A unified hoop conjecture for black holes and horizonless compact stars

Yan Peng$^{1,2*}$

$^1$ School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, China and
$^2$ Center for Gravitation and Cosmology, College of Physical Science and Technology, Yangzhou University, Yangzhou 225009, China

Abstract

We propose a unified version of hoop conjecture valid for various black holes and horizonless compact stars. This conjecture is expressed by the mass to circumference ratio $4\pi M_{\text{in}}/C \leq 1$, where $C$ is the circumference of the smallest ring that can engulf the object in all azimuthal directions and $M_{\text{in}}$ is the mass within the engulfing sphere.

PACS numbers: 11.25.Tq, 04.70.Bw, 74.20.-z

* yanpengphy@163.com
I. INTRODUCTION

The famous hoop conjecture introduced almost five decades ago asserts that the existence of black hole horizons is characterized by the mass and circumference relation $4\pi M/C \geq 1$ \cite{1, 2}. Here $C$ is the circumference of the smallest ring that can engulf the black hole in all azimuthal directions and $M$ is usually interpreted as the asymptotically measured total ADM mass of black holes \cite{3}-\cite{29}.

For horizonless compact stars, the curved spacetimes should be characterized by the relation $4\pi M/C < 1$ \cite{1, 2}. However, if $M$ is still interpreted as the total ADM mass, this relation $4\pi M/C < 1$ is violated in the background of horizonless charged compact objects \cite{30, 31}. In fact, Thorne has not provided an explicit definition for the mass term $M$ in the mass to circumference ratio \cite{1}. And it has been proved that the relation $4\pi M/C < 1$ holds in horizonless spacetimes when $M$ is interpreted as the mass contained within the engulfing sphere (not as the total ADM mass) \cite{32, 33}.

It is natural to assume that a unified hoop conjecture may be $4\pi M/C \geq 1$ for black holes and $4\pi M/C < 1$ for horizonless compact stars, where $M$ is the total ADM mass or the mass contained in the engulfing sphere \cite{34}. In the background of horizonless stars, the calculations in \cite{32} imply that the term $M$ should be the mass contained in the sphere and cannot be the total ADM mass. However, in the black hole spacetimes, Hod found that the hoop conjecture holds if $M$ is the ADM mass and cannot be the mass contained in the sphere \cite{34}. So it seems that there is no unified hoop conjecture valid for both black holes and compact stars.

In this paper, we propose another version of unified hoop conjecture, which holds for Schwarzschild black holes, neutral Kerr black holes, RN black holes, Kerr-Newman black holes and horizonless charged compact stars. It may be a general property for both black holes and horizonless compact stars.

II. TEST THE UNIFIED HOOP CONJECTURE

In the background of horizonless charged compact stars, it has recently been proved that the relation $4\pi M/C < 1$ is violated when $M$ is interpreted as the total ADM mass and the relation holds if $M = M_{\text{in}}$, where $M_{\text{in}}$ is the mass contained within the engulfing sphere \cite{30, 33}. In addition, the numerical data in \cite{34} implies that $4\pi M_{\text{in}}/C \leq 1$ may be a general property in black hole spacetimes. With these facts, we propose
a unified version of hoop conjecture

\[ 4\pi M_{in}/C \leq 1, \tag{1} \]

where \( C \) is the circumference of the smallest ring that can engulf the object in all azimuthal directions and \( M_{in} \) is the mass within the engulfing sphere. The relation (1) is natural according to the idea that the ring of the engulfing sphere is related to the mass within the sphere (not the mass in the total spacetime).

In the following, we give examples supporting the bound (1) following studies in [30–34]. A Kerr-Newman black hole spacetime is described by the curved line element [34]

\[
ds^2 = -\frac{\Delta-a^2\sin^2 \theta}{\rho^4} dt^2 + \frac{\rho^2}{\Delta} dr^2 - \frac{2a \sin^2 \theta (2Mr-Q^2)}{\rho^4} dt d\phi + \rho^2 d\theta^2 + \frac{(r^2+a^2)^2-a^2\Delta \sin^2 \theta}{\rho^4} \sin^2 \theta d\phi^2, \tag{2} \]

where \( M \) is the ADM mass, \( Q \) is the electric charge, \( J = Ma \) is the angular momentum, \( \Delta = r^2 - 2Mr + Q^2 + a^2 \) and \( \rho^2 = r^2 + a^2 \cos^2 \theta \). Black hole horizons are expressed as

\[ r_\pm = M \pm (M^2 - Q^2 + a^2)^{1/2}, \tag{3} \]

which are determined by the zeros of the metric function \( \Delta(r) = 0 \).

Setting \( dt = dr = d\theta = 0, \theta = \frac{\pi}{2} \), we get the relation

\[ ds = \frac{r_+^2 + a^2}{r_+} d\phi. \tag{4} \]

Also taking \( \Delta \phi = 2\pi \), we obtain the equatorial circumference

\[ C = 2\pi \frac{r_+^2 + a^2}{r_+}. \tag{5} \]

For Schwarzschild black holes, there is \( Q = 0 \) and \( a = 0 \). The mass to circumference ratio is

\[ \frac{4\pi M_{in}}{C} = \frac{4\pi M}{C} = \frac{4\pi M}{2\pi r_+} = \frac{4\pi M}{2\pi (2M)} = 1. \tag{6} \]

For RN black holes with \( Q \neq 0 \) and \( a = 0 \), the ratio satisfies

\[ \frac{4\pi M_{in}}{C} = \frac{4\pi (M - \frac{Q^2}{2r_+})}{2\pi r_+} = \frac{4Mr_+ - 2Q^2}{2r_+^2} = \frac{4M(M + \sqrt{M^2 - Q^2}) - 2Q^2}{2(M + \sqrt{M^2 - Q^2})^2} = 1. \tag{7} \]

The neutral Kerr black holes correspond to \( Q = 0 \) and \( a \neq 0 \). It yields the relation

\[ \frac{4\pi M_{in}}{C} = \frac{4\pi M}{2\pi \frac{r_+^2 + a^2}{r_+}} = \frac{2Mr_+}{r_+^2 + a^2} = \frac{2M(M + \sqrt{M^2 - a^2})}{(M + \sqrt{M^2 - a^2})^2 + a^2} = 1. \tag{8} \]

In the background of Kerr-Neuman black holes, both charge and rotating parameters are nonzero as \( Q \neq 0 \) and \( a \neq 0 \). The equatorial circumference is \( C = 2\pi \frac{r_+^2 + a^2}{r_+} \) and the mass contained within the black hole
horizon is \( M_{in} = M - \frac{Q^2}{4r_+} \left[ 1 + \frac{r_+^2 + a^2}{ar_+} \cdot \arctg\left( \frac{a}{r_+} \right) \right] \). In this case of Kerr-Neunman black holes, numerical data in [34] implies an upper bound on the ratio

\[
4\pi M_{in}/C \leq 1.
\]  

(9)

The horizonless charged compact star satisfies the inequality [32, 33]

\[
4\pi M_{in}/C < 1.
\]  

(10)

According to relations (6-10), we propose the unified hoop conjecture (1), which may be a general property for both black holes and horizonless compact stars.

### III. CONCLUSIONS

We proposed a unified hoop conjecture, which is valid for various black holes and horizonless compact stars. Our conjecture is that the mass and circumference relation \( 4\pi M_{in}/C \leq 1 \) holds, where \( C \) is the circumference of the smallest ring that can engulf the object in all azimuthal directions and \( M_{in} \) is the gravitating mass within the engulfing sphere. Our statement is in accordance with the idea that the ring of the engulfing sphere should be related to the mass within the sphere (not the mass in the total spacetime).

### Acknowledgments

This work was supported by the Shandong Provincial Natural Science Foundation of China under Grant No. ZR2018QA008. This work was also supported by a grant from Qufu Normal University of China under Grant No. xkjjc201906.

[1] K.S. Thorne, in Magic Without Magic: John Archibald Wheeler, ed. by J. Klauder (Freeman, San Francisco, 1972).
[2] C.W. Misner, K.S. Thorne, J.A. Wheeler, Gravitation (Freeman, San Francisco, 1973).
[3] I.H. Redmount, Phys. Rev. D 27, 699 (1983).
[4] A.M. Abrahams, K.R. Heiderich, S.L. Shapiro, S.A. Teukolsky, Phys. Rev. D 46, 2452 (1992).
[5] S. Hod, Phys. Lett. B 751, 241 (2015).
[6] P. Bizon, E. Malec, and N. ó Murchadha, Trapped surfaces in spherical stars, Phys. Rev. Lett. 61, 1147 (1988).
[7] P. Bizon, E. Malec, and N. ó Murchadha, Class. Quantum Grav. 6, 961 (1989).
[8] D. Eardley, Gravitational collapse of vacuum gravitational field configurations, J. Math. Phys. 36, 3004 (1995).
[9] J. Guven and N. ó Murchadha, Sufficient conditions for apparent horizons in spherically symmetric initial data, Phys. Rev. D 56, 7658 (1997).
[10] J. Guven and N. ó Murchadha, Necessary conditions for apparent horizons and singularities in spherically symmetric initial data, Phys. Rev. D 56, 7666 (1997).
[11] E. Malec, Event horizons and apparent horizons in spherically symmetric geometries, Phys. Rev. D 49, 6475 (1994).
[12] E. Malec and Ó Murchadha, The Jang equation, apparent horizons, and the Penrose inequality, Class. Quantum Grav. 21, 5777 (2004).
[13] T. Zannias, Phys. Rev. D 45, 2998 (1992).
[14] T. Zannias, Phys. Rev. D 47, 1448 (1993).
[15] E. Malec, Isoperimetric inequalities in the physics of black holes, Acta Phys. Pol. B 22, 829 (1991).
[16] M. Khuri, The Hoop Conjecture in Spherically Symmetric Spacetimes, Phys. Rev. D 80, 124025 (2009).
[17] H. Bray and M. Khuri, Asian J. Math. 15, 557 (2011).
[18] R. Schoen and S.-T. Yau, Commun. Math. Phys. 90, 575 (1983).
[19] Takeshi Chiba, Takashi Nakamura, Ken-ichi Nakao, Misao Sasaki, Hoop conjecture for apparent horizon formation, Class. Quant. Grav. 11(1994)431-441.
[20] Takeshi Chiba, Apparent horizon formation and hoop concept in nonaxisymmetric space, Phys. Rev. D 60(1999)044003.
[21] Ken-ichi Nakao, Kouji Nakamura, Takashi Mishima, Hoop conjecture and cosmic censorship in the brane world, Phys. Lett. B 564(2003)143-148.
[22] G.W. Gibbons, Birkhoff’s invariant and Thorne’s Hoop Conjecture, [arXiv:0903.1580, gr-qc].
[23] M. Cvetic, G.W. Gibbons, C.N. Pope, More about Birkhoff’s Invariant and Thorne’s Hoop Conjecture for Horizons, Class. Quant. Grav. 28(2011)195001.
[24] John D. Barrow, G. W. Gibbons, Maximum Tension: with and without a cosmological constant, Mon. Not. Roy. Astron. Soc. 446(2014)3874-3877.
[25] John D. Barrow, G.W. Gibbons, A maximum magnetic moment to angular momentum conjecture, Phys. Rev. D 95(2017)064040.
[26] Edward Malec, Naqing Xie, Brown-York mass and the hoop conjecture in nonspherical massive systems, Phys. Rev. D 91(2015)no.8,081501.
[27] Fabio Anzà, Goffredo Chirco, Fate of the Hoop Conjecture in Quantum Gravity, Phys. Rev. Lett. 119(2017)no.23, 231301.
[28] Shahar Hod, Bekenstein’s generalized second law of thermodynamics: The role of the hoop conjecture, Phys. Lett. B 751(2015)241-245.
[29] Shahar Hod, The gravitational two-body system: The role of the Thorne hoop conjecture, Eur. Phys. J. Plus 134(2019)no.3, 106.
[30] J.P. de Leön, Gen. Relativ. Grav. 19, 289 (1987).
[31] W.B. Bonnor, Phys. Lett. A 99, 424 (1983).
[32] Shahar Hod, On the status of the hoop conjecture in charged curved spacetimes, Eur. Phys. J. C (2018)78:1013.
[33] Yan Peng, Analytical studies on the hoop conjecture in charged curved spacetimes, Eur. Phys. J. C 79(2019)11,943.
[34] Shahar Hod, Further evidence for the non-existence of a unified hoop conjecture, Eur. Phys. J. C 80(2020)10,982.
[35] J.M. Aguirregabiria, A. Chamorro, K.S. Virbhadra, Gen. Relativ. Gravit. 28,1393(1996).