NECESSITY OF WEAK SUBORDINATION FOR SOME STRONGLY SUBORDINATED LÉVY PROCESSES

BORIS BUCHMANN,* AND KEVIN W. LU,** Australian National University

Abstract

Consider the strong subordination of a multivariate Lévy process with a multivariate subordinator. If the subordinate is a stack of independent Lévy processes and the components of the subordinator are indistinguishable within each stack, then strong subordination produces a Lévy process; otherwise it may not. Weak subordination was introduced to extend strong subordination, always producing a Lévy process even when strong subordination does not. Here we prove that strong and weak subordination are equal in law under the aforementioned condition. In addition, we prove that if strong subordination is a Lévy process then it is necessarily equal in law to weak subordination in two cases: firstly when the subordinator is deterministic, and secondly when it is pure-jump with finite activity.

Keywords: Lévy process; subordinator; multivariate subordination; weak subordination; Poisson point process; Poisson random measure

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1. Introduction

Let \( X = (X_1, \ldots, X_n) \) be an \( n \)-dimensional Lévy process and let \( T = (T_1, \ldots, T_n) \) be an \( n \)-dimensional subordinator independent of \( X \). The operation that evaluates the process \( X \) at times given by the subordinator \( T \) is defined by

\[
X \circ T = (X_1(T_1(t)), \ldots, X_n(T_n(t)))_{t \geq 0}
\]

and known as strong subordination. This creates a ‘time-changed’ process. The study of the multivariate subordination of Lévy processes originated with the work of Barndorff-Nielsen, Pedersen, and Sato [1]. It is well known that strong subordination produces a Lévy process in the following cases.

(C1) \( T \) has indistinguishable components.

(C2) \( X \) has independent components.

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* Postal address: Research School of Finance, Actuarial Studies & Statistics, Australian National University, ACT 0200, Australia. Email address: boris.buchmann@anu.edu.au

** Current address: Department of Applied Mathematics, University of Washington, Seattle, WA 98195-3925, USA. Email address: kwlu@uw.edu

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Necessity of weak subordination for some strongly subordinated Lévy processes

(C3) $T$ and $X$ satisfy the stacked univariate subordination condition: for some $1 \leq d \leq n$ and $n_1 + \cdots + n_d = n$,

$$X = (Y_1, \ldots, Y_d), \quad T = (R_1e_1, \ldots, R_de_d),$$

where $Y_1, \ldots, Y_d$ are independent Lévy processes, $Y_m$ is $n_m$-dimensional, $(R_1, \ldots, R_d)$ is a $d$-dimensional subordinator, and $e_m = (1, \ldots, 1) \in \mathbb{R}^{n_m}$, $1 \leq m \leq d$.

For the proof of sufficiency under conditions (C1) and (C3), see [14, Theorem 30.1] and [1, Theorem 3.3], respectively, though the origin of the former case goes back to [16]. Both conditions (C1) and (C2) are implied by condition (C3). It is condition (C2), as opposed to condition (C3), that more commonly appears in financial applications [3,10,15], since it has a more intuitive interpretation. Outside of these sufficient conditions, strong subordination does not necessarily produce a Lévy process [5, Proposition 3.9].

These restrictions on $X$ and $T$ for $X \circ T$ to remain in the well-understood Lévy process framework are problematic in applications because they severely limit the dependence structure of $X \circ T$. To address this shortcoming, Buchmann, Lu, and Madan [5] introduced a new operation for constructing general time-changed multivariate Lévy processes $X \circ T$, known as weak subordination, without any restriction on the subordinate $X$ or the subordinator $T$.

Weak subordination is based on the idea, roughly speaking, of constructing the Lévy process that has the distribution of $X(t) := (X_1(t_1), \ldots, X_n(t_n))$ conditional on $T(t) = t := (t_1, \ldots, t_n)$, $t \geq 0$. To be more precise, the idea is to decompose the subordinator $T(t) = d + S(t)$, $t \geq 0$, into deterministic and pure-jump parts. For the deterministic subordinator part, $X(d)$ is infinitely divisible and hence associated with a Lévy process, while for the pure-jump subordinator part, a marked Poisson process can be constructed such that it jumps with the distribution of $X(t)$ when the subordinator jumps by $\Delta T(t) = t$ and then associated with a Lévy process by the Lévy–Itô decomposition, and finally the two Lévy processes are combined by convolution. This allows for more flexible dependence modelling while remaining in the class of Lévy processes, a closure property not enjoyed by strong subordination. Weak subordination coincides with strong subordination under conditions (C1) or (C2) in the sense that $(T, X \circ T) \overset{D}{=} (T, X \circ T)$ [5, Proposition 3.3], and it also reproduces many analogous properties [5, Propositions 3.3, 3.7].

In this paper we show that the more general case (C3) of stacked univariate subordination considered in [1] also satisfies $(T, X \circ T) \overset{D}{=} (T, X \circ T)$, which unifies it under weak subordination (see Theorem 1). This raises the question of whether there are alternative definitions of weak subordination that are also consistent with strong subordination in this way. We partially address this by showing that if $(T, X \circ T)$ is a Lévy process, then $(T, X \circ T) \overset{D}{=} (T, X \circ T)$ in two cases: $T$ is deterministic (see Proposition 1), or $T$ is a pure-jump subordinator with finite activity (see Theorem 2). In the former case we can weaken the assumption to $X \circ T$ being a Lévy process. Our proof tracks the construction of weak subordination in the deterministic subordinator and pure-jump subordinator cases mentioned above, and in the latter case the theory of marked Poisson point processes is used to verify that the relevant characteristics coincide.

We briefly mention some applications. The subordination of Lévy processes is used in mathematical finance to create time-changed models of stock prices. This idea began with the work of Madan and Seneta [12], who introduced the variance gamma (VG) process for modelling stock prices, created by subordinating a Brownian motion with a gamma subordinator. Subordination can also be applied to model dependence in multivariate price processes.
The multivariate VG process in [12] was created by subordinating multivariate Brownian motion with a univariate gamma subordinator, so the components cannot have idiosyncratic time changes and must have equal kurtosis when there is no skewness. These deficiencies were addressed by the use of an alpha-gamma subordinator, resulting in the variance alpha-gamma process which was introduced in [15] and also considered in [7] and [10]. However, in this case the Brownian motion subordinates must have independent components. In both models, the use of strong subordination to create a Lévy process restricts the dependence structure. By using weak subordination instead, and a general Brownian motion, Buchmann, Lu, and Madan [4,5] introduced the weak variance alpha-gamma (WV AG) process to provide additional flexibility compared to the strong variance alpha-gamma (SV AG) process [13] and instantaneous portfolio theory [11].

The paper is structured as follows. In Section 2 we review some notations, definitions, and preliminary results related to Lévy processes, weak subordination, and Poisson random measures. In Section 3 we state and prove the main results, namely that weak subordination is consistent with strong subordination under condition (C3), and that if strong subordination produces a Lévy process it is necessarily equal in law to weak subordination when the subordinator is deterministic or pure-jump with finite activity. We conclude in Section 4 with a brief discussion placing this work in the context of open questions relating to the subordination of multivariate Lévy processes.

2. Preliminaries

2.1 Lévy processes

We write \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) as a row vector. For \( A \subseteq \mathbb{R}^n \), let \( A_* := A \setminus \{0\} \) and let \( 1_A \) denote the indicator function for \( A \). Let \( \mathbb{D} := \{x \in \mathbb{R}^n : \|x\| \leq 1\} \) be the Euclidean unit ball centred at the origin. Let \( \|x\|_{\Sigma}^2 := x\Sigma x^* \), where \( x \in \mathbb{R}^n \) and \( \Sigma \in \mathbb{R}^{n \times n} \). Let \( I : [0, \infty) \rightarrow [0, \infty) \) be the identity function.

For references on Lévy processes, see [2] and [14]. The law of an \( n \)-dimensional Lévy process \( X = (X_1, \ldots, X_n) = (X(t))_{t \geq 0} \) is determined by its characteristic function \( \Phi_X := \Phi_{X(1)} \) with

\[
\Phi_{X(t)}(\theta) := \mathbb{E} \exp \left(i(\theta, X(t))\right) = \exp \left(i \Psi_X(\theta)\right), \quad t \geq 0, \quad \theta \in \mathbb{R}^n,
\]

and characteristic exponent

\[
\Psi_X(\theta) := i(\mu, \theta) - \frac{1}{2} \|\theta\|_{\Sigma}^2 + \int_{\mathbb{R}^n} \left(e^{i(\theta, x)} - 1 - i(\theta, x)1_{\mathbb{D}}(x)\right) \lambda(dx),
\]

where \( \mu \in \mathbb{R}^n \), \( \Sigma \in \mathbb{R}^{n \times n} \) is a covariance matrix, and \( \lambda \) is a Lévy measure, that is, a non-negative Borel measure on \( \mathbb{R}^n \) such that \( \int_{\mathbb{R}^n} (1 \wedge \|x\|^2) \lambda(dx) < \infty \). We write \( X \sim L^n(\mu, \Sigma, \lambda) \) (or \( X \sim L^n \) for short) to mean that \( X \) is an \( n \)-dimensional Lévy process with characteristic triplet \((\mu, \Sigma, \lambda)\).

An \( n \)-dimensional Lévy process \( T \) with almost surely non-decreasing sample paths is called a subordinator, and it is denoted by \( T \sim S^n(d, T) := L^n(\mu, 0, T) \) (or \( T \sim S^n \) for short), where \( d := \mu - \int_{\mathbb{D}} t \mathcal{T}(dt) \in [0, \infty)^n \) is the drift, and \( \mathcal{T} \) is the Lévy measure. The law of \( T \) is characterised by its Laplace exponent \( \Lambda_T \), which satisfies \( \mathbb{E}[\exp(-\langle \lambda, T(1)\rangle)] = \exp(-\Lambda_T(\lambda)) \), where \( \langle \cdot, \cdot \rangle \) denotes the inner product.
\[ \lambda \in [0, \infty)^n. \] For \( w, z \in \mathbb{C}^n \), let \( \langle w, z \rangle := \sum_{k=1}^{n} w_k z_k \), noting that there is no conjugation. The domain of \( \Lambda_T \) can be extended, giving

\[
\Lambda_T(z) = \langle d, z \rangle + \int_{[0, \infty)^n} (1 - e^{-\langle z, t \rangle}) \mathcal{T}(dt), \quad \forall z \in [0, \infty)^n
\] (2)

(see the proof of [1, Theorem 3.3]).

We have the following characterisation of piecewise constant Lévy processes from [14, Theorem 21.2].

**Lemma 1.** Let \( X \sim L^n(\mu, \Sigma, \mathcal{X}) \). The following are equivalent:

(i) \( X \) is piecewise constant a.s.,

(ii) \( X \) is driftless, \( \Sigma = 0 \) and \( \mathcal{X}(\mathbb{R}^n_+) < \infty \),

(iii) \( X \) is a compound Poisson process or \( X \) is the zero process.

### 2.2. Weak subordination

Let \( X \sim L^n \) and \( T \sim S^n(d, \mathcal{T}) \). Let \( t = (t_1, \ldots, t_n) \in [0, \infty)^n \) and \( \langle (1), \ldots, (n) \rangle \) be a permutation of \( \{1, \ldots, n\} \) such that \( t(1) \leq \cdots \leq t(n) \), and define \( \Delta t(k) := t(k) - t(k-1) \), \( 1 \leq k \leq n \), with \( t(0) := 0 \). Further, let \( \pi_J : \mathbb{R}^n \to \mathbb{R}^n \) be the projection onto the coordinate axes in \( J \subseteq \{1, \ldots, n\} \). For all \( t \in [0, \infty)^n \), by [5, Proposition 2.1], the random vector \( X(t) := (X_1(t_1), \ldots, X_n(t_n)) \) is infinitely divisible with characteristic exponent

\[
(t \odot \Psi_X)(\theta) := \sum_{k=1}^{n} \Delta t(k) \Psi_X(\pi_{\{k, \ldots, (n)\}}(\theta)), \quad \theta \in \mathbb{R}^n,
\] (3)

and characteristic function

\[
\Phi_{X(t)}(\theta) = \exp(t \odot \Psi_X(\theta)).
\] (4)

It is convenient to consider both the subordinator and the subordinated process together as a joint \( 2n \)-dimensional Lévy process. In this form, we can define weak subordination as follows (see [5, Proposition 3.1]).

**Definition 1.** The **weak subordination** of \( X \sim L^n \) and \( T \sim S^n(d, \mathcal{T}) \) is the joint Lévy process \( Z \overset{D}{=} (T, X \odot T) \) with characteristic exponent

\[
\Psi_Z(\theta) = i\langle d, \theta_1 \rangle + (d \odot \Psi_X)(\theta_2) + \int_{[0, \infty)^n} \Phi_{(t, X(t))}(\theta) - 1 \mathcal{T}(dt),
\] (5)

where \( \theta = (\theta_1, \theta_2), \theta_1, \theta_2 \in \mathbb{R}^n \).

This is a valid characteristic exponent for a Lévy process. Weak subordination can equivalently be defined in terms of a characteristic triplet. From [5, Definition 2.1], if \( d = 0 \), then \( Z \sim L^{2n}(m, \Theta, \mathcal{Z}) \), where

\[
m = \int_{D_+} (t, x) \mathcal{Z}(dt, dx),
\] (6)

\[
\Theta = 0,
\] (7)

\[
\mathcal{Z}(dt, dx) = 1_{[0, \infty)^n} \otimes \mathcal{P}(X(t) \in dx) \mathcal{T}(dt).
\]

For additional details on weak subordination, see [5].
2.3. Poisson random measures

For references on Poisson random measures and their relationship to the jumps of Lévy processes, see [2], [6], and [9]. A Poisson random measure (PRM) $Z$ with intensity measure $\mu$ on a measurable space $(E, \mathcal{E})$ is a random measure such that $\mathbb{Z}(A) \sim \text{Poisson}(\mu(A))$ for all $A \in \mathcal{E}$, and $\mathbb{Z}(A_1), \ldots, \mathbb{Z}(A_m)$ are independent for all disjoint $A_1, \ldots, A_m \in \mathcal{E}$. In general, a PRM has the form $Z = \sum_{i=1}^{\infty} \delta_{Z_i}$, where $\delta_{Z_i}, i \in \mathbb{N}$, is the Dirac measure at the random vector $Z_i$ taking values in $(E, \mathcal{E})$. Define the random variable $Z_f := \int_E f(x) \mathbb{Z}(dx)$, where $f$ is a non-negative, $\mathcal{E}$-measurable real function. The Laplace functional of $Z$ is

$$L(f) := \mathbb{E}[e^{-Z_f}] = \mathbb{E}\left[ \prod_{i=1}^{\infty} e^{-f(Z_i)} \right] = \exp\left(-\int_E (1-e^{-f(x)}) \mu(dx)\right)$$

(see [9, equation (3.35)]). The Laplace functional is well-defined with $L(f) \in [0, 1]$, where this equality can be interpreted as 0 if $Z_f < \infty$ a.s. fails. Two PRMs are equal if their Laplace functionals are equal [6, Chapter VI, Proposition 1.4].

Let $X \sim L^\mu(\mu, \Sigma, \mathcal{X})$ with $\mathcal{X} \neq 0$. For a fixed sample path, a time $t$ is a jumping time of $X$ if the jump $\Delta X(t) := X(t) - X(t^-) \neq 0$. The following result is [14, Theorem 21.3].

**Lemma 2.** Let $X \sim L^\mu(\mu, \Sigma, \mathcal{X})$ with $\mathcal{X} \neq 0$; then its jumping times are countably infinite. Denoting these jumping times as $S = (S_i)_{i \in \mathbb{N}}$, we have in addition:

1. if $\mathcal{X}(\mathbb{R}^n) < \infty$, then $S$ is countable in increasing order, or
2. if $\mathcal{X}(\mathbb{R}^n) = \infty$, then $S$ is dense in $[0, \infty)$.

For the Lévy process $X$, the countable sequence of random vectors giving the time and size of the jumps, $(Z_i)_{i \in \mathbb{N}} := (t, \Delta X(t))_{t>0, \Delta X(t)\neq 0}$, is a Poisson point process. Consequently, $Z = \sum_{i=1}^{\infty} \delta_{Z_i}$ is the PRM of $X$ (or of the jumps of $X$), defined on the Borel space $([0, \infty) \times \mathbb{R}^n, \mathcal{B}([0, \infty) \times \mathbb{R}^n))$ with intensity measure $dt \otimes \mathcal{X}$ [2, Chapter I, Theorem 1].

3. Main results

3.1. Consistency of weak subordination for stacked univariate subordination

Here we show that the laws of weak and strong subordination coincide when the latter satisfies the stacked univariate subordination property in condition (C3). The proof follows along the lines of [5, Proposition 3.3].

**Theorem 1.** Let $T \sim S^n$ and $X \sim L^n$ be independent. If $T$ and $X$ satisfy the stacked univariate subordination condition in (1), then $(T, X \circ T) \overset{D}{=} (T, X \otimes T)$.

**Proof.** Recall that $n_1 + \cdots + n_d = n$ and let $\theta = (\theta_1, \theta_2) = (\theta_{11}, \ldots, \theta_{1d}, \theta_{21}, \ldots, \theta_{2d})$, $\theta_1, \theta_2 \in \mathbb{R}^n$, $\theta_{1m}, \theta_{2m} \in \mathbb{R}^{n_m}$ for all $1 \leq m \leq d$. Since $T$ and $X$ are independent processes, using (4) and conditioning on $T$, we get

$$\Phi_T(T, X \circ T)(\theta) = \mathbb{E}[\exp(i\langle \theta_1, T(1) \rangle + (T(1) \circ X)(\theta_2))].$$

(9)

Let $e = (1, \ldots, 1) \in \mathbb{R}^n$. Since $T$ and $X$ satisfy the stacked univariate subordination condition for $n$-dimensional processes, the subordinator $(T, T)$ and the subordinate $(e, X)$ satisfy this condition for $2n$-dimensional processes. Thus $(e, X) \circ (T, T) = (T, X \circ T)$ is a Lévy process by [1, Theorem 3.3], so it suffices to show that $\Psi_T(T, X \circ T) = \Psi_T(T, X \otimes T)$.
Noting that \( X = (Y_1, \ldots, Y_d) \), where \( Y_1 \sim L^{n_1}, \ldots, Y_d \sim L^{n_d} \) are independent Lévy processes, Kac’s theorem gives

\[
\Psi_X(\theta) = \sum_{m=1}^{d} \Psi_{Y_m}(\theta_{2m}).
\]

Form the partition \( \{1, \ldots, n\} = J_1 \cup \cdots \cup J_d \), where

\[
J_1 := \{1, \ldots, n_1\}, J_2 := \{n_1 + 1, \ldots, n_1 + n_2\}, \ldots, J_d := \{n_1 + \cdots + n_{d-1} + 1, \ldots, n\}.
\]

Let \( r = (r_1, \ldots, r_d) \in [0, \infty)^d \) and \( (1), (d) \) be a permutation of \( \{1, \ldots, d\} \) such that \( r_{(1)} \leq \cdots \leq r_{(d)} \). Define the projections \( \pi_m := \pi_{J(m) \cup \cdots \cup J(d)}, 1 \leq m \leq d \). Thus, for all \( 1 \leq m \leq d \),

\[
\Psi_X(\pi_m(\theta)) = \sum_{k=m}^{d} \Psi_{Y_{(k)}}(\theta_{(k)}).
\]

Next, due to (1), we can write \( T = RA \) for some \( A \in \mathbb{R}^{d \times n} \), where \( R = (R_1, \ldots, R_d) \sim S^d(d, \mathcal{R}) \). Then (3) and (10) give

\[
(rA) \circ \Psi_X(\theta) = \sum_{m=1}^{d} (r(m) - r(m-1)) \Psi_X(\pi_m(\theta))
\]

\[
= \sum_{m=1}^{d} r(m)(\Psi_X(\pi_m(\theta)) - \Psi_X(\pi_{m+1}(\theta))) + r(d) \Psi_X(\pi_d(\theta))
\]

\[
= \sum_{m=1}^{d} r(m) \Psi_{Y_m}(\theta_{2m})
\]

\[
= \sum_{m=1}^{d} r_m \Psi_{Y_m}(\theta_{2m}).
\]

Let \( z := (z_1, \ldots, z_d) \in \mathbb{C}^d \), where \( z_m = -i(\theta_{1m}, e_m) - \Psi_{Y_m}(\theta_{2m}), 1 \leq m \leq d \). Using (11), we have

\[
-\langle z, r \rangle = \sum_{m=1}^{d} i(\theta_{1m}, r_m e_m) + r_m \Psi_{Y_m}(\theta_{2m})
\]

\[
= i(\theta_1, RA) + (rA) \circ \Psi_X(\theta).
\]

Thus (9) becomes \( \Phi_{(T,X,T)}(\theta) = \mathbb{E}[\exp(-\langle z, R(1) \rangle)] \). By noting that \( \exists ! z \in [0, \infty)^d \) and using (2) to obtain the Laplace exponent of \( R \), we have \( \Phi_{(T,X,T)}(\theta) = -\Lambda_R(z) \), where

\[
\Lambda_R(z) = \langle d, z \rangle + \int_{[0,\infty)^{d_+}} (1 - e^{-\langle z, r \rangle}) \mathcal{R}(dr).
\]

Using (4) and (12), we have \( e^{-\langle z, r \rangle} = \Phi_{(rA,X(T))}(\theta) \) for \( r \in [0, \infty)^d \). Thus

\[
\Psi_{(T,X,T)}(\theta) = i(\theta_1, dA) + (dA) \circ \Psi_X(\theta) + \int_{[0,\infty)^{d_+}} (\Phi_{(rA,X(T))}(\theta) - 1) \mathcal{R}(dr)
\]

\[
= i(\theta_1, dA) + (dA) \circ \Psi_X(\theta) + \int_{[0,\infty)^{d_+}} (\Phi_{(rX(t))}(\theta) - 1) (R \circ A^{-1})(dt)
\]
by the transformation theorem. This matches the right-hand side of (5) because $T \sim S^n(dA, R \circ A^{-1})$. Therefore $(T, X \circ T) \overset{D}{=} (T, X \odot T)$. $\square$

3.2. Necessity of weak subordination for deterministic and pure-jump, finite activity subordinators

In this subsection we assume that $(T, X \circ T) \sim \mathcal{L}^n$, and under some conditions we show that it is equal in law to $(T, X \odot T)$. This is in contrast to Theorem 1, where we proved, under a condition for which it is known that $(T, X \circ T) \sim \mathcal{L}^n$, that it is equal in law to $(T, X \odot T)$. The conditions we consider in this subsection are that the subordinator $T$ is deterministic, or that $T$ is pure-jump with finite activity, which are dealt with in Proposition 1 and Theorem 2, respectively. In the former case we use the weaker assumption $X \circ T \sim \mathcal{L}^n$. We now consider the case where $T$ is a pure-jump subordinator. The following lemma establishes the Laplace functional corresponding to the PRM of the weakly subordinated process $(T, X \odot T)$ without the finite activity assumption on $T$. It is based on the marked Poisson point process of jumps of $(T, X \odot T)$ given in the proof of [5, Theorem 2.1 (ii)], and closely follows the arguments in the proof of [6, Chapter VI, Theorem 3.2].

Throughout the rest of this section, we let $E := [0, \infty) \times [0, \infty)^* \times \mathbb{R}^n$, and let $Q$ be the mapping $(t^*, B) \mapsto \mathbb{P}(X(t) \in B)$ for $t^* := (t, y) \in [0, \infty) \times [0, \infty)^*$ and Borel sets $B \subseteq \mathbb{R}^n$.

Lemma 3. Assume $T \sim S^n(0, \mathcal{T})$, $T \neq 0$, $X \sim \mathcal{L}^n$. The PRM of the Lévy process $(T, X \odot T)$, denoted $Z_{T,\odot}$, has Laplace functional

$$
\mathbb{E}
\left[
\exp
\left(-\int_E f(t^*, y) Z_{T,\odot}(dt^*, dy)\right)
\right]
= \prod_{i=1}^{\infty} e^{-g(T_i^*)},
$$

where $f$ is a non-negative, measurable real function,

$$
e^{-g(t^*)} = \int_{\mathbb{R}^n} e^{-f(t^*, y)} Q(t^*, dy),$$

$$(T_i^*)_{t \in \mathbb{N}} := (t, \Delta(T) t > 0, \Delta(T(t)) \neq 0).$$
Proof. Noting that the Lévy process \((T, X \circ T)\) has only countably many jumps byLemma 2, the Poisson point process of jumps can be written as
\[
(t, \Delta T(t), \Delta(X \circ T)(t))_{t > 0, \Delta T(t) \neq 0} \equiv (T^*_i, Y_i)_{i \in \mathbb{N}}.
\]
Here \(T^*_i\) and \(Y_i\) are random vectors that take values in \([0, \infty) \times [0, \infty)^n\) and \(\mathbb{R}^n\), respectively. So the PRM of \((T, X \circ T)\) is
\[
Z_\odot(dt^*, dy) = \sum_{i=1}^{\infty} \delta_{(T^*_i, Y_i)}(dt^*, dy).
\]

Now \(Q\) is a probability kernel, and \(Z_\odot\) corresponds to a marked Poisson point process with marks \((Y_i)_{i \in \mathbb{N}}, \) and \(Y_i, i \in \mathbb{N},\) are conditionally independent given \(T^* := (T^*_i)_{i \in \mathbb{N}}\) with probability distribution \(Q(T^*_i, dy)\) (see the proof of [5, Theorem 2.1 (ii)]). Consequently, the Laplace functional of \(Z_\odot\) is
\[
\mathbb{E}\left[\exp\left(-\int_E f(t^*, y) Z_\odot(dt^*, dy)\right)\right] = \mathbb{E}\left[\prod_{i=1}^{\infty} e^{-f(T^*_i, Y_i)} \mid T^*\right]
\]
\[
= \mathbb{E}\left[\prod_{i=1}^{\infty} \mathbb{E}[e^{-f(T^*_i, Y_i)} \mid T^*]\right]
\]
\[
= \mathbb{E}\left[\prod_{i=1}^{\infty} \int_{\mathbb{R}^n} e^{-f(T^*_i, y)} Q(T^*_i, dy)\right]
\]
\[
= \mathbb{E}\left[\prod_{i=1}^{\infty} e^{-g(T^*_i)}\right],
\]
as required.

Theorem 2. Let \(T \sim S^n(0, T)\) and \(X \sim L^n\) be independent, with \(T \neq 0\) and \(T([0, \infty]^n) < \infty.\)
If \((T, X \circ T) \sim L^{2n}\), then \((T, X \circ T) \overset{D}{=} (T, X \circ T).\)

Proof. Let
\[
(t, \Delta T(t), \Delta(X \circ T)(t))_{t > 0, \Delta T(t, X \circ T)(t) \neq 0} \equiv (T^*_i, Y^*_i)_{i \in \mathbb{N}}
\]
with \(T^*_i := (S_i, \Delta T(S_i))\) and \(S^* := (S_i)_{i \in \mathbb{N}}\). The PRM of \((T, X \circ T)\) is
\[
Z_\odot(dt^*, dy) = \sum_{i=1}^{\infty} \delta_{(T^*_i, Y^*_i)}(dt^*, dy).
\]

Since \(T([0, \infty)^n) < \infty\) by assumption, the jumps of \(T\) are countable in increasing order by Lemma 2 (i). Therefore the sample paths \(t \mapsto (T, X \circ T)(t)\) are piecewise constant a.s., which implies by Lemma 1 (ii) that the Lévy process \((T, X \circ T) \sim L^{2n}(m, \Theta, \mathcal{Z})\) must have \(m = \int_{\mathbb{R}_+ \times \mathbb{R}^n} (t, x) \mathcal{Z}(dt, dx)\) and \(\Theta = 0\). These are the same \(m\) and \(\Theta\) as \((T, X \circ T)\) in (6)–(7) provided that \((T, X \circ T)\) and \((T, X \circ T)\) have the same Lévy measure, which we now verify by showing they have the same PRM.

Let \(S_0 := 0\). Recalling that \((T, X \circ T)\) has càdlàg and piecewise constant sample paths a.s., and that \((S_i)_{i \in \mathbb{N}}\) is countable in increasing order, we have
\[
(\Delta T(S_i), Y^*_i) = (T(S_i) - T(S_i -), X \circ T(S_i) - X \circ T(S_i -))
\]
\[
= (T, X \circ T)(S_i) - (T, X \circ T)(S_{i-1}), \quad i \in \mathbb{N}.
\]
For \( i \in \mathbb{N} \), let \( \mathbb{P}((\Delta T(S_i), Y_1^\ast) \in (dt, dy) \mid S^\ast) \) denote the conditional distribution of \((\Delta T(S_i), Y_1^\ast)\) given \( S^\ast \). Since \((\Delta T(S_i), Y_1^\ast)\) takes values on the Borel space \((0, \infty)^n \times \mathbb{R}^n, \mathcal{B}((0, \infty)^n \times \mathbb{R}^n))\), there exists a regular version of the conditional distribution (see [8, Theorem 5.3]), so we can assume that \( \mathbb{P}((\Delta T(S_i), Y_1^\ast) \in (dt, dy) \mid S^\ast) \) is a probability kernel.

Consequently, \( \mathbb{P}((\Delta T(S_i), Y_1^\ast) \in (dt, dy) \mid S^\ast) \) is a probability measure for each value of \( S^\ast \), so it is determined by its characteristic function, which by the disintegration theorem (see [8, Theorem 5.4]) is

\[
\mathbb{E}[\exp(i((\Delta T(S_i), Y_1^\ast), \theta)) \mid S^\ast] = \mathbb{E}[\exp(i((T, X \circ T)(S_i - S_{i-1}), \theta)) \mid S^\ast] = \mathbb{E}[\exp(i(T(S_i - S_{i-1}), \theta_1)) \exp(T(S_i - S_{i-1}) \circ \Psi_X(\theta_2)) \mid S^\ast] = \mathbb{E}[\exp(i((\Delta T(S_i), \theta_1)) \exp(\Delta T(S_i) \circ \Psi_X(\theta_2)) \mid S^\ast],
\]

using the stationary increment property of \((T, X \circ T), (4)\), and the stationary increment property of \( T \), where \( \theta = (\theta_1, \theta_2), \theta_1, \theta_2 \in \mathbb{R}^n \). Similarly, the conditional distribution \( \mathbb{P}((\Delta T(S_i), X(\Delta T(S_i))) \in (dt, dy) \mid S^\ast) \) is also a probability kernel, and using the tower law, its characteristic function is

\[
\mathbb{E}[\exp(i((\Delta T(S_i), X(\Delta T(S_i))), \theta)) \mid S^\ast] = \mathbb{E}[\mathbb{E}[\exp(i((\Delta T(S_i), X(\Delta T(S_i))), \theta)) \mid \Delta T(S_i), S^\ast] \mid S^\ast],
\]

which matches (15). Thus we have

\[
\mathbb{P}((\Delta T(S_i), Y_1^\ast) \in (dt, dy) \mid S^\ast) = \mathbb{P}((\Delta T(S_i), X(\Delta T(S_i))) \in (dt, dy) \mid S^\ast) = \mathbb{P}((\Delta T(S_i), X(\Delta T(S_i))) \in (dt, dy)) = \mathbb{P}(X(\Delta T(S_i)) \in dy \mid \Delta T(S_i) = t)\mathbb{P}(\Delta T(S_i) \in dt) = \mathbb{P}(X(t) \in dy)\mathbb{P}(\Delta T(S_i) \in dt) = Q((t, t), dy)\mathbb{P}(\Delta T(S_i) \in dt), \quad i \in \mathbb{N}, \ t > 0,
\]

where the third line is obtained by noting that the conditioning on \( S^\ast \) can be dropped because, by applying Lemma 1 (iii) to \((T, X \circ T), (4)\), we see that \( S^\ast \) are the times of the jumps of a compound Poisson process, which are independent of the size of the jumps with distribution \((\Delta T(S_i), X(\Delta T(S_i)))\), the fourth line is obtained by [8, Chapter 5, equation (7)], and the fifth line is obtained by the independence of \( T \) and \( X \).

By definition, \( S^\ast \) are the jumping times of \((T, X \circ T). \) Since \( T \) is a pure-jump subordinator with finite activity, by examining the sample paths, \( X \circ T \) cannot jump unless \( T \) does. So almost surely,

\[ \{t > 0: \Delta(X \circ T)(t) \neq 0\} \subseteq \{t > 0: \Delta T(t) \neq 0\}, \]

which implies \( S^\ast = \{t > 0: \Delta T(t) \neq 0\} \). Thus \( S^\ast \) are also the jumping times of \( T \).
Next, for any non-negative, measurable real function $f$, we have

\[
\mathbb{E}[e^{-f(S, \Delta T(S), Y^*_i)} \mid S^*] = \int_{[0, \infty)^2 \times \mathbb{R}^n} e^{-f(S, t, y)} Q((t, t), dy) \mathbb{P}(\Delta T(S) \in dt)
\]

\[
= \int_{(0, \infty)^n} e^{-g(S, t)} \mathbb{P}(\Delta T(S) \in dt)
\]

\[
= \int_{(0, \infty)^n} e^{-g(S, t)} \mathbb{P}(\Delta T(S) \in dt \mid S^*)
\]

\[
= \mathbb{E}[e^{-g(S, \Delta T(S_i))} \mid S^*], \quad i \in \mathbb{N}, \ t > 0,
\]

where the first line follows from the disintegration theorem (see [8, Theorem 5.4]) and using (16), the third line follows from a similar argument to that above, $T$ being a compound Poisson process means the time and size of the jumps are independent, and the final line follows from another application of the disintegration theorem.

Putting this together, we compute the Laplace functional of $Z_o$,

\[
\mathbb{E}\left[\exp\left(-\int_E f(t^*, y) Z_o(dt^*, dy)\right)\right] = \mathbb{E}\left[\mathbb{E}\left[\prod_{i=1}^{\infty} e^{-f(S, \Delta T(S_i), Y^*_i)} \mid S^*\right]\right]
\]

\[
= \mathbb{E}\left[\mathbb{E}\left[\prod_{i=1}^{\infty} e^{-g(S, \Delta T(S_i))} \mid S^*\right]\right]
\]

\[
= \mathbb{E}\left[\mathbb{E}\left[\prod_{i=1}^{\infty} e^{-g(S, \Delta T(S_i))} \mid S^*\right]\right]
\]

\[
= \mathbb{E}\left[\mathbb{E}\left[\prod_{i=1}^{\infty} e^{-g(S, \Delta T(S_i))} \mid S^*\right]\right]
\]

where the second line follows since $(\Delta T(S_i), Y^*_i)$, $i \in \mathbb{N}$, are conditionally independent given $S^*$ because of (14) and the independent increment property of the Lévy process $(T, X \circ T)$, the third line follows from (17), and the fourth line follows from a similar argument to the second line but applied to $T$. Thus we have proved $Z_o = Z_\circ$ from (13), and hence $(T, X \circ T)$ and $(T, X \odot T)$ have the same Lévy measure and hence the same characteristic triplet. \(\square\)

**Remark 1.** It is difficult to extend Theorem 2 to the more general case of a finite activity subordinator with non-zero drift or to all pure-jump subordinators. In the former case the proof’s reliance on the properties of the compound Poisson process would fail. In the latter case, for any pure-jump subordinator $T$, we can create a finite activity subordinator $T^{(k)}$ by truncating the size of the jumps to $\{\|t\| \in (1/k, \infty)\}, k > 0$, but it is not clear that the strongly subordinated process $(T^{(k)}, X \circ T^{(k)})$ is a Lévy process to which Theorem 2 can be applied.

**Remark 2.** It would be ideal if the assumption $(T, X \circ T) \sim L^{2n}$ in Theorem 2 could be replaced by the weaker assumption $X \circ T \sim L^n$. In the proof of Proposition 1 it is shown that $X \circ T \sim L^n$ implies $(T, X \circ T) \sim L^{2n}$ under the assumption that $T$ is deterministic. We conjecture that this result holds in general, although it is not clear how this can be proved. In Theorem 2, the assumption $(T, X \circ T) \sim L^{2n}$ on the joint process is crucial to the proof.
4. Discussion

Let $T \sim S^n$ and $X \sim L^n$ be independent with $n \geq 2$. There are some simply stated but open questions on subordination of Lévy processes.

- If $X \circ T$ is a Lévy process, then can we conclude that $X \circ T \overset{D}{=} X \odot T$?
- What are the necessary and sufficient conditions on $T$ and $X$ such that $X \circ T$ is a Lévy process?
- If $X \circ T$ is not a Lévy process, what are the necessary and sufficient conditions on $T$ and $X$ such that it can be mimicked by some Lévy process $Y$ in the sense that $(X \circ T)(t) \overset{D}{=} Y(t)$ for all $t \geq 0$?

On the first question, if the answer is yes, then there cannot exist a different way to define the law of weak subordination for the class of subordinators $T \sim S^n$ and subordinates $X \sim L^n$ such that $X \circ T \sim L^n$. Otherwise, if the answer is no, it would be interesting to determine the characteristics of $X \circ T$. Here we have shown that the answer is yes if $T$ is a deterministic subordinator, and it also holds under the stronger assumption $(T, X \circ T) \sim L^{2n}$ if $T$ is a pure-jump subordinator with finite activity.

On the second question, the sufficient conditions (C1)–(C3) are well known. A partial converse has been given in [5, Proposition 3.9]. However, there are no known examples, outside of condition (C3), where $X \circ T \sim L^n$ (besides the trivial cases where some components of $T$ are the zero process). If necessary and sufficient conditions were known, it might indeed turn out that there are no additional Lévy processes to which Proposition 1 and Theorem 2 are applicable besides those satisfying condition (C3).

We do not deal with the third question here, but [5, Proposition 3.4] shows that a sufficient condition for $(X \circ T)(t) \overset{D}{=} (X \odot T)(t)$, for all $t \geq 0$, to hold is that $T = (T_1, \ldots, T_n)$ has monotonic components, meaning that there exists a permutation $((1), \ldots, (n))$ such that $T_{(1)} \leq \cdots \leq T_{(n)}$. A partial converse is given in [5, Proposition 3.10], which suggests that outside of the monotonic assumption there may be no Lévy process mimicking $X \circ T$. This raises the conjecture that if $X \circ T$ is not itself a Lévy process, then it can be mimicked by a Lévy process if and only if $T$ has monotonic components. Furthermore, these results also suggest that the mimicking Lévy process, if it exists, may be $X \odot T$.

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