Rigidity of quasicrystallic and $\mathbb{Z}^\gamma$-circle patterns

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Abstract

The uniqueness of the orthogonal $\mathbb{Z}^\gamma$-circle patterns as studied by Bobenko and Agafonov is shown, given the combinatorics and some boundary conditions. Furthermore we study (infinite) rhombic embeddings in the plane which are quasicrystallic, that is they have only finitely many different edge directions. Bicoloring the vertices of the rhombi and adding circles with centers at vertices of one of the colors and radius equal to the edge length leads to isoradial quasicrystallic circle patterns. We prove for a large class of such circle patterns which cover the whole plane that they are uniquely determined up to affine transformations by the combinatorics and the intersection angles. Combining these two results, we obtain the rigidity of large classes of quasicrystallic $\mathbb{Z}^\gamma$-circle patterns.

1 Introduction

Circles, especially circle packings and circle patterns, have successfully been used over the past years to define and study discrete analogs of classical smooth objects. In particular, this approach leads to discrete holomorphic mappings, for example discrete analogs of the power functions $z^\gamma$, and to discrete holomorphic function theory. See for example [23, 7, 8, 11] for some of the contributions to the theory of circle patterns and [27] for results on circle packings.

In this article we focus on circle patterns which are characterized by a given combinatorics specifying which circles should intersect and by the corresponding intersection angles. Thus we associate to a circle pattern a pattern of kites corresponding to intersecting circles, see Figures 1 (right) and 6. A particularly suitable source for the required knowledge on circle patterns and their relations to consistency, integrability, and discrete holomorphic functions is the textbook [11] in discrete differential geometry.

Our main results can roughly be summarized as follows: The combinatorics (together with suitable intersection angles and boundary conditions, if necessary) determines the geometry of the circle pattern. This rigidity result can be interpreted as a discrete version of Liouville’s Theorem in complex analysis.

Our first result concerns the uniqueness of circle patterns which are discrete analogs of the power functions $z^\gamma$ for $\gamma \in (0, 2)$ as defined in [7, 8, 2]. Here, the square grid combinatorics, the orthogonal intersection angles, and the two boundary lines of a sector uniquely determine the geometry of the pattern, see Figure 2 for an illustration. The rigidity of orthogonal $\mathbb{Z}^\gamma$-circle patterns was only known for rational $\gamma$, see [2].

Furthermore, we consider the case of a circle pattern which covers the whole complex plane and for which the radii of all circles are equal and the interiors of different kites are disjoint. Then the corresponding kites form a rhombic embedding. Also, assume

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that there are only finitely many different edge directions of the kites. Such rhombic embeddings are called \textit{quasicrystalline} \cite{[9]}, additionally, orient an edge $\vec{e}$ and consider a line perpendicular to the edge. Move this line parallelly in positive and in negative direction along $\vec{e}$. We suppose that in both cases this moving line intersects infinitely often edges parallel to $\vec{e}$. We assume that this property is true at least for two edges with linearly independent directions. Such rhombic embeddings of the plane are for example given by Penrose tilings, see for example Figure 6. We show that any other embedded circle pattern with the same combinatorics and intersection angles of the kites is the image of this embedding by an affine transformation.

Rigidity of some classes of infinite circle patterns of the plane have already been studied. Schramm \cite{[23]} considers square grid combinatorics and orthogonal intersection angles. As an essential step of our proof we generalize his result to square grid circle patterns with regular intersection angles $\psi \in (0, \pi)$ and $(\pi - \psi)$. He \cite{[19]} studies disk triangulation graphs and exterior intersection angles in $[\pi/2, \pi]$ which does not cover the class of isoradial quasicrystallic circle patterns defined above.

As observed in \cite{[9]} isoradial quasicrystallic rhombic embeddings can be used to define corresponding quasicrystallic $Z^\gamma$-circle patterns, see also \cite{[11]}. Examples of quasicrystallic $Z^\gamma$-circle patterns are shown in Figures 7, 9, and 11. They have been created using software developed by Veronika Schreiber for her diploma thesis \cite{[24]}. Our rigidity result for orthogonal $Z^\gamma$-circle patterns is also generalized for large classes of quasicrystallic $Z^\gamma$-circle patterns.

There is some more literature concerning rigidity for infinite planar circle packings, that is configurations with non-overlapping touching circles, see \cite{[21, 15, 22, 20]}. Our rigidity proofs adapt some of the ideas which have been used for packings. In particular, we apply discrete potential theory.

This paper is organized as follows. First we introduce terminology and present useful facts about circle patterns. We especially focus on regular circle patterns with square grid combinatorics. Then we recall in Section 4 the definition and some properties of the orthogonal $Z^\gamma$-circle patterns for $\gamma \in (0, 2)$ as studied in \cite{[4, 1, 2]} and prove their rigidity. In Section 5 we introduce quasicrystallic circle patterns and prove uniqueness for a class of these patterns. Finally, we recall the definition and some facts about quasicrystallic $Z^\gamma$-circle patterns for $\gamma \in (0, 2)$ as studied in \cite{[3, 9, 11]} and prove rigidity for certain classes of these patterns. A more detailed version of the results can be found in \cite{[12]}.

\section{Circle patterns}

In this section we focus on a definition and some useful properties of circle patterns. We describe circle patterns using combinatorial data and intersection angles.

The combinatorics are specified by a \textit{b-quad-graph} $\mathcal{D}$, that is a strongly regular cell decomposition of a domain in $\mathbb{C}$ possibly with boundary such that all 2-cells (faces) are embedded and counterclockwise oriented. Furthermore all faces of $\mathcal{D}$ are quadrilaterals, that is there are exactly four edges incident to each face, and the 1-skeleton of $\mathcal{D}$ is a bipartite graph. We always assume that the vertices of $\mathcal{D}$ are colored white and black.

To these two sets of vertices we associate two planar graphs $G$ and $G^*$ as follows. The vertices $V(G)$ are all white vertices of $V(\mathcal{D})$. The edges $E(G)$ correspond to faces of $\mathcal{D}$, that is two vertices of $G$ are connected by an edge if and only if they are incident to the same face. The dual graph $G^*$ is constructed analogously by taking as vertices $V(G^*)$ all black vertices of $\mathcal{D}$. $\mathcal{D}$ is called \textit{simply connected} if it is the cell decomposition of a simply connected domain of $\mathbb{C}$ and if every closed chain of faces is null homotopic in $\mathcal{D}$.

For the intersection angles, we use a labelling $\alpha : F(\mathcal{D}) \to (0, \pi)$ of the faces of $\mathcal{D}$. By abuse of notation, $\alpha$ can also be understood as a function defined on $E(G)$ or on...
Figure 1: Left: An example of a b-quad-graph \( \mathcal{D} \) (black edges and bicolored vertices) and its associated graph \( G \) (dashed edges and white vertices). Right: The exterior intersection angle \( \alpha \) of two intersecting circles and the associated kite built from centers and intersection points. \( \beta = \pi - \alpha \) is the interior intersection angle.

\[ E(G^*). \] The labelling \( \alpha \) is called admissible if it satisfies the following condition at all interior black vertices \( v \in V_{\text{int}}(G^*) \):

\[ \sum_{f \text{ incident to } v} \alpha(f) = 2\pi. \] (1)

Definition 2.1. Let \( \mathcal{D} \) be a b-quad-graph with associated graph \( G \) and let \( \alpha : E(G) \rightarrow (0, \pi) \) be an admissible labelling. An (immersed planar) circle pattern for \( \mathcal{D} \) (or \( G \)) are an indexed collection \( C = \{C_z : z \in V(G)\} \) of circles and an indexed collection \( \mathcal{K} = \{K_e : e \in E(G)\} = \{K_f : f \in F(\mathcal{D})\} \) of closed kites, which all carry the same orientation, such that the following conditions hold.

1. If \( z_1, z_2 \in V(G) \) are incident vertices in \( G \), the corresponding circles \( C_{z_1}, C_{z_2} \) intersect with exterior intersection angle \( \alpha([z_1, z_2]) \). Furthermore, the kite \( K_{[z_1, z_2]} \) is bounded by the centers of the circles \( C_{z_1}, C_{z_2} \), the two intersection points, and the corresponding edges, as in Figure 1 (right). The intersection points are associated to black vertices of \( V(\mathcal{D}) \) or to vertices of \( V(G^*) \).

2. If two faces are incident in \( \mathcal{D} \), the corresponding kites share a common edge.

3. Let \( f_1, \ldots, f_n \in F(\mathcal{D}) \) be the faces incident to an interior vertex \( v \in V_{\text{int}}(\mathcal{D}) \). Then the kites \( K_{f_1}, \ldots, K_{f_n} \) have mutually disjoint interiors. The union \( K_{f_1} \cup \cdots \cup K_{f_n} \) is homeomorphic to a closed disk and contains the point \( p(v) \) corresponding to \( v \) in its interior.

The circle pattern is called embedded if all kites of \( \mathcal{K} \) have mutually disjoint interiors. The circle pattern is called isoradial if all circles of \( \mathcal{C} \) have the same radius.

Note that we associate a circle pattern \( \mathcal{C} \) to an immersion of the kite pattern \( \mathcal{K} \) corresponding to \( \mathcal{D} \) where the edges incident to the same white vertex are of equal length. The kites may also be non-convex and can be constructed from a (suitable) given set of circles and from the combinatorics of \( G \).

As all circle patterns considered in this article will be planar and immersed, the notion “circle pattern” will include these properties in the following.

For our study of a circle pattern \( \mathcal{C} \) we will use the radius function \( r_\mathcal{C} = r \) which assigns to every vertex \( z \in V(G) \) the radius \( r_\mathcal{C}(z) = r(z) \) of the corresponding circle \( C_z \). The index \( \mathcal{C} \) will be dropped whenever there is no confusion likely. The following proposition specifies a condition for a radius function to originate from a planar circle pattern, see [10] for a proof.
Proposition 2.2. Let $G$ be associated to a b-quad-graph $\mathcal{D}$ and let $\alpha$ be an admissible labelling.

Suppose that $\mathcal{C}$ is a planar circle pattern for $\mathcal{D}$ and $\alpha$ with radius function $r = r_\mathcal{D}$. Then for every interior vertex $z_0 \in V_{\text{int}}(G)$ we have

$$\left( \sum_{[z,z_0] \in E(G)} f_\alpha([z,z_0])(\log r(z) - \log r(z_0)) \right) = \pi = 0, \quad (2)$$

where

$$f_\theta(x) := \frac{1}{2\pi} \log \frac{1 - e^{x-i\theta}}{1 - e^{x+i\theta}},$$

and the branch of the logarithm is chosen such that $0 < f_\theta(x) < \pi$.

Conversely, suppose that $\mathcal{D}$ is simply connected and that $r : V(G) \to (0,\infty)$ satisfies (2) for every $z \in V_{\text{int}}(G)$. Then there is a planar circle pattern for $G$ and $\alpha$ with radius function $r$. This pattern is unique up to isometries of $\mathcal{C}$.

For the special case of orthogonal circle patterns with the combinatorics of the square grid, there are also other characterizations, see for example [23].

Note that $2f_\alpha([z,z_0])(\log r(z) - \log r(z_0))$ is the angle at $z_0$ of the kite with edge lengths $r(z)$ and $r(z_0)$ and angle $\alpha([z,z_0])$, as in Figure 1 (right). Equation (2) is the closing condition for the closed chain of kites which correspond to the edges incident to $z_0$. This corresponds to condition (3) of Definition 2.1.

For further use we mention some properties of $f_\theta$, see [26, Lemma 2.2].

Lemma 2.3. (i) The derivative of $f_\theta$ is $f'_\theta(x) = \frac{\sin \theta}{2(\cosh x - \cos \theta)} > 0$.

(ii) The function $f_\theta$ satisfies the functional equation $f_\theta(x) + f_\theta(-x) = \pi - \theta$.

If there exists an isoradial circle pattern, we can obtain another circle pattern from a given radius function.

Lemma 2.4. Let $G$ be a graph constructed from a b-quad-graph $\mathcal{D}$ and let $\alpha$ be an admissible labelling. Suppose that there exists an isoradial circle pattern for $G$ and $\alpha$. Let $r$ be the radius function of a planar circle pattern for $\mathcal{D}$ and $\alpha$. Then there is a circle pattern $\mathcal{C}$ for $G$ and $\alpha$ with radius function $r_\mathcal{D} = 1/r$.

Proof. By Lemma 2.3 (ii) the function $1/r$ satisfies condition (2) of Proposition 2.2 for all interior vertices $z_0 \in V_{\text{int}}(G)$. In particular,

$$\sum_{[z,z_0] \in E(G)} f_\alpha([z,z_0])(\log \frac{1}{r(z)} - \log \frac{1}{r(z_0)}) = \sum_{[z,z_0] \in E(G)} \left( \pi - \alpha([z,z_0]) \right) = \sum_{[z,z_0] \in E(G)} f_\alpha([z,z_0])(\log \frac{r(z)}{r(z_0)}) = 2\pi - \pi = \pi.$$

Here we have also used that $\sum_{[z,z_0] \in E(G)} \left( \pi - \alpha([z,z_0]) \right) = 2\pi$ since there is an isoradial circle pattern for $G$ and $\alpha$ and the assumption that $r$ is the radius function of a circle pattern for $G$ and $\alpha$.

Let $G$ be a graph constructed from a b-quad-graph $\mathcal{D}$ and let $\alpha$ be an admissible labelling. Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are planar circle patterns for $\mathcal{D}$ and $\alpha$ with radius functions $r_1 = r_{\mathcal{D}_1}$ and $r_2 = r_{\mathcal{D}_2}$ respectively. Define a comparison function $w : V(\mathcal{D}) \to \mathbb{C}$ by

$$w(y) = r_2(y)/r_1(y) \quad \text{for } y \in V(G),$$

$$w(x) = e^{i\theta(x)} \in \mathbb{S}^1 \quad \text{for } x \in V(G^*).$$  \quad (3)
Here $\delta(x) \in \mathbb{R}$ or $w(x) = e^{i\delta(x)}$ is defined to be the rotation angle or the rotation respectively of the edge-star at $x \in V(G^*)$ when changing from the circle pattern $\mathcal{C}_1$ to $\mathcal{C}_2$. Note that $w(y)$ is the scaling factor of the circle corresponding to $y \in V(G)$. Then $w$ satisfies the following Hirota Equation for all faces $f \in F(G)$.

$$w(x_0)w(y_0)a_0 - w(x_1)w(y_0)a_1 - w(x_1)w(y_1)a_0 + w(x_0)w(y_1)a_1 = 0$$ (4)

Here $x_0, x_1 \in V(G^*)$ and $y_0, y_1 \in V(G)$ are the black and white vertices incident to $f$ and $a_0 = x_0 - y_0$ and $a_1 = x_1 - y_0$ are the directed edges. Thus equation (4) is the closing condition for the kite corresponding to the face $f$. Furthermore, the Hirota Equation is 3D-consistent; see Sections 10 and 11 of [9] or [11] for more details. This property will be used in Section 5.

3 \ SG-circle patterns

In this paper we are particularly interested in the special case of regular circle patterns with square grid combinatorics. First, we fix some notation. Let $SGD$ be the regular square grid cell decomposition of the complex plane, that is the vertices are $V(SGD) = \mathbb{Z} + i\mathbb{Z}$ and the edges are given by pairs of vertices $[z, z']$ with $z, z' \in V(SGD)$ and $|z - z'| = 1$. The 2-cells are squares $\{z + a + ib : a, b \in [0, 1]\}$ for $z \in V(SGD)$. As $SGD$ is a b-quad-graph, the vertices

$$V(SG) = \{n + im \in \mathbb{Z} + i\mathbb{Z} : n + m = 0 \pmod{2}\}$$

are assumed to be colored white. As above, $SG$ and its dual $SG^*$ are defined as the associated graphs to $SGD$. Furthermore, $SG(n, v)$ with $n \in \mathbb{N}$ and $v \in V(SG)$ denotes the subgraph of all vertices with combinatorial distance at most $n$ from $v$ in $SG$.

Let $\psi \in (0, \pi)$ be a fixed angle. Define the following regular labelling $\alpha_\psi$ on $E(SG)$. Let $[z_1, z_2]$ be an edge connecting the vertices $z_1, z_2 \in V(SG)$. Without loss of generality, we assume that $\text{Re}(z_1) < \text{Re}(z_2)$. Then

$$\alpha_\psi([z_1, z_2]) = \begin{cases} \psi & \text{if } \text{Im}(z_1) < \text{Im}(z_2), \\ \pi - \psi & \text{if } \text{Im}(z_1) > \text{Im}(z_2). \end{cases}$$ (5)

If $G$ is a subgraph of $SG$, a circle pattern for $G$ and $\alpha_\psi$ is called $SG$-circle pattern. The choice $\psi = \pi/2$ leads to orthogonal $SG$-circle patterns as considered by Schramm in [23].

**Theorem 3.1** (Rigidity of $SG$-circle patterns). *Suppose that $\mathcal{C}$ is an embedded planar circle pattern for $SG$ and $\alpha_\psi$. Then $\mathcal{C}$ is the image of a regular isoradial circle pattern for $SG$ and $\alpha_\psi$ under a similarity.*

The proof is a suitable adaption of the corresponding proof for orthogonal $SG$-circle patterns given by Schramm using suitable Möbius invariants, see [23, Theorem 7.1] or [12]. This adaption needs the following generalization of the Ring Lemma of [21], which is also useful in the following.

**Lemma 3.2.** Let $r$ be the radius function of an embedded circle pattern for $SG(3, 0)$ and $\alpha_\psi$. There is a constant $C = C(\psi) > 0$, independent of $r$, such that for $k = 0, 1, 2, 3$ there holds

$$r(\frac{i^k(1 + i)}{r(0)}) > C.$$
Proof. Assume the contrary. Then there is a sequence of embedded circle patterns for $SG(3,0)$ and $\alpha_\psi$ such that $r_n(0) = 1$ and $r_n(i^k(1 + i)) \to 0$ as $n \to \infty$ for some $k \in \{0,1,2,3\}$. Without loss of generality we assume that $k = 0$. We also may assume that the circle $C_0$ corresponding to the vertex $0 \in V(SG)$ and the intersection point corresponding to $1 \in V(SG^*)$ are fixed for the whole sequence. Then there is a subsequence such that all the circles converge to circles or lines, that is converge in the Riemann sphere $\hat{\mathbb{C}} \cong S^2$. Now equation (2) implies that there exist some kites which intersect in the limit in their interiors. But this is a contradiction to the embeddedness of the sequence.

If the number of surrounding generations is big enough, there is the following useful estimation on the quotient of radii of incident vertices.

**Lemma 3.3.** There is an absolute constant $C > 0$ such that the following holds. Let $G$ be a subgraph of $SG$ and let $\mathcal{C}$ be an embedded circle pattern for $G$ and $\alpha_\psi$ with radius function $r$. Let $v \in V(G)$ be a vertex and suppose that $SG(n,v) \subset G$, that is $G$ contains $n$ generations of $SG$ around $v$, for some $n \geq 3$. Then for all vertices $w$ incident to $v$ there holds

$$1 - \frac{C}{n} \leq \frac{r(w)}{r(v)} \leq 1 + \frac{C}{n}. \quad (6)$$

The proof is an adaption of the corresponding proof for hexagonal circle packings given by Aharonov in [5, 6] and uses Lemma 3.2, see [12] for more details. The necessary results on discrete potential theory can be found in the appendix of [13] and in [12].

4 Uniqueness of orthogonal $Z^\gamma$-circle patterns

An orthogonal circle pattern with the combinatorics of the square grid associated to the map $z^\gamma$ was introduced by Bobenko in [7]. Further development of the theory is due to Agafonov and Bobenko [4, 1, 2].

4.1 Definition and useful properties

In the following we briefly summarize the definition and some known facts about orthogonal $Z^\gamma$-circle patterns, see [4, 1, 2] for more details.

**Definition 4.1.** Let $D \subset \mathbb{Z}^2$. A map $f : D \to \mathbb{C}$ is called discrete conformal if all its elementary quadrilaterals are conformal squares, i.e. their cross-ratios are equal to $-1$:

$$q(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1}) := \frac{(f_{n,m} - f_{n+1,m})(f_{n+1,m+1} - f_{n,m+1})}{(f_{n+1,m} - f_{n+1,m+1})(f_{n,m+1} - f_{n,m})} = -1. \quad (7)$$

Here and below we abbreviate $f_{n,m} = f(n,m)$.

A discrete conformal map $f_{n,m}$ is called embedded if the interiors of different elementary quadrilaterals ($f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1}$) are disjoint.

Note that the definition of a discrete conformal map is Möbius invariant and is motivated by the following characterization for smooth mappings: A smooth map $f : \mathbb{C} \subset D \to \mathbb{C}$ is called conformal (holomorphic or antiholomorphic) if and only if for all $z = x + iy \in D$ there holds

$$\lim_{\varepsilon \to 0} q(f(x,y), f(x + \varepsilon, y), f(x + \varepsilon, y + \varepsilon), f(x, y + \varepsilon)) = -1.$$
Figure 2: Illustration of the discrete conformal map $Z^{3/2}$ and the orthogonal $Z^{3/2}$-circle pattern (right).

In order to construct an embedded discrete analog of $z^\gamma$ the following approach is used. Equation (7) can be supplemented with the nonautonomous constraint

$$
\gamma f_{n,m} = 2n\left(\frac{(f_{n+1,m} - f_{n,m})(f_{n,m} - f_{n-1,m})}{(f_{n+1,m} - f_{n-1,m})}\right)
+ 2m\left(\frac{(f_{n,m+1} - f_{n,m})(f_{n,m} - f_{n,m-1})}{(f_{n,m+1} - f_{n,m-1})}\right).
$$

(8)

This constraint, as well as its compatibility with (7), is derived from some monodromy problem; see [4]. We assume that $0 < \gamma < 2$ and denote $Z_{2}^{\pm} = \{(n,m) \in Z^2 : n, m \geq 0\}$.

The asymptotics of the constraint (8) for $n,m \to \infty$ and the properties $z^\gamma(R_{\pm}) = R_{\pm}$ and $z^\gamma(iR_{\pm}) = e^{\gamma\pi i/2}R_{\pm}$ of the holomorphic mapping $z^\gamma$ motivate the following definition of the discrete analog.

**Definition 4.2.** For $0 < \gamma < 2$, the discrete conformal map $Z^\gamma : Z_{2}^{+} \to \mathbb{C}$ is the solution of equations (7) and (8) with the initial conditions $Z^\gamma(0,0) = 0$, $Z^\gamma(1,0) = 1$, $Z^\gamma(0,1) = e^{\gamma\pi i/2}$.

From this definition, the properties $Z^\gamma(n,0) \in \mathbb{R}_{+}$ and $Z^\gamma(0,m) \in e^{\gamma\pi i/2}\mathbb{R}_{+}$ are obvious for all $n, m \in \mathbb{N}$. Furthermore, the discrete conformal map $Z^\gamma$ from Definition 4.2 determines an SG-circle pattern. Indeed, by Proposition 1 of [4] all edges at the vertex $f_{n,m}$ with $n + m = 0$ (mod 2) have the same length and all angles between neighboring edges at the vertex $f_{n,m}$ with $n + m = 1$ (mod 2) are equal to $\pi/2$. Thus, all elementary quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ build orthogonal kites and for any $(n,m) \in Z_{2}^{+}$ with $n + m = 0$ (mod 2) the points $f_{n+1,m}, f_{n,m+1}, f_{n-1,m}, f_{n,m-1}$ lie on a circle with center $f_{n,m}$. Therefore, we consider the sublattice $\{(n,m) \in Z_{2}^{+} : n + m = 0 \text{ (mod 2)}\}$ and denote by $V$ the quadrant

$$
V = \{z = N + iM : N, M \in \mathbb{Z}, M \geq |N|\} \subset \mathbb{Z}^2,
$$

where $N = (n - m)/2$, $M = (n + m)/2$. Two vertices $z_1, z_2 \in V$ are connected by an edge if and only if $|z_1 - z_2| = 1$.

**Theorem 4.3** ([3][1][2]). (i) If $R(z)$ denotes the radius function corresponding to the discrete conformal map $Z^\gamma$ for some $0 < \gamma < 2$, then it holds that

$$
(\gamma - 1)(R(z)^2 - R(z - i)R(z + 1)) \geq 0
$$

(9)

for all $z \in V \setminus \{\pm N + iN|N \in \mathbb{N}\}$. 7
(ii) For $0 < \gamma < 2$, the discrete conformal maps $Z^\gamma$ given by Definition 4.2 are embedded. Consequently, the corresponding circle patterns are also embedded.

In the following section and in Section 6 we continue to use the notation of this section. In particular the radius function is denoted by $R$ and we have the normalization $R(0) = 1$.

### 4.2 Uniqueness of the orthogonal $Z^\gamma$-circle patterns

This section is devoted to the proof of following uniqueness result.

**Theorem 4.4** (Rigidity of orthogonal $Z^\gamma$-circle patterns). For $\gamma \in (0, 2)$ the infinite orthogonal embedded circle pattern corresponding to $Z^\gamma$ is the unique embedded orthogonal circle pattern (up to global scaling) with the following two properties.

(i) The union of the corresponding kites of the $Z^\gamma$-circle pattern covers the infinite sector $\{ z = \rho e^{i\beta} \in \mathbb{C} : \rho \geq 0, \beta \in [0, \gamma \pi/2] \}$ with angle $\gamma \pi/2$.

(ii) The centers of the boundary circles lie on the boundary half lines $R_+$ and $e^{i\gamma \pi/2}R_+$.

Our proof uses results of discrete potential theory or of the theory of random walks which can be found in standard textbooks, for example by Doyle and Snell [14] or by Woess [28]. We recall some basic terminology and notation and cite adapted versions of a few theorems which will be useful for our argumentation.

By abuse of notation, we denote by $Z^2$ the points $(a, b) \in \mathbb{R}^2 \cong \mathbb{C}$ with $a, b \in \mathbb{Z}$ as well as the graph with vertices at these points and edges $e = [z_1, z_2]$ if $|z_1 - z_2| = 1$. The meaning will be clear from the context.

Consider the network $(Z^2, c)$ with conductances $c(e) > 0$ and resistances $1/c(e)$ on the undirected edges $e \in E(Z^2)$. Then a transition probability function $p$ is given by

$$p(z_1, z_2) := \begin{cases} c([z_1, z_2]) / \left( \sum_{e=\{z_1, z_2\}\in E(Z^2)} c(e) \right) & \text{if } [z_1, z_2] \in E(Z^2) \\ 0 & \text{otherwise} \end{cases}.$$  

The stochastic process on $Z^2$ given by this probability function $p$ is a reversible random walk or a reversible Markov chain on $Z^2$. The simple random walk on $Z^2$ is given by specifying $c(e) = 1$ for all edges which leads to $p(z_1, z_2) = 1/4$ if $[z_1, z_2] \in E(Z^2)$.

Denote by $p_{\text{esc}}$ the probability that a random walk starting at any point will never return to this point. The network $(Z^2, c)$ is called recurrent if $p_{\text{esc}} = 0$ (and transient otherwise). Note that $p_{\text{esc}} = 1/R_{\text{eff}}$, where $R_{\text{eff}}$ denotes the effective resistance from a point to infinity.

**Theorem 4.5.** (i) The simple random walk on $Z^2$ is recurrent.

(ii) Let $(Z^2, c_1)$ and $(Z^2, c_2)$ be two networks with conductances $c_1(e) > 0$ and $c_2(e) > 0$ on the edges. If $c_2(e) \leq c_1(e)$ for all edges $e \in E(Z^2)$, then the recurrence of $(Z^2, c_2)$ implies the recurrence of $(Z^2, c_1)$.

A proof can for example be found in [14] Chapters 5, 7, and 8 or [28] Sections 1.A, 1.B, and Corollary (2.14).

A function $f : Z^2 \to \mathbb{R}$ is called superharmonic (subharmonic) with respect to the probability function $p$ or with respect to the conductances $c$ if for every vertex $v \in Z^2$ we have $\sum_{w \in Z^2} p(v, w)f(w) \leq f(v)$ (or $\sum_{w \in Z^2} p(v, w)f(w) \geq f(v)$).

The following proposition shows that the quotient of the radius functions of two orthogonal $SG$-circle patterns is subharmonic with respect to suitably chosen conductances. As the statement is a special case of Proposition 6.3 below, we omit the proof.

8
Proposition 4.6. Consider two orthogonal circle patterns for SG(1, 0). Denote the radii by \( r_j \) and \( r_j \) respectively, where \( \rho_0 \) and \( r_0 \) denote the radii of the inner circles. Then

\[
\sum_{j=1}^{4} c_j \frac{r_j}{\rho_j} \geq \sum_{j=1}^{4} c_j \frac{r_0}{\rho_0} \quad \text{and} \quad \sum_{j=1}^{4} c_j \frac{\rho_j}{r_j} \geq \sum_{j=1}^{4} c_j \frac{\rho_0}{r_0},
\]

where \( c_j = 1/(\rho_j/\rho_0 + (\rho_0/\rho_j)). \)

Our proof of rigidity is based on the following property of superharmonic functions on recurrent networks.

Theorem 4.7 ([28, Theorem (1.16)]). A network is recurrent if and only if all nonnegative superharmonic functions are constant.

Proof of Theorem 4.7. Let \( \gamma \in (0, 2) \) and denote by \( R : V \to \mathbb{R}_+ \) the radius function of the embedded \( \mathbb{Z}^2 \)-circle pattern with \( R(0) = 1 \). Let \( r : V \to \mathbb{R}_+ \) denote the radius function of an embedded orthogonal circle pattern with the same combinatorics and the same boundary conditions (orthogonal boundary circles to the half lines \( \mathbb{R}_+ \) and \( e^{i\gamma \pi/2} \mathbb{R}_+ \)). Without loss of generality we assume same normalization \( r(0) = 1 \). This can always be achieved by a suitable scaling. In the following, we will show that the radius functions \( R \) and \( r \) take the same values on all of \( V \). This implies that both circle patterns coincide.

As both circle patterns are embedded, Lemma 3.3 implies that for some constant \( A > 0 \) and \( n \geq 3 \)

\[
1 - \frac{A}{n} \leq \frac{r(z_j^{(n)})}{r(z_0^{(n)})} \leq 1 + \frac{A}{n} \tag{11}
\]

holds for all radii \( r(z_j^{(n)}) \) for vertices \( z_j^{(n)} \) of the \( n \)th generation away from the origin and their incident vertices \( z_j^{(n)} \). Here, vertices \( z \in V \) belong to the \( n \)th generation if their combinatorial distance in \( V \) to the origin is \( n \). For estimation (11) we have also used that the reflection of the circle pattern in one of the boundary lines \( \mathbb{R}_+ \) or \( e^{i\gamma \pi/2} \mathbb{R}_+ \) also leads to an embedded orthogonal \( SG \)-circle pattern. The same reasoning applies to the radii of the \( \mathbb{Z}^2 \)-circle pattern, so

\[
1 - \frac{A}{n} \leq \frac{R(z_j^{(n)})}{R(z_0^{(n)})} \leq 1 + \frac{A}{n} \tag{12}
\]

for \( n \geq 3 \) with the same constant \( A \). Estimations (11) and (12), the boundary conditions and a suitable adaptation of the Ring Lemma 5.11 for circles of generation two and three from the origin imply that there is a constant \( K > 0 \) such that

\[
\frac{1}{K} \leq \frac{r(z_j)}{r(z_0)} \leq K \quad \text{and} \quad \frac{1}{K} \leq \frac{R(z_j)}{R(z_0)} \leq K \tag{13}
\]

for all incident vertices \( z_j \) and \( z_0 \).

We now consider two undirected networks \( \mathbb{Z}^2, C \) and \( \mathbb{Z}^2, \tilde{C} \) as follows. On the edges of \( V \) we define two conductance functions \( C \) and \( \tilde{C} \) by

\[
C(e) = C(R(z_j), R(z_k)) := \left( \frac{R(z_j)}{R(z_k)} + \frac{R(z_k)}{R(z_j)} \right)^{-1},
\]

\[
\tilde{C}(e) = \tilde{C}(r(z_j), r(z_k)) := \left( \frac{r(z_j)}{r(z_k)} + \frac{r(z_k)}{r(z_j)} \right)^{-1}.
\]
where the edge $e = [z_j, z_k]$ connects the vertices $z_j, z_k \in V$. Estimations (13) imply that both positive functions $C > 0$ and $\tilde{C} > 0$ are uniformly bounded away from 0 (and from infinity). These two conductance networks on $V \subset \mathbb{Z}^2$ can be continued to all of $\mathbb{Z}^2$ by reflection in the lines $\{|M| = |N|\}$. From Theorem 4.5 we deduce that both networks $(\mathbb{Z}^2, C)$ and $(\mathbb{Z}^2, \tilde{C})$ are recurrent.

Consider the following positive functions on $V$

$$f_1(z) = r(z)/R(z) > 0 \quad \text{and} \quad f_2(z) = R(z)/r(z) = 1/f_1(z) > 0.$$ 

By Proposition 4.6 these functions are subharmonic. Using the boundary conditions of the circle patterns, this remains true if $f_1$ and $f_2$ are continued to all of $\mathbb{Z}^2$ using reflection. Consequently, $M - f_1$ and $M - f_2$ are superharmonic for all constants $M \in \mathbb{R}$. If $f_1$ or $f_2$ is bounded from above, we get a positive superharmonic function using the upper bound. Then Theorem 4.7 implies that both functions are constant. Thus $r \equiv R$ and consequently both circle patterns coincide.

Denote by $M_1(n)$ and $M_2(n)$ the maximum of $f_1$ and $f_2$, respectively, for the set of vertices of the $n$th generation about the origin. As $f_1$ and $f_2$ are subharmonic, they assume their maxima on the boundary. Therefore the functions $M_1$ and $M_2$ are monotonically increasing. The estimations (11) and (12) imply that the quotients of any two radii of one circle pattern in the $n$th generation are bounded from above for $n \geq 3$, as two vertices in the $n$th generation can be connected by at most $4n$ edges using only vertices of the $n$th and $n + 1$st generation. So their quotient is bounded by $e^{4A}$ for both radius functions $r$ and $R$. Note that with the normalization $r(0) = 1 = R(0)$, the maxima $M_1$ and $M_2$ are bounded from below by 1. Thus their product

$$M_1(n)M_2(n) = f_1(z_{M_1}^{(n)})f_2(z_{M_2}^{(n)}) = \frac{R(z_{M_1}^{(n)})}{r(z_{M_1}^{(n)})} \frac{r(z_{M_2}^{(n)})}{R(z_{M_2}^{(n)})} \leq e^{8A}$$

is bounded from above. Here $z_{M_1}^{(n)}$ and $z_{M_2}^{(n)}$ denote the vertices of the $n$th generation where $f_1$ and $f_2$ assume their maxima, respectively. Therefore $M_1$ and $M_2$ are also bounded. This finishes the proof of uniqueness.

\section{Uniqueness of isoradial quasicrystallic circle patterns}

The uniqueness result of Theorem 5.1 can be generalized for some classes of quasicrystallic circle patterns. On this basis we will then generalize Theorem 4.4 for some classes of quasicrystallic $Z^d$-circle pattern.

\subsection{Quasicrystallic circle patterns and connection to $Z^d$}

\textbf{Definition 5.1.} A \textit{rhombic embedding in} $\mathbb{C}$ of a b-quad-graph $\mathcal{D}$ is an embedding with the property that each face of $\mathcal{D}$ is mapped to a rhombus. Given a rhombic embedding of $\mathcal{D}$, consider for each directed edge $\vec{e} \in \vec{E}(\mathcal{D})$ the vector of its embedding as a complex number with length one. Half of the number of different values of these directions is called the \textit{dimension} $d$ of the rhombic embedding. If $d$ is finite, the rhombic embedding is called \textit{quasicrystallic}.

Adding circles with centers in the white vertices of the rhombic embedding and radius equal to the edge length reveals the close connection to embedded isoradial circle patterns.

A circle pattern for a b-quad-graph $\mathcal{D}$ is called a \textit{quasicrystallic circle pattern} if there exists a quasicrystallic rhombic embedding of $\mathcal{D}$ and if the intersection angles are taken
from this rhombic embedding. The comparison function of the isoradial circle pattern \( \mathcal{C}_1 \) for \( D \) and the quasicrystalline circle pattern \( \mathcal{C}_2 \) is also called comparison function for \( \mathcal{C}_2 \).

In the following we will often identify the b-quad-graph \( D \) with a rhombic embedding of \( \mathcal{C}_D \).

**Remark 5.2.** The notion “quasicrystallic” is not uniquely defined in literature. Here we adopt the definition given in [9]. Naturally, this property only makes sense for infinite graphs or sequences of graphs with growing number of vertices and edges.

Any rhombic embedding of a b-quad-graph \( D \) can be seen as a sort of projection of a certain two-dimensional subcomplex (quad-surface) \( \Omega_{\mathcal{D}} \) of the multi-dimensional lattice \( \mathbb{Z}^d \) (or of a multi-dimensional lattice \( \mathcal{L} \) which is isomorphic to \( \mathbb{Z}^d \)). An illustrating example is given in Figure 3.

The quad-surface \( \Omega_{\mathcal{D}} \) in \( \mathbb{Z}^d \) can be constructed in the following way. Denote the set of the different edge directions of \( D \) by \( A = \{ \pm a_1, \ldots, \pm a_d \} \subset S^1 \).

Figure 3: An example of a quad-surface \( \Omega_{\mathcal{D}} \subset \mathbb{Z}^3 \).

We suppose that \( d > 1 \) and that any two non-opposite elements of \( A \) are linearly independent over \( \mathbb{R} \). Let \( e_1, \ldots, e_d \) denote the standard orthonormal basis of \( \mathbb{R}^d \). Fix a white vertex \( x_0 \in V(D) \) and the origin of \( \mathbb{R}^d \). Add the edges of \( \{ \pm e_1, \ldots, \pm e_d \} \) at the origin which correspond to the edges of \( \{ \pm a_1, \ldots, \pm a_d \} \) incident to \( x_0 \) in \( D \), together with their endpoints. Successively continue the construction at the new endpoints. Also, add two-dimensional facets (quadrilateral faces) of \( \mathbb{Z}^d \) corresponding to the quadrilateral faces of \( D \), spanned by incident edges.

A quad-surface \( \Omega_{\mathcal{D}} \) in \( \mathbb{Z}^d \) corresponding to a quasicrystallic rhombic embedding can be characterized using the following monotonicity property. For a proof see [9, Section 6].

**Lemma 5.3 (Monotonicity criterion).** Any two points of \( \Omega_{\mathcal{D}} \) can be connected by a path in \( \Omega_{\mathcal{D}} \) with all directed edges lying in one \( d \)-dimensional octant, that is all directed edges of this path are elements of one of the \( 2^d \) subsets of \( \{ \pm e_1, \ldots, \pm e_d \} \) containing \( d \) linearly independent vectors.

An important class of examples of rhombic embeddings of b-quadgraphs can be constructed using ideas of the grid projection method for quasiperiodic tilings of the plane; see for example [13, 17, 25].

**Example 5.4 (Quasicrystallic rhombic embedding obtained from a plane).** Let \( E \) be a two-dimensional plane in \( \mathbb{R}^d \). Let \( e_1, \ldots, e_d \) denote the standard orthonormal basis of \( \mathbb{R}^d \) and let \( t \in E \). We assume that \( E \) does not contain any of the segments \( s_j = \{ t + \lambda e_j : \lambda \in [0,1] \} \) for \( j = 1, \ldots, d \). If \( E \) contains two different segments \( s_{j_1} \) and \( s_{j_2} \), the following construction only leads to the standard square grid pattern \( \mathbb{Z}^2 \). If \( E \) contains exactly one segment \( s_j \), the construction may be adapted for the remaining dimensions (excluding \( e_j \)). We further assume that the orthogonal projections onto \( E \) of the two-dimensional facets \( E_{j_1,j_2} = \{ \lambda_1 e_{j_1} + \lambda_2 e_{j_2} : \lambda_1, \lambda_2 \in [0,1] \} \) for \( 1 \leq j_1 < j_2 \leq d \) are non-degenerate parallelograms. Then we can choose positive numbers \( c_1, \ldots, c_d \) such that the orthogonal projections \( P_E(c_j e_j) \) have length 1.

Consider around each vertex \( p \) of the lattice \( \mathcal{L} = c_1 \mathbb{Z} \times \cdots \times c_d \mathbb{Z} \) the hypercuboid \( V = [-c_1/2, c_1/2] \times \cdots \times [-c_d/2, c_d/2] \), that is the Voronoi cell \( p + V \). These translations of \( V \) cover \( \mathbb{R}^d \). If \( E \) intersects the interior of the Voronoi cell of a lattice point (i.e. \( (p + V)^{\circ} \cap E \neq \emptyset \) for \( p \in \mathcal{L} \)), then this point belongs to \( \Omega_{\mathcal{D}}(E) \). Undirected edges
correspond to intersections of $E$ with the interior of a $(d-1)$-dimensional facet bounding two Voronoi cells. Thus we get a connected graph in $\mathcal{L}$. An intersection of $E$ with the interior of a translated $(d-2)$-dimensional facet of $V$ corresponds to a rectangular two-dimensional face of the lattice. By construction, the orthogonal projection of this graph onto $E$ results in a planar connected graph whose faces are all of even degree (= number of incident edges or of incident vertices). A face of degree bigger than 4 corresponds to an intersection of $E$ with the translation of a $(d-k)$-dimensional facet of $V$ for some $k \geq 3$. Consider the vertices and edges of such a face and the corresponding points and edges in the lattice $\mathcal{L}$. These points lie on a combinatorial $k$-dimensional hypercuboid contained in $\mathcal{L}$. By construction, it is easy to see that there are two points of the $k$-dimensional hypercuboid which are each incident to $k$ of the given vertices. Choose a point with least distance from $E$ and add it to the surface. Adding edges to neighboring vertices splits the face of degree $2k$ into $k$ faces of degree 4.

Thus we obtain an infinite monotone two-dimensional quad-surface $\Omega^C(E)$ which projects to an infinite rhombic embedding covering the whole plane $E$. Parts of such rhombic embeddings are shown in Figures 6, 8, and 10.

5.2 Quasicrystallic circle patterns and integrability

Let $\mathcal{D}$ be a quasicrystallic rhombic embedding of a b-quad-graph. The quad-surface $\Omega_{\mathcal{D}}$ in $\mathbb{Z}^d$ is important by its connection with integrability. See also [11] for a more detailed presentation and a deepened study of integrability and consistency. In particular, a function defined on the vertices of $\Omega_{\mathcal{D}}$ which satisfies some 3D-consistent equation on all faces of $\Omega_{\mathcal{D}}$ can be uniquely extended to the brick

$$\Pi(\Omega_{\mathcal{D}}) := \{n = (n_1, \ldots, n_d) \in \mathbb{Z}^d : a_k(\Omega_{\mathcal{D}}) \leq n_k \leq b_k(\Omega_{\mathcal{D}}), \ k = 1, \ldots, d\},$$

where $a_k(\Omega_{\mathcal{D}}) = \min_{n \in V(\Omega_{\mathcal{D}})} n_k$ and $b_k(\Omega_{\mathcal{D}}) = \max_{n \in V(\Omega_{\mathcal{D}})} n_k$. Note that $\Pi(\Omega_{\mathcal{D}})$ is the hull of $\Omega_{\mathcal{D}}$. A proof may be found in [3, Section 6]. Let $\mathcal{D}$ be a quasicrystallic rhombic embedding and let $\mathcal{C}_2$ be a quasicrystallic circle pattern with the same combinatorics and the same intersection angles. Denote the comparison function for $\mathcal{C}_2$ by $w$ as in (3). Since the Hirota equation (3) is 3D-consistent (see Sections 10 and 11 of [3]) $w$ considered as a function on $V(\Omega_{\mathcal{D}})$ can uniquely be extended to the brick $\Pi(\Omega_{\mathcal{D}})$ such that equation (4) holds on all two-dimensional facets. Additionally, $w$ and its extension are real valued on white points of $V(\Omega_{\mathcal{D}})$ and have values in $S^1$ for black points of $V(\Omega_{\mathcal{D}})$. This can easily be deduced from the Hirota Equation (4).

The extension of $w$ can be used to define a radius function for any rhombic embedding with the same boundary faces as $\mathcal{D}$.

**Lemma 5.5.** Let $\mathcal{D}$ and $\mathcal{D}'$ be two simply connected finite rhombic embeddings of b-quad-graphs with the same edge directions. Assume that $\mathcal{D}$ and $\mathcal{D}'$ agree on all boundary faces. Let $\mathcal{C}$ be an (embedded) planar circle pattern for $\mathcal{D}$ and the labelling given by the rhombic embedding. Then there is an (embedded) planar circle pattern $\mathcal{C}'$ for $\mathcal{D}'$ and the corresponding labelling which agrees with $\mathcal{C}$ for all boundary circles.

**Proof.** Consider the monotone quad-surfaces $\Omega_{\mathcal{D}}$ and $\Omega_{\mathcal{D}'}$. Without loss of generality, we can assume that $\Omega_{\mathcal{D}}$ and $\Omega_{\mathcal{D}'}$ have the same boundary faces in $\mathbb{Z}^d$. Thus both define the same brick $\Pi(\Omega_{\mathcal{D}}) = \Pi(\Omega_{\mathcal{D}'}) =: \Pi$. Given the circle pattern $\mathcal{C}$, define the comparison function $w$ for $\mathcal{C}$ on $V(\Omega_{\mathcal{D}'})$ by (3). Extend $w$ to the brick $\Pi$ such that condition (4) holds for all two-dimensional facets. Consider $w$ on $\Omega_{\mathcal{D}'}$ and build the corresponding pattern $\mathcal{C}'$, such that the points on the boundary agree with those of the given circle pattern $\mathcal{C}$. Equation (4) guarantees that all faces of $\Omega_{\mathcal{D}'}$ are mapped to closed kites. Due to the combinatorics, the chain of kites is closed around each vertex. Since the boundary
kites of $\mathcal{C}'$ are given by $\mathcal{C}$ which is an immersed circle pattern, at every interior white point the angles of the kites sum up to $2\pi$. Thus $\mathcal{C}'$ is an immersed circle pattern.

Furthermore, $\mathcal{C}'$ is embedded if $\mathcal{C}$ is, because $\mathcal{C}'$ is an immersed circle pattern, and $\mathcal{C}'$ and $\mathcal{C}$ agree for all boundary kites.

5.3 Local changes of rhombic embeddings

Let $\mathcal{D}$ be a rhombic embedding of a finite simply connected b-quad-graph and let $\Omega_\mathcal{D}$ be the corresponding quad-surface in $\mathbb{Z}^d$. Let $z \in V_{int}(\Omega_\mathcal{D})$ be an interior vertex with exactly three incident two-dimensional facets of $\Omega_\mathcal{D}$. Let $\hat{\mathcal{D}}$ be a rhombic embedding of a finite simply connected b-quad-graph and let $\Omega_{\hat{\mathcal{D}}}$ be the corresponding quad-surface in $\mathbb{Z}^d$. Let $\hat{z} \in V_{int}(\Omega_{\hat{\mathcal{D}}})$ be an interior vertex with exactly three incident two-dimensional facets of $\Omega_{\hat{\mathcal{D}}}$. Consider the three-dimensional cube with these boundary facets. Replace the three given facets with the three other two-dimensional facets of this cube. This procedure is called a flip; see Figure 4 for an illustration.

Figure 4: A flip of a three-dimensional cube. Only the faces bounded by solid edges are part of the quad-surface in $\mathbb{Z}^d$.

A vertex $z \in \mathbb{Z}^d$ can be reached with flips from $\Omega_{\mathcal{D}}$ if $z$ is contained in a quad-surface obtained from $\Omega_{\mathcal{D}}$ by a suitable sequence of flips. The set of all vertices which can be reached with flips, including $V(\Omega_{\mathcal{D}})$, will be denoted by $F(\Omega_{\mathcal{D}})$.

Remark 5.6. The quad-surface $\Omega_{\mathcal{E}}(E)$ can also be generalized using a finite sequence of flips. Such an infinite rhombic embedding will be called a plane based quasicrystallic rhombic embedding.

As a generalization of simple flips we define flips for simply oder doubly infinite strips of the following form. See Figure 5 for an illustration.

Figure 5: An example of a flip for an infinite strip.

Let $\Omega_{\mathcal{D}} \subset \mathbb{Z}^d$ be a simply connected monotone quad-surface. Let $\hat{z} \in V_{int}(\Omega_{\mathcal{D}})$ be a white vertex. Let $e_1, e_2, e_3$ be three different edges incident to $\hat{z}$ such that there are two-dimensional faces $f_1, f_2$ of $\Omega_{\mathcal{D}}$ incident to $e_1$ and $e_2$, and to $e_2$ and $e_3$, respectively. Let $\alpha_1 = \alpha(f_1)$ and $\alpha_2 = \alpha(f_2)$ be the intersection angles associated to these faces. Let $\alpha_3$ be the intersection angles associated to the two-dimensional facet of $\mathbb{Z}^d$ incident to $e_1$ and $e_2$. Then $\sum_{i=1}^3 \alpha_i = 2\pi$ or $\sum_{i=1}^3 (\pi - \alpha_i) + \alpha_3 = 2\pi$. In the first case, consider the half-axis $\mathcal{A}_+ = \{\hat{z} + \lambda \hat{e}_j : \lambda \geq 0\}$, where $\hat{e}_j$ is the vector corresponding to the edge $e_j$ and pointing away from $\hat{z}$ as in Figure 5. In the second case, consider the other half-axis $\mathcal{A}_- = \{\hat{z} + \lambda \hat{e}_j : \lambda \leq 0\}$. In both cases we may also consider the whole axis $\mathcal{A} = \{\hat{z} + \lambda \hat{e}_j : \lambda \in \mathbb{R}\}$. Assume that the translations of $f_1$ and $f_2$ along these (half-)axes, that is the faces $f_j + n\hat{e}_j + \hat{z}$ for $j = 1, 2$ and $n \in \mathbb{N}$, $(-n) \in \mathbb{N}$, or $n \in \mathbb{Z}$ respectively, are contained in $\Omega_{\mathcal{D}}$. We only consider the case of the positive half-axis $\mathcal{A}_+$ further. For $\mathcal{A}_-$ the argumentation is analogous and the case of the whole axis $\mathcal{A}$ is a simple consequence. Replace each face $f_1 + n\hat{e}_j + \hat{z}$ by its
translate $f_1 + n\mathbf{e}_{j_2} + \hat{z} + \hat{\mathbf{e}}_{j_3}$ for $n \in \mathbb{N}_0$ and similarly $f_2 + n\mathbf{e}_{j_2} + \hat{z}$ by $f_2 + n\mathbf{e}_{j_2} + \hat{z} + \hat{\mathbf{e}}_{j_1}$, for $n \in \mathbb{N}_0$, where $\mathbf{e}_{j_1}$ and $\hat{\mathbf{e}}_{j_2}$ are the vectors corresponding to the edges $\mathbf{e}_{j_1}$ and $\mathbf{e}_{j_3}$ respectively and pointing away from $\hat{z}$. Adding the face incident to $\hat{z}$, $\mathbf{e}_{j_1}$, and $\hat{\mathbf{e}}_{j_3}$, we obtain a different, but still monotone simply connected quad-surface.

The definition of an infinite flip for a black vertex $\hat{z} \in V_b(\Omega)$ is very similar.

**Lemma 5.7.** Let $\Omega \subset \mathbb{Z}^d$ be a simply connected monotone quad-surface and let $\Omega' \subset \mathbb{Z}^d$ be the simply connected monotone quad-surface obtained from $\Omega$ after performing a flip for a simply infinite strip. Let $C'$ be a circle pattern for $\mathcal{D}$ and the corresponding labelling and let $C'$ be the corresponding circle pattern after performing the corresponding infinite flip as for $\Omega'$. Then the resulting circle pattern $C'$ is embedded if the original one $C$ is.

The proof is based on similar arguments as the proof of Lemma 5.5 and is therefore left to the reader.

### 5.4 Uniqueness of isoradial quasicrystallic circle patterns

Let $\mathcal{D}$ be the family of all infinite quasicrystallic rhombic embeddings of connected and simply connected b-quad-graphs $G$ which cover the entire complex plane and such that the brick $\Pi(\Omega')$ of the corresponding quad-surface $\Omega'$ contains a $\mathbb{Z}^2$-sublattice, that is there are at least two different indices $j_1, j_2$ such that $\min_{n \in V(\Omega')} n_{j_k} = -\infty$ and $\max_{n \in V(\Omega')} n_{j_k} = \infty$ for $k = 1, 2$. Note that $\mathcal{D}$ contains in particular the plane based rhombic embeddings, like the Penrose tilings, for which $\Pi(\Omega') = \mathbb{Z}^d$. Now we use the uniqueness of $SG$-circle patterns of Theorem 3.1 in order to establish the uniqueness of the circle patterns of $\mathcal{D}$.

**Theorem 5.8** (Rigidity of quasicrystallic isoradial circle patterns). Let $\mathcal{D} \in \mathcal{D}$ be an infinite quasicrystallic rhombic embedding with associated graph $G$ and corresponding labelling $\alpha$. Let $C$ be an embedded circle pattern for $G$ and $\alpha$. Then $C$ is the image of the isoradial circle pattern corresponding to $\mathcal{D}$ under a similarity of the complex plane.

**Proof.** Let $\mathcal{D} \in \mathcal{D}$ be an infinite quasicrystallic rhombic embeddings with associated graph $G$ and corresponding labelling $\alpha$. Let $C'$ be an embedded circle pattern for $G$ and $\alpha$. Consider the comparison function $w$ for $C'$ defined by (3) on $\Omega'$ and extend it to $\Pi(\Omega')$. Let $\zeta \in V(\Omega')$. By assumption on $\mathcal{D}$, there is a $\mathbb{Z}^2$-sublattice $\Omega(\zeta)$ with $\hat{z} \in V(\Omega(\zeta))$ which is contained in $\Pi(\Omega')$. Furthermore, we can perform flips for $\Omega'$ and corresponding flips for the circle pattern $C'$ such that the resulting quad-surface $\Omega'$ contains an arbitrary number of generations of the lattice $\Omega(\zeta)$ about $\hat{z}$. As the corresponding circle pattern $C'$ is embedded by Lemma 5.7 and as the number of generations about $z$ can be chosen arbitrarily large, we deduce from the Rigidity Theorem 3.1 that the radius function is constant on $\Omega(\zeta)$. More precisely, the extension of $w$ is constant on white and black vertices of $\Omega(\zeta)$ respectively.

This argumentation is valid for all vertices $\hat{z} \in V(\Omega')$ and all $\mathbb{Z}^2$-sublattice $\Omega(\zeta)$ with $\hat{z} \in V(\Omega(\zeta))$ which are contained in $\Pi(\Omega')$. Therefore the extension of $w$ is constant, on white and black vertices respectively, on all $\mathbb{Z}^2$-sublattices which are contained in $\Pi(\Omega')$. Due to our assumptions on the combinatorics of $\mathcal{D}$, the radius function which is $w$ restricted to white vertices has to be constant on the whole brick $\Pi(\Omega')$. This implies in particular that $C$ is an isoradial circle pattern and thus is the image of the isoradial circle pattern corresponding to $\mathcal{D}$ under a similarity of the complex plane.

**Definition 5.9.** Let $\mathcal{D}$ be a rhombic embedding of a simply connected b-quad-graph. The **combinatorial $K$-environment** of a point $z \in V(\mathcal{D})$ is the subgraph corresponding to all vertices which have combinatorial distance at most $K$ from $z$ in $\mathcal{D}$.

14
**Corollary 5.10.** Let $\mathcal{D} \in \mathcal{E}$ be an infinite quasicrystallic rhombic embeddings with corresponding labelling $\alpha$. Let $v_0 \in V_w(\mathcal{D})$ be a white vertex. Then there are a constant $n_0 = n_0(\mathcal{D}) \in \mathbb{N}$ and a sequence $s_n(v_0, \mathcal{D})$ decreasing to 0 for $n \to \infty$ such that the following holds.

For $n \in \mathbb{N}$, $n \geq n_0$, let $\mathcal{D}_{2n}(v_0)$ be the rhombic embedding corresponding to the $2n$ environment of $v_0$. Let $G_n(v_0)$ be the associated graph. Let $\mathcal{C}_n$ be an embedded circle pattern for $G_n(v_0)$ and the labelling $\alpha$ taken from $\mathcal{D}$ with radius function $r_n$. Then there holds

$$\left| \frac{r_n(v_0)}{r_n(v_1)} - 1 \right| \leq s_n(v_0, \mathcal{D})$$

(14)

for all vertices $v_1 \in V(G_n(v_0))$ incident to $v_0$.

We omit the proof which is very similar to the proof of the Hexagonal Packing Lemma of [21]. The following lemma is similar to Lemma 3.2 and corresponds to the Ring Lemma of [21].

**Lemma 5.11.** Let $\mathcal{D}$ be a quasicrystallic rhombic embeddings with associated graph $G$ and labelling $\alpha$. Denote by $\alpha_{\min} = \min\{\alpha(e) : e \in E(G)\}$ the smallest intersection angle. Let $n_0 \in \mathbb{N}$ be such that $(n_0 - 3)\alpha_{\min} > \pi$. Let $v_0 \in V_w(\mathcal{D})$ be a white vertex. Assume that $\mathcal{D}$ contains a $(2n_0)$-environment about $v_0$. Then there is a constant $C = C(\mathcal{D}) > 0$ such that the following holds.

Let $r$ be the radius function of an embedded circle pattern for $\mathcal{D}$ and $\alpha$ and let $v_1$ be a vertex incident to $v_0$ in $G$. Then

$$\frac{r(v_1)}{r(v_0)} > C.$$

**Proof.** Suppose that there is a vertex $v_1$ incident to $v_0$ and a sequence of embedded circle patterns for $\mathcal{D}$ and $\alpha$ with radius functions $r_n$ such that $r_n(v_0) = 1$ and $r_n(v_1) \to 0$ as $n \to \infty$. Without loss of generality we may assume that the circle $C_0$ corresponding to the vertex $v_0$ and the intersection point corresponding to one fixed black vertex $w_0$ incident to $v_0$ and $v_2$ in $\mathcal{D}$ are fixed for the whole sequence. Then there is a subsequence such that all the circles converge to circles or lines, that is converge in the Riemann sphere $\hat{\mathbb{C}} \cong \mathbb{S}^2$.

Equation (2) implies that there are at least two circles corresponding to vertices incident to $v_0$ whose radii do not converge to 0. If the limit is finite, there are at least two circles whose radii do not converge to 0 corresponding to vertices incident to this vertex of the first generation and so on.

Consider the kites which contain in the limit the intersection point corresponding to $w_0$ and apply the above argument at most $n_0$ times. Then by our assumption on $n_0$ we obtain two kites whose interiors intersect in the limit. This is a contradiction to the embeddedness of the sequence. \qed

If $\mathcal{D}$ is a plane based quasicrystallic rhombic embedding we also obtain an analog of the Rodin-Sullivan Conjecture, see [21, 19, 5].

**Corollary 5.12.** Let $\mathcal{D}$ be a plane based quasicrystallic rhombic embedding. There are absolute constants $C = C(\mathcal{D}) > 0$ and $n_0 = n_0(\mathcal{D}) \in \mathbb{N}$, depending only on $\mathcal{D}$, such that for all white vertices $v_0 \in V_w(\mathcal{D})$ and all $n \geq n_0$ there holds

$$s_n(v_0, \mathcal{D}) \leq s_n(\mathcal{D}) \leq C/n.$$ 

(15)

**Proof.** Let $v_0 \in V_w(\mathcal{D})$ be any white vertex. If $n \geq n_0(\mathcal{D})$ is big enough, for each $\mathbb{Z}^2$-sublattice $\Omega(v_0)$ the set $\mathcal{F}(\Omega_\mathcal{D})$ contains a $[B(\mathcal{D})n]$-environment of $v_0$ in $\Omega(v_0)$,
where the constant $B(\mathcal{D})$ depends only on the construction parameters of $\mathcal{D}$. Here $\lfloor p \rfloor$ denotes the largest integer smaller than $p \in \mathbb{R}$. Therefore we can choose $s_n(\mathcal{D})$ to be the maximum of $s_{\lfloor nD \rfloor}(\mathcal{D}(\mathbf{v}_0))$ for all possible regular rhombic embeddings $\mathcal{D}(\mathbf{v}_0)$ corresponding to $\mathbb{Z}^2$-sublattice $\Omega(\mathbf{v}_0)$. Now the claim follows from Corollary 3.3 for $SG$-circle patterns.

\section{Uniqueness of quasicrystallic $Z^\gamma$-circle patterns}

In this section we consider quasicrystallic $Z^\gamma$-circle patterns as defined in \cite{9} and then prove their rigidity. The proofs are based on the results of the previous sections and on similar arguments as for orthogonal $Z^\gamma$-circle patterns.

\subsection{Definition and useful properties}

Let $\psi \in (0, \pi)$ be a fixed angle. Recall the definition of the labelling $\alpha_\psi$ in \cite{9}. We consider the following generalization of Definition 4.2.

**Definition 6.1** (\cite{9}). For $0 < \gamma < 2$, the discrete map $Z^\gamma : \mathbb{Z}^2_+ \to \mathbb{C}$ is the solution of

\[ q(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1}) := \frac{(f_{n,m} - f_{n+1,m})(f_{n+1,m+1} - f_{n,m+1})}{(f_{n+1,m} - f_{n+1,m+1})(f_{n,m+1} - f_{n,m})} = e^{2i(\psi - \gamma)} \quad (16) \]

and (\ref{eq:12}) with the initial conditions

\[ Z^\gamma(0,0) = 0, \quad Z^\gamma(1,0) = 1, \quad Z^\gamma(0,1) = e^{i(\pi - \psi)}. \]

As in the orthogonal case one can again associate a circle pattern (for a quadrant of SG corresponding to $\mathbb{Z}^2_+$ and related to $V$ and $\alpha_\psi$) to the map $Z^\gamma$, see \cite{9} for more details. Furthermore, the following results generalize the orthogonal case.

**Theorem 6.2** (\cite{9}). (i) If $R(z)$ denotes the radius function corresponding to the discrete conformal map $Z^\gamma$ for some $0 < \gamma < 2$, then

\[ (\gamma - 1)(R(z)^2 - R(z - i)R(z + 1)) \geq 0 \quad (17) \]

for all $z \in V \setminus \{\pm N + iN | N \in \mathbb{N}\}$.

(ii) For $0 < \gamma < 2$, the discrete conformal maps $Z^\gamma$ given by Definition 4.2 are embedded. Consequently, the corresponding circle patterns are also embedded.

Definition 6.1 can be generalized further. As explained in Section 13 of \cite{9}, discrete analogs of the power function $z^\gamma$ can also be defined for quasicrystallic rhombic embeddings $\mathcal{D}$ instead of $\mathbb{Z}^2_+$ (or $\mathbb{Z}^2$). In particular, let $\mathcal{A} = \{\pm a_1, \ldots, \pm a_d\} \subset S^1$ be the set of edge directions. Suppose that $d > 1$ and that any two non-opposite elements of $\mathcal{A}$ are linearly independent over $\mathbb{R}$. For $0 < \gamma < 2$ define the following values of the comparison function $w$ on the coordinate semi-axis of $\mathbb{Z}^d_+$:

\[ w(na_k) = \begin{cases} 1 & \text{if } n = 0, \\
^\gamma_{-1} = e^\gamma_{-1}a_k & \text{if } n \text{ is odd}, \\
_{n/2}^{m-1} \frac{m-1+\frac{1}{2}}{m-\frac{1}{2}} & \text{if } n \geq 2 \text{ and } n \text{ is even}. \end{cases} \quad (18) \]

The value of the logarithm $\log a_k$ is chosen as follows. Without loss of generality, we assume a circular order of the points of $\mathcal{A}$ on the positively oriented unit circle $S^1$ is
Figure 6: A quasicrystallic rhombic embedding with five-fold rotation symmetry.

Figure 7: Example of a quasicrystallic $Z^{5/6}$-circle pattern from the rhombic embedding of Figure 6. Construction (left) and corresponding circle pattern (right).

$a_1, \ldots, a_d, -a_1, \ldots, -a_d$. Set $a_{k+d} = -a_k$ for $k = 1, \ldots, d$ and define $a_m$ for all $m \in Z$ by $2d$-periodicity. To each $a_m = e^{i\theta_m} \in S^1$ assign a certain value of the argument $\theta_m \in R$: choose $\theta_1$ arbitrarily and then use the rule

$$\theta_{m+1} - \theta_m \in (0, \pi) \quad \text{for all } m \in Z.$$

Clearly we then have $\theta_{m+d} = \theta_m + \pi$. The points $a_m$ supplied with the arguments $\theta_m$ can be considered as belonging to the Riemann surface of the logarithmic function (i.e. a branched covering of the complex plane). Now, the branch of the logarithm is chosen such that

$$\log(a_l) \in [i\theta_m, i\theta_{m+d-1}], \quad l = m, \ldots, m+d-1.$$

Using the Hirota Equation (4), this function $w$ can be extended to the whole sector $Z_d^+$. Using suitable branches of the logarithm, $w$ may also be extended to other sectors or to a branched covering of $Z^d$.

Figure 7 shows an example of such a quasicrystallic $Z^d$-circle pattern.

Note that for $d = 2$ the boundary conditions given in (18) lead to the circle patterns specified in Definition 6.1. Therefore, by Theorem 6.2 (ii), the circle patterns corre-
Figure 8: A quasicrystalline rhombic embedding with ten-fold rotation symmetry.

Figure 9: Example of a quasicrystalline $Z^{5/4}$-circle pattern from the rhombic embedding of Figure 6. Construction (left) and corresponding circle pattern (right).

Corresponding to the restriction of $w$ to quad-surfaces $Z^d_+ \subset Z^d_d$ which are spanned by two coordinate semi-axis are embedded. We can apply finite and infinite flips to obtain other monotone quad-surfaces corresponding to rhombic embeddings. In particular, we obtain restrictions to $Z^d_+$ of the plane based quad-surfaces constructed in Example 5.4 and Remark 5.6. Lemmas 5.5 and 5.7 imply that these lead again to embedded circle patterns. Thus we have proven

**Theorem 6.3** (Embeddedness of quasicrystallic $Z^d_+$-circle patterns). Let $\Omega \subset Z^d_+$ be a simply connected monotone quad-surface. Then the circle pattern given by the function $w$ with initial values (18) is embedded.

**Example 6.4** (Construction of the examples in Figures 7, 9, and 11). The pictures in Figures 7, 9, and 11 have been obtained using a computer program implemented by Veronika Schreiber for her master thesis [24].

The rhombic embeddings in Figures 6 and 8 can be obtained by the construction
Figure 10: A quasicrystallic rhombic embedding with seven-fold rotation symmetry.

method explained in Example 5.4 using the affine planes $E_1 = \{x = t_1 + \lambda_1 u_1 + \lambda_2 u_2, \lambda_1, \lambda_2 \in \mathbb{R}\}$ and $E_2 = \{x = t_2 + \lambda_1 u_1 + \lambda_2 u_2, \lambda_1, \lambda_2 \in \mathbb{R}\}$ respectively, where $u_1, u_2, t_1, t_2$ are defined as follows. $\{u_1, u_2, u_3, u_4, u_5\}$ is an orthonormal basis such that the matrix \[
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\] takes the form \[
\begin{pmatrix}
\cos(2\pi/5) & -\sin(2\pi/5) & 0 & 0 & 0 \\
\sin(2\pi/5) & \cos(2\pi/5) & 0 & 0 & 0 \\
0 & 0 & \cos(4\pi/5) & -\sin(4\pi/5) & 0 \\
0 & 0 & \sin(4\pi/5) & \cos(4\pi/5) & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\] with respect to this basis. The translation vectors are $t_1 = (-0.2 -0.2 -0.2 -0.2 -0.2)^T$ and $t_2 = (-0.5 -0.5 -0.5 -0.5 -0.5)^T$ respectively. Here $v^T$ is the transpose of the vector $v$. Translations in direction of the vector $(1 1 1 1 1)^T$ generally lead to rotationally symmetric rhombic embeddings.

For the remaining construction it is important that the rhombic embeddings have rotational symmetry. Now consider the two sectors indicated by lines in Figures 6 and 8 or more precisely the corresponding 5-dimensional octant in $\mathbb{Z}^5$. Choose an exponent $\gamma$ of the power function $z^\gamma$. In order to construct an image circle pattern which closes up as in Figures 7 (right) and 9 (right) we need to choose an integer $p \geq 3$ and take $\gamma = 5/p$. Given these ingredients, define the values of the comparison function $w$ on the coordinate axes of the octant in $\mathbb{Z}^5$ according to (18) and calculate the missing values for the octant using the Hirota equation (4). Taking the values on the quad-surface corresponding to the original rhombic embedding, we can construct a sector of the desired quasicrystallic circle pattern, see Figures 7 (left) and 9 (left). The closed circle patterns in Figures 7 (right) and 9 (right) are obtained using the rotational symmetry.

The construction of the examples in Figures 10 and 11 is similar using a suitable affine plane in $\mathbb{Z}^7$. 
6.2 Uniqueness of quasicrystallic $Z^7$-circle patterns

In this section we prove uniqueness of quasicrystallic $Z^7$-circle patterns using the same arguments as for the orthogonal case.

We begin with a generalization of Proposition 4.6. Note that we need a (geometric) restriction which ensures that the kites corresponding to intersecting circles are convex. Unfortunately, flips (finite or infinite) may destroy convexity.

Proposition 6.5. Let $\mathcal{D}$ be a quasicrystallic rhombic embedding with associated graph $G$. Let $\alpha$ be the labelling corresponding to $\mathcal{D}$. Let $v_0$ be an interior vertex of $G$ with incident vertices $v_1, \ldots, v_m$. Consider two circle patterns for $G$ and $\alpha$ with radius functions $r$ and $\rho$. Denote $r_j = r(v_j)$ and $\rho_j = \rho(v_j)$ for $j = 0, 1, \ldots, m$ and suppose that $r_j \geq r_0 \cos \alpha_j$ and $\rho_j \geq \rho_0 \cos \alpha_j$ for $j = 1, \ldots, m$, where $\alpha_j = \alpha([v_0, v_j])$. Then

\[
\sum_{j=1}^{4} c_j r_j \rho_j \geq \sum_{j=1}^{4} c_j r_0 \rho_0 \quad \text{and} \quad \sum_{j=1}^{4} c_j \rho_j r_j \geq \sum_{j=1}^{4} c_j \rho_0 r_0,
\]

where $c_j = \sin \alpha_j / ((\rho_j / \rho_0) + (\rho_0 / \rho_j) - 2 \cos \alpha_j)$ for $j = 1, \ldots, m$.

Proof. The proof is based on a Taylor expansion of $f_\alpha(x + \log y)$ about $y = 1$.

\[
f_\alpha(x + \log y) = f_\alpha(x) + f'_\alpha(x)(y - 1) - \frac{\sin \alpha(e^{x+\log \xi} - \cos \alpha)}{2\xi^2(cosh(x + \log \xi) - \cos \alpha)^2}(y - 1)^2
\]

by Lemma 2.3 (i) with $\xi = t + (1 - t)y$ for some suitable $t \in (0, 1)$ depending on $x$ and

---

**Figure 11**: Example of a quasicrystallic $Z^{7/5}$-circle pattern from the rhombic embedding of Figure 6. Construction (left) and corresponding circle pattern (right).
Let case) using Proposition 6.5 and Lemma 6.6. 

The center of circle placed at the apex and which has only convex kites coincides with the same combinatorics and intersection angles which covers the same cone with one circle pattern on all four sectors. 

Theorem 6.7 (Rigidity of $Z^+$-circle patterns from Definition 7). If $\psi \leq \pi/2$ and $0 < \gamma < 2$ or if $\psi < \pi/2$ and $(\pi - 2\psi)/(\pi - \psi) \leq \gamma \leq \pi/(\pi - \psi)$, all kites in the $Z^+$-circle pattern given by Definition 6.1 are convex.

The proof is technical. A brief version is presented in the appendix A. More details can be found in [12].

The following lemma specifies for which parameters the kites of the $Z^+$-circle patterns provided by Definition 6.1 is the unique embedded SG-circle pattern for $Z^2_+$ and $\alpha_\psi$ (up to global scaling) with the following properties.

(i) The infinite sector $\{z = re^{i\beta} \in \mathbb{C} : r \geq 0, \beta \in [0, \gamma(\pi - \psi)]\}$ with angle $\gamma(\pi - \psi)$ is covered by the union of the corresponding kites of the circle pattern.

(ii) The centers of the boundary circles lie on the boundary half lines.

(iii) All kites corresponding to intersecting circles are convex.

The proof is a straightforward adaption of the proof of Theorem 4.4 (orthogonal case) using Proposition 6.5 and Lemma 6.6.

Theorem 6.8. Let $\psi \in (0, \pi)$ and $\gamma \in (0, 2) \cap [\frac{\pi - 2\psi}{\pi - 2\gamma}, \frac{\pi}{\pi - 2\psi}]$. Define $Z^+$-circle patterns on all four sectors $\mathbb{Z}_+ \times \mathbb{Z}_+$ according to Definition 6.7 and glue these patterns to a circle pattern $\mathcal{C}_\gamma$ on a cone with cone angle $2\pi\gamma$. Then any embedded circle pattern with the same combinatorics and intersection angles which covers the same cone with one center of circle placed at the apex and which has only convex kites coincides with $\mathcal{C}_\gamma$ (up to scaling and rotation about the apex of the cone).
The proof is a simple generalization of the proof of Theorem 6.7 to circle patterns on a cone and combinatorics of \( \mathbb{Z}^2 \) instead of \( \mathbb{Z}^2 \).

The following theorem is a direct consequence of the previous theorem and Lemmas 5.3 and 5.7 on local changes of quasicrystallic circle patterns.

**Theorem 6.9 (Rigidity of quasicrystallic \( \mathbb{Z}^2 \)-circle patterns I).** Let \( \mathcal{D} \) be a quasicrystallic rhombic embedding of a \( b \)-quad-graph covering the whole plane. Let \( \mathcal{A} = \{ \pm a_1, \ldots, \pm a_d \} \subset \mathbb{R} \) be the edge directions, where \( d > 1 \) and any two non-opposite elements of \( \mathcal{A} \) are linearly independent over \( \mathbb{R} \). Denote by \( \psi_{\min} \) the minimum of the undirected angles between any two elements of \( \mathcal{A} \). Let \( \gamma \in (0, 2) \) with \( (\pi - 2\psi_{\min})/(\pi - \psi_{\min}) \leq \gamma \leq \pi/(\pi - \psi_{\min}) \). Assume that the origin is a white vertex of \( \mathcal{D} \). Then a quasicrystallic \( \mathbb{Z}^2 \)-circle pattern \( \mathcal{C}_\gamma \) corresponding to \( \mathcal{D} \) and embedded on a cone with cone angle \( 2\pi\gamma \) can be defined using the definition of the comparison function \( w \) on the \( 2d \) sectors of \( \mathbb{Z}^d \) which contain the quad-surface \( \Omega_\mathcal{D} \); see [18] and the remarks below. Assume further that the brick \( \Pi(\Omega_\mathcal{D}) \) contains the whole lattice \( \mathbb{Z}^d \).

Let \( \mathcal{C} \) be an embedded circle pattern with the same combinatorics and the same intersection angles which covers the same cone with one center of circle placed at the apex. Extend the comparison function \( w \) for \( \mathcal{C} \) from \( \Omega_\mathcal{D} \) to \( \mathbb{Z}^d \). For each \( \mathbb{Z}^2 \)-sublattice which contains two coordinate axes suppose that the corresponding circle pattern built according to this comparison function has only convex kites. Then \( \mathcal{C} \) coincides with \( \mathcal{C}_\gamma \) up to scaling and rotation about the apex of the cone.

Note that the assumption on the convexity of the kites is only a restriction for a (small) neighborhood of the origin. This is due to Lemma 3.3 which implies that the ratio of the radii is almost one and thus the corresponding angles are almost the same in the isoradial case if the combinatorial distance to the origin is big enough.

If all intersection angles of the labelling \( \alpha \) are larger than \( \pi/2 \), then the kites of any corresponding circle pattern are convex. Examples of such rhombic embeddings are suitable regular hexagonal patterns. Hexagonal circle patterns and in particular analogs of the holomorphic mappings \( z^\gamma \) have been studied by Bobenko and Hoffmann in [5].

**Theorem 6.10 (Rigidity of quasicrystallic \( \mathbb{Z}^2 \)-circle patterns II).** Let \( \mathcal{D} \) be a quasicrystallic rhombic embedding of a \( b \)-quad-graph. Assume that the corresponding labelling \( \alpha: \mathcal{F}(\mathcal{D}) \to [\pi/2, \pi) \) has only values larger than \( \pi/2 \). Assume further that the origin is a white vertex. Let \( \gamma \in (0, 2) \).

Define a quasicrystallic \( \mathbb{Z}^2 \)-circle pattern \( \mathcal{C}_\gamma \) for \( \mathcal{D} \) and \( \alpha \) which is embedded on a cone with cone angle \( 2\pi\gamma \) using the definition of the comparison function on the \( 2d \) sectors of \( \mathbb{Z}^d \) whose union contains the quad-surface \( \Omega_\mathcal{D} \); see [18]. In particular, the circle corresponding to the origin is centered at the apex of the cone.

Let \( \mathcal{C} \) be an embedded circle pattern for \( \mathcal{D} \) and \( \alpha \) which covers the same cone. Suppose that the circle corresponding to the origin is centered at the apex and that \( \mathcal{C} \) has only convex kites. Then \( \mathcal{C} \) coincides with \( \mathcal{C}_\gamma \) up to scaling and rotation about the apex of the cone.

In this case that all intersection angles are larger than \( \pi/2 \) the proof of Theorem 4.4 can be directly adapted using Proposition 6.5 and the following lemma.

**Lemma 6.11.** Let \( \mathcal{D} \) be a quasicrystallic rhombic embedding of a \( b \)-quad-graph which covers the whole plane \( \mathcal{C} \). Let \( \mathcal{G} \) be the associated infinite graph built from white vertices. Then the simple random walk on \( \mathcal{G} \) is recurrent.

**Proof.** Recall that for the simple random walk all edges have conductance \( c(e) = 1 \) and also resistance \( r(e) = 1/c(e) = 1 \). Our aim is to prove that the effective resistance \( R_{\text{eff}} \) of the network \( \mathcal{G} \) with these unit conductances is infinite. If two incident vertices of \( \mathcal{G} \) are identified, that is the conductance of the connecting edge is increased to \( \infty \) and the
resistance decreased to 0, then the effective resistance of the new network is certainly smaller. More generally, the procedure of identifying a set of vertices of \( G \) is called shortening and reduces the effective resistance; see [14] Section 2.2.2 or [28] Theorem (2.19)]. Therefore, it is sufficient to show that we have infinite effective resistance \( R'_{\text{eff}} = \infty \) for a network \( G' \) which is obtained from \( G \) by shortening.

Without loss of generality we assume that all edges of \( \mathcal{D} \) have length one. Since \( \mathcal{D} \) is quasicrystalllic rhombic embedding the areas of the rhombi are uniformly bounded. Denote the lower bound by \( C_1 > 0 \). Let \( v_0 \in V(G) \) be any vertex and set \( g_k = 4k \) for \( k \in \mathbb{N} \). Denote by \( V_k \subset V(G) \) the set of vertices which are contained in the annulus \( A_k = \{ z \in \mathbb{C} : g_{k-1} \leq |z - v_0| < g_k \} \) for \( k \geq 1 \). Then \( V(G) = \cup_{k=1}^{\infty} V_k \). Identify the vertices of each \( V_k \) to one new vertex \( v_k \). Then by construction, \( v_k \) is only incident to \( v_{k-1} \) for \( k \geq 2 \) and to \( v_{k+1} \) for \( k \geq 1 \); see Figure 12. Denote by \( |E_k| \) the number of edges which are incident to \( v_k \) and \( v_{k+1} \) for \( k \geq 1 \). Then the effective resistance of the shortened network is \( R_{\text{eff}}' = \sum_{k=1}^{\infty} 1/|E_k| \). Furthermore \( |E_k| \leq \text{area of } \{ z \in \mathbb{C} : g_k - 2 \leq |z| < g_k + 2 \}/C_1 \). Now a simple estimation shows that \( R_{\text{eff}}' = \infty \). □

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A Appendix: Proof of Lemma 6.6

First note that kites with intersection angle larger than \( \pi/2 \) are always convex. The remaining claims on convexity are consequences of Proposition 3 of [3], namely

\[
\begin{align*}
(N+M)(R(z)^2 - R(z+1)R(z-i) - \cos \psi R(z)(R(z-i)-R(z+1)))(R(z+i)+R(z+1)) \\
+ (M-N)(R(z)^2 - R(z+1)R(z+i) - \cos \psi R(z)(R(z+i)-R(z+1)))(R(z+i)+R(z-i)) &= 0
\end{align*}
\]

for \( z \in V \setminus \{ N+iN | N \in \mathbb{N} \} \), and the results of Theorems 6.2.

In particular, for \( \pi - \psi \leq \pi/2 \) the kites with intersection angle \( \psi \leq \pi/2 \) at black vertices are always convex. This follows from equation (20) and inequality (17) by simple calculations. This shows the claim for \( \psi \geq \pi/2 \).

If \( \psi < \pi/2 \) and \( (\pi - 2\psi)/\pi - \psi \leq \gamma \leq \pi/(\pi - \psi) \), inequality (17) only implies that for all kites with white vertices \( z \) and \( z+i \) and intersection angle \( \psi \) the angle at the point corresponding to \( z \) for \( 0 < \gamma < 1 \) and to \( z+i \) for \( 1 < \gamma < 2 \) respectively is smaller than \( \pi \). This excludes some types of non-convex kites, but not all.

For fixed \( n > 0 \), let \( \Gamma_n \) be the piecewise linear curve formed by the segments \([f_{n,m}, f_{n,m+1}]\) for \( m \geq 0 \). By Theorem 6.2 and its proof in [3], these curves are embedded without self-intersections and the vector \( v_n(m) = f_{n,m+1} - f_{n,m} \) rotates clockwise for \( 0 < \gamma < 1 \) and counterclockwise for \( 1 < \gamma < 2 \) along \( \Gamma_n \).
Without loss of generality, we only consider the case $1 < \gamma < 2$ further. For $0 < \gamma < 1$ the proof is very similar. First, consider a kite on the symmetry axis, that is with white vertices corresponding to $iK$ and $i(K + 1)$. Then the assumption $\gamma(\pi - \psi) \leq \pi$ and the properties of the curves $\Gamma_n$ imply that the kites on the symmetry axis are convex. Next, we consider the intersection angles $\alpha$ and $\beta$ of the line in direction of the vector $v_n(m)$ with the oriented half lines $\mathbb{R}^+$ and $e^{i\gamma(\pi - \psi)/2}\mathbb{R}^+$ respectively. We additionally assume that $n$ is odd. As the kites on the symmetry axis are convex and $\gamma(\pi - \psi) \leq \pi$, we deduce $\alpha \leq \psi$ and $\pi/2 - \psi \leq \beta \leq \pi/2$.

Finally, consider a kite with white vertex $f_{n,m}$ for odd $n$ and $m < n$. We estimate the angles $\alpha_1$ and $\alpha_2$ at this vertex of the kites containing the points $f_{n,m-1}$, $f_{n,m}$, $f_{n+1,m}$ and $f_{n+1,m}$, $f_{n,m}$, $f_{n,m+1}$ respectively. Note that these kites both have intersection angles $\psi$. Using the above estimations on $\alpha$ and $\beta$ and the properties of the curve $\Gamma_n$ we obtain $\pi - 2\psi \leq \alpha_1$ and $\alpha_2 \leq \pi$. Therefore the angle at $f_{n-1,m-1}$ of the kite containing the points $f_{n,m-1}$, $f_{n,m}$, $f_{n+1,m}$ is $2\pi - 2\psi - \alpha_1 \leq \pi$. So this kite is convex. Furthermore, we deduce that the kite containing the points $f_{n+1,m}$, $f_{n,m}$, $f_{n,m+1}$ is also convex. Consequently all kites containing the white vertex $f_{n,m}$ are convex.

Thus the $Z^\psi$-circle pattern with $\psi < \pi/2$ and $(\pi - 2\psi)/(\pi - \psi) \leq \gamma \leq \pi/(\pi - \psi)$ only contains convex kites.

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