Harnack Inequality for SDE with Multiplicative Noise and Extension to Neumann Semigroup on Non-Convex Manifolds

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Abstract

By constructing a coupling with unbounded time-dependent drift, dimension-free Harnack inequalities are established for a large class of stochastic differential equations with multiplicative noise. These inequalities are applied to the study of heat kernel upper bound and contractivity properties of the semigroup. The main results are also extended to reflecting diffusion processes on Riemannian manifolds with non-convex boundary.

1 Introduction

Consider the following SDE on $\mathbb{R}^d$:

\begin{equation}
\begin{aligned}
    \mathrm{d}X_t &= \sigma(t, X_t)\mathrm{d}B_t + b(t, X_t)\mathrm{d}t, \\
\end{aligned}
\end{equation}

where $B_t$ is the $d$-dimensional Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}_t_{t \geq 0}, \mathbb{P})$, and

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\(\sigma : [0, \infty) \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d \otimes \mathbb{R}^d, \quad b : [0, \infty) \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d\)

are progressively measurable and continuous in the second variable. Throughout the paper we assume that for any \(X_0 \in \mathbb{R}^d\) the equation (1.1) has a unique strong solution which is non-explosive and continuous in \(t\).

Let \(X^x_t\) be the solution to (1.1) for \(X_0 = x\). We aim to establish the Harnack inequality for the operator \(P_t:\)

\[P_t f(x) := \mathbb{E} f(X^x_t), \quad t \geq 0, x \in \mathbb{R}^d, f \in \mathcal{B}^+_b(\mathbb{R}^d),\]

where \(\mathcal{B}^+_b(\mathbb{R}^d)\) is the class of all bounded non-negative measurable functions on \(\mathbb{R}^d\). To this end, we shall make use of the following assumptions.

\((\text{A1})\) There exists an increasing function \(K : [0, \infty) \to \mathbb{R}\) such that almost surely

\[\|\sigma(t, x) - \sigma(t, y)\|^2_{HS} + 2\langle b(t, x) - b(t, y), x - y \rangle \leq K_t |x - y|^2, \quad x, y \in \mathbb{R}^d, t \geq 0.\]

\((\text{A2})\) There exists a decreasing function \(\lambda : [0, \infty) \to (0, \infty)\) such that almost surely

\[\sigma(t, x)^* \sigma(t, x) \geq \lambda^2 t I, \quad x \in \mathbb{R}^d, t \geq 0.\]

\((\text{A3})\) There exists an increasing function \(\delta : [0, \infty) \to (0, \infty)\) such that almost surely

\[|(\sigma(t, x) - \sigma(t, y))(x - y)| \leq \delta_t |x - y|, \quad x, y \in \mathbb{R}^d, t \geq 0.\]

\((\text{A4})\) For \(n \geq 1\) there exists a constant \(c_n > 0\) such that almost surely

\[\|\sigma(t, x) - \sigma(t, y)\|^2_{HS} + |b(t, x) - b(t, y)| \leq c_n |x - y|, \quad |x|, |y|, t \leq n.\]

It is well known that (A1) ensures the uniqueness of the solution to (1.1) while (A4) implies the existence and the uniqueness of the strong solution (see e.g. [12] and references within for weaker conditions). On the other hand, if \(b\) and \(\sigma\) depend only on the variable \(x \in \mathbb{R}^d\), then their continuity in \(x\) implies the existence of weak solutions (see [14], Theorem 1).
so that by the Yamada-Watanabe principle [28], the uniqueness ensured by (A1) implies the existence and uniqueness of the strong solution.

Note that if \( \sigma(t, x) \) and \( b(t, x) \) are deterministic and independent of \( t \), then the solution is a time-homogeneous Markov process generated by

\[
L := \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} \partial_i \partial_j + \sum_{i=1}^{d} b_i \partial_i,
\]

where \( a := \sigma \sigma^* \). If further more \( \sigma \) and \( b \) are smooth, we may consider the Bakry-Emery curvature condition [5]:

\[
\Gamma_2(f, f) \geq -K \Gamma(f, f), \quad f \in C^\infty(\mathbb{R}^d)
\]

for some constant \( K \in \mathbb{R} \), where

\[
\Gamma(f, g) := \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(\partial_i f)(\partial_j g), \quad f, g \in C^1(\mathbb{R}^d),
\]

\[
\Gamma_2(f, f) := \frac{1}{2} L \Gamma(f, f) - \Gamma(f, Lf), \quad f \in C^\infty(\mathbb{R}^d).
\]

According to [22, Lemma 2.2] and [23, Theorem 1.2], the curvature condition (1.2) is equivalent to the dimension-free Harnack inequality

\[
(P_t f(x))^p \leq (P_t f^p(y)) \exp \left[ \frac{pp_{\rho_a}(x, y)^2}{2(p-1)(1 - e^{-2Kt})} \right], \quad t \geq 0, p > 1, f \in \mathcal{B}_b^+(\mathbb{R}^d), x, y \in \mathbb{R}^d,
\]

where

\[
\rho_a(x, y) := \sup \{|f(x) - f(y)| : f \in C^1(\mathbb{R}^d), \Gamma(f, f) \leq 1\}, \quad x, y \in \mathbb{R}^d.
\]

This type of inequality has been extended and applied to the study of heat kernel (or transition probability) and contractivity properties for diffusion semigroups, see [1, 18, 4] for diffusions on manifolds with possibly unbounded below curvature, [24, 16] for stochastic generalized porous media and fast diffusion equations, and [2, 3, 8, 17, 15, 19, 11, 29] for the study of some other SPDEs with additive noise.
If \( \sigma \) depends on \( x \), however, it is normally very hard to verify the curvature condition (1.2), which depends on second order derivatives of \( a^{-1} \), the inverse matrix of \( a \). This is the main reason why existing results on the dimension-free Harnack inequality for SPDEs are only proved for the additive noise case.

In this paper we shall use the coupling argument developed in [4], which will allow us to establish Harnack inequalities for \( \sigma(t, x) \) depending on \( x \). This method has also been applied to the study of SPDEs in the above mentioned references. To see the difficulty in the study for \( \sigma(t, x) \) depending on \( x \), let us briefly recall the main idea of this argument.

To explain the main idea of the coupling, we first consider the easy case where \( \sigma \) and \( b \) are independent of the second variable. For \( x \neq y \) and \( T > 0 \), let \( X_t \) solve (1.1) with \( X_0 = x \) and \( Y_t \) solve

\[
dY_t = \sigma(t)dB_t + b(t)dt + \frac{|x-y|(X_t - Y_t)}{T|X_t - Y_t|}dt, \quad Y_0 = y.
\]

Then \( Y_t \) is well defined up to the coupling time

\[
\tau := \inf\{t \geq 0 : X_t = Y_t\}.
\]

Let \( X_t = Y_t \) for \( t \geq \tau \). We have

\[
d|X_t - Y_t| = -\frac{|x-y|}{T}dt, \quad t \leq \tau.
\]

This implies \( \tau = T \) and hence, \( X_T = Y_T \). On the other hand, by the Girsanov theorem we have

\[
P_T f(y) = \mathbb{E}[R f(Y_T)]
\]

for

\[
R := \exp \left[ -\frac{|x-y|}{T} \int_0^T \frac{(\sigma(t)^{-1}(X_t - Y_t), dB_t)}{|X_t - Y_t|} - \frac{|x-y|^2}{2T^2} \int_0^T \frac{|\sigma(t)^{-1}(X_t - Y_t)|^2}{|X_t - Y_t|^2} dt \right].
\]

Therefore,

\[
(P_T f(y))^p = (\mathbb{E}[R f(X_T)])^p \leq (P_T f^p(x))(\mathbb{E} R^{p/(p-1)})^{p-1}.
\]
Since by (A1) and (A2) it is easy to estimate moments of $R$, the desired Harnack inequality follows immediately.

In general, if $\sigma(t,x)$ depends on $x$, then the process $X_t - Y_t$ contains a non-trivial martingale term, which can not be dominated by and bounded drift. So, in this case, any additional bounded drift put in the equation for $Y_t$ is not enough to make the coupling successful before a fixed time $T$. This is the main difficulty to establish the Harnack inequality for diffusion semigroups with non-constant diffusion coefficient.

In this paper, under assumptions (A1) and (A2), we are able to constructed a coupling with a drift which is unbounded around a fixed time $T$, such that the coupling is successful before $T$. In this case the corresponding exponential martingale has finite entropy such that the log-Harnack inequality holds; if further more (A3) holds then the exponential martingale is $L^p$-integrable for some $p > 1$ such that the Harnack inequality with power holds. More precisely, we have the following result.

**Theorem 1.1.** Let $\sigma(t,x)$ and $b(t,x)$ either be deterministic and independent of $t$, or satisfy (A4).

1. If (A1) and (A2) hold then

$$P_T \log f(y) \leq \log P_T f(x) + \frac{K_T |x - y|^2}{2\lambda^2_T (1 - e^{-K_T T})}, \quad f \geq 1, x, y \in \mathbb{R}^d, T > 0.$$  

2. If (A1), (A2) and (A3) hold, then for $p > (1 + \frac{\delta_T}{\lambda_T})^2$ and $\delta_{p,T} := \max\{\delta_T, \frac{\lambda_T}{2}(\sqrt{p} - 1)\}$, the Harnack inequality

$$\left(P_T f(y)\right)^p \leq \left(P_T f^p(x)\right) \exp \left[\frac{K_T \sqrt{p} (\sqrt{p} - 1) |x - y|^2}{4\delta_{p,T}[(\sqrt{p} - 1)\lambda_T - \delta_{p,T}](1 - e^{-K_T T})}\right]$$

holds for all $T > 0, x, y \in \mathbb{R}^d$ and $f \in \mathcal{B}_b^+(\mathbb{R}^d)$.

Theorem 1.1(1) generalizes a recent result in [20] on the log-Harnack inequality by using the gradient estimate on $P_t$.

Let $p_t(x,y)$ be the density of $P_t$ w.r.t. a Radon measure $\mu$. Then according to [26, Proposition 2.4], the above log-Harnack inequality and Harnack inequality are equivalent to the following heat kernel inequalities respectively:
\[
\int_{\mathbb{R}^d} p_T(x, z) \log \frac{p_T(x, z)}{p_T(y, z)} \mu(\mathrm{d}z) \leq \frac{K|x-y|^2}{2\lambda_T^2(1-e^{-K_TT})}, \quad x, y \in \mathbb{R}^d, T > 0
\]

and

\[
\int_{\mathbb{R}^d} p_T(x, z) \left( \frac{p_t(x, z)}{p_t(y, z)} \right)^{1/(p-1)} \mu(\mathrm{d}z) 
\leq \exp \left[ \frac{K_T\sqrt{p}|x-y|^2}{4\delta_{p,T}(\sqrt{p}+1)(\sqrt{p}-1)\lambda_T-\delta_{p,T}(1-e^{-K_TT})} \right], \quad x, y \in \mathbb{R}^d, T > 0.
\]

So, the following is a direct consequence of Theorem 1.1.

**Corollary 1.2.** Let \(\sigma(t, x)\) and \(b(t, x)\) either be deterministic and independent of \(t\), or satisfy (A4). Let \(P_t\) have a strictly positive density \(p_t(x, y)\) w.r.t. a Radon measure \(\mu\). Then (A1) and (A2) imply (1.3), while (A1)-(A3) imply (1.4).

Next, by standard applications of the Harnack inequality with power, we have the following consequence of Theorem 1.1 on contractivity properties of \(P_t\).

**Corollary 1.3.** Let \(\sigma(t, x)\) and \(b(t, x)\) be deterministic and independent of \(t\), such that (A1)-(A3) hold for constant \(K, \lambda\) and \(\delta\). Let \(P_t\) have an invariant probability measure \(\mu\).

1. If there exists \(r > K^+ / \lambda^2\) such that \(\mu(e^{r|\cdot|^2}) < \infty\), then \(P_t\) is hypercontractive, i.e. \(\|P_t\|_{L^2(\mu) \to L^4(\mu)} = 1\) holds for some \(t > 0\).

2. If \(\mu(e^{r|\cdot|^2}) < \infty\) holds for all \(r > 0\), then \(P_t\) is supercontractive, i.e. \(\|P_t\|_{L^2(\mu) \to L^4(\mu)} < \infty\) holds for all \(t > 0\).

3. If \(P_t e^{r|\cdot|^2}\) is bounded for any \(t, r > 0\), then \(P_t\) is ultracontractive, i.e. \(\|P_t\|_{L^2(\mu) \to L^\infty(\mu)} < \infty\) for any \(t > 0\).

**Remark 1.1.** To see that results in Corollary 1.3 are sharp, let \(P_t\) be symmetric w.r.t. \(\mu\). Then the hypercontractivity is equivalent to the validity of the log-Sobolev inequality

\[
\mu(f^2 \log f^2) \leq C \mu(\Gamma(f, f)), \quad f \in C^\infty_b(\mathbb{R}^d), \mu(f^2) = 1
\]

for some constant \(C > 0\). Moreover, if there exists a constant \(R > 0\) such that...
(1.5) \[ \Gamma(f, f) \leq R^2|\nabla f|^2, \quad f \in C^\infty(\mathbb{R}^d), \]

we have \( \rho_a(x, y) \geq R^{-1}|x - y| \). So, by the concentration of measure for the log-Sobolev inequality, the hypercontractivity implies \( \mu(e^{r|\cdot|^2}) < \infty \) for some \( r > 0 \), while the supercontractivity implies \( \mu(e^{r|\cdot|^2}) < \infty \) for all \( r > 0 \). Combining this with Corollary 1.3, we have the following assertions under conditions (A1)-(A3) and (1.5):

(i) Let \( K \leq 0 \). Then \( P_t \) is hypercontractive if and only if \( \mu(e^{r|\cdot|^2}) < \infty \) holds for some \( r > 0 \);

(ii) \( P_t \) is supercontractive if and only if \( \mu(e^{r|\cdot|^2}) < \infty \) holds for all \( r > 0 \);

(iii) \( P_t \) is ultracontractive if and only if \( P_te^{r|\cdot|^2} \) is bounded for any \( t, r > 0 \).

Therefore, conditions in Corollary 1.3(2) and (3) are sharp for the supercontractivity and ultracontractivity of \( P_t \). Moreover, as shown in [7] that when \( \sigma \) is constant, the sufficient condition \( \mu(e^{r|\cdot|^2}) < \infty \) for some \( r > K^+ / \lambda^2 \) is optimal for the hypercontractivity of \( P_t \). So, Corollary 1.3(1) also provides a sharp sufficient condition for the hypercontractivity of \( P_t \).

We will prove Theorem 1.1 and Corollary 1.3 in the next section. In Section 3 we extend these results to SDEs on Riemannian manifolds possibly with a convex boundary. Finally, combining results in Section 3 with a conformal change method introduced in [24], we are able to establish Harnack inequalities in Section 4 for the Neumann semigroup on a class of non-convex manifolds.

2 Proofs of Theorem 1.1 and Corollary 1.3

Let \( x, y \in \mathbb{R}^d, T > 0 \) and \( p > (1 + \delta_T / \lambda_T)^2 \) be fixed such that \( x \neq y \). We have

(2.1) \[ \theta_T := \frac{2\delta_T}{(\sqrt{p} - 1)\lambda_T} \in (0, 2). \]
For $\theta \in (0, 2)$, let

$$\xi_t = \frac{2 - \theta}{K_T} (1 - e^{K_T(t-T)}), \quad t \in [0, T].$$

Then $\xi$ is smooth and strictly positive on $[0, T)$ such that

(2.2) \hspace{1cm} 2 - K_T \xi_t + \xi'_t = \theta, \quad t \in [0, T].$

Consider the coupling

\begin{align*}
\mathrm{d}X_t &= \sigma(t, X_t) \mathrm{d}B_t + b(t, X_t) \mathrm{d}t, \quad X_0 = x, \\
\mathrm{d}Y_t &= \sigma(t, Y_t) \mathrm{d}B_t + b(t, Y_t) \mathrm{d}t + \frac{1}{\xi_t} \sigma(t, Y_t) \sigma(t, X_t)^{-1} (X_t - Y_t) \mathrm{d}t, \quad Y_0 = y.
\end{align*}

(2.3)

Since the additional drift term $\xi_t^{-1} \sigma(t, y) \sigma(t, x)^{-1} (x - y)$ is locally Lipschitzian in $y$ if (A4) holds, and continuous in $y$ when $\sigma$ and $b$ are deterministic and time independent, the coupling $(X_t, Y_t)$ is well defined continuous process for $t < T \land \zeta$, where $\zeta$ is the explosion time of $Y_t$; namely, $\zeta = \lim_{n \to \infty} \zeta_n$ for

$$\zeta_n := \inf \{ t \in [0, T) : |Y_t| \geq n \},$$

where we set $\inf \emptyset = T$. Let

$$\mathrm{d} \tilde{B}_t = \mathrm{d}B_t + \frac{1}{\xi_t} \sigma(t, X_t)^{-1} (X_t - Y_t) \mathrm{d}t, \quad t < T \land \zeta.$$

If $\zeta = T$ and

$$R_s := \exp \left[ - \int_0^s \xi_t^{-1} \sigma(t, X_t)^{-1} (X_t - Y_t) \, \mathrm{d}B_t - \frac{1}{2} \int_0^s \xi_t^{-2} |\sigma(t, X_t)^{-1} (X_t - Y_t)|^2 \, \mathrm{d}t \right]$$

is a uniformly integrable martingale for $s \in [0, T)$, then by the martingale convergence theorem, $R_T := \lim_{t \uparrow T} R_t$ exists and $\{R_t\}_{t \in [0, T]}$ is a martingale. In this case, by the Girsanov theorem $\{\tilde{B}_t\}_{t \in [0, T]}$ is a $d$-dimensional Brownian motion under the probability $R_T \mathbb{P}$. Rewrite (2.3) as

\begin{align*}
\mathrm{d}X_t &= \sigma(t, X_t) \mathrm{d}\tilde{B}_t + b(t, X_t) \mathrm{d}t - \frac{X_t - Y_t}{\xi_t} \mathrm{d}t, \quad X_0 = x, \\
\mathrm{d}Y_t &= \sigma(t, Y_t) \mathrm{d}\tilde{B}_t + b(t, Y_t) \mathrm{d}t, \quad Y_0 = y.
\end{align*}

(2.4)
Since \( \int_0^T \xi_t^{-1} dt = \infty \), we will see that the additional drift \(-\frac{X_t - Y_t}{\xi_t} dt\) is strong enough to force the coupling to be successful up to time \( T \). So, we first prove the uniform integrability of \( \{R_{s \wedge \xi}\}_{s \in [0,T]} \) w.r.t. \( P \) so that \( R_{T \wedge \xi} := \lim_{s \uparrow T} R_{s \wedge \xi} \) exists, then prove that \( \xi = T \) \( Q \)-a.s. for \( Q := R_{T \wedge \xi} P \) so that \( Q = R_T P \).

Let

\[
\tau_n = \inf \{ t \in [0,T) : |X_t| + |Y_t| \geq n \}.
\]

Since \( X_t \) is non-explosive as assumed, we have \( \tau_n \uparrow \xi \) as \( n \uparrow \infty \).

**Lemma 2.1.** Assume \((A1)\) and \((A2)\). Let \( \theta \in (0,2), x, y \in \mathbb{R}^d \) and \( T > 0 \) be fixed.

1. There holds

\[
\sup_{s \in [0,T), n \geq 1} \mathbb{E} R_{s \wedge \tau_n} \log R_{s \wedge \tau_n} \leq \frac{K_T |x - y|^2}{2 \lambda^2 \theta (2 - \theta) (1 - e^{-K_T T})}.
\]

Consequently,

\[
R_{s \wedge \xi} := \lim_{n \uparrow \infty} R_{s \wedge \tau_n \wedge (T - 1/n)}, \quad s \in [0,T], \quad R_{T \wedge \xi} := \lim_{s \uparrow T} R_{s \wedge \xi}
\]

exist such that \( \{R_{s \wedge \xi}\}_{s \in [0,T]} \) is a uniformly integrable martingale.

2. Let \( Q = R_{T \wedge \xi} P \). Then \( Q(\xi = T) = 1 \) so that \( Q = R_T P \).

**Proof.** (1) Let \( s \in [0,T) \) be fixed. By \((2.4)\), \((A1)\) and the Itô formula,

\[
\text{d}|X_t - Y_t|^2 \leq 2 \langle \sigma(t, X_t) - \sigma(t, Y_t) \rangle (X_t - Y_t, \text{d}\bar{B}_t) + K_T |X_t - Y_t|^2 \text{d}t - \frac{2}{\xi_t} |X_t - Y_t|^2 \text{d}t
\]

holds for \( t \leq s \wedge \tau_n \). Combining this with \((2.2)\) we obtain

\[
\text{d}\frac{|X_t - Y_t|^2}{\xi_t} \leq \frac{2}{\xi_t} \langle \sigma(t, X_t) - \sigma(t, Y_t) \rangle (X_t - Y_t, \text{d}\bar{B}_t) - \frac{|X_t - Y_t|^2}{\xi_t^2} (2 - K_T \xi_t + \xi'_t) \text{d}t
\]

\[
= \frac{2}{\xi_t} \langle \sigma(t, X_t) - \sigma(t, Y_t) \rangle (X_t - Y_t, \text{d}\bar{B}_t) - \frac{\theta}{\xi_t^2} |X_t - Y_t|^2 \text{d}t, \quad t \leq s \wedge \tau_n.
\]

Multiplying by \( \frac{1}{\theta} \) and integrating from 0 to \( s \wedge \tau_n \), we obtain
\[
\int_0^{s \land \tau_n} \frac{|X_t - Y_t|^2}{\xi_t^2} \, dt \leq \int_0^{s \land \tau_n} \frac{2}{\theta \xi_t} \langle (\sigma(t, X_t) - \sigma(t, Y_t))(X_t - Y_t), \, d\tilde{B}_t \rangle - \frac{|X_t - Y_t|^2}{\theta \xi_t} + \frac{|x - y|^2}{\theta \xi_0}.
\]

By the Girsanov theorem, \( \{\tilde{B}_t\}_{t \leq \tau_n \land s} \) is the \( d \)-dimensional Brownian motion under the probability measure \( R_{s \land \tau_n} \mathbb{P} \). So, taking expectation \( \mathbb{E}_{s,n} \) with respect to \( R_{s \land \tau_n} \mathbb{P} \), we arrive at

\[
(2.6) \quad \mathbb{E}_{s,n} \int_0^{s \land \tau_n} \frac{|X_t - Y_t|^2}{\xi_t^2} \, dt \leq \frac{|x - y|^2}{\theta \xi_0}, \ s \in [0, T), n \geq 1.
\]

By (A2) and the definitions of \( R_t \) and \( \tilde{B}_t \), we have

\[
\log R_r = -\int_0^r \frac{1}{\xi_t} \langle (\sigma(t, X_t)^{-1}(X_t - Y_t), \, d\tilde{B}_t \rangle + \frac{1}{2} \int_0^r \frac{|\sigma(t, X_t)^{-1}(X_t - Y_t)|^2}{\xi_t^2} \, dt
\leq -\int_0^r \frac{1}{\xi_t} \langle (\sigma(t, X_t)^{-1}(X_t - Y_t), \, d\tilde{B}_t \rangle + \frac{1}{2 \lambda_T^2} \int_0^r \frac{|X_t - Y_t|^2}{\xi_t^2} \, dt, \ r \leq s \land \tau_n.
\]

Since \( \{\tilde{B}_t\} \) is the \( d \)-dimensional Brownian motion under \( R_{s \land \tau_n} \mathbb{P} \) up to \( s \land \tau_n \), combining this with (2.6) we obtain

\[
\mathbb{E} R_{s \land \tau_n} \log R_{s \land \tau_n} = \mathbb{E}_{s,n} \log R_{s \land \tau_n} \leq \frac{|x - y|^2}{2 \lambda_T^2 \theta \xi_0}, \ s \in [0, T), n \geq 1.
\]

By the martingale convergence theorem and the Fatou lemma, \( \{R_{s \land \zeta} : s \in [0, T]\} \) is a well-defined martingale with

\[
\mathbb{E} R_{s \land \zeta} \log R_{s \land \zeta} \leq \frac{|x - y|^2}{2 \lambda_T^2 \theta (1 - e^{-K_T T})}, \ s \in [0, T].
\]

To see that \( \{R_{s \land \zeta} : s \in [0, T]\} \) is a martingale, let \( 0 \leq s < t \leq T \). By the dominated convergence theorem and the martingale property of \( \{R_{s \land \tau_n} : s \in [0, T]\} \), we have

\[
\mathbb{E}(R_{t \land \zeta}|\mathcal{F}_s) = \mathbb{E}\left( \lim_{n \to \infty} R_{t \land \tau_n \land (T - 1/n)}|\mathcal{F}_s \right) = \lim_{n \to \infty} \mathbb{E}(R_{t \land \tau_n \land (T - 1/n)}|\mathcal{F}_s) = \lim_{n \to \infty} R_{s \land \tau_n} = R_{s \land \zeta}.
\]
(2) Let \( \sigma_n = \inf\{t \geq 0 : |X_t| \geq n\} \). We have \( \sigma_n \uparrow \infty \) \( \mathbb{P} \)-a.s and hence, also \( \mathbb{Q} \)-a.s. Since \( \{\tilde{B}_t\} \) is a \( \mathbb{Q} \)-Brownian motion up to \( T \wedge \zeta \), it follows from (2.5) that

\[
\frac{(n - m)^2}{\xi_0} \mathbb{Q}(\sigma_m > t, \zeta_n \leq t) \leq \mathbb{E}_\mathbb{Q} \frac{|X_{t \wedge \sigma_m \wedge \zeta_n} - X_{t \wedge \sigma_m \wedge \zeta_n}|^2}{\xi_{t \wedge \sigma_m \wedge \zeta_n}} \leq \frac{|x - y|^2}{\xi_0}
\]

holds for all \( n > m > 0 \) and \( t \in [0, T) \). By letting first \( n \uparrow \infty \) then \( m \uparrow \infty \), we obtain \( \mathbb{Q}(\zeta \leq t) = 0 \) for all \( t \in [0, T) \). This is equivalent to \( \mathbb{Q}(\zeta = T) = 1 \) according to the definition of \( \zeta \).

Lemma 2.1 ensures that under \( \mathbb{Q} := R_{T \wedge \zeta} \mathbb{P} \), \( \{\tilde{B}_t\}_{t \in [0, T]} \) is a Brownian motion. Then by (2.4), the coupling \( (X_t, Y_t) \) is well-constructed under \( \mathbb{Q} \) for \( t \in [0, T] \). Since \( \int_0^T \xi_t^{-1} dt = \infty \), we shall see that the coupling is successful up to time \( T \), so that \( X_T = Y_T \) holds \( \mathbb{Q} \)-a.s. (see the proof of Theorem 111 below). This will provide the desired Harnack inequality for \( P_t \) as explained in Section 1 as soon as \( R_{T \wedge \zeta} \) has finite \( p/(p - 1) \)-moment. The next lemma provides an explicit upper bound on moments of \( R_{T \wedge \zeta} \).

**Lemma 2.2.** Assume (A1)-(A3). Let \( R_t \) and \( \xi_t \) be fixed for \( \theta = \theta_T \). We have

\[
\sup_{s \in [0, T]} \mathbb{E} \left\{ R_{s \wedge \zeta} \exp \left[ \frac{\theta_T^2}{8 \delta_T^2} \int_0^{s \wedge \zeta} \frac{|X_t - Y_t|^2}{\xi_t^2} dt \right] \right\} \leq \exp \left[ \frac{\theta_T K_T |x - y|^2}{4 \delta_T^2 (2 - \theta_T)(1 - e^{-K_T T})} \right].
\]

Consequently,

\[
\sup_{s \in [0, T]} \mathbb{E} R_{s \wedge \zeta}^{1 + r_T} \leq \exp \left[ \frac{\theta_T K_T (2 \delta_T + \theta_T \lambda_T) |x - y|^2}{8 \delta_T^2 (2 - \theta_T) \lambda_T (1 - e^{-K_T T})} \right]
\]

holds for

\[
r_T = \frac{\lambda_T^2 \theta_T^2}{4 \delta_T^2 + 4 \theta_T \lambda_T \delta_T}.
\]

**Proof.** Let \( \theta = \theta_T \). By (2.5), for any \( r > 0 \) we have

\[
\mathbb{E}_{s, n} \exp \left[ r \int_0^{s \wedge \tau_n} \frac{|X_t - Y_t|^2}{\xi_t^2} dt \right] \leq \exp \left[ \frac{r |x - y|^2}{\theta_T \xi_0} \right] \cdot \mathbb{E}_{s, n} \exp \left[ \frac{2r}{\theta_T} \int_0^{s \wedge \tau_n} \frac{1}{\xi_t} \left((\sigma(t, X_t) - \sigma(t, Y_t))(X_t - Y_t), dB_t\right) \right]
\]

\[
\leq \exp \left[ \frac{r K_T |x - y|^2}{\theta_T (2 - \theta_T)(1 - e^{-K_T T})} \right] \left( \mathbb{E}_{s, n} \exp \left[ \frac{8r^2 \delta_T^2}{\theta_T^2} \int_0^{s \wedge \tau_n} \frac{|X_t - Y_t|^2}{\xi_t^2} dt \right] \right)^{1/2},
\]

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where the last step is due to (A3) and the fact that

$$\mathbb{E}e^{M_t} \leq (\mathbb{E}e^{(M)_t})^{1/2}$$

for a continuous exponential integrable martingale $M_t$. Taking $r = \theta^2_T/(8\delta^2_T)$, we arrive at

$$\mathbb{E}_{s,n} \exp \left[ \frac{\theta^2_T}{8\delta^2_T} \int_0^{s\wedge \tau_n} |X_t - Y_t|^2 \frac{1}{\xi^2_t} dt \right] \leq \left[ \frac{\theta_T K_T |x - y|^2}{4\delta^2_T (2 - \theta_T) (1 - e^{-K_T})} \right]^{1/2}, \quad n \geq 1.$$

This implies (2.7) by letting $n \to \infty$.

Next, by (A2) and the definition of $R_s$, we have

$$\mathbb{E}R_{s \wedge \tau_n}^{1+r_T} = \mathbb{E}_{s,n} R_{s \wedge \tau_n}^{r_T}$$

$$= \mathbb{E}_{s,n} \exp \left[ - r_T \int_0^{s \wedge \tau_n} \frac{1}{\xi_t} \langle \sigma(t, X_t)^{-1} (X_t - Y_t), dB_t \rangle + \frac{r_T}{2} \int_0^{s \wedge \tau_n} |\sigma(t, X_t)^{-1} (X_t - Y_t)|^2 \frac{1}{\xi^2_t} dt \right].$$

(2.9)

Noting that for any exponential integrable martingale $M_t$ w.r.t. $R_{s \wedge \tau_n} \mathbb{P}$, one has

$$\mathbb{E}_{s,n} \exp [r_T M_t + r_T (M)_t/2] = \mathbb{E}_{s,n} \exp [r_T M_t - r^2_T q (M)_t/2 + r_T (q r_T + 1) (M)_t/2]$$

$$\leq \left( \mathbb{E}_{s,n} \exp [r_T q M_t - r^2_T q^2 (M)_t/2] \right)^{q/2} \left( \mathbb{E}_{s,n} \exp \left[ \frac{r_T q (q r_T + 1)}{(q - 1)} (M)_t \right] \right)^{(q-1)/q}, \quad q > 1,$$

it follows from (2.9) that

$$\mathbb{E}R_{s \wedge \tau_n}^{1+r_T} \leq \left( \mathbb{E}_{s,n} \exp \left[ \frac{q r_T (q r_T + 1)}{2(q - 1)} \int_0^{s \wedge \tau_n} |X_t - Y_t|^2 \frac{1}{\xi^2_t} dt \right] \right)^{(q-1)/q}.$$ 

(2.10)

Take

$$q = 1 + \sqrt{1 + r_T^{-1}},$$

(2.11)

which minimizes $q (q r_T + 1)/(q - 1)$ such that
\[
\frac{qr_T(qr_T + 1)}{2\lambda_T^2(q - 1)} = r_T + \frac{r_T(r_T + 1)}{2\lambda_T^2\sqrt{1 + r_T^{-1}}} \left( r_T + 1 + \sqrt{r_T(r_T + 1)} \right)
\]

(2.12)

Combining (2.10) with (2.7) and (2.12), and noting that due to (2.11) and the definition of \( r_T \)

\[
\frac{q - 1}{q} = \frac{\sqrt{1 + r_T^{-1}}}{1 + \sqrt{1 + r_T^{-1}}} = \frac{2\delta_T + \theta_T\lambda_T}{2\delta_T + 2\theta_T\lambda_T},
\]

we obtain

\[
\mathbb{E} R_{s,T}^{1+\tau} \leq \exp \left[ -\frac{\theta_T K_T (2\delta_T + \theta_T\lambda_T) |x - y|^2}{8\delta_T^2 (2 - \theta_T) (\delta_T + \theta_T\lambda_T)(1 - e^{-K_T T})} \right].
\]

According to the Fatou lemma, the proof is then completed by letting \( n \to \infty \).

**Proof of Theorem 1.1.** Since (A3) also holds for \( \delta_{p,T} \) in place of \( \delta_T \), it suffices to prove the desired Harnack inequality for \( \delta_T \) in place of \( \delta_{p,T} \).

(1) By Lemma 2.1 \( \{R_{s,T}\}_{s \leq T} \) is an uniformly integrable martingale and \( \{\tilde{B}_t\}_{t \leq T} \) is a \( d \)-dimensional Brownian motion under the probability \( Q \). Thus, \( Y_t \) can be solved up to time \( T \). Let

\[
\tau = \inf\{t \in [0, T] : X_t = Y_t\}
\]

and set \( \inf \emptyset = \infty \) by convention. We claim that \( \tau \leq T \) and thus, \( X_T = Y_T \), \( Q \)-a.s. Indeed, if for some \( \omega \in \Omega \) such that \( \tau(\omega) > T \), by the continuity of the processes we have

\[
\inf_{t \in [0, T]} |X_t - Y_t|^2(\omega) > 0.
\]

So,

\[
\int_0^T \frac{|X_t - Y_t|^2}{\xi_t^2} dt = \infty
\]
holds on the set \( \{ \tau > T \} \). But according to Lemma 2.2 we have
\[
\mathbb{E}_Q \int_0^T |X_t - Y_t|^2 \frac{dt}{\xi_t^2} < \infty,
\]
we conclude that \( Q(\tau > T) = 0 \). Therefore, \( X_T = Y_T \) \( Q \)-a.s.

Now, combining Lemma 2.1 with \( X_T = Y_T \) and using the Young inequality, for \( f \geq 1 \) we have
\[
P_T \log f(y) = \mathbb{E}_Q[\log f(Y_T) - \mathbb{E}[R_{T \wedge \zeta} \log f(X_T)] \leq \log P_T f(x) + \frac{K_T |x - y|^2}{2\lambda_T^2 \theta (2 - \theta)(1 - e^{-K_T T})}.
\]

This completes the proof of (1) by taking \( \theta = 1 \).

(2) Let \( \theta = \theta_T \). Since \( X_T = Y_T \) and \( \{ \tilde{B}_t \}_{t \in [0, T]} \) is the \( d \)-dimensional Brownian motion under \( Q \), we have
\[
(P_T f(y))^p = (\mathbb{E}_Q[f(Y_T)])^p = (\mathbb{E}[R_{T \wedge \zeta} f(X_T)])^p \leq (P_T f^p(x)) (\mathbb{E} R_{T \wedge \zeta}^{p/(p-1)})^{p-1}.
\]

Due to (2.1) we see that
\[
\frac{p}{p-1} = 1 + \frac{\lambda_T^2 \theta_T^2}{4\delta_T (\delta_T + \theta_T \lambda_T)}.
\]

So, it follows from Lemma 2.2 and (2.1) that
\[
(\mathbb{E} R_{T \wedge \zeta}^{p/(p-1)})^{p-1} = (\mathbb{E} R_{T \wedge \zeta}^{1+r_T})^{p-1} \leq \exp \left[ \frac{(p-1)\theta_T K_T (2\delta_T + \theta_T \lambda_T)|x - y|^2}{8\delta_T^2 (2 - \theta_T) (\delta_T + \theta_T \lambda_T) (1 - e^{-K_T T})} \right]
\]
\[
= \exp \left[ \frac{K_T \sqrt{p} (\sqrt{p} - 1)|x - y|^2}{4\delta_T [(\sqrt{p} - 1)\lambda_T - \delta_T] (1 - e^{-K_T T})} \right].
\]

Then the proof is finished by combining this with (2.13). \( \square \)

**Proof of Corollary 1.3.** Let \( f \in \mathcal{B}_k^+(\mathbb{R}^d) \) be such that \( \mu(f^p) \leq 1 \). Let \( p > (1 + \delta / \lambda)^2 \). By Theorem 1.1(2), we have
\[ (P_t f(y))^p \exp \left[ -\frac{K\sqrt{p}(\sqrt{p} - 1)|x-y|^2}{4\delta_p[(\sqrt{p} - 1)\lambda - \delta_p](1 - e^{-Kt})} \right] \leq P_t f^p(x), \quad x, y \in \mathbb{R}^d, \]

where \( \delta_p = \max\{\delta, \frac{1}{2}(\sqrt{p} - 1)\} \). Integrating w.r.t. \( \mu(dx) \) and noting that \( \mu \) is \( P_t \)-invariant, we obtain

\[ (P_t f(y))^p \int_{\mathbb{R}^d} \exp \left[ -\frac{K\sqrt{p}(\sqrt{p} - 1)|x-y|^2}{4\delta_p[(\sqrt{p} - 1)\lambda - \delta_p](1 - e^{-Kt})} \right] \mu(dx) \leq 1. \]

Taking \( f = n \wedge (p_t(y, \cdot))^{1/p} \) and letting \( n \uparrow \infty \), we prove the first assertion.

Next, let \( B(0,1) = \{x \in \mathbb{R}^d : |x| \leq 1\} \). Since \( \mu \) is an invariant measure, it has a strictly positive density w.r.t. the Lebesgue measure so that \( \mu(B(0,1)) > 0 \) (cf. [6]). Let \( p \geq (1 + 2\delta/\lambda)^2 \). We have \( \delta_p = (\sqrt{p} - 1)\lambda/2 \) and thus,

\[ \frac{\sqrt{p}(\sqrt{p} - 1)}{4\delta_p[(\sqrt{p} - 1)\lambda - \delta_p]} = \frac{\sqrt{p}}{\lambda^2(\sqrt{p} - 1)}. \]

Combining this with (2.14) and noting that

\[ \int_{\mathbb{R}^d} \exp \left[ -\frac{K\sqrt{p}(\sqrt{p} - 1)|x-y|^2}{4\delta_p[(\sqrt{p} - 1)\lambda - \delta_p](1 - e^{-Kt})} \right] \mu(dx) \geq \mu(B(0,1)) \exp \left[ -\frac{K\sqrt{p}(\sqrt{p} - 1)(1 + |y|)^2}{4\delta_p[(\sqrt{p} - 1)\lambda - \delta_p](1 - e^{-Kt})} \right], \]

we obtain

\[ (P_t f(y))^p \leq C_1 \exp \left[ \frac{K\sqrt{p}(1 + |y|)^2}{\lambda^2(\sqrt{p} - 1)(1 - e^{-Kt})} \right], \quad t > 0, y \in \mathbb{R}^d \]

for some constant \( C_1 > 0 \) and all \( f \in B_b^+(\mathbb{R}^d) \) with \( \mu(f^p) \leq 1 \). Since

\[ \lim_{p \to \infty} \lim_{t \to \infty} \frac{K\sqrt{p}}{\lambda^2(\sqrt{p} - 1)(1 - e^{-Kt})} = \frac{K^+}{\lambda^2}, \]

for any \( r > K^+/\lambda^2 \) there exist \( p > (1 + 2\delta r/\lambda)^2, \beta > 1 \) and \( t_1 > 0 \) such that

\[ (P_{t_1} f(y))^{\beta p} \leq C_2 e^{r|y|^2}, \quad y \in \mathbb{R}^d, f \in B_b^+(\mathbb{R}^d), \mu(f^p) \leq 1 \]
holds for some constant $C_2 > 0$. Thus, $\mu(e^{r|\cdot|^2}) < \infty$ implies that $\|P_{t_1}\|_{L^p(\mu) \to L^{p\beta}(\mu)} < \infty$. Since $\|P_s\|_{L^q(\mu)} = 1$ holds for any $q \in [1, \infty]$, by the interpolation theorem and the semigroup property one may find $t_2 > t_1$ such that

\begin{equation}
\|P_{t_2}\|_{L^2(\mu) \to L^4(\mu)} < \infty.
\end{equation}

Moreover, by [13, Theorem 3.6(ii)], there exist some constants $\eta, C_3 > 0$ such that

$$\|P_t - \mu\|_{L^2(\mu)} \leq C_3 e^{-\eta t}, \quad t \geq 0.$$ \hspace{1cm} (2.16)

Combining this with (2.16) we conclude that $\|P_t\|_{L^2(\mu) \to L^4(\mu)} \leq 1$ holds for sufficiently large $t > 0$, i.e. (2) holds.

Finally, (3) and (4) follow immediately from (2.15) and the interpolation theorem. \hfill \square

### 3 Extension to manifolds with convex boundary

Let $M$ be a $d$-dimensional complete, connected Riemannian manifold, possibly with a convex boundary $\partial M$. Let $N$ be the inward unit normal vector filed of $\partial M$ when $\partial M \neq \emptyset$. Let $P_t$ be the (Neumann) semigroup generated by

\[ L := \psi^2(\Delta + Z) \]

on $M$, where $\psi \in C^1(M)$ and $Z$ is a $C^1$ vector field on $M$. Assume that $\psi$ is bounded and

\begin{equation}
\text{Ric} - \nabla Z \geq -K_0
\end{equation}

holds for some constant $K_0 \geq 0$. Then the (reflecting) diffusion process generated by $L$ is non-explosive.

To formulate $P_t$ as the semigroup associated to a SDE like (1.1), we set

\begin{align}
\sigma &= \sqrt{2} \psi, \quad b = \psi^2 Z.
\end{align}  \hspace{1cm} (3.2)
Let $d_I$ denote the Itô differential on $M$. In local coordinates the Itô differential for a continuous semi-martingale $X_t$ on $M$ is given by (see [10] or [4])

$$(d_I X_t)^k = dX_t^k + \frac{1}{2} \sum_{i,j=1}^d \Gamma_{ij}^k(X_t) d\langle X^i, X^j \rangle_t, \quad 1 \leq k \leq d.$$ 

Then $P_t$ is the semigroup for the solution to the SDE

$$(3.3) \quad d_I X_t = \sigma(X_t) \Phi_t dB_t + b(X_t) dt + N(X_t) dl_t,$$ 

where $B_t$ is the $d$-dimensional Brownian motion on a complete filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, $\Phi_t$ is the horizontal lift of $X_t$ onto the frame bundle $O(M)$, and $l_t$ is the local time of $X_t$ on $\partial M$. When $\partial M = \emptyset$, we simply set $l_t = 0$.

To derive the Harnack inequality as in Section 2, we assume that

$$(3.4) \quad \lambda := \inf \sigma > 0, \quad \delta := \sup \sigma - \inf \sigma < \infty.$$ 

Now, let $x, y \in M$ and $T > 0$ be fixed. Let $\rho$ be the Riemannian distance on $M$, i.e. $\rho(x, y)$ is the length of the minimal geodesic on $M$ linking $x$ and $y$, which exits if $\partial M$ is either convex or empty.

Let $X_t$ solve (3.3) with $X_0 = x$. Next any strictly positive function $\xi \in C([0, T))$, let $Y_t$ solve

$$d_Y = \sigma(Y_t) P_{X_t, Y_t} \Phi_t dB_t + b(X_t) dt - \frac{\sigma(Y_t) \rho(X_t, Y_t)}{\sigma(X_t)} \nabla \rho(X_t, \cdot)(Y_t) dt + N(Y_t) d\tilde{l}_t$$

for $Y_0 = y$, where $\tilde{l}_t$ is the local time of $Y_t$ on $\partial M$, and $P_{X_t, Y_t} : T_{X_t} M \to T_{Y_t} M$ is the parallel displacement along the minimal geodesic from $X_t$ to $Y_t$, which exists since $\partial M$ is convex or empty. As explained in [4, Section 3], we may and do assume that the cut-locus of $M$ is empty such that the parallel displacement is smooth. Let

$$d\tilde{B}_t = dB_t + \frac{\rho(X_t, Y_t)}{\xi_t \sigma(X_t)} \Phi_t^{-1} \nabla \rho(\cdot, Y_t)(X_t) dt, \quad t < T.$$
By the Girsanov theorem, for any $s \in (0, T)$ the process $\{\tilde{B}_t\}_{t \in [0, s]}$ is the $d$-dimensional Brownian motion under the weighted probability measure $R_s^P$, where

\begin{equation}
R_s := \exp \left[ -\int_0^s \frac{\rho(X_t, Y_t)}{\xi_t \sigma(X_t)} \langle \nabla \rho(\cdot, Y_t)(X_t), \Phi_t d\tilde{B}_t \rangle - \frac{1}{2} \int_0^s \frac{\rho(X_t, Y_t)^2}{\xi_t^2 \sigma(X_t)^2} dt \right].
\end{equation}

Thus, by (3.2) we have

\begin{align*}
\text{d}_t X_t &= \sqrt{2} \psi(X_t) \Phi_t d\tilde{B}_t + (\psi^2 Z)(X_t) dt - \frac{\rho(X_t, Y_t)}{\xi_t} \nabla \rho(\cdot, Y_t)(X_t) dt + N(X_t) dt,
\text{d}_t Y_t &= \sqrt{2} \psi(Y_t) \Phi_t d\tilde{B}_t + (\psi^2 Z)(Y_t) dt + N(Y_t) dt.
\end{align*}

Let $\xi \in C^1([0, T))$ be strictly positive and take

$$\beta_t = -\frac{\rho(X_t, Y_t)}{\sqrt{2} \xi_t \psi(X_t)} \Phi_t^{-1} \nabla \rho(\cdot, Y_t)(X_t).$$

Repeating the proof of (4.10) in [27] we obtain

$$\text{d}_t \rho(X_t, Y_t) \leq (\sigma(X_t) - \sigma(Y_t)) \langle \nabla \rho(\cdot, Y_t)(X_t), \Phi_t d\tilde{B}_t \rangle + K_1 \rho(X_t, Y_t) dt - \frac{\rho(X_t, Y_t)}{\xi_t} dt, \quad t < T,$$

where

$$K_1 = K_0 \|\psi\|_\infty^2 + 2 \|Z\|_\infty \|\nabla \psi\|_\infty \|\psi\|_\infty.$$

This implies that

$$\frac{d}{\xi_t} \rho(X_t, Y_t)^2 \leq \frac{2}{\xi_t^2} \rho(X_t, Y_t)(\sigma(X_t) - \sigma(Y_t)) \langle \nabla \rho(\cdot, Y_t)(X_t), \Phi_t d\tilde{B}_t \rangle - \frac{\rho(X_t, Y_t)^2}{\xi_t^2} (2 - K_1 + T_1') dt$$

holds for $t < T$ and

\begin{equation}
K := 2K_1 + \|\nabla \sigma\|_\infty^2
= 2K_0 \|\psi\|_\infty^2 + 4 \|Z\|_\infty \|\nabla \psi\|_\infty \|\psi\|_\infty + 2 \|\nabla \psi\|_\infty^2.
\end{equation}

In particular, letting
\[ \xi_t = \frac{2 - \theta}{K} (1 - e^{K(t-T)}), \quad t \in [0, T], \theta \in (0, 2), \]

we have

\[ 2 - K\xi_t + \xi'_t = \theta. \]

Therefore, the following result follows immediately by repeating calculations in Section 2.

**Theorem 3.1.** Assume that \( \partial M \) is either empty or convex. Let (4.1) and \( Z, \phi \) be bounded such that

\[
K := 2K_0\|\psi\|_\infty^2 + 4\|Z\|_\infty\|\nabla\psi\|_\infty\|\psi\|_\infty + 2\|\nabla\psi\|_\infty^2 < \infty.
\]

Then all assertions in Theorem 1.1 and Corollaries 1.2, 1.3 hold for \( P_t \) the (Neumann) semigroup generated by \( L = \psi^2(\Delta + Z) \) on \( M \), and for constant functions \( K, \delta := \sup \psi - \inf \phi \) and \( \lambda := \inf |\psi| \).

## 4 Neumann semigroup on non-convex manifolds

Following the line of [24], we are able to make the boundary from non-convex to convex by using a conformal change of metric. This will enable us to extend our results to the Neumann semigroup on a class of non-convex manifolds.

Let \( \partial M \neq \emptyset \) with \( N \) the inward normal unit vector field. Then the second fundamental form of \( \partial M \) is a two-tensor on the tangent space of \( \partial M \) defined by

\[
\mathbb{I}(X,Y) := -\langle \nabla_X N, Y \rangle, \quad X, Y \in T\partial M.
\]

Assume that there exists \( \kappa > 0 \) and \( K_0 \in \mathbb{R} \) such that

\[
(4.1) \quad \text{Ric} - \nabla Z \geq -K_0, \quad \mathbb{I} \geq -\kappa
\]

holds for \( M \) and a \( C^1 \) vector field \( Z \). We shall consider the Harnack inequality for the Neumann semigroup \( P_t \) generated by
\[ L = \Delta + Z. \]

To make the boundary convex, let \( f \in C^\infty_0(M) \) such that \( f \geq 1 \) and \( N \log f|_{\partial M} \geq \kappa \). By [24, Lemma 2.1], \( \partial M \) is convex under the metric

\[ \langle \cdot, \cdot \rangle' = f^{-2} \langle \cdot, \cdot \rangle. \]

Let \( \Delta' \) and \( \nabla' \) be the Laplacian and gradient induced by the new metric. We have (see (2.2) in [21])

\[ L = f^{-2}(\Delta' + Z'), \quad Z' = f^2Z + \frac{d-2}{2}\nabla f^2. \]

Let \( \text{Ric}' \) be the Ricci curvature induced by the metric \( \langle \cdot, \cdot \rangle' \). We have (see the proof of [27, Theorem 5.1])

\[ \text{Ric}' - \nabla'Z' \geq -K_f \langle \cdot, \cdot \rangle' \]

for

\[ (4.2) \quad K_f = \sup \left\{ Kf^2 - d\Delta f + (d - 3)|\nabla f|^2 + 3|Z|f|\nabla f| \right\}. \]

Applying Theorem 3.1 to the convex manifold \( (M, \langle \cdot, \cdot \rangle'), \psi = f^{-1} \) and

\[ (4.3) \quad K = 2K_f^+ \|f^{-1}\|_{\infty} + 4\|Z\|_{\infty} \|\nabla'f^{-1}\|_{\infty} \|f^{-1}\|_{\infty} + 2\|\nabla'f^{-1}\|_{\infty}^2 \leq 2K_f^+ + 4\|fZ\|_{\infty} \|\nabla f\|_{\infty} + 2\|\nabla f\|_{\infty}^2, \]

where \( \|\cdot\|' \) is the norm induced by \( \langle \cdot, \cdot \rangle' \) and we have used that \( f \geq 1 \), we obtain the following result.

**Theorem 4.1.** Let (4.1) hold for some \( \kappa > 0 \) and \( K_0 \in \mathbb{R} \), and let \( P_t \) be the Neumann semigroup generated by \( L = \Delta + Z \) on \( M \). Then for any \( f \in C^\infty_0(M) \) such that \( \inf f = 1 \), \( N \log f|_{\partial M} \geq \kappa \) and \( K < \infty \), where \( K \) is fixed by (4.2) and (4.3), all assertions in Theorem 1.1 and Corollaries 1.2 and 1.3 hold for constant functions \( K, \delta \) := sup \( f^{-1} - \inf f^{-1} \), and \( \lambda := \inf f^{-1} \).
Remark 4.1 A simple choice of $f$ in Theorem 4.1 is $f = \phi \circ \rho_\partial$, where $\rho_\partial$ is the Riemannian distance to the boundary which is smooth on $\{\rho_\partial \leq r_T\}$ for some $r_T > 0$ provided the injectivity radius of the boundary is positive, and $f \in C_\infty^0([0, \infty))$ is such that $f(0) = 1, f'(0) = \kappa$ and $f(r) = f(r_T)$ for $r \geq r_T$. In general, $f$ is taken according to $r_T$ and bounds of the second fundamental form and sectional curvatures, see e.g. [24, 27] for details. With specific choices of $f$, Theorem 4.1 provides explicit Harnack type inequalities, heat kernels estimates and criteria on contractivity properties for the Neumann semigroup on manifolds with non-convex boundary.

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References

[1] S. Aida, Uniform positivity improving property, Sobolev inequalities, and spectral gaps, J. Funct. Anal. 158(1998), 152–185.

[2] S. Aida and H. Kawabi, Short time asymptotics of a certain infinite dimensional diffusion process, Stochastic analysis and related topics, VII (Kusadasi, 1998), Progr. Probab., vol. 48, Birkhäuser Boston, Boston, 2001, pp. 77–124.

[3] S. Aida and T. Zhang, On the small time asymptotics of diffusion processes on path groups, Potential Anal. 16(2002), 67–78.

[4] M. Arnaudon, A. Thalmaier, F.-Y. Wang, Harnack inequality and heat kernel estimates on manifolds with curvature unbounded below, Bull. Sci. Math. 130(2006), 223–233.

[5] D. Bakry, M. Emery, Hypercontractivité de semi-groupes de diffusion, C. R. Acad. Sci. Paris. Sér. I Math. 299(1984), 775–778.

[6] V. I. Bogachev, N. V. Krylov, M. Röckner, On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions, Comm. Partial Differential Equations 26 (2001), 2037–2080.
[7] X. Chen, F.-Y. Wang, *Optimal integrability condition for the log-Sobolev inequality*, Quart. J. Math. (Oxford) 58(2007), 17–22.

[8] G. Da Prato, M. Röckner, F.-Y. Wang, *Singular stochastic equations on Hilbert spaces: Harnack inequalities for their transition semigroups*, J. Funct. Anal. 257(2009), 992–1017.

[9] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge Univ. Press, 1992.

[10] M. Emery, *Stochastic calculus in manifolds*, Springer-Verlag, Berlin, 1989.

[11] A. Es-Sarhir, M.-K. von Renesse, M. Scheutzew, *Harnack inequality for functional SDEs with bounded memory*, Elect. Comm. Probab. 14(2009), 560–565.

[12] S. Fang, T.-S. Zhang, *A study of a class of stochastic differential equations with non-Lipschitzian coefficients*, Probab. Theory Related Fields 132 (2005), 356–390.

[13] M. Hino, *Exponential decay of positivity preserving semigroups on $L^p$*, Osaka J. Math. 37 (2000), 603–624.

[14] N. Ikeda, S. Watanabe, *Stochastic Differential Equations and Diffusion Processes (Second Edition)*, North-Holland, Amsterdam, 1989.

[15] W. Liu, *Fine properties of stochastic evolution equations and their applications*, Doctor-Thesis, Bielefeld University, 2009.

[16] W. Liu and F.-Y. Wang, *Harnack inequality and strong Feller property for stochastic fast-diffusion equations*, J. Math. Anal. Appl. 342(2008), 651-662.

[17] S.-X. Ouyang, *Harnack inequalities and applications for stochastic equations*, Ph.D. thesis, Bielefeld University, 2009.

[18] M. Röckner and F.-Y. Wang, *Supercontractivity and ultracontractivity for (non-symmetric) diffusion semigroups on manifolds*, Forum Math. 15(2003), 893–921.
[19] M. Röckner, F.-Y. Wang, *Harnack and functional inequalities for generalized Mehler semigroups*, J. Funct. Anal. 203(2003), 237–261.

[20] M. Röckner and F.-Y. Wang, *Log-Harnack inequality for stochastic differential equations in Hilbert spaces and its consequences*, Inf. Dim. Anal. Quant. Proba. Relat. Top. 13(2010), 27–37.

[21] A. Thalmaier, F.-Y. Wang, *Gradient estimates for harmonic functions on regular domains in Riemannian manifolds*, J. Funct. Anal. 155:1(1998),109–124.

[22] F.-Y. Wang, *Logarithmic Sobolev inequalities on noncompact Riemannian manifolds*, Probab. Theory Related Fields 109(1997), 417–424.

[23] F.-Y. Wang, *Equivalence of dimension-free Harnack inequality and curvature condition*, Integral Equations Operator Theory 48(2004), no. 4, 547–552.

[24] F.-Y. Wang, *Harnack inequality and applications for stochastic generalized porous media equations*, Ann. Probab. 35(2007), 1333–1350.

[25] F.-Y. Wang, *Estimates of the first Neumann eigenvalue and the log-Sobolev constant on nonconvex manifolds*, Math. Nachr. 280(2007), 1431–1439.

[26] F.-Y. Wang *Harnack inequalities on manifolds with boundary and applications*, J. Math. Pures Appl. 94(2010), 304–321.

[27] F.-Y. Wang *Transportation-cost inequalities on path space over manifolds with boundary*, [arXiv:0908.2891](http://arxiv.org/abs/0908.2891).

[28] T. Yamada, S. Watanabe, *On the uniqueness of solutions of stochastic differential equations*, J. Math. Kyoto Univ. 11(1971), 155–167.

[29] T.-S. Zhang, *White noise driven SPDEs with reflection: strong Feller properties and Harnack inequalities*, to appear in Potential Analysis.