GEOMETRODYNAMICS
VS.
CONNECTION DYNAMICS

Joseph D. Romano

Department of Physics
University of Maryland, College Park, MD 20742

ABSTRACT

The purpose of this review is to describe in some detail the mathematical relationship between geometrodynamics and connection dynamics in the context of the classical theories of 2+1 and 3+1 gravity. I analyze the standard Einstein-Hilbert theory (in any spacetime dimension), the Palatini and Chern-Simons theories in 2+1 dimensions, and the Palatini and self-dual theories in 3+1 dimensions. I also couple various matter fields to these theories and briefly describe a pure spin-connection formulation of 3+1 gravity. I derive the Euler-Lagrange equations of motion from an action principle and perform a Legendre transform to obtain a Hamiltonian formulation of each theory. Since constraints are present in all these theories, I construct constraint functions and analyze their Poisson bracket algebra. I demonstrate, whenever possible, equivalences between the theories.

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1romano@umdhep.umd.edu
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1. Overview

Einstein’s theory of general relativity is by far the most attractive classical theory of gravity today. By describing the gravitational field in terms of the structure of spacetime, Einstein effectively equated the study of gravity with the study of geometry. In general relativity, spacetime is a 4-dimensional manifold $M$ with a Lorentz metric $g_{ab}$ whose curvature measures the strength of the gravitational field. Given a matter distribution described by a stress-energy tensor $T_{ab}$, the curvature of the metric is determined by Einstein’s equation $G_{ab} = 8\pi T_{ab}$. This equation completely describes the classical theory.

As written, Einstein’s equation is spacetime covariant. There is no preferred time variable, and, as such, no evolution. However, as we shall see in Section 2, general relativity admits a Hamiltonian formulation. The canonically conjugate variables consist of a positive-definite metric $q_{ab}$ and a density-weighted, symmetric, second-rank tensor field $\tilde{p}^{ab}$—both defined on a 3-manifold $\Sigma$. These fields are not free, but satisfy certain constraint equations. Evolution is defined by a Hamiltonian, which (if we ignore boundary terms) is simply a sum of the constraints.

Now it turns out that the time evolved data defines a solution, $(M, g_{ab})$, of the full field equations which is unique up to spacetime diffeomorphisms. In a solution, $\Sigma$ can be interpreted as a spacelike submanifold of $M$ corresponding to an initial instant of time, while $q_{ab}$ and $\tilde{p}^{ab}$ are related to the induced metric and extrinsic curvature of $\Sigma$ in $M$. Thus, the Hamiltonian formulation of general relativity can be thought of as describing the dynamics of 3-geometries. Following Wheeler, I will use the phrase “geometrodynamics” when discussing general relativity in this form.

On the other hand, all of the other basic interactions in physics—the strong, weak, and electromagnetic interactions—are described in terms of connection 1-forms. For example, the Hamiltonian formulation of Yang-Mills theory has a connection 1-form $A_a$ (which takes values in the Lie algebra of some gauge group $G$) as its basic configuration variable. The canonically conjugate momentum (or “electric field”) $\tilde{E}^a$ is a density-weighted vector field which takes values in the dual to the Lie algebra of $G$. As in general relativity, these variables are not free, but satisfy constraint equations: The Gauss constraint $D_a \tilde{E}^a = 0$ (where $D_a$ is the generalized derivative operator associated with $A_a$) tells us to restrict attention to divergence-free electric fields. Thus, just as we can think of the Hamiltonian formulation of general relativity as describing the dynamics of 3-geometries, we can think of the Hamiltonian formulation of Yang-Mills theory as describing the dynamics of connection 1-forms. I will

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2 More precisely, $q_{ab}$ is the induced metric on $\Sigma$, while $\tilde{p}^{ab}$ is related to the extrinsic curvature $K_{ab}$ via $\tilde{p}^{ab} = \sqrt{q}(K^{ab} - K q^{ab})$. 
often use the phrase “connection dynamics” when discussing Yang-Mills theory in this form.

Despite the apparent differences between geometrodynamics and connection dynamics, many researchers have tried to recast the theory of general relativity in terms of a connection 1-form. Afterall, if the strong, weak, and electromagnetic interactions admit a connection dynamic description, why shouldn’t gravity? Early attempts in this direction used Yang-Mills type actions, but these actions gave rise, however, to new theories of gravity. A connection dynamic theory was gained, but Einstein’s theory of general relativity was lost in the process. Later attempts (like the ones I will concentrate on in this review) left general relativity alone, but tried to reinterpret Einstein’s equation in terms of the dynamics of a connection 1-form. The most familiar of these approaches is due to Palatini who rewrote the standard Einstein-Hilbert action (which is a functional of just the spacetime metric $g_{ab}$) in such a way that the spacetime metric and an arbitrary Lorentz connection 1-form are independent basic variables. However, as we shall see in Section 6, the 3+1 Palatini theory does not succeed in recasting general relativity as a connection dynamical theory. The 3+1 Palatini theory collapses back to the standard geometrical description of general relativity when one writes it in Hamiltonian form.

More recently, Ashtekar \cite{1,2,3} has proposed a reformulation of general relativity in which a real (densitized) triad $\tilde{E}^a_i$ and a connection 1-form $A^a_i$ (which takes values in the complexified Lie algebra of $SO(3)$) are the basic canonical variables. He obtained these new variables for the real theory by performing a canonical transformation on the standard phase space of real general relativity. For the complex theory, Jacobson and Smolin \cite{4} and Samuel \cite{5} independently found a covariant action that yields Ashtekar’s new variables when one performs a 3+1 decomposition. This action is the Palatini action for complex general relativity viewed as a functional of a complex co-tetrad and a self-dual connection 1-form. In one sense, it is somewhat surprising that these new variables could capture the full content of Einstein’s equation since they involve only half (i.e., the self-dual part) of a Lorentz connection 1-form. On the other hand, the special role that self-dual fields play in the theory of general relativity was already evident in the work of Newman, Penrose, and Plebanski on self-dual solutions to Einstein’s equation. In fact, much of this earlier work provided the motivation for Ashtekar’s search for the new variables.

Not only did the new variables give general relativity a connection dynamic description; they also simplified the field equations of the theory—particularly the constraints. In terms of the standard geometrodynamical variables ($q_{ab}, \tilde{p}^{ab}$), the constraint equations are non-polynomial. However, in terms of the new variables, the constraint equations become

\[^3\text{To recover the phase space variables for the real theory, one must impose reality conditions to select a real section of the complex phase space.}\]
This result has rekindled interest in the canonical quantization program for 3+1 gravity. Due to the simplicity of the constraint equations in terms of these new variables, Jacobson, Rovelli, and Smolin and a number of other researchers have been able to solve the quantum constraints exactly. Although the quantization program has not yet been completed, the above results constitute promising first steps in that direction.

The Palatini and self-dual theories described above were attempts to give general relativity in 3+1 dimensions a connection dynamic description. A few years later, Witten considered the 2+1 theory of gravity. He was able to show that this theory simplifies considerably when expressed in Palatini form. In fact, Witten demonstrated that the 2+1 Palatini theory for vacuum 2+1 gravity was equivalent to Chern-Simons theory based on the inhomogeneous Lie group $ISO(2,1)$.

He then used this fact to quantize the theory. This result startled both relativists and field theorists alike: relativists, since the Wheeler-DeWitt equation in geometrodynamics is as hard to solve in 2+1 dimensions as it is in 3+1 dimensions; field theorists, since a simple power counting argument had shown that perturbation theory for 2+1 gravity around a flat background metric is non-renormalizable—just as it is for the 3+1 theory. The success of canonical quantization and failure of perturbation theory in 2+1 dimensions came as a welcome surprise. Despite key differences between 2+1 and 3+1 gravity (in particular, the lack of local degrees of freedom for 2+1 vacuum solutions), Witten’s result has proven to be useful to non-perturbative approaches to 3+1 quantum gravity. In particular, since the overall structure of 2+1 and 3+1 gravity are the same (e.g., they are both diffeomorphism invariant theories, there is no background time, and the dynamics is generated in both cases by 1st class constraints), researchers have been able to use 2+1 gravity as a “toy model” for the 3+1 theory.

Finally, the most recent developments relating geometrodynamics and connection dynamics involve formulations of general relativity that are independent of any metric variable. This idea for 3+1 gravity dates back to Plebanski, and was recently developed fully by Capovilla, Dell, and Jacobson (CDJ). Shortly thereafter, Peldán provided a similar formulation for 2+1 gravity. These pure spin-connection formulations of general relativity are defined by actions that do not involve the spacetime metric $g_{ab}$ in any way whatsoever—the action for the complex 3+1 theory depends only on a connection 1-form (which takes values in the complexified Lie algebra of $SO(3)$) and a scalar density of weight $-1$. Moreover, the Hamiltonian formulation of this theory is the same as that of the self-dual theory; and by using their approach, CDJ have been able to write down the most

4Chern-Simons theory, like Yang-Mills theory, is a theory of a connection 1-form. However, unlike Yang-Mills theory, it is defined only in odd dimensions and does not require the introduction of a spacetime metric.
general solution to the 4 diffeomorphism constraint equations. Whether or not these results will lead to new insights for the quantization of the 3+1 theory remains to be seen.

With this brief history of geometrodynamics and connection dynamics as background, the purpose of this review can be stated as follows: It is to describe in detail the theories mentioned above, and, in the process, clarify the mathematical relationship between geometrodynamics and connection dynamics in the context of the classical theories of 2+1 and 3+1 gravity. While preparing the text, I made a conscious effort to make the presentation as self-contained and internally consistent as possible. The calculations are somewhat technical and rather detailed, but I have included many footnotes, parenthetical remarks, and mathematical digressions to fill various gaps. I felt that this style of presentation (as opposed to relegating the necessary mathematics to appendices at the end of the paper) was more in keeping with the natural interplay between mathematics and physics that occurs when one works on an actual research problem. Also, I felt that the added details would be of value to anyone interested in working in this area.

In Section 2, I recall the standard Einstein-Hilbert theory and take some time to introduce the notation and mathematical techniques that I will use repeatedly throughout the text. Although this section is a review of fairly standard material, readers are encouraged to at least skim through the pages to acquaint themselves with my style of presentation. In Sections 3 and 4, I restrict attention to 2+1 dimensions and describe the 2+1 Palatini and Chern-Simons theories and demonstrate the relationship between them. In Section 5, I couple a cosmological constant and a massless scalar field to the 2+1 Palatini theory. 2+1 Palatini theory coupled to a cosmological constant \( \Lambda \) is of interest since we shall see that the equivalence between the 2+1 Palatini and Chern-Simons theories continues to hold even if \( \Lambda \neq 0 \); 2+1 Palatini theory coupled to a massless scalar field is of interest since it is the dimensional reduction of 3+1 vacuum general relativity with a spacelike, hypersurface-orthogonal Killing vector field (see, e.g., Chapter 16 of [10]). In fact, recent work in progress (by Ashtekar and Varadarajan) in the hamiltonian formulation of this reduced theory indicates that its non-perturbative quantization is likely to be successful. In Sections 6 and 7, I turn my attention to 3+1 dimensions and describe the 3+1 Palatini and self-dual theories. In Section 8, I couple a cosmological constant and a Yang-Mills field to 3+1 gravity. Section 9 describes a pure spin-connection formulation of 3+1 gravity, and Section 10 concludes with a brief summary and discussion of the results. All of the above theories are specified by an action. I obtain the Euler-Lagrange equations of motion by varying the action and perform a Legendre transform to put each theory in Hamiltonian form. I emphasize the similarities, differences, and equivalences of the various theories whenever possible. While this paper is primarily a review, some of the material is in fact new, or at least has not appeared in the literature in
the form given here. Much of Sections 3, 4, and 5 on the 2+1 theory fall in this category.

I should also list a few of the topics that are not covered in this review. First, I have restricted attention to the more “standard” theories of 2+1 and 3+1 gravity. I have made no attempt to treat higher-derivative theories of gravity, supersymmetric theories, or any of their equivalents. Second, I have chosen to omit any discussion of quantum theory, although it is here, in quantum theory, that the change in emphasis from geometrodynamics to connection dynamics has had the most success. All of the theories described in this paper are treated at a purely classical level; issues related, for instance, to quantum cosmology and the non-perturbative canonical quantization program for 3+1 gravity are not dealt with. This review serves, instead, as a thorough pre-requisite for addressing the above issues. Moreover, many books and review articles already exist which discuss the quantum theory in great detail. Interested readers should see, in addition to the text books [2, 3], review articles [17, 18, 19, 20] and references mentioned therein. Third, in 2+1 dimensions, I have chosen to concentrate on the relationship between the 2+1 Palatini and Chern-Simons theories, and have all but ignored the equally interesting relationships between these formulations and the standard 2+1 dimensional Einstein-Hilbert theory. Fortunately, other researchers have already addressed these issues, so interested readers can find details in [21, 22, 23]. Also, since Chern-Simons theory is not available in 3+1 dimensions, the equivalence of the 2+1 Palatini and Chern-Simons theories does not have a direct 3+1 dimensional analog. However, recent work by Carlip [24, 25] and Anderson [26] on the problem of time in 2+1 quantum gravity may shed some light on the corresponding issue facing the 3+1 theory. Finally, Section 9 on general relativity without-the metric deals exclusively with 3+1 gravity. Readers interested in a pure spin-connection formulation of 2+1 gravity should see [13].

Penrose’s abstract index notation will be used throughout. Spacetime and spatial tensor indices are denoted by latin letters from the beginning of the alphabet $a, b, c, \cdots$, while internal indices are denoted by latin letters from the middle of the alphabet $i, j, k, \cdots$ or $I, J, K, \cdots$. The signature of the spacetime metric $g_{ab}$ is taken to be $(- + +)$ or $(- + + +)$, depending on whether we are working in 2+1 or 3+1 dimensions. If $\nabla_a$ denotes the unique, torsion-free spacetime derivative operator compatible with the spacetime metric $g_{ab}$, then $R_{abc}^d k_d := 2\nabla_a \nabla_b k_c, R_{ab} := R_{acb}^c$, and $R := R_{ab} g^{ab}$ define the Riemann tensor, Ricci tensor, and scalar curvature of $\nabla_a$.

Finally, since I eventually want to obtain a Hamiltonian formulation for each theory, I will assume from the beginning that the spacetime manifold $M$ is topologically $\Sigma \times R$. If the theory depends on a spacetime metric, I assume $\Sigma$ to be spacelike; if the theory does not depend on a spacetime metric, I assume $\Sigma$ to be any (co-dimension 1) submanifold of $M$. In either case, I ignore all surface integrals and avoid any discussion of boundary conditions.
In this sense, the results I obtain are rigorous only for the case when $\Sigma$ is compact. Readers interested in a detailed discussion of the technically more difficult asymptotically flat case (in the context of the standard Einstein-Hilbert or self-dual theories) should see Chapters II.2 and III.2 of [2].

2. Einstein-Hilbert theory

In this section, we will describe the standard Einstein-Hilbert theory. We obtain the vacuum Einstein’s equation starting from an action principle and perform a Legendre transform to put the theory in Hamiltonian form. We shall see that the phase space variables consist of a positive-definite metric $g_{ab}$ and a density-weighted, symmetric, second-rank tensor field $\tilde{p}^{ab}$. These are the standard geometricodynamical variables of general relativity. We will also analyze the motions on phase space generated by the constraint functions and evaluate their Poisson bracket algebra. This section is basically a review of standard material. Our treatment will follow that given, for example, in Appendix E of [27] or Chapter II.2 of [2].

The standard Einstein-Hilbert theory is, of course, valid in $n+1$ dimensions. Everything we do in this section will be independent of the dimension of the spacetime manifold $M$. This is an important feature which will allow us to compare the standard Einstein-Hilbert theory with the Chern-Simons and self-dual theories. Unlike the standard Einstein-Hilbert theory, Chern-Simons theory is defined only in odd dimensions, while the self-dual theory is defined only in 3+1 dimensions.

2.1 Euler-Lagrange equations of motion

Let us begin with the well-known Einstein-Hilbert action

$$S_{EH}(g^{ab}) := \int_M \sqrt{-g} R. \quad (2.1)$$

Here $g$ denotes the determinant of the covariant metric $g_{ab}$, and $R$ denotes the scalar curvature of the unique, torsion-free spacetime derivative operator $\nabla_a$ compatible with $g_{ab}$. I have taken the basic variable to be the contravariant spacetime metric $g^{ab}$ for convenience when performing variations of the action. The Einstein-Hilbert action is second-order since $R$ contains second derivatives of $g_{ab}$.

To obtain the Euler-Lagrange equations of motion, we vary the action with respect to the field variable $g^{ab}$. If we write the integrand as $\sqrt{-g} R_{ab}g^{ab}$ and use the fact that $\delta g = -g_{ab} \delta g^{ab}$, we get

$$\delta S_{EH} = \int_M \sqrt{-g} (R_{ab} - \frac{1}{2} R g_{ab}) \delta g^{ab} + \int_M \sqrt{-g} \delta R_{ab} g^{ab}. \quad (2.2)$$
The first integral is of the desired form, while the second integral requires us to evaluate the variation of the Ricci tensor $R_{ab}$. Since one can show that
\[ \delta R_{ab} g^{ab} = \nabla_a v^a \] (2.3)
(where $v^a = \nabla^b (g_{bc} \delta g^{bc} - \nabla_b \delta g^{ab})$), we see that modulo a surface integral, $\delta S_{EH} = 0$ if and only if
\[ G_{ab} := R_{ab} - \frac{1}{2} R g_{ab} = 0. \] (2.4)
This is the desired result: The vacuum Einstein’s equation can be obtained starting from an action principle.

I should note that, strictly speaking, the variation of (2.1) with respect to $g_{ab}$ does not yield the vacuum Einstein’s equation $G_{ab} = 0$. The surface integral does not vanish since $v^a$ involves derivatives of the variation $\delta g^{ab}$. Even though $\delta g^{ab}$ is required to vanish on the boundary, these derivatives need not vanish. This seems to pose a potential problem, but it can handled by simply adding to (2.1) a boundary term which will (upon variation) exactly cancel the surface integral. As shown in Appendix E of [27], this boundary term involves the trace of the extrinsic curvature of the boundary of $M$. For the sake of simplicity, however, we will continue to use the unmodified Einstein-Hilbert action (2.1) and ignore all surface integrals as mentioned at the end of Section 1.

2.2 Legendre transform

To put the standard Einstein-Hilbert theory in Hamiltonian form, we will follow the usual procedure: We assume that $M = \Sigma \times R$ for some spacelike submanifold $\Sigma$ and assume that there exists a time function $t$ (with nowhere vanishing gradient $(dt)_a$) such that each $t = \text{const}$ surface $\Sigma_t$ is diffeomorphic to $\Sigma$. To talk about evolution from one $t = \text{const}$ surface to the next, we introduce a future-pointing timelike vector field $t^a$ satisfying $t^a (dt)_a = 1$. $t^a$ is the “time flow” vector field that defines the same point in space at different instants of time. We will treat $t^a$ and the foliation of $M$ by the $t = \text{const}$ surfaces as kinematical (i.e., non-dynamical) structure. Evolution will be given by the Lie derivative with respect to $t^a$.

Since we have a spacetime metric $g_{ab}$ as one of our field variables, we can also introduce a unit covariant normal $n_a$ and its associated future-pointing timelike vector field $n^a = g^{ab} n_b$.

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5To obtain this result, consider a 1-parameter family of spacetime metrics $g_{ab}(\lambda)$ and their associated spacetime derivative operators $\nabla_a$. Define $C_{ab}^c$ by $\nabla_a k_b := \nabla_a k_b + \lambda C_{ab}^c k_c$ and differentiate $\nabla_a g_{bc}(\lambda) = 0$ with respect to $\lambda$. Evaluating this expression at $\lambda = 0$ gives $C_{ab}^c = -\frac{1}{2} g^{cd} (\nabla_a \delta g_{bd} + \nabla_b \delta g_{ad} - \nabla_d \delta g_{ab})$, where $g_{ab} := g_{ab}(0)$ and $\delta g_{ab} := \frac{d}{d\lambda} \bigg|_{\lambda=0} g_{ab}(\lambda)$. Since $R_{abc}^d(\lambda) = R_{abc}^d + \lambda \left[ 2 \nabla_{[a} C_{b]c}^d + \lambda^2 [C_a, C_b]_c^d \right]$, it follows that $\delta R_{ac} := \frac{d}{d\lambda} \bigg|_{\lambda=0} R_{abc}^b(\lambda) = 2 \nabla_{[a} C_{b]c}^c$. Contracting with $g^{ac}$ (using $\delta g^{ab} = -g^{ac} g^{bd} \delta g_{cd})$ yields the above result.
Note that since $n_a n_a = -1$, $q^a_b := \delta^a_b + n^a n_b$ is a projection operator into the $t = \text{const}$ surfaces. We will construct the configuration variables associated with the field variable $g^{ab}$ by contracting with $n^a$ and $q^a_b$. We define the \textit{induced metric} $q^{ab}$, the \textit{lapse} $N$, and \textit{shift} $N^a$ via

$$q^{ab} := q^{m}_a q^n_b \ g_{mn} (= g^{ab} + n^a n_b),$$

$$N := -n^a t^b \ g_{ab},$$

and

$$N^a := q^a_b \ t^b.$$  \hspace{1cm} (2.5)

Note that in terms of $N$ and $N^a$, we can write $t^a = N n^a + N^a$. Furthermore, since $N^a n_a = 0$ and $q_{ab} n^a = 0$, $N^a$ and $q_{ab}$ are (in 1-1 correspondence with) tensor fields defined intrinsically on $\Sigma$.

The next step in constructing a Hamiltonian formulation of the Einstein-Hilbert theory is to decompose the Einstein-Hilbert action and write it in the form

$$S_{EH}(g^{ab}) = \int dt \ L_{EH}(q, \dot{q}).$$  \hspace{1cm} (2.6)

$L_{EH}$ will be the Einstein-Hilbert Lagrangian provided it depends only on $(q^{ab}, N, N^a)$ and their first time derivatives. But, as written, (2.1) is not convenient for such a decomposition. The integrand $\sqrt{-g} R$ contains second time derivatives of the configuration variable $q^{ab}$. However, as we will now show, these terms can be removed from the integrand by subtracting off a total divergence.

To see this, let us write the scalar curvature $R$ as $R = 2(G_{ab} - R_{ab}) n^a n^b$. Then the differential geometric identities

$$G_{ab} n^a n^b = \frac{1}{2} (\mathcal{R} - K_{ab} K^{ab} + K^2)$$

and

$$R_{ab} n^a n^b = -K_{ab} K^{ab} + K^2 + \nabla_b(n^a \nabla_a n^b - n^b \nabla_a n^a)$$  \hspace{1cm} (2.7)

(where $K_{ab} := q^m_a q^n_b \ \nabla_m n_n$ is the \textit{extrinsic curvature} of the $t = \text{const}$ surfaces and $\mathcal{R}$ is the scalar curvature of the unique, torsion-free spatial derivative operator $D_a$ compatible with the induced metric $q^{ab}$) imply

$$R = (\mathcal{R} + K_{ab} K^{ab} - K^2) + (\text{total divergence term}).$$  \hspace{1cm} (2.8)

Using the fact that $\sqrt{-g} = N \sqrt{q} \ dt$ (where $q$ denotes the determinant of $q^{ab}$), the Einstein-Hilbert action (2.1) becomes

$$S_{EH}(g^{ab}) = \int dt \int_\Sigma \sqrt{q} N (\mathcal{R} + K_{ab} K^{ab} - K^2) + (\text{surface integral}).$$  \hspace{1cm} (2.9)

If we ignore the surface integral, we get

$$L_{EH} = \int_\Sigma \sqrt{q} N (\mathcal{R} + K_{ab} K^{ab} - K^2).$$  \hspace{1cm} (2.10)
This is the desired Einstein-Hilbert Lagrangian first proposed by Arnowitt, Deser, and Misner (ADM) \[28\]. The identity \( K_{ab} = \frac{1}{2N}(\mathcal{L}_q q_{ab} - 2D_i N_0) \) allows us to express \( L_{EH} \) in terms of only \((q_{ab}, N, N^a)\) and their first time derivatives.

Given the Einstein-Hilbert Lagrangian, we are now ready to perform the Legendre transform. But before we do this, it is probably worthwhile to make a detour and first review the standard Dirac constraint analysis for a theory with constraints and recall some basic ideas of symplectic geometry. I propose to examine, in detail, a simple finite-dimensional system described by a Lagrangian

\[
L(q, \dot{q}) := \frac{1}{2} \dot{q}_1^2 + q_3 \dot{q}_2 - q_4 f(q_2, q_3). \tag{2.11}
\]

Here \((q_1, \ldots, q_4) \in \mathcal{C}_0\) are the configuration variables and \((\dot{q}_1, \ldots, \dot{q}_4)\) are their associated time derivatives (or velocities). \(f(q_2, q_3)\) can be any (smooth) real-valued function of \((q_2, q_3)\). The techniques that arise when analyzing this simple system will apply not only to the standard Einstein-Hilbert theory but to many other constrained theories as well. Readers interested in a more detailed description of the general Dirac constraint analysis and symplectic geometry should see \[29\] and Appendix B of \[3\], respectively. Readers already familiar with the standard Dirac constraint analysis may skip to the paragraph immediately following equation (2.25).

To perform the Legendre transform for our simple system, we first define the momentum variables \((p_1, \ldots, p_4)\) via

\[
p_{\dot{q}_i} := \frac{\delta L}{\delta \dot{q}_i} \quad (i = 1, \ldots, 4). \tag{2.12}
\]

For the special form of the Lagrangian given above, they become

\[
p_1 = \dot{q}_1, \quad p_2 = q_3, \quad p_3 = 0, \quad \text{and} \quad p_4 = 0. \tag{2.13}
\]

Since only the first equation can be inverted to give \(\dot{q}_1\) as a function of \((q, p)\), there are constraints: Not all points in the phase space \(\Gamma_0 = T^*\mathcal{C}_0 = \{(q_2, p_2) : \vec{\imath} = 1, \ldots, 4\}\) are accessible to the system. Only those \((q, p) \in \Gamma_0\) which satisfy

\[
\phi_1 := p_2 - q_3 = 0, \quad \phi_2 := p_3 = 0, \quad \text{and} \quad \phi_3 := p_4 = 0 \tag{2.14}
\]

are physically allowed. The \(\phi_{\vec{\imath}}\)’s are called primary constraints and the vanishing of these functions define a constraint surface in \(\Gamma_0\). It is the presence of these constraints that complicates the standard Legendre transform.

Following the Dirac constraint analysis, we now must now write down a Hamiltonian for the theory. But due to (2.14), the Hamiltonian will not be unique. The usual definition \(H_0(q, p) := \sum_{\vec{\imath}}^4 p_{\dot{q}_{\vec{\imath}}} - L(q, \dot{q})\) does not work, since there exist \(\dot{q}_{\vec{\imath}}\)’s which cannot be written
as functions of \( q \) and \( p \). If, however, we restrict ourselves to the constraint surface defined by (2.14), we have
\[
H_0(q, p) = \frac{1}{2}p_1^2 + q_4 f(q_2, q_3).
\]
(2.15)
Since the right hand side of (2.15) makes sense on all of \( \Gamma_0 \), \( H_0(q, p) \) actually defines one possible choice of Hamiltonian. However, as we will show below, this Hamiltonian is definitely not the only one.

For suppose \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) are three arbitrary functions on \( \Gamma_0 \). Then
\[
H_T(q, p) : = H_0(q, p) + \lambda_1 \phi_1 + \lambda_2 \phi_2 + \lambda_3 \phi_3
\]
(2.16)
is another function (defined on all of \( \Gamma_0 \)) that agrees with \( H_0(q, p) \) on the constraint surface. \( H_T(q, p) \) is called the total Hamiltonian, and it differs from \( H_0(q, p) \) by terms that vanish on the constraint surface. This non-uniqueness of the total Hamiltonian exists for any theory that has constraints.

Given \( H_T(q, p) \), the next step in the Dirac constraint analysis is to require that the primary constraints (2.14) be preserved under time evolution—i.e., that
\[
\dot{\phi}_i := \{ \phi_i, H_T \}_0 \approx 0 \quad (i = 1, 2, 3).
\]
(2.17)
Here \( \approx \) means equality on the constraint surface defined by (2.14) and \( \{ , \}_0 \) denotes the Poisson bracket defined by the natural symplectic structure\(^6\)
\[
\Omega_0 = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + dp_3 \wedge dq_3 + dp_4 \wedge dq_4
\]
(2.18)
on \( \Gamma_0 \). Equation (2.17) is equivalent to the requirement that the evolution of the system take place on the constraint surface.

Evaluating (2.17) for the primary constraints (2.14), we find that \( \{ \phi_3, H_T \}_0 \approx 0 \) implies
\[
\phi_4 := f(q_2, q_3) \approx 0.
\]
(2.19)

\(^6\)A symplectic manifold (or phase space) consists of a pair \((\Gamma_0, \Omega_0)\), where \( \Gamma_0 \) is an even dimensional manifold and \( \Omega_0 \) is a closed and non-degenerate 2-form. (i.e., \( d\Omega_0 = 0 \) and \( \Omega_0(v, w) = 0 \) for all \( v \) implies \( w = 0 \).) \( \Omega_0 \) is called the symplectic structure and it allows us to define Hamiltonian vector fields and Poisson brackets: Given any real-valued function \( f : \Gamma_0 \to R \), the Hamiltonian vector field \( X_f \) is defined by \(-i_{X_f} \Omega_0 := df\). Given any two real-valued functions \( f, g : \Gamma_0 \to R \), the Poisson bracket \( \{ f, g \}_0 \) is defined by \( \{ f, g \}_0 := -\Omega(X_f, X_g) = -X_f(g) \). As a special case, if \( \Gamma_0 = T^*C_0 \) is the cotangent bundle over some \( n \)-dimensional configuration space \( C_0 \), then \( \Omega_0 = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n \) is the natural symplectic structure on \( \Gamma_0 \) associated with the chart \((q, p)\). It follows that \( \{ f, g \}_0 = \sum_{i=1}^n (\frac{\partial f}{\partial q^i} \frac{\partial}{\partial p^i} - \frac{\partial f}{\partial p^i} \frac{\partial}{\partial q^i}) \), which is the standard textbook expression for the Poisson bracket of \( f \) and \( g \).
The other Poisson brackets yield conditions on $\lambda_1$ and $\lambda_2$. $\phi_4$ is called a secondary constraint, and for consistency we must also require that

$$\dot{\phi}_4 := \{\phi_4, H_T\}_0 \approx 0. \tag{2.20}$$

Here $\approx$ now means equality on the constraint surface defined by (2.14) and (2.19). Since one can show that (2.20) follows from the earlier conditions on $\lambda_1$ and $\lambda_2$, (2.14) and (2.19) constitute all the constraints of the theory.

The final step in the Dirac constraint analysis is to take all the constraints $\phi_1, \ldots, \phi_4$ and evaluate their Poisson brackets. If a constraint $\phi_\perp$ satisfies $\{\phi_\perp, \phi_j\}_0 \approx 0$ for all $\phi_j$, then $\phi_\perp$ is said to be 1st class. If, however, $\{\phi_\perp, \phi_j\}_0 \not\approx 0$ for some $\phi_j$, then $\phi_\perp$ and $\phi_j$ are said to form a 2nd class pair. (In terms of symplectic structures and Hamiltonian vector fields, a constraint $\phi_\perp$ is 1st class with respect to the symplectic structure $\Omega_0$ if and only if the Hamiltonian vector field $X_{\phi_\perp}$ defined by $\Omega_0$ is tangent to the constraint surface defined by the vanishing of all the constraints.) The goal is to solve all the 2nd class constraints (and possibly some 1st class constraints) and obtain a new phase space $(\Gamma, \Omega)$ where the remaining constraints (pulled-back to $\Gamma$) are all 1st class with respect to the Poisson bracket defined by $\Omega$. Evaluating $\{\phi_\perp, \phi_j\}_0$ for our simple system, we find that $\phi_3$ is the only 1st class constraint with respect to $\Omega_0$. By solving the second class pair $\phi_1 = 0$ and $\phi_2 = 0$, we get

$$\Omega_0 \big|_{\phi_1=0, \phi_2=0} = dp_1 \wedge dq_1 + dq_3 \wedge dq_2 + dp_4 \wedge dq_4 \tag{2.21}$$

and

$$H_T \big|_{\phi_1=0, \phi_2=0} = \frac{1}{2} p_1^2 + q_4 f(q_2, q_3) + \lambda_3 p_4. \tag{2.22}$$

The remaining constraints $\phi_3$ and $\phi_4$ are now both 1st class with respect to this new symplectic structure.

Although we have successively eliminated all the 2nd class constraints, we can go one step further. We can solve the 1st class constraint $\phi_3 := p_4 = 0$ by gauge fixing the configuration variable $q_4$. Even though this step is not required by the Dirac constraint analysis, it simplifies the final phase space structure somewhat. Solving $\phi_3 = 0$ and pulling-back (2.21) and (2.22) to this new constraint surface $\Gamma$ (coordinatized by $(q_1, q_2, q_3, p_1)$), we obtain

$$\Omega := dp_1 \wedge dq_1 + dq_3 \wedge dq_2 \tag{2.23}$$

and

$$H(q_1, q_2, q_3, p_1) := \frac{1}{2} p_1^2 + q_4 f(q_2, q_3). \tag{2.24}$$

Here $q_4$ is no longer thought of as a dynamical variable—it is a Lagrange multiplier of the theory associated with the 1st class constraint $f(q_2, q_3) = 0$. 12
To summarize: Given a Lagrangian of the form

\[ L(q, \dot{q}) := \frac{1}{2} \dot{q}_1^2 + q_3 \dot{q}_2 - q_4 f(q_2, q_3), \]  

(2.25)

the Dirac constraint analysis says that the momentum \( p_1 \) is unconstrained, while \( p_2 = q_3 \) and \( p_3 = p_4 = 0 \). Demanding that the constraints be preserved under evolution, we obtain a secondary constraint \( f(q_2, q_3) = 0 \). The constraints \( p_2 - q_3 = 0 \) and \( p_3 = 0 \) form a 2nd class pair and are easily solved; the remaining constraints \( p_4 = 0 \) and \( f(q_2, q_3) = 0 \) now form a 1st class set. By gauge fixing \( q_4 \) we can solve \( p_4 = 0 \), and thus obtain a new phase space \( (\Gamma, \Omega) \) coordinatized by \( (q_1, q_2, q_3, p_1) \) with symplectic structure (2.23) and Hamiltonian (2.24). We are left with a single 1st class constraint, \( f(q_2, q_3) = 0 \).

Let us now return to our analysis of the standard Einstein-Hilbert theory. Given \( L_{EH} \), we find that the momentum \( \tilde{p}^{ab} \) canonically conjugate to \( q_{ab} \) is given by

\[ \tilde{p}^{ab} := \frac{\delta L_{EH}}{\delta \dot{q}_{ab}} = \sqrt{q} (K^{ab} - K q^{ab}), \]  

(2.26)

while the momenta canonically conjugate to \( N \) and \( N^a \) are zero. Since equation (2.26) can be inverted to give

\[ \mathcal{L}_{\tilde{q}} q_{ab} = 2N q^{-1/2} (\tilde{p}_{ab} - \frac{1}{2} \tilde{p} q_{ab}) + 2 D_{(a} N_{b)}, \]  

(2.27)

it does not define a constraint. However, \( N \) and \( N^a \) play the role of Lagrange multipliers.

Thus, by following the Dirac constraint analysis we find that the phase space \( (\Gamma_{EH}, \Omega_{EH}) \) of the standard Einstein-Hilbert theory is coordinatized by the pair \( (q_{ab}, \tilde{p}^{ab}) \) and has symplectic structure\footnote{I use \( \mathbf{d} \) and \( \wedge \) to denote the infinite-dimensional exterior derivative and infinite-dimensional wedge product of forms on \( \Gamma_{EH} \). They are to be distinguished from \( d \) and \( \wedge \) which are the finite-dimensional exterior derivative and finite-dimensional wedge product of forms on \( \Sigma \). Note that in terms of the Poisson bracket \( \{, \} \) defined by \( \Omega_{EH} \), we have \( \{q_{ab}(x), \tilde{p}^{cd}(y)\} = \delta_{(a}^e \delta_{b)}^d \delta(x, y) \).}

\[ \Omega_{EH} = \int_{\Sigma} \mathbf{d} \tilde{p}^{ab} \wedge \mathbf{d} q_{ab}. \]  

(2.28)

The Hamiltonian is given by

\[ H_{EH}(q, \tilde{p}) = \int_{\Sigma} N \left( - q^{1/2} \mathcal{R} + q^{-1/2} (\tilde{p}^{ab} \tilde{p}_{ab} - \frac{1}{2} \tilde{p}^2) \right) - 2 N^a q_{ab} D_c \tilde{p}^{bc}. \]  

(2.29)

As we shall see in the next subsection, this is just a sum of 1st class constraint functions associated with

\[ \tilde{C}(q, \tilde{p}) := -q^{1/2} \mathcal{R} + q^{-1/2} (\tilde{p}^{ab} \tilde{p}_{ab} - \frac{1}{2} \tilde{p}^2) \approx 0 \quad \text{and} \quad (2.30a) \]

\[ \tilde{C}_a(q, \tilde{p}) := -2 q_{ab} D_c \tilde{p}^{bc} \approx 0. \]  

(2.30b)
Note that constraint equation (2.30a) is non-polynomial in the canonically conjugate variables due to the dependence of $\mathcal{R}$ on the inverse of $q_{ab}$. This is a major stumbling block for the canonical quantization program in terms of $(q_{ab}, \tilde{p}^{ab})$. To date, there exist no exact solutions to the quantum version of this constraint in full (i.e., non-truncated) general relativity.

2.3 Constraint algebra

To evaluate the Poisson brackets of the constraints and to determine the motions they generate on phase space, we must first construct constraint functions (i.e., mappings $\Gamma_{EH} \rightarrow \mathbb{R}$) associated with the constraint equations (2.30a) and (2.30b). To do this, we smear $\tilde{C}(q, \tilde{p})$ and $\tilde{C}_a(q, \tilde{p})$ with test fields $N$ and $N^a$ on $\Sigma$—i.e., we define

$$C(N) := \int_\Sigma N \left( -q^{1/2} \mathcal{R} + q^{-1/2} (\tilde{p}^{ab} \tilde{p}_{ab} - \frac{1}{2} \tilde{p}^2) \right)$$

and

$$C(\vec{N}) := \int_\Sigma -2N^a q_{ab} D_c \tilde{p}^{bc}.$$  

They are called the scalar and vector constraint functions of the standard Einstein-Hilbert theory.

The next step is to evaluate the functional derivatives of $C(N)$ and $C(\vec{N})$. For recall that if $f, g : \Gamma_{EH} \rightarrow \mathbb{R}$ are any two real-valued functions on phase space, the Hamiltonian vector field $X_f$ (defined by the symplectic structure (2.28)) is given by

$$X_f = \int_\Sigma \frac{\delta f}{\delta \tilde{p}^{ab}} \frac{\delta}{\delta q_{ab}} - \frac{\delta f}{\delta q_{ab}} \frac{\delta}{\delta \tilde{p}^{ab}}$$

and the Poisson bracket $\{f, g\}$ (defined by $\{f, g\} := -X_f(g)$) is given by

$$\{f, g\} = \int_\Sigma \frac{\delta f}{\delta q_{ab}} \frac{\delta g}{\delta \tilde{p}^{ab}} - \frac{\delta f}{\delta \tilde{p}^{ab}} \frac{\delta g}{\delta q_{ab}}.$$  

Note that under the 1-parameter family of diffeomorphisms on $\Gamma_{EH}$ associated with $X_f$,

$$q_{ab} \mapsto q_{ab} + \epsilon \frac{\delta f}{\delta \tilde{p}^{ab}} + O(\epsilon^2) \quad \text{and}$$

$$\tilde{p}^{ab} \mapsto \tilde{p}^{ab} - \epsilon \frac{\delta f}{\delta q_{ab}} + O(\epsilon^2).$$  

We will use (2.33) to determine the various Poisson brackets between $C(N)$ and $C(\vec{N})$; we will use (2.34a) and (2.34b) to determine the motions that they generate on phase space.

Let us begin with the vector constraint $C(\vec{N})$. Integrating (2.31b) by parts and noting that $2D_{(a} N_{b)} = \mathcal{L}_{\vec{N}} q_{ab}$, we get

$$C(\vec{N}) = \int_\Sigma (\mathcal{L}_{\vec{N}} q_{ab}) \tilde{p}^{ab} = -\int_\Sigma q_{ab} (\mathcal{L}_{\vec{N}} \tilde{p}^{ab}).$$  

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By inspection,
\[
\frac{\delta C(N)}{\delta q_{ab}} = -\mathcal{L}_N \tilde{p}^{ab} \quad \text{and} \quad \frac{\delta C(N)}{\delta \tilde{p}_{ab}} = \mathcal{L}_N q_{ab}.
\] (2.36)

Thus, we see that
\[
q_{ab} \mapsto q_{ab} + \epsilon \mathcal{L}_N q_{ab} + O(\epsilon^2) \quad \text{and} \quad \tilde{p}_{ab} \mapsto \tilde{p}_{ab} + \epsilon \mathcal{L}_N \tilde{p}_{ab} + O(\epsilon^2)
\] (2.37a, 2.37b)
is the motion on \( \Gamma_{EH} \) generated by \( C(N) \). Note that (2.37a) and (2.37b) are the maps on the tensor fields \( q_{ab} \) and \( \tilde{p}_{ab} \) induced by the 1-parameter family of diffeomorphisms on \( \Sigma \) associated with the vector field \( N^a \). In other words, the Hamiltonian vector field \( X_{C(N)} \) on \( \Gamma_{EH} \) is the lift of the vector field \( N^a \) on \( \Sigma \).

Let us now consider the scalar constraint \( C(N) \). Due to the non-polynomial dependence of \( \mathcal{R} \) on \( q_{ab} \), the functional derivative \( \delta C(N)/\delta q_{ab} \) is much harder to evaluate. After a fairly long calculation, one finds that
\[
\frac{\delta C(N)}{\delta q_{ab}} = -\frac{1}{2} N \tilde{C}(q, \tilde{p}) q^{ab} + 2 N q^{-1/2}(\tilde{p}^{ac} \tilde{p}_{c} - \frac{1}{2} \tilde{q}^{ab})
\]
\[
\quad + N q^{1/2}(\mathcal{R}^{ab} - \mathcal{R} q^{ab}) - q^{1/2}(D^a D^b N - q^{ab} D_c D_c N).
\] (2.38)

A much simpler calculation gives
\[
\frac{\delta C(N)}{\delta \tilde{p}_{ab}} = 2 N q^{-1/2}(\tilde{p}_{ab} - \frac{1}{2} \tilde{p} q_{ab}).
\] (2.39)

Recall that for the vector constraint function \( C(N) \), the motion on \( \Gamma_{EH} \) along \( X_{C(N)} \) corresponded to the Lie derivative of \( q_{ab} \) and \( \tilde{p}_{ab} \) with respect to \( N^a \). Thus, one might expect the motion on \( \Gamma_{EH} \) along \( X_{C(N)} \) to correspond to the Lie derivative with respect to \( t^a := Nn^a \). We will now show that if we restrict ourselves to the constraint surface \( \Gamma_{EH} \subset \Gamma_{EH} \) (defined by (2.30a) and (2.30b)), then this is actually the case.

Comparing (2.39) with equation (2.27) (setting \( N_a = 0 \)), we see that \( \delta C(N)/\delta \tilde{p}_{ab} = \mathcal{L}_{\tilde{t}} q_{ab} \), so
\[
q_{ab} \mapsto q_{ab} + \epsilon \mathcal{L}_{\tilde{t}} q_{ab} + O(\epsilon^2)
\] (2.40)
as conjectured. Similarly, writing \( \tilde{p}_{ab} = \sqrt{q}(K^{ab} - K q^{ab}) \) and using the differential geometric identity
\[
\mathcal{L}_{N\tilde{t}} K_{ab} = -N R_{ab} + 2 N K_{a}^{c} K_{bc} - N K K_{ab} + D_a D_b N
\] (2.41)

\footnote{To obtain this result we used the facts that \( \delta q = q q^{ab} \delta q_{ab} \) and \( \delta R_{ab} q^{ab} = D_a v^a \) for \( v^a = -D^a(q^{bc} \delta q_{bc}) + D^b(q^{ac} \delta q_{ac}) \). These are just the spatial analogs of the results used in subsection 2.1 when we varied the Einstein-Hilbert action with respect to \( g^{ab} \).}
(which holds in this form when $R_{ab} = 0$), we see that

$$\frac{\delta C(N)}{\delta q_{ab}} = - \left( \frac{1}{2} N \bar{C}(q, \bar{p}) q^{ab} + \mathcal{L}_{\vec{t}} \bar{p}^{ab} \right).$$

(2.42)

Thus,

$$\bar{p}^{ab} \mapsto \bar{p}^{ab} + \epsilon \left( \frac{1}{2} N \bar{C}(q, \bar{p}) q^{ab} + \mathcal{L}_{\vec{t}} \bar{p}^{ab} \right) + O(\epsilon^2).$$

(2.43)

If we now restrict ourselves to $\Gamma_{EH}$ (so that $\bar{C}(q, \bar{p}) = 0$), we get

$$\bar{p}^{ab} \mapsto \bar{p}^{ab} + \epsilon \mathcal{L}_{\vec{t}} \bar{p}^{ab} + O(\epsilon^2).$$

(2.44)

This is the desired result: When restricted to $\Gamma_{EH} \subset \Gamma_{EH}$, the Hamiltonian vector field $X_{C(N)}$ on $\Gamma_{EH}$ is the lift of the vector field $t^a := N n^a$ on $\Sigma$.

We are now ready to evaluate the Poisson brackets between the constraint functions. But first note that if $f(M) : \Gamma_{EH} \to R$ is any real-valued function on phase space of the form

$$f(M) := \int_\Sigma M^{a\cdots b}_{c\cdots d} \tilde{f}^{a\cdots b}_{c\cdots d}(q, \bar{p})$$

(2.45)

(2.46)

(2.47)

(2.48a)

(2.48b)

where $[\vec{N}, \vec{M}] := \mathcal{L}_{\vec{N}} M^a$ is the commutator of the vector fields $N^a$ and $M^a$ on $\Sigma$. Note that (2.48a) tells us that the subset of vector constraint functions is closed under Poisson brackets. In fact, $N^a \mapsto C(\vec{N})$ is a representation of the Lie algebra of vector fields on $\Sigma$. The commutator of vector fields on $\Sigma$ is mapped to the Poisson bracket of the corresponding vector constraint functions.
We are left with only the Poisson bracket \( \{C(N), C(M)\} \) of two scalar constraint functions to evaluate. Using (2.38) and (2.39) (and eliminating all terms symmetric in \( M \) and \( N \)), we get

\[
\{C(N), C(M)\} = \int_{\Sigma} -2M(D^a D^b N - q^{ab} D^c D_c N) \left( \tilde{p}_{ab} - \frac{1}{2} \tilde{p} q_{ab} \right) - (N \leftrightarrow M) = C(K),
\]

where \( K^a := (N \partial^a M - M \partial^a N) = q^{ab} (N \partial_b M - M \partial_b N) \). Thus, the Poisson bracket of two scalar constraints is a vector constraint. Although this implies that the subset of scalar constraint functions is not closed under Poisson bracket, the totality of constraint functions (scalar and vector) is—i.e., the constraint functions form a 1st class set as claimed in subsection 2.2. Note, however, that since the vector field \( K^a \) depends on the phase space variable \( q_{ab} \) (through its inverse), the Poisson bracket (2.49) involves structure functions. The constraint functions do not form a Lie algebra.

### 3. 2+1 Palatini theory

In this section, we will describe the 2+1 Palatini theory which, as we shall see at the end of subsection 3.2, is defined for any Lie group \( G \). We will discuss the relationship between the Palatini and Einstein-Hilbert actions, and show how the 2+1 Palatini theory based on \( SO(2,1) \) reproduces the standard results of 2+1 gravity. After performing a Legendre transform to put this theory in Hamiltonian form, we shall see that the phase space variables consist of a connection 1-form \( A^I_a \) (which takes values in the Lie algebra of \( G \)) and its canonically conjugate momentum (or “electric field”) \( \tilde{E}^I_a \). Thus, for \( G = SO(2,1) \), the 2+1 Palatini theory gives us a connection dynamic description of 2+1 gravity. The constraint equations are polynomial in the basic variables and the constraint functions form a Lie algebra with respect to Poisson bracket.

Once we write the 2+1 Palatini action in its generalized form, we will let \( G \) be an arbitrary Lie group. To reproduce the results of 2+1 gravity, we simply take \( G \) to be \( SO(2,1) \). Note that much of the material in subsections 3.2 and 3.3 can also be found in [30].

#### 3.1 Euler-Lagrange equations of motion

Recall the standard Einstein-Hilbert action of Section 2,

\[
S_{EH}(g^{ab}) = \int_{\Sigma} \sqrt{-g} R.
\]
To define the 2+1 Palatini action, it is convenient to first rewrite the integrand $\sqrt{-g}R$ in triad notation. But in order to do this, we will have to make a short mathematical digression. Readers interested in a more detailed discussion of what follows should see [31]. Readers already familiar the method of orthonormal bases may skip to the paragraph immediately following equation (3.9).

Consider an $n$-dimensional manifold $M$, and let $V$ be a fixed $n$-dimensional vector space with Minkowski metric $\eta_{IJ}$ having signature ($- + \cdots +$). A soldering form at $p \in M$ is an isomorphism $e_a^I(p) : T_pM \rightarrow V$. (Here $T_pM$ denotes the tangent space to $M$ at $p$.) Although an $n$-manifold does not in general admit a globally defined soldering form $e_a^I$, we can use $e_a^I$ to define tensor fields locally on $M$. For instance,

$$g_{ab} := e_a^I e_b^J \eta_{IJ} \quad (3.2a)$$

is a (locally defined) spacetime metric having the same signature as $\eta_{IJ}$. The inverse of $e_a^I$ will be denoted by $e^a_I$; it satisfies

$$g_{ab} e^a_I e^b_J = \eta_{IJ}. \quad (3.2b)$$

Spacetime tensor fields with additional internal indices $I, J, K, \cdots$ will be called generalized tensor fields on $M$. Spacetime indices are raised and lowered with the spacetime metric $g_{ab}$; internal indices are raised and lowered with the Minkowski metric $\eta_{IJ}$.

If one introduces a standard basis $\{b^I \mid I = 1, \cdots, n\}$ in $V$, then the vector fields $e_a^I := e_a^I b^I$ form an orthonormal basis of $g_{ab}$. These $n$-vector fields will be called a triad when $n = 3$ and a tetrad when $n = 4$. The dual co-vector fields, $e^a_I := g_{ab} \eta^{IJ} e^b_J$, will be called a co-triad and a co-tetrad when $n = 3$ and 4, respectively. I should note, however, that from now on I will ignore the distinction between a soldering form $e_a^I$ and the co-vector fields $e^a_I$. I will call a 3-dimensional soldering form $e_a^I$ a co-triad and a 4-dimensional soldering form $e^a_I$ a co-tetrad in what follows.

To do calculus with these generalized tensor fields, it is necessary to extend the definition of spacetime derivative operators so that they also “act” on internal indices. We require (in addition to the usual properties that a spacetime derivative operator satisfies) that a generalized derivative operator obey the linearity, Leibnitz, and commutativity with contraction rules with respect to the internal indices. Furthermore, we require that all generalized derivative operators be compatible with $\eta_{IJ}$. Given these properties, it is straightforward to show that the set of all generalized derivative operators has the structure of an affine space. In other words, if $\partial_a$ is some fiducial generalized derivative operator (which we treat as an origin in the space of generalized derivative operators), then any other generalized derivative operator $D_a$ is completely characterized by a pair of generalized tensor fields $A_{ab}^c$ and $A_{al}^J$. 

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defined by
\[ \mathcal{D}_a k_{bI} =: \partial_a k_{bI} + A_{ab}^c k_{cI} + A_{aI}^J k_{bJ} \] (3.3)

We will call \( A_{ab}^c \) and \( A_{aI}^J \) the **spacetime connection 1-form** and **internal connection 1-form** of \( \mathcal{D}_a \). It is easy to show that

\[ A_{aIJ} = A_{a[IJ]} \quad \text{and} \quad A_{abc} = A_{(abc)} \] (3.4)

These conditions follow from the requirements that all generalized derivative operators be compatible with \( \eta_{IJ} \) and that they be torsion-free. Later in this section, we will consider what happens if we allow derivative operators to have non-zero torsion—i.e., if \( A_{[ab]^c} \neq 0 \).

Finally, note that \( A_{abc} \) need not equal \( A_{a[IJ]} e_I e_J \), in general.

As usual, given a generalized derivative operator \( \mathcal{D}_a \), we can construct curvature tensors by commuting derivatives. The **internal curvature tensor** \( F_{abc} \) and the **spacetime curvature tensor** \( F_{abcd} \) are defined by

\[
2 \mathcal{D}_a \mathcal{D}_b k_I =: F_{abc}^d k_d \quad \text{and} \quad 2 \mathcal{D}_a \mathcal{D}_b k_c =: F_{abcd}^d k_d. \] (3.5a, 3.5b)

If our fiducial generalized derivative operator is chosen to be flat on both spacetime and internal indices, then

\[
F_{abl}^J = 2 \partial_a A_{bIJ} + [A_a, A_b]^J \quad \text{and} \quad F_{abc}^d = 2 \partial_a A_{b[c}^d + [A_a, A_b]_c^d. \] (3.6a, 3.6b)

Here \([A_a, A_b]^J := (A_a^K A_{bK}^J - A_{bK}^K A_{aK}^J)\) and \([A_a, A_b]_c^d := (A_{ac} A_{be}^d - A_{bc} A_{ae}^d)\) are the commutators of linear operators.

Just as a compatibility with a spacetime metric \( g_{ab} \) defines a unique, torsion-free spacetime derivative operator \( \nabla_a \), compatibility with an orthonormal basis \( e_I^a \) (and thus with \( g_{ab} \)) defines a unique torsion-free generalized derivative operator, which we also denote by \( \nabla_a \). The **Christoffel symbols** \( \Gamma_{ab}^c \) and \( \Gamma_{aI}^J \) are defined by

\[ \nabla_a k_{bI} =: \partial_a k_{bI} + \Gamma_{ab}^c k_{cI} + \Gamma_{aI}^J k_{bJ} \] (3.7)

and satisfy

\[
\Gamma_{aI}^J = -g_b^J (\partial_a e_{bI} + \Gamma_{ab}^c e_{cI}) \quad \text{and} \quad \Gamma_{ab}^c = \frac{1}{2} g^{cd} (\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}). \] (3.8a, 3.8b)
It also follows that internal and spacetime curvature tensors $R_{abI}^J$ and $R_{abc}^d$ of $\nabla_a$ are related by
\[ R_{abI}^J = R_{abc}^d \epsilon_c^d \epsilon_e^J. \tag{3.9} \]
We will need the above result in this and later sections to show that the Palatini and self-dual actions reproduce Einstein’s equation.

Now let us return to our discussion of the 2+1 Palatini theory. Recall that we wanted to write the integrand $\sqrt{-g} R$ in triad notation. Using
\[ R_{abI}^J = R_{abc}^d \epsilon_c^d \epsilon_e^J, \tag{3.10} \]
(which is equation (3.9) written in terms of a triad $\epsilon_I^a$) and
\[ \epsilon_{abc} = \epsilon_c^I \epsilon_a^b \epsilon_b^c \epsilon_e^J \epsilon_{ijk}, \tag{3.11} \]
(which relates the volume element $\epsilon_{abc}$ of $g_{ab} = \epsilon_I^a \epsilon_b^c \epsilon_e^J \eta_{IJ}$ to the volume element $\epsilon_{ijk}$ of $\eta_{IJ}$), we find that
\[
\sqrt{-g} R = \sqrt{-g} \delta^{bc} \epsilon_e^c \epsilon_a^b R_{bc}^{de} \\
= \frac{1}{2} \eta^{abc} \epsilon_{ade} R_{bc}^{de} \\
= \frac{1}{2} \eta^{abc} \epsilon_{ade} R_{bc}^{de} \\
= \frac{1}{2} \eta^{abc} \epsilon_{ade} R_{bc}^{de} \\
= \frac{1}{2} \eta^{abc} \epsilon_{ade} R_{bc}^{de}.
\tag{3.12}
\]
Thus, viewed as a functional of a co-triad $\epsilon_I^a$, the standard Einstein-Hilbert action is given by
\[ S_{EH} (\epsilon_I^a) = \frac{1}{2} \int \Sigma \eta^{abc} \epsilon_{ade} R_{bc}^{de}. \tag{3.13} \]

To obtain the 2+1 Palatini action, we simply replace $R_{abI}^J$ in (3.13) with the internal curvature tensor $3F_{abI}^J$ of an arbitrary generalized derivative operator $3D_a$ defined by
\[ 3D_a k_I := \partial_a k_I + [3A_a, 3A_I]_J k_J. \tag{3.14} \]
We define the 2+1 Palatini action based on $SO(2,1)$ to be
\[ S_P (\epsilon_I^a, 3A) := \frac{1}{4} \int_M \eta^{abc} \epsilon_{ade} R_{bc}^{de}. \tag{3.15} \]
to use this action so that the expressions for our canonically conjugate variables agree with those used in the literature (see, e.g., [30]).

As defined above, \( S_P(\mathcal{e}_a, \mathcal{A}) \) is a functional of both a co-triad \( \mathcal{e}_a^I \) and a connection 1-form \( \mathcal{A}_{aI}^J \) which takes values in the defining representation of the Lie algebra of \( SO(2,1) \). Note also that \( \mathcal{D}_a \) as defined by (3.14) knows how to act only on internal indices. We do not require that \( \mathcal{D}_a \) know how to act on spacetime indices. However, when performing calculations, we will find that it is often convenient to consider a torsion-free extension of \( \mathcal{D}_a \) to spacetime tensor fields. It turns out that all calculations and all results will be independent of our choice of torsion-free extension. In fact, we will see that these results hold for extensions of \( \mathcal{D}_a \) that have non-zero torsion as well.

Since the 2+1 Palatini action is a functional of both a co-triad and a connection 1-form, we will obtain two Euler-Lagrange equations of motion. When we vary \( \mathcal{e}_a^I \), we get

\[
\tilde{\eta}^{abc} \epsilon_{IJK} \mathcal{F}_{bc}^{JK} = 0. \tag{3.16}
\]

When we vary \( \mathcal{A}_{aIJ} \), we get

\[
\mathcal{D}_b(\tilde{\eta}^{abc} \epsilon_{IJK} \mathcal{e}_c^K) = 0. \tag{3.17}
\]

To arrive at (3.17), we considered a torsion-free extension of \( \mathcal{D}_a \) to spacetime tensor fields (so that \( \delta^3 \mathcal{F}_{bc}^{JK} = 2 \mathcal{D}_b \delta^3 \mathcal{A}_{c}^{JK} \)) and then integrated by parts. The surface integral vanished since \( \delta^3 \mathcal{A}_{c}^{JK} = 0 \) on the boundary, while the remaining term gave (3.17). Note that since the left hand side of (3.17) is the divergence of a skew spacetime tensor density of weight +1 on \( M \), it is independent of the choice of torsion-free extension of \( \mathcal{D}_a \). Since \( \tilde{\eta}^{abc} \epsilon_{IJK} \mathcal{e}_c^K = 2(\mathcal{e}) \mathcal{e}_I^a \mathcal{e}_J^b \) (where \( \mathcal{e} := \sqrt{-g} \)), we can rewrite (3.17) as

\[
\mathcal{D}_b \left( \mathcal{e} \mathcal{e}_I^a \mathcal{e}_J^b \right) = 0. \tag{3.18}
\]

We shall see in Section 6 that the form of equation (3.18) holds for the 3+1 Palatini theory as well.

To determine the content of equation (3.18), let us express \( \mathcal{D}_a \) in terms of the unique, torsion-free generalized derivative operator \( \nabla_a \) compatible with \( \mathcal{e}_a^I \), and \( \mathcal{C}_{aI}^J \) defined by

\[
\mathcal{D}_a k_I =: \nabla_a k_I + \mathcal{C}_{aI}^J k_J. \tag{3.19}
\]

Since (3.18) is the divergence of a skew spacetime tensor density of weight +1 on \( M \), and since \( \nabla_a \) is compatible with \( \mathcal{e}_a^I \), we get

\[
\mathcal{C}_{bI}^K \mathcal{e}_K^a \mathcal{e}_J^b + \mathcal{C}_{bJ}^K \mathcal{e}_I^a \mathcal{e}_K^b = 0. \tag{3.20}
\]

This is equivalent to the statement that the (internal) commutator of \( \mathcal{C}_{bIJ} \) and \( \mathcal{e}_I^a \mathcal{e}_J^b \) vanishes. We will now show that (3.20) implies that \( \mathcal{C}_{aI}^J = 0 \).
To see this, define a spacetime tensor field \( 3S_{abc} \) via

\[
3S_{abc} := 3C_{aI} 3e^I_b 3e^J_c. \tag{3.21}
\]

(Note, incidently, that \( 3S_{abc} \) is not the spacetime connection of \( 3D_a \) relative to \( \nabla_a \).) Then the condition \( 3C_{aI} 3e^I_a 3e^J_b 3e^J_c \) is not equivalent to \( 3S_{abc} = 3S_{a[bc]} \). Now contract equation (3.20) with \( 3e^I_a 3e^J_c \). This yields \( 3S_{bc} 3e^I_a = 0 \), so \( 3S_{abc} \) is trace-free on its first and last indices. Using this result, (3.20) reduces to

\[
3C_{bI} 3e^I_K 3e^J_K 3e^J_I - 3C_{bJ} 3e^I_K 3e^J_K 3e^I_K = 0. \tag{3.22}
\]

If we now contract (3.22) with \( 3e^I_c 3e^J_d \), we get

\[
3S_{cd} 3e^I_a = 3S_{(cd)} 3e^I_a. \tag{3.23}
\]

Thus, \( 3S_{abc} \) is symmetric in its first two indices. Since \( 3S_{abc} = 3S_{a[bc]} \) and \( 3S_{abc} = 3S_{(ab)c} \), we can successively interchange the first two and last two indices (with the appropriate sign changes) to show \( 3S_{abc} = 0 \). Furthermore, since \( e^I_a \) are invertible, we get \( 3C_{aI} 3e^I_a = 0 \). This is the desired result.\[\[\]

Since \( 3C_{aI} 3e^I_a = 0 \), we can conclude that the generalized derivative operator \( 3D_a \) must agree with \( \nabla_a \) when acting on internal indices. Thus, although the Palatini action started as a functional of a co-triad and an arbitrary generalized derivative operator \( 3D_a \), we find that one equation of motion implies that \( 3D_a = \nabla_a \). In terms of connection 1-forms, \( 3C_{aI} 3e^I_a = 0 \) implies that \( 3A_{aI} 3e^I_a = \Gamma_{aI} 3e^I_a \), where \( \Gamma_{aI} 3e^I_a \) is the internal Christoffel symbol of \( \nabla_a \). Using this result, the remaining Euler-Lagrange equation of motion (3.16) becomes

\[
\tilde{\eta}^{abc} 3e^I_a 3e^J_K R_{bc} 3e^J_K = 0. \tag{3.24}
\]

When (3.24) is contracted with \( 3e^a_d 3e^b_I 3e^c_J \), we get \( G^{ad} = 0 \). Thus, the Palatini action based on \( SO(2,1) \) reproduces the standard 2+1 vacuum Einstein’s equation.

It is interesting to note that to show that the Palatini action reduces to the standard Einstein-Hilbert action in 2+1 dimensions, we need only vary the connection 1-form \( 3A_{aI} 3e^I_a \). Since we found that (3.17) could be solved uniquely for \( 3A_{aI} 3e^I_a \) in terms of the remaining basic variables \( 3e^I_a \), we can pull-back \( S_P(3e,3A) \) to the solution space \( 3A_{aI} 3e^I_a = \Gamma_{aI} 3e^I_a \) and obtain a new action \( S_P(3e) \), which depends only on a co-triad. This pulled-back action is just 1/2 times the standard Einstein-Hilbert action \( S_{EH}(3e) \) given by (3.13). But what about the boundary term that one should strictly include in the standard Einstein-Hilbert action? It looks as if \( S_P(3e) \) is missing this needed term.

\[\footnote{This method of proving \( 3C_{aI} 3e^I_a = 0 \)—which generalizes to the 3+1 Palatini and self-dual actions—was shown to me by J. Samuel and A. Ashtekar.} \]

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The answer to this question is the following: Whereas the standard Einstein-Hilbert action is a second-order action, the 2+1 Palatini action is first-order. As mentioned at the beginning of Section 2, varying the standard Einstein-Hilbert action (3.1) with respect to $g^{ab}$ produces a surface integral involving derivatives of the variation $\delta g^{ab}$. Since we are allowed only to keep $g^{ab}$ fixed on the boundary, this surface integral is non-vanishing and must be compensated for by adding a boundary term to (3.1). This is also the case if we vary $S_{EH}(3e)$ given by (3.13) with respect to $3e^{I}a$. On the other hand, when we vary the Palatini action (3.15) with respect to $3A^{I}aJ$, we hold $3A^{I}aJ$ fixed on the boundary and $3e^{I}a$ fixed throughout. Then by solving (3.17) uniquely for $3A^{I}aJ$, we can pull-back $S_{P}(3e,3A)$ to the solution space $3A^{I}aJ = \Gamma^{I}aJ$. But now when we vary $S_{P}(3e)$ with respect to $3e^{I}a$ which lie entirely in the solution space, fixing $3e^{I}a$ on the boundary also fixes certain derivatives of $3e^{I}a$ on the boundary. This is a reflection of the fact that the reduction procedure comes with a prescription on how to do variations. It is precisely the vanishing of these derivatives of $\delta 3e^{I}a$ which eliminates the need of a boundary term for $S_{P}(3e)$.

It is also interesting to note that we could obtain the same result ($3A^{I}aJ = \Gamma^{I}aJ$) by considering an extension of $3D_{a}$ to spacetime tensor fields with non-zero torsion $3T^{c}ab$. (Recall that if $3A_{ab}^{c}$ denotes the spacetime connection 1-form of the extension of $3D_{a}$, then the torsion tensor $3T^{c}ab$ is defined by $2^{3}D_{[a}3D_{b]}f = 3T^{c}ab 3D_{c}f$ and satisfies $3T^{c}ab = 2 3A_{[ab]}^{c}$.) By varying the 2+1 Palatini action (3.15) with respect to $3A^{I}aJ$, we would find

$$2 3D_{[a}3e^{I}b] - 3T^{c}ab 3e^{I}c = 0.$$  \hspace{1cm} (3.25)

This is the field equation for $3A^{I}aJ$ which holds for any extension of $3D_{a}$ to spacetime tensor fields. If we restrict ourselves to torsion-free extensions, we get back equation (3.17). Then by following the argument given there, we would find $3A^{I}aJ = \Gamma^{I}aJ$ as before.

However, there exists an alternative approach to solving equation (3.25) which is often used by particle physicists. Namely, instead of considering a torsion-free extension of $3D_{a}$ to spacetime tensor fields, one considers an extension of $3D_{a}$ to spacetime tensor fields which is compatible with the co-triad $3e^{I}a$. This can always be done, but the price of such an extension is in general a non-zero torsion tensor $3T^{c}ab$. But since we now have $3D_{a}3e^{I}b = 0$, equation (3.25) implies

$$3T^{c}ab 3e^{I}c = 0.$$  \hspace{1cm} (3.26)

Invertibility of $3e^{I}b$ then implies that $3T^{c}ab = 0$. Since there exists only one torsion-free derivative operator compatible with $3e^{I}a$, we can conclude that $3D_{a} = \nabla_{a}$ (or equivalently, $3A^{I}aJ = \Gamma^{I}aJ$). This is the desired result.

Finally, to conclude this section, let us write the 2+1 Palatini action (3.15) in a form which is valid for any Lie group $G$. Recall that the connection 1-form $3A^{I}aJ$—being a linear
operator on the internal 3-dimensional vector space (equipped with the Minkowski metric \( \eta_{IJ} \) and satisfying \( 3A_{aIJ} = 3A_{a[IJ]} \)— takes values in the defining representation of the Lie algebra of SO(2, 1). Since \( \dim(\text{SO}(2, 1)) = 3 \) (which is the same as the dimension of the internal vector space), we can define an SO(2, 1) Lie algebra-valued connection 1-form, \( 3A^I_a \), via

\[
3A^I_a = 3A^K_a \epsilon^I_J K.
\] (3.27)

This is just the adjoint representation of the Lie algebra of SO(2, 1) with respect to the structure constants \( \epsilon^I_J K \). That the defining representation and adjoint representation agree is a property that holds only in 2+1 dimensions since \( \dim(\text{SO}(n, 1)) = n + 1 \) if and only if \( n = 2 \). In terms of \( 3A^I_a \), the generalized derivative operator \( 3D_a \) satisfies

\[
3D_a v^I = \partial_a v^I + [3A_a, v]^I,
\] (3.28)

where \([3A_a, v]^I := \epsilon^I_J K 3A^I_a \epsilon^K[J \). From (3.27), it also follows that the Lie algebra valued-curvature tensor \( 3F^I_{ab} \) (which is related to \( 3F_{abI}^J \) via \( 3F_{abI}^J = 3F^K_a \epsilon^I_J K \) ) can be written as

\[
3F^I_{ab} = 2\partial_a 3A^I_b + [3A_a, 3A_b]^I.
\] (3.29)

Thus, in terms \( 3A^I_a \) and \( 3F^I_{ab} \), the Palatini action becomes

\[
S_P(3\epsilon, 3A) = \frac{1}{4} \int_M \tilde{\eta}^{abc} \epsilon_{IJK} 3\epsilon^I_a 3F^J_{bc} 3F^K_{IJ}
\]

\[
= \frac{1}{4} \int_M \tilde{\eta}^{abc} \epsilon_{IJK} 3\epsilon^I_a 3F^L_{bc} 3F^K_{IJ} L
\]

\[
= \frac{1}{2} \int_M \tilde{\eta}^{abc} 3\epsilon_{aI} 3F^I_{bc}.
\] (3.30)

But now note that the last line above suggests a natural generalization. Namely, let \( G \) be any Lie group with Lie algebra \( \mathcal{L}_G \), and let \( 3A^I_a \) and \( 3\epsilon_{aI} \) be \( \mathcal{L}_G \) and \( \mathcal{L}_G^* \)-valued 1-forms, respectively. Although the action given by (3.30) was originally defined for the Lie group SO(2, 1), it is well-defined in the above sense for any Lie group \( G \). \( 3F^I_{ab} \) is still the curvature tensor of \( 3A^I_a \), but \( 3\epsilon_{aI} \) can no longer be thought of as a co-triad. In fact, since \( G \) is now

10Given a Lie algebra \( \mathcal{L} \) with structure constants \( C^I_{JK} \), the adjoint representation of \( \mathcal{L} \) by linear operators on \( \mathcal{L} \) is defined by the mapping \( v^I \in \mathcal{L} \mapsto (ad_v)^I_J := v^K C^I_{JK} \). Under \( ad \), the Lie bracket \( [v, w]^I := C^I_{JK} v^J w^K \in \mathcal{L} \) maps to the commutator of linear operators \( (ad_v, ad_w)^I_J := (ad_v)^I_K (ad_w)^K_J - (ad_w)^I_K (ad_v)^K_J \). I should note that since \( (ad_v)^I_J w^I = -[v, w]^J \), the above definition of the adjoint representation differs in sign from that given in most math and physics textbooks. The sign difference can be traced to my definition of the commutator of linear operators, which also differs in sign from the standard definition.
arbitrary, the index $I$ can take any value $1, 2, \cdots, \dim(G)$. Nonetheless, we can still define the Palatini action based on $G$ via

$$G_{SP}(\tilde{e}, 3A) := \frac{1}{2} \int_M \tilde{\eta}^{abc} 3e_{aI} \ 3F_{bc}^I,$$  \hspace{1cm} (3.31)

which we treat it as a functional of an $\mathcal{L}_G$-valued connection 1-form $3A_a^I$ and an $\mathcal{L}_G^*$-valued covector field $3\epsilon_{aI}$. The equations of motion we obtain by varying $3\epsilon_{aI}$ and $3A_a^I$ are

$$\tilde{\eta}^{abc} 3F_{bc}^I = 0 \quad \text{and} \quad 3\mathcal{D}_b (\tilde{\eta}^{abc} 3\epsilon_{cI}) = 0,$$  \hspace{1cm} (3.32)

which are the analogs of equations (3.16) and (3.17). As before, the second equation requires a torsion-free extension of $3\mathcal{D}_a$ to spacetime tensor fields, but again, all results will be independent of this choice.

### 3.2 Legendre transform

Given the action (3.31), it is a straightforward exercise to put the 2+1 Palatini theory based on $G$ in Hamiltonian form. We will assume that $M$ is topologically $\Sigma \times R$ and that there exists a function $t$ (with nowhere vanishing gradient $(dt)_a$) such that each $t = \text{const}$ surface $\Sigma_t$ is diffeomorphic to $\Sigma$. As usual, $t^a$ will denote the flow vector field satisfying $t^a (dt)_a = 1$. Since the Lie group $G$ is arbitrary, the 2+1 Palatini theory based on $G$ is not a theory of a spacetime metric; it does not involve a spacetime metric in any way whatsoever. Thus, in particular, $t$ does not necessarily have the interpretation of time. Nonetheless, we can still define “evolution” from one $t = \text{const}$ surface to the next using the Lie derivative with respect to $t^a$.

To write (3.31) in 2+1 form, we decompose $\tilde{\eta}^{abc}$ in terms of $t^a$ and $\tilde{\eta}^{ab}$ (the Levi-Civita tensor density of weight +1 on $\Sigma$). Using $\tilde{\eta}^{abc} = 3t^a [\tilde{\eta}^{bc}] dt$, we get

$$G_{SP}(\tilde{e}, 3A) = \frac{1}{2} \int_M \tilde{\eta}^{abc} 3e_{aI} \ 3F_{bc}^I$$

$$= \frac{1}{2} \int dt \int_\Sigma (t^a \tilde{\eta}^{bc} + t^b \tilde{\eta}^{ca} + t^c \tilde{\eta}^{ab}) 3e_{aI} \ 3F_{bc}^I$$

$$= \int dt \int_\Sigma \frac{1}{2} (3e \cdot t)_I \tilde{\eta}^{bc} F_{bc}^I + \tilde{\mathcal{E}}_I^c \mathcal{L}_c A_c^I - \tilde{\mathcal{E}}_I^c \mathcal{D}_c (3A \cdot t)^I,$$

where $(3e \cdot t)_I := t^a 3e_{aI}$, \  $\tilde{\mathcal{E}}_I^a := \tilde{\eta}^{ab} 3e_{bI}$, \  $(3A \cdot t)^I := t^a 3A_a^I$, and $A_a^I := t_a 3A_a^I$ are the configuration variables which specify all the information contained in the field variables $3\epsilon_{aI}$ and $3A_a^I$. Note that:

1. Since $G$ is an arbitrary Lie group, the internal index $I$ can take any value $I = 1, 2, \cdots, \dim(G)$. Thus, $\tilde{\mathcal{E}}_I^a$ cannot in general be interpreted as a dyad. In fact, this is true even when $G = SO(2, 1)$, since $\dim(SO(2, 1)) = 3$. However, for $SO(2, 1)$ we have $\tilde{\mathcal{E}}_I^a \tilde{\mathcal{E}}_b^a = \tilde{q}^{ab} (= qq^{ab})$. 

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2. $t^a F^I_{ab} = L^I_{\gamma} A^I_b - 3\Delta^a (3A \cdot t)^I$, which follows from a generalization of Cartan’s identity $L_{\gamma} = i_{\gamma} d\alpha + d(i_{\gamma} \alpha)$. The Lie derivative $L^I_{\gamma}$ treats fields with only internal indices as scalars.

3. $L^I_{\gamma} t^a_b = 0$, where $t^a_b := \delta^a_b - t^a (dt)_b$ is the natural projection operator into the $t = \text{const}$ surfaces defined by $t$ and $t^a$.

4. $D^I_a := t^c_a t^d_c F^I_{cd}$ is the generalized derivative operator on $\Sigma$ associated with $A^I_a$.

5. $F^I_{ab} := t^c_a t^d_c F^I_{cd}$ is the curvature tensor of $D^I_a$ and satisfies $F^I_{ab} = 2\partial_{[a} A^I_{b]} + [A_a, A_b]^I$.

From (3.33), we see that (modulo a surface integral) the Lagrangian $gL_P$ of the 2+1 Palatini theory based on $G$ is given by

$$GL_P = \int_{\Sigma} \frac{1}{2} (3e \cdot t)_I \tilde{\eta}^{ab} F^I_{ab} + \tilde{E}^a_I L^I_{\gamma} A^I_a + (D_a \tilde{E}^a_I)(3A \cdot t)^I. \quad (3.34)$$

By inspection, we see that the momentum conjugate to $A^I_a$ is $\tilde{E}^a_I$, while $(3e \cdot t)_I$ and $(3A \cdot t)^I$ both play the role of Lagrange multipliers. Thus, the Dirac constraint analysis says that the phase space $(G^T_P, G^\Omega_P)$ is coordinatized by the pair $(A^I_a, \tilde{E}^a_I)$ and has symplectic structure

$$G^\Omega_P = \int_{\Sigma} d\tilde{E}^a_I \wedge dA^I_a. \quad (3.35)$$

The Hamiltonian is given by

$$GH_P(A, \tilde{E}) = \int_{\Sigma} -\frac{1}{2} (3e \cdot t)_I \tilde{\eta}^{ab} F^I_{ab} - (D_a \tilde{E}^a_I)(3A \cdot t)^I. \quad (3.36)$$

As we shall see in the next subsection, this is just a sum of 1st class constraint functions associated with

$$\tilde{\eta}^{ab} F^I_{ab} \approx 0 \quad \text{and} \quad D_a \tilde{E}^a_I \approx 0. \quad (3.37)$$

Note that these equations are the field equations (3.32) pulled-back to $\Sigma$ with $\tilde{\eta}^{ab}$. Note also that they are polynomial in the canonically conjugate variables $(A^I_a, \tilde{E}^a_I)$. This is to be contrasted with the constraint equations for the standard Einstein-Hilbert theory. Recall that the scalar constraint of that theory depended non-polynomially on $q_{ab}$.

### 3.3 Constraint algebra

As usual, to evaluate the Poisson brackets of the constraints and to determine the motions they generate on phase space, we must first construct constraint functions associated with

\footnote{Note that in terms of the Poisson bracket $\{ , \}$ defined by $G^\Omega_P$, we have $\{ A^a_b(x), \tilde{E}^b_j(y) \} = \delta^b_a \delta^b_j \delta(x, y)$.}
Given test fields $v^I$ and $\alpha_I$, which take values in the Lie algebra $\mathcal{L}_G$ and its dual $\mathcal{L}_G^*$, we define

\[ F(\alpha) := \frac{1}{2} \int_\Sigma \alpha_I \tilde{\eta}^{ab} F^I_{ab} \quad \text{and} \quad G(v) := \int_\Sigma v^I (D_a \tilde{E}^a_I) \]  

\[ (3.38) \]

We will call $G(v)$ the Gauss constraint function since it will play the same role as the Gauss constraint of Yang-Mills theory. We will see that $G(v)$ generates the usual gauge transformations of the connection 1-form $A^I_a$ and it conjugate momentum (or “electric field”) $\tilde{E}^a_I$.

We are now ready to evaluate the functional derivatives of $F(\alpha)$ and $G(v)$. Since $F(\alpha)$ is independent of the momentum $\tilde{E}^a_I$, and since $\delta F^I_{ab} = 2D_a \delta A^I_b$, we find

\[ \frac{\delta F(\alpha)}{\delta \tilde{E}^a_I} = 0 \quad \text{and} \quad \frac{\delta F(\alpha)}{\delta A^I_a} = \tilde{\eta}^{ab} D_b \alpha_I. \]  

\[ (3.39) \]

Similarly, if we vary $G(v)$ with respect to $\tilde{E}^a_I$ and $A^I_a$, we find

\[ \frac{\delta G(v)}{\delta \tilde{E}^a_I} = -D_a v^I \quad \text{and} \quad \frac{\delta G(v)}{\delta A^I_a} = \{ v, \tilde{E}^a \}_I \bigg( := C^K_{IJ} v^J \tilde{E}^a_K \bigg). \]  

\[ (3.40) \]

Here $C^I_{JK}$ denote the structure constants of the Lie algebra $\mathcal{L}_G$ and $\{ , \} : \mathcal{L}_G \times \mathcal{L}_G \rightarrow \mathcal{L}_G^*$ denotes the co-adjoint bracket. $\{ , \}$ is defined in terms of the Lie bracket $[ , ] : \mathcal{L}_G \times \mathcal{L}_G \rightarrow \mathcal{L}_G$ via $\{ v, \alpha \}_I w^I := \alpha_K [v, w]^K$.

Given (3.39) and (3.40), we can now write down the Hamiltonian vector fields $X_{F(\alpha)}$ and $X_{G(v)}$ associated with $F(\alpha)$ and $G(v)$. They are

\[ X_{F(\alpha)} = \int_\Sigma -\tilde{\eta}^{ab} (D_b \alpha_I) \frac{\delta}{\delta \tilde{E}^a_I} \quad \text{and} \quad \]  

\[ X_{G(v)} = \int_\Sigma -(D_a v^I) \frac{\delta}{\delta A^I_a} - \{ v, \tilde{E}^a \}_I \frac{\delta}{\delta \tilde{E}^a_I}. \]  

\[ (3.41a) \]

\[ (3.41b) \]

Thus, under the 1-parameter family of diffeomorphisms on $^G \Gamma_P$ associated with $X_{F(\alpha)}$, we have

\[ A^I_a \mapsto A^I_a \quad \text{and} \quad \tilde{E}^a_I \mapsto \tilde{E}^a_I - \epsilon (\tilde{\eta}^{ab} D_b \alpha_I) + O(\epsilon^2). \]  

\[ (3.42a) \]

\[ (3.42b) \]

Similarly, under the 1-parameter family of diffeomorphisms on $^G \Gamma_P$ associated with $X_{G(v)}$, we have

\[ A^I_a \mapsto A^I_a - \epsilon D_a v^I + O(\epsilon^2) \quad \text{and} \quad \tilde{E}^a_I \mapsto \tilde{E}^a_I - \epsilon \{ v, \tilde{E}^a \}_I + O(\epsilon^2). \]  

\[ (3.43a) \]

\[ (3.43b) \]
Note that \((3.43a)\) and \((3.43b)\) are the usual gauge transformations of the connection 1-form \(A^a_I\) and its conjugate momentum \(\tilde{E}^a_i\) that we find in Yang-Mills theory. Equations \((3.42)\) and \((3.43)\) are the motions on phase space generated by \(F(\alpha)\) and \(G(v)\).

Given \((3.39)\) and \((3.40)\), we can also evaluate the Poisson brackets of the constraint functions. Since the \(G(v)\)'s play the same role as the Gauss constraint functions of Yang-Mills theory, we would expect their Poisson bracket algebra to be isomorphic to the Lie algebra \(\mathcal{L}_G\). This is indeed the case. We find

\[
\{G(v), G(w)\} = G([v, w]),
\]

where \([v, w]^I = C^I_{JK}v^Jw^K\) is the Lie bracket of \(v^I\) and \(w^I\). Thus, \(v^I \mapsto G(v)\) is a representation of the Lie algebra \(\mathcal{L}_G\). The Lie bracket in \(\mathcal{L}_G\) is mapped to the Poisson bracket of the corresponding Gauss constraint functions.

Since \(F(\alpha)\) is independent of \(\tilde{E}^a_i\), it follows trivially that

\[
\{F(\alpha), F(\beta)\} = 0.
\]

With only slightly more effort, we can show that

\[
\{G(v), F(\alpha)\} = -F(\{v, \alpha\}),
\]

where \(\{v, \alpha\}_I = C^K_{JI}v^J\alpha_K\) is the co-adjoint bracket of \(v^I\) and \(\alpha_I\). Thus, the totality of constraint functions \((G(v)\) and \(F(\alpha)\)) is closed under Poisson bracket —i.e., they form a 1st class set. Furthermore, since \((3.44)\), \((3.45)\), and \((3.46)\) do not involve any structure functions (unlike the constraint algebra of the Einstein-Hilbert theory discussed in subsection 2.3), the set of constraint functions form a Lie algebra with respect to Poisson bracket. In fact, \((\alpha, v) \in \mathcal{L}_G^* \times \mathcal{L}_G \mapsto (F(\alpha), G(v))\) is a representation of the Lie algebra \(\mathcal{L}_{IG}\) of the inhomogeneous Lie group \(IG\) associated with \(G\). The action \(\tau_{v}(\alpha) := -\{v, \alpha\}_I\) of \(v^I \in \mathcal{L}_G\) on \(\alpha_I \in \mathcal{L}_G^*\) is mapped to the Poisson bracket \(\{G(v), F(\alpha)\}\) of the corresponding constraint functions. The \(F(\alpha)\)'s play the role of “translations” and the \(G(v)\)'s play the role of “rotations” in the inhomogeneous group.

4. Chern-Simons theory

So far in this review, we have written down two different actions for 2+1 gravity: The standard Einstein-Hilbert action, which gave us a description of 2+1 gravity in terms of a

\[\text{We will discuss the construction of the inhomogeneous Lie group } IG \text{ and its Lie algebra } \mathcal{L}_{IG} \text{ in subsection 4.4 where we show the equivalence between the 2+1 Palatini theory based on } G \text{ and Chern-Simons theory based on } IG. \text{ When } G \text{ is the Lorentz group } SO(2,1), \text{ } IG \text{ is the corresponding Poincaré group } ISO(2,1).\]
spacetime metric (or equivalently, a co-triad); and the 2+1 Palatini action based on \( SO(2, 1) \), which gave us a description in terms of a co-triad and a connection 1-form. In this section, we shall see that 2+1 gravity can be described by an action that depends only on a connection 1-form. We shall see that the 2+1 Palatini action based on \( SO(2, 1) \) is equal to the Chern-Simons action based on \( ISO(2, 1) \), modulo a surface integral that does not affect the equations of motion. This result was first shown by A. Achucarro and P.K. Townsend [32]; it was later rediscovered and used by Witten [8] to quantize 2+1 gravity. In this section, we will follow the treatment of [30] in which the result for 2+1 gravity follows as a special case. We will show that the 2+1 Palatini theory based on any Lie group \( G \) is equivalent to Chern-Simons theory based on the inhomogeneous Lie group \( IG \) associated with \( G \).

The above equivalence between the 2+1 Palatini and Chern-Simons theories is at the level of actions. The gauge groups, \( G \) and \( IG \), are different, but the actions are the same. It is interesting to note that the 2+1 Palatini and Chern-Simons theories based on the same Lie group \( G \) are also related, but this time at the level of their Hamiltonian formulations. We shall see that the reduced phase space of the Chern-Simons theory based on \( G \) is the reduced configuration space of the 2+1 Palatini theory based on the same \( G \). Since Chern-Simons theory is not available in 3+1 dimensions, the relationships that we find in this section do not, unfortunately, extend to 3+1 theories of gravity.

### 4.1 Euler-Lagrange equations of motion

Unlike the standard Einstein-Hilbert and Palatini theories which are well-defined in \( n+1 \) dimensions, Chern-Simons theory is defined only in odd dimensions. In 2+1 dimensions, the basic variable is a connection 1-form \( 3A^a_i \) which takes values in a Lie algebra \( \mathcal{L}_G \) equipped with an invariant, non-degenerate bilinear form \( k_{ij} \). The Chern-Simons action based on \( G \) is defined by

\[
G_{CS}(3A) := \frac{1}{2} \int_M \tilde{\eta}^{abc} k_{ij} \left( 3A^a_i \partial^b 3A^c_j + \frac{1}{3} 3A^a_i [3A^b_j, 3A^c_j]^j \right),
\]

where \([3A^b_j, 3A^c_j]^j := C^j_{mn} 3A^m_b 3A^n_c\) denotes the Lie bracket of \( 3A^b_j \) and \( 3A^c_j \). It is important to note that Chern-Simons theory is not defined for arbitrary Lie groups—we need the additional structure provided by the invariant, non-degenerate bilinear form \( k_{ij} \).

To obtain the Euler-Lagrange equations of motion, we vary \( G_{CS}(3A) \) with respect to \( 3A^a_i \).

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13 We will require that \( k_{ij} \) be invariant under the **adjoint action** of the Lie algebra \( \mathcal{L}_G \) on itself—i.e., that \( k_{ij}[x, v]^j w^j + k_{ij} v^i[x, w]^j = 0 \) for all \( v^i, w^i, x^i \in \mathcal{L}_G \). If \( C_{ijk} \) is defined in terms of the structure constants \( C^i_{jk} \) via \( C_{ijk} := k_{im} C^m_{jk} \), then invariance of \( k_{ij} \) under the adjoint action is equivalent to \( C_{ijk} = C_{[ijk]} \). If the Lie group is semi-simple (i.e., if it does not admit any non-trivial abelian normal subgroup), then we are guaranteed that such a \( k_{ij} \) exists. This is just the **Cartan-Killing metric** defined by \( k_{ij} := C^m_{ni} C^n_{mj} \). Invariance of \( k_{ij} \) is equivalent to the invariance of \( C^i_{jk} \)—that is, the **Jacobi identity** \( C^m_{[ij} C^n_{k]} m = 0 \).
Using the fact that $C_{ijk} := k_{im} C^{m}_{jk}$ is totally anti-symmetric, we obtain

$$\tilde{\eta}^{abc} k_{ij} F^j_{bc} = 0,$$  \hspace{1cm} (4.2)

where $F^i_{ab} = 2 \partial_a [3 A^i_b] + [3 A^i_a, 3 A^i_b]$ is the Lie algebra-valued curvature tensor of the generalized derivative operator $^3D_a$ defined by

$$^3D_a v^i := \partial_a v^i + [3 A^i_a, v]^i.$$  \hspace{1cm} (4.3)

If we also use the fact that $k_{ij}$ is non-degenerate, we get $F^i_{ab} = 0$. Thus, Chern-Simons theory is a theory of a flat connection 1-form. We will see the role that this equation plays in the next two subsections when we put the theory in Hamiltonian form.

4.2 Legendre transform

Just like the 2+1 Palatini theory based on an arbitrary Lie group $G$, Chern-Simons theory is not a theory of a spacetime metric. However, we can still put this theory in Hamiltonian form if we assume that $M$ is topologically $\Sigma \times \mathbb{R}$ for some submanifold $\Sigma$ and assume that there exists a function $t$ (with nowhere vanishing gradient $(dt)_a$) such that each $t = \text{const}$ surface $\Sigma_t$ is diffeomorphic to $\Sigma$. As usual, we let $t^a$ denote the flow vector field satisfying $t^a (dt)_a = 1$.

Given $t$ and $t^a$, we are now ready to write the Chern-Simons action (4.1) in 2+1 form. Using the decomposition $\tilde{\eta}^{abc} = 3 t^a \tilde{\eta}^{bc} dt$, we get

$$G_{CS} (3A) = \frac{1}{2} \int_M \tilde{\eta}^{abc} k_{ij} (3 A^i_a \partial_b 3 A^j_c + \frac{1}{3} 3 A^i_a [3 A^j_b, 3 A^k_c])$$

$$= \frac{1}{2} \int dt \int_{\Sigma} (3 A \cdot t)^i k_{ij} \tilde{\eta}^{bc} F^j_{bc} + \tilde{\eta}^{ca} k_{ij} A^i_a \mathcal{L}^c_t A^j_c,$$  \hspace{1cm} (4.4)

where the last equality holds modulo a surface integral. Here $(3 A \cdot t)^i := t^a 3 A^i_a$ and $A^i_a := t^b 3 A^i_b (= (\delta^b_a - t^b (dt)_a) 3 A^i_b)$ are the configuration variables which specify all the information contained in the field variable $3 A^i_a$. The Lie derivative treats fields with only internal indices as scalars, and $F^i_{ab} = 2 \partial_a [3 A^i_b] + [3 A^i_a, 3 A^i_b]$ is the curvature tensor associated with $A^i_a$.

From (4.4), it follows that the Lagrangian $^GL_{CS}$ of the Chern-Simons theory based on $G$ is given by

$$^GL_{CS} = \frac{1}{2} \int_{\Sigma} (3 A \cdot t)^i k_{ij} \tilde{\eta}^{ab} F^j_{ab} + \tilde{\eta}^{ab} k_{ij} A^i_b \mathcal{L}^c_t A^j_a.$$  \hspace{1cm} (4.5)

The momentum canonically conjugate to $A^i_a$ is $\frac{1}{2} \tilde{\eta}^{ab} k_{ij} A^j_b$, while $(3 A \cdot t)^i$ plays the role of a Lagrange multiplier. Thus, the Dirac constraint analysis says that the phase space
\( (G_{\Gamma_{CS}}, G_{\Omega_{CS}}) \) is coordinatized by \((A_i^a)\) and has symplectic structure

\[
G_{\Omega_{CS}} = -\frac{1}{2} \int_{\Sigma} \bar{\eta}^{ab} k_{ij} dA_i^a \wedge dA_j^b.
\]

(4.6)

The Hamiltonian is given by

\[
G_{H_{CS}}(A) = -\frac{1}{2} \int_{\Sigma} (3A \cdot t)^i k_{ij} \bar{\eta}^{ab} F_{ab}^j.
\]

(4.7)

As we shall see in the next subsection, this is just a 1st class constraint function associated with

\[
k_{ij} \bar{\eta}^{ab} F_{ab}^j = 0.
\]

(4.8)

Note that constraint equation (4.8) is the field equation (4.2) pulled-back to \(\Sigma\) with \(\bar{\eta}^{ab}\). Just as in the 2+1 Palatini theory, the constraint equation is polynomial in the basic variable \(A_i^a\).

Note also that although (4.8) may not look like the standard Gauss constraint of Yang-Mills theory, we shall see that its associated constraint function generates the same motion of \(A_i^a\).

4.3 Constraint algebra

Following the same procedure that we used in Sections 2 and 3, we first construct a constraint function associated with (4.8). Given a test field \(v^i\) (which takes values in the Lie algebra \(L_G\)), we define

\[
G(v) := \frac{1}{2} \int_{\Sigma} v^i k_{ij} \bar{\eta}^{ab} F_{ab}^j.
\]

(4.9)

Since the phase space is coordinatized by the single field \(A_i^a\), we need to evaluate only one functional derivative. Varying \(G(v)\) with respect to \(A_i^a\), we get

\[
\frac{\delta G(v)}{\delta A_i^a} = k_{ij} \bar{\eta}^{ab} D_b v^j,
\]

(4.10)

where \(D_a\) is any torsion-free extension of the generalized derivative operator associated with \(A_i^a\). From (4.10) it then follows that the Hamiltonian vector field \(X_{G(v)}\) is given by

\[
X_{G(v)} = \int_{\Sigma} - (D_a v^i) \frac{\delta}{\delta A_i^a},
\]

(4.11)

so that

\[
A_i^a \mapsto A_i^a - \epsilon D_a v^i + O(\epsilon^2)
\]

(4.12)

\footnote{Note that in terms of the Poisson bracket \(\{ , \} \) defined by \(G\Omega_{CS}\), we have \(\{ A_i^a(x), A_j^b(y) \} = \eta_{ab} k^{ij} \delta(x,y), \) where \(\eta_{ab}\) and \(k^{ij}\) denote the inverses of \(\bar{\eta}^{ab}\) and \(k_{ij}\). This result follows from the fact that for any \(f : G_{\Gamma_{CS}} \to R\), the Hamiltonian vector field \(X_f\) is given by \(X_f = \int_{\Sigma} \eta_{ab} k^{ij} \frac{\delta f}{\delta A_i^a} \frac{\delta x}{\delta A_j^b}\). Hence the Poisson bracket of any two functions \(f, g : G_{\Gamma_{CS}} \to R\) is \(\{ f, g \} = \int_{\Sigma} \eta_{ab} k^{ij} \frac{\delta f}{\delta A_i^a} \frac{\delta g}{\delta A_j^b}\).}
under the 1-parameter family of diffeomorphisms on $G_{\Gamma_{CS}}$ associated with $X_G(v)$. This is the usual gauge transformation of the connection 1-form that we find in Yang-Mills theory. Thus, $G(v)$ can be appropriately called a Gauss constraint function.

Given (4.10), it is also straightforward to evaluate the Poisson brackets of the constraints. We find that

$$\{G(v), G(w)\} = G([v, w]),$$

which is the expected Poisson bracket algebra of the Gauss constraint functions. The map $v^i \mapsto G(v)$ is a representation of the Lie algebra $L_G$. The Lie bracket in $L_G$ is mapped to the Poisson bracket of the corresponding Gauss constraint functions.

### 4.4 Relationship to the 2+1 Palatini theory

Before we can show the relationship between the Chern-Simons and 2+1 Palatini theories, we will first have to recall the construction of the inhomogeneous Lie group $IG$ associated with any Lie group $G$. This will allow us to generalize the equivalence of the 2+1 Palatini and Chern-Simons theories (as shown in [8, 32]) to arbitrary Lie groups $G$. We will be able to show that the 2+1 Palatini theory based on any $G$ is equivalent to Chern-Simons theory based on $IG$.

Consider any Lie group $G$ with Lie algebra $L_G$, and let $L_G^*$ denote the vector space dual of $L_G$. If $v^I$, $w^I$ denote typical elements of $L_G$ and $\alpha_I$, $\beta_I$ denote typical elements of $L_G^*$, then $(\alpha, v)^i := (\alpha_I, v^I)$ and $(\beta, w)^i := (\beta_I, w^I)$ are typical elements of the direct sum vector space $L_G^* \oplus L_G$. We can define a bracket on $L_G^* \oplus L_G$ via

$$[(\alpha, v), (\beta, w)]^i := (-\{v, \beta\} + \{w, \alpha\}, [v, w])^i,$$

where $[v, w]^I := C^I_{JK}v^Jw^K$ and $\{v, \beta\}_I := C^K_{JI}v^J\beta_K$ are the Lie bracket and co-adjoint bracket associated with $L_G$. By inspection, we see that (4.14) is linear and anti-symmetric. If we use

$$\{[v, w], \alpha\}_I = -\{v, [w, \alpha]\}_I + \{w, \{v, \alpha\}\}_I$$

(which follows as a consequence of the Jacobi identity for $L_G$), we can show that (4.14) satisfies the Jacobi identity as well. Thus, the vector space $L_{IG} := L_G^* \oplus L_G$ together with (4.14) is actually a Lie algebra. We call $L_{IG}$ the inhomogeneous Lie algebra associated with $G$; the inhomogeneous Lie group $IG$ is obtained by exponentiating the Lie algebra $L_{IG}$. As we shall see later in this subsection, $IG$ is simply the cotangent bundle over $G$.

The terminology inhomogeneous is due to the fact that $L_{IG}$ admits an abelian Lie ideal isomorphic to $L_G^*$, and that the quotient of $L_{IG}$ by this ideal is isomorphic to $L_G$.

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15A Lie ideal $\mathcal{I}$ of a Lie algebra $\mathcal{L}$ is a vector subspace $\mathcal{I} \subset \mathcal{L}$ such that $[i, x] \in \mathcal{I}$ for any $i \in \mathcal{I}$, $x \in \mathcal{L}$.

32
elements of $L_G^*$ are analogous to infinitesimal “translations,” while elements of $L_G$ are analogous to infinitesimal “rotations.” Note, however, that the space of translations and rotations have the same dimension. As a special case, if one chooses $G$ to be the 2+1 dimensional Lorentz group $SO(2, 1)$, then the above construction yields for $IG$ the 2+1 dimensional Poincaré group $ISO(2, 1)$.

In addition to the above Lie algebra structure, $L_G^* \oplus L_G$ is equipped with a (natural) invariant, non-degenerate bilinear form $k_{ij}$ defined by

\[ k_{ij}(\alpha, v)^i(\beta, w)^j := \alpha_I w^I + \beta_I v^I. \]  

(4.16)

Since $L_{IG}$ is not semi-simple (because it admits a non-trivial abelian Lie ideal), $k_{ij}$ is not the (degenerate) Cartan-Killing metric of $L_{IG}$. Nevertheless, the existence of $k_{ij}$ will allow us to construct Chern-Simons theory for $IG$. Recall that without an invariant, non-degenerate bilinear form, Chern-Simons theory could not be defined. Note also that for $G = SO(2, 1)$, the above construction of $k_{ij}$ reduces to that used by Witten [8].

Given these remarks, we can now show that the 2+1 Palatini theory based on any Lie group $G$ is equivalent to Chern-Simons theory based on $IG$. To do this, recall that for any Lie group $G$, the 2+1 Palatini action based on $G$ is given by

\[ GSP(3, 3A) = \frac{1}{2} \int_M \bar{\eta}^{abc} \bar{\epsilon}_{aI} 3F_{bc}, \]  

(4.17)

where $3A_I^i$ and $3\epsilon_{aI}$ are $L_G$- and $L_G^*$-valued 1-forms. We now construct the inhomogeneous Lie algebra $L_{IG}$ associated with $G$ and define an $L_{IG}$-valued connection 1-form $3A_a^i$ via

\[ 3A_a^i := (3\epsilon_{aI}, 3A_{aI}). \]  

(4.18)

By simply substituting this expression for $3A_a^i$ into the Chern-Simons action $IGS_{CS}(3A)$, we
find that
\[ IG_{CS}(\mathbf{3}A) = \frac{1}{2} \int_M \tilde{\eta}^{abc} \epsilon_{ij} \left( 3 A_a^i \partial_b^j A_c^i + \frac{1}{3} 3 A_a^i [3 A_b^i, 3 A_c^i] \right) \]
\[ = \frac{1}{2} \int_M \tilde{\eta}^{abc} \left( 3 e_{ai}^b \partial_b \partial^j A_c^i + (\partial_b e_{cl}^j) 3 A_a^i \right. \]
\[ + \left. \frac{1}{3} \left( 3 e_{ai}^b [3 A_b^i, 3 A_c^i] - [3 A_b^i, 3 e_c^j] I 3 A_a^i + \{ 3 A_c^j, 3 e_b^j \} I 3 A_a^i \right) \right) \]
\[ = \frac{1}{2} \int_M \tilde{\eta}^{abc} \left( 3 e_{ai}^b \partial_b \partial^j A_c^i + 3 e_{ai}^b \partial_b (3 e_{cl}^j 3 A_a^i) - [3 A_b^i, 3 A_c^i] \right) \]
\[ + \left. \frac{1}{3} \left( 3 e_{ai}^b [3 A_b^i, 3 A_c^i] - e_{cl}^j [3 A_b^i, 3 A_a^i] + e_{bl} [3 A_c^j, 3 A_a^i] \right) \right) \]
\[ = \frac{1}{2} \int_M \tilde{\eta}^{abc} \left( 3 e_{ai}^b \partial_b (3 F_{bc}^I) + 3 e_{ai}^b [3 A_b^i, 3 A_c^i] \right) \]
\[ + \left. \partial_b (3 e_{cl}^j 3 A_a^i) \right) \]
\[ = \int_M \tilde{\eta}^{abc} 3 e_{ai}^b 3 F_{bc}^I + \text{(surface integral)} \]
\[ = G_{SP}(3e, 3A) + \text{(surface integral)}, \]
where we have used definitions (4.14), (4.16), and (4.18) repeatedly. Since the surface term does not affect the Euler-Lagrange equations of motion, we can conclude that the 2+1 Palatini theory based on \( G \) is equivalent to Chern-Simons theory based on \( IG \). This is the desired result. Note that as a special case, we can conclude that 2+1 gravity as described by the 2+1 Palatini action based on \( SO(2,1) \) is equivalent to Chern-Simons theory based on \( ISO(2,1) \).

Up to now, we have only described the inhomogeneous Lie group \( IG \) in terms of its associated Lie algebra \( \mathcal{L}_{IG} \). We just exponentiated the Lie algebra \( \mathcal{L}_{IG} \) to obtain \( IG \). However, it is also instructive to give an explicit construction of \( IG \) at the level of groups and manifolds. But to do this, we will need to make another short digression, this time on \textit{semi-direct products} and \textit{semi-direct sums}. Readers already familiar with these definitions may skip to the paragraph immediately following equations (4.24).

Let \( G \) and \( H \) be any two groups. To define the semi-direct product \( H \circlearrowright G \), we need a homomorphism \( \sigma \) from the group \( G \) into the group of automorphisms of \( H \)—i.e., for each \( g, g' \in G \) and \( h, h' \in H \), the map \( \sigma_g : H \to H \) must be 1-1, onto, and satisfy
\[ \sigma_g(hh') = \sigma_g(h)\sigma_g(h') \quad \text{and} \quad \sigma_{gg'}(h) = \sigma_g(\sigma_{g'}(h)) \]
for every \( g, g' \in G \) and \( h, h' \in H \). Given this structure, one can check that
\[ (h, g)(h', g') = (h\sigma_g(h'), gg') \]
defines a group multiplication law on the set \( H \times G \). The identity element is \((e_H, e_G)\), where \( e_H, e_G \) are the identities in \( H \) and \( G \), and the inverse \((h, g)^{-1}\) of \((h, g)\) is \((\sigma_{g^{-1}}(h^{-1}), g^{-1})\).
The set $H \times G$ together with this group multiplication law defines the *semi-direct product* $H \circledast G$. Note that $H$ is homomorphic to a (not necessarily abelian) normal subgroup of $H \circledast G$, and the quotient of $H \circledast G$ by this normal subgroup is homomorphic to $G$. As a trivial example, if $\sigma_g(h) = h$ for all $g \in G$, $h \in H$, then $H \circledast G$ is the usual direct product $H \otimes G$ of groups.

Now assume that $G$ and $H$ are Lie groups with Lie algebras $\mathfrak{L}_G$ and $\mathfrak{L}_H$. If $H \circledast G$ is the semi-direct product of $G$ and $H$ with respect to some action $\sigma$ of $G$ on $H$ satisfying (4.20), we would now like to determine the relationship between the Lie algebras $\mathfrak{L}_{H \circledast G}$, $\mathfrak{L}_G$, and $\mathfrak{L}_H$. To do this, we differentiate the action $\sigma$ of $G$ on $H$ to obtain an action $\tau$ of $\mathfrak{L}_G$ on $\mathfrak{L}_H$. More precisely, if $g(\epsilon)$ is a 1-parameter curve in $G$ with $g(0) = e_G$ and tangent vector $v := \frac{d}{d\epsilon}|_{\epsilon=0}g(\epsilon)$ and $h(\lambda)$ is a 1-parameter curve in $H$ with $h(0) = e_H$ and tangent vector $\alpha := \frac{d}{d\lambda}|_{\lambda=0}h(\lambda)$, then we define

$$\tau_v(\alpha) := \frac{d}{d\epsilon}|_{\epsilon=0} \frac{d}{d\lambda}|_{\lambda=0} \sigma_{g(\epsilon)}(h(\lambda)).$$

(4.22)

In terms of $\tau$, the Lie bracket of $(\alpha, v)$ and $(\beta, w)$ in $\mathfrak{L}_{H \circledast G}$ becomes

$$[(\alpha, v), (\beta, w)] = (\tau_v(\beta) - \tau_w(\alpha)) + [\alpha, \beta], [v, w],$$

(4.23)

where $[v, w]$ and $[\alpha, \beta]$ are the Lie brackets of $v, w \in \mathfrak{L}_G$ and $\alpha, \beta \in \mathfrak{L}_H$. Note that (4.23) satisfies the Jacobi identity as a consequence of

$$\tau_v([\alpha, \beta]) = [\tau_v(\alpha), \beta] + [\alpha, \tau_v(\beta)] \quad \text{and} \quad \tau_{[v, w]}(\alpha) = \tau_v(\tau_w(\alpha)) - \tau_w(\tau_v(\alpha)),$$

(4.24a)

(4.24b)

which follow from the definition of $\tau$ and the properties (4.20) satisfied by $\sigma$. Thus, if $\mathfrak{L}_G$ and $\mathfrak{L}_H$ are two Lie algebras and $\tau$ is an action of $\mathfrak{L}_G$ on $\mathfrak{L}_H$ satisfying (4.24), then the direct sum vector space $\mathfrak{L}_H \oplus \mathfrak{L}_G$ together with the bracket defined by (4.23) is a Lie algebra. This Lie algebra, denoted $\mathfrak{L}_H \circledast \mathfrak{L}_G$, is called the *semi-direct sum* of $\mathfrak{L}_G$ and $\mathfrak{L}_H$. Note that $\mathfrak{L}_H$ is isomorphic to a (not necessarily abelian) Lie ideal of $\mathfrak{L}_H \circledast \mathfrak{L}_G$, and the quotient of $\mathfrak{L}_H \circledast \mathfrak{L}_G$ by this ideal is isomorphic to $\mathfrak{L}_G$. If $\sigma_g(h) = h$ for all $g \in G$, $h \in H$ (so that $H \circledast G = H \otimes G$), then $\mathfrak{L}_H \circledast \mathfrak{L}_G$ is the usual direct sum $\mathfrak{L}_H \oplus \mathfrak{L}_G$ of Lie algebras.

Given these general remarks, let us now return to our discussion of the inhomogeneous Lie group $IG$ and its associated Lie algebra $\mathfrak{L}_{IG}$. From the above definitions, we see that

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16To obtain this result, consider 1-parameter curves $(h(\epsilon), g(\epsilon))$ and $(h'(\epsilon), g'(\epsilon))$ in $H \circledast G$ with $(h(0), g(0)) = (h'(0), g'(0)) = (e_H, e_G)$ and tangent vectors $(\alpha, v) := \frac{d}{d\epsilon}|_{\epsilon=0}(h(\epsilon), g(\epsilon))$ and $(\beta, w) := \frac{d}{d\epsilon}|_{\epsilon=0}(h'(\epsilon), g'(\epsilon))$. Then use the definition of the Lie bracket in terms of the group multiplication law (4.21), $[(\alpha, v), (\beta, w)] := \frac{d}{d\epsilon}|_{\epsilon=0} \frac{d}{d\epsilon'}|_{\epsilon'=0}(h(\epsilon), g(\epsilon))(h'(\epsilon'), g'(\epsilon'))(h(\epsilon), g(\epsilon))^{-1}(h'(\epsilon'), g'(\epsilon'))^{-1}$. This leads to (4.23).
\( \mathcal{L}_{IG} \) is simply the semi-direct sum \( \mathcal{L}^*_G \circledast \mathcal{L}_G \). \( \mathcal{L}^*_G \) is to be thought of as a Lie algebra with the trivial Lie bracket \([\alpha, \beta] = 0\) for all \( \alpha_I, \beta_I \in \mathcal{L}^*_G \); the action \( \tau \) of \( \mathcal{L}_G \) on \( \mathcal{L}^*_G \) is given by \( \tau_v(\beta) = -\{v, \beta\}_I \). Equations (4.24) hold for this action as a consequence of the Jacobi identity in \( \mathcal{L}_G \): Equation (4.24a) is satisfied since \([\alpha, \beta] = 0\) for all \( \alpha_I, \beta_I \in \mathcal{L}^*_G \), while equation (4.24b) is equivalent to equation (4.15). Furthermore, the inhomogenized Lie group \( IG \) is simply the semi-direct product \( \mathcal{L}^*_G \circledast \mathcal{G} \). \( \mathcal{L}^*_G \) is to be thought of as an abelian group with respect to vector addition, and the action \( \sigma \) of \( \mathcal{G} \) on \( \mathcal{L}^*_G \) is induced by the adjoint action of \( \mathcal{G} \) on itself.\(^{17}\) This implies that as a manifold \( IG \) is the cotangent bundle \( T^*G \). At each point \( g \in G \), the cotangent space \( T^*_gG \) is isomorphic to \( \mathcal{L}^*_G \). Moreover, the above relationship between \( G \) and \( IG \) allows us to prove an interesting mathematical result involving the space of connection 1-forms on a 2-dimensional manifold. We can show that for any Lie group \( G \)

\[
T^* (G\mathcal{A}) = IG\mathcal{A}, \tag{4.25}
\]

where \( G\mathcal{A} \) and \( IG\mathcal{A} \) denote the space of \( \mathcal{L}_G \)- and \( \mathcal{L}_{IG} \)-valued connection 1-forms on a 2-dimensional manifold \( \Sigma \). The map

\[(A^I_i, \bar{E}^a_I) \in T^* (G\mathcal{A}) \mapsto A^I_a := (e_{aI}, A^I_a) \in IG\mathcal{A} \tag{4.26}\]

(where \( e_{aI} := -\eta^{ab} \bar{E}^b_I \)) is a diffeomorphism from the manifold \( T^* (G\mathcal{A}) \) to the manifold \( IG\mathcal{A} \) that sends the natural symplectic structure

\[
G\Omega := \int_\Sigma d\bar{E}^a_I \wedge dA^I_a \tag{4.27}
\]

on \( T^* (G\mathcal{A}) \) to the natural symplectic structure

\[
IG\Omega := -\frac{1}{2} \int_\Sigma \bar{\eta}^{ab} k_{ij} dA^a_i \wedge dA^j_b \tag{4.28}
\]

on \( IG\mathcal{A} \). (Here \( k_{ij} \) denotes the (natural) invariant, non-degenerate bilinear form on \( \mathcal{L}_{IG} \) defined by (4.16).) Note that (4.27) and (4.28) are the symplectic structures of the 2+1 Palatini theory based on \( G \) and the Chern-Simons theory based on \( IG \). However, the above result (4.25) does not require any knowledge of the 2+1 Palatini or Chern-Simons actions.

Finally, to conclude this subsection, I would like to verify the claim made at the start of Section 4 that the reduced phase space of Chern-Simons theory based on any Lie group \( G \)

\[\text{\footnotesize \textsuperscript{17}}\text{The adjoint action of } G \text{ on itself is defined by } A_g(g') = gg'g^{-1} \text{ for all } g, g' \in G. \text{ By differentiating } A_g \text{ at the identity } e, \text{ we obtain a map } Ad_g : \mathcal{L}_G \rightarrow \mathcal{L}_G \text{ via } Ad_g(v) := A_g'(e) \cdot v. \text{ Ad defines the adjoint representation of the Lie group } G \text{ by linear operators on the Lie algebra } \mathcal{L}_G. \text{ The action } \sigma \text{ of } G \text{ on } \mathcal{L}_G \text{ is then given by } (\sigma_g(\alpha))(v) := \alpha(Ad_g(v)) \text{ for any } \alpha \in \mathcal{L}_G \text{ and } v \in \mathcal{L}_G.\]
is the reduced configuration space of the 2+1 Palatini theory based on the same Lie group $G$. This result will be simpler to prove than the previous two results since most of the preliminary work has already been done.

Let $G$ be any Lie group whose Lie algebra $\mathcal{L}_G$ admits an invariant, non-degenerate bilinear form $k_{IJ}$. Then Chern-Simons theory based on $G$ is well-defined, and, as we saw in subsection 4.2, the phase space $^G\Gamma_{CS}$ is coordinatized by $\mathcal{L}_G$-valued connection 1-forms $A^I_a$ on $\Sigma$. The symplectic structure is

$$^G\Omega_{CS} = -\frac{1}{2} \int_\Sigma \tilde{\eta}^{ab} k_{IJ} dA^I_a \wedge dA^J_b.$$  \hspace{1cm} (4.29)

In subsection 4.3, we then verified that the constraint functions $G(v)$ associated with the constraint equation

$$k_{IJ} \tilde{\eta}^{ab} F^J_{ab} = 0$$  \hspace{1cm} (4.30)

formed a 1st class set and generated the usual gauge transformations

$$A^I_a \mapsto A^I_a - \epsilon D_a v^I + O(\epsilon^2).$$  \hspace{1cm} (4.31)

To pass to the reduced phase space, we must factor-out the constraint surface (defined by (4.30)) by the orbits of the Hamiltonian vector fields $X_{G(v)}$. From (4.30) and (4.31) we see that the reduced phase space $^G\hat{\Gamma}_{CS}$ of the Chern-Simons theory based on $G$ is coordinatized by equivalence classes of flat $\mathcal{L}_G$-valued connection 1-forms on $\Sigma$, where two such connection 1-forms are said to be equivalent if and only if they are related by (4.31). This space is called the moduli space of flat $\mathcal{L}_G$-valued connection 1-forms on $\Sigma$.

Now recall the Hamiltonian formulation of the 2+1 Palatini theory based on the same Lie group $G$. In subsection 3.3, we saw that the phase space $^G\Gamma_P$ was coordinatized by pairs $(A^I_a, \tilde{E}^a_I)$ consisting of $\mathcal{L}_G$-valued connection 1-forms $A^I_a$ on $\Sigma$ and their canonically conjugate momentum $\tilde{E}^a_I$. $^G\Gamma_P$ was the cotangent bundle $T^*(^G\mathcal{C}_P)$ over the configuration space $^G\mathcal{C}_P$ of $\mathcal{L}_G$-valued connection 1-forms $A^I_a$ on $\Sigma$ with symplectic structure

$$^G\Omega_P = \int_\Sigma d\tilde{E}^a_I \wedge dA^I_a.$$  \hspace{1cm} (4.32)

We also saw that the constraint equations of the 2+1 Palatini theory were

$$\tilde{\eta}^{ab} F^I_{ab} = 0 \quad \text{and} \quad D_a \tilde{E}^a_I = 0.$$  \hspace{1cm} (4.33)

\footnote{Recall that given a symplectic manifold $(\Gamma, \Omega)$, a set of constraints $\phi_k$ form a 1st class set if and only if each Hamiltonian vector field $X_{\phi_k}$ is tangent to the constraint surface $\Gamma \subset \Gamma$ defined by the vanishing of all the constraints. The pull-back, $\tilde{\Omega}$, of $\Omega$ to $\tilde{\Gamma}$ is degenerate with the degenerate directions given by the $X_{\phi_k}$. Thus, $(\tilde{\Gamma}, \tilde{\Omega})$ is not a symplectic manifold. However, by factoring-out the constraint surface by the orbits of the $X_{\phi_k}$ we obtain a reduced phase space $(\hat{\Gamma}, \hat{\Omega})$ whose coordinates are precisely the true degrees of freedom of the theory. $\hat{\Omega}$ is non-degenerate; it is the projection of $\tilde{\Omega}$ to $\hat{\Gamma}$.}

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and verified that their associated constraint functions $F(\alpha)$ and $G(v)$ formed a 1st class set. They generated the motions

\begin{align}
A^I_a &\mapsto A^I_a \quad \text{and} \\
\tilde{E}^a_I &\mapsto \tilde{E}^a_I - \epsilon(\tilde{\eta}^{ab}D_b\alpha_I) + O(\epsilon^2) \quad (4.34a) \\
A^I_a &\mapsto A^I_a - \epsilon D_a v^I + O(\epsilon^2) \quad \text{and} \\
\tilde{E}^a_I &\mapsto \tilde{E}^a_I - \epsilon\{v, \tilde{E}^a\}_I + O(\epsilon^2), \quad (4.35a)
\end{align}

respectively. Thus, the reduced phase space $\hat{\Gamma}_P$ of the 2+1 Palatini theory based on $G$ is coordinatized by equivalence classes of pairs consisting of flat $L_G$-valued connection 1-forms on $\Sigma$ and divergence-free $L^*_G$-valued vector densities of weight +1 on $\Sigma$, where two such pairs are said to be equivalent if and only if they are related by (4.34) and (4.35). Since $F(\alpha)$ and $G(v)$ are independent and linear in the momentum $\tilde{E}^a_I$, it follows that $\hat{\Gamma}_P$ is naturally the cotangent bundle $T^*(\hat{G}_P)$ over the reduced configuration space $\hat{C}_P$ of the 2+1 Palatini theory based on $G$. From (4.34a) and (4.35a) we see that $\hat{G}_P$ is again the moduli space of flat $L_G$-valued connection 1-forms on $\Sigma$. Thus, $\hat{G}_P = \hat{C}_P$ as desired. In particular, the reduced configuration space of the 2+1 Palatini theory based on $G$ has the structure of a symplectic manifold.

This last result has interesting consequences. It can be used, for example, to show the relationship between the $\hat{T}^0[\gamma]$ and $\hat{T}^1[\gamma]$ observables for 2+1 gravity. These are the 2+1 dimensional analogs of the classical $T$-observables constructed by Rovelli and Smolin [7] for the 3+1 theory. As shown in [30], $\hat{T}^0[\gamma]$ is the trace of the holonomy of the connection around a closed loop $\gamma$ in $\Sigma$, while $\hat{T}^1[\gamma]$ is the function on the reduced phase space of the 2+1 Palatini theory defined by the Hamiltonian vector field associated with $\hat{T}^0[\gamma]$. Thus, many properties satisfied by the $\hat{T}^1[\gamma]$’s can be derived from similar properties satisfied by the $\hat{T}^0[\gamma]$’s.

5. 2+1 matter couplings

In this section, we will couple various matter fields to 2+1 gravity via the 2+1 Palatini action. We will consider the inclusion of a cosmological constant and a massless scalar field. One can couple other fundamental matter fields (e.g., Yang-Mills and Dirac fields) to 2+1 gravity in a similar fashion—I have chosen to consider a massless scalar field in detail since 2+1 gravity coupled to a massless scalar field is the dimensional reduction of 3+1 vacuum
general relativity with a spacelike, hypersurface-orthogonal Killing vector field. As noted in Section 1, this is an interesting case since it appears likely that the non-perturbative canonical quantization program for 3+1 gravity can be carried through to completion for this reduced theory.

In subsection 5.1, we define the 2+1 Palatini theory based on a Lie group $G$ with cosmological constant $\Lambda$. We derive the Euler-Lagrange equations of motion and perform a Legendre transform to obtain a Hamiltonian formulation of the theory. Just as in Section 3 (when $\Lambda$ was equal to zero), we shall find that the constraint equations of the theory are polynomial in the canonically conjugate variables. We shall also find that their associated constraint functions still form a Lie algebra with respect to Poisson bracket.

In subsection 5.2, we will show that the 2+1 Palatini theory based on $G$ with cosmological constant $\Lambda$ is equivalent to Chern-Simons theory based on the $\Lambda$-deformation, $\Lambda G$, of $G$. This is a generalization of Witten’s result for $G = SO(2,1)$ (and $\Lambda G = SO(3,1)$ or $SO(2,2)$ depending on the sign of $\Lambda$) which holds for any Lie group $G$ that admits an invariant, totally anti-symmetric tensor $\epsilon^{IJK}$. This result also generalizes the $\Lambda = 0$ equivalence of the 2+1 Palatini and Chern-Simons theories given in subsection 4.4.

Finally, in subsection 5.3, we define the action for a massless scalar field and couple this field to 2+1 gravity by adding the action to the 2+1 Palatini action based on $G = SO(2,1)$. It is when we wish to couple matter with local degrees of freedom to 2+1 gravity (as it is for the case of a massless scalar field) that we are forced to take the Lie group $G$ to be such that the fields $\gamma_a^I$ have the interpretation of a co-triad. We obtain the Euler-Lagrange equations of motion for the coupled theory and then perform a Legendre transform to obtain the Hamiltonian formulation. We shall find that the constraint equations remain polynomial in the canonically conjugate variables and the associated constraint functions form a 1st class set, but they no longer form a Lie algebra with respect to Poisson bracket.

The basis for much of the material in this section can be found in [8, 30, 33].

5.1 2+1 Palatini theory coupled to a cosmological constant

Recall that the equation of motion for gravity coupled to the cosmological constant $\Lambda$ is

$$G_{ab} + \Lambda g_{ab} = 0,$$  \hfill (5.1)

where $G_{ab} := R_{ab} - \frac{1}{2}Rg_{ab}$ is the Einstein tensor of $g_{ab}$. We can obtain (5.1) via an action principle if we modify the standard Einstein-Hilbert action by a term proportional to the volume of the spacetime. Defining

$$S_\Lambda(g^{ab}) := \int_\Sigma \sqrt{-g}(R - 2\Lambda),$$  \hfill (5.2)
we find that the variation of (5.2) with respect to \( g^{ab} \) yields (5.1) (modulo the usual boundary term associated with the standard Einstein-Hilbert action). These results are valid in \( n+1 \) dimensions.

To write this action in 2+1 Palatini form, we proceed as in Section 3. We replace the spacetime metric \( g_{ab} \) with a co-triad \( \tilde{v}^I_a \) and replace the unique, torsion-free spacetime derivative operator \( \nabla_a \) (compatible with \( g_{ab} \)) with an arbitrary generalized derivative operator \( \mathcal{D}_a \). Recalling that \( \sqrt{-g} = \frac{1}{3!} \tilde{\eta}^{abc} \tilde{v}^I_a \tilde{v}^J_b \tilde{v}^K_c \), we define

\[
S_\Lambda(\tilde{v}, \Lambda) := \frac{1}{2} \int_M \tilde{\eta}^{abc} \tilde{v}^I_a \tilde{F}^I_{bc} - \frac{\Lambda}{3!} \int_M \tilde{\eta}^{abc} \epsilon^{IJK} \tilde{v}^I_a \tilde{v}^J_b \tilde{v}^K_c
\]

\[
= \frac{1}{2} \int_M \tilde{\eta}^{abc} \epsilon^{IJK} \tilde{v}^I_a \left( \tilde{F}^I_{bc} - \frac{\Lambda}{3} \epsilon^{IJK} \tilde{v}^J_b \tilde{v}^K_c \right),
\]

where \( \tilde{F}^I_{ab} = 2 \partial_a \tilde{A}^I_b + \left[ \tilde{A}^I_a, \tilde{A}^I_b \right]^I \) is the internal curvature tensor of the generalized derivative operator \( \mathcal{D}_a \) defined by

\[
\mathcal{D}_a v^I := \partial_a v^I + \left[ \tilde{A}^I_a, v \right]^I.
\]

Note that \( \left[ \tilde{A}^I_a, v \right]^I := \epsilon^{IJK} \tilde{A}^J_a v^K \) where \( \epsilon^{IJK} := \epsilon^{I'MN} \eta_{MJI} \eta_{NK} \). Just as we did for the vacuum 2+1 Palatini theory, we have included an additional overall factor of \( 1/2 \) in definition (5.3).

Although the action (5.3) was originally defined for \( G = SO(2,1) \), it is well-defined for any Lie group \( G \) that admits an invariant, totally anti-symmetric tensor \( \epsilon^{IJK} \). This is additional structure that does not naturally exist for an arbitrary Lie group \( G \), so unlike the 2+1 Palatini theory with \( \Lambda = 0 \), the 2+1 Palatini theory with non-zero cosmological constant \( \Lambda \) is not defined for arbitrary \( G \). If the Lie algebra \( \mathcal{L}_G \) admits an invariant, non-degenerate bilinear form \( k_{IJ} \), then we are guaranteed that such an \( \epsilon^{IJK} \) exists—we can take \( \epsilon^{IJK} := k^{I'MN} C^I_{MN} \). Thus, in particular, 2+1 Palatini theory with a non-zero cosmological constant is well-defined for semi-simple Lie groups. We should emphasize, however, that it is not necessary to restrict ourselves to semi-simple Lie groups. In what follows, we will only assume that \( \epsilon^{IJK} \) exists. Given such a Lie group \( G \), the action

\[
G_{S_\Lambda}(\tilde{v}, \Lambda) := \frac{1}{2} \int_M \tilde{\eta}^{abc} \epsilon^{IJK} \tilde{v}^I_a \left( \tilde{F}^I_{bc} - \frac{\Lambda}{3} \epsilon^{IJK} \tilde{v}^J_b \tilde{v}^K_c \right)
\]

will be called the 2+1 Palatini action based on \( G \) with cosmological constant \( \Lambda \). Note that

\[
\tilde{F}^I_{ab} = 2 \partial_a \tilde{A}^I_b + \left[ \tilde{A}^I_a, \tilde{A}^I_b \right]^I,
\]

\footnote{We will require that \( \epsilon^{IJK} \) be invariant under the adjoint action of the Lie algebra \( \mathcal{L}_G \) on its dual \( \mathcal{L}^*_G \)—i.e., that \( \epsilon^{IJK} \{v, \alpha\}_I \beta_J \gamma_K + \epsilon^{IJK} \alpha_I \{v, \beta\}_J \gamma_K + \epsilon^{IJK} \alpha_I \beta_J \{v, \gamma\}_K = 0 \) for all \( v^I \in \mathcal{L}_G \) and \( \alpha_I, \beta_I, \gamma_I \in \mathcal{L}^*_G \). \{v, \alpha\}_I is the co-adjoint bracket of \( v^I \) and \( \alpha_I \) defined in terms of the structure constants \( C^I_{JK} \) via \( \{v, \alpha\}_I := C^K_{IJ} v^J \alpha_K \). Invariance of \( \epsilon^{IJK} \) is equivalent to \( \epsilon^{M[IJK]}_{MN} = 0 \).}
where \([2A_a, 2A_b]^I := C^I_{JK} 2A_a^J 2A_b^K\) is the Lie bracket in \(L_G\). It is only for \(G = SO(2, 1)\) that 
\(C^I_{JK} = \epsilon^I_{JK} = \epsilon^{IMN} \eta_{MJ} \eta_{NK}\).

To obtain the Euler-Lagrange equations of motion, we vary \(G_{\Lambda}^I(\check{e}, 3A)\) with respect to 
both \(3e_a^I\) and \(3A_a^I\). We find
\[
\tilde{\eta}^{abc}(3F_{abc} - \Lambda \epsilon^{IJK} 3e_{bJ} 3e_{cK}) = 0 \quad \text{and} \quad 3D_a(3\eta^{abc} 3e_{cl}) = 0, \tag{5.7}
\]
where the second equation, as usual, requires a torsion-free extension of the generalized 
derivative operator \(3D_a\) to spacetime tensor fields, but is independent of this choice. Note 
that if \(\Lambda \neq 0\), \(3D_a\) is not flat. In fact, for the special case \(G = SO(2, 1)\), equations
(5.7) imply that the spacetime \((M, g_{ab} := 3e_a^I 3e_b^J \eta^{IJ})\) has constant curvature equal to \(6\Lambda\).

To show that the above two equations reproduce (5.1), let us restrict ourselves to \(G = SO(2, 1)\) (with \(\epsilon^{IJK}\) being the volume element of \(\eta_{IJ}\)) so that \(3e_a^I\) is, in fact, a co-triad.

Then by following the argument given in Section 3, we find that the second equation implies 
\(3A_a^I = \Gamma_a^I\), where \(\Gamma_a^I\) is the (internal) Christoffel symbol of the unique, torsion-free generalized 
derivative operator \(\nabla_a\) compatible with the co-triad \(3e_a^I\). Thus, \(3D_a\) is not arbitrary, but 
equals \(\nabla_a\) when acting on internal indices. Substituting this solution back into the first 
equation, we find
\[
\tilde{\eta}^{abc}(R_{abc}^{I} - \Lambda \epsilon^{IJK} 3e_{bJ} 3e_{cK}) = 0, \tag{5.8}
\]
where \(R_{ab}^I\) is the (internal) curvature tensor of \(\nabla_a\). Contracting (5.8) with \(3e_I^d\) gives
\[
G^{ad} + \Lambda g^{ad} = 0. \tag{5.9}
\]
This is the desired result.

To put this theory in Hamiltonian form, we will assume that \(M\) is topologically \(\Sigma \times R\), 
and assume that there exists a function \(t\) (with nowhere vanishing gradient \((dt)_a\)) such that 
each \(t = \text{const} \Sigma_t\) is diffeomorphic to \(\Sigma\). By \(t^a\) we will denote the flow vector field 
satisfying \(t^a(dt)_a = 1\). Using \(\tilde{\eta}^{abc} = 3I[a \eta^{bc}] dt\) (and our decomposition of \(G_{SP}(\check{e}, 3A)\) from 
Section 3), we obtain
\[
G_{\Lambda}^I(\check{e}, 3A) = \int \frac{1}{2}(\check{e} \cdot t)_I(\tilde{\eta}^{ab} F_{ab}^I - \Lambda \epsilon^{IJK} \eta_{gb} E_{fj}^a E_{fb}^j) 
+ \tilde{\eta}^a L_a^I A_a^I - \tilde{\eta}^a D_a(3A \cdot t)_I. \tag{5.10}
\]
The configuration variables are \((\check{e} \cdot t)_I := t^a 3e_a^I, \ E_a^I := \tilde{\eta}^{ab} 3e_{bJ}, \ (3A \cdot t)_I := t^a 3A_a^I, \) 
and \(A_a^I := t^a 3A_b^I\). Thus, (modulo a surface integral) the Lagrangian \(G_{\Lambda}\) of the 2+1 Palatini 
theory based on \(G\) with cosmological constant \(\Lambda\) is given by
\[
G_{\Lambda} = \int \frac{1}{2}(\check{e} \cdot t)_I(\tilde{\eta}^{ab} F_{ab}^I - \Lambda \epsilon^{IJK} \eta_{gb} E_{fj}^a E_{fb}^j) 
+ \tilde{\eta}^a L_a^I A_a^I + (D_a \tilde{E}_I^a)(3A \cdot t)_I. \tag{5.11}
\]
$G_L$ is to be viewed as a functional of the configuration variables and their first derivatives.

Following the standard Dirac constraint analysis, we find that the momentum canonically conjugate to $A^I_a$ is $\tilde{E}^I_a$, while $(\varepsilon \cdot t)_I$ and $(\tilde{\varepsilon} A \cdot t)^I$ both play the role of Lagrange multipliers. Thus, the phase space and symplectic structure are the same as those found for the 2+1 Palatini theory with $\Lambda = 0$, and the Hamiltonian is given by

$$G_{H_\Lambda}(A, \tilde{E}) = \int_\Sigma -\frac{1}{2}(\varepsilon \cdot t)_I (\tilde{\eta}^{ab} F^I_{ab} - \Lambda \epsilon^{IJK} \tilde{\eta}_{gh} \tilde{E}^a_J \tilde{E}^b_K) - (\mathcal{D}_a \tilde{E}^a_I)(\tilde{\varepsilon} A \cdot t)^I. \quad (5.12)$$

We will see that this is just a sum of 1st class constraint functions associated with $\tilde{\eta}^{ab} F^I_{ab} - \Lambda \epsilon^{IJK} \tilde{\eta}_{gh} \tilde{E}^a_J \tilde{E}^b_K \approx 0$ and $\mathcal{D}_a \tilde{E}^a_I \approx 0$. \quad (5.13)

By inspection, constraint equations (5.13) are polynomial in the canonically conjugate variables $(A^I_a, \tilde{E}^a_I)$. They are the field equations (5.7) pulled-back to $\Sigma$ with $\tilde{\eta}^{ab}$.

As usual, given test fields $\alpha_I$ and $v^I$, which take values in $\mathcal{L}_G^*$ and $\mathcal{L}_G$, we can define constraint functions

$$F(\alpha) := \frac{1}{2} \int_\Sigma \alpha_I (\tilde{\eta}^{ab} F^I_{ab} - \Lambda \epsilon^{IJK} \tilde{\eta}_{gh} \tilde{E}^a_J \tilde{E}^b_K) \quad \text{and} \quad G(v) := \int_\Sigma v^I (\mathcal{D}_a \tilde{E}^a_I). \quad (5.14)$$

Note that $G(v)$ is unchanged from the 2+1 Palatini theory with $\Lambda = 0$, while $F(\alpha)$ has an additional term quadratic in the momentum $\tilde{E}^a_I$. There is only one new functional derivative,

$$\frac{\delta F(\alpha)}{\delta \tilde{E}^a_I} = -\Lambda \epsilon^{IJK} \tilde{\eta}_{gh} \tilde{E}^b_J \alpha_K. \quad (5.15)$$

All the others are the same as before.

Under the 1-parameter family of diffeomorphisms associated with the Hamiltonian vector field $X_{F(\alpha)}$, we have

$$A^I_a \mapsto A^I_a - \epsilon(\Lambda \epsilon^{IJK} \tilde{\eta}_{gh} \tilde{E}^b_J \alpha_K) + O(\epsilon^2) \quad \text{and} \quad \tilde{E}^a_I \mapsto \tilde{E}^a_I - \epsilon(\tilde{\eta}^{ab} \mathcal{D}_b \alpha_I) + O(\epsilon^2). \quad (5.16a)$$

Similarly, under the 1-parameter family of diffeomorphisms associated with the Hamiltonian vector field $X_{G(v)}$, we have

$$A^I_a \mapsto A^I_a - \epsilon \mathcal{D}_a v^I + O(\epsilon^2) \quad \text{and} \quad \tilde{E}^a_I \mapsto \tilde{E}^a_I - \epsilon \{v, \tilde{E}^a\}_I + O(\epsilon^2). \quad (5.17a)$$

Comparing these results with those from the 2+1 Palatini theory with $\Lambda = 0$, we see that the motion of $\tilde{\varepsilon} A^I_a$ generated by the constraint functions no longer corresponds to the usual gauge
transformation of Yang-Mills theory. This is due to the non-zero contribution from \( F(\alpha) \). In fact, since \( F(\alpha) \) depends quadratically on the momentum \( \tilde{E}_I^a \), the reduced phase space of the 2+1 Palatini theory with non-zero cosmological constant \( \Lambda \) is not naturally a cotangent bundle over a reduced configuration space. Thus, the result of Section 4 that the reduced phase space of the Chern-Simons theory based on \( G \) equals the reduced configuration space of the 2+1 Palatini theory based on the same \( G \) does not extend in general to the case \( \Lambda \neq 0 \).

Nevertheless, we can still evaluate the Poisson brackets of the constraint functions \( F(\alpha) \) and \( G(v) \). As in the \( \Lambda = 0 \) case, we find that

\[
\{ G(v), G(w) \} = G([v, w]), 
\tag{5.18}
\]

where \([v, w]^I = C^I_{JK} v^J w^K \) is the Lie bracket of \( v^I \) and \( w^I \), so \( v^I \mapsto G(v) \) is a representation of the Lie algebra \( \mathcal{L}_G \). Although \( F(\alpha) \) has changed, we again find that

\[
\{ G(v), F(\alpha) \} = -F(\{v, \alpha\}), 
\tag{5.19}
\]

where \( \{v, \alpha\}_I = C^K_{JI} v^K \alpha_I \) is the co-adjoint bracket of \( v^I \) and \( \alpha_I \). However, the Poisson bracket of \( F(\alpha) \) with \( F(\beta) \) is no longer zero; it equals

\[
\{ F(\alpha), F(\beta) \} = -\Lambda G(\epsilon(\alpha, \beta)), 
\tag{5.20}
\]

where \( \epsilon(\alpha, \beta)^I := \epsilon^{IJK} \alpha_J \beta_K \). Thus, the totality of constraint functions is closed under Poisson bracket —i.e., they form a 1st class set. In fact, since (5.18), (5.19), and (5.20) do not involve any structure functions, the constraint functions form a Lie algebra with respect to Poisson bracket. The mapping \((\alpha, v) \in \mathcal{L}_G^* \times \mathcal{L}_G \mapsto (F(\alpha), G(v)) \) is a representation of the Lie algebra \( \mathcal{L}_\Lambda \mathcal{G} \) of the \( \Lambda \)-deformation, \( \Lambda G \), of the Lie group \( G \). The \( F(\alpha) \)'s play the role of “boosts” while the \( G(v) \)'s play the role of “rotations” in the \( \Lambda \)-deformation of \( G \).

### 5.2 Relationship to Chern-Simons theory

In a manner similar to that used in subsection 4.4, we will now show that if \( G \) is any Lie group which admits an invariant, totally anti-symmetric tensor \( \epsilon^{IJK} \), then the 2+1 Palatini theory based on \( G \) with cosmological constant \( \Lambda \) is equivalent to Chern-Simons theory based on the \( \Lambda \)-deformation, \( \Lambda G \), of the Lie group \( G \). The actions for these two theories are the same modulo a surface term that does not affect the equations of motion.

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20We will discuss the construction of the \( \Lambda \)-deformation of a Lie group \( G \) in the following section where we show the equivalence between the 2+1 Palatini theory based on \( G \) with cosmological constant \( \Lambda \) and Chern-Simons theory based on \( \Lambda G \). When \( \Lambda = 0 \), \( \Lambda G \) is just the inhomogeneous Lie group \( IG \) constructed in subsection 4.4.
Given a Lie group $G$ with an invariant, totally anti-symmetric tensor $\epsilon^{IJK}$, we first construct the $\Lambda$-deformation, $\Lambda G$, of $G$ as follows: Form the direct sum vector space $L^*_G \oplus L_G$ (having typical elements $(\alpha, v)^i := (\alpha_I, v^I)$ and $(\beta, w)^i := (\beta_I, w^I)$) and then define a bracket on $L^*_G \oplus L_G$ via
\[
[(\alpha, v), (\beta, w)]^i := (-\{v, \beta\} + \{w, \alpha\}, [v, w] - \Lambda \epsilon(\alpha, \beta))^i,
\]
where $[v, w]^I := C^I_{JK}v^Jw^K$, $\{v, \beta\}_I := C^K_{IJ}v^J\beta^K$, and $\epsilon(\alpha, \beta)^I := \epsilon^{IJK}\alpha_J\beta_K$. By inspection, (5.21) is linear and anti-symmetric. By using the Jacobi identity $C^M_{IJ}C^K_{NM} = 0$ on $L_G$ together with the anti-symmetry and invariance of $\epsilon^{IJK}$, one can show that (5.21) satisfies the Jacobi identity as well. Thus, the vector space $L^*_G \oplus L_G$ together with (5.21) is actually a Lie algebra. We call $L^*_G \oplus L_G$ the $\Lambda$-deformed Lie algebra associated with $G$. The $\Lambda$-deformation, $\Lambda G$, of $G$ is obtained by exponentiating $L^*_G \oplus L_G$. We can think of $\Lambda G$ as an extension of the inhomogeneous Lie group $IG$ in the sense that $\Lambda G$ reduces to $IG$ when $\Lambda = 0$. Note also that if $G = SO(2,1)$, then the above construction for $\Lambda G$ yields $SO(3,1)$ if $\Lambda < 0$ and $SO(2,2)$ if $\Lambda > 0$.

In addition to the above Lie algebra structure, $L^*_G \oplus L_G$ is also equipped with a (natural) invariant, non-degenerate bilinear form
\[
k_{ij}(\alpha, v)^i(\beta, w)^j := \alpha_I w^I + \beta_I v^I.
\]
This is the same $k_{ij}$ that we had when $\Lambda = 0$. As before, the existence of $k_{ij}$ will allow us to construct Chern-Simons theory for $\Lambda G$.

Given these remarks, we are now ready to verify that the 2+1 Palatini theory based on $G$ with cosmological constant $\Lambda$ is equivalent to Chern-Simons theory based on $\Lambda G$. Recall that
\[
G^S_{\Lambda}(\gamma, \gamma^i) = \frac{1}{2} \int_M \tilde{n}^{abc} \gamma_e a I (\gamma_F e b c - \frac{\Lambda}{3} \epsilon^{IJK} \gamma_e b J \gamma_e c K),
\]
where $\gamma_a^i$ and $\gamma_e a I$ are $L_G$- and $L^*_G$-valued 1-forms. If we now construct the $\Lambda$-deformed Lie algebra $L_{\Lambda G}$ associated with $G$ and define an $L_{\Lambda G}$-valued connection 1-form $\gamma_a^i$ via
\[
\gamma_a^i := (\gamma_e a I, \gamma_e a^I),
\]
then a straightforward calculation along the lines of that used in subsection 4.4 shows that the Chern-Simons action
\[
\Lambda^G S_{CS}(\gamma^i) = \frac{1}{2} \int_M \tilde{n}^{abc} k_{ij} \left( \gamma_a^i \partial_b \gamma_a^j + \frac{1}{3} \gamma_a^i [\gamma_b, \gamma_a^j] \right)
\]
equals $G^S_{\Lambda}(\gamma, \gamma^i)$ modulo a surface term which does not affect the Euler-Lagrange equations of motion. Specializing to the case $G = SO(2,1)$, we see that 2+1 gravity coupled to the
cosmological constant $\Lambda$ is equivalent to Chern-Simons theory based on $SO(3, 1)$ if $\Lambda < 0$ or $SO(2, 2)$ if $\Lambda > 0$. This was the observation of Witten [8].

5.3 2+1 Palatini theory coupled to a massless scalar field

So far, we have seen that the 2+1 Palatini theory (with or without a cosmological constant $\Lambda$) is well-defined for a wide class of Lie groups. If $\Lambda = 0$, the Lie group $G$ can be completely arbitrary; if $\Lambda \neq 0$, then $G$ has to admit an invariant, totally anti-symmetric tensor $\epsilon^{IJK}$. We are not forced to restrict ourselves to $G = SO(2, 1)$. However, in order to couple fundamental matter fields with local degrees of freedom to 2+1 gravity via the 2+1 Palatini action, we will need to take $G = SO(2, 1)$. The matter actions require the existence of a spacetime metric $g_{ab}$, and, as such, $\epsilon^{IJK}$ must have the interpretation of a co-triad. We will only consider coupling a massless scalar field to 2+1 gravity in this section—a similar treatment would work for Yang-Mills and Dirac fields as well.

Let us first recall that the theory of a massless scalar field $\phi$ can be defined in $n+1$ dimensions. If $g^{ab}$ denotes the inverse of the spacetime metric $g_{ab}$, then the Klein-Gordon action $S_{KG}(g^{ab}, \phi)$ is defined by

$$S_{KG}(g^{ab}, \phi) := -8\pi \int_M \sqrt{-g} \ g^{ab} \partial_a \phi \partial_b \phi,$$  \hspace{1cm} (5.26)

where $\partial_a \phi$ denotes the gradient of $\phi$. To couple the scalar field to gravity, we simply add the Klein-Gordon action (5.26) to the standard Einstein-Hilbert action

$$S_{EH}(g^{ab}) = \int_M \sqrt{-g} R.$$  \hspace{1cm} (5.27)

The total action $S_T(g^{ab}, \phi)$ is then given by the sum

$$S_T(g^{ab}, \phi) := S_{EH}(g^{ab}) + S_{KG}(g^{ab}, \phi),$$  \hspace{1cm} (5.28)

and the Euler-Lagrange equations of motion are obtained by varying $S_T(g^{ab}, \phi)$ with respect to both $g^{ab}$ and $\phi$. The variation of $\phi$ yields

$$g^{ab} \nabla_a \nabla_b \phi = 0,$$  \hspace{1cm} (5.29)

while the variation of $g^{ab}$ yields

$$G_{ab} = 8\pi T_{ab}(KG).$$  \hspace{1cm} (5.30)

$\nabla_a$ is the unique, torsion-free spacetime derivative operator compatible with the metric $g_{ab}$, and $T_{ab}(KG)$ is the stress-energy tensor of the massless scalar field. In terms of $g_{ab}$ and $\phi$, we have

$$T_{ab}(KG) := \partial_a \phi \partial_b \phi - \frac{1}{2} g_{ab} \partial_c \phi \partial^c \phi.$$  \hspace{1cm} (5.31)
Now let us restrict ourselves to 2+1 dimensions and rewrite the above actions and equations of motion in 2+1 Palatini form. As we saw in Section 3, the 2+1 Palatini action can be written as

\[ S_P(3e, 3A) := \frac{1}{2} \int_M \tilde{\eta}^{abc} 3e_{aI} 3F^I_{bc}, \quad (5.32) \]

where \(3e_{aI}\) is the co-triad related to the spacetime metric \(g_{ab}\) via \(g_{ab} := 3e^I_a 3e^J_b \eta_{IJ}\) and \(3F^I_{ab} = 2\partial^I_a 3A^J_b + \epsilon^I_JK 3A^J_a 3A^K_b\) is the internal curvature tensor of the generalized derivative operator \(3D_a\) defined by \(3A^I_a\). The Klein-Gordon action, viewed as a functional of \(3e_{aI}\) and the scalar field \(\phi\), is given by

\[ S_{KG}(3e, \phi) = -8\pi \int_M \sqrt{-g} g^{ab} \partial_a \phi \partial_b \phi. \quad (5.33) \]

Note that although the Klein-Gordon action depends on the co-triad \(3e_{aI}\) through its dependence on \(\sqrt{-g}\) and \(g^{ab}\), it is independent of the connection 1-form \(3A^I_a\). In fact, of all the fundamental matter couplings, only the action for the Dirac field would depend on \(3A^I_a\).

Given (5.32) and (5.33), we define the total action as the sum

\[ S_T(3e, 3A, \phi) := S_P(3e, 3A) + \frac{1}{2} S_{KG}(3e, \phi). \quad (5.34) \]

The additional factor of 1/2 is needed in front of \(S_{KG}(3e, \phi)\) so that the above definition of the total action will be consistent with the definition of \(S_P(3e, 3A)\). The Euler-Lagrange equations of motion are obtained by varying \(S_T(3e, 3A, \phi)\) with respect to each field. Varying \(\phi\) gives

\[ g^{ab} \nabla_a \nabla_b \phi = 0, \quad (5.35) \]

while varying \(3A^I_a\) and \(3e_{aI}\) imply

\[ 3D_b(\tilde{\eta}^{abc} 3e_{cI}) = 0 \quad \text{and} \quad (5.36) \]

\[ \tilde{\eta}^{abc} 3F^I_{bc} - 8\pi \sqrt{-g}(3e_{aI} g^{bc} - 2 3e^I_a g^{bc}) \partial_b \phi \partial_c \phi = 0, \quad (5.37) \]

respectively. Note that equation (5.35) is just the standard equation of motion for \(\phi\), while equation (5.36) implies that \(3A^I_a = \Gamma^I_a\), as in the vacuum case. Substituting this result for \(3A^I_a\) back into (5.37) and contracting with \(3e_{dI}\) gives

\[ G^{ad} = 8\pi T^{ad}(KG). \quad (5.38) \]

These are the desired results.

To put this theory in Hamiltonian form, we will basically proceed as we have in the past, but use additional structure provided by the spacetime metric \(g_{ab}\). We will assume that \(M\) is topologically \(\Sigma \times R\) for some spacelike submanifold \(\Sigma\) and assume that there exists a time function \(t\) (with nowhere vanishing gradient \((dt)_a\)) such that each \(t = \text{const}\) surface
\( \Sigma_t \) is diffeomorphic to \( \Sigma \). \( t^a \) will denote the time flow vector field \((t^a(dt)_a = 1)\), while \( n_a \) will denote the unit covariant normal to the \( t = \text{const} \) surfaces. \( n^a := g^{ab} n_b \) will be the associated future-pointing timelike vector field \((n^a n_a = -1)\). Given \( n_a \) and \( n^a \), it follows that \( q^a_b := \delta^a_b + n^a n_b \) is a projection operator into the \( t = \text{const} \) surfaces. We can then define the induced metric \( q_{ab} \), the lapse \( N \), and shift \( N^a \) as we did for the standard Einstein-Hilbert theory in Section 2. Recall that \( t^a = N n^a + N^a \), with \( N^a n_a = 0 \).

Now let us write the total action (5.34) in 2+1 form by decomposing each of its pieces. Using \( \tilde{\eta}^{abc} = 3 t^a \tilde{\eta}^{bc} dt \), it follows that

\[
S_P(\tilde{\mathcal{E}}, 3A) = \int dt \int_\Sigma \frac{1}{2} (\tilde{\mathcal{E}} \cdot t) I \tilde{\eta}^{ab} F^I_{ab} + \tilde{E}^a_I \mathcal{L}_I A^I_a - \tilde{E}^a_I \mathcal{D}_a (3A \cdot t)^I, \tag{5.39}
\]

where \((\mathcal{E} \cdot t) I := t^a \mathcal{E}^a_{Ia}, \tilde{E}^a_I := \tilde{\eta}^{ab} \mathcal{E}^b_{Ia}, (3A \cdot t)^I := t^a 3A^a_I \), and \( A^I_a := q^a_b 3A^I_b \). To obtain (39), we used the fact that \( \mathcal{L}_I q^a_a = 0 \). Note also that \( F^I_{ab} := q^a_d q^b_d F^{I}_{cd} \) is the curvature tensor of the generalized derivative operator \( \mathcal{D}_a (:= q^a_b \mathcal{D}_b) \) on \( \Sigma \) associated with \( A^I_a \). Since

\[
\frac{1}{2} (\tilde{\mathcal{E}} \cdot t) I \tilde{\eta}^{ab} F^I_{ab} = -\frac{1}{2} \mathcal{N}^{IJK} \tilde{E}^a_I \tilde{E}^b_J F_{abK} - N^a \tilde{E}^a_IF^I_{ab} \tag{5.40}
\]

(where \( \mathcal{N} := q^{-\frac{1}{2}} N \)), we see that (modulo a surface integral) the Lagrangian \( L_P \) of the 2+1 Palatini theory is given by

\[
L_P = \int_\Sigma -\frac{1}{2} \mathcal{N}^{IJK} \tilde{E}^a_I \tilde{E}^b_J F_{abK} - N^a \tilde{E}^a_IF^I_{ab} + \tilde{E}^a_I \mathcal{L}_I A^I_a + (\mathcal{D}_a \tilde{E}^a_I)(3A \cdot t)^I. \tag{5.41}
\]

Similarly, using \( g^{ab} = q^{ab} - n^a n^b \) and the decomposition \( \sqrt{-g} = N \sqrt{\mathcal{N}} dt \), it follows that

\[
S_{KG}(\tilde{\mathcal{E}}, \phi) = -8\pi \int dt \int_\Sigma \{ \mathcal{N} \tilde{q}^{ab} \partial_a \phi \partial_b \phi - \mathcal{N}^{-1} (\mathcal{L}_I \phi - N^a \partial_a \phi)^2 \}, \tag{5.42}
\]

where \( \tilde{q}^{ab} := q q^{ab} (= \tilde{E}^a_I \tilde{E}^{Ib}) \). Thus, the Klein-Gordon Lagrangian \( L_{KG} \) is simply given by

\[
L_{KG} = -8\pi \int_\Sigma \{ \mathcal{N} \tilde{q}^{ab} \partial_a \phi \partial_b \phi - \mathcal{N}^{-1} (\mathcal{L}_I \phi - N^a \partial_a \phi)^2 \}. \tag{5.43}
\]

The total Lagrangian \( L_T \) is the sum \( L_T = L_P + \frac{1}{2} L_{KG} \) and is to be viewed as a functional of the configuration variables \((3A \cdot t)^I, \mathcal{N}, N^a, A^I_a, \tilde{E}^a_I, \phi \) and their first time derivatives.

Following the standard Dirac constraint constraint analysis, we find that

\[
\tilde{\pi} := \frac{\delta L_T}{\delta (\mathcal{L}_I \phi)} = 8\pi \mathcal{N}^{-1} (\mathcal{L}_I \phi - N^a \partial_a \phi) \tag{5.44}
\]

is the momentum canonically conjugate to \( \phi \). Since this equation can be inverted to give

\[
\mathcal{L}_I \phi = \frac{1}{8\pi} \mathcal{N} \tilde{\pi} + N^a \partial_a \phi, \tag{5.45}
\]
it does not define a constraint. On the other hand, $\tilde{E}_a^I$ is constrained to be the momentum canonically conjugate to $A_I^a$, while $(^3A \cdot t)^I$, $N$, and $N^a$ play the role of Lagrange multipliers. The resulting total phase space $(\Gamma_T, \Omega_T)$ is coordinatized by the pairs of fields $(A_I^a, \tilde{E}_a^I)$ and $(\phi, \tilde{\pi})$ with symplectic structure

$$\Omega_T = \int_\Sigma d\tilde{E}_a^I \wedge dA_I^a + \text{Tr}(d\tilde{\pi} \wedge d\phi).$$

(5.46)

The Hamiltonian is given by

$$H_T(A, \tilde{E}, \phi, \bar{E}) = \int_\Sigma N\left(\frac{1}{2} \epsilon^{IJK} \tilde{E}_I^a \tilde{E}_J^b F_{abK} + (4\pi \tilde{q}^{ab} \partial_a \phi \partial_b \phi + \frac{1}{16\pi} \tilde{\pi}^2)\right)$$

$$+ N^a(\tilde{E}_I^b F_I^a + \tilde{\pi} \partial_a \phi) - (D_a \tilde{E}_a^I)(^3A \cdot t)^I,$$

(5.47)

We shall see that this is just a sum of 1st class constraint functions associated with

$$\frac{1}{2} \epsilon^{IJK} \tilde{E}_I^a \tilde{E}_J^b F_{abK} + (4\pi \tilde{q}^{ab} \partial_a \phi \partial_b \phi + \frac{1}{16\pi} \tilde{\pi}^2) \approx 0,$$

(5.48)

$$\tilde{E}_I^b F_{ab}^I + \tilde{\pi} \partial_a \phi \approx 0,$$

(5.49)

$$D_a \tilde{E}_a^I \approx 0.$$

(5.50)

These are the constraint equations associated with the Lagrange multipliers $N$, $N^a$, $(^3A \cdot t)^I$, respectively.

Two remarks are in order: First, note that just as we found for the 2+1 Palatini theory with or without a cosmological constant $\Lambda$, the constraint equations for the 2+1 Palatini theory coupled to a Klein-Gordon field are polynomial in the basic canonically conjugate variables $(A_I^a, \tilde{E}_a^I)$ and $(\phi, \tilde{\pi})$. Since the Hamiltonian is just a sum of these constraints, it follows that the evolution equations will be polynomial as well. Second, since the constraint equations do not involve the inverse of $\tilde{E}_a^I$, the above Hamiltonian formulation is well-defined even if $\tilde{E}_a^I$ is non-invertible. Thus, we have a slight extension of the standard 2+1 theory of gravity coupled to a massless scalar field. It can handle those cases where the spatial metric $\tilde{q}^{ab} := \tilde{E}_a^I \tilde{E}_b^I$ becomes degenerate.

Our next goal is to verify the claim that the constraint functions associated with (5.48)-(5.50) form a 1st class set. To do this, we let $v^I$ (which takes values in the Lie algebra of $SO(2,1)$), $\bar{N}$, and $N^a$ be arbitrary test fields on $\Sigma$. Then we define

$$C(N) := \int_\Sigma N\left(\frac{1}{2} \epsilon^{IJK} \tilde{E}_I^a \tilde{E}_J^b F_{abK} + (4\pi \tilde{q}^{ab} \partial_a \phi \partial_b \phi + \frac{1}{16\pi} \tilde{\pi}^2)\right),$$

(5.51)

$$C'(\bar{N}) := \int_\Sigma N^a(\tilde{E}_I^b F_{ab}^I + \tilde{\pi} \partial_a \phi),$$

(5.52)

$$G(v) := \int_\Sigma v^I(D_a \tilde{E}_a^I),$$

(5.53)
to be the scalar, vector, and Gauss constraint functions.

As we saw in subsection 3.3 for the 2+1 Palatini theory, the subset of Gauss constraint functions form a Lie algebra with respect to Poisson bracket. Given \( G(v) \) and \( G(w) \), we have

\[
\{ G(v), G(w) \} = G([v, w]), \tag{5.54}
\]

where \([v, w]^I\) is the Lie bracket in \( \mathcal{L}_{SO(2,1)} \). Thus, the mapping \( v \mapsto G(v) \) is a representation of the Lie algebra \( \mathcal{L}_{SO(2,1)} \). Given its geometrical interpretation as the generator of internal rotations, it follows that

\[
\{ G(v), C(\vec{N}) \} = 0 \quad \text{and} \quad \{ G(v), C'(\vec{N}) \} = 0, \tag{5.55, 5.56}
\]
as well.

Since one can show that the vector constraint function does not by itself have any direct geometrical interpretation (see, e.g., [34]), we will define a new constraint function \( C(\vec{N}) \) by taking a linear combination of the vector and Gauss constraints. We define

\[
C(\vec{N}) := C'(\vec{N}) - G(N), \tag{5.57}
\]

where \( N^I := N^a A^I_a \). We will call \( C(\vec{N}) \) the diffeomorphism constraint function since the motion it generates on phase space corresponds to the 1-parameter family of diffeomorphisms on \( \Sigma \) associated with the vector field \( N^a \). To see this, we can write

\[
C(\vec{N}) := C'(\vec{N}) - G(N)
= \int_\Sigma N^a (\vec{E}^b F^I_{ab} + \bar{\pi} \partial_a \phi) - \int \Sigma N^I (\mathcal{D}_a \vec{E}^a_I)
= \int \Sigma N^a (\vec{E}_I^b F^I_{ab} - A^I_a \mathcal{D}_b \vec{E}^b_I) + \bar{\pi} N^a \partial_a \phi
= \int \Sigma \vec{E}_I^a \mathcal{L}_{\vec{N}} A^I_a + \bar{\pi} \mathcal{L}_{\vec{N}} \phi, \tag{5.58}
\]

where the Lie derivative with respect to \( N^a \) treats fields having only internal indices as scalars. To obtain the last line of (5.58), we ignored a surface integral (which would vanish anyways for \( N^a \) satisfying the appropriate boundary conditions). By inspection, it follows that \( A^I_a \mapsto A^I_a + \epsilon \mathcal{L}_{\vec{N}} A^I_a + O(\epsilon^2) \), etc. Using this geometric interpretation of \( C(\vec{N}) \), it follows that

\[
\{ C(\vec{N}), G(v) \} = G(\mathcal{L}_{\vec{N}} v), \tag{5.59}
\]
\[
\{ C(\vec{N}), C(M) \} = C(\mathcal{L}_{\vec{N}} M), \quad \text{and} \quad \{ C(\vec{N}), C(\vec{M}) \} = C([\vec{N}, \vec{M}]). \tag{5.60}
\]
We are left to evaluate the Poisson bracket \( \{ C_N, C_M \} \) of two scalar constraints. After a fairly long but straightforward calculation, one can show that

\[
\{ C_N, C_M \} = C'(\vec{K}) \quad (= C(\vec{K}) + G(K, K)),
\]

(5.62)

where \( K^a := \tilde{q}^{ab}(N_\partial_b M - M_\partial_b N) \) and \( \tilde{q}^{ab} = \bar{E}_I^a \bar{E}_I^b \). This result makes crucial use of the fact that

\[
\varepsilon^{IJK} \varepsilon_I^{MN} = (\eta^{JM} \eta^{KN} - \eta^{JM} \eta^{KM}).
\]

(5.63)

This is a property of the structure constants \( \varepsilon^{IJK} := \varepsilon^{IJK} \eta_{IJ} \eta_{NK} \) of the Lie algebra of \( SO(2,1) \). Thus, the constraint functions are closed under Poisson bracket—i.e., they form a 1st class set. Note, however, that since the vector field \( K^a \) depends on the phase space variable \( \bar{E}_I^a \), the Poisson bracket (5.62) involves structure functions. The constraint functions do not form a Lie algebra. This result is similar to what we found for the standard Einstein-Hilbert theory in subsection 2.3.

It is interesting to note that even if we did not couple matter to the 2+1 Palatini theory, but performed the Legendre transform as we did above (i.e., using the additional structure provided by the spacetime metric \( g_{ab} := \bar{q}_a^{\ell} \bar{q}_b^{\ell} \)), we would still obtain the same Poisson bracket algebra. The constraint functions would still fail to form a Lie algebra due to the structure functions in (5.62). At first, something seems to be wrong with this statement, since we saw in subsection 3.3 that the constraint functions \( G(v) \) and \( F(\alpha) \) of the 2+1 Palatini theory form a Lie algebra with respect to Poisson bracket. One may ask why the constraints functions obtained via one decomposition of the 2+1 Palatini theory form a Lie algebra, while those obtained from another decomposition do not.

The answer to this question is actually fairly simple. Namely, it is easy to destroy the “Lie algebra-ness” of a set of constraint functions. If \( \phi_\iota (\iota = 1, \cdots, m) \) denote \( m \) constraint functions which form a Lie algebra under Poisson bracket (i.e., \( \{ \phi_\iota, \phi_j \} = C^k_{\iota j} \phi_k \) where \( C^k_{\iota j} \) are constants), then a linear combination of these constraints, \( \chi_\iota = \Lambda_{\iota j} \phi_j \), will not in general form a Lie algebra if \( \Lambda_{\iota j} \) are not constants on the phase space. In essence, this is what happens when one passes from the \( G(v) \) and \( F(\alpha) \) constraint functions of subsection 3.3 to the \( G(v) \), \( C(N) \), and \( C(\bar{N}) \) constraint functions of this subsection. The transition from \( F(\alpha) \) to \( C(N) \) and \( C(\bar{N}) \) involve functions of the phase space variables.

6. 3+1 Palatini theory

In this section, we will describe the 3+1 Palatini theory. In subsection 6.1, we define the 3+1 Palatini action and show that the Euler-Lagrange equations of motion are equivalent
to the standard vacuum Einstein’s equation. In subsection 6.2, we will follow the standard Dirac constraint analysis to put the 3+1 Palatini theory in Hamiltonian form. We obtain a set of constraint equations which include a 2nd class pair. By solving this pair, we find that the remaining (1st class) constraints become non-polynomial in the (reduced) phase space variables. In essence, we are forced into using the standard geometrodynamical variables of general relativity. In fact, as we shall see in subsection 6.3, the Hamiltonian formulation of the 3+1 Palatini theory is just that of the standard Einstein-Hilbert theory. Thus, the 3+1 Palatini theory does not give us a connection-dynamic description of 3+1 gravity.

Much of the material in subsection 6.2 is based on an analysis of the 3+1 Palatini theory given in Chapter 4 of [3].

6.1 Euler-Lagrange equations of motion

To obtain the Palatini action for 3+1 gravity, we will first write the standard Einstein-Hilbert action

\[ S_{EH}(g^{ab}) = \int_{\Sigma} \sqrt{-g} R \]  

(6.1)

in tetrad notation. Using

\[ R_{ab}^I J = R_{abc}^d \epsilon_c^I \epsilon_d^J \]  

(6.2)

(which relates the internal and spacetime curvature tensors of the unique, torsion-free generalized derivative operator \( \nabla_a \) compatible with the tetrad \( \epsilon^I_a \)) and

\[ \epsilon_{abcd} = \epsilon^I_a \epsilon^J_b \epsilon^K_c \epsilon^L_d \epsilon_{IJKL} \]  

(6.3)

(which relates the volume element \( \epsilon_{abcd} \) of \( g_{ab} = \epsilon^I_a \epsilon^J_b \eta_{IJ} \) to the volume element \( \epsilon_{IJKL} \) of \( \eta_{IJ} \)), we find that

\[ \sqrt{-g} R = \frac{1}{4} \eta^{abcd} \epsilon_{IJKL} \epsilon^I_a \epsilon^J_b \epsilon^K_c R_{cd} \epsilon^L_d. \]  

(6.4)

Thus, viewed as a functional of the co-tetrad \( \epsilon^I_a \), the standard Einstein-Hilbert action is given by

\[ S_{EH}(\epsilon^I) = \frac{1}{4} \int_{\Sigma} \eta^{abcd} \epsilon_{IJKL} \epsilon^I_a \epsilon^J_b \epsilon^K_c R_{cd} \epsilon^L_d. \]  

(6.5)

To obtain the 3+1 Palatini action, we simply replace \( R_{ab}^I J \) in (5) with the internal curvature tensor \( F_{ab}^I J \) of an arbitrary generalized derivative operator \( \mathcal{D}_a \) defined by

\[ \mathcal{D}_a k_I := \partial_a k_I + A_{al}^J k_J. \]  

(6.6)

We define the 3+1 Palatini action to be

\[ S_P(\epsilon, A) := \frac{1}{8} \int_M \eta^{abcd} \epsilon_{IJKL} \epsilon^I_a \epsilon^J_b \epsilon^K_c F_{cd} \epsilon^L_d, \]  

(6.7)
where $4F_{ab} = 2\partial_a 4A_{bij} + [4A_a, 4A_b]_J$. Just as we did for the 2+1 Palatini theory, we have included an additional factor of $1/2$ in definition (6.7) so our canonically conjugate variables will agree with those used in the literature (see, e.g., [3]). Note also, that as defined above, $4\mathcal{D}_a$ knows how to act only on internal indices. We do not require that $4\mathcal{D}_a$ know how to act on spacetime indices. However, we will often find it convenient to consider a torsion-free extension of $4\mathcal{D}_a$ to spacetime tensor fields. All calculations and all results will be independent of this choice of extension.

Since the 3+1 Palatini action is a functional of both a co-tetrad and a connection 1-form, there will be two Euler-Lagrange equations of motion. When we vary $S_P(4\varepsilon, 4A)$ with respect to $4\varepsilon^I_a$ and $4A_a^{IJ}$, we find

$$\bar{\eta}^{abcd}\epsilon_{IJKL} 4\varepsilon_b^J 4F_{cd}^{KL} = 0 \quad \text{and} \quad 4\mathcal{D}_b(\bar{\eta}^{abcd}\epsilon_{IJKL} 4\varepsilon_c^K 4\varepsilon_d^L) = 0,$$

respectively. The last equation requires a torsion-free extension of $4\mathcal{D}_a$ to spacetime tensor fields, but since the left hand side of (6.9) is the divergence of a skew spacetime tensor density of weight +1 on $M$, it is independent of this choice. Noting that $\bar{\eta}^{abcd}\epsilon_{IJKL} 4\varepsilon_c^K 4\varepsilon_d^L = 4(\bar{\varepsilon}) 4\varepsilon^I_a 4\varepsilon^J_b$ (where $(\bar{\varepsilon}) := \sqrt{-g}$), we can rewrite (6.9) as

$$4\mathcal{D}_b((\bar{\varepsilon}) 4\varepsilon^I_a 4\varepsilon^J_b) = 0.$$  

(6.10)

This equation is identical in form to equation (3.18) obtained in Section 3 for the 2+1 Palatini theory.

By following exactly the same argument used in subsection 3.1, equation (6.10) implies that $4A_a^I = \Gamma_a^I$, where $\Gamma_a^I$ is the internal Christoffel symbol of $\nabla_a$. Using this result, the remaining Euler-Lagrange equation of motion (6.8) becomes

$$\bar{\eta}^{abcd}\epsilon_{IJKL} 4\varepsilon_b^J 4R_{cd}^{KL} = 0.$$  

(6.11)

When (6.11) is contracted with $4\varepsilon^f$, we get $G^{ae} = 0$. Thus, we can produce the 3+1 vacuum Einstein’s equation starting from the 3+1 Palatini action given by (6.7). Note that just as in the 2+1 theory, the equation of motion (6.9) for $4A_a^I$ can be solved uniquely for $4A_a^I$ in terms of the remaining basic variables $4\varepsilon^I_a$. The pulled-back action $S_P(\bar{\varepsilon})$ defined on the solution space $4A_a^I = \Gamma_a^I$ is just 1/2 times the standard Einstein-Hilbert action $S_{EH}(\bar{\varepsilon})$ given by (6.5).

6.2 Legendre transform

To put the 3+1 Palatini theory in Hamiltonian form, we will use the additional structure provided by the spacetime metric $g_{ab}$. We will assume that $M$ is topologically $\Sigma \times R$ for
some spacelike submanifold $\Sigma$ and assume that there exists a time function $t$ (with nowhere vanishing gradient $(dt)_a$) such that each $t = \text{const}$ surface $\Sigma_t$ is diffeomorphic to $\Sigma$. $t^a$ will denote the time flow vector field $(t^a(dt)_a = 1)$, while $n_a$ will denote the unit covariant normal to the $t = \text{const}$ surfaces. $n^a := g^{ab}n_b$ will be the associated future-pointing timelike vector field $(n^an_a = -1)$. Given $n_a$ and $n^a$, it follows that $q^a_b := \delta^a_b + n^a n_b$ is a projection operator into the $t = \text{const}$ surfaces. We can then define the induced metric $g_{ab}$, the lapse $N$, and shift $N^a$ as we did for the standard Einstein-Hilbert theory in Section 2. Recall that $t^a = N n^a + N^a$, with $N^a n_a = 0$.

Now let us write (6.7) in $3+1$ form. Using the decomposition $\tilde{\eta}^{abcd} = 4 t [(\tilde{\eta})^{abcd}] dt$ (where $\tilde{\eta}^{abc}$ is the Levi-Civita tensor density of weight $+1$ on $\Sigma$), we get

$$S_P(\{\epsilon^4, A\}) = \frac{1}{8} \int_M \tilde{\eta}^{abcd} \epsilon_{IJKL} \tilde{\epsilon}^I_a \tilde{\epsilon}^J_b 4 F^{KL}_{cd}$$

$$= \int dt \int \frac{1}{4} (\epsilon \cdot t)^I \epsilon_{IJKL} \tilde{\eta}^{abcd} \epsilon^a_b F_{cd}^{KL} + \frac{1}{2} \tilde{E}^a_{IJJ} \mathcal{L}_t A^I_a - \frac{1}{2} \tilde{E}^a_{IJ} \delta_a (4A \cdot t)^{IJJ}$$

(6.12)

where $(\epsilon \cdot t)^I := t^a \epsilon^a_I$, $\tilde{E}^a_{IJJ} := \frac{1}{2} \epsilon_{IJKL} \tilde{\eta}^{abcd} \epsilon^a_b \epsilon^L_c$, $(4A \cdot t)_{IJJ} := t^a 4 A^a_{IJ}$, $A^a_{IJ} := q^a_b 4 A^I_b$, and $e^a_I := q^a_b e^b_I$. To obtain the last line of (6.12), we used the fact that $\mathcal{L}_t q^a_b = 0$. Note also that $F_{ab}^{IJ} := q^a_b q^b_d 4 F_{cd}^{IJ}$ is the curvature tensor of the generalized derivative operator $D_a := q^a_b 4 D_b$ on $\Sigma$ associated with $A^a_{IJ}$. Since

$$\frac{1}{4} (\epsilon \cdot t)^I \epsilon_{IJKL} \tilde{\eta}^{abcd} \epsilon^a_b F_{cd}^{KL} = -\frac{1}{2} N^a \text{Tr}(\tilde{E}^a_\cdot \tilde{E}^b_{ab}) + \frac{1}{2} N^a \text{Tr}(\tilde{E}^b F_{ab})$$

(6.13)

(where $\mathcal{N} := q^{-\frac{1}{2}} N$ and $\text{Tr}$ denotes the trace operation on internal indices), we see that (modulo a surface integral) the Lagrangian $L_P$ of the $3+1$ Palatini theory is given by

$$L_P = \int_{\Sigma} -\frac{1}{2} \mathcal{N} \text{Tr}(\tilde{E}^a_\cdot \tilde{E}^b_{ab}) + \frac{1}{2} N^a \text{Tr}(\tilde{E}^b F_{ab})$$

$$+ \frac{1}{2} \tilde{E}^a_{IJJ} \mathcal{L}_t A^I_a + \frac{1}{2} (D_a \tilde{E}^a_{IJ})(4A \cdot t)^{IJJ}. \quad \text{(14)}$$

The configuration variables of the theory are $(4A \cdot t)^{IJJ}, \mathcal{N}, N^a, A^a_{IJ}$, and $\tilde{E}^a_{IJJ}$.

But before we perform the Legendre transform, we should note that the configuration variable $\tilde{E}^a_{IJJ}$ is not free to take on arbitrary values. In fact, from its definition

$$\tilde{E}^a_{IJJ} := \frac{1}{2} \epsilon_{IJKL} \tilde{\eta}^{abcd} \epsilon^a_b \epsilon^K_c \epsilon^L_d,$$  \quad \text{(15)}$$

one can show that

$$\tilde{\phi}^{ab} := \epsilon^{IJKL} \tilde{E}^a_{IJ} \tilde{E}^b_{KL} = 0 \quad \text{and} \quad \text{Tr}(\tilde{E}^a \tilde{E}^b) > 0. \quad \text{(16)}$$

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The second condition follows from the fact that $\text{Tr}(\tilde{E}^a \tilde{E}^b) = 2\tilde{q}^{ab} (= 2q^{ab})$, where $q^{ab}$ is the inverse of the induced positive-definite metric $q_{ab}$ on $\Sigma$. Thus, the starting point for the Legendre transform is $L_P$ together with the primary constraint $\tilde{\phi}^{ab} = 0$. Since the inequality is a non-holonomic constraint, it will not reduce the number of phase space degrees of freedom.

If we now follow the standard Dirac constraint analysis, we find that $(4A \cdot t)^{IJ}, \tilde{N}$, and $N^a$ play the role of Lagrange multipliers. Their associated constraint equations (which arise as secondary constraints in the analysis) are

\begin{align}
\text{Tr}(\tilde{E}^a \tilde{E}^b F_{ab}) &\approx 0, \\
\text{Tr}(\tilde{E}^b F_{ab}) &\approx 0, \quad \text{and} \\
\mathcal{D}_a \tilde{E}^a_{IJ} &\approx 0.
\end{align}

There is also a primary constraint which says that $\frac{1}{2} \tilde{E}^a_{IJ}$ is the momentum canonically conjugate to $A_a^{IJ}$. By demanding that the Poisson bracket of this constraint with the total Hamiltonian and the Poisson bracket of $\tilde{\phi}^{ab}$ with the total Hamiltonian be weakly zero, we find that

\begin{equation}
\chi^{ab} := \epsilon^{IJKL}(\mathcal{D}_c \tilde{E}^a_{IJ})[\tilde{E}^b, \tilde{E}^c]_{KL} \approx 0.
\end{equation}

This is an additional secondary constraint which must be appended to constraint equations (6.16)-(6.19). In virtue of $\tilde{\phi}^{ab} = 0$, the expression for $\chi^{ab}$ is independent of the choice of torsion-free extension of $\mathcal{D}_a$ to spacetime tensor fields. If we further demand that the Poisson bracket of $\chi^{ab}$ with the total Hamiltonian be weakly zero, we find nothing new—i.e., there are no tertiary constraints.

Let us summarize the situation so far: Out of the original set of configuration variables \{$(4A \cdot t)^{IJ}, \tilde{N}, N^a, A_a^{IJ}, \tilde{E}^a_{IJ}$\}, the first three are non-dynamical. We also found that $\frac{1}{2} \tilde{E}^a_{IJ}$ is the momentum canonically conjugate to $A_a^{IJ}$. Thus, the phase space $(\Gamma'_P, \Omega'_P)$ of the 3+1 Palatini theory is coordinatized by the pair $(A_a^{IJ}, \tilde{E}^a_{IJ})$ and has the symplectic structure

\begin{equation}
\Omega'_P = \frac{1}{2} \int_\Sigma \text{d} \tilde{E}^a_{IJ} \wedge \text{d} A_a^{IJ}.
\end{equation}

The Hamiltonian is given by

\begin{equation}
H'_P(A, \tilde{E}) = \int \Sigma \left( \frac{1}{2} N \text{Tr}(\tilde{E}^a \tilde{E}^b F_{ab}) - \frac{1}{2} N^a \text{Tr}(\tilde{E}^b F_{ab}) - \frac{1}{2} (\mathcal{D}_a \tilde{E}^a_{IJ})(4A \cdot t)^{IJ} \\
+ \lambda_{ab} \epsilon^{IJKL} \tilde{E}^a_{IJ} \tilde{E}^b_{KL} \right),
\end{equation}

where $\lambda_{ab}$ is a Lagrange multiplier. The constraints of the theory are (6.16)-(6.20). Note also that at this stage of the Dirac constraint analysis all constraint (and evolution) equations are polynomial in the canonically conjugate variables.
The next step in the Dirac constraint analysis is to evaluate the Poisson brackets of the constraints and solve all 2nd class pairs. Since only 
\[ \{ \tilde{\phi}^{ab}(x), \chi^{cd}(y) \} \not\approx 0, \]  
we just have to solve constraint equations (6.16) and (6.20). As shown in Chapter 4 of [3], the most general solution to (6.16) is 
\[ \tilde{E}^{a}_{IJ} =: 2\tilde{E}^{a}_{I}[n_J], \]  
for some unit timelike covariant normal \( n_I \) \((n_I n^I = -1)\) with \( \tilde{E}^{a}_{I} \) invertible and \( \tilde{E}^{a}_{I} n^I = 0 \). By writing \( \tilde{E}^{a}_{IJ} \) in this form, we see that the original 18 degrees of freedom per space point for \( \tilde{E}^{a}_{IJ} \) has been reduced to 12. Note also that \( \tilde{E}^{a}_{I} \tilde{E}^{bl} = \tilde{q}^{ab} \), so \( \tilde{E}^{a}_{I} \) is in fact a (densitized) triad.

Given (6.24), the most convenient way to solve (6.20) is to gauge fix the internal vector \( n^I \). This will further reduce the number of degrees of freedom of \( \tilde{E}^{a}_{IJ} \) to 9, since now only \( \tilde{E}^{a}_{I} \) will be arbitrary. However, gauge fixing \( n^I \) requires us to solve the boost part of (6.19) relative to \( n_I \) as well.\footnote{The boost part of any anti-symmetric tensor \( A_{IJ} \) relative to \( n_I \) is defined to be \( A_{IJ} n^J \).} We can only keep that part of (6.19) which generates internal rotations leaving \( n^I \) invariant.

To solve these constraints, let us first define an internal connection 1-form \( K^I_{aJ} \) via 
\[ D_a k_I =: D_a k_I + K^a_{IJ} k_J, \]  
where \( D_a \) is the unique, torsion-free generalized derivative operator on \( \Sigma \) compatible with the (densitized) triad \( \tilde{E}^{a}_{I} \) and the gauge fixed internal vector \( n^I \). Then constraint equation (6.20) and the boost part of (6.19) become 
\[ \chi^{ab} = -4\epsilon^{JKL} K_{cL}^{I} \tilde{E}^{a}_{L} \tilde{E}^{b}_{J} \tilde{E}^{c}_{K} \approx 0 \quad \text{and} \quad (D_a \tilde{E}^{a}_{IJ}) n^J = -K_{aM}^{N} \tilde{E}^{a}_{N} q_I^{M} \approx 0, \]  
where \( q^J_I := \delta^J_I + n^J n_I \). By using the invertibility of the (densitized) triad \( \tilde{E}^{a}_{I} \), one can then show (again, see Chapter 4 of [3]) that (6.26) and (6.27) imply that \( K^a_{IJ} \) also be pure boost with respect to \( n_I \)—i.e., that \( K^a_{IJ} \) have the form 
\[ K^a_{IJ} =: 2K^a_{IJ} n^J, \]  
with \( K^a_{IJ} n_I = 0 \). Since \( D_a \) is determined completely by \( \tilde{E}^{a}_{I} \) and \( n_I \), the original 18 degrees of freedom for \( A^a_{IJ} \) has also been reduced to 9 degrees of freedom per space point. The information contained in \( A^a_{IJ} \) (which is independent of \( \tilde{E}^{a}_{I} \) and \( n_I \)) is completely characterized.
by $K^J_a$. To emphasize the fact that $\tilde{E}_i^a n^I = 0$ and $K^J_a n_I = 0$, we will use a 3-dimensional abstract internal index $i$ and write $\tilde{E}_i^a$ and $K^i_a$ in what follows.

Thus, after eliminating the 2nd class constraints, the phase space $(\Gamma_P, \Omega_P)$ of the 3+1 Palatini theory is coordinatized by the pair $(\tilde{E}_i^a, K^i_a)$ and has the symplectic structure

$$\Omega_P = \int_\Sigma dI K^i_a \wedge d\tilde{E}_i^a.$$ (6.29)

The Hamiltonian is given by

$$H_P(\tilde{E}, K) = \int_\Sigma \frac{1}{2} \mathcal{N} \left(- qR - 2\tilde{E}_i^a \tilde{E}_j^b K^i_a K^j_b \right) - 2N^a \tilde{E}_i^b D_{[a} K^i_{b]}$$

$$+ (qA \cdot t)^{ij} \tilde{E}_i^a K^a_{ij},$$ (6.30)

where $R$ denotes the scalar curvature of $D_a$. This is just a sum of the 1st class constraints functions associated with

$$\tilde{C}(\tilde{E}, K) := -qR - 2\tilde{E}_i^a \tilde{E}_j^b K^i_a K^j_b \approx 0,$$ (6.31)

$$\tilde{C}_a(\tilde{E}, K) := 4\tilde{E}_i^a D_{[a} K^i_{b]} \approx 0, \quad \text{and}$$ (6.32)

$$\tilde{G}_{ij}(\tilde{E}, K) := -\tilde{E}_i^a K^a_{ij} \approx 0.$$ (6.33)

(The overall numerical factors have been chosen in order to facilitate the comparison with the standard Einstein-Hilbert theory.) Note that constraint equations (6.31), (6.32), and (6.33) are the remaining constraints (6.17), (6.18), and the rotation part of (6.19) relative to $n_I$ expressed in terms of the phase space variables $(\tilde{E}_i^a, K^i_a)$. We will call (6.31), (6.32), and (6.33) the scalar, vector, and Gauss constraints for the 3+1 Palatini theory.

Note that as a consequence of eliminating the 2nd class constraints by solving (6.16), (6.20), and the boost part of (6.19) relative to $n_I$, the constraint equations (6.31)-(6.33) (and hence the evolution equations generated by the Hamiltonian) are now non-polynomial in the canonically conjugate pair $(\tilde{E}_i^a, K^i_a)$. This is due to the dependence of $R$ on the inverse of $\tilde{E}_i^a$. In fact, since $\tilde{E}_i^a$ must be invertible, we are forced to take $\tilde{E}_i^a$ as the configuration variable of the theory. We are led back to a geometrodynamical description of 3+1 gravity. Thus, the Hamiltonian formulation of the 3+1 Palatini theory has the same drawback as the Hamiltonian formulation of standard Einstein-Hilbert theory. As we shall see in the next subsection, these theories are effectively the same.

6.3 Relationship to the Einstein-Hilbert theory

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The rotation part of any anti-symmetric tensor $A_{IJ}$ relative to $n_I$ is given by $q^M_i q^N_j A_{MN}$ where $q^I_j := \delta^I_j + n^I n_j$. 

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In this subsection, we will not explicitly evaluate the Poisson bracket algebra of the constraint functions for the 3+1 Palatini theory. Rather, we will describe the relationship between the constraint equations (6.31)-(6.33) and those of the standard Einstein-Hilbert theory. We shall see that if we solve the first class constraint (6.33) by passing to a reduced phase space, we recover the Hamiltonian formulation of the standard Einstein-Hilbert theory in terms of the induced metric $q_{ab}$ and its canonically conjugate momentum $\tilde{p}^{ab}$.

To do this, let us first define a tensor field $\mathcal{M}_{ab}$ (of density weight -1 on $\Sigma$) via

$$\mathcal{M}_{ab} := K^{i}_{a} E_{bi},$$

(6.34)

where $E^{i}_{a}$ is the inverse of the (densitized) triad $\tilde{E}^{a}_{i}$. Then in terms of $\mathcal{M}_{ab}$, one can show that constraint equation (6.33) is equivalent to

$$\mathcal{M}_{[ab]} \approx 0.$$  

(6.35)

Thus, the constraint surface in $\Gamma_{P}$ defined by (6.33) will be coordinatized in part by the symmetric part of $\mathcal{M}_{ab}$—i.e., by $\tilde{K}_{ab} := \mathcal{M}_{(ab)}$.

But we are not yet finished. Since (6.33) is a 1st class constraint, we must also factor-out the constraint surface by the orbits of the Hamiltonian vector field associated with the constraint function

$$G(\Lambda) := \int_{\Sigma} \tilde{G}_{ij}(\tilde{E}, K) \Lambda^{ij} \left( = \int_{\Sigma} -\tilde{E}_{i}^{a} K^{b}_{aj} \Lambda^{ij} \right).$$

(6.36)

(Here $\Lambda^{ij} = \Lambda^{[ij]}$ denotes an arbitrary anti-symmetric test field on $\Sigma$.) Since it is fairly easy to show that $G(\Lambda)$ generates (gauge) rotations of the internal indices (i.e., $\tilde{E}_{i}^{a} \mapsto \tilde{E}_{i}^{a} + \epsilon \Lambda_{ij} \tilde{E}_{j}^{a} + O(\epsilon^{2})$ and $K^{i}_{a} \mapsto K^{i}_{a} - \epsilon \Lambda^{ij} K^{j}_{a} + O(\epsilon^{2})$), the factor space will be coordinatized by $\tilde{K}_{ab}$ and the gauge invariant information contained in $\tilde{E}_{i}^{a}$. This is precisely $\tilde{q}^{ab} = \tilde{E}_{i}^{a} \tilde{E}^{bi}$. Thus, the reduced phase space $(\hat{\Gamma}_{P}, \hat{\Omega}_{P})$ is coordinatized by the pair $(\tilde{q}^{ab}, \tilde{K}_{ab})$ and has symplectic structure

$$\hat{\Omega}_{P} = \frac{1}{2} \int_{\Sigma} dI \tilde{K}_{ab} \wedge dI \tilde{q}^{ab}.$$  

(6.37)

All we must do now is make contact with the usual canonical variables of the standard Einstein-Hilbert theory. To do this, let us work with the undensitized fields $q^{ab}$ and $K_{ab}$, and lower and raise their indices, respectively. Then in terms of $\bar{p}^{ab}$ defined by

$$\bar{p}^{ab} := \sqrt{q}(K^{ab} - Kq^{ab}),$$

(6.38)

one can show that

$$\hat{\Omega}_{P} = -\frac{1}{2} \int_{\Sigma} dI \bar{p}^{ab} \wedge dI q_{ab}.$$  

(6.39)

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and
\[
\overline{C}(q, \tilde{p}) = -q R + \left( \tilde{p}^{ab} \tilde{p}_{ab} - \frac{1}{2} \tilde{p}^2 \right) \approx 0, \quad (6.40)
\]
\[
\overline{C}_a(q, \tilde{p}) = -2q_{ab} D_c \tilde{p}^{bc} \approx 0. \quad (6.41)
\]

Up to overall factors, these are just the symplectic structure \( \Omega_{EH} \) and scalar and vector constraint equations of the standard Einstein-Hilbert theory described in Section 2. The factor of \(-1/2\) in the symplectic structure is due to the combination of using an action which is \(1/2\) the standard Einstein-Hilbert action and using a (densitized) triad \( \tilde{E}_a^i \) instead of a covariant metric \( q_{ab} \) as our basic dynamical variables. Thus, we see that the Hamiltonian formulation of the 3+1 Palatini theory is nothing more than the familiar geometrodynamical description of general relativity.

7. Self-dual theory

In this section, we will describe the self-dual theory of 3+1 gravity. This theory is similar in form to the 3+1 Palatini theory described in the previous section; however, it uses a self-dual connection 1-form as one of its basic variables. We define the self-dual action for complex 3+1 gravity and show that we still recover the standard vacuum Einstein’s equation even though we are using only half of a Lorentz connection. We then perform a Legendre transform to put the theory in Hamiltonian form. In terms of the resulting complex phase space variables, all equations of the theory are polynomial. This simplification gives the self-dual theory a major advantage over the 3+1 Palatini theory. As noted in the previous section, the Hamiltonian formulation of the 3+1 Palatini theory reduces to that of the standard Einstein-Hilbert theory with its troublesome non-polynomial constraints.

As mentioned in footnote 3 in Section 1, to obtain the phase space variables for the real theory, we must impose reality conditions to select a real section of the original complex phase space. At the end of subsection 7.2 we will describe these conditions and discuss how they are implemented. The need to use reality conditions is a necessary consequence of using an action principle to obtain the new variables for real 3+1 gravity. Although we mention here that it is possible to stay within the confines of the real theory by performing a canonical transformation on the standard phase space of real general relativity, we will not follow that approach in this paper. (Interested readers should see [1] for a detailed discussion of Ashtekar’s original approach.) Rather, we will start with an action for the complex theory and obtain the new variables as outlined above. Henceforth, the co-tetrad \( \xi^J_a \) will be assumed to be complex unless explicitly stated otherwise.
To write down the self-dual action for complex 3+1 gravity, all we have to do is replace the (Lorentz) connection 1-form $A_a^I$ of the 3+1 Palatini theory by a self-dual connection 1-form $+4A_a^I$ and let the co-tetrad $\epsilon_a^I$ become complex. We define the self-dual action to be

$$S_{SD}(\epsilon, +4A) := \frac{1}{4} \int_M \tilde{\eta}^{abcd} \epsilon_{IJKL} \epsilon_a^I \epsilon_b^J + 4F_{cd}^\pi_{KL},$$

where $+4F_{abl}^J = 2\partial_a [+4A_b^I] + [+4A_a^I, +4A_b^I]$ is the internal curvature tensor of the self-dual generalized derivative operator $+4D_a$ defined by

$$+4D_a k_I := \partial_a k_I + +4A_a^J k_J.$$

Some remarks are in order:

1. We will always take our spacetime manifold $M$ to be a real 4-dimensional manifold. Complex tensors at a point $p \in M$ will take values in the appropriate tensor product of the complexified tangent and cotangent spaces to $M$ at $p$. The fixed internal space will also be complexified; however, the internal Minkowski metric $\eta_{IJ}$ will remain real. Since the co-tetrad $\epsilon_a^I$ are allowed to be complex, the spacetime metric $g_{ab}$ defined by $g_{ab} := \epsilon_a^I \epsilon_b^J \eta_{IJ}$ will also be complex.

2. Although we can no longer talk about the signature of a complex metric $g_{ab}$, compatibility with a complex co-tetrad $\epsilon_a^I$ still defines a unique, torsion-free generalized derivative operator $\nabla_a$. Thus, the complex Einstein tensor $G_{ab} := R_{ab} - \frac{1}{2} R g_{ab}$ is well-defined; whence the complex equation of motion $G_{ab} = 0$ makes sense. It is this equation that defines for us the complex theory of 3+1 gravity.

3. When we say that the connection 1-form $+4A_a^I$ (or any other generalized tensor field) is self-dual, we will always mean with respect to its internal indices. Thus, the notion of self-duality makes sense only in 3+1 dimensions and applies only to generalized tensor fields with a pair of skew-symmetric internal indices, $T^{abc} c\cdots d IJ = T^{abc} c\cdots d[IJ]$. The dual of $T^{abc} c\cdots d IJ$, denoted by $T^{abc} c\cdots d IJ$, is defined to be

$$T^{abc} c\cdots d IJ := \frac{1}{2} \epsilon_{IJKL} T^{abc} c\cdots d KL,$$

where the internal indices of $\epsilon_{IJKL}$ are raised with the internal metric $\eta^{IJ}$. Since $\eta_{IJ}$ has signature $(- + + +)$, it follows that the square of the duality operator is minus the identity. Hence, our definition of self-duality involves the complex number $i$. We
define $T^{a\ldots b}_{c\ldots dIJ}$ to be self-dual if and only if

$$T^{a\ldots b}_{c\ldots dIJ} = iT^{a\ldots b}_{c\ldots dIJ}. \quad (7.4)$$

Thus, self-dual fields in Lorentzian 3+1 gravity are necessarily complex.

4. Given any generalized tensor field $T^{a\ldots b}_{c\ldots dIJ} = T^{a\ldots b}_{c\ldots dIJ}$, we can always decompose it as

$$T^{a\ldots b}_{c\ldots dIJ} = +T^{a\ldots b}_{c\ldots dIJ} + -T^{a\ldots b}_{c\ldots dIJ}, \quad (7.5)$$

where

$$+T^{a\ldots b}_{c\ldots dIJ} := \frac{1}{2}(T^{a\ldots b}_{c\ldots dIJ} - iT^{a\ldots b}_{c\ldots dIJ}) \quad \text{and} \quad (7.6a)$$

$$-T^{a\ldots b}_{c\ldots dIJ} := \frac{1}{2}(T^{a\ldots b}_{c\ldots dIJ} + iT^{a\ldots b}_{c\ldots dIJ}). \quad (7.6b)$$

Since $\ast(+T^{a\ldots b}_{c\ldots dIJ}) = i+T^{a\ldots b}_{c\ldots dIJ}$ and $\ast(-T^{a\ldots b}_{c\ldots dIJ}) = -i-T^{a\ldots b}_{c\ldots dIJ}$, it follows that $+T^{a\ldots b}_{c\ldots dIJ}$ and $-T^{a\ldots b}_{c\ldots dIJ}$ are the self-dual and anti self-dual parts of $T^{a\ldots b}_{c\ldots dIJ}$. Equations (7.6a) and (7.6b) define the self-duality and anti self-duality operators $+$ and $-$.

5. The generalized derivative operator $^{+4}D_a$ is said to be self-dual only in the sense that it is defined in terms of a self-dual connection 1-form $^{+4}A_{aIJ}$. As in many of the previous theories, $^{+4}D_a$ (as defined by (7.2)) knows how to act only on internal indices. But as usual, we will often find it convenient to consider a torsion-free extension of $^{+4}D_a$ to spacetime tensor fields. All calculations and all results will be independent of this choice of extension. Note also that the internal curvature tensor of the generalized derivative operator $^{+4}D_a$ is given by

$$^{+4}F_{abIJ} = 2\partial_a [^{+4}A_bIJ] + [^{+4}A_a, ^{+4}A_b]J. \quad (7.7)$$

Since one can show that the (internal) commutator of two self-dual fields is also self-dual, it follows that $^{+4}F_{abIJ}$ is self-dual with respect to its internal indices.

Given these general remarks, we are now ready to return to the self-dual action (7.1) and obtain the Euler-Lagrange equations of motion. Varying $S_{SD}(^{4}e, ^{+4}A)$ with respect to $^{4}e_a^I$ gives

$$\eta^{abcd} \epsilon_{IJKL} ^{+4}e_b^J ^{+4}F_{cd}^{KL} = 0, \quad (7.8)$$

where $^{23}T^{a\ldots b}_{c\ldots dIJ}$ is defined to be anti self-dual if and only if $T^{a\ldots b}_{c\ldots dIJ} = -iT^{a\ldots b}_{c\ldots dIJ}$. The choice of $+i$ for self-dual and $-i$ for anti self-dual is purely convention. I have chosen our conventions to agree with those of [3].
while varying $S_{SD}(\mathbf{e}, +4A)$ with respect to $+4A_{a}^{IJ}$ gives

$$+4\mathcal{D}_{b}\left(\left(\gamma^{e}ight) + \left(\gamma_{I}^{[a} \ 4 \gamma_{J]}^{b}\right)\right) = 0. \tag{7.9}$$

To obtain (7.9), we used the fact that $\Xi^{abcd} e_{IJKL} 4 c^{K} 4 e_{d}^{I} = 4(\gamma) 4 e^{[a} c^{b]} I J$ (where $\left(\gamma\right) := \sqrt{-g}$). Note also that we are forced to take the self-dual part of $4 c^{I} c^{J}$ since the variations $\delta +4A_{a}^{IJ}$ are required to be self-dual. This is the distinguishing feature between the self-dual and 3+1 Palatini equations of motion. Finally note that although (7.9) requires a torsion-free extension of $+4\mathcal{D}_{a}$ to spacetime tensor fields, it is independent of this choice since the left hand side is the divergence of a skew spacetime tensor density of weight +1 on $M$.

We would now like to show that (7.9) implies that $+4\mathcal{D}_{a}$ is the self-dual part of the unique, torsion-free generalized derivative operator $\nabla_{a}$ compatible with $\gamma_{a}^{I}$. But since we are working with self-dual fields, the argument used for the 2+1 and 3+1 Palatini theories does not yet apply. We will have to do some preliminary work before we can use those results.

If $\Gamma_{a}^{IJ}$ denotes the internal Christoffel symbol of $\nabla_{a}$, we define the self-dual part $+\nabla_{a}$ of $\nabla_{a}$ by

$$+\nabla_{a} k_{I} := \partial_{a} k_{I} + +\Gamma_{a}^{IJ} k_{J}, \tag{7.10}$$

where $+\Gamma_{a}^{IJ}$ is the self-dual part of $\Gamma_{a}^{IJ}$. The difference between $+4\mathcal{D}_{a}$ and $+\nabla_{a}$ is then characterized by a generalized tensor field $+4C_{a}^{IJ}$ defined by

$$+4\mathcal{D}_{a} k_{I} =: +\nabla_{a} k_{I} + +4C_{a}^{IJ} k_{J}. \tag{7.11}$$

Note that $+4C_{a}^{IJ}$ is self-dual as the notation suggests. In fact, $+4C_{a}^{IJ} = +4A_{a}^{IJ} - +\Gamma_{a}^{IJ}$. Now let us write (7.9) in terms of $+\nabla_{a}$ and $+4C_{a}^{IJ}$. Using (7.11) to expand the left hand side of (7.9), we get

$$+\nabla_{b}\left(\left(\gamma^{e}\right) + \left(\gamma_{I}^{[a} \ 4 \gamma_{J]}^{b}\right)\right) + +\nabla_{b}\left(4 c^{K} + \left(4 c_{K}^{[a} 4 \gamma_{J]}^{b}\right)\right) + +\nabla_{b}\left(4 c_{K}^{[a} 4 \gamma_{J]}^{b}\right) = 0. \tag{7.12}$$

Since $+\nabla_{a} k_{I} = \partial_{a} k_{I} - +\Gamma_{a}^{IJ} k_{J}$ (where $+\Gamma_{a}^{IJ}$ is the anti self-dual part of the internal Christoffel symbol $\Gamma_{a}^{IJ}$), and since the last two terms of (7.12) can be written as the (internal) commutator of $+4C_{a}^{IJ}$ and $+\left(4 c_{I}^{[a} 4 \gamma_{J]}^{b}\right)$, we get

$$- \left[-4C_{b}, +\left(4 c_{I}^{[a} 4 \gamma_{J]}^{b}\right)\right]_{IJ} + \left[+4C_{b}, +\left(4 c_{I}^{[a} 4 \gamma_{J]}^{b}\right)\right]_{IJ} = 0. \tag{7.13}$$

The first commutator vanishes since $-4C_{b}^{IJ}$ is anti self-dual while $+\left(4 c_{I}^{[a} 4 \gamma_{J]}^{b}\right)$ is self-dual; in the second commutator, $+\left(4 c_{I}^{[a} 4 \gamma_{J]}^{b}\right)$ can be replaced by $4 c_{I}^{[a} 4 \gamma_{J]}^{b}$. Thus, (7.13) reduces to

$$\left[+4C_{b}, \left(4 c_{I}^{[a} 4 \gamma_{J]}^{b}\right)\right]_{IJ} = 0. \tag{7.14}$$
This is exactly the form of the equation found in the 2+1 Palatini theory with $+\mathcal{C}_{\mu I}^J$ replacing $\mathcal{C}_{\mu I}^J$. (See equation (3.20).) We can now follow the argument given there to conclude that $+\mathcal{C}_{\mu I}^J = 0$. Thus, $+\mathcal{A}_{\mu I}^J = +\mathfrak{T}_{\mu I}^J$ as desired.

By substituting the solution $+\mathcal{A}_{\mu I}^J = +\mathfrak{T}_{\mu I}^J$ into the remaining equation of motion (7.8), we get

$$\tilde{\eta}^{abcd}\varepsilon_{IJKL}\varepsilon_b^{IJ} + \tilde{R}_{cd}^{KL} = 0 \tag{7.15}$$

where $+\tilde{R}_{cd}^{KL}$ is the self-dual part of the internal curvature tensor $R_{cd}^{KL}$ of $\nabla_a$. Then by using the definition of $+\tilde{R}_{cd}^{KL}$, we see that equation (7.15) becomes

$$0 = \frac{1}{2}\tilde{\eta}^{abcd}\varepsilon_{IJKL}\varepsilon_b^{IJ}(R_{cd}^{KL} - \frac{1}{2}\varepsilon^{KL}_{MN}R_{cd}^{MN})$$

$$= \frac{1}{2}\tilde{\eta}^{abcd}\varepsilon_{IJKL}\varepsilon_b^{IJ}\tilde{R}_{cd}^{KL}, \tag{7.16}$$

where the second term on the first line vanishes by the Bianchi identity $R_{[abc]}^d = 0$. When (7.16) is contracted with $\varepsilon^a_c$, we get $G^{ae} = 0$. Thus, the self-dual action (7.1) reproduces the vacuum Einstein’s equation for complex 3+1 gravity.

Since this is an important—yet somewhat suprising—result, it is perhaps worthwhile to repeat the above argument from a slightly different perspective. First note that the self-dual action (7.1) and the 3+1 Palatini action (6.7) differ by a term involving the dual of the curvature tensor $4F_{cd}^{KL}$. This extra term in the self-dual action is not a total divergence and thus gives rise to an additional equation of motion that is not present in the 3+1 Palatini theory. This equation of motion also involves the dual of the curvature tensor. (Compare equations (7.8) and (6.8).) However, as we showed above, if we solve (7.9) for $+\mathcal{A}_{a I}^J$ and substitute the solution $+\mathcal{A}_{a I}^J = +\mathfrak{T}_{a I}^J$ back into (7.8), the additional equation of motion is automatically satisfied as a consequence of the Bianchi identity $R_{[abc]}^d = 0$. Hence, there are no “spurious” equations of motion. Moreover, since the self-dual and 3+1 Palatini actions differ by a term that is not a total divergence, the canonically conjugate variables for the two theories will disagree. As we shall see in the following section, it is this difference that will allow us to construct a Hamiltonian formulation of 3+1 gravity with a connection 1-form as the basic configuration variable.

Finally, we conclude this subsection by showing the relationship between the self-dual and standard Einstein-Hilbert actions. To do this, note that since the equation of motion (7.9) for $+\mathcal{A}_{a I}^J$ could be solved uniquely for $+\mathcal{A}_{a I}^J$ in terms of the remaining basic variables $\varepsilon_a^I$, we can pull-back the self-dual action $S_{SD}(\varepsilon, +\mathcal{A})$ to the solution space $+\mathcal{A}_{a I}^J = +\mathfrak{T}_{a I}^J$.
and obtain a new action $\mathcal{S}_{SD}(\kappa)$. Doing this, we find

$$
\mathcal{S}_{SD}(\kappa) = \frac{1}{4} \int_M \tilde{\eta}^{abcd} \epsilon_{IJKL} \frac{4}{a} \frac{4}{b} + R_{cd}^{KL},
$$

(7.17)

where we expanded $+R_{cd}^{KL}$ and used the Bianchi identity $R_{[abc]d} = 0$ to get the last line of (7.17). Thus, $\mathcal{S}_{SD}(\kappa)$ is just 1/2 times the standard Einstein-Hilbert action $S_{EH}(\kappa)$ viewed as a functional of a complex co-tetrad $\kappa_a^I$. (See equation (6.5).) In fact, $\mathcal{S}_{SD}(\kappa) = \mathcal{S}_P(\kappa)$, where $\mathcal{S}_P(\kappa)$ is the pull-back of the 3+1 Palatini action. It was precisely to obtain this last equality that we defined the self-dual action (7.1) with an overall factor of 1/4 rather than 1/8.

### 7.2 Legendre transform

To put the self-dual theory for complex 3+1 gravity in Hamiltonian form, we will basically proceed as we did in Section 6 for the 3+1 Palatini theory. However, since the spacetime metric $g_{ab} := \kappa_a^I \kappa_b^J \eta_{IJ}$ is now complex, we can only assume that $M$ is topologically $\Sigma \times R$ for some submanifold $\Sigma$ and assume that there exists a real function $t$ whose $t = \text{const}$ surfaces foliate $M$. (We cannot assume that $\Sigma$ is spacelike, since the signature of a complex metric is not defined.) We can still introduce a real flow vector field $t^a$ (satisfying $t^a(dt)_a = 1$) and a unit covariant normal $n_a$ to the $t = \text{const}$ surfaces satisfying $n_a n^a = 1$. (We are free to choose $-1$ for the normalization of $n_a$ since $n_a$ is allowed to be complex.) $n^a := g^{ab} n_b$ is the vector field associated to $n^a$, and is related to $t^a$ by a complex lapse $N$ and complex shift $N_a$ via $t^a = N n^a + N_a$, with $N^n n_a = 0$. Finally, the induced metric $q_{ab}$ on $\Sigma$ is given by $q_{ab} = g_{ab} + n_a n_b$.

Following the same steps that we used in the previous section for the 3+1 Palatini theory, we find that (modulo a surface integral) the Lagrangian $L_{SD}$ of the self-dual theory is given by

$$
L_{SD} = \int_\Sigma -N \text{Tr}(+\tilde{E}^a + \tilde{E}^b + F_{ab}) + N^a \text{Tr}(+\tilde{E}^b + F_{ab})
$$

$$
+ (+\tilde{E}^a_{IJ}) \mathcal{L}^a + A_a^{IJ} + (+\mathcal{D}_a + \tilde{E}^a_{IJ})(+A \cdot t)^{IJ}.
$$

(7.18)

Here $+\tilde{E}^a_{IJ}$ denotes the self-dual part of $\tilde{E}^a_{IJ} := \frac{1}{2} \epsilon_{IJKL} \tilde{\eta}^{abc} \frac{4}{b^K} \frac{4}{c^L}$. Note that the transition from the 3+1 Palatini Lagrangian to the self-dual Lagrangian can be made by simply replacing all of the real fields by their self-dual parts. The configuration variables of the theory are $(+A \cdot t)^{IJ}$, $N$, $N^a$, $+A_a^{IJ}$, and $+\tilde{E}^a_{IJ}$.

Now recall that for the real 3+1 Palatini theory, the configuration variable $\tilde{E}^a_{IJ}$ was not free to take on arbitrary values. From its definition in terms of the co-tetrad $\kappa_a^I$, we saw that

$$
\overline{\phi}^{ab} := \epsilon^{IJKL} \tilde{E}^a_{IJ} \tilde{E}^b_{KL} = 0 \quad \text{and} \quad \text{Tr}(\tilde{E}^a \tilde{E}^b) > 0.
$$

(7.19)
The second condition followed from the fact that \( \text{Tr}(\tilde{E}^a \tilde{E}^b) = 2\tilde{q}^{ab} (= 2qq^{ab}) \), where \( q^{ab} \) was the inverse of the induced positive-definite metric \( q_{ab} \) on \( \Sigma \). Taking the primary constraint (7.19) together with the 3+1 Palatini Lagrangian as the starting point for the Legendre transform, we found that the standard Dirac constraint analysis gave rise to additional constraints—one of which was 2nd class with respect to \( \tilde{\phi}^{ab} = 0 \). By solving this 2nd class pair, the remaining (1st class) constraints became non-polynomial and we were forced back to the usual geometrodynamical description of real 3+1 gravity.

Similarly, we must check to see if there are any primary constraints on the configuration variables of the self-dual theory. It turns out that although \( \tilde{\phi}^{ab} : = \epsilon^{IJKL} \tilde{E}^a_{IJ} \tilde{E}^b_{KL} = 0 \) still follows from the definition of \( \tilde{E}^a_{IJ} \) in terms of the complex co-tetrad \( 4^{a}_{I} \), it does not imply a constraint on the self-dual field \( +\tilde{E}^a_{IJ} \). Equation (7.19) may be viewed, instead, as a constraint on the anti self-dual field \( -\tilde{E}^a_{IJ} \). (Recall that for complex \( \tilde{E}^a_{IJ} \), \( -\tilde{E}^a_{IJ} \) is not necessarily the complex conjugate of \( +\tilde{E}^a_{IJ} \)). Thus, \( +\tilde{E}^a_{IJ} \) is free to take on arbitrary values, and the Legendre transform for the complex self-dual theory is actually fairly simple. By following the standard Dirac constraint analysis, we find that \( +\tilde{E}^a_{IJ} \) is the momentum canonically conjugate to \( +A^a_{IJ} \), while \( (\epsilon^{I}A^{I} \cdot t)^{IJ}, N, \text{ and } N^a \) play the role of Lagrange multipliers. The complex phase space \( (\mathcal{C} \Gamma_{SD}, \mathcal{C} \Omega_{SD}) \) is coordinatized by the pair of complex fields \( (+A^a_{IJ}, +\tilde{E}^a_{IJ}) \) and has the natural complex symplectic structure.

The Hamiltonian is given by

\[
H_{SD}(+A, +\tilde{E}) = \int_{\Sigma} N \text{Tr}(+\tilde{E}^a + \tilde{E}^b + F_{ab}) - N^a \text{Tr}(+\tilde{E}^b + F_{ab}) - (\epsilon^{IJKL} + \epsilon^{I}A^{I} \cdot t)^{IJ}.
\]  

As we shall see in the next subsection, this is just a sum of 1st class constraint functions associated with

\[
\text{Tr}(+\tilde{E}^a + \tilde{E}^b + F_{ab}) \approx 0, \quad (7.22)
\]
\[ \text{Tr}(\hat{\tilde{E}}^b + F_{ab}) \approx 0, \quad \text{and} \]
\[ +D_a + \hat{\tilde{E}}^a_{IJ} \approx 0. \]

Note that all the constraints (and hence the evolution equations) are \textit{polynomial} in the canonically conjugate pair \((+A_a^{IJ}, +\hat{\tilde{E}}^a_{IJ})\). This is a simplification that we found in the 2+1 Palatini theory, but lost in the 3+1 Palatini theory when we solved the 2nd class constraints. In fact, since the constraint equations never involve the inverse of \(+\hat{\tilde{E}}^a_{IJ}\), the above Hamiltonian formulation is well-defined even if \(+\hat{\tilde{E}}^a_{IJ}\) is non-invertible. Thus, we have a slight extension of complex general relativity. The self-dual theory makes sense even when the induced metric \(\tilde{q}^{ab} = \text{Tr}(\hat{\tilde{E}}^a + \hat{\tilde{E}}^b)\) becomes degenerate.

In order to make contact with the notation used in the literature (see, e.g., [3]), let us use the fact that the covariant normal \(n_a\) to \(\Sigma\) defines a unit internal vector \(n_I\) via \(n_I := n_a \epsilon^a_J\). One can then show that
\[ +b^{K}_{IJ} := -\frac{1}{2}(\epsilon^{K}_{IJ} + i q^K_I n_J) \]
is an isomorphism from the self-dual sub-Lie algebra of the complexified Lie algebra of \(SO(3,1)\) to the complexified tangent space of \(\Sigma\). (Here \(\epsilon_{IKL} := n^I \epsilon_{JKL}, q^K_I := \delta^K_I + n_I n^K, \) and \(n_I n^I = -1\).) It satisfies
\[ \ast (+b^{K}_{IJ}) := \frac{1}{2}(+b^{K}_{IJ} - i \epsilon^{K}_{IJ MN} +b^{K}_{MN}) = i +b^{K}_{IJ}, \]
\[ [+b^{I}, +b^{J}]_{MN} = \epsilon^{IJ}_{K} +b^{K}_{MN}, \quad \text{and} \]
\[ \text{Tr}(+b^{I} +b^{J}) := - +b^{I}_{MN} +b^{J MN} = -q^{IJ}. \]

The inverse of \(+b^{K}_{IJ}\) will be denoted by \(+b_{K}^{IJ}\), and is obtained by simply raising and lowering the indices of \(+b^{K}_{IJ}\) with the internal metric \(\eta_{IJ}\). Since \(n_K +b^{K}_{IJ} = 0\), we will use a 3-dimensional abstract internal index \(i\) and write \(+b_{I}^{IJ}\) and \(+b_{I}^{IJ}\) in what follows. From property (7.27), it follows that \(+b_{I}^{IJ}\) can actually be thought of as an isomorphism from the self-dual sub-Lie algebra of the complexified Lie algebra of \(SO(3,1)\) to the complexified Lie algebra of \(SO(3)\).

Given this isomorphism, we can now define a \(\mathcal{CL}_{SO(3)}\)-valued connection 1-form \(A_a^i\) and a \(\mathcal{CL}^*_{SO(3)}\)-valued vector density \(\tilde{E}^a_i\) via
\[ +A_a^{IJ} := A_a^i +b_{I}^{IJ} \quad \text{and} \quad +\hat{\tilde{E}}^a_{IJ} := -i \hat{\tilde{E}}_i^a +b_{I}^{IJ}. \]

A straightforward calculation then shows that\(^{\text{26}}\)
\[ +F_{ab}^{IJ} = (2 \partial_a A_b^i + \epsilon^{i}_{jk} A_j^a A_k^b) +b_{I}^{IJ} = : +F_{ab}^i +b_{I}^{IJ} \quad \text{and} \]

\(^{\text{26}}\)To obtain equation (7.30), I assume that the fiducial derivative operator \(\partial_a\) has been extended to act on \(\mathcal{CL}_{SO(3)}\)-indices in such a way that \(\partial_a +b_{I}^{IJ} = 0\).
Thus, $\tilde{E}^a_i$ is a complex (densitized) triad and $F^i_{ab}$ is the Lie algebra-valued curvature tensor of the generalized derivative operator $D_a$ defined by $D_a v^i := \partial_a v^i + \epsilon^{ijk} A^j_a v^k$.

In terms of $A^i_a$ and $\tilde{E}^a_i$, the complex symplectic structure $^C\Omega_{SD}$ becomes

$$ ^C\Omega_{SD} = -i \int_{\Sigma} dI \tilde{E}^a_i \wedge dI A^i_a, $$

so $-i \tilde{E}^a_i$ is the momentum canonically conjugate to $A^i_a$. The Hamiltonian (7.21) can be written as

$$ H_{SD}(A, \tilde{E}) = \int_{\Sigma} \frac{1}{2} N \epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j F^k_{ab} - iN^a \tilde{E}^b_i F^i_{ab} + i(D_a \tilde{E}^a_i)(A \cdot t)^i, $$

while the constraint equations (7.22)-(7.24) can be written as

$$ \epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j F^k_{ab} \approx 0, $$

$$ \tilde{E}^b_i F^i_{ab} \approx 0, \quad \text{and} $$

$$ D_a \tilde{E}^a_i \approx 0. $$

We will take the constraint equations in this form when we analyze the Poisson bracket algebra of the corresponding constraint functions in the following section.

So far, all of the discussion in this section has dealt with complex 3+1 gravity. In order to recover the real theory, we must now impose reality conditions on the complex phase space variables $(A^i_a, \tilde{E}^a_i)$ to select a real section of $(^C\Sigma_{SD}, ^C\Omega_{SD})$. To do this, recall that in terms of the standard geometrodynamic variables $(q_{ab}, \tilde{p}^{ab})$, one recovers real general relativity from the complex theory by requiring that $q_{ab}$ and $\tilde{p}^{ab}$ both be real. Since equation (7.31) tells us that $\tilde{E}^a_i \tilde{E}^b_i = \bar{q}^{ab} = qq^{ab}$, the condition that $q_{ab}$ be real can be conveniently expressed in terms of $\tilde{E}^a_i$ as

$$ \tilde{E}^a_i \tilde{E}^b_i \quad \text{be real}. $$

Since we will want to ensure that this reality condition be preserved under the dynamical evolution generated by the Hamiltonian, we must also demand that

$$ (\tilde{E}^a_i \tilde{E}^b_i)^* \quad \text{be real}. $$

Since in a 4-dimensional solution of the field equations $\tilde{p}^{ab}$ is effectively the time derivative of $q_{ab}$, requirement (7.38) is equivalent to the condition that $\tilde{p}^{ab}$ be real. In addition, since the Hamiltonian of the theory is just a sum of the constraints (7.34)-(7.36) (all of which are polynomial in the canonically conjugate variables), the reality conditions (7.37) and (7.38) are also polynomial in $(A^i_a, \tilde{E}^a_i)$. 

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Finally, to conclude this subsection, I should point out that the self-dual action (7.1) viewed as a functional of a self-dual connection 1-form $^{+4}A_{aI}^J$ and a real co-tetrad $^{c}e_{a}^I$ does not yield the new variables for real 3+1 gravity when one performs a 3+1 decomposition. The definition of the configuration variable $^{+}E_{aIJ}$ in terms of a real co-tetrad $^{c}e_{a}^I$ gives rise to a primary constraint. Although the non-holonomic constraint can be expressed in terms of $^{+}E_{aIJ}$ as

$$\text{Tr}(^{+}E_{a}^{+}E_{b}) > 0, \quad (7.39)$$

the holonomic constraint $\bar{\phi}^{ab} = 0$ cannot be expressed solely in terms of $^{+}E_{aIJ}$. For real $E_{aIJ}$ we have that $^{\bar{E}}_{aIJ}$ equals the complex conjugate of $^{+}E_{aIJ}$, so

$$\bar{\phi}^{ab} = \epsilon^{IJKL}(^{+}E_{aIJ} + \overline{^{+}E_{aKL}})(^{+}E_{bKL} + \overline{^{+}E_{bKL}}) = 0. \quad (7.40)$$

But by writing $\bar{\phi}^{ab} = 0$ in this way, we have destroyed the possibility of completing the standard Dirac constraint analysis. For nowhere in the analysis have we been told how to take Poisson brackets of the complex conjugate fields. The Legendre transform of the self-dual Lagrangian for real 3+1 gravity breaks down when we try to incorporate the primary constraints into the analysis.

7.3 Constraint algebra

Given constraint equations (7.34)-(7.36) for the complex self-dual theory, we would now like to verify the claim that their associated constraint functions form a 1st class set. To do this, let $v^i$ (which takes values in $CL_{SO(3)}$, $N$, and $N^a$ be arbitrary complex-valued test fields on $\Sigma$ and define

$$C(N) := \frac{1}{2} \int_{\Sigma} N^{ijk} E_{i}^{a} E_{j}^{b} F_{abk}, \quad (7.41)$$

$$C'(\bar{N}) := -i \int_{\Sigma} N^{a} \bar{E}_{i}^{b} F_{ab}, \quad \text{and} \quad (7.42)$$

$$G(v) := -i \int_{\Sigma} v^{i}(D_{a} \bar{E}_{i}^{a}). \quad (7.43)$$

These will be called the scalar, vector, and Gauss constraint functions. As the names and notation suggest, these constraint functions will play a similar role to the constraint functions defined in subsection 5.3. Many of the calculations and results found there will apply here as well.

As usual, it is fairly easy to show that the Gauss constraint functions generate the standard gauge transformations of the connection 1-form and rotation of internal indices. Since

$$\frac{\delta G(v)}{\delta E_{i}^{a}} = iD_{a}v^{i} \quad \text{and} \quad \frac{\delta G(v)}{\delta A_{i}^{a}} = -i\{v, \bar{E}_{i}^{a}\}_{i} \quad (:= -i\epsilon_{jki}v^{j}E_{k}^{a}), \quad (7.44)$$

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it follows that $A^i_a \mapsto A^i_a - \epsilon D_a v^i + O(\epsilon^2)$ and $\tilde{E}^a_i \mapsto \tilde{E}^a_i - \epsilon \{v, \tilde{E}^a_i\} + O(\epsilon^2)$. It also follows that

$$\{G(v), G(w)\} = G([v, w]),$$

(7.45)

where $[v, w]^i := \epsilon^j k v^j w^k$ is the Lie bracket of $v^i$ and $w^i$. Thus, the mapping $v \mapsto G(v)$ is a representation of the Lie algebra $CL_{SO(3)}$. Furthermore, given its geometrical interpretation as the generator of internal rotations, we have

$$\{G(v), C(N)\} = 0 \quad \text{and} \quad (7.46)$$

$$\{G(v), C'(\tilde{N})\} = 0,$$

(7.47)

as well.

Since it is possible to show that the vector constraint function does not by itself have any direct geometrical interpretation (see, e.g., [34]) we will define a new constraint function, $C(\tilde{N})$, by taking a linear combination of the vector and Gauss constraints. We define

$$C(\tilde{N}) := C'(\tilde{N}) - G(N),$$

(7.48)

where $N^i := N^a A^i_a$. We will call $C(\tilde{N})$ the \textit{diffeomorphism constraint function} since the motion it generates on phase space corresponds to the 1-parameter family of diffeomorphisms on $\Sigma$ associated with the vector field $N^a$. To see this, we can write

$$C(\tilde{N}) := C'(\tilde{N}) - G(N)$$

$$= -i \int_{\Sigma} N^a \tilde{E}^b_i F_{ab} + i \int_{\Sigma} N^i (D_a \tilde{E}^a_i)$$

$$= -i \int_{\Sigma} N^a (\tilde{E}^b_i F_{ab} - A^i_a D_b \tilde{E}^b_i)$$

$$= -i \int_{\Sigma} \tilde{E}^a_i \mathcal{L}_{\tilde{N}} A^i_a,$$

(7.49)

where the Lie derivative with respect to $N^a$ treats fields having only internal indices as scalars. To obtain the last line of (7.49), we ignored a surface integral (which would vanish anyways for $N^a$ satisfying the appropriate boundary conditions). By inspection, it follows that $A^i_a \mapsto A^i_a + \epsilon \mathcal{L}_{\tilde{N}} A^i_a + O(\epsilon^2)$, etc. Using this geometric interpretation of $C(\tilde{N})$, it follows that

$$\{C(\tilde{N}), G(v)\} = G(\mathcal{L}_{\tilde{N}} v),$$

(7.50)

$$\{C(\tilde{N}), C(M)\} = C(\mathcal{L}_{\tilde{N}} M),$$

and

$$\{C(\tilde{N}), C(\tilde{M})\} = C([\tilde{N}, \tilde{M}]).$$

(7.52)
We are left to evaluate the Poisson bracket \( \{C(\mathcal{N}), C(\mathcal{M})\} \) of two scalar constraints. Using

\[
\frac{\delta C(\mathcal{N})}{\delta E_i^a} = \mathcal{N} \epsilon^{ijk} \tilde{E}_j^b F_{abk} \quad \text{and} \quad \frac{\delta C(\mathcal{M})}{\delta A_i^a} = \epsilon_i^{jk} D_a(\mathcal{N} \tilde{E}_j^a \tilde{E}_k^b),
\]

it follows that

\[
\{C(\mathcal{N}), C(\mathcal{M})\} = \int \Sigma \frac{\delta C(\mathcal{N})}{\delta A_i^a} \frac{\delta C(\mathcal{M})}{\delta (-iE_i^a)} - (\mathcal{N} \leftrightarrow \mathcal{M})
\]

\[
= \int \epsilon_i^{mn} D^c(\mathcal{N} \tilde{E}_m^a \tilde{E}_n^b) M^{eijk} \tilde{E}_j^b F_{abk} - (\mathcal{N} \leftrightarrow \mathcal{M})
\]

\[
= \int \epsilon_i^{ijk} \epsilon_i^{mn} \tilde{E}_m^a \tilde{E}_n^b (M \partial_c \mathcal{N} - \mathcal{N} \partial_c M) F_{abk}.
\]

If we now use the fact that

\[
\epsilon_i^{ijk} \epsilon_i^{mn} = (\delta^{jm} \delta^{kn} - \delta^{jn} \delta^{km})
\]

(which is a property of the structure constants of \(SO(3)\)), we get

\[
\{C(\mathcal{N}), C(\mathcal{M})\} = C'(\vec{K}) \left( = C(\vec{K}) + G(K) \right),
\]

where \(K^a := \tilde{q}^{ab}(N \partial_b M - M \partial_b N)\) and \(\tilde{q}^{ab} = \tilde{E}_i^a \tilde{E}_i^b\). Thus, the constraint functions are closed under Poisson bracket—i.e., they form a 1st class set. Note, however, that since the vector field \(K^a\) depends on the phase space variable \(\tilde{E}_i^a\), the Poisson bracket (7.56) involves structure functions. The constraint functions do not form a Lie algebra.

8. 3+1 matter couplings

In this section, we will couple various matter fields to 3+1 gravity. We will repeat much of what we did in Section 5, but this time in the context of the 3+1 theory, and for a Yang-Mills field instead of a massless scalar field. In subsections 8.1 and 8.2, we couple a cosmological constant \(\Lambda\) and a Yang-Mills field to complex 3+1 gravity using an action principle and the self-dual action as our starting point. We shall show that the inclusion of these matter fields does not destroy the polynomial nature of the constraint equations. This is the main result. (As usual, reality conditions should be included to recover the real theory.) As I mentioned for the 2+1 theory, it is possible to couple other fundamental matter fields (e.g., scalar and Dirac fields) to 3+1 gravity in a similar fashion and obtain the same basic results. For a more detailed discussion of this and related issues, interested readers should see, e.g., [33].

8.1 Self-dual theory coupled to a cosmological constant
To couple a cosmological constant $\Lambda$ to complex 3+1 gravity via the self-dual action, we will start with the action

$$S_\Lambda({^4}_e, {^4}_A) := \frac{1}{4} \int_M \tilde{\eta}^{abcd} \epsilon_{IJKL} \, {^4}_e^I {^4}_e^J (\Lambda + {^4}_F^{KL} - \frac{\Lambda}{3!} {^4}_e^K {^4}_e^L).$$  \hspace{1cm} (8.1)

Here $+{^4}_F^{abI J} = 2 \partial_a {^4}_A^{bI} + [+{^4}_A^a, +{^4}_A^b]^I J$ is the internal curvature tensor of the self-dual generalized derivative operator $+{^4}_D_a$ defined by the self-dual connection 1-form $+{^4}_A^a I J$, and $+{^4}_e^I$ is a complex co-tetrad which defines a spacetime metric $g_{ab}$ via $g_{ab} := +{^4}_e^a I J \eta_{IJ}$. Note that $S_\Lambda({^4}_e, {^4}_A)$ is just a sum of the self-dual action

$$S_{SD}({^4}_e, {^4}_A) := \frac{1}{4} \int_M \tilde{\eta}^{abcd} \epsilon_{IJKL} \, {^4}_e^I {^4}_e^J (\Lambda + +{^4}_F^{KL} - \frac{\Lambda}{3!} {^4}_e^K {^4}_e^L),$$  \hspace{1cm} (8.2)

and a term proportional to the volume of the spacetime. In fact,

$$\frac{\Lambda}{4!} \int_M \tilde{\eta}^{abcd} \epsilon_{IJKL} \, {^4}_e^I {^4}_e^J {^4}_e^K {^4}_e^L = \Lambda \int_M \sqrt{-g},$$  \hspace{1cm} (8.3)

where $g$ is the determinant of the covariant metric $g_{ab}$.

To show that (8.1) reproduces the standard equation of motion,

$$G_{ab} + \Lambda g_{ab} = 0,$$  \hspace{1cm} (8.4)

for gravity coupled to the cosmological constant $\Lambda$, we will first vary (8.1) with respect to the self-dual connection 1-form $+{^4}_A^a I J$. Since the second term (8.3) is independent of $+{^4}_A^a I J$, we get

$$+{^4}_D_b \left( (+^4_a)^I \begin{pmatrix} +^4_b^a \\ +^4_b^b \\ +^4_b^c \\ +^4_b^d \end{pmatrix} \right) = 0,$$  \hspace{1cm} (8.5)

which is exactly the equation of motion we obtained in Section 7 for the vacuum case. Thus, just as we saw in subsection 7.1, $+{^4}_A^a I J = +{^4}_\Gamma^a I J$ where $+{^4}_\Gamma^a I J$ is the self-dual part of the internal Christoffel symbol of $\nabla_a$ (the unique, torsion-free generalized derivative operator compatible with the co-tetrad.) Since (8.5) can be solved uniquely for $+{^4}_A^a I J$ in terms of the remaining basic variables $+{^4}_e^I$, we can pull-back $S_\Lambda({^4}_e, {^4}_A)$ to the solution space $+{^4}_A^a I J = +{^4}_\Gamma^a I J$. We obtain a new action

$$S_\Lambda({^4}_e) = \frac{1}{4} \int_M \tilde{\eta}^{abcd} \epsilon_{IJKL} \, {^4}_e^I {^4}_e^J (\Lambda + +{^4}_R^{KL} - \frac{\Lambda}{3!} {^4}_e^K {^4}_e^L),$$  \hspace{1cm} (8.6)

where $+{^4}_R^{KL}$ is the self-dual part of the internal curvature tensor defined by $\nabla_a$. Then by using the Bianchi identity $R_{[abc]d} = 0$ for the first term and (8.3) for the second, we get

$$S_\Lambda({^4}_e) = \frac{1}{2} \int_M \sqrt{-g} (R - 2\Lambda).$$  \hspace{1cm} (8.7)
As mentioned in subsection 5.1, this is (up to an overall factor of 1/2) the action that one uses to obtain (8.4) starting from an action principle. This is the desired result.

To put this theory in Hamiltonian form, we proceed as in subsection 7.2. Recall that (modulo a surface integral) the Lagrangian $L_{SD}$ of the self-dual theory is given by

$$L_{SD} = \int_{\Sigma} -N \text{Tr} (+\tilde{E}^a + \tilde{E}^b + F_{ab}) + N^a \text{Tr} (+\tilde{E}^a + F_{ab}) + \left( + \tilde{E}_{a \, IJ}^a \right) \mathcal{L}_{\mathcal{I}} + A_a^{\, IJ} + \left( + \mathcal{D}_a + \tilde{E}_{a \, IJ}^a \right) \left( + 4A \cdot t \right)^{IJ},$$

(8.8)

where $+\tilde{E}_{a \, IJ}^a$ denotes the self-dual part of $\tilde{E}_{a \, IJKL}^a := \frac{1}{2} \epsilon_{IJKL} \tilde{\eta}^{abc} \epsilon_b^c K \epsilon_c^L$. By using the isomorphism between the self-dual sub-Lie algebra of the complexified Lie algebra of $SO(3,1)$ and the complexified Lie algebra of $SO(3)$, we can rewrite (8.8) as

$$L_{SD} = \int_{\Sigma} \frac{1}{2} N \epsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b F_{abk} + i N^a \tilde{E}_i^b F_{ab} - i \tilde{E}_i^a \mathcal{L}_{\mathcal{I}} A_i^a - i (\mathcal{D}_a \tilde{E}_i^a) (4A \cdot t)^a,$$

(8.9)

where $\tilde{E}_i^a$ is a complex (densitized) triad (i.e., $\bar{E}_i^a \bar{E}^{ba} = \bar{q}^{ab} (= qq^{ab})$) and $A_i^a$ is a connection 1-form on $\Sigma$ that takes values in the complexified Lie algebra of $SO(3)$.

By using the decomposition $\sqrt{-g} = N \sqrt{q} \, dt$ together with the fact that

$$\frac{1}{3!} q_{abc} \epsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b \tilde{E}_k^c = q,$$

(8.10)

one can similarly show that

$$\frac{\Lambda}{4!} \int_M \tilde{\eta}^{abcd} \epsilon_{IJKL} \epsilon_a^i \epsilon_b^j \epsilon_c^k \epsilon_d^L = \frac{\Lambda}{3!} \int dt \int_{\Sigma} N q_{abc} \epsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b \tilde{E}_k^c.$$ 

(8.11)

Thus, the Lagrangian $L_A$ for 3+1 gravity coupled to the cosmological constant $\Lambda$ via the self-dual action is given by

$$L_A = \int_{\Sigma} -N \left( \frac{1}{2} \epsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b F_{abk} + \frac{\Lambda}{3!} q_{abc} \epsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b \tilde{E}_k^c \right)$$

$$+ i N^a \tilde{E}_i^b F_{ab} - i \tilde{E}_i^a \mathcal{L}_{\mathcal{I}} A_i^a - i (\mathcal{D}_a \tilde{E}_i^a) (4A \cdot t)^a.$$ 

(8.12)

The configuration variables of the theory are $(4A \cdot t)^a$, $N$, $N^a$, $A_i^a$, and $\tilde{E}_i^a$.

By following the standard Dirac constraint analysis, we find (as in the vacuum case) that $-i \tilde{E}_i^a$ is the momentum canonically conjugate to $A_i^a$ while $(4A \cdot t)^a$, $N$, and $N^a$ play the role of Lagrange multipliers. The complex phase space and complex symplectic structure are the same as those found for the self-dual theory with $\Lambda = 0$, while the Hamiltonian is given by

$$H_A (A, \bar{E}) = \int_{\Sigma} N \left( \frac{1}{2} \epsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b F_{abk} + \frac{\Lambda}{3!} q_{abc} \epsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b \tilde{E}_k^c \right)$$

$$- i N^a \tilde{E}_i^b F_{ab} + i (\mathcal{D}_a \tilde{E}_i^a) (4A \cdot t)^a.$$ 

(8.13)
We shall see that this is just a sum of 1st class constraint functions associated with
\[
\frac{1}{2} e^{ijk} \tilde{E}_i^a \tilde{E}_j^b F_{abk} + \frac{\Lambda}{3!} \eta_{gbe} e^{ijk} \tilde{E}_i^a \tilde{E}_j^b \tilde{E}_k^c \approx 0, \quad (8.14)
\]
\[
\tilde{E}_i^b F_{ab}^i \approx 0, \quad \text{and} \quad (8.15)
\]
\[
\mathcal{D}_a \tilde{E}_i^a \approx 0. \quad (8.16)
\]
These are the constraint equations associated with the Lagrange multipliers \( \mathcal{N}, N^a, \) and \( ^t(A \cdot t) \), respectively. Note that they are polynomial in the canonically conjugate variables \( (A^a_i, \tilde{E}_i^a) \) even when \( \Lambda \neq 0 \). In fact, only constraint equation \( (8.14) \) differs from its \( \Lambda = 0 \) counterpart.

To conclude this subsection, we will verify the claim that the constraint functions associated with \( (8.14)-(8.16) \) form a 1st class set. Since the Gauss and diffeomorphism constraint functions associated with \( (8.16) \) and \( (8.15) \) will be the same as in subsection 7.3, we need only concentrate on the scalar constraint function
\[
C(\mathcal{N}) := \int \mathcal{N} \left( \frac{1}{2} e^{ijk} \tilde{E}_i^a \tilde{E}_j^b F_{abk} + \frac{\Lambda}{3!} \eta_{gbe} e^{ijk} \tilde{E}_i^a \tilde{E}_j^b \tilde{E}_k^c \right),
\]
(8.17)
Since
\[
\frac{\delta C(\mathcal{N})}{\delta \tilde{E}_i^a} = \mathcal{N} (e^{ijk} \tilde{E}_j^b F_{abk} + \frac{\Lambda}{2} \eta_{gbe} e^{ijk} \tilde{E}_j^b \tilde{E}_k^c) \quad \text{and} \quad \frac{\delta C(\mathcal{N})}{\delta A^a_i} = \epsilon_i^{jk} \mathcal{D}_b (\mathcal{N} \tilde{E}_j^a \tilde{E}_k^b),
\]
(8.18a)
(8.18b)
it follows that
\[
\{C(\mathcal{N}), C(M)\} = \int \mathcal{N} \left( \frac{\delta C(\mathcal{N})}{\delta A^a_i} \frac{\delta C(M)}{\delta (-\delta E^n_k)} - (\mathcal{N} \leftrightarrow \mathcal{M}) \right)
\]
\[
= \int \epsilon_i^{mn} \mathcal{D}_c (\mathcal{N} \tilde{E}_m^a \tilde{E}_n^c) \mathcal{M} (e^{ijk} \tilde{E}_j^b F_{abk} + \frac{\Lambda}{2} \eta_{gbe} e^{ijk} \tilde{E}_j^b \tilde{E}_k^d) - (\mathcal{N} \leftrightarrow \mathcal{M})
\]
\[
= \int \epsilon_i^{ijk} \epsilon_i^{mn} \left( \mathcal{M} \partial_c \mathcal{N} - \mathcal{N} \partial_c \mathcal{M} \right) (\tilde{E}_m^a \tilde{E}_n^c \tilde{E}_j^b F_{abk} + \frac{\Lambda}{2} q \epsilon_m \tilde{E}_n^c).
\]
(8.19)
If we again use the fact that the structure constants of \( SO(3) \) satisfy
\[
\epsilon^{ijk} \epsilon_i^{mn} = (\delta^{jm} \delta^{kn} - \delta^{jn} \delta^{km}),
\]
(8.20)
we get
\[
\{C(\mathcal{N}), C(\mathcal{M})\} = C'((\tilde{K}) \quad \left( = C(\tilde{K}) + G(K) \right),
\]
(8.21)
where \( K^a := \tilde{q}^{ab} (\mathcal{N} \partial_b \mathcal{M} - \mathcal{M} \partial_b \mathcal{N}) \) and \( \tilde{q}^{ab} = \tilde{E}_i^a \tilde{E}_j^b \) as before. Thus, the constraint functions are closed under Poisson bracket—i.e., they form a 1st class set. The Poisson bracket algebra
of the constraint functions is exactly the same as it was for the $\Lambda = 0$ case. In particular, since the vector field $K^a$ depends on the phase space variable $\bar{E}^a_l$, the constraint functions again do not form a Lie algebra.

8.2 Self-dual theory coupled to a Yang-Mills field

To couple a Yang-Mills field (with gauge group $G$) to complex 3+1 gravity via the self-dual action, we will start with the total action

$$S_T(\bar{e}, +4A) := S_{SD}(\bar{e}, +4A) + \frac{1}{2} S_{YM}(\bar{e}, +4A),$$

(8.22)

where $S_{SD}(\bar{e}, +4A)$ is the self-dual action (8.2) and $S_{YM}(\bar{e}, +4A)$ is the usual Yang-Mills action

$$S_{YM}(\bar{e}, +4A) := -\int_M \text{Tr}(\sqrt{-g} g^{ac} b^{bd} dB_{ab} \bar{F}_{cd}).$$

(8.23)

Here $S_{YM}(\bar{e}, +4A)$ is to be viewed as a functional of a co-tetrad $\bar{e}_I^a$ and a connection 1-form $\bar{A}_a^I$ which takes values in the Lie algebra of the gauge group $G$. $\text{Tr}$ denotes the trace operation in some representation of the Yang-Mills Lie algebra, and $\bar{F}_{ab} = 2\partial_{[a} + [4A_{a], 4A_{b]}$ is the (internal) curvature tensor of the generalized derivative operator $\bar{D}_a$ defined by $\bar{A}_a^I$. The additional factor of $1/2$ is needed in front of $S_{YM}(\bar{e}, +4A)$ so that the above definition of the total action will be consistent with the definition of $S_{SD}(\bar{e}, +4A)$. The Yang-Mills action depends on the co-tetrad $\bar{e}_I^a$ through its dependence on $\sqrt{-g}$ and $g^{ab}$, but is independent of the self-dual connection 1-form $+4A_{aIJ}$. As mentioned in Section 5, out of all the fundamental matter couplings, only the action for the Dirac field would depend on $+4A_{aIJ}$.

To show that (8.22) reproduces the standard Yang-Mills coupled to gravity equations of motion

$$\bar{D}_b(\sqrt{-g} \bar{F}^{ab}) = 0 \quad \text{and} \quad G^{ad} = 8\pi T^{ad}(YM),$$

(8.24)

where

$$T_{ab}(YM) := \frac{1}{4\pi} \text{Tr}(\bar{F}_{a}^c \bar{F}_{bc} - \frac{1}{4} g_{ab} \bar{F}_{cd} \bar{F}^{cd})$$

(8.25)

is the stress-energy tensor of the Yang-Mills field, we proceed as we did in the previous subsection. Since $S_{YM}(\bar{e}, +4A)$ is independent of $+4A_{aIJ}$, the variation of (8.22) with respect to $+4A_{aIJ}$ implies

$$+4D_b \left( (\bar{e}^c) + +4A_{aIJ} \right) = 0.$$  

(8.26)

$^{27}$Yang-Mills fields will be denoted by bold face stem letters and their (internal) Lie algebra indices will be suppressed. Throughout, we will assume that we have a representation of the Yang-Mills Lie algebra $L_G$ by linear operators (on some vector space $V$) with the trace operation $\text{Tr}$ playing the role of an invariant, non-degenerate bilinear form $k$.  

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As before, this tells us that $^{+4}{A}_a{}^J = ^+ {\Gamma}_{aJ}$. Recalling that the Bianchi identity $R_{[abc]d} = 0$ implies that the pull-back of $S_{SD}(t^c, ^+A)$ to the solution space $^{+4}{A}_a{}^J = ^+ {\Gamma}_{aJ}$ is just 1/2 times the standard Einstein-Hilbert action $S_{EH}(t^c)$ for complex 3+1 gravity, we obtain

$$S_T(t^c, ^4A) = \frac{1}{2} (S_{EH}(t^c) + S_{YM}(t^c, ^4A)).$$  (8.27)

This is (up to an overall factor of 1/2) the usual total action that one uses to couple a Yang-Mills field to gravity. If we now vary $S_T(t^c, ^4A)$ with respect to $^4A_a$, and contract the second equation with $^tc$, we recover (8.24). Note that to write the first equation in (8.24), we had to consider a torsion-free extension of $D_a$ to spacetime tensor fields. But since the left hand side is the divergence of a skew spacetime tensor density of weight +1 on $M$, it is independent of this choice.

To put this theory in Hamiltonian form, we need only decompose the Yang-Mills action $S_{YM}(t^c, ^4A)$ since the self-dual Lagrangian $L_{SD}$ is given by (8.9). Using $g^{ab} = q^{ab} - n^a n^b$ and $\sqrt{-g} = N \sqrt{q} \ dt$ it follows that

$$S_{YM}(t^c, ^4A) = \int dt \int_\Sigma \tr \left\{ - \frac{1}{4} N q^{-1/2} \gamma^{a} \gamma^{1} \gamma^{2} \gamma^{3} \gamma^{4} \gamma^{5} \gamma^{6} \gamma^{7} \gamma^{8} \times (L_{t^c} A_a - D_a(^4A \cdot t) + N^e \mathcal{F}_{ae}) (L_{t^c} B_b - D_b(^4A \cdot t) + N^d \mathcal{F}_{bd}) \right\},$$  (8.28)

where $\gamma^{a}_b := g^{ab} = (\bar{E}^a_i \bar{E}^b_i)$, $(^4A \cdot t) := t^a \ ^4A_a$, and $A_a := q^{ab} \ ^4A_b$. Here $\mathcal{F}_{ab} := q^{ab} \ ^4F_{cd}$ is the curvature tensor of the generalized derivative operator $D_a := q^{ab} \ ^4D_b$ on $\Sigma$ associated with $A_a$. If we now define the “magnetic field” of $A_a$ to be $B_{ab} := 2 \mathcal{F}_{ab} (= 2q^c q^d \ ^4F_{cd})$, we see that the Yang-Mills Lagrangian $L_{YM}$ is given by

$$L_{YM} = \int_\Sigma \tr \left\{ - \frac{1}{4} N q^{-1/2} \gamma^{a} \gamma^{1} \gamma^{2} \gamma^{3} \gamma^{4} \gamma^{5} \gamma^{6} \gamma^{7} \gamma^{8} \times (L_{t^c} A_a - D_a(^4A \cdot t) + \frac{1}{2} N^e \mathcal{F}_{ae}) (L_{t^c} B_b - D_b(^4A \cdot t) + \frac{1}{2} N^d \mathcal{B}_{bd}) \right\}.$$  (8.29)

The total Lagrangian $L_T$ is the sum $L_T = L_{SD} + \frac{1}{2} L_{YM}$ and is to be viewed as a functional of the configuration variables $(^4A \cdot t)$, $(^4A \cdot t)^i$, $N^a$, $A_a^i$, $\bar{E}_i^a$, $A_a$ and their first time derivatives.

Following the standard Dirac constraint analysis, we find that

$$\bar{E}_a := \frac{\delta L_T}{\delta (L_{t^c} A_a)} = 2 q^{ab} N^{-1} q^{-1} (L_{t^c} A_b - D_b(^4A \cdot t) + \frac{1}{2} N^d \mathcal{B}_{bd})$$  (8.30)

is the momentum (or “electric field”) canonically conjugate to $A_a$. Since this equation can be inverted to give

$$L_{t^c} A_a = \frac{1}{2} q_{ab} N \bar{E}_b + D_a(^4A \cdot t) - \frac{1}{2} N^e \mathcal{B}_{ae},$$  (8.31)
it does not define a constraint. On the other hand, \(-iE_i^a\) is constrained to be the momentum canonically conjugate to \(A_i^a\), while \((\mathbf{A} \cdot t)\), \((\mathbf{A} \cdot t)^i\), \(N\), and \(N^a\) play the role of Lagrange multipliers. The resulting complex total phase space \((\mathcal{C} \Gamma_T, \mathcal{C} \Omega_T)\) is coordinatized by the pairs of fields \((A_i^a, \tilde{E}_i^a)\) and \((A_a, \tilde{E}^a)\) with symplectic structure

\[
\mathcal{C} \Omega_T = \int_\Sigma -i d\tilde{E}_i^a \wedge dA_i^a + \text{Tr}(d\tilde{E}^a \wedge dA_a). \tag{8.32}
\]

The Hamiltonian is given by

\[
H_T(A, \tilde{E}, A, \tilde{E}) = \int_\Sigma \mathcal{N} \left( \frac{1}{2} \varepsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b F_{abk} + \frac{1}{8} q^{-1} \tilde{E}_a^c \tilde{E}_b^d \tilde{E}_c \tilde{E}_d \text{Tr}(B_{ab}B_{cd} + E_{ab}E_{cd}) \right) + N^a \left( -i \tilde{E}_i^b F_{ia} + \text{Tr}(\tilde{E}_i^b F_{ia}) \right) + i(D_a \tilde{E}_i^a)(4A \cdot t)^i - \text{Tr}((\mathbf{A} \cdot t)D_a \tilde{E}^a), \tag{8.33}
\]

where \(E_{ab} := g_{abc} \tilde{E}^c\) is the dual to the Yang-Mills “electric field” \(\tilde{E}^a\). We shall see that this is just a sum of 1st class constraint functions associated with

\[
\frac{1}{2} \varepsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b F_{abk} + \frac{1}{8} q^{-1} \tilde{E}_a^c \tilde{E}_b^d \tilde{E}_c \tilde{E}_d \text{Tr}(B_{ab}B_{cd} + E_{ab}E_{cd}) \approx 0, \tag{8.34}
\]

\[
-i \tilde{E}_i^b F_{ia} + \text{Tr}(\tilde{E}_i^b F_{ia}) \approx 0, \tag{8.35}
\]

\[
D_a \tilde{E}_i^a \approx 0, \quad \text{and} \quad D_a \tilde{E}^a \approx 0. \tag{8.36}
\]

These are the constraint equations associated with the Lagrange multipliers \(\mathcal{N}\), \(N^a\), \((\mathbf{A} \cdot t)^i\), and \((\mathbf{A} \cdot t)\), respectively.

Note that by inspection (8.35) and (8.36) are polynomial in the canonically conjugate variables. However, constraint equation (8.34) fails to be polynomial due to the presence of the non-polynomial multiplicative factor \(q^{-1}\). But since \(q = \frac{1}{3!} g_{abc} \varepsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b \tilde{E}_k^c\) is polynomial in \(\tilde{E}_i^a\), we can multiply (8.34) by \(q\) and restore the polynomial nature of all the constraints. Thus, to couple a Yang-Mills field to 3+1 gravity via the self-dual action, we are led to a scalar constraint with density weight +4. This implies that the associated constraint function will be labeled by a test field (i.e., lapse function) having density weight \(-3\).

To verify the claim that the constraint functions associated with (8.34)-(8.36) form a 1st class set, let \(v^i\) and \(\mathbf{v}\) (which take values in complexified Lie algebra of \(SO(3)\) and the representation of the Lie algebra of the Yang-Mills gauge group \(G\), \(\mathcal{N}\), and \(N^a\) be arbitrary complex-valued test fields on \(\Sigma\). Then define

\[
C(\mathcal{N}) := \int_\Sigma \mathcal{N} \left( \frac{1}{2} \varepsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b F_{abk} + \frac{1}{8} q^{-1} \tilde{E}_a^c \tilde{E}_b^d \tilde{E}_c \tilde{E}_d \text{Tr}(B_{ab}B_{cd} + E_{ab}E_{cd}) \right), \tag{8.37}
\]

\[
C'(\tilde{N}) := \int_\Sigma N^a \left( -i \tilde{E}_i^b F_{ia} + \text{Tr}(\tilde{E}_i^b F_{ia}) \right), \quad \text{and} \tag{8.38}
\]

\[
G(v, \mathbf{v}) := \int_\Sigma \text{Tr}(vD_a \tilde{E}^a) - iv^i(D_a \tilde{E}_i^a), \tag{8.39}
\]
to be the scalar, vector, and Gauss constraint functions.

As usual, it is fairly easy to show that the Gauss constraint functions generate the standard gauge transformations of the connection 1-forms and rotations of internal indices. Using this information, we find that

\[
\{G(v, v), G(w, w)\} = G([v, w], [v, w]), \tag{8.40}
\]

\[
\{G(v, v), C(N)\} = 0, \quad \text{and} \tag{8.41}
\]

\[
\{G(v, v), C'(\vec{N})\} = 0, \tag{8.42}
\]

where \([v, w]\) and \([v, w]^i\) are the Lie brackets in \(\mathcal{L}_G\) and \(\mathcal{C}_{SO(3)}\). Thus, the subset of Gauss constraint functions form a Lie algebra with respect to Poisson bracket. In fact, the mapping \((v, v) \mapsto G(v, v)\) is a representation of the direct sum Lie algebra \(\mathcal{L}_G \oplus \mathcal{C}_{SO(3)}\).

Again, the the vector constraint function will not have any direct geometrical interpretation, so we define the diffeomorphism constraint function \(C(\vec{N})\) by taking a linear combination of the vector and Gauss law constraints. Setting

\[
C(\vec{N}) := C'(\vec{N}) - G(N, N), \tag{8.43}
\]

where \(N := N^a A_a\) and \(N^i := N^a A^i_a\), we can show that

\[
C(\vec{N}) = \int_\Sigma -i \vec{E}_i^a \mathcal{L}_{\vec{N}} A^i_a + \text{Tr}(\vec{E}^a \mathcal{L}_{\vec{N}} A_a), \tag{8.44}
\]

where the Lie derivative with respect to \(N^a\) treats fields having only internal indices as scalars. By inspection, \(A^i_a \mapsto A^i_a + \epsilon \mathcal{L}_{\vec{N}} A^i_a + O(\epsilon^2), \quad \text{etc.}, \) so the motion on phase space generated by \(C(\vec{N})\) corresponds to the 1-parameter family of diffeomorphisms on \(\Sigma\) associated with \(N^a\). From this geometric interpretation of \(C(\vec{N})\), it follows that

\[
\{C(\vec{N}), G(v, v)\} = G(\mathcal{L}_{\vec{N}} v, \mathcal{L}_{\vec{N}} v), \tag{8.45}
\]

\[
\{C(\vec{N}), C(M)\} = C(\mathcal{L}_{\vec{N}} M), \quad \text{and} \tag{8.46}
\]

\[
\{C(\vec{N}), C(M)\} = C([\vec{N}, \vec{M}]). \tag{8.47}
\]

Finally, we are left to evaluate the Poisson bracket \(\{C(\vec{N}), C(M)\}\) of two scalar constraint functions. After a fairly long but straightforward calculation that uses the fact that the structure constants of \(SO(3)\) satisfy

\[
\epsilon^{ijk} \epsilon_{imn} = (\delta^{jm} \delta^{kn} - \delta^{jn} \delta^{km}), \tag{8.48}
\]

one can show that

\[
\{C(\vec{N}), C(M)\} = C'(\vec{K}) \quad (= C(\vec{K}) + G(K, K)), \tag{8.49}
\]

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where $K^a := \tilde{q}^{ab}(N\partial_b M - M\partial_b N)$ and $\tilde{q}^{ab} = \tilde{E}_a^b \tilde{E}^{ba}$. Thus, the constraint functions are again closed under Poisson bracket—i.e., they form a 1st class set. Just as we saw in subsection 8.1 for the cosmological constant $\Lambda$, the Poisson bracket algebra of the constraint functions for complex 3+1 gravity coupled to a Yang-Mills field via the self-dual action is exactly the same as it was for the vacuum case.

9. General relativity without-the-metric

To conclude this review, we will describe a theory of 3+1 gravity without a metric. This will complete the transition from geometrodynamics to connection dynamics in 3+1 dimensions. Although we saw in Section 7 that the Hamiltonian formulation of the self-dual theory for complex 3+1 gravity could be described in terms of a connection 1-form $A^a_i$ and its canonically conjugate momentum (or “electric field”) $\tilde{E}_a^i$, the action for the self-dual theory depended on both a self-dual connection 1-form $+4 A_a^i J^J$ and a complex co-tetrad $4^J_a$. Since the co-tetrad defines a spacetime metric $g_{ab}$ via $g_{ab} := \tilde{e}_a^I \tilde{e}_b^J \eta_{IJ}$, the self-dual action had an implicit dependence on $g_{ab}$. The purpose of this section is to show that (modulo an important degeneracy) complex 3+1 gravity can be described by an action which does not depend on a spacetime metric in any way whatsoever. We shall see in subsection 9.1 that this action depends only on a connection 1-form $4_A^i$ (which takes values in the complexified Lie algebra of $SO(3)$) and a scalar density $\Phi$ of weight -1 on $M$. Hence we obtain a pure spin-connection formulation of gravity. We shall also see how this pure spin-connection action is related to the self-dual action in the non-degenerate case.

In subsection 9.2, we will analyze the constraint equations for this theory. Since we will have shown in subsection 9.1 that the self-dual action and the pure spin-connection action are equivalent when the self-dual part of the Weyl tensor is non-degenerate, the constraint equations of this theory are the same as the the constraint equations for the self-dual theory found in subsection 7.2. However, we will now be able to write down the most general solution to the four diffeomorphism constraint equations (the scalar and vector constraints of the self-dual theory) when the “magnetic field” $\tilde{B}_i^a$ associated with the connection 1-form $A^i_a$ is non-degenerate. This is a new result for the Hamiltonian formulation of the 3+1 theory that was made manifest by working in the pure spin-connection formalism.

I should emphasize here that all of the results in this section are taken from previous work of Capovilla, Dell, Jacobson, Mason, and Plebanski. I am not adding anything new in this section; rather, I am reporting their results to bring the discussion of geometrodynamics versus connection dynamics for 3+1 gravity to its logical conclusion. Readers interested in a more detailed discussion of the general relativity without-the-metric theory (including matter
couplings and an extension of this theory to a class of generally covariant gauge theories) should see [10, 11, 12, 13, 14] and references mentioned therein. In addition, Peldán has recently provided a similar pure spin-connection formulation of 2+1 gravity. Interested readers should see [15].

9.1 A pure spin-connection formulation of 3+1 gravity

The pure spin-connection action for complex 3+1 gravity is defined to be

\[ S(\bar{\Phi}, A) := \frac{1}{8} \int_M \bar{\Phi} (\bar{\eta} \cdot 4F^i \wedge 4F^j)(\bar{\eta} \cdot 4F^k \wedge 4F^l) h_{ijkl}, \]  

(9.1)

where \( 4A_a^i \) is a connection 1-form which takes values in the complexified Lie algebra of \( SO(3) \), \( \bar{\Phi} \) is a scalar density of weight -1 on \( M \), and \( h_{ijkl} \) and \( (\bar{\eta} \cdot 4F^i \wedge 4F^j) \) are shorthand notations for

\[ h_{ijkl} := (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}) \quad \text{and} \]

\[ (\bar{\eta} \cdot 4F^i \wedge 4F^j) := \bar{\eta}^{abcd} 4F^i_{ab} 4F^j_{cd}. \]

(9.2)

(9.3)

As usual, \( 4F^i_{ab} = 2\partial[a 4A^i_b] + [4A^i_a, 4A^i_b] \) is the Lie algebra-valued curvature tensor of the generalized derivative operator \( 4D_a \) defined by

\[ 4D_a v^i := \partial_a v^i + [4A^i_a, v]^i, \]

(9.4)

where \( [4A^i_a, v]^i := \epsilon^i_{jk} 4A^j_a v^k \) denotes the Lie bracket of \( 4A^i_a \) and \( v^i \) in \( CL_{SO(3)} \). Although \( 4D_a \) defined by (9.4) knows how to act only on internal indices, we will often find it convenient to consider a torsion-free extension of \( 4D_a \) to spacetime tensor fields. All results and all calculations will be independent of this choice.

To show that the pure spin-connection action reproduces the standard results of complex 3+1 gravity, one could vary (9.1) with respect to \( \bar{\Phi} \) and \( 4A^i_a \) and analyze the resulting Euler-Lagrange equations of motion. Instead, we will start with the self-dual action

\[ S_{SD}(\bar{\epsilon}, 4A) := \frac{1}{4} \int_M \bar{\eta}^{abcd} \epsilon_{IJKL} 4\epsilon^I_a 4\epsilon^J_b 4F^c_{cd} k^{KL}, \]

(9.5)

for complex 3+1 gravity and show that (modulo an important degeneracy) the self-dual action (9.5) is actually equivalent to (9.1). Basically, we will eliminate from (9.5) the field variables which pertain to the spacetime metric by solving their associated Euler-Lagrange equations of motion. This will require that a certain symmetric trace-free tensor \( \psi_{ij} \) be invertible as a 3 \times 3 matrix. By substituting these solutions back into the original action (9.5), we will eventually obtain (9.1). We should point out that, in a solution, \( \psi_{ij} \) corresponds
to the self-dual part of the Weyl tensor associated with the connection 1-form $4A_a^I$. Thus, the equivalence between the two actions breaks down whenever the self-dual part of the Weyl tensor is degenerate. Note also that the pure spin-connection action describes complex 3+1 gravity. To recover the real theory, one would have to impose reality conditions similar to those used in Section 7 for the self-dual theory. For a detailed discussion of $\psi_{ij}$ and the reality conditions see, e.g., [11].

Since the self-dual action (9.5) depends on both a self-dual connection 1-form $+4A_a^I$ and a complex co-tetrad $+_4\epsilon_a^i$, it has an implicit dependence on the spacetime metric $g_{ab} := +_4\epsilon_a^i+\iota_i^K\eta_{IJ}$. Thus, it should come as no surprise that the first step in obtaining a metric-independent action for 3+1 gravity involves the elimination of $+_4\epsilon_a^i$ from (9.5). To do this, let us define

$$\Sigma_{abIJ} := \epsilon_{IJKL} +_4\epsilon_a^K +_b^L$$

(9.6)

and $+_4\Sigma_{abIJ}$ to be its self-dual part. Then we can write the self-dual action as

$$S_{SD}(^4\epsilon, +_4A) = \frac{1}{4} \int_M \tilde{\eta}^{abcd} +_4\Sigma_{abIJ} +_4 F_{cd}^{IJ},$$

(9.7)

where we have used the fact that $\Sigma_{abIJ} +_4 F_{cd}^{IJ} = +_4\Sigma_{abIJ} +_4 F_{cd}^{IJ}$. To simplify the notation somewhat, let us recall that the self-dual sub-Lie algebra of the complexified Lie algebra of $SO(3,1)$ is isomorphic to the complexified Lie algebra of $SO(3)$. Using the isomorphism described in Section 7, we can define a $\mathcal{CL}_{SO(3)}$-valued connection 1-form $+_4A_a^i$ and a $\mathcal{CL}_{SO(3)}^*$-valued 2-form $\Sigma_{abi}$ via

$$+_4A_a^I := +_4A_a^i +_b^I$$

and $+_4\Sigma_{abIJ} := \Sigma_{abi} +_b^I +_j^J.$

(9.8)

Then

$$S_{SD}(^4\epsilon, +_4A) = \frac{1}{4} \int_M \tilde{\eta}^{abcd} +_4\Sigma_{abi} +_4 F_{cd}^{i},$$

(9.9)

where $+_4F_{ab}^{ij} = 2\partial_a +_4A_b^i +_a^j +_b^i +_4A_a^k +_j^k$ is the Lie algebra-valued curvature tensor of the generalized derivative operator $+_4D_a$ defined by $+_4A_a^i$. It is related to $+_4F_{ab}^{IJ}$ via $+_4F_{ab}^{IJ} = +_4F_{ab}^{ij} +_b^i +_j^J.$

Although the right hand side of (9.9) involves just $\Sigma_{abi}$ and $+_4A_a^i$, the action is still a functional of $+_4A_a^i$ and $\psi_a^i$ since $\Sigma_{abi}$ depends on $\psi_a^i$ through equation (9.6). To eliminate $\psi_a^i$ from the action, we must use the result (see, e.g., [12]) that (9.6) holds for some $\psi_a^i$ if and only if the trace-free part of $\Sigma_i \wedge \Sigma^j$ equals zero—i.e., $\Sigma_{abIJ} = \epsilon_{IJKL} _4\epsilon_a^K +_d^L$ for some $\psi_a^i$ if and only if

$$\tilde{\eta}^{abcd}(\Sigma_{ab}^{ij} - \frac{1}{3} \delta^{ij} \Sigma_{cdk}^{ab}) = 0.$$  

(9.10)

28Recall that the self-dual part of $\Sigma_{abIJ}$ is defined by $+_4\Sigma_{abIJ} := \frac{1}{2}(\Sigma_{abIJ} - \frac{i}{2} \eta_{IJKL} \Sigma_{abKL}).$
Thus, the self-dual action can be viewed as a functional of $\Sigma_{ab\bar{i}}$ instead of $4^i_a$ provided we include in the action a term which gives back (9.10) as one of its Euler-Lagrange equations of motion. More precisely, let us define

$$S(\psi, \Sigma, 4A) := \frac{1}{4} \int_M \tilde{\eta}^{abcd}(\Sigma_{ab\bar{i}} 4F^i_{cd} - \frac{1}{2} \psi_{ij} \Sigma_{ab\bar{i}} \Sigma_{cd\bar{j}}),$$

(9.11)

where $\psi_{ij}$ is a symmetric trace-free tensor which will play the role of a Lagrange multiplier of the theory. Then the variation of $S(\psi, \Sigma, 4A)$ with respect to $\psi_{ij}$ will yield (9.10). Solving this equation and pulling-back the action (9.11) to this solution space gives back (9.9).

But instead of varying $S(\psi, \Sigma, 4A)$ with respect to $\psi_{ij}$, let us vary this action with respect to $\Sigma_{ab\bar{i}}$ and solve the resulting Euler-Lagrange equation of motion for $\Sigma_{ab\bar{i}}$ in terms of $\psi_{ij}$ and $4^i_a$. Varying (9.11) with respect to $\Sigma_{ab\bar{i}}$, we find

$$4F^i_{ab} - \psi^{ij} \Sigma_{abj} = 0,$$

(9.12)

where $\psi^{ij} := \delta^{ik} \delta^{jl} \psi_{kl}$. This equation can be solved for $\Sigma_{ab\bar{i}}$ in terms of the remaining field variables provided the inverse $(\psi^{-1})_{ij}$ of $\psi_{ij}$ exists. Assuming that it does, we get

$$\Sigma_{ab\bar{i}} = (\psi^{-1})_{ij} 4F^j_{ab},$$

(9.13)

If we now pull-back (9.11) to the solution space defined by (9.13), the resulting action becomes

$$S(\psi, 4A) = \frac{1}{8} \int_M \text{Tr}(4F^i_{ab} 4F^i_{cd}).$$

(9.14)

This is to be viewed as a functional of only the symmetric trace-free tensor $\psi_{ij}$ and the connection 1-form $4A^i_a$.

We are almost finished. What remains to be shown is that $\psi_{ij}$ can be eliminated from the action (9.14) in lieu of a scalar density $\bar{\Phi}$ of weight -1 on $M$. To do this, let us write the action in matrix notation and introduce another Lagrange multiplier $\tilde{\mu}$ to guarantee that $\psi_{ij}$ is trace-free. Then (9.14) can be written as

$$S(\tilde{\mu}, \psi, 4A) = \frac{1}{8} \int_M \text{Tr}(4F^i_{ab} 4F^i_{cd}).$$

(9.15)

where $\psi_{ij}$ is now assumed to be only symmetric (and invertible) and $\tilde{M}^{ij}$ is defined by

$$\tilde{M}^{ij} := \tilde{\eta}^{abcd} 4F^i_{ab} 4F^j_{cd}.$$

(9.16)

Varying $S(\tilde{\mu}, \psi, 4A)$ with respect to $\psi_{ij}$, we find

$$-\psi^{-1} \tilde{M} \psi^{-1} + \tilde{\mu} I = 0.$$

(9.17)

By introducing $\tilde{\mu}$, we can consider arbitrary symmetric variations of $\psi_{ij}$ rather than symmetric and trace-free variations.
By multiplying on the left and right by \( \psi \), we see that (9.17) is equivalent to
\[
\tilde{M} = \tilde{\mu} \psi^2.
\] (9.18)
This equation can be solved for \( \psi_{ij} \) in terms of \( \tilde{M}_{ij} \) and \( \tilde{\mu} \) provided the square-root of \( \tilde{M}_{ij} \) exists. Then
\[
\psi = \tilde{\mu}^{-1/2} \tilde{M}^{1/2},
\] (9.19)
so the action (9.15) pulled-back to this solution space equals
\[
S(\tilde{\mu}, 4^A) = \frac{1}{4} \int_M \tilde{\mu}^{1/2} \text{Tr} \tilde{M}^{1/2}.
\] (9.20)
The variation of \( S(\tilde{\mu}, 4^A) \) with respect to \( \tilde{\mu} \) now implies that \( \text{Tr} \tilde{M}^{1/2} = 0 \). From (9.19) we see that this is nothing more than the tracelessness of \( \psi_{ij} \).

In order to write the action in its final form (9.1), recall that the characteristic equation obeyed by any \( 3 \times 3 \) matrix is
\[
B^3 - (\text{Tr} B)B^2 + \frac{1}{2} \left( (\text{Tr} B)^2 - \text{Tr} B^2 \right) B - (\text{det} B) I = 0.
\] (9.21)
Multiplying by \( B \) and setting \( B^2 = \tilde{M} \) (i.e., \( B = \tilde{M}^{1/2} \)), we get
\[
(\text{det} B) B = \tilde{M}^2 - \frac{1}{2} (\text{Tr} \tilde{M}) \tilde{M},
\] (9.22)
(Here we have used the fact that \( \text{Tr} B = \text{Tr} \tilde{M}^{1/2} = 0 \).) Using \( (\text{det} B)^2 = \text{det} \tilde{M} \) and assuming invertibility of \( B \) (so that \( \text{det} B \neq 0 \)), we can write this last equation as
\[
B = (\text{det} \tilde{M})^{-1/2} (\tilde{M}^2 - \frac{1}{2} (\text{Tr} \tilde{M}) \tilde{M}).
\] (9.23)
By substituting this expression for \( B \) (\( = \tilde{M}^{1/2} \)) back into (9.20), we find
\[
S(\tilde{\mu}, 4^A) = \frac{1}{4} \int_M \tilde{\mu}^{1/2} (\text{det} \tilde{M})^{-1/2} \text{Tr} (\tilde{M}^2 - \frac{1}{2} (\text{Tr} \tilde{M}) \tilde{M}).
\] (9.24)
Finally, if we define
\[
\Phi = \tilde{\mu}^{1/2} (\text{det} \tilde{M})^{-1/2}
\] (9.25)
(which is a scalar density of weight -1 on \( M \)) and use the definition (9.16) of \( \tilde{M}^{ij} \) in terms of \( 4^F_{ab} \), we see that
\[
S(\Phi, 4^A) = \frac{1}{8} \int_M \Phi (\bar{\eta} \cdot 4^F^i \wedge 4^F^j)(\bar{\eta} \cdot 4^F^k \wedge 4^F^l) h_{ijkl}
\] (9.26)
when viewed as a functional of \( \Phi \) and \( 4^A_i \). Note that \( h_{ijkl} \) and \( (\bar{\eta} \cdot 4^F^i \wedge 4^F^j) \) are given as before by equations (9.2) and (9.3). This is the desired result.
9.2 Solution of the diffeomorphism constraints

Given that the self-dual and pure spin-connection actions are equivalent when the self-dual part of the Weyl tensor is non-degenerate, it follows that the constraint equations of the theory can be written as

\[ \epsilon^{ijk} \bar{E}_i^a \bar{E}_j^b F_{abk} \approx 0, \]  
(9.27)

\[ \bar{E}_i^b F_{ab}^i \approx 0, \quad \text{and} \]

\[ \mathcal{D}_a \bar{E}_i^a \approx 0. \]  
(9.28)

These are just the constraint equations that we found in subsection 7.2 when we put the self-dual theory in Hamiltonian form. As before, the canonically conjugate variables consist of a pair of complex fields \((A_i^a, \bar{E}_i^a)\), where \(A_i^a\) is the pull-back of the connection 1-form \(4A_i^a\) to the submanifold \(\Sigma\) and \(\bar{E}_i^a\) is a complex (densitized) triad which may or may not define an invertible induced metric \(\bar{q}^{ab} := \bar{E}_i^a \bar{E}_i^b\). However, by working in the pure spin-connection formalism, we will obtain a new result. We will be able to write down the most general solution to the four diffeomorphism constraints (9.27)-(9.28) when the “magnetic field” \(\bar{B}_i^a\) associated with \(A_i^a\) is non-degenerate.

To see this, recall that in the self-dual theory

\[ +\bar{E}_i^{aIJ} = -i \bar{E}_i^a + \psi_i^{aIJ}, \]  
(9.30)

where \(+\bar{E}_i^{aIJ}\) was the self-dual part of

\[ \bar{E}_i^{aIJ} := \frac{1}{2} \epsilon_{IJKL} \bar{q}^{abc} \bar{e}_K^a \bar{e}_L^b. \]  
(9.31)

Note that in terms of \(\Sigma_{abIJ}\) defined by (9.6), we have \(\bar{E}_i^{aIJ} = \frac{1}{2} \bar{q}^{abc} \Sigma_{bcIJ}\), so that

\[ -i \bar{E}_i^a = \frac{1}{2} \bar{q}^{abc} \Sigma_{bcI}. \]  
(9.32)

If we now use the result that an invertible symmetric trace-free tensor \(\psi_{ij}\) implies

\[ \Sigma_{ab} = (\psi^{-1})_{ij} 4F^{aj}_{ab}, \]  
(9.33)

it follows that

\[ \bar{E}_i^a = i(\psi^{-1})_{ij} \bar{B}_i^a, \]  
(9.34)

where \(\bar{B}_i^a := \frac{1}{2} \bar{q}^{abc} 4F_{bc}^i = \frac{1}{2} \bar{q}^{abc} F_{bc}^i\) is the “magnetic field” of \(A_i^a\). We will now show that by taking \(\bar{E}_i^a\) of this form, the four diffeomorphism constraints (9.27)-(9.28) are automatically satisfied.
Substituting (9.34) into the vector constraint (9.28), we get

$$\tilde{E}_i^b F_{i}^b = i(\psi^{-1})_{ij} \tilde{B}_{ij}^b \eta_{abc} \tilde{B}_c^i = 0,$$

(9.35)

where we have used the fact that $(\psi^{-1})_{ij}$ is symmetric in $i$ and $j$ while $\eta_{abc}$ is anti-symmetric in $b$ and $c$. Similarly, substituting (9.34) into the scalar constraint (9.27), we get

$$\varepsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b F_{abk} = \varepsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b \eta_{abc} \psi_{lk} \tilde{B}_c^k$$

$$= -i\varepsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b \eta_{abc} \psi_{lk} \tilde{B}_c^k$$

$$= -i\eta \varepsilon^{ijk} \psi_{lk}$$

$$= -2i\eta \psi_{lk}$$

$$= 0,$$

(9.36)

where we have used the fact that $\psi_{ij}$ is trace-free. Thus, the four diffeomorphism constraints (9.27)-(9.28) are automatically satisfied for $\tilde{E}_i^a$ having the form given by (9.34). That this is the most general solution follows if $\tilde{B}_i^a$ is non-degenerate. Then for a given $A_i^a$, $\tilde{E}_i^a$ will have 5 degrees of freedom (per space point) corresponding to the 5 degrees of freedom of the symmetric trace-free tensor $\psi_{ij}$.

What remains to be solved is the Gauss constraint (9.29), which in terms of $\tilde{B}_i^{ai}$ and $(\psi^{-1})_{ij}$ can be written as

$$0 = \mathcal{D}_a \tilde{E}_i^a$$

$$= i\mathcal{D}_a ((\psi^{-1})_{ij} \tilde{B}_i^{aj})$$

$$= i\tilde{B}_i^{aj} \mathcal{D}_a (\psi^{-1})_{ij}.$$  

(9.37)

To obtain the last line of (9.37), we used the Bianchi identity $\mathcal{D}_a \tilde{B}_i^{aj} = \frac{1}{2} \eta^{abc} \mathcal{D}_a F_{bc} = 0$.

10. Discussion

Let me begin by briefly summarizing the main results reviewed in this paper.

1. The standard Einstein-Hilbert theory is a geometrodynamical theory of gravity in which a spacetime metric is the fundamental field variable. The phase space variables consist of a positive-definite metric $q_{ab}$ and its canonically conjugate momentum $\tilde{p}^{ab}$. These variables are subject to a set of 1st class constraints, which are non-polynomial in $q_{ab}$ and which have a Poisson bracket algebra involving structure functions. This theory is valid in $n + 1$ dimensions.
2. The 2+1 Palatini theory is a connection dynamical theory defined for any Lie group $G$. The fundamental field variables consist of a $\mathcal{L}_G$-valued connection 1-form and a $\mathcal{L}_G^*$-valued covector field. The phase space is coordinatized by a connection 1-form $A_a^I$ and its canonically conjugate momentum (or “electric field”) $\tilde{E}_I^a$. These are fields defined on a 2-manifold $\Sigma$, and they are subject to a set of 1st class constraints. The constraints are polynomial in $(A_a^I, \tilde{E}_I^a)$ and provide a representation of the Lie algebra of the inhomogeneous Lie group associated with $G$. One recovers 2+1 gravity by taking $G = SO(2,1)$.

3. Chern-Simons theory is a connection dynamical theory defined for any Lie group that admits an invariant, non-degenerate bilinear form. In 2+1 dimensions, the fundamental field variable is a Lie algebra-valued connection 1-form, and the phase space variables are $A_i^a$—the pull-back of the field variable to the 2-dimensional hypersurface $\Sigma$. There are 1st class constraints, which are polynomial in $A_i^a$ and provide a representation of the defining Lie algebra. Chern-Simons theory is related to 2+1 Palatini theories as follows: (i) 2+1 Palatini theory based on any Lie group $G$ is equivalent to Chern-Simons theory based on the inhomogeneous Lie group $IG$ associated with $G$; and (ii) the reduced phase space of Chern-Simons theory based on a Lie group $G$ is the same as the reduced configuration space of the 2+1 Palatini theory based on the same $G$. As a special case of (i), 2+1 gravity is equivalent to Chern-Simons theory based on the 2+1 dimensional Poincaré group $ISO(2,1)$.

4. One can couple matter to 2+1 gravity via the 2+1 Palatini action. 2+1 Palatini theory coupled to a cosmological constant $\Lambda$ is defined for any Lie group $G$ that admits an invariant, totally antisymmetric tensor $\epsilon^{IJK}$. This theory is equivalent to 2+1 dimensional Chern-Simons theory based on the $\Lambda$-deformation of $G$. As a special case, 2+1 gravity coupled to a cosmological constant is equivalent to Chern-Simons theory based on $SO(3,1)$ or $SO(2,2)$ (depending on the sign of $\Lambda$). 2+1 Palatini theory can also be coupled to matter fields with local degrees of freedom provided $G = SO(2,1)$. The constraints remain polynomial in the canonically conjugate variables and form a 1st class set. However, due to the presence of structure functions, they no longer form a Lie algebra.

5. The 3+1 Palatini theory is a geometrodynamical theory of 3+1 gravity in which a co-tetrad and a Lorentz connection 1-form are the fundamental field variables. Due to the presence of 2nd class constraints, the Hamiltonian formulation of this theory reduces to that of the standard Einstein-Hilbert theory in 3+1 dimensions. Unlike the
2+1 Palatini theory, the 3+1 Palatini theory does not provide a connection dynamical theory of 3+1 gravity.

6. The self-dual theory is a connection dynamical theory of complex 3+1 gravity in which a complex co-tetrad and a self-dual connection 1-form are the fundamental field variables. The phase space variables consist of an $\mathbb{C}\mathcal{L}_{SO(3)}$-valued connection 1-form $A^i_a$ and its canonically conjugate momentum (or “electric field”) $\tilde{E}_i^a$, both defined on a 3-manifold $\Sigma$. These variables are subject to a set of 1st class constraints, which are polynomial in $(A^i_a, \tilde{E}_i^a)$ but which have a Poisson bracket algebra involving structure functions. In a solution, $\tilde{E}_i^a$ is a (densitized) spatial triad. Since none of the equations involve the inverse of $\tilde{E}_i^a$, the self-dual theory makes sense even if $\tilde{E}_i^a$ is non-invertible. Thus, the self-dual theory provides an extension of complex general relativity that is valid even when the induced spatial metric $\tilde{q}^{ab} (= \tilde{E}_i^a \tilde{E}_b^i)$ is degenerate. One must impose reality conditions to recover real general relativity.

7. One can couple matter to complex 3+1 gravity via the self-dual action. The constraints remain polynomial in the canonically conjugate variables and form a 1st class set. Since none of the equations involves the inverse of $\tilde{E}_i^a$, the self-dual theory coupled to matter provides an extension of complex general relativity coupled to matter that includes degenerate spatial metrics. Reality conditions must be imposed to recover the real theory.

8. The pure spin-connection formulation of general relativity is a connection dynamical theory of complex 3+1 gravity in which a $\mathbb{C}\mathcal{L}_{SO(3)}$-valued connection 1-form and a scalar density of weight $-1$ are the fundamental field variables. The Hamiltonian formulation of this theory is equivalent to that of the self-dual theory provided the self-dual part of the Weyl tensor is non-degenerate. When the “magnetic” field associated with the connection 1-form $A^i_a$ is non-degenerate, one can write down the most general solution to the four diffeomorphism constraints. Reality conditions must be imposed to recover the real theory.

So what can we conclude from all these results? Is gravity a theory of a metric or a connection? In other words, is gravity a theory of geometry, where the fundamental variable is a spacetime metric which specifies distances between nearby events, or is it a theory of curvature, where the fundamental variable is a connection 1-form which tells us how to parallel propagate vectors around closed loops? The answer: Either. As far as the classical equations of motion are concerned, both a metric and a connection describe gravity equally well in 2+1 and 3+1 dimensions. Neither metric nor connection is preferred.
As we have shown in this review and have summarized above, 2+1 and 3+1 gravity admit formulations in terms of metrics and connections. But despite the apparent differences (i.e., the different actions and field variables; the different Hamiltonian formulations and canonically conjugate momenta; and the possibility of extending the theories to include arbitrary gauge groups and solutions with degenerate spatial metrics), we have seen that the classical equations of motion for all these formulations are the same. For instance, we saw that the 2+1 Palatini theory reproduces vacuum 2+1 gravity when we choose \( G = SO(2, 1) \) and solve the equation of motion for the connection. Similarly, we saw that Chern-Simons theory reproduces 2+1 gravity coupled to a cosmological constant \( \Lambda (> 0) \) when we choose the gauge group to be \( SO(2, 2) \). At the level of field equations, all the theories are mathematically equivalent. The difference between the theories is, instead, one of emphasis.

Now such a small change may not seem, at first, to be worth all the effort. Recall that the shift in emphasis from metric to connection came only after we successively analyzed the Einstein-Hilbert, Palatini, and Chern-Simons theories in 2+1 dimensions, and the Einstein-Hilbert, Palatini, self-dual, and pure spin-connection theories in 3+1 dimensions. This analysis required a fair amount of work and, as we argued in the previous paragraph, did not lead to anything particularly new at the classical level modulo, of course, the extensions of the theories to include arbitrary gauge groups and solutions with degenerate spatial metrics. But as soon as we turn to quantum theory and consider the recent results that have been obtained there, the question as to whether the shift in emphasis from metric to connection was worth the effort has a simple affirmative answer. Yes! Indeed, almost all of the recent advances in quantum general relativity can be traced back to this change of emphasis. As mentioned in the introduction, Witten \[8\] used the equivalence of the 2+1 Palatini theory based on \( SO(2, 1) \) with Chern-Simons theory based on \( ISO(2, 1) \) to quantize 2+1 gravity. Others (e.g., Carlip \[24, 25\] and Anderson \[26\]) are now using Witten’s quantization to analyze the problem of time in the 2+1 theory. In 3+1 dimensions, Jacobson, Rovelli, and Smolin \[6, 7\] took advantage of the simplicity of the constraint equations in the self-dual formulation of 3+1 gravity to solve the quantum constraints exactly—something that nobody could accomplish for the quantum version of the scalar constraint in the traditional metric variables. And the list goes on. (See, e.g., \[2, 3\] and \[17, 18, 19, 20\] for more details.) Where this list will end, and whether or not the change in emphasis from metric to connection will lead to a mathematically consistent and physically reasonable quantum theory of 3+1 gravity, remains to be seen.

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