First-Order Methods with Increasing Iterate Averaging for Solving Saddle-Point Problems

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Abstract

First-order methods are known to be among the fastest algorithms for solving large-scale convex-concave saddle-point problems. Algorithms that achieve a theoretical convergence rate on the order of $1/T$ are known, but these often rely on uniformly averaging iterates in order to get the guaranteed rate. In contrast, using the last iterate has repeatedly been found to perform better in practice, but with no guarantee on convergence rate. In this paper we propose using averaging schemes with increasing weight on recent iterates, which leads to a guaranteed $1/T$ convergence rate, while capturing the practical performance of using the last iterate. We show this for Chambolle and Pock’s primal-dual algorithm, and mirror prox. We present numerical results on computing Nash equilibria in matrix games, competitive equilibria in Fisher markets, and image denoising via total-variation minimization under the $\ell_1$ norm. In all cases we find that our averaging schemes lead to much better performance than uniform averaging, and sometimes even better performance than using the last iterate.

1 Introduction

This paper focuses on solving convex-concave saddle-point problems of the form

$$\min_{x \in X} \max_{y \in Y} L(x, y) = \langle Kx, y \rangle + f(x) + g(x) - h^*(y),$$  \hspace{1cm} (1)

which is the convex-concave saddle-point formulation of the primal problem

$$\min_{x \in X} f(x) + g(x)h(Kx).$$  \hspace{1cm} (2)

In particular we will show that a simple modification of the iterate-averaging schemes of two well-known algorithms, Chambolle and Pock’s primal-dual algorithm ($\mathcal{PD}$) [Chambolle and Pock, 2011, 2016] and Nemirovski’s mirror prox ($\mathcal{MP}$) [Nemirovski, 2004] lead to much better practical performance, while retaining the same worst-case $O(1/T)$ dependence on $T$ in the convergence rate. Ergodic rates for $1/T$ methods for solving saddle-point problems often rely on uniformly averaging all $T$ iterates (for example in the case of $\mathcal{PD}$ and $\mathcal{MP}$), whereas in practice the last iterate is known to often converge at a much faster rate than the uniform average [Chambolle and Pock, 2016]. However, the last iterate has no guarantee on the convergence rate. This drawback is mentioned, for example, by Tran-Dinh et al. [2018]. To ameliorate this issue, we suggest averaging schemes that put increasing weight on later iterates, for example by weighting the $t$’th iterate according to $w(t) = t\tau_t$ (linear averaging) or $w(t) = t^2\tau_t$ (quadratic averaging), where $\tau_t$ is the stepsize of the algorithm. This is inspired by the practical success of the CFR$^+$ algorithm [Tammelin et al., 2015], which uses linear averaging, and has been used for several recent breakthroughs in the solution of large-scale sequential games [Bowling et al., 2015, Moravčík et al., 2017, Brown and Sandholm, 2018].

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We show that linear and quadratic averaging achieve the same theoretical performance as uniform averaging in both PD and MP. Perhaps more importantly, we point out that their convergence rate depends on the sum of iterate distances \(d(x^0, y^0), \ldots, d(x^T, y^T)\) (letting \(d(\cdot)\) be a suitably-defined stand-in for distance to the saddle point), rather than just on the distance \(d(x^0, y^0)\) as is achieved via the telescoping argument for uniform averaging. In practice we expect \(d(x^t, y^t)\) to decrease as \(t\) increases, and in that case a dependence on the sum (with a correspondingly-larger sum-of-weights denominator) is much better than a dependence on just \(d(x^0, y^0)\).

In numerical experiments we show that linear and quadratic averaging achieves strong practical performance across three different domains of saddle-point problems, and for three algorithms: PD, MP, and a linesearch variant PDELS of PD. First, in the case of matrix games, we show that quadratic averaging leads to stronger performance than the last iterate in all three algorithms, and we even find that it converges faster than the CFR algorithm [Tammelin et al. 2015], the practical state-of-the-art for solving large-scale sequential games [Brown et al., 2017, Burch, 2017, Kroer et al., 2018]. Second, we investigate denoising of images via \(l_1\)-based total-variation minimization. Again we find that, especially quadratic averaging, performs better than, or roughly as well as, the last iterate. Finally, we consider the solution of a convex program for computing a competitive equilibrium in a Fisher market, a problem that has application in Internet auction markets [Conitzer et al., 2018b, a] and fair division [Varian, 1974, Cole and Gkatzelis, 2015, Caragiannis et al., 2016]. Again we find that linear and quadratic averaging perform much better than uniform averaging, although in this case the last iterate performs somewhat better in late stages of the optimization.

The use of linear averaging in the present paper is inspired by the practical performance of linear averaging in CFR+. Similar to our setting, linear averaging provides no worst-case theoretical benefit in CFR+, but it does lead to much stronger practical performance (quadratic averaging has also been shown to work well in CFR settings [Brown and Sandholm, 2019]). Another technique from CFR+ that is important for its practical performance is alternation, where \(x^{t+1}\) is computed based on \((x^t, y^t)\), but \(y^{t+1}\) is computed based on \((x^{t+1}, y^t)\). In our setting PD has a form of alternation in that \(y^{t+1}\) is computed based on the extrapolation \(2x^{t+1} - x^t\), but MP does not have any form of alternation. Yet we find that the algorithms have near-identical performance, and so while the performance of linear (or greater) iterate averaging generalizes to our setting, alternation does not.

2 Notation

We will be solving problems of the form (1), where \(\mathcal{X}, \mathcal{Y}\) are finite-dimensional real vector spaces equipped with norms \(\|\cdot\|_x, \|\cdot\|_y\) (we will generally refrain from using the subscripts when e.g. \(\|x\|\) clearly refers to \(\|x\|_x\)). We will also need the dual norm for \(x\): \(\|g\|_x = \max_{\|x\| \leq 1} \langle g, x \rangle\). We make the following further assumptions (which are identical to those of [Chambolle and Pock, 2016]):

- We have access to distance-generating functions \(\psi_x, \psi_y\) which are 1-strongly convex wrt. the respective norms \(\|\cdot\|_x, \|\cdot\|_y\), and continuously differentiable on their domains. \(D_x, D_y\) are the corresponding Bregman divergences defined as
  \[
  D_x(x, x') = \psi_x(x) - \psi_x(x') - \langle \nabla \psi_x(x'), x - x' \rangle,
  \]
  \[
  D_y(y, y') = \psi_y(y) - \psi_y(y') - \langle \nabla \psi_y(y'), y - y' \rangle.
  \]
- \(K : \mathcal{X} \rightarrow \mathcal{Y}\) is a bounded linear operator, with operator norm \(L = \|K\| = \sup_{\|x\|, \|y\| \leq 1} \langle Kx, y \rangle\) of \(Kx, y\). This implies \(\langle Kx, y \rangle \leq L \|x\| \|y\|\)
- \(f\) is a proper, lower semicontinuous (l.s.c.) convex function. Its gradient is Lipschitz continuous on \(\mathcal{X}\), i.e. \(\|\nabla f(x) - \nabla f(x')\| \leq L_f \|x - x'\|, \forall x, x' \in \mathcal{X}\)
- \(g\) and \(h\) are proper, l.s.c. convex functions with simple structure such that their prox-mappings can be computed for any \(\tau, \sigma > 0\)
  \[
  \text{Prox}_x(\xi, x') = \arg \min_{\mathcal{X}} \langle \xi, x \rangle + g(x) + \frac{1}{\tau} D_x(x, x'),
  \]
  \[
  \text{Prox}_y(\xi, y') = \arg \min_{\mathcal{Y}} \langle \xi, y \rangle + h(y) + \frac{1}{\sigma} D_y(y, y').
  \]

We assume that \(\text{dom } f \cap \text{dom } g \subseteq \text{dom } \psi_x\) and \(\text{dom } h^* \subseteq \text{dom } \psi_y\).
We now give examples of two of the most common choices of distance-generating functions $\psi_x, \psi_y$. The squared Euclidean norm $\frac{1}{2} \| \cdot \|_2^2$ leads to the Bregman divergence $D(x, x') = \frac{1}{2} \| x - x' \|_2^2$, and is often called the “Euclidean setting.” This leads to 1-strong convexity over all of $\mathbb{R}^n$ with respect to the $\ell_2$ norm. The negative entropy $\sum_i x_i \log x_i$ is 1-strongly convex over the probability simplex $\{ x : \| x \|_1 = 1, x \geq 0 \}$ wrt. the $\ell_1$ norm, and we will refer to this as the entropy distance.

We will measure error according to the saddle-point residual

$$\epsilon_{\text{sad}}(\bar{x}, \bar{y}) = \max_{y \in \mathcal{Y}} \mathcal{L}(\bar{x}, y) - \min_{x \in \mathcal{X}} \mathcal{L}(x, \bar{y}).$$

### 3 Algorithms

We will investigate two specific algorithms from the literature, that have strong performance both in theory and practice. Both algorithms rely on ergodic convergence rates, usually achieved by uniformly averaging iterates. We first introduce the alternative averaging schemes that we will use.

#### 3.1 Averaging schemes

We assume that we have a sequence of iterates $x_1, \ldots, x_T$ and $y_1, \ldots, y_T$. We will construct the ergodic solution according to the scheme

$$\bar{x}^T = \frac{1}{T} \sum_{t=1}^T \tau_t w(t) x_t, \quad \bar{y}^T = \frac{1}{T} \sum_{t=1}^T \tau_t w(t) y_t, \quad S_T = \sum_{t=1}^T \tau_t w(t),$$

where $\tau_t$ is the stepsize used at iteration $t$ of a given algorithm, and $w(t)$ is a weakly-increasing function in $t$. If the given algorithm uses the same stepsize at all iterations then we can set $\tau_t = 1$ in the averaging scheme, even if the stepsize is not 1 (this simplifies calculations without changing the algorithm). We will be particularly interested in the following variants:

- **Uniform averaging**: $w(t) = 1$
- **Linear averaging**: $w(t) = t$
- **Quadratic averaging**: $w(t) = t^2$
- **Last iterate**: $w(t) = 0$ for $t < T$, $w(T) = 1$

We will say that $w(\cdot)$ constitutes an increasing averaging scheme if $w(t+1) \geq w(t)$, with $w(0) = 0$.

#### 3.2 Chambolle & Pock’s primal-dual algorithm

The first algorithm we will describe is Chambolle & Pock’s primal dual algorithm PD [Chambolle and Pock [2011]], using the more recent description given by Chambolle and Pock [2016]. The algorithm is given in Algorithm 1. It repeatedly updates $x$ based on proximal steps using a linearization of $f$, and then computes the proximal update for $y$ based on an extrapolation of $x^t$ and $x^{t+1}$. At each iteration two proximal mappings are computed. First $x^{t+1}$ is computed based on the proximal mapping wrt. the previous iterates $x^t, y^t$, and secondly $y^{t+1}$ is computed, but using a gradient based on an extrapolation of $x^t$ and $x^{t+1}$. Chambolle and Pock [2016] show that the uniform average of iterates converges to a saddle point at a rate of $1/T$. We will use their Lemma 1 [Chambolle and Pock]
2016, which is a descent lemma for the primal-dual iteration. Here we specialize it to the iterates of PD, whereas Chambolle and Pock [2016] show a more general variant with flexible choices of how gradients are computed:

**Lemma 1** (Chambolle and Pock [2016]). If $x^t, y^t, x^{t+1}, y^{t+1}$ are generated according to PD, then for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ we have

$$
\mathcal{L}(x^{t+1}, y) - \mathcal{L}(x, y^t) \leq \frac{1}{\tau} \left(D_x(x, x^t) - D_x(x, x^{t+1}) - D_x(x^{t+1}, x^t)\right) + \frac{L_f}{2} \|x^{t+1} - x^t\|^2 + \frac{1}{\sigma} \left(D_y(y, y^t) - D_y(y, y^{t+1}) - D_y(y^{t+1}, y^t)\right) + \left(K(x - x^{t+1}, y - y^{t+1}) - \langle K(x^{t+1} - x^t), y - y^{t+1} \rangle\right).
$$

With this lemma we can show that PD achieves a form of ergodic rate that depends on the distance to optimality over the sequence of iterates (rather than the usual dependence on the distance at $x^0, y^0$).

**Theorem 1.** Let the stepsize parameters $\tau, \sigma > 0$ in PD be such that for all $x, x' \in \text{dom } g$ and $y, y' \in \text{dom } h^*$ it holds that

$$
\left(\frac{1}{\tau} - L_f\right) D_x(x, x') + \frac{1}{\sigma} D_y(y, y') - \langle K(x - x'), y - y' \rangle \geq 0. \tag{3}
$$

Then for any $(x, y) \in \mathcal{X} \times \mathcal{Y}$ we have

$$
\bar{\varepsilon}_{sad}(\bar{x}^T, \bar{y}^T) \leq \frac{1}{S_T} \sum_{t=0}^{T-1} (w(t + 1) - w(t)) \left(\frac{1}{\tau} D_x(x, x^t) + \frac{1}{\sigma} D_y(y, y^t) - \langle K(x - x^t), y - y^t \rangle\right),
$$

where $\bar{x}^T, \bar{y}^T$ are constructed according to some increasing averaging scheme $w(\cdot)$.

**Proof.** Chambolle and Pock [2016] observe that Lemmas [1] and (3) imply the following inequality, where all bracketed terms are nonnegative

$$
\mathcal{L}(x^{t+1}, y) - \mathcal{L}(x, y^t) \leq \left[\frac{1}{\tau} D_x(x, x^t) + \frac{1}{\sigma} D_y(y, y^t) - \langle K(x - x^t), y - y^t \rangle\right] - \left[\frac{1}{\tau} D_x(x, x^{t+1}) + \frac{1}{\sigma} D_y(y, y^{t+1}) - \langle K(x - x^{t+1}), y - y^{t+1} \rangle\right] - \left[\frac{1}{\tau} D_x(x^{t+1}, x^t) + \frac{1}{\sigma} D_y(y^{t+1}, y^t) - \langle K(x^{t+1} - x^t), y^{t+1} - y^t \rangle - \frac{L_f}{2} \|x^{t+1} - x^t\|^2\right]. \tag{4}
$$

Now we sum this estimate from 1 to $T$, where each term is weighted by $w(t)$

$$
\sum_{t=1}^{T} w(t) \left(\mathcal{L}(x^t, y) - \mathcal{L}(x, y^t)\right) \leq \sum_{t=1}^{T} \left(w(t) - w(t-1)\right) \left[\frac{1}{\tau} D_x(x, x^{t-1}) + \frac{1}{\sigma} D_y(y, y^{t-1}) - \langle K(x - x^{t-1}), y - y^{t-1} \rangle\right], \tag{6}
$$

where we have dropped the last negative term in (4), performed a variant of the standard telescoping trick, but using our increasing averaging scheme, and dropped the last negative term in the sequence at $T$.

Now we note that (5) can be lower-bounded using the convex-concave structure of $\mathcal{L}(\cdot, \cdot)$

$$
\sum_{t=1}^{T} w(t) \left(\mathcal{L}(x^t, y) - \mathcal{L}(x, y^t)\right) \geq \frac{1}{S_T} \left(\mathcal{L}\left(\sum_{t=1}^{T} \frac{w(t)x^t}{S_T}, y\right) - \mathcal{L}\left(x, \sum_{t=1}^{T} \frac{w(t)y^t}{S_T}\right)\right) = \frac{1}{S_T} \left(\mathcal{L}(\bar{x}^T, y) - \mathcal{L}(x, \bar{y}^T)\right).
$$

□
Chambolle and Pock [2016] note that (5) is satisfied as soon as $L^2 \leq \frac{1}{\sigma} (\frac{1}{\tau} - L_f)$. A natural choice is $\frac{1}{\tau} = L + L_f$, $\frac{1}{\sigma} = L$.

If we have an upper bound $\Omega_{pd}$ on each distance term $D_x(x^*, x^t), D_y(y^*, y^t), \|x^* - x^t\|^2, \|y^* - y^t\|^2$, where $(x^*, y^*)$ is a saddle point then we can use Theorem 1 to get simpler bounds when using increasing averaging schemes. Here we show two special cases: linear and quadratic averaging. Similar results for higher-order odd polynomials can be derived via Faulhaber’s formula.

**Theorem 2.** $\mathcal{PD}$ with the linear averaging scheme and stepsize according to the condition in Theorem 1 satisfies

$$
\bar{\epsilon}_{sad}(\bar{x}^T, \bar{y}^T) \leq \frac{2\Omega_{pd} \left( \frac{1}{\tau} + \frac{1}{\sigma} + L \right)}{T + 1}.
$$

**Proof.** By Theorem 1 and the fact that for linear averaging $S_T = \frac{T(T+1)}{2}$ we have

$$
\epsilon_{sad}(\bar{x}^T, \bar{y}^T) \leq \frac{1}{S_T} \sum_{t=0}^{T-1} (w(t+1) - w(t)) \left( \frac{1}{\tau} D_x(x^*, x^t) + \frac{1}{\sigma} D_y(y^*, y^t) - \langle K(x^* - x^t), y^* - y^t \rangle \right)
$$

$$
\leq \frac{1}{S_T} \sum_{t=1}^{T} (w(t) - w(t-1)) \left( \Omega_{pd} \left( \frac{1}{\tau} + \frac{1}{\sigma} \right) + L \|x^* - x^t\| \|y^* - y^t\| \right)
$$

$$
\leq \frac{1}{S_T} \sum_{t=1}^{T} (w(t) - w(t-1)) \Omega_{pd} \left( \frac{1}{\tau} + \frac{1}{\sigma} + L \right)
$$

$$
\leq \frac{\Omega_{pd} \left( \frac{1}{\tau} + \frac{1}{\sigma} + L \right) T}{S_T} \sum_{t=1}^{T} (t - t + 1) \leq \frac{\Omega_{pd} \left( \frac{1}{\tau} + \frac{1}{\sigma} + L \right) T}{S_T} = \frac{2\Omega_{pd} \left( \frac{1}{\tau} + \frac{1}{\sigma} + L \right)}{T + 1}.
$$

**Theorem 3.** $\mathcal{PD}$ with the quadratic averaging scheme and stepsize according to the condition in Theorem 1 satisfies

$$
\epsilon_{sad}(\bar{x}^T, \bar{y}^T) \leq \frac{6\Omega_{pd} \left( \frac{1}{\tau} + \frac{1}{\sigma} + L \right)}{2T + 1}.
$$

**Proof.** By Theorem 1, $S_T = \frac{T(T+1)(2T+1)}{6}$, and using the same steps as in Theorem 2 we have

$$
\epsilon_{sad}(\bar{x}^T, \bar{y}^T) \leq \frac{\Omega_{pd} \left( \frac{1}{\tau} + \frac{1}{\sigma} + L \right)}{S_T} \sum_{t=1}^{T} (w(t) - w(t-1)) \leq \frac{\Omega_{pd} \left( \frac{1}{\tau} + \frac{1}{\sigma} + L \right)}{S_T} \sum_{t=1}^{T} (t^2 - (t-1)^2)
$$

$$
= \frac{\Omega_{pd} \left( \frac{1}{\tau} + \frac{1}{\sigma} + L \right)}{S_T} \sum_{t=1}^{T} (2t - 1) \leq \frac{2\Omega_{pd} \left( \frac{1}{\tau} + \frac{1}{\sigma} + L \right)}{S_T} \sum_{t=1}^{T} t
$$

$$
\leq \frac{2\Omega_{pd} \left( \frac{1}{\tau} + \frac{1}{\sigma} + L \right) T(T+1)}{2} \leq \frac{6\Omega_{pd} \left( \frac{1}{\tau} + \frac{1}{\sigma} + L \right)}{2T + 1}.
$$

Theorems 2 and 3 lead to the same asymptotic rate as uniform averaging, and are even slightly worse, in that factors of 2 and 6 are lost, respectively. Nonetheless we will show later that these averaging schemes achieve practical state-of-the-art performance, while retaining the $1/T$ rate. Uniform averaging does not do that.

From a theoretical perspective, Theorems 2 and 3 might not be the right way to motivate the superiority of linear or quadratic averaging. Instead, Theorem 1 itself is quite interesting: if we use linear averaging (chosen as an example, these observations apply to e.g. quadratic averaging as well) and apply Theorem 1 directly we get a rate

$$
\epsilon_{sad}(\bar{x}^T, \bar{y}^T) \leq \frac{1}{S_T} \sum_{t=0}^{T-1} \left( \frac{1}{\tau} D_x(x, x^t) + \frac{1}{\sigma} D_y(y, y^t) - \langle K(x - x^t), y - y^t \rangle \right).
$$

(7)
In uniform averaging the comparable result is that we get a rate which depends only on the $t = 1$ part of this equation. Having only $t = 1$ is nice from an interpretability perspective: our error depends only on how far away we start, rather than depending on intermediate distances. If each term $t = 1, \ldots, T$ in (7) is upper bounded by some constant, then the bound is basically the same for uniform and linear averaging, as Theorem 2 shows. However, in practice we expect $(x^t, y^t)$ to move closer to $(x^*, y^*)$ as $t$ increases, and in that case (7) gives a much stronger bound than the one achieved by uniform averaging. For example, if the error at $(x^t, y^t)$ goes down at a rate of $1/t$ then we get a convergence rate $O(H_T/T^2)$, where $H_T$ is the $T$th harmonic number. An extremely interesting direction of future work would be to theoretically justify cases where the error at $(x^t, y^t)$ goes down; in practice it happens very frequently, for example in all our experiments. Last-iterate convergence has been shown for some specific algorithms [Malitsky and Pock, 2018, Daskalakis et al. 2018, Daskalakis and Panagiotis2019], but these results are for the limit as $t \to \infty$, and so cannot be used to reason about our setting, where we need some guarantee on the rate as well. Furthermore, in order to match our experimental findings, as well as those of Chambolle and Pock [2016] for last-iterate performance, the theory should most likely be specific to the Euclidean distance, since the entropy distance does not perform well under increasing averaging or in the most-recent iterate.

### 3.3 Mirror prox

Next we describe the mirror prox (MP) algorithm [Nemirovski 2004], which has arguably simpler updates in that there is no alternation, and it avoids the interpolation of $x^{t+1}$ and $x^t$. On the other hand MP further assumes that $X$ and $Y$ are compact convex sets, unlike $PD$.

It is convenient, and customary, to work with the product space $Z = X \times Y$ in the context of $MP$. Distance functions, Bregman divergences, and norms, can be constructed for $Z$ based on those given for $X, Y$ (see e.g. Nemirovski2004, Juditsky and Nemirovski2011a,b for details). Mirror prox considers saddle-point problems of the form

$$\min_{x \in X} \max_{y \in Y} \phi(x, y),$$

under the assumptions that $\phi$ is convex-concave, $X, Y$ are convex and compact, and that $\phi$ is smooth in the sense that the gradient mapping $F(z) = F(x, y) = [\nabla_x \phi(x, y), -\nabla_y \phi(x, y)]$ is Lipschitz continuous with constant $L$ with respect to the norm $\| \cdot \|_z$ associated to the distance-generating function used. The pseudocode is given in Algorithm 2.

**Algorithm 2 Mirror prox algorithm**

**Input:** Bregman divergences $D_x, D_y$, and stepsizes $\tau_t$.

1. **function** $MP$
2. \hspace{1em} Choose $x_1 \in X, y_1 \in Y$
3. \hspace{1em} for $t = 1$ to $T$ do
4. \hspace{2em} $z^t = (\hat{x}^t, \hat{y}^t) = \arg \min_{z \in Z} (\tau_t F(z^t), z) + D_z(z, z^t)$
5. \hspace{2em} $z^{t+1} = (x^{t+1}, y^{t+1}) = \arg \min_{z \in Z} (\tau_t F(z^{t+1}), z) + D_z(z, z^{t+1})$
6. \hspace{1em} return averaged iterates $\bar{z}^T = (\bar{x}^T, \bar{y}^T) = \frac{1}{S_T} \sum_{t=1}^T \tau_t w(t) z^t$

[Nemirovski2004] proves the following descent lemma (see e.g. Bubeck2015 at the end of the proof of Theorem 4.4 for a direct statement of this lemma)

**Lemma 2.** For any $(x, y) \in X \times Y$ the iterates of $MP$ with stepsizes $\tau_t \leq \frac{1}{T}$ satisfy

$$\tau_t (\phi(\hat{x}^t, y) - \phi(x, \hat{y}^t)) \leq D_z(z, z^t) - D_z(z, z^{t+1})$$

With this lemma we can show a result similar to that for $PD$, using the exact same steps.

**Theorem 4.** For any $(x, y) \in X, Y$, the averaged solution from $MP$ with stepsize $\tau_t \leq \frac{1}{T}$ satisfies

$$\phi(\bar{x}^T, y) - \phi(x, \bar{y}^T) \leq \frac{1}{S_T} \sum_{t=1}^T (w(t) - w(t-1)) D_z(z, z^t).$$

With $MP$ we know that all iterates satisfy $D_z(z, z^t) \leq \Omega := \max_{z, z' \in Z} D_z(z, z')$, and so we can bound (9) for specific increasing averaging schemes.
Theorem 5. \( \text{MP} \) with the linear averaging scheme and \( \tau_t = \frac{1}{L} \) satisfies
\[
\phi(\bar{x}^T, y) - \phi(x, \bar{y}^T) \leq \frac{2\Omega L}{T + 1}.
\]

Proof. By Theorem 4 and the fact that for linear averaging \( S_T = \frac{T(T+1)}{2L} \) we have
\[
\phi(\bar{x}^T, y) - \phi(x, \bar{y}^T) \leq \frac{1}{S_T} \sum_{t=1}^{T} (w(t) - w(t-1))D_z(z, z') \\
\leq \frac{1}{S_T} \sum_{t=1}^{T} (t - t-1)\Omega \leq \frac{\Omega}{S_T} T = \frac{2\Omega L}{T + 1}.
\]
\[\square\]

Theorem 6. \( \text{MP} \) with the quadratic averaging scheme and \( \tau_t = \frac{1}{L} \) satisfies
\[
\phi(\bar{x}^T, y) - \phi(x, \bar{y}^T) \leq \frac{6\Omega L}{2T + 1}.
\]

Proof. By Theorem 4 and the fact that for quadratic averaging \( S_T = \frac{T(T+1)(2T+1)}{6L} \) we have
\[
\phi(\bar{x}^T, y) - \phi(x, \bar{y}^T) \leq \frac{1}{S_T} \sum_{t=1}^{T} (w(t) - w(t-1))D_z(z, z') \\
\leq \frac{\Omega}{S_T} \sum_{t=1}^{T} (t^2 - (t-1)^2)\Omega \\
\leq \frac{2\Omega}{S_T} T(T+1) \leq \frac{\Omega}{S_T} T(T+1) = \frac{6\Omega L}{2T + 1}.
\]
\[\square\]

All our comments on the convergence of \( \text{PD} \) with the different averaging schemes at the end of the previous section apply here as well: in practice we suspect that Theorem 4 is most informative due to the dependence on all distances rather than just \( (x^0, y^0) \), but Theorems 5 and 6.

4 Numerical experiments

Our experiments will show several variants of mirror prox (denoted MP in plots) and \( \text{PD} \) (denoted PD in plots). We also show experiments for a linesearch variant \( \text{PDLS} \) of \( \text{PD} \) (denoted PD ls in plots); which is like \( \text{PD} \) except that a linesearch is performed to find the stepsize for \( y_{t+1} \) [Malitsky and Pock, 2018], and we do not average the extrapolated \( x \) as Malitsky and Pock [2018] suggest. For variants using the Euclidean distance function we will add “l2” to the name of the algorithm, and for variants using the entropy distance (when applicable), we add “entropy” to the name. Additionally we try four averaging schemes: “last” uses the most recent iterate, while uniform, linear, and quadratic denotes three variants of ergodic averaging. We also tried cubic averaging but do not include those experiments here. It performed similarly to quadratic averaging, sometimes being better, and sometimes worse.

4.1 Matrix games

We start with the simplest case, computing a Nash equilibrium of a matrix game. This can be done by solving the saddle-point problem
\[
\min_{x \in \Delta^n} \max_{y \in \Delta^m} x^T Ay,
\] (10)
where \( A \) is the payoff matrix for the \( y \) player. Relating (10) to (1) we see that the functions \( g, h^* \) simply play the role of indicator functions for the two simplexes, and \( K = A \). As mentioned previously, the entropy distance is known to be 1-strongly convex wrt. the \( \ell_1 \) norm over the simplex, and this leads to a convergence rate with logarithmic dependence on the dimensions \( n, m \). If we use
We run simulations on three classes of random matrix games: 100 × 100 normally-distributed payoff matrices, 500 × 100 normally-distributed payoff matrices, and 100 × 100 uniformly-distributed payoff matrices. For each setting we sample 50 games at random, and run each algorithm for 2000 iterations. We also report standard deviations, but the standard errors are so small that they are hidden by the plot points in most cases.

Figure 1 shows the results in four different comparisons. First, the upper-left plots shows performance for our variants of PD. When using the entropy distance linear averaging performs worse than uniform averaging. In contrast, for the Euclidean distance both linear and quadratic averaging perform much better than uniform averaging. We also see that they perform even better than the last iterate on all three game classes. Secondly, the upper-right plot shows the performance of PDLS with the four averaging schemes. Again the last iterate performs better than uniform averaging as expected, and again quadratic and linear averaging perform significantly better. Thirdly, on the lower-left we plot the performance of MP with entropy and Euclidean distances, as well as our averaging schemes. The relative performances closely mirror those for PD. Finally, on the lower-right we plot the performance of two well-known algorithms for solving games in practice: regret matching [Hart and Mas-Colell 2000] and CFR+ [Tammelin et al. 2015] (specialized to matrix games; see Farina et al. [2019] for a generalization of CFR+ that is more similar to the setting in this paper), along with the quadratic-averaging variants of PD, PDLS, and MP (the plot for MP is hard to see because it is near-identical to that of PD). The FOMs all have performance that exceeds that of CFR+ across

1Our results suggest that hand-tuning stepsizes is not necessary. However, we use payoff distributions centered at zero; if one centers the payoffs at a positive number then the theoretical stepsize seems to be too conservative, and hand-tuning is necessary in order to achieve comparable performance.
all three games. In the two 100 × 100 game classes our algorithms perform significantly better than CFR$^+$.

4.2 Total variation-L1 minimization

The TV-$\ell_1$ model is a convex optimization formulation of an imaging problem, often used for reconstruction when an image has been subjected to salt-and-pepper noise. We follow the presentation and notation of Chambolle and Pock [2011]. In this model we are given some corrupted image $g \in \mathcal{X}$, where $\mathcal{X}$ is the image domain. Images in $\mathcal{X}$ are represented by a grid in $\mathbb{R}^{m,n}$, with each entry representing the pixel value at that particular location, and $\mathcal{X}$ is equipped with the standard inner product. The model minimizes

$$\min_{u \in \mathcal{X}} \|\nabla u\|_1 + \lambda \|u - g\|_1,$$

(11)

where $\nabla u \in \mathcal{Y} \in \mathbb{R}^{m,n,2}$ is the discrete finite-differences gradient of all zeros except that $(\nabla u)_i^j = u_{i+1,j} - u_{i,j}$ if $i < m$ and $(\nabla u)_i^j = u_{i,j+1} - u_{i,j}$ if $j < n$. The inner product in the dual space $\mathcal{Y}$ is defined as $(p, q)_{\mathcal{Y}} = \sum_{i,j} p_{i,j}^1 q_{i,j}^1 + p_{i,j}^2 q_{i,j}^2$. We let div be the adjoint to $\nabla$ defined as $(\nabla u, p)_X = -\langle u, \text{div} p \rangle_X$. The Lipschitz constant $L = \|\nabla\|_{op} = \|\text{div}\|_{op}$ can be bounded as $\sqrt{8}$ [Chambolle, 2004]. The saddle-point variant of the problem looks as follows

$$\min_{u \in \mathcal{X}} \max_{p \in \mathcal{Y}} -\langle u, \text{div} p \rangle + \lambda \|u - g\|_1 - \delta_{\mathcal{P}}(p),$$

(12)

where $\lambda$ is a hyperparameter, and $\delta_{\mathcal{P}}$ is the indicator function for pointwise $\ell_2$-norm balls at each $i,j$, that is: $(p_{i,j}^1)^2 + (p_{i,j}^2)^2 \leq 1$ for all $i,j$. To solve this model we use the Euclidean distance as our distance metric, and in terms of (1) we let $g(u) = \lambda \|u - g\|_1$ and $h^*(p) = \delta_{\mathcal{P}}(p)$. With this setup the proximal mappings can be written as

$$\text{Prox}_X(\xi, u) = \text{Shrink}_{\tau \lambda}(u - \xi), \quad \text{Prox}_Y(\xi, p) = \Pi_{\{\|\cdot\|_{\infty} \leq 1\}}(p - \xi),$$

where $\text{Shrink}$ is the pointwise shrinkage operator

$$\text{Shrink}_{\tau \lambda}(u)_{i,j} = \begin{cases} u_{i,j} - \tau \lambda, & \text{for } u_{i,j} - g_{i,j} > \tau \lambda \\ u_{i,j} + \tau \lambda, & \text{for } u_{i,j} - g_{i,j} < -\tau \lambda \\ g_{i,j}, & \text{for } |u_{i,j} - g_{i,j}| \leq \tau \lambda \end{cases}$$

Following Chambolle and Pock [2011] we set $\lambda = 1.5$.

Figure 2 shows two original pictures, along with their corrupted variants, and the reconstructed solutions based on the TV-$\ell_1$ model.
To generate a ground-truth solution, \( \mathcal{PD} \) was run for 5000 iterations with both last-iterate and quadratic averaging, and the best solution across all the 5000 iterations of the two algorithms was used as a ground truth. The convergence-rate plots (Figure 3) show the relative error \( \frac{\epsilon - \epsilon^{*}}{\epsilon} \), where \( \epsilon \) is the current value of the primal objective \( (11) \), and \( \epsilon^{*} \) is the value at the ground-truth solution. Figure 4 shows the performance of different stepsizing variants on the two images: the left plot shows results for \( \mathcal{PD} \) and the right plot shows results for \( \mathcal{PDLS} \). As expected uniform averaging performs much worse than the last iterate for both algorithms. For \( \mathcal{PD} \) linear and quadratic averaging achieves performance similar to that of the last iterate, with linear performing slightly worse, and quadratic averaging performing about the same. For \( \mathcal{PDLS} \) linear and quadratic averaging leads to better performance than using the last iterate, although the last iterate eventually overtakes on the “cameraman” image.

4.3 Competitive equilibrium in Fisher markets

In the third set of experiments we compute competitive equilibria in Fisher markets. A Fisher market consists of a set of \( n \) buyers and \( m \) goods. Each buyer has a valuation vector \( v_{i} \in \mathbb{R}^{m_{i}} \) describing their value for each good, and an allocation \( x_{i} \) of goods to buyer \( i \). A competitive equilibrium is a set of prices \( p \in \mathbb{R}^{m} \) for the goods, and an allocation \( x_{1}, \ldots, x_{n} \) such that \( x_{i} = \arg \max \{ \langle v_{i}, x_{i} \rangle : (p, x_{i}) \leq B_{i} \} \) for each buyer, and \( \sum x_{ij} = s_{j} \) for each item \( j \). The famous Eisenberg-Gale convex program [Eisenberg and Gale [1959]] showed that a competitive equilibrium can be computed via convex programming. Here we use the saddle-point formulation of that program:

\[
\min_{p \geq 0} \sum_{i} \max_{x_{i} \geq 0} B_{i} \log(\langle v_{i}, x_{i} \rangle) - (p, x_{i}) + \langle s, p \rangle.
\] (13)

This formulation was previously considered by [Kroer et al. [2019]], where they use \( \mathcal{PD} \) in order to compute competitive equilibria. [Kroer et al. [2019]] note that while the gradients of the above formulation are not Lipschitz, each buyer is guaranteed at least their proportional allocation under any feasible set of prices, and so we can add lower bounds on buyer utilities according to each buyer’s proportional share \( PS_{i} \). We add that here, and thus when relating (13) to (11), we let \( g(x) = \sum_{i} B_{i} \log(\langle v_{i}, x_{i} \rangle) + \delta_{\langle v_{i}, x_{i} \rangle \geq PS_{i}, x_{i} \geq 0} \), and \( h^{*} = \delta_{p \geq 0} \), while \( K \) is the matrix encoding \( \sum_{i} (p, x_{i}) \). The Lipschitz constant can be bounded as \( \sqrt{n} + \max_{1 \leq j \leq m} \left| v_{ij} \right| \frac{B_{j}}{PS_{j}} \) [Kroer et al. [2019]].

We generate Fisher market instances according to three settings: 1) 60 buyers and 20 goods, with values generated by a truncated normal distribution with mean 5, standard deviation 2, and truncation at 0 and 10. 2) 20 buyers and 20 goods, with the same truncated normal values. 3) 20 buyers and 20 goods with uniformly distributed values in 0, 1. We generate a total of 50 instances for each setting. The results of running \( \mathcal{PD} \) on this setting are shown in Figure 4. Again we find that linear and quadratic averaging performs much better than uniform averaging, though the last iterate performs even better after a few hundred iterations. For this problem we ran some preliminary experiments suggesting that using \( t^{5} \) in the averaging scheme can lead to performance similar to that of the last iterate.
Figure 4: Convergence rate to competitive equilibrium in the Fisher-market model for PD.

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