Generalized Busemann inequality

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Abstract

We present a result which simultaneously extends the Busemann intersection inequality to the case of non-integer moments of the corresponding volumes and the Busemann random simplex inequality to the case of simplices of smaller dimensions.

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1 Introduction

1.1 Notation

We start with basic notation of integral geometry following [15]. Let $d \geq 1$ be an integer. A compact convex subset in $\mathbb{R}^d$ with non-empty interior is called a convex body. The unit Euclidean ball in $\mathbb{R}^k$ is denoted by $B^k$. By $|\cdot|$ we denote the $d$-dimensional volume. Given $k \leq d$, slightly abusing notation, considering the sets intersected with $k$-dimensional affine subspaces or the convex hulls of $k+1$ points, we denote the $k$-dimensional volume by $|\cdot|$ as well.

For $p > 0$ denote

$$
\kappa_p := \frac{\pi^{p/2}}{\Gamma\left(\frac{p}{2} + 1\right)} \quad \text{and} \quad \omega_p = p\kappa_p.
$$

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Note that for an integer \( k \) one has \( \kappa_k = |B^k| \) and \( \omega_k = |\partial B^k| \). We will also need the following numbers
\[
b_{q,k} := \frac{\omega_{q-k+1} \cdots \omega_q}{\omega_1 \cdots \omega_k}.
\] (2)

For \( k \in \{0, \ldots, d\} \), the linear (resp., affine) Grassmannian of \( k \)-dimensional linear (resp., affine) subspaces of \( \mathbb{R}^d \) is denoted by \( G_{d,k} \) (resp., \( A_{d,k} \)) and is equipped with a unique rotation invariant (resp., rigid motion invariant) Haar measure \( \nu_{d,k} \) (resp., \( \mu_{d,k} \)), normalized by
\[
\nu_{d,k}(G_{d,k}) = 1 \quad \text{and} \quad \mu_{d,k} \left( \{ E \in A_{d,k} : E \cap \mathbb{B}^d \neq \emptyset \} \right) = \kappa_{d-k},
\]
respectively. For \( L \in G_{d,k} \) (resp., \( E \in A_{d,k} \)) we denote by \( \lambda_L \) (resp., \( \lambda_E \)) the \( k \)-dimensional Lebesgue measures on \( L \) (resp., \( E \)).

### 1.2 Busemann intersection inequality

The seminal Busemann intersection inequality [1] states that for any convex body \( K \subset \mathbb{R}^d \) and for \( k = d - 1 \) one has
\[
\int_{G_{d,k}} |K \cap L|^d \nu_{d,k}(dL) \leq \frac{\kappa_d}{\kappa_k^d} |K|^k.
\] (3)

This inequality was later generalized in [3, 10] for all \( k = 1, \ldots, d - 1 \). Using the polar coordinates, it is easy to see that for \( k = 1 \) the inequality turns to the equality. Moreover, the equality in the case \( k = 1 \) can be generalized to other moments as follows:
\[
\int_{G_{d,1}} |K \cap L|^{d+p} \nu_{d,k}(dL) = \frac{(d+p)2^{d+p}}{d \kappa_d} \int_{K^k} |x|^p \, dx, \quad p \geq -d + k + 1.
\] (4)

If \( k \geq 2 \) then the equality holds if and only if \( K \) is an ellipsoid centered at the origin, and in this case (3) turns to the classical Fustenberg–Tskoni formula [7].

The affine counterpart of (3) was obtained by Schneider [11], namely
\[
\int_{A_{d,k}} |K \cap E|^{d+1} \mu_{d,k}(dE) \leq \frac{\kappa_{d+1} \kappa_{d(k+1)}}{\kappa_{d+1} \kappa_{k+1}} |K|^{k+1}.
\] (5)
As above, for $k = 1$ the inequality turns to equality, although it is not as trivial as in the linear case, see [5] for $d = 2$ and [12] for any $d$. As in the linear case, this equality can be generalized to other moments. It was done independently in [4, Eq. (21)] and [13, Eq. (34)]:

$$
\int_{A_{d,1}} |K \cap E|^{p+d+1} \mu_{d,1}(dE) = \frac{(d + p)(d + p + 1)}{2 d \kappa_d} \int_{K^2} |x_0 - x_1|^p \, dx_0 \, dx_1.
$$

If $k \geq 2$, then equality holds if and only if $K$ is an ellipsoid.

Gardner [8] generalized (3) and (5) to bounded Borel sets and characterized the equality cases. Recently Dann, Paouris, and Pivovarov [6] extended (3), (5) to bounded integrable functions.

Given that (4) and (5) for $k = 1$ (which are the equalities in this case) can be generalized to other moments, the following question arises naturally.

**Question I. Is it possible to generalize (3) and (5) to the case of non-integer moments of $|K \cap L|$?**

### 1.3 Busemann random simplex inequality

Another group of inequalities deals with the volume of the random simplex in a body. The classical Busemann random simplex inequality states that

$$
|K|^{d+1} \leq (d + 1)! \frac{k_{d+1}^{d+1}}{2 k_{d+1}^{d+1}} \int_{K^d} |\text{conv}(0, x_1, \ldots, x_d)| \, dx_1 \ldots dx_d.
$$

This inequality can be generalized (see, e.g., [15, Theorem 8.6.1.]) as follows: for every $p \geq 1$,

$$
|K|^{p+d} \leq (d!)^p \frac{k_{d+1}^{p+d}}{2 k_{d+1}^{p+d}} \frac{b_{d+p,d}}{b_{d+p,d}} \int_{K^d} |\text{conv}(0, x_1, \ldots, x_d)|^p \, dx_1 \ldots dx_d. \quad (6)
$$

The equality holds if and only if $K$ is a centered ellipsoid.

The affine counterpart of (6) is known as the Blaschke-Grömer inequality [11]: for every $p \geq 1$,

$$
|K|^{p+d+1} \leq (d!)^p \frac{k_{d+1}^{p+d+1}}{k_{d+1}^{d+1}} \frac{K(d+1)(d+p)}{K(d+p+1)} \int_{K^{d+1}} |\text{conv}(x_0, \ldots, x_d)|^p \, dx_0 \ldots dx_d. \quad (7)
$$
As before, the equality holds if and only if $K$ is an ellipsoid.

The following question arises naturally.

**Question II.** *Is it possible to generalize (6) and (7) to the case of random simplices of all dimensions $k = 1, \ldots, d$?*

The aim of this note is to positively answer Questions I, II presenting inequalities that generalizes both the Busemann intersection inequality and the Busemann random simplex inequality.

## 2 Main results

Our first theorem generalizes (3) and (6).

**Theorem 2.1.** For any convex body $K \subset \mathbb{R}^d$, $k \in \{0, 1, \ldots, d\}$, and any real number $p \geq -d + k + 1$,

$$\int_{G_{d,k}} |K \cap L|^{p+d} \nu_{d,k}(dL) \leq (k!)^p \frac{k^{d+p}}{k^{d+p}_d} \frac{b_{d+p,k}}{b_{d,k}} \int_{K^k} |\text{conv}(0, x_1, \ldots, x_k)|^p \, dx_1 \ldots dx_k.$$

(8)

For $k \geq 2$ the equality holds if and only if $K$ is a non-degenerate ellipsoid centered at the origin.

**Remarks.**

1. Applying (8) with $p = 0$ we obtain (3), while applying it with $k = d$ we obtain (6).

2. It was shown in [9, Theorem 1.6] that if $K$ is a non-degenerate ellipsoid centered at the origin, then one has the equality in (8).

3. In the probabilistic language it may be formulated as

$$\mathbb{E} |K \cap \xi|^{p+d} \leq (k!)^p \frac{k^{d+p}}{k^{d+p}_d} \frac{b_{d+p,k}}{b_{d,k}} |K|^k \mathbb{E} |\text{conv}(0, X_1, \ldots, X_k)|^p$$

where $X_1, \ldots, X_k$ are i.i.d. copies of a random variable uniformly distributed in $K$ and $\xi$ is uniformly distributed in $G_{d,k}$.

Our second theorem generalizes (5) and (7).
Theorem 2.2. For any convex body $K \subset \mathbb{R}^d$, $k \in \{0, 1, \ldots, d\}$, and any real number $p \geq -d + k + 1$,
\[ \int_{A_{d,k}} |K \cap E|^{p+d+1} \mu_{d,k}(dE) \leq C(k, p, d) \int_{K^{k+1}} |\text{conv}(x_0, \ldots, x_k)|^p \, dx_0 \ldots dx_k, \]  
(9)
where
\[ C(k, p, d) = (k!)^p \frac{k_{k+1}^{p+d+1} \kappa_{d+p}^{(k+1)(d+p)} b_{d+p,k}}{\kappa_{d+k}^{k+1} \kappa_{k(d+p)+k} b_{d,k}^p}. \]

For $k \geq 2$ the equality holds if and only if $K$ is a non-degenerate ellipsoid.

Remarks.

1. Applying (9) with $p = 0$ we obtain (5), while applying it with $k = d$ we obtain (7).

2. It was shown in [9, Theorem 1.4] that if $K$ is a non-degenerate ellipsoid, then one has the equality in (9).

3. In probabilistic language it may be formulated as
\[ \mathbb{E} \left| K \cap \eta \right|^{p+d+1} \leq C'(k, p, d) \frac{|K|^{k+1}}{V_{d-k}(K)} \mathbb{E} \left| \text{conv}(X_0, X_1, \ldots, X_k) \right|^p, \]
where
\[ C'(k, p, d) = \frac{d!}{(d-k)!} \frac{(k!)^{p+1}}{(d-k)!(d-k)!} \frac{k_d^{p+d} \kappa_{d-k}^{k+1} \kappa_{k(d+p)+k} b_{d+p,k}}{\kappa_{k}^{k+1} \kappa_{d-k}^{k+1} \kappa_{k(d+p)+k} b_{d,k}}. \]

$X_0, X_1, \ldots, X_k$ are i.i.d. copies of a random variable uniformly distributed in $K$, $\eta$ is uniformly distributed among all affine $k$-planes intersected $K$, and $V_{d-k}$ is the $(d-k)$-th intrinsic volume of $K$ defined by the Crofton formula [15, Theorem 5.1.1] as the normalized measure of all affine $k$-planes intersected $K$:
\[ V_{d-k} := \binom{d}{k} \frac{k_d}{\kappa_k \kappa_{d-k}} \mu_{d,k}(\{E \in A_{d,k} : E \cap K \neq \emptyset\}). \]
3 Proofs

3.1 Blaschke–Petkantschin formula

Recall that $b_{d,k}$ is defined by (2). Given points $x_0, x_1, \ldots, x_k \in \mathbb{R}^d$ we denote

$$V_k = V(x_0, x_1, \ldots, x_k) := |\text{conv}(x_0, x_1, \ldots, x_k)|$$

and

$$V_{0,k} = V(x_1, \ldots, x_k) := |\text{conv}(0, x_1, \ldots, x_k)|.$$

In our further calculations we will need to integrate some non-negative measurable function $h$ of $k$-tuples of points in $\mathbb{R}^d$. To this end, we first integrate over the $k$-tuples of points in a fixed $k$-dimensional linear subspace $L$ with respect to the product measure $\lambda_L^k$ and then we integrate over $G_{d,k}$ with respect to $\nu_{d,k}$. The corresponding transformation formula is known as the linear Blaschke–Petkantschin formula (see [15, Theorem 7.2.1]):

$$\int_{(\mathbb{R}^d)^k} h \, dx_1 \ldots dx_k = (k!)^{d-k} b_{d,k} \int_{G_{d,k}} \int_{L^k} h V_{0,k}^{d-k} \lambda_L(dx_1) \ldots \lambda_L(dx_k) \nu_{d,k}(dL),$$

where $h = h(x_1, \ldots, x_k)$. The following is an affine counterpart of (10),

$$\int_{(\mathbb{R}^d)^{k+1}} h \, dx_0 \ldots dx_k = (k!)^{d-k} b_{d,k} \int_{A_{d,k}} \int_{E^{k+1}} h V_k^{d-k} \lambda_E(dx_0) \ldots \lambda_E(dx_k) \mu_{d,k}(dE),$$

where $h = h(x_0, x_1, \ldots, x_k)$ (see [15, Theorem 7.2.7]).

3.2 Proof of Theorem 2.1

Let

$$J := \int_{K^k} V_{0,k}^p \, dx_1 \ldots dx_k = \int_{(\mathbb{R}^d)^k} V_{0,k}^p \prod_{i=1}^k 1_K(x_i) \, dx_1 \ldots dx_k.$$

Applying the linear Blaschke–Petkantschin formula (10) with the function

$$h(x_1, \ldots, x_k) := V_{0,k}^p \prod_{i=1}^k 1_K(x_i),$$
we observe

\[ J = (k!)^{d-k} b_{d,k} \int_{G_{d,k}} \int_{L^k} V_{0,k}^{p+d-k} \prod_{i=1}^{k} \mathbb{1}_{K}(x_i) \lambda_L(dx_1) \ldots \lambda_L(dx_k) \nu_{d,k}(dL) \]

\[ = (k!)^{d-k} b_{d,k} \int_{G_{d,k} \cap (K \cap L)^k} V_{0,k}^{p+d-k} \lambda_L(dx_1) \ldots \lambda_L(dx_k) \nu_{d,k}(dL). \quad (12) \]

Fix \( L \in G_{d,k} \). Applying (11) with \( p+d-k \) and \( k \) instead of \( p \) and \( d \), we obtain

\[ (k!)^{p+d-k} \frac{k^{d+p}}{k^{d+p}} b_{d+p,k} \int_{(K \cap L)^k} V_{0,k}^{p+d-k} \lambda_L(dx_1) \ldots \lambda_L(dx_k) \geq |K \cap L|^{p+d}, \quad (13) \]

which together with (12) implies (8).

Finally we consider the equality case. As was mentioned above, the equality holds for ellipsoids, see [9, Theorem 1.6]. Conversely, suppose that (8) turns to equality. Then it follows from (12) that (13) turns to equality for almost all \( L \in G_{d,k} \) which, in fact, means that it is true for all \( L \in G_{d,k} \). Indeed, if for some \( L \in G_{d,k} \) we had a strict inequality in (13), then the same would be true for some neighborhood of \( L \) which would contradict to the fact that (13) turns to equality for almost all \( L \in G_{d,k} \). Thus, according to the equality case in (11), \( K \cap L \) is a centered ellipsoid for all \( L \in G_{d,k} \). Now it remains to apply the following lemma from [2 (16.12)]: if for any \( E \in A_{d,k} \) passing through some fixed point from the interior of \( K \) the intersection \( K \cap E \) happens to be a \( k \)-dimensional ellipsoid, then \( K \) is an ellipsoid itself. \( \square \)

### 3.3 Proof of Theorem 2.2

The proof is similar to the previous one. Let

\[ J := \int_{K^{k+1}} V_k^p \, dx_0 \ldots dx_k = \int_{(\mathbb{R}^d)^{k+1}} V_k^p \prod_{i=0}^{k} \mathbb{1}_E(x_i) \, dx_0 \ldots dx_k. \]

Applying the affine Blaschke–Petkantschin formula (11) with the function

\[ h(x_0, \ldots, x_k) := |\text{conv}(x_0, \ldots, x_k)|^p \prod_{i=0}^{k} \mathbb{1}_E(x_i), \]
we observe

\[ J = (k!)^{d-k} b_{d,k} \int_{A_{d,k}} \int_{E^{k+1}} V_{k}^{p+d-k} \prod_{i=0}^{k} 1_{K}(x_i) \lambda_E(dx_0) \ldots \lambda_E(dx_k) \mu_{d,k}(dE) \]

\[ = (k!)^{d-k} b_{d,k} \int_{A_{d,k}} \int_{(K \cap E)^{k+1}} V_{k}^{p+d-k} \lambda_E(dx_0) \ldots \lambda_E(dx_k) \mu_{d,k}(dE). \tag{14} \]

Fix \( E \in A_{d,k} \). Applying (7) with \( p + d - k \) and \( k \) instead of \( p \) and \( d \), we obtain

\[ (k!)^{d-k+p} b_{d+p,k} \frac{k^{p+1}}{k^{k+1}} \frac{k^{(k+1)(d+p)}}{k^{d+p+1}} \int_{(K \cap E)^{k+1}} V_{k}^{p+d-k} \lambda_E(dx_0) \ldots \lambda_E(dx_k) \]

\[ \geq |K \cap E|^{d+p+1} \]

which together with (14) implies (9).

The equality case is treated the same way as in the linear case. \( \square \)

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