Complete hyperentangled state analysis using weak cross-Kerr nonlinearity and auxiliary entanglement

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Abstract
We present a new method for the complete analysis of hyperentangled Bell state and Greenberger–Horne–Zeilinger state in polarization and spatial-mode degrees of freedom, resorting to weak cross-Kerr nonlinearity and auxiliary frequency entanglement. The weak cross-Kerr nonlinearity with small phase shift is used to construct quantum nondestructive detector, and it is realizable with the current technology. Compared with the previous schemes, our scheme largely reduces the requirement on nonlinearity with the help of auxiliary entanglement in the third degree of freedom. Our method provides an efficient avenue for the hyperentangled state analysis, and will be useful for high-capacity quantum information processing.

1. Introduction
The Bell state analysis (BSA) for two-photon system is essential to many quantum communication schemes, such as quantum teleportation [1], quantum dense coding [2] and quantum entanglement swapping [3]. As we know, it is impossible to distinguish the four polarization Bell states by just using linear optics, and only two of them can be determined [4, 5]. Similarly, the complete Greenberger–Horne–Zeilinger (GHZ) state analysis (GSA) for multi-photon system is also cannot be accomplished if just linear optics is used, and only two of the eight GHZ states can be determined [6]. To solve these problems, there are two basic methods that are mostly used. One method is using auxiliary entanglement in another degree of freedom (DOF) [7–12], and the other method is using quantum nondestructive detector (QND), which can be realized by the cross-Kerr nonlinearity [13–16].

Hyperentanglement that entangled in more than one DOFs has attracted much attention in recent years [17], and the hyperentangled Bell state analysis (HBSA) and hyperentangled Greenberger–Horne–Zeilinger state analysis (HGSA) play important roles in high-capacity quantum information processing [18–28]. Research has shown that the 16 hyperentangled Bell states in two DOFs only can be separated into 7 distinguished classes, resorting to linear optical elements [18, 19]. If auxiliary entanglement in the third DOF is utilized, then the 16 hyperentangled states can be separated into 12 groups [20]. In 2017, Li et al showed that given n DOFs, the 4^n hyperentangled Bell states can be classified into 2^n+k+1 − 2^k groups via linear optics assisted by k(k ≤ n) ancillary entangled states [20]. When the condition k = n is satisfied, the complete HBSA in n DOFs can be achieved. According to this important conclusion, the complete HBSA in two DOFs is achievable with the help of hyperentanglement in the third and fourth DOFs. Like the conventional BSA and GSA in one DOF, QND can also be used for HBSA and HGSA. In 2010, Sheng et al presented the first complete HBSA scheme in polarization and spatial-mode DOFs using QND, which is constructed by the cross-Kerr nonlinearity [21]. In 2012, Xia et al proposed an efficient HGSA scheme using a similar principle with the help of more QNDS [22]. In 2016, Li et al presented the first self-assisted complete HBSA and HGSA schemes via the cross-Kerr nonlinearity, and their schemes can greatly simplify the discrimination process and reduce the requirement on nonlinearities by using the
self-assisted mechanism [25]. Compared with the scheme of Sheng et al that contains three QNDs [21], only two QNDs are used in the HBSA scheme of Li et al [25].

Inspired by the previous works, we present a new method for the complete HBSA and HGSA in polarization and spatial-mode DOFs, resorting to QND and auxiliary frequency entanglement. In our complete HBSA scheme, only one QND is used by using auxiliary entanglement in the third DOF. Our method is also suitable for the complete three-photon HGSA, and can be generalized to the complete N-photon HGSA directly. Compared with the previous HGSA schemes based on QND [21, 25], our scheme just requires one QND, which reduces the difficulty of the experiment. Compared with the HGSA scheme of Li et al [25], our scheme is more feasible with current technology. In their self-assisted complete HGSA scheme, the π phase shift of cross-Kerr nonlinearity is required to construct QND, which is hard to realize in experiment [25]. However, we still just require the weak cross-Kerr nonlinearity with small phase shift to construct QND in our complete HGSA scheme, which will make our method for hyperentangled state analysis more useful in quantum information processing.

2. Complete hyperentangled Bell state analysis

The two-photon hyperentangled Bell state in polarization and spatial-mode DOFs can be expressed as

$$|Ψ⟩_{AB} = |ζ⟩_{P} ⊗ |η⟩_{S}. \quad (1)$$

Here, A and B denote the two photons in our system. P and S denote the polarization DOF and spatial-mode DOF, respectively. $|ζ⟩_{P}$ is one of the four polarization Bell states,

$$|φ^±⟩_{P} = \frac{1}{\sqrt{2}} (|HH⟩ ± |VV⟩)_{AB}, \quad (2)$$

$$|ψ^±⟩_{P} = \frac{1}{\sqrt{2}} (|HV⟩ ± |VH⟩)_{AB},$$

where H and V are the horizontal and vertical polarization states, respectively. $|η⟩_{S}$ is one of the four spatial-mode Bell states,

$$|φ^±⟩_{S} = \frac{1}{\sqrt{2}} (|a_1b_1⟩ ± |a_2b_2⟩)_{AB}, \quad (3)$$

$$|ψ^±⟩_{S} = \frac{1}{\sqrt{2}} (|a_1b_2⟩ ± |a_2b_1⟩)_{AB}.$$

Here, $a_1$ ($b_1$) and $a_2$ ($b_2$) are the two possible spatial-modes of photon A (B). In this section, our task is to unambiguously distinguish the 16 orthogonal hyperentangled Bell states simultaneously entangled in polarization and spatial-mode DOFs.

We firstly introduce the principle of photon number QND constructed by cross-Kerr nonlinearity, and then describe our HBSA scheme in detail. The interaction between a signal state $|φ⟩_{S}$ and a probe coherent state $|α⟩_{P}$ in nonlinear medium can be described with the Hamiltonian [13]

$$H = \hbar χ|a⟩_{P}a^†_{P}a^†_{P}a^†_{P}.$$

Here, $a^†_{P}$ and $a_{P}$ are the creation and destruction operator of the signal (probe) state, respectively. χ is the coupling strength of the nonlinearity and it depends on the property of nonlinear material. After the interaction with signal state in the nonlinear medium, the probe coherent state will get a phase shift, which is proportional to the photon number $N$ of the signal state,

$$|α⟩_{P} → |αe^{iNθ}⟩.$$

Here $θ = χ t$ ($t$ is the interaction time). By using the X-quadrature measurement on the phase shift of probe coherent state, we can read out the photon number of signal state without destroying the photons. In our scheme, this photon number QND is used to get the parity information of the spatial-mode entanglement, and only one QND is required in our complete HBSA scheme.

The auxiliary frequency entanglement we utilized is in a fixed Bell state,

$$|φ^+⟩_{F} = \frac{1}{\sqrt{2}} (|ω_1ω_1⟩ + |ω_2ω_2⟩)_{AB}, \quad (6)$$

where $ω_1$ and $ω_2$ are the two different frequencies of photon. This frequency entanglement is used to get the parity information of the polarization Bell states, and the frequency distinguishability will be eliminated with the help of frequency shifter.
spatial-mode DOF will not be changed, and the hyperentanglement in polarization and frequency DOFs evolve as the collective system composed of spatial-mode entanglement and coherent state evolve as coherent state $|\alpha\rangle$. Thus, the spatial-mode Bell states $|\phi^{+}\rangle_S$, $|\phi^{-}\rangle_S$, $|\psi^{+}\rangle_S$, and $|\psi^{-}\rangle_S$ can be nondestructively distinguished by using the X-quadrature measurement on coherent state $|\alpha\rangle$, and the detection result is shown in Table 1.

Table 1. Corresponding relations between the initial spatial-mode Bell states and phase shifts of the coherent state.

| Spatial-mode state    | $|\alpha\rangle$ |
|-----------------------|-------------------|
| $|\phi^{+}\rangle_S$  | $0$               |
| $|\phi^{-}\rangle_S$  | $0$               |
| $|\psi^{+}\rangle_S$  | $\pm\theta$       |
| $|\psi^{-}\rangle_S$  | $\pm\theta$       |

The setup of our complete HBSA scheme is shown in Figure 1. After photons $A$ and $B$ interact with the coherent state $|\alpha\rangle$, the hyperentanglement in polarization and frequency DOFs will not be changed, and the collective system composed of spatial-mode entanglement and coherent state evolve as

$$
|\phi^{+}\rangle_S|\alpha\rangle \rightarrow |\phi^{+}\rangle_S|\alpha\rangle, \quad |\psi^{+}\rangle_S|\alpha\rangle \rightarrow |\psi^{+}\rangle_S|\alpha e^{i\theta}\rangle.
$$

(7)

Thus, the spatial-mode Bell states $|\phi^{+}\rangle_S$ and $|\psi^{+}\rangle_S$ can be nondestructively distinguished by using the X-quadrature measurement on coherent state $|\alpha\rangle$, and the detection result is shown in Table 1.

After photons $A$ and $B$ pass through the FBSSs, FSs, PBSs and HWP sequentially, the entanglement in spatial-mode DOF will not be changed, and the hyperentanglement in polarization and frequency DOFs evolve as

$$
|\phi^{+}\rangle_P \otimes |\phi^{+}\rangle_E \rightarrow |\phi^{+}\rangle_P \otimes \frac{1}{\sqrt{2}}(|x_1x_1\rangle + |x_2x_2\rangle)_{AB},
$$

$$
|\phi^{-}\rangle_P \otimes |\phi^{+}\rangle_E \rightarrow |\psi^{+}\rangle_P \otimes \frac{1}{\sqrt{2}}(|x_1x_1\rangle + |x_2x_2\rangle)_{AB},
$$

$$
|\psi^{+}\rangle_P \otimes |\phi^{+}\rangle_E \rightarrow |\phi^{+}\rangle_P \otimes \frac{1}{\sqrt{2}}(|x_1x_1\rangle + |x_2x_2\rangle)_{AB},
$$

$$
|\psi^{-}\rangle_P \otimes |\phi^{+}\rangle_E \rightarrow |\psi^{-}\rangle_P \otimes \frac{1}{\sqrt{2}}(|x_1x_2\rangle + |x_2x_1\rangle)_{AB}.
$$

(8)

From the above evolutions, we can see that the four polarization Bell states have been completely separated from each other. Specifically, the parity information of polarization entanglement can be determined by
using the auxiliary frequency entanglement, which has been transformed into the path entanglement represented by $x_1$ and $x_2$. The phase information of polarization entanglement can be determined by the orthogonal measurement in $\{H, V\}$ basis, resorting to the Hadamard operation on polarization state and single-photon detectors.

Subsequently, after photons A and B pass through the BSs, the entanglement in polarization DOF will not be changed, and the entanglement in spatial-mode DOF evolve as

$$|\phi^+\rangle_S \rightarrow |\phi^+\rangle_S, \quad |\phi^-\rangle_S \rightarrow |\psi^-\rangle_S,$$

$$|\psi^+\rangle_S \rightarrow |\phi^-\rangle_S, \quad |\psi^-\rangle_S \rightarrow |\psi^-\rangle_S.$$  \hspace{1cm} (9)

Therefore, the phase information of spatial-mode entanglement can be determined by using the Hadamard operation on spatial-mode state. With these steps, the BSA in polarization DOF and BSA in spatial-mode DOF can be accomplished independently.

As shown in table 2, the 16 hyperentangled Bell states can be classified into 8 groups and each group contains two initial states, only according to the detection results of single-photon detectors (equations (8) and (9)). In each group, the two hyperentangled Bell states own different spatial-mode states, which can be identified by using table 1 (equation (7)). Thus, the complete HBSA is accomplished by using tables 1 and 2 simultaneously.

### 3. Complete hyperentangled Greenberger–Horne–Zeilinger state analysis

The three-photon GHZ state in polarization DOF can be written as

$$|\Phi^+_{000}\rangle_P = \frac{1}{\sqrt{2}}(|HHH\rangle \pm |VVV\rangle)_{ABC},$$

$$|\Phi^+_{001}\rangle_P = \frac{1}{\sqrt{2}}(|HHV\rangle \pm |VHV\rangle)_{ABC},$$

$$|\Phi^+_{010}\rangle_P = \frac{1}{\sqrt{2}}(|HVV\rangle \pm |VHH\rangle)_{ABC},$$

$$|\Phi^+_{100}\rangle_P = \frac{1}{\sqrt{2}}(|VHH\rangle \pm |HHH\rangle)_{ABC}.\hspace{1cm} (10)$$

Likely, the three-photon GHZ state in spatial-mode DOF can be written as

$$|\Phi^+_{000}\rangle_S = \frac{1}{\sqrt{2}}(|a_1b_1c_1 \pm a_2b_2c_2\rangle)_{ABC},$$

$$|\Phi^+_{001}\rangle_S = \frac{1}{\sqrt{2}}(|a_1b_1c_2 \pm a_2b_2c_1\rangle)_{ABC},$$

$$|\Phi^+_{010}\rangle_S = \frac{1}{\sqrt{2}}(|a_1b_2c_1 \pm a_2b_1c_2\rangle)_{ABC},$$

$$|\Phi^+_{100}\rangle_S = \frac{1}{\sqrt{2}}(|a_2b_1c_1 \pm a_1b_2c_2\rangle)_{ABC}.\hspace{1cm} (11)$$

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Table 2. Relations between the initial hyperentangled Bell states and the possible detections of single-photon detectors.

| Initial states | Possible detections |
|----------------|---------------------|
| $|\phi^+\rangle_S, |\phi^-\rangle_S,$ | $|\phi^+\rangle_S, |\psi^+\rangle_S,$ |
| $|\phi^-\rangle_S, |\phi^+\rangle_S,$ | $|\phi^-\rangle_S, |\psi^-\rangle_S,$ |
| $|\psi^+\rangle_S, |\phi^-\rangle_S,$ | $|\psi^+\rangle_S, |\psi^-\rangle_S,$ |
| $|\psi^-\rangle_S, |\phi^+\rangle_S,$ | $|\psi^-\rangle_S, |\psi^-\rangle_S,$ |
Figure 2. Schematic diagram of our complete three-photon HGSA scheme for photons in polarization and spatial-mode DOFs. Two coherent states are used to determine the parity information of spatial-mode GHZ state. With the help of auxiliary frequency GHZ state, the parity information of polarization GHZ state is obtained. The phase information of hyperentanglement can be determined successfully, resorting to HWP, BS and single-photon detector. With this device, the 64 hyperentangled GHZ states can be completely distinguished.

Table 3. Corresponding relations between the initial spatial-mode GHZ states and phase shifts of the two coherent states.

| Spatial-mode GHZ state | \(|\alpha_1\rangle\) | \(|\alpha_2\rangle\) |
|------------------------|------------------|------------------|
| \(|\Phi^{\pm}_{000}\rangle\) | 0 | 0 |
| \(|\Phi^{\pm}_{001}\rangle\) | \(\pm\theta\) | \(\pm\theta\) |
| \(|\Phi^{\pm}_{010}\rangle\) | \(\pm\theta\) | 0 |
| \(|\Phi^{\pm}_{100}\rangle\) | \(\pm\theta\) | \(\pm\theta\) |

The auxiliary frequency entanglement we utilized is in a fixed GHZ state,

\[ |\Phi^+\rangle_F = \frac{1}{\sqrt{2}} \left( |\omega_1\omega_1\omega_1\rangle + |\omega_2\omega_2\omega_2\rangle \right)_{ABC}. \] (12)

The setup of our complete three-photon HGSA scheme is shown in figure 2. After photons \(A, B\) and \(C\) interact with the coherent states \(|\alpha_1\rangle\) and \(|\alpha_2\rangle\), the hyperentanglement in polarization and frequency DOFs is invariant, and the collective system composed of spatial-mode GHZ state and two coherent states will evolve as

\[ |\Phi^+_{000}\rangle |\alpha_1\rangle |\alpha_2\rangle \rightarrow |\Phi^+_{000}\rangle |\alpha_1\rangle |\alpha_2\rangle, \]
\[ |\Phi^+_{001}\rangle |\alpha_1\rangle |\alpha_2\rangle \rightarrow |\Phi^+_{001}\rangle |\alpha_1\rangle e^{\pm i\theta} |\alpha_2\rangle, \]
\[ |\Phi^+_{010}\rangle |\alpha_1\rangle |\alpha_2\rangle \rightarrow |\Phi^+_{010}\rangle |\alpha_1\rangle e^{\pm i\theta} |\alpha_2\rangle, \]
\[ |\Phi^+_{100}\rangle |\alpha_1\rangle |\alpha_2\rangle \rightarrow |\Phi^+_{100}\rangle |\alpha_1\rangle e^{\pm i\theta} |\alpha_2\rangle e^{\pm i\theta}. \] (13)

Based on the measurements on the two coherent states \(|\alpha_1\rangle\) and \(|\alpha_2\rangle\), the eight GHZ states in spatial-mode DOF can be classified into four groups. In other words, the parity information of the spatial-mode GHZ state is obtained, and the result is shown in table 3.
Table 4. The classification of the 64 hyperentangled GHZ states based on the possible detections of single-photon detectors.

| Group | Possible initial states |
|-------|------------------------|
| 1     | $|\Phi^+_{000}\rangle\otimes|\Phi^+_{000}\rangle_F \rightarrow |\Phi^+_{000}\rangle_p \otimes \frac{1}{\sqrt{2}}(|x_1x_1x_1\rangle + |x_2x_2x_2\rangle)_ABC,$ |
| 2     | $|\Phi^+_{001}\rangle\otimes|\Phi^+_{001}\rangle_F \rightarrow |\Phi^+_{001}\rangle_p \otimes \frac{1}{\sqrt{2}}(|x_1x_1x_2\rangle + |x_2x_2x_1\rangle)_ABC,$ |
| 3     | $|\Phi^+_{010}\rangle\otimes|\Phi^+_{010}\rangle_F \rightarrow |\Phi^+_{010}\rangle_p \otimes \frac{1}{\sqrt{2}}(|x_1x_2x_1\rangle + |x_2x_1x_2\rangle)_ABC,$ |
| 4     | $|\Phi^+_{100}\rangle\otimes|\Phi^+_{100}\rangle_F \rightarrow |\Phi^+_{100}\rangle_p \otimes \frac{1}{\sqrt{2}}(|x_1x_1x_2\rangle + |x_2x_2x_1\rangle)_ABC.$ |

After the three photons pass through the FBs, FSs and PBSs, the entanglement in spatial-mode DOF is invariant, and the hyperentanglement in polarization and frequency DOFs will evolve as

$$|\Phi^+_{000}\rangle_p \rightarrow \frac{1}{2}([HHH] + [HHV] + [VHV] + [VHH])_ABC,$$

$$|\Phi^+_{000}\rangle_p \rightarrow \frac{1}{2}([HHV] + [HVV] + [VHV] + [VVV])_ABC.$$  \hspace{1cm} (15)

We can find that if the relative phase information of polarization state is ‘+’, the detection result will be $HHH$ or $HVV$ or $VHV$, in which the number of $V$ is even. If the relative phase information is ‘-‘, the detection result will be $HHV$ or $HVH$ or $VHH$ or $VVV$, in which the number of $V$ is odd. The phase information of spatial-mode state can be obtained with the same idea, which is realized by the BSs. For example,

$$|\Phi^+_{000}\rangle_S \rightarrow \frac{1}{2}(|a_1b_1c_1\rangle + |a_1b_2c_2\rangle + |a_2b_1c_1\rangle + |a_2b_2c_2\rangle)_ABC,$$

$$|\Phi^+_{000}\rangle_S \rightarrow \frac{1}{2}(|a_1b_2c_2\rangle + |a_1b_1c_1\rangle + |a_2b_2c_1\rangle + |a_2b_1c_2\rangle)_ABC.$$

The ‘+’ relative phase information of spatial-mode state corresponds to $a_1b_1c_1, a_1b_2c_2, a_2b_1c_2$ and $a_2b_2c_1$, in which the number of $m_2(m = a, b, c)$ is even. While the odd number of $m_2$ indicates the ‘-‘ relative phase information for the spatial-mode entanglement. With these four steps, the three-photon GSA in polarization DOF and GSA in spatial-mode DOF have been accomplished independently.

Based on equations (14)–(16), the 64 hyperentangled GHZ states can be classified into 16 groups with the detection results of single-photon detectors, as shown in table 4. In each group, the four hyperentangled GHZ states own different spatial-mode states, which can be identified by using table 3 (equation (13)).

Thus, the complete three-photon HGS is accomplished by using tables 3 and 4 simultaneously.
Using more QNDs, our scheme can be extended to the complete \( N \)-photon HGSA in polarization and spatial-mode DOFs. In our \( N \)-photon HGSA scheme, \( N - 1 \) two-photon QNDs are required, and the auxiliary frequency entanglement can be set to the \( N \)-qubit frequency GHZ state,

\[ |\Phi_F\rangle = \frac{1}{\sqrt{2}}(|\omega_1\omega_2\cdots\omega_N\rangle + |\omega_2\omega_2\cdots\omega_2\rangle)_{AB\cdots N} \]  

(17)

The complete analysis of the \( 4^N \) hyperentangled GHZ states can be accomplished via four steps, which is similar to the three-photon HGSA. First, the \( 2^N \) spatial-mode GHZ states are classified into \( 2^{N-1} \) groups using X-quadrature measurements on the \( N - 1 \) coherent states, thus the parity information of spatial-mode entanglement is obtained. Second, auxiliary frequency entanglement is used to obtain the parity information of polarization \( N \)-photon GHZ state, identified by the path entanglement represented by \( x_1, x_2, \cdots \) and \( x_N \). Subsequently, the relative phase information of polarization entanglement is obtained by using HWPs and orthogonal polarization measurement. At last, BSs are used to distinguish the spatial-mode \( N \)-photon GHZ states with different phase information. After the \( N \)-photon GSA in polarization and spatial-mode DOFs are accomplished, the \( 4^N \) hyperentangled GHZ states in two DOFs have been distinguished unambiguously.

4. Discussion and summary

In this paper, we develop a new method for the complete analysis of hyperentangled Bell state and hyperentangled GHZ state, based on the previous schemes for state analysis. The weak cross-Kerr nonlinearity is utilized to construct the two-photon QND, which can determine the parity information of spatial-mode entanglement. In the process of obtaining the parity information of polarization entanglement, the auxiliary frequency Bell state or GHZ state is used, and it has been transformed into the path entanglement. Then the linear optical elements HWP and BS are used to determine the phase information of polarization and spatial-mode state, respectively. In essence, the manipulation and measurement of the two different DOFs in hyperentangled state are independent, that is why our method can work in a successful way. It is interesting that the QND and hyperentangled Bell state simultaneously entangled in polarization, spatial-mode and frequency DOFs have been used for deterministic entanglement purification protocol and nonlocal BSA in 2010 [29]. Recently, BSA and HBSA are widely used in measurement-device-independent quantum key distribution, and measurement-device-independent quantum secure direct communication [30–36].

The generation of hyperentangled photons in polarization, spatial-mode and frequency DOFs is required in our scheme. Therefore, it is necessary to discuss the possible generation with current technology. In 2005, Barbieri et al presented an experimental method to engineer polarization-momentum hyperentangled two-photon states [37]. From the laser cooled atoms, Yan et al reported the generation of nondegenerate narrow-bandwidth paired photons with time-frequency and polarization entanglement in 2011 [38]. In 2015, Shu et al demonstrated the generation of narrowband biphotons with polarization-frequency-coupled entanglement from spontaneous four-wave mixing in cold atoms [39]. In 2019, Kaneda et al reported an efficient scheme for direct generation of frequency-bin entangled photon pairs via spontaneous parametric downconversion [40]. Inspired by these important advances, we are confident that the hyperentangled photons in polarization, spatial-mode and frequency DOFs can be realized in experiment. In our scheme, the frequency DOF is used as the auxiliary DOF to assist the HBSA or HGSA in polarization and spatial-mode DOFs. Actually, our method reveals a general conclusion that the third DOF in hyperentanglement can be useful for the HBSA or HGSA in the first and second DOFs. Therefore, our method is not only suitable for polarization–spatial-mode-frequency hyperentanglement, but also suitable for other hyperentanglement in three DOFs. Fortunately, some significant experiments have already reported the hyperentangled photons in three DOFs. In 2005, Barreiro et al produced and characterized pairs of single photons simultaneously entangled in polarization, spatial-mode and time-bin DOFs [41]. In 2009, Vallone et al presented an optical scheme enabling the simultaneous entanglement of two photons in three independent DOFs [42]. In 2018, Wang et al experimentally demonstrated an 18-qubit GHZ entanglement by exploiting three different DOFs of six photons, including their paths, polarization and orbital angular momentum [43].

In our scheme, the QND constructed by weak cross-Kerr nonlinearity is utilized, so it is important for us to discuss the feasibility of such QND in current experiment condition. We should acknowledge that although many works have been studied on cross-Kerr nonlinearity, a clean cross-Kerr nonlinearity in the optical single-photon regime is still a challenging task with current technology [44–50]. In 2010, Gea-Banacloche showed that the large phase shift via the giant Kerr effect with single-photon wave packets is impossible at present [51]. For a weak cross-Kerr nonlinearity, it is possible to distinguish the small phase
shift in the coherent state from the 0 phase shift, when a sufficiently large amplitude of the coherent state satisfies $\alpha \theta^2 \gg 1$ (where $\theta$ is the cross-phase shift). Recent research works have shown that the practical use of cross-Kerr effect is promising in the near future. In 2011, Feizpour et al showed that the weak measurement can be used to ‘amplify’ optical nonlinearities at the single-photon level [52]. In the same year, Zhu et al showed that giant Kerr nonlinearity of the probe and the signal pulses may be achieved with nearly vanishing optical absorption [53]. In 2013, Hoi et al investigated the interaction between the fields and atom, which can produce an effective cross-Kerr coupling, and they also demonstrated average cross-Kerr phase shifts of up to 20 degrees per photon with both coherent microwave fields at the single-photon level [54]. In 2015, Feizpour et al presented an implementation of a strong optical nonlinearity using electromagnetically induced transparency, and a direct measurement of the resulting nonlinear phase shift for single post-selected photons [55]. In addition, the improvement of the measurement on the coherent state can also promote the feasibility of the QND. In 2010, Wittmann et al showed that the performance of the displacement-controlled photon number resolving detector is better than that of the homodyne detector [56]. Fortunately, the QND in our complete HBSP scheme and HGSA scheme only need a small phase shift that can be distinguished from zero. In our scheme, the conditional phase shift $-\theta$ is used. However, the conditional phase shift $-\theta$ is equivalent to the conditional phase shift $2\pi - \theta$, which is impossible with the current experimental technology because the giant cross-Kerr nonlinearity is required. In order to replace the conditional phase shift $-\theta$, we can use the approach called double cross phase modulation without conditional phase shift $-\theta$, which had been developed in 2009 [57–59]. Moreover, there are some other kinds of interaction that can also provide feasible ways to construct the QND we need, such as cavity-assisted interactions [60] and quantum dot spin in optical microcavity [61]. In our protocol, the FBS and FS are used to operate the frequency DOF of photon, both of which can be implemented experimentally [62–67].

From the schemes of Sheng et al [21] and Li et al [25], we can find that two QNDs are required at least for a complete HBSP. In the theory presented by Li et al in 2017 [20], they showed that the 16 hyperentangled Bell states in two DOFs can only be separated into 12 groups if auxiliary entanglement in the third DOF is used, thus the complete HBSP cannot be accomplished. Combining these two ideas, we develop an efficient method for the complete HBSP, which can be extended to the complete $N$-photon HGSA. In our complete HBSP scheme, just one QND and one auxiliary Bell state in frequency DOF are utilized, which is the optimal result when QND and auxiliary entanglement are both used for the complete analysis of hyperentangled Bell state. Compared with the two previous HBSP schemes [21, 25], our scheme is more efficient and realizable because we use less nonlinearity (only one QND is required). In our complete $N$-photon HGSA scheme, we use $N - 1$ QNDs and one auxiliary $N$-photon GHZ state in frequency DOF to distinguish the $4^N$ hyperentangled GHZ states. The QND we required still can be realized by the weak cross-Kerr nonlinearity with a small phase shift, which is a significant advantage of our HGSA scheme. In the complete HGSA scheme proposed by Li et al, the giant cross-Kerr nonlinearity that can generate a $\pi$ shift is necessary [25]. However, as we know, it is very hard to obtain this giant nonlinear effect in experiment. Therefore, our HGSA scheme can largely reduce the requirement on nonlinear strength, and become more practical. In summary, we have presented a new method for the complete analysis of hyperentangled state that is feasible with current technology, and we think this approach provides an efficient avenue for hyperentangled state analysis.

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