Weak factorization of the Hardy space $H^p$ for small values of $p$, in the multilinear setting

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Abstract

We give a weak factorization proof of the Hardy space $H^p(\mathbb{R}^n)$ in the multilinear setting, for $\frac{n}{n+1} < p < 1$. As a consequence, we obtain a characterization of the boundedness of the commutator $[b, T]$ from $L^{r_1}(\mathbb{R}^n) \times \cdots \times L^{r_m}(\mathbb{R}^n)$ to $L^{q'}(\mathbb{R}^n)$, where $b \in \text{Lip}_\alpha(\mathbb{R}^n)$, and $\frac{2}{n} = \sum_{i=1}^m \frac{1}{r_i} + \frac{1}{q} - 1$.

Keywords: Weak factorization; Hardy space; Lipschitz space; commutator.

1 Introduction

It is well-known that any function $f$ in the Hardy space of the disc $H^r(\mathbb{D})$ can be decomposed into a product of functions in $H^p(\mathbb{D})$ and $H^q(\mathbb{D})$, where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. When it comes to the real variable Hardy Spaces, things become different.

In 1976, Coifman, Rochberg and Weiss [2] presented a weak factorization result of the Hardy space $H^1(\mathbb{R}^n)$ through commutators. Their proof was based upon the duality between $H^1$ and BMO, a result by Fefferman and Stein [3], and the characterization of BMO in terms of the boundedness of the commutator of a Calderón-Zygmund operator with multiplication operator.

In 1981, Uchiyama [8] proved a weak factorization of the Hardy space $H^p$ in the space of homogeneous type, for $p \leq 1$. His approach allows one to obtain a weak factorization result directly, without assuming any boundedness of the commutator. In fact, the boundedness of the commutator comes as a result of weak factorization. In 2016, Chaffee [1] provided a proof of the boundedness of the commutator in the multilinear setting. As a consequence of this result, one gets a weak factorization of the Hardy space $H^p(\mathbb{R}^n)$. In 2017, Li and Wick [6] adopted Uchiyama’s method to show weak factorization of $H^1(\mathbb{R}^n)$, in the multilinear setting.

In this paper, we extend Uchiyama’s method and provide a proof of the weak factorization of $H^p(\mathbb{R}^n)$, in the multilinear setting, for $\frac{n}{n+1} < p < 1$. As an application, one obtains a characterization of the boundedness of the commutator $[b, T]$ from $L^{r_1}(\mathbb{R}^n) \times \cdots \times L^{r_m}(\mathbb{R}^n)$ to $L^{q'}(\mathbb{R}^n)$, where $b \in \text{Lip}_\alpha(\mathbb{R}^n)$, and $\frac{2}{n} = \frac{1}{p} - 1$.

We first introduce some definitions.

**Definition 1.1.** A bounded tempered distribution $f$ is in the Hardy space $H^p(\mathbb{R}^n)$ if the Poisson maximal function

$$M(f; P) = \sup_{t \geq 0} |(P_t * f)(x)|$$

lies in $L^p(\mathbb{R}^n)$.

**Definition 1.2.** A function $f \in \mathbb{R}^n$ is Lipschitz continuous of order $\alpha > 0$ if there is a constant $C < \infty$ such that for all $x, y \in \mathbb{R}^n$, we have

$$|f(x + y) - f(x)| \leq C|y|^\alpha.$$ 

In this case, we write $f \in \text{Lip}_\alpha(\mathbb{R}^n)$.  

We now state the main result of this paper.
We now review the notion of multilinear Calderón-Zygmund theory studied in [5].

**Definition 1.3.** Let \( 0 < \epsilon, A < \infty \). A locally integrable function \( K(y_0, y_1, \cdots, y_m) \) defined away from the diagonal \( \{ y_0 = y_1 = \cdots = y_m \} \) in \((\mathbb{R}^n)^{m+1}\) is said to be an \( m \)-linear Calderón-Zygmund kernel with constants \( \epsilon, A \) if

- \( K \) satisfies a size condition:
  \[
  |K(y_0, y_1, \cdots, y_m)| \leq \frac{A}{(\sum_{k,l=0}^{m} |y_k - y_l|)^{mn}},
  \]
  and

- \( K \) satisfies a smoothness condition:
  \[
  |K(y_0, \cdots, y_j, \cdots, y_m) - K(y_0, \cdots, y'_j, \cdots, y_m)| \leq \frac{A|y_j - y'_j|}{(\sum_{k,l=0}^{m} |y_k - y_l|)^{mn+\epsilon}},
  \]
  whenever \( 0 \leq j \leq m \) and \( |y_j - y'_j| \leq \frac{1}{2} \max_{0 \leq k \leq m} |y_j - y_k| \).

**Definition 1.4.** Let \( 0 < \epsilon, A < \infty \). An \( m \)-linear operator \( T : L^{r_1}(\mathbb{R}^n) \times \cdots \times L^{r_m}(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \) is said to be a Calderón-Zygmund operator if \( T \) is associated with the \( m \)-linear Calderón-Zygmund kernel \( K \), i.e.,

\[
T(f_1, \cdots, f_m)(x) = \int_{\mathbb{R}^n} K(x, y_1, \cdots, y_m) \prod_{j=1}^{m} f_j(y_j) \, dy_1 \cdots dy_m,
\]

for all \( x \notin \cap_{j=1}^{m} \text{supp}(f_j) \), where \( f_1, \cdots, f_m \) are \( m \) functions on \( \mathbb{R}^n \) with \( \cap_{j=1}^{m} \text{supp}(f_j) \neq \emptyset \), and

\[
\frac{1}{p} = \sum_{i=1}^{m} \frac{1}{r_i}.
\]

The \( l \)th partial adjoint of \( T \) is

\[
T_l^*(f_1, \cdots, f_m)(x) = \int_{\mathbb{R}^n} K(x, y_1, \cdots, y_{l-1}, x, y_{l+1}, \cdots, y_m) \prod_{j=1}^{m} f_j(y_j) \, dy_1 \cdots dy_m.
\]

Note that Calderón-Zygmund operators are originally defined on Schwartz function spaces \( S(\mathbb{R}^n) \).
In [5], the authors show that a Calderón-Zygmund operator \( T \) indeed extends to a bounded operator from \( L^{r_1}(\mathbb{R}^n) \times \cdots \times L^{r_m}(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \), provided \( \frac{1}{p} = \sum_{i=1}^{m} \frac{1}{r_i} \).

**Definition 1.5.** We say that a Calderón-Zygmund operator \( T \) is \( mn \)-homogeneous if the kernel \( K \) of \( T \) satisfies

\[
|K(x_0, \cdots, x_m)| \geq \frac{C}{N^{mn}},
\]

for \( m+1 \) pairwise disjoint balls \( B_0(x_0, r), \cdots, B_m(x_m, r) \) satisfying \( |x_0 - x_l| \approx Nr \) for all \( x_l, l = 1, \cdots, m \), where \( r > 0 \) and \( N \) a large number.

## 2 Statement of Main Results

**Theorem 2.1.** Let \( T \) be an \( m \)-linear Calderón-Zygmund operator that is \( mn \)-homogeneous in the \( l \)th component, with \( \frac{n}{n+1} \) < \( \epsilon < 1 \), where \( 1 \leq l \leq m \). Then, for every \( f \in H^p(\mathbb{R}^n) \), there exist sequences...
\{f_k^1 \subseteq L^p, \{g_j^k \subseteq L^q(\mathbb{R}^n), \{h_{j,m}^k \subseteq L^{r_m}(\mathbb{R}^n), \ldots, \{h_{j,m}^k \subseteq L^{r_m}(\mathbb{R}^n), \text{ with } \frac{1}{s} + \frac{1}{t} + \ldots + \frac{1}{r_m} = \frac{1}{s}, \text{ such that} \}

f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_j^k \Pi_i (g_j^k, h_{j,1}^k, \ldots, h_{j,m}^k) \text{ in } H^p(\mathbb{R}^n), \quad (2.1)

where

\Pi_i (g_j^k, h_{j,1}^k, \ldots, h_{j,m}^k) = h_{j,l}^k \cdot T^*_i (h_{j,1}^k, \ldots, h_{j,l-1}^k, g_j^k, h_{j,l+1}^k, \ldots, h_{j,m}^k) - g_j^k \cdot T^*(h_{j,1}^k, \ldots, h_{j,m}^k).

Moreover, we have

\|f\|_{H^p(\mathbb{R}^n)} \approx \inf \left( \left\{ \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\lambda_j^k|^p \|g_j^k\|_{L^q(\mathbb{R}^n)} \|h_{j,1}^k\|_{L^{r_1}(\mathbb{R}^n)} \cdots \|h_{j,m}^k\|_{L^{r_m}(\mathbb{R}^n)} \right\} \right)^{1/p}, \quad (2.2)

where the infimum is taken over all possible representations of \(f\) that satisfy (2.1).

As a consequence of this Theorem, we obtain a new characterization of \(\text{Lip}_a(\mathbb{R}^n)\) in terms of the boundedness of the commutators with the multilinear Riesz transforms, where the commutator operator (of some function \(b\) against an operator \(T\)) is defined by

\[ [b, T]_l (f_1, \ldots, f_m) = T(f_1, \ldots, bf_1, \ldots, f_m) - bT(f_1, \ldots, f_m). \]

Notice that the multiplication operator \(\Pi_l\) defined in Theorem 2.1 is the dual of the commutator operator \([b, T]_l\).

**Theorem 2.2.** Let \(T\) be an \(m\)-linear Calderón-Zygmund operator, that is \(m\)-homogeneous in the \(l\)th component, with \(\frac{n}{m+1} < \epsilon < 1, b \in L^1_{lo} (\mathbb{R}^n)\) and \(\alpha > 0\) such that \(\frac{2}{\alpha} = \frac{1}{p} - 1\). Suppose that \(b \in \text{Lip}_a(\mathbb{R}^n)\), then

\[ \|b\|_{L^{p, \alpha}(\mathbb{R}^n)} \approx \|[b, T]_l\|_{L^{r_1(\mathbb{R}^n)} \times \cdots \times L^{r_m(\mathbb{R}^n)} \rightarrow L^{q'}(\mathbb{R}^n)}, \]

where \(\frac{1}{s} + \frac{1}{t_1} + \ldots + \frac{1}{r_m} = \frac{1}{s} \) and \(q'\) is the dual exponent of \(q\).

**3 Preliminaries**

In this section, we recall some definitions and theorems we need in order to prove our main results.

**Definition 3.1.** A function \(a\) is called an \(L^\infty\)-atom for \(H^p(\mathbb{R}^n)\) (or simply an \(H^p(\mathbb{R}^n)\)-atom) if there exists a ball \(B \subset \mathbb{R}^n\) such that \(\text{supp } a \subset B, \|a\|_\infty \leq |B|^{-1/p} \) and \(\int_0^\infty x^\gamma a(x) dx = 0 \) for all multi-indices \(\gamma\) with \(|\gamma| \leq \left[ \frac{n}{p} - n \right]\).

Notice that since \(p > \frac{n}{n+1}\) in our case, then we have \(\gamma = 0\).

**Lemma 3.2.** Let \(f\) be a function on \(\mathbb{R}^n\) satisfying:

- \(\int_{\mathbb{R}^n} f(x) dx = 0\)

- there exist \(y_1, y_2 \in \mathbb{R}^n, r \in \mathbb{R}\) and \(N\) large such that

\[ |f(x)| \lesssim C_1 \chi_{B(y_1, r)}(x) + C_2 \chi_{B(y_2, r)}(x) \text{ for } |y_1 - y_2| = Nr. \]

Then,

\[ f \in H^p(\mathbb{R}^n) \text{ and } \|f\|_{H^p(\mathbb{R}^n)} \lesssim N^{n(1-p)} \log_2 N(C_1 |B(y_1, r)| + C_2 |B(y_2, r)|) \]
Proof. We will proceed with the proof of this lemma using the atomic decomposition characterization of $H^p(\mathbb{R}^n)$. Since $|f(x)| \lesssim C_1\chi_{B(y_1, r)}(x) + C_2\chi_{B(y_2, r)}(x)$, we write

$$f = f_1 + f_2$$

where $\text{supp } f_i \subseteq B(y_i, r)$ for $i = 1, 2$.

Let $J_0$ be the smallest integer larger than $\log_2 \frac{|y_1 - y_2|}{r}$ and for $k = 1, \cdots, J_0$, let

$$\alpha^k_i = \frac{|B(y_i, r)|}{|B(y_i, 2^k r)|} \langle f_i \rangle_{B(y_i, r)},$$

and

$$f^k_i = \alpha^{k-1}_i \chi_{B(y_i, 2^{k-1} r)} - \alpha^k_i \chi_{B(y_i, 2^k r)},$$

where $\alpha^0_i = f_i$ and $\langle f_i \rangle_{B(y_i, r)} = \frac{1}{|B(y_i, r)|} \int_{B(y_i, r)} f_i(x) dx$, the average of $f_i$ on $B(y_i, r)$. Then,

$$f = f_1 + f_2 = f_1 + f_2 - \sum_{i=1}^{2} \alpha^1_i \chi_{B(y_i, 2 r)} + \sum_{i=1}^{2} \alpha^1_i \chi_{B(y_i, 2 r)}$$

$$= \sum_{i=1}^{2} (f_i - \alpha^1_i \chi_{B(y_i, 2 r)}) + \sum_{i=1}^{2} \alpha^1_i \chi_{B(y_i, 2 r)}$$

$$= \sum_{i=1}^{2} f^1_i + \sum_{i=1}^{2} \alpha^1_i \chi_{B(y_i, 2 r)}$$

$$= \sum_{i=1}^{2} f^1_i + \sum_{i=1}^{2} \alpha^1_i \chi_{B(y_i, 2 r)} - \alpha^2_i \chi_{B(y_i, 2^2 r)} + \alpha^2_i \chi_{B(y_i, 2^2 r)}$$

$$= \sum_{i=1}^{2} f^1_i + \sum_{i=1}^{2} f^2_i + \sum_{i=1}^{2} \alpha^2_i \chi_{B(y_i, 2^2 r)} - \alpha^3_i \chi_{B(y_i, 2^3 r)} + \alpha^3_i \chi_{B(y_i, 2^3 r)}$$

$$= \sum_{i=1}^{2} (\sum_{k=1}^{J_0} f^k_i) + \sum_{i=1}^{2} \alpha^0_i \chi_{B(y_i, 2^0 r)}.$$

Now for $k = 1, \cdots, J_0$, let

$$\alpha^k_i = \frac{f^k_i}{\|f^k_i\|_{L^\infty}} |B(y_i, 2^k r)|^{-1/p}.$$

Then, $\text{supp } a^k_i \subseteq B(y_i, 2^k r)$, $\|a^k_i\|_{\infty} = |B(y_i, 2^k r)|^{-1/p}$, and,

$$\int a^k_i(x) dx = \frac{1}{\|f_i\|_{L^\infty}} |B(y_i, 2^k r)|^{-1/p} \int f^k_i(x) dx$$

$$= \frac{|B(y_i, 2^k r)|^{-1/p}}{\|f_i\|_{L^\infty}} |B(y_i, 2^k r)|^{-1/p} \int \alpha^{k-1}_i \chi_{B(y_i, 2^{k-1} r)} - \alpha^k_i \chi_{B(y_i, 2^k r)} dx$$

$$= \frac{|B(y_i, 2^k r)|^{-1/p}}{\|f_i\|_{L^\infty}} \left( \int_{B(y_i, 2^{k-1} r)} |B(y_i, 2^{k-1} r)| \langle f_i \rangle_{B(y_i, r)} - \int_{B(y_i, 2^k r)} |B(y_i, 2^k r)| \langle f_i \rangle_{B(y_i, r)} \right)$$

$$= 0.$$

Thus, $a^k_i$ is a $p$-atom, and

$$f = \sum_{i=1}^{2} \left( \sum_{k=1}^{J_0} \|f^k_i\|_{L^\infty} |B(y_i, 2^k r)|^{1/p} a^k_i \right) + \sum_{i=1}^{2} \alpha^0_i \chi_{B(y_i, 2^0 r)}.$$
It remains to estimate
\[ \sum_{i=1}^{2} \alpha_{y_i} J_{0} \chi_{B(y_i, 2^{-0} r)}. \]
To do that, let
\[ \alpha_{y_i} J_{0} = \frac{|B(y_i, r)|}{|B(y_i, 2^{-0} r)|} \langle f_1 \rangle_{B(y_i, r)}. \]
Notice that, since \( f = f_1 + f_2 \) and \( \int f(x) dx = 0 \), we also have
\[ \alpha_{y_i} J_{0} = -\frac{|B(y_2, r)|}{|B(y_i, 2^{-0} r)|} \langle f_2 \rangle_{B(y_2, r)}. \]
Let
\[ f_i J_{0}^{+1} = \alpha_{y_i} J_{0} \chi_{B(y_i, 2^{-0} r)} + (-1)^i \alpha_{y_i} J_{0} \chi_{B(\frac{y_i+y_2}{2}, 2^{-0} r)} + \alpha_{y_i} J_{0} \chi_{B(\frac{y_i+y_2}{2}, 2^{-0} r + 1)} \]
then,
\[ \sum_{i=1}^{2} \alpha_{y_i} J_{0} \chi_{B(y_i, 2^{-0} r)} = f_1 J_{0}^{+1} + f_2 J_{0}^{+1} = f_i J_{0}^{+1}. \]
Let
\[ a_{y_i} J_{0}^{+1} = \frac{f_i J_{0}^{+1}}{\| f_i J_{0}^{+1} \|_{L^\infty} |B(\frac{y_1+y_2}{2}, 2^{-0} r + 1)|^{-1/p}}, \]
then it is easy to see that \( a_{y_i} J_{0}^{+1} \) is a \( p \)-atom, (the only tricky part would be to show \( a_{y_i} J_{0}^{+1} \) has mean value zero but that is implied by the above remark on \( \alpha_{y_i} J_{0} \)), and
\[ f = \sum_{i=1}^{2} \sum_{k=1}^{J_{0}^{+1}} a_{y_i}^{k} \gamma_{y_i}^{k}, \]
where
\[ \gamma_{y_i}^{k} = \begin{cases} \| f_i \|_{L^\infty} |B(y_i, 2^{-k} r)|^{1/p} & \text{for } k = 1, \cdots, J_{0} \\ \| f_i \|_{L^\infty} |B(y_i, 2^{-k} r + 1)|^{1/p} & \text{for } k = J_{0} + 1. \end{cases} \]
Note that, for \( k = 1, \cdots, J_{0} + 1 \), by doubling condition, we have
\[ |\gamma_{y_i}^{k}| = \| f_i \|_{L^\infty} |B(y_i, 2^{-k} r)|^{1/p} \]
\[ \leq \frac{|B(y_i, r)|}{|B(y_i, 2^{-k} r)|} \| f_i \|_{L^\infty} |B(y_i, 2^{-k} r)|^{1/p} \]
\[ \leq C_i (2^{(k-1)n} \| 2^{kn} |B(y_i, r)|^{1/p} \]
\[ \leq C_i 2^{kn(\frac{k}{k-1})} |B(y_i, r)|^{1/p}. \]
Thus, \( f \in H^p(\mathbb{R}^n) \), and
\[ \| f \|_{H^p(\mathbb{R}^n)} \leq \sum_{i=1}^{2} \sum_{k=1}^{J_{0}^{+1}} |\gamma_{y_i}^{k}|^p. \]
Next, we will recall the notion of atomic decomposition. Any $H^p(\mathbb{R}^n)$ function can be decomposed into an infinite sum of $H^p(\mathbb{R}^n)$-atoms. We refer the reader to [Theorem 2.3.12, [4]] for the proof of the theorem.

**Theorem 3.3.** Given a distribution $f \in H^p(\mathbb{R}^n)$, there exists $\{a_j\}_{j=1}^\infty$, a sequence of $H^p(\mathbb{R}^n)$-atoms, and $\{\lambda_j\}_{j=1}^\infty$ such that

$$f = \sum_{j=1}^\infty \lambda_j a_j \text{ in } H^p(\mathbb{R}^n).$$

Moreover, we have

$$\|f\|_{H^p(\mathbb{R}^n)} \approx \inf \left\{ \left( \sum_{j=1}^\infty |\lambda_j|^p \right)^{1/p} : f = \sum_{j=1}^\infty \lambda_j a_j \right\}.$$ 

**Theorem 3.4.** Let $T$ be a bilinear Calderón-Zygmund operator, with $\frac{n}{n+1} < \epsilon < 1$, that is 2n-homogeneous in the $l$th component where $1 \leq l \leq 2$. Then for all $\epsilon > 0$, there exist $N > 0$ and $C > 0$ such that for any $H^p(\mathbb{R}^n)$-atom $a(x)$, there exist $g \in L^q(\mathbb{R}^n)$, $h_1 \in L^{r_1}(\mathbb{R}^n)$ and $h_2 \in L^{r_2}(\mathbb{R}^n)$, with $\frac{1}{q} + \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{p}$, such that

$$\|a - \Pi_l(g, h_1, h_2)\|_{H^p(\mathbb{R}^n)} < \epsilon$$

and

$$\|g\|_{L^q(\mathbb{R}^n)} \|h_1\|_{L^{r_1}(\mathbb{R}^n)} \|h_2\|_{L^{r_2}(\mathbb{R}^n)} \leq CN^{2n}.$$ 

**Proof.** Let $\epsilon > 0$ be given and $a(x)$ be an $H^p(\mathbb{R}^n)$-atom with $\supp a \subset B(x_0, r)$ for some $x_0 \in \mathbb{R}^n, r > 0$. Fix $1 \leq l \leq 2$ and choose $N$ sufficiently large such that $\frac{\log N}{N^{1-p/2}} < \epsilon^p$. Choose $y_i \in \mathbb{R}^n$ such that $y_{l, l} - x_{0, l} = \frac{N}{\sqrt{n}}$, $i = 1, \ldots, n$, where $x_{0, l}, y_{l, i}$ represent the $l$th coordinate of $x_0, y_i$, respectively.

Now choose $y_l$ so that $y_l$ and $y_{l, i}$ satisfy the same relationship as $x_0$ and $y_i$, i.e. $|y_{l, l} - y_{l, i}| = \frac{N}{\sqrt{n}}$, where $l = 1$ if $l = 2$ and $l = 2$ otherwise.

In this proof, we will take the case $l = 2$ and $\tilde{l} = 1$. The other case is similar. Let

$$g(x) = \chi_{B(y_{l, i})}(x)$$

$$h_1(x) = \chi_{B(y_{l, i})}(x)$$

$$h_2(x) = \frac{a(x)}{T_2^*(h_1, g)(x_0)}$$

Then,

$$\supp g = B(y_{l, i}), \supp h_1 = B(y_{l, i}) \text{ and } \supp h_2 = B(x_0, r).$$

Moreover, we have

$$\|g\|_{L^q(\mathbb{R}^n)} = |B(y_{l, i})|^{1/q} \approx r^{n/q}, \|h_1\|_{L^{r_1}(\mathbb{R}^n)} = |B(y_{l, i})|^{1/r_1} \approx r^{n/r_1}$$

and,

$$\|h_2\|_{L^{r_2}(\mathbb{R}^n)} = \frac{\|a\|_{L^{r_2}(\mathbb{R}^n)}}{|T_2^*(h_1, g)(x_0)|} \leq \frac{\|a\|_{L^{r_2}} |B(x_0, r)|^{1/r_2}}{CN^{2n}} \lesssim \frac{r^{-n/p} r^{n/r_2}}{CN^{2n}}.$$
Let \( W \) be a function such that \( T^*_2 \) is \( 2n \)-homogeneous, since \( T \) is.
Thus,
\[
\| g \|_{L^p(\mathbb{R}^n)} \| h_1 \|_{L^{r_1}(\mathbb{R}^n)} \| h_2 \|_{L^{r_2}(\mathbb{R}^n)} \lesssim CN^{2n}.
\]

Let
\[
f(x) := a(x) - \pi_i(g, h_1, h_2)(x)
\]
\[
= a(x) - (h_2 T^*_2(h_1, g)(x) - gT(h_1, h_2)(x))
\]
\[
= a(x) \left( \frac{T^*_2(h_1, g)(x_0) - T^*_2(h_1, g)(x)}{W_1(x)} \right) - g(x) T(h_1, h_2)(x),
\]
and so,
\[
|f(x)| \leq |W_1(x)| + |W_2(x)|.
\]

Our goal is to get the desired bound for each of \( W_1 \) and \( W_2 \). First note that \( \text{supp} \, W_1 \subseteq B(x_0, r) \) and \( \text{supp} \, W_2 \subseteq B(y_2, r) \). So, for \( x \in B(x_0, r) \), by using \( 2n \)-homogeneity of \( T^*_2 \), we have
\[
|W_1(x)| \leq \| a \|_{L^\infty(\mathbb{R}^n)} CN^{2n} |T^*_2(h_1, g)(x_0) - T^*_2(h_1, g)(x)|
\]
\[
= r^{-n/p} CN^{2n} \left| \int K(z_2, z_1, x_0) h_1(z_1) g(z_2) dz_2 dz_1 - \int K(z_2, z_1, x) h_1(z_1) g(z_2) dz_2 dz_1 \right|
\]
\[
\leq C r^{-n/p} N^{2n} \left( \int_{B(y_1, r) \times B(y_2, r)} |K(z_2, z_1, x_0) - K(z_2, z_1, x)| dz_2 dz_1 \right)
\]
\[
\leq C r^{-n/p} N^{2n} \left( \int_{B(y_1, r) \times B(y_2, r)} \left( \| z_2 - z_1 \| + |z_2 - x_0| \right)^{2n+\varepsilon} dz_2 dz_1 \right)
\]
\[
\leq C \frac{1}{N^{\varepsilon n/p}} r^{n}.
\]

Now, for \( x \in B(y_2, r) \), we have
\[
|W_2(x)| = |T(h_1, h_2)(x)|
\]
\[
= \left| \int_{\mathbb{R}^{2n}} K(x, y_1, y_2) h_1(y_1) h_2(y_2) dy_1 dy_2 \right|
\]
\[
= \left| \int_{B(y_1, r) \times B(x_0, r)} K(x, y_1, y_2) a(x) T^*_2(h_1, g)(x_0) dy_1 dy_2 \right|
\]
\[
\leq C r^{-n/p} CN^{2n} \left( \int_{B(y_1, r) \times B(x_0, r)} \left( \| y_2 - x_0 \| + |y_1 - x_0| \right)^{2n+\varepsilon} dy_1 dy_2 \right)
\]
\[
\leq r^{-n/p} CN^{2n} \left( \int_{B(y_1, r) \times B(x_0, r)} \left( \| y_2 - x_0 \| + |y_1 - x_0| \right)^{2n+\varepsilon} dy_1 dy_2 \right)
\]
\[
\leq C \frac{1}{N^{\varepsilon n/p}} r^{n}.
\]

Therefore,
\[
|f(x)| \leq |W_1(x)| + |W_2(x)| \lesssim C \frac{1}{N^{\varepsilon n/p}} (\chi_{B(x_0, r)} + \chi_{B(y_2, r)}).
\]
By previous lemma,
\[ \|f\|_{H^p(R^n)} \leq C \frac{\log N}{N^{\frac{n}{p} - n(1-p)}} < \varepsilon. \]

\[\square\]

4 Proof of Main Results

For simplicity of notations, we will state and prove Theorems 2.1 and 2.2 in the bilinear case.

**Theorem 4.1.** Let \( T \) be a bilinear Calderón-Zygmund operator that is \( 2n \)-homogeneous in the \( l \)-th component, with \( \frac{1}{n+1} < \varepsilon < 1 \), where \( 1 \leq l \leq 2 \). Then, for every \( f \in H^p(R^n) \), there exist sequences \( \{ \lambda_j \} \subseteq \ell, \{ g_j \} \subseteq L^2(R^n), \{ h_{j,1}^k \} \subseteq L^{r_1}(R^n), \{ h_{j,2}^k \} \subseteq L^{r_2}(R^n) \), with \( \frac{1}{q} + \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{p} \), such that

\[ f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_j^k \Pi_1(g_j^k, h_{j,1}^k, h_{j,2}^k) \text{ in } H^p(R^n), \quad (4.1) \]

where

\[ \Pi_1(g_j^k, h_{j,1}^k, h_{j,2}^k) = h_{j,1}^k T_1^*(g_j^k, h_{j,2}^k) - g_j^k T(h_{j,1}^k, h_{j,2}^k) \]

and

\[ \Pi_2(g_j^k, h_{j,1}^k, h_{j,2}^k) = h_{j,2}^k T_2^*(h_{j,1}^k, g_j^k) - g_j^k T(h_{j,1}^k, h_{j,2}^k). \]

Moreover, we have

\[ \|f\|_{H^p(R^n)} \approx \inf \left\{ \left( \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\lambda_j^k|^p \|g_j^k\|_{L^q(R^n)} \|h_{j,1}^k\|_{L^{r_1}(R^n)} \|h_{j,2}^k\|_{L^{r_2}(R^n)} \right)^{1/p} \right\}, \]

such that \( f \) satisfies (2.1).

**Theorem 4.2.** Let \( T \) be a bilinear Calderón-Zygmund operator, that is \( 2n \)-homogeneous in the \( l \)-th component, with \( \frac{1}{n+1} < \varepsilon < 1 \), \( b \in L^1_{loc}(R^n) \) and \( \alpha > 0 \) such that \( \frac{\varepsilon}{n} = \frac{1}{p} - 1 \). Suppose that \( b \in \text{Lip}_\alpha(R^n) \), then

\[ \|b\|_{\text{Lip}_\alpha(R^n)} \approx ||[b, T]||_{L^{r_1}(R^n) \times L^{r_2}(R^n) \rightarrow L^{r'}(R^n)}, \]

where \( \frac{1}{q} + \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{p} \) and \( r' \) is the dual exponent of \( q \).

To prove Theorem 4.1, we will follow Uchiyama’s algorithm. Given a function in \( H^p \), one can use atomic decomposition to decompose this function into atoms. On the other hand, given an atom, one can prove that it can indeed be approximated by the multiplication operator \( \Pi \) in \( H^p \). Using this result and atomic decomposition, one can decompose \( f \) into terms and use an iterative argument to get the desired form (2.1).

**Proof of Theorem 4.1.** Let \( f \in H^p(R^n) \). By Theorem 3.3, there exist some \( \{ \lambda_j \} \subseteq \ell^p \), and a sequence \( \{ a_j \} \) of \( p \)-atoms and some constant \( C \) such that

\[ f = \sum_{j=1}^{\infty} \lambda_j^1 a_j^1 \text{ with } \left( \sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} \leq C \|f\|_{H^p(R^n)}. \]

Now let \( \varepsilon > 0 \) so that \( \varepsilon C < 1 \). By Theorem 4.3, there exist \( \{ g_j \} \subseteq L^q(R^n), \{ h_{j,1}^1 \} \subseteq L^{r_1}(R^n), \{ h_{j,2}^1 \} \subseteq L^{r_2}(R^n) \) with

\[ \|g_j^1\|_{L^q(R^n)} \|h_{j,1}^1\|_{L^{r_1}(R^n)} \|h_{j,2}^1\|_{L^{r_2}(R^n)} < C' N^{2n}, \]

such that

\[ \|a_j^1 - \Pi(g_j^1, h_{j,1}^1, h_{j,2}^1)\|_{H^p(R^n)} < \varepsilon \text{ for all } j. \]
Now,
\[-\begin{align*}
  f &= \sum_{j=1}^{\infty} \lambda_j^1 a_j^1 \\
  &= \sum_{j=1}^{\infty} \lambda_j^1 \Pi_t(g_j^1, h_{j,1}^1, h_{j,2}^1) + \sum_{j=1}^{\infty} \lambda_j^1 (a_j^1 - \Pi_t(g_j^1, h_{j,1}^1, h_{j,2}^1)) \\
  &=: M_1 + E_1.
\end{align*}\]

Notice that
\[-\begin{align*}
  \|E_1\|_{H^p(\mathbb{R}^n)} &\leq \left( \sum_{j=1}^{\infty} |\lambda_j^1|^p \|a_j^1 - \Pi_t(g_j^1, h_{j,1}^1, h_{j,2}^1)\|_{H^p(\mathbb{R}^n)}^p \right)^{1/p} \\
  &\leq C \|f\|_{H^p(\mathbb{R}^n)} < \infty,
\end{align*}\]
so that \(E_1 \in H^p(\mathbb{R}^n)\) and so Theorem 3.3 implies there exist \(\{\lambda_j^2\} \in l^p\) and \(\{a_j^2\}\), a sequence of \(p\)-atoms such that
\[-\begin{align*}
  E_1 &= \sum_{j=1}^{\infty} \lambda_j^2 a_j^2 \text{ with } \|\lambda_j^2\|_p = \left( \sum_{j=1}^{\infty} |\lambda_j^2|^p \right)^{1/p} \leq C \|E_1\|_{H^p(\mathbb{R}^n)} \leq C^2 \varepsilon \|f\|_{H^p(\mathbb{R}^n)},
\end{align*}\]
and now by Theorem 3.4 for each \(p\)-atom \(a_j^2\), there exist \(g_j^2 \in L^q(\mathbb{R}^n), h_{j,1}^2 \in L^{r_1}(\mathbb{R}^n)\) and \(h_{j,2}^2 \in L^{r_2}(\mathbb{R}^n)\) with
\[-\begin{align*}
  &\|g_j^2\|_{L^q(\mathbb{R}^n)} \|h_{j,1}^2\|_{L^{r_1}(\mathbb{R}^n)} \|h_{j,2}^2\|_{L^{r_2}(\mathbb{R}^n)} < C' N^{2n} \text{ such that}
  \|a_j^2 - \Pi_t(g_j^2, h_{j,1}^2, h_{j,2}^2)\|_{H^p(\mathbb{R}^n)} < \varepsilon.
\end{align*}\]

We apply the same iterative argument used on \(f\) but this time to \(E_1\), to get
\[-\begin{align*}
  E_1 &= \sum_{j=1}^{\infty} \lambda_j^2 a_j^2 \\
  &= \sum_{j=1}^{\infty} \lambda_j^2 \Pi_t(g_j^2, h_{j,1}^2, h_{j,2}^2) + \sum_{j=1}^{\infty} \lambda_j^2 (a_j^2 - \Pi_t(g_j^2, h_{j,1}^2, h_{j,2}^2)) \\
  &=: M_2 + E_2,
\end{align*}\]
which implies,
\[-\begin{align*}
  f &= M_1 + M_2 + E_2 = \sum_{k=1}^{2} M_k + E_2.
\end{align*}\]

Similar to what we did before, we get
\[-\begin{align*}
  \|E_2\|_{H^p(\mathbb{R}^n)} &\leq \left( \sum_{j=1}^{\infty} |\lambda_j^2|^p \|a_j^2 - \Pi_t(g_j^2, h_{j,1}^2, h_{j,2}^2)\|_{H^p(\mathbb{R}^n)}^p \right)^{1/p} \\
  &\leq C^2 \varepsilon^2 \|f\|_{H^p(\mathbb{R}^n)},
\end{align*}\]
and so \(E_2 \in H^p(\mathbb{R}^n)\). We keep repeating the same iteration process to get that, for \(M \in \mathbb{N}\), \(f\) can be represented as
\[-\begin{align*}
  f &= \sum_{k=1}^{M} \sum_{j=1}^{\infty} \lambda_j^k \Pi_t(g_j^k, h_{j,1}^k, h_{j,2}^k) + E_M,
\end{align*}\]
where, for \(j, k\), \(\lambda_j^k \in l^p, g_j^k \in L^q(\mathbb{R}^n), h_{j,1}^k \in L^{r_1}(\mathbb{R}^n), h_{j,2}^k \in L^{r_2}(\mathbb{R}^n)\), with
\[-\begin{align*}
  \|g_j^k\|_{L^q(\mathbb{R}^n)} \|h_{j,1}^k\|_{L^{r_1}(\mathbb{R}^n)} \|h_{j,2}^k\|_{L^{r_2}(\mathbb{R}^n)} \lesssim CN^{2n},
\end{align*}\]
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and

$$E_M = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_j^k (a_j^k - \Pi_t(g_j^k, h_{j,1}^k, h_{j,2}^k)),$$

with

$$\left( \sum_{j=1}^{\infty} |\lambda_j^k|^p \right)^{1/p} \leq C^k \varepsilon^{k-1} \|f\|_{H^p(\mathbb{R}^n)}$$

and

$$\|a_j^2 - \Pi_t(g_j^k, h_{j,1}^k, h_{j,2}^k)\|_{H^p(\mathbb{R}^n)} < \varepsilon.$$

This implies $E_M \in H^p(\mathbb{R}^n)$ with $\|E_M\|_{H^p(\mathbb{R}^n)} \leq (\varepsilon C)^M \|f\|_{H^p(\mathbb{R}^n)}$. Letting $M \to \infty$, we get that $\|E_M\|_{H^p(\mathbb{R}^n)} \to 0$, with

$$f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_j^k \Pi_t(g_j^k, h_{j,1}^k, h_{j,2}^k).$$

Moreover, since $\varepsilon C < 1$, we have

$$\left( \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\lambda_j^k|^p \right)^{1/p} \leq \left( \sum_{k=1}^{\infty} \varepsilon^{k-1} C^k \|f\|_{H^p(\mathbb{R}^n)} \right)^{1/p} \lesssim \|f\|_{H^p(\mathbb{R}^n)}.$$

This, together with [4.1] gives us

$$\|f\|_{H^p(\mathbb{R}^n)} \gtrsim \inf \left\{ \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\lambda_j^k|^p \|g_j^k\|_{L^q(\mathbb{R}^n)} \|h_{j,1}^k\|_{L^{r_1}(\mathbb{R}^n)} \|h_{j,2}^k\|_{L^{r_2}(\mathbb{R}^n)} : f \text{ satisfies } [4.1] \right\}.$$

On the other hand, by a result of [7], we have that, for any $g \in L^q(\mathbb{R}^n)$, $h_1 \in L^{r_1}(\mathbb{R}^n)$, $h_2 \in L^{r_2}(\mathbb{R}^n)$,

$$\|\Pi_t(g, h_1, h_2)\|_{H^p(\mathbb{R}^n)} \lesssim \|g\|_{L^q(\mathbb{R}^n)} \|h_1\|_{L^{r_1}(\mathbb{R}^n)} \|h_2\|_{L^{r_2}(\mathbb{R}^n)}.$$

Then, for any $f \in H^p(\mathbb{R}^n)$ having the representation [4.1], we have that

$$\|f\|_{H^p(\mathbb{R}^n)} = \| \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_j^k \Pi_t(g_j^k, h_{j,1}^k, h_{j,2}^k) \|_{H^p(\mathbb{R}^n)}$$

$$\leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\lambda_j^k|^p \|\Pi_t(g_j^k, h_{j,1}^k, h_{j,2}^k)\|_{H^p(\mathbb{R}^n)}$$

$$\lesssim \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\lambda_j^k|^p \|g_j^k\|_{L^q(\mathbb{R}^n)} \|h_{j,1}^k\|_{L^{r_1}(\mathbb{R}^n)} \|h_{j,2}^k\|_{L^{r_2}(\mathbb{R}^n)}.$$

This implies,

$$\|f\|_{H^p(\mathbb{R}^n)} \lesssim \inf \left\{ \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\lambda_j^k|^p \|g_j^k\|_{L^q(\mathbb{R}^n)} \|h_{j,1}^k\|_{L^{r_1}(\mathbb{R}^n)} \|h_{j,2}^k\|_{L^{r_2}(\mathbb{R}^n)} : f \text{ satisfies } [4.1] \right\}.$$

This finishes our proof. \hfill \Box

Proof of Theorem 4.2. Let $b \in Lip_p(\mathbb{R}^n)$. A more general case of the first inequality was proved in [7, Theorem 1.1]. For the second part, suppose $f \in H^p(\mathbb{R}^n)$ such that $\frac{1}{p} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{q}$, and that $[b, T]$ maps $L^{r_1}(\mathbb{R}^n) \times L^{r_2}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $q'$ is the dual exponent of $q$, then by Theorem 4.1 there exist sequences $\{\lambda_j^k\} \subseteq L^p$, $\{g_j^k\} \subseteq L^q(\mathbb{R}^n)$, $\{h_{j,1}^k\} \subseteq L^{r_1}(\mathbb{R}^n)$, $\{h_{j,2}^k\} \subseteq L^{r_2}(\mathbb{R}^n)$ such that

$$f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_j^k \Pi_t(g_j^k, h_{j,1}^k, h_{j,2}^k)$$

in $H^p(\mathbb{R}^n)$, (4.3)
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and so,

$$
\langle b, f \rangle_{L^2(\mathbb{R}^n)} = \langle b, \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_j^k \Pi_l(g_j^k, h_{j,1}^k, h_{j,2}^k) \rangle_{L^2(\mathbb{R}^n)}
$$

$$
= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_j^k \langle b, \Pi_l(g_j^k, h_{j,1}^k, h_{j,2}^k) \rangle_{L^2(\mathbb{R}^n)}
$$

$$
= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_j^k \langle b, [b, T]^l(h_{j,1}^k, h_{j,2}^k) \rangle_{L^2(\mathbb{R}^n)}
$$

This implies, by Holder’s inequality,

$$
|\langle b, f \rangle| \leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\lambda_j^k||g_j^k||_{L^q(\mathbb{R}^n)}||[b, T]^l(h_{j,1}^k, h_{j,2}^k)||_{L^s(\mathbb{R}^n)}
$$

$$
\leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\lambda_j^k||g_j^k||_{L^q(\mathbb{R}^n)}||[b, T]^l||_{L^r(\mathbb{R}^n) \times L^r(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)}||h_{j,1}^k||_{L^r(\mathbb{R}^n)}||h_{j,2}^k||_{L^r(\mathbb{R}^n)}
$$

$$
\lesssim \|f\|_{H^p(\mathbb{R}^n)}||[b, T]^l||_{L^r(\mathbb{R}^n) \times L^r(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)}.
$$

By duality between $H^p(\mathbb{R}^n)$ and $\text{Lip}_\alpha(\mathbb{R}^n)$, we have that,

$$
\|b\|_{\text{Lip}_\alpha(\mathbb{R}^n)} \approx \sup_{\|f\|_{H^p(\mathbb{R}^n)} \leq 1} |\langle b, f \rangle|_{L^2(\mathbb{R}^n)}
$$

$$
\lesssim ||[b, T]^l||_{L^r(\mathbb{R}^n) \times L^r(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)}.
$$

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