On the exact Foldy-Wouthuysen transformation for a Dirac spinor in torsion and other CPT and Lorentz violating backgrounds

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Abstract. We discuss the possibility to perform and use the exact Foldy-Wouthuysen transformation (EFWT) for the Dirac spinor coupled to different CPT and Lorentz violating terms. The classification of such terms is performed, selecting those of them which admit EFWT. For the particular example of an axial vector field, which can be associated with the completely antisymmetric torsion, we construct an explicit EFWT in the case when only a timelike component of this axial vector is present. In the cases when EFWT is not possible, one can still use the corresponding technique for deriving the perturbative Foldy-Wouthuysen transformation, as is illustrated in a particular example in the Appendix.

Keywords: Foldy-Wouthuysen transformation, CPT and Lorentz violating terms, Torsion, Magnetic Field.

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1 Introduction

One of the most natural extensions of General Relativity is related to the inclusion of the spacetime torsion which is supposed to describe, along with the metric, the physical properties of the spacetime geometry. The study of the physical aspects of the torsion gravity has a long history (see [1, 2, 3, 4, 5, 6] for extensive reviews and references). The issue which always attracted a special attention was the interaction of the spacetime torsion with the spinor field and with the spinning particle [7, 8, 9, 10]. In particular, the papers [11, 12, 13] were devoted to the nonrelativistic approximation of Dirac equation and in [12, 13], correspondingly, the Pauli equation and Foldy-Wouthuysen transformation have been obtained for the fermion field coupled...
to the combined electromagnetic and torsion fields. One can use these results for the investigation of the possible manifestations of torsion in the domain of the atomic physics [12, 14, 15].

The Foldy-Wouthuysen transformation provides, in general, more detailed information about the nonrelativistic approximation [16], especially if the exact version of this transformation is constructed [17, 18, 19, 20, 21] (see also recent works [22]). It is, in principle, safer to perform the exact transformation, since otherwise there is a certain risk of missing some important terms. Recently it has been shown that this is the case for the spinor field in the weak gravitational field [21]. Therefore it is worthwhile to construct EFWT for the case of torsion and electromagnetic background. One can imagine, for instance, the situation when the magnetic field could amplify the effect of torsion and thus make the upper bound for the torsion more precise. Recently, we have used this approach for the case of a fermion on the combined background of the gravitational wave and magnetic field and found that indeed there are potentially interesting nonlinear effects [23]. In the present paper we mainly consider the case of torsion. In fact, the same approach can be used also for other Lorentz and CPT violating terms [24, 25]. Although the main aim of our work is to study the torsion effects, in section 2 we present a table which shows the possible CPT and Lorentz violation terms that could be treated using this technique.

The usual perturbative Foldy-Wouthuysen transformation can be constructed for the Dirac field interacting with the variety of external fields, including the torsion [13]. However, the possibility to have an exact FW transformation depends on a special condition (the existence of the involution operator) on the external classical fields. The construction of an exact transformation is more complicated and more interesting from the mathematical point of view [17, 18]. As it was already mentioned above, in this paper we are interested in the set of two external fields - one is the torsion and another one is a constant and uniform magnetic field. One can safely assume that torsion is very weak, since otherwise it would be easy to detect [6], while the magnetic field of our interest should be very strong. Therefore, our goal should be to find the transformation which is exact in magnetic field but may be just linear in torsion. Actually, EFWT with a general torsion is not always possible because the corresponding Hamiltonian has a term that does not admit the involution operator. So we construct the transformation using only the scalar part of the torsion field to describe it.

In Appendix, we take into account the vector part of the torsion, introducing some ad hoc modification of the torsion-dependent term in the Hamiltonian, such that the modified expression admits the involution operator. Then one can use the known technique developed for EFWT. The main point is that, in the linear approximation, the mentioned modification can be easily removed from the final result. In this way we can reproduce the known perturbative result [13] in a technically much more economic way and also to get the Foldy-Wouthuysen Hamiltonian with the terms which show explicitly the mixture between the torsion and magnetic field. In other words, we have derived a Hamiltonian which is exact in magnetic field and linear in torsion. After performing the transformation we derive the non-relativistic equations of motion for the particle with spin $\frac{1}{2}$.

The paper is organized as follows. In the next section we study the possibility to apply EFWT to the CPT and Lorentz violating terms. In section 3 we consider an example of EFWT in the torsion case. In section 4 we draw our conclusions, and in Appendix we discuss the linear expansion in the torsion field. Throughout the paper we use Greek letters for the indices which run from 0 to 3. Latin indices are used for the space coordinates and run from 1 to 3.
2 EFWT for Dirac equation with CPT and Lorentz Violating Terms

Let us start with the action describing a Dirac fermion with Lorentz and CPT symmetry breaking terms. For the sake of generality, we include also minimal interaction to gravity.

\[ S = \int d^4x \sqrt{-g} \left\{ \frac{i}{2} \bar{\psi} \Gamma^\mu D_\mu \psi - \frac{i}{2} D^*_\mu \bar{\psi} \Gamma^\mu \psi - \bar{\psi} M \psi \right\}, \]

where we use the following classification for the possible Lorentz and CPT symmetry breaking terms [25]

\[ D_\mu = \nabla_\mu - ieA_\mu; \quad D^*_\mu = \nabla_\mu + ieA_\mu; \quad \Gamma^\nu = \gamma^\nu + \Gamma^\nu_1; \quad M = m + M_1. \]

Here \( \nabla_\mu \) is the operator of the covariant derivative, \( F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \) and the quantities \( \Gamma^\nu_1 \) and \( M_1 \) are given by

\[ \Gamma^\nu_1 = c_{\mu\nu} \gamma_\mu + d_{\mu\nu} \gamma_5 \gamma_\mu + e^\nu + if^\nu \gamma_5 + \frac{1}{2} g^{\lambda\mu\nu} \sigma_{\lambda\mu}, \]

\[ M_1 = a_\mu \gamma^\mu + b_\mu \gamma_5 \gamma^\mu + i m_5 \gamma_5 + \frac{1}{2} H_{\mu\nu} \sigma^{\mu\nu}. \]

The quantities \( a_\mu, b_\mu, m_5, c_{\mu\nu}, d_{\mu\nu}, e^\mu, f^\mu, g^{\lambda\mu\nu} \) and \( H_{\mu\nu} \) are CPT and/or Lorentz violating parameters. An extensive discussion of the possible origin of these parameters and also their numerous phenomenological implications can be found in [26, 27] and we will not consider these aspects here. From now on we are going to treat these terms as constants, so it is possible to rewrite (1) in the following way

\[ S = \int d^4x \sqrt{-g} \left\{ i \bar{\psi} \Gamma^\mu D_\mu \psi - \bar{\psi} M \psi \right\}. \]

As a result, the equations of motion for \( \psi \) can be written as \( i \Gamma^\mu D_\mu \psi = M \psi \). In order to perform EFWT we put this equation into the Schrödinger form, \( i \partial_t \psi = H \psi \), to get the Hamiltonian

\[ i \Gamma^0 \nabla_0 \psi = (M + \Gamma^\mu P^*_\mu) \psi. \]

Here we introduced the useful notations

\[ \mathcal{P}_\nu = (0, P_\nu) \quad \text{and} \quad P^*_\nu = \mathcal{P}_\nu - eA_\nu \]

and use the standard representation for the Dirac matrices (see, for example, [28])

\[ \beta = \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha_i = \beta \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \]

\[ \gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3, \quad \sigma_{\mu\nu} = \frac{i}{2} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu). \]

Let us denote \( \Gamma^0 = \gamma^0 + \Gamma^0_1 \) and introduce \( \Gamma^0_1 \) such that \( (\Gamma^0)^{-1} = \gamma^0 - \Gamma^0_1 \). If one assume that the Hamiltonian is linear in the CPT/Lorentz violating terms present in \( \Gamma^0_1 \), it is straightforward to check that

\[ \Gamma^0_1 = \gamma^0 \Gamma^0_1 \gamma^0. \]
\[ i\nabla_0\psi = \left[ \gamma_0 - \gamma_0 (e^{\mu_0} \gamma_\mu + d^{\mu_0} \gamma_5 \gamma_\mu + e^0 + i f^0 \gamma_5 + \frac{1}{2} g^{\lambda\nu_0} \sigma_{\lambda\mu}) \gamma_0 \right] \times (M + \Gamma^\nu P_{\nu}^*) \psi. \quad (9) \]

It is possible to construct an exact FW transformation if

\[ JH + HJ = 0, \quad \text{where} \quad J = i\gamma_5 \beta \quad (10) \]

is the involution operator. Only those theories where the Hamiltonian admits the involution operator enable one to perform EFWT. One thus can formulate the natural question: Which is the most general form of equation (10) that admits the involution operator? In order to answer this question, one has to check whether the criterion (14) is satisfied for the terms in the general Hamiltonian presented above in the right hand side of (10). The result of this procedure is given in the Table.

**Table 1: Interaction coefficients**

|       | \( m \) | \( e^\nu P_{\nu}^* \) | \( a_l \) | \( d^0 P_{\nu}^* \) | \( H^{ij}_\beta \) | \( m_5 \) | \( b_l \) | \( d^0 P_{\nu}^* \) | \( H^{0\mu} \) |
|-------|--------|----------------|--------|----------------|----------------|--------|--------|----------------|----------------|
| \( \gamma^0 \) | 1 | \( \gamma^l \) | \( -\gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^5 \) | \( \gamma^0 \) | \( \frac{1}{2} \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \frac{1}{2} \gamma^0 \gamma^5 \) |
| \( \gamma^0 \) | \( -\gamma^0 \) | \( -\alpha^l \) | \( \gamma^5 \) | \( -\frac{1}{2} \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) |
| \( f^0 \) | \( i\gamma^5 \) | \( i\gamma^5 \gamma^l \) | \( i\gamma^5 \gamma^l \) | \( \frac{1}{2} \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) |
| \( d^0 \) | \( -i\gamma^5 \gamma^5 \) | \( -i\gamma^5 \gamma^5 \gamma^l \) | \( \alpha^l \) | \( -\frac{1}{2} \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) |
| \( g^{00} \) | 2\( \alpha^l \) | 2\( \alpha^l \gamma^l \) | 2\( \gamma^5 \gamma^l \) | \( \alpha^l \gamma^l \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) |
| \( e^{00} \) | \( -i\gamma^5 \gamma^5 \) | \( -i\gamma^5 \gamma^5 \gamma^l \) | \( \alpha^l \gamma^l \) | \( -\gamma^5 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) |
| \( e^{00} \) | \( -i\gamma^5 \gamma^5 \) | \( -i\gamma^5 \gamma^5 \gamma^l \) | \( \alpha^l \gamma^l \) | \( -\gamma^5 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) |
| \( g^{ij} \) | \( \frac{1}{2} \gamma^0 \gamma^5 \) | \( \frac{1}{2} \gamma^0 \gamma^5 \gamma^l \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) | \( \gamma^0 \gamma^5 \) |

The Table specifies the 80 cases of CPT and Lorentz violating terms in the modified Dirac equation which admit EFWT. The form of the corresponding EFWT-positive term in the Hamiltonian, is obtained by multiplying the terms in the row and in the line. For example, the coefficient 1 in the first row and first column means that for \( \gamma^0 \) and \( m \) the Hamiltonian contains the term \( \gamma^0 \times m \times 1 = \beta m \). Of course, this term is the most trivial one as it corresponds to the free Dirac equation.

Another example is for the 8-th line with \( c^{i0} \) and the 8-th row with \( g^{0\mu\nu} P_{\nu}^* \). Taking the coefficient inside the Table into account, we arrive at the EFWT-admitting term

\[ c^{i0} \times g^{0\mu\nu} P_{\nu}^* \times \frac{1}{2} \gamma^i \sigma_{0\mu} = \frac{i}{2} \gamma^i c^{i0} \alpha^l g^{0\mu\nu} P_{\nu}^* \]

Each term in the zero line must be taken separately, e.g., in the first row there are two different terms \( m \) and \( e^\nu P_{\nu}^* \). The filled blocks with nonzero coefficients show the terms which allow EFWT. As we have just mentioned, there are 80 such terms which means the corresponding number of the modified Dirac theories admitting EFWT.
Furthermore, if some space in the table is empty, this means that EFWT is not allowed to the given pair of terms in the corresponding row and line. The same is true if a component of one term is not present on the table. Let us note that even in those cases when the Hamiltonian does not satisfy the equation (10), EFWT technique is not useless. In fact, there is a possibility to apply the EFWT prescription to perform a perturbative Foldy-Wouthuysen transformation and to achieve a reliable qualitative analysis of the even transformed Hamiltonian. For the product of the terms $\gamma^0$ and $P^*_0$ (which has an empty site in the table), the corresponding calculation has been performed in [29]. In Appendix, we analyze another interesting example, namely the case of the product of $\gamma^0$ and $b_l$ (space-like component of the axial vector, dual to the completely antisymmetric torsion).

3 Example of exact Foldy-Wouthuysen transformation

In this section we consider in details the EFWT for one of those cases which admit this exact transformation. Namely, we construct the EFWT for the purely timelike axial vector field which is dual to the completely antisymmetric torsion of the spacetime. Let us start with some necessary details about the gravity theory with torsion. We shall use the notations of [6].

In the spacetime with torsion $T^\alpha_{\beta\gamma}$, the connection $\tilde{\Gamma}^\alpha_{\beta\gamma}$ is not symmetric, $\tilde{\Gamma}^\alpha_{\beta\gamma} - \tilde{\Gamma}^\alpha_{\gamma\beta} = T^\alpha_{\beta\gamma}$. It proves useful to divide torsion $T^\alpha_{\beta\gamma}$ into the following irreducible components: the trace $S^\nu = \varepsilon_{\alpha\beta\mu\nu} T^\alpha_{\mu\nu}$ and the pure tensor part $q^\alpha_{\beta\gamma}$, satisfying the conditions $q^\alpha_{\beta\alpha} = \varepsilon_{\alpha\beta\mu\nu} q_{\alpha\beta\mu} = 0$. Then torsion can be written in the form

$$T^\alpha_{\beta\mu} = \frac{1}{3} (T^\alpha_{\beta\gamma} g_{\alpha\mu} - T^\alpha_{\mu\gamma} g_{\alpha\beta}) - \frac{1}{6} \varepsilon_{\alpha\beta\mu\nu} S^\nu + q^\alpha_{\beta\mu}. \quad (11)$$

In what follows we shall consider only the $S^\mu$-component, that is equivalent to taking completely antisymmetric torsion.

Since the Dirac fermion is in an external gravitational field with the torsion, we can perform the minimal covariant generalization of the flat-space action by replacing the Minkowski metric by a general one and the partial derivative by the covariant one. However it is somehow more interesting to consider a general non-minimal action [30, 5, 6], which includes all terms compatible with the covariance and with no inverse-mass parameters,

$$S = \int d^4 x \sqrt{-g} \left\{ i \bar{\psi} \gamma^\mu (\nabla_\mu + i \eta_1 \gamma_5 S_\mu) \psi + m \bar{\psi} \psi \right\}. \quad (12)$$

Here $\eta_1 = 1/8$ corresponds to the minimal action case [30]. According to [30, 6] (see also further references therein) the consistent quantum field theory with the torsion can be constructed only for the nonminimal interaction of Dirac field with the external torsion field. Therefore in what follows we shall keep the parameter $\eta_1$ arbitrary. Now if we put (4) into (1) and compare the result with (12), it is possible to see that $b_\mu = -\eta_1 S_\mu$.

At this point, we are in a position to develop the calculations of EFWT with one of the CPT/Lorentz violating terms. Consider the spin-1/2 particle in an external torsion and electromagnetic fields. We are going to consider constant magnetic and torsion fields. The equation of motion which follows from the action (12) has the form

$$i \hbar \frac{\partial \psi}{\partial t} = \left( e \vec{\alpha} \cdot \vec{p} - e \vec{\alpha} \cdot \vec{A} - \eta_1 \vec{\alpha} \cdot \vec{S} \gamma_5 + e \Phi + \eta_1 \gamma_5 S_0 + mc^2 \beta \right) \psi, \quad (13)$$
In case of a constant magnetic field one can set \( \Phi = 0 \). However, a direct inspection shows that the term \( \eta_1 \alpha S^5 \) in (13) does not satisfy the condition (10). So let us first treat the case when \( S = 0 \) and EFWT can be derived. The complete Hamiltonian is studied in Appendix. Here we work with the following Hamiltonian

\[
H = c \bar{\alpha} \cdot \vec{p} - e \bar{\alpha} \cdot \vec{A} + \eta_1 \gamma_5 S_0 + mc^2 \beta. \tag{14}
\]

An interesting point that has to be emphasized here is that the above Hamiltonian could have been constructed from the table of section 2 without using the arguments of the last two paragraphs. If we look to the \( \gamma^0 \) line of the table and we want to consider only the \( b_\mu \) field, we conclude that only the component \( b_0 \) is allowed. Therefore, the most general Hamiltonian to torsion field using the table scheme would be \( \gamma^0 \times (m + \gamma^1 P^\tau_1 - \gamma_0 \gamma^5 b_0) \), that has the same form of (14).

According to the standard prescription [17], the next step is to obtain \( H^2 \). Direct calculations yield the result

\[
H^2 = (c \bar{\rho} - e \bar{A} - \eta_1 \bar{S} S_0)^2 + m^2 c^4 - 2 \beta \eta_1^2 \beta_1 S_0^2. \tag{15}
\]

In order to get the transformed Hamiltonian \( H^{tr} \) we rewrite \( H^2 = A^2 + B \) with \( A \) being \( m \)-dependent terms in \( H^2 \), whereas the terms in \( B \) do not depend on the mass. In the present case \( A = mc^2 \). Then, we search for an operator \( K \) in the form

\[
K = A + \frac{1}{A} K_1 + \frac{1}{A} + \vartheta(\frac{1}{A^2}), \tag{16}
\]

such that \( K^2 = H^2 \). Finally, using (15) and the fact that

\[
H^{tr} = U H U^* = \beta[\sqrt{H^2}]^{EVEN} + J[\sqrt{H^2}]^{ODD}. \tag{17}
\]

Here the even (odd) terms in (17) are the ones that commute (anticommute) with the matrix \( \beta \). We thus get

\[
H^{tr} = \beta mc^2 + \frac{\beta}{2mc^2}(c \bar{\rho} - e \bar{A} - \eta_1 \bar{S} S_0)^2 - \beta \frac{\eta^2}{mc^2} S_0^2. \tag{18}
\]

The next step is to present the Dirac fermion field \( \psi \) in the form

\[
\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} e^{-imc^2t}, \tag{19}
\]

and to use the equation \( i \hbar \partial_t \psi = H \psi \) to derive the Hamiltonian for the two-spinor \( \varphi \). We obtain the two-component equation

\[
i \hbar \frac{\partial}{\partial t} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = (-mc^2 + H) \begin{pmatrix} \varphi \\ \chi \end{pmatrix}. \tag{20}
\]

Using the fact that the transformed Hamiltonian is an even function, we obtain, in the \( \varphi \) sector, the same nonrelativistic Hamiltonian of [12].
4 Discussion and conclusion

The new classification of the most general CPT and Lorentz violating terms in the Dirac equation with respect to an exact Foldy-Wouthuysen transformation was developed. We found 80 examples of the terms which admit such a transformation.

We have derived the exact Foldy-Wouthuysen transformation for the Dirac spinor field on the combined background of the torsion and the constant uniform magnetic fields. We have constructed this for the fermion interacting with the scalar part of torsion field $S_\mu$. Using the method of [21, 17, 18, 19] we were able to reproduce known results [12, 13] in a much more general and economic way. We also constructed a table which gives the most general Hamiltonian for each of the CPT and Lorentz violating terms that admit EFWT.

Although the vector part of torsion field does not admit the exact transformation, in Appendix we present a qualitative analysis of the term in the initial Hamiltonian that allows for such a transformation. After a proper modification of it, it is possible to find a transformed Hamiltonian which is linear in torsion ($S_\mu$) and is non-perturbative in the external constant magnetic field. The same structure was obtained for the non-relativistic equations of motion of a spinning particle. This qualitative analysis demonstrates that in this case there is a mixing between the magnetic and torsion fields terms.

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The Hamiltonian (14) does not allow for an EFWT. However, due to the weakness of the torsion field we are really interested only in the linear order in torsion while the magnetic field should be treated exactly.

Let us make an ad hoc modification of the term \( \eta_1 \vec{\alpha} \cdot \vec{S} \gamma_5 \), that is multiply it by the \( \beta \)-matrix. The modified term satisfies the condition (10) and now the EFWT is perfectly possible. The main point is that, in the linear order in the torsion field, an extra \( \beta \) has no effect. The reason is that, after deriving the final Hamiltonian operator, it will have the block diagonal structure. We are interested only in the upper block of Hamiltonian which is even (after transformation) to perform the physical analysis. At least in the first order in \( 1/m \), it does not matter if this term is multiplied by \( \beta \) or not, because beta has the form (8) and its upper block is just the unity matrix. As a result, we arrive at what one can call semi-exact Foldy-Wouthuysen transformation, because it is exact in only part of external fields and linear in other external fields. This technique was already been applied for a Dirac Hamiltonian including scalar electromagnetic potential \([29]\).

For the sake of completeness we include also the timelike component of the axial vector, \( S_\mu \). After all, the Hamiltonian we are going to deal with has the form

\[
H = e \vec{\alpha} \cdot \vec{p} - e \vec{\alpha} \cdot \vec{A} - \eta_1 \vec{\alpha} \cdot \vec{S} \gamma_5 \beta + \eta_1 \gamma_5 S_0 + mc^2 \beta .
\]  

(21)

In this case \( H^2 \) has the form

\[
H^2 = (e \vec{p} - e \vec{A} - \eta_1 \vec{S} S_0)^2 + m^2 c^4 + 2\eta_1 mc^2 \vec{S} \cdot \vec{S} + \eta_1^2 \vec{S}^2 + h c e \vec{S} \cdot \vec{B} - 2\eta_1^2 S_0^2 + 2in\eta_5 \beta \vec{S} \cdot \vec{S} \times \left( (e \vec{p} - e \vec{A}) \right). 
\]  

(22)

The last term in (22) is odd, and its presence looks somehow naturally, since we have used the artificial procedure in (21). At the same time, if we do drop this term, the rest is exactly the Hamiltonian which follows from the usual perturbative Foldy-Wouthuysen transformation with torsion \([13]\). An obvious advantage of the present method is its technical simplicity compared to the perturbative one.

If we apply the procedure described between (15) and (20) to the above equation, we find the nonrelativistic limit which is almost (but not completely) equal to the conventional one \([12]\),

\[
H^{tr} = \frac{1}{2m} \overline{\Pi}^2 + B_0 + \overline{\sigma} \cdot \overline{Q},
\]  

(23)

where

\[
\overline{\Pi} = \overline{p} - e \overline{A} - \frac{\eta_1}{c} S_0 \overline{\sigma}, \quad B_0 = -\frac{\eta_1^2}{mc^2} S_0^2,
\]

\[
\overline{Q} = \eta_1 \overline{S} + \frac{he}{2mc} \overline{B} + \frac{\eta_1}{mc} \overline{S} \times \left( \overline{p} - \frac{e}{c} \overline{A} \right).
\]  

(24)

The very last term in \( \overline{Q} \) originates from the odd term in (15) which we already discussed above. This term is new in comparison with the expressions derived in \([12]\) and in \([13]\) through the usual perturbative Foldy-Wouthuysen transformation. The fact that the exact transformation gives a new term in comparison with the perturbative transformation is analogous to the gravitational case in \([21]\), described by the appearance of the "gravitational Darwin" term.
The canonical quantization of [24] gives us the (quasi)classical equations of motion

$$\frac{dx_i}{dt} = \frac{1}{m} \left( p_i - eA_i - \frac{\eta}{c} \sigma_i S_0 \right) + \frac{\eta}{mc} \left[ \sigma^i \times \vec{S} \right]_i = v_i,$$

$$\frac{dp_i}{dt} = \frac{1}{m} \left( p^j - eA^j - \frac{\eta}{c} \sigma^j S_0 \right) \frac{e}{c} \frac{\partial A_j}{\partial x^i} + \frac{\eta}{mc} \left[ \sigma^i \times \vec{S} \right]^j \frac{e}{c} \frac{\partial A_j}{\partial x^i},$$

$$\frac{d\sigma_i}{dt} = \left[ \vec{R} \times \vec{\sigma} \right]_i, \quad \vec{R} = \frac{2\eta}{\hbar} \left[ \vec{S} - \frac{1}{c} \vec{v} S_0 + \vec{S} \times \frac{\vec{v}}{c} + \frac{2\eta}{\hbar} \vec{S} \frac{\partial S \times \vec{\sigma}}{\partial t} \right] + \frac{e}{mc} \vec{B}. \tag{25}$$

The last equations are very similar to the ones derived previously in [12] and [13] on the basis of Pauli equation and perturbative Foldy-Wouthuysen transformation. At the same time there are some extra terms due to the nonlinear approximation in the external fields which we use here.

The first two equations of (25) give

$$m \frac{dv_i}{dt} = -\frac{e}{c} \frac{\partial A_i}{\partial t} + \frac{\eta}{c} \frac{\partial S_0 \sigma_i}{\partial t} - \frac{\eta}{c} \frac{\partial (\vec{S} \times \vec{\sigma})_i}{\partial t}. \tag{26}$$

Now we can rewrite the equation (22) using the linear approximation in $S_\mu$. From now on, all the terms that have power greater than two in $S_\mu$ will be neglected. We find

$$H^2 = H_0^2 + 2\eta_1 mc^2 \sum \cdot \vec{S} + 2\eta_1 \gamma_5 S_0 \vec{\alpha} \cdot (c\vec{p} - e\vec{A}), \tag{27}$$

where

$$H_0^2 = (c\vec{p} - e\vec{A})^2 + \hbar c \sum \cdot \vec{B} + m^2 c^4. \tag{28}$$

The idea now is to consider the expansion of $\sqrt{H^2}$ not only in terms of the parameter $m$, but also in terms of $S_\mu$. To perform this, we present the equation (27) in the symmetric form

$$H^2 = \frac{H_0^2}{2} \left( 1 + \frac{1}{H_0^2} \left[ 2\eta_1 mc^2 \sum \cdot \vec{S} + 2\eta_1 \gamma_5 S_0 \vec{\alpha} \cdot (c\vec{p} - e\vec{A}) \right] \right) + \left( 1 + \left[ 2\eta_1 mc^2 \sum \cdot \vec{S} + 2\eta_1 \gamma_5 S_0 \vec{\alpha} \cdot (c\vec{p} - e\vec{A}) \right] \right) \frac{1}{H_0^2} \frac{H_0^2}{2}. \tag{29}$$

This symmetrization is an important step of the procedure which includes multiplication by $\beta$ in the equation [13]. The next step is to extract the square root of (29). We expand the term $H_0^2$ in the power series in $1/m$ (going to the second order in $1/m$) and we obtain the same result of [17] which we call $H_0^{EK}$

$$H_0^{EK} = \sqrt{H_0^2} = mc^2 + \frac{(c\vec{p} - e\vec{A})^2}{2mc^2} + \frac{\hbar c \sum \cdot \vec{B}}{2mc^2}. \tag{30}$$

We also expand the term $1/H_0^2$ in the power series in $1/m$ as well as the term in the brackets in (29) in the power series of $S_\mu$, so that the result is of the first order in $S_\mu$ and of the second order in $1/m$,

$$\sqrt{H^2} = H_0^{EK} + \frac{\eta}{2m^2 c^4} \sum \cdot \vec{S} (c\vec{p} - e\vec{A})^2 -$$
\[ -\frac{\hbar c e m}{2m^2 c^4} \mathbf{S} \cdot \mathbf{B} - \frac{\eta_1}{mc^2} (c \mathbf{p} - e \mathbf{A}) \cdot (S_0 \mathbf{S} + i \gamma_5 \beta \mathbf{S} \times \mathbf{S}). \] (31)

The last term in (31) is odd, and using (17) we derive the final Hamiltonian for this case

\[ H'_{tr} = \beta mc^2 + \beta \frac{(c \mathbf{p} - e \mathbf{A} - \eta_1 S_0 \mathbf{S} - \beta \eta_1 \mathbf{S} \times \mathbf{S})^2}{2mc^2} - \beta \eta_1 \frac{2mc^2}{2mc^2} (c \mathbf{p} - e \mathbf{A})^2 \mathbf{S} \cdot \mathbf{S} + \beta \frac{\hbar c e m}{2m^2 c^4} \mathbf{S} \cdot \mathbf{B} + \beta \eta_1 \mathbf{S} \cdot \mathbf{B} - \beta \frac{\hbar c e m}{2m^2 c^4} \mathbf{S} \cdot \mathbf{B}. \] (32)

Here, we used prime in \( H \) in order to distinguish between the Hamiltonians (9) and (32). For the Hamiltonian (32) we apply the same algorithm used between equations (19) and (20) and then finally we obtain the Hamiltonian for the two-spinor \( \varphi \). The result can be expressed in the form

\[ H'_{\varphi} = \left( 1 - \frac{\mathbf{S} \cdot \mathbf{S}}{2mc^2} \right) H_{\varphi} \left( 1 - \frac{\mathbf{S} \cdot \mathbf{S}}{2mc^2} \right), \] (33)

where \( H'_{\varphi} \) is given by the equation (9). The next step is to derive the equations of motion using the same procedure as was applied in (12). We perform the canonical quantization of the theory introducing the operators of coordinate \( \hat{x}_i \), momenta \( \hat{p}_i \) and spin \( \hat{\sigma}_i \) and implement the equal-time commutation relations of the usual way. These operators yield the equations of motion

\[ i\hbar \frac{d\hat{x}_i}{dt} = [\hat{x}_i, H], \quad i\hbar \frac{d\hat{p}_i}{dt} = [\hat{p}_i, H], \quad i\hbar \frac{d\hat{\sigma}_i}{dt} = [\hat{\sigma}_i, H]. \] (34)

After the computation of the commutators in (34), we arrive at the explicit form of the operator equations of motion. Now we can omit all the terms which vanish when \( \hbar \to 0 \). Thus we obtain the classical equations which can be interpreted as the (quasi)classical equations of motion for the particle in an external torsion and electromagnetic fields. In this case the equations of motion are

\[ \frac{dx_i}{dt} = (1 - \frac{\eta_1}{mc^2} \bar{\sigma} \cdot \mathbf{S}) \frac{1}{m} (p_i - \frac{e}{c} A_i - \frac{\eta_1}{c} S_0 \sigma_i) + \frac{\eta_1}{mc} (\bar{\sigma} \times \mathbf{S})_i, \]

\[ \frac{dp_i}{dt} = (1 - \frac{\eta_1}{mc^2} \bar{\sigma} \cdot \mathbf{S}) \frac{1}{m} \left( p_j - \frac{e}{c} A_j - \frac{\eta_1}{c} \sigma_j S_0 \right) \frac{e \partial A_i}{c \partial x^j} + \frac{\eta_1 e}{mc^2} \frac{\partial A_j}{\partial x^i} (\bar{\sigma} \times \mathbf{S})_j, \]

\[ \frac{d\sigma_i}{dt} = [\bar{\tau} \times \bar{\sigma}]_i, \quad \bar{\tau} = \frac{2\eta_1}{\hbar} \left[ (1 - \frac{v^2}{2c^2}) \mathbf{S} + \mathbf{S} \times \frac{\bar{\tau}}{c} - \frac{1}{c} \bar{\tau} S_0 \right] + \frac{e}{mc} \frac{\bar{B}}{c}. \] (35)

Using the first two equations of (35), we write

\[ m \frac{d\mathbf{v}}{dt} = \left( \frac{e}{c} \frac{\partial A_i}{\partial t} + \frac{e}{c} \frac{\partial \bar{\tau}}{\partial t} \right)_i \left( 1 - \frac{\eta_1}{2mc^2} \bar{\sigma} \cdot \mathbf{S} \right) - \frac{\eta_1}{c} \frac{d}{dt} \left( S_0 \sigma_i \right) + \frac{\eta_1}{c} \frac{d}{dt} (\bar{\sigma} \times \mathbf{S})_i - \frac{\eta_1 v_i}{c^2} \frac{d}{dt} (\bar{\sigma} \cdot \mathbf{S}). \] (36)

As compared to (25), the new terms in the equation (35) are of the order \( 1/m^2 \). The equation (35) has the two important points. The first is that the last term is of the order \( 1/m^2 \).
and it was not present in equation (26). This result shows that the fact that we used only the parameter $1/m$ in expansion of $H^2$ did not give us all the possible linear terms with $S_\mu$ in the final Hamiltonian, as it should be. The second point to note, is that the second term in the equation (36) shows an interesting effect. This equation is analogous to the Lorentz force acting on a particle that interacts with an external electromagnetic field. The term where $S_\mu$ appears can be seen as a correction for this case. Thinking along these lines, this term shows an explicit mixing between the torsion and the magnetic field. One can imagine a situation when the magnetic field is strong enough to compensate weakness of the spacetime torsion $S_\mu$ so that this term would affect particle’s motion in a notable way.
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