Mixture representations of noncentral distributions

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Abstract

With any symmetric distribution $\mu$ on the real line we may associate a parametric family of noncentral distributions as the distributions of $(X + \delta)^2$, $\delta \neq 0$, where $X$ is a random variable with distribution $\mu$. The classical case arises if $\mu$ is the standard normal distribution, leading to the noncentral chi-squared distributions. It is well-known that these may be written as Poisson mixtures of the central chi-squared distributions with odd degrees of freedom. We obtain such mixture representations for the logistic distribution and for the hyperbolic secant distribution. We also derive alternative representations for chi-squared distributions and relate these to representations of the Poisson family. While such questions originated in parametric statistics they also appear in the context of the generalized second Ray-Knight theorem, which connects Gaussian processes and local times of Markov processes.

Keywords: Noncentral distribution; mixture distribution; Poisson family; Ray-Knight theorem

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1 Introduction

Many of the classical parametric distributions, such as the normal and Poisson families, arise in the context of limit theorems, whereas others, such as the exponential distributions, are characterized by certain properties or by invariance under specific transformations. In view of this origin direct distributional relations between these families are often found to be surprising. A particularly intriguing example is the following, which is also the starting point for the present paper:

\[(X + \delta)^2 \overset{D}{=} X^2 + 2 \sum_{j=1}^{N} E_j.\] (1)

Here \(\delta \neq 0\) is a real number, the random variables \(N, X, E_1, E_2, \ldots\) are independent, \(X\) has the standard normal distribution, \(E_1, E_2, \ldots\) are exponentially distributed with mean 1, \(N\) has the Poisson distribution with parameter \(\delta^2/2\), and ‘\(\overset{D}{=}\)’ denotes equality in distribution.

In statistics, the distribution in (1) is known as the noncentral chi-squared distribution with one degree of freedom and noncentrality parameter \(\delta^2\), and commonly denoted by \(\chi^2_1(\delta^2)\). We recall that the central chi-squared distributions \(\chi^2_k\) with \(k\) the degree of freedom, \(k \in \mathbb{N}\), the set of natural numbers not including 0, and the exponential distributions are both subfamilies of the family \(\Gamma(\alpha, \lambda)\), \(\alpha, \lambda > 0\), of gamma distributions. Specifically, \(\chi^2_k = \Gamma(k/2, 1/2)\) and \(\Gamma(1, \lambda)\) is the exponential distribution with mean \(1/\lambda\). In view of the convolution property of the gamma family,

\[\Gamma(\alpha, \lambda) \star \Gamma(\beta, \lambda) = \Gamma(\alpha + \beta, \lambda) \quad \text{for all } \alpha, \beta, \lambda > 0,\]

we may thus rewrite (1) as a mixture representation,

\[\chi^2_1(\delta^2) = e^{-\delta^2/2} \sum_{n=0}^{\infty} \frac{\delta^{2n}}{2^n n!} \chi^2_{2n+1} \quad \text{for all } \delta \neq 0,\] (2)

and indeed, it often appears in this form; see e.g. pp. 382, 458 in Johnson et al. (1994). Interestingly, the equivalent relations (1) and (2) appear under two quite different circumstances: In statistics in connection with the power of statistical tests, see Liese and Miescke (2008), and in probability theory in connection with the local times of Markov processes, see Lemma 6.33 in Mörters and Peres (2010).
Note that the base family \( \{ \chi^2_k : k \in \mathbb{N} \} \) of mixture components is the same for all noncentrality parameters, so that \((2)\) may be regarded as separating the two variables, the argument of the probability measure and its parameter. A third way to write the representation is in terms of random variables \( Y_k \) with distribution \( \chi^2_{2k+1} \), \( k \in \mathbb{N}_0 \), the set of natural numbers including 0, as

\[
(X + \delta)^2 = \mathcal{D} Y_N,
\]

where the random variable \( N \) is independent of the \( Y \)-variables and Poisson distributed with parameter \( \delta^2/2 \).

In the present paper we obtain similar results for two other families of noncentral distributions. For example, if \( X \) has the hyperbolic secant distribution then, with \( \phi : [0, \infty) \to [0, 1) \), \( \phi(x) = 2/(e^{2x} + e^{-2x}) \),

\[
(X + \delta)^2 = \mathcal{D} \phi^{-1}
\left( \frac{X_0^2}{X_0^2 + X_1^2 + 2 \sum_{j=1}^{N} E_j} \right)^2.
\]

Here, similar to \((1)\), the random variables \( X_0, X_1, N, E_1, E_2, \ldots \) are independent, \( X_0 \) and \( X_1 \) are standard normal, \( E_1, E_2, \ldots \) are exponentially distributed with mean 1, but \( N \) now has the negative binomial distribution with parameters \( 1/2 \) and \( 2/(e^\delta + e^{-\delta})^2 \).

We also reconsider the classical case, where \( \mu \) is the standard normal distribution. We find alternative representations for \( \chi^2_{\delta^2} \) and relate these to mixture representations of the Poisson distributions.

Section 2 contains the main results, proofs are given in Section 3. In Section 4 we collect various remarks; in particular, we expand on the connections to tests and local times mentioned above.

### 2 Main results

In the first two subsections we obtain mixture representations for the noncentral distributions associated with two non-normal distributions that are symmetric about 0. In the third subsection we return to the normal distribution and construct a general family of representations that contains \((1)\) as a special case. These turn out to be related to mixture representations of the Poisson distributions.
2.1 The logistic distribution

The distribution function $F$ of the logistic distribution is given by

$$F(x) = \frac{1}{1 + e^{-x}}, \quad x \in \mathbb{R}. \quad (5)$$

In view of $F(x) = 1 - F(-x)$ this distribution is symmetric about 0. It will be convenient to rescale the noncentrality parameter via

$$\theta = \theta(\delta) := F(\delta) = \frac{e^\delta}{1 + e^\delta}. \quad (6)$$

We consider two types of noncentrality.

**Theorem 1.** Suppose that $X$ has the distribution function $F$ given in (5). Then, for all $\delta \neq 0$ and with $\theta$ as in (6),

$$|X + \delta| =_{D} \sum_{j=1}^{N} \frac{1}{j} E_j, \quad (7)$$

$$(X + \delta)^2 =_{D} \max\{E^2_1, E^2_2, \ldots, E^2_N\}, \quad (8)$$

where $N, E_1, E_2, \ldots$ are independent random variables, all $E_j$ are exponentially distributed with mean 1, and

$$P(N = n) = \theta (1 - \theta)^n + (1 - \theta) \theta^n \quad \text{for all } n \in \mathbb{N}. \quad (9)$$

**Remark 2.** (a) The distribution of the random index $N$ given in (9) may be regarded as a mixture with weights $\theta$ and $1 - \theta$ of the geometric distributions with parameters $1 - \theta$ and $\theta$ respectively.

(b) The distributional equation (8) may be rewritten as

$$(X + \delta)^2 =_{D} \max\{W_1, W_2, \ldots, W_N\}, \quad (10)$$

with the same independence structure but where the $W$-variables now have a Weibull distribution. Incidentally, the maximum in (8) resp. (10) is the analogue of the sum in an area that is known as tropical arithmetic, see e.g. Speyer and Sturmfels (2009).
2.2 The hyperbolic secant distribution

The continuous density \( f \) of the hyperbolic secant distribution is given by

\[
f(x) = \frac{1}{\pi} \text{sech}(x) = \frac{1}{\pi \cosh(x)} = \frac{2}{\pi (e^x + e^{-x})}, \quad x \in \mathbb{R}.
\]  

(11)

It will now be convenient to rescale the noncentrality parameter via

\[
\theta = \theta(\delta) := \frac{2}{(e^{-\delta} + e^{\delta})^2}.
\]  

(12)

We note in passing that \( \theta(-\delta) = \theta(\delta) \) and that \( 0 < \theta(\delta) < 1/2 \) for all \( \delta \neq 0 \).

**Theorem 3.** Suppose that \( X \) has the density function \( f \) given in (11). Let \( \delta \neq 0 \) and let \( \theta \) be as in (12). Then the density \( f_\delta \) of \( |X + \delta| \) can be written as

\[
f_\delta(x) = \sum_{k=0}^{\infty} w_\theta(k) g_k(x) \quad \text{for all } x > 0,
\]  

(13)

where, for all \( k \in \mathbb{N}_0 \) and \( x > 0 \),

\[
w_\theta(k) = \frac{(2k)}{2^{2k}} (1 - \theta)^k \theta^{1/2},
\]  

(14)

\[
g_k(x) = \frac{2^{2k+3/2} e^x + e^{-x}}{\pi (2k)} \frac{e^{2x} + e^{-2x}}{e^{2x} + e^{-2x}} \left( \frac{(e^x - e^{-x})^2}{e^{2x} + e^{-2x}} \right)^k.
\]  

(15)

**Remark 4.** (a) In (14) and (15) the terms have been normalized so that \( w_\theta \) is a probability mass function and \( g_k \) is a probability density. In fact, the mixing distribution is the negative binomial distribution with parameters \( 1/2 \) and \( \theta \), and a random variable \( Y \) with density \( g_k \) can be obtained from a random variable \( Z \) that has the beta distribution with parameters \( 1/2 \) and \( k + 1/2 \) via \( Y = \psi(Z) \), where \( \psi = \frac{1}{2} \text{sech}^{-1} \) with \( \text{sech}^{-1} \) as the inverse of the hyperbolic secant function, restricted to the positive half-line. The well-known relationship between the beta and gamma distributions now implies that we may rewrite the mixture representation of the noncentral hyperbolic secant distributions as

\[
|X + \delta| = \psi\left(\frac{Y_0^2}{Y_0^2 + \sum_{j=1}^{2N+1} Y_j^2}\right);
\]
with \( N, Y_0, Y_1, Y_2, \ldots \) independent, \( Y_j \) standard normal for all \( j \in \mathbb{N}_0 \), and the negative binomial distribution with parameters \( 1/2 \) and \( \theta \) as the law of \( N \); obviously, this implies \[ (4) \].

(b) A similar result holds for the density of \( (X + \delta)^2 \) instead of \(|X + \delta|\). Indeed, writing temporarily \( \tilde{f}_\theta \) for the density of \( (X + \theta)^2 \), we get

\[
\tilde{f}_\theta(x) = \frac{1}{2\sqrt{x}} f_\theta(\sqrt{x}) = \sum_{k=0}^{\infty} w_\theta(k) \tilde{g}_k(x) \quad \text{for all } x > 0,
\]

with \( \tilde{g}_k(x) := g_k(\sqrt{x})/(2\sqrt{x}) \) for all \( k \in \mathbb{N}_0, x > 0 \).

(c) In the standard literature on probability and statistics the hyperbolic secant distribution is not as widely discussed as other distributions that are symmetric about zero, such as the normal, the logistic, the two-sided exponential, or the Cauchy distribution, for example. For various fascinating connections of the hyperbolic secant distribution with the one-dimensional Brownian motion, planar Brownian motion, and the three-dimensional Bessel process the interested reader is referred to Lévy (1951) and Rogers and Williams (1994). As an example, we specially mention the distributional identity

\[
\frac{2}{\pi} X \overset{D}{=} \sup_{0 \leq t \leq 1} |W(t)|, \tag{16}
\]

where \( W = (W(t), t \geq 0) \) is a standard Brownian motion and \( Z \) is a standard normal random variable independent of \( W \). Some statistical applications of the hyperbolic secant distribution are given in Ding (2014).

### 2.3 Alternative representations in the normal case

Here we present a family of mixture representations for the situation where we start with the normal distribution, i.e. we consider the distribution \( \chi^2_1(\delta^2) \) of \( (X + \delta)^2 \) where \( X \) is standard normal. In this section we write \( f_\delta \) for the density of \( \chi^2_1(\delta^2) \), and \( g_k \) for the density associated with the central chi-squared distribution with \( k \) degrees of freedom, \( k \in \mathbb{N} \). The classical representation \[ (1) \] may then be written as

\[
f_\delta = e^{-\eta} \sum_{k=0}^{\infty} \frac{\eta^k}{k!} g_{2k+1}
\]

with \( \eta = \delta^2/2 \); this equation will appear as a special case of Theorem \[ 5 \] below.
The new representation families are indexed by two sequences \( p = (p_n)_{n \in \mathbb{N}_0} \) and \( q = (q_n)_{n \in \mathbb{N}_0} \) of non-negative real numbers where we assume that the corresponding power series

\[
u(t) = \sum_{n=0}^{\infty} q_n t^n, \quad \lim_{t \uparrow t_0} \frac{u(t)}{v(t)} = \infty.
\]

converge to finite values on some nonempty interval \([0, t_0)\), with \( t_0 \leq \infty \), and where \( p_0, q_0, u, v \) satisfy

\[p_0 = 0, \quad v \neq 0, \quad \lim_{t \uparrow t_0} u(t) = \infty.\]

The sequences \( p \) and \( q \) can be used to obtain two families of discrete distributions via ‘exponential tilting’: For each \( \theta \in (0, t_0) \),

\[
p_\theta(n) := \frac{1}{u(\theta)} p_n \theta^n, \quad q_\theta(n) := \frac{1}{v(\theta)} q_n \theta^n, \quad n \in \mathbb{N}_0,
\]

are probability mass functions, and the associated probability generating functions are given by

\[
t \mapsto \frac{u(\theta t)}{u(\theta)}, \quad t \mapsto \frac{v(\theta t)}{v(\theta)}, \quad 0 \leq t \leq 1.
\]

The assumptions on \( p \) imply that \( u : [0, t_0) \rightarrow [0, \infty) \) is a bijection, which we may use to transform the noncentrality parameter via

\[
\theta := u^{-1}(|\delta|).
\]

(18)

In order to define the family of densities for the mixture components and the parametric family of mixture distributions we need some more definitions.

We first note that, for all \( k \in \mathbb{N} \),

\[
u^k(t) = \sum_{n=k}^{\infty} p_n^n t^n \quad \text{with} \quad p_n^n = \sum_{(j_1, \ldots, j_k) \in \mathbb{N}_0^k, \ j_1 + \cdots + j_k = n} p_{j_1} \cdots p_{j_k}.
\]

We put \( p_0^{\ast 0} := 1 \) and \( p_j^{\ast 0} := 0 \) for all \( j \in \mathbb{N} \), so that this also holds for \( k = 0 \). Further let

\[
b_n := \sum_{k=0}^{[n/2]} \left( \sum_{\ell=2k}^{n} p_{\ell}^{2k} q_{n-\ell} \right) \frac{1}{k!2^k}, \quad n \in \mathbb{N}_0,
\]

(19)
\[ B := \{ n \in \mathbb{N}_0 : b_n > 0 \}, \]

\[ h_n(x) := b_n^{-1} \sum_{k=0}^{\lfloor n/2 \rfloor} \left( \sum_{\ell=2k}^{n} p^{2k}_{\ell} q_{n-\ell} \right) \frac{1}{k! 2^k} g_{2k+1}(x), \quad x > 0, \quad n \in B, \]

\[ w_{\theta}(n) := (v(\theta))^{-1} \exp \left( -\frac{u^2(\theta)}{2} \right) b_n \theta^n, \quad n \in \mathbb{N}_0. \]

It is clear that the functions \( h_n, n \in B \), are probability densities on \((0, \infty)\). Further, it is part of the assertion of the next theorem that, for each \( \theta \in (0, t_0) \), \( w_{\theta} \) is a probability mass function with support \( B \).

**Theorem 5.** Let \( \theta, B, h_n \) and \( w_{\theta} \) be as in (18), (20), (21) and (22) respectively. For each \( n \in B \) let \( Y_n \) be a random variable with density \( h_n \) and let \( N \) be a \( B \)-valued random variable with probability mass function \( w_{\theta} \) that is independent of the \( Y \)-sequence. Then

\[ (X + \delta)^2 =_D Y_N. \]

**Remark 6.** As in the logistic and the hyperbolic secant case, more can be said about the distribution of the random summation index \( N \) in (23): it follows from the computations in the proof of Theorem 5 in Section 3.3 that the generating function of \( N \) is given by

\[ \frac{v(\theta t)}{v(\theta)} \exp \left( \frac{1}{2} u^2(\theta) \left[ \frac{u^2(\theta t)}{u^2(\theta)} - 1 \right] \right), \quad 0 \leq t \leq 1. \]

Based on this explicit formula standard calculations show that

\[ N =_D K + \sum_{j=0}^{M} (L_{j,1} + L_{j,2}). \]

Here the \( \mathbb{N}_0 \)-valued random variables \( K, M, L_{j,1}, L_{j,2}, j \in \mathbb{N} \), are independent, \( K \) has the generating function \( v(\theta t)/v(\theta) \), \( 0 \leq t \leq 1 \), \( M \) has the Poisson distribution with parameter \( \delta^2/2 \), and the \( L_{j,1}, L_{j,2}, j \in \mathbb{N} \), have the distribution with generating function \( u(\theta t)/u(\theta) \), \( 0 \leq t \leq 1 \); see also the above remarks on exponential tilting.

Theorem 5 is quite general; as a result, the formulas for the mixture components and the mixture distribution are somewhat involved. Specializing \( u \) and \( v \) we obtain several interesting subrepresentations where some of these ingredients can be made more explicit.
Example 7. (a) The classical case (1) resp. (17) appears if we take \( v(t) \equiv 1 \), \( u(t) = t \) for all \( t \in [0, t_0) \) with \( t_0 := \infty \), or, equivalently, \( p_1 = 1 \) and \( p_j = 0 \) for \( j \neq 1 \), \( q_0 = 1 \) and \( q_j = 0 \) for \( j \neq 0 \).

(b) Let \( \alpha \in (-1, \infty) \). With \( v(\theta) = (1 - \theta)^{-(1 + \alpha)} \) and \( u(\theta) = \theta/(1 - \theta) \), \( \theta \in (0, 1) \), some simplifications occur. First, we have, for \( n \geq 2k \),

\[
\sum_{\ell=2k}^{n} p_{\ell} q_{n-\ell} = \binom{n + \alpha}{n - 2k},
\]

and the distributions of \( K \) and the \( L \)-variables in \ref{24} turn out to be negative binomial with parameters \( 1 + \alpha \) and \( 1 - \theta \), and geometric with parameter \( 1 - \theta \), respectively.

(c) The following may be regarded as the \( \alpha = -1 \) version of the previous part. Again we take \( u(\theta) = \theta/(1 - \theta) \), but now let \( v \equiv 1 \). We exploit the identity

\[
\exp\left( \frac{x\theta}{1 - \theta} \right) = 1 + \sum_{n=1}^{\infty} \theta^n \sum_{k=1}^{n} x^k \frac{(n - 1)}{k!} (k - 1), \quad x \in \mathbb{R}, \ \theta \in (-1, +1)
\]

to obtain \( b_0 = 1 \), \( b_1 = 0 \), and, for \( n \geq 2 \),

\[
b_n = \sum_{k=1}^{[n/2]} \binom{n - 1}{2k - 1} \frac{1}{2^k k!} \cdot \tag{25}
\]

Further, recalling that \( g_k \) denotes the density of \( \chi^2_k \) for all \( k \in \mathbb{N} \), \( h_0(x) = g_1(x) \) and, for \( n \geq 2 \) and \( x > 0 \),

\[
h_n(x) = b_n^{-1} \sum_{k=1}^{[n/2]} \binom{n - 1}{2k - 1} \frac{1}{2^k k!} g_{2k+1}(x). \tag{26}
\]

In order to obtain a closed-form expression for the sums appearing in \ref{25} and \ref{26} and/or a probabilistic interpretation of the mixture distribution with the density \( h_n \) for \( n \geq 2 \) it may be helpful to write

\[
b_n = \sum_{k=1}^{[n/2]} L(n, 2k) \frac{(2k - 1)!!}{n!}, \quad n \geq 2,
\]
and

\[ h_n(x) = b_n^{-1} \sum_{k=1}^{[n/2]} L(n, 2k) \frac{(2k - 1)!!}{n!} g_{2k+1}(x), \quad x > 0, \ n \geq 2, \]

where

\[ L(n, \ell) = \frac{n!}{\ell!} \left( \frac{n - 1}{\ell - 1} \right), \quad n, \ell \in \mathbb{N}, \ \ell \leq n, \]

are the special (unsigned) Lah numbers: \( L(n, \ell) \) is the number of ways a set of \( n \) elements can be partitioned into \( \ell \) non-empty linearly ordered subsets. Additionally, \( L(0, 0) = 0 \), \( L(n, 0) = 0 \) for \( n > 0 \), and \( L(n, k) = 0 \) for \( k > n \); see [Lah (1955)] or, e.g., [Cereceda (2015)] for recent research on Lah numbers.

(d) With \( v \equiv 1 \) and \( u(\theta) = -\log(1 - \theta), \ \theta \in (0, 1), \)

\[ \frac{u(\theta t)}{u(\theta)} = \frac{\log(1 - \theta t)}{\log(1 - \theta)}, \quad 0 \leq t \leq 1. \]

This is the generating function of the logarithmic series distribution with parameter \( \theta \), with probability mass function \( n^{-1} \theta^n / ( -\log(1 - \theta)) \), \( n \in \mathbb{N} \).

Because of

\[ \frac{1}{2} (\log(1 - x))^2 = \sum_{n=2}^{\infty} \frac{H_{n-1}}{n} x^n, \quad |x| < 1, \]

where \( H_m \) the \( m \)th harmonic number \( H_m = \sum_{j=1}^{m} \frac{1}{j}, \ m \in \mathbb{N}, \) see e.g. formula 1.516 in [Gradshteyn and Ryzhik (2007)], we get

\[ \left( \frac{\log(1 - \theta t)}{\log(1 - \theta)} \right)^2 = \sum_{n=2}^{\infty} 2 \frac{H_{n-1}}{n} \frac{1}{(\log(1 - \theta))^2} \theta^n t^n, \quad 0 \leq t \leq 1, \]

as the generating function of the distribution of \( L_{j,1} + L_{j,2}, \ j \in \mathbb{N} \).

The family \( h_n, \ n \in B \), of probability densities in our alternative representations for noncentral chi-squared distributions consists of mixtures of central chi-squared distributions with odd degrees of freedom. The transition from the classical representation [2] may thus be regarded as a linear change of the mixture basis. We now relate such changes to non-canonical mixture representations of the Poisson family.

Of course, any distribution with support \( \mathbb{N}_0 \) can canonically be seen as a mixture of the one-point masses concentrated at \( k, \ k \in \mathbb{N}_0 \). More general
mixture representations of the Poisson distributions consist of two families of distributions on $\mathbb{N}_0$, which we specify by their probability mass functions $\{c_n : n \in \mathbb{N}_0\}$ and $\{m_\eta : \eta > 0\}$, requiring

$$e^{-\eta \frac{k}{k!}} = \sum_{n=0}^{\infty} m_\eta(n) c_n(k) \quad \text{for all } k \in \mathbb{N}_0, \eta > 0. \quad (27)$$

If $B_n$, $n \in \mathbb{N}_0$, are random variables with $P(B_n = k) = c_n(k)$ for all $k \in \mathbb{N}_0$ and $M_\eta$ is a random variable, independent of the $B$-family, with $P(M_\eta = n) = m_\eta(n)$ for all $n \in \mathbb{N}_0$, then (27) implies that $N := B_{M_\eta}$ is Poisson distributed with parameter $\eta$. Further, starting with the classical representation $(X + \delta)^2 = Y_N$, see (3) with $\delta^2/2 = \eta$, assuming the independence of the random variables $M_\eta, B_n, n \in \mathbb{N}_0, Y_k, k \in \mathbb{N}_0$, we then obtain $(X + \delta)^2 = Y_{B_{M_\eta}}$, which can be rewritten as

$$(X + \delta)^2 = Z_N, \quad Z_j := Y_{B_j} \quad \text{for all } j \in \mathbb{N}_0.$$ 

This shows that any mixture representation of the Poisson family leads to a mixture representation of the noncentral chi-squared distributions.

Interestingly, there is a converse.

**Theorem 8.** Let $b_n$ and $B$ be as in (19) and (20) respectively. For $n \in B$ and $k \in \mathbb{N}_0$ put

$$c_n(k) := \frac{1}{b_n k!2^k} \sum_{\ell=2k}^{n} \ell^\ast q_{n-\ell}^t \quad \text{if } n \geq 2k, \text{ and } c_n(k) = 0 \text{ otherwise.}$$

For $n \in \mathbb{N}_0, n \notin B$, let $c_n$ be some arbitrary probability mass function on $\mathbb{N}_0$. Further, let $\theta$ and $w_\theta(n)$ be as in (18) and (22) respectively. Put $\eta = u^2(\theta)/2$ and define the probability mass function $m_\eta$ by $m_\eta(n) = w_\theta(n), n \in \mathbb{N}_0$. Then the families $\{c_n : n \in \mathbb{N}_0\}$ and $\{m_\eta : \eta > 0\}$ provide a mixture representation of the Poisson family.

**3 Proofs**

Before we begin with the proofs of the theorems we briefly recall a specific argument for the classical case as it also underlies our approach in the present paper.
With $f_{\delta}$ the density of $(X + \delta)^2$, $X$ standard normal, we get
\[
\frac{d}{dx} \left( P(X \leq \sqrt{x} - \delta) - P(X \leq -\sqrt{x} - \delta) \right) = \frac{1}{\sqrt{2\pi x}} \left( e^{-\left(\sqrt{x} - \delta\right)^2/2} + e^{-\left(-\sqrt{x} - \delta\right)^2/2} \right)
\]
\[
= \frac{1}{\sqrt{2\pi x}} e^{-\delta^2/2} e^{-x/2} \frac{1}{2} \left( e^{\delta\sqrt{x}} + e^{-\delta\sqrt{x}} \right)
\]
\[
= \sum_{n=0}^{\infty} e^{-\delta^2/2} \left( \frac{\delta^2}{2} \right)^n \frac{2^n}{\sqrt{2\pi(2n)!}} x^{n-\frac{1}{2}} e^{-x/2}. \]

Using the duplication formula for the gamma function,
\[
n! 2^{2n} \Gamma \left( n + \frac{1}{2} \right) = \pi^{1/2} (2n)!, \quad n \in \mathbb{N}_0,
\]
this is easily identified as a Poisson mixture with parameter $\delta^2/2$ of the densities
\[
g_{2n+1}(x) = \frac{1}{2^{n+1} \Gamma \left( n + \frac{1}{2} \right)} x^{n-\frac{1}{2}} e^{-x/2}, \quad x > 0,
\]
(28)

of the distributions $\chi^2_{2n+1}, n \in \mathbb{N}_0$, and (1) follows.

In order to indicate how this can be extended to obtain a proof of the representation mentioned in Example 7(b) we replace the standard series representation of the exponential function used above by
\[
(1 - \theta)^{-1-\alpha} \exp \left( \frac{x\theta}{1-\theta} \right) = \sum_{n=0}^{\infty} L_n^\alpha(-x) \theta^n
\]
for all $\theta \in (-1, 1)$ and all $x \in \mathbb{R}$, where
\[
L_n^\alpha(x) := \sum_{k=0}^{n} \binom{n + \alpha}{n-k} \frac{1}{k!} (-x)^k
\]
are the generalized Laguerre polynomials; see e.g. [Erdélyi et al. (1953)]. With $\theta := |\delta|/(1 + |\delta|)$ we get
\[
e^{\delta\sqrt{x}} + e^{-\delta\sqrt{x}} = \exp \left( \frac{\theta\sqrt{x}}{1-\theta} \right) + \exp \left( -\frac{\theta\sqrt{x}}{1-\theta} \right)
\]
so that again the odd terms in the expansion disappear, and a brute force approach finally yields the desired representation.
3.1 Proof of Theorem

For $\delta \neq 0$ the distribution function $F_\delta$ of $|X + \delta|$ is given by

$$F_\delta(x) = F(x - \delta) - F(-x - \delta) = \frac{1}{1 + e^{-x + \delta}} - \frac{1}{1 + e^{x + \delta}}, \quad x \geq 0.$$  

With $\theta$ as in (6) this leads to

$$\frac{1}{1 + e^{-x + \delta}} = \frac{\frac{1}{1 + e^\delta}}{1 - \frac{e^\delta}{1 + e^\delta} (1 - e^{-x})} = \frac{1 - \theta}{1 - \theta (1 - e^{-x})} = \sum_{n=0}^{\infty} (1 - \theta) \theta^n (1 - e^{-x})^n$$

and similarly,

$$\frac{1}{1 + e^{x + \delta}} = 1 - \theta - \sum_{n=1}^{\infty} \theta (1 - \theta)^n (1 - e^{-x})^n,$$

both for all $x \geq 0$. Combining these we obtain

$$F_\delta(x) = \sum_{n=1}^{\infty} ((1 - \theta) \theta^n + \theta(1 - \theta)^n)(1 - e^{-x})^n \quad \text{for all } x \geq 0.$$  

(29)

Now let $E_1, E_2, \ldots$ be independent random variables, where for each $j \in \mathbb{N}$ the random variable $E_j$ has the exponential distribution with mean 1. Then, for each $n \in \mathbb{N}$

$$P(\max(E_1, \ldots, E_n) \leq x) = (1 - e^{-x})^n, \quad x \geq 0,$$  

(30)

and (29) translates into

$$|X + \delta| = \mathcal{D} \max(1, \ldots, E_N)$$  

(31)
with $N$ as in the theorem and independent of $E_j, j \in \mathbb{N}$. By the Rényi-Sukhatme representation, see e.g. p. 721 in Shorack and Wellner (1986),

$$\max(E_1, \ldots, E_n) =_{\mathbb{P}} \sum_{j=1}^{n} \frac{1}{j} E_j \quad \text{for all } n \in \mathbb{N}.$$ 

Thus, using (31),

$$|X + \delta| =_{\mathbb{P}} \sum_{j=1}^{N} \frac{1}{j} E_j.$$ 

Also, (31) immediately leads to the representation of the distribution of $(X + \delta)^2$ as a maximum

$$(X + \delta)^2 =_{\mathbb{D}} \max(E_1^2, \ldots, E_N^2)$$

of a random number of squared exponentials. The latter are easily seen to have the Weibull distribution with distribution function $1 - e^{-\sqrt{x}}, x \geq 0$.

### 3.2 Proof of Theorem 3

With $f$ as in (11) and $\delta \neq 0$ the density $f_{\delta}$ of $|X + \delta|$ is given by

$$f_{\delta}(x) = f(x - \delta) + f(x + \delta)$$

$$= \frac{2}{\pi} \frac{(e^x + e^{-x})(e^\delta + e^{-\delta})}{e^{2x} + e^{-2x} + e^{2\delta} + e^{-2\delta}}$$

$$= \frac{2}{\pi} \frac{(e^x + e^{-x})(e^\delta + e^{-\delta})}{e^{2x} + e^{-2x} + e^{2\delta} + e^{-2\delta}} \frac{2}{e^{2x} + e^{-2x} + e^{2\delta} + e^{-2\delta}}$$

$$= \frac{2}{\pi} \frac{(e^x + e^{-x})(e^\delta + e^{-\delta})}{e^{2x} + e^{-2x}} \frac{2}{e^{2x} + e^{-2x} + e^{2\delta} + e^{-2\delta}}$$

$$= \frac{4}{\pi} \frac{(e^x + e^{-x})(e^\delta + e^{-\delta})}{e^{2x} + e^{-2x}} \frac{1}{e^{2x} + e^{-2x} + e^{2\delta} + e^{-2\delta}} \frac{2}{1 - \frac{2}{e^{2x} + e^{-2x} + e^{2\delta} + e^{-2\delta}}}$$

$$= \frac{4}{\pi} \frac{e^{x} + e^{-x}}{e^{2x} + e^{-2x}} \frac{1}{e^{\delta} + e^{-\delta}} \int_{0}^{\infty} \left( \frac{e^x - e^{-x}}{e^{2x} + e^{-2x}} \right)^k \left( \frac{e^{\delta} + e^{-\delta}}{e^{\delta} + e^{-\delta}} \right)^k, x > 0.$$ 

With $\theta$ as in (12) this can be written as

$$f_{\delta}(x) = \sum_{k=0}^{\infty} \frac{23/2}{\pi} \frac{e^{x} + e^{-x}}{e^{2x} + e^{-2x}} \left( \frac{e^{x} - e^{-x}}{e^{2x} + e^{-2x}} \right)^k (1 - \theta)^k \theta^{1/2}, x > 0. \quad (32)$$
For each $k \in \mathbb{N}_0$ let

$$a_k := \frac{2^{3/2}}{\pi} \int_0^\infty \frac{e^x + e^{-x}}{e^{2x} + e^{-2x}} \left( \frac{(e^x - e^{-x})^2}{e^{2x} + e^{-2x}} \right)^k \, dx.$$  \hfill (33)

Because of \int f_\delta(x) \, dx = 1 we have $1 = \sum_{k=0}^{\infty} a_k (1 - \theta)^k \theta^{1/2}$, or, equivalently,

$$\sum_{k=0}^{\infty} a_k (1 - \theta)^k = \theta^{-1/2} = (1 - (1 - \theta))^{-1/2} = \sum_{k=0}^{\infty} (-1)^k \binom{-1/2}{k} (1 - \theta)^k,$$

from which we deduce that

$$a_k = (-1)^k \binom{-1/2}{k} = 2^{-2k} \binom{2k}{k} \text{ for all } k \in \mathbb{N}_0.$$  

The assertion \ref{13} of the theorem now follows from \ref{32} and the definitions \ref{14} and \ref{15}.

The proof of the distributional fact noted in Remark 4 (a) that $\frac{1}{2} \text{sech}^{-1}(Y)$ has the density $g_k$ given in \ref{15}, if $Y$ has the beta distribution with parameters $1/2$ and $k + 1/2$, is easily carried out by straightforward calculation.

### 3.3 Proof of Theorem 5

Again, we begin with a suitable calculation,

$$v(\theta) \exp (xu(\theta)) = \sum_{n=0}^{\infty} q_n \theta^n + \left( \sum_{\ell=0}^{\infty} q_\ell \theta^\ell \right) \left( \sum_{k=1}^{\infty} \frac{1}{k!} x^k u(\theta)^k \right)$$

$$= \sum_{n=0}^{\infty} q_n \theta^n + \left( \sum_{\ell=0}^{\infty} q_\ell \theta^\ell \right) \left( \sum_{k=1}^{\infty} \frac{1}{k!} x^k \sum_{\ell=k}^{\infty} p^*_{\ell_k} \theta^\ell \right)$$

$$= \sum_{n=0}^{\infty} q_n \theta^n + \left( \sum_{\ell=0}^{\infty} q_\ell \theta^\ell \right) \left( \sum_{\ell=1}^{\infty} \left[ \sum_{k=1}^{\ell} \frac{1}{k!} p^*_{\ell_k} x^k \right] \theta^\ell \right)$$

$$= \sum_{n=0}^{\infty} q_n \theta^n + \sum_{\ell=1}^{n} \left( \sum_{k=1}^{\ell} \frac{x^k}{k!} p^*_{\ell_k} q_{n - \ell} \right) \theta^n \right) \right).$$

With

$$L_n(x) := \sum_{\ell=0}^{n} \left( \sum_{k=0}^{\ell} \frac{x^k}{k!} p^*_{\ell_k} \right) q_{n - \ell} = \sum_{k=0}^{n} \left( \sum_{\ell=k}^{n} p^*_{\ell_k} q_{n-\ell} \right) \frac{x^k}{k!}.$$
this may be rewritten as
\[ v(\theta) \exp(xu(\theta)) = \sum_{n=0}^{\infty} L_n(x) \theta^n. \]

If \( q_0 \neq 0 \) and \( p_1 \neq 0 \) the polynomials \( L_n(x), n \in \mathbb{N}_0 \), are of the generalized Appell type; more specifically, the \( n!L_n(x) \) arise as Sheffer polynomials, see Roman (1982). Further, with
\[ M_n(x) := \sum_{k=0}^{\lfloor n/2 \rfloor} \left( \sum_{\ell=2k}^{n} p_{\ell} q_{n-\ell} \right) \frac{x^k}{(2k)!}, \]
we get
\[ M_n(\sqrt{x}) = \frac{1}{2} \left[ L_n(\sqrt{x}) + L_n(-\sqrt{x}) \right] \text{ for } x > 0 \]
and therefore
\[ \frac{1}{2} \left( \exp(\sqrt{x}\delta) + \exp(-\sqrt{x}\delta) \right) = \frac{1}{v(\theta)} \sum_{n=0}^{\infty} M_n(x) \theta^n. \]

Using the definitions in (19) to (22) we may now write the density \( f_\delta \) of \( \chi_1^2(\sigma^2) \) as
\[ f_\delta(x) = \sum_{n \in B} h_n(x) w_\theta(n), \quad x > 0. \] (34)

It is clear that \( h_n \) is a probability density for all \( n \in B \). Finally, to see that \( w_\theta \) is a probability mass function, we first note that \( w_\theta(n) \geq 0 \) and then use (34) to obtain
\[ 1 = \int_{0}^{\infty} f_\delta(x) \, dx = \sum_{n \in B} w_\theta(n). \]

### 3.4 Proof of Theorem 8

Taken together, and with \( g_k \) the density of \( \chi_k^2 \), the classical representation (17) and the alternative representation given in Theorem 5 imply that
\[ \sum_{k=0}^{\infty} e^{-n \eta k} g_{2k+1} = f_\delta = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} m_{\eta}(n)c_n(k) g_{2k+1}. \]
As all the individual terms are nonnegative we may rearrange the right hand side. Multiplying both sides by $\sqrt{2x}e^{x/2}$ we then obtain

$$
\sum_{k=0}^{\infty} e^{-\eta} \frac{\eta^k}{k!} \frac{1}{2^k \Gamma(k + \frac{1}{2})} x^k = \sum_{k=0}^{\infty} \left( \sum_{n=0}^{\infty} m_{\eta}(n)c_n(k) \right) \frac{1}{2^k \Gamma(k + \frac{1}{2})} x^k,
$$

and

$$
e^{-\eta} \frac{\eta^k}{k!} = \sum_{n=0}^{\infty} m_{\eta}(n)c_n(k)
$$

follows by comparison of coefficients.

### 4 Comments

#### 4.1 The connection to statistics

So far the only noncentral chi-squared distributions considered were those with one degree of freedom. In statistics, the distribution of $Z := \sum_{i=1}^{k} (X_i + \mu_i)^2$ is important, where $X_1, \ldots, X_k$ are independent standard normals and $\mu_1, \ldots, \mu_k$ are arbitrary real numbers. If $\delta^2 := \sum_{i=1}^{k} \mu_i^2 > 0$ this is the noncentral chi-squared distribution $\chi_k^2(\delta^2)$ with $k$ degrees of freedom and noncentrality parameter $\delta^2$, and the following generalization of (2) holds:

$$
\chi_k^2(\delta^2) = e^{-\delta^2/2} \sum_{n=0}^{\infty} \frac{\delta^{2n}}{2^{n!}} \chi_{2n+k}^2
$$

for all $\delta \neq 0$, $k \in \mathbb{N}$.

In contrast to this, the known mixture representations for noncentral $F$-distributions, see e.g. Stuart and Ord (1991), do not arise as the distributions of a function of $|X + \delta|$, $\delta \neq 0$, for some random variable $X$ satisfying $X = \mathcal{D} - X$. Noncentral $t$-distributions are not even concentrated on the nonnegative half-line, but see Stuart and Ord (1991) again and Section 4.3 below.

Special quadratic forms of multivariate normal random vectors have noncentral chi-squared distributions; see, e.g., Rao (1965) for a thorough discussion. Additionally, the power function of various important statistical tests can be expressed in terms of noncentral chi-squared distributions. To give a simple specific example, let $X$ be a $d$-variate normal random vector with unknown mean vector $a \in \mathbb{R}^d$ and the $d \times d$ identity matrix as covariance
matrix. Consider testing the hypothesis $H : a = 0$ against the general alternative $K : a \neq 0$. The testing problem is invariant under the group of orthogonal transformations on $\mathbb{R}^d$. For given $\alpha \in (0, 1)$, there exists a uniformly most powerful invariant test at level $\alpha$. With the euclidean norm $\|X\|^2$ of $X$ there is a simple maximal invariant test statistic. Denoting by $\chi^2_{d,1-\alpha}$ the $1-\alpha$ quantile of $\chi^2_d$, the test rejects the hypothesis iff $\|X\|^2 > \chi^2_{d,1-\alpha}$. If $X$ has the mean $a \neq 0$ the distribution of the test statistic $\|X\|^2$ is the noncentral $\chi^2$ distribution with noncentrality parameter $\|a\|^2$. Thus, denoting by $G_{d,\|a\|^2}$ the distribution function of the $\chi^2_d(\delta^2)$ distribution, the power function of the test is given by $1 - G_{d,\|a\|^2}(\chi^2_{d,1-\alpha})$, $a \in \mathbb{R}^d$. See, e.g., Liese and Miescke (2008) for other testing problems where the power function can be expressed in terms of noncentral chi-squared distributions.

4.2 The connection to stochastic processes

Let $X = (X_t)_{t \geq 0}$ be an irreducible Markov chain with finite state space $S$ and symmetric jump rates. For $i \in S$ and $t \geq 0$ let

$$L^0_i(t) := \int_0^t 1\{X_u = i\} \, du$$

be the amount of time spent by $X$ in state $i$ up to time $t$. For a fixed state $i_0$ and a given $\eta > 0$ let

$$\tau := \inf\{t \geq 0 : L_{i_0}(t) \geq \eta\},$$

and let $L = (L_i)_{i \in S'}$ with $S' := S \setminus \{i_0\}$ be given by $L_i := L^0_i(\tau)$. Suppose that $X$ starts at $i_0$. The symmetry assumption implies that the matrix

$$G = (g_{ij})_{i,j \in S'}, \quad g_{ij} := EL_i L_j,$$

is positive semidefinite and symmetric; in particular, it can serve as the covariance matrix of a central normal random vector $Y = (Y_i)_{i \in S'}$. The generalized second Ray-Knight theorem says that, for $X$ and $Y$ independent,

$$\frac{1}{2} Y^2 + L = \mathcal{D} \left( \frac{1}{2} (Y + \sqrt{2\eta})^2 \right).$$

The standard reference for this circle of ideas is the book by Marcus and Rosen (2006), but see also the very recent approach in Bauerschmidt et al. (2019).
The simplest nontrivial example has \( S = \{0, 1\} \) and jump rates 1. We take \( i_0 = 0 \) and glue the time intervals in 0 together, which results in a Poisson process with rate 1. This implies that the number \( N \) of visits to 1 up to time \( \tau \) has a Poisson distribution with parameter \( \eta \). The \( L \)-vector consists of one component only, it is the sum of \( N \) independent random variables \( E_1, E_2, \ldots \) that are all exponentially distributed with mean 1. The \( G \)-matrix consists of the single number 1, which means that \( Y \) is a standard normal random variable. Taken together we see that (4.2) leads to

\[
(Y + \sqrt{2\eta})^2 =_D Y^2 + 2 \sum_{j=1}^{N} E_j,
\]

which is (1).

The classical second Ray-Knight theorem refers to Brownian motion where a discrete time approximation leads to the simple symmetric random walk. In the approach given by Mörters and Peres (2010) the proof is reduced to the two-states situation and (1). The authors of the book ask for a probabilistic argument (as the authors of the present paper did more than 20 years ago, coming from the statistical side). In fact, in the much more general circle of ideas known as the Dynkin isomorphism, the lack of a probabilistic explanation of the basic distributional equalities is often mentioned; see e.g. the preface to Marcus and Rosen (2006).

### 4.3 A geometric outlook

The mixture representation \( \mu = \sum_{k=0}^{\infty} a_k \nu_k \) of a probability distribution \( \mu \) in terms of a mixing distribution \( a_k \) and a mixing base \( \nu_k \) of probability distributions, \( k \in \mathbb{N}_0 \), may equivalently be seen as a representation of \( \mu \) as a convex combination of the \( \nu_k \)'s. In particular, (11) leads to

\[
\{ \chi_1^2(\delta) : \delta \neq 0 \} \subset \overline{\text{conv}\{ \chi_{2k+1}^2 : k \in \mathbb{N}_0 \}},
\]

where the notation on the right refers to the closure of the convex hull, and closure in turn refers to total variation norm (or \( L^1 \)-convergence if there are densities). Similarly, (4.11) implies

\[
\{ \chi_k^2(\delta) : \delta \neq 0, k \in \mathbb{N} \} \subset \overline{\text{conv}\{ \chi_k^2 : k \in \mathbb{N} \}}.
\]

Of course, the mixture results go beyond this geometric interpretation as they also provide the mixing coefficients \( a_k(\delta), k \in \mathbb{N}_0 \), for each \( \delta \neq 0 \).
Moving from convex to general linear combinations \( \mu = \sum_{k=0}^{\infty} a_k \nu_k \) we obtain

\[
\mu \in \overline{\text{Lin}\{\nu_k : k \in \mathbb{N}_0\}},
\]

where the notation now refers to the closed linear span, and the topology may be the \( L^1 \)- or \( L^2 \)-distance in the density case (we temporarily do not distinguish between measures and their densities in this case). For example, the generalized Laguerre polynomials with \( \alpha = 1/2 \) can easily be transformed into an orthonormal basis of the Hilbert space \( L^2 := L^2(\mathbb{R}, \mathcal{B}_+, \chi_2^1) \) of all measurable functions \( f : (0, \infty) \to \mathbb{R} \) satisfying \( \int f^2 \, d\chi_2^1 < \infty \). In view of

\[
\frac{d\chi_{2k+1}^2}{d\chi_1^2}(x) = c_k x^k \quad \text{with} \quad c_k := 2^{-k} \frac{\Gamma(1/2)}{\Gamma(k + 1/2)} = 1/(2k - 1)!!
\]

the central chi-squared densities with degrees \( 2k + 1, k \in \mathbb{N} \), are finite linear combinations of these basis elements, and vice versa. This implies

\[
\overline{\text{Lin}\{\chi_{2k+1}^2 : k \in \mathbb{N}\}} = L^2
\]

so that, for example, the uniform distributions on \([0, \theta], \theta > 0\), can be written as infinite linear combinations of the central chi-squared distributions with odd degrees of freedom, but it is easy to see that they are not contained in their closed convex hull. Also, the representation for noncentral \( t \)-distributions given in Stuart and Ord (1991) is of the general linear and not of the convex type.

In both cases the representations lead to approximations by finite sums. For the statistician and before computers became widely available this aspect was of particular importance, and it generated a sizable literature. For example, Tiku (1965) used Laguerre polynomials in the context of approximating noncentral chi-squared distributions, but in contrast to our approach the coefficients can be negative. An advantage of a mixture representation over a general linear representation is that the corresponding approximations for the distribution function are monotone increasing; also, tail estimates for the mixture distribution immediately lead to error bounds. It is therefore of considerable interest to obtain similar mixture representations for other symmetric distributions, such as the Cauchy distribution and the two-sided exponential distribution. Finally, the convexity aspect raises some interesting questions of its own. For example, we may pass to the closed convex hull on the left hand side of both (35) and (36) without destroying the inclusion
relation. Does this operation lead to both sets being equal? (It is easy to see that this is indeed the case for 436). Is there a general and usable description of the closed convex hull of general parametric families? We postpone these questions to future research work.

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