Almost zip Bezout domain

Bohdan Zabavsky, Oleh Romaniv

Department of Mechanics and Mathematics, Ivan Franko National University
Lviv, 79000, Ukraine
zabavskii@gmail.com, oleh.romaniv@lnu.edu.ua

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Abstract: J. Zelmanowitz introduced the concept of ring, which we call zip rings. In this paper we characterize a commutative Bezout domain whose finite homomorphic images are zip rings modulo its nilradical.

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1 Introduction

All rings considered will be commutative with identity. A ring is a Bezout ring if every finitely generated ideal is principal. I. Kaplansky [4] defined the class of elementary divisor rings as rings $R$ for which every matrix $A$ over $R$ admits a diagonal reduction, that is there exist invertible matrices $P$ and $Q$ such that $PAD$ is a diagonal matrix $D = (d_i)$ with the property that every $d_i$ is a divisor $d_{i+1}$. B. Zabavsky defined fractionally regular rings as rings $R$ such that for which every nonzero and nonunit element $a$ from $R$ the classical quotient ring $Q_{\text{cl}}(R/\text{rad}(aR))$ is regular, where $\text{rad}(aR)$ is nilradical of $aR$ [7]. We say that the ring $R$ has stable range 2 if whenever $aR + bR + cR = R$, then there are $\lambda, \mu \in R$ such that $(a + c\lambda)R + (b + c\mu)R = R$. We say $R$ is semi-prime if $\text{rad}(R) = \{0\}$, where $\text{rad}(R)$ is the nilradical of the ring $R$. Obviously, rings in which nonzero principal ideal has only finitely many minimal prime are examples of fractionally regular rings [1].

An ideal $I$ of a ring $R$ is called a $J$-radical if it is an intersection of maximal ideals, or, equivalently, if $R/I$ has zero Jacobson radical. We call $R$
J-Noetherian if it satisfies the ascending chain condition on J-radical ideals.

For every ideal \( I \) in \( R \) we define the annihilator of \( I \) by \( I^\perp = \{ x \in R \mid ix = 0 \ \forall i \in I \} \).

Following C. Faith [3] a ring \( R \) is zip if \( I \) is an ideal and if \( I^\perp = \{ 0 \} \) for a finitely generated ideal \( I_0 \subset I \). An ideal \( I \) of a ring \( R \) is dense if its annihilator is zero. Thus \( I \) is a dense ideal if and only if it is a faithful \( R \)-module. A ring \( R \) is a Kasch ring if \( I^\perp \neq \{ 0 \} \) for any ideal \( I \neq R \).

Let \( R \) be a ring. Then the ring \( R \) has finite Goldie dimension if it contains a direct sum of finite number of nonzero ideals. A ring \( R \) is called a Goldie ring if it has finite Goldie dimension and satisfies the ascending chain condition for annihilators [3 5 8]. By [3] we have the following result.

**Theorem 1.** [3] Semiprime commutative ring \( R \) is zip if and only if \( R \) is a Goldie ring.

**Proposition 1.** [3] A commutative Kasch ring is zip.

**Proposition 2.** [3] If \( Qcl(R) \) is a Kasch ring then \( R \) is zip.

For further research we will need the following results.

**Theorem 2.** [3] A commutative ring \( R \) is zip if and only if its classical ring of quotients \( Qcl(R) \) is zip.

**Theorem 3.** Let \( R \) be a commutative Bezout domain and \( 0 \neq a \in R \), then \( R/aR \) is a Kasch ring if and only if \( R \) is a ring in which any maximal ideal is principal.

**Proof.** First we will prove that the annihilator of any principal ideal of \( R/aR \) is a principal ideal.

Suppose \( b \in R \) and \( aR \subseteq bR \). Then \( (b:a) = \{ r \in R \mid br \in aR \} = sR \), where \( a = bs \), so \( (b:a) = aR \). We can also show that every principal ideal of \( R/aR \) is an annihilator of a principal ideal. Moreover, if \( I_1 = \text{Ann}(J_1) \), \( I_2 = \text{Ann}(J_2) \), where \( I_1, J_i, i = 1,2 \), are principal ideals, then

\[
\text{Ann}(I_1 \cap I_2) = \text{Ann}(\text{Ann}(J_1) \cap \text{Ann}(J_2)) = \\
= \text{Ann}(\text{Ann}(J_1 + J_2)) = J_1 + J_2 = \text{Ann}(J_1) + \text{Ann}(J_2).
\]

Let \( R/aR \) be a Kasch ring. Let \( \overline{M} \) be a maximal ideal in \( R/aR \). Denote \( R/aR = \overline{R} \). Then \( \text{Ann}(\overline{M}) = \overline{H} \), where \( \overline{H} \) is an ideal in \( \overline{R} = R/aR \) and
\( \overline{H} = \{0\} \). Since \( \overline{H} \) annihilates the maximal ideal \( \overline{M} \) then \( \overline{H} \cdot \overline{M} = \{0\} \).

Since the maximal ideal \( \overline{M} \) belongs to \( \text{Ann}(\overline{H}) \), then by maximality of \( \overline{M} \), \( \overline{M} = \text{Ann}(\overline{H}) \neq R/aR \).

Since \( \overline{M} \) is a maximal ideal, then for every element \( \overline{d} \neq 0 \), which belongs to \( \overline{H} \). We have the equality \( \overline{d} \overline{M} = \{0\} \). Thus, the maximal ideal \( \overline{M} \) belongs to \( \text{Ann}(\overline{d}) \), where \( \overline{d} \) is a nonunit.

Hence \( \overline{M} = \text{Ann}(\overline{d}) = \overline{bR} \). Therefore, \( \overline{M} = \overline{bR} \) and \( \overline{M} = bR + aR = cR \), because \( R \) is a commutative Bezout domain for some \( c \in R \). Hence \( M \) is a maximal ideal which is a principal ideal.

Suppose that a maximal \( M \) contains an element \( a \), is a principal one considering its homomorphic image we have \( \overline{M} = \overline{mR} = \text{Ann}(\overline{nR}) \). Since \( \overline{m} \notin U(\overline{R}) \) then we have \( \text{Ann}(\overline{nR}) \neq \overline{R} \) and hence \( \overline{nR} \neq \{0\} \).

As a result \( \text{Ann}(\overline{M}) = \text{Ann}(\text{Ann}(\overline{nR})) = \overline{nR} \neq (0) \). Therefore, \( \text{Ann}(\overline{M}) \) is a nonzero principal ideal. This proves the fact that \( \overline{R} \) is a Kasch ring.

\[ \Box \]

2 Our results

Note that

**Proposition 3.** Let \( R \) be a Bezout ring. Then \( R \) is zip if and only if every dense ideal contains a regular element.

**Proof.** If \( I \) is a dense ideal of a zip ring, and if \( I \) is principal dense ideal contained in \( I \), hence \( I \) is generated by a regular element. \[ \Box \]

**Theorem 4.** Let \( R \) be a semiprime commutative Bezout ring which is a Goldie ring. Then any minimal prime ideal of \( R \) is principal, generated by an idempotent, and there are only finitely many minimal prime ideals.

**Proof.** The restrictions on \( R \) imply that the classical quotient ring \( Q_{cl}(R) \) is an Artinian regular ring with finitely many minimal prime ideals. Let \( P \) be a minimal prime ideal of \( R \). Consider the ideal \( P_Q = \{ \frac{p}{x} \mid p \in P \} \). It is obvious that \( P_Q \) is a prime ideal of \( Q_{cl}(R) \). Since \( Q_{cl}(R) \) is an Artinian regular ring, there exists an idempotent \( e \in Q_{cl}(R) \) such that \( P_Q = eQ_{cl}(R) \).

Since \( R \) is arithemical ring, then we have \( e \in R \) \[ 2 \]. For any \( p \in P \) we obtain that \( p = er \), where \( r \) is a von Neumann regular element, i.e. \( rxr = r \) for some \( x \in R \). Hence \( ep = e^2r = er = p \), we have \( P \subset eR \), \( e \in P \), so \( eR \subset P \) and \( P = eR \). Since any minimal prime ideal of \( R \) is principal by \[ 1 \], we have that \( R \) have finitely many minimal prime ideals. \[ \Box \]
Definition 1. Let $R$ be a commutative Bezout domain. Nonzero and nonunit element $a \in R$ is said to be almost zip element if $R/\text{rad}(aR)$ is a zip ring. Commutative Bezout domain is said to be almost zip ring if any nonzero nonunit element of $R$ is almost zip element.

Theorem 5. Let $R$ be a commutative Bezout domain and a almost zip element of $R$. Then there are only finitely many prime ideals minimal over $aR$.

Proof. Since $R/\text{rad}(aR)$ is semiprime zip ring then by Theorem 1 we have that $R/\text{rad}(aR)$ is a Goldie Bezout ring. By Theorem 4 we have that any minimal prime ideal of $R/\text{rad}(aR)$ is principal and is generated by an idempotent. Then there are only finitely many minimal prime ideals. Obvious then $aR$ has finitely many minimal prime ideals. □

Consequently we have the following results.

Theorem 6. Almost zip commutative Bezout domain is $J$-Noetherian domain (i.e. Noetherian maximal spectrum).

Proof. By Theorem 6 we have that any nonzero and nonunit element has finitely many minimal ideals. By 2 $R$ is a $J$-Noetherian domain. □

Since a commutative $J$-Noetherian Bezout domain is an elementary divisor ring by Theorem 6, we have the following results.

Theorem 7. A commutative almost zip Bezout domain is an elementary divisor domain.

Since $J$-Noetherian Bezout domain is fractionally regular ring. We have the following result.

Theorem 8. Almost zip Bezout domain is fractionally regular domain.

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