Separation induced resonances in quasi-one-dimensional ultracold atomic gases

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We study the effective one-dimensional (1D) scattering of two distinguishable atoms confined individually by separated transverse harmonic traps. With equal trapping frequency for two s-wave interacting atoms, we find that by tuning the trap separations, the system can undergo double 1D scattering resonance, named as the separation induced resonance (SIR), when the ratio between the confinement length and s-wave scattering length is within (0.791, 1.46). Near SIR, the scattering property shows unique dependence on the resonance position. The universality of a many-body system on scattering branch near SIR is demonstrated by studying the interaction effect of a localized impurity coupled with a Fermi sea of light atoms in a quasi-1D trap.

I. INTRODUCTION

Ultracold atomic gases have provided unprecedented accesses to fascinating strongly interacting many-body systems, especially those in unitary limit with resonant scattering. Besides Feshbach resonances, various external confinements are recognized as another efficient way to achieve resonances in all dimensions1–3, and there have been successful explorations of resonance scattering properties in experiments4–10.

The mechanism of confinement induced resonance was first explored by Bergeman et al.2. A bound state constructed in the Hilbert space spanned by the excited transverse states of non-interacting Hamiltonian was introduced as a closed channel bound state (CCBS). By tuning the confinements the resonance occurs when CCBS touches the scattering threshold. This class of resonance can also be understood as the consequence of modified low-energy scattering theory by properly renormalizing all virtual scatterings to high-energy states11. The properties of these resonances closely depend on the type of confinement potentials. Previous studies have shown that within the contact interaction model, all the induced resonances fall into two classes. If the trapping potential decouples relative motion(r) from center-of-mass(R), the only one CCBS would induce a single resonance such as in Ref.1,2; if not, there would be an infinite number of CCBS due to the coupling between r and all R–channels, resulting in an infinite number of resonances such as in Ref.4–5. Therefore an interesting question is whether these two classes have covered all the possible resonances under external confinements. In this paper, by providing an alternative class of scattering resonance induced by trap separations (see below), we show the answer is no.

We consider two distinguishable atoms confined individually by transverse harmonic traps with tunable separations (see Fig.1(a)). We find two resonances of one-dimensional(1D) scattering by tuning the separation, which we name as “separation induced resonance” (SIR). It is the non-monotonic evolution of CCBS with the separation that gives rise to the emergence of two resonances, and also leads to new features in parameters describing the effective 1D scattering. By introducing such a system, we show an interesting class of induced resonances as SIR. For such resonances, the number of resonances is not solely determined by the number of CCBS, and the effective 1D scattering strength exhibits exotic dependence on the tunable parameter as shown by Fig.3. All these features are qualitatively different from those of Feshbach resonances and previously studied confinement induced resonances. Our finding substantially enriches the existing understanding of the induced resonance physics. In addition, we study the many-body physics across SIR. By employing an impurity problem, we show that a many-body system on metastable scattering branch can go across double SIRs and exhibit universal properties at each resonance. The two-body bound state is also studied, and experimental realizations and detections are discussed finally.

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FIG. 1: (Color online) Schematic plot of system setup. (a) Two interacting atoms(A, B) are separately confined in transverse harmonic traps with characteristic length a0 and distance d. (b) Potentials V(x, 0, 0) in the center-of-mass frame, including trapping potential centered at r = (d, 0, 0) (1) and short-range (r0 ≈ 0) interaction at r = 0(2).

The rest of the paper is organized as follows. In section II, we present the formalism for effective 1D scattering across SIR. In section III we discuss the origin of double resonances and show the unique features of
effective scattering strength near resonances. The two-body bound state is studied in Section IV. Section V is contributed to the impurity problem, from which we address the universal property of a many-body system at metastable scattering branch across SIR. We discuss the experimental realization and finally remark on the generalization of SIR to other systems in Section VI.

II. EFFECTIVE SCATTERING IN 1D

We consider two distinguishable atoms, A and B, respectively trapped by transverse potentials $V_t(r_A) = m_A \omega^2 ((x_A + d/2)^2 + y_A^2)/2$ and $V_t(r_B) = m_B \omega^2 ((x_B - d/2)^2 + y_B^2)/2$, which decouples the center-of-mass and relative motions. The Hamiltonian for the relative motion is $H_{rel}(r) = H_0 + U(r)$, where (we take $\hbar = 1$ throughout the paper)

$$H_0 = -\frac{\nabla^2}{2\mu} + \frac{\mu}{2} \omega^2 ((x - d)^2 + y^2),$$

$$U(r) = \frac{2\pi \alpha}{\mu \omega} \delta(r - r_0),$$

with the reduced mass $\mu = m_A m_B / (m_A + m_B)$ and the s-wave scattering length $a_s$. $H_0$ determines the spectrum as $E = E_{n_x,n_y} + k^2/(2\mu)$, with $E_{n_x,n_y} = (n_x + n_y + 1/2)\omega$ the eigen-energies for the transverse eigen-states $|\phi_{n_x,n_y}(x,y)\rangle = \psi_{n_x}(x - d)\psi_{n_y}(y)$, $n_x,n_y = 0, 1, 2, ...$ ($\psi_n$ is the 1D harmonic oscillator wavefunctions).

In this system, the low energy scattering processes for incoming wave functions with $n_x = n_y = 0$ and $k^2/2\mu \ll \omega$ can be described by a 1D Hamiltonian for the relative motion (along z-direction) as $\hat{H}_{rel} = -(1/(2\mu))(d^2/dz^2) + g_{1D}\delta(z)$, with $g_{1D}$ the 1D coupling strength.$^1$ To derive $g_{1D}$ in terms of $a_s$, we study the scattering wave function at low energy $E = \omega + k^2/(2\mu)$,

$$\Psi(r) = \phi_0,0(x,y)e^{ikz} + fG(r,0),$$

where the Green function, $G(r,0) = \langle r | \frac{1}{E - H_0 + i\delta} | 0 \rangle$, is expressed as

$$G(r,0) = \sum_{n_x,n_y} \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipz} \phi_{n_x,n_y}(x,y) \psi_{n_x,n_y}(0,0).$$

To solve the problem, it is essential to write the asymptotic form of Eq.3 at $r \to 0$ as $G(r,0) = -\frac{\mu}{2\pi r} + C(ka_0,\delta) + o(r)$, where $a_0 = \sqrt{1/\mu\omega}$ is the characteristic length of the harmonic oscillator, $\delta = d/\alpha_0$. With the help of the imaginary-time evolution operator of the harmonic oscillator,$^4$ we obtain $C = \frac{\mu}{2\pi a_0}(rG(r,0))_{r\to 0}$ as

$$C(ka_0,\delta) = \frac{\mu}{\pi a_0} \int_{i\chi a_0}^{1/k\alpha_0} e^{-a^2} - \frac{1}{\sqrt{2\pi}} A(ka_0,\delta),$$

and

$$A(ka_0,\delta) = \int_{-\infty}^{\infty} dr e^{i^2 \chi^2 r^2} \left( \frac{e^{-r^2} \tanh \frac{\xi}{2}}{1 - e^{-2r^2}} - \frac{e^{-i^2 \chi^2 r^2}}{2\tau} - e^{-r^2} \right).$$

According to the Schrodinger equation $H_{rel}\Psi = E\Psi$, we find the scattering amplitude as

$$f = \phi_0,0(0,0)\left(\frac{\mu}{2\pi a_0} - C(ka_0,\delta)\right)^{-1},$$

thus we obtain the closed form of Eq.2. When $d = 0$, all equations reproduce the well-known quasi-1D results.$^1$ More detailed derivation of Eq.4 is given in Appendix A.

At large distances along z-direction, all terms in the Green function (Eq.3) decays except for the lowest transverse mode with $n_x = n_y = 0$. Since $U(r)$ only takes effect at $r = 0$ but vanishes otherwise, the part of $\Psi(r)$ with even-parity of $z$, $\Psi_{even}(r) = \frac{1}{2}(\Psi(x,y,z) + \Psi(x,y,-z))$, is scattered while the odd-parity part remains unaltered. By Eq. 2, $\Psi_{even}(r) = \phi_0,0(x,y)e^{i\delta_k}\cos(|k|z + \delta_k)$, where the phase shift $\delta_k$ satisfies

$$\tan \delta_k = -\frac{2}{ka_0} e^{-a^2} a_0 + \sqrt{\frac{2}{\pi}} A(ka_0,\delta)^{-1}.$$
III. BASIC FEATURES OF SIR

In this section we address the basic features of SIR. First we explore the origin of double resonances, and secondly we study the resulted structure of effective scattering strength across SIR and corresponding resonance width.

A. Origin of double resonances

The origin of double resonances can be understood by studying the CCBS that is constructed by all excited transverse modes of $H_0$. The binding energy of the CCBS, $E^c_b = E - \omega$, is given by

$$\mu \left( \frac{2}{\pi a_s} \right) = \lim_{r \to 0} (G^c(r, 0) + \frac{\mu}{2\pi r}),$$

where $G^c$ is the closed channel Green function which follows Eq. 3 but excludes $n_x = n_y = 0$ in the summation. Thus Eq. 1 is equivalent to

$$\frac{a_0}{a_s} = -\sqrt{2} A \sqrt{\frac{E^c_b}{\omega}} \	ilde{d},$$

Compared with Eq. [2], we see that SIR occurs when $E^c_b = 0$, i.e., when the CCBS touches the threshold.

The intriguing dependence of the resonance position on $\tilde{d}$ is a direct consequence of that of $E^c_b$ on $\tilde{d}$. In Fig. 3(a), we show that for given $a_0/a_s$, $E^c_b$ decreases with $\tilde{d}$ at small $\tilde{d}$ but increases at large $\tilde{d}$. Mathematically this is attributed to the non-monotonic behavior of $A-\text{function}$ (Eq. [5]) when increasing $\tilde{d}$. For fixed $E^c_b$, $A-\text{function}$ increases/decreases with $\tilde{d}$ in the small/large $\tilde{d}$ limit (particularly these properties are shown in Eqs. [13, 15] for $A(0, \tilde{d})$); while for fixed $\tilde{d}$, $A-\text{function}$ always increases with the energy. These properties together give rise to the non-trivial dependence of $E^c_b$ on $\tilde{d}$ as solved from Eq. (10) for each given $a_0/a_s$. Physically, the non-trivial behavior of $E^c_b$ can be explained in the following way. The explicit form of $G^c$ in Eq. (9) (cf. Eq. (3)) indicates that $\tilde{d}$ affects $E^c_b$ only through the coupling weight $\alpha_{n_x,n_y}(\tilde{d}) = |\phi_{n_x,n_y}(0,0)|^2 = \psi_{n_x}^2(-\tilde{d}) \psi_{n_y}^2(0)$, particularly the $\psi_{n_x}^2(-\tilde{d})$ part. Unlike $d = 0$ case where the interaction only couples even-parity states ($n_x = 2, 3, ...$), non-zero $d$ additionally mix all odd-parity states ($n_x = 1, 3, ...$) into $\Psi(r)$. When $\tilde{d} \ll 1$, $\alpha_{n_x,n_y}$ increases with $\tilde{d}$ for all odd $n_x$, while decreases for all even $n_x$. We find that the former effect dominates so that small nonzero $\tilde{d}$ facilitates the formation of CCBS and gives lower $E^c_b(\tilde{d}) \approx E^c_b(0) - 3\tilde{d}^2/(2\mu a_0^2)$. When $\tilde{d} \gg 1$, the coupling for both even and odd $n_x$ decays exponentially with $\tilde{d}$. $E^c_b$ in this limit approaches $\omega$ from below, implying two uncorrelated atoms when trapped far apart.

As analyzed above, for any given $a_0/a_s$, $E^c_b$ as a function of $d$ first decreases and then increases. As shown in Fig. 3, for small $a_0/a_s < 0.791$ the interaction is not strong enough to make $E^c_b$ even touch the threshold and there is no resonance; for large $a_0/a_s > 1.46$, the interaction is so strong that at $d = 0$ the CCBS is already below the threshold, and therefore the resonance is only possible at large $d$; however, for intermediate $a_0/a_s \approx (0.791, 1.46)$, $E^c_b$ is able to cross zero twice due to its non-monotonic behavior, and correspondingly $g_{1D}$ diverges whenever $E^c_b$ across zero. This is the origin of double resonance feature.

![Figure 3](image-url)  
**FIG. 3:** The binding energy of CCBS($E^c_b/\omega$) and the effective 1D coupling strength($g_{1D}/(\mu a_0)$) as functions of $d/a_0$, for several typical values of $a_0/a_s$ that correspond to five horizontal lines in Fig. 2a. Vertical blue lines denote the resonance positions $d_{res}$, where $E^c_b = 0$ and $g_{1D} = \infty$. 

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![Figure 2](image-url)  
**FIG. 2:** Separation induced resonance(SIR) in quasi-1D system. (a)$a_0/a_s$ as a function of $d = d/a_0$ at SIR. Dashed lines are the functional fit to $1.46 - 1.39d^2$ in small $d$, and $d - 1/d$ in large $d$ limit. The horizontal lines label five coupling strengths (from bottom to top): $a_0/a_s = 0.4, 0.791, 1.2, 1.46, 1.8$, with further properties shown in Fig. 3a. (b)Resonance width $W_d$ versus resonance position $d$ and corresponding $a_0/a_s$(inset).
Near SIR, $g_{1D}$ can be parameterized as
\[ g_{1D} = W_d/(\tilde{d} - \tilde{d}_{res}), \]
with
\[ W_d = \sqrt{2}d e^{-\tilde{d}^2} \left[ \frac{\partial A(0, \tilde{d})}{\partial \tilde{d}} \right]_{res}^{-1}. \] (11)

Here $W_d$ is the width of SIR, analogous to that defined in Feshbach resonance \[12\]. In our system, $W_d$ reflects the coupling strength between the open and closed channel responsible for SIR, and also determines how well SIR can be accessed in experiment due to the limited resolution of trap separations.

An important feature of SIR is that the resonance width strongly depends on the resonance position, which changes sign at $d_{res} = 1.123$. We show in Fig.2(b) how $W_d$ changes with $d_{res}$ and corresponding $a_0/a_s$. Due to the vanishing $\frac{\partial A(0, \tilde{d})}{\partial \tilde{d}}$ around $d_{res} = 0$ and 1.123, $W_d$ also experiences divergent behavior asymptotically as $W_d \sim 1/d_{res}$ and $W_d \sim -1/(d_{res} - 1.123)$ respectively. At the left(right) side of $d_{res} = 1.123$, $W_d$ is positive(negative).

Particularly, for a given $a_0/a_s \in (0.791, 1.46]$ the system accomplishes two SIRs: one of them is with large positive $a_s$ and the other with small negative $a_s$ (see the inset of Fig.2(b)): the opposite signs of these two $W_d$ determine that $g_{1D}$ keeps sign between two SIRs (see Fig.3), in contrast with Feshbach resonance where $a_s$ crosses zero between two adjacent resonances.

In the limit of $a_0/a_s$, $d_{res} \gg 1$, $W_d$ exponentially decays as $W_d = -2e^{-\tilde{d}^2}$. Physically such a narrow width corresponds to the very weak overlap of the wavefunctions of interacting particles, i.e., A and B atoms have little probability to collide with each other when they are trapped far apart. Note that the width defined in Eq.11 is meaningful for the realistic detection of SIR in experiment, but should not be confused with the narrow width effect in a magnetic Feshbach resonance \[13\]. In fact, as shown in Appendix B, the $k$-dependence of $g_{1D}$ is always very weak and even negligible in $d \gg 1$ limit although the resonance width is exponentially small.

IV. TWO-BODY BOUND STATE

The true two-body bound state is given by the pole of the scattering amplitude, $f(i\kappa) = \infty$, i.e.,
\[ \frac{\mu}{2\pi a_s} = C(i\kappa a_0, \tilde{d}). \] (12)

In Fig.4(a) the binding energy, $E_b = E - \omega = -\kappa^2/(2\mu)$, is plotted as a function of $\tilde{d}$. We see that $E_b$ always exists below zero for any $a_s$ and $\tilde{d}$, due to the inclusion of $n_x = n_y = 0$ mode and the effective 1D density of state at low energies. Moreover, $E_b$ monotonically decreases with $\tilde{d}$, which is dramatically different from $E_b^0$. In weak coupling limit($a_s/a_0 \to 0^+$), the bound state is merely the 1D consequence which gives $E_b = -\frac{\mu \omega^2}{2} = -\frac{2a_s^2}{\mu a_0^2}e^{-2\tilde{d}^2}$. In the strong coupling limit($a_s/a_0 \to 0^+$), for small $\tilde{d}$ it is straightforward to obtain the binding energy $E_b(\tilde{d}) = E_b(0) + \mu \omega^2 \tilde{d}^2/2$, which is shifted up exactly by the potential barrier; for large $\tilde{d}$, $E_b$ would be very small and exponentially decay as $E_b = -2[\mu a_s^2(\tilde{d} - a_0/a_s)]^2 e^{-2\tilde{d}^2}$. To explore the difference between the CCBS and true two-body bound state, we plot in Fig.4(b) their energy difference, $\Delta = E_b - E_b$, right at SIR as a function of $\tilde{d}$. When $\tilde{d} = 0$, $\Delta = 2\omega$ reproduces the result in Ref.\[1\]. As $\tilde{d}$ increases, $\Delta$ decreases and becomes exponentially small in the large $\tilde{d}$ limit.

V. UNIVERSALITY OF SCATTERING BRANCH AT SIR

The universal property of a many-body system at SIR can be effectively explored by considering the following exactly solvable impurity problem. Consider that atoms A($m_A = m$) form a Fermi-sea with Fermi energy $k_F$ and interact with a localized impurity B($m_B = \infty$) at the origin. In this case the Hamiltonian for A is exactly $H_{rel}$ with $\mu = m$ (cf. Fig.1(b)). The same phenomenon of double SIRs can be deduced for A moving in the effective 1D tube.

Suppose the tube is with boundary $[-L, L]$, the allowed wave vectors are given by $kL + \delta_k = (n + \frac{1}{2})\pi$ ($n = 0, 1, \ldots$). Given that there is no occupation of possible bound states, the non-zero phase shifts $\delta_k$ give rise to the interaction energy (or the energy difference from non-interacting case) as
\[ E_{int} = -\frac{1}{m \pi} \int_0^{k_F} k\delta_k dk. \] (13)

Note that the scattering states contributing to $E_{int}$ here
belong to the metastable scattering branch in quasi-1D in which \(-\pi < \delta_k < 0\) for all \(k\). Explicitly, the metastable branch corresponds to a Hilbert space expanded by scattering states, i.e., without the occupation of molecules. One typical example is the repulsive Fermi gas in 3D with positive \(a_s\), and recently there have been extensive studies of single impurity problem in such system both theoretically[15] and experimentally[16]. Moreover, the scattering branch has also been realized in the bosonic quasi-1D system[7] in the absence of trap separations. In our system with trap separations, as shown in Fig.4, a two-body bound state always exists below the threshold for any \(a_s\) and \(d\), whose binding energy monotonically decreases as \(d\) increases. This indicates that in a many-body system the universality is only possible for the metastable scattering branch which excludes the Hilbert space of molecules.

According to Eq.7 in Fig.5(a) we plot \(\delta_k\) as a function of \(d\) for different momenta \(k\). Considering \(ka_0 \ll 1\), we have used \(k\)-independent \(A(0, \hat{d})\) in Eq.4 which will bring a negligible correction of the order of \(o((ka_0)^2)\). We see that all the curves with different \(k\) cross exactly at the location of SIR. This is due to the universal phase shift as \(-\pi/2\) right at SIR for all values of \(k\), and according to Eq.13 this further leads to the universal interaction energy as half of the Fermi energy, \(E_{\text{int}} = E_F/2\), at any position of SIR (see Fig.5(b)). According to Eq.7 and Eq.14 the slopes of \(\delta\) and \(E_{\text{int}}\) across SIR are both inversely proportional to the resonance width \(W_d\).

**FIG. 5:** Universality at SIR in an impurity system for given \(a_0/a_s = 1.2\) (\(a_0 = 1/\sqrt{m\omega}\)). (a)\(\delta_k\) in terms of \(d = \hat{d}/a_0\) at given \(ka_0 = 0.05, 0.1, 0.2\). The double SIR occur at \(d = 0.47, 1.77\) (as shown by red arrows), where \(\delta_k = -\pi/2\) (gray line) for any \(k\). (b)\(E_{\text{int}}/E_F\) as a function of \(d\) for different fermi momentum \(k_Fa_0 = 0.05, 0.1, 0.2\). \(E_{\text{int}}\) shows universal value as \(E_F/2\) (gray line) at SIR.

Above impurity problem explicitly shows that the universal behavior of a many-body system is a direct consequence of the divergent two-body coupling strength and the resulted uniform phase shift in k-space. This conclusion should generally apply to a wide range of cases, e.g., arbitrary numbers and mass ratios of two-component fermions. In our system, we conclude that within \(a_0/a_s \in (0.791, 1.46]\), by increasing \(\hat{d}\) the metastable many-body scattering state would go across two consecutive universal regimes. At small \(\hat{d}\) it is a crossover from the Fermionic super-Tonks-Girardeau(TG)[17] to Fermionic TG regime, and at large \(\hat{d}\) a reverse process.

**VI. EXPERIMENTAL REALIZATION AND FINAL REMARKS**

The probing of SIR and its physical consequences can be realized by taking advantage of sophisticated optical techniques to manipulate cold atoms. For two spin-components of the same isotope \(m_A = m_B\), separated harmonic traps can be generated in the setup of spin-dependent optical lattices[18, 19]. The separation \(d\) can be tuned by adjusting the polarization angles of two linearly polarized and counterpropagating laser beams (with wavelength \(\lambda\)), which create the lattices. In Ref.[18], the maximum separation between the nearest two species is \(d_{\text{max}} = 3\lambda/16 \sim 150nm\), comparing to the typical confinement length \(a_0 \sim 50nm\). The ratio, \(d/a_0 = 0 \sim 3\), is of most interest as shown by Fig.2 and Fig.3. For different isotopes \(m_A \neq m_B\), the separated traps can be achieved by fine tuning the laser frequency according to different atomic transition lines for different atoms[3, 20]. In such cold atom systems, the position of SIR can be pinned down by the maximum of atom loss rate[2]: \(g_{1D}\) can be mapped out from the frequencies of collective modes[2]; the binding energy or interaction energy can be deduced from the shift of peak frequency of atomic transition using rf spectroscopy[10].

Before closing, we emphasize that SIR should fall into a new class of resonance different from magnetic Feshbach resonances and confinement induced resonances. The effect brought about by the trap separation is so generic that it should also hold for other trap geometries or interaction types. Our further studies find that the basic features of SIR, i.e., the non-monotonic evolution of CCBS with the separation and thus the resulted exotic properties of effective scattering strength near resonance, still persist in quasi-2D geometry or in the presence of p-wave interaction[21]. Even in the case of \(\omega_A \neq \omega_B\) in our quasi-1D system, where the center-of-mass and relative motions can not be separated, each CCBS emerging from different center-of-mass channels will evolve non-monotonically with \(\hat{d}\). It is conceivable that for given \(a_s/a_0\) several CCBS could go across the zero threshold energy. This will bring extra interesting resonance properties, such as arbitrarily finite resonance number and novel structure of \(g_{1D}\) near resonance. In contrast, for Feshbach resonance or confinement induced resonances, each CCBS monotonically evolves with the magnetic field or confinement length, resulting in infinite number of resonances and the similar structure of effective scattering strength near each resonance[4, 5, 8]. The SIR found at the two-body level also has strong indication for intriguing many-body phenomena, which await full exploration in the future.
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Appendix A: Derivation of Eq.(4)

We expand the Green function (Eq.(3) in the text) as

\[
G(r, 0) = \phi_{0,0}(x, y)\phi^*_0(0, 0)e^{ik|z|} - \frac{\mu}{i k} \int_0^\infty \frac{dt}{t} \left[ \sum_{n=0}^\infty e^{-n\omega t} \phi_n(x)\phi^*_n(0) \right] \left( \sum_{n=0}^\infty e^{-n^2\omega t} \phi_n(y)\phi^*_n(0) \right) - \phi_{0,0}(x, y)\phi^*_0(0, 0), \tag{A1}
\]

where we have used imaginary-time integration for the low-energy scattering (\(E = \omega + \frac{k^2}{2\mu} < 2\omega\)); \(\mu\) is the reduced mass of two atoms A and B; \(\phi_n(x) = \psi_n(x - d), \phi_n(y) = \psi_n(y), \) \(\psi_n(x-d),\) and \(\psi_n(x)\) is the eigen-state for 1D harmonic oscillator centered at \(x = 0\) and with characteristic length \(a_0.\)

Further by utilizing the single-particle imaginary-time propagator \((a_0 = 1/\sqrt{\mu\omega})\)

\[
\sum_{n=0}^\infty e^{-n\omega t} \psi_n(x)\psi^*_n(x') = \frac{1}{\sqrt{\pi a_0}} \frac{1}{\sqrt{1 - e^{-2\omega t}}} e^{-\frac{x^2 + x'^2}{2a_0^2} \coth(\omega t) + \frac{x - x'}{a_0^2} \frac{1}{\sinh(\omega t)}}, \tag{A2}
\]

the second term in Eq.(A1) is reduced to

\[
- \frac{1}{\pi a_0^2} \sqrt{\frac{\mu}{2\pi w}} \int_0^\infty \frac{d\tau}{\sqrt{\tau}} e^{\frac{\mu a_0^2 \tau}{2}} e^{-\frac{x^2 + x'^2}{2a_0^2} \coth \tau + \frac{(x-d)^2 + (-d)^2}{2a_0^2} \coth \tau} 1 - e^{-2\tau} \frac{1}{1 - e^{-2\tau}} - e^{-\frac{(x-d)^2}{2a_0^2} - \frac{d^2}{2a_0^2} + \frac{x'^2}{2a_0^2}}, \tag{A3}
\]

which implicitly includes a divergence as \(-\frac{\mu}{2\tau}\) at small \(r = \sqrt{x^2 + y^2 + z^2}\) due to the integration at \(\tau \to 0.\)

Using the exact relation

\[
\int_0^\infty \frac{d\tau}{\tau} e^{-\frac{\mu a_0^2 \tau}{2}} = \sqrt{2\pi a_0} \frac{1}{r}, \tag{A4}
\]

we extract the divergent term from the Green function and rewrite it as

\[
G(r, 0) = \phi_{0,0}(x, y)\phi^*_0(0, 0)e^{ik|z|} - \frac{\mu a_0}{\pi a_0^2} \sqrt{2\pi} \int_0^\infty \frac{d\tau}{\sqrt{\tau}} e^{\frac{\mu a_0^2 \tau}{2}} e^{-\frac{x^2 + x'^2}{2a_0^2} \coth \tau + \frac{(x-d)^2 + (-d)^2}{2a_0^2} \coth \tau} 1 - e^{-2\tau} \frac{1}{1 - e^{-2\tau}} - e^{-\frac{(x-d)^2}{2a_0^2} - \frac{d^2}{2a_0^2} + \frac{x'^2}{2a_0^2}} - \frac{\mu}{2\pi r}. \tag{A5}
\]

Now it is straightforward to obtain \(C(ka_0, \vec{d}) = \frac{\partial}{\partial r}(rG(r, 0))|_{r \to 0}\) as Eq.(4) and further \(A(ka_0, \vec{d})\) as Eq.(5) in the text.
Appendix B: Property of $A(k_{0}, \tilde{d})$

To see the energy-dependence of $A(k_{0}, \tilde{d})$, we expand it in terms of small $k_{0} \ll 1$ as $A(k_{0}, \tilde{d}) = A(0, \tilde{d}) + \lambda_{d}(k_{0})^{2} + o(k^{4}a_{0}^{4})$, with

$$\lambda_{d} = \int_{0}^{\infty} d\tau \sqrt{\frac{2}{\tau}} \left( \frac{e^{-\tilde{d}^{2}\tanh \frac{\tilde{d}}{2}} - e^{-\tilde{d}^{2}}}{1 - e^{-\tilde{d}^{2}}} \right) = \begin{cases} 0.409, & \tilde{d} = 0 \\ \sqrt{\frac{2\pi}{3d}}, & \tilde{d} \to \infty \end{cases} \quad (B1)$$

Note that due to $\partial \lambda_{d}/\partial \tilde{d} > 0$ at $\tilde{d} \ll 1$, $\lambda_{d}$ increases with $\tilde{d}$ at small $\tilde{d}$. However, $\lambda_{d}$ decrease as $1/\tilde{d}$ at large $\tilde{d}$. Therefore in the intermediate $\tilde{d}$, $\lambda_{d}$ should reach a maximum determined by $\partial \lambda_{d}/\partial \tilde{d} = 0$, i.e.,

$$\int_{0}^{\infty} d\tau \sqrt{\frac{2}{\tau}} \left( \frac{e^{-\tilde{d}^{2}\tanh \frac{\tilde{d}}{2}} \tanh \frac{\tilde{d}}{2} - e^{-\tilde{d}^{2}}}{1 - e^{-\tilde{d}^{2}}} \right) = 0, \quad (B2)$$

and this gives $\tilde{d} = 1.023$ and $(\lambda_{d})_{\text{max}} = 0.666$.

![Graph](image)

**FIG. 6:** $\lambda_{d}(\text{Eq. B1})$ as functions of the separation $\tilde{d}$.

The small $\lambda_{d}$ obtained for all $\tilde{d}$ justify us to only consider the $k$-independent part of $A(k_{0}, \tilde{d})$ in the scattering problem.

$$A(0, \tilde{d}) = \int_{0}^{\infty} \frac{d\tau}{\sqrt{\tau}} \left( \frac{e^{-\tilde{d}^{2}\tanh \frac{\tilde{d}}{2}} - 1}{2\tau} - e^{-\tilde{d}^{2}} \right). \quad (B3)$$

Next we analyze it in two limits.

1. when $\tilde{d} \to 0$, $A(0, \tilde{d}) = a + b\tilde{d}^{2} + o(\tilde{d}^{4})$, with

$$a = \int_{0}^{\infty} \frac{d\tau}{\sqrt{\tau}} \left( \frac{1}{1 - e^{-2\tau}} - \frac{1 - 1}{2\tau} \right) \approx -1.83,$$

$$b = \int_{0}^{\infty} \frac{d\tau}{\sqrt{\tau}} \left( 1 - \frac{\tanh \frac{\tilde{d}}{2}}{1 - e^{-2\tilde{d}^{2}}} \right) \approx 1.75. \quad (B4)$$

This gives the resonance value of $[a_{s}/a_{s}]_{\text{res}} = 1.46 - 1.39\tilde{d}^{2}$ at small $\tilde{d}$.

2. when $\tilde{d} \to \infty$, Eq. [B3] is mostly contributed by small $\tau$. In this limit, Eq. [B3] is equivalent to

$$A(0, \tilde{d}) \approx \int_{0}^{\infty} \frac{d\tau}{\sqrt{\tau}} \left( \frac{e^{-\tilde{d}^{2}\tanh \frac{\tilde{d}}{2}} - 1}{2\tau} \right) \approx \int_{0}^{\infty} \frac{e^{-\tilde{d}^{2}\tanh \frac{\tilde{d}}{2}} - 1}{2\tau^{1/2}} = \sqrt{\frac{\pi}{2}} (-\tilde{d} + 1/\tilde{d}), \quad (B5)$$

which gives $[a_{0}/a_{s}]_{\text{res}} = \tilde{d} - 1/\tilde{d}$ at large $\tilde{d}$.

Eq. [B3] and Eq. [B5] show that $A(0, \tilde{d})$ increases with $\tilde{d}$ at small $\tilde{d}$ while decreases at large $\tilde{d}$. The turning point is given by $\partial A(0, \tilde{d}) / \partial \tilde{d} = 0$, i.e.,

$$\int_{0}^{\infty} \frac{d\tau}{\sqrt{\tau}} \left( \frac{e^{-\tilde{d}^{2}\tanh \frac{\tilde{d}}{2}} \tanh \frac{\tilde{d}}{2} - e^{-\tilde{d}^{2}}}{1 - e^{-\tilde{d}^{2}}} \right) = 0, \quad (B6)$$

and this gives $\tilde{d} = 1.123$, corresponding to the resonance value of $[a_{0}/a_{s}]_{\text{res}} = 0.791$ (see also Fig. 2(a)).

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