Branch and Bound Method to Resolve Non-Convex Quadratic Problems Over a Boxed Set

Boutheina Gasmi1*, Rachid Benacer2

1 Laboratoire of (LTM), University Batna 2, Faculty of Maths and Informatic 05078Fesdis, Batna, Algeria.
2 Laboratoire of (LTM), University Batna 2, Faculty of Maths and Informatic 05078Fesdis, Batna, Algeria.
*Corresponding author E-mail: gasmi.boutheina2@gmail.com

Abstract

We present in this paper the technique Branch and Bound with new quadratic approach over a boxed arrangement of \( R^n \). We develop an inexact arched quadratic capacity of the target capacity to decide a lower bound of the worldwide ideal estimation of the first non raised issue (NQP) over every subset of this boxed set. We connected a segment and specialized lessening on the feasible area of (NQP) to quicken the intermingling of the proposed calculation. Finally, we think about the assembly of the proposed calculation and we give a straightforward examination between this strategy and another technique which have a similar guideline.

Keywords: Non Convex Quadratic Programming, Global Optimization, Optimization Methods, Branch and Bound Method, Bilinear 0-1 programming.

1. Introduction

We consider the accompanying non raised quadratic programming issues:

\[
\begin{align*}
\min f(x) &= \frac{1}{2} x^T Q x + d^T x \\
\text{s.t.} & \quad x \in S \cap (D_f)
\end{align*}
\]  
\text{(NQP)}

where:

\[
S^0 = \left\{ x \in \mathbb{R}^n : L^0_i \leq x_i \leq U^0_i : i = 1, n \right\}
\]

\( D_f \) = \{ \( x \in \mathbb{R}^n : A x \leq b ; x \geq 0 \} \)

\( Q \) : is a real \((n \times n)\) non positive symetric matrix

\( A \) : is a real \((n \times n)\) symmetric matrix

\( dT \) = \((d_1, d_2, \ldots, d_n) \in \mathbb{R}^n \)

In our life, each thing, each issue is make as a mathematic issues [5], we can likewise take the statements of Gualili "The word is made at numerical language or scientific issues", specially "quadratic one".

In this paper we present another square shape Branch and Headed methodology for taking care of non arched quadratic programming issues were we construct a lower rough raised quadratic elements of the target quadratic capacity \( f \) over a boxed arrangement of \( R^n \) introduced as a \( n \)-rectangle [2].

This lower inexact capacity is given to decide a lower bound of the worldwide ideal estimation of the first issue (NQP) over every square shape.

The paper is sorted out as follows:

Area 1: In this segment we give a basic presentation of our examinations; in which we give and characterize the standard type of our concern.

Area 2: another proportional type of the target work proposed as an lower rough straight elements of the quadratic structure over the \( n \)-square shape [6]. We can likewise proposed as an upper inexact direct capacities, however we should regard the procedure of ascertain the lower and the upper bound of the first central square shape \( S^0 \) which noted by \( S^k \) in the \( k \)-step [4].

Area 3: In this segment we characterize another lower surmise quadratic elements of the quadratic non raised capacity over a square shape as for a square shape to ascertain a lower bound on the worldwide ideal estimation of the first no arched issue (NQP) [7].

Segment 4: We give another straightforward square shape apportioning strategy and depict square shape lessening strategies [3].

Segment 5: Gives another Branch and Decrease Calculation so as to take care of the first non raised issue (NQP).

Segment 6: We examine the intermingling of the proposed Calculation and we give a straightforward examination between this technique and different strategies which have a similar rule [1].

At long last, a finish of the paper is given to appear and explain the proficiency of the proposed Calculation.

2. The Equivalent forms of \( f \) over the \( n \)-rectangle

In this area we build and characterize the equal type of the non raised quadratic capacity which proposed as a lower inexact straight capacities over an \( n \)-rectangle \( S^0 = \left[ L^k, U^k \right] \).
this work is proposed to decide the lower bound of the worldwide ideal estimation of the first issue (NQP).

Let $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$ be the min eigenvalue and the maximum eigenvalue of the framework $Q$ respectively, and we demonstrate the number $\theta$ that $\theta \geq |\lambda_{\text{min}}|$. The equivalent linear form of the objective function $f$ is given by:

$$f(x) = (x - U^K)^T(Q + \theta I)(x - U^K) + d^Tx - \theta \sum_{i=1}^n x_i^2$$

$$+ 2(U^K)^T(Q + \theta I)x - (U^K)^T(Q + \theta I)U^K$$

(2)

by the use of the lower bound $L_i$, and is given by:

$$f(x) = (x - U^K)^T(Q + \theta I)(x - U^K) + d^Tx - \theta \sum_{i=1}^n x_i^2$$

$$+ 2(U^K)^T(Q + \theta I)x - (U^K)^T(Q + \theta I)U^K$$

(3)

by the use of the upper bound $U_i$ of the rectangle $S_i$.

In the other hand, we have the following definition:

**Definition 1:** Let the function $f : C \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ and the rectangle $S_i \subseteq C \subseteq \mathbb{R}^n$, the convex envelope of the function $f$ is given by:

$$f_{\text{conv}}(x) = \delta_i x_i + \eta_i : i = 1, n$$

with:

$$\delta_i = \frac{f_i(U_i^0) - f_i(L_i^0)}{U_i - L_i} : i = 1, n$$

$$\eta_i = f_i(L_i^0) - \delta_i L_i^0 : i = 1, n$$

So, by the use of this definition the convex envelope of the function $h(x) = (-x^2_j)$ over the interval $S_{ij} = [L_j^0, U_j^0]$ is given by the function:

$$h_{\text{conv}}(x) = -(U_j + L_j)xi + L_j U_j$$

which is a linear function, then we get the best linear lower bound of $h(x) = x^2_j$ given by:

$$f_{\text{conv}}(x) = \sum_{i=1}^n \delta_i x_i + \sum_{i=1}^n \eta_i$$

(4)

$$f_{\text{conv}}(x) = -(U_j^0 + L_j^0)x_i + (U_j^0)^T U_j$$

(5)

### 3. Lower Approximate functions and Error Calculation

By definition, the initial rectangle $S_i$ is given by:

$$S_i = \{x \in \mathbb{R}^n : L_i^0 \leq x_i \leq U_i^0 : i = 1, n\}$$

We subdivide this rectangle into two sub-rectangles defined by:

$$S_{ij} = \{x \in \mathbb{R}^n : L_i^0 \leq x_i \leq b_i^j : b_i^j \leq x_j \leq L_j^0 : j = 1, n : j \neq s\}$$

$$S_{isk} = \{x \in \mathbb{R}^n : b_i^j \leq x_i \leq L_i^0 : L_i^0 \leq x_j \leq U_j^0 : j = 1, n : j \neq s\}$$

(6)

Where, we calculate the point $h_i$ by a normal rectangular subdivision ($\omega$-subdivision).

#### 3.1. The lower approximate linear function of $f$ over the rectangle $S_i^k$:

The best lower inexact direct capacity of the target non raised capacity $f$ over the square shape $S_i^k$ is given in the accompanying hypothesis:

**Theorem 1:** Let the function $f : C \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ and the rectangle $S_i^k \subseteq \mathbb{R}^n$ where $C \subseteq S_i^k \subseteq \mathbb{R}^n$, the lower approximate linear function of $f$ is given by:

$$L_{S_i^k}(x) = (a_{S_i^k})^T x + b_{S_i^k}$$

(7)

where:

$$a_{S_i^k} = d + 2(Q + \theta I)L_i^k - \theta(L_i^k + U_i^k)$$

$$b_{S_i^k} = -(L_i^k)^T(Q + \theta I)L_i^k + \theta(L_i^k)^T(U_i^k)$$

$$a_{S_i^k} = d + 2(Q + \theta I)U_i^k - \theta(L_i^k + U_i^k)$$

$$b_{S_i^k} = -(U_i^k)^T(Q + \theta I)U_i^k + \theta(L_i^k)^T(U_i^k)$$

(8)

#### 3.2. The new lower approximate quadratic convex function of $f$ over the rectangle $S_i^k$:

We utilize the former lower estimated direct capacity of $f$ over the square shape $S_i^k$ to characterize the new lower inexact quadratic capacity of $f$ over a similar square shape by:

**Definition 2:**

$$L_{\text{quad}}(x) := L_{S_i^k}(x) - \frac{1}{2}K(U_i^k - x)(x - L_i^k)$$

$$U_{\text{quad}}(x) := U_{S_i^k}(x) - \frac{1}{2}K(U_i^k - x)(x - L_i^k)$$

and:

$$L_{\text{quad}}(x) := L_{S_i^k}(x) - \frac{1}{2}K(U_i^k - x)(x - L_i^k)$$

(9)

where:

- $K$ is a positive real number given by the spectral radius of the matrix $(Q + \theta I)$$$
- $\theta \geq |\lambda_{\text{min}}|$ is the best lower approximate lineair function of $f$ over the rectangle $S_i^k$.

#### 3.3. The New Lower Approximate Linear Function of $f$ over the Rectangle $S_i^k$:

By the utilization of the first new lower inexact quadratic capacity of $f$ over the square shape $S_i^k$, we can characterize the new lower estimated straight capacity of $f$ over a similar square shape by:

**Definition 3:**

$$L_{\text{quad}}(x) := L_{S_i^k}(x) - \frac{1}{2}K(U_i^k - x)(x - L_i^k)$$

$$U_{\text{quad}}(x) := U_{S_i^k}(x) - \frac{1}{2}K(U_i^k - x)(x - L_i^k)$$

and:

$$L_{\text{quad}}(x) := L_{S_i^k}(x) - \frac{1}{2}K(U_i^k - x)(x - L_i^k)$$

(10)

**with:**

$$h := \|U_i^k - L_i^k\|$$

#### 3.3.1. The relation between the convex quadratic approximation and the linear one

We have the following theorem:
Theorem 3.4: The tow following inequality are satisfied:
\[ \tilde{L}_{\text{quad}}(x) = L_{SK}(x) - \frac{1}{2}K h^2 \leq L_{\text{quad}}(x) \leq f(x) \]
\[ \tilde{U}_{\text{quad}}(x) = U_{SK}(x) - \frac{1}{2}K h^2 \leq U_{\text{quad}}(x) \leq f(x) \]
for all \( x \in (D) \cap S^K \) and \( h := ||U^K - L^K|| \) and
\[ \left\| \frac{\partial^2 f(x)}{\partial x^2} \right\| \leq K \]
(The normality condition).

Proof: Let the function \( g_1: \mathbb{R}^n \rightarrow \mathbb{R} \) defined by:
\[ g_1(x) = \frac{1}{2}K \left( -2x + (L^K + U^K) \right) \]
Thus:
\[ \left( \frac{\partial g_1}{\partial x} \right)(x) = 0 \iff \left( x = \frac{1}{2} \left( L^K + U^K \right) \right) \]
The critical point of the function \( g_1 \) is the middle point of the edge \( [L^K, U^K] \), in the other hand, the function \( g_1 \) is concave, immediately, at the middle point max \( x^* = \frac{(L + U)}{2} \) of \( [L^K, U^K] \), then we have:
\[ g_1(x) = \max \{ g_1(x) : x \in (D) \cap S^K \} = g_1(x^*) \]
Then:
\[ L_{\text{quad}}(x) - L_{\text{quad}}(x^*) \leq 0 \]
In the other hand, we define the function \( g_2: \mathbb{R}^n \rightarrow \mathbb{R} \) given by:
\[ g_2(x) = f(x) - L_{\text{quad}}(x) \]
Passing to the first derivation of \( g_2 \), then, we get:
\[ \frac{\partial g_2}{\partial x}(x) = \frac{\partial f}{\partial x}(x) - \frac{\partial L_{SK}}{\partial x}(x) + \frac{1}{2}K \frac{\partial}{\partial x} \left( [(U^K - x)(x - L^K)] \right) \]
\[ = \frac{\partial f}{\partial x}(x) - a_{SK} + \frac{1}{2}K \frac{\partial}{\partial x} \left( -x^2 + (U^K + L^K)x - L^K U^K \right) \]
\[ = \frac{\partial f}{\partial x}(x) - a_{SK} + \frac{1}{2}K \frac{\partial}{\partial x} \left( -2x + (U^K + L^K) \right) \]
Then, passing to the second derivation:
\[ \frac{\partial^2 g_2}{\partial x^2}(x) = \frac{\partial^2 f}{\partial x^2}(x) - K \]
We have the condition:
\[ \frac{\partial^2 f}{\partial x^2}(x) \leq K \]
(The normality condition)
Then, we obtain:
\[ 0 \leq \frac{\partial^2 g_2}{\partial x^2}(x) \]
Thus, the function \( g_2 \) is concave over \( S^K \) and by this we have:
\[ g_2(x) \geq \max \{ g_2(x) : x \in S^K \} = \min \{ g_2(L^K), g_2(U^K) \} = 0 \]
Then:
\[ (g_2(x) = f(x) - L_{\text{quad}}(x) \geq 0 \Rightarrow L_{\text{quad}}(x) \leq f(x) \]
Finally, we get:
\[ L_{\text{quad}}(x) \leq L_{\text{quad}}(x) \leq f(x) : x \in S^K \]
3.4. Approximation errors

We can assess the guess mistake by the separation between the non arched target work and here lower approximation capacities.

3.4.1. The linear approximation error

Is induced by the separation between the capacity \( f \) and here lower surmise direct capacity \( \tilde{L}_{\text{quad}} \) over the boxed set \( SK \), then we have the accompanying suggestion:

\[ \max_{x \in (D) \cap S^K} \left\{ f(x) - L_{\text{quad}}(x) \right\} \leq \left( \rho(Q + \Theta) + \theta + \frac{1}{2}K \right) ||U^K - L^K||^2 \]

\[ \max_{x \in (D) \cap S^K} \left\{ f(x) - L_{\text{quad}}(x) \right\} \leq \left( \rho(Q + \Theta) + \theta + \frac{1}{2}K \right) ||U^K - L^K||^2 \]

Proof: Let the function \( f: C \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \) where \( C \subseteq S^0 \subseteq \mathbb{R}^n \) and \( \theta \geq |\lambda_{\text{min}}| \) for this the matrix \( (Q + \Theta) \) be semi-positive, then we have:

\[ f(x) - L_{\text{quad}}(x) = (x - L^K)^T (Q + \Theta)(x - L^K) + \theta^T (x - L^K) + \frac{1}{2}K \theta^T \]

The same thing when we use the upper bound \( U_{\text{quad}}(x) \) with the equivalent linear form of the objective function \( f \) and we obtain:

\[ L_{\text{quad}}(x) - U_{\text{quad}}(x) = (L^K - U^K)^T x - L^K x - (L^K)^T x \]

Proposition 3.5: Let the function \( f: C \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \) where \( C \subseteq S^0 \subseteq \mathbb{R}^n \) and \( \theta \geq |\lambda_{\text{min}}| \) for this the matrix \( (Q + \Theta) \) be semi-positive, then we have:

\[ \max_{x \in (D) \cap S^K} \left\{ f(x) - L_{\text{quad}}(x) \right\} \leq \left( \rho(Q + \Theta) + \theta + \frac{1}{2}K \right) ||U^K - L^K||^2 \]

3.4.2. The quadratic approximation error

Is exhibited by the separation between the capacity \( f \) and here lower rough quadratic capacity \( \tilde{L}_{\text{quad}} \) over the square shape \( SK \), at that point we have the accompanying recommendation:

\[ \max_{x \in (D) \cap S^K} \left\{ f(x) - L_{\text{quad}}(x) \right\} \leq \left( \rho(Q + \Theta) + \theta + \frac{1}{2}K \right) ||U^K - L^K||^2 \]

Proof: By the definition of the function \( L_{\text{quad}}(x) \) as well as the meaning of \( \phi_k(x) \), we have:

\[ f(x) - L_{\text{quad}}(x) = (x - L^K)^T (Q + \Theta)(x - L^K) + \frac{1}{2}K \theta^T \]

The same thing when we use the upper bound \( U_{\text{quad}}(x) \) with the equivalent linear form of the objective function \( f \) and we obtain:

\[ U_{\text{quad}}(x) \leq f(x) : x \in S^K \]
\[
\|f(x) - L_{\text{quad}}(x)\|_\infty = \max \{\|f(x) - L_{\text{quad}}(x)\| : x \in S^K \cap (D_i)\}
\leq \|x - L_{\text{quad}}(x)\|_\infty + \|f(x) - \theta_1(x)(x - L_{\text{quad}}(x))\|_\infty
\leq (\|f(x) - \theta_1(x)(x - L_{\text{quad}}(x))\|_\infty
The same thing when we use the lower bound \(U_{\text{quad}}(x)\) with the equivalent linear form of the objective function \(f\) and we obtain:
\[
\|f(x) - U_{\text{quad}}(x)\|_\infty \leq (\|f(x) - \theta_1(x)(x - L_{\text{quad}}(x))\|_\infty
\]
So, the proof is complete.

3.5. The quadratic approximate problem (QAP)

3.5.1. Construction of the interpolate problem (IP)

It’s clear that:
\[
f(\bar{x}) \geq \max \{L_{\text{quad}}(x) : x \in (D_i) \cap S^K\} = \gamma(x)
\]  
This function present the best quadratic lower bound of \(f\), similarly, we construct the following interpolate problem by:
\[
\begin{align*}
\min \alpha & & \\
\text{b} \quad \text{(LBP)} & & \\
x \in \{L_{\text{quad}}(x) : x \in (D_i) \cap S^K\} & & \\
\end{align*}
\]
And the convex quadratic problem define by:
\[
\begin{align*}
\min \alpha & & \\
\text{b} \quad \text{(AQP)} & & \\
x \in \{x \in (X) : x \in (D_i) \cap S^K\} & & \\
\end{align*}
\]
The question is: what’s the relation between the optimal values
\[
f(x), f(x^*) \text{ and } L_{\text{quad}}(x^*)
\]
We have the following proposition:

**Proposition 3.7**: Let the function \(f : C \subseteq \mathbb{R}^n \rightarrow \mathbb{R}\) and \(S^\alpha \subseteq \mathbb{R}^n\) where \(C \subseteq S^\alpha \subseteq \mathbb{R}^n\), we have:
\[
0 \leq f(\bar{x}) - f(x^*) \leq (\rho\|Q + \theta I\| + \theta + \frac{1}{2}K) \|U^K - L^K\|^2
\]
and:
\[
L_{\text{quad}}(\bar{x}) \leq f^* \leq f(\bar{x})
\]
with \(f^* = f(x^*)\) is the global optimal value of the original problem (NQP) and \(x^*\) is the \(e\) optimal solution of (ACQP)

**Proof**: From the previous proposition, we have:
\[
f(x) - L_{\text{quad}}(x) \leq (\rho\|Q + \theta I\| + \theta + \frac{1}{2}K) \|U^K - L^K\|^2 ; x \in S^k \cap (D_i)
\]
And for:
\[
\begin{align*}
\bar{x} & = \bar{x} : \\
& \\
& \end{align*}
\]
Thus:
\[
f(\bar{x}) - f^* \leq (\rho\|Q + \theta I\| + \theta + \frac{1}{2}K) \|U^K - L^K\|^2
\]
And:
\[
f(\bar{x}) - f^* \leq (\rho\|Q + \theta I\| + \theta + \frac{1}{2}K) \|U^K - L^K\|^2 + (L_{\text{quad}}(\bar{x}) - f^*)
\]
As well as \(L_{\text{quad}}(x) - f^* \leq 0\), we have:
\[
0 \leq f(\bar{x}) - f^* \leq (\rho\|Q + \theta I\| + \theta + \frac{1}{2}K) \|U^K - L^K\|^2
\]
In the other hand, we have:
\[
\begin{align*}
L_{\text{quad}}(\bar{x}) - f^* & \leq 0 \\
f(\bar{x}) - f^* & \geq 0
\end{align*}
\]
\[
\iff (L_{\text{quad}}(\bar{x}) \leq f^* \leq f(\bar{x})
\]
Then, the proof is complete.

3.5.2. Question: is the solution \(x^*\) present the best lower bound of the global

**optimal solution of (NQP)?**

We have the following proposition:

**Proposition 3.8**: Let take the estimate function noted by:
\[
E(x) := f(x) - L_{\text{quad}}(x)
\]
For all \(x \in S^k \cap (D_i)\), the following inequality is satisfied:
\[
E(x) \geq (\gamma(x) - f^*) e \quad e
\]

**Proof**: We have:
\[
f(x) - f^* = f(x) - L_{\text{quad}}(x) + L_{\text{quad}}(x) - f^* e \quad e
\]
And, from the previous proposition we have:
\[
L_{\text{quad}}(x) \leq f^* \leq f(\bar{x}) e \quad e
\]
So:
\[
L_{\text{quad}}(x) - f^* \leq 0 \quad e
\]
Then:
\[
f(x) - f^* \leq E(x) e \quad e
\]

**Lemma**: If \(E(x)\) is a small value, then \(f(x)\) is an acceptable approximate value to \(e \quad e\) the global optimal value \(f^* = f(x^*)\) over the rectangle \(S^k\). Similarly, we can find that the point \(x\) is the global approximate solution of the global optimal solution \(x^*\) of the original problem (NQP) over \(S^k\).

**Proof**: We have:
\[
f(x) - f^* = E(x) e \quad e
\]
So, let take that \(E(x)\) is a small value we get:
\[
f(\bar{x}) - f^* \leq E(\bar{x}) \quad e \quad e
\]
Then:
\[
k(x) - f^* \quad e \quad e
\]
And:
\[
\lim_{x \to \text{quad}} \|f(\bar{x}) - f^*\| = 0
\]
Immediately, we get that \(f(x)\) is an acceptable approximate value to the global optimal \(e\) value \(f^* = f(x^*)\). Similarly, the point \(x\) is a global approximate solution of the original problem (NQP) over \(S^k\).

In the other hand, the rank of the non convex function \(f\) over the new rectangle (subrectangle) \(S^k\) is small then here rank over the initial rectangle \(S^k\) by this, the value \(E(x) e\) will be very small.

4. The technical reduction (technical eliminate)

We get to describe the rectangle partition by the following steps:

**Step(0)**:
Let:
\[
S^K = \{x^k \in \mathbb{R}^n : L_i^K \leq x^k_i \leq U_i^K : i = 1, n\}
\]
with \(x^k \in S^k\).
Step(1): We find a partition information point:
\[ h_s = \max \left\{ (x_i - L_i^K)(U_i^K - x_i) : i = 1, n \right\} \quad (23) \]

Step(2): If \( h_s = 0 \) then we partition the rectangle \( S^K \) into two sub-rectangles by the point \( h_s \), else, we partition the rectangle \( S^K \) into two sub-rectangles on the longest edge \( \left[ L_i^K, U_i^K \right] \) by the middle point \( \frac{L_i^K + U_i^K}{2} \) which is yet noted as \( h_s \).

Step(3): The rest rectangle is yet noted as \( S^k \).

Now, we describe the rectangle reducing tactics to accelerate the convergence of the proposed global optimization algorithm (ARSR).

Remarks:
1. All linear constraints of the problem (NQP) are expressed as:
   \[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i : i = 1, n \]
2. The rectangle \( S^K \) be also recorded as constraint to be added to the problem (NQP).
3. The minimum and the maximum of each function:
   \[ \begin{align*}
   \psi(x_i) &= a_{i\alpha} x_{\alpha} : i = 1, n \\
   x_i &\in \left[ L_i^K, U_i^K \right]
   \end{align*} \]

Are obtaind at the extremes points of the same interval.

Linearity Based Range Reduction Algorithm:
This algorithm is given to reduce and delete the rectangle \( S^K \).

Program (LBRA)
Let \( J := \left\{ 1, 2, 3, \ldots, n \right\} \) the set of the index, \( P_k := P \) for \( 1 \leq i \leq n \) do
\[ r U_i := \sum_{j=1}^{n} \max \left\{ a_{ij} L_j^K, a_{ij} U_j^K \right\} \]
compute \( r L_i := \sum_{j=1}^{n} \min \left\{ a_{ij} L_j^K, a_{ij} U_j^K \right\} \)
compute if \( r U_i > b_i \) then stop.
the problem (NQP) is infeasible over \( S^K \) (there are no solution of (NQP) over \( S^K \) because, \( S^K \) is deleted. From the subrectangle set produced through partitioning of the rectangle \( S^K \) else if \( r U_i < b_i \) then the constraint is redundant.
\[ I_i := I_i \setminus \{ i \} \]
\[ P_i := P_i \setminus \{ x \in R^n : (a_i)^T x \geq b_i \} \]
else for \( 1 \leq j \leq n \) do \( a_{ij} > 0 \) then
\[ U_j^{k+1} := \min \left\{ U_j^{k}, b_j - r L_i + \min \left\{ a_{ij} L_j^{k}, a_{ij} U_j^{k} \right\} \right\} \]
else \[ U_j^{k+1} := \max \left\{ U_j^{k}, b_j - r U_i + \max \left\{ a_{ij} L_j^{k}, a_{ij} U_j^{k} \right\} \right\} \]
end if and do end if
end if

5. Algorithm (ARSR): Branch and Bound

5.1 Algorithm (ARSR): Branch and Bound program (ARSR)

Initialization: Determine the initial rectangle \( S^0 \) where \( (\gamma) \subset S^0 \) and suppose that:
\[ \text{QLBP}_0 := S^0 \cap (\gamma) \]
iteration \( k \):
if \( \text{QLBP}_k = \emptyset \) then solve the quadratic problem (LBLP) when \( k = 0 \)
Let \( x^K \) be an optimal solution of (LBLP) and \( a(S^0) \) be the optimal value acompanied to \( x^K H := S^K \) (the set of the subrectangle of the initial rectangle \( S^K \)) \( a_0 := \min \{ a(S^0), b_0 := f(x^K) \} \) (the upper bound of \( f(x^K) \)) \( k := 0 \) while \( \text{Stop} = \text{false} \) do if \( a_0 = b_0 \) then \( \text{Stop = true} \)
\( x^* \) is a global optimal solution of the problem (NQP).

If \( LBP \) is added to the quadratic one. In the other hand, we give some guide to explain the proposed calculation.

6. The convergence of the Algorithm (ARSR)

In this area, we examine the union of the proposed calculation (ARSR) and we give a straightforward examination between the direct surmised and the quadratic one. In the other hand, we give some guide to explain the proposed calculation.

6.1. The convergence of the proposed algorithm

The proposed calculation in area 5 is not the same as the one in ref [3] in lower-jumping (quadratic estimate), and added to the square shape decreasing methodology. We will demonstrate that the proposed calculation be united.

Hypothesis 6.1 : On the off chance that the proposed calculation ends in limited advances, at that point a worldwide ideal arrangement of the issue (NQP) is gotten when the calculation ends.

Confirmation: Let the outcome out coming when the calculation end be \( x^k \), at that point, quickly we have \( x = Bx \) while ending at the \( k \)-step, so \( x^k \) be a global optimal solution of the problem (NQP).

Theorem 6.2 If the algorithm generates an infinite sequence \( \{ x^K \}_k \subset N^*, \) then every accumulation piont \( x^* \) of this sequence is a global optimal solution of the problem (NQP)(i.e: the global optimal solution is not unique).
7. comparison between “Branch and Bound” and “Method (DCT)”

7.1 Method (DCT)

We should include the ordinarity condition characterize by the decision of the parameter µ > 0 so as to garante the presence of the worldwide ideal arrangement, this condition is given by:

7.1.1 Introduction

Let take the non convex quadratic optimization problem given by:

\[
\begin{align*}
\min f(x) &= \frac{1}{2}x^TQx - d^Tx \\
Ax &\leq b; \quad x \geq 0
\end{align*}
\]

where \( A \in \mathbb{R}^{m \times n} \) arbitrary matrix, \( b \) vertex of \( \mathbb{R}^n \)

The fundamental idea of this method is in the choice of the operator:

\[ \Lambda(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m \]

By this the objective function be write as the following canonical form:

\[ f(x) = \Phi(x, \Lambda(x)) \]

Define over the set \( \mathbb{R}^n \times \mathbb{R}^n \) to R in the condition that the function be canonical at every unone (point) \( x \) and \( y \).

We need the following definitions:

**Remark:** The canonical function \( \Phi(x, \Lambda(x)) \) can represent by:

\[ \Phi(x, \Lambda(x)) = W(y) - F(y) : y \in \mathbb{R}^m \]

This function is defined over \( \mathbb{R}^n \times \mathbb{R}^n \) to R.

In the other hand, we use the dual \( \Lambda^- \) – canonical transformation to calculate the conjugate

function of \( F(y) \) given by:

\[ \bar{F}^A(y^*) = \{ (\Lambda(x))^T y^* - \bar{F}(x) : \Lambda^T(x)y^* - D\bar{F}(x) = 0 \} \]

with:

\[ \Lambda^T(x) = DA(x) \]

By the use of this notions, we can construct the associate dual function of \( f \) by:

\[ f^d(y^*) = \bar{F}^A(y^*) - \bar{W}^A(y^*) \]

7.1.2 Method (DCT) for the non convex quadratic problems

At that point, we connected the strategy (DCT) over the partner parametric issue (PQP) in the spot of the non convex quadratic issue (NQP) like follow:

\[ |x|^2 \leq 2\mu \]

Then, we have:

\[
\begin{align*}
\min f(x) &= \frac{1}{2}x^TQx - d^Tx \\
Ax &\leq b; \quad x \geq 0 ; |x|^2 \leq 2\mu
\end{align*}
\]

We can transform the problem (PQP) as:

\[
\begin{align*}
\min f(x) &= \frac{1}{2}x^TQx - d^Tx \\
Ax &\leq b; \quad \frac{1}{2}|x|^2 \leq \mu
\end{align*}
\]

With:

\[ A = \begin{pmatrix} n+1 \end{pmatrix} a \quad b \quad n+1 \]

\[ a \in \mathbb{R}^n \quad b \in \mathbb{R}^n \]

At that point, we connected the strategy (DCT) over the partner parametric issue (PQP) in the spot of the non convex quadratic issue (NQP) like follow:
Step(1): The form of the operator \( \Lambda(x) \):
For this type of problem the canonical geometric operator:
\( \Lambda(x) : \mathbb{R}^m \rightarrow \mathbb{R}^m \times \mathbb{R} \)
Is defined by:
\[
y = \Lambda(x) = (Ax, \frac{1}{2} |x|^2) = (\varepsilon, \rho) \in \mathbb{R}^m \times \mathbb{R}
\]
And, it’s presented as an Vertex-Value application.
By this, the realisable domain of \((PQP)\) will be define by:
\[
D_{PQP} = \{ y = (\varepsilon, \rho) \in \mathbb{R}^m \times \mathbb{R} : \varepsilon \leq b, \rho \leq \mu \}
\]

Step(2): The structure of the function \( W(y) \):
In this case, the function \( W(y) \) is given by the Indicative function of the realisable domain \( D_{PQP} \) like follows:
\[
W : \mathbb{R}^m \rightarrow \mathbb{R}
\]
\[
\{ (y, y^* ) : \mathbb{R}^m \rightarrow \mathbb{R} \}
\]
\[
\varepsilon \leq y \leq \infty \quad \text{else}
\]
Then, it’s clear that the function \( W(y) \) is always convex from the propriety of the indicative function. In the other hand, the function \( W(y) \) is proper and s-lower continuous over the set \( D_{PQP} \).
By this we have:
\[
\star \star \star \star
\]
Step(3): The structure of the function \( W(y^*) \):
\[
W^*(y^*) = \sup_{y \in D_{PQP}} \{ (y, y^*) - W(y) \}
\]
\[
= \sup_{\varepsilon \leq y \leq \infty} \{ (\varepsilon, \rho)^T (y^*, \rho^*) - W(y^*) : y \in D_{PQP} \}
\]
\[
= \sup_{\varepsilon \leq y \leq \infty} \{ (\varepsilon^T \varepsilon^* + \rho^T \rho^*) : y \in D_{PQP} \}
\]
\[
\star \star \star \star
\]
Step(4): The structure of the function \( F(y) \):
The function \( F(y) \) is a linear function, and we have:
\[
F(y) = \Phi(x, \Lambda(x)) = W(y) - F(y) : y \in \mathbb{R}^m \times \mathbb{R}
\]
And for \( y \in D_{PQP} \) we have:
\[
-\bar{f}(x) = F(y)
\]
Immediately, the \( \Lambda \)-canonical conjugate of the function \( F(y) \) is defined by:
\[
\bar{F}(y^*) = \sup_{y \in D_{PQP}} \{ (\varepsilon^T \varepsilon^* + \rho^T \rho^*) : y \in D_{PQP} \}
\]
And, from the first step we have:
\[
y = \Lambda(x) = (Ax, \frac{1}{2} |x|^2) = (\varepsilon, \rho) \in \mathbb{R}^m \times \mathbb{R}
\]
Thus:
\[
\bar{F}(y^*) = \sup_{x \in D_{PQP}} \{ \Lambda(x) \varepsilon^T + \bar{\Phi}(\Lambda(x)) : \Lambda(x) y^* - DF(x) = 0 : x \in D_{PQP} \}
\]
\[
= \sup_{x \in D_{PQP}} \{ \frac{1}{2} \varepsilon^T (Q + \rho^T) y^* - (d - A^T \varepsilon^*) x : x \in D_{PQP} \}
\]
\[
= \frac{1}{2} (d - A^T \varepsilon^*)^T (Q + \rho^T) (d - A^T \varepsilon^*) - \varepsilon^T \varepsilon^* - \rho^T \rho^* : (\varepsilon, \rho) \in \mathbb{R}^m \times \mathbb{R}
\]
Step(5): The structure of the dual canonical function \( F^d(y^*) \):
Finally, and from the forth step, we define the dual canonical function by:
\[
F^d(y^*) = F^\Gamma(y^*) = \frac{1}{2} (d - A^T \varepsilon^*)^T (Q + \rho^T) (d - A^T \varepsilon^*) - \varepsilon^T \varepsilon^* - \rho^T \rho^*
\]
Then, the parametric dual problem is given by:
\[
(\max_{\varepsilon, \rho^*}) \left\{ \begin{array}{l}
\varepsilon^T \varepsilon^* + \rho^T \rho^* \\
\geq 0, \rho^* \geq 0, \det(Q + \rho^T I) = 0 \quad \text{(CPD)}
\end{array} \right.
\]
We can find an equivalence between the primal problem and the dual one, that’s given by the following theorem:

**Theorem 7.1 [1]**: \( \bar{F}(y^*) = (\varepsilon^*, \rho^*) \) be a (K.K.T) point of the parametric dual problem (CPD) then the vertex \( x = (Q + \rho^T I)^{-1}(d - A^T \varepsilon^*) \) is a (K.K.T) point of the parametric primal problem (PQP), and we have:

\[
f^d(y^*) = f(x) \varepsilon
\]

**Remark**: Let take \( id \) be the number of the negative distincts eigenvalues of the matrix \( Q \) then, the quadratic problem be non convex if \( id > 0 \).

### 7.2 Convergence Theorem of the method (DCT)
We can assume the inquiry "what’s the connection between the ideal arrangements of the parametric issue (PQP), the base issue (NPQ) and the parametric double issue (CPD)?

To give the appropriate response we have this hypothesis:

**Theorem 7.2 [1]**: Let \( Q \) a matrix with the index \( id > 0 \) and \( \{ \lambda_i \}_{i=1}^p \) : \( p \leq n \) a distincts eigenvalues in the order:
\[
\lambda_1 < \lambda_2 < ... < \lambda_{id} < 0 \leq \lambda_{id+1} < \lambda_{id+2} < ... < \lambda_p
\]
and let \( (\varepsilon^*, \rho^*) \) be a K.K.T point of the parametric dual problem (CPD), and let:
\[
x = (Q + \rho^T I)^{-1}(d - A^T \varepsilon^*) \varepsilon
\]
be a K.K.T point of the prametric primal problem (PQP), then we have:

1. If \( \rho^*_{id} > -\lambda_1 > 0 \) then, the vertex \( (\varepsilon^*, \rho^*) \) is a maximum of \( f^d(y^*) \) over \( D_{PQP} \) if and only if the vertex \( x \) is a minimum of \( f(x) \) over \( D_{PQP} \), and we write:
   \[
f^d(\bar{z}_i) = \min_{x \in D_{PQP}} f(x) = \max_{(\varepsilon, \rho^*) \in D_{PQP}} f^d(\varepsilon, \rho^*) = f^d(\varepsilon^*, \rho^*)
   \]
2. If \( 0 < \rho^*_{id} < -\lambda_1 \) then, the vertex \( (\varepsilon^*, \rho^*) \) is a maximum of \( f^d(y^*) \) over \( D_{PQP} \) if and only if the vertex \( x \) is a global maximum of \( f(x) \) over \( D_{PQP} \), and we write:
   \[
f^d(\bar{z}_i) = \max_{x \in D_{PQP}} f(x) = \max_{(\varepsilon, \rho^*) \in D_{PQP}} f^d(\varepsilon, \rho^*) = f^d(\varepsilon^*, \rho^*)
   \]
3. If \( 0 < \rho^*_{id} < -\lambda_1 \) then, the vertex \( (\varepsilon^*, \rho^*) \) is a minimum of \( f^d(y^*) \) over \( D_{PQP} \) if and only if the vertex \( x \) is a global minimum of \( f(x) \) over \( D_{PQP} \), and we write:
   \[
f^d(\bar{z}_i) = \min_{x \in D_{PQP}} f(x) = \max_{(\varepsilon, \rho^*) \in D_{PQP}} f^d(\varepsilon, \rho^*) = f^d(\varepsilon^*, \rho^*)
   \]
\[ f(\xi) = \min_{x \in D_{QP}} f(x) = \min_{(\xi^*, \rho^*) \in D_{\overline{QP}}} f^d(\xi^*, \rho^*) = f^d(\bar{\xi}^*, \rho^*) \]

7.3 Examples

7.3.1 Example 1

Let the non convex quadratic function defined by:
\[ f(x) = (x_1 + 1)^2 + (x_2 + 1)^2 - \frac{5}{2}(x_1 + x_2) - 3(x_1^2 + x_2^2) - 2 \tag{32} \]
So, we have:
\[ L_{quad}(x) = (x_1^2 + x_2^2) + \frac{3}{2}(x_1 + x_2) - 2 \]
\[ L_{quad}(x) = \frac{1}{2}(x_1 + x_2) - 2 - \frac{1}{8}(3) \]
Figure 1: Figure 1
With:
\[ f(x) \text{ : broun with black} \]
\[ L_{quad}(x) \text{ : red with yellow} \]
\[ L_{quad}(x) \text{ : darkgray with navy} \]

The graphic representation of the non convex quadratic function \( f \), the linear approximate function and the convex quadratic lower bound function over the rectangle \([-1, 0] \subseteq \mathbb{R}^n \).

Plainly the raised quadratic inexact capacity is between the target work and the straight rough one of a similar capacity over the square shape \( S^0 = [-1, 0] \subseteq \mathbb{R}^n \).

7.3.2 Example 2

Let take the following quadratic programming problem:
\[ \begin{align*}
\min f(x) &= \frac{1}{2}ax^2 + dx \\
|x| &\leq r
\end{align*} \]
So, if \( a \geq 0 \) then, the problem be convex and this case is simple to resolve, however, if \( a < 0 \).
Let \( a = -6 \), \( d = 4 \) and \( r = 1.5 \), then:
\[ (\min f(x)) = -3x^2 - 4x \]
\[ |x| \leq 1.5 \]
Figure 2: Figure 2
This function accept one and only extrema in the point \( x = -\frac{2}{3} \) with the associate value
\[ f(\bar{x}) = \frac{3}{4} \]
And, by the use of the dual canonical transformation, we can define the associate dual form of \( f \) by:
\[ f^d(\rho^*) = -\frac{1}{2}d(x + \rho^*)^{-1}d - \mu \rho^* \]
\[ = -\frac{1}{2}(16)(-6 + \rho^*)^{-1} - \frac{1}{2}(1.5)^2 \rho^* \]
\[ = -(1.125\rho^* + (\frac{8}{\rho^*} - 6)) \]
In the other part, the dual canonical problem is given by:
\[ \begin{align*}
\max f^d(\rho^*) &= -(1.125\rho^* + (\frac{8}{\rho^*} - 6)) \\
\rho^* &\geq 6
\end{align*} \]  
(DCP)

In the other hand, we find the following results:
\[ \bar{\rho}_1^* = 3,333 < a = 6 \]
With:
\[ f(\bar{x}_1) = \min_{x \in D_{\overline{QP}}} f(x) = \min_{(\rho^*) \in D_{\overline{QP}}} f^d(\rho^*) = f^d(\bar{\rho}_1^*) = -0.75 \]
And:
\[ \bar{\rho}_2^* = 8,666 > a = 6 \]
With:
\[ f(\bar{x}_2) = \min_{x \in D_{\overline{QP}}} f(x) = \max_{(\rho^*) \in D_{\overline{QP}}} f^d(\rho^*) = f^d(\bar{\rho}_2^*) = -12.75 \]
So, by the use of the "Branch and Bound method" the convex approximate quadratic form of \( f \) is given by:
\[ L_{quad}(x) = \frac{1}{2}x^2 + \frac{7}{4}x \]
And the convex approximate quadratic problem associate to the non convex one is given by:
\[ \begin{align*}
\min L_{quad}(x) &= \frac{1}{2}x^2 + \frac{7}{4}x \\
x &\in [0, \frac{1}{2}] 
\end{align*} \]
Where we applied the reducing and eliminate technic over the initial rectangle
\[ S^0 := \left[ -\frac{1}{2}, \frac{1}{2} \right] \]
And we find that the rest rectangle is:
\[ S^1 := \left[ 0, \frac{1}{2} \right] \]
So, we have this graph:
\[ f(x) \text{ : black} \]
\[ f^d(\rho^*) \text{ : broun} \]
\[ (-12.75: \text{lightred}) \]
\[ (-0.75: \text{lightblue}) \]
The graphic representation of the primal function \( f \), the associate dual function \( f^* \) and the convex quadratic approximate function \( L_{\text{quad}}(x) \):

\[
S^1 := \begin{bmatrix} 0, 1 \end{bmatrix}
\]

So, over the rectangle \( \left( \frac{1}{2}, \frac{1}{2} \right) \) we find that:

- \((0)\) is the minimum point of \( f \) and it is the maximum point of the convex quadratic function \( L_{\text{quad}} \) and the minimum point of the associate dual function \( f^* \) over the rectangle

\[
S^1 := \begin{bmatrix} 0, 1 \end{bmatrix}
\]

- \((0)\) is the maximum point of \( f \) and the minimum point of the convex quadratic function \( L_{\text{quad}} \) and: \( f(0) = L_{\text{quad}}(0) < f^*(0) \)

### 8. Conclusion

In this paper we present another square shape Branch and Headed methodology for taking care of non curved quadratic programming issues were we propose another lower rough raised quadratic elements of the target quadratic capacity \( f \) over a \( n \)-rectangle.

This lower surmise is given to decide a lower bound of the worldwide ideal estimation of the first issue (NQP) over every square shape.

To quicken the combination of the proposed calculation we utilized a basic two-segment and decreasing method over the subrectangles \( SK \) in the k - step [3].

In the other hand, we presente an other worldwide technique to determine the issue (NQP), this strategy is "the double standard change (DCT)". This strategy change a non raised quadratic issue to an Algebraic framework.

It's dependably unite to the worldwide ideal arrangement over the feasible space wich is a minimal arrangement of \( R^n \).

Figure 4: Figure 4

The new calculation B&B where we utilized the curved quadratic estimate of the non arched quadratic capacity \( f \) over a rectangle \( K = [L, U] \subseteq R^n \) with \( \theta \geq \theta_{\text{min}} \) and it is not impty, convex, close, and bounded (compact) of \( R^n \) is best at that point the strategy (DCT) over the relative Intrior of the feasible space of the capacity wich we streamlined.

We can utilize the Branch and Bound technique (Partition and assessment) where we compose the capacity \( f \) like a (DC) structure (reverence of tow arched capacities) and we estimated the curved part by a raised quadratic capacity by the utilization of the lower bound or the upper bound of the feasible square shape \( SK \) wich have a very little position and it's considered as a confianced locale, and by this we guaranteed the existence of the ideal worldwide arrangement of the first issue (NQP).

In the other hand, the "Branch and Bound strategy" acquire the rough ideal arrangement of the ideal worldwide arrangement of the first issue (NPQ) with a quadratic vitesse of union over the feasible set \( SK \), yet the (DCT) technique locate the ideal worldwide arrangement over the Sphar of this feasible set \( SK \).

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