Mean Field Portfolio Games in Incomplete Markets: 
— Nonconstant Equilibria Do Not Exist in $L^\infty$ —

Guanxing Fu* Chao Zhou†

June 14, 2021

Abstract

We study mean field portfolio games in incomplete markets with random market parameters, where each player is concerned with not only her own wealth but also the relative performance to her competitors. We use the martingale optimality principle approach to characterize the unique Nash equilibrium in terms of a mean field FBSDE with quadratic growth, which is solvable under a weak interaction assumption. Motivated by the weak interaction assumption, we establish an asymptotic expansion result in powers of the competition parameter. When the market parameters do not depend on the Brownian paths, we get the Nash equilibrium in closed form. Moreover, when all the market parameters become time-independent, we revisit the games in [21] and our analysis shows that nonconstant equilibria do not exist in $L^\infty$, and the constant equilibrium obtained in [21] is unique in $L^\infty$, not only in the space of constant equilibria.

AMS Subject Classification: 93E20, 91B70, 60H30

Keywords: mean field game, portfolio game, martingale optimality principle, FBSDE

1 Introduction

Mean field games (MFGs) are a powerful tool to study large population games, where each player has negligible influence on the outcome of the game. Introduced independently by [18] and [22], MFGs have received considerable attention in probability and financial mathematics literature. In this paper, we study a class of mean field portfolio games in incomplete markets with random market parameters by the martingale optimality principle (MOP) approach.

Assume the market environment has $N$ risky assets, e.g. stocks, with price dynamics of asset $i \in \{1, \cdots, N\}$ follows

\[
    dS^i_t = S^i_t \left( h^i_t \, dt + \sigma^i_t W^i_t + \sigma^{i0}_t \, dW^0_t \right),
\]

where the return rate $h^i$ and the volatility $(\sigma^i, \sigma^{i0})$ are assumed to be bounded progressively measurable stochastic processes. $W^i$ is a Brownian motion describing the private noise to the asset $i$ while $W^0$ is

*Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong. Email: guanxing.fu@polyu.edu.hk. G. Fu’s research is supported by the Start-up Fund P0035348 from The Hong Kong Polytechnic University.
†Department of Mathematics, The City University of Hong Kong, Kowloon Tong, Hong Kong, on leave from National University of Singapore. Email: matzc@nus.edu.sg. C. Zhou’s research is supported by Singapore MOE (Ministry of Educations) AcRF Grants R-146-000-271-112 and R-146-000-284-114 as well as NSFC Grant No. 11871364.
a Brownian motion that is independent of $W^i$, describing the common noise to all risky assets. The interest rate of the riskless asset is assumed to be zero for simplicity. Let $X^i$ be the wealth process of player $i$, who trades asset $i$, and $\overline{X}^i$ be the “average” wealth of player $i$’s competitors. Here, the exact form of $\overline{X}^i$ depends on the type of risk preference, or utility functions. Each player solves a utility maximization problem and she is concerned with not only her own wealth $X^i$ but also the relative performance from her competitors $X^i - \overline{X}^i$. We analyze the cases of exponential utility functions and power utility functions.

In the case of exponential utility functions, we assume each player $i$ chooses the amount of money invested in the risky asset $i$ to

$$\text{maximize } \mathbb{E} \left[ -\alpha^i \left( X_T^i - \theta^i \overline{X}_T^i \right) \right],$$  \hspace{1cm} (1.2)

where the wealth process $X^i$ follows

$$dX^i_t = \pi^i_t \left( h^i_t \, dt + \sigma^i_t \, dW^i_t + \sigma^0_t \, dW^0_t \right), \quad X^i_0 = x^i,$$ \hspace{1cm} (1.3)

and $\overline{X}^i := \frac{1}{T-0} \sum_{j \neq i} X^j$ is the average wealth of all players except player $i$. Moreover, $\alpha^i > 0$ is the degree of risk aversion and $\theta^i \in [0, 1]$ is the absolute competition parameter; player $i$ is more concerned with her own terminal wealth $X_T^i$, if $\theta^i$ is closer to 0 and more concerned with the relative performance $X_T^i - \overline{X}_T^i$ if $\theta^i$ is closer to 1.

In the case of power utility functions, player $i$ chooses the fraction of her wealth invested in the risky asset $i$ to

$$\text{maximize } \mathbb{E} \left[ \frac{1}{\gamma^i} \left( X_T^i (\overline{X}^i)^{-\gamma^i} \right) \right],$$ \hspace{1cm} (1.4)

where the wealth process $X^i$ follows

$$dX^i_t = \pi_t^i X^i_t \left( h^i_t \, dt + \sigma^i_t \, dW^i_t + \sigma^0_t \, dW^0_t \right), \quad X^i_0 = x^i,$$ \hspace{1cm} (1.5)

and $\overline{X}^i = (\Pi_{j \neq i} X^j)^{\frac{1}{T-0}}$ is the geometric average of all players’ wealth except for player $i$, $\theta^i \in [0, 1]$ is the relative competition parameter and $\gamma^i \in (-\infty, 1)/[0]$ is the degree of risk aversion.

By MOP as in [17, 24] for the single player’s utility maximization problems, the unique Nash equilibrium (NE) for each $N$-player game can be characterized by a multidimensional FBSDE with quadratic growth; see Section 5. Although such FBSDE is solvable, the equations are tedious. MFGs have the advantage to solve large population games in a concise manner. The corresponding MFGs associated with exponential utility functions and power utility functions are given by

1. Fix $\mu$ in some suitable space;
2. Solve the optimization problem:
   \[ \mathbb{E} \left[ e^{-\alpha(X_T - \theta \mu_T)} \right] \rightarrow \text{max over } \pi \]
   such that $dX_t = \pi_t (h_t \, dt + \sigma_t \, dW_t + \sigma_0 t \, dW^0_t)$, $X_0 = x_{exp}$;
3. Search for the fixed point $\mu_t = \mathbb{E}[X_t | \mathcal{F}_t^0]$, $t \in [0, T]$,
   where $X^*$ is the optimal wealth from 2 and $\mathcal{F}^0$ is the filtration generated by $W^0$.

\begin{align*}
\text{maximize } \mathbb{E} \left[ \frac{1}{\gamma} \left( X_T \mu_T^{-\theta} \right)^\gamma \right] \rightarrow \text{max over } \pi \\
\text{such that } dX_t = \pi_t X_t (h_t \, dt + \sigma_t \, dW_t + \sigma_0 \, dW^0_t), \quad X_0 = x_{po};
\end{align*}

3. Search for the fixed point $\mu_t = \exp \left( \mathbb{E}[\log X_t | \mathcal{F}_t^0] \right)$, $t \in [0, T]$,
   $X^*$ is the optimal wealth from 2 and $\mathcal{F}^0$ is the filtration generated by $W^0$. 

2
In (1.6), $\pi$ is the amount of money invested in the risky asset while $\pi$ in (1.7) is the fraction of money invested in the risky asset. In MFGs (1.6) and (1.7), by the approach introduced in [18, 22], we only need to consider a representative player’s utility maximization problem with $\mu$ fixed, which in turn should be consistent with the aggregation of the optimal wealth. By MOP, the solvability of MFGs can be reduced to the solvability of mean field FBSDEs with quadratic growth, whose dimension is significantly reduced compared with the multidimensional FBSDE in the $N$-player games; see Section 2. The mean field FBSDEs characterizing the NE for (1.6) and (1.7) can be unified into one and we solve the unified mean field FBSDE under a weak interaction assumption, that is the competition parameter $\theta$ is assumed to be small. Such assumption is widely used in the game theory and financial mathematics literature; see [4, 5, 10, 11, 13, 15] among others. In order to do so, we consider the difference between the unified FBSDE with the benchmark one when $\theta = 0$, and transform the resulting FBSDE into a mean field BSDE. Motivated by the weak interaction assumption, we establish an asymptotic expansion result in powers of the competition parameter $\theta$. Similar results of the asymptotic expansion in the literature are [4, 5], where Chan and Sircar studied competition between different energy sources by a PDE approach.

The games introduced above are a reminiscence of [21]. Indeed, when the market parameters $(h, \sigma, \sigma^0)$ become time-independent, our games reduce to the ones in [21], where Lacker and Zariphopoulou used PDE approach to establish a constant equilibrium (see [21, Definition 1]) that was proved to be unique among all constant ones. Our approach, which shares the same idea as [21] and reduces the solvability of the MFGs to the solvability of mean field FBSDEs, is more powerful, not only because it makes the games tractable in more general and practical frameworks, but also because by our approach we obtain two stronger results than the one in [21]. First, when the market parameters $(h, \sigma, \sigma^0)$ are time-independent, we revisit the model of [21] and conclude that the constant equilibrium obtained in [21] is unique among all bounded ones, not only constant ones. It implies among bounded equilibria, nonconstant equilibria do not exist. Second, when the market parameters $(h, \sigma, \sigma^0)$ are time-dependent but do not depend on the Brownian paths, we can still get the unique bounded equilibrium in closed form.

In addition to [21], many player portfolio games with mean field interaction have been studied in [6, 7, 8, 9, 16, 20], where in [6, 7] Reis and Platonov studied many player games with forward utilities, in [20] Lacker and Soret extended the CRRA model in [21] to include consumption, by using PDE approaches. In [16] Hu and Zariphopoulou studied the many player games in Itô-diffusion environment. The same as [21], the market incompleteness of our paper is due to the individual noise driven the stock price while the market incompleteness of [8, 9] is due to the trading constraint, which makes each player does not have access to the entire financial market, and the incompleteness in [16] is due to stock coefficients driven by a Brownian motion that is correlated with the one driving the stock price. The methodology of our paper shares more similarities with [8, 9], where Espinosa and Touzi, Frei and Reis studied games without individual noise but with trading constraint, also by a BSDE approach. In particular, we extend the argument in [8, 9] to establish an equivalent relationship between the existence of NE and the solvability of FBSDEs. Such equivalence result is necessary to show the uniqueness result of constant NE in Section 3.5 and Section 5.1.

The rest of the paper is organized as follows.

- After the introduction of notation, in Section 2 we establish an equivalent relationship between the existence of NE of MFGs (1.6) and (1.7) and the solvability of mean field FBSDEs.
- In Section 3 we study MFGs (1.6) and (1.7) in great detail; in particular, Section 3 addresses the wellposedness of the MFGs with general market parameters by solving the FBSDEs introduced in Section 2. Moreover, when the market parameters do not depend on the Brownian paths, we find the NE in closed form and in the case of time-independent market parameters, the uniqueness of the constant equilibrium is discussed.
- In Section 4 the asymptotic result is established. In particular, the logarithm of value functions are expanded into any order in powers of $\theta$.
- In Section 5 we comment on the result of $N$-player games.
Note on the probability space $(\Omega, \mathbb{P}, \mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T})$, a two dimensional Brownian motion $\mathbf{W} = (W, W^0)^T$ is defined, where $W$ is the individual noise for the representative player and $W^0$ is the common noise for all players. Moreover, $\mathcal{G} = \{\mathcal{G}_t, t \in [0, T]\}$ is assumed to be the augmented natural filtration of $\mathbf{W}$. The augmented natural filtration of $W^0$ is denoted by $\mathbb{F}^0 = \{\mathcal{F}_t^0, t \in [0, T]\}$. Let $\mathcal{A}$ be a $\sigma$-algebra that is independent of $\mathcal{G}$. Let $\mathbb{F} = \{\mathcal{F}_t, t \in [0, T]\}$ be the $\sigma$-algebra generated by $\mathcal{A}$ and $\mathcal{G}$.

For a random variable $\xi$, let $\|\xi\|$ be its essential upper bound of its absolute value $|\xi|$. For a sub-$\sigma$-algebra $\mathcal{H}$ of $\mathcal{F}$, let $\text{Prog}(\Omega \times [0, T]; \mathcal{H})$ be the space of all stochastic processes that are $\mathcal{H}$-progressively measurable. For each $\eta \in \text{Prog}(\Omega \times [0, T]; \mathcal{H})$, define $\|\eta\|_\infty = \text{ess sup}_{\omega \in \Omega, t \in [0, T]} |\eta(\omega)|$. Let $L^\infty_\mathcal{H}$ be the space of all essentially bounded stochastic processes, i.e.,

$$L^\infty_\mathcal{H} = \{\eta \in \text{Prog}(\Omega \times [0, T]; \mathcal{H}) : \|\eta\|_\infty < \infty\}.$$

Let $S^p_{\mathcal{H}}$ be the subspace of $L^\infty_\mathcal{H}$, where the trajectories of all processes are continuous. Moreover, for a probability measure $\mathbb{Q}$ and $p > 1$, define

$$S^p_{\mathcal{H}} = \left\{ \eta \in \text{Prog}(\Omega \times [0, T]; \mathcal{H}) : \eta \text{ has continuous trajectory and } \|\eta\|_{S^p_{\mathcal{H}}} := \left( \mathbb{E}^\mathbb{Q} \left[ \sup_{0 \leq t \leq T} |\eta|^p \right] \right)^{1/p} < \infty \right\}$$

and

$$M^p_{\mathcal{H}} = \left\{ \eta \in \text{Prog}(\Omega \times [0, T]; \mathcal{H}) : \|\eta\|_{M^p_{\mathcal{H}}} := \left( \mathbb{E}^\mathbb{Q} \left[ \left( \int_0^T |\eta|^2 \, dt \right)^{p/2} \right] \right)^{1/p} < \infty \right\}.$$

Define the BMO space under $\mathbb{Q}$ as

$$H^2_{\text{BMO}, \mathcal{H}} = \left\{ \eta \in \text{Prog}(\Omega \times [0, T]; \mathcal{H}) : \|\eta\|^2_{H^2_{\text{BMO}, \mathcal{H}}} := \sup_{\tau : \mathcal{H}-\text{stopping time}} \mathbb{E}^\mathbb{Q} \left[ \left( \int_{\tau}^T |\eta|^2 \, dt \right)^{1/2} \right] < \infty \right\}.$$

In particular, if $\mathbb{Q} = \mathbb{P}$, which is the physical measure, and/or $\mathcal{H} = \mathbb{F}$, we drop the dependence on $\mathbb{Q}$ and/or $\mathcal{H}$ in the definition of the above spaces.

**Assumption 1.** The initial wealth $x_{\text{exp}}$ and $x_{\text{po}}$, risk aversion parameters $\alpha$ and $\gamma$, and competition parameter $\theta$ are assumed to be bounded $\mathcal{A}$-random variables. In particular, $x_{\text{exp}}$ is $\mathbb{R}$-valued, $x_{\text{po}}$, $\alpha$ are valued in $[0, \infty)$, $\gamma$ is valued in $(-\infty, 1)/\{0\}$ and $\theta$ is valued in $[0, 1]$.

Assume the return rate $h \in L^\infty$ and the volatilities $\tilde{\sigma} := (\sigma, \sigma^0)^T \in L^\infty \times L^\infty$. Moreover, $\alpha, |\gamma|, |\sigma| + |\sigma^0|$ are bounded away from 0, i.e., there exist positive constants $\underline{\alpha}$, $\underline{\gamma}$ and $\underline{a}$ such that $\alpha \geq \underline{\alpha} > 0$ a.s., $|\gamma| \geq \underline{\gamma} > 0$ a.s. and $|\sigma| + |\sigma^0| \geq \underline{a} > 0$ a.s. a.e.

**Space of Admissible Strategies.** We assume the space of admissible strategies for the representative player is $H^2_{\text{BMO}, \mathcal{H}}$.

**Definition of NE.**

(1) We say the pair $(\mu^*, \pi^*)$ is an NE of (1.6), if $\pi^*$ is admissible, $\mu^*_t = \mathbb{E}[X^*_t | \mathcal{F}_t^0]$ for each $t \in [0, T]$ and $\mathbb{E}[\exp(-\alpha(X^*_t - \theta \mu^*_t))] \geq \mathbb{E}[\exp(-\alpha(X^*_t - \theta \pi^*_t))]$ for each admissible strategy $\pi$. In particular, $(\mu^*_t)_{0 \leq t \leq T}$ is a solution of (1.6).

(2) We say the pair $(\mu^*, \pi^*)$ is an NE of (1.7), if $\pi^*$ is admissible, $\mu^*_t = \exp \left( \mathbb{E} \log X^*_t | \mathcal{F}_t^0 \right)$, $t \in [0, T]$ and $\mathbb{E} \left[ \frac{1}{2} \left( X^*_t (\mu^*_t)^{-\theta} \right)^2 \right] \geq \mathbb{E} \left[ \frac{1}{2} \left( X^*_t (\pi^*_t)^{-\theta} \right)^2 \right]$ for each admissible strategy $\pi$. In particular, $(\mu^*_t)_{0 \leq t \leq T}$ is a solution of (1.6).

1This assumption is consistent with the random type in [21] and we refer to [21] for details of the random type.

2This admissible space is consistent with the one in [23].

3The NE in this definition is open loop. We make a remark on the closed loop NE in Remark 5.8.

4
2 NE of MFGs and Mean Field FBSDEs Are Equivalent

In this section, we prove that the solvability of a mean field FBSDE is sufficient and necessary for the existence of an NE of MFGs (1.6) and (1.7). The sufficient part is proved by MOP as in [17] and the necessary part is proved by dynamic programming principle from [8] Lemma 4.4 and [9] Lemma 3.2, where the N-player game with exponential utility functions and trading constraint but without individual noise is considered. In the next two lemmas, we adapt the argument to our MFGs (1.6) and (1.7).

Lemma 2.1 (Exponential Utility). (1) If an NE \((\mu^*, \pi^*) \in S_{\mu^0}^2 \times H_{BMO}^2\) of the MFG (1.6) exists, with

\[
\mathbb{E}\left[-e^{-\alpha(X_t^\pi - \theta t)} \mid \mathcal{F}_t\right] \quad \text{satisfying } R_p \text{ for some } p > 1 \quad (\text{see Appendix B}),
\]

then the following mean field FBSDE admits a solution such that \((Z, Z^0) \in H_{BMO}^2 \times H_{BMO}^2\)

\[
\begin{align*}
    dx_t &= \frac{\alpha \sigma Z_t + \alpha \sigma_0 Z_t^0 + h_t}{\alpha (\sigma_t^2 + (\sigma_t^0)^2)} \left( \beta_t dt + \sigma_t dW_t + \sigma_t^0 dW_t^0 \right), \\
    dy_t &= \left( \frac{\alpha \sigma Z_t + \alpha \sigma_0 Z_t^0 + h_t}{2 \alpha (\sigma_t^2 + (\sigma_t^0)^2)} \right) dt + Z_t dW_t + Z_t^0 dW_t^0, \\
    X_0 &= x_{\text{exp}},
\end{align*}
\]

(2.2)

\[
X_0 = x_{\text{exp}}, \quad Y_T = \theta \mathbb{E}[X_T | \mathcal{F}_T].
\]

(2.1)

(2) If the FBSDE (2.1) admits a solution such that \((Z, Z^0) \in H_{BMO}^2 \times H_{BMO}^2\), then MFG (1.6) admits an NE \((\mu^*, \pi^*) \in S_{\mu^0}^2 \times H_{BMO}^2\) such that (2.1) holds.

The relationship is given by \(\pi^* = \frac{\alpha Z + \alpha Z^0 + h}{\alpha (\sigma^2 + (\sigma^0)^2)}\).

Proof. (1) For each \(\pi \in H_{BMO}^2\), define

\[
M_t^\pi = e^{-\alpha X_t^\pi} \underbrace{\text{ess sup}}_{\kappa \in H_{BMO}^2} \mathbb{E}\left[-e^{-\alpha(X_t^\pi - \theta t)} \mid \mathcal{F}_t\right].
\]

By [9] Lemma 3.3, \(M^\sigma\) has a continuous version which is a supermartingale for all \(\pi\) and a martingale for \(\pi^*\) and moreover there exists \(\hat{Z} \in H_{BMO}^2 \times H_{BMO}^2\) such that

\[
M_t^\pi = M_0^\pi \mathcal{E}\left( \int_0^t \hat{Z}_s^\top dW_s \right).
\]

Straightforward calculation implies

\[
M_t^\pi = e^{\alpha (X_t^\pi - X_t^\mu)} M_t^\pi^* = M_0^\pi^* \mathcal{E}\left( \int_0^t \left( \alpha \sigma_s^\top \sigma_s - \alpha \sigma_s^\top \sigma_s^\top + \hat{Z}_s^\top \right) dW_s \right) e^{\int_0^t f_s ds},
\]

(2.3)

where

\[
f = \alpha \mu^* h - \alpha \pi h + \frac{1}{2} \alpha^2 (\pi^*)^2 \sigma^\top \sigma + \frac{1}{2} \alpha^2 \sigma^2 \sigma^\top \sigma - \alpha^2 \pi^* \pi^\top \sigma + \alpha \pi^* \sigma^\top \hat{Z} - \alpha \pi^\top \hat{Z}.
\]

Define \(\tilde{Z} = \frac{\hat{Z}}{\sigma^\top} + \pi^* \tilde{\sigma}\). Then \(f\) can be rewritten as

\[
f = \frac{1}{2 \sigma^\top \sigma} \left| \alpha \pi^* \sigma^\top \hat{Z} - \alpha \sigma^\top \hat{Z} - h \right|^2 - \frac{1}{2 \sigma^\top \sigma} \left| \alpha \pi^* \sigma^\top \hat{Z} - \alpha \sigma^\top \hat{Z} - h \right|^2.
\]

Since \(M^\pi\) is a supermartingale, \(\exp(\int_0^t f_s ds)\) should be nondecreasing. As a result, \(f \geq 0\) for all \(\pi\), which implies \(\pi^* = \frac{\alpha \sigma^\top \hat{Z} + h}{\alpha \sigma^\top \sigma}\). Let \(Y := \frac{1}{\alpha} \log \left( -M^\pi^* \exp(\alpha X^\pi^*) \right)\). Then \((X^\pi^*, Y, \hat{Z})\) satisfies (2.2).
(2) The second statement is proved by MOP. To do so, we construct stochastic processes \( R^\pi \) such that

- \( R^\pi_0 \) is independent of \( \pi \);
- \( R^\pi_\pi \) is a supermartingale for all \( \pi \) and a martingale for some \( \pi^* \);

\[ R^\pi_\pi = -e^{-\alpha(X^\pi_\pi - \theta \hat{\mu}^\pi)}, \]

which implies \( \mathbb{E}[R^\pi_\pi] = \mathbb{E}[R^\pi_0] = \mathbb{E}[R^\pi_\pi] \leq \mathbb{E}[R^\pi_\pi] \) for all \( \pi \). Thus, \( \pi^* \) is a candidate of the optimal strategy. In order to implement the MOP scheme (2.4), let \( R^\pi_\pi = -e^{-\alpha(X^\pi_\pi - \gamma \pi)} \), where \( X^\pi \) is the wealth process in (1.6) associated with the strategy \( \pi \) and \( Y \) satisfies the BSDE

\[ dY_t = f_t(Z_t, Z^0_t) dt + Z_t dW_t + Z^0_t dW^0_t, \quad Y_T = \theta \hat{\mu}^\pi, \]

with \( f \) to be determined. By construction,

\[
R^\pi_\pi = -e^{-\alpha(x - Y_0)} e^{-\alpha \int_0^t \alpha \sigma_s Z_s - \gamma_s \pi_s - Z^0_s ds} \frac{\alpha \sigma_s Z_s + \sigma^0_s Z^0_s + h}{\alpha(\sigma^2 + (\sigma^0)^2)} - e^{-\alpha \int_0^t \alpha \sigma_s Z_s - \gamma_s \pi_s - Z^0_s ds} \frac{\alpha \sigma_s Z_s + \sigma^0_s Z^0_s + h}{\alpha(\sigma^2 + (\sigma^0)^2)} + \frac{\alpha^2}{2} \int_0^t \left( Z^2 + Z^0_s(\delta + (\sigma^0)^2) \right) ds,
\]

which implies

\[
\pi^* = \frac{\alpha \sigma Z_s + \sigma^0_s Z^0_s + h}{\alpha(\sigma^2 + (\sigma^0)^2)}
\]

and

\[
f(z, Z^0) = \frac{(\alpha \sigma z + \sigma^0 z + h)^2}{2\alpha(\sigma^2 + (\sigma^0)^2)} - \frac{\alpha}{2}(z^2 + (Z^0)^2) \cdot \]

By (2.4), if \( (X^\pi, Y, Z, Z^0) \) satisfies (2.2), \( (\mu^*, \pi^*) \) is an NE of (1.6). \( \square \)

The next lemma is the power utility counterpart of Lemma 2.1.

**Lemma 2.2** (Power Utility). (1) If an NE \((\mu^*, \pi^*) \in S^\mu_\pi \times H^2_{BMO} \) of the MFG (1.7) exists, with

\[
\mathbb{E}\left[ \frac{1}{\gamma} e^{(X^\pi_\pi - \theta \hat{\mu}^\pi)} \right] \text{satisfying } R_p \text{ for some } p > 1,
\]

where \( X^\pi_\pi = \log X^\pi_\pi \) is the log-wealth and \( \hat{\mu}^\pi = \log \mu^* \), then the following mean field FBSDE admits a solution such that \((Z, Z^0) \in H^2_{BMO} \times H^2_{BMO} \)

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{dX_t}{\gamma} = \frac{h_t + \sigma_t Z_t + \sigma^0_t Z^0_t}{(1 - \gamma)(\sigma^2_t + (\sigma^0)^2)} \left( h_t - \frac{\sigma_t Z_t + \sigma^0_t Z^0_t}{2(1 - \gamma)} \right) dt + \sigma_t dW_t + \sigma^0_t dW^0_t, \\
-dY_t = \frac{Z^2 + (Z^0)^2}{2} + \gamma \frac{(h_t + \sigma_t Z_t + \sigma^0_t Z^0_t)^2}{(1 - \gamma)(\sigma^2_t + (\sigma^0)^2)} dZ_t - Z_t dW_t - Z^0_t dW^0_t, \\
X_0 = \log(x_{0\mu}), \quad Y_T = -\gamma \theta \mathbb{E}[X_T],
\end{array} \right.
\end{align*}
\]
(2) If the FBSDE (2.4) admits a solution such that $(Z, Z^0) \in H^2_{BMO} \times H^2_{BMO}$, then MFG (1.7) admits an NE $(\mu^*, \pi^*) \in S^*_{0} \times H^2_{BMO}$ such that (2.6) holds.

The relationship is given by $\pi^* = \frac{h+\sigma + \sigma^0 Z^0}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)}$.

Proof. (1) If $(\mu^*, \pi^*)$ is an NE of (1.7) such that (2.6) holds. Define

$$M^*_t = \mathcal{E} \left( \int_t^T (\sigma^0_{\gamma\mu} + \gamma\sigma_{\gamma\mu}^0 + \gamma Z^0_{\gamma\mu}) d\tilde{W}_s \right),$$

where $\mathcal{E}$ is the expectation. Following the argument in [8, Lemma 4.4] and [9, Lemma 3.2], $M^*$ has a continuous version which is a supermartingale for all $\pi$ and a martingale for $\pi^*$, and there exists a $\tilde{Z} \in H^2_{BMO} \times H^2_{BMO}$ such that

$$M^*_t = M_0^* e^{\int_0^t \tilde{Z}_s^\top d\tilde{W}_s - \frac{1}{2} \int_0^t \tilde{Z}_s^\top \tilde{Z}_s ds}.$$

Straightforward calculation implies that

$$M^*_t = e^{-\gamma(\tilde{X}^*_t - \tilde{X}^*_0)} M^*_0.$$

(2) For each strategy $\pi \in H^2_{BMO}$, define $R^\pi_T = \frac{1}{\gamma} X_T^\gamma e^{\gamma Y_T}$, where $dY_t = f_t(Z_t, Z^0_t) dt + Z_t dW_t + Z^0_t dW^0_t$. Let $f$ be determined such that (2.4) is satisfied with the last point replaced by $R^\pi_T = \frac{1}{\gamma} (X_T(\mu^*_T)^{-\gamma})$. Note that

$$R^\pi_T = \frac{1}{\gamma} x^{\gamma}_{\pi} \exp(Y_0) \exp \left( \int_0^T \left( \gamma \pi_s h_s - \frac{\gamma^2}{2} (\sigma^2 + (\sigma^0)^2) + f_s(Z_s, Z^0_s) + \frac{1}{2} (\gamma \pi_s\sigma_s + Z_s)^2 + \frac{1}{2} (\gamma \pi_s\sigma^0_s + Z^0_s)^2 \right) ds \right)$$

$$\times \mathcal{E} \left( \int_0^T (\gamma \pi_s\sigma_s + Z_s) dW_s + \int_0^T (\gamma \pi_s\sigma^0_s + Z^0_s) dW^0_s \right)$$

$$:= \frac{1}{\gamma} x^{\gamma}_{\pi} \exp(Y_0) \exp \left( \int_0^T \tilde{f}(\pi_s, Z, Z^0_s) ds \right) \mathcal{E} \left( \int_0^T (\gamma \pi_s\sigma_s + Z_s) dW_s + \int_0^T (\gamma \pi_s\sigma^0_s + Z^0_s) dW^0_s \right).$$

Since $\pi \in H^2_{BMO}$, $\mathcal{E} \left( \int_0^T (\gamma \pi_s\sigma_s + Z_s) dW_s + \int_0^T (\gamma \pi_s\sigma^0_s + Z^0_s) dW^0_s \right)$ is a martingale. In order to make $R^\pi_T$ satisfy the second point of (2.4), we choose $f$ such that $\tilde{f}(\pi, Z, Z^0)$ is nonpositive for all $\pi$ and zero for some $\pi^*$. By rearranging terms we have

$$\tilde{f}(\pi, Z, Z^0) = -\frac{\gamma - \gamma^2}{2} (\sigma^2 + (\sigma^0)^2) + (\gamma h + \sigma\gamma Z + \gamma\sigma^0 Z^0) \pi + \frac{Z^2 + (Z^0)^2}{2} + f(Z, Z^0).$$
Let $(\mu, FBSDE$ with the benchmark one when the competition parameter $\theta$ and is unique since the log utility function is concave. Thus, following FBSDE admits a solution with $(\mu, \tilde{f}(X, Z^0) = \frac{Z^2 + (Z^0)^2}{2} - \frac{\gamma}{2(1 - \gamma)} \frac{(h + \sigma Z + \sigma^0 Z^0)^2}{\sigma^2 + (\sigma^0)^2}$, it holds that $\tilde{f}$ is nonpositive for all $\pi$ and $\tilde{f}(\pi^*, Z^0) = 0$. Thus, by (2.4), an NE of (1.7) exists if the following FBSDE admits a solution with $(Z, Z^0) \in H^2_{BMO} \times H^2_{BMO}$.

By choosing $\hat{X}$ becomes $\pi^* \in \arg \max_{\pi} \mathbb{E}[\log X_T | F^0_T]$, given by

$$
\pi^* = \frac{h + \sigma Z + \sigma^0 Z^0}{(1 - \gamma)(\sigma^2 + (\sigma^0)^2)}
$$

and

$$
f(Z, Z^0) = \frac{Z^2 + (Z^0)^2}{2} - \frac{\gamma}{2(1 - \gamma)} \frac{(h + \sigma Z + \sigma^0 Z^0)^2}{\sigma^2 + (\sigma^0)^2},
$$

Let $\hat{X} = \log X$. (2.9) is equivalent to (2.7).

Remark 2.3 (MFGs with Log Utility Functions). If each player uses log utility criterion, then the MFG becomes

1. Fix $\mu$ in some suitable space;
2. Solve the optimization problem:
   $$
   \mathbb{E}[\log (X_T H_T^{-\theta})] \rightarrow \max \text{ over } \pi
   $$
   such that $dX_t = \pi_t X_t (h_t dt + \sigma_t dW_t + \sigma^0_t dW^0_t)$, $X_0 = x_{\log}$;
3. Search for the fixed point $\mu_t = \exp (\mathbb{E}[\log X_t | F^0_t]), t \in [0, T]$.

$X^*$ is the optimal wealth from 2 and $F^0$ is the filtration generated by $W^0$.

Note that $\arg \max_{\pi} \mathbb{E}[\log (X_T^\pi H_T^{-\theta})] = \arg \max_{\pi} \mathbb{E}[\log X_T^\pi]$. Thus, the MFG with log utility criterion is decoupled; each player makes her decision by disregarding her competitors. By [17], the NE of (2.10) is given by

$$
\mu^* = \exp \left( \mathbb{E}[\log X_T | F^0_T] \right), \quad \pi^* = \frac{h}{\sigma^2 + (\sigma^0)^2},
$$

where $X$ together with some $(Y, Z)$ is the unique solution to the (trivially solvable) FBSDE

$$
\begin{align*}
  dX_t &= \frac{h_t}{\sigma^2_t + (\sigma^0_t)^2} X_t (h_t dt + \sigma_t dW_t + \sigma^0_t dW^0_t), \\
  -dY_t &= \frac{h^2}{2(\sigma^2_t + (\sigma^0_t)^2)} dt + Z_t dW_t + Z^0_t dW^0_t, \\
  X_0 &= x_{\log}, \quad Y_T = -\theta \mathbb{E}[\log X_T | F^0_T].
\end{align*}
$$

Let $(\mu', \pi')$ be any other NE of (2.10). Given $\mu'$, by MOP in [17], the optimal response is $\frac{h}{\sigma^2_t + (\sigma^0_t)^2}$, which is unique since the log utility function is concave. Thus, $\pi' = \frac{h}{\sigma^2_t + (\sigma^0_t)^2}$ and $\mu'_t = \exp \left( \mathbb{E}[\log X_t | F^0_t] \right), t \in [0, T]$ and $X$ is the unique solution of (2.12). Thus, $(\mu^*, \pi^*) = (\mu', \pi')$ and the NE of (2.10) is unique.

In Section 8 we will study MFGs (1.6) and (1.7) by studying the FBSDEs (2.2) and (2.7). In particular, we will merge (2.2) and (2.7) into a unified FBSDE and consider the difference between the unified FBSDE with the benchmark one when the competition parameter $\theta = 0$. 

8
3 Wellposedness of FBSDEs (2.2) and (2.7), MFGs (1.6) and (1.7).

3.1 The Unified Mean Field FBSDE

To solve the MFGs (1.6) and (1.7), it is equivalent to solve the mean field FBSDEs (2.2) and (2.7), respectively. The FBSDEs (2.2) and (2.7) can be unified into the following one

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{dX_t}{dt} = \frac{h_t + \beta_1 \sigma_t Z_t + \beta_1 \sigma_t^0 Z_t^0}{\beta_2 (\sigma_t^2 + (\sigma_t^0)^2)} \\
-\frac{dY_t}{dt} = \frac{\beta_4 (h_t + \beta_1 \sigma_t Z_t + \beta_1 \sigma_t^0 Z_t^0)^2}{2(\sigma_t^2 + (\sigma_t^0)^2)} \\
X_0 = x, \quad Y_T = \theta \beta_4 \mathbb{E}[X_T | F_T^0],
\end{array} \right.
\end{align*}
\]

(3.1)

When \( x = x_{exp} \), \( \beta_1 = \alpha \), \( \beta_2 = \alpha \), \( \beta_3 = 0 \), \( \beta_4 = -\frac{1}{\alpha} \) and \( \beta_5 = 1 \), (3.1) reduces to (2.2). When \( x = \log(x_{po}) \), \( \beta_1 = 1 \), \( \beta_2 = 1 - \gamma \), \( \beta_3 = 1 \), \( \beta_4 = \frac{1}{1-\gamma} \) and \( \beta_5 = -\gamma \), (3.1) reduces to (2.7).

To solve (3.1), we compare (3.1) with the benchmark FBSDE associated with the single player’s utility maximization problem, i.e., the utility game with \( \theta = 0 \). When \( \theta = 0 \), (3.1) is decoupled into

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{dX_t}{dt} = \frac{h_t + \beta_1 \sigma_t Z_t + \beta_1 \sigma_t^0 Z_t^0}{\beta_2 (\sigma_t^2 + (\sigma_t^0)^2)} \\
-\frac{dY_t}{dt} = \frac{\beta_4 (h_t + \beta_1 \sigma_t Z_t + \beta_1 \sigma_t^0 Z_t^0)^2}{2(\sigma_t^2 + (\sigma_t^0)^2)} \\
X_0 = x, \quad Y_T = 0.
\end{array} \right.
\end{align*}
\]

(3.2)

The solvability of the FBSDE (3.2) is summarized in Lemma A.1 where \( (X, Y, Z, Z^0) \) denote the unique solution of (3.2) by (\( X^0, Y^0, Z^0, Z^0^0 \)). Let \( (X, Y, Z, Z^0) \) be a solution of (3.1) and we consider the difference

\[
(X, Y, Z, Z^0) := (X - X^0, Y - Y^0, Z - Z^0, Z^0 - Z^0^0),
\]

(3.3)

which satisfies

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{dX_t}{dt} = \frac{\beta_1 \sigma_t Z_t + \beta_1 \sigma_t^0 Z_t^0}{\beta_2 (\sigma_t^2 + (\sigma_t^0)^2)} \left( h_t - \frac{\beta_3}{\beta_2} (h_t + \beta_1 \sigma_t Z_t + \beta_1 \sigma_t^0 Z_t^0) \right) \\
-\frac{dY_t}{dt} = \frac{\beta_4 (h_t + \beta_1 \sigma_t Z_t + \beta_1 \sigma_t^0 Z_t^0)^2}{2(\sigma_t^2 + (\sigma_t^0)^2)} (\sigma_t Z_t + \sigma_t^0 Z_t^0) \\
X_0 = 0, \quad Y_T = \theta \beta_4 \mathbb{E}[X_T | F_T^0] + \theta \beta_4 \mathbb{E}[Y_T | F_T^0].
\end{array} \right.
\end{align*}
\]

(3.4)

Remark 3.1. There are two reasons why we consider the difference instead of the original dynamics. First, we want to solve (3.1) under a weak interaction assumption. In order to avoid any unreasonable assumption on \( (h, \alpha, \gamma, \sigma, \sigma^0) \), we need to drop the non-homogenous terms without \( \theta \). This can be done by considering the difference. Second, (3.4) is the starting point of the asymptotic expansion result of the value functions in powers of \( \theta \); see Section 4.
3.2 The Equivalent BSDE

In this section we transform the FBSDE (3.34) to a BSDE. From now on, we use the following notation

\[
\begin{align*}
\tilde{f}^\sigma &= \frac{\sigma h}{\beta_2(\sigma^2 + (\sigma)^2)}, & \tilde{f}^{\sigma_0} &= \frac{\sigma_0 h}{\beta_2(\sigma^2 + (\sigma_0)^2)}, & \tilde{f}^h &= \frac{h^2}{\beta_2(\sigma^2 + (\sigma)^2)}, \\
\psi &= \frac{\beta_1 \sigma_0}{\beta_2(\sigma^2 + (\sigma)^2)}, & \psi^{\sigma_0} &= \frac{\beta_1 \sigma_0}{\beta_2(\sigma^2 + (\sigma_0)^2)}, & \psi^h &= \frac{\beta_1 (\sigma_0)^2}{\beta_2(\sigma^2 + (\sigma)^2)}, \\
\phi^{(1)} &= \frac{1}{\beta_2(\sigma^2 + (\sigma)^2)} + \frac{1}{\beta_2(\sigma^2 + (\sigma_0)^2)} + \frac{1}{\beta_1 (\sigma_0)^2} Z_{\sigma_0}^0, \\
\phi^{(2)} &= \frac{1}{\beta_2(\sigma^2 + (\sigma)^2)} + \frac{1}{\beta_2(\sigma^2 + (\sigma_0)^2)} + \frac{1}{\beta_1 (\sigma_0)^2} Z_{\sigma_0}^0, \\
\phi^{(3)} &= \beta_1 Z_{\sigma_0}^0 + \frac{1}{\beta_1 \sigma_0 (h + \beta_1 \sigma Z_{\sigma}^0 + \beta_1 \sigma_0 Z_{\sigma_0}^0)} , \\
\phi^{(4)} &= \beta_1 Z_{\sigma_0}^0 + \frac{1}{\beta_1 \sigma_0 (h + \beta_1 \sigma Z_{\sigma}^0 + \beta_1 \sigma_0 Z_{\sigma_0}^0)} , \\
\phi &= \beta_1 + \frac{\beta_4 \beta_5 \sigma_0}{\sigma^2 + (\sigma)^2}, & \phi^{\sigma_0} &= \beta_1 + \frac{\beta_4 \beta_5 \sigma_0}{\sigma^2 + (\sigma)^2}, & g &= \mathbb{E}\left[\frac{\theta \beta_5 (\sigma_0)^2}{\beta_2(\sigma^2 + (\sigma)^2)}|F_0\right].
\end{align*}
\]

With the new notation, we introduce the following BSDE

\[
\tilde{Y}_t = \theta \mathbb{E}[x] + \int_t^T \mathcal{I}(s; \tilde{Z}, \tilde{Z}) ds - \int_t^T \tilde{Z}_s dW_s - \int_t^T \tilde{Z}_s^0 dW_s,
\]

where \( \mathcal{I}(; \tilde{Z}, \tilde{Z}) = \mathcal{I}_1(; \tilde{Z}) + \mathcal{I}_2(; \tilde{Z}) + \mathcal{I}_3(; \tilde{Z}, \tilde{Z}) + \mathcal{I}_4(;), \) and the terms involving \( \tilde{Z} \) are given by

\[
\begin{align*}
\mathcal{I}_1(t; \tilde{Z}) &= \frac{\phi^{\sigma_0}(\tilde{Z})^2}{2} + \frac{\phi^{\sigma_0}(\beta_5 \sigma_0)^2}{2(1 - g_t)} \mathbb{E}[\psi^{\sigma_0} \tilde{Z}_t|F_t^0] \mathbb{E}[\psi^{\sigma_0} \tilde{Z}_t|F_t^0] + \frac{\beta_3 \beta_4 \beta_5 \sigma_0}{1 - g_t} \mathbb{E}[\psi^{\sigma_0} \tilde{Z}_t|F_t^0] + \frac{(\beta_3 \sigma_0)^2}{1 - g_t} \mathbb{E}[\psi^{\sigma_0} \tilde{Z}_t|F_t^0] \\
+ \left\{ \phi^{(4)} + \frac{\beta_3 \sigma_0}{1 - g_t} \phi^{(2)} \right\} \tilde{Z}_t \\
- \theta \beta_5 \mathbb{E}\left[\frac{\beta_1 \beta_3 \beta_5 \sigma_0}{\beta_2(1 - g_t)} \tilde{Z}_t \right] \mathbb{E}[\psi^{\sigma_0} \tilde{Z}_t|F_t^0] \\
- \theta \beta_5 \mathbb{E}\left[\frac{\beta_1 \beta_3 \beta_5 \sigma_0}{\beta_2(1 - g_t)} \tilde{Z}_t \right] \mathbb{E}[\psi^{\sigma_0} \tilde{Z}_t|F_t^0] \\
- \theta \beta_5 \mathbb{E}\left[\frac{\beta_1 \beta_3 \beta_5 \sigma_0}{\beta_2(1 - g_t)} \tilde{Z}_t \right] \mathbb{E}[\psi^{\sigma_0} \tilde{Z}_t|F_t^0] \\
\end{align*}
\]

the terms involving \( \tilde{Z}_0 \) are given by

\[
\begin{align*}
\mathcal{I}_2(t; \tilde{Z}_0) &= \frac{\phi^{(2)}(\tilde{Z})^2}{2} + \frac{\phi^{(2)}(\beta_5 \sigma_0)^2}{2(1 - g_t)} \mathbb{E}[\psi^{\sigma_0} \tilde{Z}_0|F_t^0] \mathbb{E}[\psi^{\sigma_0} \tilde{Z}_0|F_t^0] + \frac{\beta_3 \beta_5 \sigma_0}{1 - g_t} \mathbb{E}[\psi^{\sigma_0} \tilde{Z}_0|F_t^0] + \frac{(\beta_3 \sigma_0)^2}{1 - g_t} \mathbb{E}[\psi^{\sigma_0} \tilde{Z}_0|F_t^0] \\
+ \left\{ \phi^{(3)} + \frac{\beta_3 \sigma_0}{1 - g_t} \phi^{(1)} \right\} \tilde{Z}_0
\end{align*}
\]
and the remaining terms are given by

\[ I_3(t; \tilde{Z}, \tilde{Z}^0) = \beta_1 \beta_2 \beta_3 \psi_1 Z^0_t Z^0_t + \frac{\beta_1 \beta_2 \beta_3 \beta_4 \psi_0}{\beta_2 (1 - g_t)} Z^0_t E[\psi^0_0 Z^0_t | \mathcal{F}_t^0] - \theta \beta_5 \mathbb{E} \left[ \frac{\theta_3_1 \beta_3 \beta_4 \beta_5 \psi_0^0}{\beta_2 (1 - g_t)} \right] \mathcal{F}_t^0 \]

and the remaining terms are given by

\[ I_4(t) = \frac{(\theta_3_1)^2 \phi_0^0}{2(1 - g_t)} (E[\phi_1^0(\mathcal{F}_t^0)])^2 + \theta \beta_5 \left[ \frac{\theta_3_1 \beta_3 \beta_4 \beta_5 \psi_0^0}{\beta_2 (1 - g_t)} \right] \mathcal{F}_t^0 \]

The following lemma establishes an equivalent relationship between the FBSDE (3.4) and the BSDE (3.6).

**Lemma 3.2.** The wellposedness of FBSDE (3.4) is equivalent to the wellposedness of the BSDE (3.6).

**Proof.** Let \((\bar{X}, \bar{Y}, \bar{Z}, \bar{Z}^0)\) satisfy (3.4). Define \(\bar{Y}_t\) as

\[ \bar{Y}_t = \bar{Y}_t - \theta \beta_5 \int_0^t \mathbb{E} \left[ \frac{\beta_1 \beta_2 \beta_3 \beta_4 \beta_5 \psi_0^0}{\beta_2 (1 - g_t)} \right] \mathcal{F}_s^0 ds \]

(3.7)
Then taking the expression of $E[X_T|\mathcal{F}_T^0]$ into the dynamics of $\hat{Y}$, from the definition (3.4) we have

$$
\hat{Y}_t = \theta \beta T_x + \beta \Sigma + T + E \left[ \frac{\beta T_T}{\beta T_T^2 + (\sigma^2 T_T^2)} \right] ds
$$

(3.8)

For each $t \in [0, T]$, let

$$
\hat{Z}_t = \mathcal{Z}_t, \quad \hat{Z}_t^0 = \mathcal{Z}_t^0 - \frac{\beta T_T}{\beta T_T^2 + (\sigma^2 T_T^2)} ds
$$

Given the notation in (3.5), (3.9) can be rewritten as

$$
\hat{Z}_t^0 = \mathcal{Z}_t^0 - \frac{\beta T_T}{\beta T_T^2 + (\sigma^2 T_T^2)} ds
$$

(3.10)

Multiplied by $\psi^0$ on both sides of the second equality in (3.10) and taking conditional expectations $E[\cdot|\mathcal{F}_T^0]$, we get an equality for $E[\psi^0 \hat{Z}_t^0|\mathcal{F}_T^0]$ in terms of $E[\psi^0 \hat{Z}_t^0|\mathcal{F}_T^0]$, from which we get for each $t \in [0, T]

$$
E \left[ \psi^0 \hat{Z}_t^0 \right] = E \left[ \psi^0 \hat{Z}_t^0 \right] + \frac{\beta T_T}{\beta T_T^2 + (\sigma^2 T_T^2)} ds
$$

(3.11)

Taking (3.11) back into (3.10) and rearranging terms, we obtain $\hat{Z}^0$ in terms of $\hat{Z}^0$ and $\hat{Z}^0$

$$
\hat{Z}_t^0 = \mathcal{Z}_t^0 + \frac{\beta T_T}{\beta T_T^2 + (\sigma^2 T_T^2)} ds
$$

(3.12)

Taking (3.12) into (3.8) and straightforward calculation implies that $(\hat{Y}, \hat{Z}, \hat{Z}^0)$ solves (3.9).

On the other hand, let $(\tilde{Y}, \tilde{Z}, \tilde{Z}^0)$ solves (3.6). Define $\tilde{Z} = \tilde{Z}$ and define $\tilde{Z}^0$ as (3.12). Define $\hat{Y}$ in terms of $\hat{Y}, \hat{Z}$ and $\hat{Z}^0$ through (3.7). Moreover, define $\hat{X}$ in terms of $\hat{Z}$ and $\hat{Z}^0$ as in (3.4). Then one can verify that the tuple of stochastic processes $(\hat{X}, \hat{Y}, \hat{Z}, \hat{Z}^0)$ satisfies (3.4).

3.3 Wellposedness of the BSDE (3.6) and FBSDEs (2.7) and (2.7)

The BSDE (3.6) is a quadratic one of conditional mean field type and it does not satisfy the assumptions in [14]. In particular, the quadratic growth in (3.6) comes from both $(\tilde{Z}, \tilde{Z}^0)$ and the conditional expectation.
of \((\bar{Z}, \bar{Z}^0)\). Given the background of portfolio games with competition, we will solve (3.16) under a weak interaction assumption, where the competition parameter \(\theta\) is assumed to be small enough.

To obtain the wellposedness result under a weak interaction assumption, we divide the driver of (3.16) into two parts; one part depends on \(\theta\), which is denoted by \(J_1(\cdot; \bar{Z}, \bar{Z}^0)\), and the other part does not depend on \(\theta\), which is denoted by \(J_2(\cdot; \bar{Z}, \bar{Z}^0, \theta)\). In particular, the terms without \(\theta\) are given by

\[
J_1(t; \bar{Z}, \bar{Z}^0) = \frac{\phi_0^\sigma}{2} Z_t^2 + \frac{\phi_0^{\sigma_0}}{2} (Z_t^0)^2 + \phi_1^4 \bar{Z}_t + \phi_1^3 \bar{Z}_t^0 + \beta_1 \beta_2 \beta_4 \psi_1 \bar{Z}_t \bar{Z}_t^0.
\]

Define

\[
d\mathbb{P}^0 = \mathcal{E} \left( \int_0^T \phi_s^{(4)} dW_s + \int_0^T \phi_s^{(3)} dW_s^0 \right) := \mathcal{E} \left( \int_0^T \mathcal{M}_s d(W_s, W_s^0)^\top \right),
\]

where

\[
\mathcal{M} = \left( \phi^{(4)}, \phi^{(3)} \right).
\]

By Lemma [A.1] and [19] Theorem 2.3, it holds \(\mathbb{P}^0\) is a probability measure and Girsanov theorem yields that

\[
\tilde{Y}_t = \theta \beta_3 \mathbb{E}[x] + \int_t^T \left( J_1^0(s; \tilde{Z}, \tilde{Z}^0) + J_2(s; \tilde{Z}, \tilde{Z}^0, \theta) \right) ds - \int_t^T \tilde{Z}_s d\tilde{W}_s - \int_t^T \tilde{Z}_s^0 d\tilde{W}_s^0,
\]

where

\[
W_t^0 := (\bar{W}_t, \bar{W}_t^0) = \left( W_t - \int_t^T \phi_s^{(4)} ds, W_t^0 - \int_t^T \phi_s^{(3)} ds \right)
\]

is a two dimensional Brownian motion under \(\mathbb{P}^0\), and

\[
J_1^0(t; \tilde{Z}, \tilde{Z}^0) = \frac{\phi_0^\sigma}{2} \tilde{Z}_t^2 + \frac{\phi_0^{\sigma_0}}{2} (\tilde{Z}_t^0)^2 + \beta_1 \beta_2 \beta_4 \psi_1 \tilde{Z}_t \tilde{Z}_t^0.
\]

Thus, solving (3.16) under \(\mathbb{P}\) is equivalent to solving (3.15) under \(\mathbb{P}^0\). To do so, we use a fixed point argument as in [25]. First, for fixed \((z, z^0) \in H^2_{BMO, \mathbb{P}^0} \times H^2_{BMO, \mathbb{P}^0}\), where \(\|z\|_{BMO, \mathbb{P}^0} \leq R\) and \(\|z^0\|_{BMO, \mathbb{P}^0} \leq R\) with \(R\) to be determined, we consider the following quadratic BSDE without mean field terms

\[
\tilde{Y}_t = \theta \beta_3 \mathbb{E}[x] + \int_t^T \left( J_1^0(s; \tilde{Z}, \tilde{Z}^0) + J_2(s; z, z^0, \theta) \right) ds - \int_t^T \tilde{Z}_s d\tilde{W}_s - \int_t^T \tilde{Z}_s^0 d\tilde{W}_s^0.
\]

The following Lemma establishes the wellposedness of (3.17) and a priori estimate, which is necessary for the subsequent analysis.

**Lemma 3.3.** Let Assumption 1 hold. There exists a positive constant \(\theta^*_1\) only depending on \(R, T, \alpha, \gamma, \sigma, \sigma^0\) and \(h\) such that for \(0 \leq \|\theta\| \leq \theta^*_1\), there exists a unique \((\tilde{Y}, \bar{Z}, \bar{Z}^0) \in S^\infty \times H^2_{BMO, \mathbb{P}^0} \times H^2_{BMO, \mathbb{P}^0}\) satisfying (3.17). Moreover, it admits the following estimate

\[
\|\tilde{Y}\|_\infty \leq \|\theta\| \|\mathbb{E}[x]\| - \frac{1}{2\alpha} \log \left( 1 - \left( 2\alpha \left| J_2(\cdot; z, z^0, \theta) \right| \right) \right)^{\frac{1}{2}} \leq \|\mathbb{E}[x]\| \leq BMO, \mathbb{P}^0
\]

and

\[
\|\bar{Z}\|_{BMO, \mathbb{P}^0} + \|\bar{Z}^0\|_{BMO, \mathbb{P}^0} \leq \frac{1}{2} \frac{\epsilon^2}{\mathcal{L}^2} \left( \|\mathbb{E}[x]\| - \frac{1}{2} \right) \mathbb{E}[x] + \frac{\epsilon^2 \|\mathbb{E}[x]\|}{\mathcal{L}} \left( \left| J_2(\cdot; z, z^0, \theta) \right| \right)^{\frac{1}{2}} \leq \|\mathbb{E}[x]\| \leq BMO, \mathbb{P}^0,
\]

where \(c = \beta_1 + \frac{\beta_2 \beta_4}{4}\) and \(\mathcal{L} > 0\) is a constant such that \(c \geq \mathcal{L}\).
Proof. Let $\tilde{Z} = (\tilde{Z}, \tilde{Z}^0)$. Then $\mathcal{J}^\phi(t; \tilde{Z}, \tilde{Z}^0)$ can be rewritten as

$$
\mathcal{J}^\phi_t(\cdot; \tilde{Z}) = \tilde{Z} \mathcal{J}^\phi_t \tilde{Z}^T + \frac{\beta_1 \beta_2 \beta_3 \psi_t}{2} |\tilde{Z}|^2 - \frac{\beta_1 \beta_2 \beta_3 \psi_t}{2} \tilde{Z} \tilde{Z}^T,
$$

where

$$
\mathcal{J}^\phi_{11} = \left(\begin{array}{cc}
\frac{\phi}{2} & 0 \\
0 & \frac{\phi}{2^n}
\end{array}\right), \quad \mathcal{J}^\phi_{12} = \left(\begin{array}{c}
1 \\
1
\end{array}\right).
$$

Thus, (3.17) is a quadratic BSDE. In particular, by noting $\phi^0 + \phi^0 = 2\beta_1 + \beta_3 \beta_1^2$ and $|\psi| \leq \frac{1}{2}$ we have

$$
|\mathcal{J}^\phi_t(\cdot; \tilde{Z})| \leq \left(\beta_1 + \frac{\beta_3 \beta_1^2}{2} + \frac{\beta_1}{4}\right)|\tilde{Z}|^2 = C |\tilde{Z}|^2.
$$

Thus, [2] Assumption A.1(iii) is satisfied. By [12] Lemma A.1, it can be verified that there exists a positive constant $C_1$ depending only on $R, T, C, \gamma, \sigma, \sigma^0$ and $h, \|Z^0\|_{\text{BMO}}$ and $\|Z^0,\sigma\|_{\text{BMO}}$ such that

$$
\| (2c \mathcal{J}_2(\cdot; z, z^0, \theta))^{\frac{1}{2}} \|_{\text{BMO}} \leq \|\theta\| C_1 \left( T, \alpha, \gamma, \sigma, \sigma^0, h, \|Z^0\|_{\text{BMO}}, \|Z^0,\sigma\|_{\text{BMO}} \right).
$$

By [13] Lemma A.1, we have

$$
\| (2c \mathcal{J}_2(\cdot; z, z^0, \theta))^{\frac{1}{2}} \|_{\text{BMO}^{p_0}} \leq \|\theta\| \left( \|M\|_{\text{BMO},p_0} + 1 \right)^2 C_1 \left( T, \alpha, \gamma, \sigma, \sigma^0, h, \|Z^0\|_{\text{BMO}}, \|Z^0,\sigma\|_{\text{BMO}} \right) < \infty,
$$

where $M$ is defined in (3.14). Thus, there exists $\theta_1^*$ such that for each $0 \leq \|\theta\| \leq \theta_1^*$ it holds that

$$
\| (2c \mathcal{J}_2(\cdot; z, z^0, \theta))^{\frac{1}{2}} \|_{\text{BMO}^{p_0}} \leq \|\theta\| C_1 \left( T, \alpha, \gamma, \sigma, \sigma^0, h, \|Z^0\|_{\text{BMO}}, \|Z^0,\sigma\|_{\text{BMO}} \right) < 1.
$$

Note that $\theta_1^*$ only depends on $R, T, \alpha, \gamma, \sigma, \sigma^0$ and $h$. Then [19] Theorem 2.2 implies

$$
\mathbb{E}^{p_0} \left[ e^{2c \int_0^T \mathcal{J}_2(s; z, z^0, \theta) ds} \right] \leq \frac{1}{1 - \left( (2c \mathcal{J}_2(\cdot; z, z^0, \theta))^{\frac{1}{2}} \right)^2_{\text{BMO}^{p_0}}} < \infty.
$$

Thus, [2] Proposition 3] implies that (3.14) has a solution $(\tilde{Y}, \tilde{Z})$ such that

$$
|\tilde{Y}_t| \leq \|\theta\| \mathbb{E}[x] + \frac{1}{2} \log \left( \frac{1}{1 - \left( (2c \mathcal{J}_2(\cdot; z, z^0, \theta))^{\frac{1}{2}} \right)^2_{\text{BMO}^{p_0}}} \right);
$$

Let $\mathcal{T}(y) = \frac{1}{e^{|y|}-1} \frac{1}{|y|}$. Itô’s formula implies that

$$
\mathcal{T}(\tilde{Y}_t) = \mathcal{T}(\theta \mathbb{E}[x]) + \int_t^T \mathcal{T}'(\tilde{Y}_s) \left( \mathcal{J}^\phi_t(\cdot; \tilde{Z}) + \mathcal{J}_2(s; z, z^0, \theta) \right) ds
$$
$$
- \frac{1}{2} \int_t^T \mathcal{T}''(\tilde{Y}_s) |\tilde{Z}_s|^2 ds - \int_t^T \mathcal{T}'(\tilde{Y}_s) \tilde{Z}_s d(W_s^0)
$$
$$
\leq \mathcal{T}(\theta \mathbb{E}[x]) + \int_t^T \left( c|\mathcal{T}'(\tilde{Y}_s)| - \frac{1}{2} \mathcal{T}''(\tilde{Y}_s) \right) |\tilde{Z}_s|^2 ds
$$
$$
+ \int_t^T \mathcal{T}'(\tilde{Y}_s) \mathcal{J}_2(s; z, z^0, \theta) ds - \int_t^T \mathcal{T}'(\tilde{Y}_s) \tilde{Z}_s d(W_s^0)
$$
$$
= \mathcal{T}(\theta \mathbb{E}[x]) - \int_t^T |\tilde{Z}_s|^2 ds + \int_t^T \mathcal{T}'(\tilde{Y}_s) \mathcal{J}_2(s; z, z^0, \theta) ds - \int_t^T \mathcal{T}'(\tilde{Y}_s) \tilde{Z}_s d(W_s^0).
$$

14
Note that \( T \) is increasing for \( y > 0 \). Thus, for any stopping time \( \tau \) it holds

\[
E^{\mathbb{P}} \left[ \int_0^T |\hat{Z}_s|^2 \, ds \bigg| \mathcal{F}_\tau \right]
\leq T(\|\theta\|) + \int_0^T T'(\hat{Y}_s)J_2(s; z, z^0, \theta) \, ds
\leq \frac{1}{\bar{c}}c^2E^2|\theta||E[|x|]| - \frac{1}{\bar{c}} - \|\theta\||E[x]| + c^2|\theta||E|\|1 - \frac{1}{\bar{c}} - \|J_2(s; z, z^0, \theta)\|_{1,2}^2\|^2_{BMO, \bar{c}^2},
\]

which implies (3.19).

Moreover, for any two solutions \((\hat{Y}, \hat{Z})\) and \((\hat{Y}', \hat{Z}')\) in \( S^\infty \times H^2_{BMO, p^0} \times H^2_{BMO, p^0}, (\Delta Y, \Delta Z) := (\hat{Y} - \hat{Y}', \hat{Z} - \hat{Z}') \) follows for some stochastic process \( L \)

\[
\Delta Y_t = \int_t^T L_s \Delta Z_s \, ds - \int_t^T \Delta Z_s \, d(W_s^\alpha)^{\top},
\]

where \( |L| \leq c(|\hat{Z}| + |\hat{Z}'|) \). Define \( Q \) by \( \frac{dQ}{d\mathbb{P}} = \mathcal{E} \left( \int_0^T L_s \, d(W_s^\alpha)^{\top} \right) \). Theorem 2.3] implies \( Q \) is a probability measure. Thus, we can rewrite the above equation as

\[
\Delta Y_t = -\int_t^T \Delta Z_s \, d(\hat{W}_s)^{\top},
\]

where \( \hat{W} = W^\alpha - \int_0^T L_s \, ds \) is a two-dimensional Brownian motion under \( Q \). Obviously it holds that \( \Delta Y = \Delta Z = 0 \) and the uniqueness result follows.

**Theorem 3.4.** Let Assumption 1 hold. Let \( c = \beta_1 + \frac{\beta_2 R_1^2}{4}, \frac{\beta_4 \beta_3 |\beta_1|}{4} \) and choose \( R \) such that \( R \leq \frac{1}{4|\bar{c}|} \). Then there exists a positive constant \( \theta^*_2 < \theta^*_1 \) only depending on \( R, \alpha, \gamma, \sigma, \sigma^0 \), and \( s \) such that for \( \|\theta\| \leq \theta^*_2 \), there exists a unique \((\hat{Y}, \hat{Z}, \hat{Z}^0) \in S^\infty \times H^2_{BMO, p^0} \times H^2_{BMO, p^0} \) satisfying \( 3.19 \).

**Proof.** For each \((z, z^0) \in H^2_{BMO, p^0} \times H^2_{BMO, p^0} \) such that \( \|z\|_{BMO, p^0} \leq R \) and \( \|z^0\|_{BMO, p^0} \leq R \), Lemma 3.3 implies there is a unique \((\hat{Y}, \hat{Z}, \hat{Z}^0) \in S^\infty \times H^2_{BMO, p^0} \times H^2_{BMO, p^0} \) satisfying \( 3.17 \).

From (3.18) and (3.19) we can choose \( \theta \) small enough such that

\[
\|\hat{Z}\|^2_{BMO, p^0} + \|\hat{Z}^0\|^2_{BMO, p^0} \leq R^2.
\]

Thus, the mapping \((z, z^0) \mapsto (\hat{Z}, \hat{Z}^0)\) from a \( R \)-ball of \( H^2_{BMO, p^0} \times H^2_{BMO, p^0} \) to itself is well defined. It remains to show the mapping is a contraction. For \((z^{(1)}, z^{0,(1)})\) and \((z^{(2)}, z^{0,(2)})\) in the \( R \)-ball of \( H^2_{BMO, p^0} \times H^2_{BMO, p^0} \), let \((\hat{Y}^{(1)}, \hat{Z}^{(1)}, \hat{Z}^{0,(1)}) \in S^\infty \times H^2_{BMO, p^0} \times H^2_{BMO, p^0}\) and \((\hat{Y}^{(2)}, \hat{Z}^{(2)}, \hat{Z}^{0,(2)}) \in S^\infty \times H^2_{BMO, p^0} \times H^2_{BMO, p^0}\) be the corresponding solutions. Applying Itô’s formula and [12] Lemma A.1 to the system of \((\Delta \hat{Y}, \Delta \hat{Z}, \Delta \hat{Z}^0) := (\hat{Y}^{(1)} - \hat{Y}^{(2)}, \hat{Z}^{(1)} - \hat{Z}^{(2)}, \hat{Z}^{0,(1)} - \hat{Z}^{0,(2)}) \), we have

\[
\|\Delta \hat{Y}\|^2_{L^\infty} + \|\Delta \hat{Z}\|^2_{BMO, p^0} + \|\Delta \hat{Z}^0\|^2_{BMO, p^0}
\leq 2\|c\|_{\mathbb{R}} \|\Delta \hat{Y}\|_{L^\infty} (\|\Delta \hat{Z}\|_{BMO, p^0} + \|\Delta \hat{Z}^0\|_{BMO, p^0})
\]

\[
+ C_2 \|\theta\| \|\Delta \hat{Y}\|_{L^\infty} (\|\Delta z\|_{BMO, p^0} + \|\Delta z^0\|_{BMO, p^0})
\]

\[
+ C_2 \|\theta\| \|\Delta \hat{Y}\|_{L^\infty} (\|\Delta z\|_{BMO, p^0} + \|\Delta z^0\|_{BMO, p^0})
\]

\[
+ C_2 \|\theta\| \|\Delta \hat{Y}\|_{L^\infty} (\|\Delta z\|_{BMO, p^0} + \|\Delta z^0\|_{BMO, p^0}) (\|\Delta z\|_{BMO, p^0} + \|\Delta z^0\|_{BMO, p^0})
\]

\[
\leq \frac{1}{4} \|\Delta \hat{Y}\|^2_{L^\infty} + 8\|c\|^2R^2 (\|\Delta \hat{Z}\|^2_{BMO, p^0} + \|\Delta \hat{Z}^0\|^2_{BMO, p^0})
\]
The wellposedness of (3.1), (2.2) and (2.7) follow from (3.3), Corollary 3.5, (3.9) and (3.12).

Thus, the contraction property follows.

Theorem 3.4, Lemma A.1, (3.9) and (3.12) imply that (3.18) holds.

This result is used to establish the convergence result in Corollary 3.6, which will be used in Section 4.

Corollary 3.5. Let Assumption 1 hold and let \((\widetilde{Y}, \widetilde{Z}, \widetilde{Z}^0)\) be the unique solution of (3.6). Then it holds

\[
\lim_{\|\theta\| \to 0} \left( \|\widetilde{Y}\|_{BMO} + \|\widetilde{Z}\|_{BMO} + \|\widetilde{Z}^0\|_{BMO} \right) = 0.
\]

and

\[
\lim_{\|\theta\| \to 0} \left( \|\widetilde{Z}\|_{M_p} + \|\widetilde{Z}^0\|_{M_p} \right) = 0.
\]

Proof. Note that \(\widetilde{Z}\) and \(\widetilde{Z}^0\) belong to the R-ball of \(H^2_{BMO}\), where \(R\) is independent of \(\theta\) by Theorem 3.3. Thus, by (3.18) with \((z, z^0)\) replaced by \((\widetilde{Z}, \widetilde{Z}^0)\), and letting \(\|\theta\| \to 0\) we get \(\|\widetilde{Z}\|_{\infty} \to 0\). Taking this convergence into (3.19) with \((z, z^0)\) replaced by \((\widetilde{Z}, \widetilde{Z}^0)\) we get the convergence of \((\widetilde{Z}, \widetilde{Z}^0)\) in \(H^2_{BMO}\). The convergence in \(M_p\) is obtained by the energy inequality; see [19, P.26].

Another corollary of Theorem 3.4 is that the FBSDEs (3.1), and thus (2.2) and (2.7) are wellposed.

Corollary 3.6. Let Assumption 1 hold and \(\|\theta\| \leq 6_2\). The FBSDE (3.1) has a unique solution \((X, Y, Z, Z^0)\) in \(S^2 \times S^2 \times H^2_{BMO} \times H^2_{BMO}\) with the convergence

\[
\lim_{\|\theta\| \to 0} \left( \|Z\|_{M_p} + \|Z^0\|_{M_p} \right) = 0
\]

and the FBSDE (3.1) has a unique solution in \(S^2 \times S^2 \times H^2_{BMO} \times H^2_{BMO}\). Consequently, the FBSDEs (2.2) and (2.7) are wellposed in \(S^2 \times S^2 \times H^2_{BMO} \times H^2_{BMO}\).

Proof. Theorem 3.4 and Lemma 3.2 imply that there exists a unique \((X, Y, Z, Z^0)\) satisfying (3.3). Theorem 3.4, Lemma A.1, (3.9) and (3.12) imply that \((Z, Z^0)\) is \(H^2_{BMO} \times H^2_{BMO}\). Moreover, Theorem 3.3, Lemma A.1 and (5.7) imply that \(Y \in S^2\). As a result, \(X \in S^2\). The convergence is obtained by Corollary 3.6, (3.9) and (3.12).

The wellposedness of (3.1), (2.2) and (2.7) follow from (3.3).
3.4 Wellposedness of MFGs (1.6) and (1.7)

The main result in Section 3 is the following wellposedness result of MFGs (1.6) and (1.7).

**Theorem 3.7.** Let Assumption 1 hold and \(|\theta| \leq \theta^*_t\).

1. Let \((X^*, Y^*, Z^*, Z^0^*)\) be the unique solution of (2.3) and let \(\mu^\text{exp,*}_t = \mathbb{E}[X^*_t | \mathcal{F}^0_t], t \in [0, T]\). Under \(\mu^\text{exp,*}\) the optimal strategy for the representative player in the case of exponential utility functions is

\[
\pi^\text{exp,*} = \frac{\alpha \sigma Z^* + \alpha \sigma^0 Z^0^* + h}{\alpha (\sigma^2 + (\sigma^0)^2)}
\]

and the value function given one realization of \((\theta, \alpha, x_{\text{exp}})\) follows

\[
V^\text{exp}(\theta, \alpha, x_{\text{exp}}, \mu^\text{exp,*}) = -e^{-\alpha(x_{\text{exp}} - Y^*_0)}.
\]

Moreover, \(\mu^\text{exp,*} \in S^2_{\mathbb{F}}\) is the unique solution of the MFG (1.6) and \((\mu^\text{exp,*}, \pi^\text{exp,*}) \in S^2_{\mathbb{F}} \times H^2_{\text{BMO}}\) is the unique NE of (1.6).

2. Let \((X^*, Y^*, Z^*, Z^0^*)\) be the unique solution of (2.7) and let \(\mu^\text{po,*}_t = \exp\left(\mathbb{E}[X^*_t | \mathcal{F}^0_t]\right), t \in [0, T]\). Under \(\mu^\text{po,*}\) the optimal strategy for the representative player in the case of power utility functions is

\[
\pi^\text{po,*} = \frac{h + \alpha Z^* + \sigma^0 Z^0^*}{(1 - \gamma)(\sigma^2 + (\sigma^0)^2)}
\]

and the value function given one realization of \((\theta, \gamma, x_{\text{po}})\) follows

\[
V^\text{po}(\theta, \gamma, x_{\text{po}}, \mu^\text{po,*}) = \frac{1}{\gamma}x_{\text{po}}^\gamma e^{Y^*_0}.
\]

Moreover, \(\mu^\text{po,*} \in S^2_{\mathbb{F}}\) is the unique solution of the MFG (1.7) and \((\mu^\text{po,*}, \pi^\text{po,*}) \in S^2_{\mathbb{F}} \times H^2_{\text{BMO}}\) is the unique NE of (1.7).

**Proof.** The desired results are obtained by Lemma 2.1, Lemma 2.2, and Corollary 3.6.

**Remark 3.8.** By [17] Proposition 9 and Proposition 15, for each realization of the random variables \((\theta, \gamma, x_{\text{exp}}, x_{\text{po}})\), the value functions satisfy the dynamic version

\[
V^\text{exp}(t, X^*_t; \theta, \alpha, x_{\text{exp}}, \mu^\text{exp,*}) = -e^{-\alpha(X^*_t - Y^*_t)}, \quad V^\text{po}(t, X^*_t; \theta, \gamma, x_{\text{po}}, \mu^\text{po,*}) = \frac{1}{\gamma}(X^*_t)^\gamma e^{Y^*_t}
\]

where

\[
V^\text{exp}(t, X^*_t; \theta, \alpha, x_{\text{exp}}, \mu^\text{exp,*}) := \max_{\pi} \mathbb{E}\left[-e^{-\alpha(X^*_T - \theta \mu^\text{exp,*}_T)} | \mathcal{F}_t\right]
\]

and

\[
V^\text{po}(t, X^*_t; \theta, \gamma, x_{\text{po}}, \mu^\text{po,*}) := \max_{\pi} \mathbb{E}\left[\frac{1}{\gamma}(X^*_T (\mu^\text{po,*}_T)^\gamma) | \mathcal{F}_t\right].
\]

The dynamic version of value functions (3.24) will be used in Section 4.

3.5 Nonconstant Equilibria Do Not Exist

As a byproduct of our analysis, this sections shows that the constant equilibrium obtained in [21] is the unique one when all the coefficients are time-independent. Indeed, from the FBSDEs (2.3) and (2.7) we can get a stronger result than [21] by assuming all coefficients are time-dependent and \(\mathcal{A}\)-measurable.

As a corollary, the constant equilibrium can be obtained.
Assumption 2. For each $t \in [0, T]$, the return rate $h_t$ and the volatilities $(\sigma_t, \sigma_0^t)$ are measurable w.r.t. $\mathcal{F}_t$.

The following proposition shows the closed form solution of the FBSDE (3.1) under Assumption 2.

Proposition 3.9. Under Assumption 1 and Assumption 2, there exists a unique tuple $(X, Y, Z, Z^0) \in S^2 \times S^2 \times L^\infty \times L^\infty$ satisfying (3.1). In particular, the $Z$-component of the solution has the following closed form expression

\[
Z = 0, \quad Z^0 = \frac{\theta \beta_3 E[f^{o\theta}]}{1 - E[\theta \beta_3 \psi^{o\theta}]}, \quad (3.25)
\]

Proof. We first verify (3.25) together with some $X$ and $Y$ satisfies (3.1) by construction. Our goal is to construct $(Z, Z^0)$ such that $(Z_t, Z^0_t)$ is $\mathcal{A}$-measurable for each $t \in [0, T]$. Assuming $(Z_t, Z^0_t)$ is $\mathcal{A}$-measurable for each $t \in [0, T]$ and taking the forward dynamics in (3.1) into the backward one in (3.1), we get

\[
Y_t = \theta \beta_3 E[x] + \theta \beta_3 \int_0^T \mathbb{E} \left[ h_s + \beta_1 \sigma_s Z_s + \beta_3 \sigma_0^s Z^0_s \right] \frac{h_s - \beta_3 h_s + \beta_1 \sigma_s Z_s + \beta_3 \sigma_0^s Z^0_s}{2 \beta_2} \, ds
\]

Proving uniqueness result, by Lemma 3.2 it is sufficient to show the solution of (3.1) is unique in $S^2 \times S^2 \times L^\infty \times L^\infty$ and $S^2 \times L^\infty \times L^\infty$ and $(\tilde{Y}', \tilde{Z}, \tilde{Z}^0) \in S^2 \times L^\infty \times L^\infty$ be two solutions of (3.1). Define $(\Delta Y, \Delta Z, \Delta Z^0) := (\tilde{Y} - \tilde{Y}', \tilde{Z} - \tilde{Z}, \tilde{Z}^0 - \tilde{Z}^0)$. Then by the dynamics of (3.1) and by noting that $(\tilde{Z}, \tilde{Z}^0) \in L^\infty \times L^\infty$ the tuple $\Delta Y, \Delta Z, \Delta Z^0$ satisfies a conditional mean field BSDE with Lipschitz functions. Standard arguments imply that the unique solution is $(\Delta Y, \Delta Z, \Delta Z^0) = (0, 0, 0)$.

With the result in Proposition 3.9 we can get the optimal strategy for the representative player in closed form.

Theorem 3.10. Let Assumption 1 and Assumption 2 hold. Then in $L^\infty$ the unique optimal response of the representative player in the case of exponential utility functions (resp. in the case of power utility functions) is given by

\[
\pi_t^{exp,*} = \frac{h_t}{\alpha(\sigma_t^2 + (\sigma_0^t)^2)} + \frac{\sigma_0^t}{\alpha(\sigma_t^2 + (\sigma_0^t)^2)} \frac{E \left[ \frac{\sigma_0^t h_s}{\alpha(\sigma_t^2 + (\sigma_0^t)^2)} \right]}{1 - E \left[ \frac{\sigma_0^t h_s}{\alpha(\sigma_t^2 + (\sigma_0^t)^2)} \right]}, \quad t \in [0, T], \text{ if } E \left[ \frac{(\sigma_0^t)^2}{\alpha(\sigma_t^2 + (\sigma_0^t)^2)} \right] < 1, \quad (3.26)
\]

respectively,

\[
\pi_t^{pow,*} = \frac{h_t}{(1 - \gamma)(\sigma_t^2 + (\sigma_0^t)^2)} - \frac{\theta \gamma_0^t}{(1 - \gamma)(\sigma_t^2 + (\sigma_0^t)^2)} \frac{E \left[ \frac{h_s (1 - \gamma)}{(1 - \gamma)(\sigma_t^2 + (\sigma_0^t)^2)} \right]}{1 - E \left[ \frac{h_s (1 - \gamma)}{(1 - \gamma)(\sigma_t^2 + (\sigma_0^t)^2)} \right]}, \quad t \in [0, T]. \quad (3.27)
\]
Proof. Taking (3.25) into (3.21) and (3.23) we get (3.26) and (3.27), respectively.

As a corollary, when all coefficients become time-independent, we revisit the MFGs in [21].

Corollary 3.11 (Lacker & Zariphopoulou’s MFGs Revisited). Let Assumption 1 and Assumption 2 hold, and the return rate \( h \) and the volatilities \( (\sigma, \sigma^0) \) be time-independent. Then the optimal responses in the case of exponential utility functions and in the case of power utility functions are

\[
\pi^{\exp, \ast} = \frac{h}{\alpha(\sigma^2 + (\sigma^0)^2)} + \frac{\sigma^0 \theta}{\sigma^2 + (\sigma^0)^2} \frac{\mathbb{E}\left[ \frac{\sigma^0 h}{\alpha(\sigma^2 + (\sigma^0)^2)} \right]}{1 - \mathbb{E}\left[ \frac{\sigma^0 h}{\alpha(\sigma^2 + (\sigma^0)^2)} \right]}, \quad \text{if } \mathbb{E}\left[ \frac{(\sigma^0)^2}{\sigma^2 + (\sigma^0)^2} \right] < 1, \tag{3.28}
\]

respectively,

\[
\pi^{\po, \ast} = \frac{h}{(1 - \gamma)(\sigma^2 + (\sigma^0)^2)} - \frac{\theta \gamma \sigma^0}{(1 - \gamma)(\sigma^2 + (\sigma^0)^2)} \frac{\mathbb{E}\left[ \frac{\sigma^0 h}{1 - \gamma(\sigma^2 + (\sigma^0)^2)} \right]}{1 + \mathbb{E}\left[ \frac{\sigma^0 h}{1 - \gamma(\sigma^2 + (\sigma^0)^2)} \right]}, \tag{3.29}
\]

The constant equilibria (3.28) and (3.29) are identical to [21, Theorem 10] and [21, Theorem 19], which are unique in \( L^\infty \). Moreover, in the case of exponential utility functions, when \( \mathbb{E}\left[ \frac{(\sigma^0)^2}{\sigma^2 + (\sigma^0)^2} \right] = 1 \), there is no bounded equilibrium, not only no constant equilibrium as in [21, Theorem 10].

Remark 3.12. (1) Note that every strategy in \( L^\infty \) makes (2.1) and (2.6) hold. (2) If \( \|\theta\| \leq \bar{\theta}^2 \) holds, (3.25) is also unique in \( H^2_{BMO} \times H^2_{BMO} \) by Corollary 3.6. Thus, (3.28) and (3.29) are the unique optimal responses in \( H^2_{BMO} \) and there is no nonconstant equilibrium in the subspace of \( H^2_{BMO} \) with (2.1) or (2.6) being true.

4 Asymptotic Expansion in Terms of \( \theta \)

Motivated by the weak interaction assumption, we develop an asymptotic expansion result of the value functions; in particular, we expand the logarithm of the value functions in powers of \( \theta \) into any order.

Let \( V^\exp, \theta \) and \( V^\po, \theta \) be the value functions of the MFGs (1.6) and (1.7), respectively. Let \( V^\exp, \delta \) and \( V^\po, \delta \) be the value functions of the benchmark portfolio optimization problems in the cases of exponential utility functions and power utility functions (i.e. when \( \theta = 0 \) in (1.6) and (1.7)), respectively. Then by Remark 3.8, we have

\[
\begin{align*}
V^\exp, \theta_t &= -e^{-\alpha(X^\exp, \theta_t - Y^\exp, \theta_t)}, & V^\exp, \delta_t &= -e^{-\alpha(X^\exp, \delta_t - Y^\exp, \delta_t)}, \\
V^\po, \theta_t &= \frac{1}{\gamma} e^{\alpha(X^\po, \theta_t - Y^\po, \theta_t)} & V^\po, \delta_t &= \frac{1}{\gamma} e^{\alpha(X^\po, \delta_t - Y^\po, \delta_t)},
\end{align*}
\]

where \((X^\exp, \theta, Y^\exp, \theta, Z^\exp, \theta, Z^\exp, 0, \theta)\) and \((X^\exp, \delta, Y^\exp, \delta, Z^\exp, \delta, Z^\exp, 0, \delta)\) are the unique solutions of (2.2) with \( \theta \) and \( \theta = 0 \), respectively, and \((X^\po, \theta, Y^\po, \theta, Z^\po, \theta, Z^\po, 0, \theta)\) and \((X^\po, \delta, Y^\po, \delta, Z^\po, \delta, Z^\po, 0, \delta)\) are the unique solutions of (2.2) with \( \theta \) and \( \theta = 0 \), respectively. As a result, it holds

\[
\begin{align*}
\log \frac{V^\exp, \theta_t}{V^\exp, \delta_t} &= -\alpha(X^\exp, \theta_t - X^\exp, \delta_t) + \alpha(Y^\exp, \theta_t - Y^\exp, \delta_t), \\
\log \frac{V^\po, \theta_t}{V^\po, \delta_t} &= \gamma(X^\po, \theta_t - X^\po, \delta_t) + (Y^\po, \theta_t - Y^\po, \delta_t).
\end{align*}
\]

Let \((X^\theta, Y^\theta, Z^\theta, Z^0, \theta), (X^\delta, Y^\delta, Z^\delta, Z^0, \delta)\) and \((\bar{X}^\theta, \bar{Y}^\theta, \bar{Z}^\theta, \bar{Z}^0, \theta)\) be the unique solutions of (3.11), (3.2) and (3.3), respectively. Note that both \((X^\exp, \theta - X^\exp, \delta, Y^\exp, \theta - Y^\exp, \delta)\) and \((X^\po, \theta - X^\po, \delta, Y^\po, \theta - Y^\po, \delta)\) are special cases of \((\bar{X}^\theta, \bar{Y}^\theta)\).
Our goal is to prove that there exist \((\hat{X}^{(i)}, \hat{Y}^{(i)})\) such that for any \(n \geq 1\) it holds:

\[
X^\theta - X^\theta_0 = \sum_{i=1}^{n} \theta_i \hat{X}^{(i)} + o(\theta^n) \quad \text{and} \quad Y^\theta - Y^\theta_0 = \sum_{i=1}^{n} \theta_i \hat{Y}^{(i)} + o(\theta^n).
\]

For \(\varphi = X, Y, Z, Z^0\), define

\[
\varphi^{\theta,(1)} = \frac{\varphi^\theta - \varphi^\theta_0}{\theta} = \frac{\varphi^\theta}{\theta},
\]

which implies

\[
d\hat{X}^{\theta,(1)}_t = \left\{ \begin{array}{l}
\frac{\beta_1 \sigma_1 \hat{Z}^{\theta,(1)}_t + \beta_1 \sigma_0 \hat{Z}^{\theta,0,(1)}_t}{\beta_2 (\sigma_t^2 + (\sigma_t^0)^2)} \left( h_t - \frac{\beta_3}{\beta_2} (h_t + \beta_1 \sigma_t Z^\theta_0 + \beta_1 \sigma_t Z^{0,\theta}_t) \right) + \beta_t \left( \beta_1 \sigma_1 \hat{Z}^{\theta,(1)}_t + \beta_1 \sigma_0 \hat{Z}^{\theta,0,(1)}_t \right) \frac{\sigma_t dW_t + \sigma_t^0 dW^0_t}{2(\sigma_t^2 + (\sigma_t^0)^2)} \\
- \frac{1}{2} \beta_4 (2h_t + 2\beta_1 \sigma_t Z^\theta_0 + \beta_1 \sigma_t Z^{0,\theta}_t + \beta_1 \sigma_t Z^{0,\theta}_t + \beta_1 \sigma_t Z^{0,\theta}_t) \right) dt \\
\end{array} \right.
\]

\[
- \hat{Z}^{\theta,(1)}_t \frac{\sigma_t dW_t - \sigma_t^0 dW^0_t}{\sigma_t^2 + (\sigma_t^0)^2} = 0, \quad \hat{Y}^{\theta,(1)}_t = \beta_3 \mathbb{E}[X^\theta_t | \mathcal{F}^\theta_T] + \beta_3 \mathbb{E}[X^\theta_T | \mathcal{F}^\theta_T].
\]

We now establish the convergence of \((\hat{X}^{\theta,(1)}, \hat{Y}^{\theta,(1)}, \hat{Z}^{\theta,(1)}, \hat{Z}^{0,\theta,(1)})\) in a suitable sense as \(\theta \to 0\). Let the candidate limit of (4.1) satisfy

\[
\left\{ \begin{array}{l}
d\hat{X}^{(1)}_t = \frac{\beta_1 \sigma_1 \hat{Z}^{(1)}_t + \beta_1 \sigma_0 \hat{Z}^{0,(1)}_t}{\beta_2 (\sigma_t^2 + (\sigma_t^0)^2)} \left( h_t - \frac{\beta_3}{\beta_2} (h_t + \beta_1 \sigma_t Z^\theta_0 + \beta_1 \sigma_t Z^{0,\theta}_t) \right) dt \\
- \frac{1}{2} \beta_4 (2h_t + 2\beta_1 \sigma_t Z^\theta_0 + \beta_1 \sigma_t Z^{0,\theta}_t) \right) dt \\
\end{array} \right.
\]

\[
- \hat{Z}^{(1)}_t \frac{\sigma_t dW_t - \sigma_t^0 dW^0_t}{\sigma_t^2 + (\sigma_t^0)^2} = 0, \quad \hat{Y}^{(1)}_t = \beta_3 \mathbb{E}[X^\theta_t | \mathcal{F}^\theta_T].
\]

The following lemma establishes the wellposedness result of (4.2) and the convergence from (4.1) to (4.3).

**Lemma 4.1.** Let Assumption 1 hold.

1. There exists a unique \((\hat{X}^{(1)}, \hat{Y}^{(1)}, \hat{Z}^{(1)}, \hat{Z}^{0,(1)})\) \(\in \cap_{p>1} \mathcal{S}^p \times \cap_{p>1} \mathcal{S}^p \times \cap_{p>1} \mathcal{M}_p \times \cap_{p>1} \mathcal{M}_p\) satisfying (4.2).  

2. For each \(p > 1\), the following convergence holds

\[
\lim_{\theta \to 0} \left( \|\hat{X}^{\theta,(1)} - \hat{X}^{(1)}\|_{\mathcal{S}^p} + \|\hat{Y}^{\theta,(1)} - \hat{Y}^{(1)}\|_{\mathcal{S}^p} + \|\hat{Z}^{\theta,(1)} - \hat{Z}^{(1)}\|_{\mathcal{M}_p} + \|\hat{Z}^{\theta,0,\theta} - \hat{Z}^{0,\theta} \|_{\mathcal{M}_p} \right) = 0.
\]
Proof. (1). Recall $\mathbb{P}^o$ defined in (3.13). Then the backward dynamics in (4.2) can be rewritten as
\[
    d\tilde{Y}_{t}^{(1)} = \tilde{Z}_{t}^{(1)} \, d\tilde{W}_{t} + \tilde{Z}_{t}^{0,(1)} \, d\tilde{W}_{t}^{0}, \quad \tilde{Y}_{T}^{(1)} = \beta_{0} \mathbb{E}[X_{T}^{0}|\mathcal{F}_{T}^{p}],
\]
where $(\tilde{W}, \tilde{W}^{0})$ is defined in (3.10). In order to apply [1] Theorem 3.5], it suffices to prove for each $p > 1$
\[
    \mathbb{E}^p_o((\mathbb{E}[X_{T}^{0}|\mathcal{F}_{T}^{p}])^p) < \infty.
\]
In fact, by the notation $\mathcal{M}$ in (3.13) and $p_{\mathcal{M}}$ in Appendix [3] for any $p > 1$ and any $q \in (1, p_{\mathcal{M}})$$\mathbb{E}^p_o((\mathbb{E}[X_{T}^{0}|\mathcal{F}_{T}^{p}])^p) \leq \mathbb{E}^p_o((\mathbb{E}[X_{T}^{0}|\mathcal{F}_{T}^{p}])^q)^{\frac{p}{q}} \leq K^{\frac{1}{q}}(q, \|\mathcal{M}\|_{BMO}) \mathbb{E}^p_o((\mathbb{E}[X_{T}^{0}|\mathcal{F}_{T}^{p}])^q)^{\frac{p}{q}} < \infty \quad \text{by Lemma [A.1]}
\]
Thus, by [1] Theorem 3.5], there exists a unique $(\tilde{Y}^{(1)}, \tilde{Z}^{(1)}, \tilde{Z}^{0,(1)}) \in S_{p_o}^2 \times \bigcap_{p > 1} M_{p, p_o} \times \bigcap_{p > 1} M_{p, p_o}$, which implies $(\tilde{Y}^{(1)}, \tilde{Z}^{(1)}, \tilde{Z}^{0,(1)}) \in S^2 \times \bigcap_{p > 1} M_{p} \times \bigcap_{p > 1} M_{p}$. Indeed, by the definition of $\mathbb{P}^o$ in (3.13), it holds
\[
    \frac{d\mathbb{P}^o}{d\mathbb{P}^o} = \mathcal{E} \left( -\int_{0}^{T} \mathcal{M}_{s} \, dW_{s}^{0} \right),
\]
where $W^{o} = (\tilde{W}, \tilde{W}^{0})$ is defined in (3.10). Thus, by the same argument as (4.4) we have for any $p > 1$
\[
    \mathbb{E} \left[ \left( \int_{0}^{T} (\tilde{Z}^{(1)}_{s})^2 \, ds \right)^{\frac{p}{2}} \right] = \mathbb{E}^p_o \left[ \mathcal{E} \left( -\int_{0}^{T} \mathcal{M}_{s} \, dW_{s}^{0} \right) \left( \int_{0}^{T} (\tilde{Z}^{(1)}_{s})^2 \, ds \right)^{\frac{p}{2}} \right] < \infty.
\]
The same result holds for $\tilde{Z}^{0,(1)}$. By the dynamics of $\tilde{Y}^{(1)}$ and standard estimate, we have $\tilde{Y}^{(1)} \in \bigcap_{p > 1} S^p$.

(2). Let $\Delta \varphi = \tilde{\varphi}^{(1)} - \hat{\varphi}^{(1)}$ for $\varphi = X, Y, Z, Z^{0}$. Then $(\Delta Y^{\theta}, \Delta Z^{\theta}, \Delta Z^{0, \theta})$ satisfies
\[
    -d\Delta Y_{t}^{\theta} = \left\{ \begin{array}{l}
    \frac{\beta_{1}}{2} Z_{t}^{\theta, (1)} + \frac{\beta_{1}}{2} Z_{t}^{0, \theta, (1)} + \frac{\beta_{1}^2}{2} \beta_{4} \left( \sigma_{t} Z_{t}^{\theta, (1)} + \sigma_{t} Z_{t}^{0, \theta, (1)} \right) \left( \sigma_{t} Z_{t}^{\theta} + \sigma_{t} Z_{t}^{0, \theta} \right) \\
    \phi_{t}^{(4)} \Delta Z_{t}^{\theta} + \phi_{t}^{(3)} \Delta Z_{t}^{0, \theta} \end{array} \right\} \, dt - \Delta Z_{t}^{\theta} \, dW_{t} - \Delta Z_{t}^{0, \theta} \, dW_{t}^{0}.
\]
(4.5)
The conditions in [1] Corollary 3.4] are satisfied. Indeed, by Corollary [3.6] and the energy inequality (see [19] P. 26), it holds that for each $p > 1$, $\mathbb{E} \left[ (\mathbb{E}[X_{T}^{0}|\mathcal{F}_{T}^{p}])^p \right] < \infty$. The same reason together with the result in (1) implies that for each $p > 1$ the non-homogenous term in the driver of $\Delta Y^{\theta}$ is in $M_{p}$. Thus, [1] Assumption A3] is satisfied. [1] Assumption A2] holds due to Corollary [3.3] and Lemma [A.1] By [1] Corollary 3.4], it holds that for each $p > 1$ there exists $p' > 1$ and $p'' > 1$ such that
\[
    \|\Delta Y^{\theta}\|_{S^p} + \|\Delta Z^{\theta}\|_{M_{p}} + \|\Delta Z^{0, \theta}\|_{M_{p}} \\
    \leq C \left\{ \mathbb{E} \left[ \left| X_{T}^{0} \right|^p \right] + \left( \|\tilde{\varphi}^{(1)}\|_{M_{p}} + \|\tilde{Z}^{0, \theta, (1)}\|_{M_{p}} \right) \left( \|\tilde{Z}^{\theta}\|_{M_{p}} + \|\tilde{Z}^{0, \theta}\|_{M_{p}} \right) \right\} \\
    \times \left( 1 + \|\tilde{\varphi}^{(3)}\|_{M_{p''}} + \|\tilde{\varphi}^{(4)}\|_{M_{p''}} \right) \\
    \rightarrow 0,
\]
where $C$ does not depend on $\theta$. \[\square\]
Define
\[
\begin{align*}
K^{(\theta,n-1),(\theta,1)} & = \frac{(\sigma Z^\theta,(n-1) + \sigma^0 Z^{0,\theta,(n-1)}) (\sigma Z^\theta,(1) + \sigma^0 Z^{0,\theta,(1)})}{2(\sigma^2 + (\sigma^0)^2)}, \\
K^{(i),(\theta,n-i)} & = \frac{(\sigma Z^\theta,(i) + \sigma^0 Z^{0,\theta,(i)}) (\sigma Z^\theta,(n-i) + \sigma^0 Z^{0,\theta,(n-i)})}{2(\sigma^2 + (\sigma^0)^2)}, \\
K^{(i),(\theta,n-i)} & = \frac{(\sigma Z^\theta,(i) + \sigma^0 Z^{0,\theta,(i)}) (\sigma Z^{(n-i)}(i) + \sigma^0 Z^{0,(n-i)})}{2(\sigma^2 + (\sigma^0)^2)}.
\end{align*}
\]

For each \( n \geq 2 \), we introduce two (coupled) FBSDE systems recursively
\[
\begin{align*}
\dot{X}^\theta_T & = \left\{ \frac{\beta_1 \sigma t Z^\theta_t + \sigma^0 Z^{0,\theta,(n)}_t}{\sigma^2_t + (\sigma^0)^2} \right\} dt + \frac{\beta_1 \sigma t Z^\theta_t + \sigma^0 Z^{0,\theta,(n)}_t}{\sigma^2_t + (\sigma^0)^2} \left( \sigma_t dW_t + \sigma^0_t dW^0_t \right) \\
-\dot{Y}^\theta_T & = \left\{ \frac{\beta_1 \beta_3 Z^\theta_T(n-1) + \beta_1 \sigma^0 Z^{0,\theta,(n)}_T}{\sigma^2_t + (\sigma^0)^2} \right\} dt + \frac{\beta_1 \beta_3 Z^\theta_T(n-1) + \beta_1 \sigma^0 Z^{0,\theta,(n)}_T}{\sigma^2_t + (\sigma^0)^2} \left( \sigma_t dW_t + \sigma^0_t dW^0_t \right)
\end{align*}
\]
\[
\dot{X}^\theta_0 = 0, \quad \dot{Y}^\theta_T = \beta_3 E[X^{(n-1)}_T | \mathcal{F}_T^0]
\]
(4.6)

and
\[
\begin{align*}
\dot{X}^n & = \left\{ \frac{\beta_1 \sigma t Z^{(n)}_t + \sigma^0 Z^{0,\theta,(n)}_t}{\sigma^2_t + (\sigma^0)^2} \right\} dt + \frac{\beta_1 \sigma t Z^{(n)}_t + \sigma^0 Z^{0,\theta,(n)}_t}{\sigma^2_t + (\sigma^0)^2} \left( \sigma_t dW_t + \sigma^0_t dW^0_t \right) \\
-\dot{Y}^n & = \left\{ \frac{\beta_1 Z^{(n)}_t + \sigma^0 Z^{0,\theta,(n)}_t}{\sigma^2_t + (\sigma^0)^2} \right\} dt + \frac{\beta_1 Z^{(n)}_t + \sigma^0 Z^{0,\theta,(n)}_t}{\sigma^2_t + (\sigma^0)^2} \left( \sigma_t dW_t + \sigma^0_t dW^0_t \right)
\end{align*}
\]
\[
\dot{X}^n_0 = 0, \quad \dot{Y}^n_T = \beta_3 E[X^{(n-1)}_T | \mathcal{F}_T^0]
\]
(4.7)

In the FBSDEs (4.6) and (4.7), we use the convention that the sum vanishes whenever the lower bound of the index is larger than the upper bound of the index.
Theorem 4.2. Let Assumption 1 hold. We have for any $n \geq 1$

$$X^n = X^0 + \sum_{i=1}^{n} \theta^i \hat{X}^{(i)} + o(\theta^n), \quad Y^n = Y^0 + \sum_{i=1}^{n} \theta^i \hat{Y}^{(i)} + o(\theta^n),$$

(4.8)

where $X^{(n)}$ and $Y^{(n)}$ together with some $Z^{(n)}$ and $Z^{0,(n)}$ are the unique solution of (4.7). In particular,

$$\log \frac{V^{\exp,\theta}}{V^{\exp,0}} = \sum_{i=1}^{n} \theta^i \left(-\alpha \hat{X}^{\exp,(i)} + \alpha \hat{Y}^{\exp,(i)}\right) + o(\theta^n)$$

where $(\hat{X}^{\exp,(i)}, \hat{Y}^{\exp,(i)})$ satisfies (4.7) with $\beta_1 = \beta_2 = \alpha, \beta_3 = 0, \beta_4 = -\frac{1}{\alpha}, \beta_5 = 1$. Moreover,

$$\log \frac{V^{po,\theta}}{V^{po,0}} = \sum_{i=1}^{n} \theta^i \left(\gamma \hat{X}^{po,(i)} + \hat{Y}^{po,(i)}\right) + o(\theta^n),$$

where $(\hat{X}^{po,(i)}, \hat{Y}^{po,(i)})$ satisfies (4.7) with $\beta_1 = 1, \beta_2 = 1 - \gamma, \beta_3 = 1, \beta_4 = \frac{1}{1-\gamma}, \beta_5 = -\gamma$.

Proof. The proof is done by induction. Lemma 4.1 implies that

$$X^n = X^0 + \theta \hat{X}^{(1)} + o(\theta), \quad Y^n = Y^0 + \theta \hat{Y}^{(1)} + o(\theta),$$

which verifies (4.8) for $n = 1$. Moreover, by definition it holds $\phi^{\theta,(1)} = \frac{\phi^{\theta} - \phi^0}{\theta}$, $\varphi = X, Y, Z, Z^0$.

Now assume the result holds for $n \geq 2$, namely, there exists a unique tuple $(\hat{X}^{\theta,(n)}, \hat{Y}^{\theta,(n)}, \hat{Z}^{\theta,(n)}, \hat{Z}^{0,\theta,(n)}) \in \bigcap_{p>1} S^p \times \bigcap_{p>1} S^p \times \bigcap_{p>1} M_p \times \bigcap_{p>1} M_p$ and a unique tuple $(\hat{X}^{(n)}, \hat{Y}^{(n)}, \hat{Z}^{(n)}, \hat{Z}^{0,(n)}) \in \bigcap_{p>1} S^p \times \bigcap_{p>1} S^p \times \bigcap_{p>1} M_p \times \bigcap_{p>1} M_p$ satisfying (4.6) and (4.7), respectively, and for each $p > 1$

$$\lim_{\theta \to 0} \|\hat{X}^{\theta,(n)} - \hat{X}^{(n)}\|_{S^p} + \|\hat{Y}^{\theta,(n)} - \hat{Y}^{(n)}\|_{S^p} + \|\hat{Z}^{\theta,(n)} - \hat{Z}^{(n)}\|_{M_p} + \|\hat{Z}^{0,\theta,(n)} - \hat{Z}^{0,(n)}\|_{M_p} = 0,$$

(4.9)

and

$$\varphi^{\theta,(n)} = \frac{\phi^{\theta} - \phi^0 - \sum_{i=1}^{n-1} \theta^i \hat{\varphi}^{(i)}}{\theta}, \quad \varphi = X, Y, Z, Z^0.$$

(4.10)

It remains to show that the above results also hold for $n + 1$. Define

$$\varphi^{\theta,(n+1)} = \frac{\phi^{\theta,(n+1)} - \phi^{(n)}}{\theta}, \quad \varphi = X, Y, Z, Z^0,$$

which implies by (4.10)

$$\varphi^{\theta,(n+1)} = \frac{\varphi^{\theta} - \varphi^0 - \sum_{i=1}^{n+1} \theta^i \hat{\varphi}^{(i)}}{\theta}, \quad \varphi = X, Y, Z, Z^0.$$

Moreover, it can be verified directly that $(\hat{X}^{\theta,(n+1)}, \hat{Y}^{\theta,(n+1)}, \hat{Z}^{\theta,(n+1)}, \hat{Z}^{0,\theta,(n+1)})$ satisfies (4.6) with $n$ replaced by $n + 1$. By the argument in the proof of Lemma 4.1 (4.7) is wellposed with $n$ replaced by $n + 1$. Denote by $(\hat{X}^{(n+1)}, \hat{Y}^{(n+1)}, \hat{Z}^{(n+1)}, \hat{Z}^{0,(n+1)})$ the unique solution of (4.7). By (4.11) and the same argument in the proof of Lemma 4.1 (ii), we have for each $p > 1$

$$\lim_{\theta \to 0} \|\hat{X}^{\theta,(n+1)} - \hat{X}^{(n+1)}\|_{S^p} + \|\hat{Y}^{\theta,(n+1)} - \hat{Y}^{(n+1)}\|_{S^p} + \|\hat{Z}^{\theta,(n+1)} - \hat{Z}^{(n+1)}\|_{M_p} + \|\hat{Z}^{0,\theta,(n+1)} - \hat{Z}^{0,(n+1)}\|_{M_p} = 0.$$

Thus, there exists $\Delta_{\theta}$ with $\lim_{\theta \to 0} \Delta_{\theta} = 0$ such that

$$\varphi^{\theta,(n+1)} = \varphi^{(n+1)} + \Delta_{\theta}, \quad \varphi = X, Y, Z, Z^0,$$

23
which implies that by the definition of $\tilde{\varphi}^{(n)}$

$$\varphi^{(n)} = \varphi^{(n)} + \theta \tilde{\varphi}^{(n+1)} + \theta \Delta\theta.$$ 

By the induction assumption (4.10), it holds

$$\varphi^\theta = \varphi^0 + \sum_{i=1}^{n-1} \theta^i \varphi^i + \theta^n \varphi^{(n)} + \theta^{n+1} \tilde{\varphi}^{(n+1)} + \theta^{n+1} \Delta\theta,$$

which implies (4.8) with $n + 1$.

5 Comments on $N$-Player Games in Incomplete Markets

In this section, we comment on $N$-player games in the cases of exponential utility functions and power utility functions, as introduced in the Introduction. By the same argument as in Lemma 2.1 and Lemma 2.2 the NE of the $N$-player game is equivalent to a multidimensional FBSDE, whose wellposedness can be obtained by the same argument in Section 3 and poses no essential difference other than notational complexity. So we will omit the detailed proof for the solvability of the FBSDE with general market parameters, but instead, we discuss the case when all market parameters are deterministic functions. In the special case when all coefficients are constants, we revisit the $N$-player models in [21] and conclude that the constant equilibrium in [21] is unique in $L^\infty$.

The $N$-Player Game with Exponential Utility Functions (1.2)-(1.3). By the same argument as in Lemma 2.1 the NE $(\pi^1, \cdots, \pi^N, *)$ of the $N$-player game (1.2)-(1.3) is equivalent to the following multidimensional FBSDE

$$\left\{
\begin{array}{l}
\frac{dX_i^t}{dt} = \frac{\alpha^i \sigma_i^1 Z_i^t + \sigma_i^0 Z_i^0 + h_i^t}{\alpha^i \sigma^2 + (\sigma_i^0)^2} (h_i^t dt + \sigma_i^i dW^i_t + \sigma_i^0 dW^0_t) \\
\frac{dY_i^t}{dt} = \frac{(\alpha^i \sigma_i^1 Z_i^t + \sigma_i^0 Z_i^0 + h_i^t)^2}{2 \alpha^i \sigma^2 + (\sigma_i^0)^2} - \frac{1}{2} \alpha^i (Z_i^t)^2 - \frac{1}{2} \alpha^i \sigma^2 (Z_i^t)^2 - \frac{\alpha^i}{2} \sum_{j \neq i} (Z_j^t)^2 dt,
\end{array}\right. \quad (5.1)$$

and for each $i = 1, \cdots, N$

$$\pi_i^* = \frac{\alpha^i \sigma_i^1 Z_i^t + \sigma_i^0 Z_i^0 + h_i^t}{\alpha^i \sigma^2 + (\sigma_i^0)^2}$$

(5.2)

The $N$-Player Game with Power Utility Functions (1.4)-(1.5). By the same argument as in Lemma 2.2 the NE $(\pi^1, *, \cdots, \pi^N, *)$ of the $N$-player game with power utility functions (1.4)-(1.5) is equivalent to the following multidimensional FBSDE

$$\left\{
\begin{array}{l}
\frac{d\tilde{X}_i^t}{dt} = \frac{h_i^t + \sigma_i^1 Z_i^t + \sigma_i^0 Z_i^0}{(1 - \gamma^i)(\sigma_i^2 + (\sigma_i^0)^2)} \left\{ h_i^t dt + \sigma_i^i dW^i_t + \sigma_i^0 dW^0_t \right\}, \\
-dY_i^t = \frac{(Z_i^t)^2 + (Z_i^0)^2}{2} + \sum_{j \neq i} (Z_j^t)^2 + \gamma^i h_i^t \sigma_i^1 Z_i^t + \sigma_i^0 Z_i^0 \sigma_i^2 (1 - \gamma^i) \left( \frac{(\sigma_i^0)^2}{\sigma_i^2 + (\sigma_i^0)^2} \right) \right\} dt, \\
\frac{d\tilde{X}_0^t}{dt} = \log x^*_{p0}, \quad Y_i^t = \frac{\theta^i \gamma^i}{N - 1} \sum_{j \neq i} \tilde{X}_j^t,
\end{array}\right. \quad (5.3)$$
with $\tilde{X}^i = \log X^i$ denoting the logarithm of the wealth process, and

$$\pi^{i,*}_t = \frac{h^i_t + \alpha^i_t Z^{i0} + \sigma^{i0}_t Z^{i0}}{(1 - \gamma^i)((\sigma^i)_t)^2}.$$  \hfill (5.4)

The proof of the wellposedness for (5.1) and (5.3) is the same as that for (3.1); we compare it with the system when $\theta^i = 0$ and transform the resulting FBSDE to a BSDE by the approach in Lemma 3.2, followed by a priori estimate and fixed point argument as in Lemma 3.3 and Theorem 3.4. Instead of the detailed proof, in the next subsection we obtain the explicit expression of the NE when the market parameters are deterministic, and prove the constant equilibrium obtained in [21] is unique in $L^\infty$.

### 5.1 The Constant Equilibrium of the $N$-Player Game Is Unique in $L^\infty$

**Theorem 5.1.** When all the coefficients $h^i$, $\alpha^i$ and $\sigma^{i0}$, $i = 1, \cdots, N$, are deterministic functions of the time, the NE is $(\pi^{1,*}, \cdots, \pi^{N,*})$, where the unique bounded optimal strategy for player $i$ in the case of exponential utility functions and power utility functions is

$$\pi^{i,*}_t = \frac{\sigma^{i0}_t (\sigma^i_t)^2 + (\sigma^{i0}_t)^2}{(\sigma^i_t)^2 + (\sigma^{i0}_t)^2} - \sum_{j=1}^N N - 1 + \frac{\theta^j (\sigma^j_t)^2}{(\sigma^j_t)^2 + (\sigma^{i0}_t)^2} \sum_{j=1}^N N - 1 + \frac{\theta^j (\sigma^{i0}_j)^2}{(\sigma^{i0}_j)^2 + (\sigma^{i0}_t)^2}$$

$$t \in [0, T], \quad t \in [0, T], \quad i = 1, \cdots, N,$$

respectively,

$$\pi^{i,*}_t = \frac{h^i_t (\sigma^i_t)^2 + (1 - \gamma^i)(\sigma^{i0}_t)^2}{(1 - \gamma^i)(\sigma^i_t)^2 + (1 - \gamma^i)(\sigma^{i0}_t)^2} - \sum_{j=1}^N N - 1 + \frac{\theta^j (\sigma^i_t)^2}{(1 - \gamma^i)(\sigma^{i0}_t)^2} \sum_{j=1}^N N - 1 + \frac{\theta^j (\sigma^{i0}_j)^2}{(1 - \gamma^i)(\sigma^{i0}_t)^2}$$

$$t \in [0, T], \quad i = 1, \cdots, N,$$

where

$$\varphi^{(1),N} = \frac{1}{N - 1} \sum_{j=1}^N (1 - \gamma^j)(\sigma^j)^2 + \frac{h^j (\sigma^{i0}_t)^2}{(1 - \gamma^j)(\sigma^{i0}_t)^2}$$

and

$$\varphi^{(2),N} = \frac{1}{N - 1} \sum_{j=1}^N (1 - \gamma^j)(\sigma^j)^2 + \frac{\theta^j (\sigma^{i0}_j)^2}{(1 - \gamma^j)(\sigma^{i0}_t)^2}.$$

Moreover, if $\{(x^i, \theta^i, \alpha^i, \gamma^i, h^i, \sigma^i, \sigma^{i0})\}_{i=1}^N$ are realizations from independent copies of $(x, \theta, \alpha, \gamma, h, \sigma, \sigma^{i0})$ in Theorem 3.10, then (5.5) converges to (3.26) and (5.6) converges to (3.24).

**Proof.** One can show (5.1) admits at most one solution in $(S^2 \times S^2 \times L^\infty \times L^\infty \times L^\infty)^N$. The unique bounded optimal strategy is constructed as follows. From the forward dynamics of (5.1), it holds that

$$X^{-1} = x^{-1} + \frac{1}{N - 1} \sum_{j \neq i}^{N} \int_0^T \frac{\alpha^j Z^{ij} + \alpha^i Z^{ij} + h^j}{\alpha^j ((\sigma^j)^2 + (\sigma^{i0})^2)} ds.”
\[
\sum_{j \neq i} \int_0^T \frac{\alpha^i \sigma^i_j Z^i_j + \alpha^j \sigma^0_j Z^0_j + \sigma^i_j h^i_j}{\alpha^j ((\sigma^j)^2 + (\sigma^0)^2)^2} \sigma^i_j dW^j
\]
\[
+ \frac{1}{N-1} \sum_{j \neq i} \int_0^T \frac{\alpha^i \sigma^i_j Z^i_j + \alpha^j \sigma^0_j Z^0_j + h^i_j}{\alpha^j ((\sigma^j)^2 + (\sigma^0)^2)^2} \sigma^0_j dW^0_j,
\]

where \( \overline{X}^{-i} = \frac{1}{N-1} \sum_{j \neq i} X^j \) and \( \overline{Z}^{-i} = \frac{1}{N-1} \sum_{j \neq i} Z^j \). Taking it into the backward dynamics of \( \overline{Z}^{-i} \), one has

\[
Y^i_t = \theta^t \overline{X}^{-i} + \frac{\theta^t}{N-1} \sum_{j \neq i} \int_0^T \frac{\alpha^i \sigma^i_j Z^i_j + \alpha^j \sigma^0_j Z^0_j + h^i_j}{\alpha^j ((\sigma^j)^2 + (\sigma^0)^2)^2} \sigma^i_j dW^j
\]

\[
- \int_t^T \left( \frac{\alpha^i \sigma^i_j Z^i_j + \alpha^j \sigma^0_j Z^0_j + h^i_j}{2\alpha^j ((\sigma^j)^2 + (\sigma^0)^2)} \right)^2 - \frac{1}{2} \alpha^j (Z^j_t)^2 - \frac{1}{2} \alpha^i (Z^i_t)^2 - \frac{\alpha^j}{2} \sum_{j \neq i} (Z^j_t)^2 \right) ds
\]

\[
+ \frac{\theta^t}{N-1} \sum_{j \neq i} \int_0^T \frac{\alpha^i \sigma^i_j Z^i_j + \alpha^j \sigma^0_j Z^0_j + h^i_j}{\alpha^j ((\sigma^j)^2 + (\sigma^0)^2)^2} \sigma^0_j dW^0_j
\]

\[
+ \frac{\theta^t}{N-1} \sum_{j \neq i} \int_t^T \sum_{j \neq i} \left\{ \frac{\theta^t}{N-1} \frac{\alpha^i \sigma^i_j Z^i_j + \alpha^j \sigma^0_j Z^0_j + h^i_j}{\alpha^j ((\sigma^j)^2 + (\sigma^0)^2)^2} \sigma^i_j - Z^i_j \right\} dW^j
\]

\[
+ \int_t^T \left\{ \frac{\theta^t}{N-1} \sum_{j \neq i} \frac{\alpha^i \sigma^i_j Z^i_j + \alpha^j \sigma^0_j Z^0_j + h^i_j}{\alpha^j ((\sigma^j)^2 + (\sigma^0)^2)^2} \sigma^0_j - Z^0_j \right\} dW^0_j.
\]

To construct an adapted \( Y^i \), let

\[
\begin{cases}
0 = Z^i \\
0 = \frac{\theta^t}{N-1} \frac{\alpha^i \sigma^i_j Z^i_j + \alpha^j \sigma^0_j Z^0_j + h^i_j}{\alpha^j ((\sigma^j)^2 + (\sigma^0)^2)} - Z^i \\
0 = \frac{\theta^t}{N-1} \sum_{j \neq i} \frac{\alpha^i \sigma^i_j Z^i_j + \alpha^j \sigma^0_j Z^0_j + h^i_j}{\alpha^j ((\sigma^j)^2 + (\sigma^0)^2)} - Z^0_j
\end{cases}
\]

(5.7)

The first and the third equalities in (5.7) yield that

\[
\left( 1 + \frac{\theta^t}{N-1} \frac{(\sigma^0)^2}{(\sigma^j)^2 + (\sigma^0)^2} \right) Z^0 = \frac{\theta^t}{N-1} \sum_{j=1}^N \frac{(\sigma^0)^2 Z^0_j}{(\sigma^j)^2 + (\sigma^0)^2} + \sum_{j \neq i} \frac{h^j \sigma^j}{\alpha^j ((\sigma^j)^2 + (\sigma^0)^2)}.
\]

(5.8)

which further implies by multiplying \( \frac{(\sigma^0)^2}{1 + \frac{(\sigma^0)^2}{(\sigma^j)^2 + (\sigma^0)^2}} \) and taking sum from 1 to \( N \) on both sides

\[
\sum_{i=1}^N \frac{(\sigma^0)^2 Z^0_i}{(\sigma^j)^2 + (\sigma^0)^2} = \sum_{i=1}^N \frac{\theta^t}{N-1} \frac{(\sigma^0)^2}{(\sigma^j)^2 + (\sigma^0)^2} \sum_{j=1}^N \frac{(\sigma^0)^2 Z^0_j}{(\sigma^j)^2 + (\sigma^0)^2} + \sum_{j \neq i} \frac{h^j \sigma^j}{\alpha^j ((\sigma^j)^2 + (\sigma^0)^2)}.
\]

26
From the above linear equation for \(\sum_{i=1}^{N} \frac{(\sigma^0)^2 Z^i}{(\sigma^i)^2 + (\sigma^0)^2}\), we get

\[
\sum_{i=1}^{N} \frac{(\sigma^0)^2 Z^i}{(\sigma^i)^2 + (\sigma^0)^2} = \frac{1}{1 - \sum_{i=1}^{N} \frac{\theta^i}{(\sigma^i)^2 + (\sigma^0)^2}} \sum_{i=1}^{N} \frac{\theta^i}{N-1 (\sigma^i)^2 + (\sigma^0)^2} \sum_{j \neq i} \alpha^j (\sigma^j)^2 + (\sigma^0)^2).
\]

Taking the equality back into (5.8), we obtain

\[
Z^0 = \frac{\theta^i}{N-1 + \frac{\theta^i}{(\sigma^i)^2 + (\sigma^0)^2}} \frac{1}{1 - \sum_{i=1}^{N} \frac{\theta^i}{(\sigma^i)^2 + (\sigma^0)^2}} \sum_{j \neq i} \alpha^j (\sigma^j)^2 + (\sigma^0)^2). \tag{5.9}
\]

Thus, the optimal strategy (5.5) can be obtained. The convergence from (5.5) to (5.26) is obtained by law of large numbers.

Similarly, we get obtain (5.6) and its convergence to (5.27).

**Remark 5.2.** The N-player game with exponential utility functions, deterministic market parameters and trading constraint but without individual noise was studied in [8, 9]. The NE obtained in [8, 9] was called deterministic Nash equilibrium.

**Remark 5.3** (Comments on the closed loop NE). The closed loop NE may not exist. However, once it exists, the closed loop NE must be identical to the open loop NE. Indeed, if we assume all competing players except player \(i\) use a closed loop strategy \(\pi^i(t, \theta, \alpha, X)\) with \(X = (X^1, \ldots, X^N)\), then the same argument leading to (5.1) or (5.3) implies \(\pi^{i*}\) has the same expression as (5.2) or (5.4). If the market parameters \((h^i, \sigma^i, \sigma^0)\) are progressively measurable with respect to the filtration generated by the Brownian motions, \(\{Z^i\}_{i=0,1,\ldots,N}\) are not necessarily deterministic functions of \(X\) since the nonsingularity of \((\pi^{i*})^2 (\sigma^i)^2 + (\sigma^0)^2\) cannot be guaranteed. Consequently, the closed loop equilibrium may not exist. Moreover, under the assumptions in Theorem 4.1, the open loop NE (5.3) and (5.4) is also closed loop.

This comment also applies to MFGs (1.6) and (1.7).

As a corollary of Theorem 5.1 we revisit the N-player games in [21] and conclude the constant equilibrium is unique in \(L^\infty\).

**Corollary 5.4** (Lacker and Zariphopoulou’s N-Player Games Revisited). Assume all the coefficients \(h^i, \sigma^i\) and \(\sigma^0\) are constants, the NE obtained in (5.5) and (5.6) are unique in \(L^\infty\). Indeed, (5.5) and (5.6) are consistent with [21] Theorem 3] and [21] Theorem 14] by taking [21] Remark 5] and [21] Remark 16] into account.

6 Conclusion

In this paper we study mean field portfolio games with random market parameters. By MOP we characterize the NE by mean field FBSDE. The FBSDE is solved under a weak interaction assumption, motivated by which, we establish an asymptotic expansion result in powers of the competition parameter \(\theta\). In particular, we expand the log-value function in powers of \(\theta\) into any order. When the market parameters do not depend on the Brownian paths, we get the NE in closed form. Moreover, when the market parameters become time-independent, we revisit the model in [21]. We fill in the gap in the literature by concluding the constant NE obtained in [21] is unique in \(L^\infty\).

A The Benchmark BSDE
Lemma A.1. The BSDE in (3.2) has a unique solution \((Y^\circ, Z^\circ, Z^{0,\circ}) \in S^\infty \times H^2_{BMO} \times H^2_{BMO}\). As a result, there exists a unique \(X^\circ \in \bigcap_{p>1} S^p\) satisfying the forward dynamics of (3.2).

Proof. Theorem 7 in [17] implies there exists a unique \((Y, Z, Z^0) \in S^\infty \times H^2_{BMO} \times H^2_{BMO}\) satisfying the BSDE in (3.2). By the energy inequality ([19, P.26]) it holds \((Z^\circ, Z^{0,\circ}) \in \bigcap_{p>1} M_p \times \bigcap_{p>1} M_p\), which implies \(X^\circ \in \bigcap_{p>1} S^p\).

B Reverse Hölder’s Inequality

We summarize the reverse Hölder’s inequality for a general stochastic process and for the stochastic exponential of a stochastic process in the BMO space ([19, Theorem 3.1]), which is used in the main text.

For some \(1 < p < p_0\) and any stopping time \(\tau\), it holds that \(E(\Theta)\) satisfies \(R_p\). In particular,

\[
E \left[ \left| \frac{D_T}{D_\tau} \right|^p \bigg| \mathcal{F}_\tau \right] \leq C(p, \|\Theta\|_{BMO}),
\]

where

\[
K(p, N) = \frac{2}{1 - 2(N^2 + 2N)}.\]

References

[1] P. Briand and F. Confortola. BSDEs with stochastic lipschitz condition and quadratic PDEs in Hilbert spaces. Stochastic Processes and their Applications, 118(5):818–838, 2008.

[2] P. Briand and Y. Hu. Quadratic BSDEs with convex generators and unbounded terminal conditions. Probability Theory and Related Fields, 141(3-4):543–567, 2008.

[3] R. Carmona and F. Delarue. Probabilistic analysis of mean-field games. SIAM Journal on Control and Optimization, 51(4):2705–2734, 2013.

[4] P. Chan and R. Sircar. Bertrand and cournot mean field games. Applied Mathematics & Optimization, 71(3):533–569, 2015.
[5] P. Chan and R. Sircar. Fracking, renewables, and mean field games. *SIAM Review*, 59(3):588–615, 2017.

[6] G. dos Reis and V. Platonov. Forward utilities and mean-field games under relative performance concerns. *to appear in Particle Systems and Partial Differential Equations*, 2021.

[7] G. dos Reis and V. Platonov. Forward utility and market adjustments in relative investment-consumption games of many players. *arXiv:2012.01235*, 2021.

[8] G. Espinosa and N. Touzi. Optimal investment under relative performance concerns. *Mathematical Finance*, 25(2):221–257, 2015.

[9] C. Frei and G. dos Reis. A financial market with interacting investors: does an equilibrium exist? *Mathematics and Financial Economics*, 4:161–182, 2011.

[10] G. Fu, P. Graewe, U. Horst, and A. Popier. A mean field game of optimal portfolio liquidation. *to appear in Mathematics of Operations Research*, 2021.

[11] G. Fu and U. Horst. Mean field leader follower games with terminal state constraint. *SIAM Journal on Control and Optimization*, 58(4):2078–2113, 2020.

[12] G. Fu, X. Su, and C. Zhou. Mean field exponential utility games: a probabilistic approach. *arXiv:2006.07684*, 2020.

[13] M. Herdegen, J. Muhle-Karbe, and D. Possamaï. Equilibrium asset pricing with transaction costs. *to appear in Finance and Stochastics*, 2020.

[14] H. Hibon, Y. Hu, and S. Tang. Mean-field type quadratic BSDEs. *arXiv:1708.08784*, 2017.

[15] U. Horst. Stationary equilibria in discounted stochastic games with weakly interacting players. *Games and Economic Behavior*, 52(1):83–108, 2005.

[16] R. Hu and T. Zariphopoulou. N-player and mean-field games in Itô-diffusion markets with competitive or homophilous interaction. *arXiv:2106.00581*, 2021.

[17] Y. Hu, P. Imkeller, and M. Müller. Utility maximization in incomplete markets. *The Annals of Applied Probability*, 15(3):1691–1712, 2005.

[18] M. Huang, R. Malhamé, and P. Caines. Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Communications in Information and Systems*, 6(3):221–252, 2006.

[19] N. Kazamaki. *Continuous Exponential Martingales and BMO*. Springer, 2006.

[20] D. Lacker and A. Soret. Many-player games of optimal consumption and investment under relative performance criteria. *Mathematics and Financial Economics*, 14(2):263–281, 2020.

[21] D. Lacker and T. Zariphopoulou. Mean field and n-agent games for optimal investment under relative performance criteria. *Mathematical Finance*, 29(4):1003–1038, 2019.

[22] J.-M. Lasry and P.-L. Lions. Mean field games. *Japanese Journal of Mathematics*, 2(1):229–260, 2007.

[23] A. Matoussi, D. Possamaï, and C. Zhou. Robust utility maximization in non-dominated models with 2BSDEs. *Mathematical Finance*, 25(2):258–287, 2015.

[24] R. Rouge and N. El Karoui. Pricing via utility maximization and entropy. *Mathematical Finance*, 10(2):259–276, 2000.

[25] R. Tevzadze. Solvability of backward stochastic differential equations with quadratic growth. *Stochastic processes and their Applications*, 118(3):503–515, 2008.