Financial Time Series Analysis of SV Model by Hybrid Monte Carlo

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Abstract. We apply the hybrid Monte Carlo (HMC) algorithm to the financial time series analysis of the stochastic volatility (SV) model for the first time. The HMC algorithm is used for the Markov chain Monte Carlo (MCMC) update of volatility variables of the SV model in the Bayesian inference. We compute parameters of the SV model from the artificial financial data and compare the results from the HMC algorithm with those from the Metropolis algorithm. We find that the HMC decorrelates the volatility variables faster than the Metropolis algorithm. We also make an empirical analysis based on the Yen/Dollar exchange rates.

Key words: Hybrid Monte Carlo Algorithm, Stochastic Volatility Model, Markov Chain Monte Carlo, Bayesian Inference, Financial Data Analysis

1 Introduction

It is well known that financial time series of asset returns shows various interesting properties which can not be explained from the assumption of that the time series obeys the Brownian motion. Those properties are classified as stylized facts\[1\]. Some examples of the stylized facts are (i) fat-tailed distribution of return (ii) volatility clustering (iii) slow decay of the autocorrelation time of the absolute returns. The true dynamics behind the stylized facts is not fully understood. There are some attempts to construct physical models based on spin dynamics\[2\] - \[5\] and they are able to capture some of the stylized facts.

The volatility is an important value to measure the risk in finance. The volatilities of asset returns change in time and shows the volatility clustering. In order to mimic these empirical properties of the volatility Engle advocated the autoregressive conditional heteroskedasticity (ARCH) model\[6\] where the volatility variable changes deterministically depending on the past squared value of the return. Later the ARCH model is generalized by adding also the past volatility dependence to the volatility change. This model is known as the generalized ARCH (GARCH) model\[7\]. The parameters of the GARCH model applied to financial time series are easily determined by the maximum likelihood method.

The stochastic volatility (SV) model\[8\] is another model which captures the properties of the volatility. Contrast to the GARCH model of which the volatility change is deterministic, the volatility of the SV model changes stochastically in...
time. As a result the likelihood function of the SV model is given as a multiple integral of the volatility variables. Such an integral in general is not analytically calculable and thus the determination of the parameters in the SV model by the maximum likelihood method becomes ineffective.

For the SV model the MCMC method based on the Bayesian approach is developed. In the MCMC of the SV model one has to update the parameter variables and also the volatility ones. The number of the volatility variables to be updated increases with the data size of the time series. Usually the update scheme of the volatility variables is based on the local one such as the Metropolis-type algorithm[8]. It is however known that when the local update scheme is done for the volatility variables which have interactions to their neighbor variables in time, the autocorrelation time of sampled volatility variables becomes high and thus the local update scheme is not effective. In order to improve the efficiency of the local update method the blocked scheme which updates several variables at once is also proposed[10].

In this paper we use the HMC algorithm[11] to update the volatility variables. There exists an application of the HMC algorithm to the GARCH model[12] where three GARCH parameters are updated by the HMC scheme. It is more interesting to apply the HMC for update of the volatility variables because the HMC algorithm is a global update scheme which can update all variables at once.

To examine the HMC we calculate the autocorrelation function of the volatility variables and compare the result with that of the Metropolis algorithm.

2 Stochastic Volatility Model and its Bayesian inference

2.1 Stochastic Volatility Model

The standard version of the SV model[8,9] is given by

\[ y_t = \sigma_t \epsilon_t = \exp(h_t/2)\epsilon_t, \]  
\[ h_t = \mu + \phi(h_{t-1} - \mu) + \eta_t, \]  

where \( y_t = (y_1, y_2, ..., y_n) \) represents the time series data, \( h_t \) is defined by \( h_t = \ln \sigma_t^2 \) and \( \sigma_t \) is called volatility. The error terms \( \epsilon_t \) and \( \eta_t \) are independent normal distributions \( N(0, 1) \) and \( N(0, \sigma^2_{\eta}) \) respectively.

For this model the parameters to be determined are \( \mu, \phi \) and \( \sigma^2_{\eta} \). Let us use \( \theta \) as \( \theta = (\mu, \phi, \sigma^2_{\eta}) \). The likelihood function \( L(\theta) \) for the SV model is written as

\[ L(\theta) = \int \prod_{t=1}^{n} f(\epsilon_t|\sigma_t^2) f(h_t|\theta) dh_1 dh_2 ... dh_n, \]  

where

\[ f(\epsilon_t|\sigma_t^2) = (2\pi\sigma_t^2)^{-\frac{1}{2}} \exp(-\frac{\epsilon_t^2}{2\sigma_t^2}), \]  
\[ f(h_t|\theta) = \left( \frac{2\pi\sigma^2_{\eta}}{1-\phi^2} \right)^{-\frac{1}{2}} \exp(-\frac{(h_t - \mu)^2}{2\sigma^2_{\eta}/(1-\phi^2)}), \]
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\[ f(h_\ell|\theta) = (2\pi \sigma^2 \eta)^{-\frac{1}{2}} \exp\left(-\frac{[h_\ell - \mu - \phi(h_{\ell-1} - \mu)]^2}{2\sigma^2 \eta}\right). \] (6)

2.2 Bayesian inference for the SV model

In the Bayesian theorem, the probability distributions of the parameters to be estimated are given by

\[ f(\theta|y) = \frac{1}{Z} L(\theta) \pi(\theta), \] (7)

where \(Z\) is the normalization constant \(Z = \int L(\theta) \pi(\theta) d\theta\) and \(\pi(\theta)\) is a prior distribution of \(\theta\) for which we make a certain assumption. The values of the parameters are inferred as the expectation values of \(\theta\) given by \(\langle \theta \rangle = \int \theta f(\theta|y) d\theta\). In general this integral can not be performed analytically. For that case, one can use the MCMC method to estimate the expectation values numerically. In the MCMC method, we first generate a series of \(\theta\) with a probability \(P(\theta) = f(\theta|y)\). Let \(\theta^{(i)} = (\theta^{(1)}, \theta^{(2)}, ..., \theta^{(k)})\) be values of \(\theta\) generated by a MCMC sampling. Then using these values the expectation value of \(\theta\) is estimated by \(\langle \theta \rangle = \frac{1}{k} \sum_{i=1}^{k} \theta^{(i)}\).

For the SV model, in addition to \(\theta\), volatility variables \(h_\ell\) also have to be updated since they are integrated out as in eq.(3). Let \(P(\theta, h_\ell)\) be the joint probability distribution of \(\theta\) and \(h_\ell\). Then \(P(\theta, h_\ell)\) is given by \(P(\theta, h_\ell) \sim L(\theta, h_\ell) \pi(\theta)\), (8)

where \(L(\theta, h_\ell) = \prod_{t=1}^{n} f(\epsilon_t|h_\ell)f(h_\ell|\theta)\).

For the prior \(\pi(\theta)\) we assume that \(\pi(\sigma^2 \eta) \sim (\sigma^2 \eta)^{-n/2-1}\) and for others \(\pi(\mu) = \pi(\phi) = constant\). The probability distributions for the parameters and the volatility variables are derived from eq.(8)\[8,9\]. The probability distributions and their update schemes are given in the followings.

- **\(\sigma^2 \eta\) update scheme.**
  The probability distribution of \(\sigma^2 \eta\) is given by

  \[ P(\sigma^2 \eta) \sim (\sigma^2 \eta)^{-\frac{n}{2}-1} \exp\left(-\frac{A}{\sigma^2 \eta}\right), \] (9)

  where \(A = \frac{1}{2}\{(1 - \phi^2)(h_1 - \mu)^2 + \sum_{t=2}^{n}[h_t - \mu - \phi(h_{t-1} - \mu)]^2\}\). Since eq.(9) is an inverse gamma distribution we can easily draw a value depending on eq.(9) by using an appropriate statistical library.

- **\(\mu\) update scheme.**
  The probability distribution of \(\mu\) is given by

  \[ P(\mu) \sim \exp\left(-\frac{B}{2\sigma^2 \eta}(\mu - C)^2\right), \] (10)

  where \(B = (1 - \phi^2) + (n - 1)(1 - \phi)^2\), and \(C = (1 - \phi^2)h_1 + (1 - \phi)\sum_{t=2}^{n} (h_t - \phi h_{t-1})\). Eq.(10) is a Gaussian distribution. Again we can easily update \(\mu\).
• $\phi$ update scheme.
  The probability distribution of $\phi$ is given by

$$P(\phi) \sim (1 - \phi^2)^{1/2} \exp\{-\frac{D}{2\sigma_\eta^2}(\phi - \frac{E_D}{D})^2\},$$  \hspace{1cm} (11)$$

where $D = -(h_1 - \mu)^2 + \sum_{t=2}^n (h_{t-1} - \mu)^2$ and $E = \sum_{t=1}^n (h_t - \mu)(h_{t-1} - \mu)$. In order to update $\phi$ with eq. (11), we use the Metropolis-Hastings algorithm[13,14]. Let us write eq. (11) as $P(\phi) \sim P_1(\phi)P_2(\phi)$ where

$$P_1(\phi) = (1 - \phi^2)^{1/2},$$  \hspace{1cm} (12)$$

and

$$P_2(\phi) \sim \exp\{-\frac{D}{2\sigma_\eta^2}(\phi - \frac{E_D}{D})^2\}.$$  \hspace{1cm} (13)$$

Since $P_2(\phi)$ is a Gaussian distribution we can easily draw $\phi$ from eq. (13). Let $\phi_{\text{new}}$ be a candidate given from eq. (13). Then in order to obtain the correct distribution, $\phi_{\text{new}}$ is accepted with the following probability $P_{\text{MH}}$.

$$P_{\text{MH}} = \min\left\{ \frac{P(\phi_{\text{new}})P_2(\phi)}{P(\phi)P_2(\phi_{\text{new}})}, 1 \right\} = \min\left\{ \frac{(1 - \phi_{\text{new}}^2)}{(1 - \phi^2)}, 1 \right\}.$$  \hspace{1cm} (14)$$

In addition to the above step we restrict $\phi$ within $[-1, 1]$ to avoid a negative value in the calculation of square root.

• Probability distribution for $P(h_t)$.
  The probability distribution of the volatility variables $h_t$ is given by

$$P(h_1, h_2, ..., h_n) \sim \exp\left(-\sum_{i=1}^n \left(\frac{h_i^2}{2\sigma_i^2} + \frac{\sigma_i^2}{2} e^{-h_i}\right) - \sum_{i=2}^n \frac{|h_i - \mu - \phi(h_{i-1} - \mu)|^2}{2\sigma_i^2}\right).$$  \hspace{1cm} (15)$$

This probability distribution is not a simple function for drawing values of the volatility variables $h_t$. A conventional method is the Metropolis method[13] which updates the variables locally. Here we use the HMC algorithm which updates the volatility variables globally.

3 Hybrid Monte Carlo Algorithm

The HMC algorithm is originally developed for the MCMC of the lattice Quantum Chromo Dynamics (QCD) calculations[11] where local type update algorithms are not effective. The notable feature of the HMC algorithm is that it updates a number of variables simultaneously.

Here we briefly describe the HMC algorithm. The HMC algorithm combines molecular dynamics (MD) simulations and the Metropolis test. Let $f(x)$ be a probability density and $O(x)$ a function of $x = (x_1, x_2, ..., x_n)$. We determine the expectation value of $O(x)$ with the probability density $f(x)$ which is given by

$$\langle O(x) \rangle = \int O(x)f(x)dx = \int O(x)e^{lnf(x)}dx.$$  \hspace{1cm} (16)$$
Now let us introduce momentum variables \( p = (p_1, p_2, ..., p_n) \) conjugate to the variables \( x \) and rewrite eq.(16) as

\[
\langle O(x) \rangle = \frac{1}{Z} \int O(x)e^{-\frac{1}{2}p^2 + \ln f(x)} \, dx \, dp = \frac{1}{Z} \int O(x)e^{-H(p,x)} \, dx \, dp. \tag{17}
\]

where \( Z = \int e^{-\frac{1}{2}p^2} \, dp \). \( H(p,x) \) is the Hamiltonian defined by

\[ H(p,x) = \frac{1}{2}p^2 - \ln f(x) \]

where \( p^2 \) stands for \( \sum_{i=1}^{n} p_i^2 \). The introduction of \( p \) does not change the value of \( \langle O(x) \rangle \).

In the HMC algorithm, new candidates of the variables are drawn by integrating the Hamilton’s equations of motions. The Hamilton’s equations of motions are solved numerically by doing the MD simulation with a fictitious time. To solve the equations we use the standard 2nd order leapfrog integrator. One could use improved integrators[15] or higher order integrators[16,17] if necessary.

Let \( (p', x') \) be the new candidates given by the MD simulation. The new candidates are accepted with a probability \( \min\{1, \exp(-\Delta H)\} \) where \( \Delta H = H(p', x') - H(p, x) \). Since the Hamilton’s equation of motions are not solved exactly \( \Delta H \) deviates from zero. The magnitude of the deviation is tuned by the discrete time step size in the MD simulation such that the acceptance of the new candidates becomes high.

For the volatility variables of the SV model, from eq.(15) the Hamiltonian can be defined by

\[
H(p_t, h_t) = \sum_{i=1}^{n} \frac{1}{2} p_i^2 + \sum_{i=1}^{n} \left( \frac{h_i}{2} e^{-h_i} \right) + \frac{[h_1 - \mu]^2}{2\sigma^2/(1 - \phi^2)} + \sum_{i=2}^{n} \frac{[h_i - \mu - \phi(h_{i-1} - \mu)]^2}{2\sigma^2}, \tag{18}
\]

where \( p_i \) is defined as a conjugate momentum to \( h_i \).

### 4 Numerical Test

In this section we investigate the HMC algorithm for the SV model with artificial financial data. The artificial data is generated with a set of known parameters. We try to infer the values of those parameters by the HMC and Metropolis algorithms and compares the results.

Using eq.(11) with \( \phi = 0.97, \sigma^2 = 0.05 \) and \( \mu = -1 \) we have generated 2000 data. To this data we made the Bayesian inference with the HMC and Metropolis algorithms. The initial parameters are set to \( \phi = 0.5, \sigma^2 = 1.0 \) and \( \mu = 0 \). The first 10000 samples are discarded as thermalization or burn-in process. Then 200000 samples are recorded for analysis. The acceptance of the volatility variables is tuned to be about 50%.

Fig.4 shows the history of the volatility variable \( h_{100} \). We use \( h_{100} \) as the representative one of the volatility variable. We observe the similar behavior for other volatility variables. As seen in Fig.4 the correlation of the volatility variable from the HMC algorithm is smaller than that from the Metropolis algorithm. To
quantify this we calculated the autocorrelation function (ACF) of the volatility variable shown in Fig. 2. The ACF is defined as

$$ACF(t) = \frac{1}{N} \sum_{j=1}^{N} (x(j) - < x >)(x(j + t) - < x >) \frac{1}{\sigma_x^2},$$  \hspace{1cm} (19)

where $< x >$ and $\sigma_x^2$ are the average value and the variance of $x$ respectively.

The autocorrelation time $\tau_{int}$ of the volatility variables is given in Table 1. The values in the parentheses represent the errors estimated by the jackknife method. The autocorrelation time is defined by $\tau_{int} = \frac{1}{2} + \sum_{t=1}^{\infty} ACF(t)$.

The HMC algorithm gives a smaller autocorrelation time than the Metropolis algorithm, which means that the HMC algorithm samples the volatility variables more effectively than the Metropolis algorithm.
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Table 1. Results estimated by the HMC and Metropolis algorithms.

|        | $\phi$     | $\mu$     | $\sigma^2$ | $h_{100}$ |
|--------|------------|-----------|------------|-----------|
| true   | 0.97       | -1        | 0.05       |           |
| HMC    | 0.978(7)   | -0.92(26) | 0.053(12)  |           |
| $\tau_{int}$ | 540(60)    | 3(1)      | 1200(150)  | 18(1)     |
| Metropolis | 0.978(7)   | -0.92(26) | 0.052(11)  |           |
| $\tau_{int}$ | 400(100)   | 13(2)     | 1000(270)  | 210(50)   |

Table 2. Results estimated by the HMC to the Yen/Dollar exchange rates.

|        | $\phi$     | $\mu$     | $\sigma^2$ | $h_{100}$ |
|--------|------------|-----------|------------|-----------|
| HMC    | 0.960(12)  | -1.13(8)  | 0.014(4)   |           |
| $\tau_{int}$ | 610(300)   | 14(6)     | 1400(800)  | 55(11)    |

The autocorrelation times for the parameters of the SV model are summarized in Table 1. The autocorrelation times from the HMC algorithm are similar to those of the Metropolis algorithm except for $\tau_{int}$ of $\mu$.

The values of the SV parameters estimated by the HMC and the Metropolis algorithms are given in Table 1. The values in the parentheses represent the standard deviations of the sampled data. The results from the both algorithms well reproduce the true values used for the generation of the artificial financial data. Furthermore for each parameter two values obtained by the HMC and the Metropolis algorithms agree well. This is not surprising because the same data is used for the both calculations by the HMC and Metropolis algorithms.

5 Empirical Study

We have also made an empirical study of the SV model by the HMC. The empirical study is based on daily data of the exchange rates for Japanese Yen and US dollar. The sampling period is 1 March 2000 to 29 February 2008, which has 2007 observations. The exchange rates $p_i$ are transformed to $r_i = 100 \ln(p_i/p_{i-1} - \bar{s})$ where $\bar{s}$ is the average value of $\ln(p_i/p_{i-1})$. The MCMC sampling is performed as in the previous section. The first 10000 MC samples are discarded and then 20000 samples are recorded for the analysis. The estimated values of the parameters are summarized in Table 2. The estimated value of $\phi$ is close to one, which means the persistency of the volatility shock. The similar values are obtained in the previous studies[8,9].

6 Summary

The HMC algorithm is applied for the Bayesian inference of the SV model. It is found that the correlations of the volatility variables sampled by the HMC algorithm are much reduced. On the other hand we observe no significant improvement on the correlations of the sampled parameters of the SV model. Thus it is concluded that the HMC algorithm has a similar efficiency to the Metropolis
algorithm and it is an alternative algorithm for the Bayesian inference of the SV model.

If one needs to calculate a certain quantity depending on the volatility variables, then the HMC algorithm may serve as a good algorithm which samples the volatility variables effectively because the HMC algorithm decorrelates the sampled volatility variables faster than the Metropolis algorithm.

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