CLASSIFICATION OF JACOBIAN ELLIPTIC K3 SURFACES WITH 2-TORSION AND FINITE AUTOMORPHISM GROUP

ADRIAN CLINGHER AND ANDREAS MALMENDIER

Abstract. We study the complex algebraic K3 surfaces of finite automorphism group that admit a Jacobian elliptic fibration that also has a 2-torsion section. We give a complete classification for these K3 surfaces, including a description of all supported Jacobian elliptic fibrations as well as birational models as double sextics and quartic projective surfaces and their coarse moduli spaces.

1. Introduction

Let \( X \) be a smooth algebraic K3 surface defined over the field of complex numbers. Denote by \( \text{NS}(X) \) the Néron-Severi lattice of \( X \). A lattice polarization on \( X \) is, by definition \([11, 22–25]\), a primitive lattice embedding \( i : L \rightarrow \text{NS}(X) \), whose image \( i(L) \) contains a pseudo-ample class. Here, \( L \) is a choice of even indefinite lattice of signature \((1, \rho_L - 1)\), with \( 1 \leq \rho_L \leq 20 \). Two \( L \)-polarized K3 surfaces \((X, i)\) and \((X', i')\) are said to be isomorphic\(^1\), if there exists an isomorphism \( \alpha : X \rightarrow X' \) and a lattice isometry \( \beta \in O(L) \), such that \( \alpha^* \circ i' = i \circ \beta \), where \( \alpha^* \) is the appropriate morphism at cohomology level.

It is known \([12]\) that \( L \)-polarized K3 surfaces are classified, up to isomorphism, by a coarse moduli space \( \mathcal{M}_L \), which is a quasi-projective variety of dimension \( 20 - \rho_L \). A general \( L \)-polarized K3 surface \((X, i)\) satisfies \( i(L) = \text{NS}(X) \).

A first case with interesting geometry occurs when \( L = H \), where \( H \) stands for the standard rank-two hyperbolic lattice. An \( H \)-polarization on \( X \) uniquely determines \([5, \text{Lemma 3.6}]\) an elliptic fibration \( X \rightarrow \mathbb{P}^1 \), together with a choice of section in this fibration. Hence, an \( H \)-polarization on \( X \) is equivalent with a Jacobian elliptic fibration on \( X \). One obtains therefore a coarse moduli space for Jacobian elliptic fibrations by just considering the 18-dimensional variety \( \mathcal{M}_H \).

Consider the situation \( L = H \oplus N \), where \( N \) is the negative definite rank-eight Nikulin lattice (see \([21, \text{Def. 5.3}]\), for a definition). In this case, an \( L \)-polarization uniquely determines \([35]\) a two-fold geometrical structure: a Jacobian elliptic fibration \( \pi_X : \mathcal{X} \rightarrow \mathbb{P}^1 \), as well as an element of order-two in the associated Mordell-Weil group \( \text{MW}(\mathcal{X}, \pi_X) \). The latter element can be seen geometrically as a second section in the elliptic fibration \( \pi_X \), which determines an order-two point in each smooth fiber.

\(^1\)Our definition for isomorphic lattice polarizations coincides with the one used by Vinberg \([36–38]\) but is slightly more general than the one used in \([12, \text{Sec. 1}]\).
Fiber-wise translations by these order-two points determine furthermore a canonical symplectic involution $j_X: \mathcal{X} \to \mathcal{X}$; see [35]. Such involutions are referred to in the literature as van Geemen-Sarti involutions. The underlying elliptic fibration $\pi_\mathcal{X}$ is called the alternate fibration. A coarse moduli space for K3 surfaces endowed with van Geemen-Sarti involutions is given by the 10-dimensional variety $\mathcal{M}_{H \oplus N}$.

Continuing in the above context, if one factors the surface $\mathcal{X}$ by the action of the involution $j_\mathcal{X}$ and then resolves the eight occurring $A_1$-type singularities, a new K3 surface $\mathcal{Y}$ is obtained, related to $\mathcal{X}$ via a rational double cover map $\mathcal{X} \to \mathcal{Y}$. This construction is referred to in the literature as the Nikulin construction. The surface $\mathcal{Y}$ inherits a canonical van Geemen-Sarti involution $j_\mathcal{Y}$, as well as an underlying alternate elliptic fibration. Moreover, if one repeats the Nikulin construction on $\mathcal{Y}$, the original K3 surface $\mathcal{X}$ is recovered. The two surface $\mathcal{X}$ and $\mathcal{Y}$ are related via dual birational double covers:

\[
\begin{array}{cc}
\mathcal{X} & \mathcal{Y} \\
\xrightarrow{j_\mathcal{X}} & \xleftarrow{j_\mathcal{Y}} \\
\mathcal{X} & \mathcal{Y}
\end{array}
\]

We shall refer to this correspondence as the van Geemen-Sarti-Nikulin duality. It determines an interesting involution, at the level of moduli spaces:

\[
\iota_{\text{vgsn}}: \mathcal{M}_{H \oplus N} \to \mathcal{M}_{H \oplus N}, \quad \text{with} \quad \iota_{\text{vgsn}} \circ \iota_{\text{vgsn}} = \text{id}.
\]

In this article, we study several specific families of lattice polarized K3 surfaces with Picard ranks eleven and higher. The K3 surfaces in question admit canonical van Geemen-Sarti involutions. As an additional condition, we assume that the general members of the family have finite automorphism groups. Rephrasing more precisely the above, we shall study families of $L$-polarized K3 surfaces $\mathcal{X}$ satisfying the following conditions:

- \(a\) $L \simeq H \oplus K$, with $K$ a negative-definite lattice of ADE type,
- \(b\) $L$ admits a canonical primitive lattice embedding $H \oplus N \hookrightarrow L$,
- \(c\) $L$ is of finite automorphism group type\(^2\), in the sense of Nikulin [26,27].

An elliptic fibration associated with the $H$ embedding of condition (a) will be referred to as standard. The elliptic fibration associated with the $H$-embedding of condition (b) will be referred to as alternate. The alternate fibration is the fibration underlying the van Geemen-Sarti involution. We also note that there may be multiple non-equivalent decompositions, as in condition (a). However, using a natural representation of $\mathcal{X}$ as the minimal resolution of a double quadric surface a particular standard decomposition can be chosen that we will refer to as the standard fibration.

Using classification results for lattices [26], elliptic fibrations [31], and work by the authors [3], one can narrow down the list of lattices $L$ satisfying (a)-(c). As it turns out, we have one such instance of lattice $L$ in each rank $11 \leq \rho_L \leq 19$, with the exception of rank $\rho_L = 14$, for which two inequivalent instances exist. The complete classification result is as follows:

\[
\begin{align*}
(i) & \quad \rho_L = 11, \quad L = H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 5} \\
(ii) & \quad \rho_L = 12, \quad L = H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4} \\
(iii) & \quad \rho_L = 13, \quad L = H \oplus E_7(-1) \oplus A_1(-1)^{\oplus 4} \\
(iv) & \quad \rho_L = 14, \quad L = H \oplus D_8(-1) \oplus D_4(-1)
\end{align*}
\]

\(^2\)This condition asserts $\text{Aut}(\mathcal{X})$ is a finite group, for any K3 surface $\mathcal{X}$ with $\text{NS}(\mathcal{X}) = L$. 

2.1. Lattice theoretic considerations. For a lattice \( L \) let \( D(L) = L^\vee/L \) be the discriminant group with the associated discriminant form \( q_L \). A lattice \( L \) is then called 2-elementary if \( D(L) \) is a 2-elementary abelian group, i.e., \( D(L) \cong (\mathbb{Z}/2\mathbb{Z})^\ell \) where \( \ell = \ell_L \) is the length of \( L \), i.e., the minimal number of generators of the group \( D(L) \). One also has the parity \( \delta_L \in \{0,1\} \). By definition, \( \delta_L = 0 \) if \( q_L(x) \) takes values in \( \mathbb{Z}/2\mathbb{Z} \subset \mathbb{Q}/2\mathbb{Z} \) for all \( x \in D(L) \), and \( \delta_L = 1 \) otherwise. Following Kondo [17], we define the dual graph of the smooth rational curves to be the simplicial complex whose set of vertices is the set of all smooth rational curves on a K3 surface such that the vertices \( \Sigma, \Sigma' \) are joint by an \( m \)-fold edge if and only if their intersection product satisfies \( \Sigma \cdot \Sigma' = m \). We have the following:

**Theorem 2.1.** A general \( H \oplus N \)-polarized K3 surface \( X \) satisfies the following:

\[
\begin{align*}
\text{(v)} & \quad \rho_L = 14, \ L = H \oplus E_8(-1) \oplus A_1(-1)^{64} \\
\text{(vi)} & \quad \rho_L = 15, \ L = H \oplus E_8(-1) \oplus D_4(-1) \oplus A_1(-1) \\
\text{(vii)} & \quad \rho_L = 16, \ L = H \oplus E_8(-1) \oplus D_6(-1) \\
\text{(viii)} & \quad \rho_L = 17, \ L = H \oplus E_8(-1) \oplus E_7(-1) \\
\text{(ix)} & \quad \rho_L = 18, \ L = H \oplus E_8(-1) \oplus E_8(-1) \\
\text{(x)} & \quad \rho_L = 19, \ L = H \oplus E_8(-1) \oplus E_8(-1) \oplus A_1
\end{align*}
\]
Figure 1. Rational curves on $\mathcal{X}$ with Néron-Severi lattice $H \oplus N$

(1) the automorphism group of $\mathcal{X}$ is finite, i.e., $|\text{Aut}(\mathcal{X})| < \infty$.
(2) $\mathcal{X}$ has exactly 18 smooth rational curves and the dual graph of rational curves is given by Figure 1.

Proof. It is well known that for a general $H \oplus N$-polarized K3 surface $\mathcal{X}$ the automorphism group is finite [16, 17], i.e., $|\text{Aut}(\mathcal{X})| < \infty$. Moreover, the Jacobian elliptic fibration constructed above is the only Jacobian elliptic fibration supported on $\mathcal{X}$. We then construct the dual graph of smooth rational curves and their intersection properties on $\mathcal{X}$. From the alternate fibration the graph in Figure 1 is obtained where the black nodes represent the reducible fiber of type $A_1$ and the yellow nodes are the classes of the section and 2-torsion section. □

A result of Nikulin [26, Thm. 4.3.2] asserts that even, indefinite, 2-elementary lattices are uniquely determined by their rank $\rho_L$, length $\ell_L$, and parity $\delta_L$. Based on the classification of 2-elementary lattices the authors classified in [3] all lattices $L$ such that the general $L$-polarized K3 surface $\mathcal{X}$ satisfies $\text{Aut}(\mathcal{X}) < \infty$ and admits a Jacobian elliptic fibration:

Theorem 2.2 ([3]). All lattices of the form $L = H \oplus K$ such that a general $L$-polarized K3 surfaces $\mathcal{X}$ satisfies $\text{Aut}(\mathcal{X}) < \infty$ and admits a Jacobian elliptic fibration with 2-torsion section are given in Table 1. Moreover, $\mathcal{X}$ admits exactly the Jacobian elliptic fibrations with root lattices and Mordell-Weil groups $(K_{\text{root}}, W)$ given in Table 1.

Proof. In [3, Table 4] one selects all lattices admitting $\mathbb{Z}/2\mathbb{Z} \leq \text{MW}(\mathcal{X}, \pi_\mathcal{X})$. □

Remark 2.3. In Table 1 references are provided for the construction of the dual graph of smooth rational curves. We note that results in [16] are only partial results. In particular, the dual graphs of smooth rational curves in [16] for Picard rank $p_\mathcal{X} < 15$ only contain subsets of all smooth rational curves on the K3 surface.

Remark 2.4. In Table 1 a standard and alternate fibration are indicated. The alternate fibration is given by Equation (2.7), the construction of the standard fibration is explained in Section 2.3. The standard fibration admits a section for the lattice polarizations in Table 1 of rank $\rho_L \geq 12$; see Proposition 2.21.

Let us also investigate the number of distinct primitive lattice embeddings $H \twoheadrightarrow L$ in Theorem 2.2, following the approach in [13]. Assume $j: H \twoheadrightarrow L$ is such a primitive embedding. Denote by $K$ the orthogonal complement of $j(H)$ in $L$ and denote by
the sub-lattice spanned by the roots of $K$, i.e., the lattice elements of self-intersection $-2$ in $K$. The factor group is denoted by $\mathcal{W} = K/K^{\text{root}}$. It follows that $L = j(H) \oplus K$. Moreover, the lattice $K$ is negative-definite of rank $\rho_L - 2$, and its discriminant groups and pairing satisfy

$$
(2.1) \quad \left( D(K), q_K \right) \cong \left( D(L), q_L \right),
$$

with $D(L) \cong \mathbb{Z}_{27}^2$. Now assume that for a K3 surface $\mathcal{X}$ with $\text{NS}(\mathcal{X}) = L$ we have a second primitive embedding $j': H \hookrightarrow L$, such that the orthogonal complement of the image $j'(H)$, denoted $K'$, is isomorphic to $K$. We would like to see under what conditions $j$ and $j'$ correspond to Jacobian elliptic fibrations isomorphic under $\text{Aut}(\mathcal{X})$. By standard lattice-theoretic arguments (see [25, Prop. 1.15.1]), there will exist an isometry $\gamma \in O(L)$ such that $j' = \gamma \circ j$. The isometry $\gamma$ has a counterpart $\gamma^* \in O(D(K))$ obtained as image of $\gamma$ under the group homomorphism

$$
(2.2) \quad O(L) \to O(D(L)) \cong O(D(K)).
$$

The isomorphism in (2.2) is due to the decomposition $L = j(H) \oplus K$ and, as such, it depends on the lattice embedding $j$.

Denote the group $O(D(K))$ by $\mathcal{A}$. There are two subgroups of $\mathcal{A}$ that are relevant to our discussion. The first subgroup $\mathcal{B} \leq \mathcal{A}$ is given as the image of the following group homomorphism:

$$
(2.3) \quad O(K) \cong \{ \varphi \in O(L) \mid \varphi \circ j(H) = j(H) \} \hookrightarrow O(L) \to O(D(L)) \cong O(D(K)).
$$

The second subgroup $\mathcal{C} \leq \mathcal{A}$ is obtained as the image of following group homomorphism:

$$
(2.4) \quad O_h(T_X) \to O(T_X) \to O(D(T_X)) \cong O(D(L)) \cong O(D(K)).
$$

Here $T_X$ denotes the transcendental lattice of the K3 surface $\mathcal{X}$ and $O_h(T_X)$ is given by the isometries of $T_X$ that preserve the Hodge decomposition. Furthermore, one has $D(\text{NS}(\mathcal{X})) \cong D(T_X)$ with $q_L = -q_{T_X}$, as $\text{NS}(\mathcal{X}) = L$ and $T_X$ is the orthogonal complement of $\text{NS}(\mathcal{X})$ with respect to an unimodular lattice.

Consider then the correspondence

$$
(2.5) \quad H \hookrightarrow L \leadsto \mathcal{C} \gamma^* \mathcal{B},
$$

that associates to a lattice embedding $H \hookrightarrow L$ a double coset in $\mathcal{C}\mathcal{A}/\mathcal{B}$. As proved in [13, Thm 2.8], the map (2.5) establishes a one-to-one correspondence between Jacobian elliptic fibrations on $\mathcal{X}$ with $j(H)^{\perp} \cong K$, up to the action of the automorphism group $\text{Aut}(\mathcal{X})$ and the elements of the double coset set $\mathcal{C}\mathcal{A}/\mathcal{B}$. The number of elements in the double coset is referred by Festi and Veniani as the *multiplicity* associated with the frame $(K^{\text{root}}, \mathcal{W})$. Table 1 lists for the lattices $L$ of Theorem 2.2 all distinct possible frames. We have the following:

**Theorem 2.5.** For the lattices $L = H \oplus K$ in Table 1, the multiplicities associated with $(K^{\text{root}}, \mathcal{W})$ and $\mathcal{W} = \mathbb{Z}/2\mathbb{Z}$ equal one.

**Proof.** In [2] the authors showed that the multiplicity associated with each frame $K^{\text{root}}$ and $\mathcal{W}$ in Table 1 for $\rho_L \geq 16$ equals one. In [35, Sec. 1.10] it was shown that for $\rho_L = 10$, $L = H \oplus N$, $K = N$, and $K^{\text{root}} = A_1^{\oplus 8}$ the map (2.3) is surjective whence the multiplicity associated with the alternate fibration for $\rho_L = 10$ equals one.
For each frame $K$ with $\mathcal{W} = \{\mathbb{I}\}$, the Gram matrix of $K$, the Gram matrix of $D(K)$, and the group morphism $O(K) \to O(D(K))$ can be found in [31, Sec. 6]. For the alternate fibration, the lattice $K$ is the overlattice spanned by $K^{\text{root}}$ and one additional lattice vector $\vec{v}_{\text{max}}$. Using the same notation for the bases of root lattices as in [31, Sec. 6], each vector $\vec{v}_{\text{max}}$ is in the orthogonal complement of the lattice spanned by the section and the smooth fiber class. The lattice vector $\vec{v}_{\text{max}}$ can then be computed using the properties of the alternate fibration. The vectors are given in (2.6). The overlattices are computed using the command \texttt{overlattice} of the Sage class \texttt{QuadraticForm}. The corresponding Gram matrix is computed using the command \texttt{gram_matrix}.

\begin{equation}
\begin{array}{c|c|c}
\rho_L & K^{\text{root}} & \vec{v}_{\text{max}} \\
\hline
10 & 8A_1 & \frac{1}{2} \{1,1,1,1,1,1,1,1\} \\
11 & A_1 + 8A_1 & \frac{1}{2} \{0,1,1,1,1,1,1,1\} \\
12 & 6A_1 + D_4 & \frac{1}{2} \{1,1,1,1,1,1,1,0,1,0\} \\
13 & 5A_1 + D_6 & \frac{1}{2} \{1,1,1,1,1,1,0,1,0,0,0\} \\
14 & 5A_1 + E_7 & \frac{1}{2} \{1,1,1,1,1,0,0,0,1,0,1\} \\
15 & 4A_1 + D_8 & \frac{1}{2} \{1,1,1,1,1,0,1,0,1,0,0,0\} \\
16 & 3A_1 + D_{10} & \frac{1}{2} \{1,1,1,1,0,1,0,1,0,0,0,1\} \\
17 & 2A_1 + D_{12} & \frac{1}{2} \{1,1,1,0,1,0,1,0,1,0,1,0\} \\
18 & A_1 + D_{14} & \frac{1}{2} \{1,0,1,0,1,0,1,0,1,1,0,1\} \\
19 & D_{16} & \frac{1}{2} \{1,0,1,0,1,0,1,0,1,1,0,1\} \\
20 & A_1 + D_{16} & \frac{1}{2} \{0,1,0,1,0,1,0,1,0,1,0,1\} \\
21 & D_{10} + E_7 & \frac{1}{2} \{1,0,1,0,1,0,1,0,1,0,1,0\} \\
\end{array}
\end{equation}

For these lattices, the multiplicity associated with $K^{\text{root}}$ and $\mathcal{W} = \mathbb{Z}/2\mathbb{Z}$ is then shown to equal one by checking that the images of the generators for $O(K)$ also generate $O(D(K))$. For a lattice $K$, the discriminant group is computed using $D = K.\text{discriminant\_group()}$. The automorphism groups are computed using $O = K.\text{orthogonal\_group()}$ and $\mathcal{O}D = D.\text{orthogonal\_group()}$.

Images of the generators are computed as $\mathcal{O}bar = D.\text{orthogonal\_group}(O.gens())$.

\[ \square \]

Theorems 2.2 and 2.5 imply the following:

**Corollary 2.6.** For $10 \leq \rho_L \leq 18$ there is a canonical lattice embedding $H \oplus N \to L$, and any $L$-polarized $K3$ surface carries a unique $H \oplus N$-polarization. In particular, there is a canonical embedding $\mathcal{M}_L \to \mathcal{M}_{H \oplus N}$.

2.2. Construction of coarse moduli spaces. Coarse moduli spaces $\mathcal{M}_L$ for the lattices $L$ in Table 1 can now be constructed using the alternate fibration, which is unique due to Theorem 2.5. Concretely, a Weierstrass model for the Jacobian elliptic fibration $\pi_X : \mathcal{X} \to \mathbb{P}^1$ induced by the polarization with fibers in $\mathbb{P}^2 = \mathbb{P}(X,Y,Z)$ varying over $\mathbb{P}^1 = \mathbb{P}(u,v)$ is given by

\begin{equation}
\mathcal{X} : \quad Y^2Z = X(X^2 + a(u,v)XZ + b(u,v)Z^2),
\end{equation}
where \(a\) and \(b\) are homogeneous polynomials of degree four and eight, respectively. The alternate fibration admits a section \(\sigma : [X : Y : Z] = [0 : 1 : 0]\) and 2-torsion section \([X : Y : Z] = [0 : 0 : 1]\), and it has the discriminant
\[
\Delta_{\text{alt}} = b(u, v)^2 (a(u, v)^2 - 4b(u, v)).
\]
For the elliptic fibration (2.7) the translation by 2-torsion acts fiberwise as
\[
J_X : \left[ X : Y : Z \right] \mapsto \left[ b(u, v) X Z : -b(u, v) Y Z : X^2 \right]
\]
for \([X : Y : Z] \neq [0 : 1 : 0], [0 : 0 : 1]\), and by swapping \([0 : 1 : 0] \leftrightarrow [0 : 0 : 1]\). This is easily seen to be a Nikulin involution as it leaves the holomorphic 2-form invariant. Thus, \(J_X\) is a van Geemen-Sarti involution. Similarly, the K3 surface \(Y\) has the Weierstrass model
\[
Y : Y^2 Z = X \left( X^3 - 2a(u, v) X Z + (a(u, v)^2 - 4b(u, v)) Z^2 \right).
\]
The general K3 surfaces \(X\) in Equation (2.7) and \(Y\) in Equation (2.10) have 8 fibers of type \(I_1\) over the zeroes of \(a^2 - 4b = 0\) (resp. \(b = 0\)) and 8 fibers of type \(I_2\) over the zeroes of \(b = 0\) (resp. \(a^2 - 4b = 0\)) with \(\text{MW}(X, \pi_X) \cong \text{MW}(Y, \pi_Y) \cong \mathbb{Z}/2\mathbb{Z}\). It follows that
\[
\text{NS}(X) \cong \text{NS}(Y) \cong H \oplus N, \quad T_X \cong T_Y \cong H^2 \oplus N.
\]
Corollary 2.6 implies that the moduli space \(\mathcal{M}_L\) for any lattice \(L\) in Table 1 is a subspace of the 10-dimensional moduli space \(\mathcal{M}_{H \oplus N}\) and therefore admits a unique action of the van Geemen-Sarti-Nikulin duality \(\iota_{\text{vgsn}}\). We have the following:

**Proposition 2.7.** The van Geemen-Sarti-Nikulin duality \(\iota_{\text{vgsn}}\) restricts to an involution on the moduli spaces \(\mathcal{M}_L\) for the following lattices \(L\) in Table 1:
\[
L = H \oplus N, \quad H \oplus D_6(-1) \oplus A_1(-1)^\oplus 4, \quad H \oplus D_8(-1) \oplus D_4(-1).
\]
In Equation (2.7) the involution fixes the moduli defining the polynomial \(a(u, v)\) and maps the moduli defining \(b(u, v)\) according to
\[
b(u, v) \mapsto \frac{1}{4} a(u, v)^2 - b(u, v).
\]
**Proof.** For the cases in the statement one checks that the van Geemen-Sarti-Nikulin dual of a given \(L\)-polarized K3 surface \(X\) is again an \(L\)-polarized surface \(Y\). This is seen by computing a Weierstrass model for \(Y\). Equation (2.10) becomes after normalization
\[
Y : Y^2 Z = X^3 + a(u, v) X^2 Z + \frac{1}{4} (a(u, v)^2 - 4b(u, v)) X Z^2.
\]
One checks that Equation (2.14) has the same singular fibers as the ones present in the Weierstrass model of \(X\).

Before we state our results characterizing the various moduli spaces, we establish a minimality condition for the Weierstrass equations in question:

**Lemma 2.8.** If \(b \neq 0\), \(b \neq a^2/4\) and if there is no polynomial \(c \in \mathbb{C}[u, v]\) so that \(c^2\) divides \(a\) and \(c^4\) divides \(b\), then the minimal resolution of Equation (2.7) is a K3 surface.
Table 1. All 2-elementary polarization of Eq. (2.7) with Aut(\(\mathcal{X}\)) < \(\infty\)

\begin{tabular}{|c|c|c|c|c|c|}
    \hline
    \(\rho_L\) & \(L\) & \(K^{\text{root}}\) & \(\mathcal{W}\) & construction & dual graph \\
    \hline
    \((\ell, t_{\mathbb{J}})\) & & & & & \\
    \hline
    \((6, 0)\) & \(H(2) \oplus D_4(-1)^{\oplus 2} \cong H \oplus N\) & \(8\mathbb{A}_1\) & \(Z/22\) & alternate & [35] \ Figr. 1 \\
    \hline
    \((7, 1)\) & \(H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 5}\) & \(D_3 \oplus 5\mathbb{A}_1\) & \{4\} & no section & standard \[1, 4, 16\] \\
    \hline
    \((6, 1)\) & \(H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 4}\) & \(9\mathbb{A}_1\) & \(Z/22\) & alternate & Thrm. 4.9 \ Figr. 2 \\
    \hline
    \((5, 1)\) & \(H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 3}\) & \(D_3 \oplus 4\mathbb{A}_1\) & \{3\} & standard & Thrm. 4.10 \ Figr. 5 \\
    \hline
    \((4, 0)\) & \(H \oplus D_8(-1) \oplus A_1(-1)^{\oplus 2}\) & \(D_3 \oplus 4\mathbb{A}_1\) & \{2\} & standard & [2] \ [2, 16] \\
    \hline
    \((4, 1)\) & \(H \oplus E_7(-1) \oplus A_1(-1)^{\oplus 4}\) & \(E_7 \oplus D_2\) & \{2\} & standard & Thrm. 4.13 \ Figr. 9 \\
    \hline
    \((3, 1)\) & \(H \oplus E_9(-1) \oplus D_5(-1) \oplus A_1(-1)\) & \(E_9 \oplus D_3\) & \{2\} & standard & [2] \ [2, 16, 26] \\
    \hline
    \((2, 1)\) & \(H \oplus E_8(-1) \oplus D_6(-1)\) & \(E_8 \oplus D_3\) & \{2\} & standard & [8, 9] \ [8, 16, 26] \\
    \hline
    \((1, 1)\) & \(H \oplus E_8(-1) \oplus A_1(-1)\) & \(E_8 \oplus D_3\) & \{2\} & standard & [6, 7, 19] \ [6, 16, 26] \\
    \hline
    \((0, 0)\) & \(H \oplus E_8(-1) \oplus A_1(-1)\) & \(E_8 \oplus D_3\) & \{2\} & standard & [5, 32] \ [5, 16, 26] \\
    \hline
    \((1, 1)\) & \(H \oplus E_8(-1) \oplus A_1(-1)\) & \(E_8 \oplus D_3\) & \{2\} & standard & [15] \ [16, 26] \\
    \hline
  \end{tabular}

Proof. For \(b = 0\) or \(b = a^2/4\) Equation (2.7) becomes \(Y^2Z = X^2(X + aZ)^2\), \(Y^2Z = X(X + aZ/2)^2\), respectively. Otherwise, Equation (2.7) can be brought into the Weierstrass normal form

(2.15) \[Y^2Z = X^3 + f(u, v)XZ + g(u, v)Z^3,\]

with coefficients

(2.16) \[f(u, v) = \frac{3b(u, v) - a(u, v)^2}{3}, \quad g(u, v) = \frac{a(u, v)(2a(u, v)^2 - 9b(u, v))}{27},\]

and discriminant \(\Delta_{ab} = 4f^3 + 27g^2 = b^2(a^2 - 4b)\). Equation (2.15) has a non-minimal singularity if and only if \((f, g, \Delta_{ab})\) has a \((4, 6, 12)\)-point. This occurs if and only if there is a polynomial \(c\) so that \(c^2\) divides \(a\) and \(c^4\) divides both \(b\). \(\Box\)

Next, we choose 12 distinct points, grouped into 4 plus 8 points, to fix the polynomials \(a\) and \(b\) for \(\mathcal{X}\) in Equation (2.7). Since permutations of roots determining either \(a\) or \(b\) give isomorphic K3 surfaces, we will use the coefficients of the polynomials \(a\)
and \(b\) as projective coordinates instead, i.e.,
\[
\begin{align*}
    a(u, v) &= a_0 u^4 + a_1 u^3 v + a_2 u^2 v^2 + a_3 uv^3 + a_4 v^4, \\
b(u, v) &= b_0 u^8 + b_1 u^7 v + b_2 u^6 v^2 + b_3 u^5 v^3 + b_4 u^4 v^4 + b_5 u^3 v^5 + b_6 u^2 v^6 + b_7 u v^7 + b_8 v^8.
\end{align*}
\]

We have the following:

**Proposition 2.9.** The 10-dimensional open space

\[
\mathcal{M}_{H\oplus N} = \left\{ \left[ (a_0, \ldots, a_4, b_1, \ldots, b_7) \right| b(u, v) \neq 0, a(u, v)^2/4, \exists c \in \mathbb{C}[u, v]: c^2 | a, c^4 | b \right\}
\]

is the moduli space of \(H \oplus N\)-polarized K3 surfaces \(X\) with \(p_X \leq 17\). Here, a K3 surface \(X \in \mathcal{M}_{H\oplus N}\) is the minimal resolution of Equation (2.7) for \(a, b\) in Equation (2.17) with \(b_0 = b_8 = 0\).

**Proof.** It follows from Theorem 2.2 that every \(H \oplus N\)-polarized K3 surface, up to isomorphism, admits an alternate fibration that can be brought into the form of Equation (2.7). Using the action of Aut(\(\mathbb{P}^1\)) we move two roots of \(b\) that is, the base points of two fibers of type \(I_2\) to \(v = 0\) and \(u = 0\), respectively. The polynomial \(b\) is then of the form \(b(u, v) = w^2(\ldots)\), and in Equation (2.17) we have \(b_0 = b_8 = 0\). This is possible since the polynomial \(b\) must have at least two distinct roots as long as \(p_X \leq 17\). Moreover, one can tell precisely when two members of the family in Equation (2.7) are isomorphic. Writing \(b\) in the form \(b(u, v) = w^2(\ldots)\) fixes the coordinates \([u : v] \in \mathbb{P}^1\) up to scaling. Thus, two members are isomorphic if and only if their coefficient sets are related by a scaling \((u, v, X, Y, Z) \mapsto (u, v, \lambda^2 X, \lambda^3 Y, Z)\) for \(\lambda \in \mathbb{C}^\times\) or \((u, v, X, Y, Z) \mapsto (\mu^2 u, v, \mu^4 X, \mu^6 Y, Z)\) for \(\mu \in \mathbb{C}^\times\) since such rescalings are holomorphic isomorphisms of Equation (2.7). The other conditions in Equation (2.18) follow from Lemma 2.8. \(\square\)

**Remark 2.10.** If roots of the polynomials \(a\) and \(b\) in Equation (2.7) coincide, fibers of type \(I_1\) and \(I_2\) coalesce and form fibers of type \(III\). For example, for \(a_0 = b_0 = 0\) Equation (2.7) has the singular fibers \(III + 7I_2 + 6I_1\) and \(p_X = 10\).

**Remark 2.11.** For \(a \equiv 0\) Equation (2.7) has 8 singular fibers of type \(III\). The corresponding K3 surfaces have a non-symplectic automorphism of order 4, given by \((x, y) \mapsto (-x, iy)\). The associated 5-dimensional moduli space was constructed by Kondo [18]. A compactification of the moduli space by KSBA stable pairs was subsequently described in [20].

As explained above, the lattice \(H \oplus N\) admits the extension \(H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 5}\) of rank 11. We have the following:

**Proposition 2.12.** The 9-dimensional open space

\[
\mathcal{M}_{H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 5}} = \left\{ \left[ a_0 : \cdots : a_4 : b_1 : \cdots : b_5 \right] \in \mathbb{WP}(2, \ldots, 2, 4, \ldots, 4) \therefore b(u, v) \neq 0, a(u, v)^2/4, \exists c \in \mathbb{C}[u, v]: c^2 | a, c^4 | b \right\}
\]
is the moduli space of $H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 5}$-polarized K3 surfaces $X$ with $p_X < 17$. Here, a K3 surface $X \in \mathcal{M}_{H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 5}}$ is the minimal resolution of Equation (2.7) with

$$a(u, v) = a_0 u^4 + a_1 u^3 v + a_2 u^2 v^2 + a_3 u v^3 + a_4 v^4,$$

$$b(u, v) = \frac{1}{4} a(u, v)^2 - (u - v)^2 \left( \frac{a_0^2}{4} u^6 + b'_1 u v + \cdots + b'_5 u v^5 + \frac{a_4^2}{4} v^6 \right).$$

(2.20)

**Proof.** It follows from Theorem 2.2 that every $H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 5}$-polarized K3 surface, up to isomorphism, admits an alternate fibration that can be brought into the form of Equation (2.7) with the singular fibers $9I_2 + 6I_1$. Using suitable transformations in $\text{Aut}(\mathbb{P}^1$) we move the base points of 3 fibers of type $I_2$ to $[0 : 1], [1 : 0], [u_0 : v_0]$. Since $b$ is then of the form $b(u, v) = uv(\ldots)$ we must have $a_0 a_4 \neq 0$ because of Remark 2.10. Equation (2.7) then has the singular fibers $9I_2 + 6I_1$ if and only if $b$ satisfies

$$b(u, v) = \frac{1}{4} a(u, v)^2 - (u_0 v - u v)^2 \left( b'_0 u^6 + b'_1 u v + \cdots + b'_5 u v^5 + b'_6 v^6 \right),$$

(2.21)

with $\frac{a_0^2}{4} - b'_0 = 0$ and $\frac{a_4^2}{4} - b'_6 = 0$ for $u_0 v_0 \neq 0$. To establish Equation (2.21) one computes the discriminant of the corresponding Weierstrass model. The polynomial $b$ must have at least 3 distinct roots if $p_X < 17$. We then fix $u_0 = v_0 = 1$, and moduli are the coefficients $a_0, \ldots, a_4$ and $b'_1, \ldots, b'_5$ which have only single weights with respect to one remaining scaling; the other scaling is fixed by the constraint $u_0 = v_0 = 1$. □

Next we consider the lattices in Table 1 of Picard rank greater than 11. We have the following:

**Proposition 2.13.** In Proposition 2.9 one has

$$\mathcal{M}_{H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}} \cong \mathcal{M}_{H \oplus \mathcal{N}} \bigg|_{a_0 = b_1 = 0}$$

(2.22)

where $\mathcal{M}_{H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}}$ is the 8-dimensional moduli space of $H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}$-polarized K3 surfaces $X$ with $p_X \leq 17$. Here, a K3 surface $X \in \mathcal{M}_{H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}}$ is the minimal resolution of Equation (2.7) for $a, b$ in Equation (2.17) with $a_0 = b_0 = b_1 = b_8 = 0$.

**Proof.** It follows from Theorem 2.2 that every $H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}$-polarized K3 surface, up to isomorphism, admits an alternate fibration that can be brought into the form of Equation (2.7) with a fiber of type $I_0^*$ over $[u : v] = [1 : 0]$. In particular, Equation (2.7) has a singular fiber of type $I_0^*$ at $v = 0$ if and only if $a_0 = b_0 = b_1 = 0$ and $b_2 \neq 0, a_2^2/4$ and $a(u, v) \neq 0$. The rest of the proof is analogous to the proof of Proposition 2.9. □

In the following, we set $L = H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}$ for convenience. We decompose the moduli space in Proposition 2.13 further, into the two components $\mathcal{M}^{(0)}_L = \{(a_1, \ldots, b_7) \in \mathcal{M}_L \mid (a_1, b_2) = (0, 0)\}$ and $\mathcal{M}^{(1)}_L = \{(a_1, \ldots, b_7) \in \mathcal{M}_L \mid (a_1, b_2) \neq (0, 0)\}$ such that

$$\mathcal{M}_L = \mathcal{M}^{(0)}_L \cup \mathcal{M}^{(1)}_L.$$

We have the following:
Proposition 2.14. One has \( \mathcal{M}_L^{(0)} \cong \mathcal{M}_{H \oplus D_8(-1) \oplus D_4(-1)} \) where

\[
\mathcal{M}_{H \oplus D_8(-1) \oplus D_4(-1)} = \left\{ \begin{array}{c} a_2 : a_3 : a_4 : b_5 : b_6 : b_7 : b_8 \\ \in \mathbb{WP}(2, 6, 10, 8, 12, 16, 20) \end{array} : \begin{array}{c} \exists r, s \in \mathbb{C}: (a_3, a_4) = (2a_2r, a_2s^2) \text{ and} \\ (b_5, b_6, b_7, b_8) = (-10r^2, -20r^3, -15r^4, -4r^5) \end{array} \right\}
\]

is the 6-dimensional moduli space of \( H \oplus D_8(-1) \oplus D_4(-1) \)-polarized K3 surfaces. Here, a K3 surface \( X \in \mathcal{M}_{H \oplus D_8(-1) \oplus D_4(-1)} \) is the minimal resolution of Equation (2.7) for \( a, b \) in Equation (2.17) with \( a_0 = a_1 = b_0 = b_1 = b_2 = b_3 = 0 \) and \( b_3 = 1 \).

Proof. In the situation of Proposition 2.13 the scaling \((v, x, y) \mapsto (\lambda^2v, \lambda^2x, \lambda^3y)\) leaves the coefficients \(a_1, b_2\) invariant. For \( a_1 = b_2 = 0 \) we must have \( b_3 \neq 0 \) otherwise the Weierstrass model has a \((4, 6, 12)\)-point. For \( b_3 \neq 0 \) the polynomial \( b(u, v) \) must have at least 2 distinct roots. We then use a scaling to fix \( b_3 = 1 \) and shift variables so that \( b_4 = 0 \) (and possibly \( b_8 \neq 0 \)). Accordingly, we always have \( b \neq 0, a^2/4 \). The weights of the remaining moduli are obtained by considering the scaling \((v, x, y) \mapsto (\lambda^4v, \lambda^6x, \lambda^9y)\) which leaves \( b_3 = 1 \) fixed. We then check that \( c(u, v) = u + rv \) is the only other polynomial for which \( c^2 \mid a \) and \( c^4 \mid b \) is possible. This yields the stated minimality condition. In particular, \( u^2 \mid a \) and \( u^4 \mid b \) if and only if \((a_3, a_4, b_5, b_6, b_7, b_8) = 0\). Finally, one checks that Equation (2.7) for polynomials \( a, b \) in Equation (2.17) with \( a_1 = b_2 = 0 \) and \( b_3 \neq 0 \) gives the alternate fibration for \( H \oplus D_8(-1) \oplus D_4(-1) \)-polarized K3 surfaces in Theorem 2.2. \( \Box \)

The elliptic curve

(2.23) \[ E_{[a_1:b_2]} : \quad Y^2Z = X^3 + a_1X^2Z + b_2XZ^2 \]

with \([a_1 : b_2] \in \mathbb{WP}(2, 4)\) is the modular family for the Hecke congruence subgroup \( \Gamma_0(4) \leq \text{SL}(2, \mathbb{Z}) \) where the \( j \)-invariant satisfies

(2.24) \[ j(E_{[a_1:b_2]}) = \frac{256(a_1^2 - 3b_2)^3}{b_2^2(a_1^2 - 4b_2)} \quad \Leftrightarrow \quad j(E_{[a_1:b_2]}) - 12^3 = \frac{64a_1^2(2a_1^2 - 9b_2)^2}{b_2^2(a_1^2 - 4b_2)}. \]

The 8-dimensional open space \( \mathcal{M}_L^{(1)} \) is then fibered over \( \mathbb{WP}(2, 4) \) by means of the holomorphic map

(2.25) \[ p : \quad \mathcal{M}_L^{(1)} \to \mathbb{WP}(2, 4), \quad [(a_1, \ldots, b_7)] \mapsto [a_1 : b_2], \]

such that the \( j \)-invariant of the fiber \( \mathcal{X}_{[v]} \) over \([u : v] = [1 : 0]\) for \( \mathcal{X} \in \mathcal{M}_L^{(1)} \) is \( j(\mathcal{X}_{[v]}) = j(E_{[a_1:b_2]}) \). We consider the neighborhood \( \mathcal{U} = \{[1 : b_2] \in \mathbb{WP}(2, 4) \mid b_2 \in \mathbb{C}\} \) of \([a_1 : b_2] = [1 : 0]\) and its complement, and write

(2.26) \[ \mathcal{M}_L^{(1)} = \mathcal{M}_L^{(1)} \big|_{\mathbb{P}^{-1}(\mathcal{U})} \bigcup \mathcal{M}_L^{(1)} \big|_{\mathbb{P}^{-1}(\mathcal{U}')} \]

We have the following:

Lemma 2.15. One has \( \mathcal{M}_L^{(1)} \big|_{\mathbb{P}^{-1}(\mathcal{U})} \cong \mathcal{M}'_L \) where \( \mathcal{M}'_L \) is the 7-dimensional open space

\[
\left\{ \begin{array}{c} a_2 : a_3 : a_4 : b_5 : b_6 : b_7 : b_8 \\ \in \mathbb{WP}(2, 4, 6, 6, 8, 10, 12) \end{array} : \begin{array}{c} \exists r, s \in \mathbb{C}: (a_3, a_4) = (2a_2r, a_2s^2) \text{ and} \\ (b_5, b_6, b_7, b_8) = (20r^3 + 4rb_4, 45r^4 + 6r^2b_4, 36r^5 + 4r^3b_4, 10r^6 + r^4b_4) \end{array} \right\}.
\]
Here, a K3 surface $\mathcal{X} \in \mathcal{M}_L$ is the minimal resolution of Equation (2.7) for $a, b$ in Equation (2.17) with $a_0 = a_1 = b_0 = b_1 = b_3 = 0$ and $b_2 = 1$ and has $j(\mathcal{X}_{[1:0]}) = 12^3$.

**Proof.** For $b_2 \neq 0$ the polynomial $b(u, v)$ has at least 2 distinct roots. We can then use a scaling to fix $b_2 = 1$ and shift variables so that $b_3 = 0$ (and possibly $b_8 \neq 0$). The scaling $(v, x, y) \mapsto (\lambda^2 v, \lambda^2 x, \lambda^3 y)$ leaves the coefficients $b_2 = 1$ invariant and scales the remaining coefficients as stated. By construction, we always have $b \neq 0, a^2/4$. The statement $j(\mathcal{X}_{[1:0]}) = 1$ follows from Equation (2.24). The rest of the proof is analogous to the proof of Proposition 2.14. \qed

For $L = H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}$ we introduce the 8-dimensional open space

$$
(2.27) \quad \overline{\mathcal{M}}_L = \left\{ \begin{bmatrix} 1 : b_2 \\ a_3 : a_4 : b_3 : b_4 : b_5 : b_6 : b_7 : b_8 \end{bmatrix} \in \mathbb{WP}(2, 4) \times \mathbb{WP}(4, 6, 2, 4, 6, 8, 10, 12) \, | \, b(u, v) \neq 0, a(u, v)^2/4, \exists c \in \mathbb{C}[u, v]: c^2 \mid a, c^4 \mid b \right\}.
$$

Here, a K3 surface $\mathcal{X} \in \overline{\mathcal{M}}_{H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}}$ is the minimal resolution of Equation (2.7) for $a, b$ in Equation (2.17) with $a_0 = a_2 = b_0 = b_1 = 0$ and $a_1 = 1$. For general $\mathcal{X}$, the singular fibers are $I_5 \oplus 6I_2 + 6I_1$ and the Mordell-Weil group is $\mathbb{Z}/2\mathbb{Z}$. In particular, every such K3 surface $\mathcal{X}$ is polarized by $H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}$ and has $j(\mathcal{X}_{[1:0]}) = j(\mathcal{E}_{[1:0]})$ in Equation (2.24). We have the following:

**Proposition 2.16.** Let $L = H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}$. For $\mathcal{X} \in \mathcal{M}_L^{(1)}|_{p^1(\mathcal{X})}$ one has $\mathcal{X} \in \overline{\mathcal{M}}_L$. Conversely, for $\mathcal{X} \in \overline{\mathcal{M}}_L$ and $p_{\mathcal{X}} \leq 17$ one has $\mathcal{X} \in \mathcal{M}_L^{(1)}|_{p^1(\mathcal{X})}$.

**Proof.** For every $\mathcal{X} \in \mathcal{M}_L^{(1)}|_{p^1(\mathcal{X})}$ we have $a_1 \neq 0$. We can use a scaling to fix $a_1 = 1$ and shift variables so that $a_2 = 0$ (and $b_8 \neq 0$ generically). The scaling $(v, x, y) \mapsto (\lambda^2 v, \lambda^2 x, \lambda^3 y)$ leaves the coefficients $a_1 = 1$ invariant and scales the remaining coefficients with weights as stated. Accordingly, we have $\mathcal{X} \in \overline{\mathcal{M}}_L$. Conversely, for $\mathcal{X} \in \overline{\mathcal{M}}_L$ the polynomial $b$ is of the form $b(u, v) = uv(\ldots)$ if it has at least two distinct roots. This is true as long as $p_{\mathcal{X}} \leq 17$. One then has $\mathcal{X} \in \mathcal{M}_L^{(1)}|_{p^1(\mathcal{X})}$. \qed

The space $\overline{\mathcal{M}}_{H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}}$ is well suited to embed all moduli spaces of Picard rank $\rho_L \geq 13$ from Table 1 into it. We have the following:

**Theorem 2.17.** For every lattice polarization $L$ in Table 2 with $\rho_L \geq 13$, the moduli space $\mathcal{M}_L$ of $L$-polarized K3 surfaces is the subspace of $\overline{\mathcal{M}}_{H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}}$ of dimension $20 - \rho_L$, with defining equations given in Table 2.

**Remark 2.18.** The minimality conditions for $\mathcal{X} \in \overline{\mathcal{M}}_{H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}}$, ensuring that the minimal resolution of Equation (2.7) is a K3 surface, are $b \neq 0, a^2/4$ and

$$
\mathfrak{d} \mathbf{r}, b_2, b_3, b_4 \in \mathbb{C} : \begin{bmatrix} a_3 \\ a_4 \\ b_5 \\ b_6 \\ b_7 \\ 10r^6b_2 - 4r^4b_3 + r^4b_4 \end{bmatrix} = \begin{bmatrix} -3r^2 \\ -2r^3 \\ 20r^3b_2 - 10r^2b_3 + 4rb_4 \\ 45r^4b_2 - 20r^3b_3 + 6r^2b_4 \\ 36r^5b_2 - 15r^4b_3 + 4rb_4 \\ 10r^6b_2 - 4r^4b_3 + r^4b_4 \end{bmatrix}.
$$
Proof. One checks for every lattice polarization $L$ that Equation (2.7) for $a, b$ in Equation (2.17) (using the defining equations in Table 2) gives the corresponding alternate equation in Table 1. Let us discuss the minimality conditions: For $b(u, v) = a(u, v)^2/4$ we have $a_0 = b_1 = 0$ and $b_2 = a_1^2/4$. Thus, $b(u, v) ≠ a(u, v)^2/4$ is always satisfied for the lattices with $ρ_L ≥ 13$ in Table 2. As for the minimality condition, one checks that because of $a_1 = 1$ the condition $∃c ∈ \mathbb{C}[u, v]: c^2 | a, c^4 | b$ can never be satisfied with $c = v$. Moreover, for $ρ_L ≥ 15$ in Table 2 the condition $∃c ∈ \mathbb{C}[u, v]: c^2 | a, c^4 | b$ can only be satisfied for $c = u$. However, this would imply $b = 0$, and $b ≠ 0$ is assumed. For $ρ_L ≤ 14$ one checks that $c(u, v) = u + rv$ so that $c^2 | a$ and $c^4 | b$ is possible. This yields the stated minimality conditions in Remark 2.18. □

Remark 2.19. For each lattice polarization $L$ in Table 2 with $ρ_L ≥ 13$ the corresponding $K3$ surfaces $X ∈ \mathcal{M}_L$ have $b_2 = 0$ and $j(X_{[1:0]}) = \infty$. In Equation (2.24) one observes that $j(X_{[1:0]}) = \infty$ for $b_2 = \frac{1}{4}$ as well. In latter case, the standard fibration has the singular fibers $2I^2_0 + I_4 + 8I_1$. Thus, the subspace $b_2 = \frac{1}{2}$ of $\tilde{\mathcal{M}}_{H^0D_6\oplus A_1(-1)}$ is in fact the moduli space of $H \oplus D_4(-1)^{\oplus 2} \oplus A_3(-1)$-polarized $K3$ surfaces.

Remark 2.20. In [14] the authors constructed the compactification of the moduli space of $H(2) \oplus D_6 \oplus A_1(-1)^{\oplus 2}$-polarized $K3$ surfaces. The moduli spaces $\mathcal{M}_L$ for $L$ in Table 1 with $ρ_L ≥ 11$ embed into this moduli space since the corresponding families of $K3$ surfaces are naturally subfamilies of the family considered in [14].

2.3. Related double quadrics. Equation (2.7) is naturally a double cover of the Hirzebruch $\mathbb{F}_4$. This is seen by constructing the Hirzebruch surfaces $\mathbb{F}_n$ for $n ∈ \mathbb{N}_0$ as the GIT quotients

\begin{equation}
\mathbb{F}_n \cong (\mathbb{C}^2 - \{0\}) \times (\mathbb{C}^2 - \{0\}) / (\mathbb{C}^* \times \mathbb{C}^*),
\end{equation}

where the action of $\mathbb{C}^* \times \mathbb{C}^*$ on the coordinates $(u, v)$ and $(x, z)$ is given by

\begin{equation}
(λ, μ) \cdot (u, v, x, z) = (λu, λv, μx, λ^{-n}μz).
\end{equation}

In this way, the branch loci in Equation (2.7) is contained in $\mathbb{F}_4$. Suitable blowups transform Equation (2.7) into a double cover of $\mathbb{F}_0 = \mathbb{P}(s, t) \times \mathbb{P}(u, v)$ branched along a curve of bi-degree $(4, 4)$, i.e., along a section in the line bundle $O_{\mathbb{F}_0}(4, 4)$. Such a cover is known as double quadric surface and has two natural elliptic fibrations corresponding to the two rulings of the quadric $\mathbb{F}_0$ coming from the projections $π_i: \mathbb{F}_0 → \mathbb{P}^1$ for $i = 1, 2$. The latter elliptic fibration, i.e., the projection onto $\mathbb{P}(u, v)$ recovers the alternate fibration from above. The former fibration is the standard fibration.

Concretely, if we factor the polynomial $b$ in Equation (2.7) into homogenous polynomials $c, d$ of degree 4, i.e., $b(u, v) = c(u, v) d(u, v)$, blowups are performed by writing $[x : z] = [c(u, v) s : t]$. A double quadric, birational to $X$, is then given by

\begin{equation}
y^2 = st \left( c(u, v) s^2 + a(u, v) st + d(u, v) t^2 \right).
\end{equation}

Equation (2.31) is the double cover of $\mathbb{F}_0 = \mathbb{P}(s, t) \times \mathbb{P}(u, v)$ branched along $ℓ_0 + ℓ_1 + D$. Here, we introduced the lines $ℓ_0 = V(s)$ and $ℓ_1 = V(t)$, and the curve $D = V(F)$, with

\begin{equation}
F(s, t, u, v) = c(u, v) s^2 + a(u, v) st + d(u, v) t^2,
\end{equation}

ON JACOBIAN ELLIPTIC K3 SURFACES WITH 2-TORSION 13
the roots of $\Delta$

$X$

Equation $(\mathcal{D}(\oplus a, a_8) - 2)$ may be brought into the form $(\mathcal{D}(\oplus a, a_8) - 2)$.

### Table 2. Natural subspaces of $\overline{\mathcal{M}}_{H \oplus D_4(-1) \oplus A_1(-1)}^{\oplus 4}$

which is a $(2,4)$-class on $\mathbb{F}_0 = \mathbb{P}(s,t) \times \mathbb{P}(u,v)$. Equation (2.31) has 8 rational double points of type $A_1$ and its minimal resolution is the K3 surface $\mathcal{X}$. The isomorphism class of $\mathcal{X}$ is independent of the factorization because elementary transformations change the order of the 8 roots defining $b(u,v) = c(u,v) d(u,v)$; see [4].

The curve $\mathcal{D} = V(F)$ is a binary quartic in $\{u : v\}$ whose coefficients are themselves binary forms $q_i$ of degree two, i.e.,

$$F(s,t,u,v) = q_0(s,t) u^4 + q_1(s,t) u^3 v + q_2(s,t) u^2 v^2 + q_3(s,t) u v^3 + q_4(s,t) v^4.$$  

The discriminant of $F$ is $\Delta = 4f(s,t)^3 + 27g(s,t)^2$ with

$$f = q_1 q_3 - 4 q_0 q_4 - \frac{1}{3} g^2, \quad g = q_0 q_2^2 + \frac{8}{3} q_0 q_2 q_4 - \frac{1}{3} q_1 q_2 q_3 + \frac{2}{27} g^3.$$  

In particular, $\mathcal{D}$ is a hyperelliptic curve of genus 3 mapping 4-to-1 onto $\mathbb{P}(u,v)$ and the roots of $\Delta = 4f(s,t)^3 + 27g(s,t)^2$ are 12 simple branching points in $\mathbb{P}(s,t)$.

Equation (2.31) may be brought into the form

$$\tilde{y}^2 = st \left( q_0(s,t) u^4 + q_1(s,t) u^3 v + q_2(s,t) u^2 v^2 + q_3(s,t) u v^3 + q_4(s,t) v^4 \right).$$
The standard fibration is the elliptic fibration, given by the projection onto \( \mathbb{P}(s,t) \). Its relative Jacobian fibration is given by
\[
\eta^2 = \xi^3 + s^2 t^2 f(s,t) \xi + s^3 t^3 g(s,t).
\]
Here, we are using the affine coordinates \((\xi, \eta) \in \mathbb{C}^2\) for the elliptic fibers. The discriminant of Equation (2.35) is \( \Delta_{\text{std}} = s^6 t^6 (4f(s,t)^3 + 27g(s,t)^2) \). In particular, the fibration has the singular fibers \( 2I_0^* + 12I_1 \) and a trivial Mordell-Weil group. Conversely, 12 points in \( \mathbb{P}(s,t) \) are given by a degree-12 binary form that is expressed as a cube and a square. The structure of Equation (2.35) fixes the coordinates \([s : t] \in \mathbb{P}^1\) up to scaling. As projective coordinates \( f_0, \ldots, f_4 \) and \( g_0, \ldots, g_6 \) we use the coefficients of polynomials \( f \) and \( g \) with
\[
f = f_0 s^4 + f_1 s^3 t + \cdots + f_4 t^4, \quad g = g_0 s^6 + g_1 s^5 t + \cdots + g_6 t^6,
\]
and consider the open subscheme where the polynomials \( f \) and \( g \) define a rational elliptic surface such that the vanishing degrees of \((f, g)\) at \( s = 0 \) (and \( t = 0 \)) are less than (2, 3) or, equivalently, Equation (2.36) is a minimal Weierstrass equation for a Jacobian elliptic K3 surface.

In [34] the author considers a locus of codimension 1 in the space of sets of 12 points in \( \mathbb{P}^1 \) which can be defined in several equivalent ways, including (i) the discriminant set of a general rational elliptic fibration, (ii) binary forms of degree 12 admitting a representation as a sum \( 4f^3 + 27g^2 \), where \( f, g \) are binary forms of degree 4 and 6, respectively, and (iii) the branch locus of a general canonical pencil of a non-hyperelliptic genus 3 curve. In terms of Proposition 2.9, the result implies that there is a finite étale morphism \( \mathcal{M}_{H\oplus N} \to \mathcal{M}_{H\oplus D_4(-1)\oplus 2} \). In [4] the authors showed that this morphism is the mathematical underpinning of the duality between F-theory and the heterotic CHL string theory.

A sufficient criterion for the standard fibration on the double quadric (2.31) to admit a section is that the polynomials \( a, c, d \) contain a common linear factor over the function field of the base curve. That is, after suitable normalization, we may assume that
\[
a(u, v) = v a'(u, v), \quad c(u, v) = v c'(u, v), \quad d(u, v) = v d'(u, v),
\]
where \( a', c', d' \) are generic homogeneous polynomials of degree 3. In this case, the double quadric (2.31) has a total branch locus of type \((1, 0) + (0, 1) + (0, 1) + (3, 2)\). We have the following:

**Proposition 2.21.** Assuming Equation (2.38) the standard fibration is a Jacobian elliptic fibration on \( \mathcal{X} \), necessarily of Picard rank \( p_\mathcal{X} \geq 12 \).

**Proof.** Assuming Equation (2.38) the statement follows by computing the Weierstrass normal forms corresponding to the two rulings of the quadric \( \mathbb{F}_0 \) coming from the projections \( \pi_i : \mathbb{F}_0 \to \mathbb{P}^1 \) for \( i = 1, 2 \). Details of this construction can be found in [9, 10]. One then checks that the standard fibration has the singular fibers \( 2I_0^* + 2I_2 + 8I_4 \) and a trivial Mordell-Weil group. This proves that a general surface \( \mathcal{X} \) is \( H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4} \)-polarized. In particular, we have \( p_\mathcal{X} \geq 12 \). \( \square \)
3. Double sextics

Let us consider K3 surfaces that are the minimal resolution of double coverings of $\mathbb{P}^2 = \mathbb{P}(u, v, w)$ branched along plane sextic curves. Such K3 surfaces are called double sextic surfaces or double sextics for short. In general, such K3 surfaces have Picard rank greater than or equal to one. The Picard rank of a double sextic increases if there exist curves in special position. For example, a sextic might possess tritangents or contact conics. In [28] this approach was used to construct a number of examples for sextics defining K3 surfaces with the maximal Picard rank 20.

In this section we will construct double sextics whose minimal resolutions are K3 surfaces of lower Picard rank. We will do so by considering double covers branched over the strict transform of a (reducible) sextic, given as the union of a nodal quartic $\mathcal{N}$ and a conic $\mathcal{C}$ (or two lines $\ell_1, \ell_2$). We will write such a double sextic in the form

$$S: \quad \ddot{y}^2 = C(u, v, w) \cdot Q(u, v, w),$$

where $C$ and $Q$ are homogeneous polynomials of degree 2 and 4, respectively, such that $\mathcal{C} = V(C)$ and $\mathcal{N} = V(Q)$ and $\ddot{y}$ has weight 3. We will show that these double sextics give birational models for all $L$-polarized K3 surfaces with $L$ in Table 1.

3.1. Double sextics of Picard rank 10. We will first consider the double sextic

$$S: \quad \ddot{y}^2 = \left(j_0 u^2 + v w + h_0 w^2\right)\left(c_2(u, v) w^2 + e_3(u, v) w + d_4(u, v)\right).$$

Here, $c_2, d_4, e_3$ are homogeneous polynomials of the degree indicated by the subscript and $h_0, j_0 \in \mathbb{C}$. The branch locus in Equation (3.2) has two irreducible components: a conic $\mathcal{C} = V(C)$ and a quartic $\mathcal{N} = V(Q)$ with a node.

The curve $\mathcal{N}$ has the node $n : [u : v : w] = [0 : 0 : 1]$, and $c_2(u, v) = 0$ is the equation of the two tangents at $n$. Moreover, $e_3(u, v)^2 - 4c_2(u, v)d_4(u, v) = 0$ is the equation of 6 lines which touch the quartic in 6 more points. The first polar which passes through these 6 points is the nodal cubic $2c_2(u, v)w + e_3(u, v) = 0$. As a reminder, the first polar of an algebraic plane curve $\mathcal{N}$ of degree $k$ with respect to a point $n$ is an algebraic curve of degree $k - 1$ which contains every point of $\mathcal{N}$ whose tangent line passes through $n$. The arithmetic genus of the curve $\mathcal{N}$ is three, and its geometric genus is two. The curve $\mathcal{N}$ projects from $n$ onto a hyperelliptic curve of genus 3, given by

$$\eta^2 = e_3(u, v)^2 - 4c_2(u, v)d_4(u, v),$$

with $\eta = 2c_2(u, v)w - e_3(u, v)$.

Conversely, if an irreducible quartic curve has one node, it and the two nodal tangents provide a point and two lines that uniquely determine the curve. We call such a curve a uninodal quartic curve. It was proved in [30] that every uninodal quartic can be brought into the form $V(Q)$ with

$$Q(u, v, w) = uvw^2 + \left(u^3 + v^3\right)w + \left(s_1 u^4 + s_2 u^3 v + s_3 u^2 v^2 + s_4 u v^3 + s_5 v^4\right),$$

where $s_1, \ldots, s_5$ are natural geometric invariants, and $u = 0$ and $v = 0$ are the tangents at the node $n$.

Note that the conic $\mathcal{C}$ in Equation (3.2) was obtained from a general conic $h_0 u^2 + k_1(u, v)w + j_2(u, v)$ by simple coordinates shifts of the form $w \mapsto w + \rho_1 u + \rho_2 v$ and
Proposition 3.1. Let $c_2, c_3, d_4$ general polynomials and $h_0, j_0 \in \mathbb{C}$ with $h_0 \neq 0$ and $S$ the double sextic in Equation (3.2). The pencil of lines through the node $n \in N$ induces an elliptic fibration without section on $S$. The relative Jacobian fibration is Equation (2.7) with

$$a = ve_3 - 2j_0w^2c_2 - 2h_0d_4,$$

and has the Mordell-Weil group $\mathbb{Z}/2\mathbb{Z}$ and the singular fibers $8I_2 + 8I_1$.

Proof. The pencil of lines through $n$ induces an elliptic fibration given by the projection onto $\mathbb{P}(u, v)$ in Equation (3.2). We then write Equation (3.2) in the form

$$\tilde{y}^2 = q_0 w^4 + q_1(u, v) w^3 + q_2(u, v) w^2 + q_3(u, v) w + q_4(u, v),$$

with $q_0 = c_2h_0, q_1 = c_2k_1 + c_3h_0$, etc. The relative Jacobian of Equation (3.2) is

$$\eta^2 = \xi^3 + f(u, v) \xi + g(u, v),$$

where $f, g$ are given by Equation (2.34), and $(\xi, \eta) \in \mathbb{C}^2$ are affine coordinates on the elliptic fibers. One checks that Equation (3.7) can be brought into the form of Equation (2.10) with $a, b$ given by Equation (3.5). \hfill \Box

We will refer to the fibration obtained from the pencil of lines through the node again as the alternate fibration on $S$. We have the following:

Proposition 3.2. A general $H \oplus N$-polarized K3 surface is the relative Jacobian of the alternate fibration on a double sextic $S$ that is branched on a uninodal quartic $N$ and a conic $C$ not coincident with the node. Conversely, the relative Jacobian fibration of the alternate fibration of such a double sextic is an $H \oplus N$-polarized K3 surface.

Proof. From the relative Jacobian fibration in a Proposition 3.1 an elliptic K3 surface can be constructed. This family of Jacobian elliptic K3 surfaces defines a 10-dimensional family of $H \oplus N$-polarized K3 surfaces. Conversely, if we change coordinates $u, v, w$ such that the conic is $C : w^2 + uv = 0$, we have $b(u, v) - a(u, v)^2/4 = wvb'(u, v)$ in Equation (3.5) for a homogeneous polynomial $b'$ of degree 6. Then, the coefficients in Equation (3.5) determine 12 polynomial equations for 12 unknown coefficients defining the nodal quartic $Q$ that have a finite number of solutions for general polynomials $a, b'$.

\hfill \Box

3.2. Double sextics of Picard rank 11. Next we consider the case when the conic in Equation (3.2) splits into 2 lines. After a suitable shift in the coordinates $v, w$, the double sextic becomes

$$S : \tilde{y}^2 = w\left(v + h_0w\right)\left(c_2(u, v) w^2 + e_3(u, v) w + d_4(u, v)\right).$$

The branch locus in Equation (3.8) has three irreducible components: the quartic curve $N = V(Q)$ with the node $n = [0 : 0 : 1]$ and the lines $\ell_1 = V(v + h_0w), \ell_2 = V(w)$ which are not coincident with $n$ (for $h_0 \neq 0$) and satisfy $\ell_1 \cap \ell_2 \cap N = \varnothing$. We have the following:
Proposition 3.3. Let $c_2, e_3, d_4$ general polynomials and $h_0 \in \mathbb{C}^*$ and $S$ the double sextic in Equation (3.8). The following holds:

(i) the pencil of lines through the node $n \in N$ induces the Jacobian elliptic fibration (2.7) with

$$a = ve_3 - 2h_0d_4, \quad b = \frac{1}{4}a^2 - \frac{1}{4}v^2(e_3^2 - 4c_2d_4),$$

and the Mordell-Weil group $\mathbb{Z}/2\mathbb{Z}$ and the singular fibers $9I_2 + 6I_1$.

(ii) a pencil of lines through a point in $(\ell_1 \cap N) \cup (\ell_2 \cap N)$ induces a Jacobian elliptic fibration with a trivial Mordell-Weil group and the singular fibers $I_0^* + 5I_2 + 8I_1$.

(iii) the pencil of lines through $\ell_1 \cap \ell_2$ induces an elliptic fibration without section and the singular fibers $2I_0^* + I_2 + 10I_1$.

Proof. For (i) we follow the proof of Proposition 3.1; it is obvious that the pencil of lines now induces an elliptic fibration with two sections. We recover the normal form for the alternate fibration with the singular fibers $9I_2 + 6I_1$ established in Proposition 2.12. For (ii) we construct a pencil through a point in $\ell_2 \cap N$, the case $\ell_1 \cap N$ will then be analogous. To do so, we set $d_3(u, v) = (v_0u - u_0v)d_3(u, v)$ where $d_3$ is a polynomial of degree 3 with $d_3(u_0, v_0) \neq 0$. One point in $N \cap \ell_2$ is $[u_0 : v_0 : 0]$. A pencil of lines through this intersection point is

$$0 = s(v_0u - u_0v) + tw$$

with $[s : t] \in \mathbb{P}^1$. Upon eliminating $u$ in Equation (3.8) we obtain a Jacobian elliptic fibration over $\mathbb{P}(s, t)$ with the singular fibers $I_0^* + 5I_2 + 8I_1$. For (iii) we use the pencil $0 = sv - tw$ and proceed as in (ii). However, we only obtain a genus-one fibration over $\mathbb{P}(s, t)$. We compute the relative Jacobian elliptic fibration as in (i), confirming the statement.

We have the following:

Proposition 3.4. A general $H \oplus D_4(-1) \oplus A_1(-1)^{\otimes 5}$-polarized K3 surface is birational to a double sextic $S$ that is branched on a uninodal quartic $N$ and 2 lines $\ell_1, \ell_2$ not coincident with the node and $\ell_1 \cap \ell_2 \cap N = \emptyset$. Conversely, every such double sextic is birational to an $H \oplus D_4(-1) \oplus A_1(-1)^{\otimes 5}$-polarized K3 surface.

Proof. Due to Theorem 2.5 a general $H \oplus D_4(-1) \oplus A_1(-1)^{\otimes 5}$-polarized K3 surface admits a unique Jacobian elliptic fibration (2.7) with singular fiber $9I_2 + 6I_1$. Then, after a suitable choice of coordinates, one has $a(u, v)^2/4 - b(u, v) = v^2b'(u, v)$ where $b'$ is a homogeneous polynomial of degree 6. Given $a, b'$ we choose a factorization of $b(u, v) = a(u, v)^2/4 - v^2b'(u, v) = d_4(u, v)d_4'(u, v)$ into homogeneous polynomials of degree 4. It follows that the coefficients are related as follows:

$$\frac{1}{4} \text{coeff}_{u^4}(a) = \text{coeff}_{u^4}(d_4) \cdot \text{coeff}_{u^4}(d'_4),$$

$$\frac{1}{2} \text{coeff}_{u^4}(a) \cdot \text{coeff}_{u^3}(a) = \text{coeff}_{u^4}(d_4) \cdot \text{coeff}_{u^3}(d'_4) + \text{coeff}_{u^3}(d'_4) \cdot \text{coeff}_{u^4}(d_4).$$

We then find $h_0 \in \mathbb{C}^*$ so that $e_3 = (a + 2h_0d_4)/v$ is a polynomial of degree 3 by solving $\text{coeff}_{u^4}(a) = -2h_0 \text{coeff}_{u^4}(d_4)$. It follows that $c_2 = (h_0ve_3 - h_0^2d_4 + d'_4)/v^3$ is a
homogeneous polynomials of degree two. Then, \( h_0, c_2, e_3, d_4 \) are a solution of Equation (3.9) for a given pair \( a, b' \) of polynomials and determine a double sextic \( S \) via the birational transformation \( X = h_0 d_4 + d_4 v/w \). The other direction follows from Proposition 3.3.

We also consider the case when the conic in Equation (3.8) specializes to a conic that is coincident with the node (rather than a conic decomposing into two lines), i.e.,

\[
(3.11) \quad S : \quad y^2 = \left( v w + j_0 u^2 \right) \left( c_2(u, v) w^2 + e_3(u, v) w + d_4(u, v) \right).
\]

Here, \( j_0 \in \mathbb{C}^* \) and \( C : v w + j_0 u^2 = 0 \) satisfies \( n \in C \). We have the following:

**Proposition 3.5.** Let \( c_2, e_3, d_4 \) general polynomials and \( j_0 \in \mathbb{C}^* \) and \( S \) the double sextic in Equation (3.11). The following holds:

(i) the pencil of lines through the node \( n \in N \cap C \) induces the Jacobian elliptic fibration (2.7) with

\[
a = v e_3 - 2 j_0 u^2 c_2, \quad b = \frac{1}{4} a^2 - \frac{1}{4} v^2 (e_3^2 - 4 c_2 d_4),
\]

and the Mordell-Weil group \( \mathbb{Z}/2\mathbb{Z} \) and the singular fibers \( 9 I_2 + 6 I_1 \),

(ii) a pencil of lines through a point in \( N \cap C - \{ n \} \) induces a Jacobian elliptic fibration with a trivial Mordell-Weil group and the singular fibers \( I_0^* + 5 I_2 + 8 I_1 \).

**Proof.** For (i) we follow the proof of Proposition 3.3. For (ii) one explicitly constructs the minimal resolution of the double sextic \( S \) in Equation (3.11). One checks that one obtains the reducible fibers \( D_4 + 5 A_1 \) and an elliptic fibration with section.

---

### 3.3. Double sextics of Picard rank 12 and 13.

Next we consider the case when one line in Equation (3.8) becomes coincident with the node by setting \( h_0 = 0 \), i.e.,

\[
(3.12) \quad S : \quad y^2 = v w \left( c_2(u, v) w^2 + e_3(u, v) w + d_4(u, v) \right).
\]

Equivalently, Equation (3.12) is obtained when the conic in Equation (3.11) splits into two lines by setting \( j_0 = 0 \). The branch locus in Equation (3.12) then has three irreducible components: the quartic curve \( N = V(Q) \) with the node \( n = [0 : 0 : 1] \) and the lines \( \ell_1 = V(v), \ell_2 = V(w) \) such that \( n \in \ell_1 \cap N \) and \( \ell_1 \cap \ell_2 \cap N = \emptyset \). We have the following:

**Proposition 3.6.** Let \( c_2, e_3, d_4 \) general polynomials and \( S \) the double sextic in Equation (3.12). The following holds:

(i) the pencil of lines through the node \( n \in N \) induces the Jacobian elliptic fibration (2.7) with

\[
a = v e_3, \quad b = v^2 c_2 d_4,
\]

and the Mordell-Weil group \( \mathbb{Z}/2\mathbb{Z} \) and the singular fibers \( I_0^* + 6 I_2 + 6 I_1 \),

(ii) a pencil of lines through a point in \( \ell_2 \cap N \) induces a Jacobian elliptic fibration with a trivial Mordell-Weil group and the singular fibers \( 2 I_0^* + 2 I_2 + 8 I_1 \),

(iii) a pencil of lines through a point in \( \ell_1 \cap N - \{ n \} \) induces a Jacobian elliptic fibration with a trivial Mordell-Weil group and the singular fibers \( I_2^* + 4 I_2 + 8 I_1 \),
(iv) the pencil of lines through $\ell_1 \cap \ell_2$ induces an elliptic fibration without section and the singular fibers $I_2^* + I_0^* + 10I_1$.

Proof. The proof is analogous to the proof of Proposition 3.3. □

Remark 3.7. It follows from Plücker’s formula that a general nodal quartic $N$ has 10 tangents that intersect at a given point on the plane $\mathbb{P}(u, v, w)$. In Proposition 3.6(iv) these lines form the singular fibers of type $I_1$.

We have the following:

Proposition 3.8. A general $H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}$-polarized K3 surface is birational to a double sextic $S$ that branched on a uninodal quartic $N$ (with node $n$) and 2 lines $\ell_1, \ell_2$ so that $n \in \ell_1 \cap N$ and $\ell_1 \cap \ell_2 \cap N = \emptyset$. Conversely, every such double sextic is birational to an $H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}$-polarized K3 surface.

Proof. Due to Theorem 2.5 a general $H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}$-polarized K3 surface admits a unique Jacobian elliptic fibration (2.7) with a fiber of type $I_0^*$. This is precisely the case considered in Proposition 2.21. We have $a(u, v) = v e_3(u, v)$ and group the base points of the six fibers of type $I_2$ over $b = 0$ into sets of 2 and 4 elements by writing $b(u, v) = v^2 c_2(u, v) d_4(u, v)$ where $c_2, e_3, d_4$ are homogenous polynomials of degree 2, 3, and 4, respectively. We obtain a double sextic by blowing up in Equation (2.7) via $[x : y] = [v c_2(u, v) x : y]$ with $[(u, v, x, z)] \in F_4$ and $[(u, v, x, z)] \in F_1$ (using the notation of Equation (2.29)) and identifying $F_1 \cong \mathbb{P}(u, v, w)$ via $w = \bar{x}/\bar{z}$ for $\bar{z} \neq 0$. The double sextic is given by

$$y^2 = v w (c_2(u, v) w^2 + e_3(u, v) w + d_4(u, v)).$$

The branch locus in Equation (3.14) has three irreducible components: the lines $\ell_1 = V(v)$ and $\ell_2 = V(w)$ and a nodal quartic. One checks that $\ell_1$ is coincident with the node whereas $\ell_2$ is not. The other direction follows from Proposition 3.6. □

We also have the following:

Proposition 3.9. An $H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}$-polarization extends to a $H \oplus D_8(-1) \oplus A_1(-1)^{\oplus 3}$-polarization if and only if for the corresponding double sextic $S$ in Equation (3.12) either $\ell_1$ is tangent to $n$ or $\ell_1 \cap \ell_2 \cap N \neq \emptyset$.

Proof. As in the proof of Proposition 3.8, we plug Equation (2.38) into Equation (2.7). The fiber of type $I_0^*$ extends to a fiber of type $I_2^*$ if and only if $b(u, v) = v^3 b'(u, v)$. For the double sextic in Equation (3.14) we then have either $c_2(u, v) = v c_1(u, v)$ or $d_4(u, v) = v d_3(u, v)$ where $c_1$ and $d_3$ have degree 1 and 3, respectively. In the first case, the line $V(v)$ in the branch locus is tangent to the node of the quartic. In the second case, the two lines in the branch locus, i.e., $V(v)$ and $V(w)$, which intersect at $[1 : 0 : 0]$, now intersect on the quartic. □

Remark 3.10. The fibration in Proposition 3.6(iv) extends in situation of Proposition 3.9 as follows: when $\ell_1$ is tangent to $n$, the fibration without section has the singular fibers $II^* + I_0^* + 9I_1$. For $\ell_1 \cap \ell_2 \cap N \neq \emptyset$ the fibration has a section and is the standard fibration.
4. Projective quartic models

In this section we consider the projective quartic hypersurface $K \subset \mathbb{P}^3 = \mathbb{P}(u, v, w, y)$ given by
\begin{equation}
K : 0 = -y^2C(u, v, w) + Q(u, v, w),
\end{equation}
where $C$ and $Q$ are the same polynomials as in Equation (3.1). In this way, for every quartic surface $K$ in Equation (4.1) one has an associated double sextic $S$ in Equation (3.1) and vice versa. In turn, the pencil of lines through the node $n \in N = V(Q)$ induces the Jacobian elliptic fibration (2.7) that is the alternate fibration.

We will also establish particular normal forms for these projective quartic hypersurface and denote them by $Q \subset \mathbb{P}^3 = \mathbb{P}(X, Y, Z, W)$. They will turn out to be generalizations of the famous Inose quartic, given by
\begin{equation}
0 = 2Y^2ZW - 8X^2Z + 6aXZW^2 + 2bZW^3 - (Z^2W^2 + W^4).
\end{equation}
The two-parameter family was first introduced by Inose in [15]. In [5,32] it was proved that a complex algebraic K3 surface with Picard lattice $H \oplus E_8(-1) \oplus E_8(-1)$ admits a birational model isomorphic to an Inose quartic.

4.1. Quartic surfaces associated with double sextics. Consider the family of quartic surfaces
\begin{equation}
K : 0 = -(j_0u^2 + vw + h_0w^2)y^2 + c_2(u, v)w^2 + e_3(u, v)w + d_4(u, v),
\end{equation}
where $c_2, d_4, e_3$ are general homogeneous polynomials of the degree indicated by the subscript and $h_0, j_0 \in \mathbb{C}$. It has the associated double sextic $S$ in Equation (3.2). It follows from Proposition 3.2 that the minimal resolution of the general quartic surface $K$ in Equation (4.1) has Picard rank 10. We have the following:

**Proposition 4.1.** The minimal resolution of $K$ in Equation (4.1) is a smooth K3 surface if and only if Equation (2.7) with $a, b$ in (3.5) defines a minimal Weierstrass equation. In particular, $K$ has exactly 2 singularities at points $[u : v : w : y]$, given by
\begin{equation}
p_1 = [0 : 0 : 0 : 1], \quad p_2 = [0 : 0 : 1 : 0],
\end{equation}
which are rational double points. The minimal resolution of $K$ is isomorphic to the Jacobian elliptic K3 surface $\mathcal{X}$ if $h_0 = 0$.

**Proof.** The double sextic in Equation (3.1) is birational to the quartic projective hypersurface in Equation (4.1) in $\mathbb{P}^3 = \mathbb{P}(u, v, w, y)$. This can be seen by setting $\tilde{y} = C(u, v, w)y$ in Equation (3.1). It follows from Proposition 3.1 that the pencil of lines through the node of the quartic induces an elliptic fibration on this double sextic whose relative Jacobian fibration is Equation (2.7). It follows that $K$ is a smooth if and only if Equation (2.7) is a minimal Weierstrass equation. Since the degree of Equation (4.1) is four, the minimal resolution of $K$ is a K3 surface. Hence, the singularities of $K$ must be rational double points. One checks that the points in Equation (4.4) are the only singularities. It follows from Propositions 3.3, 3.5, 3.6 that the fibration induced by the pencil of lines through the node has a section if $h_0 = 0$. Then it is isomorphic to its relative Jacobian elliptic fibration. $\square$
4.1.1. **Quartic surfaces of Picard rank 11.** Next we set $j_0 = 0$, $h_0 = -\rho$ in Equation (4.3) and consider the family

$$
K: \quad 0 = -(v - \rho w)wy^2 + c_2(u,v)w^2 + c_3(u,v)w + d_4(u,v).
$$

The associated double sextic $S$ is Equation (3.8) with $\rho = -h_0$. It follows from Proposition 3.4 that the minimal resolution of $S$ is polarized by $H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 5}$. We set

$$
c_2(u,v) = (\gamma u - \delta v)(\varepsilon u - \zeta v), \quad d_4(u,v) = (\eta u - \nu v)(\kappa u - \lambda v)(\mu u - \nu v)(\xi u - \nu v),
$$

where we introduced complex coefficients $\gamma, \delta, \varepsilon, \zeta, \eta, \kappa, \lambda, \mu, \nu, \xi, \eta$. The roots of the polynomial $d_4$ determine the intersection points of $\ell_2 = V(w)$ and the nodal quartic $N$. For $\rho \neq 0$ the line $\ell_1 = V(v - \rho v)$ is not coincident with the node $n \in N$. Moreover, we can assume $\gamma \in \eta \kappa \mu \xi \neq 0$ in the general case. To make the intersection points $\ell_1 \cap N$ explicit, we set $w = (v + \bar{w})/\rho$ in Equation (4.5) and obtain an equivalent equation of the form

$$
K: \quad 0 = -\rho(v + \bar{w})\bar{w}y^2 + c_2(u,v)\bar{w}^2 + c_3'(u,v)\bar{w} + d_4'(u,v),
$$

where $\ell_1 = V(\bar{w})$ and

$$
e_3'(u,v) = \rho c_3(u,v) + 2c_2(u,v)v,
$$

$$
d_4'(u,v) = \rho^2d_4(u,v) + \rho e_3(u,v)v + c_2(u,v)v^2.
$$

The roots of $d_4'$ determine the intersection points of $\ell_1 \cap N$, and we write

$$
d_4'(u,v) = (\eta' u - \nu' v)(\kappa' u - \lambda' v)(\mu' u - \nu' v)(\xi' u - \nu' v).
$$

The coefficient set $(\gamma, \delta, \varepsilon, \zeta, \eta, \ldots, \eta', \ldots, \eta')$ determines the polynomials $c_2, d_4, d_4'$ and $\rho, e_3, e_3'$, using the equations

$$
\rho^2 = \frac{\eta'\kappa'\mu'\xi'}{\eta\kappa\mu\xi}, \quad e_3(u,v) = \frac{d_4'(u,v) - \rho^2d_4(u,v) - c_2(u,v)v^2}{\rho v},
$$

$$
e_3'(u,v) = \frac{d_4'(u,v) - \rho^2d_4(u,v) + c_2(u,v)v^2}{v}.
$$

The pencil of lines through $n \in N$ induces the Jacobian elliptic fibration (2.7) with

$$
a(u,v) = \frac{d_4'(u,v) + \rho^2d_4(u,v) - c_2(u,v)v^2}{\rho}, \quad b(u,v) = d_4'(u,v)d_4'(u,v).$$

**Remark 4.2.** The different sign choices $\pm \rho$ in Equation (4.10) (resulting in sign changes $\pm e_3$ and $\pm a$ in Equation (4.11)) yield isomorphic surfaces in Equation (4.5) (resp. in Equation (2.7)) related by $[u : v : w : y] \mapsto [u : v : -w : iy]$ (resp. $[X : Y : Z] \mapsto [-X : iY : Z]$).

Conversely, due to Theorem 2.5 a general $H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 5}$-polarized K3 surface $\mathcal{X}$ admits a Jacobian elliptic fibration (2.7) with the singular fiber $9I_2 + 6I_1$. Thus, after a suitable change of coordinates, one has $a(u,v)^2/4 - b(u,v) = v^2b'(u,v)$ where $b'$ is a homogeneous polynomial of degree 6. We choose a factorization of $b(u,v) = a(u,v)^2/4 - v^2b'(u,v) = d_4(u,v)d_4'(u,v)$ into homogeneous polynomials of degree 4. We find $\rho \in \mathbb{C}^*$ so that $e_3 = (a - 2pd_4)/v$ and $c_2 = (d_4'(u,v) + \rho^2d_4'(u,v) - \rho a(u,v))/v^2$ are polynomials of degree 3 and 2, respectively. Surfaces $K$ and $S$ are then obtained by using $h_0 = -\rho, c_2, e_3, d_4$ in Equation (4.7) and Equation (3.8), respectively. We have the following:
Proposition 4.3. Assume that the conditions of Proposition 4.1 are satisfied. The singularities $p_1$ and $p_2$ are rational double points of type $A_3$ and $A_1$ on $K$.

Proof. The type of rational double point is determined by an explicit computation. \hfill \Box

Proposition 4.4. Assume that the conditions of Proposition 4.1 are satisfied. A Nikulin involution on the minimal resolution of $K$ is induced by the projective automorphism $\mathbb{P}^3 \to \mathbb{P}^3$ given by

\begin{equation}
(4.12) \quad \left[ u : v : w : y \right] \rightarrow \left[ \tilde{d}_4(u,v,w) u : \tilde{d}_4(u,v,w) v : d_4(u,v)(v - \rho w) : \tilde{d}_4(u,v,w) y \right]
\end{equation}

with $\tilde{d}_4(u,v,w) = \rho d_4(u,v) + \rho wc_3(u,v) + vwc_2(u,v)$.

Proof. The Jacobian elliptic fibration (2.7) carries the van Geemen-Sarti involution in Equation (2.9). One checks that it induces (after composition with the hyperelliptic involution) on the quartic surface $K$ the involution in Equation (4.20). \hfill \Box

4.1.2. Quartic surfaces of Picard rank 12. Next we consider the quartic surface

\begin{equation}
(4.13) \quad K : \quad 0 = -vwy^2 + c_2(u,v) w^2 + e_3(u,v) w + d_4(u,v).
\end{equation}

The quartic has the associated double sextic $S$ in Equation (3.12). It follows from Proposition 3.8 that the minimal resolution of $S$ is polarized by the lattice $H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}$. We also assume that the leading coefficients of $c_2, e_3, d_4$ do not vanish since $K$ is general. We then write the polynomials as follows:

\begin{equation}
(4.14) \quad c_2(u,v) = (\gamma u - \delta v)(\varepsilon u - \zeta v), \quad e_3(u,v) = u^3 - 3\alpha uv^2 - 2\beta v^3, \quad d_4(u,v) = (\eta u - \nu v)(\kappa u - \lambda v)(\mu u - \nu v)(\xi u - \nu v).
\end{equation}

Here, we introduced the coefficient set $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \iota, \kappa, \lambda, \mu, \nu, \xi, o) \in \mathbb{C}^{14}$ and assumed $\gamma \varepsilon \eta \kappa \mu \xi \neq 0$ in the general case. The pencil of lines through the node $n \in N$ induces the Jacobian elliptic fibration (2.7) with

\begin{equation}
(4.15) \quad a(u,v) = v e_3(u,v), \quad b(u,v) = v^2 c_2(u,v) d_4(u,v)
\end{equation}

in (3.13), and the corresponding Jacobian elliptic K3 surface denoted by $\mathcal{X}$.

Conversely, a K3 surface $\mathcal{X} \in \mathcal{M}_{H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}}$ admits a unique Jacobian elliptic fibration (2.7); see Theorem 2.5. We then match the moduli in Equation (4.14) with the coefficients of the polynomials $a, b$ by writing

\begin{equation}
(4.16) \quad a(u,v) = v\left(u^3 + 3a_2uv^2 + a_4v^3\right), \quad b(u,v) = v^2\left(b_2u^6 + b_3u^5v + b_4u^4v^2 + b_5u^3v^3 + b_6u^2v^4 + b_7uv^5 + b_8v^6\right)
\end{equation}

so that the moduli of the K3 surface $\mathcal{X}$ are $([1:b_2], [a_3:a_4:b_3: \cdots : b_8])$ with

\begin{equation}
(4.17) \quad a_3 = -3\alpha, \quad a_4 = -2\beta, \quad b_2 = \gamma \varepsilon \eta \kappa \mu \xi, \quad b_k = (-1)^k \gamma \varepsilon \eta \kappa \mu \xi \cdot \sigma_{k-2}^{(6)}(\frac{\delta}{\gamma}, \ldots, \frac{o}{\xi}), \quad b_8 = \delta \xi \iota \nu o.
\end{equation}

Here, $\sigma_{k-2}^{(6)}$ for $k = 2, \ldots, 8$ are the elementary symmetric polynomials in 6 variables of degree $k-2$. Thus, given a K3 surface $\mathcal{X}$ with a Jacobian elliptic fibration (2.7) for $a, b$
in (4.16), surfaces $K$ and $S$ are obtained by using Equation (4.14) in Equation (4.13) and Equation (3.12), respectively.

We introduce the vanishing order $N_r \in \{0, 1, \ldots, 6\}$ of $b(u, v) = 0$ at $[u : v] = [1 : -r]$. Using the notation of Equation (4.16), $N_r$ equals the number of pairs of parameters $(u, v) \neq 0$ such that $[u : v] = [1 : -r]$ and

\begin{equation}
[u : v] \in \left\{ \left[ \gamma, \delta \right], \left[ \varepsilon, \zeta \right], \left[ \eta, \iota \right], \left[ \kappa, \lambda \right], \left[ \mu, \nu \right], \left[ \xi, o \right] \right\}.
\end{equation}

We have the following:

**Proposition 4.5.** Assume that $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \iota, \kappa, \lambda, \mu, \nu, \xi, o) \in \mathbb{C}^{14}$ is a coefficient set so that for $a, b$ in Equation (4.16) one has $b(u, v) \neq 0, \frac{1}{4}a(u, v)^2$ and $r \in \mathbb{C}$: $\alpha = r^2, \beta = r^3, N_r \geq 4$, and the coefficients are generic otherwise. The singularities $p_1$ and $p_2$ are rational double points of type $A_3$ on $K$. For $(\xi, o) = (0, 1)$ the singularity $p_1$ is a rational double point of type $A_5$ and $p_2$ is of type $A_3$.

**Proof.** Using Lemma 2.8, Remark 2.18, and Proposition 4.1 it follows that the points $p_1$ and $p_2$ are rational double points. The type of rational double point is determined by an explicit computation.

Notice that starting with Equation (4.16) and then using Equation (4.14) involves a choice of how the factors of $b(u, v)$ are distributed between $c_2(u, v)$ and $d_4(u, v)$. However, it is easy to show that different choices are connected by projective automorphisms between the corresponding quartic surfaces. We have the following:

**Lemma 4.6.** Assume that the conditions of Proposition 4.5 are satisfied. The K3 surfaces obtained as the minimal resolution of quartic surfaces $K$ are isomorphic if and only if one of the following conditions is satisfied:

(a) the coefficient sets are related by pairwise interchanges of elements in

\[ \{ (\gamma, \delta), (\varepsilon, \zeta), (\eta, \iota), (\kappa, \lambda), (\mu, \nu), (\xi, o) \} \],

(b) the coefficient sets are related by

\[ (\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \iota, \kappa, \lambda, \mu, \nu, \xi, o) \mapsto (\Lambda^4 \alpha, \Lambda^6 \beta, \Lambda^{10} \gamma, \Lambda^{12} \delta, \Lambda^{-2} \varepsilon, \zeta, \Lambda^{-2} \eta, \iota, \Lambda^{-2} \kappa, \lambda, \Lambda^{-2} \mu, \nu, \Lambda^{-2} \xi, o) \]

for $\Lambda \in \mathbb{C}^\times$.

**Proof.** Let $K$ and $\tilde{K}$ be two quartic surfaces satisfying the conditions of Proposition 4.5. For the quartic surfaces we consider the associated double sextics in Equation (3.1). Using Proposition 3.6 it follows that each of them admits a Jacobian elliptic fibration (2.7) with polynomials $a, b$ and $\tilde{a}, \tilde{b}$, respectively, that define points in the moduli space $\mathcal{M}_{\tilde{b} + P_{\mathrm{d}_{-1}(1)}(\tilde{a})} (\tilde{a} + P_{\mathrm{d}_{-1}(1)})$ in Equation (2.27). Upon bringing these polynomials into the standard form it follows that the K3 surfaces are isomorphic if and only if

\begin{equation}
\Lambda^2 a(u, v) = \tilde{a}(u, \Lambda^2 v), \quad \Lambda^4 b(u, v) = \tilde{b}(u, \Lambda^2 v),
\end{equation}

for $\Lambda \in \mathbb{C}^\times$. Note that the coordinates in Equation (4.16) are such that for the polynomial $a(u, v)$ the coordinate $u$ is fixed, up to scaling, and the leading coefficient is fixed. Every interchange of pairs of parameters in part (a) leaves invariant the
polynomial \( b(u, v) \) and thus the period point in \( \mathcal{H}_{\mathbb{P}^6} \). Moreover, the rescaling in part (b) leads to a rescaling of the coefficients in the polynomials \( a, b \) when using Equation (4.17) which in turn yields exactly Equation (4.19). Thus, the action also leaves the the period point in \( \mathcal{H}_{\mathbb{P}^6} \) invariant. □

**Proposition 4.7.** Assume that the conditions of Proposition 4.5 are satisfied. A Nikulin involution on the minimal resolution of \( \mathcal{K} \) is induced by the projective automorphism \( \mathbb{P}^3 \to \mathbb{P}^3 \) given by

\[
(4.20) \quad [u : v : w : y] \mapsto [c_2(u, v)uw : c_2(u, v)vw : d_4(u, v) : c_2(u, v)wy]
\]

**Proof.** We obtain the result by setting \( \rho = 0 \) in Equation (4.12). □

4.2. **Inose-type quartic surfaces.** We will now introduce multiparameter generalizations of the Inose quartic in Equation (4.2). For Picard number 11, we consider the quartic surface \( \mathcal{Q} \) in \( \mathbb{P}^3 \) with equation

\[
(4.21) \quad \mathcal{Q} : \quad 0 = 2Y^2Z(W - \rho Z) - c_3(2X, W)Z - c_2(2X, W)Z^2 - d_4(2X, W).
\]

Here, a given coefficient set \((\gamma, \delta, \varepsilon, \zeta, \eta, \ldots, \alpha, \eta', \ldots, \alpha')\) determines \( c_2, d_4 \) using Equation (4.6) and \( c_3 \) and \( \rho \) using Equation (4.10).

For Picard number 12 we will also consider the case when \( \rho = 0, c_3(u, 1) = u^3 - 3\alpha u - 2\beta \) in Equation (4.21). That is, we consider the quartic surface \( \mathcal{Q} \) in \( \mathbb{P}^3 \) with equation

\[
(4.22) \quad \mathcal{Q} : \quad 0 = 2Y^2ZW - 8X^3Z + 6\alpha XZW^2 + 2\beta ZW^3 - (2\gamma X - \delta W)(2\varepsilon X - \zeta W)Z^2 - (2\eta Y - \iota W)(2\kappa X - \lambda W)(2\mu X - \nu W)(2\xi X - \omega W)
\]

for a coefficient set \((\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \iota, \kappa, \lambda, \mu, \nu, \xi, \omega) \in \mathbb{C}^{14}\). One checks the following:

**Lemma 4.8.** Assume that the conditions of Proposition 4.1 are satisfied. The surfaces \( \mathcal{Q} \) in Equation (4.5) and (4.13) are isomorphic to the surfaces \( \mathcal{K} \) in Equation (4.21) and (4.22), respectively, with

\[
(4.23) \quad [u : v : w : y] = [2X : W : Z : \sqrt{2}Y].
\]

In particular, \( \mathcal{Q} \) has two singularities at points \([W : X : Y : Z]\), given by

\[
(4.24) \quad P_1 = [0 : 0 : 1 : 0], \quad P_2 = [0 : 0 : 0 : 1].
\]

which are rational double points.

For \( \mathcal{Q} \) in Equation (4.21) we introduce the following lines, denoted by \( L_4, L_5, L_6, L_7, L'_4, L'_5, L'_6, L'_7 \) with

\[
L_4: \quad Z = 2\eta X - \iota W = 0, \quad L'_4: \quad W - \rho Z = 2\eta' X - \lambda' W = 0,
\]

and equations for \( L_5, L_6, \) and \( L_7 \) that are obtained from \( L_4 \) by interchanging parameters according to \((\eta, \iota) \leftrightarrow (\kappa, \lambda), (\eta', \iota) \leftrightarrow (\mu, \nu), \) and \((\eta, \iota) \leftrightarrow (\xi, \omega)\), respectively. Similarly, equations for \( L'_5, L'_6, \) and \( L'_7 \) are obtained from \( L'_4 \) by interchanging parameters according to \((\eta', \iota') \leftrightarrow (\kappa', \lambda'), (\eta', \iota') \leftrightarrow (\mu', \nu'), \) and \((\eta', \iota') \leftrightarrow (\xi', \omega')\), respectively. The lines are readily checked to lie on the quartic surface \( \mathcal{Q} \) in Equation (4.21). For general parameters, the lines are distinct and concurrent, meeting at \( P_1 \). Each line induces a pencil; they are denoted by \( L_i(u, v), L_i(u, v) \) and \( L'_i(u, v) \) for \( i = 4, 5, 6, 7 \). For example, we set \( L_4(u, v) = uZ - v(2\eta X - \iota W) \). We have the following:
Theorem 4.9. Assume that the conditions of Proposition 4.1 are satisfied and let $L = H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 5}$. The minimal resolution of $\mathcal{Q}$ in Equation (4.21) is a K3 surface endowed with a canonical $L$-polarization. Conversely, a general $L$-polarized K3 surface has a birational projective model (4.21). In particular, the Jacobian elliptic fibrations of the type determined in Theorem 2.2 are attained as follows:

| # | singular fibers | MW | reducible fibers | pencil |
|---|----------------|----|-----------------|--------|
| 1 | $9I_2 + 6I_1$ | $\mathbb{Z}/2\mathbb{Z}$ | $\tilde{A}_1 \oplus \tilde{A}_1$ | family of quartic curves through $P_1$ and $P_2$, intersection of $2vX + uW = 0$ and $\mathcal{Q}$ |
| 2 | $I_0^* + 5I_2 + 8I_1$ | $\{1\}$ | $\tilde{D}_4 + \tilde{A}_1 \oplus \tilde{A}_1$ | residual surface intersection of $L_n(u, v) = 0$ or $L_n'(u, v) = 0$ ($i = 4, 5, 6, 7$) and $\mathcal{Q}$ |

Proof. To obtain the fibration (1) we make the substitution

\[ W = 2v^2X(X + \rho d_4(u, v)Z), \quad X = uvX(X + \rho d_4(u, v)Z), \]
\[ Y = \sqrt{2}Y(X + \rho d_4(u, v)Z), \quad Z = 2v^2d_4(u, v)XZ, \]

in Equation (4.21). This determines the Jacobian elliptic fibration (2.7) where $a, b$ are given by Equation (3.9) with $h_0 = -\rho$. Fibration (1) is induced by family of quartic curves through $P_1$ and $P_2$, which is obtained as the intersection of then pencil of hyperplanes $2vX + uW = 0$ and $\mathcal{Q}$. In general, $X = W = 0$ does not define a line on $\mathcal{Q}$ if $\rho \neq 0$.

The fibrations induced by the pencils $L_n(u, v) = 0$ and $L_n'(u, v) = 0$ for $n = 4, 5, 6, 7$ are all found in the same way. For example, if one substitutes $L_4(u, v) = 0$ into Equation (3.9), one obtains a genus-one fibration. The fibration has a section since $2\eta X - tW = 0$ is a root of the polynomial $d_4(2X, W)$. By bringing the equation into Weierstrass normal form one obtains fibration (2). The fiber of type $I_0^*$ is located over $u = 0$, one fiber of type $I_2$ over $v = 0$, and the remaining fibers of type $I_2$ over

\[ \frac{u}{v} = \frac{\rho(\eta' - \nu'v^3)}{\eta'}, \quad \frac{\rho(\eta' - \nu'v^3)}{\kappa'}, \quad \frac{\rho(\eta' - \nu'v^3)}{\mu'}, \quad \frac{\rho(\eta' - \nu'v^3)}{\xi'}. \]

Similar results are obtained for the pencils $L_i(u, v)$ for $i = 5, 6, 7$ by interchanging parameters according to $(\eta, \nu) \leftrightarrow (\kappa, \lambda)$, $(\eta, \nu) \leftrightarrow (\mu, \gamma)$, and $(\eta, \nu) \leftrightarrow (\xi, \rho)$, respectively. Using Lemma 4.8 and Equation (4.7) as an equivalent way of writing Equation (4.5), it follows that the same results hold if we interchange the lines $L_n \leftrightarrow L_n'$ for $n = 4, 5, 6, 7$.

Conversely, Theorem 2.5 proves that every general $H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 5}$-polarized K3 surface admits a unique Jacobian elliptic fibration (2.7) with the singular fibers $9I_2 + 6I_1$. Thus, after a suitable change of coordinates, one has $a(u, v)^2/4 - b(u, v) = v^2b'(u, v)$ where $b'$ is a homogeneous polynomial of degree 6. We choose a factorization of $b(u, v) = a(u, v)^2/4 - b^2(u, v) = d_4(u, v)d_4'(u, v)$ into homogeneous polynomials of degree 4. We can then find $\rho \in \mathbb{C}$ so that $e_3 = (a - 2pd_4)/v$ and $c_2 = (d_4'(u, v) + \rho^2d_4(u, v) - \rho d_4(u, v))/v^2$ are polynomials of degree 3 and 2, respectively. Upon solving Equations (4.25) for $u, v, X, Y$ and plugging the result into Equation (2.7), the proper transform is a quartic surface $\mathcal{Q}$, given by an equation of the form (4.21).

Because of Theorem 2.5 the constructed pencil (1) is the only way of realizing the alternate Jacobian elliptic fibration up to the action of the Nikulin involution in Equation (4.4). I follows immediately that the K3 surface $\mathcal{X}$ is endowed with a canonical $H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 5}$-polarization. \(\square\)
For \( Q \) in Equation (4.22) we find additional lines, denoted by \( L_1, L_2^+ \): 
\[
L_1 : W = X = 0, \quad L_2^+ : W = (1 \pm \chi)X + \gamma \varepsilon Z = 0,
\]
where we introduced a parameter \( \chi \) with \( 4\gamma \varepsilon \eta \kappa \mu \varepsilon = 1 - \chi^2 \). The lines \( L_1, L_2^+, L_4, L_5, L_6, L_7 \) then lie on the quartic surface \( Q \) in Equation (4.22). For general parameters, the lines are distinct and concurrent, meeting at \( P_1 \). The line \( L_1 \) is also coincident with \( P_2 \). We have the following:

**Theorem 4.10.** Assume that the conditions of Proposition 4.5 are satisfied and let \( L = H \oplus D_0(-1) \oplus A_1(-1)^{\oplus 4} \). The minimal resolution of \( Q \) in Equation (4.22) is a K3 surface endowed with a canonical \( L \)-polarization. Conversely, a general \( L \)-polarized K3 surface \( X \in \bar{\mathcal{M}}_L \) has a birational projective model (4.22). In particular, the Jacobian elliptic fibrations of the type determined in Theorem 2.2 are attained as follows:

| \# | singular fibers | MW | reducible fibers | pencil |
|----|----------------|-----|-----------------|--------|
| 1  | \( I_0^* + 6I_2 + 6I_1 \) | \( \mathbb{Z}/2\mathbb{Z} \) | \( \bar{D}_4 + \bar{A}_1^{\oplus 6} \) | residual surface intersection of \( L_1(u, v) = 0 \) and \( Q \) |
| 2  | \( 2I_0^* + 2I_2 + 8I_1 \) | \( \{1\} \) | \( \bar{D}_4 + \bar{D}_4 + \bar{A}_1^{\oplus 2} \) | residual surface intersection of \( L_i(u, v) = 0 \) \( (i = 4, 5, 6, 7) \) and \( Q \) |
| 3  | \( I_2^* + 4I_2 + 8I_1 \) | \( \{1\} \) | \( \bar{D}_6 + \bar{A}_1^{\oplus 4} \) | residual surface intersection of \( L_2^+(u, v) = 0 \) and \( Q \) |

**Remark 4.11.** The fibrations in Theorem 4.10 correspond precisely to the pencils in Proposition 3.6: the pencil through the node \( n \) for (1), the pencils through any point chosen from \( \ell_2 \cap \mathcal{N} \) for (2), and the pencils through any point chosen from \( \ell_1 \cap \mathcal{N} - \{n\} \) for (3).

**Proof.** The proof is analogous to the proof of Theorem 4.9. To obtain the fibration (1) we make the substitution
\[
W = 2v^2 x, \quad X = uwx, \quad Y = \sqrt{2}y, \quad Z = 2v^2(\eta u - \nu v)(\kappa u - \lambda v)(\mu u - \nu v)(\xi u - \omega v)z,
\]
in Equation (4.22), compatible with the pencil \( L_1(u, v) = 0 \). This determines the Jacobian elliptic fibration (2.7) where \( a, b \) are given by Equation (4.16). Conversely, Theorem 2.2 proves that every general \( H \oplus D_0(-1) \oplus A_1(-1)^{\oplus 4} \)-polarized K3 surface admits a Jacobian elliptic fibration (2.7). If \( X \in \bar{\mathcal{M}}_L \) in Equation (2.27), then the polynomials \( a \) and \( b \) are of the form given by Equation (4.16). Upon solving Equations (4.28) for \( u, v, X, Y \) and plugging the result into Equation (2.7), the proper transform is a quartic surface \( Q \) with an equation of the form (4.22).

The fibrations induced by the pencils \( L_n(u, v) = 0 \) for \( n = 4, 5, 6, 7 \) are all found in the same way as they were in the proof of Theorem 4.9: if one substitutes \( L_4(u, v) = 0 \) into Equation (3.13), one obtains a genus-one fibration with section. By bringing the equation into Weierstrass normal form one obtains fibration (2). The corresponding equations for the other pencils are then generated through the symmetries in Lemma 4.6. Similarly, if one substitutes \( L_n'(u, v) = 0 \) for \( n = 4, 5, 6, 7 \) into Equation (3.13), one obtains a genus-one fibration with section. By bringing the equation into Weierstrass normal form one obtains Jacobian elliptic fibration (3). The rest of the proof is as in the proof of Theorem 4.9. \( \square \)
Next we investigate the situation for $(\xi, \eta) = (0, 1)$ (or, equivalently, $\chi^2 = 1$). In this case, instead of the four lines $L_1$, $L_2^*$, and $L_7$, there only remain three lines, namely,
\begin{equation}
L_1 : W = X = 0, \quad L_2 : W = Z = 0, \quad L_2 : W = 2X + \gamma \epsilon Z = 0,
\end{equation}
with the corresponding pencils $L_1(u, v)$, $L_2(u, v)$, $\tilde{L}_2(u, v)$ on the quartic surface $Q$.

**Remark 4.12.** A different exceptional case is given by $\chi = 0$ or, equivalently, $b_2 = \frac{1}{4}$ in Equation (4.16). In this situation one has $L_2^* = L_2$ and the polarizing lattice extends to $H \oplus D_4(-1)^{\oplus 2} \oplus A_3(-1)$; see Remark 2.19. It was proved in [3] that the associated $K3$ surface $X$ satisfies $\text{Aut}(X) = \infty$. We will not consider this case any further.

We have the following:

**Theorem 4.13.** Assume that the conditions of Proposition 4.5 are satisfied with $(\xi, \eta) = (0, 1)$ and let $L = H \oplus D_8(-1) \oplus A_1(-1)^{\oplus 3}$. The minimal resolution of $Q$ in Equation (4.22) is a $K3$ surface endowed with a canonical $L$-polarization. Conversely, a general $L$-polarized $K3$ surface has a birational projective model (4.22) with $(\xi, \eta) = (0, 1)$. In particular, the Jacobian elliptic fibrations of the type determined in Theorem 2.2 are attained as follows:

| # | singular fibers | MW | reducible fibers | pencil |
|---|----------------|----|-----------------|-------|
| 1 | $I_2^* + 5I_2 + 6I_1$ | $\mathbb{Z}/2\mathbb{Z}$ | $\tilde{D}_6 + \tilde{A}_1^{\oplus 5}$ | residual surface intersection of $L_1(u, v) = 0$ and $Q$ |
| 2 | $I_0^* + I_2^* + I_2 + 8I_1$ | $\{1\}$ | $\tilde{D}_4 + \tilde{D}_6 + \tilde{A}_1$ | residual surface intersection of $L_i(u, v) = 0$ ($i = 2, 4, 5, 6$) and $Q$ |
| 3 | $I_4^* + 3I_2 + 8I_1$ | $\{1\}$ | $\tilde{D}_8 + \tilde{A}_1^{\oplus 3}$ | residual surface intersection of $\tilde{L}_2(u, v) = 0$ and $Q$ |
| 4 | $III^* + 4I_2 + 7I_1$ | $\{1\}$ | $\tilde{E}_7 + \tilde{A}_1^{\oplus 4}$ | intersection of quadric surfaces $C_i(u, v) = 0$ ($i = 4, 5, 6$) and $Q$ |

**Proof.** The statements concerning fibrations (1) and (2) are proved in the same way as in proof of Theorem 4.10 after setting $(\xi, \eta) = (0, 1)$. Similarly, if one substitutes $\tilde{L}_2(u, v) = 0$ into Equation (3.13) with $(\xi, \eta) = (0, 1)$, one obtains a genus-one fibration with section. By bringing the equation into Weierstrass normal form one obtains Jacobian elliptic fibration (3).

Pencils of quadratic surfaces, denoted by $C_n(u, v) = 0$ with $[u : v] \in \mathbb{P}^1$ for $n = 4, 5, 6$, are constructed as follows. We set
\begin{equation}
C_4(u, v) = \mu(\kappa u - \lambda v)(2\mu X - \nu W)(2\kappa X - \lambda W + \gamma \epsilon \kappa Z) - \kappa(\mu u - \nu v)(2\kappa X - \lambda W)(2\mu X - \nu W + \gamma \epsilon \mu Z).
\end{equation}
The pencil is invariant under the permutation of parameters $(\gamma, \delta) \leftrightarrow (\epsilon, \zeta)$. Replacing parameters $((\kappa, \lambda), (\mu, \nu))$ by $((\eta, \nu), (\mu, \nu))$ or $((\eta, \nu), (\kappa, \lambda))$ then yields the pencils $C_5(u, v)$ and $C_6(u, v)$, respectively. Making the substitutions
\begin{equation}
\begin{align*}
W &= 2\gamma^2 \epsilon^2 \kappa^2 \mu^2 v^2 (\gamma u - \delta v)(\epsilon u - \zeta v)xz, \\
X &= \gamma^2 \epsilon^2 \kappa^2 \mu^2 v(\gamma u - \delta v)(\epsilon u - \zeta v)q_1(x, z, u, v), \\
Y &= \sqrt{2}\gamma \kappa \mu(\gamma u - \delta v)(\epsilon u - \zeta v)y, \\
Z &= 2q_2(x, z, u, v)q_3(x, z, u, v),
\end{align*}
\end{equation}
in Equation (4.22), compatible with \( C_4(u, v) = 0 \), and using the polynomials
\[
q_1(x, z, u, v) = ux - \gamma \xi \mu \nu (\gamma u - \delta v)(\varepsilon u - \zeta v)(\kappa u - \lambda v)(\mu u - \nu v)z,
\]
\[
q_2(x, z, u, v) = x - \gamma \xi \mu^2 \nu (\gamma u - \delta v)(\varepsilon u - \zeta v)(\mu u - \nu v)z,
\]
\[
q_3(x, z, u, v) = x - \gamma \xi \mu v (\gamma u - \delta v)(\varepsilon u - \zeta v)(\kappa u - \lambda v)z,
\]
yields the Jacobian elliptic fibration (4). Jacobian elliptic fibrations with the same singular fibers are obtained from \( C_5(u, v) \) and \( C_6(u, v) \).

Theorems 4.9, 4.10, and 4.13 establish birational models isomorphic to quartic projective surfaces for the lattices of rank 11, 12, and 13 in Theorem 1. The constraints \( (\mu, \nu) = (\xi, \sigma) = (0, 1) \) restrict the family further to quartic surface whose associated K3 surfaces \( X \) have Picard rank \( p_X = 14 \). Similarly, the constraints \( (\kappa, \lambda) = (\mu, \nu) = (\xi, \sigma) = (0, 1) \) and \( (\eta, \iota) = (\kappa, \lambda) = (\mu, \nu) = (\xi, \sigma) = (0, 1) \) restrict the family to K3 surfaces with \( p_X = 15 \) and \( p_X = 16 \), respectively. These cases were discussed by the authors in [2, 8]. In particular, theorems analogous with Theorems 4.9, 4.10, 4.13, but for Picard rank 14, 15, and 16 were established in [2] and [8]. Restricting even further yields K3 surfaces with Picard rank \( p_X = 17, 18, 19 \). For these cases, similar results were proven in [5, 6, 15].

4.3. The dual graph of rational curves. In this section we will construct the dual graphs of smooth rational curves for the K3 surfaces of Theorems 4.9, 4.10, and 4.13 with \( \rho_L = 11, 12, \) and 13. For each graph we also provide an enlarged copy at the end of the article. The graphs will only show some 2-fold edges. The complete lists of intersections products are then given as tables at the end of this article.

Below we will also provide the embeddings of the reducible fibers from Theorems 4.9, 4.10, and 4.13 into the dual graph of smooth rational curves for \( \rho_L = 11, 12, \) and 13. In these figures, the components of the reducible fibers, given by extended Dynkin diagrams, are distinguished by the same colors as in the theorems, and the classes of the section (and a 2-torsions section if applicable) are represented by yellow nodes.

We determine the dual graph of all smooth rational curves and their intersection numbers for the K3 surfaces \( X \) with Néron-Severi lattice \( H \oplus D_4(-1) \oplus A_1(-1)^{65} \) in Theorem 4.9. In this case, one has the following rational curves:
\[
a_1, a_2, a_3, b_1, R_0, R_{13}, \tilde{R}_1, L_n, R_n, L'_n, R'_n, S_{n,n'}, S_{n,n'} \text{ with } n, n' = 4, 5, 6, 7.
\]
The curves \( R_*, R'_*, S_*, \tilde{S}_* \) are obtained from singular complete intersection curves of arithmetic genus \( g \geq 2 \) on the quartic surface \( Q \). When resolving the quartic surface (4.21), these curves lift to smooth rational curves on a K3 surface, which by a slight abuse of notation we shall denote by the same symbol. The two sets \( \{a_1, a_2, a_3\} \) and \( \{b_1\} \) denote the curves appearing when resolving the rational double point singularities at \( P_1 \) and \( P_2 \), respectively; see Proposition 4.3. The lines \( L_4, \ldots, L_7 \) were already determined in Section 4.2. Explicit equations for the remaining divisors are given in Appendix A.1. We have the following:
The dual graph of smooth rational curves is Figure 2 (and Figure 14). The complete list of multi-fold edges (2-fold, 4-fold, 6-fold) is given in Table 3.

Proof. The proof follows the strategy of a similar proof given by the authors in [2]. First, one establishes the symmetries among the divisors in Equation (4.33). Divisors $L_n$ and $R_n$, $L'_n$ and $R'_n$, $S_{n,n'}$ and $\tilde{S}_{n,n'}$ for $n,n' = 4,5,6,7$ are related by the action of the Nikulin involution in Proposition 4.4. Moreover, pairs of divisors within $\{(L_4, R_4), (L_5, R_5), (L_6, R_6), (L_7, R_7)\}$ are interchanged when changing parameters according to $(\eta, \iota) \leftrightarrow (\kappa, \lambda)$, $(\eta, \iota) \leftrightarrow (\mu, \nu)$, and $(\eta, \iota) \leftrightarrow (\xi, \varnothing)$, respectively. Similarly, pairs within $\{(L'_4, R'_4), (L'_5, R'_5), (L'_6, R'_6), (L'_7, R'_7)\}$ are interchanged when changing parameters according to $(\eta', \iota') \leftrightarrow (\kappa', \lambda')$, $(\eta', \iota') \leftrightarrow (\mu', \nu')$, and $(\eta', \iota') \leftrightarrow (\xi', \varnothing')$, respectively. Equations for $S_{5,4}$, $S_{6,4}$, and $S_{7,4}$ are obtained from $S_{4,4}$ by interchanging $(\eta, \iota) \leftrightarrow (\kappa, \lambda)$, $(\eta, \iota) \leftrightarrow (\mu, \nu)$, and $(\eta, \iota) \leftrightarrow (\xi, \varnothing)$, respectively. Equations for $S_{n,5}$, $S_{n,6}$, and $S_{n,7}$ are obtained from $S_{n,4}$ for $n = 4,5,6,7$ by interchanging $(\eta, \iota) \leftrightarrow (\kappa', \lambda')$, $(\eta', \iota') \leftrightarrow (\mu', \nu')$, and $(\eta', \iota') \leftrightarrow (\xi', \varnothing')$, respectively. Analogous statements hold for $\tilde{S}_{n,n'}$ with $n,n' = 4,5,6,7$. In this way, the curve configuration exhibits the action of the finite automorphism group determined by Kondo [16].

Figure 2 is then uniquely determined by the entries in Table 3. Using the explicit equations for the divisors, their intersection numbers in Table 3 are readily computed. In order to check that all smooth rational curves are given by Equation (4.33) one constructs the embeddings of the reducible fibers, i.e., the corresponding extended Dynkin diagrams determined in Theorem 4.9, into the graph given by Figure 2. In this way, the elliptic fibrations of Theorem 2.2 determine what rational curves are contained in the reducible fibers. On the graph in Figure 2 (and Figure 14) the action of the Nikulin involution is represented by a horizontal flip, exchanging the section and the two-torsion section in the alternate fibration. There is one way of embedding the reducible fibers of fibration (1) into the graph; it is depicted in Figure 3. For the pencils corresponding to $L_4, L_5, L_6, L_7$ and $L'_4, L'_5, L'_6, L'_7$, the embeddings for the reducible fibers of fibration (2) are depicted in Figure 4. For the latter Jacobian fibrations, there is a second embedding of the reducible fibers, related by the action of the Nikulin involution.

Next we determine the dual graph of all smooth rational curves and their intersection numbers for the K3 surfaces in Theorem 4.10 with Néron-Severi lattice $H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}$. In this situation one has the following rational curves:

$$a_1, a_2, a_3, b_1, b_2, b_3, L_1, L_2, L_3, L_4, L_5, L_6, L_7, R_1, R_3, R_n, S_n, S_n^\pm$$

with $n = 4,5,6,7$.

The lines $L_1, \ldots, L_7$ were determined in Section 4.2. The two sets $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ denote the curves appearing when resolving the rational double point singularities at $P_1$ and $P_2$, respectively; see Proposition 4.5. Equations for the remaining divisors are given in Appendix A.2. We have the following:

**Theorem 4.15.** For the K3 surface in Theorem 4.10 the dual graph of smooth rational curves is Figure 5 (and Figure 15). The complete list of multi-fold edges (2-fold, 4-fold, 6-fold) is given in Table 4.
For the pencils corresponding to each Jacobian elliptic fibration, related by the action of the Nikulin involution.

There is one way of embedding the reducible fibers of fibration (1) into the graph given by Figure 1. It is depicted in Figure 2, and is induced by the pencil $L_1(u,v)$. For the remaining fibrations, there are always two distinct embeddings of the reducible fibers for each Jacobian elliptic fibration, related by the action of the Nikulin involution. For the pencils corresponding to $L_4, L_5, L_6, L_7$, the embeddings for the reducible fibers

\[ \text{Figure 2. Rational curves for } \text{NS}(\mathcal{X}) = H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 5} \]

**Proof.** Divisors $S_n^+$ and $S_n^-$, as well as $\tilde{S}_n^+$ and $\tilde{S}_n^-$ for $n = 4, 5, 6, 7$ are interchanged by the sign change $\chi \mapsto -\chi$ with $4\gamma_\eta \mu \kappa \xi = 1 - \chi^2$. Here, the fact that the K3 surface is assumed to be general implies $\chi^2 \neq 0, 1$. Divisors $S_n^+$ and $\tilde{S}_n^+$ as well as $L_n$ and $R_n$ for $n = 4, 5, 6, 7$ are related by the action of the Nikulin involution in Proposition 4.7. Moreover, the elements within each set $\{L_4, L_5, L_6, L_7\}$, $\{R_4, R_5, R_6, R_7\}$, $\{S_n^+, S_n^-, S_6^+, S_7^+\}$, and $\{\tilde{S}_4^+, \tilde{S}_5^-, \tilde{S}_6^-, \tilde{S}_7^+\}$ are related by the symmetries in Lemma 4.6. That is, equations for $R_5$, $R_6$, and $R_7$ are obtained from $R_4$ by $(\eta, \iota) \leftrightarrow (\kappa, \lambda)$, $(\eta, \iota) \leftrightarrow (\kappa, \lambda)$, and $(\eta, \iota) \leftrightarrow (\xi, o)$, respectively. Similarly, $S_5^-$, $S_6^*$, and $S_7^*$ are obtained from $S_4^+$ by interchanging parameters according to $(\eta, \iota) \leftrightarrow (\kappa, \lambda)$, $(\eta, \iota) \leftrightarrow (\mu, \nu)$, and $(\eta, \iota) \leftrightarrow (\xi, o)$, respectively; in the same way, $\tilde{S}_4^+$, $\tilde{S}_5^-$, and $\tilde{S}_7^+$ are obtained from $\tilde{S}_4^+$. The equation for $R_3$ is obtained from $R_1$ by interchanging $(\gamma, \delta) \leftrightarrow (\varepsilon, \zeta)$. 

Next one constructs the embeddings of the reducible fibers, i.e., the corresponding extended Dynkin diagrams determined in Theorem 4.10, into the graph in Figure 5.
of fibration (2) are depicted in Figure 7. For the pencils $L_2^\pm (u,v)$, the embeddings for the reducible fibers of fibration (3) are shown in Figure 8.

Finally, we determine the dual graph of all smooth rational curves and their intersection numbers for the K3 surfaces in Theorem 4.13 with Néron-Severi lattice $H \oplus D_8(-1) \oplus A_1(-1)^3$. In this situation one finds the following rational curves:

$$a_1, \ldots, a_5, b_1, b_2, b_3, L_1, L_2, \tilde{L}_2, L_n, R_n, S_2, S_n, \tilde{S}_n, T_{m,n}, \tilde{T}_{m,n}$$

with $m = 1, 3, n = 4, 5, 6$.

The lines $L_1, L_2, \tilde{L}_2, L_4, L_5, L_6$ were already determined in Section 4.2. The two sets $\{a_1, a_2, a_3, a_4, a_5\}$ and $\{b_1, b_2, b_3\}$ denote the curves appearing when resolving the rational double point singularities at $P_1$ and $P_2$, respectively. Equations for the divisors are determined in Appendix A.3. We have the following:

**Theorem 4.16.** For the K3 surface in Theorem 4.13 the dual graph of smooth rational curves is Figure 9 (and Figure 16). The complete list of multi-fold edges (2-fold, 4-fold, 6-fold) is given in Table 5.

**Proof.** We follow the same strategy as in the proof of Theorem 4.15. For $\chi = 1$ the equations for $S_n^+$ with $n = 4, 5, 6$ in Equation (4.34) define divisors that will be denoted by $S_n$. The equations for $S_n^-$ become reducible combinations of $L_4, L_5, L_6$. Similarly,
the equations for $\tilde{S}_n^+$ define divisors that will be denoted by $\tilde{S}_n$, whereas the equations for $\tilde{S}_n^-$ become reducible. The same divisors are obtained if one sets $\chi = -1$ in the equations for $S_n^+$ and $\tilde{S}_n^-$. The curves $S_n$ and $\tilde{S}_n$ are related by the action of the Nikulin involution in Equation (4.20), so are the pairs of divisors $T_{1,4}$ and $\tilde{T}_{1,4}$. Equations for $T_{3,4}$ and $\tilde{T}_{3,4}$ are obtained from $T_{1,4}$ and $\tilde{T}_{1,4}$ by interchanging parameters according to $(\gamma, \delta) \leftrightarrow (\varepsilon, \zeta)$, respectively. Moreover, equations for $T_{m,5}$ and $\tilde{T}_{m,6}$ are obtained from $T_{m,4}$ and $\tilde{T}_{m,4}$ for $m = 1, 3$ by interchanging parameters according to $(\eta, \iota) \leftrightarrow (\kappa, \lambda)$ and $(\eta, \iota) \leftrightarrow (\mu, \nu)$, respectively. These divisors appeared above in connection with the pencils $C_n(u, v)$ for $n = 4, 5, 6$ in the proof of Theorem 4.13.

The next step is to construct the embeddings of the reducible fibers, i.e., the corresponding extended Dynkin diagrams determined in Theorem 4.13, into the graph
Figure 5. Rational curves for $\text{NS}(\mathcal{X}) = H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}$

Figure 6. Fibration with reducible fibers $\widetilde{D}_4 + \widetilde{A}_1^{\oplus 6}$ and sections

Figure 7. Fibrations with reducible fibers $\widetilde{D}_4 + \widetilde{D}_4 + \widetilde{A}_1^{\oplus 2}$ and section

given by Figure 9. There is one way of embedding the reducible fibers of fibration (1). It is depicted in Figure 10. For the remaining fibrations there are two embeddings in each case, related by the action of the Nikulin involution. For the pencils corresponding to $L_4, L_5, L_6$ and $L_7$, the embeddings for the reducible fibers of fibration (2) are depicted in Figure 11. For the pencil $L_2(u,v)$ the embedding for the reducible fibers
Thus, the polarizing divisor can be written as a linear combination

\[ \mathcal{H} = f F_{\text{std}} + l_2 L_2^+ + \sum_{i=4}^{7} l_i L_i + \sum_{i=1}^{3} \beta_i b_i + \alpha_1 a_1 + \alpha_3 a_3. \]
We use $\mathcal{H} \circ a_i = \mathcal{H} \circ b_i = 0$ for $i = 1, 2, 3$, $\mathcal{H} \circ L_1 = \mathcal{H} \circ L_2 = 1$, and $\mathcal{H} \circ L_j = 1$ for $j = 4, \ldots, 7$. We obtain a linear system of equations for the coefficients in Equation (4.38) whose unique solution is given by Equation (4.36). □
Figure 11. Fibrations with reducible fibers \( \bar{D}_4 + \bar{D}_6 + \bar{A}_1 \) and section.

Figure 12. Fibration with reducible fibers \( \bar{D}_8 + \bar{A}_1^3 \) and section.

Figure 13. Fibration with reducible fibers \( \bar{E}_7 + \bar{A}_1^4 \) and section.
Appendix A. Rational curves as complete intersections

A.1. Equations of rational curves for Picard rank 11. On the quartic hypersurface $Q$ in Equation (4.21) one has the following rational curves:

\[ a_1, a_2, a_3, b_1, R_0, R_{13}, L_n, R_n, L'_n, R'_n, S_{n,n'}, \tilde{S}_{n,n'} \text{ with } n, n' = 4, 5, 6, 7. \]  

The curves $R_*, R'_*, S_*, \tilde{S}_*$ are obtained from the singular complete intersections of higher (arithmetic) genus on the quartic surface $Q$. The two sets \{a_1, a_2, a_3\} and \{b_1\} will denote the curves appearing when resolving the rational double point singularities at $P_1$ and $P_2$, respectively; see Proposition 4.3. Equations for the lines $L_n, L'_n$ for $n = 4, 5, 6, 7$ were already determined in Section 4.2.

The following curves are given as complete intersections:

\[
R_0: \begin{cases} 
0 = W \\
0 = 2\rho Y^2Z^2 + 4X^2(c_2(1,0)Z^2 + 2e_3(1,0)XZ + 4d_4(1,0)X^2),
\end{cases}
\]
and

\[
R_{13}: \begin{cases} 
0 = WZc_2(2X,W) + \rho Ze_3(2X,W) + \rho d_4(2X,W) \\
0 = 2\rho Y^2 + c_2(2X,W),
\end{cases}
\]
and

\[
R_4: \begin{cases} 
0 = 2\eta X - i'W \\
0 = 2\eta^3(W - \rho Z)Y^2 - \eta c_2(\nu,\eta)W^2Z - e_3(\nu,\eta)W^3,
\end{cases}
\]
and

\[
R'_4: \begin{cases} 
0 = 2\eta' X - i'W \\
0 = 2(\eta')^4 WZc_2(2X,W) + (\eta')^2 c_2(\nu',\eta')W^2Z - \rho d_4(\nu',\eta')W^3.
\end{cases}
\]

Equations for $R_5, R_6,$ and $R_7$ are obtained from $R_4$ by interchanging $(\eta, i) \leftrightarrow (\kappa, \lambda), (\eta, i) \leftrightarrow (\mu, \nu),$ and $(\eta, i) \leftrightarrow (\xi, o),$ respectively. Similarly, the equations for $R'_5, R'_6,$ and $R'_7$ are obtained from $R'_4$ by interchanging $(\eta', i') \leftrightarrow (\kappa', \lambda'), (\eta', i') \leftrightarrow (\mu', \nu'),$ and $(\eta', i') \leftrightarrow (\xi', o'),$ respectively.

We also have the following complete intersection curve:

\[
S_{4,4}: \begin{cases} 
0 = \eta'((W - \rho Z) - \eta(2\eta' X - \rho i'Z)) \\
0 = 2\rho\eta' Y^2 + (\eta')^2(2X - W)c_2(2X,W) \frac{2\eta' X - i'W}{2\eta' X - i'W} + \frac{\rho^2(\eta' - \eta')e_3(2X,W)}{(2\eta' X - i'W)(2\eta' X - i'W)},
\end{cases}
\]
where the second equation is a homogeneous polynomial of degree three. Equations for $S_{5,4}, S_{6,4},$ and $S_{7,4}$ are obtained from $S_{4,4}$ by interchanging $(\eta, i) \leftrightarrow (\kappa, \lambda), (\eta, i) \leftrightarrow (\mu, \nu),$ and $(\eta, i) \leftrightarrow (\xi, o),$ respectively. The equations for $S_{n,5}, S_{n,6},$ and $S_{n,7}$ are obtained from $S_{n,4}$ for $n = 4, 5, 6, 7$ by interchanging $(\eta', i') \leftrightarrow (\kappa', \lambda'), (\eta', i') \leftrightarrow (\mu', \nu'),$ and $(\eta', i') \leftrightarrow (\xi', o'),$ respectively. This procedure generates the 16 rational curves $S_{4,4}, S_{4,5}, \ldots, S_{7,6}, S_{7,7}.$
Applying the Nikulin involution in Equation (4.4), we obtain from $S_{4,4}$ the complete intersection curve:

\[
\begin{aligned}
0 &= \frac{1}{2\eta X - \tau W}(\eta' WZ_{2}(2X, W) + \rho \eta' Z_{3}(2X, W) \\
&+ \rho \eta' (2\eta X - \tau Z)(2\eta X - \tau W)_{d_{3}}(2X, W)) \\
0 &= 2\rho \eta' Y^{2} + \left(\pm \right)\left(2\eta X - \tau W\right)_{d_{2}}(2X, W) \\
&+ \rho \eta' (\eta' - \eta')_{d_{3}}(2X, W) + \rho^{2} (\eta' - \eta')_{d_{4}}(2X, W)
\end{aligned}
\]

where the first and second equation are again homogeneous polynomials. Equations for $\tilde{S}_{5,4}, \ldots, \tilde{S}_{7,7}$ are obtained from $\tilde{S}_{4,4}$ in the same way as before.

A.2. Equations of rational curves for Picard rank 12. On the quartic hypersurface $Q$ in Equation (4.22) one has the following curves:

\[
\begin{aligned}
a_{1}, a_{2}, a_{3}, b_{1}, b_{2}, b_{3}, L_{1}, L_{2}, L_{n}, R_{1}, R_{3}, R_{n}, S_{n}, \tilde{S}_{n}, S_{n}^{+} \text{ with } n = 4, 5, 6, 7.
\end{aligned}
\]

Equations for the lines $L_{1}, L_{2}, L_{n}$ with $n = 4, 5, 6, 7$ were already determined in Section 4.2. Here, we use the polynomials

\[
\begin{aligned}
c_{2}(u, v) &= (\gamma u - \delta v)(\epsilon u - \zeta v), \\
c_{3}(u, v) &= u^{3} - 3\alpha u v^{2} - 2\beta v^{3}, \\
d_{4}(u, v) &= (\eta u - \nu v) / (\kappa u - \lambda v) / (\mu u - \nu v) / (\xi u - \eta v),
\end{aligned}
\]

and the parameter $\chi$ with $4\gamma \epsilon \eta \kappa \mu \xi = 1 - \chi^{2}$. The following curves are given as complete intersections:

\[
\begin{aligned}
R_{1}^{+}: \left\{ \begin{array}{l}
0 = 2\gamma X - \delta W \\
0 = 2\gamma^{4} Y^{2} Z - \gamma e_{3}(\delta, \gamma) W^{2} Z - d_{4}(\delta, \gamma) W^{3},
\end{array} \right.
\end{aligned}
\]

and

\[
\begin{aligned}
R_{1}^{-}: \left\{ \begin{array}{l}
0 = 2\eta X - \tau W \\
0 = 2\eta^{2} Y^{2} - \eta c_{2}(\nu, \eta) W Z - e_{3}(\nu, \eta) W^{2}.
\end{array} \right.
\end{aligned}
\]

An equation for $R_{3}$ is obtained from $R_{1}$ by interchanging $(\gamma, \delta) \leftrightarrow (\epsilon, \zeta)$, and $R_{5}$, $R_{6}$, and $R_{7}$ are obtained from $R_{4}$ by $(\eta, \iota) \leftrightarrow (\kappa, \lambda)$, $(\eta, \iota) \leftrightarrow (\mu, \nu)$, and $(\eta, \iota) \leftrightarrow (\xi, \eta)$, respectively. We also have the complete intersection

\[
\begin{aligned}
S_{4}^{+}: \left\{ \begin{array}{l}
0 = -\iota(1 \pm \chi) W + 2\eta(1 \pm \chi) X + 2\gamma \epsilon \eta Z \\
0 = 4\gamma \epsilon \eta(1 \pm \chi) Y^{2} + \left(1 \pm \chi\right)^{2}(2\eta X - \tau W)_{c_{2}}(2X, W) \\
&- 2\gamma \epsilon \eta(1 \pm \chi) e_{3}(2X, W) + 4\gamma^{2} \epsilon^{2} \eta^{2} d_{4}(2X, W) / (2\eta X - \tau W),
\end{array} \right.
\end{aligned}
\]

where the second equation is a homogeneous polynomial of degree two. Applying the Nikulin involution in Equation (4.20), we obtain the complete intersection

\[
\begin{aligned}
\tilde{S}_{4}^{+}: \left\{ \begin{array}{l}
0 = (1 \pm \chi) c_{2}(2X, W) Z + 2\gamma \epsilon \eta d_{4}(2X, W) / (2\eta X - \tau W) \\
0 = 4\gamma \epsilon \eta(1 \pm \chi) Y^{2} + \left(1 \pm \chi\right)^{2}(2\eta X - \tau W)_{c_{2}}(2X, W) \\
&- 2\gamma \epsilon \eta(1 \pm \chi) e_{3}(2X, W) + 4\gamma^{2} \epsilon^{2} \eta^{2} d_{4}(2X, W) / (2\eta X - \tau W). 
\end{array} \right.
\end{aligned}
\]

Equations for $S_{4}^{+}, S_{6}^{+}$, and $S_{7}^{+}$ are obtained from $S_{4}^{+}$ by interchanging parameters according to $(\eta, \iota) \leftrightarrow (\kappa, \lambda)$, $(\eta, \iota) \leftrightarrow (\mu, \nu)$, and $(\eta, \iota) \leftrightarrow (\xi, \eta)$, respectively; in the same way, $\tilde{S}_{5}^{+}, \tilde{S}_{6}^{+}$, and $\tilde{S}_{7}^{+}$ are obtained from $\tilde{S}_{4}^{+}$. 
A.3. Equations of rational curves for Picard rank 13. On the quartic hypersurface $Q$ in Equation (4.22) with $(\xi, o) = (0, 1)$ and $\chi^2 = 1$ one has the following rational curves:

\begin{equation}
\begin{aligned}
a_1, \ldots, a_5, b_1, b_2, b_3, L_1, L_2, L_2, L_n, R_n, S_1, S_2, S_n, \tilde{S}_2, \tilde{S}_n, T_{m,n}, \tilde{T}_{m,n} \text{ with } m = 1, 3, n = 4, 5, 6.
\end{aligned}
\end{equation}

Equations for the lines $L_1, L_2, \tilde{L}_2, L_n$ for $n = 4, 5, 6$ were determined in Section 4.2. For $(\xi, o) = (0, 1)$ the curves $R_1$ and $R_3$ in Section A.2 restrict to curves of the same degree which we will denote by the same symbol. For $(\xi, o) = (0, 1)$ we use the polynomials

\begin{equation}
\begin{aligned}
c_2(u, v) &= (\gamma u - \delta v)(\epsilon u - \zeta v), \\
c_3(u, v) &= u^3 - 3\alpha uv^2 - 2\beta v^3,
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
d_4(u, v) &= - (\eta u - \nu v)(\kappa u - \lambda v)(\mu u - \nu v)v.
\end{aligned}
\end{equation}

We have the complete intersection

\begin{equation}
\begin{aligned}
S_3: \quad & 0 = 2\eta X - iW + \gamma \epsilon \eta Z \\
& 0 = 2\gamma \epsilon \eta Y^2 + \frac{1}{W}\left((2\eta X - iW)c_2(2X, W) - \gamma \epsilon \eta e_3(2X, W) + \gamma^2 \epsilon^2 \eta d_4(2X, W)\right),
\end{aligned}
\end{equation}

where the second equation is a homogeneous polynomial of degree two. Applying the Nikulin involution in Equation (4.20), we obtain the complete intersection

\begin{equation}
\begin{aligned}
\tilde{S}_3: \quad & 0 = c_2(2X, W)Z + \frac{\gamma \epsilon \eta d_4(2X, W)}{2\epsilon \gamma X - iW} \\
& 0 = 2\gamma \epsilon \eta Y^2 + \frac{1}{W}\left((2\eta X - iW)c_2(2X, W) - \gamma \epsilon \eta e_3(2X, W) + \gamma^2 \epsilon^2 \eta d_4(2X, W)\right).
\end{aligned}
\end{equation}

Equations for $S_5$ and $S_6$ are obtained from $S_4$ by interchanging parameters according to $(\eta, \iota) \leftrightarrow (\kappa, \lambda)$ and $(\eta, \iota) \leftrightarrow (\mu, \nu)$, respectively; in the same way, $\tilde{S}_5$ and $\tilde{S}_6$ are obtained from $\tilde{S}_4$. Moreover, curves $S_2$, $\tilde{S}_2$ are given as the following complete intersections:

\begin{equation}
\begin{aligned}
S_2: \quad & 0 = \eta \kappa \mu W - Z \\
& 0 = \eta^2 \kappa^2 \mu^2 c_2(2X, W) + \eta \kappa \mu e_3(2X, W)W + d_4(2X, W) - 2\eta \kappa \mu Y^2,
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
\tilde{S}_2: \quad & 0 = \eta \kappa \mu c_2(2X, W)Z - \frac{d_4(2X, W)}{W} \\
& 0 = \eta^2 \kappa^2 \mu^2 c_2(2X, W) + \frac{2\eta \kappa \mu e_3(2X, W)}{W} + \frac{d_4(2X, W)}{W^2(2\eta X - iW)} - 2\eta \kappa \mu Y^2,
\end{aligned}
\end{equation}

where all equations are polynomial by construction, and the divisors are related by the action of the Nikulin involution. We also have the following complete intersections:

\begin{equation}
\begin{aligned}
\tilde{T}_{1.4}: \quad & 0 = \frac{(2\gamma \kappa - \delta W)Z - \gamma \kappa \mu (2\eta X - iW)W}{2\gamma X - iW} \\
& 0 = \frac{\gamma \kappa^2 \mu^2 (2\eta X - iW)c_2(2X, W)}{2\gamma X - iW} + \frac{\gamma \kappa \mu e_3(2X, W)}{W^2(2\eta X - iW)} - 2\gamma \kappa \mu Y^2,
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
T_{1.4}: \quad & 0 = \gamma \kappa \mu \frac{(2\epsilon X - \zeta W)Z - (2\kappa X - \lambda W)(2\mu X - \nu W)}{W(2\epsilon \kappa X - \zeta W)} \\
& 0 = \frac{(2\epsilon X - \zeta W)(2\kappa X - \lambda W)c_2(2X, W)}{W(2\epsilon \kappa X - \zeta W)} + \frac{\gamma \mu e_3(2X, W)}{W^2(2\epsilon \kappa X - \zeta W)} - 2\gamma \kappa \mu Y^2,
\end{aligned}
\end{equation}

and the corresponding divisors are related by the action of the Nikulin involution. Equations for $T_{3.4}$ and $\tilde{T}_{3.4}$ are obtained from $T_{1.4}$ and $\tilde{T}_{1.4}$ by interchanging parameters according to $(\gamma, \delta) \leftrightarrow (\epsilon, \zeta)$, respectively. Moreover, equations for $T_{m,5}$ and $\tilde{T}_{m,6}$ are
obtained from $T_{m,4}$ and $\tilde{T}_{m,4}$ for $m = 1, 3$ by interchanging parameters according to
$(\eta, \iota) \leftrightarrow (\kappa, \lambda)$ and $(\eta, \iota) \leftrightarrow (\mu, \nu)$, respectively.
Figure 14. Dual graph of rational curves with 1-fold (thin, black) and some 2-fold (thick, color) edges for $\text{NS}(\mathcal{X}) = H \oplus D_4(-1) \oplus A_1(-1)^5$. 
Table 3. Intersection numbers of rational curves for $\text{NS}(\Omega) = H \oplus D_1(-1) \oplus D_2(-1)^{\oplus 2}$.
Figure 15. Dual graph of rational curves with 1-fold (thin, black) and some 2-fold (thick, color) edges for $\text{NS}(\mathcal{X}) = H \oplus D_6(-1) \oplus A_1(-1)^{\otimes 4}$
Table 4. Intersection numbers of rational curves for $\text{NS}(\mathcal{X}) = H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}$
Figure 16. Dual graph of rational curves with 1-fold (thin, black) and some 2-fold (thick, color) edges for $\text{NS}(\mathcal{X}) = H \oplus D_8(-1) \oplus A_1(-1)^{63}$
Table 5. Intersection numbers of rational curves for NS(\(\mathcal{X}\)) = \(H \oplus D_8(-1) \oplus A_1(-1)^{\otimes 3}\)
References

[1] Chiara Camere and Alice Garbagnati, *On certain isogenies between K3 surfaces*, Trans. Amer. Math. Soc. 373 (2020), no. 4, 2913–2931. MR4069236
[2] Adiran Clingher and Andreas Malmendier, *On K3 surfaces of Picard rank 14*, available at arXiv:2009.09635.
[3] ________, *On Picard lattices of Jacobian elliptic K3 surfaces*, available at arXiv:2109.01929.
[4] ________, *On the duality of F-theory and the CHL string in eight dimensions*, available at arXiv:2111.15125.
[5] Adrian Clingher and Charles F. Doran, *Modular invariants for lattice polarized K3 surfaces*, Michigan Math. J. 55 (2007), no. 2, 355–393. MR2369941 (2009a:14049)
[6] ________, *Note on a geometric isogeny of K3 surfaces*, Int. Math. Res. Not. IMRN 16 (2011), 3657–3687. MR2824841 (2012f:14072)
[7] ________, *Lattice polarized K3 surfaces and Siegel modular forms*, Adv. Math. 231 (2012), no. 1, 172–212. MR2935386
[8] Adrian Clingher, Thomas Hill, and Andreas Malmendier, *Jacobian elliptic fibrations on a special family of K3 surfaces of Picard rank sixteen*, arXiv:1908.09578 [math.AG] (2019).
[9] ________, *The duality between F-theory and the heterotic string in D = 8 with two Wilson lines*, Lett. Math. Phys. 110 (2020), no. 11, 3081–3104. MR4160930
[10] Adrian Clingher and Andreas Malmendier, *Nikulin involutions and the CHL string*, Comm. Math. Phys. 370 (2019), no. 3, 959–994. MR3995925
[11] Igor Dolgachev, *Integral quadratic forms: applications to algebraic geometry (after V. Nikulin)*, Bourbaki seminar, Vol. 1982/83, 1983, pp. 251–278. MR728992
[12] Igor V. Dolgachev, *Mirror symmetry for lattice polarized K3 surfaces*, J. Math. Sci. 81 (1996), no. 3, 2599–2630. Algebraic geometry, 4. MR1420220 (97i:14024)
[13] Dino Festi and Davide Cesare Veniani, *Counting elliptic fibrations on K3 surfaces*, 2020, arXiv:2102.09411.
[14] Patricio Gallardo, Jesus Martinez-Garcia, and Zheng Zhang, *Compactifications of the moduli space of plane quartics and two lines*, Eur. J. Math. 4 (2018), no. 3, 1000–1034. MR3851127
[15] Hiroshi Inose, *Defining equations of singular K3 surfaces and a notion of isogeny*, Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977), 1978, pp. 495–502. MR578868
[16] Shigeyuki Kondo, *Algebraic K3 surfaces with finite automorphism groups*, Nagoya Math. J. 116 (1989), 1–15. MR1029967
[17] ________, *Automorphisms of algebraic K3 surfaces which act trivially on Picard groups*, J. Math. Soc. Japan 44 (1992), no. 1, 75–98. MR1139659
[18] ________, *The moduli space of 8 points of P^1 and automorphic forms*, Algebraic geometry, 2007, pp. 89–106. MR2296434
[19] Abhinav Kumar, *K3 surfaces associated with curves of genus two*, Int. Math. Res. Not. IMRN 6 (2008), Art. ID rnm165, 26. MR2427457 (2009d:14044)
[20] Han-Bom Moon and Luca Schaffler, *KSBA compactification of the moduli space of K3 surfaces with a purely non-symplectic automorphism of order four*, Proc. Edinb. Math. Soc. (2) 64 (2021), no. 1, 99–127. MR4249842
[21] David R. Morrison, *On K3 surfaces with large Picard number*, Invent. Math. 75 (1984), no. 1, 105–121. MR728142 (85j:14071)
[22] V. V. Nikulin, *An analogue of the Torelli theorem for Kummer surfaces of Jacobians*, Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974), 22–41. MR0357410
[23] ________, *Kummer surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), no. 2, 278–293, 471. MR0429917
[24] ________, *Finite groups of automorphisms of Kählerian K3 surfaces*, Trudy Moskov. Mat. Obshch. 38 (1979), 75–137. MR544937
ON JACOBIAN ELLIPTIC K3 SURFACES WITH 2-TORSION

[25] Integer symmetric bilinear forms and some of their geometric applications, Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), no. 1, 111–177, 238. MR525944

[26] Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections. Algebrogeometric applications, Journal of Soviet Mathematics 22 (1983), no. 4, 1401–1475.

[27] Elliptic fibrations on K3 surfaces, Proc. Edinb. Math. Soc. (2) 57 (2014), no. 1, 253–267. MR3165023

[28] Ulf Persson, Double sextics and singular K-3 surfaces, Algebraic geometry, Sitges (Barcelona), 1983, 1985, pp. 262–328. MR805337

[29] I. I. Pjatecki-Sapiro and I. R. Safarevic, Torelli’s theorem for algebraic surfaces of type K3, Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971), 530–572. MR0284440

[30] HW Richmond, On the uninodal quartic curve, Proceedings of the Edinburgh Mathematical Society 1 (1927), no. 1, 31–38.

[31] Ichiro Shimada, On elliptic K3 surfaces, Michigan Math. J. 47 (2000), no. 3, 423–446. MR1813537

[32] Tetsuji Shioda, Kummer sandwich theorem of certain elliptic K3 surfaces, Proc. Japan Acad. Ser. A Math. Sci. 82 (2006), no. 8, 137–140. MR2279280

[33] Hans Sterk, Finiteness results for algebraic K3 surfaces, Math. Z. 189 (1985), no. 4, 507–513. MR786280

[34] Ravi Vakil, Twelve points on the projective line, branched covers, and rational elliptic fibrations, Math. Ann. 320 (2001), no. 1, 33–54. MR1835061

[35] Bert van Geemen and Alessandra Sarti, Nikulin involutions on K3 surfaces, Math. Z. 255 (2007), no. 4, 731–753. MR2274533

[36] E. B. Vinberg, On automorphic forms on symmetric domains of type IV, Uspekhi Mat. Nauk 65 (2010), no. 3(393), 193–194. MR2682724

[37] Some free algebras of automorphic forms on symmetric domains of type IV, Transform. Groups 15 (2010), no. 3, 701–741. MR2718942

[38] On the algebra of Siegel modular forms of genus 2, Trans. Moscow Math. Soc. (2013), 1–13. MR3235787

Department of Mathematics and Statistics, University of Missouri - St. Louis, St. Louis, MO 63121

Email address: clinghera@umsl.edu

Department of Mathematics & Statistics, Utah State University, Logan, UT 84322
Current address: Department of Mathematics, University of Connecticut, Storrs, CT 06269
Email address: andreas.malmendier@usu.edu
This figure "csvsimple-title.png" is available in "png" format from:

http://arxiv.org/ps/2206.00269v1