CHIRAL SYMMETRY ON $S^2_F$

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Abstract

In this talk we give a brief description of the formulation of chiral and gauge symmetries on the fuzzy sphere. In particular fermion doublers are shown to be absent and the correct anomaly equation in two dimensions is obtained in the corresponding continuum limit.

A fuzzy space is by construction a discrete lattice-like structure which serves to regularize [2], it allows for an exact chiral invariance to be formulated, but still the fermion-doubling problem is completely avoided [4]. Global chiral anomaly was also, along with other topological non-trivial field configurations, formulated in [5, 6].

In this talk we sketch how we can go beyond global considerations and define a "fuzzy" chiral anomaly associated with a "fuzzy" $U(1)$ global chiral symmetry. This "fuzzy" chiral anomaly has the correct continuum limit in the sense that it approaches, in the limit of large IRR’s $l$ of $SU(2)$, the "local" axial anomaly on continuum $S^2$ with a corresponding canonical theta term.

1 Continuum Action

The free fermionic action on continuum $S^2$ is given by [7, 8]

$$S = \int_{S^2} \frac{d\Omega}{4\pi} \sum_{\alpha \beta} \bar{\chi}^\alpha D^{\alpha \beta} \chi^\beta,$$

where $\chi^\alpha$, $\alpha = 1, 2$, are the two components of the continuum spinor which are both smooth functions on $S^2$. $D$ is the Dirac operator on $S^2$ given by [2, 4, 5, 6]

$$D = \mathcal{P}_{ij} \sigma_i (\mathcal{L}_j + \frac{1}{2} \sigma_j) + \bar{\sigma} \tilde{\mathcal{L}} + 1, \quad \mathcal{L}_k = -i\epsilon_{kij} n_i \partial_j.$$
This Dirac operator admits the chirality operator \( \gamma = \sigma \cdot \vec{n} \), \( \gamma^2 = 1 \), \( \gamma^+ = \gamma \) and such that \( \{ \gamma, D \} = 0 \).

The projector \( \mathcal{P} \) which defines the tangent bundle \( TS^2 \) of \( S^2 \) is given by \( \mathcal{P} = (\vec{n} \cdot \vec{\theta})^2 \), where \( (\theta_i)_{jk} = -i\epsilon_{ijk} \), and satisfies \( \mathcal{P}^2 = \mathcal{P} \) and \( \mathcal{P}^+ = \mathcal{P} \). So given any vector \( \vec{A} \) in \( \mathbb{R}^3 \), \( \mathcal{P} \vec{A} \) defines a vector tangent to \( S^2 \). Indeed \( \mathcal{P} \vec{A} \) has only two independent components tangent to \( S^2 \) as one can check if we rewrite it in terms of its components, namely \( \mathcal{P}_{ij} = \delta_{ij} - n_i n_j \), and then compute that \( \vec{n} \cdot \mathcal{P} \vec{A} = n_i \mathcal{P}_{ij} A_j = 0 \).

Gauging the Dirac operator (2) means invoking the minimal replacement \( L_i \rightarrow \nabla_i = L_i + A_i \), where \( \vec{A} \) is a gauge field which is generally in \( \mathbb{R}^3 \) satisfying \( A_i^+ = A_i \). The gauged Dirac operator is therefore

\[
D_G = \mathcal{P}_{ij} \sigma_i (\nabla_j + \frac{1}{2} \sigma_j) \equiv \sigma \bar{\nabla} + 1 + \sigma_i \hat{A}_i. \tag{3}
\]

The gauge connection on \( S^2 \) is defined by \( \vec{\hat{A}} = \mathcal{P} \vec{A} \) with components \( \hat{A}_i = \mathcal{P}_{ij} A_j \), and correspondingly, the gauged fermionic action is given by

\[
S_G = \int_{S^2} \frac{d\Omega}{4\pi} \sum_{\alpha\beta} \bar{\chi}_\alpha D^\alpha_{G} \chi_\beta; \tag{4}
\]

The corresponding pure \( U(1) \) Yang-Mills action is of the form

\[
S_{YM} = -\frac{1}{4e^2} \int_{S^2} \frac{d\Omega}{4\pi} F_{ij} F_{ij}, \tag{5}
\]

where the \( U(1) \) curvature \( F_{ij} \) is defined by

\[
F_{ij} = [\nabla_i, \nabla_j] - i\epsilon_{ijk} \nabla_k \equiv [\mathcal{L}_i, A_j] - [\mathcal{L}_j, A_i] - i\epsilon_{ijk} A_k. \tag{6}
\]

\( U(1) \) gauge transformations, \( U = U(\vec{n}) = e^{i\Lambda(\vec{n})} \), act on the spinor \( \chi \), on the covariant derivative \( \nabla_i \) and on the curvature \( F_{ij} \) in the usual fashion

\[
\chi \rightarrow \chi' = U \chi \quad \nabla_i \rightarrow \nabla'_i = U \nabla_i U^+ \quad F_{ij} \rightarrow F'_{ij} = F_{ij}. \tag{7}
\]

2 Regularization via Fuzzification

Instead of replacing Euclidean space-time \( S^2 \) by a lattice, we will implement in the following the regularization prescription given by the substitution \( S^2 \rightarrow S^2_F \), where \( S^2_F \) is the fuzzy sphere [see \( \cite{2} \) and references therein]. We first make the following replacement

\[
n_i \rightarrow n_i^F = \frac{L_i}{\sqrt{l(l+1)}}, \tag{8}
\]
where $L_i$'s are the generators of the IRR $l$ of $SU(2)$, i.e. they satisfy $\sum_{i=1}^{3} L_i^2 = l(l + 1)$ and $[L_i, L_j] = i\epsilon_{ijk} L_k$. Loosely speaking, $S^2_F$ is the algebra $A$ of all $(2l + 1) \times (2l + 1)$ matrices which is generated by the $L_i$'s, i.e $A = Mat_{2l+1}$.

At the level of the action we also make the following replacements

$$
\int d\Omega \rightarrow \frac{1}{2(2l+1)} Tr_{H^{(2)}_l} D = \tilde{\sigma} \vec{L} + 1 \rightarrow D_F = \tilde{\sigma}[\vec{L}, \cdot] + 1
$$

$\chi^\alpha \rightarrow \tilde{\psi}^\alpha_F$

$A_i \rightarrow A_i^F$. (9)

$H^{(2)}_l$ is the Hilbert space generated by the $SU(2)$ coherent states $|\vec{n}, l >$. In particular we have the identity

$$
\frac{1}{2l+1} Tr_{H^{(2)}_l} X = \int d\Omega \frac{1}{4\pi} <\vec{n}, l | X |\vec{n}, l >. \tag{10}
$$

$\psi^\alpha_F$, $\alpha = 1, 2$, as well as $A_i^F$, $i = 1, 2, 3$, are now $(2l + 1) \times (2l + 1)$ matrices. The fuzzy action is therefore

$$
S_{GF} = \frac{1}{2(2l+1)} Tr_{H^{(2)}_l} \sum_{\alpha\beta} \tilde{\psi}^\alpha_F D_{GF}^{\alpha\beta} \psi^\beta_F
$$

$$
D_{GF} = D_F + \sigma_i A_i^F. \tag{11}
$$

$D_F$ above is the Grosse-Klimčík-Prešnajder Dirac operator on fuzzy $S^2$, it can be rewritten in the form $D_F = \tilde{\sigma}. \vec{L} - \tilde{\sigma}. \vec{R} + 1$, where $L_i^L$'s and $-L_i^R$'s are the generators of the IRR $l$ of $SU(2)$ which act respectively on the left and on the right of the algebra $A$. (2, 4, 5, 6). This Dirac operator has the correct continuum limit in the sense that $D_F \rightarrow D$.

The chirality operator on $S^2_F$ was, on the other hand, first found in [11], it is given by

$$
\Gamma^{R} = \frac{1}{l + \frac{3}{2}} [-\tilde{\sigma}. \vec{R} + \frac{1}{2}], \tag{12}
$$

which also satisfies

$$
\Gamma^{R} D_F + D_F \Gamma^{R} = \frac{1}{l + \frac{3}{2}} D_F^2. \tag{13}
$$

Despite the fact that this anticommutation relation is not exact, one can show that $(D_F, \Gamma^{R})$ defines a chiral structure on fuzzy $S^2$ which satisfies a) the Ginsparg-Wilson relation, b) is without fermion doubling and c) has the correct continuum limit (4, 4).

The absence of fermion doubling can be easily seen from comparing the spectrum of $D_F$ which is given by [6]

$$
D_F(j) = \pm (j + \frac{1}{2}), j = \frac{3}{2}, \frac{5}{2}, ..., 2l - \frac{1}{2}, \frac{3}{2}
$$

$$
= j + \frac{1}{2} \text{ for } j = 2l + \frac{1}{2}, \tag{14}
$$
with the spectrum of $D$ given by

$$D(j) = \pm(j + \frac{1}{2}), j = \frac{1}{2}, \frac{3}{2}, \ldots, \infty. \quad (15)$$

As one can immediately see, there is no fermion doubling and the spectrum of $D_F$ is simply cut-off at the top eigenvalue $j = 2l + \frac{1}{2}$ if compared to the continuum spectrum $[2, 4]$.

Similarly to the continuum case, the fuzzy gauge potential $A^F_i$ must be projected onto the sphere in order to have a fuzzy gauge theory localized in some appropriate sense on $S^2_F$. The simplest way to this end is to construct the fuzzy analogue $P$ of the continuum projector $\mathcal{P}$ which will define the fuzzy tangent bundle $TS^2_F$. It can be derived without any difficulty and it turns out to be given by

$$P_{ij} = \delta_{ij} - n^F_i n^F_j. \quad (16)$$

The vector $P_{ij} A^F_j$ is indeed normal to the fuzzy sphere in the sense that we have $n^F_i \hat{A}^F_i = 0$. Beside the fact that the components $\hat{A}^F_i$'s are not self-adjoint, the transversality condition $n^F_i \hat{A}^F_i = 0$ is not stable under fuzzy gauge transformations, and this, in turn, introduces a lot of complications into the problem which are addressed in great detail in [4]. In particular, we showed there that the normal component of the fuzzy gauge potential can be projected out by making it infinitely heavy through a gauge-invariant mass term in the action, and hence it effectively decouples.

Finally, and as was shown in [4], a fuzzy gauge principle can also be written down, under which the fuzzy gauged fermionic action (11) will indeed be invariant. This fuzzy $U(1)$ gauge theory will have as a continuum limit the ordinary $U(1)$ gauge theory defined earlier in section 1.

3 Chiral Symmetry and Axial Anomaly

On continuum $S^2$ exact chiral invariance of the classical action is expressed by the anti-commutation relation

$$\gamma D + D \gamma = 0. \quad (17)$$

It is a well known fact that this symmetry will be broken by quantum effects, and different methods of regularization are shown to give rise to the same topological action. We showed in [4] that the fuzzy sphere is a novel regularization scheme which leads also to the canonical theta term. This result will now be sketched.

In spite of the Ginsparg-Wilson relation (13), exact chiral invariance on fuzzy $S^2$ in the sense of (17) can also be constructed as was shown first in [4]. Following [4] we adopt here a different route in defining chiral invariance. First we start by rewriting the Ginsparg-Wilson relation (13) in the suggestive form [12]

$$-\Gamma^L D_F + D_F \Gamma^R = 0, \quad (18)$$
where $\Gamma^R$ is given by equation (12), and $\Gamma^L$ is obtained from (12) through the substitution $-L_i^R \rightarrow L_i^L$. As in the continuum, Fuzzy chiral transformations will be defined by

$$\psi_F \rightarrow \psi_F' = \psi_F + \delta \psi_F, \quad \delta \psi_F = \Gamma^R \psi_F \lambda^L$$

$$\bar{\psi}_F \rightarrow \bar{\psi}_F' = \bar{\psi}_F + \delta \bar{\psi}_F, \quad \delta \bar{\psi}_F = -\lambda^L \bar{\psi}_F \Gamma^L.$$  \hfill (19)

These transformations, as it turns out, do not leave the action (11) invariant but instead leave invariant the action

$$S_{CF} = \frac{1}{2(2l + 1)} \text{Tr}_{H^{(2)}} \sum_{a\beta} \left[ \bar{\psi}^\alpha_F D^\alpha_{CF} \psi^\beta_F \right]$$

$$D_{CF} = D_F + \epsilon_{ijk} Z^F_i n^F_k A^F_i.$$ \hfill (20)

Indeed, the change of this action under these transformations is given by

$$\Delta S_{CF} = -\frac{1}{2(2l + 1)} \text{Tr}_{H^{(2)}} \lambda^L \left[ L_i, \bar{\psi}_F \sigma_i \Gamma^R \psi_F \right].$$ \hfill (21)

The connection between (11) and (20) is discussed thoroughly in [1]. In particular $Z^F_k$ is given by

$$Z^F_k = \frac{i}{2} \left[ \Gamma^L \sigma_k + \sigma_k \Gamma^R \right]$$

and hence both actions tend in the large $l$ limit to the same continuum action (4). $\lambda^L$ in all the above equations is a test $(2l + 1) \times (2l + 1)$ matrix which is infinitesimal in some appropriate sense [1].

The continuum limit of (21) was computed in [4] and is given by

$$\Delta S_{CF} \rightarrow \Delta S_C = \frac{2 \ln l}{(4\pi)^5} \int_{S^2} \frac{d\Omega}{4\pi} \lambda(\vec{n}) L_i \left[ \bar{\chi} \sigma_i \gamma \chi \right](\vec{n}).$$ \hfill (22)

### 3.1 The Theta Term

Since we are dealing with a matrix model, manipulations on the quantum measure will all have a well defined meaning. Following [9], we first expand the fuzzy spinors $\psi_F$ and $\bar{\psi}_F$ in terms of the eigen-tensors $\psi_F(a, A)$ of the gauged Dirac operator $D_{GF}$ as follows

$$\psi_F = \sum_a \theta_a \psi_F(a, A), \quad \bar{\psi}_F = \sum_a \bar{\theta}_a \psi_F^+(a, A),$$ \hfill (23)

where $\theta_a$’s and $\bar{\theta}_a$’s are two independent sets of Grassmanian variables, and $\psi_F(a, A)$’s are defined by

$$D_{GF} \psi_F(a, A) = e_a(A) \psi_F(a, A),$$ \hfill (24)

and are normalized such that

$$\frac{1}{2(2l + 1)} \text{Tr}_{H^{(2)}} \psi_F^+(a, A) \psi_F(b, A) = \delta_{ab}.$$ \hfill (25)

$a$ stands for all the quantum numbers needed to characterize the eigenvalues of the Dirac operator $D_{GF}$. For weak fuzzy gauge fields $A^F_i$’s, $a$ stands for $j$, $k$ and $m$ which are the eigenvalues of $\vec{J}^2 = \left( \vec{K} + \frac{\sigma}{2} \right)^2$, $\vec{K}^2 = (\vec{L}^L - \vec{L}^R)^2$ and $J_3$ respectively.
As it turns out \cite{1}, and similarly to \cite{9}, what really matters in the calculation is the asymptotic behaviour of $\psi_F(a, A)'s$ and $e_a(A)'s$ given by \cite{2}

$$
e_a(A)_{A^F \to 0} \longrightarrow e_a = j(j + 1) - k(k + 1) + \frac{1}{4}
$$

$$
\psi_F(a, A)_{A^F \to 0} \longrightarrow \psi_F(a) = \sqrt{2(2l + 1)} \sum_{k_3, \sigma} C^m_{k_3, \sigma} T_{k_3}(l) \chi_{\frac{1}{2} \sigma}.
$$

The sum over $\sigma$ in (23) is therefore finite and given by

$$
\sum_{\sigma} = \sum_{k=0}^{2l} \sum_{j=k-\frac{1}{2}}^{k+\frac{1}{2}} \sum_{m=-j}^{j},
$$

and hence the quantum measure is also well defined

$$
D\psi_F D\bar{\psi}_F = \prod_a d\theta_a d\bar{\theta}_a \longrightarrow \prod_{k=0}^{2l} \prod_{j=k-\frac{1}{2}}^{k+\frac{1}{2}} \prod_{m=-j}^{j} d\theta_{kjm} d\bar{\theta}_{kjm},
$$

A canonical calculation shows that the above quantum measure changes under the fuzzy chiral transformations \cite{19} as follows

$$
\int D\psi_F D\bar{\psi}_F e^{-S^{\prime}_{CF}} = \int D\psi_F D\bar{\psi}_F e^{S_{CF}} e^{-S^{\prime}_{CF} - \Delta S_{CF}}.
$$

The theta term on $S^2_F$ is therefore given by

$$
S_{\theta F} = -\frac{1}{2(2l + 1)} \sum_a Tr_{H_i^{(2)}} \lambda^L \psi_F^+(a, A)(\Gamma^R - \Gamma^L) \psi_F(a, A).
$$

We will now extract the large $l$ behaviour of this last formula, i.e. we will carefully analyze the spectrum of the theory and derive the continuum limit of the axial anomaly. We first start with the free theory and rewrite the Ginsparg-Wisor relation \cite{13} in the form

$$
(\Gamma^R - \Gamma^L)D_F + D_F(\Gamma^R - \Gamma^L) = 0,
$$

which means that in the absence of gauge field we must have

$$
tr[\Gamma^R - \Gamma^L] = 0.
$$

The trace is meant to be in the space of spinors. This is also true in the naive continuum free theory, i.e. $\gamma D + D\gamma = 0$, $tr\gamma = 0$. As we will show and if we have to be precise this result is only valid for the infrared sector of the theory. However if we include the gauge field we can compute instead that

$$
(\Gamma^R - \Gamma^L)D_{GF} + D_{GF}(\Gamma^R - \Gamma^L) = \{\Gamma^R - \Gamma^L, \sigma_i A_i^F \}.
$$
The naive limit of this equation is \( \gamma D_G + D_G^\gamma = 2\phi \) where \( \phi \) is the normal component of the gauge field, namely \( \phi = \vec{n}.\vec{A} \), and hence it will be eventually set equal to zero by means of the projector \( \mathcal{P} \) for example [see \( \mathcal{P} \) for a different method of projecting the gauge potential onto the sphere]. In other words, in the continuum interacting theory one might be tempted to conclude that \( Tr\gamma = 0 \) which we know is wrong in the presence of gauge fields. Noncommutative geometry, as it will be obvious from equation (33), gives us immediately the structure of the chiral anomaly, indeed (33) can be rewritten in the form

\[
\left[ \Gamma^R - \Gamma^L - \frac{2}{2l+1} D_G \right]^2 - 4 = \frac{2i}{(2l+1)^2} \epsilon_{ijk} \sigma_k (F_{ij} + F_{ij}^F) + \frac{8\sqrt{l(l+1)}}{2l+1} \left[ \phi^F + \frac{\vec{A}^F}{2\sqrt{l(l+1)}} \right],
\]

(34)

where we have used extensively the identities

\[
D_F^2 = (2l+1)^2 - \frac{1}{4}(2l+1)^2(\Gamma^R - \Gamma^L)^2
\]

\[
D_G^2 = D_G + (\mathcal{L}_i + A_i^F)^2 + \frac{i}{2} \epsilon_{ijk} A_{ij}^F.
\]

\( F_{ij} \) in (34) denotes on the other hand the \textit{would-be} continuum curvature

\[
F_{ij}^F = F_{ij} + [A_i^F, A_j^F], \quad F_{ij} = [L_i, A_j^F] - [L_j, A_i^F] - i\epsilon_{ijk} A_{kj}^F
\]

whereas \( \phi^F \) denotes the fuzzy normal component of the gauge field, namely \( \phi^F = n_i^F A_i^F + A_i^F n_i^F \).

Now if we multiply both sides of equation (34) by \( \psi_F(a, A) \) from the right and \( \psi_F(a, A)^\dagger \) from the left, we obtain the exact and elegant answer

\[
- tr(\Gamma^R - \Gamma^L) + \frac{2l+1}{4} tr\left[\left(\Gamma^R - \Gamma^L\right)^2 - 4\right] \frac{1}{D_G} + \frac{1}{2l+1} trD_G = \frac{i}{2(2l+1)} \epsilon_{ijk} tr\left[\sigma_k (F_{ij} + F_{ij}^F) \frac{1}{D_G}\right] + \frac{2\sqrt{l(l+1)}}{2l+1} tr\left[\phi^F + \frac{\vec{A}^F}{2\sqrt{l(l+1)}} \right] \frac{1}{D_G}.
\]

(35)

The symbol \( tr \) here denotes the \textit{fuzzy} trace in the space of fuzzy spinors, i.e \( tr(X) = \sum_a \psi^*_F(a, A) X \psi_F(a, A) \). In particular the cyclic property of the trace has to be used with care because of the non-commutativity of the different ingredients, and therefore the order in (35) is important. This is more clear from the fact that the result of this operation is still an element in the algebra \( \mathbf{A} \). The limit is of course an ordinary spin trace.

We will now show that most contributions to the chiral anomaly (34) are coming from high frequency modes of the spectrum. First we have in the large \( l \) limits, \( \vec{n}^R = \vec{L}^R / \sqrt{l(l+1)} \rightarrow \vec{n} \), \( \vec{n}^L = \vec{L}^L / \sqrt{l(l+1)} \rightarrow \vec{n} \) and hence the different chiralities must have the continuum limits

\[
\Gamma^R = \frac{1}{l + \frac{1}{2}} (-\vec{\sigma}.\vec{L}^R + \frac{1}{2}) \rightarrow -\gamma, \quad \Gamma^L = \frac{1}{l + \frac{1}{2}} (\vec{\sigma}.\vec{L}^L + \frac{1}{2}) \rightarrow \gamma.
\]

(36)
But from the free fuzzy eigenvalues equation

$$\frac{D_F}{2l+1} \psi_F(a, 0) = \frac{\Gamma^R + \Gamma^L}{2} \psi_F(a, 0),$$  \hspace{1cm} (37)

we can easily see that for all the infrared modes \(a << 2l\), the above limits (36) are indeed satisfied since

$$\lim_{l \to \infty} \left[ \frac{D_F}{2l+1} \psi(a, 0) \right] = 0.$$  \hspace{1cm} (38)

But for ultraviolet modes \(a \sim 2l\) we have instead

$$\lim_{l \to \infty} \left[ \frac{D_F}{2l+1} \psi(a, 0) \right] = \pm 1.$$  \hspace{1cm} (39)

The sign is +1 if \(e_a\) is a positive energy eigenvalue and −1 if \(e_a\) is a negative energy eigenvalue. This means in particular that the limits (38) are not valid in the UV domain and therefore the statement \(\text{tr} [\Gamma^R - \Gamma^L] = 0\) itself which is a consequence of (31) is in fact a statement about the IR modes only of the theory.

Equation (39) also means that in the limit the operator \(\frac{D_F}{2l+1}\) restricted to the above UV modes is acting as the sign operator \(\widetilde{F}_F = \frac{D_F}{|D_F|}\) and is not identically zero as we would have expected. So if we are only restricted to the high frequency part of the spectrum we can, for all practical purposes, set in the large \(l\) limit

$$\frac{D_F}{2l+1}|_{a \sim 2l} \longleftrightarrow \widetilde{F}_F = -i\Gamma'_F.$$  \hspace{1cm} (40)

In the continuum limit the above equation reduces to the identity \(F = -i\gamma \widetilde{F}_w\) where \(F\) is the sign of the Dirac operator \(D\), while \(\widetilde{F}_w\) is the sign of the Watamura Dirac operator \(D_w\), i.e \(\widetilde{F}_w = \frac{D_w}{|D_w|} [11]\). \(\Gamma'\) on the other hand is another chirality operator which is different from both \(\Gamma^R\) and \(\Gamma^L\), but has the same continuum limit and that is all we will need. It was however constructed explicitly in [2]. The fuzzy version of the Watamuras Dirac operator \(D_w\) annihilates the top modes \(j = 2l + \frac{1}{2}\) and hence a proper regularization of \(\widetilde{F}_w = \frac{D_w}{|D_w|}\) is required. We made therefore the natural replacement

$$\widetilde{F}_w \longrightarrow F_{\Lambda w} = \frac{D_w}{|D_w|} \text{ for all } j < 2l + \frac{1}{2}$$

$$= +1 \text{ for } j = 2l + \frac{1}{2}.$$  \hspace{1cm} (41)

As \(\widetilde{F}_w\) is a sign operator, the only other completely equivalent choice will be \(F_{\Lambda w} = -1\) on the modes \(j = 2l + \frac{1}{2}\).

This funny behaviour in the UV is the source of the anomaly which is essentially a quantum mechanical effect as of the presence of the full propagator \(\frac{1}{|D_{GF}|}\) in (35). Now we expect that the behaviour of the IR modes will not change as compared to the free
case and therefore all the anomaly will be captured by the UV modes. Indeed for any field configuration we will have in the continuum limit the identity

$$-\frac{1}{2(2l + 1)} Tr \lambda^L tr_{a<<2l} (\Gamma^R - \Gamma^L) = -\frac{1}{2} \int_{S^2} \frac{d\Omega}{4\pi} \lambda(\Omega) tr_{a<<2l} (-2\gamma) = 0.$$  

(42)

We know that the trace is 0 because we know that all the infrared gauge-invariant modes are paired.

The leading contribution to the chiral anomaly will be given in the large $l$ limit by

$$i \frac{\epsilon_{ijk}}{4(2l + 1)^2} \epsilon_{ijk} Tr \lambda^L tr \left[ \sigma_k(F_{ij} + F_{ij}^E) \frac{1}{D_{GF}} \right] \simeq i \frac{\epsilon_{ijk}}{2(2l + 1)} \epsilon_{ijk} Tr \lambda^L tr_{a \sim 2l} \left[ \sigma_k F_{ij} \frac{1}{D_{F}^2} F_{ij} \right],$$  

(43)

where the approximation $\simeq$ clearly becomes exact in the strict large $l$ limit, i.e all ignored terms in (43) are subleading and vanishes identically in this strict limit. We have also used the fact that $F_{ij}^E$ for large $l$ is essentially $F_{ij}$, and that the gauge field can always be treated as a weak perturbation in the continuum limit in the sense that $<\vec{n}, l|A_i^F|\vec{n}, l> \ll l$.

By using the crucial result (40) we can rewrite the above axial anomaly as

$$i \frac{\epsilon_{ijk}}{4(2l + 1)^2} \epsilon_{ijk} Tr \lambda^L tr \left[ \sigma_k(F_{ij} + F_{ij}^E) \frac{1}{D_{GF}} \right] \simeq i \frac{\epsilon_{ijk}}{2(2l + 1)} \epsilon_{ijk} Tr \lambda^L tr_{j \sim 2l} \left[ \sigma_k F_{ij} \frac{1}{D_{F}^2} \Gamma' \right]$$

$$+ i \frac{\epsilon_{ijk}}{2(2l + 1)} \epsilon_{ijk} Tr \lambda^L tr_{a \sim 2l} \left[ \sigma_k F_{ij} \frac{1}{D_{F}^2} \Gamma' F_{\Lambda w} \right],$$  

(44)

where we have separated the top modes $j = 2l + \frac{1}{2}$ from the rest of the UV’s as they are not paired to anything else which is clear from equation (41). It is this same property which allows us to conclude that the second term above vanishes identically and we are left with

$$i \frac{\epsilon_{ijk}}{4(2l + 1)^2} \epsilon_{ijk} Tr \lambda^L tr \left[ \sigma_k(F_{ij} + F_{ij}^E) \frac{1}{D_{GF}} \right] \simeq i \frac{\epsilon_{ijk}}{2(2l + 1)} \epsilon_{ijk} Tr \lambda^L tr_{j \sim 2l} \left[ \Gamma' \sigma_k F_{ij} \frac{1}{D_{F}^2} \right],$$  

(45)

where we have now used the fact that in the large $l$ limit the trace $tr$ is almost cyclic and therefore all corrections are subleading. Since we are analyzing the continuum limit and since $\Gamma'$ has the canonical continuum limit $\gamma$ we can set $\Gamma' = \vec{\sigma}.\vec{n}^L$ without any loss of generality. The actual expression is of course much more complicated as was shown in [2]. We can then use the identity

$$\epsilon_{ijk} \Gamma' \sigma_k F_{ij} = \epsilon_{ijk} n_k^L F_{ij} - 2i \sigma_j n_i^L F_{ij}.$$  

(46)

The last term above is identically zero in the strict $l \rightarrow \infty$ limit provided the gauge field $A_i$ is projected appropriately onto the sphere, i.e we have $n_i F_{ij} = 0$ and we end up with
the final result
\[
\frac{i}{4(2l+1)^2} \epsilon_{ijk} \text{Tr} \lambda^k \text{tr} \left[ \sigma_k (F_{ij} + \bar{F}_{ij}) \frac{1}{D_{GF}} \right] \approx \frac{1}{2(2l+1)} \epsilon_{ijk} \text{Tr} \lambda^k \text{tr} \left[ \frac{1}{D_{GF}} \right] 
\]
\[
\approx \epsilon_{ijk} \text{Tr} \lambda^k n_k^l F_{ij} \frac{1}{|D_F|^2}.
\] (47)

We have now used that \( \frac{1}{|D_F|^2} \) will behave exactly as a volume form, i.e. it does not contain spin indices and hence \( n_k^l F_{ij} \) acts as an identity in the space of spinors. This identity is exactly \( 2(2l + \frac{1}{2}) + 1 = 2(2l + 1) \) - dimensional in the top mode sector \( j = 2l + \frac{1}{2} \).

The last step is to convert the above result into an integral. To this end we use the generalized coherent states and the answer is
\[
S_{\theta F} \rightarrow S_{\theta} = \epsilon_{ijk} \text{Tr} \lambda^k n_k^l F_{ij} \frac{1}{|D_F|^2} = \frac{2\ln l}{(4\pi)^5} \int_S \frac{d\Omega}{4\pi} \lambda(n) \epsilon_{ijk} n_k^l F_{ij}(\vec{n}).
\] (48)

The last thing to do is to verify that the high-frequency contribution of the other terms in (35) are vanishing in the continuum limit. This is indeed a trivial exercise to do so we skip here the detail.

Putting together this last result (48) with the result (22) we obtain exactly the local chiral anomaly equation
\[
\mathcal{L}_i (\bar{\chi} \gamma^i \gamma^j) = \epsilon_{ijk} n_k^l F_{ij}.
\] (49)

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