The Relationship between the Maximum and Minimum Principal Stresses as a Strength Criterion of Brittle Materials

T B Duishenaliev¹, V A Khokhlov¹, S V Ushanov¹
¹National Research University "Moscow Power Engineering Institute",
14 Krasnokazarmennaya street, 111250, Moscow, Russia

E-mail: DuyshenaliyevT@mpei.ru

Abstract. An equation is obtained that expresses the quantitative relationship between the limiting values of the smallest and largest principal stresses. Using this equation, it is possible to determine all those stress states that lead to the destruction of brittle (semi-brittle) materials, as well as to construct an envelope line of limiting stress circles of the German scientist O. Mohr. The experimental consistency of this criterion is checked in comparison with the experimental data of T. Karman, A.N. Stavrogin, M.I. Koifman and M.F. Kuntysh, obtained by testing cylindrical samples of various rocks (brittle materials) under triaxial compression.

1. Introduction
The development of mineral deposits at great depths of the earth's crust, where rocks are under triaxial compression, requires careful study and prediction of the behavior of such materials at various levels of stress. This problem is closely related to the theory of strength by O. Mohr [8], the main task of which is a quantitative description of destructive forces under conditions of triaxial stress states.

Mohr's assumption that the largest stress circles should form a regular family was brilliantly confirmed by subsequent experimental studies [4]. However, it cannot be seen as a solution to the problem.

What type is this envelope line, can it be represented as a parabola, hyperbola, ellipse, cycloid, etc.? Finally, through what constants of the material can its equation be expressed? These are the questions that are being addressed in this work.

2. An equation of the envelope line of Mohr's limit circles in parametric form
Any stress state can be depicted on the plane of normal \( \sigma \) and tangential \( \tau \) stresses. According to the Mohr's assumption, the stress states that destroy the material, depicted on the plane \( \sigma, \tau \), should form a regular family and have a common envelope line.

Assuming that the compressive stresses are positive, as is customary in rock mechanics, we denote by \( \sigma_1, \sigma_3 \), respectively, the largest and the smallest principal stresses. The equation of limit circles is:

\[
\sigma^2 + \tau^2 - (\sigma_1 + \sigma_3)\sigma + \sigma_1 \sigma_3 = 0. \tag{1}
\]

The expression (1) implies those values of stresses that correspond to the moment of destruction of materials. The circles described by the equation (1) are called the limit stress circles, or Mohr limit circles.
For a family of circles (1) to have a common envelope line, it must be defined by one parameter. As a parameter, we will choose one of the main stresses, for example, \( \sigma_1 \). For the presence of an envelope line of limit stress circles, it is necessary that the partial derivative of equation (1) with respect to the selected parameter be equal to zero.

\[-(1 + \sigma_3') \sigma + (\sigma_3 + \sigma_1 \sigma_3') = 0.\]

From here we find:

\[\sigma = \frac{\sigma_3 + \sigma_1 \sigma_3'}{1 + \sigma_3'}.\]  
(2)

Substitute (2) into the equation (1):

\[\tau = \pm \frac{\sigma_1 - \sigma_3}{1 + \sigma_3'} \sqrt{\sigma_3'}.\]  
(3)

The expressions (2), (3) represent the equation of the envelope of Mohr's limit circles in parametric form.

3. Determination of the function serving as a parameter in the equation of the envelope line of Mohr's limit circles

Let’s assume that it was possible to exclude the parameter \( \sigma_1 \) from the equations (2) and (3), as well as the functions \( \sigma_3(\sigma_1), \sigma_3'(\sigma_1) \) determined by this parameter. This would lead to an obvious envelope equation:

\[\tau = \tau(\sigma).\]  
(4)

Substitute (4) into the equation (1):

\[\sigma^2 + \tau(\sigma)^2 - (\sigma_1 + \sigma_3) \sigma + \sigma_1 \sigma_3 = 0\]  
(5)

Now the expression (5) can be viewed as the equation between the principal stresses \( \sigma_1, \sigma_3 \), which contains a parameter \( \sigma \). By setting this parameter to different values, we get a curve on the plane \( \sigma_1, \sigma_3 \), i.e.:

\[\sigma_3 = \sigma_3(\sigma_1).\]  
(6)

This curve is of the second order, because the expression (5) is an algebraic equation of the second degree with respect to variables \( \sigma_1, \sigma_3 \). Curve (6) has an asymptote, the slope of which is equal to unity element. This follows from the fact that the envelope line of the limiting circles in the region of high pressures becomes parallel to the normal stress axis (the radius of the circles \( (\sigma_1 - \sigma_3)/2 \) tends to a constant value). Of the curves of the second order, only the hyperbola has an asymptote. This is the basis for the search for dependence (6) in the form of a hyperbola, the slope coefficient of the asymptote of which is equal to unity element. The equation of such a hyperbola was obtained in the work [1] and has the form:

\[\sigma_3 = \frac{a + b}{2} + \sqrt{\frac{(a - b)^2}{8} + \sigma_1^2},\]  
(7)

where \( a, b \) are constants. This equation is a quantitative description of Mohr's theory of strength and can be regarded as a strictly mathematically formulated strength criterion.
Let's say the values of the constants \( a, b \) for a given material are known. In this case, setting the values \( \sigma_1 \) within \( 0 \leq \sigma_1 < \infty \) and calculating the corresponding values \( \sigma_3 \) it is possible to determine all limiting stress states. It is also possible to build an envelope line using the following equations:

\[
\sigma = \frac{\sigma_3 + \sigma_1 \sigma_3}{1 + \sigma_3}, \quad \tau = \pm \frac{\sigma_1 - \sigma_3}{1 + \sigma_3} \sqrt{\sigma_3}.
\]

\[
\sigma_3 = \frac{a + b}{2} + \sqrt{\frac{(a - b)^2}{8} + \sigma_1^2}, \quad \sigma_3 = \frac{\sigma_1}{\sqrt{\frac{8}{8} + \sigma_1^2}}.
\] (8)

4. Determination of constants and comparison with experimental data

The immediate results on triaxial compression of materials are the measured values of the principal stresses at which fracture occurred. Below are the results of Karman's experiments (stresses in kgf/cm\(^2\)) [4]:

\[
\sigma_1 = (1360 \ 2350 \ 3150 \ 3565 \ 4055 \ 5550) \quad \sigma_3 = (0 \ 200 \ 500 \ 685 \ 845 \ 1650). \] (9)

By substituting the experimental values \( \sigma_1, \sigma_3 \) into the equation (7), we determine the values of the constants \( a = 3043, b = -18560 \).

Almost always in triaxial compression tests, the axial tensile strength of the material is not determined. But there is a great opportunity here. The value of the axial tensile strength can be determined theoretically. For this, in the equation (7) it is necessary to set \( \sigma_1 = 0 \) and determine the value of this limit. In this case, we get \( \sigma_p = -120.378 \).

Experimental data (9) are plotted on the graph of the function (7) (Fig. 1). Here the graph is built according to the values of the function (7) at the following values of the main stress \( \sigma_1 \):

\[
x = (0 \ 120 \ 240 \ 360 \ ... \ 6000).
\] (10)

**Figure 1.** Plot of a function \( \sigma_3 = \frac{a + b}{2} + \sqrt{\frac{(a - b)^2}{8} + \sigma_1^2} \) (solid line) and experimental data (points).
Figure 2 shows stress circles according to the experimental data (9) and an envelope line constructed according to the equations (8).

Figure 2. Experimental stress circles and envelope line constructed according to the equations (8).

Below are the experimental data, the values of the constants a, b and the calculated values of the axial tensile strength (dimension in kgf/cm²) for some rocks.

Sandstone (Karman's experiments) [4]:

\[ \sigma_1 = (690 \ 2320 \ 3135 \ 4850 \ 6475), \ \sigma_3 = (0 \ 280 \ 555 \ 1550 \ 2475), \]

\[ a = 3120, \ b = -18400, \ \sigma_p = -31.6. \]

Pyroxenite (experiments of A.N. Stavrogin) [10]:

\[ \sigma_1 = (3560 \ 7760 \ 8880 \ 11200), \ \sigma_3 = (0 \ 720 \ 1040 \ 1960), \]

\[ a = 12370, \ b = -73530, \ \sigma_p = -209. \]

Melomergel (experiments of M.I. Koifman) [5]:

\[ \sigma_1 = (40 \ 257 \ 505 \ 790 \ 1005 \ 1405), \ \sigma_3 = (0 \ 50 \ 100 \ 200 \ 300 \ 600), \]

\[ a = 510, \ b = -2977, \ \sigma_p = -0.6. \]

Gabbro (experiments of M.F. Kuntys) [7]:

\[ \sigma_1 = (2006 \ 3510 \ 4180 \ 4820 \ 5250 \ 5580), \ \sigma_3 = (0 \ 250 \ 400 \ 600 \ 750 \ 900), \]

\[ a = 6339, \ b = -37820, \ \sigma_p = -129. \]

By constructing the curves according to the equation (7) and plotting the experimental stress values \( \sigma_1, \ \sigma_3 \) on them, we see that the experimental points almost lie on these curves. The stress circles, constructed according to the given experimental values of the principal stresses, are also bent around with a high degree of accuracy by the lines determined by the equation (8).
5. References

[1] Duishenaliev T B 2017 Non-classical solutions of the mechanics of a deformable body (Moscow: MPEI Publishing House) 400

[2] Duishenaliev T B, Ushanov S V & Salimov M S 2021 Boundary - Value static problem and modeling of elastic deformations of structures Proceedings of the 3rd 2021 International Youth Conference on Radio Electronics, Electrical and Power Engineering, REEPE 2021

[3] Duishembiev A S, Duishenaliev T B & Talipov D R 2020 Virtualization of the Behavior of Structures from Rubber-like and Metal Composites 2020 5th International Conference on Information Technologies in Engineering Education Inforino 2020

[4] Karman T 1915 Experiments on all-round compression New ideas in technology Digest of articles 1 pp 51-102

[5] Koifman M I 1964 Strength of rocks in a volumetric stress state (Moscow: Nauka) 34

[6] Kovalenko M D, Menshova I V, Kerzhaev A P & Yu G 2020 A boundary value problem in the theory of elasticity for a rectangle: exact solutions Zeitschrift fur Angewandte Mathematik und Physik vol 71 6

[7] Kuntysh M F 1964 Research of the method for determining the main physical and mechanical characteristics of rocks used in solving rock pressure problems Abstract of the dissertation of the candidate of technical sciences (Moscow) 24

[8] Mohr O 1915 Causes the elastic limit and temporary resistance of materials New ideas in technology: Collection 1 Education Publishing House p 50

[9] Revenko V P 2007 Investigation of the distribution of stresses in a rectangular plate under the action of distributed loads Materials Science vol 43 3 pp 300-309

[10] Stavrogin A N 1961 Research of rocks in complex stress states Mining journal 3 pp 34-39

[11] Stavrogin A N 1965 On the limiting states and deformation of rocks Mountain pressure. Collectoin VNIMI (Leningrad) 59 pp 33–62

Acknowledgments

The investigation was carried out within the framework of the project “Quantitative expression of Mohr's theory of strength” with the support of a grant from NRU "MPEI" for implementation of scientific research programs "Energy", "Electronics, Radio Engineering and IT", and “Industry 4.0, Technologies for Industry and Robotics in 2020-2022".