Normal-Ordered Wave-Function Factorization of the 1D Hubbard Model for Finite Values of the On-site Repulsion $U$

J. M. P. Carmelo  
GCEP-Center of Physics, University of Minho, Campus Gualtar, P-4710-057 Braga, Portugal  
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In this paper we find that in the thermodynamic limit and for the the ground-state normal-ordered 1D Hubbard model the wave function of excited states contained in few-electron excitations factorizes for all values of the on-site Coulombian repulsion $U$. This factorization results from the non-interacting character of the pseudofermions whose occupancy configurations describe these excited states. Our study includes the introduction of the pseudoparticle - pseudofermion unitary transformation and of an operator algebra for both the pseudoparticles and the pseudofermions. The pseudofermion description takes into account the relationship between the rotated electrons and the holons, spinons, and c0 pseudoparticles. [Rotated electrons are related to the electrons by a canonical transformation.] As the corresponding pseudoparticles, the $c\nu$ pseudofermions (and $s\nu$ pseudofermions) are $s$-spin zero 2$s$-holon composite quantum objects (and spin zero 2$s$-spinon composite quantum objects) where $\nu = 1, 2, \ldots$. The pseudofermions are non-interacting and thus have no residual interactions, in contrast to the corresponding pseudoparticles, whose statistics we classify according to the generalized Pauli principle. The physics behind the invariance of the pseudofermions under the above transformations for specific values of the bare momentum is also studied and discussed.

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I. INTRODUCTION

Recently there has been a renewed experimental interest in the exotic one-electron and two-electron spectral properties of quasi-1D materials \[ 1 \leq \nu \leq \Omega \] . Some of these experimental studies observed unusual finite-energy/frequency spectral properties, which are far from being well understood. For low values of the energy, the microscopic electronic properties of these materials are usually described by systems of coupled chains. For finite values of the energy larger than the transfer integrals for electron hopping between the chains, the one-electron (1D) Hubbard model is expected to provide a good description of the physics of these materials \[ 2 \leq \nu \leq \Omega \] . This is confirmed by the recent quantitative studies of Refs. \[ 1, 10 \]. Similar unusual spectral properties observed in two-dimensional (2D) high-$T_c$ superconductors could result from effective quasi-1D charge and spin transport \[ 2 \leq \nu \leq \Omega \] . However, the non-perturbative nature of the 1D Hubbard model implies that the electronic creation and annihilation operators do provide a suitable operational description for the study of the finite-energy spectral properties. Thus, the first step for the study of these properties is the introduction of a suitable operational description. Except in the limit of infinite on-site Coulombian repulsion $U \rightarrow \infty$, the introduction of such a description is an open problem of great physical interest. For low values of energy useful information about the effects of the non-perturbative electronic correlations is provided by two-component conformal-field theory \[ 21, 22 \]. Unfortunately, that method does not apply for finite values of energy.

In view of the above-mentioned unusual finite-energy/frequency spectral properties observed in real experiments, which are far from being well understood, efforts towards the introduction of a suitable operational description to deal with the finite-energy problem are welcome. In this paper we introduce an operational representation for the 1D Hubbard model \[ 2 \leq \nu \leq \Omega \] in terms of non-interacting pseudofermions. We find that in the thermodynamic limit the wave function of excited states contained in the few-electron excitations factorizes into separated contributions corresponding to different pseudofermion branches. (A few-electron excitation is generated by application onto the ground state of operators whose expression involves the product of a few electronic creation and/or annihilation operators.) Such factorization occurs for all values of the on-site Coulombian repulsion $U$ of the ground-state normal-ordered 1D Hubbard model. The pseudofermion operational description is closely related to the pseudoparticle representation previously considered in the literature \[ 22, 23 \], and is the natural starting point for studies of the finite-energy/frequency few-electron spectral properties. As a result of the wave-function factorization, the pseudofermion description is more suitable for the study of the overlap between few-electron excitations and the energy eigenstates than the pseudoparticle representation.

Our starting point is a holon, spinon, and c0 pseudoparticle representation, which refers to the whole Hilbert space of the model \[ 31 \]. The relation between the original electrons and these elementary quantum objects involves the concept of rotated electron. The rotated electrons are related to the electrons by a unitary transformation first introduced in Refs. \[ 32, 33 \]. For such rotated electrons double occupation is a good quantum number for all values of $U$. Except
for the c0 pseudoparticles, all pseudoparticle branches introduced in Ref. 29 have a composite character in terms of holons or spinons 31. The pseudofermions are related to the pseudoparticles by a second unitary transformation. The charge or spin carried by the pseudoparticle and its holon or spinon contents remains invariant under such a transformation. It follows that except for the c0 pseudofermions, all remaining pseudofermion branches are composite objects of holons or spinons. The concepts of local pseudoparticle and effective lattice widely used in this paper are introduced in Ref. 35.

The pseudofermion operational description introduced in this paper and the associated factorization of the wave function of energy eigenstates contained in few-electron excitations are used elsewhere in the study of the finite-energy spectral properties 1, 10, 34. The theoretical predictions of Refs. 1, 10 seem to describe both qualitatively and quantitatively the one-electron removal spectral lines observed by photoemission experiments for finite values of the energy in real quasi-1D materials. We thus expect that our pseudofermion operational description is useful for the further understanding of the exotic properties displayed by low-dimensional materials.

The paper is organized as follows: In Sec. II we introduce the 1D Hubbard model and the rotated electrons. The elementary holon, spinon, and c0 pseudoparticle description and associated αν pseudoparticle representation are summarized in Sec. III. Moreover, in that section we introduce the operator algebra for the pseudoparticles and the statistics of these quantum objects according to the generalized Pauli principle 36. In Sec. IV we consider the ground-state normal-ordered pseudoparticle operator description and introduce useful ground-state quantities. The pseudofermion description and the relationship between pseudoparticle and pseudofermion operators are introduced and discussed in Sec. V. This includes the introduction of the pseudofermion anticommutator algebra. In Sec. VI we study the pseudofermion energy and momentum spectra and introduce and discuss the factorization of the few-electron Hilbert subspace of the ground-state normal-ordered 1D Hubbard model. The investigation of the laws under the pseudoparticle - pseudofermion transformation of several quantum objects and quantities is the subject of Sec. VII. Finally, in Sec. VIII we present the discussion and the concluding remarks.

II. THE 1D HUBBARD MODEL AND ROTATED ELECTRONS

In a chemical potential μ and magnetic field H the 1D Hubbard Hamiltonian can be written as,

\[ \hat{H} = \hat{H}_{SO(4)} + \sum_{\alpha = c, s} \mu_\alpha 2 \hat{S}_\alpha^z, \]  

where the Hamiltonian

\[ \hat{H}_{SO(4)} = \hat{H}_H - \frac{U}{2} \hat{N} + \frac{U}{4} N_a; \quad \hat{H}_H = \hat{T} + U \hat{D}, \]  

has SO(4) symmetry. Here \( \hat{H}_H \) is the “simple” Hubbard model,

\[ \hat{T} = -t \sum_{\sigma = \uparrow, \downarrow} \sum_{j=1}^{N_a} \left[ c_{j, \sigma}^\dagger c_{j+1, \sigma} + h.c. \right], \]  

is the kinetic-energy operator, and

\[ \hat{D} = \sum_{j=1}^{N_a} c_{j, \uparrow}^\dagger c_{j, \uparrow} + c_{j, \downarrow}^\dagger c_{j, \downarrow} = \sum_{j=1}^{N_a} \hat{n}_{j, \uparrow} + \hat{n}_{j, \downarrow}, \]  

is the electron double-occupation operator. On the right-hand side of Eq. 1 we have that \( \mu_c = \mu, \mu_s = \mu_0 H, \mu_0 \) is the Bohr magneton, and the number operators,

\[ \hat{S}_c^z = -\frac{1}{2} [N_a - \hat{N}]; \quad \hat{S}_s^z = -\frac{1}{2} [\hat{N}_\uparrow - \hat{N}_\downarrow], \]  

are the diagonal generators of the η-spin and spin SU(2) algebras 37, 38, 39, respectively. We consider that the number of lattice sites \( N_a \) is large and even and that \( N_a/2 \) is odd. The electronic number operators on the right-hand
side of Eq. 6 read \( \hat{N} = \sum_{\sigma=\uparrow, \downarrow} \hat{N}_\sigma \) and \( \hat{N}_\sigma = \sum_{j=1}^{N_a} \hat{N}_{j, \sigma} \), where the operator \( \hat{N}_{j, \sigma} = c_{j, \sigma}^\dagger c_{j, \sigma} \) counts the number of spin \( \sigma \) electrons at real-space lattice site \( j \). On the right-hand side of Eqs. (3)–(5) the operator \( c_{j, \sigma}^\dagger \) (and \( c_{j, \sigma} \)) creates (and annihilates) a spin \( \sigma \) electron at lattice site \( j = 1, 2, ..., N_a \). We denote the lattice constant by \( a \) and the lattice length by \( L = N_a a \).

The momentum operator is given by,

\[
\hat{P} = \sum_{\sigma=\uparrow, \downarrow} \sum_k \hat{N}_\sigma(k) k = \frac{L}{2\pi} \sum_{\sigma=\uparrow, \downarrow} \int_{-\pi/a}^{+\pi/a} dk \, \hat{N}_\sigma(k) k .
\] (6)

Here the spin \( \sigma \) momentum distribution operator reads \( \hat{N}_\sigma(k) = c_{k, \sigma}^\dagger c_{k, \sigma} \), where the operator \( c_{k, \sigma}^\dagger \) (and \( c_{k, \sigma} \)) creates (and annihilates) a spin \( \sigma \) electron at momentum \( k \). The operators \( c_{k, \sigma}^\dagger \) and \( c_{k, \sigma} \) are related to the above operators \( c_{j, \sigma}^\dagger \) and \( c_{j, \sigma} \) by the following Fourier transforms,

\[
c_{k, \sigma}^\dagger = \frac{1}{\sqrt{L}} \sum_{j=1}^{N_a} e^{ikaj} c_{j, \sigma}^\dagger ; \quad c_{k, \sigma} = \frac{1}{\sqrt{L}} \sum_{j=1}^{N_a} e^{-ikaj} c_{j, \sigma} .
\] (7)

The Hamiltonian \( \hat{H}_{SU(2)} \), given in Eq. (4) commutes with the six generators of the \( \eta \)-spin and spin \( SU(2) \) algebras and has \( SO(4) \) symmetry. While the expressions of the two corresponding diagonal generators are given in Eq. (5), the off-diagonal generators of these two \( SU(2) \) algebras read

\[
\hat{S}_c^\dagger = \sum_{j=1}^{N_a} (-1)^j c_{j, \downarrow}^\dagger c_{j, \uparrow} ; \quad \hat{S}_c = \sum_{j=1}^{N_a} (-1)^j c_{j, \uparrow}^\dagger c_{j, \downarrow},
\] (8)

and

\[
\hat{S}_s = \sum_{j=1}^{N_a} c_{j, \downarrow}^\dagger c_{j, \uparrow} ; \quad \hat{S}_s = \sum_{j=1}^{N_a} c_{j, \uparrow}^\dagger c_{j, \downarrow},
\] (9)

respectively.

Throughout this paper we use units of Planck constant one and denote the electronic charge by \(-e\). The Bethe-ansatz solvability of the 1D Hubbard model (1) is restricted to the Hilbert subspace spanned by the lowest-weight states (LWSs) or highest-weight states (HWSs) of the \( \eta \)-spin and spin algebras, that is by the states whose \( S_\alpha \) and \( S_\alpha^\dagger \) numbers are such that \( S_\alpha = -S_\alpha^\dagger \) or \( S_\alpha = S_\alpha^\dagger \), respectively, where \( \alpha = c \) for charge and \( \alpha = s \) for spin. In this paper we choose the \( \eta \)-spin and spin LWSs description of the Bethe-ansatz solution. In this case, that solution describes energy eigenstates with electronic densities \( n = N/L \) and spin densities \( m = [N_\uparrow - N_\downarrow]/L \) in the domains \( 0 \leq n \leq 1/a \) and \( 0 \leq m \leq n \), respectively. Some of our results correspond to the ranges \( 0 < n < 1/a \) and \( 0 < m < n \). The description of the states corresponding to the extended domains \( 0 \leq n \leq 1/a \) and \( -n \leq m \leq n \) is achieved by application onto the latter states of off-diagonal generators of the \( \eta \)-spin and spin \( SU(2) \) algebras.

Each lattice site \( j = 1, 2, ..., N_a \) of the model (1) can either be doubly occupied, empty, or singly occupied by a spin-down or spin-up electron. The maximum number of electrons is \( 2N_a \) and corresponds to density \( n = 2/a \). Besides the \( N \) electrons, it is useful to consider \( 2N_a - N \) electronic holes. (Here we use the designation "electronic hole" instead of "hole", in order to distinguish this type of hole from the pseudoparticle hole and pseudofermion hole.) Our definition of electronic hole is such that when a lattice site is empty, we say that it is occupied by two electronic holes. If a lattice site is singly occupied, we say that it is occupied by an electron and an electronic hole. If a lattice site is doubly occupied, it is unoccupied by electronic holes. The same definition holds for the rotated-electronic holes. We note that the lattice occupied by rotated electrons is identical to the original electronic lattice.

The electron - rotated-electron unitary transformation maps the electrons onto rotated electrons such that rotated-electron double occupation, no occupation, and spin-up and spin-down single occupation are good quantum numbers for all values of \( U/t \). We call \( c_{j, \sigma}^\dagger \) the electrons that occur in the 1D Hubbard model (1) and (2), while the operator \( c_{j, \sigma}^\dagger \) such that \( c_{j, \sigma}^\dagger = \hat{V}(U/t) c_{j, \sigma}^\dagger \hat{V}(U/t) \) represents the rotated electrons, where the electron - rotated-electron unitary
The operator $\hat{V}(U/t)$ is defined below. Similarly, $c_{j,\sigma}^\dagger = \hat{V}(U/t)c_{j,\sigma}^\dagger \hat{V}^\dagger(U/t)$. Note that $c_{j,\sigma}^\dagger$ and $\tilde{c}_{j,\sigma}^\dagger$ are only identical in the $U/t \to \infty$ limit where electron double occupation becomes a good quantum number.

The operators $\hat{V}^\dagger(U/t)$ and $\hat{V}(U/t)$ associated with the electron - rotated-electron unitary transformation can be written as,

$$\hat{V}^\dagger(U/t) = e^{-\hat{S}}; \quad \hat{V}(U/t) = e^{\hat{S}}.$$  \hfill (10)

The operator $\hat{S}$ of Eq. (10) is uniquely defined by the following two equations,

$$\hat{H}_H = \hat{V}^\dagger(U/t) \hat{H} \hat{V}(U/t) = \hat{H}_H + [\hat{H}_H, \hat{S}] + \frac{1}{2} [[\hat{H}_H, \hat{S}], \hat{S}] + ..., \hfill (11)$$

and

$$[\hat{H}_H, \hat{V}^\dagger(U/t) \hat{D} \hat{V}(U/t)] = [\hat{H}_H, \hat{D}] = 0, \hfill (12)$$

where the Hamiltonian $\hat{H}_H$ is given in Eq. (2) and the rotated-electron double occupation operator $\hat{D}$ reads,

$$\hat{D} \equiv \hat{V}^\dagger(U/t) \hat{D} \hat{V}(U/t) = \sum_j c_{j,\uparrow}^\dagger \tilde{c}_{j,\uparrow}^\dagger c_{j,\downarrow} \tilde{c}_{j,\downarrow}. \hfill (13)$$

Here $\hat{D}$ is the electron double occupation operator given in Eq. (4). The operator $\hat{D}$ commutes with the 1D Hubbard model. We denote the rotated-electron double occupation by $D$. It is a good quantum number for all values of $U/t$.

The transformation associated with the electron - rotated-electron unitary operator $\hat{V}(U/t)$ was introduced in Ref. 32. The studies of that reference referred to large values of $U/t$ and did not clarify for arbitrary values of $U/t$ the relation of rotated-electron double occupation to the quantum numbers provided by the Bethe-ansatz solution. However, this transformation is uniquely defined for all values of $U/t$ by Eqs. (10)-(12). Equations (11) and (12) can be used to derive an expression for the unitary operator order by order in $t/U$. The authors of Ref. 32 carried out this expansion up to eighth order (see foot note [12]).

III. THE PSEUDOPARTICLE OPERATORS AND THE HOLON, SPINON, AND $c^0$

PSEUDOPARTICLE BASIC DESCRIPTION

According to the studies of Ref. 29, there is an infinite number of pseudoparticle branches: the $c^0$ pseudoparticles and the $\alpha\nu$ pseudoparticles such that $\alpha = c, s$ and $\nu = 1, 2, ...$. The $\alpha\nu$ pseudoparticle notation considered in this paper is related to the $c$ pseudoparticle and $\alpha, \gamma$ pseudoparticle notation of Ref. 29 as follows: $c^0 \equiv c$ for $\nu = 0$, $\nu = \gamma$ and $\alpha\nu \equiv c, \gamma$ for $\nu = 1, 2, ...$, and $\nu = \gamma + 1$ and $\alpha\nu \equiv s, \gamma + 1$ for $\nu = 1, 2, ...$. Moreover, we denote by bare momentum $q$ the momentum carried by the pseudoparticles, while in Ref. 29 it was called band momentum. Our designation is justified by the form of the pseudofermion momentum, as discussed below.

Note that within our notation, the general designation of $\alpha\nu$ pseudoparticle refers to the $\alpha = c$ branches such that $\nu = 0, 1, 2, ...$ and $\alpha = s$ branches such that $\nu = 1, 2, ...$. Elsewhere it is shown that the $\alpha\nu$ pseudoparticles and $s\nu$ pseudoparticles are $2\nu$-holon and $2\nu$-spinon composite objects, respectively, where $\nu = 1, 2, ...$

The introduction of the pseudofermion operational description studied in this paper requires the use of an operator representation for the pseudoparticles. Thus, in this section we introduce an operational description for the $\alpha\nu$ pseudoparticles and the holons and spinons which are not part of composite pseudoparticles. An operational description for the pseudoparticles of bare-momentum $q$ was introduced in Ref. 29. However, such a description did not take into account the holon (and spinon) composite character of the $\alpha\nu$ pseudoparticles (and $s\nu$ pseudoparticles). Moreover, the spinon description used in the studies of that reference is not valid for the whole Hilbert space. This affected the values of the entries of the pseudoparticle statistical-interaction matrix 30. In this section we provide the correct values for the entries of such a matrix. Another limitation was the lack of a representation for the pseudoparticle operators in terms of spatial coordinates. The concepts of local $\alpha\nu$ pseudoparticle and effective $\alpha\nu$ lattice are introduced in Ref. 34. The studies of that reference include a description of the local pseudoparticles in terms of the rotated-electron site distribution configurations. The absence of such a local operational description is one of the reasons why the studies of Ref. 29 did not provide useful information about the main issue of the relation between the pseudoparticle
and electronic operators. Also the concept of rotated electron is important for the study of that issue and was not considered in the studies of such a reference. All these concepts are valuable and necessary for the application of the operational pseudofermion representation introduced in this paper to the evaluation of few-electron spectral functions

In Appendix A we summarize the basic properties of the $\alpha\nu$ pseudoparticles which are needed for our studies. This includes introduction to the $\alpha\nu$ pseudoparticle bare momentum, effective $\alpha\nu$ lattice, and useful ground-state quantities.

A. THE PSEUDOPARTICLE OPERATORS AND PSEUDOPARTICLE STATISTICS ACCORDING TO THE GENERALIZED PAULI PRINCIPLE

Generation and removal of pseudoparticles is in general associated with creation and/or annihilation of electrons. Yet there are also transitions which change the numbers of these quantum objects at constant spin $\sigma$ electron numbers. One can introduce elementary operators for creation or annihilation of $\alpha\nu$ pseudoparticles. In this subsection, we introduce two alternative representations corresponding to pseudoparticle operators, both in terms of the bare-momentum $q$ and spatial coordinates. These representations refer to the bare-momentum pseudoparticles and local pseudoparticles

Let us introduce the bare-momentum $\alpha\nu$ pseudoparticle creation (and annihilation) operator $b_{q,\alpha\nu}^\dagger$ (and $b_{q,\alpha\nu}$) which creates (and annihilates) a $\alpha\nu$ pseudoparticle of bare momentum $q$. In addition, we introduce the local $\alpha\nu$ pseudoparticle creation operator $b_{x_j,\alpha\nu}^\dagger$ and annihilation operator $b_{x_j,\alpha\nu}$. These bare-momentum and local pseudoparticle operators are related as follows,

$$b_{q,\alpha\nu}^\dagger = \frac{1}{\sqrt{L}} \sum_{j=1}^{N_{\alpha\nu}^q} e^{iqx_j} b_{x_j,\alpha\nu}^\dagger; \quad b_{q,\alpha\nu} = \frac{1}{\sqrt{L}} \sum_{j=1}^{N_{\alpha\nu}^q} e^{-iqx_j} b_{x_j,\alpha\nu}.$$  

The local $\alpha\nu$ pseudoparticle creation (and annihilation) operator $b_{x_j,\alpha\nu}^\dagger$ (and $b_{x_j,\alpha\nu}$) creates (and annihilates) a $\alpha\nu$ pseudoparticle at the effective $\alpha\nu$ lattice site of spatial coordinate $x_j = a_{\alpha\nu} j$, where $j = 1, 2, ..., N_{\alpha\nu}$. The effective $\alpha\nu$ lattice constant $a_{\alpha\nu}$ is defined in Eq. (A7) of Appendix A, where the concept of an effective $\alpha\nu$ lattice, introduced in Ref. 35, is described. The conjugate variable of the bare-momentum $q_j$ of the $\alpha\nu$ pseudoparticle branch is the space coordinate $x_j$ of the corresponding effective $\alpha\nu$ lattice. This is different to the electronic operators of Eq. (7), where the conjugate variable of the momentum $k_j$ is the space variable of the original electronic lattice. In reference 35, the pseudoparticle site distribution configurations in the effective $\alpha\nu$ lattices are related to the corresponding rotated-electron site distribution configurations.

The pseudoparticles obey a Pauli principle relative to the bare-momentum occupancy configurations, i.e. a discrete bare-momentum value $q_j$ can either be unoccupied or singly occupied by a pseudoparticle. For $\nu > 0$ composite $\alpha\nu$ pseudoparticles, the number of discrete momentum values $N_{\alpha\nu}^q$ is not the same for all energy eigenstates. Thus, these objects cannot be classified as fermions or bosons. In order to classify the $\alpha\nu$ pseudoparticle according to the generalized Pauli principle introduced in Ref. 34, we consider the $\alpha\nu$ dimensions,

$$d_{\alpha\nu} \equiv 1 + N_{\alpha\nu}^+ - N_{\alpha\nu} = 1 + N_{\alpha\nu}^h,$$  

where according to Eqs. (A3)-(A6) of Appendix A, $N_{\alpha\nu}^+$ and $N_{\alpha\nu}^h$ are the number of $\alpha\nu$ pseudoparticles and $\alpha\nu$ pseudoparticle holes, respectively, and $N_{\alpha\nu} = N_{\alpha\nu}^+ + N_{\alpha\nu}^h$. A transition to an excited energy eigenstate producing deviations $\Delta N_{\alpha\nu}$ in the $\alpha\nu$ pseudoparticle numbers, leads to the following deviations in the corresponding $\alpha\nu$ dimension,

$$\Delta d_{\alpha\nu} = - \sum_{\alpha' = \alpha, s} \sum_{\nu = \nu'} g_{\alpha\nu, \alpha'\nu'} \Delta N_{\alpha'\nu'}.$$  

According to the generalized Pauli principle 36, the parameters $g_{\alpha\nu, \alpha'\nu'}$ on the right-hand side of this equation are the entries of the statistical-interaction matrix. From the use of Eqs. (A3)-(A6) of Appendix A, we find that for the $\alpha\nu$ pseudoparticles such a statistical-interaction matrix has infinite dimension and its entries are given by,

$$g_{\alpha\nu, \alpha'\nu'} = \delta_{\alpha, \alpha'} \delta_{\nu, 0}; \quad g_{\alpha\nu, \alpha'\nu} = \delta_{\alpha, \alpha'} - \delta_{\alpha, \nu}; \quad g_{\alpha\nu, \alpha'\nu'} = \delta_{\alpha, \alpha'} (\nu + \nu' - |\nu - \nu'|); \quad \nu, \nu' > 0.$$  

This fully defines the statistics of the $\alpha \nu$ pseudoparticles. We emphasize that the $c0$ pseudoparticle entry $g_{c0, \alpha \nu}$ gives in Eq. (17) has Fermionic character. This is related to the fact that the number of sites of the effective $c0$ lattice is constant and given by $N_\alpha$, and that the corresponding effective lattice constant $a_{c0}$ equals the electronic lattice constant $a$. Thus, the effective $c0$ lattice and electronic lattice are identical. Furthermore, it is found in Ref. [32] that the sites occupied by $c0$ pseudoparticles (and $c0$ pseudoparticle holes) are the same as the sites singly occupied by rotated electrons (and doubly occupied and unoccupied by rotated electrons). While the $c0$ pseudoparticles do not feel the statistical-interactions of the remaining pseudoparticles, the $\nu > 0$ composite $\alpha \nu$ pseudoparticles feel the statistical interactions of the $c0$ pseudoparticles, as confirmed by Eq. (17). The form of the entries given in Eq. (17) confirms that the latter composite pseudoparticles are neither fermions nor bosons [30]. The statistical interactions of the composite pseudoparticles result in part from the property that the width of the bare-momentum domain $2q_{c0} = 2\pi[1/a_{c0} - 1/L]$ is different for different values of the pseudoparticle numbers.

The $\alpha \nu$ pseudoparticle bare-momentum distribution functions $N_{\alpha \nu}(q)$ play an important role in the pseudoparticle description [30, 31]. These functions are for all energy eigenstates the eigenvalues of the following pseudoparticle bare-momentum distribution operators,

$$N_{\alpha \nu}(q) = b_{q, \alpha \nu}^\dagger b_{q, \alpha \nu}.$$

The bare-momentum distribution functions $N_{\alpha \nu}(q)$ read $N_{\alpha \nu}(q_j) = 1$ for occupied discrete bare-momentum values $q_j$ and $N_{\alpha \nu}(q_j) = 0$ for unoccupied discrete bare-momentum values $q_j$. Each LWS is uniquely specified by the values of the set of distribution functions $\{N_{\alpha \nu}(q)\}$ such that $\nu = 0, 1, 2, ...$ for $\alpha = c$ and $\nu = 1, 2, ...$ for $\alpha = s$. Physical quantities such as the energy, depend on the values of these distribution functions and numbers through the rapidity momentum functional $k(q)$ and rapidity functionals $\Lambda_{\alpha \nu}(q)$ and $\Lambda_{\alpha \nu}(q')$. The value of these functionals is uniquely provided by solution of the following functional integral equations [34, 38].

$$k(q) = q - \frac{1}{\pi} \sum_{\nu=1}^{\infty} q_{c0, \alpha \nu} \int_{-q_{c0}}^{q_{c0}} dq' N_{c0}(q') \arctan \left( \frac{\sin k(q) - \Lambda_{\alpha \nu}(q')}{\nu U/4t} \right)$$

$$- \frac{1}{\pi} \sum_{\nu=1}^{\infty} q_{c0, \alpha \nu} \int_{-q_{c0}}^{q_{c0}} dq' N_{\alpha \nu}(q') \arctan \left( \frac{\sin k(q) - \Lambda_{\alpha \nu}(q')}{\nu U/4t} \right),$$

$$k_{c\nu}(q) = q + \frac{1}{\pi} \int_{q_{c0}}^{q_{c0}'} dq' N_{c0}(q') \arctan \left( \frac{\Lambda_{\alpha \nu}(q) - \sin k(q')}{\nu U/4t} \right)$$

$$+ \frac{1}{2\pi} \sum_{\nu' = 1}^{\infty} \int_{-q_{c, \nu'}}^{q_{c, \nu'}} dq' N_{\alpha \nu}(q') \Theta_{c, \nu'} \left( \frac{\Lambda_{\alpha \nu}(q) - \Lambda_{\alpha \nu}(q')}{U/4t} \right); \quad \nu > 0,$$

and

$$0 = q - \frac{1}{\pi} \int_{q_{c0}}^{q_{c0}'} dq' N_{c0}(q') \arctan \left( \frac{\Lambda_{\alpha \nu}(q) - \sin k(q')}{\nu U/4t} \right)$$

$$+ \frac{1}{2\pi} \sum_{\nu' = 1}^{\infty} \int_{-q_{c, \nu'}}^{q_{c, \nu'}} dq' N_{\alpha \nu}(q') \Theta_{c, \nu'} \left( \frac{\Lambda_{\alpha \nu}(q) - \Lambda_{\alpha \nu}(q')}{U/4t} \right).$$

Here

$$k_{c\nu}(q) = 2 \Re \{ \arcsin(\Lambda_{c\nu}(q) + i\nu U/4t) \}; \quad \nu > 0,$$

is the $c\nu$ rapidity-momentum functional and the limiting bare-momentum values $q_{c0}^{\pm}$ and $q_{c\nu}$ where $\alpha = c, s$ and $\nu = 1, 2, ...$ are given in Eqs. (A10) and (A11) and in Eq. (A11) of Appendix A. The function $\Theta_{c, \nu'}(x)$ appearing in Eqs. (20) and (21) is given in Eq. (117) of Appendix B. The equations (19), (20), and (21) correspond to a functional representation of the thermodynamic Bethe-ansatz equations introduced by Takahashi [26]. The rapidity-momentum functional is real and the rapidity functionals are the real part of Takahashi’s ideal strings [26, 31]. It is useful to introduce the following $c0$ rapidity functional,

$$\Lambda_{c0}(q) = \sin k(q),$$

where $k(q)$ is the rapidity-momentum functional.
B. HOLONS, SPINONS, Ρ PSEUDOPARTICLES AND THE YANG HOLON AND HL SPINON ELEMENTARY OPERATORS

In this subsection we consider a holon, spinon, and ρ pseudoparticle description for the whole Hilbert space of the 1D Hubbard model whose validity is shown elsewhere [31]. Such elementary quantum objects correspond to specific rotated-electron site occupations. This description of all energy eigenstates in terms of occupancy configurations of three elementary quantum objects only, is useful for the studies of this paper.

Let us distinguish the total η spin (and spin) value, which we denote by $S_c$ (and $S_s$) and the corresponding η-spin (and spin) projection, which we denote by $S_c^z$ (and $S_s^z$), from the η spin (and spin) carried by the elementary quantum objects. We call $s_c$ (and $s_s$) the η spin (and spin) carried by the holons, spinons, and other elementary objects and $\sigma_c$ (and $\sigma_s$) their η-spin (and spin) projection. The operators $M_c, \sigma_c,$ and $M_s, \sigma_s$ which count the number of the $\sigma_c = \pm 1/2$ holons and $\sigma_s = \pm 1/2$ spinons have the following form [31],

\[
\hat{M}_{c,-1/2} = \hat{V}^\dagger(U/t) \sum_j c_j^\dagger c_j^\dagger c_j^\dagger c_j^\dagger \hat{V}(U/t);
\]

\[
\hat{M}_{c,+1/2} = \hat{V}^\dagger(U/t) \sum_j c_j^\dagger c_j^\dagger c_j^\dagger c_j^\dagger \hat{V}(U/t),
\]

and

\[
\hat{M}_{s,-1/2} = \hat{V}^\dagger(U/t) \sum_j c_j^\dagger c_j^\dagger c_j^\dagger c_j^\dagger \hat{V}(U/t);
\]

\[
\hat{M}_{s,+1/2} = \hat{V}^\dagger(U/t) \sum_j c_j^\dagger c_j^\dagger c_j^\dagger c_j^\dagger \hat{V}(U/t),
\]

respectively. Here, the operator $\sum_j c_j^\dagger c_j^\dagger c_j^\dagger c_j^\dagger$ counts the number of electron doubly-occupied sites, $\sum_j c_j^\dagger c_j^\dagger c_j^\dagger c_j^\dagger$ counts the number of electron empty sites, $\sum_j c_j^\dagger c_j^\dagger c_j^\dagger c_j^\dagger$ counts the number of spin-down electron singly-occupied sites, and $\sum_j c_j^\dagger c_j^\dagger c_j^\dagger c_j^\dagger$ counts the number of spin-up electron singly-occupied sites. The operator $\hat{V}(U/t)$ on the right-hand side of Eqs. (24) and (25) is uniquely defined for all values of $U/t$ by Eqs. (10)–(12). The new physics brought about by the relations of Eqs. (24) and (25) is that the electron - rotated-electron unitary transformation generates the holons and spinons whose occupancy configurations describe the exact energy eigenstates. Throughout this paper we denote such holons and spinons according to their value of $\sigma_c = \pm 1/2$ and $\sigma_s = \pm 1/2$, respectively. The unitary rotation is such that for all values of $U/t$ the number of emerging $-1/2$ holons, $1/2$ holons, $-1/2$ spinons, and $+1/2$ spinons equals precisely the number of rotated-electron doubly occupied sites, empty sites, spin-down singly occupied sites, and spin-up singly occupied sites, respectively.

The holons have $s_c = 1/2, s_s = 0$, and $\sigma_s = 0$. Thus, there are two types of holons, which have $\sigma_c = -1/2$ and $\sigma_c = +1/2$ and carry charge $-2e$ and $+2e$, respectively. These objects are on-site spin-singlet rotated-electron pairs and rotated-electron unoccupied sites, respectively. An important property is that rotated-electron double occupation $D_r$ equals the value $M_{c,-1/2}$ for the number of $-1/2$ holons. The rotated-electron double occupation $D_r = M_{c,-1/2}$ plays an important role in the description of the few-electron spectral properties of the quantum problem. For few-electron spectral functions the weight distribution resulting from transitions to the Hilbert subspace spanned by excited states of rotated-electron double occupation $D_r$ corresponds to the $D_r^{\text{upper}}$ upper Hubbard band, where $D_r = 1, 2, ...$ [34]. Moreover, the spin-up and spin-down rotated-electron singly occupied sites correspond to the $\sigma_s = +1/2$ and $\sigma_s = -1/2$ spinons, respectively. The $s_s = 1/2$ spinons have no charge degrees of freedom and thus describe the spin degrees of freedom of rotated-electron singly occupied sites only. The charge degrees of freedom of the rotated electrons (and rotated-electron holes) of these singly occupied sites are described by the chargeons of charge $-e$ (and antichargeons of charge $+e$) [31]. These quantum objects are part of the charge degrees of freedom, but do not contribute to the η-spin and spin $SU(2)$ algebras. In the case of the description of the transport of charge in terms of electrons (and electronic holes) the elementary carriers of charge are the chargeons of charge $-e$ and $-1/2$ holons of charge $-2e$ (and the antichargeons of charge $+e$ and $+1/2$ holons of charge $+2e$). The ρ pseudoparticle has no η-spin and no spin degrees of freedom and is a composite quantum object which contains a chargeon and an antichargeon. However, in how transport of charge is concerned, the chargeon and antichargeon correspond to alternative descriptions of the ρ pseudoparticle. When the transport of charge is described in terms of electrons (and electronic holes) the ρ pseudoparticle couples to charge probes through the chargeon (and antichargeon) and carries elementary charge $-e$ (and $+e$).
All energy eigenstates of the 1D Hubbard model can be described in terms of occupancy configurations of holons, spinons, and c0 pseudoparticles. On the other hand, in Ref. 20 it was found that all energy eigenstates associated with the 1D Hubbard model Bethe-ansatz solution 25, 26, can be described in terms of occupancy configurations of αν pseudoparticles. By merging these two representations, one finds that the αν pseudoparticles such that $\nu \neq 0$ and the $\nu s$ pseudoparticles are $2 \nu$-holon and $2 \nu$-spinon composite objects, respectively 31. The composite αν pseudoparticles (and $\nu s$ pseudoparticles) are $s_c = \sigma_c = 0$ (and $s_s = \sigma_s = 0$) objects without spin (and charge) degrees of freedom. These composite quantum objects contain an equal number $\nu$ of $-1/2$ holons and $+1/2$ holons (and $-1/2$ spinons and $+1/2$ spinons). When the transport of charge is described in terms of electrons (and electronic holes), the $2 \nu$-holon composite αν pseudoparticles couple to charge probes through their $\nu -1/2$ holons of charge $-2e$ (and their $\nu +1/2$ holons of charge $+2e$). Thus these composite quantum objects carry elementary charge $-2\nu e$ (and $+2\nu e$).

The $\pm 1/2$ holons (and $\pm 1/2$ spinons) which are not part of $2 \nu$-holon composite αν pseudoparticles (and $2 \nu$-spinon composite $\nu s$ pseudoparticles) are called $\pm 1/2$ Yang holons (and $\pm 1/2$ HL spinons). In the designations $HL$ spinon and Yang holon, HL stands for Heilmann and Lieb and Yang refers to C. N. Yang, respectively, who are the authors of Refs. 37, 38. Note that the holons (and spinons) which are part of $s_c = 0$ (and $s_s = 0$) composite αν pseudoparticles (and $\nu s$ pseudoparticles) remain invariant under application of the off-diagonal generators of the $\eta$-spin (and spin) $SU(2)$ algebras given in Eq. 8 (and 9). Application of these generators produces $\eta$-spin (and spin) flips in $\pm 1/2$ Yang holons (and $\pm 1/2$ HL spinons). Thus, the $-1/2$ Yang holons and $-1/2$ HL spinons are created by application onto the LWSs of suitable off-diagonal generators of the $\eta$-spin and spin $SU(2)$ algebras, respectively, given in Eqs. 8 and 9. The generators 8 and 9 can be written in terms of rotated-electron operators as,

$$d_{q_c, -1/2}^\dagger = \sum_{j=1}^{N_a} (-1)^j \hat{c}_{j, \downarrow} \hat{c}_{j, \uparrow} \quad \quad d_{q_s, -1/2}^\dagger = \sum_{j=1}^{N_a} (-1)^j \hat{c}_{j, \uparrow} \hat{c}_{j, \downarrow}; \quad \quad d_{q_c, -1/2} = \sum_{j=1}^{N_a} (-1)^j \hat{c}_{j, \downarrow} \hat{c}_{j, \uparrow} \quad \quad d_{q_s, -1/2} = \sum_{j=1}^{N_a} \hat{c}_{j, \uparrow} \hat{c}_{j, \downarrow},$$

respectively. In these equations $q_c = \pi/a$ and $q_s = 0$ is the momentum. The operators 26 and 27 are invariant under the electron - rotated-electron unitary transformation 31. Therefore, they have precisely the same expression in terms of electron and rotated-electron creation and annihilation operators.

Note that within the operational representation of Eqs. 26 and 27 the Yang $+1/2$ holons and $+1/2$ spinons are not explicitly considered. Indeed, the numbers $L_{c, +1/2}$ of $+1/2$ Yang holons and $L_{s, +1/2}$ of $+1/2$ HL spinons are fully determined by the numbers $L_{c, -1/2}$ of Yang $-1/2$ holons, $L_{s, -1/2}$ of HL $-1/2$ spinons, and the set $\{N_{\nu s}\}$ of the different $\alpha\nu$ pseudofermion branches. This justifies why here we consider the operators of Eqs. 26 and 27 as creation and annihilation operators for $-1/2$ Yang holons and $-1/2$ HL spinons, respectively. When applied onto LWSs, the operators $d_{q_s, -1/2}^\dagger$ produce energy eigenstates with finite values for the numbers $L_{c, -1/2}$ and/or $L_{s, -1/2}$ of the following form,

$$|\{L_{c, -1/2}, L_{s, -1/2}\} = \prod_{\alpha = c, s} \frac{(d_{q_{\alpha s}, -1/2})^{L_{\alpha, -1/2}}}{\sqrt{L_{\alpha}}}|LWS\rangle.$$

Here $|LWS\rangle$ is the LWS that corresponds to the state $|\{L_{c, -1/2}, L_{s, -1/2}\}$ and $L_{\alpha} = L_{\alpha, +1/2} + L_{\alpha, -1/2} = 2S_{\alpha}$ where $S_{\alpha}$ is the state $\eta$-spin value ($\alpha = c$) and spin value ($\alpha = s$). The energy eigenstates 28 are not described by the Bethe-ansatz solution 31, 32.

The pseudoparticle bare momentum $q$ is such that $q \in (-q_{\alpha s}, +q_{\alpha s})$ where the limiting bare-momentum values $\pm q_{\alpha s}$ associated with the limits of the $\alpha\nu$ pseudoparticle Brillouin zone are given in Eq. A11 of Appendix A. According to the results of Ref. 30, the $\pm 1/2$ Yang holons, $\pm 1/2$ HL spinons, and $\nu > 0$ composite $\alpha\nu$ pseudoparticles of bare-momentum values $q = \pm q_{\alpha s}$ are non-interacting and localized quantum objects. Such behavior results from the invariance of these quantum objects under the electron - rotated-electron unitary transformation. This means that the $\pm 1/2$ Yang holons, $\pm 1/2$ HL spinons, and $\nu > 0$ composite $\alpha\nu$ pseudoparticles of bare-momentum values $q = \pm q_{\alpha s}$ are the same quantum objects as the corresponding rotated objects. Thus, these objects are localized and do not contribute to the transport of charge or spin. However, in general a $\nu > 0$ composite $\alpha\nu$ pseudoparticle is different from a $\nu > 0$ rotated composite $\alpha\nu$ pseudoparticle. The only exception is precisely for bare momentum values $q$
such that $q \rightarrow \pm q_{\alpha \nu}$. As the bare momentum approaches its limiting values, $q \rightarrow \pm q_{\alpha \nu}$, the $\nu > 0$ composite $\alpha \nu$ pseudoparticle and the $\nu > 0$ rotated composite $\alpha \nu$ pseudoparticle become the same quantum object. Importantly, the electrons and rotated electrons involved in processes associated with creation of $\pm 1/2$ Yang holons, $\pm 1/2$ HL spinons, and $\nu > 0$ composite $\alpha \nu$ pseudoparticles of bare-momentum values $q = \pm q_{\alpha \nu}$ are also the same quantum object, i.e. remain also invariant under the electron - rotated-electron unitary transformation. This always refers to localized electrons. It follows that the transport of charge (and spin) is associated with the $c0$ pseudoparticle and $q \neq \pm q_{\alpha \nu}$ composite $\nu c$ pseudoparticle quantum charge fluids (and $q \neq \pm q_{\alpha \nu}$ composite $\nu s$ pseudoparticle quantum spin fluids) where $\nu = 1, 2, \ldots$

Within the pseudoparticle, Yang holon, and HL spinon operational description, the $\pm 1/2$ holon ($\alpha = c$) and $\pm 1/2$ spinon ($\alpha = s$) number operators $\hat{M}_{\alpha, \pm 1/2}$ given in Eqs. (24) and (25) are written in terms of pseudoparticle operators and $\pm 1/2$ Yang holon ($\alpha = c$) or $\pm 1/2$ HL spinon ($\alpha = s$) number operators $\hat{L}_{\alpha, \pm 1/2}$ as follows,

$$\hat{M}_{\alpha, \pm 1/2} = \hat{L}_{\alpha, \pm 1/2} + \sum_{\nu=1}^{\infty} \frac{q_{\alpha \nu}}{q_{\alpha, \nu}} \nu \hat{N}_{\alpha \nu}(q),$$

(29)

where the pseudoparticle bare-momentum distribution operators $\hat{N}_{\alpha \nu}(q)$ are provided in Eq. (18). The operator $\hat{L}_{\alpha, \pm 1/2}$ can be written as,

$$\hat{L}_{\alpha, \pm 1/2} = \sqrt{\hat{S}_{\alpha z}^2 - 1/4 - 1/2} \mp \hat{S}_{\alpha z}^z; \quad \alpha = c, s.$$

(30)

Here $\hat{S}_{\alpha z}^z$ is the diagonal generator of the $\eta$-spin ($\alpha = c$) and spin ($\alpha = s$) algebras whose expression in terms of electron operators is provided in Eq. (9). The spin $\sigma$ electron number operator commutes with the electron - rotated-electron unitary operator. Thus, the operator $\hat{S}_{\alpha z}^z$ has the same expression in terms of electron and rotated-electron creation and annihilation operators. The same occurs for the $\eta$-spin ($\alpha = c$) and spin ($\alpha = s$) operator $\hat{S}_{\alpha z}^z$. This operator can be expressed in terms of rotated-electron creation and annihilation operators by use of Eqs. (26) and (27). Therefore, Eq. (30) provides the expression of the operator $\hat{L}_{\alpha, \pm 1/2}$ both in terms of electron and rotated-electron creation and annihilation operators.

An electronic ensemble space is spanned by all energy eigenstates with the same values for the numbers $N^\uparrow$, and $N^\downarrow$. An important concept is that of CPHS ensemble space [31]. This is a subspace spanned by all energy eigenstates with the same values for the numbers of $\pm 1/2$ holons and $\pm 1/2$ spinons $\{M_{\alpha, \pm 1/2}\}$, where $\alpha = c, s$. In general, an electronic ensemble space contains several CPHS ensemble spaces. Moreover, usually a CPHS ensemble space includes different CPHS ensemble subspaces. A CPHS ensemble subspace is spanned by all energy eigenstates with the same values for the sets of numbers $N_{\alpha}, \{N_{\alpha \nu}\}$, and $\{L_{\alpha, \nu\downarrow}\}$ such that $\alpha = c, s$ and $\nu = 1, 2, \ldots$ (According to the notation of Ref. [31], CPHS stands for $c$ pseudoparticle, holon, and spinon.)

C. THE GENERAL ENERGY SPECTRUM AND THE MOMENTUM OPERATOR

The generators of the $\eta$-spin and spin $SU(2)$ algebras given in Eqs. (20) and (24) commute with the $c0$ pseudoparticle and composite $\alpha \nu$ pseudoparticle creation and annihilation operators of Eq. (18). We emphasize that this property has the following important effect: All $2S_{\alpha}$ energy eigenstates obtained from a given regular energy eigenstate have the same pseudoparticle momentum distribution functions $N_{\alpha \nu}(q)$ and $\{N_{\alpha \nu}(q)\}$ for all branches $\alpha = c, s$, and $\nu = 1, 2, \ldots$. Thus these states are described by similar pseudoparticle occupancy configurations and only differ in the relative numbers of $+1/2$ Yang holons and $-1/2$ Yang holons ($\alpha = c$), or/and in the relative numbers of $+1/2$ HL spinons and $-1/2$ HL spinons ($\alpha = s$). This reveals that the coupled functional equations (19)-(21), which involve the pseudoparticle momentum distribution functions and do not depend on the $L_{\alpha, \nu \downarrow}$ numbers, describe both LWSs and non-LWSs. The eigenvalues $E$ of the energy eigenstates of the Hamiltonian (11) can be written in the following form,

$$E = E_{SO(4)} + \sum_{\alpha = c, s} \mu_{\alpha} S_{\alpha z}^z,$$

(31)

where

$$E_{SO(4)} = E_H + \frac{U}{2} \left[ M_c - 2M_{c, -1/2} - \frac{N_a}{2} \right],$$

(32)
and

\[ E_H = -2t \frac{L}{2\pi} \int_{q_{c,0}}^{q_{c,0}^+} dq \, N_{c,0}(q) \cos q k(q) \]

\[ + 4t \frac{L}{2\pi} \sum_{\nu = 1}^{\infty} \int_{q_{c,\nu}}^{q_{c,\nu}^+} dq \, N_{c,\nu}(q) \text{Re} \left\{ \sqrt{1 - (\Lambda_{c,\nu}(q) + i\nu U/4t)^2} + U L_{c,-1/2} \right\}. \]  

(33)

On the right-hand side of Eq. (31) the numbers \( S^z = -\frac{1}{2}[N_{\uparrow} - N_{\downarrow}] \) and \( S^z_s = -\frac{1}{2}[N_{\uparrow} - N_{\downarrow}] \) are the eigenvalues of the diagonal generators given in Eq. (30) and \( \mu_c = 2\mu \) and \( \mu_s = 2\mu_0 H \) are the same quantities as on the right-hand side of Eq. (11).

The values of the rapidity-momentum functional \( k(q) \) and rapidity functionals \( \Lambda_{c,\nu}(q) \) are the same for all \( 2S_c + 1 \) states in the same tower. Such functionals are eigenvalues of operators which commute with the off-diagonal generators of \( \eta \)-spin and spin algebras. This is consistent with the \( \eta \)-spin and spin \( SU(2) \) symmetries, which imply that the Hamiltonian (2) commutes with these generators and thus the energy (32) is the same for the set of \( 2S_c + 1 \) states belonging to the same \( \eta \)-spin (\( \alpha = c \)) or spin (\( \alpha = s \)) tower. The above operators obey equations similar to Eqs. (19)-(21), with the \( \alpha \nu \) bare-momentum distribution functions \( N_{c,\nu}(q) \) replaced by the corresponding operators \( \hat{N}_{c,\nu}(q) \) given in Eq. (18). The pseudoparticle Hamiltonian expression is also obtained by replacing in the energy expressions (31)-(33) the distribution functions \( N_{c,\nu}(q) \) by the operators \( \hat{N}_{c,\nu}(q) \), and the rapidity-momentum and rapidity functionals by the corresponding operators.

The momentum operator given in Eq. (31) can be expressed in terms of pseudoparticle and \(-1/2\) holon operators as follows (31),

\[ \hat{P} = \sum_{q=q_{c,0}}^{q_{c,0}^+} \hat{N}_{c,0}(q) q + \sum_{\nu = 1}^{\infty} \sum_{q = q_{c,\nu}}^{q_{c,\nu}^+} \hat{N}_{c,\nu}(q) q + \sum_{\nu = 1}^{\infty} \sum_{q = q_{c,\nu}}^{q_{c,\nu}^+} \hat{N}_{c,\nu}(q) \left[ \frac{\pi}{\alpha} - q \right] + \frac{\pi}{\alpha} L_{c,-1/2} \]

\[ = \sum_{q=q_{c,0}}^{q_{c,0}^+} \hat{N}_{c,0}(q) q + \sum_{\nu = 1}^{\infty} \sum_{q = q_{c,\nu}}^{q_{c,\nu}^+} \hat{N}_{c,\nu}(q) q + \sum_{\nu = 1}^{\infty} \sum_{q = q_{c,\nu}}^{q_{c,\nu}^+} \hat{N}_{c,\nu}(q) \left[ (1 + \nu) \frac{\pi}{\alpha} - q \right] + \frac{\pi}{\alpha} L_{c,-1/2} \].

(34)

The momentum operator (34) commutes with the electron - rotated-electron unitary operator \( V(U/t) \). The momentum eigenvalues can be straightforwardly written by replacing on the right-hand side of Eq. (34) the pseudoparticle and holon number operators by the corresponding eigenvalues.

Note that the number \( L_{c,-1/2} \) and corresponding operator \( \hat{L}_{c,-1/2} \) appearing on the right-hand side of Eqs. (38) and (44), respectively, are beyond the Bethe-ansatz solution. Indeed, the states which span the Hilbert subspace associated with that solution have no \(-1/2\) Yang holons and thus \( L_{c,-1/2} = 0 \) in Eq. (38) for these states. However, the energy spectrum defined by Eqs. (31)-(33) and the momentum-operator expression (44) refer to the whole Hilbert space of the \( 1D \) Hubbard model. On the other hand, as a result of the \( SO(4) \) symmetry of the Hamiltonian (2), the corresponding energy spectrum (32) does not depend on the value of the number \( L_{c,-1/2} \). This property is confirmed by analysis of expressions (38) and (33) and by noting that \( M_{c,-1/2} = L_{c,-1/2} + \sum_{\nu = 1}^{\infty} \nu N_{c,\nu} \).

IV. THE GROUND-STATE NORMAL-ORDERED PSEUDOPARTICLE OPERATIONAL DESCRIPTION

In this section we consider the ground-state normal-ordered pseudoparticle operational description which is needed for the introduction of the pseudofermion description. We are mostly interested in excited states generated from the ground state by processes involving changes in the occupancy configurations of a finite number of \( \alpha \nu \) pseudoparticles, \(-1/2\) Yang holons, and \(-1/2\) HL spinons. The pseudofermion description introduced in the ensuing section corresponds to the Hilbert subspace spanned by such excited states. In the thermodynamic limit the few-electron excitations are contained in that subspace.

A. NORMAL-ORDERED NUMBER OPERATORS AND USEFUL GROUND-STATE QUANTITIES

Throughout this paper the symbol : \( \hat{O} \) : refers to the ground-state normal-ordered expression of a general operator \( \hat{O} \). Such a ground-state normal-ordered expression is given by that operator minus its ground-state expectation value.
We introduce the ground-state normal-ordered \( \alpha \nu \) pseudoparticle bare-momentum distribution operator,

\[
\hat{N}_{\alpha \nu}(q) := \hat{N}_{\alpha \nu}(q) - N_{\alpha \nu}^0(q),
\]
and the \(-1/2\) Yang holon \((\alpha = c)\) and \(-1/2\) HL spinon \((\alpha = s)\) ground-state normal-ordered number operator,

\[
\hat{L}_{\alpha, -1/2} := \hat{L}_{\alpha, -1/2} - L_{\alpha, -1/2}^0 = \hat{L}_{\alpha, -1/2}.
\]

Here the operators \(\hat{N}_{\alpha \nu}(q)\) and \(\hat{L}_{\alpha, -1/2}\) are defined in Eqs. (13) and (30), respectively, \(N_{\alpha \nu}^0(q)\) is the ground-state \(\alpha \nu\) pseudoparticle distribution function whose expressions are given in Eqs. (A12)–(A14) of Appendix A, and \(L_{\alpha, -1/2}^0 = 0\) is the \(-1/2\) Yang holon \((\alpha = c)\) and \(-1/2\) HL spinon \((\alpha = s)\) ground-state number. It follows from the expressions given in Eq. (A15) of Appendix A for the ground-state values of the number \(N_{\alpha \nu}^0\), whose general expressions are given in Eqs. (A3), (A4), and (A6) of the same Appendix, that in the case of the ground state the effective \(\alpha \nu\) lattice constants \(\Lambda\) are given by (35).

\[
a_{c0}^0 = a; \quad a_{cv}^0 = \frac{1}{\delta}; \quad a_{s1}^0 = \frac{1}{n}; \quad a_{sv}^0 = \frac{1}{m},
\]

where \(\delta = (1/a - n)\) is the doping concentration. We note that the meaning of the divergences in the value of the constants \(a_{\nu}^0\) defined in Eq. (37) is that the corresponding effective \(\alpha \nu\) lattice has no sites, i.e. \(N_{\alpha \nu}^0 = 0\) and, therefore, does not exist for the ground state. This is the case of the effective \(cw\) lattices for half filling when \(\nu > 0\) and of the effective \(sv\) lattices for zero spin density when \(\nu > 1\). It follows that such singularities just indicate the collapse of the corresponding effective \(\alpha \nu\) lattice. This is one of the reasons why some of our expressions refer to electronic densities \(0 < n < 1/a\) and spin densities \(0 < m < n\), such that all ground-state effective \(\alpha \nu\) lattice constants \(\Lambda\) have finite values.

From use of expressions (37) one can write the ground-state number \(N_{\alpha \nu}^{0, \nu}\) given in Eq. (A15) of Appendix A as,

\[
N_{\alpha \nu}^{0, \nu} = \frac{L}{a_{\alpha \nu}^0}.
\]

Except for terms of order \(1/L\), the limiting bare-momentum values defined in Eqs. (A8)—(A11) of Appendix A simplify and are given by,

\[
q_{\alpha \nu}^0 = \frac{\pi}{a_{\alpha \nu}^0},
\]

where the ground-state effective-lattice constants \(a_{\alpha \nu}^0\) are these given in Eq. (37). Comparison of Eqs. (37) and (39) leads to the following ground-state expressions for the \(\alpha \nu\) pseudoparticle limiting bare-momentum values,

\[
q_{c0}^0 = \frac{\pi}{a}; \quad q_{s1}^0 = k_{F \uparrow};
\]
\[
q_{cv}^0 = \left[\pi/a - 2k_F\right], \quad \nu > 0; \quad q_{sv}^0 = \left[k_{F \uparrow} - k_{F \downarrow}\right], \quad \nu > 1.
\]

In most situations, one can disregard the \(1/L\) corrections and use the bare-momentum limiting values given in Eq. (40).

For each specific energy eigenstate the rapidity-momentum functional \(k(q)\) and the rapidity functionals \(\Lambda_{\alpha \nu}(q)\) become mere functions of \(q\) and are called rapidity-momentum and rapidity functions, respectively. We denote the ground-state rapidity-momentum function by \(k^0(q)\) and we call \(\Lambda_{\alpha \nu}^0(q)\) and \(\Lambda_{\alpha \nu}(q)\) the ground-state rapidity functions. These ground-state functions are computed by solution of the integral equations obtained by introducing in Eqs. (14)–(21) the ground-state distribution functions (A12)–(A14) given in Appendix A. The solution of these equations can be written in terms of the inverse functions of \(k^0(q)\) and \(\Lambda_{\alpha \nu}^0(q)\) which we call \(\tilde{q}_c(k)\) and \(\tilde{q}_{\alpha \nu}(\Lambda)\), respectively. Here \(k\) and \(\Lambda\) are the rapidity-momentum coordinate and rapidity coordinate, respectively. For the pseudoparticle description these functions refer to the ground state only. However, in the case of the pseudofermion description introduced in the ensuing section these functions apply both to the ground state and excited states. It follows that these functions play an important role in the latter description. These are odd functions such that,

\[
\tilde{q}_c(k) = -\tilde{q}_c(-k); \quad \tilde{q}_{\alpha \nu}(\Lambda) = -\tilde{q}_{\alpha \nu}(-\Lambda); \quad \alpha = c, s; \quad \nu = 1, 2, \ldots.
\]
The domains of the rapidity-momentum coordinate \( k \) and rapidity coordinate \( \Lambda \) are such that \(-\pi/a \leq k \leq +\pi/a\) and \(-\infty \leq \Lambda \leq \infty\), respectively. If follows that,

\[
k^0\left(\pm \frac{\pi}{a}\right) = \pm \frac{\pi}{a}; \quad \Lambda^0_{c0}\left(\pm \frac{\pi}{a}\right) = 0; \quad \Lambda^0_{\alpha\nu}\left(\pm q^0_{\alpha\nu}\right) = \pm \infty; \quad \alpha = e, s; \quad \nu = 1, 2, \ldots, \tag{42}\]

where we introduced the ground-state \( c_0 \) rapidity function,

\[
\Lambda^0_{c0}(q) = \sin k^0(q). \tag{43}\]

The \( c_0 \) rapidity function \( \text{Eq. (43)} \) is the ground-state value of the rapidity functional defined in Eq. \( \text{Eq. (23)} \). The relations \( \text{Eq. (42)} \) are equivalent to,

\[
\bar{q}_{c}\left(\pm \frac{\pi}{a}\right) = \pm \frac{\pi}{a}; \quad \bar{q}_{c\nu}\left(\pm \infty\right) = \pm \left[\frac{\pi}{a} - 2k_F\right]; \quad \bar{q}_{s\nu}\left(\pm \infty\right) = \pm |k_{F\uparrow} - k_{F\downarrow}|; \quad \nu = 1, 2, \ldots. \tag{44}\]

The Fermi points \( \pm 2k_F \) and \( \pm k_{F\downarrow} \) of the \( c_0 \) pseudoparticles and \( s1 \) pseudoparticles, respectively, correspond to the following rapidity Fermi points,

\[
Q \equiv k^0(2k_F), \quad B \equiv \Lambda^0_{s1}(k_{F\downarrow}). \tag{45}\]

Then,

\[
\bar{q}_{c}(\pm 2k_F) = \pm Q; \quad \bar{q}_{s1}(\pm k_{F\downarrow}) = \pm B. \tag{46}\]

Interestingly, one can write the above functions \( \bar{q}_{c}(k), \bar{q}_{c\nu}(\Lambda), \) and \( \bar{q}_{s\nu}(\Lambda) \) in terms of the two-pseudofermion phase shifts \( \Phi_{\alpha\nu, \alpha'\nu'} \) expressed in terms of the rapidity-momentum coordinate \( k \) and rapidity coordinate \( \Lambda \), whose physical meaning is discussed in the ensuing section. These functions read,

\[
\bar{q}_{c}(k) = k + \int_{-Q}^{+Q} dk' \Phi_{c_0, c0}(k', k), \tag{47}\]

\[
\bar{q}_{c\nu}(\Lambda) = 2 \text{Re} \left\{ \text{arcsin} \left( \Lambda + i\nu U / 4t \right) \right\} - \int_{-Q}^{+Q} dk' \Phi_{c_0, c0}(k', \Lambda); \quad \nu = 1, 2, \ldots, \tag{48}\]

\[
\bar{q}_{s\nu}(\Lambda) = \int_{-Q}^{+Q} dk' \Phi_{s0, s\nu}(k', \Lambda); \quad \nu = 1, 2, \ldots, \tag{49}\]

respectively. The two-pseudofermion phase shifts play an important role in the few-electron spectral properties and appear in the expression of the momentum carried by the pseudofermions. The two-pseudofermion phase shifts \( \Phi_{\alpha\nu, \alpha'\nu'} \) expressed in terms of the rapidity-momentum coordinate \( k \) and rapidity coordinate \( \Lambda \) are mathematically defined by the following equations,

\[
\Phi_{c0, c0}(k, k') = \Phi_{c0, c0} \left( \frac{4t \sin k}{U}, \frac{4t \sin k'}{U} \right); \quad \Phi_{c0, \alpha\nu}(k, \Lambda') = \Phi_{c0, \alpha\nu} \left( \frac{4t \sin k}{U}, \frac{4t \Lambda'}{U} \right), \tag{50}\]

\[
\Phi_{\alpha\nu, c0}(\Lambda, k') = \Phi_{\alpha\nu, c0} \left( \frac{4t \Lambda}{U}, \frac{4t \sin k'}{U} \right); \quad \Phi_{\alpha\nu, \alpha'\nu'}(\Lambda, \Lambda') = \Phi_{\alpha\nu, \alpha'\nu'} \left( \frac{4t \Lambda}{U}, \frac{4t \Lambda'}{U} \right), \tag{51}\]

where the two-pseudofermion phase shifts \( \Phi_{\alpha\nu, \alpha'\nu'}(r, r') \) are defined by the integral equations \( \text{Eq. (152)} - \text{Eq. (153)} \) of Appendix B. These are expressed in terms of the variable \( r \) defined in the arguments of the functions on the right-hand side of
Eqs. 65-67. The values of the parameters $Q$ and $B$ introduced in Eq. 68 are controlled by the two-pseudofermion phase shifts and are computed by self-consistent solution of the following equations,

$$2k_F = Q + \int_{-Q}^{+Q} dk \bar{\Phi}_{c\nu,0}\,(k,Q) ; \quad k_{F\perp} = \int_{-Q}^{+Q} dk \bar{\Phi}_{c\nu,1}\,(k,B) ,$$

(52)

respectively. At zero spin density the parameter $Q$ changes from $Q = k_F$ in the limit $U/t \to 0$ to $Q = 2k_F$ as $U \to \infty$. At half filling and zero spin density, the $U/t$ dependence of the parameter $Q$ is singular at $U/t = 0$. It reads $Q = \pi/2a$ at $U/t = 0$ and is given by $Q = \pi/a$ for all finite values of $U/t$. The $U/t > 0$ value, $Q = 2k_F = \pi/a$, is associated with a full $c\nu$ pseudoparticle band both in bare-momentum $q$ and in rapidity-momentum $k$ spaces and thus implies insulator behavior. In contrast, the $U/t = 0$ value $Q = \pi/2a$ corresponds to a metallic band which is half filled in the rapidity-momentum space. Such a singular behavior is associated with the Mott-Hubbard transition 10, 25. At zero spin density the parameter $B$ is given by $B = \infty$ and vanishes in the limit of spin density $m \to n$, as the fully polarized ferromagnetic state is approached.

Unfortunately, it is in general difficult to obtain closed form expressions for the ground-state functions $k^0(q)$ and $\Lambda^0_{\nu\nu}(q)$ by inverting the functions defined in Eqs. (17)-(19). This can be achieved for specific limits of the parameter space only. For instance, from the use of Eqs. 17-19 we can obtain for zero spin density $m = 0$, values of the electronic density $0 \leq n \leq 1/a$, and limiting on-site repulsion values $U/t \to 0$ and $U/t >> 1$ the following closed form expressions for the ground-state functions $k^0(q)$, $\Lambda^0_{c\nu}(q)$, $\Lambda^0_{\nu\nu}(q)$, and $\Lambda^0_{s1}(q)$,

$$k^0(q) = \begin{cases} \frac{q}{2} & |q| \leq 2k_F, \quad U/t \to 0; \\ \text{sgn}(q) (|q| - k_F) & 2k_F \leq |q| < \pi/a, \quad U/t \to 0; \\ |q| = \pi/a, \quad U/t \to 0; \\ q - \frac{4\pi n}{U} \ln(2) \sin(qa) & |q| \leq \pi/a, \quad U/t >> 1, \end{cases}$$

(53)

$$\Lambda^0_{c\nu}(q) = \begin{cases} \sin\left(\frac{qa}{2}\right) & |q| \leq 2k_F, \quad U/t \to 0; \\ \text{sgn}(q) \sin\left(\frac{(|q| - k_F)a}{2}\right) & 2k_F \leq |q| < \pi/a, \quad U/t \to 0; \\ |q| = \pi/a, \quad U/t \to 0; \\ \sin(qa) - \frac{2\pi n}{U} \ln(2) \sin(2qa) & |q| \leq \pi/a, \quad U/t >> 1, \end{cases}$$

(54)

$$\Lambda^0_{\nu\nu}(q) = \begin{cases} \text{sgn}(q) \sin\left(\frac{(|q| + \pi n)a}{2}\right) & 0 < |q| < (\pi/a - 2k_F), \quad U/t \to 0 \\ 0 & q = 0, \quad U/t \to 0 \\ \pm \infty & q = \pm (\pi/a - 2k_F), \quad U/t \to 0 \\ \frac{4\nu t}{U} \tan\left(\frac{qa}{2n}\right) & 0 \leq |q| \leq (\pi/a - 2k_F), \quad U/t >> 1, \end{cases}$$

(55)

for $\nu > 0$, and

$$\Lambda^0_{s1}(q) = \begin{cases} \sin(qa) & |q| \leq k_F, \quad U/t \to 0 \\ \pm \infty & q = \pm k_F, \quad U/t \to 0 \\ \frac{8t}{\pi U} \arcsinh\left(\tan\left(\frac{qa}{2n}\right)\right) & |q| \leq k_F, \quad U/t >> 1, \end{cases}$$

(56)

respectively. We note that for zero-spin density the $\nu > 1$ ground-state rapidity function $\Lambda^0_{\nu\nu}(q)$ vanishes for all values of $n$ and $U/t$. For $m = 0$, the bare-momentum domain width of the $s\nu$ pseudoparticles belonging to branches such that $\nu > 1$ vanishes and the corresponding effective $s\nu$ lattice has no sites and thus collapses. This collapsing is associated with the divergence of the effective $s\nu$ lattice constant given in Eq. 47 as $m \to 0$ when $\nu > 1$. The same occurs for the $c\nu$ pseudoparticles belonging to branches such that $\nu > 0$ when half filling is approached.
For any energy eigenstate $|\psi\rangle$ the normal-ordered bare-momentum distribution and number operators defined in Eqs. (A12) and (A13), respectively, obey the following eigenvalue equations,

$$
: \hat{N}_{\alpha\nu}(q) : |\psi\rangle = \Delta N_{\alpha\nu}(q)|\psi\rangle; \quad : \hat{L}_{\alpha, -1/2} : |\psi\rangle = \Delta L_{\alpha, -1/2}|\psi\rangle.
$$

(57)

Here $\Delta N_{\alpha\nu}(q)$ is the $\alpha\nu$ pseudoparticle bare-momentum distribution function deviation and $\Delta L_{\alpha, -1/2}$ is the deviation in the number of $-1/2$ Yang holons ($\alpha = c$) or of $-1/2$ HL spinons ($\alpha = s$). These deviations are given by

$$
\Delta N_{\alpha\nu}(q) \equiv N_{\alpha\nu}(q) - N_{\alpha\nu}^0(q); \quad \Delta L_{\alpha, -1/2} \equiv L_{\alpha, -1/2} - L_{\alpha, -1/2}^0.
$$

(58)

These values describe deviations of occupancy configurations of excited states relative to the ground-state occupancy configurations described by the bare-momentum distribution functions and numbers given in Eqs. (A12) and (A13) of Appendix A. For these excited states, the $\alpha\nu$ pseudoparticle bare-momentum distribution function and the $-1/2$ Yang holon and $-1/2$ HL spinon numbers read,

$$
N_{\alpha\nu}(q) = N_{\alpha\nu}^0(q) + \Delta N_{\alpha\nu}(q); \quad L_{\alpha, -1/2} = L_{\alpha, -1/2}^0 + \Delta L_{\alpha, -1/2}.
$$

(59)

From use of the ground-state distribution and number values given in Eqs. (A12) and (A13) of Appendix A, we find the following operational relations,

$$
: \hat{N}_{c\nu}(q) := \hat{N}_{c\nu}(q); \quad \nu > 0; \quad : \hat{N}_{s\nu}(q) := \hat{N}_{s\nu}(q); \quad \nu > 1; \quad : \hat{L}_{\alpha, -1/2} := \hat{L}_{\alpha, -1/2}.
$$

(60)

Relations (60) are justified by the absence of the corresponding quantum objects in the initial ground state.

**B. THE NORMAL-ORDERED PSEUDOPARTICLE MOMENTUM AND ENERGY FUNCTIONALS**

Let us introduce the momentum and energy functionals associated with the Hilbert subspace spanned by excited states described by small deviations $|\psi\rangle$. The bare-momentum distribution deviations of these states involve a small density of pseudoparticles, $-1/2$ Yang holons, and $-1/2$ HL spinons. In the case of momentum, such a functional is obtained directly from use of the operator expression (58). Introduction of the corresponding normal-ordered expression and replacement of the bare-momentum number operators and holon number operators by their eigenvalues leads to the following momentum functional,

$$
\Delta P = \sum_{q=q^+_{\alpha\nu}} \Delta N_{c\nu}(q) + \sum_{\nu=1}^{\infty} \sum_{q=-q_{\alpha\nu}}^{+q_{\alpha\nu}} \Delta N_{s\nu}(q) + \sum_{\nu=1}^{\infty} \sum_{q=-q_{\alpha\nu}}^{+q_{\alpha\nu}} \Delta N_{c\nu}(q) [(1 + \nu) \frac{\pi}{a} - q] + \frac{\pi}{a} \Delta L_{c, -1/2}.
$$

(61)

The momentum functional (61) is linear in the bare-momentum distribution function deviations for all excited states independently of whether the values of these deviations are small or large. Thus, this expression is valid even for large values of these deviations.

As for the Fermi-liquid quasiparticles $[12, 13]$, while the momentum functional (61) is linear in the bare-momentum distribution function deviations, the corresponding energy functional includes linear, quadratic, and higher-order terms in these deviations. Such an energy functional is derived by solution of the integral equations (19) and (21) for bare-momentum distribution functions of the general form (58). This leads to deviation expansion expressions for the rapidity-momentum and rapidity functionals in terms of the set of bare-momentum distribution function deviations $\Delta N_{c\nu}(q)$, $\{\Delta N_{s\nu}(q)\}$, and $\{\Delta N_{c\nu}(q)\}$ where $\nu = 1, 2, \ldots$. According to Eqs. (51) and (53), the energy spectrum of the excited states can be expressed in terms of these functionals. Use in the general energy expressions (51) and (53) of the systematic expansion in the pseudoparticle bare-momentum distribution deviations and holon and spinon number deviations of the above functionals leads to a general finite-energy Landau-liquid energy functional of the following form,

$$
\Delta E = \sum_{i=1}^{\infty} \Delta E_i.
$$

(62)
The terms of order \( i \) larger than one describe the residual interactions of the pseudoparticles. In contrast, the energy functional (62) is linear in the \(-1/2\) Yang holon and \(-1/2\) HL spinon number deviations for all excited states independently of whether the values of these deviations are small or large. Such behavior follows from the non-interacting character of these quantum objects. In the case of excited states whose bare-momentum distribution function deviations involve a small but finite density of pseudoparticles, the physics is described by the energy terms on the right-hand side of Eq. (62) of all scattering orders \( i = 1, 2, \ldots \). However, in the limit when the density of these pseudoparticles becomes vanishing, only the first two terms become relevant. According to the results obtained in Refs. [21], [29], [30], the first and second-order terms on the right-hand side of Eq. (62) are of the following general form,

\[
\Delta E_1 = \omega_0 + \sum_{q=-q_0^0}^{q_0^0} dq \epsilon_{\alpha}(q) \Delta N_{\alpha}(q) + \sum_{q=-q_{s1}^0}^{q_{s1}^0} dq \epsilon_{s1}(q) \Delta N_{s1}(q) + \sum_{\alpha=e,s} \sum_{\nu=1+\delta_{\alpha,s}} \sum_{q=-q_{0}^0}^{q_{0}^0} dq \epsilon_{\alpha \nu}(q) \Delta N_{\alpha \nu}(q), \tag{63}
\]

and

\[
\Delta E_2 = \frac{1}{L} \sum_{\alpha=e,s} \sum_{\nu=1-\delta_{\alpha,s}} \sum_{q=-q_{0}^0}^{q_{0}^0} \sum_{\alpha'=e,s} \sum_{\nu'=1-\delta_{\alpha',s'}} \sum_{q'=q} dq dq' \sum_{\alpha''} \sum_{\nu''} \frac{1}{2} f_{\alpha \alpha' \nu \nu'}(q, q') \Delta N_{\alpha''}(q') \Delta N_{\alpha''}(q'), \tag{64}
\]

where \( \Delta N_{\nu}(q) \), \( \Delta N_{\alpha \nu}(q) \), and \( \Delta N_{\alpha \nu'}(q) \) are the pseudoparticle bare-momentum distribution function deviations given in Eq. (63) and \( \omega_0 \) is an energy functional linear in the \(-1/2\) holon, \(-1/2\) spinon, and \( s1 \) pseudoparticle number deviations \( \Delta M_{\nu}, -1/2 \Delta M_{s}, -1/2 \Delta M_{s} \), \( \Delta N_{s1} \), respectively. (The latter functional is given below in Eq. (114) of Sec. V, with the \( s1 \) pseudofermion number deviation equaling the corresponding \( s1 \) pseudoparticle number deviation.) The coefficients of the \( i = 1 \) linear terms are the \( \alpha \nu \) pseudoparticle energy bands \( \epsilon_{\alpha}(q), \epsilon_{s1}(q), \) and \( \epsilon_{0}^0(\nu) \) studied in Refs. [24], [31]. The coefficients of the \( i = 2 \) quadratic terms are the pseudoparticle \( f \) functions \( f_{\alpha \alpha' \nu \nu'}(q, q') \) studied in Ref. [29].

In contrast to what the energy terms (63)-(64) may suggest, we emphasize that the energy (62) is not an expansion in \( 1/L \). The small parameters in such an expansion are the pseudoparticle bare-momentum distribution function deviations, as in the case of the quasiparticle Fermi-liquid energy functional [12]. Indeed, when these function deviations involve a small but finite density of pseudoparticles, the energy terms of order \( i \) on the right-hand side of Eq. (62) are not of order \( [1/L]^i \). Instead, in that case all terms of the energy expansion (62) are of the same order \( [1/L]^{-1} = L \). As discussed in the ensuing section, few-electron excitations are associated with excited states generated from the ground state by a finite number of pseudoparticle, holon, and spinon processes. The deviations of these states involve a vanishing density of pseudoparticles, holons, and spinons. However, in order to achieve the correct microscopic description of the few-electron spectral properties, one must consider the limit where a vanishing density of pseudoparticles is approached rather than considering that such a density is zero from the very beginning. As the density of pseudoparticles involved in the bare-momentum distribution deviations decreases and approaches zero, the energy terms on the right-hand side of Eq. (62) of large scattering order \( i \), also vanish and do not not contribute as they become of order \( [1/L]^i \).

The \( f \) functions on the right-hand side of Eq. (63), are associated with the two-pseudoparticle residual interactions and have the same role as those of Fermi-liquid theory [12], [13]. Indeed, for small values of the energy \( \omega \) and electronic densities \( n \) and spin densities \( m \) such that \( 0 < n < 1/a \) and \( 0 < m < n \), respectively, the few-electron spectral properties are controlled by the residual two-pseudoparticle interactions described by the \( i = 2 \) terms (64) of the energy functional (62). In the corresponding low-energy Hilbert subspace, only the \( c0 \) and \( s1 \) pseudoparticle branches have finite occupancies. In this case, as the limit of vanishing density of pseudoparticles is approached, the general energy functional (62) acquires the form of the energy spectrum of a two-component \( c0 \) and \( s1 \) conformal field theory [21]. For the relation of the \( c \equiv e, 0 \) and \( s \equiv s, 1 \) pseudoparticle description to two-component conformal-field theory see Refs. [22], [23]. The weight distribution of the few-electron spectral functions can in this case be studied by conformal field theory [24].

We thus conclude that the pseudoparticles have residual interactions associated with the terms of the general energy functional (62) of order \( i > 1 \). The pseudoparticle residual interactions control the low-energy few-electron spectral properties of the quantum liquid, as confirmed by the studies of Refs. [22], [23]. In this paper, we introduce an alternative representation in terms of non-interacting pseudofermions. For such pseudofermions the bare momentum is replaced by a momentum functional whose coefficients are two-pseudofermion phase shifts. In this case, the relevant energy spectrum is non-interacting and thus has only linear terms in the momentum distribution function deviations. For the pseudofermion description the few-electron spectral properties are controlled by such two-pseudofermion phase shifts [24]. Such shifts are associated with first-order momentum distribution function deviations only, whereas the
pseudoparticle residual interactions involve scattering orders \( i \) associated with terms of the energy functional such that \( i > 1 \).

C. THE CONCEPT OF J-CPHS ENSEMBLE SUBSPACE

Let us introduce the quantum number \( \iota = \text{sgn}(q) \pm 1 \) which refers to the number of right (\( \iota = +1 \)) and left (\( \iota = -1 \)) pseudoparticle movers. The number \( N_{\alpha \nu, \iota} \) of \( \alpha \nu \) pseudoparticles of \( \iota \) character is a good quantum number. We thus introduce the \( \alpha \nu \) pseudoparticle current number \( J_{\alpha \nu} \) such that,

\[
J_{\alpha \nu} = \frac{1}{2} \sum_{\iota = \pm 1} (\iota) N_{\alpha \nu, \iota}; \quad N_{\alpha \nu, \iota} = \frac{N_{\alpha \nu}}{2} + \iota J_{\alpha \nu}.
\]  

(65)

Each CPHS ensemble subspace contains in general several subspaces with different values for the sets of current numbers \( J_0 \) and \( \{ J_{\alpha \nu} \} \) such that \( \alpha = c, s \) and \( \nu = 1, 2, \ldots \). We call these subspaces \( J \)-CPHS ensemble subspaces. For a given \( J \)-CPHS ensemble subspace one can introduce the \( J \)-CPHS ground state. This energy eigenstate has compact bare-momentum occupancies for the \( \alpha \nu \) pseudoparticle bands. Moreover, a partial \( J \)-CPHS ground state is a state with such a type of occupancy configuration for the \( \alpha \nu \) pseudoparticle branches such that \( \alpha \nu \neq c0 \) and \( \alpha \nu \neq s1 \) only. These concepts are associated with the right (\( \iota = +1 \)) and left (\( \iota = -1 \)) Fermi points, which separate the \( \alpha \nu \) pseudoparticle occupied from the unoccupied regions of bare-momentum space. Ignoring terms of order \( 1/L \), these Fermi bare-momentum values read,

\[
q_{F \alpha \nu, \iota} = \iota q_{0F \alpha \nu} + \Delta q_{F \alpha \nu, \iota}.
\]  

(66)

The ground state is a particular case of \( J \)-CPHS ground state such that the Fermi points of Eq. (66) read,

\[
q_{0F \alpha \nu, \iota} = \iota q_{0F \alpha \nu}^0,
\]  

(67)

where

\[
\begin{align*}
q_{0F c0}^0 &= 2k_F; & q_{0F s1}^0 &= k_F \downarrow; & q_{0F c\nu}^0 &= 0; & q_{0F s\nu}^0 &= 0, & \nu > \delta_{\alpha, s}.
\end{align*}
\]  

(68)

In most situations one can disregard the \( 1/L \) corrections and use the bare-momentum Fermi values provided in Eqs. (66)-(68).

The pseudofermion description introduced in the ensuing section, corresponds to \( J \)-CPHS ensemble subspaces spanned by energy eigenstates differing from the ground-state by the occupancy configuration of a finite number of \( \alpha \nu \) pseudoparticles, \( -1/2 \) Yang holons, and \( -1/2 \) HL spinons. For such subspaces the pseudoparticle Fermi band momenta given in Eq. (66) can be written as follows,

\[
q_{F \alpha \nu, \iota} = \iota q_{0F \alpha \nu}^0 + \Delta q_{F \alpha \nu, \iota}
\]  

(69)

where \( q_{0F \alpha \nu}^0 \) is the \( \alpha \nu \) ground-state Fermi bare momentum whose values are given in Eq. (68) and thus,

\[
\begin{align*}
q_{F c0, \iota} &= \iota 2k_F + \Delta q_{F c0, \iota}; & q_{F s1, \iota} &= \iota k_F \downarrow + \Delta q_{F s1, \iota}; \\
q_{F c\nu, \iota} &= \Delta q_{F c\nu, \iota}, & \nu > 0; & q_{F s\nu, \iota} &= \Delta q_{F s\nu, \iota}, & \nu > 1.
\end{align*}
\]  

(70)

Here

\[
\Delta q_{F \alpha \nu, \iota} = \iota \frac{2\pi}{L} \Delta N_{\alpha \nu, \iota} = \iota \frac{2\pi}{L} \left[ \frac{\Delta N_{\alpha \nu}}{2} + \iota \Delta J_{\alpha \nu} \right].
\]  

(71)
V. THE PSEUDOFERMION DESCRIPTION

In this section we introduce the pseudofermion operational description. This includes introduction of the pseudoparticle - pseudofermion unitary transformation and of the Hilbert subspace associated with that transformation. In such a subspace, the rapidity functionals have a particular form. Moreover, we obtain the anticommutation relations of the pseudofermions.

A. THE FUNCTIONAL CHARACTER OF THE PSEUDOFERMION MOMENTUM

The $\alpha\nu$ pseudofermion carries momentum $\bar{q}_j$ given by,

$$\bar{q}_j = q(q_j) = q_j + \frac{Q_{\alpha\nu}(q_j)}{L} = \frac{2\pi}{L} I^\nu_j + \frac{Q_{\alpha\nu}(q_j)}{L}; \quad j = 1, 2, \ldots, N_{\alpha\nu}^*.$$  \hfill (72)

Here $Q_{\alpha\nu}(q_j)/L$ is the momentum functional,

$$\frac{Q_{\alpha\nu}(q_j)}{L} = \frac{2\pi}{L} \sum_{\alpha'=c,s} \sum_{\nu'=\delta,\gamma} \sum_{j'=-1}^{N^*_{\alpha\nu'}} \Phi_{\alpha\nu,\alpha'\nu'}(q_j, q_{j'}) \Delta N_{\alpha'\nu'}(q_{j'}).$$  \hfill (73)

The functions $\Phi_{\alpha\nu,\alpha'\nu'}(q, q')$ on the right-hand side of this equation are the two-pseudofermion phase shifts expressed in terms of the bare-momentum $q$. These phase shifts are given by,

$$\Phi_{\alpha\nu,\alpha'\nu'}(q, q') = \Phi_{\alpha\nu,\alpha'\nu'}(\left(\frac{4t \Lambda^0_{\alpha\nu}(q)}{U}, \frac{4t \Lambda^0_{\alpha'\nu'}(q')}{U}\right),$$  \hfill (74)

where $\Phi_{\alpha\nu,\alpha'\nu'}(r, r')$ are the two-pseudofermion phase shifts expressed in terms of the variable $r$. The latter phase shifts are the unique solutions of the integral equations (B2)-(B15) of Appendix B. The value of the momentum functional given in Eq. (72) is controlled by shake-up two-pseudofermion phase-shift processes resulting from the changes in the quantum-object occupancy configurations described by the deviations $\Delta N_{\alpha'\nu'}(q_{j'})$ on the right-hand side of that equation.

The $\alpha\nu$ pseudofermion is related to the corresponding $\alpha\nu$ pseudoparticle by a mere unitary transformation involving the discrete bare-momentum values $q_j$ and such that,

$$q_j \to \bar{q}_j,$$  \hfill (75)

where $\bar{q}_j$ is the discrete momentum defined in Eq. (72). The number of $\alpha\nu$ pseudoparticles, $N_{\alpha\nu}$, equals that of $\alpha\nu$ pseudofermions, which we denote by $N_{\alpha\nu}^*$. Moreover, we introduce the $\alpha\nu$ pseudofermion momentum distribution function $\mathcal{N}_{\alpha\nu}(q_j)$ such that $\mathcal{N}_{\alpha\nu}(\bar{q}_j) = N_{\alpha\nu}(q_j, q_j)$, where $q_j = q_j(\bar{q}_j)$ is the inverse function of (72).

Note that the momentum functional (72) is such that $Q_{\alpha,\nu}(q)/L = 0$ for the initial ground state. Indeed, the pseudofermion description refers to the ground-state normal ordered 1D Hubbard model. Thus, there is a specific $\alpha\nu$ pseudofermion description for each initial ground state. For the latter state the bare momentum $q_j = [2\pi/L] I^\nu_j$ equals the pseudofermion momentum $\bar{q}_j = q_j + Q_{\alpha\nu}(q_j)/L$. This justifies the designation bare momentum for $\bar{q}_j$.

Thus, in the case of the ground state the $\alpha\nu$ pseudoparticles are identical to the $\alpha\nu$ pseudofermions. It follows that the ground state is invariant under the pseudoparticle - pseudofermion unitary transformation and plays the role of the vacuum of the pseudofermion theory.

As for the case of the pseudoparticle representation, the pseudofermion description corresponds to large values of the Hubbard chain length $L$ and is thus compatible with Takahashi’s string hypothesis [20, 31]. However, while the pseudoparticle representation corresponds to the whole Hilbert space, the pseudofermion description refers to a Hilbert subspace spanned by the initial ground state and the following types of states:

(A) - Excited states generated from the initial ground state by a finite number of pseudofermion processes involving creation or annihilation of $c0$ and $s1$ pseudofermions, creation of pseudofermions belonging to other $\alpha\nu \neq c0$ and $\alpha\nu \neq s1$ branches, and creation of $-1/2$ Yang holons and $-1/2$ HL spinons. In the thermodynamic limit, the
momentum and bare-momentum distribution function deviations associated with such excited states obey the following relations,

\[
\sum_{j=1}^{N_{\nu}} \Delta N_{\alpha\nu}(\bar{q}_j)/L = \sum_{j=1}^{N_{\nu}} \Delta N_{\alpha\nu}(\bar{q}_j)/L = \Delta N_{\alpha\nu}/L = 0; \quad \nu \geq \delta_{\alpha,s}, \quad \alpha = c,s;
\]

\[
\Delta L_{\alpha,s,1/2}/L = L_{\alpha,s,1/2}/L \to 0; \quad \alpha = c,s.
\]

(B) - Excited states generated from the initial ground state by a finite number of \(c0\) and \(s1\) pseudofermion particle-hole processes.

Throughout this paper, we call such states, excited states A and excited states B, respectively. We emphasize that in general the excited states involve both. However, for few-electron excitations which do not change the electronic numbers the low-energy states are of type B only. Moreover, for simplicity we are considering densities in the ranges \(0 < n < 1/\alpha\) and \(0 < m < n\). Our analysis also holds for other values of the densities, yet for the half-filling \(n = 1/\alpha\) or zero-magnetization \(m = 0\) phases the excitation subspace is more reduced.

The unitary transformation associated with Eq. (74), is well defined for the Hilbert subspace spanned by the initial ground state and the excited states A and B. Fortunately, in the thermodynamic limit application onto the ground state of a few-electron operator generates excited states A and B only. By few-electron operators, we mean here operators which can be written as a product of \(N\) electron creation and/or annihilation operators and \(N/N_a \to 0\) as \(N_a \to \infty\). The pseudofermion functional theory introduced in this paper, can be applied to the study of the few-electron spectral weight distributions for all values of energy \([1, 10, 23]\). We find below that the pseudofermions have a non-interacting character. This allows the evaluation of few-electron spectral functions for all values of energy \([10, 23]\). The \(\alpha\nu\) pseudoparticle residual interactions are cancelled by the momentum transfer term \(Q_{\alpha,\nu}(q)/L\) of Eq. (76). The information recorded in the pseudoparticle interactions is transferred over to the momentum two-pseudofermion phase shifts of the momentum term \(Q_{\alpha\nu}(q)/L\). The pseudofermion theory is of first order in \(1/L\) both for the energy and momentum spectra. In contrast, the methods that use the pseudoparticle residual interactions lead to expressions for these spectral functions which are valid for small values of the energy \(\omega\) only \([22, 23, 22]\). Therefore, the pseudofermion description introduced below corresponds to a breakthrough in what the study of the finite-energy spectral properties of the 1D Hubbard model is concerned.

That only excited states A and B have finite overlap with few-electron excitations, is confirmed by direct evaluation of matrix elements between few-electron excitations and excited states \([34]\). For instance, if \(Q_{\alpha\nu}(q)/L = 0\) for the excited states B, the contribution to the one-electron spectral function would come from one particle-hole pseudofermion processes only, and the spectral functions would have the familiar \(\delta\)-function structure. This is not true for our case where \(Q_{\alpha\nu}(q)/L \neq 0\) for these states. In this case, there are contributions from many particle-hole processes in the \(c0\) and \(s1\) pseudofermion bands as well. The largest weight comes from the one-particle-hole pseudofermion processes, and increasing the number of particle-hole processes, the additional weight decreases rapidly. Most of the weight associated with the excited states B is generated by one, two, and three particle-hole pseudofermion processes \([34]\). In the general case of few-electron spectral functions, the contribution from excited states B involving an infinite number of particle-hole processes in the \(c0\) and \(s1\) pseudofermion bands vanishes in the thermodynamic limit. The same holds for the excited states A. The excited states generated from the ground state by an infinite number of processes of the types reported in (A) and (B) have vanishing overlap with few-electron excitations as \(L \to \infty\).

The cancellation of the \(\alpha\nu\) pseudoparticle residual interactions by the momentum transfer term \(Q_{\alpha,\nu}(q)/L\) of Eq. (76) is related to the form of the rapidity functionals \(\Lambda_{\alpha\nu}(q)\) and rapidity-momentum functional \(k(q)\) in the Hilbert subspace spanned by the excited states A and B. Introduction of the pseudoparticle bare-momentum distribution functions of general form given in Eq. (75) in the rapidity functional integral equations (19)-(21) and their expansion in the small deviations \([55]\), permits explicit solution of these equations. This procedure leads to expressions for the rapidity functionals \(\Lambda_{\alpha\nu}(q)\) and rapidity-momentum functional \(k(q)\) in terms of the bare-momentum distribution function deviations introduced in Eq. (76). Solution of the integral equations (19)-(21) for distributions of the general form \([70]\) leads to first-order in the deviations to expressions for the rapidity-momentum functional and rapidity functionals of the following form,

\[
k(q) = k^0(\bar{q}(q)); \quad \Lambda_{\alpha\nu}(q) = \Lambda_{\alpha\nu}^0(\bar{q}(q)); \quad \alpha = c, \quad \nu = 0, 1, 2, \ldots; \quad \alpha = s, \quad \nu = 1, 2, \ldots
\]
Here $\Lambda_{\alpha\nu}(q)$ and $k^0(q)$ are the corresponding ground state functions. On the right-hand side of Eq. (72), $\bar{q}(q)$ is the "momentum functional" given in Eq. (2) with $q_j$ replaced by the continuum momentum $q$.

It is remarkable that in the Hilbert subspace spanned by the excited states $A$ and $B$ the functionals $\Lambda_{\alpha\nu}(q)$ and $k(q)$ equal the corresponding ground-state functions $\Lambda_{\alpha\nu}(q)$ and $k^0(q)$, respectively, with the bare momentum $q$ replaced by the momentum functional (2). This property is behind the non-interacting character of the pseudofermions. The two-pseudofermion phase shifts $\Phi_{\alpha\nu,\alpha'\nu'}(r, r')$ defined by Eqs. (32)-(34) of Appendix B which appear in the expression of the momentum functional (23) through the relation (24) play a central role in the pseudofermion description of the few-electron spectral properties. Within that description the expression of many physical quantities involve such two-pseudofermion phase shifts. For instance, the functions defined by Eqs. (37) and the expressions of the pseudofermion energy bands given below involve the two-pseudofermion phase shifts. Furthermore, these phase shifts control the matrix elements between few-electron excitations and the excited states and thus also the few-electron spectral weight distributions (34).

According to Eq. (33) the energy spectrum depends on the quantum object occupancy configurations through the rapidity and rapidity-momentum functionals. It is this functional character that is behind the pseudoparticle residual interactions and leads to the energy expansion given in Eqs. (31)-(33). However, by re-expression of these functional in terms of the pseudofermion momentum $\bar{q}$, the general energy spectrum (31)-(33) can be written in terms of pseudofermion momentum distribution functions $N_{\alpha\nu}(\bar{q}_j)$ and $N_{\alpha\nu'}(\bar{q}_j)$ as,

$$
E = -2\sum_{j=1}^{N_a} N_{\alpha\nu}(\bar{q}_j) \cos k^0(\bar{q}_j) + 4t \sum_{\nu=1}^{\infty} \sum_{j=1}^{N_{\alpha\nu}} N_{\alpha\nu'}(\bar{q}_j) \text{Re} \left\{ \sqrt{1 - (\Lambda_{\alpha\nu}(\bar{q}_j) + i\nu U/4t)^2} \right\}
$$

$$
+ \frac{U}{2} \left[ M - \sum_{\nu=1}^{\infty} 2\nu N_{\alpha\nu} - \frac{N_a}{2} \right] + \sum_{\alpha} \mu_\alpha S_\alpha^\alpha.
$$

(78)

This expression is valid for the Hilbert subspace spanned by the initial ground state and the excited states $A$ and $B$. The term $\sum_{\alpha} \mu_\alpha S_\alpha^\alpha$ is the same as on the right-hand side of Eq. (31) and $N_{\alpha\nu} = N_{\alpha\nu'}$ is the number of $\alpha\nu$ pseudofermions. The form of the general energy spectrum (78) justifies why the shake-up effects associated with the phase shifts of the functional (23) occur in the case of the pseudofermions in the momentum instead of in the energy. The dependence of the general energy spectrum (78) on the momentum occupancy configuration values occurs through the arguments of the ground-state rapidity and rapidity-momentum functions. Thus these functions play the role of non-interacting spectra, since they have the same form both for the initial ground state and excited states $A$ and $B$. The shake-up effects associated with the two-pseudofermion phase shifts are thus felt by the non-interacting pseudofermions as mere changes in the momentum occupancies through the twisted boundary conditions imposed by the elementary processes which generate the excited state from the ground state. This property is behind the non-interacting character of the pseudofermions. It is also behind the fact that within the pseudofermion description the functions $\bar{q}_j(k)$ and $\bar{q}_{\alpha\nu}(\Lambda)$ defined in Eqs. (37) and (39) refer to both the ground state and the excited states. In contrast, within the pseudoparticle representation these functions refer to the ground state only.

The pseudoparticle bare-momentum $q_j$ description is naturally provided by the Bethe-ansatz equations within Takahashi’s string hypothesis (26, 28, 31). We recall that the pseudoparticle discrete bare-momentum values $q_j$ are of form given in Eq. (A2) of Appendix A and according to Eq. (A1) are such that $q_{j+1} - q_j = 2\pi/L_j$. The single discrete bare-momentum values $q_j$ are integer multiples of $2\pi/L$ or of $\pi/L$ (31) and bare-momentum contributions of order $[1/L]^j$ such that $j > 1$ are outside the validity of the pseudofermion description. These bare-momentum contributions have no physical meaning and must be considered as zero. Importantly, the same is required for the pseudofermion momentum discrete values $\bar{q}_j$ given in Eq. (23). These discrete values are also at least of order of $1/L$ and contributions of order $[1/L]^j$ such that $j > 1$ must be considered as equaling zero. For instance, as confirmed in Appendix C, the discrete momentum level separation,

$$
\bar{q}_{j+1} - \bar{q}_j = \frac{2\pi}{L} + \frac{Q_{\alpha\nu}(q_{j+1}) - Q_{\alpha\nu}(q_j)}{L} \approx \frac{2\pi}{L},
$$

(79)

is such that the second term on the right-hand side of Eq. (79) is of order $[1/L]^2$, where $Q_{\alpha\nu}(q)/L$ is the momentum functional given in Eq. (23). Thus, to first order in $1/L$ one finds that $\bar{q}_{j+1} - \bar{q}_j = 2\pi/L$, as for the corresponding discrete bare-momentum level separation given in Eq. (A1) of Appendix A. However, this does not imply that to first order in $1/L$ the pseudofermion momentum equals the bare-momentum. Indeed, we emphasize that the momenta $Q_{\alpha\nu}(q_j)/L$ on the right-hand side of Eq. (79) are of order $1/L$ and play a central role in the control of the few-electron spectral weight distribution by the non-perturbative many-electron shake-up effects (34). We note that the level separation $\bar{q}_{j+1} - \bar{q}_j = 2\pi/L$ is valid locally in the discrete momentum space. By that we mean the following: If
in the present thermodynamic limit two momentum values \( \bar{q}_j \) and \( \bar{q}_{j'} \) differ by a small yet finite momentum difference \( \Delta \bar{q} = \bar{q}_j - \bar{q}_{j'} \), then in general \( \Delta \bar{q} \neq \frac{2\pi}{L} [j - j'] \). In contrast, for the corresponding bare-momentum values it holds that \( \Delta q = \frac{2\pi}{L} [j - j'] \). Therefore, for small but non-vanishing momentum separation the difference \( [Q_{\alpha\nu}(q_j) - Q_{\alpha\nu}(q_{j'})]/L \) is not anymore of order \( [1/L]^2 \) and has physical significance.

**B. PSEUDOFOERMIAN OPERATOR ALGEBRA**

The elementary creation and annihilation operators of the \( \alpha\nu \) pseudoparticles can be expressed in terms of the corresponding operators of the \( \alpha\nu \) pseudoparticles as follows,

\[
\hat{f}^\dagger_{q_j, \alpha\nu} = \hat{V}_{\alpha\nu}^\dagger q_j, \alpha\nu \hat{V}_{\alpha\nu}; \quad \hat{f}_{q_j, \alpha\nu} = \hat{V}_{\alpha\nu} q_j, \alpha\nu \hat{V}_{\alpha\nu}.
\]  

(80)

Here \( \hat{V}_{\alpha\nu} \) is a unitary operator that we call the \( \alpha\nu \) pseudoparticle \(-\alpha\nu \) pseudofermion unitary operator. This operator acts onto the Hilbert subspace spanned by the initial ground state and excited states A and B. It shifts the bare-momentum value \( q_j \) by \( Q_{\alpha\nu}(q_j)/L \) and thus is given by,

\[
\hat{V}_{\alpha\nu}^\dagger = \exp\left(\sum_{j=1}^{N^*_{\alpha\nu}} b_j^\dagger Q_{\alpha\nu}(q_j)/L, \alpha\nu b_j, \alpha\nu\right); \quad \hat{V}_{\alpha\nu} = \exp\left(\sum_{j=1}^{N^*_{\alpha\nu}} b_j - Q_{\alpha\nu}(q_j)/L, \alpha\nu b_j, \alpha\nu\right).
\]  

(81)

The momentum distribution function \( \hat{N}_{\alpha\nu}(\bar{q}_j) \) is the eigenvalue of the operator,

\[
\hat{N}_{\alpha\nu}(\bar{q}_j) = f^\dagger_{q_j, \alpha\nu} f_{q_j, \alpha\nu}; \quad \hat{N}_{\alpha\nu}(\bar{q}_j) = N_{\alpha\nu}(q_j(\bar{q}_j)).
\]  

(82)

Here \( N_{\alpha\nu}(q_j) \) stands for the \( \alpha\nu \) pseudoparticle bare-momentum distribution function and \( q_j(\bar{q}_j) \) is the inverse of the function \( (85) \). The function \( N_{\alpha\nu}(q_j) \) is the eigenvalue of the corresponding bare-momentum number operator \( (18) \). Keeping only the physical contributions that correspond to terms up to first order in \( 1/L \), the function \( q_j(\bar{q}_j) \) is given by,

\[
q_j = q_j(\bar{q}_j) = \bar{q}_j - \frac{Q_{\alpha\nu}(q_j)/L}{L} = \bar{q}_j - \frac{2\pi}{L} \sum_{\alpha' = \alpha, \nu} \sum_{\nu' = \alpha', \nu} \sum_{j = 1}^{N^*_{\alpha'\nu'}} \phi_{\alpha\nu, \alpha'\nu'}(\bar{q}_j, \bar{q}_{j'}) \Delta N_{\alpha'\nu'}(\bar{q}_{j'}). \quad \Delta N_{\alpha'\nu'}(\bar{q}_{j'})
\]  

(83)

Here, \( \Delta N_{\alpha'\nu'}(\bar{q}_{j'}) \) is the deviation of the function \( (82) \) relative to its ground-state value. Since the functional \( (\bar{q}_j, q_j \) vanishes for the ground state, note that \( q_j = \bar{q}_j \) for that state. Moreover, on the right-hand side of Eq. \( (83) \), \( \phi_{\alpha\nu, \alpha'\nu'}(\bar{q}, \bar{q}') \) is the two-pseudoparticle phase shift. It is defined as,

\[
\phi_{\alpha\nu, \alpha'\nu'}(\bar{q}, \bar{q}') = \Phi_{\alpha\nu, \alpha'\nu'}(q(\bar{q}), q(q')) = \bar{\Phi}_{\alpha\nu, \alpha'\nu'} \left( \frac{4t \Lambda^0_{\alpha\nu}(q(\bar{q})), \frac{4t \Lambda^0_{\alpha\nu}(q(q'))}{U}, \frac{4t \Lambda^0_{\alpha\nu}(q(q'))}{U}, \right)
\]  

(84)

where the function \( q = q(\bar{q}) \) is the continuum version of the function \( (82) \). \( \Phi_{\alpha\nu, \alpha'\nu'}(q, q') \) is given in Eq. \( (84) \), and \( \Phi_{\alpha\nu, \alpha'\nu'}(r, r') \) is the two-pseudoparticle phase shift expressed in the variable \( r \) defined by the integral equations \( (85) \) of Appendix B.

Often in the expressions given in previous sections we replaced the pseudoparticle bare-momentum summations by integrals and the corresponding discrete bare-momentum values \( q_j \) by a continuum bare-momentum variable \( q \). Since according to Eq. \( (86) \) of Appendix A, the difference \( q_{j+1} - q_j = 2\pi/L \) is constant for all values of \( j \), the use of such a continuum representation involves the replacement,

\[
\sum_{j=1}^{N^*_{\alpha\nu}} \equiv \sum_{q} \sum_{-q_{\alpha\nu}} \rightarrow \frac{L}{2\pi} \int_{-q_{\alpha\nu}}^{+q_{\alpha\nu}} dq,
\]  

(85)

In the Hilbert subspace spanned by the excited states A and B, the rapidity functional \( \Lambda_{\alpha\nu}(q) \) and rapidity-momentum functional \( k(q) \) equal the corresponding ground-state rapidity function \( \Lambda^0_{\alpha\nu}(q) \) and rapidity-momentum functional \( k(\bar{q}) \).
function $k^0(q)$, respectively, with the bare-momentum $q$ replaced by the momentum $\tilde{q}$. It follows that in such a subspace the limiting values of the continuum momentum $q$ are given by the ground-state limiting values $\pm q^0_{\alpha\nu}$, given in Eqs. (83) and (84). (This is confirmed in Sec. VII.) Thus, to replace the discrete momentum values by a continuum momentum variable $\tilde{q}$, one must replace the summations by the following integrals,

$$\sum_{j=1}^{N^*_{\alpha\nu}} \equiv \sum_{\tilde{q}=-q^0_{\alpha\nu}}^{+q^0_{\alpha\nu}} \rightarrow \frac{L}{2\pi} \int_{-q^0_{\alpha\nu}}^{+q^0_{\alpha\nu}} dq \frac{dq(\tilde{q})}{d\tilde{q}}. \quad (86)$$

We then introduce the momentum distribution function,

$$\mathcal{N}_{\alpha\nu}(\tilde{q}) = \frac{dq(\tilde{q})}{d\tilde{q}} \mathcal{N}_{\alpha\nu}(\tilde{q}). \quad (87)$$

The function $q = q(\tilde{q})$ on the right-hand side of Eq. (87) is the inverse of the function given in Eq. (86) and is given in Eq. (86) with $\tilde{q}_j$ replaced by $\tilde{q}$. Its derivative is given by,

$$\frac{dq(\tilde{q})}{d\tilde{q}} = 1 - \sum_{\alpha'=c,s} \left[ \sum_{q=-q^0_{\alpha\nu}}^{+q^0_{\alpha\nu}} \int_{-q^0_{\alpha\nu}}^{+q^0_{\alpha\nu}} dq' \frac{d\Phi_{\alpha\nu,\alpha'\nu'}(\tilde{q}, q')}{dq'} \mathcal{N}_{\alpha'\nu'}(q') \right]. \quad (88)$$

The second term on the right-hand side of this equation is of first order in the momentum distribution function deviations. For the non-interacting pseudofermion description only momentum and energy contributions up to first order in these deviations are physical. As a result, in the case of momentum distribution function deviations $\Delta \mathcal{N}_{\alpha\nu}(\tilde{q})$ one can consider that,

$$\Delta \mathcal{N}_{\alpha\nu}(\tilde{q}) = \Delta \mathcal{N}_{\alpha\nu}(\tilde{q}), \quad (89)$$

where in contrast to the case of Eq. (87) we used $dq(\tilde{q})/d\tilde{q} = 1$.

The $\alpha\nu$ pseudoparticle number operator,

$$\hat{N}_{\alpha\nu} = \sum_{j=1}^{N^*_{\alpha\nu}} b^\dagger_{q_j,\alpha\nu} b_{q_j,\alpha\nu} = \sum_{q=-q^0_{\alpha\nu}}^{+q^0_{\alpha\nu}} b^\dagger_{q,\alpha\nu} b_{q,\alpha\nu} = \frac{L}{2\pi} \int_{-q^0_{\alpha\nu}}^{+q^0_{\alpha\nu}} dq b^\dagger_{q,\alpha\nu} b_{q,\alpha\nu}, \quad (90)$$

is invariant under the pseudoparticle - pseudofermion transformation. It equals the number of $\alpha\nu$ pseudofermions operator,

$$\hat{N}_{\alpha\nu} = \sum_{j=1}^{N^*_{\alpha\nu}} f^\dagger_{\tilde{q}_j,\alpha\nu} f_{\tilde{q}_j,\alpha\nu} = \sum_{\tilde{q}=-q^0_{\alpha\nu}}^{+q^0_{\alpha\nu}} f^\dagger_{\tilde{q},\alpha\nu} f_{\tilde{q},\alpha\nu} = \frac{L}{2\pi} \int_{-q^0_{\alpha\nu}}^{+q^0_{\alpha\nu}} dq \frac{dq(\tilde{q})}{d\tilde{q}} f^\dagger_{\tilde{q},\alpha\nu} f_{\tilde{q},\alpha\nu}. \quad (91)$$

The $c\nu$ pseudoparticle charge and $s_c$ and $\sigma_c$ values considered in Sec. III are also invariant under the same transformation. The same occurs for the $sv$ pseudoparticle $s_s$ and $\sigma_s$ values. The pseudoparticle - pseudofermion transformation also leaves invariant the $\pm 1/2$ Yang holons and $\pm 1/2$ HL spinons. The $\pm 1/2$ holon (and $\pm 1/2$ spinon) composite character of the $c\nu$ pseudoparticles (and $sv$ pseudoparticles) also remains invariant under that transformation. It follows that the $c\nu$ pseudofermions (and $sv$ pseudofermions) are $s_c = 0$ (and $s_s = 0$) composite objects of an equal number of $\nu = 1, 2, \ldots$ of $-1/2$ holons and $+1/2$ holons (and $-1/2$ spinons and $+1/2$ spinons). Consistently, the $\pm 1/2$ holon ($\alpha = c$) and $\pm 1/2$ spinon ($\alpha = s$) number operators $\hat{M}_{\alpha,\pm 1/2}$ given in Eq. (92) can be rewritten in terms of pseudofermion operators as follows,

$$\hat{M}_{\alpha,\pm 1/2} = \hat{L}_{\alpha,\pm 1/2} + \sum_{\nu=1}^{\infty} \sum_{q=-q^0_{\alpha\nu}}^{+q^0_{\alpha\nu}} \nu \hat{N}_{\alpha\nu}(\tilde{q}), \quad (92)$$
where the pseudofermion momentum distribution operators $\hat{N}_{\alpha\nu}(\bar{q})$ are given in Eq. (32). Like in Eq. (32), the operator $\hat{L}_{\alpha, \pm 1/2}$ on the right-hand side of Eq. (32) is the $\pm 1/2$ Yang holon ($\alpha = c$) and $\pm 1/2$ HL spinon ($\alpha = s$) number operator given in Eq. (33). All results reported in Sec. III concerning pseudoparticle charge and spin transport are also valid for the corresponding pseudofermions. For instance, for finite values of $U/t$ the transport of charge (and spin) is associated with the $c\rho$ pseudofermion and $\nu\rho$ pseudofermion quantum charge fluids (and $s\nu$ pseudofermion quantum spin fluids).

It is found in Ref. [33] and discussed in Appendix A that the bare-momentum $q$ is the conjugate of the spatial coordinate $x_{j} = a_{\alpha\nu, j}$ associated with the effective $\nu\rho$ lattice, where $j = 1, 2, \ldots, N_{\alpha\nu}$. As in the case of the charge (or spin) carried by the pseudoparticles and of their composite character in terms of chargeons and antichargeons, $\pm 1/2$ holons, or $\pm 1/2$ spinons, also the effective $\nu\rho$ momentum remains invariant under the $\nu\rho$ pseudofermion unitary transformation. Indeed, the momentum functional $Q_{\alpha\nu}(q)/L$ which controls the pseudoparticle - pseudofermion transformation (34) does not affect the underlying effective $\nu\rho$ lattice. That momentum functional just imposes a twisted boundary condition such that each $\alpha\nu$ pseudofermion hopping from site $N_{\alpha\nu} - 1$ to site 0 of such an effective lattice will acquire a phase $e^{iQ_{\alpha\nu}(q)}$. From combination of that analysis with the expression for the general momentum functional $Q_{\alpha\nu}(q)/L$ given in Eq. (35), we find that the phase shift $\Phi^{\nu\nu', \alpha\nu'}(\bar{q}, \bar{q}')$ is such that creation into the system of a $\alpha'\nu'$ pseudofermion of momentum $\bar{q}'$ imposes a twisted boundary condition on the wave function of the $\nu\rho$ pseudofermion branch. This condition is such that when the $\alpha\nu$ pseudofermion of momentum $\bar{q}$ hops from site $N_{\alpha\nu} - 1$ to site 0 of the effective $\nu\rho$ lattice, it will acquire a phase $e^{i2\pi Q_{\alpha\nu}(\bar{q})/L}$. The studies of Ref. [34] reveal that within the pseudofermion description the few-electron spectral properties are fully controlled by such two-pseudofermion twisted boundary conditions associated with the phase shifts (34).

As for the case of the pseudoparticles, it is useful to introduce the local $\alpha\nu$ pseudofermion creation operator $f_{\alpha\nu}^{\dagger}$ and annihilation operator $f_{\alpha\nu}$. These operators are related to the operators $f_{\alpha\nu}^{\dagger}$ and $f_{\nu\rho}$, respectively, obtained from the corresponding pseudoparticle operators through the relations given in Eq. (36), as follows,

$$f_{\bar{q}, \alpha\nu}^{\dagger} = \frac{1}{\sqrt{L}} \sum_{j=1}^{N_{\alpha\nu}} e^{iqx_{j}} f_{x_{j}, \alpha\nu}^{\dagger}; \quad f_{\bar{q}, \alpha\nu} = \frac{1}{\sqrt{L}} \sum_{j=1}^{N_{\alpha\nu}} e^{-iqx_{j}} f_{x_{j}, \alpha\nu}, \quad (93)$$

where the summations refer to the sites of the effective $\nu\rho$ lattice. The local $\alpha\nu$ pseudofermion creation (and annihilation) operator $f_{x_{j}, \alpha\nu}^{\dagger}$ (and $f_{x_{j}, \alpha\nu}$) creates (and annihilates) a $\nu\rho$ pseudofermion at the effective $\nu\rho$ lattice site of spatial coordinate $x_{j} = a_{\alpha\nu, j}$, where $j = 1, 2, \ldots, N_{\alpha\nu}$ and $a_{\alpha\nu}$ is the effective $\nu\rho$ lattice constant given in Eq. (37). (Since the pseudofermion representation refers to the Hilbert subspace spanned by the initial ground state and the excited states A and B, except for $1/L$ corrections we can consider that the corresponding effective $\alpha\nu$ lattice constants are the ground-state constants $a_{\alpha\nu}^{0}$.) Thus, the conjugate variable of the momentum $\bar{q}$ of the $\nu\rho$ pseudofermion branch is the space coordinate $x_{j}$ of the corresponding effective $\nu\rho$ lattice. The local $\alpha\nu$ pseudoparticles and corresponding local $\alpha\nu$ pseudofermions have the same effective $\nu\rho$ lattice. It follows that the local pseudoparticle and local pseudofermion site distribution configurations which describe the ground state and the excited states A and B are the same. These configurations are expressed in terms of rotated-electron site distribution configurations in Ref. [35].

While the local $\alpha\nu$ pseudoparticles and corresponding local $\alpha\nu$ pseudofermions live in the same effective $\nu\rho$ lattice, the values of the set of discrete bare-momentum values $\{\bar{q}_{j}\}$ and momentum values $\{q_{j}\}$ such that $j = 1, 2, \ldots, N_{\alpha\nu}$ are different and related by Eq. (38). There is an one-to-one relation between these two sets of discrete values, which keep the same order because there is no level crossing. This property follows from the discrete bare-momentum and momentum separation given in Eq. (A11) of Appendix A and Eq. (79), respectively.

Finally, we consider the anticommutation relations of the pseudofermion operators. It is confirmed in Ref. [32] that such relations play a major role in the evaluation of matrix elements between energy eigenstates. Let us consider the general situation when the momenta $\bar{q}$ and $\bar{q}'$ of the operators $f_{\bar{q}, \alpha\nu}^{\dagger}$ and $f_{\bar{q}', \alpha\nu'}$, respectively, correspond to different J-CPHS ensemble subspaces. The anticommutator $\{f_{\bar{q}, \alpha\nu}^{\dagger}, f_{\bar{q}', \alpha\nu'}\}$ can be expressed in terms of the local-pseudofermion anticommutators $\{f_{x_{j}, \alpha\nu}^{\dagger}, f_{x_{j'}, \alpha\nu'}\}$ associated with spatial coordinates $x_{j}$ and $x_{j'}$ of the effective $\alpha\nu$ and $\alpha'\nu'$ lattices, respectively, as follows,

$$\{f_{\bar{q}, \alpha\nu}^{\dagger}, f_{\bar{q}', \alpha\nu'}\} = \frac{1}{L} \sum_{j=1}^{N_{\alpha\nu}} \sum_{j'=1}^{N_{\alpha'\nu'}} e^{i(qx_{j} - q'x_{j'})} \{f_{x_{j}, \alpha\nu}^{\dagger}, f_{x_{j'}, \alpha\nu'}\}. \quad (94)$$
After performing the $j$ and $j'$ summations we find that the $\alpha \nu$ pseudofermion operators obey the following algebra,

$$\{ f_{q, \alpha \nu}^\dagger, f_{q', \alpha' \nu'} \} = \delta_{\alpha, \alpha'} \delta_{\nu, \nu'} \frac{1}{L} e^{-i(\bar{q} - \bar{q}') a/2} e^{i(Q_{\alpha \nu}(q) - Q'_{\alpha' \nu'}(q'))/2} \sin \left( \frac{|Q_{\alpha \nu}(q) - Q'_{\alpha' \nu'}(q')|}{a/2} \right),$$

(95)

and the anticommutators between two $\alpha \nu$ pseudofermion creation or annihilation operators vanish. On the right-hand side of Eq. (95) the functionals $Q_{\alpha \nu}(q)$ and $Q'_{\alpha' \nu'}(q')$ whose general expression is given in Eq. (93) refer to the J-CHPES ensemble subspaces where the momenta $\bar{q}$ and $\bar{q}'$ refer to, respectively. Note that when $\sin(|Q_{\alpha \nu}(q) - Q'_{\alpha' \nu'}(q')|/2)$ vanishes the anticommutation relation (95) is the usual one, $\{ f_{q, \alpha \nu}^\dagger, f_{q', \alpha' \nu'} \} = \delta_{\alpha, \alpha'} \delta_{\nu, \nu'} \delta_{\bar{q}, \bar{q}'}$. In contrast, for finite values of that quantity there is an overall two-pseudofermion phase shift which arises from a non-perturbative shake-up effect associated with the functional character of the momentum $Q_{\alpha \nu}(q)/L$ given in Eq. (93).

A case of particular importance is when one of the two J-CPHES ensemble subspaces associated with the momenta $\bar{q}$ and $\bar{q}'$ is that of the initial ground state. In such a case $Q'_{\alpha' \nu'}(q') = 0$ for the ground-state J-CPHES ensemble subspace and thus the anticommutation relation (95) simplifies to,

$$\{ f_{q, \alpha \nu}^\dagger, f_{q', \alpha' \nu'} \} = \delta_{\alpha, \alpha'} \delta_{\nu, \nu'} \frac{1}{L} e^{-i(\bar{q} - \bar{q}') a/2} e^{iQ_{\alpha \nu}(q)/2} \sin \left( \frac{Q_{\alpha \nu}(q)/2}{(q - q') a/2} \right).$$

(96)

VI. THE PSEUDOFERMION ENERGY AND MOMENTUM SPECTRA AND THE WAVE-FUNCTION FACTORIZATION OF THE NORMAL-ORDERED 1D HUBBARD MODEL

In this section, we find that the description of the quantum problem in terms of the non-interacting pseudofermions leads in the thermodynamic limit to a wave-function factorization for the excited states A and B contained in the excitations generated by application onto the ground state of few-electron operators. Such a factorization refers to the ground-state normal-ordered 1D Hubbard model.

A. THE PSEUDOFERMION ENERGY AND MOMENTUM SPECTRA

Provided that all scattering orders are considered, the pseudoparticle energy functional (62) describes excitations which involve a small, but finite density of pseudoparticles. However, in the thermodynamic limit, the few-electron excitations are contained in a smaller Hilbert subspace. This subspace is spanned by the excited states A and B. These states are generated from the ground state by processes which involve a finite number of pseudofermions (and pseudoparticles). In the thermodynamic limit, this corresponds to a vanishing density of these objects rather than to a small but finite density. In the Hilbert subspace spanned by the ground state and the excited states A and B the energy spectrum is of the form given in Eq. (78). This spectrum can be expressed in terms of continuum momentum integrals with the result,

$$E = -2t \frac{L}{2\pi} \int_{-q_{\alpha \alpha_0}^c}^{q_{\alpha \alpha_0}^c} dq N_{c0}(\bar{q}) \cos k(\bar{q}) + 4t \frac{L}{2\pi} \sum_{\nu=1}^{\infty} \int_{-q_{\alpha \nu}^c}^{q_{\alpha \nu}^c} dq N_{c\nu}(\bar{q}) \Re \left\{ \sqrt{1 - (\Lambda_{\alpha \nu}(\bar{q}) + i\nu U/4t)^2} \right\}$$

$$+ \frac{U}{2} \left[ M_c - \sum_{\nu=1}^{\infty} 2\nu N_{c\nu} - \frac{N_a}{2} \right] + \sum_{\alpha} \mu_{\alpha} S_{\alpha z},$$

(97)

where $N_{c0}(\bar{q})$ and $N_{c\nu}(\bar{q})$ are the pseudofermion momentum distribution functions of Eq. (87).

In contrast to the pseudoparticle energy functional (62), (64) in terms of the pseudoparticle bare-momentum distribution function deviations, the energy spectrum derived from expressions (78) and (87) only includes first-order terms in the pseudofermion momentum distribution function deviations. In the Hilbert subspace that the energy spectrum (78) refers to, only such first-order pseudofermion momentum distribution function deviation contributions have physical meaning. The information recorded in the pseudoparticle interactions is transferred over to the momentum two-pseudofermion phase shifts of the momentum term $Q_{\alpha \nu}(q)/L$. The pseudofermion discrete momentum values are of $1/L$ order and store this information.
According to Eq. (98), the processes which generate the excited states A and B from the initial ground state lead to a collective momentum shift of all the \( c0 \) pseudofermions and \( s1 \) pseudofermions of the initial-state Fermi sea. That momentum shift reads,

\[
\left[ \pi_{\alpha\nu}^0 + Q_{\alpha\nu}(q_{\bar{q}}) \right]/L; \quad \pi_{\alpha\nu}^0 = 0, \pm \pi; \quad \alpha = c0, s1,
\]

where the bare-momentum shift \( \pi_{\alpha\nu}^0/L \) is given by,

\[
\pi_{\alpha\nu}^0 = 0; \quad \sum_{\alpha = c, s} \sum_{\nu = 1}^\infty \Delta N_{\alpha\nu} \text{ even}; \quad \pi_{\alpha\nu}^0 = \pm \pi; \quad \sum_{\alpha = c, s} \sum_{\nu = 1}^\infty \Delta N_{\alpha\nu} \text{ odd};
\]

\[
\pi_{\alpha\nu}^0 = 0; \quad \Delta N_{c0} + \Delta N_{s1} \text{ even}; \quad \pi_{\alpha\nu}^0 = \pm \pi; \quad \Delta N_{c0} + \Delta N_{s1} \text{ odd}.
\] (99)

The excited states A and B are generated from the ground state by two virtual excitations: (1) a collective momentum shift of the Fermi sea \( c0 \) pseudofermions and \( s1 \) pseudofermions; (2) for the states A, a finite number of \( c0 \) pseudofermion and \( s1 \) pseudofermion creation or annihilation processes and of \( \alpha\nu \neq c0 \) and \( \alpha\nu \neq s1 \) pseudofermion, \(-1/2\) Yang holon, and \(-1/2\) HL spinon creation processes; for the states B, a finite number of \( c0 \) pseudofermion and \( s1 \) pseudofermion particle-hole processes. Consideration of both the energy associated with the virtual excitations (1) and (2), gives the energy spectrum of few-electron excitations associated with the energy (78)-(97). Such an energy spectrum corresponds to the ground-state normal-ordered 1D Hubbard model and is additive in the \(-1/2\) holon, \(-1/2\) spinons, and \( \alpha\nu \) pseudofermion energies. It reads,

\[
\Delta E = \omega_0 + \sum_{j=1}^{N_s} \Delta N_{c0}(\bar{q}_j) \epsilon_{c0}(\bar{q}_j) + \sum_{j=1}^{N_{s1}} \Delta N_{s1}(\bar{q}_j) \epsilon_{s1}(\bar{q}_j) + \sum_{\alpha = c, s} \sum_{\nu = 1}^\infty \sum_{j=1}^{N_{\alpha\nu}} \Delta N_{\alpha\nu}(\bar{q}_j) \epsilon_{\alpha\nu}^0(\bar{q}_j)
\]

\[
= \omega_0 + \frac{L}{2\pi} \int_{-q_{c0}^0}^{+q_{c0}^0} d\bar{q} \Delta N_{c0}(\bar{q}) \epsilon_{c0}(\bar{q}) + \frac{L}{2\pi} \sum_{\nu = 1}^\infty \int_{-q_{s1}^0}^{+q_{s1}^0} d\bar{q} \Delta N_{s1}(\bar{q}) \epsilon_{s1}(\bar{q})
\]

\[
+ \frac{L}{2\pi} \sum_{\alpha = c, s} \sum_{\nu = 1}^\infty \sum_{j=1}^{N_{\alpha\nu}} \int_{-q_{\alpha\nu}^0}^{+q_{\alpha\nu}^0} d\bar{q} \Delta N_{\alpha\nu}(\bar{q}) \epsilon_{\alpha\nu}^0(\bar{q}),
\] (100)

where the energy parameter \( \omega_0 \) is given by,

\[
\omega_0 = 2\mu \Delta M_{c, -1/2} + 2\mu_0 H \left[ \Delta M_{s, -1/2} - \Delta N_{s1} \right].
\] (101)

Here \( \Delta M_{c, -1/2} \) and \( \Delta N_{s1} = \Delta N_{s1} \) are the deviations in the numbers of \(-1/2\) holons (\( \alpha = c \)) and of \(-1/2\) spinons (\( \alpha = s \)) and \( \Delta N_{s1} \) is the deviation in the number of \( s1 \) pseudofermions. The limiting-momentum values \( q_{c0}^0, q_{s1}^0, \) and \( q_{\alpha\nu}^0 \) appearing in the integrations of Eqs. (97) and (100) are given in Eqs. (99) and (100). Moreover, on the right-hand side of Eq. (100), the functions \( \epsilon_{c0}(\bar{q}), \epsilon_{s1}(\bar{q}), \) and \( \epsilon_{\alpha\nu}^0(\bar{q}) \) are the pseudofermion energy bands. These energy bands equal the corresponding pseudoparticle energy bands appearing on the right-hand side of Eq. (103), provided that the bare momentum \( q \) is replaced by the momentum \( \bar{q} \). The latter bands are plotted in Figs. 6 to 9 of Ref. 30 as a function of the bare momentum for zero spin density and several values of electronic density and on-site Coulombian repulsion.

The shape of the pseudofermion energy bands is controlled by the two-pseudofermion phase shifts and reads,

\[
\epsilon_{c0}(\bar{q}) = \frac{U}{2} - 2t \cos k^0(\bar{q}) + 2t \int_{-Q}^{+Q} dk \tilde{\Phi}_{c0, c0}(k, k^0(\bar{q})) \sin k + \mu - \mu_0 H,
\] (102)

\[
\epsilon_{c\nu}^0(\bar{q}) = -\nu U + 4t \text{Re} \left\{ \sqrt{1 - \left[ \Lambda_{c\nu}(\bar{q}) - iU \right] \frac{1}{4t}} \right\} + 2t \int_{-Q}^{+Q} dk \tilde{\Phi}_{c0, c\nu}(k, \Lambda_{c\nu}(\bar{q})) \sin k,
\] (103)

\[
\epsilon_{s\nu}^0(\bar{q}) = 2t \int_{-Q}^{+Q} dk \tilde{\Phi}_{s0, s\nu}(k, \Lambda_{s\nu}(\bar{q})) \sin k,
\] (104)
and

\[ \epsilon_{s1}(\bar{q}) = \epsilon_{s1}^0(\bar{q}) + 2\mu_0 H. \]

The two-pseudofermion phase shifts on the right-hand side of these equations are defined in Eqs. \ref{eq:100} and \ref{eq:101}. The parameter \( Q \) is given in Eq. \ref{eq:115} and the functions \( k^0(\bar{q}) \) and \( \Lambda_{s1}^0(\bar{q}) \) are the inverse of the functions defined in Eqs. \ref{eq:117} and \ref{eq:118}, respectively. The zero-energy levels of the energy dispersions \ref{eq:102}-\ref{eq:105} are such that,

\[ \epsilon_s(\pm 2k_F) = \epsilon_{s1}(\pm k_{F1}) = \epsilon_{s1}^0(\pm |\pi/a - 2k_F|) = \epsilon_{s1}^0(\pm |k_F1 - k_{F1}|) = 0. \]

The dependence on the electronic density \( n \), spin density \( m \), and \( U/t \) of the chemical potential \( \mu \) and magnetic field \( H \) appearing in Eqs. \ref{eq:102} and \ref{eq:105} can be expressed in terms of the two-pseudofermion phase shifts as,

\[ \mu = \frac{U}{2} + 2t \cos Q - 2t \int_{-Q}^{+Q} dk \Phi_{s0,c0}(k, Q) \sin k - t \int_{-Q}^{+Q} dk \Phi_{s0,s1}(k, B) \sin k, \]

and

\[ H = -\frac{t}{\mu_0} \int_{-Q}^{+Q} dk \Phi_{s0,s1}(k, B) \sin k, \]

respectively.

We emphasize that the virtual-excitation (1) energy spectrum vanishes and thus the general deviation-linear energy spectrum \ref{eq:111} amounts to the contributions from the excitation (2). By construction of the pseudofermion description subspace, the virtual excitation (2) of the states A and B, involves changes in the occupancy configurations of a finite number of quantum objects. However, the excitation (1) is a collective momentum shift of all the ground-state \( c0 \) pseudofermions and \( s1 \) pseudofermions. In the thermodynamic limit, the value of the corresponding ground-state numbers \( N_{c0} = N \) and \( N_{s1} = N \) approaches infinity. Thus, the self-consistency of the non-interacting pseudofermion theory implies that the energy of the excitation (1) vanishes in that limit, so that the total energy is additive in the corresponding pseudofermion energies. Let us confirm that as \( L \to \infty \) this holds true. The \( c0 \) and \( s1 \) pseudofermion momentum distribution function deviations of Eq. \ref{eq:110} can be written as,

\[ \Delta N_{av}(q_j) = \Delta N_{av}^{(1)}(q_j) + \Delta N_{av}^{(2)}(q_j); \]

\[ \alpha \nu = c0, s1, \]

where \( \Delta N_{av}^{(1)}(q_j) \) and \( \Delta N_{av}^{(2)}(q_j) \) are associated with the collective pseudofermion Fermi sea excitation (1) and excitation (2), respectively. Within the first order 1/L pseudofermion description the former deviation reads,

\[ \Delta N_{av}^{(1)}(q) = N_{av}^0(q + \frac{\pi_{av}^0 + Q_{av}(q)}{L}) - N_{av}^0(q) \]

\[ = \frac{[Q_{av}(q) + \pi_{av}^0]}{L} \partial N_{av}^0(\bar{q}) = -\text{sgn}(q) \frac{[\pi_{av}^0 + Q_{av}(\text{sgn}(\bar{q})q_{Fav})]}{N_a} \delta(q_{Fav} - |\bar{q}|); \]

\[ \alpha = c0, s1. \]

Here, the momentum Fermi value \( q_{Fav}^0 \) of the \( c0 \) and \( s1 \) branches is given in Eq. \ref{eq:105} and \( \pi_{av}^0/L \) is the corresponding bare-momentum shift of Eq. \ref{eq:111}. To first order in 1/L, use of Eq. \ref{eq:110} in the energy spectrum \ref{eq:111} leads to the following excitation (1) energy spectrum,

\[ \Delta E_{c0,s1}^{(1)} = \sum_{\alpha\nu = c0,s1} \sum_{q_j = -q_{Fav}^0}^{+q_{Fav}^0} \Delta N_{av}^{(1)}(q) \epsilon_{av}(q) = -\sum_{\nu = \pm 1} \sum_{\alpha \nu = c0,s1} \frac{[\pi_{av}^0 + Q_{av}(\nu q_{Fav})]}{N_a} \epsilon_{av}(q_{Fav}) = 0. \]

In order to obtain the result \ref{eq:111} we have used the symmetry \( \epsilon_{av}(q) = \epsilon_{av}(-q) \) and Eq. \ref{eq:105} such that \( \epsilon_{av}(q_{Fav}) = 0 \) for \( \alpha \nu = c0, s1 \). We recall that the occupancies of other \( \alpha \nu \neq c0, s1 \) pseudofermion branches vanish for the ground state. Since the excitation (1) involves all ground-state \( c0 \) and \( s1 \) pseudofermions, the evaluation of the 1/L higher-order contributions to the corresponding energy spectrum involves non-linear deviation terms. We find that such 1/L
higher-order energy contributions also vanish as \( L \to \infty \). Thus, the self-consistency condition that the excitation (1) energy spectrum vanishes as the system length \( L \) approaches infinity is fulfilled.

The energy \( (10) \) on the right-hand side of Eq. (10) controls the finite-energy physics. The remaining energy terms correspond to gapless contributions provided that the involved pseudofermions correspond to the momentum values of the energy-band arguments of Eq. (10). For most excited states, the latter terms also lead to finite-energy contributions. The typical value of such energy contributions is of the order of the pseudofermion energy dispersion band-width per pseudofermion involved in the excited states. We note that the energy spectrum (10) of the excited states \( A \) and \( B \), can have any finite value associated with the regions of the \((k, \omega)\) plane where the few-electron spectral functions have finite spectral weight [10, 54].

Also, the excitation momentum spectrum can be written in terms of the pseudofermion momentum distribution function deviations. It is given by,

\[
\Delta P = \frac{\pi}{a} \Delta M_{c,-1/2} + \sum_{j=1}^{N_a} \Delta N_{c0}(\bar{q}_j) \bar{q}_j + \sum_{\nu=1}^{N^*_a} \sum_{j=1}^{N^*_a} \Delta N_{sv}(\bar{q}_j) \bar{q}_j + \sum_{\nu=1}^{N^*_a} \sum_{j=1}^{N^*_a} \Delta N_{cv}(\bar{q}_j) \left( \frac{\pi}{a} - \bar{q}_j \right) 
\]

\[
= \frac{\pi}{a} \Delta M_{c,-1/2} + \frac{L}{2\pi} \int_{q_{c0}}^{q_0} dq \Delta N_{c0}(\bar{q}) \bar{q} + \frac{L}{2\pi} \sum_{\nu=1}^{N^*_a} \sum_{j=1}^{N^*_a} \int_{q_{cvj}}^{q_{cvj0}} dq \Delta N_{cv}(\bar{q}) \left( \frac{\pi}{a} - \bar{q}_j \right). 
\]

The important point is that the large-\( L \) pseudofermion operational description introduced in this paper, associated with the momentum distribution function deviation first-order energy spectrum (10) and momentum spectrum (112), contains full information about the few-electron spectral properties. Besides the expression in terms of pseudofermion operators of the diagonal (in the basis of the energy eigenstates) operators associated with these spectra, such a description is suitable for the evaluation of few-electron spectral functions for finite values of energy [34].

When acting in the Hilbert subspace spanned by the initial ground state and excited states \( A \) and \( B \), the ground-state normal-ordered 1D Hubbard model and momentum operator can be written in terms of pseudofermion, \(-1/2\) Yang holon, and \(-1/2\) HL spinon operators as follows,

\[
: \hat{H} := \sum_{\alpha=c, s} \sum_{\nu=\delta_{\alpha, s}} \sum_{j=1}^{N^*_\alpha} \epsilon_{\alpha\nu}(\bar{q}_j) : f^\dagger_{\bar{q}_j, c\alpha} : f_{\bar{q}_j, c\alpha} + \sum_{\alpha=c, s} \epsilon_{L\alpha,-1/2} \hat{L}_{\alpha,-1/2} , 
\]

and

\[
: \hat{P} := \sum_{j=1}^{N_a} \bar{q}_j : f^\dagger_{\bar{q}_j, c0} : f_{\bar{q}_j, c0} + \sum_{\nu=1}^{N^*_a} \sum_{j=1}^{N^*_a} \bar{q}_j : f^\dagger_{\bar{q}_j, sv} : f_{\bar{q}_j, sv} + \sum_{\nu=1}^{N^*_a} \sum_{j=1}^{N^*_a} [ (1 + \nu) \frac{\pi}{a} - \bar{q}_j ] : f^\dagger_{\bar{q}_j, cv} : f_{\bar{q}_j, cv} + \frac{\pi}{a} \hat{L}_{c,-1/2} , 
\]

respectively, where \( N^*_\alpha = N_\alpha \) and the operator \( \hat{L}_{\alpha,-1/2} \) is given in Eq. (80). On the right-hand side of Eq. (113), the \( c0 \) and \( s1 \) pseudofermion energy bands are given in Eqs. (102) and (105), respectively, and the \( \alpha\nu \neq c0 \) and \( \alpha\nu \neq s1 \) pseudofermion energy bands and \(-1/2\) Yang holon and \(-1/2\) HL spinon energies read [34],

\[
\epsilon_{c0}(\bar{q}) = \epsilon^0_{c0}(\bar{q}) + 2\nu\mu ; \quad \epsilon_{sv}(\bar{q}) = \epsilon^0_{sv}(\bar{q}) + 2\nu\mu_0 H ; \quad \nu \geq \delta_{\alpha, s} , 
\]

and

\[
\epsilon_{Lc,-1/2} = 2\mu ; \quad \epsilon_{Ls,-1/2} = 2\mu_0 H , 
\]

respectively. The energy dispersions \( \epsilon^0_{c0}(\bar{q}) \) and \( \epsilon^0_{sv}(\bar{q}) \) of Eq. (115) are defined in Eqs. (103) and (104), respectively. The ground-state normal-ordered Hamiltonian (113) and momentum operator (114) correspond to the energy and momentum spectra given in Eqs. (100) and (112), respectively.
B. WAVE-FUNCTION FACTORIZATION OF THE NORMAL-ORDERED 1D HUBBARD MODEL

It is well known that both the ground state wave function and the wave function of the excited states of the 1D Hubbard model can in the \( U/t \to \infty \) limit be constructed as a product of a spin-less fermion wave function and a squeezed spin wave function \cite{16, 18, 19}. In our pseudofermion, Yang holon, and HL spinon language this factorization means that in such a limit the expression of the momentum and energy of these states is linear in the \( \alpha \nu \) pseudofermion momentum distribution functions and in the \(-1/2\) Yu holon and \(-1/2\) HL spinon numbers.

Let us show that for finite values of \( U/t \) the energy \cite{31} is not linear in the \( \alpha \nu \) pseudofermion momentum distribution functions \( N_{\alpha \nu}(q) \). Our analysis refers to the few-electron excitation subspace. This confirms that the above type of factorization does not occur for finite values of the on-site repulsion. For the pseudofermion Hilbert subspace spanned by the ground state and excited states A and B the form of the general energy spectrum \cite{31} simplifies to,

\[
E = E^0 + \Delta E; \quad E^0 = E^0_H + \frac{U}{2} \left[ \frac{N}{2} - N_{c0} \right] + \sum_{\nu} \mu_{\nu} S^\nu_0 ,
\]

where \( \Delta E \) is the general energy excitation spectrum given in Eq. \ref{eq:energy_excitation}, \( E^0 \) stands for the ground-state energy, \( N_{c0} \) is the number of \( c \) pseudofermions, and the term \( \sum_{\nu} \mu_{\nu} S^\nu_0 \) is the same as on the right-hand side of Eq. \ref{eq:energy_spectrum}. From combination of the ground-state occupancy configurations provided in Appendix A with the general energy spectrum given in Eq. \ref{eq:energy_spectrum} we find that the energy \( E^0_H \) on the right-hand side of Eq. \ref{eq:energy_spectrum} reads,

\[
E^0_H = -2t \frac{L}{2\pi} \int_{-\pi}^{+\pi} dq N^0_{c0}(q) \cos k^0(q) ,
\]

where \( N^0_{c0}(q) \) is the ground-state \( c \) pseudofermion momentum distribution function. (We recall that \( q = \tilde{q} \) for the ground state.) From use of Eqs. \ref{eq:energy_spectrum}, \ref{eq:K0}, and \ref{eq:phi_0} we arrive to the following expressions,

\[
k^0(q) = \tilde{q} - \int_{-\pi}^{+\pi} dq \frac{\partial k^0(q')}{\partial q'} N^0_{c0}(q') \Phi^f_{c0,c0}(q', \tilde{q}) ; \quad \frac{\partial k^0(q)}{\partial q} = 1 - \int_{-\pi}^{+\pi} dq \frac{\partial k^0(q')}{\partial q'} N^0_{c0}(q') \frac{\partial \Phi^f_{c0,c0}(q', \tilde{q})}{\partial q} .
\]

By iterative solution of the equations given in \ref{eq:phi_0} one can derive a functional representation for the rapidity-momentum function \( k^0(q) \) in terms of the ground-state distribution function \( N^0_{c0}(q) \). Use of such a functional on the right-hand side of Eq. \ref{eq:energy_spectrum} leads to the following non-linear energy functional in the ground-state momentum distribution function \( N^0_{c0}(q) \),

\[
E^0_H = -2t \frac{L}{2\pi} \int_{-\pi}^{+\pi} dq N^0_{c0}(q) \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} \left[ \tilde{q} - \int_{-\pi}^{+\pi} dq_1 N^0_{c0}(q_1) \Phi^f_{c0,c0}(q_1, \tilde{q}) \times \left( 1 + \sum_{i=1}^{\infty} \prod_{i=1}^{l} dq_i N^0_{c0}(q_{i+1}) \frac{\partial \Phi^f_{c0,c0}(q_{i+1}, \tilde{q}_i)}{\partial \tilde{q}_i} \right) \right]^{2j}.
\]

Here the indices \( i = 1, 2, ... \) of the momentum \( \tilde{q}_i \) label independent integration continuum variables rather than discrete momentum values. The form of expression \ref{eq:energy_functional} confirms that for finite values of \( U/t \) the energy \ref{eq:energy_spectrum} is highly non-linear in the ground-state pseudofermion momentum distribution function \( N^0_{c0}(q) \). Therefore, the above type of factorization does not occur in general for the 1D Hubbard model. However, one can confirm that in the limit \( U/t \to \infty \) the energy functional \ref{eq:energy_functional} becomes linear in \( N^0_{c0}(q) \). Indeed, one finds from analysis of the integral equations provided in Appendix B that for spin density \( m = 0 \) the two-pseudofermion phase shift on the right-hand side of Eq. \ref{eq:energy_functional} is such that \( \Phi^f_{c0,c0}(\tilde{q}, \tilde{q}') \to 0 \) as \( U/t \to \infty \). Thus, in that limit the ground-state expression \ref{eq:energy_functional} simplifies to,

\[
E^0_H = -2t \frac{L}{2\pi} \int_{-\pi}^{+\pi} dq N^0_{c0}(q) \cos(q) ; \quad U/t \to \infty .
\]

This property also holds for the excited states and is behind the full factorization of the wave functions used in the studies of Refs. \cite{16, 18, 19}.
Fortunately, the evaluation of few-electron spectral functions can be achieved without the full factorization of the wave functions. Such a problem can be solved by use of the ground-state normal-ordered 1D Hubbard model. When expressed in terms of pseudofermion operators, that normal-ordered Hamiltonian and associated momentum operator are given in Eqs. (13) and (14), respectively. These quantum problems correspond to the energy and momentum spectra given in Eqs. (10) and (12), respectively. The ground-state normal-ordered description is equivalent to specifying the energy eigenstates by means of the momentum distribution function deviations \( \Delta \bar{N}_{\alpha\nu}(\vec{q}) = \Delta \bar{N}_{\alpha\nu}(\vec{q}) \) instead of in terms of the corresponding full momentum distribution functions \( \bar{N}_{\alpha\nu}(\vec{q}) \).

In the thermodynamic limit, there is a wave-function factorization for the excited states belonging to the Hilbert subspace of the pseudofermion normal-ordered Hamiltonian (15). This factorization follows from the form of the expressions (10) and (12) for the energy and momentum, respectively. The energy spectrum defined in Eqs. (10) and (11) can be written as,

\[
\Delta E = \sum_{\alpha=c,s} \sum_{\nu=\delta_{\alpha,s}} \infty \Delta E_{\alpha\nu} + 2\mu \Delta L_{c,-1/2} + 2\mu_0 H \Delta L_{s,-1/2},
\]

where the contributions from each branch are linear in the \( \alpha \nu \) pseudofermion momentum distribution function deviations and read,

\[
\Delta E_{\alpha\nu} = \sum_{j=1}^{N_{\alpha\nu}^*} \Delta \bar{N}_{\alpha\nu}(\vec{q}_j) \epsilon_{\alpha\nu}(\vec{q}) = \frac{L}{2\pi} \int_{-\bar{q}_{\alpha\nu}^*}^{\bar{q}_{\alpha\nu}^*} d\vec{q} \Delta \bar{N}_{\alpha\nu}(\vec{q}) \epsilon_{\alpha\nu}(\vec{q}).
\]  

Here the pseudofermion energy bands are given in Eqs. (12), (10) and (11). The \(-1/2\) Yang holons, \(-1/2\) HL spinons, and each pseudofermion branch leads to a different energy term linear in the corresponding number deviation or momentum distribution function deviation. We emphasize that the virtual-excitation (1) energy \( \Delta E_{c0,1}^{(1)} \) of Eq. (10) associated with the momentum shift of Eq. (9) which involves all \( c0 \) pseudofermions and \( s1 \) pseudofermions of the ground-state Fermi sea, is also additive in the energy contributions from each of these quantum objects. Indeed, according to Eqs. (10) and (11), the energy contribution from each \( c0 \) pseudofermion and \( s1 \) pseudofermion involved in such a collective excitation vanishes. As the total energy \( \Delta E_{c0,1}^{(1)} \) of the same excitation also vanishes, it is indeed additive in the energies of all involved quantum objects. Therefore, the wave function of the energy eigenstates of the normal-ordered Hamiltonian can be expressed in the pseudofermion subspace as a product of wave functions. Each wave function corresponds to a different pseudofermion branch. In excited states with finite \(-1/2\) Yang holon and \(-1/2\) HL spinon occupancy there is also a wave function for these objects. This factorization is associated with the deep physical meaning of the pseudo-particle - pseudofermion transformation (58): it transfers the information recorded in the pseudoparticle interactions over to the pseudofermion momentum, providing a non-interacting character to the latter objects.

In contrast, for the pseudoparticle representation the energy functional (62-64) includes bare-momentum distribution function deviation non-linear terms associated with the pseudoparticle residual interactions. For instance, the quadratic energy term (51) contains bare-momentum summations over products of deviations of the form \( f_{\alpha\nu;\alpha'\nu'}(q, q') \Delta \bar{N}_{\alpha\nu}(q) \Delta \bar{N}_{\alpha'\nu'}(q') \). Such \( f \) function terms are associated with the residual two-pseudoparticle interactions. The occurrence of these energy terms mixes contributions from different branches. It follows that the pseudofermion energy spectrum is not additive in the \( \alpha \nu \) pseudoparticle branch contributions, in contrast to the pseudofermion energy spectrum given in Eqs. (10) and (12). Thus, in this case the wave function of the excited states does not factorize in the form of a product of pseudoparticle wave functions.

The number of wave functions contributing to the factorized wave function of a given energy eigenstate depends on the occupancy configurations of that state. Only the \( \alpha \nu \) pseudofermion branches with finite occupancy in the state contribute the wave function. This contribution is in the form of a \( \alpha \nu \) wave function factor. The same applies to the occupancy of \(-1/2\) Yang holons and \(-1/2\) HL spinons.

**VII. TRANSFORMATION LAWS UNDER THE PSEUDOPARTICLE - PSEUODOFERMIION SUBSPACE UNITARY ROTATION**

In this section, we find the transformation laws under the pseudoparticle - pseudofermion Hilbert subspace unitary rotation of several quantum objects and quantities and discuss the physical meaning of these laws. For instance, the \( \alpha \nu \) pseudoparticle number operator (50), is invariant under such a transformation and thus equals the corresponding
αν pseudoparticle number operator \[^{[1]}\]. Also the Yang holons and HL spinons, the charge and spin carried by the pseudoparticles, and the effective pseudoparticle lattices remain invariant under such a transformation.

The pseudoparticle - pseudofermion unitary transformation is generated by the operator \[^{[3]}\], which only shifts the pseudoparticle bare-momentum values. Thus, given a pseudoparticle of bare-momentum \( q \), the transformation law \( q \to \tilde{q} \) of its bare momentum also defines the transformation law of such a quantum object under the pseudoparticle - pseudofermion unitary transformation. For instance, if a specific value of the bare momentum \( q \) remains invariant under that transformation, then the pseudoparticle carrying bare momentum \( q \) is the same quantum object as the corresponding pseudofermion.

A. TRANSFORMATION LAWS OF PSEUDOPARTICLES AT THE BARE-MOMENTUM LIMITING VALUES

The transformation laws of the limiting pseudoparticle bare-momentum values \( \pm q_{\alpha, \nu} \) under the pseudoparticle - pseudofermion transformation defined in Eq. \[^{[5]}\] provide interesting information about the physics described by the \( \alpha\nu \) pseudofermions. For the excited states A and B, the limiting \( \alpha\nu \) pseudoparticle bare-momentum values \( \iota q_{\alpha\nu} \), where \( \iota = \pm 1 \), can be written as,

\[
\iota q_{\alpha\nu} = \iota q_{\alpha\nu}^0 + \iota \Delta q_{\alpha\nu} ; \quad \alpha = c, s ; \quad \nu = 1, 2, \ldots ; \quad \iota = \pm 1 .
\] (124)

Here \( q_{\alpha\nu}^0 \) is the ground-state limiting bare-momentum value given in Eqs. \[^{[8]}\] and \[^{10}\].

We start by considering the case of \( \nu > 0 \) pseudoparticle branches. In that case the deviations \( \Delta q_{\alpha\nu} \) have the following form,

\[
\iota \Delta q_{\alpha\nu} = \iota \frac{\pi}{L} \left[ \Delta N_{\alpha\nu} + \Delta M_\alpha - \sum_{\nu' = 1}^{-\infty} (\nu + \nu' - |\nu - \nu'|) \Delta N_{\alpha\nu'} \right] ; \quad \alpha = c, s ; \quad \nu = 1, 2, \ldots ; \quad \iota = \pm 1 ,
\] (125)

where \( \Delta N_{\alpha\nu} \) (and \( \Delta N_{\alpha\nu'} \)) is the deviation in the number of \( \alpha\nu \) pseudoparticles (and \( \alpha\nu' \) pseudoparticles) and \( \Delta M_\alpha \) stands for the deviation in the holon number \( M_\alpha = [N_\alpha - N_{\alpha'}] \) or spinon number \( M_\alpha = N_{\alpha'} \). These expressions are easily obtained by combination of Eqs. \[^{[21]}\], \[^{23}\], \[^{27}\], and \[^{11}\] of Appendix A.

In Appendix C we find that at \( q = \pm q_{\alpha\nu} \) and for \( \nu > 0 \) the functional defined in Eq. \[^{[29]}\] equals the following value for all excited states A and B,

\[
\frac{Q_{\alpha\nu}(\iota q_{\alpha\nu})}{L} = \sum_{\alpha' = c, s} \sum_{\nu' = 1}^{\infty} \int_{-q_{\alpha\nu}}^{q_{\alpha\nu}} dq' \Phi_{\alpha, \alpha' \nu'}(\iota q_{\alpha\nu}, q') \Delta N_{\alpha' \nu'}(q') \Delta q_{\alpha\nu} ; \quad \alpha = c, s ; \quad \nu = 1, 2, \ldots ; \quad \iota = \pm 1 ,
\] (126)

where the quantity \( \Delta q_{\alpha\nu} \) is given in Eq. \[^{[23]}\]. Here it is assumed that the bare momentum \( q' \) associated with the pseudoparticle bare-momentum distribution function deviations \( \Delta N_{\alpha' \nu'}(q') \) on the right-hand side of Eq. \[^{26}\] belongs to the domain \( q' \in (-q_{\alpha' \nu'}, +q_{\alpha' \nu'}) \) and can be such that \( q' \to \pm q_{\alpha' \nu'} \) but \( q' \neq q_{\alpha' \nu'} \). Interestingly, the value of the momentum functional \( Q_{\alpha\nu}(q)/L \) at \( q = \iota q_{\alpha\nu} \) is such that it precisely cancels the term \( \iota \Delta q_{\alpha\nu} \) appearing in the limiting bare-momentum expression on the right-hand side of Eq. \[^{24}\]. Thus at the limiting bare-momentum values \( q = \pm q_{\alpha\nu} \) given in Eq. \[^{24}\] the value of the momentum functional \( q(q) \) defined in Eq. \[^{22}\] is independent of the value of the deviation \( \Delta q_{\alpha\nu} \) on the right-hand side of Eq. \[^{24}\] which defines the final excited state. By use of this result in Eq. \[^{24}\] we find that,

\[
\iota q_{\alpha\nu} = \iota q_{\alpha\nu}^0 + \iota \Delta q_{\alpha\nu} \to \iota q_{\alpha\nu} = \iota q_{\alpha\nu}^0 ; \quad \alpha = c, s ; \quad \nu = 1, 2, \ldots ; \quad \iota = \pm 1 .
\] (127)

Thus, we conclude that for \( \nu > 0 \) pseudoparticle branches the limiting momentum values of the corresponding \( \alpha\nu \) pseudofermion bands equal for all excited states A and B the ground-state limiting bare-momentum values given in Eqs. \[^{8}\] and \[^{10}\], i.e. \( \pm q_{\alpha\nu} = \pm q_{\alpha\nu}^0 \). While the width of the bare-momentum \( \text{Brillouin zone} \) of the \( \alpha\nu \) pseudoparticles, \( 2q_{\alpha\nu} \), has according to Eqs. \[^{24}\] and \[^{23}\] an exotic dependence on the state occupancy configurations, the value of the corresponding momentum width of the \( \alpha\nu \) pseudofermion \( \text{Brillouin zone} \) is constant and equals \( 2q_{\alpha\nu} = 2q_{\alpha\nu}^0 \), where \( q_{\alpha\nu}^0 \) is the ground-state limiting bare momentum given in Eqs. \[^{8}\] and \[^{10}\].
This confirms that the width of the \( \alpha \nu \) pseudofermion momentum domain remains unchanged, as in the case of the momentum bands of non-interacting particles. This result is consistent with the expressions given in Eq. (73) for the rapidity and rapidity-momentum functions, respectively, of the excited states. When expressed in terms of the pseudofermion momentum \( \bar{q} \) such functions have the same form both for the ground state and excited states. This introduces the requirement that the width of the momentum domain where these functions are defined must be the same for the ground state and the excited states. Thus we have just confirmed that this requirement is fulfilled through the cancelling associated with the value of the momentum functional (126).

On the other hand, the bare-momentum width of the \( c0 \) pseudoparticles is independent of the pseudoparticle occupancy configurations and reads \( 2q^0_{c0} = 2\pi/a \). In Appendix C we find that the transformation (86) leaves this width invariant and shifts the bare-momentum values \( q = 0 \) and \( q = \pm q^0_{c0} \) by the same amount as follows,

\[
0 \to \frac{Q_{c0}(0)}{L}; \quad \pm q^0_{c0} \to \pm q^0_{c0} + \frac{Q_{c0}(0)}{L},
\]

where \( Q_{c0}(0)/L \) is the \( q = 0 \) value of the momentum functional \( Q_{c0}(q)/L \) given in Eq. (123). Thus, for the \( c0 \) band the shift is the same for these three bare-momentum values. As a result the width of the corresponding pseudofermion momentum domain remains unchanged and is given by \( 2q^0_{c0} = 2\pi/a \). Note that the value of such a shift is a functional which depends on the actual occupancy configurations of the excited states pseudoparticle bare-momentum distribution function deviations.

In general, a \( \alpha \nu \) pseudoparticle is different from the corresponding \( \alpha \nu \) pseudofermion. Below we find that for \( \alpha \nu \) pseudoparticle branches such that \( \alpha \nu \neq c0 \) and \( \alpha \nu \neq s1 \), the only exception is for bare-momentum values \( q \), such that \( q \to 0 \) in the particular case when these pseudoparticles are involved in the bare-momentum distribution function deviations of zero-momentum excited states. According to the above results concerning the \( \nu > 0 \) branches, as \( q \to \pm q_{\alpha \nu} \) the excited-state \( \alpha \nu \) pseudoparticle limiting bare-momentum values (124) map onto the same momentum domain remains unchanged, as in the case of the composite \( \alpha \nu \) pseudofermion and the rotated \( \alpha \nu \) pseudofermion become the same quantum object. In that limit, the energy \( \epsilon_{\alpha \nu}(\bar{q}) = 2\mu + \epsilon^0_{\alpha \nu}(\bar{q}) \) (and \( \epsilon_{\alpha \nu}(\bar{q}) = 2\mu H \nu + \epsilon^0_{\alpha \nu}(\bar{q}) \)) of the composite \( \alpha \nu \) pseudofermion (and \( s \) pseudoparticle) is additive in the \( -1/2 \) Yang holon energy \( 2\mu \) (and \( -1/2 \) HL spinon energy \( 2\mu H \nu \)) because \( \epsilon^0_{\alpha \nu}(\bar{q}) \to 0 \) (and \( \epsilon^0_{\alpha \nu}(\bar{q}) \to 0 \)) as \( \bar{q} \to \pm \bar{q}_{\alpha \nu} = \pm q^0_{\alpha \nu} \) (and \( \bar{q} \to \pm \bar{q}_{s \nu} = \pm q^0_{s \nu} \)). It follows that in that limit the \( \nu -1/2 \) holons \((\alpha = c)\) or \(-1/2 \) spinons \((\alpha = s)\) contained in the composite \( \alpha \nu \) pseudofermion acquire the same localized character as the \(-1/2 \) Yang holons \((\alpha = c)\) or \(-1/2 \) HL spinons \((\alpha = s)\) respectively.

\section*{B. TRANSFORMATION LAWS OF FERMI-POINT PSEUDOPARTICLES}

Let us next consider excitations corresponding to J-CPHS ground states whose Fermi points are defined in Eqs. (69)-(71). Our goal is to find out how the Fermi bare momentum \( q_{F \alpha \nu, \iota} \) given in these equations transforms under the pseudoparticle - pseudofermion transformation (86). In this case, such a transformation can formally be written as,

\[
q_{F \alpha \nu, \iota} = \iota q^0_{F \alpha \nu} + \Delta q_{F \alpha \nu, \iota} \rightarrow q'_{F \alpha \nu, \iota} = \iota q^0_{F \alpha \nu} + \Delta q'_{F \alpha \nu, \iota},
\]

where according to Eq. (71), \( \Delta q_{F \alpha \nu, \iota} = \iota [2\pi/L] N_{\alpha \nu, \iota} \). In Appendix C, we find that the deviation \( \Delta q'_{F \alpha \nu, \iota} \) on the right-hand side of Eq. (129) is given by,

\[
\Delta q'_{F \alpha \nu, \iota} = \frac{2\pi}{L} \Delta N^f_{\alpha \nu, \iota}; \quad \Delta N^f_{\alpha \nu, \iota} = \frac{\Delta N^f_{\alpha \nu}}{2} + \iota \Delta J^f_{\alpha \nu},
\]

Here

\[
\frac{\Delta N^f_{\alpha \nu}}{2} = \sum_{\alpha' = c,s} \sum_{\nu' = \delta_{\alpha'}, s} \Delta N^f_{\alpha' \nu'}; \quad \Delta J^f_{\alpha \nu} = \sum_{\alpha' = c,s} \sum_{\nu' = \delta_{\alpha'}, s} \epsilon^1_{\alpha, \alpha' \nu'} \Delta J_{\alpha' \nu'},
\]

and

\[
\epsilon^1_{\alpha, \alpha' \nu'} = \delta_{\alpha, \alpha'} \delta_{\nu, \nu'} + \sum_{\iota' = \pm 1} (\iota')^j \Phi^f_{\alpha, \alpha' \nu'}(q^0_{F \alpha \nu}, \iota' q^0_{F \alpha' \nu'}); \quad j = 0, 1.
\]
pseudoparticles) of bare momentum values at the Fermi points such that there are both pseudoparticle and pseudoparticle holes in the densities the low-energy excited states are described by pseudoparticle bare-momentum occupancy configurations associated with few-electron excitations. However, the pseudoparticle - pseudofermion unitary transformation is more general and refers to all values of energy where $\Phi_{\alpha, \alpha'}(\bar{q}, \bar{q}')$ is the two-pseudofermion phase shift given in Eq. (84).

Let us show that in the specific limit of small energy $\omega \rightarrow 0$ that unitary transformation is directly related to the primary-field conformal dimensions of the 1D Hubbard model two-component conformal field theory [21, 22, 23]. We consider the particular case of excited states with finite occupancies in the low-energy Hilbert subspace considered in the studies of Refs. [22, 23]. For simplicity, in this specific case we use the notation of these references such that $c \equiv c, 0$ and $s \equiv s, 1$. The indices $\alpha \nu$ of all quantities are replaced by the index $\alpha$ such that $\alpha = c, s$. In the present low-energy Hilbert subspace the above general expressions simplify to,

$$N_{\alpha}^f = \frac{\Delta N_{\alpha} f}{2} + i \Delta J_{\alpha}^f,$$

and

$$N_{\alpha, \iota}^f = \frac{\Delta N_{\alpha, \iota} f}{2} + i \Delta J_{\alpha, \iota}^f,$$

Here

$$\xi_{\alpha, \alpha'}^f = \delta_{\alpha, \alpha'} + \sum_{\iota' = \pm 1} (\iota')^j \Phi_{\alpha, \alpha'}(q_{F\alpha, \iota}', q_{F\alpha'}).$$

Let us combine Eqs. (134) and (135) and rewrite the number $\Delta N_{\alpha, \iota}^f$ as,

$$\Delta N_{\alpha, \iota}^f = \left[ \sum_{\alpha' = c, s} \xi_{\alpha, \alpha'}^f \frac{\Delta N_{\alpha'}}{2} + i \sum_{\alpha' = c, s} \xi_{\alpha, \alpha'}^f \Delta J_{\alpha'} \right].$$

Based on the form (136) of the parameters $\xi_{\alpha, \alpha'}^f$ one can by manipulation of the integral equations given in Appendix B, define these parameters in terms of related integral equations. Importantly, comparison of these latter equations and associated expressions (136) and (137) with the results of Refs. [22, 23] reveals that the number $\Delta N_{\alpha, \iota}^f$ given in Eq. (137) is such that,

$$[\Delta N_{\alpha, \iota}^f]^2 = 2 \Delta_0^c = \left[ \sum_{\alpha' = c, s} \xi_{\alpha, \alpha'}^f \frac{\Delta N_{\alpha'}}{2} + i \sum_{\alpha' = c, s} \xi_{\alpha, \alpha'}^f \Delta J_{\alpha'} \right]^2,$$

where the quantity $\Delta_0^c$ is the $\alpha$ conformal dimension of the primary fields [20, 21] associated with the $\alpha = c$ and $\alpha = s$ excitation branches. Moreover, the parameters $\xi_{\alpha, \alpha'}^c$ and $\xi_{\alpha, \alpha'}^s$ are entries of the transpose of the dressed charge matrix and of the inverse of the transpose of the dressed charge matrix, respectively, of the low-energy two-component 1D Hubbard model conformal field theory [21, 22, 23].

These results confirm that for the low-energy subspace defined above and for electronic densities and spin densities such that $0 < n < 1/a$ and $0 < m < n$, respectively, the conformal invariance of the 1D Hubbard model is directly related to the pseudoparticle - pseudofermion unitary transformation [58]. For these values of the electronic and spin densities the low-energy excited states are described by pseudoparticle bare-momentum occupancy configurations such that there are both pseudoparticle and pseudoparticle holes in the $c \equiv c, 0$ and $s \equiv s, 1$ pseudoparticle bands. However, the pseudoparticle - pseudofermion unitary transformation is more general and refers to all values of energy associated with few-electron excitations.

The above results provide interesting information about the transformation laws of $c0$ pseudoparticles (and $s1$ pseudoparticles) of bare momentum values at the Fermi points, $q = i 2k_F + i \frac{2\pi}{L} \Delta N_{\alpha, \iota}$ (and $q = i k_F + i \frac{2\pi}{L} \Delta N_{s, \iota}$)
where \( \nu = \pm 1 \). Under the pseudoparticle - pseudofermion transformation \(^{88}\), these pseudoparticles are mapped onto \( c0 \) pseudofermions (and \( s1 \) pseudofermions) with momentum values at the Fermi points \( \tilde{q} = i 2 k_F + i \frac{2 \pi}{L} \sqrt{2 \Delta^s_c} \) (and \( \tilde{q} = i 2 k_F + i \frac{2 \pi}{L} \sqrt{2 \Delta^s_s} \)) where \( \Delta^s_c \) (and \( \Delta^s_s \)) is the conformal dimension of the \( c0 \) (and \( s1 \)) excitation branch primary field \(^{21, 22, 23}\). We then conclude that these conformal dimensions are such that the quantities \( \sqrt{2 \Delta^s_c} \) are nothing but deviations in the values of \( c0 \) (\( \alpha = c \)) and \( s1 \) (\( \alpha = s \)) pseudofermion momentum Fermi points resulting from low-energy excitations. (We note that the positive and negative value of the root \( \sqrt{2 \Delta^s_c} \) refers to pseudofermion creation and annihilation, respectively.) Conformal-field theory can be used to evaluate expressions for low-energy few-electron spectral functions \(^{21, 24}\). We recall that our general pseudofermion description provides the momentum deviation values resulting from finite-energy excitations for all values of pseudofermion momentum and for all pseudofermion branches. Then it is not unexpected that the general pseudofermion description introduced in this paper can be used in the evaluation of few-electron spectral function expressions for all values of energy \(^{1, 10, 34}\).

**C. INVARIANCE UNDER THE PSEUDOPARTICLE - PSEUDOFEVRMION UNITARY TRANSFORMATION**

According to the results of Appendix A, there are no \( c\nu \) pseudoparticles and \( s\nu \) pseudoparticles belonging to \( \nu > 0 \) and \( \nu > 1 \) branches, respectively, in the ground state. This is consistent with the Fermi values of Eq. \(^{88}\), such that \( q_\text{F, c\nu} = 0 \) for these \( c\nu \) pseudoparticle branches. In Appendix C, it is shown that the momentum functional \( Q_{c\nu}(0)/L \) given in Eq. \(^{88}\) vanishes at \( q_\text{F, c\nu} = 0 \) for these \( c\nu \) pseudoparticle branches,

\[
\frac{Q_{c\nu}(0)}{L} = \sum_{\alpha = c, s} \sum_{\nu = 1}^{\infty} \int_{-q_{c\nu}}^{q_{c\nu}} dq \Phi_{\alpha\nu, \alpha'\nu'}(0, q') \Delta N_{\alpha'\nu'}(q) = 0 \tag{139}
\]

provided that the excited states are partial J-CPHS ground states and the bare-momentum distribution function deviations on the right-hand side of Eq. \(^{139}\) are for the \( c0 \) and \( s1 \) pseudoparticle branches such that,

\[
\Delta N_{c0}(q) = \Delta N_{c0}(-q) ; \quad \Delta N_{s1}(q) = \Delta N_{s1}(-q) \tag{140}
\]

Such excited states have vanishing momentum. We recall that partial J-CPHS ground-state occupancy configurations for all the \( c\nu \) pseudoparticle branches other than the \( c0 \) and \( s1 \) pseudoparticle branches. It follows that for the former branches the vanishing pseudoparticle Fermi points,

\[
q_{F, c\nu, \nu} = i q_{F, c\nu} + \Delta q_{F, c\nu, \nu} = 0 + i \frac{2\pi}{L} \Delta N_{c\nu, \nu} \tag{141}
\]

remain invariant under the pseudoparticle - pseudofermion transformation \(^{88}\), or

\[
q_{F, c\nu, \nu} \rightarrow \tilde{q}_{F, c\nu, \nu} = q_{F, c\nu, \nu} \tag{142}
\]

This means that a \( c\nu \) pseudoparticle belonging to a branch other than the \( c0 \) and \( s1 \) branches, carrying bare momentum \( q \rightarrow 0 \), and involved in excitation processes associated with zero-momentum excited states whose \( c0 \) and \( s1 \) bare-momentum distribution function deviations obey Eq. \(^{141}\) remains invariant under the pseudoparticle - pseudofermion transformation \(^{88}\). It follows that in this limiting case the \( q \rightarrow 0 \) \( c\nu \) pseudoparticle is the same quantum object as the corresponding \( \tilde{q} \rightarrow 0 \) \( c\nu \) pseudofermion and then,

\[
q \rightarrow \tilde{q} = q ; \quad \text{as} \quad q \rightarrow 0 \tag{143}
\]

This invariance occurring as \( q \rightarrow 0 \) for \( c\nu \) pseudoparticles belonging to branches such that \( c\nu \neq c0 \) and \( c\nu \neq s1 \), is associated with the free, non-interacting, and delocalized character of these quantum objects. Indeed, in this limit the residual interactions vanish and these pseudoparticles are free, non interacting, and delocalized quantum objects provided that the excitation processes they are involved in correspond to excited states of zero momentum. On the other hand, according to the discussions presented in Sec. III, as \( q \) approaches the limiting bare-momentum values \(^{12, 24}\), \( q \rightarrow \pm q_{\nu} \), the \( c\nu \) pseudoparticles belonging to branches other than the \( c0 \) pseudoparticle branch become
non-interacting and localized. Such a $q \rightarrow \pm q_{\alpha \nu}$ non-interacting and localized behavior is associated with another invariance: As $q \rightarrow \pm q_{\alpha \nu}$, these $\alpha \nu$ pseudoparticles become invariant under the electron - rotated-electron unitary transformation $[33, 34]$. Thus, as the bare momentum $q$ approaches the limiting values $\pm q_{\alpha \nu}$, the $\alpha \nu$ pseudoparticle and the rotated $\alpha \nu$ pseudoparticle become the same quantum object. In contrast, according to Eq. $(124)$ in that limit, these $\alpha \nu$ pseudoparticles are not invariant under the pseudoparticle - pseudofermion unitary transformation. Their non-interacting and localized behavior is achieved by a cancellation of the term $\epsilon \Delta q_{\alpha \nu}$ appearing in the limiting bare-momentum expression on the right-hand side of Eq. $(124)$ by the momentum functional $Q_{\alpha \nu}(\epsilon q_{\alpha \nu})/L$ of Eq. $(126)$. Moreover, for other bare momentum values $q$ such that $q \neq 0$ and $q \neq \pm q_{\alpha \nu}$ the $\alpha \nu$ pseudoparticle is different both from the $\alpha \nu$ pseudofermion and rotated $\alpha \nu$ pseudoparticle.

We thus conclude that as $q \rightarrow 0$ a $\alpha \nu$ pseudoparticle involved in deviations associated with zero-momentum excited states becomes free, non-interacting, delocalized, and the same quantum object as the corresponding $\alpha \nu$ pseudofermion. As the limiting values are approached and thus $q \rightarrow \pm q_{\alpha \nu}$, the $\alpha \nu$ pseudoparticle becomes non-interacting, localized, and the same quantum object as the corresponding rotated $\alpha \nu$ pseudoparticle. In this latter case, the energy of this composite object becomes additive in the energy of its $\nu - 1/2$ holons ($\alpha = c$) or $\nu - 1/2$ spinons ($\alpha = s$). Note that one of the effects of increasing the bare-momentum absolute value from $|q| = 0$ to $|q| = q_{\alpha \nu}$ is to enhance the localization character of the $\alpha \nu$ pseudoparticles. If a pseudoparticle has bare-momentum $q$ and $q + \Delta q$ for an initial ground state and an excited state, respectively, then the limiting free, non-interacting, and delocalized character and non-interacting and localized character are achieved when $Q_{\alpha \nu}(q)/L = 0$ and $Q_{\alpha \nu}(q)/L = -\Delta q$, respectively, where $Q_{\alpha \nu}(q)/L$ is the momentum functional $[33]$. For other values of that momentum functional the pseudoparticle is interacting. The pseudoparticles are never invariant under both the pseudoparticle - pseudofermion and pseudoparticle - rotated-pseudoparticle unitary transformations. This is because invariance under the pseudoparticle - pseudofermion unitary transformation is associated with a free, non-interacting, and delocalized character, whereas invariance under the pseudoparticle - rotated-pseudoparticle unitary transformation corresponds to a non-interacting and localized character. Therefore the impossibility of simultaneous invariance under these two transformations is related to Heisenberg’s uncertainty relation.

Another interesting example of invariance under the pseudoparticle - pseudofermion transformation $[33]$ concerns the bare-momentum difference $(q - q')$ when $q$ and $q'$ differ by a finite multiple of $2\pi/L$. This case is of physical importance because $(q - q')$ corresponds to the momentum of a pseudoparticle - pseudoparticle hole excitation in the $c0$ or $s1$ band. This type of process is associated with the excited states B and plays a key role in the few-electron spectral-weight distribution $[34]$. In this case, we find in Appendix C that in the thermodynamic limit, $(q - q')$ is indeed invariant under the pseudoparticle - pseudofermion transformation and thus,

\[(q - q') \rightarrow (q - q') = (q - q').\] (144)

VIII. CONCLUDING REMARKS

In this paper we introduced a pseudofermion operational description for the 1D Hubbard model. We found that in the thermodynamic limit the wave function of excited states belonging to the Hilbert subspace of the ground-state normal-ordered 1D Hubbard model associated with few-electron excitations factorizes for all values of $U/t$. This factorization results from the non-interacting character of the pseudofermions whose occupancy configurations describe these excited states. While the pseudofermion description studied in Refs. $[33, 34, 35]$ corresponds to the whole Hilbert space, the pseudofermions are defined in the Hilbert subspace associated with few-electron excitations. In such a subspace, the pseudofermions are related to the pseudoparticles by a unitary transformation. We classified the statistics of the latter quantum objects according to a generalized Pauli principle $[36]$.

Our study included the introduction of the pseudoparticle - pseudofermion unitary transformation and of an operator algebra for both the pseudoparticles and pseudofermions. Such a transformation removes the residual interactions of the $\alpha \nu$ pseudoparticles, which are mapped onto the non-interacting $\alpha \nu$ pseudofermions. The $\alpha \nu$ pseudoparticle residual interactions are cancelled by the momentum functional $Q_{\alpha \nu}(q)/L$ of Eq. $(123)$. The information recorded in the pseudoparticle interactions is transferred over to the momentum two-pseudofermion phase shifts of that functional. These phase shifts control the few-electron spectral properties through the same unconventional momentum functional.

We introduced creation and annihilation operators for both the pseudoparticles and pseudofermions and derived the anticommutation relations of these pseudofermion operators. We also studied and discussed the transformation laws of the pseudoparticles under the pseudoparticle - pseudofermion transformation. This study included the discussion of the physics behind both these transformation laws and the invariance under the pseudoparticle - pseudofermion and pseudoparticle - rotated-pseudoparticle unitary transformations of the pseudoparticles for some specific bare-momentum values. Invariance under the former (and the latter) transformation is associated with a free, non-interacting, and delocalized character (and non-interacting and localized character) for these quantum objects. Thus
as a result of Heisenberg’s uncertainty relation, these two invariances never occur simultaneously. We also find that under the pseudoparticle - pseudofermion unitary transformation, the c0 and s1 pseudoparticles of bare momentum at the Fermi points are mapped onto corresponding pseudofermions whose momentum Fermi points expressions are related to the conformal dimensions of the two-component conformal-field theory primary fields \[20, 21\]. It follows that in the limit of low energy the conformal invariance of the 1D Hubbard model is related to the general pseudoparticle - pseudofermion unitary transformation.

The pseudofermion algebra introduced in this paper is used elsewhere in the evaluation of finite-energy few-electron spectral-function expressions \[10, 34\]. Fortunately, as a consequence of the above wave function factorization, the few-electron spectral functions can be written as a convolution of pseudofermion, \(-1/2\) Yang holon, and/or \(-1/2\) HL spinon spectral functions for all values of energy and on-site repulsion \(U\) \[34\].

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**APPENDIX A: PSEUDOPARTICLE REPRESENTATION, EFFECTIVE PSEUDOPARTICLE LATTICE, AND GROUND-STATE QUANTITIES**

In this Appendix, we summarize some aspects of the pseudoparticle, description which are needed for the studies of this paper. This includes a brief summary of the basic pseudoparticle properties, introduction to the pseudoparticle bare momentum \[31\], local pseudoparticle concept, and effective lattices \[35\]. In addition, we provide the ground-state pseudoparticle bare-momentum distribution functions, as well as other ground-state quantities.

According to the results of Ref. \[35\], the \(\alpha\nu\) pseudoparticle bare-momentum description obtained naturally from the Bethe-ansatz solution in Refs. \[22, 23, 29, 31\] is related by Fourier transform to a local \(\alpha\nu\) pseudoparticle description in terms of spatial coordinates of an effective \(\alpha\nu\) lattice. These concepts are needed and useful for both the operational pseudoparticle and pseudofermion representations introduced in this paper.

The \(\alpha\nu\) pseudoparticles carry bare momentum \(q\). This is the continuum bare momentum associated with the discrete bare-momentum values \(q_j\) such that,

\[
q_{j+1} - q_j = \frac{2\pi}{L}.
\]

(A1)

These discrete values read \[31\],

\[
q_j = \frac{2\pi}{L} I_j^{\alpha\nu},
\]

(A2)

where the \(I_j^{\alpha\nu}\) numbers are integers or half-odd integers \[30, 31\]. The index number \(j\) can have the values \(j = 1, 2, ..., N_{\alpha\nu}^*\). Here the number \(N_{\alpha\nu}^*\) equals the number of discrete bare-momentum values in the \(\alpha\nu\) pseudoparticle band and is such that,

\[
N_{\alpha\nu}^* = N_{\alpha\nu} + N_{\alpha\nu}^h,
\]

(A3)

where \(N_{\alpha\nu}\) is the number of \(\alpha\nu\) pseudoparticles and \(N_{\alpha\nu}^h\) is the number of \(\alpha\nu\) pseudoparticle holes. The latter number is given by,

\[
N_{c0}^h = N_a - N_{c0}; \quad N_{\alpha\nu}^h = 2 S_\alpha + 2 \sum_{\nu' = \nu + 1}^{\infty} (\nu' - \nu) N_{\alpha\nu'\nu}; \quad \alpha = c, s, \quad \nu > 0, \quad (A4)
\]

where \(S_c\) and \(S_s\) are the values of \(\eta\)-spin and spin, respectively, which read

\[
S_c = \frac{1}{2}[N_a - N_{c0}] - \sum_{\nu = 1}^{\infty} \nu N_{c\nu}; \quad S_s = \frac{1}{2} N_{c0} - \sum_{\nu = 1}^{\infty} \nu N_{s\nu}.
\]

(A5)
Combination of Eqs. (A7) and (A6) leads to,

\[ N_{\alpha\nu}^h = N_a - N_{c0} - \sum_{\nu' = 1}^{\infty} \left( \nu + \nu' - |\nu - \nu'| \right) N_{s\nu'}; \quad N_{s\nu}^h = N_{c0} - \sum_{\nu' = 1}^{\infty} \left( \nu + \nu' - |\nu - \nu'| \right) N_{s\nu'}; \quad \nu > 0. \tag{A6} \]

For \( \alpha \nu \neq \alpha 0 \) (and \( \alpha \nu = \alpha 0 \)) the numbers \( I_{\alpha\nu}^0 \) on the right-hand side of Eq. (A2) are integers (half-odd integers), if \( N_{\alpha\nu}^0 \) (and \( N_a/2 - \sum_{\alpha=\epsilon, s} \sum_{\nu'=1}^{\infty} N_{s\nu'} \)) is odd (even). This non-perturbative shake-up effect is related to the so-called orthogonality catastrophe \[^{41}\]. It reveals that the values of the available bare-momentum values \( q_j \) of each \( \alpha \nu \) pseudoparticle band might be different for different CPHS ensemble subspaces.

The holons, spinons, and pseudoparticles are related to the electrons through the rotated electrons, as discussed in Sec. III and in Ref. \[^{31},^{35}\]. The unitary transformation which maps electrons onto rotated electrons is defined in Sec. II. Since the momentum operator is invariant under such a transformation, also the lattice occupied by rotated electrons might be different for different CPHS ensemble subspaces. For \( \alpha \nu \neq \alpha 0 \) and \( \alpha \nu = \alpha 0 \) the numbers \( I_{\alpha\nu}^0 \) defined in Eqs. (A3), (A4), and (A6) might be different for different CPHS ensemble subspaces. This follows from Eq. (A7) that the value of the corresponding effective \( \alpha \nu \) pseudoparticle lattice. This lattice has the same length \( L = N_{\alpha\nu}^0 a_{\alpha\nu} \) as the original real-space lattice, where

\[ a_{\alpha\nu} = a \frac{N_a}{N_{\alpha\nu}} = \frac{L}{N_{\alpha\nu}^0}, \tag{A7} \]

is the effective \( \alpha \nu \) lattice constant. Since \( N_{\alpha\nu}^0 \) is different for different CPHS ensemble subspaces, it follows from Eq. (A7) that the value of the corresponding effective \( \alpha \nu \) lattice constant \( a_{\alpha\nu} \) also changes. The spatial coordinates of the effective \( \alpha \nu \) pseudoparticle lattice are \( x_j = a_{\alpha\nu} j \) where \( j = 1, 2, 3, ..., N_{\alpha\nu}^0 \). The value of \( N_{\alpha\nu}^0 \) defined in Eqs. (A3) - (A6) determines the corresponding limiting bare-momentum values of the \( \alpha \nu \) pseudoparticle Brillouin zone.

In the case of the \( \alpha 0 \) pseudoparticle band, the discrete bare-momentum values \( q_j \) belong to the following range,

\[ q_{\tilde{c}0} \leq q_j \leq q_{\tilde{c}0}^+, \tag{A8} \]

Here,

\[ q_{\tilde{c}0} = -q_{\tilde{c}0} = \frac{\pi}{a} \left[ 1 - \frac{1}{N_a} \right], \tag{A9} \]

for \( \tilde{N}_{c0} \) even and

\[ q_{\tilde{c}0}^+ = \frac{\pi}{a}; \quad q_{\tilde{c}0}^- = -\frac{\pi}{a} \left[ 1 - \frac{2}{N_a} \right], \tag{A10} \]

for \( \tilde{N}_{c0} \) odd. Note that since \( N_{\alpha\nu}^0 = N_a \), the lattice constant of the effective \( \alpha 0 \) lattice equals the lattice constant \( a \) of the original electronic lattice, that is \( a_{\alpha 0} = a \). Thus, for the \( \alpha 0 \) band the lattice constant \( a_{\alpha 0} \) has the same value for all CPHS ensemble subspaces.

For the \( \alpha \nu \) pseudoparticle bands such that \( \nu > 0 \), the discrete bare-momentum values \( q_j \) are distributed symmetrically relative to zero, and are such that \( |q_j| \leq q_{\alpha\nu} \). The two band momenta \( \pm q_{\alpha\nu} \) are the limiting bare-momentum values associated with the limits of the \( \alpha \nu \) pseudoparticle Brillouin zone. The limiting bare-momentum \( q_{\alpha\nu} \) reads,

\[ q_{\alpha\nu} = \frac{\pi}{a_{\alpha\nu}} \left[ 1 - \frac{1}{N_{\alpha\nu}} \right], \tag{A11} \]

where the lattice constant \( a_{\alpha\nu} \geq a \) is given in Eq. (A7).

We close this Appendix by providing some useful ground-state quantities. Following the results of Refs. \[^{27},^{30}\], the ground-state pseudoparticle bare-momentum distribution functions and \(-1/2\) Yang holon and \(-1/2\) HL spinon numbers read,

\[ N_{\alpha\nu}^0(q) = \Theta \left( q_{\tilde{c}0}^+ - q \right), \quad 0 \leq q \leq q_{\tilde{c}0}^+; \quad N_{\alpha\nu}^0(q) = \Theta \left( q - q_{\tilde{c}0}^- \right), \quad q_{\tilde{c}0}^- \leq q \leq 0; \tag{A12} \]
\[ N^0_{s_1}(q) = \Theta(q_{Fs_1} - q), \quad 0 \leq q \leq q_{s_1}; \quad N^0_{s_2}(q) = \Theta(q + q_{Fs_2}) , \quad -q_{s_1} \leq q \leq 0; \quad (A13) \]

\[ N^0_{\alpha\nu}(q) = L^0_{\alpha,-1/2} = 0; \quad -q_{\alpha\nu} \leq q \leq q_{\alpha\nu}; \quad \alpha = c, s; \quad \nu \geq 1 + \delta_{\alpha,s}. \quad (A14) \]

Except for 1/L corrections, the Fermi values on the right-hand side of Eq. (A12) are given by \( q^F_{\alpha\nu} = \pm q^0_{F\alpha}, \) where the ground-state values \( q^0_{F\alpha} \) and \( q^0_{Fc} \) are given in Eq. (68). Here we have ignored terms of order of 1/L and used the expressions \( k_{F\alpha} = \pi n_{\alpha} \) and \( 2k_F = \pi n. \) The expressions of the Fermi values including 1/L contributions are provided in Appendix C of Ref. [31].

For the ground-state CPHS ensemble subspace the expression of the number \( N^*_{\alpha\nu} \) given in Eqs. (A3), (A4), and (A6) simplifies. Let us denote by \( N^0_{\alpha\nu} \) the ground-state value of the numbers \( N^*_{\alpha\nu}. \) These numbers read,

\[ N^0_{\alpha\nu} = (N_{\alpha} - N); \quad N^0_{s_1} = N_{\uparrow}; \quad N^0_{s_2} = (N_{\downarrow} - N_1), \quad \nu > 1, \quad (A15) \]

whereas \( N^0_{c0} = N^*_{c0} \) is given by \( N^*_{c0} = N_{\alpha} \) for the whole Hilbert space.

In addition to the \(-1/2\) Yang holon and \(-1/2\) HL spinon numbers already given in Eq. (A14), the ground state belongs to a CPHS ensemble subspace with the following values for the pseudoparticle, \pm1/2 holon, and \pm1/2 spinon numbers,

\[ M^0_{c,-1/2} = 0; \quad M^0_{c,+1/2} = L^0_{c,+1/2} = N_{\alpha} - N; \quad N^0_{c} = N; \quad N^0_{c\nu} = 0, \quad (A16) \]

in the charge sector and

\[ M^0_{s,-1/2} = N^0_{s_1} = N_{\downarrow}; \quad M^0_{s,+1/2} = N_{\uparrow}; \quad N^0_{s\nu} = 0, \quad \nu \geq 2; \quad L^0_{s,+1/2} = N_{\uparrow} - N_\downarrow, \quad (A17) \]

in the spin sector.

Finally, note that for \( \nu > 0 \) the number \( A3 \) can be expressed in terms of the ground-state values \( A15 \) as follows,

\[ N^*_{\alpha\nu} = N^0_{\alpha\nu} + \Delta L_\alpha + 2 \sum_{\nu' = \nu+1}^\infty (\nu' - \nu) N_{\alpha\nu'}; \quad \alpha = c, s, \quad \nu > 0. \quad (A18) \]

Moreover, for \( \alpha\nu = s1 \) the pseudofermion hole number defined in Eqs. (A4) and (A6) can be written as,

\[ N^h_{s_1} = N_{c0} - 2 \sum_{\nu' = 1}^\infty N_{s\nu'}. \quad (A19) \]

**APPENDIX B: THE TWO-PSEUDOFERMION PHASE SHIFTS \( \Phi_{\alpha\nu,\alpha'\nu'}(r,r') \)**

Here we provide the set of integral equations which define the two-pseudofermion phase shifts \( \Phi_{\alpha\nu,\alpha'\nu'}(r,r') \) on the right-hand side of Eqs. (50) - (51), (74), and (84).

Let us start by introducing the following Fermi surface parameters,

\[ r^0_c = \frac{4t \sin Q}{U}; \quad r^0_s = \frac{4t B}{U}, \quad (B1) \]

where the parameters \( Q \) and \( B \) are defined in Eq. (15).

In order to derive the integral equations which define the two-pseudofermion phase shifts \( \Phi_{\alpha\nu,\alpha'\nu'}(r,r') \), we first use in Eqs. (19) - (21) the bare-momentum distribution functions of the general form (59) as well as the rapidity functional expressions given in Eq. (77) and rapidity-momentum functional expression provided in Eq. (78). Such functionals are written in terms of corresponding ground-state rapidity and rapidity-momentum functions whose argument is the
momentum functional. The expression of this functional is given in Eq. \((B2)\) with the momentum \(Q_{\alpha \nu}(q)/L\) provided in Eq. \((B3)\). Expansion of the obtained equations up to first order in the bare-momentum distribution function deviations on the right-hand side of Eqs. \((B3)\) and \((B4)\) leads to expression \((B4)\) with the two-pseudofermion phase shift \(\Phi_{\alpha \nu, \alpha' \nu'}(r, r')\) uniquely defined by the integral equations given below. A first group of two-pseudofermion phase shifts obey integral equations by their own. These equations read,

\[
\Phi_{s1, c0} (r, r') = -\frac{1}{\pi} \arctan (r - r') + \int_{-r_0^c}^{r_0^c} dr'' G(r, r'') \Phi_{s1, c0} (r'', r') , \tag{B2}
\]

\[
\Phi_{s1, c\nu} (r, r') = -\frac{1}{\pi^2} \int_{-r_0^c}^{r_0^c} dr'' \frac{\arctan \left( \frac{r'' - r'}{\nu} \right)}{1 + (r'' - r')^2} + \int_{-r_0^c}^{r_0^c} dr'' G(r, r'') \Phi_{s1, c\nu} (r'', r') , \tag{B3}
\]

and

\[
\Phi_{s1, s\nu} (r, r') = \frac{\delta_{1, \nu}}{\pi} \arctan \left( \frac{r - r'}{2} \right) + \left( 1 - \delta_{1, \nu} \right) \left\{ \arctan \left( \frac{r - r'}{\nu - 1} \right) + \arctan \left( \frac{r - r'}{\nu + 1} \right) \right\}
- \frac{1}{\pi^2} \int_{-r_0^c}^{r_0^c} dr'' \frac{\arctan \left( \frac{r'' - r'}{\nu} \right)}{1 + (r'' - r')^2} + \int_{-r_0^c}^{r_0^c} dr'' G(r, r'') \Phi_{s1, s\nu} (r'', r') . \tag{B4}
\]

Here the kernel \(G(r, r')\) is given by,

\[
G(r, r') = -\frac{1}{2\pi} \left[ \frac{1}{1 + ((r - r')^2/2)^2} \right] \left[ 1 - \frac{1}{2} \left( t(r) + t(r') + \frac{l(r) - l(r')}{r - r'} \right) \right] , \tag{B5}
\]

where

\[
t(r) = \frac{1}{\pi} \left[ \arctan(r + r_0^c) - \arctan(r - r_0^c) \right] , \tag{B6}
\]

and

\[
l(r) = \frac{1}{\pi} \left[ \ln(1 + (r + r_0^c)^2) - \ln(1 + (r - r_0^c)^2) \right] . \tag{B7}
\]

The kernel defined in Eqs. \((B5)-(B7)\) was first introduced in Ref. \[25\] within the low-energy two-component \(c \equiv c0\) and \(s \equiv s1\) pseudoparticle theory studied in that reference.

A second group of two-pseudofermion phase shifts are expressed in terms of the basic functions given in Eqs. \((B2)-(B4)\) as follows,

\[
\Phi_{c0, c0} (r, r') = \frac{1}{\pi} \int_{-r_0^c}^{r_0^c} dr'' \frac{\Phi_{s1, c0} (r'', r')}{1 + (r'' - r')^2} , \tag{B8}
\]

\[
\Phi_{c0, c\nu} (r, r') = -\frac{1}{\pi} \arctan \left( \frac{r - r'}{\nu} \right) + \frac{1}{\pi} \int_{-r_0^c}^{r_0^c} dr'' \frac{\Phi_{s1, c\nu} (r'', r')}{1 + (r'' - r')^2} , \tag{B9}
\]

and

\[
\Phi_{c0, s\nu} (r, r') = -\frac{1}{\pi} \arctan \left( \frac{r - r'}{\nu} \right) + \frac{1}{\pi} \int_{-r_0^c}^{r_0^c} dr'' \frac{\Phi_{s1, s\nu} (r'', r')}{1 + (r'' - r')^2} . \tag{B10}
\]
Finally, the remaining two-pseudofermion phase shifts can be expressed either in terms of the functions \( \Theta \) only,

\[
\phi_{c\nu, \nu'} (r, r') = \frac{1}{\pi} \arctan \left( \frac{r - r'}{\nu} \right) - \frac{1}{\pi} \int_{-r_x}^{r_x} dr'' \frac{\bar{\Phi}_{c\nu, c\nu'} (r'', r')}{\nu \left[ 1 + \left( \frac{r - r''}{\nu} \right)^2 \right]},
\]

(B11)

and

\[
\phi_{s\nu, \nu'} (r, r') = \frac{1}{\pi} \frac{\arctan \left( \frac{r - r'}{\nu} \right) + \int_{-r_x}^{r_x} dr'' \frac{\bar{\Phi}_{s\nu, c\nu'} (r'', r')}{\nu \left[ 1 + \left( \frac{r - r''}{\nu} \right)^2 \right]} - \int_{-r_x}^{r_x} dr'' \bar{\Phi}_{s\nu, c\nu'} (r'', r')}{\nu \left[ 1 + \left( \frac{r - r''}{\nu} \right)^2 \right]},
\]

(B13)

or both in terms of the basic functions \( \Theta \) and of the phase shifts \( \Theta \).

\[
\phi_{s\nu, c\nu'} (r, r') = \frac{1}{\pi} \frac{\arctan \left( \frac{r - r'}{\nu} \right) + \int_{-r_x}^{r_x} dr'' \frac{\bar{\Phi}_{s\nu, c\nu'} (r'', r')}{\nu \left[ 1 + \left( \frac{r - r''}{\nu} \right)^2 \right]} - \int_{-r_x}^{r_x} dr'' \bar{\Phi}_{s\nu, c\nu'} (r'', r') \Theta_{\nu, 1} (r - r'')}{\nu \left[ 1 + \left( \frac{r - r''}{\nu} \right)^2 \right]} ; \quad \nu > 1, \quad (B14)
\]

\[
\phi_{s\nu, c\nu'} (r, r') = \frac{1}{\pi} \frac{\arctan \left( \frac{r - r'}{\nu} \right) + \int_{-r_x}^{r_x} dr'' \frac{\bar{\Phi}_{s\nu, s\nu'} (r'', r')}{\nu \left[ 1 + \left( \frac{r - r''}{\nu} \right)^2 \right]} - \int_{-r_x}^{r_x} dr'' \bar{\Phi}_{s\nu, s\nu'} (r'', r') \Theta_{\nu, 1} (r - r'')}{\nu \left[ 1 + \left( \frac{r - r''}{\nu} \right)^2 \right]} ; \quad \nu > 1, \quad (B15)
\]

and

\[
\phi_{s\nu, s\nu'} (r, r') = \frac{\Theta_{\nu, \nu'} (r - r')}{2\pi} + \frac{1}{\pi} \int_{-r_x}^{r_x} dr'' \frac{\bar{\Phi}_{s\nu, s\nu'} (r'', r')}{\nu \left[ 1 + \left( \frac{r - r''}{\nu} \right)^2 \right]} - \int_{-r_x}^{r_x} dr'' \bar{\Phi}_{s\nu, s\nu'} (r'', r') \Theta_{\nu, 1} (r - r'')}{\nu \left[ 1 + \left( \frac{r - r''}{\nu} \right)^2 \right]} ; \quad \nu > 1. \quad (B16)
\]

In the above two-pseudofermion phase shift expressions the functions \( \Theta_{\nu, \nu'} (x) \) and \( \Theta_{\nu, 1} (x) \) read,

\[
\Theta_{\nu, \nu'} (x) = \delta_{\nu, \nu'} \left\{ 2 \arctan \left( \frac{x}{2\nu} \right) + \sum_{l=1}^{\nu - 1} 4 \arctan \left( \frac{x}{2l} \right) \right\} + (1 - \delta_{\nu, \nu'}) \left\{ 2 \arctan \left( \frac{x}{\nu - \nu'} \right) \right\},
\]

(B17)

and

\[
\Theta_{\nu, 1} (x) = \frac{d\Theta_{\nu, \nu'} (x)}{dx} = \delta_{\nu, \nu'} \left\{ \frac{1}{\nu \left[ 1 + \left( \frac{x}{\nu} \right)^2 \right]} + \sum_{l=1}^{\nu - 1} \frac{2}{l \left[ 1 + \left( \frac{x}{\nu} \right)^2 \right]} \right\} + (1 - \delta_{\nu, \nu'}) \left\{ \frac{2}{\nu - \nu'} \left[ 1 + \left( \frac{x}{\nu - \nu'} \right)^2 \right] \right\},
\]

(B18)

respectively. Note that the latter function is the \( x \) derivative of the function defined in Eq. (B17).

In spite of the different notation and except for simplifications introduced here as a result of some integrations performed analytically, the integral equations (B30)-(B40) of Ref. 29 are equivalent to the system of coupled integral equations (B30)-(B40) of Ref. 29.
The bare-momentum two-pseudofermion phase shifts are defined by Eq. (B4), in terms of the above phase shifts $\Phi(x, y)$ associated with the integral equations (B2)-(B5). In applications of the pseudofermion description to the study of few-electron spectral properties the phase shifts $\Phi_{s1, s1}(q, q'), \Phi_{s1, c0}(q, q'), \Phi_{c0, c0}(q, q'), \Phi_{c0, c1}(q, q'), \Phi_{o1, c0}(q, q'), \Phi_{e0, c0}(q, q'), \Phi_{e0, c1}(g, g')$ with $q$ at the Fermi points play a major role [31]. By manipulation of the above integral equations, we find that in the limits of zero spin density $m \to 0$ and $U/t \to 0$, these two-pseudofermion phase shifts with $q$ at the Fermi points and the second bare-momentum denoted by $q$ are given by,

$$
\Phi_{s1, s1}(\tau k_F, q) = \frac{\tau}{2\sqrt{2}}, \quad q \neq \tau k_F;
\Phi_{s1, c0}(\tau k_F, q) = \frac{1}{2\sqrt{2}} \left( -\Theta(2k_F - |q|) + \text{sgn}(q) \Theta(|q| - 2k_F) \right), \quad q \neq \tau 2k_F;
\Phi_{e0, c0}(\tau 2k_F, q) = \frac{1}{2\sqrt{2}} \left( -\Theta(2k_F - |q|) + \text{sgn}(q) \Theta(|q| - 2k_F) \right), \quad q \neq \tau 2k_F;
\Phi_{e0, c1}(\tau 2k_F, q) = \frac{\text{sgn}(q)}{\sqrt{2}}, \quad q \neq 0;
$$

$$
\Phi_{s1, c1}(\tau k_F, q) = 0, \quad \tau = \pm 1,
$$

respectively, where $\Theta(x) = 1$ for $x \geq 0$, and $\Theta(x) = 0$ for $x < 0$.

For $m \to 0$ and $U/t \to \infty$ the phase-shift expression (B19) for $\Phi_{s1, s1}(\tau k_F, q)$ remains valid, whereas the two-pseudofermion phase shifts $\Phi_{s1, c0}(q, q'), \Phi_{c0, c0}(q, q'), \Phi_{s1, c1}(q, q'), \Phi_{s1, c1}(q, q')$, and $\Phi_{e0, c1}(q, q')$ with $q$ at the Fermi points and the second bare-momentum denoted by $q$ read,

$$
\Phi_{s1, c0}(\tau k_F, q) = \frac{\tau}{2\sqrt{2}}, \quad \tau = \pm 1,
\Phi_{c0, c0}(\tau 2k_F, q) = \Phi_{s1, c1}(\tau k_F, q) = 0, \quad \tau = \pm 1,
\Phi_{e0, c1}(\tau 2k_F, q) = \frac{q}{4k_F}, \quad |q| \leq k_F, \quad \tau = \pm 1,
$$

and

$$
\Phi_{e0, c1}(\tau 2k_F, q) = \frac{q}{2|\tau - 2k_F|}, \quad |q| \leq \pi - 2k_F, \quad \tau = \pm 1.
$$
APPENDIX C: TRANSFORMATION LAWS UNDER THE PSEUDOPARTICLE - PSEUDOFERMION SUBSPACE UNITARY ROTATION

Here we study the momentum functional \( Q_{\alpha\nu}(q)/L \) given in Eq. (123) for specific values of \( q \). In some cases we consider general excited states obeying relations (140), whereas in other cases we consider particular cases of such general excited states.

We start by confirming that the second term on the right-hand side of Eq. (123) is of \([1/L]^2\) order. Let us consider a more general situation and also confirm the validity of Eq. (141) and show that \((q - q')\) is invariant under the pseudoparticle - pseudofermion transformation when \( q = q' + N_{ph}[2\pi/L] \) and \( N_{ph} = \pm 1, \pm 2, \ldots \) is a finite integer number. Equation (123) corresponds to the particular case when \( N_{ph} = 1 \). Thus we want to show that the quantity \([Q_{\alpha\nu}(q + N_{ph}[2\pi/L]) - Q_{\alpha\nu}(q)]/L\) involving the functional (123) is of \((1/L)^2\) order. By expressing this quantity in terms of the derivative \( \partial Q_{\alpha\nu}(q)/\partial q \) one finds,

\[
\frac{Q_{\alpha\nu}(q + N_{ph}[2\pi/L]) - Q_{\alpha\nu}(q)}{2\pi/L N_{ph}^2} \frac{\partial Q_{\alpha\nu}(q)}{\partial q}.
\]

(C1)

Analysis of the form of the derivative \( \partial Q_{\alpha\nu}(q)/\partial q \) reveals that it is of \([1/L]^1\) order. This is the confirmation that the momentum contribution (C1) is of second order in \( 1/L \) order. Let us consider that \( \Lambda_{\alpha\nu}(\pm q_{\alpha\nu}) = \pm \infty \) and can be such that \( q' \to \pm q_{\alpha\nu} \) but \( q' \neq q_{\alpha\nu} \).

First we note that according to Eq. (42), the two-pseudofermion phase shift \( \Phi_{\alpha\nu, \alpha'\nu'}(t; q_{\alpha\nu}, q') \Delta N_{\alpha\nu}(q') \); \( \alpha = c, s; \; \nu > 0; \; t = \pm 1 \),

(C2)

and confirm that it is given by expression (124) where the deviation \( \Delta q_{\alpha\nu} \) is provided in Eq. (125). It is assumed that the bare momentum \( q' \) belongs to the domain \( q' \in (-q_{\alpha\nu} - q_{\alpha'\nu'}, +q_{\alpha'\nu'}) \) and can be such that \( q' \to \pm q_{\alpha\nu} \) but \( q' \neq q_{\alpha'\nu'} \).

We start by confirming that the second term on the right-hand side of Eq. (79) is of \([1/L]^2\) order. By expressing this quantity in terms of the pseudoparticle bare momentum on the right-hand side of Eq. (C2) leads to,

\[
\Phi_{\alpha\nu, \alpha'\nu'}(\pm q_{\alpha\nu}, q') = \Phi_{\alpha\nu, \alpha'\nu'}\left(\pm \infty, \frac{\Lambda_{\alpha\nu}(q')}{u}\right),
\]

(C3)

where the two-pseudofermion phase shift \( \Phi_{\alpha\nu, \alpha'\nu'}(r, r') \) is defined in Appendix B. In order to achieve this result we used that \( \Lambda_{\alpha\nu}(\pm q_{\alpha\nu}) = \pm \infty \), as given in Eq. (12).

By manipulation of the integral equations given in Appendix B, we find that all the two-pseudofermion phase shifts of form (C3) vanish except the following ones,

\[
\Phi_{\alpha\nu, c0}(\pm q_{\alpha\nu}, q') = \pm \frac{[\delta_{\alpha, c} - \delta_{\alpha, s}]}{2}; \quad \Phi_{\alpha\nu, \alpha'\nu'}(\pm q_{\alpha\nu}, q') = \pm \frac{\delta_{\nu, \nu'} - |\nu - \nu'|}{2}; \quad \alpha = c, s; \; \nu, \nu' > 0.
\]

(C4)

Use of Eq. (C4) on the right-hand side of Eq. (C2) leads to,

\[
\frac{Q_{\alpha\nu}(\pm q_{\alpha\nu})}{L} = \pm \frac{1}{2} \left[ \int_{-q_{\alpha\nu}}^{q_{\alpha\nu}} dq' \Delta N_{\alpha\alpha}(q') - [\delta_{\alpha, c} - \delta_{\alpha, s}] \int_{q_{\alpha\nu}}^{q_{\alpha\nu}^+} dq' \Delta N_{c0}(q') - \sum_{\nu = 1}^{\infty} \int_{-q_{\alpha\nu}}^{q_{\alpha\nu}^+} dq' \Delta N_{\alpha\nu}(q') \right] \]

\[
= \pm \frac{\pi}{L} \left[ \Delta N_{\alpha\nu} - [\delta_{\alpha, c} - \delta_{\alpha, s}] \Delta N_{c0} - \sum_{\nu = 1}^{\infty} (\nu + \nu' - |\nu - \nu'|) \Delta N_{c\nu'} \right] = \mp \Delta q_{\alpha\nu}; \quad \alpha = c, s; \; \nu > 0,
\]

(C5)

where we also used expression (124). Finally, note that expression (C5) is equivalent to Eq. (126).

Let us now consider the quantity,

\[
\frac{Q_{\alpha\nu}(\pm q_{\alpha\nu})}{L} = \sum_{\alpha' = c, s} \sum_{\nu' = 1 - \delta_{\alpha', c}}^{\infty} \int_{-q_{\alpha'\nu'}}^{q_{\alpha'\nu'}} dq' \Phi_{\alpha\nu, \alpha'\nu'}(q_{\alpha\nu}, q') \Delta N_{\alpha'\nu'}(q'),
\]

(C6)
and confirm that it obeys relation (128). This is easily confirmed by noting that according to Eq. (22) the two-pseudofermion phase shifts $\Phi_{c\alpha, \alpha'}(q_{L\alpha}^+)$ on the right-hand side of Eq. (22) are such that,

$$\Phi_{c\alpha, \alpha'}^{\pm}(q_{L\alpha}^+) = \Phi_{c\alpha, \alpha'}(0, q^+)' = \Phi_{c\alpha, \alpha'}^{\pm}(0, \frac{\Lambda_{0}\alpha'(q^+)}{u}),$$  \hspace{1cm} (C7)

where the two-pseudofermion phase shifts $\Phi_{c\alpha, \alpha'}^{\pm}(r, r')$ are defined in Appendix B. The result (C7) implies the validity of Eq. (128).

Our next task involves evaluation of the momentum functional $Q_{\alpha\nu}(q)/L$ given in Eq. (73) for $q = \iota q_{F\alpha\nu}$ in the particular case when the excited states associated with the deviations $\Delta\mathcal{N}_{\alpha'}(q^+)$ on the right-hand side of that equation are J-CPHS ground states. The Fermi bare momentum values that limit the compact pseudoparticle bare-momentum occupancy configurations of these states are given in Eqs. (70) and (71). While the above expressions derived in this Appendix refer to general excited states whose deviations obey relations (76), the following results are valid only for the particular case when these excited states are J-CPHS ground-states. Our goal is to arrive to expressions (130)-(132). This implies evaluation of the following quantity,

$$\Delta \bar{q}_{F\alpha\nu, \iota} = \iota \frac{2\pi}{L} \Delta \mathcal{N}_{\alpha\nu, \iota} + \frac{Q_{\alpha\nu}(\iota q_{F\alpha\nu}^0)}{L},$$ \hspace{1cm} (C8)

where

$$\frac{Q_{\alpha\nu}(\iota q_{F\alpha\nu}^0)}{L} = \sum_{\alpha' = c, s} \sum_{\nu' = 1}^{Q_{\alpha\nu}} \int_{-q_{F\alpha\nu}^{\nu'}}^{q_{F\alpha\nu}^{\nu'}} dq^+ \Phi_{\alpha\nu, \alpha'}(\iota q_{F\alpha\nu}^0, q^+) \Delta \mathcal{N}_{\alpha'}(q^+),$$ \hspace{1cm} (C9)

and the bare-momentum distribution function deviations $\Delta \mathcal{N}_{\alpha'}(q^+)$ correspond to final J-CPHS ground states.

The phase shifts $\Phi_{\alpha\nu, \alpha'}(q, q^+)$ and $\Phi_{\alpha\nu, \alpha'}(r, r')$ have the following property,

$$\Phi_{\alpha\nu, \alpha'}(q, q^+) = -\Phi_{\alpha\nu, \alpha'}(-q, -q^+), \hspace{1cm} \Phi_{\alpha\nu, \alpha'}(r, r') = -\Phi_{\alpha\nu, \alpha'}(-r, -r').$$ \hspace{1cm} (C10)

This symmetry is found by analysis of the integral equations given in Appendix B, which define the two-pseudofermion phase shifts $\Phi_{c\alpha, \alpha'}^{\pm}(q_{L\alpha}^+)$ associated with the phase shifts $\Phi_{c\alpha, \alpha'}(q_{L\alpha}^+, q^+)$ through Eq. (22). Once the ground-state rapidity functions that appear on the right-hand side of Eq. (74) in the argument of the two-pseudofermion phase shifts $\Phi_{c\alpha, \alpha'}^{\pm}(r, r')$ are odd functions of the bare momentum, the second relation given in Eq. (C10) implies the validity of the first relation of the same equation.

Since the pseudoparticle bare-momentum distribution function deviations of the J-CPHS ground-states include creation or annihilation of pseudoparticles in the vicinity of the Fermi points only, we can replace $q^+$ by $\text{sgn}(q^+) q_{F\alpha\nu}^0$ in the argument of the phase shift $\Phi_{\alpha\nu, \alpha'}(\iota q_{F\alpha\nu}^0, q^+)$ on the right-hand side of Eq. (C9). This leads to,

$$\Delta \bar{q}_{F\alpha\nu, \iota} = \iota \frac{2\pi}{L} \Delta \mathcal{N}_{\alpha\nu, \iota} + \sum_{\alpha' = c, s} \sum_{\nu' = 1}^{Q_{\alpha\nu}} \int_{-q_{F\alpha\nu}^{\nu'}}^{q_{F\alpha\nu}^{\nu'}} dq^+ \Delta \mathcal{N}_{\alpha'}(q^+)(\iota q_{F\alpha\nu}^0, \text{sgn}(q^+) q_{F\alpha\nu}^0).$$ \hspace{1cm} (C11)

By performing the $q^+$ integrations and using Eq. (C10) we arrive to,

$$\Delta \bar{q}_{F\alpha\nu, \iota} = \iota \frac{2\pi}{L} \sum_{\alpha' = c, s} \sum_{\nu' = 1}^{Q_{\alpha\nu}} \sum_{\iota'' = \pm 1} \left[ \iota'' \delta_{\alpha, \alpha'} \delta_{\nu, \nu'} \delta_{\iota, \iota''} + \iota \Phi_{\alpha\nu, \alpha'}(0, \iota' q_{F\alpha\nu}^0, \text{sgn}(q^+) q_{F\alpha\nu}^0) \right] \frac{\Delta \mathcal{N}_{\alpha'}(q^+)}{2} + \iota'' \Delta J_{\alpha'}.$$ \hspace{1cm} (C12)

After performing the $\iota''$ summation and using again Eq. (C10) we find,

$$\Delta \bar{q}_{F\alpha\nu, \iota} = \iota \frac{2\pi}{L} \sum_{\alpha' = c, s} \sum_{\nu' = 1}^{Q_{\alpha\nu}} \left[ \delta_{\alpha, \alpha'} \delta_{\nu, \nu'} + \sum_{\iota'' = \pm 1} \iota' \Phi_{\alpha\nu, \alpha'}(0, \iota' q_{F\alpha\nu}^0) \right] \frac{\Delta \mathcal{N}_{\alpha'}(q^+)}{2} + \frac{2\pi}{L} \sum_{\alpha' = c, s} \sum_{\nu' = 1}^{Q_{\alpha\nu}} \sum_{\iota'' = \pm 1} \left[ \delta_{\alpha, \alpha'} \delta_{\nu, \nu'} + \sum_{\nu'' = \pm 1} \iota' \Phi_{\alpha\nu, \alpha'}(0, \iota' q_{F\alpha\nu}^0) \right] \Delta J_{\alpha'}.$$ \hspace{1cm} (C13)
Note that since $\Delta f_{\alpha\nu,t} = \frac{2\pi}{\sqrt{\lambda}} \Delta N_{\alpha\nu}$, this result is equivalent to Eqs. (30)–(32).

Finally, let us confirm the validity of Eq. (33) and show that the momentum functional $Q_{\alpha\nu}(0)/L$ given in Eq. (26) vanishes for $\alpha\nu$ pseudoparticle branches such that $\alpha, \nu \neq c0$ and $\alpha, \nu \neq s1$ provided that the deviations $\Delta N_{\alpha0}(q)$ and $\Delta N_{s1}(q)$ on the right-hand side of that equation obey Eq. (140). In this case, the excited states are partial J-CPHS ground states. These are states that have J-CPHS ground-state occupancy configurations for $\alpha\nu$ pseudoparticles such that $\alpha, \nu \neq c0$ and $\alpha, \nu \neq s1$.

There are no $\alpha\nu$ pseudoparticles belonging to branches other than $c0$ and $s1$ in the initial ground state. It follows that excited states obeying relations (17) and having J-CPHS ground-state occupancy configurations for $\alpha\nu$ pseudoparticles such that $\alpha, \nu \neq c0$ and $\alpha, \nu \neq s1$ have only a vanishing density of these quantum objects. Such J-CPHS ground state occupancy configurations correspond to compact bare-momentum domains of vanishing width centered at $q = q_{\alpha\nu}^{0} = 0$. The generation from the ground state of these excited states includes creation of $\alpha\nu$ pseudoparticles such that $\alpha, \nu \neq c0$ and $\alpha, \nu \neq s1$, bare momentum $q \rightarrow 0$. A simple and useful example, is the creation of a single $\alpha\nu$ pseudoparticle at bare momentum $q = q_{\alpha\nu}^{0} = 0$. The use of Eq. (141) in expression (132) leads to

$$\Phi_{\alpha\nu,\alpha'\nu'}(0, t') = 0 ; \quad \nu, \nu' > 0 \quad \text{for} \quad \alpha, \alpha' = c ; \quad \nu, \nu' > 1 \quad \text{for} \quad \alpha, \alpha' = s .$$

(C14)

Use of Eq. (C14) in expression (132) leads to

$$\xi_{\alpha\nu,\alpha'\nu'} = \delta_{\alpha,\alpha'} \delta_{\nu,\nu'} ; \quad \nu, \nu' > 0 \quad \text{for} \quad \alpha, \alpha' = c ; \quad \nu, \nu' > 1 \quad \text{for} \quad \alpha, \alpha' = s ; \quad j = 0, 1 .$$

(C15)

Moreover, it follows from Eq. (140) that

$$\Phi_{\alpha\nu, c0}(0, q') = -\Phi_{\alpha\nu, c0}(0, -q') ; \quad \Phi_{\alpha\nu, s1}(0, q') = \Phi_{\alpha\nu, s1}(0, -q') ; \quad \nu > \delta_{\alpha,s} .$$

(C16)

Let us next consider the same type of excited states but with small deviations $\Delta N_{\alpha0}(q)$ and $\Delta N_{s1}(q)$ obeying Eq. (140). These excited states have vanishing momentum and their $c0$ and $s1$ current number deviations $\Delta J_{c0}$ and $\Delta J_{s1}$, respectively, vanish, i.e. $\Delta J_{c0} = \Delta J_{s1} = 0$. In this case it follows from Eqs. (C14)–(C15) that

$$\frac{Q_{\alpha\nu}(0)}{L} = \sum_{\alpha' = c, s} \sum_{\nu' = 1}^{\infty} \int_{-q_{\alpha'0}}^{q_{\alpha'0}} dq' \Phi_{\alpha\nu, \alpha'\nu'}(0, q') \Delta N_{\alpha'\nu'}(q')$$

$$= \int_{-q_{c0}}^{q_{c0}} dq' \Phi_{\alpha\nu, c0}(0, q') \Delta N_{c0}(q') + \int_{-q_{s1}}^{q_{s1}} dq' \Phi_{\alpha\nu, s1}(0, q') \Delta N_{s1}(q')$$

$$+ \frac{2\pi}{L} \sum_{\alpha' = c, s} \sum_{\nu' = 1}^{\infty} \left[ \int_{t=\pm1}^{\infty} \Phi_{\alpha\nu, \alpha'\nu'}(0, t'0) \frac{\Delta N_{\alpha'\nu'}}{2} + \sum_{t'=\pm1} \Phi_{\alpha\nu, \alpha'\nu'}(0, t'0) \Delta J_{\alpha'\nu'} \right] = 0 .$$

(C17)

This result confirms the validity of Eq. (33).

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