FINITE GROUP ACTIONS ON 4-MANIFOLDS WITH NONZERO EULER CHARACTERISTIC

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ABSTRACT. We prove that if $X$ is a compact, oriented, connected 4-dimensional smooth manifold, possibly with boundary, satisfying $\chi(X) \neq 0$, then there exists an integer $C \geq 1$ such that any finite group $G$ acting smoothly and effectively on $X$ has an abelian subgroup $A$ generated by at most two elements which satisfies $[G : A] \leq C$ and $\chi(X^A) = \chi(X)$. Furthermore, if $\chi(X) < 0$ then $A$ is cyclic. This proves, for any such $X$, a conjecture of Ghys. We also prove an analogous result for manifolds of arbitrary dimension and non-vanishing Euler characteristic, but restricted to pseudofree actions.

1. Introduction

1.1. The results. In this paper we prove two results on smooth finite group actions on compact, connected manifolds with non-vanishing Euler characteristic, and possibly with boundary. Our main result is on actions on 4-dimensional manifolds:

Theorem 1.1. Let $X$ be a compact, oriented, connected 4-dimensional smooth manifold, possibly with boundary, satisfying $\chi(X) \neq 0$. There exists an integer $C \geq 1$ such that any finite group $G$ acting smoothly and effectively on $X$ has an abelian subgroup $A$ satisfying $[G : A] \leq C$ and $\chi(X^A) = \chi(X)$. Furthermore, if $\chi(X) > 0$ then $A$ can be generated by at most 2 elements, and if $\chi(X) < 0$ then $A$ is cyclic.

This proves, for the manifolds satisfying the hypothesis of the theorem, a conjecture of Ghys stating that any finite group acting smoothly and effectively on a compact manifold has an abelian subgroup whose index is bounded above by a constant depending only on the manifold (see Question 13.1 in [5], and [15, 16] for some other partial results).

McCooey has proved in [10, 11] stronger restrictions than ours on finite groups acting effectively and homologically trivially on general compact, oriented, connected and closed 4-manifolds satisfying $\chi \neq 0$. However, McCooey’s results require some technical restrictions on the manifold, or on the finite group which acts on it, or on the action, and for this reason they do not imply Ghys’ conjecture for all closed oriented 4-manifolds with nonzero Euler characteristic.

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Concretely, [10] applies to 4-manifolds $X$ satisfying $H_1(X;\mathbb{Z}) = 0$ and $b_2(X) \geq 3$ or, if $b_2(X) = 2$, to actions with nonempty fixed point set, and it proves that finite groups acting smoothly and homologically trivially on these manifolds are isomorphic to subgroups of $S^1 \times S^1$. [11] applies to 4-manifolds $X$ satisfying $H_1(X;\mathbb{Z}) \cong \mathbb{Z}^r$ for some $r \geq 1$, $b_2(X) \notin \{0, 2\}$, and $\chi(X) \neq 0$, and the group which acts is required to have 2-rank at most 1; the conclusion is that if the action is smooth and homologically trivial then the group is cyclic; assuming either that $X$ has a Spin structure (and that the group action is compatible with it in a weak sense, see p. 10 in [11]), or that $X$ and the action are symplectic, [11] also proves that $(\mathbb{Z}/2\mathbb{Z})^2$ cannot act homologically trivially on $X$.

By a classic result of Minkowski (see Lemma 2.3 below), [10] implies Ghys’ conjecture for simply connected closed 4-manifolds satisfying $b_2 \geq 3$. Using the arguments in Subsection 2.3 of [15], this automatically extends to closed 4-manifolds satisfying $b_1 = 0$ and $b_2 \geq 3$ (note that if $X$ is a closed 4-manifold satisfying $b_1(X) = 0$ then its universal cover $\tilde{X}$ satisfies $2 + b_2(\tilde{X}) = \chi(\tilde{X}) = |H_1(X;\mathbb{Z})| \cdot \chi(X) = |H_1(X;\mathbb{Z})| \cdot (2 + b_2(X))$, so if $b_2(X) \geq 3$ then $b_2(\tilde{X}) \geq 3$). However, due to the technical assumptions, the results in [11] only imply Ghys’ conjecture for closed spin 4-manifolds with $\chi \neq 0$ or for symplectic closed 4-manifolds with $\chi \neq 0$ and symplectic actions.

For other results on finite group actions on 4-manifolds, not covered by McCooey’s results, and implying particular cases of Ghys’ conjecture, see e.g. [6, 12, 13, 19].

Theorem 1.1 extends the 4-dimensional case of the main result in [16], which only applies to simply connected 4-manifolds. In fact, the proof of Theorem 1.1 follows quite closely the scheme of [16], but there are some important differences, the most remarkable being that, in contrast to [16], we do not use here the classification of finite simple groups. See Subsection 1.2 for some details.

Our second result is analogous to the first one. Whereas the class of manifolds to which it applies is much wider, it is limited to pseudofree actions.

**Theorem 1.2.** Let $X$ be a compact connected manifold, possibly with boundary, satisfying $\chi(X) \neq 0$. There exists an integer $C \geq 1$ such that, if a finite group $G$ acts pseudofreely, smoothly and effectively on $X$, then $G$ has an abelian subgroup $A$ satisfying $[G : A] \leq C$ and $\chi(X^A) = \chi(X)$, and $A$ can be generated by at most $\lfloor \dim X/2 \rfloor$ elements.

Theorems 1.1 and 1.2 extend automatically in two directions. First, we can remove the condition that the manifold is connected using the arguments in Sections 6 and 8 of [16]. Second, if $X$ is a compact and unorientable 4-manifold satisfying $\chi(X) \neq 0$, then Ghys’ conjecture for $X$ can be deduced from that of the orientation two sheeted covering of $X$, via the arguments in Subsection 2.3 of [15].

1.2. Some ideas of the proofs. The proofs of Theorems 1.1 and 1.2 are based on estimating the Euler characteristic of the singular set or of some approximation of it. Recall that if a group $G$ acts effectively on a manifold $X$, the singular set of the action
is defined as
\[
S_X = \bigcup_{g \in G \setminus \{1\}} X^g = \{ x \in X \mid G_x \neq \{1\} \},
\]
where \(X^g\) is the set of points in \(X\) which are fixed by \(g\) and \(G_x\) denotes the isotropy group of \(x\). Using Minkowski’s theorem on finite subgroups of \(GL(n, \mathbb{Z})\) (see Lemma 2.3 below) the proofs of both theorems can be reduced to the case of actions which are trivial on cohomology. A variant of Lefschetz’s fixed point theorem (Lemma 2.4) implies that, for these actions, we have
\[
\chi(X^g) = \chi(X) \quad \text{for all } g \in G.
\]
In the case of pseudofree actions on manifolds satisfying \(\chi(X) < 0\) this implies that \(X^g = X\), so if the action of \(G\) is effective then we must have \(G = \{1\}\); this proves the statement of Theorem 1.2 for manifolds with negative Euler characteristic.

To complete the proof of Theorem 1.2 for manifolds with positive Euler characteristic we proceed as follows. Using (2) we estimate \(\chi(S_X) \leq \chi(X)(|G| - 1)\). This implies, via an argument relating the cohomologies of \(X\) and the orbit space \(X/G\), that the projection of \(S_X\) to \(X/G\) contains less than \(2\chi(X)\) elements, so the number of \(G\)-orbits in \(S_X\) is uniformly bounded. Next, an argument involving the cardinals of the orbits implies (by Lemma 2.7) the existence of some \(x \in S_X\) such that \([G : G_x]\) is smaller than a constant depending on \(X\) (this is similar to the usual proof of Hurwitz’s theorem on automorphisms of curves of genus \(\geq 2\), see e.g. §V.1.3 in [FK]). The proof of Theorem 1.2 is finished, with some easy arguments, by applying the classic Jordan theorem for finite subgroups of \(GL(n, \mathbb{C})\) (see Theorem 2.1) to \(G_x\).

We now sketch the proof of Theorem 1.1. Consider an arbitrary (cohomologically trivial) effective action of a group \(G\) on an oriented 4-manifold \(X\) satisfying \(\chi(X) \neq 0\). As soon as the action is not pseudofree, the problem of estimating in a useful way the Euler characteristic of the singular set becomes very difficult. Certainly we have \(\chi(X^g) = \chi(X)\) for every \(g \in G\), but to compute the Euler characteristic of the singular set (say, using the inclusion-exclusion principle) one needs to control \(\chi(X^{g_1} \cap \cdots \cap X^{g_k})\) for different \(g_1, \ldots, g_k \in G\), and there is no general formula for this quantity.\footnote{However, for some restricted classes of groups acting on \(X\) one can study in detail the topology of the singular set; in the case of minimal non-abelian groups, this is done in [10, 11], and it is the crucial ingredient of the proofs.}

To circumvent this difficulty we replace the singular set \(S_X\) by a set \(S'_X \subset X\) whose Euler characteristic is much easier to compute and which is in some sense a uniform approximation of \(S_X\); by the latter we mean that there exist constants \(1 < C_1 \leq C_2\), which are independent of \(G\), such that the isotropy group of any point in \(S'_X\) (resp. in the complementary of \(S'_X\)) has at least \(C_1\) (resp. at most \(C_2\)) elements. The actual definition of \(S'_X\) uses the notion of \(C\)-rigid abelian subgroup of \(G\), which we next explain.
Let $C \geq 1$ be an integer. We say that a subgroup $A \subseteq G$ is $C$-rigid if $A$ is abelian and for any subgroup $A' \subseteq A$ satisfying $[A : A'] \leq C$ we have $X^{A'} = X^A$. In this paper we prove the following properties of $C$-rigid subgroups:

(a) given $C$ there is a number $\Lambda(C)$ such that any abelian subgroup $A \subseteq G$ has a $C$-rigid subgroup $A' \subseteq A$ satisfying $[A : A'] \leq \Lambda(C)$ (Lemma 5.4);
(b) there exists some number $\chi$ such that for any $C\chi$-rigid subgroup $A \subseteq G$ we have $\chi(X^A) = \chi(X)$ (Lemma 5.5);
(c) given $C$ there is a number $I(C)$ such that if $A_1, \ldots, A_s \subseteq G$ are $I(C)$-rigid subgroups satisfying $\bigcap_j X^{A_j} \neq \emptyset$, then there is a $C$-rigid subgroup $A \subseteq G$ such that $\bigcap_j X^{A_j} = X^A$ (Lemma 5.8).

Both functions $C \mapsto \Lambda(C)$, $C \mapsto I(C)$ and the constant $C\chi$ depend only on $X$.

To define $S'_X$, we take a big number $C(X)$ depending on $X$ but independent of $G$, and we set $S'_X = \bigcup_A X^A$, where $A$ runs over the set of nontrivial $C(X)$-rigid subgroups of $G$. This is indeed an approximation of $S_X$ in the preceding sense: indeed, property (a) and Jordan’s theorem guarantees that if $x \in X \setminus S'_X$ then $G_x$ can not be too big, whereas the definition of rigidity implies that if $A$ is nontrivial and $C(X)$-rigid then $|A| > C(X)$, from which we deduce that if $x \in S'_X$ then $|G_x| > C(X)$.

The choice of $C(X)$ guarantees that each connected component of $S'_X$ has the same Euler characteristic as $X$ (this follows from Lemmas 5.5, 5.7, 5.8 and 5.9). The group $G$ acts on the set $\pi_0(S'_X)$ of connected components of $S'_X$, and we prove that the number of $G$-orbits in $\pi_0(S'_X)$ is uniformly bounded (Lemma 6.2). From this we deduce, using the same arithmetic arguments as in the proof of Theorem 1.2, that there is some point in $X$ satisfying $[G : G_x] \leq C'$, where $C'$ only depends on $X$. The proof is finished like that of Theorem 1.2 with the extra ingredient of properties (a) and (b).

These arguments are very similar to those of Theorem 1.4 in [16], which is one of the two principal ingredients in the proof of the main theorem of [16] (the other ingredient is the main result in [17], which we don’t use in the present paper), but there are some differences.

First of all, the notion of $K$-stable subgroup in [16] is replaced here by the similar but better behaved notion of $C$-rigid subgroup (this is possible because we only consider actions on 4-dimensional manifolds). A more superficial but important difference is that the inductive structure in the proof of Theorem 1.4 in [16] becomes almost trivial in the present paper, because in order to prove our theorems in 4 dimensions we rely to their analogues in dimensions 0, 1 and 2, which are proved by direct methods. It is because of this that the constant $C$ in Theorem 1.1, unlike the constant of Theorem 1.4 in [16], does not depend on the number of prime divisors of the order of the group. This allows us to avoid using the result in [17], which is based on the classification of finite simple groups, and makes our arguments much more elementary than those in [16].
1.3. Conventions and contents. All groups appearing in this paper will be assumed by default to be finite, all manifolds will be smooth and compact (but possibly with boundary), and all group actions on manifolds will be smooth.

Section 2 contains several unrelated results which will be used in the subsequent sections. Section 3 contains the proof of Theorem 1.2. Section 4 contains some results on finite group actions on surfaces, Section 5 is devoted to proving the main results on C-rigid (sub)groups and finally Section 6 contains the proof of Theorem 1.1.

2. Preliminaries

2.1. Jordan’s theorem. The conjecture of Ghys which we mentioned in the introduction is inspired by the following classic theorem due to C. Jordan (see for example [2, 3, 9, 15]).

Theorem 2.1 (Jordan). For any natural \( n \) there exists a constant \( C_{\text{Jor}}(n) \) such that any finite subgroup of \( \text{GL}(n, \mathbb{C}) \) has an abelian subgroup of index at most \( C_{\text{Jor}}(n) \).

This result will be used in the proofs of Theorems 1.1 and 1.2.

2.2. Linearizing group actions. The following is part of Lemma 2.1 of [16]. It implies in particular that the fixed point set of any (smooth) finite group action on a manifold with boundary is a neat submanifold in the sense of §1.4 in [7].

Lemma 2.2. Let a finite group \( G \) act smoothly on a manifold \( X \), and let \( x \in X^G \). The tangent space \( T_xX \) carries a linear action of \( G \), defined as the derivative at \( x \) of the action on \( X \), satisfying the following properties.

(1) There exist neighborhoods \( U \subset T_xX \) and \( V \subset X \), of \( 0 \in T_xX \) and \( x \in X \) respectively, such that:
   (a) if \( x \notin \partial X \) then there is a \( G \)-equivariant diffeomorphism \( \phi : U \to V \);
   (b) if \( x \in \partial X \) then there is \( G \)-equivariant diffeomorphism \( \phi : U \cap \{ \xi \geq 0 \} \to V \), where \( \xi \) is a nonzero \( G \)-invariant element of \( (T_xX)^* \) such that \( \text{Ker} \xi = T_x\partial X \).

(2) If the action of \( G \) is effective and \( X \) is connected then the action of \( G \) on \( T_xX \) is effective, so it induces an inclusion \( G \hookrightarrow \text{GL}(T_xX) \).

2.3. Regular triangulations. Let a group \( G \) act on a manifold \( X \). A \( G \)-regular triangulation of \( X \) is a pair \((\mathcal{C}, \phi)\) consisting of a \( G \)-regular finite simplicial complex \( \mathcal{C} \) (in the sense of Definition 1.2 of [11, Chapter III] — note that the \( G \)-regularity of \( \mathcal{C} \) implies that \( \mathcal{C}/G \) is a simplicial complex) and a \( G \)-equivariant homeomorphism \( \phi : X \to \mathcal{C} \). Such triangulations always exist: one can begin with an arbitrary equivariant triangulation of \( X \) (see e.g. [8]) and then take its second barycentric subdivision, which is automatically regular (see Proposition 1.1 in [11, Chapter III]).
2.4. Cohomologically trivial (CT) actions. We say that the action of a group $G$ on a space $X$ is cohomologically trivial (CT for short) if the induced action of $G$ on $H^*(X; \mathbb{Z})$ is trivial (equivalently, we say that $G$ acts on $X$ in a CT way).

The following result generalizes Lemma 2.3 of [16].

**Lemma 2.3.** For any manifold $X$ there exists some $C \in \mathbb{N}$ such that any finite group $G$ acting on $X$ has a subgroup $G_0 \subseteq G$ of index at most $C$ whose action on $X$ is CT.

**Proof.** Since $X$ is implicitly assumed to be compact, its cohomology is finitely generated as an abelian group. Let $T \subseteq H^*(X; \mathbb{Z})$ be the torsion. A classic result of Minkowski states that, given any integer $k$, the size of any finite subgroup of $\text{GL}(k; \mathbb{Z})$ is bounded above by a number depending only on $k$ (see [14, 18], or [15] for a proof using Jordan’s theorem). So if $G$ is a finite group acting on $X$, there is a subgroup $G' \subseteq G$, of index bounded above by a constant depending only on $X$, whose action on $H^*(X; \mathbb{Z})/T$ is trivial. There is also a subgroup $G'' \subseteq G'$ of index at most $|\text{Aut}(T)|$ which acts trivially on $T$. Let $F := H^*(X; \mathbb{Z})/T$. In terms of a splitting $H^*(X; \mathbb{Z}) \simeq F \oplus T$, the action of $G''$ on $H^*(X; \mathbb{Z})$ is through lower triangular matrices with ones in the diagonal, so it factors through the group $\text{Hom}(F, T)$, which is finite; hence, there is a subgroup $G_0 \subseteq G''$ of index at most $|\text{Hom}(F, T)|$ whose action on $H^*(X; \mathbb{Z})$ is trivial. □

In the following two lemmas we denote by $b_j(Y; k)$ the $j$-th Betti number of a space $Y$ with coefficients in a field $k$.

**Lemma 2.4.** Let $\Gamma$ be a (finite) cyclic group acting on a manifold $X$ and let $\gamma \in \Gamma$ be a generator. We have $\chi(X^\Gamma) = \sum_j (-1)^j \text{Tr}(H^j(\gamma) : H^j(X; \mathbb{Q}) \to H^j(X; \mathbb{Q}))$. In particular, if the action of $\Gamma$ on $X$ is CT, then $\chi(X^\Gamma) = \chi(X)$. In general, we can estimate

$$\left| \chi(X^\Gamma) \right| \leq \sum_j b_j(X; \mathbb{Q}).$$

**Proof.** The first part of the proof is the same as the proof of Lemma 2.4 in [16]. Since $\gamma$ has finite order all the eigenvalues of $H^j(\gamma) : H^j(X; \mathbb{Q}) \to H^j(X; \mathbb{Q})$ have modulus one, so $|\text{Tr}(H^j(\gamma) : H^j(X; \mathbb{Q}) \to H^j(X; \mathbb{Q}))| \leq b_j(X; \mathbb{Q})$. This proves (3). □

The following is a particular case of Theorem 3.1 in [16].

**Lemma 2.5.** Let $\Gamma$ be cyclic and of primer order $p$. Assume that $\Gamma$ acts on a manifold $X$ in a CT way. Then

$$\sum_j b_j(X^\Gamma; \mathbb{F}_p) \leq \sum_j b_j(X; \mathbb{F}_p).$$

**Proof.** This is a classical result, which can be proved using localization in $\mathbb{Z}/p\mathbb{Z}$-equivariant cohomology. Alternatively, using a $\Gamma$-regular triangulation (e.g. as in the proof of Lemma 3.2 below), it follows from Theorem 4.1 in [11, Chapter III]. □
2.5. **Fixed point set of an orientation preserving finite order diffeomorphism.** Let a group \( G \) act smoothly and effectively on a connected manifold \( X \). For any \( g \in G \) the fixed point set \( X^g \) is a (non-necessarily connected) neat submanifold of \( X \) (see Subsection 2.2). The following result is a consequence of Lemma 2.2 and the fact that for any \( A \in \text{SO}(n, \mathbb{R}) \) the difference \( n - \dim \text{Ker}(A - \text{Id}) \) is even.

**Lemma 2.6.** Suppose that \( X \) is oriented and that the action of \( G \) preserves the orientation. For any \( \gamma \in G \), any connected component of the fixed point set \( X^\gamma \) has even codimension in \( X \).

2.6. **An arithmetic lemma.** The following is Lemma 7.10 in [16] (or Lemma 4.1 in [2]). The proof is an easy exercise.

**Lemma 2.7.** There exists a function \( C_\Delta : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) with the following property. Suppose that \( d, e_1, \ldots, e_l, a \) are positive integers satisfying: \( e_1 \geq \cdots \geq e_l \), each \( e_j \) divides \( d \), and \( de_1^{-1} + \cdots + de_l^{-1} - 1 = dta^{-1} \) for some integer \( t \). Then \( e_1 \geq d/C_\Delta(l, a) \).

3. **Pseudofree actions: proof of Theorem 1.2**

3.1. **The singular set and its projection to the orbit space.** In this and the next subsection we consider an arbitrary action of a (finite) group \( G \) on a (compact) manifold \( X \), so pseudofreeness is not essential here. Let \( S_X \subset X \) the singular set \( (\mathcal{I}) \) of the action of \( G \), let \( \pi : X \to Y := X/G \) denote the projection to the orbit space, and let \( S_Y := \pi(S_X) \).

**Lemma 3.1.** The cohomologies of the spaces \( Y, S_X \) and \( S_Y \) are finitely generated abelian groups, so \( \chi(Y), \chi(S_X) \) and \( \chi(S_Y) \) are well defined. Furthermore, we have

\[
\chi(X) - \chi(S_X) = |G|(\chi(Y) - \chi(S_Y)).
\]

**Proof.** Let \( (\mathcal{C}, \phi) \) be a \( G \)-regular triangulation of \( X \). Since \( \mathcal{C} \) is \( G \)-regular, \( \mathcal{C}/G \) is a simplicial complex, and the homeomorphism \( \phi : X \to |\mathcal{C}| \) descends to a homeomorphism \( \phi_Y : Y \to |\mathcal{C}/G| \) (here we use the homeomorphism \( |\mathcal{C}|/G \cong |\mathcal{C}/G| \) described at the end of Section 1 in [11], Chapter III). Hence, \( H^*(Y; \mathbb{Z}) \) is a finitely generated abelian group and so \( \chi(Y) \) is well defined. Let \( \mathcal{C}' = \{ \sigma \in \mathcal{C} \mid G_\sigma \neq \{1\} \} \). The regularity of \( \mathcal{C} \) also implies that \( \phi(S_X) = |\mathcal{C}'| \) and \( \phi_Y(S_Y) = |\mathcal{C}'/G| \), which imply that \( \chi(S_X) \) and \( \chi(S_Y) \) are well defined. Since Euler characteristics can be computed counting simplices in triangulations, we have

\[
\chi(X) - \chi(S_X) = \sum_{\sigma \in \mathcal{C} \setminus \mathcal{C}'} (-1)^{\dim \sigma}, \quad \chi(Y) - \chi(S_Y) = \sum_{[\sigma] \in (\mathcal{C}/G) \setminus (\mathcal{C}'/G)} (-1)^{\dim \sigma}.
\]

Since \( G \) acts freely on \( \mathcal{C} \setminus \mathcal{C}' \) (and, of course, preserving dimensions), we have

\[
\sum_{\sigma \in \mathcal{C} \setminus \mathcal{C}'} (-1)^{\dim \sigma} = |G| \sum_{[\sigma] \in (\mathcal{C}/G) \setminus (\mathcal{C}'/G)} (-1)^{\dim \sigma},
\]

\[
= |G| \left( \sum_{[\sigma] \in (\mathcal{C}/G) \setminus (\mathcal{C}'/G)} (-1)^{\dim \sigma} \right),
\]
which proves the lemma. 

3.2. Rational cohomology of the quotient space.

Lemma 3.2. The morphism

$$H_*(X; \mathbb{Q})^G \rightarrow H_*(X/G; \mathbb{Q})$$

induced by the quotient map \(X \rightarrow X/G\) is an isomorphism.

Proof. Take a \(G\)-regular triangulation \((\mathcal{C}, \phi)\) of \(X\). Since \(\mathcal{C}\) is \(G\)-regular, \(\mathcal{C}/G\) is a simplicial complex, and \(\phi\) descends to a homeomorphism \(\psi : X/G \rightarrow |\mathcal{C}/G| \simeq |\mathcal{C}|/G\), in such a way that the following diagram (in which vertical arrows are projections) is commutative

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & |\mathcal{C}| \\
| & \downarrow \pi & \\
X/G & \xrightarrow{\psi} & |\mathcal{C}/G|
\end{array}
\]

Hence it suffices to prove that \(\pi\) induces an isomorphism \(H_*(\mathcal{C}; \mathbb{Q})^G \xrightarrow{\simeq} H_*(\mathcal{C}/G; \mathbb{Q})\) in simplicial homology with rational coefficients. This is a particular case of Theorem 2.4 in [1, Chapter III] (the proof is based on a standard averaging argument).

3.3. Proof of Theorem 1.2

By Lemma 2.3 it suffices to consider CT actions. Let a group \(G\) act smoothly on a connected manifold \(X\) satisfying \(\chi(X) \neq 0\). Suppose furthermore that the action is effective, pseudofree, and trivial on cohomology. By Lemma 2.4, for any \(\gamma \in G \setminus \{1\}\) the set \(X^\gamma\) consists of \(\chi(X)\) points. This implies, if \(\chi(X) < 0\), that \(G = \{1\}\), so Theorem 1.2 is true in this case.

Let us, for the rest of the proof, assume that \(\chi := \chi(X)\) is positive. Denote for convenience \(d = |G|\). Since \(S_X = \bigcup_{\gamma \in G \setminus \{1\}} X^\gamma\),

\[|S_X| \leq (d - 1)\chi.\]

By Lemma 3.2 we have \(\chi(Y) = \chi\). Lemma 3.1 gives

\[|S_Y| = \frac{(d - 1)\chi + |S_X|}{d} \leq \frac{2(d - 1)\chi}{d} \leq 2\chi.\]

This implies that the number \(r\) of \(G\)-orbits in \(S_X\) is at most \(2\chi\). Let \(d/a_1, \ldots, d/a_r\) be the number of elements of the \(G\)-orbits in \(S_X\), and assume that \(a_1 \geq \cdots \geq a_r\). Then \(|S_X| = \sum d/a_j\), so Lemma 3.1 implies that \(\chi - (d/a_1^{-1} + \cdots + d/a_r^{-1}) = d(\chi - r)\), which is equivalent to

\[
\frac{d}{\chi a_1} + \cdots + \frac{d}{\chi a_r} - 1 = \frac{d(r - \chi)}{\chi}.
\]
Lemma 2.7 implies that $\chi a_1 \geq d/C_\Delta(r, \chi)$ for some universal function $C_\Delta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Since $r$ is bounded, we deduce that there exists some constant $C_f$, depending only on $X$, with the property that $S_X$ contains a point $x$ satisfying $[G : G_x] \leq C_f$.

By (2) in Lemma 2.2, there is an injective morphism $G_x \hookrightarrow \text{GL}(T_x X)$, so by Jordan’s Theorem [21] there is an abelian subgroup $G_a \subseteq G_x$ satisfying $[G_x : G_a] \leq C_{\text{Jor}}(\dim X)$. By Theorem 2.2 in [16], there exists a subgroup $G_b \subseteq G_a$ which can be generated by at most $[\dim X/2]$ elements and which satisfies $[G_a : G_b] \leq C'(\dim X)$.

Let $\gamma \in G_b$ be any nontrivial element. Then $X^{G_b} \subseteq X^\gamma$, and $X^\gamma$ consists of $\chi$ points. Let $\langle \gamma \rangle \subseteq G_b$ be the subgroup generated by $\gamma$. Then $G_b/\langle \gamma \rangle$ acts on $X^\gamma$, so there is a subgroup $\overline{A} \subseteq G_b/\langle \gamma \rangle$ satisfying $[G_b/\langle \gamma \rangle : \overline{A}] \leq \chi!$ whose action on $X^\gamma$ is trivial. Let $A \subseteq G_b$ be the preimage of $\overline{A}$ by the projection map $G_b \to G_b/\langle \gamma \rangle$. Then $X^\gamma \subseteq X^A$ because $\overline{A}$ acts trivially on $X^\gamma$, but we also have $X^A \subseteq X^\gamma$ because $\langle \gamma \rangle \subseteq A$. It follows that $X^A = X^\gamma$, hence $\chi(X^A) = \chi$. Since $A$ is a subgroup of an abelian subgroup which can be generated by at most $[\dim X/2]$ elements, $A$ can also be generated by at most $[\dim X/2]$ elements. We have 

$$[G : A] \leq C_f C(\dim X) C'(\dim X) \chi!,$$

and the right hand side only depends on $X$, so the proof of Theorem 1.2 is complete.

4. Results on surfaces

**Lemma 4.1.** Let $Y$ be either the circle or a closed interval. Let a group $G$ act nontrivially on $Y$. We have $|Y^G| \leq b_0(Y; \mathbb{Q}) + b_1(Y; \mathbb{Q})$.

**Proof.** Let $g \in G$ be an element acting nontrivially on $Y$. Then $Y^g$ is a finite set of points, and $|Y^g| \leq b_0(Y; \mathbb{Q}) + b_1(Y; \mathbb{Q})$ by (3). Since $Y^G \subseteq Y^g$, the result follows. \qed

**Lemma 4.2.** Let $\Sigma$ be a compact connected surface. There exists a constant $C_F(\Sigma)$ such that for any abelian group $A$ acting on $\Sigma$ the number of connected components of $\Sigma^A$ is at most $C_F(\Sigma)$.

**Proof.** It clearly suffices to consider nontrivial actions. So let $A$ be an abelian group acting on $\Sigma$ and assume that there is an element $a \in A$ acting nontrivially on $\Sigma$. We distinguish two possibilities.

If all connected components of $\Sigma^a$ are zero dimensional then, by (3), $|\Sigma^a| \leq b(\Sigma; \mathbb{Q}) := \sum_{j=0}^{\dim \Sigma} b_j(\Sigma; \mathbb{Q})$. Since $\Sigma^A \subseteq \Sigma^a$, the result follows immediately.

Now assume that $\Sigma^a$ contains some one-dimensional component. Any such component is (diffeomorphic to) either a circle or a closed interval. For $j = 0, 1$ let $\Sigma_j^a \subseteq \Sigma^a$ denote the union of the connected components whose Euler characteristic is $j$. The estimate (3) gives $|\pi_0(\Sigma_j^a)| \leq b(\Sigma; \mathbb{Q})$. Let us now bound $|\pi_0(\Sigma_0^a)|$, which is equal to the number of circles in $\Sigma^a$. The fact that $\Sigma^a$ has a codimension one connected component implies, by
Lemma 2.2. Let \( a \) have order 2. Then \( \langle a \rangle = \{1, a\} \). Then \( \Sigma' := \Sigma/\langle a \rangle \) is a surface with corners, so it is homeomorphic to a surface with boundary. We may bound
\[
\chi(\Sigma') = (\chi(\Sigma) + |\pi_0(\Sigma'_a)|)/2 \geq \chi(\Sigma)/2
\]
(this can be proved using an \( A \)-regular triangulation on \( \Sigma \) and computing Euler characteristics in terms of counting simplices). As a topological surface, \( \Sigma' \) is the result of removing in a compact connected surface \( S \) without boundary a number of disjoint open discs; \( \chi(\Sigma') \) is equal to \( \chi(S) \) minus the number of discs, and the latter can be identified with \( |\pi_0(\partial \Sigma')| \). By the classification of compact connected surfaces we have \( \chi(\Sigma) \leq 2 \); this gives \( \chi(\Sigma') \leq 2 - |\pi_0(\partial \Sigma')| \) or, equivalently,
\[
|\pi_0(\partial \Sigma')| \leq 2 - \chi(\Sigma')
\]
On the other hand, each connected component of \( \Sigma^a \) which is a circle contributes to a connected component of \( \partial \Sigma' \). We deduce that \( |\pi_0(\Sigma_a)| \leq 2 - \chi(\Sigma)/2 \).

To complete the argument in this case, note that \( \Sigma^A \subseteq \Sigma^a \). This implies that \( \Sigma^A \) contains at most as many one-dimensional connected components as \( \Sigma^a \), so we only need to bound the number of zero dimensional connected components (i.e., the isolated points) of \( \Sigma^A \). Each isolated point in \( \Sigma^A \) is either an isolated point in \( \Sigma^a \) or belongs to a one-dimensional connected component of \( \Sigma^a \). Since we have a bound on \( |\pi_0(\Sigma^a)| \), it suffices to bound uniformly the number of isolated points in \( \Sigma^A \) which can belong to a given one-dimensional component of \( \Sigma^a \). Now, if \( Y \subseteq \Sigma^a \) is one such component and \( Y \) contains an isolated point of \( \Sigma^A \), then the action of \( A \) on \( \Sigma^a \) (which is naturally induced from the action of \( A \) on \( \Sigma \), since \( A \) is abelian) fixes \( Y \). Furthermore, we can identify \( \Sigma^A \cap Y \) with the fixed point set of the action of \( A \) on \( Y \). By Lemma 4.1, the latter contains at most 2 points, so the proof is now complete.

Lemma 4.3. For any compact connected surface \( \Sigma \) there is a constant \( C_G(\Sigma) \) with the property that any finite group acting effectively on \( \Sigma \) has an abelian subgroup of index at most \( C_G(\Sigma) \).

Proof. Suppose first that \( \partial \Sigma \) is empty. If \( \Sigma \) is orientable, then the lemma is Theorem 1.3 in \cite{15} (if furthermore \( \chi(\Sigma) \neq 0 \) then it also follows from Theorem 1.2 and Lemma 2.6 of the present paper). If \( \Sigma \) is not orientable, then the arguments of Section 2.3 in \cite{15} allow to deduce the lemma from the orientable case. Now suppose that \( \partial \Sigma \) is nonempty, say with \( k \) connected components. Let a finite group \( G \) act on \( \Sigma \). Replacing \( G \) by a subgroup of index at most \( k \), we can assume that \( G \) fixes one connected component \( Y \subseteq \partial \Sigma \). Considering the restriction of the action to \( Y \) we get a morphism of groups \( G \to \text{Diff}(Y) \) which we claim to be injective. This follows from the fact that a finite order diffeomorphism of \( \Sigma \) which is the identity on \( Y \) is automatically the identity on the whole \( \Sigma \), which in turn is a consequence of (1.b) in Lemma 2.2. So to finish the proof we need to prove that a finite subgroup of \( \text{Diff}(S^1) \) has an abelian subgroup of uniformly bounded index. This the simplest case of Theorem 1.4 in \cite{15}, but it can also be proved
directly observing that, since all metrics in \( S^1 \) are isometric up to rescaling, choosing an invariant metric on \( S^1 \) gives an embedding of the group in a dihedral group. \( \square \)

**Lemma 4.4.** For any compact connected surface \( \Sigma \) there exists a constant \( C_\chi(\Sigma) \) with the property that if an abelian group \( A \) acts on \( \Sigma \) then there is an abelian subgroup \( A_0 \subseteq A \) satisfying \( [A : A_0] \leq C_\chi(\Sigma) \) and \( \chi(\Sigma^{A_0}) = \chi(\Sigma) \).

*Proof.* Let \( \Sigma \) be a compact connected surface, and let an abelian group \( A \) act on \( \Sigma \). By Lemma 2.3 there exists a subgroup \( A' \subseteq A \) whose action on \( \Sigma \) is CT and such that \([A : A']\) is not bigger than a constant depending only on \( \Sigma \). If the action of \( A' \) on \( \Sigma \) is trivial, then we set \( A_0 := A' \) and we are done. Otherwise, there exists some \( a \in A' \) acting nontrivially on \( \Sigma \). By Lemma 2.3 there exists a subgroup \( A_0 \subseteq A \) whose action on \( \Sigma^{A_0} \) fixes the connected components. Define \( A_0 \subseteq A'' \) as the subgroup of elements whose action on the one-dimensional components of \( \Sigma^{a_0} \) is by orientation preserving diffeomorphisms. Since \( |\pi_0(\Sigma^{a_0})| \leq C := C_F(\Sigma) \), where \( C_F(\Sigma) \) is the constant given by Lemma 4.2 for \( \Sigma \). So there is a subgroup \( A'' \subseteq A' \) of index \([A' : A''] \leq C! \) whose action on \( \Sigma^{a_0} \) fixes the connected components. Define \( A_0 \subseteq A'' \) as the subgroup of elements whose action on the one-dimensional components of \( \Sigma^{a_0} \) is by orientation preserving diffeomorphisms. Since \( |\pi_0(\Sigma^{a_0})| \leq C := C_F(\Sigma) \), we have \([A'' : A_0] \leq C! \). We claim that \( \chi(\Sigma^{A_0}) = \chi(\Sigma^{a_0}) \). We have \( \Sigma^{A_0} \subseteq \Sigma^{a_0} \), because \( a \in A_0 \), so it is enough to prove that \( \sum_Y \chi(\Sigma^{A_0} \cap Y) = \sum_Y \chi(Y) \), where \( Y \) runs over the set of connected components of \( \Sigma^{a_0} \). Each isolated point in \( \Sigma^{a_0} \) belongs to \( \Sigma^{A_0} \), because the action of \( A_0 \) preserves the connected components of \( \Sigma^{a_0} \). If \( Y \subseteq \Sigma^{a_0} \) is a circle then, since \( A_0 \) acts preserving the orientation of \( Y \), for any element \( b \in A_0 \) the fixed point set \( Y^b \) is either \( Y \) or the empty set. This implies that \( \Sigma^{A_0} \cap Y \) is either \( \emptyset \) or \( Y \), so \( \chi(\Sigma^{A_0} \cap Y) = \chi(Y) \) in any case. Finally, if \( Y \subseteq \Sigma^{a_0} \) is a closed interval then, since \( A_0 \) acts preserving the orientation of \( Y \), for any element \( b \in A_0 \) we have \( Y^b = Y \). Hence, \( \Sigma^{A_0} \cap Y = Y \), so \( \chi(\Sigma^{A_0} \cap Y) = \chi(Y) \). This completes the proof that \( \chi(\Sigma^{A_0}) = \chi(\Sigma^{a_0}) \).

\( \square \)

**Lemma 4.5.** For any compact connected surface \( \Sigma \) there exists a constant \( C_{\chi}(\Sigma) \) with the property that if a finite group \( G \) acts effectively on \( \Sigma \) then there is an abelian subgroup \( A \subseteq G \) satisfying \([G : A] \leq C_{\chi}(\Sigma) \) and \( \chi(\Sigma^A) = \chi(\Sigma) \).

*Proof.* Combine Lemmas 4.3 and 4.4. \( \square \)

5. \( C \)-rigid abelian group actions

### 5.1. Bounding the number of components of fixed point sets.

For any space \( Y \) with finitely generated homology we define

\[
\begin{align*}
    b_+(Y) := \sum_{j \geq 0} \max \{ b_j(Y; \mathbb{F}_p) \mid p \text{ prime} \}, \\
    b_-(Y) := \sum_{j \geq 0} \min \{ b_j(Y; \mathbb{F}_p) \mid p \text{ prime} \}.
\end{align*}
\]

For any 4-dimensional oriented manifold \( X \) we denote by \( S(X) \) any collection of compact connected surfaces such that any compact connected surface \( \Sigma \) satisfying \( b_-(\Sigma) \leq b_+(X) \) is diffeomorphic to exactly one element of \( S \). By the classification theorem of compact connected surfaces with boundary, \( S(X) \) is a finite set.
**Lemma 5.1.** Let $X$ be a 4-dimensional connected oriented manifold $X$, and let $H$ be a group acting nontrivially on $X$ preserving the orientation. The connected components of $X^H$ are neat submanifolds of dimensions 0, 1 or 2. Any two-dimensional connected component of $X^H$ is diffeomorphic to an element of $S(X)$.

**Proof.** That $X^H$ is a (non-necessarily connected) neat submanifold of $X$ follows from (1.b) in Lemma 2.2. By Lemma 2.6, for any $h \in H$ the connected components of $X^h$ are zero or two-dimensional; hence, the dimension of any connected component of $X^H$ is at most two. To prove the last statement, suppose that $Y \subset X^H$ is a two-dimensional connected component. Let $h \in H$ be an element acting nontrivially; replacing $h$ by a power $h^r$ we may assume that the diffeomorphism of $X$ induced by the action of $h$ has primer order, say $p$. Since the connected components of $X^h$ have dimension at most 2, the inclusion $X^H \subset X^h$ implies that $Y$ is a connected component of $X^h$. Then, by Lemma 2.5, $b^-(Y) \leq b^+(X)$, so $Y$ is diffeomorphic to an element of $S(X)$. \[\square\]

**Lemma 5.2.** For any 4-dimensional oriented manifold $X$ there exists a constant $C_F(X)$ such that for any abelian group $A$ acting on $X$ in a CT way and preserving the orientation the number of connected components of $X^A$ is at most $C_F(X)$.

**Proof.** Let $X$ be a 4-dimensional oriented manifold. Define

$$C_F(X, \Sigma) := \sup \{C_F(\Sigma) \mid \Sigma \in S(X)\},$$

where $C_F(\Sigma)$ is defined in Lemma 4.2.

Let $A$ be a group acting on $X$ in a CT way and preserving the orientation. If the action of $A$ is trivial then there is nothing to prove. Otherwise, let $a \in A$ be an element acting nontrivially on $X$ through a diffeomorphism of order $p$, where $p$ is any prime. By Lemma 2.5 we have

$$\sum_j b_j(X^a; F_p) \leq \sum_j b_j(X; F_p) \leq b_+(X),$$

so $X^a$ has at most $b_+(X)$ connected components, and each connected component $Y \subseteq X^a$ satisfies $b_-(Y) \leq b_+(X)$. Since $X^A \subseteq X^a$, it suffices to prove that there is some constant $C$ depending only on $X$ such that that each connected component of $X^a$ contains at most $C$ connected components of $X^A$. By Lemma 2.6 the connected components of $X^a$ are either points or surfaces. Of course each isolated point in $X^a$ contains at most one connected component of $X^A$. Now suppose that $Y \subseteq X^a$ is a surface. Then $Y$ is diffeomorphic to some element of $S(X)$. If $Y \cap X^a = \emptyset$, then there is nothing to be proved. Otherwise, the natural action of $A$ on $X^a$ (which exists because $A$ is abelian, so its action on $X$ preserves $X^a$) leaves $Y$ fixed. This means that $A_a := A/\langle a \rangle$ (where $\langle a \rangle \subseteq A$ is the subgroup generated by $a$) acts on $Y$, and in fact we can identify $Y \cap X^A$ with $Y^{A_a}$. By Lemma 1.2 $Y^{A_a}$ has at most $C_F(X, \Sigma)$ connected components, so the proof is now complete. \[\square\]
Lemma 5.3. For any 4-dimensional oriented manifold $X$ and any integer $k$ there exists a constant $C_M(X, k)$ such that if $\emptyset \neq Y_1 \subsetneq Y_2 \subsetneq \cdots \subsetneq Y_r$ are neat submanifolds of $X$ (in the sense of Section 1.4 in [7]) which satisfy $|\pi_0(Y_j)| \leq k$ for each $j$, then $r \leq C_M(X, k)$.

Proof. This is a particular case of Lemma 4.1 in [16] (the proof is easy and elementary). \qed

5.2. Definition and basic results on $C$-rigid abelian group actions. Let $A$ be an abelian group acting on a manifold $X$ in a CT way and let $C \geq 1$ be an integer. We say that the action of $A$ on $X$ (or, abusing language, the group $A$) is $C$-rigid if for any subgroup $A_0 \subseteq A$ satisfying $[A : A_0] \leq C$ we have $X^{A_0} = X^A$.

Lemma 5.4. Let $X$ be a compact connected 4-manifold. For any $C$ there exists a constant $\Lambda(C)$, depending only on $X$ and $C$, such that any abelian group $A$ acting on $X$ in a CT way has a subgroup of index at most $\Lambda(C)$ whose action on $X$ is $C$-rigid.

Proof. Let $C' := C_M(X, C_F(X))$. We prove that $\Lambda(C) := C^{C'-1}$ has the stated property. Let $A$ be an abelian group acting on $X$ in a CT way and assume by contradiction that no subgroup of $A$ of index at most $\Lambda(C)$ is $C$-rigid. Then we may construct recursively a sequence of subgroups $A := A_0 \supset A_1 \supset \cdots \supset A_{C'}$ satisfying $[A_i : A_{i+1}] \leq C$ and $X^{A_i} \subsetneq X^{A_{i+1}}$ for each $i$; indeed, once $A_0, A_1, \ldots, A_i, i < C'$, have been constructed we have $[A : A_i] \leq C^i \leq C^{C'-1}$ so by our initial assumption on $A$ the group $A_i$ is not $C$-rigid; hence, we may pick a subgroup $A_{i+1} \subset A_i$ such that $[A_i : A_{i+1}] \leq C$ and $X^{A_i} \subsetneq X^{A_{i+1}}$. By Lemma 5.2 each $X^{A_i}$ has at most $C_F(X)$ connected components, so we obtain a contradiction with Lemma 5.3. \qed

Lemma 5.5. Let $X$ be a compact connected 4-manifold. There exists a constant $C_\chi(X)$ such that any abelian group $A$ acting on $X$ in a CT and $C_\chi(X)$-rigid way satisfies $\chi(X^A) = \chi(X)$.

Proof. Let $C_\chi(X, \Sigma) := \sup\{C_\chi(\Sigma) \mid \Sigma \in S(X)\}$ and let $C_\chi(X) := b_+(X)!C_\chi(X, \Sigma)$. We claim that this value of $C_\chi(X)$ has the stated property. Indeed, suppose that an abelian group $A$ acts on $X$ in a CT and $C_\chi(X)$-rigid way. If the action of $A$ is trivial, then there is nothing to prove. Otherwise there exists some $a \in A$ acting nontrivially on $X$ through a diffeomorphism of prime order. Then, by Lemma 2.3 $X^a$ contains at most $b_+(X)$ connected components, and by Lemma 2.6 each connected is either a point or a surface, which belongs to $S(X)$ (see the proof of Lemma 5.2). By Lemma 2.4 we have $\chi(X^a) = \chi(X)$.

Since $A$ is abelian, its action on $X$ preserves $X^a$. There is a subgroup $A' \subseteq A$ of index at most $b(X)!$ whose action on $X^a$ preserves the connected components. Since $A$ is $C_\chi(X)$-rigid, we have $X^{A'} = X^A$, so what we want to prove is equivalent to $\chi(X^{A'}) = \chi(X^a)$. Furthermore, $a \in A'$, so $X^{A'} \subseteq X^a$. To prove that $\chi(X^{A'}) = \chi(X^a)$ it suffices to check that for any connected component $Y \subseteq X^a$ we have $\chi(X^{A'} \cap Y) = \chi(Y)$. If $Y$
is an isolated point of \( X^a \), then \( Y \) is also contained in \( X^{A'} \), because \( A' \) preserves the connected components of \( X^a \), so \( \chi(X^{A'} \cap Y) = \chi(Y) \) trivially in this case. Now suppose that \( Y \) is a surface. By Lemma 4.3 there exists a subgroup \( A'' \subseteq A' \) of index at most \( C_\chi(Y) \leq C_\chi(X, \Sigma) \) so that \( \chi(Y^{A''}) = \chi(Y) \). Since \([A : A''] = [A : A'][A' : A''] \leq C_\chi(X)\), and \( A \) is \( C_\chi(X) \)-rigid, we have \( X^{A''} = X^{A'} = X^A \), so \( Y^{A''} = Y^{A'} \cap Y = X^{A'} \cap Y \). Hence, \( \chi(X^{A'} \cap Y) = \chi(Y) \) as we wanted to prove. \( \square \)

**Lemma 5.6.** For any \( 4C_{\text{Jor}}(4) \)-rigid action of an abelian group \( A \) on a 4-manifold \( X \), all connected components of \( X^A \) are even dimensional.

**Proof.** Suppose that \( A \) is an abelian group acting on a 4-manifold \( X \), and that the action is \( 4C_{\text{Jor}}(4) \)-rigid (recall that \( C_{\text{Jor}}(4) \) is defined in Theorem 2.1). We assume that \( X^A \) is nonempty. Let \( x \in X^A \) be any point. By (2) in Lemma 2.2 the isotropy group \( G_x \) injects in \( GL(T_x X) \), so by Jordan’s Theorem 2.1 there exists an abelian subgroup \( A_x \subseteq G_x \) of index at most \( C_{\text{Jor}}(4) \). By simultaneous diagonalization there is an \( A_x \)-invariant splitting \( T_x X = L_1 \oplus L_2 \), where \( L_1, L_2 \subseteq T_x X \) are two-dimensional subspaces. There is a subgroup \( A_x' \subseteq A_x \) of index at most 4 whose action on each \( L_j \) is orientation preserving, hence given by rotations. This implies that for any subgroup \( B \subseteq A_x' \) the fixed point set \( (T_x X)^B \) is of the four spaces \( 0, L_1, L_2 \) or \( T_x X \), all of which are even dimensional. Now, \( A \) is a subgroup of \( G_x \), so \([A : A \cap A_x'] \leq 4C_{\text{Jor}}(4) \), and since \( A \) is \( 4C_{\text{Jor}}(4) \)-rigid we have \( X^{A \cap A_x'} = X^A \). In particular, \( T_x (X^A) = T_x (X^{A \cap A_x'}) \) and the latter is equal, by (1) in Lemma 2.2, to \( (T_x X)^{A \cap A_x'} \). Since obviously \( A \cap A_x' \subseteq A_x' \), the space \( (T_x X)^{A \cap A_x'} \) is even dimensional. \( \square \)

5.3. Further results on \( C \)-rigid abelian group actions. Assume, throughout this subsection, that \( X \) is a compact oriented connected 4-manifold satisfying \( \chi(X) \neq 0 \).

As usual we denote the centralizer of a subgroup \( H \subseteq G \) as \( Z_G(H) \).

**Lemma 5.7.** There exists a constant \( C_Z(X) \) such that if a group \( G \) acts on \( X \) in a \( CT \) way and preserving the orientation, and two \( C_Z(X) \)-rigid abelian groups \( A_1, A_2 \subseteq G \) satisfy \( Z_G(A_1) \cap Z_G(A_2) \neq \{1\} \), then \( X^{A_1} \cap X^{A_2} \neq \emptyset \).

**Proof.** Let \( C_Z(X) := C_F(X) \sup \{ C_G(\Sigma) \mid \Sigma \in S(X) \} \), where \( C_G(\Sigma) \) is defined in Lemma 4.5. Suppose that \( A_1, A_2 \subseteq G \) are \( C_Z(X) \)-rigid abelian subgroups and that \( G \) acts on \( X \) satisfying the conditions of the statement. Let \( g \in Z_G(A_1) \cap Z_G(A_2) \) be a nontrivial element. Since \( g \) centralizes both \( A_1 \) an \( A_2 \), the actions of \( A_1 \) and \( A_2 \) fix \( X^g \). By Lemma 2.1 we have \( \chi(X^g) = \chi(X) \neq 0 \), so there is a connected component \( Y \subseteq X^g \) such that \( \chi(Y) \neq 0 \). By Lemma 5.2 \( X^g \) has at most \( C_F(X) \) connected components. Hence, for \( j = 1, 2 \) there exist subgroups \( A_j' \subseteq A_j \) which fix \( Y \) and which satisfy \([A_j : A_j'] \leq C_F(X) \). By Lemma 2.6 \( Y \) can be either a point or a surface. If \( Y \) is a point, then \( Y \subseteq X^{A_j'} \cap X^{A_j} \). Since \( C_Z(X) \geq C_F(X) \), \( X^{A_j'} = X^{A_j} \) for \( j = 1, 2 \), so we are done. Suppose now that \( Y \) is a surface. Let \( H \subseteq G \) be the group generated by \( A_1' \) and \( A_2' \). Then \( H \) acts on \( Y \),
so by Lemma 5.3 there is an abelian subgroup $A \subseteq H$ such that $[H : A] \leq C_G \chi(Y)$ and $\chi(Y^A) = \chi(Y) \neq 0$, so $Y^A$ is nonempty. Let $A''_j := A'_j \cap H$. Then $[A_j : A''_j] \leq C_F(X) C_G \chi(Y) \leq C_Z(X)$, because by Lemma 5.1 $Y$ is diffeomorphic to an element of $S(X)$. It follows that $X^{A_j} = X^{A''_j}$. But both $X^{A''_j}$ contain $Y^A$, so we are done. □

**Lemma 5.8.** For any $C$ there exists a constant $I(C)$, depending only on $X$ and $C$, such that if a group $G$ acts on $X$ in a CT way and preserving the orientation and $A_1, \ldots, A_s \subseteq G$ are $I(C)$-rigid abelian subgroups (with $s$ arbitrary) satisfying $X^{A_1} \cap \cdots \cap X^{A_s} \neq \emptyset$, then there is a $C$-rigid abelian subgroup $A \subseteq G$ such that $X^{A_1} \cap \cdots \cap X^{A_s} = X^A$.

**Proof.** Define $I(C) := C_{\text{Jor}}(4) \Lambda(C)$, where $C_{\text{Jor}}(4)$ is the constant given by Theorem 2.1. Suppose that $G$ acts on $X$ satisfying the hypothesis of the statement, and that $A_1, \ldots, A_s \subseteq G$ are $I(C)$-rigid abelian subgroups satisfying $X^{A_1} \cap \cdots \cap X^{A_s} \neq \emptyset$. Let $H \subseteq G$ be the subgroup generated by $A_1, \ldots, A_s$. Then $H$ fixes pointwise $X^{A_1} \cap \cdots \cap X^{A_s}$. Choose a point $x \in X^{A_1} \cap \cdots \cap X^{A_s}$. By (2) in Lemma 2.2 the derivative of the action gives an inclusion $H \hookrightarrow \text{GL}(T_x X) \cong \text{GL}(4, \mathbb{R})$. By Jordan’s Theorem 2.1 there exists an abelian subgroup $B \subseteq H$ satisfying $[H : B] \leq C_{\text{Jor}}(4)$. By Lemma 5.4 there is a $C$-rigid subgroup $A \subseteq B$ of index $[B : A] \leq \Lambda(C)$. We now prove that

$$X^{A_1} \cap \cdots \cap X^{A_s} = X^A. \tag{4}$$

Since $A \subseteq H$ and $H$ fixes pointwise the left hand side, we have $X^{A_1} \cap \cdots \cap X^{A_s} \subseteq X^A$. To prove the reverse inclusion, define $A'_j := A_j \cap A$ for every $j$. Since $A_j \subseteq H$ and $[H : A] \leq C_{\text{Jor}}(4) \Lambda(C)$, we have $[A_j : A'_j] \leq C_{\text{Jor}}(4) \Lambda(C) = I(C)$. Since each $A_j$ is $I(C)$-rigid, we have $X^{A_j} = X^{A'_j}$, and, since $A'_j \subseteq A$, we have $X^A \subseteq X^{A'_j}$ for every $j$. Putting everything together we get

$$X^A \subseteq X^{A'_1} \cap \cdots \cap X^{A'_s} = X^{A_1} \cap \cdots \cap X^{A_s},$$

so the proof of (4) is complete. □

**Lemma 5.9.** There exists a constant $C_t(X)$ such that if a group $G$ acts on $X$ in a CT way and preserving the orientation, and three $C_t(X)$-rigid abelian groups $A_1, A_2, A_3 \subseteq G$ satisfy $X^{A_1} \cap X^{A_3} \neq \emptyset$, $X^{A_2} \cap X^{A_3} \neq \emptyset$ and $X^{A_3} \neq X$, then $X^{A_1} \cap X^{A_2} \neq \emptyset$.

**Proof.** Let $C_t(X) := \max\{C_Z(X) C_{\text{Jor}}(4), C_{\text{Jor}}(4)^2\}$. Suppose that a group $G$ acts on $X$ satisfying the hypothesis in the statement. Let $A_1, A_2, A_3 \subseteq G$ be $C_t(X)$-rigid abelian subgroups satisfying

$$X^{A_1} \cap X^{A_3} \neq \emptyset, \quad X^{A_2} \cap X^{A_3} \neq \emptyset. \tag{5}$$

Let $H_j$ be the subgroup of $G$ generated by $A_j$ and $A_3$, where $j = 1, 2$. By an argument in the proof of Lemma 5.8 the hypothesis (5) imply (using Lemma 2.2) that we can identify $H_1$ and $H_2$ with (finite) subgroups of $\text{GL}(4, \mathbb{R})$, so by Jordan’s Theorem 2.1 there are abelian subgroups $H'_j \subseteq H_j$ satisfying $[H_j : H'_j] \leq C_{\text{Jor}}(4)$ for $j = 1, 2$. Let $A'_j := A_j \cap H'_j$
for \( j = 1, 2 \), and let \( A_3' := A_3 \cap H_1' \cap H_2' \). Then \( |A_j : A_j'| \leq C_{\text{Jor}}(4) \) for \( j = 1, 2 \), which imply that \( A_1' \) and \( A_2' \) are \( C_Z(X) \)-rigid, and \( |A_3 : A_3'| \leq C_{\text{Jor}}(4)^2 \). The latter implies that \( X^{A_1'} = X^{A_3} \neq X \), so there is some \( a \in A_3' \) which acts nontrivially on \( X \). By construction, \( a \) commutes with \( A_1' \) and \( A_2' \), and since \( A_1' \) and \( A_2' \) are \( C_Z(X) \)-rigid Lemma 5.7 implies that \( X^{A_1'} \cap X^{A_2'} \neq \emptyset \). Since obviously \( C_{\text{Jor}}(4) \geq 1 \), the estimate \( |A_j : A_j'| \leq C_{\text{Jor}}(4) \) and the fact that \( A_j \) is \( C_{\text{Jor}}(4)^2 \)-rigid imply that \( X^{A_j} = X^{A_j} \) for \( j = 1, 2 \), which in view of the preceeding conclusion implies that \( X^{A_1} \cap X^{A_2} \neq \emptyset \).

6. Proof of Theorem 1.1

By Lemma 2.3 it suffices to consider CT actions. Let \( X \) be a compact, oriented, connected 4-dimensional smooth manifold, possibly with boundary, satisfying \( \chi(X) \neq 0 \). Using Theorem 2.1 and Lemmas 5.5, 5.7, 5.8 and 5.9 we define

\[ C_1(X) := \max\{C_{\text{Z}}(X), C_{\chi}(X), 4C_{\text{Jor}}(4)\}, \quad C(X) := \max\{4, I(C_1(X)), C_t(X)\}. \]

Suppose that a group \( G \) acts effectively on \( X \) in a CT way. Let \( \mathcal{F} \) be the collection of nonempty subsets of \( X \) of the form \( X^A \) where \( A \) is a \( C(X) \)-rigid nontrivial abelian subgroup of \( G \). The action of \( G \) on \( X \) induces an action on \( \mathcal{F} \), since \( gX^A = X^{gAg^{-1}} \) for any \( g \in G \) and \( A \subseteq G \). Let \( \approx \) be the relation between elements of \( \mathcal{F} \) such that \( F \approx F' \) if and only if \( F \cap F' \neq \emptyset \). By Lemma 5.9, \( \approx \) is an equivalence relation. Let \( \mathcal{H} := \mathcal{F}/\approx \).

The action of \( G \) on \( \mathcal{F} \) preserves the relation \( \approx \), so it descends to an action on \( \mathcal{H} \).

For any \( H \in \mathcal{H} \) define \( X_H := \bigcap_{F \in \mathcal{F}, [F] = H} F \). Then \( \{X_H \mid H \in \mathcal{H}\} \) is a collection of disjoint (non necessarily connected) submanifolds of \( X \). Since each \( F \) is of the form \( X^A \) with \( A \) an \( I(C_1(X)) \)-rigid abelian group, Lemma 5.5 implies that, for any \( H \in \mathcal{H} \), \( X_H \) is of the form \( X^A \) for some \( C_1(X) \)-rigid abelian subgroup \( A \subseteq G \); this has three consequences: \( X_H \) has at most \( C_F(X) \) connected components (Lemma 5.2); \( \chi(X_H) = \chi(X) \) (Lemma 5.5); and all connected components of \( X_H \) are even-dimensional (Lemma 5.6).

As usual, we denote by \( G_H \subseteq G \) the isotropy group of an element \( H \in \mathcal{H} \). Define also

\[ G(H) \subset G \]

as the subset consisting of the elements \( g \in G \) such that there is at least one connected component of \( X_H \) which is a connected component of \( X^g \).

Define

\[ C_{G\chi}(X, \Sigma) := \sup\{C_{G\chi}(\Sigma) \mid \Sigma \in \mathcal{S}(X)\} \]

and let also

\[ C_2(X) := C_F(X)C_{G\chi}(X, \Sigma)C_{\text{Jor}}(4)\Lambda(C_1(X)), \]

\[ C_3(X) := \max\{16C_{\text{Jor}}(4)C_F(X), 2C_2(X)\}. \]

**Lemma 6.1.** Let \( H \in \mathcal{H} \) be any element. If \( |G_H| > C_2(X) \), then we have \( |G(H)| \geq |G_H|/C_3(X) \).
Proof. Since $\chi(X_H) = \chi(X) \neq 0$, there is a connected component $Y \subseteq X_H$ with $\chi(Y) \neq 0$. Since $X_H$ has at most $C_F(X)$ connected components, there is a subgroup $G'_H \subseteq G_H$ whose action on $X_H$ fixes $Y$ and such that $[G_H : G'_H] \leq C_F(X)$. By Lemma 5.6, $Y$ is a manifold of dimension 0 or 2.

Suppose that $\dim Y = 0$, so $Y = \{y\}$ is a point which is fixed by the action of $G'_H$. As in the proof of Lemma 5.6, there is an abelian subgroup $A'_y \subseteq G_y$ of index at most $4C_{\text{Jor}}(4)$ and a $A'_y$-invariant splitting $T_yX = L_1 \oplus L_2$, where $L_1, L_2 \subseteq T_yX$ are two-dimensional subspaces, and $A'_y$ acts on each $L_j$ preserving the orientation. Let $\rho_j : A'_y \to \text{GL}(L_j)$ the morphism given by the action of $A'_y$ on $L_j$. We claim that $[A'_y : \text{Ker} \rho_j] \geq 4$ for $j = 1, 2$. Indeed, if we had say $[A'_y : \text{Ker} \rho_j] \leq 4$ for some $j$ then for any 4-rigid abelian subgroup $A \subseteq G$ we would have $L_j \subset T_yX^A$, by the same argument as in the proof of Lemma 5.6, which would imply $L_j \subset T_yX_H$, contradicting the fact that $y$ is an isolated point of $X_H$. So the claim has been proved. We deduce that

$$|A'_y \setminus (\text{Ker} \rho_1 \cup \text{Ker} \rho_2)| \geq |A'_y|/4 \geq |G_y|/(16C_{\text{Jor}}(4))$$

because $G'_H \subseteq G_y$

$$\geq |G_H|/(16C_{\text{Jor}}(4))C_F(X)).$$

Since for any $g \in A'_y \setminus (\text{Ker} \rho_1 \cup \text{Ker} \rho_2)$ the fixed point set $X^g$ contains $y$ as a fixed point set, we have $A'_y \setminus (\text{Ker} \rho_1 \cup \text{Ker} \rho_2) \subseteq G(H)$, so the lemma follows in this case without using the bound $|G_H| > C_2(X)$.

If $\dim Y = 2$ then, by Lemma 5.1, $Y$ is diffeomorphic to an element of $\mathcal{S}(X)$. By Lemma 4.5 there is a subgroup $G''_H \subseteq G'_H$ satisfying $[G'_H : G''_H] \leq C_{\text{G}\chi}(Y) \leq C_{\text{G}\chi}(X, \Sigma)$ and $\chi(Y^{G''_H}) = \chi(Y) \neq 0$. In particular, there exists some $y \in Y$ such that $G''_H \subseteq G_y$, so (by (1) of Lemma 2.2) we can identify $G''_H$ with a subgroup of $\text{GL}(T_yX)$. By Jordan's Theorem 2.1 and Lemma 5.1, there exists a $C(X)$-rigid abelian subgroup $A_y \subseteq G''_H$ satisfying $[G''_H : A_y] \leq C_{\text{Jor}}(4)\Lambda(C_1(X))$. Then $X^{A_y}$ is an element of $\mathcal{F}$ containing $y \in Y \subseteq X_H$, so $[X^{A_y}] \in H$, and it follows that $X_H \subseteq X^{A_y}$. In particular $Y \subseteq X^{A_y}$, which implies that for any $g \in A_y$ we have $Y \subseteq X^g$. By Lemma 2.6, this implies that $A_y \setminus \{1\} \subseteq G(H)$. Since we have

$$[G_H : A_y] \leq C_F(X)C_{\text{G}\chi}(X, \Sigma)C_{\text{Jor}}(4)\Lambda(C_1(X)) = C_2(X),$$

we deduce that if $|G_H| > C_2(X)$ then $|A_y \setminus \{1\}| \geq |G_H|/(2C_2(X))$, so the lemma also follows in this case. \[\square\]

Lemma 6.2. $|\mathcal{H}/G| \leq C_2 + C_3C_F(X)$.

Proof. Let $H_1, \ldots, H_s$ be the elements of $\mathcal{H}$. Choose for each $j$ a $C_1(X)$-rigid abelian subgroup $A_j \subseteq G$ such that $X_{H_j} = X^{A_j}$. Since $C_1(X) \geq C_2(X)$, Lemma 5.1 implies that $A_i \cap A_j = \{1\}$ for $i \neq j$, because $X_{H_i} \cap X_{H_j} = \emptyset$. Consequently, $|\mathcal{H}| = s \leq |G|$. Let

$$\mathcal{H}_\text{small} = \{H \in \mathcal{H} \mid |G_H| \leq C_2\}, \\
\mathcal{H}_\text{big} = \{H \in \mathcal{H} \mid |G_H| > C_2\}.$$


Both subsets $\mathcal{H}_{\text{small}}, \mathcal{H}_{\text{big}} \subseteq \mathcal{H}$ are $G$-invariant. Each $G$-orbit in $\mathcal{H}_{\text{small}}$ has at least $|G|/C_2$ elements, so the bound $|\mathcal{H}_{\text{small}}| \leq |\mathcal{H}| \leq |G|$ implies that $\mathcal{H}_{\text{small}}$ contains at most $C_2$ orbits, i.e., $|\mathcal{H}_{\text{small}}/G| \leq C_2$. To estimate the number of orbits in $\mathcal{H}_{\text{big}}$ we use the following:

$$|G| \cdot |\mathcal{H}_{\text{big}}/G| = \sum_{H \in \mathcal{H}_{\text{big}}} |G_H| \leq C_3 \sum_{H \in \mathcal{H}_{\text{big}}} |G(H)| \leq C_3 C_F(X)|G|.$$  

The equality follows from a simple counting argument, the first inequality follows from Lemma 6.1, and the second inequality follows from the fact that the submanifolds $\{X_H\}$ are disjoint and that for any $g \in G$ the number of connected components of $X^g$ is at most $C_F(X)$. Dividing both extremes by $|G|$ we deduce $|\mathcal{H}_{\text{big}}/G| \leq C_3 C_F(X)$ which combined with the estimate on $|\mathcal{H}_{\text{small}}/G|$ proves the lemma. □

For any $H \in \mathcal{H}$ let $Y_H = \bigcup_{F \in F, [F] = H} H$. By Lemma 5.8 and the inclusion-exclusion principle, we have

$$\chi(Y_H) = \chi(X).$$

Let $H_1, \ldots, H_r \in \mathcal{H}$ be representatives of the orbits of the action of $G$ on $\mathcal{H}$. Let $d = |G|$ and let $e_j = |G_{H_j}|$. Since for different $H, H' \in \mathcal{H}$ we have $Y_H \cap Y_{H'} = \emptyset$, we have

$$\chi\left( \bigcup_{F \in F} F \right) = \left( \frac{d}{e_1} \chi(Y_{H_1}) + \cdots + \frac{d}{e_r} \chi(Y_{H_r}) \right) = \chi(X) \left( \frac{d}{e_1} + \cdots + \frac{d}{e_r} \right).$$

Lemma 6.3. The difference $\chi(X) - \chi\left( \bigcup_{F \in F} F \right)$ is divisible by

$$\frac{d}{\text{GCD}(d, (C_{\text{Jor}}(4) \Lambda(C(X)))!).}$$

Proof. Let $x \in X$. If $|G_x| \geq C_{\text{Jor}}(4) \Lambda(C(X))$ then, by Jordan’s Theorem 2.1 and Lemma 5.4, there is a $C(X)$-rigid abelian subgroup $A \subseteq G_x$, so $x \in X^A \subseteq \bigcup_{F \in F} F$. So the isotropy group of any point in $X \setminus \bigcup_{F \in F} F$ has less than $C_{\text{Jor}}(4) \Lambda(C(X))$ elements. Now take a $G$-regular triangulation of $X$ (see Subsection 2.3). The fact that the triangulation is $G$-regular implies that the isotropy group of any simplex is contained in the isotropy group of any of its points. The difference $\chi(X) - \chi\left( \bigcup_{F \in F} F \right)$ can be computed as the alternate sum of numbers of simplexes of each dimension in the triangulation which are not contained in $\bigcup_{F \in F} F$. Grouping the simplexes in $G$-orbits and observing that by the previous observations each orbit of simplexes in $X \setminus \bigcup_{F \in F} F$ has size $d/e$ for some $e < C_{\text{Jor}}(4) \Lambda(C(X))$ dividing $d$ (so that $e$ is a divisor of $\text{GCD}(d, (C_{\text{Jor}}(4) \Lambda(C(X)))!))$, the result follows. □
Combining the previous lemma with (6) we obtain the following equality:
\[
\frac{d}{e_1} + \cdots + \frac{d}{e_r} - 1 = \frac{dt}{a},
\]
where \( t \) is an integer and \( a \) is a positive integer bounded above by a constant depending only on \( X \). By Lemma [6.2], \( r \) is also bounded above by a constant depending only on \( X \). Assuming that \( e_1 \geq \cdots \geq e_r \) (which we can, up to reordering), Lemma 2.4 gives
\[ |G_{H_i}| = e_1 \geq d/K \]
for some constant \( K \) depending only on \( X \). It follows that \([G : G_{H_1}] \leq K\).

By the arguments in the proof of Lemma 6.1, there is a subgroup \( G_2 \subseteq G_{H_1} \) satisfying \([G_{H_1} : G_2] \leq C_F(X)C_{G_X}(X, \Sigma) \) such that \( X_{H_2}^{G_2} \neq \emptyset \). By (2) in Lemma 2.2, for any \( x \in X_{H_2}^{G_2} \) we have an inclusion \( G_2 \hookrightarrow \text{GL}(T_xX) \simeq \text{GL}(4, \mathbb{R}) \). Jordan’s Theorem 2.1, Theorem 2.2 in [16], and Lemmas 5.4 and 5.5 imply the existence of an abelian subgroup \( A \subseteq G_2 \) of index \([G_2 : A] \leq C_{\text{Jor}}(4)\Lambda(C_1(X))C_4 \) such that \( \chi(X^A) = \chi(X) \), and \( A \) can be generated by at most 2 elements.

If \( \chi(X) < 0 \) then, since \( \chi(X^A) < 0 \), there is at least one connected component of \( X^A \) which is a surface. If \( \Sigma \subseteq X^A \) is one such component and \( x \in \Sigma \), then the linearization of the action of \( A \) near \( x \) gives an embedding \( A \hookrightarrow \text{GL}(T_xX/T_x\Sigma) \) preserving the orientation and a metric (see Lemma 2.2), so we may identify \( A \) with a subgroup of \( \text{SO}(2, \mathbb{R}) \); hence \( A \) is cyclic.

Combining the previous estimates, we have
\[ [G : A] \leq KC_F(X)C_{G_X}(X, \Sigma)C_{\text{Jor}}(4)\Lambda(C_1(X))C_4, \]
so the proof of Theorem 1.1 is now complete.

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