The lifting bifurcation problem on feed-forward networks

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Abstract
We consider feed-forward networks, that is, networks where cells can be divided into layers, such that every edge targeting a layer, excluding the first one, starts in the prior layer. A feed-forward system is a dynamical system that respects the structure of a feed-forward network. The synchrony subspaces for a network, are the subspaces defined by equalities of some cell coordinates, that are flow-invariant by all the network systems. The restriction of a network system to each of its synchrony subspaces is a system associated with a smaller network, which may be, or not, a feed-forward network. The original network is then said to be a lift of the smaller network. We show that a feed-forward lift of a feed-forward network is given by the composition of two types of lifts: lifts that create new layers and lifts inside a layer. Furthermore, we address the lifting bifurcation problem on feed-forward systems. More precisely, the comparison of the possible codimension-one local steady-state bifurcations of a feed-forward system and those of the corresponding lifts is considered. We show that for most of the feed-forward lifts, the increase of the center subspace is a sufficient condition for the existence of additional bifurcating branches of solutions, which are not lifted from the restricted system. However, when the bifurcation condition is associated with the internal dynamics and the lift occurs inside an intermediate layer, we prove that the existence of a bifurcation branch not lifted from the restricted system does depend generically on the chosen feed-forward system.

Keywords: feed-forward networks, steady-state bifurcations, lifting bifurcation problem
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1. Introduction

Coupled cell networks describe influences between cells and are represented by directed graphs with possible multiple arrows. Examples of modeling through networks include biological, computational and physical real-world applications, see e.g. [4, 12, 16, 25]. In applications, networks are commonly used to describe properties of dynamical systems formed by interacting individual dynamical systems. We consider coupled cell systems which are given by vector fields that respect the structure of the network [15, 24]. Briefly, a vector field respects the structure of a network if each cell corresponds to an individual dynamical system that depends on its own state and on the state of the cells in its input set. Examples of dynamics that can be observed in coupled cell systems include full-synchronized attractors, synchrony-breaking bifurcations and heteroclinic networks, see e.g. [1, 4, 13].

In this work, we consider networks such that all cells are identical and receive exactly one edge of each type. The different edge’s types are graphically represented by different arrowheads. For example, considering the networks in figure 1, every cell receives two edges of a different type.

Two cells are synchronized if their dynamics agree for trajectories with the same initial condition. In [15, 24], the authors showed that there is an intrinsic relation between cells’ synchronization and colorings of the network set of cells. More precisely, it is shown that, given a network, a subspace defined by equalities of some of the network cell coordinates is an invariant subspace for any coupled cell system if and only if the coloring of the cells determined by those equalities is balanced. Given a balanced coloring, the correspondent quotient network is obtained by merging cells with the same color. Moreover, the dynamics associated with the quotient network is the restriction of the dynamics on the original network to the correspondent invariant subspace. The original network is then said to be a lift of the smaller network.

Consider the networks in figure 1. As the cells 2 and 3 of the network C receive their inputs from cell 1, the coloring in C that identify cells 2 and 3 is balanced. The quotient network associated to this coloring is the network B. By merging cells 2 and 3, we have that each edge of C starting at cell 2 or 3 has a corresponding edge in B starting at the merged cell. In the same way, we can see that the network A is obtained from B by merging cells 1 and 2. Hence the network C is a lift of B which is itself a lift of A.

In general real-world networks, due to their complexity, are big in size. Quotient networks are a way of reducing the size of a network and study, at least a fraction of, the dynamics associated with a big network. If a dynamical property in a lifted network follows from the study of that dynamical property on a smaller network, then the dimension of the problem can be reduced. Given a bifurcation problem on a feed-forward network. The main goal of this paper is to study the lifting bifurcation problem for feed-forward networks, i.e. investigate when every bifurcation branch in a lift feed-forward network is obtained by (lifting) a bifurcation branch in a quotient feed-forward network. As we explain below, the answer to this problem can help us, for example, to understand which synchronization patterns are broken via bifurcation. Synchrony-breaking events can be observed, for example, in biological networks, [25], and neuronal networks, [18]. In the following paragraphs, we describe the lifting bifurcation problem for feed-forward networks, our approach to it and the results obtained.

Feed-forward networks are a class of networks where the set of cells can be partitioned into disjoint sets called layers. We consider feed-forward networks where each cell in the first layer only receives inputs from itself and cells in the other layers only receive inputs from cells in the previous layer. Feed-forward networks have been used, for example, to design machine learning networks and neuronal networks, [12, 16]. In those applications, the cells emulate neurons and each edge corresponds to a unidirectional connection between two neurons.

P Soares
Nonlinearity 31 (2018) 5500
cells in the first layer are viewed as the original information receptors. The following layers receive and transform this information and then transmit it to the next layer. In the last layer, we can find the outcome of processing the original information. This process is usually assumed to be discrete. Nevertheless some features of neuronal networks have been observed in continuous models, such as binocular rivalry [7–9]. The feed-forward structure allows us to reason layer by layer and we use this to study feed-forward networks and the associated feed-forward systems.

Three examples of feed-forward networks appear in figure 1. The network C has three layers: {1}, {2, 3} and {4, 5, 6}. The networks A and B have, respectively, two and three layers.

The lift network of a feed-forward network does not need to be a feed-forward network. Our first goal is to understand which lifts of feed-forward networks also have a feed-forward structure. Previous work about balanced colorings on feed-forward networks can be found in [2]. We introduce two basic lifts on feed-forward networks, lifts inside a layer and lifts that create new layers. A lift that creates new layers is the replication of the first layer into consecutive layers. A lift inside a layer is given by the split of some cells within a layer. In proposition 3.1, we prove that the lifts of feed-forward networks that also have a feed-forward structure are given by the composition of basic lifts. The definition of the basic lifts is crucial to the study of the lifting bifurcation problem, since it reduces our study to the two cases corresponding to the basic lifts.

Returning to the networks in figure 1. The network C is obtained from the network B by splitting the cell 2 of B into the cells 2 and 3. That is, C is a lift of B inside the second layer. Also, the network B is a lift of the network A with one more layer. Thus the network B is a lift of A that creates a new layer. We have then that the network C is a lift of A given by the composition of a lift that creates a new layer with a lift inside a layer.

Our second goal is to address the (codimension-one steady-state) bifurcations from a full synchrony equilibrium on feed-forward systems. Previous works have addressed bifurcations on feed-forward systems with one cell per layer, [5, 10, 11, 14, 20], and have identified a surprising phenomenon of bifurcation on feed-forward systems: the layers of the feed-forward network act as amplifiers for the growth rate of the bifurcation branches. We generalize that study of steady-state bifurcations to general feed-forward networks. In this work, we assume that the phase space of each cell is real and one dimensional and that the feed-forward system

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{networks}
\caption{Feed-forward networks with two types of inputs that are distinguished by their arrow heads. The network A has two layers and the networks B and C have three layers. The network A is the quotient network of B obtained by merging cells 1 and 2, that is, B is a lift of A that creates a new layer. The network B is the quotient network of C got by merging cells 2 and 3, C is a lift of B inside the second layer.}
\end{figure}
has a steady-state solution with full synchrony. It follows from the feed-forward structure
that the linearization of a feed-forward system at a full synchrony equilibrium has only two
eigenvalues: the valency and the internal dynamics. This leads to two types of bifurcation con-
ditions from a full synchrony equilibrium on feed-forward systems. One bifurcation condition
is given by the linearization of the self input, that we call internal dynamics. The other bifurca-
tion condition is given by the sum of all inputs’ linearization, that we call valency.

In this work, we study bifurcations on feed-forward systems given by the two different
bifurcation conditions. A direct application of the implicit function theorem describes the
bifurcation branches of a feed-forward system with a bifurcation condition associated to the
valency. Furthermore, we give in proposition 6.1 a full characterization of the bifurcation
branches of a feed-forward system with a bifurcation condition associated to the internal
dynamics in terms of their square-root-orders and slopes. In order to obtain this characteriza-
tion, we follow the technique used in [20] for feed-forward networks with one cell per layer.
In [20], the authors used a suitable change of coordinates to see how the growth rate of a
bifurcation branch propagates from one layer to the next. Using this change of coordinates and
exploiting the layer structure, we obtain the complete description of the bifurcation branches
on any feed-forward system.

As an example, we present some conclusions that follow, using our results, from the char-
acterization of the bifurcation branches of a feed-forward system on the network $C$ in figure 1
with a bifurcation condition associated to the internal dynamics. The full characterization of
the bifurcation branches is given in example 6.2. There are eight bifurcation branches with a
linear growth rate and eight bifurcation branches with a square root growth. The cells 2 and
3 of $C$ are in synchrony for any of those bifurcation branches. Moreover, there are 16 more
bifurcation branches with a square root growth if and only if the feed-forward system has the
same sign for the linearizations of the first and second inputs. In the latter case, for the extra
bifurcation branch solutions, cells 2 and 3 of $C$ are not synchronized.

Last, we study our main goal, the lifting bifurcation problem for feed-forward networks.
The restriction of a coupled cell system to a synchrony subspace is a coupled cell system of
the correspondent quotient network. Thus any bifurcation branch for the quotient system lifts
to a bifurcation branch on the lift system. The lifting bifurcation problem asks if there are
more bifurcation branches on the lift system. Note that the bifurcation branches lifted from a
quotient network preserve the synchrony associated to the quotient network. So any bifurca-
tion branch not lifted from a quotient network breaks the synchrony associated to that quotient
network. The lifting bifurcation problem was first raised in [3] where the authors proved that
there are networks which have more bifurcation branches on some lift systems than the ones
lifted from the quotient network. This problem was also studied in [17, 19]. A well-known
result gives a necessary condition for the lifting bifurcation problem: there can be additional
bifurcation branches on a lifted network than the ones lifted only if the center subspace of the
coupled cell systems associated to the original network and the lift network have different
dimensions. We refer to corollary 7.1. As we show, the study of the lifting bifurcation problem
for feed-forward networks reduces to its analysis on the basic lifts: lifts that create new layers
and lifts inside a layer. We describe now the results that we obtain about the lifting bifurca-
tion problem on feed-forward networks. In order to obtain those results, we use the characteriza-
tion of the bifurcation branches on feed-forward systems, stated above.

Frequently, the aforementioned necessary condition for the lifting bifurcation problem
is also sufficient. In propositions 7.2–7.4, we prove this in the following cases. For feed-
forward systems with a bifurcation condition associated to the valency. For lifts that create
new layers, inside the first layer and inside a layer which has only one cell in the next layer,
and feed-forward systems with a bifurcation condition associated with the internal dynamics. Moreover, those cases do not depend generically on the feed-forward system.

Consider the networks in figure 1 and the correspondent feed-forward systems. If the feed-forward systems have a bifurcation condition associated to the valency, then the center subspaces associated to \( A \), \( B \) and \( C \) have dimension 1. So every bifurcation branch on the network \( C \) is lifted from \( A \). On the other hand, if the feed-forward systems have a bifurcation condition associated to the internal dynamics then there exists a bifurcation branch on \( B \) which is not lifted from \( A \). This follows from the fact that the network \( B \) is a lift of \( A \) that creates a new layer and the center subspaces associated to the network \( A \) and \( B \) are, respectively, one and two dimensional.

For lifts inside an intermediate layer and feed-forward systems with a bifurcation condition associated to the internal dynamics, the center subspaces of the feed-forward system and of the lift system have different dimensions. For lifts inside an intermediate layer, we show in proposition 7.5 that there exists an open set of feed-forward systems with a bifurcation condition associated to the internal dynamics for which there are more bifurcation branches on the lifted network than the ones lifted from the quotient network. Remarkably, we see in propositions 7.6 and 7.7 that, for a class of lifts inside an intermediate layer, there is also an open set of feed-forward systems with a bifurcation associated to the internal dynamics such that there are no more bifurcation branches on the lift system.

Consider the network \( C \) in figure 1 and a feed-forward system with a bifurcation condition associated to the internal dynamics. Note that the feed-forward network \( C \) is a lift of the network \( B \) inside the second layer and the center subspace associated to \( C \) is bigger than the center subspace associated to \( B \). We stated before that all bifurcation branches of the feed-forward system associated with \( C \) have synchrony between the cells 2 and 3 if and only if the linearization of the two inputs have opposite signs. We can show that a bifurcation branch of \( C \) is lifted from \( B \) only if it has synchrony between the cells 2 and 3. Therefore every bifurcation branch on the network \( C \) is lifted from \( B \) if and only if the linearization of the two inputs have the opposite signs. This condition provides the open sets of feed-forward systems mentioned in the previous paragraph.

The paper is organized as follows. In section 2, we recall the definitions of coupled cell networks and feed-forward networks. Next, we study lifts of a feed-forward network which preserve the feed-forward structure (section 3). Coupled cell systems and feed-forward systems are recalled in section 4. Then we analyze the steady-state bifurcations in feed-forward systems with bifurcation conditions associated to the valency (section 5) and the internal dynamics (section 6). Finally, we study the lifting bifurcation problem for feed-forward systems.
with bifurcation conditions associated to the valency (section 7.1) and the internal dynamics (section 7.2).

2. Feed-forward networks

In this section, we recall a few facts concerning coupled cell networks, following [15, 21], and define feed-forward networks.

**Definition 2.1.** A network $N$ is defined by a directed graph with a finite set of cells $C$ and a finite sets of directed edges divided by types $E_1, \ldots, E_k$. We assume that each cell $c$ is target by one and only one edge of each type. We denote by $|N|$ the number of cells in the network $N$. ♦

Figure 2 displays an example of a network with two types of edges.

We say that the networks $N$ and $N'$ are equal and write $N = N'$ if they only differ by a relabel of cells, edges and types. In this paper, we consider connected networks, i.e. networks such that every two distinct cells have an undirected path between them.

Following [21], we can regard each type of edge as a function from the target cell to the input cell. Let $(\sigma_i : C \rightarrow C)_{i=1}^k$ be the collection of functions such that there exists an edge $e \in E_i$ from $\sigma_i(c)$ to $c$, for every $c \in C$ and $1 \leq i \leq k$. We write that $N$ is represented by the functions $(\sigma_i : C \rightarrow C)_{i=1}^k$.

**Example 2.1.** Consider the functions $\sigma_1$ and $\sigma_2$ defined by

\[
\begin{align*}
\sigma_1(1) &= \sigma_1(2) = \sigma_1(3) = \sigma_1(4) = 1, & \sigma_1(5) &= 3, & \sigma_1(6) &= 4, & \sigma_1(7) &= 2, \\
\sigma_2(1) &= \sigma_2(2) = \sigma_2(3) = \sigma_2(4) = 1, & \sigma_2(5) &= 2, & \sigma_2(6) &= 3, & \sigma_2(7) &= 4.
\end{align*}
\]

The network in figure 2 is represented by $(\sigma_1, \sigma_2)$, where $\sigma_1$ represents the edges with one head and $\sigma_2$ represents the edges with two heads. ♦

In feed-forward networks, the cells can be partition into layers. Each cell in the first layer only receives inputs from itself. Cells in the other layers only receive inputs from cells in the previous layer. The network in figure 2 is an example of a feed-forward network with three layers.

**Definition 2.2.** Let $N$ be a network represented by the functions $(\sigma_i)_{i=1}^k$. We say that $N$ is a feed-forward network (FFN), if there exists a partition of the set of cells of $N$ into subsets $C_1, \ldots, C_m$ such that $\bigcup_{i=1}^k \sigma_i(C_j) = C_{j-1}$, for every $2 \leq j \leq m$ and $\sigma_i(c) = c$, for every $c \in C_1$.
and $1 \leq i \leq k$. The subset $C_j$ is called the $j$th layer of $N$.

Backward connectivity of a network is an important concept for the results obtained in this paper. Roughly speaking, a network is backward connected for some cell if there exists a directed path starting in any other cell and ending in that cell.

**Definition 2.3.** We say that a network $N$ is backward connected for a cell $c$ if for every cell $c'$ different from $c$ there exists a sequence of cells $c_0, c_1, \ldots, c_{l-1}, c_l$ in $N$ such that $c' = c_0$, $c = c_l$ and there is an edge from $c_{a-1}$ to $c_a$ for every $1 \leq a \leq l$. The network $N$ is backward connected if it is backward connected for some cell.

The network in figure 3 is an example of a backward connected network for the cell 10. An example of a network which is not backward connected is the one pictured in figure 2, as there is no directed path between the cells 5, 6 and 7. Observe that, by definition, a feed-forward network is backward connected if and only if it has only one cell in the last layer.

### 3. Lifts of feed-forward networks

The main goal of this section is to show how the feed-forward lifts can be decomposed. We recall the notions of balanced colorings, quotient networks and lifts of networks, following [15, 21, 22, 24], with emphasis on feed-forward networks. Roughly speaking, in a balanced coloring, cells with the same color receive each type of input from cells with the same color. And the associated quotient network is obtained by merging cells with the same color.

Let $N$ be a network represented by the functions $(\sigma_i : C \to C_{[i]}^1)_{i=1}^{k}$. A coloring of the set of cells of $N$ is an equivalence relation on the set of cells. A coloring $\bowtie$ is balanced if $\sigma_i(c) \bowtie \sigma_i(c')$, for every $1 \leq i \leq k$ and $c, c' \in C$ such that $c \bowtie c'$. Given a subset of cells $S$ in $N$, we denote by $[S]_{\bowtie}$ the set of $\bowtie$-classes of the cells in $S$, i.e. $[S]_{\bowtie} = \{[c]_{\bowtie} : c \in S\}$.

**Definition 3.1 ([15, section 5]).** Let $N$ be a network represented by the functions $(\sigma_i : C \to C_{[i]}^1)_{i=1}^{k}$ and $\bowtie$ a balanced coloring in $N$. The quotient network of $N$ associated to $\bowtie$ is the network where the set of cells is $[C]_{\bowtie}$ and there is an edge of type $i$ from $[\sigma_i(c)]_{\bowtie}$ to $[c']_{\bowtie}$ for every $1 \leq i \leq k$ and $c \in C$. We denote by $N/\bowtie$ the quotient network of $N$ associated to $\bowtie$. We also say that a network $L$ is a lift of $N$, if $N$ is a quotient of $L$ for some balanced coloring in $L$.

Let $N$ be a network represented by the functions $(\sigma_i : C \to C_{[i]}^1)_{i=1}^{k}$ and $\bowtie$ a balanced coloring in $N$. The quotient network $N/\bowtie$ is represented by the functions $(\sigma_i^{[\bowtie]} : [C]_{\bowtie} \to [C]_{\bowtie})_{i=1}^{k}$, where $\sigma_i^{[\bowtie]}$ is given by $\sigma_i^{[\bowtie]}([c]_{[\bowtie]}) = [\sigma_i(c)]_{[\bowtie]}$, for every $1 \leq i \leq k$ and $c \in C$. Note that the quotient network of a backward connected network is also backward connected.

**Example 3.1.** Let $N$ be the network in figure 2. Consider the coloring $\bowtie$ in $N$ given by $2 \bowtie 3 \bowtie 4$ and $5 \bowtie 6$. Note that the cells 5 and 6 receive inputs from the cells 2, 3 and 4 that have the same color. Moreover, the cells 2 and 3 receive inputs from a unique cell 1. So the coloring is balanced and the network in figure 4 is the quotient of $N$ associated to $\bowtie$.

In [2], the authors studied and described the balanced colorings of feed-forward networks. The set of balanced colorings forms a partial order set as studied in [23] given by the refinement relation. Given two balanced colorings $\bowtie', \bowtie$ of a network $N$, we say that $\bowtie'$ refines $\bowtie$ and we write $\bowtie' \bowtie \bowtie$ if $c \bowtie' d$ implies that $c \bowtie d$, for every cells $c$ and $d$ of $N$. We have that if $\bowtie' \bowtie \bowtie$, then $N/\bowtie$ is a quotient of $N/\bowtie'$. 

5506
If \( L, N \) and \( Q \) are networks such that \( L \) is a lift of \( N \) and \( N \) is a lift of \( Q \), then \( L \) is a lift of \( Q \). Moreover, we say that the lift \( L \) of \( Q \) is given by the composition of the lift \( N \) of \( Q \) and the lift \( L \) of \( N \). In some cases, a lift can be seen as the composition of two lifts, see [6, theorem 2.4]. In the next result, we give a sufficient condition for the existence of an intermediate quotient which only merges cells in an independent subset of cells.

**Lemma 3.1.** Let \( L \) be a network represented by the functions \( (\sigma_i : C \rightarrow C)^{k}_{i=0} \) a balanced coloring in \( L \) and \( S \subseteq C \) such that \( \sigma_i(S) \subseteq S \), for \( 1 \leq i \leq k \). Then, there exists a balanced coloring \( \succcurlyeq \) in \( L \) such that \( L / \succcurlyeq \) is a lift of \( L / \succcurlyeq, [C \setminus S]_{\succcurlyeq} = C \setminus S \) and there exists a bijection between \([S]_{\succcurlyeq} \) and \([S]_{\succcurlyeq} \).

**Proof.** Let \( L \) be a network represented by the functions \( (\sigma_i : C \rightarrow C)^{k}_{i=0} \) a balanced coloring in \( L \) and \( S \subseteq C \) such that \( \sigma_i(S) \subseteq S \), for \( 1 \leq i \leq k \).

Define \( \succcurlyeq \) as the coloring of \( L \) such that \( c \succcurlyeq c' \) if \( c \succcurlyeq c' \) and \( c, c' \in S \). Let \( c, c' \in S \) such that \( c \succcurlyeq c' \). Then \( c \succcurlyeq c' \). \( \sigma_i(c) \sigma_i(c') \in S \) and \( \sigma_i(c), \sigma_i(c') \in S \), for every \( 1 \leq i \leq k \). Hence \( \sigma_i(c) \succcurlyeq \sigma_i(c') \), for every \( 1 \leq i \leq k \), and \( \succcurlyeq \) is a balanced coloring of \( L \). Note that \( \succcurlyeq \preceq \succcurlyeq \) and so \( L / \succcurlyeq \) is a quotient of \( L / \prec \).

The \( \succcurlyeq \)-class of any cell in \( C \setminus S \) is singular, so \( [C \setminus S]_{\succcurlyeq} = C \setminus S \).

Let \( \alpha : [S]_{\succcurlyeq} \rightarrow [S]_{\succcurlyeq} \) be given by \( \alpha([c]_{\succcurlyeq}) = [c']_{\succcurlyeq} \), where \( c \in S \). Let \( c, c' \in S \) such that \( [c]_{\succcurlyeq} \equiv [c']_{\succcurlyeq} \). Then \( c \equiv c' \) and \( c \succcurlyeq c' \). \( \sigma_i(c), \sigma_i(c') \in S \) and \( \sigma_i(c) \sigma_i(c') \in S \), for every \( 1 \leq i \leq k \). Hence \( \sigma_i(c) \succcurlyeq \sigma_i(c') \), for every \( 1 \leq i \leq k \), and \( \succcurlyeq \) is a balanced coloring of \( L \). Note that \( \succcurlyeq \preceq \succcurlyeq \) and so \( L / \succcurlyeq \) is a quotient of \( L / \succcurlyeq \).

We exemplify the use of the previous lemma in the next example.

**Example 3.2.** Consider the network \( L \) in figure 2 and the balanced coloring defined by \( 2 \succcurlyeq 3 \succcurlyeq 4 \) and \( 5 \succcurlyeq 6 \). Every input of a cell in the subset \( S = \{1, 2, 3, 5\} \) is from a cell in By lemma 3.1, there exists a network between \( L \) and \( L / \succcurlyeq \). Let \( \succcurlyeq \) be the balanced coloring such that \( 2 \succcurlyeq 3 \). The quotient \( L / \succcurlyeq \) is the network \( C \) in figure 1. And \( L / \succcurlyeq \) is a lift of \( L / \prec \).

However, if we consider the subset \( S' = \{1, 2, 3, 5, 6\} \), then does not exist an intermediate balanced coloring as stated in lemma 3.1. The cell 6 belongs to \( S' \) but it receives an input from cell 4 which does not belong to \( S' \). If there was such an intermediate balanced coloring, then cells 5 and 6 would have the same color. If the coloring is balanced and cells 5 and 6 have the same color, then their inputs must have the same color and cells 3 and 4 must have the same
color. In lemma 3.1, cells not in $S'$ do not share the color with any other cell. If there was a balanced coloring as stated in lemma 3.1, then cells 3 and 4 would not have the same color. This contradiction implies that lemma 3.1 does not hold for $S'$.

There are lift networks of feed-forward networks that are not feed-forward networks. For example, the network with exactly one cell and $k$ types of edges is a feed-forward network and every other network with $k$ types of edges is a lift of that network. We study feed-forward lift networks of feed-forward networks.

We define two types of basic lifts for feed-forward networks: lifts inside a layer and lifts that create new layers. A lift is inside a layer if the corresponding balanced coloring only identifies cells within some layer. And a lift that replicates the first layer is a lift that creates new layers. A lift is inside a layer if the corresponding balanced coloring only identifies cells within some layer. And a lift that replicates the first layer is a lift that creates new layers. In figure 1, we present examples of those basic lifts.

**Definition 3.2.** Let $N$ be a feed-forward network and $L$ a feed-forward lift of $N$. Denote the layers of $N$ and $L$ by $C_1, \ldots, C_m$ and $C'_1, \ldots, C'_n$, respectively.

We say that $L$ is a lift inside the layer $C_j$ where $1 \leq j \leq m$, if $m = n$, $|C'_j| \neq |C_j|$ and $|C'| = |C|$ for every $i \neq j$. Moreover, the number of cells in each layer of $N$ is backward connected, then the networks $N$ and $Qm$ coincide, except in the first layer. Hence $L$ is a lift of $Q1$ inside the first layer (or $N = Q_{m-1}$).

We can repeat the previous application of lemma 3.1 to the subset $S_j = C_j$ and to the lift $L$ of $N$. We have that there exists a network $Qj$ between the lift $L$ and the network $N$. The network $Qj$ is given by the merge of cells in the first layer of $L$. So $Qj$ is also a feed-forward network with the same number of layers $N$ and $L$. Moreover, the number of cells in each layer of $L$ and $Qj$ coincide, except in the first layer. Hence $L$ is a lift of $Qj$ inside the first layer (or $N = Q_{m-1}$).

Let $N$ be a feed-forward network and $L$ a feed-forward lift of $N$. Suppose that $N$ and $L$ have the same number of layers and denote the layers of $L$ by $C_1, \ldots, C_m$.

Consider lemma 3.1 applied to the subset $S_1 = C_1$ and to the lift $L$ of $N$. We have that there exists a network $Q1$ between the lift $L$ and the network $N$. The network $Q1$ is given by the merge of cells in the first layer of $L$. So $Q1$ is also a feed-forward network with the same number of layers $N$ and $L$. Moreover, the number of cells in each layer of $L$ and $Q1$ coincide, except in the first layer. Hence $L$ is a lift of $Q1$ inside the first layer (or $N = Q_{m-1}$).

We can repeat the previous application of lemma 3.1 to the subset $S_j = C_j$ and the lift $Q_{j-1}$ of $N$, for each $2 \leq j \leq m - 1$. Thus we obtain a sequence of networks $L = Q_0, Q_1, \ldots, Q_{m-1}, Q_m = N$ such that $Q_{j-1}$ is a lift of $Q_j$ inside the $j$th layer (or $Q_{j-1} = Q_j$) for $j = 1, \ldots, m - 1$ and $Q_{m-1}$ is a lift of $N$. Therefore the lift of $N$ to $L$ is the composition of lifts inside the layers and the lift of $N$ to $Q_{m-1}$.

If $L$ is backward connected, then the networks $N$ and $Q_{m-1}$ have only one cell in the last layer. By the previous construction, we already knew that $N$ and $Q_{m-1}$ are equal in every layer, except the last. Thus $N = Q_{m-1}$ and the lift of $L$ from $N$ is the composition of lifts inside a layer.

When the lifted network is not backward connected, we can have lifts which do not decompose into basic lifts.

**Example 3.3.** Let $N$ be a feed-forward network and $L$ the feed-forward network on the right of figure 5. The network $L$ is a lift of $N$, considering the coloring in $L$ given by the classes $\{1a, 1b\}, \{2\}, \{3\}$. This lift cannot be obtained by a composition of
lifts that create new layers and lifts inside the layers. Note that the lift network is not backward connected.

For backward connected lifts, we show the following. The first layer of a feed-forward quotient network is given by the merge of all the cells in a number of consecutive layers, starting in the second layer, with the cells in the first layer, eventually with the merge of some cells in the first layer. Moreover, the next layers of the feed-forward quotient network are given by merging cells in a specific layer of the lift network respecting the layers order.

**Lemma 3.2.** Let $N$ be a feed-forward network with layers $C_1, \ldots, C_m$ and $L$ a feed-forward lift of $N$ with layers $C'_1, \ldots, C'_k$ such that $L$ is backward connected. Denote by $\triangleright$ the balanced coloring of $L$ such that $L/ \triangleright = N$. Then

$$[C'_{n+m+1}]_{\triangleright} = \cdots = [C'_{1}]_{\triangleright} = C_1, \quad [C'_{n-j}]_{\triangleright} = C_{m-j}, \quad 0 \leq j \leq m - 2.$$

**Proof.** Let $N$ be a feed-forward network with layers $C_1, \ldots, C_m$ and $L$ a feed-forward lift of $N$ with layers $C'_1, \ldots, C'_k$ such that $L$ is backward connected. Assume that $N$ and $L$ are represented by $(\sigma_i)^{j=1}_{j=1}$ and $(\sigma'_i)^{j=1}_{j=1}$, respectively. Let $\triangleright$ be a balanced coloring such that $L/ \triangleright = N$. Then $n \geq m$ and $[\sigma'_i(e)]_{\triangleright} = \sigma_i([e]_{\triangleright})$, for every cell $e$ in $L$ and $1 \leq i \leq k$.

Since $L$ is backward connected, we have that $N$ is backward connected, $C'_n = \{e_1\}$, $C_m = \{\{e_1\}_{\triangleright}\}$ and $[C'_n]_{\triangleright} = C_m$. If $m = 1$, then $N$ has only one cell and there is only one equivalence class of $\triangleright$. Hence $[C'_n]_{\triangleright} = \cdots = [C'_1]_{\triangleright} = C_1$.

Now, suppose that $m > 1$. Assuming that $[C'_{n+1-j}]_{\triangleright} = C_{m+1-j}$, we see that $[C_{n-j}]_{\triangleright} = C_{m-j}$ for $j = 1, \ldots, m - 1$. Let $d_1 \in C_{m-j}$ where $1 \leq j \leq m$. Then there exist $1 \leq i \leq k$ and $d_2 \in C_{m+1-j}$ such that $d_1 = \sigma(d_2)$. By assumption, there exists $d'_2 \in C'_{m+1-j}$ such that $d'_2 = [d_2]_{\triangleright}$. Then there exist $1 \leq i \leq k$ and $d'_2 \in C'_{m+1-j}$ such that $d'_2 = \sigma(d_2)$. By assumption, we have that $[d'_2]_{\triangleright} = C_{m+1-j}$ and $[d'_2]_{\triangleright} = [\sigma'_i(d'_2)]_{\triangleright} \in C_{m-j}$. Therefore $[C'_{n-j}]_{\triangleright} = C_{m-j}$. It follows inductively from $[C'_n]_{\triangleright} = C_m$ that

$$[C'_{n-j}]_{\triangleright} = C_{m-j} \quad 1 \leq j \leq m - 1.$$

In particular, $[C'_{n-m+1}]_{\triangleright} = C_1$ and $\sigma(C_1) = C_{m-1}$. Using the same argument, we conclude that

$$[C'_{n-m}]_{\triangleright} = \cdots = [C'_1]_{\triangleright} = C_1.$$

**Example 3.5.** Let $L$ be the network in figure 3 which is backward connected for the cell 10. The network $L$ has five layers that we denote by $C'_1, C'_2, C'_3, C'_4$ and $C'_5$. Consider the balanced

---

**Figure 5.** The feed-forward network on the right is a lift of the feed-forward network on the left. This lift is not given by the composition of lifts that create new layers and lifts inside layers. Note that the lift network is not backward connected.
coloring $\bowtie$ in $L$ given by $1 \bowtie 2 \bowtie 3$ and $4 \bowtie 5$. Note that $L/\bowtie$ is a feed-forward network with four layers. Denote the layers of $L/\bowtie$ by $C_1$, $C_2$, $C_3$ and $C_4$. Lemma 3.2 states that $\{C_1\}_{[n]} = [C_2]_{[n]} = C_1$, $\{C_3\}_{[n]} = C_2$, $\{C_4\}_{[n]} = C_3$, $\{C_5\}_{[n]} = C_4$. $\diamondsuit$

Example 3.4 shows that lemma 3.2 does not hold if the lift network is not backward connected. The lift in figure 5 merges a cell in the second layer with one in the first layer and does not merge the other cell in the second layer with a cell on the first layer. The lift network in figure 5 is not backward connected and this lift does not satisfy the conclusion of the previous lemma.

Using the method described in example 3.3 and lemma 3.2, we state the main result of this section. This result shows how to decompose a feed-forward lift into lifts that create new layers and lifts inside a layer.

**Proposition 3.1.** Let $N$ be a feed-forward network and $L$ a feed-forward lift of $N$ such that $L$ is backward connected. Then, the lift of $N$ to $L$ is the composition of a lift that creates new layers with lifts inside the layers.

**Proof.** Let $N$ be a feed-forward network with layers $C_1, \ldots, C_m$ and $L$ a feed-forward lift of $N$ with layers $C'_1, \ldots, C'_n$ such that $L$ is a backward connected and $L$ is represented by $(\sigma_i)_{i=1}^n$. Denote by $\bowtie$ the balanced coloring in $L$ such that $N$ is the quotient network of $L$ associated to $\bowtie$.

Define the coloring $\bowtie_1$ in $L$ such that $c \bowtie_1 d$ if $c \bowtie d$ and $c, d \in C'_i$ for $1 \leq j \leq n$. Let $c \bowtie_1 d$. Since $\bowtie$ is balanced, we have that $\sigma_i(c) \bowtie \sigma_i(d)$ and $\sigma_i(c), \sigma_i(d) \in C'_{j'}$, where $j' = \max\{1, j-1\}$ and $1 \leq i \leq k$. Then $\sigma_i(c) \bowtie_1 \sigma_i(d)$ for $1 \leq i \leq k$. Hence $\bowtie_1$ is balanced.

Define the network $Q_1 = L/\bowtie_1$ and the set of cells $A_j = [C'_j]_{[\bowtie_1]}$ for $1 \leq j \leq n$. The network $Q_1$ is a feed-forward network with the layers $A_1, \ldots, A_n$. Let $c \in A_1$. There exists $d \in C'_1$ such that $c = [d]_{[\bowtie_1]}$. Then $\sigma_1(c) = \sigma_1([d]_{[\bowtie_1]}) = [\sigma_1(d)]_{[\bowtie_1]} = [d]_{[\bowtie_1]} = c$, for $1 \leq i \leq k$. We have that $A_{i-1} = [C'_{i-1}]_{[\bowtie_1]} = [\cup_{1 \leq j \leq k} \sigma_i(C'_j)]_{[\bowtie_1]} = [\cup_{1 \leq j \leq k} \sigma_i^{\bowtie_1}(A_j)]$, for $j = 2, \ldots, m$. Therefore $Q_1$ is a feed-forward network with the layers $A_1, \ldots, A_n$.

Note that $\bowtie_1 \bowtie_2 \bowtie$. Hence $Q_1$ is a lift of $N$, if $\bowtie_2 \bowtie_3$, and $Q_1 = N$, if $\bowtie_1 \bowtie_2 \bowtie_3$. It follows from lemma 3.2 that

$$|A_{n-m+j}| = |[C'_{n-m+j}]_{[\bowtie_1]}| = |[C'_j]_{[\bowtie]}| = |C_j|, \quad 1 \leq j \leq m,$$

and

$$|A_j| = |[C'_j]_{[\bowtie_1]}| = |[C'_j]_{[\bowtie]}| = |C_j|, \quad 1 \leq j \leq n - m.$$

Hence $Q_1$ is a lift of $N$ that creates $n - m$ new layers or $Q_1 = N$.

The networks $Q_1$ and $L$ have the same number of layers and $L$ is a backward connected lift of $Q_1$. Following example 3.3, we see that the lift of $Q_1$ to $L$ is the composition of lifts inside the layers. Thus the lift of $N$ to $L$ is the composition of a lift that creates new layers and lifts inside the layers.

Example 3.6. Let $L$ be the feed-forward network in figure 3 which is backward connected. Consider the balanced coloring $\bowtie$ in $L$ given by $1 \bowtie 2 \bowtie 3$ and $4 \bowtie 5$. Note that $L/\bowtie$ is a feed-forward network.
Using proposition 3.1, we know that \( L \) can be obtained from \( L/\approx\) by a lift that creates a new layer and lifts inside the layers. In fact, we can see that \( L \) is given by a lift of \( L/\approx \) that creates a new layer, then a lift on the second layer and finally a lift inside the third layer. \( \diamond \)

The lift in figure 5 shows that proposition 3.1 does not hold if the lift is not backward connected.

In the rest of this section, we make two remarks that will be useful in the next sections. First, we see that a lift inside a layer can be further decomposed into simpler lifts. This will allow us to consider simpler lifts.

**Definition 3.3.** Let \( N \) be a network and \( L \) a lift of \( N \). We say that \( L \) is the split of a cell \( c \) in \( N \) into cells \( c_1, c_2, \ldots, c_l \) in \( L \), if the coloring \( \approx \) in \( L \) given by \( c_i \approx c_j \), for \( 1 \leq i, j \leq l \), is balanced, \( L/\approx = N \) and \([c_i]_{\approx} = c\). \( \diamond \)

The network in figure 2 is a split of the cell 2 in the network \( B \) of figure 1 into the cells 2, 3 and 4. The split of the cell 2 in \( B \) into the cells 2 and 4 gives the network \( C \) of figure 1. And the split of the cell 2 in \( C \) into the cells 2 and 3 return back the network in figure 2. Hence the lift inside the second layer from \( B \) to the lift network in figure 2 is the composition of splits of a cell into two cells. Using lemma 3.1, we can easily see that this is the case for every lift inside a layer.

**Remark 3.1.** A lift inside a layer is the composition of splits of a cell into two cells. \( \diamond \)

Second, we prove that each lift of a feed-forward network is given by a unique balanced coloring, if the lifted network is backward connected. This statement will be useful to understand which solutions are lifted from the quotient system to the lift system. We start by looking at an example of a lift given by more than one balanced colorings. In this example, the lift is not backward connected.

**Example 3.7.** Consider the feed-forward network \( N \) in figure 4 and the feed-forward network \( B \) in figure 1. Take the following three balanced colorings in \( B \): \( \approx_1 \) given by \( 4 \approx_1 5 ; \approx_2 \) given by \( 4 \approx_2 6 \); and \( \approx_3 \) given by \( 5 \approx_3 6 \). Then \( N = B/\approx_1 = B/\approx_2 = B/\approx_3 \). \( \diamond \)

Using the backward connectedness of the lifted network, we have the following lemma.

**Lemma 3.3.** Let \( N \) be a feed-forward network and \( L \) a lift of \( N \) such that \( L \) is a backward connected feed-forward network. Let \( \approx_1, \approx_2 \) be two balanced colorings in \( L \). If \( L/\approx_1 = L/\approx_2 = N \), then \( \approx_1 = \approx_2 \).

**Proof.** Let \( N \) be a feed-forward network and \( L \) a lift of \( N \) such that \( L \) is a backward connected feed-forward network. Let \( \approx_1, \approx_2 \) be two balanced colorings in \( L \). Denote by \( C_1, \ldots, C_m \) the layers of \( N \), by \( (\sigma_i^k)_{k=1}^m \) the representative functions of \( N \), by \( C'_1, \ldots, C'_n \) the layers of \( L \) and by \( (\sigma^k_i)_{i=1}^m \) the representative functions of \( L \). Suppose that \( L/\approx = L/\approx_2 = N \).

Since \( L \) is backward connected, we know that \( N \) is backward connected and \( |C_m| = |C'_m| = 1 \). So for \( c \in C'_m \), we have that \([c]_{\approx} = [c]_{\approx_2} \). Suppose that \([c]_{\approx} = [c]_{\approx_2} \), for every \( c \in C'_j \) where \( j > 1 \). Let \( d \in C'_{j-1} \). Then there exist \( 1 \leq i \leq k \) and \( c \in C'_j \) such that \( \sigma_i^k(c) = d \). Thus

\[
[d]_{\approx} = [\sigma_i^k(c)]_{\approx} = \sigma_i^N([c]_{\approx_1}) = \sigma_i^N([c]_{\approx_2}) = [\sigma_i^k(c)]_{\approx_2} = [d]_{\approx_2}.
\]

And \([d]_{\approx_1} = [d]_{\approx_2} \), for every \( d \in C'_{j-1} \). By induction, \([c]_{\approx} = [c]_{\approx_2} \), for every \( c \in C'_j \) and \( 1 \leq j \leq n \). Hence \( \approx_1 = \approx_2 \). \( \square \)
Example 3.8. Let $\mathcal{L}$ be the network in figure 3. Consider the balanced coloring $\otimes \triangleleft$ in $\mathcal{L}$ given by $1 \otimes 2 \otimes 3$ and $4 \otimes 5$, and the quotient network $\mathcal{N} = \mathcal{L} / \otimes \triangleleft$. Note that $\mathcal{L}$ is backward connected for the cell 10. By lemma 3.3, we know that $\otimes \triangleleft$ is the unique balanced coloring such that $\mathcal{N} = \mathcal{L} / \otimes \triangleleft$. ♦

4. Feed-forward systems

Given a network, a coupled cell system admissible by that network is a system that respects the network structure. In a coupled cell system, we view each cell of the network as a dynamical system whose dynamics depends on its own state and on the state of the cells that are coupled to it. In this section, we formalize coupled cell systems associated to a network, synchrony subspaces and steady-state bifurcations, following [15, 24].

Let $\mathcal{N}$ be a network represented by the functions $(\sigma_i)_{i=1}^k$. For each cell $c$ of the network, we associate a coordinate $x_c \in \mathbb{R}$. We say that $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ is an admissible vector field for $\mathcal{N}$, if there is $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ such that

$$(F(x))_c = f(x_c, x_{\sigma_1(c)}, \ldots, x_{\sigma_k(c)}),$$

for every cell $c$ of $\mathcal{N}$. The admissible vector fields for $\mathcal{N}$ are defined by the functions $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$. We denote by $f^\mathcal{N}$ the admissible vector field for $\mathcal{N}$ defined by $f$.

A coupled cell system associated to a network $\mathcal{N}$ is a system of ordinary differential equations

$$\dot{x} = f^\mathcal{N}(x), \quad x \in \mathbb{R}^{\left|\mathcal{N}\right|},$$

where $f^\mathcal{N} : \mathbb{R}^{\left|\mathcal{N}\right|} \to \mathbb{R}^{\left|\mathcal{N}\right|}$ is an admissible vector field for $\mathcal{N}$. When $\mathcal{N}$ is a feed-forward network, we refer to a coupled cell system associated to $\mathcal{N}$ as a feed-forward system.

Example 4.1. Let $\mathcal{N}$ be the feed-forward network in figure 6. A feed-forward system associated to $\mathcal{N}$ has the following form

$$\begin{aligned}
\dot{x}_1 &= f(x_1, x_1, x_1) \\
\dot{x}_2 &= f(x_2, x_1, x_1) \\
\dot{x}_3 &= f(x_3, x_1, x_1) \\
\dot{x}_4 &= f(x_4, x_2, x_3)
\end{aligned}$$
where \( f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \) is a smooth function that defines the dynamics of each cell. For example the variable \( x_4 \), that corresponds to the cell 4, depends on the variable \( x_2 \) and \( x_3 \), which correspond to its input cells 2 and 3.

In order to study steady-state bifurcation, we will assume that the system has an equilibrium at a full-synchronized equilibrium. Without loss of generality, we assume that this equilibrium is the origin.

Let \( C_1, \ldots, C_m \) be the layers of \( N \). For every feed-forward system \( f^N \) associated to \( N \), the Jacobian matrix at the origin has the form

\[
J_f^N = \begin{bmatrix}
(\sum_{i=0}^{k} f_i) I_1 & 0 & 0 & \cdots & 0 & 0 \\
R_2 & f_0 I_2 & 0 & \cdots & 0 & 0 \\
0 & R_3 & f_0 I_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & f_0 I_{m-1} & 0 \\
0 & 0 & 0 & \cdots & R_m & f_0 I_m
\end{bmatrix},
\]

where

\[
f_i := \frac{\partial f}{\partial x_i} (0, 0, \ldots, 0),
\]

\( 0 \leq i \leq k \), \( I_j \) is the identity matrix of size \( |C_j| \times |C_j| \), \( j = 1, \ldots, m \), \( R_l \) is a \( |C_l| \times |C_{l-1}| \)-matrix, \( l = 2, \ldots, m \). The eigenvalues of \( J_f^N \) are

\[
\sum_{i=0}^{k} f_i \quad \text{and} \quad f_0.
\]

The Jacobian matrix of a feed-forward system at a full synchrony equilibrium has only those two eigenvalues which we call the valency and the internal dynamics, respectively.

A \textit{polydiagonal subspace} is a subspace of \( \mathbb{R}^{|N|} \) given by the equalities of some cell coordinates. Given a coloring \( \bowtie \) on the set of cell of \( N \), the polydiagonal subspace associated to \( \bowtie \) is

\[
\Delta_{\bowtie} := \{ x : c \bowtie d \Rightarrow x_c = x_d \} \subseteq \mathbb{R}^{|N|}.
\]

Note that any polydiagonal subspace of \( \mathbb{R}^{|N|} \) defines a unique coloring on the set of cells of \( N \).

Given a function \( G : \mathbb{R}^{|N|} \to \mathbb{R}^{|N|} \) and a subset \( \Delta \subseteq \mathbb{R}^{|N|} \), we say that \( \Delta \) is \textit{invariant} by \( G \) if \( G(\Delta) \subseteq \Delta \). A \textit{synchrony subspace} of a network \( N \) is a polydiagonal subspace of \( \mathbb{R}^{|N|} \) that is invariant by any admissible vector field of \( N \). There is a one-to-one correspondence between balanced colorings \( \bowtie \) and synchrony subspaces \( \Delta_{\bowtie} \). See [15, theorem 4.3]. More specifically, the polydiagonal \( \Delta_{\bowtie} \) associated to a coloring \( \bowtie \) is a synchrony subspace of \( N \) if and only if the coloring \( \bowtie \) is balanced.

Since a synchrony subspace \( \Delta_{\bowtie} \) is invariant by every admissible vector field \( f^N \) of \( N \), every coupled cell system of \( N \) given by \( f^N \) can be restricted to \( \Delta_{\bowtie} \). Each restricted system is a coupled cell system of \( N/\bowtie \) given by \( f^{N/\bowtie} \). Moreover, given a solution \( y(t) \in \mathbb{R}^{|N/\bowtie|} \) of the coupled cell system of \( N/\bowtie \) given by \( f^{N/\bowtie} \), we have that \( x(t) = (x_c(t)) \), where \( x_c(t) = y_{c/\bowtie}(t) \) is a solution of the coupled cell system of \( N \) given by \( f^N \). See [15, theorem 5.2].

\textbf{Example 4.2.} Consider the network \( N \) in figure 6 and the general form of a feed-forward system associated to \( N \) given in example 4.1. It is easy to see that the polydiagonal subspace \( \Delta_1 = \{ x_2 = x_3 \} \) is flow-invariant for every such feed-forward system. This synchro-
ny subspace corresponds to the balanced coloring given by $2 \triangleright 3$ and the quotient network $Q = \mathbb{N}/\triangleright$ is the feed-forward network with three layers and one cell in each layer.

Let $y = (y_1, y_2, y_3) : \mathbb{R} \rightarrow \mathbb{R}^3$ be a solution of $\dot{y} = f^0(y)$, a feed-forward system of $Q$. Then we can lift this solution to a solution of the corresponding feed-forward system of $N$, $\dot{x} = f^N(x)$. The lifted solution has the form $x(t) = (y_1(t), y_2(t), y_3(t), y_4(t))$. \hfill \Box$

Now we define the classes of feed-forward systems with a steady-state bifurcation from a full synchronized equilibrium.

Let $G : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ be a family of smooth vector fields, $d > 0$ and the corresponding dynamical systems, depending on a parameter $\lambda$,

$$\dot{x} = G(x, \lambda),$$

(1)

Consider an equilibrium $(x^*, \lambda^*)$ of (1), i.e. $G(x^*, \lambda^*) = 0$. The family of dynamical systems (1) suffers a local bifurcation at $(x^*, \lambda^*)$ if for every neighborhoods $U_1$ and $U_2$ of $x^*$ and $\lambda^*$, respectively, there exists $\lambda_1, \lambda_2 \in U_\lambda$ such that the family (1) at $\lambda_1$ and $\lambda_2$ have different topological structures (different stability/number of equilibrium points or periodic orbits, etc). A necessary condition for a local bifurcation to occur is that the Jacobian of $G$ at $(x^*, \lambda^*)$, $DG_{(x^*, \lambda^*)}$, has an eigenvalue with zero real part. We focus on steady-state bifurcations and we say that a steady-state bifurcation at $(x^*, \lambda^*)$ occurs if the number of equilibrium points in a neighborhood of $x^*$ changes when the parameter $\lambda$ crosses $\lambda^*$. A necessary condition for the occurrence of a steady-state bifurcation at $(x^*, \lambda^*)$ is that 0 is an eigenvalue of $DG_{(x^*, \lambda^*)}$.

In order to study the steady-state bifurcations of a family of coupled cell systems associated to $N$ from a fully synchronous equilibrium at $\lambda = 0$, we consider a family of smooth functions $f : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f((0, 0, \ldots, 0), \lambda) = 0,$$

for every $\lambda \in \mathbb{R}$. We denote by $\mathcal{V}(N)$ the set of those functions. The set of functions $f \in \mathcal{V}(N)$ such that a steady-state bifurcation occurs at $(0, 0)$ for $f^0$ is given by the union of the following sets

$$\mathcal{V}_k(N) := \{ f \in \mathcal{V}(N) : \sum_{i=0}^k f_i = 0 \}, \quad \mathcal{V}_0(N) := \{ f \in \mathcal{V}(N) : f_0 = 0 \}.$$

Thus $\mathcal{V}_k(N)$ denotes the set of functions with a bifurcation condition associated with the valency of $N$ and $\mathcal{V}_0(N)$ the set of functions with a bifurcation condition associated with the internal dynamics of the cells.

Next, we define equilibrium branches of a coupled cell system. Since our study is local, we define branches using germs. A germ is a class of functions with the same values in some neighborhood of the origin.

We say that $D \subseteq \mathbb{R}$ is a domain if $D$ has one of the following forms: $]- \lambda_0, 0]$; $]- \lambda_0, 0]$; or $[0, \lambda_0]$, for some $\lambda_0 > 0$. Let $D_1, D_2$ be domains. We say that two smooth functions $b_1 : D_1 \rightarrow \mathbb{R}^{|N|}$ and $b_2 : D_2 \rightarrow \mathbb{R}^{|N|}$ are germ equivalents if there exists an open neighborhood $U$ of 0 such that $U \cap D_1 \cap D_2 \neq \{0\}$ and $b_1(\lambda) = b_2(\lambda)$, for every $\lambda \in U \cap D_1 \cap D_2$. The previous relation is not transitive, so we consider its closer by transitivity. Given a smooth function $b$, we use the term germ $b$ to refer to a representative element of the equivalence class of $b$ with respect to germ equivalence.

Let $D$ be a domain. We say that a germ $b : D \rightarrow \mathbb{R}^{|N|}$ is an equilibrium branch of $f$ on $N$, if

$$f^N(b(\lambda), \lambda) = 0,$$

$$5514$$
for every $\lambda \in D$. Since $f(0,0,\ldots,0,\lambda) = 0$ for every $\lambda$, we have that $x(\lambda) = (0,\ldots,0)$ is an equilibrium branch of $f$ on $N$, called the trivial branch of $f$ on $N$. The equilibrium branches of $f$ on $N$ different from trivial branch are called the bifurcation branches of $f$ on $N$. We define the set of equilibrium branches of $f$ on $N$

$$B(N,f) = \{ b : D \to \mathbb{R}^{|N|} : b \text{ is an equilibrium branch of } f \text{ on } N \}.$$

5. Steady-state bifurcations for FFNs associated with the valency

In this section, we study the bifurcation branches on a feed-forward system with a bifurcation condition associated with the valency. This corresponds to solve the equation

$$f^N(x, \lambda) = 0$$

in a neighborhood of the origin. By the feed-forward structure, we know that the solution for each cell in the first layer is independent of the others cells. Using the implicit function theorem, we see that the solution on the first layer determinates the solution on the other layers, and the dynamics in each cell of the first layer has a transcritical bifurcation.

**Proposition 5.1.** Let $N$ be a feed-forward network with layers $C_1, \ldots, C_m$. Let $f \in \mathcal{V}_k(N)$. Then, generically, there are $2^{|C_1|}$ equilibrium branches of $f$ on $N$. Moreover, every equilibrium branch is uniquely determined by its value at the cells of the first layer $C_1$.

**Proof.** Let $N$ be a feed-forward network with layers $C_1, \ldots, C_m$. Let $f \in \mathcal{V}_k(N)$. Generically, assume that $f_0 \neq 0$, $\sum_{i,j=0}^k f_{ij} \neq 0$ and $\sum_{i=0}^k f_{i\lambda} \neq 0$, where $f_{ij}$ is the second order partial derivatives of $f(x_0, x_1, \ldots, x_k, \lambda)$ at $(0,0,\ldots,0,0)$ with respect to $x_i$ and $x_j$, and $f_{i\lambda}$ is the second order partial derivatives of $f$ at $(0,0,\ldots,0,0)$ with respect $x_i$ and $\lambda$, for $0 \leq i,j \leq k$.

The equilibrium branches of $f$ on $N$ are given by the solutions of

$$f^N(x, \lambda) = 0,$$

in a neighborhood of the origin. The Taylor expansion of $f$ at $(0,0,\ldots,0,0)$ is given by
\[ f(x, x_1, \ldots, x_k, \lambda) = \sum_{i=0}^{k} f_i x_i + \sum_{i=0}^{k} f_{i\lambda} x_i \lambda + \sum_{i,j=0}^{k} \frac{f_{ij}}{2} x_i x_j + h.o.t., \]

where \( h.o.t \) denotes high order terms.

For \( c \in C_1 \), we have that

\[ f_c^N(x, \lambda) = 0 \iff f(x_c, x_c, \ldots, x_c, \lambda) = 0. \]

\[ \iff x_c \lambda \sum_{i=0}^{k} f_{i\lambda} + x_c^2 \sum_{i,j=0}^{k} \frac{f_{ij}}{2} + h.o.t. = 0 \]

\[ \iff x_c = 0 \lor \lambda \sum_{i=0}^{k} f_{i\lambda} + x_c \sum_{i,j=0}^{k} \frac{f_{ij}}{2} + h.o.t. = 0. \]

Using the implicit function theorem, there exist \( \lambda_0 > 0 \) and a germ \( \beta : [0, \lambda_0) \to \mathbb{R} \) such that \( \beta(0) = 0 \) and

\[ f(x_c, x_c, \ldots, x_c, \lambda) = 0 \iff x_c = 0 \lor x_c = \beta(\lambda), \quad -\lambda_0 < \lambda < \lambda_0. \]

Denote by \( D \) the set of cells \( C_2 \cup \cdots \cup C_m \). Since \( f_0 \neq 0 \), the matrix \( \partial f^N_d / \partial x^d \bigl|_{d,d' \in D} \) is invertible. By the implicit function theorem, there exist \( \lambda_0' > 0 \) and \( W : \mathbb{R}^{|C_1|} \times C_0 \to \mathbb{R}^{|D|} \) such that \( \lambda_0' \leq \lambda_0 \) and

\[ f^N(x, \lambda) = 0 \iff \left( \bigwedge_{c \in C_1} f(x_c, x_c, \ldots, x_c, \lambda) = 0 \right) \land x_D = W(x_{C_1}, \lambda) \]

\[ \iff \left( \bigwedge_{c \in C_1} [x_c = 0 \lor x_c = \beta(\lambda)] \right) \land x_D = W(x_{C_1}, \lambda) \]

for \( -\lambda_0' < \lambda < \lambda_0' \).

Therefore any equilibrium branch is uniquely determined by its value at the cells of the first layer \( C_1 \) and each cell of \( C_1 \) has one of two possible values. So there are \( 2^{|C_1|} \) equilibrium branches.

**Example 5.1.** Let \( L \) be the feed-forward network on the right of figure 7. Consider a generic feed-forward system with a bifurcation condition associated to the valency. By proposition 5.1, there are eight equilibrium branches. The equilibrium branches must synchronize for at least two cells in the first layer. Since each cell in the first layer has two possible values and there are 3 cells in the first layer.

Consider the following colorings in \( L \): \( \triangleright\triangleright_1 \) given by 1 \( \triangleright\triangleright_2 \) 2; \( \triangleright\triangleright_3 \) given by 2 \( \triangleright\triangleright_2 \) 3; and \( \triangleright\triangleright_3 \) given by 1 \( \triangleright\triangleright_2 \) 3. They are balanced and the network on the left of figure 7 is the quotient of \( L \) by any of the colorings.

Every equilibrium branch in \( L \) corresponds to a bifurcation branch in the quotient network. There are more bifurcation branches in \( L \), however, they are copies of bifurcation branches in the quotient network. \( \Diamond \)
6. Steady-state bifurcations for FFNs associated with the internal dynamics

Now, we study the bifurcation branches of a feed-forward system with a bifurcation condition associated to the internal dynamics. Layer by layer, we solve the equation

\[ f^N(x, \lambda) = 0 \]

in a neighborhood of the origin. In the first layer, there exists no bifurcation and the trivial solution is the unique solution. The inputs of each cell on the second layer are from cells in the first layer. Using the implicit function theorem, we can see that the cells in the second layer have a transcritical bifurcation. Fixing a solution in the first two layers, we solve the equation for the cells in the following layer

\[ f(x_c, x_d, \ldots, x_e, \lambda) = 0, \]

where \( c_1, \ldots, c_k \) are the input cells of \( c \) which belong to the previous layer and have a fixed solution value. In order to solve this equation, we use an appropriate change of coordinates that was used in [20]. The solutions of this equation have a predictable growth-rate and slopes.

**Definition 6.1 ([20, Definition 2.2]).** Let \( b : D \to \mathbb{R} \) be a germ and \( D = [0, \lambda_0] \) or \( D = ] - \lambda_0, 0[ \). If \( b = 0 \), we say that \( b \) has square-root-order \(-1\) and slope \( 0 \). Otherwise, we say that \( b \) has square-root-order \( p \) and slope \( s \) and write that \( b \sim O(2^{-p}) \), if \( p \) is the smallest non-negative integer such that there is a smooth function \( b^* : [0, \lambda^2 \sqrt{2^p}] \to \mathbb{R} \) satisfying

\[ b(\lambda) = b^*(|\lambda|^{2^{-p}}), \quad s = \lim_{|\lambda| \to 0} \frac{b^*(|\lambda|)}{\lambda} \neq 0. \]

In previous studies of bifurcation in feed-forward networks, different authors have noticed that the layers act as amplifiers, [14, 20]. We will also see that the square-root-order of a bifurcation branch increases from a layer to the next layer.

In the next two lemmas, we determine the square-root-order and slope of a solution to \( f(x, x_1, \ldots, x_k, \lambda) = 0 \), when the square-root-orders and slopes of the inputs \( x_1, \ldots, x_k \) are known. In the first lemma, we consider inputs that are defined for positive values of the parameter \( \lambda \).

**Lemma 6.1.** Let \( f \in \mathcal{V}_0(N) \) generic, \( y : [0, \lambda_0] \to \mathbb{R} \) a germ, \( p_1, \ldots, p_k \) and \( s_1, \ldots, s_k \) such that \( y_i \) has square-root-order \( p_i \) and slope \( s_i \) for \( 1 \leq i \leq k \). Suppose that \( p := \max\{p_1, \ldots, p_k\} \geq 0 \) and define

\[ A := \{ i : y_i \sim O(2^{-p}) \}, \quad Z = \sum_{i \in A} \frac{f_i s_i}{1} \].

(i) If \( Z < 0 \), then there exist \( 0 < \lambda^*_0 < \lambda_0 \) and germs \( b^+, b^- : [0, \lambda^*_0] \to \mathbb{R} \) such that \( b^\pm \) have square-root-order \( p + 1 \) and slope \( \pm \sqrt{-Z} \), and

\[ f(x, y, \lambda) = 0 \iff x = b^\pm(\lambda), \quad 0 < \lambda < \lambda^*_0. \]

(ii) If \( Z > 0 \), then the equation \( f(x, y, \lambda) = 0 \) has only the trivial solution \((x, \lambda) = (0, 0)\).

**Proof.** Let \( f \in \mathcal{V}_0(N), y : [0, \lambda_0] \to \mathbb{R} \) \( p_1, \ldots, p_k \) and \( s_1, \ldots, s_k \) such that \( y_i \) has square-root-order \( p_i \) and slope \( s_i \) for \( 1 \leq i \leq k \). Suppose that \( p := \max\{p_1, \ldots, p_k\} \geq 0 \) and define
\[ A := \{ i : y_i \sim O(2^{-p}) \}, \quad Z = \sum_{i \in A} \frac{f_i}{f_0}. \]

The Taylor expansion of \( f \) at the origin is
\[
f(x, x_1, \ldots, x_k, \lambda) = \sum_{i=1}^{k} f_i x_i + \frac{f_0}{2} x^2 + f_0 x \lambda + \sum_{i=1}^{k} f_{i \lambda} x_i \lambda + \sum_{i=1}^{k} f_{i \lambda} x_i \lambda \]
\[
+ \sum_{i=1}^{k} f_{0i} x_i + \sum_{i,j=1}^{k} f_{ij} x_i x_j + h.o.t. \]

For \( \lambda \geq 0 \), consider the following transformation of variables
\[
\mu = \lambda^{2^{-p}}, \quad x = \mu z, \quad y_i(\lambda) = \lambda^{2^{-p}} w_i(\mu). \]

Then
\[
w_i(0) = \lim_{\lambda \to 0} \frac{y_i(\lambda)}{\lambda^{2^{-p}}} = s_i, \quad \lambda = \mu^{2^{-(p+1)}}, \quad y_i(\lambda) = \mu^{2^{-(p+1)-p}} w_i(\mu).
\]

Moreover \( p - p_i = 0 \), if \( i \in A \), and \( p - p_i > 0 \), otherwise. Using the transformation of variables and the Taylor expansion of \( f \), we obtain that
\[
f(x, y(\lambda), \lambda) = 0 \Leftrightarrow \sum_{i=1}^{k} f_{i y_i}(\lambda) + \frac{f_0}{2} x^2 + f_0 x \lambda + \sum_{i=1}^{k} f_{i \lambda} y_i(\lambda) \lambda
\]
\[
+ \sum_{i=1}^{k} f_{0i} y_i(\lambda) x + \sum_{i,j=1}^{k} f_{ij} y_i(\lambda) y_j(\lambda) + h.o.t. = 0 \]
\[
\Leftrightarrow \sum_{i \in A} f_i \mu^2 w_i(0) + \frac{f_0}{2} \mu^2 z^2 + h.o.t. = 0 \]
\[
\Leftrightarrow \mu^2 \left( \sum_{i \in A} f_i w_i(0) + \frac{f_0}{2} z^2 + h.o.t. \right) = 0 \]
\[
\Leftrightarrow \mu = 0 \lor \sum_{i \in A} f_i s_i + \frac{f_0}{2} z^2 + h.o.t. = 0.
\]

Define
\[
h(z, \mu) = \sum_{i \in A} f_i s_i + \frac{f_0}{2} z^2 + h.o.t..\]

If \( Z < 0 \), we have that \( h(\pm \sqrt{2Z}, 0) = 0 \) and \( h(\pm \sqrt{2Z}, 0) \neq 0 \). By the implicit function theorem, there exist a neighborhood \( U \) of 0 and functions \( z^+, z^- : U \to \mathbb{R} \) such that
\[
h(z, \mu) = 0 \Leftrightarrow z = z^+(\mu), \quad z^+(\mu) = \pm \sqrt{2Z} + h.o.t..\]

Let \( 0 < \lambda^*_0 < \lambda_0 \) and \( b^+, b^- : [0, \lambda^*_0] \to \mathbb{R} \) such that \( [0, (\lambda^*_0)^{2^{-(p+1)}}] \subseteq U \) and
Then $b^\pm$ have square-root-order $p + 1$ and slope $\pm \sqrt{-2Z}$, and
\[
f(x, y(\lambda), \lambda) = 0 \iff x = b^\pm(\lambda).
\]

This proves (i).

If $Z > 0$, then $h(z, 0)$ is always positive, when $f_{00} > 0$, or it is always negative, when $f_{00} < 0$. So there is no solution to the equation $h(z, 0) = 0$. And the equation $f(x, y(\lambda), \lambda) = 0$ has only the trivial solution $(x, \lambda) = (0, 0)$, proving (ii).

In the second lemma, we look to the solution of $f(x, x_1, \ldots, x_k, \lambda) = 0$ when the inputs solutions $(x_1, \ldots, x_k)$ are defined for $\lambda < 0$. The proof is very similar to the previous one and it is omitted.

**Lemma 6.2.** Let $f \in \mathcal{V}_0(N)$ generic, $y : ] - \lambda_0, 0] \to \mathbb{R}^k$ a germ, $p_1, \ldots, p_k$ and $s_1, \ldots, s_k$ such that $y_i$ has square-root-order $p_i$ and slope $s_i$ for $1 \leq i \leq k$. Suppose that $p := \max\{p_1, \ldots, p_k\} \geq 0$ and define
\[
A := \{i : y_i \sim O(2^{-p})\}, \quad Z = \sum_{i \in A} \frac{f_i}{f_{00}}.
\]

(i) If $Z > 0$, then there exist $0 < \lambda_0 < 0$ and germs $b^+, b^- : ] - \lambda_0, 0] \to \mathbb{R}$ such that $b^\pm$ have square-root-order $p + 1$ and slope $\pm \sqrt{-2Z}$, and
\[
f(x, y(\lambda), \lambda) = 0 \iff x = b^\pm(\lambda), \quad 0 > \lambda > \lambda_0^*.
\]

(ii) If $Z < 0$, then the equation $f(x, y(\lambda), \lambda) = 0$ has only the trivial solution $(x, \lambda) = (0, 0)$.

We return to the network $C$ of figure 1 and calculate some bifurcation branches of a feed-forward system associated to this network with a bifurcation condition associated with the internal dynamics.

**Example 6.1.** Let $C$ be the feed-forward network in figure 1 with three layers $\{1\}$, $\{2, 3\}$, $\{4, 5, 6\}$ and $f^C$ a feed-forward system with a bifurcation condition associated with the internal dynamics. In order to study the bifurcation branches, we need to solve the equation
\[
f^C(x, \lambda) = 0
\]
in a neighborhood of the origin. We solve this equation layer by layer and start by the first layer. For the cell 1, we need to solve the equation
\[
f(x_1, x_1, x_1, \lambda) = 0.
\]

By the implicit function theorem, there exists only the trivial branch
\[
x_1(\lambda) = 0,
\]
for $\lambda$ in a neighborhood of 0. This completes the study of the first layer and we proceed to the second layer.

Fixing the unique solution on the first layer, we look to the equation in one of the cells in the second layer. Take for example the cell 2, we have the following equation
\[
f(x_1, x_1, x_1, \lambda) = 0.
\]
The study of this equation is similar to the bifurcations with a condition associated to the valency. Assuming that $f_00 \neq 0$ and $f_{0\lambda} \neq 0$, we conclude that there are two solutions

$$x_2(\lambda) = 0 \lor x_2(\lambda) = b^0(\lambda) = -\frac{2f_{0\lambda}}{f_00^2} \lambda + h.o.t. \quad (2)$$

for $\lambda$ in a neighborhood of 0. Repeat the same procedure for the other cell in the second layer. Then there are four solutions on the second layer

$$\begin{cases} x_2(\lambda) = 0 & \lor x_2(\lambda) = b^0(\lambda) \\ x_3(\lambda) = 0 & \lor x_3(\lambda) = b^0(\lambda). \end{cases}$$

Before we study the next layer, note that the branch $b^0(\lambda)$ has square-root-order 0 and slope $-\frac{2f_{0\lambda}}{f_00^2}$.

Now, we fix one of the solutions for the previous layer and solve the equation in the next layer using lemmas 6.1 and 6.2. For example we fix the solution $(x_1, x_2, x_3) = (0, 0, b^0)$ and look to the equation of the cell 5

$$f(x_5, x_2, x_3, \lambda) = 0 \iff f(x_5, 0, b^0(\lambda), \lambda) = 0.$$  

Remember that $x_2$ has square-root-order $-1$ and $x_3$ has square-root-order 0. By the lemmas 6.1 and 6.2, we know that any solution has square-root-order 1. If $f_2f_00 > 0$, lemma 6.1 implies that there are two solutions with square-root-order 1

$$x_5 = s_1^+(\lambda) \lor x_5 = s_1^-(\lambda),$$

where $s_1^\pm$ has slope $\pm2\sqrt{f_2f_00}/f_00$ and $\lambda$ is restricted to positive values, $\lambda \geq 0$. If $f_2f_00 < 0$, we apply lemma 6.2 and obtain two solutions with square-root-order 1 and slope $\pm2\sqrt{|f_2f_00|}/f_00$, however the solutions will be defined for negative values of $\lambda$, $\lambda \leq 0$.

We still need to solve the equations for the cell 4 and 6

$$\begin{cases} f(x_4, x_2, x_3, \lambda) = 0 \\ f(x_6, x_2, x_3, \lambda) = 0 \end{cases} \iff \begin{cases} f(x_4, 0, 0, \lambda) = 0 \\ f(x_6, b^0, 0, \lambda) = 0 \end{cases}.$$  

As before and using the implicit function theorem, we note that the first equation has two solutions

$$x_4(\lambda) = 0 \lor x_4(\lambda) = b^0(\lambda),$$

defined in a neighborhood of 0. For the second equation, we need to use one of the previous lemmas. If $f_1f_00 > 0$, we obtain two solutions with square-root-order 1

$$x_6 = s_2^+(\lambda) \lor x_6 = s_2^-(\lambda),$$

slopes $\pm2\sqrt{f_1f_00}/f_00$ which are defined for $\lambda$ positive. If $f_1f_00 < 0$, we also have two solutions but they are defined for $\lambda$ negative.

We have computed the possible solutions on each cell of the third layer when we fix a solution on the first two layers. Now, we need to patch the solutions into a solution of the network system. In order to do that we define the solution in the intersection of the domain of each cell.
solution. However, the solutions can be defined on different sides of $\lambda = 0$. When this occurs, it does not correspond to a solution of the whole system. For example take $f_{20} < 0$ and $f_{10} > 0$, then the solutions for $x_5$ are defined for $\lambda$ positive and $x_6$ is defined for $\lambda$ negative.

Thus, it does not correspond to a solution of the whole system. Fixing the solution $(x_1, x_2, x_3) = (0, 0, b^0)$. If $f_{12} > 0$, we have the following solutions

$$
\begin{align*}
  x_4 &= 0 \lor x_4 = b^0 \\
  x_5 &= s_1^+(\lambda) \lor x_5 = s_1^-(\lambda) \\
  x_6 &= s_2^+(\lambda) \lor x_6 = s_2^-(\lambda),
\end{align*}
$$

where $s_i^+$ have square-root-order 1 and it is defined for $\lambda \geq 0 (\lambda \leq 0)$, when $f_{10} > 0 (f_{10} < 0)$, respectively. If $f_{12} < 0$, then there is no solution to the equation in the next layer and there is no bifurcation branch $b$ of $f$ on $C$ such that $(b_1, b_2, b_3) = (0, 0, b^0)$.

Repeating this process for the other solutions on the first two layers, we find all bifurcation branches of $f$ on $C$. We can check that there is no bifurcation branch of $f$ on $C$ without synchrony on the cells 2 and 3, if $f_{12} < 0$.

Finally, we represent all bifurcation branches using their growth-rates and slopes. Let $N$ be a feed-forward network and $f \in \mathcal{V}_0(N)$. We define the function that assign for each bifurcation branch a symbol

$$
\Theta : \mathcal{B}(N, f) \rightarrow \{-1, 0, 1\} \times \mathbb{Z}^{[N]} \times \mathbb{R}^{[N]}$

$$
where $b_c$ has square-root-order $p_c$ and slope $s_c$, for each cell $c$ of $N$, and $\delta$ indicates the domain of the branch $b$. If $b$ is defined in a neighborhood of 0, then $\delta = 0$. If $b$ is only defined for $\lambda > 0$, then $\delta = 1$. Finally, if $b$ is only defined for $\lambda < 0$, then $\delta = 1$.

**Example 6.2.** Returning to the feed-forward network $C$ of figure 1, we give the complete description of the bifurcation branches using the symbols $(\delta, (p_c)_c, (s_c)_c)$. There is the trivial branch

$$
\Theta(0, 0, 0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0, 0, 0).
$$

There are seven branches with square-root-order 0

$$
\begin{align*}
  &\Theta(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}.
\end{align*}
$$

where $s_0 = -2f_{0\lambda}/f_{00}$. There are eight branches with square-root-order 1
where $\delta = \text{sign}(f_1 + f_2) f_{0\lambda}$ and $s_1 = 2\sqrt{|(f_1 + f_2) f_{0\lambda}| / f_{00}}$. If $ff_2 > 0$, there are 16 more branches with square-root-order 1

$$(\delta, (-1, 0, -1, 1, 1), (0, s_0, 0, \pm s_2, \pm s_3)),$$

$$(\delta, (-1, -1, 0, 0, 1, 1), (0, s_0, 0, \pm s_3, \pm s_2)),$$

$$(\delta, (1, -1, 0, 0, 1, 1), (0, s_0, 0, \pm s_3, \pm s_2)),$$

where $s_2 = 2\sqrt{|f_{0\lambda}| / f_{00}}$ and $s_3 = 2\sqrt{|f_{0\lambda}| / f_{00}}$. \hfill \Box

The symbols which are the image of some bifurcation branch respect some rules. If $(\delta, (p_c)_c, (s_c)_c) \in \Theta(B(N,f))$, then

$\Omega.1 \quad p_c = -1 \iff s_c = 0.$

This follows from definition 6.1.

$\Omega.2 \quad \delta = 0 \Rightarrow \forall_c p_c \leq 0.$

The bifurcation branch is defined on an open neighborhood of 0 if and only if $\delta = 0$. By lemmas 6.1 and 6.2, we know that every branch with a square-root-order greater than 0 is defined only on one side of $\lambda = 0$ and $\delta \neq 0$. So $\delta = 0$ implies that $p_c \leq 0$ for every cell $c$.

$\Omega.3 \quad p_c = -1 \Rightarrow \forall_c p_{\sigma(c)} = -1.$

Suppose that the branch is not trivial on one input of $c$, i.e. $p_{\sigma(c)} > -1$ for some $i$. By lemmas 6.1 and 6.2, we have that $p_c > -1$.

$\Omega.4 \quad p_c > -1 \Rightarrow \forall_i p_{\sigma(c)} \leq p_c - 1 \land \exists_c p_{\sigma(c)} = p_c - 1.$

Suppose that the branch has a square-root-order greater than $p_c - 1$ for some input of $c$, i.e. $p_{\sigma(c)} > p_c - 1$ for some $i$. By lemmas 6.1 and 6.2, we obtain the contradiction that $b_c$ has square-root-order greater than $p_c$. Supposing that every input of $c$ has square-root-order less than $p_c - 1$, lemmas 6.1 and 6.2 lead to an absurd.

$\Omega.5 \quad p_c = 0 \Rightarrow s_c = -\frac{2f_{0\lambda}}{f_{00}}.$

Assume that $p_c = 0$. By $\Omega.4$, we know that $p_{\sigma(c)} = -1$ for every $1 \leq i \leq k$ and $b_{\sigma(c)} = 0$.

In the running example, we saw that if a cell has trivial inputs, then there are two options. See equation (2). Since $p_c = 0$, we know that $b_c \neq 0$ and $b_c$ has slope $-2f_{0\lambda}/f_{00}$. So $s_c = -\frac{2f_{0\lambda}}{f_{00}}$.

$\Omega.6 \quad p_c > 0 \Rightarrow s_c = \pm \sqrt{\frac{\lambda}{2f_{0\lambda}}} \sum_{i \in A_c} f_i s_{\sigma(c)},$ where $A_c = \{ i : p_{\sigma(c)} = p_c - 1 \}.$

This follows from lemma 6.1, if $\delta = 1$, and lemma 6.2, if $\delta = -1$.

Let $\Omega(N,f) \subseteq \{-1, 0, 1\} \times \mathbb{Z}^{|N|} \times \mathbb{R}^{|N|}$ be the set of symbols $(\delta, (p_c)_c, (s_c)_c) \in \{-1, 0, 1\} \times \mathbb{Z}^{|N|} \times \mathbb{R}^{|N|}$ satisfying $\Omega.1$–$\Omega.6$. Next, we prove that $\Theta$ is a one-to-one correspondence between $B(N,f)$ and $\Omega(N,f)$.

**Proposition 6.1.** Let $N$ be a feed-forward network and $f \in \mathcal{V}_0(N)$ generic. If $(\delta, (p_c)_c, (s_c)_c) \in \Omega(N,f)$, then there exists a unique $b \in B(N,f)$ such that

$$\Theta(b) = (\delta, (p_c)_c, (s_c)_c).$$
Proof. Let \( N \) be a feed-forward network represented by the function \((\sigma_i)_{i=1}^k \) and \( f \in V_0(N) \) generic.

Let \( (\delta, (p_c)_c, (s_c)_c) \in \Omega(N, f) \). We construct the equilibrium branch \( b \) of \( f \) on \( N \) such that \( \Theta(b) = (\delta, (p_c)_c, (s_c)_c) \). It follows from \( \Omega.4 \) that \( p_c = -1 \) for every cell \( c \) in the first layer and \(-1 \leq p_c \leq m-2 \), for every cell \( c \) of \( N \).

Let \( c \) be a cell of \( N \) such that \( p_c = -1 \). Then \( s_c = 0 \), by \( \Omega.1 \). Define \( b_c \) as the germ defined on an open neighborhood of \( 0 \) such that \( b_c = 0 \). Then \( b_c \) has square-root-order \( p_c \) and slope \( s_c \).

It follows from \( \Omega.3 \) that
\[
\begin{align*}
  f(b_c, b_{\sigma(c)}, \ldots, b_{\sigma(c)}, \lambda) &= f(0, 0, \ldots, 0) = 0 .
\end{align*}
\]

Let \( c \) be a cell of \( N \) such that \( p_c = 0 \). Then \( s_c = -2f_{00}/f_{00} \), by \( \Omega.5 \), and \( p_{\sigma(c)} = -1 \) for \( 1 \leq i \leq k \), by \( \Omega.4 \). Define \( b_c \) as the germ \( b^d \) defined in \( (2) \) on an open neighborhood of \( 0 \). Then \( b_c \) has square-root-order \( p_c \), slope \( s_c \) and
\[
\begin{align*}
  f(b_c, b_{\sigma(c)}, \ldots, b_{\sigma(c)}, \lambda) &= f(b^0, 0, \ldots, 0) = 0 .
\end{align*}
\]

The following germs are defined by induction on \( p \geq 1 \). Assuming that \( b_c \) is defined for every cell \( c' \) of \( N \) such that \( p_{c'} < p \) and \( b_c \) has square-root-order \( p_{c'} \) and slope \( s_{c'} \), we define the germ \( b_c \) which has square-root-order \( p_c \) and slope \( s_c \), for each cell \( c \) of \( N \) such that \( p_c = p \). Since \( p_c \leq m-2 \), this process must terminate.

Let \( p \geq 1 \) and \( c \) be a cell of \( N \) such that \( p_c = p \). By \( \Omega.4 \) and the assumption, \( p_{\sigma(c)} < p \) and \( b_{\sigma(c)} \) is defined for every \( 1 \leq i \leq k \). Consider the germ \( y : D \to \mathbb{R}^k \) such that \( y_i = b_{\sigma(c)} \) for every \( 1 \leq i \leq k \), and let \( \lambda \) be the germ obtained in lemma \( 6.1 \) (\( 6.2 \), if \( \delta = 1 \) (\(-1 \), respectively), such that \( b_c \) has square-root-order \( p_c \) slope \( s_c \) and it is defined for positive (negative) values. It follows from \( \Omega.6 \) that there exists such germ and it is unique. Moreover,
\[
  f(b_c, b_{\sigma(c)}, \ldots, b_{\sigma(c)}, \lambda) = 0 .
\]

Define the germ \( b = (b_c)_c : D \to \mathbb{R}^{|N|} \), where \( D \) is the intersection of the domains of each \( b \). By construction \( f^N(b(\lambda), \lambda) = 0 \), so \( b \) is an equilibrium branch of \( f \) on \( N \). Let \( (\delta', (p'_c)_c, (s'_c)_c) := \Theta(b) \). By construction, \( p'_c = p_c \) and \( s'_c = s_c \), for every cell \( c \). If \( \delta = 0 \), then \( p_c \leq 0 \) and \( \delta' = 0 \), by \( \Omega.2 \). If \( \delta = \pm 1 \), then there exists \( p_c > 0 \), by \( \Omega.1 \) and \( \Omega.6 \). If \( \delta = 1 \), then \( b_c \) is defined for positive values and \( \delta' = 1 \). Similar, if \( \delta = -1 \), then \( \delta' = 1 \). Therefore
\[
  \Theta(b) = (\delta, (p_c)_c, (s_c)_c) .
\]

We can see in each step of the construction of \( b \) that we choose the unique germ that respects the conditions of square-root-order, slope and be a solution to the equation. \( \square \)

Next, we see how to use this result to find some bifurcation branches.

Let \( N \) be a feed-forward network with layers \( C_1, \ldots, C_m \) and \( f : \mathbb{R}^{k+1} \times \mathbb{R} \to \mathbb{R} \in V_0(N) \) generic. Define
\[
\tilde{\delta} = \text{sign}(f_{00}) \sum_{i=1}^k f_i = \frac{f_{00} \sum_{i=1}^k f_i}{|f_{00} \sum_{i=1}^k f_i|}, \quad \tilde{p}_1 = -1, \quad \tilde{s}_1 = 0 , \quad (3)
\]
and
\[
\tilde{p}_j = j - 2, \quad \tilde{s}_j = -\text{sign}(f_{00}) \frac{2f_{00}^{1-\frac{j}{2}}}{f_{00}} \left( \sum_{i=1}^k \left| f_i \right| \right)^{1-2^{-(j-2)}} , \quad (4)
\]
for $2 \leq j \leq m$. Now, for each $3 \leq r \leq m$, define $\delta^r = \tilde{\delta}$,

$$p_c^r = \tilde{p}_r, \quad s_c^r = \tilde{s}_r, \quad c \in C_1 \cup \cdots \cup C_{m-r+1},$$

$$p_c^{r+} = \tilde{p}_r, \quad s_c^{r+} = \tilde{s}_r, \quad c \in C_{m-r+1}, 2 \leq l \leq r - 1,$$

$$p_c^{r-} = \tilde{p}_r, \quad s_c^{r-} = \pm \tilde{s}_r, \quad c \in C_m.$$

We also define $\delta^2 = 0$,

$$p_c^2 = -1, \quad s_c^2 = 0, \quad c \in C_1 \cup \cdots \cup C_{m-1}, \quad p_c^2 = 0, \quad s_c^2 = \frac{2f_{01}}{f_{00}}, \quad c \in C_m.$$

$$\delta^1 = 0, \quad p_c^1 = -1, \quad s_c^1 = 0, \quad c \in C_1 \cup \cdots \cup C_m.$$

We can see that the symbols

$$(\delta^1, (p_c^1), (s_c^1)),
$$

$$(\delta^2, (p_c^2), (s_c^2)),
$$

$$(\delta^r, (p_c^r), (s_c^r)),
$$

respect the rules $\Omega_1$–$\Omega_6$, for $3 \leq r \leq m$. Therefore, they belong to $\Omega(N, f)$ and correspond to equilibrium branches of $f$ on $N$, proposition 6.1.

This means that the set $B(N, f)$ contains the trivial equilibrium branch $b^1$, a bifurcation branch $b^2$ which have a square-root-order 0 and two bifurcation branches $b^{r+}, b^{r-}$ with square-root-order $r - 2$ for every $3 \leq r \leq m$. Moreover, those equilibrium branches have synchrony inside each layer. We summarize the previous in the following result.

**Corollary 6.1.** Let $N$ be a feed-forward network with $m$ layers and $f \in V_0(N)$ generic. For every $1 \leq r \leq m$, there exists $b \in B(N, f)$ such that $b$ has square-root order $r - 2$. If $b \in B(N, f)$, then $b$ has square-root-order less or equal to $m - 2$.

In example 6.2, we saw that the network $C$ in figure 1 respects the previous corollary.

**Example 6.3.** Let $N$ be the feed-forward network $C$ in figure 1 with three layers. In example 6.2, we saw that any feed-forward system on $N$ with a bifurcation condition associated to the internal dynamics has equilibrium branches with square-root-order $-1$, 0 and 1. Moreover, every equilibrium branch has square-root-order less or equal to one.

7. **Lifting bifurcation problem on FFNs**

The bifurcation branches occurring in a quotient system are lifted to bifurcation branches occurring in a lift system. In this section, we study the lifting bifurcation problem which consists on understanding if every bifurcation branch occurring in a coupled cell system associated to a lift network is lifted from a bifurcation branch occurring in the coupled cell system associated to the quotient network.
Definition 7.1. Let \( N \) be a network and \( L \) a lift of \( N \). We say that a bifurcation branch \( b \) of \( f \) on \( L \) is lifted from \( N \), if there exists a balanced coloring \( \triangleright \) in \( L \) such that \( b \in \Delta_{\triangleright} \) and \( N = L/\triangleright \).

In the next proposition, we recover a well-known result about the bifurcation branches being inside a flow-invariant space which contains the center subspace. We present the proof here for completeness. Let \( A : \mathbb{R}^d \to \mathbb{R}^d \) be a linear operator from \( \mathbb{R}^d \) to itself and \( d > 0 \). The center subspace of \( A \) is given by

\[
\ker^*(A) = \{ v \in \mathbb{R}^d : A^k v = 0 \text{ for some } k \}.
\]

We denote the orthogonal complement with respect to the usual inner product of a subspace \( B \subseteq \mathbb{R}^d \) by \( B^\perp \).

Proposition 7.1. Let \( F : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d \) be a smooth function and \( K \subseteq \mathbb{R}^d \) such that \( \ker^*(DF_{(0,0)}) \subseteq K \), \( F(0,0) = 0 \) and \( F(K, \lambda) \subseteq K \) for every \( \lambda \in \mathbb{R} \). Suppose that there exists a function \( x : D \to \mathbb{R}^d \) defined in a domain \( D \) such that \( F(x(\lambda), \lambda) = 0 \) for \( \lambda \in D \). Then there exists a neighborhood \( U \) of \( 0 \) such that \( x(\lambda) \in K \) for every \( \lambda \in U \cap D \).

Proof. Let \( F : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d \) be a smooth function and \( K \subseteq \mathbb{R}^d \) such that \( \ker^*(DF_{(0,0)}) \subseteq K \), \( F(0,0) = 0 \) and \( F(K, \lambda) \subseteq K \) for every \( \lambda \in \mathbb{R} \). Note that \( \mathbb{R}^d = K \oplus K^\perp \). Writing every element of \( \mathbb{R}^d \) in its decomposition in \( K \) and \( K^\perp \), \( v = y + w \), where \( y \in K \) and \( w \in K^\perp \), there are \( g : K \times K^\perp \to K \) and \( h : K \times K^\perp \times \mathbb{R} \to K^\perp \) such that

\[
\dot{v} = F(v, \lambda) \iff \begin{cases} \dot{y} = g(y, w, \lambda) \\ \dot{w} = h(y, w, \lambda). \end{cases}
\]

Hence

\[
DF_{(0,0)} = \begin{bmatrix} D_y g_{(0,0)} & D_w g_{(0,0)} \\ D_y h_{(0,0)} & D_w h_{(0,0)} \end{bmatrix}.
\]

Observe that \( h(y, 0, \lambda) = 0 \), because \( K \) is invariant. Then \( D_y h_{(0,0)} = 0 \) and \( D_w h_{(0,0)} \) is invertible, since \( \ker^*(DF_{(0,0)}) \subseteq K \). By the implicit function theorem, there is \( W : K \times \mathbb{R} \to K^\perp \) such that \( W(0,0) = 0 \) and \( h(y, w, \lambda) = 0 \) if and only if \( w = W(y, \lambda) \).

From \( h(y, 0, \lambda) = 0 \), we have that \( W(y, \lambda) = 0 \). Therefore

\[
F(y, 0, \lambda) = 0 \iff g(y, 0, \lambda) = 0 \land w = 0.
\]

Supposing that \( x \) is a solution to \( F(v, \lambda) = 0 \), we have that \( x \in K \).

It follows that a necessary condition for the existence of a bifurcation branch on a lift network not lifted from the original network is that the center subspace of the coupled cell systems associated to the original network and the lift network have different dimensions.

Corollary 7.1. Let \( N \) be a network, \( L \) a lift of \( N \) associated to the coloring \( \triangleright \triangleright \) and \( f \in V(N) \). If \( \ker^*(J_f^R) \) and \( \ker^*(J_f^L) \) have the same dimension, then every bifurcation branch of \( f \) in \( L \) belongs to \( \Delta_{\triangleright} \) and it is lifted from \( N \).

We recall the dimension of the center subspace of \( J_f^R \) for feed-forward systems with a bifurcation condition associated to the valency and internal dynamics.
Remark 7.1. Let $N$ be a feed-forward network with layers $C_1, \ldots, C_m$.

(i) If $f \in V_k(N)$, then the dimension of $\ker^*(J_N^f)$ is $|C_1|$.  
(ii) If $f \in V_0(N)$, then the dimension of $\ker^*(J_N^f)$ is $|C_2| + \cdots + |C_m|$.  

Example 7.1. Let $N$ be the network on the left of figure 7 and $L$ the lift of $N$ on the right of figure 7.

Consider $f \in V_0(N)$ with a bifurcation condition associated to the internal dynamics. The spaces $\ker^*(J_N^f)$ and $\ker^*(J_L^f)$ have the same dimension, the conclusion of corollary 7.1 holds and every bifurcation branch of $f$ on $L$ is lifted from $N$.

Consider $f \in V_k(N)$ with a bifurcation condition associated to the valency. Note that $\ker^*(J_N^f)$ has dimension 2 and $\ker^*(J_L^f)$ has dimension 3. We saw in example 5.1 that every bifurcation branch of $f$ on $L$ is lifted from $N$. This example shows that the condition in corollary 7.1 is sufficient but not necessary.

7.1. Lifting bifurcation problem on FFNs associated with the valency

In this section, we study the lifting bifurcation problem for feed-forward systems determined by a regular function that has a bifurcation condition associated to the valency. We prove that a lift has an extra bifurcation branch if and only if the center subspace is bigger on the lift than on the quotient network.

Proposition 7.2. Let $N$ be a feed-forward network, $f \in V_k(N)$ generic and $L$ a feed-forward lift of $N$.

(i) If $L$ is a lift that creates new layers or a lift inside a layer, except the first layer, then every bifurcation branch of $f$ on $L$ is lifted from $N$.

(ii) If $L$ is a lift inside the first layer and $L$ is backward connected, then there is at least one bifurcation branch of $f$ on $L$ which is not lifted from $N$.

Proof. Let $N$ be a feed-forward network, $f \in V_k(N)$ generic and $L$ a feed-forward lift of $N$. Denote by $C_1$ and by $C_1'$ the first layer of $N$ and $L$, respectively.

If $L$ is a lift that creates new layers or a lift inside a layer, except the first, then $\ker^*(J_N^f)$ and $\ker^*(J_L^f)$ have the same dimension. Recall remark 7.1. By corollary 7.1, every bifurcation branch of $f$ on $L$ is lifted from $N$.

Suppose that $L$ is a lift inside the first layer and $L$ is backward connected. By remark 3.1, we assume that $L$ is the split of a cell $c \in C_1$ into two cells $c_1, c_2 \in C_1'$ and denote by $\gg$ the balanced coloring in $L$ given by $c_1 \gg c_2$. By lemma 3.3, $\gg$ is the unique balanced coloring such that $L/\gg = N$. By the proof of proposition 5.1, we know that there exists a bifurcation branch $b \in B(L,f)$ such that $b_{c_1} \neq b_{c_2}$. So $b \notin \Delta_\gg$ and it is not lifted from $N$.

We give two examples where the previous result can be applied.

Example 7.2. Consider the networks in figure 1 and $f \in V_k(A)$ generic. The network $B$ is a lift that creates a new layer from $A$. Moreover, the network $C$ is a lift inside the second layer of $B$. 

P Soares
Nonlinearity 31 (2018) 5500
By proposition 7.2 (i), every bifurcation branch of \( f \) on \( B \) is lifted from \( A \). Again by proposition 7.2 (i), every bifurcation branch of \( f \) on \( C \) is lifted from \( B \). Thus every bifurcation branch of \( f \) on \( C \) is lifted from \( A \).

Example 7.3. Let \( N \) be the network on the left of figure 8, \( L \) the lift inside the first layer described on the right of figure 8 and \( f \in \mathcal{V}_k(L) \). Consider the balanced coloring \( \triangledown \triangleleft \) in \( L \) given by \( 2 \triangledown 3 \). By lemma 3.3, this is the unique balanced coloring in \( L \) such that \( L / \triangledown = N \), as \( L \) is backward connected. There exists a bifurcation branch \( b \in B(L,f) \) such that \( b_2 \neq b_3 \) and thus it is not lifted from \( N \). This agrees with the proposition 7.2 (ii).

In examples 5.1 and 7.1, we saw that proposition 7.2 (ii) is not valid if the lift is not backward connected. In the above example, the lifted network has more bifurcation branches but they are copies of some bifurcation branches on the smaller network. The bifurcation branches of the smaller network can be lifted in multiple ways because there is more than one balanced coloring on the lift network that corresponds to the quotient network.

Example 7.4. Let \( N \) be the feed-forward network on the left of figure 7, \( L \) the lift inside the first layer of \( N \) represented in the right of figure 7 and \( f \in \mathcal{V}_k(N) \) generic. In example 5.1, we saw that \( L \) has three balanced colorings \( \triangledown \triangledown_1, \triangledown \triangledown_2 \) and \( \triangledown \triangledown_3 \) such that \( L / \triangledown_1 = L / \triangledown_2 = L / \triangledown_3 = N \). Every bifurcation branch \( b \in B(L,f) \) is lifted from \( N \) using one of the previous colorings.

7.2. Lifting bifurcation problem on FFNs associated with the internal dynamics

In this section, we study the lifting bifurcation problem for feed-forward systems determined by a regular function that has a bifurcation condition associated to the internal dynamics. We start by the lifts that create new layers and lifts inside the first layer. These cases do not depend on the feed-forward system and every bifurcation branch is lifted if and only if the center subspace in the lifted network is equal to the center subspace on the smaller network.

Proposition 7.3. Let \( N \) be a feed-forward network, \( f \in \mathcal{V}_0(N) \) generic and \( L \) a feed-forward lift of \( N \).

(i) If \( L \) is a lift inside the first layer, then every bifurcation branch of \( f \) on \( L \) is lifted from \( N \).

(ii) If \( L \) is a lift that creates new layers, then there is a bifurcation branch of \( f \) on \( L \) which is not lifted from \( N \).
Proof. Let $N$ be a feed-forward network, $f \in \mathcal{V}_0(N)$ generic and $L$ a feed-forward lift of $N$. Suppose that $L$ is a lift inside the first layer. Then the center subspace of $J^L_1$ and $J^L_2$ have the same dimension. By corollary 7.1, every bifurcation branch of $f$ on $L$ is lifted from $N$.

Suppose that $L$ is a lift that creates new layers. By corollary 6.1, there exists a bifurcation branch $b \in \mathcal{B}(L, f)$ having square-root-order greater than any bifurcation branch of $f$ on $N$. Hence there is a bifurcation branch of $f$ on $L$ which is not lifted from $N$. \hfill \Box

We apply the previous result to the networks in figures 1 and 7.

Example 7.5. Let $N$ the feed-forward network on the left of figure 7 and $L$ the lift inside the first layer of $N$ given on the right of figure 7. Consider $f \in \mathcal{V}_0(N)$. Using proposition 7.3 (i), we know that every bifurcation branch of $f$ on $L$ is lifted from $N$. \hfill \Diamond

Example 7.6. Consider the networks $A$ and $B$ in figure 1 and $f \in \mathcal{V}_0(A)$ generic. The network $B$ is a lift that creates a new layer from $A$. Proposition 7.3 (ii) states that there exists a bifurcation branch of $f$ on $B$ not lifted from $A$. In fact, the feed-forward system $J^B$ has a bifurcation branch with square-root-order 1 but every bifurcation branch of $f$ on $A$ has square-root-order less or equal to 0, because $A$ has only two layers. \hfill \Diamond

There is one more special case that does not depend on the specific feed-forward system considered. For lifts inside a layer such that the next layer only has one cell, there is a bifurcation branch on the lift network not lifted from the original network.

Proposition 7.4. Let $N$ be a feed-forward network with layers $C_1, \ldots, C_m$, $f \in \mathcal{V}_0(N)$ generic and $L$ a feed-forward lift of $N$.

If $L$ is a lift inside $C_j$ for $1 < j < m$ and $|C_{j+1}| = 1$, then there is a bifurcation branch of $f$ on $L$ which is not lifted from $N$.

Proof. Let $N$ be a feed-forward network with layers $C_1, \ldots, C_m$, $f \in \mathcal{V}_0(N)$ generic and $L$ a feed-forward lift of $N$. Suppose that $L$ is a lift inside $C_j$, $1 < j < m$ and $C_{j+1} = \{d\}$. Denote by $C_j'$ the $j$-layer of $L$ and by $(\sigma^L_i)_{i=1}^l$ the representative functions of $L$. By remark 3.1, we assume that $L$ is the split of a cell $c \in C_j'$ into two cells $c_1, c_2 \in C_j$ and denote by $\gg$ the balanced coloring in $L$ given by $c_1 \gg c_2$. Since $|d|_{\gg} = d$ and $C_{j+1} = \{d\}$, $\gg$ is the unique balanced coloring such that $L/\gg \cong N$.

Using proposition 6.1, we construct a bifurcation branch $b \in \mathcal{B}(L, f)$ such that $b_{c_1} \neq b_{c_2}$. Let $A = \{i : \sigma^A_i(d) = c_2\}$, $\delta = \text{sign}(f_{\text{origin}} \sum_{i \in A} f_i)$ and

\[
p_a = -1, \quad s_a = 0, \quad a \in C_1 \cup \cdots \cup C_{j-1} \cup C_j \setminus \{c_2\},
\]

\[
p_{c_2} = 0, \quad s_{c_2} = -\frac{2f_{\text{origin}}}{f_0}, \quad p_d = 1, \quad s_d = -\text{sign} \left( \delta \sum_{i=1}^k f_i \right) \frac{2}{f_0} \left| f_{\text{origin}} \sum_{i \in A} f_i \right|^{2^{-1}},
\]

\[
p_a = 1, \quad s_a = -\text{sign} \left( \delta \sum_{i=1}^k f_i \right) \frac{2}{f_0} \left| f_{\text{origin}} \sum_{i \in A} f_i \right|^{1-2^{-(l-1)}} \left| f_{\text{origin}} \sum_{i \in A} f_i \right|^{2^{-l}},
\]

for $a \in C_{j+1}$ and $2 \leq l \leq m - j$. We have that $(\delta, (p_c)_c, (s_c)_c) \in \Omega(L, f)$. By proposition 6.1, there exists a bifurcation branch $b \in \mathcal{B}(L, f)$ such that $b_{c_1} \neq b_{c_2}$, since $p_{c_1} \neq p_{c_2}$. Thus $b \notin \Delta_{\text{origin}}$ and $b$ is not lifted from $N$. \hfill \Box
Example 7.7. Let \( L \) be the network in figure 6 and \( \triangledown \) the balanced coloring in \( L \) given by \( 2 \triangledown 3 \). The network \( L \) is a lift inside the second layer of \( L/\triangledown \) and \( L/\triangledown \) has only one cell in the third layer. Taking \( f \in \mathcal{V}_0(L) \), we know that there exists a bifurcation branch of \( f \) on \( L \) not lifted from \( L/\triangledown \). \( \diamondsuit \)

The last layer of a backward connected feed-forward network has only one cell. So we can apply proposition 7.4 for backward connected lifts inside the last but one layer.

We look next to a lift inside a layer such that the next layer has more than one cell. The example is similar to example 5.1 and it exemplifies how multiple balanced colorings can allow for the existence of new bifurcation branches which are copies of bifurcation branches on the quotient network. In this case the conclusion of proposition 7.4 does not hold.

Example 7.8. Consider the network \( C \) in figure 1 and the network \( L \) in figure 2. The network \( L \) is a lift inside the second layer of \( C \). Note that the third layer of \( L \) has three cells.

The lift network has three balanced colorings \( \triangledown_1, \triangledown_2 \) and \( \triangledown_3 \) such that \( C = L/\triangledown_1 = L/\triangledown_2 = L/\triangledown_3 \). The balanced colorings in \( L \) are defined by \( 2 \triangledown 1 \), \( 3 \triangledown 2 \) and \( 2 \triangledown 3 \).

Let \( f \in \mathcal{V}_0(C) \) generic and \( b \in \mathcal{B}(L,f) \). We know that \( b_2, b_3 \) and \( b_4 \) have square-root-order \(-1\) or \( 0\) and only two possible values. Thus \( b_2 = b_3, b_2 = b_4 \) or \( b_3 = b_4 \) and \( b \) is lifted from \( C \). \( \diamondsuit \)

We focus on backward connected lifts, since every backward connected lift has a unique corresponding balanced coloring, see lemma 3.3. Lifts that create new layers and lifts inside the first layer are covered by proposition 7.3. A backward connected feed-forward network has only one cell in the last layer. So a backward connected lift cannot be a lift inside the last layer. And proposition 7.4 includes backward connected lifts inside the last but one layer.

The following results consider lifts inside an intermediate layer. This result depends on the specific feed-forward system consider. The first result shows that there exists an open set of feed-forward systems in \( \mathcal{V}_0(N) \) such that every lift inside an intermediate layer has a bifurcation branch which is not lifted.

Proposition 7.5. Let \( N \) be a feed-forward network with layers \( C_1,\ldots,C_m \), \( f \in \mathcal{V}_0(N) \) generic and \( L \) a feed-forward lift of \( N \) such that \( L \) is backward connected and a lift inside a layer \( C_p \) where \( 1 < j < m - 1 \).

If \( f_i > 0 \) for every \( 1 \leq i \leq k \) (or \( f_i < 0 \) for every \( 1 \leq i \leq k \)), then there is a bifurcation branch of \( f \) on \( L \) which is not lifted from \( N \).

Proof. Let \( N \) be a feed-forward network with layers \( C_1,\ldots,C_m \), \( f \in \mathcal{V}_0(N) \) generic and \( L \) a feed-forward lift of \( N \) such that \( L \) is backward connected and a lift inside a layer \( C_p \) where \( 1 < j < m - 1 \). Denote by \( C_j \) the \( j \)-layer of \( L \) and by \( (\sigma^i_j)_{i=1}^4 \) the representative functions of \( L \). By remark 3.1, we assume that \( L \) is the split of a cell \( c \in C_j \) into two cells \( c_1, c_2 \in C_j \) and denote by \( \triangledown \) the balanced coloring in \( L \) given by \( c_1 \triangledown c_2 \). By lemma 3.3, \( \triangledown \) is the unique balanced coloring such that \( L/\triangledown = N \).

Assuming that \( f_j > 0 \) for every \( 1 \leq i \leq k \), we use proposition 6.1 to construct a bifurcation branch \( b \in \mathcal{B}(L,f) \) such that \( b \notin \Delta_\infty \). Define \( \delta = \text{sign}(f_0), p_a = -1 \) and \( s_a = 0 \), for \( a \in C_1 \cup \cdots \cup C_j \setminus \{c_1\} \), \( p_{c_1} = 0 \) and \( s_{c_1} = -2f_0/\delta \). We define the value of \( p \) and \( s \) by induction in the layers \( C_{j+1},\ldots,C_m \) in the following way: for \( a \in C_l \), \( j < l \leq m \), if \( p_{\sigma^i_l(a)} = \cdots = p_{\sigma^m_l(a)} = -1 \) define \( p_a = -1 \) and \( s_a = 0 \), otherwise define \( p_a = \max\{p_{\sigma^i_l(a)},\cdots,p_{\sigma^m_l(a)}\} + 1 \) and
\[ s_a = -\text{sign}(f_0f_0) \sqrt{-\frac{2\delta}{f_0} \sum_{i \in A(a)} f_i s_{\sigma_i(a)}}, \]

where \( A(a) = \{ i : p_{\sigma_i(a)} = p_a - 1 \} \).

We have that \((\delta, (p_a)_{\sigma_i}, (s_a)_a) \in \Omega(L, f)\) and \(p_{c_1} \neq p_{c_2}\). By proposition 6.1, there exists \(b \in B(L, f)\) such that \(b \notin \Delta_{\infty}\). Thus there is a bifurcation branch of \(f\) on \(L\) not lifted from \(N\).

The case \(f_i < 0\) for every \(1 \leq i \leq k\) is analogous. \(\Box\)

Example 7.8 shows that the previous result does not always hold if the lift is not backward connected.

**Example 7.9.** Let \(L\) be the feed-forward network of figure 3, \(\infty\) the balanced coloring in \(L\) given by \(2 \otimes 3\) and \(N\) the quotient network of \(L\) associated to \(\infty\). The network \(N\) is a feed-forward network and \(L\) is a lift inside the second layer of \(N\). And \(L\) is backward connected for the cell 10. Take \(f \in V_0(N)\) generic such that \(f_1 > 0, f_2 > 0\) and \(f_3 > 0\).

Proposition 7.5 states that there exists a bifurcation branch of \(f\) on \(L\) not lifted from \(N\). In example 7.13, we will see that this is not true for every generic \(f \in V_0(N)\). In fact, there exists an open set of functions in \(V_0(N)\) such that every bifurcation branch of \(f\) on \(L\) is lifted from \(N\). \(\checkmark\)

Finally, we give sufficient conditions on a feed-forward lift, with a bigger central subspace, and on a feed-forward system for every bifurcation branch on the lift be lifted from the quotient network. First, we look to lifts inside the second layer. By remark 3.1, we assume that the lift inside the second layer is a split of two cells.

**Proposition 7.6.** Let \(N\) be a feed-forward network with layers \(C_1, \ldots, C_m\), \(f \in V_0(N)\) generic and \(L\) a feed-forward lift of \(N\). Denote by \(C_2\) the second layer of \(L\) and by \((\sigma_i^L)_{i=1}^k\) the representative function of \(L\). Assume that \(L\) is the split of \(c \in C_2^c\) into \(c_1, c_2 \in C_2\) (and a lift inside \(C_2\)).

If for every \(I \subseteq C_2^c \setminus \{ c_1, c_2 \} \) there exist \(d', d'' \in C_3\) such that

\[
(w_1^d + w_2^d)(w_1^{d'} + w_2^{d'}) < 0 \land (w_1^d + w_2^d)(w_1^{d''} + w_2^{d''}) < 0,
\]

where \(w_1^d = \sum_{\sigma_i^L(d) = c_1} f_i\), \(w_2^d = \sum_{\sigma_i^L(d) = c_2} f_i\), and \(w_1^{d'} = \sum_{\sigma_i^L(d) = c_1} f_i\) and \(w_2^{d'} = \sum_{\sigma_i^L(d) = c_2} f_i\), then every bifurcation branch of \(f\) on \(L\) is lifted from \(N\).

**Proof.** Let \(N\) be a feed-forward network with layers \(C_1, \ldots, C_m\), \(f \in V_0(N)\) generic and \(L\) a feed-forward lift of \(N\). Denote by \(C_2\) the second layer of \(L\) and by \((\sigma_i^L)_{i=1}^k\) the representative function of \(L\). Assume that \(L\) is the split of \(c \in C_2\) into \(c_1, c_2 \in C_2\).

We prove the result by contraposition. Suppose that there exists \(b \in B(L, f)\) not lifted from \(N\). Then \(b_{c_1} \neq b_{c_2}\). Let \((\delta, (p_a)_{\sigma_i}, (s_a)_a) = \Theta(b) \in \Omega(L, f)\). For every \(a \in C_2^r\) we have that

\[ p_a \in \{-1, 0\}, \quad s_a = -(p_a + 1) \frac{2f_0\lambda}{f_0}. \]

Let \(I = \{ a \in C_2^c \setminus \{ c_1, c_2 \} : p_a = 0 \} \subseteq C_2^c \setminus \{ c_1, c_2 \}\). By 7.6, for \(d \in C_3\) such that \(p_d = 1\) we have that
\[ s_d = \pm \frac{2}{f_{00}} \sqrt{\delta_{00} \sum_{i \in A(d)} f_i}, \]

where \( A(d) = \{ i : p_i^f(d) = 0 \} \). Then \( \left( \sum_{i \in A(d')} f_i \right) \left( \sum_{i \in A(d'')} f_i \right) > 0 \), if \( p_{d'} = p_{d''} = 1 \), \( \left( \sum_{i \in A(d')} f_i \right) \left( \sum_{i \in A(d'')} f_i \right) = 0 \), if \( p_{d'} < 1 \) or \( p_{d''} < 1 \), for every \( d', d'' \in C_3 \). Thus

\[
\left( \sum_{i \in A(d')} f_i \right) \left( \sum_{i \in A(d'')} f_i \right) \geq 0,
\]

for every \( d', d'' \in C_2 \). Since \( b_{c_1} \neq b_{c_2}, -1 \leq p_{c_1} \neq p_{c_2} \leq 0 \). If \( p_{c_1} = 0 \) and \( p_{c_2} = -1 \), then \( \sum_{i \in A(d')} f_i = w_1^d + w_1^d \). If \( p_{c_1} = -1 \) and \( p_{c_2} = 0 \), then \( \sum_{i \in A(d')} f_i = w_1^d + w_1^d \). So

\[
(w_1^d + w_1^d) (w_1^d + w_1^d) \geq 0 \quad \text{or} \quad (w_1^d + w_1^d) (w_1^d + w_1^d) \geq 0,
\]

for every \( d', d'' \in C_3 \). By contraposition, we obtain the result.

**Example 7.10.** Consider the networks \( B \) and \( C \) in figure 1. Consider \( f \in V_0(C) \) such that \( f_{12} < 0 \). The lift \( C \) of \( B \) and the function \( f \) satisfy the conditions of proposition 7.6. As we saw in example 6.2, and accordingly with proposition 7.6, every bifurcation branch of \( f \) on \( C \) is lifted from \( B \).

Last, we consider lifts inside the other intermediate layers. We give sufficient conditions on a lift network and feed-forward systems such that there is no new bifurcation branch. We will assume that the lift is given by the split of a cell into two cells which are the unique inputs cells of two other cells in the next layer.

**Proposition 7.7.** Let \( N \) be a feed-forward network with layers \( C_1, \ldots, C_m \) \( f \in V_0(N) \) generic, \( L \) a feed-forward lift of \( N \) and \( 2 < j < m - 1 \). Denote by \( C_j \) the \( j \)-layer of \( L \) and by \( (\sigma^j_i)_{i=1}^k \) the representative function of \( L \). Assume that \( L \) is the split of \( c \in C_j \) into \( c_1, c_2 \in C_{j+1} \) (and a lift inside \( C_i \)).

If there exist \( d', d'' \in C_{j+1} \) such that \( \sigma^j_i(d'), \sigma^j_i(d'') \in \{ c_1, c_2 \} \), for every \( 1 \leq i \leq k \), and

\[
w_1^d w_1^d < 0 \quad \text{and} \quad w_2^d w_2^d < 0 \quad \text{and} \quad w_1^d w_1^d + w_2^d w_2^d < w_1^d w_1^d + w_1^d w_1^d + w_2^d w_2^d,
\]

where \( w_i^d = \sum_{i \in A(d) = c_i} f_i \) and \( w_2^d = \sum_{i \in A(d) = c_2} f_i \), then every bifurcation branch \( b \) of \( f \) on \( L \) is lifted from \( N \).

**Proof.** Let \( N \) be a feed-forward network with layers \( C_1, \ldots, C_m \) \( f \in V_0(N) \) generic, \( L \) a feed-forward lift of \( N \) and \( 2 < j < m - 1 \). Denote by \( C_j \) the \( j \)-layer of \( L \) and by \( (\sigma^j_i)_{i=1}^k \) the representative function of \( L \). Assume that \( L \) is the split of \( c \in C_j \) into \( c_1, c_2 \in C_{j+1} \). Suppose that there exist \( d', d'' \in C_{j+1} \) such that \( \sigma^j_i(d'), \sigma^j_i(d'') \in \{ c_1, c_2 \} \), for \( 1 \leq i \leq k \). Let \( b \in B(L, f) \) be a bifurcation branch and \( (\delta, (p_{b_{j+1}}), (s_{b_{j+1}})) = \theta(b) \in \Omega(L, f) \) the correspondent symbol.

We know that \( b \in B(L, f) \) is not lifted from \( N \) if and only if \( b_{c_1} \neq b_{c_2} \). We assume that \( b_{c_1} \neq b_{c_2} \) and obtain a contradiction with the given conditions.

Suppose that \( b_{c_1} \neq b_{c_2} \). Then \( p_{c_1} = 0 \) and \( p_{c_2} = -1 \) or \( p_{c_1} = -1 \) and \( p_{c_2} = 0 \) or \( p_{c_1} = 0 \) and \( p_{c_2} = -1 \) or \( p_{c_1} = -1 \) and \( p_{c_2} = 0 \). We have that \( w_1^d w_1^d > 0 \), if \( p_{c_1} = 0 \) and \( p_{c_2} = -1 \). And \( w_2^d w_2^d > 0 \), if \( p_{c_1} = -1 \) and \( p_{c_2} = 0 \). If \( p_{c_1} = p_{c_2} > 0 \) and \( s_{c_1} = -s_{c_2} \), then \( p_{d'} = p_{d'} = p_{d'} + 1 \) and
Thus \((w_1^2 - w_2^2)(w_1^3 - w_2^3) > 0\). Generically, \((w_1^2 - w_2^2)(w_1^3 - w_2^3) \neq 0\).

Therefore

\[
\begin{align*}
    w_1^2 w_2^3 < 0 & \wedge w_2^3 w_2^2 < 0 \wedge w_1^2 w_1^3 w_2^2 w_2^2 < w_1^3 w_2^3 + w_2^3 w_2^3
\end{align*}
\]

implies that \(b_{c_1} = b_{c_2}\) and that \(b \in B(L, f)\) is lifted from \(N\).

Example 7.11. Let \(L\) be the network in figure 9 and \(\triangleright\triangleright\) the balanced coloring in \(L\) given by \(4 \triangleright\triangleright 5\). Consider a function \(f \in V_0(L)\) such that \(f^L\) has a bifurcation condition associated to the internal dynamics. Denote by \(N\) the quotient network \(L/\triangleright\triangleright\).

The network \(L\) is a split of a cell in \(N\) into the cells 4 and 5 and the cells 6 and 7 only receive inputs from those cells. Denote by \(w^c_d\) the sum of the linear inputs from \(c\) to \(d\), where \(c = 4, 5\) and \(d = 6, 7\). We have that

\[
    w_4^c = f_1, \quad w_5^c = f_2, \quad w_6^d = w_7^d = f_3.
\]

By proposition 7.7, we know that every bifurcation branch of \(f\) on \(L\) is lifted from \(N\) if

\[
    f_1(f_1 + f_1) < 0 \wedge f_2(f_2 + f_3) < 0 \wedge (f_1 - f_3)^2 < f_3^2.
\]

In order to see that the previous inequalities are satisfied by some function \(f\), take \(f_1 = 2, f_2 = 1\) and \(f_3 = -3\).

The previous result does not necessary hold if the splitted cells only target one cell or if the splitted cells are not the unique inputs of two cells in the next layer. We present next an example where the splitted cells are not the unique inputs and the conclusion does not hold.

Example 7.12. Let \(L\) be the feed-forward network of figure 10, \(\triangleright\triangleright\) the balanced coloring in \(L\) given by \(5 \triangleright\triangleright 6\) and \(N\) the quotient network of \(L\) associated to \(\triangleright\triangleright\). The network \(N\) is a feed-forward network and \(L\) is a split of one cell inside the third layer. Let \(f \in V_0(N)\) generic. We show that there exists a bifurcation branch of \(f\) on \(L\) not lifted from \(N\).

Let 

\[
\delta = \text{sign}(f_0, f_3), \quad p_1 = p_3 = p_6 = -1, \quad p_2 = p_5 = 0, \quad p_4 = 1, \quad p_7 = p_8 = 2, \quad p_9 = 3, \quad s_1 = s_3 = s_6 = 0,
\]

\[
\begin{align*}
    s_2 &= s_4 = -\text{sign}(f_1) \delta \frac{2|f_{0,3}|}{f_{0,0}}, \quad s_4 = -\text{sign}(f_0, f_3) \frac{2\sqrt{|f_{0,3}|/f_{0,0}}}{\sqrt{|f_1| + |f_3|}}.
\end{align*}
\]

Figure 9. Feed-forward network with five layers.
Figure 10. A network \( L \) with a quotient network \( N \) obtained by the balanced coloring \( \triangleright \triangleright \) given by \( 5 \triangleright \triangleright 6 \). If \( f \in \mathcal{V}_0(N) \), then there exists a bifurcation branch of \( f \) on \( L \) not lifted from \( N \).

\[
\begin{align*}
\sin(f_{0\lambda}) & = \frac{2 \sqrt{|f_{0\lambda}| \sqrt{|f_1| \sqrt{|f_1 + f_2|}}}}{f_{00}} \\
\text{and} \\
\sin(f_{0\lambda}) & = \frac{2 \sqrt{|f_{0\lambda}| \sqrt{|f_1 + f_2| \sqrt{|f_1 + f_2|}}}}{f_{00}}.
\end{align*}
\]

Note that \((\delta, (p_\alpha)_\alpha, (s_\alpha)_\alpha) \in \Omega(L, f)\). Let \( b \in \mathcal{B}(L, f) \) be the bifurcation branch associated to \((\delta, (p_\alpha)_\alpha, (s_\alpha)_\alpha)\). Since \( L \) is backward connected, a bifurcation branch \( b \) on \( L \) can be lifted from \( N \) if and only if \( b_5 = b_6 \). However \( p_5 \neq p_6 \) and \( b \) is not lifted from \( N \).

In the next example, we see that the lifting bifurcation problem goes beyond the one layer to the next layer reasoning. As the example shows, the network structure can further restrict the possible bifurcation branches. In this case, we need to look for the next two layers to understand the possible bifurcation branches.

**Example 7.13.** Let \( L \) be the feed-forward network of figure 3, \( \triangleright \triangleright \) the balanced coloring in \( L \) given by the class \( \{2, 3\} \) and \( N \) the quotient network of \( L \) associated to \( \triangleright \triangleright \). The network \( N \) is a feed-forward network and \( L \) is a lift inside the second layer.

Let \( f \in \mathcal{V}_0(L) \) be generic such that \( f_2(f_2 + f_3) < 0 \). As we show next, there is no bifurcation branch \( b \in \mathcal{B}(L, f) \) such that \( b_2 \neq b_3 \) and every bifurcation branch of \( f \) on \( L \) is lifted from \( N \).

Let \( b \in \mathcal{B}(L, f) \). Suppose by contradiction that \( b_2 \neq b_3 \). We know that \( b_2 \) and \( b_3 \) are the trivial branch or the branch \( b^0_\lambda \) with square-root-order 0 and defined in (2). There are two options: \( b_2 = b^0_\lambda \) and \( b_3 = 0 \); or \( b_2 = 0 \) and \( b_3 = b^0_\lambda \). If \( b_2 = b^0_\lambda \) and \( b_3 = 0 \), we have that \( b_5 \) and \( b_6 \) are defined on different sides of \( \lambda = 0 \), like we saw in proposition 7.6. Thus there is no bifurcation branch of \( f \) on \( L \) such that \( b_2 = b^0_\lambda \) and \( b_3 = 0 \). If \( b_2 = 0 \) and \( b_3 = b^0_\lambda \), then \( b_4 \) has square-root-order \( -1 \) or 0 and \( b_5 \) has square-root-order 1. Now, we look to the next layer, in particular to the cells 7 and 8. We know that \( b_7 \) and \( b_8 \) have square-root-order 2 and that the side of \( \lambda = 0 \) where they are defined depend on \( b_5 \). Since \( f_3(f_3 + f_3) < 0 \), the cell 7 receives inputs of type 2 and of type 3 from the cell 5 and the cell 8 receives an input of type 3 from cell 5, we know that \( b_7 \) and \( b_8 \) are defined on different sides of \( \lambda = 0 \). So there is no bifurcation branch of \( f \) on \( L \) such that \( b_2 \neq b_3 \).\( \diamond \)
8. Discussion

The main goal of this work is to address the lifting bifurcation problem in the context of feed-forward systems. We identify two important types of lifts in feed-forward networks, lifts that create new layers and lifts inside a layer. We show that every backward connected lift is given by the composition of basic lifts. When studying codimension-one steady-state bifurcations on feed-forward systems, we identify two possible bifurcations conditions, that we call valency and internal dynamics. For both conditions, we give a complete description of the bifurcation branches. For bifurcations associated to the internal dynamics, we introduce a new symbolic set which describes all bifurcation branches. The symbols represent the growth rate and slope of the bifurcation branches and the symbolic set is given by the symbols that satisfy some given rules. Finally, we study the lifting bifurcation problem for feed-forward systems. For a fixed bifurcation condition, when the lifted network has a center subspace bigger than the center subspace associated to the smaller network, we expect the existence of new synchrony-breaking bifurcation branches. We prove this for different cases, including lifts that create new layers. From the lifting bifurcation problem point of view, it can be important when no new bifurcation branches appear and so the study of the quotient network is sufficient to understand the bifurcations on the lift network. For some lifts inside an intermediate layer, we prove that there is a class of feed-forward systems without new bifurcation branches. This depends on the correct balance between the signs of each of the input’s linearization.

We stress that some of the restrictions that we impose at the class of feed-forward systems in this work can, in fact, be easily removed. If the dimension of each cell phase space is bigger than 1, then the steady-state bifurcation analysis is generically the same, using the Lyapunov–Schmidt reduction. If every layer has a different type of cells, then we only have lifts inside a layer and the steady-state bifurcation study is similar to the case of feed-forward systems with a bifurcation condition associated with the valency. Also, if we do not impose that the origin is an equilibrium for every value of the parameter, then the steady-state bifurcation with a condition associated to the internal dynamics is the same. This follows from the fact that the full-synchronized subspace does not support a bifurcation and a full-synchronized equilibrium will still exist for every value of the parameter which can be assumed to be the origin. For the steady-state bifurcations with a condition associated to the valency, we will have a fold bifurcation instead of a transcritical bifurcation. However, the lifting bifurcation problem with a condition associated to the valency will be essentially the same, because we still have two options for each cell in the first layer.

Example 7.13 shows that the study of the lifting bifurcation problem for feed-forward networks is not limited to the next layer reasoning. Examples that depend on the next two (or more) layers are not covered by our results on the lifting bifurcation problem. It would be interesting to have results that include those type of examples.

We could consider the lifting bifurcation problem for feed-forward networks with a fixed number of layers. And we can ask if there exists a minimal feed-forward network such that all bifurcation branches of any feed-forward lift, with the same number of layers, are lifted from that minimal network.

We can also ask the same questions about the lifts, the bifurcation branches and the lifting bifurcation problem for feed-forward networks with intra-layer connections.

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