Parafermi Algebra and Interordinality

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Abstract

Starting from the observation that for neighboring orders \( p = 2^n - 1 \), \( p' = 2^{n+1} - 1 \) of the well-known Green's representations of parafermi algebra there exists a specifiable interordinal relationship, matrices with similar interordinality properties and intrinsic Catalan structure are constructed which seem to have a bearing on Euclidean geometry and applications via Nebe kissing numbers.

Key words: parafermi algebra, paraorder, Mersenne numbers, interordinality, Catalan numbers, span parameter, stenoscopy, kissing numbers, topological/geometrical operator, preon model, cardioidic transformation, crossing ellipses, continued fractions, sine-like and cosine-like structural relationships, positional number systems, qphyletics

1 Introduction – parafermi operator and root-of-nilpotent sequences

Parafermi structures have been studied both in modern quantum field theory, following H.S. Green, and quantum information theory [Green98]. The term parafermion is specifically used for the generalization of a spin-\( \frac{1}{2} \) particle (fermion) to spin \( \frac{p}{2} \).

Translated to operator language,

\[ b^{p+1} = (b^+)_{p+1} = 0. \]  

(1)

In his original paper [Green53], Green supplied a \((p + 1) \times (p + 1)\) matrix representation for \( b \),

\[ b_{\alpha\beta} = B_{\beta} \delta_{\alpha, \beta+1}, \quad (b^+)_{\alpha\beta} = B_{\alpha} \delta_{\alpha+1, \beta}, \quad B_{\beta} = \sqrt{\beta(p - \beta + 1)}, \]

(2)

which realizes the spin-\( \frac{p}{2} \) representation

\[ \frac{1}{2} [b^+, b] = \text{diag}(\frac{p}{2}, \frac{p}{2}, \cdots, -\frac{p}{2} + 1, -\frac{p}{2}) \]

(3)

and the characteristic trilinear relations of parafermi algebra

\[ [[b^+, b], b] = -2b, \quad [[b^+, b], b^+] = 2b^+. \]

(4)

For the least paraorder, the parafermi operator coincides with the fermi operator \( f^{(1)} \) and satisfies the well-known algebra

\[ \{ f^{(1)}, (f^{(1)})^+ \} = 1, \quad (f^{(1)})^2 = 0 = ((f^{(1)})^+)^2. \]

(5)

One fact that seems to have been neglected, or overlooked, is that those representations, when of order \( p = 2^n - 1 \) and tensorially extended by \( \mathbf{1} \), are related to those of order \( p' = 2p + 1 = 2^{n+1} - 1 \) by an operator identity that could be named the Mersennian of parafermi algebra, for the 17th-century scholar Marin Mersenne who studied the properties of \( 2^n - 1 \):

\[ \frac{1}{2} \{ b^{(p')}, \mathbf{1} \otimes b^{(1)} \} = b^{(p)} \otimes \mathbf{1}. \]

(6)
For example,

\[
\left\{ b^{(7)}, 1^{\otimes 2} \otimes b^{(1)} \right\} =
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{15} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{15} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{7} & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{15} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{15} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{7} & 0
\end{pmatrix}
\]

= 2

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{15} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{15} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{7} & 0
\end{pmatrix}
\times
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
= 2b^{(3)} \otimes 1.
\]

Since numbers of the form \( p = 2^n - 1 \) have the binary representation 1, 11, 111, …, we say that the above paraorders \( p' \) and \( p \) are in a carry-bit neighborhood to one another. While its physical and information-theoretical meaning remain unclear, the operator identity (6) neatly carries over to nilpotent operators \( f^{(p')} \) which are obtained by “extracting the square root” of \( f^{(p)} \otimes 1^{[1]} \) in a recursive process beginning with the fermi operator \( f^{(1)} \). To allow \( f^{(p')} \) squared to act as a normalized-anticommutator analog of Eq. (6),

\[
(f^{(p')})^2 = f^{(p)} \otimes 1,
\]

the structure of \( f^{(p')} \) has to be amalgamated with \( 1^{\otimes n} \otimes f^{(1)} \), as we shall see. In matrix form, the structural parts are blockwise composed of elements of the Clifford algebra \( Cl(2, 1) \) with basis

\[
\left\{ c_1 = (1, 0, 0), c_2 = (0, 1, 0), c_3 = (0, 0, 1) \right\}.
\]

The simplest representation of the initial operator in the recursive process consists of a linear combination of one basis element per signature, usually

\[
f^{(1)} = \frac{1}{2}(c_2 - c_3).
\]

Equivalently, and closer to physics, one may start by Clifford algebra \( Cl(3) \) which has the set of Pauli matrices as basis, where \( f^{(1)} \) is represented by combining one real basis element of grade 1 – vector \( \sigma_1 \) or \( \sigma_3 \) – with the only real basis element of

\[1\text{ note that } f^{(p)} \text{ untensorized is not a proper exponentiation of an operator.}\]
grade 2 – bivector $\sigma_{31}$, the preferred choice being

$$f^{(1)} = \frac{1}{2}(\sigma_1 - \sigma_{31}) = \frac{1}{2}(\sigma^+_1 + \sigma^+_{31}).$$

The simplification achieved is that the conjugations $^+$ and $^T$ coincide.

Solving Eq. (7) for $f^{(p)}$ is made easy by requiring the main block diagonal of the $f^{(p)}$ matrix to coincide with $1^\otimes n \otimes f^{(1)}$, and the triangular matrix below it (LTM) to mutually exclusively consist of blocks $G_{\mu\nu}c_3$, $E_{\mu\nu}(f^{(1)})^+$ or $J_{\mu\nu}c_2$ ($\mu > \nu$; $G_{\mu\nu}, E_{\mu\nu}, J_{\mu\nu} \in \mathbb{Z}$). As the action below the main diagonal shows,

$$\begin{pmatrix}
0 & 0 & \cdots \\
1 & 0 & \\
0 & x_1 & 0 & 0 \\
x_3 & y & 0 & 1 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}^2 = \begin{pmatrix}
0 & 0 & \cdots \\
0 & 0 & \\
1 & 0 & \\
0 & 1 & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
\end{pmatrix} \sim x_1 = 1, \quad (y \in \{-1, 0, 1\})$$

the whole task is thereby effectively reduced to a linear problem. To distinguish the resulting matrix sequences from one another, we call them root-$f$ sequence, root-$d$ sequence and root-$h$ sequence according to their LTM content:

$$f^{(p)} = 1^\otimes n - 1 \otimes f^{(1)} + (G^{(p)}_{\mu\nu}) \otimes c_3,$$

$$d^{(p)} = 1^\otimes n - 1 \otimes f^{(1)} + (E^{(p)}_{\mu\nu}) \otimes (f^{(1)})^+,$$

$$h^{(p)} = 1^\otimes n - 1 \otimes f^{(1)} + (J^{(p)}_{\mu\nu}) \otimes c_2.$$

The solutions to the LPs start out behaving as expected:

$$f^{(1)} = \begin{pmatrix} \hfill 0 & 0 \hfill \\
\hfill 1 & 0 \hfill \\
\end{pmatrix},$$

$$\sqrt{f^{(1)} \otimes 1} = f^{(3)} = \begin{pmatrix} f^{(1)} & 0 \\
c_3 & f^{(1)} \\
\end{pmatrix},$$

$$\sqrt{f^{(3)} \otimes 1} = f^{(7)} = \begin{pmatrix} f^{(1)} & 0 & 0 & 0 \\
c_3 & f^{(1)} & 0 & 0 \\
c_3 & c_3 & f^{(1)} & 0 \\
c_3 & c_3 & c_3 & f^{(1)} \\
\end{pmatrix};$$

thereafter, however, the bulk of the coefficients $G_{\mu\nu}$ ($\mu > \nu$) begin to deviate from 1, as a snapshot of the LP $(f^{(15)})^2 = f^{(7)} \otimes 1$ when the upper left and lower right quadrants are already computed –

$$f^{(1)} \quad 0 \quad 0 \quad 0 \\
c_3 \quad f^{(1)} \quad 0 \quad 0 \\
c_3 \quad c_3 \quad f^{(1)} \quad 0 \\
c_3 \quad c_3 \quad c_3 \quad f^{(1)}$$

shows:
row 5/col 4 downto 1 : \( G_{54} = 1 \)
\[
G_{53} - G_{54} = 0 \; \leadsto \; G_{53} = 1 \\
G_{52} - G_{53} - G_{54} = 1 \; \leadsto \; G_{52} = 3 \\
G_{51} - G_{52} - G_{53} - G_{54} = 0 \; \leadsto \; G_{51} = 5
\]
row 6/col 4 downto 1 : \( G_{64} - G_{54} = 0 \; \leadsto \; G_{64} = 1 \)
\[
G_{63} - G_{64} - G_{53} = -1 \; \leadsto \; G_{63} = 1 \\
G_{62} - G_{63} - G_{64} - G_{52} = 0 \; \leadsto \; G_{62} = 5 \\
G_{61} - G_{62} - G_{63} - G_{64} - G_{51} = -1 \; \leadsto \; G_{61} = 11
\]
row 7/col 4 downto 1 : \( G_{74} - G_{54} - G_{64} = 1 \; \leadsto \; G_{74} = 3 \)
\[
G_{73} - G_{74} - G_{53} - G_{63} = 0 \; \leadsto \; G_{73} = 5 \\
G_{72} - G_{73} - G_{74} - G_{52} - G_{62} = 1 \; \leadsto \; G_{72} = 17 \\
G_{71} - G_{72} - G_{73} - G_{74} - G_{51} - G_{61} = 0 \; \leadsto \; G_{71} = 41
\]
row 8/col 4 downto 1 : \( G_{84} - G_{54} - G_{64} - G_{74} = 0 \; \leadsto \; G_{84} = 5 \)
\[
G_{83} - G_{84} - G_{53} - G_{63} - G_{73} = -1 \; \leadsto \; G_{83} = 11 \\
G_{82} - G_{83} - G_{84} - G_{52} - G_{62} - G_{72} = 0 \; \leadsto \; G_{82} = 41 \\
G_{81} - G_{82} - G_{83} - G_{84} - G_{51} - G_{61} - G_{71} = -1 \; \leadsto \; G_{81} = 113.
\]

Thus,
\[
\sqrt{f^{(7)} \otimes 1} = f^{(15)} = \begin{pmatrix}
  f^{(1)} & 0 & \ldots & 0 \\
  c_3 & f^{(1)} \\
  c_3 & c_3 & f^{(1)} \\
  c_3 & c_3 & c_3 & f^{(1)} \\
  5c_3 & 3c_3 & c_3 & c_3 & f^{(1)} \\
  11c_3 & 5c_3 & c_3 & c_3 & c_3 & f^{(1)} \\
  41c_3 & 17c_3 & 5c_3 & c_3 & c_3 & c_3 & f^{(1)} \\
  113c_3 & 41c_3 & 11c_3 & 5c_3 & c_3 & c_3 & c_3 & f^{(1)} \\
\end{pmatrix},
\]

Mutatis mutandis for the root-\( d \) and root-\( h \) sequences:
\[
d^{(1)} = f^{(1)},
\]
\[
\sqrt{d^{(1)} \otimes 1} = d^{(3)} = \begin{pmatrix}
  f^{(1)} & 0 \\
  (f^{(1)})^+ & f^{(1)} \\
\end{pmatrix},
\]
\[
\sqrt{d^{(3)} \otimes 1} = d^{(7)} = \begin{pmatrix}
  f^{(1)} & 0 & 0 & 0 \\
  (f^{(1)})^+ & f^{(1)} & 0 & 0 \\
  0 & (f^{(1)})^+ & f^{(1)} & 0 \\
  0 & 0 & (f^{(1)})^+ & f^{(1)} \\
\end{pmatrix} \quad (16)
\]

4
and similar identities hold for the $J_{\mu\nu}$ by Catalan-type bookkeeping identities: Where $C_G$ evolving away from 1, While constant recurrence of $E$, even though not immediately apparent, the way it happens is controlled by Catalan-type bookkeeping identities: Where $C_k$ denotes the $k$th Catalan number, in Eqs. (14)-(15) there hold the identities $G_{51} = G_{62} = G_{73} = G_{84} = C_3$, $G_{52} + G_{61} = G_{74} + G_{83} = C_4$, $G_{53} + G_{71} = G_{64} + G_{82} = C_5$, 

$$\sum_{i=0}^{3} G_{5+i,4-i} = C_6,$$

and similar identities hold for the $J_{\mu\nu}$ of the lower left quadrant of $h^{(7)}$:

$$-1 = -C_1,$$

$$1 - 3 = -C_2,$$
as well as for those of the lower left quadrant of $h^{(15)}$:

$$-5 = -C_3,$$

$$1 - 15 = -C_4,$$

$$-1 + 43 = C_5,$$

$$1 - 3 - 15 + 149 = C_6.$$  

At the same time, the coefficients $G_{\mu\nu}$ (or $J_{\mu\nu}$) are linked to an important characteristics of Euclidean $D$-space, namely the kissing number $L_D$ for densest packing of (hyper)spheres of equal size in that space. For $f^{(7)}$, we have

$$L_1 = G_{4,1} + G_{4,2} = 1 + 1 = 2,$$

and for $f^{(15)}$:

$$L_2 = G_{5,1} + G_{5,2} - G_{5,3} - G_{5,4} = 5 + 3 - 1 - 1 = 6,$$

$$L_3 = G_{7,2} - G_{7,3} = 17 - 5 = 12,$$

$$L_4 = G_{7,1} - G_{7,2} = 41 - 17 = 24,$$

$$L_5 = G_{7,1} - G_{7,2} + G_{8,3} + G_{8,4} = 41 - 17 + 11 + 5 = 40,$$

$$L_6 = G_{8,1} + G_{8,2} = 113 - 41 = 72,$$

$$L_7 = G_{7,1} - C_{7,2} + G_{8,1} - G_{8,3} = 41 - 17 + 113 - 11+ = 126.$$  

Thus, the members of the root-$f$- and the root-$h$ sequences first of all are partitioners of Catalan numbers in unfamiliar environs, and $f^{(p)}$ and $h^{(p)}$, as they climb up their parental sequences, traverse all of these. That evolutionary behavior, and the lack thereof in the root-$d$ sequence, calls for an examination of whether and how it relates to the very different given of parafermi algebra. Not surprisingly, the closest resemblance to the Green ansatz is found among the members of root-$d$ sequence. Not only is the nilpotence property $(d^{(p)})^{p+1} = ((d^{(p)})^+)^{p+1} = 0$ satisfied, the structure as well is analogous: $d_{\alpha,\beta} = E_{\beta} \delta_{\alpha,\beta+1}$ with $E_{\beta} = 1$. However, the spin-$(p/2)$ representation, as we have learned, demands $b_{\alpha,\beta} = \sqrt{\beta(p - \beta + 1)} \delta_{\alpha,\beta+1}$, a condition that counts our premise that the main diagonal of $d^{(p)}$ ($y' = 2^{n+1} - 1$) carries solely $f^{(1)}$ blocks. So the scope of reference to parafermi algebra is quickly cut down to members of the root-$f$- and root-$h$-sequences, and it’s foremost the former that we will pick up for exemplarily scrutinizing a possible relationship with the Green ansatz.

The rest of the paper is organized as follows. In Sect. 2 we outline the tenets of $f$-parafermi algebra for order $p \in \{3, 7\}$. In Sect. 3, an alternative ansatz called heterotic $f$-parafermi algebra, again of order $p \in \{3, 7\}$, is discussed. In both sections, parenthesized paraorder superscripts are used only where needed. The structure of the members of the root-$f$ sequence is considered from various perspectives in Sect. 4. Specifically in Sect. 4.1, the interordinal aspect between carry-bit neighbors $f^{(p)}$ and $f^{(p)}$, is elucidated and contrasted with the intraordinal aspect, and the partial sequences of $G_{\mu\nu}$ representatives, $(G^{(3)}) = (1), (G^{(7)}) = (1), (G^{(15)}) = (3, 5, 11, 17, 41, 113), \ldots$, which already showed up more or less as a curiosity in [Merk89], are examined for their intrinsic Catalan-number related properties in Sect. 4.2 and kissing-number related properties in Sect. 4.3; in Sect. 4.4, the members of $(G^{(p)})$ are further examined under the factorization aspect, with an aside to the factorization of the Catalan number $C_{(p-3)/4}$. As shown in Sect. 4.5, the symmetries underlying the $(\frac{1}{2})^p \times (\frac{1}{2})^p$ LTM $(G^{(p)})$ allow a generalization of the results of Sects. 2 and 3 to orders $p \in \{15, 31, \ldots\}$; advantage is thereby taken of the persistence of symmetry properties in residues left by the coefficients $G_{\mu\nu}$ after division by eight. In Sect. 5, the analysis is extended to differences derivable from the members of $(B^{(p)})$ and $(G^{(p)})$; the concept of interordinal differences is developed in Sect. 5.3 and applied to the kissing number problem in Sect. 5.4. Even though cursory, in Sect. 6 a glance is taken at the root-$h$ sequence and its partial sequences of representatives of $J_{\mu\nu}$: $(J^{(3)}) = (1), (J^{(7)}) = (-1, 3), (J^{(15)}) = (1, -5, 15, -43, 149), \ldots$; so-called synoptic differences derivable from these sequence members and those of $(G^{(p)})$ shed light on the periodicity aspect of kissing number representation (Sect. 7). After a brief outline of an interordinal preon model (Sect. 8), Sect. 9 adds a proposal for a planar geometric model that nicely fits with the root-$f$- and root-$h$-sequences, and an outline of the connection between Catalan structure and the model’s characteristic feature, the cardioidic arclength, as well as continued fraction representations – some of these in extension of one discussed earlier in Sect. 4.2, some addressing the kissing number problem from a qphyletic perspective – close this article.
2 $f$-parafermi algebra

To begin with, for $p = 2^n - 1$ ($n > 1$), with the exception of the nilpotence property
\[ f^{p+1} = (f^+)^{p+1} = 0, \]
no relation from the Green ansatz is satisfied after substituting $f$ for $b$. This necessitates an adaptation in form of an orthogonal decomposition
\[ f = \sum_v f_v \]
such that
\[ f^+_v f_v = \begin{cases} \text{diag} \{0,1\}, & v = 0, \\ \text{diag} \{0\} \cup \{G^{\frac{2}{n}}\}_{\mu,s_v(\mu)}), & v = 1, \ldots, (p - 1)/2, \quad \mu > s_v(\mu). \end{cases} \]

How a $2^n \times 2^n$ matrix (here with a granularity of $2^{(n-1)}$ blocks $A_{\mu,\nu}$, rather than $2^{2n}$ matrix elements $m_{\alpha,\beta}$) is orthogonally decomposed into (here $2^{n-1} = (p + 1)/2$) basis elements, whilst delineated in literature, is discovered each time anew. Key part of the decomposition procedure is the index permutations $s_v(\mu) \cong Z_2^{n-1}$. In Table 1, the permutations $s_v(\mu) \cong Z_2^{n-1}$ are shown, which are known under various isomorphic maps from other fields of mathematics (octonions, Fano plane). For the basis-element characterizations
\[ f_0: \quad (a_{1,1})_{\alpha,\lambda} + (a_{2,2})_{\alpha,\lambda} + \cdots = A_{\alpha,\lambda}(\delta_{1,\alpha}\delta_{\lambda,1} + \delta_{2,\alpha}\delta_{\lambda,2} + \cdots) \]
\[ \Rightarrow f_0 = a_{1,1} + a_{2,2} + \cdots \]
eq \begin{pmatrix} A_{1,1} & 0 & \cdots \\ 0 & A_{1,1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}

etc. we use the shorthand $0: 11 + 22 + \ldots$ etc.

Table 1

| $v$ | $\sum_{\mu} a_{\mu,s_v(\mu)}$ | $\{s_v\} \cong Z_2 \times Z_2 \times Z_2$ |
|-----|-------------------------------|---------------------------------|
| 0   | $11 + 22 + 33 + 44 + 55 + 66 + 77 + 88$ |                                 |
| 1   | $12 + 21 + 34 + 43 + 56 + 65 + 78 + 87$ |                                 |
| 2   | $13 + 24 + 31 + 42 + 57 + 68 + 75 + 86$ |                                 |
| 3   | $14 + 23 + 32 + 41 + 58 + 67 + 76 + 85$ |                                 |
| 4   | $15 + 26 + 37 + 48 + 51 + 62 + 73 + 84$ |                                 |
| 5   | $16 + 25 + 38 + 47 + 52 + 61 + 74 + 83$ |                                 |
| 6   | $17 + 28 + 35 + 46 + 53 + 64 + 71 + 82$ |                                 |
| 7   | $18 + 27 + 36 + 45 + 54 + 63 + 72 + 81$ |                                 |

Under the delineated proviso we get an $f$-parafermi algebra
\[ \frac{1}{2} [f_0^+, f_0] + \frac{(p-1)/2}{2} \sum_{v=1}^{(p-1)/2} [f^+_v, f_v] = \text{diag} \left( \frac{p-1}{2}, -1, \ldots, -\frac{p}{2} + 1, -\frac{p}{2} \right), \]
\[ \sum_{v=0}^{(p-1)/2} ([f^+_v, f_v], f_v] = -2f^+, \quad \sum_{v=0}^{(p-1)/2} ([f^+_v, f_v], f^+_v] = 2f^+. \]

Note that the above equations hold for $p \in \{3, 7\}$; how they can be generalized for $p \in \{15, 31, \ldots\}$ will be shown in Section 4.5.

For $p = 3$, the orthogonal decomposition reads $f = f_0 + f_1$ where, according to the shorthand prescription $0: 11 + 22, 1: 12 + 21$, we have
\[ (f_0)_{1,1} = (f_0)_{2,2} = (c_2 - c_3)/2, \quad (f_1)_{1,2} = 0, \quad (f_1)_{2,1} = G_{2,1}c_3 = c_3. \]

Mutatis mutandis for case $p = 7$. 

7
The spin arithmetics differ in one respect: in Green’s algebra (Eq. 3), spin values emerge as a half times differences of squares $B_2^2 - B_{B-1}^2 (\beta \in \{1, \ldots, p+1\}, B_0, B_{p+1} \equiv 0)$,

$$\begin{pmatrix}
\frac{1}{2} & (7 - 0) & (12 - 7) & (15 - 12) & (16 - 15) & \cdots & (0 - 7) \\
\frac{7}{2} & \frac{5}{2} & \frac{3}{2} & \frac{1}{2} & \cdots & -\frac{7}{2}.
\end{pmatrix}$$

in $f$-parafermi algebra (Eq. (23)), as sums of linear terms,

$$\begin{pmatrix}
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{2} \\
3 & 3 & 1 & 1 & \cdots & -3 \\
\frac{7}{2} & \frac{5}{2} & \frac{3}{2} & \frac{1}{2} & \cdots & -\frac{7}{2}.
\end{pmatrix}$$

3 A variant of $f$-parafermi algebra

Even though it seems unlikely it can reveal parafermionic aspects of the structure of $(G^{(p)}_{\mu \nu})$, a second version of $f$-parafermi algebra is worth reviewing. One always finds a $g$ (a matrix with free parameters in general), for which

$$[[f^+, f], g] = -2f, \quad [[f^+, f], g^+] = 2f^+.$$

(25)

As the system of linear equations embraced by $g$ is underdetermined, one has to constrain the block structure of $g$ to – compared

with $f$’s – slightly relaxed linear combinations $H_{\mu \nu} c_2 + K_{\mu \nu} c_3 (\mu, \nu = 1, \ldots, (p+1)/2)$ to get the solutions unique, or their range narrowed by further constraints, and $g$ thus constructed. The spin-$p/2$ representation is recovered by imposing the requirement $g = \sum_v g_v, (g_v)_{\mu, s_{v, (\mu)}} = H_{\mu, s_{v, (\mu)}} c_2 + K_{\mu, s_{v, (\mu)}} c_3 \quad (\{s_c\} \cong \mathbb{Z}_2^{q-1})$ and choosing the ansatz

$$\sum_{\nu=0}^{(p-1)/2} \left( \chi[f_v^+, f_v] + \sigma([f_v^+, g_v] + [g_v^+, f_v]) + \tau([f_v, g_v] + [g_v^+, f_v^+]) + \gamma[g_v, g_v^+] \right)$$

$$= \text{diag}(\frac{1}{2}, \frac{1}{2}, \ldots, -\frac{1}{2} + 1, -\frac{1}{2})$$

so that Eqs. (25)–(26) may be slated as a heterotic version of $f$-parafermi algebra. Whatever relationship there might exist between $B_3$ and $G_{\mu \nu} (\mu > \nu)$, by the additional quantities $H_{\mu \nu} K_{\mu \nu} (\mu, \nu = 1, \ldots, (p+1)/2)$ and $\chi, \sigma, \tau, \gamma$ it is rather concealed than revealed. The steps of computation to be taken shall nevertheless briefly be expounded for paraorders 3 and 7, whilst postponing the question of how a generalization for $p \in \{15, 31, \ldots\}$ might look like. The LSE for $g^{(3)}$ has a unique solution which reads

$$g^{(3)} = \begin{pmatrix}
0 & 1 & 0 & \frac{1}{2} \\
0 & 0 & -\frac{3}{2} & 0 \\
0 & \frac{3}{2} & 0 & 1 \\
-\frac{1}{2} & 0 & 0 & 0
\end{pmatrix}.$$

In a similar way as $f^{(3)}$ was treated, $g^{(3)}$ is orthogonally decomposed by following the prescription $0: 11+22, 1: 12+21$, which yields $g = g_0 + g_1$, with block structures

$$(g_0)_{1,1} = (g_0)_{2, 2} = \frac{1}{2} c_2 + \frac{1}{2} c_3, \quad (g_1)_{1, 2} = c_3 - \frac{1}{2} c_2, \quad (g_1)_{2, 1} = c_3 + \frac{1}{2} c_2.$$

By the LSE of the spin-3/2 representation (Eq. (26)) we then obtain a parametrized set of solutions for the normalizing factors,

$$\chi^{(3)} = (4r_2 + 2r_1 + 2)/3,$$

$$\sigma^{(3)} = (-10r_2 - 2r_1 + 1)/2, \quad (r_1 \text{ free parameters})$$

$$\tau^{(3)} = r_2,$$

$$\gamma^{(3)} = r_1.$$

Solving the LSE for $g^{(7)}$ raises a matrix with no less than four free parameters! Of which we may free us – not arbitrarily, but by implying on $g^{(7)}$ the very same symmetries that govern $g^{(3)}$. Three types of these can be read from the above representation of $g^{(3)}$ ($A^T$ transposed matrix, $A^S$ matrix reflected in secondary diagonal): 1) $(A^T B^A)$; 2) $(A^T B^B)$; 3) $(A^C B^B)$, where the subscript in $A_0$ is indicative of a “zero block” in the lower left part of the secondary diagonal: $A_0 = \begin{pmatrix} 0 & V \\ W & 0 \end{pmatrix}$. In fact, each of the
which those belonging to 1) can be made to conform to by the choice $r_1 = 0$ and which those belonging to 3) differ from by no more than $\approx 2\%$. Viewed in this light, \{ $\chi^{(7)} = 1, \sigma^{(7)} = \frac{-1}{10}, \tau^{(7)} = \frac{1}{4}, \gamma^{(7)} = 0$ \} can be considered the standard normalizing factor set for $p = 3$. It cannot be excluded that other types of symmetries expand the range of viable solutions; lack of symmetry however – by simply setting all four parameters in the general matrix of $g^{(7)}$ equal to zero – only results in \{} for the set of normalizing factors.

Conspicuously, variant 2) seems to bring out a “standard set” of normalizing factors \{ $\chi^{(7)} = 1, \sigma^{(7)} = -\frac{(r_1 + 2)}{8}, \tau^{(7)} = \frac{r_1 + 2}{8}, \gamma^{(7)} = r_1$ \};

After this aside, we turn to interordinality as a way of exploring the putative parafermionic nature of the coefficients $G_{\mu \nu}$ by relating them directly to the Green coefficients $B_3$. 

$$g^{(7)} = \begin{pmatrix}
0 & 5 & 0 & 2 & 0 & 9/20 & 0 & 1/10 \\
3/8 & 0 & -1/4 & 0 & -1/8 & 0 & 1/8 & 0 \\
0 & 1/4 & 0 & 5/8 & 0 & 9/20 & 0 & 9/20 \\
-2/3 & 0 & 5/8 & 0 & 1/3 & 0 & 5/8 & 0 & 2/3 \\
-2/3 & 0 & -1/20 & 0 & 3/8 & 0 & -1/4 & 0 \\
0 & -1/4 & 0 & 1/8 & 0 & 1/4 & 0 & 5/8 \\
-1/10 & 0 & -1/20 & 0 & 2/5 & 0 & 3/8 & 0 \\
\end{pmatrix}$$

$$\chi^{(7)} = 1, \sigma^{(7)} = -\frac{(r_1 + 2)}{8}, \tau^{(7)} = \frac{r_1 + 2}{8}, \gamma^{(7)} = r_1$$

$$g^{(7)} = \begin{pmatrix}
0 & 1/24 & 0 & -1/60 & 0 & -1/40 & 0 & 1/60 \\
23/24 & 0 & 0 & 0 & -5/24 & 0 & -1/6 & 0 \\
0 & 0 & 0 & 3/8 & 0 & 1/5 & 0 & -1/40 \\
1/60 & 0 & 5/8 & 0 & -7/60 & 0 & -5/27 & 0 \\
0 & 5/24 & 0 & 7/60 & 0 & 3/8 & 0 & -1/40 \\
1/20 & 0 & -5/24 & 0 & 5/8 & 0 & 0 & 0 \\
0 & 1/8 & 0 & 5/27 & 0 & 0 & 0 & 1/27 \\
-1/60 & 0 & 1/20 & 0 & 1/60 & 0 & 1/27 & 0 \\
\end{pmatrix}$$

$$\chi^{(7)} = 1, \sigma^{(7)} = -1/4, \tau^{(7)} = 1/4, \gamma^{(7)} = 0$$

$$g^{(7)} = \begin{pmatrix}
0 & 1/24 & 0 & -1/60 & 0 & -1/40 & 0 & 1/60 \\
187/200 & 0 & -1/100 & 0 & -41/200 & 0 & -3/20 & 0 \\
0 & 0 & 0 & 3/8 & 0 & 1/5 & 0 & -1/40 \\
0 & 0 & 123/200 & 0 & -3/25 & 0 & -41/200 & 0 \\
0 & 5/24 & 0 & 7/60 & 0 & 3/8 & 0 & -1/40 \\
3/200 & 0 & -21/200 & 0 & 123/200 & 0 & -1/100 & 0 \\
0 & 1/8 & 0 & 5/27 & 0 & 0 & 0 & 1/27 \\
-1/60 & 0 & 3/200 & 0 & 0 & 0 & 187/200 & 0 \\
\end{pmatrix}$$

$$\chi^{(7)} = \frac{14183539}{14137018}, \sigma^{(7)} = -\frac{1737725}{7068509}, \tau^{(7)} = \frac{1738225}{7068509}, \gamma^{(7)} = \frac{1475000}{7068509}$$
4 Structure of the members of root-\(f\) sequence

4.1 Interordinal aspect vs. intraordinal aspect

We first encountered the structural interordinal aspect in links between the paraorder \(p\) and its upper carry-bit neighbor \(p'\) (Eqs. (6) and (7) which pair \(b^{(p)}\), \(b^{(p')}\) and \(f^{(p)}\), \(f^{(p')}\) respectively) and with its lower carry-bit neighbor (the \((p+1)\times(p+1)\) array of \(f^{(p)}\) and \(h^{(p)}\) vs. the \((\frac{p+1}{2})^2 + 1\) \((\frac{p+1}{2})^2 + 1\) structure of \((G_{ll}^{(p)}_{f})\) and \((J_{ll}^{(p)}\)). We briefly touched a further link existing between \(p\) and its next but one lower carry-bit neighbor (the Catalan “accounting identities” controlling the \((\frac{p+3}{2})^2 + 1\) \((\frac{p+3}{2})^2 + 1\) structure of the lower left quadrant of \((G_{ll}^{(p)}_{f})\) and \((J_{ll}^{(p)}\)), and we shall soon find it necessary to enlarge the picture by one or several more (higher or lower) carry-bit neighbors, so it seems suitable to adopt a shorthand for them. In analogy to the denotation of Mersenne numbers \(M_n = 2^n - 1\), we occasionally find it convenient to write \(p_n\) for \(p = 2^n + 1\), \(p_{n+1} = 2p + 1\) and so on, also \(q_a\) for \(q = (p-3)/4\) or \(q_{n+1} = (p-7)/8\) or \(q_{n+1} = q' = (p-1)/2\).

Let us begin with the link between \(b^{(p)}\) and \(b^{(p')}\) where the structural interordinal aspect enters in a basic way; every coefficient that falls within order \(p\) is echoed by every second coefficient that falls within order \(p'\) via the relation \(B_{2b}^{(p')} = 2B_{2b}^{(p)}\) : for instance

\[
\begin{align*}
p' = 15: & & \sqrt{15} \sqrt{28} \sqrt{39} \sqrt{48} \sqrt{55} \sqrt{60} \sqrt{63} \sqrt{64} \ldots, \\
p = 7: & & \sqrt{7} \sqrt{12} \sqrt{15} \sqrt{16} \ldots.
\end{align*}
\]

A variant of this doubling effect then likely is to be found when ascending from \(f^{(p)}\) to \(f^{(p')}\). Before elaborating, let us address the exponential nature of the objects we deal with. \(f^{(31)} = \sqrt{\sqrt{\sqrt{\sqrt{1 \otimes 1 \otimes 1 \otimes 1}}} = \text{the first of root-}f \text{ sequence members that is too wide to fit standard paper size} – \text{as we need navigation of some form for them, however, we introduce a minimum of new notation, speaking of upper/lower left, or upper/lower right, parts to retain some depictability. Specific quadrants are determined by one-place navigation} || = \text{arg, subquadrants by} || = (|| = \text{arg}), \text{and so on. The first observation worth a mention is that all subquadrants (and quadrants, as well as} f^{(p)} \text{itself) show invariance under reflection in the secondary diagonal – sometimes called secondary symmetry [ALee76]:}
\]

\[
|| = (|| = f^{(p)}) = (|| = (|| = f^{(p)}))^S,
\]

where the sequence of symbols \(|| = \) is the same on both sides of the equation. One further is that identical subquadrant content appears at different places, namely at \(UL(ULf)\) and \(LR(LLf)\). Also, at \(LL(ULf)\) and \(LL(LRf)\) and, flanked by these, at \(UR(LLf)\):

\[
\text{Figure 1. Identical subquadrants}
\]

\[
f^{(7)} = \begin{pmatrix} f^{(1)} & 0 & 0 & 0 \\ c_3 & c_3 \end{pmatrix} \quad , \quad f^{(15)} = \begin{pmatrix} f^{(1)} & 0 & \ldots & 0 \\ c_3 & c_3 & f^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ c_3 & c_3 & 5c_3 & 3c_3 \\ 11c_3 & 5c_3 & 17c_3 & 11c_3 \\
41c_3 & 17c_3 & 5c_3 & 3c_3 & 11c_3 & 5c_3 & 11c_3 & 5c_3 \\
5c_3 & 3c_3 & c_3 & c_3 & f^{(1)} & c_3 & c_3 & c_3 & c_3 & c_3 & c_3 & f^{(1)} \\
c_3 & c_3 & c_3 & c_3 & f^{(1)} & c_3 & c_3 & c_3 & c_3 & c_3 & c_3 & f^{(1)} \\
c_3 & c_3 & c_3 & c_3 & f^{(1)} & c_3 & c_3 & c_3 & c_3 & c_3 & c_3 & f^{(1)} \\
c_3 & c_3 & c_3 & c_3 & f^{(1)} & c_3 & c_3 & c_3 & c_3 & c_3 & c_3 & f^{(1)} \\
\end{pmatrix}
\]

Now the content of the single framed blocks in \(f^{(p')}\) is identical to the quadrant \(LLf^{(p)}\). Thus, if the double framed blocks are suggestive of the notion of subquadrantal intraordinality, the single framed ones (shown bracketed in what follows) may be attributed to what we here call subquadrantal interordinality (not to be confused with the term used in statistics).

This property of two types of ordinality governing the root-\(f\) structure is further refined on the subsquadrantal level, emerging first with \(p = 7, p' = 15:\)
Figure 2. Interordinally related subsubquadrants

interordinal identity:

\[ UR(UL(LLf^{(p'}))) = LL(LLf^{(p)}) + 2 UR(LLf^{(p)}) \]  \hspace{1cm} \text{(28)}

\[
LL f^{(15)} = \begin{pmatrix}
5c_3 & 3c_3 \\
11c_3 & 5c_3 \\
41c_3 & 17c_3 \\
113c_3 & 41c_3
\end{pmatrix}
\]

\[
LL f^{(31)} = \begin{pmatrix}
429c_3 & 155c_3 & 43c_3 & 19c_3 \\
1275c_3 & 429c_3 & 115c_3 & 43c_3 \\
4819c_3 & 1595c_3 & 429c_3 & 155c_3 \\
15067c_3 & 4819c_3 & 1275c_3 & 429c_3
\end{pmatrix}
\]

Figure 3. Intraordinally related subsubquadrants

intraordinal identity:

\[ UR(LL(LLf^{(p'}))) = LL(UL(LLf^{(p'}))) + 2 UR(LLf^{(p')}) \]  \hspace{1cm} \text{(29)}

\[
LL f^{(15)} = \begin{pmatrix}
5c_3 & 3c_3 \\
11c_3 & 5c_3 \\
41c_3 & 17c_3 \\
113c_3 & 41c_3
\end{pmatrix}
\]
The logical consequence of the structural interordinal aspect is that it restricts the domain from which to select specific identities (29,31) as cosine-like.

As will be explained in Sect. 9.2, the interordinal identities (28,30) can be classified as sine-like, and the intraordinal identities (29,31) as cosine-like.

The logical consequence of the structural interordinal aspect is that it restricts the domain from which to select specific coefficients as representatives falling within order \( p \). We define the representatives \( G^{(p)}_{\mu\nu} \) as to be taken from those \( G^{(p)}_{\mu\nu} \) that spring from the nonbracketed part of \( LLf^{(p)} \), denoted \( -\text{UR}(LLf^{(p)}) \). To find out how many such representatives \( G^{(p)}_{\mu\nu} \) exist, we have to address Catalan structure next.

4.2 Secondary trace structure vs. stub structure

As mentioned in the introduction, the bookkeeping on the coefficients of \( LL(G^{(p)}_{\mu\nu}) \) is done by way of \( \text{str()} \), the symbol signifying traces over the secondary and adjacent diagonals, in that quadrant. Taking these as gross traces,

\[
gstr G^{(p)}_{q+1+\xi-\zeta,\xi+\zeta} (\xi, \zeta + 1 \in \{1, \ldots, q + 1\}),
\]

yields sums of \( C_{q-1+\xi} \) in general. For \( LL(G^{(31)}_{\mu\nu}) \) for instance:

\[
\begin{align*}
\Sigma_{\zeta=0}^{1}G_{10-\zeta,1+\zeta} &= C_{8}, \\
\Sigma_{\zeta=0}^{3}G_{12-\zeta,1+\zeta} &= C_{10}, \\
\Sigma_{\zeta=0}^{5}G_{14-\zeta,1+\zeta} &= C_{12}, \\
\Sigma_{\zeta=0}^{7}G_{16-\zeta,1+\zeta} &= C_{14}.
\end{align*}
\]

An alternative way to keep records is with net secondary traces, \( \text{nstr()} \):

\[
\text{nstr} G^{(p)}_{q-n+1+\xi-\zeta,\xi+\zeta} = C_{q-n-1+\xi},
\]

where

\[
\xi, \zeta + 1 \in \{1, \ldots, q_n + 1\}; G^{(p)}_{q_n+1+\xi-\zeta,\xi+\zeta} \neq G^{*(p)}_{\mu\nu}.
\]
This one requires a preprocessing where each main-type diagonal, the main diagonal itself and its adjacents, are traversed from the upper leftmost entry to the lower rightmost and checked for duplicates which are marked when identified. The \texttt{nsr()} then simply skips marked entries (see Fig. 4 where duplicates are marked by an asterisk).

Figure 4. Secondary trace structure of $LL(G^{(31)}_{\mu \nu})$

\[
\begin{array}{cccccccc}
429 & 155 & 43 & 19 & 5 & 3 & 1 & 1 \\
1275 & 429* & 115 & 43* & 11 & 5* & 1 & \\
4819 & 1595 & 429* & 155* & 41 & 17 & & \\
15067 & 4819* & 1275* & 429* & 113 & & & \\
58781 & 18627 & 4905 & 1633 & & & & \\
189371 & 58781* & 15297 & & & & & \\
737953 & 227089 & & & & & & \\
2430289 & & & & & & & \\
\end{array}
\]

Now, any summand lying in a (net secondary) trace $r$ positions away from that of $C_q$ obeys an upper bound $4rC_q$ because \[\lim_{n \to \infty} C_{n+1}/C_n = 4.\] Thus $G^{(p)}_{\text{max}} = G^{(p)}_{2q+2,1}$, though the summand largest in its trace, does stay well below this bound: $113 \approx 2.82^3 \cdot 5$; and $2430289 \approx 3.43^7 \cdot 429$. From $G^{(p)}_{\text{max}} = (\Phi^{(p)})^q C_q$ we may define a span parameter $\Phi^{(p)}$ with continued fraction representation (CFR)

\[\Phi^{(p)} = \left( \frac{G^{(p)}_{\text{max}}}{C_q} \right)^{1/q} = \phi_{0}^{(p)} + \frac{1}{\phi_{1}^{(p)} + \frac{1}{\phi_{2}^{(p)} + \frac{1}{\phi_{3}^{(p)} + \ddots}}} \equiv [\phi_{0}^{(p)}; \phi_{1}^{(p)}, \phi_{2}^{(p)}, \phi_{3}^{(p)}, \ldots]. \tag{32}\]

The $LL(G^{(p)}_{\mu \nu})$ coefficients $< G^{(p)}_{\text{max}}$ then become coefficients $\phi_{\alpha+j(2q+2)-[(j+1)/2]}^{(p)}$ of the span parameter for some start value $\alpha$. Those of $LL(G^{(15)}_{\mu \nu})$ take the form $\phi_{5+8j-[(j+1)/2]}^{(15)}$ for $j = 0, 1, \ldots, 5$, while the special form $G^{(15)}_{\text{max}} = G^{(15)}_{8,1} = \phi_{50}^{(15)} - \sum_{i=1}^{q} D_i$ is assumed for $j = 6$:

\[\Phi^{(15)} = [2; \ldots, 1, \ldots, 3, \ldots, 5, \ldots, 11, \ldots, 17, \ldots, 41, \ldots, 139, \ldots] \]

\[0 \quad 5 \quad 12 \quad 20 \quad 27 \quad 35 \quad 42 \quad 50 \]

2 Some conspicuous near matches are springing up incidentally: Five of kissing numbers with deviations $\Delta_L < G^{(15)}_{\text{max}}$, viz. $\phi_{50}^{(15)} = 139 = L_7 + 13$, $\phi_{173}^{(15)} = 230 = L_8 - 10$, $\phi_{403}^{(15)} = 431 = L_{11} - 7$, $\phi_{508}^{(15)} = 1432 = L_{14} + 10$, $\phi_{128}^{(15)} = 10558 = L_{19} - 110$; and two of Catalan numbers with $\Delta_C < C_3$, viz. $\phi_{308}^{(15)} = 1432 = C_8 + 2$, $\phi_{403}^{(15)} = 431 = C_7 + 2$. We shall come back to the relationship between kissing numbers and Catalan numbers before long.
A corollary to its secondary symmetry is what we call stub structure of \( LL \left( G_{\mu \nu}(p) \right) \), defined by the coefficient-wise homogeneity of its main diagonal and the main diagonals of its subquadrants, subsubquadrants etc. to either side. It may also be recognized that the subquadrants \( UL(LL(G_{\mu \nu}(p))) \) and \( LL(LL(G_{\mu \nu}(p))) \) suffice as sources for \( G_{\rho \sigma}^p \), and what is more, intraordinal relation (29) guarantees them to be independent. The number of different \( G_{\rho \sigma}^p \), denoted \( T_p \) here, is readily computed by inspecting these subquadrants using their secondary symmetry and stub structure. As square matrices of dimension \( \frac{q+1}{2} \times \frac{q+1}{2} \) with secondary symmetry, they have at most \( \frac{(q+1)(q+3)}{8} \) distinct elements, the entries to a secondary, skewed triangular matrix each:

Redundant copies of entries on the subquadrantal main-diagonal, and subsubquadrantal main-diagonal etc. stubs to either side (see Fig. 5) have to be subtracted yet. For \( p = 15 \), \( \frac{(q+1)(q+3)}{8} = 3 \), there are no subtractions, hence \( T_{15} = 6 \). For \( p = 31 \), \( \frac{(q+1)(q+3)}{8} = 10 \), there’s one subtraction for either subquadrant, hence \( T_{31} = 18 \). For \( p = 63 \), \( \frac{(q+1)(q+3)}{8} = 36 \), tedious but straightforward calculations show that nine upper-subquadrantal entries and nine lower-subquadrantal entries are redundant, hence \( T_{63} = 54 \). Thus, while the subtractions seem nontrivial, the result boils down to a simple formula for the number of distinct representatives,

\[
T_p = 2 \cdot 3^{\log_2(p+1) - 3} \quad (p = 15, 31, 63, \ldots).
\]  

(33)

See Table 2 for a summary.
The stub structure-based ansatz takes account of subtractions (row 2) and reads

$$2 \left( \frac{(q+1)(q+3)}{8} - s \right).$$  

The subtractions \( s_n, n = \log_2(p_n + 1), \) build from two types of atoms, \( m_\mu \equiv p_\mu + p_{\mu-1}, \) and \( o_\nu \equiv (p_\nu + p_{\nu-2})/2. \) For \( p_5 = 31, \) \( s_5 = m_1 = 1. \) For \( p_6 = 63, \) \( s_6 = o_4 = 9. \) For \( p_7 = 127, \) the first more complex case, a mix shows up, \( s_7 = m_5 + o_4 = 55. \) This mix begins to branch in increasingly complex ways, \( s_8 = \sum_{n=1}^{2} (m_7 + m_5 + m_1) + (o_6 + o_4) = 285, \) \( s_9 = \ldots, \) but should ultimately lead to a proof of the following conjecture: Let \( L_s \) be the set of numbers \( \left\lfloor \log_2(C_q C_{2q+1}) \right\rfloor > s, \) where \( q^* \in \{1, 3, 7, 17, \ldots\}. \) Then the least element \( l_{\min} \in L_s \) satisfies \( l_{\min} - s \equiv u \pmod{16} \) where \( u \in \{1, 5, 9, 13\}. \) For \( \left(\frac{q+1}{8}\right)^{(q+3)} = 10, \) \( s = 1, \) we find \( \left\lfloor \log_2(C_3C_5) \right\rfloor - s \equiv 1 \pmod{16}; \) also for \( \left(\frac{q+1}{8}\right)^{(q+3)} = 36, \) \( s = 9, \) \( \left\lfloor \log_2(C_3C_7) \right\rfloor - s \equiv 1 \pmod{16}; \) the next instances are \( \left(\frac{q+1}{8}\right)^{(q+3)} = 136, s = 55 \) with \( \left\lfloor \log_2(C_3C_{31}) \right\rfloor - s \equiv 5 \pmod{16}, \) \( \left(\frac{q+1}{8}\right)^{(q+3)} = 528, s = 285 \) with \( \left\lfloor \log_2(C_3C_{127}) \right\rfloor - s \equiv 9 \pmod{16}, \) and so on.

4.3 Row (column) structure by way of kissing-number stenoscop eye

Much in the same way as the (secondary-diagonal) trace structure and (diagonal) stub structure of \( LL(G^{(p)}_{\mu'}) \) are governed by Catalan numbers, rows or columns, by way of additive partitions of their elements which in the simplest case are of length \( q \pm 1, \) harbor kissing numbers that are sandwiched between Catalan numbers in a stenoscopic way as shown in Table 3.

For \( p = 15, \) the partitions in question have already been presented in Eqs.(18); of these, those which are of length 4 are

| \( L_2 = 6 \) | 5 + 3 - 1 - 1 | \( G_{5,1}, G_{5,2}, G_{5,3}, G_{5,4} \) | (UL(LL(\( G^{(p)}_{\mu'} \))), UR(LL(\( G^{(p)}_{\mu'} \)))) |
| \( L_5 = 40 \) | 41 - 17 + 11 + 5 | \( G_{7,1}, G_{7,2}, G_{8,3}, G_{8,4} \) | (LL(LL(\( G^{(p)}_{\mu'} \))), LR(LL(\( G^{(p)}_{\mu'} \)))) |
| \( L_7 = 126 \) | 41 - 17 + 113 - 11 | \( G_{7,1}, G_{7,2}, G_{8,3}, G_{8,4} \) | (LL(LL(\( G^{(p)}_{\mu'} \))), LR(LL(\( G^{(p)}_{\mu'} \)))) |

and those of length 2

| \( L_3 = 12 \) | 17 - 5 | \( G_{7,1}, G_{7,2}, G_{7,3} \) | (LL(LL(\( G^{(p)}_{\mu'} \))), LR(LL(\( G^{(p)}_{\mu'} \)))) |
| \( L_4 = 24 \) | 41 - 17 | \( G_{7,1}, G_{7,2}, G_{7,3} \) | (LL(LL(\( G^{(p)}_{\mu'} \))), LR(LL(\( G^{(p)}_{\mu'} \)))) |
| \( L_6 = 72 \) | 113 - 41 | \( G_{7,1}, G_{7,2} \) | (LL(LL(\( G^{(p)}_{\mu'} \))), LR(LL(\( G^{(p)}_{\mu'} \)))) |
For $p' = 31$, there is an interordinal corridor in which the remaining kissing numbers $\sum_{i=1}^{q+1} G_{2q+2,i}^{(p)} > L_2 < C_{q'}(= C_5)$ reside; they are given by additive partitions of length $\rho \in \{q \pm 1, q' \pm 1\}$.  

$$L_8 = 240 \colon 429-155-(43-19)-[5+3+1+1] \quad G_{0,1} \quad G_{0,2} \quad G_{0,3} \quad G_{0,4} \quad G_{0,5} \quad G_{0,6} \quad G_{0,7} \quad G_{0,8} \quad \left(UL(\text{LL}(\text{G}_{^{(p')}^{(p'^*)}})), \text{UR}(\text{LL}(\text{G}_{^{(p')}^{(p'^*)}}))\right)$$

$$L_9 = 272 \colon 429-115-[41+17]+[11+5] \quad G_{10,2} \quad G_{10,3} \quad G_{11,5} \quad G_{11,6} \quad G_{12,7} \quad G_{12,8} \quad \left(UL(\text{LL}(\text{G}_{^{(p')}^{(p'^*)}})), \text{UR}(\text{LL}(\text{G}_{^{(p')}^{(p'^*)}}))\right)$$

$$L_{10} = 336 \colon 429-155+(43+19) \quad G_{13,5} \quad G_{13,6} \quad G_{13,7} \quad G_{13,8} \quad LR(\text{LL}(\text{G}_{^{(p')}^{(p'^*)}})).$$

We adopt the name corridor G-set for the collection of $G_{\mu \nu}^{(p)}$ that potentially partake in additive partitions realizing kissing numbers that reside in the interordinal corridor $\sigma_{t=1}^{q+1/2} G_{t+1,i}^{(p-1/2)}$, $C_q$. For $p = 15, 31$, the corridor G-set is $G_{\text{cor}}^{(15)} = \{1, 3\}$, $G_{\text{cor}}^{(31)} = \{1, 3, 5, 11, 17, 41\}$. For $p = 15, 31$, the corridor G-set is $G_{\text{cor}}^{(15)} = \{1, 3\}$, $G_{\text{cor}}^{(31)} = \{1, 3, 5, 11, 17, 41\}$. Little is known for certain about higher kissing numbers, but Table 3 gives valuable hints on not directly accessible details especially about how many of them would belong to LL($G_{\text{cor}}^{(31)}$) and how many to LL($G_{\text{cor}}^{(63)}$) etc.

**Conjecture 1** Let $R_{a}^{(n)}$ ($n = \log_2(p + 1)$) be the # of kissing numbers in $\sigma_{t=1}^{q+1/2} G_{t+1,i}^{(p-1/2)}$, $C_q$, $R_{b}^{(n)}$ the # of those in $|C_{2q}, G_{2q}|$, and $R_{c}^{(n)}$ the # of those in $|C_{2q}, \Sigma_{t=1}^{q+1/2} G_{2q+2,i}^{(p)}|$, all representable by suitably chosen additive partitions from rows (columns) in LL($G_{\mu \nu}^{(p)}$) (or LL($J_{\mu \nu}^{(p)}$)). Then

$$R_{a}^{(n)} = R_{c}^{(n)} = \frac{7}{2} T_{(p-1)/2} (n > 4),$$

(35)

where $R_{a}^{(n)}$ (marked by dashed-line, dotted-line delimiters in Table 3) determines the corridor G- (J)-set, $G_{\text{cor}}^{(p)}$, $J_{\text{cor}}^{(p)}$. 

Case $p = 7$ is degenerate, with interval $[1, 2]$ harboring one kissing number, $L_1 = 2$. We set $R_{a}^{(3)} = R_{c}^{(3)} = 0$, $R_{b}^{(3)} = 1$.

Case $p = 15$: No kissing number lives in $[2, 5]$ : $R_{a}^{(4)} = 0$ – which is equivalent to saying the interordinal corridor G-set $G_{\text{cor}}^{(15)}$ has no unbracketed, unparenthesized entries –, and in $[132, 170]$ live none either: $R_{c}^{(4)} = 0$, thus the case is degenerate, too. Since interval $[5, 132]$ harbors six kissing numbers – 6, 12, 24, 40, 72, 126 –, we set $R_{b}^{(4)} = 6$.

Case $p = 31$: In $[170, 429]$ we find $R_{a}^{(5)} = 3$ kissing numbers, indicated by the presence of unbracketed, unparenthesized entries in the corridor G-set $G_{\text{cor}}^{(31)}$. In $[429, 2674440]$ , $R_{b}^{(5)} = 18$, and in $[2674440, 3437984]$ , $R_{b}^{(5)} = 3$ (most of them uncertified).

Case $p = 63$: In $[3437984, 9694845]$ live $R_{a}^{(6)} = 9$ kissing numbers, in $[9694845, C_{30}] R_{b}^{(6)} = 54$ and in $|C_{30}, \Sigma_{i=1}^{16} G_{32,1}^{(63)}| R_{b}^{(6)} = 9$.

And so on for the cases $p = 127, 255, \ldots$

All partitions realizing kissing numbers may equivalently be defined over columns, and the reader is invited to identify the corresponding rows and columns in LL($J_{\mu \nu}^{(p)}$), too.

---

3 entries $G_{\mu \nu}^{(p')}$ that owe their existence to the subsubquadrantal identity (28) are set in parentheses, while those springing from UR(UL($G_{\mu \nu}^{(p')}$)) are set in brackets

4 the table favors the latest results by Nebe (summarized in [Nebe11]) over older ones regarding Mordell-Weil lattice, Barnes-Wall lattice
Table 3

Stenoscropy of kissing numbers of Euclidean $D$-space relative to Catalan numbers and least-row-of-LL sums $\sum_{i=1}^{q+1} c_{2q+2i}^{(p)}$

| $D$ | $L_D$ | bounding Catalan numbers and least-row-of-LL sums |
|-----|-------|-------------------------------------------------|
| 1   | 2     | $\leq C_2 < C_3 = C_4 = 5$                     |
| 2   | 6     | $> C_3 = 5$                                     |
| 3   | 12    | $< C_4 = 14$                                    |
| 4   | 24    | $< C_5 = 42$                                    |
| 5   | 40    |                                                 |
| 6   | 72    |                                                 |
| 7   | 126   | $\leq C_6 (= 132) \leq \sum_{i=1}^{4} G_{8,i}^{(15)} (= 170)$ |
| 8   | 240   | $> \sum_{i=1}^{4} G_{8,i}^{(13)}$               |
| 9   | (Leech lattice) 272 |                                           |
| 10  | 336   | $< C_7' = C_7 = 429$                             |
| 11  | 438   | $> C_7 = 429$                                    |
| 12  | 648   | $< C_8 = 5$                                     |
| 13  | 906   |                                                 |
| 14  | 1422  | $< C_9 = 1430$                                  |
| 15  | 2340  |                                                 |
| 16  | (certified) 4320 | $< C_9 = 4862$                                 |
| ?   | ?     | $< C_{10} = 16796$                               |
| ?   | ?     | $< C_{11} = 58786$                               |
| 24  | (certified) 196560 | $< C_{12} = 208012$                             |
| ?   | ?     | $< C_{13} = 742900$                              |
| 28  | ?     | $\leq C_{14} = 2674440$                          |
| 29  | ?     | $< C_{14} = 2674440$                             |
| 30  | ?     | $> C_{14} = 2674440$                             |
| 31  | ?     | $\leq \sum_{i=1}^{8} C_{16,i}^{(31)} (= 3437984)$ |
| 32  | ?     | $> \sum_{i=1}^{8} G_{16,i}^{(31)}$              |
| ?   | ?     |                                                   |
| 40  | ?     | $< C_{15}'' = C_{15} = 9694845$                  |
| 41  | ?     | $> C_{15}'' = C_{15} = 9694845$                  |
| ?   | ?     |                                                   |
| 48  | (Nebe) 52416000 | $< C_{17} = 129644790$                          |
| ?   | ?     |                                                   |
| 72  | (Nebe) 6218175600 | $< C_{20} = 6564120420$                        |
| ?   | ?     |                                                   |
| 102 | ?     | $\leq C_{30} = 3814986502092304$                |
| 103 | ?     | $> C_{30}$                                       |
| ?   | ?     |                                                   |
4.4 The factorization aspect

The question of whether the partial sequences \(G^{(3)} = 1\), \(G^{(7)} = 1\), \(G^{(15)} = \{3, 5, 11, 17, 41, 113\}\), ... always consist of prime-numbered representatives at paraorder fifteen or higher may have seemed intriguing at first: after all, the expressions \(B_k^2 = 2\) themselves started out with primes for \(p = 7, 15, 31, 63\). Notable as these facts may be, we have afterwards seen that the bulk of \(G_p^{(p)}\) do not stay prime as we go along with the computation of the next higher members of the root-\(f\) sequence. In fact, the prime-numbered \(G_p^{(p)}\) decrease in number – six in \(\neg \text{UR}(LL(G^{(15)}_{\mu\nu}))\), four in \(\neg \text{UR}(LL(G^{(31)}_{\mu\nu}))\) –, circumstantial evidence that might indicate a trend. For a first, identity (28) provides an opportunity for a quick prime number test. For \(p = 15, p' = 31\), Eq. (28) furnishes 19 und 43, the paraorder-thirty-one twins of the paraorder-fifteen primes 17 und 41. And for paraorders thirty-one and sixty-three, this equation yields

\[
\text{UR}(\text{UL}(LL(G^{(63)}_{\mu\nu'}))) = \left( G^{(63)}_{16+\xi,4+\zeta} \right) = \begin{pmatrix}
58791 & 18633 & 4907 & 1635 \\
189393 & 58791 & 15299 & 4907 \\
738035 & 227123 & 58791 & 18633 \\
2430515 & 738035 & 189393 & 58791 \\
\end{pmatrix},
\]

adding one additional prime, \(\pi_{1787} = 15299\). But prime-numbered \(G^{(63)}_{\mu'\nu'}\) can spring from any other quadrant of \(\neg \text{UR}(LL(G^{(63)}_{\mu\nu}))\). However small the increase in knowledge to expect from the endeavor, we thought the question intriguing enough to undertake a complete int64 computation of all 54 representatives of \(\neg \text{UR}(LL(G^{(63)}_{\mu\nu}))\) – and found three more primes, \(\pi_{2\ 364\ 489} = 38\ 792\ 251\), \(\pi_{7} = 69\ 531\ 783\ 535\ 237\) and \(\pi_{7} = 283\ 858\ 869\ 110\ 417\). If primes refuse to go, maybe their number settles on 4 from Mersennian paraorder 31 on.

More generally, the quantities \(G^{(p)} > 1\) can be classified by their factorization. We distinguish pure prime numbers \(\pi_r\), factorization into two or three prime factors, \(\pi_r \cdot \pi_s\) and \(\pi_r \cdot \pi_s \cdot \pi_t\), as well as factorization into one or more prime factors exponentiated \(\pi_r^z \cdot \pi_s^z \cdot \pi_t^z \cdot \ldots\), \(z_r > 1 \ (\forall z_s > 1 \ldots)\). Only the conditions up to \(p = 63\) have been analyzed; so far, there’s nothing that contradicts the assumption that the \# of factorization types, with more complex ones pooled as \(\pi_r \cdot \pi_s \cdot \pi_t \cdot \ldots\) and \(\pi_r^z \cdot \pi_s^z \cdot \pi_t^z \cdot \ldots\), respectively, stays as even-numbered as it turns out to be for \(\neg \text{UR}(LL(G^{(p)}_{\mu\nu}), p = 15, 31, 63):\)

\[
\begin{array}{cccc}
\# & G_p & \text{fact’ed as} & \neg \text{UR}(LL(G^{(15)}_{\mu\nu})) & \neg \text{UR}(LL(G^{(31)}_{\mu\nu})) & \neg \text{UR}(LL(G^{(63)}_{\mu\nu})) \\
\hline
\pi_r & 6 & 4 & 4 \\
\pi_r \cdot \pi_s & - & 6 & 16 \\
\pi_r \cdot \pi_s \cdot \pi_t & - & 6 & 14 \\
\pi_r \cdot \pi_s \cdot \pi_t \cdot \ldots & - & - & 8 \\
\pi_r^z & - & 2 & 12 \\
\sum & 6 & 18 & 54 \\
\end{array}
\]

One further observation is that composite \(G^{(31)}_{\mu\nu}\) – indeed the bulk of them – are missing pure prime numbers with minimal spacings that lie in the same range as the \# of \(G^{(31)}_{\mu\nu}\) congruent with \((7 - 2k) \text{mod} \ 8\) \((k = 1, 2, 3)\) – see Table 6 for a summary. Conversely, just as a \(G_{\mu\nu}\) congruent with 7 modulo 8 is absent from \(\neg \text{UR}(LL(f^{(31)}))\), so is the minimal spacing 14 involving the factor 7.
At the interval in question, prime numbers are relatively close to one another, so instead of surmising some lawfulness behind this phenomenon, suffice it to say in this section that the $G^{(31)}_\rho \equiv \pi_7$ (mod 8)$(k = 0, 1, 2, 3)$ (as shown in Table 6) and the prime-number interpolations seem to follow a common structural pattern.

Let us now address a phenomenon that we considered important enough to assign its characteristic order a separate letter, $q$, consistently meaning $(p - 3)/4$. As we have seen, there are places where the coefficients $G^{(p)}_\rho$ directly intersect with the root-$h$ associated coefficients $J^{(p)}_\omega$ times minus one – demonstrated in Fig. 7 as framed entries for $LL(G^{(31)}_{\mu\nu})$ and $LL(J^{(31)}_{\mu\nu})$ respectively; the appendant entries for $LL(G^{(p)}_{\mu\nu})$ and $LL(J^{(p)}_{\mu\nu})$ for $p = 15, p = 7$ are set off in the bracketed parts, upper right:
The associated Catalan number $C_q$ takes a special role here, viz.$$
abla^{(p)}_{q+1+\xi,\xi} + \nabla^{(p)}_{q+1+\zeta,\zeta} = C_q - C_q = 0 \quad (\xi, \zeta \in \{1, \ldots, q+1\}). \tag{37}$$

This relation is part of a larger underlying symmetry: while the quadrant sum $\LL(G^{(p)}_{\mu \nu}) + \LL(J^{(p)}_{\mu \nu})$ is secondary symmetric and $(n-2)$-fold traceless, one trace vanishing main-, the next to the right submain-, and so on, the Lie bracket of the summands is secondary antisymmetric,$$[\LL(G^{(p)}_{\mu \nu}), \LL(J^{(p)}_{\mu \nu})] = -[\LL(G^{(p)}_{\mu \nu}), \LL(J^{(p)}_{\mu \nu})]^S,$$and $(p_n-1)$-fold traceless, with $p_n-1-2$ traces vanishing main- and adjacent- on either side due to secondary antisymmetry, and one secondary- due to secondary-diagonal zeroing. We may single out the upper left coefficient $G^{(p)}_{q+2,1}$, say, to get what may be called an overarching Catalan representative of $(G^{(p)}_q)$ and $(J^{(p)}_q)$. What makes the latter unique is that, from $p = 15$ on, it displays a peculiar type of factorization that involves a suffix of consecutive prime factors (SCPF) lying in the interval $]q+1, 2q[$. To wit,$$G^{(15)}_{5,1} = C_3 \text{ corresponds to the suffix (underlined) } \underline{5},$$
while to $G^{(31)}_{9,1} = C_7$ there belongs$$429 = 3 \cdot 11 \cdot 13,$$and to $G^{(63)}_{17,1} = C_{15}$
$$9694845 = 3^2 \cdot 5 \cdot 17 \cdot 19 \cdot 23 \cdot 29,$$
followed by$$G^{(127)}_{33,1} = C_{31} = 7 \cdot 11 \cdot 17 \cdot 19 \cdot \underline{37} \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61,$$
followed further by $G^{(255)}_{65,1} = C_{63} =$

\[5\]

special thanks go to wolframalpha.com through whose good offices larger Catalan numbers have now become widely accessible.
and so on, which can be summarized in the multiplicative Euler-product partition
\[
2 \cdot \prod_{p \text{ Mersenne prime}} p \cdot \prod_{p=15,31,\ldots} \text{SCPF}(\mathcal{G}_{q+2,1}^{(p)}) = \prod \pi_r.
\]

With the denotation $\text{SCPF}_p$: the set of prime factors contained in $\text{SCPF}(\mathcal{G}_{q+2,1}^{(p)})$ ($p = 15, 31, \ldots$), the set of all prime numbers becomes the disjoint union of the singleton $\{2\}$, the Mersenne prime numbers and the $\text{SCPF}_p$'s. According to the prime-number density theorem, the number of factors contained in $\text{SCPF}_p$, denoted $S_p$ here, is of order $\frac{2q}{\log(2q)} - \frac{q+1}{\log(q+1)}$. In Table 5, the values of $S_p$ for $p = 15, 31, \ldots$ are listed together with two other order-$\frac{2q}{\log(2q)} - \frac{q+1}{\log(q+1)}$ numbers. Where $n_q = \log_2(q+1)$: the Catalan numbers of half-integer index, $C_{1+n_q/2}$, defined by $\frac{q^{2(1+n_q/2)^2} \Gamma(1+1+n_q/2)}{\sqrt{3.14} \cdot \Gamma(3+n_q/2)}$, and the kissing numbers $L_{n_q-3}$.

| $q$ | $\frac{2q}{\log(2q)} - \frac{q+1}{\log(q+1)}$ | $S_p$ | $C_{1+n_q/2}$ | $L_{n_q-3}$ |
|-----|---------------------------------|-------|----------------|-------------|
| 3   | 0.46                            | 1     | 2              | –           |
| 7   | 1.45                            | 2     | 3.10           | –           |
| 15  | 3.04                            | 4     | 5              | 2           |
| 31  | 5.78                            | 7     | 8.27           | 6           |
| 63  | 10.66                           | 12    | 14             | 12          |
| 127 | 19.48                           | 23    | 24.08          | 24          |
| 255 | 35.63                           | 43    | 42             | (40, 44)    |
| 511 | 65.41                           | 75    | 74.09          | (72, 78)    |
| 1023| 120.64                          | 137   | 132            | (126, 134)  |
| 2047| 223.62                          | 255   | 237.11         | 240         |
| 4095| 419.48                          | 463   | 429            | ×           |

With the definition
\[
\text{SCPF primes} : \bigcup_{p=15,31,\ldots} \text{SCPF}_p
\]
we face a dilemma: they almost form the class of all prime numbers that lie between two consecutive Mersenne numbers, just like, where $C_{1/2} = 0$, $\mathcal{C} = (C_{1/2}; C_{1/2}; C_{1/2}; C_{1/2}; C_{1}; C_{2}; C_{2}; \ldots; C_6; C_8; \ldots; C_{14}; \ldots)$ almost forms the class of all Catalan numbers of non-Mersenne-numbered index that ensue from net traces over the secondary and adjacent diagonals of $LL(G^3_{\mu\nu})$. There clearly is one element missing in either case. Regarding the $\text{SCPF}_p$'s: the prime number 2 lying between the Mersenne numbers 1 and 3, and regarding the sequence $\mathcal{C}$: the one net trace belonging to $LL(G^3_{\mu\nu})$. Incorporating the missing items, we respectively get
\[
\text{SCPF}^+ \text{ prime numbers} : \bigcup_{p=15,31,\ldots} \text{SCPF}_p
\]
and
\[
\mathcal{C}^+ = (C_{1/2}; C_{1/2}; C_{1/2}; C_{1}; C_{2}; C_{2}; \ldots; C_6; C_8; \ldots; C_{14}; \ldots),
\]
with the effect that one even number is included among otherwise odd numbers in the former case, and one odd number among even numbers in the latter.

\[\text{Strictly speaking, only lower and upper bounds are known for them in some places – e.g. (40,44), (72,78) and (126,134); but, according to prevailing knowledge, they do stop being order-}
\]

\[\frac{2q}{\log(2q)} - \frac{q+1}{\log(q+1)}\] \] numbers after dimension eight.
The distinction created between SCPF and Mersenne primes becomes vital when it comes to determining a possible set membership of \(G_{\mu\nu}^{(p)}\) in SCPF\(_{p_N}\), where \(N\) denotes the identity
\[
N = - [(n + 1)/2] + \Sigma_{i=1}^n p_i = [\log_2 C'] .
\] (40)
Then Mersenne primes \(> 3\), it turns out, are not among the factors of \(p_N\), but the factors of \(q_N = p_{N-2}\). Let us, for example, check the membership of \(G_{\text{max}}^{(p)}\) in SCPF\(_{p_N}\) for the few known cases:
\[
G_{\text{max}}^{(15)} = \pi_{6.5} = 113 \in \text{SCPF}_{2^8-1} \Rightarrow p_N = 2^8 - 1 = 3 \cdot 5 \cdot 17, \quad q_N = 2^6 - 1 = 3^2 \cdot 7 ,
\]
and
\[
G_{\text{max}}^{(31)} = \pi_{6.29724} = 2430289 \in \text{SCPF}_{2^{23}-1} \Rightarrow p_N = 2^{23} - 1 = 47 \cdot 178481, \quad q_N = 2^{21} - 1 = 7^2 \cdot 127 \cdot 337 ,
\]
where Mersenne primes \(> 3\) are marked with underbraces. The next instance, \(G_{\text{max}}^{(63)}\), is a composite. We expect its three prime factors 3, 613 and 1,910,047,210,943 to be elements of SCPF\(_{2^{23}-1}\) because we find the conjectured pattern
\[
p_N = 2^{53} - 1 = 6361 \cdot 69431 \cdot 20394401, \quad q_N = 2^{51} - 1 = 7 \cdot 103 \cdot 2143 \cdot 11119 \cdot 131071 .
\]

4.5 The modulo-eight aspect

Taking the modulo-eight aspect into account allows us briefly to resume the subject of Sects. 4.3-4 to show how \(f\)-parafermi algebra (Eqs. (23)-(24)) can be made to hold for \(p \in \{15,31,\ldots\}\). Thus, as odd numbers \(2n + 1\) come with the identity \((2n + 1)^2 = 8 \sum n + 1\) and with all \(G_{\mu\nu}\) odd-numbered, one has \(G_{\mu\nu}^2 = 1\mod 8\). Hence, by Eq. (21),
\[
\frac{1}{2} [(f_0^+, f_0) + \sum_{v=1}^{(p-1)/2} [(f_v^+, f_v)]_{\text{mod8}} = \text{diag}(\frac{p}{2}, \frac{p}{2} - 1, \cdots, \frac{p}{2} - 1, -\frac{p}{2} + 1, -\frac{p}{2}) ,
\]
\[
\sum_{v=0}^{(p-1)/2} [(f_v^+, f_v)_{\text{mod8}}, f_v] = -2f, \quad \sum_{v=0}^{(p-1)/2} [(f_v^+, f_v)_{\text{mod8}}, f_v^+] = 2f^+ .
\] (42)
But it’s worthwhile to have a look at the very arrangement of residues left by \(G_{\mu\nu}\) after division by eight
\[
\left( G_{\mu\nu}^{(7)} \right)_{\text{mod8}} = \left( G_{\mu\nu}^{(7)} \right) = \begin{pmatrix}
0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 3 & 5 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 5 & 3 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 5 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0
\end{pmatrix}
\]
which shows that underneath the overt secondary symmetry of \(f^{(7)}\) the original main symmetry of LL\((G_{\mu\nu}^{(7)})\) exerts its influence on paraorders beyond that mark. Its persistence in modulo-8 form, quadrantwise in LL, subquadrantwise in LLUL, LLLR, URLL etc., makes clear how the heterotic variant of \(f\)-parafermi algebra (Eqs. 25-26) may be reshaped in order to have it hold for \(p \in \{15,31,\ldots\}\), namely:
\[
[(f^o)^+, f^o], g] = -2f^o, \quad [(f^o)^+, f^o], g^+] = 2(f^o)^+, \quad \sum_{v=0}^{(p-1)/2} \left( \chi[(f^o)_v^+, (f^o)_v] + \sigma[(f^o)^+_v, g_v] + [g^+_v, (f^o)_v] + \tau[(f^o)_v, g_v] + [g^+_v, (f^o)_v^+] + \gamma[g_v, g_v^+] \right) = \text{diag}(\frac{p}{2}, \frac{p}{2} - 1, \cdots, -\frac{p}{2} + 1, -\frac{p}{2}) ,
\] (44)
where \(f \equiv f^o \mod 8\), or explicitly, \((f^{(p)}) = 1^{\otimes n-1} \otimes f^{(1)} + (G_{\mu\nu}^{(p)})_{\text{mod8}} \otimes c_3 \).

This is not to say that larger moduli are less important. One can e.g. notice the interesting fact that \(G_{\mu\nu}^{(15)} = 113 \equiv 7^2 \mod 64\) and \(G_{\mu\nu}^{(31)} = 2430289 \equiv 9^2 \mod 128\). The modulo-8 approach is chosen here because it is in agreement with the closure effect that can spring from the group \(\mathbb{Z}_2^3\) through its various isomorphic maps. For octonions, it marks the loss of associativity of the hypercomplex number system; for \(f\)-parafermi algebra bar the modulo-8 approach, the loss of consistency. Plus, kising numbers \(L_{n_q-3}\) with \(3 < n_q = \log_2 (q+1)\) lose their order-\((\frac{2^q}{\log_2 (q+1)} - \frac{2^{q+1}}{\log_2 (q+1)})\)-number characteristics after dimension eight (see Table 5).
Apart from its consequences for $f$-parafermi algebra, the persistence even of main symmetry in modulo-8 form allows a very compact way of describing the LL part:

$$LL(G_{u\nu}^{(p)})_{\text{mod 8}} = \text{sym}(d_m(\_\_), ..., d_s(\_\_)).$$

Applied to the case $p = 31$, say, the expression

$$LL(G_{u\nu}^{(31)})_{\text{mod 8}} = \text{sym}(d_m(\frac{5}{3}, \frac{3}{5}), (\frac{1}{3}, \frac{3}{1}), d_s(\frac{1}{1}))$$

(45)

can be read as a shorthand for the evolution

$$\begin{pmatrix}
5 & 3 & 3 & 5 & 3 & 1 & 1 \\
3 & 5 & 3 & 3 & 5 & 1 & 1 \\
5 & 3 & 3 & 5 & 1 & 1 \\
3 & 5 & 5 & 3 & 1 & 1 \\
3 & 5 & 1 & 1 & 5 & 3 \\
3 & 5 & 1 & 1 & 3 & 5 \\
1 & 1 & 5 & 3 \\
1 & 1 & 3 & 5
\end{pmatrix} \rightarrow \begin{pmatrix}
5 & 3 & 3 & 5 & 3 & 1 & 1 \\
3 & 5 & 3 & 3 & 5 & 1 & 1 \\
3 & 3 & 5 & 3 & 1 & 1 \\
5 & 3 & 1 & 1 & 5 & 3 \\
3 & 5 & 1 & 1 & 3 & 5 \\
5 & 3 & 1 & 1 & 5 & 3 \\
3 & 5 & 1 & 1 & 3 & 5 \\
1 & 1 & 5 & 3 \\
1 & 1 & 3 & 5
\end{pmatrix}$$

where it is understood that the last two steps are recursively repeated on subquadrants etc. in case of positions left blank – such as would be the case with paraorder 63, 127 etc.

We have argued that representatives $G_{\rho}^{(p)}$ are to be sought among those $G_{u\nu}^{(p)}$ that originate from $-\text{UR}(LLf^{(p)})$. So far, this yielded $G^{(3)} = 1, G^{(7)} = 1, (G_{u\nu}^{(31)}) = (3, 5, 11, 17, 41, 113), (7 \text{mod 8})$-congruence did not occur among them, nor does it by the new arrivals from $-\text{UR}(LLf^{(31)})$: only $(7 - 2k) \text{mod 8}$ ($k = 1, 2, 3$)-congruence is to be found among these. So the question arises: Does $7 \text{mod 8}$-congruence finally show up in $(G_{\rho}^{(63)})$? We stop short of listing the entire $64 \times 64$ matrix $f^{(63)}$ as a quick inspection of the first row of the LL part of $(G_{u\nu}^{(63)})$ already answers the question in the affirmative:

$$\begin{pmatrix}
G_{17,1} & G_{17,2} & G_{17,3} & G_{17,4} & G_{17,5} & G_{17,6} & G_{17,7} & G_{17,8} & G_{17,9} & G_{17,10} & G_{17,11} & G_{17,12} & G_{17,13} & G_{17,14} & G_{17,15} & G_{17,16}
\end{pmatrix}_{\text{mod 8}} = \begin{pmatrix}
9694845 & 2926323 & 747891 & 230395 & 58791 & 18633 & 4907 & 1635 & 429 & 155 & 43 & 19 & 5 & 3 & 1 & 1
\end{pmatrix}_{\text{mod 8}}$$

(46)

The reality of $G_{\rho}^{(p)}$ $\equiv (7 - 2k) \text{mod 8}$ ($k = 0, 1, 2, 3$) ($p = 63, 127, \ldots$) simply is a consequence of interordinal identity (28) applied modulo eight:

$$\text{UR}(LL(G_{u\nu}^{(63)})_{\text{mod 8}})) = \text{sym}(\frac{5}{3}, \frac{3}{5}), d_s(\frac{1}{1}), + 2\text{sym}(\frac{5}{3}, \frac{3}{5}), d_s(\frac{1}{1})) \equiv \text{sym}(\frac{7}{1}, \frac{3}{3}, \frac{3}{3}) \text{ (mod 8).}$$

(47)

The numbers of $G_{\rho}^{(p)}$ partitioned by congruence with $(7 - 2k) \text{mod 8}$ ($k = 0, 1, 2, 3$) up to $p = 63$ are listed in Table 6.
Table 6
\(G_{p}\) according to congruence with \((7 - 2k) (\mod 8) \quad (k = 0, 1, 2, 3)\) up to \(p = 63\)

| \# \(G_{p}\) cong’t w/ | \(\neg \text{UR}(LLf^{(15)})\) | \(\neg \text{UR}(LLf^{(31)})\) | \(\neg \text{UR}(LLf^{(63)})\) |
|-------------------|----------------|----------------|----------------|
| 1 (mod 8)         | 3              | 6              | 16             |
| 3 (mod 8)         | 2              | 10             | 36             |
| 5 (mod 8)         | 1              | 2              | 6              |
| 7 (mod 8)         | -              | -              | 4              |
| \(\sum\)          | 6              | 18             | 62             |

The composite map \(\Lambda = (\mod 8) \circ (3)\) ensuing from

\[
\text{LL}(\text{LL}(G^{(p)}_{\mu \nu}) \mod 8) + 2 \text{UR}(\text{LL}(G^{(p)}_{\mu \nu}) \mod 8) = \left(\text{LL}(\text{LL}(G^{(p)}_{\mu \nu}))^T \mod 8\right) + 2 \text{UR}(\text{LL}(G^{(p)}_{\mu \nu}) \mod 8) = 3 \text{UR}(\text{LL}(G^{(p)}_{\mu \nu}) \mod 8),
\]

offers an illuminating side to it, as can be gleaned from Table 7 where we list arguments and outputs to emphasize two things:

Table 7
Structural interordinality under \(\Lambda\) up to \(p = 127, p' = 255\)

| \(\Lambda : \text{UR}(\text{LL}(G^{(p)}_{\mu \nu}) \mod 8)) \mapsto \text{UR}(\text{UL}(\text{LL}(G^{(p')}_{\mu \nu'}) \mod 8))\) |
|-----------------------------------|
| \(p = 7\)                        | (1)                          |
| \(p' = 15\)                      | (3)                          |
| \(p = 15\)                       | \text{sym}(1 1)              |
| \(p' = 31\)                      | \text{sym}(3 3)              |
| \(p = 31\)                       | \text{sym}(\frac{5}{3}, \frac{3}{5}, \text{d}_{\nu}(\frac{1}{1}))\) |
| \(p' = 63\)                      | \text{sym}(\frac{7}{1}, \frac{3}{5}, \frac{3}{5})\) |
| \(p = 63\)                       | \text{sym}(\frac{5}{3}, \frac{3}{5})(\frac{3}{5}, \frac{5}{3}, \text{d}_{\nu}(\frac{1}{1}))\) |
| \(p' = 127\)                     | \text{sym}(\frac{7}{1}, \frac{3}{5}, \frac{3}{5})\) |
| \(p = 127\)                      | \text{sym}(\frac{5}{3}, \frac{3}{5}, \frac{3}{5}, \frac{5}{3}, \text{d}_{\nu}(\frac{1}{1}))\) |
| \(p' = 255\)                     | \text{sym}(\frac{7}{1}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{5}{3})\) |

i) Secondary diagonal patterns, among others, are left intact upon crossing the paraparorder boundary \(p\) to \(p'\), as can be seen from the tail \(\text{d}_{\nu}(\ell)\) in the respective arguments;

ii) patterns subject to the map, at least with the values we know of, do oscillate: \((\frac{5}{3}, \frac{3}{5}) \leftrightarrow (\frac{7}{1}, \frac{1}{7}), (\frac{3}{5}, \frac{3}{3}) \leftrightarrow (\frac{1}{1}, \frac{1}{1})\).

The oscillatory appearance is corroborated by the observation that the \(\text{LL}_f(\mod 8)\) determinant (rank) alternates between 0 and a nonzero (deficient and a complete) value among neighboring orders \(p\) and \(p'\):

\[
\det(\text{LL}_f^{(3)}(\mod 8)) = 1, \quad \det(\text{LL}_f^{(7)}(\mod 8)) = 0, \quad \det(\text{LL}_f^{(15)}(\mod 8)) = 240^2, \quad \det(\text{LL}_f^{(31)}(\mod 8)) = 0, \ldots
\]
Differences have thus far arisen at two stages in our analysis: in Green’s model, differences of squares $\beta(p - \beta + 1)$ are responsible for the capture of spin values; and, in the context of $f$- (or $h$-) parafermions, coefficient differences constitute in part, by fitting certain kissing numbers in the simplest case of length-2 additive partitions, the row (column) structure of $LL(G_{\mu\nu}^{(p)})$. It is therefore natural to ask which types of differences else might be structurally constitutive, the first time so with paraorder fifteen.

5.1 Naive differences

Let the members of the partial sequence $\left(G_{\rho}^{(p)}\right)$ be arranged in ascending order and differences springing from member and predecessor denoted by $\Delta G_{\rho^*}^{(p)}$. One runs across a peculiarity then. For $p = 15$, one gets a monotonously nondecreasing sequence of differences,

$$\left(\Delta G_{\rho^*}^{(15)}\right) = (2, 6, 6, 24, 72),$$

whereas the related sequence for $p = 31$ misses monotonicity of nondecrease:

$$\left(\Delta G_{\rho^*}^{(31)}\right) = ([24, 72, 40, 274, 846, 320, 38, 3186, 86, 10162, 230, 3330, 40154, 130590, 37718, 510864, 1692336]).$$

Part of the order clash goes to the account of overlap of sequence members entangled in interordinality (bracketed terms), the remaining warps are due to intraordinal effects.

One way out is to economize on the number of selectable differences, as expounded in Sect. 5.2. Another way out is to follow the opposite track, as our demonstration in Sect. 5.3 aims to achieve. Different from Eqs. (49)-(50) though they may look, the types of differences thereby earmarked are clearly linked to one another. Of particular interest turn out to be differences derivable from Green’s squares along a succession of individual carry-bit neighbors as they shed light on the themes of Sects. 4.4 to 4.5; bar none so those derivable from Green’s squares for an enlarged neighborhood which lead us to believe that $b$- and $f$-parafermions may with their inherent interordinal maps blend into a topological operator (Section 5.4). But even differences with no seeming coming from Green’s squares but combining $f$- and $h$-parafermion lineage would hold topological information, as our comments in Sects. 6 and 7 try to make clear.

5.2 Skewed differences

Recalling the way Catalan numbers were partitioned (see Fig. 4), namely in form of net traces over the secondary diagonal and adjacent diagonals in $LL(G_{\mu\nu}^{(p)})$, one would expect more meaningful differences to spring from a skewed pairing of coefficients. There indeed exists a reduced set of $\frac{(p_{\rho}+1)(p_{\rho}+2)}{2} \cdot 4$ differences, $\partial G_{\kappa}^{(p)}$ ($p = p_n = 15, 31, \ldots)$, that increase monotonously when subtraction is performed subsubquadrantwise along a tilted path from the upper right to the lower left.

---

8 As we have seen, at paraorders three and seven the respective partial sequences are monomial
9 we omit configurations which stay the same upon reflection in the secondary diagonal
An alternative to economizing on differences is to dovetail ones nondecreasingly into an enlarged sequence \( (\Delta G_{\mu(15)}^p) \), taking from \( \{G_r^p\} \cup \{G_r^{p'}\} \); for the paraorder window (15,31), this procedure yields a sequence \( (2, 6, 22, 40, 70, 274, \ldots, 1692336) \).

The conjecture thereby dawning on the scrutator is that in order for structural consistency with \( \Delta G_{\zeta}^p \) to be achieved, not so much intra- as interordinal differences of Green squares are of importance. Those we first define with a single index \( \alpha \) by

\[
\varphi_{\alpha}^{(p,p')} = \alpha(p' - \alpha + 1) - \alpha(p - \alpha + 1) = \alpha(p' - p) \quad (\alpha = 1, \ldots, p) : \tag{54}
\]
Figure 9. Interordinal differences of Green’s squares

\[
p' = 3: \quad 3 \quad p = 1: \quad 1 \quad p = 3: \quad 3 \quad 4 \quad 3
\]

\[
p' = 7: \quad 7 \quad 12 \quad 15
\]

\[
p' = 15: \quad 15 \quad 28 \quad 39 \quad 48 \quad 55 \quad 60 \quad 63
\]

\[
p = 7: \quad 7 \quad 12 \quad 15 \quad 16 \quad 15 \quad 12 \quad 7
\]

\[
p = 15: \quad 15 \quad 28 \quad 39 \quad 48 \quad 55 \quad 60 \quad 63 \quad 55 \quad 48 \quad 39 \quad 28 \quad 15
\]

\[
p' = 31: \quad 31 \quad 60 \quad 87 \quad 112 \quad 135 \quad 156 \quad 175 \quad 192 \quad 207 \quad 220 \quad 231 \quad 240 \quad 247 \quad 252 \quad 255
\]

\[
p = 15: \quad 15 \quad 28 \quad 39 \quad 48 \quad 55 \quad 60 \quad 63 \quad 64 \quad 63 \quad 60 \quad 55 \quad 48 \quad 39 \quad 28 \quad 15
\]

\[
\vartheta: \quad 8 \quad 16 \quad 24 \quad 32 \quad 40 \quad 48 \quad 56
\]

Compatibility is now achieved in that each \( \Delta G'_\zeta \) allows for a parafermial representation \( \sum_{\vartheta} \vartheta^{(q, q')} \) \( (q' = 2q + 1) \). For instance, for \( \Omega_\zeta \) and \( g_q \), the ansatz

\[
\Omega_\zeta = \{2^\iota - 1 \mid \iota \in \Omega \zeta \subset \{1, \ldots, p\}\},
\]

\[
g_q \in (\Phi_\zeta \subset (G_\rho) \mid \text{thru to order } p),
\]

can be probed to achieve representations like that of \( \Delta G_{\max}^{(15,31)} \),

\[
1692336 = 41 \cdot 2^{14} + 113 \cdot 2^{13} + 17 \cdot 2^{12} + 11 \cdot 2^{11} + 5 \cdot 2^9 + 1 \cdot 2^6 + 3 \cdot 2^5 + 1 \cdot 2^4.
\]

Whereas both for the \# of factorization types (Table 4) and for the spacings in prime interpolations (Fig. 6) linear parafermial expressions

\[
p.e. = \sum_{i \in I_q} \vartheta_i^{(q, q')} + \sum_{j \in I_s} \vartheta_j^{(s, s')} \cdots \quad (I_q \subset \{1, \ldots, q\}, I_s \subset \{1, \ldots, s\}, \ldots; q' = 2q + 1, s' = 2s + 1) \quad (55)
\]

should be as significant as for the partitions of the \# of \( G^{(p)}_\rho \) according to their congruence with \((7 - 2k)(\mod 8) \) \( (k = 0, 1, 2, 3) \) (Table 6).

5.4 Kissing numbers – the parafermion as a topological operator

In the introduction and in Sect. 4.3, it was noted that kissing numbers are tied to the row (column) structure of \( LL(G^{(p)}_{\parallel}) \). This connection is borne out by the front members of the naive partial sequences \( (\Delta G^{(15)}_\rho) = (2, 6, 6, 24, 72), (\Delta G^{(31)}_\rho) = (24, 72, 40, \ldots) \) and the skewed partial sequence \( (\vartheta G^{(15)}_\rho) = (2, 6, 12, 24) \), which yield the first six of them: \( L_1 = 2, L_2 = 6, L_3 = 12, L_4 = 24, L_5 = 40, L_6 = 72 \). Yet, there also exists a spin-based connection which ties the spin-defining Green square differences – now generalized with two indices \( \alpha \) and \( \beta \) to \( \vartheta^{(p, p_u)}_{\alpha \beta} \equiv \beta(p_u - \beta + 1) - \alpha(p_u - \alpha + 1) \) where \( p_l \) and \( p_u \) are Mersenne numbers defined by \( p_l \in \{1, 3, 7, \ldots, 2^l - 1 \mid 2^l \leq 2D\}, \) \( p_u \in \{3, 7, 15, \ldots, 2^u - 1 \mid 2^u \leq 32D\} \) \( (l, u, D \in \mathbb{N}; p_l < p_u) \) and \( \alpha \) and \( \beta \) are respectively running from 1 to \( p_l \) and \( p_u \) – to kissing numbers.\(^10\)

\(^10\) As the bulk of kissing numbers shown in Table 8 are not certified, we are on risky ground here.
| \( p_l \) | \( p_u \) | \( \vartheta^{(p_l,p_u)}_{\alpha \beta} \) (\( \alpha \) odd) | \( \vartheta^{(p_l,p_u)}_{\alpha \alpha} \) \( \equiv \vartheta^{(p_l,p_u)}_{\alpha} \) * | \( \vartheta^{(p_l,p_u)}_{\alpha \beta} \) (\( \alpha \) even) | \( \vartheta^{(p_l,p_u)}_{\alpha \beta} \) (\( \alpha \neq \beta \)) |
|---|---|---|---|---|---|
| 1 | 3 | \( \vartheta^{(1,3)}_{1} = L_1 = 2 \) | - | - | |
| 1 | 7 | \( \vartheta^{(1,7)}_{1} = L_2 = 6 \) | \( \vartheta^{(3,7)}_{3} = L_3 = 12 \) | - | - |
| 3 | 7 | \( \vartheta^{(7,15)}_{3} = L_4 = 24 \) | \( \vartheta^{(7,15)}_{5} = L_5 = 40 \) | - | 0, \( L_3 = 12 \) |
| 1 | 15 | - | - | - | |
| 3 | 15 | - | \( L_4 = 24 \) | \( L_3 = 12 \) | |
| 7 | 15 | \( \vartheta^{(7,15)}_{3} = L_4 = 24 \) | - | 0, \( L_3 = 12 \) | |
| 1, 3 | 31 | - | - | - | |
| 7 | 31 | \( \vartheta^{(7,31)}_{1,1} = L_4 = 24 \) | \( \vartheta^{(7,31)}_{3} = L_6 = 72 \) | - | \( L_8 = 240 \) |
| 15 | 31 | - | - | - | 0, \( L_3 = 12 \), \( L_4 = 24 \), \( L_6 = 72 \), \( L_8 = 240 \) |
| 1, 3 | 63 | - | - | - | |
| 7 | 63 | - | \( L_{10} = 336 \) | - | \( L_{12} = 756 \) |
| 15 | 63 | \( \vartheta^{(15,63)}_{5,3} = L_8 = 240 \) | - | 0, \( L_4 = 24 \) | |
| 31 | 63 | - | - | 0, \( L_3 = 12 \), \( L_5 = 40 \), \( L_8 = 240 \), \( L_{10} = 336 \) | |
| 1 | 127 | \( \vartheta^{(1,127)}_{1} = L_7 = 126 \) | - | - | |
| 3 | 127 | - | - | - | |
| 7 | 127 | - | \( L_8 = 240 \) | - | |
| 15 | 127 | \( \vartheta^{(15,127)}_{3,3} = L_{10} = 336 \) | - | 0, \( L_3 = 12 \), \( L_5 = 40 \), \( L_8 = 240 \), \( L_{15} = 2340 \) | |
| 31 | 127 | - | - | 0, \( L_3 = 12 \), \( L_6 = 72 \), \( L_9 = 272 \), \( L_{10} = 336 \) | |
| 63 | 127 | - | - | 0, \( L_3 = 12 \), \( L_5 = 40 \), \( L_8 = 240 \), \( L_{10} = 336 \) | |
| 1 | 255 | - | - | - | |
| 3 | 255 | \( \vartheta^{(3,255)}_{3} = L_{12} = 756 \) | - | - | |
| 7 | 255 | - | - | \( L_8 = 240 \) | |
| 15 | 255 | - | - | - | |
| 31 | 255 | - | - | 0, \( L_4 = 24 \), \( L_{12} = 756 \) | |
| 63 | 255 | - | - | 0, \( L_4 = 24 \), \( L_6 = 72 \), \( L_8 = 240 \), \( L_{16} = 4320 \) | |
| 127 | 255 | - | - | 0, \( L_3 = 12 \), \( L_4 = 24 \), \( L_{10} = 336 \), \( L_{12} = 756 \), \( L_{16} = 4320 \) | |
| 1, ..., 15 | 511 | - | - | - | |
| 31 | 511 | \( \vartheta^{(31,511)}_{3} = L_8 = 4320 \) | - | \( L_{10} = 336 \) | |
| 63 | 511 | - | - | 0, \( L_3 = 12 \), \( L_{16} = 4320 \) | |
| 127 | 511 | - | - | 0, \( L_3 = 12 \), \( L_9 = 240 \), \( L_{10} = 336 \), \( L_{16} = 4320 \) | |
| 255 | 511 | - | - | 0, \( L_3 = 12 \), \( L_4 = 24 \), \( L_9 = 272 \), \( L_{10} = 336 \), \( L_{12} = 756 \), \( L_{15} = 2340 \) | |

*) first appearance only
The Green squares \( \alpha(p_l - \alpha + 1) \) and \( \beta(p_u - \beta + 1) \) define points on two sets of parabolas – with extremal ordinates \( 2^{2l-2} \) and \( 2^{2u-2} \), respectively, and general ordinate differences divided by two defining (generalized) spin. Vanishing ordinate differences define a spin-0 subclass which is given by \( \alpha = (2^l - 1)2^m, \beta = 2^m, p_l = 2^{l+m+1} - 1, p_u = 2^{2l+m-1} \) \( (l = 2, 3, 4, \ldots; m = 0, 1, 2, \ldots) \).

With no larger spin than two, Green squares can be subdivided into two classes – a degenerate one, with remainder 1 under \( \Lambda = (\text{mod } 8) \) alternation map. Stated that this sequence bears a resemblance to the root-

Out of the first sixteen kissing numbers containing base prime two non-exponentiated – only three conform with this condition via the representation \( \sigma^{(p_l,p_u)} \). In other words, they are equivalent to the Mesenne number differences \( L_1 = 3 - 1, L_2 = 7 - 1, L_7 = 127 - 1 \). Together with the kissing numbers \( L_x = 2^{z_1} \cdot \pi_2^{z_2} \cdot \ldots \) \( (z_1 > 1, \pi_2 > 2) \) and zeroing differences, they are shown in Table 8. A surprising feature can be read from that table: If only a kissing number’s first appearance as \( \sigma^{(p_l,p_u)} \) \( (\alpha \text{ odd}) \) or \( \sigma^{(p_l,p_u)} \) \( (\alpha \neq \beta) \) is considered (framed items), first appearances as \( \sigma^{(p_l,p_u)} \) \( (\alpha \text{ odd}) \) lead to an \( \alpha \) sequence 1, 1, 3, 5, 3, 7, 1. Taken pairwise, the \( \alpha \)’s follow the alternation map \( \Lambda = (\text{mod } 8) \circ (\times 3) \) characteristic of LL\((G_{\mu}^{(p)^{\nu}}) \) structure – which is the reason for our usage of the single index \( \lambda \) in the table. We arrive at the unlooked-for topologic

Conjecture 2

\[ f^{(p)} = \text{(or } h^{(p)} \text{-) parafermions are containers of hypersphere configurations of densest packing, in Euclidean } D^{(p)} \text{-space down to those in } D^{(1)} \text{-space, where } D^{(p)} \text{ is the largest dimension for which the kissing number } L_{D^{(p)}} \text{ determines the row (column) structure of } \text{LL}(G_{\mu}^{(p)^{\nu}}) \text{ (or } \text{LL}(J_{\mu}^{(p)^{\nu}}) \text{)} - D^{(1)} = 7, \text{ for instance.} \]

Dual to this inner structural connection is the spin-based connection relating densest-packing hypersphere configurations to Green square differences of parafermions of Mersennian order such that either \( L_D = \sigma^{(p_l,p_u)} \) or \( L_D = \sigma^{(p_l,p_u-1)} \pm \sigma^{(p_l+2,p_u-1)} \pm \ldots \). This latter connection is exterior in the sense that \( p_u \) may become larger than \( p_l \).

The exterior connection can be delineated as follows:

Corollary 3

\[ \text{For each pair } p_l, p_u, \text{ interordinal differences of Green squares } \beta(p_u - \beta + 1) - \alpha(p_l - \alpha + 1) \text{ form a distinct rectangular matrix } \sigma^{(p_l,p_u)}_{\alpha\beta}. \text{ Among its entries, which include zeroing differences leading to spin } 0, \text{ of particular interest are those coinciding with members of the class } L_x = 2^{z_1} \cdot \pi_2^{z_2} \text{ \ldots \ldots \ldots } (z_1 > 1, \pi_2 > 2) \text{ of kissing numbers. If only } \sigma^{(p_l,p_u)}_{\alpha\alpha} \text{ (odd) and } \sigma^{(p_l,p_u)}_{\alpha\beta} \text{ (odd) are considered, the first appearances of these kissing number in representations } \sigma^{(p_l,p_u)}_{\alpha\alpha} \text{ (odd) are characterized by indices } \lambda \text{ that (pairwise, in ascending order) are in one-to-one correspondence with the patterns } (1^1,1^1), (1^3,3^1), (5^3,3^5) \text{ where } \Lambda = (\text{mod } 8) \circ (\times 3). \]

One approach to narrowing the range of pairs \( p_l, p_u \) leans on paraoformer sums which are endowed with the identity

\[ \Sigma_{i=1}^{2n-1} p_i = \frac{1}{C_n B(p_n, p_n + 1)} = \Sigma_{i=1}^{2n-1} p_i - p_n(p_n + 1) = -n + \Sigma_{i=1}^{n-1} p_i \]

(where \( B(\_\) is the beta function) and allow taking three different paraoformers into account on each assignment:

- \( p_l := p_n - 1 \)
- \( p_u := p_n \)
- \( p_l := p_n - 1 \)
- \( p_u := p_{2n-1} \)
- \( p_l := p_n \)
- \( p_u := p_{2n-1} \)

It is easily shown that these choices include \( L_x = \sigma^{(p_l,p_u)}_{\alpha\alpha} \) (\( \alpha \) even) and thus are a broader approach than the \( \Lambda \)-approach.

6 Synopsis of root-\( f \) and root-\( h \) related coefficient differences

Making the review more complete by a further sideglance to the root-\( h \) sequence is overdue. In the introduction it was already stated that this sequence bears a resemblance to the root-\( f \) sequence. The kinship tellingly expresses itself in the relations, starting with \( p = 7, p' = 15, \)

\[ \text{UR(LL(h\( p' \))) = LL(LL(h\( p \))) - 2 \text{UR(LL(h\( p \))),} \quad (56) \]

\[ \text{UR(LL(h\( p' \))) = LL(LL(h\( p' \))) - 2 \text{UR(LL(h\( p' \))) + 2 \text{UR(LL(h\( p \))),} \quad (57) \]

\[ \text{http://www.math.rwth-aachen.de/\ Gabriele.Nebe/LATTICES/kiss.html}; \] the numbers in parentheses spring from nonlattice calculations.

\[ \text{As Conjecture 1 suggests, } D^{(31)} = 31. \]

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\[ \text{\textsuperscript{11}http://www.math.rwth-aachen.de/\ Gabriele.Nebe/LATTICES/kiss.html} \]

\[ \text{\textsuperscript{12}As Conjecture 1 suggests, } D^{(31)} = 31. \]
only the first of which is purely interordinal, while the second is a mixture of intra- and interordinal relationship. Juxtaposing these opposite (28)-(29) – identities that we remember are purely interordinal and intraordinal respectively –, one is not surprised to find that the partial sequences \( J_\omega^{(p)} \) – with \( J_\omega^{(p)} \) as representatives of \( J_\mu^{(p)} \) – cease being monomial already at paraorder seven.\(^{13}\) Starting out with that order, partial sequences with differences \( \Delta J_\omega^{(p)}, \partial J_\mu^{(p)} \)\(^{14}\) and \( \Delta J_b^{(p,p')} \) then are readily formable. Briefly expounding to what extent a synopsis between them and their \( G \) counterparts on the one hand and \( \vartheta_\lambda^{(p,p \omega)} \) on the other can be used to the kissing-number problem is the subject of this section.

We have already learned three ways of expressing kissing numbers:

a) in the Introduction and in Sect. 4.3, by additive partitions within rows, or groups of rows, of \( LL(G^{(p)}) \);

b) in Sect. 5.4, by interordinal differences \( \vartheta_\alpha^{(p_1,p \omega_1)} \pm \vartheta_\alpha^{(p_2,p \omega_2)} \pm \ldots \).

c) same place, by higher-order parafermial differences \( \vartheta_\lambda^{(p_1,p \omega_1)} \pm \vartheta_\lambda^{(p_2,p \omega_2)} \pm \ldots \).

It was also remarked upon the connection of the naive partial sequences of Eq. \((50)\)

\[
(\Delta G_{\rho'^{\omega \ast}}^{(15)}) = (2, 6, 6, 24, 72),
\]
\[
(\Delta G_{\rho''^{\omega \ast}}^{(31)}) = (24, 72, 40, 274, 846, 320, \ldots)
\]

as well as the skewed sequence of scheme \((51)\)

\[
(\partial G_{\kappa'}^{(15)}) = (2, 6, 12, 24)
\]

with kissing numbers. What remained to be checked is whether \( G_{\rho'^{\omega \ast}}^{(p)} \) and \( J_\rho^{(p)} \)-derived differences, \( \tilde{p} \in \{p, p'\} \), have a way of jointly determining these numbers. We therefore computed certain \( J_\rho^{(p)} \)-derived partial sequences of differences for the occasion: the naive

\[
(\Delta J_\omega^{(7)}) = (4), \quad (\Delta J_\omega^{(15)}) = (38,6,14,134),
\]
\[
(\Delta J_{\omega' finance}^{(31)}) = (688974, 53888, 4474, 388, 54, 104, 26, 1176, 24, 204, 14000, 2722, 176724, 28580, 2662662),
\]

and the skewed

\[
(\partial J_{\kappa'}^{(15)}) = (-6, 20, -58),
\]
\[
(\partial J_{\kappa'}^{(31)}) = (104, -388, 1404, 1226, 1202, -4474, -4394, 16722, 16442, 14228, -53968, 205584).
\]

With the root- \( f \) related sequence \( (\partial G_{g' \omega}^{(31)}) \) of scheme \((52)-(53)\) computed to

\[
(\partial G_{g' \omega}^{(31)}) = (136, 386, 1160, 1440, 1478, 4390, 4476, 13792, 14022, 16994, 53886, 174074),
\]

we’ve actually found a scheme construing the values \( L_D \) \((D \leq 8)\) as second-order synoptic differences:

those linked to odd-dimensional Euclidean spaces in representations that mix \( \Delta \)- and \( \partial \) terms,

\[
L_1 = 2 = \Delta J_3^{(15)} - \partial G_3^{(15)} = 14 - 12,
\]
\[
L_3 = 12 = \Delta G_7^{(31)} - \partial J_2^{(31)} = 38 - 26,
\]
\[
L_5 = 40 = \partial J_1^{(15)} + \Delta G_1^{(15)} = 38 + 2,
\]
\[
L_7 = 126 = \Delta G_5^{(15)} + \partial J_5^{(31)} = 72 + 54,
\]

and those linked to even-dimensional spaces in representations that are homogeneous in either \( \Delta \) or \( \partial \):

\(^{13}\) As opposed to the partial sequences \( (G_r^{(p)}) \), which do not move on from monomiality until paraorder fifteen

\(^{14}\) where we again encounter a reduced set of \( g_{(q+1)(q+2)} \cdot 4 \) differences \( \partial J_{(p)} \), \( p = 15, 31, \ldots ; q = 1, 3, \ldots \), based on subsubquadrantwise subtraction performed along a tilted path that pairs distinct \( J_{(p)} \) from upper right to lower left; even though performed in the same way, the subtraction process does not automatically lead to a monotonously increasing sequence of differences such as \( (\partial G_r^{(p)}) \).
The kissing number associated with a hypersphere configuration of densest packing in Euclidean $D$-space is representable both by a 2nd-order syncopic difference and an interordinal difference $\partial G^{(63)}_{\alpha^\prime \beta^\prime}$ of Green squares for $D \leq 8$. As $D > 8$, representations can be assigned accordingly and either are pairwise 2nd-order syncopic based on paraorders $2m - 1$, $2^{m+1} - 1$ and associated with dimensions $D_1 = i$, $D_3 = k$, or returning interordinal $\partial G^{(63)}_{\alpha^\prime \beta^\prime}$, $\partial G^{(63)}_{\alpha^\prime \beta^\prime}$ based on paraorders $2^{m+1} - 1$, $2^{m+5} - 1$ and associated with dimensions $D_2 = j$, $D_4 = l$, while consisting of higher-order syncopic / otherwise interordinal differences at interstitial dimensions. $i, j, k, l$ are determined by the above system of equations, and the span of dimensions taken is one full 8-period for $(\lambda_1, \lambda_2) \cong ((^{(1,1)}_{1,1}))$ and two successive 8-periods for $(\lambda_1, \lambda_2) \cong ((^{5,3}_{3,5}))$.

Table 9
Kissing numbers $L_9$ to $L_{16}$ as 2nd-order syncopic / returning interordinal (or higher-order syncopic / otherwise interordinal) differences

| $L_9$  | $L_{10}$ | $L_{11}$ | $L_{12}$ | $L_{13}$ | $L_{14}$ | $L_{15}$ | $L_{16}$ |
|-------|---------|---------|---------|---------|---------|---------|---------|
| 272   | 336     | 438     | 756     | 918     | 1422    | 2340    | 4320    |

Now the alternation map $\Lambda$ is closely related to the odd-integer partitions of the number 8: the quadripartite $1+1+3+3=8$ and the bipartite $5+3=8$ and $7+1=8$. Thus the action of $\Lambda$ can be put in one-to-one correspondence with either the alternation of the halves of the quadripartite or the alternation of the full bipartite partition(s). The alternation of the halves of the quadripartite partition fits in one eight-period of dimensions and in fact is in one-to-one correspondence with the action of $\Lambda$ on characteristic increments in that eight-period; conversely, the alternation of the full bipartite partitions should fit in two such periods and also be in one-to-correspondence with characteristic index increments in there. Writing

$$i = 2^m + \lambda_1,$$
$$j = 2^m + \lambda_1 + \lambda_2,$$  
(for some $m \geq 3$)  
$$k = 2^m + \lambda_1 + \lambda_2 + \lambda_3,$$
$$l = 2^m + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4,$$

we arrive at the following conjecture which supplements Corollary 3:

**Conjecture 4** The kissing number associated with a hypersphere configuration of densest packing in Euclidean $D$-space is representable both by a 2nd-order syncopic difference and an interordinal difference $\partial G^{(63)}_{\alpha^\prime \beta^\prime}$ of Green squares for $D \leq 8$. As $D > 8$, representations can be assigned accordingly and either are pairwise 2nd-order syncopic based on paraorders $2m - 1$, $2^{m+1} - 1$ and associated with dimensions $D_1 = i$, $D_3 = k$, or returning interordinal $\partial G^{(63)}_{\alpha^\prime \beta^\prime}$, $\partial G^{(63)}_{\alpha^\prime \beta^\prime}$ based on paraorders $2^{m+1} - 1$, $2^{m+5} - 1$ and associated with dimensions $D_2 = j$, $D_4 = l$, while consisting of higher-order syncopic / otherwise interordinal differences at interstitial dimensions. $i, j, k, l$ are determined by the above system of equations, and the span of dimensions taken is one full 8-period for $(\lambda_1, \lambda_2) \cong ((^{(1,1)}_{1,1}))$ and two successive 8-periods for $(\lambda_1, \lambda_2) \cong ((^{5,3}_{3,5}))$, $(\lambda_3, \lambda_4) \cong \Lambda ((^{5,3}_{3,5}))$.  

31
A natural question to ask is if Conjecture 4 allows instantiations of \( \lambda_1 \) etc. to repeat periodically. If \( L_{2^c+b_0} ((2^s+b_0) \text{mod } 8 = \lambda_1) \) is representable 2nd-order synoptic, so too could \( L_{2^c+b_0} ((2^s+c^t+b_t) \text{mod } 8 = \lambda_1, c = \text{const.}) \). While the input/output entries of Table 7 and the \( \lambda \)-sequence of Table 8, which share with \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) the \( \Lambda \)-mapping precept, signal nothing of the kind, the stenography of kissing numbers described in Sect. 4.3 hints at such a possibility. The argument proceeds as follows. Via the stenoscopic coupling

\[
D^{(p)} = \max(D) \mid L_D \leq \sum_{i=1}^{(q+1)/2} G_{q+1,i}^{(p-1)/2}
\]  

(dotted underlined in Table 3), a least dimension falling within the subsequent interordinal corridor is being defined:

\[
D_{\text{lowest}}^{(p')} = D^{(p)} + 1.
\]

For reasons given below, we relate to the latter the dimension \( N \) taking values

\[
N = - \left\lfloor \frac{(n+1)/2}{2} \right\rfloor \sum_{i=1}^{n-1} p_i = \lfloor \log_2 C_{q'} \rfloor
\]

(an identity that first sprang up in Eq. (40)). We have assembled a selection of the numbers \( D_{\text{lowest}}^{(p')} \) and \( N \) in Table 10:

| \( p' \) | 2 | 2^2 | 2^3 | 2^4 | 2^5 | 2^6 | 2^7 | 2^8 | 2^9 | 2^10 | 2^11 | 2^12 | 2^13 | 2^14 | 2^15 | 2^16 | 2^17 | ... |
|-----|---|----|----|----|----|----|----|----|----|-----|----|-----|----|-----|----|-----|----|----|-----|
| \( D_{\text{lowest}}^{(p')} \) | 2 | 8 | 32 | 112 | 416 | 1664 | 6586 | 25504 | 101132 | 407154 | 1642292 | 6618374 | 26638982 | 107107722 | ... |
| \( N \) | 2 | 8 | 23 | even | even | 497 | 1007 | even | even | 8171 | 16361 | even | even | ... |

As far as that Table 10 goes, \( D_{\text{lowest}}^{(p')} \) runs from 2 to 107 107 722, and \( N \) from 2 to the two even numbers following 16361. From Table 11 we see two possibilities for follow-up. If only \( D_{\text{lowest}}^{(p')} - N \) (\( N \) odd-numbered) is being realized, Table 11 suggests the possibility of second-order synoptic (or returning interordinal) kissing number representability for

\[
L_{2^s+16t+b_t} \quad ((2^s+16t+b_t) \text{mod } 8 = \lambda; \ t = 0, 1, 2, \ldots),
\]

with \( c \) coinciding with a key figure of Table 8, \( \max(p_n+1) = 16 \). If, on the other hand, \( D_{\text{lowest}}^{(p')} - N' \) (\( N' \) too odd-numbered) is being realized as well, then representations with multiple periodicities might come to light. Either way, if so, relating \( D_{\text{lowest}}^{(p')} \) to \( N \) (and \( N' \)) would lie at the heart of the effectiveness of the interordinal map \( \Lambda \) for the kissing number representations in question.
Obviously, our $\lambda$'s 3, 5 and 7 and Mersenne primes > 7 represent the only base primes that are not SCPF primes in the factorization of $C_q (q = 7, 15, 31, \ldots)$. When all base primes defining the infix (Mersenne primes > 7) are mapped to the remaining value $\lambda = 1$, the factorization has a prefix that can be defined by

$$P(p) = (\{\text{base primes} > 7\} \to 1) \circ (\text{factorization of } C_q/\text{SCP}(G_{q+2,1}^p), \quad p = 31, 63, \ldots \quad (65)$$

The resulting factorization beginnings are given in

| $p'$ | $p_{ns+1}$ | $D_{\text{lowest}}^{p'(r)} - N$ | $D_{\text{lowest}}^{p'(r)} - N'$ | $(N, N'$ odd-numbered) |
|------|-------------|-----------------|-----------------|---------------------|
| $2^7 - 1$ | $112 - 23 = 2^5 + 57, \quad (2^5 + 57) \mod 8 = 1$ | $112 - 53 = 2^5 + 27, \quad (2^5 + 27) \mod 8 = 3$ |
| $2^{11} - 1$ | $25,504 - 497 = 2^9 + 24,495, \quad (2^9 + 24,495) \mod 8 = 7$ | $25,504 - 1,007 = 2^9 + 23,985, \quad (2^9 + 23,985) \mod 8 = 1$ |
| $2^{15} - 1$ | $6,618,374 - 8,171 = 2^{13} + 6,602,011, \quad (2^{13} + 6,602,011) \mod 8 = 3$ | $6,618,374 - 16,361 = 2^{13} + 6,593,821, \quad (2^{13} + 6,593,821) \mod 8 = 5$ |
| $2^{19} - 1$ | $598,753,098 - 131,045 = 2^{17} + 598,490,981, \quad (2^{17} + 598,490,981) \mod 8 = 5$ | $598,753,098 - 262,115 = 2^{17} + 598,359,911, \quad (2^{17} + 598,359,911) \mod 8 = 7$ |
| $2^{23} - 1$ | $54,868,958,480 - 2,097,119 = 2^{21} + 54,864,764,209, \quad (2^{21} + 54,864,764,209) \mod 8 = 1$ | $54,868,958,480 - 4,194,269 = 2^{21} + 54,862,667,059, \quad (2^{21} + 54,862,667,059) \mod 8 = 3$ |

To understand why, we have to recall the definition of the suffix of consecutive prime factors, SCPF($G_{q+2,1}^p$) from Sect. 4.4. Now consider the paraorder products $\prod_{r=1}^{n+1} p_r$ (which as we shall see in the next section play a vital role in the interordinal preon model to be presented there) and extract from them factorization beginnings in analogous fashion, namely

$$\prod_{r=1}^{n+1} (p') = (\{\text{base primes} > 7\} \to 1) \circ (\text{factorization of } \prod_{r=1}^{n+1} p_r), \quad (66)$$

as summarized in

| $p'$ | 2^2-1 | 2^3-1 | 2^4-1 | 2^5-1 | 2^6-1 | 2^7-1 | 2^8-1 | 2^9-1 | 2^10-1 | 2^11-1 | 2^12-1 | \ldots |
|------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $P(p')$ | 3 | 3\cdot5 | 7 | 3\cdot5 | 7 | 3\cdot5 | 7 | 3\cdot5 | 7 | 3\cdot5 | 7 | \ldots |

Then we find that $\prod (p')$ can be expressed as a multiplicative partition $P^{(r)}P^{(s)} \ldots$ ($r, s, \ldots \in \{2q + 1, 4q + 3, 8q + 7, \ldots\}$) of
Interordinal preon model

To our knowledge, Oscar Wallace Greenberg was the first to recognize that quarks can be viewed as parafermions of order 3. But with the advent of QCD, and the experimental findings to date that quarks are pointlike down to 10⁻²⁰ m, preons, parafermionic or otherwise, have not found much acclaim among physicists. This is not the place to review the variously theorized preon types in the literature, including Green’s own proposal, nor is the following meant to be a worked out physical model of the subatomic onion – it rather tries to point the way to the putative mathematical structure of hadronic matter, which likely is interordinal in the Mersemian sense. A more self-contained elaboration of the present ideas will appear in a separate paper. Here, we choose the symbols \( p_{\text{up}}^{(p)} \) and \( p_{\text{down}}^{(p)} \) to denote up-type and down-type preons of paraorder \( p \) respectively.

**Conjecture 5** Preons of order \( p_{n+1} \) are either up-type or down-type, \( p_{\text{up}}^{(p_{n+1})} \) or \( p_{\text{down}}^{(p_{n+1})} \). The electric charge (in e) of up-type items is given by expressions \( c_{\text{up}}^{(p_{n+1})} = p_{n+1} - \sum_{r=0}^{n} p_r \prod_{r=1}^{n+1} p_r \) and the charge of down-type items by \( c_{\text{down}}^{(p_{n+1})} = -\sum_{r=0}^{n} p_r \prod_{r=1}^{n+1} p_r \).

The charge of up-type items transforms as \( c_{\text{up}}^{(p_n)} = (p_{n+1} - 1)c_{\text{up}}^{(p_{n+1})} + c_{\text{down}}^{(p_{n+1})} \) and the charge of down-type items as \( c_{\text{down}}^{(p_n)} = (p_n + 1)c_{\text{down}}^{(p_{n+1})} + p_n c_{\text{up}}^{(p_{n+1})} \).

See the tabularized values:

| \( p_{n+1} \) | up-type charge | down-type charge |
|---------------|----------------|------------------|
| 1             | 1              | 0                |
| 3             | \( \frac{2}{3} \) | \( -\frac{1}{3} \) |
| 7             | \( \frac{3}{7} \) | \( -\frac{4}{7} \) |
| 15            | \( \frac{6}{315} \) | \( -\frac{11}{315} \) |
| 31            | \( \frac{6}{255} \) | \( -\frac{35}{255} \) |
| 63            | \( \frac{6}{615195} \) | \( -\frac{57}{615195} \) |
| ...           | ...            | ...              |

Let us start with \( p_n = 7 \). According to Conjecture 5, level-7 preons will contain fourteen \( p_{\text{up}}^{(15)} \) and one \( p_{\text{down}}^{(15)} \) in the up-type case, and eight \( p_{\text{down}}^{(15)} \) plus seven \( p_{\text{up}}^{(15)} \) in the down-type case. For sublevel-3 preons, the constituents are \( 6p_{\text{up}}^{(7)} + p_{\text{down}}^{(7)} \) in the up-type and \( 4p_{\text{down}}^{(7)} + 3p_{\text{up}}^{(7)} \) in the down-type case, and for sublevel-1 items, the constituents are valence quarks, \( 2p_{\text{up}}^{(3)} + p_{\text{down}}^{(3)} \) for the proton and \( 2p_{\text{down}}^{(3)} + p_{\text{up}}^{(3)} \) for the neutron. Bearing the lesson of Sect. 4.3 in mind, a larger confining space – only the 3D lower end of which is familiar to us – is required to implement the Russian-doll preon structure. The operators \( f^{(15)} \) and \( h^{(15)} \) encode the necessary topological information to successively guide configurations across extra-dimensional (Euclidean) space. This is propounded in our second

**Conjecture 6** To qualify as constituents of a superordinate preon, hyperspheres must be used in \#’s that divide the kissing number of the space they live in.

\(^{15}\) the prefixes \( P^{(65535)} \) and beyond are not easily assessable.
Conjecture 6 implies that down-type constituents are subspace inhabitants relative to the space that up-type constituents live in. Since there are at least three generations of quarks, there must be room for enlarged configurations. For instance, while the down quark with electric charge $-\frac{2}{3}$ is a level-3 configuration on its own, an “uptype-preon plus downtype-antipreon” configuration formed by level-15 constituents, plus an “uptype antipreon” configuration formed by level-7 constituents, is required to yield the same result for the strange quark:
\[
[(14p^{(15)}_{\text{up}} + p^{(15)}_{\text{down}}) + (8p^{(15)}_{\text{down}} + p^{(15)}_{\text{up}})] + (6p^{(7)}_{\text{up}} + p^{(7)}_{\text{down}}),
\]
combines to yield
\[
\frac{14 \cdot 4 + 1 \cdot (-11) + 8 \cdot 11 + 7 \cdot (-4)}{315} + \frac{6 \cdot (-3) + 1 \cdot (4)}{21} = \frac{1}{3}.
\]
Correspondingly, level-31 constituents are part of configurations that reproduce the charges of quarks of the third generation. Just as level-15 constituents require $14_{\text{up}} \mid L_x, 8_{\text{down}} \mid L_y \ (x > y)$, level-31 constituents require $30_{\text{up}} \mid L_w, 16_{\text{down}} \mid L_z \ (w > z)$. According to Table 3, the only certified kissing numbers covered by $f^{(31)}$ and $h^{(31)}$ that are divisible by 30 are also divisible by 16: $L_8 = 240, L_{16} = 4320$ and $L_{24} = 196560$. Close to the confining $D^{(31)} = 31$, there is a further, if uncertified, one: $L_{29} = 207930$. It’s divisible by 30 but not by 16. There is the faint hope that this kissing number will be certified one day.

### 9 A proposal for a planar geometric model

#### 9.1 The cardioid and her arclength

Apart from the conjectured connection with sphere packing in Euclidean $D$-space, nilpotent operators such as $f^{(p)}$ and $h^{(p)}$ have interesting representations in ordinary plane trigonometry, involving cardioids whose cusps are located at the origin. Consider the rightmost cardioid in Fig. 10 which has the polar representation (a a parameter)
\[
r = a(1 + \cos \theta),
\]
and compare it to the cardioid left to it with polar representation
\[
r = a(1 + \sin \theta).
\]

Figure 10. $f$- ($h$-) parafermion in cardiodic representation

\[
r_n = a \left(1 + \cos (\theta + \frac{n-1}{p+1} \cdot \pi)\right), \quad n = \log_2(p + 1), \ p \in \{1, 3, 7, \ldots\},
\]

Obviously the transformation implies a quarter-turn around the origin, and the transformation into the leftmost cardioid a half-turn or flip-over. Recalling that $c_3$ and $c_2$, the basic building blocks of $f^{(p)}$ and $h^{(p)}$ for $p > 1$, realize such transformations in matrix form, we may conclude they lie at the basis of planar representations of $f^{(p)}$ and $h^{(p)}$. Whichever of the two one uses, they should be made an infinite process to mirror in a geometric spirit the forming of the root-$f$- and root-$h$ sequence. Now reflection is an operation indivisible within the framework of plane trigonometry, so we are left with rotation as a vehicle to express the infinite sequence. It would consist, first of no turn, followed by a quarter-turn, followed by further turns of ever-halving angle measured in radians as shown in Fig. 10 (see the second cardioid from the left as an example of an intermediate stage in the process corresponding to $p = 7$):

\[16\text{http://www.math.rwth-aachen.de/~Gabriele.Nebe/LATTICES/kiss.html}
whence in the limit – as though it was effected by \( c_2 \) – the rightmost cardioid would take the position of the leftmost:

\[
\lim_{p \to \infty} \frac{1}{a} \left( \frac{\cos \theta + \frac{1}{p+1} \cdot \pi}{1 + \cos \theta + \frac{1}{p+1} \cdot \pi} \right) = a(1 + \cos \theta + \pi) = a(1 - \cos \theta). \tag{70}
\]

Let us expound the details of the envisioned representation. The arclength of the cardioid is determined by the integral

\[
\int_{a_1}^{a_2} ds,
\]

where \( ds = \sqrt{r^2 d\theta^2 + dr^2} \). For the cardioid at rest, \( ds = 2a \cos \frac{\theta}{2} d\theta \). This function remains valid if the arc traced between the limits doesn’t cross the cusp or its antipodal point. The maximal admissible interval for the lower and upper limits thus is \([0, \pi]\), and the circumference becomes

\[
C = 2 \int_{0}^{\pi} ds = 8a; \tag{72}
\]

otherwise the circumference turns 0, a result which is in accord with the nullity of an order-\( p \) nilpotent operator for exponents larger than \( p \), and immediately makes clear that this operator exponentiated has to be represented by a compound of arclengths which eventually transgresses the \( 4\alpha \) boundary. In the case of the cardioid set into motion, the total angle accruing from counterclockwise rotations according to Eq. 69 toward its end position does not exceed \( \pi \). To properly map the nilpotence condition, however, we must after each step use the \( z \)-axis as an equator and 1) separate arc parts from the upper half-plane with lower and upper azimuths \( u \) and \( v \), from those of the lower half-plane with azimuths \( u \) and \( v \), and 2) keep track of the gap left behind in the upper half-plane by the moving cardioid with lower and upper azimuths \( co-u \) and \( co-v \). As for actual arclength computations, an option has to be taken of either using the cardioid-at-rest arclength function or its cardioid-in-motion counterpart. Let us first examine option one according to which we have to compare those arclength parts of the cardioid in motion and the cardioid at rest that are in correspondence with each other. It turns out that, to accommodate cardioidic motion, the lower and upper integral limits have to be determined by the coordinate transformations

\[
\begin{align*}
\text{upper half-plane:} & \quad (u_n, v_n) = \left( \frac{p-1}{p+1}, \pi \right) \rightarrow (w_n, z_n) = \left( 0, \left( 1 - \frac{p-1}{p+1} \right) \pi \right), \\
\text{co-u_n, co-v_n} & = (co-w_n, co-z_n) = \left( 0, \frac{p-1}{p+1} \pi \right),
\end{align*}
\]

\[
\begin{align*}
\text{lower half-plane:} & \quad (\bar{u}_n, \bar{v}_n) = \left( \pi, \left( 1 + \frac{p-1}{p+1} \right) \pi \right) \rightarrow (\bar{w}_n, \bar{z}_n) = \left( \left( 1 - \frac{p-1}{p+1} \right) \pi, \pi \right),
\end{align*}
\]

where of course \( n = \log_2(p+1), \quad p \in \{1, 3, 7, \ldots \} \) both times. Labeling the corresponding arclengths \( A_n \) and \( \bar{A}_n \) and computing them for the first four cardioid stops \( r_1 = a(1 + \cos \theta) \), \( r_2 = a(1 + \cos(\theta + \frac{\pi}{8})) \), \( r_3 = a(1 + \cos(\theta + \frac{7}{8}\pi)) \), \( r_4 = a(1 + \cos(\theta + \frac{3}{8}\pi)) \), one finds

\[
\begin{align*}
A_1 &= \int_0^\pi 2a \cos \frac{\theta}{2} d\theta = 4a; \\
A_2 &= \int_0^\pi 2a \cos \frac{\theta}{2} d\theta = 4a \cdot \sqrt{2}; \\
A_3 &= \int_0^\pi 2a \cos \frac{\theta}{2} d\theta = 4a \cdot \sqrt{2 - \sqrt{2}}; \\
A_4 &= \int_0^\pi 2a \cos \frac{\theta}{2} d\theta = 4a \cdot \sqrt{2 - \sqrt{2 + \sqrt{2}}}; \\
&\cdots
\end{align*}
\]

and for the subequatorials,

\[
\begin{align*}
\bar{A}_1 &= \int_{\pi}^\pi 2a \cos \frac{\theta}{2} d\theta = 0; \\
\bar{A}_2 &= \int_{\pi}^\pi 2a \cos \frac{\theta}{2} d\theta = 4a \left( 1 - \frac{\pi}{4} \right); \\
\bar{A}_3 &= \int_{\pi}^\pi 2a \cos \frac{\theta}{2} d\theta = 4a \left( 1 - \sqrt{2 - \sqrt{2}} \right); \\
\bar{A}_4 &= \int_{\pi}^\pi 2a \cos \frac{\theta}{2} d\theta = 4a \left( 1 - \sqrt{2 - \sqrt{2 + \sqrt{2}}} \right); \\
&\cdots
\end{align*}
\]

\[17\] In case of a process \( r_i = a(1 + \sin(\theta + \frac{p-1}{p+1} \cdot \pi)) \) the \( y \)-axis had to be used as equator.

36
A nilpotent operator of order $p \in \{1, 3, 7, \ldots\}$ then is representable by $A_n$ ($n = \log_2(p + 1)$), and the action on itself by the operation
\[
(A_n, A_n) \equiv 2 \frac{A_n \cdot \text{co-}A_n}{A_n + A_n} = A_{n-1}.
\] (74)
The auxiliary expressions co-$A_n$ (for co-arclength on the cardioid fixed at rest) are given by
\[
\begin{align*}
\text{co-}A_1 &= \int_0^\pi 2a \cos \frac{u}{2} d\theta = 0; \\
\text{co-}A_2 &= \int_0^\pi 2a \cos \frac{u}{2} d\theta = 4a \cdot 2\sqrt{3}; \\
\text{co-}A_3 &= \int_0^\pi 2a \cos \frac{u}{2} d\theta = 4a \cdot 2\sqrt{2+\sqrt{3}}; \\
\text{co-}A_4 &= \int_0^\pi 2a \cos \frac{u}{2} d\theta = 4a \cdot 2\sqrt{2+\sqrt{2+\sqrt{3}}}; \\
&\vdots
\end{align*}
\]
and obey the Vieta condition, i.e., for $a = \frac{1}{4}$ form the Euler product
\[
\lim_{n \to \infty} \prod_{i=2}^{n} \text{co-}A_i = \frac{2}{\pi}.
\] (75)
The “unmoved-mover” representation constructed this way seems to be akin to the root-$f$ sequence since the integration limits are derived by coordinate rotation (see transformations (73)). Following this reasoning, a “moving-mirror” representation that was akin to the root-$h$ sequence would be expected to ensue from option two with integration limits derived by coordinate reflection. To see if this is true we introduce the arclength function of the cardioid in motion, $ds = 2a \cos(\frac{\theta}{2} + \frac{p-1}{p+1} \cdot \frac{\pi}{3}) d\theta$, and look what else is needed to reproduce identical results in terms of arclengths as this function is used. It turns out that in all azimuth-to-integration limit transformations there is a flip over the equator ($x$-axis) in this case, confirming the expectation that the corresponding set of arclength formulae is akin to the root-$h$ sequence. This conclusion is also supported by interordinality considerations. Recalling that for the root-$f$ sequence the carry-bit neighborhood link is characterized by identities that either are purely interordinal (Eq. (28)) or intraordinal (Eq. (29)) whereas for the root-$h$ sequence the corresponding link mixes interordinal and intraordinal relationship (Eq. 57), we can observe a similar phenomenon in the present planar representations: For the upper definite-integral limit co-$z_n'$, e.g., we find
\[
\begin{align*}
\text{co-}z_n' &= \frac{p}{p+1} \cdot \pi, \\
\text{and in the “moving-mirror” representation:} \\
\text{co-}z_n' &= \left(\frac{p'+1}{p+1} - \frac{p}{p+1}\right) \pi,
\end{align*}
\] (77)
where $n' = \log_2(p'+1)$, $p' = 2p + 1$ and $p \in \{1, 3, 7, 15, \ldots\}$, which completes the analogy.

9.2 Cardioidic arclength and its relation to Catalan structure

When focusing on cardioidic (co-)arclength per se:
\[
A_n = 4a \sin \frac{\pi}{p+1},
\]
\[
\text{co-}A_n = 4a \cos \frac{\pi}{p+1},
\] (78)
no such distinction can be made. This aside, simple features of the Catalan structure such as the departure from $G_{\mu\nu} = 1$ and onset of $(q,p)$-interordinality at $p = 15$ still find a planar analogy, namely in the loss of homogeneity for $n > 3$ of the factoring of $x^{2n} + y^{2n}$ – whose limit contour is a square centered at the origin of the $x$-$y$ plane – into polynomials whose contours are
diagonally-oriented crossing ellipses\textsuperscript{18} with eccentricities rooted in Eqs. (78). In what follows we always assume \( a = \frac{1}{4} \). Then the first approximation to the square is an inscribed circle:

\[
x^2 + 2 \text{co-A}_1 xy + y^2
\]

(co-A\(_1\) = 0 marks the degeneracy of the case); the second approximation is given by

\[
x^4 + y^4 = (x^2 + 2 \text{co-A}_2 xy + y^2)(x^2 - 2 \text{co-A}_2 xy + y^2) = (x^2 + 2A_2 xy + y^2)(x^2 - 2A_2 xy + y^2);
\]

and the third one by

\[
x^8 + y^8 = (x^2 + 2 \text{co-A}_3 xy + y^2)(x^2 - 2 \text{co-A}_3 xy + y^2)(x^2 + 2A_3 xy + y^2)(x^2 - 2A_3 xy + y^2).
\]

As early as in the next higher instance, however, interpolating (co-)arclengths – as a footprint of the first approximation to the square is an inscribed circle:

\[
F or further insight is gained by considering the continued fraction expansions
\]

\[
A_n = a_0^{(n)} + \frac{1}{a_1^{(n)} + \frac{1}{a_2^{(n)} + \frac{1}{a_3^{(n)} + \ldots}}}
\]

in conjunction with the accordingly defined continued fractions

\[
\text{co-A}_n \equiv \{a_0^{(n)}, a_1^{(n)}, a_2^{(n)}, \ldots\}, \quad \tilde{A}_n = \{\tilde{a}_0^{(n)}, \tilde{a}_1^{(n)}, a_2^{(n)}, \ldots\},
\]

\[
\text{co-A}_2 \equiv \{\text{sqco-}a_0^{(n)}, \text{sqco-}a_1^{(n)}, \text{sqco-}a_2^{(n)}, \ldots\}, \quad A_n^2 = 1 - \text{co-A}_n^2 \equiv \{\text{sqco-}\tilde{a}_0^{(n)}, \text{sqco-}\tilde{a}_1^{(n)}, \text{sqco-}a_2^{(n)}, \ldots\},
\]

and their associated identities for \( n > 2 \),

\[
(1 + \text{co-a}_2^{(n)})/2 = \text{sqco-a}_2^{(n)} \quad \text{for} \quad n \equiv 1 \text{ mod } 4 \quad \text{else} \quad (1 + \text{co-a}_2^{(n)})/2 = \text{sqco-a}_2^{(n)}.
\]

as well as the special cases

\[
A_2 = \text{co-A}_2 = [0; 1, \frac{1}{2}], \quad \tilde{A}_2 = [0; 3, \frac{2}{2}].
\]

Then the leading \( A_n \) CF expansion coefficient will be found to mimic a carry-bit neighborhood \( p' = 2p + 1 \):

\[
a_1^{(n+1)} = 2a_1^{(n)} + 1 + \delta_1^{(n)}
\]

where \( \delta_1^{(n)} \in \{-3, -1, 0\} \); and the next-to-leading co-A\(_n\) CF expansion coefficient a second-closest-carry-bit neighborhood \( p'' = 4p + 3 \):

\[
\text{co-a}_2^{(n+1)} = 4\text{co-a}_2^{(n)} + 3 + \delta_2^{(n)}
\]

where \( \delta_2^{(n)} \in \{-1, 0, 1, 2, 3\} \). As we shall see, these expansion coefficients can directly be tied to the intrinsic Catalan structure of \((C_\mu^\nu)\) (or \((j_\mu^\nu)\)). This is because the carry-bit neighborhood \( p' = 2p + 1 \) and its extension, the second-closest-carry-bit neighborhood \( p'' = 4p + 3 = 2p' + 1 = 2(2p + 1) + 1 \), have structural analogs in the interordinal identities (28,30) and intraordinal identities (29,31): In the light of Eqs. (82,83), these identities can be dubbed sine-like and cosine-like respectively\textsuperscript{19}. We conveniently harmonize Eqs. (82,83) by constraining Eq. (82) to its second-closest-neighbor form:

\[
a_1^{(2n+r)} = 4a_1^{(2n+r-2)} + 3 + 2\delta_1^{(2n+r-2)} + \delta_1^{(2n+r-1)} \quad (n > 2; \ r \in \{0, 1\}).
\]

\textsuperscript{18} see [ShuKo10] for further study of crossing ellipses

\textsuperscript{19} For \( p'' = 31 \), for instance, the assignments \( \tilde{p} \) := \((4905c_3 \ 1633c_3)\), \( \bar{p} := (c_3 \ c_3) \), \( 1_\bar{p} := (41c_3 \ 17c_3) \) and

\[
1_\bar{p} := \left(\begin{array}{c}
4819c_3 \\
15067c_3
\end{array}\right) 
\]

furnish the structural analog of second-closest-carry-bit neighborhood: \( \tilde{p} \) := 2(2\( \bar{p} + 1\)) + 1\( 1_\bar{p} \).
The last two terms in Eq. (84) suggest that, in Eq. (83), $\delta_2^{(n)}$ may be resolved in a cognate way, viz.

$$\delta_2^{(n)} = 2\delta_{1,co-a_2^{(n)}} + \varepsilon_{bool_1^{(n)},bool_2^{(n)}}$$  \hfill (85)

where $\delta_{ab}$ is the Kronecker symbol and $\varepsilon_{bool_1, bool_2}$ a Levi-Civita symbol with $\varepsilon_{FF} = \varepsilon_{TT} = 0$, $\varepsilon_{FT} = 1$, $\varepsilon_{TF} = -1$. It turns out that the boolean data in question ensue from the truth values of distinct inequalities $co-a_i^{(n)} \geq co-a_{i+1}^{(n)}$, $co-a_{i+2}^{(n)} \geq co-a_{i+3}^{(n)}$ in modulo-8 arithmetics. Their onset at $co-a_1^{(n)}$ is determined by the index of the Fibonacci number $F_n$ as $n$ progresses and is preserved $F_i$ times. Following the pattern $F_3, F_4, F_5, \ldots$, we for $F_i \neq 0 \pmod{8}$ find unidirectionality of inequality pairs whose sense alternates with progressing $n$, whereas for $F_i = 0 \pmod{8}$, we see same-sense bidirectionality within subdivisions $F_{i-1}$, $F_{i-2}$ and sense reversal at the boundary. See Table below where onset is emphasized by a vertical bar:

| $F_i$ | $n = \log_2(p + 1)$ | $\delta_2^{(n)}$ | co-$A_i$ | $\delta_{1,co-a_2^{(n)}} + \varepsilon_{bool_1^{(n)},bool_2^{(n)}}$ |
|-------|---------------------|------------------|---------|--------------------------------------------------|
| $F_3 = 2$ | 3                  | 0                | $[0;1,12,7,3,2,1,\ldots]$ | $2\delta_{1,7} + \varepsilon_{7<3,2<1} = 0$ |
|        | 4                  | -1               | $[0;1,51,1,23,43,8,1,\ldots]$ | $2\delta_{1,7} + \varepsilon_{7>3,0>1} = -1$ |
| $F_4 = 3$ | 5                  | 2                | $[0;1,206,1,2,1,8,1,\ldots]$ | $2\delta_{1,1} + \varepsilon_{2<2,1<1} = 2$ |
|        | 6                  | 0                | $[0;1,829,5,3,1,2,1,\ldots]$ | $2\delta_{1,5} + \varepsilon_{3>1,2>1} = 0$ |
|        | 7                  | 0                | $[0;1,3319,3,1,6,32,1,\ldots]$ | $2\delta_{1,3} + \varepsilon_{1<6,0<1} = 0$ |
| $F_5 = 5$ | 8                  | 1                | $[0;1,13279,1,1,6,3,1,1,\ldots]$ | $2\delta_{1,1} + \varepsilon_{6>3,1>1} = 1$ |
|        | 9                  | 2                | $[0;1,53120,1,1,1,5,10,13,\ldots]$ | $2\delta_{1,1} + \varepsilon_{1<5,2<5} = 2$ |
|        | 10                 | -1               | $[0;1,1212485,11,6,6,1,2,2,\ldots]$ | $2\delta_{1,3} + \varepsilon_{6>1,2>2} = -1$ |
|        | 11                 | 2                | $[0;1,849942,1,6,16,1,13,7,\ldots]$ | $2\delta_{1,1} + \varepsilon_{0<1,5<7} = 2$ |
|        | 12                 | 3                | $[0;1,3399773,1,14,225,1,5,2,\ldots]$ | $2\delta_{1,1} + \varepsilon_{1<5,1>2} = 3$ |
| $F_6 = 8$ | 13                 | 0                | $[0;1,13599098,4,3,1,1,1,1,\ldots]$ | $2\delta_{1,4} + \varepsilon_{1>5,1<1} = 0$ |
|        | 14                 | 1                | $[0;1,54396395,2,3,3,1,2,1,3,\ldots]$ | $2\delta_{1,2} + \varepsilon_{1>2,1<3} = 1$ |
|        | 15                 | 0                | $[0;1,217585584,4,3,1,4,6,11,1,\ldots]$ | $2\delta_{1,4} + \varepsilon_{4>3,1<0} = 1$ |
|        | 16                 | 1                | $[0;1,870342339,2,3,1,1,3,1,14,\ldots]$ | $2\delta_{1,2} + \varepsilon_{1>1,1<6} = 1$ |
|        | 17                 | 0                | $[0;1,3481369360,3,1,17,4,7,2,17,\ldots]$ | $2\delta_{1,3} + \varepsilon_{4,7<1,2<0} = 0$ |
| $F_7 = 13$ | 18                | 1                | $[0;1,1392547744,1,1,17,2,30,2,4,\ldots]$ | $2\delta_{1,1} + \varepsilon_{2<6,2>4} = 1$ |
|        | 19                 | 1                | $[0;1,55701909776,1,1,1,1,4,94,8,1,\ldots]$ | $2\delta_{1,1} + \varepsilon_{1<6,0<1} = 1$ |
|        | 20                 | 2                | $[0;1,222807639108,1,2,1,1,1,1,30,\ldots]$ | $2\delta_{1,1} + \varepsilon_{1<1,1<6} = 2$ |
|        | 21                 | 1                | $[0;1,891230556437,2,1,1,3,7,2,5,23,\ldots]$ | $2\delta_{1,2} + \varepsilon_{7<2,5<7} = 1$ |

On closer inspection, it appears that $\delta_2^{(n)}$ provides the clue to the envisioned continued-fraction Catalan-structure link:

**Conjecture 7** Let $LL(G^{(p)}_{\mu \nu})$ be the product of a $(\frac{p+1}{3}) \times m_c^{(n)}$ matrix $(e_{rs})$ and a $m_c^{(n)} \times (\frac{p+1}{3})$ matrix $(\chi_{sr})$. Then $e_{rs} \in \{\delta_2^{(2)}, \ldots, \delta_2^{(n)}\}$ and $\chi_{sr} \in \{C_{min(1,q)} \ldots C_{2q}\}$, where $p = 3, 7, 15, \ldots, q = \frac{p-3}{4}$, $n = \log_2(p + 1)$.

This is trivially true in the case $p = 3$, where $LL(G^{(3)}_{\mu \nu}) = C_0 = 1$ coincides with $\delta_2^{(2)} = 1$ in the computation: $co-a_2^{(3)} = 12 = 4co-a_2^{(2)} + 3 + \delta_2^{(2)} = 4 \cdot 2 + 3 + 1$. For paraorder seven, or $n = 3$, every $G^{(7)}_{\mu \nu}$ from $LL(G^{(7)}_{\mu \nu})$ can be represented by a dot product of a vector containing two elements $\in \{1\} \cup \{\delta_2^{(3)}\}$, where $\delta_2^{(3)} = 0$, and the vector $(C_1, C_1)$. Hence also trivially:

$$LL(G^{(7)}_{\mu \nu}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C_1 & C_1 \\ C_1 & C_1 \end{pmatrix};$$  \hfill (86)
and the $\delta_2^{(3)}$ value added here coincides with the one used in the computation: \(\text{co-a}_2^{(4)} = 4\text{co-a}_2^{(3)} + 3 + \delta_2^{(3)} = 4 \cdot 12 + 3 + 0 = 51\). For paraorder fifteen, every $G_{\mu\nu}^{(15)}$ from $\text{LL}(G_{\mu\nu}^{(15)})$ can be represented by a dot product of a vector with elements $\in \{0, 1\} \cup \{\delta_2^{(4)}\}$, where $\delta_2^{(4)} = -1$, and vector $(C_1, C_2, \ldots, C_6)$. It turns out that we have to use $4 \times 11$ and $11 \times 4$ matrices to, for the first time nontrivially, epitomize $\text{LL}(G_{\mu\nu}^{(15)})$ in product form:

\[
\text{LL}(G_{\mu\nu}^{(15)}) = \begin{pmatrix}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1
\end{pmatrix}
\]

(87)

here, the $\delta_2^{(4)}$ coincides with the one used in $\text{co-a}_2^{(5)} = 4\text{co-a}_2^{(4)} + 3 + \delta_2^{(4)} = 4 \cdot 51 + 3 - 1 = 206$. Continuing for paraorder thirty-one, every $G_{\mu\nu}^{(31)}$ can be represented by a dot product of a vector with elements $\in \{-1, 0, 1\} \cup \{\delta_2^{(5)}\}$, where $\delta_2^{(5)} = 2$, and vector $(C_1, C_2, \ldots, C_{14})$. We find that it takes $8 \times 48$ and $48 \times 8$ matrices to render $\text{LL}(G_{\mu\nu}^{(31)})$ in product form; saving space, we name only those $G_{\mu\nu}^{(31)}$ that require $\delta_2^{(5)} = 2$, namely the secondary-diagonal entries from $\text{LL}(\text{LL}(G_{\mu\nu}^{(31)}))$; but its presence benefits other entries and has the dimensions of the factor matrices shrink from $8 \times 51$ and $51 \times 8$ to $8 \times 48$ and $48 \times 8$. And so on.

The number of columns in the left (rows in the right) matrix factor, $m_c^{(n)}$, can now be linked to both leading/next-to-leading continued-fraction coefficients, $a_1^{(n)}$ and $\text{co-a}_2^{(n)}$, via the harmonized second-closest-carry-bit neighborhood equation

\[
m_c^{(n+2)} = \text{co-a}_2^{(n+1)} - a_1^{(2n-2)} + n - 1 \quad (n > 1).
\]

(88)

Thus $m_c^{(4)} = 12 - 2 + 1 = 11$; $m_c^{(5)} = 51 - 5 + 2 = 48$; also $m_c^{(6)} = 206 - 20 + 3 = 189$; and so on. We further note that the associated paraorder is always contained among the infinity of coefficients $a_1^{(n)}$ or $\text{co-a}_2^{(n)}$ for some $\alpha$, where the connection between value and least place is provided by $m_c^{(n)}$. It appears that relation (81) decides which of the two is the representative of $p$ with $\alpha$ least. Where $\alpha_{\text{least}} = m_c^{(n)}$, $n > 3$; if $\left[(1 + \text{co-a}_2^{(n-1)}/2\right] = \text{sqco-a}_2^{(n-1)}$, which is the case for $n - 1 \equiv 1 \mod 4$, then

\[
p = a_1^{(n)}\text{else } \left[(1 + \text{co-a}_2^{(n-1)}/2\right] = \text{sqco-a}_2^{(n-1)}
\]

and $p = \text{co-a}_{\alpha_{\text{least}}^{(n)}}$, as shown in Table 16:

| $n$ | $m_c^{(n)}$ | $p$ |
|-----|-------------|-----|
| 3   | $\text{co-a}_2^{(3)} = 7$ |     |
| 4   | $12 - 2 + 1$ | $\text{co-a}_2^{(4)} = 15$ |
| 5   | $51 - 5 + 2$ | $\text{co-a}_{48} = 31$ |
| 6   | $206 - 20 + 3$ | $a_{189}^{(6)} = 63$ |
| 7   | $829 - 81 + 4$ | $\text{co-a}_{792}^{(7)} \approx 127$ |
|     |             |     |
We have seen the continued-fraction representation (CFR) of cardioid arclength is closely bound up with basic properties of the Catalan structure of \( LL(G^{(p)}_{max}) \). Since we are also interested in the finer points of Catalan structure – such as were hinted at in the CFR \( \Phi^{(p)} = (G^{(p)}_{max}/C_q)^{1/9} \) of Sect. 4.2 –, we are searching here for specially shaped numbers whose CFR properties would help examine them side by side with the cardioidic arclength case. As a starting point, we have a closer look at a conspicuous region where \( \delta_1^{(n)} \) first begins constant and then resumes deviating shortly after:

\[
\delta_1^{(3)} \rightarrow a_1^{(4)} = 5 \quad \delta_1^{(4)} = -1 \rightarrow a_1^{(5)} = 10 \quad \delta_1^{(5)} = -1 \rightarrow a_1^{(6)} = 20 \quad \delta_1^{(6)} = -1 \rightarrow a_1^{(7)} = 40 \quad \delta_1^{(7)} = 0 \rightarrow a_1^{(8)} = 81 \ldots \tag{89}
\]

The ensuing \( a_1^{(n)} \) hints at a base-5: \( 2^{n-4} \) positional number system origin, but the question is, if there are deviations pending, is the relationship between \( a_1^{(n)} \) and \( 5 \cdot 2^{n-4} \) (\( n \geq 4 \)) stable enough so that \( 5 \cdot 2^{n-4} \) could be called their base? The answer is in the affirmative since the ratio \( a_1^{(n)}/(5 \cdot 2^{n-4}) \) is fast approaching the constant 1.01859... There is a neat interpretation at hand for this phenomenon in terms of two rival angular measurement systems – the SI system, in which there are 2000 π milliradians in the circle, and the NATO system, with 6400 angular mil in the circle. Their conversion ratio coinciding with 1.01859..., we can set

\[
\frac{a_1^{(n)}}{5 \cdot 2^{n-4}} \sim \frac{6400}{2000 \pi} = \frac{2^5}{2 \cdot 5 \cdot \pi}
\]

to see that \( a_1^{(n)} \) is just the integer approximation of \( 2^n/\pi \) and the change in \( \delta_1^{(n)} \) due to decimal switching < 0.5 \( \leftrightarrow \) > 0.5, viz.

\[
\begin{align*}
a_1^{(4)} &= 5 & a_1^{(5)} &= 10 & a_1^{(6)} &= 20 & a_1^{(7)} &= 40 & a_1^{(8)} &= 81 \\
\frac{2^4}{\pi} &= 5.09 & \frac{2^5}{\pi} &= 10.18 & \frac{2^6}{\pi} &= 20.37 & \frac{2^7}{\pi} &= 40.74 & \frac{2^8}{\pi} &= 81.48
\end{align*}
\]

Candidate numbers that would allow taking such features into account are the special Catalan’s \( C_{-1/k} = \frac{2^{2-k} \Gamma(1/2-1/k)}{\sqrt{\pi} \Gamma(2-1/k)} \),

\[
C_{-1/k} = t_0^{(k)} + \frac{1}{l_1^{(k)} + \frac{1}{l_2^{(k)} + \frac{1}{l_3^{(k)} + \ldots}} = [t_0^{(k)} ; t_1^{(k)} , t_2^{(k)} , t_3^{(k)} , \ldots]}, \tag{90}
\]

which satisfy the successor axiom

\[
l_1^{(k+1)} = l_0^{(k)} + l_1^{(k)} = 1 + l_1^{(k)} \quad (k > 3), \tag{91}
\]

and possess incidences of \( l_\kappa^{(k+1)} = 1 + l_\kappa^{(k)} \) for \( \kappa > 1 \) where the successor relation breaks after a few increases in \( k \), viz.

\[
l_4^{(6)} = 1, l_4^{(7)} = 2, l_4^{(8)} = 3, l_4^{(9)} = 4 \quad (\text{but } l_4^{(10)} = 7). \tag{92}
\]

Apparently there is successorship in example (92), expressed by \( k - l_4^{(k)} = \text{const.} \), which lasts until carry occurs in the quinary or the decimal system, as the case may be. This suggests arranging \( k \) such that the numbers \( C_{-1/k} \) are indexed by pairs of alternating Mersenne numbers, arithmetically-averaged and organized as negative reciprocals, or “intensional”, for short:

\[
C_{-1/k} \quad \text{with} \quad k = \frac{p+q}{2} = 4, 9, 19, 39, \ldots \quad \text{for} \quad p = 7, 15, 31, 63, \ldots \quad q = (p-3)/4. \tag{93}
\]

In terms of positional number systems, the first member of the above \( k \) sequence, 4, is about to carry in the quinary system; the second, 9, is about to carry in the decimal or the quinary system, and mutatis mutandis for the further members with respect to the vigesimal, quadragesimal, etc. systems. Thus, \( k \) and \( k' \), endowed with the relation

\[
k' = 2k + 1,
\]
form carry-digit neighborhoods in all base-5·2ⁿ systems, and the original carry-bit neighborhood of p and p’ is recovered cutting by the rightmost digit, most easily recognizable for the decimal system:

\[ 19 \ 39 \ 79 \ 159 \ \ldots \]

As Catalan structure is characterized by the way the Catalan numbers \( C_q \) thru to \( C_{2q} \) are partitioned in \( LL(G^{(p)}_{t\mu}) \) (or \( LL(J^{(p)}_{t\mu}) \) for that matter), we first have to search for an algorithm that looks for an intensional-Catalan-number CFR for \( C_q = G^{(p)}_{q+2,1} \).

Determining intensional-Catalan-number CFR for the remaining entries \( G^{(p)}_{t\mu} \) then consists in further refinement steps. In other words, where \( \varphi^{(k)} : \mathbb{N} \to \mathbb{N} \) is defined by \( \varphi^{(k)}(x) := l^{(k)}_{q} \) \((k > 3)\), we search for a partial inverse map \( (\varphi^{-1})^{(k)} : \mathbb{N} \to \mathbb{N} \) defined by \( (\varphi^{-1})^{(k)}(y) = \nu \), for select values \( y \) and begin with \( y = G^{(p)}_{q+2,1} \). The first algorithm we propose embodies an interordinal relationship:

**Algorithm 1**

Where \( j \) and \( m \) are natural numbers, pick the paraorders \( p \) and \( p' = 2p+1 \), with \( q = (p-3)/4 \) and \( q' = (p'-3)/4 \), and initialize \( j \) with \( q \) and \( m \) with \( \max(2^4 C_q,20) \). Vary \( m \) by successive increases or decreases, and if needed reinitialize \( m \) and increase \( j \), until for some pair \( (j,m) \) and for some prime \( \pi_s > 2 \) the condition \( 4^j - 1 \cdot m - \pi_s = G^{(p)}_{3q+5} \) is fulfilled under the constraint \( (2) \ j < q' \). Then \( \nu = 2^j m + 2^j - 1 + C^{2}_{q+1} \).

Case \( p' = 15 \): This is an example where with any contfrac calculator we can find \( l^{(9)}_{47} = C_3 = G^{(15)}_{5,1} = 5 \) and check Algorithm 1 for this solution.

Given are \( q' = 3 \mapsto C_{q'} = 5; \ q = 1 \mapsto C_q = 1, \max(2^4 C_q,20) = 20, C_{q+1} = 2; \) and \( p = 7 \mapsto G^{(p)}_{3q+5} = 1 \).

With \( j = 1, m = 20 \), right from the start we have \( \nu = 20 \cdot 2 + 1 + 2^2 = 47 \), and the pair \( (j,m) \) fulfills condition (1) \( 4^j - 1 - m - \pi_s = 20 - 19 = 11 \) as well as constraint (2).

Case \( p' = 31 \): Given are \( q' = 7 \mapsto C_{q'} = 429; \ q = 3 \mapsto C_q = 5, \max(2^4 C_q,20) = 40, C_{q+1} = 14; \) and \( p = 15 \mapsto G^{(p)}_{3q+5} = 41 \).

Then \( \nu = 13 \cdot 2^4 + 2^4 - 1 + 14^2 = 419 \) and the pair \( (j,m) \) fulfills condition (1) \( 4^j - 1 - m - \pi_s = 52 - 11 = 41 \) as well as constraint (2).

Case \( p' = 63 \): Given are \( q' = 15 \mapsto C_{15} = 9694845; \ q = 7 \mapsto C_q = 429, \max(2^4 C_q,20) = 54912, C_{q+1} = 1430; \) and \( p = 31 \mapsto G^{(p)}_{3q+5} = 58781 \).

Then \( \nu = 59764 \cdot 2^7 + 2^7 - 1 + 1430^2 = 9694819 \) and the pair \( (j,m) \) fulfills condition (1) \( 4^j - 1 - m - \pi_s = 59764 - 983 = 58781 \) as well as constraint (2).

We have found a second algorithm that delivers identical results for \( p = 15, 31 \), but differs for \( p = 63 \).

**Algorithm 2**

Where \( \bar{C}(p) \) is the largest even Catalan number \( C_r < p \) \((p = 15, 31, \ldots; q = (p-3)/4)\), choose a prime number \( \pi_{2m} \) such that \( m \geq q \) is least under the constraint \( \pi_{2m} > C_q \). Then \( \nu = \pi_{2m} - \bar{C}(p) \).

Case \( p = 15 \): Given are \( q = 3, C_3 = 5, \bar{C}(15) = C_4 = 14; \) then \( \nu = \pi_{6,3} - 14 = 61 - 14 = 47 \).

Case \( p = 31 \): Given are \( q = 7, C_7 = 429, \bar{C}(31) = C_4 = 14; \) then \( \nu = \pi_{6,14} - 14 = 433 - 14 = 419 \).

Case \( p = 63 \): Given are \( q = 15, C_{15} = 9694845, \bar{C}(63) = C_5 = 42; \) then \( \nu = \pi_{6,107624} - 42 = 9694877 - 42 = 9694835 \).

Proving one of these algorithms wrong lies beyond the scope of present-day online computing capabilities yet. In what follows we stick to \( \nu \leq 500 \) to address finer points of intensional-Catalan-number CFR, and also attempt disambiguating the result for \( p = 63 \). An important aid in this enterprise is supplied by the \( \text{parorder} \) sums \( S_{i=1}^{n+1} p_i = 2^{n+1} - n - 2 \), and their \( \text{entourage} \)

---

²⁰ The naturalness of \( k = \frac{p+q}{2} = 9,19,\ldots \) can be recognized by the behavior of the first nontrivial representative sequence \( (G^{(p)}_{15}) = (3, 5, 11, 17, 41, 113) \). By constructing the tuples \((3, 5), (3, 11), \ldots, (3, 5, 11), (3, 5, 17), \ldots, (3, 5, 11, 17, 41, 113) \) and checking their scalar products (in absolute value) with appropriately-sized tuples \((1, 1)^T, (1, -1)^T, \ldots, (1, 1, 1)^T, (1, -1)^T, \ldots, (1, -1, -1, -1, -1, -1)^T \), one sees that the natural numbers in the range 1 to 190 = \( \sum G^{(p)}_{15} \) are covered, leaving but an unrepresentable rest of twenty numbers: 7, 34, 48, \ldots, 189. Nineteen of these can be lifted by adding the first member of the representative sequence \( (G^{(31)}_{p}) = (19, \ldots) \), adapting the tuples appropriately; but at the expense of - now nine - new exceptions 183, 191, \ldots, 208 within the enlarged (interordinal) range 1 to 209.
\[
\text{pose}_1(p) = 2^\Theta n \sum_{i=1}^n p_i - (\varphi^{-1})^{(k)}(C_q), \\
\text{pose}_2(p) = 2^{\Theta n - 1} \sum_{i=1}^{n+1} p_i - (\varphi^{-1})^{(k)}(C_q),
\]

\[n > 2\]

where \( \Theta_n = \mathcal{C}_n^+ + \sum_{i=1}^n \Delta r_i \), \( \mathcal{C}_n^+ \) being the \( n \)-th member of the ordered sequence \( \mathcal{C}^+ \) (see Eq. (39)) and \( \sum_{i=2}^n \Delta r_i \) the sum of index increments \( \Delta r_i = |r_i| - |r_{i-1}| \) for \( C_{r_{i-1}} = \mathcal{C}_{i-1}^+ \) and \( C_{r_i} = \mathcal{C}_i^+ \) \( n > 2 \).

### Table 17
Paraorder sums and their entourage up to \( n = 8 \)

| \( p_n \) | \( k \) | \( \sum_{i=1}^n p_i \) | \( \text{pose}_1(p) \) | \( \text{pose}_2(p) \) |
|---|---|---|---|---|
| 1 | - | 1 | - | - |
| 3 | - | 4 | - | - |
| 7 | 4 | 11 | \( 2^{10} \cdot 11 - 1 = 10 \) | \( 2^{-1} \cdot 26 - 1 = 12 \) |
| 15 | 9 | 26 | \( 2^{10} \cdot 26 - 47 = 5 \) | \( 2^0 \cdot 57 - 47 = 10 \) |
| 31 | 19 | 57 | \( 2^{2^2+1} \cdot 57 - 419 = 37 \) | \( 2^2 \cdot 120 - 419 = 61 \) |
| 63 | 39 | 120 | \( 2^{14+3} \cdot 120 - (\varphi^{-1})^{(39)}(C_{15}) \geq 6033821 \) | \( 2^{16} \cdot 247 - (\varphi^{-1})^{(39)}(C_{15}) \geq 6492573 \) |
| 127 | 79 | 247 | \( 2^{42+4} \cdot 247 - (\varphi^{-1})^{(79)}(C_{31}) = ? \) | \( 2^{45} \cdot 502 - (\varphi^{-1})^{(79)}(C_{31}) = ? \) |
| 255 | 159 | 502 | ? | ? |

We recall: \( C_7 = 429 \) is the constitutive Catalan representative \( G_{9,1}^{(31)} \) of \( (\mathcal{C}_7^{(31)}) \). It is one of the results predicted by Algs. 1 and 2 that this number is matched by the 419th expansion coefficient of \( C_{1/19} \)

\[ l_{1/19}^{(19)} = 429. \]

Out of the remaining \( G_{9,2}^{(31)} \), only those that belong to the nonbracketed, nonparenthesized part of the corridor \( G \)-set \( G_{\text{cor}}^{(31)} = \{1, 3, 5, 11, 17, 41, 19, 43, 115, 155, 429\} \), are allocated in the vicinity of \( \kappa = 419 \), which means Eqs. (95) and (96) constitute an intensional Catalan-number CF description of that part of the corridor \( G \)-set. Thus, \( G_{9,2}^{(31)} = 155 \) is matched by the 408th expansion coefficient, and \( G_{10,3}^{(31)} = 115 \) by the 397th,

\[ l_{408}^{(19)} = 155, \quad l_{397}^{(19)} = 115. \]

\[ Viz. \]

\[ C_{1/19} = [1; 16, 2, 4, \ldots, 115, 2, 13, \ldots, 155, 97, 1, \ldots, 429, 2, 4, \ldots]. \]

\[ 0 1 2 3 \ldots 397 398 399 \ldots 408 409 410 \ldots 419 420 421 \ldots \]

It turns out that \( G_{9,2}^{(31)} = 155 \) as a CF denominator of \( C_{1/19} \) occurs at a distance \( 11 = \sum_{i=1}^3 p_i \) off \( \kappa = 419 \), and the same distance lies between \( (\varphi^{-1})^{(19)}(G_{9,2}^{(31)}) \) and \( (\varphi^{-1})^{(19)}(G_{10,3}^{(31)}) \),

\[ (l_{1/19}^{(19)} = 429) \quad \underline{\varphi^{(19)} = 11} \quad l_{408}^{(19)} = 155 \quad \underline{\varphi^{(19)} = 11} \quad l_{397}^{(19)} = 115, \]

so that the distance (edge length) between the entries (nodes), \( D^{(19)} \), in this case coincides with the paraorder sum

\[ 11 = \sum_{i=1}^{n-2} p_i. \]

\[ ^{21} \text{for a definition, and the meaning of the parentheses and brackets, see Sect. 4.3} \]
It’s interesting to compare this pattern with that corresponding to the alternating-sign corridor $J$-set of $LL(J^{(31)}_{\ell\nu})$, although a slightly different methodology is required to this end. Let $C_{1/k}$ alternatively be given by the expansion\textsuperscript{22}

$$C_{1/k} = \ell_0^{(k)} - \frac{1}{\ell_1^{(k)}} = \frac{1}{\ell_2^{(k)}} - \frac{1}{\ell_3^{(k)}} + \ldots \equiv [\ell_0^{(k)} : \ell_1^{(k)} , \ell_2^{(k)} , \ell_3^{(k)} , \ldots] ,$$

(99)

and the associated map $\psi^{(k)} : N_0 \to \mathbb{N}$ by $\psi^{(k)}(\kappa) := \ell_2^{(k)} (k > 3)$ with partial inverse $\left(\psi^{-1}\right)^{(k)} : \mathbb{N} \to \mathbb{N}$. Then, for entry $J^{(19)}_9$, there exists a denominator whose place $\left(\psi^{-1}\right)^{(19)}(\epsilon_9^{(19)})$ lies in the vicinity of the place $\left(\phi^{-1}\right)^{(19)}(G^{(31)}_9)$ predicted by Algs. 1 and 2, namely:

$$J^{(31)}_9 = 429 = \ell^{(19)}_{438} ;$$

but that vicinity by necessity now leads to branchings for the remaining two entries which do not obey a strict ordering, $J^{(31)}_9, J^{(31)}_{10,3} \neq J^{(15)}_{max}$ instead of $G^{(31)}_9, G^{(31)}_{10,3} > G^{(15)}_{max} :$

$$J^{(31)}_9 = \psi^{(19)}(\kappa_a) + \psi^{(19)}(\kappa_b) , \quad J^{(31)}_{10,3} = \psi^{(19)}(\kappa_c) + \psi^{(19)}(\kappa_d).$$

(100)

Thus,

$$J^{(31)}_9 = 117 = \ell^{(19)}_{411} + \ell^{(19)}_{409} = 116 + 1 ,$$

and

$$J^{(31)}_{10,3} = 143 = \ell^{(19)}_{425} - \ell^{(19)}_{414} = 156 - 13 .$$

Including the sublevels created, in contradistinction to the edge length $11 = \Sigma_{i=1}^3 p_i$ of example (97), the average edge length now equals $((438 - 425) + (425 - 414) + (425 - 411) + (411 - 409))/4 = 10 = \text{pose}_{19}(7)$:

$$\left(\ell^{(19)}_{438} = 429 \right) \xlongequal{\ell^{(19)}_{425} = 156} \ell^{(19)}_{411} = 116 \xlongequal{\ell^{(19)}_{414} = 13} \ell^{(19)}_{409} = 1 .$$

(101)

Now the restriction of $C_{1/19}$ CF denominators to those qualifying as representatives of nonbracketed, nonparenthesized corridor $G$-set entries recalls a similar one of kissing number representatives to those qualifying as simple interordinal $(\phi^{(p_i,p_i)})$ or second-order synaptic differences at paraorder 31: the entries $G^{(31)}_9 = 429$ and $G^{(31)}_2 = 155$ are $C_{1/19}$ CF represented by $l^{(19)}_{419}$ and $l^{(19)}_{408}$, respectively; their pendants (in the kissing-number representative sense) from Table 8 are given by $L_9 = 240 = \phi^{(15,255)}_2$ and $L_{10} = 336 = \phi^{(15,63)}_2 = \phi^{(15,127)}_3$. Also, $G^{(31)}_{10,3} = 115, C_{1/19}$ CF represented by $l^{(19)}_{397}$, has a pendant from Table 9, $L_9 = 272 = \partial G^{(31)}_{10,3} - \partial J^{(31)}_{10}$, the simple-interordinal/second-order synaptic difference representability desert following $L_{10}$, first ending at $L_{13}$ for Table 9, and at $L_{16}$ for Table 8, should be accompanied by a similar desert in $C_{1/19}$ CF representability, whose discovery yet awaits improved CFR computability conditions.

Before looping back to the case $k = 9$, let us make it clear that the ensuing places discussed this far are least, that is, the $t$- or $l$-values they map may reappear at higher places. Thus the results of Algs. 1 and 2 for case $p = 63$, if meaningful, need not be conflicting: $\kappa = 9694835$ could be a place of recurrence of $C_{15}$, as suggested by $C_{15} - (\phi^{-1})^{(39)}(C_{15}) = \text{pose}_{15}(15)$, while $\kappa = 9694819$ would be the least and supported by $C_{15} - (\phi^{-1})^{(39)}(C_{15}) = \Sigma_{i=1}^3 p_i$. Keeping this in mind, we can now turn to entries which, unavailable though they seem for $k = 19$, are reachable for $k = 9$. Thus, the place of $G^{(15)}_{8,1} = 113$ can be computed as $\text{pose}_{19}(31) \cdot \Sigma_{i=1}^3 p_i = 37 \cdot 11$:

$$l^{(9)}_{37,11} = 113 ,$$

(102)

viz.

$$C_{1/9} = [1 ; 6, 1, 1, 4, \ldots, 5, 3, 113, 1, 1, \ldots]$$

0 1 2 3 4 ... 405 406 407 408 409 ...

\textsuperscript{22} for further details, see the contrfrac-routine options provided by wims.unice.fr
and there is no lower place than 407 with this property. Yet, there is another occurrence of 113, close to the first,
\[ t_{414}^{(9)} = 113, \]

viz.
\[ C_{1/9} = [1; 6, 1, 1, 4, \ldots, 6, 1, 113, 1, 15, \ldots] \]
\[ 0 \ 1 \ 2 \ 3 \ 4 \ \ldots \ 412 \ 413 \ 414 \ 415 \ 416 \ \ldots \]

which falls into place in that \( G_{8,1}^{(15)} = 113 \) recurs interordinally as (non-corridor-G set entry) \( G_{12,5}^{(31)} = 113 \). Subtracting \( \text{pose}_1(15) = 5 \) from place \( (\varphi^{-1})_{\text{least}}^{(19)}(429) = 419 \), we get
\[ (\varphi^{-1})_{\text{least}}^{(9)}(113) = (\varphi^{-1})_{\text{least}}^{(19)}(429) - \text{pose}_1(15). \tag{103} \]

And there is a third occurrence of 113, doubled in value, and computable using \( (\varphi^{-1})_{\text{least}}^{(9)}(5), \) but \( \text{pose}_2(15) = 10: \)
\[ (\varphi^{-1})_{\text{least}}^{(19)}(226) = (\varphi^{-1})_{\text{least}}^{(9)}(5) - \text{pose}_2(15), \tag{104} \]

viz.
\[ C_{1/19} = [1; 16, 2, 4, 1, \ldots, 2, 1, 226, 3, 1, \ldots] . \]
\[ 0 \ 1 \ 2 \ 3 \ 4 \ \ldots \ 35 \ 36 \ 37 \ 38 \ 39 \ \ldots \]

So the preliminary interpretation of these observations would read: if the map \( \varphi^{-1}(y_1) \mapsto \varphi^{-1}(y_2) - \text{pose}(p) \) for key values \( y_1, y_2 \) is associated with a context change \( k' \mapsto k \) and the subtrahend is \( \text{pose}_1(p) \), the result is non-minimal \( \kappa \), and, conversely, if this map is associated with a context change \( k \mapsto k' \) and \( \text{pose}_2(p) \) is subtracted, the result is minimal \( \kappa \), but with a doubled reference outcome. Further study is required to corroborate this point.

### 9.4 The kissing number aspect revisited

Figures linked to Catalan numbers in a fundamental way like the kissing numbers can be expected to be present in more overt form in the current framework. They lay hidden in cardioid-arclength CFR, where \( C_7 - m_c^{(6)} \) yields the eighth kissing number, \( 429 - 189 = L_8 \), and \( C_6 - m_c^{(5)} \) equals the third plus the sixth, \( 132 - 48 = 12 + 72 = L_3 + L_6 \). They’re also an implicit part of the workings of our algorithms, where \( |(\varphi^{-1})_{\text{least}}^{(9)}(5) - 5| = L_5 + L_1 \), \( |(\varphi^{-1})_{\text{least}}^{(19)}(429) - 429| = L_3 - L_1 \), and
\[ |(\varphi^{-1})_{\text{least}}^{(39)}(9694845) - 9694845| = L_4 + L_1 \] according to Algorithm 1, and \( L_3 - L_1 \) according to Algorithm 2. Plus, they led to a salient interordinal corridor aspect in the previous section. Changing that perspective of inner regulative to its dual – exterior connection of densest-packing hypersphere configurations with Green’s parafermions of Merseennian order such that \( L_D = \vartheta_{\lambda_{p_1+p_2}} + \vartheta_{\lambda_{p_2-p_1}} + \ldots \) as outlined in Conjecture 2 –, there is no a priori reason why kissing numbers should not occur overtly as expansion coefficients of suitably chosen irrationals in imitation of this connection. This may be tested using the assessable case when simple interordinal differences \( \vartheta_{\lambda_{p_2-p_1}} \) of Green’s squares suffice to represent \( L_D \).
9.4.1 Detuning intensional Catalan numbers

From Eq. (93) it follows that \( k \sim \frac{5}{6} p \) for large \( p \), we might therefore take the integer approximation of the mean value \( \frac{5}{6} (p + p_u) \) as a target index \( k \) in \( C_{1/k} \) and look for occurrences of \( l_{\text{detunt}}^{(k)} = \vartheta_{\lambda}^{(p_i+p_u)} = L_D \), keeping the pairing that used in Table 8:

| \( D \) | \( \frac{5}{6} (p_i + p_u) \) | \( \vartheta_{\lambda}^{(p_i+p_u)} \) | \( k \) | \( C_{1/k} \) |
|---|---|---|---|---|
| 1 | 1.25 | \( \vartheta_{4}^{(1,3)} = 2 \) | 4* | \( l_{3}^{(4)} = 2 \) |
| 2 | 2.5 | \( \vartheta_{4}^{(1,7)} = 6 \) | 4* | \( l_{4}^{(4)} = 6 \) |
| 3 | 3.125 | \( \vartheta_{3}^{(5,7)} = 12 \) | 4* | \( l_{i_{42}}^{(4)} = 12 \) |
| 4 | 6.87 | \( \vartheta_{3}^{(7,15)} = 24 \) | 7 | \( l_{i_{78}}^{(7)} = 24 \) |
| 5 | 6.87 | \( \vartheta_{5}^{(7,15)} = 40 \) | 8 | \( l_{i_{148}}^{(9)} = 40 \) |
| 6 | 11.87 | \( \vartheta_{3}^{(7,31)} = 72 \) | 11 | \( l_{i_{151}}^{(11)} = 72 \) |
| 7 | 40 | \( \vartheta_{4}^{(1,127)} = 126 \) | 40 | \( l_{n/a}^{(40)} \) (but \( l_{4}^{(42)} = 126 \)) |
| 8 | 84.37 | \( \vartheta_{4}^{(15,255)} = 240 \) | 84 | \( l_{n/a}^{(84)} \) (but \( l_{401}^{(91)} = 240 \)) |

* the case \( k < 4 \) is outside the domain of successor relation (91)
† not available due to limitation to \(< 500 \) contfrac steps

The meaningfulness of detuning \( k \) to \( \frac{5}{6} (p_i + p_u) \) is apparently limited: a) for the first three dimensions, the Catalan numbers \( C_{1/k} \) fall out of the range [1, \( C_{1/4} \)] = [1, 1.57...], obeyed for finite \( k \geq 4 \): \( C_{1} = -0.5 \), \( C_{1/2} = 0 \), \( C_{1/3} = 0.11... \); and b) for dimensions seven and eight, the CFRs of \( C_{1/40} \) and \( C_{1/84} \) respectively fail to include 126 or 240 among their (first 500) denominators. The basic idea of incorporating \( p_i \) and \( p_u \) in the irrationals’ gradation yet seems sound and just calling for a different implementation.

9.4.2 A qphyletic approach

What looks more promising is finding ways to exploit the identity \(- \lfloor n/2 \rfloor + \sum_{i=1}^{n-2} p_i = \lfloor \log_2 C_q \rfloor \) \((n > 3)\). Setting \( q_u = (p_u - 3)/4 \), we may construct irrationals from \( \log_2 C_{q_u} \), \( 2^i/\pi \) and \( e^{(i)} \) for \( i = 1, 2, \ldots, \lfloor \log_2 C_{q_u} \rfloor \), and mould them into graded sequences, obvious candidates being

\[
\left( \left( \frac{2^i}{\pi} \right)^{-1} \lfloor \log_2 C_{q_u} \rfloor \right),
\left( \left( \frac{2^i}{\pi} \right)^{-1} \log_2 C_{q_u} \right),
\left( \left( e^{(i)} \right)^{-1} \log_2 C_{q_u} \right),
q_u = 3, 7, 15, 31, 63, 127;
\]
\[
i = 1, 2, \ldots, \lfloor \log_2 C_{q_u} \rfloor.
\]

We further introduce a regularized range \([0, 1]\) for candidates to be admissible, constraining the gradings to

\[
\left( \left( \frac{2^{n_i}}{\pi} \right)^{-1} \lfloor \log_2 C_{q_u} \rfloor \right) \text{ with regular CFR } \chi_{\lambda}^{(p_i,q_u)}
\left( \left( \frac{2^{n_i}}{\pi} \right)^{-1} \log_2 C_{q_u} \right) \text{ with regular CFR } \chi_{\lambda}^{(n_i,q_u)}
\left( \left( e^{(n_i)} \right)^{-1} \log_2 C_{q_u} \right) \text{ with regular CFR } \chi_{\lambda}^{(n_i,q_u)}.
\]

\[
q_u = 7, 15, 31, 63, 127;
n_i = \log_2 (p_u + 1), \log_2 (p_u + 1) + 1, \ldots, \lfloor \log_2 C_{q_u} \rfloor.
\]

The results of this program are summarized in the table below and compared to those of Table 8 (column labeled \( \Lambda \) approach):

46
The first sixteen kissing numbers $L_D$ represented by contfrac expansion coefficients $\gamma_{\text{nleast}}^{(n_1, q_u)}$, $z_{\text{nleast}}^{(n_1, q_u)}$ and $(\gamma_A)^{n_1, q_u}$

| $D$ | $L_D$ (A approach) | $L_D$ as $\gamma_{\text{nleast}}^{(n_1, q_u)}$ | $L_D$ as $z_{\text{nleast}}^{(n_1, q_u)}$ | $L_D$ as $(\gamma_A)^{n_1, q_u}$ |
|-----|------------------|------------------|------------------|------------------|
| 1   | $\vartheta_1^{(1.3)} = 2$ | $\gamma_1^{(5.7)}$ | $z_1^{(5.7)}$ | $(\gamma_A)_A^{(5.7)}$ |
| 2   | $\vartheta_1^{(1.7)} = 6$ | $\gamma_2^{(5.7)}$ | $z_2^{(5.7)}$ | $(\gamma_A)_2^{(5.7)}$ |
| 3   | $\vartheta_3^{(3.7)} = 12$ | $\gamma_3^{(5.7)}$ | $z_3^{(5.7)}$ | $(\gamma_A)_A^{(5.7)}$ |
| 4   | $\vartheta_3^{(7.15)} = 24$ | $\gamma_4^{(5.7)}$ | $z_4^{(5.7)}$ | $(\gamma_A)_3^{(5.7)}$ |
| 5   | $\vartheta_3^{(7.31)} = 40$ | $\gamma_5^{(5.7)}$ | $z_5^{(5.7)}$ | $(\gamma_A)_2^{(5.7)}$ |
| 6   | $\vartheta_3^{(1.127)} = 72$ | $\gamma_6^{(5.7)}$ | $z_6^{(9.7)}$ | $(\gamma_A)_4^{(11.7)}$ |
| 7   | $\vartheta_1^{(15.255)} = 240$ | $\gamma_7^{(15.15)}$ | $z_7^{(10.31)}$ | n/a |
| 8   | 2nd o.p.e. = 272 | n/a | n/a | $(\gamma_A)^{(8)}$ |
| 9   | $\vartheta_7^{(15.63)} = 336$ | n/a | n/a | $(\gamma_A)^{(15.63)}$ |
| 10  | 2nd o.p.e. = 438 | n/a | n/a | $(\gamma_A)^{(17.31)}$ |
| 11  | 2nd o.p.e. = 648 | $\gamma_7^{(8.7)}$ | $z_7^{(12.127)}$ | n/a |
| 12  | 2nd o.p.e. = 906 | $\gamma_7^{(16.15)}$ | n/a | n/a |
| 13  | 2nd o.p.e. = 1422 | n/a | n/a | n/a |
| 14  | 2nd o.p.e. = 2340 | n/a | n/a | n/a |
| 15  | 2nd o.p.e. = 4320 | n/a | n/a | n/a |
| 16  | $\vartheta_9^{(31.511)} = 4320$ | n/a | n/a | n/a |

* these denominators are related to $t_408$, $t_397$ (see comparison of Diagrams (97) and (106) in main text)

† these denominators form the $L_D$ divisor hierarchy up to $L_{12}$ (see discussion of Diagram (108) in main text)

‡ these are nonlattice kissing numbers, their lattice counterparts being $L_{12} = 756$ and $L_{13} = 918$ respectively

For dimensions one to eight, in the chosen range, all three types of irrational numbers have denominators that cover (nearly) all of the corresponding kissing numbers. While for dimensions nine and thirteen, the irrationals $(a_1^{(8)})^{-1} \log_2 C_7$ and $(2^{16}/\pi)\nuch (2^{16}/\pi)\nuch$ supplement the picture with denominators respectively representing $L_9 = 272$ and $L_{13} = 906$, the two of them last seen in the synoptic Table 9. The representations of $L_{11} = 438$ and $L_{12} = 648$ are entirely new additions. The phenomenon of apparent non-representability of $L_D$ for dimensions fourteen to sixteen in the chosen range could prove to be real, but the artificial limitation to 500 contfrac steps, accounted for by a “n/a” in the table, casts doubts on that conclusion – at least $L_{16} = 4320$ should show up somewhere, judging from parafermial and synoptic representability considerations.

The strength of some of the figures in Table 19 can be acknowledged on the basis of the kinship to base-5·2⁻₄ positional number systems they reveal in much the same way as intensional Catalan numbers do. We can recognize a similarity between structure diagram (97) and

$$(L_{11} = 438) \xrightarrow{D = 31} \gamma^{(8,7)}_4 = 408 \xrightarrow{D = 9} z^{(10,31)}_{398} = 240,$$

which has its origin in the branching $l_1^{(19)} = \gamma^{(8,7)}_1 + z^{(10,31)}_1$.

While the average edge length in Dgs. (97) and (101) equals 11 and 10, with a margin reflecting the swings 408 ↔ 407, 397 ↔ 398, in Dg. (106) it equals $(31 + 9)/2 = 20$, and in the following hierarchy diagram of $L_D$ divisors up to $L_{12} = 648$ it...
amounts to \((43 + 40 + 40 + 37)/4 = 40\):

\[
\begin{align*}
\mathcal{Z}_{236}^{(32,127)} &= 648 \quad \mathcal{P} = 43 \\
\mathcal{Z}_{279}^{(11,7)} &= 72 \\
\mathcal{P} = 40 \quad \uparrow \quad \mathcal{P} = 40 \quad \downarrow \\
\mathcal{Z}_{239}^{(5,7)} &= 24.
\end{align*}
\]

What is novel about these diagrams is that they seem to impart a terminologic shift on the labels intra- and interordinality. Borrowing from systematic biology, we may quasi identify structures arising in \(l^{(k)}\) (or in \(\ell^{(k)}\)) with “phyla”, and Dgs. (97) and (101) phyletically intraordinal, accordingly. By contrast, Dgs. (106) and (108), by displaying parallelism between \(\mathcal{Z}_{\lambda}^{(n_1,q_1)}\) and \(\mathcal{Z}_{\mu}^{(n_1,q_1)}\), are phyletically interordinal. As the latter radiate along distinct \(n_1\) and/or \(q_1\), we could, in contradistinction, call them qphyletic, qphyla becoming the name for superordinate structures fed from both \(\mathcal{Z}_{\lambda}^{(n_1,q_1)}\) and \(\mathcal{Z}_{\mu}^{(n_1,q_1)}\).

We have found that Eq. (107) is only one of three possible solutions to the “ancestral branching” aspect

\[
l_1^{(p+q)/2} = \mathcal{Z}_1^{(n_1,q)} + \mathcal{Z}_1^{(n_2,p)},
\]

the other two being

\[
l_1^{(9)} = \mathcal{Z}_1^{(5,3)} + \mathcal{Z}_1^{(7,15)}
\]

and

\[
l_1^{(39)} = \mathcal{Z}_1^{(10,15)} + \mathcal{Z}_1^{(13,63)}.
\]

The latter is strong – and independent – evidence that in the neighborhood of \((\varphi^{-1})_{\text{least}}^{(39)}(9694845)\) analogs of Dgs. (97), (101) and (106) could be hiding, which requires for its verification knowing those irrationals to a precision that would allow for a computation scope of roughly \(10^7\) contfrac denominators.

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