On the density of polynomials in some $L^2(M)$ spaces.

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1 Introduction.

In this paper we shall study the density of polynomials in some $L^2(M)$ spaces. Two choices of the measure $M$ and polynomials will be considered:

(A) a $\mathbb{C}^N_{N \times N}$-valued measure $M$ on $\mathcal{B}(\mathbb{R})$ and vector-valued polynomials:

$$p(x) = (p_0(x), p_1(x), \ldots, p_{N-1}(x)), \quad (1)$$

where $p_j(x)$ are complex polynomials, $0 \leq j \leq N - 1$; $N \in \mathbb{N}$;

(B) a scalar non-negative Borel measure $\sigma$ in a strip

$$\Pi = \{(x, \varphi) : x \in \mathbb{R}, \varphi \in [-\pi, \pi]\}, \quad (2)$$

and power-trigonometric polynomials:

$$p(x, \varphi) = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \alpha_{m,n} x^m e^{in\varphi}, \quad \alpha_{m,n} \in \mathbb{C}, \quad (3)$$

where all but finite number of coefficients $\alpha_{m,n}$ are zeros.

The case (A) is closely related to the matrix Hamburger moment problem which consists of finding a left-continuous non-decreasing matrix function $M(x) = (m_{k,l}(x))_{k,l=0}^{N-1}$ on $\mathbb{R}$, $M(-\infty) = 0$, such that

$$\int_{\mathbb{R}} x^n dM(x) = S_n, \quad n \in \mathbb{Z}_+, \quad (4)$$

where $\{S_n\}_{n=0}^{\infty}$ is a prescribed sequence of Hermitian $(N \times N)$ complex matrices, $N \in \mathbb{N}$. In the scalar case ($N = 1$) it is well known that polynomials are dense in $L^2(M)$ on the real line if and only if $M$ is a canonical solution of the corresponding moment problem [1].

In the case of an arbitrary $N$ and if the matrix Hamburger moment problem is completely indetermined, the density of polynomials is equivalent to the fact that $M$ is a canonical solution of the moment problem (4) (i.e. it corresponds to a constant unitary matrix in the Nevanlinna type parameterization for solutions of [1]) [2].
On the other hand, the case (B) is related to the Devinatz moment problem: to find a non-negative Borel measure \( \mu \) in a strip \( \Pi \) such that

\[
\int_{\Pi} x^m e^{inz} d\mu = s_{m,n}, \quad m \in \mathbb{Z}_+, n \in \mathbb{Z},
\]

where \( \{s_{m,n}\}_{m \in \mathbb{Z}_+, n \in \mathbb{Z}} \) is a prescribed sequence of complex numbers \([3]\).

In the both cases, we shall prove that polynomials are dense in \( L^2(M) \) if and only if \( M \) is a canonical solution of the corresponding moment problem, without any additional assumptions (definitions of the canonical solutions shall be given below). For this purpose, we derive a model for a finite set of commuting self-adjoint and unitary operators with a spectrum of a finite multiplicity (precise definitions shall be stated below). The latter is a generalization of the canonical model for a self-adjoint operator with a spectrum of a finite multiplicity \([4]\). Using known descriptions of canonical solutions, we shall obtain conditions for the density of polynomials in \( L^2(M) \).

**Notations.** As usual, we denote by \( \mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+ \) the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively; \( \mathbb{C}_+ := \{z \in \mathbb{C} : 1/z (z - \bar{z}) \geq 0\} \). By \( \mathbb{C}_{n \times n} \) we denote a set of all \((n \times n)\) matrices with complex elements; \( \mathbb{C}_n := \mathbb{C}_{1 \times n}, n \in \mathbb{N} \). By \( \mathbb{C}_{n \times n}^\geq \) we mean a set of all nonnegative Hermitian matrices from \( \mathbb{C}_{n \times n}, n \in \mathbb{N} \). By \( \mathbb{P} \) we denote a set of all complex polynomials. By \( \mathbb{P}^N \) we mean a set of vector-valued polynomials: \( p(z) = (p_0(z), p_1(z), \ldots, p_{N-1}(z)); p_j \in \mathbb{P}, 0 \leq j \leq N - 1; N \in \mathbb{N} \). For a subset \( S \) of the complex plane we denote by \( \mathcal{B}(S) \) the set of all Borel subsets of \( S \). Everywhere in this paper, all Hilbert spaces are assumed to be separable. By \( (\cdot, \cdot)_H \) and \( \|\cdot\|_H \) we denote the scalar product and the norm in a Hilbert space \( H \), respectively. The indices may be omitted in obvious cases. For a set \( M \) in \( H \), by \( M \) we mean the closure of \( M \) in the norm \( \|\cdot\|_H \). For \( \{x_k\}_{k \in S}, x_k \in H \), we write \( \text{Lin}\{x_k\}_{k \in S} \) for a set of linear combinations of vectors \( \{x_k\}_{k \in S} \) and \( \text{span}\{x_k\}_{k \in S} = \text{Lin}\{x_k\}_{k \in S} \). Here \( S \) is an arbitrary set of indices. The identity operator in \( H \) is denoted by \( E = E_H \). For an arbitrary linear operator \( A \) in \( H \), the operators \( A^*, \overline{A}, A^{-1} \) mean its adjoint operator, its closure and its inverse (if they exist). By \( D(A) \) and \( R(A) \) we mean the domain and the range of the operator \( A \). We denote by \( R_z(A) \) the resolvent function of \( A \), where \( z \) belongs to the resolvent set of \( A \). If \( A \) is bounded, then the norm of \( A \) is denoted by \( \|A\| \). If \( A \) is symmetric, we denote \( \Delta_A(z) := (A - zE_H)D(A), z \in \mathbb{C}; \) and \( N_\lambda = N_\lambda(A) = H \ominus \Delta_A(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R} \). By \( P_{H_1}^H = P_{H_1} \) we mean the operator of orthogonal projection in \( H \) on a subspace \( H_1 \) in \( H \). We denote \( D_{r,l} = \mathbb{R}^r \times [-\pi, \pi]^l = \{(x_1, x_2, \ldots, x_r, \varphi_1, \varphi_2, \ldots, \varphi_l), x_j \in \mathbb{R}, \varphi_k \in [-\pi, \pi), 1 \leq j \leq r, 1 \leq k \leq l\}, r, l \in \mathbb{Z}_+ \). Elements \( u \in D_{r,l} \) we briefly...
denote by \( u = (x, \varphi), x = (x_1, x_2, \ldots, x_r), \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_l) \). We mean 
\( D_{r,0} = \mathbb{R}^r; D_{0,l} = [-\pi, \pi]^l \).

Let \( M(\delta) = (m_{i,j}(\delta))_{i,j=0}^{N-1} \) be a \( \mathbb{C}_{N \times N} \)-valued measure on \( \mathfrak{B}(D_{r,l}) \), and \( \tau = \tau_{M}(\delta) := \sum_{k=0}^{N-1} m_{k,k}(\delta); M' = (m'_{k,l})_{k,l=0}^{N-1} = (dm_{k,l}/d\tau_{M})_{k,l=0}^{N-1}; N \in \mathbb{N} \). We denote by \( L^2(M) \) a set (of classes of equivalence) of vector-valued functions 
\( f : D_{r,l} \to \mathbb{C}_N, f = (f_0, f_1, \ldots, f_{N-1}) \), such that (see, e.g., [5],[6])
\[
\|f\|_{L^2(M)}^2 := \int_{D_{r,l}} f(u)\Psi(u)f^*(u)d\tau_M < \infty.
\]
The space \( L^2(M) \) is a Hilbert space with the scalar product

\[
(f,g)_{L^2(M)} := \int_{D_{r,l}} f(u)\Psi(u)g^*(u)d\tau_M, \quad f,g \in L^2(M).
\]

Set
\[
W_n f(x,\varphi) = e^{i\varphi_n} f(x,\varphi), \quad f \in L^2(M); 1 \leq n \leq l;
\]
and
\[
X_m f(x,\varphi) = x_m f(x,\varphi), \quad f(x,\varphi) \in L^2(M); x_m f(x,\varphi) \in L^2(M); 1 \leq m \leq r.
\]

Operators \( W_n \) are unitary. In the usual manner [7], one can check that operators \( X_m \) are self-adjoint.

## 2 A set of commuting self-adjoint and unitary operators with a spectrum of a finite multiplicity.

It is well known that a self-adjoint operator with a spectrum of a finite multiplicity in a Hilbert space \( H \) has a canonical model as a multiplication by an independent variable in \( L^2(M) \). Here \( M \) is a \( \mathbb{C}_{N \times N} \)-valued measure on \( \mathfrak{B}(\mathbb{R}) \), and \( N \) is the multiplicity of the spectrum of \( A \) [4]. For our investigation on the density of polynomials, mentioned in the Introduction, we shall use a generalization of this result to the case of an arbitrary finite set of commuting self-adjoint and unitary operators. Moreover, we shall need a result which is a little more general even in the classical case. Our method of proof is little different from the classical one (we shall not use Lemma in [4] p.287).

Consider a set
\[
\mathcal{A} = (S_1, S_2, \ldots, S_r, U_1, U_2, \ldots, U_l), \quad r, l \in \mathbb{Z}_+ : \quad r + 1 \neq 0, \quad (6)
\]
where $S_j$ are self-adjoint operators and $U_k$ are unitary operators in a Hilbert space $H$, $1 \leq j \leq r$, $1 \leq k \leq l$. In the case $r = 0$ operators $S_j$ disappear. Analogously, for $l = 0$ we only have operators $S_j$. The set $\mathcal{A}$ is said to be a $\textbf{SU-set of order } (r, l)$.

The set $\mathcal{A}$ is called \textbf{commuting} if operators $S_j, U_k$ pairwise commute. This mean that

$$U_k U_m = U_m U_k, \quad 1 \leq k, m \leq l; \quad (7)$$

$$U_k S_j \subset S_j U_k, \quad 1 \leq j \leq r; \quad 1 \leq k \leq l; \quad (8)$$

and the spectral measures of $S_j$ pairwise commute \cite{7}. In this case, there exists a spectral measure $E(\delta)$, $\delta \in \mathfrak{B}(D_{r, l})$, such that \cite{7}:

$$S_j = \int_{D_{r, l}} x_j dE, \quad 1 \leq j \leq r; \quad (9)$$

$$U_k = \int_{D_{r, l}} e^{i\varphi_k} dE, \quad 1 \leq k \leq l. \quad (10)$$

We shall call $E$ the spectral measure of the commuting $\textbf{SU-set } \mathcal{A}$ of \textbf{order } $(r, l)$.

We shall say that a commuting $\textbf{SU-set } \mathcal{A}$ of order $(r, l)$ has a \textbf{spectrum of multiplicity} $d$, if

1) there exist vectors $h_0, h_1, \ldots, h_{d-1}$ in $H$ such that

$$h_i \in D(S_1^{m_1} S_2^{m_2} \ldots S_r^{m_r}), \quad m_1, m_2, \ldots, m_r \in \mathbb{Z}_+, \quad 0 \leq i \leq d - 1; \quad (11)$$

$$\text{span}\{U_1^{n_1} U_2^{n_2} \ldots U_r^{n_r} S_1^{m_1} S_2^{m_2} \ldots S_r^{m_r} h_i, \quad n_1, n_2, \ldots, n_r \in \mathbb{Z}; \quad 0 \leq i \leq d - 1\} = H; \quad (12)$$

2) \textbf{(minimality)} For arbitrary $\tilde{d} \in \mathbb{Z}_+: \tilde{d} < d$, and arbitrary $\tilde{h}_0, \tilde{h}_1, \ldots, \tilde{h}_{\tilde{d}-1}$ in $H$, at least one of conditions (11), (12), with $\tilde{d}$ instead of $d$, and $\tilde{h}_i$ instead of $h_i$, is not satisfied.

In the case $r = 0$, condition (11) is redundant. Condition (12) in cases $r = 0$, $l = 0$, has no $U_k$ or $S_j$, respectively.

Set

$$\tilde{e}_i = (\delta_{0,i}, \delta_{1,i}, \ldots, \delta_{N-1,i}), \quad 0 \leq i \leq N - 1.$$
Theorem 1. Let $A$ be a commuting SU-set of order $(r, l)$ in a Hilbert space $H$ which has a spectrum of multiplicity $d$. Let $x_0, x_1, ..., x_{N-1}$, $N \geq d$, be elements of $H$ such that

$$x_i \in D(S_1^m S_2^{m_2} ... S_r^{m_r}), \quad m_1, m_2, ..., m_r \in \mathbb{Z}_+, \quad 0 \leq i \leq N - 1; \quad (13)$$

$$\text{span}\{U_1^{m_1} U_2^{m_2} ... U_r^{m_r} S_1^{m_1} S_2^{m_2} ... S_r^{m_r} x_i, \quad m_1, m_2, ..., m_r \in \mathbb{Z}_+, \quad n_1, n_2, ..., n_r \in \mathbb{Z}; \quad 0 \leq i \leq N - 1\} = H. \quad (14)$$

Set

$$M(\delta) = ((E(\delta)x_i, x_j)_H)_{i,j=0}^{N-1}, \quad \delta \in \mathfrak{B}(D_{r,l}), \quad (15)$$

where $E$ is the spectral measure of $A$.

Then there exists a unitary transformation $V$ which maps $L^2(M)$ onto $H$ such that:

$$V^{-1} S_j V = X_j, \quad 1 \leq j \leq r; \quad (16)$$

$$V^{-1} U_k V = W_k, \quad 1 \leq k \leq l. \quad (17)$$

Moreover, we have

$$V \vec{e}_s = x_s, \quad 0 \leq s \leq N - 1. \quad (18)$$

Remark. In the case $r = 0$ relations $(13), (16)$ should be removed, and in $(14)$ operators $S_j$ disappear. In the case $l = 0$ relation $(17)$ should be removed and in $(14)$ operators $U_k$ disappear.

Proof. Let $\chi_\delta(u)$ be the characteristic function of a set $\delta \in \mathfrak{B}(D_{r,l})$. In the space $L^2(M)$ consider the following set:

$$L := \text{Lin}\{\chi_\delta(u) \vec{e}_s, \quad \delta \in \mathfrak{B}(D_{r,l}), \quad 0 \leq s \leq N - 1\}. \quad (19)$$

Choose two arbitrary functions

$$f(u) = \sum_{j=0}^{N-1} \sum_{\delta \in I_j} \alpha_j(\delta) \chi_\delta(u) \vec{e}_j, \quad \alpha_j(\delta) \in \mathbb{C}, \quad (20)$$

$$g(u) = \sum_{s=0}^{N-1} \sum_{\delta' \in J_s} \beta_s(\delta') \chi_{\delta'}(u) \vec{e}_s, \quad \beta_s(\delta') \in \mathbb{C}, \quad (21)$$

where $I_j, J_s$ are some finite subsets of $\mathfrak{B}(D_{r,1})$. We may write

$$(f(u), g(u))_{L^2(M)} = \sum_{j,s=0}^{N-1} \sum_{\delta \in I_j} \sum_{\delta' \in J_s} \alpha_j(\delta) \beta_s(\delta') \int_{D_{r,1}} \chi_{\delta \cap \delta'}(u) \vec{e}_j^\dagger M'_r(u) \vec{e}_s^\dagger d\tau_M$$
\[\sum_{j,s=0}^{N-1} \sum_{\delta \in I_j \delta' \in J_s} \alpha_j(\delta) \beta_s(\delta') m_{j,s}(\delta \cap \delta'). \]

Set
\[x_f = \sum_{j=0}^{N-1} \sum_{\delta \in I_j} \alpha_j(\delta) E(\delta) x_j, \quad x_g = \sum_{s=0}^{N-1} \sum_{\delta' \in J_s} \beta_s(\delta') E(\delta') x_s. \]

Then
\[(x_f, x_g)_H = \sum_{j,s=0}^{N-1} \sum_{\delta \in I_j \delta' \in J_s} \alpha_j(\delta) \beta_s(\delta')(E(\delta)x_r, E(\delta')x_s)_H\]
\[= \sum_{j,s=0}^{N-1} \sum_{\delta \in I_j \delta' \in J_s} \alpha_j(\delta) \beta_s(\delta') m_{j,s}(\delta \cap \delta'). \]

Comparing relations (22) and (24) we obtain:
\[(f, g)_{L^2(M)} = (x_f, x_g)_H. \]

Now assume that \(f\) and \(g\) belong to the same class of equivalence in \(L^2(M)\):
\[\|f - g\|_{L^2(M)} = 0. \]

Then
\[\|x_f - x_g\|_H^2 = \left\| \sum_{j=0}^{N-1} \left( \sum_{\delta \in I_j} \alpha_j(\delta) E(\delta) - \sum_{\delta' \in J_j} \beta_j(\delta) E(\delta) \right) x_j \right\|_H^2\]
\[= \left\| \sum_{j=0}^{N-1} \sum_{\delta \in I_j \cup J_j} c_j(\delta) E(\delta) x_j \right\|_H^2, \]
where
\[c_j(\delta) = \begin{cases} \alpha_j(\delta), & \delta \in I_j \backslash J_j \\ -\beta_j(\delta), & \delta \in J_j \backslash I_j \\ \alpha_j(\delta) - \beta_j(\delta), & \delta \in I_j \cap J_j \end{cases}. \]

Set
\[w(u) = \sum_{j=0}^{N-1} \sum_{\delta \in I_j \cup J_j} c_j(\delta) \chi_\delta(u) \bar{e}_j. \]

Applying relation (25) with \(f = g = w\) we obtain:
\[\|x_f - x_g\|_H^2 = \|x_w\|_H^2 = \|w\|_{L^2(M)}^2. \]
\[
\sum_{j=0}^{N-1} \left( \sum_{\delta \in I_j} \alpha_j(\delta) \chi_\delta(u) - \sum_{\delta \in J_j} \beta_j(\delta) \chi_\delta(u) \right) \vec{e}_j \right\|_{L^2(M)} = \|f - g\|_{L^2(M)}^2 = 0.
\]

Therefore a transformation \( V: Vf = x_f \), is correctly defined on \( L \), and \( R(V) \subseteq H \). Moreover, relation (25) shows that \( V \) is an isometric transformation. Since simple functions are dense in \( L^2(M) \) ([5, Theorem 3.11]), we have \( \mathcal{T} = L^2(M) \). By continuity we extend \( V \) on the whole \( L^2(M) \).

Suppose that \( R(V) \neq H \). Then there exists \( 0 \neq h \in H \), such that
\[
(E(\delta)x_s, h)_H = 0, \quad \delta \in \mathfrak{B}(D_{r,1}), \ 0 \leq s \leq N - 1.
\]

Therefore we may write
\[
(U_1^{m_1}U_2^{m_2}...U_1^{m_1}S_1^{m_1}S_2^{m_2}...S_r^{m_r}x_s, h)_H = \int_{D_{r,1}} x_1^{m_1}x_2^{m_2}...x_r^{m_r} e^{im_1\phi_1} e^{im_2\phi_2}...e^{im_r\phi_1} d(Ex_s, h)_H = 0,
\]
\[
m_1, m_2, ..., m_r \in \mathbb{Z}_+, \ n_1, n_2, ..., n_l \in \mathbb{Z}.
\]

By (14) we get \( h = 0 \). This contradiction proves that \( R(V) = H \). Thus, \( V \) is a unitary transformation of \( L^2(M) \) onto \( H \). Observe that relation (18) holds. Set
\[
L_i^2(M) = \{ f(u) = (f_0(u), f_1(u), ..., f_{N-1}(u)) \in L^2(M) : \int_{D_{r,1}} |f_s(u)|^2 dm_{s,s} < \infty, \ 0 \leq s \leq N - 1 \}.
\]

(28)

Here, as usual, we mean that \( L_i^2(M) \) consists of classes of equivalence from \( L^2(M) \), which have at least one representative \( f \) with square integrable components. Observe that simple functions belong to \( L_i^2(M) \) and therefore \( L_i^2(M) \) is dense in \( L^2(M) \). Let us check that
\[
Vf = \sum_{s=0}^{N-1} \int_{D_{r,1}} f_s(u)dEx_s, \quad f = (f_0, f_1, ..., f_{N-1}) \in L_i^2(M).
\]

(29)

Choose an arbitrary function \( f = (f_0, f_1, ..., f_{N-1}) \in L_i^2(M) \). Let
\[
f_s^k(u) = \sum_{\delta \in L_{s,k}} \alpha_{s,k}(\delta) \chi_\delta(u), \quad 0 \leq s \leq N - 1; \ k \in \mathbb{N},
\]

(30)
where \( I_{s,k} \) is a finite subset of \( \mathfrak{B}(D_{r,1}) \), be simple functions such that

\[
\int_{D_{r,1}} |f_s(u) - f^k_s(u)|^2 \, dm_{s,s} \leq \frac{1}{k^2}, \quad 0 \leq s \leq N - 1; \ k \in \mathbb{N}.
\]

(31)

Then

\[
\|f(u) - \sum_{s=0}^{N-1} f^k_s(u)\vec{e}_s\|_{L^2(M)} \leq \frac{N}{k}, \quad k \in \mathbb{N}.
\]

(32)

Set

\[
f^k(u) = \sum_{s=0}^{N-1} f^k_s(u)\vec{e}_s = \sum_{s=0}^{N-1} \sum_{\delta \in I_{s,k}} \alpha_{s,k}(\delta) \chi_{\delta}(u)\vec{e}_s, \quad k \in \mathbb{N}.
\]

Then

\[
\|f - f^k\|_{L^2(M)} \to 0, \quad \text{as} \ k \to \infty.
\]

(33)

Therefore

\[
\|Vf - Vf^k\|_H \to 0, \quad \text{as} \ k \to \infty.
\]

(34)

Observe that

\[
Vf^k(u) = \sum_{s=0}^{N-1} \sum_{\delta \in I_{s,k}} \alpha_{s,k}(\delta) E(\delta)x_s, \quad k \in \mathbb{N}.
\]

(35)

We may write

\[
\left\| \sum_{s=0}^{N-1} \int_{D_{r,1}} f_s(u) \, dE x_s - \sum_{s=0}^{N-1} \sum_{\delta \in I_{s,k}} \alpha_{s,k}(\delta) E(\delta)x_s \right\|_H \leq \sum_{s=0}^{N-1} \left\| \int_{D_{r,1}} \left( f_s(u) - \sum_{\delta \in I_{s,k}} \alpha_{s,k}(\delta) \chi_{\delta}(u) \right) \, dE x_s \right\|_H
\]

\[
\leq \sum_{s=0}^{N-1} \left\| \int_{D_{r,1}} \left( f_s(u) - \sum_{\delta \in I_{s,k}} \alpha_{s,k}(\delta) \chi_{\delta}(u) \right) \, dE x_s \right\|_H
\]

\[
= \sum_{s=0}^{N-1} \left\{ \int_{D_{r,1}} \left| f_s(u) - \sum_{\delta \in I_{s,k}} \alpha_{s,k}(\delta) \chi_{\delta}(u) \right|^2 \, d(E x_s, x_s)_H \right\}^\frac{1}{2} \leq \frac{N}{k}, \quad k \in \mathbb{N}.
\]

By the uniqueness of the limit we conclude that relation (29) holds.
In the case \( r = 0 \), the following considerations until relations (40),(41) are redundant, and in these relations one should choose \( f \in L^2_i(M) \).

Set
\[ L^2_{i;2}(M) = \{ f(x, \varphi) = (f_0(x, \varphi), f_1(x, \varphi), \ldots, f_{N-1}(x, \varphi)) \in L^2(M) : \]
\[ \int_{D_{r,1}} |f_s(x, \varphi)|^2 dm_{s,s} < \infty, \int_{D_{r,1}} |x_kf_s(x, \varphi)|^2 dm_{s,s} < \infty, \]
\[ 1 \leq k \leq r, \ 0 \leq s \leq N - 1 \}. \] (36)

Of course, \( L^2_{i;2}(M) \subseteq L^2_i(M) \), and \( L^2_{i;2}(M) \subseteq D(X_k), \ 1 \leq k \leq r \). Moreover, we have
\[ X_mL^2_{i;2}(M) \subseteq L^2_i(M), \quad 1 \leq m \leq r. \] (37)

Observe that functions
\[ \chi_{\delta_k \cap \delta_k}(x, \varphi)e_s, \quad \delta \in \mathfrak{B}(D_{r,1}), \ 0 \leq s \leq N - 1, \] (38)
\[ \delta_k = \{(x, \varphi) \in D_{r,1} : |x_m| \leq k, \ 1 \leq m \leq r\}, \ k \in \mathbb{N}, \] (39)
belong to \( L^2_{i;2}(M) \). Therefore \( L^2_{i;2}(M) \) is dense in \( L^2_i(M) \).

Choose an arbitrary function \( f \in L^2_{i;2}(M) \). By virtue of relation (29) we may write:
\[ Vf = \sum_{s=0}^{N-1} \int_{D_{r,1}} f_s(x, \varphi)dEx_s, \] (40)
\[ VX_mf = \sum_{s=0}^{N-1} \int_{D_{r,1}} x_mf_s(x, \varphi)dEx_s = \sum_{s=0}^{N-1} S_m \int_{D_{r,1}} f_s(x, \varphi)dEx_s = S_mVf, \]
\[ VW_nf = \sum_{s=0}^{N-1} \int_{D_{r,1}} e^{i\varphi_n}f_s(x, \varphi)dEx_s = \sum_{s=0}^{N-1} U_n \int_{D_{r,1}} f_s(x, \varphi)dEx_s = U_nVf, \] (41)
where \( 1 \leq m \leq r, \ 1 \leq n \leq l \). By continuity, from the latter relation we obtain that relation (17) holds. In the case \( r = 0 \) this completes the proof. In the opposite case we may write
\[ X_mf = V^{-1}S_mVf, \quad f \in L^2_{i;2}(M), \ 1 \leq m \leq r. \] (42)

Let us prove that
\[ L^2_{i;2}(M) \subseteq (X_m \pm iE_{L^2_i(M)})L^2_{i;2}(M). \] (43)
Choose an arbitrary function \( f = (f_0, f_1, ..., f_{N-1}) \in L^2_{i;2}(\mathcal{M}) \). Observe that
\[
g_{\pm}(x, \varphi) := \frac{1}{x_m \pm i}(f_0(x, \varphi), f_1(x, \varphi), ..., f_{N-1}(x, \varphi)) \in L^2_{i;2}(\mathcal{M}). \tag{44}
\]
Therefore \((X_m \pm iE_{L^2(\mathcal{M})})g_{\pm}(x, \varphi) = f\). Thus, relation (13) is true. This relation means that operators \(X_m\) and \(V^{-1}S_mV\), restricted to \(L^2_{i;2}(\mathcal{M})\), are essentially self-adjoint. Therefore they have a unique self-adjoint extension. Since operators \(X_m\) and \(V^{-1}S_mV\) are self-adjoint extensions, we conclude that relation (16) holds.

\section{Density of polynomials: the case (A).}

Let \( M = (m_{k,l})_{k,l=0}^{N-1} \) be a \( C^\infty_{N \times N} \)-valued measure on \( \mathfrak{B}(\mathbb{R}) \), \( N \in \mathbb{N} \), such that
\[
\int x^ndm_{k,l} \text{ exist, } \quad n \in \mathbb{Z}_+; \quad 0 \leq k, l \leq N - 1. \tag{45}
\]
In this section, we shall use the same notation for matrix-valued measures \( M(\delta) \) on \( \mathfrak{B}(\mathbb{R}) \) and their distribution functions \( M(x), x \in \mathbb{R} \). Set
\[
S_n := \int x^ndM, \quad n \in \mathbb{Z}_+, \tag{46}
\]
and consider the matrix Hamburger moment problem with moments \( \{S_n\}_{n \in \mathbb{Z}_+} \).
Set
\[
\Gamma_n = (S_{k+l})_{k,l=0}^n, \quad n \in \mathbb{Z}_+; \quad \Gamma = (S_{k+l})_{k,l=0}^\infty = (\Gamma_{n,m})_{n,m=0}^\infty, \quad \Gamma_{n,m} \in \mathbb{C}. \tag{47}
\]
Since the moment problem has a solution we have
\[
\Gamma_n \geq 0, \quad n \in \mathbb{Z}_+. \tag{48}
\]
There exists a Hilbert space \( H \) and a sequence \( \{x_n\}_{n=0}^\infty \) in \( H \), such that \( \text{span}\{x_n\}_{n \in \mathbb{Z}_+} = H \), and
\[
(x_n, x_m)_H = \Gamma_{n,m}, \quad n, m \in \mathbb{Z}_+. \tag{49}
\]
Let \( A \) be a linear operator with \( D(A) = \text{Lin}\{x_n\}_{n \in \mathbb{Z}_+} \), defined by equalities
\[
Ax_k = x_{k+N}, \quad k \in \mathbb{Z}_+. \tag{50}
\]
In [8] it was shown that $A$ is a correctly defined symmetric operator in $H$. Denote by $F = F(A)$ a set of all analytic in $C_+$ operator-valued functions $F(\lambda)$, which values are contractions which map $N_i(A)$ into $N_{-i}(A)$ ($\|F(\lambda)\| \leq 1$). In [8, Theorem 4] it was proved that all solutions of the moment problem have the following form:

$$M(x) = (m_{k,j}(x))_{k,j=0}^{N-1},$$

where $m_{k,j}$ satisfy the following relation

$$\int_{\mathbb{R}} \frac{1}{x - \lambda} dm_{k,j}(x) = ((A_F(\lambda) - \lambda E_H)^{-1} x_k, x_j)_H, \quad \lambda \in \mathbb{C}_+,$$

where $A_F(\lambda)$ is the quasiself-adjoint extension of $\widetilde{A}$ defined by $F(\lambda) \in F(\widetilde{A})$.

On the other hand, to any operator function $F(\lambda) \in F(A)$ there corresponds by relation (50) a solution of the matrix Hamburger moment problem. The correspondence between all operator functions $F(\lambda) \in F(A)$ and all solutions of the moment problem, established by relation (50), is bijective.

Relation (50) may be written in the following form:

$$\int_{\mathbb{R}} \frac{1}{x - \lambda} dm_{k,j}(x) = (R(\lambda, A) x_k, x_j)_H, \quad \lambda \in \mathbb{C}_+,$$

where $R(\lambda, A)$ is a generalized resolvent of $\widetilde{A}$. The correspondence between all generalized resolvents and all solutions of the moment problem is bijective. From relation (51) it follows that (8, Theorem 2)

$$M(t) = (m_{k,j}(t))_{k,j=0}^{N-1}, \quad m_{k,j}(t) = (E(t x_k, x_j))_H, \quad t \in \mathbb{R},$$

where $E_t$ is a spectral function of $\widetilde{A}$. The latter means that $E_t = P_H \hat{E}_t$, where $\hat{E}_t$ is the orthogonal resolution of unity of a self-adjoint operator $\hat{A} \supseteq A$ in a Hilbert space $\hat{H} \supseteq H$. The correspondence between all spectral functions and all solutions of the moment problem is bijective, as well.

**Definition 1** A solution $M(t) = (m_{k,j}(t))_{k,j=0}^{N-1}$ of the matrix Hamburger moment problem [8] is said to be canonical, if it corresponds by relation (52) to an orthogonal spectral function of $\widetilde{A}$, i.e. to a spectral function generated by a self-adjoint extension $\hat{A} \supseteq A$ inside $H$.

From this definition we see that canonical solutions exist if and only if the defect numbers of $A$ are equal. Observe that $M(t) = (m_{k,j}(t))_{k,j=0}^{N-1}$ is a canonical solution of the matrix Hamburger moment problem [8] if and only
if it corresponds to an orthogonal resolvent of \( \overline{A} \), i.e. to a usual resolvent of a self-adjoint extension \( \hat{A} \supseteq A \) inside \( H \), in relation (51). Assume that the defect numbers of \( A \) are equal. From the Shtraus formula for generalized resolvents [9, Theorem 7], it easily follows that the orthogonal resolvents of \( A \) correspond to \( F(\lambda) \equiv C \), \( C \) is a unitary operator from \( N_i(\overline{A}) \) onto \( N_{-i}(\overline{A}) \).

Consequently, canonical solutions of the moment problem correspond in relation (50) to functions \( F(\lambda) \equiv C \), \( C \) is a unitary operator from \( N_i(\overline{A}) \) onto \( N_{-i}(\overline{A}) \).

**Theorem 2** Let \( M = (m_{k,l})_{k,l=0}^{N-1} \) be a \( C_{N \times N}^{\geq} \)-valued measure on \( \mathcal{B}(\mathbb{R}) \), \( N \in \mathbb{N} \), such that relation (45) holds. Let \( L^2_0(M) \) be the closure in \( L^2(M) \) of a set of all vector-valued polynomials \( p \in \mathbb{P}_N \). Consider the matrix Hamburger moment problem with moments \( \{S_n\}_{n \in \mathbb{Z}_+} \) defined by (46). Consider a Hilbert space \( H \) and a sequence \( \{x_n\}_{n=0}^\infty \) in \( H \), such that \( \text{span}\{x_n\}_{n \in \mathbb{Z}_+} = H \), and relation (48) holds. Let \( A \) be a linear operator with \( D(A) = \text{Lin}\{x_n\}_{n \in \mathbb{Z}_+} \), defined by equalities

\[
Ax_k = x_{k+N}, \quad k \in \mathbb{Z}_+.
\]

The following conditions are equivalent:

(i) \( L^2_0(M) = L^2(M) \);

(ii) \( M \) is a canonical solution of the corresponding matrix Hamburger moment problem;

(iii) \( M(x) = (m_{k,j}(x))_{k,j=0}^{N-1} \) satisfy the following relation:

\[
\int_{\mathbb{R}} \frac{1}{x - \lambda} dm_{k,j}(x) = ((A_U - \lambda E_H)^{-1}x_k, x_j)_H, \quad \lambda \in \mathbb{C}_+,
\]  

where \( A_U \) is a quasiself-adjoint extension of \( \overline{A} \) defined by a unitary operator \( U \) from \( N_i(\overline{A}) \) onto \( N_{-i}(\overline{A}) \). The latter is equivalent to the fact that \( A_U \) is a self-adjoint extension of \( A \) inside \( H \).

(iv) For every \( \lambda \in \mathbb{C}_+ \), there exists a linear bounded operator \( D_\lambda \) in \( H \) such that

\[
(D_\lambda x_{Nk+r}, x_{Nl+s})_H = \int_{\mathbb{R}} \frac{x^{k+l}}{x - \lambda} dm_{r,s}, \quad 0 \leq r, s \leq N - 1; \quad k, l \in \mathbb{Z}_+,
\]  

which is invertible and

\[
D^{-1}_\lambda + \lambda E_H \equiv A_U,
\]

where \( A_U \) is a self-adjoint extension of \( A \) inside \( H \).
Proof. (i)⇒(ii): Repeating arguments from [8, pp.276-278] we construct a self-adjoint extension \( \hat{A} \) of \( A \), which acts in \( H \oplus (L^2(M) \ominus L^2_0(M)) = H \), and
\[
m_{k,j}(t) = (\hat{E}_t x_k, x_j)_H,
\]
where \( \hat{E}_t \) is a left-continuous resolution of unity of \( \hat{A} \). Thus, \( M \) is a canonical solution of the moment problem.

(ii)⇒(i): Let \( M = (m_{k,j})_{k,j=0}^{N-1} \) has form (56), where \( \hat{E}_t \) is a left-continuous resolution of unity of a self-adjoint operator \( \hat{A} \supset A \) in \( H \). Since \( \hat{A}x_n = Ax_n = x_{n+N}, n \in \mathbb{Z}_+ \), then by the induction argument we get
\[
\hat{A}^r x_s = x_{rN+s}, \quad 0 \leq s \leq N - 1; \ r \in \mathbb{Z}_+.
\] (57)
Therefore
\[
\text{span}\{A^r x_s, \ 0 \leq s \leq N - 1; \ r \in \mathbb{Z}_+\} = H.
\] (58)
Thus, \( \hat{A} \) has a spectrum of multiplicity \( d \leq N \). By Theorem 1 there exists a unitary transformation \( W \) which maps \( L^2(M) \) onto \( H \) such that:
\[
W^{-1}\hat{A}W = X,
\] (59)
\[
W\vec{e}_s = x_s, \quad 0 \leq s \leq N - 1,
\] (60)
where \( X \) is the operator of multiplication by an independent variable in \( L^2(M) \). Let us check that
\[
W x^k \vec{e}_s = x_{kN+s}, \quad 0 \leq s \leq N - 1; \ k \in \mathbb{Z}_+.
\] (61)
Fix an arbitrary \( s: 0 \leq s \leq N - 1 \). Let us use the induction argument. For \( k = 0 \) relation (61) holds. Assume that it is true for \( k = r \in \mathbb{Z}_+ \). Then
\[
W x^{r+1} \vec{e}_s = W X W^{-1} W x^r \vec{e}_s = \tilde{A} x_{rN+s} = x_{(r+1)N+s}.
\]
Therefore relation (61) is true.

Repeating arguments from [8, pp.276-277] we construct a unitary transformation \( V \) which maps \( L^2_0(M) \) onto \( H \), such that
\[
V x^k \vec{e}_s = x_{kN+s}, \quad 0 \leq s \leq N - 1; \ k \in \mathbb{Z}_+.
\] (62)
By (61), (62) we conclude that \( W f = V f, f \in L^2_0(M) \). Therefore \( WL^2_0(M) = H \), and \( L^2_0(M) = W^{-1}H = L^2(M) \).

(ii)⇔(iii): This equivalence was established before the statement of the Theorem.
(ii)⇒(iv): Let \( M = (m_{k,j})^{N-1}_{k,j=0} \) has form (60) where \( \hat{E}_t \) is a left-continuous resolution of unity of a self-adjoint operator \( \hat{A} \supseteq A \) in \( H \). Then
\[
(R_{\lambda}(\hat{A})x_{Nk+r}, x_{Nl+s})_H = (R_{\lambda}(\hat{A})\hat{A}^kx_r, \hat{A}^l x_s)_H = (\hat{A}^{k+l}R_{\lambda}(\hat{A})x_r, x_s)_H
\]
\[
= \int_{\mathbb{R}} \frac{t^{k+l}}{t-\lambda} d(\hat{E}x_r, x_s)_H = \int_{\mathbb{R}} \frac{t^{k+l}}{t-\lambda} dm_{r,s}, \quad 0 \leq r, s \leq N-1; \ k, l \in \mathbb{Z}_+.
\]
(63)
Therefore for \( D_{\lambda} := R_{\lambda}(\hat{A}) \) condition (iv) holds.

(iv)⇒(ii): Let \( E_{U,t} \) be the left-continuous orthogonal resolution of unity of \( A_U \). Observe that \( D_{\lambda} \) is the resolvent function of the self-adjoint operator \( A_U \supseteq A \) in \( H \). Using (54) we may write
\[
\int_{\mathbb{R}} \frac{1}{x-\lambda} d(E_{U,t}x_r, x_s)_H = (R_{\lambda}(A_U)x_r, x_s)_H = (D_{\lambda}x_r, x_s)_H
\]
\[
= \int_{\mathbb{R}} \frac{1}{x-\lambda} dm_{r,s}, \quad 0 \leq r, s \leq N-1.
\]
(65)
Therefore \( M = ((E_{U,t}x_r, x_s)_H)^{N-1}_{r,s=0} \). Hence, \( M \) is a canonical solution of the moment problem. \( \square \)

4 Density of polynomials: the case (B).

Let \( \sigma \) be a non-negative measure on \( \mathfrak{B}(\Pi) \), such that
\[
\int_{\Pi} x^m d\sigma < \infty, \quad m \in \mathbb{Z}_+.
\]
(66)
Set
\[
s_{m,n} := \int_{\Pi} x^m e^{in\varphi} d\sigma, \quad m \in \mathbb{Z}_+, \ n \in \mathbb{Z},
\]
(67)
and consider the Devinatz moment problem with moments \( \{s_{m,n}\}_{m \in \mathbb{Z}_+, n \in \mathbb{Z}} \). Since the moment problem has a solution, for arbitrary complex numbers \( \alpha_{m,n} \) (where all but finite numbers are zeros) we have [3]
\[
\sum_{m,k=0}^{\infty} \sum_{n,l=-\infty}^{\infty} \alpha_{m,n} \alpha_{k,l} s_{m+k,n-l} \geq 0.
\]
(68)
There exists a Hilbert space \( H \) and a sequence \( \{x_{m,n}\}_{m \in \mathbb{Z}_+, n \in \mathbb{Z}} \) in \( H \), such that \( \text{span}\{x_{m,n}\}_{m \in \mathbb{Z}_+, n \in \mathbb{Z}} = H \), and [3]
\[
(x_{m,n}, x_{k,l})_H = s_{m+k,n-l}, \quad m, k \in \mathbb{Z}_+, \ n, l \in \mathbb{Z}.
\]
(69)
Let $A_0$, $B_0$ be linear operators and $J_0$ be an antilinear operator, with $D(A_0) = D(B_0) = D(J_0) = \text{Lin}\{x_{m,n}\}_{m \in \mathbb{Z}_+, n \in \mathbb{Z}}$, defined by equalities:

$$A_0 x_{m,n} = x_{m+1,n}, \quad B_0 x_{m,n} = x_{m,n+1}, \quad J_0 x_{m,n} = x_{m,-n}, \quad m \in \mathbb{Z}_+, n \in \mathbb{Z}.$$ 

In [3] it was shown that these operators are correctly defined, $A_0$ is symmetric and $B_0$ is isometric. Operators $A = A_0$ and $B = B_0$ are commuting closed symmetric and unitary operators, respectively. The operator $J_0$ extends by continuity to a conjugation $J$ in $H$.

In [3] it was proved that an arbitrary solution $\mu$ of the Devinatz moment problem has the following form:

$$\mu(\delta) = ((E \times F)(\delta) x_{0,0}, x_{0,0})_H, \quad \delta \in \mathcal{B}(\Pi), \quad (70)$$

where $F$ is the spectral measure of $B$, $E$ is a spectral measure of $A$ which commutes with $F$. By $((E \times F)(\delta) x_{0,0}, x_{0,0})_H$ we mean the non-negative Borel measure on $\Pi$ which is obtained by the Lebesgue continuation procedure from the following non-negative measure on rectangles

$$((E \times F)(I_x \times I_{\varphi}) x_{0,0}, x_{0,0})_H := (E(I_x) F(I_{\varphi}) x_{0,0}, x_{0,0})_H, \quad (71)$$

where $I_x \subset \mathbb{R}$, $I_{\varphi} \subseteq [-\pi, \pi)$ are arbitrary intervals.

On the other hand, for an arbitrary spectral measure $E$ of $A$ which commutes with the spectral measure $F$ of $B$, by relation (70) there corresponds a solution of the Devinatz moment problem. The correspondence between the spectral measures of $A$ which commute with the spectral measure of $B$ and solutions of the Devinatz moment problem is bijective.

Recall the following definition [3]:

**Definition 2** A solution $\mu$ of the Devinatz moment problem (7) is said to be canonical if it is generated by relation (70) where $E$ is an orthogonal spectral measure of $A$ which commutes with the spectral measure of $B$. Orthogonal spectral measures are those measures which are the spectral measures of self-adjoint extensions of $A$ inside $H$.

We also need some objects introduced in [3] to formulate a description of all canonical solutions. Set $V_A := (A + iE_H)(A - iE_H)^{-1}$, and

$$H_1 := \Delta_A(i), \quad H_2 := H \ominus H_1, \quad H_3 := \Delta_A(-i), \quad H_4 := H \ominus H_3. \quad (72)$$

The restriction $B_{H_2}$ of $B$ to $H_2$ is unitary, and by the Godič-Lucenko Theorem it has a representation: $B_{H_2} = KL$, where $K$ and $L$ are some conjugations in $H_2$. Set $U_{2,4} := JK$. Let $F_2 = F_2(\delta), \delta \in \mathcal{B}([-\pi, \pi))$, be the
spectral measure of the operator $B_{H_2}$ in $H_2$. Let $\mu$ be a scalar non-negative measure with a type which coincides with the spectral type of the measure $F_2$. Let $N_2$ be the multiplicity function of the measure $F_2$. Then there exists a unitary transformation $W$ of the space $H_2$ on the direct integral $\mathcal{H} = \mathcal{H}_\mu, N_2$ such that

$$WB_{H_2}W^{-1} = Q_{e^{iy}},$$

(73)

where $Q_{e^{iy}} : g(y) \mapsto e^{iy}g(y)$. Denote by $D(B; H_2)$ a set of all unitary decomposable operators in $\mathcal{H}$.

In relation (70), canonical solutions correspond to those spectral measures $E$ which are spectral measures of self-adjoint operators $\hat{A}$ of the following form:

$$\hat{A} = iE_H + 2(V_\Lambda \oplus U_{2,4}W^{-1}V_2W - E_H)^{-1},$$

(74)

where $V_2 \in D(B; H_2)$. The correspondence between all operators $V_2 \in D(B; H_2)$ and all canonical solutions is bijective [3].

**Theorem 3** Let $\sigma$ be a non-negative measure on $\mathcal{B}(\Pi)$, such that relation (66) holds. Let $L^2_0(\sigma)$ be the closure in $L^2(\sigma)$ of a set of all power-trigonometric polynomials (3). Consider the Devinatz moment problem with moments $\{s_{m,n}\}_{m \in \mathbb{Z}_+, n \in \mathbb{Z}}$ defined by (67). Consider a Hilbert space $H$ and a sequence $\{x_{m,n}\}_{m \in \mathbb{Z}_+, n \in \mathbb{Z}}$ in $H$, such that $\text{span}\{x_{m,n}\}_{m \in \mathbb{Z}_+, n \in \mathbb{Z}} = H$, and relation (63) holds. The following conditions are equivalent:

(i) $L^2_0(\sigma) = L^2(\sigma)$;

(ii) $\sigma$ is a canonical solution of the Devinatz moment problem;

(iii) $\sigma$ is generated by relation (70), where $E$ is the spectral function of $\hat{A}$ which has the form (74) with an operator $V_2 \in D(B; H_2)$.

(iv) For every $\lambda \in \mathbb{C}_+$, there exists a linear bounded operator $D_\lambda$ in $H$ such that

$$\langle D_\lambda x_{m,n}, x_{m',n'} \rangle_H = \int_\Pi \frac{x^{m+m'}e^{i(n-n')\varphi}}{x - \lambda} d\sigma, \quad m, m' \in \mathbb{Z}_+, \, n, n' \in \mathbb{Z},$$

(75)

which is invertible, and

$$\langle (E_H + 2iD_\lambda)^k x_{0,n}, x_{0,0} \rangle_H = \int_\Pi \left(\frac{x + i}{x - i}\right)^k e^{in\varphi} d\sigma, \quad n, k \in \mathbb{Z};$$

(76)

$$D_\lambda^{-1} + \lambda E_H \equiv \hat{A},$$

(77)

where $\hat{A}$ has the form (74) with an operator $V_2 \in D(B; H_2)$.  

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Proof. (i)⇒(ii): This implication was proved in [3] (see considerations before References).

(ii)⇒(i): Let σ has form (70), where E is the spectral function a self-adjoint operator \( \hat{A} \supseteq A \) in \( H \), which commutes with \( B \). Since \( \hat{A}x_{m,n} = Ax_{m,n} = x_{m+1,n}, m \in \mathbb{Z}_+, n \in \mathbb{Z} \), by an induction argument we get

\[
\hat{A}^r x_{m,n} = x_{m+r,n}, \quad m, r \in \mathbb{Z}_+, n \in \mathbb{Z}.
\]  

Therefore

\[
\hat{A}^r B^l x_{0,0} = \hat{A}^r x_{0,l} = x_{r,l}, \quad r, l \in \mathbb{Z}_+.
\]

We conclude that

\[
\text{span}\{\hat{A}^m B^n x_{0,0}, m \in \mathbb{Z}_+, n \in \mathbb{Z}\} = H.
\]  

Thus, \( (\hat{A}, B) \) has a spectrum of multiplicity 1. By Theorem II there exists a unitary transformation \( W \) which maps \( L^2(\sigma) \) onto \( H \) such that:

\[
W^{-1} \hat{A} W = X, \quad W^{-1} B W = U
\]

where \( X : f(x, \varphi) \mapsto x f(x, \varphi) \) and \( U : f(x, \varphi) \mapsto e^{i\varphi} f(x, \varphi) \) in \( L^2(\sigma) \). Let us check that

\[
W x^m = x_{m,0}, \quad m \in \mathbb{Z}_+. \tag{82}
\]

For \( m = 0 \) it is true. Assume that it is true for \( n = r \in \mathbb{Z}_+ \). Then

\[
W x^{r+1} = W X W^{-1} W x^r = \hat{A} x_{r,0} = x_{r+1,0},
\]

and therefore (82) holds. Let us prove that

\[
W x^m e^{in\varphi} = x_{m,n}, \quad m \in \mathbb{Z}_+, n \in \mathbb{Z}. \tag{83}
\]

Fix an arbitrary \( m \in \mathbb{Z}_+ \). For \( n = 0 \) relation (83) holds. Assume that it is true for \( n = r \in \mathbb{Z}_+ \). Then

\[
W e^{i(r+1)\varphi} x^m = W U W^{-1} W e^{i\varphi} x^m = B x_{m,r} = x_{m,r+1}.
\]

On the other hand, assume that (83) holds for \( n = -r, r \in \mathbb{Z}_+ \). Then

\[
W e^{i(-r-1)\varphi} x^m = W U^{-1} W^{-1} W e^{-i\varphi} x^m = B^{-1} x_{m,-r} = x_{m,-r-1}.
\]

Therefore relation (83) is true.
Repeating arguments from the beginning of the Proof of Theorem 3.1 in [3] we construct a unitary transformation \( V \) which maps \( L^2_0(\sigma) \) onto \( H \), such that
\[
V x^m e^{i\varphi} = x_{m,n}, \quad m \in \mathbb{Z}_+, \ n \in \mathbb{Z}.
\]
By (83), (84) we conclude that \( W f = V f, \ f \in L^2_0(\sigma) \). Therefore \( WL^2_0(\sigma) = H \), and \( L^2_0(\sigma) = W^{-1} H = L^2(\sigma) \).

(ii)\(\Leftrightarrow\)(iii): This equivalence was established in [3, Theorem 3.2] and discussed before the statement of the Theorem.

(ii)\(\Rightarrow\)(iv): Let \( \sigma \) has form (70), where \( \pi \) is the spectral function of a self-adjoint operator \( \hat{A} \supset A \) in \( H \), which commutes with \( B \). By considerations before the statement of the Theorem we obtain that \( \hat{A} \) has the form (74) with an operator \( V_2 \in \mathcal{D}(B; H_2) \). Then
\[
\left( R_\lambda(\hat{A}) x_{m,n}, x_{m',n'} \right)_H = \left( R_\lambda(\hat{A}) \hat{A}^m B^n x_{0,0}, \hat{A}^{m'} B^{n'} x_{0,0} \right)_H
\]
\[
= \left( B^{n-n'} \hat{A}^{m+m'} R_\lambda(\hat{A}) x_{0,0}, x_{0,0} \right)_H = \int_\Pi \frac{x^{m+m'} e^{i(n-n') \varphi}}{t-\lambda} d(\pi \times F) x_{0,0}, x_{0,0})_H
\]
\[
= \int_\Pi \frac{x^{m+m'} e^{i(n-n') \varphi}}{t-\lambda} d\sigma, \quad m, m' \in \mathbb{Z}_+, \ n, n' \in \mathbb{Z};
\]
\[
\left( (E_H + 2iR_i(\hat{A}))^k x_{0,n}, x_{0,0} \right)_H = \left( (E_H + 2iR_i(\hat{A}))^k B^n x_{0,0}, x_{0,0} \right)_H
\]
\[
= \int_\Pi \left( \frac{x + i}{x - i} \right)^k e^{i\varphi} d(\pi \times F) x_{0,0}, x_{0,0})_H
\]
\[
= \int_\Pi \left( \frac{x + i}{x - i} \right)^k e^{i\varphi} d\sigma, \quad k, n \in \mathbb{Z}.
\]

Therefore for \( D_\lambda := R_\lambda(\hat{A}) \) condition (iv) holds.

(iv)\(\Rightarrow\)(ii): Observe that \( D_\lambda \) is the resolvent function of the self-adjoint operator \( \hat{A} \supset A \) in \( H \) which commutes with \( B \). Let \( E \) be the spectral function of \( \hat{A} \). Using (70) we may write
\[
\int_\Pi \left( \frac{x + i}{x - i} \right)^k e^{i\varphi} d((E \times F) x_{0,0}, x_{0,0})_H = ((E_H + 2iR_i(\hat{A}))^k B^n x_{0,0}, x_{0,0})_H
\]
\[
= (D_\lambda(E_H + 2iD_i)^k x_{0,n}, x_{0,0})_H = \int_\Pi \left( \frac{x + i}{x - i} \right)^k e^{i\varphi} d\sigma, \quad n, k \in \mathbb{Z}.
\]

Repeating arguments from the Proof of Theorem 3.1 [3], we easily obtain that \( \sigma = ((E \times F) x_{0,0}, x_{0,0})_H \). Hence, \( \sigma \) is a canonical solution of the moment problem. \( \square \)
On the density of polynomials in some $L^2(M)$ spaces.

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In this paper we study the density of polynomials in some $L^2(M)$ spaces. Two choices of the measure $M$ and polynomials are considered: 1) a $(N \times N)$ matrix non-negative Borel measure on $\mathbb{R}$ and vector-valued polynomials $p(x) = (p_0(x), p_1(x), ..., p_{N-1}(x))$. $p_j(x)$ are complex polynomials, $N \in \mathbb{N}$; 2) a scalar non-negative Borel measure in a strip $\Pi = \{(x, \varphi) : \varphi \\leq \text{constant}\}$.
\( x \in \mathbb{R}, \varphi \in [-\pi, \pi) \), and power-trigonometric polynomials: 
\[ p(x, \varphi) = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \alpha_{m,n} x^m e^{in\varphi}, \quad \alpha_{m,n} \in \mathbb{C}, \]
where all but finite number of \( \alpha_{m,n} \) are zeros. We prove that polynomials are dense in \( L^2(M) \) if and only if \( M \) is a canonical solution of the corresponding moment problem. Using descriptions of canonical solutions, we get conditions for the density of polynomials in \( L^2(M) \). For this purpose, we derive a model for commuting self-adjoint and unitary operators with a spectrum of a finite multiplicity.