ASYMPTOTIC ANALYSIS FOR A VERY FAST DIFFUSION EQUATION ARISING FROM THE 1D QUANTIZATION PROBLEM

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(Communicated by José A. Carrillo)

Abstract. In this paper we study the asymptotic behavior of a very fast diffusion PDE in 1D with periodic boundary conditions. This equation is motivated by the gradient flow approach to the problem of quantization of measures introduced in [3]. We prove exponential convergence to equilibrium under minimal assumptions on the data, and we also provide sufficient conditions for $W^2$-stability of solutions.

1. Introduction. During the last years, asymptotic analysis for solutions of nonlinear parabolic equations have attracted a lot of attention, also in connection with gradient flows and entropy methods.

The aim of the present paper is to investigate the dynamics of the PDE

$$
\begin{cases}
\partial_t f(t,x) = -r \partial_x \left( f(t,x) \partial_x \left( \frac{\rho(x)}{f(t,x)^{r+1}} \right) \right) & \text{in } (0,\infty) \times [0,1], \\
f(t,0) = f(t,1) & \text{on } (0,\infty),
\end{cases}
$$

(1)

where $r > 1$, and $\rho > 0$ and $f(t,\cdot)$ are probability densities on $[0,1]$ with periodic boundary conditions. When $\rho = 1$, this equation takes the form

$$
\partial_t f = -(r+1) \partial_x^2 (f^{-r}),
$$

(2)

which belongs to the general class of fast diffusion equations

$$
\partial_t u = \text{div}(u^{m-1} \nabla u), \quad m < 1.
$$

We recall that, when the problem is set on the whole space $\mathbb{R}^n$, the value of $m$ plays a crucial role: solutions are smooth and exist for all times if $m > m_c := (n-2)/2$, while they vanish in finite time if $m \leq m_c$ (the existence of such an extinction time motivates the name “very fast diffusion equations”). There is a huge literature on the subject, and we refer the interested reader to the monograph [16] for a comprehensive overview and more references.

Our case corresponds to the range $m = -r < -1$. It is interesting to point out that (2) set on the whole space $\mathbb{R}$ or with zero Dirichlet boundary conditions has no solutions, since all the mass instantaneously disappear [14, Theorem 3.1] (see also

2010 Mathematics Subject Classification. Primary: 35K15, 35K65; Secondary: 70F45.

Key words and phrases. Very fast diffusion, quantization of measures, gradient flows, asymptotic.
for related results). It is therefore crucial that in our setting the equation has periodic boundary conditions, so that the mass is preserved.

We observe that this kind of equations has the property of diffusing extremely fast. In particular, if $f_0$ is a non-negative and bounded initial datum, the solution becomes instantaneously positive. As we are only interested in the long time behaviour of solutions, to simplify the presentation we will only consider initial data that are bounded away from zero and infinity.

Our equation (1) is motivated by the so-called quantization problem. The term quantization refers to the process of finding the best approximation of a $d$-dimensional probability distribution by a convex combination of a finite number $N$ of Dirac masses. This problem arises in several contexts and has applications in information theory (signal compression), numerical integration, and mathematical models in economics (optimal location of service centers). In order to explain the meaning of the equation (1), we now briefly recall the gradient flow approach to the quantization problem introduced in [3], and further investigated in [4].

Given $r \geq 1$, consider $\mu = \rho(x) \, dx$ a probability measure on an open set $\Omega \subset \mathbb{R}^n$. Given $N$ points $x^1, \ldots, x^N \in \Omega$, one wants to find the best approximation of $\mu$, in the Wasserstein distance $W_r$, by a convex combination of Dirac masses centered at $x^1, \ldots, x^N$. Hence one minimizes

$$
\inf \left\{ W_r \left( \sum_i m_i \delta_{x^i}, \mu \right)^r : m_1, \ldots, m_N \geq 0, \sum_i m_i = 1 \right\},
$$

with

$$W_r(\nu_1, \nu_2) := \inf \left\{ \left( \int_{\Omega \times \Omega} |x - y|^r \, d\gamma(x, y) \right)^{1/r} : (\pi_1)_\# \gamma = \nu_1, (\pi_2)_\# \gamma = \nu_2 \right\},$$

where $\gamma$ varies among all probability measures on $\Omega \times \Omega$, and $\pi_i : \Omega \times \Omega \to \Omega$ ($i = 1, 2$) denotes the canonical projection onto the $i$-th factor. See [1, 17] for more details on Wasserstein distances.

As explained in [7, Chapter 1, Lemmas 3.1 and 3.4], this problem is equivalent to minimizing the functional

$$F_{N,r}(x^1, \ldots, x^N) := \int_{\Omega} \min_{1 \leq i \leq N} |x^i - y|^r \, d\mu(y).$$

To find a minimizer to this function, in [3] the authors introduce a dynamical approach where they study the dynamics of the gradient flow induced by $F_{N,r}$. Since the main goal is to understand the structure of minimizers in the limit as $N$ tends to infinity, in [3, Section 1] and in [4, Sections 2 and 3] the authors are able to find a formula for the “limit” of $F_{N,r}$ when $N \to \infty$.

As shown in [3], when $n = 1$ this limit is given by the functional

$$\mathcal{F}[X] := \int_0^1 |\partial_\theta X|^{r+1} \rho(X) \, d\theta,$$

defined on periodic maps $X : [0, 1] \to [0, 1]$, and its $L^2$-gradient flow is given by the following non-linear parabolic equation

$$\partial_t X = (r + 1) \partial_\theta (\rho(X) |\partial_\theta X|^{r-1} \partial_\theta X) - \rho'(X) |\partial_\theta X|^{r+1},$$

coupled with the Dirichlet boundary condition. This equation provides a Lagrangian description of the evolution of our system of particles in the limit $N \to \infty$. We can
also study the Eulerian picture for (3). Indeed, if we denote by \( f(t, x) \) the image of the Lebesgue measure through the map \( X \), i.e.,

\[
f(t, x) dx = X(t, \theta) dy d\theta,
\]

then the PDE satisfied by \( f \) takes the form (see [1])

\[
\partial_t f(t, x) = -r \partial_x \left( f(t, x) \partial_x \left( \frac{\rho(x)}{f(t, x)^{r+1}} \right) \right)
\]

with periodic boundary conditions, and in view of the results in [7, 3] we expect the following long time behavior:

\[
f(t, x) \rightarrow \gamma \rho(x)^{1/(r+1)}(x) \quad \text{as} \quad t \rightarrow \infty,
\]

where \( \gamma := \frac{1}{\int_0^1 \rho(y)^{1/(r+1)} dy} \).

More precisely, the results in [3] show the validity of the limit only when \( r = 2 \) and under the assumption that \( \rho \) is close to 1 in \( C^2 \). The goal here is to generalize and improve this result.

Our starting point for studying the asymptotic behavior of (1) is the observation that this equation can be seen as the gradient flow of the functional

\[
\mathcal{F}_\rho[f] := \int_0^1 \frac{\rho(x)}{f(x)^r} dx.
\]

with respect to the \( W_2 \) distance.

In a first step, by exploiting a modulated \( L^2 \) energy method, we obtain exponential convergence to equilibrium under minimal assumptions on the density \( \rho \). Then, we investigate the displacement convexity of the functional \( F_\rho \). Notice that, as we shall prove in Proposition 1 below, if \( \rho \) and \( f(0) \) are bounded away from zero, then \( f(t) \) remains uniformly away from zero for all \( t \geq 0 \). In particular (1) is uniformly parabolic, and \( f(t) \) is smooth if \( \rho \) is so.

Since our focus is on the asymptotic behavior, we shall assume that \( \rho \) is of class \( C^{2,\alpha} \) for some \( \alpha \in (0, 1) \), so that parabolic regularity theory ensures that \( f(t) \) is of class \( C^{2,\alpha} \) for all times, hence \( f \) is a classical solution. However our results are independent of the smoothness of \( \rho \) and can be thought as a priori estimates. In particular, one could extend them to the setting of weak solutions by using the general theory of minimizing movements in [2] (see also [5]). Since our main goal is to understand the general asymptotic properties of the equation (1) we shall not investigate this here, and this will be the focus of a future work [8].

Before stating our results, we make an important comment: whenever we say that \( \rho : [0, 1] \to \mathbb{R} \) is a periodic function of class \( C^{2,\alpha} \), we mean that \( \rho \) is \( C^{2,\alpha} \) when seen as a function on the 1-dimensional torus. In particular the values of \( \rho, \partial_x \rho, \partial_{xx} \rho \) are the same at 0 and 1.

Our first result is the following:

**Theorem 1.1.** Let \( \lambda \in (0, 1) \), and assume that \( \rho : [0, 1] \to [\lambda, 1/\lambda] \) is a periodic function of class \( C^{2,\alpha} \) for some \( \alpha \in (0, 1) \). Let \( f(0, \cdot) : [0, 1] \to \mathbb{R} \) satisfy \( 0 < a_1 \leq f(0, \cdot) \leq A_1 \), and let \( f \) solve (1). Then

\[
a := a_1 \lambda^{2/(r+1)} \leq f(t, x) \leq \frac{A_1}{\lambda^{2/(r+1)}} =: A \quad \text{for all} \quad t \geq 0, \quad x \in [0, 1],
\]

and there exist positive constants \( c_0, c_0 \) depending only on \( \lambda, a_1, \) and \( A_1 \), such that

\[
\| f(t) - \gamma \rho(x)^{1/(r+1)} \|_{L^2([0,1])} \leq c_0 \ e^{-c_0 t} \quad \text{for all} \quad t \geq 0.
\]
The result above shows the exponential convergence to equilibrium with a rate independent of the smoothness of \( \rho \). However, it does not say anything about stability of solutions. For this, we investigate the convexity of the functional \( F \) with respect to the 2-Wasserstein distance of the smoothness of \( \rho \) solutions in stability of solutions. For this, we investigate the convexity of the functional \( F \).

\[ \text{Proof.} \]

It is enough to prove the bound

\[ \parallel \rho' \parallel_\infty + \parallel \rho'' \parallel_\infty \leq C \]

such that

\[ \text{Theorem 1.2. Let } \lambda \in (0, 1], \text{ and assume that } \rho : [0, 1] \to [\lambda, 1/\lambda] \text{ is a periodic function of class } C^{2,\alpha} \text{ for some } \alpha \in (0, 1). \text{ Let } f_1(0, \cdot), f_2(0, \cdot) : [0, 1] \to \mathbb{R} \text{ satisfy } 0 < a_1 \leq f_1(0, \cdot), f_2(0, \cdot) \leq A_1, \text{ let } f_1, f_2 \text{ solve (1), and let } a, A > 0 \text{ be as in Theorem 1.1. Assume that } \parallel \rho' \parallel_\infty \leq \eta_1 \text{ and } \parallel \rho'' \parallel_\infty \leq \eta_2 \text{ for some } \eta_1, \eta_2 > 0, \text{ and define}
\]

\[ \mu := \left( \frac{r(r + 1)\lambda}{A^r} - \frac{2\eta_1^2(r + 1)A^r}{r\lambda a^{2r}} - \frac{\eta_2}{a^r} \right) \]

Then

\[ W_2(f_1(t), f_2(t)) \leq e^{-\min(\mu/A, \mu/a)t} W_2(f_1(0), f_2(0)) \quad \text{for all } t \geq 0. \]

In particular, if \( \eta_1 \) and \( \eta_2 \) are small enough then \( \mu > 0 \) and solutions converge exponentially fast to equilibrium.

The arguments used to prove Theorem 1.1 is very general, and could be applied also to the n-dimensional version of (1). On the other hand, Theorem 1.2 holds only in one dimension. Indeed, in the proof we shall need that the space of probability densities bounded from below by a constant is convex with respect to Wasserstein geodesics, and this is true in dimension one (see Lemma 4.1 below), while it is false in higher dimension (this is a consequence, for instance, of the results in [12]).

In the next section we shall first show some preliminary results, and then prove our main Theorems 1.1 and 1.2.

2. Maximum principle. The goal of this section is to prove a maximum-type principle for (1) which shows that, if \( \rho \) and \( f(0) \) are bounded away from zero, then \( f(t) \) remains uniformly away from zero for all \( t \geq 0 \). In particular (1) is uniformly parabolic, and \( f(t) \) is smooth if \( \rho \) is so. Note that the following proposition corresponds to the first part of Theorem 1.1.

\[ \text{Proposition 1. Let } \lambda \in (0, 1], \text{ and assume that } \rho : [0, 1] \to [\lambda, 1/\lambda] \text{ is periodic and of class } C^{k,\alpha} \text{ for some } k \geq 0 \text{ and } \alpha \in (0, 1). \text{ Let } f(0, \cdot) : [0, 1] \to \mathbb{R} \text{ be a periodic function of class } C^{k,\alpha} \text{ satisfying } 0 < a_1 \leq f(0, \cdot) \leq A_1, \text{ and let } f \geq 0 \text{ solve (1) with periodic boundary conditions. Then}
\]

\[ \lambda^{2/(r+1)}a_1 \leq f(t, x) \leq \frac{A_1}{\lambda^{2/(r+1)}} \quad \text{for all } t \geq 0, \]

\[ f(t, \cdot) \text{ is of class } C^{k,\alpha} \text{ for all } t \geq 0, \text{ and there exists a constant } C, \text{ depending only on } \lambda, \parallel \rho \parallel_{C^{k,\alpha}}, k, \alpha, a_1, \text{ and } A_1, \text{ such that } \parallel f(t, \cdot) \parallel_{C^{k,\alpha}([0,1])} \leq C, \text{ for all } t \geq 0. \]

\[ \text{Proof.} \]

It is enough to prove the bound

\[ \lambda^{2/(r+1)}a_1 \leq f(t, x) \leq \frac{A_1}{\lambda^{2/(r+1)}} \quad \text{for all } t \geq 0, \]

since then, once these bounds are proved, the rest of the proposition follows by standard parabolic regularity.
To prove the result, we set
\[ m(x) := \rho(x)^{1/(r+1)}, \quad u(t, x) := \frac{f(t, x)}{m(x)}. \]

With these new unknowns (1) becomes
\[ \partial_t u = - \frac{r + 1}{m} \partial_x \left( m \partial_x \left( \frac{1}{u^r} \right) \right) \quad \text{on } [0, \infty) \times [0, 1] \]
with periodic boundary conditions. The advantage of this form is that constants are solutions and we can prove a comparison principle with them. More precisely, we set \( c_0 := \lambda^{1/(r+1)} a_1 \) and \( C_0 := \frac{A}{\lambda^{1/(r+1)}}. \)

Recalling the notation \( s_+ = \max\{s, 0\} \) and \( s_- = \max\{-s, 0\} \), we claim that the maps
\[ t \mapsto \int_0^1 (u(t, x) - c_0)_- m \, dx \quad \text{and} \quad t \mapsto \int_0^1 (u(t, x) - C_0)_+ m \, dx \]
are nonincreasing functions. Since \( u(0, x) := \frac{f(0, x)}{m(x)}, a_1 \leq f(0) \leq A_1 \), and \( \lambda^{1/(r+1)} \leq m \leq \lambda^{-1/(r+1)} \), it follows that
\[ \int_0^1 (u(0, x) - c_0)_- m \, dx = \int_0^1 (u(0, x) - C_0)_+ m \, dx = 0. \]

Hence, thanks to the claim
\[ \int_0^1 (u(t, x) - c_0)_- m \, dx = \int_0^1 (u(t, x) - C_0)_+ m \, dx = 0 \quad \forall t \geq 0, \]
therefore \( c_0 \leq u(t, x) \leq C_0 \) for all times. Recalling that \( u(t, x) = \frac{f(t, x)}{m(x)} \) and that \( \lambda^{1/(r+1)} \leq m(x) \leq \lambda^{-1/(r+1)} \), this proves the result. Hence, we only need to prove the claim.

To this aim, we only show that
\[ t \mapsto \int_0^1 (u(t, x) - c_0)_- m \, dx \]
is nonincreasing (the other statement being analogous).

Since constants are solutions of (6), it holds
\[ \partial_t (u - c_0) = - \frac{r + 1}{m} \partial_x \left( m \partial_x \left( \frac{1}{u^r} - \frac{1}{c_0^r} \right) \right). \]

We now multiply the above equation by \(-m \phi_\varepsilon \left( \frac{1}{u^r} - \frac{1}{c_0^r} \right)\), with \( \phi_\varepsilon \) a smooth approximation of the indicator function of \( \mathbb{R}_+ \) satisfying \( \phi'_\varepsilon \geq 0 \). Integrating by parts we get
\[ \frac{d}{dt} \int_0^1 \Psi_\varepsilon (u - c_0) m \, dx = - \int_0^1 \phi_\varepsilon \left( \frac{1}{u^r} - \frac{1}{c_0^r} \right) \partial_t (u - c_0) m \, dx \]
\[ = -(r + 1) \int_0^1 \partial_x \left( \frac{1}{u^r} - \frac{1}{c_0^r} \right) \phi'_\varepsilon \left( \frac{1}{u^r} - \frac{1}{c_0^r} \right) m \, dx \leq 0, \]
where we set
\[ \Psi_\varepsilon (s) := - \int_0^s \phi_\varepsilon \left( \frac{1}{(s + c_0)^r} - \frac{1}{c_0^r} \right) \, ds \quad \text{for all } s \geq -c_0. \]
(note that, since \( u \geq 0 \), the function \( \Psi_\varepsilon(u - c_0) \) is well defined). Letting \( \varepsilon \to 0 \) we see that \( \Psi_\varepsilon(s) \to s^- \) for \( s \geq -c_0 \), hence

\[
\frac{d}{dt} \int_0^1 (u - c_0)_- m \, dx \leq 0,
\]

proving the result. \( \square \)

3. **Exponential convergence to equilibrium:** Proof of Theorem 1.1. We begin by observing that, thanks to Proposition 1, \( f(t) \) satisfies (5). Also, recalling the definition of \( F_\rho \) (see (4)), a direct computation gives then

\[
\frac{d}{dt} F_\rho[f(t)] = -r^2 \int_0^1 f(t, x) \left| \partial_x \left( \frac{\rho(x)}{f(t, x)^{r+1}} \right) \right|^2 \, dx.
\]

Given \( x \in [0, 1] \), let us define the function

\[
(0, \infty) \ni s \mapsto F_x[s] := \frac{\rho(x)}{s^r},
\]

so that

\[
F_\rho[f(t)] = \int_0^1 F_x(f(t, x)) \, dx.
\]

Then,

\[
F_x[\gamma \rho(x)^{1/(r+1)}] = \frac{1}{\gamma^r} \rho(x)^{1/(r+1)},
\]

with \( \gamma \) the renormalization constant of the stationary solution (so that \( \gamma \rho(x)^{1/(r+1)} \) is a probability density).

Now we introduce a function \( G_x[f] \) that, up to translation, has the same integral as \( F_x[f] \), and such that \( G_x[f] \) can be used to perform an \( L^2 \) Gronwall estimate. We define

\[
G_x[s] := F_x[s] - F_x[\gamma \rho(x)^{1/(r+1)}] - F_x[\gamma \rho(x)^{1/(r+1)}](s - \gamma \rho(x)^{1/(r+1)}).
\]

Then,

\[
G_x[s] = \left[ \int_0^1 (1 - \tau) F_x''[\gamma s + (1 - \gamma) \rho(x)^{1/(r+1)}] \, d\tau \right] (s - \gamma \rho(x)^{1/(r+1)})^2.
\]

By Proposition 1 we have that \( f \) is bounded away from zero and infinity, see (5). Therefore, since \( s \mapsto F_x[s] \) is uniformly convex on \([a, A]\), it holds

\[
b |f(t, x) - \gamma \rho(x)^{1/(r+1)}|^2 \leq G_x[f(t, x)] \leq B |f(t, x) - \gamma \rho(x)^{1/(r+1)}|^2
\]

for all times, with \( b, B \) positive constants.

Moreover,

\[
G_x[f(t, x)] = \frac{\rho(x)}{f(t, x)^r} - \frac{\rho(x)^{1/(r+1)}}{\gamma^r} + \frac{r}{\gamma^{r+1}} (f(t, x) - \gamma \rho(x)^{1/(r+1)}),
\]

thus, since \( f \) and \( \gamma \rho(x)^{1/(r+1)} \) are two probability densities, \( G_x \) and \( F_x \) have the same integral up to an additive constant:

\[
\int_0^1 G_x[f(t, x)] \, dx = \int_0^1 F_x[f(t, x)] \, dx - \int_0^1 \frac{\rho(x)^{1/(r+1)}}{\gamma^r} \, dx.
\]
Therefore
\[
\frac{d}{dt} \int_0^1 G_x[f(t, x)] \, dx = \frac{d}{dt} \int_0^1 F_x[f(t, x)] \, dx
\]
\[
= -r^2 \int_0^1 f(t, x) \left| \partial_x \left( \frac{\rho(x)}{f(t, x)^{r+1}} \right) \right|^2 \, dx
\]
\[
\leq -r^2 a \int_0^1 \left| \partial_x \left( \frac{\rho(x)}{f(t, x)^{r+1}} \right) \right|^2 \, dx.
\]
Notice that
\[
\partial_x \left( \frac{\rho(x)}{f(t, x)^{r+1}} \right) = \partial_x \left( \frac{\rho(x)^{1/(r+1)}}{f(t, x)} \right)^{r+1}
\]
\[
= (r + 1) \left( \frac{\rho(x)^{1/(r+1)}}{f(t, x)} \right)^r \partial_x \left( \frac{\rho(x)^{1/(r+1)}}{f(t, x)} \right).
\]
Thus, denoting by \( c \) and \( C \) positive constants depending only on \( \lambda, a, A, r \), and that \( c \) and \( C \) may change from line to line, we have:
\[
\int_0^1 \left| \partial_x \left( \frac{\rho(x)}{f(t, x)^{r+1}} \right) \right|^2 \, dx = (r + 1)^2 \int_0^1 \left( \frac{\rho(x)^{1/(r+1)}}{f(t, x)} \right)^2 \partial_x \left( \frac{\rho(x)^{1/(r+1)}}{f(t, x)} \right) \, dx
\]
\[
\geq c \int_0^1 \left| \partial_x \left( \frac{\rho(x)^{1/(r+1)}}{f(t, x)} \right) \right|^2 \, dx.
\]
Hence, by the Poincaré inequality
\[
\frac{1}{2} \int_0^1 \left| \partial_x g \right|^2 \, dx \geq \int_0^1 \left| g - c_g \right|^2 \, dx, \quad c_g := \int_0^1 g \, dx,
\]
applied to \( g = \rho^{1/(r+1)}/f \), we get
\[
\int_0^1 \left| \partial_x \left( \frac{\rho(x)}{f(t, x)^{r+1}} \right) \right|^2 \, dx \geq c \int_0^1 \left| \frac{\rho(x)^{1/(r+1)}}{f(t, x)} - \frac{1}{\alpha(t)} \right|^2 \, dx
\]
\[
= c \int_0^1 \frac{1}{\alpha^2(t) f(t, x)^2} \left| \alpha(t) \rho(x)^{1/(r+1)} - f(t, x) \right|^2 \, dx,
\]
where
\[
\alpha(t) = \left( \int_0^1 \frac{\rho(x)^{1/(r+1)}}{f(t, x)} \, dx \right)^{-1}
\]
is bounded away from zero and infinity for all times (thanks to (5) and the bound \( \lambda \leq \rho \leq 1/\lambda \)):
\[
0 < c \leq \alpha(t) \leq C < \infty.
\]
Therefore
\[
\int_0^1 \left| \partial_x \left( \frac{\rho(x)}{f(t, x)^{r+1}} \right) \right|^2 \, dx \geq c \int_0^1 \frac{1}{\alpha^2(t) f(t, x)^2} \left| \alpha(t) \rho(x)^{1/(r+1)} - f(t, x) \right|^2 \, dx
\]
\[
\geq c \int_0^1 \left| \alpha(t) \rho(x)^{1/(r+1)} - f(t, x) \right|^2 \, dx.
\]
Now, the problem is that $\alpha(t)$ a priori does not coincide with $\gamma$. For this reason, we first suitably modify the expression appearing in the right hand side as follows:

$$\int_0^1 \left| \partial_x \left( \frac{\rho(x)}{f(t,x)^{r+1}} \right) \right|^2 dx \geq c \int_0^1 \left| \alpha(t)\rho(x)^{1/(r+1)} - f(t,x) \right|^2 dx$$

$$= c \int_0^1 \rho(x)^{1/(r+1)} \left| \alpha(t)\rho^{1/2(r+1)} - \frac{f(t,x)}{\rho(x)^{1/2(r+1)}} \right|^2 dx$$

$$\geq c \int_0^1 \alpha(t)\rho^{1/2(r+1)} - \frac{f(t,x)}{\rho(x)^{1/2(r+1)}} \right|^2 dx.$$

Now, we note that by replacing $\alpha(t)$ with $\gamma$ we lower the value of the expression in the right hand side:

$$\int_0^1 \left| \alpha(t)\rho^{1/2(r+1)} - \frac{f(t,x)}{\rho(x)^{1/2(r+1)}} \right|^2 dx \geq \min_{\beta} \int_0^1 \left| \beta\rho^{1/2(r+1)} - \frac{f(t,x)}{\rho(x)^{1/2(r+1)}} \right|^2 dx$$

$$= \int_0^1 \gamma\rho^{1/2(r+1)} - \frac{f(t,x)}{\rho(x)^{1/2(r+1)}} \right|^2 dx.$$

Hence, recalling (7), this proves that

$$\int_0^1 \left| \partial_x \left( \frac{\rho(x)}{f(t,x)^{r+1}} \right) \right|^2 dx \geq c \int_0^1 \left| \gamma\rho^{1/(r+1)} - f(t,x) \right|^2 \frac{1}{\rho(x)^{1/(r+1)}} dx$$

$$\geq c \int_0^1 \left| \gamma\rho^{1/(r+1)} - f(t,x) \right|^2 dx$$

$$\geq \frac{c}{B} \int_0^1 G_x[f](t,x)dx.$$ 

Therefore, by Gronwall Lemma, there exists a constant $\hat{c}$ such that

$$\int_0^1 G_x[f](t,x)dx \leq e^{-\hat{c}t} \int_0^1 G_x[f(0,x)]dx.$$ 

Since $G_x[f(t,x)]$ is comparable to $|f(t,x) - \gamma\rho(x)^{1/(r+1)}|^2$ (see (7)), this Gronwall estimate implies the exponential convergence of $f$ to the stationary solution $\gamma\rho(x)^{1/(r+1)}$, namely

$$\int_0^1 \left| f(t,x) - \gamma\rho(x)^{1/(r+1)} \right|^2 \leq \hat{C} e^{-\hat{c}t},$$

as desired.

4. Stability in $W_2$: Proof of Theorem 1.2. To prove Theorem 1.2, we shall first compute the Hessian of $F_\rho[f]$ at a fixed probability density $f$, and then we apply this estimate to prove the contraction along two solutions of (1). Since, under our assumptions, solutions are of class $C^{2,\alpha}$, in the next section we assume that $f \in C^2$. 
4.1. **Hessian of** $\mathcal{F}_\rho[f]$. In this section we compute the Hessian of

$$
\mathcal{F}_\rho[f] = \int_0^1 \frac{\rho(x)}{f(x)^r} \, dx
$$

with respect to $W_2$. For this, we use the Riemannian formalism introduced in [9].

Our state space $\mathcal{M}$ is the space of positive functions $f : (0, 1) \to (0, \infty)$ with unit integral:

$$
\int_0^1 f \, dx = 1.
$$

We may think of infinitesimal perturbations $\delta f \in T_f \mathcal{M}$ of a state $f \in \mathcal{M}$ as functions $\delta f : (0, 1) \to \mathbb{R}$ with

$$
\int_0^1 \delta f \, dx = 0. \tag{9}
$$

For given $f \in \mathcal{M}$ we define the scalar product $g_f$ on $T_f \mathcal{M}$ via

$$
g_f(\delta f_0, \delta f_1) := \int_0^1 \partial_x \phi_0 \partial_x \phi_1 f \, dx,
$$

where, up to additive constants, the functions $\phi_i : (0, 1) \to \mathbb{R}$ are defined by

$$
\delta f_i - \partial_x (f \partial_x \phi_i) = 0. \tag{10}
$$

Observe that, as a consequence of (9) and the periodicity of $f$, it follows by (10) that

$$
0 = \int_0^1 \partial_x (f \partial_x \phi_i) = \left[ f \partial_x \phi_i \right]_0^1,
$$

hence the functions $\phi_i$ satisfy the boundary conditions

$$
\partial_x \phi_i(1) = \partial_x \phi_i(0). \tag{11}
$$

Note that, since the variational derivative of $\mathcal{F}_\rho[f]$ is given by

$$
\frac{\delta \mathcal{F}_\rho[f]}{\delta f} = -r \frac{\rho(x)}{f(x)^{r+1}},
$$

the equation (1) can be interpreted as the gradient flow of the functional $\mathcal{F}_\rho[f]$ in the 2-Wasserstein metric:

$$
\partial_t f(t, x) = -\text{grad}_W \mathcal{F}_\rho[f(t)] = \partial_x \left( f(t, x) \partial_x \left( \frac{\delta \mathcal{F}_\rho[f(t)]}{\delta f} \right) \right). \tag{12}
$$

Now, given a periodic probability density $f : [0, 1] \to (0, \infty)$ of class $C^2$, let the function $\delta f$ satisfy (9), and let $\phi$ be related to $\delta f$ by (10).

We compute the first derivative of $\mathcal{F}_\rho[f]$. Using that

$$
\frac{\partial_x f}{f^{r+1}} = -\frac{1}{r} \partial_x \left( \frac{1}{f^r} \right),
$$
we have:
\[
\langle \delta F_\rho \frac{\delta f}{\delta f}, \delta f \rangle = -r \int_0^1 \frac{\rho}{f^{r+1}} \delta f \, dx \overset{(10)}{=} -r \int_0^1 \frac{\rho}{f^{r+1}} \partial_x (f \partial_x \phi) \, dx \\
= -r \int_0^1 \frac{\rho}{f^{r}} \partial_{xx} \phi \, dx - r \int_0^1 \frac{\rho}{f^{r+1}} \partial_x f \partial_x \phi \, dx \\
= -r \int_0^1 \frac{\rho}{f^{r}} \partial_{xx} \phi \, dx + \int_0^1 \rho \partial_x \left( \frac{1}{f^r} \right) \partial_x \phi \, dx \\
= - \int_0^1 \frac{\rho}{f^{r}} \partial_x \phi \, dx - (r + 1) \int_0^1 \frac{\rho}{f^{r+1}} \partial_{xx} \phi \, dx.
\]

Now, to compute the Hessian of \( F_\rho \), we consider a geodesic \( f : [0, 1] \rightarrow \mathcal{M} \) such that \( f(0) = f \). Then the Hessian of \( F_\rho \) at \( f \) is computed by considering
\[
\frac{d^2}{ds^2} \bigg|_{s=0} F_\rho[f(s)].
\]
Recall that the geodesic equation is given by the system (see for instance \([10, \text{Sections 2 and 3.2}]\))
\[
\partial_s f - \partial_x (f \partial_x \phi) = 0 \tag{13}
\]
\[
\partial_s \phi - \frac{1}{2} (\partial_x \phi)^2 = 0, \tag{14}
\]
coupled with the periodicity conditions
\[
f(s, 0) = f(s, 1), \quad \phi(s, 0) = \phi(s, 1), \quad \partial_x \phi(s, 0) = \partial_x \phi(s, 1)
\]
(the periodicity of \( \phi \) is a consequence of the fact that we are considering densities on the 1-dimensional torus, so also \( \phi \) is a function on the torus, while the periodicity of \( \partial_x \phi \) follows by \((11))

With this notation,
\[
\frac{d}{ds} F_\rho[f(s)] = \langle \frac{\delta F_\rho[f(s)]}{\delta f}, \delta f(s) \rangle \\
= - \int_0^1 \frac{\partial_x \rho}{f^r} \partial_x \phi \, dx - (r + 1) \int_0^1 \frac{\rho}{f^{r+1}} \partial_{xx} \phi \, dx,
\]
where \( \delta f(s) \) is related to \( \phi(s) \) by \((10))

Recalling that \( \rho \) is a periodic function of class \( C^{2,\alpha} \) (hence, the values of \( \rho, \partial_x \rho, \partial_{xx} \rho \) at 0 and 1 coincide), we now compute the second derivative of \( F_\rho[f(s)] \):
\[
\frac{d^2}{ds^2} \bigg|_{s=0} F_\rho[f(s)] \\
= \frac{d}{ds} \left( - \int_0^1 \frac{\partial_x \rho}{f^r} \partial_x \phi \, dx - (r + 1) \int_0^1 \frac{\rho}{f^{r+1}} \partial_{xx} \phi \, dx, \right) \\
= r \int_0^1 \frac{\partial_x \rho}{f^{r+1}} \partial_s f \partial_x \phi \, dx - \int_0^1 \frac{\partial_x \rho}{f^r} \partial_x (\partial_s \phi) \, dx \\
+ r (r + 1) \int_0^1 \frac{\rho}{f^{r+1}} \partial_s f \partial_{xx} \phi \, dx - (r + 1) \int_0^1 \frac{\rho}{f^{r+1}} \partial_{xx} (\partial_s \phi) \, dx.
\]
\[(13)+(14) = r \int_0^1 \frac{\partial_x \rho}{f r^{r+1}} \partial_x (f \partial_x \phi) \partial_x \phi \, dx - \int_0^1 \frac{\partial_x \rho}{f r} \partial_x \left( \frac{1}{2} |\partial_x \phi|^2 \right) \, dx + r(r + 1) \int_0^1 \frac{\partial_x \rho}{f r} \partial_x f \partial_x \phi \partial_x \phi \, dx - (r + 1) \int_0^1 \frac{\partial_x \rho}{f r} \partial_x \left( \frac{1}{2} |\partial_x \phi|^2 \right) \, dx \]

Using again that \(\frac{\partial f}{\partial x} = -\frac{1}{r} \partial_x \left( \frac{1}{f r} \right)\), and integrating by parts, we get (note that all the boundary terms vanish by periodicity)

\[
\begin{align*}
\frac{d^2}{ds^2} \bigg|_{s=0} \mathcal{F}_\rho[f(s)] &= -\int_0^1 \partial_x \rho \partial_x \left( \frac{1}{f r} \right) (\partial_x \phi)^2 \, dx + r \int_0^1 \frac{\partial_x \rho}{f r} \partial_x \phi \partial_x \phi \, dx \\
&\quad - \int_0^1 \frac{\partial_x \rho}{f r} \partial_x \left( \frac{1}{2} |\partial_x \phi|^2 \right) \, dx + r(r + 1) \int_0^1 \frac{\rho}{f r} (\partial_x \phi)^2 \, dx \\
&\quad - (r + 1) \int_0^1 \rho \partial_x \left( \frac{1}{f r} \right) \partial_x \phi \partial_x \phi \, dx - (r + 1) \int_0^1 \frac{\rho}{f r} \partial_x \phi \left( \frac{1}{2} |\partial_x \phi|^2 \right) \, dx \\
&= \int_0^1 \frac{\partial_x \rho}{f r} (\partial_x \phi)^2 \, dx + \int_0^1 \frac{\partial_x \rho}{f r} \partial_x \left( (\partial_x \phi)^2 \right) \, dx \\
&\quad + r \int_0^1 \frac{\partial_x \rho}{f r} \partial_x \phi \partial_x \phi \, dx - \int_0^1 \frac{\partial_x \rho}{f r} \partial_x \left( \frac{1}{2} |\partial_x \phi|^2 \right) \, dx \\
&\quad + (r + 1) \int_0^1 \frac{\rho}{f r} (\partial_x \phi)^2 \, dx + (r + 1) \int_0^1 \frac{\partial_x \rho}{f r} \partial_x \phi \partial_x \phi \, dx \\
&\quad + (r + 1) \int_0^1 \frac{\rho}{f r} \partial_x \phi (\partial_x \phi \partial_x \phi) \, dx - (r + 1) \int_0^1 \frac{\rho}{f r} \partial_x \phi \left( \frac{1}{2} |\partial_x \phi|^2 \right) \, dx \\
&= 2(r + 1) \int_0^1 \frac{\partial_x \rho}{f r} \partial_x \phi \partial_x \phi \, dx + (r + 1)^2 \int_0^1 \frac{\rho}{f r} (\partial_x \phi)^2 \, dx \\
&\quad + \int_0^1 \frac{\partial_x \rho}{f r} (\partial_x \phi)^2 \, dx - (r + 1) \int_0^1 \frac{\rho}{f r} \left[ -\partial_x \phi \partial_x \phi \partial_x \phi \partial_x \phi \partial_x \phi + \partial_x \phi \left( \frac{1}{2} |\partial_x \phi|^2 \right) \right] \, dx.
\end{align*}
\]

We now notice that

\[-\partial_x \phi \partial_x \phi \partial_x \phi \partial_x \phi + \partial_x \phi \left( \frac{1}{2} |\partial_x \phi|^2 \right) = (\partial_x \phi)^2,\]

so we get

\[
\begin{align*}
\frac{d^2}{ds^2} \bigg|_{s=0} \mathcal{F}_\rho[f(s)] &= 2(r + 1) \int_0^1 \frac{\partial_x \rho}{f r} \partial_x \phi \partial_x \phi \, dx \\
&\quad + r(r + 1) \int_0^1 \frac{\rho}{f r} (\partial_x \phi)^2 \, dx + \int_0^1 \frac{\partial_x \rho}{f r} (\partial_x \phi)^2 \, dx.
\end{align*}
\]
We now want to investigate the $\mu$-convexity of the functional $F_\rho$ in terms of the assumptions on $\rho$ and $f$.

Assume that $\rho$ is a periodic probability density of class $C^{2,\alpha}$ with $\|\rho\|_\infty \leq \eta_1$ and $\|\rho''\|_\infty \leq \eta_2$. We assume also that $0 < \lambda \leq \rho \leq 1/\lambda$ and $0 < a \leq f \leq A$. Then

$$\frac{d^2}{ds^2} \bigg|_{s=0} F_\rho[f(s)] \geq -\frac{2\eta_1(r+1)}{a^r} \int_0^1 |\partial_x \phi| |\partial_{xx} \phi| \, dx$$

$$+ \frac{r(r+1)\lambda}{A^r} \int_0^1 (\partial_{xx} \phi)^2 \, dx - \frac{\eta_2}{a^r} \int_0^1 (\partial_x \phi)^2 \, dx.$$

By Young inequality we have, for any $\varepsilon > 0$,

$$\frac{d^2}{ds^2} \bigg|_{s=0} F_\rho[f(s)] \geq -\frac{\eta_1(r+1)}{\varepsilon a^r} \int_0^1 |\partial_x \phi|^2 \, dx - \varepsilon \frac{\eta_1(r+1)}{a^r} \int_0^1 |\partial_{xx} \phi|^2 \, dx$$

$$+ \frac{r(r+1)\lambda}{A^r} \int_0^1 (\partial_{xx} \phi)^2 \, dx - \frac{\eta_2}{a^r} \int_0^1 (\partial_x \phi)^2 \, dx.$$

Choosing $\varepsilon = \frac{r\lambda a^r}{2\eta_1 A^r}$, we get

$$\frac{d^2}{ds^2} \bigg|_{s=0} F_\rho[f(s)] \geq \frac{r(r+1)\lambda}{2A^r} \int_0^1 |\partial_x \phi|^2 \, dx - \left( \frac{2\eta_1^2(r+1)A^r}{r\lambda a^{2r}} - \frac{\eta_2}{a^r} \right) \int_0^1 |\partial_{xx} \phi|^2 \, dx.$$

Using Poincaré inequality on $[0,1]$ (recall (8) and note that, by periodicity, $\int_0^1 \partial_x \phi = 0$), we obtain

$$\frac{d^2}{ds^2} \bigg|_{s=0} F_\rho[f(s)] \geq \left( \frac{r(r+1)\lambda}{A^r} - \frac{2\eta_1^2(r+1)A^r}{r\lambda a^{2r}} - \frac{\eta_2}{a^r} \right) \int_0^1 |\partial_x \phi|^2 \, dx$$

$$= \mu \int_0^1 |\partial_x \phi|^2 \, dx,$$

where

$$\mu := \left( \frac{r(r+1)\lambda}{A^r} - \frac{2\eta_1^2(r+1)A^r}{r\lambda a^{2r}} - \frac{\eta_2}{a^r} \right).$$

Hence, recalling that $a \leq f \leq A$, if $\mu \geq 0$ we have

$$\frac{d^2}{ds^2} \bigg|_{s=0} F_\rho[f(s)] \geq \frac{\mu}{A} \int_0^1 f |\partial_x \phi|^2 \, dx,$$

while if $\mu < 0$ we have

$$\frac{d^2}{ds^2} \bigg|_{s=0} F_\rho[f(s)] \geq \frac{\mu}{a} \int_0^1 |\partial_x \phi|^2 \, dx.$$

This proves that the Hessian of $F_\rho$ at $f$ is bounded from below by $\min\{\mu/A, \mu/a\}$.

4.2. Application to stability of solutions to (1). As we shall explain in the next section, to ensure that the above convexity results can be applied to equation (1), one needs to know that if $f_1(t,x)$, $f_2(t,x)$ are solutions of (1), and if

$$[0,1] \ni s \mapsto f^s(t,x)$$

is a Wasserstein geodesic such that $f^0(t,x) = f_1(t,x)$ and $f^1(t,x) = f_2(t,x)$, then there exist constants $a, A > 0$ such that

$$0 < a \leq f^s(t,x) \leq A \quad \forall s \in [0,1], \forall t \geq 0, \forall x.$$
Thanks to Theorem 1.1, we know that the above bounds hold at $s = 0, 1$, for all $t, x$.

We now fix $t \geq 0$ and consider $s \mapsto f^s$ the geodesic connecting $f_1(t)$ to $f_2(t)$ on $(\mathcal{M}, W_2)$.

The goal is to show that

$$\text{Hess}_{W_2} F_{\rho}[f^s] \geq \mu \quad \text{for all } s \in [0, 1],$$

and as explained above, to prove this result it is enough to prove the following 1-dimensional result about Wasserstein geodesics, and apply it to $f_0 = f_1(t)$ and $f_1 = f_2(t)$.

**Lemma 4.1.** Let $f_0, f_1$ be two probability densities on $[0, 1]$ satisfying $0 \leq a \leq f_0, f_1 \leq A$. Also, let $s \mapsto f_s$ denote the Wasserstein geodesic connecting them. Then

$$a \leq f_s \leq A.$$

**Proof.** Let $T$ be the optimal transport map from $f_0$ to $f_1$. By definition $f_s$ is given by

$$(T_s)' f_0 = f^s \quad \text{where} \quad T_s(x) = (1 - s)x + sT(x).$$

By definition of push-forward we have

$$T_s' = \frac{f_0}{f_s \circ T_s},$$

and

$$T' = \frac{f_0}{f_1 \circ T}.$$  \hfill (15)

Hence, by (15) and (16),

$$f_s \circ T_s = \frac{f_0}{T_s'} = \frac{f_0}{sT' + (1 - s)} = \frac{s f_0 + (1 - s)f_1 \circ T}{s f_0 + (1 - s)f_1 \circ T} = \frac{f_0 f_1 \circ T}{s f_0 + (1 - s)f_1 \circ T}.$$  \hfill (16)

Noticing that

$$\min \{f_0; f_1 \circ T\} \leq \frac{f_0 f_1 \circ T}{s f_0 + (1 - s)f_1 \circ T} \leq \max \{f_0; f_1 \circ T\}$$

we obtain the result. \hfill \qed

In the next subsection, we briefly summarize the general consequences of $\mu$-convexity and we conclude the proof of Theorem 1.2.

4.3. $W_2$-stability. In this section we use Otto’s formalism to deduce convergence and stability of solutions. Although these computations are formal, we present them as they show in a very elegant way why convexity of $F$ implies such stability. For a rigorous proof, the reader may look at the paper [10, Section 4].

Recall that, formally, our equation (1) can be written as

$$\dot{f} = -\nabla W_2 F_{\rho}[f],$$

where

$$\nabla W_2 F_{\rho}[f] = r \text{div} \left( f \nabla \left( \frac{\rho}{f^{r+1}} \right) \right).$$
Now, given two solutions $f_1$ and $f_2$ as in the statement of the theorem, and denoting by $f^*$ the geodesic connecting them, we compute
\[
\frac{d}{dt} W_2(f_1, f_2)^2 = g_{f_1} \left( \dot{f}_1, \partial_s f^* \bigg|_{s=0} \right) - g_{f_2} \left( \dot{f}_2, \partial_s f^* \bigg|_{s=1} \right)
\]
\[= -g_{f_1} \left( \nabla W_2 \mathcal{F}_\rho [f_1], \partial_s f^* \bigg|_{s=0} \right) + g_{f_2} \left( \nabla W_2 \mathcal{F}_\rho [f_2], \partial_s f^* \bigg|_{s=1} \right)
\]

Now, since $f^*$ is a geodesic,
\[
\frac{d}{ds} g_{f^*} \left( \nabla W_2 \mathcal{F}_\rho [f^*], \partial_s f^* \right) = g_{f^*} \left( \text{Hess} W_2 \mathcal{F}_\rho [f^*] \partial_s f^*, \partial_s f^* \right).
\]

Thus
\[
-g_{f_1} \left( \nabla W_2 \mathcal{F}_\rho [f_1], \partial_s f^* \bigg|_{s=0} \right) + g_{f_2} \left( \nabla W_2 \mathcal{F}_\rho [f_2], \partial_s f^* \bigg|_{s=1} \right)
\]
\[= -\int_0^1 g_{f^*} \left( \text{Hess} W_2 \mathcal{F}_\rho [f^*] \partial_s f^*, \partial_s f^* \right) ds
\]
\[\leq -\mu \int_0^1 g_{f^*} (\partial_s f^*, \partial_s f^*) ds = -\mu W_2(f_1, f_2)^2,
\]

where in the last inequality we used again that $f^*$ is a geodesic. Hence, combining these two equations we get
\[
\frac{d}{dt} W_2(f_1, f_2)^2 \leq -2\mu \frac{W_2(f_1, f_2)^2}{2},
\]

which gives the result.

**Acknowledgments.** The author is grateful to Prof. José Antonio Carrillo and to Francesco Saverio Patacchini for useful comments and for carefully reading this manuscript; and to Prof. Matteo Bonforte for suggesting references on very fast diffusion equations. Also, the author thanks the anonymous referees for their useful comments.

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Received for publication February 2016.

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