Optimal control in an inventory management problem considering replenishment lead time based upon a non-diffusive stochastic differential equation

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Abstract
An inventory management problem is theoretically discussed for a factory having effects of lead times in replenishing the inventory, where it stocks materials used for its products. It is assumed that the factory can dynamically control the size of ordering materials. By applying the stochastic control theory, the optimal control of the ordering size is derived, in which the expected total cost up to an expiration time is minimized. First, a new stochastic model is constructed for describing an inventory fluctuation of the factory by the use of a non-diffusive stochastic differential equation, where an analytic time is introduced so that the inventory process can be a Markov process even though it is affected by lead times. Next, an optimal control is formulated by introducing an evaluation function quantifying total costs. Based upon them, the Hamilton-Jacobi-Bellman (HJB) equation is derived, whose solution gives the optimal control. Finally, the optimal control is quantitatively examined through numerical solutions of the HJB equation. Numerical results indicate that if time up to an expiration time is short then the optimal control is affected by it, otherwise, the optimal control does not depend on it.

Keywords : Inventory, Optimal control, Lead time, Stochastic differential equation, Stochastic control

1. Introduction

For factories and retail stores, it is undoubtedly important to keep an adequate amount of inventories, which are supposed to be materials for products or products themselves. They generally have to spend higher cost holding too large amount of inventory, whereas they might lose opportunities for selling their products owing to shortages of the inventory. We should always perform the best control of the inventory level.

In the mathematical theory of inventory management, we can roughly classify continuous-time models describing temporal variation of the inventory level into two types. One is suitable for expressing the inventory level of a mass production factory (Rempała 2005), which manufactures a large amount of products whereas it receives a large orders from customers such as orders from wholesale stores. Thus, the increase of its inventory level can be treated as continuous variation whereas the decrease can be treated as discontinuous variation with discrete jumps. The other type is used for representing that of a retail store (Feng and Rao 2007), which receives small orders from many customers, such as orders from consumers whereas it makes a large order for products to suppliers at some intervals. Thus, the decrease of its inventory level can be treated as continuous variation whereas the increase can be treated as discontinuous variation with discrete jumps.

However, there are some factories having intermediate characteristic between these two types (Chen, Feng and Ou 2006). One example is a factory which manufactures steel products for construction of buildings. The outline of work in the factory is as follows;
(i) The factory occasionally receives orders for steel products from several customers. The factory ordinarily manufactures some components for ordered products by cutting steel plates stocked in its warehouse, which is called shearing procedure. Then it assembles the components into the ordered products.

(ii) When stocked plates get fewer, the factory orders some steel plates to a steel plate supplier. The factory has to wait until it gets the ordered plates, which is the so-called lead time.

(iii) When the current inventory is under the amount of steel plates necessary for an order from a customer, the factory immediately compensates the shortage of components by ordering them to external components suppliers.

Such an intermediate characteristic expressing a diagram of works is schematically illustrated in Fig. 1.

![Fig. 1 Schematic diagram of the system of a factory and its related factors.](image)

Taking such characteristics of the factory into consideration, we construct a stochastic model describing the inventory fluctuation so that it is also applicable to several factories whose inventory fluctuations are similar to the intermediate characteristic as illustrated in Fig. 1. If we rigorously formulate the inventory fluctuation reflecting characteristics above, we have a serious problem when we apply the stochastic control theory, which is often used in the inventory management problem to obtain the optimal control, i.e., the stochastic process describing the inventory level is not a Markov process because of the above-mentioned lead time. In order to remove the problem, we newly propose an improved model of the inventory fluctuation by introducing a new time variable, in which a stochastic differential equation driven by a compound Poisson process is applied (Øksendal and Sulem 2007, Applebaum 2009 and Kanekiyo 2014). Based upon the improved model, we introduce an evaluation function, which consists of two kinds of costs, to formulate the optimal control. One is the cost for holding the inventory and the other is the cost for ordering to external components suppliers.

The main purpose of this paper is to derive the optimal control minimizing the evaluation function. Here, we suppose that the factory orders to a material supplier when the inventory level falls below a predetermined level, and that the factory can dynamically control the amount of materials ordered to a material supplier. Based upon the improved model and the evaluation function, we introduce the well-known Hamilton-Jacobi-Bellman (HJB) equation (Bellman 1957) giving the optimal control. Further, we numerically derive the optimal control by solving the HJB equation.

Recently, many studies have been reported on an application of stochastic differential equations to the inventory problem, for instance, Li, Y. et. al. 2013, Li, S. et. al. 2015, Weerasinghe and Zhu 2016 and Ouaret et.al. 2018. A good review of recent developments on such studies has been given by Huang et. al. 2012. Most of such studies are based upon stochastic differential equations of diffusive type, i.e., driven by Brownian motion processes and having continuous path. Although a precise structure of stochastic evolution of inventory level can be effectively taken into analysis by the use of stochastic differential equations, discontinuous variation mentioned above can not be reproduced as far as using a diffusive stochastic differential equation.

On the other hand, studies on a probabilistic model whose state shows discontinuous variation have been also reported so far such as Markov chain models (for example, Nasr and Elshar 2018 and Schlosser 2016). Further, discussions on inventory problems having such an intermediate structure from a viewpoint of mechanical engineering have been recently reported, for instance, studies on supply chain management (Fatrias and Shimizu 2010, Grewal and Rogers 2015).

However, very few studies have been reported so far with respect to stochastic differential equations of non-diffusive type such as driven by stochastic processes having discontinuous path. Probabilistic models using non-diffusive stochastic differential equations have great advantages such that (i) they can reproduce discontinuous increase and/or decrease of inventory level, which enable us to apply to the inventory problem having the intermediate characteristic discussed in this paper and (ii) precise structure of inventory system can be naturally taken into models. Add to them, discussions can be
easily extend to probabilistic models using a stochastic differential equation driven by noises of more general type such as Lévy processes, which leads to very flexible probabilistic models applicable to quite many fields.

2. Model of inventory fluctuation and formulation of optimal ordering policy

2.1. Basic assumptions on a proposed model

As mentioned in the previous section, we consider a factory which receives orders for products from many customers (it is called customer order hereinafter in this study) whose schematic diagram is given by Fig.1. Each product consists of some components, which the factory manufactures by the use of materials stocked in its warehouse. The factory can make an order of materials to a material supplier at the time when the shortage of the inventory occurs. We call the order from the factory to the material supplier ordinary order. The factory has to wait from the time when a decision is made for an ordinary order to the time of warehousing the materials, which is the so-called lead time. As the factory makes an ordinary order for more amount of materials, the lead time generally becomes longer.

When the current inventory is under the amount of the materials which is necessary for a customer order, the factory manufactures the possible amount of components using the current inventory and immediately orders the shortage components to external components suppliers. We call the order from the factory to the components suppliers urgent order. The factory can make an urgent order in order only to compensate the shortage of the inventory caused by customer orders, since components necessary for a customer order are usually different from components for others. Although urgent orders need higher costs, a lead time in an urgent order is usually much shorter than that in ordinary orders.

2.2. Model of inventory fluctuation

We denote an inventory process of a factory, i.e., an inventory level that the factory possesses at time \( \tilde{t} \), by \( \tilde{X}_t \). Its temporal variation consists of three parts: decrease caused by customer orders, increase caused by urgent orders and increase caused by ordinary orders. Hence, \( \tilde{X}_t \) is given as follows:

\[
\tilde{X}_t = \tilde{X}_0 - C_{(c)}^t + C_{(u)}^t + C_{(o)}^t,
\]

where \( \tilde{X}_0 \) is an initial inventory. In this study, we suppose that \( \tilde{X}_t \) is a stochastic process expressing temporally random variation of the inventory level. The process \( C_{(c)}^t \) in Eq.(1), which we call customer order process, represents the cumulative amount of materials needed for customer orders up to time \( \tilde{t} \) satisfying \( C_{(c)}^0 = 0 \) (a.s.). The process \( C_{(u)}^t \) in Eq.(1), which we call urgent order process, represents the cumulative amount of materials corresponding to the amount needed for manufacturing the components due to urgent orders up to time \( \tilde{t} \) satisfying \( C_{(u)}^0 = 0 \) (a.s.). The process \( C_{(o)}^t \) in Eq.(1), which we call ordinary order process, represents the cumulative amount of materials due to ordinary orders, defined in Section 2.1, up to time \( \tilde{t} \) satisfying \( C_{(o)}^0 = 0 \) (a.s.).

The cumulative amount of materials needed for customer orders \( C_{(c)}^t \) is here mathematically modeled by a compound Poisson process, i.e.,

\[
C_{(c)}^t = \sum_{k=1}^{N_{(c)}^t} \tilde{Y}_k,
\]

where \( N_{(c)}^t \) denotes the total number of customer orders up to time \( \tilde{t} \) and is described by a Poisson process with an intensity \( \lambda \) and \( \tilde{Y}_k \) denotes the amount of materials required by the \( k \)-th customer order and \( \{ \tilde{Y}_k \}_{k=1,2,\ldots} \) is a sequence of independently and identically distributed (i.i.d.) positive random variables. According to the standard description style in modern probability theory, we assume, in what follows, that all processes appearing in our inventory model have right-continuous path for describing discrete jumps occurring at customer orders as well as ordinary and urgent orders. For example, the value of \( C_{(c)}^t \) at the instance of customer order represents the cumulative amount of customer orders just after the customer order occurrence, i.e., its left-limit \( C_{(c)}^{t-} = \lim_{u \to t^-} C_{(c)}^u \) represents the cumulative amount just before the customer order occurrence.

The factory can control the inventory level only to decide when and how much amount of materials it orders on ordinary orders. We assume that the factory makes an ordinary order with an ordering size \( \tilde{H}_t \), which we call ordinary order size process, when \( \tilde{X}_t \) falls below a predetermined threshold level \( b (\geq 0) \). Here we assume that \( \tilde{H}_t \) is bounded, i.e.,

\[
b \leq \tilde{H}_t \leq H_{\text{max}},
\]

where \( H_{\text{max}} \) represents an upper bound of the ordering size.
The factory compensates the shortage of materials needed for customer orders by applying urgent orders. We assume that we can ignore the length of the lead time from the urgent order until the ordered components arrive, i.e.,

\[
\Delta C_{\tilde{t}}^{(o)} = \begin{cases} 
\tilde{Y}_k - \tilde{X}_{\tilde{t}}, & (\tilde{t} = \tilde{\Gamma}_k \quad (k = 1, 2, \ldots)) \\
0 & \text{(otherwise)},
\end{cases}
\]

provided that neither of customer order nor ordinary order is arrived at \( t = \Gamma_k \), where \( \Gamma_k \) denotes the time when the factory makes the \( k \)-th urgent order.

It should be noted that we have a serious problem when we apply the stochastic control theory to the above model. The temporal variation of the ordinary order process \( C_{\tilde{t}}^{(o)} \) depends on the information both on the past behavior of the inventory process \( \tilde{X}_t \) and the ordinary order size process \( H_t \), i.e., when the factory made an ordinary order and how much amount of materials the factory ordered. Hence, considering Eq. (1), we find that the inventory process \( \tilde{X}_t \) is not a Markov process, which generally makes it very difficult to derive optimal control in applying the stochastic control theory. Thus, we provide an improved model of inventory fluctuation in the next section.

2.3. Improved model of inventory fluctuation

In order to remove the problem indicated in the previous section, we introduce a new time variable \( \tilde{t} \) which measures the elapsed time except for lead time of the ordinary order, i.e., the difference between \( t \) and \( \tilde{t} \) corresponds to the cumulative length of the lead time. We call the time variables \( \tilde{t} \) in the previous section and \( t \) in this section actual time variable and analytic time variable, respectively. Similarly, the model in the previous section and the improved model in this section are called actual time model and analytic time model, respectively.

Figure 2 is a schematic diagram of each sample behavior of the inventory process in the actual time model (Fig.2(a)) and the newly introduced analytic time model (Fig.2(b)). A similar approach compressing lead times is reported by Jiang et.al. 2015. In what follows, we use \( t \) as a time variable.

Corresponding to the time variable changing from \( t \) to \( \tilde{t} \), we introduce some new stochastic processes related to the inventory fluctuation. Let \( C_t \) be the cumulative amount of materials needed for customer orders arriving in the time except for lead times up to time \( t \). Since increments of \( \tilde{C}_{\tilde{t}}^{(c)} \) are independent of the past, \( C_t \) can be expressed as

\[
C_t = \sum_{k=1}^{N_t^{(c)}} Y_k,
\]

where \( N_t^{(c)} \) is a Poisson process with the same intensity \( \lambda \) as \( \tilde{N}_t^{(c)} \) and \( Y_k \) denotes the amount of materials required by the \( k \)-th (counted in the time except for lead time) customer order and \( \{Y_k\}_{k=1,2,\ldots} \) is a sequence of i.i.d. positive random variables whose probability distribution is the same as that of \( \{\tilde{Y}_k\}_{k=1,2,\ldots} \). Further, let \( H_t \) be the ordinary order size process under the new setting of time \( \tilde{t} \), i.e., the factory makes an ordinary order with an ordering size \( H_t \) when the inventory level falls below \( b \) at time \( \tilde{t} \). Let \( N_t \) be the total number of ordinary orders up to time \( t \), which is a stochastic process that is fully dependent on the compound process \( C_t \).

In order to take customer order behavior into the analytic time model, we newly introduce the following quantity. We assume that the length of lead time of the ordinary order depends only on how much amount of materials the factory
orders and let \( L(h_0) \) be the length of lead time under its ordering size \( h_0 \). Further, stochastic temporal variation of the inventory level within a lead time is caused only by the customer order, which is an external noise. Thus, letting \( \Xi \) be the inventory level at the end of a lead time (i.e., at the time when ordinary order arrives at the warehouse), we can express it as \( \Xi(x, h_0) \) under the condition that the inventory level is \( x \) when the factory makes ordinary order and the ordering size is \( h_0 \), which can be described as follows:

\[
\Xi(x, h_0) = \left(x - \int_{0}^{L(h_0)} d\tilde{Z}_t \right)^{(+)}.
\]

(6)

where \( \tilde{Z}_t \) is a compound Poisson process which has the same stochastic property as \( \tilde{C}^{(c)}_t \) and \( (x)^{(+)} \equiv \max(x, 0) \). However, using the expression Eq.(6), we have a serious difficulty to derive optimal control in applying the stochastic control theory. Hence, we approximately evaluate it in terms of an expected variation of the customer order process, i.e., we redefine \( \Xi(x, h_0) \) as

\[
\Xi(x, h_0) = \left(x - E\left\{ \int_{0}^{L(h_0)} d\tilde{Z}_t \mid h_0 \text{ is given} \right\} \right)^{(+)}.
\]

(7)

We denote an inventory process under the new time variable \( t \) by \( X_t \), whose evolution is described by the following stochastic differential equation;

\[
dX_t = -dc_t - \left[X_t - \Xi(X_t - dc_t, H_t)\right]dN_t + H_{ct}dN_t + dc_t dN_t,
\]

(8)

where the first, second and third terms in the right-hand side represent decrease of inventory due to customer order, decrease of inventory during the lead time and increase of inventory due to ordinary order, respectively. The forth term in the right-hand side needs to be introduced so that the first term must be removed when the ordinary order occurs. Here, we define a constant \( x_{max} \) as

\[
x_{max} = \sup_{h_0 \in [0, H_{max}]} \left\{ h_0 + \sup_{x \in [0, h_0]} \Xi(x, h_0) \right\},
\]

(9)

which gives the upper bound of the inventory level in our model. If we suppose that the initial inventory level \( X_0 \) satisfies \( X_0 \leq x_{max} \), we can easily show that an inequality \( X_t \leq x_{max} \) \((\forall t \geq 0)\) holds. Hence, we assume \( X_0 \leq x_{max} \) in what follows.

Let us consider temporal behavior in a time interval \([s, T]\) \((0 \leq s \leq T)\) and let \( H = \{H_t; s \leq t \leq T\} \) be total behavior of the ordinary order size process in the interval. Since the solution process \( X_t \) as well as the process \( N_t \) describing ordinary order occurrences generally depend on the total variation of the ordinary order, such dependence should be taken into notations. Thus, we newly introduce \( X^H_t \) representing the solution process under the effect of \( H \) and \( N^H_t \) representing the ordinary order occurrence under the effect of \( H \). Further, we rewrite Eq.(8) as follows;

\[
dX_t^H = -dc_t - \left[X_t^H - \Xi(X_t^H - dc_t, H_t)\right]dN_t^H + H_{ct}dN_t^H + dc_t dN_t^H.
\]

(10)

It should be noted that the solution of Eq.(10) is defined as satisfying the following stochastic integral equation;

\[
X_t^H = X_s^H - \int_s^t dc_r - \int_s^t \left[X_r^H - \Xi(X_r^H - dc_r, H_r)\right]dN_r^H + \int_s^t H_{cr}dN_r^H + \int_s^t dc_r dN_r^H.
\]

(11)

Since both of \( C_t \) and \( N_t^H \) show only discrete jumps, Eq.(12) can be rewritten as

\[
X_t^H = X_s^H - \sum_{s < r \leq t} C_r - \sum_{s < r \leq t} \left[X_r^H - \Xi(X_r^H - DC_r, H_r)\right]dN_r^H + \sum_{s < r \leq t} H_{cr}dN_r^H + \sum_{s < r \leq t} \Delta C_r \Delta N_r^H,
\]

(12)

where \( \Delta C_r = C_r - C_{r^-}, \Delta N_r^H = N_r^H - N_{r^-} \), and summation is taken for all discrete jumps occurred in the time interval \((s, t]\).

2.4. Formulation of the optimal ordinary order control

As mentioned earlier, we suppose that the factory can dynamically control the ordinary order size, described by the process \( H_t \), so that it can obtain the best performance. The performance is usually quantified as costs necessary for the factory managing its inventory level. We consider two kinds of cost; one is the cost which is spent for the factory holding materials in the warehouse and the other is the cost for urgent orders. The former cost is the so-called holding cost and we call the latter cost urgent cost. Then, we introduce the following evaluation function to quantify the performance of ordinary order size control;

\[
J^H(s, x) = E_{x^s}[K^H(s, x)],
\]

(13)
where \( \mathbb{E}^x\cdot \cdot \cdot = \mathbb{E}\cdot \cdot \cdot |X_s^H = x \) and

\[
K^H(s, x) = \int_s^T G(X_t^H)dt + \int_s^T G^{(0)}(dC_t - X_{t-}^H)dN_t^H + \int_s^T \beta(X_t^H - dC_t, H_{t-})dN_t^H
\]

where our control is supposed to be executed in the time interval \([s, T]\). We call the total behavior \( H \) in \([s, T]\) simply a control hereinafter. In Eq.(14)), the first term of the right hand side is the total holding cost in the time except for lead times. The second term is the total urgent cost due to urgent orders caused by customer orders at the time of ordinary orders. The third term is the total cost in lead times, which consists of both the holding cost and the urgent cost. The functions \( G(x) \) and \( G^{(a)}(y) \) express costs per unit time for the factory holding its inventory level \( x \) and costs for an urgent order whose size is \( y \) respectively, which satisfy

\[
G(x) \equiv 0 \quad \text{(for } \forall x \leq 0), \quad G^{(a)}(y) \equiv 0 \quad \text{(for } \forall y \leq 0).
\]

The random variable \( \hat{\beta}(x, h_0) \) represents costs in a lead time provided that the factory has made the ordinary order with size \( h_0 \) and the inventory level has been \( x \) at that time. Hence, we can express \( \hat{\beta}(x, h_0) \) as

\[
\hat{\beta}(x, h_0) = \int_0^{L(h_0)} G(x - \tilde{Z}_t)dt + G^{(a)}(\tilde{Z}_t - (x)^+)) + \int_{(\gamma \geq L(h_0))}\gamma G^{(a)}(d\tilde{Z}_t),
\]

where

\[
\gamma = \inf_{\tilde{Z}_t \geq 0} [\tilde{Z}_t > x].
\]

Let \( \mathbb{E}^x \cdot \cdot \cdot | x, h_0 \) be an operator to take expectation with respect to \( \hat{\beta} \) under the condition that the factory has made the ordinary order with size \( h_0 \) and the inventory level has been \( x \) at that time. Then, the expectation of the third term of the right hand side in Eq.(14) can be express as

\[
\mathbb{E}^x\cdot \cdot \cdot \left\{ \int_s^T \hat{\beta}(X_t^H - dC_t, H_{t-})dN_t^H \right\} = \mathbb{E}^x\cdot \cdot \cdot \left\{ \int_s^T \mathbb{E}^\hat{\beta}(X_t^H - dC_t, H_{t-})dN_t^H \right\}.
\]

Therefore, we can replace \( \hat{\beta}(x, h_0) \) by its expectation with respect to the random behavior in the lead time, denoted by \( \hat{\beta}(x, h_0), \) i.e., Eq.(14) can be replaced as follows;

\[
K^H(s, x) = \int_s^T G(X_t^H)dt + \int_s^T G^{(0)}(dC_t - X_{t-}^H)dN_t^H + \int_s^T \beta(X_t^H - dC_t, H_{t-})dN_t^H.
\]

We define the optimal control \( H^* \), which minimizes the evaluation function given by Eq.(13), as follows;

\[
J^H(s, x) = \inf_H J^H(s, x) \equiv V(s, x).
\]

### 2.5. Markov control

Since we improve the mathematical model of the inventory process so that the inventory process \( X_t^H \) is a Markov process if \( H_t \) is a so-called Markov control, we consider only Markov controls in deriving the optimal control hereinafter. That is, we do not need to examine controls that depend on all the past information on the inventory system, i.e., we have only to examine controls that are determined by the current information. Hence, by the use of a deterministic function \( h, \) \( H_t \) is assumed to be expressed as

\[
H_t = h(t, X_t^s),
\]

where \( X_t^s \) means the inventory process under the Markov control by the use of \( h, \) simply called Markov control \( h, \) which is given as a solution of the following stochastic differential equation;

\[
dX_t^h = -dC_t - \left\{ X_t^h - \Xi(X_t^h, dC_t, h(t-, X_t^h)) \right\} dN_t^h + h(t-, X_t^h)dN_t^h + dC_t dN_t^h.
\]

In what follows, we use a notation, for example, \( J^h(s, x) \) instead of \( J^H(s, x) \) provided that the control is restricted to a Markov control described by a function \( h(s, x) \).
3. HJB equation for optimal control

3.1. Necessary condition for the optimality

To find the optimal control, we first derive a necessary condition for the optimality by applying the well-known Bellman principle.

Suppose that the optimal control $H^*_t$ is given. We define a control $\hat{H}_t$ as

$$\hat{H}_t = \begin{cases} 0 & \text{for } s \leq t < s + \Delta s \\ h^*(t, X_t) & \text{for } s + \Delta s \leq t \leq T , \end{cases}$$

with a smallness interval $\Delta s$ and an arbitrary constant $h_0 \in [b, H_{\text{max}}]$. Because of the optimality, the evaluation function under $\hat{H}_t$, denoted by $J^\hat{H}(s, x)$, cannot be smaller than the evaluation function under the optimal control $V(s, x)$, i.e., an inequality $J^\hat{H}(s, x) \geq V(s, x)$ holds, which leads to

$$\lim_{\Delta s \to 0} \frac{J^\hat{H}(s, x) - V(s, x)}{\Delta s} \geq 0.$$  

(24)

Minimization of the left hand side in Eq. (24), the inequality is reduced to an equality as

$$\inf_{h_0 \in [b, H_{\text{max}}]} \left[ \lim_{\Delta s \to 0} \frac{J^\hat{H}(s, x) - V(s, x)}{\Delta s} \right] = 0.$$  

(25)

Using the Bayes formula, we can obtain the following relation within first order of $\Delta s$;

$$J^\hat{H}(s, x) = E[^{s,t}\{K^\hat{H}(s, x)A_0\}P(A_0) + E[^{s,t}\{K^\hat{H}(s, x)A_1\}P(A_1)]$$

$$+E[^{s,t}\{K^\hat{H}(s, x)A_2\}P(A_2) + E[^{s,t}\{K^\hat{H}(s, x)A_3\}P(A_3)] + o(\Delta s),$$

(26)

where events $A_i$ ($i = 0, 1, 2, 3$) mean as follows;

$$\begin{align*}
A_0 & : \text{No customer order occurs in } [s, s + \Delta s] \\
A_1 & : \text{One customer order occurs and an ordinary order is not made in } [s, s + \Delta s] \\
A_2 & : \text{One customer order occurs and an ordinary order is made in } [s, s + \Delta s] \\
A_3 & : \text{More than two customer orders occur in } [s, s + \Delta s] 
\end{align*}$$

Substituting Eq.(26) into Eq.(25) and calculating the left hand side of Eq.(24) (see, for example, Højgaard 2002 or Øksendal and Sulem 2009), we can finally obtain the following equation;

$$\inf_{h \in [b, H_{\text{max}}]} \left[ G(s) - \lambda V(s, x) + \frac{\partial V}{\partial s}(s, x) + \lambda \int_0^{s-b} V(s, x - y) dF(y) + \lambda \int_x^{\infty} G^{(0)}(y - x) dF(y) \right.$$  

$$\left. + \lambda \int_{x-b}^{\infty} \beta(x - y, h) dF(y) + \lambda \int_{s-b}^{\infty} V(s, h + \Xi(x - y, h)) dF(y) \right] = 0,$$  

(27)

where $F(\cdot)$ is a probability distribution function of each customer order size. Equation (27) is called HJB (Hamilton-Jacobi-Bellman) equation.

According to the cost function given by Eq.(19), we can easily derive the following terminal condition for $V(s, x)$;

$$\lim_{s \to T} V(s, x) = 0.$$  

(28)

In this paper, the existence of the solution of the HJB equation (27) is verified through numerical approach in Section 4.

3.2. Sufficient condition for the optimality

Next, we show that the HJB equation also gives a sufficient condition for the optimality.

Suppose that $W(s, x)$ is a solution of the HJB equation (27) satisfying the terminal condition Eq.(26), provided that $W(s, x)$ is differentiable with respect to $s$ and integrable so that integral terms in Eq.(27) exist. We construct a stochastic process $D_r$ as

$$D_r = W(t, X^b_t) + \int_s^t G(X^b_r) dr + \sum_{r < r'} G^{(0)}(C_r - X^b_{r'}) dN^b_r + \sum_{r < r'} \beta(X^b_r - C_r, h(r, X^b_{r'})) dN^b_r \quad (s \leq t \leq T),$$  

(29)
where \( X_t \) is assumed to be equal to \( x \). Applying the well-known Itô formula (Itô 1942) for an increment of \( D_t \), we obtain

\[
D_T - D_s = \int_s^T \frac{d}{ds} W(r, X_t^h) dr + \sum_{r < s < T} \Delta W(r, X_t^h) + \int_s^T G(X_t^h) dr
\]

Taking expectation of Eq.(30) by paying attention to that a jump of \( C_r \) in \( (r + dr) \) occurs with probability \( \lambda dr \), we obtain

\[
J(h, s) - W(s, x) = \mathbb{E} \left\{ \int_s^T Q^h(s, x; W) dr \right\},
\]

where

\[
Q^h(s, x; W) = G(X_t^h) - \lambda W(r, X_t^h) + \frac{\partial}{\partial s} W(r, X_t^h) + \lambda \int_0^{x-b} W(r, X_t^h - y) dF(y) + \lambda \int_r^{x-b} G^0(y - X_t^h) dF(y) + \lambda \int_r^{x-b} \beta(X_t^h - y, h(r, X_t^h)) dF(y) + \frac{\partial}{\partial s} V(s, h(y, r, X_t^h)) + \frac{\partial}{\partial y} \mathbb{E}(X_t^h - y, h(r, X_t^h)) dF(y).
\]

Since \( W(s, x) \) is a solution of Eq.(25), an inequality \( Q^h(s, x; W) \geq 0 \) holds for any function \( h(s, x) \) specifying a corresponding Markov control. Therefore, we can conclude that \( W(s, x) \) equals to \( \text{inf}_h J^h(s, x) \), i.e., \( W(s, x) \) coincides with \( V(s, x) \) and a control obtained as a solution of the HJB equation gives an optimal control.

### 3.3. Basic algorithm for solving HJB equation

In this paper, we discuss an optimal control based upon a numerical solution of the HJB equation. To construct a numerically solving algorithm for Eq.(25), we rewrite Eq.(25), by paying attention to that the derivative \( \partial V/\partial s \) is independent of the control variable \( h \), as

\[
\frac{\partial V}{\partial s} (s, x) = -G(x) + \lambda V(s, x) + \frac{\partial}{\partial s} V(s, x - y) dF(y) - \lambda \int_y^{(x-b)} G^0(y - x) dF(y)
\]

\[
- \mathbb{E} \left[ \left. \int_{x-b}^{\infty} \beta(x - y, h(y, r, X_t^h)) dF(y) + \int_{x-b}^{\infty} V(s, h(y, r, X_t^h)) dF(y) \right| X_t^h \right).
\]

Approximating the left hand side by the use of a backward difference scheme with a small mesh size \( \Delta x \), we can obtain the following backward recurrence formula;

\[
V(s - \Delta x) = V(s) + \left( G(x) - \lambda V(s, x) + \lambda \int_0^{x-b} V(s, x - y) dF(y) + \lambda \int_r^{x-b} G^0(y - x) dF(y) \right) \Delta s
\]

\[
+ \mathbb{E} \left[ \left. \left( \int_{x-b}^{\infty} \beta(x - y, h(y, r, X_t^h)) dF(y) + \int_{x-b}^{\infty} V(s, h(y, r, X_t^h)) dF(y) \right) \right| X_t^h \right] \Delta s.
\]

Starting from the terminal condition given by Eq. (26), we can calculate \( V(s, x) \) by the use of Eq.(34).

### 4. Numerical examples

In this section, we give some numerical examples to examine optimal controls of the inventory by solving the HJB equation given by Eq.(25) based upon the recurrence formula derived in Section 3.3.

In what follows, we fix the intensity \( \lambda \) of the Poisson process \( N_t^c \) as \( \lambda = 1 \), i.e., we select a time variable so that mean customer orders in unit time is unity. Further, we assume the probability distribution function of customer order size \( F(y) \) as an exponential distribution, i.e.,

\[
F(y) = \begin{cases} 
1 - \exp\left(-\frac{y}{\mu}\right) & \text{for } y \geq 0 \\
0 & \text{for } y < 0
\end{cases},
\]

where \( \mu \) is a positive constant representing mean size of each customer order. Similarly as selecting the time variable, we fix \( \mu = 1 \), i.e., inventory level is here supposed to be dimensionless by a normalization by the mean size of customer order.

Based upon such setting, we fix the expiration time \( T \) and the upper bound of ordinary order size \( H_{\text{max}} \) as \( T = 100.0 \), and \( H_{\text{max}} = 30.0 \), respectively. The function of the holding cost \( G(\cdot) \) is supposed to be the following bi-linear form;

\[
G(x) = \begin{cases} 
0 & \text{for } x \leq 0 \\
g_1 x & \text{for } 0 < x \leq x_G \\
g_2(x - x_G) + g_1 x_G & \text{for } x_G < x
\end{cases}
\]

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where $x_0$, $g_1$ and $g_2$ are positive constants satisfying $b < x_0 < x_{\text{max}}$ and $g_1 < g_2$. The supposed function means that the factory incurs higher cost for excess inventory larger than $x_0$. We further suppose that the function of cost for an urgent order $G^{(u)}(\cdot)$ is linear, i.e.,

$$G^{(u)}(y) = \begin{cases} 0 & \text{for } y \leq 0 \\ g^{(u)}y & \text{for } y > 0 \end{cases},$$

(37)

where $g^{(u)}$ is a positive constant. Here, we fix the above constants as

$$g_1 = 0.01, \quad g_2 = 0.02, \quad x_0 = 20.0, \quad g^{(u)} = 0.5.$$

4.1. Case in which the length of lead time is independent of ordinary order size

First we consider a case in which the length of lead time is independent of ordinary order size, i.e., we assume

$$L(h_0) = L_0 \quad \text{(const.)} \quad \forall h_0 \in [b, H_{\text{max}}].$$

(38)

Here we set as $L_0 = 12.0$ and $x_{\text{max}} = 30.0$. and both $\Xi(x, h_0)$ and $\beta(x, h_0)$ are independent of the ordinary order size $h_0$ under the assumption.

In Fig.3, we show results of Monte Carlo simulation for estimating $\beta$ for several values of $x$, denoted by solid square, and obtained regression curve denoted by solid curve, where we apply a fifth order polynomial for the regression curve as shown in Fig.3.

![Fig. 3 Simulation results and a regression curve of $\beta(x, h_0)$ (independent of $h_0$).](image)

In Fig.4, we show the optimal control $h^*(s, x)$ obtained by solving the HJB equation (27) as a function of $x$ for several values of $s$, where we set the time mesh $\Delta s$ in Eq.(34) as $\Delta s/T = 10^{-3}$. The convergence of numerical solution with respect to discrete time mesh size is verified by checking results for several values of the mesh size. Since Fig.4(a) indicates that $h^*(s, x)$ does not depend on $x$, we plot $h^*(s, x)$ as a function of $s$ for $\forall x \in [b, x_{\text{max}}]$.

![Fig. 4 Optimal ordinary order $h^*(s, x)$ obtained by solving the HJB equation.](image)
Although Fig. 4(a) shows that the optimal control \( h^*(s, x) \) is independent of \( x \), it depends on time \( s \) only in the region in which the rest time \( T - s \) is relatively small as shown in Fig. 4(b). Further, we can see that the optimal control \( h^*(s, x) \) shows slightly oscillating behavior for \( 50 \leq s \leq 90 \). Here, we should note that costs in a lead time is immediately added without the elapse of the lead time with respect to analytic time \( t \). Therefore, we can expect that the optimal control tends to reduce the frequency of ordinary orders so that the factory can reduce the total costs due to lead times. When time \( s \) becomes close to the expiration time \( T \), i.e., \( 80 \leq s \leq 100 \) in Fig. 4(b), \( h^*(s, x) \) monotonically decreases as \( s \) increases so that no ordinary order occurs after \( s \).

On the other hand, when \( s \) is slightly smaller than 80, \( h^*(s, x) \) shows slightly increasing behavior as a function of \( s \), which is considered to be due to a kind of balance between the costs in lead times and the holding costs. It is clear that if the rest time \( T - s \) is large then we can not effectively increase the probability that no ordinary order occurs after \( s \). Thus, the optimal control abandons reducing the total costs due to lead times for reducing the total holding costs after \( s \), though we can not necessarily assert that the factory can reduce the total holding costs by making a relatively small ordinary order.

For \( s \approx 65 \), \( h^*(s, x) \) similarly has a slightly large value as \( s \approx 80 \) in order to reduce the probability that two (or more) ordinary orders occurs after \( s \). The reason of these tendencies of \( h^*(s, x) \) can be verified by examining sensitivity to \( L_0 \), as shown in Fig. 5. As \( L_0 \) becomes smaller, the costs in a lead time correspondingly become smaller. Thus, we can not observe the remarkable oscillating behavior of \( h^*(s, x) \). Such oscillating behavior is gradually weakened as \( T - s \) becomes even larger.

Finally, the optimal control \( h^*(s, x) \) almost converges to a balanced value realizing the trade-off between the total holding costs and the total costs in lead times. Figure 6 shows the evaluation function \( V(s, x) \) under the optimal control as a function of \( x \) for several values of \( s \). We can see that \( V(s, x) \) monotonically increases as the rest time \( T - s \) increases and that the “optimal” inventory level minimizing \( V(s, x) \) exits.

### 4.2. Case in which the length of lead time depends on ordinary order size

Next, we consider a case in which the length of lead time depends on ordinary order size. Although the lead time generally becomes larger as the ordering size becomes large, such an effect is expected to be small when the ordering size
is small. Hence, we here assume that (i) the length of lead time is independent of ordinary order size if the ordinary order size is less than a threshold value and (ii) the length of lead time linearly increases as ordinary order size increases when the ordinary order size is larger than the threshold value, for the function \( L(h_0) \), i.e., we suppose the following form:

\[
L(h_0) = \begin{cases} 
L_0 & \text{for } b \leq h_0 < h_L \\
L_0 + k(h_0 - h_L) & \text{for } h_L \leq h_0 \leq H_{\max}.
\end{cases}
\]

(39)

where we set as \( b = 10.0, L_0 = 8.0, l = 1.0 \) and \( h_L = 20.0 \). Correspondingly, the upper bound of the inventory level \( x_{\max} \) is determined as \( x_{\max} = 30.0 \). We can easily show that the expectation of costs in a lead time \( \beta(x, h) \) monotonically increases as \( h_0 \) increases for \( h_0 \geq h_L \).

As for the cost function \( \beta(x, h_0) \), setting mesh points for \( h_0 \) as \( h_0^{(k)} = k\Delta h_0 \) (\( k = 0, 1, \cdots, 1000 \); \( \Delta h_0 = (H_{\max} - b)/1000 \)), we derive a regression curve of fifth order polynomial for each \( h_0^{(k)} \) based upon Monte Carlo estimations similarly as in the previous subsection. Then, \( \beta(x, h_0) \) is approximated by a linear interpolation for \( \beta(x, h_0^{(k)}) \) (\( k = 0, 1, \cdots, 1000 \)). Figure 7 shows simulation results and regression curves for several values of \( h_0 \), where coefficients in the regression curve \( \beta(x, h_0) = \sum_{j=0}^{5} a_jx^j \) are listed in Table 1.

![Simulation results and regression curves of \( \beta(x, h_0) \) for several values of \( h_0 \).](image1)

**Table 1** Estimated coefficients of the regression curves of \( \beta(x, h_0) \) for several values of \( h_0 \)

| \( h_0 \) | \( a_0 \) | \( a_1 \) | \( a_2 \) | \( a_3 \) | \( a_4 \) | \( a_5 \) |
|---|---|---|---|---|---|---|
| 20.0 | 4.000 | -4.852 \times 10^{-1} | 6.040 \times 10^{-3} | 2500 \times 10^{-3} | -4.559 \times 10^{-2} | -3.253 \times 10^{-6} |
| 25.0 | 6.500 | -4.925 \times 10^{-1} | 7.420 \times 10^{-3} | -8.326 \times 10^{-4} | 1.431 \times 10^{-3} | -4.566 \times 10^{-6} |
| 30.0 | 9.000 | -4.904 \times 10^{-1} | 5.108 \times 10^{-3} | 3.354 \times 10^{-5} | -1.237 \times 10^{-5} | 1.507 \times 10^{-6} |

![Fig. 8 Optimal control \( h^*(s, x) \) obtained by solving the HJB equation.](image2)

In Fig.8(a), we show the optimal control \( h^*(s, x) \) as a function of \( x \) for several values of \( s \). Based upon the result that \( h^*(s, x) \) does not depend on \( x \), we plot \( h^*(s, x) \) as a function of \( s \) for \( \forall x \in [h, x_{\max}] \) in Fig.8(b), where we additionally...
plot $h'(s, x)$ obtained in the case of $L(h_0) = 8.0$ (const.), denoted by dashed curve, for comparison. Further in Fig.9, the optimal evaluation function $V(s, x)$ is plotted as a function of $x$ for several values of $s$. As shown in Fig.8(b), there is a remarkable difference between numerical results in the previous subsection and those in this subsection. That is, the optimal control $h'(s, x)$ is bounded by $h_L$, which indicates that we should not select an ordinary order size larger than $h_L$, so that the expected costs in a lead time is reduced.

5. Conclusions

In this paper, we have newly constructed a mathematical model using a non diffusive stochastic differential equation for discussing the inventory management problem of a factory having effects of lead times in replenishing the inventory, in which a Markov property can be realized by effectively introducing a concept of analytical time. Consequently, we have clarified that a technique of Markov control can be applied to such an inventory problem and thus the HJB equation can be formulated to derive the optimal control of the ordering size for an evaluation function expressing the expected total costs.

Through numerical examples, we have shown that (i) The optimal control $h'(s, x)$ is almost independent of an inventory level $x$ just before an ordinary order, (ii) The optimal ordinary order size drastically decreases as the rest time up to the expiration time becomes small and (iii) There exists an ‘optimal’ inventory level minimizing the optimal evaluation function.

We have clarified that the stochastic control approach can be applied to a probabilistic inventory model using stochastic differential equations driven by a compound Poisson process, where the HJB equation is derived and numerically solved. Combining our result with results using diffusive model obtained so far, we can extend our discussion to more extended and generalized probabilistic inventory models, since stochastic processes having independent increments can be constructed as a sum of diffusive noise and a superposition of compound Poisson processes. That is, such a feature enables us to construct a probabilistic inventory model using stochastic differential equations driven by noise of more general type, which has a very important meaning in the point that the discussion given by this paper can be applied to inventory problems of various types. Further, we should discuss a new mathematical model in which the factory can also dynamically control the threshold level $b$ in addition to its size.

References

Applebaum, D., Lévy Processes and Stochastic Calculus (2009), Cambridge Univ. Press.
Bellman, R., Dynamic programming (1957), Princeton Univ. Press.
Chen, L., Feng, Y. and Ou, J., Joint Management of Finished Goods Inventory and Demand Process for a Make-to-Stock Product: A Computational Approach, IEEE Transactions on Automatic Control, Vol.52, No.2 (2006), pp.258-273.
Fatrias, D. and Y. Shimizu, Multi-objective analysis of periodic review inventory problem with coordinated replenishment in two-echelon supply chain system through differential evolution, J. of Advanced Mechanical Design, Systems, and Manufacturing, Vol.4, No.3 (2010), pp.637-650.
Feng, K. and Rao, U. S., Echelon-stock $(R, nT)$ control in two-stage serial stochastic inventory systems, Operations Research Letters, Vol.35 (2007), 95-104.
Grewal, C. S., S. T. Enns and P. Rogers, Duymetric reorder point replenishment strategies for a capacitated supply chain with seasonal demand, Computers & Industrial Engineering, Vol. 80 (2015), pp.97-110.
Huang, J., M. Leng and L. Liang, Recent developments in dynamic advertising research, European J. of Operational Research, Vol. 220 (2012), pp.591-609.
Højgaard, B., Optimal dynamic premium control in non-life insurance. Maximizing dividend payouts, Scandinavian Actuarial Journal, Vol. 4 (2002), pp. 225-245.
Ito, K., On Stochastic processes II: infinitely divisible laws of probability, Japanese Journal of Mathematics, Vol.18 (1942), pp.261-301.
Jian, M., X. Fang, L.-q. Jin and A. Rajapov, The impact of lead time compression on demand forecasting risk and production cost: A newsvendor model, Transportation Research, Part E, Vol. 84 (2015), pp.61-72.
Kanekiyo, H., Proposal of a New Probabilistic Model for Random Fatigue Crack Growth Using a Noise of Poisson Type, Journal of The Society of Materials Science, Japan, Vol.63, No.2 (2014), pp.92-97 (in Japanese).
Li, S., J. Zhang and W. Tang, Joint dynamic pricing and inventory control policy for a stochastic inventory system with perishable products, Int. J. Production Research, Vol. 53, No.10 (2015), pp.2937-2950.
Li, Y., S. Zhang and J. Han, Dynamic pricing and periodic ordering for a stochastic inventory system with deteriorating items, Automatica, Vol. 76 (2017), pp.200-213.
Nasr, W. W. and I. J. Elshar, Continuous inventory control with stochastic and non-stationary Markovian demand, European J. of Operational Research, Vol. 270 (2018), pp.198-217.
Oksendal, B. and A. Sulem, Applied Stochastic Control of Jump Diffusions (2007), Springer-Verlag, Berlin Heidelberg.
Ouaret, S., J.-P. Kennes and A. Gharbi, Production and replacement policies for a deteriorating manufacturing system under random demand and quality, European J. of Operational Research, Vol. 264 (2018), pp.623-636.
Rempała, R., A continuous production-inventory problem with regeneration cycles, International Journal of Production Economics, Vol.93-94 (2005), pp.447-454.
Schlosser, R. Joint stochastic dynamic pricing and advertising with time-dependent demand, J. of Economic Dynamics & Control, Vol. 73 (2016), pp.439-452.
Weerasinghe, A. and C. Zhu, Optimal inventory control with path-dependent cost criteria, Stochastic Processes and their Applications, Vol. 126 (2016), pp.1585-1621.