Classification of linear operators satisfying
\((Au, v) = (u, A^r v)\) or \((Au, A^r v) = (u, v)\) on a vector
space with indefinite scalar product

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Abstract
We classify all linear operators \(A : V \to V\) satisfying \((Au, v) = (u, A^r v)\) and all
linear operators satisfying \((Au, A^r v) = (u, v)\) with \(r = 2, 3, \ldots\) on a complex,
real, or quaternion vector space with scalar product given by a nonsingular
symmetric, skew-symmetric, Hermitian, or skew-Hermitian form.

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1. Introduction

Let \(F\) be \(\mathbb{C}\), \(\mathbb{R}\), or the skew field of quaternions \(\mathbb{H}\). Let \(V\) be a finite
dimensional right vector space over \(F\) with scalar product given by a nonsingular form \(\mathcal{F} : V \times V \to F\) that is symmetric or skew-symmetric if \(F \in \{\mathbb{C}, \mathbb{R}\}\),
and Hermitian or skew-Hermitian if \(F \in \{\mathbb{C}, \mathbb{H}\}\). Let \(r \in \{1, 2, \ldots\}\). A linear
operator \(A : V \to V\) is \(r\)-selfadjoint if

\[ \mathcal{F}(Au, v) = \mathcal{F}(u, A^r v) \quad \text{for all } u, v \in V; \]

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\( A : V \rightarrow V \) is \( r \)-unitary if it is nonsingular and
\[
\mathcal{F}(Au, A^r v) = \mathcal{F}(u, v) \quad \text{for all } u, v \in V.
\]

The 1-selfadjoint operators are selfadjoint operators on spaces with indefinite scalar product; their classification is given in [2, 6, 7, 11, 17, 21, 22]. The 1-unitary operators are unitary operators on spaces with indefinite scalar product; their classification is given in [1, 6, 7, 11, 20, 22, 24].

We give canonical matrices of \( r \)-selfadjoint operators and \( r \)-unitary operators for \( r \geq 2 \). We use the method developed in [22], which reduces the problem of classifying systems of forms and linear mappings to the problem of classifying systems of linear mappings. This method allows to consider the problems of classifying \( r \)-selfadjoint operators and \( r \)-unitary operators as the same classification problem.

Later on, we use the term “\((-r)\)-selfadjoint operators” instead of “\(r\)-unitary operators” and solve the problem of classifying \( r \)-selfadjoint operators for each \( r \in \mathbb{Z} \setminus \{-1,0,1\} \). In matrix form, this problem is formulated as follows: we consider pairs \((A,F)\) of \( n \times n \) matrices over \( \mathbb{C} \) or \( \mathbb{R} \) satisfying
\[
A^T F = FA^r, \quad F^T = F \quad \text{is nonsingular} \tag{1}
\]
and give their canonical form with respect to transformations
\[
(A,F) \mapsto (S^{-1}AS, S^T FS), \quad S \text{ is nonsingular}; \tag{2}
\]
we also consider matrix pairs \((A,F)\) over \( \mathbb{C} \) or \( \mathbb{H} \) satisfying
\[
A^* F = FA^r, \quad F^* = F \quad \text{is nonsingular} \tag{3}
\]
and give their canonical form with respect to transformations
\[
(A,F) \mapsto (S^{-1}AS, S^* FS), \quad S \text{ is nonsingular} \tag{4}
\]
(A is nonsingular if \( r < 0 \), and \( S^* := \overline{S^T} \)).

This research was inspired by the articles [3, 4, 5, 12, 13, 14, 15, 16], in which Catral, Lebtahi, Romero, Stuart, Thome, and Weaver study \( \{R, s+1, k\}\)-potent (respectively, \( \{R, s+1, k, *\}\)-potent) matrices; i.e., those matrices \( A \in \mathbb{C}^{n \times n} \) that satisfy \( RA = A^{s+1}R \) (respectively, \( RA^* = A^{s+1}R \)), in which \( R \in \mathbb{C}^{n \times n} \) is a given matrix satisfying \( R^k = 1 \) and \( s, k \) are positive integers; compare with [1] and [3].
Each sesquilinear form $\mathcal{F} : V \times V \to \mathbb{F}$ that we consider is semilinear in the first argument and linear in the second; $\mathcal{F} : V \to V$ is skew-Hermitian if $\mathcal{F}(u, v) = -\mathcal{F}(v, u)$ for all $u, v \in V$. We do not consider skew-Hermitian forms over $\mathbb{C}$ since if $\mathcal{F}(u, v)$ is skew-Hermitian, then $i\mathcal{F}(u, v)$ is Hermitian.

Define the matrix
\[
(a + bi)^\mathbb{R} := \begin{bmatrix} a & -b \\ b & a \end{bmatrix}
\]
for each $a + bi \in \mathbb{C} (a, b \in \mathbb{R})$,

and the direct sum of matrix pairs
\[
(A_1, F_1) \oplus (A_2, F_2) := \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}.
\]

The notation $\lambda \sim \mu$ means that a parameter $\lambda$ is determined up to replacement by $\mu$. We write $\lambda \in \mathbb{C}_\ell$ if $\lambda$ is a complex parameter that is determined up to replacement by its complex conjugate $\overline{\lambda}$. We write "$(A, \pm F)$" instead of "$(A, F)$ and $(A, -F)$". We denote by $0_n$ and $I_n$ the $n \times n$ zero and identity matrices.

Our main result is the following theorem.

**Theorem 1.** Let $r \in \mathbb{Z} \setminus \{-1, 0, 1\}$.

(A) Let $V_\mathbb{C}$ be a vector space over $\mathbb{C}$.

(a) Let $A$ be an $r$-selfadjoint operator on $V_\mathbb{C}$ with a nonsingular symmetric form $\mathcal{F}$. Then there exists a basis of $V_\mathbb{C}$ in which the pair $(A, \mathcal{F})$ is given by a direct sum, uniquely determined up to permutations of summands, of pairs of the form
\[
([\lambda], [1]), \quad \left( \begin{bmatrix} \mu & 0 \\ 0 & \mu^r \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right),
\]
in which $\lambda, \mu \in \mathbb{C}$, $\lambda^r = \lambda$, $\mu^r = \mu$, $\mu^r \neq \mu$, $\mu \sim \mu^r$.

(a) Let $A$ be an $r$-selfadjoint operator on $V_\mathbb{C}$ with a nonsingular Hermitian form $\mathcal{F}$. Then there exists a basis of $V_\mathbb{C}$ in which the pair $(A, \mathcal{F})$ is given by a direct sum, uniquely determined up to permutations of summands, of pairs of the form
\[
([\lambda], \pm [1]), \quad \left( \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right),
\]
in which $\lambda, \mu \in \mathbb{C}$, $\lambda^r = \lambda$, $\mu^r = \mu$, $\mu^r \neq \mu$, $\mu \sim \mu^r$. 

3
(a3) Let $A$ be an $r$-selfadjoint operator on $V_\mathbb{C}$ with a nonsingular skew-symmetric form $F$. Then there exists a basis of $V_\mathbb{C}$ in which the pair $(A, F)$ is given by a direct sum, uniquely determined up to permutations of summands, of pairs of the form

$$\left( \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^r \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right),$$

in which $\lambda \in \mathbb{C}$, $\lambda^2 = \lambda$, and $\lambda \not\sim \lambda^r$.

(B) Let $V_\mathbb{R}$ be a vector space over $\mathbb{R}$.

(b1) Let $A$ be an $r$-selfadjoint operator on $V_\mathbb{R}$ with a nonsingular symmetric form $F$. Then there exists a basis of $V_\mathbb{R}$ in which the pair $(A, F)$ is given by a direct sum, uniquely determined up to permutations of summands, of pairs of the form

$$\left( \begin{bmatrix} 0 & \pm 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} \mu^\mathbb{R} & \pm I_2 \\ 0 & (\mu^r)^\mathbb{R} \end{bmatrix}, \begin{bmatrix} 0 & I_2 \\ 0 & I_2 \end{bmatrix} \right),$$

in which $\lambda, \mu, \nu \in \mathbb{C}^\mathbb{i} \setminus \mathbb{R}$, $\lambda^r = \lambda$, $\mu^r = \overline{\mu}$, $\nu^r = \nu$, $\nu^r \not\sim \nu$, $\nu^r \neq \overline{\nu}$, $\nu \not\sim \nu^r$.

(b2) Let $A$ be an $r$-selfadjoint operator on $V_\mathbb{R}$ with a nonsingular skew-symmetric form $F$. Then there exists a basis of $V_\mathbb{R}$ in which the pair $(A, F)$ is given by a direct sum, uniquely determined up to permutations of summands, of pairs of the form

$$\left( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}, \begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix} \right),$$

in which $\lambda, \mu \in \mathbb{C}^\mathbb{i} \setminus \mathbb{R}$, $\lambda^r = \overline{\lambda}$, $\mu^r = \mu$, $\mu^r \neq \overline{\mu}$, $\mu \not\sim \mu^r$.

(C) Let $V_\mathbb{H}$ be a right vector space over $\mathbb{H}$. 

4
Let $A$ be an $r$-selfadjoint operator on $V_{\mathbb{H}}$ with a nonsingular Hermitian form $F$ with respect to quaternion conjugation.

$$h = a + bi + cj + dk \mapsto \bar{h} = a - bi - cj - dk, \quad a, b, c, d \in \mathbb{R}. \quad (6)$$

Then there exists a basis of $V_{\mathbb{H}}$ in which the pair $(A, F)$ is given by a direct sum, uniquely determined up to permutations of summands, of pairs of the form

$$([\lambda], \pm [1]), \quad \left( \begin{bmatrix} \mu & 0 \\ 0 & \bar{\mu^r} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right),$$

in which $\lambda, \mu \in \mathbb{C}^1$, $\lambda^r = \bar{\lambda}$, $\mu^r = \bar{\mu}$, $\mu \not= \bar{\mu}$.

(c) Let $A$ be an $r$-selfadjoint operator on $V_{\mathbb{H}}$ with a nonsingular Hermitian form $F$ with respect to quaternion semiconjugation.

$$h = a + bi + cj + dk \mapsto \hat{h} = a - bi + cj + dk, \quad a, b, c, d \in \mathbb{R}. \quad (7)$$

Then there exists a basis of $V_{\mathbb{H}}$ in which the pair $(A, F)$ is given by a direct sum, uniquely determined up to permutations of summands, of pairs of the form

$$([\lambda], \pm [1]) \text{ if } \lambda \not\in \mathbb{R}, \quad ([\lambda], [1]) \text{ if } \lambda \in \mathbb{R},$$

$$([\mu], [j]), \quad \left( \begin{bmatrix} \nu & 0 \\ 0 & \bar{\nu^r} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right),$$

in which $\lambda, \mu, \nu \in \mathbb{C}^1$, $\nu = \nu^r \not= \bar{\nu}$, $\nu \not= \bar{\nu}$.

(c) Let $A$ be an $r$-selfadjoint operator on $V_{\mathbb{H}}$ with a nonsingular form $F$ that is skew-Hermitian with respect to quaternion conjugation.

Then there exists a basis of $V_{\mathbb{H}}$ in which the pair $(A, F)$ is given by a direct sum, uniquely determined up to permutations of summands, of pairs of the form

$$([\lambda], \pm [i]) \text{ if } \lambda \not\in \mathbb{R}, \quad ([\lambda], [i]) \text{ if } \lambda \in \mathbb{R},$$

$$([\mu], [j]), \quad \left( \begin{bmatrix} \nu & 0 \\ 0 & \bar{\nu^r} \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right),$$

in which $\lambda, \mu, \nu \in \mathbb{C}^1$, $\lambda^r = \bar{\lambda}$, $\mu \not\in \mathbb{R}$, $\nu = \nu^r \not= \bar{\nu}$.
Let $A$ be an $r$-selfadjoint operator on $V_H$ with a nonsingular form $F$ that is skew-Hermitian with respect to quaternion semiconjugation \((7)\). Then there exists a basis of $V_H$ in which the pair $(A, F)$ is given by a direct sum, uniquely determined up to permutations of summands, of pairs of the form

\[
([\lambda], \pm [i]), \quad \left(\begin{bmatrix} \mu & 0 \\ 0 & \mu^r \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right),
\]

in which $\lambda, \mu \in \mathbb{C}^1$, $\lambda^r = \bar{\lambda}$, $\mu^{r^2} = \mu$, $\mu^r \neq \bar{\mu}$, $\mu \neq \mu^r$.

Each condition $\lambda^{r^2} = \lambda$, $\lambda^r = \lambda$, or $\lambda^r = \bar{\lambda}$ implies that $\lambda \neq 0$ if $r < 0$. Theorem 1 remains true if $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{H}$ are replaced by an algebraically closed field of zero characteristic, a real closed field, and the skew field of quaternions over a real closed field, respectively.

An involution $a \mapsto \bar{a}$ on a field or skew field $\mathbb{F}$ is a bijection $\mathbb{F} \rightarrow \mathbb{F}$ satisfying

\[
\bar{a + b} = \bar{a} + \bar{b}, \quad \bar{ab} = \bar{b} \bar{a}, \quad \bar{\bar{a}} = a \quad \text{for all } a, b \in \mathbb{F}.
\]

If an involution on $\mathbb{H}$ is not quaternion conjugation \((6)\), then it is quaternion semiconjugation \((7)\) in a suitable set of the fundamental units $i, j, k$; see [24, Lemma 2.2].

2. Reduction of the problem of classifying $r$-selfadjoint operators to the problem of classifying matrices under similarity

We prove Theorem 1 in the next section by the method that is developed in [22]. It reduces the problem of classifying systems of linear mappings and forms to the problem of classifying systems of linear mappings. Bilinear and sesquilinear forms, pairs of symmetric, skew-symmetric, and Hermitian forms, unitary and selfadjoint operators on a vector space with indefinite scalar product are classified in [22] over a field $\mathbb{K}$ of characteristic not 2 up to classification of Hermitian forms over finite extensions of $\mathbb{K}$ (and so they are fully classified over $\mathbb{R}$ and $\mathbb{C}$).

The reader is expected to be familiar with this method; it is described in details in [24] and is used in [9, 10, 19, 23]. In this section, we sketchily describe it in a special case: for the problem of classifying $r$-selfadjoint operators.
Systems consisting of vector spaces and of linear mappings and forms on them are considered as *representations of mixed graphs*; i.e., graphs with undirected and directed edges. Its vertices represent vector spaces, its undirected edges represent forms, and its directed edges represent linear mappings.

In particular, each pair \((A, F)\) from Theorem 1 defines the representation

\[
\begin{align*}
\begin{array}{c}
\text{A} \\
\text{\bigcirc} \\
\text{V} \\
\text{\bigcirc} \\
\text{F}
\end{array},
\end{align*}
\]

\[
\begin{align*}
F(Au, v) &= F(u, Av) \text{ if } r \geq 2, \\
F(Au, A^{-r}v) &= F(u, v) \text{ if } r \leq -2, \\
F(u, v) &= \varepsilon F(v, u) \text{ is nonsingular}
\end{align*}
\]  

of the mixed graph \(\bigcirc \bullet \bigcirc\) over \(\mathbb{F}\) with involution \(a \mapsto \overline{a}\), in which \(\varepsilon \equiv 1\) if \(F\) is symmetric or Hermitian, and \(\varepsilon \equiv -1\) if \(F\) is skew-symmetric or skew-Hermitian. Choosing a basis in \(V\), we give (8) by its matrices

\[
\begin{align*}
\begin{array}{c}
\text{A} \\
\text{\bigcirc} \\
\text{n} \\
\text{\bigcirc} \\
\text{F}
\end{array},
\end{align*}
\]

\[A^\wedge F = FA^r, \quad F^\wedge = \varepsilon F \text{ is nonsingular,} \tag{9}\]

in which \(n := \dim V\), \(A\) and \(F\) are \(n \times n\) matrices over \(\mathbb{F}\), and \(A^\wedge := \overline{A}^T\) (\(A^\wedge = A^T\) if \(a \mapsto \overline{a}\) is the identity involution, and \(A^\wedge = A^*\) otherwise). Changing the basis in \(V\), we can reduce \((A, F)\) by transformations

\[
\begin{align*}
(A, F) &\mapsto \left(S^{-1}AS, S^\wedge FS\right), \quad S \text{ is nonsingular} \tag{10}
\end{align*}
\]

(see (2) and (11)). We say that the pairs \((A, F)\) and \((S^{-1}AS, S^\wedge FS)\) are *isomorphic via S*.

Replacing \(F : V \times V \to \mathbb{F}\) in (8) by the pair of mutually adjoint linear mappings \(F : v \mapsto \overline{F}(?, v)\) and \(F^\wedge : u \mapsto \overline{F}(u, ?)\), we obtain the system of linear mappings

\[
\begin{align*}
\begin{array}{c}
\text{A} \\
\text{\bigcirc} \\
\text{V} \\
\text{\bigcirc} \\
\text{\overline{F}} \\
\text{\overline{F}^\wedge}
\end{array},
\end{align*}
\]

\[
\begin{align*}
A^\wedge F &= FA^r, \quad F^\wedge = \varepsilon F \text{ is nonsingular,} \tag{11}
\end{align*}
\]

in which \(V^\wedge\) is the *dual space* (with respect to the involution \(a \mapsto \overline{a}\)) consisting of semilinear forms on \(V\), and \(A^\wedge : V^\wedge \to V^\wedge\) is the *dual mapping* defined by \(\varphi \mapsto \overline{\varphi A}\). In the matrix form,

\[
\begin{align*}
\begin{array}{c}
\text{A} \\
\text{\bigcirc} \\
\text{n} \\
\text{\bigcirc} \\
\text{\overline{F}} \\
\text{\overline{F}^\wedge}
\end{array},
\end{align*}
\]

\[A^\wedge F = FA^r, \quad F^\wedge = \varepsilon F \text{ is nonsingular.} \tag{12}\]
Thus, there is the bijective correspondence

\[ A(F) \mapsto A^c(F) \]  

between the matrix sets of systems (8) and (11).

Let us consider a system of linear mappings over \( F \):

\[ M : A_1 \xrightarrow{F_1} V_1 \xrightarrow{F_2} V_2 \xrightarrow{A_2} F_1, \]  
\[ A_2 F_1 = F_1 A_1', A_2' F_2 = F_2 A_1, \]  
\[ F_1 = \epsilon F_2 \text{ is nonsingular}, \]  

which is a representation of the quiver \( A \). Choosing bases in \( V_1 \) and \( V_2 \), we give it by a system of \( n \times n \) matrices \((n := \dim V_1 = \dim V_2)\)

\[ M : A_1 \xrightarrow{n} F_1 \xrightarrow{n} F_2 \xrightarrow{n} A_2 \]  
\[ A_2 F_1 = F_1 A_1', A_2' F_2 = F_2 A_1, \]  
\[ F_1 = \epsilon F_2 \text{ is nonsingular}. \]  

Changing bases in \( V_1 \) and \( V_2 \), we can reduce (15) by transformations

\[ M' : RA_1 R^{-1} \xrightarrow{n} SF_1 R^{-1} \xrightarrow{n} SA_2 S^{-1} \]  
\[ R, S \text{ are nonsingular}. \]  

We say that the matrix sets (15) and (16) are isomorphic via \( R \) and \( S \) and write \( M \simeq M' \). This isomorphism can be shown by the commutative diagram

The direct sum of systems \( M \) and \( M' \) is the system

\[ M \oplus M' : A_1 \oplus A_1' \xrightarrow{n + n'} F_1 \oplus F_1' \xrightarrow{n + n'} A_2 \oplus A_2' \]  

A system is indecomposable if it is not isomorphic to a direct sum of the form (17) with nonzero \( n \) and \( n' \).
For each system (15), we define the dual system

\[ M^\circ : \ A_2 \bigcirc n \bigcirc n \bigcirc A_1 \]

A system \( M \) is selfdual if \( M = M^\circ \), which means that it has the form (12).

Suppose we know the following sets of systems of the form (15):

\[ M \in \mathcal{M}(\mathbb{F}) \] which is a set of nonisomorphic indecomposable systems such that every indecomposable system (15) is isomorphic to exactly one system from \( \mathcal{M}(\mathbb{F}) \),

\[ M' \in \mathcal{M}(\mathbb{F}) \] which is a set of nonisomorphic indecomposable selfdual systems such that every indecomposable selfdual system is isomorphic to exactly one system from \( \mathcal{M}(\mathbb{F}) \),

\[ M'' \in \mathcal{M}(\mathbb{F}) \] which is a set of nonisomorphic indecomposable systems that are not isomorphic to selfdual such that every indecomposable system that is not isomorphic to selfdual is isomorphic to exactly one system from \( \mathcal{M}(\mathbb{F}) \).

For each \( M \in \mathcal{M}(\mathbb{F}) \) of the form (12), we define the matrix pairs

\[ \tilde{M} : \ A \bigcirc n \bigcirc F \quad \tilde{M}^- : \ A \bigcirc n \bigcirc -F \] (18)

and for each \( N \in \mathcal{M}(\mathbb{F}) \) of the form (15), we define the matrix pair

\[ N^+ : \begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix} \bigcirc 2n \bigcirc \begin{bmatrix} 0 & F_2^\circ \\ F_1 & 0 \end{bmatrix} \]

Note that the natural bijection (13) takes \( \tilde{M} \) into \( M \), and \( N^+ \) into a selfdual system that is isomorphic to \( N \oplus N^\circ \).

The following lemma reduces the problem of classifying matrix pairs (9) up to transformations (10) to the problem of classifying systems (15) up to transformations (16). This lemma is a special case of [24, Theorem 3.2] about arbitrary systems of linear mappings and forms.

**Lemma 1.** Each pair (9) over \( \mathbb{F} \in \{ \mathbb{C}, \mathbb{R}, \mathbb{H} \} \) is isomorphic to a direct sum of pairs of the types

\[ N^+ \quad \text{and} \quad \begin{cases} \tilde{M} & \text{if } \tilde{M}^- \text{ and } \tilde{M} \text{ are isomorphic,} \\ \tilde{M}, \tilde{M}^- & \text{if } \tilde{M}^- \text{ and } \tilde{M} \text{ are not isomorphic,} \end{cases} \]
in which $M \in \mathcal{M}_r(F)$ and $N \in \mathcal{M}_c(F)$. This sum is uniquely determined, up to permutations of direct summands and replacements of $N \in \mathcal{M}_c(F)$ by $N^\circ$.

**Lemma 2.** If a system of the form (15) is isomorphic to a selfdual system via $R$ and $S$, then it is isomorphic to some selfdual system via $I$ and $R^\ast S$.

**Proof.** The corresponding selfdual system is constructed as follows:

3. **Proof of Theorem 1**

**Lemma 3.** Each square matrix over $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{H}$ is similar to a direct sum, uniquely determined up to permutations of summands, of matrices from the following matrix sets:

(a) $\mathcal{C}(\mathbb{C}) := \{J_n(\lambda) \mid \lambda \in \mathbb{C}\}$, in which

\[
J_n(\lambda) := \begin{bmatrix}
\lambda & 1 & \cdots & 0 \\
\lambda & \ddots & \ddots & \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & 1 & \lambda
\end{bmatrix} \quad (n\text{-by-}n).
\]

(b) $\mathcal{C}(\mathbb{R}) := \{J_n(a) \mid a \in \mathbb{R}\} \cup \{J_n(\lambda)^R \mid \lambda \in \mathbb{C}^\ast \setminus \mathbb{R}\}$, in which

\[
J_n(\lambda)^R := \begin{bmatrix}
\lambda^R & I_2 & \cdots & 0 \\
\lambda^R & \ddots & \ddots & \\
\vdots & \ddots & \ddots & I_2 \\
0 & \cdots & I_2 & \lambda^R
\end{bmatrix} \quad (2n\text{-by-}2n),
\]

$\lambda^R$ is determined in (5), and $\mathbb{C}^\ast = \{\lambda \in \mathbb{C} \mid \lambda \not\in \mathbb{R}\}$.

(c) $\mathcal{C}(\mathbb{H}) := \{J_n(\lambda) \mid \lambda \in \mathbb{C}^\ast\}$.
Proof. The statement (a) is the Jordan theorem; (b) and (c) are given in [8, Theorem 3.4.1.5] and [21, Theorem 5.5.3].

Each system (15) is reduced by transformations (16) with $R = F_1$ and $S = I_n$ to a system of the form

\[ M_\varepsilon(A) : A \begin{array}{c} \varepsilon I_n \\ n \end{array} \begin{array}{c} \varepsilon I_n \\ n \end{array} A^r \quad A^{r^2} = A, \]

whose dual system is

\[ M_\varepsilon(A)^\circ : (A^r)^+ \begin{array}{c} \varepsilon I_n \\ n \end{array} \begin{array}{c} \varepsilon I_n \\ n \end{array} A^\lambda \]

Clearly, $M_\varepsilon(A) \simeq M_\varepsilon(B)$ if and only if $A$ and $B$ are similar; and so

\[ \mathfrak{M}_\varepsilon(\mathbb{F}) = \{ M_\varepsilon(A) | A \in \mathfrak{C}(\mathbb{F}) \text{ such that } A^{r^2} = A \}; \quad (19) \]

\[ M_\varepsilon(B) \simeq M_\varepsilon(A)^\circ \iff B \text{ is similar to } (A^r)^+. \quad (20) \]

3.1. Case (A): $\mathbb{F} = \mathbb{C}$

Let $a \mapsto \bar{a}$ be the identity involution or complex conjugation. Suppose that $J_n(\lambda)^{r^2} = J_n(\lambda)$ with $\lambda \in \mathbb{C}$. If $\lambda = 0$, then $n = 1$ since $r^2 \geq 4$. If $\lambda \neq 0$, then $J_n(\lambda)^{r^2-1} = I_n$, all entries of the first over-diagonal of $J_n(\lambda)^{r^2-1}$ are $(r^2-1)\lambda^{r^2-2}$, and so $n = 1$ too. Thus,

\[ J_n(\lambda)^{r^2} = J_n(\lambda) \quad \implies \quad n = 1 \quad (21) \]

and

\[ \mathfrak{M}_\varepsilon(\mathbb{C}) = \{ M_\varepsilon(\lambda) | \lambda \in \mathbb{C}, \ \lambda^{r^2} = \lambda \} \]

(to simplify notation, we write $\lambda$ instead of $\lambda I_1$).

Since

\[ M_\varepsilon(\lambda) : \lambda \begin{array}{c} 1 \\ \varepsilon \end{array} \begin{array}{c} 1 \\ \varepsilon \end{array} \lambda \]

and $\lambda^{r^2} = \lambda$, we have $M_\varepsilon(\lambda)^\circ \simeq M_\varepsilon(\bar{\lambda}^r)$. Hence,

\[ M_\varepsilon(\mu) \simeq M_\varepsilon(\lambda)^\circ \iff \mu = \bar{\lambda}^r. \quad (22) \]

The following cases are possible:
(a₁): \( \varepsilon = 1 \) and the involution on \( \mathbb{C} \) is the identity. Then
\[
M_1(\lambda) \simeq M_1(\lambda)^\circ \iff M_1(\lambda) = M_1(\lambda)^\circ \iff \lambda = \lambda^r,
\]
and so
\[
\mathcal{M}'_1(\mathbb{C}) = \{M_1(\lambda) \mid \lambda \in \mathbb{C}, \ \lambda^r = \lambda \},
\]
\[
\mathcal{M}''_1(\mathbb{C}) = \{M_1(\mu) \mid \mu \in \mathbb{C}, \ \mu^{r^2} = \mu, \ \mu^r \neq \mu \}.
\]
Lemma \( \text{I} \) and (22) ensure (a₁).

(a₂): \( \varepsilon = 1 \) and the involution on \( \mathbb{C} \) is complex conjugation. Then
\[
M_1(\lambda) \simeq M_1(\lambda)^\circ \iff M_1(\lambda) = M_1(\lambda)^\circ \iff \lambda = \bar{\lambda}^r,
\]
and so
\[
\mathcal{M}'_1(\mathbb{C}) = \{M_1(\lambda) \mid \lambda \in \mathbb{C}, \ \lambda^r = \bar{\lambda} \},
\]
\[
\mathcal{M}''_1(\mathbb{C}) = \{M_1(\mu) \mid \mu \in \mathbb{C}, \ \mu^{r^2} = \mu, \ \mu^r \neq \bar{\mu} \}.
\]
Lemma \( \text{I} \) and (22) ensure (a₂).

(a₃): \( \varepsilon = -1 \) and the involution on \( \mathbb{C} \) is the identity. The system \( M_{-1}(\lambda) \) is not isomorphic to a selfdual system since there are no nonsingular \( 1 \times 1 \) matrices \( R \) and \( S \) such that \( SI_1R^{-1} = S(-I_1)R^{-1} \) (see (16)). Therefore,
\[
\mathcal{M}'_{-1}(\mathbb{C}) = \emptyset,
\]
\[
\mathcal{M}''_{-1}(\mathbb{C}) = \{M_{-1}(\lambda) \mid \lambda \in \mathbb{C}, \ \lambda^{r^2} = \lambda \}.
\]
Lemma \( \text{I} \) and (22) ensure (a₃).

3.2. Case (B): \( \mathbb{F} = \mathbb{R} \)

The set \( \mathfrak{C}(\mathbb{R}) \) is given in Lemma \( \text{3(b)} \). The equality
\[
\mathcal{M}_\varepsilon(\mathbb{R}) = \{M_\varepsilon(0), \ M_\varepsilon(1)\} \cup \{M_\varepsilon(-1) \mid \text{if } r \text{ is odd} \}
\]
\[
\cup \{M_\varepsilon(\lambda^R) \mid \lambda \in \mathbb{C}^\bigcup \ \mathbb{R}, \ \lambda^{r^2} = \lambda \}
\]  \hspace{1cm} (23)

is proved as follows:
Consider \( J_n(a) \in \mathcal{C}(\mathbb{R}) \) with \( a \in \mathbb{R} \) and \( J_n(a)^{r^2} = J_n(a) \). By (21), \( n = 1 \). Since \( a \) is real, \( a^{r^2} = a \) implies that either \( a = 0 \), or \( a = \pm 1 \) if \( r \) is odd and \( a = 1 \) if \( r \) is even. Note that each system \( M_\varepsilon(\lambda) \) with \( \lambda \in \{0, 1, -1\} \) is selfdual if \( \varepsilon = 1 \); it is not isomorphic to selfdual if \( \varepsilon = -1 \).

Consider \( J_n(\lambda) \in \mathcal{C}(\mathbb{R}) \) with \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) and \((J_n(\lambda))^{r^2} = J_n(\lambda)\mathbb{R}^\varepsilon\). The matrix \( J_n(\lambda) \) is similar over \( \mathbb{C} \) to \( J_n(\lambda) \oplus J_n(\bar{\lambda}) \). Hence \( J_n(\lambda)^{r^2} = J_n(\lambda), n = 1, \) and \( \lambda^{r^2} = \lambda^r \).

For every \( M_\varepsilon(\lambda^R), M_\varepsilon(\mu^R) \in \mathcal{M}_\varepsilon(\mathbb{R}) \), we have
\[
M_\varepsilon(\mu^R) \simeq M_\varepsilon(\lambda^R)^\varepsilon \iff \mu = \lambda^r \text{ or } \mu = \bar{\lambda}^r
\]

(25) since \( M_\varepsilon(\mu^R) \simeq M_\varepsilon(\lambda^R)^\varepsilon \) if and only if \( \mu^R \) is similar to \((\lambda^R)^r\), if and only if \( \text{diag}(\mu, \bar{\mu}) \) is similar to \( \text{diag}(\lambda, \bar{\lambda})^r \), if and only if \( \mu = \lambda^r \) or \( \mu = \bar{\lambda}^r \).

Thus, if \( M_\varepsilon(\lambda^R) \) is isomorphic to a selfdual system, then \( \lambda^r = \lambda \) or \( \lambda^r = \bar{\lambda} \).

Write \( \lambda = a + bi \ (a, b \in \mathbb{R}, \ b \neq 0) \), then \( \lambda^R = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \) and \( \lambda^r = \lambda \iff \begin{bmatrix} a & -b \\ b & a \end{bmatrix}^r = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \), \( \lambda^r = \bar{\lambda} \iff \begin{bmatrix} a & -b \\ b & a \end{bmatrix}^r = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \).

The following two cases are possible:

(b1): \( \varepsilon = 1 \). Let \( \lambda = a + bi \ (a, b \in \mathbb{R}, \ b \neq 0) \). If \( M_1(\lambda^R) \) is isomorphic to a selfdual system, then \( \lambda^r = \lambda \) or \( \lambda^r = \bar{\lambda} \). If \( \lambda^r = \lambda \), then \( \begin{bmatrix} a & -b \\ b & a \end{bmatrix} Z = Z \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \) with \( Z := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \), and so \( M_1(\lambda^R) \) is isomorphic to a selfdual system:

\[
\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{array}{c} 1 \\ 1 \end{array} = \begin{array}{c} 2 \\ 2 \end{array} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}
\]

If \( \lambda^r = \bar{\lambda} \), then the system \( M_1(\lambda^R) \) is selfdual.

Thus,
\[
\mathcal{M}_1'(\mathbb{R}) = \{M_1(0), M_1(1)\} \cup \{M_1(-1)\} \cup \{M_1(a) \mid a \in \mathbb{C} \setminus \mathbb{R}, \lambda^r = \lambda \} \cup \{M_1(\mu^R) \mid \mu \in \mathbb{C} \setminus \mathbb{R}, \mu^r = \bar{\mu} \},
\]
\[
\mathcal{M}_1''(\mathbb{R}) = \{M_1(\nu^R) \mid \nu \in \mathbb{C} \setminus \mathbb{R}, \nu^r = \nu, \nu^r = \nu \}.
\]

13
Each system $\lambda^R \bigcirc \bigcirc 2 \xrightarrow{\frac{Z}{Z}} 2 \bigcirc \bigcirc \lambda^R$ from $\mathcal{M}_1'(\mathbb{R})$ defines the pairs $(\lambda^R, Z)$ and $(\lambda^R, -Z)$ of the form (18); they are isomorphic via $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (see (10)). Each system $M_1(\mu^R) \in \mathcal{M}_1'(\mathbb{R})$ defines the pairs $(\mu^R, I_2)$ and $(\mu^R, -I_2)$; they are not isomorphic since $I_2$ and $-I_2$ are not congruent over $\mathbb{R}$.

Lemma 11 and (25) ensure (b_1) (we do not write $\nu \sim \bar{\nu}'$ since $\nu$ is determined up to replacement by $\bar{\nu}$).

(b_2): $\varepsilon = -1$. Let $\lambda = a + bi$ ($a, b \in \mathbb{R}$, $b \neq 0$). If $M_{-1}(\lambda^R)$ is isomorphic to a selfdual system, then $\lambda^r = \lambda$ or $\lambda^r = \bar{\lambda}$. If $\lambda^r = \lambda$, then $M_{-1}(\lambda^R)$ is not isomorphic to a selfdual system; otherwise by Lemma 2 there is a nonsingular $P$ such that

$$
\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \bigcirc \bigcirc 2 \xrightarrow{L_2} 2 \bigcirc \bigcirc \begin{bmatrix} a & -b \\ b & a \end{bmatrix}
$$

Then $P = \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}$ for some $x \neq 0$ since $P^T = -P$. The equality $P \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} P$ implies that $b = 0$, which contradicts our assumption that $\lambda \notin \mathbb{R}$.

If $\lambda^r = \bar{\lambda}$, then $\begin{pmatrix} a & b \\ b & a \end{pmatrix} L = L \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ with $L := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and so $M_{-1}(\lambda^R)$ is isomorphic to a selfdual system:

$$
\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \bigcirc \bigcirc 2 \xrightarrow{L} 2 \bigcirc \bigcirc \begin{bmatrix} a & -b \\ b & a \end{bmatrix}
$$

Using (24), we obtain

$\mathcal{M}_1'(\mathbb{R}) = \{ \lambda^R \bigcirc \bigcirc 2 \xrightarrow{L} 2 \bigcirc \bigcirc \lambda^R \mid \lambda \in \mathbb{C}^\dagger \setminus \mathbb{R}, \ \lambda^r = \bar{\lambda} \}$,

$\mathcal{M}''_1'(\mathbb{R}) = \{ M_{-1}(0), \ M_{-1}(1) \} \cup \{ M_1(-1) \} \text{ if } r \text{ is odd} \cup \{ M_{-1}(\mu^R) \mid \mu \in \mathbb{C}^\dagger \setminus \mathbb{R}, \ \mu^{r^2} = \mu, \ \mu^r \neq \bar{\mu} \}$.

Each system from $\mathcal{M}_1'(\mathbb{R})$ defines the pairs $(\lambda^R, L)$ and $(\lambda^R, -L)$; they are isomorphic via $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (see (10) and (18)).

Lemma 11 and (25) ensure (b_2).
3.3. Case (C): \( \mathbb{F} = \mathbb{H} \).

In this case, \( \varepsilon = \pm 1, a \mapsto \overline{a} \) is quaternion conjugation (6) or quaternion semiconjugation (7), and \( \mathcal{C}(\mathbb{H}) \) is given in Lemma 3(c). By (19) and (21),

\[
\mathcal{M}_\varepsilon(\mathbb{H}) = \{ M_\varepsilon(\lambda) | \lambda \in \mathbb{C}, \lambda^r = \lambda \}
\]

If \( M_\varepsilon(\lambda), M_\varepsilon(\mu) \in \mathcal{M}_\varepsilon(\mathbb{H}) \), then

\[
M_\varepsilon(\mu) \cong M_\varepsilon(\lambda)^{\circ} \iff \mu = \lambda^r \text{ or } \mu = \overline{\lambda}^r.
\]

(26)

Indeed, if \( M_\varepsilon(\mu) \cong M_\varepsilon(\lambda)^{\circ} \), then the \( 1 \times 1 \) matrix \([\mu]\) is similar to \([\lambda^r] = [\overline{\lambda}^r]\). By Lemma 3(c), \( \mu = \lambda^r \) or \( \mu = \overline{\lambda}^r \). Conversely, let \( \mu = \lambda^r \) or \( \mu = \overline{\lambda}^r \).

We can take \( \mu = \overline{\lambda}^r \) since \( \lambda \) is determined up to replacement by \( \overline{\lambda} \).

If \( M_\varepsilon(\lambda) \in \mathcal{M}_\varepsilon(\mathbb{H}) \) is isomorphic to a selfdual system, then by (26) \( \lambda^r = \lambda \) or \( \lambda^r = \overline{\lambda} \). Conversely,

- if \( \lambda^r = \overline{\lambda} \), then \( M_\varepsilon(\lambda) \) is isomorphic to a selfdual system:

\[
\begin{array}{c}
\lambda
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\delta_\varepsilon := \begin{cases} 1 & \text{if } \varepsilon = 1, \\ i & \text{if } \varepsilon = -1; \end{cases}
\end{array}
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\lambda^r = \overline{\lambda}
\end{array}
\end{array}
\end{array}
\end{array}
\]

(27)

- if \( \lambda^r = \lambda \) and \( \lambda \notin \mathbb{R} \) (the case \( \lambda \in \mathbb{R} \) is considered in the previous paragraph), then \( M_\varepsilon(\lambda) \in \mathcal{M}_\varepsilon(\mathbb{H}) \) is isomorphic to a selfdual system if and only if either \( \varepsilon = 1 \) and the involution is (7), or \( \varepsilon = -1 \) and the involution is (6). Indeed, suppose that \( M_\varepsilon(\lambda) \) is isomorphic to a selfdual system. By Lemma 2 there exists \( h \in \mathbb{H} \) such that

\[
\begin{array}{c}
\lambda
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\lambda^r = \lambda
\end{array}
\end{array}
\end{array}
\end{array}
\]

(28)
is an isomorphism. If either \( \varepsilon = 1 \) and the involution is \((7)\), or \( \varepsilon = -1 \) and the involution is \((6)\), then \((28)\) holds for \( h = j \). If \( \varepsilon = 1 \) and the involution is \((6)\), then \((28)\) implies \( h = \bar{h} \), \( h \in \mathbb{R} \), \( h\lambda = \bar{\lambda}h \), and so \( \lambda \in \mathbb{R} \), a contradiction. If \( \varepsilon = -1 \) and the involution is \((7)\), then \((28)\) implies \( -h = \bar{h} \), \( h \in \mathbb{R}i \), \( h\lambda = \bar{\lambda}h \), and so \( \lambda \in \mathbb{R} \), a contradiction.

The following cases are possible:

\((c_1)\): \( \varepsilon = 1 \) and the involution is quaternion conjugation \((6)\). Then

\[
\mathcal{M}'_1(\mathbb{H}) = \{ \lambda \quad \text{trapezoid} \mid \lambda \in \mathbb{C}^1, \quad \lambda' = \bar{\lambda} \},
\]

\[
\mathcal{M}''_1(\mathbb{H}) = \{ M_1(\mu) \mid \mu \in \mathbb{C}^2, \quad \mu^2 = \mu, \quad \mu' \neq \bar{\mu} \}.
\]

Each system from \( \mathcal{M}'_1(\mathbb{H}) \) defines the pairs \((\lambda, 1)\) and \((\lambda, -1)\); they are not isomorphic since \( \bar{c}1c \neq -1 \) for all \( c \in \mathbb{H} \).

Lemma \((11)\) and \((22)\) ensure \((c_1)\).

\((c_2)\): \( \varepsilon = 1 \) and the involution is quaternion semiconjugation \((7)\). Then

\[
\mathcal{M}'_1(\mathbb{H}) = \{ \lambda \quad \text{trapezoid} \mid \lambda \in \mathbb{C}^1, \quad \lambda' = \bar{\lambda} \}
\]

\[
\cup \{ \mu \quad \text{trapezoid} \mid \mu \in \mathbb{C}^1 \setminus \mathbb{R}, \quad \mu^2 = \mu, \quad \mu' = \bar{\mu} \},
\]

\[
\mathcal{M}''_1(\mathbb{H}) = \{ M_1(\nu) \mid \nu \in \mathbb{C}^3, \quad \nu^2 = \nu, \quad \nu' \neq \nu, \quad \nu' = \bar{\nu} \}.
\]

Each system \( \lambda \quad \text{trapezoid} \) from \( \mathcal{M}'_1(\mathbb{H}) \) defines the pairs \((\lambda, 1)\) and \((\lambda, -1)\).

- If \( \lambda \notin \mathbb{R} \), then \((\lambda, 1)\) and \((\lambda, -1)\) are not isomorphic. On the contrary, suppose that there is a nonzero \( c \in \mathbb{H} \) such that

\[
c^{-1}\lambda c = \lambda, \quad \bar{c}1c = -1 \tag{29}
\]

(see \((10)\)). By \( c^{-1}\lambda c = \lambda \), we have \( c \in \mathbb{C} \), which contradicts \( \bar{c}1c = -1 \).

- If \( \lambda \in \mathbb{R} \), then \((\lambda, 1)\) and \((\lambda, -1)\) are isomorphic since \((29)\) holds for \( c = j \).
The pairs \((\mu, j)\) and \((\mu, -j)\) constructed by \(\mu \xrightarrow{j} 1 \xrightarrow{\pi} \mu\) from \(\mathcal{M}'(\mathbb{H})\) are isomorphic via \(i\) since \(i\mu = \mu i\) and \(iji = -ki = -j\).

Lemma 1 and (22) ensure (c_2).

(c_3): \(\varepsilon = -1\) and the involution is quaternion conjugation (6). Then
\[
\mathcal{M}'_1(\mathbb{H}) = \left\{ \lambda \xrightarrow{-i} 1 \xrightarrow{\pi} \lambda \mid \lambda \in \mathbb{C}^1, \; \lambda^r = \bar{\lambda} \right\} \\
\cup \left\{ \mu \xrightarrow{j} 1 \xrightarrow{\pi} \mu \mid \mu \in \mathbb{C}^1 \setminus \mathbb{R}, \; \mu^r = \mu \right\},
\]
\[
\mathcal{M}''_1(\mathbb{H}) = \{ M_{-1}(\nu) \mid \nu \in \mathbb{C}^1, \; \nu^{r^2} = \nu, \; \nu^r \neq \nu, \; \nu^{r^2} \neq \bar{\nu} \}.
\]

If \(\lambda \notin \mathbb{R}\), then the pairs \((\lambda, i)\) and \((\lambda, -i)\) constructed by a system from \(\mathcal{M}'_1(\mathbb{H})\) are not isomorphic. On the contrary, suppose there exists \(c \in \mathbb{H}\) such that
\[
c^{-1}\lambda c = \lambda, \quad \bar{c}ic = -i \tag{30}
\]
(see (11)). Since \(\lambda = \lambda_1 + \lambda_2 i\) with \(\lambda_1, \lambda_2 \in \mathbb{R}\) and \(\lambda_2 \neq 0\), the equality \(c^{-1}\lambda c = \lambda\) implies that \(ic = ci\), and so \(c \in \mathbb{C}\), which contradicts \(\bar{c}ic = -i\).

If \(\lambda \in \mathbb{R}\), then \((\lambda, i)\) and \((\lambda, -i)\) are isomorphic since (30) holds for \(c = j\).

The pairs \((\mu, j)\) and \((\mu, -j)\) are isomorphic via \(i\) since \(i\mu = \mu i\) and \(iji = -ki = -j\).

Lemma 1 and (22) ensure (c_3).

(c_4): \(\varepsilon = -1\) and the involution is quaternion semiconjugation (7). Then
\[
\mathcal{M}'_1(\mathbb{H}) = \left\{ \lambda \xrightarrow{-i} 1 \xrightarrow{\pi} \lambda \mid \lambda \in \mathbb{C}^1, \; \lambda^r = \bar{\lambda} \right\},
\]
\[
\mathcal{M}''_1(\mathbb{H}) = \{ M_{-1}(\mu) \mid \mu \in \mathbb{C}^1, \; \mu^{r^2} = \mu, \; \mu^r \neq \bar{\mu} \}.
\]

The pairs \((\lambda, i)\) and \((\lambda, -i)\) constructed by a system from \(\mathcal{M}'_1(\mathbb{H})\) are not isomorphic since \(\bar{c}ic = i\bar{c}c \neq -i\) for all \(c \in \mathbb{H}\).

Lemma 1 and (22) ensure (c_4).
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