Presentations of homotopy skein modules of oriented 3-manifolds

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ABSTRACT. A new method to derive presentations of skein modules is developed. For the case of homotopy skein modules it will be shown how the topology of a 3-manifold is reflected in the structure of the module. The freeness problem for $q$-homotopy skein modules is solved, and a natural skein module related to linking numbers is computed.

INTRODUCTION

Let $R$ be a commutative ring. A skein module of an oriented 3-manifold $M$ is defined from the free $R$-module on a set of isotopy classes of links (possibly decorated by orientation or framing) in $M$ by dividing out a submodule generated by local skein relations [P-1]. The skein modules of 3-manifolds $F \times I$ for $F$ a compact oriented surface play a central role in the study of quantum invariants, topological quantum field theory and finite type invariants of links in 3-manifolds. For important skein relations it is known that the modules of $F \times I$ can be derived through quantization of algebras of homotopy classes of loops on $F$ [B-F-K, H-P, P-3, P-4, T]. Moreover several recent results have shown that skein modules of 3-manifolds should be interpreted as deformations of algebras emerging from the theory of linear representations of the fundamental group [B, P-S]. A theory of skein modules for arbitrary 3-manifolds beyond $F \times I$ seems to be crucial in understanding the quantum topology of 3-dimensional manifolds. But it is difficult to derive global results about skein modules, and complete structure results are known only for simple skein relations [P-1, P-5]. A workable description of the structure of a class of skein modules always has to face two different problems:

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**Problem 1:** Find a suitable set of generators for the skein module.

**Problem 2:** Describe the relations with respect to this generating set in terms of the topology of $M$.

Problem 1 is related with subtle finiteness questions concerning topology and combinatorics of links in 3-manifolds [Ka-L]. With respect to minimal generating sets, Problem 2 asks how the topology obstructs the existence of certain skein invariants. It is known that skein invariants are related to finite type invariants, which have been studied intensively. Thus it is natural to study problem 2 modulo problem 1. This has been carried out in various approaches in terms of (completed) isotopy skein modules [B-F-K, Ka-2].

We will show that a direct approach to global results concerning skein modules is possible along the ideas of [Ka-L], [K] and [Ki-L]. Our main philosophy is: The obstructions to the existence of finite type invariants are relations in a presentation of skein modules. For homotopy skein modules problem 1 has a natural and easy answer. So it is possible to focus on problem 2 from scratch (without completion). We will give a quite complete general answer to problem 2 for this case, though often it is still difficult to solve explicit problems by our method. A discussion of presentations for the (completed) universal Jones-Conway skein module as defined in [T, P-4] and for Kauffman skein modules [P-1, H-P] will be left to future work.

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1. STATEMENT OF THE MAIN RESULTS

Throughout let $R = \mathbb{Z}[q^{\pm 1}, z]$ and let $M$ be a compact oriented 3-manifold. Let $\mathcal{H}(M)$ denote the set of link homotopy classes of oriented links in $M$, including the empty link $\emptyset$. Two oriented links are link homotopic if they are homotopic through a deformation, which keeps different components disjoint.

Following J. Przytycki [P-2] the $q$-homotopy skein module $\mathcal{H}(M)$ is the quotient of $R\mathcal{H}(M)$ by the submodule, which is generated by all skein elements $q^{-1}K_+ - qK_- - zK_0$ for all crossings of different components of links (i.e. mixed crossings).

\[
\begin{align*}
K_+ & \quad K_- & \quad K_0 \\
\end{align*}
\]

The modules $\mathcal{H}(M)$ are universal with respect to the Homfly homotopy
skein relation [P-3]. There are two obvious ways to simplify the relation. Let \( \mathcal{C}(M) \) resp. \( \mathcal{L}(M) \) denote the \( \mathbb{Z}[z] \)- resp. \( \mathbb{Z}[q^{\pm 1}] \)-modules resulting from \( \mathcal{H}(M) \) by the ring homomorphisms \( q \mapsto 1 \) resp. \( z \mapsto 0 \). It follows from Przytycki's Universal Coefficient Theorem [P-3] that \( \mathcal{C}(M) \cong \mathcal{H}(M) \otimes_R \mathbb{Z}[z] \) resp. \( \mathcal{L}(M) \cong \mathcal{H}(M) \otimes_R \mathbb{Z}[q^{\pm 1}] \) with the \( R \)-module structures on \( \mathbb{Z}[z] \) and \( \mathbb{Z}[q^{\pm 1}] \) given by the ring homomorphisms above. In this way results about \( \mathcal{H}(M) \) also give rise to results about the derived skein modules.

For \( M \) a connected 3-manifold, \( \mathcal{H}(M) \cong R \oplus \tilde{\mathcal{H}}(M) \), where \( \tilde{\mathcal{H}}(M) \) is the module defined from \( \tilde{\mathfrak{S}}(M) \setminus \emptyset \). In general, \( \mathcal{H}(M) \cong R \oplus \oplus_i \tilde{\mathcal{H}}(M_i) \), where the sum is over the set of path components \( M_i \) of \( M \). Throughout the following we assume that \( M \) is connected and we fix a basepoint \(*\). All homotopy groups will be defined with respect to \(*\). Let \( \tilde{\pi}(M) \) denote the set of conjugacy classes of elements of \( \pi_1(M) \). Let \( \mathfrak{b}(M) \) denote the set of unordered sequences of elements of \( \tilde{\pi}(M) \) of length \( \geq 0 \). The surjective wrapping invariant

\[
\omega : \mathfrak{S}(M) \to \mathfrak{b}(M)
\]

is defined by assigning to each oriented link in \( M \) the unordered sequence of free homotopy classes of its components.

A section \( \sigma \) of \( \omega \) defines a set of standard links \( K_\alpha := \sigma(\alpha) \), and induces the epimorphism of \( R \)-modules

\[
\sigma : SR\tilde{\pi}(M) \cong R\mathfrak{b}(M) \twoheadrightarrow \mathcal{H}(M).
\]

Here, for each module \( A \) we let \( SA \) denote the symmetric algebra on \( A \). (We let \( \sigma \) also denote the corresponding \( \mathbb{Z}[z] \)-epimorphism onto \( \mathcal{C}(M) \).) The surjectivity of the homomorphism \( \sigma \) is easily proved by induction on the number of components of links. It follows from the simple observation that smoothing a mixed crossing reduces this number.

A set of generators \( \mathfrak{S} \) of a module \( H \) is minimal, if for each \( g \in \mathfrak{S} \) the set \( \mathfrak{S} \setminus g \) does not generate \( H \). The image of \( \sigma \) is a minimal set of generators of \( \mathcal{H}(M) \) (and similarly for \( \mathcal{C}(M) \) and \( \mathcal{L}(M) \)). This can be proved by applying the ring homomorphism \( q \mapsto 1 \), \( z \mapsto 0 \). It maps \( \mathcal{H}(M) \) onto the free abelian group generated by the set of oriented links in \( M \) with relations \( K_+ = K_- \) for all crossings. But this free abelian group is isomorphic to \( \mathbb{Z}\tilde{\pi}(M) \).

Let \( \hat{M} \) be the 3-manifold defined from \( M \) by capping off all 2-spheres in the boundary by standard 3-balls. It is well-known [P-1] that the inclusion \( M \to \hat{M} \) induces isomorphisms of skein modules. A 3-manifold \( M \) is atoroidal [J-S] if each map of a torus \( S^1 \times S^1 \to M \), which induces an injective homomorphism of fundamental groups, can be homotoped into the boundary of \( M \). Let \( b_1(X) \) denote the first Betti number of a complex \( X \).

**Theorem 1.1.**

a) \( \mathcal{H}(M) \) is free if and only if \( \pi_1(M) \) is abelian and \( 2b_1(M) = b_1(\partial M) \). Then \( \sigma \) is an isomorphism. Otherwise \( \mathcal{H}(M) \) has torsion.
b) If $\pi_2(\hat{M}) = 0$ and $M$ (or $\hat{M}$) is atoroidal then $C(M)$ is free (and $\sigma$ is an isomorphism).

Theorem 1.1 generalizes the results of J. Przytycki [P-3] and J. Hoste and J. Przytycki [H-P] for the case $M = F \times I$ and $F$ a compact oriented surface. Our proofs do not rely on their methods or results but are derived by conceptually new and independent ideas. In particular we prove freeness of $C(M)$ for all 3-manifolds with complete hyperbolic structures. (It is known that the modules $C(M)$ are free if and only if they are torsion-free [K-1].) It also follows that q-homotopy skein modules of Lens spaces are free as conjectured by J. Przytycki in [P3].

The oriented intersection number between oriented closed surfaces and loops defines the intersection pairing (see [K-2] for details):  
\[ \iota : H_2(M) \otimes H_1(M) \rightarrow \mathbb{Z}. \]

Let $LM$ denote the free loop space of $M$ and let $f : S^1 \rightarrow M$ be a basepoint. Elements of $\pi_1(LM, f)$ can represented by maps $S^1 \times S^1 \rightarrow M$. The map $\iota$ composes with natural maps $\pi_1(LM, f) \rightarrow H_2(M)$ and $b(M) \rightarrow H_1(M)$ (adding the homology classes of the elements of a sequence) to define  
\[ \iota_f : \pi_1(LM, f) \times b(M) \rightarrow \mathbb{Z}. \]

Let $f$ represent some $a \in \hat{\pi}(M)$, which appears in $\alpha$. Let $\alpha \setminus a$ be defined by omitting some element $a$ from the sequence. Then let $\iota(\alpha, a)$ be the positive generator of the subgroup $\iota_f(\pi_1(LM, f), \alpha \setminus a) \subset \mathbb{Z}$. Let $\lambda(\alpha)$ be the greatest common divisor of all $\iota(\alpha, a)$ for all $a$ in $\alpha$.

**Theorem 1.2.** The module $\mathcal{L}(M)$ is isomorphic to  
\[ \bigoplus_{\alpha \in b(M)} \mathbb{Z}[q^{\pm 1}] / (q^{2\lambda(\alpha)} - 1) \mathbb{Z}[q^{\pm 1}] . \]

The structure of $\mathcal{L}(M)$ can be interpreted as a statement about linking numbers. In fact, by 1.2 we can define linking numbers (relative to the choice of standard links) $\ell_\alpha : \omega^{-1}(\alpha) \rightarrow \mathbb{Z}/\lambda(\alpha)\mathbb{Z}$ satisfying the property: $\ell_\alpha$ changes by $\pm 1$ through $\pm 1$-crossings of different components of a link. For given $\alpha$, the linking number is defined in $\mathbb{Z}$, if and only if intersection numbers of singular tori, defined by self-homotopies of a component of $\alpha$, with the remaining components of $\alpha$, vanish. It follows that linking numbers are globally defined, or equivalently, $\mathcal{L}(M)$ is free if and only if singular tori in $M$ are homologous into the boundary of $M$ (see also [K-2]).

The results above will be deduced from a general presentation for homotopy skein modules, which we describe now. A set of standard links $\{K_\alpha\}$ is geometric if the following conditions hold for each $\alpha$:

1. $K_\alpha$ is an oriented embedded link in $M$.  


2. Each sublink of $K_\alpha$ is a standard link.

3. If $a^{\pm 1} \in \hat{\pi}(M)$ appears $k$ times in $\alpha$ then $K_\alpha$ contains $k$ parallel copies of a knot in $M$ with the suitable orientations. (Here $a^{-1}$ is the inverse conjugacy class, and parallel refers to some framing).

4. The sublink of null-homotopic components of $K_\alpha$ is a link with unknotted and unlinked components contained in a 3-ball in $M$.

It is not hard to prove that each 3-manifold admits a geometric set of standard links. In fact, in a handle-body such a set can easily be constructed by using monotonicity with respect to some $I$-structure. For general $M$ choose a Heegaard decomposition. Then consider a suitable subset of a set of geometric standard links for the skein module of one of the handle-bodies $F$ of the decomposition. The subset can be defined from a section of the surjection $\pi(F) \to \hat{\pi}(M)$.

Given a geometric set of standard links we choose orderings and basings of its components. The basings are paths, which join $\ast \in M$ with points on the components. Because of condition 2 in the definition above we can chose the basings compatible up to isotopy with sublinks. Thus in particular we have chosen a section of the projection map $\pi_1(M) \to \hat{\pi}(M)$ and an ordering of the sequences in $b(M)$, which will apply to all following constructions.

\[ \Delta - \text{Construction.} \] Let $c = (a, b)$ be a pair of elements of $\pi_1(M)$, which appears in $\alpha$ (in some order). Let $K^c_\alpha$ be the oriented link, which results from $K_\alpha$ by connecting two components with homotopy classes $a, b$ by a band, which follows the given basings. We can change the homotopy class of the band by a loop in $\ast$ representing some $g \in \pi_1(M)$. This defines a new banded link $K^c_{\alpha, g}$.

The difference $K^c_{\alpha, g} - K^c_\alpha$ can be expanded (not uniquely) in terms of standard links with less components than $K_\alpha$. Let $\Delta(c, a, g) \in SR\hat{\pi}(M)$ be some element, which results from such an expansion. Let $\Delta(c, a, g) := z(q^{-1} - q)\Delta(a, c, g)$.

\[ \Theta - \text{Construction.} \] Let $a \in \pi_1(M)$ appear in $\alpha$ and let $f : S^1 \to M$ be a parametrized component of $K_\alpha$ with homotopy class $a$. Each element $h \in \pi_1(LM, f)$ defines a self-homotopy $f_t$ of $f$ in $M$ (which defines a singular torus in $M$). We can assume that $f_t$ intersects $K_{\alpha, a}$ for only a finite number of $t$ transversely and in a single point for each $t$. Here $K_{\alpha, a}$ is the link, which results by forgetting a component with homotopy class $a$. Each intersection of $f_t$ with $K_{\alpha, a}$ comes naturally with a sign and a smoothing thus defining a linear combination of links. We can expand each term in this linear combination in terms of standard links, again not uniquely of course, to define some element $\Theta(a, a, h) \in SR\hat{\pi}(M)$.

**Theorem 1.3.** Let $\sigma$ define a geometric set of standard links. Then the kernel of $\sigma$ is generated by the elements $\Delta(c, a, g)$ and $\Theta(a, a, g)$, which result from the $\Delta$/$\Theta$-construction.

By 1.3, the relations of the homotopy skein module are parametrized by copies of the fundamental group of $M$ and of its free loop spaces (singular
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tori in $M$). This reduces problem 2 of the introduction to questions about curves and singular tori in 3-manifolds. Though the homotopy classification of singular tori in 3-manifolds is well-understood (see e. g. [J-S]), explicit problems about skein modules can still be difficult to handle. In [K-4] we discuss how torsion problems of homotopy skein modules are interpreted in terms of non-commutative intersection maps.

Theorem 1.3 will be deduced from a skein theory of paths in mapping spaces with restricted singularities, which is the central point of this paper.

Here is the plan for the rest of the paper:

In section 2 we state two refined versions of theorem 1.3: Presentations for homotopy skein modules defined from restricted sets of links, and presentations of $H(M)$ as module over the homotopy skein algebra of $S^3$. The first presentations are interesting from the viewpoint of a study of link homotopy invariants in 3-manifolds. In section 3 we state the necessary background about mapping spaces related with the Lin approach and show how the ideas apply to skein theory. In section 4 we describe our basic construction of a skein theory of paths. This is applied in section 5 to prove theorem 1.3. In section 6 we develop some homotopy theory of free loop spaces. In section 7 we show that relations given by the $\Delta/\Theta$-construction vanish for many situations. This allows to conclude theorems 1.1 and 1.2 in section 8.

2. THE REFINED PRESENTATION RESULTS

Following [K-1] we introduce homotopy skein modules over restricted sets of links. Let $K = K_+$ be a link with $\omega(K_+) = \alpha$. Then we describe the wrapping invariant of a smoothing $K_0$ in the following way: Choose an ordering and basing of the components of $K$. This induces an ordering of $\omega(K) = \langle \alpha_1, \alpha_2, \ldots, \alpha_r \rangle$ to a sequence $(a_1, a_2, \ldots, a_r)$ of elements of $\pi_1(M)$. Consider the ball in $M$, where two components of $K$ are smoothed. Let us assume that these are the first two components and the overcrossing arc comes from the first component. Choose a path $\gamma$, which joins $\ast \in M$ to a point in the boundary of the 3-ball. The homotopy class of the smoothed component (with new basepoint on the right arc of the smoothed link in the ball) is given by $g_1 a_1 g_1^{-1} g_2 a_2 g_2^{-1}$ for suitable elements $g_i \in \pi_1(M)$ and $i = 1, 2$. E. g. the element $g_1$ is defined by juxtaposing $\gamma$ with a path in the smoothing ball to the crossing point, then with the segment on $K_1$ from the crossing point on $K_1$ to the basepoint $\ast_1$ on $K_1$, and finally with the basing arc for $K_1$ from $\ast_1$ back to $\ast$. The corresponding wrapping invariant

$$\langle (a_1 g a_2 g^{-1})^\circ, \alpha_3, \ldots, \alpha_r \rangle$$

with $g := g_1^{-1} g_2$, is called a (first order) descendant of $\alpha$. Here $a^\circ$ denotes the conjugacy class of $a \in \pi_1(M)$. By iteration this defines the set of descendants of $\alpha$. 
A subset \( C \subset \mathfrak{b}(M) \) is called skein closed if it contains the descendants of its elements. The skein closure \( \bar{C} \) of \( C \) is the intersection of all skein closed sets containing \( C \). Let \( \mathcal{H}_C(M) := \omega^{-1}(C) \). We define the \( q \)-homotopy skein module \( \mathcal{H}_C(M) \) to be the quotient of \( R\mathcal{H}_C(M) \) by the submodule generated by all skein relations \( q^{-1}K_+ - qK_- - zK_0 \) for all \( K_+ \in \omega^{-1}(\bar{C}) \). It follows from the definitions that \( \sigma \) restricts to epimorphisms \( R\bar{C} \to \mathcal{H}_C(M) \).

We need further notation. Let \( \mathfrak{b}_r(M) \) denote the set of elements \( \alpha \in \mathfrak{b}(M) \) of fixed length \( r =: |\alpha| \). The unique sequence of length 0 is denoted \( \langle \rangle \). For \( \alpha \in \mathfrak{b}_r(M) \) and a subsequence \( \alpha' \in \mathfrak{b}_s(M) \) let \( \alpha \setminus \alpha' \in \mathfrak{b}_{r-s}(M) \) be the sequence defined by eliminating the elements of \( \alpha' \) from \( \alpha \). For \( C \) skein closed let \( C^* \) be the set of those sequences, which result from adjoining arbitrary trivial conjugacy classes to sequences of \( C \). The sets \( C^* \) are obviously skein closed. Let \( t(M) := \langle \rangle \) be the set of all sequences with only trivial conjugacy classes.

For \( g \in \pi_1(M) \) let \( \langle g \rangle \) denote the cyclic subgroup generated by \( g \). Given \( a, b \in \pi_1(M) \), let \( D(a, b) \) denote the set of non-trivial double left \( \langle a \rangle \) and right \( \langle b \rangle \) cosets of elements of \( \pi_1(M) \). For \( a \in \pi_1(M) \) let \( T(a) \) denote a set of generators of \( \pi_1(LM, f) \) for \( f \) a parametrized component of standard links representing \( a \). For \( \alpha \in C \), and \( (a, b) \) resp. \( a \) contained in \( \alpha \), the \( \Delta \) resp. \( \Theta \)-constructions define maps \( D(a, b) \to SR\bar{C} \) resp. \( T(a) \to SR\bar{C} \). (It will be shown in section 5 that the \( \Delta \)-construction is defined on cosets as above.) For each \( \alpha \in C \) we define the structure set:

\[
S(\alpha) := \bigcup_a T(a) \cup \bigcup_{(a,b)} D(a, b),
\]

with the union over all distinct \( a \) in \( \alpha \) resp. distinct pairs \( (a, b) \) in \( \alpha \) with both \( a, b \) non-trivial, such that \( \alpha \setminus a^\circ \) resp. \( \alpha \setminus (a^\circ, b^\circ) \) are not contained in \( t(M) \).

The maps so defined combine to define

\[
\chi_\alpha: S(\alpha) \longrightarrow SR\bar{C}
\]

by \( \chi_\alpha(g) = \Delta(\alpha, (a, b), g) \) for \( g \in D(a, b) \), and \( \chi_\alpha(h) = \Theta(\alpha, a, h) \) for \( h \in T(a) \).

So we have defined the homomorphism

\[
\chi = \sum_{\alpha \in C} \chi_\alpha : \bigoplus_{\alpha \in C} RS(\alpha) \longrightarrow R\bar{C}.
\]

The definition involves choices of expansions of links and is not defined in a unique way.

**Theorem 2.1.** Let \( C \) be a subset of \( \mathfrak{b}(M) \) and let \( \sigma \) define a geometric set of standard links. Then the sequence

\[
\bigoplus_{\alpha \in C} RS(\alpha) \longrightarrow^{\chi} R\bar{C} \longrightarrow^{\sigma} \mathcal{H}_C(M) \longrightarrow 0
\]

is exact for all choices of \( \chi \) determined by the \( \Delta/\Theta \)-construction.

For \( C = \mathfrak{b}(M) \) theorem 2.1 reduces to 1.3.
Example. For each 3-manifold $M$ the module $H_{\hat{\Delta}}(M)$ is free. On the other hand it is not known whether in general the image of $R\hat{\Delta}(M)$ in $\hat{H}(M)$, which is also the natural image of $H(D^3)$ in $\hat{H}(M)$, is free.

Remark 2.2. The above presentations are not always natural with respect to oriented embeddings of 3-manifolds $j : M \rightarrow N$. Let $j_* : \hat{H}(M) \rightarrow \hat{H}(N)$ be the homomorphism induced by $j$. Let $\sigma_M$ and $\sigma_N$ be any choices of standard links. Then there is a homomorphism $c(\sigma_M, \sigma_N)$, which makes the following diagram commute:

$$
\begin{array}{c}
SR\hat{\pi}(M) \xrightarrow{\sigma_M} \hat{H}(M) \\
\downarrow c(\sigma_M, \sigma_N) \quad \downarrow j_* \\
SR\hat{\pi}(N) \xrightarrow{\sigma_N} \hat{H}(N)
\end{array}
$$

The homomorphism $c(\sigma_M, \sigma_N)$ expands the standard links given by $\sigma_M$ in terms of standard links from $\sigma_N$. But in general this homomorphism is not induced from the corresponding induced maps on fundamental groups and sets of conjugacy classes. On the other hand, let $j^* : \hat{\Delta}(M) \rightarrow \hat{\Delta}(N)$ be injective. The $j$ induces injective maps $j_* : b(M) \rightarrow b(N)$ and, for $\alpha \in b(M)$, injective maps $j_* : S(\alpha) \rightarrow S(j_*(\alpha))$. Finally we have induced homomorphisms

$$
\begin{array}{c}
\bigoplus_{\alpha \in C} RS(\alpha) \xrightarrow{\bigoplus_{\beta \in (j(C))} RS(\beta)}
\end{array}
$$

for $C \subset b(M)$ skein closed. In this case it is easy to choose geometric standard links such that $c(\sigma_M, \sigma_N) = j_*$. There is the commutative diagram for all sets $C \subset b(M)$ and $j_*(C) \subset D \subset b(N)$:

$$
\begin{array}{c}
\bigoplus_{\alpha \in (C)} S(\alpha) \xrightarrow{x_M} SR\hat{C} \xrightarrow{\sigma_M} \hat{H}(C)(M) \xrightarrow{j_*} 0 \\
\downarrow j_* \quad \downarrow j_* \\
\bigoplus_{\beta \in (D)} S(\beta) \xrightarrow{x_N} SR(D) \xrightarrow{\sigma_N} \hat{H}(D)(N) \xrightarrow{j_*} 0
\end{array}
$$

This applies in particular to $C = b(M)$ and $D = b(N)$, but also to $M = N$ and $C \subset D$.

It is immediate from 2.1 that $\hat{H}(M)$ is a free module for each homotopy 3-sphere (by 6.1. or [L], for $M$ a homotopy sphere, $\pi_1(LM, f)$ is the trivial group for each map $f : S^1 \rightarrow M$). For $S^3$ the result has been proven by J. Przytycki [P-3]. Moreover, $\sigma$ is an isomorphism and $\hat{H}(S^3) \cong \hat{H}(D^3)$ is isomorphic to the polynomial algebra:

$$
\mathcal{R} := \mathbb{Z}[q^{\pm 1}, z, u],
$$

the multiplication defined by stacking links.

We want to describe $\hat{H}(M)$ as module over $\hat{H}(D^3)$.
Lemma. The inclusion of an oriented 3-ball in $M$ induces the canonical map
\[ \mathcal{S}(D^3) \times \mathcal{S}(M) \to \mathcal{S}(M), \]
which defines the structure of a $\mathcal{R}$-module on $\mathcal{H}(M)$, i.e. a homomorphism of $R$-modules:
\[ \mathcal{H}(D^3) \otimes \mathcal{H}(M) \to \mathcal{H}(M). \]

Proof. We can replace $M$ by $M \setminus \text{int}(D^3)$. Then define the map of link homotopy sets by forming the union of $D^3$ and $M \setminus \text{int}(D^3)$. The homomorphism of $R$-modules is deduced in the obvious way. Since any two 3-balls in $M$ are isotopic the choice of 3-ball does not change the structure of module. \( \square \)

Remark. The $\mathcal{R}$-module $\mathcal{H}(M)$ is $\mathcal{R}$-isomorphic to the following skein module: The generating set is given by the set of all link homotopy classes of oriented links in $M$ and the relations are given by

i) $q^{-1}K_+ - qK_- - zK_0$ for all skein triples, and

ii) $tK - (K \sqcup U)$, where $U$ is the unknot contained in a 3-ball in $M$ separated from $K$.

Recall that the sublink of trivial components of each link $K_\alpha$ is a standard unlink contained in a 3-ball in $M$. Thus $K_\alpha$ is the product of this unlink and some standard link $K_\alpha'$ with only non-trivial free homotopy classes in $\alpha'$. It follows that the $\mathcal{R}$-module $\mathcal{H}(M)$ is generated by standard links indexed by $\hat{b}(M) = b(M) \setminus t(M)$.

Now we define the structure sets $\mathcal{S}_\mathcal{R}(\alpha)$ for the presentations over $\mathcal{R}$. Let $\mathcal{S}$ be a set of generators of $\pi_2(M)$. For $\alpha \in \hat{b}(M)$ let
\[ \mathcal{S}_\mathcal{R}(\alpha) := \mathcal{S} \cup \bigcup_a T(a) \cup \bigcup_{(a,b)} D(a,b), \]
where the unions are over all distinct $a$ in $\alpha$ and $|\alpha| \geq 2$ resp. distinct pairs $a,b$ of elements in $\alpha$ and $|\alpha| \geq 3$.

Theorem 2.3 Let $\sigma$ be a geometric set of standard links. Then the sequence of $\mathcal{R}$-modules:
\[ \bigoplus_{\alpha \in \hat{b}(M)} \mathcal{R}\mathcal{S}_\mathcal{R}(\alpha) \xrightarrow{\chi_{\mathcal{R}} = \sum_{\alpha \in \hat{b}(M)} \chi_{\mathcal{R}}(\alpha)} \mathcal{R}\hat{b}(M) \xrightarrow{\sigma} \mathcal{H}(M) \to 0. \]
is exact for all choices of $\chi_{\mathcal{R}}$ determined by the $\Delta/\Theta$-construction.

Proof. Let $C = \hat{b}(M)$ in theorem 2.1. Let $u$ act on the set $\hat{b}(M)$ by defining $u(\alpha_1, \ldots, \alpha_r) := (1, \alpha_1, \ldots, \alpha_r)$. Let $u^i$ act by composition for $i > 1$ and let $u^0 := \text{id}$. It follows that $\{u^i(\cdot) | i \geq 0\} = t(M)$, and $u(b_r(M)) \subset b_{r+1}(M)$. 

The action defines the $R$-module isomorphism $\mathcal{R}\hat{b}(M) \cong R\hat{b}(M)$. The set of relations in theorem 2.1 is indexed by the union of the sets

$$S(\alpha) = \bigcup_a T(a) \cup \bigcup_{(a,b)} D(a,b),$$

where the unions are over all distinct $a$ in $\alpha$ with $a \setminus a^0 \notin t(M)$ resp. all distinct pairs of non-trivial elements $(a,b)$ in $\alpha$ with $a \setminus \langle a^0, b^0 \rangle \notin t(M)$.

Consider the contributions from $T(1)$. These can be non-trivial only if $\alpha$ also contains a non-trivial conjugacy class. By definition $T(1)$ is a set of generators of the group $\pi_1(LM, f_1)$ for a null homotopic map $f_1 : S^1 \to M$, which represents a trivial component of $K_\alpha$. Consider a corresponding self homotopy of this trivial component of $K_\alpha$ to itself and the resulting map $f : S^1 \times S^1 \to M$ with $f|S^1 \times * = f_1$. We can change the map by homotopy (compare section 6) such that $S^1 \vee S^1 \to M$ maps into a circle (the image of $f(*) \times S^1$) without intersecting other components of $K_\alpha$. There exists an extension to a map of torus into that circle. The corresponding self homotopy of $K_\alpha$ will give no contribution since the singular torus can be assumed not intersecting with any other components of $K_\alpha$ by transversality. Now $\pi_2(M)$ acts on the set $\pi_1(LM, f_1)$ by changing maps on the top cell. So we can replace $T(1) = \mathcal{G}$ without changing the image of $\chi$. Consider the decomposition

$$S(\alpha) = U(\alpha) \cup \bigcup_{a \neq 1} T(a) \cup \bigcup_{(a,b)} D(a,b),$$

where $U(\alpha)$ is equal to $T(1)$ if $\alpha$ contains both a trivial conjugacy class and a non-trivial conjugacy class, and is empty otherwise.

Now reorder the set $\cup_\alpha S(\alpha)$ into $\cup_\alpha S_R^\prime(\alpha)$, where

$$S_R^\prime(\alpha) := U(\alpha) \cup \bigcup_{a \neq 1} T(a) \cup \bigcup_{(a,b)} D(a,b).$$

The unions are restricted by the same conditions on $\alpha$ and $a, b$ resp. $a$ as before. Let the homomorphism $\chi_R = \sum \chi_{R, \alpha}$ be defined on $\cup_\alpha S_R^\prime(\alpha)$ by the $\Delta/\Theta$-construction as before with values now in $SR\hat{\pi}(M)$.

The module $\oplus_{\alpha \in \hat{b}(M)} R S_R^\prime(\alpha)$ admits the canonical action of $u$ by shift of basis elements. Note that the non-triviality restrictions in the definition of $S_R^\prime(\alpha)$ are preserved under multiplication by $u$. Also for $\alpha \in \hat{b}(M)$ we have that $S_R(\alpha) = S_R^\prime(\alpha)$.

The main point is that the homomorphism $\chi_R$ is compatible with the actions, i.e. $\chi_{R, u\alpha} = u \chi_{R, \alpha}$ for all $\alpha$. This follows because of the definition of $\chi_R$ and condition 2 in the definition of geometric standard links. So we can replace the free $R$-module with basis $\cup_\alpha S_R^\prime(\alpha)$ by the free $R$-module with basis $\cup_\alpha S_R(\alpha)$. Obviously the restrictions that $\alpha \setminus a^0$ resp. $\alpha \setminus \langle a^0, a^2 \rangle$ are not in $t(M)$ for $\alpha \in \hat{b}(M)$ translate into $|\alpha| \geq 2$ resp. $|\alpha| \geq 3$ for $\alpha \in \hat{b}(M)$.

**Remark.** The proof shows that there are analogous presentations for $R$-modules $\mathcal{H}_{C^*}(M)$ and skein closed sets $C$. 
For later purpose we state the following simple result at this point.

**Lemma 2.4.** The natural map $\mathcal{H}(M) \to \mathcal{H}(M)$ induces the injective homomorphism of abelian groups $\mathbb{Z}\mathcal{H}(M) \to \mathcal{H}(M)$.

**Proof.** Consider the composition with the homomorphism $\mathcal{H}(M) \to S\mathbb{Z}\hat{\pi}(M)$ induced by the epimorphism $R \to \mathbb{Z}$, which is defined by $q \mapsto 1, z \mapsto 0$ (compare section 1). This composition is injective. □

### 3. HOMOTOPIES OF PATHS IN MAPPING SPACES

First we describe simple standard ideas from homotopy theory, which in fact are the basic ingredients for 2.1.

For $r \geq 1$ let $\tilde{\mathcal{M}}_r$ denote the space of differentiable maps $P = \bigsqcup_r S^1 \to M$. The space $\mathcal{M}_r$ is equipped with a topology in the following way: A map $X \to \tilde{\mathcal{M}}_r$ is continuous if the corresponding map $X \times P \to M$ is continuous. Let $\mathcal{M}_r$ denote the quotient of this mapping space by the permutation action of the symmetric group. Let $\mathcal{M} = \cup_{r \geq 1} \mathcal{M}_r$. A set of parametrized unordered standard links $K_\alpha$ can be considered as a set of base points in the components of this space. The link homotopy space $\mathcal{M}_{\ell h}$ is the subspace of those maps in $\mathcal{M}$ without intersections of different components.

For each $\alpha$ there is the exact homotopy sequence of pointed sets:

$$\pi_1(\mathcal{M}_{\ell h}, K_\alpha) \to \pi_1(\mathcal{M}, K_\alpha) \to \pi_1(\mathcal{M}, \mathcal{M}_{\ell h}, K_\alpha) \xrightarrow{\partial} \mathcal{H}_\alpha(M)$$

with $\partial$ surjective. By definition, $\pi_1(\mathcal{M}, \mathcal{M}_{\ell h}, K_\alpha)$ is the set of homotopy classes of paths in $\mathcal{M}$, which start at a point of $\mathcal{M}_{\ell h}$ (i.e. a link map) and end in the standard link $K_\alpha$. The action of $\pi_1(\mathcal{M}, K_\alpha)$ on $\pi_1(\mathcal{M}, \mathcal{M}_{\ell h}, K_\alpha)$ is transitive on the fibres $\partial^{-1}([f])$ for all $[f] \in \mathcal{H}_\alpha(M)$. If $\gamma_0$ is a loop in $K_\alpha$ and $\gamma$ is a path ending in $K_\alpha$ then the action is defined by juxtaposition. This defines a 1-1 correspondence between the orbit space and $\mathcal{H}_\alpha(M)$ and so the $R$-isomorphism (for each $C \subset b(M)$)

$$\bigoplus_{\alpha \in C} R(\pi_1(\mathcal{M}, \mathcal{M}_{\ell h}, K_\alpha) / \pi_1(\mathcal{M}, K_\alpha)) \xrightarrow{\partial} R\mathcal{H}_C(M).$$

By pull-back of the skew submodule $U_C(M)$ of skew triples of links with wrapping invariant in $C$ we derive the isomorphism:

$$\bigoplus_{\alpha \in C} R(\pi_1(\mathcal{M}, \mathcal{M}_{\ell h}, K_\alpha) / \pi_1(\mathcal{M}, K_\alpha)) / (\partial^{-1}(U_C)) \xrightarrow{\partial'} \mathcal{H}_C(M).$$

The relation sets $\partial^{-1}(U_C)$ are generated by all $q^{-1}\gamma_+ - q\gamma_- - z\gamma_0$. Here $\gamma$ is a homotopy class of paths in $\mathcal{M}$ (representing an element in $\pi_1(\mathcal{M}, \mathcal{M}_{\ell h}, K_\alpha)$ up to the action of $\pi_1(\mathcal{M}, K_\alpha)$). From the definition, $\gamma_\pm$ resp. $\gamma_0$ are any paths from $K_\pm$ resp. $K_0$ to $K_\alpha$ resp. $K_\alpha$. For any skew triples $K_\pm, K_0$ in $M$ ($\omega(K_0) = \alpha'$).
In section 5 we will use $\partial'$ to deduce the presentation 2.1 for the skein modules $\mathcal{H}_C(M)$. For this we need further notions and results about homotopies of paths in the mapping spaces. The following is due to X. S. Lin [L].

**Definition.** Let $j : P = \coprod_e S^1 \to M$ be a differentiable map.

(a) $j$ is called 1-generic if it is an immersion with at most a single doublepoint, which is rigid vertex (this means that the tangent vectors in the double point span a two-dimensional oriented subspace of the tangent space of $M$ in that point).

(b) $j$ is called 2-generic if it is either an immersion with at most two rigid vertex doublepoints or it is an embedding except in a single point, where the derivative vanishes.

Let $\tilde{M}_1$ resp. $\tilde{M}_2$ denote the subspaces of $\tilde{M}$ of 1-generic resp. 2-generic differentiable maps. Note that $\tilde{M}_1 \subset \tilde{M}_2$. Let $\tilde{M}_0$ denote the subspace of differentiable embeddings. We consider maps $j_i : V^i \to \tilde{M}_i$ for compact oriented $i$-dimensional manifolds $V_i$ and $i = 1, 2$. The singularity set $\mathcal{S}_i$ of such a map is defined by $\mathcal{S}_i := j_i^{-1}(\tilde{M}_i \setminus \tilde{M}_0)$.

**Definition.** Let $i \in \{1, 2\}$. A map $j_i : V^i \to \tilde{M}_i$ is called in almost general position if

(i) $\mathcal{S}_1$ is a finite collection of points in the interior of $V^1$, resp.

(ii) $\mathcal{S}_2$ is a compact immersed 1-manifolds with transverse double points and $\mathcal{S}_2 \cap \partial V^2 \subset j_2^{-1}(\tilde{M}_1 \setminus \tilde{M}_0)$, $j_2$ maps double points of $\mathcal{S}_2$ to immersions with two rigid vertex double points. Each boundary point of $\mathcal{S}_2$ in the interior of $V^2$ is mapped to a non-immersion in $\tilde{M}_2$. All embedded points, or boundary points of $\mathcal{S}_2$ in $\partial V^2$, are mapped into $\tilde{M}_1$.

Let $\mathcal{S}' \subset \mathcal{S}_2$ denote the set of non-embedding points or boundary points in the interior of $V^2$.

**Remarks.**

(a) The restriction of a map $V^2 \to \tilde{M}_2$ in almost general position to the boundary is a map $\partial V^2 \to \tilde{M}_1$ in almost general position.

(b) All differentiable maps in the image of a component of $V^i \setminus \mathcal{S}_i$ are isotopic embeddings, $i = 1, 2$.

(c) All differentiable maps in the image of a component of $\mathcal{S}_2 \setminus \mathcal{S}'$ are differentiable isotopic immersions with a single rigid vertex double point.

(d) A neighbourhood of an interior boundary point of $\mathcal{S}_2$ in $\mathcal{S}_2$ describes the shrinking of a kink in a component to a point (see [L]).

**Theorem 3.1.** (Lin [L]) Let $j_i : V^i \to \tilde{M}$ be a map, whose restriction to the boundary is in almost general position (in particular maps into $\tilde{M}_i \setminus 1$). Then $j_i$ can be approximated rel boundary by a map $j'_i : V^i \to \tilde{M}_i$ in almost general position.

We are only interested in that part of the singularity sets $\mathcal{S}_i$, which maps to immersions with only double points of different components. This defines submanifolds $\mathcal{D}_i \subset \mathcal{S}_i \subset V_i$. It follows from Lin’s work that the immersed
singularity manifold $D_2$ satisfies the following properties.

**Proposition 3.2.** Let $j_2 : V^2 \rightarrow \tilde{M}_2$ be a map in almost general position. Then the following holds:

a) The set $D_2$ is a properly immersed 1-dimensional manifold in $V^2$.

b) $j_2$ maps all points in a component of $V^2 \setminus D_2$ to link homotopic link maps (i.e. without singularities of different components, but with possible singularities of type $\tilde{M}_2$ of same components).

c) If $D'$ denotes the double point set of $D_2$ then all points in a component of $D_2 \setminus D'$ are mapped to immersions with a single double point of different components and possibly a further double point of same components. □

**Definition.** Let $j_2 : V^2 \rightarrow \tilde{M}_2$ be in almost general position. A point in $D'$ maps to an immersion with two double points of different components. We call such a point paired if both double points are intersection points of the same pair of components. Otherwise the point in $D'$ is called unpaired.

A map $V^i \rightarrow M$ is in almost general position if it lifts to a map $V^i \rightarrow \tilde{M}$ in almost general position. The definition of singularity sets $D_i$ and $D'$ obviously applies to all maps $V^i \rightarrow M$, which can be lifted.

Let $\mathcal{M}^*$ denote the set of those maps $P \rightarrow M$ with a single rigid vertex intersection point of different components and possible arbitrary self-intersections of components disjoint from the intersection point. Let $\tilde{\mathcal{H}}^1(M) := \pi_0(\mathcal{M}^*)$. We can obviously replace $\mathcal{M}^*$ by $\mathcal{M}^* \cap \mathcal{M}_2$ without changing the set of path components.

We prove a few results about link homotopy and skein theory related to the ideas above.

**Proposition.** Let $j : M \hookrightarrow N$ be a inclusion of oriented 3-manifolds. Assume that $j$ induces a bijection of link homotopy sets $j^0 : \tilde{\mathcal{H}}(M) \rightarrow \tilde{\mathcal{H}}(N)$ and a surjection $j^1 : \tilde{\mathcal{H}}^1(M) \rightarrow \tilde{\mathcal{H}}^1(N)$. Then the induced homomorphism $j_* : \mathcal{H}(M) \rightarrow \mathcal{H}(N)$ is an isomorphism.

**Proof.** The isomorphism $j_* : R\tilde{\mathcal{H}}(M) \rightarrow R\tilde{\mathcal{H}}(N)$ descends to an epimorphism $\mathcal{H}(M) \rightarrow \mathcal{H}(N)$, which is injective if $U_N = j_*(U_M)$ (note that $j_*(U_M) \subset U_N$ always holds). But this follows from the surjectivity of $j^1$. □

**Corollary.** Assume that $j_* : \tilde{\mathcal{H}}(M) \rightarrow \tilde{\mathcal{H}}(N)$ is bijective and $j$ induces an isomorphism of fundamental groups. Then the induced homomorphism of q-homotopy skein modules is an isomorphism. □

**Examples.** The inclusion $M \hookrightarrow \tilde{M}$ induces the isomorphism $\mathcal{H}(M) \rightarrow \mathcal{H}(N)$. Let $e \subset M$ be a (possibly) fake open 3-cell. It is known that $M \setminus e \hookrightarrow M$ induces the bijection $\tilde{\mathcal{H}}(M \setminus e) \rightarrow \tilde{\mathcal{H}}(M)$ for $M = S^3$ (see [H]). The corresponding result for $M \neq S^3$ is not known (the surjectivity part is obvious). In section 8 we will show that $\mathcal{H}(M \setminus e) \rightarrow \mathcal{H}(M)$ is an isomorphism.
4. THE BASIC CONSTRUCTION

For $\alpha \in b(M)$ let $P(\alpha)$ denote the set of paths in almost general position $\gamma : I \to M_1 \subset M$ in the component of $M$ determined by $\alpha$. In particular, $\gamma \in P(\alpha)$ implies $\gamma(0), \gamma(1)$ are embeddings and $\gamma(t)$ has free homotopy classes given by $\alpha$ and is not an embedding for only a finite number of points.

**Definitions 4.1.** Let $t_1, \ldots, t_k$ be the parameters for which $\gamma(t_i)$ is an immersion with a single intersection point of different components (the singularity parameters for link homotopy). Let $K_{t_i}$ be the result of smoothing the corresponding rigid vertex immersion in the usual way. We define $\varepsilon_i \in \{\pm 1\}$ by $\varepsilon_i = \pm 1$ if $K_{t_i-\delta} = K_\pm$ and $K_{t_i+\delta} = K_\mp$ for $\delta > 0$ small enough ($K_\pm$ are the links defined by the double point). Let $\varepsilon(\gamma) := \sum_{i=1}^k \varepsilon_i$ be the index and let $\sum_{i=1}^k |\varepsilon_i|$ be the length of $\gamma$.

**Remark.** By 3.2 the index of paths in almost general position is invariant under homotopies in $M$ with endpoints in $M_{th}$. This can be shown by lifting to a homotopy in the covering space $\tilde{M}$ and applying 3.1.

For $\alpha \in b(M)$ let $\lfloor \alpha \rfloor \subset b(M)$ denote the set of all first order descendants of $\alpha$ (compare section 2).

**Definition.** For each elementary path $\gamma$ with a single singularity point $t_1$ of index $\varepsilon_1$ define

$$s : P(\alpha) \to \mathbb{Z}[q^{\pm 1}]\delta_{\lfloor \alpha \rfloor}(M),$$

by $s(\gamma) := \varepsilon_1 q^{\varepsilon_1}K_{t_1}$. Extend the definition by the juxtaposition formula (assuming $\gamma_1(1) = \gamma_2(0)$)

$$s(\gamma_1 \gamma_2) := s(\gamma_1) + q^{2\varepsilon(\gamma_1)}s(\gamma_2),$$

where $\gamma_1 \gamma_2$ is the usual product of paths.

We need some elementary properties of the map $s$. Throughout, if not indicated otherwise, path means path in almost general position. Note that $s$ does not change if we pre- or post compose by a path (in almost general position) in $M_{th}$.

**Lemma 4.2.** For $\gamma$ a path with singularities $0 < t_1 < \ldots < t_k < 1$ of indices $\varepsilon_1, \ldots, \varepsilon_r$ the following holds:

$$s(\gamma) = \varepsilon_1 q^{\varepsilon_1}K_{t_1} + \varepsilon_2 q^{2\varepsilon_1+\varepsilon_2}K_{t_2} + \varepsilon_3 q^{2(\varepsilon_1+\varepsilon_2)+\varepsilon_3}K_{t_3} + \ldots + \varepsilon_k q^{2(\varepsilon_1+\ldots+\varepsilon_{k-1})+\varepsilon_k}K_{t_k}.$$

**Remark.** If $\gamma$ has non-trivial index then $s(\gamma) \neq 0$. Also the image of $s(\gamma)$ in $\mathcal{H}(M)$ cannot be trivial. In order to show this use 2.4. to see that $\varepsilon(\gamma)$ can be computed from the image in the homotopy skein module.

**Lemma 4.3.** Let $\gamma^{-1}$ the inverse path of $\gamma$. Then

$$s(\gamma^{-1}) = s(\gamma^{-1}) = 0.$$
Proof. This is proved by induction on the length of paths. □

The following is checked by direct computation.

**Lemma 4.4.**

a) For each path \( \gamma \): 
\[
s(\gamma^{-1}) = -q^{-2\varepsilon(\gamma)}s(\gamma)
\]

b) For two paths \( \gamma_1, \gamma_2 \) with the same initial and end points we have
\[
s(\gamma_2) - s(\gamma_1) = q^{2\varepsilon(\gamma_1)}s(\gamma_1^{-1}\gamma_2)
\]

c) Let \( \gamma_1, \gamma_2 \) be paths with \( \gamma_1(1) = \gamma_2(0) \), then
\[
s(\gamma_1\gamma_2) - s(\gamma_2\gamma_1) = (1 - q^{e(\gamma_2)})s(\gamma_1) - (1 - q^{e(\gamma_1)})s(\gamma_2) \quad □
\]

The map \( s \) is not always commutative with respect to composition. It is commutative, if \( \varepsilon(\gamma_1) = \varepsilon(\gamma_2) = 0 \).

An important consequence of 4.4 c) is the following result:

**Corollary 4.5.** Let \( \gamma_1 \) be a loop in \( \gamma_0(1) \). Then the following holds:
\[
s(\gamma_0\gamma_1\gamma_0^{-1}) = (1 - q^{2\varepsilon(\gamma_1)})s(\gamma_0) + q^{2\varepsilon(\gamma_0)}s(\gamma_1). \quad □
\]

If \( \varepsilon(\gamma_1) = 0 \) then conjugation by some path \( \gamma_0 \) (this is changing the base point) just amounts to multiplication by \( q^{2\varepsilon(\gamma_0)} \). In particular this applies when the base point on a loop with trivial index is moved on the loop.

The map \( s \) measures the **skein difference** between initial and end point of a path. In order to define absolute skein invariants we need to include the links \( \gamma(0) \) and \( \gamma(1) \) into the definition.

**Definitions 4.6.** Let \( \gamma \in \mathcal{P}(\alpha) \). Then we define
\[
s_f(\gamma) = q^{2\varepsilon(\gamma)}\gamma(1) + zs(\gamma) \in R\delta_\alpha(M)
\]
and
\[
s_{if}(\gamma) = q^{2\varepsilon(\gamma)}\gamma(1) - \gamma(0) + zs(\gamma) \in R\delta_\alpha(M).
\]

Let \( \mathcal{P}_\alpha \) be the set of paths in almost general position with end point in \( K_\alpha \). Recall that \( \pi_1(\mathcal{M}, \mathcal{M}_{th}, K_\alpha) \) is the set of homotopy classes of paths with end point \( K_\alpha \) and initial point in \( \mathcal{M}_{th} \).  

The map
\[
s_f : \mathcal{P}_\alpha \rightarrow R(\delta_{|\alpha}^j(M) \cup \{K_\alpha\})
\]
assigns to each link with wrapping invariant \( \alpha \) a linear combination of \( K_\alpha \) and links with fewer components. It describes the skein expansion of a link along a path \( \gamma \) in terms of a standard link and links with fewer components.

The properties of the map \( s \) translate to important properties of \( s_f \) and \( s_{if} \).

**Proposition 4.7.** Let \( \gamma_1, \gamma_2 \) with \( \gamma_1(1) = \gamma_2(0) \). Then
\[
s_{if}(\gamma_1\gamma_2) = s_{if}(\gamma_1) + q^{2\varepsilon(\gamma_1)}s_{if}(\gamma_2), \quad \text{and}
\]
\[ s_f(\gamma_1 \gamma_2) = s_f(\gamma_1) + q^{2\varepsilon(\gamma_1)} s_f(\gamma_2). \]

**Proof.** Both results are immediate by computation. For the second formula use that \( s_f(\gamma) = s_i f(\gamma) + \gamma(0). \)

**Remark 4.8.**

a) The second formula generalizes the skein relations \( K_+ = q^2 K_- + zqK_0 \) and \( K_- = q^{-2} K_+ - zq^{-1} K_0 \) in the following way: Let \( \gamma_1 \) be an elementary path with singularity parameter \( t_1 \) of index \( \varepsilon(\gamma_1) = \varepsilon_1 \). Then it follows from the above

\[ s_f(\gamma_1 \gamma_2) = s_f(\gamma_1) + q^{2\varepsilon(\gamma_1)} (s_f(\gamma_2) - \gamma_2(0)) \]
\[ = q^{2\varepsilon(\gamma_1)} s_f(\gamma_2) + q^{2\varepsilon(\gamma_1)} (\gamma_1(1) - \gamma_2(0)) + zs_f(\gamma_1) \]
\[ = q^{2\varepsilon_1} s_f(\gamma_2) + \varepsilon_1 zq^{\varepsilon_1} K_{t_1}. \]

The initial point of \( \gamma_1 \gamma_2 \) is a link \( K_{\pm} \), the initial point of \( \gamma_2 \) is \( K_{\mp} \) and the smoothing \( K_0 \) is \( K_{t_1} \).

b) The result holds in particular for all compositions with \( \gamma_1(1) = \gamma_2(0) = \gamma_2(1) \). Thus it describes the action of the loops in \( K_\alpha \) on paths joining links with \( K_\alpha \).

To combine the results of this section with section 3 define the homomorphism

\[ R\bar{C} \xrightarrow{\sigma'} \bigoplus_{\alpha \in \bar{C}} R(\pi_1(M, M_{th}, K_\alpha)/\pi_1(M, K_\alpha))/\tilde{\partial}^{-1}(U_C)), \]

by mapping the basis element determined by \( \alpha \in \bar{C} \) to the constant loop in \( K_\alpha \).

**Proposition.** The homomorphism \( \sigma' \) above is surjective.

**Proof.** Represent elements of \( \pi_1(M, M_{th}, L_\alpha)/\pi_1(M, K_\alpha) \) by paths in almost general position, which join a link \( K \) to the basepoint \( K_\alpha \). Then proceed by induction on the complexity \( \ell(\gamma) = (|\alpha|, \text{length of } \gamma) \) (see 4.1). If there are no singular parameters then \( \gamma \) can be homotoped into the constant map. Otherwise consider the first \( t_1 > 0 \) with \( \gamma(t_1) \in D_1 \). Let \( \gamma = \gamma_1 \gamma_2 \), where \( \gamma_1 \) is an elementary path starting in \( K \) and ending in \( K_{t_1+\delta} \) for \( \delta > 0 \) sufficiently small. Choose a path \( \gamma_0 \), which joins \( K_{t_1} \) to its standard link. The triple \( \gamma, \gamma_2, \gamma_0 \) is a skein triple of paths. So we can replace \( \gamma \) by paths of smaller complexities \( \gamma_2, \gamma_0 \).

5. PROOF OF THEOREM 2.1.

We can assume that \( C \) is a skein closed subset of \( b(M) \). The set \( C \) decomposes into the union \( C = \cup_{i \geq 1} C(i) \), where \( C(i) \) is the set of all elements of \( C \) of length \( \leq i \).

The idea of the proof is to define a homomorphism

\[ \rho : \mathcal{H}_C(M) \longrightarrow RC/W \]
for the submodule \( W := \text{im}(\chi) \). This will be done such that

1. the composition

\[
RC \xrightarrow{\sigma} \mathcal{H}(M) \xrightarrow{\rho} RC/W
\]

is the natural projection (this proves \( \ker(\sigma) \subset W \)), and

2. \( \sigma(W) = 0 \) (or \( W \subset \ker(\sigma) \) for \( W \subset RC \)).

It follows that \( \ker(\sigma) = W \), which is the assertion of theorem 2.1.

The homomorphism \( \rho \) will be defined using the isomorphism \( \partial' \) from section 3: First define a skein invariant map on \( P_\alpha \). Then prove that it only depends on the link homotopy class of initial points of paths in \( P_\alpha \).

We will construct \( \rho \) and \( W \) inductively through the sequence of homomorphisms:

\[
\rho_r : \mathcal{H}(r)(M) \longrightarrow A(r)
\]

for the modules \( A(r) := RC(r)/W(r) \), where \( W(r) \) is the submodule of \( RC(r) \), which is generated by images of \( \chi_\alpha \) for all \( \alpha \in C(r) \). The result follows from the fact that \( \mathcal{H}(M) \) is the direct limit of the modules \( \mathcal{H}(r)(M) \) for \( r \to \infty \), which is obvious from the definitions.

Define \( \rho_1(K) = \rho_1(\gamma) := \alpha \in RC(1) \) for each knot \( K \) with \( \omega(K) = \alpha \) resp. path joining \( K \) with \( K_\alpha \). This is well defined and extends to the homomorphism

\[
\mathcal{H}(1)(M) \longrightarrow RC(1) =: A(1).
\]

There are no skein relations on this level, which need to be considered.

We assume inductively that we have defined

\[
\rho_{r-1} : \mathcal{H}(r-1)(M) \longrightarrow A(r-1).
\]

For links with wrapping invariant in \( C(r-1) \) we define \( \rho_r \) by composition of \( \rho_{r-1} \) with the natural homomorphism:

\[
j_{r-1,r} : A(r-1) = RC(r-1)/W(r-1) \to RC(r)/W(r) = A(r).
\]

We have to define \( \rho_r \) on \( \mathcal{H}(r)(M) \). Choose a path \( \gamma \) in almost general position, which joins \( K \) with \( K_\alpha \) for \( \alpha \in C(r) \). Consider

\[
s_f(\gamma) = q^{2z(\gamma)} K_\alpha + zs(\gamma) \in R(\mathcal{H}(r-1)(M) \cup \{K_\alpha\}) \subset R(\mathcal{H}(r-1)(M) \cup \{K_\alpha\}).
\]

Define the \( R \)-homomorphism \( p(\alpha) : R(\mathcal{H}(r-1)(M) \cup \{K_\alpha\}) \to R(C(r-1) \cup \{\alpha\})/W(r-1) =: A'(r-1) \)

by composition with \( \rho_{r-1} \) on \( \mathcal{H}(r-1)(M) \) (actually we first have to project this into \( \mathcal{H}(r-1)(M) \)), and by mapping \( K_\alpha \) to \( \alpha \). This is well defined since \( W(r-1) \) does not contain any terms involving \( K_\alpha \).
Finally let $\rho'(\alpha) := p(\alpha) \circ s_f : \mathcal{P}_\alpha \to A'(r-1)$. The next step is to change the module $A'(r-1)$ to the module $A(r)$ in such a way that the map $\rho'$ does not depend on the choice of path but only on the initial point.

We have to consider the following two ways of changing $\gamma$:

1. The path $\gamma$ is changed in its homotopy class in $\pi_1(\mathcal{M}, \mathcal{M}_{th}, L_\alpha)$
2. The path is changed by the action of $\pi_1(\mathcal{M}, K_\alpha)$.

**Claim.** The contributions of $\rho'(\alpha)$ defined by (1) resp. (2) are contained in $pr(\chi_\alpha(S(\alpha)))$, where $pr$ is the projection $R(C(r-1) \cup \{\alpha\}) \to A'(r-1)$.

Let $K \in H_\alpha(M)$ and $\gamma \in \mathcal{P}_\alpha$ with $\gamma(0) = K$. Then define

$$\rho_r(K) = pr' \circ p(\alpha) \circ s_f(\gamma)$$

for some path $\gamma \in \mathcal{P}_\alpha$ with $\gamma(0) = K$. Here we use the canonical projection $pr' : A'(r-1) \to A(r)$.

**Remarks.**

a) The contributions to $W$ from 1) resp. 2) are local resp. global obstructions in the construction of finite type link homotopy invariants of Homfly type [Ka-L].

b) If the independence of the choice of paths has been established then skein invariance is provided just by the very definition of the map $s_f$ and its skein property stated in 4.8).

c) Obviously $\sigma(W) = 0$ because $W$ is defined from terms $s_{if}(\gamma)$ for loops in $K_\alpha$.

i.) Discussion of the local contributions.

Now a homotopy of some representative path in $\pi_1(\mathcal{M}, \mathcal{M}_{th}, L_\alpha)$ can be split up in homotopies, which move the initial point in $\mathcal{M}_{th}$, and homotopies relative to the initial and end point. Obviously $s_f$ is invariant in the first case. So we only need to consider the second case. Let $\gamma'$ be a path homotopic to $\gamma$, both paths in almost general position. Consider the loop $\gamma^{-1} \circ \gamma'$ in $K_\alpha$. This loop is null-homotopic in $\mathcal{M}$ thus it bounds a disk in $\mathcal{M}$, which lifts to $\tilde{\mathcal{M}}$. So we can apply 3.1 and approximate the map of this disk into $\tilde{\mathcal{M}}$ by an map in almost general position and project this to $\mathcal{M}$. The singularity set $\mathcal{D}_2$ of this map of a disk (the set of points, for which we have immersions with double points of different components) is a properly embedded immersion by 3.2.

Now consider the deformation of $\gamma$ into $\gamma'$ defined by the disk map in almost general position. Consider the disk bounding the union of two intervals identified in their end points. Homotope the two bounding intervals into each other across the disk. By 3.2.c) the invariant $s(\gamma)$ can only change when we homotope the path across a point of $\mathcal{D}'$. In fact note that in the homotopy pairs of intersection points with $\mathcal{D}_2$ can appear or vanish if we move the path over a critical point of $\mathcal{D}_2$ with respect to a suitable height function. But the two contributions here are easily seen to cancel. We have to consider the situation in the picture
for $\varepsilon(\gamma_1) = \varepsilon(\gamma_2) \in \{-2,0,+2\}$. Up to cyclic permutation the indices of the singular points on the circle are given by $(+1,+1,-1,-1)$.

Using 4.7. we compute the difference $s_f(\gamma_0 \gamma_1 \gamma_0') - s_f(\gamma_0 \gamma_2 \gamma_0')$:

$$s_f(\gamma_0 \gamma_1 \gamma_0') = s_f(\gamma_0) + q^{2\varepsilon(\gamma_0)} s_f(\gamma_1) + q^{2\varepsilon(\gamma_0) + \varepsilon(\gamma_1)} s_f(\gamma_0'),$$

so

$$s_f(\gamma_0 \gamma_2 \gamma_0') - s_f(\gamma_0 \gamma_1 \gamma_0') = q^{2\varepsilon(\gamma_0)} (s_f(\gamma_2) - s_f(\gamma_1)) - q^{2\varepsilon(\gamma_0) + \varepsilon(\gamma_1)} s(\gamma_2^{-1} \gamma_1).$$

It follows that we can focus on computing the value of $s(\gamma_2^{-1} \gamma_1)$. By 4.5, a change of basepoint on the loop alters the contribution only by multiplication with a power of $q$.

Each point of $\mathcal{D}'$ is an immersion with two double points of different components. Thus we can compute the contribution $\Delta(L)$ from a loop in some link $K = K_{+}$ (the two lower indices corresponding to the two double points of the immersion $L$ in the crossing, the signs over the arrows indicating the indices):

$$K_{+} \xrightarrow{+1} K_{--} \xrightarrow{-1} K_{-+} \xrightarrow{-1} K_{++} \xrightarrow{+1} K_{+-}$$

By 4.2. we compute: $\Delta(L) = qK_{0-} - qK_{-0} - q^{-1}K_{0+} + q^{-1}K_{+0}$.

**Case 1.** Assume that the point in $\mathcal{D}'$ is *unpaired* (compare 4.3). Then the $+$ resp. $-$ crossings in the formula are still crossings of different components and the computation can be proceeded:

$$\Delta(L) = qK_{0-} - qK_{-0} - q^{-1}(q^2K_{0-} + qzK_{00}) + q^{-1}(q^2K_{-0} + qzK_{00}) = 0.$$  

**Case 2.** Assume that the point in $\mathcal{D}'$ is *paired*. Then we can simplify:

$$\Delta(L) = z(q - q^{-1})(K_{0+} - K_{+0}).$$

Those are local contributions, which are not trivial in general. These are precisely the terms, which are taken care of in the submodule $T'$. We have to show that $j_{r-1,r} \circ \rho_{r-1}(\Delta(L)) = 0$. Note that $\rho_{r-1}$ is well defined on the links $K_{0+}, K_{+0} \in \mathcal{D}_{C(r-1)}(M)$.

Note that the links $K_{+0}, K_{0+}$ result from $K = K_{++}$ by attaching bands to $K$, possibly in different ways. Assume that the crossings, which are smoothed, are crossings of the first two components of $K$. Consider a path in almost general position joining $K$ and $K_{0}$. This path can be followed by a homotopy of the
bands in the complement of the last \( r - 2 \) components of \( K \), with the endpoints of the bands possibly moving in the first two components. This follows from transversality. In fact all crossing changes of components can be avoided just by moving the bands aside or the band is isotoped in \( M \) with the link. So the path above also provides two paths, which join \( K_{+0} \) resp. \( K_{0+} \) with two links resulting from \( K_\alpha \) by attaching bands. Consider the crossing changes for the two deformations. This means applying e.g. relations

\[
K_{0+,+} = q^2 K_{0+,-} + qz K_{0+,0} \\
K_{+,0+} = q^2 K_{+,0-} + qz K_{+,0+}
\]

(and similarly for \( K_{0+,-} \mapsto K_{+,0+} \)). So the difference terms will still be a linear combination of terms \( K'_{0+} - K_{0+,0} \) for links of \( \leq r - 2 \) components resulting from the construction above on links with \( r - 1 \) components. Note that crossing changes of the first two components are not contributing anything. So the additional contributions are already taken care of in the induction.

Finally we have to show that the contribution \( \Delta(L) \) does not depend on the explicit choice of bands. Consider a homotopy of the bands relative to the attaching arcs. Of course crossings of the band with any of the first two components do not contribute. Consider the crossing of a band with any of the last \( r - 2 \) components. The resulting difference in the expansion of \( K_{0+} - K_{+,0} \) is easily computed. In fact, at such a crossing we have to consider e.g. \( K' := K_{0+} \) the change from \( K'_{+,+} \) to \( K'_{+,0} \) (these are different crossings now).

We compute:

\[
K_{+,+} = q^2 K_{+,+} + qz K_{+,0} = K'_{+,+} + qz (K'_{+,+} - K'_{+,0}),
\]

where again we have smoothed a pair of components. The difference just appears in the form

\[
z(q - q^{-1})qz (K'_{+,+} - K'_{+,0})
\]

in \( \Delta(L) \). Since \( K' = K_{0+} \) has \( r - 1 \) components this term is already contained in the submodule \( W(r-1) \). The same argument works for \( K_{+,0} \). Thus we see that the band can be homotoped freely in any position with endpoints on the first two components. So only this homotopy class has to be considered. Also any kind of twisting or knotting or linking of the band with other components can be neglected by the arguments above.

We only have to account differences of expansions in terms of standard links resulting from attaching two bands to a standard link. So it suffices to consider all differences with respect to a fixed choice determined by the basing. Recall that the basing of the two first components determines a band (which runs from the first component to the basepoint along the basing arc of the first component, then along the basing arc of the second component from the basepoint to the second component). The basing is changed by composition with loops in \( * \). In this way all possible bands (up to homotopy) appear. Moreover, we can slide the band along the first component resp. along the second component. So we can change the group element by multiplication with powers of \( a \) from the left.
and powers of \( b \) from the right. For a more detailed consideration of basings and band operations see [K3]. So the \( \Delta \)-construction generates the module \( W'(r) \) of \( W(r) \) (modulo lower order contributions) of local contributions \( z(q - q^{-1})(K_{0+} - K_{+0}) \).

ii) Discussion of global contributions.

Up to this point we have defined a map:
\[
P_\alpha \to \mathbb{RC}(r)/(W(r - 1) + W'(r)),
\]
which now only depends on the image of \( \gamma \in P_\alpha \) in \( \pi_1(\mathcal{M}, K_\alpha) \). In order to define the map \( \rho_r \) we consider the change of \( s_f(\gamma) \) by the action of \( \pi_1(\mathcal{M}, K_\alpha) \).

By 4.7, for \( \gamma, \gamma' \in P_\alpha \) with \( \gamma'(0) = K_\alpha \):
\[
s_f(\gamma \gamma') = s_f(\gamma) + q^{\varepsilon(\gamma)} s_f(\gamma')
\]
So let \( W''(r) \) (mod \( W(r - 1) + W'(r) \)) denote the submodule generated by the images \( s_{i_f}(\gamma') \) for all loops in \( K_\alpha \) in almost general position.

Then theorem 2.1 follows if we can prove that \( W''(r) \) is generated by the \( \Theta \)-construction.

By 4.7 we only have to consider a generating set of \( \pi_1(\mathcal{M}, K_\alpha) \). Recall that \( \pi_1(\mathcal{M}, K_\alpha) \) is a subgroup of \( \pi_1(\tilde{\mathcal{M}}, K_\alpha) \) for some lift \( K_\alpha \) into the covering space. There are loops in \( \mathcal{M} \), which do not lift to a loop in the covering space. But such a lift is a path joining two different orderings \( K_\alpha \) and \( K''_\alpha \) of \( K_\alpha \). Obviously the corresponding permutation of components of \( K_\alpha \) can only permute components with the same homotopy classes. Now recall condition 3 in the definition of a geometric standard basis. We can always compose with a loop of embeddings in \( \mathcal{M}_0 \subset \mathcal{M} \), which permutes the components in arbitrary possible ways, and gives no contributions to \( s_f \) (use 4.7). Thus, in order to compute \( W''(r) \), we can restrict to the subgroup \( \pi_1(\tilde{\mathcal{M}}, K_\alpha') \), which can be easier described.

Free loop spaces are related with our problem because of the following well known result. See also the work of Lin [L] for the following considerations.

**Proposition.** For each map \( f : P = \coprod_i S^1 \to M \) there is a natural isomorphism of groups:
\[
\pi_1(\tilde{\mathcal{M}}, f) \to \times_{i=1}^r \pi_1(LM, f_i),
\]
where \( f_i \) is the restriction of \( f \) to the \( i \)-th component.

By 4.7 we can restrict to sets of generators. So the submodule \( W''(r) \) can be computed from the contributions of the groups \( \pi_1(LM, f_i) \) for components \( f_i \) of standard links \( K_\alpha \). We have to compute the image \( W''(r) \) in \( \mathbb{RC}(r)/(W(r - 1) + W'(r)) \) or the corresponding lift to \( \mathbb{RC}(r) \). Each element of \( \pi_1(LM, f_i) \) determines a homotopy class of loops in \( K_\alpha \) (after projecting to \( \mathcal{M} \)), defined by a self homotopy of a component with homotopy class \( f_i \). We choose a representative path \( \gamma \in P_\alpha \) and compute \( s_{i_f}(\gamma) \). The result corresponds to the definition of \( \chi \) by the \( \Theta \)-construction. Because of condition 3 for geometric
standard links, the contributions for different components but with the same homotopy class coincide.

6. EXPANSION OF RELATIONS

We need a workable algebraic description of the fundamental group of free loop spaces. The results for $\pi_2(M) = 0$ are in \[L\].

**Lemma 6.1.** Let $f : S^1 \to M$ be a map with homotopy class $[f] \in \pi_1(M)$. Then there is the exact sequence of groups

$$\pi_2(M) \xrightarrow{\pi_2^*} \pi_1(LM, f) \xrightarrow{i_1^*} Z([f]) \to 1,$$

where for each element $b \in \pi_1(M)$ we let $Z(b)$ denote the centralizer of $b$. The basepoint for the definition of $\pi_2(M)$ and $Z(b) \subset \pi_1(M)$ is $f(1)$.

**Proof.** The homomorphism $\pi_2^*$ is defined as follows: Consider the map $\tilde{f} : S^1 \vee S^1 \to M$, which restricts to $f$ on $S^1 \times *$ and is constant on $* \times S^1$. Consider the attaching map of the 2-cell $c : S^1 \to S^1 \vee S^1$. Then $\tilde{f} \circ c$ is canonically null-homotopic in $M$ (contracting the path determined by $f$ to the initial point). So each element of $\pi_2(M)$ determines a unique up to homotopy extension of $\tilde{f}$ over the attached 2-cell. Thus we have defined a map $S^1 \times S^1 \to M$, which represents an element of $\pi_1(LM, f)$. It is easy to see that this actually defines a homomorphism with image the kernel of $i_1^*$. Here $i_1^*$ is defined by composition of the natural map $\pi_1(LM, f) \to [S^1 \times S^1, M]$ with the map induced by the inclusion $* \times S^1 \subset S^1 \times S^1$. Here $[S^1 \times S^1, M]$ denotes the set of all based homotopy classes of maps $S^1 \times S^1 \to M$. The exactness follows by obstruction theory. □

For $\alpha \in b(M)$ and $f$ representing a component of $K_\alpha$ with homotopy class $a$, define

$$\iota_{f, \alpha} : \pi_1(LM, f) \to Z$$

by $\iota_{f, \alpha}(h) = \iota_f(h, \alpha \setminus a)$ (compare section 1). Let $f_1, \ldots, f_r$ be maps representing the components of $K_\alpha$. For $h \in \times_{i=1}^r \pi_1(LM, f_i)$ define

$$\varepsilon(h) = \sum_{i=1}^r \iota_{f_i, \alpha}(h_i).$$

The proposition in section 5 identifies $\varepsilon$ with the the index function on $\pi_1(M, K_\alpha)$ defined in 4.1.

A homomorphism $\varepsilon : G \to Z$ is called an **indexed group.** Let $A$ be a $Z[q^{\pm 1}]$-module and $\varepsilon : G \to Z$ be an indexed group. A map $\chi : G \to A$ is called a **homomorphism** if $\chi(1) = 0$ and $\chi(g_1g_2) = \chi(g_1) + q^{2\varepsilon(g_1)} \chi(g_2)$. For an indexed group $\varepsilon : G \to Z$ and $i$ a non-negative integer let $G^i := \varepsilon^{-1}(i)$.

**Lemma 6.2.** Let $\varepsilon : G \to Z$ be an indexed group and $\chi : G \to A$ be a homomorphism. Then $\chi$ restricts to a group homomorphism $\chi^0 : G^0 \to A$, where $G^0 := \varepsilon^{-1}(0)$. \[Q.E.D.\]
which factors factors through $G^0/[G^0, G^0]$. Here $[G^0, G^0]$ is the commutator subgroup.

**Lemma 6.3.** For each indexed group $G$ and exact sequence of groups:

$$
\begin{array}{c}
H \rightarrow^j G \rightarrow Q \rightarrow 1
\end{array}
$$

there exists the commutative diagram of groups:

$$
\begin{array}{c}
H^0 \rightarrow^j G^0 \rightarrow Q^0 \rightarrow 1 \\
\downarrow \subset \downarrow \subset \downarrow \subset \\
H \rightarrow^j G \rightarrow Q \rightarrow 1 \\
\downarrow \varepsilon \downarrow \bar{\varepsilon} \\
Z \rightarrow Z/\varepsilon \circ j(H)
\end{array}
$$

with the canonical isomorphism $G^0/[j([H^0])] \rightarrow Q^0$. Here $H^0 := \ker(\varepsilon \circ j)$, $\bar{\varepsilon}$ is the canonical homomorphism induced by $\varepsilon$ and $Q^0 = \ker(\bar{\varepsilon})$. Moreover, there is the exact sequence

$$
\begin{array}{c}
H^0 \rightarrow G^0/[G^0, G^0] \rightarrow Q^0/[Q^0, Q^0] \rightarrow 0
\end{array}
$$

**Proof.** This is proved by standard diagram chase arguments. □

We deduce the following consequence:

**Proposition 6.4.** Let $a \in \pi_1(M)$ be contained in $\alpha$. Then the module generated by $\chi(T(a))$ is also generated by the images of $\chi$ on generators of abelian groups

$$
\pi_2(M), Z(a)/Z(a)^0 \text{ and } Z(a)^0/[Z(a)^0, Z(a)^0].
$$

**Proof.** Apply 6.3 and 6.2 to the exact sequence of 6.1. Define $\chi$ on $Z(a)$ by a section of maps $Z(a) \rightarrow \pi_1(LM, f)$, which maps into $\pi_1(LM, f)^0$. □

Now we describe first order expansions of the elements $\Delta(\alpha, (a, b), g)$ and $\Theta(\alpha, a, h)$ from section 1 and compute important examples. The ordering and basing of $K_\alpha$ lifts $\alpha = \langle \alpha_1, \ldots, \alpha_r \rangle$ to $(a_1, \ldots, a_r)$. Then for each $g \in \pi_1(M)$ define

$$
\alpha(i, j; g) := \langle (a_i ga_j g^{-1})^\circ, \alpha_1, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_r \rangle.
$$

Assume that the based first based component of $K_\alpha$ has homotopy class $a \in \pi_1(M)$. For $2 \leq i \leq r$ and $h \in T(a)$ let $\varepsilon_i = \varepsilon_i(h)$ denote the intersection number of the map of the torus given by $h$ with the $i$-th component of $K_\alpha$. Note that $\varepsilon(h) = \sum_{i=2}^r \varepsilon_i(h)$.

**Proposition 6.5.** Let $|\alpha| = r$ and $h \in T(a)$ for some $a$ in $\alpha$ as above. Then we can choose the expansion in the $\Theta$-construction so that

$$
\Theta(\alpha, a, h) = (q^{2\varepsilon(h)} - 1)\alpha + z_{p_1}(h).
$$
Moreover, for \(2 \leq i \leq r\) there exist natural numbers \(n_i \geq 0\) and for \(0 \leq j \leq n_i\) numbers \(\varepsilon_{i,j} \in \{+1, -1\}\) with \(\sum_{j=0}^{n_i} \varepsilon_{i,j} = \varepsilon_i(h)\). Also there exist elements \(g_{i,j} \in \pi_1(M)\) and integer numbers \(\eta_{i,j}\) (all numbers and group elements depending on \(h\)) such that

\[
p_1(h) = \sum_{i=2}^{r} q^{2(\varepsilon_2 + \ldots + \varepsilon_{i-1})} \sum_{j=0}^{n_i} \varepsilon_{i,j} q^{2(\varepsilon_{i,j} + \ldots + \varepsilon_{i-1}) + \varepsilon_{i,j} + \eta_{i,j}} \alpha(1, i; g_{i,j}) + z\tilde{p}_1(h)
\]

for some \(\tilde{p}_1(h) \in SR\pi(M)\).

**Proof.** The self-homotopy of \(f\) with \([f] = a\) defines \(f' : S^1 \times S^1 \to M\). Assume that \(f'\) intersects \(K_{\alpha, \alpha}\) transversely. Recall that \(f'[S^1 \times * = f\). Homotope the inclusion \(S^1 \times * \subset S^1 \times S^1\) by pushing arcs successively across intersection points. First move arcs across intersections with the second component of \(K_{\alpha}\), then with the third component, and so on (the first component is deformed). Finally we have deformed \(S^1 \times *\) across all intersection points and can isotope the result back to the original inclusion without further intersections. The composition of this deformation with \(f'\) defines a self-homotopy of \(f\) homotopic to the given one. □

**Example 6.6.** Let \(M = S^1 \times S^1 \times S^1\) with \(\pi_1(M) \cong \mathbb{Z}^3\) generated by elements \(b_i, i = 1, 2, 3\). Represent \(M\) as quotient of \(I^3\) by identification of opposite sides. Let \(\alpha = \langle b_1, b_2, b_3 \rangle\). The standard link \(K_{\alpha}\) can be represented in \(I^3\) by intervals parallel to the faces of \(I^3\). Consider the self homotopy \(\tilde{f}_{\alpha}\), which is determined by \(h \in \pi_1(LM, f_1)\) with \([f_1] = b_1\) as follows: Keep fixed all but the first component with homotopy class \(b_1\). The first component crosses the \(b_2\)-component and then the link is isotoped back to the original standard link \(b_1\) and \(b_3\). We have \(\varepsilon(h) = \varepsilon_2(h) = 1\) and compute:

\[
\Theta(\alpha, a, h) = (q^2 - 1)\alpha + zq\langle b_1 b_2, b_3 \rangle.
\]

Now, after the deformation above, let the first component additionally crosses the third component with negative sign. Let \(h'\) be the resulting element of \(\pi_1(LM, f_1)\). The restriction of the corresponding map of the torus to \(S^1 \times S^1\) has homotopy classes given by \(b_1\) and \(b_3b_2^{-1}\). We have \(\varepsilon(h') = 0\) and \(\varepsilon_2(h') = 1\), \(\varepsilon_3(h') = -1\), and compute:

\[
\Theta(\alpha, a, h') = zq(\langle b_1 b_2, b_3 \rangle - \langle b_1b_3, b_2 \rangle).
\]

The element \(\sigma(\langle b_1 b_2, b_3 \rangle) - \sigma(\langle b_1b_3, b_2 \rangle)\) is a torsion element in \(H(S^1 \times S^1 \times S^1)\) by 2.4.

**\(\Theta\)-Construction for the image of \(\pi_1^*\):** Let \(f : S^1 \to M\) represent a component of \(K_{\alpha}\) and \([f] = a\). By the homomorphism \(\pi_1^* : \pi_2(M) \to \pi_1(LM, f)\) each element \(h \in \pi_2(M)\) defines a self homotopy of \(f\). Then the construction of \(\Theta(\alpha, a, \pi_1^*(h))\) for \(h \in \pi_2(M)\) can be described as follows: Let \(f' : S^2 \to M\) represent \(h\) with \(f'(*) = f(1)\). We can change \(f'\) by homotopy such that \(f'\)
restricts to \( f \) on some closed interval \( I \) containing \( * \) in its interior by using some homeomorphism \( j \) between \( I \) and the closed half circle \( I' \) containing \( 1 \in S^1 \). Now let \( H \) be a homotopy of the interval \( I \) across the 2-sphere fixing \( \partial I \) such that the image of the homotopy covers the 2-sphere with degree 1. This defines a self homotopy of \( f \) by \( f|I' = f' \circ H \circ j^{-1} \) and \( f \) is constant on \( S^1 \setminus I' \).

Recall the action of \( u \) on \( b(M) \) defined in the proof of 2.3.

**Proposition 6.7.** For \( \alpha \in b(M) \) and \( h \in \pi_2(M) \), the \( \Theta \)-construction can be chosen such that

\[
\Theta(\alpha, a, \pi_j^1(h)) = (u(q^{2\varepsilon(h)} - 1) + zq\frac{1 - q^{2\varepsilon(h)}}{1 - q})\alpha + z^2p_0(h)
\]

for some \( p_0(h) \in \mathcal{R}b(M) \).

**Example 6.8.** Let \( M = S^2 \times S^1 \) and let \( h \in \pi_2(M) \) be a generator. We represent \( h \) by the standard embedded sphere \( S := S^2 \times \{1\} \subset S^2 \times S^1 \). Let \( a \in \pi_1(M) \cong \mathbb{Z} \) be a generator and \( \alpha = \langle a, a, a^{-1} \rangle \). We assume that the ordering and basing of \( K_\alpha \) lifts \( \alpha \) to \( \langle a, a, a^{-1} \rangle \). We can assume that the first component intersects \( S \) in a closed interval in the complement of the two transverse intersection points of the last two components with \( S \). Then we apply the \( \Theta \)-construction for a self-homotopy of the first component. Note that \( \varepsilon(h) = 0 \).

We compute \( \chi(h) \) in this case. By expanding at the crossing points we find that the contributions add up to \( zq(K - K') \) for two links with \( \omega(K) = \langle a^2, a^{-1} \rangle \) and \( \omega(K') = \langle 1, a, a \rangle \). If we expand \( K, K' \) in terms of standard links we compute

\[
\Theta(\alpha, a, \pi_j^1(h)) = zq(q^{\eta_1} \langle a^2, a^{-1} \rangle - q^{\eta_2} \langle 1, a, a \rangle) + z^2\hat{p}_1(h)
\]

for numbers \( \eta_1, \eta_2 \in \mathbb{Z} \). It follows that \( S^2 \times S^1 \) has torsion.

Now we consider the \( \Delta \)-construction.

**Proposition 6.9.** Let \( \alpha \in b(M) \) and \( (a, b) \) a pair of elements (w. r. lifting the first two components of the ordered link) in \( \alpha \) and \( g \in \pi_1(M) \). Then there exists \( \delta(g) \in \mathbb{Z} \) and \( p_2(g) \in SR\hat{\pi}(M) \) (containing only elements of \( b(M) \) of length \( \leq |\alpha| - 2 \)) such that

\[
\Delta(\alpha, (a, b), g) = z(q^{-1} - q)(q^{2\delta(g)}\alpha(1, 2; g) - \alpha(1, 2; 1)) + z^2(q^{-1} - q)p_2(g).
\]

**Example.** Let \( M = S^1 \times S^1 \times S^1 \) and let \( b_i, i = 1, 2, 3 \) be the natural generators of \( \pi_1(M) \). Let \( \alpha = \langle b_1, b_1^{-1}, b_2 \rangle \) and \( c = \langle b_1, b_1^{-1} \rangle \). Then \( \alpha(1, 2; 1) = \alpha(1, 2; b_3) = \langle 1, b_2 \rangle \) because \( \pi_1(M) \) is commutative. We can choose the basings such that \( K_\alpha \) is the standard link with wrapping invariant \( \langle 1, b_2 \rangle \). The result of attaching the band determined by \( b_3 \) is a link, which can be deformed into the standard link through a single crossing change with the component of homotopy class \( b_2 \). We compute

\[
\hat{\Delta}(\alpha, c, b_3) = (1 - q^2)\langle 1, b_2 \rangle - qa\langle b_2 \rangle.
\]

The element \( (1 - q^2)\langle 1, b_2 \rangle - qa\langle b_2 \rangle \) is not trivial in \( \mathcal{H}(M) \). In order to show this apply the homomorphism \( z \mapsto 0 \) and consider the resulting element of
The result follows because \( \lambda(1, b_2) = 0 \) (compare theorem 1.2). In fact \( \iota_f(b, b_2) = 0 \) for \( b \in Z(1) \cong \pi_2(M) = 0 \) and \( f \) representing the trivial homotopy class, and \( \iota_f(b, 1) = 0 \) for all \( b \in Z(b_2) \) and \( f' \) representing \( b_2 \) (since intersections with a trivial homology class vanish). Thus \( \mathcal{H}(S^1 \times S^1 \times S^1) \) has torsion coming from the \( \Delta \)-construction, even though the fundamental group is abelian.

7. REDUCTION OF STRUCTURE SETS

We prove that \( \chi \) is trivial in certain situations. This appears in connection with cyclic and peripheral elements for both the \( \Delta \)- and \( \Theta \)-construction.

Throughout let \( T_1, \ldots, T_n \) be the tori in \( \partial M \). We choose paths from \( * \in M \) to fixed points in \( T_i \) for each \( i \), thus define homomorphisms \( j_* : \pi_1(T_i) \to \pi_1(M) \) for \( 1 \leq i \leq n \).

**Proposition 7.1.** Let \( M \) be a 3-manifold with \( \pi_1(M) \) cyclic. Then for all \( g \in D(a, b) \) and \( (a, b) \) a pair in \( \alpha \) we have \( \Delta(\alpha, (a, b), g) = 0 \).

**Proof.** Let \( T \to \text{int}(M) \) be an embedded solid torus, which induces an epimorphism of fundamental groups. Let \( K_\alpha \) be a link and let \( (a, b) \) be the homotopy classes of the first two components. Consider a band determined by \( g \) (compare the \( \Delta \)-construction). We only need to consider the arc (joining the two components) up to homotopy with endpoints on the components. So we can assume that the first two components are contained in \( T \) while all the other components are in \( M \setminus \text{int}(T) \). There could be intersections of different components during such a homotopy. But we can assume that the smoothings will contribute only terms, which have already been taken care of by induction, similar to the proof of 2.1 in section 5. The band will be homotoped along. Then we can homotope the band into \( T \), again intersections with the link can be neglected by the arguments from section 6. But since \( \pi_1(T) \) is abelian the results of the standard banding and the one determined by \( g \) are homotopic in \( T \). So the skein difference actually vanishes. In general the resulting link is not a standard link. But for both ways of smoothing it is the same link, so the expansions in terms of standard links can be assumed equal. \( \square \)

**Proposition 7.2.** Let \( M = S^1 \times S^1 \times I \). Then \( \Delta(\alpha, a, g) = 0 \) for all \( g \in D(a, b) \) and for all pairs \( (a, b) \) in \( \alpha \).

**Proof.** The idea is similar as in 8.1. We can assume that the two distinguished components and the attaching band are contained in \( S^1 \times S^1 \times [0, \frac{1}{2}] \) while the remaining components of \( K_\alpha \) are contained in \( S^1 \times S^1 \times [\frac{1}{2}, 1] \). Because \( \pi_1(S^1 \times S^1) \) is abelian, the links which result from attaching the bands are homotopic to each other in \( S^1 \times S^1 \times [0, \frac{1}{2}] \). The rest of the argument is similar to 7.1. \( \square \)

The two geometric observations above generalize to arbitrary structure sets.

1. Assume that \( a, b \in \langle h \rangle \) for some \( h \in \pi_1(M) \). Then let \( R_1(a, b) \) denote the set of cosets of elements \( h^i \) for all \( i \in \mathbb{Z} \).
2. Assume that $a$ or $b$ is conjugate to the image of a torus $T_i \subset \partial M$, e. g.
$h^{-1}ah \in j_*(\pi_1(T_i))$ for some $h \in \pi_1(M)$. Let $\Pi_i$ be the union of all
corresponding subgroups $h^{-1}j_*(\pi_1(T_i))h$. Then let $R_2(a, b)$ be the set of
cosets $\langle a \rangle g(b)$ with $g \in \Pi_i$ for some $1 \leq i \leq n$.

**Theorem 7.3** \(\chi_\alpha(R_1(a, b) \cup R_2(a, b)) = 0\) for all \(\alpha\) and \((a, b)\) in \(\alpha\).

**Proof.** For cosets from $R_1(a, b)$ we use the torus argument as before. We can
homotope both $a, b$ into the torus determined by $h$ as before. If the band is
determined by some element in $\pi_1(M)$, which can be homotoped into the torus,
then the argument follows 7.1. For $R_2(a, b)$ assume that $a$ is freely homotopic
into some boundary torus $T_i$ and let $h$ be the conjugating element. The band is
determined by some element in $\pi_1(M)$ and by assumption can be homotoped,
such that it runs inside a collar neighbourhood of the torus and otherwise follows
the standard band. The rest of the argument is like in 7.2. (use the basepoint
in $T_i$). \(\square\)

Next we consider $T(a)$ for some $a \in \pi_1(M)$. Again we can reduce in peripheral
and a cyclic situations.

**Proposition 7.4.** Let $M = F \times I$ for $F$ a compact oriented surface. Then
\(\chi_\alpha(T(a)) = 0\) for all $a$ in $\alpha$.

**Proof.** We can homotope singular tori into $F \times [0, \frac{1}{2}]$ while keeping the rest of
the link in $F \times [\frac{1}{2}, 1]$. In general such a homotopy of a singular torus is not a
homotopy of loops in $LM$ rel $f$. Let $\gamma$ denote a loop in $M$ in $K_\alpha$ in almost
general position, which represents the given element in $\pi_1(LM, f)$. Consider
the homotopy as a map $S^1 \times I \rightarrow M$, which is in almost general position on
the boundary. By 3.1 we can approximate by a map $S^1 \times I \rightarrow M_2$ in almost general
position. Moreover we can assume that the restriction to $\{1\} \times I$ is in almost
general position. Let $\gamma_0$ be the path defined by the restriction to $\{1\} \times I$ and
let $\gamma'$ be the loop in $\gamma_0(1)$, which is determined by the restriction to $S^1 \times \{1\}$.
We have to compare the two terms $s_{if}(\gamma)$ and $s_{if}(\gamma_0 \gamma' \gamma_0^{-1})$. Since there are no
singular parameters on $\gamma'$ we know that $s(\gamma') = 0$, in particular $\varepsilon(\gamma') = 0$. It
follows that also $\varepsilon(\gamma_0 \gamma' \gamma_0^{-1}) = \varepsilon(\gamma) = 0$. Because of 4.6 it suffices to consider
the difference between $s(\gamma)$ and $s(\gamma_0 \gamma' \gamma_0)$, which has been computed in 4.5. It
follows that $s_{if}(\gamma) = 0$ and so the assertion. \(\square\)

**Proposition 7.5.** Assume that $\pi_2(M) = 0$. Let $a \in \pi_1(M)$, $a = b^k$ for
some $k \in \mathbb{Z}$ and and $h \in Z(a) \cap \langle b \rangle$. Then $\chi(h) = 0$ (use the isomorphism
$Z(a) \cong \pi_1(LM, f)$ for $[f] = a$ from 6.1).

**Proof.** Find a knot in $M$ with homotopy class $b$. There is a regular
neighbourhood $T$ disjoint from the link $K_\alpha$. Because of the assumptions we can first
homotope the map of the torus restricted to its 1-skeleton $T$. But then because
of $\pi_2(M) = 0$ the map of torus is homotopic into $T$, because $T$ is disjoint from
other components of $K_\alpha$. Since $\pi_2(M) = 0$ the map of torus can be homotoped
into $T$. Then we can approximate the resulting map in $M$ by a map in almost
general position in $T$. The arguments from 7.4 also apply in this case. The
intersections with the components of $K_{\alpha,a}$ are trivial so $\chi(h) = 0$. □

Again the two ways of reductions generalize.

1. Note that $\langle a \rangle \subset Z^0(a)$. If $a = b^k$ for some $k \in \mathbb{Z}$ then let $\langle a \rangle_c$ denote the subgroup $Z(a) \cap \langle b \rangle$.

2. Assume that $a$ is conjugate to some element in $j_*(\pi_1(T_i))$ for some $1 \leq i \leq n$. Then let $g \in \pi_1(M)$ such that $gag^{-1} \in j_*(\pi_1(T_i))$ for some $i$, $1 \leq i \leq n$. Let $\Pi_i := gj_*(\pi_1(T_i))g^{-1} \subset Z^0(a)$, and let $\Pi$ be the product of the $\Pi_i$.

**Theorem 7.6.** The image of the homomorphism $\chi$ is generated by its images on generating sets of the three abelian groups:

$$\pi_2(M), \ Z^0(a)/(\langle a \rangle_c[Z^0(a), Z^0(a)]\Pi) \text{ and } Z(a)/Z^0(a).$$

**Proof.** By 6.3 it suffices to consider sets of generators of $\pi_2(M)$, $Z(a)/Z^0(a)$ and $Z^0(a)/(Z^0(a), Z^0(a)]$. By the argument from 7.5, $\chi$ is trivial on $\langle a \rangle_c$. A singular torus, which is determined by some element of $\Pi$, can be homotoped into a collar neighborhood of a boundary torus. The rest of the argument is analogous to the proof of 7.4. □.

8. _Proofs of Theorem 1.1 and Theorem 1.2_

For standard results about 3-manifolds we refer to [He].

Theorem 1.2 follows from 1.3 and the results of section 6. For the proof of 1.1 b), by the universal coefficient result [P-3] and right-exactness of the tensor product, it suffices to show $\chi(T(a)) = 0$, if $M$ is atoroidal and $\pi_2(M) = 0$. By [He] 9.13, $f_*(\pi_1(S^1 \times S^1))$ is cyclic or $\mathbb{Z}^2$ for each map $f$ of a torus in $M$. If it is cyclic then apply 7.5 or 7.6. If the torus map induces an injection of fundamental groups then by assumption the corresponding element $h \in Z(a)$ is contained in a subgroup $\Pi_i$, compare 7.6. So $\chi(T(a) = 0$ for all $a \in \alpha$. So we only have to prove theorem 1 a).

For each 3-manifold $M$ let $\mathcal{P}(M)$ denote the _Poincare associate_ of $M$, which is defined by first capping off all 2-spheres in the boundary of $M$ and then replacing a possibly fake 3-cell in $M$ by a standard 3-cell (w. r. w. we can assume that there is at most one fake 3-cell in $M$).

**Lemma.** For each 3-manifold $M$ there is a natural isomorphism

$$\mathcal{H}(M) \cong \mathcal{H}(\mathcal{P}(M)),$$

induced by inclusions.

**Proof.** The inclusions $M \hookrightarrow \hat{M}$ and $M \setminus e \hookrightarrow \mathcal{P}(M)$ ($e$ is the interior of a fake 3-cell in $M$) induce isomorphisms because of the same reason. We consider the inclusion $M \setminus e \hookrightarrow \hat{M}$. Since the induced map of fundamental groups is
an isomorphism it also induces bijections between the sets of conjugacy classes. So we can choose a geometric set of standard links in \( \hat{M} \setminus e \), which is also a set of geometric standard links for \( M \) (see 2.2). But band constructions and homotopies of singular tori can be assumed in \( \hat{M} \setminus e \). So the homomorphisms \( \chi \) for \( \hat{M} \setminus e \) and \( \hat{M} \) are compatible. The result follows by comparison of the presentation sequences in 2.2.  □.

Assume that \( \pi_1(M) \) not abelian. Let \( a, g \in \pi_1(M) \) be elements with \( aga^{-1}g^{-1} \neq 1 \) and \( \alpha := \langle a^\circ, (a^{-1})^\circ \rangle \). We can choose basings of the standard links such that \( a^\circ \) lifts to \( a \). Then \( \alpha(1, 2; 1) = 1^\circ \) and \( \alpha(1, 2; g) = (aga^{-1}g^{-1})^\circ \neq 1^\circ \). It follows from 1.3 and 2.4 that \( \sigma \) maps \( \tilde{\Delta}(\alpha, (a,a^{-1}), g) \) to a torsion element of \( H(M) \).

Assume that \( \pi_1(M) \) is abelian. Then by [He, 9.13], \( \pi_1(M) \) is cyclic or free abelian of rank 2 or 3. Moreover, if \( M \) is free abelian of rank 2 resp. 3 then \( \mathcal{P}(M) \) is is homeomorphic to \( S^1 \times S^1 \times I \) or \( S^1 \times S^1 \times S^1 \) (see [He 5.2 and 11.11]). If \( M \) is free abelian of rank 1 then \( \mathcal{P}(M) \) is homeomorphic to \( S^1 \times D^2 \) or \( S^1 \times S^2 \). Note that the condition \( 2h_1(M) = h_1(\partial M) \) (\( \Leftrightarrow M \) is submanifold of a rational homology 3-sphere [K-2]) is unchanged if fake 3-cells are replaced by standard 3-cells. In particular, among the 3-manifolds with \( \pi_1(M) \) abelian precisely those with finite cyclic fundamental group or \( \mathcal{P}(M) \) homeomorphic to \( S^1 \times D^2 \) or \( S^1 \times S^1 \times I \) are those, which are submanifolds of rational homology 3-spheres. Note that by the lemma above we can replace \( \hat{M} \) by \( \mathcal{P}(M) \). So assume that \( \pi_1(M) \) is cyclic, but \( \mathcal{P}(M) \) is not homeomorphic to \( S^1 \times S^2 \). Then it follows from 7.1, 7.4 and 7.5 that \( \chi \) is the trivial homomorphism and \( H(M) \) is free. If \( \mathcal{P}(M) \) is homeomorphic to \( S^1 \times S^1 \times I \) then the same follows from 7.2 and 7.4. In 6.6 and 6.8 we have shown that \( H(S^1 \times S^1 \times S^1) \) and \( H(S^1 \times S^2) \) have torsion. This completes the proof of 1.1.  □

Remark. It is immediate from 6.5 that \( C(M) \) has torsion, if \( M \) contains a singular torus, which has non-trivial intersection number with some oriented loop in \( M \). But in general it is difficult to construct torsion in \( C(M) \) [K-4].

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