A NOTE ON MIXED GRAPHS AND MATROIDS

J. ORESTES CERDEIRA AND RAUL CORDOVIL

Abstract. A mixed graph is a graph with some directed edges and some undirected edges. We introduce the notion of mixed matroids as a generalization of mixed graphs. A mixed matroid can be viewed as an oriented matroid in which the signs over a fixed subset of the ground set have been forgotten. We extend to mixed matroids standard definitions from oriented matroids, establish basic properties, and study questions regarding the reorientations of the unsigned elements. In particular we address in the context of mixed matroids the $P$-connectivity and $P$-orientability issues which have been recently introduced for mixed graphs.

1. Introduction

A mixed graph is a graph with a set of directed edges and a set of undirected edges $A$. Recently, Arkin and Hassin [1] consider the following question regarding the orientation of the edges of $A$. Given a set of ordered pair of vertices $P = (s_j, t_j), j = 1, ..., m$, is there an orientation of the edges in $A$ such that in the resulting digraph there is a (directed) $s_j \rightarrow t_j$ path, for every $j = 1, ..., m$? When such a $P$-orientation exists the mixed graph is called $P$-orientable.

Arkin and Hassin prove that recognizing whether a mixed graph has a $P$-orientation is NP-complete, and give a characterization of $P$-orientable graphs, for $|P| = 2$, in terms of what they call $P$-essential edges. The mixed graph is $P$-connected if, for each pair $(s_j, t_j) \in P$, there is an orientation of the edges of $A$ such that in the resulting digraph there is a (directed) $s_j \rightarrow t_j$ path. An edge $e \in A$ is $P$-essential if the graph obtained from any of its two orientations is not $P$-connected. They prove the following.

Theorem 1.1 (Arkin and Hassin [1]). A $P$-connected mixed graph, with $|P| = 2$, has a $P$-orientation if and only if it has no $P$-essential edges.

In this note we introduce the notion of mixed matroid as a generalization of a mixed graph. The concept of mixed matroid, basic definitions and properties are given in section 2. In section 3 we address the $P$-connectivity and $P$-orientability questions for mixed matroids, and extend Theorem 1.1. We end in section 4 with a remark concerning the possibility to obtain strong connected digraphs from orientations of the undirected edges of mixed graphs.

As an overall conclusion we trust that mixed matroids, and in particular duality, give a consistent framework for the study of issues such as connectivity in mixed graphs.
2. Mixed matroids

We refer readers to [3] as a standard source for graphs and matroids, and [2] for oriented matroids.

A signed set (or vector) $X$, $X \in \ell^n$, is a (signed) mapping

$$\text{sg}_X : [n] \to \{\pm 1, 0\}.$$ 

Define $X^+ := \{\ell \in [n] : \text{sg}_X(\ell) = +1\}$, $X^- := \{\ell \in [n] : \text{sg}_X(\ell) = -1\}$, and let $X := X^+ \cup X^- \subseteq \ell^n$ be the support of the signed set $X$. We use the notation $\ell \in X$ meaning that $\ell$ is in the support of $X$. If $X$ and $Y$ are two signed sets, in order to simplify notation, $X \cap Y$ will be denoted by $X \cap Y$. If $X$ is a signed set and $Y$ is a subset of $[n]$, define $X \cap Y := X \cap Y$, and let $\text{sg}_{X \setminus Y}$ be the map from $[n]$ to $\{\pm 1, 0\}$ where

$$\text{sg}_{X \setminus Y}(\ell) = \begin{cases} \text{sg}_X(\ell) & \text{if } \ell \in X \setminus Y, \\ 0 & \text{otherwise}. \end{cases}$$

Let $\mathcal{M}([n])$ [resp. $\mathcal{M}((n))$] denote a matroid [resp. oriented matroid] on the ground set $[n]$. To simplify we do not explicit refer the ground set $[n]$ simply writing $\mathcal{M}$ or $\mathcal{M}$. As usual $\mathcal{M}^*$ denotes the dual of $\mathcal{M}$. Finally, we denote by $\mathcal{C}(\mathcal{M})$ [resp. $\mathcal{C}(\mathcal{M})$] the set of circuits [resp. signed circuits] of $\mathcal{M}$ [resp. $\mathcal{M}$].

**Definition 2.1.** A mixed matroid is a pair $\mathcal{M} = (\mathcal{M}, A)$, where $\mathcal{M}$ is an oriented matroid, $A$ is a subset of the ground set $[n]$, and to which we associate the set

$$\mathcal{C}(\mathcal{M}) := \{\text{sg}_{C \setminus A} : \text{sg}_C \in \mathcal{C}(\mathcal{M})\},$$

that will be called the set of the circuits $\mathcal{M}$.

It is useful to interpret $\mathcal{M}$ as an oriented matroid whose signs of a fixed subset $A$ have been forgotten in every signed circuit. The set $A$ will be referred as the set of unsigned elements of the mixed matroid $\mathcal{M}$. Note that in the definition above we fixed an orientation $\mathcal{M}$ of the underlying (non-oriented) matroid because, as opposed to what happen with graphs, an orientable matroid may have different orientations not in the same class of reorientations.

Examples of mixed matroids are obtained in the extreme cases $A = \emptyset$ and $A = [n]$ where $\mathcal{M}$ is, respectively, an oriented matroid and an orientable matroid.

**Definition 2.2.** Any oriented matroid $\mathcal{M}'$ on the class of reorientations of $\mathcal{M}$ over $A$ is said to be coherent with the mixed matroid $\mathcal{M} = (\mathcal{M}, A)$.

Consider two mixed matroids $\mathcal{M} = (\mathcal{M}, A)$ and $\mathcal{M}' = (\mathcal{M}', A')$ with the same underlying matroid. If $A' \subseteq A$, and $\mathcal{M}'$ is coherent with $\mathcal{M}$, we say that $\mathcal{M}'$ is obtained from $\mathcal{M}$ by a permissible signature of the elements of $A \setminus A'$, and write $\mathcal{M}' := \mathcal{M}(A \setminus A')$.

The operations deletion and contraction on oriented matroids naturally extend to a mixed oriented matroid $\mathcal{M} = (\mathcal{M}, A)$. The deletion $\mathcal{M} \setminus X$ is the mixed matroid $(\mathcal{M} \setminus X, A \setminus X)$ where

$$\mathcal{C}(\mathcal{M} \setminus X) := \{\text{sg}_{C \setminus A} : \text{sg}_C \in \mathcal{C}(\mathcal{M} \setminus X)\}.$$ 

The contraction $\mathcal{M}/X$ is the mixed matroid $(\mathcal{M}/X, A \setminus X)$ where

$$\mathcal{C}(\mathcal{M}/X) := \{\text{sg}_{C \setminus A} : \text{sg}_C \in \mathcal{C}(\mathcal{M}/X)\}.$$ 

The mixed matroids support a notion of duality similar to that of oriented matroids. The dual or orthogonal of $\mathcal{M}$ is the mixed matroid $\mathcal{M}^* = (\mathcal{M}^*, A)$ whose set of circuits is

$$\mathcal{C}^*(\mathcal{M}) := \{\text{sg}_{C, \setminus A} : \text{sg}_C \in \mathcal{C}(\mathcal{M}^*)\}.$$
Example 2.3. Consider a mixed graph $G$ with a set $A$ of undirected edges. Fix labels $e_1, e_2, \ldots, e_n$ on the edges of $G$, and identify the subsets of edges with the subsets of indices of the corresponding labels. The graphic (non oriented) matroid $\mathcal{M}(G)$ is a matroid on the ground set $[n]$ and such that $C$ is a circuit of $\mathcal{M}(G)$ if and only if it is a circuit of the graph $G$. The partial signatures of the edges of $G$ determine a mixed graphic matroid. Indeed, every partially signed circuit $C$ of the graph $G$ fixes a pair of opposite maps $\pm \sg_C \setminus A \[\pm \sg_C \setminus A\]$. Thus, the mixed graph $G$ determines a pair of dual mixed oriented matroids $\mathcal{M}(G)$ and $\mathcal{M}^*(G)$. It is clear that every possible orientation of the edges of $A$ determines a digraph $\tilde{G}$, and the oriented matroid $\mathcal{M}(\tilde{G})$ and its dual $\mathcal{M}^*(\tilde{G})$ are coherent with the mixed matroids $\mathcal{M}(G)$ and $\mathcal{M}^*(G)$, respectively.

The following are dual propositions.

Proposition 2.4. Let $\mathcal{M} = (\mathcal{M}, A)$ be a coloop free mixed matroid. The two conditions are equivalent:
\begin{itemize}
  \item $\mathcal{M}/A$ is a totally cyclic oriented matroid;
  \item There is a totally cyclic reorientation of $\mathcal{M}$ coherent with $\mathcal{M}$.
\end{itemize}

Proposition 2.5. Let $\mathcal{M} = (\mathcal{M}, A)$ be a loop free mixed matroid. The two conditions are equivalent:
\begin{itemize}
  \item $\mathcal{M} \setminus A$ is an acyclic oriented matroid.
  \item There is an acyclic reorientation of $\mathcal{M}$ coherent with $\mathcal{M}$.
\end{itemize}

Proof of Proposition 2.4. If a reorientation of $\mathcal{M}$ coherent with $\mathcal{M}$ is totally cyclic, then it is a consequence of the definitions that $\mathcal{M}/A$ is also a totally cyclic oriented matroid.

Suppose that $\mathcal{M}/A$ is a totally cyclic oriented matroid, and $\mathcal{M}$ is not. Given that the contraction of any element of a coloop free matroid does not create coloops, we can consider without loss of generality that $A = \{e\}$. Since $\mathcal{M}^*$ is not acyclic and $\mathcal{M}$ is coloop free, there is a positive cocircuit $D$ with at least two elements that includes the element $e$. By hypothesis $\mathcal{M}^* \setminus e$ is an acyclic oriented matroid, and the elimination axiom for cocircuits ensures that $\mathcal{M}^*$ has no signed cocircuit $D_1$ with $D_1^+ = \{e\}$. Indeed, the existence of $D$ and $D_1$ together would imply the existence of a positive cocircuit not containing $e$, which is a contradiction. It then follows that $-(e)\mathcal{M}^*$ is acyclic, and by duality $\tilde{\mathcal{M}} := -(e)\mathcal{M}$ is a totally cyclic oriented matroid coherent with $\mathcal{M}$.

3. P-orientation

This section extends Theorem 1.1 to matroids.

Let $\mathcal{M} = (\mathcal{M}, A)$ be a mixed matroid. The circuit $C \in \mathcal{C}(\mathcal{M})$ is said to be positive if $\sg_{C \setminus A}(C \setminus A) = \{+1\}$.

The following two definitions are generalizations (with a slight modification) of the definitions introduced by Arkin and Hassin \[1\], and which we presented in section 1.

Let $P$ be a fixed set of signed elements of the mixed matroid $\mathcal{M}$.

Definition 3.1. The mixed matroid $\mathcal{M}$ is P-connected if the two following conditions are verified:
\begin{itemize}
  \item[(P.1)] For every element $p \in P$, there is a positive circuit $C$ of $\mathcal{M}$ such that $C \cap P = \{p\}$;
  \item[(P.2)] $\mathcal{M}$ is totally cyclic.
\end{itemize}
The condition \( \text{3.4}(2) \) is technical and not essential to the definition. Without it the main results would still hold, however, the proofs would be more involved.

Note that if \( A\) is a basis of \( \mathcal{M} \) and \( P = [n] \setminus A \), then \( \mathcal{M} = (\mathcal{M}, A) \) is clearly \( P \)-connected.

**Definition 3.2.** The mixed matroid \( \mathcal{M} \) has a \( P \)-orientation if there exists a \( P \)-connected oriented matroid coherent with \( \mathcal{M} \).

**Lemma 3.3.** Let \( \mathcal{M} = (\mathcal{M}, A) \) be a \( P \)-connected mixed matroid. Then \( P \) is an independent set of the matroid \( \mathcal{M}^* \). Moreover, for every \( a \in A \), \( \mathcal{M}/a \) is also a \( P \)-connected matroid. If in addition \( \mathcal{M} \) is \( P \)-orientable, then \( \mathcal{M}/a \) is also \( P \)-orientable.

**Proof.** Note that if \( \mathcal{M} \) is \( P \)-connected the condition \( \text{3.4}(1) \) implies, by the orthogonality property of circuits and cocircuits, that \( P \) is an independent set of the matroid \( \mathcal{M}^* \). The other results are direct consequences of the definitions. \( \square \)

We can give to Definition \( \text{3.2} \) a more geometrical interpretation. Recall that a set (hyperplane) \( F \subset [n] \) is a facet of the oriented matroid \( \mathcal{M} \) if there is a positive cocircuit \( C \in \mathcal{C}(\mathcal{M}^*) \) such that \( F = [n] \setminus C \).

**Proposition 3.4.** Let \( \mathcal{M} \) be a totally cyclic mixed \( P \)-connected matroid, and suppose that \( P \) has at least two elements. The two conditions are equivalent:

\[ \begin{align*}
\text{3.4}(1) & \quad \text{\( \mathcal{M} \) has a \( P \)-orientation;} \\
\text{3.4}(2) & \quad \text{There is a totally cyclic oriented matroid \( \mathcal{M} \) coherent with \( \mathcal{M} \) such that, for every \( p \in P \), the set \( P \setminus p \) is on a facet of \( \mathcal{M}^* \).}
\end{align*} \]

**Proof.** This is the dual of Definition \( \text{3.2} \). \( \square \)

The definition below extends the definition in [1] of \( P \)-essential edges of a mixed graph.

**Definition 3.5.** Let \( \mathcal{M} \) be a \( P \)-connected mixed matroid, and \( e \) an unsigned element of \( \mathcal{M} \). The element \( e \) is \( P \)-essential if the two possible opposite permissible signatures \( \mathcal{M}(\vec{e}) \) and \( -_e \mathcal{M}(\vec{e}) \) are not \( P \)-connected.

The following technical proposition is fundamental in this section. See the mixed graph \( G \) of Figure 1 for an illustration of the result for mixed graphs.

**Proposition 3.6.** Let \( \mathcal{M} \) be a \( P \)-connected mixed matroid. Suppose that \( P = \{p_1, p_2\} \) and neither \( p_1 \) nor \( p_2 \) is a loop. The conditions \( \text{3.6}(1) \) and \( \text{3.6}(2) \) are equivalent:

\[ \begin{align*}
\text{3.6}(1) & \quad \text{The unsigned element } e \text{ is } P \text{-essential;} \\
\text{3.6}(2) & \quad \text{a) If } C \text{ is a positive circuit and } |C \cap P| = 1 \text{ then } e \in C. \\
& \quad b) \text{If } C_1 \text{ and } C_2 \text{ are two positive circuits with } p_i \in C_i, i = 1, 2, \text{ there is positive circuit } C_{12} \text{ containing } p_1 \text{ and } p_2 \text{ such that } C_{12} \subset C_1 \cup C_2 \setminus e.
\end{align*} \]

**Proof.** We first prove that \( \text{3.6}(1) \Rightarrow \text{3.6}(2) \). The condition \( \text{3.6}(2a) \) is a direct consequence of the cardinality of \( P \). To prove \( \text{3.6}(2b) \) consider two positive circuits \( C_1 \) and \( C_2 \) of \( \mathcal{M} \), such that \( C_i \cap P = \{p_i\}, i = 1, 2 \). Let \( \mathcal{M}(\vec{e}) \) be the mixed matroid obtained from \( \mathcal{M} \) by a permissible signature of the element \( e \), and let \( C_i(\vec{e}), i = 1, 2 \), be the circuits of \( \mathcal{M}(\vec{e}) \) corresponding to \( C_i \). As \( e \) is \( P \)-essential it necessarily has opposite signs in the circuits \( C_1(\vec{e}) \) and \( C_2(\vec{e}) \). The circuit elimination axiom ensures that there is a circuit \( C_{12} \) containing \( p_1 \) and contained in \( C_1 \cup C_2 \setminus e \). The signed circuit axiom ensures that \( C_{12} \) is a positive circuit of the mixed matroid \( \mathcal{M} \). Since \( p_1 \in C_{12} \) and \( e \notin C_{12} \), we conclude from \( \text{3.6}(2a) \) that necessarily \( p_2 \in C_{12} \) and that \( \text{3.6}(2b) \) is true.

We now prove that \( \text{3.6}(2) \Rightarrow \text{3.6}(1) \). Suppose that \( e \) is not a \( P \)-essential element of \( \mathcal{M} \), and let \( C_1(\vec{e}) \) and \( C_2(\vec{e}) \) be positive circuits of a permissible signature
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Figure 1. The mixed graph $G$.

$M(\vec{e})$, such that $C_i(\vec{e}) \cap P = \{p_i\}$ and $e \in C_i(\vec{e})$, $i = 1, 2$. The condition (3.6.2) ensures the existence of a positive circuit $C_{12}$ of $M(\vec{e})$ such that

$p_1, p_2 \in C_{12} \subset C_1(\vec{e}) \cup C_2(\vec{e}) \setminus e$.

Note that as $e \notin C_{12}$ this is also a positive circuit of $M$. By the elimination of $e$ in $C_1(\vec{e})$ and $C_2(\vec{e})$ it follows from the signed elimination axiom that $p_1$ and $p_2$ have opposite signs in the signed circuit $C_{12}$, which is a contradiction. Hence, $e$ is a $P$-essential element of $M$. □

We can now generalize Theorem 1.1. We highlight that the definition of $P$-essential elements given by the Proposition 3.6 simplifies the proof in [1].

Theorem 3.7. A $P$-connected mixed matroid $M = (\mathcal{M}, A)$, with $|P| = 2$, has a $P$-orientation if and only if it has no $P$-essential elements.

Proof. The necessary condition is clear. To prove that it is sufficient we use induction on the cardinal of $A$. For $|A| = 0$ there is nothing to prove. If $A = \{e\}$, the result follows directly from the hypothesis that $e$ is not $P$-essential. Suppose now that the result is true for all sets of unsigned elements with cardinality at most $m-1$, and let $2 \leq |A| = m$. Since contracting any unsigned element of a mixed oriented matroid with no $P$-essential elements does not create $P$-essential elements, the induction hypothesis implies that, for every $e \in A$, $M/e$ is $P$-orientable. Hence, there is a signature of the elements of $A \setminus e$ such that the mixed matroid $M' := M(A \setminus e)$ is $P$-connected. A contradiction results if $M'$ is not $P$-orientable, i.e., the element $e$ is $P$-essential in $M'$. In fact, let $P = \{p_1, p_2\}$. Proposition 3.6 ensures that, for every pair of positive circuits $C_1$ and $C_2$ of $M'$ with $p_i \in C_i$, $i = 1, 2$, necessarily $e \in C_1 \cap C_2$, and there is a positive circuit $C_{12}$ such that $p_1, p_2 \in C_{12} \subset C_1 \cup C_2 \setminus e$. We can now forget about the signs over $A$, and use Proposition 3.6 to conclude that element $e$ is also $P$-essential in $M$, which is a contradiction. Thus, $M$ has a $P$-orientation. □

4. Final Remark

We point out that the graph version of Proposition 2.4 is a well known result of mixed graphs.

Observe that coloop free matroids correspond to graphs where each component is 2-edge connected, and totally cyclic oriented matroids correspond to digraphs where each component is strongly connected. Thus, it follows from the Proposition 2.4 that a 2-edge connected mixed graph has an orientation of the undirected edges so that the resulting digraph is strongly connected if and only if the digraph obtained contracting the undirected edges is strongly connected. Since 2-edge connectivity
is clearly a necessary condition to turn a mixed graph into a strongly connected digraph, this is precisely the following well known result.

**Theorem 4.1** ([3], Theorem 61.4). Let \( G = (V, E) \) be a graph in which part of the edges is oriented (a mixed graph). Then the remainder of the edges can be oriented so as to obtain a strongly connected digraph if and only if \( G \) is 2-edge connected and there is no non empty proper subset \( U \) of \( V \) such that all edges in \( \delta(U) \) are oriented from \( U \) to \( V \setminus U \).

□

From the Proposition 2.5 we can establish the dual of Theorem 4.1.

**Corollary 4.2.** Let \( G = (V, E) \) be a mixed graph. Then the non oriented edges can be oriented so as to obtain an acyclic digraph if and only for every oriented edge \( e \) there is a non empty proper subset \( U \) of \( V \) such that \( e \in \delta(U) \) and all edges in \( \delta(U) \setminus e \) are non oriented or oriented from \( U \) to \( V \setminus U \) and \( e \in \delta(U) \).

□

**Acknowledgements**

We are grateful to Paulo Barcia and David Forge for assistance and discussion.

**References**

[1] Arkin, Esther M.; Hassin, Refael: A note on orientations of mixed graphs. *Discrete Appl. Math.* 116 (2002), 271-278.

[2] Björner, Anders; Las Vergnas, Michel; Sturmfels, Bernd; White, Neil; Ziegler, Günter: Oriented matroids. Second edition. *Encyclopedia of Mathematics and its Applications*, 46. Cambridge University Press, Cambridge, 1999.

[3] Schrijver, Alexander: Combinatorial optimization. Polyhedra and efficiency (3 volumes). *Algorithms and Combinatorics*, 24. Springer, Berlin, 2003.