGTs shown in Figure 3 and Figure 4, the corresponding Modified Path Matrix and train values are given as follows

Modified Path Matrix for Figure 3:
\[
\begin{bmatrix}
00 & 16 & 36 & 18 & 24 & 42 \\
16 & 00 & 28 & 10 & 12 & 34 \\
36 & 28 & 00 & 34 & 36 & 60 \\
18 & 10 & 34 & 00 & 14 & 30 \\
24 & 12 & 36 & 14 & 00 & 32 \\
42 & 34 & 60 & 30 & 32 & 00
\end{bmatrix} = \frac{1}{144}
\]

Train values of above two PGTs are identical; hence both the PGTs are isomorphic. As a second example consider two PGTs shown in Figure 3 and Figure 4, the corresponding Modified Path Matrix and train values are given as follows

Modified Path Matrix for Figure 4:
\[
\begin{bmatrix}
136 \\
100 \\
194 \\
106 \\
118 \\
198
\end{bmatrix} = \frac{1}{144}[852] = 5.91
\]
Modifg4g30g1321g1313g130g231 = 1

\[
\begin{bmatrix}
00 & 10 & 14 & 18 & 34 & 30 \\
10 & 00 & 12 & 16 & 28 & 34 \\
14 & 12 & 00 & 24 & 36 & 32 \\
18 & 16 & 24 & 00 & 36 & 42 \\
34 & 28 & 36 & 30 & 00 & 60 \\
30 & 34 & 32 & 42 & 60 & 00 \\
\end{bmatrix} = \frac{1}{144}
\]

\[
\begin{bmatrix}
106 & 100 \\
118 & 136 \\
194 & 198 \\
\end{bmatrix} = \frac{1}{144}(852) = 5.91
\]

Train values of above two PGTs are identical; hence both the PGTs are isomorphic. If the train values are not identical then the PGTs are said to be non-isomorphic PGTs.

4. Conclusion
Unique Hamming strings of both the PGTs confirm the isomorphism among the PGTs. Characteristic polynomial and Eigenvectors methods, Modified Path Matrix methods are used to detect only isomorphism in the PGTs. Hamming strings detects isomorphism, symmetry and number of possible levels of PGTs. All these four methods are feasible to the researchers. However, Hamming method is the simplest among all these methods to identify isomorphism. Hamming number technique requires low computational efforts and results are accurate. Results are compared with various other methods and all the results are accurate. Except isomorphism, no other information is known from the remaining three methods. With the same effort one can easily identify the symmetry and levels using Hamming number technique. Hence Hamming number technique is definitely superior to other approaches including characteristic polynomial, Eigen value methods and Modified Path Matrix method.

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