RANDOM BALL-POLYTOPES IN SMOOTH CONVEX BODIES

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ABSTRACT. We study approximations of sufficiently smooth convex bodies by random ball-polytopes. We examine the following probability model: let $K \subset \mathbb{R}^d$ be a convex body such that $K$ slides freely in a ball of radius $R > 0$ and has $C^2$ smooth boundary. Let $x_1, \ldots, x_n$ be i.i.d. uniform random points in $K$. For $r \geq R$, let $K_{r(n)}$ denote the intersection of all radius $r$ closed balls that contain $x_1, \ldots, x_n$. We study the asymptotic properties of the expectation of the number of proper facets of $K_{r(n)}$. While sufficiently round convex bodies behave in a similar way with respect to random approximation by ball-polytopes as to classical polytopes, an interesting phenomenon can be observed when, say, a unit ball is approximated by unit radius random ball-polytopes: the expected number of proper facets approaches a finite limit as the number of points tends to infinity.

1. INTRODUCTION AND RESULTS

In the theory of random polytopes, one of the oldest and probably most frequently investigated model is when one selects a sample of $n$ independent and identically distributed random points from a convex body chosen according to the uniform probability distribution. The convex hull of these random points is a (uniform) random polytope contained in the body. A sequence of random polytopes obtained this way tends to the convex body as $n \to \infty$. Since the classical papers of Rényi and Sulanke, a large part of results concerning this probability model has been in the form of asymptotic formulae (as $n$ tends to infinity) about the behaviour of various geometric quantities of the random polytopes, such as, for example, the number of $i$-dimensional faces, intrinsic volumes, etc. Here we do not wish to give a detailed overview of this extensive topic, instead we refer to the surveys by Bárány [3], Hug [13], Reitzner [21], Schneider [24,26], Schneider and Weil [27], Weil and Wieacker [29].

In this paper we investigate a variant of this much studied model where polytopes are replaced by so-called ball-polytopes, which are (non-finite) intersections of congruent closed balls. We describe the the probability model below.

Let $K \subset \mathbb{R}^d$ be a convex body (compact convex set with non-empty interior) in Euclidean $d$-space, and let $r > 0$ be a real number. Let $x_1, \ldots, x_n$ be i.i.d. random points from $K$ chosen according to the uniform probability distribution. Let $K_{r(n)}$ denote the intersection of all radius $r$ closed balls. We call $K_{r(n)}$ a uniform random
ball-polytope. What convex bodies can be approximated by such ball-polytopes? It is known that $K_{(n)}^r \subset K$ for any choice of $x_1, \ldots, x_n \in K$ precisely when $K$ slides freely in a ball of radius $r$. We say that $K$ slides freely in a ball $B$ if for each point $x \in \text{bd} B$, there exist a vector $v \in \mathbb{R}^d$ such that $x \in K + v \subset B$, cf. \cite{24}. So, naturally, in this probability model we examine approximations of only such convex bodies.

An ordinary polytope is the convex hull of a finite number of points, which is the same as the intersection of all closed half-spaces containing the points. In the construction of $K_{(n)}^r$, the role of closed half-spaces is played by radius $r$ closed balls. This explains the the use of the term ball-polytope for $K_{(n)}^r$. Note that unless some of the points are on a unique sphere of radius $r$, the ball-polytope $K_{(n)}^r$ is not the intersection of a finite number of balls. A ball $B$ of radius $r$ is a supporting ball of $K_{(n)}^r$ if $K_{(n)}^r \subset B$ and $K_{(n)}^r \cap \text{bd} B = \emptyset$. We call a set $F \subset \text{bd} K_{(n)}^r$ a proper facet if $F = K_{(n)}^r \cap \text{bd} B$ for a supporting ball $B$ of radius $r$ and $F$ contains at least $d$ affinely independent points. Note that, in general, the union of proper facets may not cover the whole boundary of $K_{(n)}^r$, which would normally be the case for an ordinary polytope. In this paper we only deal with the number $f_{d-1}^r(K_{(n)}^r)$ of proper facets of $K_{(n)}^r$, so we do not embark on a detailed investigation of the general facial structure of $K_{(n)}^r$. We note, however, that if the points $x_1, \ldots, x_n \in K$ are uniform i.i.d. random points, then $K_{(n)}^r$ is simplicial with probability one, in the sense that all of its proper facets are spherical $(d-1)$-simplices.

Intersections of congruent balls and the associated notion of spindle convexity (or hyperconvexity) have played important roles in the study of several problems recently, such as, for example, the Kneser-Poulsen conjecture, bodies of constant width, diametrically complete bodies, randomized isoperimetric inequalities, etc. For a more complete overview and references see, for example, \cite{9,15,14}. Random approximations with intersections of congruent circles were treated, for example, in \cite{11,12}. This new probability model can be considered as a generalization of the classical model with random polytopes, however, some of the phenomena that can be observed for balls is different from the classical model.

It is known that a convex body slides freely in a ball of radius $r > 0$ if and only if $K$ is a Minkowski summand of a ball of radius $r$, cf. Theorem 3.2.2 in \cite{24}. Furthermore, it is the intersection of all balls of radius $r$ which contain $K$, cf. Corollary 3.4 in \cite{9}. In particular, a convex body $K$ with $C^2$ boundary (a class $C^2$ submanifold of $\mathbb{R}^d$ with strictly positive Gaussian curvature everywhere) slides freely in a ball of some radius that only depends on $K$, cf. \cite{24} Theorem 3.2.12, Corollay 3.2.13).

In this paper we are concerned with the expectation $E f_{d-1}^r(K_{(n)}^r)$ of the number of proper facets of random ball-polytopes in sufficiently smooth convex bodies. If the convex body $K \subset \mathbb{R}^d$ with $C^2$ smooth boundary slides freely in a ball of radius $R > 0$, and $r > R$, then $E f_{d-1}^r(K_{(n)}^r)$ is expected to behave in a similar manner (in the sense of the order of magnitude in $n$) as uniform random polytopes do in the ordinary convex case. The major difference occurs when $K$ is a ball of radius $r$. Then $E f_{d-1}^r(K_{(n)}^r)$ tends to a finite limit as $n \to \infty$. The main result of this paper is the following theorem.
Theorem 1.1. For $r > 0$ and $K = rB^d$, it holds that

\begin{equation}
\lim_{n \to \infty} \mathbb{E} f^*_{d-1}(K^r_{(n)}) = \frac{\pi^{d-1} \kappa_d}{\kappa_{d-1}}.
\end{equation}

Here $\kappa_d = V(B^d) = \pi^{d/2} / \Gamma(d/2 + 1)$.

Theorem 1.1 also yields that, in the case when $K = rB^d$, the limit of the expectation of the number of vertices of $K^r_{(n)}$ is also bounded above by a constant,

\begin{equation}
\lim_{n \to \infty} \mathbb{E} f^*_0(K^r_{(n)}) \leq c^*(0,d),
\end{equation}

where $c^*(0,d)$ depends only on the dimension of the space. From the bounds on $\mathbb{E} f^*_0(B^d_{(n)})$ it follows that

\begin{equation}
\mathbb{E} V(B^d \setminus B_{(n)}) \approx \frac{1}{n},
\end{equation}

by the ball-convex version of Efron’s identity [8];

\begin{equation}
\mathbb{E} f^*_0(K^r_{(n)}) = \frac{n \mathbb{E} V(K \setminus K^r_{(n)})}{V(K)},
\end{equation}

whose two-dimensional version was proved in [11, p. 911]. The proof of its $d$-dimensional version is completely analogous.

We note that the fact that the limit in Theorem 1.1 is finite was announced (without proof) in [10], and the special case of (1.1), when $d = 2$, was proved earlier by Fodor, Kevei and Vígh, cf. [11, Theorem 3]. The phenomenon described in Theorem 1.1 has no analogue in the probability model of uniform random polytopes in convex bodies.

Furthermore, a similar phenomenon was described by Bárány, Hug, Reitzner and Schneider in [5] in a different probability model. They proved, cf. [5, Theorem 3.1], that the expected number of facets of random spherical polytopes generated as the spherical convex hull of $n$ i.i.d. uniform random points from a half-sphere tends to a finite limit as $n \to \infty$.

In the case when $K \subset \mathbb{R}^d$ has $C^2_+$ smooth boundary, then the following upper and lower bounds hold for the expected number of proper facets.

Theorem 1.2. Let $K \subset \mathbb{R}^d$ be a convex body with $C^2_+$ smooth boundary such that $K$ slides freely in a ball of radius $R > 0$. For $r > R$, there exist positive constants $\bar{c}_K(r,d)$ and $\bar{C}_K(r,d)$, depending on $K$, $d$ and $r$, such that for sufficiently large $n$, it holds that

\begin{equation}
\bar{c}_K(r,d) \cdot n^{\frac{d+1}{d+2}} \leq \mathbb{E} (f_{d-1}^{r*}(K^r_{(n)})) \leq \bar{C}_K(r,d) \cdot n^{\frac{d+1}{d+2}}.
\end{equation}

We note that the order of magnitude of $\mathbb{E} (f_{d-1}^{r*}(K^r_{(n)}))$ is the same in the ordinary convex case. When $K_n$ is the convex hull of $n$ i.i.d. uniform random points from the convex body $K$ with $C^2_+$ smooth boundary, and $f_i(K_n)$ denotes the number of $i$-dimensional faces of $K_n$, then for $0 \leq i \leq d - 1$,

\begin{equation}
\lim_{n \to \infty} \mathbb{E} (f_i(K_n)) \cdot n^{-\frac{d-i}{d+1}} = c_{d,i} \Omega(K),
\end{equation}

where $\Omega(K)$ denotes the affine surface area of $K$, and $c_{d,i}$ is a constant that depends only on the dimension of space. The (1.5) asymptotic formula was proved by Rényi and Sulanke [22] in the two-dimensional case. Bárány [2] established lower and upper bounds of the correct order of magnitude for $\mathbb{E} (f_{d-1}^{r*}(K^r_{(n)}))$ for general $d$. This
exact form of (1.5) is due to Wieacker [30] for \( i = d - 1 \), and for general \( i \) to Reitzner [20]. The method of convex floating bodies, that was used by Bárány [2], could naturally be employed in the case of random ball-polytopes as well provided we had the equivalent of the Economic Cap Covering Theorem of Bárány and Larman [4]. Therefore, it would of major interest to prove the ball-convex equivalent of the Economic Cap Covering Theorem. However, it seems to the author that this may require new ideas compared to the Euclidean case.

When convenient, we use the Landau symbols to indicate the relation between two functions: If for \( f, g : \mathbb{N} \to \mathbb{R}_+ \), there exists a constant \( \gamma > 0 \) and a number \( n_0 \in \mathbb{N} \) such that \( f(n) < \gamma g(n) \) for all \( n > n_0 \), then we write \( f \ll g \). If \( g \ll f \ll g \), then we denote this by \( f \approx g \).

The outline of the rest of the paper is the following: In Section 2, we collect some necessary general tools for our arguments. In Section 3, we study some important properties of ball caps of convex bodies. Section 4 contains the detailed proof of Theorem 1.1. In Section 5, we give an outline of the proof of Theorem 1.2.

2. Tools

Let \( V(\cdot) \) be the \( d \)-dimensional Lebesgue measure or volume. Integration in \( \mathbb{R}^d \) is always with respect to the \( d \)-dimensional Lebesgue measure unless noted otherwise. We denote by \( S^{d-1} \) the \( d \)-dimensional unit sphere centred at \( o \). It is well known that \( \kappa_d := V(B^d) = \pi^{d/2}/\Gamma(d/2 + 1) \), where \( \Gamma \) denotes Euler’s gamma function, cf. [1]. The spherical Lebesgue measure is denoted by \( \sigma_{d-1} \). Then \( \omega_d = \sigma_{d-1}(S^{d-1}) = d\kappa_d \), cf. [24].

We start with the following statement, which is a Blaschke–Petkantschin type transformation formula involving \( d \)-dimensional spheres of radius \( r > 0 \). It is similar to, for example, [27, Theorem 7.3.1., p. 287], which was originally proved by Miles [16].

For \( v_1, \ldots, v_d \in \mathbb{R}^d \), let \( \nabla_d(v_1, \ldots, v_d) \) denote the \( d \)-dimensional volume of the parallelotope spanned by the vectors \( v_1, \ldots, v_d \). Let \( r > 0 \) be fixed, and consider the differentiable map

\[
(2.1) \quad T : \mathbb{R}^d \times (S^{d-1})^d \to (\mathbb{R}^d)^d, \quad T(z, u_1, \ldots, u_d) = (z + ru_1, \ldots, z + ru_d).
\]

Let \( D \subset \mathbb{R}^d \times (S^{d-1})^d \) be a measurable set such that the restriction of \( T \) to \( D \) is bijective with the possible exception of a set of measure zero. Then the following holds.

**Lemma 2.1.** If \( f : (\mathbb{R}^d)^d \to \mathbb{R} \) is a non negative measurable function, then

\[
(2.2) \quad \int_{T(D)} \cdots \int_{T(D)} f(x_1, \ldots, x_d) \, dx_1 \cdots dx_d
= r^{d(d-1)} \int_{\mathbb{R}^d} \int_{S^{d-1}} \cdots \int_{S^{d-1}} 1((z, u_1, \ldots, u_d) \in D) \times f(z + ru_1, \ldots, z + ru_d) \nabla_d(u_1, \ldots, u_d) \, du_1 \cdots du_d dz.
\]

Here \( \int_{S^{d-1}} \ldots du \) denotes integration with respect to the spherical Lebesgue measure on \( S^{d-1} \). We note that the two-dimensional version of (2.2) was already known to Santaló [23], and was recently used in [11], where a short proof of it was also provided for \( d = 2 \).
Proof. We need to show that the Jacobian $|dT|$ of $T$ is

$$|dT| = r^{d(d-1)} \cdot \nabla_d(u_1, \ldots, u_d).$$

We follow a similar argument and notation as in the proof of Theorem 7.3.1 in Schneider and Weil [27, pp. 287-288] whose idea goes back to Møller [18].

Vectors of $\mathbb{R}^d$ are considered columns, and $I_d$ is the $d \times d$ identity matrix. For a vector valued differentiable function $v$, the symbol $\dot{v}$ denotes its derivative.

We assume that in a neighbourhood of the $u_i$ the local coordinate system is chosen such that the $d \times d$ matrix $(u_i \dot{u}_i)$ is orthogonal for all $i$. We recall from [27] p. 287 that for a vector $u \in S^{d-1}$, where the matrix $(u \dot{u})$ is orthogonal, the following hold

$$\dot{u}^t u = 0, \quad \dot{u}^t \dot{u} = I_{d-1}, \quad I_d - \dot{u} \dot{u}^t = uu^t.$$

The Jacobian of $T$ can be written in the following block matrix form

$$dT = \begin{vmatrix} I_d & r\dot{u}_1 & 0 & \cdots & 0 \\ I_d & 0 & r\dot{u}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_d & 0 & 0 & \cdots & r\dot{u}_d \end{vmatrix}.$$

Then it follows that

$$r^{-2d(d-1)}(dT)^2 = \begin{vmatrix} I_d & \dot{u}_1 & \dot{u}_2 & \cdots & \dot{u}_d \\ \dot{u}_1^t & I_{d-1} & 0 & \cdots & 0 \\ \dot{u}_2^t & 0 & I_{d-1} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \dot{u}_d^t & 0 & 0 & \cdots & I_{d-1} \end{vmatrix}$$

$$= \left| dI_d - \sum_{i=1}^d \dot{u}_i \dot{u}_i^t \right| = \sum_{i=1}^d u_i u_i^t$$

$$= \left| (u_1 \ldots u_d) \begin{pmatrix} u_1^t \\ \vdots \\ u_d^t \end{pmatrix} \right| = \left| (u_1 \ldots u_d) \right|^2 = \nabla_d(u_1, \ldots, u_d),$$

which finishes the proof of the lemma. $\square$

Let $K \subset \mathbb{R}^d$ be a convex body with $C^2$ smooth boundary. Then there exist positive constants $R \geq \varrho > 0$ such that $K$ slides freely in a ball of radius $R$ and a ball of radius $\varrho$ rolls freely in $K$, cf. [24] Theorem 3.2.12, Corollary 3.2.13]. Let $R_K$ denote the smallest number such that $K$ slides freely in a ball of radius $R_K$.

Let $\sigma_K : \text{bd} K \to S^{d-1}$ denote the spherical image map which assigns to each $x \in \text{bd} K$ the unique outer unit normal $\sigma_K(x) \in S^{d-1}$ to $\text{bd} K$ at $x$. In this particular case, the inverse $\sigma_K^{-1} : S^{d-1} \to \text{bd} K$ of the spherical map $\sigma_K$ is also well-defined and bijective between $S^{d-1}$ and $\text{bd} K$, and to a unit vector $u \in S^{d-1}$ it assigns the unique boundary point $x \in \text{bd} K$ where the outer unit normal to $\text{bd} K$
is exactly \( u \). It is known that both \( \sigma_K \) and \( \sigma_K^{-1} \) are \( C^1 \) functions in this particular case, see [24, pp. 113–115].

Let \( r \geq R_K \), and define the differentiable map \( \Phi_r : S^{d-1} \times \mathbb{R}_+ \to \mathbb{R}^d \) as
\[
\Phi_r(u, t) := u_K^{-1}(u) - (r + t)u.
\]

**Lemma 2.2.** For the Jacobian \( |d\Phi_r| \) it holds that
\[
|d\Phi_r(u, t)| = \left| \sum_{i=0}^{d-1} (-1)^i \binom{d-1}{i} s_{d-i-1}(u)(r + t)^i \right|,
\]
where
\[
s_j(u) = \left( \frac{d-1}{j} \right)^{-1} \sum_{1 \leq i_1 < \cdots < i_j \leq d-1} r_{i_1}(u) \cdots r_{i_j}(u)
\]
are the normalized elementary symmetric functions of the principal radii of curvature \( r_1(u), \ldots, r_{d-1}(u) \) at \( \sigma_K^{-1}(u) \in \partial K \).

The proof of Lemma 2.2 is quite standard, in fact, using Lemma 3.1, it is essentially the same as the one presented on page 122 of Section 2.5 in [24] with the substitution \( \lambda = -(r + t) \).

Next, we quote ([11, (5.6) on page 909], see also [7, (11) on page 229]) the following asymptotic formula.

**Lemma 2.3.** For any \( \beta \geq 0, \omega > 0 \) and \( \alpha > 0 \), it holds that
\[
\int_0^{g(n)} t^{\beta}(1 - \omega t^\alpha)^n dt \sim \frac{1}{\alpha \omega^{\frac{\alpha + 1}{\alpha}}} \cdot \Gamma\left( \frac{\beta + 1}{\alpha} \right) \cdot n^{-\frac{\beta + 1}{\alpha}},
\]
as \( n \to \infty \), assuming that
\[
\left( \frac{(\beta + \alpha + 1) \ln n}{\alpha \omega n} \right) \frac{1}{\alpha} < g(n) < \omega^{-\frac{1}{\alpha}}
\]
for sufficiently large \( n \).

We will also use the following result from [5]. Let \( S^{d-1}_+ \) denote the closed half-sphere which is above the coordinate hyperplane \( x_d = 0 \). Then
\[
\int_{S^{d-1}_+} \cdots \int_{S^{d-1}_+} \nabla(u_1, \ldots, u_d) \, du_1 \cdots du_d = \left( \frac{\omega_{d+1}}{2} \right)^{d-1}.
\]

We note that (2.5) is a special case of a more general formula, cf. [5] pages 7–8.

3. **Properties of ball caps**

In this section we assume that \( \partial K \) is a \( C^2 \) smooth such that all of the sectional curvatures at each point of \( \partial K \) are strictly greater than 1. Then for the Gaussian curvature it holds that \( \kappa(x) > 1 \) for all \( x \in \partial K \), and \( K \) slides freely in the unit ball \( B^d \). Furthermore, in this case, \( K \) has the property that for any points \( x, x' \in K \), the shorter arc of any unit circle passing through \( x \) and \( x' \) is contained in \( K \). This property is called \textit{spindle convexity} or \textit{hyperconvexity}, for more information see, for example, [6] or [9].

We will call the closure of the intersection of \( K \) and the complement of a unit ball a \textit{ball cap}. Ball caps play a similar role in our arguments to usual (linear) caps of convex bodies cut off by hyperplanes. We need to establish some basic facts about ball caps that are in analogy with linear caps, most importantly, that each
such cap has a well-defined vertex and height. We note that the two-dimensional case was already treated in [11]. Here we extend the planar statements of [11] (cf. Lemmas 4.1–4.3, pp. 905–906) to d-dimensions.

Lemma 3.1. Let $C = \text{cl}(K \setminus (B^d + p))$, $p \in \mathbb{R}^d$ be a non-empty ball cap. Then there exists a unique point $x = x(p) \in C \cap \text{bd} K$ (the vertex) and a positive real number $t = t(p) > 0$ (the height) such that $p = x - (1 + t)\sigma_K(x)$.

Proof. Following Lutwak, Yang and Zhang [14, Section 3], we define the reverse radial Gauss map $\alpha^* : S^{d-1} \to S^{d-1}$ as

$$\alpha^*(u) := \frac{\sigma_K^{-1}(u)}{\|\sigma_K^{-1}(u)\|},$$

which is known to be continuous. It is easy to check that

$$\alpha^*(\sigma_K(C \cap \text{bd} K)) \subset \sigma_K(C \cap \text{bd} K).$$

Since $\sigma_K(C \cap \text{bd} K)$ is a topological ball on $S^{d-1}$ and $\alpha^*$ is a continuous function for a fixed $K$, the Lefschetz fixed point theorem, see [28, Theorem 7 and Corollary 8, pp. 195–196], guarantees the existence of a fixed point $u_1 \in \sigma_K(C \cap \text{bd} K)$ of $\alpha^*$, that is, $\alpha^*(u_1) = u_1$. Then, clearly, $x_1 = \sigma_K^{-1}(u_1)$ is a vertex of $C$.

We need only check the uniqueness of the vertex. Assume, on the contrary, that there are two such points, say, $x_1$ and $x_2$ with $x_1 \neq x_2$. Let $\gamma$ denote the curve which is the intersection of $\text{bd} C \cap \text{bd} K$ with the 2-dimensional flat $S$ spanned by $p, x_1$ and $x_2$, and whose endpoints are $x_1$ and $x_2$. Due to the spindle convex property of $K$, the curvature of $\gamma$ is greater than 1 for each $x \in \gamma$.

Without loss of generality, we may assume that $\gamma(s)$ is parametrized with its arc-length over the interval $[0, L]$ with $\gamma(0) = x_1$ and $\gamma(L) = x_2$. Let $\varphi$ denote the angle of the outer unit normal vectors $u_1 = \sigma_K(x_1)$ and $u_2 = \sigma_K(x_2)$ of $\text{bd} K$ at $x_1$ and $x_2$, respectively. We assume that $\gamma$ is oriented in such a way that $\varphi > 0$. On the one hand, it is clear that $L > \varphi$. Let $k(s)$ denote the curvature of the plane curve $\gamma$ at $\gamma(s)$. Since $k(s) > 1$ for all $s \in [0, L]$, we obtain

$$L > \varphi = \int_0^L k(s)ds > \int_0^L ds = L,$$

a contradiction. \hfill \Box

We introduce the following notations. For $u \in S^{d-1}$ and $t \geq 0$, let $C(u, t)$ denote the cap of height $t$ and vertex $x = \sigma_K^{-1}(u)$, and let $V(u, t) = V(C(u, t))$.

Let us fix $u \in S^{d-1}$ and assume that $x = \sigma_K^{-1}(u) = o$ such that the (unique) supporting hyperplane of $K$ at $x$ is identified with $\mathbb{R}^{d-1}$. Since $\text{bd} K$ is $C^2_\star$, there exists a convex function $f$ in a sufficiently small open ball around $o$ in $\mathbb{R}^{d-1}$ such that $\text{bd} K$ is the the graph of $f$ in above this neighbourhood. Then

$$f(z) = \frac{1}{2}Q(z) + o(\|z\|^2) \text{ as } z \to 0$$

in this small neighbourhood. The quadratic form $Q$ is the second fundamental form of $\text{bd} K$ at $x$. Under the above hypotheses, $Q$ is positive definite. It is well-known that if we choose a suitable orthonormal basis $e_1, \ldots, e_{d-1}$ in $\mathbb{R}^{d-1}$, then

$$Q(z) = k_1z_1^2 + \cdots + k_{d-1}z_{d-1}^2$$

for $z = z_1e_1 + \cdots + z_{d-1}e_{d-1} \in \mathbb{R}^{d-1}$, where the quantities $k_1, \ldots, k_{d-1}$ are the principal curvatures of $\text{bd} K$ at $x = \sigma_K^{-1}(u)$, and the directions determined by the
orthonormal basis vectors $e_1, \ldots, e_{d-1}$ are the principal directions. In particular, if $w \in S^{d-2}$, then $k(w) = Q(w)$ is the sectional curvature of $\text{bd} K$ at $x$ in the direction of $w$. Of course, $k_i = Q(e_i)$ for $i = 1, \ldots, d - 1$.

**Lemma 3.2.** With the same hypotheses and notation as above, it holds

$$\lim_{{t \to 0^+}} V(u, t) \cdot t^{-\frac{d+1}{2}} = \frac{2^{d+1} \kappa_{d-1}}{d+1} \int_{{S^{d-2}}} (Q(w) - 1)^{-\frac{d-1}{2}} dw$$

**Proof.** Now, let $z = \tau w$, where $\tau \geq 0$ and $w \in S^{d-2}$. Assume that $\mathbb{R}^d = \mathbb{R}^{d-1} \times \mathbb{R}$ and that $e_d$ is unit vector such that $e_1, \ldots, e_d$ is an orthonormal basis of $\mathbb{R}^d$ that has a positive orientation, and $e_d = -\sigma_K(x)$, that is, $e_d$ is the inner unit normal of $\text{bd} K$ at $x$. For $t > 0$, the sphere $S^{d-1} + (1 + t)e_d$ determines a cap of $K$, that has height $t$ and vertex $x$. In a sufficiently small neighbourhood of $o$, the lower hemisphere of $S^{d-1} + (1 + t)e_d$ is the graph of the function

$$g_t(\tau w) = \tau w + (1 + t - \sqrt{1 - \tau^2})e_d.$$

It is not difficult to check that for a fixed $w \in S^{d-2}$, the intersection point $\tau^*(w)$ of $\text{bd} K$ and the sphere $S^{d-1} + (1 + t)e_d$ satisfies

$$\tau^*(w) = \sqrt{\frac{2}{k(w) - 1}} t^{1/2} + o(t^{1/2}) \text{ as } \tau \to 0^+.$$

Thus,

$$V(u, t) = \int_{{S^{d-2}}} \int_0^{\tau^*(w)} (g_t(\tau w) - f(\tau w)) \tau^{d-2} \omega_{d-1} d\tau dw.$$

Using Taylor’s theorem, we obtain that

$$V(u, t) = \omega_{d-1} \int_{{S^{d-2}}} \int_0^{\tau^*(w)} \left(t + \frac{\tau^2}{2} - \frac{k(w)\tau^2}{2} + o(\tau^2)\right) \tau^{d-2} d\tau dw$$

$$= \omega_{d-1} \int_{{S^{d-2}}} \left(\frac{\tau^{d-1}}{d-1} + \frac{k(w)\tau^{d+1}}{2(d+1)} + o(\tau^{d+1})\right) d\tau dw$$

$$= \frac{2^{d+1} \kappa_{d-1}}{d+1} t^{\frac{d+1}{2}} \int_{{S^{d-2}}} (k(w) - 1)^{-\frac{d-1}{2}} dw + o\left(t^{\frac{d+1}{2}}\right) \text{ as } t \to 0^+,$$

which completes the proof. \qed

We note that (3.1) is a generalization of (10) on page 2290 in \[7\] as \(\int_{{S^{d-2}}} Q(w)^{-(d-1)/2} dw = \kappa(x)^{-1/2}\).

Let $x_1, \ldots, x_d \in K$ be points which determine exactly two distinct (ball) caps of $K$, namely $C_-(x_1, \ldots, x_d) = \text{cl}(K \setminus (B^d + p_-))$, and $C_+(x_1, \ldots, x_d) = \text{cl}(K \setminus (B^d + p_+))$ such that $x_1, \ldots, x_d \in S^{d-1} + p_-$ and $x_1, \ldots, x_d \in S^{d-1} + p_+$ with the extra assumption that $V(C_-(x_1, \ldots, x_d)) \leq V(C_+(x_1, \ldots, x_d))$. For the sake of brevity, henceforth, we will use the following shorthand notations $V_-(x_1, \ldots, x_d) = V(C_-(x_1, \ldots, x_d))$ and $V_+(x_1, \ldots, x_d) = V(C_+(x_1, \ldots, x_d))$.

**Lemma 3.3.** With the above hypotheses and notation, there exists a constant $\delta > 0$, depending only on $d$ and $K$, such that $V_+(x_1, \ldots, x_d) > \delta$.

**Proof.** Since $\text{bd} K$ is $C^2_2$ and all principal curvatures are strictly larger than 1 at each $x \in \text{bd} K$, the intersection $(B^d + p_-) \cap (B^d + p_+)$ can never cover $K$. By compactness, there exists $\delta > 0$, depending only on $K$, such that $V(K \setminus ((B^d + p_-) \cap (B^d + p_+))) > 2\delta$, from which the statement of the lemma follows easily. \qed
4. Proof of Theorem [11] — The case of the unit ball

We start this section with some general statements about the expectation of facet numbers that will also be used in the proof of Theorem [12] in Section 5.

Let $X_n = \{x_1, \ldots, x_n\}$ be a sample of $n$ independent random points from $K$ selected according to the uniform probability distribution. The intersection

$$K_{(n)}^r = \bigcap_{x_n \in rB^d + y} (rB^d + y)$$

is a random ball-polytope contained in $K$. Let $f_{d-1}^*(K_{(n)}^r)$ denote the number of facets of $K_{(n)}^r$ whose vertices form a $(d-1)$-dimensional simplex or, equivalently, a $(d-1)$-dimensional spherical simplex.

Let $V = V(K)$. Notice that we cannot assume that $V = 1$ since a homothety would change the radius of the generating balls of $K_{(n)}^r$. However, without loss of generality, we may and do assume that $r = 1$. Subsequently, we use the simplified notation $K_{(n)}$ for the random ball-polytope $K_{(n)}^1$. The results for general $r$ follow by scaling.

For $1 \leq i_1 < \cdots < i_d \leq n$, the subset $\{x_{i_1}, \ldots, x_{i_d}\} \subset X_n$ determines exactly two unit spheres, $S^{d-1} + p_-$ and $S^{d-1} + p_+$, with probability 1. Notice that $x_{i_1}, \ldots, x_{i_d}$ determine a simplicial facet of $K_{(n)}$ only if all other points of $X_n$ fall into one of the caps $C_+(i_1, \ldots, i_d)$ and $C_-(i_1, \ldots, i_d)$. Due to the independence of the points in $X_n$, it holds that

$$E(f_{d-1}^*(K_{(n)})) = \binom{n}{d} \frac{1}{V^d} \int_{K^d} \left( 1 - \frac{V_+(x_1, \ldots, x_d)}{V} \right)^{n-2} \left( 1 - \frac{V_-(x_1, \ldots, x_d)}{V} \right)^{n-2} dx_1 \ldots dx_d$$

The unit ball $B^d$ is special in the sense that the $E(f_{d-1}^*(B_{(n)}^d))$ approaches a finite number as $n$ tends to infinity. A similar phenomenon was pointed out in the paper by Fodor, Kevei and Vígh, cf. [11] Theorem 1.3 (1.7) on p. 902 in the case when $d = 2$. Here we treat the general case.

Consider the cap $C(u, t)$ of $B^d$ with vertex $u \in S^{d-1}$ and height $0 < t < 2$. In this case $C(u, t) = cl(B^d \setminus (B^d - tu))$. Let $u = e_d$. Elementary geometry shows that the distance of the set $S^{d-1} \cap (S^{d-1} - tu)$ (which is a $(d-2)$-sphere) from the hyperplane $x_d = 0$ is $t/2$. Therefore, the volume $V(u, t)$ of $C(u, t)$ is equal to the volume of a centred spherical plank, that is, the volume of the intersection of $B^d$ with a plank (the part of space between two parallel hyperplanes) of width $t$ and symmetric to $o$. Therefore, it is easy to see that

$$\lim_{t \to 0^+} \frac{1}{t} \cdot V(u, t) = \kappa_{d-1}.$$  

We will use that in the case when $K = B^d$ the Jacobian of the map $\Phi = \Phi_1$ is $|d\Phi(u, t)| = t^{d-1}$.

Now, from [11], we obtain

$$\lim_{n \to \infty} E(f_{d-1}^*(B_{(n)}^d)) = \lim_{n \to \infty} \frac{1}{\kappa_d^d} \binom{n}{d} \int_{B^d} \ldots \int_{B^d} \left( 1 - \frac{V_+(x_1, \ldots, x_d)}{\kappa_d} \right)^{n-d}$$
+ \left(1 - \frac{V(x_1, \ldots, x_d)}{\kappa_d}\right)^{n-d} dx_1 \ldots dx_d.

Let \( S(u, t) = B^d \cap (S^{d-1} - tu) \). Using \( \Phi(u, t) \) and the symmetries of \( S^{d-1} \), we get

\[
\lim_{n \to \infty} \mathbb{E}(f_{d-1}^n(B_d)) = \lim_{n \to \infty} \frac{n}{\kappa_d^d} \int_{S^{d-1}} \int_{0}^{2} \int_{S(u,t)} \cdots \int_{S(u,t)} \left(1 - \frac{V(u,t)}{\kappa_d}\right)^{n-d} \times t^{d-1} \nabla_d(u_1, \ldots, u_d) du_1 \ldots du_d du.
\]

We now split the domain of integration in the variable \( t \). It turns out that it is sufficient to integrate on the interval \([0, h(n)]\), where \( h(n) \) is a sequence defined below. This is a standard technique in such approximation problems. For a similar argument see, for example, [11, Lemma 5.1 on p. 908]. Let \( h(n) = c \log n/n \), where \( c > d\kappa_d/\gamma_1 \) is a constant. There exist an \( n_0 \in \mathbb{N} \) and \( \gamma_1 > 0 \) such that for \( n > n_0 \) it holds that \( h(n) < 2 \) and for all \( n \leq t < 2 \), \( V(u,t) > \gamma_1 h(n) \) for any \( u \in S^{d-1} \).

Since \( \nabla_d(u_1, \ldots, u_d) \leq 1 \) for any \( u_1, \ldots, u_d \in S^{d-1} \), it follows that

\[
\int_{h(n)}^{t_1} \int_{S(u,t)} \left(1 - \frac{V(u,t)}{\kappa_d}\right)^{n-d} t^{d-1} \nabla_d(u_1, \ldots, u_d) du_1 \ldots du_d dt \\
\leq 2^{d-1} \omega_d \int_{h(n)}^{t_1} \left(1 - \frac{\gamma_1 h(n)}{\kappa_d}\right)^{n-d} dt \\
= 2^{d-1} \omega_d \int_{h(n)}^{t_1} \left(1 - \frac{\gamma_1 c \log n/n}{\kappa_d}\right)^{n-d} dt \\
\leq 2^{d-1} \omega_d n^{-\gamma_1 c/\kappa_d}.
\]

Thus,

\[
\lim_{n \to \infty} \frac{1}{\kappa_d^d} \int_{S^{d-1}} \int_{h(n)}^{t_1} \int_{S(u,t)} \left(1 - \frac{V(u,t)}{\kappa_d}\right)^{n-d} t^{d-1} \nabla_d(u_1, \ldots, u_d) du_1 \ldots du_d du \\
\leq \lim_{n \to \infty} \frac{n}{\kappa_d^d} \omega_d^{d+1} n^{-\gamma_1 c/\kappa_d} \\
= 0.
\]

We define the sequence

\[
\theta_n(u) = \frac{1}{\kappa_d^d} \int_{h(n)}^{t_1} \int_{S(u,t)} \left(1 - \frac{V(u,t)}{\kappa_d}\right)^{n-d} t^{d-1} \nabla_d(u_1, \ldots, u_d) du_1 \ldots du_d dt.
\]

Clearly \( \theta_n(u) \) independent of \( u \in S^{d-1} \), so we use the simplified notation \( \theta_n(u) = \theta_n \).

Then

\[
\lim_{n \to \infty} \mathbb{E}(f_{d-1}^n(K_n)) = \lim_{n \to \infty} \int_{S^{d-1}} \theta_n(u) du = \omega_d \lim_{n \to \infty} \theta_n.
\]

Let \( \varepsilon \in (0, 1) \). Then, using (2.5) and (4.2), there exists a \( 0 < t_2 < 2 \) such that

\[
(1 - \varepsilon) \left(\frac{\omega_d + 1}{2}\right)^{d-1} < \int_{S(u,t)} \nabla_d(u_1, \ldots, u_d) du_1 \ldots du_d < (1 + \varepsilon) \left(\frac{\omega_d + 1}{2}\right)^{d-1},
\]

\[
(1 - \varepsilon) t \kappa_{d-1} < V(u, t) < (1 + \varepsilon) t \kappa_{d-1}
\]
for all $t \in (0,t_2)$. Since $\varepsilon$ is arbitrary, we get that
\[
\omega_d \lim_{n \to \infty} \theta_n = \omega_d \frac{\omega_{d+1}}{\kappa_d} \left( \lim_{n \to \infty} \frac{n}{d} \int_0^{\theta(n)} \left( 1 - t \frac{\kappa_{d-1}}{\kappa_d} \right)^{n-d} dt \right)
\]
\[
= \frac{1}{\kappa_d} \left( \omega_{d+1} \right)^{d-1} \frac{1}{(d-1)!} \lim_{n \to \infty} n^d \int_0^{\theta(n)} \left( 1 - t \frac{\kappa_{d-1}}{\kappa_d} \right)^{n-d} dt.
\]
Now, with $\alpha = 1$, $\beta = d - 1$ and $\omega = \kappa_d / \kappa_{d-1}$, we obtain from (2.4) that
\[
\omega_d \lim_{n \to \infty} \theta_n = \frac{1}{\kappa_d^{d-1}} \left( \frac{\omega_{d+1}}{2} \right)^{d-1} \left( \frac{\kappa_d}{\kappa_{d-1}} \right)^d \frac{1}{(d-1)!} \Gamma(d)
\]
\[
= \frac{\kappa_d^{d-1}}{\kappa_{d-1}^{d-1}} \frac{(d+1)^{d-1}}{2^{d-1}}
\]
\[
= \frac{\kappa_d^{d-1}}{\kappa_{d-1}^{d-1}} \frac{2^d}{2^{d-1}}.
\]
Taking into account that $\kappa_{d+1} / \kappa_{d-1} = 2 \pi / (d+1)$, we get that
\[
\lim_{n \to \infty} \mathbb{E}(f_{d-1}(K_n)) = \frac{\pi^{d-1} \kappa_d}{\kappa_{d-1}}
\]
which finishes the proof of Theorem 1.1.

5. OUTLINE OF THE PROOF OF THEOREM 1.2

Since some of the arguments are similar to those in the proof of Theorem 1.1, we only give a limited amount of details.

First, we note that it is enough to consider the term (4.1) which contains $V_-(x_1, \ldots, x_d)$. Indeed, for any fixed $\alpha$, it follows from Lemma 3.3 that
\[
\lim_{n \to \infty} n^\alpha \left( \frac{n}{d} \right) \frac{1}{V^d} \int_{K^d} \left( 1 - \frac{V_-(x_1, \ldots, x_d)}{V} \right)^{n-2} dx_1 \ldots dx_d
\]
\[
\leq \lim_{n \to \infty} n^\alpha \left( \frac{n}{d} \right) \frac{1}{V^d} \int_{K^d} e^{-\frac{1}{2} \left( n-2 \right)} dx_1 \ldots dx_d
\]
\[
= 0.
\]
By a similar argument, one can easily verify that it is sufficient to integrate over such $d$-tuples $x_1, \ldots, x_d$ for which $V_-(x_1, \ldots, x_d) < \delta$. Thus,
\[
\lim_{n \to \infty} \mathbb{E}(f_{d-1}^*(K_n)) \frac{n^{-\frac{d-1}{d+1}}}{d} = \lim_{n \to \infty} \frac{n^{-\frac{d-1}{d+1}}}{d} \left( \frac{n}{d} \right) \frac{1}{V^d} \int_{K^d} \left( 1 - \frac{V_-(x_1, \ldots, x_d)}{V} \right)^{n-2}
\]
\[
\times \mathbf{1}(V_-(x_1, \ldots, x_d) < \delta) dx_1 \ldots dx_d.
\]
Now, we reparametrize the $d$-tuple $(x_1, \ldots, x_d)$ using the function according to
\[
(x_1, \ldots, x_d) = T(\Phi(u, t), u_1, \ldots, u_d),
\]
for $u, u_1, \ldots, u_d \in S^{d-1}$ and $t \in \mathbb{R}_+$. For $u \in S^{d-1}$ and $t > 0$, let
\[
C(u, t) = \text{cl}(K \setminus (B^d + \Phi(u, t))),
\]
\[
S(u, t) = K \cap (S^{d-1} + \Phi(u, t)),
\]
and \( V(u, t) = V(C(u, t)) \). Let
\[
\psi(u, t) = \int_{S(u, t)} \cdots \int_{S(u, t)} \nabla_d(u_1, \ldots, u_d) du_1 \cdots du_d.
\]
Further, let
\[
s(u, t) = \sum_{i=0}^{d-1} (-1)^i \binom{d-1}{i} s_{d-1}(u)(1 + t)^i.
\]
By the \( C^2 \) property of \( \text{bd} K \), there exists a \( 0 < t_1 \) such that \( V_-(u, t) \geq \delta \) for all \( t_1 \leq t \) and \( u \in S^{d-1} \). Using Lemmas 2.1 and 2.2 we have that
\[
\lim_{n \to \infty} \mathbb{E}(f_{d-1}^*(K(n))) \cdot n^{-\frac{d-1}{d+1}}
= \lim_{n \to \infty} n^{-\frac{d-1}{d+1}} \left( \frac{n}{d} \right) \frac{1}{V} \int_{S^{d-1}} \int_{h(n)}^{t_1} \left( 1 - \frac{\psi(u, t)}{V} \right)^{n-2} s(u, t) \psi(u, t) du dt
\]
The domain of integration with respect to \( t \) can be split further in a standard way similarly to the case of the unit ball. Let \( h(n) = (c \ln n/n)^{2/(d+1)} \), for some constant \( c \geq (V/\gamma_2)(d^2 + 1)/(d(d+1)) \). There exist \( \gamma_2 > 0 \) and \( n_0 \in \mathbb{N} \) such that for all \( n > n_0 \), \( h(n) < t_1 \), and \( V(u, t) > \gamma_2 h(n)^{d+1}/2 \) for all \( h(n) \leq t \leq t_1 \) and \( u \in S^{d-1} \). Since \( \nabla_d(u_1, \ldots, u_d) \leq 1 \) for all \( u_1, \ldots, u_d \in S^{d-1} \), and all the \( s_i \) functions are uniformly bounded above, we get that for some suitable constants \( \gamma_3, \gamma_4 > 0 \),
\[
\lim_{n \to \infty} n^{-\frac{d-1}{d+1}} \left( \frac{n}{d} \right) \frac{1}{V} \int_{S^{d-1}} \int_{h(n)}^{t_1} \left( 1 - \frac{\psi(u, t)}{V} \right)^{n-2} s(u, t) \psi(u, t) du dt
\leq \gamma_3 \lim_{n \to \infty} n^{-\frac{d-1}{d+1}} \left( \frac{n}{d} \right) \frac{1}{V} \int_{S^{d-1}} \int_{h(n)}^{t_1} \left( 1 - \frac{\gamma_2 \ln n/n}{V} \right)^{n-2} ddu
\leq \gamma_4 \lim_{n \to \infty} n^{-\frac{d-1}{d+1}} \left( \frac{n}{d} \right) n^{-\gamma_2 c/V}
= 0.
\]
Then
\[
\lim_{n \to \infty} \mathbb{E}(f_{d-1}^*(K(n))) \cdot n^{-\frac{d-1}{d+1}}
= \lim_{n \to \infty} n^{-\frac{d-1}{d+1}} \left( \frac{n}{d} \right) \frac{1}{V} \int_{S^{d-1}} \int_{h(n)}^{0} \left( 1 - \frac{\psi(u, t)}{V} \right)^{n-2} s(u, t) \psi(u, t) du dt.
\]
For \( u \in S^{d-1} \) and \( n \in \mathbb{N} \), introduce
\[
\theta_n(u) = n^{-\frac{d-1}{d+1}} \left( \frac{n}{d} \right) \frac{1}{V} \int_{0}^{h(n)} \left( 1 - \frac{\psi(u, t)}{V} \right)^{n-2} s(u, t) \psi(u, t) dt.
\]
Therefore
\[
\lim_{n \to \infty} \mathbb{E}(f_{d-1}^*(K(n))) \cdot n^{-\frac{d-1}{d+1}} = \lim_{n \to \infty} \int_{S^{d-1}} \theta_n(u) du.
\]
In order to be able to change the limit and the integral using Lebesgue’s dominated convergence theorem, we must show that the functions \( \theta_n(u) \) are uniformly bounded. This follows from Lemmas 2.3 and 3.1 in a quite standard way using the fact that both \( s(u, t) \) and \( \psi(u, t) \) are uniformly bounded. For an analogous argument, see, for example, [11] p. 909. Thus,
\[
\lim_{n \to \infty} \mathbb{E}(f_{d-1}^*(K(n))) \cdot n^{-\frac{d-1}{d+1}} = \int_{S^{d-1}} \lim_{n \to \infty} \theta_n(u) du.
\]
Note that due to the \( C^2 \) smoothness of \( \text{bd} \ K \), \( s(u, t) = s(u, 0) + O(t) \) as \( t \to 0^+ \) uniformly for \( u \in S^{d-1} \). Thus, for an \( \varepsilon > 0 \), there exists \( 0 < t_\varepsilon < t_2 \) such that for all for all \( u \in S^{d-1} \) and any \( 0 < t < t_\varepsilon \) it holds that

\[
(1 - \varepsilon)s(u, 0) < s(u, t) < (1 + \varepsilon)s(u, 0), \quad \text{and}
\]

\[
(1 - \varepsilon)c(K, u)t^\frac{d+1}{d} < V(u, t) < (1 + \varepsilon)c(K, u)t^\frac{d+1}{d},
\]

where \( c(K, u) \) is the quantity on the right-hand-side of (3.1).

For \( x_0, \ldots, x_d \in B^d \), and let \( \Delta_d(x_0, \ldots, x_d) \) denote the volume of the simplex with vertices \( x_0, \ldots, x_d \). Miles [17, (29)], among many other things, determined the smoothness of \( \text{bd} \ K \), and let \( \Delta \) denote the quantity on the right-hand-side of (3.1). Thus, using (5.1), (5.2), (5.3) and Lemma 2.4, we obtain (disregarding the implied constants) that the following holds uniformly for \( u \in S^{d-1} \):

\[
\frac{1}{2} \int_{\text{rd}(d+2)} I(d) \leq \int_{rB^d} \Delta_d(x_0, \ldots, x_d) dx_0 \ldots dx_d.
\]

If \( 0 < \theta < 1 \) is a number such that a ball of radius \( \theta \) rolls freely in \( K \), and \( 0 < R < 1 \) is such that \( K \) slides freely in \( RB^d \), then let

\[
S_\theta(u, t) = (S^{d-1} + x - (1 + t)) \cap (\theta B^d + x - \theta u),
\]

\[
S_R(u, t) = (S^{d-1} + x - (1 + t)) \cap (RB^d + x - Ru).
\]

Then

\[
S_\theta(u, t) \subset S(u, t) \subset S_R(u, t).
\]

Let

\[
\psi_\theta(u, t) = \int_{S_\theta(u, t)} \int_{S(u, t)} \nabla_d(u_1, \ldots, u_d) du_1 \ldots du_d,
\]

and let \( \psi_R(u, t) \) be defined similarly on \( S_R(u, t) \). Then

\[
\psi_\theta(u, t) \leq \psi(u, t) \leq \psi_R(u, t).
\]

Using (5.2) it is not difficult to see that

\[
\lim_{t \to 0^+} \frac{\psi_\theta(u, t)}{\psi_R(u, t)} = 1, \quad \text{and} \quad \lim_{t \to 0^+} \frac{\psi_R(u, t)}{\theta^d I(d - 1)} = 1.
\]

Therefore, there exist positive constants \( \gamma_5 \) and \( \gamma_6 \) such that for any \( u \in S^{d-1} \) it holds that for sufficiently small \( t \),

\[
\gamma_5 t^\frac{d+1}{d} \leq s(u, t) \leq \gamma_6 t^\frac{d+1}{d}.
\]

Thus, using (5.1), (5.2), (5.3) and Lemma 2.4 we obtain (disregarding the implied constants) that the following holds uniformly for \( u \in S^{d-1} \):

\[
\theta_n(u) \ll \lim_{n \to \infty} n^{-\frac{d+1}{d+2}}(n) \frac{1}{Vd} \int_0^{b(n)} \left( 1 - \frac{c(K, u)t^\frac{d+1}{d}}{V} \right)^{(n-2)} s(u, 0)t^\frac{d+1}{d} dt
\]

\[
\ll n^{-\frac{d+1}{d+2}} \frac{s(u, 0)}{Vd} \int_0^{b(n)} \left( 1 - \frac{c(K, u)t^\frac{d+1}{d}}{V} \right)^{(n-2)} t^\frac{d+1}{d} dt
\]

\[
\ll n^{-\frac{d+1}{d+2}} n^d n^{-\frac{d+1}{d+2}}.
\]
Therefore,
\[
Ef_{d-1}(K(n)) \ll n^{-\frac{d+1}{d+2}}.
\]

The lower bound can be proved similarly. This finishes the proof of Theorem 1.2.

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