Entanglement Rényi $\alpha$-entropy

Yu-Xin Wang,\(^1\) Liang-Zhu Mu,\(^1\) Vlatko Vedral,\(^2\) and Heng Fan\(^3\),\(^4\)

\(^1\)School of Physics, Peking University, Beijing 100871, China
\(^2\)Department of Atomic & Laser Physics, Clarendon Laboratory, University of Oxford, Parks Road, Oxford OX1 3PU, UK
\(^3\)Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117543 and Department of Physics, National University of Singapore, 2 Science Drive 3, Singapore 117542
\(^4\)Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China

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Entanglement can be well quantified by Rényi $\alpha$-entropy which is a generalization of the standard von Neumann entropy. Here we study the measure of entanglement Rényi $\alpha$-entropy for arbitrary two-qubit states. We show that entanglement of two states may be incomparable, contrary to other well-accepted entanglement measures. These facts impose constraint on the convertibility of entangled states by local operations and classical communication. We find that when $\alpha$ is larger than a critical value, the entanglement measure by Rényi $\alpha$-entropy is determined solely by concurrence which is a well accepted measure of entanglement. When $\alpha$ is small, the entanglement Rényi $\alpha$-entropy of Werner state is obtained. Interestingly, we show that entanglement Rényi $\alpha$-entropy of Werner state is always less than any pure entangled state when $\alpha$ is close to zero, even this Werner state is close to a maximally entangled state and the concurrence is larger. We also conclude that the optimal decomposition of a general mixed state cannot be the same for all $\alpha$. 

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Introduction.— Entanglement is a valuable resource for quantum information processing \(\cite{1}\). Quantification of entanglement is therefore a fundamental problem in quantum information science and quantum physics. Various measures of entanglement have been proposed such as for instance, entanglement of formation (EoF), entanglement of distillation, relative entropy of entanglement \(\cite{2,3}\), etc. These measures have different, yet closely related, physical interpretations. In general, they can be associated respectively with different protocols for quantum information processing. Several well-accepted measures of entanglement such as EoF \(\cite{4}\), the entanglement of distillation and the relative entropy of entanglement converge to the same quantity for pure bipartite states, which is the von Neumann entropy of the reduced density operator of this bipartite state. For a given state, which is the von Neumann entropy of the reduced density operator of this bipartite state. For a given state, entanglement measures are not the same in general, nor unique for one kind of measure. A class of measures may constitute the entanglement monotones with physical significance in the framework of local operations and classical communication (LOCC) \(\cite{5,6}\). Rényi $\alpha$-entropy is a natural generalization of von Neumann entropy and it reduces to the latter when $\alpha$ is approaching 1. We thus can consider the quantity entanglement Rényi $\alpha$-entropy (ERoE). These entropies parameterized by continuous variable $\alpha$ can be a class of entropy monotones. Important applications are found by using Rényi $\alpha$-entropy in describing entanglement of ground states of many-body systems \(\cite{9,10}\).

In contrast to the relatively simple case of pure entangled state, the quantification of mixed state entanglement is still challenging due to the need for hard optimization procedures \(\cite{17}\). However, we know that the problem of quantifying two-qubit state entanglement can be well solved by EoF based on the analytical formula for concurrence in the seminal work of Wootters \(\cite{18}\). Additionally, the entanglement of pure two-qubit states depends on only one free parameter with the help of Schmidt decomposition, so the local convertibility of any two pure states by LOCC is completely determined. One may expect simply that the problem of quantification of arbitrary two-qubit entanglement is closed. However, we will show that interesting phenomena can be found for two-qubit states by using ERoE which is much more complex than generally expected.

Definition.—Suppose we have a composite system with subsystems A and B in a pure state, $|\psi\rangle = \sum_{i=1}^{d} \sqrt{\mu_i} |a_i, b_i\rangle_{AB}$, which is written in the form of Schmidt decomposition with Schmidt vector $\mu$. For simplicity, we denote the density matrix as $\psi \equiv |\psi\rangle\langle\psi|$, and let $\rho_B(A) = \text{tr}_{A(B)}(\psi)$ be the reduced density matrix of subsystem B or A. The entanglement of pure state $|\psi\rangle$ can be quantified by the Rényi $\alpha$-entropy of one of the reduced density operators, for example $\rho_B$, defined as,

$$R_\alpha(\psi) \equiv (1 - \alpha)^{-1} \log (\text{tr} \rho_B^\alpha),$$

(1)

This measure of entanglement can be easily generalized to mixed states using the so-called convex roof construction \(\cite{8,17}\). For a mixed state with density matrix $\rho$, it has pure state decomposition $\rho = \sum_k p_k \psi_k$, we can define ERoE as,

$$R_\alpha(\rho) \equiv \min_{\{p_k, \psi_k\}} \sum_k p_k R_\alpha(\psi_k),$$

(2)

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where the minimization is over all possible pure state en-
sembles \( \{ p_k, \psi_k \} \). As in most cases of mixed state entan-
glement measure, the evaluation of ERoE of mixed states is
much more difficult to carry out due to the complexity
involving in the optimization.

It is well known that EoF corresponding to \( \alpha \to 1 \) for
ERoE depends only on concurrence which has an ana-
lytic form for arbitrary two-qubit state and itself can act
as a measure of entanglement [8]. For \( \alpha \in (1, +\infty) \),
ERoE depends similarly only on concurrence [19]. It is
then possible that concurrence, in principle, might be
the only essential measure of entanglement for two-qubit
state even ERoE for \( \alpha \in (0, 1) \) can act as entanglement monotonies [8]. We will show in this Letter that this is
not the case. We remark that ERoE satisfies monogamy
inequality [25] for multiqubit states when \( \alpha = 2 \) [19].
EoF or concurrence of higher-dimensional system are known for classes of states with special sym-
metry such as the Werner states [22] and isotropic states
[23,24].

The critical value of \( \alpha \).— Suppose we have a compos-
ite system of two-qubit in a pure state \( |\psi\rangle \). Given the
spin flip operation on this state, \( |\tilde{\psi}\rangle = \sigma_y |\psi\rangle \), its
concurrence can be defined as, \( C(\psi) = \langle \langle \tilde{\psi} | \tilde{\psi} \rangle \rangle \). Without
confusion, we sometimes simplify \( C(\psi) \) or \( \tilde{C}(\psi) \)
the notation \( C \). It is easy to see that the Schmid coeffi-
cients \( \lambda_\pm \), i.e., the eigenvalues of the reduced density
matrix \( \rho_{\psi} \) of \( |\tilde{\psi}\rangle \), are uniquely related with the value of
\( C \), \( \lambda_\pm = (1 \pm \sqrt{1 - C^2})/2 \). By direct substitution of \( \lambda_\pm \)
into the definition [11] of Rényi \( \alpha \)-entropy, we obtain
\[
R_\alpha(\psi) = (1 - \alpha)^{-1} \log(\lambda_+^\alpha + \lambda_-^\alpha)
= \Omega(C, \alpha),
\] (3)
where we have introduced the definition of func-
tion \( \Omega(C, \alpha) \). The concurrence of arbitrary mixed
state \( \rho \) can be similarly defined on convex roof for-
mulae, \( C(\rho) = \min_{\{ p_k, \psi_k \}} \sum_k p_k C(\psi_k) \), where the mini-
mization is over all possible pure state decomposi-
tion of \( \rho \). An important observation made in [18],
[20] is that this measure is computable for two-qubit
state. If we generalize the spin flip operation to any
mixed state \( \rho \) by \( \tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y) \), and let
\( \lambda_1^2 (\Lambda_1 \geq \Lambda_2 \geq \Lambda_3 \geq \Lambda_4 \geq 0) \) denote the eigenvalues of
\( \tilde{\rho} \), then the concurrence can be calculated as \( C(\rho) =
\max\{\Lambda_1 - 2 \Lambda_2 - 3 \Lambda_3 - 4 \Lambda_4, 0\} \).

The ERoE of an arbitrary qubit-qubit state \( \rho \) now be-
comes,
\[
R_\alpha(\rho) = \min_{\{ p_k, \psi_k \}} \sum_k p_k \Omega(C(\psi_k), \alpha).
\] (4)
Up to now, we largely only use various known definitions.
We also know the following facts: When \( \alpha \to 1 \), ERoE
in Eq. (4) is simplified as \( R_{\alpha \to 1}(\rho) = \Omega(C(\rho), \alpha \to 1) \).
This fact means that the optimization for ERoE, which is
EoF since \( \alpha \to 1 \), can be realized by the optimization for
concurrence which is simply computable [20]. Another
fact is that when \( \alpha \in (1, +\infty) \), we similarly have,
\[
R_\alpha(\rho) = \Omega(C(\rho), \alpha),
\] (5)
the ERoE is essentially based on concurrence [19]. We
next try to investigate the question in what region equa-
tion (5) holds, we can see that only the unknown region
\( \alpha \in (0, 1) \) need be considered. We shall examine when the
function \( \Omega(C, \alpha) \) is convex and monotonically increasing
with respect to \( C \).

Let us present the first derivative of \( \Omega(C, \alpha) \) with
respect to \( C \), in terms of the Schmid coefficients \( \lambda_\pm \) by
using Eq. (3),
\[
\frac{\partial \Omega(C, \alpha)}{\partial C} = \frac{\alpha}{(1 - \alpha)(\lambda_+^\alpha + \lambda_-^\alpha)} [\lambda_+^{\alpha - 1} \frac{d \lambda_+}{dC} + \lambda_-^{\alpha - 1} \frac{d \lambda_-}{dC}].
\]
The derivatives of \( \lambda_\pm \) are,
\[
\frac{d \lambda_\pm}{dC} = \frac{C}{2(1 - 2 \lambda_\pm)}, \quad \frac{d^2 \lambda_\pm}{dC^2} = \frac{2}{C^2(1 - 2 \lambda_\pm)} \frac{d \lambda_\pm}{dC}.
\]
Also we know, \( d \lambda_+ / dC + d \lambda_- / dC = 0 \), and we introduce the
notations, \( D_1 = |d \lambda_+ / dC| = C/2\sqrt{1 - C^2} \) and \( x \equiv \lambda_- / \lambda_+ \). Now we know,
\[
\frac{\partial \Omega(C, \alpha)}{\partial C} = \frac{\alpha \lambda_+^{\alpha - 1} D_1}{(\alpha - 1)(\lambda_+^\alpha + \lambda_-^\alpha)} (1 - x^\alpha - 1) \geq 0.
\] (6)
So function \( \Omega(C, \alpha) \) is monotonically increasing with
respect to \( C \).

The evaluation of the second derivative of \( \Omega(C, \alpha) \) is
more complicated,
\[
\frac{\partial^2 \Omega(C, \alpha)}{\partial C^2} = -\frac{\alpha \lambda_+^{2\alpha - 2} D_1^2}{(1 - \alpha)(\lambda_+^\alpha + \lambda_-^\alpha)} K,
\] (7)
where
\[
K = (1 - x^\alpha - 1)^2 + \frac{(1 + x)^2}{2x(1 - x)} g(x, \alpha),
\] (8)
\[
g(x, \alpha) = 1 - x^{2\alpha - 1} - (2\alpha - 1)(1 - x)x^{\alpha - 1},
\] (9)
the restrictions \( x \in (0, 1) \) and \( \alpha \in [0, 1] \) are assumed. By
calculation, we then can observe the fact,
\[
\begin{cases}
K \geq (1 - x^\alpha - 1)^2 \geq 0, & \alpha \in [0, \frac{1}{2}), \\
K \leq (1 - x^\alpha - 1)^2, & \alpha \in (\frac{1}{2}, 1].
\end{cases}
\] (10)
Thus we conclude that the function \( \Omega(C, \alpha) \) is concave
with respect to \( C \) for \( \alpha \in [0, \frac{1}{2}) \),
\[
\frac{\partial^2 \Omega(C, \alpha)}{\partial C^2} \leq -\frac{\alpha \lambda_+^{2\alpha - 2} D_1^2 (1 - x^\alpha - 1)^2}{(1 - \alpha)(\lambda_+^\alpha + \lambda_-^\alpha)^2} \leq 0.
\] (11)
This result is actually opposite of the convexity of the function \( \Omega (C, \alpha) \) for \( \alpha \in (1, +\infty) \). We thus give a negative answer for the holding of Eq. (5) for calculation shows that the value of \( C \) increases monotonically with respect to \( \alpha \), see FIG. 1 for the dependence of \( C_0 \) on \( \alpha \). So there may exist a critical value of \( \alpha \) corresponding to \( C_1 \) such that the second derivative of \( \Omega (C, \alpha) \) is zero. Such a critical value \( \alpha_c \) does exist, such that the simplification, meaning the holding of Eq. (5), is still valid for any \( \alpha \) larger than this value. In fact, it is not difficult to obtain the value of \( \alpha_c \) analytically. One simply considers the limit \( C \to 1 \) and the requirement that,

\[
\lim_{C \to 1} \frac{\partial^2 \Omega (C, \alpha)}{\partial C^2} \geq 0. \tag{12}
\]

This condition is equivalent to that, \( \lim_{x \to 1} K \leq 0 \). With the help of the definitions of \( K \) and \( g(x, \alpha) \), we need the following inequality,

\[
\frac{(\alpha - 1)}{3} [3 (\alpha - 1) + 2 (\alpha - 1) \alpha] \leq 0. \tag{13}
\]

The value of \( \alpha_c \) can be solved by considering the equality, which gives us,

\[
\alpha_c = \frac{\sqrt{7} - 1}{2} \approx 0.82. \tag{14}
\]

This solution can also be confirmed in FIG. 1.

We summarize the above results into the expression,

\[
\frac{\partial^2 \Omega (C, \alpha)}{\partial C^2} \begin{cases} 
\leq 0, & \alpha \in [0, \frac{1}{\sqrt{7}}], \\
\geq 0, & \alpha \in [\frac{\sqrt{7} - 1}{2}, 1].
\end{cases} \tag{15}
\]

So when \( \alpha \geq \alpha_c \approx 0.82 \), the ER\( \alpha \)E can be calculated analytically based on concurrence by Eq. (13). The ER\( \alpha \)E for two-qubit states when \( \alpha < \alpha_c \) in general is still a challenging problem. However, we may consider the Werner state which possesses special symmetry \( \Omega (C, \alpha) \) and \( g(x, \alpha) \). We thus give a negative answer for the holding of Eq. (5) for calculation shows that the value of \( C \) increases monotonically with respect to \( \alpha \), see FIG. 1 for the dependence of \( C_0 \) on \( \alpha \). So there may exist a critical value of \( \alpha \) corresponding to \( C_1 \) such that the second derivative of \( \Omega (C, \alpha) \) is zero. Such a critical value \( \alpha_c \) does exist, such that the simplification, meaning the holding of Eq. (5), is still valid for any \( \alpha \) larger than this value. In fact, it is not difficult to obtain the value of \( \alpha_c \) analytically. One simply considers the limit \( C \to 1 \) and the requirement that,

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steps. The first step is to calculate the function $s(F, \alpha, d)$ defined as,

$$s(F, \alpha, d) \equiv \inf \{\mathcal{R}_\alpha(\psi)\left| \sum_{i,j=1}^{d} \sqrt{\mu_i \mu_j} v_{ij} v_{ij}^* = -F \right.\} \quad (23)$$

If we can calculate this function, we can easily carry out the second step to obtain the ER\alphaE,

$$\mathcal{R}_\alpha(p_w) = co(s(F, \alpha, d)) \quad (24)$$

where we define the convex hull of a function $f(x)$, defined on $\mathcal{D}$, to be,

$$co(f(x)) = \inf \left\{ \sum_k p_k f(x_k) \left| \sum_k p_k x_k = x, \sum_k p_k = 1, x_k \in \mathcal{D} \right. \right\} \quad (25)$$

While the optimization in Eq. (23) for general $d$ is not quite simple, it can be carried out for qubit-qubit state. We also note that, since the class of Werner state is equivalent to isotropic states in this case, this calculation can alternatively be done for the set of isotropic states, and the results will be identical. To complete the first step in our two-step procedure, we note that from the constraint for the parameters $\{\mu_i, v_{ij}\}$ and setting $d = 2$, we have that for $F > 0$,

$$\sqrt{\mu_1 \mu_2} \geq F, \quad (26)$$

where equality holds if and only if $\mu_{1,2} = \lambda_\pm$. One easily finds that,

$$s(F, \alpha, d) = \begin{cases} \Omega(F, \alpha), & F \geq 0, \\ 0, & F < 0. \end{cases} \quad (27)$$

Thus we obtain a computable expression for the ER\alphaE of Werner states $p_w$ for two-qubit states,

$$\mathcal{R}_\alpha(p_w) = \begin{cases} co(\Omega(F, \alpha)), & F \geq 0, \\ 0, & F < 0. \end{cases} \quad (28)$$

The second step, namely to calculate the convex hull of the function $\Omega(F, \alpha)$, can always be done numerically.

As an example, we can compare the ER\alphaE of a Werner state with $F = 0.8$ and a pure state with $F = 0.5$, as shown in FIG. 2. Apparently, by the entanglement measure EoF corresponding to $\alpha = 1$, the entanglement of the Werner state is larger than this pure state. This result seems natural and well-accepted. Surprisingly, when $\alpha$ is small and is approaching 0, the order of entanglement for those two states is reversed. One can find that the entanglement of the pure state is smaller than that of the Werner state. Actually, the entanglement of Werner states will always less than that of an arbitrary pure entangled state when $\alpha \to 0$. In this case, the ER\alphaE for any entangled pure states will be 1, while the optimal pure states decomposition for a Werner state will include a maximally entangled state with probability $F$ and the identity operator, resulting in that ER\alphaE equals to $F$. In this sense, we may find that ER\alphaE for two-qubit states may have the possibility to be incomparable.

**Discussions.**—ER\alphaE quantifies entanglement. For two-qubit states, when $\alpha \geq \alpha_c$, ER\alphaE can be obtained analytically based on the well-known concurrence. This result implies that the pure states in the optimal decomposition for ER\alphaE when $\alpha \geq \alpha_c$ possess the same Schmidt vector similar as that for concurrence. The general analytical formula of ER\alphaE even for the simplest two-qubit states is still a challenging problem, also the pure states decomposition in general will not possess the same Schmidt vector. However, ER\alphaE for Werner states can be obtained. Interestingly, we notice that ER\alphaE for the simplest two-qubit states may be incomparable, implying that they are not local convertible by local quantum operations and classical communication. This phenomenon is previously only known for higher-dimensional systems.

In summary, quantification of entanglement is challenging. In this Letter, we find that entanglement for two-qubit states is still an open problem. Some interesting phenomena are found by using ER\alphaE. Those results may stimulate more interests in studying quantum entanglement.

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