Stochastic Calculus with Jumps Processes : Theory and Numerical Techniques
(Master Thesis)

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13 June 2015
Submitted in Partial Fulfillment of a Masters II at AIMS Senegal
Abstract

In this work we consider a stochastic differential equation (SDEs) with jump. We prove the existence and the uniqueness of solution of this equation in the strong sense under global Lipschitz condition. Generally, exact solutions of SDEs are unknowns. The challenge is to approach them numerically. There exist several numerical techniques. In this thesis, we present the compensated stochastic theta method (CSTM) which is already developed in the literature. We prove that under global Lipschitz condition, the CSTM converges strongly with standard order 0.5. We also investigated the stability behaviour of both CSTM and stochastic theta method (STM). Inspired by the tamed Euler scheme developed in [8], we propose a new scheme for SDEs with jumps called compensated tamed Euler scheme. We prove that under non-global Lipschitz condition the compensated tamed Euler scheme converges strongly with standard order 0.5. Inspired by [11], we propose the semi-tamed Euler for SDEs with jumps under non-global Lipschitz condition and prove its strong convergence of order 0.5. This latter result is helpful to prove the strong convergence of the tamed Euler scheme. We analyse the stability behaviours of both tamed and semi-tamed Euler scheme. We present also some numerical experiments to illustrate our theoretical results.

Key words: Stochastic differential equation, strong convergence, mean-square stability, Euler scheme, global Lipschitz condition, polynomial growth condition, one-sided Lipschitz condition.

Declaration

I, the undersigned, hereby declare that the work contained in this essay is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Jean Daniel Mukam, 25 May 2015
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INTRODUCTION

In many branches of sciences like finance, economics, biology, engineering, ecology one often encountered some problems influenced by uncertainties. For example, in finance, the unpredictable nature of events such as markets crashes and booms may have significant and sudden impact on the stock price fluctuations. Therefore, in order to have more realistic prediction of these phenomena, it is natural to model them with equations which involves the deterministic part and the random part including jump. The SDEs with jumps is the generalization of both deterministic part and random part with jumps. SDEs with jumps have probability theory and stochastic process as prerequisites. We refer to [2], [3], [4] for general notions in probability theory and stochastic process.

In this thesis, under global Lipschitz condition, we prove the existence and uniqueness of solution of SDEs with jumps. We focus on the strong convergence of the compensated stochastic theta methods (CSTM) of these equations under global Lipschitz condition. In particular, we prove that CSTM have strong convergence of order 0.5. We investigate the stability of both CSTM and stochastic theta method (STM). For the linear case, we prove that under the assumption \( \frac{1}{2} \leq \theta \leq 1 \), CSTM holds the A-stability property. For the general nonlinear problem, we study the stability for \( \theta = 1 \). In this case, when the drift coefficient have a negative one-sided Lipschitz coefficient, the diffusion coefficient and the jump coefficient satisfy the global Lipschitz condition, we prove that STM reproduce stability under certain step-size and the CSTM is stable for any step-size.

Most phenomena are modelised by SDEs with jumps where the drift coefficient is one-sided Lipschitz and satisfies the polynomial growth condition. For such equations, it is proved in [10] that Euler explicit method fails to converge strongly to the exact solution while Euler implicit method converges strongly, but requires much computational efforts. Recently, a new explicit and efficient method was developed in [8] called tamed Euler scheme. In [8], the authors proved that the tamed Euler converges strongly with order 0.5 to the exact solution of SDEs under non-global Lipschitz condition. In this thesis, we extend the tamed Euler scheme by introducing a compensated tamed Euler scheme for SDEs with jumps. We prove that this scheme converges strongly with standard order 0.5. We also extend the semi-tamed Euler developed in [11] and we prove that this scheme converge strongly with order 0.5 for SDEs with jumps. As a consequence of this latter result, we prove the strong convergence of the tamed Euler scheme for SDEs with jumps. The stability analysis of both tamed Euler and semi-tamed Euler are done in this thesis.

This thesis is organized as follows. In chapter 1, we recall some basic notions in probability theory and stochastic process. Chapter 2 is devoted to the proof of the existence and uniqueness of SDEs with jumps under global Lipschitz condition. In chapter 3, we focus on the strong convergence of the CSTM and the stability analysis of both CSTM and STM. In chapter 4, under non-global Lipschitz condition we investigate the strong convergence of the compensated tamed Euler scheme. In chapter 5, under non-global Lipschitz condition, we investigate the strong convergence and stability of both semi-tamed Euler scheme and tamed Euler scheme. Our theoretical results are illustrated by numerical examples at the end of chapter 3, chapter 4 and chapter 5.
Chapter 1

Basic notions in probability theory and stochastic process

1.1 Basic notions in probability theory

In this chapter, we present some basic concepts and results in probability theory and stochastic process useful to understand the notion of stochastic differential equations.

More details for this chapter can be found in [2], [3] and [4].

1.1.1 Basic notions in probability theory

Definition 1.1.2 [σ-algebra]
Let \( \Omega \) be a non-empty set.

1. A \( \sigma \)-algebra (or \( \sigma \)-field) \( \mathcal{F} \) on \( \Omega \) is a family of subsets of \( \Omega \) satisfying
   
   (i) \( \Omega \in \mathcal{F} \).
   
   (ii) \( \forall A \in \mathcal{F}, A^c \in \mathcal{F} \).
   
   (iii) If \( (A_i)_{i \in I} \) is a countable collection of set in \( \mathcal{F} \), then \( \bigcup_{i \in I} A_i \in \mathcal{F} \).

2. Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be two \( \sigma \)-algebra on \( \Omega \). \( \mathcal{F}_1 \) is said to be a sub-\( \sigma \)-algebra of \( \mathcal{F}_2 \) if \( \mathcal{F}_1 \subset \mathcal{F}_2 \).

Remark 1.1.3
1. Given any family \( \mathcal{B} \) of subset of \( \Omega \), we denote by

   \[ \sigma(\mathcal{B}) := \cap \{ \mathcal{C} : \mathcal{C} \text{, } \sigma \text{-algebra of } \Omega, \mathcal{B} \subset \mathcal{C} \} \]

   the smallest \( \sigma \)-field of \( \Omega \) containing \( \mathcal{B} \), \( \sigma(\mathcal{B}) \) is called the \( \sigma \)-field generated by \( \mathcal{B} \).

   When \( \mathcal{B} \) is a collection of all open sets of a topological space \( \Omega \), \( \sigma(\mathcal{B}) \) is called the Borel \( \sigma \)-algebra on \( \Omega \) and the elements of \( \sigma(\mathcal{B}) \) are called Borel sets.

2. If \( X : \Omega \rightarrow \mathbb{R}^n \) is a function, then the \( \sigma \)-algebra generated by \( X \) is the smallest \( \sigma \)-algebra on \( \Omega \) containing all the sets of the form

   \[ \{ X^{-1}(U) : U \subset \mathbb{R}^n, \text{ open} \} \]

Definition 1.1.4 [Probability measure].
Let \( \mathcal{F} \) be a \( \sigma \)-field on \( \Omega \). A probability measure is an application \( P : \mathcal{F} \rightarrow [0, 1] \) satisfying

(i) \( P(\Omega) = 1 - P(\emptyset) = 1 \).

(ii) If \( (A_i)_{i \in I} \) is a countable collection of elements of \( \mathcal{F} \) pairwise disjoints, then

   \[ P(\bigcup_{i \in I} A_i) = \sum_{i \in I} P(A_i) \].
Definition 1.1.5 [Probability space].
Let \( \Omega \) be a non-empty set, \( \mathcal{F} \) a \( \sigma \)-field on \( \Omega \) and \( \mathbb{P} \) a probability measure on \( \mathcal{F} \).
The triple \((\Omega, \mathcal{F}, \mathbb{P})\) is called a probability space.

Definition 1.1.6 [Negligeable set]
(i) Given a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), \( A \subset \Omega \) is said to be \( \mathbb{P} \)-null or negligeable if \( \mathbb{P}(A) = 0 \)
(ii) A property is said to be true almost surely (a.s) if the set on which this property is not true is negligeable.

Definition 1.1.7 [Measurability and random variable]
(i) Let \((\Omega, \mathcal{F}, \mathbb{P})\) and \((\Omega', \mathcal{F}', \mathbb{P}')\) be two probability spaces. A function \( X : \Omega \rightarrow \Omega' \) is said to be \( \mathcal{F} \)-measurable if and only if
\[
X^{-1}(U) := \{ \omega \in \Omega : X(\omega) \in U \} \subset \mathcal{F}, \quad \forall U \in \mathcal{F}'
\]
(ii) A random variable \( X \) is a function \( X : \Omega \rightarrow \Omega' \) \( \mathcal{F} \)-measurable.
(iii) If \( \Omega' = \mathbb{R} \), then \( X \) is called a real random variable.
(iv) If \( \Omega' = \mathbb{R}^n, n > 1 \) then \( X \) is called a vector random variable.

In the following, unless otherwise state, \((\Omega, \mathcal{F}, \mathbb{P})\) denote a probability space and \( X \) a random variable, \( X : \Omega \rightarrow \mathbb{R}^n \).

Remark 1.1.8 .
Every random variable induces a probability measure on \( \mathbb{R}^n \) denoted \( \mu_X \) and define by
\[
\mu_X(B) := \mathbb{P}(X^{-1}(B)), \quad \forall B \text{ open set of } \mathbb{R}^n. \mu_X \text{ is called the distribution function of } X.
\]

Definition 1.1.9 [Expected value]
(i) If \( X \) is a random variable such that \( \int_{\Omega} ||X(\omega)|| d\mathbb{P}(\omega) < \infty \) almost surely, the quantity
\[
\mathbb{E}(X) := \int_{\Omega} X(\omega) d\mathbb{P}(\omega) = \int_{\mathbb{R}^n} d\mu_X(x)
\]
is called the expected value of \( X \), where \( ||.|| \) denote the euclidean norm on \( \mathbb{R}^n \).
(ii) In general, if \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is measurable and \( \int_{\Omega} ||f(X(\omega))|| d\mathbb{P}(\omega) < \infty \) almost surely, then the quantity \( \mathbb{E}(f(X)) \) define by
\[
\mathbb{E}(f(X)) := \int_{\Omega} f(X(\omega)) d\mathbb{P}(\omega) = \int_{\mathbb{R}^n} f(x) d\mu_X(x)
\]
is called expected value of \( f(X) \).

Definition 1.1.10 [Independent random variables]
Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space.

1. Two elements \( A \) and \( B \) of \( \mathcal{F} \) are independent if
\[
\mathbb{P}(A \cap B) = \mathbb{P}(A) \cap \mathbb{P}(B).
\]
2. Two random variables \( X_1 \) and \( X_2 \) of \((\Omega, \mathcal{F}, \mathbb{P})\) are independent if for every choice of different borel sets \( B_1 \) and \( B_2 \) the following holds :
\[
\mathbb{P}(X_1 \in B_1, X_2 \in B_2) = \mathbb{P}(X_1 \in B_1) \times \mathbb{P}(X_2 \in B_2).
\]
The following proposition is from [2].

**Proposition 1.1.11** Two random variables \( X_1 \) and \( X_2 \) are independent if and only if for any measurable positive functions \( f_1 \) and \( f_2 \), the following equality holds

\[
E(f_1(X_1)f_2(X_2)) = E(f_1(X_1))E(f_2(X_2)).
\]

**Definition 1.1.12** [Conditional probability]

For any event \( A \) such that \( P(A) > 0 \), the conditional probability on \( A \) is the probability measure defined by:

\[
P(B/A) := \frac{P(A \cap B)}{P(A)}, \quad \forall B \in \mathcal{F}.
\]

**1.1.13 Conditional expectation**

**Definition 1.1.14** Let \( X \) be a random variable such that \( \int \Omega |X(\omega)|dP(\omega) < \infty \) almost surely. Let \( \mathcal{G} \) a sub-\( \sigma \)-algebra of \( \mathcal{F} \). The conditional expectation of \( X \) relative to the \( \sigma \)-algebra \( \mathcal{G} \) is a random variable denoted by \( E(X/\mathcal{G}) \) satisfying

(i) \( E(X/\mathcal{G}) \) is \( \mathcal{G} \)-measurable.

(ii) \( \int \mathcal{G} E(X/\mathcal{G})dP = \int \mathcal{G} XdP, \quad \forall \mathcal{G} \in \mathcal{G}. \)

In the literature, \( E(X/\mathcal{G}) \) is called the projection of \( X \) upon \( \mathcal{G} \).

The proof of the following theorem can be seen in [4].

**Proposition 1.1.15**

(i) \( E(E(X/\mathcal{G})) = E(X) \).

(ii) If \( X \) is \( \mathcal{G} \)-measurable, then \( E(X/\mathcal{G}) = X \).

(iii) \( E((X + Y)/\mathcal{G}) = E(X/\mathcal{G}) + E(Y/\mathcal{G}) \).

(iv) If \( \mathcal{G} \subset \mathcal{G}' \) then \( E(X/\mathcal{G}') = E(E(X/\mathcal{G})/\mathcal{G}') \).

(v) If \( \sigma(X) \) and \( \mathcal{G} \) are independent, then \( E(X/\mathcal{G}) = E(X) \).

(vi) If \( X \leq Y \) a.s, then \( E(X/\mathcal{G}) \leq E(Y/\mathcal{G}) \).

(vii) If \( X \) is \( \mathcal{G} \) measurable, then \( E(XY/\mathcal{G}) = EX(Y/\mathcal{G}) \).

**1.1.16 Convergence of random variables**

**Definition 1.1.17** Let \( p \in [1, \infty) \), we denote by \( L^p(\Omega, \mathbb{R}^n) \) the equivalence class of measurable functions \( X : \Omega \to \mathbb{R}^n, \mathcal{F}_1 \)-measurable such that

\[
||X||_{L^p(\Omega, \mathbb{R}^n)}^p := E(||X||^p) = \int\Omega ||X(\omega)||^p dP(\omega) < +\infty.
\]

Let \( (X_n) \subset L^p(\Omega, \mathbb{R}^n) \) be a sequence of random variables and \( X \in L^p(\omega, \mathbb{R}^n) \) a random variable. Let

\[
N := \{\omega : \lim_{n \to \infty} X_n(\omega) = X(\omega)\}
\]

1. \( (X_n) \) converges to \( X \) almost surely if \( N^c \) is negligible.

2. \( (X_n) \) converges in probability to \( X \) if

\[
\forall \epsilon > 0 \lim_{n \to \infty} P(||X_n - X|| > \epsilon) = 0.
\]
3. \((X_n)\) converges in \(L^p\) to \(X\) if
\[
\lim_{n \to +\infty} \mathbb{E}(|X_n - X|^p) = 0.
\]

**Definition 1.1.18**: [Frobenius norm]

The Frobenius norm of an \(m \times n\) matrix \(A = (a_{ij})\) is defined by
\[
\|A\| := \sqrt{\sum_{j=1}^{n} \sum_{i=1}^{n} |a_{ij}|^2}.
\]

**Remark 1.1.19** Frobenius norm and euclidean norm are the same for vectors.

**Proposition 1.1.20** [Minkowski inequality: Integral form]

Let \(1 \leq p < +\infty\) and let \((X, \mathcal{A}, dx)\) and \((Y, \mathcal{B}, dy)\) be \(\sigma\)-finite measures spaces. Let \(F\) be a measurable function on the product space \(X \times Y\). Then
\[
\left( \int_X \left( \int_Y |F(x, y)|^p\,dy \right)^{1/p}\,dx \right)^{1/p} \leq \int_Y \left( \int_X |F(x, y)|^p\,dx \right)^{1/p}\,dy.
\]

The above inequality can be written as
\[
\left\| \int_Y F(\cdot, y)\,dy \right\|_{L^p(X, \mathcal{A}, dx)} \leq \int_Y \left\| F(\cdot, y) \right\|_{L^p(X, \mathcal{A}, dx)}\,dy.
\]

**Proposition 1.1.21** [Gronwall inequality: Continuous form]

Let \(a(t)\) and \(b(t)\) be two continuous and positive functions defined on \(\mathbb{R}_+\) such that
\[
a(t) \leq b(t) + c \int_0^t a(s)\,ds, \quad \forall t \in \mathbb{R}_+,
\]
then
\[
a(t) \leq b(t) + c \int_0^t b(s)e^{c(t-s)}\,ds, \quad \forall t \in \mathbb{R}_+.
\]

**Gronwall inequality: Discrete form.**

Let \(\theta\) and \(K\) be two constants and \((v_n)\) be a sequence satisfying :
\[
v_{n+1} \leq (1 + \theta)v_n + K,
\]
then
\[
v_n \leq e^{\theta n}v_0 + K\frac{e^{\theta n} - 1}{e^\theta - 1}.
\]

**Proof 1.1.22** [16].

**Lemma 1.1.23** [Borel Cantelli]

Let \((A_n)_{n \in \mathbb{N}}\) be a family of subset of \(\Omega\).

1. If \(\sum_{n \in \mathbb{N}} \mathbb{P}(A_n) < \infty\), then \(\mathbb{P}\left( \limsup_{n \to \infty} A_n \right) = 0\).

2. If the events \((A_n)\) are independent and \(\sum_{n \in \mathbb{N}} \mathbb{P}(A_n) = 0\), then \(\mathbb{P}\left( \limsup_{n \to \infty} A_n \right) = 1\).

**Proof 1.1.24** [4].
1.2 Stochastic processes

Definition 1.2.1 Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. A family \((\mathcal{F}_t)_{t \geq 0}\) of sub \(\sigma\)-algebra of \(\mathcal{F}\) is called filtration if \(\mathcal{F}_s \subset \mathcal{F}_t, \forall \ 0 \leq s \leq t\).

If \((\mathcal{F}_t)\) is such that \(\mathcal{F}_t = \cap_{s \geq t} \mathcal{F}_s\), then \((\mathcal{F}_t)_{t \geq 0}\) is said to be right continuous.

Definition 1.2.2 A stochastic process is a family of vector random variables \((X_t)_{t \geq 0}\). That is for all \(t > 0\), the application
\[
X_t : \Omega \rightarrow \mathbb{R}^n
\]
\[w \mapsto X_t(\omega)\]
is measurable.

If \((X_t)_{t \geq 0}\) is a stochastic process, then for all \(t \geq 0\), the application \(t \mapsto X_t\) is called sample path.

Definition 1.2.3 Let \((\mathcal{F}_t)\) be a filtration on \((\Omega, \mathcal{F}, \mathbb{P})\). A stochastic process \((X_t)\) is said to be \(\mathcal{F}_t\)-adapted if \(\forall \ t \geq 0\) \(X_t\) is \(\mathcal{F}_t\)-measurable.

Definition 1.2.4 [Martingale] Let \((\mathcal{F}_t)_{t \geq 0}\) be a filtration on \((\Omega, \mathcal{F}, \mathbb{P})\). A stochastic process \((M_t)_{t \geq 0}\) is called \(\mathcal{F}_t\)-martingale if the following properties holds

(i) \((M_t)\) is \(\mathcal{F}_t\)-adapted.

(ii) \(\mathbb{E}|M_t| < \infty, \forall \ t \geq 0\).

(iii) \(\mathbb{E}(M_t / \mathcal{F}_s) = M_s, \forall \ 0 \leq s \leq t\).

Remark 1.2.5 (i) If the condition (iii) of the previous definition is replaced by \(\mathbb{E}(M_t / \mathcal{F}_s) \geq M_s, \forall \ 0 \leq s \leq t\), then \((M_t)\) is called submartingale.

(ii) If the condition (iii) of the previous definition is replaced by \(\mathbb{E}(M_t / \mathcal{F}_s) \leq M_s, \forall \ 0 \leq s \leq t\), then \((M_t)\) is called supermartingale.

(iii) A positive submartingale is a submartingale \((X_t)_{t \geq 0}\) satisfying \(X_t \geq 0\) for all \(t \geq 0\).

Definition 1.2.6 [Predictable process]
Let \((\mathcal{F}_t)_{t \geq 0}\) be a filtration on \((\Omega, \mathcal{F}, \mathbb{P})\). A stochastic process \((X_t)_{t \geq 0}\) is called \(\mathcal{F}_t\)-predictable process if for all \(t > 0\), \(X_t\) is measurable with respect to the \(\sigma\)-algebra generated by \(\{X_s, \ s < t\}\).

Proposition 1.2.7 Let \(M = (M_t)\) be a submartingale. Then for \(1 < p < \infty\), we have

(i) Markov’s inequality
\[
\mathbb{P} \left( \sup_{0 \leq s \leq t} ||M_s|| \geq \alpha \right) \leq \frac{\mathbb{E}(||M_t||)}{\alpha}, \ \forall \ \alpha > 0.
\]

(ii) Doob’s maximal inequality
\[
\mathbb{E} \left[ \left( \sup_{0 \leq s \leq t} ||M_s|| \right)^p \right]^{1/p} \leq \frac{p}{p-1} \mathbb{E}[||M_t||^p]^{1/p}.
\]

Proof 1.2.8 [4].

Definition 1.2.9 [Wiener process or Brownian motion]
Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \((\mathcal{F}_t)_{t \geq 0}\) a filtration on this space. A \(\mathcal{F}_t\)-adapted stochastic process \((W_t)_{t \geq 0}\) is called Wiener process or Brownian motion if:

(i) \(W_0 = 0\).

(ii) \(t \mapsto W_t\) is almost surely continous.

(iii) \((W_t)_{t \geq 0}\) has independent increments (i.e \(W_t - W_s\) is independent of \(W_r, \ r \leq s\)).
(iv) $W_t - W_s \sim \mathcal{N}(0, t - s)$, for $0 \leq s \leq t$. Usually, this property is called stationarity.

**Proposition 1.2.10** If $(W_t)$ is an $\mathcal{F}_t$- Brownian motion, then the following process are $\mathcal{F}_t$- martingales

(i) $W_t$.

(ii) $W_t^2 - t$.

(iii) $\exp \left( \gamma W_t - \frac{\gamma^2 t}{2} \right)$, $\forall \gamma \in \mathbb{R}$.

**Proof 1.2.11** Let $0 \leq s \leq t$, then

(i)

$$E(W_t/\mathcal{F}_s) = E(W_t - W_s + W_s/\mathcal{F}_s)$$

$$= W_s + E(W_t - W_s/\mathcal{F}_s) \text{ since } W_s \text{ is } \mathcal{F}_s \text{- measurable}$$

$$= W_s + E(W_t - W_s) \text{ (since the increments are independents)}$$

$$= W_s (\text{since } W_t - W_s \sim \mathcal{N}(0, t - s)).$$

(ii)

$$E(W_t^2 - t/\mathcal{F}_s) = E(W_t^2 + W_s^2 - 2W_s W_t + 2W_s W_t - W_s^2/\mathcal{F}_s) - t$$

$$= E((W_t - W_s)^2/\mathcal{F}_s) + W_s E((2W_t - W_s)/\mathcal{F}_s) - t$$

(since $W_s$ is $\mathcal{F}_s$- measurable)

$$= E((W_t - W_s)^2) + W_s E(W_t - W_s) + W_s E(W_t/\mathcal{F}_s) - t$$

(since the increments are independents)

$$= t - s + 0 + W_s^2 - t \text{ since } W_t - W_s \sim \mathcal{N}(0, t - s)$$

$$= W_s^2 - s.$$

(iii) Using the same argument as above, we have :

$$E(e^{\gamma W_t}/\mathcal{F}_s) = e^{\gamma W_s} E(e^{\gamma (W_t - W_s)}/\mathcal{F}_s)$$

$$= e^{\gamma W_s} E(e^{\gamma W_{t-s}})$$

$$= e^{\gamma W_s} \int_{-\infty}^{+\infty} e^{-x^2/2(t-s)} dx$$

$$= e^{\gamma W_s} e^{\gamma^2 (t-s)/2} = e^{\gamma W_s + \gamma^2 (t-s)/2}.$$

Therefore,

$$E \left( \exp \left( \gamma W_t - \frac{\gamma^2 t}{2} \right)/\mathcal{F}_s \right) = E(e^{\gamma W_t}/\mathcal{F}_s)e^{-\gamma t/2}$$

$$= e^{\gamma W_s + \gamma^2 (t-s)/2}e^{-\gamma^2 t/2}$$

$$= \exp(\gamma W_s - \gamma^2 s/2).$$

The following proposition is from [3].

**Proposition 1.2.12** Almost all sample paths of a Brownian motion are nowhere differentiable.

**Definition 1.2.13** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_t)$ a filtration on this space. Let $(S_k)_{k \geq 1}$ be an $\mathcal{F}_t$-adapted stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ with $0 \leq S_1(\omega) \leq S_2(\omega) \leq ...$ for all $k \geq 1$ and $\omega \in \Omega$. The $\mathcal{F}_t$- adapted process $N = (N_t)_{t \geq 0}$ defined by :

$$N_t := \sum_{k \geq 1} 1_{\{S_k \leq t\}}$$

is called counting process with jump times $S_k$. 
Definition 1.2.14 Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \((\mathcal{F}_t)\) a filtration on this space. A counting process \((N_t, \mathcal{F}_t)\)-adapted is called poisson process of intensity \(\lambda > 0\) if:

(i) \(N_0 = 0\).

(ii) \(\forall 0 \leq t_0 < t_1 < \ldots < t_n\), the random variables \(\{N_{t_j} - N_{t_{j-1}} \mid 1 \leq j \leq n\}\) are independent.

(iii) For \(0 \leq s \leq t\), \(N_t - N_s \approx N_{t-s}\), where \(\approx\) stand for the equality in probability law.

(iv) For all \(t > 0\), \(N_t\) follows a poisson law with parameter \(\lambda t\) (and we denote \(N_t \sim \mathcal{P}(\lambda t)\)). That is
\[
\mathbb{P}(N_t = k) = e^{-\lambda t} \frac{\lambda^k t^k}{k!}, \quad k \in \mathbb{N}.
\]

Definition 1.2.15 [Compound poisson process]

Let \((Z_n)\) be a sequence of discrete independent identically distributed random variables with probability law \(\nu_Z\). Let \(N = (N_t)\) be a poisson process with parameter \(\lambda\). Let’s assume that \((N_t)\) and \((Z_n)\) are independent. A compound poisson process with intensity \(\lambda > 0\) with a jump law \(\nu_Z\) is a \(\mathcal{F}_t\)-adapted stochastic process \((Y_t)\) defined by:
\[
Y_t := \sum_{k=1}^{N_t} Z_k.
\]

Definition 1.2.16 [Compensated poisson process]

A compensated poisson process associated to a poisson process \(N\) with intensity \(\lambda\) is a stochastic process \(\overline{N}\) defined by:
\[
\overline{N}(t) := N(t) - \lambda t.
\]

Proposition 1.2.17 Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \((\mathcal{F}_t)\) a filtration on this space.

If \((N_t)\) is a \(\mathcal{F}_t\)-adapted poisson process with intensity \(\lambda\), then

1. \(\overline{N}\) is a \(\mathcal{F}_t\)-adapted martingale.

2. \(\mathbb{E}(\overline{N}(t+s) - \overline{N}(t)) = 0\).

3. \(\mathbb{E}[(\overline{N}(t+s) - \overline{N}(t))^2] = \lambda s, \quad \forall t, s \geq 0\).

4. \(\overline{N}_t^2 - \lambda t\) is a martingale.

Proof 1.2.18 1. Let \(s \leq t\), then
\[
\mathbb{E}(\overline{N}_t / \mathcal{F}_s) = \mathbb{E}(N_t - \overline{N}_s + \overline{N}_s / \mathcal{F}_s)
\]
\[
= \mathbb{E}(N_t - \overline{N}_s / \mathcal{F}_s) + \overline{N}_s
\]
\[
= \mathbb{E}(N_t - N_s - \lambda t + \lambda s / \mathcal{F}_s) + N_s - \lambda s
\]
\[
= \mathbb{E}(N_t - N_s) - \lambda t + \lambda s + N_s - \lambda s
\]

since the increments of the poisson process are independents
\[
= \lambda(t - s) - \lambda t + N_s \quad (\text{since } N_t - N_s \sim \mathcal{P}(\lambda(t - s)))
\]
\[
= N_s - \lambda s
\]
\[
= \overline{N}(s).
\]

2.
\[
\mathbb{E}(\overline{N}(t+s) - \overline{N}(t)) = \mathbb{E}(N(t+s) - N(t) - \lambda s)
\]
\[
= \lambda(t + s - t) - \lambda s = 0.
\]
3. \[
(N(t + s) - N(t))^2 = [N(t + s) - N(t) - \lambda s]^2 \\
= [N(t + s) - N(t)]^2 + \lambda^2 s^2 - 2\lambda s(E(t + s) - N(t)).
\]

Since \(N(t) \sim \mathcal{P}(\lambda t)\), using the relation \(\mathbb{E}(N_t) = \text{var}(N_t) = \lambda t\), it follows that:
\[
\mathbb{E}(N(t + s) - N(s))^2 = \lambda(t + s - t) + \lambda^2(t + s - t)^2 + \lambda^2 s^2 - 2\lambda s(\lambda s) = \lambda s.
\]

4. \[
\mathbb{E}[\overline{N}_t^2 - \lambda t|\mathcal{F}_s] = \mathbb{E}[\overline{N}_t^2|\mathcal{F}_s] - \lambda t \\
= \mathbb{E}[(\overline{N}_t - \overline{N}_s)^2|\mathcal{F}_s] + \mathbb{E}[\overline{N}_s(\overline{N}_t - \overline{N}_s)|\mathcal{F}_s] - \lambda t \\
= \lambda(t - s) + 0 + \overline{N}_s \mathbb{E}[\overline{N}_s - \overline{N}_s|\mathcal{F}_s] - \lambda t \\
= \lambda(t - s) + 0 + \overline{N}_s - \lambda t \\
= \overline{N}_s^2 - \lambda s
\]

This complete the proof.

1.3 Stochastic integral

Definition 1.3.1 Let \(\mathcal{M}^p([0, T], \mathbb{R})\) be the subspace of \(\mathcal{L}^p([0, T], \mathbb{R})\) such that for any process \((X_t) \in \mathcal{M}^p([0, T], \mathbb{R})\) we have
\[
\mathbb{E}\left(\int_0^T |X(t)|^p dt\right) < \infty.
\]

Consider a Brownian motion \(W\) and a stochastic process \((X_t)\) both adapted to a given filtration \((\mathcal{F}_t)\). We will define the following expression called stochastic integral
\[
I_t(X) = \int_0^t X(s)dW(s).
\]

We will also give some of its properties.

Let’s start with the stochastic integral of simple process.

Definition 1.3.2 [Elementary process or simple process]
A process \((X_t)_{t \in \mathbb{R}} \in \mathcal{L}^p([0, T], \mathbb{R})\) is called simple or elementary process if there exist a partition \(0 = t_0 < t_1 < ... < t_n = T\) such that
\[
X_s(\omega) = \sum_{j=0}^n 1_{[t_j, t_{j+1}]}(\theta_j(\omega),
\]
where \(\theta_j\) is a bounded \(\mathcal{F}_{t_j}\)-measurable random variable.

Definition 1.3.3 [Itô’s integral]
The Itô’s Integral of the simple process \((X_t)_{t \in \mathbb{R}} \in \mathcal{L}^2([0, T], \mathbb{R})\) is defined by
\[
I_t(X) = \int_0^t X(s)dW(s) := \sum_{j=0}^{n-1} \theta_j(W_{t_{j+1}} - W_{t_j}).
\]
Lemma 1.3.4 If \( f \) is an elementary function in \( L^2([a,b], \mathbb{R}) \) and \( W_t \) a Brownian motion, then:

1. \( \mathbb{E} \left( \int_a^b f(t) dW_t \right) = 0. \)
2. \( \mathbb{E} \left( \int_a^b f(t) dW_t \right)^2 = \int_a^b \mathbb{E}(f^2(t)) dt. \)

Proof 1.3.5 1. By definition we have
\[
\int_a^b f(t) dW_t = \sum_{j=0}^{n-1} f_j (W_{t_{j+1}} - W_{t_j}).
\]

By taking expectation in both sides, we obtain
\[
\mathbb{E} \left[ \int_a^b f(t) dW_t \right] = \sum_{j=0}^{n-1} \mathbb{E}(f_j) \mathbb{E}(W_{t_{j+1}} - W_{t_j}) = 0,
\]
since \( W_{t_{j+1}} - W_{t_j} \) is a normal distribution with mean 0 and standard deviation \( \sqrt{t_{j+1} - t_j} \).

2. 
\[
\left( \int_a^b f(t) dW_t \right)^2 = \left[ \sum_{j=0}^{n-1} f_j (B_{t_{j+1}} - B_{t_j}) \right]^2
\]
\[
= \sum_{j=0}^{n-1} (f_j)^2 (W_{t_{j+1}} - W_{t_j})^2 + \sum_{j=0}^{n-1} \sum_{k=0, k \neq l}^{n-1} f_j f_k (W_{t_{j+1}} - W_{t_j}) (W_{t_{k+1}} - W_{t_k}).
\]

Taking expectation in both sides and using independence of the increments of Brownian motion, we get
\[
\mathbb{E} \left( \int_a^b f(t) dW_t \right)^2 = \sum_{j=0}^{n-1} \mathbb{E}(f_j)^2 \mathbb{E} (W_{t_{j+1}} - W_{t_j})^2
\]
\[
= \sum_{j=0}^{n-1} \mathbb{E}(f_j)^2 (t_{j+1} - t_j)
\]
\[
= \int_a^b \mathbb{E}(f^2(t)) dt.
\]

The following proposition can be seen in [3].

Proposition 1.3.6 For any process \( X = (X_t)_{t \geq 0} \in \mathcal{M}^2([0,T], \mathbb{R}) \), such that \( \mathbb{E}|X_t|^2 < \infty \) for all \( t \geq 0 \), there exist a sequence \( (f^{(n)}_t)_{t \geq 0} \) of simple processes such that \( \mathbb{E}|f^{(n)}_t|^2 < \infty \) and
\[
\lim_{n \to \infty} \mathbb{E} \left[ \int_0^t |X_s - f^{(n)}_s|^2 ds \right] = 0.
\]

Definition 1.3.7 For any process \( X = (X_t)_{t \geq 0} \in \mathcal{M}^2([0,T], \mathbb{R}) \), we define a stochastic integral of \( X \) with respect to a Brownian motion \( W \) by:
\[
\int_0^t X_s dW(s) = \lim_{n \to \infty} \int_0^t f^{(n)}_s dW(s),
\]

where \( (f^{(n)}_t) \) is the sequence of simple processes converging almost surely to \( X \) according to the previous proposition. Moreover, using Itô isometry for elementary functions one can prove that the limit on this definition does not depend on the actual choice of \( (f^{(n)}_t) \).
1.3. STOCHASTIC INTEGRAL

Proposition 1.3.8 [Properties of Itô integral].
For any process \( X = (X_t)_{t \geq 0} \in \mathbb{M}^2([0,T], \mathbb{R}) \) such that \( \mathbb{E}|X_t|^2 < \infty \), for any functions \( f, g \in \mathbb{M}^2([0,T], \mathbb{R}) \) and \( 0 \leq S < U < T \), the following holds:

(i) \( \int_S^T f dW(t) = \int_S^U f dW(t) + \int_U^T f dW(t) \) almost surely.

(ii) \( \int_S^T (cf + g)dW(t) = c \int_S^T f dW(t) + \int_S^T g dW(t) \), for any constant \( c \).

(iii) \( \int_S^T f dW(t) \) is \( \mathcal{F}_T \)-measurable.

(iv) \( \mathbb{E} \left( \int_0^t X_s dW(s) \right) = 0 \).

(v) \( \mathbb{E} \left( \int_0^t X_s dW(s) \right)^2 = \int_0^t \mathbb{E}(X_s^2) ds \).

Proof 1.3.9 [3]

Proposition 1.3.10 [3] For any elementary function \( f^{(n)} \) \( \mathcal{F}_t \)-adapted, the integral

\[
I_n(t, \omega) = \int_0^t f^{(n)} dW(r)
\]

is a martingale with respect to \( \mathcal{F}_t \).

Proof 1.3.11 For \( t \leq s \), we have:

\[
\mathbb{E}[I_n(s, \omega) / \mathcal{F}_t] = \mathbb{E} \left[ \left( \int_0^s f^{(n)} dW(r) \right) / \mathcal{F}_t \right] = \mathbb{E} \left[ \left( \int_0^t f^{(n)} dW(r) \right) / \mathcal{F}_t \right] + \mathbb{E} \left[ \left( \int_t^s f^{(n)} dW(r) \right) / \mathcal{F}_t \right]
\]

\[
= \int_0^t f^{(n)} dW(r) + \mathbb{E} \left[ \sum_{t \leq t_j^{(n)} \leq t_{j+1}^{(n)} \leq s} f_j^{(n)} \Delta W_j / \mathcal{F}_t \right]
\]

\[
= \int_0^t f^{(n)} dW(r) + \sum_{t \leq t_j^{(n)} \leq t_{j+1}^{(n)} \leq s} \mathbb{E}[f_j^{(n)} \Delta W_j / \mathcal{F}_t]
\]

\[
= \int_0^t f^{(n)} dW(r) + \sum_{t \leq t_j^{(n)} \leq t_{j+1}^{(n)} \leq s} \mathbb{E}[f_j^{(n)} \mathbb{E}[\Delta W_j / \mathcal{F}_t] / \mathcal{F}_t]
\]

\[
= \int_0^t f^{(n)} dW(r) + \sum_{t \leq t_j^{(n)} \leq t_{j+1}^{(n)} \leq s} \mathbb{E}[f_j^{(n)} \mathbb{E}[\Delta W_j / \mathcal{F}_t] / \mathcal{F}_t]
\]

\[
= \int_0^t f^{(n)} dW(r), \text{ since } \mathbb{E}[\Delta W_j / \mathcal{F}_t] = \mathbb{E}[\Delta W_j] = 0
\]

\[
= I_n(t, \omega).
\]

Proposition 1.3.12 [Generalisation]
Let \( f(t, \omega) \in \mathbb{M}^2([0,T], \mathbb{R}) \) for all \( t \). Then the integral

\[
M_t(\omega) = \int_0^t f(s, \omega) dW(s)
\]

is a martingale with respect to \( \mathcal{F}_t \) and

\[
\mathbb{P} \left[ \sup_{0 \leq t \leq T} |M_t| \geq \lambda \right] \leq \frac{1}{\lambda^2} \mathbb{E} \left[ \int_0^T f^2(s, \omega) ds \right], \quad \forall \lambda > 0
\]

Proof 1.3.13 [3].
1.3.14 One dimensional Itô Formula

**Definition 1.3.15 [1-dimensional Itô process]**

Let \( W_t \) be a 1-dimensional Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P})\). An Itô process (or Stochastic integral) is any stochastic process \( X_t \) of the form

\[
X_t = X_0 + \int_0^t u(s, \omega) \, ds + \int_0^t v(s, \omega) \, dW(s),
\]

where \( u \in L^1([0, T], \mathbb{R}) \) and \( v \in L^2([0, T], \mathbb{R}) \).

1.3.16 First 1-dimensional Itô Formula

If \( X_t \) is any process of the form (1.2), then \( f(X_t) \) is an Itô process and

\[
f(X_t) = f(X_0) + \int_0^t f'(X_s) u_s \, ds + \frac{1}{2} \int_0^t f''(X_s) v_s^2 \, ds + \int_0^t f'(X_s) v_s \, dW_s.
\]

**Proof 1.3.17 [2].**

1.3.17 Second 1-dimensional Itô Formula

If in the previous proposition we consider \( f : [0, \infty) \times \mathbb{R} \to \mathbb{R} \) such that \( f \) is once differentiable with respect to the first variable \( t \) and twice differentiable with respect to the second variable \( x \), then \( f(t, X_t) \) is an Itô process and

\[
f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s) \, ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s) u_s \, ds + \int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s) v_s^2 \, ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s) v_s \, dW_s.
\]

or in its differential form:

\[
df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t) \, dt + \frac{\partial f}{\partial x}(t, X_t) \, dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X_t) (dX_t)^2.
\]

\((dX_t)^2 = dX_t dX_t\) is computed according to the rules

\[
dt dt = dW_t dt = dt dW_t = 0, \quad dW_t dW_t = dt.
\]

**Proof 1.3.19 [2].**

1.3.20 Multi-dimensional Itô integral

**Definition 1.3.21 [m-dimensional Brownian motion]**

Let \( W_1, \ldots, W_m \) be \( m \) Brownian motions. The random variable \( W = (W_1, W_2, \ldots, W_m) \) is called a \( m \)-dimensional Brownian motion. Let \( \mathbb{L}^{n \times m}([0, T], \mathbb{R}^{n \times m}) \) denotes the set of \( n \times m \) matrices \( v = [v_{ij}(t, \omega)] \), \( 1 \leq i \leq n, 1 \leq j \leq m \). Where \( v_{ij}(t, \omega) \in L^2([0, T], \mathbb{R}) \). \( \int_0^T v_s \, dW_s \) denotes the Itô integral of \( v \) with respect to the \( m \)-dimensional Brownian motion \( W \). It can be written into its matrix form

\[
\int_0^T v \, dW(s) = \int_0^T \begin{pmatrix} v_{11} & \cdots & v_{1m} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nm} \end{pmatrix} \begin{pmatrix} dW_1(s) \\ \vdots \\ dW_m(s) \end{pmatrix},
\]

which is a \( n \times 1 \) matrix (column vector) whose \( i^{th} \) components are given by

\[
\sum_{j=1}^m \int_0^T v_{ij}(s, \omega) \, dW_j(s).
\]
**Definition 1.3.22 [n-dimensional Itô process]**

Let $W$ be an $m$-Brownian motion and $v = [v_{i,j}, 1 \leq i \leq n, 1 \leq j \leq m]$ an element of $L^{n \times m}([0, t], \mathbb{R}^{n \times m})$. Let $u = (u_i)_{i=1}^n$ such that $u_i \in L^2([0, T])$ for all $1 \leq i \leq n$.

The $n$-dimensional Itô process is any stochastic process of the form

$$dX(t) = u dt + v dW(t),$$

which is a system of $n$ Itô processes, where the $i^{th}$ process is given by:

$$dX_i(t) = u_i dt + \sum_{j=1}^m v_{ij} dW_j(t).$$

**Proposition 1.3.23 [General Itô formula]

Let $dX(t) = u dt + v dW(t)$ be an $n$-dimensional Itô process. Let $g(t, x) = (g_1(t, x), ..., g_p(t, x))$ be a function once differentiable with respect to $t$ and twice differentiable with respect to $x$.

Then the process $Y(t) = g(t, X(t))$ is also a $p$-dimensional Itô process, whose component $Y_k$ are given by:

$$Y_k(t) = \frac{\partial g_k}{\partial t}(t, X_t) dt + \sum_{i=1}^n \frac{\partial g_k}{\partial x_i}(t, X_t) dX_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 g_k}{\partial x_i \partial x_j}(t, X_t) dX_i dX_j,$$

where $dW_i dW_j = \delta_{ij} dt$ and $dW_i dt = dtdW_i = 0$.

**Proof 1.3.24 [3].

### 1.4 Stochastic process with jumps and Stochastic integral with jumps**

**Definition 1.4.1**

(i) A function $f : [0, T] \to \mathbb{R}^n$ is said to be right continuous with left limit at $t \in [0, T]$ if

$$f(t^+) := \lim_{s \to t^+} f(s) \text{ and } f(t^-) := \lim_{s \to t^-} f(s) \text{ exist and } f(t^+) = f(t).$$

(ii) A function $f : [0, T] \to \mathbb{R}^n$ is said to be left continuous with right limit if

$$f(t^+) := \lim_{s \to t^+} f(s) \text{ and } f(t^-) := \lim_{s \to t^-} f(s) \text{ exist and } f(t^-) = f(t).$$

In the litterature, the french short forms ”cádlág” and ”cágład” denote respectively functions which are right continous with left limit and left continous with right limit.

**Remark 1.4.2**

- If $f$ is right continous with left limit at $t$, then $\Delta f(t) = f(t) - f(t^-)$ is called the jump of $f$ at $t$.

- If $f$ is left continous with right limit at $t$, then $\Delta f(t) = f(t^+) - f(t)$ is called the jump of $f$ at $t$.

**Definition 1.4.3** A stochastic process $X = (X_t)_{t \geq 0}$ is called jump process if the sample path $s \mapsto X_s$ is left continuous (cáglád) or right continuous (cádlág) for all $s \geq 0$.

**Definition 1.4.4 [Lévy process]

A stochastic process $X = \{X_t, \ t \geq 0\}$ is a Lévy process if the following conditions are fulfilled

(i) The increments on disjoint time intervals are independent. That is for $0 \leq t_0 < t_1 < ... < t_n$

$$\{X_{t_j} - X_{t_{j-1}}, 1 \leq j \leq n\} \text{ are independent.}$$

(ii) The increments of sample paths are stationary: $X_t - X_s \approx X_{t-s}$ for $0 \leq t \leq s$.
The sample paths are right continuous with left limit.

**Remark 1.4.5** The Brownian motion and the Poisson process starting at 0 are Lévy processes.

**Definition 1.4.6** [2] Let $D_{ucp}$ denote the space of càdlàg adapted process equipped with the topology of the uniform convergence in probability (ucp) on compact sets. $ucp : H_n \rightarrow H$ if $\forall \ t \geq 0 \ \sup_{0 \leq s \leq t} |H_n(s) - H(s)| \rightarrow 0$ in probability ($A_n \rightarrow A$ in probability if $\forall \ \epsilon > 0, \exists \ n_\epsilon \in \mathbb{N}$ such that $n > n_\epsilon \implies P(|A_n - A| > \epsilon) < \epsilon$).

In the sequel $L_{ucp}$ denote the space of adapted càdlàg processes (left continuous with right limit) equipped with the ucp topology.

**Definition 1.4.7** [2] Let $H$ be an elementary function. i.e. there exist a partition $0 = t_0 < t_1 < \ldots < t_n = T$ such that

$$H = \sum_{j=0}^{n} H_j 1_{[t_j, t_{j+1})},$$

where $H_j$ are $\mathcal{F}_{t_j}$-measurable. Let $(X_t)$ be a Lévy process. The stochastic integral $\int_0^t H(s) dX(s)$ is defined by

$$J_X H(t) := \int_0^t H(s) dX(s) := \sum_{j=0}^{n} H_j (X(t_{j+1}) - X(t_j)) \quad t \geq 0.$$

**Proposition 1.4.8** [2] Let $X$ be a semimartingale, then the mapping $J_X$ can be extended to the continuous linear map

$$J_X : L_{ucp} \rightarrow D_{ucp}.$$

The above proposition allows us to define a stochastic integral of the form

$$\int_0^t H(s) dX(s),$$

where $H \in L_{ucp}$.

**Definition 1.4.9** [2] For $H \in L_{ucp}$ we define $\int_0^t H(s) dX(s) :$

$$\int_0^t H(s) dX(s) := \lim_{n \rightarrow +\infty} \int_0^t H^n(s) dX(s),$$

where $(H^n)$ is a sequence of simple process converging to $H$.

**Proposition 1.4.10** Let $f \in L_{ucp}$ and $(\overline{N}_t)$ be a compensated Poisson process. The following holds

(i) $E \left( \int_0^t f(s) d\overline{N}(s) \right) = 0.$

(ii) $E \left( \int_0^t f(s) d\overline{N}(s) \right)^2 = \lambda \int_0^t E(f(s))^2 ds.$

**Proof 1.4.11** [2]
1.4.12 Itô formula for jump process

Definition 1.4.13 [Itô jump-diffusion process]

The Itô jump-diffusion process is any process of the form
\[ X_t = X_0 + \int_0^t a(X_s)ds + \int_0^t b(X_s)dW_s + \int_0^t c(X_s)dN_s. \] (1.3)

The coefficient \( a \in L^1([0,T],\mathbb{R}^n) \) is called the drift coefficient, \( b \in L^2([0,T],\mathbb{R}^{n\times m}) \) is called the diffusion coefficient and \( c(X_s) \in L^2([0,T],\mathbb{R}^n) \) is called the jump coefficient. \( W \) is a \( m \)-dimensional Brownian motion and \( N \) a one dimensional poisson process.

Proposition 1.4.14 [2, pp 6] [Itô formula for jump process]

If \( (X_t) \) is a jump-diffusion process of the form (1.3) and \( f : [0,\infty) \rightarrow \mathbb{R}^n \) any function twice derivable, then \( f(X_t) \) is a jump-diffusion process and satisfies the following equation
\[
\frac{d}{dt}f(X_t) = f(X_0) + \int_0^t \frac{\partial f}{\partial x}(X_s) a_s ds + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(X_s) b_s^2 ds + \int_0^t \frac{\partial f}{\partial x}b_s dW_s + \int_0^t (f(X_s) + c(X_s)) - f(X_s) dN_s.
\]

Lemma 1.4.15 [Itô’s lemma for product]

If \( X_t \) and \( Y_t \) are two Itô’s jump-diffusion process, Then \( X_t Y_t \) is an Itô jump-diffusion process and
\[
d(X_t Y_t) = Y_t dX_t + X_t dY_t + dX_t dY_t.
\]

\( dX_1 dX_2 \) is called the Itô’s corrective term and it is computed according to the relations
\[
dt dt = dN_t dt = dt dN_t = dW_t dN_t = dN_t dW_t = 0, \quad dN_t dN_t = dN_t.
\]

Proof 1.4.16 [2]

After being familiar with some basic notions in probability theory and stochastic process, we are now ready to provide in the following chapter the proof of the existence and uniqueness of solution of SDEs with jumps under global Lipschitz conditions.
Chapter 2

Existence and uniqueness of solution of the jump-diffusion Itô’s stochastic differential equations

In this chapter, we give the general formulation of the compensated stochastic differential equation (CSDE) with jumps which will be helpful to prove the existence and the uniqueness solutions of the stochastic differential equation with jump.

2.1 General formulation

Along this work, \( \| \cdot \| \) denote the Frobenius matrix norm, \((\Omega, \mathcal{F}_t, \mathbb{P})\) denote a complete probability space. For all \( x, y \in \mathbb{R}^n, \langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n \) is the inner product. For all \( a, b \in \mathbb{R} \),
\[
\max(a, b) := a \lor b.
\]
Throughout this work, we consider a jump diffusion Itô’s stochastic differential (SDEs) of the form
\[
dX(t) = f(X(t^-))dt + g(X(t^-))dW(t) + h(X(t^-))dN(t), \quad X(0^-) = X_0,
\]
where \( X_0 \) is the initial condition, \( X(t^-) = \lim_{s \to t^-} X(s) \), \( W(t) \) is an \( m \)-dimensional Brownian motion and \( N(t) \) is a 1-dimensional poisson process with intensity \( \lambda > 0 \). We assume \( W_t \) and \( N_t \) to be both \( \mathcal{F}_t \)-measurable. \( f : \mathbb{R}^n \to \mathbb{R}^n, g : \mathbb{R}^n \to \mathbb{R}^{n \times m} \) and \( h : \mathbb{R}^n \to \mathbb{R}^n \).

Our aim in this chapter is to prove the existence and the uniqueness of solution of equation (2.1) in the strong sense.

Definition 2.1.1 [Strong solution]
A stochastic process \( X = \{X_t\}_{t \in [0,T]} \) is called strong solution of Itô jump-diffusion differential equation (2.1) if :
1. \( X_t \) is \( \mathcal{F}_t \)-measurable \( \forall t \in [0, T] \).
2. \( \mathbb{P} \left[ \int_0^t |f(X_s, s)| ds < \infty \right] = \mathbb{P} \left[ \int_0^t |g(X_s, s)|^2 ds < \infty \right] = \mathbb{P} \left[ \int_0^t |h(X_s, s)|^2 ds < \infty \right] = 1 \).
3. \( X \) satisfies equation (2.1) almost surely.

Assumptions 2.1.2 Throughout this chapter, we make the following hypothesis :
There exist positive constants \( L \) and \( K \) such that for all \( x, y \in \mathbb{R}^n \),
1. \( \mathbb{E}||X(0)||^2 < +\infty \) and \( X(0) \) is independent of the Weiner process \( W(t) \) and of the poisson process \( N(t) \).
2. \( f, g \) and \( h \) satisfy the global Lipschitz condition :
\[
||f(x) - f(y)||^2 \lor ||g(x) - g(y)||^2 \lor ||h(x) - h(y)||^2 \leq L||x - y||^2
\]
3. $f$, $g$ and $h$ satisfy the linear growth condition:

$$||f(x)||^2 \vee ||g(x)||^2 \vee ||h(x)||^2 \leq K(1 + ||x||^2).$$

(2.3)

**Remark 2.1.3** The globally Lipschitz condition implies the linear growth condition. So it is enough to make the assumptions only on the global Lipschitz condition.

**Definition 2.1.4** A solution $\{X_t\}$ of (2.1) is said to be pathwise unique if any other solution $\overline{X}$ is stochastically indistinguishable from it, that is $\mathbb{P}\{X(t) = \overline{X}(t)\} = 1$, $\forall t \in [0, T]$ almost surely.

In order to prove the existence and uniqueness solution of (2.1), it is useful to write (2.1) in its compensated form.

**2.1.5 Compensated stochastic differential equation (CSDE)**

From the relation $\overline{N}(t) = N(t) - \lambda t$, we have $d\overline{N}(t) = dN_t - \lambda dt$. Substituting this latter relation in (2.1) leads to:

$$dX(t) = f\lambda(X(t^-))dt + g(X(t^-))dW(t) + h(X(t^-))d\overline{N}(t),$$

(2.4)

where

$$f\lambda(x) := f(x) + \lambda h(x).$$

(2.5)

(2.4) can be rewritten into its integral form

$$X(t) = X_0 + \int_0^t f\lambda(X(s^-))ds + \int_0^t g(X(s^-))dW(s) + \int_0^t h(X(s^-))d\overline{N}(s),$$

(2.6)

**Lemma 2.1.6** If assumptions 2.1.2 are satisfied, then the function $f\lambda$ satisfies the global Lipschitz and the linear growth conditions with constants $L_\lambda = (1 + \lambda)^2L$ and $K_\lambda = (1 + \lambda)^2K$ respectively.

**Proof 2.1.7**

1. Using the global lipschitz condition satisfied by $f$ and $h$, it follows from (2.5) that:

$$||f\lambda(x) - f\lambda(y)||^2 = ||f(x) - f(y) + \lambda(h(x) - h(y))||^2 \leq \left(\sqrt{L} ||x - y|| + \lambda \sqrt{L} ||x - y||\right)^2 = (1 + \lambda)^2L||x - y||^2.

2. Along the same lines as above, we obtain the linear growth condition satisfies by $f\lambda$.

**2.2 Well-posedness problem**

Based on Lemma 2.1.6 and using the fact that equations (2.1) and (2.4) are equivalent, the existence and uniqueness of solution of equation (2.1) is equivalent to the existence and uniqueness of solution of equation (2.4).

**Theorem 2.2.1** If Assumptions 2.1.2 are fulfilled, then there exist a unique strong solution of equation (2.4).

In order to prove Theorem 2.2.1, we need the following lemma.

**Lemma 2.2.2** let $X^0(t) = X_0(t)$, $\forall t \in [0, T]$ and

$$X^{n+1}(t) = X_0 + \int_0^t f(X^n(s^-))ds + \int_0^t g(X^n(s^-))dW(s) + \int_0^t h(X^n(s^-))d\overline{N}(s)ds,$$

(2.7)

then

$$\mathbb{E}||X^{n+1}(t) - X^n(t)||^2 \leq \frac{(Mn)^{n+1}}{(n+1)!},$$

(2.8)

where $M$ is a positive constant depending on $\lambda, K, L, X_0$. 

Proof 2.2.3: By induction

1. For $n = 0$

$$
\|X^1(t) - X^0(t)\|^2 = \left\| \int_0^t f(X_0(s^-)) ds + \int_0^t g(X_0(s^-)) dW(s) + \int_0^t h(X_0(s^-)) d\mathbb{N}(s) \right\|^2 \\
\leq 3 \left\| \int_0^t f(X_0(s^-)) ds \right\|^2 + 3 \left\| \int_0^t g(X_0(s^-)) dW(s) \right\|^2 + 3 \left\| \int_0^t h(X_0(s^-)) d\mathbb{N}(s) \right\|^2.
$$

Using Cauchy-Schwartz inequality and the linear growth condition, it follows that:

$$
\mathbb{E}(I_1) := \mathbb{E}\left(3 \left\| \int_0^t f(X_0(s^-)) ds \right\|^2\right) \leq 3T \int_0^t \mathbb{E}\|f(X_0(s^-))\|^2 ds \\
\leq 3TK\int_0^t (1 + E\|X_0\|^2) ds.
$$

From the martingale property of $\mathbb{N}(t)$ and the linear growth condition, it follows that:

$$
\mathbb{E}(I_2) := \mathbb{E}\left(3 \left\| \int_0^t h(X_0(s^-)) d\mathbb{N}(s) \right\|^2\right) = 3\int_0^t \mathbb{E}\|h(X_0(s^-))\|^2 ds \\
\leq 3\lambda K\int_0^t (1 + E\|X_0\|^2) ds.
$$

Using the martingale property of $W(t)$ and the linear growth condition, it follows that:

$$
\mathbb{E}(I_3) := \mathbb{E}\left(3 \left\| \int_0^t g(X_0(s^-)) dW(s) \right\|^2\right) \leq 3K \int_0^t (1 + E\|X_0\|^2) ds.
$$

Taking expectation in both sides of (2.9) and using estimations (2.10), (2.11) and (2.12) leads to:

$$
\mathbb{E}\|X^1(t) - X^0(t)\|^2 \leq (3TK + 3K + 3\lambda K) \int_0^t (1 + E\|X_0\|^2) ds \leq Mt,
$$

where $M = (3TK + 3K + 3\lambda K) \vee (3TL + 3L + 3\lambda L)$.

2. Let’s assume that inequality (2.8) holds up to a certain rank $n \geq 0$. We have to show that it remains true for $n + 1$. That is, we have to prove that

$$
\mathbb{E}\|X^{n+2}(t) - X^{n+1}(t)\|^2 \leq \frac{(Mt)^{n+2}}{(n+2)!}.
$$

$$
X^{n+2}(t) = X_0 + \int_0^t f(X^{n+1}(s^-)) ds + \int_0^t g(X^{n+1}(s^-)) dW(s) + \int_0^t h(X^{n+1}(s^-)) d\mathbb{N}(s),
$$

$$
X^{n+1}(t) = X_0 + \int_0^t f(X^n(s^-)) ds + \int_0^t g(X^n(s^-)) dW(s) + \int_0^t h(X^n(s^-)) d\mathbb{N}(s)
$$

$$
\|X^{n+2}(t) - X^{n+1}(t)\|^2 = \left\| \int_0^t [f(X^{n+1}(s^-) - f(X^n(s^-))] ds \\
+ \int_0^t [g(X^{n+1}(s^-)) - g(X^n(s^-))] dW(s) \\
+ \int_0^t [h(X^{n+1}(s^-) - h(X^n(s^-))] d\mathbb{N}(s) \right\|^2.
$$
Using the inequality \((a + b + c)^2 \leq 3a^2 + 3b^2 + 3c^2\) for all \(a, b, c \in \mathbb{R}\), it follows that:

\[
||X^{n+2}(t) - X^{n+1}(t)||^2 \leq 3 \left( \int_0^t [f_\lambda(X^{n+1}(s^-)) - f_\lambda(X^n(s^-))]ds \right)^2 \\
+ 3 \left( \int_0^t [g(X^{n+1}(s^-)) - g(X^n(s^-))]dW \right)^2 \\
+ 3 \left( \int_0^t [h(X^{n+1}(s^-)) - h(X^n(s^-))]d\mathcal{N} \right)^2.
\]

Using the martingale properties of \(W(s)\) and \(\mathcal{N}(s)\) and the global Lipschitz condition satisfied by \(f_\lambda, g\) and \(h\), it follows that:

\[
\mathbb{E}[||X^{n+2}(t) - X^{n+1}(t)||^2] \leq (3TL_\lambda + 3L + 3\lambda L) \int_0^t \mathbb{E}[||X^{n+1}(s^-) - X^n(s^-)||^2]ds.
\]

Using the hypothesis of induction, it follows that:

\[
\mathbb{E}[||X^{n+2}(t) - X^{n+1}(t)||^2] \leq M \int_0^t \frac{(Ms)^{n+1}}{(n+1)!}ds = \frac{(Mt)^{n+2}}{(n+2)!}.
\]

This complete the proof of the lemma.

**Proof 2.2.4 [Theorem 2.2.1]**

**Uniqueness**: Let \(X_1\) and \(X_2\) be two solutions of (2.6). Then:

\[
X_1(t) = X_0 + \int_0^t f_\lambda(X_1(s^-))ds + \int_0^t g(X_1(s^-))dW(s) + \int_0^t h(X_1(s^-))d\mathcal{N}(s),
\]

\[
X_2(t) = X_0 + \int_0^t f_\lambda(X_2(s^-))ds + \int_0^t g(X_2(s^-))dW(s) + \int_0^t h(X_2(s^-))d\mathcal{N}(s).
\]

Therefore,

\[
||X_1(t) - X_2(t)||^2 = \left\| \int_0^t [f_\lambda(X_1(s^-)) - f_\lambda(X_2(s^-))]ds + \int_0^t [g(X_1(s^-)) - g(X_2(s^-))]dW(s) \\
+ \int_0^t [h(X_1(s^-)) - h(X_2(s^-))]d\mathcal{N}(s) \right\|^2 \\
\leq 3 \left\| \int_0^t [f_\lambda(X_1(s^-)) - f_\lambda(X_2(s^-))]ds \right\|^2 \\
+ 3 \left\| \int_0^t [g(X_1(s^-)) - g(X_2(s^-))]dW(s) \right\|^2 \\
+ 3 \left\| \int_0^t [h(X_1(s^-)) - h(X_2(s^-))]d\mathcal{N}(s) \right\|^2.
\]

Using Cauchy-Schwartz inequality and the globaly Lipschitz condition, it follows that:

\[
\mathbb{E}(I_1(t)) := \mathbb{E}\left( 3 \left\| \int_0^t [f_\lambda(X_1(s^-)) - f_\lambda(X_2(s^-))]ds \right\|^2 \right) \\
\leq 3t \int_0^t \mathbb{E}\left\| f_\lambda(X_1(s^-)) - f_\lambda(X_2(s^-)) \right\|^2 \\
\leq 3tL_\lambda \int_0^t \mathbb{E}[||X_1(s^-) - X_2(s^-)||^2]ds.
\]
Using the martingale property of $W(s)$ and the global Lipschitz condition, it follows that:

\[
\mathbb{E}(I_2(t)) := \mathbb{E} \left( 3 \left\| \int_0^t [g(X_1(s^-)) - g(X_2(s^-))]dW(s) \right\|^2 \right) \\
\leq 3 \int_0^t \mathbb{E} \left\| g(X_1(s^-)) - g(X_2(s^-)) \right\|^2 ds \\
\leq 3L \int_0^t \mathbb{E} \left\| X_1(s^-) - X_2(s^-) \right\|^2 ds.
\]

(2.16)

Along the same lines as above, we obtain:

\[
\mathbb{E}(I_3(t)) := \mathbb{E} \left( 3 \left\| \int_0^t [h(X_1(s^-)) - h(X_2(s^-))]d\overline{N}(s) \right\|^2 \right) \\
\leq 3L \int_0^t \mathbb{E} \left\| X_1(s^-) - X_2(s^-) \right\|^2 ds.
\]

(2.17)

Taking expectation in both sides of (2.14) and using estimations (2.15), (2.16) and (2.17) leads to:

\[
\mathbb{E}\left\| X_1(t) - X_2(t) \right\|^2 \leq (3tL\lambda + 3L + 3\lambda L) \int_0^t \mathbb{E}\left\| X_1(s^-) - X_2(s^-) \right\|^2 ds.
\]

(2.18)

Applying Gronwall lemma (continuous form) to inequality (2.18) leads to:

\[
\mathbb{E}\left\| X_1(t) - X_2(t) \right\|^2 = 0, \quad \forall t \in [0, T].
\]

It follows from Markov’s inequality that:

\[
\forall a > 0, \quad \mathbb{P}(\|X_1 - X_2\|^2 > a) = 0.
\]

Therefore,

\[
\mathbb{P}(\{X_1(t) = X_2(t), \ t \in [0, T]\}) = 1 \ a.s.
\]

**Existence:**

From the sequence $X^n(t)$ defined in Lemma 2.2.2, it follows that:

\[
\left\| X^{n+1}(t) - X^n(t) \right\|^2 = \left\| \int_0^t [f_n(X^n(s^-)) - f_n(X^{n-1}(s^-))]ds + \int_0^t [g(X^n(s^-)) - g(X^{n-1}(s^-))]dW(s) \\
+ \int_0^t [h(X^n(s^-)) - h(X^{n-1}(s^-))]d\overline{N}(s) \right\|^2.
\]

(2.19)

Taking expectation and the supremum in the both sides of inequality (2.19) and using inequality $(a + b + c)^2 \leq 3a^2 + 3b^2 + 3c^2$ for all $a, b, c \in \mathbb{R}$ leads to:

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \left\| X^{n+1}(t) - X^n(t) \right\|^2 \right) \leq 3T \sup_{0 \leq t \leq T} \left( \mathbb{E} \int_0^t \left\| f_n(X^n(s^-)) - f_n(X^{n-1}(s^-)) \right\|^2 ds \right) \\
+ 3\mathbb{E} \left( \sup_{0 \leq t \leq T} M_1(t) \right) + 3\mathbb{E} \left( \sup_{0 \leq t \leq T} M_2(t) \right),
\]

(2.20)

where

\[
M_1(t) = \left\| \int_0^t [g(X^n(t)) - g(X^{n-1}(t))]dW(s) \right\|^2 \\
M_2(t) = \left\| \int_0^t [h(X^n(t)) - h(X^{n-1}(t))]d\overline{N}(s) \right\|^2.
\]
Using Lemma 2.2.2, it follows that
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \int_0^t \| f_\lambda(X^n(s^-)) - f_\lambda(X^{n-1}(s^-)) \|^2 ds \right) \leq L_\lambda \int_0^T \mathbb{E} \| X^n(s^-) - X^{n-1}(s^-) \|^2 ds. \tag{2.21}
\]

Using respectively Doop’s maximal inequality, martingale property of \( W(s) \) and global Lipschitz condition satisfied by \( g \), it follows that:
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} M_1(t) \right) \leq 4M_1(t) = 4 \mathbb{E} \left| \int_0^T [g(X^n(s^-)) - g(X^{n-1}(s^-))]dW(s) \right|^2 = 4 \int_0^T \mathbb{E} \| g(X^n(s^-)) - g(X^{n-1}(s^-)) \|^2 ds \leq 4L \int_0^T \mathbb{E} \| X^n(s^-) - X^{n-1}(s^-) \|^2 ds. \tag{2.22}
\]

Along the same lines as above, we obtain
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} M_2(t) \right) \leq 4\lambda L \int_0^T \mathbb{E} \| X^n(s^-) - X^{n-1}(s^-) \|^2 ds. \tag{2.23}
\]

Inserting (2.21), (2.22) and (2.23) in (2.20) leads to:
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \| X^{n+1}(t) - X^n(t) \|^2 \right) \leq (3L_\lambda + 12L + 12\lambda L) \int_0^T \mathbb{E} \| X^n(s^-) - X^{n-1}(s^-) \|^2 ds.
\]

Using Lemma 2.2.2, it follows that:
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \| X^{n+1}(t) - X^n(t) \|^2 \right) \leq C \int_0^T \frac{(Ms^-)^n}{n!} ds = C \frac{(MT^-)^{n+1}}{(n+1)!},
\]
where \( C = 3L_\lambda + 12L + 12\lambda L \).

It follows from Doop’s and Markov’s inequalities that:
\[
\mathbb{P} \left( \sup_{0 \leq t \leq T} \| X^{n+1}(t) - X^n(t) \| > \frac{1}{2^{n+1}} \right) \leq \mathbb{E} \| X^{n+1}(t) - X^n(t) \|^2 \left( \frac{1}{(2^{n+1})^2} \right) \leq C \frac{(2^2 MT)^{n+1}}{(n+1)!}.
\]

Moreover,
\[
\sum_{n=0}^{\infty} \frac{(2^2 MT)^{n+1}}{(n+1)!} = e^{2MT} - 1 < \infty.
\]

Using Borel Cantelli’s Lemma, it follows that:
\[
\mathbb{P} \left( \sup_{0 \leq t \leq T} \| X^{n+1}(t) - X^n(t) \| > \frac{1}{2^{n+1}} \right) = 0, \quad \text{almost surely.}
\]

Therefore, for almost every \( \omega \in \Omega \), \( \exists n_0(\omega) \) such that
\[
\sup_{0 \leq t \leq T} \| X^{n+1}(t) - X^n(t) \| \leq \frac{1}{2^{n+1}}, \quad \forall n \geq n_0(\omega).
\]

Therefore \( X^n \) converge uniformly on \([0, T]\). Let \( X \) be its limit.

Furthermore, since \( X^n_t \) is continuous and \( \mathcal{F}_t \)-measurable for all \( n \in \mathbb{N} \), it follows that \( X(t) \) is continuous and \( \mathcal{F}_t \)-measurable. It remains to prove that \( X \) is a solution of equation 2.1.
One can see that \((X^n)\) converges in \(L^2([0,T] \times \Omega, \mathbb{R}^n)\). Indeed,
\[
||X^m_t - X^n_t||_{L^2}^2 \leq \sum_{k=n}^{m-1} ||X^{k+1}_t - X^k_t||_{L^2} \leq \sum_{k=n}^{\infty} \frac{(MT)^{k+1}}{(k+1)!} \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.
\]
Therefore, \((X^n)\) is a Cauchy sequence in a Banach \(L^2([0,T] \times \Omega, \mathbb{R}^n)\), so it converges to \(X\).

Using Fatou’s lemma, it follows that:
\[
\mathbb{E} \left[ \int_0^T ||X_t - X^n_t||^2 dt \right] \leq \liminf_{m \to +\infty} \mathbb{E} \left[ \int_0^T ||X^m_t - X^n_t||^2 dt \right] \longrightarrow 0, \quad \text{when} \quad m \longrightarrow +\infty.
\]

Using the global Lipschitz condition and the Ito isometry, it follows that
\[
\mathbb{E} \left[ \int_0^t |g(X_s) - g(X^n_s)| dW_s \right]^2 \leq L \mathbb{E} \int_0^t ||X_s - X^n_s||^2 ds \longrightarrow 0 \quad \text{when} \quad n \longrightarrow +\infty.
\]

So we have the following convergence in \(L^2([0,T] \times \mathbb{R}^n)\)
\[
\int_0^t g(X^n_s) dW_s \longrightarrow \int_0^t g(X_s) dW_s.
\]

Along the same lines as above, the following holds convergence holds in \(L^2([0,T] \times \mathbb{R}^n)\)
\[
\int_0^t g(X^n_s) d\mathcal{N}_s \longrightarrow \int_0^t g(X_s) d\mathcal{N}_s.
\]

Using Holder inequality and the global Lipschitz condition, it follows that:
\[
\mathbb{E} \left[ \int_0^t |f_\lambda(X_s) - f_\lambda(X^n_s)| ds \right]^2 \leq L_\lambda \mathbb{E} \int_0^t ||X_s - X^n_s||^2 ds \longrightarrow 0 \quad \text{when} \quad n \longrightarrow +\infty.
\]

So the following convergence holds in \(L^2([0,T] \times \mathbb{R}^n)\)
\[
\int_0^t f_\lambda(X^n_s) d\mathcal{N}_s \longrightarrow \int_0^t f_\lambda(X_s) d\mathcal{N}_s.
\]

Therefore, taking the limit in the sense of \(L^2([0,T] \times \mathbb{R}^n)\) in the both sides of the following equality:

\[
X^{n+1}(t) = X_0 + \int_0^t f(X^n(s^-)) ds + \int_0^t g(X^n(s^-)) dW(s) + \int_0^t h(X^n(s^-)) d\mathcal{N}(s) ds
\]

leads to:
\[
X(t) = X_0 + \int_0^t f(X(s^-)) ds + \int_0^t g(X(s^-)) dW(s) + \int_0^t h(X(s^-)) d\mathcal{N}(s) ds.
\]

So \(X(t)\) is a strong solution of (2.1). This complete the proof of Theorem 2.2.1.

Generally, analytical solutions of SDEs are unknows. Knowing that the exact solution exist, one tool to approach it, is the numerical resolution. In the following chapters, we provide some numerical schemes for SDEs with jumps.
Chapter 3

Strong convergence and stability of the compensated stochastic theta methods

Our goal in this chapter is to prove the strong convergence of the compensated stochastic theta method (CSTM) and to analyse the stability behavior of both stochastic theta method (STM) and CSTM under global Lipschitz conditions. The strong convergence and stability of STM for SDEs with jumps has been investigated in [5], while the strong convergence and stability of the CSTM for SDEs with jumps has been investigated in [7]. Most results presented in this chapter are from [5] and [7]. In the following section, we recall the theta method which will be used to introduce the STM and the CSTM.

3.1 Theta Method

Let’s consider the following deterministic differential equation

\[
\begin{align*}
  \frac{du}{dt} &= f(t, u(t)) \\
  u(t_0) &= u_0,
\end{align*}
\]

which can be written into its following integral form:

\[u(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds.\] (3.1)

3.1.1 Euler explicit method

This method uses the following approximation:

\[\int_{a}^{b} f(s)ds \simeq (b - a)f(a).\]

So for a constant step \(\Delta t\), the Euler explicit approximation of (3.1) is given by:

\[u_{k+1} = u_k + \Delta t f(t_k, u_k),\]

where \(u_k := u(t_k)\).

3.1.2 Euler implicit method

This method uses the following approximation:

\[\int_{a}^{b} f(s)ds \simeq (b - a)g(b).\]

Therefore, for a constant step \(\Delta t\), the Euler implicit approximation of (3.1) is:

\[u_{k+1} = u_k + \Delta t f(t_k, u_{k+1}).\]
3.1.3 Theta Euler method

In order to have a better approximation of the integral, we can take a convex combination of Euler explicit and Euler implicit method. So we have the following approximation

$$\int_a^b f(s)ds \simeq (b - a)[(1 - \theta)f(a) + \theta f(b)],$$

where $\theta$ is a constant satisfying $0 \leq \theta \leq 1$.

Hence, for a constant step $\Delta t$, the Euler theta approximation of (3.1) is:

$$u_{k+1} = u_k + \Delta t[(1 - \theta)f(t_k, u_k) + \theta f(t_k, u_{k+1})].$$

For $\theta = 1$, the Euler theta method is called Euler backward method, which is also the Euler implicit method.

3.1.4 Stochastic theta method and compensated stochastic theta method

(STM and CSTM)

In order to have an approximate solution of equation (2.1), we use the theta Euler method for the deterministic integral and the Euler explicit method for the two random parts. So we have the following approximate solution of (2.1) called Stochastic theta method (STM):

$$Y_{n+1} = Y_n + (1 - \theta)f(Y_n)\Delta t + \theta f(Y_{n+1})\Delta t + g(Y_n)\Delta W_n + h(Y_n)\Delta N_n, \quad (3.2)$$

where $Y_n := X(t_n)$, $\Delta W_n := W(t_{n+1}) - W(t_n)$ and $\Delta N_n := N(t_{n+1}) - N(t_n)$.

Applying the same rules as for the STM to equation (2.6) leads to:

$$Y_{n+1} = Y_n + (1 - \theta)f_{\lambda}(Y_n)\Delta t + \theta f_{\lambda}(Y_{n+1})\Delta t + g(Y_n)\Delta W_n + h(Y_n)\Delta \bar{N}_n, \quad (3.3)$$

where

$$f_{\lambda}(x) = f(x) + \lambda h(x).$$

The numerical approximation (3.3) is called compensated stochastic theta method (CSTM).

3.2 Strong convergence of the CSTM on a finite time interval $[0,T]$ 

In this section, we prove the strong convergence of order 0.5 of the CSTM. Throughout, $T$ is a fixed constant. For $t \in [t_n, t_{n+1})$ we define the continuous time approximation of (3.3) as follows:

$$\bar{Y}(t) := Y_n + (1 - \theta)(t - t_n)f_{\lambda}(Y_n) + \theta(t - t_n)f_{\lambda}(Y_{n+1}) + g(Y_n)\Delta W_n(t) + h(Y_n)\Delta \bar{N}_n(t), \quad (3.4)$$

where $\Delta W_n(t) := W(t) - W(t_n)$, $\Delta \bar{N}_n(t) := \bar{N}(t) - \bar{N}(t_n)$.

The continuous approximation (3.4) can be written into its following integral form:

$$\bar{Y}(t) = Y_0 + \int_0^t [(1 - \theta)f_{\lambda}(Y(s)) + \theta f_{\lambda}(Y(s + \Delta t))] ds + \int_0^t g(Y(s))dW(s) + \int_0^t h(Y(s))d\bar{N}(s), \quad (3.5)$$

where

$$Y(s) := \sum_{n=0}^{\infty} 1_{\{t_n \leq s < t_{n+1}\}} Y_n.$$

It follows from (3.4) that $\bar{Y}(t_n) = Y_n$. In others words, $\bar{Y}(t)$ and $Y_n$ coincide at the grid points. The main result of this section is formulated in the following theorem.
3.2. STRONG CONVERGENCE OF THE CSTM ON A FINITE TIME INTERVAL \([0,T]\)

**Theorem 3.2.1** Under Assumptions 2.1.2, the continuous time approximation solution \(\overline{Y}(t)\) given by (3.5) converges to the true solution \(X(t)\) of (2.1) in the mean square sense. More precisely,

\[
E\left(\sup_{0 \leq t \leq T} ||\overline{Y}(t) - X(t)||^2\right) \leq C(1 + E||X_0||^2)\Delta t,
\]

where \(C\) is a positive constant independent of the stepsize \(\Delta t\).

In order to prove Theorem 3.2.1, we need the following two lemmas.

**Lemma 3.2.2** Under Assumptions 2.1.2, there exist a fixed constant \(\Delta t_0\) such that for any stepsize \(\Delta t\) satisfying \(0 < \Delta t < \Delta t_0 = \frac{1}{K_\lambda + 1}\), the following bound of the numerical solution holds

\[
\sup_{0 \leq n\Delta t \leq T} E||Y_n||^2 \leq C_1(1 + E||X_0||^2),
\]

where \(C_1\) is a positive constant independent of \(\Delta t\).

**Proof 3.2.3** From (3.3), it follows that :

\[
||Y_{n+1} - \theta \Delta t f_\lambda(Y_{n+1})||^2 = ||Y_n + (1 - \theta) f_\lambda(Y_n) \Delta t + g(Y_n) \Delta W_n + h(Y_n) \Delta \overline{N}_n||^2.
\]

Taking expectation in both sides of (3.6) leads to :

\[
E||Y_{n+1} - \theta \Delta t f_\lambda(Y_{n+1})||^2 = E||Y_n||^2 + (1 - \theta)^2(\Delta t)^2 E||f_\lambda(Y_n)||^2 + E||g(Y_n) \Delta W_n||^2 + E||h(Y_n) \Delta \overline{N}_n||^2
\]

\[+ 2E\langle Y_n, (1 - \theta) f_\lambda(Y_n) \Delta t \rangle.
\]

(3.7)

Since \(W\) is a Brownian motion, \(\Delta W_n = W_{t_{n+1}} - W_{t_n} \sim \mathcal{N}(0, t_{n+1} - t_n)\). So \(E(\Delta W_n) = 0\).

Using the properties \(E(\Delta W_n) = 0\) and \(E(\Delta \overline{N}_n) = 0\), we have

\[
E(Y_n, g(Y_n) \Delta W_n) = E(f_\lambda(Y_n) \Delta, g(Y_n) \Delta W_n) = E(f_\lambda(Y_n) \Delta t, h(Y_n) \Delta \overline{N}_n) = 0.
\]

The martingale properties of \(\Delta W_n\) and \(\Delta \overline{N}_n\) leads to :

\[
E||g(Y_n) \Delta W_n||^2 = E||g(Y_n)||^2 \Delta t \quad \text{and} \quad E||h(Y_n) \Delta \overline{N}_n||^2 = \lambda \Delta t E||h(Y_n)||^2.
\]

Hence equality (3.7) becomes :

\[
E||Y_{n+1} - \theta \Delta t f_\lambda(Y_{n+1})||^2 = E||Y_n||^2 + (1 - \theta)^2(\Delta t)^2 E||f_\lambda(Y_n)||^2 + E||g(Y_n)||^2 \Delta t
\]

\[+ 2(1 - \theta) \lambda \Delta t E\langle Y_n, f_\lambda(Y_n) \rangle + \lambda \Delta t E||h(Y_n)||^2.
\]

(3.8)

Using Cauchy-Schwartz inequality and the linear growth condition, it follows that :

\[
E(Y_n, f_\lambda(Y_n)) = E||Y_n f_\lambda(Y_n)|| \leq \sqrt{E||Y_n||^2 E||f_\lambda(Y_n)||^2} \leq \frac{1}{2} E||Y_n||^2 + \frac{1}{2} E||f_\lambda(Y_n)||^2 \leq \frac{K_\lambda}{2} + \frac{1}{2}(1 + K_\lambda) E||Y_n||^2.
\]

By the same arguments as above, it follows that :

\[
E(Y_{n+1}, f_\lambda(Y_{n+1})) \leq \frac{K_\lambda}{2} + \frac{1}{2}(1 + K_\lambda) E||Y_{n+1}||^2.
\]

Since \(\theta \in [0,1]\) and \(\Delta t \in [0,1]\), it follows from (3.8) that :

\[
E||Y_{n+1}||^2 \leq 2\Delta t E||Y_{n+1}, f_\lambda(Y_{n+1})|| + E||Y_n||^2 + \Delta t E||f_\lambda(Y_n)||^2 + \Delta t E||g(Y_n)||^2
\]

\[+ 2\Delta t E\langle Y_n, f_\lambda(Y_n) + \lambda \Delta t E||h(Y_n)||^2 \rangle \leq 2\Delta t \left[\frac{1}{2}(1 + K_\lambda) E||Y_{n+1}||^2 + \frac{1}{2} K_\lambda\right] + E||Y_n||^2 + \Delta t K_\lambda (1 + E||Y_n||^2) + \Delta t K(1 + E||Y_n||^2)
\]

\[+ 2\Delta t \left[\frac{1}{2}(1 + K_\lambda) E||Y_n||^2 + \frac{1}{2} K_\lambda\right] + \lambda \Delta t K(1 + E||Y_n||^2).
\]
Therefore, from the above inequality the following holds:

\[(1 - \Delta t - \Delta t K)E||Y_{n+1}||^2 \leq (1 + \Delta t K + \Delta t + \Delta t K + \lambda \Delta t K)E||Y_n||^2 + \Delta t K + \Delta t K + \lambda \Delta t K \leq (1 + 2\Delta t K + \Delta t K + \Delta t \lambda K)E||Y_n||^2 + 3\Delta t K + \Delta t K + \lambda \Delta t K.\]

Then it follows from the previous inequality that:

\[E||Y_{n+1}||^2 \leq \left(1 + \frac{3\Delta t K + K + \lambda K t + 2\Delta t t}{1 - \Delta t - K \Delta t}\right)E||Y_n||^2 + \frac{3\Delta t K + K + \lambda K t + 2\Delta t t}{1 - \Delta t - K \Delta t t}E||Y_n||^2 + \frac{3\Delta t K + K + \lambda K t + 2\Delta t t}{1 - \Delta t - K \Delta t t}E||Y_n||^2.\]

Since \(\Delta t - \Delta t _0 < \frac{1}{K + 1}\), we have \(1 - \Delta t - K \Delta t > 1 - \Delta t _0 - K \Delta t _0 > 0\) and then

\[E||Y_{n+1}||^2 \leq \left(1 + \frac{3\Delta t K + K + \lambda K t + 2\Delta t t}{1 - \Delta t - K \Delta t t}\right)E||Y_n||^2 + \frac{3\Delta t K + K + \lambda K t + 2\Delta t t}{1 - \Delta t - K \Delta t t}E||Y_n||^2 + \frac{3\Delta t K + K + \lambda K t + 2\Delta t t}{1 - \Delta t - K \Delta t t}E||Y_n||^2.\]

In the short form, we have

\[E||Y_{n+1}||^2 \leq (1 + A)E||Y_n||^2 + B,\]

where

\[A = \frac{3\Delta t K + K + \lambda K + 2\Delta t t}{1 - \Delta t - K \Delta t t} \quad \text{and} \quad B = \frac{3\Delta t K + K + \lambda K + 2\Delta t t}{1 - \Delta t - K \Delta t t}E||Y_n||^2.\]

Applying Gronwall lemma (discrete form) to (3.9) leads to:

\[E||Y_n||^2 < e^{nA}E||X_0||^2 + B\frac{e^{nA} - 1}{e^A - 1},\]

(3.10)

\[nA = \frac{3\Delta t K + K + \lambda K + 2\Delta t t}{1 - \Delta t - K \Delta t t} \quad \text{and} \quad B = \frac{3\Delta t K + K + \lambda K + 2\Delta t t}{1 - \Delta t - K \Delta t t}E||Y_n||^2.\]

Therefore, it follows from (3.10) that:

\[E||Y_n||^2 \leq e^{C}E||X_0||^2 + B\frac{e^{C} - 1}{e^D - 1},\]

(3.11)

where

\[C = \frac{3\Delta t K + K + \lambda K + 2\Delta t t}{1 - \Delta t - K \Delta t t} \quad \text{and} \quad D = \frac{3\Delta t K + K + \lambda K + 2\Delta t t}{1 - \Delta t - K \Delta t t}E||Y_n||^2.\]

It is straightforward to see that \(B, C\) and \(D\) are independents of \(\Delta t\).

(3.11) can be rewritten into the following appropriate form:

\[E||Y_n||^2 \leq C_1(1 + E||X_0||^2) \quad C_1 = \max\left(e^C, B\frac{e^{C} - 1}{e^D - 1}\right).\]

This complete the proof of the lemma.

**Lemma 3.2.4** If the conditions of Lemma 3.2.2 are satisfied, then there exist a positive constant \(C_2\) independent of \(\Delta t\) such that for \(s \in [t_n, t_{n+1})\)

\[E||y(s) - y(s + \Delta t)||^2 \leq C_2(1 + E||X_0||^2)\Delta t.\]
3.2. STRONG CONVERGENCE OF THE CSTM ON A FINITE TIME INTERVAL \([0,T]\)

1. The continuous interpolation of the numerical solution (3.3) is given by

\[
\bar{Y}(s) = Y_n + (1 - \theta)(s - t_n)f_\lambda(Y_n) + \theta(s - t_n)f_\lambda(Y_{n+1}) + g(Y_n)\Delta W_n(s) + h(Y_n)\Delta N(s),
\]

where

\[
Y(s) = \sum_{n=0}^{\infty} 1_{\{t_n \leq s < t_{n+1}\}} Y_n.
\]

For \(s \in [t_n, t_{n+1})\), we have \(Y(s) = Y_n\). Then, we have the following equality:

\[
\bar{Y}(s) - Y(s) = (1 - \theta)(s - t_n)f_\lambda(Y_n) + \theta(s - t_n)f_\lambda(Y_{n+1}) + g(Y_n)\Delta W_n(s) + h(Y_n)\Delta N(s).
\]

By squaring both sides of (3.12) and taking expectation, using the martingale properties of \(\Delta W_n\) and \(\Delta N_n\) leads to:

\[
\mathbb{E}[|\bar{Y}(s) - Y(s)|^2] \leq 3(1 - \theta)^2(s - t_n)^2\mathbb{E}[|f_\lambda(Y_n)|^2] + 3\theta^2(s - t_n)^2\mathbb{E}[|f_\lambda(Y_{n+1})|^2] + 3\mathbb{E}[|g(Y_n)|^2|\Delta W_n(s)|^2] + 3\mathbb{E}[|h(Y_n)|^2|\Delta N_n(s)|^2] \\
\leq 3(1 - \theta)^2\Delta t^2\mathbb{E}[|f_\lambda(Y_n)|^2] + 3\theta^2\Delta t^2\mathbb{E}[|f_\lambda(Y_{n+1})|^2] + 3\mathbb{E}[|g(Y_n)|^2] + 3\Delta t\mathbb{E}[|h(Y_n)|^2].
\]

By using the linear growth condition and the fact that \(\theta \in [0,1]\), it follows from (3.13) that:

\[
\mathbb{E}[|\bar{Y}(s) - Y(s)|^2] \leq 3\Delta tK_\lambda(1 + \mathbb{E}[|Y_n|^2]) + 3\Delta tK_\lambda(1 + \mathbb{E}[|Y_{n+1}|^2]) + 3\mathbb{E}[|h(Y_n)|^2].
\]

Now by application of Lemma 3.2.4 to (3.14), it follows that there exist a constant \(C_1 > 0\) independent of \(\Delta t\) such that:

\[
\mathbb{E}[|\bar{Y}(s) - Y(s)|^2] \leq C_1(1 + \mathbb{E}[|X_0|^2])\Delta t.
\]

2. For \(s \in [t_n, t_{n+1})\), \(s + \Delta t \in [t_{n+1}, t_{n+2})\) and then \(Y(s + \Delta t) = Y_{n+1}\). So it follows from (3.4) that:

\[
Y(s + \Delta t) = Y_{n+1} = Y_n + (1 - \theta)(t_{n+1} - t_n)f_\lambda(Y_n) + \theta(t_{n+1} - t_n)f_\lambda(Y_{n+1}) + g(Y_n)\Delta W_n + h(Y_n)\Delta N_n.
\]

So we have

\[
\bar{Y}(s) - Y(s + \Delta t) = (1 - \theta)(s - t_{n+1})f_\lambda(Y_n) + \theta(s - t_{n+1})f_\lambda(Y_{n+1}) + g(Y_n)(W(s) - W(t_{n+1})) + h(Y_n)(N(s) - N(t_{n+1})).
\]

By squaring both sides of (3.15), taking expectation and using martingale properties of \(\Delta W_n\) and \(\Delta N_n\), it follows that:

\[
\mathbb{E}[|\bar{N}(s) - Y(s + \Delta t)|^2] \leq 3\Delta t\mathbb{E}[|f_\lambda(Y_n)|^2] + 3\Delta t\mathbb{E}[|f_\lambda(Y_{n+1})|^2] + 3\Delta t\mathbb{E}[|g(Y_n)|^2] + 3\Delta t\mathbb{E}[|h(Y_n)|^2].
\]

Applying respectively the linear growth condition and Lemma 3.2.4 to the previous inequality, it follows that there exist a positive constant \(C_2\) independent of \(\Delta t\) such that:

\[
\mathbb{E}[|\bar{Y}(s) - Y(s + \Delta t)|^2] \leq C_2(1 + \mathbb{E}[|X_0|^2])\Delta t.
\]

This complete the proof of Lemma 3.2.4.

Now, we are ready to give the proof of Theorem 3.2.1.
3.2. STRONG CONVERGENCE OF THE CSTM ON A FINITE TIME INTERVAL [0, T]

Proof 3.2.6 [Theorem 3.2.1]

From equations (2.6) and (3.5), it follows that:

\[
\| \hat{Y}(s) - X(s) \|^2 = \left\| \int_0^s [(1 - \theta)(f_\lambda(Y(r)) - f_\lambda(X(r^-)) + \theta(f_\lambda(Y + \Delta t)) - f_\lambda(X(r^-))]dr \right. \\
+ \left. \int_0^s (g(Y(r)) - g(X(r^-)))dW(r) + \int_0^s (h(Y(r)) - h(X(r^-)))d\mathcal{N}(r) \right\|^2.
\]

Taking expectation in both sides of the above equality and using the inequality
\[ (a + b + c)^2 \leq 3a^2 + 3b^2 + 3c^2 \] for all \(a, b, c \in \mathbb{R}\) leads to:

\[
E \left[ \sup_{0 \leq s \leq t} \| \hat{Y}(s) - X(s) \|^2 \right] \leq 3E \left[ \sup_{0 \leq s \leq t} M_1(t) \right] + 3E \left( \sup_{0 \leq s \leq t} M_2(t) \right) + 3E \left( \sup_{0 \leq s \leq t} M_3(t) \right), \quad (3.16)
\]

where
\[
M_1(t) = \left\| \int_0^s [(1 - \theta)(f_\lambda(Y(r)) - f_\lambda(X(r^-)) + \theta(f_\lambda(Y + \Delta t)) - f_\lambda(X(r^-))]dr \right\|^2, \quad M_2(s) = \left\| \int_0^s (g(Y(r)) - g(X(r^-)))dW(r) \right\|^2
\]

and
\[
M_3(s) = \left\| \int_0^s (h(Y(r)) - h(X(r^-)))d\mathcal{N}(r) \right\|^2.
\]

Using Holder inequality, it follows that:

\[
M_1(s) \leq s \int_0^s \| (1 - \theta)(f_\lambda(Y(r)) - f_\lambda(X(r^-)) + \theta(f_\lambda(Y + \Delta t)) - f_\lambda(X(r^-)) \|^2 dr. \quad (3.17)
\]

Using the convexity of the application \( x \mapsto \| x \|^2 \), it follows from (3.17) that

\[
M_1(s) \leq s \int_0^s (1 - \theta)\| f_\lambda(Y(r)) - f_\lambda(X(r^-)) \|^2 dr + s \int_0^s \theta \| f_\lambda(Y + \Delta t) - f_\lambda(X(r^-)) \|^2 dr.
\]

Taking the supremum in both sides of the above inequality and using the global Lipschitz condition satisfied by \( f_\lambda \), and then taking expectation it follows that

\[
E \left[ \sup_{0 \leq s \leq t} M_1(s) \right] \leq t(1 - \theta)\lambda \int_0^t E\| Y(r) - X(r^-) \|^2 dr + t\theta \lambda \int_0^t E\| Y(r) - X(r^-) \|^2 dr. \quad (3.18)
\]

Using Doop’s inequality, it follows that:

\[
E \left[ \sup_{0 \leq s \leq t} M_2(s) \right] \leq 4 \sup_{0 \leq s \leq t} E[M_2(s)] = 4 \sup_{0 \leq s \leq t} \int_0^s E\| g(Y(r)) - g(X(r^-)) \|^2 dr.
\]

Using the global Lipschitz condition satisfied by \( g \), it follows that:

\[
E \left[ \sup_{0 \leq s \leq t} M_2(s) \right] \leq 4L \int_0^t E\| Y(r) - X(r^-) \|^2 dr. \quad (3.19)
\]

Along the same lines as above, we have

\[
E \left[ \sup_{0 \leq s \leq t} M_3(s) \right] = 4\lambda \int_0^t E\| Y(r) - X(r^-) \|^2 dr. \quad (3.20)
\]

Inserting (3.18), (3.19) and (3.20) in (3.16) leads to:

\[
E \left[ \sup_{0 \leq s \leq t} \| \hat{Y}(s) - X(s) \|^2 \right] \leq 3T(1 - \theta)\lambda \int_0^t E\| Y(r) - X(r^-) \|^2 dr \\
+ 3T\theta \lambda \int_0^t E\| Y(r) - X(r^-) \|^2 dr \\
+ 12L \int_0^t E\| Y(r) - X(r^-) \|^2 dr \\
+ 12\lambda L \int_0^t E\| Y(r) - X(r^-) \|^2 dr. \quad (3.21)
\]
Using the fact that
\[ ||Y(r) - X(r^-)||^2 = ||(Y(r) - \overline{Y}(r)) - (X(r^-) - \overline{Y}(r))||^2 \leq 2||Y(r) - \overline{Y}(r)||^2 + 2||X(r^-) - \overline{Y}(r)||^2, \]
it follows from (3.21) that:
\[
E \left[ \sup_{0 \leq s \leq t} ||\overline{Y}(s) - X(s)||^2 \right] \leq 6T(1 - \theta)L_\lambda \int_0^t \left[ E||Y(r) - \overline{Y}(r)||^2 + E||\overline{Y}(r) - X(r^-)||^2 \right] dr 
+ 6T\theta L_\lambda \int_0^t \left[ E||Y(r) - \overline{Y}(r)||^2 + E||\overline{Y}(r) - X(r^-)||^2 \right] dr 
+ 24L(1 + \lambda) \int_0^t \left[ E||Y(r) - \overline{Y}(r)||^2 + E||\overline{Y}(r) - X(r^-)||^2 \right] dr.
\]
Using lemma 3.2.4 in the above inequality, it follows that
\[
E \left[ \sup_{0 \leq s \leq t} ||\overline{Y}(s) - X(s)||^2 \right] \leq [6TL_\lambda + 24L(1 + \lambda)] \int_0^t E \left[ \sup_{0 \leq r \leq s} ||\overline{Y}(r) - X(r^-)||^2 \right] ds 
+ [6T^2L_\lambda + 24TL(1 + \lambda)]C_2(1 + E||X_0||^2)\Delta t. \tag{3.22}
\]
Applying Gronwall lemma (continuous form) to (3.22) leads to the existence of a positive constant \( C \) independent of \( \Delta t \) such that
\[
E \left[ \sup_{0 \leq s \leq t} ||\overline{Y}(s) - X(s)||^2 \right] \leq C(1 + E||X_0||^2)\Delta t.
\]
This complete the proof of Theorem 3.2.1.

The strong convergence of the STM has been studied in [5]. Since STM and CSTM convergence strongly to the exact solution, it is interesting to study their stability behaviours.

### 3.3 Linear mean-square stability of the CSTM

In this section, we focus on the linear mean-square stability. Let’s consider the following linear test equation with real coefficients
\[
dX(t) = aX(t^-)dt + bX(t^-)dW(t) + cX(t^-)dN(t), \quad X(0) = X_0. \tag{3.23}
\]

**Definition 3.3.1** The exact solution \( X \) of SDEs is said to be exponentially mean-square stable if there exist constants \( \alpha > 0 \) and \( L > 0 \) such that:
\[
E||X(t)||^2 \leq Le^{-\alpha t}E||X(0)||^2.
\]

**Definition 3.3.2**
1. The numerical solution \( X_n \) of SDEs is said to be exponentially mean-square stable if there exist constants \( \alpha > 0 \) and \( L > 0 \) such that:
\[
E||X_n||^2 \leq Le^{-\alpha t}E||X(0)||^2.
\]
2. The numerical solution \( X_n \) of SDEs is said to be mean-square stable if there exist constants
\[
0 < L < 1 \text{ such that : for all } n \in [0, T]
\]
\[
E||X_{n+1}||^2 \leq LE||X_n||^2.
\]
3. A numerical method is said to be A-stable if it is stable for any stepsize.

It is proved in [5] that the exact solution of (3.23) have the following stability property:
\[
\lim_{t \to \infty} E||X(t)||^2 = 0 \Leftrightarrow l := 2a + b^2 + \lambda c(2 + c) < 0, \tag{3.24}
\]
where \( \lambda \) is the intensity of the poisson precess \( (N_t)_{t \geq 0} \).
Remark 3.3.9 If the exact solution of the problem satisfying:

\[ \frac{1}{2} \leq \theta \leq 1. \]

For \( 0 \leq \theta < \frac{1}{2} \) this numerical solution is mean-square stable for any stepsize \( \Delta t > 0 \) satisfying:

\[ \Delta t < \frac{-l}{(1 - 2\theta)(a + \lambda c)^2}. \]

**Proof 3.3.4** Applying the compensated theta method to (3.23) gives:

\[ Y_{n+1} = Y_n + (1 - \theta)\Delta t(a + \lambda c)Y_n + \theta \Delta t(a + \lambda c)Y_{n+1} + bY_n \Delta W_n + cY_n \Delta N_n. \]

So we have:

\[ (1 - \theta \Delta t a - \theta \Delta t \lambda c)Y_{n+1} = (1 - \theta)\Delta t(a + \lambda c)Y_n + bY_n \Delta W_n + c\Delta N_n. \]

It follows that:

\[ (1 - \theta \Delta t a - \theta \Delta t \lambda c)^2 \mathbb{E}\|Y_{n+1}\|^2 = [1 + (1 - \theta)\Delta t(a + \lambda c)]^2 \mathbb{E}\|Y_n\|^2 + b^2 \Delta t \mathbb{E}\|Y_n\|^2 + c^2 \lambda \Delta t \mathbb{E}\|Y_n\|^2. \]

Therefore,

\[ \mathbb{E}\|Y_{n+1}\|^2 = \frac{1 + [2(1 - \theta)(a + \lambda c) + b^2 + c^2 \lambda] \Delta t + (1 - \theta)^2(a + \lambda c)^2 \Delta t^2}{(1 - \theta \Delta t a - \theta \Delta t \lambda c)^2} \mathbb{E}\|Y_n\|^2. \]

It follows that \( \mathbb{E}\|Y_n\|^2 \) is a geometric sequence which converges if and only if

\[ \frac{1 + [2(1 - \theta)(a + \lambda c) + b^2 + c^2 \lambda] \Delta t + (1 - \theta)^2(a + \lambda c)^2 \Delta t^2}{(1 - \theta \Delta t a - \theta \Delta t \lambda c)^2} < 1. \]

That is if and only if

\[ (1 - 2\theta)(a + \lambda c)^2 \Delta t < -l. \] (3.25)

It follows that:

- If \( \frac{1}{2} \leq \theta \leq 1 \), then the condition (3.25) is satisfied for any stepsize. And then the numerical solution is mean-square stable for any stepsize.
- If \( 0 \leq \theta < \frac{1}{2} \), then it follows from (3.25) that if \( 0 < \Delta t < \frac{-l}{(1 - 2\theta)(a + \lambda c)^2} \), the numerical method is stable.

**Remark 3.3.5** Changing \( c \) to \( -2 - c \) does not affect the mean-square stability condition (3.24). Hence the exact solution of (3.23) have the same stability property under this transformation. It is interesting to look for what happens to the numerical solution under this transformation.

**Definition 3.3.6** A numerical method applied to (3.23) is said to be jump symmetric if whenever stable (unstable) for \( \{a, b, c, \lambda, \Delta t\} \) it is also stable (unstable) for \( \{a, b, -2 - c, \lambda, \Delta t\} \).

**Corollary 3.3.7** The compensated stochastic theta method applied to (3.23) is jump symmetric if and only if \( \theta = \frac{1}{2} \).

**Proof 3.3.8**

1. For \( \theta = \frac{1}{2} \), clearly the stability condition (3.25) of the numerical solution is equivalent to the stability condition (3.24) of the exact solution. Since the stability condition (3.24) is invariant under the transformation \( c \mapsto -2 - c \), it follows that the jump symmetric property holds.

2. If \( \theta \neq \frac{1}{2} \), the right hand side of (3.24) remains the same under the transformation \( c \mapsto -2 - c \), but the left hand side changes. Therefore the jump symmetry property does not holds.

**Remark 3.3.9** If the exact solution of the problem (3.23) is mean-square stable, then for \( \frac{1}{2} < \theta \leq 1 \), it follows from (3.25) the stability property of the CSTM is preserved under the transformation \( c \mapsto -2 - c \).
3.4 Nonlinear mean-square stability

This section is devoted to the nonlinear mean-square analysis.

Throughout, this section, we make the following assumptions.

Assumptions 3.4.1 We assume that there exist constants \( \mu, \sigma, \gamma \) such that for all \( x, y \in \mathbb{R}^n \)

\[
(x - y, f(x) - f(y)) \leq \mu \|x - y\|^2 \tag{3.26}
\]

\[
\|g(x) - g(y)\| \leq \sigma \|x - y\|^2
\]

\[
\|h(x) - h(y)\| \leq \gamma \|x - y\|^2.
\]

Usually, condition (3.26) is called "one-sided Lipschitz condition".

3.4.2 Nonlinear mean-square stability of the exact solution

Theorem 3.4.3 \([6, \text{Theorem 4, pp 13}]\)

Under assumptions 3.4.1, any two solutions \( X(t) \) and \( Y(t) \) of the SDEs with jumps (2.1) with \( E\|X_0\|^2 < \infty \) and \( E\|Y_0\|^2 < \infty \) satisfy the following property:

\[
E\|X(t) - Y(t)\|^2 \leq E\|X_0 - Y_0\|^2 e^{\alpha t},
\]

where \( \alpha := 2\mu + \sigma + \lambda \sqrt{\gamma} (\sqrt{\gamma} + 2) \).

Hence, the condition \( \alpha < 0 \) is sufficient for the exponential mean-square stability property.

Proof 3.4.4 The two solutions \( X(t) \) and \( Y(t) \) of (2.1) satisfy respectively

\[
\begin{align*}
\text{d}X(t) &= f(X(t^-))\text{d}t + g(X(t^-))\text{d}W(t) + h(X(t^-))\text{d}N(t) \\
\text{d}Y(t) &= f(Y(t^-))\text{d}t + g(Y(t^-))\text{d}W(t) + h(Y(t^-))\text{d}N(t).
\end{align*}
\]

Applying Itô's lemma for product (Lemma 1.4.15) to the stochastic process \( Z(t) = \|X(t) - Y(t)\|^2 \) leads to

\[
\begin{align*}
\text{d}Z(t) &= 2\langle X(t^-) - Y(t^-), d(X(t^-)) - d(Y(t^-)) \rangle + \|d(X(t^-)) - d(Y(t^-))\|^2 \\
&= [2\langle X(t^-) - Y(t^-), f(X(t^-)) - f(Y(t^-)) \rangle + 2\lambda \|X(t^-) - Y(t^-)\|^2] \text{d}t + \text{d}M_t,
\end{align*}
\]

where \( M_t \) is a martingale and where we used the following rule of calculation

\[
\text{d}t \text{d}t = \text{d}t \text{d}W(t) = 0, \quad \text{d}N_t \text{d}N(t) = \text{d}N_t, \quad \text{d}W_t \text{d}W_t = \text{d}t \quad \text{and} \quad \text{d}t \text{d}N(t) = \text{d}W_t \text{d}N_t = 0.
\]

Using Assumptions 3.4.1, one gets:

\[
\begin{align*}
\text{d}\|X(t) - Y(t)\|^2 &\leq [2\mu \|X(t^-) - Y(t^-)\|^2 + 2\lambda \sqrt{\gamma} \|X(t^-) - Y(t^-)\|^2 + \sigma \|X(t^-) - Y(t^-)\|^2] \text{d}t + \text{d}M(t),
\end{align*}
\]

So we have

\[
\begin{align*}
\text{d}\|X(t^-) - Y(t^-)\|^2 &\leq [2\mu + \sigma + \lambda \sqrt{\gamma} (\sqrt{\gamma} + 2)] \|X(t^-) - Y(t^-)\|^2 \text{d}t + \text{d}M(t),
\end{align*}
\]

which can be written into its following integral form:

\[
\|X(t^-) - Y(t^-)\|^2 \leq [2\mu + \sigma + \lambda \sqrt{\gamma} (\sqrt{\gamma} + 2)] \int_0^t \|X(s^-) - Y(s^-)\|^2 \text{d}s + \int_0^t \text{d}M(s). \tag{3.27}
\]
Taking expectation in both sides of (3.27) and using the fact that $\mathbb{E}\left(\int_0^t dM(s)\right) = 0$ leads to:

$$\mathbb{E}|X(t^-) - Y(t^-)|^2 \leq [2\mu + \sigma + \lambda\sqrt{\lambda}(\sqrt{\lambda} + 2)] \int_0^t \mathbb{E}|X(s^-) - Y(s^-)|^2 ds. \quad (3.28)$$

Applying Gronwall lemma (continuous form) to (3.28) leads to:

$$\mathbb{E}|X(t^-) - Y(t^-)|^2 \leq \mathbb{E}|X_0 - Y_0|^2 e^{2\mu + \sigma + \lambda\sqrt{\lambda}(\sqrt{\lambda} + 2)t}. \quad (3.29)$$

This complete the proof of Theorem 3.4.3.

**Remark 3.4.5** For the linear test equation (3.23), the one-sided Lipschitz and the global Lipschitz condition become:

$$(x - y, f(x) - f(y)) = a|x - y|^2$$

$$|g(x) - g(y)|^2 = b^2|x - y|^2$$

$$|h(x) - h(y)|^2 = c^2|x - y|^2.$$ 

Along the same lines as for the nonlinear case, we obtain:

$$\mathbb{E}|X(t^-)|^2 = \mathbb{E}|X_0|^2 e^{(2a + b^2 + \lambda c(2 + c))t}.$$ 

Therefore, we have the following equivalence for the linear mean-square stability:

$$\lim_{t \to +\infty} \mathbb{E}|X(t)|^2 = 0 \iff l := 2a + b^2 + \lambda c(2 + c) < 0. \quad (3.29)$$

Based on Theorem 3.4.3, it is interesting to analyse whether or not the numerical solution of (2.1) reproduce the mean-square stability of the exact solution.

### 3.4.6 Nonlinear mean-square stability of the numerical solutions

**Theorem 3.4.7** [Stability of the stochastic theta method]

Under Assumptions 3.4.1 and the further hypothesis $\alpha < 0$, for

$$\Delta t < \frac{-\alpha}{\lambda^2\gamma},$$

the Euler backward method (STM with $\theta = 1$) applied to equation (2.1) is mean-square stable in the sense that

$$\mathbb{E}|X_n - Y_n|^2 \leq \mathbb{E}|X_0 - Y_0|^2 e^{\beta_1(\Delta t)n\Delta t},$$

where

$$\beta_1(\Delta t) := \frac{1}{\Delta t} \ln \left( \frac{1 + (\sigma + \lambda\gamma + 2\lambda\sqrt{\gamma})\Delta t + \lambda^2\gamma\Delta t^2}{1 - \mu}\right).$$

**Proof 3.4.8** Let’s introduce the following notations:

$$\Delta Z_n = X_n - Y_n, \quad \Delta f_n = f(X_n) - f(Y_n), \quad \Delta g_n = g(X_n) - g(Y_n), \quad \Delta h_n = h(X_n) - h(Y_n).$$

If $\theta = 1$, the numerical approximation (3.2) applied to $X$ and $Y$ gives:

$$Y_{n+1} = Y_n + f(Y_n)\Delta t + g(Y_n)\Delta W_n + h(Y_n)\Delta N_n$$

$$X_{n+1} = X_n + f(X_n)\Delta t + g(X_n)\Delta W_n + h(X_n)\Delta N_n.$$

So we have:

$$||\Delta Z_{n+1} - \Delta f_{n+1}\Delta t||^2 = ||\Delta Z_n + \Delta g_n\Delta W_n + \Delta h_n\Delta N_n||^2. \quad (3.30)$$
Using the independence of $\Delta W_n$ and $\Delta N_n$ and the fact that
\[
E|\Delta N_n|^2 = \text{var}(\Delta N_n) + (E(\Delta N_n))^2 = \lambda \Delta t + \lambda^2 \Delta t^2
\]
we obtain from (3.30) the following estimation :
\[
E|\Delta Z_{n+1}|^2 - 2\Delta tE(\Delta Z_{n+1}, \Delta f_{n+1}) \leq E|\Delta Z_n|^2 + \Delta tE|\Delta g_n|^2 + \lambda \Delta t(1 + \lambda \Delta t)E|\Delta h_n|^2 + 2\lambda \Delta tE(\Delta Z_n, \Delta h_n).
\]
Using the one-sided Lipschitz condition and the global Lipschitz condition, it follows that :
\[
E|\Delta Z_{n+1}|^2 \leq 2\Delta t\mu E|\Delta Z_{n+1}|^2 + E|\Delta Z_n|^2 + \sigma \Delta tE|\Delta Z_n|^2 + \lambda \Delta t(1 + \lambda \Delta t)\gamma E|\Delta Z_n|^2
\]
(1 - 2\mu \Delta t)E|\Delta Z_{n+1}|^2 \leq \left[1 + (\sigma + \lambda \gamma + 2\lambda \sqrt{\gamma})\Delta t + \lambda^2 \gamma \Delta t^2\right]E|\Delta Z_n|^2.
\]
The latter inequality leads to :
\[
E|\Delta Z_n|^2 \leq \left[1 + (\sigma + \lambda \gamma + 2\lambda \sqrt{\gamma})\Delta t + \lambda^2 \gamma \Delta t^2\right]^{\frac{n}{2}} E|\Delta Z_0|^2.
\]
Therefore, we have :
\[
E|\Delta Z_n|^2 \leq \left[1 + (\sigma + \lambda \gamma + 2\lambda \sqrt{\gamma})\Delta t + \lambda^2 \gamma \Delta t^2\right]^{\frac{n}{2}} E|\Delta Z_0|^2.
\] (3.31)
In order to have stability, we impose the following condition :
\[
1 + (\sigma + \lambda \gamma + 2\lambda \sqrt{\gamma})\Delta t + \lambda^2 \gamma \Delta t^2 \leq 1 - 2\mu \Delta t < 1.
\] (3.32)
The hypothesis $\alpha < 0$ implies that $\mu < 0$. So $1 - 2\mu \Delta t > 0$, for all positive stepsize. It follows that (3.32) is equivalent to
\[
\Delta t < \frac{-\alpha}{\lambda^2 \gamma}.
\]
Applying the equality $a^n = e^{\ln a}$, for all $a > 0$ and all $n \in \mathbb{N}$ to (3.31) complete the proof of Theorem 3.4.9.

**Theorem 3.4.9 [A-stability of the compensated Euler backward method]**

Under Assumptions 3.4.1 and the further hypothesis $\alpha < 0$, for any stepsize, the compensated backward Euler method (CSTM with $\theta = 1$) for equation (2.1) is mean square stable in the sense that :
\[
E||X_n - Y_n||^2 \leq E||X_0 - Y_0||^2 e^{\beta_2(\Delta t)n\Delta t},
\]
where
\[
\beta_2(\Delta t) := \frac{1}{\Delta t} \ln \left(\frac{1 + (\sigma + \lambda \gamma)\Delta t}{1 - 2(\mu + \lambda \sqrt{\gamma})\Delta t}\right).
\]

**Proof 3.4.10** We use the same notations as for the proof of Theorem 3.4.9 except for $\Delta f_n^\lambda$ for which we have $\Delta f_n^\lambda = f_\lambda(X_n) - f_\lambda(Y_n)$.

Along the same line as for the proof of Theorem 3.4.9, we obtain :
\[
||\Delta Z_{n+1} - \Delta t \Delta f_{n+1}^\lambda||^2 = ||\Delta Z_n + \Delta g_n \Delta W_n + \Delta h_n \Delta N_n||^2.
\] (3.33)
Furthermore, the relations $E|\Delta N_n| = 0$ and $E|\Delta N_n|^2 = \lambda \Delta t$ leads to :
\[
\langle x - y, f_\lambda(x) - f_\lambda(y) \rangle = \langle x - y, f(x) - f(y) \rangle + \lambda \langle x - y, h(x) - h(y) \rangle \leq (\mu + \lambda \sqrt{\gamma})||x - y||^2.
\]
Using the independence of $\Delta W_n$ and $\Delta N_n$, it follows from (3.33) that
\[ \mathbb{E}[|\Delta Z_{n+1}|^2] \leq 2\Delta t \mathbb{E}[|\Delta Z_n|^2] + \mathbb{E}[|\Delta f_{n+1}|^2 + \Delta t \mathbb{E}[|\Delta g_n|^2] + \lambda \Delta t \mathbb{E}[|\Delta h_n|]. \]
Using the one-sided Lipschitz and the global Lipschitz condition, it follows (3.34) that:
\[ (1 - 2(\mu + \lambda \sqrt{\gamma}) \Delta t) \mathbb{E}[|\Delta Z_{n+1}|^2] \leq (1 + \sigma \Delta t + \lambda \gamma \Delta t) \mathbb{E}[|\Delta Z_n|^2]. \]
Therefore,
\[ \mathbb{E}[|\Delta Z_n|^2] \leq \left[ \frac{1 + \sigma \Delta t + \lambda \gamma \Delta t}{1 - 2(\mu + \lambda \sqrt{\gamma}) \Delta t} \right] \mathbb{E}[|Z_0|^2]. \]

In order to have stability, we need the following condition to be fulfilled
\[ \frac{1 + \sigma \Delta t + \lambda \gamma \Delta t}{1 - 2(\mu + \lambda \sqrt{\gamma}) \Delta t} < 1. \]

From the hypothesis $\alpha < 0$, we have $2(\mu + \lambda \sqrt{\gamma}) < 0$ and then $1 - 2(\mu + \lambda \sqrt{\gamma}) \Delta t > 0$ for any stepsize. Hence condition (3.36) is equivalent to $\alpha \Delta t < 0$, which is satisfied for any stepsize.

Applying the relation $a^n = e^{n \ln a}$ to (3.35) complete the proof of the theorem.

### 3.5 Numerical Experiments

The purpose of this section is to illustrate our theoretical results of strong convergence and stability. We will focus in the linear case. We consider the linear jump-diffusion Itô’s stochastic integral (SDEs)
\[ \begin{cases} 
  dX(t) = aX(t^-)dt + bX(t^-)dW(t) + cX(t^-)dN(t), & t \geq 0, \quad c > -1, \\
  X(0) = 1.
\end{cases} \]

#### 3.5.1 Strong convergence illustration

In order to illustrate the strong convergence result, we need the exact solution of problem (3.37).

**Proposition 3.5.2** The problem (3.37) has the following process as a unique solution
\[ X(t) = X_0 \exp \left[ \left( a - \frac{b^2}{2} \right) t + bW(t) \right] (1 + c)^{N(t)}, \]
which can be written in the following equivalent form
\[ X(t) = X_0 \exp \left[ \left( a - \frac{b^2}{2} \right) t + bW(t) + \ln(1 + c)N(t) \right]. \]

**Proof 3.5.3**

1. Obviously, the functions $f(x) = ax$, $g(x) = bx$ and $h(x) = cx$ satisfy the global Lipschitz condition and the linear growth condition. Therefore from Theorem 2.2.1, it follows that the problem (3.37) admit a unique solution.

2. Let’s consider the following Itô’s jump-diffusion process
\[ Z(t) = \left( a - \frac{b^2}{2} \right) t + bW(t) + N(t) \ln(1 + c). \]

The function $f : [0, \infty) \to \mathbb{R}, \quad x \mapsto x_0 \exp(x)$ is infinitely differentiable. Then applying Itô formula for jump process to the process $Z(t)$ leads to:
\[ f(Z_t) = f(Z_0) + \int_0^t \left( a - \frac{b^2}{2} \right) f'(Z_{s^-})ds + \frac{1}{2} \int_0^t b^2 f''(Z_{s^-})ds + \int_0^t b f'(Z_{s^-})dW(s) + \int_0^t (f(Z_s) - f(Z_{s^-}))dN(s), \]

(3.39)
where

\[ f(Z_s) - f(Z_{s^-}) = X_0 \exp[Z_{s^-} + \ln(1 + c)] - X_0 \exp(Z_{s^-}) \]
\[ = (1 + c)X_0 \exp(Z_{s^-}) - X_0 \exp(Z_{s^-}) \]
\[ = cX_0 \exp(Z_{s^-}) = cf(Z_{s^-}) \]  \hspace{1cm} (3.40)

and

\[ X(s^-) = f(Z_{s^-}) = f'(Z_{s^-}) = f''(Z_{s^-}) \].  \hspace{1cm} (3.41)

Substituting (3.40) and (3.41) in (3.39) and rewriting the result into its differential form leads to

\[ dX(t) = aX(t^-)dt + bX(t^-)dW(t) + cX(t^-)dN(t). \]

So \( X(t) \) satisfies the desired equation.

For the numerical simulation, we take \( a = b = 1, c = 0.5 \) and \( \lambda = 1 \). We have the following graphs for the strong error. We use 5000 sample paths. The algorithms for simulation are based on [17]. We take \( dt = 2^{\frac{1}{14}} \) and \( \Delta t = 2^{p-1} \) for \( p = 1, 2, 3, 4, 5 \). The error is computing at the end point \( T = 1 \).

![Figure 3.1: Mean square error of the CSTM with \( \theta = 0 \)](image)

### 3.5.4 Mean-square stability illustration

In order to illustrate our theoretical result of A-stability, we first consider two examples

**Example I** \( a = b = 2, c = -0.9 \) and \( \lambda = 9 \).

**Example II** \( a = -7, b = c = 1 \) and \( \lambda = 4 \).

In both examples, the stability condition (3.24) is satisfied. So exact solutions of both examples are mean-square stable. For \( \theta \) slightly less than 0.5 (for instance \( \theta = 0.495 \)) both solutions may be unstable for a large stepsize (\( \Delta t = 60, 25 \)), but for \( \frac{1}{2} \leq \theta \leq 1 \), numerical solutions of both examples are stable. From the top to the bottom, we present numerical examples of example I and example II respectively.
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The following curves provide the stability comparison between CSTM and STM. We focus on the example I. Here, $a > 0$ and $c < 0$. So the jump part can stabilise the problem. In this case, from the theoretical result the STM is stable for $\Delta t < 0.0124829$. For $\Delta t = 0.005$, both CSTM and STM stabilities behavior look the same. But for $\Delta t = 0.5$ CSTM is stable while STM produce an oscillation. For $\Delta t = 0.1$, numerical solution of STM grows rapidly to the scale $10^7$ and is unstable while the numerical solution of CSTM is stable. So CSTM works better than STM.

In this chapter, we provided the proof of the strong convergence of order 0.5 of the CSTM under global Lipschitz condition. We also studied the stability behaviour of both STM and CSTM. We proved that the CSTM works better than the STM. Some situations in real life are modelised by SDEs with jumps, where the drift coefficient does not satisfy the global Lipschitz condition. It is proved in [10] that the Euler explicit method for such equations diverge strongly. The tamed Euler scheme for SDEs without jump is the currently investigated by many authors. The compensated tamed Euler scheme for SDEs with jumps is not yet well developped in the litterature. In the following chapter, we establish the strong convergence of the compensated tamed Euler scheme for SDEs with jumps under non-global Lipschitz condition. This scheme is slightly different to what is already done in the litterature.

Figure 3.2: Mean square error of the CSTM with $\theta = 0.5$
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Figure 3.3: Mean square error of the CSTM with $\theta = 1$

Figure 3.4: A-stability for example I
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Figure 3.5: A-stability for example II

Figure 3.6: Stability behavior of the STM
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Figure 3.7: Stability behavior of the CSTM

Figure 3.8: Stability behavior of the STM
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Figure 3.9: Stability behavior of the CSTM

Figure 3.10: Stability behavior of the STM
Figure 3.11: Stability behavior of the CSTM
Chapter 4

Strong convergence of the compensated tamed Euler scheme for stochastic differential equation with jump under non-global Lipschitz condition

Under non-global Lipschitz condition, Euler Explicit method fails to converge strongly to the exact solution, while Euler implicit method converges but requires more computational efforts. The strong convergence of the tamed Euler scheme has been investigated in [8]. This scheme is explicit and requires less computational efforts than the Euler implicit method. In this chapter, we extend the strong convergence of the tamed Euler scheme by introducing its compensated form for stochastic differential equations with jumps. More precisely, we prove that under non-global Lipschitz condition, the compensated tamed Euler scheme converges strongly with order 0.5 to the exact solution of the SDEs with jumps. This scheme is different to the one proposed in [19]. As opposed to what is done in [19], here we obtain the strong convergence and the rate of convergence simultaneously under more relaxed conditions. The contents of this chapter can also be found in [20].

4.1 Compensated tamed Euler scheme

In this chapter, we still consider the jump-diffusion Itô’s stochastic differential equations (SDEs) of the form

\[ dX(t) = f(X(t^-))dt + g(X(t^-))dW(t) + h(X(t^-))dN(t), \quad X(0) = X_0, \]  

(4.1)

where \( W_t \) is an \( m \)-dimensional Brownian motion, \( f : \mathbb{R}^d \to \mathbb{R}^d \) satisfies the one-sided Lipschitz condition and the polynomial growth condition. The functions \( g : \mathbb{R}^d \to \mathbb{R}^{d \times m} \) and \( h : \mathbb{R}^d \to \mathbb{R}^d \) satisfy the global Lipschitz condition, \( N_t \) is a one dimensional poisson process with parameter \( \lambda \).

We recall that the compensated poisson process \( \overline{N}(t) := N(t) - \lambda t \) is a martingale satisfying the following properties:

\[ E(\overline{N}(t+s) - \overline{N}(t)) = 0, \quad E[|\overline{N}(t+s) - \overline{N}(t)|^2] = \lambda s, \quad s,t \geq 0. \]  

(4.2)

We can rewrite the jump-diffusion SDEs (4.1) in the following equivalent form

\[ dX(t) = f_{\lambda}(X(t^-))dt + g(X(t^-))dW(t) + h(X(t^-))d\overline{N}(t), \]  

(4.3)

where

\[ f_{\lambda}(x) = f(x) + \lambda h(x). \]  

(4.4)

To easy notation, we will use \( X(t) \) instead of \( X(t^-) \).
If $T$ is the final time, the tamed Euler scheme is defined by:

$$X_{n+1}^M = X_n^M + \frac{\Delta t f(X_n^M)}{1 + \Delta t \|f(X_n^M)\|} + g(X_n^M)\Delta W_n + h(X_n^M)\Delta N_n$$

(4.5)

and the compensated tamed Euler scheme is given by:

$$Y_{n+1}^M = Y_n^M + \frac{\Delta t f(Y_n^M)}{1 + \Delta t \|f(Y_n^M)\|} + g(Y_n^M)\Delta W_n + h(Y_n^M)\Delta \overline{N}_n,$$

(4.6)

where $M \in \mathbb{N}$ is the number of steps and $\Delta t = \frac{T}{M}$ is the stepsize.

Inspired by [8], we prove the strong convergence of the numerical approximation (4.6) to the exact solution of (4.1).

### 4.2 Moments bounded of the numerical solution

**Notation 4.2.1** Throughout this chapter $(\Omega, \mathcal{F}, \mathbb{P})$ denote a complete probability space with a filtration $(\mathcal{F}_t)_{t \geq 0}$, $\|X\|_{L^p(\Omega, \mathbb{R}^d)}$ is equal to $(\mathbb{E}[\|X^p\|])^{1/p}$, for all $p \in [1, +\infty)$ and for all $(\mathcal{F}_t)$-adapted process $X$. For all $x, y \in \mathbb{R}^d$, we denote by $(x, y) = x, y = x_1y_1 + x_2y_2 + \cdots + x_ny_n$. $||x|| = ((x, x))^{1/2}$ and $||A|| = \sup_{x \in \mathbb{R}^d, ||x|| \leq 1} ||Ax||$ for all $A \in \mathbb{R}^{m \times d}$. We use also the following convention: $\sum_{i=u}^n a_i = 0$ for $u > n$.

We define the continuous time interpolation of the discrete numerical approximation of (4.6) by the family of processes $(\overline{Y}_t^M)^\nu_M, \overline{Y}^M : [0, T] \times \Omega \longrightarrow \mathbb{R}^d$ such that:

$$\overline{Y}_t^M = Y_n^M + \frac{(t - n\Delta t) f(Y_n^M)}{1 + \Delta t \|f(Y_n^M)\|} + g(Y_n^M)(W_t - W_{n\Delta t}) + h(Y_n^M)(\overline{N}_t - \overline{N}_{n\Delta t}),$$

(4.7)

for all $M \in \mathbb{N}$, all $n \in \{0, \cdots, M - 1\}$, and all $t \in [n\Delta t, (n + 1)\Delta t]$.

**Assumptions 4.2.2** Throughout this chapter, We make the following assumptions:

(A.1) $f, g, h \in C^1$.

(A.2) For all $p > 0$, there exist a finite $M_p > 0$ such that $\mathbb{E}[\|X_0\|] \leq M_p$.

(A.3) $g$ and $h$ satisfy the global Lipschitz condition:

$$||g(x) - g(y)|| \vee ||h(x) - h(y)|| \leq C||x - y||, \quad \forall x, y \in \mathbb{R}^d.$$

(A.4) $f$ satisfies the one-sided Lipschitz condition:

$$(x - y, f(x) - f(y)) \leq C||x - y||^2, \quad \forall x, y \in \mathbb{R}^d.$$

(A.5) $f$ satisfies the superlinear growth condition:

$$||f(x) - f(y)|| \leq C(K + ||x||^c + ||y||^c)||x - y||, \quad \forall x, y \in \mathbb{R}^d,$$

where $K$, $C$ and $c$ are strictly positive constants.

**Remark 4.2.3** Under conditions (A.1), (A.2) and (A.3) of Assumptions 4.2.2, it is proved in [6, Lemma 1] that (4.1) has a unique solution with all moments bounded.

**Remark 4.2.4** We note that if Assumptions 4.2.2 are satisfied, the function $f_\lambda$ defined in (4.4) satisfies the one-sided Lipschitz condition and the superlinear growth condition with constants $C_\lambda := C(1 + \lambda)$ and $K_\lambda := K + \lambda$. 


Indeed, for all \( x, y \in \mathbb{R}^d \),

\[
\langle x - y , f(x) - f(y) \rangle = \langle x - y , f(x) \rangle + \lambda \langle x - y , h(x) - h(y) \rangle \\
\leq C(1 + \lambda)\| x - y \|,
\]

\[
\| f(x) - f(y) \| \leq \| f(x) - f(y) \| + \lambda \| h(x) - h(y) \| \\
\leq C( K + \lambda + \| x \|^c + \| y \|^c)\| x - y \| \\
= C( K + \| x \|^c + \| y \|^c)\| x - y \|.
\]

Since the value of the constant does not matter too much, we will use \( C \) of \( K \) throughout this work, the generic constants \( C_p \) may change the value from one line to another one. We will sometimes use \( Y_n^M \) instead of \( Y_n^M(\omega) \) to simplify notations.

The main result of this section is given in the following theorem.

**Theorem 4.2.5** Let \( Y_n^M : \Omega \rightarrow \mathbb{R}^d \) be defined by (4.6) for all \( M \in \mathbb{N} \) and all \( n \in \{0, \cdots , M \} \). Then the following inequality holds:

\[
\sup_{M \in \mathbb{N}} \sup_{n \in \{0, \cdots , M \}} \mathbb{E}\left[ \| Y_n^M \| \right] < +\infty,
\]

for all \( p \in [1, \infty) \).

In order to prove Theorem 4.2.5 we introduce the following notations facilitating computations.

**Notation 4.2.6**

\[
\alpha_k^M := 1_{\{\| Y_k^M \| \geq 1\}} \left\langle \frac{Y_k^M}{\| Y_k^M \|} , \frac{g(Y_k^M)}{\| Y_k^M \|} \Delta W_k \right\rangle ,
\]

\[
\beta_k^M := 1_{\{\| Y_k^M \| \geq 1\}} \left\langle \frac{Y_k^M}{\| Y_k^M \|} , \frac{h(Y_k^M)}{\| Y_k^M \|} \Delta \mathcal{N}_k \right\rangle ,
\]

\[
\beta := (1 + K + 2c + KTc + TC + \| f_0(0) \| + \| g(0) \| + \| h(0) \|)^4 ,
\]

\[
D_n^M := (\beta + \| \varepsilon \|) \exp \left( \frac{3\beta}{2} + \sup_{u \in \{0, \cdots , n\}} \sum_{k = u}^{n-1} \left[ \frac{3\beta}{2} \| \Delta W_k^M \|^2 + \frac{3\beta}{2} \| \Delta \mathcal{N}_k^M \| + \alpha_k^M + \beta_k^M \right] \right) ,
\]

\[
\Omega_n^M := \{ \omega \in \Omega : \sup_{k \in \{0,1,\cdots ,n-1\}} D_k^M(\omega) \leq M^{1/2c} , \sup_{k \in \{0,1,\cdots ,n-1\}} \| \Delta W_k^M(\omega) \| \leq 1 , \sup_{k \in \{0,1,\cdots ,n-1\}} \| \Delta \mathcal{N}_k^M(\omega) \| \leq 1 \}.
\]

In order to prove Theorem 4.2.5, we need the following lemmas.

**Lemma 4.2.7** For all positive real numbers \( a \) and \( b \), the following inequality holds

\[
1 + a + b^2 \leq e^{a+\sqrt{2}b}.
\]

**Proof 4.2.8** For \( a \geq 0 \) fixed, let’s define the function \( f(b) = e^{a+\sqrt{2}b} - 1 - a - b^2 \). It can be easily checked that \( f'(b) = \sqrt{2}e^{a+\sqrt{2}b} - 2b \) and \( f''(b) = 2(e^{a+\sqrt{2}b} - 1) \). Since \( a \) and \( b \) are positive, it follows that \( f''(b) \geq 0 \) for all \( b \geq 0 \). So \( f' \) is a non-decreasing function. Therefore, \( f'(b) \geq f'(0) = \sqrt{2}e^a > 0 \) for all \( b \geq 0 \). This implies that \( f \) is a non-decreasing function. Hence \( f(b) \geq f(0) = e^a - 1 - a \) for all \( b \geq 0 \). Since \( 1 + a \leq e^a \) for all positive number \( a \), it follows that \( f(b) \geq 0 \) for all positive number \( b \). i.e. \( 1 + a + b^2 \leq e^{a+\sqrt{2}b} \), \( \forall b \geq 0 \). Therefore for all \( a \geq 0 \) fixed, \( 1 + a + b^2 \leq e^{a+\sqrt{2}b} \), \( \forall b \geq 0 \).

The proof of lemma is complete.
Following closely [8, Lemma 3.1, pp 15], we have the following main lemma.

**Lemma 4.2.9** The following inequality holds for all $M \in \mathbb{N}$ and all $n \in \{0, 1, \cdots, M\}$

$$1_{\Omega_n^M}||Y_n^M|| \leq D_n^M,$$

(4.9)

where $D_n^M$ and $\Omega_n^M$ are given in Notation 4.2.6.

**Proof 4.2.10** Using the inequality $\frac{\Delta t}{1 + \Delta t||f_n(x)||} \leq T$, the global Lipschitz condition of $g$ and $h$ and the polynomial growth condition of $f_\lambda$ we have the following estimation on $\Omega_{n+1}^M \{ \omega \in \Omega : ||Y_n^M(\omega)|| \leq 1 \}$, for all $n \in \{0, 1, \cdots, M - 1\}$

$$||Y_{n+1}^M|| \leq ||Y_n^M|| + \frac{\Delta t ||f_n(Y_n^M)||}{1 + \Delta t ||f_n(Y_n^M)||} + ||g(Y_n^M)||||\Delta W_n^M|| + ||h(Y_n^M)||||\Delta N_n^M||$$

$$\leq ||Y_n^M|| + T||f_\lambda(Y_n^M) - f_\lambda(0)|| + T||f_\lambda(0)|| + ||g(Y_n^M) - g(0)|| + ||g(0)||$$

$$+ ||h(Y_n^M) - h(0)|| + ||h(0)||$$

$$\leq ||Y_n^M|| + TC(K + ||Y_n^M||^c + ||0||^c)||Y_n^M - 0|| + T||f_\lambda(0)|| + C||Y_n^M|| + C||Y_n^M|| + ||g(0)|| + ||h(0)||.$$

Since $||Y_n^M|| \leq 1$, it follows that:

$$||Y_{n+1}^M|| \leq 1 + KTC + TC + 2T||f_\lambda(0)|| + ||g(0)|| + ||h(0)|| \leq \beta.$$  

Further more, from the numerical approximation (4.6), we have

$$||Y_{n+1}^M||^2 = ||Y_n^M||^2 + \frac{\Delta t^2 ||f_n(Y_n^M)||^2}{(1 + \Delta t||f_n(Y_n^M)||)^2} + ||g(Y_n^M)||||\Delta W_n^M||^2 + ||h(Y_n^M)||||\Delta N_n^M||^2$$

$$+ \frac{2\Delta t ||f_n(Y_n^M)||}{1 + \Delta t||f_n(Y_n^M)||} + 2||Y_n^M, g(Y_n^M)||||\Delta W_n^M|| + 2||Y_n^M, h(Y_n^M)||||\Delta N_n^M||$$

$$+ \frac{2\langle \Delta t f_n(Y_n^M), Y_n^M, \Delta W_n^M \rangle}{1 + \Delta t||f_n(Y_n^M)||} + \frac{2\langle \Delta t f_n(Y_n^M), Y_n^M, \Delta N_n^M \rangle}{1 + \Delta t||f_n(Y_n^M)||}$$

$$+ 2\langle g(Y_n^M), \Delta W_n^M, h(Y_n^M) \Delta N_n^M \rangle.$$  

(4.11)

Using Cauchy-Schwartz inequality and the estimation

$$\frac{1}{1 + \Delta t||f_n(Y_n^M)||} \leq 1,$$

from inequality from (4.11) :

$$||Y_{n+1}^M||^2 \leq ||Y_n^M||^2 + \Delta t^2 ||f_n(Y_n^M)||^2 + ||g(Y_n^M)||^2||\Delta W_n^M||^2 + ||h(Y_n^M)||^2||\Delta N_n^M||^2$$

$$+ 2\Delta t \langle f_n(Y_n^M), Y_n^M \rangle + 2||Y_n^M, g(Y_n^M)||||\Delta W_n^M|| + 2||Y_n^M, h(Y_n^M)||||\Delta N_n^M||$$

$$+ 2\Delta t \langle f_n(Y_n^M), g(Y_n^M) \Delta W_n^M \rangle + 2\Delta t \langle f_n(Y_n^M), h(Y_n^M) \Delta N_n^M \rangle$$

$$+ 2\langle g(Y_n^M), \Delta W_n^M, h(Y_n^M) \Delta N_n^M \rangle.$$  

(4.12)

Using the estimation $2ab \leq a^2 + b^2$, inequality (4.12) becomes:

$$||Y_{n+1}^M||^2 \leq ||Y_n^M||^2 + \Delta t^2 ||f_n(Y_n^M)||^2 + ||g(Y_n^M)||^2||\Delta W_n^M||^2 + ||h(Y_n^M)||^2||\Delta N_n^M||^2$$

$$+ 2\Delta t \langle f_n(Y_n^M), Y_n^M \rangle + 2||Y_n^M, g(Y_n^M)||||\Delta W_n^M|| + 2||Y_n^M, h(Y_n^M)||||\Delta N_n^M||$$

$$+ \Delta t^2 ||f_n(Y_n^M)||^2 + ||g(Y_n^M)||^2||\Delta W_n^M||^2 + \Delta t^2 ||f_n(Y_n^M)||^2$$

$$+ ||h(Y_n^M)||^2||\Delta N_n^M||^2 + ||g(Y_n^M)||^2||\Delta W_n^M||^2 + ||h(Y_n^M)||^2||\Delta N_n^M||^2.$$  

(4.13)

Putting similars terms of inequality (4.13) together, we obtain:

$$||Y_{n+1}^M||^2 \leq ||Y_n^M||^2 + 3\Delta t^2 ||f_n(Y_n^M)||^2 + 3||g(Y_n^M)||^2||\Delta W_n^M||^2 + 3||h(Y_n^M)||^2||\Delta N_n^M||^2$$

$$+ 2\Delta t \langle f_n(Y_n^M), Y_n^M \rangle + 2||Y_n^M, g(Y_n^M)||||\Delta W_n^M||$$

$$+ 2||Y_n^M, h(Y_n^M)||||\Delta N_n^M||.$$  

(4.14)
on $\Omega$, for all $M \in \mathbb{N}$ and all $n \in \{0, 1, \ldots, M - 1\}$.

In addition, for all $x \in \mathbb{R}^d$ such that $||x|| \geq 1$, the global Lipschitz condition satisfied by $g$ and $h$ leads to:

$$
||g(x)||^2 \leq (||g(x) - g(0)|| + ||g(0)||)^2 \\
\leq (C||x|| + ||g(0)||)^2 \\
\leq (C + ||g(0)||)^2||x||^2 \\
\leq \beta||x||^2.
$$

(4.15)

Along the same lines as above, for all $x \in \mathbb{R}^d$ such that $||x|| \geq 1$, we have:

$$
||h(x)||^2 \leq \beta||x||^2.
$$

(4.16)

Also, for all $x \in \mathbb{R}^d$ such that $||x|| \geq 1$, the one-sided Lipschitz condition satisfied by $f_\lambda$ leads to:

$$
\langle x, f_\lambda(x) \rangle = \langle x, f_\lambda(x) - f_\lambda(0) + f_\lambda(0) \rangle = \langle x, f_\lambda(x) - f_\lambda(0) \rangle + \langle x, f_\lambda(0) \rangle \\
\leq C||x||^2 + ||x||||f_\lambda(0)|| \\
\leq (C + ||f_\lambda(0)||)||x||^2 \\
\leq \sqrt{\beta}||x||^2.
$$

(4.17)

Furthermore, for all $x \in \mathbb{R}^d$ such that $1 \leq ||x|| \leq M^{1/\alpha}$ and for all $M \in \mathbb{N}$, using the polynomial growth condition of $f_\lambda$, the following inequality holds:

$$
||f_\lambda(x)||^2 \leq (||f_\lambda(x) - f_\lambda(0)|| + ||f_\lambda(0)||)^2 \\
\leq (C(K + ||x||^p)||x|| + ||f_\lambda(0)||)^2 \\
\leq (C(K + 1)||x||^{p+1} + ||f_\lambda(0)||)^2 \\
\leq (KC + C + ||f_\lambda(0)||)||x||^{2(p+1)} \\
\leq M^{\sqrt{\beta}||x||^2}.
$$

(4.18)

Now combining inequalities (4.14), (4.15), (4.16), (4.17) and (4.18), we obtain:

$$
||Y_{n+1}^M||^2 \leq ||Y_n^M||^2 + \frac{3T\sqrt{\beta}}{M}||Y_n^M||^2 + 3\beta||Y_n^M||^2||\Delta W_n^M||^2 + 3\beta||Y_n^M||^2||\Delta \bar{N}_n^M||^2 \\
+ \frac{2T\sqrt{\beta}}{M}||Y_n^M||^2 + 2\langle Y_n^M, g(Y_n^M)\Delta W_n^M \rangle + 2\langle Y_n^M, h(Y_n^M)\Delta \bar{N}_n^M \rangle \\
\leq ||Y_n^M||^2 + \left(\frac{3T^2 + 2T\sqrt{\beta}}{M}||Y_n^M||^2 + 3\beta||Y_n^M||^2||\Delta W_n^M||^2 + 3||Y_n^M||^2||\Delta \bar{N}_n^M||^2 \right) \\
+ 2\langle Y_n^M, g(Y_n^M)\Delta W_n^M \rangle + 2\langle Y_n^M, h(Y_n^M)\Delta \bar{N}_n^M \rangle.
$$

(4.19)

Using the inequality $3T^2 + 2T \leq 3\sqrt{\beta}$, it follows that:

$$
||Y_{n+1}^M||^2 \leq ||Y_n^M||^2 + \frac{3\beta}{M}||Y_n^M||^2 + 3\beta||Y_n^M||^2||\Delta W_n^M||^2 + 3\beta||Y_n^M||^2||\Delta \bar{N}_n^M||^2 \\
+ 2\langle Y_n^M, g(Y_n^M)\Delta W_n^M \rangle + 2\langle Y_n^M, h(Y_n^M)\Delta \bar{N}_n^M \rangle \\
= ||Y_n^M||^2 \left(1 + \frac{3\beta}{M} + 3\beta||\Delta W_n^M||^2 + 3\beta||\Delta \bar{N}_n^M||^2 \right) + 2\left< \frac{Y_n^M}{||Y_n^M||}, \frac{g(Y_n^M)}{||Y_n^M||} \Delta W_n^M \right> \\
+ 2\left< \frac{Y_n^M}{||Y_n^M||}, \frac{h(Y_n^M)}{||Y_n^M||} \Delta \bar{N}_n^M \right> \\
= ||Y_n^M||^2 \left(1 + \frac{3\beta}{M} + 3\beta||\Delta W_n^M||^2 + 3\beta||\Delta \bar{N}_n^M||^2 + 2\Delta \bar{N}_n^M \right).
$$

(4.20)

Using Lemma 4.2.7 for $a = \frac{3\beta}{M} + 3\beta||\Delta W_n^M||^2 + 2\alpha_n^M + 2\beta_n^M$ and $b = \sqrt{3\beta}||\Delta \bar{N}_n^M||$ it follows from (4.20) that:

$$
||Y_{n+1}^M||^2 \leq ||Y_n^M||^2 \exp \left(\frac{3\beta}{M} + 3\beta||\Delta W_n^M||^2 + 3\beta||\Delta \bar{N}_n^M|| + 2\alpha_n^M + 2\beta_n^M \right)
$$

(4.21)
on \{w \in \Omega : 1 \leq ||Y_n^M(\omega)|| \leq M^{1/2c}\}, for all \(M \in \mathbb{N}\) and all \(n \in \{0, 1, \cdots, M - 1\}\).

In order to complete our proof, we need the following map

\[ \tau_l^M : \Omega \rightarrow \{-1, 0, 1, \cdots, l\}, \quad l \in \{0, 1, \cdots, M\}, \]

such that :

\[ \tau_l^M(\omega) := \max \left( \{-1\} \cup \{n \in \{0, 1, \cdots, l - 1\} : ||Y_n^M(\omega)|| \leq 1\} \right), \]

for all \(\omega \in \Omega\), \(M \in \mathbb{N}\) and all \(l \in \{0, 1, \cdots, M\}\).

For \(M \in \mathbb{N}\) fixed we prove by induction on \(n \in \{0, 1, \cdots, M\}\) that

\[ 1_{\Omega_n^M}||Y_n^M|| \leq D_n^M. \tag{4.22} \]

- For \(n = 0\), \(D_0^M = (\beta + ||X_0||) \exp(\beta)\) and \(||Y_0^M|| = ||X_0||\). Since \(\beta \geq 1\) we have \(\exp(\beta) \geq 1\). So the following inequality holds

\[ 1_{\Omega_0^M}||Y_0^M|| \leq D_0^M. \]

- Let \(l \in \{0, 1, \cdots, M - 1\}\) be arbitrary and let’s assume that

\[ 1_{\Omega_n^M}||Y_n^M|| \leq D_n^M \text{ for all } n \in \{0, 1, \cdots, l\}. \]

We want to prove that inequality (4.9) holds for \(n = l + 1\).

Let \(\omega \in \Omega_{l+1}^M\) we have to prove that \(||Y_{l+1}^M(\omega)|| \leq D_{l+1}^M(\omega)\).

Since \((\Omega_{n+1}^M)\) is a decreasing sequence and \(\omega \in \Omega_{l+1}^M\), we have \(\omega \in \Omega_k^M\) and it follows from the hypothesis of induction that : \(||Y_k^M(\omega)|| \leq D_k^M(\omega)\), for all \(k \in \{0, \cdots, l\}\).

Also, since \(\omega \in \Omega_{k+1}^M\), by definition of \(\Omega_{k+1}^M\) it follows that \(D_k^M(\omega) \leq M^{1/2c}\), for all \(k \in \{0, \cdots, l\}\).

So for all \(k \in \{0, 1, \cdots, l\}\),

\[ ||Y_k^M(\omega)|| \leq D_k^M(\omega) \leq M^{1/2c}. \]

For all \(k \in \{\tau_{l+1}^M(\omega) + 1, \tau_{l+1}^M(\omega) + 2, \cdots, l\}\) we have

\[ 1 \leq ||Y_k^M(\omega)|| \leq M^{1/2c}. \tag{4.23} \]

Since (4.23) holds, it follows from (4.21), that

\[ ||Y_{k+1}^M(\omega)|| \leq ||Y_k^M(\omega)|| \exp \left( \frac{3\beta}{2M} + \frac{3\beta}{2} ||\Delta W_k^M(\omega)||^2 \right. \]

\[ + \left. \frac{3\beta}{2} |\Delta N_k^M(\omega)| + \alpha_k^M(\omega) + \beta_k^M(\omega) \right) \]

for all \(k \in \{\tau_{l+1}^M(\omega) + 1, \tau_{l+1}^M(\omega) + 2, \cdots, l\}\).

For \(k = l\) from the previous inequality, we have :

\[ ||Y_{l+1}^M(\omega)|| \leq ||Y_l^M(\omega)|| \exp \left( \frac{3\beta}{2M} + \frac{3\beta}{2} ||\Delta W_l^M(\omega)||^2 \right. \]

\[ + \left. \alpha_l^M(\omega) + \beta_l^M(\omega) \right). \tag{4.24} \]

Iterating (4.24) \(l - \tau_{l+1}^M(\omega)\) times leads to

\[ ||Y_{l+1}^M(\omega)|| \leq ||Y_{\tau_{l+1}^M(\omega)+1}^M(\omega)|| \exp \left( \sum_{m=\tau_{l+1}^M(\omega)+1}^{l} \left[ \frac{3\beta}{2M} + \frac{3\beta}{2} ||\Delta W_m^M(\omega)||^2 \right. \right. \]

\[ + \left. \frac{3\beta}{2} |\Delta N_m^M(\omega)| + \alpha_m^M(\omega) + \beta_m^M(\omega) \right]. \]
By definition of $\tau^M_l(\omega)$, we have $||Y^M_{\tau^M_l+1}(\omega)|| \le 1$.

Then it follows from (4.10) that $||Y^M_{\tau^M_l+1+1}(\omega)|| \le \beta$. So the above estimation of $||Y^M_{l+1}(\omega)||$ becomes:

$$||Y^M_{l+1}(\omega)|| \le \beta \exp \left( \sum_{m=\tau^M_l+1+1}^{l} \frac{3\beta}{2M} \right) + \frac{3\beta}{2} ||\Delta W^M_m(\omega)||^2 + \frac{3\beta}{2} ||\Delta \mathcal{N}^M_m(\omega)|| + \alpha^M_m(\omega) + \beta^M_m(\omega))$$

$$\le (\beta + ||X_0||) \exp \left( \frac{3\beta}{2} \sup_{u \in \{0,1,\ldots,l+1\}} \sum_{m=\tau^M_l+1+1}^{l} \frac{3\beta}{2} ||\Delta W^M_m(\omega)||^2 \right) + \frac{3\beta}{2} ||\Delta \mathcal{N}^M_m(\omega)|| + \alpha^M_m(\omega) + \beta^M_m(\omega)$$

Therefore $||Y^M_{l+1}(\omega)|| \le D^M_{l+1}(\omega)$. This complete the proof of Lemma 4.2.9.

The following is from [8, Lemma 3.2 pp 15].

**Lemma 4.2.11** Let $n \in \mathbb{N}$ and $Z : \Omega \rightarrow \mathbb{R}^m$ be an $m-$dimensional standard normal random variable. Then for all $a \in [0, \frac{1}{4}]$ the following inequality holds

$$E\left[\exp(a||Z||^2)\right] = (1 - 2a)^{-m/2} \le e^{2am}.$$

**Proof 4.2.12** Using the relation $||Z||^2 = |Z_1|^2 + |Z_2|^2 + \cdots + |Z_n|^2$ and the fact that $(Z_i)$ are independent and identically distributed, we have :

$$E\left[\exp(a||Z||^2)\right] = E\left[\exp \left( \sum_{i=1}^{m} a|Z_i|^2 \right) \right] = E\left[\prod_{j=1}^{m} \exp \left( a|Z_j|^2 \right) \right] = \left[ E\left(\exp(a|Z_1|^2)\right) \right]^m. \quad (4.25)$$

From the definition of the expected value of the standard normal random variable, we have :

$$E[\exp(a|Z_1|^2)] = \int_{-\infty}^{+\infty} e^{ax^2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{1}{\sqrt{1 - 2a}}.$$  

Using the inequality $\frac{1}{1-x} \le e^{2x} \forall x \in \left[0, \frac{1}{2}\right]$, it follows that

$$E[\exp(a|Z_1|^2)] = \frac{1}{\sqrt{1 - 2a}} \le e^{2a}, \quad \forall a \in \left[0, \frac{1}{4}\right]. \quad (4.26)$$

Combining (4.25) and (4.26) leads to :

$$E\left[\exp(a||Z||^2)\right] = (1 - 2a)^{-m/2} \le e^{2am}, \quad \forall a \in \left[0, \frac{1}{4}\right].$$

The following lemma and its proof are based on [8, Lemma 3.3, pp 15] with only different value of the coefficient $\beta$.

**Lemma 4.2.13** The following inequality holds :

$$\sup_{M \in \mathbb{N}, M \ge 4\beta^2 T} E\left[\exp \left( \beta p \sum_{k=0}^{M-1} ||\Delta W^M_k||^2 \right) \right] < \infty.$$
Proof 4.2.14 Let \( Z = N(0,1) \) be an \( m \)-dimensional standard normal random variable. Since for \( k = 0, \cdots, M - 1 \), \( \Delta W^M_k \) are independent, stationary and follows the normal distribution with mean 0 and variance \( \frac{\Delta}{M} ||Z||^2 \), it follows that:

\[
E \left( \exp \left[ \beta p \sum_{k=0}^{M-1} ||\Delta W^M_k||^2 \right] \right) = \prod_{k=0}^{M-1} E \left( \exp \left( \beta p ||\Delta W^M_k||^2 \right) \right) = \left( E \left( \exp \left( \beta p \frac{\Delta}{M} ||Z||^2 \right) \right) \right)^M \leq \left[ \exp \left( \frac{2\beta p}{M} \right) \right]^M \leq \exp \left( \frac{2\beta p}{T} \right) < \infty,
\]

for all \( p \in [1, +\infty) \) and all \( M \in \mathbb{N} \cap [4\beta p, \infty) \).

Lemma 4.2.15 Let \( Y \) be a standard normal random variable of dimension \( m \) and \( c \in \mathbb{R}^m \), then

\[
E[\exp(cY)] = \exp \left( \frac{c^2}{2} \right).
\]

Proof 4.2.16 \( E[\exp(cY)] \) is the moment generating function of \( Y \) at \( c \). Since the mean \( \mu = 0 \) and the standard deviation \( \sigma = 1 \), it follows directly that

\[
E[\exp(cY)] = \exp \left( \mu + \frac{1}{2}\sigma^2 c^2 \right) = \exp \left( \frac{c^2}{2} \right).
\]

The following lemma is from [9, Lemma 5.7, pp 15].

Lemma 4.2.17 The following inequality holds

\[
E \left[ \left| p z 1_{\{||x|| \geq 1\}} \left( \frac{x}{||x||} \cdot g(x) \Delta W^M_k \right) \right|^2 \right] \leq \exp \left[ \frac{p^2 (C + ||g(0)||)^2}{M} \right],
\]

for all \( x \in \mathbb{R}^d \), \( k \in \{0, 1, \cdots, M - 1\} \), \( p \in [1, \infty) \) and all \( z \in \{-1, 1\} \).

Proof 4.2.18 Let the notation \( a^\top \) stand for the transposed of a vector \( a \) and \( Y \) the \( m \) column vector define by \( Y = \sqrt{\frac{T}{M}} (1, \cdots, 1) \). Then we have :

\[
E \left[ \exp \left( p z \left( \frac{x}{||x||} \cdot g(x) \Delta W^M_k \right) \right) \right] = E \left[ \exp \left( p z \frac{g(x)^\top x}{||x||^2} \Delta W^M_k \right) \right] = E \left[ \exp \left( p z \frac{g(x)^\top x}{||x||^2} \sqrt{\frac{T}{M}} N(0,1) \right) \right] = E \left[ \exp \left( p z Y \frac{g(x)^\top x}{||x||^2} \right) \right]
\]

Using Lemma 4.2.17, it follows that

\[
E \left[ \exp \left( p z \left( \frac{x}{||x||} \cdot g(x) \Delta W^M_k \right) \right) \right] \leq \exp \left[ \frac{1}{2} p z \frac{g(x)^\top x}{||x||^2} Y \right] \leq \exp \left[ \frac{p^2 ||g(x)||^2}{M ||x||^2} ||Y||^2 \right]
\]
Using the global Lipschitz condition and the fact that \( \|x\| \geq 1 \), we have:

\[
\frac{\|g(x)\|^2}{\|x\|^2} \leq \frac{(||g(x) - g(0)|| + \|g(0)||)^2}{\|x\|^2} \leq \frac{(C + \|g(0)||)^2}{\|x\|^2} \leq (C + \|g(0)||)^2.
\]

Therefore, for all \( x \in \mathbb{R}^d \) such that \( \|x\| \geq 1 \), we have

\[
\mathbb{E} \left[ \exp \left( p z \frac{x}{\|x\|} \frac{g(x)}{\|x\|} \Delta W_k^M \right) \right] \leq \exp \left( \frac{p^2 T (C + \|g(0)||)^2}{M} \right),
\]

for all \( M \in \mathbb{N}, k \in \{0, \ldots, M - 1\} \), all \( p \in [1, \infty) \) and \( z \in \{-1, 1\} \).

Following closely [8, Lemma 3.4, pp 15] we have the following lemma.

**Lemma 4.2.19** Let \( \alpha_n^M : \Omega \to \mathbb{R} \) for \( M \in \mathbb{N} \) and \( n \in \{0, 1, \ldots, M\} \) defined in Notation 4.2.6, then the following inequality holds:

\[
\sup_{z \in \{-1, 1\}} \sup_{M \in \mathbb{N}} \left\| \sup_{n \in \{0, 1, \ldots, M\}} \exp \left( z \sum_{k=0}^{n-1} \alpha_k^M \right) \right\|_{L^p(\Omega, \mathbb{R})} < \infty,
\]

for all \( p \in [2, +\infty) \).

**Proof 4.2.20** The time discrete stochastic process \( z \sum_{k=0}^{n-1} \alpha_k^M, n \in \{0, 1, \ldots, M\} \) is an \((\mathcal{F}_{nt/M})_{n \in \{0, \ldots, M\}}\) martingale for every \( z \in \{-1, 1\} \) and \( M \in \mathbb{N} \). So \( \exp \left( z \sum_{k=0}^{n-1} \alpha_k^M \right) \) is a positive \((\mathcal{F}_{nt/M})_{n \in \{0, \ldots, M\}}\) submartingale for every \( z \in \{-1, 1\} \) and \( M \in \mathbb{N} \) since \( \exp \) is a convex function.

Applying Doop’s maximal inequality leads to:

\[
\left\| \sup_{n \in \{0, \ldots, M\}} \exp \left( z \sum_{k=0}^{n-1} \alpha_k^M \right) \right\|_{L^p(\Omega, \mathbb{R})} = \left( \mathbb{E} \sup_{n \in \{0, \ldots, M\}} \exp \left( p z \sum_{k=0}^{n-1} \alpha_k^M \right) \right)^{1/p} \leq \left( \frac{p}{p - 1} \right) \left( \mathbb{E} \left( \sum_{k=0}^{M-1} \alpha_k^M \right)^{1/p} \right) \leq \frac{p}{p - 1} \left\| \exp \left( z \sum_{k=0}^{M-1} \alpha_k^M \right) \right\|_{L^p(\Omega, \mathbb{R})}. \tag{4.27}
\]

Using Lemma 4.2.17, it follows from the previous inequality that:

\[
\mathbb{E} \left[ \exp(p z \alpha_k^M) / \mathcal{F}_{kt/M} \right] \leq \exp \left( \frac{p^2 T (C + \|g(0)||)^2}{M} \right). \tag{4.28}
\]

Using inequality (4.28), it follows that:

\[
\mathbb{E} \left[ \exp \left( p z \sum_{k=0}^{M-1} \alpha_k^M \right) \right] = \mathbb{E} \left[ \exp \left( p z \sum_{k=0}^{M-2} \alpha_k^M \right) \mathbb{E} \left[ \exp(p \alpha_k^M M^{-1} / \mathcal{F}_{(M-1)t/M} \right] \right] \leq \mathbb{E} \left[ \exp \left( p z \sum_{k=0}^{M-2} \alpha_k^M \right) \mathbb{E} \left[ \exp \left( \frac{p^2 T (C + \|g(0)||)^2}{M} \right) \right] \right].
\]

Iterating the previous inequality \( M \) times gives:

\[
\mathbb{E} \left[ \exp \left( p z \sum_{k=0}^{M-1} \alpha_k^M \right) \right] \leq \exp(p^2 T (C + \|g(0)||)^2). \tag{4.29}
\]

Now combining inequalities (4.27) and (4.29) leads to:

\[
\sup_{z \in \{-1, 1\}} \sup_{M \in \mathbb{N}} \left\| \sup_{n \in \{0, \ldots, M\}} \exp \left( z \sum_{k=0}^{n-1} \alpha_k^M \right) \right\|_{L^p(\Omega, \mathbb{R})} \leq 2 \exp(p^2 T (C + \|g(0)||)^2) < \infty,
\]

for all \( p \in [2, \infty) \).
Lemma 4.2.21 For all \( c \in \mathbb{R} \), we have:
\[
\mathbb{E}[\exp(c\Delta N_n^M)] = \exp\left[\frac{(e^c + c - 1)\lambda T}{M}\right],
\]
for all \( M \in \mathbb{N} \) and all \( n \in \{0, \cdots, M\} \).

Proof 4.2.22 It is known that if \( Y \) is a random variable following the poisson law with parameter \( \lambda \), then its moment generating function is given by:
\[
\mathbb{E}[\exp(cY)] = \exp(\lambda(e^c - 1)).
\]
Since \( \Delta N_n \) follows a poisson law with parameter \( \lambda t \), it follows that
\[
\mathbb{E}[\exp(c\Delta N_n^M)] = \mathbb{E}[\exp(c\Delta N_n^M + c\lambda t)] = \mathbb{E}\left[\exp\left(\frac{\lambda t}{M}\exp(c\Delta N_n^M)\right)\right] = \exp\left(\frac{c\lambda t}{M}(e^c - 1)\right) = \exp\left(\frac{(e^c + 1 - 1)\lambda t}{M}\right).
\]

Lemma 4.2.23 The following inequality holds
\[
\mathbb{E}\left[\exp\left(pz\mathbf{1}_{\{\|x\|\geq 1\}}\left(\frac{x}{\|x\|}, \frac{h(x)}{\|x\|} \Delta N_n^M\right)\right)\right] \leq \mathbb{E}\left[\exp\left(pz\frac{\|x\|\|h(x)\|}{\|x\|^2} \Delta N_n^M\right)\right] \leq \exp\left(\frac{\lambda(p(C + \|h(0)\|) + p(C + \|h(0)\|)}{M}\right),
\]
for all \( M \in \mathbb{N} \), \( z \in \{-1, 1\} \), all \( p \in [1, +\infty) \) and all \( n \in \{0, \cdots, M\} \).

Proof 4.2.24 For \( x \in \mathbb{R}^d \) such that \( \|x\| \neq 0 \), we have:
\[
\mathbb{E}\left[\exp\left(pz\left(\frac{x}{\|x\|}, \frac{h(x)}{\|x\|} \Delta N_n^M\right)\right)\right] \leq \mathbb{E}\left[\exp\left(pz\frac{\|x\|\|h(x)\|}{\|x\|^2} \Delta N_n^M\right)\right] = \mathbb{E}\left[\exp\left(pz\frac{\|h(x)\|}{\|x\|} \Delta N_n^M\right)\right].
\]

For all \( x \in \mathbb{R}^d \) such that \( \|x\| \geq 1 \), using the global Lipschitz condition satisfied by \( h \), we have:
\[
\|h(x)\| \leq \|h(x) - h(0)\| + \|h(0)\| \leq C + \|h(0)\|. \tag{4.30}
\]
So from inequality (4.30) and using Lemma 4.2.21 it follows that:
\[
\mathbb{E}\left[\exp\left(pz\mathbf{1}_{\{\|x\|\geq 1\}}\left(\frac{x}{\|x\|}, \frac{h(x)}{\|x\|} \Delta N_n^M\right)\right)\right] \leq \mathbb{E}[\exp(pz(C + \|h(0)\|)\Delta N_n^M)] \leq \exp\left(\frac{(e^{p(C + \|h(0)\|) + p(C + \|h(0)\| - 1)\lambda T)}{M}\right) \leq \exp\left(\frac{(e^{p(C + \|h(0)\|) + p(C + \|h(0)\|)\lambda T)}{M}\right).
\]

Lemma 4.2.25 Let \( \beta_n^M : \Omega \rightarrow \mathbb{R} \) define as in Notation 4.2.6 for all \( M \in \mathbb{N} \) and all \( n \in \{0, \cdots, M\} \), then we have the following inequality:
\[
\sup_{z \in \{-1, 1\}} \sup_{M \in \mathbb{N}} \left\| \sup_{n \in \{0, \cdots, M\}} \exp\left(z \sum_{K=0}^{n-1} \beta_k^M\right) \right\|_{L^p(\Omega, \mathbb{R})} < +\infty.
\]
Proof 4.2.26 For the same reason as for $\alpha_k^M, \beta_k^M$ is a positive \((\mathcal{F}_{nT/M})\)-martingale. So $\exp \left( p z^{n-1} \beta_k^M \right)$ is a positive \((\mathcal{F}_{nT/M})\)-submartingale for all $M \in \mathbb{N}$ and all $n \in \{0, \cdots, M\}$. Using Doob’s maximal inequality we have:

\[
\left\| \sup_{n \in \{0, \cdots, M\}} \exp \left( z \sum_{k=0}^{n-1} \beta_k^M \right) \right\|_{L^p(\Omega, \mathbb{R})} \leq \left( \frac{p}{p-1} \right) \left\| \exp \left( z \sum_{k=0}^{M-1} \beta_k^M \right) \right\|_{L^p(\Omega, \mathbb{R})},
\]

(4.31)

Using Lemma 4.2.23 it follows that

\[
\left\| \exp \left( z \sum_{k=0}^{M-1} \beta_k^M \right) \right\|^p_{L^p(\Omega, \mathbb{R})} = E \left[ \exp \left( p z \sum_{k=0}^{M-1} \beta_k^M \right) \right] = E \left[ \exp \left( p z \left( \sum_{k=0}^{M-2} \beta_k^M \right) + p z \beta_{M-1}^M \right) \right]
\]

\[
= E \left[ \exp \left( p z \sum_{k=0}^{M-2} \beta_k^M \right) \exp \left( p z \beta_{M-1}^M \right) \right] E \left[ \frac{e^{p(C+||h(0)||)} + p(C + ||h(0)||)}{M} \right].
\]

Iterating this last inequality $M$ times leads to:

\[
\left( E \left[ \exp \left( p z \sum_{k=0}^{M-1} \beta_k^M \right) \right] \right)^p \leq \exp \left[ \lambda T \left( e^{p(C+||h(0)||)} + T p(C + ||h(0)||) \right) \right],
\]

(4.32)

for all $M \in \mathbb{N}$, all $p \in (1, \infty)$ and all $z \in \{-1, 1\}$.

Combining inequalities (4.31) and (4.32) complete the proof of Lemma 4.2.25

Lemma 4.2.27 The following inequality holds

\[
\sup_{M \in \mathbb{N}} E \left[ \exp \left( p \beta \sum_{k=0}^{M-1} ||\Delta N_k^M|| \right) \right] < +\infty,
\]

for all $p \in [1, +\infty)$.

Proof 4.2.28 Using independence, stationarity of $\Delta N_k^M$ and Lemma 4.2.21, it follows that:

\[
\sup_{M \in \mathbb{N}} E \left[ \exp \left( p \beta \sum_{k=0}^{M-1} ||\Delta N_k^M|| \right) \right] = \prod_{k=0}^{M-1} E[\exp(p \beta ||\Delta N_k^M||)]
\]

\[
= \left( E[\exp(p \beta ||\Delta N_k^M||)] \right)^M
\]

\[
= \left( \exp \left[ \frac{(e^{p\beta} + p\beta - 1)^{-1}}{M} \right] \right)^M
\]

= $\exp[e^{p\beta} + p\beta - 1] < +\infty$

\]

for all $p \in [1, +\infty)$.

Inspired by [8, Lemma 3.5, pp 15], we have the following estimation.

Lemma 4.2.29 [Uniformly bounded moments of the dominating stochastic processes].

Let $M \in \mathbb{N}$ and $D_n^M : \Omega \rightarrow [0, \infty)$ for $n \in \{0, 1, \cdots, M\}$ be define as above, then we have:

\[
\sup_{M \in \mathbb{N}, M \geq 8\lambda T} \sup_{n \in \{0, 1, \cdots, M\}} \left\| D_n^M \right\|_{L^p(\Omega, \mathbb{R})} < \infty,
\]

for all $p \in [1, \infty)$. 
4.2. MOMENTS BOUNDED OF THE NUMERICAL SOLUTION

Proof 4.2.30 Let’s recall that:

\[ D_n^M = (\beta + \|\varepsilon\|) \exp \left( \frac{3\beta}{2} + \sup_{u \in \{0, \ldots, n\}} \sum_{k=u}^{n-1} \frac{3\beta}{2} |\Delta W_k^M|^2 + \frac{3\beta}{2} |\Delta N_k^M| + \alpha_k^M + \beta_k^M \right). \]

Using Holder inequality, it follows that:

\[ \sup_{M \in \mathbb{N}, M \geq 8\lambda p T} \left\| \sup_{n \in \{0, \ldots, M\}} D_n^M \right\|_{L^p(\Omega, \mathbb{R})} \leq e^{3\beta/2} (\beta + \|\varepsilon\|) \exp \left( \frac{3\beta}{2} \sum_{k=0}^{M-1} |\Delta W_k^M|^2 \right) \exp \left( \frac{3\beta}{2} \sum_{k=0}^{M-1} |\Delta N_k^M| \right) \exp \left( \sum_{k=0}^{M-1} \alpha_k^M \right) \exp \left( \sum_{k=0}^{M-1} \beta_k^M \right) \leq A_1 \times A_2 \times A_3 \times A_4 \times A_5. \]

By assumption A_1 is bounded. Lemma 4.2.13 and 4.2.27 show that A_2 and A_3 are bounded. Using again Holder inequality and Lemma 4.2.19 it follows that:

\[ A_4 = \left\| \sup_{n \in \{0, \ldots, M\}} \exp \left( \sup_{u \in \{0, \ldots, n\}} \sum_{k=u}^{n-1} \alpha_k^M \right) \right\|_{L^{16p}(\Omega, \mathbb{R})} \leq \left\| \sup_{n \in \{0, \ldots, M\}} \exp \left( \sum_{k=0}^{n-1} \alpha_k^M \right) \right\|_{L^{32p}(\Omega, \mathbb{R})} \times \left\| \sup_{u \in \{0, \ldots, M\}} \exp \left( -\sum_{k=0}^{n-1} \alpha_k^M \right) \right\|_{L^{32p}(\Omega, \mathbb{R})} < +\infty, \]

for all M \in \mathbb{N} and all p \in [1, \infty).

Along the same lines as above, we prove that A_5 is bounded.

Since each of the terms A_1, A_2, A_3, A_4 and A_5 is bounded, this complete the proof of Lemma 4.2.29.

The following lemma is an extension of [8, Lemma 3.6, pp 16]. Here, we include the jump part.

Lemma 4.2.31 Let M \in \mathbb{N} and \Omega_M^M \in \mathcal{F}. The following holds:

\[ \sup_{M \in \mathbb{N}} (M^p \mathbb{P}[\Omega_M^M]) < +\infty, \]

for all p \in [1, \infty).

Proof 4.2.32 Using the subadditivity of the probability measure and the Markov’s inequality, it follows that

\[ \mathbb{P}[(\Omega_M^M)^c] \leq \mathbb{P} \left[ \sup_{n \in \{0, \ldots, M-1\}} D_n^M > M^{1/2e} \right] + M \mathbb{P} \left[ \|W_T/M\| > 1 \right] + M \mathbb{P} \left[ |\overline{N}_T/M| > 1 \right] \]

\[ \leq \mathbb{P} \left[ \sup_{n \in \{0, \ldots, M-1\}} |D_n^M| > M^{9/2e} \right] + M \mathbb{P} \left[ \|W_T\| > \sqrt{M} \right] + M \mathbb{P} \left[ |\overline{N}_T| > M \right] \]

\[ \leq \mathbb{P} \left[ \sup_{n \in \{0, \ldots, M-1\}} |D_n^M| > M^{9/2e} \right] + M \mathbb{P} \left[ \|W_T\|^q > M^{9/2} \right] + M \mathbb{P} \left[ |\overline{N}_T|^q > M^q \right] \]

\[ \leq \mathbb{E} \left[ \sup_{n \in \{0, \ldots, M-1\}} |D_n^M|^q \right] M^{-q/2e} + \mathbb{E}[\|W_T\|^q] M^{1-q/2} + \mathbb{E}[|\overline{N}_T|^q] M^{1-q}, \]
for all \( q > 1 \).

Multiplying both sides of the above inequality by \( M^p \) leads to

\[
M^p P[(\Omega_M^M)^c] \leq E \left[ \sup_{n \in \{0, \ldots, M-1\}} |D_n^M|^p \right] M^{p-q/2c} + E[||W_T||^q] M^{p+1-q/2} + E[||\overline{N}_T||^q] M^{p+1-q}
\]

for all \( q > 1 \).

For \( q > \max\{2pc, 2p + 2\} \), we have \( M^{p+1-q/2} < 1, M^{p-q/2c} < 1 \) and \( M^{p+1-q} < 1 \). It follows for this choice of \( q \) that

\[
M^p P[(\Omega_M^M)^c] \leq E \left[ \sup_{n \in \{0, \ldots, M-1\}} |D_n^M|^p \right] + E[||W_T||^q] + E[||\overline{N}_T||^q].
\]

Using Lemma 4.2.29 and the fact that \( W_T \) and \( \overline{N}_T \) are independents of \( M \), it follows that

\[
\sup_{M \in \mathbb{N}} \left( M^p P[(\Omega_M^M)^c] \right) < +\infty.
\]

The following lemma can be found in [15, Theorem 48 pp 193] or in [18, Theorem 1.1, pp 1].

**Lemma 4.2.33** [Burkholder–Davis–Gundy inequality]

Let \( M \) be a martingale with càdlàg paths and let \( p \geq 1 \) be fixed. Let \( M^*_t = \sup_{s \leq t} ||M_s|| \). Then there exist constants \( c_p \) and \( C_p \) such that for any \( M \)

\[
c_p \left[ E \left[ (|M_t|)^{p/2} \right] ^{1/p} \right] \leq \left[ E \left[ (M_t^*)^{p/2} \right] ^{1/p} \right] \leq C_p \left[ E \left[ (|M_t|)^{p/2} \right] ^{1/p} \right],
\]

for all \( 0 \leq t \leq \infty \), where \( |M, M| \) stand for the quadratic variation of the process \( M \). The constants \( c_p \) and \( C_p \) are universal : They does not depend on the choice of \( M \).

The following lemma can be found in [8, Lemma 3.7, pp 16].

**Lemma 4.2.34** Let \( k \in \mathbb{N} \) and let \( Z : [0, T] \times \Omega \rightarrow \mathbb{R}^{k \times m} \) be a predictable stochastic process satisfying \( P \left[ \int_0^T ||Z_s||^2 ds < +\infty \right] = 1 \). Then we have the following inequality

\[
\left\| \sup_{s \in [0, t]} \left\| \int_0^s Z_u dW_u \right\| \right\|_{L^p(\Omega, \mathbb{R})} \leq C_p \left( \int_0^t \left( \sum_{i=1}^m ||Z_s \vec{e}_i||^2_{L^l(\Omega, \mathbb{R}))} ds \right)^{1/2}
\]

for all \( t \in [0, T] \) and all \( p \in [1, \infty] \). Where \( (\vec{e}_1, \cdots, \vec{e}_m) \) is the canonical basis of \( \mathbb{R}^m \).

**Proof 4.2.35** Since \( W \) is a continuous martingale satisfying \( d[W, W]_s = ds \), it follows from the property of the quadratic variation (see [14, 8.21, pp 219]) that

\[
\left[ \int_0^t Z_u dW_s, \int_0^t Z_u dW_s \right] = \int_0^t ||Z_s||^2 d[W, W]_s = \int_0^t ||Z_s||^2 ds.
\]  

Applying Lemma 4.2.33 for \( M_t = \sup_{0 \leq s \leq t} \int_0^s Z_u dW_s \) and using (433) leads to:

\[
E \left[ \left( \sup_{0 \leq s \leq t} \left\| \int_0^s Z_u dW_u \right\|^p \right)^{1/p} \right] \leq C_p \left( E \left( \int_0^t ||Z_s||^2 ds \right)^{p/2} \right)^{1/p}, \tag{4.34}
\]

where \( C_p \) is a positive constant depending on \( p \):
Using the definition of \( ||X||_{L^p(\Omega, \mathbb{R})} \) for any random variable \( X \), it follows that

\[
\left\| \sup_{s \in [0, T]} \left\| \int_0^s Z_u dW_u \right\|_{L^p(\Omega, \mathbb{R})} \right\| \leq C_p \left( \int_0^T \left\| |Z_s|^2 \right\|_{L^{p/2}(\Omega, \mathbb{R})}^2 ds \right)^{1/2}
\]

\[
\leq C_p \left( \int_0^T \sum_{i=1}^m \left\| |Z_s\bar{e}_i|^2 \right\|_{L^{p/2}(\Omega, \mathbb{R}^k)}^2 ds \right)^{1/2}.
\]

Using Minkowski inequality in its integral form (see Proposition 1.1.20) yields:

\[
\left\| \sup_{s \in [0, T]} \left\| \int_0^s Z_u dW_u \right\|_{L^p(\Omega, \mathbb{R})} \right\| \leq C_p \left( \int_0^T \sum_{i=1}^m \left\| |Z_s\bar{e}_i|^2 \right\|_{L^{p/2}(\Omega, \mathbb{R}^k)}^2 ds \right)^{1/2}.
\]

Using Holder inequality, it follows that:

\[
\left\| \sup_{s \in [0, T]} \left\| \int_0^s Z_u dW_u \right\|_{L^p(\Omega, \mathbb{R})} \right\| \leq C_p \left( \int_0^T \sum_{i=1}^m \left\| |Z_s\bar{e}_i|^2 \right\|_{L^{p/2}(\Omega, \mathbb{R}^k)}^2 ds \right)^{1/2}.
\]

This complete the proof of the lemma.

The following lemma and its proof can be found in [8, Lemma 3.8, pp 16].

**Lemma 4.2.36** Let \( k \in \mathbb{N} \) and let \( Z_l^M : \Omega \rightarrow \mathbb{R}^{k \times m} \), \( l \in \{0, 1, \cdots, M - 1\} \), \( M \in \mathbb{N} \) be a family of mappings such that \( Z_l^M \) is \( \mathcal{F}_{IT/M} / \mathcal{B}(\mathbb{R}^{k \times m}) \)-measurable for all \( l \in \{0, 1, \cdots, M - 1\} \) and \( M \in \mathbb{N} \). Then the following inequality holds:

\[
\left\| \sup_{j \in \{0, 1, \cdots, n\}} \left\| \sum_{i=0}^{j-1} Z_l^M \Delta W_i^M \right\|_{L^p(\Omega, \mathbb{R})} \right\| \leq C_p \left( \sum_{i=0}^{n-1} \sum_{l=0}^m \left\| Z_l^M \bar{e}_i \right\|_{L^{p}(\Omega, \mathbb{R}^k)}^2 \frac{T}{M} \right)^{1/2}.
\]

**Proof 4.2.37** Let \( Z_l^M : [0, T] \times \Omega \rightarrow \mathbb{R}^{k \times m} \) such that \( Z_s := Z_l^M \) for all \( s \in \left[ \frac{IT}{M}, \frac{(l+1)T}{M} \right] \), \( l \in \{0, 1, \cdots, M - 1\} \) and all \( M \in \mathbb{N} \).

Using Lemma 4.2.34, one obtain:

\[
\left\| \sup_{j \in \{0, 1, \cdots, n\}} \left\| \sum_{i=0}^{j-1} Z_l^M \Delta W_i^M \right\|_{L^p(\Omega, \mathbb{R})} \right\| = \left\| \sup_{j \in \{0, 1, \cdots, n\}} \left\| \int_0^{jT/M} Z_u^M dW_u \right\|_{L^p(\Omega, \mathbb{R})} \right\|
\]

\[
\leq \left\| \sup_{s \in \left[ \frac{nT}{M} \right]} \left\| \int_0^s Z_u^M dW_u \right\|_{L^p(\Omega, \mathbb{R})} \right\|
\]

\[
\leq C_p \left( \int_0^{nT/M} \sum_{i=1}^m \left\| Z_s\bar{e}_i \right\|^2_{L^{p}(\Omega, \mathbb{R}^k)} ds \right)^{1/2}
\]

\[
= C_p \left( \sum_{i=0}^{n-1} \sum_{l=0}^m \left\| Z_l^M \bar{e}_i \right\|^2_{L^{p}(\Omega, \mathbb{R}^k)} \frac{T}{M} \right)^{1/2}.
\]

**Lemma 4.2.38** Let \( k \in \mathbb{N} \) and \( Z : [0, T] \times \Omega \rightarrow \mathbb{R}^k \) be a predictable stochastic process satisfying

\[\mathbb{E} \left( \int_0^T ||Z_s||^2 ds \right) < +\infty.\]

Then the following inequality holds:

\[
\left\| \sup_{s \in [0, T]} \left\| \int_0^s Z_u dN_u \right\|_{L^p(\Omega, \mathbb{R})} \right\| \leq C_p \left( \int_0^T \left\| Z_s \right\|^2_{L^{p}(\Omega, \mathbb{R}^k)} ds \right)^{1/2},
\]

for all \( t \in [0, T] \) and all \( p \in [1, +\infty) \).
Proof 4.2.39 Since $\overline{N}$ is a martingale with càdlàg paths satisfying $d[\overline{N}, \overline{N}] = \lambda s$ (see Proposition 1.2.17), it follows from the property of the quadratic variation (see [14, 8.21, pp 219]) that

$$\left[ \int_0^t Z_s d\overline{N}_s, \int_0^t Z_s d\overline{N}_s \right]_t = \int_0^t \|Z_s\|^2 \lambda ds. \tag{4.36}$$

Applying Lemma 4.2.33 for $M_t = \sup_{0 \leq s \leq T} \int_0^s Z_s d\overline{N}_s$ and using (4.36) leads to:

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left\| \int_0^t Z_u d\overline{N}_u \right\|^p \right]^{1/p} \leq C_p \left[ \mathbb{E} \left( \int_0^T \|Z_s\|^2 ds \right)^{p/2} \right]^{1/p}, \tag{4.37}$$

where $C_p$ is a positive constant depending on $p$ and $\lambda$.

Using the definition of $\|X\|_{L^p(\Omega, \mathbb{R})}$ for any random variable $X$, it follows that

$$\left\| \sup_{s \in [0,T]} \left\| \int_0^s Z_u d\overline{N}_u \right\|_{L^p(\Omega, \mathbb{R})} \right\| \leq C_p \left( \int_0^T \|Z_s\|^2_{L^p(\Omega, \mathbb{R})} ds \right)^{1/2}. \tag{4.38}$$

Using Minkowski inequality in its integral form (see Proposition 1.1.20) yields:

$$\left\| \sup_{s \in [0,T]} \left\| \int_0^s Z_u d\overline{N}_u \right\|_{L^p(\Omega, \mathbb{R})} \right\| \leq C_p \left( \int_0^T \|Z_s\|^2_{L^p(\Omega, \mathbb{R})} ds \right)^{1/2}. \tag{4.39}$$

This completes the proof of the lemma.

Lemma 4.2.40 Let $k \in \mathbb{N}$, $M \in \mathbb{N}$ and $Z^M_l : \Omega \rightarrow \mathbb{R}^k, l \in \{0, 1, \ldots, M-1\}$ be a family of mappings such that $Z^M_l$ is $\mathcal{F}_{t/M}/\mathcal{B}(\mathbb{R}^k)$-measurable for all $l \in \{0, 1, \ldots, M-1\}$, then for all $n \in \{0, 1, \ldots, M\}$ the following inequality holds:

$$\left\| \sup_{j \in \{0, 1, \ldots, n\}} \left\| \sum_{l=0}^{j-1} Z^M_l \Delta N^M_l \right\|_{L^p(\Omega, \mathbb{R})} \right\| \leq C_p \left( \sum_{j=0}^{n-1} \|Z^M_j\|^2_{L^p(\Omega, \mathbb{R}^k)} \frac{T}{M} \right)^{1/2},$$

where $C_p$ is a positive constant independent of $M$.

Proof 4.2.41 Let's define $\overline{Z}^M : [0, T] \times \Omega \rightarrow \mathbb{R}^k$ such that $\overline{Z}^M_s := Z^M_l$ for all $s \in \left[ \frac{t}{M}, \frac{(l+1)T}{M} \right)$, $l \in \{0, 1, \ldots, M-1\}$.

Using the definition of stochastic integral and Lemma 4.2.38, it follows that:

$$\left\| \sup_{j \in \{0, 1, \ldots, n\}} \left\| \sum_{l=0}^{j-1} Z^M_l \Delta N^M_l \right\|_{L^p(\Omega, \mathbb{R})} \right\| \leq \left\| \sup_{s \in [0,nT/M]} \left\| \int_0^s Z^M_u d\overline{N}_u \right\|_{L^p(\Omega, \mathbb{R}^k)} \right\| \leq C_p \left( \int_0^{nT/M} \|Z^M_s\|^2_{L^p(\Omega, \mathbb{R}^k)} ds \right)^{1/2} = C_p \left( \sum_{j=0}^{n-1} \|Z^M_j\|^2_{L^p(\Omega, \mathbb{R}^k)} \frac{T}{M} \right)^{1/2}. \tag{4.39}$$

This completes the proof of the lemma.
Now we are ready to prove Theorem 4.2.5.

**Proof 4.2.42 [Theorem 4.2.5]**

Let’s first represent the numerical approximation $Y_n^M$ in the following appropriate form:

$$Y_n^M = Y_{n-1}^M + \frac{\Delta t f_{\lambda}(Y_{n-1}^M)}{1 + \Delta t ||f_{\lambda}(Y_{n-1}^M)||} + g(Y_{n-1})\Delta W_{n-1}^M + h(Y_{n-1}^M)\Delta N_{n-1}^M$$

$$= X_0 + \sum_{k=0}^{n-1} \frac{\Delta t f_{\lambda}(Y_k^M)}{1 + \Delta t ||f_{\lambda}(Y_k^M)||} + \sum_{k=0}^{n-1} g(Y_k^M)\Delta W_k^M + \sum_{k=0}^{n-1} h(Y_k^M)\Delta N_k^M$$

$$= X_0 + \sum_{k=0}^{n-1} g(0)\Delta W_k^M + \sum_{k=0}^{n-1} h(0)\Delta N_k^M + \sum_{k=0}^{n-1} \frac{\Delta t f_{\lambda}(Y_n^M)}{1 + \Delta t ||f_{\lambda}(Y_n^M)||}$$

$$+ \sum_{k=0}^{n-1} (g(Y_k^M) - g(0))\Delta W_k^M + \sum_{k=0}^{n-1} (h(Y_k^M) - h(0))\Delta N_k^M,$$

for all $M \in \mathbb{N}$ and all $n \in \{0, \ldots, M\}$.

Using the inequality

$$\left\| \frac{\Delta t f_{\lambda}(Y_n^M)}{1 + \Delta t ||f_{\lambda}(Y_n^M)||} \right\|_{L^p(\Omega, \mathbb{R}^d)} < 1$$

it follows that:

$$||Y_n^M||_{L^p(\Omega, \mathbb{R}^d)} \leq ||X_0||_{L^p(\Omega, \mathbb{R}^d)} + \sum_{k=0}^{n-1} g(0)\Delta W_k^M + \sum_{k=0}^{n-1} h(0)\Delta N_k^M + M$$

Using Lemma 4.2.36 and Lemma 4.2.40, it follows that:

$$||Y_n^M||_{L^p(\Omega, \mathbb{R}^d)} \leq ||X_0||_{L^p(\Omega, \mathbb{R}^d)} + C_p \left( \sum_{k=0}^{n-1} \sum_{i=1}^{m} ||g_i(0)||^2 \frac{T}{M} \right)^{1/2} + C_p \left( \sum_{k=0}^{n-1} ||h(0)||^2 \frac{T}{M} \right)^{1/2}$$

$$+ M + C_p \left( \sum_{k=0}^{n-1} \lambda \left\| (h(Y_k^M) - h(0))\Delta W_k^M \right\|_{L^p(\Omega, \mathbb{R}^d)} \frac{T}{M} \right)^{1/2}$$

$$\leq ||X_0||_{L^p(\Omega, \mathbb{R}^d)} + C_p \left( \frac{nT}{M} \sum_{i=1}^{m} ||g_i(0)||^2 \right)^{1/2} + C_p \left( \frac{nT}{M} ||h(0)||^2 \right)^{1/2}$$

$$+ M + C_p \left( \sum_{k=0}^{n-1} \sum_{i=1}^{m} ||g_i(Y_k^M) - g_i(0)||^2 \frac{T}{M} \right)^{1/2}$$

$$+ C_p \left( \sum_{k=0}^{n-1} ||h(Y_k^M) - h(0)||^2 \frac{T}{M} \right)^{1/2}. \quad (4.40)$$

From $||g_i(0)||^2 \leq ||g(0)||^2$ and the global Lipschitz condition satisfied by $g$ and $h$, we obtain $||g_i(Y_k^M) - g_i(0)||_{L^p(\Omega, \mathbb{R}^d)} \leq C||Y_k^M||_{L^p(\Omega, \mathbb{R}^d)}$ and $||h(Y_k^M) - h(0)||_{L^p(\Omega, \mathbb{R}^d)} \leq C||Y_k^M||_{L^p(\Omega, \mathbb{R}^d)}$. So using (4.40), we obtain

$$||Y_n^M||_{L^p(\Omega, \mathbb{R}^d)} \leq ||X_0||_{L^p(\Omega, \mathbb{R}^d)} + C_p \sqrt{Tm}||g(0)|| + C_p \sqrt{T}||h(0)|| + M$$

$$+ C_p \left( \frac{Tm}{M} \sum_{k=0}^{n-1} ||Y_k^M||^2_{L^p(\Omega, \mathbb{R}^d)} \right)^{1/2} + C_p \left( \frac{T}{M} \sum_{k=0}^{n-1} ||Y_k^M||^2_{L^p(\Omega, \mathbb{R}^d)} \right)^{1/2}.$$
Using the inequality \((a + b + c)^2 \leq 3a^2 + 3b^2 + 3c^2\), it follows that:

\[
\|Y_n^M\|_{L^p(\Omega, \mathbb{R}^d)}^2 \leq 3 \left( \|X_0\|_{L^p(\Omega, \mathbb{R}^d)} + C_p \sqrt{Tm} \|g(0)\| + C_p \sqrt{T} \|h(0)\| + M \right)^2 + 3T(C_p \sqrt{m} + C_p)^2 \sum_{k=0}^{n-1} \|Y_k^M\|_{L^p(\Omega, \mathbb{R}^d)}^2
\]

for all \(p \in [1, \infty)\).

Using the fact that \(\frac{3T(p\sqrt{m} + C_p)^2}{M} < 3(p\sqrt{m} + C_p)^2\) we obtain the following estimation

\[
\|Y_n^M\|_{L^p(\Omega, \mathbb{R}^d)}^2 \leq 3 \left( \|X_0\|_{L^p(\Omega, \mathbb{R}^d)} + C_p \sqrt{Tm} \|g(0)\| + C_p \sqrt{T} \|h(0)\| + M \right)^2 + 3T(C_p \sqrt{m} + C_p)^2 \sum_{k=0}^{n-1} \|Y_k^M\|_{L^p(\Omega, \mathbb{R}^d)}^2
\]

(4.41)

Applying Gronwall lemma to (4.41) leads to

\[
\|Y_n^M\|_{L^p(\Omega, \mathbb{R}^d)}^2 \leq 3e^{3(C_p \sqrt{m} + C_p)} \left( \|X_0\|_{L^p(\Omega, \mathbb{R}^d)} + C_p \sqrt{Tm} \|g(0)\| + C_p \sqrt{T} \|h(0)\| + M \right)^2.
\]

(4.42)

Taking the square root and the supremum in the both sides of (4.42) leads to:

\[
\sup_{n \in \{0, \ldots, M\}} \|Y_n^M\|_{L^p(\Omega, \mathbb{R}^d)} \leq \sqrt{3e^{3(C_p \sqrt{m} + C_p)}} \left( \|X_0\|_{L^p(\Omega, \mathbb{R}^d)} + C_p \sqrt{Tm} \|g(0)\| + C_p \sqrt{T} \|h(0)\| + M \right)
\]

(4.43)

Unfortunately, (4.43) is not enough to conclude the proof of the lemma due to the term \(M\) in the right hand side. Using the fact that \((\Omega_n^M)_n\) is a decreasing sequence and by exploiting H"older inequality, we obtain:

\[
\sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left\| \mathbf{1}_{\{(\Omega_n^M)^c\}} Y_n^M \right\|_{L^p(\Omega, \mathbb{R}^d)} \leq \sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left\| \mathbf{1}_{\{(\Omega_n^M)^c\}} \right\|_{L^2(\Omega, \mathbb{R}^d)} \|Y_n^M\|_{L^{2p}(\Omega, \mathbb{R}^d)}
\]

\[
\leq \left( \sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left( M \left\| \mathbf{1}_{\{(\Omega_n^M)^c\}} \right\|_{L^2(\Omega, \mathbb{R}^d)} \right) \right) \times \left( \sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left( M^{-1} \|Y_n^M\|_{L^{2p}(\Omega, \mathbb{R}^d)} \right) \right).
\]

(4.44)

Using inequality (4.43) we have

\[
\left( \sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left( M^{-1} \|Y_n^M\|_{L^{2p}(\Omega, \mathbb{R}^d)} \right) \right) \leq \sqrt{3e^{3(C_p \sqrt{m} + C_p)}} \left( \|X_0\|_{L^{2p}(\Omega, \mathbb{R}^d)} + C_p \sqrt{Tm} \|g(0)\| + C_p \sqrt{T} \|h(0)\| + 1 \right)
\]

\[
\leq \sqrt{3e^{3(C_p \sqrt{m} + C_p)}} \left( \|X_0\|_{L^{2p}(\Omega, \mathbb{R}^d)} + C_p \sqrt{Tm} \|g(0)\| + C_p \sqrt{T} \|h(0)\| + 1 \right) < +\infty,
\]

(4.45)

for all \(p \geq 1\). From the relation

\[
\left\| \mathbf{1}_{\{(\Omega_n^M)^c\}} \right\|_{L^2(\Omega, \mathbb{R}^d)} = \mathbb{E} \left[ \left( \mathbf{1}_{\{(\Omega_n^M)^c\}} \right)^{1/2p} \right]^{1/2} = \mathbb{P} \left[ \left( \Omega_n^M \right)^c \right]^{1/2p},
\]

it follows using Lemma 4.2.31 that:

\[
\sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left( M \left\| \mathbf{1}_{\{(\Omega_n^M)^c\}} \right\|_{L^p(\Omega, \mathbb{R}^d)} \right) = \sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left( M^{2p} \mathbb{P} \left[ \left( \Omega_n^M \right)^c \right]^{1/2p} \right) < +\infty,
\]

(4.46)
for all $p \geq 1$.

So plugging (4.45) and (4.46) in (4.44) leads to:

$$\sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left\| \mathbf{1}_{(\Omega^M_n)} Y_n^M \right\|_{L^p(\Omega, \mathbb{R}^d)} < +\infty. \tag{4.47}$$

Furthermore, we have

$$\sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left\| Y_n^M \right\|_{L^p(\Omega, \mathbb{R}^d)} \leq \sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left\| \mathbf{1}_{(\Omega^M_n)} Y_n^M \right\|_{L^p(\Omega, \mathbb{R}^d)} + \sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left\| D_n^M \right\|_{L^p(\Omega, \mathbb{R}^d)}. \tag{4.48}$$

From (4.47), the second term of inequality (4.48) is bounded, while using Lemma 4.2.9 and Lemma 4.2.29 we have:

$$\sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left\| D_n^M \right\|_{L^p(\Omega, \mathbb{R}^d)} < +\infty. \tag{4.49}$$

Finally plugging (4.47) and (4.49) in (4.48) leads to:

$$\sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left\| Y_n^M \right\|_{L^p(\Omega, \mathbb{R}^d)} < +\infty. \tag{4.50}$$

### 4.3 Strong convergence of the compensated tamed Euler scheme

The main result of this chapter is given in the following theorem.

**Theorem 4.3.1** Under Assumptions 4.2.2, for all $p \in [1, +\infty)$ there exist a positive constant $C_p$ such that:

$$\left( \mathbb{E} \left[ \sup_{t \in [0, T]} \left\| X_t - \bar{Y}_t^M \right\|^p \right] \right)^{1/p} \leq C_p \Delta t^{1/2}, \tag{4.51}$$

for all $M \in \mathbb{N}$.

Where $X : [0, T] \times \Omega \rightarrow \mathbb{R}^d$ is the exact solution of equation (4.1) and $\bar{Y}_t^M$ is the time continuous approximation defined by (4.7).

In order to prove Theorem 4.3.1, we need the following two lemmas.

Following closely [8, Lemma 3.10, pp 16], we have the following lemma.

**Lemma 4.3.2** Let $Y_n^M$ be defined by (4.6) for all $M \in \mathbb{N}$ and all $n \in \{0, 1, \ldots, M\}$, then we have

$$\sup_{M \in \mathbb{N}} \sup_{n \in \{0, 1, \ldots, M\}} \left( \mathbb{E} \left[ \left\| f_n(Y_n^M) \right\|^p \right] \vee \mathbb{E} \left[ \left\| q(Y_n^M) \right\|^p \right] \vee \mathbb{E} \left[ \left\| h(Y_n^M) \right\|^p \right] \right) < +\infty,$n \in \{0, 1, \ldots, M\}$, then we have

for all $p \in [1, +\infty)$.

**Proof 4.3.3** From the polynomial growth condition of $f_\lambda$, for all $x \in \mathbb{R}^d$ we have

$$\| f_\lambda(x) \| \leq C(1 + |x|^p) |x| + |f_\lambda(0)| = CK|x| + C|x|^{c+1} + |f_\lambda(0)|.$$

- If $|x| \leq 1$, then $CK|x| \leq CK$, hence

$$\begin{align*}
\| f_\lambda(x) \| & \leq CK + C|x|^{c+1} + |f_\lambda(0)| \\
& \leq KC + KC|x|^{c+1} + C + C|x|^{c+1} + |f_\lambda(0)| + |f_\lambda(0)||x|^{c+1} \\
& = (KC + C + |f_\lambda(0)|)(1 + |x|^{c+1}). \tag{4.51}
\end{align*}$$
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Proof 4.3.5

• If \( ||x|| \geq 1 \), then \( C||x|| \leq C||x||^{c+1} \), hence

\[
||f_\lambda(x)|| \leq KC||x||^{c+1} + C||x||^{c+1} + ||f_\lambda(0)|| \\
\leq KC + KC||x||^{c+1} + C||x||^{c+1} + ||f_\lambda(0)|| + ||f_\lambda(0)|| ||x||^{c+1} \\
= (KC + C + ||f_\lambda(0)||)(1 + ||x||^{c+1}).
\] (4.52)

So it follows from (4.51) and (4.52) that

\[
||f_\lambda(x)|| \leq (KC + C + ||f_\lambda(0)||)(1 + ||x||^{c+1}), \quad \text{for all } x \in \mathbb{R}^d.
\] (4.53)

Using inequality (4.53) and Theorem 4.2.5, it follows that:

\[
\sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \|f_\lambda(Y_n^M)\|_{L^p(\Omega, \mathbb{R}^d)} \leq (KC + C + ||f_\lambda(0)||) \\
\times \left( 1 + \sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \|Y_n^M\|_{L^{p(c+1)}(\Omega, \mathbb{R}^d)} \right)^{c+1} \\
< +\infty,
\]

for all \( p \in [1, \infty) \).

In other hand, using the global Lipschitz condition satisfied by \( g \) and \( h \), it follows that:

\[
||g(x)|| \leq C||x|| + ||g(0)|| \quad \text{and} \quad ||h(x)|| \leq C||x|| + ||h(0)||.
\] (4.54)

Using once again Theorem 4.2.5, it follows from (4.54) that:

\[
\sup_{M \in \mathbb{N}, n \in \{0, \ldots, M\}} \|g(Y_n^M)\|_{L^p(\Omega, \mathbb{R}^d)} \leq ||g(0)|| + C \sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \|Y_n^M\|_{L^p(\Omega, \mathbb{R}^d)} < +\infty,
\]

for all \( p \in [1, \infty) \).

Using the same argument as for \( g \) the following holds

\[
\sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \|h(Y_n^M)\|_{L^p(\Omega, \mathbb{R}^d)} < +\infty,
\]

for all \( p \in [1, +\infty) \).

This complete the proof of Lemma 4.3.2.

For \( s \in [0, T] \) let \( \lfloor s \rfloor \) be the greatest grid point less than \( s \). We have the following lemma.

Lemma 4.3.4 For any stepsize \( \Delta t \), the following inequalities holds

\[
\sup_{t \in [0, T]} \left\| Y_t^M - Y_t^M \right\|_{L^p(\Omega, \mathbb{R}^d)} \leq C_p \Delta t^{1/2},
\]

\[
\sup_{M \in \mathbb{N}} \sup_{t \in [0, T]} \left\| Y_t^M \right\|_{L^p(\Omega, \mathbb{R}^d)} < \infty,
\]

\[
\sup_{t \in [0, T]} \left\| f_\lambda(Y_t^M) - f_\lambda(Y_t^M) \right\|_{L^p(\Omega, \mathbb{R}^d)} \leq C_p \Delta t^{1/2}.
\]

Proof 4.3.5

• Using Lemma 4.2.38, Lemma 4.2.34 and the time continuous approximation (4.7), it follows that:
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\[
\sup_{t \in [0,T]} \left\| \tilde{Y}_t^M - \tilde{Y}_t^M \right\|_{LP(\Omega,\mathbb{R}^d)}
\leq \frac{T}{M} \left( \sup_{t \in [0,T]} \left\| \frac{f(\tilde{Y}_t^M)}{1 + \Delta t} \right\|_{L^2(\Omega,\mathbb{R}^d)} \right) + \sup_{t \in [0,T]} \left\| \int_t^T g(\tilde{Y}_s^M) dW_s \right\|_{LP(\Omega,\mathbb{R}^d)}
\leq \frac{T}{\sqrt{M}} \left( \sup_{n \in \{0,\ldots,M\}} \left\| f(\tilde{Y}_n^M) \right\|_{LP(\Omega,\mathbb{R}^d)} \right)
+ \frac{1}{\sqrt{M}} \left( \sup_{t \in [0,T]} \left\| \frac{h(\tilde{Y}_s^M)}{\sqrt{\Delta t}} \right\|_{LP(\Omega,\mathbb{R}^d)} \right)
\leq \frac{T}{\sqrt{M}} \left( \sup_{n \in \{0,\ldots,M\}} \left\| f(\tilde{Y}_n^M) \right\|_{LP(\Omega,\mathbb{R}^d)} \right)
+ \frac{\sqrt{T}m}{\sqrt{M}} \left( \sup_{t \in [1,\ldots,m]} \sup_{n \in \{0,\ldots,M\}} \left\| g(\tilde{Y}_n^M) \right\|_{LP(\Omega,\mathbb{R}^d)} \right)
\leq C_p \Delta t^{1/2},
\] (4.55)

for all \( M \in \mathbb{N} \).

Using inequality (4.55) and Lemma 4.3.2, it follows that :

\[
\left( \sup_{t \in [0,T]} \left\| \tilde{Y}_t^M - \tilde{Y}_t^M \right\|_{LP(\Omega,\mathbb{R}^d)} \right) < C_p \Delta t^{1/2},
\] (4.56)

for all \( p \in [1,\infty) \) and all stepsize \( \Delta t \).

- Using the inequalities (4.56), \( ||a|| \leq ||a - b|| + ||b|| \) for all \( a, b \in \mathbb{R}^d \) and Theorem 4.2.5 it follows that

\[
\sup_{t \in [0,T]} \left\| \tilde{Y}_t^M - \tilde{Y}_t^M \right\|_{LP(\Omega,\mathbb{R}^d)} \leq \left( \sup_{t \in [0,T]} \left\| \tilde{Y}_t^M - \tilde{Y}_t^M \right\|_{LP(\Omega,\mathbb{R}^d)} \right) + \sup_{t \in [0,T]} \left\| \tilde{Y}_t^M \right\|_{LP(\Omega,\mathbb{R}^d)}
\leq \frac{C_p}{M^{1/2}} + \sup_{t \in [0,T]} \left\| \tilde{Y}_t^M \right\|_{LP(\Omega,\mathbb{R}^d)}
\leq C_p T^{1/2} + \sup_{t \in [0,T]} \left\| \tilde{Y}_t^M \right\|_{LP(\Omega,\mathbb{R}^d)}
< \infty,
\]

for all \( p \in [1,\infty) \) and all \( M \in \mathbb{N} \).

- Further, using the polynomial growth condition :

\[
||f(x) - f(y)|| \leq C(K + ||x||^c + ||y||^c)||x - y||,
\]

for all \( x, y \in \mathbb{R}^d \), it follows using Holder inequality that :

\[
\sup_{t \in [0,T]} \left\| f(\tilde{Y}_t^M) - f(\tilde{Y}_t^M) \right\|_{LP(\Omega,\mathbb{R}^d)} \leq C \left( K + 2 \sup_{t \in [0,T]} \left\| \tilde{Y}_t^M \right\|_{LP(\Omega,\mathbb{R}^d)} \right)
\times \left( \sup_{t \in [0,T]} \left\| \tilde{Y}_t^M - \tilde{Y}_t^M \right\|_{LP(\Omega,\mathbb{R}^d)} \right)
\leq C \Delta t^{1/2},
\] (4.57)

Using (4.57) and the first part of Lemma 4.3.4, the following inequality holds

\[
\left[ \sup_{t \in [0,T]} \left\| f(\tilde{Y}_t^M) - f(\tilde{Y}_t^M) \right\|_{LP(\Omega,\mathbb{R}^d)} \right] < C_p \Delta t^{1/2},
\] (4.58)

for all \( p \in [1,\infty) \) and for all stepsize \( \Delta t \).
Now we are ready to give the proof of Theorem 4.3.1.

**Proof 4.3.6 [Theorem 4.3.1]**

Let’s recall that for \( s \in [0, T] \), \( \lfloor s \rfloor \) denote the greatest grid point less than \( s \). The time continuous solution (4.7) can be written into its integral form as follows:

\[
Y_s^M = X_0 + \int_0^s \frac{f_{\lambda}(\bar{Y}_{[u]})}{1 + \Delta t||f_{\lambda}(\bar{Y}_{[u]})||} \, du + \int_0^s g(\bar{Y}_{[u]}) \, dW_u + \int_0^s h(\bar{Y}_{[u]}) \, dN_u,
\]

for all \( s \in [0, T] \) almost surely and all \( M \in \mathbb{N} \).

Let’s estimate first the quantity \( ||X_s - Y_s^M||^2 \)

\[
X_s - Y_s = \int_0^s \left( f_{\lambda}(X_u) - \frac{f_{\lambda}(\bar{Y}_{[u]})}{1 + \Delta t||f_{\lambda}(\bar{Y}_{[u]})||} \right) \, du + \int_0^s \left( g(X_u) - g(\bar{Y}_{[u]}) \right) \, dW_u
\]

Using the relation \( dN_u = dN_u - \lambda \, du \), it follows that

\[
X_s - Y_s = \int_0^s \left[ \left( f_{\lambda}(X_u) - \frac{f_{\lambda}(\bar{Y}_{[u]})}{1 + \Delta t||f_{\lambda}(\bar{Y}_{[u]})||} \right) - \lambda \left( h(X_u) - h(\bar{Y}_{[u]}) \right) \right] \, du
\]

The function \( k : \mathbb{R}^m \rightarrow \mathbb{R}, \ x \mapsto ||x||^2 \) is twice differentiable. Applying Itô’s formula for jumps process to the process \( X_s - Y_s^M \) leads to:

\[
||X_s - Y_s^M||^2 = 2 \int_0^s \left( X_u - Y_u^M, f_{\lambda}(X_u) - \frac{f_{\lambda}(\bar{Y}_{[u]})}{1 + \Delta t||f_{\lambda}(\bar{Y}_{[u]})||} \right) \, du
\]

Using again the relation \( dN_u = dN_u - \lambda \, du \) leads to:

\[
||X_s - Y_s^M||^2 = 2 \int_0^s \left( X_u - Y_u^M, f_{\lambda}(X_u) - \frac{f_{\lambda}(\bar{Y}_{[u]})}{1 + \Delta t||f_{\lambda}(\bar{Y}_{[u]})||} \right) \, du
\]

\[
- 2\lambda \int_0^s \left( X_u - Y_u^M, h(X_u) - h(\bar{Y}_{[u]}) \right) \, du + \sum_{i=1}^m \int_0^s \left( ||g_i(X_u) - g_i(\bar{Y}_{[u]})||^2 \right) \, du
\]

\[
+ 2 \sum_{i=1}^m \int_0^s \left( X_u - Y_u^M, g_i(X_u) - g_i(\bar{Y}_{[u]}) \right) \, dW_u^i
\]

\[
+ \int_0^s \left[ ||X_u - Y_u^M + h(X_u) - h(\bar{Y}_{[u]})||^2 - ||X_u - Y_u^M||^2 \right] \, dN_u
\]

\[
+ \lambda \int_0^s \left[ ||X_u - Y_u^M + h(X_u) - h(\bar{Y}_{[u]})||^2 - ||X_u - Y_u^M||^2 \right] \, du
\]

\[
= A_1 + A_2 + A_3 + A_4 + A_5 + A_6.
\]
In the next step, we give some useful estimations of $A_1, A_2, A_3$ and $A_6$.

$$A_1 := 2 \int_0^s \left\langle X_u - \bar{Y}_u^M, f_\lambda - \frac{f_\lambda(\bar{Y}_{[u]})}{1 + \Delta t ||f_\lambda(\bar{Y}_{[u]})||} \right\rangle du$$

$$= 2 \int_0^s \left\langle X_u - \bar{Y}_u^M, f_\lambda(X_u) - f_\lambda(\bar{Y}_u^M) \right\rangle du$$

$$+ \int_0^s \left( f_\lambda(\bar{Y}_{[u]}) ||f_\lambda(\bar{Y}_{[u]})|| \right) du.$$

$$= A_{11} + A_{12}$$

Using the one-sided Lipschitz condition satisfied by $f_\lambda$ leads to:

$$A_{11} := 2 \int_0^s \left\langle X_u - \bar{Y}_u^M, f_\lambda(X_u) - f_\lambda(\bar{Y}_u^M) \right\rangle du$$

$$\leq 2C \int_0^s ||X_u - \bar{Y}_u^M||^2 du.$$  \hspace{1cm} (4.61)

Moreover, using the inequality $(a, b) \leq |a||b| \leq \frac{a^2}{2} + \frac{b^2}{2}$ leads to:

$$A_{12} = 2 \int_0^s \left\langle X_u - \bar{Y}_u^M, f_\lambda(\bar{Y}_u^M) - \frac{f_\lambda(\bar{Y}_{[u]})}{1 + \Delta t ||f_\lambda(\bar{Y}_{[u]})||} \right\rangle du$$

$$= 2 \int_0^s \left\langle X_u - \bar{Y}_u^M, f_\lambda(\bar{Y}_u^M) - f_\lambda(\bar{Y}_{[u]}) \right\rangle ds$$

$$+ 2\Delta t \int_0^s \left\langle X_u - \bar{Y}_u^M, \frac{f_\lambda(\bar{Y}_{[u]}) ||f_\lambda(\bar{Y}_{[u]})||}{1 + \Delta t ||f_\lambda(\bar{Y}_{[u]})||} \right\rangle du$$

$$\leq \int_0^s ||X_u - \bar{Y}_u^M||^2 du + \int_0^s ||f_\lambda(\bar{Y}_u^M) - f_\lambda(\bar{Y}_{[u]})||^2 du$$

$$+ \int_0^s ||X_u - \bar{Y}_u^M||^2 du + \frac{T^2}{M^2} \int_0^s ||f_\lambda(\bar{Y}_{[u]})||^4 du$$

$$\leq 2 \int_0^s ||X_u - \bar{Y}_u^M||^2 du + \int_0^s ||f_\lambda(\bar{Y}_u^M) - f_\lambda(\bar{Y}_{[u]})||^2 du$$

$$+ \frac{T^2}{M^2} \int_0^s ||f_\lambda(\bar{Y}_{[u]})||^4 du.$$  \hspace{1cm} (4.62)

Combining (4.61) and (4.62) give the following estimation of $A_1$:

$$A_1 \leq (2C + 2) \int_0^s ||X_u - \bar{Y}_u^M||^2 du + \int_0^s ||f_\lambda(\bar{Y}_u^M) - f_\lambda(\bar{Y}_{[u]})||^2 du$$

$$+ \frac{T^2}{M^2} \int_0^s ||f_\lambda(\bar{Y}_{[u]})||^4 du.$$  \hspace{1cm} (4.63)

Using again the inequality $2(a, b) \leq 2|a||b| \leq a^2 + b^2$ and the global Lipschitz condition satisfied by $h$ leads to:

$$A_2 := -2\lambda \int_0^s \left\langle X_u - \bar{Y}_u^M, h(X_u) - h(\bar{Y}_u^M) \right\rangle du$$

$$= -2\lambda \int_0^s \left\langle X_u - \bar{Y}_u^M, h(X_u) - h(\bar{Y}_u^M) \right\rangle du - 2\lambda \int_0^s \left\langle X_u - \bar{Y}_u^M, h(\bar{Y}_u^M) - h(\bar{Y}_{[u]}) \right\rangle du$$

$$\leq (2\lambda + \lambda C^2) \int_0^s ||X_u - \bar{Y}_u^M||^2 du + \lambda C^2 \int_0^s ||\bar{Y}_u^M - \bar{Y}_{[u]}^M||^2 du.$$  \hspace{1cm} (4.64)
4.3. STRONG CONVERGENCE OF THE COMPENSATED TAMMED EULER SCHEME

Using the inequalities $||g_i(x) - g_i(y)|| \leq ||g(x) - g(y)||$ and $(a + b)^2 \leq 2a^2 + 2b^2$ and the global Lipschitz condition we have

$$A_3 := \sum_{i=1}^{m} \int_0^s ||g_i(X_u) - g_i(Y^M_{[u]})||^2 du$$

$$\leq m \int_0^s ||g(X_u) - g(Y^M_{[u]})||^2 du$$

$$\leq 2m \int_0^s ||g(X_u) - g(Y^M_{[u]})||^2 du + 2m \int_0^s ||g(Y^M_{[u]}) - g(Y^M_{[u]})||^2 du$$

$$\leq 2mC^2 \int_0^s ||X_u - Y^M_{[u]}||^2 du + 2mC^2 \int_0^s ||Y^M_{[u]} - Y^M_{[u]}||^2 du. \quad (4.65)$$

Using the same idea as above we obtain the following estimation of $A_6$ :

$$A_6 := \lambda \int_0^s \left[ X_u - Y^M_{[u]} + h(Y^M_{[u]}) - h(Y^M_{[u]}) \right]^2 du$$

$$\leq 3\lambda \int_0^s ||X_u - Y^M_{[u]}||^2 du + 2\lambda \int_0^s ||h(X_u) - h(Y^M_{[u]})||^2 du$$

$$\leq 3\lambda \int_0^s ||X_u - Y^M_{[u]}||^2 du + 4\lambda \int_0^s ||h(X_u) - h(Y^M_{[u]})||^2 du$$

$$+ 4\lambda \int_0^s ||h(Y^M_{[u]}) - h(Y^M_{[u]})||^2 du$$

$$\leq (3\lambda + 4\lambda C^2) \int_0^s ||X_u - Y^M_{[u]}||^2 du + 4\lambda C^2 \int_0^s ||Y^M_{[u]} - Y^M_{[u]}||^2 du. \quad (4.66)$$

Inserting (4.63), (4.64), (4.65) and (4.66) in (4.60) we obtain :

$$||X_s - Y^M_s||^2 \leq (2C + 2 + 2mC^2 + 5\lambda + 5\lambda C^2) \int_0^s ||X_u - Y^M_{[u]}||^2 du$$

$$+ (2mC^2 + 5\lambda C^2) \int_0^s ||Y^M_{[u]} - Y^M_{[u]}||^2 du$$

$$+ \int_0^s ||f_{\lambda}(Y^M_{[u]}) - f_{\lambda}(Y^M_{[u]})||^2 du + \frac{T^2}{M^2} \int_0^s ||f_{\lambda}(Y^M_{[u]})||^4 du$$

$$+ 2 \sum_{i=1}^{m} \int_0^s \left\langle X_u - Y^M_{[u]}, g_i(X_u) - g_i(Y^M_{[u]}) \right\rangle dW^i_u$$

$$+ \int_0^s \left[ ||X_u - Y^M_{[u]} + h(X_u) - h(Y^M_{[u]})||^2 - ||X_u - Y^M_{[u]}||^2 \right] dN_u.$$

Taking the supremum in both sides of the previous inequality leads to

$$\sup_{s \in [0,t]} ||X_s - Y^M_s||^2 \leq (2C + 2 + 2mC^2 + 5\lambda + 5\lambda C^2) \int_0^t ||X_u - Y^M_{[u]}||^2 du$$

$$+ (2mC^2 + 5\lambda C^2) \int_0^t ||Y^M_{[u]} - Y^M_{[u]}||^2 du$$

$$+ \int_0^t ||f_{\lambda}(Y^M_{[u]}) - f_{\lambda}(Y^M_{[u]})||^2 du + \frac{T^2}{M^2} \int_0^t ||f_{\lambda}(Y^M_{[u]})||^4 du$$

$$+ 2 \sup_{s \in [0,t]} \left| \sum_{i=1}^{m} \int_0^s \left\langle X_u - Y^M_{[u]}, g_i(X_u) - g_i(Y^M_{[u]}) \right\rangle dW^i_u \right|$$

$$+ \sup_{s \in [0,t]} \int_0^s \left[ ||X_u - Y^M_{[u]} + h(X_u) - h(Y^M_{[u]})||^2 \right] dN_u$$

$$+ \sup_{s \in [0,t]} \int_0^s ||X_u - Y^M_{[u]}||^2 d\bar{N}_u \quad (4.67)$$
Using Lemma 4.2.34 we have the following estimation for all \( p \geq 2 \)

\[
B_1 := \left\| 2 \sup_{s \in [0,t]} \sum_{i=1}^{m} \int_{0}^{s} \left( X_u - \bar{Y}_u^M, g_i(X_u) - g_i(\bar{Y}_{[u]}^M) \right) dW_u^i \right\|_{L^{p/2}(\Omega, \mathbb{R})}
\]

\[
\leq C_p \left( \int_{0}^{t} \sum_{i=1}^{m} \left\| \left( X_u - \bar{Y}_u^M, g_i(X_u) - g_i(\bar{Y}_{[u]}^M) \right) \right\|_{L^{p/2}(\Omega, \mathbb{R})}^2 \right)^{1/2}
\]

Moreover, using Holder inequality, the inequalities \( ab \leq \frac{a^2}{2} + \frac{b^2}{2} \) and \( (a+b)^2 \leq 2a^2 + 2b^2 \) we have the following estimations for all \( p \geq 2 \)

\[
B_1 \leq C_p \left( \int_{0}^{t} \sum_{i=1}^{m} \left\| X_u - \bar{Y}_u^M \right\|_{L^{p}(\Omega, \mathbb{R})}^2 \right)^{1/2}
\]

\[
\leq C_p \left( \int_{0}^{t} \sum_{i=1}^{m} \left\| X_u - \bar{Y}_u^M \right\|_{L^{p}(\Omega, \mathbb{R})}^2 \right)^{1/2}
\]

\[
\leq \frac{C_p}{\sqrt{2}} \left( \sup_{s \in [0,t]} \left\| X_s - \bar{Y}_s^M \right\|_{L^{p}(\Omega, \mathbb{R})}^2 \right)^{1/2}
\]

\[
\leq \frac{1}{4} \sup_{s \in [0,t]} \left\| X_s - \bar{Y}_s^M \right\|_{L^{p}(\Omega, \mathbb{R})}^2 + 2p^2C_p\int_{0}^{t} \left\| X_s - \bar{Y}_s^M \right\|_{L^{p}(\Omega, \mathbb{R})}^2 ds
\]

\[
\leq \frac{1}{4} \sup_{s \in [0,t]} \left\| X_s - \bar{Y}_s^M \right\|_{L^{p}(\Omega, \mathbb{R})}^2 + 2p^2C_p\int_{0}^{t} \left\| X_s - \bar{Y}_s^M \right\|_{L^{p}(\Omega, \mathbb{R})}^2 ds
\]

\[
+ 2p^2C_p\int_{0}^{t} \left\| Y_s - \bar{Y}_{[u]}^M \right\|_{L^{p}(\Omega, \mathbb{R})}^2 ds,
\]

(4.68)

Using Lemma 4.2.38 and the inequality \( (a+b)^2 \leq 16a^4 + 16b^4 \), it follows that

\[
B_2 := \left\| \sup_{s \in [0,t]} \left( \int_{0}^{s} \left\| X_u - \bar{Y}_u^M + h(X_u) - h(\bar{Y}_{[u]}^M) \right\|_{L^{p/2}(\Omega, \mathbb{R})}^2 d\mathbb{N}_u \right) \right\|_{L^{p/2}(\Omega, \mathbb{R})}
\]

\[
\leq C_p \left( \int_{0}^{t} \left\| X_u - \bar{Y}_u^M + h(X_u) - h(\bar{Y}_{[u]}^M) \right\|_{L^{p/2}(\Omega, \mathbb{R})}^2 du \right)^{1/2}
\]

\[
\leq C_p \left( \int_{0}^{t} \left\| X_u - \bar{Y}_u^M \right\|_{L^{p/2}(\Omega, \mathbb{R})}^4 du \right)^{1/2}
\]

\[
\leq 16 \left\| h(X_u) - h(\bar{Y}_{[u]}^M) \right\|_{L^{p/2}(\Omega, \mathbb{R})}^4 du \right)^{1/2}
\]

for all \( p \geq 2 \).

Using the inequality \( \sqrt{a+b} \leq \sqrt{a} + \sqrt{b} \), it follows that :

\[
B_2 \leq 2C_p \left( \int_{0}^{t} \left\| X_u - \bar{Y}_u^M \right\|_{L^{p/2}(\Omega, \mathbb{R})}^4 du \right)^{1/2} + 2C_p \left( \int_{0}^{t} \left\| h(X_u) - h(\bar{Y}_{[u]}^M) \right\|_{L^{p/2}(\Omega, \mathbb{R})}^4 du \right)^{1/2}
\]

\[
= B_{21} + B_{22}.
\]

(4.69)

Using Holder inequality, it follows that

\[
B_{21} := 2C_p \left( \int_{0}^{t} \left\| X_u - \bar{Y}_u^M \right\|_{L^{p/2}(\Omega, \mathbb{R})}^4 du \right)^{1/2}
\]

\[
\leq 2C_p \left( \int_{0}^{t} \left\| X_u - \bar{Y}_u^M \right\|_{L^{p}(\Omega, \mathbb{R})}^2 \right)^{1/2} \left\| X_u - \bar{Y}_u^M \right\|_{L^{p}(\Omega, \mathbb{R})}^2 du \right)^{1/2}
\]

\[
\leq \frac{1}{4} \sup_{u \in [0,t]} \left\| X_u - \bar{Y}_u^M \right\|_{L^{p/(2)}(\Omega, \mathbb{R})} B C_p \left( \int_{0}^{t} \left\| X_u - \bar{Y}_u^M \right\|_{L^{p}(\Omega, \mathbb{R})}^2 du \right)^{1/2}.
\]

(4.70)
Using the inequality $2ab \leq a^2 + b^2$ leads to:

$$B_{21} \leq \frac{1}{16} \sup_{u \in [0,t]} \|X_u - \bar{Y}_u^M\|_{L^p(\Omega,\mathbb{R}^d)}^2 + 16C_p^2 \int_0^t \|X_u - \bar{Y}_u^M\|_{L^p(\Omega,\mathbb{R}^d)}^2 du. \quad (4.70)$$

Using the inequalities $(a + b)^4 \leq 4a^4 + 4b^4$ and $\sqrt{a} + \sqrt{b} \leq \sqrt{a + b}$, we obtain

$$B_{22} := 2C_p \left( \int_0^t \left[ 4||h(X_u) - h(\bar{Y}_u^M)||_{L^{p/2}(\Omega,\mathbb{R}^d)}^4 + 4||h(\bar{Y}_u^M) - h(\bar{Y}_u^M)||_{L^{p/2}(\Omega,\mathbb{R}^d)}^4 \right] du \right)^{1/2} \leq 2C_p \left( \int_0^t \left[ 4||h(X_u) - h(\bar{Y}_u^M)||_{L^{p/2}(\Omega,\mathbb{R}^d)}^4 \right] du \right)^{1/2} \leq 4C_p \left( \int_0^t \left[ 4||h(X_u) - h(\bar{Y}_u^M)||_{L^{p/2}(\Omega,\mathbb{R}^d)}^4 \right] du \right)^{1/2} + 4C_p \left( \int_0^t ||h(\bar{Y}_u^M) - h(\bar{Y}_u^M)||_{L^{p/2}(\Omega,\mathbb{R}^d)}^4 du \right)^{1/2}.$$

Using the global Lipschitz condition, leads to:

$$B_{22} \leq 4C_p \left( \int_0^t C||X_u - \bar{Y}_u^M||_{L^{p/2}(\Omega,\mathbb{R}^d)}^4 du \right)^{1/2} + 4C_p \left( \int_0^t C||Y_u^M - \bar{Y}_u^M||_{L^{p/2}(\Omega,\mathbb{R}^d)}^4 du \right)^{1/2}.$$

Using the same estimations as for $B_{21}$, it follows that:

$$B_{22} \leq \frac{1}{16} \sup_{u \in [0,t]} \|X_u - \bar{Y}_u^M\|_{L^p(\Omega,\mathbb{R}^d)}^2 + 64C_p \int_0^t \|X_u - \bar{Y}_u^M\|_{L^p(\Omega,\mathbb{R}^d)}^2 du + \frac{1}{4} \sup_{u \in [0,t]} \|\bar{Y}_u^M - \bar{Y}_u^M\|_{L^p(\Omega,\mathbb{R}^d)}^2 + 64C_p \int_0^t \|\bar{Y}_u^M - \bar{Y}_u^M\|_{L^p(\Omega,\mathbb{R}^d)}^2 du.$$

Taking the supremum under the integrand in the last term of the above inequality and using the fact that we don’t care about the value of the constant leads to:

$$B_{22} \leq \frac{1}{16} \sup_{u \in [0,t]} \|X_u - \bar{Y}_u^M\|_{L^p(\Omega,\mathbb{R}^d)}^2 + 64C_p \int_0^t \|X_u - \bar{Y}_u^M\|_{L^p(\Omega,\mathbb{R}^d)}^2 du + C_p \sup_{s \in [0,t]} \|\bar{Y}_s^M - \bar{Y}_{[s]}^M\|_{L^p(\Omega,\mathbb{R}^d)}^2. \quad (4.71)$$

Inserting (4.70) and (4.71) into (4.69) gives:

$$B_2 \leq \frac{1}{8} \sup_{u \in [0,t]} \|X_u - \bar{Y}_u^M\|_{L^p(\Omega,\mathbb{R}^d)}^2 + C_p \int_0^t \|X_u - \bar{Y}_u^M\|_{L^p(\Omega,\mathbb{R}^d)}^2 du + C_p \sup_{s \in [0,t]} \|\bar{Y}_s^M - \bar{Y}_{[s]}^M\|_{L^p(\Omega,\mathbb{R}^d)}^2. \quad (4.72)$$

Using again Lemma 4.2.38 leads to:

$$B_3 := \left\| \sup_{u \in [0,t]} \left( \int_0^s \|X_u - \bar{Y}_u^M\|_{L^p(\Omega,\mathbb{R}^d)}^4 dN_u \right)^{1/2} \right\|_{L^{p/2}(\Omega,\mathbb{R}^d)} \leq C_p \left( \int_0^t \|X_u - \bar{Y}_u^M\|_{L^{p/2}(\Omega,\mathbb{R}^d)}^4 du \right)^{1/2}.$$

Using the same argument as for $B_{21}$, we obtain

$$B_3 \leq \frac{1}{8} \sup_{u \in [0,t]} \|X_u - \bar{Y}_u^M\|_{L^p(\Omega,\mathbb{R}^d)}^2 + C_p \int_0^t \|X_u - \bar{Y}_u^M\|_{L^p(\Omega,\mathbb{R}^d)}^2 du. \quad (4.73)$$
4.3. STRONG CONVERGENCE OF THE COMPENSATED TAMMED EULER SCHEME

Taking the $L^p$ norm in both side of (4.67), inserting inequalities (4.68), (4.72), (4.73) and using Holder inequality in its integral form (see Proposition 1.1.20) leads to:

$$\left\| \sup_{s \in [0,t]} \| X_s - \bar{Y}_s^M \| \right\|_{L^p(\Omega, \mathcal{F}_t)}^2 = \left\| \sup_{s \in [0,t]} \| X_s - \bar{Y}_s^M \| \right\|_{L^{p/2}(\Omega, \mathcal{F}_t)}^2$$

$$\leq C_p \int_0^t \| X_s - \bar{Y}_s^M \|_{L^p(\Omega, \mathcal{F}_t)}^2 ds + C_p \int_0^t \| \bar{Y}_s^M - \bar{Y}_s \|_{L^p(\Omega, \mathcal{F}_t)}^2 ds$$

$$+ \int_0^t |f(X_s) - f(\bar{Y}_s^M)|_{L^p(\Omega, \mathcal{F}_t)}^2 ds + C_p \sup_{s \in [0,t]} |\bar{Y}_u^M - \bar{Y}_u|_{L^p(\Omega, \mathcal{F}_t)}^2$$

$$+ \frac{T^2}{M^2} \int_0^t \| f(\bar{Y}_s^M) \|_{L^{2p}(\Omega, \mathcal{F}_t)}^4 ds + 2C_p \int_0^t \| \bar{Y}_s^M - \bar{Y}_s \|_{L^p(\Omega, \mathcal{F}_t)}^2 ds$$

Applying Gronwall lemma to the previous inequality leads to:

$$\left\| \sup_{s \in [0,t]} \| X_s - \bar{Y}_s^M \| \right\|_{L^p(\Omega, \mathcal{F}_t)}^2 \leq \begin{cases} C_p e^{C_p} \left( \int_0^T |f(\bar{Y}_s^M) - f(\bar{Y}_s)|_{L^p(\Omega, \mathcal{F}_t)}^2 ds + \sup_{t \in [0,t]} |\bar{Y}_u^M - \bar{Y}_u|_{L^p(\Omega, \mathcal{F}_t)}^2 \right) \\
+ \frac{T^2}{M^2} \int_0^T \| f(\bar{Y}_s^M) \|_{L^{2p}(\Omega, \mathcal{F}_t)}^4 ds + C_p \int_0^T \| \bar{Y}_s^M - \bar{Y}_s \|_{L^p(\Omega, \mathcal{F}_t)}^2 ds \end{cases}.$$  

From the inequality $\sqrt{a + b + c} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}$, it follows that

$$\left\| \sup_{t \in [0,T]} \| X_t - \bar{Y}_t^M \| \right\|_{L^p(\Omega, \mathcal{F}_t)} \leq C_p e^{C_p} \left( \sup_{t \in [0,T]} |f(\bar{Y}_s^M) - f(\bar{Y}_s)|_{L^p(\Omega, \mathcal{F}_t)} + \sup_{t \in [0,t]} |\bar{Y}_u^M - \bar{Y}_u|_{L^p(\Omega, \mathcal{F}_t)} \right)$$

$$+ \frac{T}{M} \left[ \sup_{n \in \{1, \ldots, M\}} |f(\bar{Y}_n^M)|_{L^p(\Omega, \mathcal{F}_t)}^2 \right] + C_p \sup_{t \in [0,T]} \| \bar{Y}_t^M - \bar{Y}_t \|_{L^p(\Omega, \mathcal{F}_t)}$$

(4.74)

for all $p \in [2, \infty)$.

Using Lemma 4.3.2 and Lemma 4.3.4 it follows from 4.74 that

$$\left( \mathbb{E} \left[ \sup_{t \in [0,T]} \| X_t - \bar{Y}_t^M \|^p \right] \right)^{1/p} \leq C_p (\Delta t)^{1/2},$$

(4.75)
for all $p \in [2, \infty)$. Using Holder inequality, one can prove that (4.75) holds for all $p \in [1, 2]$. The proof of the theorem is complete.

4.4 Numerical Experiments

In order to illustrate our theoretical result, we consider the following stochastic differential equation

$$dX_t = -X_t^4 dt + X_t dW_t + X_t dN_t$$

$X_0 = 1$, $\lambda = 1$. It is straightforward to verify that Assumptions 4.2.2 are satisfied. We use Monte carlo method to evaluate the error. The exact solution is consider as the numerical one with small stepsize $dt = 2^{-14}$. We have the following curve for 5000 paths.

![Figure 4.1: Strong error of the compensated tamed Euler scheme](image)

In this chapter, we proposed a compensated tamed Euler scheme to solve numerically SDEs with jumps under non-global Lipschitz condition. We proved the strong convergence of order 0.5 of the compensated Euler scheme. This scheme is explicit and then requires less computational efforts than the implicit scheme. In some situations, the drift part can be equipped with the Lipschitz continuous part and the non-Lipschitz continuous part. In the following chapter, we combine the tamed Euler scheme and the Euler scheme and obtain another scheme called semi-tamed Euler scheme in order to solve numerically the kind of equation mentioned above.
Chapter 5

Strong convergence and stability of the semi-tamed Euler and the tamed Euler scheme for stochastic differential equations with jumps, under non-global Lipschitz continuous coefficients

Explicit numerical method called compensated tamed Euler scheme is developed in the previous chapter. More precisely, it is proved that such numerical approximation have strong convergence of order 0.5 for stochastic differential equations with jumps under non-global Lipschitz condition. In this chapter, following the idea of [11], we propose a semi-tamed Euler scheme to solve stochastic differential equations with jumps, where the drift coefficient is equipped with the Lipschitz continuous part and the non-Lipschitz continuous part. We prove that for SDEs with jumps, the semi-tamed Euler scheme converges strongly with order 0.5. We use this result to deduce a strong convergence of order 0.5 of the tamed Euler scheme for SDEs with jumps, where the drift coefficient satisfies the non-global Lipschitz condition. We also investigate the stability analysis of both semi-tamed Euler scheme and tamed Euler scheme. The contents of this chapter can also be found in [20] and [21].

5.1 Semi-tamed Euler scheme

In this chapter, we consider again a jump-diffusion Itô’s stochastic differential equations (SDEs) of the form:

$$dX(t) = f((X(t^-))dt + g((X(t^-))dW(t) + h(X(t^-))dN(t), \quad X(0) = X_0, \quad (5.1)$$

where $W_t$ is a $m$-dimensional Brownian motion, $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $f(x) = u(x) + v(x)$ satisfies the global one-sided Lipschitz condition. $u, v: \mathbb{R}^d \rightarrow \mathbb{R}^d$, $u$ is the global Lipschitz continuous part while $v$ is the non-global Lipschitz continuous part. The functions $g: \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ and $h: \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfy the global Lipschitz condition, $N_t$ is a one dimensional poisson process with parameter $\lambda$. Using the relation $f = u + v$, equation (5.1) can be rewritten into its equivalent form:

$$dX(t) = u(X(t^-))dt + v(X(t^-))dt + g(X(t^-))dW(t) + h(X(t^-))dN(t), \quad X(0) = X_0. \quad (5.2)$$

We can rewrite the jump-diffusion SDEs (5.1) in the following equivalent form

$$dX(t) = f_\lambda(X(t^-))dt + g(X(t^-))dW(t) + h(X(t^-))d\bar{N}(t), \quad (5.3)$$

where

$$f_\lambda(x) = f(x) + \lambda h(x) = u(x) + \lambda h(x) + v(x). \quad (5.4)$$
5.2. Moments bounded of the numerical solution

If $T$ is the final time we consider the tamed Euler scheme

$$X^M_{n+1} = X^M_n + \frac{\Delta tf(X^M_n)}{1 + \Delta t||f(X^M_n)||} + g(X^M_n)\Delta W_n + h(X^M_n)\Delta N_n$$

and the semi-tamed Euler scheme

$$Y^M_{n+1} = Y^M_n + u(Y^M_n)\Delta t + \frac{\Delta tv(Y^M_n)}{1 + \Delta t||v(Y^M_n)||} + g(Y^M_n)\Delta W_n + h(Y^M_n)\Delta N_n,$$

where $\Delta t = \frac{T}{M}$ is the time step-size, $M \in \mathbb{N}$ is the number of steps.

Inspired by [11] and [8] we prove the strong convergence of the numerical approximation (5.7) and deduce the strong convergence of (5.5) to the exact solution of (5.1).

5.2 Moments bounded of the numerical solution

Throughout this chapter, we use Notations 4.2.1.

Remark 5.2.1 Note that the numerical approximation (5.6) can be written into its following equivalent form

$$Y^M_{n+1} = Y^M_n + u(Y^M_n)\Delta t + \lambda h(Y^M_n)\Delta t + \frac{\Delta tv(Y^M_n)}{1 + \Delta t||v(Y^M_n)||} + g(Y^M_n)\Delta W_n + h(Y^M_n)\Delta N_n.$$  

We define the continuous time interpolation of the discrete numerical approximation of (5.7) by the family of processes $(\overline{Y}^M)^M : [0, T] \times \Omega \rightarrow \mathbb{R}^d$ such that:

$$\overline{Y}^M_t = Y^M_n + u(Y^M_n)(t - n\Delta t) + \lambda h(Y^M_n)(t - n\Delta t) + \frac{(t - n\Delta t)v(Y^M_n)}{1 + \Delta t||v(Y^M_n)||} + g(Y^M_n)(W_t - W_{n\Delta t}) + h(Y^M_n)(\overline{N}_t - \overline{N}_{n\Delta t}),$$

for all $M \in \mathbb{N}$, all $n \in \{0, \cdots, M - 1\}$, and all $t \in [n\Delta t, (n+1)\Delta t]$.

Assumptions 5.2.2 We assume that:

(A.1) $f, g, h \in C^1$.

(A.2) For all $p > 0$, there exist a finite $M_p > 0$ such that $\mathbb{E}||X_0||^p \leq M_p$.

(A.3) $g, h$ and $u$ satisfy global Lipschitz condition:

$$||g(x) - g(y)|| \vee ||h(x) - h(y)|| \vee ||u(x) - u(y)|| \leq C||x - y||, \quad \forall x, y \in \mathbb{R}^d.$$

(A.4) $v$ satisfies the one-sided Lipschitz condition:

$$(x - y, f(x) - f(y)) \leq C||x - y||^2, \quad \forall x, y \in \mathbb{R}^d,$$

(A.5) $v$ satisfies the superlinear growth condition:

$$||v(x) - v(y)|| \leq C(K ||x||^c + ||y||^c)||x - y||, \quad \forall x, y \in \mathbb{R}^d,$$

where $K$, $C$ and $c$ are constants strictly positives.

Remark 5.2.3 Under conditions (A.1), (A.2) and (A.3) of Assumptions 5.2.2 it is proved in [6, Lemma 1] that (5.1) has a unique solution with all moments bounded.

Remark 5.2.4 Let’s define $u_\lambda(x) = u(x) + \lambda h(x)$. From Assumptions 5.2.2, it is straightforward to prove that $u_\lambda$ satisfies the global Lipschitz condition with constant $C_\lambda = (1 + \lambda)C$ and $v$ satisfies the one-sided Lipschitz condition. We denote by $C_p$ a generic constant. Throughout this work, this constant may change the value from one line to another one. We will sometimes use $Y^M_n$ instead of $Y^M_n(\omega)$ to simplify notations.
The main result of this section is formulated in the following theorem, which is based on [8, Lemma 3.9 pp 16]. Here, we include the jump part.

**Theorem 5.2.5** Let \( Y_n^M : \Omega \rightarrow \mathbb{R}^d \) be defined by (5.7) for all \( M \in \mathbb{N} \) and all \( n \in \{0, \cdots, M\} \). The following inequality holds:

\[
\sup_{M \in \mathbb{N}} \sup_{n \in \{0, \cdots, M\}} \mathbb{E} \left[ ||Y_n^M||^p \right] < +\infty,
\]

for all \( p \in [1, \infty) \).

In order to prove Theorem 5.2.5 we introduce the following notations facilitating computations.

**Notation 5.2.6**

\[
\begin{align*}
\alpha_k^M &:= 1_{\{||Y_k^M|| \geq 1\}} \left\langle \frac{Y_k^M + u_\lambda(Y_k^M) \Delta t}{||Y_k^M||}, g(Y_k^M) \Delta W_k^M \right\rangle, \\
\beta_k^M &:= 1_{\{||Y_k^M|| \geq 1\}} \left\langle \frac{Y_k^M + u_\lambda(Y_k^M) \Delta t}{||Y_k^M||}, h(Y_k^M) \Delta \nabla_k^M \right\rangle,
\end{align*}
\]

\[
\beta = (1 + K + 3C_\lambda + KTC_\lambda + TC_\lambda + ||u_\lambda(0)|| + ||g(0)|| + ||h(0)||)^4,
\]

\[
D_n^M := (\beta + ||\varepsilon||) \exp \left( 4\beta + \sup_{u \in \{0, \cdots, n\}} \sum_{k=0}^{n-1} \left[ 2\beta||\Delta W_k^M||^2 + 2\beta||\Delta \nabla_k^M|| + \alpha_k^M + \beta_k^M \right] \right),
\]

\[
\Omega_n^M := \{ \omega \in \Omega : \sup_{k \in \{0,1,\cdots,n-1\}} D_k^M(\omega) \leq M^{1/2e}, \sup_{k \in \{0,1,\cdots,n-1\}} ||\Delta W_k^M(\omega)|| \leq 1, \sup_{k \in \{0,1,\cdots,n-1\}} ||\Delta \nabla_k^M(\omega)|| \leq 1 \},
\]

for all \( M \in \mathbb{N} \) and \( k \in \{0, \cdots M\} \).

Following closely [11, Lemma 2.1] we have the following main lemma.

**Lemma 5.2.7** The following inequality holds for all \( M \in \mathbb{N} \) and all \( n \in \{0,1,\cdots,M\} \)

\[
1_{\Omega_n^M} ||Y_n^M|| \leq D_n^M.
\]

**Proof 5.2.8** Using the inequality \( \frac{\Delta t}{1 + \Delta t||u_\lambda(x)||} \leq T \), the global Lipschitz condition of \( g \) and \( h \) and the polynomial growth condition of \( v \) we have the following estimation on \( \Omega_{n+1}^M \cap \{ \omega \in \Omega : ||Y_n^M(\omega)|| \leq 1 \} \), for all \( n \in \{0,1,\cdots,M-1\} \)

\[
\begin{align*}
||Y_{n+1}^M|| &\leq ||Y_n^M|| + ||u_\lambda(Y_n^M)|| \Delta t + \frac{\Delta t||v(Y_n^M)||}{1 + \Delta t||v(Y_n^M)||} + ||g(Y_n^M)||||\Delta W_n^M|| + ||h(Y_n^M)||||\Delta \nabla_n^M|| \\
&\leq ||Y_n^M|| + T||u_\lambda(Y_n^M) - u_\lambda(0)|| + T||u_\lambda(0)|| + T||v(Y_n^M) - v(0)|| + T||v(0)|| \\
&+ ||g(Y_n^M) - g(0)|| + ||g(0)|| + ||h(Y_n^M) - h(0)|| + ||h(0)|| \\
&\leq 1 + C||Y_n^M|| + T||u_\lambda(0)|| + TC(K + ||\nabla Y_n^M|| + ||0||)||Y_n^M - 0|| + T||v(0)|| \\
&+ C||Y_n^M|| + ||g(0)|| + ||h(0)|| \\
&\leq 1 + KTC + TC + 3C + T||u_\lambda(0)|| + T||v(0)|| + ||g(0)|| + ||h(0)|| \leq \beta.
\end{align*}
\]
Furthermore, from the numerical approximation (5.7), we have

\[
||Y_{n+1}^M||^2 = ||Y_n^M||^2 + ||u_\lambda(Y_n^M)||^2 t^2 + \frac{\Delta t^2 ||v(Y_n^M)||^2}{(1 + \Delta t)||f_\lambda(Y_n^M)||^2} + ||g(Y_n^M)\Delta W_n^M||^2 \\
+ ||h(Y_n^M)\Delta \nabla_n^M||^2 + 2\Delta t\langle Y_n^M, u_\lambda(Y_n^M) \rangle + \frac{2\Delta t\langle Y_n^M, v(Y_n^M) \rangle}{1 + \Delta t||f_\lambda(Y_n^M)||} \\
+ 2\langle Y_n^M, g(Y_n^M)\Delta W_n^M \rangle + 2\langle Y_n^M, h(Y_n^M)\Delta \nabla_n^M \rangle + \frac{2\Delta t^2\langle u_\lambda(Y_n^M), v(Y_n^M) \rangle}{1 + \Delta t||f_\lambda(Y_n^M)||} \\
+ 2\Delta t\langle u_\lambda(Y_n^M), g(Y_n^M)\Delta W_n^M \rangle + 2\Delta t\langle u_\lambda(Y_n^M), h(Y_n^M)\Delta \nabla_n^M \rangle \\
+ \frac{2\Delta t\langle v(Y_n^M), g(Y_n^M)\Delta W_n^M \rangle}{1 + \Delta t||f_\lambda(Y_n^M)||} + \frac{2\Delta t\langle v(Y_n^M), h(Y_n^M)\Delta \nabla_n^M \rangle}{1 + \Delta t||f_\lambda(Y_n^M)||} \\
+ 2\langle g(Y_n^M)\Delta W_n^M, h(Y_n^M)\Delta \nabla_n^M \rangle. \tag{5.12}
\]

Using the estimations \(a \leq ||a|| \) and \(\frac{1}{1 + \Delta t||f_\lambda(Y_n^M)||} \leq 1\), we obtain the following inequality from equation (5.12):

\[
||Y_{n+1}^M||^2 \leq ||Y_n^M||^2 + ||u_\lambda(Y_n^M)||^2 t^2 + \frac{2\Delta t\langle Y_n^M, u_\lambda(Y_n^M) \rangle + 2\Delta t\langle Y_n^M, v(Y_n^M) \rangle}{1 + \Delta t||f_\lambda(Y_n^M)||} \\
+ 2\langle Y_n^M, g(Y_n^M)\Delta W_n^M \rangle + 2\langle Y_n^M, h(Y_n^M)\Delta \nabla_n^M \rangle + \frac{2\Delta t^2\langle u_\lambda(Y_n^M), v(Y_n^M) \rangle}{1 + \Delta t||f_\lambda(Y_n^M)||} \\
+ 2\Delta t\langle u_\lambda(Y_n^M), g(Y_n^M)\Delta W_n^M \rangle + 2\Delta t\langle u_\lambda(Y_n^M), h(Y_n^M)\Delta \nabla_n^M \rangle \\
+ \frac{2\Delta t\langle v(Y_n^M), g(Y_n^M)\Delta W_n^M \rangle}{1 + \Delta t||f_\lambda(Y_n^M)||} + \frac{2\Delta t\langle v(Y_n^M), h(Y_n^M)\Delta \nabla_n^M \rangle}{1 + \Delta t||f_\lambda(Y_n^M)||} \\
+ 2\langle g(Y_n^M)\Delta W_n^M, h(Y_n^M)\Delta \nabla_n^M \rangle. \tag{5.13}
\]

Using the estimation \(2ab \leq a^2 + b^2\), the inequality (5.13) becomes:

\[
||Y_{n+1}^M||^2 \leq ||Y_n^M||^2 + ||u_\lambda(Y_n^M)||^2 t^2 + \frac{2\Delta t\langle Y_n^M, u_\lambda(Y_n^M) \rangle + 2\Delta t\langle Y_n^M, v(Y_n^M) \rangle}{1 + \Delta t||f_\lambda(Y_n^M)||} \\
+ 2\langle Y_n^M, g(Y_n^M)\Delta W_n^M \rangle + 2\langle Y_n^M, h(Y_n^M)\Delta \nabla_n^M \rangle \\
+ \frac{2\Delta t\langle u_\lambda(Y_n^M), g(Y_n^M)\Delta W_n^M \rangle + 2\Delta t\langle u_\lambda(Y_n^M), h(Y_n^M)\Delta \nabla_n^M \rangle}{1 + \Delta t||f_\lambda(Y_n^M)||} \\
+ ||v(Y_n^M)||^2 t^2 + \frac{2\Delta t\langle v(Y_n^M), h(Y_n^M)\Delta \nabla_n^M \rangle}{1 + \Delta t||f_\lambda(Y_n^M)||} \\
+ ||g(Y_n^M)||^2 \Delta W_n^M ||^2 + ||h(Y_n^M)||^2 ||\Delta \nabla_n^M||^2 \tag{5.14}
\]

Putting similar terms of inequality (5.14) together, we obtain:

\[
||Y_{n+1}^M||^2 \leq ||Y_n^M||^2 + 3||u_\lambda(Y_n^M)||^2 t^2 + 4||v(Y_n^M)||^2 t^2 + 3||g(Y_n^M)||^2 ||\Delta W_n^M||^2 \\
+ 4||h(Y_n^M)||^2 ||\Delta \nabla_n^M||^2 + 2\Delta t\langle Y_n^M, u_\lambda(Y_n^M) \rangle + 2\Delta t\langle Y_n^M, v(Y_n^M) \rangle \\
+ 2\langle Y_n^M, g(Y_n^M)\Delta W_n^M \rangle + 2\langle Y_n^M, h(Y_n^M)\Delta \nabla_n^M \rangle, \tag{5.15}
\]

on \(\Omega\), for all \(M \in \mathbb{N}\) and all \(n \in \{0, 1, \ldots, M - 1\}\).

In addition, for all \(x \in \mathbb{R}^d\) such that \(||x|| \geq 1\), the global Lipschitz condition satisfied by \(g, h\) and \(u_\lambda\) leads to:

\[
||g(x)||^2 \leq (||g(x) - g(0)|| + ||g(0)||)^2 \\
\leq (C||x|| + ||g(0)||)^2 \\
\leq (C + ||g(0)||^2)||x||^2 \\
\leq \beta||x||^2 \tag{5.16}
\]
Along the same lines as above, for all \( x \in \mathbb{R}^d \) such that \( ||x|| \geq 1 \), we have:

\[
||h(x)||^2 \leq \beta ||x||^2 \quad \text{and} \quad ||u_\lambda(x)|| \leq \beta ||x||. \tag{5.17}
\]

Also, for all \( x \in \mathbb{R}^d \) such that \( ||x|| \geq 1 \), the one-sided Lipschitz condition satisfied by \( v \) leads to:

\[
\langle x, v(x) \rangle = \langle x, v(x) - v(0) + v(0) \rangle = \langle x - 0, v(x) - v(0) \rangle + \langle x, v(0) \rangle \leq C||x||^2 + ||x||||v(0)||
\]

\[
\leq (C + ||v(0)||)||x||^2
\]

\[
\leq \sqrt{\beta}||x||^2. \tag{5.18}
\]

Along the same lines as above, \( \forall x \in \mathbb{R}^d \) such that \( ||x|| \geq 1 \), we have

\[
\langle x, u(x) \rangle \leq \sqrt{\beta}||x||^2. \tag{5.19}
\]

Furthermore, using the polynomial growth condition satisfied by \( v \), the following inequality holds for all \( x \in \mathbb{R}^d \) with \( 1 \leq ||x|| \leq M^{1/2c} \) and for all \( M \in \mathbb{N} \)

\[
||v(x)||^2 \leq (||v(x) - v(0)|| + ||v(0)||)^2
\]

\[
\leq (C(K + ||x||^c)||x|| + ||v(0)||)^2
\]

\[
\leq (C(K + 1)||x||^{c+1} + ||v(0)||)^2
\]

\[
\leq (KC + C + ||v(0)||)^2||x||^{2(c+1)}
\]

\[
\leq M \sqrt{\beta}||x||^2. \tag{5.20}
\]

Using the global-Lipschitz condition of \( u_\lambda \) leads to

\[
||u_\lambda(x)||^2 \leq \sqrt{\beta}||x||^2. \tag{5.21}
\]

Now combining inequalities (5.15), (5.16), (5.17), (5.18),(5.19), (5.20) and (5.21) we obtain

\[
||Y_{n+1}^M||^2 \leq ||Y_n^M||^2 + \frac{3T^2 \sqrt{\beta}}{M}||Y_n^M||^2 + \frac{4T^2 \sqrt{\beta}}{M}||Y_n^M||^2 + 3\beta ||Y_n^M||^2 ||\Delta W_n^M||^2
\]

\[
+ \frac{4\beta ||Y_n^M||^2 ||\Delta N_n^M||^2 + \frac{2T^2 \sqrt{\beta}}{M} ||Y_n^M||^2 + \frac{2T^2 \sqrt{\beta}}{M} ||Y_n^M||^2
\]

\[
+ 2\langle Y_n^M + \Delta tu_\lambda(Y_n^M), g(Y_n^M) \Delta W_n^M \rangle + 2\langle Y_n^M + \Delta tu_\lambda(Y_n^M), h(Y_n^M) \Delta N_n^M \rangle
\]

\[
\leq ||Y_n^M||^2 + \left( \frac{8T^2 \sqrt{\beta}}{M} + \frac{4T^2 \sqrt{\beta}}{M} \right) ||Y_n^M||^2 + 4\beta ||Y_n^M||^2 ||\Delta W_n^M||^2 + 4\beta ||Y_n^M||^2 ||\Delta N_n^M||^2
\]

\[
+ 2\langle Y_n^M + \Delta tu_\lambda(Y_n^M), g(Y_n^M) \Delta W_n^M \rangle + 2\langle Y_n^M + \Delta tu_\lambda(Y_n^M), h(Y_n^M) \Delta N_n^M \rangle.
\]

Using the inequality \( 8T^2 + 4T \leq 8\sqrt{\beta} \), it follows that:

\[
||Y_{n+1}^M||^2 \leq ||Y_n^M||^2 + \frac{8\beta}{M} ||Y_n^M||^2 + 4\beta ||Y_n^M||^2 ||\Delta W_n^M||^2 + 4\beta ||Y_n^M||^2 ||\Delta N_n^M||^2
\]

\[
+ 2\langle Y_n^M + \Delta tu_\lambda(Y_n^M), g(Y_n^M) \Delta W_n^M \rangle + 2\langle Y_n^M + \Delta tu_\lambda(Y_n^M), h(Y_n^M) \Delta N_n^M \rangle
\]

\[
= ||Y_n^M||^2 \left( 1 + \frac{8\beta}{M} + 4\beta ||\Delta W_n^M||^2 + 4\beta ||\Delta N_n^M||^2 \right)
\]

\[
+ \left( \frac{Y_n^M + \Delta tu_\lambda(Y_n^M), g(Y_n^M) \Delta W_n^M}{||Y_n^M||} + 2 \left( \frac{Y_n^M + \Delta tu_\lambda(Y_n^M), h(Y_n^M) \Delta N_n^M}{||Y_n^M||} \right) \right)
\]

\[
= ||Y_n^M||^2 \left( 1 + \frac{8\beta}{M} + 4\beta ||\Delta W_n^M||^2 + 4\beta ||\Delta N_n^M||^2 + 2\alpha_n^M + 2\beta_n^M \right). \tag{5.22}
\]

Using Lemma 4.2.7 for \( a = \frac{8\beta}{M} + 4\beta ||\Delta W_n^M||^2 + 2\alpha_n^M + 2\beta_n^M \) and \( b = 2\sqrt{\beta} ||\Delta N_n^M|| \) it follows from (5.22) that:

\[
||Y_{n+1}^M||^2 \leq ||Y_n^M||^2 \exp \left( \frac{8\beta}{M} + 4\beta ||\Delta W_n^M||^2 + 4\beta ||\Delta N_n^M||^2 + 2\alpha_n^M + 2\beta_n^M \right), \tag{5.23}
\]
of Lemma 5.2.7.

and of the coefficient \( \beta \). The following inequality holds:

**Lemma 5.2.9** The following inequality holds:

\[
\sup_{M \in \mathbb{N}, M \geq A \beta} \mathbb{E} \left[ \exp \left( \beta p \sum_{k=0}^{M-1} ||\Delta W^M_k||^2 \right) \right] < \infty.
\]

The following lemma is based on [9, Lemma 5.7].

**Lemma 5.2.10** The following inequality holds:

\[
\mathbb{E} \left[ \exp \left( pz \frac{\left( x + u(x)T/M, g(x)\Delta W^M_k \right)}{||x||} \right) \right] \leq \exp \left[ \frac{p^2 T (1 + TC + T||u(0)||^2 (C + ||g(0)||^2)}{M} \right].
\]

**Proof 5.2.11** Let \( a^T \) stand for the transposed of a vector \( a \), let \( Y \) be the \( m \) column vector defined by \( : Y = \sqrt{\frac{T}{M}} (1, \cdots, 1) \) and let \( N(0, 1) \) be a 1-dimensional standard normal random variable. Then we have

\[
\mathbb{E} \left[ \exp \left( pz \frac{\left( x + u(x)T/M, g(x)\Delta W^M_k \right)}{||x||} \right) \right] = \mathbb{E} \left[ \exp \left( pz \frac{\left( x + u(x)T/M, g(x)\Delta W^M_k \right)}{||x||} \right) \right] = \mathbb{E} \left[ \exp \left( pz \frac{(x + u(x)T/M, g(x)\Delta W^M_k) - \left( x + u(x)T/M, g(x)\Delta W^M_k \right)}{||x||} \right) \right] = \mathbb{E} \left[ \exp \left( pz \frac{(x + u(x)T/M, g(x)\Delta W^M_k) - \left( x + u(x)T/M, g(x)\Delta W^M_k \right)}{||x||} \right) \right].
\]

Using Lemma 4.2.15 it follows that:

\[
\mathbb{E} \left[ \exp \left( pz \frac{\left( x + u(x)T/M, g(x)\Delta W^M_k \right)}{||x||} \right) \right] \leq \exp \left[ \frac{p^2 z^2 (g(x)^T(x + u(x)T/M)Y)^2}{||x||^2} \right] = \exp \left[ \frac{p^2 T (g(x)^T(x + u(x)T/M))^2}{M ||x||^4} \right] \leq \exp \left[ \frac{p^2 T (g(x)^2 ||x + u(x)T/M||^2}{M ||x||^4} \right].
\]

From the global Lipschitz condition, for all \( x \in \mathbb{R}^d \) such that \( ||x|| \geq 1 \) we have

\[
||g(x)||^2 \leq (||g(x) - u(0)|| + ||g(0)||^2) \leq (C||x||^2 + ||g(0)||^2)^2 \leq (C + ||g(0)||^2)||x||^2
\]

\[
||x + u(x)T/M|| \leq ||x|| + T/M||u(x)|| \leq ||x|| + T/M||u(0)|| \leq ||x|| + TC||x|| + T||u(0)|| \leq (1 + TC + T||u(0)||)||x||.
\]

Therefore, it follows that:

\[
\mathbb{E} \left[ \exp \left( pz \frac{\left( x + u(x)T/M, g(x)\Delta W^M_k \right)}{||x||} \right) \right] \leq \exp \left[ \frac{p^2 T (1 + TC + T||u(0)||^2 (C + ||g(0)||^2)}{M} \right].
\]

for all \( M \in \mathbb{N}, k \in \{0, \cdots, M - 1\}, p \in [1, \infty) \) and \( z \in \{-1, 1\} \).

Following closely [11, Lemma 2.3] we have the following lemma.

**Lemma 5.2.12** Let \( a^M_n : \Omega \rightarrow \mathbb{R} \) for \( M \in \mathbb{N} \) and \( n \in \{0, 1, \cdots, M\} \) define as in notation (??). Then the following inequality holds:

\[
\sup_{z \in \{-1, 1\}} \sup_{M \in \mathbb{N}} \left\| \sum_{n \in \{0, 1, \cdots, M\}} \exp \left( z \sum_{k=0}^{n-1} a^M_k \right) \right\|_{L^p(\Omega, \mathbb{R})} < \infty,
\]

for all \( p \in [2, +\infty) \).
Proof 5.2.13 The time discrete stochastic process \(z \sum_{k=0}^{n-1} \alpha_k^M, \ n \in \{0, 1, \cdots, M\}\) is an \((\mathcal{F}_{nT/M})_{n \in \{0, \cdots, M\}}\) martingale for every \(z \in \{-1, 1\}\) and \(M \in \mathbb{N}\). So \(\exp \left(z \sum_{k=0}^{n-1} \alpha_k^M \right)\) is a positive \((\mathcal{F}_{nT/M})_{n \in \{0, \cdots, M\}}\) submartingale for every \(z \in \{-1, 1\}\) and \(M \in \mathbb{N}\) since \(\exp\) is a convex function.

Applying Doop's maximal inequality leads to:

\[
\| \sup_{n \in \{0, \cdots, M\}} \exp \left( z \sum_{k=0}^{n-1} \alpha_k^M \right) \|_{L^p(\Omega, \mathbb{R})} = \left( E \| \sup_{n \in \{0, \cdots, M\}} \exp \left( p z \sum_{k=0}^{n-1} \alpha_k^M \right) \|_{L^p(\Omega, \mathbb{R})} \right)^{1/p} \leq \left( \frac{1}{p} \right) \left( E \| \exp \left( p z \sum_{k=0}^{M-1} \alpha_k^M \right) \|_{L^p(\Omega, \mathbb{R})} \right)^{1/p} = \frac{1}{p} \| \exp \left( z \sum_{k=0}^{M-1} \alpha_k^M \right) \|_{L^p(\Omega, \mathbb{R})}.
\]

Using Lemma 5.2.10, it follows that:

\[
E \left[ \exp(p z \alpha_k^M) / \mathcal{F}_{kT/M} \right] \leq \exp \left( \frac{p^2 T (C + ||g(0)||)^2 (1 + TC + T ||u(0)||)^2}{M} \right).
\]

Using inequality (5.25), it follows that:

\[
E \left[ \exp \left( p z \sum_{k=0}^{M-1} \alpha_k^M \right) \right] = E \left[ \exp \left( p z \sum_{k=0}^{M-2} \alpha_k^M \right) \right] E \left[ \exp \left( p \alpha_k^M / \mathcal{F}_{(M-1)T/M} \right) \right] \leq E \left[ \exp \left( p z \sum_{k=0}^{M-2} \alpha_k^M \right) \right] \exp \left( \frac{p^2 T (C + ||g(0)||)^2 (1 + TC + T ||u(0)||)^2}{M} \right).
\]

Iterating the previous inequality \(M\) times gives:

\[
E \left[ \exp \left( p z \sum_{k=0}^{M-1} \alpha_k^M \right) \right] \leq \exp \left( p^2 T (C + ||g(0)||)^2 (1 + TC + T ||u(0)||)^2 \right).
\]

Now combining inequalities (5.24) and (5.26) leads to

\[
\sup_{z \in \{-1, 1\}} \sup_{M \in \mathbb{N}} \left\| \sup_{n \in \{0, \cdots, M\}} \exp \left( z \sum_{k=0}^{n-1} \alpha_k^M \right) \right\|_{L^p(\Omega, \mathbb{R})} \leq 2 \exp \left( p^2 T (C + ||g(0)||)^2 (1 + TC + T ||u(0)||)^2 \right) < +\infty,
\]

for all \(p \in [2, \infty)\).

Lemma 5.2.14 The following inequality holds

\[
E \left[ \exp \left( p z 1_{\{||x|| \geq 1\}} \left( \frac{x + u(x)T/M}{||x||}, \frac{h(x)}{||x||} \Delta N_n^M \right) \right) \right] \leq \exp \left( \left[ p^2 (1 + ||h(0)||) (1 + TC + T ||u(0)||) + p(C + ||h(0)||) (1 + TC + T ||u(0)||) \right] \lambda T \right),
\]

for all \(M \in \mathbb{N}\), all \(p \in [1, +\infty)\) and all \(n \in \{0, \cdots, M\}\), \(z \in \{-1, 1\}\).

Proof 5.2.15 For \(x \in \mathbb{R}^d\) such that \(||x|| \neq 0\), we have:

\[
E \left[ \exp \left( p z \left( \frac{x + u(x)T/M}{||x||}, \frac{h(x)}{||x||} \Delta N_n^M \right) \right) \right] \leq E \left[ \exp \left( p z \frac{||x + u(x)T/M|| ||h(x)|| \Delta N_n^M}{||x||^2} \right) \right].
\]
5.2. MOMENTS BOUNDED OF THE NUMERICAL SOLUTION

Using the global Lipschitz condition on $h$, for all $x \in \mathbb{R}^d$ such that $||x|| \geq 1$, we have:

$$||h(x)|| \leq ||h(x) - h(0)|| + ||h(0)|| \leq (C + ||h(0)||)||x||. \quad (5.27)$$

$$||x + u(x)T/M|| \leq (1 + TC + T||u(0)||)||x|| \quad (5.28)$$

So using inequalities (5.27) and (5.28), it follows that:

$$\mathbb{E}\left[\exp\left(\sum_{0 \leq t \leq T} \frac{x + u(x)T/M, h(x)}{||x||, ||x||} \Delta N_n^M\right)\right] = \mathbb{E}\left(\exp[pz(C + ||h(0)||)(1 + TC + T||u(0)||)\Delta N_n^M]\right).$$

Using Lemma 4.2.21, it follows that:

$$\mathbb{E}\left[\exp\left(pz1_{\{||x|| \geq 1\}} \left(\frac{x + u(x)T/M, h(x)}{||x||, ||x||} \Delta N_n^M\right)\right)\right] \leq \mathbb{E}\left[\exp[pz(C + ||h(0)||)(1 + TC + T||u(0)||)\Delta N_n^M]\right].$$

The following lemma is similar to [11, Lemma 2.3].

**Lemma 5.2.16** Let $\beta_n^M : \Omega \rightarrow \mathbb{R}$ defined in Notation 5.2.6 for all $M \in \mathbb{N}$ and all $n \in \{0, \ldots, M\}$ then we have the following inequality:

$$\sup_{x \in (-1, 1)} \sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left\| \exp\left(-\frac{1}{M} \sum_{k=0}^{n-1} \beta_k^M\right) \right\|_{L^p(\Omega, \mathbb{R})} < +\infty.$$

**Proof 5.2.17** Following the proof of [8, Lemma 3.4], the result is straightforward using lemmas 5.2.14 and 5.2.10.

**Lemma 5.2.18** The following inequality holds:

$$\sup_{M \in \mathbb{N}} \mathbb{E}\left[\exp\left(p\beta \sum_{k=0}^{M-1} ||\Delta N_k^M||\right)\right] < +\infty,$$

for all $p \in [1, +\infty)$.

**Proof 5.2.19** The proof is similar to the proof of Lemma 4.2.27 with only different value of $\beta$.

The following lemma is based on Lemma 5.2.16.

**Lemma 5.2.20** [Uniformly bounded moments of the dominating stochastic processes].

Let $M \in \mathbb{N}$ and $D_n^M : \Omega \rightarrow [0, \infty)$ for $n \in \{0, 1, \ldots, M\}$ be define as in notation (5.2.6). Then we have:

$$\sup_{M \in \mathbb{N}, M \geq 8\lambda pT} \sup_{n \in \{0, 1, \ldots, M\}} \left\| D_n^M\right\|_{L^p(\Omega, \mathbb{R})} < +\infty,$$

for all $p \in [1, \infty)$.

The following lemma is an extension of [8, Lemma 3.6, pp 16]. Here, we include the jump part.

**Lemma 5.2.21** Let $M \in \mathbb{N}$ and $\Omega_M^M \in \mathcal{F}$. The following inequality holds:

$$\sup_{M \in \mathbb{N}} \left(M^{pp}[\Omega_M^M]\right) < +\infty,$$

for all $p \in [1, \infty)$. 

Proof 5.2.22 [Theorem 5.2.5].

Let’s first represent the numerical approximation $Y_n^M$ in the following appropriate form:

$$
Y_n^M = Y_{n-1}^M + u(Y_{n-1}^M)T/M + \frac{\Delta t v(Y_{n-1}^M)}{1 + \Delta t \|v(Y_{n-1}^M)\|} + g(Y_{n-1}^M)\Delta W_{n-1}^M + h(Y_{n-1}^M)\Delta N_{n-1}^M + u(0) + \sum_{k=0}^{n-1} u(Y_k^M)T/M + \sum_{k=0}^{n-1} \frac{\Delta t v(Y_k^M)}{1 + \Delta t \|v(Y_k^M)\|} + g(Y_k^M)\Delta W_k^M + h(Y_k^M)\Delta N_k^M
$$

for all $M \in \mathbb{N}$ and all $n \in \{0, \cdots, M\}$.

Using the inequality

$$
\left\| \frac{\Delta t v(Y_k^M)}{1 + \Delta t \|v(Y_k^M)\|} \right\|_{L^p(\Omega, \mathbb{R}^d)} < 1
$$

it follows that:

$$
\|Y_n^M\|_{L^p(\Omega, \mathbb{R}^d)} \leq \|X_0\|_{L^p(\Omega, \mathbb{R})} + \|u(0)\|nT/M + \left\| \sum_{k=0}^{n-1} g(Y_k^M)\Delta W_k^M \right\|_{L^p(\Omega, \mathbb{R}^d)} + M
$$

Using Lemma 4.2.36 and Lemma 4.2.40, it follows that

$$
\|Y_n^M\|_{L^p(\Omega, \mathbb{R}^d)} \leq \|X_0\|_{L^p(\Omega, \mathbb{R})} + \|u(0)\|nT/M + C_p \left( \sum_{k=0}^{n-1} \sum_{i=1}^{m} \|g_i(0)\|^2 \frac{T}{M} \right)^{1/2} + C_p \left( \sum_{k=0}^{n-1} \|h(0)\|^2 \frac{T}{M} \right)^{1/2}
$$

From $\|g_i(0)\|^2 \leq \|g(0)\|^2$ and the global Lipschitz condition satisfied by $g$ and $h$, we obtain $\|g_i(Y_k^M) - g_i(0)\|_{L^p(\Omega, \mathbb{R}^d)}^2 + C_p \left( \sum_{k=0}^{n-1} \sum_{i=1}^{m} \|g_i(Y_k^M) - g_i(0)\|_{L^p(\Omega, \mathbb{R}^d)}^2 \frac{T}{M} \right)^{1/2}$.

(5.29)
$g_i(0) \leq C\|Y^M_k\|_{L^p(\Omega,\mathbb{R}^d)}$ and \(h(Y^M_k) - h(0)\) \leq C\|Y^M_k\|_{L^p(\Omega,\mathbb{R}^d)}$. So using (5.29), we have:

\[
|Y_n^M|_{L^p(\Omega,\mathbb{R}^d)}^2 \leq |X_0|_{L^p(\Omega,\mathbb{R}^d)} + T\|u(0)\| + C_p\sqrt{Tm}|g(0)|| + C_p\sqrt{T}|h(0)| + M
\]

Using the inequality \((a + b + c)^2 \leq 3a^2 + 3b^2 + 3c^2\), it follows that:

\[
|Y_n^M|_{L^p(\Omega,\mathbb{R}^d)}^2 \leq 3\left(|X_0|_{L^p(\Omega,\mathbb{R}^d)} + T\|u(0)\| + C_p\sqrt{Tm}|g(0)|| + C_p\sqrt{T}|h(0)| + M\right)^2
\]

Taking the square root and the supremum in both sides of the previous inequality leads to:

\[
\sup_{n \in \{0,\ldots,M\}} \|Y_n^M\|_{L^p(\Omega,\mathbb{R}^d)} \leq \sqrt{3}e^{3(C_p\sqrt{\bar{m}} + C_p)^2} \left(|X_0|_{L^p(\Omega,\mathbb{R}^d)} + T\|u(0)\| + C_p\sqrt{Tm}|g(0)|| + C_p\sqrt{T}|h(0)| + M\right).
\]
From the relation
\[
\left\| 1_{\Omega^M_n} \right\|_{L^p(\Omega,\mathbb{R}^d)} = \mathbb{E} \left[ \left( 1_{\Omega^M_n} \right)^{1/2p} \right] = \mathbb{P} \left[ \left( \Omega^M_n \right)^{1/2p} \right],
\]
it follows using Lemma 5.2.21 that:
\[
\sup_{M \in \mathbb{N}} \sup_{n \in \{0,\ldots,M\}} \left( M \left\| 1_{\Omega^M_n} \right\|_{L^p(\Omega,\mathbb{R}^d)} \right) = \sup_{M \in \mathbb{N}} \sup_{n \in \{0,\ldots,M\}} \left( M^{2p} \mathbb{P} \left[ \left( \Omega^M_n \right)^{1/2p} \right] \right)^{1/2p} < +\infty. \tag{5.34}
\]
So plugging (5.33) and (5.34) in (5.32) leads to:
\[
\sup_{M \in \mathbb{N}} \sup_{n \in \{0,\ldots,M\}} \left\| 1_{\Omega^M_n} Y^M_n \right\|_{L^p(\Omega,\mathbb{R}^d)} < +\infty. \tag{5.35}
\]
Furthermore, we have
\[
\sup_{M \in \mathbb{N}} \sup_{n \in \{0,\ldots,M\}} \left\| Y^M_n \right\|_{L^p(\Omega,\mathbb{R}^d)} \leq \sup_{M \in \mathbb{N}} \sup_{n \in \{0,\ldots,M\}} \left\| 1_{\Omega^M_n} Y^M_n \right\|_{L^p(\Omega,\mathbb{R}^d)} + \sup_{M \in \mathbb{N}} \sup_{n \in \{0,\ldots,M\}} \left\| 1_{\Omega^M_n} Y^M_n \right\|_{L^p(\Omega,\mathbb{R}^d)}. \tag{5.36}
\]
From (5.35) the second term of inequality (5.36) is bounded, while using Lemma 5.2.7 and Lemma 5.2.20 we have:
\[
\sup_{M \in \mathbb{N}} \sup_{n \in \{0,\ldots,M\}} \left\| 1_{\Omega^M_n} Y^M_n \right\|_{L^p(\Omega,\mathbb{R}^d)} \leq \sup_{M \in \mathbb{N}} \sup_{n \in \{0,\ldots,M\}} \left\| D^M_n \right\|_{L^p(\Omega,\mathbb{R}^d)} < +\infty. \tag{5.37}
\]
Finally plugging (5.35) and (5.37) in (5.36) leads to:
\[
\sup_{M \in \mathbb{N}} \sup_{n \in \{0,\ldots,M\}} \left\| Y^M_n \right\|_{L^p(\Omega,\mathbb{R}^d)} < +\infty.
\]
This complete the proof of Theorem 5.2.5.

5.3 Strong convergence of the semi-tamed Euler scheme

**Theorem 5.3.1** Under Assumptions 5.2.2, for all \( p \in [1, +\infty) \) there exist a positive constant \( C_p \) such that:
\[
\left( \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| X_t - \overline{Y}_t^M \right\|^p \right] \right)^{1/p} \leq C_p \Delta t^{1/2}, \tag{5.38}
\]
for all \( M \in \mathbb{N} \).

Where \( X : [0,T] \times \Omega \rightarrow \mathbb{R}^d \) is the exact solution of equation (5.1) and \( \overline{Y}_t^M \) is the time continuous solution defined in (5.8).

In order to prove Theorem 5.3.1, we need the following two lemmas.

**Lemma 5.3.2** [Based on [8, Lemma 3.10, pp 16]].

Let \( Y^M_n \) be defined by (5.7) for all \( M \in \mathbb{N} \) and all \( n \in \{0,1,\ldots,M\} \). Then the following inequalities holds:
\[
\sup_{M \in \mathbb{N}} \sup_{n \in \{0,1,\ldots,M\}} \left( \mathbb{E} \left[ |u\lambda(Y^M_n)|^p \right] + \mathbb{E} \left[ |v(Y^M_n)|^p \right] + \mathbb{E} \left[ |g(Y^M_n)|^p \right] + \mathbb{E} \left[ |h(Y^M_n)|^p \right] \right) < +\infty,
\]
for all \( p \in [1, \infty) \).

**Proof 5.3.3** The proof is similar to the proof [8, Lemma 3.10, pp 16].
For $s \in [0, T]$ let $\lfloor s \rfloor$ be the greatest grid point less than $s$. We have the following lemma.

**Lemma 5.3.4** The following inequalities holds for any stepsize $\Delta t$.

\[
\sup_{t \in [0, T]} \left\| \frac{Y^M_t - Y^M_{t|t}}{L_p(\Omega, \mathbb{R}^4)} \right\| \leq C_p \Delta t^{1/2},
\]

\[
\sup_{M \in \mathbb{N}} \sup_{t \in [0, T]} \left\| \frac{Y^M_t}{L_p(\Omega, \mathbb{R}^4)} \right\| < \infty,
\]

\[
\sup_{t \in [0, T]} \left\| \frac{v(Y^M_t) - v(Y^M_{t|t})}{L_p(\Omega, \mathbb{R}^4)} \right\| \leq C_p \Delta t^{1/2}.
\]

**Proof 5.3.5**

- Using the time continuous approximation (5.8), Lemma 4.2.40 and Lemma 4.2.36, it follows that:

\[
\sup_{t \in [0, T]} \left\| \frac{Y^M_t - Y^M_{t|t}}{L_p(\Omega, \mathbb{R}^4)} \right\|
\leq \frac{T}{M} \sup_{t \in [0, T]} \left\| \frac{u_{\lambda}(Y^M_{t|t})}{L_p(\Omega, \mathbb{R}^4)} \right\|
+ \frac{T}{M} \left( \sup_{t \in [0, T]} \left\| \frac{v(Y^M_{t|t})}{1 + \Delta t \| v(Y^M_{t|t}) \|_{L_p(\Omega, \mathbb{R}^4)}} \right\| \right)
+ \sup_{t \in [0, T]} \left\| \frac{g(Y^M_{t|t})}{L_p(\Omega, \mathbb{R}^4)} \right\|
+ \sup_{t \in [0, T]} \left\| \frac{h(Y^M_{t|t})}{L_p(\Omega, \mathbb{R}^4)} \right\|
\leq \frac{T}{\sqrt{M}} \left( \sup_{n \in \{0, \ldots, M\}} \left\| \frac{u_{\lambda}(Y^M_n)}{L_p(\Omega, \mathbb{R}^4)} \right\| \right)
+ \frac{T}{\sqrt{M}} \left( \sup_{n \in \{0, \ldots, M\}} \left\| \frac{v(Y^M_n)}{L_p(\Omega, \mathbb{R}^4)} \right\| \right)
+ \frac{C_p \sqrt{T}}{\sqrt{M}} \left( \sup_{n \in \{0, \ldots, M\}} \left\| \frac{g(Y^M_n)}{L_p(\Omega, \mathbb{R}^4)} \right\| \right)
+ \frac{C_p \sqrt{T}}{\sqrt{M}} \left( \sup_{n \in \{0, \ldots, M\}} \left\| \frac{h(Y^M_n)}{L_p(\Omega, \mathbb{R}^4)} \right\| \right)
\]

for all $p \in [1, +\infty)$ and all $M \in \mathbb{N}$. Using inequality (5.39) and Lemma 5.3.2 it follows that:

\[
\left[ \sup_{t \in [0, T]} \left\| \frac{Y^M_t - Y^M_{t|t}}{L_p(\Omega, \mathbb{R}^4)} \right\| \right] < C_p \Delta t^{1/2},
\]

for all $p \in [1, +\infty)$ and for all stepsize $\Delta t$.

- Using inequality (5.40), inequality $\|a\| \leq \|a - b\| + \|b\|$ for all $a, b \in \mathbb{R}^d$ and Theorem 5.2.5 it follows that

\[
\sup_{t \in [0, T]} \left\| \frac{Y^M_t}{L_p(\Omega, \mathbb{R}^4)} \right\|
\leq \sup_{t \in [0, T]} \left\| \frac{Y^M_t - Y^M_{t|t}}{L_p(\Omega, \mathbb{R}^4)} \right\|
+ \sup_{t \in [0, T]} \left\| \frac{Y^M_{t|t}}{L_p(\Omega, \mathbb{R}^4)} \right\|
\leq C_p \Delta t^{1/2} + \sup_{t \in [0, T]} \left\| \frac{Y^M_{t|t}}{L_p(\Omega, \mathbb{R}^4)} \right\|
< C_p \Delta t^{1/2} + \sup_{t \in [0, T]} \left\| \frac{Y^M_{t|t}}{L_p(\Omega, \mathbb{R}^4)} \right\| < \infty,
\]

for all $p \in [1, +\infty)$ and all $M \in \mathbb{N}$.
• Further, using the polynomial growth condition:

\[ \|v(x) - v(y)\| \leq C(K + ||x||^\gamma + ||y||^\gamma)||x - y||, \]

for all \( x, y \in \mathbb{R}^d \), it follows using Holder inequality that:

\[
\sup_{t \in [0,T]} \|v(\overline{Y}_M^t) - v(\overline{Y}_{[t]}^M)\|_{L^p(\Omega, \mathbb{R}^d)} \leq C \left( K + 2 \sup_{t \in [0,T]} \|\overline{Y}_M^t\|_{L^{2p}(\Omega, \mathbb{R}^d)} \right) \times \left( \sup_{t \in [0,T]} \|\overline{Y}_M^t - \overline{Y}_{[t]}^M\|_{L^{2p}(\Omega, \mathbb{R}^d)} \right) (5.41)
\]

Using (5.41) and the first part of Lemma 5.3.4, the following inequality holds for all \( p \in [1, +\infty) \)

\[
\sup_{t \in [0,T]} \|v(\overline{Y}_M^t) - v(\overline{Y}_{[t]}^M)\|_{L^p(\Omega, \mathbb{R}^d)} < C_p \Delta t^{1/2},
\]

for all \( p \in [1, \infty) \) and all \( M \in \mathbb{N} \).

Now we are ready to prove Theorem 5.3.1.

**Proof 5.3.6 [ Theorem 5.3.1]**

Let's recall that for \( z \in \{0, T\}, \lfloor z \rfloor \) is the greatest grid point less than \( z \). The time continuous solution (5.8) can be written into its integral form as below:

\[
\overline{Y}_s^M = \varepsilon + \int_0^s u(\overline{Y}_s^M)dz + \int_0^s \frac{v(\overline{Y}_s^M)}{1 + \Delta t||v(\overline{Y}_s^M)||}dz + \int g(\overline{Y}_s^M)dWz + \int h(\overline{Y}_s^M)d\overline{N}, (5.43)
\]

for all \( z \in [0, T] \) almost sure (a.s) and all \( M \in \mathbb{N} \).

Let's estimate first the quantity \( \|X_s - \overline{Y}_s^M\|^2 \)

\[
X_s - \overline{Y}_s = \int_0^s \left( u_\lambda(X_s) - u_\lambda(\overline{Y}_s^M) \right)dz + \int_0^s \left( v(X_s) - \frac{v(\overline{Y}_s^M)}{1 + \Delta t||v(\overline{Y}_s^M)||} \right)dz + \int_0^s \left( g(X_s) - g(\overline{Y}_s^M) \right) dWz + \int_0^s \left( h(X_s) - h(\overline{Y}_s^M) \right) d\overline{N}.
\]

Using the relation \( d\overline{N} = dN - \lambda dz \), it follows that

\[
X_s - \overline{Y}_s = \int_0^s \left[ \left( v(X_s) - \frac{v(\overline{Y}_s^M)}{1 + \Delta t||v(\overline{Y}_s^M)||} \right) + \left( u(X_s) - u(\overline{Y}_s^M) \right) + \lambda \left( h(X_s) - h(\overline{Y}_s^M) \right) \right]dz + \int_0^s \left( g(X_s) - g(\overline{Y}_s^M) \right) dWz + \int_0^s \left( h(X_s) - h(\overline{Y}_s^M) \right)d\overline{N}.
\]

The function \( k : \mathbb{R}^n \rightarrow \mathbb{R}, x \mapsto ||x||^2 \) is twice differentiable. Applying Itô's formula for jump process to the process \( X_s - \overline{Y}_s^M \) with the function \( k \) leads to:

\[
\|X_s - \overline{Y}_s^M\|^2 = 2\int_0^s \left[ X_s - \overline{Y}_s^M, v(\overline{Y}_s^M) - \frac{v(\overline{Y}_s^M)}{1 + \Delta t||v(\overline{Y}_s^M)||} \right]dz + 2\lambda \int_0^s \left( X_s - \overline{Y}_s^M, h(X_s) - h(\overline{Y}_s^M) \right)dz + 2\int_0^s \left( X_s - \overline{Y}_s^M, g(X_s) - g(\overline{Y}_s^M) \right) dWz + \sum_{i=1}^m \int_0^s \left( X_s - \overline{Y}_s^M, g_i(X_s) - g_i(\overline{Y}_s^M) \right) dWz^i + \int_0^s \left[ X_s - \overline{Y}_s^M + h(X_s) - h(\overline{Y}_s^M) \right]dNz.
\]
Using again the relation \(d\bar{N}_z = dN_z - \lambda dz\) leads to

\[
||X_s - \bar{Y}_s^M||^2 = 2 \int_0^s \left\langle X_z - \bar{Y}_z^M, v(X_z) - \frac{v(\bar{Y}_z^M)}{1 + \Delta t||v(\bar{Y}_z^M)||} \right\rangle dz + 2\lambda \int_0^s \left\langle X_z - \bar{Y}_z^M, h(X_z) - h(\bar{Y}_z^M) \right\rangle dz
+ 2 \int_0^s \left\langle X_z - \bar{Y}_z^M, u(X_z) - u(\bar{Y}_z^M) \right\rangle dz + \sum_{i=1}^m \int_0^s ||g_i(X_z) - g_i(\bar{Y}_z^M)||^2 dz
+ 2 \sum_{i=1}^m \int_0^s \left\langle X_z - \bar{Y}_z^M, g_i(X_z) - g_i(\bar{Y}_z^M) \right\rangle dW^i_z
+ \int_0^s \left[ ||X_z - \bar{Y}_u^M + h(X_z) - h(\bar{Y}_z^M)||^2 - ||X_z - \bar{Y}_z^M||^2 \right] d\bar{N}_z
+ \lambda \int_0^s \left[ ||X_z - \bar{Y}_z^M + h(X_z) - h(\bar{Y}_z^M)||^2 - ||X_z - \bar{Y}_z^M||^2 \right] dz
= A_1 + A'_1 + A_2 + A_3 + A_4 + A_5 + A_6. \tag{5.44}
\]

In the next step, we give some useful estimations of \(A_1, A'_1, A_2, A_3\) and \(A_6\).

\[
A_1 = 2 \int_0^s \left\langle X_z - \bar{Y}_z^M, v(X_z) - \frac{v(\bar{Y}_z^M)}{1 + \Delta t||v(\bar{Y}_z^M)||} \right\rangle dz
= 2 \int_0^s < X_s - \bar{Y}_z^M, v(X_z) - v(\bar{Y}_z^M) > dz
+ 2 \int_0^s \left\langle X_s - \bar{Y}_z^M, v(\bar{Y}_z^M) - \frac{v(\bar{Y}_z^M)}{1 + \Delta t||v(\bar{Y}_z^M)||} \right\rangle dz
= A_{11} + A_{12}.
\]

Using the one-sided Lipschitz condition satisfied by \(v\) leads to

\[
A_{11} = 2 \int_0^s (X_s - \bar{Y}_z^M, v(X_z) - v(\bar{Y}_z^M)) dz
\leq 2C \int_0^s ||X_z - \bar{Y}_z^M||^2 dz. \tag{5.45}
\]

Moreover, using the inequality \((a, b) \leq |a||b| \leq \frac{a^2}{2} + \frac{b^2}{2}\) leads to :

\[
A_{12} = 2 \int_0^s \left\langle X_z - \bar{Y}_z^M, v(\bar{Y}_z^M) - \frac{v(\bar{Y}_z^M)}{1 + \Delta t||v(\bar{Y}_z^M)||} \right\rangle dz
= 2 \int_0^s \left\langle X_z - \bar{Y}_z^M, v(\bar{Y}_z^M) - v(\bar{Y}_z^M) \right\rangle dz
+ 2\Delta t \int_0^s \left\langle X_z - \bar{Y}_z^M, \frac{v(\bar{Y}_z^M)||v(\bar{Y}_z^M)||}{1 + \Delta t||v(\bar{Y}_z^M)||} \right\rangle dz
\leq \int_0^s ||X_z - \bar{Y}_z^M||^2 dz + \int_0^s ||v(\bar{Y}_z^M) - v(\bar{Y}_z^M)||^2 dz
+ \int_0^s ||X_z - \bar{Y}_z^M||^2 dz + \frac{T^2}{M^2} \int_0^s ||v(\bar{Y}_z^M)||^4 dz
\leq 2 \int_0^s ||X_z - \bar{Y}_z^M||^2 dz + \int_0^s ||v(\bar{Y}_z^M) - v(\bar{Y}_z^M)||^2 dz
+ \frac{T^2}{M^2} \int_0^s ||v(\bar{Y}_z^M)||^4 dz \tag{5.46}
\]
Combining (5.45) and (5.46) give the following estimation for $A_1$:

$$A_1 \leq (2C + 2) \int_0^s \|X_z - \overline{Y}_z^M\|^2 dz + \int_0^s \|v(\overline{Y}_z^M) - v(\overline{Y}_{[z]}^M)\|^2 dz + \frac{T^2}{M^2} \int_0^s \|v(\overline{Y}_{[z]}^M)\|^4 dz. \tag{5.47}$$

Using again the inequality $\langle a, b \rangle \leq |a||b| \leq \frac{a^2}{2} + \frac{b^2}{2}$ and the global-Lipschitz condition satisfied by $u$ leads to:

$$A_2 = 2 \int_0^s \left\langle X_z - \overline{Y}_z^M, u(X_z) - u(\overline{Y}_{[z]}^M) \right\rangle dz$$

$$= 2 \int_0^s \left\langle X_z - \overline{Y}_z^M, u(X_z) - u(\overline{Y}_z^M) \right\rangle dz + \int_0^s \left\langle X_z - \overline{Y}_z^M, u(\overline{Y}_z^M) - u(\overline{Y}_{[z]}^M) \right\rangle dz$$

$$\leq 2C \int_0^s \|X_z - \overline{Y}_z^M\|^2 dz + 2C \int_0^s \|\overline{Y}_z^M - \overline{Y}_{[z]}^M\|^2 dz. \tag{5.48}$$

Using the same arguments as for $A_2$ leads to the following estimation of $A'_1$:

$$A'_1 = 2\lambda \int_0^s \left\langle X_z - \overline{Y}_z^M, h(X_z) - h(\overline{Y}_{[z]}^M) \right\rangle dz$$

$$\leq 2\lambda C \int_0^s \|X_z - \overline{Y}_z^M\|^2 dz + 2\lambda C \int_0^s \|\overline{Y}_z^M - \overline{Y}_{[z]}^M\|^2 dz. \tag{5.49}$$

Using the inequalities $\|g_i(x) - g_i(y)\| \leq \|g(x) - g(y)\|$ and $(a + b)^2 \leq 2a^2 + 2b^2$ and the global Lipschitz condition satisfied by $g$, we have:

$$A_3 = \sum_{i=1}^m \int_0^s \|g_i(X_z) - g_i(\overline{Y}_z^M)\|^2 dz$$

$$\leq m \int_0^s \|g(X_z) - g(\overline{Y}_z^M)\|^2 dz$$

$$= m \int_0^s \|g(X_z) - g(\overline{Y}_z^M) + g(\overline{Y}_z^M) - g(\overline{Y}_{[z]}^M)\|^2 dz$$

$$\leq 2m \int_0^s \|g(X_z) - g(\overline{Y}_z^M)\|^2 dz + 2m \int_0^s \|g(\overline{Y}_z^M) - g(\overline{Y}_{[z]}^M)\|^2 dz$$

$$\leq 2mC^2 \int_0^s \|X_z - \overline{Y}_z^M\|^2 dz + 2mC^2 \int_0^s \|\overline{Y}_z^M - \overline{Y}_{[z]}^M\|^2 dz. \tag{5.50}$$

Using the same reasons as above we obtain the following estimation for $A_6$:

$$A_6 = \lambda \int_0^s \left[ X_z - \overline{Y}_z^M + h(X_z) - h(\overline{Y}_{[z]}^M) \right]^2 dz$$

$$\leq 3\lambda \int_0^s \|X_z - \overline{Y}_z^M\|^2 dz + 2\lambda \int_0^s \|h(X_z) - h(\overline{Y}_{[z]}^M)\|^2 dz$$

$$\leq 3\lambda \int_0^s \|X_z - \overline{Y}_z^M\|^2 dz + 4\lambda \int_0^s \|h(X_z) - h(\overline{Y}_{[z]}^M)\|^2 dz$$

$$+ 4\lambda \int_0^s \|h(\overline{Y}_{[z]}^M) - h(\overline{Y}_{[z]}^M)\|^2 dz$$

$$\leq (3\lambda + 4\lambda C^2) \int_0^s \|X_z - \overline{Y}_z^M\|^2 dz + 4\lambda C^2 \int_0^s \|\overline{Y}_z^M - \overline{Y}_{[z]}^M\|^2 dz. \tag{5.51}$$
Inserting (5.47), (5.48), (5.49), (5.50) and (5.51) in (5.44) we obtain:

\[ ||X_s - \overline{Y}_s^M||^2 \leq (4C + 2 + 2mC^2 + 3\lambda + 4\lambda C^2 + 2\lambda C) \int_0^s ||X_z - \overline{Y}_z^M||^2 dz \]

\[ + (2C + 2mC^2 + 4\lambda C^2 + 2\lambda C) \int_0^s ||Y_z^M - \overline{Y}_z^M||^2 dz \]

\[ + \int_0^s ||v(Y_z^M) - v(\overline{Y}_z^M)||^2 dz + \frac{T^2}{M^2} \int_0^s ||v(\overline{Y}_z^M)||^4 dz \]

\[ + 2 \sum_{i=1}^m \int_0^s \left< X_z - \overline{Y}_z^M, g_i(X_z) - g_i(\overline{Y}_z^M) \right> dW_i^z \]

\[ + \int_0^s \left[ ||X_z - \overline{Y}_z^M + h(X_z) - h(\overline{Y}_z^M)||^2 - ||X_z - \overline{Y}_z^M||^2 \right] d\overline{N}_z. \]

Taking the supremum in both sides of the previous inequality leads to

\[ \sup_{s \in [0,t]} ||X_s - \overline{Y}_s^M||^2 \leq (4C + 2 + 2mC^2 + 3\lambda + 4\lambda C^2 + 2\lambda C) \int_0^t ||X_z - \overline{Y}_z^M||^2 dz \]

\[ + (2C + 2mC^2 + 4\lambda C^2 + 2\lambda C) \int_0^t ||Y_z^M - \overline{Y}_z^M||^2 dz \]

\[ + \int_0^t ||v(Y_z^M) - v(\overline{Y}_z^M)||^2 dz + \frac{T^2}{M^2} \int_0^t ||v(\overline{Y}_z^M)||^4 dz \]

\[ + 2 \sup_{s \in [0,t]} \left| \sum_{i=1}^m \int_0^s \left< X_z - \overline{Y}_z^M, g_i(X_z) - g_i(\overline{Y}_z^M) \right> dW_i^z \right| \]

\[ + \sup_{s \in [0,t]} \left| \int_0^s \left[ ||X_z - \overline{Y}_z^M + h(X_z) - h(\overline{Y}_z^M)||^2 - ||X_z - \overline{Y}_z^M||^2 \right] d\overline{N}_z \right|. \]

Using Lemma 4.2.34 we have the following estimations

\[ B_1 := \left\| 2 \sup_{s \in [0,t]} \left| \sum_{i=1}^m \int_0^s \left< X_z - \overline{Y}_z^M, g_i(X_z) - g_i(\overline{Y}_z^M) \right> dW_i^z \right| \right\|_{L^{p/2}(\Omega;\mathbb{R})} \]

\[ \leq C_p \left( \int_0^t \sum_{i=1}^m \left\| \left< X_z - \overline{Y}_z^M, g_i(X_z) - g_i(\overline{Y}_z^M) \right> \right\|_{L^{p/2}(\Omega;\mathbb{R})}^2 dz \right)^{1/2}, \]

for all \( p \in [2, \infty) \).

Moreover, using Holder inequality, the inequality \( ab \leq \frac{a^2}{2} + \frac{b^2}{2} \) and the global Lipschitz condition
Using the inequality \( g \) satisfied by (5.3).

\[
B_1 \leq C_p \left( \int_0^t \sum_{i=1}^m \|X_z - \mathbf{Y}_z^M, g_i(X_z) - g_i(\mathbf{Y}_z)\|^2_{L^{p/2}(\Omega, \mathbb{R}^d)} \right)^{1/2} \\
\leq C_p \left( \int_0^t \sum_{i=1}^m \|X_z - \mathbf{Y}_z^M\|^2_{L^p(\Omega, \mathbb{R}^d)} \|g_i(X_z) - g_i(\mathbf{Y}_z)\|^2_{L^p(\Omega, \mathbb{R}^d)} \right)^{1/2} \\
\leq C_p \left( \frac{1}{2} \int_0^t \|X_z - \mathbf{Y}_z\|_{L^p(\Omega, \mathbb{R}^d)}^2 \|g_i(X_z) - g(\mathbf{Y}_z)\|_{L^p(\Omega, \mathbb{R}^d)}^2 \right)^{1/2} \\
\leq C_p \left( \frac{\sup_{s \in [0, t]} \|X_s - \mathbf{Y}_s\|^2_{L^p(\Omega, \mathbb{R}^d)} + C_p^3 \m 0 \|X_s - \mathbf{Y}_s\|^2_{L^p(\Omega, \mathbb{R}^d)} \right)^{1/2} \\
\leq C_p \left( \frac{1}{4} \sup_{s \in [0, t]} \|X_s - \mathbf{Y}_s\|^2_{L^p(\Omega, \mathbb{R}^d)} + \frac{1}{4} \sup_{s \in [0, t]} \|X_s - \mathbf{Y}_s\|^2_{L^p(\Omega, \mathbb{R}^d)} + 2C_p \|h(X_z) - h(\mathbf{Y}_z)\|^4_{L^p(\Omega, \mathbb{R}^d)} \right)^{1/2} \\
+ 2C_p \|h(X_z) - h(\mathbf{Y}_z)\|^4_{L^p(\Omega, \mathbb{R}^d)} dz. \tag{5.53}
\]

Using Lemma 4.2.38 and the inequality \((a + b)^4 \leq 4a^4 + 4b^4\), it follows that :

\[
B_2 = \left( \sup_{s \in [0, t]} \left[ \int_0^t \|X_z - \mathbf{Y}_z\|^4_{L^p(\Omega, \mathbb{R}^d)} dz \right] \right)^{1/2} \\
\leq C_p \left( \frac{1}{2} \int_0^t \|X_z - \mathbf{Y}_z\|^4_{L^p(\Omega, \mathbb{R}^d)} dz + 2C_p \left( \int_0^t \|h(X_z) - h(\mathbf{Y}_z)\|^4_{L^p(\Omega, \mathbb{R}^d)} dz \right)^{1/2} \right) \\
= B_{21} + B_{22} \tag{5.54}
\]

Using Holder inequality, it follows that :

\[
B_{21} := 2C_p \left( \int_0^t \|X_z - \mathbf{Y}_z\|^4_{L^p(\Omega, \mathbb{R}^d)} dz \right)^{1/2} \\
\leq 2C_p \left( \int_0^t \|X_z - \mathbf{Y}_z\|^4_{L^p(\Omega, \mathbb{R}^d)} \|X_z - \mathbf{Y}_z\|^4_{L^p(\Omega, \mathbb{R}^d)} dz \right)^{1/2} \\
= 2C_p \left( \int_0^t \frac{1}{16} \|X_z - \mathbf{Y}_z\|^2_{L^p(\Omega, \mathbb{R}^d)} 16 \|X_z - \mathbf{Y}_z\|^4_{L^p(\Omega, \mathbb{R}^d)} dz \right)^{1/2} \\
\leq \frac{1}{4} \sup_{s \in [0, t]} \|X_s - \mathbf{Y}_s\|^2_{L^p(\Omega, \mathbb{R}^d)} 8C_p \left( \int_0^t \|X_z - \mathbf{Y}_z\|^2_{L^p(\Omega, \mathbb{R}^d)} dz \right)^{1/2} . \tag{5.55}
\]

Using the inequality \(2ab \leq a^2 + b^2\) leads to :

\[
B_{21} \leq \frac{1}{16} \sup_{s \in [0, t]} \|X_s - \mathbf{Y}_s\|^2_{L^p(\Omega, \mathbb{R}^d)} + 16C_p \int_0^t \|X_z - \mathbf{Y}_z\|^2_{L^p(\Omega, \mathbb{R}^d)} dz. \tag{5.55}
\]
Using the inequalities \((a + b)^4 \le 4a^4 + 4b^4\) and \(\sqrt{a + b} \le \sqrt{a} + \sqrt{b}\), we have the following bound for \(B_{22}\)

\[
B_{22} := 2C_p \left( \int_0^t \| h(X_z) - h(Y_{[z]}^M) \|_{L^{p/2}([0,1])}^2 dz \right)^{1/2}
\]

\[
\le 2C_p \left( \int_0^t \left[ 4\| h(X_z) - h(Y_{[z]}^M) \|_{L^{p/2}([0,1])}^4 + \| h(Y_{[z]}^M) \|_{L^{p/2}([0,1])}^4 \right] dz \right)^{1/2}
\]

\[
\le 4C_p \left( \int_0^t \| h(X_z) - h(Y_{[z]}^M) \|_{L^{p/2}([0,1])}^4 dz \right)^{1/2}
\]

\[
+ 4C_p \left( \int_0^t \| h(Y_{[z]}^M) - h(Y_{[z]}^M) \|_{L^{p/2}([0,1])}^4 dz \right)^{1/2}
\].

Using the global Lipschitz condition satisfied by \(h\) leads to:

\[
B_{22} \le 4C_p \left( \int_0^t C^4 \| X_z - Y_{[z]}^M \|_{L^{p/2}([0,1])}^4 dz \right)^{1/2} + 4C_p \left( \int_0^t C^4 \| Y_{[z]}^M - Y_{[z]}^M \|_{L^{p/2}([0,1])}^4 dz \right)^{1/2}
\].

Using the same idea as for \(B_{21}\), it follows that:

\[
B_{22} \le \frac{1}{16} \sup_{s \in [0,t]} \| X_s - Y_s^M \|_{L^p([0,1])}^2 + 64C_p \int_0^t \| X_z - Y_{[z]}^M \|_{L^p([0,1])}^2 dz
\]

\[
+ \frac{1}{16} \sup_{s \in [0,t]} \| Y_{[z]}^M - Y_{[z]}^M \|_{L^p([0,1])}^2 + 64C_p \int_0^t \| Y_{[z]}^M - Y_{[z]}^M \|_{L^p([0,1])}^2 dz
\].

Taking the supremum under the integrand of the last term in the above inequality and using the fact that we don’t care about the value of the constant leads to:

\[
B_{22} \le \frac{1}{16} \sup_{s \in [0,t]} \| X_s - Y_s^M \|_{L^p([0,1])}^2 + 64C_p \int_0^t \| X_z - Y_{[z]}^M \|_{L^p([0,1])}^2 ds
\]

\[
+ C_p \sup_{s \in [0,t]} \| Y_{[z]}^M - Y_{[z]}^M \|_{L^p([0,1])}^2 \tag{5.56}
\].

Inserting (5.55) and (5.56) into (5.54) gives:

\[
B_2 \le \frac{1}{8} \sup_{s \in [0,t]} \| X_s - Y_s^M \|_{L^p([0,1])}^2 + C_p \int_0^t \| X_z - Y_{[z]}^M \|_{L^p([0,1])}^2 ds
\]

\[
+ C_p \sup_{s \in [0,t]} \| Y_{[z]}^M - Y_{[z]}^M \|_{L^p([0,1])}^2 \tag{5.57}
\].

Using again Lemma 4.2.38 leads to:

\[
B_3 := \left\| \sup_{s \in [0,t]} \int_0^s \| X_z - Y_{[z]}^M \|_{L^{p/2}([0,1])}^2 d\tilde{N}_z \right\|_{L^{p/2}([0,1])}^{1/2}
\]

\[
\le C_p \left( \int_0^t \| X_z - Y_{[z]}^M \|_{L^{p/2}([0,1])}^4 dz \right)^{1/2} \tag{5.58}
\].

Using the same argument as for \(B_{21}\), we obtain:

\[
B_3 \le \frac{1}{8} \sup_{s \in [0,t]} \| X_s - Y_s^M \|_{L^p([0,1])}^2 + C_p \int_0^t \| X_z - Y_{[z]}^M \|_{L^p([0,1])}^2 ds
\].

Taking the \(L^p\) norm in both side of (5.52) and inserting inequalities (5.53), (5.57) and (5.58) leads to:

\[
\left\| \sup_{s \in [0,t]} \| X_s - Y_s^M \|_{L^p([0,1])}^2 \right\|_{L^{p/2}([0,1])} = \left\| \sup_{s \in [0,t]} \| X_s - Y_s^M \|_{L^p([0,1])}^2 \right\|_{L^{p/2}([0,1])}
\]
\[ \begin{align*}
\int_0^t & \|X_s - \overline{Y}_s\|_{L^p(\Omega, \mathbb{R}^d)}^2 ds + C_p \int_0^t \|\overline{Y}_s - \overline{Y}_s\|_{L^p(\Omega, \mathbb{R}^d)}^2 ds \\
+ & \int_0^t \|v(X_s) - v(\overline{Y}_s)\|_{L^p(\Omega, \mathbb{R}^d)}^2 ds + C_p \sup_{s \in [0,t]} \|\overline{Y}_s - \overline{Y}_s\|_{L^p(\Omega, \mathbb{R}^d)}^2 \\
+ & \frac{T^2}{M^2} \int_0^t \|v(\overline{Y}_s)\|_{L^{2p}(\Omega, \mathbb{R}^d)}^4 ds + 2C_p \int_0^t \|\overline{Y}_s - \overline{Y}_s\|_{L^p(\Omega, \mathbb{R}^d)}^2 ds \\
+ & \frac{1}{2} \sup_{s \in [0,t]} \|X_s - \overline{Y}_s\|_{L^{p/2}(\Omega, \mathbb{R}^d)}^2,
\end{align*} \]

for all \( t \in [0,T] \) and all \( p \in [2, +\infty) \). Where \( C_p \) is the generic constant.

The previous inequality can be written in the following appropriate form

\[ \frac{1}{2} \left\| \sup_{s \in [0,t]} \|X_s - \overline{Y}_s\|_{L^p(\Omega, \mathbb{R}^d)} \right\|^2 \leq C_p \left( \int_0^T \|v(X_s) - v(\overline{Y}_s)\|_{L^p(\Omega, \mathbb{R}^d)}^2 ds + C_p \sup_{s \in [0,t]} \|\overline{Y}_s - \overline{Y}_s\|_{L^p(\Omega, \mathbb{R}^d)}^2 \\
+ \frac{T^2}{M^2} \int_0^T \|v(\overline{Y}_s)\|_{L^{2p}(\Omega, \mathbb{R}^d)}^4 ds + 2C_p \int_0^T \|\overline{Y}_s - \overline{Y}_s\|_{L^p(\Omega, \mathbb{R}^d)}^2 ds \right). \]

Applying Gronwall lemma to the previous inequality leads to :

\[ \frac{1}{2} \left\| \sup_{s \in [0,t]} \|X_s - \overline{Y}_s\|_{L^p(\Omega, \mathbb{R}^d)} \right\|^2 \leq C_p e^{C_p} \left( \int_0^T \|v(X_s) - v(\overline{Y}_s)\|_{L^p(\Omega, \mathbb{R}^d)}^2 ds + C_p \sup_{u \in [0,T]} \|\overline{Y}_u - \overline{Y}_u\|_{L^p(\Omega, \mathbb{R}^d)}^2 \\
+ \frac{T^2}{M^2} \int_0^T \|v(\overline{Y}_s)\|_{L^{2p}(\Omega, \mathbb{R}^d)}^4 ds + 2C_p \int_0^T \|\overline{Y}_s - \overline{Y}_s\|_{L^p(\Omega, \mathbb{R}^d)}^2 ds \right). \]

Using the inequality \( \sqrt{a + b + c} \leq \sqrt{a} + \sqrt{b} + \sqrt{c} \), it follows that

\[ \left\| \sup_{t \in [0,T]} \|X_t - \overline{Y}_t\|_{L^p(\Omega, \mathbb{R}^d)} \right\| \leq C_p e^{C_p} \left( \sup_{s \in [0,T]} \|v(\overline{Y}_s) - v(\overline{Y}_s)\|_{L^p(\Omega, \mathbb{R}^d)} + C_p \sup_{s \in [0,T]} \|\overline{Y}_s - \overline{Y}_s\|_{L^p(\Omega, \mathbb{R}^d)} \\
+ \frac{T^2}{M} \left[ \sup_{n \in \{0, \ldots, M\}} \|v(Y_n)\|_{L^{2p}(\Omega, \mathbb{R}^d)} \right] + C_p \sup_{s \in [0,T]} \|\overline{Y}_s - \overline{Y}_s\|_{L^p(\Omega, \mathbb{R}^d)} \right), \]

(5.59)

for all \( p \geq 2 \). Using Lemma 5.3.2 and Lemma 5.3.4, it follows that :

\[ \mathbb{E} \left[ \sup_{t \in [0,T]} \|X_t - \overline{Y}_t\|^p \right]^{1/p} \leq C_p (\Delta t)^{1/2}, \]

for all \( M \in \mathbb{N} \) and all \( p \in [2, +\infty) \). The application of Holder inequality shows that the latter inequality is satisfied for all \( p \in [1, \infty) \), and this complete the proof of Theorem 5.3.1.
5.4 Strong convergence of the tamed Euler Scheme

Theorem 5.4.1 Under Assumptions 5.2.2, for all \( p \in [1, +\infty) \) there exist a constant \( C_p > 0 \) such that

\[
\left( E \left[ \sup_{t \in [0,T]} \| X_t - \bar{X}^M_t \|^p \right] \right)^{1/p} \leq C_p \Delta t^{1/2},
\]

(5.60)

for all \( M \in \mathbb{N} \). Where \( X : [0,T] \times \Omega \rightarrow \mathbb{R}^d \) is the exact solution of (5.1) and \( \bar{X}^M_t \) the continuous interpolation of the numerical solution (5.5) defined by:

\[
\bar{X}^M_t = X^M_n + \frac{(t - n\Delta t)f(X^M_n)}{1 + \| f(X^M_n) \|} + g(X^M_n)(W_t - W_{n\Delta t}) + h(X^M_n)(N_t - N_{n\Delta t}),
\]

for all \( M \in \mathbb{N} \), all \( n \in \{0, \cdots, M\} \) and all \( t \in [n\Delta t, (n+1)\Delta t) \).

Proof 5.4.2 Using the relation \( \Delta \bar{N}_n^M = \Delta N_n^M - \lambda \Delta t \), the continuous interpolation of (5.5) can be express in the following form

\[
\bar{X}^M_t = X^M_n + \lambda(t - n\Delta t)h(X^M_n) + \frac{(t - n\Delta t)f(X^M_n)}{1 + \| f(X^M_n) \|} + g(X^M_n)(W_t - W_{n\Delta t}) + h(X^M_n)(N_t - N_{n\Delta t}),
\]

for all \( t \in [n\Delta t, (n+1)\Delta t] \).

From the numerical solution (5.5) and using the relation \( \Delta N_n^M = \Delta \bar{N}_n^M + \lambda \Delta t \), it follows that:

\[
X^M_{n+1} = X^M_n + \frac{\Delta tf(X^M_n)}{1 + \Delta t\| f(X^M_n) \|} + g(X^M_n)\Delta W_n^M + h(X^M_n)\Delta N_n^M + \lambda h(X^M_n)\Delta T/M + \frac{\Delta tf(X^M_n)}{1 + \Delta t\| f(X^M_n) \|} + g(X^M_n)\Delta W_n^M + h(X^M_n)\Delta \bar{N}_n^M.
\]

(5.61)

The functions \( \lambda h \) and \( f \) satisfy the same conditions as \( u_\lambda \) and \( v \) respectively. So from (5.5) it follows that the numerical solution (5.5) satisfied the same hypothesis as the numerical solution (5.7). Hence, it follows from Theorem 5.3.1 that there exist a constant \( C_p > 0 \) such that

\[
\left( E \left[ \sup_{t \in [0,T]} \| X_t - \bar{X}^M_t \|^p \right] \right)^{1/p} \leq C_p \Delta t^{1/2},
\]

(5.62)

for all \( p \in [1, \infty) \).

5.5 Linear mean-square stability

The goal of this section is to find a stepsize for which the tamed euler scheme and the semi-tamed Euler scheme are stable. The first approach to the stability analysis of a numerical method is to study the stability behavior of the method for a scalar linear equation. So we will focus in the linear case. Let’s consider a linear test equation with real and scalar coefficients.

\[
dX(t) = aX(t^-)dt + bX(t^-)dW(t) + cX(t^-)dN(t), \quad X(0) = X_0.
\]

(5.63)

It is proved in [6] that the exact solution of (5.63) is mean-square stable if and only if

\[
l := 2a + b^2 + \lambda c(2 + c) < 0.
\]

That is:

\[
\lim_{t \to \infty} |X(t)|^2 = 0 \iff l := 2a + b^2 + \lambda c(2 + c) < 0.
\]

(5.64)
Remark 5.5.1 In this section, to easy notations, the tamed Euler approximation $X^M$ will be replaced by $X$ and the semi-tamed Euler approximation $Y^M$ will be replaced by $Y$.

We have the following result for the numerical method (5.7).

Theorem 5.5.2 Under Assumption 5.64, the semi-tamed Euler (5.7) is mean-square stable if and only if

$$\Delta t < \frac{-l}{(a + \lambda c)^2}.$$  

Proof 5.5.3 Applying the semi-tamed Euler scheme to (5.63) leads to

$$Y_{n+1} = Y_n + aY_n \Delta t + \lambda cY_n \Delta t + bY_n \Delta W_n + cY_n \Delta N_n.$$  

(5.65)

Squaring both sides of (5.65) leads to

$$Y_{n+1}^2 = Y_n^2 + (a + \lambda c)^2 \Delta t^2 + b^2 \Delta W_n^2 + c^2 \Delta N_n^2 + 2(a + \lambda c)Y_n \Delta t + 2bY_n \Delta W_n$$

$$+ 2cY_n \Delta N_n + 2(b + \lambda c)Y_n \Delta t \Delta W_n + 2c(a + \lambda c)Y_n \Delta t \Delta N_n + 2bcY_n \Delta W_n \Delta N_n.$$  

(5.66)

Taking expectation in both sides of (5.66) and using the relations $E(\Delta W_n^2) = \Delta t$, $E(\Delta N_n^2) = \lambda \Delta t$ and $E(\Delta W_n) = E(\Delta N_n) = 0$ leads to

$$E[Y_{n+1}]^2 = (1 + (a + \lambda c)^2 \Delta t^2 + (b^2 + \lambda c^2 + 2a + 2\lambda c)\Delta t)E[Y_n]^2.$$  

So the numerical method is stable if and only if

$$1 + (a + \lambda c)^2 \Delta t^2 + (b^2 + \lambda c^2 + 2a + 2\lambda c)\Delta t < 1.$$  

That is if and only if

$$\Delta t < \frac{-l}{(a + \lambda c)^2}.$$  

Theorem 5.5.4 Under Assumption 5.64, the tamed Euler scheme (5.5) is mean-square stable if one of the following conditions is satisfied:

- $a(1 + \lambda c \Delta t) \leq 0$, $2a - l > 0$ and $\Delta t < \frac{2a - l}{a^2 + \lambda^2 c^2}$.
- $a(1 + \lambda c \Delta t) > 0$ and $\Delta t < \frac{-l}{(a + \lambda c)^2}$.

Proof 5.5.5 Applying the tamed Euler scheme (5.5) to equation (5.63) leads to:

$$X_{n+1} = X_n + \frac{aX_n \Delta t}{1 + \Delta t[aX_n]} + bX_n \Delta W_n + cX_n \Delta N_n.$$  

(5.67)

By squaring both sides of (5.67) leads to:

$$X_{n+1}^2 = X_n^2 + \frac{a^2 X_n^2 \Delta t^2}{(1 + \Delta t[aX_n])^2} + b^2 X_n^2 \Delta W_n^2 + c^2 X_n^2 \Delta N_n^2 + \frac{2aX_n^2 \Delta t}{1 + \Delta t[aX_n]} + 2bX_n^2 \Delta W_n$$

$$+ 2cX_n^2 \Delta N_n + \frac{2abX_n^2}{1 + \Delta t[aX_n]} \Delta W_n + \frac{2acX_n^2 \Delta t}{1 + \Delta t[aX_n]} \Delta N_n + 2bcX_n^2 \Delta W_n \Delta N_n.$$  

Using the inequality $\frac{a^2 \Delta t^2}{1 + \Delta t[aX_n]} \leq a^2 \Delta t^2$, the previous equality becomes

$$X_{n+1}^2 \leq X_n^2 + a^2 X_n \Delta t^2 + b^2 X_n \Delta W_n^2 + c^2 X_n \Delta N_n^2 + \frac{2aX_n^2 \Delta t}{1 + \Delta t[aX_n]} + 2bX_n^2 \Delta W_n$$

$$+ 2cX_n^2 \Delta N_n + \frac{2abX_n^2}{1 + \Delta t[aX_n]} \Delta W_n + \frac{2acX_n^2 \Delta t}{1 + \Delta t[aX_n]} \Delta N_n + 2bcX_n^2 \Delta W_n \Delta N_n.$$
Taking expectation in both sides of the previous equality and using independence and the fact that 
\( \mathbb{E}(\Delta W_n) = 0, \mathbb{E}(\Delta W_n^2) = \Delta t, \mathbb{E}(\Delta N_n) = \lambda \Delta t, \mathbb{E}(\Delta N_n^2) = \lambda \Delta t + \lambda^2 \Delta t^2 \) leads to:
\[
\mathbb{E}|X_{n+1}|^2 \leq \left[1 + a^2 \Delta t^2 + b^2 \Delta t + \lambda^2 c^2 \Delta t^2 + (2 + c)\lambda c \Delta t\right] \mathbb{E}|X_n|^2 + \mathbb{E}\left(\frac{2aX_n^2 \Delta t (1 + \lambda c \Delta t)}{1 + \Delta t |a X_n|}\right). \tag{5.68}
\]

- If \( a(1 + \lambda c \Delta t) \leq 0 \), it follows from (5.68) that
\[
\mathbb{E}|X_{n+1}|^2 \leq \left[1 + (a^2 + \lambda^2 c^2)\Delta t^2 + |b^2 + \lambda c(2 + c)|\Delta t\right] \mathbb{E}|X_n|^2.
\]
Therefore, the numerical solution is stable if
\[
1 + (a^2 + \lambda^2 c^2)\Delta t^2 + |b^2 + \lambda c(2 + c)|\Delta t < 1.
\]
That is \( \Delta t < \frac{2a - l}{a^2 + \lambda^2 c^2} \).

- If \( a(1 + \lambda c \Delta t) > 0 \), using the fact that \( \frac{2aX_n^2 \Delta t (1 + \lambda c \Delta t)}{1 + \Delta t |a X_n|} < 2aX_n^2 \Delta t (1 + \lambda c \Delta t) \), inequality (5.68) becomes
\[
\mathbb{E}|X_{n+1}|^2 \leq \left[1 + a^2 \Delta t^2 + b^2 \Delta t + \lambda^2 c^2 \Delta t^2 + 2\lambda ac \Delta t^2 + (2 + c)\lambda c \Delta t + 2a \Delta t\right] \mathbb{E}|X_n|^2. \tag{5.69}
\]
Therefore, it follows from (5.69) that the numerical solution is stable if
\[
1 + a^2 \Delta t^2 + b^2 \Delta t + \lambda^2 c^2 \Delta t^2 + 2\lambda ac \Delta t^2 + (2 + c)\lambda c \Delta t + 2a \Delta t < 1. \quad \text{That is} \quad \Delta t < \frac{-l}{(a + \lambda c)^2}.
\]

Remark 5.5.6 In Theorem 5.5.4, we can easily check that if \( l < 0 \), we have:
\[
\begin{align*}
\begin{cases}
 a(1 + \lambda c \Delta t) \leq 0, \\
 \Delta t < \frac{-2a - l}{a^2 + \lambda^2 c^2}
\end{cases} & \iff \begin{cases}
 a \in (l/2, 0], c \geq 0, \\
 \Delta t < \frac{2a - l}{a^2 + \lambda^2 c^2}
\end{cases} \quad \text{or} \quad \begin{cases}
 a \in (l/2, 0), c < 0, \\
 \Delta t < \frac{2a - l}{a^2 + \lambda^2 c^2}
\end{cases} \quad \text{or} \quad \begin{cases}
 a \in (l/2, 0), c < 0, \\
 \Delta t \leq \frac{-1}{\lambda c}
\end{cases}
\end{align*}
\] (5.70)

\[
\begin{align*}
\begin{cases}
 a(1 + \lambda c \Delta t) > 0, \\
 \Delta t < \frac{-2a - l}{(a + \lambda c)^2}
\end{cases} & \iff \begin{cases}
 a > 0, c \geq 0, \\
 \Delta t < \frac{-l}{(a + \lambda c)^2}
\end{cases} \quad \text{or} \quad \begin{cases}
 a > 0, c < 0, \\
 \Delta t < \frac{-l}{(a + \lambda c)^2} \wedge \frac{-1}{\lambda c}
\end{cases} \quad \text{or} \quad \begin{cases}
 a < 0, c < 0, \\
 \Delta t < \frac{-l}{(a + \lambda c)^2} \wedge \frac{-1}{\lambda c}
\end{cases} \quad \text{or} \quad \begin{cases}
 a < 0, c < 0, \\
 \Delta t > \frac{-1}{\lambda c}
\end{cases}
\end{align*}
\] (5.72)

5.6 Nonlinear mean-square stability

In this section, we focus on the mean-square stability of the approximation (5.6). It is proved in [6] that under the following conditions,
\[
\langle x - y, f(x) - f(y) \rangle \leq \mu||x - y||^2,
\]
\[
||g(x) - g(y)||^2 \leq \sigma||x - y||^2,
\]
\[
||h(x) - h(y)||^2 \leq \gamma||x - y||^2,
\]
where \( \mu, \sigma \) and \( \gamma \) are constants, the exact solution of SDE (2.1) is mean-square stable if
\[
\alpha := 2\mu + \sigma + \lambda \sqrt{\gamma(\sqrt{\gamma} + 2)} < 0.
\]
Remark 5.6.1 In this section, to easy notations, the tamed Euler approximation $X^M$ will be replaced by $X$ and the semi-tamed Euler approximation $Y^M$ will be replaced by $Y$.

Following the literature of [11], in order to examine the mean-square stability of the numerical solution given by (5.7), we assume that $f(0) = g(0) = h(0) = 0$. Also we make the following assumptions.

**Assumptions 5.6.2** There exist a positive constants $\rho$, $\beta$, $\theta$, $K$, $C$ and $a > 1$ such that

\[
\begin{aligned}
\langle x - y, u(x) - u(y) \rangle &\leq -\rho |x - y|^2, \\
\langle x - y, v(x) - v(y) \rangle &\leq -\beta |x - y|^{a+1}, \\
\langle g(x) - g(y) \rangle &\leq \theta |x - y|, \\
\langle h(x) - h(y) \rangle &\leq C|x - y|.
\end{aligned}
\]

We define $\alpha_1 = -2\rho + \theta^2 + \lambda C(2 + C)$.

**Theorem 5.6.3** Under Assumptions 5.6.2 and the further hypothesis $2\beta - \beta > 0$, for any stepsize $\Delta t < \frac{2\beta}{(K + \lambda C)^2} \wedge \frac{2(2 + \beta)}{2(K + \lambda C)^2}$, the numerical solution (5.7) is exponentially mean-square stable.

**Proof 5.6.4** The numerical solution (5.7) is given by

\[
Y_{n+1} = Y_n + \Delta t u_\lambda(Y_n) + \frac{\Delta tv(Y_n)}{1 + \Delta t \|v(Y_n)\|^a} + g(Y_n)\Delta W_n + h(Y_n)\Delta N_n,
\]

where $u_\lambda = u + \lambda h$.

Taking the inner product in both sides of the previous equation leads to

\[
\begin{aligned}
||Y_{n+1}||^2 &= ||Y_n||^2 + \Delta t^2 ||u_\lambda(Y_n)||^2 + \frac{\Delta t^2 |v(Y_n)|^2}{(1 + \Delta t \|v(Y_n)\|^a)^2} + ||g(Y_n)||^2 \Delta W_n^2 + ||h(Y_n)||^2 \Delta N_n^2 \\
&+ 2\Delta t \langle Y_n, u_\lambda(Y_n) \rangle + 2\Delta t \left\langle Y_n, \frac{v(Y_n)}{1 + \Delta t \|v(Y_n)\|^a} \right\rangle + 2\Delta t \langle g(Y_n), \Delta W_n \rangle + 2\Delta t \langle h(Y_n), \Delta N_n \rangle \\
&+ 2\Delta t \left\langle u_\lambda(Y_n), \frac{v(Y_n)}{1 + \Delta t \|v(Y_n)\|^a} \right\rangle + 2\Delta t \langle u_\lambda(Y_n), g(Y_n)\Delta W_n \rangle + 2\Delta t \langle u_\lambda(Y_n), h(Y_n)\Delta N_n \rangle \\
&+ 2\Delta t \left\langle g(Y_n), \Delta W_n \right\rangle + 2\Delta t \left\langle \frac{v(Y_n)}{1 + \Delta t \|v(Y_n)\|^a}, h(Y_n)\Delta N_n \right\rangle \\
&+ 2\langle g(Y_n)\Delta W_n, h(Y_n)\Delta N_n \rangle.
\end{aligned}
\]

Using Assumptions 5.6.2, it follows that:

\[
2\Delta t \left\langle Y_n, \frac{v(Y_n)}{1 + \Delta t \|v(Y_n)\|^a} \right\rangle \leq \frac{-2\beta \Delta t ||Y_n||^{a+1}}{1 + \Delta t \|v(Y_n)\|^a}
\]

\[
2\Delta t^2 \left\langle u_\lambda(Y_n), \frac{v(Y_n)}{1 + \Delta t \|v(Y_n)\|^a} \right\rangle \leq \frac{2\Delta t^2 ||u_\lambda(Y_n)|| \|v(Y_n)\|}{1 + \Delta t \|v(Y_n)\|^a} \leq \frac{2\Delta t^2 (K + \lambda C)\beta ||Y_n||^{a+1}}{1 + \Delta t \|v(Y_n)\|^a}.
\]

Let’s define $\Omega_n = \{\omega \in \Omega : ||Y_n|| > 1\}$.

- On $\Omega_n$ we have

\[
\frac{\Delta t^2 ||v(Y_n)||^2}{(1 + \Delta t \|v(Y_n)\|^a)^2} \leq \frac{\Delta t \|v(Y_n)\|}{1 + \Delta t \|v(Y_n)\|^a} \leq \frac{\beta \Delta t ||Y_n||^{a+1}}{1 + \Delta t \|v(Y_n)\|^a}.
\]

Therefore using (5.75), (5.76) and (5.77), equality (5.74) becomes:

\[
2\Delta t \left\langle Y_n, \frac{v(Y_n)}{1 + \Delta t \|v(Y_n)\|^a} \right\rangle \leq \frac{-2\beta \Delta t ||Y_n||^{a+1}}{1 + \Delta t \|v(Y_n)\|^a} \leq \frac{2\Delta t^2 (K + \lambda C)\beta ||Y_n||^{a+1}}{1 + \Delta t \|v(Y_n)\|^a}.
\]
The hypothesis $\Delta t < \frac{2\beta - \bar{\beta}}{2(K + \lambda C)} \bar{\beta}$ implies that $-2\beta \Delta t + 2(K + \lambda C)\bar{\beta} \Delta t^2 + \bar{\beta} \Delta t < 0$.  Therefore, (5.78) becomes

\[
\begin{align*}
||Y_{n+1}||^2 &\leq ||Y_n||^2 + \Delta t^2 ||u_\lambda(Y_n)||^2 + 2\Delta t < Y_n, u_\lambda(Y_n) > + ||g(Y_n)||^2 ||\Delta W_n||^2 + ||h(Y_n)||^2 ||\Delta \bar{N}_n||^2 \\
&+ 2\Delta t < Y_n, g(Y_n) \Delta W_n > + 2\Delta t < Y_n, h(Y_n) \Delta \bar{N}_n > \\
&+ 2\Delta t < u_\lambda(Y_n), g(Y_n) \Delta W_n > + 2\Delta t < u_\lambda(Y_n), h(Y_n) \Delta \bar{N}_n > \\
&+ 2\Delta t \left( \frac{v(Y_n)}{1 + \Delta t||v(Y_n)||}, g(Y_n) \Delta W_n \right) + 2\Delta t \left( \frac{v(Y_n)}{1 + \Delta t||v(Y_n)||}, h(Y_n) \Delta \bar{N}_n \right) \\
&+ 2(g(Y_n) \Delta W_n, h(Y_n) \Delta \bar{N}_n) \\
&+ \left[ -2\beta \Delta t + 2(K + \lambda C)\bar{\beta} \Delta t^2 + \bar{\beta} \Delta t \right] ||Y_n||^{a+1} \\
\end{align*}
\]

On $\Omega^*_n$ we have

\[
\begin{align*}
\frac{\Delta t^2 ||v(Y_n)||^2}{(1 + \Delta t||v(Y_n)||)^2} &\leq \frac{\Delta t^2 ||v(Y_n)||^2}{1 + \Delta t||v(Y_n)||} \leq \frac{\bar{\beta}^2 \Delta t^2 ||Y_n||^{2a}}{1 + \Delta t||v(Y_n)||} \leq \frac{\bar{\beta}^2 \Delta t^2 ||Y_n||^{a+1}}{1 + \Delta t||v(Y_n)||}.
\end{align*}
\]

Therefore, using (5.75), (5.76) and (5.80), equality (5.74) becomes

\[
\begin{align*}
||Y_{n+1}||^2 &\leq ||Y_n||^2 + \Delta t^2 ||u_\lambda(Y_n)||^2 + 2\Delta t < Y_n, u_\lambda(Y_n) > + ||g(Y_n)||^2 ||\Delta W_n||^2 \\
&+ ||h(Y_n)||^2 ||\Delta \bar{N}_n||^2 + 2\Delta t < Y_n, g(Y_n) \Delta W_n > + 2\Delta t < Y_n, h(Y_n) \Delta \bar{N}_n > \\
&+ 2\Delta t < u_\lambda(Y_n), g(Y_n) \Delta W_n > + 2\Delta t < u_\lambda(Y_n), h(Y_n) \Delta \bar{N}_n > \\
&+ 2\Delta t \left( \frac{v(Y_n)}{1 + \Delta t||v(Y_n)||}, g(Y_n) \Delta W_n \right) + 2\Delta t \left( \frac{v(Y_n)}{1 + \Delta t||v(Y_n)||}, h(Y_n) \Delta \bar{N}_n \right) \\
&+ 2\Delta t \left( \frac{v(Y_n)}{1 + \Delta t||v(Y_n)||}, g(Y_n) \Delta W_n \right) + 2\Delta t \left( \frac{v(Y_n)}{1 + \Delta t||v(Y_n)||}, h(Y_n) \Delta \bar{N}_n \right) \\
&+ 2(g(Y_n) \Delta W_n, h(Y_n) \Delta \bar{N}_n) \\
&+ \left[ -2\beta \Delta t + 2(K + \lambda C)\bar{\beta} \Delta t^2 + \bar{\beta}^2 \Delta t^2 \right] ||Y_n||^{a+1} \\
\end{align*}
\]

The hypothesis $\Delta t < \frac{2\beta}{2(K + \lambda C) + \bar{\beta} \beta}$ implies that $-2\beta \Delta t + 2(K + \lambda C)\bar{\beta} \Delta t^2 + \bar{\beta} \Delta t < 0$.  Therefore, (5.81) becomes

\[
\begin{align*}
||Y_{n+1}||^2 &\leq ||Y_n||^2 + \Delta t^2 ||u_\lambda(Y_n)||^2 + 2\Delta t < Y_n, u_\lambda(Y_n) > + ||g(Y_n)||^2 ||\Delta W_n||^2 \\
&+ ||h(Y_n)||^2 ||\Delta \bar{N}_n||^2 + 2\Delta t < Y_n, g(Y_n) \Delta W_n > + 2\Delta t < Y_n, h(Y_n) \Delta \bar{N}_n > \\
&+ 2\Delta t < u_\lambda(Y_n), g(Y_n) \Delta W_n > + 2\Delta t < u_\lambda(Y_n), h(Y_n) \Delta \bar{N}_n > \\
&+ 2\Delta t \left( \frac{v(Y_n)}{1 + \Delta t||v(Y_n)||}, g(Y_n) \Delta W_n \right) + 2\Delta t \left( \frac{v(Y_n)}{1 + \Delta t||v(Y_n)||}, h(Y_n) \Delta \bar{N}_n \right) \\
&+ 2(g(Y_n) \Delta W_n, h(Y_n) \Delta \bar{N}_n) \\
\end{align*}
\]
Finally, from the discussion above on $\Omega_n$ and $\Omega_n^c$, it follows that on $\Omega$, the following inequality holds for all $\Delta t < \frac{2\beta}{2(K + \lambda C) + \beta} \wedge \frac{2\beta - \beta^*}{2(K + \lambda C)^2}$

\[
\begin{align*}
||Y_{n+1}||^2 &\leq ||Y_n||^2 + \Delta t^2||\lambda(Y_n)||^2 + 2\Delta t < Y_n, u_\lambda(Y_n) > + ||g(Y_n)||^2||\Delta W_n||^2 \\
&+ ||h(Y_n)||^2 ||\Delta N_n||^2 + 2(Y_n, g(Y_n)\Delta W_n) + 2(Y_n, h(Y_n)\Delta \overline{N}_n) \\
&+ 2\Delta t\langle u_\lambda(Y_n), g(Y_n)\Delta W_n \rangle + 2\Delta t\langle u_\lambda(Y_n), h(Y_n)\Delta \overline{N}_n \rangle \\
&+ 2\langle g(Y_n)\Delta W_n, h(Y_n)\Delta \overline{N}_n \rangle.
\end{align*}
\]

(5.83)

Taking expectation in both sides of (5.83) and using the martingale properties of $\Delta W_n$ and $\Delta \overline{N}_n$ leads to:

\[
E||Y_{n+1}||^2 \leq E||Y_n||^2 + \Delta t^2 E||\lambda(Y_n)||^2 + 2\Delta t E\langle Y_n, u_\lambda(Y_n) \rangle + \Delta t E||g(Y_n)||^2 \\
+ \lambda \Delta t E||h(Y_n)||^2.
\]

(5.84)

From Assumptions 5.6.2, we have

\[
||\lambda(Y_n)||^2 \leq (K + \lambda C)^2 ||Y_n||^2 \quad \text{and} \quad \langle Y_n, u_\lambda(Y_n) \rangle \leq (-\rho + \lambda C)||Y_n||^2.
\]

So inequality (5.84) gives

\[
E||Y_{n+1}||^2 \leq E||Y_n||^2 + (K + \lambda C)^2 \Delta t^2 E||Y_n||^2 + 2(-\rho + \lambda C)\Delta t E||Y_n||^2 + \theta^2 \Delta t E||Y_n||^2 \\
+ \lambda^2 \Delta t E||Y_n||^2 \\
= \left[1 - 2\rho \Delta t + (K + \lambda C)^2 \Delta t^2 + 2\lambda C \Delta t + \theta^2 \Delta t + \lambda C^2 \Delta t\right] E||Y_n||^2.
\]

Iterating the previous inequality leads to

\[
E||Y_n||^2 \leq \left[1 - 2\rho \Delta t + (K + \lambda C)^2 \Delta t^2 + 2\lambda C \Delta t + \theta^2 \Delta t + \lambda C^2 \Delta t\right]^n E||Y_0||^2.
\]

In order to have stability, we impose:

\[
1 - 2\rho \Delta t + (K + \lambda C)^2 \Delta t^2 + 2\lambda C \Delta t + \theta^2 \Delta t + \lambda C^2 \Delta t < 1.
\]

That is

\[
\Delta t < \frac{-[-2\rho - \theta^2 + \lambda C(2 + C)]}{(K + \lambda C)^2} = \frac{-\alpha_1}{(K + \lambda C)^2}.
\]

(5.85)

In the following, we analyse the mean-square stability of the tamed Euler. We make the following assumptions which are essentially consequences of Assumptions 5.6.2.

**Assumptions 5.6.5** we assume that there exists positive constants $\beta$, $\beta^*$, $\theta$, $\mu$, $K$, $C$, $\rho$, and $a > 1$ such that:

\[
\langle x - y, f(x) - f(y) \rangle \leq -\rho||x - y||^2 - \beta||x - y||^a + 1,
\]

\[
||f(x)|| \leq \beta||x||^a + K||x||,
\]

\[
||g(x) - g(y)|| \leq \theta||x - y||,
\]

\[
||h(x) - h(y)|| \leq C||x - y||,
\]

\[
\langle x - y, h(x) - h(y) \rangle \leq -\mu||x - y||^2.
\]

(5.86)

**Remark 5.6.6** Apart from (5.86), Assumption 5.6.5 is a consequence of Assumption 5.6.2.
Theorem 5.6.7  Under Assumptions 5.6.5 and the further hypothesis $\beta - C\beta > 0$, $\beta (1 + 2C) - 2\beta < 0$, $K + \theta^2 + \lambda C^2 - 2\mu \lambda + 2\lambda CK < 0$, the numerical solution (5.5) is mean-square stable for any stepsize $\Delta t < \frac{|K + \theta^2 + \lambda C^2 - 2\mu \lambda + 2\lambda CK|}{2K^2 + \lambda^2 C^2} \wedge \beta - \frac{1}{\beta^2}$.

Proof 5.6.8  From equation (4.5), we have
\begin{align*}
||X_{n+1}||^2 & = ||X_n||^2 + \frac{\Delta t^2}{(1 + \Delta t||f(X_n)||)} ||f(X_n)||^2 + ||g(X_n)\Delta W_n||^2 + ||h(X_n)\Delta N_n||^2 \\
+ 2 \left< X_n, \frac{\Delta f(X_n)}{1 + \Delta t||f(X_n)||} \right> + 2 \left< X_n + \frac{\Delta f(X_n)}{1 + \Delta t||f(X_n)||}, g(X_n)\Delta W_n \right> \\
+ 2 \left< X_n + \frac{\Delta f(X_n)}{1 + \Delta t||f(X_n)||}, h(X_n)\Delta N_n \right> + 2 \left< g(X_n)\Delta W_n, h(X_n)\Delta N_n \right>. \quad (5.87)
\end{align*}

Using assumptions 5.6.5, it follows that :
\begin{align*}
2 \left< X_n, \frac{\Delta f(X_n)}{1 + \Delta t||f(X_n)||} \right> & \leq -2\Delta t|\beta||X_n||^2 + \frac{2\beta \Delta t||X_n||^2}{1 + \Delta t||f(X_n)||} \leq -2\beta \Delta t||X_n||^2 + \frac{2\beta \Delta t||X_n||^2}{1 + \Delta t||f(X_n)||}, \\
||g(X_n)\Delta W_n||^2 & \leq \theta^2||X_n||^2||\Delta W_n||^2 \quad \text{and} \quad ||h(X_n)\Delta N_n||^2 \leq C^2||X_n||^2||\Delta N_n||^2.
\end{align*}

So from Assumptions 5.6.5, we have
\begin{align*}
\left\{ \begin{array}{l}
\left< X_n, \frac{\Delta f(X_n)}{1 + \Delta t||f(X_n)||} \right> \leq \frac{2\beta \Delta t||X_n||^2}{1 + \Delta t||f(X_n)||} \\
||g(X_n)\Delta W_n||^2 \leq \theta^2||X_n||^2||\Delta W_n||^2 \\
||h(X_n)\Delta N_n||^2 \leq C^2||X_n||^2||\Delta N_n||^2 \\
2 \left< Y_n, h(X_n)\Delta N_n \right> \leq -2\mu||X_n||^2||\Delta N_n|| \\
2 \left< \frac{\Delta f(X_n)}{1 + \Delta t||f(X_n)||}, h(X_n)\Delta N_n \right> \leq \frac{2\Delta tC^2||X_n||^2||\Delta N_n||}{1 + \Delta t||f(X_n)||} + 2CK||X_n||^2||\Delta N_n||
\end{array} \right. \quad (5.88)
\end{align*}

Let's define $\Omega_n := \{ w \in \Omega : ||X_n(\omega)|| > 1 \}$.

- On $\Omega_n$ we have :
\begin{align*}
\frac{\Delta t^2||f(X_n)||^2}{(1 + \Delta t||f(X_n)||)^2} & \leq \frac{\Delta t||f(X_n)||}{1 + \Delta t||f(X_n)||} \leq \frac{\Delta t||X_n||^2}{1 + \Delta t||f(X_n)||} + K\Delta t||X_n|| \\
& \leq \frac{\Delta t\beta||X_n||^2}{1 + \Delta t||f(X_n)||} + K\Delta t||X_n||^2. \quad (5.89)
\end{align*}

Therefore using (5.88) and (5.89), equality (5.87) becomes
\begin{align*}
||X_{n+1}||^2 & \leq ||X_n||^2 + K\Delta t||X_n||^2 + \theta^2||X_n||^2||\Delta W_n||^2 + C^2||X_n||^2||\Delta N_n||^2 \\
+ 2 \left< X_n + \frac{\Delta f(X_n)}{1 + \Delta t||f(X_n)||}, g(X_n)\Delta W_n \right> - 2\mu||X_n||^2||\Delta N_n|| + 2CK||X_n||^2||\Delta N_n|| \\
+ 2 \left< g(X_n)\Delta W_n, h(X_n)\Delta N_n \right> + [\frac{-2\beta \Delta t + \beta \Delta t + 2\beta CK\Delta t}{1 + \Delta t||f(X_n)||}]||X_n||^2 + 1. \quad (5.90)
\end{align*}
Using the hypothesis $\beta(1+2C) - 2\beta < 0$, (5.90) becomes
\[
||X_{n+1}||^2 \leq ||X_n||^2 + K\Delta t||X_n||^2 + \theta^2||X_n||^2||\Delta W_n||^2 + C^2||X_n||^2||\Delta N_n||^2
+ 2\left( X_n + \frac{\Delta tf(X_n)}{1 + \Delta t||f(X_n)||}, g(X_n)\Delta W_n \right) - 2\mu||X_n||^2||\Delta N_n|| + 2CK||\Delta N_n||
+ 2\langle g(X_n)\Delta W_n, h(X_n)\Delta N_n \rangle. \tag{5.91}
\]

• On $\Omega'_n$, we have:
\[
\frac{\Delta t^2||f(X_n)||^2}{(1 + \Delta t||f(X_n)||)^2} \leq \frac{\Delta t^2||f(X_n)||^2}{1 + \Delta t||f(X_n)||} \leq \frac{2\Delta t^2\beta^2||X_n||^{2n}}{1 + \Delta t||f(X_n)||} + 2K^2\Delta t^2||X_n||^2 \tag{5.92}
\]
\[
\leq \frac{2\Delta t^2\beta^2||X_n||^{n+1}}{1 + \Delta t||f(X_n)||} + 2K^2\Delta t^2||X_n||^2. \tag{5.93}
\]

Therefore, using (5.88), and (5.93), (5.87) becomes
\[
||X_{n+1}||^2 \leq ||X_n||^2 + 2K^2\Delta t^2||X_n||^2 + \theta^2||X_n||^2||\Delta W_n||^2 + C^2||X_n||^2||\Delta N_n||^2
+ 2\left( X_n + \frac{\Delta tf(X_n)}{1 + \Delta t||f(X_n)||}, g(X_n)\Delta W_n \right) - 2\mu||X_n||^2||\Delta N_n|| + 2CK||\Delta N_n||
+ 2\langle g(X_n)\Delta W_n, h(X_n)\Delta N_n \rangle + \frac{[2C\beta\Delta t - 2\beta\Delta t + 2\beta^2\Delta t^2]||X_n||^{n+1}}{1 + \Delta t||f(X_n)||}. \tag{5.94}
\]

The hypothesis $\Delta t < \frac{\beta - C\beta}{\beta^2}$ implies that $2C\beta\Delta t - 2\beta\Delta t + 2\beta^2\Delta t^2 < 0$. Therefore, (5.94) becomes
\[
||X_{n+1}||^2 \leq ||X_n||^2 + 2K^2\Delta t^2||X_n||^2 + \theta^2||X_n||^2||\Delta W_n||^2 + C^2||X_n||^2||\Delta N_n||^2
+ 2\left( X_n + \frac{\Delta tf(X_n)}{1 + \Delta t||f(X_n)||}, g(X_n)\Delta W_n \right) - 2\mu||X_n||^2||\Delta N_n|| + 2CK||\Delta N_n||
+ 2\langle g(X_n)\Delta W_n, h(X_n)\Delta N_n \rangle. \tag{5.95}
\]

From the above discussion on $\Omega_n$ and $\Omega'_n$, the following inequality holds on $\Omega$ for all $\Delta t < \frac{\beta - C\beta}{\beta^2}$ and $\beta(1+2C) - \beta < 0$
\[
||X_{n+1}||^2 \leq ||X_n||^2 + K\Delta t||X_n||^2 + 2K^2\Delta t^2||X_n||^2 + \theta^2||X_n||^2||\Delta W_n||^2 + C^2||X_n||^2||\Delta N_n||^2
+ 2\langle X_n + \frac{\Delta tf(X_n)}{1 + \Delta t||f(X_n)||}, g(X_n)\Delta W_n \rangle - 2\mu||X_n||^2||\Delta N_n|| + 2CK||\Delta N_n||
+ 2\langle g(X_n)\Delta W_n, h(X_n)\Delta N_n \rangle. \tag{5.96}
\]

Taking Expectation in both sides of (5.96), using the relation $\mathbb{E}||\Delta W_n|| = 0$, $\mathbb{E}||\Delta W_n||^2 = \Delta t$, $\mathbb{E}||\Delta N_n|| = \lambda\Delta t$ and $\mathbb{E}||\Delta N_n||^2 = \lambda^2\Delta t^2 + 2\lambda\Delta t$ leads to:
\[
\mathbb{E}||X_{n+1}||^2 \leq \mathbb{E}||X_n||^2 + 2K^2\Delta t^2\mathbb{E}||X_n||^2 + \theta^2\mathbb{E}||X_n||^2||\Delta W_n||^2 + \lambda^2C^2\Delta t^2\mathbb{E}||X_n||^2 + \lambda\mathbb{E}||\Delta t||\mathbb{E}||X_n||^2
- 2\mu\lambda\mathbb{E}||X_n||^2 + \lambda\mathbb{E}||\Delta N_n||^2
= \left[ 1 + (2K^2 + \lambda^2C^2)\Delta t^2 + (K + \theta^2 + \lambda\mathbb{E}||X_n||^2 + 2\mu\lambda + 2\lambda\mathbb{E}||\Delta N_n||^2 \right] \mathbb{E}||X_n||^2.
\]

Iterating the last inequality leads to
\[
\mathbb{E}||X_n||^2 \leq \left[ 1 + (2K^2 + \lambda^2C^2)\Delta t^2 + (K + \theta^2 + \lambda\mathbb{E}||X_n||^2 + 2\mu\lambda + 2\lambda\mathbb{E}||\Delta N_n||^2 \right] \mathbb{E}||X_0||^2.
\]

In order to have stability, we impose
\[
1 + (2K^2 + \lambda^2C^2)\Delta t^2 + (K + \theta^2 + \lambda\mathbb{E}||X_n||^2 + 2\mu\lambda + 2\lambda\mathbb{E}||\Delta N_n||^2 \Delta t < 1.
\]

That is
\[
\Delta t < \frac{-[K + \theta^2 + \lambda\mathbb{E}||X_n||^2 + 2\mu\lambda + 2\lambda\mathbb{E}||\Delta N_n||^2]}{2K^2 + \lambda^2C^2}.
\]
5.7 Numerical Experiments

In this section, we present some numerical experiments that illustrate our theoretical strong convergence and stability results. For the strong convergence illustration of (5.5) and (5.7), let’s consider the stochastic differential equation
\[
dX_t = (-4X_t - X_t^3)dt + X_t dW_t + X_t dN_t,
\]
with initial \( X_0 = 1 \). \( N \) is the scalar Poisson process with parameter \( \lambda = 1 \). Here \( u(x) = -4x \), \( v(x) = -x^3 \), \( g(x) = h(x) = x \). It is easy to check that \( u, v, g \) and \( h \) satisfy the Assumptions 5.2.2.

For the illustration of the linear mean-square stability, we consider a linear test equation
\[
\begin{align*}
\frac{dX(t)}{dt} &= aX(t^-) + bX(t^-)dW(t) + cX(t^-)dN(t) \\
X(0) &= 1
\end{align*}
\]
We consider the particular case \( a = -1 \), \( b = 2 \), \( c = -0.9 \) and \( \lambda = 9 \). In this case \( l = -0.91 \), \( \frac{-l}{(a + \lambda c)^2} < 0.084 \) and \( \frac{2a - l}{a^2 + \lambda c^2} < 0.074 \). \( a(1 + \lambda c\Delta t) > 0 \) for \( \Delta t < 0.124 \). We test the stability behaviour of semi-tamed and of tamed Euler for different step-size, \( \Delta t = 0.02, 0.05 \) and \( 0.08 \). We use \( 7 \times 10^3 \) sample paths. For all step-size \( \Delta t < 0.083 \) the semi-tamed Euler is stable. But for the step-size \( \Delta t = 0.08 > 0.074 \), the tamed Euler scheme is unstable while the semi-tamed Euler scheme is stable. So the semi-tamed Euler scheme works better than the tamed Euler scheme.

![Mean square error for the tamed Euler scheme](image)

**Figure 5.1: Error of the tamed Euler scheme**
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Figure 5.2: Error of the semi-tamed Euler scheme

Figure 5.3: Stability tamed Euler
Figure 5.4: Stability semi-tamed Euler

Figure 5.5: Stability tamed Euler
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Figure 5.6: Stability semi-tamed Euler

Figure 5.7: Stability tamed Euler
Figure 5.8: Stability semi-tamed Euler
Conclusion

In this thesis, we provided an overview in probability theory which allowed us to define some basic concepts in stochastic process. Under global Lipschitz condition, we proved the existence and the uniqueness of solution of stochastic differential equation (SDEs) with jumps. In general, it is difficult to find the exact solution of most SDEs. One tool to approach the exact solution is the numerical resolution. We provided in this dissertation some numerical techniques to solve SDEs with jumps. More precisely, we investigated the strong convergence of the compensated stochastic theta method (CSTM) under global Lipschitz condition. We investigated the stability of both CSTM and stochastic theta method (STM) and we proved that for the linear test equation, when $\frac{1}{2} \leq \theta \leq 1$ the CSTM is A-stable while the STM is not. So CSTM works better than STM.

Under non-global Lipschitz condition Euler explicit method fails to converge strongly while Euler implicit method converges strongly but requires much computational efforts. We extended the tamed Euler scheme by introducing the compensated tamed Euler scheme for SDEs with jumps, which converges strongly with standard order 0.5 to the exact solution. We also extended the semi-tamed Euler scheme proposed in [11] for SDEs with jumps under non-global Lipschitz condition and proved his strong convergence. This latter enable us define the tamed Euler scheme and to prove his strong convergence which was not yet done in the literature. In this thesis, we also analysis the stability behaviours of both tamed and semi-tamed Euler schemes in the linear and the nonlinear case. We proved that these two numerical schemes reproduce the exponentially mean-square property of the exact solution.

All the numerical scheme presented in this work are of rate of convergence 0.5. The tamed Milstein scheme was introduced in [12], where the authors proved the strong convergence of order 1 of this scheme for SDEs without jumps. The case with jumps is not yet well developed in the literature. The weak convergence under non-global Lipschitz condition is not yet investigated. In the future, We would like to focus on the opened topics mentioned above.
Appendix

The goal of this section is to present some Scilab codes for simulations.

A.1 Code for simulation of the mean square error

```scilab
lambda = 1; Xzero = 1;
T = 1; N = 2^(14); dt = T/N; a=1, b=1, c=0.5, theta=1, A=a+lambda*c
M = 5000;
Xerr = zeros(M,5);
for s = 1:M,
    dW = sqrt(dt)*grand(1,N,'nor',0,1)
    W = cumsum(dW);
    dN=grand(1,N,'poi',dt*lambda)-dt*lambda*ones(1,N);
    W(1)=0
    dP=dN+dt*lambda*ones(1,N);
    P=cumsum(dP,'c');
    P(1)=0;
    X=linspace(0,1,N);
    Xtrue=exp((a-1/2*b^2)*X+b*W+log(1+c)*P);
    for p = 1:5
        R = 2^(p-1); Dt = R*dt; L = N/R;
        Xtemp = Xzero;
        for j = 1:L
            Winc = sum(dW(R*(j-1)+1:R*j));
            Ninc=sum(dN(R*(j-1)+1:R*j));
            Xtemp=(Xtemp+(1-theta)*A*Xtemp*Dt+b*Xtemp*Winc+c*Xtemp*Ninc)/(1-A*theta*Dt);
        end
        Xerr(s,p) = abs(Xtemp - Xtrue(N))^2;
    end
end
Dtvals = dt*(2.^([0:4]));
T=mean(Xerr,'r')^(1/2);
disp(T);
plot2d('ll',Dtvals,T,[5])
plot2d('ll',Dtvals,Dtvals^(1/2),[3])
legends([ 'mean square error for CSTM for theta=1','Reference line' ],[5,3,2],2)
xtitle("Mean square stability for CSTM");
xlabel("Time")
ylabel("E|Y_L-X(T)|^0.5")
```

A.2 Code for the simulation of the mean square stability

```scilab
T=2500, M=5000, Xzero=1,
a=-7,b=1, c=1, lambda=4, Dt=25,
```
A=a+lambda*c // drift coefficient for compensated equation
N=T/Dt;
theta=[0.4995, 0.50, 0.51]; // different value for theta
Xms=zeros(3,N); // initialisation of the mean
for i=1:3
Xtemp=Xzero*ones(M,1); // initialization of the solution
for j=1:N
Winc=grand(M,1, 'nor', 0, sqrt(Dt)); // generation of random variables following the normal distribution
Ninc=grand(M,1, 'poi', Dt*lambda)-Dt*lambda*ones(M,1); // generation of compensated poisson process
B=1-theta(i)*A*Dt
Xtemp=(Xtemp+(1-theta(i))*A*Xtemp*Dt+b*Xtemp.*Winc+c*Xtemp.*Ninc)/B;
Xms(i,j)=mean(Xtemp^2);
end
end
X=linspace(0,T,N);
plot(X,Xms(1,:), 'b', X,Xms(2,:), 'r', X,Xms(3,:), 'g')
legends(['Theta=0.499' 'theta=0.50' 'Theta=0.51'], [2,5,3],2)
xtitle("Mean-square stability for CSTM");
xlabel("tn")
ylabel("E|Yn|^2")
Acknowledgements

This dissertation would not have been possible without the guidance and the help of several individuals who in one way or another contributed and extended their valuable assistance in the preparation and completion of this study.

I would like to take this opportunity to express my gratitude to my supervisor Dr. Antoine Tambue to give me the chance to work with him; for his availability, his time to guide my researches, for sincerity, kindness, patience, understanding and encouragement that I will never forget. I am very happy for the way you introduced me in this nice topic which gave us two preprints papers.

I would also like to thank Dr. Moustapha Sene, my tutor for the time he took to read this work and give a valuable suggestions.

Special thanks to Neil Turok for his wonderful initiative. Thanks to all staff of AIMS-Senegal, tutors and my classmates. I would like to express my gratitude to all lecturers who taught me at AIMS-Senegal, I would like to mention here Pr. Dr. Peter Stollmann. I would like to extend my thanks to the Ph.D. students at the chair of Mathematics of AIMS-senegal for their moral support during my stay in Senegal.

Many thanks to all lecturers in my country (Cameroon) who have being training me since my undergraduate.

Big thanks to my family, to my friends and to everyone who supported me during my studies.
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