UNIFORM ATTRACTORS AND THEIR CONTINUITY FOR THE NON-AUTONOMOUS KIRCHHOFF WAVE MODELS

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Abstract. The paper investigates the existence and the continuity of uniform attractors for the non-autonomous Kirchhoff wave equations with strong damping: $u_{tt} - (1 + \epsilon \|\nabla u\|^2)\Delta u - \Delta u_t + f(u) = g(x, t)$, where $\epsilon \in [0, 1]$ is an extensibility parameter. It shows that when the nonlinearity $f(u)$ is of optimal supercritical growth $p : \frac{N+2}{N-2} = p^* < p < p^{**} = \frac{N+4}{(N-4)^+}$: (i) the related evolution process has in natural energy space $\mathcal{H} = (H_0^1 \cap L^{p+1}) \times L^2$ a compact uniform attractor $\mathcal{A}_\epsilon$ for each $\epsilon \in [0, 1]$; (ii) the family of compact uniform attractor $\{\mathcal{A}_\epsilon\}_{\epsilon \in [0, 1]}$ is continuous on $\epsilon$ in a residual set $I^* \subset [0, 1]$ in the sense of $\mathcal{H}_{p^*} = (H_0^1 \cap L^{p+1.\infty}) \times L^2$-topology; (iii) $\{\mathcal{A}_{\epsilon}\}_{\epsilon \in [0, 1]}$ is upper semicontinuous on $\epsilon \in [0, 1]$ in $\mathcal{H}_{p^*}$-topology.

1. Introduction. In this paper, we are concerned with the existence and the continuity of uniform attractors for the non-autonomous Kirchhoff wave equations with strong damping:

\begin{align*}
    u_{tt} - (1 + \epsilon \|\nabla u\|^2)\Delta u - \Delta u_t + f(u) &= g(x, t), \quad x \in \Omega, \quad t > \tau, \quad (1) \\
    u|_{\partial \Omega} &= 0, \quad u(x, \tau) = u_0, \quad u_t(x, \tau) = u_1, \quad \tau \in \mathbb{R}, \quad (2)
\end{align*}

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ ($N \geq 3$) with the smooth boundary $\partial \Omega$, $\epsilon \in [0, 1]$ is an extensibility parameter.

When $N = 1$, Eq. (1), without strong damping $-\Delta u_t$, was originally introduced by Kirchhoff [10] to describe the nonlinear vibrations of an elastic stretched string. After that, the well-posedness and asymptotic behavior of solutions to the Kirchhoff wave models with dissipation $-\Delta u_t$ or $u_t$ or $h(u_t)$ (with $h(s)s \geq 0$) have been well

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investigated by many authors in the case that the nonlinearity \( f(u) \) is of at most critical growth (see [16, 18, 19, 25, 26] and references therein).

In 2012, Chueshov [4] studied the well-posedness and longtime dynamics for the autonomous Kirchhoff wave model with strong nonlinear damping

\[
u_{tt} - \phi(\|\nabla u\|^2)\Delta u - \sigma(\|\nabla u\|^2)\Delta u_t + f(u) = h(x). \tag{3}
\]

A major advance is that he finds a supercritical exponent \( p^* := \frac{N+4}{(N-4)\tau} \), which breaks through the long-standing restriction for the critical exponent \( p^* := \frac{N+2}{(N-2)\tau} \), and shows that when the nonlinearity \( f(u) \) is of the supercritical growth: \( 1 \leq p < p^* \), the corresponding autonomous problem is still well-posed and the related solution semigroup has in phase space \( \mathcal{H} = (H^1_0 \cap L^{p+1}) \times L^2 \) a partially strong global attractor \( \mathcal{A} \), where ‘partially strong’ means that the compactness and the attractiveness of \( \mathcal{A} \) are in the slightly weaker space

\[ \mathcal{H}_{ps} = (H^1_0 \cap L^{p+1},w) \times L^2 \]

equipped with the partially strong topology \( (\mathcal{H}_{ps},\text{-topology}) \):

\[
(u^n,v^n) \to (u,v) \text{ in } \mathcal{H}_{ps} \text{ if and only if } \\
(u^n,v^n) \to (u,v) \text{ in } H^1_0 \times L^2 \text{ and } u^n \rightharpoonup u \text{ in } L^{p+1}, \tag{4}
\]

where the sign “\( \rightharpoonup \) ” denotes weak convergence. After that, Ding, Yang and Li [5] removed the restriction of ‘partially strong’ for the global attractor in [4]. Ma and Zhong [13] showed that in non-supercritical case: \( 1 \leq p \leq p^* \), the related solution semigroup has a strong \( (H^1_0 \times L^2, H^1_0 \times H^1_0) \)-global attractor \( \mathcal{A} \), that is, the compactness and the attractiveness of \( \mathcal{A} \) are in the topology of the stronger space \( H^1_0 \times H^1_0 \) where \( \mathcal{A} \) lies. For the related researches on this topic, one can see [14, 15].

Uniform attractor (see Def. 2.2 below) and pullback attractor are two basic concepts to study the longtime behavior of non-autonomous evolution equations with various dissipations (cf. [3, 23]).

Given a non-autonomous dynamical process \( U(t,\tau) \), there are two common ways to characterize its asymptotic behavior: roughly speaking, the limit of \( U(t,\tau) \) for a fixed \( t \) as \( \tau \to -\infty \) leads to the definition of the pullback attractor, while the limit of \( U(t,\tau) \) as \( t \to +\infty \) (given sufficient uniformity in \( t \)) (or equivalently, the limit \( U_\sigma(t,\tau) \) as \( t \to +\infty \) (uniformity in \( \sigma \in \Sigma \) (the sign space)) leads to the uniform attractor. Although both of them give rise to global attractor for autonomous case, they are completely different in non-autonomous case.

Recently, Li and Yang [11] investigated the existence and robustness of pullback attractors and pullback exponential attractors of non-autonomous problem (1)-(2) (replacing the strong damping \( -\Delta u_t \) there by more general nonlocal one \(-\sigma(\|\nabla u\|^2)\Delta u_t \)) and obtained the following results: under the assumption that \( f(u) \) is of the supercritical growth: \( 1 \leq p < p^* \), (i) the corresponding dynamical process has a pullback attractor \( \mathcal{A}_\epsilon \) in natural energy space \( \mathcal{H} \) for each \( \epsilon \), and it is upper semicontinuous on \( \epsilon \in [0,1] \); (ii) the related process has a partially strong pullback exponential attractor for each \( \epsilon \in [0,1] \), and it is Hölder continuous on \( \epsilon \in [0,1] \) in the sense of \( \mathcal{H}_{ps} \)-topology. For the related research on this topic, one can see [6, 24].

However, what about the existence of the uniform attractor of problem (1)-(2)? What about the robustness of the uniform attractors on the perturbed parameter \( \epsilon \)? These questions are still open.
Let \((U^\lambda(t, \tau), Y), \sigma \in \Sigma\) be a family of non-autonomous dissipative dynamical systems parameterized by \(\Lambda\) such that for each \(\lambda \in \Lambda\) the corresponding dynamical system possesses a uniform attractor \(A^\Lambda_{\Sigma} \subset Y\), where \(Y\) is a complete metric space with metric \(d_Y\). There are some methods and criteria to show that the map \(\lambda \mapsto A^\Lambda_{\Sigma}\) is upper semicontinuous according to the Hausdorff semi-distance, i.e.,
\[
\lim_{\lambda \to \lambda_0} \text{dist}_Y(A^\lambda_{\Sigma}, A^{\lambda_0}_{\Sigma}) = 0, \quad \forall \lambda_0 \in \Lambda.
\]
However, the lower semicontinuity and hence the full continuity with respect to the Hausdorff metric \(\text{dist}^{\text{sym}}_Y\), is much harder to prove because it requires strict conditions on the structure of the unperturbed attractor, which are rarely satisfied even for slightly simpler global attractor of complicated systems (see Hale and Raugel [7] and Stuart and Humphries [21]). However, recently some advances have been made on this topic. Hoang, Olson and Robinson [8, 9] showed, using the theory of Baire category, that continuity holds for \(\lambda_0\) in a residual set \(\Lambda^* \subseteq \Lambda\) under natural conditions when \(\Lambda\) is a complete metric space. For the related research on this topic, one can see [1].

The purpose of the present paper is to probe into the existence of the uniform attractors and their continuity on the extensibility parameter, in the concrete, motivated by the idea in [9], we shall discuss the residual continuity and the upper semicontinuity of the uniform attractors, respectively. The main results are that under the assumptions that the external force \(g\) is translation bounded and the nonlinearity \(f(u)\) is of optimal supercritical growth \(p : p^* < p < p^{**}\),

(i) the related family of processes \(\{U^\epsilon_g(t, \tau)\}, g \in \Sigma\) has in \(\mathcal{H}\) a compact uniform attractor \(A^\epsilon_{\Sigma}\) for each \(\epsilon \in [0, 1]\); (see Theorem 4.4)

(ii) the family of compact uniform attractor \(\{A^\epsilon_{\Sigma}\}_{\epsilon \in [0, 1]}\) is continuous on the perturbed parameter \(\epsilon\) in a residual set \(I^* \subset [0, 1]\) in the sense of \(\mathcal{H}_{\text{ps}}\)-topology (i.e., partially strong topology); (see Theorem 5.2)

(iii) \(\{A^\epsilon_{\Sigma}\}_{\epsilon \in [0, 1]}\) is upper semicontinuous on \(\epsilon \in [0, 1]\) in \(\mathcal{H}_{\text{ps}}\)-topology. (see Theorem 6.1)

Especially when the nonlinearity \(f(u)\) is of non-supercritical growth \(p : 1 \leq p \leq p^* = \frac{N+2}{(N-2)^*}(N \geq 1)\), \(H^1_0 \hookrightarrow L^{p+1}\) is valid and \(\mathcal{H}_{\text{ps}} = \mathcal{H}\), and all the above conclusions (i)-(iii) still hold. (see Remark 6.2)

We mention that many mathematicians recently concern the uniform attractor of non-autonomous dissipative PDEs with non translation compact external forces. They introduce several new classes of external forces that are not translation compact, but nevertheless allow the attraction in a strong topology of the phase space and give some criteria on this kind of uniform attractor and applications of them (cf. [12, 17, 22, 27, 28]).

We show in the present paper that the weak solutions of non-autonomous Kirchhoff wave model (1)-(2) are of higher partial regularity when \(t > \tau\), which results in that the requirement for the external force \(g : g \in H^1_0(\mathbb{R}; L^2)\) is natural and just allows non translation compact external forces \(g\).

The paper is organized as follows. In Section 2, we state some preliminaries. In Section 3, we give some results on the well-posedness. In Section 4, we discuss the existence of uniform attractors. In Sections 5 and 6, we study the residual continuity and the upper semicontinuity of uniform attractors on the perturbed parameter \(\epsilon\) in \(\mathcal{H}_{\text{ps}}\)-topology, respectively.
2. Preliminaries. In this section, we recall some basic concepts and results of uniform attractor theory which can be found in literatures [3, 12, 17, 22].

Definition 2.1. (i) A family of sets \{U_\sigma(t, \tau) | t \geq \tau, \tau \in \mathbb{R}\}, \sigma \in \Sigma (parameter set) is said to be a family of processes acting on metric space \( M \) if for each \( \sigma \in \Sigma \), \( \{U_\sigma(t, \tau) | t \geq \tau, \tau \in \mathbb{R}\} \) is a process acting on \( M \), i.e., the two-parameter mappings from \( M \) to \( M \) satisfying
\[
U_\sigma(t,s)U_\sigma(s,\tau) = U_\sigma(t,\tau), \quad \forall t \geq s \geq \tau, \tau \in \mathbb{R}, \\
U_\sigma(\tau,\tau) = I \quad \text{(identity operator)}, \quad \tau \in \mathbb{R}.
\]
And the set \( \Sigma \) is said to be the symbol space and \( \sigma \in \Sigma \) is said to be a family of processes acting on metric space \( M \).

(ii) A family of processes \( \Sigma \) is said to be uniformly absorbing set of the family of processes \( \{U_\sigma(t, \tau) | t \geq \tau, \tau \in \mathbb{R}\} \) if for each \( \sigma \in \Sigma \) and \( t, \tau \in \mathbb{R} \), bounded sequences \( \{x_n\}_n \subset E \) and \( \{\xi \}_\tau \in \Sigma \) implies that \( \lim_{n \to \infty} U_\sigma(t,\tau)x_n \to U_\sigma(t,\tau)\xi \in E \).

(iii) A bounded subset \( B_0 \subset M \) is said to be a bounded uniformly \( (w.r.t. \ \sigma \in \Sigma) \) absorbing set of the family of processes \( \{U_\sigma(t, \tau) | t \geq \tau, \tau \in \mathbb{R}\} \) absorbing set of the family of processes \( \{U_\sigma(t, \tau) | t \geq \tau, \tau \in \mathbb{R}\} \).

Definition 2.2. A closed set \( A_\Sigma \subset M \) is said to be the uniform \( (w.r.t. \ \sigma \in \Sigma) \) attractor of the family of processes \( \{U_\sigma(t, \tau) | t \geq \tau, \tau \in \mathbb{R}\} \).

(i) (Uniform attractiveness) \( A_\Sigma \) uniformly \( (w.r.t. \ \sigma \in \Sigma) \) attracts all the bounded subsets in \( M \), i.e., for every bounded subset \( B \subset M \) and \( \tau \in \mathbb{R} \),
\[
\lim_{t \to \infty} \sup_{\sigma \in \Sigma} \text{dist}_M \{U_\sigma(t,\tau)B, A_\Sigma\} = 0,
\]
where, \( \text{dist}_M \{\cdot, \cdot\} \) is the Hausdorff semi-distance in \( M \), i.e.,
\[
\text{dist}_M \{A, B\} = \sup_{x \in A} \inf_{y \in B} d_M(x,y), \quad A, B \subset M.
\]

(ii) (Minimality) For any closed set \( A' \subset M \), if \( A' \) is of property (i), then \( A_\Sigma \subset A' \).

Definition 2.3. [17] (i) A family of processes \( \{U_\sigma(t, \tau) | t \geq \tau, \tau \in \mathbb{R}\} \), \( \sigma \in \Sigma \) acting on Banach space \( E \) is said to be uniformly \( (w.r.t. \ \sigma \in \Sigma) \) asymptotically compact, if for any \( \tau \in \mathbb{R} \), bounded sequences \( \{\xi_n\}_n \subset E \), \( \{\sigma_n\} \subset \Sigma \) and sequence \( \{t_n\}_n \subset \mathbb{R} \) with \( t_n \geq \tau \) and \( t_n \to +\infty \), the sequence \( \{U_{\sigma_n}(t_n, \tau)\xi_n\} \) is precompact in \( E \).

(ii) A family of processes \( \{U_\sigma(t, \tau) | t \geq \tau, \tau \in \mathbb{R}\} \), \( \sigma \in \Sigma \) acting on Banach space \( E \) is said to be norm-to-weak continuous, if for any fixed \( t \) and \( \tau \in \mathbb{R} \) with \( t \geq \tau \), for any sequence \( \{(x_n, \sigma_n)\} \subset E \times \Sigma \), \( (x_n, \sigma_n) \to (x, \sigma) \) in \( E \times \Sigma \) implies that \( U_{\sigma_n}(t, \tau)x_n \to U_{\sigma}(t, \tau)x \) in \( E \).

Lemma 2.4. [22] Assume that \( \Sigma \) is a compact metric space, the translation semigroup \( \{T(t)\}_{t \geq 0} \) in \( \Sigma \), the family of processes \( \{U_\sigma(t, \tau) | t \geq \tau, \tau \in \mathbb{R}\} \), \( \sigma \in \Sigma \) acting on Banach space \( E \) satisfies the translation identity (5) and

(1) it is norm-to-weak continuous;
(2) it has a bounded uniformly \( (w.r.t. \ \sigma \in \Sigma) \) absorbing set \( B_0 \) in \( E \);
(3) it is uniformly \( (w.r.t. \ \sigma \in \Sigma) \) asymptotically compact in \( E \).

Then it has a compact uniform \( (w.r.t. \ \sigma \in \Sigma) \) attractor \( A_\Sigma \), and
\[
A_\Sigma = \omega_{0,\Sigma}(B_0) = \bigcap_{t \geq 0} \left[ \bigcup_{\sigma \in \Sigma} \bigcup_{s \geq t} U_\sigma(s,0)B_0 \right]_E.
\]
Remark 2.5. \[ y \in \omega_0, \Sigma(B_0) \text{ if and only if there exist } \{x_n\} \subset B_0, \{\sigma_n\} \subset \Sigma, \{t_n\} \subset \mathbb{R}^+, \text{ with } t_n \to \infty \text{ such that } U_{\sigma_n}(t_n,0)x_n \to y \text{ in } E, \text{ where } E \text{ is a metric space.}
\]

Definition 2.6. Let \( \Sigma \) be a symbol space and \( B \) be a bounded subset in Banach space \( E \). A function \( \phi(\cdot, \cdot, \cdot) \) defined on \((\Sigma \times \Sigma)\) is said to be a contractive function if for any sequences \( \{x_n\} \subset B \) and \( \{\sigma_n\} \subset \Sigma, \text{ there exist subsequences} \{x_{n_k}\} \subset \{x_n\} \text{ and } \{\sigma_{n_k}\} \subset \{\sigma_n\} \text{ such that} \)
\[
\lim_{k \to \infty} \lim_{l \to \infty} \phi(x_{n_k}, x_{n_l}; \sigma_{n_k}, \sigma_{n_l}) = 0.
\]

Lemma 2.7. \[ \text{Assume that the family of processes } \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \text{ acting on Banach space } E \text{ satisfies translation identity (5), and the following conditions holds:}
\]
\[(i) \text{ it has a bounded uniformly (w.r.t. } \sigma \in \Sigma) \text{ absorbing set } B_0 \subset E;
(ii) \text{ for any } \delta > 0 \text{ there exist } T = T(B_0, \delta) > 0 \text{ and a contractive function } \phi_T \text{ defined on } (B_0 \times B_0) \times (\Sigma \times \Sigma) \text{ such that}
\]
\[
\|U_\sigma(T,0)x - U_\sigma(T,0)y\|_E \leq \delta + \phi_T(x,y; \sigma_1, \sigma_2), \quad \forall x, y \in B_0, \sigma_1, \sigma_2 \in \Sigma.
\]
Then the family of processes \( \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \) is uniformly (w.r.t. \( \sigma \in \Sigma \)) asymptotically compact on \( E \).

Lemma 2.8. \[ \text{Assume that the family of processes } \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \text{ satisfy the translation identity (5) and the symbol space } \Sigma \text{ be translation invariant, i.e., } T(h)\Sigma = \Sigma \text{ for all } h \geq 0. \text{ Then for every } \tau \geq 0 \text{ and } \sigma \in \Sigma, \text{ there exists at least one } \sigma' \in \Sigma \text{ satisfying}
\]
\[
U_\sigma(t, \tau) = U_{\sigma'}(t - \tau + \tau_0, \tau_0), \quad \forall t \geq \tau, \tau_0 \in \mathbb{R}.
\]

Lemma 2.9. \[ \text{Let } X, B, \text{ and } Y \text{ be Banach spaces, } X \hookrightarrow B \hookrightarrow Y,
\]
\[
W = \{u \in L^p(0, T; X)|u_t \in L^2(0, T; Y)\}, \text{ with } 1 \leq p < \infty,
W_1 = \{u \in L^\infty(0, T; X)|u_t \in L^r(0, T; Y)\}, \text{ with } r > 1.
\]
Then
\[
W \hookrightarrow L^p(0, T; B), \quad W_1 \hookrightarrow C([0, T]; B),
\]
where and in the following the sign \( \hookrightarrow \) denotes the continuous embedding, and the sign \( \hookrightarrow \) denotes the compact embedding.

3. Assumptions and well-posedness. In this section, we discuss the well-posedness of problem (1)-(2). We begin with the following notations:
\[
L^p = L^p(\Omega), \quad H^k = W^{k,2}(\Omega), \quad H_0^k = W_0^{k,2}(\Omega), \quad \| \cdot \|_p = \| \cdot \|_{L^p}, \quad \| \cdot \| = \| \cdot \|_{L^2},
\]
with \( p \geq 1 \). We denote the phase spaces
\[
\mathcal{H}_1 = (H_0^1 \cap L^{p+1}) \times H_0^1, \quad \mathcal{H} = (H_0^1 \cap L^{p+1}) \times L^2, \quad \mathcal{H} = H_0^1 \times H^{-1},
\]
which are equipped with usual graph norms. For example,
\[
\|(u, v)\|_{\mathcal{H}}^2 = \|u\|_{H^1}^2 + \|u\|_{L^{p+1}}^2 + \|v\|^2.
\]

Assumption 3.1. (i) \( f \in C^1(\mathbb{R}) \) and
\[
c_0|s|^{p-1} - c_1 \leq f'(s) \leq c_2(1 + |s|^{p-1}), \quad \forall s \in \mathbb{R},
\]
with some \( \frac{N+2}{N-2} = p^* < p < p^{**} = \frac{N+4}{(N-4)^2} \), where \( c_i \) are positive constants and \( a^+ = \max\{a, 0\} \);
(ii) \( (u_0^\tau, u_1^\tau) \in \mathcal{H} \text{ with } \|(u_0^\tau, u_1^\tau)\|_{\mathcal{H}} \leq R.\)
We first define a symbol space generated by the given external force term \( g_0 \), with
\[
g_0, \partial_t g_0 \in L^2_0(\mathbb{R}; L^2) = \{ \phi \in L^2_0(\mathbb{R}; L^2) ||\phi||^2_{L^2_0(\mathbb{R}; L^2)} = \sup_{t \in \mathbb{R}} \int_t^{t+1} ||\phi(s)||^2 ds < +\infty \}.
\]
Define the translation operator
\[
K \text{ where } L^2_0(\mathbb{R}; L^2) \text{ and } \Sigma \text{ be equipped with }
\{
\]
Then \( \Sigma \) is a compact metric space, and
\[
\text{Obviously, } \{ T(h) \}_{h \in \mathbb{R}} \text{ constitutes a translation group on } L^2_0(\mathbb{R}; L^2).
\]
Let \( \text{Assumption 3.1 be valid. Then for each } \}
\[
\text{the treatment of } g \text{ problem (1) (Partial regularity when } t > \tau
\]
\[
\text{(ii) (Energy identity)}
\]
\[
E(\xi^t_0(t)) + \int_s^t \left[ \left\| \nabla u^t_0(s) \right\|^2 - (g, u^t_0) \right] \, ds = E(\xi^t_0(s)), \quad \forall t > s \geq \tau,
\]
where
\[
E(\xi^t_0) = \frac{1}{2} \left[ ||u^\epsilon||^2 + ||\nabla u^\epsilon||^2 + \frac{\epsilon}{2} ||\nabla u^\epsilon||^4 + 2(F(u^\epsilon), 1) \right]
\]
with \( F(s) = \int_0^s f(r) \, dr \).
(iii) (Stability and quasi-stability in $H_{-1}$) the following Lipschitz stability
\[
\| (z(t), z_t(t)) \|_{H_{-1}}^2 \\
\leq K \left[ \| (z(\tau), z_t(\tau)) \|_{H_{-1}}^2 + \| g_1 - g_2 \|_{L^2(\tau, t; H^{-1})}^2 \right], \quad t \in [\tau, T],
\]
and quasi-stability
\[
\| (z(t), z_t(t)) \|_{H_{-1}}^2 \\
\leq e^{-\kappa(t-\tau)} \| (z(\tau), z_t(\tau)) \|_{H_{-1}}^2 \\
+ K \int_{\tau}^{t} \left[ \| (z(s), z_t(s)) \|_{L^2(H^{-2})}^2 + \| (g_1(s) - g_2(s)) \|_{H_{-1}}^2 \right] ds, \quad t \in [\tau, T],
\]
hold for $z = u_{\epsilon} - u_{\epsilon'}$, where $u_{\epsilon}, u_{\epsilon'}$ are two weak solutions of problem (1)-(2) corresponding to initial data $(u_{\epsilon}^0(\tau), u_{\epsilon'}^0(\tau)) \in H$, with $\| (u_{\epsilon}^0(\tau), u_{\epsilon'}^0(\tau)) \|_H \leq R$, and $g_1, g_2 \in \Sigma$, respectively.

For any $g \in \Sigma$, we define the solution operator
\[
U_g(t, \tau) : H \to H, \quad U_g(t, \tau)(u_0^0, u_0^1) = \xi_\epsilon(t), \quad t \geq \tau,
\]
where $u_\epsilon$ is a weak solution of problem (1)-(2). Theorem 3.2 shows that $\{U_g(t, \tau), g \in \Sigma, \epsilon \in [0, 1]\}$ is a family of processes acting on the phase space $H$. The uniqueness of weak solutions implies the translation identity
\[
U_g(t + s, \tau + s) = U_{T(s)g}(t, \tau), \quad \forall t \geq \tau, \quad \tau \in \mathbb{R}, \quad s \geq 0, \quad \epsilon \in [0, 1].
\]

4. Existence of uniform attractors. In this section, we show the existence of the uniform attractor of the process $U_g(t, \tau)$ for each $\epsilon \in [0, 1]$. For simplicity, we omit the superscript $\epsilon$ and denote $u = u_\epsilon$ in the following.

**Lemma 4.1.** Let Assumption 3.1 be valid. Then

(i) For any sequence $\{(\xi_n, g_n)\} \subset H \times \Sigma$ with $(\xi_n, g_n) \to (\xi, g)$ in $H_{-1} \times \Sigma$, we have
\[
U_{g_n}(t, \tau) \xi_n \to U_{g}(t, \tau) \xi \quad \text{in} \quad H_{-1}, \quad \forall \epsilon \in [0, 1].
\]

(ii) The family of processes $\{U_g(t, \tau), g \in \Sigma\}$ is norm-to-weak continuous for each $\epsilon \in [0, 1]$. 

**Proof.** (i) The fact $g_n \to g$ in $\Sigma$ implies that
\[
g_n \to g \quad \text{in} \quad L^2(\tau, t; H^{-1}), \quad \forall t > \tau.
\]
Indeed, it follows from estimate (9) that both the sequences $\{g_n\}$ and $\{\partial_t g_n\}$ are bounded in $L^2(\tau, t; L^2)$, which implies that $\{g_n\}$ is precompact in $L^2(\tau, t; H^{-1})$ for $L^2 \hookrightarrow H^{-1}$ (see Lemma 2.9). So formula (17) holds. The combination of (17) and stability estimate (13) yields (16).

(ii) Let $(\xi_n, g_n) \to (\xi, g)$ in $H \times \Sigma$. For any given $t > \tau$, by (16) and the fact that $H \hookrightarrow H_{-1}$,
\[
U_{g_n}(t, \tau) \xi_n = \xi_{u_n}(t) \to \xi_u(t) = U_{g}(t, \tau) \xi \quad \text{in} \quad H_{-1}.
\]
It follows from estimate (10) that $\{\xi_{u_n}(t)\}$ is bounded in $H$. Thus,
\[
u_n(t) \to u(t) \quad \text{in} \quad L^{p+1}, \quad u^\epsilon(t) \to u(t) \quad \text{in} \quad L^2,
\]
which combining with (18) shows that
\[
\xi_{u_n}(t) \to \xi_u(t) \quad \text{in} \quad H.
\]
This completes the proof. \qed
Lemma 4.2. Let Assumption 3.1 be valid. Then the family of processes \{U^\tau_g(t, \tau)\}, \(g \in \Sigma, \epsilon \in [0, 1]\) has a uniformly (w.r.t. \(g \in \Sigma\) and \(\epsilon \in [0, 1]\)) absorbing set \(B_0\), which is closed in \(H_{-1}\) and bounded in \(H_1\). Moreover, there is a positive constant \(T_0\) such that
\[
\bigcup_{\epsilon \in [0, 1]} \bigcup_{g \in \Sigma} U^\tau_g(t, \tau) B_0 \subset B_0, \quad \forall t > T_0 + \tau.
\]

Proof. Using the multiplier \(u_t + \delta u (= u_t^\epsilon + \delta u^\epsilon)\) in Eq. (1), we obtain
\[
\frac{d}{dt} \Gamma(\xi_u(t)) + \Psi(\xi_u(t)) = 0,
\]
where \(\xi_u = (u, u_t)\),
\[
\Gamma(\xi_u) = \frac{1}{2} \left( \|u_t\|^2 + \frac{\epsilon}{2} \|\nabla u\|^2 + 2(f(u), 1) \right) + \delta \left( \frac{1}{2} \|\nabla u\|^2 + (u_t, u) \right),
\]
\[
\Psi(\xi_u) = \|\nabla u_t\|^2 - \delta \|u_t\|^2 + \delta \left( \|\nabla u\|^2 + c \|\nabla u\|^4 + (f(u), u) - (g, u_t + \delta u) \right).
\]
Assumption (7) implies that
\[
\frac{c_0}{2p} |u|^{p+1} - C \leq f(u)u \leq C(1 + |u|^{p+1}),
\]
\[
\frac{c_0}{2p(p+1)} |u|^{p+1} - C \leq F(u) \leq C(1 + |u|^{p+1}),
\]
\[
f(u)u - F(u) + \frac{c_1}{2} |u|^2 \geq 0.
\]
Thus a simple calculation shows that
\[
\Gamma(\xi_u) \geq \frac{1}{4} \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 + \frac{c_0}{2p(p+1)} \|u\|_{p+1}^{p+1} - C \geq \kappa \|\xi_u\|^2_{\mathcal{H}} - C
\]
\[
\Gamma(\xi_u) \leq C \left( \|u_t\|^2 + \|\nabla u\|^2 + \|u\|_{p+1}^{p+1} + \epsilon \|\nabla u\|^4 + 1 \right),
\]
\[
\Psi(\xi_u) \geq \left( 1 - \frac{2\delta}{\lambda_1} \right) \|u_t\|^2 + \delta \left( \frac{1}{2} \|\nabla u\|^2 + \epsilon \|\nabla u\|^4 + \frac{c_0}{2p} |u|^{p+1} \right) - C(1 + \|g\|^2)
\]
\[
\geq \kappa \Gamma(\xi_u) - C(1 + \|g\|^2)
\]
for \(\delta > 0\) suitably small, where \(\lambda_1\) is the first eigenvalue of \(-\Delta\) with Dirichlet boundary condition and \(\kappa\) is a small positive constant. Hence, we have
\[
\frac{d}{dt} \Gamma(\xi_u(t)) + \kappa \Gamma(\xi_u(t)) \leq C \left( 1 + \|g(t)\|^2 \right),
\]
which implies that
\[
\|\xi_u(t)\|^2_{\mathcal{H}} \leq Q(\|\xi_u(\tau)\|_{\mathcal{H}}) e^{-\kappa(t-\tau)} + C \left( 1 + \|g(t)\|^2_{L^2(\mathbb{R};L^2)} \right), \quad \forall t \geq \tau,
\]
for all \(\epsilon \in [0, 1], g \in \Sigma\) and \(\xi_u(\tau) \in \mathcal{H}\), where \(\xi_u(t) = U^\tau_g(t, \tau) \xi_u(\tau)\), \(Q\) is a monotone positive function.

Let
\[
\mathcal{B} = \{ \xi \in \mathcal{H} | \|\xi\|_{\mathcal{H}} \leq R_0 \} \quad \text{with} \quad R_0^2 = 2C \left( 1 + \|g_0\|^2_{L^2(\mathbb{R};L^2)} \right).
\]
Estimate (20) shows that there is a \(T > 0\) such that
\[
\bigcup_{\epsilon \in [0, 1]} \bigcup_{g \in \Sigma} U^\tau_g(t, \tau) \mathcal{B} \subset \mathcal{B}, \quad \forall t \geq T + \tau.
\]
Define
\[ B_0 = \left( \bigcup_{c \in [0,1]} \bigcup_{g \Sigma} U_g^c(t,0)B \right)_{\mathcal{H}_1} \] (19)
Then \( B_0 \) is the desired absorbing set. Indeed, for any bounded set \( D \subset \mathcal{H} \), by (20), there exists a \( t_D \geq 0 \) such that
\[ \bigcup_{c \in [0,1]} \bigcup_{g \Sigma} U_g^c(t,0)D \subset B \text{ as } t \geq t_D. \]
When \( t \geq t_D + T + 1 + \tau \), by Lemma 2.8, there exist at least one \( g' \in \Sigma \) for any \( \epsilon \in (0,1] \) and \( \tau \in \mathbb{R} \), such that
\[ U_g^c(t,\tau)D = U_g^c(t-\tau,0)D = U_g^c(t-\tau,t_D)U_g^c(t_D,0)D \]
\[ \subset U_g^c(t-\tau,t_D)B = U_{g(t_D)}^c(t-\tau-t_D,0)B \subset B_0, \]
where we have used translation identity (15). Due to
\[ \bigcup_{c \in [0,1]} \bigcup_{g \Sigma} U_g^c(t-1,t-1)U_g^c(t,0)B \subset \bigcup_{c \in [0,1]} \bigcup_{g \Sigma} U_g^c(t-1,t)B, \]
we infer from estimates (11) and (20) that \( B_0 \) is bounded in \( \mathcal{H}_1 \). Taking \( T_0 = T + 1 \), the combination of Lemma 2.8 and formula (21) gives (19).

**Lemma 4.3.** Let Assumption 3.1 be valid. Then for every \( \epsilon \in [0,1] \), any sequences \( \{g_n\} \subset \Sigma, \{\xi_n\} \subset B_0 \) and \( \tau_n \to -\infty \), the sequence \( \{U_{g_n}^\epsilon(0,\tau_n)\xi_n\} \) is precompact in \( \mathcal{H} \).

**Proof.** Let
\[ \xi_n^u(t) = (u^n(t), u^n(t)) = U_{g_n}^\epsilon(t,\tau_n)\xi_n, \quad \forall t \geq \tau_n, \quad n \in \mathbb{N}. \] (22)

For any fixed \( T \in \mathbb{N} \), there exist \( N \in \mathbb{N}^+ \) such that \( -T \geq \tau_n + T_0 + 1 \) for all \( n \geq N \). Hence when \( n \geq N \), by (19) and (22),
\[ \xi_n^u(t) \in U_{g_n}^\epsilon(t,\tau_n)B_0 = U_{g_n}^\epsilon(t,-T-1)U_{g_n}^\epsilon(-T-1,\tau_n)B_0 \]
\[ \subset U_{g_n}^\epsilon(t,-T-1)B_0, \quad t \in [-T,0]. \]
Therefore, it follows from (10)-(11) that
\[ \{(u^n, u^n, u^n)\}_{n \geq N} \text{ is bounded in } L^\infty(-T,0; (H^1_0 \cap L^{p+1}) \times H^1_0 \times H^{-1}). \]
Then, there exist \( \xi_u \in L^\infty(-T,0; \mathcal{H}_1) \) and \( g \in \Sigma \) such that (subsequence if necessary)
\[ \xi_n^u(\cdot) \to \xi_u(\cdot) \text{ weakly* in } L^\infty(-T,0; \mathcal{H}_1), \]
\[ \xi_n^u(\cdot) \to \xi_u(\cdot) \text{ in } C([-T,0]; H^{1-\delta} \times H^{1-\delta}), \quad \text{with } \delta \in (0,1), \]
\[ \xi_n^u(t) \to \xi_u(t) \text{ in } \mathcal{H}_1, \quad t \in [-T,0], \]
\[ g_n \to g \text{ in } \Sigma, \]
where we have used the compactness of \( \Sigma \) and Lemma 2.9. It follows from estimate (14) that
\[ \|U_{g_1}^\epsilon(t,0)x - U_{g_2}^\epsilon(t,0)y\|_{\mathcal{H}_1}^2 \]
\[ \leq e^{-\eta t}\|x - y\|_{\mathcal{H}_1}^2 \]
\[ + C \int_0^t \|g_1(s) - g_2(s)\|_{H^{-2}}^2 + \|U_{g_1}^\epsilon(s,0)x - U_{g_2}^\epsilon(s,0)y\|_{L^2 \times H^{-2}}^2 ds, \quad \forall t \geq 0, \] (23)
for any $\epsilon \in [0, 1]$, $x, y \in B_0$ and $g_1, g_2 \in \Sigma$, where $C = C(t, R_0, \|g_0\|_{L^2(R; L^2)})$. By the compactness of $\Sigma$, estimate (10), Lemma 2.9 and the similar arguments as (17), we obtain that

$$\Sigma \text{ is precompact in } L^2(0, t; H^{-1}),$$

$$\bigcup_{g \in \Sigma} U^e_g(\cdot, 0) B_0 \text{ is precompact in } L^2(0, t; L^2 \times H^{-2}), \ \forall t > 0.$$

Thus, it follows from (24) that for any $\delta > 0$, there exist $T = T(B_0, \delta) > 0$ and a contractive function

$$\Psi_T(x, y; g_1, g_2)$$

defined on $(B_0 \times B_0) \times (\Sigma \times \Sigma)$ such that

$$\|U^e_{g_1}(T, 0)x - U^e_{g_2}(T, 0)y\|_{H^{-1}} \leq \delta + \Psi_T(x, y; g_1, g_2).$$

By Lemma 2.7, the family of processes $\{U^e_g(t, \tau)\}, g \in \Sigma$ is uniformly (w.r.t. $g \in \Sigma$) asymptotically compact in $H_{-1}$. Therefore, one can infer from Lemma 2.8 and formula (23) that (subsequence if necessary)

$$\xi^n_u(-T) = U^e_{g_n}(-T, \tau_n) \xi_n \rightarrow \xi_u(-T) \text{ in } H_{-1}. \quad (25)$$

By formulas (16), (23) and the uniqueness of the limit,

$$\xi^n_u(t) = U^e_{g_n}(t, -T) \xi^n_u(-T)$$

$$\rightarrow U^e_g(t, -T) \xi_u(-T) = \xi_u(t) \text{ in } H_{-1}, \ \forall t \in [T, 0]. \quad (26)$$

The combination of (23) and (26) yields

$$\xi^n_u(t) \rightarrow \xi_u(t) \text{ in } H_{ps} \text{ and a.e. in } \Omega, \ \forall t \in [-T, 0]. \quad (27)$$

By the standard diagonal process, we can extract a subsequence (still denoted by itself) such that (23) and (27) hold for all $T \in \mathbb{N}$. Obviously, energy identity (12) holds for $\xi^n_u$ and $\xi_u$. Thus, using Ball’s energy method [2] and repeating the similar arguments as in Theorem 3.2 of [11], we obtain

$$\lim_{n \rightarrow \infty} \|u^n(0)\|_{p+1} = \|u(0)\|_{p+1}. \quad (28)$$

By the uniform convexity of $L^{p+1}$ and (23), we have

$$\lim_{n \rightarrow \infty} \|u^n(0) - u(0)\|_{p+1} = 0. \quad (28)$$

The combination of (27)-(28) yields

$$\xi^n_u(0) = U^e_{g_n}(0, \tau_n) \xi_n \rightarrow \xi_u(0) \text{ in } H,$$

which completes the proof. \hfill \Box

**Theorem 4.4.** Let Assumption 3.1 be valid. Then the family of processes $\{U^e_g(t, \tau)\}$, $g \in \Sigma$ has in $H$ a compact uniform (w.r.t. $g \in \Sigma$) attractor $A^e_{\Sigma}$ for each $\epsilon \in [0, 1]$, and

$$A^e_{\Sigma} = \omega^e_{0, \Sigma}(B_0) = \bigcap_{t \geq 0} \left[ \bigcup_{g \in \Sigma} \bigcup_{s \geq t} U^e_g(s, 0) B_0 \right]_H \subset B_0. \quad (29)$$
Proof. Since the family of processes \( \{U_g^\epsilon(t, \tau)\} \), \( g \in \Sigma \) satisfies translation identity (15), it is norm-to-weak continuous for each \( \epsilon \in [0, 1] \) (see Lemma 4.1) and has a uniformly (w.r.t. \( g \in \Sigma \) and \( \epsilon \in [0, 1] \)) absorbing set \( \mathcal{B} \) (see Lemma 4.2), by Lemma 2.4, it is sufficient to prove Theorem 4.4 to show the precompactness of the sequence \( \{U_g^\epsilon(t_n, \tau)\} \) in \( \mathcal{H} \), where \( \{\xi_n\} \subset \mathcal{H} \) is bounded, \( \tau \leq t_n \to +\infty \) as \( n \to \infty \) (see Def. 2.3: (i)). Since \( \mathcal{B} \) is a uniformly absorbing set, there exists a \( \tau_0 \geq \tau \) such that \( U_g^\epsilon(0, \tau_0)\xi_n \in \mathcal{B} \) for all \( n \in \mathbb{N} \). Thus, by translation identity (15),

\[
U_g^\epsilon(t_n, \tau)\xi_n = U_g^\epsilon(t_n, \tau_0)U_g^\epsilon(\tau_0, \tau)\xi_n = U_g^\epsilon(0, \tau_0 - t_n)\eta_n, \quad n \geq N_0,
\]

where \( N_0 \in \mathbb{N}^+ \) such that

\[
t_n \geq \max\{0, \tau_0\}, \quad \forall n \geq N_0, \quad T(t_n)g_n \in \Sigma, \quad n \geq N_0.
\]

\[
\eta_n = U_g^\epsilon(\tau_0, \tau)\xi_n \in \mathcal{B}, \quad \tau_n = \tau_0 - t_n \to -\infty.
\]

It follows from Lemma 4.3 that the sequence \( \{U_g^\epsilon(0, \tau_0 - t_n)\eta_n\} \) is precompact in \( \mathcal{H} \). Therefore, by Lemma 2.4, we get the conclusions of Theorem 4.4. \( \square \)

5. Residual continuity of the uniform attractors in \( \mathcal{H}_{ps} \)-topology. The aim of this section is to show that the family of compact uniform attractors \( \{\mathcal{A}_\Sigma^\epsilon\}_{\epsilon \in [0, 1]} \) as shown in Theorem 4.4 is continuous on \( \epsilon \) in a residual set \( I^* \subset [0, 1] \) in the sense of \( \mathcal{H}_{ps} \)-topology.

Definition 5.1. A set \( \Lambda \) in a complete metric space \( Y \) is said to be nowhere dense if its closure contains no non-empty open sets. A set \( \Lambda \) is said to be residual if its complement is the countable union of nowhere dense sets.

Obviously, any residual subset of a complete metric space \( Y \) is dense in \( Y \).

We consider the uniformly absorbing set \( \mathcal{B} \) as a topological space equipped with \( \mathcal{H}_{ps} \)-topology as shown in (1). Since \( \mathcal{B} \) is bounded in \( \mathcal{H}_1 \), this \( \mathcal{H}_{ps} \)-topology can be induced by the following metric \( \rho \):

\[
\rho(\xi_u, \xi_v) = \| \nabla(u_0 - v_0) \| + \| u_1 - v_1 \| + \sum_{n=1}^{\infty} 2^{-n} \frac{|(u_0 - v_0, e_n)|}{1 + |(u_0 - v_0, e_n)|},
\]

where \( \xi_u = (u_0, u_1), \xi_v = (v_0, v_1) \in \mathcal{B}, \{e_n\} \subset H^{-1} \cap L^{1+\frac{1}{p}} \) such that \( \|e_n\|_{H^{-1}} = 1 \) and \( \text{span}\{e_n|n \in \mathbb{N}\} \) is dense in \( L^{1+\frac{1}{p}} \) (cf. [4]).

For any \( \xi_u = (u_0, u_1), \xi_v = (v_0, v_1) \in \mathcal{B} \), by the interpolation and the boundedness of \( \mathcal{B} \) in \( \mathcal{H}_1 \), we have

\[
\|\xi_u - \xi_v\|_{H_0^1 \times L^2} \leq \| \nabla(u_0 - v_0) \| + \| \nabla(u_1 - v_1) \|^{1/2} \| u_1 - v_1 \|^{1/2}_{H^{-1}} \leq C(\mathcal{B}) \|\xi_u - \xi_v\|_{H^{-1}}^{1/2}.
\]

Due to

\[
\frac{|(u_0 - v_0, e_n)|}{1 + |(u_0 - v_0, e_n)|} \leq \frac{\| \nabla(u_0 - v_0) \| \| e_n \|_{H^{-1}}}{1 + |(u_0 - v_0, e_n)|} \leq \| \nabla(u_0 - v_0) \|, \quad \forall n \geq 1,
\]

we see from (30)-(31) that

\[
\rho(\xi_u, \xi_v) \leq 2\|\xi_u - \xi_v\|_{H_0^1 \times L^2} \leq C(\mathcal{B}) \|\xi_u - \xi_v\|_{H^{-1}}^{1/2}.
\]
Theorem 5.2. Let Assumption 3.1 be valid. Then the compact uniform attractors $A_{\epsilon}^\Sigma$ as shown in Theorem 4.4 are continuous with respect to $\epsilon$ at a residual subset $I^*$ of $[0, 1]$ in the sense of $H_{ps}$-topology, i.e.,

$$\lim_{\epsilon \to \epsilon_0} \text{dist}^{symm}_{ps}\{A_{\epsilon}^\Sigma, A_{\epsilon_0}^\Sigma\} = 0, \ \forall \epsilon_0 \in I^*,$$

where $\text{dist}^{symm}_{ps}\{A, B\} = \max\{\text{dist}_{ps}\{A, B\}, \text{dist}_{ps}\{B, A\}\}$, and

$$\text{dist}_{ps}\{A, B\} = \sup_{x \in A} \inf_{y \in B} \rho(x, y), \ \ A, B \subset B_0.$$

To prove Theorem 5.2, we first show the following lemmas and theorem, which are indispensable for our argument.

Lemma 5.3. [8] Let $X$ be a complete metric space, $Y$ be a metric space, $f_n : X \to Y$ be a continuous map for each $n \in \mathbb{N}$, and $f$ be the pointwise limit of $f_n$, i.e.,

$$f(x) = \lim_{n \to \infty} f_n(x), \ \forall x \in X.$$

Then, there exists a residual subset $I$ of $X$ such that $f$ is continuous at every point $x \in I$.

Motivated by the idea of Hoang, Olason and Robinson (cf. Theorem 4.1 in [9]), we firstly establish an abstract criterion. For the readers' convenience, we give the details of the proof.

Theorem 5.4. Let $(I, d_I)$ be a complete metric space, the family of processes $\{U_\epsilon^\sigma(t, \tau)\}, \sigma \in \Sigma$ acting on the metric space $(M, d_M)$ satisfy the translation identity (5) for every $\epsilon \in I$, and the symbol space $\Sigma$ be translation invariant. Suppose that there is a closed bounded subset $K \subset M$ such that

(C1) for every $\epsilon \in I$, the set

$$A_{\epsilon}^\Sigma := \bigcap_{t \geq 0} \left[ \bigcup_{\sigma \in \Sigma} \bigcup_{s \geq t} U_\epsilon^\sigma(s, 0)K \right]_M$$

uniformly (w.r.t.$\sigma \in \Sigma$) attracts $K$, that is,

$$\lim_{t \to \infty} \sup_{\sigma \in \Sigma} \text{dist}_M(U_\epsilon^\sigma(t, 0)K, A_{\epsilon}^\Sigma) = 0;$$

(C2) there is a positive constant $T$ such that

$$\bigcup_{\epsilon \in [0, 1]} \bigcup_{\sigma \in \Sigma} U_\epsilon^\sigma(t, 0)K \subset K, \ \forall t \geq T;$$

(C3) for any $t \geq T$ the mapping $\epsilon \to U_\epsilon^\sigma(t, 0)x$ is uniformly (w.r.t. $\sigma \in \Sigma$ and $x \in K$) continuous on $\epsilon \in I$, that is,

$$\lim_{\epsilon \to \epsilon_0} \sup_{\sigma \in \Sigma} \sup_{x \in K} d_M(U_\epsilon^\sigma(t, 0)x, U_{\epsilon_0}^\sigma(t, 0)x) = 0, \ \forall \epsilon_0 \in I.$$

Then $A_{\epsilon}^\Sigma$ is continuous on $\epsilon$ at a residual subset $I^*$ of $I$, that is,

$$\lim_{\epsilon \to \epsilon_0} \text{dist}^{symm}_{ps}(A_{\epsilon}^\Sigma, A_{\epsilon_0}^\Sigma) = 0, \ \forall \epsilon_0 \in I^*.$$

Proof. Let the collection

$$BC(M) = \{D \subset M| D \text{ is non-empty, closed and bounded subsets of } M\}$$

equipped with the Hausdorff metric $\text{dist}^{symm}_{ps}$. Then $(BC(M), \text{dist}^{symm}_{ps})$ constitutes a metric space.
For every \( \epsilon \in I \), by condition (C_1), we have
\[
\lim_{t \to \infty} \text{dist}_M \left( [\cup_{\sigma \in \Sigma} U_\sigma^r(t,0)K]_M, \mathcal{A}_\Sigma^r \right) = \lim_{t \to \infty} \text{dist}_M \left( [\cup_{\sigma \in \Sigma} U_\sigma^r(t,0)K, \mathcal{A}_\Sigma^r \right)
\leq \lim_{t \to \infty} \sup_{\sigma \in \Sigma} \text{dist}_M \left( U_\sigma^r(t,0)K, \mathcal{A}_\Sigma^r \right) = 0. \tag{33}
\]

On the other hand, one can infer from conditions (C_1) and (C_2), the translation invariance of \( \Sigma \) that
\[
\text{dist}_M \left( [\cup_{\sigma \in \Sigma} U_\sigma^r(t,0)K]_M, \mathcal{A}_\Sigma^r \right) = 0, \quad \forall \epsilon \in I. \tag{35}
\]
Define
\[K_\epsilon(t) := \bigcup_{\sigma \in \Sigma} U_\sigma^r(t,0)K, \quad \forall \epsilon \in I, \ t \geq T.\]
It follows from condition (C_2) that \( [K_\epsilon(t)]_M \subset [K]_M \subset K \), which implies that \([K_\epsilon(t)]_M \) is a closed bounded subset of \( M \) for all \( \epsilon \in I \) and \( t \geq T \).

Given \( \epsilon_0 \in I, \ t \geq T \) and \( \eta > 0 \), by the condition (C_3), there exists a \( \delta = \delta(\epsilon_0, t, \eta) > 0 \) such that for all \( \epsilon \in I \) with \( d_I(\epsilon, \epsilon_0) < \delta \),
\[
\sup_{\sigma \in \Sigma} \text{dist}_M \left( U_\sigma^r(t,0)x, U_\sigma^{\epsilon_0}(t,0)x \right) < \eta,
\]
which implies that
\[
\text{dist}_M \left( [K_\epsilon(t)]_M, [K_{\epsilon_0}(t)]_M \right) = \text{dist}_M \left( K_\epsilon(t), K_{\epsilon_0}(t) \right)
\leq \sup_{\sigma \in \Sigma} \text{dist}_M \left( U_\sigma^r(t,0)x, U_\sigma^{\epsilon_0}(t,0)x \right) < \eta. \tag{36}
\]
Similarly,
\[
\text{dist}_M \left( [K_{\epsilon_0}(t)]_M, [K_\epsilon(t)]_M \right) < \eta \quad \text{for all} \quad \epsilon \in I \quad \text{with} \quad d_I(\epsilon, \epsilon_0) < \delta. \tag{37}
\]

The combination of (36)-(37) gives
\[
\text{dist}_M \left( [K_\epsilon(t)]_M, [K_{\epsilon_0}(t)]_M \right) < \eta \quad\text{for all} \quad \epsilon \in I \quad\text{with} \quad d_I(\epsilon, \epsilon_0) < \delta. \tag{38}
\]
that is, the mapping \( \epsilon \mapsto [K_\epsilon(t)]_M \) is continuous from \( I \) into \( BC(M) \). The result now follows from (35) and Lemma 5.3.

**Lemma 5.5.** Let Assumption 3.1 be valid. Then for any \( t \geq \tau \), we have
\[
\sup_{g \in \Sigma} \sup_{\xi \in \mathcal{B}_0} \| U_g^r(t, \tau)\xi - U_g^{\epsilon_0}(t, \tau)\xi \|_{H^{-1}}.
\]
\[ \leq C(B_0)e^{C(B_0)(t-\tau)}|\epsilon_1 - \epsilon_2|, \quad \forall \epsilon_1, \epsilon_2 \in [0,1], \]

where \( C(B_0) \) is a positive constant.

**Proof.** Let \( \xi^i_t(t) = (u^i(t), u^i_t(t)) = U^i_t(t,\tau)\xi, \quad i = 1,2. \) It follows from energy identity (12), dissipative estimate (20), the forward invariance (19) and a simple calculation that

\[ \|\xi^i_t(t)\|_{\mathcal{H}^1}^2 + \|u^i_t(t)\|_{H^{-2}}^2 + \int_\tau^t \|\nabla u^i(s)\|_{2}^2 ds \leq C(B_0), \quad \forall t \geq \tau, \quad i = 1,2. \]  

(39)

Then \( z = u^1 - u^2 \) solves

\[
\begin{aligned}
&z_{tt} - \Delta z_t - \Delta z + f(u^1) - f(u^2) \\
= &\epsilon_1\|\nabla u^1\|_2^2 \Delta u^1 - \epsilon_2\|\nabla u^2\|_2^2 \Delta u^2, \quad t > \tau, \\
&z(\tau) = 0, \quad z_t(\tau) = 0.
\end{aligned}
\]

(40)

Using the multiplier \( (-\Delta)^{-1}z_t + \delta z \) in Eq. (40), we obtain

\[ \frac{d}{dt} \Phi(z, z_t) + (1-\delta)|z_t|^2 + \delta|\nabla z|^2 + (f(u^1) - f(u^2), (-\Delta)^{-1}z_t + \delta z) = I_1 + I_2, \]

(41)

where

\[ \Phi(z, z_t) = \frac{1}{2}\left(\|z_t\|_{H^{-1}}^2 + \|z\|_2^2 + \|\nabla z\|_2^2 + \delta(z_t, z) \right) \sim \|(z, z_t)\|_{\mathcal{H}^{-1}}^2, \]

\[ I_1 = \left(\epsilon_1 - \epsilon_2\right)\|\nabla u^1\|_2^2 \Delta u^1 + \epsilon_2(u^1 + u^2, -\Delta z) \right) \left(\Delta u^1, (-\Delta)^{-1}z_t + \delta z \right), \]

\[ I_2 = \epsilon_2\|\nabla u^2\|_2^2 \left(\Delta z, (-\Delta)^{-1}z_t + \delta z \right) \]

for \( \delta > 0 \) suitably small. It follows from estimate (39) that

\[ |I_1 + I_2| \leq C(B_0) \left|\epsilon_1 - \epsilon_2\right| + \|\nabla z\| \left(\|z_t\|_{H^{-1}} + \|\nabla z\| \right) < \leq C(B_0) \|(z, z_t)\|_{\mathcal{H}^{-1}}^2 + \left|\epsilon_1 - \epsilon_2\right|^2. \]

Due to \( p < p^* \), which means \( H^{2-\theta} \rightarrow L^{p+1} \) for \( \theta : 0 < \theta \ll 1 \), by Assumption 3.1 and the interpolation, we have

\[ (f(u^1) - f(u^2), z) \geq -C\|z\|_2^2 + C \int_{\Omega} \left(|u^1|^{p-1} + |u^2|^{p-1}\right)|z|^2 dx, \]

and

\[ \left|(f(u^1) - f(u^2), (-\Delta)^{-1}z_t)\right| \]

\[ \leq C \int_{\Omega} \left(1 + |u^1|^{p-1} + |u^2|^{p-1}\right)|z|\left((-\Delta)^{-1}z_t\right) dx \]

\[ \leq \frac{\delta C}{2} \int_{\Omega} \left(1 + |u^1|^{p-1} + |u^2|^{p-1}\right)|z|^2 dx \]

\[ + C \left(1 + \|u^1\|_{p+1}^{p-1} + \|u^2\|_{p+1}^{p-1}\right) \left((-\Delta)^{-1}z_t\right)_{p+1}^2 \]

\[ \leq \frac{\delta C}{2} \int_{\Omega} \left(|u^1|^{p-1} + |u^2|^{p-1}\right)|z|^2 dx + \delta\|z_t\|_2^2 + C(B_0)(\|z\|_2^2 + \|z_t\|_{\mathcal{H}^{-1}}^2). \]

Inserting above estimates into (41) yields

\[ \frac{d}{dt} \Phi(z(t), z_t(t)) \leq |\epsilon_1 - \epsilon_2|^2 + C(B_0)\Phi(z(t), z_t(t)), \quad t > \tau. \]  

(42)

Applying the Gronwall inequality to (42) over \((\tau, t)\) gives the conclusions of Lemma 5.5. \( \square \)
Proof of Theorem 5.2. Taking $M = \mathcal{H}$ with $d_M(x, y) = \|x - y\|_{\mathcal{H}_{-1}}$, and $I = [0, 1]$ in Theorem 5.4, Lemma 4.2 shows that $K = B_0$ is a closed bounded subset of $M$. Since $B_0$ is bounded in $\mathcal{H}_1$ and $\mathcal{H}_1 \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{-1}$, one can infer from Remark 2.5, Theorem 4.4 and Lemma 4.3 that, the uniform attractor $\mathcal{A}_t$ given by formula (29) satisfies

$$\mathcal{A}_t = \bigcap_{t \geq 0} \left( \bigcup_{g \in \Sigma} \bigcup_{s \geq t} U^*_g(s, 0)B_0 \right),$$

and

$$\lim_{t \to \infty} \sup_{g \in \Sigma} \text{dist}_{\mathcal{H}_{-1}}(U^*_g(t, 0)B_0, \mathcal{A}_t) = 0$$

for all $\epsilon \in [0, 1]$, that is, the condition $(C_1)$ of Theorem 5.4 is valid. Taking $T_0 = T$, Lemma 4.2 and Lemma 5.5 imply that the conditions $(C_2)$ and $(C_3)$ hold.

By formula (32) and the fact that $\bigcup_{\epsilon \in [0, 1]} \mathcal{A}_{\Sigma}^* \subset B_0$, we have

$$\lim_{\epsilon \to \epsilon_0} \sup_{g \in \Sigma} \text{dist}_{\mathcal{H}_{-1}}(U^*_g(t, 0)B_0, \mathcal{A}_{\Sigma}^*) = 0, \quad \forall \epsilon_0 \in I^*.$$  

Thus by Theorem 5.4, there is a residual subset $I^*$ of $[0, 1]$ such that

$$\lim_{\epsilon \to \epsilon_0} \sup_{g \in \Sigma} \left[ \sup_{t \to \infty} \text{dist}_{\mathcal{H}_{-1}}(U^*_g(t, 0)B_0, \mathcal{A}_{\Sigma}^*) \right] = 0, \quad \forall \epsilon_0 \in I^*,$$

that is, the conclusion of Theorem 5.2 holds. \hfill \Box

6. Upper semicontinuity of the uniform attractors in $\mathcal{H}_{ps}$-topology. Although the family of uniform attractors $\{\mathcal{A}_{\Sigma}^*\}_{\epsilon \in [0, 1]}$ is continuous on $\epsilon$ in a residual set $I^* \subseteq [0, 1]$, and $I^*$ is dense in $[0, 1]$, we do not know whether $I^* = [0, 1]$ especially whether $0 \in I^*$. In this section, as a supplement we show that the family of uniform attractors $\{\mathcal{A}_{\Sigma}^*\}_{\epsilon \in [0, 1]}$ is upper continuous on $\epsilon \in [0, 1]$.

**Theorem 6.1.** Let Assumption 3.1 be valid. Then the uniform attractors $\mathcal{A}_{\Sigma}^*$ as shown in Theorem 4.3 are upper semicontinuous at the point $\epsilon_0 \in [0, 1]$ in the sense of $\mathcal{H}_{ps}$-topology, i.e.,

$$\lim_{\epsilon \to \epsilon_0} \sup_{g \in \Sigma} \text{dist}_{\mathcal{H}_{-1}}(\mathcal{A}_{\Sigma}^*, \mathcal{A}_{\Sigma}^{\epsilon_0}) = 0.$$  

**Proof.** We firstly show that

$$\lim_{\epsilon \to \epsilon_0} \sup_{g \in \Sigma} \text{dist}_{\mathcal{H}_{-1}}(\mathcal{A}_{\Sigma}^*, \mathcal{A}_{\Sigma}^{\epsilon_0}) = 0, \quad \forall \epsilon_0 \in [0, 1]. \quad (44)$$

Or else, there must exist $\delta > 0, \epsilon_0 \in [0, 1], \{\epsilon_n\} \subset [0, 1]$ with $\epsilon_n \to \epsilon_0$, and $\xi_n \in \mathcal{A}_{\Sigma}^{\epsilon_n}$ such that

$$\text{dist}_{\mathcal{H}_{-1}}(\xi_n, \mathcal{A}_{\Sigma}^{\epsilon_0}) > \delta, \quad \forall n \in \mathbb{N}^+.$$

Due to $\mathcal{H} \hookrightarrow \mathcal{H}_{-1}$, we have

$$\sup_{g \in \Sigma} \text{dist}_{\mathcal{H}_{-1}}(U^*_{g}(t, 0)B_0, \mathcal{A}_{\Sigma}^{\epsilon_0}) \leq C \sup_{g \in \Sigma} \text{dist}_{\mathcal{H}}(U^*_{g}(t, 0)B_0, \mathcal{A}_{\Sigma}^{\epsilon_0}) \to 0,$$

which implies that there exists a $T > T_0$ such that when $t \geq T$,

$$\sup_{g \in \Sigma} \text{dist}_{\mathcal{H}_{-1}}(U^*_{g}(t, 0)B_0, \mathcal{A}_{\Sigma}^{\epsilon_0}) \leq \frac{\delta}{3} \quad \text{and} \quad \bigcup_{\epsilon \in [0, 1]} \bigcup_{g \in \Sigma} U^*_{g}(t, 0)B_0 \subset B_0.$$

(46)
Due to $\xi_n \in A^e_{\epsilon_n} = \omega^e_{\epsilon_n}(B_0)$, we infer from Remark 2.5 that there exist $g_n \in \Gamma$, $\eta_n \in B_0$ and $t_n \geq 2T$ such that

$$\|U^{\epsilon_n}_{g_n}(t_n, 0)\eta_n - \xi_n\|_{H^{-1}} \leq C\|U^{\epsilon_n}_{g_n}(t_n, 0)\eta_n - \xi_n\|_H \leq \frac{\delta}{3}, \quad \forall n \in \mathbb{N}^+. \quad (47)$$

Since

$$U^{\epsilon_n}_{g_n}(t_n, 0)\eta_n = U^{\epsilon_n}_{g_n}(t_n, t_n - T)U^{\epsilon_n}_{g_n}(t_n - T, 0)\eta_n$$

$$= U^{\epsilon_n}_{g_n}(t_n, t_n - T)z_n$$

$$= U^{\epsilon_n}_{g_n}(t_n - T, 0)z_n, \quad \forall n \in \mathbb{N}^+,$$

where $z_n = U^{\epsilon_n}_{g_n}(t_n - T, 0)\eta_n \in B_0$ for $t_n - T \geq T$, we infer from Lemma 5.5 that there exists a $N > 0$ such that

$$\|U^{\epsilon_n}_{g_n}(t_n - T, 0)z_n - U^{\epsilon_n}_{g_n}(t_n - T, 0)z_n\|_{H^{-1}}$$

$$\leq C(B_0)e^{C(B_0)T}|\epsilon_n - \epsilon_0| \leq \frac{\delta}{3} \text{ as } n \geq N \quad (48)$$

for $\epsilon_n \rightarrow \epsilon_0$. Therefore, it follows from estimates (46)-(48) that

$$\text{dist}_{H^{-1}}(\{\xi_n, A^e_{\epsilon_n}\})$$

$$\leq \|\xi_n - U^{\epsilon_n}_{g_n}(t_n, 0)\eta_n\|_{H^{-1}}$$

$$+ \|U^{\epsilon_n}_{g_n}(t_n - T, 0)z_n - U^{\epsilon_n}_{g_n}(t_n - T, 0)z_n\|_{H^{-1}}$$

$$+ \text{dist}_{H^{-1}}(U^{\epsilon_n}_{g_n}(t_n - T, 0)z_n, A^e_{\epsilon_n}) \leq \delta, \quad \forall n \geq N,$$

which violates (45). Therefore, formula (44) holds.

By formulas (32), (44) and the fact that $\bigcup_{\epsilon \in [0, 1]} A^e_{\epsilon} \subset B_0$, we have

$$\lim_{\epsilon \rightarrow \epsilon_0} \text{dist}_p\left(A^e_{\epsilon}, A^e_{\epsilon_0}\right) \leq \lim_{\epsilon \rightarrow \epsilon_0} C(B_0)\left[\text{dist}_{H^{-1}}(A^e_{\epsilon}, A^e_{\epsilon_0})\right]^{1/2} = 0, \quad \forall \epsilon_0 \in [0, 1],$$

that is, the conclusion of Theorem 6.1 is valid.

**Remark 6.2.** When the nonlinearity $f(u)$ is of non-supercritical growth $p : 1 \leq p \leq p^* \equiv \frac{N+2}{N-2} (N \geq 1)$, $H^1_0 \hookrightarrow L^{p+1}$. Let Assumption 3.1 (ii) still be valid as $p : 1 \leq p \leq p^*$ and $g \in \Sigma$, Assumption 3.1 (i) need to be revised as $(i^*)$; $f \in C^1(\mathbb{R})$, $f(0) = 0$ and

$$\liminf_{|s| \rightarrow \infty} \frac{f(s)}{s} > -\lambda_1, \quad |f'(s)| \leq c_2(1 + |s|^{p-1}),$$

where $\lambda_1(> 0)$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition, and $1 \leq p < +\infty$ if $N = 1, 2$; $1 \leq p \leq \frac{N+2}{N-2}$ if $N \geq 3$, in particular,

$$f'(s) > -c_0, \quad \forall s \in \mathbb{R}, \text{ if } p = p^*.$$ 

We obtain that (i) the related evolution process has in natural energy space $H = H^1_0 \times L^2$ a compact uniform attractor $A^e_{\epsilon}$ for each $\epsilon \in [0, 1]$; (ii) the family of compact uniform attractor $\{A^e_{\epsilon}\}_{\epsilon \in [0, 1]}$ is continuous on $\epsilon$ in a residual set $I^* \subset [0, 1]$ in the sense of $H$-topology; (iii) $\{A^e_{\epsilon}\}_{\epsilon \in [0, 1]}$ is upper semicontinuous on $\epsilon \in [0, 1]$ in $H$-topology.
REFERENCES

[1] A. V. Babin and S. Yu. Pilyugin, Continuous dependence of attractors on the shape of domain, *J. Math. Sci.*, **87** (1997), 3304–3310.

[2] J. M. Ball, Global attractors for damped semilinear wave equations, *Discrete Contin. Dyn. Syst.*, **10** (2004), 31–52.

[3] V. V. Chepyzhov and M. I. Vishik, *Attractors for Equations of Mathematical Physics*, American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, RI, 2002.

[4] I. Chueshov, Long-time dynamics of Kirchhoff wave models with strong nonlinear damping, *J. Differential Equations*, **252** (2012), 1229–1262.

[5] P. Y. Ding, Z. J. Yang and Y. N. Li, Global attractor of the Kirchhoff wave models with strong nonlinear damping, *Appl. Math. Lett.*, **76** (2018), 40–45.

[6] X. Fan and S. Zhou, Kernel sections for non-autonomous strongly damped wave equations of non-degenerate Kirchhoff-type, *Appl. Math. Comput.*, **237** (2014), 4389–4395.

[7] J. K. Hale and G. Raugel, Lower semicontinuity of attractors of gradient systems and applications, *Ann. Mat. Pura Appl.*, **154** (1989), 281–326.

[8] L. T. Hoang, E. J. Olason and J. C. Robinson, On the continuity of global attractors, *Proc. Amer. Math. Sc.* **143** (2015), 4389–4395.

[9] L. T. Hoang, E. J. Olason and J. C. Robinson, Continuity of pullback and uniform attractors, *J. Differential Equations*, **264** (2018), 4067–4093.

[10] G. Kirchhoff, Vorlesungen über Mechanik, Lectures on Mechanics, Teubner, Stuttgart, 1883.

[11] Y. N. Li and Z. J. Yang, Robustness of attractors for non-autonomous Kirchhoff wave models with strong nonlinear damping, *Appl. Math. Optim.*, (2019).

[12] S. S. Lu, H. Q. Wu and C. K. Zhong, Attractors for non-autonomous 2D Navier-Stokes equations with normal external forces, *Discrete Contin. Dyn. Syst.*, **13** (2005), 701–719.

[13] H. L. Ma and C. K. Zhong, Attractors for the Kirchhoff equations with strong nonlinear damping, *Appl. Math. Lett.*, **74** (2017), 127–133.

[14] H. L. Ma, J. Zhang and C. K. Zhong, Global existence and asymptotic behavior of global smooth solutions to the Kirchhoff equations with strong nonlinear damping, *Discrete Contin. Dyn. Syst.-B.*, (2019).

[15] H. L. Ma, J. Zhang and C. K. Zhong, Attractors for the degenerate Kirchhoff wave model with strong damping: Existence and the fractal dimension, *J. Math. Anal. Appl.*, **484** (2020), 123670, 15 pp.

[16] T. Matsuyama and R. Ikeda, On global solution and energy decay for the wave equation of Kirchhoff-type with nonlinear damping term, *J. Math. Anal. Appl.*, **204** (1996), 729–753.

[17] I. Moise, R. Rosa and X. Wang, Attractors for noncompact non-autonomous systems via energy equations, *Discrete Contin. Dyn. Syst.*, **10** (2004), 473–496.

[18] M. Nakao, An attractor for a nonlinear dissipative wave equation of Kirchhoff type, *J. Math. Anal. Appl.*, **353** (2009), 652–659.

[19] K. Ono, Global existence, decay, and blowup of solutions for some mildly degenerate non-linear Kirchhoff strings, *J. Differential Equations*, **137** (1997), 273–301.

[20] J. Simon, Compact sets in the space $L^p(0,T; B)$, *Ann. Mat. Pura Appl.*, **146** (1986), 65–96.

[21] A. M. Stuart and A. R. Humphries, *Dynamical Systems and Numerical Analysis*, Cambridge University Press, Cambridge, 1996.

[22] C. Y. Sun, D. M. Cao and J. Q. Duan, Uniform attractors for non-autonomous wave equations with nonlinear damping, *SIAM J. Appl. Dyn. Syst.*, **6** (2007), 293–318.

[23] B. X. Wang, Uniform attractors of non-autonomous discrete reaction-diffusion systems in weighted spaces, *Int. J. Bifurcation Chaos*, **18** (2008), 659–716.

[24] Y. H. Wang and C. K. Zhong, Upper semicontinuity of pullback attractors for non-autonomous Kirchhoff wave models, *Discrete Contin. Dyn. Syst.*, **33** (2013), 3189–3209.

[25] Z. J. Yang and Y. Q. Wang, Global attractor for the Kirchhoff type equation with a strong dissipation, *J. Differential Equations*, **249** (2010), 3258–3278.

[26] Z. J. Yang and P. Y. Ding, Longtime dynamics of the Kirchhoff equation with strong damping and critical nonlinearity on $\mathbb{R}^N$, *J. Math. Anal. Appl.*, **434** (2016), 1826–1851.

[27] X.-G. Yang, Marcelo J. D. Nascimento and Mauricio L. Pelicer, Uniform attractors for non-autonomous plate equations with $p$-Laplacian perturbation and critical nonlinearities, *Discrete Contin. Dyn. Syst.*, **40** (2020), 1937–1961.
[28] S. Zelik, Strong uniform attractors for non-autonomous dissipative PDEs with non translation-compact external forces, Discrete Contin. Dyn. Syst.-B 20 (2015), 781–810.

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