INFINITESIMAL ISOMETRIES OF CONNECTION METRIC AND GENERALIZED MOMENT MAP EQUATION

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Abstract. Let \((M, g)\) be a smooth Riemannian manifold, \(K\) a compact Lie group and \(p : P \to M\) a principal \(K\)-bundle over \(M\) endowed with a connection \(A\). Fixing a bi invariant inner product on Lie algebra \(\mathfrak{k}\) of \(K\), the connection \(A\) and metric \(g\) define a Riemannian metric \(g_A\) on \(P\). Let \(X\) be the horizontal lift of vector field \(X\) on \(M\) and, let \(\xi^\nu\) be the vertical field associated with section \(\nu \in \mathfrak{A}^0(\text{ad}(P))\) of the adjoint bundle. It is proved that the connection \(A\) is invariant under the 1-parameter group of local diffeomorphisms generated by \(\tilde{X} + \xi^\nu\) if and only if \(X\) and \(\nu\) satisfy the generalized moment map equation \(\iota_X F_A = -\nabla A^\nu\). The Lie algebra of fiber preserving Killing fields of \((P, g_A)\) is studied, in the case where \(K\) is compact, connected and semisimple.

1. Lifts associated to a section of the adjoint bundle

1.1. Introduction. Let \((M, g)\) be a differentiable Riemannian manifold and \(K\) be a compact Lie group. Let \(p : P \to M\) be a principal \(K\)-bundle on \(M\) and \(A\) a connection on \(P\). The same notation \(A\) is used to denote the horizontal distribution associated to connection \(A\). Fixing an inner product \(\langle \cdot, \cdot \rangle\) on the Lie algebra \(\mathfrak{k}\) of \(K\) we can define a Riemannian metric \(g_A\) on \(P\) characterized by the following condition

1. If \(V_P := \ker p_*\) denotes the vertical bundle, then the canonical bundle isomorphism \(V_P \cong P \times \mathfrak{k}\) is an orthogonal bundle isomorphism with respect to the inner products defined by \(g_A\) and \(\langle \cdot, \cdot \rangle\).

2. The restriction of \(p_* : TP \to TM\) to the horizontal subbundle \(A \subset TP\) gives an orthogonal bundle isomorphism \(A \to p^*(TM)\).

3. By the definition, the horizontal distribution \(A \subset TP\) is \(K\)-invariant and the following short exact sequence is splitting.
   \[
   \{0\} \to V_P \to TP \to A \to \{0\}.
   \]

We require that the direct sum decomposition \(TP \cong A \oplus V_P\) of the tangent bundle \(TP\) is \(g_A\)-orthogonal.

The metric \(g_A\) on \(P\) defined in this way is called the connection metric and defines a Riemannian submersion \(p : (P, g_A) \to (M, g)\) with totally geodesic fibers \([13]\). Moreover, if \(g\) is a complete metric on \(M\), then the connection metric \(g_A\) is also complete \([14]\). We use \(I(M)\) to denote the group of isometries of \((M, g)\). It is well known that the isometry group of a compact manifold is a finite dimensional Lie group \([7]\). The group of \(K\)-equivariant covering bundle isomorphisms of \(P\) is defined by

\[
I_K(P) := \{ (\Phi, \varphi) | \varphi \in I(M), \Phi : P \to P\ is a \ \varphi\text{-covering bundle isomorphism} \}.
\]

The group \(I_K(P)\) has a natural topology (induced by the weak \(C^\infty\)-topology \([6, section 2.1]\)). A map \(\Phi : P \to P\) is called \(\varphi\)-covering isomorphism of principal
$K$-bundles, if it is a $K$-equivariant diffeomorphism of $P$ such that the induced map $\varphi : M \to M$ on the base manifold is a diffeomorphism satisfying $p \circ \Phi = \varphi \circ p$. Let $\text{Diff}^A_K(P)$ denote the space of $K$-equivariant diffeomorphism of $P$ leaving connection $A$ invariant, $I(P)$ the group of isometries of $(P, g_A)$ and $I^A_K(P)$ the stabilizer of connection $A$ in the group $I_K(P)$, then it is not difficult to prove that (Lemma 2.1)

$$I^A_K(P) = \text{Diff}^A_K(P) \cap I(P).$$

This Lie group play an important role in theory of locally homogeneous triples introduced in [4]. A bundle map $\Phi : P \to P$ is called fibre preserving if for all $x \in M$, it restricts to smooth map $\Phi_x : P_x \to P_{\Phi(x)}$. If $I_V(P)$ denotes the Lie group of all fiber preserving isometries of $(P, g_A)$ then we have $I^A_K(P) \subset I_V(P) \subset I(P)$. Use $i_K(P)$, $i_V(P)$ and $i(P)$ to denote the Lie algebras of $I^A_K(P)$, $I_V(P)$ and $I(P)$. Our goal is to describe the Lie algebra $i_V(P)$ of all fiber preserving infinitesimal isometries of $(P, g_A)$. If $K$ is compact, connected and semisimple, we will give a complete description of the structure of $i_V(P)$. If $P$ is compact, $I(P)$ is a finite dimensional Lie group and therefore $I^A_K(P)$ and $I_V(P)$ is also a finite dimensional. Any element of the Lie algebra $i_V(P)$ is an infinitesimal isometry (Killing vector field) of total space $P$ with respect to metric $g_A$.

**Definition 1.1.** A vector field $Y \in \mathcal{X}(P)$ on a principal $K$-bundle $P$ is called fiber preserving if, the 1-parameter group of local diffeomorphisms generated by $Y$ maps each fiber into another fiber.

**Remark 1.** A vector field $Y \in \mathcal{X}(P)$ is fiber preserving if and only if the Lie bracket $[Y, Z]$ is vertical for any vertical vector field $Z \in \mathcal{X}^v(P)$.

Let us recall some basic facts about infinitesimal isometries (Killing vector fields). A vector field $X \in \mathcal{X}(M)$ on Riemannian manifold $(M, g)$ is called a Killing vector field if, the 1-parameter group of local diffeomorphisms generated by it in a neighborhood of each point of $M$ are the local isometries. This is also equivalent to the following conditions

1. $L_X g = 0$ (where $L_X$ denotes the Lie derivative with respect to $X$).
2. For all vector fields $X_1, X_2 \in \mathcal{X}(M)$
   $$Xg(X_1, X_2) = g([X, X_1], X_2) + g(X_1, [X, X_2]).$$

For a compact manifold $M$, the set of all Killing vector fields on $M$ is a finite dimensional Lie subalgebra of $\mathcal{X}(M)$ and can be considered as the Lie algebra of group of isometries $I(M)$ and will be denoted by $i(M)$.

**1.2. Connections invariant by lifts.** Let $M$ be a smooth manifold of dimension $\dim M = n$. A rank $r$ distribution on $M$ is a subbundle $D \subset T_M$ of the tangent bundle with constant rank $r$. In other words $D$ is a smooth distribution if and only if for any fixed point $x_0 \in M$, there exist a family $\{X_1, ..., X_r\}$ of vector fields defined on some neighborhood $U$ of $x_0$ such that $\{X_1(x), ..., X_r(x)\}$ is a basis for $D_x$ at any point $x \in U$. We have the next result in the theory of distribution.

**Lemma 1.2.** Let $M$ be a smooth $n$-dimensional manifold. Let $D \subset T_M$ be a rank $r$ distribution on $M$ and let $X$ be a vector field on $M$. If for every vector field $Y \in \Gamma(D)$ one has $[X, Y] \in \Gamma(D)$, then the distribution $D$ is invariant under the 1-parameter group of local diffeomorphism generated by $X$. 
Proof. Fix \( x_0 \in M \) and choose an open neighborhood \( U \subset M \) of \( x_0 \) and \( \epsilon > 0 \) such that \( (-\epsilon, \epsilon) \times U \) is contained in the maximal domain of definition of the flow of the vector field \( X \). We may suppose that the subbundle \( D_U \) is trivial over \( U \). Hence, there exists a family \( \{X_1, ..., X_r\} \) of smooth vector fields defined on \( U \) such that \( \{X_1(x), ..., X_r(x)\} \) is a basis for \( D_x \) at any point \( x \in U \). Let \( \gamma_t \) denotes the flow of the vector field \( X \). We need just to prove that for every \( Y \in \Gamma(D) \), the local field \( Y_t := (\gamma_t)_* Y_x \) defined at every \( (t, x) \in (-\epsilon, \epsilon) \times U \) is a section of \( \Gamma(D) \). Let \( \{\omega_1, ..., \omega_{r-1}\} \) be a family of independent 1-forms on \( U \) which define a basis of the annihilator of \( \Gamma(D_U) \). Consider the functions

\[
(1) \quad \varphi_j(t) := ((\gamma_t)^* \omega_j)(Y_x) = (\omega_j)_{\gamma_t(x)}(Y_t).
\]

Taking derivation with respect to \( t \), we obtain

\[
\frac{d\varphi_j(t)}{dt} = (L_X \omega_j)_{\gamma_t(x)}(Y_t) = X_{\gamma_t(x)}(\omega_j(Y)) - \omega_j([X, Y]_{\gamma_t(x)}).
\]

Since \( Y \in \Gamma(D) \) and \( [X, Y] \in \Gamma(D) \), we have \( \omega_j(Y) = \omega_j([X, Y]) = 0 \). Hence \( d\varphi_j(t)/dt = 0 \) which means that \( \varphi_j(t) \) is constant. As \( \varphi_j(0) = 0 \), the right hand side of \( (1) \) is vanishes and we have \( Y_t \in \Gamma(D) \). \( \blacksquare \)

**Proposition 1.3.** Let \( p : P \to M \) be a principal \( K \)-bundle over \( n \)-dimensional manifold \( M \). For a vector field \( Y \in \mathcal{X}(P) \) the following are equivalent

1. The horizontal distribution \( A \subset T_P \) is invariant under the 1-parameter group of local diffeomorphism generated by \( Y \).
2. For every horizontal field \( Z \in \mathcal{X}^h(P) \), the Lie bracket \( [Y, Z] \) is horizontal.
3. For every horizontal lift \( \tilde{X} \in \mathcal{X}^h(P) \) of vector field \( X \in \mathcal{X}(M) \), the Lie bracket \( [Y, \tilde{X}] \) is horizontal.

Proof. (1) \( \Leftrightarrow \) (2) follows by Lemma 1.2. The implication (2) \( \Rightarrow \) (3) is evident. To prove the implication (3) \( \Rightarrow \) (2), let \( Z \in \mathcal{X}^h(P) \), so it is a section of horizontal distribution \( A \subset T_P \). There exists an open set \( U \subset M \), a family of horizontal lifts \( \tilde{X}_1, ..., \tilde{X}_n \) on \( P_U \) and a family of maps \( \varphi_1, ..., \varphi_n \) such that \( Z = \varphi_1 \tilde{X}_1 + ... + \varphi_n \tilde{X}_n \). So, we have

\[
[Y, Z] = [Y, \sum_{i=1}^{n} \varphi_i \tilde{X}_i] = \sum_{i=1}^{n} (Y(\varphi_i)) \tilde{X}_i + \varphi_i[Y, \tilde{X}_i]).
\]

Since \( \tilde{X}_i \in \mathcal{X}^h(P) \) and \( [Y, \tilde{X}_i] \in \mathcal{X}^h(P) \), we have \( [Y, Z] \in \mathcal{X}^h(P) \). \( \blacksquare \)

**Proposition 1.4.** Let \( p : P \to M \) be a principal \( K \)-bundle on \( M \) endowed with a connection \( A \). If \( a^# \) denotes the fundamental field corresponding to \( a \in \mathfrak{k} \) and \( \tilde{X} \) is the horizontal lift of \( X \in \mathcal{X}(M) \), then the vector field \( a^# \) is invariant under the 1-parameter group of local diffeomorphism generated by \( \tilde{X} \).

Proof. The vector field \( a^# \) at \( y \in P \) is defined by

\[
a^#_y := \frac{d}{dt} \bigg|_{t=0} (y \exp ta).
\]

Let \( (\beta_t) \) be the 1-parameter group of local diffeomorphism generated by \( \tilde{X} \). The vector field \( a^# \) is invariant under \( (\beta_t) \) if for all \( (t, y) \in \mathbb{R} \times P \) for which \( \beta_t \) is defined, one has

\[
((\beta_t)_y)^*(a^#_y) = a^#_{\beta_t(y)}.
\]
Using Proposition 1.3 the vector field $a^\#$ is invariant under $\beta_t$ if and only if $[\hat{X}, a^\#] = 0$. Set $k_t = \exp(ta) \in K$ and use $R_{k_t}, \tilde{X} = \tilde{X}$ to obtain

$$[a^\#, \hat{X}] = \lim_{t \to 0} \frac{1}{t} (\hat{X} - R_{k_t} \hat{X}) = 0.$$  

The canonical bundle isomorphism $V_P \simeq P \times \mathfrak{k}$ can be used to identify the Lie subalgebra of vertical fields $\mathcal{X}^\nu(P) := \Gamma(V_P)$ with the space of smooth maps $C^\infty(P, \mathfrak{k})$. This identification is given by $\nu \mapsto \xi^\nu$, where $\xi^\nu$ is used to denote the vertical field associated with $\nu \in C^\infty(P, \mathfrak{k})$ and $\xi^\nu : P \to V_P$ at a point $y \in P$ is defined by

$$\xi^\nu_y := \frac{d}{dt}{\bigg|}_{t=0} (y \exp tv(y)).$$

A smooth map $\nu : P \to \mathfrak{k}$ is called $K$-equivariant if, for every $(y, k) \in P \times K$ one has $\nu(yk) = \text{ad}_{k^{-1}} \nu(y)$, where $\text{ad} : K \to \text{End}(\mathfrak{k})$ denotes the adjoint representation. The space of all $K$-equivariant maps is denoted by $C^K(P, \mathfrak{k})$. Now, suppose $A^0(\text{ad}(P))$ denotes the space of smooth sections of the adjoint bundle $\text{ad}(P) := P \times \text{ad} \mathfrak{k} \to M$. There exists a natural identification between two spaces $C^K(P, \mathfrak{k})$ and $A^0(\text{ad}(P))$, given at a point $y \in P_x \ (x \in M)$ by

$$C^K(P, \mathfrak{k}) \to A^0(\text{ad}(P)), \ \nu \mapsto (x \mapsto [y, \nu(y)]).$$

By the above discussion it is clear that:

**Corollary 1.5.** For each $K$-invariant vertical vector field $Y \in \mathcal{X}^\nu(P)$, there exists a section $\nu \in A^0(\text{ad}(P))$ such that $Y = \xi^\nu$.

**Lemma 1.6.** Let $p : P \to M$ be a principal $K$-bundle endowed with connection $A$. If $X$ denotes the horizontal lift of a vector field $Y$ on $M$, then for any section $\nu \in A^0(\text{ad}(P))$ we have $[\hat{X}, \xi^\nu] = \xi^{\nabla^A_X \nu}$.

**Proof.** Let $t \mapsto x_t = x(t)$ be an integral curve of $X$ defined for $t \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ with the initial condition $x(0) = x_0$. The horizontal lift $t \mapsto y_t$ of the path $t \mapsto x_t$ through a point $y_0 \in P_x$ is the integral curve of $\hat{X}$ with initial condition $y(0) = y_0$. A section $\nu \in A^0(\text{ad}(P))$ can be regarded as a $K$-equivariant map $\nu : P \to \mathfrak{k}$. Put $a_t := \nu(y_t) \in \mathfrak{k}$, for all $t \in (-\varepsilon, \varepsilon)$. The vertical vector field $\xi^\nu_{y_t}$ is given by

$$\xi^\nu_{y_t} = (a_t)^\#_{y_t} = \frac{d}{ds}{\bigg|}_{s=0} (y_t \exp(sa_t)).$$

Let $(\beta_t)_{t \in (-\varepsilon, \varepsilon)}$ be the 1-parameter group of local diffeomorphism generated by $\hat{X}$. We have

$$[\hat{X}, \xi^\nu]_{y_0} = \lim_{t \to 0} \frac{1}{t} \{ (\xi^\nu - (\beta_t)_* (\xi^\nu)_{y_{-t}}) \} = \lim_{t \to 0} \frac{1}{t} \{ (a_0)^\#_{y_0} - (\beta_t)_* (a_{-t})^\#_{y_{-t}} \}.$$  

Since $(\beta_t)_* (a_{-t})^\#_{y_{-t}} = (a_{-t})^\#_{y_t}$, we have

$$[\hat{X}, \xi^\nu]_{y_0} = \lim_{t \to 0} \frac{1}{t} \{ (a_0)^\#_{y_0} - (a_{-t})^\#_{y_0} \} = \lim_{t \to 0} \frac{1}{t} \{ (a_0 - a_{-t})^\#_{y_0} \} = (a_t)_{t=0}^\#_{y_0} = \{ d\nu(\hat{X}_{y_0}) \}^\# = \{ \nabla^A_{\hat{X}_{y_0}} \nu \}^\# = \xi^{\nabla^A_{\hat{X}_{y_0}} \nu}.$$  

\[\blacksquare\]
Lemma 1.7. Let \( p : P \to M \) be a principal \( K \)-bundle. For any \( a \in \mathfrak{k} \) and any section \( \nu \in A^0(\text{ad}(P)) \) we have \([a^\#, \xi^\nu] = 0\). In particular, the vector field \( \xi^\nu \) is \( K \)-equivariant.

Proof. Let \( (\varphi_t)_{t \in \mathbb{R}} \) be the 1-parameter group of diffeomorphisms generated by the vector field \( a^\# \). The maps \( \varphi_t : P \to P \) are defined by \( y \mapsto y \exp(ta) \). Let \( (f_s)_{s \in \mathbb{R}} \) be the 1-parameter group of diffeomorphisms generated by \( \xi^\nu \). Using [7, Proposition 1.11] the Lie bracket \([a^\#, \xi^\nu] = 0\) if and only if \( \varphi_t \circ f_s = f_s \circ \varphi_t \) for all \( s, t \in \mathbb{R} \). Since \( f_s \) is a bundle isomorphism, it commutes with the right translations and for all \( y \in P \)

\[
f_s(\varphi_t(y)) = f_s(y \exp(ta)) = f_s(y) \exp(ta) = \varphi_t(f_s(y)).
\]

Suppose \( p : P \to M \) is a principal \( K \)-bundle over \( M \) with connection \( A \) and use \( \nabla^A : A^0(\text{ad}(P)) \to A^1(\text{ad}(P)) \) to denote the induced connection on \( \text{ad}(P) \). The covariant exterior derivative associated with connection \( A \) is denoted by \( d^{\nabla^A} : A^r(\text{ad}(P)) \to A^{r+1}(\text{ad}(P)) \) and the curvature 2-form \( F_A \in A^2(\text{ad}(P)) \) is defined by

\[
F_A = d^{\nabla^A} \circ \nabla^A.
\]

Let \( Y \in \mathfrak{X}(P) \) be a vector field on \( P \) and \((\beta_t)\) be the 1-parameter group of local diffeomorphism generated by it. We say that the connection \( A \) is invariant under the 1-parameter group of local diffeomorphism generated by \( Y \), if

\[
(\beta_t)_* A_y = A_{\beta_t(y)}.
\]

for all \((y,t) \in P \times \mathbb{R}\) for which \( \beta_t(y) \) is defined.

Definition 1.8. Let \( p : P \to M \) be a principal \( K \)-bundle over \( M \) endowed with a connection \( A \). Let \( X \) be a vector field on \( M \) and \( \nu \in A^0(\text{ad}(P)) \) be a section of the adjoint bundle. We say that \( X \) and \( \nu \) satisfy the generalized moment map equation if \( \iota_X F_A = -\nabla^A \nu \).

It is known [5, p. 257] that for two horizontal lifts \( \tilde{X}_1, \tilde{X}_2 \) of vector fields \( X_1, X_2 \) on \( M \) we have the following formula

\[
[X_1, X_2] = \tilde{[X_1, X_2]} - \xi^{F_A(X_1,X_2)}.
\]

Therefore, the horizontal projection of \( \tilde{[X_1, X_2]} \) is \( [X_1, X_2] \) and the vertical projection of it is \( [\tilde{X}_1, \tilde{X}_2]^{\nu} = -\xi^{F_A(X_1,X_2)} \).

Theorem 1.9. Let \( p : P \to M \) be a principal \( K \)-bundle on \( M \) with connection \( A \) and the curvature 2-form \( F_A \in A^2(\text{ad}(P)) \). Let \( \tilde{X} \) be the horizontal lift of a vector field \( X \) on \( M \) and \( \nu \in A^0(\text{ad}(P)) \). The connection \( A \) is invariant under the 1-parameter group of local diffeomorphism generated by \( \tilde{X} + \xi^\nu \) if and only if \( X \) and \( \nu \) satisfy the generalized moment map equation.

Proof. By Proposition 1.11 the connection \( A \) is invariant under the 1-parameter group of local diffeomorphism generated by \( \tilde{X} + \xi^\nu \) if and only if for every horizontal lift \( \tilde{Z} \) of vector field \( Z \) on \( M \), the Lie bracket \( [\tilde{X} + \xi^\nu, \tilde{Z}] \) is also horizontal. Using equation (2) we obtain

\[
[\tilde{X} + \xi^\nu, \tilde{Z}] = [\tilde{X}, \tilde{Z}] + [\xi^\nu, \tilde{Z}] = [\tilde{X}, \tilde{Z}] - \xi^{F_A(X,Z)} + [\xi^\nu, \tilde{Z}].
\]
The connection $A$ is invariant under the 1-parameter group of local diffeomorphism generated by $\tilde{X} + \xi^\nu$ if and only if $[\tilde{X} + \xi^\nu, Z] \in \Gamma(A)$. But, this happen if and only if

$$
\xi_{FA}(X,Z) = [\xi^\nu, Z].
$$

By Lemma 1.7 $[\xi^\nu, Z] = -\xi^{\nabla^A}(Z)$. Hence (3) is equivalent to $\iota_X F_A = -\nabla^A \nu$. \hfill \blacksquare

**Proposition 1.10.** Let $\tilde{X}$ be the horizontal lift of a vector field $X$ on $M$ and let $\nu \in \mathcal{A}^0(\text{ad}(P))$. If $(\beta_t)$ denotes the 1-parameter group of local diffeomorphism generated by $\tilde{X} + \xi^\nu$, then $(\beta_t)$ leaves invariant the vertical distribution.

**Proof.** Using Proposition 1.2 the vertical bundle $V_P$ is invariant by $(\beta_t)$ iff for any section $Z \in \Gamma(V_P)$ one has $[\tilde{X} + \xi^\nu, Z] \in \Gamma(V_P)$. But

$$
[\tilde{X} + \xi^\nu, Z] = [\tilde{X}, Z] + [\xi^\nu, Z].
$$

The integrability of vertical distribution imply $[\xi^\nu, Z] \in \Gamma(V_P)$. Moreover, $[\tilde{X}, Z]$ is also a vertical section. To see this note that the space of vertical sections is a free $C^\infty(P)$-module, generated by the fundamental vector fields. Thus, for any vertical section $Z \in \Gamma(V_P)$ there exists a family $\{a_1^\#, ..., a_r^\#\}$ of fundamental vector fields and a family $\{f_1, ..., f_r\}$ of smooth functions on $P$ such that $Z = f_1 a_1^\# + ... + f_r a_r^\#$. Therefore

$$
[\tilde{X}, Z] = [\tilde{X}, \sum_{i=1}^r f_i a_i^\#] = \sum_{i=1}^r (d f_i(\tilde{X}) a_i^\# + f_i[\tilde{X}, a_i^\#]) = \sum_{i=1}^r d f_i(\tilde{X}) a_i^\# \in \Gamma(V_P).
$$

\hfill \blacksquare

2. Fiber preserving infinitesimal isometries of connection metric

Let $p: P \to M$ be a principal $K$-bundle endowed with connection $A$. Let $g$ be a Riemannian metric on $M$ and $\langle \cdot, \cdot \rangle$ be a ad-invariant metric on the Lie algebra $\mathfrak{k}$ of $K$. We summarize these information by saying that $(g, P, \mathfrak{k} \to M, A)$ is a $K$-triple over $M$. For $y \in P$ the fibre over $x \in M$ is denoted by $P_x := p^{-1}(x)$. For all $y \in P_x$ the map $\tau_y: K \to P_x$ defined by $\tau_y(k) := yk$ is a diffeomorphism and can be used to define a metric on the fiber $P_x$. To show that this metric is well defined, let $y' \in P_x$ be another point in the fibre, then there exists $k_0 \in K$ such that $y' = y k_0$. For all $k \in K$

$$
\tau_{yk_0}(k) = (yk_0)k = y(k_0 k) = \tau_y (l_{k_0} k) = (\tau_y l_{k_0})(k),
$$

where $l_{k_0}$ denotes the left translation by $k_0$. Since the metric on $K$ is ad-invariant the induced metric on $(V_P)_y \simeq T_y P_x$ is well defined. Use the linear isomorphism $A_g \simeq T_x M$ to lift the metric $g_x$ on the base to a metric on the horizontal space $A_g$. These two metrics define a metric $g_A$ on $T_P \simeq A \oplus V_P$.

If $\omega_A \in \mathcal{A}^1(P, \mathfrak{k})$ denotes the connection 1-form associated to connection $A$, then $\omega_A$ and ad-invariant metric $\langle \cdot, \cdot \rangle$ on $\mathfrak{k}$ define a symmetric tensor $\langle \omega_A, \omega_A \rangle$ of type $(0, 2)$ acts on $(v, w) \in (T_y P) \times (T_y P)$ by

$$
\langle \omega_A, \omega_A \rangle(v, w) := \langle \omega_A(v), \omega_A(w) \rangle.
$$

Adding $p^* g$ to $\langle \omega_A, \omega_A \rangle$ we obtain a Riemannian metric $g_A := p^* g + \langle \omega_A, \omega_A \rangle$ on total space $P$ such that for all vectors $v, w \in T_y P$

$$
g_A(v, w) = g(p_*(v), p_*(w)) + \langle \omega_A(v), \omega_A(w) \rangle.
$$
It is clear that $g_A$ is the connection metric on $P$.

**Lemma 2.1.** Suppose $(g, P \rightarrow M, A)$ is a $K$-triple over $M$, then

(a) The right multiplication $R_k : P \rightarrow P$ is an isometry of $P$ with respect to metric $g_A$.

(b) Let $\Phi \in \text{Diff}_A^K(P)$ and denote the induced map on the base by $\varphi \in \text{Diff}(M)$. Then, $\Phi \in I(P)$ if and only if $\varphi \in I(M)$.

(c) $I^A_K(P) = \text{Diff}_A^K(P) \cap I(P)$.

**Proof.** (a) Since the inner product $\langle \cdot, \cdot \rangle$ is ad-invariant and for all $k \in K$: $p \circ R_k = p$, $R_k^* g_A = \text{ad}_{k^{-1}} \omega_A$ we have

$$R_k^* g_A = R_k^*(p^* g) + R_k^* \langle \omega_A, \omega_A \rangle = p^* g + \langle \text{ad}_{k^{-1}} \omega_A, \text{ad}_{k^{-1}} \omega_A \rangle = g_A.$$  

(b) Suppose $\Phi : P \rightarrow P$ is a $\varphi$-covering bundle isomorphism which leaves invariant the connection $A$, then $p \circ \Phi = \varphi \circ p$ and $\Phi^* \omega_A = \omega_A$. If $\varphi \in I(M)$, then

$$\Phi^* g_A = \Phi^*(p^* g) + \Phi^* \langle \omega_A, \omega_A \rangle = (p \circ \Phi)^* g + \langle \Phi^* \omega_A, \Phi^* \omega_A \rangle = p^* (\varphi^* g) + \langle \omega_A, \omega_A \rangle = p^* g + \langle \omega_A, \omega_A \rangle = g_A.$$  

Conversely, if $\Phi \in I(P)$ then $\Phi^* g_A = g_A$, and hence for all vector fields $X, Y$ on $M$ and their horizontal lifts $\tilde{X}, \tilde{Y} \in \mathcal{X}^h(P)$

$$g(X, Y) = g(p_\ast (\tilde{X}), p_\ast (\tilde{Y})) = (p^* g)(\tilde{X}, \tilde{Y}) = g_A(\tilde{X}, \tilde{Y}) = (\Phi^* g_A)(\tilde{X}, \tilde{Y})$$

$$= \Phi^*(p^* g)(\tilde{X}, \tilde{Y}) = (p^* \varphi^* g)(\tilde{X}, \tilde{Y}) = (\varphi^* g)(X, Y).$$

(c) It follows from (b) and the definition of $I^A_K(P)$. \hfill \blacksquare

**Remark 2.** For any $\alpha \in \mathfrak{a}$, the map $R_{\exp\alpha^\#} : P \rightarrow P$ is the 1-parameter group of diffeomorphism generated by $a^\#$. It is clear that $a^\# \in \mathfrak{iv}(P)$.

**Proposition 2.2.** Let $(g, P \rightarrow M, A)$ be a $K$-triple over $M$. Let $\tilde{X}$ denote the horizontal lift of vector field $X \in \mathcal{X}(M)$ and $\nu \in \mathcal{A}^0(\mathfrak{ad}(P))$. The vector field $\tilde{X} + \xi^\nu$ is Killing on $(P, g_A)$ if and only if $X$ is a Killing field on $(M, g)$ and $\iota_X F_A = -\nabla^A \nu$.

**Proof.** Let $(\beta_t)$ be the 1-parameter group of local diffeomorphism generated by $Y := \tilde{X} + \xi^\nu$, and let $(\alpha_t)$ denote the 1-parameter group of local diffeomorphism generated by $X$. If $X$ is Killing and $\iota_X F_A = -\nabla^A \nu$, then $(\alpha_t)$ are local isometries of $(M, g)$ and the connection $A$ is invariant under $(\beta_t)$ using Theorem [1.9]. From Lemma [2.1] we deduce that $(\beta_t)$ are local isometries of $(P, g_A)$ which means that $Y$ is Killing. Conversely, if $Y = \tilde{X} + \xi^\nu$ is Killing then $X$ is Killing and we have $\beta_t^* g_A = g_A$ for all $t$. By Proposition [1.10] the vertical distribution is invariant under $(\beta_t)$, hence the horizontal distribution is invariant under $(\beta_t)$ which is equivalent to the generalized moment map equation $\iota_X F_A = -\nabla^A \nu$ by Theorem [1.9]. \hfill \blacksquare

**Remark 3.** Let $\mathcal{G}(P)$ denote the gauge group of principal $K$-bundle $p : P \rightarrow M$ [4], [12], i.e. the group of $\text{id}_M$-covering bundle automorphisms of $P$. The space of smooth sections of adjoint bundle $\mathcal{A}^0(\mathfrak{ad}(P))$ with point wise bracket can be considered as the Lie algebra of gauge group $[3]$. If $\mathcal{G}^A(P)$ denotes the stabilizer of connection $A$ in the gauge group $\mathcal{G}(P)$, then $\mathcal{G}^A(P) \subset I^A_K(P)$. Moreover, thanks to Proposition [2.2] the Lie algebra of $\mathcal{G}^A(P)$ is given by

$$\text{Lie}(\mathcal{G}^A(P)) = \{ \nu \in \mathcal{A}^0(\mathfrak{ad}(P)) | \nabla^A \nu = 0 \}.$$
Remark 4. Using Proposition 2.2 if for a Killing vector field $X \in \mathfrak{i}(M)$ there exists a section $\nu_0 \in A^0(\mathfrak{ad}(P))$ which satisfies in the generalized moment map equation, then $\tilde{X} + \xi^{\nu_0}$ is a fiber preserving Killing field on $(P, g_A)$.

Suppose that $K$ is a connected, compact and semisimple Lie group endowed with a bi invariant metric, then any Killing vector field on $K$ is a sum of a right invariant vector field and a left invariant vector field on $K$.

**Lemma 2.3.** Let $(g, P \rightarrow M, A)$ be a $K$-triple over $M$. If $K$ is connected, compact and semisimple, then any vertical Killing field $Y$ on $(P, g_A)$ is decomposed as $Y = \nu^\# + a^\#$, where $\nu \in A^0(\mathfrak{ad}(P))$ is a $\nabla^A$-parallel section and $a \in \mathfrak{k}$.

**Proof.** For all $y \in P$ the map $\tau_y : K \rightarrow P_{p(y)}$ defined by $k \mapsto yk$ is an isometry. Denote by $T_y : \mathfrak{k} \rightarrow T_y P_{p(y)} \simeq (V_P)_y$ the derivation of $\tau_y$ at identity element of $K$. Since $Y$ is Killing and $\tau_y$ is isometry $T^{-1}_y(Y_y) \in \mathfrak{k}$ is also Killing for all $y \in P$. Using the fact that $K$ is connected, compact and semisimple, we have the following decomposition

$$T^{-1}_y(Y) = v_1(y) + v_2(y),$$

where $v_1(y)$ is a right invariant vector field and $v_2(y)$ is a left invariant vector field on $K$. Put $Y_1 := T_y(v_1(y))$, $Y_2 := T_y(v_2(y))$. One can check that $Y_1, Y_2$ are well defined vertical field on $P$. We will show that there exist a section $\nu \in A^0(\mathfrak{ad}(P))$ and an element $a \in \mathfrak{k}$ such that $Y_1 = \xi^\nu$, $Y_2 = a^\#$. Let $\nu : P \rightarrow \mathfrak{k}$ be a map defined at $y \in P$ by $\nu(y) := T^{-1}_y(Y_1)$. For any $k \in K$ we have $\tau_{yk} = \tau_y l_k$, where $l_k : K \rightarrow K$ denotes the left translation by $k \in K$. If we denote the derivation of $l_k$ by $L_k : \mathfrak{k} \rightarrow \mathfrak{k}$ then $T_{yk} = T_y \circ L_k$ and hence

$$\nu(yk) = T^{-1}_y(Y_1) = L^{-1}_k(T^{-1}_y(Y_1))$$

$$= L^{-1}_k(v_1(y)) = (L^{-1}_k R_k) v_1(y) = \text{ad}_{L^{-1}_k} \nu(y).$$

Hence $\nu \in \mathcal{C}^K(P, \mathfrak{k}) \simeq A^0(\mathfrak{ad}(P))$. So, there exist a section $\nu \in A^0(\mathfrak{ad}(P))$ such that $Y_1 = \xi^\nu$. The vertical vector field $\xi^\nu$ is Killing if and only if for all vector fields $Z_1, Z_2 \in \mathfrak{X}(P)$ one has

$$\xi^\nu g_A(Z_1, Z_2) = g_A(\xi^\nu, Z_1, Z_2) + g_A(Z_1, \xi^\nu, Z_2).$$

Taking $Z_1 = \tilde{X}$ and $Z_2 = e^\#$ where $X$ is an arbitrary vector field on $M$ and $e \in \mathfrak{k}$ and using Lemma 1.7 we see that $\nabla^A_X \nu = 0$, which means that $\nu$ is $\nabla^A$ parallel. If $y, y' \in P$ are two arbitrary points in the same fiber, then there exist an element $k \in K$ such that $y' = yk$. This implies $\tau_{y'} = \tau_y \circ l_k$ and $T_{y'} = T_y \circ L_k$, so

$$v_2(y') = T^{-1}_{y'}(Y_2(y')) = (L^{-1}_k T^{-1}_y)(Y_2(y)) = L^{-1}_k(v_2(y)) = v_2(y),$$

whence $v_2(y)$ is constant on the fiber and there exist $a \in \mathfrak{k}$ such that $Y_2 = a^\#$. ■

**Theorem 2.4.** Let $K$ be a connected, compact and semisimple Lie group. Let $(g, P \rightarrow M, A)$ be a $K$-triple over $M$. If $Y \in \mathfrak{i}_V(P)$ is a fiber preserving Killing vector field on $(P, g_A)$, then

(a) There exists a unique Killing vector field $X \in \mathfrak{i}(M)$ such that the horizontal projection of $Y$ is the horizontal lift $\tilde{X}$ of $X$.

(b) If the generalized moment map equation for $X$ admits a solution, then there exists a section $\nu \in A^0(\mathfrak{ad}(P))$ and $a \in \mathfrak{k}$ such that $Y$ decompose as

$$Y = \tilde{X} + \xi^\nu + a^\#$$

where $\iota_X F_A = -\nabla^A \nu$.  


Proof. (1) Decompose $Y$ into horizontal projection $Y^h$ and vertical projection $Y^v$. Since $Y^v$ is fiber preserving, so is $Y^h$, hence the Lie bracket $[Y^h, a^\#]$ is vertical. But the Lie bracket of a horizontal field and a fundamental field is horizontal [7, Page. 65]. Therefore $[Y^h, a^\#] = 0$ for all $a \in k$ which means that $(R_k)_*(Y^h) = Y^h$, for all $k \in K$. Using [11, Proposition 1.2] there exist a unique vector field $X$ on $M$ such that $Y^h = \dot{X}$. Because $Y$ is Killing and $p_*(Y) = X$, the vector field $X$ is also Killing. (2) If $Y$ is Killing and there exist a section $\nu_0 \in A^0(\text{ad}(P))$ such that $\iota_X F_A = -\nabla A \nu_0$, then using Proposition 2.2 we conclude that $\dot{X} + \xi^\nu_0$ is Killing. Therefore, $Y - (\dot{X} + \xi^\nu_0)$ is a vertical Killing field and Lemma 2.3 implies that there exist a $\nabla A$-parallel section $\nu_1 \in A^0(\text{ad}(P))$ and an element $a \in k$ such that

$$Y - (\dot{X} + \xi^\nu_0) = \xi^\nu_1 + a^\#.$$ 

Putting $\nu := \nu_0 + \nu_1$ we obtain $Y = \dot{X} + \xi^\nu + a^\#$, and we have $\iota_X F_A = -\nabla A \nu$.

Proposition 2.5. Let $(g, P \xrightarrow{\text{pr}} M, A)$ be a $K$-triple over $M$ with compact Lie group $K$. The Lie algebra of $K$-invariant Killing fields of $(P, g_A)$, denoted by $i_K(P)$, is a Lie subalgebra of $i(P)$ and is given by

$$i_K(P) = \{\dot{X} + \xi^\nu | \iota_X F_A = -\nabla A \nu , \text{ for } (X, \nu) \in i(M) \times A^0(\text{ad}(P))\}.$$

Proof. If $Y \in i_K(P)$, then by Theorem 2.4 there exists a unique Killing vector field $X \in i(M)$ such that the horizontal projection $Y^h$ is equal to $\dot{X}$. As $Y$ and $Y^h$ are $K$-invariant Killing fields, the vertical projection $Y^v$ is also $K$-invariant. Since the action of $K$ on $P$ is free the map $\nu : P \rightarrow k$ is equivariant if and only if $\xi^\nu$ is invariant. Using corollary 1.5 there exists a section $\nu \in A^0(\text{ad}(P))$ such that $Y = \xi^\nu$. Hence, $Y = \dot{X} + \xi^\nu$ and Proposition 2.2 implies that $\iota_X F_A = -\nabla A \nu$. Conversely, for a Killing field $X \in i(M)$ and a section $\nu \in A^0(\text{ad}(P))$ which satisfy the generalized moment map equation, we have $\dot{X} + \xi^\nu \in i_K(P)$.

## 3. Examples and applications

### 3.1. Fiber preserving Killing vector field of orthogonal frame bundle

Let $(M, g)$ be an oriented Riemannian manifold and $P = \text{SO}(M)$ its principal bundle of orthonormal frame. Let $A$ be the Levi-Civita connection on $P$ and endow it by connection metric $g_A$. The induced linear connection on associated bundle $T_M \simeq P \times_{\text{SO}(n)} \mathbb{R}^n$ coincide with the Levi-Civita connection of metric $g$ and is denoted by $\nabla : A^0(T_M) \rightarrow A^1(T_M)$. The connection $A$ induces a linear connection $\nabla A$ on the adjoint bundle $\text{ad}(P) = \text{End}(T_M)$. The Riemann curvature tensor is $F_A \in A^2(\text{End}(T_M))$. If $X \in i(M)$ is a Killing vector field on $M$ then using [10] Lemma 6.1] we have $\iota_X F_A = -\nabla A \nabla X$. Thanks to Proposition 2.2 it follows that the vector field $X^L := \dot{X} + \xi^\nu X$ is Killing and will be called the natural lift of $X$. We have the following theorem.

Proposition 3.1. [11] Theorem A.] Let $\text{SO}(M)$ be the frame bundle of the oriented Riemannian manifold $(M, g)$. If $Y$ is a fiber preserving Killing vector field on $(\text{SO}(M), g_A)$, then $Y$ can be decomposed as

$$Y = X^L + \xi^\nu + a^\#.$$ 

where $X^L$ is the natural lift of a Killing field $X$ on $M$, $\nu$ is a $\nabla A$-parallel section of $\text{End}(T_M)$ and $a \in \text{so}(n)$.
Proof. As $Y$ is a fiber preserving Killing field on $\text{SO}(M)$, there exists a Killing field $X \in \mathfrak{i}(M)$ such that $Y^h = \tilde{X}$. Using [10, Lemma 6.1], since $X \in \mathfrak{i}(M)$ is Killing we have $\iota_X F_A = -\nabla^A \nabla X$. Regarding $\nabla X$ as a section of $\text{End}(T_M) \simeq \text{ad}(\text{SO}(M))$, the generalized moment map equation $\iota_X F_A = -\nabla^A \nu$ admits a solution $\nu_0 := \nabla X \in A^0(\text{End}(T_M))$. Using Proposition 2.2 we deduce that the vector field $X^h := \tilde{X} + \xi \nabla X$ is Killing and Theorem 2.4 complete the proof.

3.2. Quantizable symplectic manifold. A symplectic manifold $(M, \omega)$ is called quantizable if and only if there is a principal circle bundle $p: P \to M$ over $M$ and a 1-form $\alpha \in A^1(P, \mathbb{R})$ which is invariant under the action of $\mathbb{S}^1$ and $d\alpha = p^* \omega$. One can prove that $(M, \omega)$ is quantizable if and only if $rac{1}{2\pi}[\omega] \in H^2(M, \mathbb{Z})$.

For a proof of this result see [1, Page. 440]. By the classical Chern-Weil theory there exist a connection $A$ on $P$ for which the connection form is given by $\theta_A = \alpha$ and the curvature form is $F_A = \omega$. The circle bundle $P$ is called the quantizing manifold. In this case, the space of sections of the adjoint bundle $A^0(\text{ad}(P))$ can be identified with $C^\infty(M)$. Let $\nu \mapsto f_\nu$, denote this identification. If $\tilde{X}$ denotes the horizontal lift of a Killing field $X$ on $M$ and $\nu \in A^0(\text{ad}(P))$, then using Proposition 2.2 we deduce that the vector field $\tilde{X} + \xi \nu$ is Killing with respect to $g_A$ if and only if $X$ and $\nu$ satisfy the generalized moment map equation. This is to say that $X$ is a Hamiltonian vector field with Hamiltonian function $H: M \to \mathbb{R}$ given by $H = -f_\nu + c$, where $c \in \mathbb{R}$ is a constant.

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