On the Tanaka formula for the derivative of self-intersection local time of fBm

Paul Jung Greg Markowsky

May 1, 2014

Abstract

The derivative of self-intersection local time (DSLT) for Brownian motion was introduced by Rosen [Ros05] and subsequently used by others to study the $L^2$ and $L^3$ moduli of continuity of Brownian local time. A version of the DSLT for fractional Brownian motion (fBm) was introduced in [YYL08]. In the present work, we further its study by proving a Tanaka-style formula for it. In the course of this endeavor we provide a Fubini theorem for integrals with respect to fBm. The Fubini theorem may be of independent interest, as it generalizes (to Hida distributions) similar results previously seen in the literature. Finally, we also give an explicit Wiener chaos expansion for the DSLT of fBm.

Contents

1 Introduction 2
2 Existence of the DSLT of fBm 4
3 A Fubini theorem for the WIS integral 7
  3.1 Classical white noise theory 7
  3.2 Fractional white noise theory and a Fubini theorem 11
4 The Wiener chaos decomposition of DSLT 14
5 The Tanaka formula 19
A Second moment bounds for fBm 20
1 Introduction

Let $B_t$ denote a Brownian motion on $\mathbb{R}$ with $B_0 = 0$ and let $L(t, x) = L^t_x$ be its local time at $x$ up to time $t$. In connection with stochastic area integrals with respect to local time [Wal83] and the Brownian excursion filtration, Rogers and Walsh [RW91a] studied the space integral of local time,

$$A(t, x) := \int_0^t 1_{[0, \infty)}(x - B_s) \, ds ; \quad \frac{\partial A}{\partial x} = L(t, x). \tag{1.1}$$

In a companion work on the Brownian local time sheet [RW91b], the occupation density of $A(t, B_t)$ was investigated. It was also shown in [RW90] that the process $A(t, B_t) - \int_0^t L(s, B_s) \, dB_s$ has finite, non-zero $4/3$-variation. An alternate proof of this fact using fractional martingales was recently given in [HNS12].

In [Ros05], Rosen developed a new approach to the study of $A(t, B_t)$ as follows. If one lets $h(x) := 1_{[0, \infty)}(x)$, then formally

$$\frac{d}{dx} h(x) = \delta(x) \quad \text{and} \quad \frac{d^2}{dx^2} h(x) = \delta'(x), \tag{1.2}$$

where $\delta$ is the Dirac delta distribution. Holding $r$ fixed and applying Itô’s formula with respect to the Brownian motion $B_s - B_r$ gives

$$1_{[0, \infty)}(B_t - B_r) - 1_{[0, \infty)}(0) = \int_r^t \delta(B_s - B_r) \, dB_s + \frac{1}{2} \int_r^t \delta'(B_s - B_r) \, ds. \tag{1.3}$$

Integrating with respect to $s$ from 0 to $t$ and interchanging the orders of integration in the integrals leads to

$$\tilde{A}(t, B_t) - \int_0^t L(s, B_s) \, dB_s = t + \frac{1}{2} \int_0^t \int_0^s \delta'(B_s - B_r) \, dr \, ds. \tag{1.4}$$

This formal identity was stated in [Ros05]. We note however, that there is some ambiguity since if we change the definition of $h$ only slightly to $h(x) := 1_{(0, \infty)}(x)$ and apply Itô’s formula in the same manner, we obtain

$$A(t, B_t) - \int_0^t L(s, B_s) \, dB_s = \frac{1}{2} \int_0^t \int_0^s \delta'(B_s - B_r) \, dr \, ds. \tag{1.5}$$

Motivated by (1.4), Rosen [Ros05] showed the existence of a process now known as the derivative of self-intersection local time (DSLT) for $B_t$. That is, he demonstrated the existence of a process $\alpha'_t(y)$ formally defined as

$$\alpha'_t(y) := - \int_0^t \int_0^s \delta'(B_s - B_r - y) \, dr \, ds. \tag{1.6}$$
This process was later used in [HN09] and [HN10] to prove Central Limit Theorems for the $L^2$ and $L^3$ moduli of continuity of Brownian local time.

In [Mar08a], it was proved that almost surely, for all $y$ and $t$,

$$
\frac{1}{2} \alpha_t'(y) + \frac{1}{2} \text{sgn}(y)t = \int_0^t L_{s-y} B_s \, dB_s - \frac{1}{2} \int_0^t \text{sgn}(B_t - B_r - y) \, dr .
$$

In particular, for $y = 0$ equation (1.5) is correct and (1.4) should not have a $t$ term. The formula (1.7) is commonly referred to as a Tanaka formula. The method of proof in [Mar08a] also serves to give an alternate proof of the existence of $\alpha_t'(y)$ and joint continuity of $\alpha_t'(y) + \text{sgn}(y)t$, which Rosen had deduced earlier by other methods.

In [Mar12], yet another existence proof for $\alpha_t'(y)$ was given using its Wiener chaos expansion. Our aim is to extend these results to the more general process of fractional Brownian motion.

Standard fractional Brownian motion (fBm), with Hurst parameter $H \in (0, 1)$, is the unique centered Gaussian process with covariance function

$$
\mathbb{E}(B^{H}_t B^{H}_s) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H}).
$$

Note that $H = 1/2$ gives us a standard Brownian motion. In [Hu01], it was shown that the self-intersection local time of fBm is differentiable in the Meyer-Watanabe sense. Using Hu’s arguments, for $H < 2/3$, [YYL08] deduced the existence of processes which are related to what we shall call the DSLT of fBm. We should note that a slightly erroneous bound is utilized in both [Hu01] and [YYL08], for which we have provided a corrected modification in Appendix A.

The work of [YYL08] shows that there is a critical value below which the p-variation of their DSLT of fBm is non-trivial. Their result was partially motivated by stochastic integrals with respect to fBm-local times which were further studied in [YLY09]. In particular, in [YYL08], two versions of a DSLT of fBm are defined and the following formal identity is stated

$$
H \tilde{\alpha}_t'(0) = t + \int_0^t L_{s-t}^{H} \, dB^H_s - \frac{1}{2} \int_0^t \text{sgn}(B^H_t - B^H_r) \, dr .
$$

This reduces to (1.4) when $H = 1/2$.

In this work, by proving an analog of (1.7) (which again shows that (1.9) is better off without a $t$ term), we modify equation (1.9) to a process formally defined by

$$
\alpha_t'(y) := -H \int_0^t \int_0^s \delta'(B^H_s - B^H_r - y)(s-r)^{2H-1} \, dr \, ds .
$$

When $H < 2/3$, we will show that such a process exists in the $L^2$ sense.
The rest of the paper is organized as follows. In the next section we give a few remarks on the existence of $\alpha'_t(0)$. In Section 3 we review some tools from Malliavin calculus needed for the sequel. One of these tools is a Fubini theorem for fractional Brownian integrals which generalizes, to Hida distributions, similar results found in [CN05] and [Mis08]. In Section 4 we present an explicit Wiener chaos expansion for $\alpha'_t(0)$ and prove the existence of DSLT for all $y \in \mathbb{R}$. We conclude in Section 5 by proving a Tanaka formula for $\alpha'_t(y)$.

2 Existence of the DSLT of fBm

Let $F \diamond G$ denote the Wick product of $F$ and $G$. Formally applying Itô’s Lemma (see Theorem 3.8 below) for fBm in a similar fashion to (1.5) gives

$$1_{(0,\infty)}(B^H_t - B^H_r) = \int_r^t \delta(B^H_s - B^H_r) \diamond dB^H_s + H \int_r^t \delta'(B^H_s - B^H_r)(s-r)^{2H-1} ds.$$ (2.1)

Integrating with respect to $r$ and switching order of integration gives

$$\int_0^t 1_{(0,\infty)}(B^H_t - B^H_r) \, dr = \int_0^t L^{BH}_s \circ dB^H_s + H \int_0^t \int_0^s \delta'(B^H_s - B^H_r)(s-r)^{2H-1} \, dr \, ds.$$ (2.2)

Here $(s-r)^{2H-1}$ is called a reproducing kernel, and is standard in integrals involving $B^H_t$ (see Chapter 2 of [BHØZ08]). Comparing the above with (1.4) and (1.6), it is natural to define the DSLT of fBm as the formal process $\alpha'_t(y)$ given in (1.10).

In order to rigorously define $\alpha'_t(y)$, let $f_1(x)$ denote the standard Gaussian density. We set $f_\varepsilon(x) := \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{x^2}{2\varepsilon}}$, so that

$$f_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{x^2}{2\varepsilon}} \quad \text{and} \quad f'_\varepsilon(x) = \frac{d}{dx} f_\varepsilon(x).$$ (2.3)

Note that as $\varepsilon \to 0$, $f_\varepsilon(x)$ converges weakly to the Dirac delta distribution, $\delta(x)$. For fixed $0 < H < 1$, let

$$\alpha'_{t,\varepsilon}(y) := -\int_0^t \int_0^s f'_\varepsilon(B^H_s - B^H_r - y)(s-r)^{2H-1} \, dr \, ds.$$ (2.4)

An analog of the following result appears in [YYL08].

Proposition 2.1. For fBm with $H < 2/3$, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the processes $\alpha'_{t,\varepsilon}(0)$ converge in $L^2(\mathbb{P})$ to a process $\alpha'_t(0)$ as $\varepsilon \to 0$. Moreover, $\alpha'_t(0)$ is continuous in $t$.\footnote{The Wick product is further explained in Section 3 or for a full treatment see [BHØZ08].}
EXISTENCE OF THE DSLT OF FBM

The proof of the above is similar to the arguments of [YYL08], except for the convergence of an integral which is given in Lemma 2.2 below. The proof presents some points of interest, so we sketch it as follows.

The key lies in computing \( \mathbb{E}[(\alpha'_{t, \varepsilon}(0))^2] \). In the sequel, \( K \) denotes a positive constant which may change from line to line. We start by expressing \( f_{\varepsilon}(x) \) in a convenient form using the Fourier identity \( f_1(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\rho x} e^{-\rho^2/2} \, dp \). This gives

\[
f_{\varepsilon}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\rho x} e^{-\rho^2/2} \, dp = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ipx} e^{-\varepsilon p^2/2} \, dp,
\]

whence

\[
f_{\varepsilon}'(x) = i \frac{p}{2\pi} \int_{\mathbb{R}} e^{ipx} e^{-\varepsilon p^2/2} \, dp. \tag{2.4}
\]

Now, for \( \mathcal{D}_t = \{0 \leq r \leq s \leq t\} \),

\[
\mathbb{E}[(\alpha'_{t, \varepsilon}(0))^2] = K \int_{\mathcal{D}_t} \int_{(p,q) \in \mathbb{R}^2} pq e^{-\varepsilon (p^2+q^2)/2} e^{-\text{Var}(\pi(B^H_s - B^H_r)+q(B^H_s - B^H_{r'}))/2} \times (s-r)^{2H-1} (s'-r')^{2H-1} \, dp \, dq \, dr' \, ds \, ds'.
\]

We will show that this can be bounded uniformly in \( \varepsilon \). Using standard notation from the literature, let

\[
\lambda := \text{Var}(B^H_s - B^H_r) = |s-r|^{2H}, \tag{2.6}
\]

\[
\rho := \text{Var}(B^H_{s'} - B^H_{r'}) = |s'-r'|^{2H},
\]

\[
\mu := \text{Cov}(B^H_s - B^H_r, B^H_{s'} - B^H_{r'}) = \frac{1}{2} (|s'-r|^{2H} - |s'-s|^{2H} - |r'-r|^{2H}).
\]

With this notation, we have

\[
\mathbb{E}[(\alpha'_{t, \varepsilon}(0))^2] = K \int_{\mathcal{D}_t} (s-r)^{2H-1} (s'-r')^{2H-1} \times \int_{\mathbb{R}^2} pq e^{-\varepsilon (p^2+q^2)/2} e^{-\rho^2 \lambda + 2p\mu + q^2 \rho/2} \, dp \, dq \, dr' \, ds \, ds'. \tag{2.7}
\]

Isolating the \( dq \) integral we have:

\[
\int_{\mathbb{R}} q e^{-pq\mu - q^2(\rho + \varepsilon)/2} \, dq = e^{\frac{\rho^2 \mu^2}{2(\rho + \varepsilon)}} \int_{\mathbb{R}} q e^{-(q + \frac{\mu}{\rho + \varepsilon})^2 (\rho + \varepsilon)/2} \, dq = e^{\frac{\rho^2 \mu^2}{2(\rho + \varepsilon)}} \left[ \int_{\mathbb{R}} q e^{-q^2(\rho + \varepsilon)/2} \, dq - \frac{\rho H}{(\rho + \varepsilon)} \int_{\mathbb{R}} e^{-q^2(\rho + \varepsilon)/2} \, dq \right]. \tag{2.8}
\]
The first term on the right side is an integral of an odd function, and thus vanishes. The second integral on the right side converges as $\varepsilon \to 0$ to

$$K e^{r^2 \mu^2 / (2 \rho)} \frac{p^2}{\rho^{3/2}}.$$  

At $\varepsilon = 0$, we therefore have for the $dp$ integral,

$$K \frac{\mu}{\rho^{3/2}} \int_{\mathbb{R}} p^2 e^{-p^2 / 2 \rho} (\lambda \rho - \mu^2) \, dp = K \frac{\mu}{(\lambda \rho - \mu^2)^{3/2}}.$$  

Thus, we have reduced the problem to determining the integrability, over $D^2_\varepsilon$, of

$$K \mu (s - r)^{2H-1} (s' - r')^{2H-1} \frac{(\lambda \rho - \mu^2)^{3/2}}{(\lambda \rho - \mu^2)^{3/2}}.$$  

The existence of $\alpha_\varepsilon'(0)$ is therefore proved by invoking the following lemma, which is proved in the appendix.

**Lemma 2.2.** If $H < 2/3$, then

$$\int_{D^2_\varepsilon} \frac{\mu (s - r)^{2H-1} (s' - r')^{2H-1}}{(\lambda \rho - \mu^2)^{3/2}} \, dr' \, ds' < \infty.$$  

A few remarks are in order. First, something close to Lemma 2.2 was proved in [YYL08], but they had a factor of $s^{2H-1}$ where we have $(s - r)^{2H-1}$. This resulted from applying Itô’s formula to $B^H_s$ as opposed to $B^H_s - B^H_r$.

Next we note that in [Ros05], Rosen states “The (DSLT of Brownian motion) in $\mathbb{R}^1$, in a certain sense, is even more singular than self-intersection local time in $\mathbb{R}^2$.” If we believe that the critical Hurst parameter $H_c$ for the DSLT to exist in $L^2$ is $2/3$, then Rosen’s statement would be supported by the fact that $2/3$ is less than the critical Hurst parameter for the self-intersection local time of planar fBm ($\tilde{H}_c = 3/4$, see [Ros87, HN05]).

Above the critical parameter $H_c$, the behavior of $\alpha_\varepsilon'(0)$ as $\varepsilon \to 0$ is also of interest. One would expect a Central Limit Theorem to exist, along the lines of Theorem 2 in [HN05] or Theorem 1 in [Mar08b], but this remains unproved. In particular, it seems as though the techniques developed in [HN05] should apply, especially since the Wiener chaos expansion for $\alpha_\varepsilon'$ is readily computed (see Section 4), but the presence of the derivative seems to complicate matters. Nevertheless, we venture the following conjecture.

**Conjecture:**

- The critical parameter is $H_c = 2/3$. At $H_c$, $\frac{1}{\log(1/\varepsilon)} \alpha_\varepsilon'(0)$ converges in distribution to a normal law for some $\gamma > 0$. 
For $H > H_c$, $e^{-\gamma(H) t} \alpha_{t,\varepsilon}(0)$ converges in distribution to a normal law for some function $\gamma(H) > 0$ which is linear in $1/H$ and for which $\gamma(2/3) = 0$.

This would mirror the behavior of the intersection local time as seen in [HN05]. We should mention that, to our knowledge, no such Central Limit Theorem has yet been proved even for intersection local time in two dimensions at $H_c = 3/4$.

3 A FUBINI THEOREM FOR THE WIS INTEGRAL

There are several different ways one can integrate with respect to fBm as can be seen in [BHØZ08] or [Mis08]. We use the integral based on the fractional white noise theory introduced in [EVDH03] (see also [BØSW04], [BHØZ08, Ch. 4]). In particular, we adopt the nomenclature of [BHØZ08] and call this stochastic integral the Wick-Itô-Skorohod (WIS) integral.

In an effort to be somewhat self-contained, in this section we summarize some results concerning white noise and the WIS integral. For more details we refer the reader to [HOUZ96, EVDH03, BØSW04]. The only new result in the section is Theorem 3.9 which is standard for other stochastic integrals (cf. [CN05, Theorem 3.7], [Mis08, Theorem 1.13.1]); however, we have found no reference for such results with respect to Hida distributions and the WIS integral. Theorem 3.9 follows easily once one has the right definitions. Its formulation may prove useful in its own right, but the main reason for its presentation here is due to its role in proving the Tanaka formula.

3.1 CLASSICAL WHITE NOISE THEORY

Let $\Lambda = \mathbb{N}_0^n$ be the set of multi-indices with finite support, $S(\mathbb{R})$ be the Schwartz space, and $S'(\mathbb{R})$ be the space of tempered distributions. By the Bochner-Minlos theorem there is a probability measure $\mathcal{P}$ on the Borel $\sigma$-field of $S'$ satisfying

\[
\int_{S'(\mathbb{R})} \exp(i\langle \omega, f \rangle) \, d\mathcal{P}(\omega) = \exp \left(-\frac{1}{2} \|f\|^2_{L^2(\mathbb{R})}\right), \quad f \in S(\mathbb{R}). \tag{3.1}
\]

This measure satisfies, for all $f \in S(\mathbb{R})$,

\[
\mathbf{E}\langle \omega, f \rangle = 0 \quad \text{and} \quad \mathbf{E}\langle \omega, f \rangle^2 = \|f\|^2_{L^2(\mathbb{R})}. \tag{3.2}
\]

If $g_n \in S(\mathbb{R})$ converge to $1_{[0, t]}$ in $L^2(\mathbb{R})$, we define $\langle \omega, 1_{[0, t]} \rangle := \lim_{n \to \infty} \langle \omega, g_n \rangle$ as a limit in $L^2(\mathcal{P})$. The family $\left(1_{[0, t]}\right)_{t \geq 0}$ now maps $S'(\mathbb{R})$ to $\mathbb{R}$, and can be identified with random variables having characteristic functions $\phi(\nu) = \exp \left(-\frac{1}{2} (t\nu)^2\right)$. Choosing a continuous version of this family gives us Brownian motion.
Recall that the Hermite polynomials given by

\[ h_n(x) := (-1)^n e^{x^2/2} \frac{d^n}{dx^n} (e^{-x^2/2}), \quad n = 0, 1, 2, \ldots \]  

are orthogonal with respect to the standard Gaussian measure on \( \mathbb{R} \). Thus multiplying by the Gaussian density and choosing a convenient normalization gives us an orthonormal basis for \( L^2(\mathbb{R}) \),

\[ \xi_n(x) := \pi^{-1/4} ((n - 1)!)^{-1/2} h_{n-1}(\sqrt{2}x)e^{-x^2/2}, \quad n = 1, 2, 3, \ldots \]  

called the Hermite functions.\(^2\)

For \( \beta = (\beta_1, \ldots, \beta_n) \in \Lambda \), we define

\[ H_\beta(\omega) := h_{\beta_1}(\langle \omega, \xi_1 \rangle)h_{\beta_1}(\langle \omega, \xi_2 \rangle) \cdots h_{\beta_n}(\langle \omega, \xi_n \rangle). \]  

(3.5)

Every \( F(\omega) \in L^2(\mathcal{P}) \) has a representation in terms of the \( H_\beta \):

\[ F(\omega) = \sum_{\beta \in \Lambda} c_\beta H_\beta(\omega) \]  

(3.6)

where the series converges in \( L^2(\mathcal{P}) \). Moreover, one has the isometry

\[ \mathbb{E}F^2 = \sum_{\beta \in \Lambda} \beta! c_\beta^2. \]  

Representation (3.6) for \( F \) is called the Hermite chaos expansion, and it is related to the Wiener-Itô chaos expansion\(^3\) via the following formula which follows from [Itô51]:

\[ \int_{\mathbb{R}^n} \xi^{\odot \beta} dB^{\odot n} = H_\beta(\omega). \]  

(3.7)

Here \( \odot \) denotes the symmetrized tensor product, and \( \xi^{\odot \beta} := \xi_1^{\odot \beta_1} \odot \cdots \odot \xi_n^{\odot \beta_n} \). In particular, the Hermite chaos is a way of writing the \( n \)th Wiener-Itô chaos in terms of \( n \)-fold products of Hermite functions, which form orthonormal bases of \( L^2(\mathbb{R}^n) \) (see [HOUZ96, pg. 30]).

The main reason for using chaos expansions in terms of Hermite polynomials instead of multiple Wiener-Itô integrals is the natural extension to distributions they have from \( L^2(\mathcal{P}) \) random variables. Let

\[ (2N)^\gamma := (2 \cdot 1)^{\gamma_1} (2 \cdot 2)^{\gamma_2} \cdots (2 \cdot n)^{\gamma_n} \quad \text{for} \quad \gamma = (\gamma_1, \ldots, \gamma_n) \in \Lambda. \]  

(3.8)

\(^2\)One may substitute any orthonormal basis of \( L^2(\mathbb{R}) \) whose elements possess decay properties such that Lemma 4.1 of [EVDH03] holds. See also Theorem 3.1 in [Itô51].

\(^3\)For a full treatment of multiple Wiener-Itô integrals and their related chaos expansions, see [Nua95].
Definition 3.1 (Hida test functions and distributions). Given the probability measure $\mathcal{P}$ on $S'$, the Hida test function space $(S)$ is the set of all $\psi \in L^2(\mathcal{P})$ given by

$$\psi(\omega) = \sum_{\beta \in \Lambda} a_\beta \mathcal{H}_\beta(\omega)$$

satisfying

$$\sum_{\beta \in \Lambda} a_\beta^2 \beta!(2N)^k \beta \text{ for all } k = 1, 2, \ldots$$

The Hida distribution space $(S)^*$ is the set of all formal expansions

$$F(\omega) = \sum_{\beta \in \Lambda} b_\beta \mathcal{H}_\beta(\omega)$$

satisfying

$$\sum_{\beta \in \Lambda} b_\beta^2 \beta!(2N)^{-q} \beta \text{ for some } q \in \mathbb{R}.$$  

It was shown in [Zha92] that $(S)^*$ is the dual of $(S)$. Moreover, by Corollary 2.3.8 of [HOUZ96],

$$(S) \subset L^2(\mathcal{P}) \subset (S)^*.$$ 

This should be thought of as analogous to the triplet $S \subset L^2(\mathbb{R}) \subset S^*$. Note that if $F \in L^2(\mathcal{P})$, then

$$\langle\langle F, \psi \rangle\rangle = \langle F, \psi \rangle_{L^2(\mathcal{P})} = \mathbb{E}(F\psi).$$

Thus, for $\psi = \sum_{\beta \in \Lambda} a_\beta \mathcal{H}_\beta \in (S)$ and $F = \sum_{\beta \in \Lambda} b_\beta \mathcal{H}_\beta \in (S)^*$, the duality inherited from $L^2(\mathcal{P})$ is given by

$$\langle\langle F, \psi \rangle\rangle := \sum_{\beta \in \Lambda} \beta! a_\beta b_\beta. \quad (3.9)$$

Lemma 3.2. Suppose $F(x)$ is an $(S)^*$-valued function on a $\sigma$-finite measure space $(X, \mathcal{B}, \nu)$ and that

$$\langle\langle F(x), \psi \rangle\rangle \in L^1(X, \nu) \ \forall \ \psi \in (S). \quad (3.10)$$

Then there is a unique $G$ in $(S)^*$ such that

$$\langle\langle G, \psi \rangle\rangle = \int_X \langle\langle F(x), \psi \rangle\rangle \, d\nu \ \forall \ \psi \in (S). \quad (3.11)$$

We write $\int_X F(x) \, d\nu := G$.

Proof. See Theorem 3.7.1 in [HP57] or Proposition 8.1 in [HKPS93].
Let $T \subseteq \mathbb{R}$ be a time interval. In light of the above result, we say an $(S)^*$-valued process is in $L^1(T)$ if
\[
\langle\langle F(t), \psi \rangle\rangle \in L^1(T) \quad \text{for all } \psi \in (S).
\] (3.12)

The Hida distribution space $(S)^*$ is a convenient space on which to define the Wick product.

**Definition 3.3 (Wick product).** If \( F = \sum_{\beta \in \Lambda} b_{\beta} H_{\beta} \) and \( G = \sum_{\gamma \in \Lambda} c_{\gamma} H_{\beta} \) are elements of $(S)^*$, then their Wick product is defined as
\[
F \cdot G := \sum_{\beta, \gamma} b_{\beta} c_{\gamma} H_{\beta + \gamma}(\omega) = \sum_{\lambda \in \Lambda} \left( \sum_{\beta + \gamma = \lambda} b_{\beta} c_{\gamma} \right) H_{\lambda}(\omega).
\]

By Lemma 2.4.4 in [HOUZ96], $(S)^*$ is closed under this product.

Given the measure $\mathcal{P}$ on $S'$ from (3.1), we may write Brownian motion as
\[
B(t) = \langle \omega, 1_{[0,t]}(\cdot) \rangle = \left( \omega, \sum_{k=1}^{\infty} \langle 1_{[0,t]} \xi_k \rangle_{L^2(\mathbb{R})} \xi_k(\cdot) \right) = \sum_{k=1}^{\infty} \int_{0}^{t} \xi_k(s) \, ds \, H_{\varepsilon^{(k)}}(\omega).
\] (3.13)

where $\varepsilon^{(k)}$ is the multi-index with a 1 in the $k$th entry and 0’s elsewhere. This motivates the following definition:

**Definition 3.4 (White noise).** The $(S)^*$-valued process
\[
W(t) := \sum_{k \geq 1} \xi_k(t) H_{\varepsilon^{(k)}}(\omega)
\]
is called white noise.

Using the Wick product on $(S)^*$ and the definition of the integral of a Hida distribution given in Lemma 3.2 we can now integrate a Hida distribution with respect to white noise:
\[
\int_{\mathbb{R}} F(t) \, dB_t := \int_{\mathbb{R}} F(t) \cdot W(t) \, dt.
\] (3.14)

The above is called a WIS integral with respect to white noise. The following theorem shows that the WIS integral is a generalization of the Skorohod integral:
Theorem 3.5. Suppose \( F(t, \omega) : \mathbb{R} \times \Omega \rightarrow \mathbb{R} \) is Skorohod integrable. Then \( F(t, \cdot) \circ W(t) \) is \( dt \)-integrable in \((S)^*\) and

\[
\int_a^b F(t, \omega) \delta B(t) = \int_a^b F(t, \cdot) \circ W(t) \, dt. \tag{3.15}
\]

Proof. See Theorem 2.5.9 in [HOUZ96]. \( \square \)

3.2 Fractional white noise theory and a Fubini theorem

Elliot and Van Der Hoek [EVDH03] introduced the fractional white noise as an element of the Hida distribution space, and thus constructed the WIS integral\(^4\) which is valid for any \( H \in (0, 1) \). The main tool used to define the fractional white noise is the following operator for which we set:

\[
c_H := \sqrt{\sin(\pi H) \Gamma(2H + 1)}
\]

Definition 3.6 \((M_H\) operator\). The \( M_H \) operator on \( f \in S(\mathbb{R}) \) is defined by

\[
\hat{M_H} f(y) = c_H |y|^{\frac{1}{2} - H} \hat{f}(y).
\]

This operator extends to the space

\[
L^2_H(\mathbb{R}) := \{ f : M_H f \in L^2(\mathbb{R}) \}
\]

which is equipped with the inner product

\[
\langle f, g \rangle_{L^2_H(\mathbb{R})} = \langle M_H f, M_H g \rangle_{L^2(\mathbb{R})}. \tag{3.17}
\]

We note that \( L^2_H(\mathbb{R}) \) is not closed under this inner product, and that its closure contains distributions (see [Nua95, pg. 280] or [BHØZ08, Ch. 2]).

Since \( M_H f \in L^2(\mathbb{R}) \) we can moreover define \( M_H : S'(\mathbb{R}) \rightarrow S'(\mathbb{R}) \) by

\[
\langle M_H \omega, f \rangle := \langle \omega, M_H f \rangle \quad \text{for } f \in S(\mathbb{R}), \omega \in S'(\mathbb{R}). \tag{3.18}
\]

We have defined \( M_H \) for a given fixed \( H \in (0, 1) \), however, the theory extends to \( M \) operators which are linear combinations \( a_1 M_{H_1} + \cdots + a_n M_{H_n} \); for more on the \( M \) operator see [SKM82, EVDH03, LV11].

\(^4\)Compare this to the integral of [HØ03] where a fractional Brownian measure is defined directly on tempered distributions \( S' \). Integrals with respect to this measure are defined for \( H > 1/2 \).
Using the orthonormal basis $e_k := M_H^{-1} \xi_k$ of $L^2_H(\mathbb{R})$, we define (and choose a continuous version) of fBm as

$$B^H(t) := \langle \omega, M_H 1_{[0,t]}(\cdot) \rangle = \langle M_H \omega, 1_{[0,t]}(\cdot) \rangle$$  \hspace{1cm} (3.19)

which satisfies, by (3.2) and (A.10) in [EVDH03],

$$\mathbb{E}[B^H(t)B^H(s)] = \langle M_H 1_{[0,t]}, M_H 1_{[0,s]} \rangle_{L^2(\mathbb{R})} \hspace{1cm} (3.20)$$

$$= \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Note that the action of $S'$ on $S$ given by $\langle \omega, \cdot \rangle$ is still inherited from $L^2(\mathbb{R})$ and not from $L^2_H(\mathbb{R})$.

The definition in (3.19) can be further rewritten as

$$B^H(t) = = \langle M_H \omega, 1_{[0,t]}(\cdot) \rangle \hspace{1cm} (3.21)$$

$$= \left\langle M_H \omega, \sum_{k=1}^{\infty} \langle 1_{[0,t]}, e_k \rangle_{L^2_H(\mathbb{R})} e_k(\cdot) \right\rangle$$

$$= \left\langle M_H \omega, \sum_{k=1}^{\infty} \langle M_H 1_{[0,t]}, \xi_k \rangle_{L^2(\mathbb{R})} e_k(\cdot) \right\rangle$$

$$= \sum_{k=1}^{\infty} \int_0^t M_H \xi_k(s) \, ds \, \langle M_H \omega, e_k \rangle = \sum_{k=1}^{\infty} \int_0^t M_H \xi_k(s) \, ds \, \mathcal{H}_{\varepsilon(k)}(\omega)$$

which motivates the following notion of fractional white noise:

$$W^H(t) := \sum_{k \geq 1} M_H \xi_k(t) \mathcal{H}_{\varepsilon(k)}(\omega).$$  \hspace{1cm} (3.22)

By Lemma 4.1 in [EVDH03], $W^H(t)$ is an $(S)^*$-valued process. We note here that the underlying probability measure $\mathcal{P}$ on $S'$ is the same as for $W(t)$.

**Definition 3.7 (WIS integral).** Let $F(t)$ be an $(S)^*$-valued process such that $F(t) \circ W^H(t)$ is in $L^1(\mathbb{R})$ (as in (3.12)). We define

$$\int_\mathbb{R} F(t) \, dB^H_t := \int_\mathbb{R} F(t) \, dW^H_t := \int_\mathbb{R} F(t) \circ W^H(t) \, dt.$$ 

**Theorem 3.8** (fractional Itô formula). Let $f(s,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be in $C^{1,2}(\mathbb{R} \times \mathbb{R})$. If the random variables

$$f(t,B^H_t), \quad \int_0^t f_s(s,B^H_s) \, ds, \quad \text{and} \quad \int_0^t f_{xx}(s,B^H_s) s^{2H-1} \, ds$$
are in $L^2(\mathcal{P})$ then
\[
    f(t, B^H_t) - f(0, 0) = \int_0^t f_s(B^H_s) \, ds + \int_0^t f_x(s, B^H_s) \, dB^H_s + H \int_0^t f_{xx}(s, B^H_s) \, s^{2H-1} \, ds.
\]

**Proof.** See Theorem 3.8 of [BØSW04].

We now prove a result which one can compare to Fubini-type theorems in [CN05, Thm 3.7] and [Mis08, Thm 1.13.1]. Our result extends these Fubini theorems to integrals of Hida distributions.

**Theorem 3.9** (Fubini-Tonelli theorem). Let
\[
    F_{s,r} = \sum_{\beta \in \Lambda} c_\beta(s, r) \mathcal{H}_\beta
\]
be an $(S)^*$-valued process indexed by $(s, r) \in \mathbb{R} \times [0, t]$. If, for each $(\beta, k)$ pair, $c_\beta(s, r) M_H \xi_{k}(s)$ is bounded above or below by an $L^1([r, t] \times [0, t])$ function, then
\[
    \int_0^t \int_r^t F_{s,r}(\omega) \, dB^H_s \, dr = \int_0^t \left( \int_0^s F_{s,r}(\omega) \, dr \right) \, dB^H_s.
\]

The equality in (3.24) is in the sense that if one side is in $(S)^*$, then the other is as well, and they are equal. If in addition,
\[
    (F_{s,r} 1_{[r,t]}(s) \circ W^H(s)) (s, r) \in L^1(\mathbb{R} \times [0, t]),
\]
then both sides are in $(S)^*$ and $c_\beta(s, \cdot) \in L^1[0, s]$ for a.e. $s \in [0, t]$.

**Proof.** Unraveling the above definitions we have
\[
    \int_0^t \int_r^t F_{s,r}(\omega) \, dB^H_s \, dr
    \]
\[
    := \int_0^t \int_{\mathbb{R}} F_{s,r} 1_{[r,t]}(s) \circ W^H(s) \, ds \, dr
    \]
\[
    := \int_0^t \int_{\mathbb{R}} \left( \sum_{\beta \in \Lambda} c_\beta(s, r) 1_{[r,t]}(s) \mathcal{H}_\beta(\omega) \right) \circ \left( \sum_{k \geq 0} M_H \xi_{k}(s) \mathcal{H}_{\beta+k}(\omega) \right) \, ds \, dr
    \]
\[
    := \int_0^t \int_{\mathbb{R}} \left( \sum_{\beta \in \Lambda, k \in \mathbb{N}} c_\beta(s, r) 1_{[r,t]}(s) M_H \xi_{k}(s) \mathcal{H}_{\beta+k}(\omega) \right) \, ds \, dr.
\]
4 THE WIENER CHAOS DECOMPOSITION OF DSLT

Denote the right-hand side above as $G$, an $(S)^*$-valued integral. By Lemma 3.2, such integrals are characterized by their action on $(S)$:

$$
\langle \langle G, \psi \rangle \rangle = \int_0^t \int_{\mathbb{R}} \sum_{\beta \in \Lambda, k \in \mathbb{N}} \langle \langle c_\beta (s, r) 1_{[r, t]}(s) M_H \xi_k (s) \mathcal{H}_{\beta + \varepsilon(k), \psi} \rangle \rangle \, ds \, dr \tag{3.27}
$$

where equality in the second line follows since for any $\beta' \in \Lambda$ there are finitely many pairs $(\beta, k)$ such that $\beta + \varepsilon(k) = \beta'$ (recall also that the $\mathcal{H}_{\beta'}$ are orthogonal). The third equality follows from Tonelli’s Theorem for real-valued functions which is possible due to our hypothesis. If in addition (3.25) holds then both sides of this equality are in $(S)^*$ by (3.12). The final equality follows from Lemma 3.2.

Teasing apart the right side of (3.27) gives

$$
G = \int_{\mathbb{R}} \left( \sum_{\beta \in \Lambda} \left[ \int_0^s c_\beta (s, r) \, dr \right] 1_{[0, t]}(s) \mathcal{H}_\beta (\omega) \right) \diamond \left( \sum_{k \geq 0} M_H \xi_k (s) \mathcal{H}_{\varepsilon(k), \omega} \right) \, ds
$$

as needed.

\[4\] The Wiener chaos decomposition of DSLT

In this section we start by calculating the Wiener chaos expansion for $\alpha'_t (0)$ defined in (1.10). In the process, we obtain a new proof of the existence of $\alpha'_t (0)$ for $H < 2/3$. Later in the section we will adapt the arguments used in obtaining the Wiener chaos to show existence of $\alpha'_t (y)$ for all $y \in \mathbb{R}$. To reduce notation, in this sequel we write $H = L^2 (\mathbb{R})$.

Theorem 4.1. For $H < 2/3$, $\alpha'_t (0)$ is in $L^2 (\mathbb{P})$ and its Wiener chaos decomposition is

$$
\alpha'_t (0) = \sum_{m=1}^{\infty} I_{2m-1} (g(2m - 1, t)) \tag{4.1}
$$
where \( g(2m-1, t) \in \mathbf{H}^{2m-1} \) and

\[
g(2m-1, t) = g(2m-1, t; v_1, \ldots, v_{2m-1}) = \frac{(-1)^m}{(m-1)!2^{m-1}\sqrt{2\pi}} \int_0^t \int_0^s \prod_{j=1}^{2m-1} M_{H1[r,s]}(v_j) \, dr \, ds \quad (4.2)
\]

**Proof.** Recall that \( B^H_t := \langle \omega, M_{H1[0,t]} \rangle \) so that in particular, its Malliavin derivative is \( DB^H_t = M_{H1[0,t]} \).

It is not hard to see that \( f'_\varepsilon(B^H_s - B^H_r) \) is in \( \cap_{k\in\mathbb{N}}\mathbb{D}^{k,2} \), thus by Stroock’s formula, the \( n \)-th integrand in the chaos expansion of \( \alpha'_{t,\varepsilon}(0) \) is given by

\[
\frac{1}{n!} \int_0^t \int_0^s (s-r)^{2H-1} \mathbf{E}[D^n f'_\varepsilon(B^H_s - B^H_r)] \, dr \, ds \quad (4.3)
\]

\[
= \frac{1}{n!} \int_0^t \int_0^s (s-r)^{2H-1} \mathbf{E}[(\frac{d^{n+1}}{dx^{n+1}} f'_\varepsilon)(B^H_s - B^H_r)] \prod_{j=1}^n M_{H1[r,s]}(v_j) \, dr \, ds.
\]

As in Section 2 we write

\[
\frac{d^n}{dx^n} f'_\varepsilon(x) = \frac{i^n}{2\pi} \int_{\mathbb{R}} e^{ipx} p^n e^{-\varepsilon p^2/2} \, dp. \quad (4.4)
\]

Thus,

\[
\mathbf{E}[(\frac{d^{n+1}}{dx^{n+1}} f'_\varepsilon)(B^H_s - B^H_r)] = \frac{i^{n+1}}{2\pi} \int_{\mathbb{R}} \mathbf{E}[e^{ip(B^H_s - B^H_r)}] p^{n+1} e^{-\varepsilon p^2/2} \, dp
\]

\[
= \frac{i^{n+1}}{2\pi} \int_{\mathbb{R}} p^{n+1} e^{-(s-r)^{2H} + \varepsilon p^2/2} \, dp
\]

\[
= \frac{i^{n+1}}{2\pi((s-r)^{2H} + \varepsilon)^{(n/2)+1}} \int_{\mathbb{R}} p^{n+1} e^{-\varepsilon p^2/2} \, dp
\]

\[
= \frac{i^{n+1} \sqrt{2\pi}}{2\pi((s-r)^{2H} + \varepsilon)^{(n/2)+1}} \frac{(n+1)!}{2(n+1/2)((n+1)/2)!}. \quad (4.5)
\]

if \( n+1 \) is even, and 0 if \( n+1 \) is odd. Setting \( n = 2m-1 \), it follows that the chaos expansion for \( \alpha'_{t,\varepsilon}(0) \) is

\[
\sum_{m=1}^{\infty} I_{2m-1} \left( \frac{(-1)^m}{(m-1)!2^{m-1}\sqrt{2\pi}} \int_0^t \int_0^s (s-r)^{2H-1} \varepsilon + (s-r)^{2H(2m+1)/2} \prod_{j=1}^{2m-1} M_{H1[r,s]}(v_j) \, dr \, ds \right). \quad (4.6)
\]
We now need to show that as $\varepsilon \to 0$, the above converges in $L^2(\mathcal{P})$ to (4.1). We will apply the following lemma adapted from [NV92], which is a consequence of the Dominated Convergence Theorem.

**Lemma 4.2.** Let $F_\varepsilon$ be a family of $L^2(\mathcal{P})$ random variables with chaos expansions

$$F_\varepsilon = \sum_{n=0}^{\infty} I_n(f_\varepsilon^n).$$

If for each $n$, $f_\varepsilon^n$ converges in $H^n \otimes \mathcal{P}$ to $f_n$ as $\varepsilon \to 0$, and if

$$\sum_{n=0}^{\infty} \sup_{\varepsilon} \mathbb{E}[|I_n(f_\varepsilon^n)|^2] = \sum_{n=0}^{\infty} \sup_{\varepsilon} \{n!||f_\varepsilon^n||^2_{H^n \otimes \mathcal{P}}\} < \infty,$$

(4.7) then $F_\varepsilon$ converges in $L^2(\mathcal{P})$ to $F = \sum_{n=0}^{\infty} I_n(f_n)$ as $\varepsilon \to 0$.

We note that this argument has also been used in [HN05] and [Mar12]. To apply the lemma here, we calculate the $L^2(\mathcal{P})$-norms of the chaos expansions and show they are bounded uniformly in $\varepsilon$. Recall that $D_t = \{0 \leq r \leq s \leq t\}$. If we let $g(2m - 1, t, \varepsilon)$ be the integrand of $I_{2m-1}$ in (4.6), we have

$$\mathbb{E}[I_{2m-1}(g(2m - 1, t, \varepsilon))^2] = \frac{(2m - 1)!((2m)!)}{2\pi([2m - 1]![2m]!^2)^2} \int_{D_t^2} \frac{(s - r)^{2H-1}(s' - r')^{2H-1}}{(s + (s - r)^{2H})(s + (s' - r')^{2H})^2} \times \left( \int_{\mathbb{R}^{2m-1}} \prod_{j=1}^{2m-1} \langle M_{H1[r,s]}(v_j), M_{H1[r',s']}(v_j) \rangle_H \, dv_j \right) \, dr \, dr' \, ds \, ds'.

(4.8)

Maximizing by setting $\varepsilon = 0$ and using (A.10) in [EVDH03] and the notation in (2.6), this is

$$\frac{m(2m)!}{\pi(m)!^2} \int_{D_t^2} \frac{(s - r)^{2H-1}(s' - r')^{2H-1}}{\lambda^{m+1/2} \rho^{m+1/2}} \, dr \, dr' \, ds \, ds'.

(4.9)

Let $\gamma = \mu^2/(\lambda \rho)$. The $L^2(\mathcal{P})$-norm of (4.6) is then

$$\frac{1}{\pi} \int_{D_t^2} \left( \sum_{m=1}^{\infty} \frac{m(2m)!\gamma^m}{(m!)^22^{2m}} \right) \frac{(s - r)^{2H-1}(s' - r')^{2H-1}}{\mu \sqrt{\lambda \rho}} \, dr \, dr' \, ds \, ds'.

(4.10)

However, by the generalized binomial theorem,

$$\frac{\gamma}{2(1 - \gamma)^{3/2}} = \sum_{m=1}^{\infty} \frac{m(2m)!\gamma^m}{(m!)^22^{2m}}.

(4.11)$$
Thus, the \( L^2(\mathcal{P}) \)-norm of (4.6) is

\[
\frac{1}{2\pi} \int_{\mathbb{D}_t^2} \frac{\gamma(s-r)^{2H-1}(s'-r')^{2H-1}}{(1-\gamma)^{3/2} \mu \sqrt{\lambda \mu}} = \frac{1}{2\pi} \int_{\mathbb{D}_t^2} \frac{\mu(s-r)^{2H-1}(s'-r')^{2H-1}}{(\lambda \mu - \lambda \mu)^{3/2}}.
\]

By Lemma \[\text{[2.2]}\] this is finite if \( H < 2/3 \).

\[
\text{Remark: One might think of fBm as an isonormal Gaussian process } W : H' \to L^2(\mathcal{P}) \text{ on the space } H' = L^2_H(\mathbb{R}) \text{ which may contain distributions. Then}
\]

\[
\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{H'} := \langle M_H 1_{[0,t]}, M_H 1_{[0,t]} \rangle_{L^2(\mathbb{R})} = \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H}).
\]

Using this so call “twisted” inner product Hilbert space, the isonormal Gaussian process gives \( B_t^H = \langle \omega, 1_{[0,t]} \rangle_{tw} \) so that \( DB_t^H = 1_{[0,t]} \). Comparing this with the above, we see that the twisted inner product incorporates the operation of \( M_H \) in \( \langle \omega, M_H f \rangle \) into \( \langle \omega, f \rangle_{tw} \). When \( f \) is a step function, essentially nothing but notation has changed, which can be verified by the use of the \( D^M \) operator in Example 6.4 from [EVDH03].

One may then write

\[
\int_0^t \int_0^s \frac{(s-r)^{2H-1}}{((s-r)^{2H})^{(2m+1)/2}} \prod_{j=1}^{2m-1} 1_{[r,s]}(v_j) \, dr \, ds = \int_0^t \int_0^r \frac{(s-r)^{2H-1}}{((s-r)^{2H})^{(2m+1)/2}} \, dr \, ds,
\]

where

\[
\overline{v} = v_1 \vee \ldots \vee v_{2m-1}, \\
\underline{v} = v_1 \wedge \ldots \wedge v_{2m-1}.
\]

It is then straightforward to verify that Proposition 4.1 simplifies to the chaos expansion given in [Mar12] in the case \( H = 1/2 \).

We now use the methods in the above proof to show \( L^2(\mathcal{P}) \) convergence for \( \alpha'_\epsilon(y) \) as \( \epsilon \to 0 \).

\section*{Proposition 4.3} For \( H < 2/3 \) and any \( y \in \mathbb{R} \), \( \alpha'_\epsilon(y) \) is in \( L^2(\mathcal{P}) \).

\textbf{Proof.} We may follow the proof of Theorem 4.1 except that in place of (4.5) we have

\[
|\mathbf{E}[\left. (\frac{d^{n+1}}{dx^{n+1}} f_\epsilon)(B^H_s - B^H_r - y) \right.]| = \left| \frac{1}{2\pi} \int_\mathbb{R} \mathbf{E}[e^{ip(B^H_s - B^H_r)}] e^{ipy} p^{n+1} e^{-p^2/2} \, dp \right|
\leq \frac{1}{2\pi((s-r)^{2H} + \epsilon)^{(n+1)/2+1}} \int_\mathbb{R} |p|^{n+1} e^{-p^2/2} \, dp.
\]
We aim to apply Lemma 4.2 again. The arguments from Theorem 4.1 show that the sum of the odd terms in (4.7) converges. However, we can no longer argue that the even terms are 0, as we did before. Instead, we must use the identity
\[ \int_{\mathbb{R}} \left| p \right|^{n+1} e^{-p^2/2} \, dp = 2^{n/2} (n/2)! \], valid for even \( n \).
Replacing (4.8), we then have
\[ E[I_{2m}(g(2m, t, \varepsilon))^2] = (2m)! \| g(2m, t, \varepsilon) \|_2^2 \]
\[ = \frac{(2m)! (m!)^2}{(2m)!} \left( \int_{\mathbb{R}} (s-r)^{2H-1} (s'-r')^{2H-1} \varepsilon + (s-r)^{2H} + (s'-r')^{2H} \right)^{m+1} \]
\[ \times \left( \int_{\mathbb{R}^{2m}} \prod_{j=1}^{2m} (M_{H} 1_{[r,s]}(v_j), M_{H} 1_{[r',s']})(v_j) \right) \, dr \, dr' \, ds \, ds'. \]
(4.15)
We proceed through steps (4.9) and (4.10), setting \( \varepsilon = 0 \) and \( \gamma = \mu^2/(\lambda \rho) \), in order to reach a bound on the even terms of
\[ \frac{1}{\pi} \int_{D^2} \left( \sum_{m=1}^{\infty} \frac{(m!)^2 2^{2m} \gamma^m}{(2m)!} \right) \frac{(s-r)^{2H-1} (s'-r')^{2H-1}}{\lambda \rho} \, dr \, dr' \, ds \, ds'. \]
(4.16)
We now use the following identity and bound, valid for \( 0 \leq \gamma < 1 \) and with \( K \) a positive constant:
\[ \sum_{m=1}^{\infty} \frac{(m!)^2 2^{2m} \gamma^m}{(2m)!} = \frac{\gamma \sqrt{1-\gamma} + \sqrt{\gamma} \sin^{-1}(\sqrt{\gamma})}{(1-\gamma)^{3/2}} \leq \frac{K \gamma}{(1-\gamma)^{3/2}} \leq \frac{K \sqrt{\gamma}}{(1-\gamma)^{3/2}}. \]
(4.17)
Inserting this and the expression for \( \gamma \) into (4.10) gives a bound of
\[ \frac{K}{\pi} \int_{D^2} \mu (s-r)^{2H-1} (s'-r')^{2H-1} \, dr \, dr' \, ds \, ds' \]
\[ = \frac{K}{\pi} \int_{D^2} \mu (s-r)^{2H-1} (s'-r')^{2H-1} \, (\lambda \rho - \mu^2)^{3/2} \, dr \, dr' \, ds \, ds' < \infty. \]
(4.18)
By Lemma 2.2 this is finite for \( H < 2/3 \). \( \Box \)
5 The Tanaka formula

The following is what we have referred to as the Tanaka formula for the DSLT process.

**Theorem 5.1.** For $0 < H < 2/3$, the following equality holds for all $y$ and $t$ in $L^2(\mathcal{F})$:

$$H \alpha_t'(y) + \frac{1}{2} \text{sgn}(y) t = \int_0^t L_s^{B_t^H - y} dB_s^H - \frac{1}{2} \int_0^t \text{sgn}(B_t^H - B_r^H - y) \, dr.$$  

(5.1)

**Proof.** Let $f_\varepsilon$ be defined as in Section 2 and let

$$F_\varepsilon(x) = \int_0^x f_\varepsilon(u) \, du = \int_0^{x/\varepsilon} f(u) \, du.$$  

(5.2)

We apply Theorem 3.8 (Itô’s formula) to $F_\varepsilon(x - y)$ using the fractional Brownian motion $B_t^H - B_r^H$, and then integrate with respect to $r$ from 0 to $t$ to get

$$\int_0^t F_\varepsilon(B_t^H - B_r^H - y) \, dr - t F_\varepsilon(-y) =$$

$$\int_0^t \int_r^t f_\varepsilon(B_s^H - B_r^H - y) \, dB_s^H \, dr + H \int_0^t \int_r^t f_\varepsilon'(B_s^H - B_r^H - y)(s - r)^{2H-1} \, ds \, dr.$$  

(5.3)

Note that $F_\varepsilon(-y) \to -\frac{1}{2} \text{sgn}(y)$ as $\varepsilon \to 0$. Also, $|F_\varepsilon(\cdot)| \leq 1/2$ for all $\varepsilon > 0$, so by dominated convergence the integral term on the left side approaches

$$\frac{1}{2} \int_0^t \text{sgn}(B_t^H - B_r^H - y) \, dr$$

as $\varepsilon \to 0$. We now want to apply Theorem 3.9 to the first term on the left side to get

$$\int_0^t \int_0^s f_\varepsilon(B_s^H - B_r^H - y) \, dr \, dB_s^H.$$  

(5.4)

To show (5.4), we use the following Wiener-Ito chaos expansion obtained from Stroock’s formula:

$$f_\varepsilon(B_s^H - B_r^H - y) = \sum_{n \geq 0} \left( \frac{1}{n!} \mathbb{E}[\left( \frac{d^n}{dx^n} f_\varepsilon(B_s^H - B_r^H - y) \right)(M_H^1_{[r,s]} \otimes n) \right].$$  

(5.5)

As stated earlier, the Hermite chaos refines the $n$th Wiener-Ito chaos in terms of the Hermite orthonormal basis of $H^\otimes n$. In particular, the coefficient $c_\beta(s,r)$, with $|\beta| = n$, in the Hermite chaos expansion of $f_\varepsilon(B_s^H - B_r^H - y)$ is given by

$$c_\beta(s,r) = \frac{1}{n!} \mathbb{E}[\left( \frac{d^n}{dx^n} f_\varepsilon(B_s^H - B_r^H - y) \right) \langle \xi \otimes \beta, (M_H^1_{[r,s]} \otimes n) \rangle_{H^\otimes n}.$$  

(5.6)
Using (4.14) and the fact that $M_H \xi_k$ is bounded (see Lemma 4.1 of [EVDH03]), one can now easily verify the conditions of Theorem 3.9 for $f_\varepsilon(B^H_s - B^H_r - y)$ for $\varepsilon > 0$.

For fixed $s$, as $\varepsilon \to 0$ the inner integral in (5.4) converges to $L^B_H s - B^H_y$ in $(S)^*$ by Proposition 10.1.13 in [BHØZ08]. Now, if $L^B_H s - B^H_y \circ W^H(s)$ is integrable in $(S)^*$, then by the arguments of Proposition 8.1 in [HKPS93], (5.4) converges in $(S)^*$ to $\int_0^t L^B_H s - B^H_y \circ W^H(s) \, ds$. In other words, the equality in (5.1) is valid as long as one side or the other is in $(S)^*$. However, by Proposition 4.3, $\alpha'_{t}(y) \in L^2(\mathcal{P})$, for $H < 2/3$.

Thus for such $H$, (5.1) holds in $L^2(\mathcal{P})$.

Remark: One-dimensional fBm has an $L^2(\mathcal{P})$ local time for any $0 < H < 1$ (see [BHØZ08]), but it is not clear whether or not $\int_0^t L^B_H s - B^H_y \, dB^H_s$ is in $L^2(\mathcal{P})$ for $H > 2/3$. As stated in the conjecture of Section 2, we suspect that it is not. Of course, a positive answer to the conjecture does not rule out the possibility that $\int_0^t L^B_H s - B^H_y \, dB^H_s \in (S)^*$ for $H > 2/3$. If (5.7) were indeed true, then the DSLT of fBm would also be well-defined in $(S)^*$ for all $H \in (0, 1)$. For $H < 2/3$, another open problem is to prove joint continuity, in $y$ and $t$, of $\alpha'_{t}(y) + \text{sgn}(y)t$. One approach to proving this is to use the explicit chaos expansion for this integral (see Theorem 4.1) combined with Definition 3.7.

A Second moment bounds for fBm

Recall

\begin{align}
\lambda &:= \text{Var}(B^H_s - B^H_r) = |s - r|^{2H} \\
\rho &:= \text{Var}(B^H_s' - B^H_{s'}') = |s' - r'|^{2H} \\
\mu &:= \text{Cov}(B^H_s - B^H_r, B^H_s' - B^H_{s'}') \\
&= \frac{1}{2} (|s' - r'|^{2H} + |s - r'|^{2H} - |s - s'|^{2H} - |r' - r|^{2H}).
\end{align}

The following useful bounds appear as Lemma 3.1 of [Hu01]. As before, we let $K$ denote a positive constant, which may change from line to line.

**Lemma A.1.** (i) Suppose $r < r' < s < s'$. Let $a = r' - r$, $b = s - r'$, $c = s' - s$. Then

\begin{align}
\lambda \rho - \mu^2 \geq K \left( (a + b)^{2H} c^{2H} + a^{2H} (b + c)^{2H} \right).
\end{align}
(ii) Suppose \( r < r' < s' < s \). Let \( a = r' - r, b = s' - r', c = s - s' \). Then

\[
\lambda \rho - \mu^2 \geq K b^{2H} (a + b + c)^{2H}. \tag{A.3}
\]

(iii) Suppose \( r < s < r' < s' \). Let \( a = s - r, b = r' - s, c = s' - r' \). Then

\[
\lambda \rho - \mu^2 \geq K (a^{2H} c^{2H}). \tag{A.4}
\]

Unfortunately, (ii) is false, which can be seen by noting that if \( r' \searrow r, s' \nearrow s \) then the left side of (A.3) approaches 0 while the right side does not. However, (ii) can be replaced with the following, which suffices in every instance in which we have seen (ii) applied.

(ii') Suppose \( r < r' < s' < s \). Let \( a = r' - r, b = s' - r', c = s - s' \). Then

\[
\lambda \rho - \mu^2 \geq K b^{2H} (a^2H + c^{2H}). \tag{A.5}
\]

Proof. We will follow the method of proof given for (ii) to arrive at (ii'). We use the property of fBm known as local nondeterminism, which implies that the following is true for \( t_0 < \ldots < t_j \):

\[
\Var \left( \sum_{i=1}^{j} u_i (B_{t_i}^{H} - B_{t_{i-1}}^{H}) \right) \geq K \sum_{i=1}^{j} |u_i|^2 (t_i - t_{i-1})^{2H}. \tag{A.6}
\]

In our case, this implies

\[
\Var \left( u (B_{s}^{H} - B_{r}^{H}) + v (B_{s'}^{H} - B_{r'}^{H}) \right) \geq K \left( u^2 (a^{2H} + c^{2H}) + (u + v)^2 (b^{2H}) \right). \tag{A.7}
\]

However, we also have

\[
\Var \left( u (B_{s}^{H} - B_{r}^{H}) + v (B_{s'}^{H} - B_{r'}^{H}) \right) = u^2 \lambda + 2uv \mu + v^2 \rho. \tag{A.8}
\]

Combining (A.7) and (A.8) yields

\[
u^2 (\lambda - Ka^{2H} - Kb^{2H} - Kc^{2H}) + 2uv (\mu - Kb^{2H}) + v^2 (\rho - Kc^{2H}) \geq 0. \tag{A.9}
\]

The discriminant of (A.9) therefore satisfies
A SECOND MOMENT BOUNDS FOR FBM

\[
4(\mu - Kb^{2H})^2 - 4(\lambda - Ka^{2H} - Kb^{2H} - Kc^{2H})(\rho - Kb^{2H}) \leq 0
\]

which gives

\[
\lambda \rho - \mu^2 \geq K \left( (a^{2H} + b^{2H} + c^{2H})\rho + b^{2H} \lambda - 2\mu b^{2H} \right) - K^2 \left( b^{2H} (a^{2H} + b^{2H} + c^{2H}) - b^{4H} \right) .
\]

By the Cauchy-Schwarz inequality,

\[
\mu \leq \sqrt{\lambda \rho} \leq \frac{\lambda + \rho}{2}.
\]

Using this, together with \(\lambda = (a + b + c)^{2H}\) and \(\rho = b^{2H}\) allows us to reduce (A.10) to

\[
\lambda \rho - \mu^2 \geq (K - K^2)b^{2H} (a^{2H} + c^{2H}).
\]

The result follows by replacing \((K - K^2)\) with \(K\).

\[\square\]

Proof of Lemma 2.2: Recall that we must show

\[
\int_{\mathbb{P}} \frac{\mu(s-r)^{2H-1}(s'-r')^{2H-1}}{(\lambda \rho - \mu^2)^{3/2}} \, dr \, dr' \, ds \, ds' < \infty,
\]

for \(0 < H < 2/3\). We will split the range of integration into the three regions described in in Lemma A.1 with \(a, b, c\) defined accordingly on each region.

**Case 1:** \(r < r' < s < s'\). Using \(s - r = a + b, s' - r' = b + c\), and (A.2) we see that the contribution of this region to (A.11) can be bounded by

\[
K \int_{[0,1]^3} \frac{\mu(a + b)^{2H-1}(b + c)^{2H-1}}{(a + b)^{2H}c^{2H} + a^{2H}(b + c)^{2H/3}} \, da \, db \, dc \leq K \int_{[0,1]^3} \frac{\mu(a + b)^{2H-1}(b + c)^{2H-1}}{(a + b)^{3H/2}c^{3H/2}a^{3H/2}(b + c)^{3H/2}} \, da \, db \, dc.
\]

Using an idea which appears in [Hu01], we write
A SECOND MOMENT BOUNDS FOR FBM

\[ 2\mu = (a + b + c)^{2H} + b^{2H} - a^{2H} - c^{2H} \]

\[ = 2H(b + c) \int_0^1 (a + (b + c)u)^{2H-1} du + b^{2H} - c^{2H}. \]  \hspace{1cm} (A.13)

We can bound the integral in (A.13) by replacing the integrand by its maximal value. This gives a bound of \( |\mu| \leq K((b + c)a^{2H-1} + b^{2H} + c^{2H}) \) for \( H < 1/2 \), and a bound of \( |\mu| \leq K((b + c) + b^{2H} + c^{2H}) \) when \( H \geq 1/2 \).

If \( H < 1/2 \), we bound (A.12) by

\[ K \int_{[0,1]^3} \frac{(b + c)a^{2H-1} + b^{2H} + c^{2H}}{(a + b)^{1-H/2}c^{3H/2}a^{3H/2}(b + c)^{1-H/2}} da db dc. \]  \hspace{1cm} (A.14)

We need only show that each of these three terms is integrable, and we may obtain bounds by replacing nonnegative powers of \((a + b)\) in the denominator by powers of either \(a\) or \(b\), and likewise for \((b + c)\). When \( H < 1/2 \) we will also use \((a + b)^{1-H/2} \geq a^{1-2H}b^{3H/2}\). Integrability follows from:

\[
\begin{align*}
\frac{(b + c)a^{2H-1}}{(a + b)^{1-H/2}c^{3H/2}a^{3H/2}(b + c)^{1-H/2}} & \leq \frac{K}{(a + b)^{1-H/2}c^{3H/2}a^{1-H/2}}, \\
\frac{b^{2H}}{(a + b)^{1-H/2}c^{3H/2}a^{3H/2}(b + c)^{1-H/2}} & \leq \frac{K}{b^{1-H/2}c^{1-H}a^{3H/2}}, \\
\frac{c^{2H}}{(a + b)^{1-H/2}c^{3H/2}a^{3H/2}(b + c)^{1-H/2}} & \leq \frac{K}{b^{1-H/2}c^{1-H}a^{3H/2}}.
\end{align*}
\]  \hspace{1cm} (A.15)

In each case we obtain an integrable function of \(a, b, c\), i.e. the exponent of each variable in the denominator is less than one.

For \( 1/2 \leq H < 2/3 \), we have

\[
\begin{align*}
\frac{(b + c)}{(a + b)^{1-H/2}c^{3H/2}a^{3H/2}(b + c)^{1-H/2}} & \leq \frac{K}{(a + b)^{1-H/2}c^{3H/2}a^{3H/2}/2}, \\
\frac{b^{2H}}{(a + b)^{1-H/2}c^{3H/2}a^{3H/2}(b + c)^{1-H/2}} & \leq \frac{K}{b^{2-3H}c^{3H/2}a^{3H/2}/2},
\end{align*}
\]

while the third term follows exactly as in the case \( H < 1/2 \) (note the second inequality in (A.15) used a bound which is not valid when \( H > 1/2 \)). This completes Case 1.

**Case 2:** \( r < r' < s' < s \). Following [Hu01] we can write
A SECOND MOMENT BOUNDS FOR FBM

\[ 2\mu = (a + b)^{2H} - a^{2H} + (b + c)^{2H} - c^{2H} \]
\[ = 2Hb \int_0^1 ((a + bu)^{2H-1} + (c + bu)^{2H-1}) \, du. \]  

(A.16)

Replacing the integrand with its maximum over the region gives us a bound of \( \mu \leq Kb \) for \( H \geq 1/2 \) and \( \mu \leq Kb(a^{2H-1} + c^{2H-1}) \) for \( H < 1/2 \).

First consider \( H < 1/2 \). In this case, \( s - r = a + b + c \) and \( s' - r' = b \). Also note that

\[ (a + c)^{2H} \leq (2 \max(a, c))^{2H} = 2^{2H} \max(a, c)^{2H} \leq 2^{2H}(a^{2H} + c^{2H}), \]

thus \((a + c)^{2H}\) and \((a^{2H} + c^{2H})\) are equivalent up to a constant. We bound the contribution of this region to (A.11), using (A.5), by

\[
K \int_{[0,t]^3} \frac{b(a^{2H-1} + c^{2H-1})(a + b + c)^{2H-1}b^{2H-1}}{b^H(a + c)^{3H}} \, da \, db \, dc \\
\leq K \int_{[0,t]^3} \frac{a^{2H-1}b^{2H-1}}{b^H(a + c)^{3H}} \, da \, db \, dc \\
\leq K \int_{[0,t]^2} \frac{a^{2H-1}}{(a + c)^{3H}} \, da \, dc \\
\leq K \int_{[0,t]} a^{2H-1}(1 + \frac{1}{a^{3H-1}}) \, da < \infty. 
\]  

(A.17)

For \( 1/2 \leq H < 2/3 \), we have

\[
K \int_{[0,t]^3} \frac{b(a + b + c)^{2H-1}b^{2H-1}}{b^H(a + c)^{3H}} \, da \, db \, dc \\
= K \int_{[0,t]^3} \frac{(a + b + c)^{2H-1}}{b^H(a + c)^{3H}} \, da \, db \, dc \\
\leq K \int_{[0,t]^3} \frac{da \, db \, dc}{b^H(a + c)^{3H}} < \infty. 
\]  

(A.18)

This completes Case 2.

**Case 3:** \( r < s < r' < s' \). Following [Hu01] we write
\[ 2\mu = (a + b + c)^{2H} + b^{2H} - (a + b)^{2H} - (b + c)^{2H} \]
\[ = 2H(2H - 1)ac \int_0^1 \int_0^1 (b + vc + ua)^{2H-2} du dv. \] (A.19)

By Young’s inequality, applied twice, for \( \alpha, \beta > 0 \) with \( \alpha + \beta = 1 \) we have

\[ (b + vc + ua) \geq Kb^\alpha(vc + ua)^\beta \geq Kb^\alpha(vc)^{\beta/2}(ua)^{\beta/2}. \] (A.20)

This combines with (A.19) to give

\[ |\mu| \leq K(ac)^{\beta(H-1)+1}H^{2\alpha(H-1)}. \] (A.21)

Using \( s - r = a, s' - r' = c, \) (A.21), and (A.4) shows that the contribution of this region to (A.11) is bounded by

\[ K \int_{[0,t]^3} \frac{(ac)^{\beta(H-1)+1}H^{2\alpha(H-1)}a^{2H-1}c^{2H-1}}{a^{3H}c^{3H}} da db dc \]
\[ = K \int_{[0,t]^3} \frac{da db dc}{b^{2\alpha(1-H)}(ac)^{\beta+H(1-\beta)}}. \] (A.22)

Note that \( 1 > 2H(1 - H) \) so that \( \frac{1}{2(1-H)} > H \). We may therefore choose \( \alpha \) such that \( H < \alpha < \frac{1}{2(1-H)} \), implying \( \beta < 1 - H \). The exponents in the final expression in (A.22) can therefore be bounded by \( 2\alpha(1 - H) < 2(\frac{1}{2(1-H)})(1 - H) = 1 \) and \( \beta + H(1 - \beta) < (1 - H) + H = 1 \). We conclude that (A.22) is finite, which completes the final case for the proof of Lemma 2.2. \( \square \)

References

[BHØZ08] F. Biagini, Y. Hu, B. Øksendal, and T. Zhang. Stochastic calculus for fractional Brownian motion and applications. Springer Verlag, 2008.

[BØSW04] F. Biagini, B. Øksendal, A. Sulem, and N. Wallner. An introduction to white–noise theory and Malliavin calculus for fractional Brownian motion. Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences, 460(2041):347, 2004.

[CN05] P. Cheridito and D. Nualart. Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter \( H \in (0, 1/2) \). In Annales de l’IHP Probabilités et statistiques, volume 41, pages 1049–1081. Elsevier, 2005.
[EVDH03] R.J. Elliott and J. Van Der Hoek. A general fractional white noise theory and applications to finance. *Mathematical Finance*, 13(2):301–330, 2003.

[HKPS93] T. Hida, H.H. Kuo, J. Potthoff, and L. Streit. *White noise: An infinite dimensional calculus*. Mathematics and its Applications (Dordrecht), 1993.

[HN05] Y. Hu and D. Nualart. Renormalized self-intersection local time for fractional Brownian motion. *Annals of probability*, 33(3):948–983, 2005.

[HN09] Y. Hu and D. Nualart. Stochastic integral representation of the $L^2$ modulus of Brownian local time and a central limit theorem. *Electronic Communications in Probability*, 14:529–539, 2009.

[HN10] Y. Hu and D. Nualart. Central limit theorem for the third moment in space of the Brownian local time increments. *Electronic Communications in Probability*, 15:396–410, 2010.

[HNS12] Y. Hu, D. Nualart, and J. Song. The 4/3-variation of the derivative of the self-intersection Brownian local time and related processes. *Arxiv preprint arXiv:1203.1368*, 2012.

[HØ03] Y. Hu and B. Øksendal. Fractional white noise calculus and applications to finance. *Infinite Dimensional Analysis Quantum Probability and Related Topics*, 6:1–32, 2003.

[HOUZ96] H. Holden, B. Oksendal, J. Uboe, and T. Zhang. *Stochastic Partial Differential Equations: A Modeling, White Noise Approach*. Springer Verlag, 1996.

[HP57] E. Hille and R.S. Phillips. *Functional analysis and semi-groups*, volume 31. American Mathematical Society, 1957.

[Hu01] Y. Hu. Self-intersection local time of fractional Brownian motions-via chaos expansion. *Journal of Mathematics-Kyoto University*, 41(2):233–250, 2001.

[Itô51] K. Itô. Multiple wiener integral. *Journal of the Mathematical Society of Japan*, 3(1):157–169, 1951.

[LV11] J. Lebovits and J.L. Vehel. Stochastic calculus with respect to multifractional brownian motion. *Arxiv preprint arXiv:1103.5291*, 2011.

[Mar08a] G. Markowsky. Proof of a Tanaka-like formula stated by J. Rosen in Séminaire XXXVIII. *Séminaire de Probabilités XLI*, pages 199–202, 2008.

[Mar08b] G. Markowsky. Renormalization and convergence in law for the derivative of intersection local time in $\mathbb{R}^2$. *Stochastic Processes and their Applications*, 118(9):1552–1585, 2008.
REFERENCES

[Mar12] G. Markowsky. The derivative of the intersection local time of Brownian motion through Wiener chaos. *Séminaire de Probabilités XLIV*, pages 141–148, 2012.

[Mis08] Y. Mishura. *Stochastic calculus for fractional Brownian motion and related processes*. Springer Verlag, 2008.

[Nua95] D. Nualart. *The Malliavin calculus and related topics*. Springer, 1995.

[NV92] D. Nualart and J. Vives. Chaos expansions and local times. *Publicacions Matematiques*, 36(2):827–836, 1992.

[Ros87] J. Rosen. The intersection local time of fractional Brownian motion in the plane. *Journal of Multivariate Analysis*, 23(1):37–46, 1987.

[Ros05] J. Rosen. Derivatives of self-intersection local times. *Séminaire de Probabilités XXXVIII*, pages 263–281, 2005.

[RW90] L.C.G. Rogers and J.B. Walsh. $A(t, B_t)$ is not a semimartingale. In *Seminar on Stochastic Processes*, pages 275–283, 1990.

[RW91a] L.C.G. Rogers and J.B. Walsh. Local time and stochastic area integrals. *The Annals of Probability*, 19(2):457–482, 1991.

[RW91b] L.C.G. Rogers and J.B. Walsh. The intrinsic local time sheet of Brownian motion. *Probability Theory and Related Fields*, 88(3):363–379, 1991.

[SKM87] S.G. Samko, A.A. Kilbas, and O.I. Marichev. *Fractional integrals and derivatives: theory and applications*. London: Gordon and Breach, 1987.

[Wal83] JB Walsh. Stochastic integration with respect to local time, 1983.

[YLY09] L. Yan, J. Liu, and X. Yang. Integration with respect to fractional local time with Hurst index $1/2 < H < 1$. *Potential Analysis*, 30(2):115–138, 2009.

[YYL08] L. Yan, X. Yang, and Y. Lu. p-variation of an integral functional driven by fractional Brownian motion. *Statistics & Probability Letters*, 78(9):1148–1157, 2008.

[Zha92] T. Zhang. Characterizations of the white noise test functionals and Hida distributions. *Stochastics: An International Journal of Probability and Stochastic Processes*, 41(1-2):71–87, 1992.