INVARiance OF THE BFV–COMPLEX

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ABSTRACT. The BFV-formalism was introduced to handle classical systems, equipped with symmetries. It associates a differential graded Poisson algebra to any coisotropic submanifold $S$ of a Poisson manifold $(M, \Pi)$.

However the assignment (coisotropic submanifold) $\rightsquigarrow$ (differential graded Poisson algebra) is not canonical, since in the construction several choices have to be made. One has to fix: 1. an embedding of the normal bundle $NS$ of $S$ into $M$ as a tubular neighbourhood, 2. a connection $\nabla$ on $NS$ and 3. a special element $\Omega$.

We show that different choices of a connection and an element $\Omega$ – but with the tubular neighbourhood fixed – lead to isomorphic differential graded Poisson algebras. If the tubular neighbourhood is changed too, invariance can be restored at the level of germs.

1. INTRODUCTION

The Batalin-Vilkovisky-Fradkin complex (BFV-complex for short) was introduced in order to understand physical systems with complicated symmetries ([BF], [BV]). The connection to homological algebra was made explicit in [St] later on. We focus on the smooth setting, i.e. we want to consider arbitrary coisotropic submanifolds of smooth finite dimensional Poisson manifolds. Bordemann and Herbig found a convenient adaptation of the BFV-construction in this framework ([B], [He]): One obtains a differential graded Poisson algebra associated to any coisotropic submanifold. In [Sch] a slight modification of the construction of Bordemann and Herbig was presented. It made use of the language of higher homotopy structures and provided in particular a conceptual construction of the BFV-bracket.

Note that in the smooth setting the construction of the BFV-complex requires a choice of the following pieces of data: 1. an embedding of the normal bundle of the coisotropic submanifold as a tubular neighbourhood into the ambient Poisson manifold, 2. a connection on the normal bundle, 3. a special function on a smooth graded manifold, called a BFV-charge.

We apply the point of view established in [Sch] to clarify the dependence of the resulting BFV-complex on these data. If one leaves the embedding

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fixed and only changes the connection and the BFV-charge, one simply obtains two isomorphic differential graded Poisson algebras, see Theorem 1 in Section 3. Note that the dependence on the choice of BFV-charge was well understood, see [St] for instance. Dependence on the embedding is more subtle. We introduce the notion of “restriction” of a given BFV-complex to an open neighbourhood of the coisotropic submanifold inside its normal bundle (Definition 2) and show that different choices of embeddings lead to isomorphic restricted BFV-complexes – see Theorem 2 in Section 4. As a Corollary one obtains that a germ-version of the BFV-complex is independent of all the choices up to isomorphism (Corollary 4).

It turns out that the differential graded Poisson algebra associated to a fixed embedding of the normal bundle as a tubular neighbourhood, yields a description of the moduli space of coisotropic sections in terms of the BFV-complex – see [Sch2].

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2. Preliminaries

The purpose of this Section is threefold: to recollect some facts about the theory of higher homotopy structures, to recall some concepts concerning Poisson manifolds and coisotropic submanifolds and to outline the construction of the BFV-complex. More details on these subjects can be found in Sections 2 and 3 of [Sch] and in the references cited therein. We assume the reader to be familiar with the theory of graded algebras and smooth graded manifolds.

2.1. $L_{\infty}$-algebras: Homotopy Transfer and Homotopies. Let $V$ be a $\mathbb{Z}$-graded vector space over $\mathbb{R}$ (or any other field of characteristic 0); i.e., $V$ is a collection $(V_i)_{i \in \mathbb{Z}}$ of vector spaces $V_i$ over $\mathbb{R}$. The homogeneous elements of $V$ of degree $i \in \mathbb{Z}$ are the elements of $V_i$. We denote the degree of a homogeneous element $x \in V$ by $|x|$. A morphism $f : V \to W$ of graded vector spaces is a collection $(f_i : V_i \to W_i)_{i \in \mathbb{Z}}$ of linear maps. The $n$th suspension functor $[n]$ from the category of graded vector spaces to itself is defined as follows: given a graded vector space $V$, $V[n]$ denotes the graded vector space corresponding to the collection $V[n]_i := V_{n+i}$. The $n$th suspension of a morphism $f : V \to W$ of graded vector spaces is given by the collection $(f[n]_i := f_{n+i} : V_{n+i} \to W_{n+i})_{i \in \mathbb{Z}}$. The tensor product of two graded vector spaces $V$ and $W$ over $\mathbb{R}$ is the graded vector whose component in degree $k$ is given by

$$(V \otimes W)_k := \bigoplus_{r+s=k} V_r \otimes W_s.$$ 

The denote this graded vector space by $V \otimes W$.

The structure of a flat $L_{\infty}[1]$-algebra on $V$ is given by a family of multilinear maps $(\mu^k : V^{\otimes k} \to V[1])_{k \geq 1}$ that satisfies:
(1) $\mu^k(\cdots \otimes a \otimes b \otimes \cdots) = (-1)^{|a||b|} \mu^k(\cdots \otimes b \otimes a \otimes \cdots)$ holds for all $k \geq 1$ and all homogeneous elements $a, b$ of $V$.

(2) The family of Jacobiators $(J^k)_{k \geq 1}$ defined by

$$J^k(x_1 \cdots x_n) := \sum_{r+s=k} \sum_{\sigma \in \{1, \ldots, k\}_{-\text{shuffles}}} \text{sign}(\sigma) \mu^{s+1}(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(r)} \otimes x_{\sigma(r+1)} \otimes \cdots \otimes x_{\sigma(n)})$$

vanishes identically. Here sign$(\cdot)$ is the Koszul sign, i.e. the representation of $\Sigma_n$ on $V^\otimes_n$ induced by mapping the transposition $(2, 1)$ to $a \otimes b \mapsto (1)^{|a||b|} b \otimes a$. Moreover $(r, s)$-shuffles are permutations $\sigma$ of $\{1, \ldots, k \equiv r + s\}$ such that $\sigma(1) < \cdots < \sigma(r)$ and $\sigma(r + 1) < \cdots < \sigma(k)$.

Since we are only going to consider flat $L_\infty[1]$-algebras we will suppress the adjective “flat” from now on. In this case the vanishing of the first Jacobiator implies that $\mu^1$ is a coboundary operator. We remark that an $L_\infty[1]$-algebra structure on $V$ is equivalent to the more traditional notion of an $L_\infty$-algebra structure on $V[-1]$, see [MSS] for instance.

Given an $L_\infty$-algebra structure $(\mu^k)_{k \geq 1}$ on $V$, there is a distinguished subset of $V_1$ that contains elements $v \in V_1$ satisfying the Maurer-Cartan equation (MC-equation for short)

$$\sum_{k \geq 1} \frac{1}{k!} \mu^k(v \otimes \cdots \otimes v) = 0.$$

This set is called the set of Maurer-Cartan elements (MC-elements for short) of $V$.

Let $V$ be equipped with an $L_\infty$-algebra structure such that the coboundary operator $\mu^1$ decomposes into $d + \delta$ with $d^2 = 0 = \delta^2$ and $d \circ \delta + \delta \circ d = 0$. i.e. $(V, d, \delta)$ is a double complex. Then – under mild convergence assumptions – it is possible to construct an $L_\infty$-algebra structure on $H(V, d)$ that is “isomorphic up to homotopy” to the original $L_\infty$-algebra structure on $V$ ([GL]). More concretely, one has to fix an embedding $i$ of $H(V, d)$ into $V$, a projection $pr$ from $V$ to $H(V, d)$ and a homotopy operator $h$ (of degree $-1$) which satisfies

$$d \circ h + h \circ d = id_V - i \circ pr.$$

We will also impose the following side-conditions for the sake of simplicity: 1.) $h \circ h = 0$, 2.) $pr \circ h = 0$ and 3.) $h \circ i = 0$. Then explicit formulae for the structure maps for an $L_\infty$-algebras on $H(V, d)$ can be written down. These are given in terms of rooted planar trees, see [Sch] for a review. We will explain the construction in more detail later on for the examples which are relevant for our purpose.

Furthermore one obtains $L_\infty$-morphisms between $H(V, d)$ and $V$ that induce inverse maps on cohomology. Such $L_\infty$-morphisms are called $L_\infty$ quasi-isomorphisms.
Consider the differential graded algebra $(\Omega([0,1]),d_{DR},\wedge)$ of smooth forms on the interval $I := [0,1]$. The inclusions of a point $\{s\}$ as $0 \leq s \leq 1$ induces a chain map $ev_s : (\Omega(I),d_{DR}) \to (\mathbb{R},0)$ that is a morphisms of algebras. Given any $L_\infty$-algebra structure on $V$ there is a natural $L_\infty$-algebra structure on $V \otimes \Omega(I)$ defined by

$$\tilde{\mu}^1(v \otimes \alpha) := \mu^1(v) \otimes \alpha + (-1)^{|v|}v \otimes d_{DR}\alpha$$

and

$$\tilde{\mu}^k((v_1 \otimes \alpha_1) \otimes \cdots \otimes (v_k \otimes \alpha_k)) := (-1)^\# \mu^k(v_1 \otimes \cdots \otimes v_k) \otimes (\alpha_1 \wedge \cdots \wedge \alpha_k)$$

for $k \geq 2$. Here $\#$ denotes the sign one picks up by assigning $(-1)^{|v_i+1||\alpha_i|}$ to passing $\alpha_i$ from the left-hand side of $v_{i+1}$ to the right-hand side (and replacing $\alpha_{i+1}$ by $\alpha_i \wedge \alpha_{i+1}$).

Following [MSS], we call two morphisms $f$ and $g$ from an $L_\infty$-algebra $A$ to $B$ homotopic if there exists an $L_\infty$-morphism $F$ from $A$ to $B \otimes \Omega(I)$ such that

- $(\text{id} \otimes ev_0) \circ F = f$ and
- $(\text{id} \otimes ev_1) \circ F = g$ hold.

This defines an equivalence relation on the set of $L_\infty$-morphisms from $A$ to $B$.

Let $F$ be an $L_\infty$-morphism from $A$ to $B \otimes \Omega(I)$. Consequently $f_s := ev_s \circ F$ is an $L_\infty$ morphism between $A$ and $B$ for any $s \in I$. Given a MC-element $v$ in $A$ one obtains a one-parameter family of MC-elements

$$w_s := \sum_{k \geq 1} \frac{1}{n!} (f_s)_k (v \otimes \cdots \otimes v)$$

of $B$. Here $(f_s)_k$ denotes the $k$th Taylor component of $f_s$.

In the main body of this paper we are only interested in the following particular case: $B$ is a differential graded Lie algebra (i.e. only the first and second structure maps are non-vanishing). Denote the graded Lie bracket by $[\cdot, \cdot]$. Furthermore we assume that the differential $D$ is given by the adjoint action of a degree +1 element $\Gamma$ that satisfies $[\Gamma, \Gamma] = 0$. The MC-equation for an element $w$ of $(B,D = [\Gamma, \cdot],[\cdot, \cdot])$ reads

$$[\Gamma + w, \Gamma + w] = 0.$$ 

From the one-parameter family of MC-elements $w_s$ in $B$ one obtains a one-parameter family of differential graded Lie algebras on $B$ by setting

$$D_s(\cdot) := [\Gamma + w_s, \cdot]$$

while leaving the bracket unchanged.

How are the differential graded Lie algebras $(B, D_s, [\cdot, \cdot])$ related for different values of $s \in I$? To answer this question we first apply the $L_\infty$ morphism $F : A \sim B \otimes \Omega(I)$ to $v$ and obtain a MC-element $w(t) + u(t)dt$ in $B \otimes \Omega(I)$. 


It is straightforward to check that \( w(s) = w_s \) for all \( s \in I \). Moreover the MC-equation in \( B \otimes \Omega(I) \) splits up into

\[
[\Gamma + w(t), \Gamma + w(t)] = 0
\]

and

\[
\frac{d}{dt} w(t) = [u(t), \Gamma + w(t)].
\]

The second equation implies that whenever the adjoint action of \( u(t) \) on \( B \) can be integrated to a one-parameter family of automorphisms \( (U(t))_{t \in I} \), \( U(s) \) establishes an automorphism of \( (B, [\cdot, \cdot]) \) that maps \( \Gamma + w(0) \) to \( \Gamma + w(s) \) (for any \( s \in I \)). Consequently:

**Lemma 1.** Let \( A \) and \( (B, [\Gamma, \cdot], [\cdot, \cdot]) \) be differential graded Lie algebras, \( v \) an MC-element in \( A \) and \( F \) an \( L_\infty \) morphism from \( A \) to \( B \otimes \Omega(I) \) such that

\[
\sum_{k \geq 1} \frac{1}{k!} F_k(v \otimes \cdots \otimes v)
\]

is well-defined in \( B \otimes \Omega(I) \). Denote this element by \( w(t) + u(t)dt \). Furthermore the flow equation

\[
X(0) = b, \quad \left. \frac{d}{dt} \right|_{t=s} X(t) = [u(s), X(s)], \quad s \in I
\]

is assumed to have a unique solution for arbitrary \( b \in B \).

Then the one-parameter family \( U(t) \) of automorphisms of \( B \) that integrates the adjoint action by \( u(t) \) maps \( \Gamma + w(0) \) to \( \Gamma + w(t) \). In particular \( U(s) \) is an isomorphisms of differential graded Lie algebras

\[
(B, [\Gamma + w(0), \cdot], [\cdot, \cdot]) \rightarrow (B, [\Gamma + w(s), \cdot], [\cdot, \cdot])
\]

for arbitrary \( s \in I \).

### 2.2. Coisotropic Submanifolds

We essentially follow [W], where more details can be found. Let \( M \) be a smooth, finite dimensional manifold. The bivector field \( \Pi \) on \( M \) is Poisson if the binary operation \( \{\cdot, \cdot\} \) on \( C^\infty(M) \) given by \( (f, g) \mapsto \langle \Pi, df \wedge dg \rangle \) satisfies the Jacobi identity, i.e.

\[
\{\{f, g\}, h\} + \{g, \{f, h\}\} = \{f, \{g, h\}\}
\]

holds for all smooth functions \( f, g \) and \( h \). Here \( \langle -, - \rangle \) denotes the natural pairing between \( TM \) and \( T^*M \). Alternatively one can consider the graded algebra \( \mathcal{V}(M) \) of multivector fields on \( M \) equipped with the Schouten-Nijenhuis bracket \( [\cdot, \cdot]_{SN} \). A bivector field \( \Pi \) is Poisson if and only if \( [\Pi, \Pi]_{SN} = 0 \).

Associated to any Poisson bivector field \( \Pi \) on \( M \) there is a vector bundle morphism \( \Pi^\#: T^*M \rightarrow TM \) given by contraction. Consider a submanifold \( S \) of \( M \). The annihilator \( N^*S \) of \( TS \) is a subbundle of \( T^*M \). This subbundle fits into a short exact sequence of vector bundles:

\[
0 \longrightarrow N^*S \longrightarrow T^*M|_S \longrightarrow T^*S \longrightarrow 0.
\]
Definition 1. A submanifold \( S \) of a smooth, finite dimensional Poisson manifold \((M, \Pi)\) is called coisotropic if the restriction of \( \Pi^\# \) to \( N^*S \) has image in \( TS \).

There is an equivalent characterization of coisotropic submanifolds: define the vanishing ideal of \( S \) by

\[ I_S := \{ f \in \mathcal{C}^\infty(M) : f|_S = 0 \}. \]

A submanifold \( S \) is coisotropic if and only if \( I_C \) is a Lie subalgebra of \((\mathcal{C}^\infty(M), \{\cdot, \cdot\})\).

2.3. The BFV-Complex. The BFV-complex was introduced by Batalin, Fradkin and Vilkovisky with application in physics in mind ([BF], [BV]). Later on Stasheff ([St]) gave an interpretation of the BFV-complex in terms of homological algebra. The construction we present below is explained with more details in [Sch]. It uses a globalization of the BFV-complex for arbitrary coisotropic submanifolds found by Bordemann and Herbig ([B], [He]).

Let \( S \) be a coisotropic submanifold of a smooth, finite dimensional Poisson manifold \((M, \Pi)\). We outline the construction a differential graded Poisson algebra, which we call a BFV-complex for \( S \) in \((M, \Pi)\). The construction depends on the choice of three pieces of data: 1. an embedding of the normal bundle of \( S \) into \( M \) as a tubular neighbourhood, 2. a connection on \( NS \) and 3. a special smooth function, called the charge, on a smooth graded manifold.

Denote the normal bundle of \( S \) inside \( M \) by \( E \). Consider the graded vector bundle \( E^*[1] \oplus E[-1] \to S \) over \( S \) and let \( E^*[1] \oplus E[-1] \to E \) be the pull back of \( E^*[1] \oplus E[-1] \to S \) along \( E \to S \).

We define \( BFV(E) \) to be the space of smooth functions on the graded manifold which is represented by the graded vector bundle \( E^*[1] \oplus E[-1] \) over \( E \). In terms of sections one has \( BFV(E) = \Gamma(\wedge(E) \otimes \wedge(E^*)) \). This algebra carries a bigrading given by

\[ BFV^{(p,q)}(E) := \Gamma(\wedge^pE \otimes \wedge^qE^*). \]

In physical terminology \( p / q \) is referred to as the ghost degree / ghost-momentum degree respectively. One defines

\[ BFV^k(E) := \bigoplus_{p-q=k} BFV^{(p,q)}(E) \]

and calls \( k \) the total degree (in physical terminology this is the “ghost number”).

The smooth graded manifold \( E^*[1] \oplus E[-1] \) comes equipped with a Poisson bivector field \( G \) given by the natural fibre pairing between \( E \) and \( E^* \), i.e. it is defined to be the natural contraction on \( \Gamma(E) \otimes \Gamma(E^*) \) and extended to a graded skew-symmetric biderivation of \( BFV(E) \).

Choice 1. Embedding.

Fix an embedding \( \psi : E \hookrightarrow M \) of the normal bundle of \( S \) into \( M \). Hence
the normal bundle $E$ inherits a Poisson bivector field which we also denote by $\Pi$. (Keep in mind that $\Pi$ depends on $\psi$!)

**Choice 2.** Connection.

Next choose a connection on the vector bundle $E \to S$. This induces a connection on $\wedge E \otimes \wedge E^* \to S$ and via pull back one obtains a connection $\nabla$ on $\wedge \mathcal{E} \otimes \wedge \mathcal{E}^* \to E$. We denote the corresponding horizontal lift of multivector fields by

$$\iota_{\nabla}: \mathcal{V}(E) \to \mathcal{V}(\mathcal{E}^*[1] \otimes \mathcal{E}[-1]).$$

It extends to an isomorphism of graded commutative unital associative algebras

$$\varphi: A := C^\infty(T^*[1]E \oplus \mathcal{E}^*[1] \oplus \mathcal{E}[-1] \oplus \mathcal{E}[0] \oplus \mathcal{E}^*[2]) \to \mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1]).$$

Using $\varphi$ we lift $\Pi$ to a bivector field on $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]$. Since $\varphi$ fails in general to be a morphism of Gerstenhaber algebras, $\varphi(\Pi)$ is not a Poisson bivector field. Similarly the sum $G + \varphi(\Pi)$ fails to be a Poisson bivector field in general. However the following Proposition provides an appropriate correction term:

**Proposition 1.** Let $\mathcal{E}$ be a finite rank vector bundle with connection $\nabla$ over a smooth, finite dimensional manifold $E$. Consider the smooth graded manifold $\mathcal{E}^*[1] \oplus \mathcal{E}[-1] \to E$ and denote the Poisson bivector field on it coming from the natural fibre pairing between $\mathcal{E}$ and $\mathcal{E}^*$ by $G$.

Then there is an $L_\infty$ quasi-isomorphism $L_{\nabla}$ between the graded Lie algebra

$$(\mathcal{V}(E)[1], [\cdot, \cdot]_{SN})$$

and the differential graded Lie algebra

$$(\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1], [G, \cdot]_{SN}, [\cdot, \cdot]_{SN}).$$

A proof of Proposition 1 can be found in [Sch]. It immediately implies

**Corollary 1.** Let $\mathcal{E} \to E$ be a finite rank vector bundle with connection $\nabla$ over a smooth, finite dimensional Poisson manifold $(E, \Pi)$. Consider the smooth graded manifold $\mathcal{E}^*[1] \oplus \mathcal{E}[-1] \to E$ and denote the Poisson bivector field on it coming from the natural fibre pairing between $\mathcal{E}$ and $\mathcal{E}^*$ by $G$.

Then there is a Poisson bivector field $\tilde{\Pi}$ on $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]$ such that

$$\tilde{\Pi} = G + \varphi(\Pi) + \Delta$$

for $\Delta \in \mathcal{V}^{(1,1)}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$.

For a proof we refer the reader to [Sch] again.

We remark that $\mathcal{V}^{(1,1)}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$ is the ideal of $\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$ generated by multiderivations which map any tensor product of functions of total bidegree $(p, q)$ to a function of bidegree $(P, Q)$ where $P > p$ and $Q > q$. In general, let $\mathcal{V}^{(r,s)}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$ be the ideal generated by multiderivations of $C^\infty(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$ with total ghost degree larger than or equal to $r$ and total ghost-momentum degree larger than or equal to $s$, respectively.

The bivector field $\tilde{\Pi}$ from Corollary 1 equips $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]$ with the structure of a graded Poisson manifold. Consequently $BFV(E)$ inherits a graded
Poisson bracket which we denote by $[\cdot, \cdot]_{BFV}$. It is called the BFV-bracket. Keep in mind that the BFV-bracket depends on the connection on $E \to S$ we have chosen.

**Choice 3. Charge.**

The last step in the construction of the BFV-complex is to provide a special solution to the MC-equation associated to $(BFV(E), [\cdot, \cdot]_{BFV})$, i.e. one constructs a degree +1 element $\Omega$ that satisfies

$$[\Omega, \Omega]_{BFV} = 0.$$ 

Additionally, one requires this element $\Omega$ to contain the tautological section of $\mathcal{E} \to E$ as the lowest order term. To be more precise, recall that

$$BFV^1(E) = \bigoplus_{k \geq 0} \Gamma(\wedge^k \mathcal{E} \otimes \wedge^{k-1} \mathcal{E}^*).$$

Hence any element of $BFV^1(E)$ contains a (possibly zero) component in $\Gamma(\mathcal{E})$. One requires that the component of $\Omega$ in $\Gamma(\mathcal{E})$ is given by the tautological section of $\mathcal{E} \to E$. A MC-element satisfying this requirement is called a BFV-charge.

**Proposition 2.** Let $(E, \Pi)$ be a vector bundle equipped with a Poisson bivector field and denote its zero section by $S$. Fix a connection on $E \to S$ and equip the ghost/ghost-momentum bundle $\mathcal{E}^*[1] \oplus [1] \to E$ with the corresponding BFV-bracket $[\cdot, \cdot]_{BFV}$.

1. There is a degree +1 element $\Omega$ of $BFV(E)$ whose component in $\Gamma(\mathcal{E})$ is given by the tautological section $\Omega_0$ and that satisfies

$$[\Omega, \Omega]_{BFV} = 0$$

if and only if $S$ is a coisotropic submanifold of $(E, \Pi)$.

2. Let $\Omega$ and $\Omega'$ be two BFV-charges. Then there is an automorphism of the graded Poisson algebra $(BFV(E), [\cdot, \cdot]_{BFV})$ that maps $\Omega$ to $\Omega'$.

See [St] for a proof of this proposition.

Given a BFV-charge $\Omega$ one can define a differential $D_{BFV}(\cdot) := [\Omega, \cdot]_{BFV}$, called BFV-differential. It is well-known that the cohomology with respect to $D$ is isomorphic to the Lie algebroid cohomology of $S$ (as a coisotropic submanifold of $(E, \Pi)$).

By the second part of Proposition 2, different choices of the BFV-charge lead to isomorphic differential graded Poisson algebra structures on $BFV(E)$. In the next Section we will establish that different choices of connection on $E \to S$ lead to differential Poisson algebras that lie in the same isomorphism class. The dependence on the embedding of the normal bundle of $S$ is more subtle and will be clarified in Section 4.
3. Choice of Connection

Consider a vector bundle $E$ equipped with a Poisson bivector field $\Pi$ such that that zero section $S$ is coisotropic. The aim of this Section is to investigate the dependence of the differential graded Poisson algebra $(BFV(E), D_{BFV}, [\cdot, \cdot]_{BFV})$ constructed in Subsection 2.3 on the choice of a connection $\nabla$ on $E \to S$.

Recall that in order to lift the Poisson bivector field $\Pi$ to a bivector field on $E^* \oplus E[-1]$, a connection $\nabla$ on $E \to S$ was used. Furthermore the $L_\infty$ quasi-isomorphism between $(\mathcal{V}(E)[1], [\cdot, \cdot]_{SN})$ and $(\mathcal{V}(E^*[1] \oplus E[-1])[1], [G, \cdot]_{SN}, [\cdot, \cdot]_{SN})$ in Proposition 1 depends on $\nabla$ too. Consequently so does the graded Poisson bracket $[\cdot, \cdot]_{BFV}$.

Let $\nabla_0$ and $\nabla_1$ be two connections on a smooth finite rank vector bundle $E \to E$. By Proposition 1 we obtain two $L_\infty$ quasi-isomorphisms $L_{\nabla_0}$ and $L_{\nabla_1}$ from $(\mathcal{V}(E)[1], [\cdot, \cdot]_{SN})$ to $(\mathcal{V}(E^*[1] \oplus E[-1])[1], [G, \cdot]_{SN}, [\cdot, \cdot]_{SN})$. Although these morphisms depend on the connections, this dependence is very well-controlled:

**Proposition 3.** Let $E$ be a smooth finite rank vector bundle over a smooth, finite dimensional manifold $E$ equipped with two connections $\nabla_0$ and $\nabla_1$. Denote the associated $L_\infty$ quasi-isomorphisms between $(\mathcal{V}(E)[1], [\cdot, \cdot]_{SN})$ and $(\mathcal{V}(E^*[1] \oplus E[-1])[1], [G, \cdot]_{SN}, [\cdot, \cdot]_{SN})$ from Proposition 1 by $L_0$ and $L_1$ respectively.

Then there is an $L_\infty$ quasi-isomorphism

$$\hat{L} : (\mathcal{V}(E)[1], [\cdot, \cdot]_{SN}) \sim (\mathcal{V}(E^*[1] \oplus E[-1]) \otimes \Omega(I), [G, \cdot]_{SN} + d_{DR}, [\cdot, \cdot]_{SN})$$

such that $(id \otimes ev_0) \circ \hat{L} = L_0$ and $(id \otimes ev_1) \circ \hat{L} = L_1$ hold.

**Proof.** Given two connections $\nabla_0$ and $\nabla_1$, one can define a family of connections $\nabla_s := \nabla_0 + s(\nabla_1 - \nabla_0)$ parametrized by the closed unit interval $I$. Consequently we obtain a one-parameter family of isomorphisms of graded algebras

$$\varphi_s : A := \mathcal{C}^\infty(T^*[1]E \oplus E^*[1] \oplus E[-1] \oplus E[0] \oplus E^*[2]) \xrightarrow{\sim} \mathcal{V}(E^*[1] \oplus E[-1]),$$

extending the horizontal lifting with respect to the connection $\nabla_s \oplus \nabla^*_s$. Via this identification, $A$ inherits a one-parameter family of Gerstenhaber brackets which we denote by $[\cdot, \cdot]_s$. and a differential $\tilde{Q}$ which can be checked to be independet from $s$ in local coordinates.
For arbitrary $s \in I$ these structures fit into the following commutative diagram:

\[
\begin{array}{c}
\text{(A[1],} \hat{Q}, [\cdot, \cdot]_0) \\
\text{(V(E)[1],} [\cdot, \cdot]_{SN}) \\
\text{(A[1],} \hat{Q}, [\cdot, \cdot]_s) \\
\end{array}
\]

\[
\begin{array}{c}
\text{Pr} \quad \psi_s \\
\varphi_0 \\
\varphi_s \\
\end{array}
\]

where $\psi_s := \varphi_0^{-1} \circ \varphi_s$ is a morphism of differential graded algebras and of Gerstenhaber algebras. $Pr$ denotes the natural projection.

It is straightforward to show that the cohomology of $(A, \hat{Q})$ is $V(E)$ and that the induced $L_\infty$ algebra coincides with $(V(E)[1], [\cdot, \cdot])$, see the proof of Proposition 1 in [Sch]. Hence we obtain a one-parameter family of $L_\infty$ quasi-isomorphisms $\mathcal{K}_s : (V(E)[1], [\cdot, \cdot]_{SN}) \sim (A[1], \hat{Q}, [\cdot, \cdot]_s)$. Composition with $\psi_s$ yields a one-parameter family of $L_\infty$ quasi-isomorphisms

\[
\mathcal{K}_s : (V(E)[1], [\cdot, \cdot]_{SN}) \sim (A[1], \hat{Q}, [\cdot, \cdot]_0).
\]

We remark that the composition of $\mathcal{K}_s$ with $\varphi_s$ yields the $L_\infty$ quasi-isomorphism $\mathcal{L}_s$ between $(V(E), [\cdot, \cdot]_{SN})$ and $(V(E^*[1] \oplus E[-1]), [G, \cdot]_{SN}, [\cdot, \cdot]_{SN})$ associated to the connection $\nabla_s$ from Proposition 1. Consequently $\mathcal{L}_0 (\mathcal{L}_1)$ is the composition of $\mathcal{K}_0 (\mathcal{K}_1)$ with $\varphi_0$.

Next, consider the differential graded Lie algebra $(A[1] \otimes \Omega(I), \hat{Q} + d_{DR}, [\cdot, \cdot]_0)$. To prove Proposition 1, a homotopy $\tilde{H}$ for $\hat{Q}$ was constructed in [Sch] such that

\[
\hat{Q} \circ \tilde{H} + \tilde{H} \circ \hat{Q} = id - \iota \circ Pr
\]

is satisfied. Here, $\iota$ denotes the natural inclusion $V(E) \hookrightarrow A$. One defines a one-parameter family of homotopies $\tilde{H}_s := \psi_s \circ \tilde{H} \circ \psi_s^{-1}$ and checks that

\[
\hat{Q} \circ \tilde{H}_s + \tilde{H}_s \circ \hat{Q} = id - \psi_s \circ \iota \circ Pr
\]

holds.

We define $\tilde{Pr} : A \otimes \Omega(I) \to V(E) \otimes \Omega(I)$ to be $Pr \otimes id$ and $i : V(E) \otimes \Omega(I) \to A \otimes \Omega(I)$ to be $i := (\psi_s \circ \iota) \otimes id$. Clearly $\tilde{Pr} \circ i = id$ and $\tilde{H}_s$ provides a homotopy between $id$ and $i \circ \tilde{Pr}$. Moreover the side-conditions $\tilde{H}_s \circ \tilde{H}_s = 0$, $\tilde{Pr} \circ \tilde{H}_s = 0$ and $\tilde{H}_s \circ i = 0$ are still satisfied. We summarize the situation in the following diagram:

\[
(V(E) \otimes \Omega(I), 0) \xrightarrow{i_{\hat{Q}}} (A \otimes \Omega(I), \hat{Q}) \xrightarrow{\tilde{Pr}} (A \otimes \Omega(I), \tilde{H}_s).
\]
Finally, we define the straightforward to show that $(\break)

We define $C$ instead. The leaves (the exterior vertices with the root excluded) are decorated by $i$, the root by $Pr$, the interior bivalent vertices by $d_{DR}$, the interior trivalent vertices by $[,]_0$ and the interior edges (i.e. the edges not connected to any exterior vertices) by $-\tilde{H}_s$. One then composes these maps in the order given by the orientation towards the root. The associated $L_\infty$ quasi-isomorphism is constructed in the same manner, however, the root is not decorated by $Pr$ but by $-\tilde{H}_s$ instead.

Recall that $\mathcal{V}(r,s)(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$ is the ideal generated by multiderivations of $\mathcal{C}^\infty(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$ with total ghost degree larger than or equal to $r$ and total ghost-momentum degree larger than or equal to $s$, respectively. One can check inductively that trees decorated with $e$ copies of $-\tilde{H}_s$ increase the filtration index by $(e,e)$. Moreover trees containing more than one interior bivalent vertex do not contribute since $d_{DR}$ increases the form-degree by 1. These facts imply that 1. the induced structure is given by $(\mathcal{V}(E)[1] \otimes \Omega(I), d_{DR},[,\cdot]_{SN})$ and 2. there is an $L_\infty$ quasi-isomorphism

$$(\mathcal{V}(E)[1] \otimes \Omega(I), d_{DR},[,\cdot]_{SN}) \sim (\mathcal{A}[1] \otimes \Omega(I), \hat{Q} + d_{DR}, [-,-]_0).$$

We define

$$\hat{K} : (\mathcal{V}(E)[1],[,\cdot]_{SN}) \sim (\mathcal{V}(\mathcal{A}[1] \otimes \Omega(I), \hat{Q} + d_{DR},[,\cdot]_{SN})$$

to be the composition of this $L_\infty$ quasi-isomorphism and the obvious $L_\infty$ quasi-isomorphism $(\mathcal{V}(E)[1],[,\cdot]_{SN}) \sim (\mathcal{V}(E)[1] \otimes \Omega(I), d_{DR},[,\cdot]_{SN})$.

The composition of $\hat{K}$ with $id \otimes ev_s : \mathcal{A} \otimes \Omega(I) \rightarrow \mathcal{A}$ can be computed as follows: first of all only trees without any bivalent interior edges contribute since all elements of form-degree 1 vanish under $id \otimes ev_s$. Using the identities $\psi_s^{-1}([-,-])_s = [-,-]_s, \tilde{H}_s = \psi_s \circ \tilde{H} \circ \psi_s^{-1}$ and $i = \psi_s \circ i$ it is a straightforward to show that $(id \otimes ev_s) \circ \hat{K} = \psi_s \circ K_s$. Hence

$$\varphi_0 \circ (id \otimes ev_s) \circ \hat{K} = \varphi_s \circ K_s = \mathcal{L}_s.$$

Finally, we define the $L_\infty$ quasi-isomorphism $\hat{\mathcal{L}}$ between $(\mathcal{V}(E)[1],[,\cdot]_{SN})$ and $(\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1] \otimes \Omega(I), [G,\cdot]_{SN} + d_{DR},[,\cdot]_{SN})$ to be $(\varphi_0 \otimes id) \circ \hat{K}$. By construction $(id \otimes ev_0) \circ \hat{\mathcal{L}} = \mathcal{L}_0$ and $(id \otimes ev_1) \circ \hat{\mathcal{L}} = \mathcal{L}_1$ are satisfied.

We remark that Propositions 1 and 3 seem to permit “higher analogous”, where one incorporates the differential graded algebra of differential forms on the n-simplex $\Omega(\Delta^n)$ instead of just $\Omega(\{\ast\}) = \mathbb{R}$ (Proposition 1) or $\Omega(I)$ (Proposition 3) – see [Co], where this idea was worked out in the context of the BV-formalism.
Corollary 2. Let $\mathcal{E}$ be a finite rank vector bundle over a smooth, finite dimensional Poisson manifold $(E, \Pi)$. Suppose $\nabla_0$ and $\nabla_1$ are two connections on $\mathcal{E} \to E$. Denote the associated $L_\infty$ quasi-isomorphisms between $(\mathcal{V}(E)[1], [\cdot, \cdot]_{SN})$ and $(\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1], [G, \cdot]_{SN}, [\cdot, \cdot]_{SN})$ from Proposition 1 by $\mathcal{L}_0$ and $\mathcal{L}_1$, respectively. Applying these $L_\infty$ quasi-isomorphisms to $\Pi$ yields two MC-elements $\Pi_0$ and $\Pi_1$ of $(\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1], [G, \cdot]_{SN}, [\cdot, \cdot]_{SN})$. Hence $\Pi_0 := G + \Pi_0$ and $\Pi_1 := G + \Pi_1$ are MC-elements of $(\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1], [\cdot, \cdot]_{SN})$, i.e. Poisson bivector fields on $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]$.

There is a diffeomorphism of the smooth graded manifold $\mathcal{E}^*[1] \oplus \mathcal{E}[1]$ such that the induced automorphism of $\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$ maps $\Pi_0$ to $\Pi_1$. Moreover, this diffeomorphism induces a diffeomorphism of the base $E$ which coincides with the identity.

Proof. Apply the $L_\infty$ quasi-isomorphism $\hat{\mathcal{L}}$ from Proposition 3 to $\Pi$ and add $G$ to obtain a MC-element $\hat{\Pi}_s = \hat{\Pi} + \hat{Z}dt$ of $(\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1], [G, \cdot]_{SN})$. Let $\mathcal{L}_s$ denote the $L_\infty$ quasi-isomorphism from Proposition 1 constructed with the help of the connection $\nabla_0 + s(\nabla_1 - \nabla_0)$. Recall that $(id \otimes ev_s) \circ \hat{\mathcal{L}} = \mathcal{L}_s$ holds for all $s \in I$.

We set $\Pi_s := (id \otimes ev_s)(\hat{\Pi})$ and $\hat{Z}_s := (id \otimes ev_s)(\hat{Z})$. Proposition 3 implies that this definition of $\Pi_s$ is compatible with $\Pi_0$ and $\Pi_1$ defined in the Corollary.

We want to apply Lemma 1 to $A := (\mathcal{V}(E)[1], [\cdot, \cdot]_{SN}), B := (\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1], [G, \cdot]_{SN}, [\cdot, \cdot]_{SN})$ and $F := \hat{\mathcal{L}}$. To do so, it remains to show that the flow of $\hat{Z}_s$ is globally well-defined for $s \in [0, 1]$. Recall that $\hat{Z}$ is the one-form part of the MC-element constructed from the Poisson bivector field $\Pi$ on $E$ with help of the $L_\infty$ quasi-isomorphism $\hat{\mathcal{L}} : (\mathcal{V}(E)[1], [\cdot, \cdot]_{SN}) \to (\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1] \otimes \Omega(I), [G, \cdot]_{SN} + d_{DR}, [\cdot, \cdot]_{SN})$. Only trees with exactly one bivalent interior vertex give non-zero contributions because the form degree must be one. Consequently there is at least one homotopy in the diagram and by the degree estimate in the proof of Proposition 3 this implies that $\hat{Z}$ is contained in $\mathcal{V}^{(1,1)}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1]) \otimes \Omega(I)$. Hence the derivation $[\hat{Z}, -]_{SN}$ is nilpotent and can be integrated. Furthermore the degree estimate directly implies the last claim of the Corollary.

The following is an immediate consequence of the previous Corollary:

Corollary 3. Let $(E, \Pi)$ be a vector bundle $E \to S$ equipped with a Poisson structure $\Pi$ such that $S$ is a coisotropic submanifold. Fix two connections $\nabla_0$ and $\nabla_1$ on $E \to S$ and denote the corresponding graded Poisson brackets on $BFV(E)$ by $[\cdot, \cdot]_{BFV}$ and $[\cdot, \cdot]_{BFV}^1$ respectively.

There is an isomorphism of graded Poisson algebra

$$(BFV(E), [\cdot, \cdot]_{BFV}^0) \xrightarrow{\cong} (BFV(E), [\cdot, \cdot]_{BFV}^0).$$

Moreover the induced automorphism of $C^\infty(E)$ coincides with the identity.

Combining Proposition 2 and Corollary 3 we obtain
Theorem 1. Let $E$ be a vector bundle equipped with a Poisson bivector $\Pi$ such that the zero Section $S$ is a coisotropic submanifold. Recall that the pull back of $E \to S$ by $E \to S$ is denoted by $E \to E$ and

$$BFV(E) := C^\infty(E^*[-1] \oplus E[-1]) = \Gamma(\wedge E \otimes \wedge E^*)$$

Different choices of a connection $\nabla$ on $E \to S$ and of a degree $+1$ element $\Omega$ of $(BFV(S), [-, -]_{BFV})$ satisfying

1. the lowest order term of $\Omega$ is given by the tautological Section $\Omega_0$ of $E \to E$
2. $[\Omega, \Omega]^\nabla_{BFV} = 0$,

lead to isomorphic differential graded Poisson algebras $(BFV(E), [\cdot, \cdot]_{BFV}^\nabla, [-, -]_{BFV})$.

Proof. Pick two connections $\nabla_0$ and $\nabla_1$ on $E \to S$ and consider the two associated graded Poisson algebras $(BFV(E), [\cdot, \cdot]_{BFV}^0)$ and $(BFV(E), [\cdot, \cdot]_{BFV}^1)$, respectively. By Corollary 3 there is an isomorphism of graded Poisson algebras

$$\gamma : (BFV(E), [\cdot, \cdot]_{BFV}^0) \xrightarrow{\cong} (BFV(E), [\cdot, \cdot]_{BFV}^1)$$

Moreover the induced automorphism of $C^\infty(E)$ is the identity.

Assume that $\Omega$ and $\tilde{\Omega}$ are two BFV-charges of $(BFV(E), [\cdot, \cdot]_{BFV}^0)$ and $(BFV(E), [\cdot, \cdot]_{BFV}^1)$, respectively. Applying the automorphism $\gamma$ to $\Omega$ yields another element of $(BFV(E), [\cdot, \cdot]_{BFV}^1)$, which can be checked to be a BFV-charge again. By Proposition 2 this implies that there is an inner automorphism $\beta$ of $(BFV(E), [\cdot, \cdot]_{BFV}^1)$ which maps $\gamma(\Omega)$ to $\tilde{\Omega}$.

Hence

$$\beta \circ \gamma : (BFV(E), [\cdot, \cdot]_{BFV}^0) \xrightarrow{\cong} (BFV(E), [\cdot, \cdot]_{BFV}^1)$$

is an isomorphism of graded Poisson algebras which maps $\Omega$ to $\tilde{\Omega}$. □

4. Choice of Tubular Neighbourhood

Let $S$ be a coisotropic submanifold of a smooth, finite dimensional Poisson manifold $(M, \Pi)$. Throughout this Section, $E$ denotes the normal bundle of $S$ inside $M$. As explained in subsection 2.3, the first step in the construction of the BFV-complex for $S$ inside $(M, \Pi)$ is the choice of an embedding $\psi : E \hookrightarrow M$. Such an embedding equips $E$ with a Poisson bivector field $\Pi_\psi$, which is used to construct the BFV-bracket on the ghost/ghost-momentum bundle, see Subsection 2.3.

Let us first consider the case where the embedding is changed by composition with a linear automorphism of the normal bundle $E$:

Lemma 2. Let

$$(BFV(E), [\Omega, \cdot]_{BFV}, [\cdot, \cdot]_{BFV})$$
be a BFV-complex corresponding to some choice of tubular neighbourhood $\psi : E \hookrightarrow M$, while

$$(BFV(E), [\cdot, \cdot]_{BFV})$$

is a BFV-complex corresponding to the embedding $\psi \circ g : E \hookrightarrow M$, where $g : E \to E$ is a vector bundle isomorphism covering the identity.

Then there is an isomorphism of graded Poisson algebras

$$(BFV(E), [\cdot, \cdot]_{BFV}) \to (BFV(E), [\cdot, \cdot]_{BFV}^g)$$

which maps $\Omega$ to $\Omega^g$.

Proof. Let $\Pi / \Pi^g$ be the Poisson bivector field on $E$ obtained from $\psi : E \hookrightarrow M / \psi \circ g : E \hookrightarrow M$, respectively. Clearly $\Pi^g = (g)_* \Pi$.

Choose some connection $\nabla$ of $E$, which is used to construct the $L_\infty$ quasi-isomorphism

$$L : (V(\cdot), [\cdot, \cdot]_{SN}) \sim (V(\cdot)^*[1] \oplus E[-1])[1], [-, -]_{SN}, [G, -]_{SN}).$$

Plugging in $\Pi$ results into the BFV-bracket $[\cdot, \cdot]_{BFV}$. On the other hand, we can use $\nabla^g := (g^{-1})^* \nabla$ to construct another $L_\infty$ quasi-isomorphism $L^g$.

Plugging in $\Pi^g$ results into another BFV-bracket $[\cdot, \cdot]_{BFV}^g$.

We claim that $[\cdot, \cdot]_{BFV}$ and $[\cdot, \cdot]_{BFV}^g$ are isomorphic graded Poisson brackets. First, observe that the isomorphism $g : E \to E$ lifts to an vector bundle isomorphism $E^\hat{g} \to E$, such that the tautological section gets mapped to itself under $(\hat{g})^*$. We denote the induced automorphism of $E^*[1] \oplus E[-1]$ by $\hat{g}$ as well.

By naturality of the pull back of connections, we obtain the commutative diagram

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\hat{g}^*} & \mathcal{E} \\
\downarrow & & \downarrow \\
E & \xrightarrow{g} & E,
\end{array}$$

such that the tautological section gets mapped to itself under $(\hat{g})^*$. We denote the induced automorphism of $E^*[1] \oplus E[-1]$ by $\hat{g}$ as well.

By naturality of the pull back of connections, we obtain the commutative diagram

$$\begin{array}{ccc}
\mathcal{V}(E) & \xrightarrow{\iota_{\nabla}} & \mathcal{V}(E^*[1] \oplus E[-1]) \\
\downarrow & & \downarrow \\
\mathcal{V}(E) & \xrightarrow{\iota_{\nabla^g}} & \mathcal{V}(E^*[1] \oplus E[-1]),
\end{array}$$

where $\iota_{\nabla}$ ($\iota_{\nabla^g}$) is the horizontal lift induced by $\nabla$ ($\nabla^g$). Using this together with the explicit description of the $L_\infty$ quasi-isomorphism $L$ from Proposition 1 contained in [Sch], or in the proof of Proposition 3, one concludes that

$$(L^g)_k = (\hat{g})_* \circ (L)_k \circ (g^{-1}_*) \circ \cdots \circ (g^{-1}_*)$$

Here, $(L)_k$ denotes the $k$th structure map of the $L_\infty$ quasi-isomorphism $L$.  

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This immediately implies that \( \hat{g} \) induces an isomorphism between \([\cdot , \cdot ]_{BFV}^g\) and \([\cdot , \cdot ]_{BFV}^g\), respectively. Moreover, since \( \hat{g} \) maps the tautological section to itself, it maps any BFV-charge to another one.

Finally, Theorem 1 implies the statement of Lemma 2.

In general, a different choice of embedding can cause drastic changes in the associated BFV-complexes. Consider \( S = \{0\} \) inside \( M = \mathbb{R}^2 \) equipped with the smooth Poisson bivector field

\[
\Pi(x, y) := \begin{cases} 
0 & x^2 + y^2 \leq 4 \\
\exp\left(-\frac{1}{x^2+y^2-4}\right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} & x^2 + y^2 \geq 4
\end{cases}
\]

Let \( \psi_0 \) be the embedding of \( E \cong \mathbb{R}^2 \) into \( \mathbb{R}^2 \) given by the identity and \( \psi_1 \) the embedding given by \((x, y) \mapsto \frac{1}{\sqrt{1+x^2+y^2}}(x, y)\).

The image of \( \psi_1 \) is contained in the disk of radius 1. Hence \( \Pi\psi_0 \) vanishes identically whereas \( \Pi\psi_1 \) does not.

The ghost/ghost-momentum bundle \( \mathcal{E}^*[-1] \) is of the very simple form

\[
\mathbb{R}^2 \times ((\mathbb{R}^2)^*[-1] \oplus \mathbb{R}^2[-1]) \rightarrow \mathbb{R}^2.
\]

Denote the Poisson bivector field coming from the natural pairing between \((\mathbb{R}^2)^*[-1] \oplus \mathbb{R}^2[-1]\) by \( G \). We choose the standard flat connection on the bundle \( \mathbb{R}^2 \rightarrow 0 \). Then the Poisson bivector fields for the BFV-brackets \([\cdot , \cdot ]_{BFV}^0\) and \([\cdot , \cdot ]_{BFV}^1\) are simply given by the sums \( G + \Pi\psi_0 \) and \( G + \Pi\psi_1 \), respectively.

Any isomorphism of graded Poisson algebras between \((BFV(E), [\cdot , \cdot ]_{BFV}^0)\) and \((BFV(E), [\cdot , \cdot ]_{BFV}^1)\) yields an induced isomorphism of Poisson algebras between \((\mathcal{C}^\infty(\mathbb{R}^2), [\cdot , \cdot ]_{\Pi\psi_0})\) and \((\mathcal{C}^\infty(\mathbb{R}^2), [\cdot , \cdot ]_{\Pi\psi_1})\). Since \( \Pi\psi_1 \) vanishes, the induced automorphism would have to map something non-vanishing to 0, which is a contradiction. Hence there is no isomorphism of graded Poisson algebras between \((BFV(E), [\cdot , \cdot ]_{BFV}^0)\) and \((BFV(E), [\cdot , \cdot ]_{BFV}^1)\).

Although different choices of embeddings can lead to differential graded Poisson algebras that are not isomorphic, it is always possible to find appropriate “restrictions” of the BFV-complexes such that the corresponding differential graded Poisson algebras are isomorphic. To this end we define

**Definition 2.** Let \( E \) be a finite rank vector bundle over a smooth manifold \( S \). Assume \( E \) is equipped with a Poisson bivector field \( \Pi \) such that \( S \) is a coisotropic submanifold of \( E \). Moreover let \((BFV(E), D_{BFV}, [\cdot , \cdot ]_{BFV})\) be a BFV-complex for \( S \) in \((E, \Pi)\) and \( U \) an open neighbourhood of \( S \) inside \( E \).

Then the restriction of the BFV-complex on \( U \) is the differential graded Poisson algebra

\[
(BFV^U(E), D_{BFV}^U) = ([\Omega^U, \cdot ]_{BFV}^U, [\cdot , \cdot ]_{BFV}^U)
\]

given by the following data:
(a) $BFV^U(E)$ is the space of smooth functions on the graded vector bundle $(E^*[1] \oplus E[-1])|_U$ fitting into the following Cartesian square:

\[
\begin{array}{c}
E^*[1] \oplus E[-1]|_U \\
\downarrow
\end{array}
\begin{array}{c}
E^*[1] \oplus E[-1]
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
E.
\end{array}
\]

(b) $BFV^U(E)$ inherits a graded Poisson bracket $[\cdot, \cdot]_{BFV}^U$ from $BFV(E)$: one restricts the Poisson bivector field corresponding to $[\cdot, \cdot]_{BFV}$ to the graded submanifold $(E^*[1] \oplus E[-1])|_U$ of $E^*[1] \oplus E[-1]$.

(c) An element $\Omega^U$ of $BFV^U(E)$ is called a restricted BFV-charge if $[\Omega^U, \Omega^U]_{BFV} = 0$ holds and the component of $\Omega^U$ in $\Gamma(E)|_U$ is equal to the restriction of the tautological section $\Omega_0 \in \Gamma(E)$ to $U$.

**Proposition 4.** Let $S$ be a coisotropic submanifold of a smooth, finite dimensional Poisson manifold $(M, \Pi)$. Denote the normal bundle of $S$ by $E$ and fix a connection $\nabla$ on $E$. Moreover let $\psi_0$ and $\psi_1$ be two embeddings of $E$ into $M$ as tubular neighbourhoods of $S$.

Using these data one constructs two graded Poisson algebra structures on $BFV(E)$ following subsection 2.3 (in particular one applies Proposition 1). Denote the two corresponding graded Poisson brackets by $[\cdot, \cdot]_{BFV}^0$ and $[\cdot, \cdot]_{BFV}^1$ respectively.

Then there are two open neighbourhoods $A_0$ and $A_1$ of $S$ in $E$ such that an isomorphism of graded Poisson algebras

\[
(BFV^{A_0}(E), [\cdot, \cdot]_{BFV}^0) \cong (BFV^{A_1}(E), [\cdot, \cdot]_{BFV}^1)
\]

exists.

**Proof.** We make use of the fact that any two embeddings of $E$ as a tubular neighbourhood are homotopic up to inner automorphisms of $E$, i.e. given two embeddings $\psi$ and $\phi$ of $E$ into $M$ as a tubular neighbourhood, one can find

- a vector bundle isomorphism $g$ of $E$ and
- a smooth map $F : E \times I \to M$

satisfying

- $F|_{E \times \{0\}} = \psi$ and $F|_{E \times \{1\}} = \phi \circ g$,
- $\psi_s := F|_{E \times \{s\}} : E \to M$ is an embedding for all $s \in I$ and
- $\psi_s|_S = id_S$ for all $s \in I$.

The construction of $F$ can be found in [Hi] for instance.

Since vector bundle automorphisms of $E$ yield isomorphic BVF-complexes by Lemma 2, we can assume without loss of generality that the two embeddings $\psi := \psi_0$ and $\phi =: \psi_1$ are homotopic (i.e. $g = id$).

Denote the images of $\psi_s$ by $V_s$. Since $\psi_s$ is an embedding of a manifold of the same dimension as $M$, the image $V_s$ is an open subset of $M$. Moreover
$S \subset V_s$ holds for arbitrary $s \in I$, i.e. $V_s$ is an open neighbourhood of $S$ in $M$. Because $F$ is continuous, one can find an open neighbourhood $V$ of $S$ in $M$ which is contained in $\bigcap_{s \in I} V_s$.

One defines $\tilde{F} : E \times I \to M \times I$, $(e, t) \mapsto (F(e, t), t)$ and checks that $\tilde{F}$ is an embedding, hence its image is a submanifold $W$ of $M \times I$ and $\tilde{F}$ is a diffeomorphism between $E \times I$ and $W$. Consider the restriction of $\tilde{F}^{-1} : W \cong E \times I$ to $V \times I$ which we denote by $G$. If one restricts $G$ to “slices” of the form $V \times \{s\}$ one obtains $\psi_s^{-1}|_V$. The images of $\psi_s^{-1}|_V$ are denoted by $W_s$. By continuity of $G$ there is an open neighbourhood $W$ of $S$ in $E$ which is contained in $\bigcap_{s \in I} W_s$.

We define the following one-parameter family of local diffeomorphisms of $E$:

$$\phi_s : W_0 \xrightarrow{\psi_0|_W} V \xrightarrow{(\psi_s|_V)^{-1}} W_s,$$

Moreover $E$ inherits a one-parameter family of Poisson bivector fields defined by $\Pi_s := (\psi_s|_V^{-1})_* (\Pi|_V)$. The restriction $\Pi_s|_W$ is equal to $(\psi_s|_V^{-1})_* (\Pi|_V)$. Consequently

$$\Pi_s|_W = (\phi_s)_* (\Pi_0|_{W_0})$$

holds for all $s \in I$.

Differentiating $\phi_s$ yields a smooth one-parameter family of local vector fields $(Y_s)_{s \in I}$ on $E$. By (1) the smooth one-parameter family

$$\Pi_t|_W - Y_t|_W dt$$

is a MC-element of $(\mathcal{V}(W)[1] \otimes \Omega(I), d_{DR}, [\cdot, \cdot]_{SN})$.

The $L_\infty$ quasi-isomorphism

$$\mathcal{L}_\nabla : (\mathcal{V}(E)[1], [\cdot, \cdot]_{SN}) \sim (\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1], [G, \cdot]_{SN}, [\cdot, \cdot]_{SN})$$

from Proposition 1 restricts to an $L_\infty$ quasi-isomorphism

$$\mathcal{L}_\nabla|_W : (\mathcal{V}(W)[1], [\cdot, \cdot]_{SN}) \sim (\mathcal{V}((\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[W])[1], [G, \cdot]_{SN}, [\cdot, \cdot]_{SN}).$$

Hence we obtain an $L_\infty$ quasi-isomorphism

$$\mathcal{L}_\nabla|_W \otimes id : (\mathcal{V}(W)[1] \otimes \Omega(I), d_{DR}, [\cdot, \cdot]_{SN}) \sim (\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[W][1] \otimes \Omega(I), d_{DR} + [G, \cdot]_{SN}, [\cdot, \cdot]_{SN}).$$

Applying $\mathcal{L}_\nabla|_W \otimes id$ to the MC-element $\Pi_t|_W - Y_t|_W dt$ and adding $G$ yields a MC-element $\tilde{\Pi}_t - \tilde{Y}_t dt$ of $(\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[W][1] \otimes \Omega(I), d_{DR}, [\cdot, \cdot]_{SN})$.

It is straightforward to check that $\tilde{\Pi}_s$ is the restriction of $\mathcal{L}_\nabla(\sum_{k \geq 1} \frac{1}{k!} \Pi_s^k)$ to $W$ and that $\tilde{Y}_s$ is the sum of the horizontal lift $\nu\nabla(Y_s)$ of $Y_s$ with respect to $\nabla$ restricted to $W$ plus a part in $\mathcal{V}^{(1,1)}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$ (that acts as a nilpotent derivation).

Using parallel transport with respect to $\nabla$, $(\nu\nabla(Y_t))_{t \in I}$ can be integrated to a one-parameter family of vector bundle automorphisms

$$\hat{\phi}_s : \mathcal{E}^*[1] \oplus \mathcal{E}[-1]|_{W_0} \to \mathcal{E}^*[1] \oplus \mathcal{E}[-1]|_{W_s}$$
covering $\phi_s : W_0 \to W_s$ for arbitrary $s \in I$. Similar to the construction of $V$ and $W$ one finds an open neighbourhood $A_0$ of $S$ in $W$ such that $\phi_t|_{A_0} : A_0 \xrightarrow{\sim} A_t$ with $\bigcup_{s \in I} A_s \subset W$. So the restriction of $\hat{\phi}_s$ to $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]|_{A_0}$ has image $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]|_{A_s}$ which is a submanifold of $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]|_W$ for arbitrary $s \in I$.

Hence the one-parameter family of local vector fields

$$(i\nabla(Y_t))|_{(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])|_{A_t}} \in I$$

can be uniquely integrated to a one-parameter family of local diffeomorphisms $(\hat{\phi}_t)_{t \in I}$ and consequently the one-parameter family of local vector fields $(Y_t|_{A_t})_{t \in I}$ can be uniquely integrated to a one-parameter family of local diffeomorphisms which we denote by

$$\varphi_s : (\mathcal{E}^*[1] \oplus \mathcal{E}[-1])|_{A_0} \to (\mathcal{E}^*[1] \oplus \mathcal{E}[-1])|_{A_s}$$

for $s \in I$.

Applying Lemma 1 shows that $\hat{\Pi}_s|_{A_s} = (\varphi_s)_*(\hat{\Pi}_0|_{A_0})$ holds for all $s \in I$. Hence

$$(\varphi_1)_* : \mathcal{C}^\infty(\mathcal{E}^*[1] \oplus \mathcal{E}[-1]|_{A_0}) \to \mathcal{C}^\infty(\mathcal{E}^*[1] \oplus \mathcal{E}[-1]|_{A_1})$$

is an isomorphism of Poisson algebras. \hfill \Box

**Theorem 2.** Let $S$ be a coisotropic submanifold of a smooth, finite dimensional Poisson manifold $(M, \Pi)$. Suppose $(\text{BFV}(E), D^0_{\text{BFV}}, [\cdot, \cdot]_{\text{BFV}})$ and $(\text{BFV}(E), D^1_{\text{BFV}}, [\cdot, \cdot]_{\text{BFV}})$ are two BFV-complexes constructed with help of two arbitrary embeddings of $E$ into $M$, two arbitrary connections on $E \to S$ and two arbitrary BFV-charges.

Then there are two open neighbourhoods $B_0$ and $B_1$ of $S$ in $E$ such that an isomorphism of differential graded Poisson algebras

$$(\text{BFV}^{B_0}(E), D^0_{\text{BFV}}, [\cdot, \cdot]_{\text{BFV}}) \xrightarrow{\sim} (\text{BFV}^{B_1}(E), D^1_{\text{BFV}}, [\cdot, \cdot]_{\text{BFV}})$$

exists.

**Proof.** By Theorem 1 we can assume without loss of generality that the two chosen connections coincide. Furthermore it suffices to prove that there is an isomorphism of graded Poisson algebras from some restriction of $(\text{BFV}(E), [\cdot, \cdot]_{\text{BFV}})$ to some restriction of $(\text{BFV}(E), [\cdot, \cdot]_{\text{BFV}})$ which maps a restricted BFV-charge to another restricted BFV-charge. This is a consequence of the fact that Theorem 1 holds also in the restricted setting as long as the open neighbourhood $U$ of $S$ in $E$, to which we restrict, is contractible to $S$ along the fibres of $E$.

By Lemma 2, we may assume without loss of generality that the two embeddings under consideration are homotopic. Hence there is a smooth one-parameter family of isomorphisms of graded Poisson algebras

$$(\varphi_s)_* : (\text{BFV}^{A_0}(E), [\cdot, \cdot]_{\text{BFV}}) \to (\text{BFV}^{A_s}(E), [\cdot, \cdot]_{\text{BFV}}),$$

where $s$ denotes the parameter.
which we constructed in the proof of Proposition 4. The smoothness of this family and the fact that the zero section \( S \) is fixed under \((\varphi_s)s\in I\) imply that there is a open neighbourhood \( A \) of \( S \) in \( E \) satisfying \( A \subset \bigcap_{s\in I} A_s \).

Fix a restricted BFV-charge \( \Omega \) of \((BFV^A,\{\cdot,\cdot\}^A)\). The restriction of

\(\Omega(t) := (\varphi_t)_* (\Omega)\) \(t\in I\)

to \( A \) yields a smooth one-parameter family of sections of \( \bigwedge \mathcal{E} \otimes \bigwedge \mathcal{E}^*|_A \).

Although \([\Omega(s)|_A,\Omega(s)|_A]^A_{BFV} = 0\) holds for all \( s \in I \), \( \Omega(s)|_A \) is in general not a BFV-charge since its component in \( \Gamma(\mathcal{E}|_W) \) is \( \Omega_0(s) := (\varphi_s)_*(\Omega_0) \) which does not need to be equal to \( \Omega_0 \) as required – see Definition 2. In particular \( \Omega(1) \) might not be a restricted BFV-charge of \((BFV(E),\{\cdot,\cdot\}^A_{BFV})\). However we will show that \( \Omega(1) \) can be “gauged” to a BFV-charge in the remainder of the proof.

We have to recall some of the ingredients involved in the proof of Proposition 2. The first observation is that \( \delta := [\Omega_0,\cdot]_G \) is a differential. Here \( \Omega_0 \) denotes the tautological section of \( \mathcal{E} \to E \), \( G \) is the Poisson bivector field associated to the fibre pairing between \( \mathcal{E} \) and \( \mathcal{E}^* \), and \( [\cdot,\cdot]_G \) denotes the graded Poisson bracket on \( BFV(E) \) corresponding to \( G \). Second it is possible to construct a homotopy \( h \) for \( \delta \), i.e. a degree \(-1\) map satisfying

\[
\delta \circ h + h \circ \delta = id - i \circ pr
\]

where \( i \) is an embedding of the cohomology of \( \delta \) into \( BFV(E) \) and \( pr \) is a projection from \( BFV(E) \) onto cohomology. We remark that \( h \) does not restrict to arbitrary open neighbourhoods of \( S \) in \( E \). However one can check that it does restrict to open neighbourhoods that can be contracted to \( S \) along the fibres of \( E \). Without loss of generality we can assume that \( A \) has this property.

We are interested in the smooth one-parameter family

\[h(\Omega_0(s)) \in \Gamma(\mathcal{E} \otimes \mathcal{E}^*|_A) \cong \Gamma(\text{End}(\mathcal{E}|_A))\]

with \( s \in I \). Since \( \Omega_0 \) intersects the zero section of \( \mathcal{E} \to E \) transversally at \( S \), so does \( \Omega_0(s) \) for arbitrary \( s \in I \). This implies 1.) the evaluation of \( \Omega_0(s) \) at \( S \) is zero and 2.) \( h(\Omega_0(s))|_S \in \Gamma(\mathcal{E} \otimes \mathcal{E}^*|_S) \) is fibrewise invertible, i.e. it is an element of \( \Gamma(GL(\mathcal{E}|_S)) \).

For any \( s \in I \) we have \( \delta h(\Omega_0(s)) = [\Omega_0,\Omega_0(s)]_G = 0 \) since both \( \Omega_0 \) and \( \Omega_0(s) \) are sections of \( \mathcal{E}|_A \) and \( G \) is the Poisson bivector given by contraction between \( \mathcal{E} \) and \( \mathcal{E}^* \). Moreover \((i \circ pr)(\Omega_0(s)) = 0 \) since the projection \( pr \) involves evaluation of the section at \( S \), where \( \Omega_0(s) \) vanishes. Consequently (2) reduces to \( \delta h(\Omega_0(s)) = 0 \) for all \( s \in I \). However this means that if we interpret \( h(\Omega_0(s)) \) as a fibrewise endomorphism of \( \mathcal{E}|_A \) the image of \( \Omega_0 \) under \(-h(\Omega_0(s))\) is \( \Omega_0(s) \).

We define \( M_s := -h(\Omega_0(s)) \) – as already observed, \((M_t)_{t\in I} \) is a smooth one-parameter family of sections of \( \text{End}(\mathcal{E}|_A) \) and the restriction to \( S \) is
a smooth one-parameter family of $GL(\mathcal{E}|_S)$. By smoothness of the one-parameter family it is possible to find an open neighbourhood $B$ of $S$ in $E$ such that the restriction of $(M_t)_{t \in I}$ to $B$ is always fibrewise invertible. Since $M_0 = \text{id}|_A$ we know that $(M_t|_B)_{t \in I}$ is a smooth one-parameter family of sections in $GL_+(\mathcal{E}|_B)$, i.e. fibrewise invertible automorphisms of $E|_B$ with positive determinant. In particular $M_1 \in \Gamma(GL_+(\mathcal{E}|_B))$.

Consider the smooth one-parameter family $(m_t)_{t \in I}$ of sections of $\text{End}(\mathcal{E}|_B)$ given by

$$m_t := -M_t^{-1} \circ \left( \frac{d}{dt} M_t \right).$$

It integrates to a smooth one-parameter family of sections of $GL_+(\mathcal{E}|_B)$ that coincides with $(M_t)_{t \in [0,1]}$. The adjoint action of $m_t$ on $(BFV^B(E),[\cdot,\cdot]_{BFV}^B)$ can be integrated to an automorphism of $(BFV^B(E),[\cdot,\cdot]_{BFV}^B)$ and this automorphism maps the restriction of $\Omega_0(1)$ to $B$ to the restriction of $\Omega_0$ to $B$. Hence $(\exp(m) \circ (\varphi_1)_*)$ maps the restricted BFV-charge $\Omega$ to another restricted BFV-charge of $(BFV^B(E),[\cdot,\cdot]_{BFV}^B)$.

**Definition 3.** Let $(BFV(E),D_{BFV},[\cdot,\cdot]_{BFV})$ be a BFV-complex associated to a coisotropic submanifold $S$ of a smooth Poisson manifold $(M,\Pi)$. We define a differential graded Poisson algebra $(BFV^g(E),D^g_{BFV},[\cdot,\cdot]_{BFV}^g)$ as follows:

(a) $BFV^g(E)$ is the algebra of equivalence classes of elements of $BFV(E)$ under the equivalence relation: $f \sim g :\iff$ there is an open neighbourhood $U$ of $S$ in $E$ such that $f|_U = g|_U$.

(b) $D^g_{BFV}([\cdot]) := [D_{BFV}(\cdot)]$ where $[\cdot]$ denotes the equivalence class of $\cdot$ under $\sim$.

(c) $[\cdot,\cdot]_{BFV}^g := ([\cdot,\cdot]_{BFV})$.

Given a differential graded Poisson algebra with unit $(A,\wedge,d,[\cdot,\cdot])$ we define the corresponding abstract differential graded Poisson algebra with unit $[(A,\wedge,d,[\cdot,\cdot]])$ to be the isomorphism class of $(A,\wedge,d,[\cdot,\cdot])$ in the category of differential graded Poisson algebras with unit. In particular $[(A,\wedge,d,[\cdot,\cdot])]$ is an object in the category of differential graded Poisson algebras with unit up to isomorphisms.

Theorem 2 immediately implies

**Corollary 4.** Consider a coisotropic submanifold $S$ of a smooth, finite dimensional Poisson manifold $(M,\Pi)$ and let $(BFV(E),D_{BFV},[\cdot,\cdot]_{BFV})$ be a BFV-complex associated to $S$ inside $(M,\Pi)$.

The abstract differential graded Poisson algebra

$$[(BFV^g(E),D^g_{BFV},[\cdot,\cdot]_{BFV}^g)]$$

is independent of the specific choice of a BFV-complex and hence is an invariant of $S$ as a coisotropic submanifold of $(M,\Pi)$.
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