A nonlinear evolution equation for sand ripples based on geometry and conservation

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From geometry and conservation we derive two nonlinear evolution equations for sand ripples. In the case of a strong wind leading to a net erosion of the sand bed, ripples obey the Benney equation. This leads either to order or disorder depending on whether dispersion is strong or weak. In the most frequent case where erosion is counterbalanced by deposition, we derive a new one-parameter nonlinear equation. It reveals ripple structures which then undergo a coarsening process at long times, a process which then slows down dramatically with the growth of the ripple wavelength.

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Ripples on sand in desert and see are fascinating patterns. Despite the fact that sand, and granular media in general, are very familiar, in principle, to anyone, the understanding of their static and dynamical properties still continues to pose a formidable challenge to theoretical modeling. A continuum description, as those well established for simple fluids (Navier-Stokes), or solids (Hookes law), is lacking, most likely due to a strong interaction of disparate scales. Whatever complex a continuum theory might be, it should be compatible with symmetries and conservation laws. The present paper deals with derivations of two generic nonlinear equations for sand ripples based on geometry and conservation.

A model describing the physical origin of ripple formation has been presented in the early forties by Bagnold. This model is based on energetic saltating grains impacting the ripple. Since then, several contribution both analytical and numerical have allowed further elucidation of the problem. A derivation of a continuum generic nonlinear evolution equation of sand front is lacking, however. This means in particular that the question of whether ordered, or disordered pattern, would prevail is to date unanswered. It is the main goal of this Letter to address these questions on the basis of geometry and conservation. In the most interesting case where erosion is counterbalanced by deposition on the average, we show here that the generic equation in one dimension (when the ripple is translationally invariant in the y-direction) takes the following form close to the instability threshold

\[
\frac{\partial h}{\partial t} = -\frac{\partial^2 h}{\partial x^2} - \nu \frac{\partial^4 h}{\partial x^4} - \frac{\partial^2}{\partial x^2} \left[ \left( \frac{\partial h}{\partial x} \right)^2 \right]
\]

(1)

where \( h \) is the ripple profile (in a dimensionless form) which is a function of \( x \) and \( t \). As seen below this equation can always be reduced to one dimensionless parameter denoted here as \( \nu \). This equation reveals ripples. The wavelength is first dominated by that of the linearly most unstable mode. At long time they coarsen before revealing a dramatic slowing down of the wavelength increase. Note that the equation is local, which is a consequence of the proximity of the instability threshold (see below).

For ease of presentation, we consider a one dimensional front. The most natural way of representing a front is to use intrinsic coordinates, namely the curvature \( \kappa(s) \) as a function of the arclength \( s \). Each point of the front is labelled by its vector position \( r(\alpha, t) \), where \( \alpha \) is a time-independent parametrization of the curve which can be taken at liberty to lie in the interval 0 to 1. We obtain for the evolution of the arclength \( s \)

\[
\frac{\partial s}{\partial t} = v_t[s(\alpha)] - v_\zeta[s(0)] + \int_0^s ds \kappa v_n.
\]

(2)

This is simply obtained by setting \( ds = \sqrt{g} \alpha \zeta \), where \( g \) is the induced metric, and integrating \( s = \int \sqrt{g} \alpha \zeta \) by parts. Here we have used the relation \( dv_t/\zeta t = 2g(\partial v_t/\zeta s + \kappa v_n) \), where \( v_t \) and \( v_n \) are the tangential and normal velocities respectively. In order to derive the evolution equation for \( \kappa \), we first need to evaluate the commutator \( \left[ \frac{d}{dt}, \frac{\partial}{\partial s} \right] \). We obtain

\[
\left[ \frac{d}{dt}, \frac{\partial}{\partial s} \right] = -\left( \frac{\partial v_t}{\partial s} + \kappa v_n \right) \frac{\partial}{\partial s}.
\]

(3)

Applying this identity to \( \theta \) (the angle between the vertical axis and the normal vector), we arrive at

\[
\frac{\partial \kappa}{\partial t} = -\left( \frac{\partial^2}{\partial s^2} + \kappa^2 \right) v_n + v_t \frac{\partial \kappa}{\partial s}.
\]

(4)

Equations (2) and (3) constitute the evolution equations for the arclength and the curvature. These equations are general and only geometrical concepts are evoked.

The tangential velocity is a gauge. Indeed if one considers the front at some time and its state at later time there is no way which allows us to state if a point \( a \) (lying...
on the curve) at time $t$ has become point $a'$ at $t + \Delta t$, or $a''$ (obtained by displacing $a'$ tangentially). The tangential velocity is fixed once the parametrization of the curve is given \[10\]). In contrast, $v_n$ is a physical quantity.

A way of viewing that the tangential velocity disappears from the evolution equation is to work at given $s$, and not at given $\alpha$. Using \[13\], the curvature evolution equation \[14\] takes the form

$$\frac{\partial \kappa}{\partial t} \bigg|_s = -\left[\frac{\partial^2}{\partial s^2} + \kappa^2\right]v_n - \frac{\partial \kappa}{\partial s} \int_0^s ds' \kappa v_n$$

(Gauges usually introduce nonlocality. Similar formulations were used in other contexts \[11,12]. This equation constitutes a general nonlinear equation for curvature dynamics if the normal velocity is known. This formulation is powerful in numerical studies. If the normal velocity has thus the weakly nonlinear regime in terms of Cartesian coordinates where the front is represented by $z = h(x,t)$.

Let us first illustrate our analysis on a situation that leads to a well known equation in the literature. This will serve later to determine relevant nonlinearities. For a rotationally invariant system, the normal velocity must necessarily be a function of only those quantities which are invariant under any surface reparametrization. In one dimension the only quantity which is intrinsic is the curvature and its odd derivatives with respect to the arclength. Generically, the normal velocity has thus the form

$$v_n = C + a_1 \kappa + a_2 \kappa^2 + a_3 \kappa^3 + b_1 \kappa s + \ldots$$

(6)

From now on most of differentiations will be subscripted for brevity. Before proceeding further an important remark is in order concerning the assumption of locality in Eq. \[13\]. For sand ripples discussed below, there are two kinds of grains that contribute to the development of ripples \[13\]: the saltating ones (traveling on length scale $l_s \sim \lambda$, where $\lambda$ is the ripple wavelength) that have high kinetic energy, and the low-energy splashed grains traveling in reptation (or hopping) on a scale $a$ which is several (typically 6–10) times smaller than $\lambda$. The saltating grains are accelerated by the wind and this provides the driving force that governs the motion of the surface in aeolian ripple growth and in ripple translation. In the most frequent case where erosion is counterbalanced by deposition, the population of saltating grains remains almost constant (as recognized already by Bagnold \[1\] and in \[13\]). The saltating grains serve merely to bring energy into the system, the saltating population exchanges almost no grains with the reptating population. The information on the surface profile is propagated only by reptating grains. Saltating grains loose their memory in the course of their flight due to collisions and the turbulent airflow. It follows then that $a$ (reptation length) is the natural candidate \[3\] for a characteristic length scale of interaction. Since $a/\lambda \ll 1$, the local assumption is legitimate. The ratio $a/\lambda$ serves here as the small expansion parameter.

The constant $C$ in Eq. \[13\] expresses the fact that the front advances on the average at constant velocity. For a straight front the velocity is $C$ (precisely as what happens when a front is collecting particles from outside when exposed to a given flux). For a weakly curved front we have approximately $\kappa \simeq -h_{xx}$. Using this in \[14\] and upon substitution into \[13\] we obtain

$$h_t = C - a_1 h_{xx} - b_1 h_{xxxx} + \frac{C}{2} h_{xx}^2,$$

(7)

where we have truncated the expansion to leading order, as explained above. First we can have $a_1 > 0$ or $a_1 < 0$. In the former case the straight front is unstable. This is what happens in a large number of situations (see Ref. [13]). The above derivation has concentrated on the situation close enough to the threshold where $a_1$ is small (or equivalently the frequencies of interest are of order $\epsilon^2$). The truncation to make sense, $a_1$ must be of order $\epsilon$ (some appropriate small quantity). This is the case sufficiently close to the instability threshold. For the fourth derivative to be of the order of the second one, we must have $\epsilon h_{xx} \sim h_{xxxx}$, this implies in Fourier space that $q$ (the wavenumber) must be of order $\sqrt{\epsilon}$. In other words our equation is expected to be valid for slowly varying modulations on the scale of the typical length of interest in a given problem. This means that modulations in physical units occurs on scales of order $1/\sqrt{\epsilon}$. Now in order that the linear part counterbalance the nonlinear one, we must have $\epsilon h_{xx} \sim h_{xxxx}$, this implies in Fourier space that $q$ (the wavenumber) must be of order $\sqrt{\epsilon}$. Once the scalings are known, it is a simple matter to show that other permissible nonlinearities (e.g., $h_{xx}^3$) are of higher order contribution (i.e., of order $\epsilon^3$).

Note that the constant term in Eq. \[13\] is unimportant since it can be absorbed on the l.h.s. upon a transformation $h \rightarrow h - C t$. Note also (see below) that $b_1$ is generally positive in order to ensure a well behaved solution at a short scale. The sign of $C$ is however unimportant since changing it would simply correspond to the transformation $h \rightarrow -h$. The equation can be made free of parameter upon appropriate rescaling. Equation \[13\] is known under the name of Kuramoto-Sivashinsky(KS) \[14,15\]. It models pattern formation in different contexts (see Ref. [13]). For large enough extent of the front in the $x$ direction, it reveals spatio-temporal chaos. If $a_1 < 0$, there is no instability, and there would thus be no need to keep the fourth derivative in Eq. \[13\]. Adding a small stochastic force to $C$ (like shot noise in Molecular Beam Epitaxy –MBE), we obtain the well known Kardar-Parisi-Zhang
Having illustrated our study on a reference example, we are now in a position to deal with ripple formation under wind blow. On the one hand the front is not advancing on the average, in principle. Thus the constant $C$ must be set to zero (this is unimportant as seen below). On the other hand, the wind causes the normal front velocity to be orientation-dependent. We first concentrate on the situation where there is a strong erosion, so that the front is surrounded by an atmosphere of flying grains (a sort of reservoir). The front motion can thus globally loose or gain grains from the atmosphere, so that no constraint must be imposed. In the presence of anisotropy (due to the wind) the most natural way is to write

$$v_n = a_1 \kappa + a_2 \kappa^2 + b_1 \kappa_{ss} + a_1 \sin \theta + a_2 (\sin \theta)^2$$

$$+ \beta_1 \frac{\partial^2}{\partial s^2} (\sin \theta) + \beta_2 \frac{\partial^2}{\partial s^2} (\sin^2 \theta) + \gamma_1 \kappa \sin \theta + \ldots . \quad (8)$$

The terms in $\sin \theta$ express the fact that the growth velocity depends on the local slope of the front. In the present case, the direction perpendicular to the $z$ axis (from which $\theta$ is measured) is favored. Had we wanted to give a greater importance to the $x$ direction, we would then have expanded $v_n$ in power of $\cos \theta$. However, it is a simple matter to realize that this is unimportant for our purposes. In the limit of a weakly curved front, the substitution of (8) into (8) yields to leading order

$$h_t = -a_1 h_x - a_1 h_{xx} - \beta_1 h_{xxx} - b_1 h_{xxxx} + \alpha_2 h_x^2 \quad (9)$$

This equation is known in the literature under the name of Benney equation [17]. It has been derived recently from a microscopic model in the context of step-bunching dynamics during sublimation of a vicinal surface [18]. The same equation arises in other contexts such as phase dynamics for traveling modes [20], and in some models of traffic flow [21].

The first derivative term can be absorbed in the temporal derivative by means of a Galilean transformation (i.e. $x \rightarrow x - a_1 t$). The scaling of space, time and amplitude with $\epsilon$ are obviously the same as for the KS equation. The third derivative (not present in the KS equation) is of higher order $1/\epsilon^{1/2}$ if all the scales in the KS part are set to one (as seen before space scales as $1/\sqrt{\epsilon}$, and $h \sim \epsilon$, so that $a_1 h_x \sim h_{xxx} \sim h_x^2 \sim \epsilon^3$, while $h_{xxx} \sim \epsilon^{5/2}$). Thus the expansion would make sense in principle only if $\beta_1$ were small enough (of order $\sqrt{\epsilon}$), a demand whose realization depends on the system under consideration. This apparent difficulty can be circumvented by noting that $h_{xxx}$ contributes to the imaginary part of the linear growth rate (if we seek a solution in the form $e^{\omega x + i \omega t}$, where $\omega$ is the growth rate), which concerns thus propagative terms, whereas $h_{xx}$ and $h_{xxxx}$ produce real contributions. Thus one can ‘split’ time into a slow part corresponding to growth of perturbations, and a fast part corresponding to propagation.

Upon rescaling of the space, time and amplitude, the Benney equation can be rewritten in a form in which only one parameter survives. Therefore all the coefficients can be set to unity except one, let say $\beta_1$. Depending on the strength of this coefficient, the dynamics is either chaotic for $\beta_1$ of order or smaller than one (we recover the KS dynamics), or exhibits a rather ordered structure drifting sideways for $\beta_1$ of order few unities [19].

For sand ripples, it seems that in most situations an equilibrium between erosion and deposition sets in due to the drag force of the transported grains exerted on the wind (the greater the number of transported grains is, the weaker the wind gets and the less it erodes the sand bed). In other words, the number of transported grains remains constant in average; there is neither net erosion nor deposition. In order to treat that case we should impose the global conservation condition. In order to ensure this all the homogeneous terms in (8) must be left out, except the linear terms in $\kappa$ and $\sin \theta$. Indeed the area $A$ bounded by the front and some horizontal axis behaves in course of time as $\partial A/\partial t = \int ds v_n$, and as global conservation imposes $\partial A/\partial t = 0$, all terms giving a non-zero contribution to that area should be removed (the sand front moves because either particles have left the region of interest, or other grains have landed from a neighboring part). In this case, the front dynamics is governed to leading order by

$$h_t = -a_1 h_x - a_1 h_{xx} - \beta_1 h_{xxx} - b_1 h_{xxxx} + \beta_2 \frac{\partial^2}{\partial x^2} (h_x^2) \quad (10)$$

For an instability $a_1$ must be positive, $b_1$ must be positive as well in order to introduce a short wavelength cut-off, while the sign of $\beta_2$ is unimportant, since it can be changed upon the transformation $h \rightarrow -h$. Obviously $a_1 h_x$ can be absorbed in $h_t$ via a Galilean transformation, and the sign of $\beta_1$ is unimportant as well. Space and time scale with $\epsilon$ as in the KS and Benney equations, while the scale of $h$ here is of order one, in a marked contrast with the KS limit. This scaling has an important consequence, to be discussed below. What makes the other higher contribution small is precisely the scaling of space (higher and higher derivatives are of smaller and smaller contributions). After rescaling (and absorbing $h_x$ in $h_t$) only one parameter survives and the principal equation for ripples can be written in a canonical form (Eq. (1)).

The linear dispersion relation of Eq. (1) reveals the structureless state is linearly unstable against perturbations with wavenumbers smaller than $q_c = 1$. A physical origin of the instability was put forward long time ago by Baguelin [20]. Here we assume that the threshold has been reached, and thus we take a negative sign in front of the second derivative. Numerical solutions for sizes $L \geq 10 \lambda_c = 2\pi$ of Eq. (1) reveals an evolution towards a steady state with a given wavelength. In a marked contrast to the KS equation which exhibits spatiotemporal chaos (or Benney equation when dispersion is small), the new evolution equation leads
to ripple pattern (Fig.1). The wavelength is first close to that of the most dangerous mode. At long time, the structure coarsens producing thereby wider and wider dunes. Then the coarsening slows down dramatically. This feature agrees with experiments. An extensive study will be presented in the future.

Finally it is important to make some important comments. (i) Since $h$ is of order one, while $x$ is of order $1/\sqrt{\epsilon}$, the slope is of order $\sqrt{\epsilon}$, and thus remains small (see Fig.1). Had the slope been of order one (as in some nonlinear equations), avalanches would manifest themselves, and no steady-solutions would have been possible. (ii) Figure 1 corresponds to equation $(1)$. As stated above changing the sign in front of the nonlinear term corresponds to making an up-down operation on Figure 1. Inspecting several examples of sand ripples conveys the strong impression that it is the situation in Figure 1 which seems likely. (iii) Since $h \sim 1$ and $\lambda \sim 1/\sqrt{\epsilon}$, the ratio of the amplitude to wavelength close to the instability threshold scales as $\epsilon^{1/2}$. Thus close to threshold the amplitude is several times smaller than the wavelength (usually ripples have an amplitude which is approximately 10 times smaller than their wavelength). (iv) We have deliberately, for sake of simplicity, considered a one dimensional structure (that is the ripples are translationally invariant along the $y$ direction). The extension to two dimensions is feasible, and is currently under investigation. This should be crucial with regard to the study of possible secondary instabilities of ripples.

In summary we have derived a generic nonlinear equation to describe sand ripple dynamics. The study is based on geometry and conservation. While apparently close equations (e.g., the KS equation) lead to spatiotemporal chaos, the new equation reveals rather steady ripples which coarsen with time. The advantage of this study lies in the fact that no matter how complex the physics might be, the equations close to the instability threshold must be of the sort given here as long as symmetries and conservations are preserved. In turn, geometry and conservation can not, by their very nature, provide the values of coefficients. The same holds for the existence of instability. Thus there is a need in future to derive our equation starting from a given ‘microscopic’ model and to determine the dimensionless coefficient $\nu$ in terms of physical quantities. Other remarks are in order. We have limited ourselves to leading order, which is valid close enough to the instability point. Expansion to higher order is straightforward. We have assumed that the normal velocity is both local in space and in time. This is valid close to the instability threshold. If reptation length $a$ remains the relevant length scale, we may expect our assumption of locality to hold at arbitrary distance from threshold, since $a/\lambda$ is the small parameter of the expansion. Finally, it goes without saying that the present study can have impact on other systems than the sand ripples.

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