Non-reversible jump algorithms for Bayesian nested model selection

Philippe Gagnon 1, Arnaud Doucet 1

November 5, 2019

1Department of Statistics, University of Oxford, United Kingdom.

Abstract

Non-reversible Markov chain Monte Carlo methods often outperform their reversible counterparts in terms of asymptotic variance of ergodic averages and mixing properties. Lifting the state-space (Chen et al., 1999; Diaconis et al., 2000) is a generic technique for constructing such samplers. The idea is to think of the random variables we want to generate as position variables and to associate to them direction variables so as to design Markov chains which do not have the diffusive behaviour often exhibited by reversible schemes. In this paper, we explore the benefits of using such ideas in the context of Bayesian model choice for nested models, a class of models for which the model indicator variable is an ordinal random variable. By lifting this model indicator variable, we obtain non-reversible jump algorithms, a non-reversible version of the popular reversible jump algorithms introduced by Green (1995). This simple algorithmic modification provides samplers which can empirically outperform their reversible counterparts at no extra computational cost. The code to reproduce all experiments is available online.1

Keywords: Bayesian statistics; Markov chain Monte Carlo methods; non-reversible Markov chains; weak convergence.

1 Introduction

1.1 Reversible jump algorithms

Reversible jump (RJ) algorithms are a popular class of Markov chain Monte Carlo (MCMC) methods introduced by Green (1995, 2003). They are used to sample from a target distribution \( \pi \) defined on \( \bigcup_{j \in \mathcal{K}} \{j\} \times \mathbb{R}^{d_j} \), \( \mathcal{K} \) being a countable set. In the statistics applications discussed in this paper, this distribution corresponds to the joint posterior distribution of a model indicator \( k \in \mathcal{K} \) and its corresponding parameters \( x_k \in \mathbb{R}^{d_k} \). It allows us to perform simultaneously model selection and parameter estimation.

In the following, we assume for simplicity that the parameters of all models are continuous random variables and abuse notation by also using \( \pi \) to denote the target density.

Given the current state \((k, x_k)\), a RJ algorithm generates the next state by proposing a model candidate \( k' \) from some probability mass function (PMF) \( g(k, \cdot) \) then a proposal for its corresponding parameter values. This last step is usually achieved through two sub-steps:

1See the ArXiv page of this paper.
1. generate $u_{k\rightarrow k'} \sim q_{k\rightarrow k'}$ (this vector corresponds to auxiliary variables used, for instance, to propose values for additional parameters when $d_{k'} > d_k$), where $q_{k\rightarrow k'}$ is a probability density function (PDF).

2. apply the function $T_{k\rightarrow k'}$ to $(x_k, u_{k\rightarrow k'})$, $T_{k\rightarrow k'}(x_k, u_{k\rightarrow k'}) =: (y_{k'}, u'_{k\rightarrow k'})$, where the vector $y_{k'}$ represents the proposal for the parameters of model $k'$ and $T_{k\rightarrow k'}$ is a diffeomorphism.

When $k' = k$, we say that a parameter update is proposed, whereas we say that a model switch is proposed when $k' \neq k$. The proposal $(k', y_{k'})$ is accepted with probability:

$$
\alpha_{\text{RJ}}((k, x_k), (k', y_{k'})) := 1 \wedge \frac{g(k', k) \pi(k', y_{k'}) q_{k\rightarrow k'}(u'_{k\rightarrow k})}{g(k, k') \pi(k, x_k) q_{k\rightarrow k'}(u_{k\rightarrow k}) |J_{T_{k\rightarrow k'}}(x_k, u_{k\rightarrow k})|^{-1}},
$$

where $|J_{T_{k\rightarrow k'}}(x_k, u_{k\rightarrow k'})|$ is the Jacobian of the function $T_{k\rightarrow k'}$. Otherwise, the chain remains at the same state $(k, x_k)$.

In this paper, we consider the special case of nested models; i.e. $K$ is an ordinal discrete random variable that reflects the complexity of the models. For instance, it represents the number of components in mixture modelling (Richardson and Green, 1997), the number of change-points in multiple change-point problems (Section 4, Green (1995)), or the order of an autoregressive process. We restrict our attention to samplers that switch models by taking steps of $\pm 1$, i.e. $k' \in \{k - 1, k + 1\}$ when a model switch is proposed. This is a common choice which implies that the model space $\mathcal{K}$ is explored through a random walk, a process exhibiting a diffusive behaviour.

The objective of this paper is to propose sampling schemes which do not suffer from such a behaviour by exploiting the lifting idea introduced by Chen et al. (1999) and Diaconis et al. (2000) to induce persistent movement in the model indicator. In the somewhat related contexts of simulated tempering (Sakai and Hukushima, 2016) and parallel tempering (Syed et al., 2019), lifting the temperature variable provides non-reversible samplers which perform substantially better than their reversible counterparts.

The changes that we apply to the RJ sampling framework described above are remarkably simple. First, we extend the state-space by adding a direction variable $\nu \in \{-1, 1\}$ and assign it a uniform distribution $\mathcal{U}(-1, 1)$. Second, the model to explore at each iteration is selected deterministically instead of randomly by setting: $k' := k + \nu$. If the proposal for the model to explore next (model $k'$) along with its parameter values $y_{k'}$ is accepted, the next state of the chain is $(k', y_{k'}, \nu)$. The direction for the model indicator remains the same; this is what induces persistent movement. If the proposal is rejected, the next state of the chain is $(k, x_k, -\nu)$, so the direction is reversed for $K$. A proposal may be rejected because there is negligible mass beyond $k$ in the direction followed; a change in direction may thus imply a return towards the high probability area. A general non-reversible jump (NRJ) algorithm is described in detail in Algorithm 1 in Section 2.1. Such simple modifications can be very efficient as illustrated in Figure 1.

1.2 Organisation of the paper

We first introduce in Section 2 a general NRJ scheme and establish its validity. We also present its ideal version that is able to generate from the conditional distributions $\pi(\cdot | k)$ for proposals for the model parameters. This ideal algorithm is simple and allows us to exploit existing theoretical results to establish in Section 4 that NRJ can outperform the corresponding ideal RJ under some assumptions on the marginal PMF $\pi(k)$. Although such an ideal sampler cannot be implemented in practice for complex models, we show in Section 3 how we can leverage existing methods that have been developed in the
Non-reversible jump algorithms for Bayesian nested model selection

Figure 1. Trace plots for ideal RJ selecting uniformly at random which model to switch to and NRJ algorithms (we define what we mean by ideal in Section 2.2), and showing only the iterations in which model switches are proposed; the horizontal lines represent the marginal targeted PMF $\pi(k)$ which is that defined in Section 4 with $\phi := 1.3$ and $\mathcal{K} := \{1, \ldots, 11\}$; ESS stands for effective sample size.

RJ literature to approximate this ideal NRJ sampler. The weak convergence of the resulting sampler towards the ideal NRJ sampler is established as a precision parameter increases without bounds. We present in Section 5 numerical experiments to illustrate the performance of NRJ samplers on a toy example for which the ideal sampler can be implemented as well as a real multiple change-point problem. We provide a detailed discussion of implementation aspects and possible extensions in Section 6.

2 Non-reversible jump algorithms and ideal samplers

2.1 Non-reversible jump schemes

We present in Algorithm 1 the general NRJ which takes as inputs an initial state $(k, x_k, \nu)$, a total number of iterations, the functions $q_{k \rightarrow k'}$ and $T_{k \rightarrow k'}$, and $0 \leq \tau \leq 1$ which represents the probability of proposing a parameter update at a given iteration in the trans-dimensional sampler. In this type of sampler, the probability of proposing a parameter update typically depends on the current state. For ease of presentation, it is considered constant here.

Algorithm 1 NRJ

1. Generate $u \sim \mathcal{U}(0, 1)$.
2.(a) If $u \leq \tau$, attempt a parameter update using a MCMC kernel of invariant distribution $\pi(\cdot \mid k)$ while keeping the current value of the model indicator $k$ and direction $\nu$ fixed.
2.(b) If $u > \tau$, attempt a model switch from model $k$ to model $k' = k + \nu$. Generate $u_{\nu} \sim \mathcal{U}(0, 1)$, and compute $(y_{k'}, u_{k' \rightarrow k}) = T_{k \rightarrow k'}(x_k, u_{k \rightarrow k'})$. If

$$u_{\nu} \leq \alpha_{\text{NRJ}}((k, x_k), (k', y_{k'})) := 1 \wedge \frac{\pi(k', y_{k'}) q_{k' \rightarrow k}(u_{k' \rightarrow k})}{\pi(k, x_k) q_{k \rightarrow k'}(u_{k \rightarrow k'}) | J_{T_{k \rightarrow k'}}(x_k, u_{k \rightarrow k'}) |^{-1}},$$

set the next state of the chain to $(k', y_{k'}, \nu)$. Otherwise, set it to $(k, x_k, -\nu)$.
3. Go to Step 1.

Proposition 1 below ensures that Algorithm 1 targets the correct distribution. Note that $x_k$ can
be vectors containing both position and velocity variables, which allows using Hamiltonian Monte Carlo (HMC, see, e.g., Neal (2011)) and more generally discrete-time piecewise-deterministic MCMC schemes (Vanetti et al., 2017) for updating the parameters within Algorithm 1. The only prerequisite is that the method leaves the conditional distributions \( \pi(\cdot | k) \) invariant.

**Proposition 1** (Invariance). The transition kernel of the Markov chain \( \{(K, X_K, \nu)(m) : m \in \mathbb{N}\} \) simulated by Algorithm 1 admits \( \pi \otimes \mathcal{U}[-1, 1] \) as invariant distribution.

*Proof.* See Section 7.

2.2 Ideal samplers and their advantages

Ideally when switching models, the parameter proposals \( y_{k'} \) would be generated from the conditional distributions \( \pi(\cdot | k') \). In this ideal situation, one can set \( q_{k \rightarrow k'} := \pi(\cdot | k') \), \( q_{k' \rightarrow k} := \pi(\cdot | k) \), and \( T_{k' \rightarrow k} \) such that \( y_{k'} := \mathbf{u}_{k \rightarrow k'} \) (which implies that \( \mathbf{u}_{k' \rightarrow k} := \mathbf{x}_k \)), and observe that the acceptance probabilities reduce to

\[
a_{\text{NRJ}}((k, x_k), (k', y_{k'})) = 1 \wedge \frac{\pi(k')}{\pi(k)}, \tag{3}
\]

These probabilities are independent of the current and proposed parameters value: a model proposal \( k' \) is accepted solely on the basis of the ratio of marginal posterior probabilities \( \pi(k')/\pi(k) \).

In general, the acceptance probabilities are as above whenever

\[
\frac{\pi(y_{k'} | k') q_{k' \rightarrow k}(\mathbf{u}_{k' \rightarrow k})}{\pi(x_k | k) q_{k \rightarrow k'}(\mathbf{u}_{k \rightarrow k'})} \left| J_{T_{k' \rightarrow k}}(\mathbf{x}_k, \mathbf{u}_{k \rightarrow k'}) \right|^{-1} = 1, \tag{4}
\]

for any switch from model \( k \) with parameter values \( x_k \) to model \( k' \neq k \) with parameter values \( y_{k'} \), using the auxiliary variables \( \mathbf{u}_{k \rightarrow k'} \) and \( \mathbf{u}_{k' \rightarrow k} \). In this more general setting, we observe that (4) is verified if, starting with random variables distributed as \( \pi(\cdot | k) \otimes q_{k \rightarrow k'} \) and applying the function \( T_{k' \rightarrow k} \), we obtain random variables distributed as \( \pi(\cdot | k') \otimes q_{k' \rightarrow k} \).

We now highlight that the usual situation is not ideal by reexpressing the acceptance probability of model switches for NRJ (2) as a “noisy” version of that in (3):

\[
a_{\text{NRJ}}((k, x_k), (k', y_{k'})) = 1 \wedge \frac{\pi(k')}{\pi(k)} \varepsilon(k, k', x_k, q_{k \rightarrow k'}, q_{k' \rightarrow k}, T_{k' \rightarrow k}), \tag{5}
\]

where \( \varepsilon \) represents multiplicative noise given by the left-hand side (LHS) in (4).

For the NRJ to be beneficial, it will be useful to have a low variance noise. Imagine that the targeted marginal PMF is that on the right of Figure 2 and that the samplers are initialised at \((K, \nu)(0) := (1, +1)\) (as in Figure 2), the advantage of the ideal NRJ is that it continues following the direction \( +1 \) for several iterations as the ratios \( \pi(k + 1)/\pi(k) \) are greater than 1 (which implies that the proposals are accepted). If the noise fluctuations are significant, such moves might be rejected.

3 Towards ideal NRJ

We explained in the last section why it may be important to implement NRJ samplers that are close to their ideal counterparts, with low variance noise \( \varepsilon \). We present in this section methods to achieve this by adapting some developed within the RJ framework. In Section 3.1, we present and adapt for NRJ the method of Karagiannis and Andrieu (2013). We proceed similarly in Section 3.2 with the approach of Andrieu et al. (2018). In Section 3.3, we prove that as \( \varepsilon \rightarrow 1 \) in distribution the Markov chains produced by NRJ incorporating these approaches converge weakly to those produced by ideal NRJ.
the path can be generated through also a function of ρ which is at the beginning close to from model γt for t
sampling (Neal, 2001). These auxiliary distributions take the form k k′ between model k and model k′. A proposal distribution is build by sampling an inhomogeneous Markov chain which targets at each step one of these auxiliary distributions in the spirit of annealed importance sampling (Neal, 2001). These auxiliary distributions take the form

$$\rho_{k\rightarrow k'}(x_k(t), u_{k\rightarrow k'}) \propto \left[ \pi(k, x_k(t)) q_{k\rightarrow k'}(u_{k\rightarrow k'}) | J_{T_{k\rightarrow k'}}(x_k(t), u_{k\rightarrow k'}) |^{1-\gamma_t} \right]^{1-\gamma_t} \gamma_t \Rightarrow \left[ \pi(k', y_k(t)) q_{k\rightarrow k'}(u_{k\rightarrow k'}) | J_{T_{k\rightarrow k'}}(x_k(t), u_{k\rightarrow k'}) |^{1-\gamma_t} \right]^{\gamma_t},$$

for t = 0, . . . , T where T is a positive integer, γ0 := 0, γT := 1 and γt ∈ [0, 1] for t ∈ {1, . . . , T − 1}. We set γt := t/T in our numerical experiments as in Karagiannis and Andrieu (2013). When switching from model k to model k′, we thus use at time t a transition kernel K(t) k→k′, to target the distribution $P_{k\rightarrow k'}^{(t)}$, which is at the beginning close to (π(k, ·) ⊗ qk→k′)|JT_{k→k′}|, and the end close to π(k′, ·) ⊗ qk→k′. We wrote $\rho_{k\rightarrow k'}^{(t)}$ as a function of (xk(t), uk→k′) to emphasise that the starting point is (xk(0), uk→k′). It is in fact also a function of (yk(t), uk′→(0)) that can be found using (yk(t), uk′→(0)) = T_{k→k′}(yk(t), uk′→(0)).

The overall NRJ procedure incorporating such proposals is described in Algorithm 2. In Step 2.(b), the path can be generated through (yk, uk′→(0)) instead.

Karagiannis and Andrieu (2013) explain that the MH correction term in Algorithm 2, that we denote by

$$n_{NRJ2}((k, x_k(0)), (k', y_k(T−1))) := \prod_{t=0}^{T−1} \rho_{k\rightarrow k'}^{(t)}(x_k(t), u_{k\rightarrow k'}(t)),$$

represents a consistent estimator of π(k′)/π(k) as T → ∞.

Under the following two conditions, the RJ corresponding to Algorithm 2 are valid, in the sense that the target distribution is an invariant distribution.

**Symmetry condition:** For t = 1, . . . , T − 1 the pairs of transition kernels $K_{k\rightarrow k'}^{(t)}(·, ·)$ and $K_{k'\rightarrow k}^{(T−t)}(·, ·)$ satisfy

$$K_{k\rightarrow k'}^{(t)}((x_k, u_{k→k'}), ·) = K_{k'\rightarrow k}^{(T−t)}((x_k, u_{k→k'}), ·) \quad \text{for any } (x_k, u_{k→k'}).$$
**Algorithm 2** NRJ incorporating the method of Karagiannis and Andrieu (2013)

1. Generate $u \sim \mathcal{U}(0, 1)$.

2. (a) If $u \leq \tau$, attempt a parameter update using a MCMC kernel of invariant distribution $\pi(\cdot \mid k)$ while keeping the current value of the model indicator $k$ and direction $\nu$ fixed.

2. (b) If $u > \tau$, attempt a model switch from model $k$ to model $k' := k + \nu$. Generate $u_{k \rightarrow k'}^{(0)} \sim q_{k \rightarrow k'}$ and $u_{k \rightarrow k'} \sim \mathcal{U}(0, 1)$, and set $x_k^{(0)} := x_k$. Generate a path $(x_k^{(1)}, u_{k \rightarrow k'}^{(1)}), \ldots, (x_k^{(T)}, u_{k \rightarrow k'}^{(T)})$, where $(x_k^{(t)}, u_{k \rightarrow k'}^{(t)}) \sim K_{k \rightarrow k'}^{(t)}((x_k^{(t-1)}, u_{k \rightarrow k'}^{(t-1)}), \cdot)$. Compute $(y_k^{(t)}, u_{k \rightarrow k'}^{(t)}) := T_{k \rightarrow k'}(x_k^{(t)}, u_{k \rightarrow k'}^{(t)})$ for $t = 0, \ldots, T - 1$. If

$$u_u \leq \alpha_{\text{NRJ2}}((k, x_k^{(0)}), (k', y_k^{(T-1)})) := 1 \land r_{\text{NRJ2}}((k, x_k^{(0)}), (k', y_k^{(T-1)})) \quad \text{(where } r_{\text{NRJ2}} \text{ is defined in (7)),}$$

set the next state of the chain to $(k', y_k^{(T-1)}, \nu)$. Otherwise, set it to $(k, x_k, -\nu)$.

3. Go to Step 1.

**Reversibility condition**: For $t = 1, \ldots, T - 1$, and for any $(x_k, u_{k \rightarrow k'})$ and $(x_k', u_{k \rightarrow k'})$,

$$p_{k \rightarrow k'}^{(t)}(x_k, u_{k \rightarrow k'}) K_{k \rightarrow k'}^{(t-1)}((x_k', u_{k \rightarrow k'}), (x_k, u_{k \rightarrow k'})) = p_{k \rightarrow k'}^{(t-1)}(x_k', u_{k \rightarrow k'}') K_{k \rightarrow k'}^{(t)}((x_k', u_{k \rightarrow k'}'), (x_k, u_{k \rightarrow k'})). \quad (9)$$

As mentioned in Karagiannis and Andrieu (2013), (8) is verified if for all $t$, $K_{k \rightarrow k'}^{(t)}(\cdot, \cdot)$ and $K_{k \rightarrow k'}^{(t-1)}(\cdot, \cdot)$ are Metropolis–Hastings (MH) kernels sharing the same proposal distributions. Proposition 2 below indicates that under the same conditions, Algorithm 2 is valid as well.

**Proposition 2** (Invariance 2). Assume (8) and (9) are verified. The transition kernel of the Markov chain $\{(K, X_k, \nu)(m) : m \in \mathbb{N}\}$ simulated by Algorithm 2 admits $\pi \otimes \mathcal{U}[-1, 1]$ as invariant distribution.

**Proof.** Analogous to that of Proposition 1.

The use of such sophisticated proposals comes at a computational cost. As in Karagiannis and Andrieu (2013), we observe experimentally a steady increase in the ESS as $T$ increases until the samplers are close enough to the ideal sampler it approximates; after this point the increase in ESS is less marked. The strategy for users is to find the approximate location of this point using trial runs and to choose a smaller value for $T$ that they find suitable. We show in our experiments that this cost may be offset by a large enough increase in ESS resulting in a net increase in ESS per unit time.

### 3.2 NRJ additionally incorporating the method of Andrieu et al. (2018)

As mentioned previously, the estimator $r_{\text{NRJ}}$ in (7) can be interpreted as an estimator of $\pi(k') / \pi(k)$. To further reduce the variance of this estimator, one could produce in parallel $N$ inhomogeneous Markov chains ending with $N$ proposals, that we denote by $y_k^{(T-1, 1)}, \ldots, y_k^{(T-1, N)}$, and use instead the average of the $N$ estimates $r_{\text{NRJ2}}((k, x_k^{(0)}), (k', y_k^{(T-1, 1)})), \ldots, r_{\text{NRJ2}}((k, x_k^{(0)}), (k', y_k^{(T-1, N)}))$. Simplifying notation, an estimate of $\pi(k') / \pi(k)$ is thus given by

$$\hat{r}(k, k') := \frac{1}{N} \sum_{j=1}^{N} r_{\text{NRJ2}}((k, x_k^{(0)}), (k', y_k^{(T-1, j)})).$$
However, applying this method naively does not lead to valid algorithms. The approach of Andrieu et al. (2018) exploits this averaging idea while leading to valid schemes. We present in Algorithm 3 the NRJ version of this algorithm.

**Algorithm 3** NRJ additionally incorporating the method of Andrieu et al. (2018)

1. Generate \( u \sim \mathcal{U}(0, 1) \).
   
2. (a) If \( u \leq \tau \), attempt a parameter update using a MCMC kernel of invariant distribution \( \pi(\cdot | k) \) while keeping the current value of the model indicator \( k \) and direction \( \nu \) fixed.
   
2. (b) If \( u > \tau \), attempt a model switch from model \( k \) to model \( k' := k + \nu \). Generate \( u_a, u_c \sim \mathcal{U}(0, 1) \). If \( u_c \leq 1/2 \) go to Step 2.(b-i), otherwise go to Step 2.(b-ii).
   
2. (b-i) Generate \( N \) proposals \( y_k^{(T-1,1)}, \ldots, y_k^{(T-1,N)} \) as in Step 2.(b) of Algorithm 2. Generate \( j^* \) from a PMF such that \( P(J^* = j) \propto r_{NRJ2}((k, x_k), (k', y_k^{(T-1,j)})) \). If \( u_a \leq \bar{r}(k, k') \), set the next state of the chain to \( (k', y_k^{(T-1,j^*}), \nu) \). Otherwise, set it to \( (k, x_k, -\nu) \).
   
2. (b-ii) Generate one forward path as in Step 2.(b) of Algorithm 2. Denote by \( y_k^{(T-1,1)} \) the endpoint. From the endpoint, generate \( N - 1 \) reverse paths again as in Step 2.(b) of Algorithm 2, yielding \( N - 1 \) proposals for the parameters of model \( k \). If \( u_a \leq \bar{r}(k', k)^{-1} \), set the next state of the chain to \( (k', y_k^{(T-1,1)}, \nu) \). Otherwise, set it to \( (k, x_k, -\nu) \).
   
3. Go to Step 1.

Proposition 3 indicates that Algorithm 3 is valid under the same conditions as Proposition 2.

**Proposition 3** (Invariance 3). Assume (8) and (9) are verified. The transition kernel of the Markov chain \( \{(K, X_k, \nu)(m) : m \in \mathbb{N}\} \) simulated by Algorithm 3 admits \( \pi \otimes \mathcal{U}(-1, 1) \) as invariant distribution.

*Proof*. Analogous to that of Proposition 1. ~

Andrieu et al. (2018) prove that increasing \( N \) decreases the asymptotic variance of the Monte Carlo estimates produced by RJ incorporating their approach. Their proof cannot be easily extended to NRJ. However, we have observed empirically that increasing \( N \) (as increasing \( T \) in Algorithm 2) leads to a steady increase in the ESS until the samplers are close enough to be ideal. So again, the strategy is to find the approximate location of the threshold and to select a suitable smaller value for \( N \).

In a parallel computing environment, an advantage of the approach presented here is that the additional computational cost (over Algorithm 2) is negligible considering that we can generate the \( N \) proposals \( y_k^{(T-1,1)}, \ldots, y_k^{(T-1,N)} \) and compute the corresponding estimates of the acceptance ratio in parallel.

### 3.3 Convergence of Algorithms 2 and 3 towards ideal NRJ

We presented at the beginning of Section 3 intuitive reasons explaining why Algorithms 2 and 3 can be made as close as we want to their ideal counterparts. We present here theoretical arguments supporting this intuition by establishing the weak convergence of the Markov chains produced by Algorithm 2 towards those simulated by its ideal version as \( T \rightarrow \infty \). This implies that Algorithm 3 with large enough \( T \) and fixed \( N \) generates Markov chains sharing the same behaviour as its ideal counterpart.
given that the noise $\epsilon$ (recall (5)) is only made more stable around the constant 1 by additionally using the approach of Andrieu et al. (2018). The corresponding weak convergence result for RJ incorporating the method of Karagiannis and Andrieu (2013) holds under the same assumptions presented here.

The Markov kernel simulated by Algorithm 2 (when switching models) is given by:

$$P_T((k, x_k, v), (k', y_k', v'))$$

$$= q_{k \rightarrow k + v}(u_{k \rightarrow k + v}^{(0)}) \prod_{t=1}^{T-1} K_{k \rightarrow k + v}^{(t)}(y_{k + v}, u_{k + y \rightarrow k + v}^{(t)}), (y_{k + v}^{(t)}, u_{k + y \rightarrow k + v}^{(t)}) \delta_{(k + v, y_{k + v}, u_{k + y \rightarrow k + v})}(k', y_k', v') \alpha_{\text{NRJ2}}((k, x_k), (k', y_k'))$$

$$+ \delta_{(k, x_k, v)}(k', y_k', v') \int q_{k \rightarrow k + v}(u_{k \rightarrow k + v}^{(0)}) \prod_{t=1}^{T-1} K_{k \rightarrow k + v}^{(t)}(y_{k + v}, u_{k + y \rightarrow k + v}^{(t)}), (y_{k + v}^{(t)}, u_{k + y \rightarrow k + v}^{(t)})$$

$$\times (1 - \alpha_{\text{NRJ2}}((k, x_k), (k + v, y_{k + v}^{(T-1)})) d\mathbf{u}_{k \rightarrow k + v}^{(0)} dy_{k + v}^{(1)} u_{k + y \rightarrow k + v}^{(1)} \ldots dy_{k + v}^{(T-1)} u_{k + y \rightarrow k + v}^{(T-1)}).$$

Use $\{(K, X_K, v)_T(m) : m \in \mathbb{N}\}$ to denote the Markov chain associated with this kernel. The ideal version of Algorithm 2 presented in Section 2.2 sets $q_{k \rightarrow k'} := \pi(\cdot \mid k')$, the conditional distribution of the parameters of model $k'$. This is when switching models. For the parameter update step (see Step 2(a) in Algorithm 2 for instance), we assume that both samplers use the same MCMC kernels of invariant distributions $\pi(\cdot \mid k)$. The Markov kernel simulated by the ideal version (when switching models) is thus given by:

$$P_{\text{ideal}}((k, x_k, v), (k', y_k', v')) := \pi(u_{k \rightarrow k + v} \mid k + v) \delta_{(k + v, u_{k \rightarrow k + v}, v)}(k', y_k', v') \left(1 \wedge \frac{\pi(k')}{\pi(k)}\right)$$

$$+ \delta_{(k, x_k, v)}(k', y_k', v') \int \pi(u_{k \rightarrow k + v} \mid k + v) \left(1 - 1 \wedge \frac{\pi(k + v)}{\pi(k)}\right) d\mathbf{u}_{k \rightarrow k + v}$$

$$= \pi(u_{k \rightarrow k + v} \mid k + v) \delta_{(k + v, u_{k \rightarrow k + v}, v)}(k', y_k', v') \left(1 \wedge \frac{\pi(k')}{\pi(k)}\right)$$

$$+ \delta_{(k, x_k, v)}(k', y_k', v') \left(1 - 1 \wedge \frac{\pi(k + v)}{\pi(k)}\right).$$

Use $\{(K, X_K, v)_{\text{ideal}}(m) : m \in \mathbb{N}\}$ to denote the corresponding Markov chain. The transitions in the ideal case are therefore such that with probability $1 \wedge \pi(k + v)/\pi(k)$ there is a move to model $k + v$ with parameters $y_{k + v} \sim \pi(\cdot \mid k + v)$ (and the direction $v$ is conserved). Otherwise, the model and parameters stay the same (and the direction $v$ is reversed). The two distinctive elements of the ideal sampler are the form of the acceptance probability and distribution of the proposal $y_{k + v}$. Intuitively, if Algorithm 2 proposes parameters with a distribution close to $\pi(\cdot \mid k + v)$ and accept them with a probability close to $1 \wedge \pi(k + v)/\pi(k)$ (in the limit), the weak convergence should happen as the transition probabilities share the same behaviour. This is essentially what Theorem 1 in Karr (1975) indicates: if Algorithm 2 and its ideal version are initialised in the same way (i.e. $(K, X_K, v)_T(0)$ and $(K, X_K, v)_{\text{ideal}}(0)$ follow the same distribution), and $P_T \rightarrow P_{\text{ideal}}$ in some sense as $T \rightarrow \infty$, then $\{(K, X_K, v)_T(m) : m \in \mathbb{N}\}$ converges weakly towards $\{(K, X_K, v)_{\text{ideal}}(m) : m \in \mathbb{N}\}$, denoted by $\{(K, X_K, v)_T(m) : m \in \mathbb{N}\} \Rightarrow \{(K, X_K, v)_{\text{ideal}}(m) : m \in \mathbb{N}\}$, as $T \rightarrow \infty$.

We already know that the acceptance probabilities are the same in the limit for both samplers as it is mentioned in Karagiannis and Andrieu (2013) that $r_{\text{NRJ2}}((k, x_k), (k + v, y_{k + v}^{(T-1)}))$ is a consistent estimator of $\pi(k + v)/\pi(k)$ as $T \rightarrow \infty$ under realistic assumptions. For our result, we more precisely consider the following assumption.

**Assumption 1.** The random variable $r_{\text{NRJ2}}((k, x_k), (k + v, y_{k + v}^{(T-1)}))$ converges in distribution towards $\pi(k + v)/\pi(k)$ as $T \rightarrow \infty$, for any given $(k, x_k, v)$. 
The proposals for the parameters in Algorithm 2 $y^{(T-1)}_{k+v}$ should in practice be distributed in the limit as $\pi(\cdot | k + \nu)$. Indeed, consider as in our practical example in Section 5.2 and those in Karagiannis and Andrieu (2013) that $K_{t-k+v}$ are $P_{k-v}^{(t)}$-reversible MH kernels in which the proposal distributions are the same for all $t$. This more precisely means that

$$K_{t-k+v}^{(t)}((y^{(t-1)}_{k+v}, u^{(t-1)}_{k+v-y-k}), (y^{(t)}_{k+v}, u^{(t)}_{k+v-y-k}))$$

are bounded above by a positive constant that depends only on $\nu$. Theorem 1 result. We assume to simplify that it is such that the associated Markov chain is uniformly ergodic. It is close to 1, then $q^{(t)}_{k-v}$ is essentially a time-homogeneous Markov chain with $\pi(\cdot | k + \nu) \otimes q_{k+v-y-k}$ as a stationary distribution. This is why if $T$ is additionally such that $T - t^*$ is large enough, then $Y^{(T-1)}_{k+v}$ is (approximately) distributed as $\pi(\cdot | k + \nu)$. In our weak convergence result, we assume to simplify that it is such that the associated Markov chain is uniformly ergodic. It is highlighted in the proof what modifications and which additional technical conditions are required if geometric ergodicity is instead assumed.

**Assumption 2.** For all $k$ and $\nu$, the time-homogeneous $\pi(\cdot | k + \nu) \otimes q_{k+v-y-k}$-reversible Markov chain associated with the proposal distribution $q^{(t)}_{k-v}$ for all iteration $m$ with a stationary distribution that is fixed and set to be $\pi(\cdot | k + \nu) \otimes q_{k+v-y-k}$). The design of $q^{(t)}_{k-v}$ has an impact on how large the distance between $T$ and $t^*$ need to be to have $Y^{(T-1)}_{k+v}$ approximately distributed as $\pi(\cdot | k + \nu)$. In our weak convergence result, we assume to simplify that it is such that the associated Markov chain is uniformly ergodic. It is highlighted in the proof what modifications and which additional technical conditions are required if geometric ergodicity is instead assumed.

Finally, we assume regularity conditions on the PDF $q^{(t)}_{k-v}$.

**Assumption 3.** For all $k$ and $\nu$, $q^{(t)}_{k-v}(y^{(t-1)}_{k+v}, u^{(t-1)}_{k+v-y-k}), (y^{(t)}_{k+v}, u^{(t)}_{k+v-y-k}))$ and

$$q^{(t)}_{k-v}((y^{(t-1)}_{k+v}, u^{(t-1)}_{k+v-y-k}), (y^{(t-1)}_{k+v}, u^{(t-1)}_{k+v-y-k}))$$

are bounded above by a positive constant that depends only on $k$ and $\nu$.

Note that (10) is equal to 1 if $q^{(t)}_{k-v}$ is symmetric. We are now ready to present the weak convergence result.

**Theorem 1** (Weak convergence of Algorithm 2). Under Assumptions 1 to 3 and assuming that $(K, X_K, \nu)_{T}(0) \sim \pi \otimes U([-1, 1])$ and $(K, X_K, \nu)_{ideal}(0) \sim \pi \otimes U([-1, 1])$, we have

$$\{(K, X_K, \nu)_{T}(m) : m \in \mathbb{N}\} \Rightarrow \{(K, X_K, \nu)_{ideal}(m) : m \in \mathbb{N}\} \quad \text{as} \quad T \to \infty.$$
4 About NRJ performance

We have shown that it is possible to construct samplers as close as we want to their ideal counterparts, at least in the weak convergence sense. We focus in this section on the marginal ideal behaviour of $K$ associated with the different sampling strategies (ideal RJ and NRJ) to establish the dominance of the latter when the target distribution is not too concentrated. We in fact consider only iterations in which model switches are proposed to focus on this type of transition; we in particular do not study the impact of the proportion of parameter updates $\tau$, but we discuss briefly how this parameter is selected in Section 6.1.

In Section 4.1, existing results describing the behaviours of ideal RJ and NRJ when the marginal distribution is uniform or log concave are presented. NRJ outperform RJ in the former case, but not necessarily in the latter. To analyse this latter case further, we use a parameter $\phi \geq 1$ to characterise log concave distributions in Section 4.2, and present a family of “worst” (for NRJ) log concave distributions for which the larger is $\phi$ the more concentrated is the PMF. NRJ outperform RJ when $\phi$ is not too large and the target is a member of this family. We also discuss scaling limit results in Section 4.3.

4.1 Existing results

Denote by $\{K_{\text{ideal}}^{\text{RJ}}(m) : m \in \mathbb{N}\}$ and $\{K_{\text{ideal}}^{\text{NRJ}}(m) : m \in \mathbb{N}\}$ the Markov chains produced by ideal RJ and NRJ, where we highlighted the dependence of the behaviour on $g$ for RJ. We have not yet discussed $g$ and show here how it can impact performance.

We consider in this section a scenario where $\mathcal{K}$ is finite and given by $\mathcal{K} := \{1, \ldots, K_{\text{max}}\}$, where $K_{\text{max}}$ is a positive integer. Diaconis et al. (2000) proved that, when the target distribution is uniform, it takes on the order of $K_{\text{max}}$ steps for a Markov chain similar to $\{K_{\text{ideal}}^{\text{NRJ}}(m) : m \in \mathbb{N}\}$ to converge towards the target in total variation distance; the order is of $K_{\text{max}}^2$ for $\{K_{\text{ideal}}^{\text{RJ}}(m) : m \in \mathbb{N}\}$, $g^*$ being the optimal proposal distribution. The usual symmetric distribution $g^*(k, k+1) = g^*(k, k-1) = 1/2$ is the optimal (conditional) proposal distribution (given that a model switch is proposed) in this case. The associated transition matrix is indeed proved in Boyd et al. (2006) to be optimal among all symmetric stochastic tridiagonal matrices (which reflects a restriction to samplers with proposals of the form $k \mapsto k' \in \{k-1, k+1\}$).

The stochastic process $\{K_{\text{ideal}}^{\text{NRJ}}(m) : m \in \mathbb{N}\}$ is actually deterministic when the target is uniform; all proposals are accepted and it thus goes from 1 to $K_{\text{max}}$ without stopping, and changes direction at $K_{\text{max}}$ to return to 1. The distribution after $m$ steps therefore does not converge towards the uniform as it assigns a probability of 1 to some state for any $m$. This is however not an issue for Monte Carlo approximation. For this task, results like those of Diaconis et al. (2000) are interesting because they tell us how many steps it takes to traverse the state-space. For our sampler, it is thus clear that it is of order $K_{\text{max}}$ when the target is uniform. This thus establishes the superiority of NRJ over RJ in this case among samplers with proposals of the form $k \mapsto k' \in \{k-1, k+1\}$. We do not consider the sampler of Diaconis et al. (2000) as a competitor to ours because they are, in essence, the same. The difference is that instead of systematically changing direction at 1 and $K_{\text{max}}$, the sampler of Diaconis et al. (2000) changes direction probabilistically after on average $K_{\text{max}}$ steps. This difference makes it non-deterministic and aperiodic.

Superiority for uniform targets is an interesting theoretical result, but this is not a scenario of interest in Bayesian model selection. We believe that it is more likely that the posteriors reflect a balance between too simple models (that are more stable but do not capture well the dynamics in the data) and too complex models (that overfit and have less generalisation power), in the spirit of Occam’s razor.
Unimodal distributions, which are such that \( \pi(1) \leq \ldots \leq \pi(k^*) \geq \ldots \geq \pi(K_{\text{max}}) \), are in this sense more interesting to analyse. Hildebrand (2002) generalised the result of Diaconis et al. (2000) on the Markov chain similar to \( \{K_{\text{NRJ}}^\ast(m) : m \in \mathbb{N}\} \) to log concave distributions, defined as distributions such that \( \pi(k) / \pi(k-1) \geq \pi(k+1) / \pi(k) \) for all \( k \in \{2, \ldots, K_{\text{max}}-1\} \). Log concave distributions belong to the family of unimodal distributions. Indeed, if we consider for instance \( k > k^* \) (the mode), we observe that the ratios \( \pi(k+1) / \pi(k) \) are smaller and smaller as we get further away from the mode.

An adaptation of the proof of Hildebrand (2002) allows us to prove that \( O(K_{\text{max}}) \) steps are sufficient for \( \{K_{\text{ideal}}^\ast(m) : m \in \mathbb{N}\} \) to converge towards the target distribution in total variation distance, if we assume that the distribution is log concave, but not uniform. For \( \{K_{\text{ideal}}^\ast(m) : m \in \mathbb{N}\} \), no such results are available. To establish the superiority of NRJ when the target is not too concentrated, we need to identify the optimal proposal distribution \( g^\ast \) for RJ and to prove that the number of required steps is larger. We take here a step in this direction.

We choose the competitor to NRJ to be the RJ with the distribution \( g^\ast \) given by

\[
g^\ast(k, k') \propto \frac{1}{\sqrt{\pi(k')/\pi(k)}} \quad \text{for} \quad k' \in \{k-1, k+1\}. \tag{11}
\]

This choice finds its justification in Zanella (2019), in which it is shown that a class of what the author calls informed distributions with \( g^\ast \) as a special case are optimal within reversible samplers in some situations. In fact, it is possible to numerically show that the optimal distribution in terms of speed of convergence among distributions \( g(k, \cdot) \) defined on \( \{k-1, k+1\} \) is very close to \( g^\ast \) with a negligible speed difference when the log concave distribution belongs to the family defined in the next section. Note that in particular the optimal proposal distribution \( g^\ast(k, k+1) = g^\ast(k, k-1) = 1/2 \) is retrieved when the target is uniform.

For any log concave target, \( g^\ast \) is such that

\[
\frac{1}{\pi(k)/\pi(k+1) + 1} \leq g^\ast(k, k+1) \leq \frac{1}{\pi(k-1)/\pi(k) + 1}.
\]

The acceptance probabilities are controlled in the same way by ratios of posterior model probabilities. As long as the target is not too concentrated, meaning ratios not too far from 1, \( \{K_{\text{ideal}}^\ast(m) : m \in \mathbb{N}\} \) thus still has a diffusive behaviour that makes it traverse the state-space in order of \( K_{\text{max}}^2 \) steps. However, for concentrated targets, \( g^\ast(k, k+1) \) gets close to 1 when the chain is at the left of the mode as \( \pi(k)/\pi(k+1) \) and \( \pi(k-1)/\pi(k) \) are close to 0. The stochastic process thus moves persistently towards the mode and wanders around it afterwards.

### 4.2 Log concave distributions: a worst case scenario

One way to characterise any log concave distribution is through the minimum of these ratios \( \pi(k)/\pi(k+1) \) and \( \pi(k-1)/\pi(k) \). Consider that this minimum is \( 1/\phi \). The distribution with a constant decreasing factor from the mode of \( 1/\phi \) leads to RJ with \( g^\ast \) with the most significant advantage over RJ with a symmetric proposal. This is because the distribution is the most concentrated, which at the same time leaves not much room for persistent movement for NRJ.

We now introduce a class of distributions with this characteristic. This class is such that the mode \( k^* \) is at the middle of the domain:

\[
\frac{\pi(k+1)}{\pi(k)} = \frac{1}{\phi} \quad \text{for} \quad k \geq k^*, \quad \text{and} \quad \frac{\pi(k-1)}{\pi(k)} = \frac{1}{\phi} \quad \text{for} \quad k \leq k^*, \quad \text{with} \quad \phi > 1. \tag{12}
\]
Indeed, when $\phi$ in fact more efficient because there are basically 3 possible values for $K$: the mode $k^*$, $k^* + 1$ and $k^* - 1$. Beyond this threshold RJ with $g^*$ slowly starts to perform better. Beyond this threshold RJ with $g^*$ moves persistently towards the mode but is in fact more efficient because there are basically 3 possible values for $K$: the mode $k^*$, $k^* + 1$ and $k^* - 1$. Indeed, when $\phi$ is exactly 7, the total mass outside of these values is $2\sum_{k=k^*+2}^{\max} \pi(k)/\phi^{k-k^*} \approx 3.57\%$. This is essentially true for any value of $\max$ as this percentage is equal to the limiting value (to two decimal places) as $\max \to \infty$. When there are 3 possible values, starting from the mode $k^*$, both NRJ and RJ with $g^*$ go either to the right or the left with equal probability (on average for NRJ given that $\nu \sim \mathcal{U}(-1, 1)$). Let us say that they go to $k^* + 1$, the difference is that NRJ tries to go to $k^* + 2$ (because of the direction), which is likely to be rejected, and therefore, it stays for one iteration at $k^* + 1$; RJ directly goes back to $k^*$ given that $g^*(k^* + 1, k^*) = \phi/\phi + 1$ is close to 1. RJ thus seems to have an advantage in terms of required number of steps to traverse the state-space.

To summarise, NRJ is expected to perform better than RJ with $g^*$ for any log concave distribution such that the minimum of the ratios $\pi(k)/\pi(k + 1)$ and $\pi(k - 1)/\pi(k)$ is larger than $1/\phi^*$, where $\phi^* \approx 7$. We noticed that $g^*$ uses information about the target which is obviously not available prior pilot runs. We also observed that NRJ always outperform RJ with a symmetric proposal. In fact, Theorem 3.17 in Andrieu and Livingstone (2019) states that the asymptotic variance of ergodic averages is guaranteed to be smaller. As practical guidelines, we thus recommend to start by using NRJ, and if after pilot runs the target appears strongly concentrated, then it may be beneficial to switch to RJ with $g^*$. In the multiple change-point example in Section 5.2, the target is for instance not too concentrated and RJ with $g^*$ performs similarly to RJ with the symmetric proposal.

We finish this section by noting that when the target is strongly concentrated, attempts to go to models other than the mode have a high rejection rate. After the rejection, attempting a parameter update within the same iteration leaves the distribution invariant. This strategy may be implemented to improve the mixing of the parameters.

### 4.3 Scaling limits of model indicator process

Another way to evaluate the performance of algorithms is through the identification and analysis of the scaling limits of their associated stochastic processes as the dimension $d$ of the state-space goes to infinity. Roberts et al. (1997) and Roberts and Rosenthal (1998) applied this strategy to optimally tune the random walk Metropolis (RWM) and Metropolis-adjusted Langevin algorithm (MALA), but their analyses can also be used to establish that MALA is more efficient than RWM. This follows from the fact that to obtain non-trivial continuous limiting stochastic processes we need to speed up time by factors $d$ and $d^{1/3}$ for RWM and MALA, respectively. We explore such scaling limits for the processes $\{K^{\text{NRJ}}_\text{ideal}(m) : m \in \mathbb{N}\}$ and $\{K^{\text{NRJ}}_\text{ideal}(m) : m \in \mathbb{N}\}$.

In our framework, we have no guarantee that the model indicator variable will converge towards a continuous random variable as $\max$ increases. In the supplementary material (Section 8), we present strong and technical assumptions on $\pi(k)$ under which results analogous to those of Syed et al. (2019) are obtained; i.e. the reversible process suitably rescaled converges to a diffusion while the non-reversible version converges to a piecewise-deterministic Markov process. The required time rescalings lead to conclusions consistent with the results presented in the previous sections showing that $O(\max^2)$ and $O(\max)$ steps are required to explore the state-space for $\{K^{\text{NRJ}}_\text{ideal}(m) : m \in \mathbb{N}\}$ and $\{K^{\text{NRJ}}_\text{ideal}(m) : m \in \mathbb{N}\}$, respectively.
5 Numerical experiments

Recall that in the usual non-ideal situation, the acceptance ratio in $\alpha_{\text{NRJ}}$ can be viewed as the ideal ratio $\pi(k')/\pi(k)$ corrupted by some multiplicative noise; see (5). In practice, the noise fluctuates around 1. In Section 5.1, we show how the difference in performance between NRJ and RJ varies when the noise amplitude changes (in a sense made precise in that section), or in other words as we move away or towards ideal NRJ and RJ. The methods presented in Section 3 are then applied to illustrate how their beneficial effect translates in practice for different noise behaviours. We also show how performances vary when the total number of models increases and the shape of the marginal PMF $\pi(k)$ varies on a simple target distribution for which we can precisely control the noise behaviour, number of models and the shape of the PMF. In Section 5.2, we evaluate the performance of NRJ and RJ in a real multiple change-point problem.

5.1 Simulation study

Let the target distribution be

$$\pi(k, x_k) := p_{\phi, K_{\text{prior}}}(k) \prod_{i=1}^{k} f(x_{i,k}),$$

where $p_{\phi, K_{\text{prior}}}$ is the PMF defined in Section 4.2 in (12), $f$ is the density of a standard normal and $x_k := (x_{1,k}, \ldots, x_{k,k})$. When switching from model $k$ to model $k + 1$ in this case, one parameter needs to be added. It is not necessary to move the parameters that were in model $k$ given that they have the same distributions as the first $k$ parameters of model $k + 1$. In this context, it is straightforward to specify the functions $T_{k \rightarrow k+1}$ that are required for the implementation of RJ and NRJ: they are such that the proposals for the parameters of model $k + 1$ are $y_{k+1} := (x_k, u_{k \rightarrow k+1})$. This also defines the functions $T_{k+1 \rightarrow k}$ for the (deterministic) reverse moves. Note that $u_{k+1 \rightarrow k} := \emptyset$ for all $k$.

We have $\pi(k + 1)/\pi(k) = p_{\phi, K_{\text{prior}}}(k + 1)/p_{\phi, K_{\text{prior}}}(k)$, and the noise term $\epsilon$ is given by

$$\frac{\pi(y_{k+1} | k + 1) q_{k+1 \rightarrow k}(u_{k+1 \rightarrow k})}{\pi(x_k | k) q_{k \rightarrow k+1}(u_{k \rightarrow k+1}) J_{T_{k \rightarrow k+1}}(x_k, u_{k \rightarrow k+1})^{-1}} = \frac{f(u_{k \rightarrow k+1})}{q_{k \rightarrow k+1}(u_{k \rightarrow k+1})}.$$ (13)

We can therefore precisely control the noise behaviour by setting $q_{k \rightarrow k+1} = \mathcal{N}(0, \sigma^2)$, where $\sigma > 0$ is the varying parameter. Indeed, in this case

$$\frac{f(u_{k \rightarrow k+1})}{q_{k \rightarrow k+1}(u_{k \rightarrow k+1})} = \sigma \exp \left[-\frac{u_{k \rightarrow k+1}^2}{2} \left(1 - \frac{1}{\sigma^2}\right)\right],$$

which behaviour varies with $\sigma$ given that $u_{k \rightarrow k+1} \sim \mathcal{N}(0, \sigma^2)$. This is also true for the reverse move. A small $\sigma$ represents a proposal distribution that is more concentrated around the mode than the target, whereas it is less concentrated when $\sigma$ is large.

For implementing Algorithm 2 or the corresponding RJ, we only need to create paths for the proposals $u_{k \rightarrow k+1}$ used to switch from model $k$ to model $k + 1$. This is realised by looking at the noise term (13), and also by remembering that it is not necessary to move the parameters that were in model $k$. We thus essentially create a bridge between model $k$ to model $k + 1$ made of weighted geometric averages of $f$ and $q_{k \rightarrow k+1}$. The annealing intermediate distributions indeed have intuitive forms:

$$p^{(t)}_{k \rightarrow k+1}(u_{k \rightarrow k+1}) \propto \exp\left(-\frac{1}{2\sigma^2} (u_{k \rightarrow k+1})^2\right)^{1-\gamma} \left[\exp\left(-\frac{1}{2} (u_{k \rightarrow k+1}^2)\right)^{\gamma}\right] = \exp\left(-\frac{(u_{k \rightarrow k+1})^2}{2} [(1 - \gamma)\sigma^{-2} + \gamma_1]\right),$$
where $\gamma_t := t/T$; $\rho_{k+1 \rightarrow k}^{(t)}$ is defined analogously. Therefore, to go from model $k$ to model $k+1$, we target normal distributions with mean 0 and variances $[(1-t/T)\sigma^{-2} + t/T]^{-1}$; we thus start with variances close to $\sigma^2$ (corresponding to the initial proposal distribution) to finish with variances close to 1 (the target distribution). For the reverse move, we do the opposite. Because we can generate from the distributions $\rho_{k \rightarrow k+1}^{(t)}$, we in fact use them as transitions kernels: $K_{k \rightarrow k+1}(u_{k \rightarrow k+1},\cdot) := \rho_{k \rightarrow k+1}^{(t)}$ (which satisfy the symmetry (8) and reversibility (9) conditions).

The results are presented in Figure 3. They are based on 1,000 runs of 100,000 iterations for each value of $\sigma$, $K_{\text{max}}$ and $\phi$; the ESS are computed considering only the iterations in which model switches are proposed. As expected, the further $\sigma$ is from 1 (the latter corresponding to ideal samplers), the higher is the ESS. We notice that the impact is almost symmetric in $\sigma$ if we consider distances from the distribution $N(0,1)$ (e.g. the normal with $\sigma = 1/2$ is two times more concentrated, whereas the normal with $\sigma = 2$ is two times less concentrated; they both are at a distance of two, but the former is on the more concentrated side and the latter on the less concentrated one). We also notice that in extreme cases, for instance when $\sigma$ is close to 0, NRJ and RJ have similar performances. This is explained by the fact that the direction-assisted scheme characterising NRJ does not help any more; almost all moves are rejected, which implies that direction changes very often. This leads to the same diffusive behaviour as RJ. Applying Algorithms 2 and 3 improve performances. It is possible to obtain essentially flat lines around the maximum value of 0.21 ESS per iteration by increasing $T$ and $N$, leading to samplers that are at least 2.5 times more efficient than RJ for any value of $\sigma$. Note that we do not show the results for the RJ corresponding to Algorithms 2 and 3 as it does not add information to Figure 3 (a) given that the lines would be on top of each other.

The ESS also decreases as the total number of models $K_{\text{max}}$ increases (see Figure 3 (b)). This is expected as the difference between $k$ and $k'$ (representing the current model and the next one to explore) is constant, equal to 1. The exploration abilities of the stochastic processes thus diminish as a smaller fraction of the state-space is traversed at each iteration. In theory, a way to compensate is to generate a random variable at each iteration that dictates the difference between $k$ and $k'$, allowing larger jumps. However, proposal distributions for transitions to models at a distance of more than 1 are often difficult to design. Note that the total probability mass of the 15 most likely models is essentially 1 when $\phi := 2$ (and $K_{\text{max}} \geq 15$), which explains why the ESS becomes constant beyond this value. Note also that we do not show the results for Algorithms 2 and 3 as $\sigma := 1$, meaning that ideal samplers are applied.

Recall that Figure 3 (c) was discussed in Section 4.

![Figure 3](image-url)

**Figure 3.** (a) ESS as a function of $\sigma$ for NRJ (Algorithm 1, Algorithm 2 with $T := 15$, and Algorithm 3 with $T := 15$ and $N := 15$) and RJ (with optimal and symmetric $g$), when $\phi := 2$ and $K_{\text{max}} := 11$; (b) ESS as a function of $K_{\text{max}}$ for NRJ (Algorithm 1) and RJ (with optimal and symmetric $g$), when $\phi := 2$ and $\sigma := 1$; (c) ESS as a function of $\phi$ for NRJ (Algorithm 1) and RJ (with optimal and symmetric $g$), when $K_{\text{max}} := 11$ and $\sigma := 1$.
5.2 Performance evaluation in multiple change-point problems

In this section, we evaluate the performance of RJ and NRJ algorithms when applied to sample from the posterior of the model in Green (1995) for multiple change-point analysis and coal mining disaster data set detailed in Raftery and Akman (1986). The $n$ data points $t := (t_1, \ldots, t_n)$ represent times of occurrence of disasters, to which we add the “origin” $t_0 := 0$. It is assumed that they arose from a non-homogeneous Poisson process that have for intensity a step function $\lambda_k$; the model indicator $k \in K := \{0, \ldots, K_{\text{max}}\}$ reflects the number of steps $k + 1$, where $K_{\text{max}}$ is a known positive integer. It is also assumed that the Poisson process has been observed on the time interval $[0, L]$, where $L > 0$ is known. The starting point for each step is denoted $s_{j,k}$, $j = 0, \ldots, k$, to which we add the endpoint of the last step $s_{k+1,k}$, where these are subject to the constraint $0 =: s_{0,k} < s_{1,k} < \ldots < s_{k+1,k} := L$. The height of the $j$-th step is by denoted $h_{j,k}$, $j = 1, \ldots, k + 1$. The log-likelihood of model $k$ is

$$
\log L(x_k \mid k, t) := \sum_{i=1}^{n} \log(\lambda_k(t_i \mid x_k)) - \int_{0}^{L} \lambda_k(t \mid x_k) \, dt,
$$

where $\lambda_k(t \mid x_k) := \sum_{j=0}^{k} h_{j+1,k} I_{[s_{j,k},s_{j+1,k})}(t)$ for $t \in [0, L]$ and $x_k := (s_{1,k}, \ldots, s_{k,k}, h_{1,k}, \ldots, h_{k+1,k})^T$, $I$ being the indicator function.

We use the same prior structure as Green (1995). The prior on $K$ is a Poisson distribution with parameter $\lambda > 0$, but conditioned on $K \leq K_{\text{max}}$. Given $K = k$, the starting points $s_{1,k}, \ldots, s_{k,k}$ are a priori distributed as the even-numbered order statistics from $2k + 1$ points uniformly distributed on $[0, L]$, and the heights are independently and identically distributed as $\Gamma(\alpha, \beta)$, where $\alpha > 0$ and $\beta > 0$ are the shape and rate parameters, respectively. In Green (1995), the hyperparameters are set to $\alpha := 3, K_{\text{max}} := 30, \beta := 1$, and $\beta := 200$.

As done in Section 5.1, one may take advantage of the information at its disposal about the problem and model to design the sampler. Green (1995) follows this approach. We design the RJ and the corresponding Algorithm 1 as this author. For parameter updates, we randomly choose to modify either one of the heights $h_{j,k}$ or one of the starting points $s_{j,k}$. We modify a starting point $s_{j,k}$ by proposing a new value uniformly between $s_{j-1,k}$ and $s_{j+1,k}$. We modify a height $h_{j,k}$ by proposing a new value $h'_{j,k}$ that is such that $\log(h'_{j,k}/h_{j,k}) \sim U(-1/2, 1/2)$. For model switches, we randomly choose to either add or withdraw a step. When we add a step, we first generate its starting point $s^* \sim U[0, L]$. Deterministically, given $s^*$, we know which step will be splitted in two, in the sense that the proposal for the starting points is: $(s_{0,k}, \ldots, s_{j,k}, s^*, s_{j+1,k}, \ldots, s_{k+1,k})$, where $s_{0,k} < \ldots < s_{j,k} < s^* < s_{j+1,k} < \ldots < s_{k+1,k}$ (the step $(s_{j,k}, s_{j+1,k})$ is splitted in two). We perturb as follows the height of this step $h_{j+1,k}$ to obtain proposals for the two heights $h'_{j+1,k+1}$ and $h'_{j+2,k+1}$ in the proposed model: generate $u_p \sim U[0, 1]$ which is such that $h'_{j+2,k+1}/h'_{j+1,k+1} = (1 - u_p)/u_p$, and set the height proposals such that

$$
(h'_{j+1,k+1})^{s^* - j}_{j+1,k} = h'_{j+2,k+1}^{s^* - j}_{j+1,k} = h'_{j+1,k}.
$$

The height proposals are $(h_{1,k}, \ldots, h_{j,k}, h_{j+1,k+1}, h'_{j+2,k+1}, h'_{j+2,k}, h_{j+2,k}, \ldots, h_{k+1,k})$. When we withdraw a step, we proceed with the reverse move, which is deterministic after having generated $j^* \sim U[0, \ldots, k - 1]$ (starting from model $k$). See Green (1995) for the acceptance probabilities and more details.

For implementing Algorithm 3 and the corresponding RJ, we proceed as in Karagiannis and Andrieu (2013) for generating the paths. More precisely, when switching from model $k$ to model $k + 1$, we use the same strategy as above to set the starting point of the path to $(s_{0,k}, \ldots, s_{j,k}, s^*, s_{j+1,k}, \ldots, s_{k+1,k})$ and $(h_{1,k}, \ldots, h_{j,k}, h'_{j+1,k+1}, h'_{j+2,k+1}, h'_{j+2,k}, \ldots, h_{k+1,k})$, which are parameters in model $k + 1$. We next update the parameters in model $k + 1$ using blockwise MCMC sweeps. In a random order, we modify one of
the heights $h_{j,k+1}$ and one of the starting points $s_{j,k+1}$ as when updating the parameters in Algorithm 1 (as explained above), and we update $j^*$. Note that when updating $h_{j,k+1}$ and $s_{j,k+1}$, we update the corresponding parameters in model $k$ as they are linked through deterministic functions. When updating $j^*$ given the rest, the parameters in model $k$ may be updated as we may change which step is splitted in two. The intermediate distributions are\footnote{There is a typo in the definition of these functions in Karagiannis and Andrieu (2013).}

$$
\rho_{k=k+1}(x^{(t)}_k, u^{(t)}_{k'=k}) \propto \frac{\pi(k, x^{(t)}_k)}{L} \frac{h^{(t)}_{j^*+1,k}}{(h^{(t)}_{j^*+2,k+1} + h^{(t)}_{j^*+1,k+1})^2} \left[ \frac{\pi(k+1, y^{(t)}_{k+1})}{k+1} \right]^{1-t/T}.
$$

See Karagiannis and Andrieu (2013) for more details.

A comparison based on 1,000 runs of RJ and NRJ with 100,000 iterations and burn-ins of 10,000 is presented in Figure 4; the ESS are computed considering only the iterations in which model switches are proposed. Only the results for RJ with symmetric $g$ are shown as those with $g^*$ defined in (11) are similar. We notice that Algorithm 1 and the corresponding RJ offer similar performance (Figure 4 (a)). This is due to too many rejected proposals, which induces in NRJ a lot of direction changes and a diffusive behaviour (as explained in Section 5.1). The true nature of NRJ characterised by persistent movement is revealed when the proportion of rejections decreases as a by-product of applying the methods of Karagiannis and Andrieu (2013) and Andrieu et al. (2018) (Figure 4 (b)) which allow to get closer to ideal samplers (Figure 4 (c)). The ESS per iteration is (on average) 2.1 times larger for Algorithm 3 than for the corresponding RJ; this ratio is 4.0 for ideal samplers. In Karagiannis and Andrieu (2013) it is explained that applying their strategy in RJ in this example of change-point problems results in a net increase in ESS per unit time for any $T$ between 0 and 1000. This net increase is therefore even more marked for NRJ. Recall that the additional computational cost associated to the method of Andrieu et al. (2018) is negligible in a parallel computing environment.

![Figure 4](image)

**Figure 4.** ESS for NRJ and RJ (with symmetric $g$) when the samplers are: (a) vanilla samplers (i.e. Algorithm 1 and the corresponding RJ), (b) Algorithm 3 with $T := 100$ and $N := 10$ and the corresponding RJ, (c) ideal samplers

### 6 Discussion

In this paper, we have introduced non-reversible trans-dimensional samplers that can be applied to Bayesian nested model selection. They are derived from RJ algorithms by making simple modifications which require no additional computational cost during implementation; the model indicator process
now follows a direction $\nu$ which is conserved as long as the model switches are accepted, but reversed at the next rejection. Empirically, these samplers outperform their reversible counterparts when the marginal posterior distribution of $K$ is not too concentrated. We now discuss some implementation aspects that have not been addressed in previous sections and possible directions for future research.

### 6.1 Other implementation aspects

Several functions need to be specified for implementing trans-dimensional samplers: $g$ (which corresponds to the specification of $\tau$ for NRJ), $q_{k \to k'}$ and $T_{k \to k'}$. Significant amount of work has been carried out to address the specification of the last two when no prior information about the problems can be exploited (contrary to the examples in Section 5) or a more automatic perspective is adopted (see, e.g., Green (2003) and Brooks et al. (2003)). The approaches of these authors are arguably the most popular. They are directly applicable in the NRJ framework. We believe a particularly good way to proceed is to design the functions $q_{k \to k'}$ and $T_{k \to k'}$ according to the approach of Green (2003) to afterwards use them in Algorithm 3 to benefit from the strategies of Karagiannis and Andrieu (2013) and Andrieu et al. (2018) that aim to ensure good mixing properties.

Little attention has been devoted to the impact of the specification of $\tau$. Gagnon et al. (2019) prove weak convergence results for RJ and identify a range of values around 0.4 for which a suitable balance between a lot of model switches (but few parameter updates) and a lot of parameter updates (but few model switches) is reached, in the situation where $q_{k \to k'}$ and $T_{k \to k'}$ are well designed. When it is not the case, our understanding is that after a switch to model $k'$ we have to “wait” for the chain to reach “stationarity” under this model before reswitching. In the same way we set burn-ins of say $B$, we should thus set $\tau := 1 - 1/B$ to have on average $B$ parameter updates before a model switch. Methods incorporated in Algorithm 3 allow to reduce the value of $B$.

### 6.2 Possible directions for future research

We identified in Section 4 a specific ideal RJ (associated with $g^*$) as the main competitor to NRJ within all ideal RJ algorithms when the marginal posterior distribution of $K$ belongs to a family of unimodal PMF and samplers are restricted to model switching proposals of the form $k \mapsto k' \in \{k - 1, k + 1\}$. We next provided arguments explaining why ideal NRJ outperform this ideal RJ when the target is not too concentrated and numerically showed the range of concentration parameters $\phi$ in the PMF (12) for which this is the case. It would be interesting to conduct an exhaustive theoretical analysis to expand the scope of the conclusions and make more precise the expected gain. We believe it would also be interesting to develop NRJ that can be applied when the models are not nested.

### References

Andrieu, C., Doucet, A., Yildirim, S. and Chopin, N. (2018) On the utility of Metropolis–Hastings with asymmetric acceptance ratio. arXiv:1803.09527.

Andrieu, C. and Livingstone, S. (2019) Peskun-Tierney ordering for Markov chain and process Monte Carlo: beyond the reversible scenario. arXiv preprint arXiv:1906.06197.

Bierkens, J., Fearnhead, P., Roberts, G. et al. (2019) The zig-zag process and super-efficient sampling for Bayesian analysis of big data. Ann. Statist., 47, 1288–1320.
Bierkens, J. and Roberts, G. (2017) A piecewise deterministic scaling limit of lifted Metropolis–Hastings in the Curie–Weiss model. *Ann. Appl. Probab.*, **27**, 846–882.

Bouchard-Côté, A., Vollmer, S. J. and Doucet, A. (2018) The Bouncy Particle Sampler: A nonreversible rejection-free Markov chain Monte Carlo method. *J. Amer. Statist. Assoc.*, 1–13.

Boyd, S., Diaconis, P., Sun, J. and Xiao, L. (2006) Fastest mixing Markov chain on a path. *The American Mathematical Monthly*, **113**, 70–74.

Brooks, S. P., Giudici, P. and Roberts, G. O. (2003) Efficient construction of reversible jump Markov chain Monte Carlo proposal distributions. *J. R. Stat. Soc. Ser. B. Stat. Methodol.*, **65**, 3–39.

Chen, F., Lovász, L. and Pak, I. (1999) Lifting Markov chains to speed up mixing. In *Proceedings of the thirty-first annual ACM symposium on Theory of computing*, 275–281.

Diaconis, P., Holmes, S. and Neal, R. M. (2000) Analysis of a nonreversible Markov chain sampler. *Ann. Appl. Probab.*, 726–752.

Ethier, S. N. and Kurtz, T. G. (1986) *Markov Processes: Characterization and Convergence*. Wiley.

Gagnon, P., Bédard, M. and Desgagné, A. (2019) Weak convergence and optimal tuning of the reversible jump algorithm. *Math. Comput. Simulation*, **161**, 32–51.

Green, P. J. (1995) Reversible jump Markov chain Monte Carlo computation and Bayesian model determination. *Biometrika*, **82**, 711–732.

— (2003) Trans-dimensional Markov chain Monte Carlo. In: *Highly structured stochastic systems*, 179–196.

Hildebrand, M. (2002) Analysis of the Diaconis-Holmes-Neal Markov chain sampler for log concave probabilities. URL: [https://www.albany.edu/~martinhi/dvifiles/dhnlc4.dvi](https://www.albany.edu/~martinhi/dvifiles/dhnlc4.dvi).

Karagiannis, G. and Andrieu, C. (2013) Annealed importance sampling reversible jump MCMC algorithms. *J. Comp. Graph. Stat.*, **22**, 623–648.

Karr, A. F. (1975) Weak convergence of a sequence of Markov chains. *Z. Wahrsch. Verw. Gebiete*, **33**, 41–48.

Neal, R. M. (2001) Annealed importance sampling. *Stat. Comput.*, **11**, 125–139.

— (2011) MCMC using Hamiltonian dynamics. In: *Handbook of Markov Chain Monte Carlo*, 113–160.

Raftery, A. E. and Akman, V. (1986) Bayesian analysis of a Poisson process with a change-point. *Biometrika*, 85–89.

Richardson, S. and Green, P. J. (1997) On Bayesian analysis of mixtures with an unknown number of components (with discussion). *J. R. Stat. Soc. Ser. B. Stat. Methodol.*, **59**, 731–792.

Roberts, G. O., Gelman, A. and Gilks, W. R. (1997) Weak convergence and optimal scaling of random walk Metropolis algorithms. *Ann. Appl. Probab.*, **7**, 110–120.
The second term on the right-hand side (RHS) of (15) is equal to $P$ where $P$ is the transition kernel. Note that we abuse notation here by denoting the measure $d\nu$ on the LHS given that we in fact use the vector $u_{k\rightarrow k'}$ when switching models, which often do not have the same dimension as $y_{k'}$.

We consider two distinct events: a model switch is proposed, that we denote $S$, or a parameter update is proposed (therefore denoted $S^c$). We know that the probabilities of these events are $1 - \tau$ and $\tau$, respectively, regardless of the current state of the Markov chain. We rewrite the LHS of (14) as

$$
\sum_{k,v} \int \pi(k, x_k) \times (1/2) \left( \int_A P((k, x_k, v), (k', y_{k'}, v')) dy_{k'} \right) dx_k = \int_A \pi(k', y_{k'}) \times (1/2) dy_{k'},
$$

(14)

where $P$ is the transition kernel. Note that we abuse notation here by denoting the measure $d\nu$ on the LHS given that we in fact use the vector $u_{k\rightarrow k'}$ when switching models, which often do not have the same dimension as $y_{k'}$.

We consider two distinct events: a model switch is proposed, that we denote $S$, or a parameter update is proposed (therefore denoted $S^c$). We know that the probabilities of these events are $1 - \tau$ and $\tau$, respectively, regardless of the current state of the Markov chain. We rewrite the LHS of (14) as

$$
\sum_{k,v} \int \pi(k, x_k) \times (1/2) \left( \int_A P((k, x_k, v), (k', y_{k'}, v')) dy_{k'} \right) dx_k = \mathbb{P}(S) \times (1/2) \sum_{k,v} \int_A \pi(k, x_k) \left( \int_A P((k, x_k, v), (k', y_{k'}, v')) dy_{k'} \right) dx_k dy_{k'}
$$

$$
+ \mathbb{P}(S^c) \times (1/2) \sum_{k,v} \int_A \pi(k, x_k) \left( \int_A P((k, x_k, v), (k', y_{k'}, v')) dy_{k'} \right) dx_k dy_{k'},
$$

(15)

using Fubini’s theorem. We analyse the two terms separately. We know that

$$
P((k, x_k, v), (k', y_{k'}, v') | S^c) = \delta_{(k,v)}(k', v') P_{S^c}(x_{k'}, y_{k'}),
$$

where $P_{S^c}$ is the transition kernel associated with the method used to update the parameters. Therefore, the second term on the right-hand side (RHS) of (15) is equal to

$$
\mathbb{P}(S^c) \times (1/2) \sum_{k,v} \int_A \pi(k, x_k) \left( \int_A P((k, x_k, v), (k', y_{k'}, v')) dy_{k'} \right) dx_k dy_{k'}
$$
\[ P(S') \pi(k') \times (1/2) \int \pi(x_{k'} \mid k') \left( \int_A P_S(x_{k'}, y_{k'}) \, dy_{k'} \right) \, dx_{k'}. \]

We also know that \( P_{S'} \) leaves the conditional distribution \( \pi(\cdot \mid k') \) invariant, implying that

\[
\mathbb{P}(S') \pi(k') \times (1/2) \int \pi(x_{k'} \mid k') \left( \int_A P_{S'}(x_{k'}, y_{k'}) \, dy_{k'} \right) \, dx_{k'} = \mathbb{P}(S') \int \pi(k', y_{k'}) \times (1/2) \, dy_{k'}. \tag{16}
\]

For the model switching case (the first term on the RHS of (15)), we use the fact that there is a connection between \( P(k, x_k, y, k', y', | S) \) and the kernel associated to a specific RJ. Consider that \( g(k, k + 1) = g(k, k - 1) \) for all \( k \) and that all other proposal distributions are the same as the NRJ. In this case, \( \alpha_{\text{RJ}} = \alpha_{\text{NRJ}} \). Given the reversibility of RJ, the probability to go from model \( k \) with parameters in \( B \) to model \( k + 1 \) with parameters in \( A \) is

\[
\int_B \pi(k, x_k) \left( \int_A P_{\text{RJ}}((k, x_k), (k + 1, y_{k+1}) \mid S) \, dy_{k+1} \right) \, dx_k
= \int_A \pi(k + 1, y_{k+1}) \left( \int_B P_{\text{RJ}}((k + 1, y_{k+1}), (k, x_k) \mid S) \, dx_k \right) \, dy_{k+1}, \tag{17}
\]

where \( P_{\text{RJ}} \) is the transition kernel associated with the RJ. Note that

\[
P_{\text{RJ}}((k, x_k), (k + 1, y_{k+1}) \mid S) = (1/2) P((k, x_k, 1), (k + 1, y_{k+1}, 1) \mid S),
\]

given that the difference between both kernels is that in RJ, once it is decided that a model switch is attempted, there is an additional probability of \( 1/2 \) of trying model \( k + 1 \). Analogously, \( P_{\text{RJ}}((k + 1, y_{k+1}), (k, x_k) \mid S) = (1/2) P((k + 1, y_{k+1}, -1), (k, x_k, -1) \mid S) \). Using that and taking \( B \) equals the whole parameter (and auxiliary) space in (17), we have

\[
\mathbb{P}(S) \int \pi(k, x_k) \times (1/2) \left( \int_A P((k, x_k, 1), (k + 1, y_{k+1}, 1) \mid S) \, dy_{k+1} \right) \, dx_k
= \mathbb{P}(S) \int \pi(k + 1, y_{k+1}) \times (1/2) \left( \int_A P((k + 1, y_{k+1}, -1), (k, x_k, -1) \mid S) \, dx_k \right) \, dy_{k+1}.
\]

We thus analyse the probability to reach \( k + 1 \) with parameters in \( A \) and direction +1. We know that the only other way of reaching this state (other than coming from \( k \)) is by being at \( k + 1 \) with parameters in \( A \) and direction −1 and rejecting, which probability is

\[
\mathbb{P}(S) \int_A \pi(k + 1, y_{k+1}) \times (1/2) \left( 1 - \int_A P((k + 1, y_{k+1}, -1), (k, x_k, -1) \mid S) \, dx_k \right) \, dy_{k+1}.
\]

Therefore, the total probability to reach \( k + 1 \) with parameters in \( A \) and direction +1 in one step (given that a model switch is proposed) is

\[
\mathbb{P}(S) \int \pi(k, x_k) \times (1/2) \left( \int_A P((k, x_k, 1), (k + 1, y_{k+1}, 1) \mid S) \, dy_{k+1} \right) \, dx_k
+ \mathbb{P}(S) \int_A \pi(k + 1, y_{k+1}) \times (1/2) \left( 1 - \int_A P((k + 1, y_{k+1}, -1), (k, x_k, -1) \mid S) \, dx_k \right) \, dy_{k+1}
= \mathbb{P}(S) \int \pi(k + 1, y_{k+1}) \times (1/2) \, dy_{k+1}.
\]

Combining this with (16) allows to conclude the proof.
Proof of Theorem 1. We show that Algorithm 2 converges towards its ideal version as $T \to \infty$. As mentioned, for the ideal version, we consider the case where $q_{k \to k'} := \pi(\cdot | k')$, the conditional distribution of the parameters of model $k'$. In this case, we set $y_{k'} := u_{k \to k'}$ to be the proposal for the parameters of model $k'$, and thus the function $T_{k \to k'}$ to be the identity function.

To show the convergence, we use Theorem 1 in Karr (1975). We thus have to verify three assumptions, and this will allow to conclude that $\{(K, X_k, \nu)_T(m) : m \in \mathbb{N}\} \Rightarrow \{(K, X_k, \nu)_{\text{ideal}}(m) : m \in \mathbb{N}\}$ as $T \to \infty$. We focus on the movements involving model switches as the same parameter update schemes are used in both samplers. Here are the three assumptions.

1. The distributions that are used to initialise Algorithm 2 converge towards that used to initialise the ideal NRJ.

This is verified as we assume that the Markov chains produced by both Algorithm 2 and its ideal counterpart start at stationarity, i.e. $(K, X_k, \nu)_T(0) \sim \pi \otimes \mathcal{U}[-1, 1]$ and $(K, X_k, \nu)_{\text{ideal}}(0) \sim \pi \otimes \mathcal{U}[-1, 1]$.

2. For $h \in C^\infty$ (the space of bounded uniformly continuous functions), we have that

$$P_{\text{ideal}}(k, x_k, \nu) h := \sum_{k', \nu'} \int h(k', y_{k'}, \nu') P_{\text{ideal}}((k, x_k, \nu), (k', y_{k'}, \nu')) dy_{k'}$$

is a bounded continuous function.

This kernel is such that

$$P_{\text{ideal}}(k, x_k, \nu) h = \left(1 + \frac{\pi(k + \nu)}{\pi(k)}\right) \int h(k + \nu, u_{k \to k + \nu}, \nu) \pi(u_{k \to k + \nu} | k + \nu) du_{k \to k + \nu} + h(k, x_k, -\nu) \left(1 - 1 + \frac{\pi(k + \nu)}{\pi(k)}\right),$$

which is bounded and continuous.

3. For every $h \in C^\infty$, the Markov kernel associated with Algorithm 2 $P_T h$ converges towards $P_{\text{ideal}} h$ uniformly on each compact subset of the state-space as $T \to \infty$.

We first show the pointwise convergence. Let us denote the conditional joint density of all the random variables involved in the proposal $y_{k + \nu}^{(T-1)}$ given $(k, x_k, \nu)$ by

$$q(u_{k \to k + \nu}, y_{k + \nu}^{(T-1)}, y_{k + \nu}^{(1:T-1)}) := q_{k \to k + \nu}(u_{k \to k + \nu}) \prod_{t=1}^{T-1} K_{k \to k + \nu}^{(t)}((y_{k + \nu}^{(t-1)}, u_{k \to k + \nu}^{(t-1)}), (y_{k + \nu}^{(t)}, u_{k \to k + \nu}^{(t)}),$$

where $K_{k \to k + \nu}^{(t)}$ is a MH kernel reversible with respect to $\rho_{k \to k + \nu}^{(t)}$. We have that

$$P_T h(k, x_k, \nu) := \int h(k + \nu, y_{k + \nu}^{(T-1)}, \nu) q(u_{k \to k + \nu}, y_{k + \nu}^{(1:T-1)}, u_{k \to k + \nu}^{(1:T-1)}) \alpha_{\text{NRJ2}}((k, x_k), (k + \nu, y_{k + \nu}^{(T-1)})) du_{k \to k + \nu} y_{k + \nu}^{(1:T-1)} u_{k \to k + \nu}^{(1:T-1)} + h(k, x_k, -\nu) \int q(u_{k \to k + \nu}, y_{k + \nu}^{(1:T-1)}, u_{k \to k + \nu}^{(1:T-1)})(1 - \alpha_{\text{NRJ2}}((k, x_k), (k + \nu, y_{k + \nu}^{(T-1)}))) du_{k \to k + \nu} y_{k + \nu}^{(1:T-1)} u_{k \to k + \nu}^{(1:T-1)}.$$

Using the triangle inequality, we thus have that

$$|P_T h(k, x_k, \nu) - P_{\text{ideal}} h(k, x_k, \nu)| \leq \int h(k + \nu, y_{k + \nu}^{(T-1)}, \nu) q(u_{k \to k + \nu}, y_{k + \nu}^{(1:T-1)}, u_{k \to k + \nu}^{(1:T-1)}) \alpha_{\text{NRJ2}}((k, x_k), (k + \nu, y_{k + \nu}^{(T-1)})) du_{k \to k + \nu} y_{k + \nu}^{(1:T-1)} u_{k \to k + \nu}^{(1:T-1)}.$$
\[-\int h(k + v, u_{k\rightarrow k + v}, v) \pi(u_{k\rightarrow k + v} \mid k + v) \left( 1 \wedge \frac{\pi(k + v)}{\pi(k)} \right) d u_{k\rightarrow k + v}\]
\[+ \left| h(k, x_k, -v) \int q(u_{k\rightarrow k + v}^{(0)}, y_{k \rightarrow k + v}^{(1:T-1)}, u_{k\rightarrow k + v}^{(1:T-1)})(1 - \alpha_{\text{NRJ}2}(Y_{k \rightarrow k + v}^{(T)}))(1 - \alpha_{\text{NRJ}2}(k, x_k, (k + v, Y_{k \rightarrow k + v}^{(T)}))) d(u_{k\rightarrow k + v}^{(0)}, y_{k \rightarrow k + v}^{(1:T-1)}, u_{k\rightarrow k + v}^{(1:T-1)}) - h(k, x_k, -v) \left( 1 \wedge \frac{\pi(k + v)}{\pi(k)} \right) \right|.\]  

(18)

We analyse the first absolute value on the RHS. We write the integrals as (conditional) expectations (given \((k, x_k, v)\)):

\[\left| \mathbb{E} \left[ h(k + v, Y_{k \rightarrow k + v}^{(T-1)}, v) \alpha_{\text{NRJ}2}(k, x_k, (k + v, Y_{k \rightarrow k + v}^{(T-1)})) \right] - \mathbb{E} \left[ h(k + v, U_{k\rightarrow k + v}, v) \left( 1 \wedge \frac{\pi(k + v)}{\pi(k)} \right) \right] \right| \]
\[\leq \left| \mathbb{E} \left[ h(k + v, Y_{k \rightarrow k + v}^{(T-1)}, v) \alpha_{\text{NRJ}2}(k, x_k, (k + v, Y_{k \rightarrow k + v}^{(T-1)})) \right] - \mathbb{E} \left[ h(k + v, Y_{k \rightarrow k + v}^{(T-1)}, v) \left( 1 \wedge \frac{\pi(k + v)}{\pi(k)} \right) \right] \right|
\[+ \left( 1 \wedge \frac{\pi(k + v)}{\pi(k)} \right) \mathbb{E} \left[ h(k + v, Y_{k \rightarrow k + v}^{(T-1)}, v) \right] - \left( 1 \wedge \frac{\pi(k + v)}{\pi(k)} \right) \mathbb{E} \left[ h(k + v, U_{k\rightarrow k + v}, v) \right] \right|, \]

using again the triangle inequality. We now show that both absolute values on the RHS converge towards 0. For the first one, we have

\[\left| \mathbb{E} \left[ h(k + v, Y_{k \rightarrow k + v}^{(T-1)}, v) \alpha_{\text{NRJ}2}(k, x_k, (k + v, Y_{k \rightarrow k + v}^{(T-1)})) \right] - \mathbb{E} \left[ h(k + v, Y_{k \rightarrow k + v}^{(T-1)}, v) \left( 1 \wedge \frac{\pi(k + v)}{\pi(k)} \right) \right] \right| \leq M \mathbb{E} \left| \alpha_{\text{NRJ}2}(k, x_k, (k + v, Y_{k \rightarrow k + v}^{(T-1)})) - \left( 1 \wedge \frac{\pi(k + v)}{\pi(k)} \right) \right| \rightarrow 0, \]

using that there exists a positive constant \(M\) such that \(|h| \leq M\) and that \(r_{\text{NRJ}2}(k, x_k, (k + v, Y_{k \rightarrow k + v}^{(T-1)})) \rightarrow \pi(k + v)/\pi(k)\) in distribution (by assumption). The convergence of the expectation follows from the fact that if a random variable \(X_n\) converges towards a constant \(c\) in distribution, then \(X_n - c\) converges towards 0 in probability and \(\mathbb{E}[g(X_n) - g(c)] \rightarrow 0\) for any bounded uniformly continuous function \(g\) (\(\min(1, x)\) with \(x \geq 0\) is a function having these characteristics). For the second absolute value, we have

\[\left( 1 \wedge \frac{\pi(k + v)}{\pi(k)} \right) \mathbb{E} \left[ h(k + v, Y_{k \rightarrow k + v}^{(T-1)}, v) - \mathbb{E} \left[ h(k + v, U_{k\rightarrow k + v}, v) \right] \right] \leq \mathbb{E} \left[ h(k + v, Y_{k \rightarrow k + v}^{(T-1)}, v) - \mathbb{E} \left[ h(k + v, U_{k\rightarrow k + v}, v) \right] \right] \rightarrow 0, \]

if the (conditional) distribution of \(Y_{k \rightarrow k + v}^{(T-1)}\) (given \((k, x_k, v)\)) converges towards \(\pi(\cdot \mid k + v)\) given that \(h\) is a bounded continuous function.

Let us now prove this convergence in distribution. The conditional distribution of \(Y_{k \rightarrow k + v}^{(T-1)}\) given \((k, x_k, v)\) is written as

\[\mathbb{P}(Y_{k \rightarrow k + v}^{(T-1)} \in A \mid k, x_k, v) \]
\[:= \int_{(T-1) \in A} q_{k\rightarrow k + v}(u_{k\rightarrow k + v}^{(0)}, y_{k \rightarrow k + v}^{(1:T-1)}, u_{k \rightarrow k + v}^{(1:T-1)}) \prod_{t=1}^{T-1} K_{k \rightarrow k + v}^{(0)}((y_{k \rightarrow k + v}^{(t-1)}, u_{k \rightarrow k + v}^{(t-1)}), (y_{k \rightarrow k + v}^{(t)}, u_{k \rightarrow k + v}^{(t)})) d(u_{k\rightarrow k + v}^{(0)}, y_{k \rightarrow k + v}^{(1:T-1)}, u_{k \rightarrow k + v}^{(1:T-1)}) \]

\[= \int_{(T-1) \in A} q_{k\rightarrow k + v}(u_{k\rightarrow k + v}^{(0)}, y_{k \rightarrow k + v}^{(1:T-1)}, u_{k \rightarrow k + v}^{(1:T-1)}) \prod_{t=1}^{T'-1} K_{k \rightarrow k + v}^{(0)}((y_{k \rightarrow k + v}^{(t-1)}, u_{k \rightarrow k + v}^{(t-1)}), (y_{k \rightarrow k + v}^{(t)}, u_{k \rightarrow k + v}^{(t)})) \]



Under Assumption 3, one can show that \( t^* \) and \( T \) can be chosen such that \( (T - t^*)/T \) is small and

\[
\left| K_{k \to k^+}^{(t)}((y_{k^+}^{(t-1)}, u_{k^+ y \to k}^{(t-1)}), (y_{k^+ y \to k}^{(t)}, u_{k^+ y \to k}^{(t)})) - K_{k \to k^+}^{(T)}((y_{k^+}^{(t-1)}, u_{k^+ y \to k}^{(t-1)}), (y_{k^+ y \to k}^{(T)}, u_{k^+ y \to k}^{(T)})) \right| < \frac{1}{T - t^*} \epsilon,
\]

for all \( t \geq t^* \) and any \( \epsilon > 0 \), where \( K_{k \to k^+}^{(T)} \) is the MH kernel for which \( \rho_{k \to k^+}^{(T)} := \pi(\cdot \mid k + \nu) \otimes q_{k \to k^+} \) is used instead in the acceptance probability. One can thus show that

\[
\left| \prod_{t=t^*}^{T-1} K_{k \to k^+}^{(t)}((y_{k^+}^{(t-1)}, u_{k^+ y \to k}^{(t-1)}), (y_{k^+ y \to k}^{(t)}, u_{k^+ y \to k}^{(t)})) - \prod_{t=t^*}^{T-1} K_{k \to k^+}^{(T)}((y_{k^+}^{(t-1)}, u_{k^+ y \to k}^{(t-1)}), (y_{k^+ y \to k}^{(T)}, u_{k^+ y \to k}^{(T)})) \right| < \epsilon,
\]

and therefore,

\[
\left| q_{k \to k^+}(u_{k \to k^+}^{(0)}) \prod_{t=t^*}^{T-1} K_{k \to k^+}^{(t)}((y_{k^+}^{(t-1)}, u_{k^+ y \to k}^{(t-1)}), (y_{k^+ y \to k}^{(t)}, u_{k^+ y \to k}^{(t)})) - q_{k \to k^+}(u_{k \to k^+}^{(0)}) \prod_{t=t^*}^{T-1} K_{k \to k^+}^{(T)}((y_{k^+}^{(t-1)}, u_{k^+ y \to k}^{(t-1)}), (y_{k^+ y \to k}^{(T)}, u_{k^+ y \to k}^{(T)})) \right| < \epsilon.
\]

We have that the integral of the two functions in the absolute value converges towards 0 as well as a result of Scheffé’s lemma (see Scheffé (1947)):

\[
\left| \mathbb{P}(Y_{k^+}^{T-1} \in A \mid k, x_k, \nu) - \int_{y_{k^+}^{T-1} \in A} q_{k \to k^+}(u_{k \to k^+}^{(0)}) \prod_{t=t^*}^{T-1} K_{k \to k^+}^{(t)}((y_{k^+}^{(t-1)}, u_{k^+ y \to k}^{(t-1)}), (y_{k^+ y \to k}^{(t)}, u_{k^+ y \to k}^{(t)})) \right| < \epsilon. \tag{19}
\]

We also have that

\[
\int_{y_{k^+}^{T-1} \in A} \prod_{t=t^*}^{T-1} K_{k \to k^+}^{(T)}((y_{k^+}^{(t-1)}, u_{k^+ y \to k}^{(t-1)}), (y_{k^+ y \to k}^{(t)}, u_{k^+ y \to k}^{(t)})) d(y_{k^+}^{(T-1)}, u_{k^+ y \to k}^{(T-1)}) - \mathbb{P}_{\rho_{k \to k^+}^{(T)}}(Y_{k^+}^{T-1} \in A) \leq \left| \int_{y_{k^+}^{T-1} \in A} \prod_{t=t^*}^{T-1} K_{k \to k^+}^{(T)}((y_{k^+}^{(t-1)}, u_{k^+ y \to k}^{(t-1)}), (y_{k^+ y \to k}^{(t)}, u_{k^+ y \to k}^{(t)})) d(y_{k^+}^{(T-1)}, u_{k^+ y \to k}^{(T-1)}) - \mathbb{P}_{\rho_{k \to k^+}^{(T)}}(Y_{k^+}^{T-1} \in A) \right| \tag{20}
\]

where \( \mathbb{P}_{\rho_{k \to k^+}^{(T)}} \) is the probability measure using the density \( \rho_{k \to k^+}^{(T)} \). We choose \( t^* \) and \( T \) such that the absolute value above is smaller than \( \epsilon \) which does not depend on \( (y_{k^+}^{(T-1)}, u_{k^+ y \to k}^{(T-1)}) \). This is possible given that the time-homogeneous \( \pi(\cdot \mid k + \nu) \otimes q_{k \to k^+} \) is reversible Markov chain associated with the proposal distribution \( q_{k \to k^+}^{m}((y_{k^+}^{m}, u_{k^+ y \to k}^{m}))(m : m \in \mathbb{N}) \), is uniformly ergodic (by assumption). This yields the convergence of the (conditional) distribution of \( Y_{k^+}^{(T-1)} \) (given \( (k, x_k, \nu) \)) towards \( \pi(\cdot \mid k + \nu) \).
It is proved that the second absolute value in (18) converges towards 0 using the same arguments, which allows to establish the pointwise convergence $P_T h(k, x_k, \nu) \longrightarrow P_{\text{ideal}} h(k, x_k, \nu)$. The uniform convergence on each compact subset of the state-space follows from the uniform ergodicity of the MH kernels.

We now highlight what modifications and which additional technical conditions are required if geometric ergodicity is instead assumed. The absolute value on the RHS in (20) is in this case bounded above by $M(y_{k+v}^{(t'-1)}, u_{k+v-y-k}^{(t'-1)}) \rho^{t'-1}$, where $M(y_{k+v}^{(t'-1)}, u_{k+v-y-k}^{(t'-1)})$ is finite for all $(y_{k+v}^{(t'-1)}, u_{k+v-y-k}^{(t'-1)})$ and $\rho < 1$. If the following integral is finite

$$\int q_{k-v+y}(u_{k-v-y-k}^{(0)}) \prod_{t=1}^{t'-1} K_{k-v+y}^{(t)}((y_{k+y}^{(t-1)}, u_{k+y-y-k}^{(t-1)}), (y_{k+y}^{(t)}, u_{k+y-y-k}^{(t)})) M(y_{k+y}^{(t'-1)}, u_{k+y-y-k}^{(t'-1)}) d(u_{k-v+y}^{(0)}, y_{k+y}^{(t)}, u_{k+y-y-k}^{(t)}),$$

then we know that we have the same conclusion as above, i.e. we can choose $t'$ and $T$ such that the absolute value on the RHS in (20) is smaller than $\varepsilon$. That integral shall be finite when the process associated with the kernels $K_{k-v+y}^{(t)}$ do not reach states $(y_{k+y}^{(t'-1)}, u_{k+y-y-k}^{(t'-1)})$ such that $M(y_{k+y}^{(t'-1)}, u_{k+y-y-k}^{(t'-1)})$ is extremely large (or at least if it does, it is with small enough probability).

This condition thus suffices to show the pointwise convergence $P_T h(k, x_k, \nu) \longrightarrow P_{\text{ideal}} h(k, x_k, \nu)$. To establish the uniform convergence under geometric ergodicity, we use the same strategy as that applied to show (19). We can choose $t'$ and $T$ such that the first $t'$ steps (after having generated $u_{k-v+y}^{(0)}$) with density $\prod_{t=1}^{t'} K_{k-v+y}^{(t)}((y_{k+y}^{(t-1)}, u_{k+y-y-k}^{(t-1)}), (y_{k+y}^{(t)}, u_{k+y-y-k}^{(t)}))$ are essentially MH steps with an invariant distribution given by $\rho_{k-v+y} := \pi(\cdot | k) \times q_{k-v+y} \times |J_{T_{k-v+y}}|^{-1}$. This implies that

$$\left| \int \prod_{t=1}^{t'} K_{k-v+y}^{(t)}((y_{k+y}^{(t-1)}, u_{k+y-y-k}^{(t-1)}), (y_{k+y}^{(t)}, u_{k+y-y-k}^{(t)}) ) d(y_{k+y}^{(1:t')}, u_{k+y-y-k}^{(1:t')}) - \int \prod_{t=1}^{t'} K_{k-v+y}^{(t)}((y_{k+y}^{(t-1)}, u_{k+y-y-k}^{(t-1)}), (y_{k+y}^{(t)}, u_{k+y-y-k}^{(t)})) d(y_{k+y}^{(1:t')}, u_{k+y-y-k}^{(1:t')}) \right| < \varepsilon,$$

which in turns implies that

$$\left| \int \prod_{t=1}^{t'} K_{k-v+y}^{(t)}((y_{k+y}^{(t-1)}, u_{k+y-y-k}^{(t-1)}), (y_{k+y}^{(t)}, u_{k+y-y-k}^{(t)})) d(y_{k+y}^{(1:t')}, u_{k+y-y-k}^{(1:t')}) - \int \rho_{k-v+y}^{(t')} d(y_{k+y}^{(t')}, u_{k+y-y-k}^{(t')}) \right| < M_1(y_{k+y}^{(0)}, u_{k+y-y-k}^{(0)}) \rho_1^{t'},$$

where $M(y_{k+y}^{(0)}, u_{k+y-y-k}^{(0)})$ is finite for all $(y_{k+y}^{(0)}, u_{k+y-y-k}^{(0)})$ and $\rho_1 < 1$. The uniform convergence $P_T h \longrightarrow P_{\text{ideal}} h$ on each compact subset of the state-space as $T \longrightarrow \infty$ follows if

$$\int q_{k-v+y}(u_{k-v+y}) M_1(y_{k+v}^{(0)}, u_{k+v-y-k}^{(0)}) d(u_{k-v+y}^{(0)})$$

is finite and continuous in $x$ (recall that $x$ and $u_{k-v+y}$ are mapped to $(y_{k+v}^{(0)}, u_{k+v-y-k}^{(0)})$ using $T_{k-v+y}$).

8 Supplementary material

We present in Section 8.1 weak convergence results for the ideal samplers as the size of the state-space increases.
8.1 Weak convergence results for the ideal samplers

We analyse the asymptotic scenario in which the number of models grows to infinity. It will be noticed that the reversible and non-reversible Markov chains produced respectively by ideal RJ and NRJ have two distinct asymptotic behaviours which are consistent with what is observed for fixed numbers of models (see, e.g., Figure 1), explaining their different state-space exploration speed.

We prove convergence towards continuous-time stochastic processes that take values on the real line. We thus need to consider functions of $K$ to achieve that. Firstly, we consider that the model indicator $K$ takes values in $\mathcal{K}^n := \{1, \ldots, \lfloor \sqrt{n} \log n \rfloor \}$, where $\lfloor \cdot \rfloor$ is the floor function. We added the superscript $n$ to highlight the dependence on this variable. We select $\mathcal{K}^n$ in this way to obtain a random variable $S^n_K := (K^n - \psi(n))/\sqrt{n}$ that is (in the limit) continuous in addition to taking values on the real line, for a given function $\psi$ (which can be thought of as the mean that can be for instance $\lfloor \sqrt{n} \log n \rfloor / 2$). Imagine that the mode is around $\lfloor \sqrt{n} \log n \rfloor / 2$ (so the mass is moving towards infinity), this transformation puts the mass around 0 and makes the different values of the centred variable $(-1, 0, 1$ and so on) close to each other (e.g. $|1 - 0|/\sqrt{n} \to 0$). We squeeze the state-space as in the proof of existence of Brownian motion from random walks. We assume that $\pi^n(k) > 0$ for all $k \in \mathcal{K}^n$. For $t \geq 0$, we define the following rescaled stochastic process:

$$Z^n_{RJ}(t) := \frac{K^n_{RJ}([nt]) - \psi(n)}{\sqrt{n}},$$

where $\{K^n_{RJ}(m) : m \in \mathbb{N}\}$ is a Markov chain produced by the ideal RJ corresponding to the ideal NRJ described in Section 2.2. We consider that this RJ updates parameters and switches models with probabilities $\tau$ and $1 - \tau$, respectively, and that $g(k, k + 1) = g(k, k - 1) = 0.5(1 - \tau)$, so it proposes to increase or decrease the model indicator with the same probability and $\alpha_{RJ} = \alpha_{NRJ}$. The continuous-time stochastic process $\{Z^n_{RJ}(t) : t \geq 0\}$ is a sped up and modified version of $\{K^n_{RJ}(m) : m \in \mathbb{N}\}$. The decreasing size of the jumps of $\{Z^n_{RJ}(t) : t \geq 0\}$ as $n$ increases (the size is $1/\sqrt{n}$), combined with its time acceleration, result in a continuous and non-trivial limiting process, as specified in Theorem 2. This time acceleration can be thought of as squeezing the time axis to make the iterations close to each other, again as in the proof of existence of Brownian motion from random walks.

**Theorem 2** (Weak convergence of RJ). Assume that:

(a) the function $\psi$ can be chosen such that $S^n_K$ is asymptotically distributed as $f_S \in C^1(\mathbb{R})$, a strictly positive probability density function (PDF), where $C^1(\mathbb{R})$ denotes the space of real-valued functions on $\mathbb{R}$ with continuous first derivative;

(b) the function $(\log f_S(\cdot))'$ is Lipschitz continuous;

(c) $\psi$ can be chosen such that

$$\frac{1}{\pi^n(k)} \frac{\pi^n(k + 1) - \pi^n(k)}{1/\sqrt{n}} - (\log f_S(S^n_K))' \tag{21}$$

is bounded for all $n$ and converges towards 0 as $n \to \infty$, for all $k$;

(d) $\lim_{n \to \infty} \sqrt{n} \pi^n(1) = \lim_{n \to \infty} \sqrt{n} \pi^n(\lfloor \sqrt{n} \log n \rfloor) = 0$.

If $K^n_{RJ}(0) \sim \pi^n$, then $\{Z^n_{RJ}(t) : t \geq 0\}$ converges weakly towards a Langevin diffusion as $n \to \infty$, i.e.

$$\{Z^n_{RJ}(t) : t \geq 0\} \Rightarrow \{Z_{RJ}(t) : t \geq 0\} \text{ as } n \to \infty,$$
where the process $\{Z_{RJ}(t) : t \geq 0\}$ is such that $Z_{RJ}(0) \sim f_S$ and
\[
dZ_{RJ}(t) = \frac{1 - \tau}{2} (\log f_S(Z_{RJ}(t)))' dt + \sqrt{1 - \tau} dB(t),
\]
with $\{B(t) : t \geq 0\}$ being a Wiener process.

**Proof.** It is a straightforward adaptation of Theorem 1 in Gagnon et al. (2019). For sake of completeness, it is detailed in Section 8.2. \qed

The notation “$\Rightarrow$” represents here weak convergence of processes in the Skorokhod topology (see Section 3 of Ethier and Kurtz (1986) for more details about this type of convergence).

The two main assumptions are (a) and (c). The former requires to find a transformation of $K^n$ such that the limit in distribution of the transformed random variable is a continuous random variable with density $f_S$. The latter requires that the “discrete version” of the derivative of $\log \pi^n$ share the same asymptotic behaviour as the derivative of $\log f_S$. Indeed, in Gagnon et al. (2019), it is explained that the left term in (21) can be seen as the discrete version of the derivative of $\log \pi^n$ because $\pi^n$ is also the PMF of $S^n_k$ (evaluated at a different point) and $S^n_{k+1} - S^n_k = 1/\sqrt{n}$. Assumption (b) is standard in the weak convergence literature; it ensures the existence of a unique strong solution to the stochastic differential equation given above. Assumption (d) is a regularity condition. In Gagnon et al. (2019), to illustrate how a PMF that satisfies the conditions looks like, the authors show one that is such that $S^n_k$ converges in distribution towards a standard normal.

We now analyse the behaviour of the stochastic process produced by the ideal NRJ algorithm. We consider as before that $K^n = \{1, \ldots, \lfloor \sqrt{n} \log n\rfloor\}$ and $\pi^n(k) > 0$ for all $k \in K^n$. For $t \geq 0$, we define the following rescaled stochastic process:
\[
Z^\nu_{NRJ}(t) := \left(\frac{K^n_{NRJ}(\lfloor \sqrt{n}t\rfloor) - \psi(n)}{\sqrt{n}}, \nu(\lfloor \sqrt{n}t\rfloor)\right),
\]
where $\{(K^n_{NRJ}, \nu)(m) : m \in \mathbb{N}\}$ is a Markov chain produced by ideal NRJ described in Section 2.2. Note that the distribution of $\nu$ does not change with $n$.

**Theorem 3** (Weak convergence of NRJ). Assume that the same conditions (a)-(d) as in Theorem 2 are satisfied. Assume additionally that there exist two positive constants $c$ and $x_0$ such that $|\log f_S(x)|' \geq c$ for all $|x| \geq x_0$. If $(K^n_{NRJ}, \nu)(0) \sim \pi^n \otimes \mathcal{U}\{-1, 1\}$, then $\{Z^\nu_{NRJ}(t) : t \geq 0\}$ converges weakly towards a piecewise deterministic Markov process (PDMP) as $n \to \infty$, i.e.
\[
\{Z^n_{NRJ}(t) : t \geq 0\} \Rightarrow \{Z_{NRJ}(t) : t \geq 0\} \text{ as } n \to \infty,
\]
where the process $\{Z_{NRJ}(t) : t \geq 0\}$ is such that $Z_{NRJ}(0) \sim f_S \otimes \mathcal{U}\{-1, 1\}$ with generator
\[
Gh(x, y) := (1 - \tau)yh_x(x, y) + \max\{0, -y (\log f_S(x))'(1 - \tau)(h(x, -y) - h(x, y)),
\]
where $h(\cdot, y) \in C^1(\mathbb{R})$ and such that itself and $h_x(\cdot, y)$ vanish at infinity, for $y \in \{-1, 1\}$, $h_x$ denoting the first derivative of $h$ with respect to its first argument.

**Proof.** See Section 8.2. \qed
The additional regularity condition on \( f_S \) in Theorem 3 essentially ensures that outside of a bounded set, this PDF decreases sufficiently quickly. Indeed, given that \((\log f_S(x))' = f_S'(x)/f_S(x)\) and \( f_S \) is strictly positive, it is required that the tail decay is bounded from below (relatively to \( f_S \)). This guarantees that the limiting PDMP has some important properties (e.g. non-explosiveness and \( f_S \otimes \mathcal{U}[-1, 1] \) is an invariant distribution, see Bierkens and Roberts (2017)).

The PDMP in Theorem 3 corresponds to a zig-zag Markov process (Bierkens et al. (2019)), and in fact, a bouncy particle sampler (BPS, Bouchard-Côté et al. (2018)) given that they both coincide when the position variable is unidimensional. This position variable evolves with constant drift \( 1 - \tau \) either to the right or left of the real line depending on the direction variable, and changes direction with rate \( \max(0, -y(\log f_S(x))')(1 - \tau) \) when the position is \( x \) and direction \( y \). PDMP are known for being non-diffusive and having persistency-driven paths. We constructed NRJ to induce such a behaviour, but we do not know \textit{a priori} when this will happen and how this will translate. An analysis was conducted in Section 4 to provide some answers. Theorem 3 and Theorem 2 indicate that in the (asymptotic) theoretical framework considered, the model indicator’s paths produced by RJ and NRJ behave exactly as expected; the former show diffusive patterns and the latter not. This suggests that (at least under those conditions) NRJ outperform RJ. We even have a guarantee for the speed of convergence towards the target distribution for NRJ: Bierkens and Roberts (2017) prove that the PDMP in Theorem 3 is exponentially ergodic. We additionally know that the convergence is an order of magnitude slower for \( \{K_{RJ}^n(m) : m \in \mathbb{N}\} \). Indeed, the different behaviour of \( \{K_{NRJ}^n(m) : m \in \mathbb{N}\} \) compared with \( \{K_{RJ}^n(m) : m \in \mathbb{N}\} \) requires to accelerate the time by a factor of only \( \sqrt{n} \) in the definition of \( \{Z_{NRJ}^n(t) : t \geq 0\} \) comparatively to \( n \) in that of \( \{Z_{RJ}^n(t) : t \geq 0\} \) to obtain non-trivial limiting stochastic processes. This highlights again that \( \{K_{NRJ}^n(m) : m \in \mathbb{N}\} \) explores its state-space more quickly.

### 8.2 Proofs of Theorems 2 and 3

**Proof of Theorem 2.** In order to prove the result, we demonstrate the convergence of the finite-dimensional distributions of \( \{Z_{RJ}^n(t) : t \geq 0\} \) to those of \( \{Z_{RJ}^n(t) : t \geq 0\} \). To achieve this, we verify Condition (c) of Theorem 8.2 from chapter 4 of Ethier and Kurtz (1986). The weak convergence then follows from Corollary 8.6 of Chapter 4 of Ethier and Kurtz (1986). The remaining conditions of Theorem 8.2 and the conditions specified in Corollary 8.6 are either straightforward or easily derived from the proof given here.

The proof of the convergence of the finite-dimensional distributions relies on the convergence of (what we call) the “pseudo-generator”, a quantity that we define as:

\[
\xi_{RJ}^n(t) := n \mathbb{E}[h(Z_{RJ}^n(t + 1/n)) - h(Z_{RJ}^n(t)) \mid \mathcal{F}^{\overline{Z}_{RJ}^n}(t)],
\]

where \( h \in C^\infty_c(\mathbb{R}) \), the space of infinitely differentiable functions on \( \mathbb{R} \) with compact support. Theorem 2.1 from Chapter 8 of Ethier and Kurtz (1986) allows us to restrict our attention to this set of functions when studying the limiting behaviour of the pseudo-generator. In our situation, the pseudo-generator has a more precise expression:

\[
\xi_{RJ}^n(t) = \frac{n(1 - \tau)}{2} \left( h(S_{RJ}^n_{K+1}) - h(S_{RJ}^n_K) \right) \left( 1 \wedge \frac{\pi^n(K^n + 1)}{\pi^n(K^n)} \right)
+ \frac{n(1 - \tau)}{2} \left( h(S_{RJ}^n_{K-1}) - h(S_{RJ}^n_K) \right) \left( 1 \wedge \frac{\pi^n(K^n - 1)}{\pi^n(K^n)} \right),
\]

(23)

Note that the Markov process \( \{K_{RJ}^n(m) : m \in \mathbb{N}\} \) is time-homogeneous, and because of this we replaced the random variable \( Z_{RJ}^n(t) \) by \( S_{RJ}^n(t + 1/n) \) by \( S_{RJ}^n_{K+1} \) or \( S_{RJ}^n_{K-1} \) given that we will work under ex-
where $G$ is the generator of the limiting diffusion with

$$Gh(Z^n_{R^f}(t)) := \frac{1 - \tau}{2} (\log f_\beta(Z^n_{R^f}(t)))'h'(Z^n_{R^f}(t)) + \frac{1 - \tau}{2} h''(Z^n_{R^f}(t)).$$

Note that there exists a positive constant $M$ such that $h$ and all its derivatives are bounded in absolute value by this constant. We choose $M$ such that it is a Lipschitz constant for the function $(\log f_\beta(\cdot))'$. The key here is to use Taylor expansions in (23) to obtain derivatives of $h$ as in $G$. By noting that $S^n_{k+1} = S^n_k + 1/\sqrt{n}$ and $S^n_{k-1} = S^n_k - 1/\sqrt{n}$, and using Taylor expansions of $h$ around $S^n_k$, we obtain

$$h(S^n_k + 1/\sqrt{n}) - h(S^n_k) = \frac{1}{\sqrt{n}} h'(S^n_k) + \frac{1}{2n} h''(S^n_k) + \frac{1}{6n^{3/2}} h'''(W),$$

$$h(S^n_k - 1/\sqrt{n}) - h(S^n_k) = -\frac{1}{\sqrt{n}} h'(S^n_k) + \frac{1}{2n} h''(S^n_k) - \frac{1}{6n^{3/2}} h'''(T),$$

where $W$ and $T$ belong to $(S^n_k, S^n_k + 1/\sqrt{n})$ and $(S^n_k - 1/\sqrt{n}, S^n_k)$, respectively. We also note that the first term on the RHS of (23) equals 0 when $K^n = [\sqrt{n} \log n]$ because $\pi^n([\sqrt{n} \log n] + 1) = 0$. For the analogous reason, the second term on the RHS of (23) equals 0 when $K^n = 1$. Therefore,

$$\mathbb{E} \left[ |e^n_R(t) - Gh(Z^n_{R^f}(t))| \right] \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,$$

where $G$ is the generator of the limiting diffusion with

$$Gh(Z^n_{R^f}(t)) := \frac{1 - \tau}{2} (\log f_\beta(Z^n_{R^f}(t)))'h'(Z^n_{R^f}(t)) + \frac{1 - \tau}{2} h''(Z^n_{R^f}(t)).$$

We now prove that expectation of the absolute value of each term on the RHS in (24) converges towards 0 as $n \rightarrow \infty$. We start with the last terms and make our way up. It is clear that the expectation
of the absolute value of each of the last two terms converges towards 0 as \( n \to \infty \) given that \(|h''| \leq M\) and \(0 \leq 1 \wedge x \leq 1\) for positive \( x \). We now analyse the fourth one (starting from the bottom). As \( n \to \infty \),

\[
\mathbb{E} \left[ \mathbb{I}(K^n = 1) \frac{1 - \tau}{2} h''(S^n_k) \left(1 \wedge \frac{\pi^n(K^n + 1)}{\pi^n(K^n)} - 2\right) \right] \leq \frac{(1 - \tau)M}{2} \mathbb{P}(K^n = 1) \to 0,
\]

using \(|h''| \leq M\) and

\[
0 \leq 1 \wedge \frac{\pi^n(K^n + 1)}{\pi^n(K^n)} - 2 \leq 2.
\]

Recall that \( \mathbb{P}(K^n = 1) \to 0 \) by assumption. The proof for the third term (starting from the bottom) is similar.

Applying Lemmas 1 to 3 (that follow), each of the remaining terms is seen to converge towards 0 in \( L^1 \) as \( n \to \infty \), which concludes the proof.

\[\text{Lemma 1. As } n \to \infty, \text{ we have}\]

\[
\mathbb{E} \left[ \mathbb{I}(2 \leq K^n \leq \lfloor \sqrt{n} \log n \rfloor - 1) \frac{1 - \tau}{4} h''(S^n_k) \left(1 \wedge \frac{\pi^n(K^n + 1)}{\pi^n(K^n)} + 1 \wedge \frac{\pi^n(K^n - 1)}{\pi^n(K^n)} - 2\right) \right] \to 0.
\]

\[\text{Proof. We have}\]

\[
\mathbb{E} \left[ \mathbb{I}(2 \leq K^n \leq \lfloor \sqrt{n} \log n \rfloor - 1) \frac{1 - \tau}{4} h''(S^n_k) \left(1 \wedge \frac{\pi^n(K^n + 1)}{\pi^n(K^n)} + 1 \wedge \frac{\pi^n(K^n - 1)}{\pi^n(K^n)} - 2\right) \right] \\ \leq \frac{(1 - \tau)M}{4} \mathbb{E} \left[ \mathbb{I}(2 \leq K^n \leq \lfloor \sqrt{n} \log n \rfloor - 1) \left(1 \wedge \frac{\pi^n(K^n + 1)}{\pi^n(K^n)} + 1 \wedge \frac{\pi^n(K^n - 1)}{\pi^n(K^n)} - 2\right) \right],
\]

because \(|h''| \leq M\). We show that

\[
\frac{\pi^n(k + 1)}{\pi^n(k)} \to 1 \quad \text{for all } k \in \{1, \ldots, \lfloor \sqrt{n} \log n \rfloor - 1\},
\]

which allows to conclude using the triangle inequality, the continuity of the function \(1 \wedge x\), and the Lebesgue’s dominated convergence theorem. We have

\[
\left| \frac{\pi^n(k + 1)}{\pi^n(k)} - 1 \right| = \left| \frac{1}{\pi^n(k)} \frac{\pi^n(k + 1) - \pi^n(k)}{1/\sqrt{n}} - (\log f_S(S^n_k))' + (\log f_S(S^n_k))' \right| \frac{1}{\sqrt{n}} \\ \leq \left| \frac{1}{\pi^n(k)} \frac{\pi^n(k + 1) - \pi^n(k)}{1/\sqrt{n}} - (\log f_S(S^n_k))' \right| \frac{1}{\sqrt{n}} \\ + \left| (\log f_S(S^n_k))' \right| \frac{1}{\sqrt{n}},
\]

using again the triangle inequality. By assumption, we have that

\[
\left| \frac{1}{\pi^n(k)} \frac{\pi^n(k + 1) - \pi^n(k)}{1/\sqrt{n}} - (\log f_S(S^n_k))' \right| \frac{1}{\sqrt{n}} \to 0.
\]

We also have that

\[
\left| (\log f(S^n_k))' \right| \frac{1}{\sqrt{n}} = \left| (\log f_S(S^n_k))' - (\log f_S(0))' + (\log f_S(0))' \right| \frac{1}{\sqrt{n}}.
\]
Proof. We have that $|\frac{k - \psi(n)}{\sqrt{n}}| \leq 2 \sqrt{n} \log n \to 0$, using the triangle inequality and the fact that $k, \psi(n) \leq \sqrt{n} \log n$. 

Lemma 2. As $n \to \infty$, we have

$$
\mathbb{E} \left[ \left| \mathbb{I}(K^n = 1) \frac{1 - \tau}{2} h'(S^n_K) \left( \sqrt{n} \left( 1 \wedge \frac{\pi^n(K^n + 1)}{\pi^n(K^n)} \right) - (\log f_S(S^n_K))^\tau \right) \right| \right] \to 0,
$$

and

$$
\mathbb{E} \left[ \left| \mathbb{I}(K^n = \lfloor \sqrt{n} \log n \rfloor) \frac{1 - \tau}{2} h'(S^n_K) \left( \sqrt{n} \left( 1 \wedge \frac{\pi^n(K^n - 1)}{\pi^n(K^n)} \right) - (\log f_S(S^n_K))^\tau \right) \right| \right] \to 0.
$$

Proof. We have that

$$
\mathbb{E} \left[ \left| \mathbb{I}(K^n = 1) \frac{1 - \tau}{2} h'(S^n_K) \left( \sqrt{n} \left( 1 \wedge \frac{\pi^n(K^n + 1)}{\pi^n(K^n)} \right) - (\log f_S(S^n_K))^\tau \right) \right| \right] \leq \frac{(1 - \tau)M}{2} \mathbb{E} \left[ \mathbb{I}(K^n = 1) \sqrt{n} \left( 1 \wedge \frac{\pi^n(K^n + 1)}{\pi^n(K^n)} \right) \right] + \frac{(1 - \tau)M}{2} \mathbb{E} \left[ \left| \mathbb{I}(K^n = 1) (\log f_S(S^n_K))^\tau \right| \right],
$$

using that $|h'| \leq M$ and the triangle inequality. The first term on the RHS converges towards 0 by assumption because $0 \leq 1 \wedge x \leq 1$ for positive $x$. Using the same mathematical arguments as in the proof of Lemma 1, we have that

$$
|\log f_S(S^n_K)| \leq 2M \sqrt{n} \log n + |\log f_S(0)|.
$$

Therefore, using the triangle inequality

$$
\mathbb{E} \left[ \mathbb{I}(K^n = 1) \left| (\log f_S(S^n_K))^\tau \right| \right] \leq \mathbb{P}(K^n = 1) \left( 2M \frac{\sqrt{n} \log n}{\sqrt{n}} + |\log f_S(0)| \right) \to 0,
$$

by assumption (and because $f_S \in C^1(\mathbb{R})$). The proof that

$$
\mathbb{E} \left[ \left| \mathbb{I}(K^n = \lfloor \sqrt{n} \log n \rfloor) \frac{1 - \tau}{2} h'(S^n_K) \left( \sqrt{n} \left( 1 \wedge \frac{\pi^n(K^n - 1)}{\pi^n(K^n)} \right) - (\log f_S(S^n_K))^\tau \right) \right| \right] \to 0
$$

is similar.
Lemma 3. As $n \to \infty$, we have

$$
\mathbb{E} \left[ \left| \mathbb{I}(2 \leq K^n \leq \lceil \sqrt{n} \log n \rceil - 1) \frac{1 - \tau}{2} h'(S^n_K) \right| \times \left( \sqrt{n} \left( 1 - \frac{\pi^n(K^n + 1)}{\pi^n(K^n)} \right) - \frac{\pi^n(K^n - 1)}{\pi^n(K^n)} \right) - (\log f_S(S^n_K))' \right] \to 0.
$$

Proof. First, we have that

$$
\mathbb{E} \left[ \left| \mathbb{I}(2 \leq K^n \leq \lceil \sqrt{n} \log n \rceil - 1) \frac{1 - \tau}{2} h'(S^n_K) \right| \times \left( \sqrt{n} \left( 1 - \frac{\pi^n(K^n + 1)}{\pi^n(K^n)} \right) - \frac{\pi^n(K^n - 1)}{\pi^n(K^n)} \right) - (\log f_S(S^n_K))' \right] \leq \frac{(1 - \tau)M}{2} \mathbb{E} \left[ \left| \mathbb{I}(2 \leq K^n \leq \lceil \sqrt{n} \log n \rceil - 1) \frac{1 - \tau}{2} h'(S^n_K) \right| \times \left( \sqrt{n} \left( 1 - \frac{\pi^n(K^n + 1)}{\pi^n(K^n)} \right) - \frac{\pi^n(K^n - 1)}{\pi^n(K^n)} \right) - (\log f_S(S^n_K))' \right],
$$

because $|h'| \leq M$. We now consider four cases for $K^n$:

1. $\pi^n(K^n + 1)/\pi^n(K^n) < 1$ and $\pi^n(K^n - 1)/\pi^n(K^n) \geq 1$,
2. $\pi^n(K^n + 1)/\pi^n(K^n) \geq 1$ and $\pi^n(K^n - 1)/\pi^n(K^n) < 1$,
3. $\pi^n(K^n + 1)/\pi^n(K^n) \geq 1$ and $\pi^n(K^n - 1)/\pi^n(K^n) \geq 1$,
4. $\pi^n(K^n + 1)/\pi^n(K^n) < 1$ and $\pi^n(K^n - 1)/\pi^n(K^n) < 1$.

In Case 1, we have that

$$
\sqrt{n} \left( 1 - \frac{\pi^n(K^n + 1)}{\pi^n(K^n)} \right) - \frac{\pi^n(K^n - 1)}{\pi^n(K^n)} - (\log f_S(S^n_K))' = \sqrt{n} \left( \frac{\pi^n(K^n + 1)}{\pi^n(K^n)} - 1 \right) - (\log f_S(S^n_K))' = \frac{1}{\pi^n(K^n)} \frac{\pi^n(K^n + 1) - \pi^n(K^n)}{1/\sqrt{n}} - (\log f_S(S^n_K))' \to 0,
$$

by assumption. We can prove that it converges towards 0 in Case 2 in the same way. Case 3 corresponds to a local minimum. In this case,

$$
\sqrt{n} \left( 1 - \frac{\pi^n(K^n + 1)}{\pi^n(K^n)} \right) - \frac{\pi^n(K^n - 1)}{\pi^n(K^n)} - (\log f_S(S^n_K))' = 0,
$$

for all $n$, and $(\log f_S(S^n_K))' = f'_Z(S^n_K)/f_S(S^n_K) \to 0$. Case 4 corresponds to a local (or global) maximum. Again, $(\log f_S(S^n_K))' \to 0$. Additionally,

$$
\sqrt{n} \left( 1 - \frac{\pi^n(K^n + 1)}{\pi^n(K^n)} \right) - \frac{\pi^n(K^n - 1)}{\pi^n(K^n)} - (\log f_S(S^n_K))' = \sqrt{n} \left( \frac{\pi^n(K^n + 1) - \pi^n(K^n)}{\pi^n(K^n)} - \frac{\pi^n(K^n - 1) - \pi^n(K^n)}{\pi^n(K^n)} \right) - (\log f_S(S^n_K))'.
$$
but both terms converge towards 0. Consequently, Lebesgue’s dominated convergence theorem allows to conclude the proof.

**Proof of Theorem 3.** Analogously to the proof of Theorem 2, we demonstrate the convergence of the finite-dimensional distributions of \( \{Z_{\text{NRJ}}^n(t) : t \geq 0 \} \) to those of \( \{Z_{\text{NRJ}}^n(t) : t \geq 0 \} \). The same strategy as in that proof is employed: we verify Condition (c) of Theorem 8.2 from chapter 4 of Ethier and Kurtz (1986). The weak convergence then follows from Corollary 8.6 of Chapter 4 of Ethier and Kurtz (1986). The remaining conditions of Theorem 8.2 and the conditions specified in Corollary 8.6 are satisfied. In that paper, it is proved that it implies that the PDMP defined in Theorem 3 is a non-explosive strong Markov process. The authors also demonstrate that the Markov transition semigroup to which the generator corresponds is Feller.

Beforehand, we note that the additional assumption on \( f_S \) (about the lower bound on \(|(\log f_S(.))^\prime|\) outside of a bounded set) implies that Assumption 3 in Section 5 of Bierkens and Roberts (2017) is satisfied. In that paper, it is proved that it implies that the PDMP defined in Theorem 3 is a non-explosive strong Markov process. The authors also demonstrate that the Markov transition semigroup to which the generator corresponds is Feller.

For this proof, the time acceleration factor is different, and accordingly, the pseudo-generator is defined as:

\[
\tilde{g}_{\text{NRJ}}^n(t) := \sqrt{n} \mathbb{E}[h(Z_{\text{NRJ}}^n(t + 1/ \sqrt{n})) - h(Z_{\text{NRJ}}^n(t)) \mid \mathcal{F}_{\text{NRJ}}^n(t)]
\]

\[
= \sqrt{n}(1 - \tau)(h(S_{K,\nu}^n, \nu) - h(S_{K}^n, \nu)) \left(1 \wedge \frac{\pi^n(K^n + \nu)}{\pi^n(K^n)}\right) + \sqrt{n}(1 - \tau)(h(S_{K}^n, -\nu) - h(S_{K}^n, \nu)) \left(1 - 1 \wedge \frac{\pi^n(K^n + \nu)}{\pi^n(K^n)}\right).
\]

As in the proof of Theorem 2, we replaced \( Z_{\text{NRJ}}^n(t) \) by \( (S_{K,\nu}^n, \nu) \) and \( Z_{\text{NRJ}}^n(t + 1/ \sqrt{n}) \) by \( (S_{K,\nu}^n, \nu) \) or \( (S_{K}^n, -\nu) \) given that the Markov process \( \{(K_{\nu}^n, \nu)(m) : m \in \mathbb{N}\} \) is time-homogeneous and we will work under expectations. Recall that Condition (c) of Theorem 8.2 from chapter 4 of Ethier and Kurtz (1986) is essentially

\[
\mathbb{E} \left[ |\tilde{g}_{\text{NRJ}}^n(t) - Gh(Z_{\text{NRJ}}^n(t))| \right] \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty,
\]

where \( G \) is in this case the generator expressed in Theorem 3. We have that

\[
\mathbb{E} \left[ |\tilde{g}_{\text{NRJ}}^n(t) - Gh(Z_{\text{NRJ}}^n(t))| \right] \leq \mathbb{E} \left[ \sqrt{n}(1 - \tau)(h(S_{K,\nu}^n, \nu) - h(S_{K}^n, \nu)) \left(1 \wedge \frac{\pi^n(K^n + \nu)}{\pi^n(K^n)}\right) - (1 - \tau)\nu h_s(S_{K}^n, \nu) \right] + \mathbb{E} \left[ \sqrt{n}(1 - \tau)(h(S_{K}^n, -\nu) - h(S_{K}^n, \nu)) \left(1 - 1 \wedge \frac{\pi^n(K^n + \nu)}{\pi^n(K^n)}\right) - \max\{0, -\nu(\log f_S(S_{K}^n))^\prime(1 - \tau)(h(S_{K}^n, -\nu) - h(S_{K}^n, \nu))\} \right],
\]

using the triangle inequality. We analyse the two terms separately. We start with the first one. By the mean value theorem and using that \( S_{K,\nu}^n - S_{K}^n = \nu/ \sqrt{n} \), we have that

\[
h(S_{K,\nu}^n, \nu) - h(S_{K}^n, \nu) = \frac{\nu}{\sqrt{n}} h_s(T, \nu),
\]
where $T$ is in $(S_n^+, S_{K+}^+)$ or $(S_n^{-}, S_{K+}^+)$. We therefore also know that $T \to S_K^n$ with probability 1. In the proof of Lemma 1, it is shown that

$$1 \land \frac{\pi^n(K^n + \nu)}{\pi^n(K^n)} \to 1 \quad \text{with probability 1,}$$

and consequently,

$$\mathbb{E}\left[\sqrt{n}(1 - \tau)(h(S_K^n, \nu) - h(S_K^n, \nu))\left(1 \land \frac{\pi^n(K^n + \nu)}{\pi^n(K^n)}\right) - (1 - \tau)\nu h(S_K^n, \nu)\right] \to 0,$$

using Lebesgue’s dominated convergence theorem (given that the quantity in the expectation is bounded, because $h(\cdot, \nu)$ is bounded for $\nu \in [-1, 1]$). For the second term in (25), we have

$$\mathbb{E}\left[\sqrt{n}(1 - \tau)(h(S_K^n, -\nu) - h(S_K^n, \nu))\left(1 \land \frac{\pi^n(K^n + \nu)}{\pi^n(K^n)}\right) - \max\{0, -\nu (\log f_5(S_K^n)'(1 - \tau)(h(S_K^n, -\nu) - h(S_K^n, \nu))\right\} \leq (1 - \tau)2M\mathbb{E}\left[\sqrt{n}(1 - \land \frac{\pi^n(K^n + \nu)}{\pi^n(K^n)}) - \max\{0, -\nu (\log f_5(S_K^n)'\right\} \right], \quad (26)$$

because there exists a positive constant $M$ such that $|h(\cdot, \nu)| \leq M$ for $\nu \in [-1, 1]$ (recall that $h(\cdot, \nu)$ is continuous and vanishes at infinity for any value of $\nu$).

We now consider four cases for $K^n$ and $\nu$:

1. $\nu = +1$ and $\pi^n(K^n + \nu)/\pi^n(K^n) \geq 1$ (we are going to the right on the real line and in this direction the PMF increases),
2. $\nu = +1$ and $\pi^n(K^n + \nu)/\pi^n(K^n) < 1$ (we are going to the right on the real line and in this direction the PMF decreases),
3. $\nu = -1$ and $\pi^n(K^n + \nu)/\pi^n(K^n) \geq 1$ (we are going to the left on the real line and in this direction the PMF increases),
4. $\nu = -1$ and $\pi^n(K^n + \nu)/\pi^n(K^n) < 1$ (we are going to the left on the real line and in this direction the PMF decreases).

In Case 1,

$$\left(1 - 1 \land \frac{\pi^n(K^n + \nu)}{\pi^n(K^n)}\right) = 0,$$

for all $n$ and $-\nu (\log f_5(S_K^n)') = -(\log f_5(S_K^n)')$ is negative in the limit because $f_5(S_K^n)$ is positive in the limit. Therefore, $\max\{0, -\nu (\log f_5(S_K^n)')\} \to 0$. Using Lebesgue’s dominated convergence theorem, we thus know that the expectation at the RHS in (26) converges towards 0 when restricted to Case 1. We can prove that it converges towards 0 in Case 3 in the same way. In Case 2,

$$\sqrt{n}\left(1 - 1 \land \frac{\pi^n(K^n + \nu)}{\pi^n(K^n)}\right) = -\frac{1}{\pi^n(K^n)} \frac{\pi^n(K^n + \nu) - \pi^n(K^n)}{1/\sqrt{n}}.$$ 

By assumption, we know that this behaves asymptotically like $(\log f_5(S_K^n)')'$. We also know that $-\nu (\log f_5(S_K^n)')' = -(\log f_5(S_K^n)')'$ is positive in the limit because $f_5(S_K^n)$ is negative in the limit. Therefore, $\max\{0, -\nu (\log f_5(S_K^n)')\}$ behaves like $(\log f_5(S_K^n)')'$ in the limit. Using Lebesgue’s dominated convergence theorem, we thus know that the expectation at the RHS in (26) converges towards 0 when restricted to Case 2 (recall the assumed boundedness of the limiting quantity in the expectation). We can prove that it converges towards 0 in Case 4 in the same way. \hfill \blacksquare