Let $R$ be a commutative (Noetherian) local ring of prime characteristic $p$ that is $F$-pure. This paper is concerned with comparison of three finite sets of radical ideals of $R$, one of which is only defined in the case when $R$ is $F$-finite (that is, is finitely generated when viewed as a module over itself via the Frobenius homomorphism). Two of the afore-mentioned three sets have links to tight closure, via test ideals. Among the aims of the paper are a proof that two of the sets are equal, and a proposal for a generalization of I. M. Aberbach’s and F. Enescu’s splitting prime.

0. Introduction

Throughout the paper, let $(R, \mathfrak{m})$ be a commutative (Noetherian) local ring of prime characteristic $p$ having maximal ideal $\mathfrak{m}$. In recent years, the study of $R$-modules with a Frobenius action has assisted in the development of the theory of tight closure over $R$. An $R$-module with a Frobenius action can be viewed as a left module over the Frobenius skew polynomial ring over $R$, and such left modules will play a central role in this paper.

The Frobenius skew polynomial ring over $R$ is described as follows. Throughout, $f : R \to R$ denotes the Frobenius ring homomorphism, for which $f(r) = r^p$ for all $r \in R$. The Frobenius skew polynomial ring over $R$ is the skew polynomial ring $R[x,f]$ associated to $R$ and $f$ in the indeterminate $x$; as a left $R$-module, $R[x,f]$ is freely generated by $(x^i)_{i \geq 0}$, and so consists of all polynomials $\sum_{i=0}^{n} r_i x^i$, where $n \geq 0$ and $r_0, \ldots, r_n \in R$; however, its multiplication is subject to the rule $xr = f(r)x = r^px$ for all $r \in R$.

We can think of $R[x,f]$ as a positively-graded ring $R[x,f] = \bigoplus_{n=0}^{\infty} R[x,f]_n$, where $R[x,f]_n = Rx^n$ for $n \geq 0$. The graded annihilator of a left $R[x,f]$-module $H$ is the largest graded two-sided ideal of $R[x,f]$ that annihilates $H$; it is denoted by $\text{gr-ann}_{R[x,f]} H$.

Let $G$ be a left $R[x,f]$-module that is $x$-torsion-free in the sense that $xg = 0$, for $g \in G$, only when $g = 0$. Then $\text{gr-ann}_{R[x,f]} G = bR[x,f]$, where $b = (0 :_R G)$ is a radical ideal. See [11, Lemma 1.9]. We shall use $\mathcal{I}(G)$ (or $\mathcal{I}_R(G)$) to denote the set of $R$-annihilators of the $R[x,f]$-submodules of $G$; we shall refer to the members of $\mathcal{I}(G)$
as the \(G\)-special \(R\)-ideals. For a graded two-sided ideal \(\mathfrak{B}\) of \(R[x, f]\), we denote by \(\text{ann}_G(\mathfrak{B})\) or \(\text{ann}_G\mathfrak{B}\) the \(R[x, f]\)-submodule of \(G\) consisting of all elements of \(G\) that are annihilated by \(\mathfrak{B}\). Also, we shall use \(\mathcal{A}(G)\) to denote the set of special annihilator submodules of \(G\), that is, the set of \(R[x, f]\)-submodules of \(G\) of the form \(\text{ann}_G(\mathfrak{B})\), where \(\mathfrak{B}\) is a graded two-sided ideal of \(R[x, f]\). In [11 §1], the present author showed that there is a sort of ‘Galois’ correspondence between \(\mathcal{I}(G)\) and \(\mathcal{A}(G)\). In more detail, there is an order-reversing bijection, \(\Delta : \mathcal{A}(G) \rightarrow \mathcal{I}(G)\) given by

\[
\Delta : N \mapsto (\text{gr-ann}_{R[x, f]} N) \cap R = (0 :_R N).
\]

The inverse bijection, \(\Delta^{-1} : \mathcal{I}(G) \rightarrow \mathcal{A}(G)\), also order-reversing, is given by

\[
\Delta^{-1} : b \mapsto \text{ann}_G(\mathfrak{b}R[x, f]).
\]

We shall be mainly concerned in this paper with the situation where \(R\) is \(F\)-pure. We remind the reader what this means. For \(j \in \mathbb{N}\) (the set of positive integers) and an \(R\)-module \(M\), let \(M^{(j)}\) denote \(M\) considered as a left \(R\)-module in the natural way and as a right \(R\)-module via \(f^j\), the \(j\)th iterate of the Frobenius ring homomorphism. Then \(R\) is \(F\)-pure if, for every \(R\)-module \(M\), the natural map \(M \rightarrow R^{(1)} \otimes_R M\) (which maps \(m \in M\) to \(1 \otimes m\)) is injective.

Note that \(R^{(j)} \cong Rx^j\) as \((R, R)\)-bimodules. Let \(i \in \mathbb{N}_0\), the set of non-negative integers. When we endow \(Rx^i\) and \(Rx^j\) with their natural structures as \((R, R)\)-bimodules (inherited from their being graded components of \(R[x, f]\)), there is an isomorphism of \((\text{left})\) \(R\)-modules \(\phi : Rx^{i+j} \otimes_R M \xrightarrow{\cong} Rx^i \otimes_R (Rx^j \otimes_R M)\) for which \(\phi(rx^{i+j} \otimes m) = rx^i \otimes (x^j \otimes m)\) for all \(r \in R\) and \(m \in M\). It follows that \(R\) is \(F\)-pure if and only if the left \(R[x, f]\)-module \(R[x, f] \otimes_R M\) is \(x\)-torsion-free for every \(R\)-module \(M\). This means that, when \(R\) is \(F\)-pure, there is a good supply of natural \(x\)-torsion-free left \(R[x, f]\)-modules.

In fact, we shall use \(\Phi\) (or \(\Phi_R\) when it is desirable to specify which ring is being considered) to denote the functor \(R[x, f] \otimes_R \bullet\) from the category of \(R\)-modules (and all \(R\)-homomorphisms) to the category of all \(\mathbb{N}_0\)-graded \(R[x, f]\)-modules (and all homogeneous \(R[x, f]\)-homomorphisms). For an \(R\)-module \(M\), we shall identify \(\Phi(M)\) with \(\bigoplus_{n\in\mathbb{N}_0} Rx^n \otimes_R M\), and (usually) identify its 0th component \(R \otimes_R M\) with \(M\), in the obvious ways.

Let \(E\) be the injective envelope of the simple \(R\)-module \(R/\mathfrak{m}\). We shall be concerned with \(\Phi(E)\), the \(\mathbb{N}_0\)-graded left \(R[x, f]\)-module \(\bigoplus_{n\in\mathbb{N}_0} Rx^n \otimes_R E\). Assume now that \(R\) is \(F\)-pure. In [12, Corollary 4.11], the present author proved that the set \(\mathcal{I}(\Phi(E))\) is a finite set of radical ideals of \(R\); in [11 Theorem 3.6 and Corollary 3.7], he proved that \(\mathcal{I}(\Phi(E))\) is closed under taking primary (prime in this case) components; and in [14, Corollary 2.8], he proved that the big test ideal \(\overline{\tau}(R)\) of \(R\) (for tight closure) is equal to the smallest member of \(\mathcal{I}(\Phi(E))\) that meets \(R^e\), the complement in \(R\) of the union of the minimal prime ideals of \(R\).

Let \(a \in \mathcal{I}(\Phi(E))\) (with \(a \neq R\)), still in the \(F\)-pure case. The special annihilator submodule \(\text{ann}_R(\mathfrak{a}\Phi(E)) \cap \mathfrak{a}R[x, f]\) of \(\Phi(E)\) corresponding to \(a\) inherits a natural structure as a graded left module over the Frobenius skew polynomial ring \((R/\mathfrak{a})[x, f]\), and its 0th component is contained in \((0 :_E \mathfrak{a})\). As \(R/\mathfrak{a}\)-module, the latter is isomorphic to the injective envelope of the simple \(R/\mathfrak{a}\)-module. Motivated by results in [14 §3] in the case where \(R\) is complete, and by work of K. Schwede in [10 §5] in the \(F\)-finite
case, we say that $a$ is fully $\Phi(E)$-special if (it is $\Phi(E)$-special and) its 0th component is exactly $(0 :_E a)$. The main result of this paper is that a $\Phi(E)$-special ideal of $R$ is always fully $\Phi(E)$-special provided that $R$ is an ($F$-pure) homomorphic image of an excellent regular local ring of characteristic $p$. When $R$ satisfies this condition, corollaries can be drawn from that main result: we shall establish an analogue of [14, Theorem 3.1] and, in particular, show that $R/a$ is $F$-pure whenever $a$ is a proper $\Phi(E)$-special ideal of $R$.

Along the way, we shall show that, in the case where $R$ is $F$-finite as well as $F$-pure, the set $\mathcal{I}(\Phi(E))$ of $\Phi(E)$-special ideals of $R$ is equal to the set of uniformly $F$-compatible ideals of $R$, introduced by K. Schwede in [10, §3]. An ideal $b$ of $R$ is said to be uniformly $F$-compatible if, for every $j > 0$ and every $\phi \in \text{Hom}_R(R^{(j)}, R)$, we have $\phi(b^{(j)}) \subseteq b$. In [10, Corollary 5.3 and Corollary 3.3], Schwede proved that there are only finitely many uniformly $F$-compatible ideals of $R$ and that they are all radical; in [10, Proposition 4.7 and Corollary 4.8], he proved that the set of uniformly $F$-compatible ideals is closed under taking primary (prime in this case) components; in [10, Theorem 6.3], Schwede proved that the big test ideal $\bar{\tau}(R)$ of $R$ is equal to the smallest uniformly $F$-compatible ideal of $R$ that meets $R^c$; and in [10, Remark 4.4 and Proposition 4.7], he proved that there is a unique largest proper uniformly $F$-compatible ideal of $R$, and that that is prime and equal to the splitting prime of $R$ discovered and defined by I. M. Aberbach and F. Enescu [11, §3].

Thus, in the $F$-finite $F$-pure case, the set of uniformly $F$-compatible ideals of $R$ has properties similar to some properties of $\mathcal{I}(\Phi(E))$. Are the two sets the same? We shall, during the course of the paper, show that the answer is ‘yes’. It should be emphasized, however, that Schwede only defined uniformly $F$-compatible ideals in the $F$-finite case, whereas the majority of this paper is devoted to the study of fully $\Phi(E)$-special ideals in the ($F$-pure but) not necessarily $F$-finite case.

We shall use the notation of this Introduction throughout the remainder of the paper. In particular, $R$ will denote a local ring of prime characteristic $p$ having maximal ideal $m$. We shall sometimes use the notation $(R, m)$ just to remind the reader that $R$ is local. The completion of $R$ will be denoted by $\widehat{R}$. We shall only assume that $R$ is reduced, or $F$-pure, or $F$-finite, when there is an explicit statement to that effect; also $E$ will continue to denote $E_R(R/m)$. We continue to use $\mathbb{N}$, respectively $\mathbb{N}_0$, to denote the set of all positive, respectively non-negative, integers.

For $j \in \mathbb{N}_0$, the $j$th component of an $\mathbb{N}_0$-graded left $R[x,f]$-module $G$ will be denoted by $G_j$.

1. Fully $\Phi(E)$-special ideals

We remind the reader that we usually identify the 0th component of $\Phi(E) = \bigoplus_{a \in \mathbb{N}_0} R a^n \otimes_R E$ with $E$ in the obvious natural way. For an ideal $a$ of $R$, we have, with this convention, that the 0th component of $\text{ann}_{\Phi(E)}(aR[x,f])$ is contained in $(0 :_E a)$.

1.1. Lemma. Assume that $(R, m)$ is $F$-pure; let $a$ be an ideal of $R$. Then the 0th component $(\text{ann}_{\Phi(E)}(aR[x,f]))_0$ of $\text{ann}_{\Phi(E)}(aR[x,f])$ contains $(0 :_E a)$ if and only if $a$ is $\Phi(E)$-special and $(\text{ann}_{\Phi(E)}(aR[x,f]))_0 = (0 :_E a)$.

Proof. Only the implication ‘$\Rightarrow$’ needs proof.
Assume that \((0 :_E a) \subseteq (\ann_{\Phi(E)}(aR[x,f]))_0\). Since \(\ann_{\Phi(E)}(aR[x,f])\) is an \(R[x,f]\)-submodule of \(\Phi(E)\), it follows that \(\ann_{\Phi(E)}(aR[x,f])\) contains the image \(J\) of the map

\[
\Phi((0 :_E a)) = R[x,f] \otimes_R (0 :_E a) \longrightarrow R[x,f] \otimes_R E = \Phi(E)
\]
induced by inclusion. Let \(b\) be the radical ideal of \(R\) for which \(\gr-\ann_{R[x,f]} J = bR[x,f]\), so that \(b = (0 :_R J)\). As \(J \subseteq \ann_{\Phi(E)}(aR[x,f])\), we must have \(a \subseteq b\). Furthermore, \(b\) annihilates \((0 :_E a) \cong \text{Hom}_R(R/a, E)\), and since an \(R\)-module and its Matlis dual have the same annihilator, we also have \(b \subseteq a\). Thus \(a = b\) is the \(R\)-annihilator of an \(R[x,f]\)-submodule of \(\Phi(E)\), and so \(a \in \mathcal{I}(\Phi(E))\).

Finally, note that an \(e \in (\ann_{\Phi(E)}(aR[x,f]))_0\) must be annihilated by \(a\), and so lies in \((0 :_E a)\). \(\square\)

1.2. Definition. Assume that \((R, m)\) is \(F\)-pure; let \(a\) be an ideal of \(R\). We say that \(a\) is fully \(\Phi(E)\)-special if the equivalent conditions of Lemma 1.1 are satisfied.

Thus \(a\) is fully \(\Phi(E)\)-special if and only if \((0 :_E a) \subseteq (\ann_{\Phi(E)}(aR[x,f]))_0\), and, then, \(a\) is \(\Phi(E)\)-special and we have the equality \((0 :_E a) = (\ann_{\Phi(E)}(aR[x,f]))_0\).

To facilitate the presentation of some examples of \(\Phi(E)\)-special ideals that are fully \(\Phi(E)\)-special, we review next the theory of \(S\)-tight closure, where \(S\) is a multiplicatively closed subset of \(R\). This theory was developed in [14]. The special case of the theory in which \(S = R^\circ\) is the ‘classical’ tight closure theory of M. Hochster and C. Huneke [2].

1.3. Reminders. Let \(H\) be a left \(R[x,f]\)-module and let \(S\) be a multiplicatively closed subset of \(R\).

(i) We define the internal \(S\)-tight closure of zero in \(H\), denoted \(\Delta^S(H)\), to be the \(R[x,f]\)-submodule of \(H\) given by

\[
\Delta^S(H) = \{ h \in H : \text{there exists } s \in S \text{ with } sx^nh = 0 \text{ for all } n \gg 0 \}.
\]

When \(M\) is an \(R\)-module and we take the graded left \(R[x,f]\)-module \(\Phi(M) = R[x,f] \otimes_R M\) for \(H\), the \(R[x,f]\)-submodule \(\Delta^S(\Phi(M))\) of \(\Phi(M)\) is graded, and we refer to its 0th component as the \(S\)-tight closure of 0 in \(M\), or the tight closure with respect to \(S\) of 0 in \(M\), and denote it by \(0^*_M S\). See [14] §1.

(ii) By [14] Example 1.3(ii)], we have, for an \(R\)-module \(M\),

\[
\Delta^S(R[x,f] \otimes_R M) = 0^*_M \oplus 0^*_{R \otimes_R M} \oplus \cdots \oplus 0^*_{R^e \otimes_R M} \oplus \cdots.
\]

(iii) Recall that an \(S\)-test element for \(R\) is an element \(s \in S\) such that, for every \(R\)-module \(M\) and every \(j \in \mathbb{N}_0\), the element \(sx^j\) annihilates \(1 \otimes m \in (\Phi(M))_0\) for every \(m \in 0^*_M S\). The ideal of \(R\) generated by all the \(S\)-test elements for \(R\) is called the \(S\)-test ideal of \(R\), and denoted by \(\tau^S(R)\).

1.4. Reminders. Suppose that \((R, m)\) is \(F\)-pure. Let \(S\) be a multiplicatively closed subset of \(R\). Recall that the set \(\mathcal{I}(\Phi(E))\) of \(\Phi(E)\)-special \(R\)-ideals is finite; let \(b^{S, \Phi(E)}\) denote the intersection of all the minimal members of the set

\[
\{ p \in \text{Spec}(R) \cap \mathcal{I}(\Phi(E)) : p \cap S \neq \emptyset \}.
\]

Thus \(b^{S, \Phi(E)}\) is the smallest member of \(\mathcal{I}(\Phi(E))\) that meets \(S\).
(i) By [14, Theorem 2.6], the set \( S \cap b^{s,\Phi(E)} \) is (non-empty and) equal to the set of \( S \)-test elements for \( R \).

(ii) Thus there exists an \( S \)-test element for \( R \).

(iii) Furthermore, \( \Delta^S(\Phi(E)) = \text{ann}_{\Phi(E)}(b^{s,\Phi(E)}R[x,f] \cap (0 :_R \Delta^S(\Phi(E))) = b^{s,\Phi(E)} \), by [14, Proposition 1.5].

(iv) By [14, Proposition 2.10(v)], we have \( b^{s,\Phi(E)} = (0 :_R 0^{*s}_E) \).

1.5. Lemma ([Sharp 14 Corollary 2.8]). Suppose that \((R, m)\) is \( F \)-pure. Let \( S \) be the complement in \( R \) of the union of finitely many prime ideals. Then the \( S \)-test ideal \( \tau^S(R) \) is equal to \( b^{s,\Phi(E)} \), the smallest member of the finite set \( \mathcal{I}(\Phi(E)) \) that meets \( S \).

We shall also use the following result from [14].

1.6. Theorem ([Sharp 14, Theorem 2.12]). Suppose that \((R, m)\) is \( F \)-pure. Let \( \mathfrak{a} \in \mathcal{I}(\Phi(E)) \). Then there exists a multiplicatively closed subset \( S \) of \( R \) such that \( \mathfrak{a} \) is the \( S \)-test ideal of \( R \). Moreover, \( S \) can be taken to be the complement in \( R \) of the union of finitely many prime ideals.

We are now able to give examples of fully \( \Phi(E) \)-special ideals, because the next result shows that, when \((R, m)\) is complete and \( F \)-pure, every \( \Phi(E) \)-special ideal of \( R \) is automatically fully \( \Phi(E) \)-special.

1.7. Proposition. Suppose that \((R, m)\) is complete and \( F \)-pure. Then every \( \Phi(E) \)-special ideal of \( R \) is fully \( \Phi(E) \)-special.

Proof. Let \( \mathfrak{a} \) be a \( \Phi(E) \)-special ideal of \( R \). If \( \mathfrak{a} = R \), then

\[
(0 :_E \mathfrak{a}) = 0 \subseteq \text{ann}_{\Phi(E)}(\mathfrak{a}R[x,f])
\]

and \( \mathfrak{a} \) is fully \( \Phi(E) \)-special. We therefore assume that \( \mathfrak{a} \) is proper.

By Theorem [13] and [14] Corollary 2.8, there exist finitely many prime ideals \( p_1, \ldots, p_n \) of \( R \) such that, if we set \( S := R \setminus \bigcup_{i=1}^n p_i \), then \( \mathfrak{a} \) is the \( S \)-test ideal of \( R \), that is \( \mathfrak{a} = \tau^S(R) = b^{s,\Phi(E)} \), where the notation is as in [13] and [14]. Therefore, by [13](ii) and [14](iii),

\[
0^{*s}_E \oplus 0^{*s}_{Rx \oplus R} \oplus \cdots \oplus 0^{*s}_{Rx \oplus R} \oplus \cdots = \Delta^S(\Phi(E)) = \text{ann}_{\Phi(E)}(b^{s,\Phi(E)}R[x,f]).
\]

Now we know that \( b^{s,\Phi(E)} = (0 :_R 0^{*s}_E) \), by [14](iv). Since \( R \) is complete, it follows from Matlis duality (see, for example, [15, p. 154]) that \( 0^{*s}_E = (0 :_E b^{s,\Phi(E)} \). We have thus shown that \( (0 :_E \mathfrak{a}) = R \oplus (0 :_E \mathfrak{a}) \subseteq (\text{ann}_{\Phi(E)}(\mathfrak{a}R[x,f]))_0 \). Thus \( \mathfrak{a} \) is fully \( \Phi(E) \)-special.

Next, we develop some theory for fully \( \Phi(E) \)-special ideals.

1.8. Lemma. Suppose that \((R, m)\) is \( F \)-pure, and let \( \mathfrak{a} \) be a fully \( \Phi(E) \)-special ideal of \( R \). Then \( \mathfrak{a} \) is radical and every associated prime of \( \mathfrak{a} \) is also fully \( \Phi(E) \)-special.

Proof. We can assume that \( \mathfrak{a} \) is proper. Since \( \mathfrak{a} \) is \( \Phi(E) \)-special, it must be radical. Let \( \mathfrak{a} = p_1 \cap \cdots \cap p_t \) be the minimal primary (prime in this case) decomposition of \( \mathfrak{a} \), and let \( i \in \{1, \ldots, t\} \).
Since \( a \) is fully \( \Phi(E) \)-special, we have \((0 :_E a) \subseteq (\operatorname{ann}_{\Phi(E)}(aR[x,f]))_0\). Let \( e \in (0 :_E p_i) \) and let \( r \in p_i \). We show that \( rx^n \) annihilates the element \( 1 \otimes e \) of the 0th component of \( \Phi(E) \). There exists
\[
a \in \bigcap_{j=1 \atop j \ne i}^t p_j \setminus p_i.
\]
Now \((0 :_E p_i) = a(0 :_E p_i)\), because multiplication by \( a \) provides a monomorphism of \( R/p_i \) into itself and \( E \) is injective. Therefore \( e = ae' \) for some \( e' \in (0 :_E p_i) \). Therefore \( rx^n \otimes e = rx^n \otimes ae' = ra^{p^n} x^n \otimes e' = 0 \) since \( ra^{p^n} \in a \) and
\[
(0 :_E p_i) \subseteq (0 :_E a) \subseteq \operatorname{ann}_{\Phi(E)}(aR[x,f]).
\]
Therefore \((0 :_E p_i) \subseteq (\operatorname{ann}_{\Phi(E)}(p_iR[x,f]))_0\) and \( p_i \) is fully \( \Phi(E) \)-special.

1.9. Proposition. Suppose that \((R, \mathfrak{m})\) is \( F \)-pure. Let \( (a_\lambda)_{\lambda \in \Lambda} \) be a non-empty family of fully \( \Phi(E) \)-special ideals of \( R \). Then \( \sum_{\lambda \in \Lambda} a_\lambda \) is again fully \( \Phi(E) \)-special.

Proof. Set \( a := \sum_{\lambda \in \Lambda} a_\lambda \), and observe that \( aR[x,f] = \sum_{\lambda \in \Lambda} (a_\lambda R[x,f]) \). By assumption, we have \((0 :_E a) \subseteq \operatorname{ann}_{\Phi(E)}(a_\lambda R[x,f]) \) for all \( \lambda \in \Lambda \). It follows that
\[
(0 :_E a) = (0 :_E \sum_{\lambda \in \Lambda} a_\lambda) = \bigcap_{\lambda \in \Lambda} (0 :_E a_\lambda) \\
\subseteq \bigcap_{\lambda \in \Lambda} (\operatorname{ann}_{\Phi(E)}(a_\lambda R[x,f]))_0 \\
= (\operatorname{ann}_{\Phi(E)}(\sum_{\lambda \in \Lambda} (a_\lambda R[x,f])))_0 = (\operatorname{ann}_{\Phi(E)}(aR[x,f]))_0.
\]

Therefore \( a := \sum_{\lambda \in \Lambda} a_\lambda \) is fully \( \Phi(E) \)-special.

1.10. Corollary. Suppose that \((R, \mathfrak{m})\) is \( F \)-pure. Then \( R \) has a unique largest fully \( \Phi(E) \)-special proper ideal, and this is prime.

Proof. The zero ideal is fully \( \Phi(E) \)-special, and so it follows from Proposition 1.9 that the sum \( \mathfrak{b} \) of all the fully \( \Phi(E) \)-special proper ideals of \( R \) is fully \( \Phi(E) \)-special (and contained in \( \mathfrak{m} \)), and so is the unique largest fully \( \Phi(E) \)-special proper ideal of \( R \). Also \( \mathfrak{b} \) must be prime, since all the associated primes of \( \mathfrak{b} \) are fully \( \Phi(E) \)-special, by Lemma 1.8.

In what follows, we shall have cause to pass between \( R \) and its completion. Note that if \( R \) is \( F \)-pure, then so too is \( \hat{R} \), by Hochster and Roberts [3] Corollary 6.13. The following technical lemma will be helpful.

1.11. Lemma. (See [13] Lemma 4.3.) There is a unique way of extending the \( R \)-module structure on \( E := E_R(R/\mathfrak{m}) \) to an \( \hat{R} \)-module structure. Recall that, as an \( \hat{R} \)-module, \( E \cong E_{\hat{R}}(\hat{R}/\hat{\mathfrak{m}}) \).

Since each element of \( \Phi_R(E) = R[x,f] \otimes_R E \) is annihilated by some power of \( \mathfrak{m} \), the left \( R[x,f] \)-module structure on \( \Phi_R(E) \) can be extended in a unique way to a left \( \hat{R}[x,f] \)-module structure.

The map \( \beta : \Phi_R(E) = R[x,f] \otimes_R E \to \hat{R}[x,f] \otimes_{\hat{R}} E = \Phi_{\hat{R}}(E) \) for which
\[
\beta(rx^ih) = rx^ih \quad \text{for all } r \in R, \ i \in \mathbb{N}_0 \text{ and } h \in E
\]
is a homogeneous \( \hat{R}[x,f] \)-isomorphism.
Since each element of $\Phi_R(E)$ is annihilated by some power of $m$, it follows that a subset of $\Phi_R(E)$ is an $R[x, f]$-submodule if and only if it is an $\hat{R}[x, f]$-submodule. Consequently,

$$\mathcal{I}_R(\Phi_R(E)) = \{B \cap R : B \in \mathcal{I}_{\hat{R}}(\Phi_{\hat{R}}(E))\}.$$  

1.12. **Lemma.** Suppose that $(R, m)$ is $F$-pure, and let $a$ be an ideal of $R$. Then $a\hat{R}$ is a fully $\Phi_{\hat{R}}(E)$-special ideal of $\hat{R}$ if and only if $a$ is a fully $\Phi_R(E)$-special ideal of $R$.

**Proof.** By Lemma 1.11 when we extend the left $R[x, f]$-module structure on $\Phi_R(E)$, in the unique way possible, to a left $\hat{R}[x, f]$-module structure, $E \cong E_{\hat{R}}(\hat{R}/\hat{m})$ as $\hat{R}$-modules and $\Phi_R(E) \cong \Phi_{\hat{R}}(E)$ as left $\hat{R}[x, f]$-modules. The claim therefore follows from the facts that

$$\text{ann}_{\Phi_R(E)}(aR[x, f]) = \text{ann}_{\Phi_{\hat{R}}(E)}((a\hat{R})\hat{R}[x, f])$$

and $(0 :_E a) = (0 :_{\hat{R}} a\hat{R})$. \hfill \Box

2. **The case where $R$ is an $F$-pure homomorphic image of an excellent regular local ring of characteristic $p$**

The main aim of this section is to prove that, when $R$ is an $F$-pure homomorphic image of an excellent regular local ring of characteristic $p$, every $\Phi(E)$-special ideal of $R$ is fully $\Phi(E)$-special ideal. This will enable us to extend some results obtained in [14, §3] about an $F$-pure complete local ring to an $F$-pure homomorphic image of an excellent regular local ring of characteristic $p$. We begin the section with a lemma that is derived from a result of G. Lyubeznik [5, Lemma 4.1].

2.1. **Lemma.** Let $(S, \mathfrak{M})$ be a complete regular local ring of characteristic $p$, and let $\mathfrak{B}$ be a proper, non-zero ideal of $S$. Denote $E_S(S/\mathfrak{M})$ by $E_S$, and let $S[x, f]$ denote the Frobenius skew polynomial ring over $S$. Let $n \in \mathbb{N}$.

Since $S$ is regular, $S^{(n)}$ is faithfully flat over $S$, and we identify $Sx^n \otimes_S (0 :_{E_S} \mathfrak{B})$ as an $S$-submodule of $Sx^n \otimes_S E_S$ in the natural way. Let $a_1, \ldots, a_d$ be a regular system of parameters for $S$. Consider the $S$-isomorphism $\delta_n : Sx^n \otimes_S E_S \xrightarrow{\cong} E_S$ of [11, 4.2(iii)], for which (with the notation used in the statement of that result)

$$\delta_n \left( bx^n \otimes \left[ \frac{s}{(a_1 \ldots a_d)^j} \right] \right) = \left[ \frac{bs^{p^n}}{(a_1 \ldots a_d)^{jp^n}} \right]$$

for all $b, s \in S$ and $j \in \mathbb{N}_0$.

The isomorphism $\delta_n$ maps

(i) $Sx^n \otimes_S (0 :_{E_S} \mathfrak{B})$ onto $(0 :_{E_S} \mathfrak{B}^{[p^n]})$, and
(ii) $\mathfrak{B}(Sx^n \otimes_S (0 :_{E_S} \mathfrak{B}))$ onto $(0 :_{E_S} (\mathfrak{B}^{[p^n]} : \mathfrak{B}))$.

**Proof.** (i) Use of the analogue of Lyubeznik [5] Lemma 4.1 for the functor $Sx^n \otimes_S \mathfrak{B}$. It shows that the Matlis dual of $Sx^n \otimes_S (0 :_{E_S} \mathfrak{B})$ is $S$-isomorphic to $Sx^n \otimes_S (S/\mathfrak{B}^{[p^n]})$. Since each $S$-module has the same annihilator as its Matlis dual, we thus see that $Sx^n \otimes_S (0 :_{E_S} \mathfrak{B})$ has annihilator $\mathfrak{B}^{[p^n]}$. Since $S$ is complete, we have $T = (0 :_{E_S} (0 : S) T)$ for each submodule $T$ of $E_S$, by Matlis duality (see, for example, [15, p. 154]). It therefore follows that

$$\delta_n(Sx^n \otimes_S (0 :_{E_S} \mathfrak{B})) = (0 :_{E_S} \mathfrak{B}^{[p^n]}).$$
(ii) Set $N := Sx^n \otimes_S (0 :_{ES} \mathfrak{B})$. Similar reasoning shows that
$$\delta_n(\mathfrak{B}N) = (0 :_{ES} (0 : S \mathfrak{B}N)) = (0 :_{ES} ((0 : S N) : \mathfrak{B})) = (0 :_{ES} (\mathfrak{B}[p^n] : \mathfrak{B})).$$

\[\square\]

2.2. Proposition. Suppose that $R = S/\mathfrak{A}$, where $(S, \mathfrak{M})$ is a regular local ring of characteristic $p$, and $\mathfrak{A}$ is a proper ideal of $S$. Assume also that $R$ is $F$-pure. Let $\mathfrak{b}$ be a proper ideal of $R$; let $\mathfrak{B}$ be the unique ideal of $S$ that contains $\mathfrak{A}$ and is such that $\mathfrak{B}/\mathfrak{A} = \mathfrak{b}$.

Then $\mathfrak{b}$ is fully $\Phi(E)$-special if and only if $(\mathfrak{A}[p^n] : \mathfrak{A}) \subseteq (\mathfrak{B}[p^n] : \mathfrak{B})$ for all $n \in \mathbb{N}$.

Note. In the $F$-finite case, this result is already known and due to K. Schwede [10, Proposition 3.11 and Lemma 5.1].

Proof. If $\mathfrak{A} = 0$, then $R$ is regular, so that its big test ideal is $R$ itself (by [6, Theorem 8.8], for example) and the only proper $\Phi(E)$-special ideal of $R$ is $0$; also, $(0 :_{ES} 0) = S$, and the only proper ideal $\mathfrak{B}$ of $S$ satisfying $(0 :_{ES} 0) \subseteq (\mathfrak{B}[p^n] : \mathfrak{B})$ for all $n \in \mathbb{N}$ is the zero ideal. Thus the result is true when $\mathfrak{A} = 0$; we therefore assume for the remainder of this proof that $\mathfrak{A} \neq 0$.

Note that $\hat{R} = \hat{S}/\mathfrak{A}\hat{S}$ is again $F$-pure and that $\hat{S}$ is an excellent complete regular local ring of characteristic $p$, with maximal ideal $\mathfrak{M}\hat{S}$.

We also note that $\mathfrak{b}$ is a fully $\Phi_{\hat{R}}(E)$-special ideal of $\hat{R}$ if and only if $\mathfrak{b}\hat{R}$ is a fully $\Phi_{\hat{R}}(E)$-special ideal of $\hat{R}$, by Lemma [1.12]. Furthermore, by the faithful flatness of $\hat{S}$ over $S$, we have, for $n \in \mathbb{N}$,
$$(\mathfrak{A}\hat{S})[p^n] : \mathfrak{A}\hat{S}) = (\mathfrak{A}[p^n] : \mathfrak{A})\hat{S} \subseteq (\mathfrak{B}[p^n] : \mathfrak{B})\hat{S} = (\mathfrak{B}\hat{S})[p^n] : \mathfrak{B}\hat{S})$$
if and only if $(\mathfrak{A}[p^n] : \mathfrak{A}) \subseteq (\mathfrak{B}[p^n] : \mathfrak{B})$. Therefore, we can, and do, assume henceforth in this proof that $S$ is complete.

Let $E_S := E_S(S/\mathfrak{M})$. Now $(0 :_{ES} \mathfrak{A}) = E := E_R(R/\mathfrak{m})$ and $(0 :_{ES} \mathfrak{B}) = (0 :_E \mathfrak{b})$. Note that $\mathfrak{b}$ is fully $\Phi_R(E)$-special if and only if, for each $n \in \mathbb{N}$ and each $r \in \mathfrak{b}$, the element $rx^n \in Rx^n$ annihilates the $R$-submodule $(0 :_E \mathfrak{b})$ of the 0th component $E$ of $\Phi_R(E)$.

Let $n \in \mathbb{N}$. There is an exact sequence of $(S,S)$-bimodules
$$0 \rightarrow \mathfrak{A}Sx^n \xrightarrow{\subseteq} Sx^n \xrightarrow{\nu} Rx^n \rightarrow 0,$$
where $\nu(sx^n) = (s + \mathfrak{A})x^n$ for all $s \in S$. The map
$$Sx^n \otimes_S (0 :_{ES} \mathfrak{A}) \rightarrow Rx^n \otimes_S (0 :_{ES} \mathfrak{A}) = Rx^n \otimes_R (0 :_{ES} \mathfrak{A}) = Rx^n \otimes_R E$$
induced by $\nu$ therefore has kernel $\mathfrak{A}(Sx^n \otimes_S (0 :_{ES} \mathfrak{A}))$.

It follows that $\mathfrak{b}$ is fully $\Phi_R(E)$-special if and only if, for all $n \in \mathbb{N}$, $s \in \mathfrak{B}$ and $g \in (0 :_{ES} \mathfrak{B}) = (0 :_E \mathfrak{b})$, the element $sx^n \otimes g$ of $Sx^n \otimes_S (0 :_{ES} \mathfrak{A})$ lies in
$$\mathfrak{A}(Sx^n \otimes_S (0 :_{ES} \mathfrak{A})).$$
In other words, $\mathfrak{b}$ is fully $\Phi_R(E)$-special if and only if, for all $n \in \mathbb{N}$, we have
$$\mathfrak{B}(Sx^n \otimes_S (0 :_{ES} \mathfrak{B})) \subseteq \mathfrak{A}(Sx^n \otimes_S (0 :_{ES} \mathfrak{A})).$$
(We are here identifying $Sx^n \otimes_S (0 :_{ES} \mathfrak{B})$ and $Sx^n \otimes_S (0 :_{ES} \mathfrak{A})$ with submodules of $Sx^n \otimes_S E_S$ in the obvious ways, using the faithful flatness of $S^{(n)}$ over $S$.)
By [11, 4.2(iii)], we have \( Sx^n \otimes S E_S \cong E_S \). Since \( S \) is complete, each submodule of \( E_S \) satisfies \( T = (0 : E_S (0 : S T)) \). Set \( N : = Sx^n \otimes S E_S \). Thus

\[ \mathcal{A}(Sx^n \otimes S (0 : E_S \mathcal{A})) = (0 : N (0 : S (\mathcal{A}(Sx^n \otimes S (0 : E_S \mathcal{A})))) = (0 : N (\mathcal{A}^{[n]} : \mathcal{A})), \]

by Lemma [2, 1]. Similarly, \( \mathcal{B}(Sx^n \otimes S (0 : E_S \mathcal{B})) = (0 : N (\mathcal{B}^{[n]} : \mathcal{B})) \). It follows that \( b \) is fully \( \Phi_R(E) \)-special if and only if

\[ (0 : N (\mathcal{B}^{[n]} : \mathcal{B})) \subseteq (0 : N (\mathcal{A}^{[n]} : \mathcal{A})) \text{ for all } n \in \mathbb{N}, \]

that is (since \( N \cong E_S \)), if and only if \( (\mathcal{A}^{[n]} : \mathcal{A}) \subseteq (\mathcal{B}^{[n]} : \mathcal{B}) \) for all \( n \in \mathbb{N} \).

\[ \square \]

2.3. **Theorem.** Suppose that \( R = S/\mathfrak{A} \) is a homomorphic image of an excellent regular local ring \((S, \mathfrak{M})\) of characteristic \( p \), modulo a proper ideal \( \mathfrak{A} \). Assume that \( R \) is \( F \)-pure.

Then each \( \Phi(E) \)-special ideal of \( R \) is fully \( \Phi(E) \)-special.

**Proof.** Once again, the claim is easy to prove if \( \mathfrak{A} = 0 \), and so we assume henceforth in this proof that \( \mathfrak{A} \neq 0 \).

Note that \( \hat{R} = \hat{S}/\mathfrak{A}\hat{S} \) is again \( F \)-pure and that \( \hat{S} \) is an excellent complete regular local ring of characteristic \( p \), with maximal ideal \( \hat{\mathfrak{M}} \).

Let \( b \) be a \( \Phi(E) \)-special \( R \)-ideal with \( b \neq R \). Then \( b = c \cap R \) for some \( \Phi_R(E) \)-special \( \hat{R} \)-ideal \( c \). (We have used Lemma [11] here.) Let \( \mathcal{C} \) be the unique ideal of \( \hat{S} \) that contains \( \mathfrak{A}\hat{S} \) and is such that \( \mathcal{C}/\mathfrak{A}\hat{S} = c \). By Proposition [2, 7] the ideal \( c \) of \( \hat{R} \) is fully \( \Phi_{\hat{R}}(E) \)-special, and so, by Proposition 2.2, we have

\[ (\mathcal{A}^{[n]} : \mathcal{A})\hat{S} = \left( (\mathcal{A}\hat{S})^{[n]} : \mathcal{A}\hat{S} \right) \subseteq (\mathcal{C}^{[0]} : \mathcal{C}) \]

for all \( n \in \mathbb{N} \).

Set \( \mathcal{C} \cap S : = \mathcal{B} \), so that \( \mathcal{B}/\mathcal{A} = \mathcal{B} \).

Let \( n \in \mathbb{N} \) and \( s \in (\mathcal{A}^{[n]} : \mathcal{A}) \). Therefore \( s \in (\mathcal{C}^{[n]} : \mathcal{C}) \). It follows from G. Lyubeznik and K. E. Smith [6, Lemma 6.6] that \( \mathcal{C}^{[n]} \cap S = (\mathcal{C} \cap S)^{[n]} \). (Lyubeznik’s and Smith’s proof of this result uses work of N. Radu [7, Corollary 5], which, in turn, uses D. Popescu’s general Néron desingularization [7, 8].) We can now deduce that

\[ s(\mathcal{C} \cap S) \subseteq s\mathcal{C} \cap S \subseteq (\mathcal{C}^{[n]} : \mathcal{C}) \cap S = (\mathcal{C} \cap S)^{[n]}, \]

so that \( s \in ((\mathcal{C} \cap S)^{[n]} : \mathcal{C} \cap S) = (\mathcal{B}^{[n]} : \mathcal{B}) \).

We have thus shown that \( (\mathcal{A}^{[n]} : \mathcal{A}) \subseteq (\mathcal{B}^{[n]} : \mathcal{B}) \) for all \( n \in \mathbb{N} \), so that \( b = \mathcal{B}/\mathcal{A} \) is fully \( \Phi(E) \)-special by Proposition 2.2.

\[ \square \]

In the case where \( R \) is an \( F \)-pure homomorphic image of an excellent regular local ring of characteristic \( p \), the characterization of \( \mathcal{I}(\Phi(E)) \) afforded by Proposition 2.2 and Theorem 2.3 enables us to see that \( \mathcal{I}(\Phi(E)) \) is fully \( \Phi(E) \)-special. The ideals in \( \mathcal{I}(\Phi(E)) \) are precisely those that can be expressed as intersections of finitely many prime members of \( \mathcal{I}(\Phi(E)) \), it is of interest to examine the behaviour of \( \mathcal{I}(\Phi(E)) \cap \text{Spec}(R) \) under localization. The next proposition, which is an extension of part of [12, Proposition 2.8], is in preparation for this investigation.

2.4. **Proposition.** Let \( S \) be a regular local ring of characteristic \( p \), and let \( n \in \mathbb{N} \). Let \( \mathfrak{A}, \mathfrak{B}_1, \ldots, \mathfrak{B}_t, \mathcal{C} \) be ideals of \( S \) with \( 0 \neq \mathfrak{A} \neq S \), and let \( \mathfrak{A} = \mathfrak{A}_1 \cap \ldots \cap \mathfrak{A}_t \) be a minimal primary decomposition of \( \mathfrak{A} \).

(i) We have \( (\mathfrak{B}_1 \cap \ldots \cap \mathfrak{B}_t)^{[n]} = \mathfrak{B}_1^{[n]} \cap \ldots \cap \mathfrak{B}_t^{[n]} \).
(ii) If $\mathfrak{N}$ is a $\mathfrak{P}$-primary ideal of $S$, then $\mathfrak{N}^{[p^n]}$ is also $\mathfrak{P}$-primary.

(iii) The equation $\mathfrak{A}^{[p^n]} = \mathfrak{N}_1^{[p^n]} \cap \cdots \cap \mathfrak{N}_t^{[p^n]}$ provides a minimal primary decomposition of $\mathfrak{A}^{[p^n]}$.

(iv) We have $(\mathfrak{A} : \mathfrak{C})^{[p^n]} = (\mathfrak{A}^{[p^n]} : \mathfrak{C}^{[p^n]})$ and $(\mathfrak{A}^{[p^n]} : \mathfrak{A}) \subseteq (\mathfrak{A} : \mathfrak{C})^{[p^n]} : (\mathfrak{A} : \mathfrak{C})$.

(v) If $\mathfrak{P}$ is an associated prime ideal of $\mathfrak{A}$, then $(\mathfrak{A}^{[p^n]} : \mathfrak{A}) \subseteq (\mathfrak{P}^{[p^n]} : \mathfrak{P})$.

(vi) Since $0 \neq \mathfrak{A} \neq S$, we have $(\mathfrak{A}^{[p^n]} : \mathfrak{A}) \neq S$. If $\mathfrak{P}_1 := \sqrt{\mathfrak{P}_1}$ is a minimal prime ideal of $\mathfrak{A}$, then $\mathfrak{P}_1$ is a minimal prime ideal of $(\mathfrak{A}^{[p^n]} : \mathfrak{A})$ and the unique $\mathfrak{P}_1$-primary component of $(\mathfrak{A}^{[p^n]} : \mathfrak{A})$ is $(\mathfrak{A}_1^{[p^n]} : \mathfrak{P}_1)$.

Proof. Parts (i), (ii) and (iii) were essentially proved in [12, Proposition 2.8], while parts (iv), (v) and (vi) can be proved by obvious modifications of the arguments used to prove the corresponding parts of [12, Proposition 2.8].

2.5. Corollary. Suppose that $R$ is $F$-pure and a homomorphic image of an excellent regular local ring $S$ of characteristic $p$ modulo a proper ideal $\mathfrak{A}$. Let $\mathfrak{p} \in \text{Spec}(R)$. Then

$$\mathcal{I}_{R_p}(\Phi_{R_p}(E_{R_p}(R_p/pR_p))) \cap \text{Spec}(R_p) = \{qR_p : q \in \mathcal{I}(\Phi(E)) \cap \text{Spec}(R) \text{ and } q \subseteq \mathfrak{p}\}.$$

Proof. Note that, by M. Hochster and J. L. Roberts [3, Lemma 6.2], the localization $R_p$ is again $F$-pure. The claim is easy to prove when $\mathfrak{A} = 0$, and so we assume that $\mathfrak{A} \neq 0$.

For each lower case fraktur letter that denotes an ideal of $R$, let the corresponding upper case fraktur letter denote the unique ideal of $S$ that contains $\mathfrak{A}$ and has quotient modulo $\mathfrak{A}$ equal to the specified ideal of $R$. For example, $\mathfrak{P}$ denotes the unique ideal of $S$ that contains $\mathfrak{A}$ and is such that $\mathfrak{P}/\mathfrak{A} = \mathfrak{p}$.

Note that $R_p \cong S_p/\mathfrak{A}S_p$ is again a homomorphic image of an excellent regular local ring $S$ of characteristic $p$. Let $q \in \text{Spec}(R)$ with $q \subseteq \mathfrak{p}$.

Suppose first that $q \in \mathcal{I}(\Phi(E)) \cap \text{Spec}(R)$. By Theorem 2.3, we see that $q$ is fully $\Phi(E)$-special; use of Proposition 2.2 shows that $(\mathfrak{A}^{[p^n]} : \mathfrak{A}) \subseteq (\mathfrak{N}^{[p^n]} : \mathfrak{N})$ for all $n \in \mathbb{N}$. Therefore

$$(\mathfrak{A}S_p)^{[p^n]} : \mathfrak{A}S_p \subseteq (\mathfrak{N}S_p)^{[p^n]} : \mathfrak{N}S_p$$

for all $n \in \mathbb{N}$.

Since the standard isomorphism $S_p/\mathfrak{A}S_p \cong R_p$ maps $\mathfrak{N}S_p/\mathfrak{A}S_p$ onto $qR_p$, it follows from Proposition 2.2 that $qR_p$ is fully $\Phi_{R_p}(E_{R_p}(R_p/pR_p))$-special.

Conversely, suppose that $qR_p$ is $\Phi_{R_p}(E_{R_p}(R_p/pR_p))$-special, so that, by Theorem 2.3, it is fully $\Phi_{R_p}(E_{R_p}(R_p/pR_p))$-special. By Proposition 2.2 this means that

$$(\mathfrak{A}S_p)^{[p^n]} : \mathfrak{A}S_p \subseteq (\mathfrak{N}S_p)^{[p^n]} : \mathfrak{N}S_p$$

for all $n \in \mathbb{N}$.

Let $^e$ and $^c$ denote extension and contraction of ideals under the natural ring homomorphism $S \rightarrow S_p$. Contract the last displayed inclusion relations back to $S$ to see that

$$(\mathfrak{A}^{[p^n]} : \mathfrak{A}) \subseteq (\mathfrak{A}^{[p^n]} : \mathfrak{A})^e \subseteq (\mathfrak{N}^{[p^n]} : \mathfrak{N})^e = (\mathfrak{N}^{[p^n]} : \mathfrak{N})$$

for all $n \in \mathbb{N}$ because $(\mathfrak{N}^{[p^n]} : \mathfrak{N})$ is $\mathfrak{N}$-primary (for all $n \in \mathbb{N}$), by Proposition 2.4(vi). It follows from Proposition 2.2 that $\mathfrak{N}/\mathfrak{A} = q$ is fully $\Phi(E)$-special.

We can now recover a special case of a result of Lyubeznik and Smith.
2.6. Corollary (G. Lyubeznik and K. E. Smith [6, Theorem 7.1]). Suppose that \( R \) is \( F \)-pure and a homomorphic image of an excellent regular local ring \( S \) of characteristic \( p \) modulo a proper ideal \( \mathfrak{A} \). Let \( \mathfrak{p} \in \text{Spec}(R) \). Then the big test ideal of \( R_\mathfrak{p} \) is the extension to \( R_\mathfrak{p} \) of the big test ideal of \( R \). In symbols, \( \tilde{\tau}(R_\mathfrak{p}) = \tilde{\tau}(R)R_\mathfrak{p} \).

**Proof.** The big test ideal \( \tilde{\tau}(R) \) of \( R \) is equal to the intersection of the (finitely many) members of \( \mathcal{I}(\Phi(E)) \cap \text{Spec}(R) \) of positive height, and a similar statement holds for \( R_\mathfrak{p} \). The claim therefore follows from Corollary 2.5. \( \square \)

Some results were obtained in [14, Theorem 3.1] for an \( F \)-pure complete local ring of characteristic \( p \). We can now use Theorem 2.3 to establish analogous results for an \( F \)-pure homomorphic image of an excellent regular local ring of characteristic \( p \).

2.7. Theorem. Suppose \( (R, \mathfrak{m}) \) is \( F \)-pure and that every \( \Phi(E) \)-special ideal of \( R \) is fully \( \Phi(E) \)-special. (For example, by Theorem 2.3, this would be the case if \( R \) were a homomorphic image of an excellent regular local ring of characteristic \( p \).) Let \( \mathfrak{c} \) be a proper ideal of \( R \) that is \( \Phi(E) \)-special. In the light of Theorem 1.6 let \( \mathfrak{p}_1, \ldots, \mathfrak{p}_m \) be prime ideals of \( R \) for which the multiplicatively closed subset \( S \) of \( R \) of \( c \)-special, we have \( \mathfrak{J}_0 \subseteq (0 :_E \mathfrak{c}) \); as \( V/E \)-module, this is isomorphic to \( E_{R/\mathfrak{c}}/(R/\mathfrak{c}) \).

(iii) The 0th component \( J_0 \) of \( J \) is \( (0 :_E \mathfrak{c}) \); as \( V/E \)-module, this is isomorphic to \( E_{R/\mathfrak{c}}/(R/\mathfrak{c}) \).
(iv) The ring \( R/\mathfrak{c} \) is \( F \)-pure.
(v) We have \( \mathcal{I}(\Phi_{R/\mathfrak{c}}(J_0 \subseteq \mathcal{I}(R/\mathfrak{c}) \), so that

\[ \{ \mathfrak{d} : \mathfrak{d} \text{ is an ideal of } R \text{ with } \mathfrak{d} \supseteq \mathfrak{c} \text{ and } \mathfrak{d}/\mathfrak{c} \in \mathcal{I}(\Phi_{R/\mathfrak{c}}(J_0) \subseteq \mathcal{I}(\Phi_{R/\mathfrak{c}}(J_0)) \subseteq \mathcal{I}(\Phi_{R/\mathfrak{c}}(J_0)) \} \subseteq \mathcal{I}(\Phi_{R/\mathfrak{c}}(J_0)). \]

**Proof.** Since the \( \Phi(E) \)-special ideal \( \mathfrak{c} \) is fully \( \Phi(E) \)-special, we have \( J_0 = (0 :_E \mathfrak{c}) \). Given this observation, one can now use the arguments employed in the proof of [14, Theorem 3.1] to furnish a proof of this theorem. \( \square \)

The next corollary follows from Theorem 2.7 just as in [14], Corollary 3.2 follows from Theorem 3.1.

2.8. Corollary. Suppose that \( (R, \mathfrak{m}) \) is local, \( F \)-pure and that every \( \Phi(E) \)-special ideal of \( R \) is fully \( \Phi(E) \)-special. (For example, by Theorem 2.3, this would be the case if \( R \) were a homomorphic image of an excellent regular local ring of characteristic \( p \).) Let \( \mathfrak{c} \) be a proper ideal of \( R \) that is \( \Phi(E) \)-special. Denote \( R/\mathfrak{c} \) by \( \overline{R} \), and note that \( \overline{R} \) is \( F \)-pure, by Theorem 2.7(iv). Let \( T \) be a multiplicatively closed subset of \( \overline{R} \) which is the complement in \( \overline{R} \) of the union of finitely many prime ideals. The finitistic \( T \)-test ideal \( \tau_{fg,T}(\overline{R}) \) of \( \overline{R} \) is defined to be \( \bigcap_L(0 :_{\overline{R}} 0_{\overline{R}}) \), where the intersection is taken over all finitely generated \( \overline{R} \)-modules \( L \).

(i) If \( \mathfrak{h} \) denotes the unique ideal of \( R \) that contains \( \mathfrak{c} \) and is such that \( \mathfrak{h}/\mathfrak{c} = \tau_{fg,T}(\overline{R}) \), the finitistic \( T \)-test ideal of \( \overline{R} \), then \( \mathfrak{h} \in \mathcal{I}(\Phi(E)) \).

(ii) In particular, if \( \mathfrak{h}' \) denotes the unique ideal of \( R \) that contains \( \mathfrak{c} \) and is such that \( \mathfrak{h}'/\mathfrak{c} = \tau(\overline{R}) \), the test ideal of \( \overline{R} \), then \( \mathfrak{h}' \in \mathcal{I}(\Phi(E)) \).
(iii) If \( g \) denotes the unique ideal of \( R \) that contains \( c \) and is such that \( g/c = \tau_T(\overline{R}) \), the \( T \)-test ideal of \( \overline{R} \), then \( g \in \mathcal{I}(\Phi(E)) \).

(iv) In particular, if \( g' \) denotes the unique ideal of \( R \) that contains \( c \) and is such that \( g'/c = \tau(\overline{R}) \), the big test ideal of \( \overline{R} \), then \( g' \in \mathcal{I}(\Phi(E)) \).

Proof. Straightforward modifications of the arguments given in the proof of [14, Corollary 3.2] will provide a proof for this. \( \square \)

2.9. Lemma. Assume that \((R, m)\) is local, \( F \)-pure and a homomorphic image of an excellent regular local ring of characteristic \( p \).

(i) There is a strictly ascending chain \( 0 = \tau_0 \subset \tau_1 \subset \cdots \subset \tau_t = R \) of radical ideals of \( R \) such that, for each \( i = 0, \ldots, t \), the reduced local ring \( R/\tau_i \) is \( F \)-pure and its test ideal is \( \tau_{i+1}/\tau_i \). We call this the test ideal chain of \( R \). All of \( \tau_0 = 0, \tau_1, \ldots, \tau_t \), and all their associated primes, belong to \( \mathcal{I}(\Phi(E)) \).

(ii) There is a strictly ascending chain \( 0 = \tilde{\tau}_0 \subset \tilde{\tau}_1 \subset \cdots \subset \tilde{\tau}_w \subset \tilde{\tau}_{w+1} = R \) of radical ideals in \( \mathcal{I}(\Phi(E)) \) such that, for each \( i = 0, \ldots, w \), the reduced local ring \( R/\tilde{\tau}_i \) is \( F \)-pure and its big test ideal is \( \tilde{\tau}_{i+1}/\tilde{\tau}_i \). We call this the big test ideal chain of \( R \). All of \( \tilde{\tau}_0 = 0, \tilde{\tau}_1, \ldots, \tilde{\tau}_w \), and all their associated primes, belong to \( \mathcal{I}(\Phi(E)) \).

Note. In the case when \( R \) is an \((F \text{-pure})\) homomorphic image of an \( F \)-finite regular local ring, part (i) of this result is known and due to Janet Cowden Vassilev [16, §3].

Proof. (i) Set \( \tau_1 := \tau(R) \), and note that \( \tau(R) \in \mathcal{I}(\Phi(E)) \). If \( \tau_1 \neq R \), apply Theorem 2.7 with the choice \( c = \tau(R) = \tau_1 \). That shows that \( R/\tau_1 \) is \( F \)-pure. Now argue by induction on \( \dim R \), noting that \( R/\tau_1 \) is a homomorphic image of an excellent regular local ring of characteristic \( p \). Use Theorem 2.7(v) to show that all of \( \tau_0, \tau_1, \ldots, \tau_t \) belong to \( \mathcal{I}(\Phi(E)) \).

(ii) This is proved similarly. \( \square \)

3. The \( F \)-finite case

In the \( F \)-finite case, the results above have strong connections with work of K. Schwede in [10], and the purpose of this section is to explore some of those connections. The introduction contains a description of certain properties of the set of all uniformly \( F \)-compatible ideals in an \( F \)-finite, \( F \)-pure local ring \( R \), and some of these are similar to properties of the set of all fully \( \Phi(E) \)-special ideals of \( R \): we shall show in this section that, in this special case, an ideal of \( R \) is uniformly \( F \)-compatible if and only if it is \( \Phi(E) \)-special, and that this is the case if and only if it is fully \( \Phi(E) \)-special.

3.1. Definition. Suppose that \( R \) is \( F \)-finite; let \( b \) be an ideal of \( R \). Then \( b \) is said to be \textit{uniformly \( F \)-compatible} if, for every \( n > 0 \) and every \( \phi \in \text{Hom}_R(R^n, R) \), we have \( \phi(b^{(n)}) \subseteq b \).

3.2. Proposition (Schwede [10, Lemma 5.1]). Suppose that \((R, m)\) is \( F \)-finite; let \( b \) be an ideal of \( R \). Then \( b \) is uniformly \( F \)-compatible if and only if \((0 :_E b) \subseteq (\text{ann}_{\Phi(E)}(bR[x, f]))_0 \).

Thus when \( R \) is \( F \)-finite and \( F \)-pure, \( b \) is uniformly \( F \)-compatible if and only if it is fully \( \Phi(E) \)-special.
Proposition 3.3. Let $n \in \mathbb{N}$ and $r \in R$. Multiplication by $r$ yields an $R$-homomorphism of $R^{(n)}$, which, strictly speaking, we should denote by $r \text{Id}_{R^{(n)}}$. Also $f^n : R \to R^{(n)}$ is an $R$-homomorphism. Thus we can consider the composition of $R$-homomorphisms $R \xrightarrow{f^n} R^{(n)} \xrightarrow{r} R^{(n)}$.

Application of the functor $\cdot \otimes_R E$ yields a composition of $R$-homomorphisms

$$ R \otimes_R E \to R^{(n)} \otimes_R E \xrightarrow{r} R^{(n)} \otimes_R E, $$

where the ‘$r$’ over the second arrow is an abbreviation for $r \text{Id}_{R^{(n)} \otimes_R E}$. But $R^{(n)} \cong Rx^n$ as $(R, R)$-bimodules; furthermore, $(0 : E b) \cong \text{Hom}_R(R/b, E)$. It follows that $(0 : E b) \subseteq (\text{ann}_{\Phi(E)}(bR[x, f]))_0$ if and only if, for all $n \in \mathbb{N}$ and all $r \in \mathfrak{b}$, the composition

$$ (0 : E b) \xrightarrow{\cong} E \xrightarrow{\cong} R \otimes_R E \to R^{(n)} \otimes_R E \xrightarrow{r} R^{(n)} \otimes_R E $$

(in which the second map is the natural isomorphism) is zero.

Let $M$ be an $R$-module. Recall that there is an $R$-homomorphism

$$ \xi_M : M \otimes_R E \to \text{Hom}_R(\text{Hom}_R(M, R), E) $$

such that, for $m \in M$, $e \in E$ and $g \in \text{Hom}_R(M, R)$, we have $(\xi_M(m \otimes e))(g) = g(m)e$. Furthermore, as $M$ varies, the $\xi_M$ constitute a natural transformation of functors; also $\xi_M$ is an isomorphism whenever $M$ is finitely generated. We shall use $D$ to denote the functor $\text{Hom}_R(\cdot, E)$.

Since $R^{(n)}$ is a finitely generated $R$-module, $(0 : E b) \subseteq (\text{ann}_{\Phi(E)}(bR[x, f]))_0$ if and only if, for all $n \in \mathbb{N}$ and all $r \in \mathfrak{b}$, the composition

$$ D(R/b) \to D(R) \xrightarrow{\cong} D(\text{Hom}_R(R, R)) \to D(\text{Hom}_R(R^{(n)}, R)) \xrightarrow{r} D(\text{Hom}_R(R^{(n)}, R)) $$

is zero. (Here, the first map is induced from the natural epimorphism $R \to R/b$, the second map is the natural isomorphism, and the sequence from the middle term rightwards is the result of application of the functor $\text{Hom}_R(\cdot, R)$ to the composition $R \xrightarrow{f^n} R^{(n)} \xrightarrow{r} R^{(n)}$ described at the beginning of the proof.)

Since $D$ is a faithful functor (because $E$ is an injective cogenerator for $R$), we can deduce that $(0 : E b) \subseteq (\text{ann}_{\Phi(E)}(bR[x, f]))_0$ if and only if, for all $n \in \mathbb{N}$ and all $r \in \mathfrak{b}$, the composition

$$ \text{Hom}_R(R^{(n)}, R) \xrightarrow{r} \text{Hom}_R(R^{(n)}, R) \to \text{Hom}_R(R, R) \xrightarrow{\cong} R \to R/b $$

is zero, that is, if and only if $\mathfrak{b}$ is uniformly $F$-compatible.

3.3. Proposition (Schwede [10]). Suppose that $(R, \mathfrak{m})$ is $F$-finite, and let $\mathfrak{a}$ be an ideal of $R$. Note that the completion $\hat{R}$ of $R$ is again $F$-finite.

(i) If $\mathfrak{a}$ is a uniformly $F$-compatible ideal of $R$, then $\mathfrak{a}R$ is a uniformly $F$-compatible ideal of $\hat{R}$. See Schwede [10] Lemma 3.9].

(ii) If $\mathfrak{c}$ is a uniformly $F$-compatible ideal of $\hat{R}$, then $\mathfrak{c} \cap R$ is a uniformly $F$-compatible ideal of $R$. See Schwede [10] Lemma 3.8].

Proof. For a finitely generated $R$-module $M$, we identify $\hat{M}$ with $M \otimes_R \hat{R}$ in the usual way, and we note that there is a natural $\hat{R}$-isomorphism $\psi_M : \text{Hom}_R(M, R) \otimes_R \hat{R} \xrightarrow{\cong} \text{Hom}_{\hat{R}}(M \otimes_R \hat{R}, R \otimes_R \hat{R})$ for which $\psi_M(g \otimes \hat{r}) = \hat{r}(g \otimes \text{Id}_R)$ for all $g \in \text{Hom}_R(M, R)$.
and $\hat{r} \in \hat{R}$. Let $n \in \mathbb{N}$. Consideration of Cauchy sequences shows that $\widehat{M(n)} = \widehat{M(n)}$.

In particular, $\widehat{R(n)} = \widehat{R(n)}$ and $\widehat{a}(n) = (\widehat{a}(\hat{a})(n) = (\hat{a})(\hat{a})(n)$.

There is an $\hat{R}$-isomorphism $\gamma : R(n) \otimes_R \hat{R} \xrightarrow{\cong} \hat{R}(n)$ which maps $a(n) \otimes_R \hat{R}$ onto $(\hat{a}(\hat{a})(n))$. Also, the natural $\hat{R}$-isomorphism $\delta : R \otimes_R \hat{R} \xrightarrow{\cong} \hat{R}$ maps $a \otimes_R \hat{R}$ onto $a(\hat{a})$.

(i) Let $\theta \in \text{Hom}_R(R(n) \otimes_R \hat{R}, R \otimes_R \hat{R})$. By the above, there exist $\phi_1, \ldots, \phi_t \in \text{Hom}_R(R(n), R)$ and $\hat{r}_1, \ldots, \hat{r}_t \in \hat{R}$ such that $\theta = \hat{r}_1(\phi_1 \otimes \text{Id}_R) + \cdots + \hat{r}_t(\phi_t \otimes \text{Id}_R)$. Since $\phi_i(a(n)) \subseteq a$ for all $n \in \mathbb{N}$ and $i = 1, \ldots, t$, we see that $\theta(a(n) \otimes_R \hat{R}) \subseteq a \otimes_R \hat{R}$ for all $n \in \mathbb{N}$. Use of the above-mentioned isomorphisms $\gamma$ and $\delta$ now enables us to conclude that $a(\hat{R})$ is a uniformly $F$-compatible ideal of $\hat{R}$.

(ii) Let $\phi \in \text{Hom}_R(R(n), R)$, and set $\mathfrak{c} := \mathfrak{c} \cap R$. Then

$$\phi \otimes \text{Id}_R \in \text{Hom}_R(R(n) \otimes_R \hat{R}, R \otimes_R \hat{R})$$

and $\delta \circ (\phi \otimes \text{Id}_R) \circ \gamma^{-1}$ maps $\mathfrak{c}(n)$ into $\mathfrak{c}$, and therefore maps $(\mathfrak{c}(\mathfrak{c})(n) \otimes_R \hat{R})$ into $\mathfrak{c}$. Therefore $\delta \circ (\phi \otimes \text{Id}_R) \otimes R$ maps $\mathfrak{c}(n) \otimes_R \hat{R}$ into $\mathfrak{c}$, so that $\phi(a) \in \mathfrak{c} \cap R = \mathfrak{c}$ for all $a \in \mathfrak{c}(n)$. Therefore $\mathfrak{c}$ is a uniformly $F$-compatible ideal of $R$. \hfill $\square$

3.4. Theorem. Suppose that $(R, m)$ is $F$-pure and $F$-finite. Then each $\Phi(E)$-special ideal $a$ of $R$ is automatically fully $\Phi(E)$-special.

**Proof.** Note that $\hat{R}$ is also $F$-pure, by Hochster and Roberts [3 Corollary 6.13]. Also, $\hat{R}$ is $F$-finite, because the completion of the finitely generated $R$-module $R^{(1)}$ is $\hat{R}^{(1)}$.

Thus, by definition, $a$ is the $R$-annihilator of an $R[x, f]$-submodule of $\Phi(E)$. It follows from Lemma 1.11 that $a = \mathfrak{a} \cap R$ for some ideal $\mathfrak{a}$ of $\hat{R}$ that is the $\hat{R}$-annihilator of an $\hat{R}[x, f]$-submodule of $\Phi(R(E))$. Thus $\mathfrak{a}$ is $\Phi(R(E))$-special. It follows from Proposition 1.7 that $\mathfrak{a}$ is a fully $\Phi(R(E))$-special ideal of $\hat{R}$, and so is uniformly $F$-compatible, by Proposition 3.2 Therefore, by Proposition 3.3(ii), the contraction $\mathfrak{a} \cap R = a$ is a uniformly $F$-compatible ideal of $R$, and is therefore fully $\Phi(E)$-special, by Proposition 3.2 again. \hfill $\square$

3.5. Corollary. Suppose that $(R, m)$ is $F$-pure and $F$-finite; let $a$ be an ideal of $R$. Then the following statements are equivalent:

(i) $a$ is uniformly $F$-compatible;
(ii) $a$ is $\Phi(E)$-special;
(iii) $a$ is fully $\Phi(E)$-special.

**Proof.** This is now immediate from Proposition 3.2 and Theorem 3.4 \hfill $\square$

3.6. Question. Suppose that $(R, m)$ is $F$-pure.

We have seen that each $\Phi(E)$-special ideal of $R$ is fully $\Phi(E)$-special if $R$ is complete (by Proposition 1.7) or if $R$ is a homomorphic image of an excellent regular local ring of characteristic $p$ (by Theorem 2.3) or if $R$ is $F$-finite (by Theorem 3.4).

Note that each complete local ring is excellent, and that each $F$-finite local ring of characteristic $p$ is excellent (by E. Kunz [1 Theorem 2.5]). The above results raise the following question. If the $F$-pure local ring $R$ is excellent, is it the case that every $\Phi(E)$-special ideal of $R$ is fully $\Phi(E)$-special?
4. A generalization of Aberbach’s and Enescu’s splitting prime

Recall from [3] Remark 2.8 and Proposition 2.9 that G. Lyubeznik and K. E. Smith defined \((R, \mathfrak{m})\) to be strongly \(F\)-regular (even in the case where \(R\) is not \(F\)-finite) precisely when the zero submodule of \(E\) is tightly closed in \(E\). See M. Hochster and C. Huneke [2, §8].

4.1. Theorem. Suppose that \((R, \mathfrak{m})\) is \(F\)-pure and that every \(\Phi(E)\)-special ideal of \(R\) is fully \(\Phi(E)\)-special. (For example, by Theorem 2.3 this would be the case if \(R\) were a homomorphic image of an excellent regular local ring of characteristic \(p\); it would also be the case if \(R\) were \(F\)-finite, by Theorem 3.4)

(i) There exists a unique largest \(\Phi(E)\)-special proper ideal, \(\mathfrak{c}\) say, of \(R\) and this is prime. Furthermore, \(R/\mathfrak{c}\) is strongly \(F\)-regular.

(ii) Let \(T\) be the \(R[x,f]\)-submodule of \(\Phi(E)\) generated by \((0 :_E \mathfrak{m}) \subseteq R \otimes_R E\). Then \(\text{gr-ann}_{R[x,f]} T = \mathfrak{c} R[x,f]\).

Proof. (i) By Corollary 1.10 there is a unique largest \(\Phi(E)\)-special proper ideal \(\mathfrak{c}\) of \(R\), and this is prime. By Corollary 2.3(iv), the big test ideal of \(R/\mathfrak{c}\) is \(R/\mathfrak{c}\) itself, so that \(1_{R/\mathfrak{c}}\) is a big test element for \(R/\mathfrak{c}\). Therefore the zero submodule of \(E_{R/\mathfrak{c}}(R/\mathfrak{m})\) is tightly closed in \(E_{R/\mathfrak{c}}(R/\mathfrak{m})\), and so \(R/\mathfrak{c}\) is strongly \(F\)-regular.

(ii) Note that \(T\) is the image of the \(R[x,f]\)-homomorphism

\[ R[x,f] \otimes_R (0 :_E \mathfrak{m}) \longrightarrow R[x,f] \otimes_R E = \Phi(E) \]

induced by the inclusion map \((0 :_E \mathfrak{m}) \subseteq E\). Let \(\mathfrak{d}\) be the \(\Phi(E)\)-special ideal of \(R\) for which \(\text{gr-ann}_{R[x,f]} T = \mathfrak{d} R[x,f]\). Since \(\mathfrak{d}\) annihilates \((0 :_E \mathfrak{m})\), we see that \(\mathfrak{d}\) is proper. Suppose that there exists \(h \in \mathcal{I}(\Phi(E))\) such that \(\mathfrak{d} \subseteq h \subseteq \mathfrak{m}\). (The symbol ‘\(\subset\)’ is reserved to denote strict inclusion.) Thus we have \((0 :_E \mathfrak{m}) \subseteq (0 :_E h) \subseteq (0 :_E \mathfrak{d})\).

But we know that every \(\Phi(E)\)-special ideal of \(R\) is \(\Phi(E)\)-special, and therefore \((0 :_E h) \subseteq (\text{ann}_{\Phi(E)}(h R[x,f]))_0\). Since \(\text{ann}_{\Phi(E)}(h R[x,f])\) is an \(R[x,f]\)-submodule of \(\Phi(E)\), it follows that

\[ T \subseteq \text{ann}_{\Phi(E)}(h R[x,f]) \subseteq \text{ann}_{\Phi(E)}(\mathfrak{d} R[x,f]). \]

Now take graded annihilators: in view of the bijective correspondence between the sets \(\mathcal{I}(\Phi(E))\) and \(\mathcal{A}(\Phi(E))\) alluded to in the Introduction, we have

\[ \mathfrak{d} R[x,f] = \text{gr-ann}_{R[x,f]}(\text{ann}_{\Phi(E)}(\mathfrak{d} R[x,f])) \]
\[ \subseteq \text{gr-ann}_{R[x,f]}(\text{ann}_{\Phi(E)}(h R[x,f])) = h R[x,f] \]
\[ \subseteq \text{gr-ann}_{R[x,f]} T = \mathfrak{d} R[x,f]. \]

Hence \(h = \mathfrak{d}\) and we have a contradiction.

Thus \(\mathfrak{d}\) is a maximal member of the set of proper \(\Phi(E)\)-special ideals of \(R\); therefore \(\mathfrak{d} = \mathfrak{c}\). \(\square\)

4.2. Definition (I. M. Aberbach and F. Enescu [1, Definition 3.2]). Suppose \((R, \mathfrak{m})\) is \(F\)-finite and reduced. Let \(u\) be a generator for the socle \((0 :_E \mathfrak{m})\) of \(E\). Aberbach and Enescu defined

\[ \mathfrak{P} = \{ r \in R : r \otimes u = 0 \text{ in } R^{(n)} \otimes_R E \text{ for all } n \gg 0 \}, \]

an ideal of \(R\).
In [1 §3], Aberbach and Enescu showed that in the case where \((R, \mathfrak{m})\) is \(F\)-finite and \(F\)-pure, and with the notation of \([4.2]\) the ideal \(\mathfrak{P}\) is prime and is equal to the set of elements \(c \in R\) for which, for all \(e \in \mathbb{N}\), the \(R\)-homomorphism \(\phi_{c,e} : R \rightarrow R^{1/p^e}\) for which \(\phi_{c,e}(1) = c^{1/p^e}\) does not split over \(R\). Aberbach and Enescu call this \(\mathfrak{P}\) the splitting prime for \(R\). By [1] Theorem 4.8(i)], the ring \(R/\mathfrak{P}\) is strongly \(F\)-regular.

4.3. Proposition. Suppose that \((R, \mathfrak{m})\) is \(F\)-finite and \(F\)-pure. Let \(\mathfrak{P}\) be Aberbach’s and Enescu’s splitting prime, as in \([4.2]\). Let \(q\) be the unique largest \(\Phi(E)\)-special proper ideal of \(R\), as in Theorem \([1.1]\). Then \(\mathfrak{P} = q\).

Proof. Let \(u\) be a generator for the socle \((0 :_E \mathfrak{m})\) of \(E\). We can write

\[
\mathfrak{P} = \{ r \in R : rx^n \otimes u = 0 \text{ in } Ra^n \otimes_R E \text{ for all } n \gg 0 \}.
\]

Now for a positive integer \(j\) and \(r \in R\), if \(rx^j \otimes u = 0\) in \(\Phi(E)\), then \(x(rx^{j-1} \otimes u) = rx^j \otimes u = 0\), so that \(rx^{j-1} \otimes u = 0\) because the left \(R[x,f]\)-module \(\Phi(E)\) is \(x\)-torsion-free. Therefore

\[
\mathfrak{P} = \{ r \in R : rx^n \otimes u = 0 \text{ in } Ra^n \otimes_R E \text{ for all } n \geq 0 \}.
\]

Let \(T\) be the \(R[x,f]\)-submodule of \(\Phi(E)\) generated by \((0 :_E \mathfrak{m}) \subseteq R \otimes_R E\). We thus see that \(\mathfrak{P}R[x,f] = \text{gr-ann}_{R[x,f]} T\), and this is \(qR[x,f]\) by Theorem \([4.1]\). Hence \(\mathfrak{P} = q\). \(\square\)

4.4. Remarks. Suppose that \((R, \mathfrak{m})\) is \(F\)-pure and a homomorphic image of an excellent regular local ring \(S\) of characteristic \(p\) modulo an ideal \(\mathfrak{A}\). By Theorem \([1.1](i)\), there exists a unique largest \(\Phi(E)\)-special proper ideal, \(q\), say, of \(R\) and this is prime. Let \(\mathfrak{Q}\) be the unique ideal of \(S\) containing \(\mathfrak{A}\) for which \(\mathfrak{Q}/\mathfrak{A} = q\).

(i) The results of this section suggest that \(q\) can be viewed as a generalization of Aberbach’s and Enescu’s splitting prime: for example, Proposition \([4.3]\) shows that \(q\) is that splitting prime in the case where \(R\) is, in addition, \(F\)-finite.

(ii) Note that \(R/q\) is strongly \(F\)-regular (in the sense of Lyubeznik and Smith mentioned at the beginning of the section).

(iii) By Proposition \([2.2]\), we have \((\mathfrak{A}[p^n] : \mathfrak{A}) \subseteq (\mathfrak{Q}[p^n] : \mathfrak{Q})\) for all \(n \in \mathbb{N}\). In the special case in which \(S\) is \(F\)-finite, this result was obtained by Aberbach and Enescu [1, Proposition 4.4].

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