The QCD Membrane

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Abstract. In this paper we study spatially quenched, $SU(N)$ Yang–Mills theory in the large-$N$ limit. The resulting reduced action shows the same formal look as the Banks–Fischler–Shenker–Susskind $M$–theory action. The Weyl–Wigner–Moyal symbol of this matrix model is the Moyal deformation of a $p(=2)$–brane action. Thus, the large-$N$ limit of the spatially quenched $SU(N)$ Yang–Mills is seen to describe a dynamical membrane. By assuming spherical symmetry we compute the mass spectrum of this object in the WKB approximation.

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1. Introduction

The pivotal role played by gauge field theories in a unified description of fundamental interactions proposed one of the most challenging questions of modern high energy theoretical physics: if Nature likes so much gauge symmetry why does gravity cannot fit into such an elegant and, would be, universal blueprint?

Before the advent of string theory this was a question without an answer. After that, it became clear that all field theories, including Yang–Mills type models, must be seen as low energy, effective approximations of some more fundamental theory where the dynamical degrees of freedom are carried by relativistic extended objects. Furthermore, even the low–energy effective gauge theories require a “stringy” approach in the strong coupling regime, where standard perturbation series breaks down. Color confinement in QCD is a remarkable example of a phenomenon where the string tenet meets gauge symmetry. The stringy aspects of confinement have long been studied, but are not yet fully understood. Several different models have been proposed as phenomenological descriptions of the quark–gluon bound states, including color flux tubes, three–string of various shapes, bag–models. To promote some of them to a deeper status one would like to derive extended objects as non-perturbative excitations of an underlying gauge theory. The first remarkable achievement of this program was to obtain stringy objects from $SU(N)$ Yang–Mills theory in the large–$N$ limit.

These results have been extended to the case of a self–dual membrane in the $SU(\infty)$ Toda model. In a nutshell, the problem is to establish a formal correspondence between a Yang–Mills connection, $A^i_{j\mu}(x)$, and the string coordinates, $X^\mu(\sigma^0, \sigma^1)$, i.e. one has to “get rid of” the internal, non-Abelian, indices $i, j$ and replace the spacetime coordinates $x^\mu$ with two continuous coordinates $(\sigma^0, \sigma^1)$. With hindsight, the “recipe” to turn a non-Abelian gauge field into a set of Abelian functions describing the embedding of the string world-sheet into target spacetime can be summarized as follows:

a) transform the original field theory into a sort of “matrix quantum mechanics”;
b) use the Wigner-Weyl-Moyal map to build up the symbol associated to the above matrix model;
c) take the “classical limit” of the theory obtained in b).

Stage a) requires two sub–steps:
a1) take the large–$N$ limit, i.e. let the row and column labels $i, j$ to range over arbitrarily large values; a2) dispose of the spacetime coordinate dependence through the so called “quenching approximation”, i.e. a technical manipulation which is formally equivalent to collapse the whole spacetime to a single point.

After stage a) the original gauge theory is transformed into a quantum mechanical model where the physical degrees of freedom are carried by large coordinate independent matrices. Then, stage b) associates to each of such big matrices its corresponding

+ There are many interesting reviews on this problem, as [1], [2], [3] . . . and we apologize for omitting many other good ones.
symbol, i.e. a function defined over an appropriate non-commutative phase space. The resulting theory is a deformation of an ordinary field theory, where the ordinary product between functions is replaced by a non-commutative $\ast$-product. The deformation parameter, measuring the amount of non-commutativity, results to be $1/N$, and the classical limit corresponds exactly to the large-$N$ limit. The final result, obtained at the stage c), is a string action of the Schild type, which is invariant under area-preserving reparametrization of the world-sheet.

More recently, we have also shown that bag-like objects fit the large-$N$ spectrum of Yang–Mills type theories as well, both in four \cite{8} and higher dimensions. We started from the Yang–Mills action for an $SU(N)$ gauge theory supplemented by a topological term

$$S \equiv -\int d^4x \left( \frac{N}{4g_Y^2} \text{Tr} F_{\mu\nu} F^{\mu\nu} + \frac{\theta N g_Y^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} F_{\mu\nu} F_{\rho\sigma} \right)$$

and went through the steps from a) to c). As a final result we obtained the following action

$$W = -\frac{\mu_0^4}{16} \int_{\Sigma} d^4\sigma \left\{ X^\mu , X^\nu \right\}_{PB} \left\{ X_\mu , X_\nu \right\}_{PB} +$$

$$-\kappa \epsilon_{\mu\nu\rho\sigma} \int_{\partial\Sigma} d^3s X^\mu \left\{ X^\nu , X^\rho , X^\sigma \right\}_{NPB} ,$$

where

$$\{ X^\mu , X^\nu \}_{PB} \equiv \epsilon^{mn} \partial_m X^\mu \partial_n X^\nu$$

is the Poisson Bracket and

$$\{ X^\nu , X^\rho , X^\sigma \}_{NPB} \equiv \epsilon^{ijk} \partial_i X^\nu \partial_j X^\rho \partial_k X^\sigma$$

is the Nambu–Poisson Bracket. The first term in (2) describes a bulk three–brane, or bag, which in four dimensions is a pure volume term characterized by a pressure $\mu_0^4$. All the dynamical degrees of freedom are carried by the second term in (2), where $\kappa$ is the membrane tension; this term encodes the dynamics of the boundary, Chern–Simons membrane, enclosing the bag. Tracing back the bulk and boundary terms in the original action (1) it is possible to establish the following formal correspondence

“glue” $\leftrightarrow$ bulk 3-brane

“instantons” $\leftrightarrow$ Chern–Simons boundary 2-brane.

This scheme, which has been generalized to Yang–Mills theories in higher dimensional spacetime \cite{9}, points out that not only strings but bag-like objects fit the large-$N$ spectrum of $SU(N)$ gauge theories. However, a non-trivial dynamics for these spacetime filling objects comes only from boundary effects, described by Chern–Simons terms in the original gauge action. Against this background, we would like to investigate the existence of non-topological membrane-like excitations in the large large–$N$ spectrum of a $SU(N)$ Yang–Mills theory in four dimensions. Clues suggesting the existence of these objects come from earlier
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Abelian models [10] and the recent conjectures about M–Theory. M(atrix) theory is
the, alleged, ultimate non–perturbative formulation of string theory. More in detail, two
models have been constructed as possible non–perturbative realization of Type IIA [11]
and Type IIB [12] string theory. The matrix formulation of Type IIB strings is provided
by a large–N, 10-dimensional super Yang-Mills theory reduced to a single point

\[ S_{IKKT} = -\frac{\alpha}{4} \text{Tr} \left[ A_\mu, A_\nu \right]^2 + \ldots \]  

(3)

The dots refers to the fermionic part of the action which is not relevant to our discussion.
The model (3) has a rich spectrum of extended objects. Our investigations in [8] and
[9] has been initially triggered by the formal analogy between (3) in 10D and quenched
Yang–Mills theory in 4D.

Matrix description of Type IIA strings is given in terms of 0-branes quantum mechanics

\[ S_{BFSS} = \frac{1}{2g_s} \text{Tr} \left( \frac{dX^i}{dt} \frac{dX_i}{dt} - \frac{1}{2} \left[ X^i, X^j \right]^2 + \ldots \right) , \]  

(4)

where \( i = 1, \ldots, 9 \). Again, the 0-branes matrix coordinates can be seen as Yang–Mills
fields reduced to a line.

In this paper we would like to “reverse the path” leading Type IIA strings to the matrix
model (4) and show how a 3D version of (4) can be obtained from the canonical
formulation of an SU( N ) Yang–Mills theory through a modified quenching prescription.

Then, we are going to extract a non–relativistic, dynamical 2–brane from the large–N
spectrum of the model by following the procedure introduced in [8] and [9].

The paper is organized as follows: in section 2 we start from SU(N) Yang–Mills
theory in four dimensions and obtain a corresponding matrix theory through the
quenching approximation; two different type of quenching are discussed in section 2.1
and section 2.2; in section 3 we study the large-N limit of the matrix model introduced
in section 2.2 and use the Weyl–Wigner–Moyal map to get the action for a membrane;
we conclude the paper by computing the mass spectrum of this membrane in the WKB
approximation.

2. Yang–Mills Theory as a Matrix model

In the introduction we referred to the supposed relation between confinement and
extended excitations of the Yang–Mills field. Recently, an even deeper and more
fundamental relation between branes and Yang–Mills fields has been conjectured in
the framework of M-theory [13]. Non-perturbative formulations of string theory require
the introduction of higher dimensional solitonic objects satisfying Dirichlet boundary
conditions [14]. Dirichlet-branes are formally described by non-commuting matrix
coordinates. Thus, non-perturbative string theory, or M-theory, is conveniently written
in terms of matrix dynamical variables. The corresponding low energy, effective,
supersymmetric Yang–Mills theory is derived through an appropriate compactification
procedure of the original matrix model [14], [15].
In this section we are going to follow a similar path, but in the opposite direction: we shall start from a $SU(N)$ gauge theory in four dimensional spacetime and build a matrix model. Our final purpose is to show that the spectrum of such a QCD matrix model contains dynamical membrane type objects. We thus start from the Yang–Mills action

$$S_{YM} = \int dt \int_{V_H} d^3x \mathcal{L}_{YM}(F, A)$$

(5)

defined in terms of the Lagrangian density

$$\mathcal{L}_{YM}(F, A) \equiv \frac{N}{4g_0^2} \text{Tr} (F_{\mu\nu})^2 - \frac{1}{2g_0^2} \text{Tr} F_{\mu\nu} D_{[\mu} A_{\nu]}.$$ (6)

The covariant derivative is defined as usual,

$$D_{[\mu} A_{\nu]} \equiv \partial_{[\mu} A_{\nu]} + i [A_{\mu}, A_{\nu]},$$ (7)

but the $SU(N)$ Yang–Mills Lagrangian $\mathcal{L}_{YM}(F, A)$ is written in the first order formulation: thus

$$A_{\mu} \equiv A^a_{\mu} T^a, F_{\mu\nu} \equiv F^a_{\mu\nu} T^a$$

are independent vector and tensor fields respectively valued in the Lie algebra defined by the commutation relations

$$[T^a, T^b] = i f^{abc} T^c.$$

The integration volume $V_H$ will be specified later on. The form (5) is appropriate for an Hamiltonian formulation of the action, as it is required by the quenching approximation that we shall apply in the next section.

Starting from the Lagrangian density we can, as usual, introduce the canonical momentum and Hamiltonian

$$E^i \equiv \partial \mathcal{L}_{YM} \partial t A^i = -\frac{1}{2g_0^2} F^{tm}$$

(8)

$$H_0 \equiv \text{Tr} (E^i \partial_t A_i) - \mathcal{L}_0$$

(9)

and rewrite (5), in terms of phase space variables, as:

$$\mathcal{L}_{YM} = -\frac{N}{2g_0^2} \text{Tr} (F^{tm})^2 - \frac{N}{2g_0^2} \text{Tr} F^{tm} (\partial_t A_i + \partial_i A_t - i [A_t, A_i])$$

$$+ \frac{N}{4g_0^2} \text{Tr} (F_{mn})^2 - \frac{N}{2g_0^2} \text{Tr} F^{mn} D_{[m} A_{nu]}$$

$$= -\frac{g_0^2}{2N} \text{Tr} (E^i)^2 + \frac{1}{2} \text{Tr} (E^i \partial_t A_i) - \frac{1}{2} \text{Tr} (A_t D_i E^i) - \frac{N}{4g_0^2} \text{Tr} (D_{[m} A_{nu]})^2.$$

(10)

Accordingly, the phase space action reads

$$S = \int dt \int_{V_H} d^3x \left[ E^i \partial_t A_i - H_0 \right]$$

(11)

$$= \int dt \int_{V_H} d^3x \left[ \frac{g_0^2}{2} \text{Tr} (E^i)^2 - \frac{1}{2} \text{Tr} (E^i \partial_t A_i) + \frac{1}{2} \text{Tr} (A_t D_i E^i) + \frac{1}{4g_0^2} \text{Tr} (D_{[m} A_{nu]})^2 \right].$$

(12)

* The metric signature is $(− + + +)$ and in our notation matrices are denoted by boldface letters.
Let us remark that $A_t$ enters linearly in the canonical form of the action (12) and plays the role of Lagrange multiplier enforcing the (non-Abelian) Gauss Law:

$$\frac{\delta S}{\delta A_t} = 0 \quad \Rightarrow \quad D_i E^i = 0.$$  

Thus, solving the classical field equation for $A_t$ is equivalent as requiring $E^i$ to be covariantly divergence free in vacuum. Thus, inserting the solution of the Gauss constraint (13), the action for $E^i$ becomes

$$S_{\text{YM}} = \int dt \int_{\mathcal{V}} d^3x \left[ \frac{g_0^2}{2N} \text{Tr} (E^i)^2 - \frac{1}{2} \text{Tr} (E^i \partial_t A_i) + \frac{N}{4g_0^2} \text{Tr} (D_m A_n)^2 \right].$$  

We will now go on with the quenching procedure.

### 2.1. Spacetime Quenching $\rightarrow$ IKKT–type model

In order to provide the reader a self-contained derivation of our model, let us briefly review how the quenching approximation works in a simple toy model [16]. Consider the following two-dimensional model of matrix non-relativistic quantum mechanics

$$S \equiv \int d^2x \text{Tr} \left[ \frac{1}{2} (\partial_0 M)^2 - \frac{1}{2} (\partial_1 M)^2 - V(M) \right],$$  

where $M(x^0, x^1)$ is an Hermitian, $2 \times 2$ matrix and $V(M)$ is an appropriate potential term and we suppose that the system described by $M$ is enclosed in a (one-dimensional) “spatial box” of size $a$. Quenching is an approximation borrowed from the theory of spin glasses where only “slow” momentum modes are kept to compute the spectrum of the model. Slow modes are described by the eigenvalues of the linear momentum matrix $P$, while the off-diagonal, “fast” modes can be thought as being integrated out. Thus, $M(x)$ can be written as

$$M(x^0, x^1) = \exp \left( i P x^1 \right) M(x^0) \exp \left( -i P x^1 \right)$$  

and the spatial derivative $\partial_1 M(x^0, x^1)$ becomes the the commutator of $P$ and $M$

$$\partial_1 M = i \left[ P, M \right]$$  

so that the action becomes

$$S \equiv a \int dx^0 \text{Tr} \left[ \frac{1}{2} (\partial_0 M)^2 + \frac{1}{2} \left[ P, M \right]^2 - V(M) \right],$$  

in which the the $x^1$ dependence of the original matrix field has been removed.

Quenching can be applied to a Yang–Mills gauge theory by taking into account that in the large-$N$ limit $SU(N) \rightarrow U(N)$ and the group of spacetime translations fits into the diagonal part of $U(\infty)$. By neglecting off-diagonal components, spacetime dependent dynamical variables can be shifted to the origin by means of a translation operator $U(x)$: since the translation group is Abelian one can choose the matrix $U(x)$ to be a plane wave diagonal matrix [17]

$$U_{ab}(x) = \delta_{ab} \exp \left( iq^a \mu x^\mu \right),$$  

where \( q^a_\mu \) are the eigenvalues of the four-momentum \( q_\mu \). Then

\[
A_\mu(x) = \exp \left( -i q_\mu x^\mu \right) A_\mu(0) \exp \left( i q_\mu x^\mu \right) \equiv U^\dagger(x) A_\mu^{(0)} U(x)
\]

and in view of the equality

\[
D_\mu A_\nu = i U^\dagger(x) \left[ q_\mu + A_\mu^{(0)}, A_\nu \right] U(x),
\]

which when antisymmetrized yields

\[
D_{[\mu} A_{\nu]} = i U^\dagger(x) \left[ q_\mu + A_\mu^{(0)}, q_\nu + A_\nu^{(0)} \right] U(x) \equiv i U^\dagger(x) \left[ A_\mu^{(q)}, A_\nu^{(q)} \right] U(x),
\]

we can see that the translation is compatible with the covariant differentiation, so that

\[
F_{\mu\nu}(x) = \exp \left( -i q_\mu x^\mu \right) F_{\mu\nu}(0) \exp \left( i q_\mu x^\mu \right) \equiv U^\dagger(x) F_{\mu\nu}^{(0)} U(x).
\]

Once the original gauge field theory is turned into a constant matrix model, we still need to dispose of the spacetime volume integration. The gluon field is spatially confined inside a volume \( V_H \) comparable with the typical size of an hadron. Thus, for any finite time interval \( T \) we can replace the four-volume integral by

\[
\int_0^T dt \int_{V_H} d^3x \rightarrow TV_H
\]

and the quenched action becomes

\[
S_{\text{YM-red}}^{(q)} = TV_H \frac{N}{g_0^2} \text{Tr} \left( \frac{1}{4} \left( F_{\mu\nu}^{(0)} \right)^2 - i \frac{1}{2} F_{\mu(0)\nu} \left[ A_{\mu}^{(q)}, A_{\nu}^{(q)} \right] \right), \tag{20}
\]

which is the first order formulation of the IKKT-type action in four spacetime dimensions \([12]\). The usual second order formulation is readily obtained by solving for \( F_{\mu\nu}^{(0)} \) in terms of \( A_\mu^{(q)} \)

\[
F_{\mu\nu}^{(0)} = i \left[ A_{\mu}^{(q)}, A_{\nu}^{(q)} \right]
\]

and substituting back this result into (20)

\[
S_{\text{YM-red}}^{(q)} \rightarrow S_{(4)}^{\text{IKKT}} = \beta V_H \frac{N}{4g_0^2} \text{Tr} \left( \left[ A_{\mu}^{(q)}, A_{\nu}^{(q)} \right] \right)^2. \tag{21}
\]

The string-like excitations of this model and the relation between large-\( N \) gauge symmetry and area-preserving diffeomorphism have been investigated in several papers \([3]\). More recently, we found that not only strings are present in the large-\( N \) spectrum of (21) but also spacetime filling, bag-like objects \([3]\), for which a non-trivial boundary dynamics was found through the addition of topological terms to the original Yang–Mills action. Here, we would like to explore a different route leading in a more straightforward way to a dynamical brane action. From this purpose we need to introduce a different quenching approximation, which we discuss in the next section.
2.2. Spatial Quenching $\rightarrow$ BFSS-type model

Instead of shifting $A_{\mu}(t, \vec{x})$ to a single point, as we did in the final part of the previous section, we translate the matrix gauge field to a fixed time slice by means of the conserved three–momentum $\vec{q}$. As in the previously discussed case

$$A_{i}(t, \vec{x}) = \exp \left(-i \vec{q} \cdot \vec{x} \right) A_{i}(t) \exp \left(i \vec{q} \cdot \vec{x} \right) \equiv U^{\dagger}(\vec{x}) A_{i}(t) U(\vec{x})$$

the translation operation commutes with the covariant differentiation since

$$D_{i} A_{n} = i U^{\dagger}(\vec{x}) \left[ q_{i} + A_{i}(t), A_{n}(t) \right] U(\vec{x})$$

implies

$$D_{[m} A_{n]} = i U^{\dagger}(\vec{x}) \left[ q_{i} + A_{i}(t), q_{n} + A_{\nu}(t) \right] U(\vec{x})$$

and for the conjugate momentum we also get

$$U^{\dagger}(\vec{x}) E_{i}(t) U(\vec{x}) \equiv i U^{\dagger}(\vec{x}) \left[ A^{(q)}_{i}(t), A^{(q)}_{n}(t) \right] U(\vec{x}).$$

Enclosing the system in a proper quantization volume $V_{H}$ while keeping the time integration free

$$\int_{V_{H}} d^{3} x \rightarrow V_{H}$$

we get

$$S = V_{H} \int dt \left[ \frac{g_{0}^{2}}{2N} \text{Tr} \left( E^{i}(t) \right)^{2} - \frac{1}{2} \text{Tr} \left( E^{i} \frac{d A_{i}}{dt} \right) - \frac{N}{4g_{0}^{2}} \text{Tr} \left( \left[ A^{(q)}_{i}, A^{(q)}_{n} \right] \right)^{2} \right],$$

which is the action in the first order formulation; by substituting for $E^{i}(t)$ its expression in terms of the vector potential we obtain, in the second order formulation, an action quite similar to the one for the bosonic sector of the BFSS model describing a system of $N$ D0-branes in the gauge $A_{0} = 0$:

$$S^{\text{BFSS}} = V_{H} \frac{N}{g_{0}^{2}} \int dt \left[ \frac{1}{2} \text{Tr} \left( \frac{d A^{(q)}_{i}}{dt} \right)^{2} - \frac{1}{4} \text{Tr} \left( \left[ A^{(q)}_{i}, A^{(q)}_{n} \right] \right)^{2} \right]; \quad (22)$$

the only difference is the range of the spatial indices: we are working in three rather than nine spatial dimensions. An action of the form (22) can also be obtained from monopole condensation and toroidal compactification [18].

3. Non-commutative Phase Space

To match the large-$N$ $SU(N)$ gauge theory with some appropriate brane model we have to bridge the gap between non-commuting Yang–Mills matrices and commuting brane coordinates. Since the world–trajectory of a $p$-brane is the target spacetime image $x^{\mu} = X^{\mu}(\sigma^{0}, \sigma^{1}, \ldots, \sigma^{p})$ of the world manifold $\Sigma : \sigma^{m} = (\sigma^{0}, \sigma^{1}, \ldots, \sigma^{p})$, $X^{\mu}$ belonging
to the algebra $\mathcal{A}$ of $C^\infty$ functions over $\Sigma$, to realize our program we must deform $\mathcal{A}$ to a non-commutative “starred” algebra by introducing a $\ast$-product. The general rule is to define the new product between two functions as (for a recent review see [19]):

$$f \ast g = fg + \hbar P_h(f, g),$$

(23)

where $P_h(f, g)$ is a bilinear map $P_h : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$. $\hbar$ is the deformation parameter which is often denoted by the same symbol as the Planck constant to stress the analogy with quantum mechanics, where classically commuting dynamical variables are replaced by non-commuting operators. In our case the role of deformation parameter is played by

$$\hbar \equiv \frac{2\pi}{N}. \quad (24)$$

For our purposes, we select $\Sigma = \mathbb{R}^{2n}$ and choose the Moyal product as the deformed $\ast$-product

$$f(\sigma) \ast g(\sigma) \equiv \exp \left[ i \frac{\hbar}{2} \omega^{mn} \frac{\partial^2}{\partial \sigma^m \partial \xi^n} f(\sigma) g(\xi) \right]_{\xi = \sigma}, \quad (25)$$

where $\omega^{mn}$ is a non-degenerate, antisymmetric matrix, which can be locally written as

$$\omega^{mn} = \begin{pmatrix} \mathbb{O}_{n \times n} & \mathbb{I}_{n \times n} \\ -\mathbb{I}_{n \times n} & \mathbb{O}_{n \times n} \end{pmatrix}. \quad (26)$$

The Moyal product (25) takes a simple looking form in Fourier space

$$F(\sigma) \ast G(\sigma) = \int \frac{d^{2n} \xi}{(2\pi)^n} \exp \left( i \frac{\hbar}{2} \omega^{mn} \sigma^m \xi^n \right) F \left( \frac{\sigma}{2} + \xi \right) G \left( \frac{\sigma}{2} - \xi \right), \quad (27)$$

where $F$ and $G$ are the Fourier transform of $f$ and $g$. Let us consider the Heisenberg algebra

$$[K^m, P^l] = i\hbar \delta^{ml}; \quad (28)$$

Weyl suggested, many years ago, how an operator $O_F(K, P)$ can be written as a sum of algebra elements as

$$O_F = \frac{1}{(2\pi)^n} \int d^n p d^n k F(p, k) \exp \left( ip_m K^m + ik_l P^l \right). \quad (29)$$

The Weyl map (29) can be inverted to associate functions, or more exactly symbols, to operators

$$F(q, k) = \int \frac{d^n \xi}{(2\pi)^n} \exp \left( -ik \xi \right) \left\langle q + h \frac{\xi}{2} \left| O_F(K, P) \right| q - h \frac{\xi}{2} \right\rangle; \quad (30)$$

moreover it translates the commutator between two operators $U, V$ into the Moyal Bracket between their corresponding symbols $U(\sigma), V(\sigma)$

$$\frac{1}{i\hbar} [U, V] \longleftrightarrow \{U, V\}_{\text{MB}} \equiv \frac{1}{i\hbar} (U \ast V - V \ast U)$$

and the quantum mechanical trace into an integral over Fourier space

$$(2\pi)^n \text{Tr}_H O_F(K, P) \longleftrightarrow \int d^n p d^n k F(q, k) \equiv \int d^{2n} \sigma F(\sigma). \quad (31)$$
A concise but pedagogical introduction to the deformed differential calculus and its application to the theory of integrable system can be found in [20].

We are now ready to formulate the alleged relationship between the quenched model (22) and membrane model: the symbol of the matrix $A^{(q)}_j$ is proportional to the 2n-brane coordinate $X^j(\sigma^1, \ldots, \sigma^{2n})$. Going through the steps discussed above the action $S_{BFSS}$ transforms into its symbol $W_{BFSS}$:

$$S_{BFSS} \rightarrow W_{BFSS} = \frac{NV_H}{2\pi g_0^2} \int dt\int d^{2n}\sigma \left[ \frac{1}{2} \frac{dA^i}{dt} \frac{dA^i}{dt} + \frac{\{A_i, A_n\}_{MB}}{4} \right].$$

The action (32) is manifestly Lorentz non-covariant, as it is expected (the covariant, supersymmetric, higher dimensional version of the action (32) is discussed in [21]). The adopted quenching scheme explicitly breaks the equivalence between spacelike and timelike coordinates. Accordingly, our final result takes a typical “non-relativistic” look.

Up to now we have not fixed the Fourier space dimension $n$. To give $A^i$ the meaning of embedding function, we have to choose $2n \leq D - 1$, where $D$ is the target spacetime dimension. To match QCD in four spacetime dimensions we set $n = 1$. In this case

$$\omega^{mn} = \epsilon^{mn} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and we rescale the Yang–Mills charge and field\footnote{For the sake of clarity, let us summarize the canonical dimensions in natural units of various quantities:

$[A_\mu^{(q)}(x)] = [A_\mu] = (\text{length})^{-1}$

$[F_{\mu\nu}^{(q)}(x)] = [F_{\mu\nu}] = (\text{length})^{-2}$

$[\sigma^m] = (\text{length})^0 = 1$ ,  \quad [t] = \text{length} ,  \quad [\beta] = \text{length} $

$[g_0] = [g_{YM}] = (\text{length})^0 = 1$

$[V_H] = (\text{length})^4$

$[X^i] = \text{length}$

$[\mu_0] = (\text{length})^{-1}$ ,  \quad [\alpha] = (\text{length})^{-5}.$

} as

$$\frac{N}{g_0^2} \quad \mapsto \quad \frac{1}{g_{YM}^2}$$

$$A^i \quad \mapsto \quad V_H^{-2/3} X^i.$$  

Since the “glue” is supposed to be confined inside an hadronic size volume $V_H$, we can assign to $g_{YM}$ the standard value at the confinement scale

$$\frac{g_{YM}^2}{4\pi} \simeq 0.18.$$  

Finally, if $N \gg 1$ the Moyal bracket can be approximated by the Poisson bracket

$$\{X^i, X^j\}_{MB} \mapsto \{X^i, X^j\}_{PB}.$$
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and \((32)\) takes the form \([22]\)

\[
S_{NR}^{p=2} = \int dt d^2 \sigma \left[ \frac{1}{2} \mu_0 \frac{dX_k}{dt} \frac{dX^k}{dt} + \frac{\alpha}{4} \{X_i, X_j\}_{PB} \{X^i, X^j\}_{PB} \right],
\]

(37)

where \(\mu_0\) and \(\alpha\) are defined by

\[
\mu_0 \equiv \frac{V H^{-1/3}}{2\pi g_{YM}^2}, \quad \alpha \equiv \frac{V H^{-5/3}}{2\pi g_{YM}^2}.
\]

Moreover from the definition of the Poisson bracket

\[
\{X^i, X^j\}_{PB} \equiv \epsilon^{mn} \partial_m X^i \partial_n X^j
\]

we can compute

\[
\{X^i, X^j\}_{PB} \{X_i, X_j\}_{PB} = 2 \cdot \det \left( \partial_m X^k \partial_n X_k \right) \equiv 2\gamma
\]

so that the action \((37)\) can be rewritten as

\[
S_{NR}^{p=2} = \int dt d^2 \sigma \left[ \frac{1}{2} \mu_0 \frac{dX_k}{dt} \frac{dX^k}{dt} + \frac{\alpha}{2} \det \left( \partial_m X^k \partial_n X_k \right) \right].
\]

(38)

The first term in \((38)\) is a straightforward generalization of the kinetic energy of a non-relativistic particle; the second term represents the “potential energy” associated to the elastic deformations of the membrane. The action \((38)\) still displays a residual symmetry under area preserving diffeomorphisms, leaving only one dynamical degree of freedom describing transverse oscillations of the membrane surface\[^{†}\]. If the action \((38)\) has any chance to provide a membrane model of hadronic objects, then it must be able to provide at least the correct order of magnitude of hadronic masses. Our model does not take into account spin effects, therefore it is consistent to look for spherically symmetric configurations. Again this is a sort of quenching even if of a more geometric type. Infinite vibration modes of the brane, corresponding to local shape deformations, are frozen and the dynamics is reduced to the “radial” breathing mode alone. This kind of approximation, commonly called “minisuperspace” approximation, is currently adopted in Quantum Cosmology, where it amounts, in practice, to quantize a single scale factor (thereby selecting a class of cosmological models, for instance, the Friedman–Robertson–Walker spacetimes) while neglecting the quantum fluctuations of the full metric. The effect is to turn the exact, but intractable, Wheeler–DeWitt functional equation \([23]\) into an ordinary quantum mechanical wave equation \([24]\). As a matter of fact, the various forms of the “wave function of the universe” that attempt to describe the quantum birth of the cosmos are obtained through this kind of approximation \([25]\) or modern refinements of it \([26]\). This non-standard approximation scheme was applied

\[^{†}\] We have assumed that \(X^i\) are three spacelike coordinates. By relaxing this assumptions we can give \((38)\) a slightly different physical interpretation. If all three \(X^i\) are considered as transverse directions, then \((38)\) can be seen as the light–cone gauge action for a bosonic brane in a 5-dimensional target spacetime.
to a relativistic membrane in the seminal paper by Collins and Tucker [27], and since then it has been used several times [28], including the case of self-gravitating objects [29].

Following [27], we parametrize the membrane coordinates as follows

\[ X^1 \equiv R(t) \sin \theta \cos \phi \]
\[ X^2 \equiv R(t) \sin \theta \sin \phi \]
\[ X^3 \equiv R(t) \cos \theta \]

and the transverse, dynamical degree of freedom corresponds to \( R \). The metric \( \gamma_{ab} \) induced on the membrane by the embedding (39) is:

\[ \gamma_{ab} = \text{diag} \left( R^2(t), R^2(t) \sin^2 \theta \right), \quad \det (\gamma_{mn}) = R^4(t) \sin^2 \theta. \]

The corresponding action turns out to be

\[ S = \int L dt = \pi^2 \int dt \left[ \frac{1}{2} \mu_0 \left( \frac{dR}{dt} \right)^2 + \frac{\alpha}{4} R^4 \right]; \]

accordingly the momentum conjugated to the only dynamical degree of freedom is

\[ P_R \equiv \frac{\partial L}{\partial (dR/dt)} = \pi^2 \mu_0 \frac{dR}{dt} \]

from which the Hamiltonian can be calculated as

\[ H \equiv P_R \frac{dR}{dt} - L = \frac{1}{2\pi^2 \mu_0} P_R^2 + \frac{\alpha \pi^2}{4} R^4. \]

Then the action in Hamiltonian form is

\[ S = \int dt \left[ P_R \frac{dR}{dt} - \left( \frac{1}{2\pi^2 \mu_0} P_R^2 + \frac{\alpha \pi^2}{4} R^4 \right) \right]. \]

The above results allow one to compute the hadronic mass spectrum from the spherical membrane Schrödinger equation

\[ \left[ -\frac{1}{2\pi^2 \mu_0} \frac{d^2}{dR^2} + \frac{\alpha \pi^2}{4} R^4 \right] \Psi(R) = M_n \Psi(R) \]

with the following boundary conditions:

\[ \Psi(0) = 0 \]
\[ \lim_{R \to \infty} \Psi(R) = 0. \]

The lowest mass eigenvalues can be evaluated numerically [30] or through WKB approximation [28]:

\[ (2\pi^2 \mu_0 M_n)^{1/2} = \frac{4\pi}{\beta(1/4, 3/2)} \left( \frac{1}{8 \sqrt{2} V_{Hg_{YM}}} \right)^{1/3} \left( n + \frac{3}{4} \right)^{2/3}. \]
where, $\beta$ is the Euler $\beta$–function: $\beta(1/4, 3/2) \equiv \frac{\Gamma(1/4)\Gamma(3/2)}{\Gamma(7/4)}$. The WKB formula (43) gives a mass scale of the correct order of magnitude, $M_n \propto 4\pi g_{YM}^2 V^{-1/3} \sim 1\text{GeV}$. More sophisticated estimates of the glueball mass spectrum, including topological corrections [31], are not very different from the values given by (43). Thus, we conclude that the QCD membrane action (38) encodes, at least the dominant contribution, to the gluon bound states spectrum. Hopefully, an improvement of this result will come from an extension of the minisuperspace approximation along the line discussed in [32], where a new form of the p–brane propagator has been obtained.

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