Algebraic Properties of Einstein Solutions in Ghost-Free Bimetric Theory

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A known fact is that an Einstein solution in one sector in ghost-free bimetric theory implies an Einstein solution in the other sector. Earlier studies have also shown that some classes of bimetric models necessitate proportional solutions between the sectors. Here we consider a general setup of the parameters in the theory as well as the general algebraic form of the potential. We show that, if one sector has an Einstein solution, the solutions are either proportional or block proportional with at most two different eigenvalues in the square root governing metric interactions.

I. INTRODUCTION AND SUMMARY

The understanding of classical interacting massive spin-2 theories has seen a significant progress in recent years. Several consistent theories have emerged, free of the instability known as the Boulware-Deser ghost [1]. In particular, de Rham-Gabadadze-Tolley (dRGT) massive gravity was introduced in [2, 3] as a nonlinear theory of a massive spin-2 field (shown to be ghost-free in [4]), and the Hassan-Rosen (HR) bimetric theory was introduced in [5, 6] as a ghost-free nonlinear theory of two interacting spin-2 fields. Recent reviews of these theories and their extensions (like vielbein formulation, multiple interacting spin-2 fields etc.), can be found in [7, 8].

In this paper we focus on the HR bimetric theory, which schematically consists of two GR sectors governed by their own metric, here denoted $g$ and $f$, coupled through a ghost-free bimetric interaction term that involves the square root of the two metrics, $S = \sqrt{g^{-1}f}$. In addition, the interaction term is parameterized by real constants, $\beta_n$, where $n = 0, \ldots, 4$ in four dimensions (more details are in Section II).

As shown in [10], an Einstein solution in one sector of the HR bimetric theory is a necessary condition to have an Einstein solution in the other sector. This bi-Einstein setup, if attainable, imposes certain algebraic conditions on the bimetric interaction term and limits the structure of the square root matrix depending on the $\beta$-parameters. In [10], it was also shown that the two metrics are proportional for some classes of bimetric models.

Assuming a general form of the square root, we provide the following statement.

**Proposition.** In four dimensions, unless algebraically decoupled, Einstein solutions to the bimetric equations are either proportional or block proportional with at most two different eigenvalues in the square root.

The proof is given in Section IV. The algebraically decoupled cases are handled in Section V.

The proof is based on the algebraic classification of the square roots of $g^{-1}f$ given in [11]. By this classification, we get: for Type I (a diagonal square root) either one or two eigenvalues, for Type IIa (Jordan block of size 2) a single eigenvalue but with the constrained $\beta$-parameters, for Type IIb (a complex block) it either falls back to Type I or has no solutions, for Type III (Jordan block of size 3) a single eigenvalue but with the constrained $\beta$-parameters. Type IV (a non-primary square root) has the same structure as Type IIb.

A comprehensive list of all possible square roots is summarized in Table I in terms of Segre characterization.$^1$

| Type | Segre char. | Possible cases | D | Constraints on $\beta_n$ |
|------|-------------|---------------|---|--------------------------|
| I    | [1111]      |               |   |                          |
|      | [11]        |               |   |                          |
|      | [111]1      | $\Delta = 0$, $A \neq 0$ |   |                          |
|      | [11]1       | $\Delta = 0$, $A = 0$ |   |                          |
|      | [111]       | $\beta_1 = \beta_2 = \beta_3 = 0$ |   |                          |
| IIa  | [211]       | $\Delta = 0$ |   |                          |
|      | [21]1       | $\Delta = 0$, $A = 0$ |   |                          |
|      | [211]       | $\beta_1 = \beta_2 = \beta_3 = 0$ |   |                          |
| IIb  | $[z\bar{z}]_1$ | $\rightarrow$ Type I |   |                          |
| III  | [31]        | $\Delta = 0$ |   | $\beta_1 = \beta_2 = \beta_3 = 0$ |
|      | [31]        |               |   |                          |

TABLE I. The Segre characteristics of all possible square roots $S$. The highlighted eigenvalues are functions of the $\beta$-parameters. The asterisk * in the D-column indicates a decoupled case with some of the eigenvalues being arbitrary. The constraints are given in terms of $\Delta := B^2 - 4AC$, $A := \beta_2^2 - \beta_1\beta_3$, $B := \beta_1\beta_2 - \beta_0\beta_3$, and $C := \beta_0^2 - \beta_0\beta_2$.

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$^1$ The Segre characteristic is a set of integers listed in descending order that give the sizes of the blocks in a Jordan normal form. Complex Jordan blocks are denoted by $z\bar{z}$ instead. The integers corresponding to submatrices containing the same eigenvalue are grouped together in parentheses. For example, $[(21)1]$ is a class of matrices which have two different eigenvalues, where the first is in a sequence of Jordan blocks of sizes 2 and 1, and the second is in a Jordan block of size 1.
II. BACKGROUND: BIMETRIC FIELD EQUATIONS

The Hassan-Rosen (HR) action reads [5],
\[
S_{HR} = \frac{1}{2} M_g^{-d-2} \int d^d x \sqrt{-g} \, R_g + \frac{1}{2} M_f^{-d-2} \int d^d x \sqrt{-f} \, R_f - m^d \int d^d x \sqrt{-g} \, V(S).
\]
(1)

It consists of two ordinary Einstein-Hilbert terms with Planck masses \(M_g\) and \(M_f\), and the interaction term with the potential,
\[
V(S) := \sum_{n=0}^d \beta_n e_n(S).
\]
(2)

Here, \(S\) is the square root matrix function of the (1,1) tensor field \(g^{\mu\nu} f_{\mu\nu}\) (in matrix notation, \(S = \sqrt{g^{-1} f}\)). The scalar invariant coefficients \(e_n(S)\) in (2) are the elementary symmetric polynomials obtained through the generating function [9],
\[
E(t, S) = \det(I + tS) = \sum_{n=0}^{\infty} e_n(S) t^n,
\]
(3)

where \(e_n > d(S) = 0\) due to the Cayley-Hamilton theorem. In light of the \(E(t, S)\), the potential \(V(S)\) can be seen as a linear combination of the span of the \(\beta\)-parameters \(\{\beta_n\}\), which are the free parameters of the theory.

Variation of the action (1) yield two sets of equations of motion in operator form [10],
\[
\begin{align*}
G_g^\mu_\nu + \frac{m^d}{M_g^{-d-2}} V_g^\mu_\nu(S) &= 0, \quad (4a) \\
G_f^\mu_\nu + \frac{m^d}{M_f^{-d-2}} V_f^\mu_\nu(S) &= 0, \quad (4b)
\end{align*}
\]

where \(G_g\) and \(G_f\) denote the Einstein tensors of \(g\) and \(f\), respectively, and the stress-energy-like contributions \(V_g\) and \(V_f\) of the potential (2) have the structure,
\[
\begin{align*}
V_g(S) &= \sum_{n=0}^{d-1} \beta_n \sum_{k=0}^n (-1)^{n+k} e_k(S) S^{n-k}, \quad (5a) \\
V_f(S) &= \sum_{n=0}^{d-1} \beta_{d-n} \sum_{k=0}^n (-1)^{n+k} e_k(S^{-1}) S^{-n+k}. \quad (5b)
\end{align*}
\]

We say that \(g\) is an Einstein solution iff it satisfies the Einstein field equations,
\[
G_g^\mu_\nu + \Lambda_g \delta^\mu_\nu = 0,
\]
(6)

where \(\Lambda_g = \text{const}\); this implies \(V_g^\mu_\nu(S) = m^d M_g^{2-d} \Lambda_g \delta^\mu_\nu\). Similarly, we say that \(f\) is an Einstein solution iff \(G_f^\mu_\nu + \Lambda_f \delta^\mu_\nu = 0\) for some \(\Lambda_f = \text{const}\), which implies \(V_f^\mu_\nu(S) = m^d M_f^{2-d} \Lambda_f \delta^\mu_\nu\). Notably, \(V_g\) and \(V_f\) are not independent as they obey the algebraic identity [10],
\[
V_g(S) + \det(S) V_f(S) = V(S).
\]
(7)

III. DEFINITIONS

For a set of variables \(x_1, \ldots, x_d\), define the symmetric function,
\[
\langle x_1, \ldots, x_d \rangle^n_k := \sum_{i=0}^n \beta_i k e_i(x_1, \ldots, x_d).
\]
(8)

This function shifts the degree of homogeneity of the elementary symmetric polynomials for \(k\), and truncates the generating function (3) at \(n\). The subscript \(k\) is accordingly called an offset.

For a repeated single variable \(x_1 = \ldots = x_n =: \lambda\), we introduce the following convention,
\[
\langle \lambda \rangle^n_k := \langle \lambda, \ldots, \lambda \rangle^n_k = \sum_{i=0}^n \left(\begin{array}{c} n \\ i \end{array}\right) \beta_i k^i \lambda^i.
\]
(9)

Also, if an argument of \(\langle \cdot \rangle^n_k\) is a matrix \(X\), the eigenvalues of \(X\) are used instead.

In this notation, obviously \(V(S) = \langle S \rangle_d^d\), and after some simplification, (5a) can be expressed,
\[
V_g(S) = \sum_{n=0}^d (-1)^n \langle S \rangle_{d-n}^d S^n.
\]
(10)

IV. ALGEBRAIC FORMS OF THE POTENTIAL

In general, the square root matrix \(S\) is not always diagonalizable; yet it can always be put into the Jordan normal form. As shown in [11], the matrix \(S\) will contain at most one Jordan block up to size three, or at most one complex block, regardless of spacetime dimension, which enables the algebraic classification of bimetric solutions.

An Einstein solution in the \(g\)-sector implies \(V_g(S) = \text{const}\). Any such constant can be absorbed in \(\beta_0\) allowing one to consider only the case \(V_g(S) = 0\). Including a minimal matter coupling to one of the metrics does not introduce any further complication; adding a stress-energy contribution on the left-hand side of (4a) and setting \(G_g^\mu_\nu = M_g^{2-d} T^\mu_\nu\), again implies a constant \(V_g\).

Since we are dealing with a system of algebraic equations, it is possible to encounter degenerate cases for certain values of \(\beta\)-parameters so that some of the eigenvalues of \(S\) can be freely chosen (they will not be functions of the \(\beta\)-parameters). Such cases will be denoted as \emph{algebraically decoupled} and treated in Section V. Another categorization can be done according to the imposed conditions on the \(\beta\)-parameters by the very algebraic structure of the square root. If all \(\beta\)-parameters are independent of each other, such a case will be called \emph{unconstrained}.

As we shall see, these two properties are expressible in terms of the following auxiliary variables, for convenience defined here,
\[
\begin{align*}
A &:= \beta_2^2 - \beta_1 \beta_3, & B &:= \beta_1 \beta_2 - \beta_0 \beta_3, \\
C &:= \beta_1^2 - \beta_0 \beta_2, & \Delta &:= B^2 - 4AC.
\end{align*}
\]
In the rest of this section, we explicitly solve this equation for different types of the non-singular square roots matrix $S$ according to the algebraic classification from [11] in four dimensions.

### Type I

The square root $S$ of Type I has the diagonal form,

$$S_1 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \text{diag}(\lambda_1, ..., \lambda_4)$$

where $\lambda_i$ are real eigenvalues. Expanding the equation $V_0(S_1) = 0$ yields,

$$\langle \lambda_2, \lambda_3, \lambda_4 \rangle^3_0 = 0, \quad \langle \lambda_1, \lambda_3, \lambda_4 \rangle^3_0 = 0, \quad \langle \lambda_1, \lambda_2, \lambda_3 \rangle^3_0 = 0.$$  

(13a)

(13b)

These equations can also be seen as a homogeneous linear system in $\beta_0 \leq \beta_1 \leq \beta_2$. Unless $\beta_2 = 0$, the discriminant of the linear system must vanish identically,

$$\prod_{1 \leq i < j \leq 4} (\lambda_i - \lambda_j) = 0.$$  

This means that at least two eigenvalues must be equal. Without loss of generality, we can select $\lambda_3 = \lambda_4 = \lambda$ (the permutation of the eigenvalues is a similarity transformation); consequently,

$$2 \langle a \rangle_0^2 + \langle a \rangle_1^2 (\lambda_1 + \lambda_2) = 0,$$  

(14)

$$\langle a \rangle_1 (\lambda_1 - \lambda_2) = 0,$$  

(15)

$$\langle a \rangle_0^2 + a \langle b \rangle_0^2 = 0.$$  

(16)

From (15), we can have $\lambda_1 = \lambda_2$ or $\langle a \rangle_1^2 = 0$. Let us first consider the case when $\lambda_1 = \lambda_2 = b$, which gives,

$$\langle a \rangle_0^2 + b \langle a \rangle_1^2 = 0, \quad \langle b \rangle_0^2 + a \langle b \rangle_1^2 = 0.$$  

(17)

Adding and subtracting (17), then redefining $a = u + v$ and $b = u - v$ in terms of two independent variables $u$ and $v$, yields,

$$\langle u \rangle_0^3 - \beta_2 u^2 - \beta_3 u v^2 = 0,$$  

(18)

$$v \left( \langle u \rangle_1^2 - \beta_3 v^2 \right) = 0.$$  

(19)

Let $v \neq 0$, i.e., $\lambda_3 = a \neq b = \lambda_2$, and $\langle u \rangle_1^2 - \beta_3 v^2 = 0$. By using $\langle u \rangle_0^3 = \langle u \rangle_0^2 + u \langle u \rangle_1^2$, (18)–(19) become,

$$\langle u \rangle_1^2 = \beta_3 v^2, \quad \langle u \rangle_0^2 = \beta_2 v^2.$$  

(20)

The case $\beta_2 = \beta_3 = 0$ implies $\beta_0 = \beta_1 = 0$. For $\beta_2 \neq 0$ or $\beta_3 \neq 0$, we get the solutions,

$$u = -B/(2A), \quad v = \pm \sqrt{\Delta}/(2A),$$  

(21)

where the variables $A, B$, and the discriminant $\Delta$ are given in (11). Hence, (14)–(16) behaves as a single quadratic equation with the solutions $\lambda_1 = \lambda_2 = u \mp v$ and $\lambda_3 = \lambda_4 = u \pm v$.

On the other hand, let $v = 0$ in (18)–(19), that is $a = b = u$. Then,

$$\langle u \rangle_0^3 = \beta_0 + 3\beta_1 u + 3\beta_2 u^2 + \beta_3 u^3 = 0.$$  

(22)

Depending on the $\beta$’s, (22) is at most cubic in $u$ with up to three different solutions. For $\beta_3 \neq 0$, the solutions to (22) are given by the Cardano formula,

$$u_1 = -\beta_2 + (s + t)\sqrt{\frac{A}{B}}/\beta_3,$$  

(23)

$$u_{2,3} = \left[ -\beta_2 - \frac{1}{2}(s + t) \pm \sqrt{\Delta_3} \right]/\beta_3,$$  

(24)

where,

$$t = \left( \frac{1}{2}\beta_3 B - \beta_2 A \pm \sqrt{-\Delta_3} \right)^{1/3}, \quad s = A/t,$$  

(25)

$$\Delta_3 = A^3 - \left( \frac{1}{2}\beta_3 B - \beta_2 A \right)^2.$$  

(26)

Here, $\Delta_3$ is the discriminant of (22); in particular, (22) has three real roots for $\Delta_3 \geq 0$ (where at least two roots are equal for $\Delta_3 = 0$), otherwise it has one real and two complex roots ($u_1$ is always real in all cases).

Notably, the discriminant of (22) can be expressed in terms of $\Delta$ from (11b) as $\Delta_3 = -\Delta \beta_3^2/4$. This relation implies that, if all the cubic solutions $u$ of (22) are real, then all $u \pm v$ from (21) are complex. Also, if one of the cubic solution is real and two complex conjugate, then all $u \pm v$ from (21) are real.

Now, we return to (14)–(16) and consider the case $\langle a \rangle_1^2 = 0$, which implies $\langle a \rangle_0^2 = 0$ with the condition $\Delta = 0$. Then, if any of $\beta_2, \beta_3$ is vanishing, all the eigenvalues are equal; otherwise, when $\beta_2 \neq 0, \beta_3 \neq 0$, we have a decoupled case with an arbitrary $\lambda_1$ direct summed with a block of three equal eigenvalues,

$$\lambda_2 = a = (-\beta_2 \pm \sqrt{A})/\beta_3 = -B/(2A).$$  

(27)

where the last equality follows from $\Delta = B^2 - 4AC = 0 \iff A^3 - \left( \frac{1}{2}\beta_3 B - \beta_2 A \right)^2 = 0$. In the limit $A \to 0$ we have $B \to 0$, and $a = -B/(2A) \to \beta_0/\beta_1 = \beta_2/\beta_3$, rendering both $\lambda_1$ and $\lambda_3$ arbitrary.

Note that requiring $\Delta = 0$ would constrain the $\beta$-parameters. Another way to constrain the $\beta$-parameters is to impose a condition on some of the eigenvalues in (22). For example, requiring $u = 1$ to be one of the solutions imposes,

$$\beta_0 + 3\beta_1 + 3\beta_2 + \beta_3 = 0,$$  

(28)

known as the asymptotic flatness condition for $f = g$ at infinity, appearing, e.g., in [12].
Concluding this section, we count the number of solutions in Type I. In general, we have 3 branches in the cubic solution for $u$, and 6 branches in different combinations of $a$ and $b$: $2 \times aabb$, $2 \times abab$ and $2 \times baab$. Note the symmetry in exchange $a \leftrightarrow b$, so we do not calculate 6 branches in $a$ and $b$ twice. This yields 9 solutions in total of which some are necessarily complex.

The cubic solutions $\lambda_1 = \ldots = \lambda_4 = u \neq 0$, $v = 0$ imply proportional metrics, and at least one of the cubic solutions is real. The quadratic solutions $\lambda_1 = \lambda_2 = u - v$, $\lambda_3 = \lambda_4 = u + v$ imply block proportional metrics. As noted, if all the cubic solutions are real, then there are no quadratic solutions enabling the block proportional metrics. As a consequence, we have either 3 real solutions yielding proportional metrics, or 7 real solutions of which one is giving proportional metric and 6 giving different combinations of block proportional metrics.

**Type IIa**

The square root of Type IIa has the block form,

$$ S_{IIa} = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \oplus \lambda_2 \oplus \lambda_3, $$

(29)

where $\lambda_i$ are real eigenvalues.

Expanding the equation $V_9(S_{IIa}) = 0$, yields

$$ \langle \lambda_1, \lambda_2, \lambda_3 \rangle_0^3 = 0, \quad \langle \lambda_2, \lambda_3 \rangle_0^2 = 0, \quad \langle \lambda_1, \lambda_2, \lambda_3 \rangle_0 = 0. \quad \text{(30a)} $$

$$ \langle \lambda_1, \lambda_1, \lambda_3 \rangle_0^3 = 0, \quad \langle \lambda_1, \lambda_1, \lambda_2, \lambda_3 \rangle_0^2 = 0. \quad \text{(30b)} $$

The first equation in (30a) can be expanded $\langle \lambda_2, \lambda_3 \rangle_0^2 + \lambda_1 \langle \lambda_2, \lambda_3 \rangle_1 = 0$, thus $\langle \lambda_2, \lambda_3 \rangle_0^2 = 0$. Also, the sum and the difference of the equations in (30b), gives

$$ \langle \lambda_2, \lambda_3 \rangle_0^2 = \langle \lambda_2, \lambda_3 \rangle_1^2 = 0, $$

(31)

$$ 2 \langle \lambda_1 \rangle_0^2 + \langle \lambda_1 \rangle_1^2 \langle \lambda_3 + \lambda_2 \rangle = 0, $$

(32)

$$ \langle \lambda_1 \rangle_1^2 \langle \lambda_3 - \lambda_2 \rangle = 0. $$

(33)

Consider the case $\lambda_2 = \lambda_3 = a$. Then,

$$ \langle a \rangle_0^2 = \langle a \rangle_1^2 = 0. $$

(34)

For $\beta_2 = \beta_3 = 0$ we have $\beta_0 = \beta_1 = 0$. For $\beta_2 = 0$, $\beta_3 \neq 0$ we get $\beta_0^2 \beta_3 + 4 \beta_0^3 = 0$ and the solution $\lambda_1 = a = -\beta_0/(2 \beta_1)$. For $\beta_3 = 0$, $\beta_2 \neq 0$, we get $4 \beta_0 \beta_2 = 3 \beta_1^2$ and the solution $\lambda_1 = a = -\beta_1/(2 \beta_2)$. Finally, for $\beta_2 \neq 0$, $\beta_3 \neq 0$, we get $\Delta = 0$, and the solutions

$$ \lambda_1 = a = (-\beta_2 \pm \sqrt{A})/\beta_3 = -B/(2A). $$

(35)

A similar analysis as in Type I, gives a decoupled case with $a = -\beta_2/\beta_3$ and an arbitrary $\lambda_1$ in the limit $A \to 0$.

Now, consider the case $\langle \lambda_1 \rangle_1^2 = 0$. This implies $\langle \lambda_1 \rangle_0^2 = \langle \lambda_1 \rangle_1^2 = 0$, which produces the same constraints as in the case above for $\lambda_2 = \lambda_3 = a$ but with $\lambda_1$ in place of $a$ in (34). On the other hand, defining $\lambda_2 = u + v$, $\lambda_3 = u - v$ again yields the same $u$ and $v$ as in (21). Substituting the constraint on $\beta_0$ from $\langle \lambda_1 \rangle_0^2 = \langle \lambda_1 \rangle_1^2 = 0$ into $u$ and $v$ yields $v = 0$, and $\lambda_2 = \lambda_3 = u$. Hence we again get (35).

Therefore, we always have $\lambda_1 = \lambda_2 = \lambda_3$ in Type IIa, with the constraint $\Delta = 0$ on the $\beta$-parameters.

**Type IIb**

The square root of Type IIb has the block form,

$$ S_{IIb} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \oplus \lambda_1 \oplus \lambda_2, $$

(36)

where $\lambda_1$ and $\lambda_2$ are two real eigenvalues, while $a$ and $b$ are two real numbers representing a complex conjugate eigenvalue pair $a \pm ib$. In the limit $a \to 0$, one gets Type IV as a branch-cut, while $b \to 0$ produces Type I.

The equation $V_9(S_{IIb}) = 0$ yields,

$$ \langle \lambda_1, \lambda_2 \rangle_0^2 + a \langle \lambda_1, \lambda_2 \rangle_1^2 = 0, $$

(37a)

$$ b \langle \lambda_1, \lambda_2 \rangle_0^2 = 0. $$

(37b)

$$ \langle a \rangle_0^2 + \beta_2 b^2 + \langle \lambda_1 \rangle_1^2 + 2 \lambda_2 = 0. $$

(37c)

$$ \langle \lambda_1, \lambda_2 \rangle_0^2 + \beta_2 b^2 + \langle \lambda_1, \lambda_2 \rangle_1^2 = 0. $$

(37d)

Multiplying (37b) by $a/b$ and subtracting from (37a), then adding and subtracting (37c) and (37d) gives,

$$ 2 \langle a \rangle_0^2 + \beta_2 b^2 + \langle \lambda_1 \rangle_1^2 = 0. $$

(38)

$$ \langle \lambda_1, \lambda_2 \rangle_0^2 = \langle \lambda_1, \lambda_2 \rangle_1^2 = 0. $$

(39)

$$ (\langle a \rangle_0^2 + \beta_2 b^2) (\lambda_1 - \lambda_2) = 0. $$

(40)

For $\lambda_1 = \lambda_2 = c$ we have,

$$ \langle c \rangle_0^2 = \langle c \rangle_1^2 = 0, $$

(41)

$$ \langle a \rangle_0^2 + \beta_2 b^2 + \langle \lambda_1 \rangle_1^2 = 0. $$

(42)

As earlier, the first equation imposes a condition on the $\beta$-parameters, fixing $c$. Setting $a = u + c$, gives $u^2 + b^2(\beta_2 - c \beta_3) = 0$. The case $\beta_2 + c \beta_3 = 0$ is equivalent to $\beta_2^2 - \beta_1 \beta_3 = 0$, yielding arbitrary $a$ and $b$. Otherwise, $\beta_2 + c \beta_3 \neq 0$ implies $a + ib = c$. This is only possible if $a = c$ and $b = 0$, which is then a subset of Type I.

For $\lambda_1 \neq \lambda_2$ and $\langle a \rangle_1^2 + \beta_2 b^2 = 0$, the solutions $\lambda_1$ and $\lambda_2$ are obtained from (38), so that $\lambda_1 = u + v$ and $\lambda_2 = u - v$ with $u$ and $v$ again given in (21). On the other hand, we have $\langle a \rangle_0^2 + \beta_2 b^2 = 0$ and $b^2 = -\beta_3/\langle a \rangle_1^2 = -\beta_2/\langle a \rangle_0^2$, implying $\langle a \rangle_0^2 \beta_3 = \beta_2 \langle a \rangle_0^2$ and,

$$ a = -B/(2A), \quad b^2 = -\langle a \rangle_0^2/\beta_2 = -v^2. $$

(43)

This shows that either $b$ or $v$ must be imaginary, unless $b = 0$, which is a subset of Type I. This is not surprising as Type IIb can be expressed as Type I having a pair of complex conjugate eigenvalues. Since the Lorentzian
signature of the metrics forbids the existence of two complex blocks, Type I cannot be block proportional with two pairs of complex numbers. Therefore, there are no solutions of Type Ib which are different than of Type I.

### Type III

The square root of Type III has the form,

\[
S_{III} = \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix} \oplus \lambda_2, \quad (44)
\]

where \(\lambda_1\) and \(\lambda_2\) are two real eigenvalues. The equation \(V_g(S_{III}) = 0\) yields,

\[
\begin{align*}
\langle \lambda_1, \lambda_1, \lambda_2 \rangle_0^3 &= 0, \quad \langle \lambda_2 \rangle_2^1 = 0, \quad (45a) \\
\langle \lambda_1, \lambda_2 \rangle_1^1 &= 0, \quad \langle \lambda_1 \rangle_0^1 = 0. \quad (45b)
\end{align*}
\]

Expanding \(\langle \lambda_1, \lambda_1, \lambda_2 \rangle_0^3 = \langle \lambda_1, \lambda_2 \rangle_0^2 + \lambda_1 \langle \lambda_1, \lambda_2 \rangle_1^1\) gives \(\langle \lambda_1, \lambda_2 \rangle_0^2 = 0\) that together with \(\langle \lambda_1, \lambda_2 \rangle_1^1 = 0\) completely determines \(\lambda_1\) and \(\lambda_2\) so that \(\lambda_1 = u + v\) and \(\lambda_2 = u - v\) with \(u\) and \(v\) given in (21). On the other hand, \(\langle \lambda_2 \rangle_2^1 = 0\) is a linear equation in \(\lambda_2\) with the solution \(\lambda_2 = -\beta_2/\beta_3\).

Substituting \(\langle \lambda_1, \lambda_2 \rangle_1^1 = 0\) gives \(A = 0\), and further \(\langle \lambda_1, \lambda_2 \rangle_2^0 = 0\) yields \(\beta_0 = \beta_3^2 = \beta_2^2\). Finally, \(\langle \lambda_1 \rangle_0 = 0\) gives \(\lambda_1 = \lambda_2\) so that both \(v = 0\) and the discriminant of the cubic equation vanishes. In conclusion, we have only a single-eigenvalue solution \(\lambda_1 = \lambda_2 = -\beta_2/\beta_3\) with the constraint \(\beta_0/\beta_1 = \beta_1/\beta_2 = \beta_2/\beta_3\).

### Type IV

This is the case of a non-primary square root that has the block form as in Type Ib (36), but with \(\alpha = 0\) so that the complex block represents a pair of imaginary eigenvalues \(\pm ib\). The solutions are \(-b = \lambda_3^2 = \lambda_2^2\), which cannot be satisfied for any non-vanishing real \(b\) and \(\lambda_1\) (unless \(A = A = 0\)).

### V. Algebraically Decoupled Cases

In particular for \(\beta_1 = \beta_2 = \beta_3 = 0\), the kinetic terms in (1) decouple, so there is no algebraic restriction on \(S\) imposed by \(V_g(S) = 0\) (dynamical restrictions can, however, come out from the field equations). A similar degeneracy occurs when the \(\beta\)-parameters form a geometric progression, \(\beta_{k+1}/\beta_k = \text{const}\).

During the enumeration of all possible solutions of the equation \(V_g = 0\), we have encountered three possible decoupled cases:

\[
\begin{align*}
&\text{(D1) } \beta_1 = \beta_2 = \beta_3 = 0, \\
&\text{(D2) } \Delta = 0, \ A = 0, \\
&\text{(D3) } \Delta = 0, \ A \neq 0,
\end{align*}
\]

where (D2) and (D3) only occur for non-vanishing \(\beta\)-parameters. The condition (D2) states a geometric progression \(\beta_{k+1}/\beta_k = \text{const}\).

For Type I and Type IIb, the conditions (D1), (D2) or (D3) render four, three or two arbitrary eigenvalues, respectively.

For Type IIa, the conditions (D1) or (D2) make two or one of the eigenvalues arbitrary, respectively. However, (D3) is mandatory for Type IIa; it must be satisfied for Type IIa to exist, in which case all the eigenvalues are given by the \(\beta\)-parameters.

For Type III the eigenvalues are determined by the \(\beta\)-parameters. Moreover, the condition (D2) must be satisfied for Type III to exist.

### VI. An Example Solution

In the analysis above, we assessed the algebraic restrictions on the bimetric field equations. As a dynamical example, let us consider a spherically symmetric Einstein solution in one sector, let’s say \(g\).

By Birkhoff’s theorem, any spherically symmetric solution is locally isometric to a subset of the Schwarzschild solution. Therefore, without loss of generality, we can consider the line element of \(g\) in the standard Schwarzschild chart \((t, r, \theta, \phi)\),

\[
ds_g^2 = -F dt^2 + F^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,
\]

where \(F \equiv 1 - r_u/r\). Including a cosmological constant \(\Lambda\), we have \(F = 1 - r_u/r - \Lambda r^2/3\), yielding \(G_{\mu \nu} + \Lambda \delta_{\mu \nu} = 0\). In the limit \(r_u \to 0\), \(\Lambda \to 0\), we recover the Minkowski solution. Since the metric is diagonal, the square root is of Type I as in (12) (if \(g\) was in the Eddington-Finkelstein chart, the square root could be of Type IIa or IIb). This implies,

\[
ds_I^2 = -\lambda_1^2 F dt^2 + \lambda_2^2 F^{-1} dr^2 + \lambda_3^2 r^2 d\theta^2 + \lambda_4^2 r^2 \sin^2 \theta d\phi^2,
\]

resulting in the Einstein operator for \(f\),

\[
G_{f \mu \nu} = \left[ -\Lambda \delta_2^{-2} + r^2 \left( \lambda_2^{-2} - \lambda_3^{-2} \right) \right] \delta_\mu^\nu. \quad (46)
\]

Hence, the dynamics imposes \(\lambda_3^2 = \lambda_3^2\) for \(f\) to satisfy the Einstein field equations. This constraint is absent for the Minkowski solution; nevertheless, in general, for a more complicated \(g\), constraining \(S\) this way might not always be possible.

On the other hand, algebraically, the proved proposition states that either \(\lambda_1 = \lambda_2\) and \(\lambda_3 = \lambda_4\), or \(\lambda_1 = \lambda_3\) and \(\lambda_2 = \lambda_4\), or \(\lambda_1 = \lambda_4\) and \(\lambda_2 = \lambda_3\), or that all eigenvalues are equal. Therefore, (depending on the \(\beta\)-parameters),

\[
ds_I^2 = -\lambda_1^2 F dt^2 + \lambda_2^2 F^{-1} dr^2 + \lambda_3^2 r^2 d\theta^2 + \lambda_4^2 r^2 \sin^2 \theta d\phi^2,
\]

is a healthy Einstein solution in the \(f\)-sector. This solution may or may not be spherically symmetric.
After the chart transition $t' = \lambda_1 t, \ r' = \lambda_2 r, \ \theta' = \lambda_2 \theta, \ \phi' = \lambda_1 \phi$, one can show that $\lambda_2 = 1$ is needed for $f$ to be spherically symmetric. This condition (put by hand) leaves an arbitrary $\lambda_1 = c$. Then, in the setup with $S = \text{diag}(c, 1, 1, c)$, there are two sets of Killing vector fields (KVF) generating $SO(3)$. In particular, in the original chart $(t, r, \theta, \phi)$, the KVFs that generate a separate spherical symmetry for $f$ are,

$$\eta_1 = c^{-1} \partial_\phi,$$

$$\eta_2 = \cos(c \phi) \partial_\theta - \cot(\theta) \sin(c \phi) c^{-1} \partial_\phi,$$

$$\eta_3 = -\sin(c \phi) \partial_\theta - \cot(\theta) \cos(c \phi) c^{-1} \partial_\phi. \quad (49)$$

In comparison, the KVFs of $g$ have a similar structure, but without the presence of $c$. Finally, we take into account the topology of the Schwarzschild solution, which is diffeomorphic to $\mathbb{R}^2 \times S^2$. This further imposes $c = \pm 1$, which can be shown by considering all possible scalars created from the KVFs of $g$ and $f$, for example, $g(\xi_2, \eta_2)$. Namely, all of these are the invariant scalar fields in $\phi$ involving the mixings of the trigonometric functions of $\phi$ and $c \phi$. The $S^2$ subspace topology in an atlas adapted to $g$ requires the scalars to have the same value at $\phi = 0$ and $\phi = 2\pi$, which constrains $c$ to be a non-vanishing integer (a winding number of the orbits of $f$’s KVFs).

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[1] D. G. Boulware and S. Deser, Phys. Rev. D 6 (1972) 3368, doi:10.1103/PhysRevD.6.3368.
[2] C. de Rham and G. Gabadadze, Phys. Rev. D 82 (2010) 044020, doi:10.1103/PhysRevD.82.044020 arXiv:1007.0443 [hep-th].
[3] C. de Rham, G. Gabadadze, and A.J. Tolley, Phys. Rev. Lett. 106, 231101 (2011), doi:10.1103/PhysRevLett.106.231101 arXiv:1011.1232 [hep-th].
[4] S.F. Hassan, R.A. Rosen, Phys. Rev. Lett. 108 (2012), doi:10.1103/PhysRevLett.108.041101 arXiv:1106.3344 [hep-th].
[5] S.F. Hassan, R.A. Rosen, JHEP 1202 (2012) 126, doi:10.1007/JHEP02(2012)126 arXiv:1109.3515 [hep-th].
[6] S.F. Hassan, R.A. Rosen, JHEP 1204 (2012), doi:10.1007/JHEP04(2012)123 arXiv:1111.2070 [hep-th].
[7] A. Schmidt-May, M. von Strauss, J. Phys. A 49 (2016) no.18, 183001, doi:10.1088/1751-8113/49/18/183001 arXiv:1512.00021 [hep-th].
[8] C. de Rham, Living Rev. Rel. 17 (2014) 7, doi:10.12942/lrr-2014-7 arXiv:1401.4173 [hep-th].
[9] I.G. Macdonald, Symmetric Functions and Orthogonal Polynomials, (American Mathematical Society, 1998).
[10] S.F. Hassan, A. Schmidt-May, and M. von Strauss, Int. J. Mod. Phys. D 23 (2014) no.13, 1443002, doi:10.1142/S0218271814430020 arXiv:1407.2772 [hep-th].
[11] S.F. Hassan, M. Kocic, (in preparation).
[12] F. Torsello, M. Kocic, E. Mortsell, On the classification and asymptotic structure of black holes in bimetric theory, submitted to JHEP, arXiv:1703.07787 [gr-qc].