Generalized fractional integral operators on Campanato spaces and their bi-preduals

Dedicated to Professor Toshio Horiuchi on his 65th birthday

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Abstract

In this paper we prove the boundedness of the generalized fractional integral operator $I_\rho$ on generalized Campanato spaces with variable growth condition, which is a generalization and improvement of previous results, and then, we establish the boundedness of $I_\rho$ on their bi-preduals. We also prove the boundedness of $I_\rho$ on their preduals by the duality.

1. Introduction

Let $I_\alpha$ be the fractional integral operator of order $\alpha > 0$, that is,

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy, \quad x \in \mathbb{R}^n.$$  

It is well known as the Hardy-Littlewood-Sobolev theorem that $I_\alpha$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if $\alpha \in (0,n)$, $p,q \in (1,\infty)$ and $-n/p + \alpha = -n/q$. In this paper we consider the generalized fractional integral operator $I_\rho$. For a function $\rho : (0,\infty) \to (0,\infty)$, the operator $I_\rho$ is defined by

$$I_\rho f(x) = \int_{\mathbb{R}^n} \rho(|x-y|/|x-y|^\alpha) f(y) \, dy, \quad x \in \mathbb{R}^n,$$  

where we always assume that

$$\int_0^1 \frac{\rho(t)}{t} \, dt < \infty.$$  

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If \( \rho(r) = r^\alpha \), \( 0 < \alpha < n \), then \( I_{\rho} \) is the usual fractional integral operator \( I_{\alpha} \). The condition (1.2) is needed for the integral in (1.1) to converge for bounded functions \( f \) with compact support. In this paper we also assume that there exist positive constants \( C, K_1 \) and \( K_2 \) with \( K_1 < K_2 \) such that, for
\[
\sup_{r \leq t \leq 2r} \rho(t) \leq C \int_{K_1 r}^{K_2 r} \frac{\rho(t)}{t} \, dt. \tag{1.3}
\]
The condition (1.3) was considered in [29]. If \( \rho \) satisfies the doubling condition (2.1) below, then \( \rho \) satisfies (1.3). Let \( \rho(r) = \min(r^\alpha, e^{-r/2}) \) for \( 0 < \alpha < n \), which controls the Bessel potential. Then \( \rho \) also satisfies (1.3).

The operator \( I_{\rho} \) was introduced in [15, 16, 17] (2000, 2001) to extend the Hardy-Littlewood-Sobolev theorem to Orlicz spaces. For example, let
\[
\rho(r) = \begin{cases} 
1/(\log(1/r))^{\alpha+1} & \text{for small } r, \\
(\log r)^{\alpha-1} & \text{for large } r, 
\end{cases} \tag{1.4}
\]
p, \( q \in (0, \infty) \), \( -1/p + \alpha = -1/q \). Then \( I_{\rho} \) is bounded from \( \exp L^p(\mathbb{R}^n) \) to \( \exp L^q(\mathbb{R}^n) \), where \( \exp L^p(\mathbb{R}^n) \) is the Orlicz space \( L^\Phi(\mathbb{R}^n) \) with
\[
\Phi(t) = \begin{cases} 
1/\exp(1/t^p) & \text{for small } t, \\
\exp(t^p) & \text{for large } t. 
\end{cases} \tag{1.5}
\]

Since then, the boundedness of the operator \( I_{\rho} \) was established on other function spaces also, for example, Morrey and Campanato spaces in [4, 5, 6, 10, 18, 26, 23], Orlicz-Morrey spaces in [12, 19, 21, 32], Hardy and Orlicz-Hardy spaces in [2, 24], etc. In this paper we first prove the boundedness of the operator \( I_{\rho} \) on generalized Campanato spaces with variable growth condition, which is a generalization and improvement of previous results, and then, we establish the boundedness of \( I_{\rho} \) on their bi-preduals.

It is known that the dual of \( H^1(\mathbb{R}^n) \) is \( \text{BMO}(\mathbb{R}^n) \), that is,
\[
(\text{H}^1(\mathbb{R}^n))^* = \text{BMO}(\mathbb{R}^n). \tag{1.6}
\]
It is also known that
\[
\left( C^\infty_{\text{comp}}(\mathbb{R}^n)^* \right)^{\text{BMO}(\mathbb{R}^n)} = \text{H}^1(\mathbb{R}^n), \tag{1.7}
\]
where \( C^\infty_{\text{comp}}(\mathbb{R}^n) \) is the set of all infinitely differentiable functions with compact support and \( C^\infty_{\text{comp}}(\mathbb{R}^n)^{\text{BMO}(\mathbb{R}^n)} \) is the closure of \( C^\infty_{\text{comp}}(\mathbb{R}^n) \) with respect to \( \text{BMO}(\mathbb{R}^n) \). The space \( C^\infty_{\text{comp}}(\mathbb{R}^n)^{\text{BMO}(\mathbb{R}^n)} \) is often referred to as \( \text{CMO}(\mathbb{R}^n) \) or \( \text{VMO}(\mathbb{R}^n) \). For (1.6) and (1.7), see [7, 8] and [3, 9], respectively.

These dualities were extended to generalized Campanato spaces \( L_{(\phi, \varphi)}(\mathbb{R}^n) \) and atomic Hardy spaces \( H^{(\phi, \psi)}(\mathbb{R}^n) \) with variable growth function \( \phi : \mathbb{R}^n \times (0, \infty) \to (0, \infty) \), which were introduced by [13, 27] and [22], respectively. More precisely, the second author [22] extended the duality (1.6) to
\[
(\text{H}^{(\phi, \infty)}(\mathbb{R}^n))^* = L_{1, \varphi}(\mathbb{R}^n). \tag{1.8}
\]
Recently, the first author [34] extended the duality (1.7) to
\[
\left( C^\infty_{\text{comp}}(\mathbb{R}^n) \right)^*_{L^{1,\alpha}(\mathbb{R}^n)} = H^{[\phi,\infty]}(\mathbb{R}^n),
\]  
(1.9)
where \( C^\infty_{\text{comp}}(\mathbb{R}^n) \) is the closure of \( C^\infty_{\text{comp}}(\mathbb{R}^n) \) with respect to \( L_{1,\alpha}(\mathbb{R}^n) \). If \( \phi \equiv 1 \), then \( L_{1,\alpha}(\mathbb{R}^n) = \text{BMO}(\mathbb{R}^n) \) and \( H^{[\phi,\infty]}(\mathbb{R}^n) = H^1(\mathbb{R}^n) \). If \( \phi(x, r) = r^\alpha \), \( 0 < \alpha < 1 \), then \( L_{1,\alpha}(\mathbb{R}^n) = \text{Lip}_\alpha(\mathbb{R}^n) \). The dualities (1.8) and (1.9) also cover the case \( \phi(x, r) = r^n(x) \).

The boundedness of the operators \( I_\omega \) and \( I_\rho \) on \( L_{1,\alpha}(\mathbb{R}^n) \) was studied in [5, 16, 18, 23]. In this paper we first generalize and improve the previous results, and then, we establish the boundedness of \( I_\rho \) on \( C^\infty_{\text{comp}}(\mathbb{R}^n) \) whose bidual is \( L_{1,\alpha}(\mathbb{R}^n) \). We also prove the boundedness of \( I_\rho \) on \( H^{[\phi,\infty]}(\mathbb{R}^n) \) by the duality.

The organization of this paper is as follows: In the next section we state definitions of \( L_{p,\phi}(\mathbb{R}^n) \) and \( H^{[\phi,\infty]}(\mathbb{R}^n) \) with variable growth function \( \phi : \mathbb{R}^n \times (0, \infty) \to (0, \infty) \). We also state the results on the duality. Then we give the main results in Section 3 and prove them in Section 4.

At the end of this section, we make some conventions. Throughout this paper, we always use \( C \) to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as \( C_p \), are dependent on the subscripts. If \( f \leq C g \), we then write \( f \lesssim g \) or \( g \gtrsim f \); and if \( f \lesssim g \lesssim f \), we then write \( f \sim g \).

2. Definitions

We denote by \( B(a, r) \) the open ball centered at \( a \in \mathbb{R}^n \) and of radius \( r \). For a function \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) and a ball \( B \), let
\[
f_B = \int_B f = \int_B f(y) \, dy = \frac{1}{|B|} \int_B f(y) \, dy,
\]
where \( |B| \) is the Lebesgue measure of the ball \( B \).

First we recall the definition of generalized Campanato spaces \( L_{p,\phi}(\mathbb{R}^n) \) for \( p \in [1, \infty) \) and variable growth function \( \phi : \mathbb{R}^n \times (0, \infty) \to (0, \infty) \). For a ball \( B = B(x, r) \) we write \( \phi(B) = \phi(x, r) \).

**Definition 2.1.** For \( p \in [1, \infty) \) and \( \phi : \mathbb{R}^n \times (0, \infty) \to (0, \infty) \), let \( L_{p,\phi}(\mathbb{R}^n) \) be the set of all functions \( f \) such that the following functional is finite:
\[
\|f\|_{L_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left( \int_B |f(y) - f_B|^p \, dy \right)^{1/p},
\]
where the supremum is taken over all balls \( B \) in \( \mathbb{R}^n \).

Then \( \|f\|_{L_{p,\phi}} \) is a norm modulo constant functions and thereby \( L_{p,\phi}(\mathbb{R}^n) \) is a Banach space. Generalized Campanato spaces \( L_{p,\phi}(\mathbb{R}^n) \) with variable growth condition were introduced in [27] to characterize pointwise multipliers on \( \text{BMO}(\mathbb{R}^n) \) and studied
in [13, 20, 23], etc. Moreover, it has been proved that $L_{p,\phi}(\mathbb{R}^n)$ is the dual space of the Hardy space $H^{p,\ominus}(\mathbb{R}^n)$ with variable exponent in [25]. That is, $H^{p,\ominus}(\mathbb{R}^n)$ is another predual of $L_{p,\phi}(\mathbb{R}^n)$.

We say that a function $\theta : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ satisfies the doubling condition if there exists a positive constant $C$ such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0, \infty)$,

$$\frac{1}{C} \leq \frac{\theta(x, r)}{\theta(x, s)} \leq C, \quad \text{if } \frac{1}{2} \leq \frac{r}{s} \leq 2.$$  

We say that $\theta$ is almost increasing (resp. almost decreasing) if there exists a positive constant $C$ such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0, \infty)$,

$$\theta(x, r) \leq C \theta(x, s) \quad \text{(resp. } \theta(x, s) \leq C \theta(x, r)\text{)}, \quad \text{if } r < s.$$  

We also consider the following nearness condition; there exists a positive constant $C$ such that, for all $x, y \in \mathbb{R}^n$ and $r \in (0, \infty)$,

$$\frac{1}{C} \leq \frac{\theta(x, r)}{\theta(y, r)} \leq C, \quad \text{if } |x - y| \leq r.$$  

For two functions $\theta, \kappa : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$, we write $\theta \sim \kappa$ if there exists a positive constant $C$ such that, for all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$,

$$\frac{1}{C} \leq \frac{\theta(x, r)}{\kappa(x, r)} \leq C.$$  

Let $1 \leq p < \infty$ and $\phi, \tilde{\phi} : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$. If $\phi \sim \tilde{\phi}$, then $L_{p,\phi}(\mathbb{R}^n) = L_{p,\tilde{\phi}}(\mathbb{R}^n)$ with equivalent norms.

In this paper we consider the following class of $\phi$:

**Definition 2.2.**

(i) Let $\mathcal{G}$ be the set of all functions $\phi : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ such that $r \mapsto \phi(x, r)r^n$ is almost increasing and that $r \mapsto \phi(x, r)/r$ is almost decreasing. That is, there exists a positive constant $C$ such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0, \infty)$,

$$\phi(x, r)r^n \leq C \phi(x, s)s^n, \quad C \phi(x, r)/r \geq \phi(x, s)/s, \quad \text{if } r < s.$$  

(ii) Let $\mathcal{G}^{\text{inc}}$ be the set of all functions $\phi \in \mathcal{G}$ such that $\phi$ is almost increasing. That is, there exists a positive constant $C$ such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0, \infty)$,

$$\phi(x, r) \leq C \phi(x, s), \quad C \phi(x, r)/r \geq \phi(x, s)/s, \quad \text{if } r < s.$$  

If $\phi \in \mathcal{G}$, then $\phi$ satisfies the doubling condition (2.1).

**Remark 2.1.** It is known that, if $\phi \in \mathcal{G}^{\text{inc}}$ and $\phi$ satisfies (2.3), then $L_{p,\phi}(\mathbb{R}^n) = L_{1,\phi}(\mathbb{R}^n)$ with equivalent norms for each $p \in [1, \infty)$, see [22, Theorem 3.1]. In particular, for each $p \in [1, \infty)$, $L_{p,\phi}(\mathbb{R}^n) = \text{BMO}(\mathbb{R}^n)$ if $\phi \equiv 1$, $L_{p,\phi}(\mathbb{R}^n) = \text{Lip}_a(\mathbb{R}^n)$ if $\phi(x, r) = r^a, \; 0 < a \leq 1$, and $L_{p,\phi}(\mathbb{R}^n)$ coincides with $L_{p,\lambda}(\mathbb{R}^n)$ modulo constant functions if $\phi(x, r) = r^\lambda, -n \leq \lambda < 0$. For the relation between generalized Campanato spaces $L_{p,\phi}(\mathbb{R}^n)$, generalized Morrey spaces $L_{p,\phi}(\mathbb{R}^n)$ and Hölder (Lipschitz) spaces $\Lambda_\phi(\mathbb{R}^n)$ with variable growth condition, see [20, Theorem 2.4].
We also consider the following condition: There exists a positive constant $C$ such that, for all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$,

$$
\int_r^\infty \frac{\phi(x,t)}{t^2} \, dt \leq C \frac{\phi(x,r)}{r},
$$

(2.5)

**Remark 2.2.** Let $\phi$ be almost increasing or satisfy the doubling condition. If $\phi$ satisfies (2.5), then $t \mapsto \phi(x,t)/t$ is almost decreasing and $\phi(x,t)/t \to 0$ as $t \to \infty$, see [11, Lemma 6]. Indeed, we have

$$
\frac{\phi(x,r)}{r} \lesssim \int_r^{2r} \frac{\phi(x,t)}{t^2} \, dt \leq \int_r^\infty \frac{\phi(x,t)}{t^2} \, dt \lesssim \frac{\phi(x,r)}{r}.
$$

The condition (2.5) used by [14, 23, 28], etc.

Next we state the definition of the atomic Hardy space $H^{\phi,q}(\mathbb{R}^n)$.

**Definition 2.3 ([\phi,q]-atom).** Let $\phi : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ and $1 < q \leq \infty$. A function $a$ on $\mathbb{R}^n$ is called a $[\phi,q]$-atom if there exists a ball $B$ such that

1. $\text{supp } a \subset B$,
2. $\|a\|_q \leq \frac{1}{|B|^{1/q'} \phi(B)}$,
3. $\int_{\mathbb{R}^n} a(x) \, dx = 0$,

where $\|a\|_q$ is the $L^q$ norm of $a$ and $1/q + 1/q' = 1$. We denote by $A[\phi,q]$ the set of all $[\phi,q]$-atoms.

If $a$ is a $[\phi,q]$-atom and a ball $B$ satisfies (i)–(iii), then

$$
\left| \int_{\mathbb{R}^n} a(x)g(x) \, dx \right| = \left| \int_B a(x)(g(x) - g_B) \, dx \right|
\leq \|a\|_q \left( \int_B |g(x) - g_B|^q' \, dx \right)^{1/q'}
\leq \frac{1}{\phi(B)} \left( \frac{1}{|B|} \int_B |g(x) - g_B|^q' \, dx \right)^{1/q'}
\leq \|g\|_{L_{q',\phi}}.
$$

That is, the mapping $g \mapsto \int a \, dg$ is a bounded linear functional on $L_{q',\phi}(\mathbb{R}^n)$ with norm not exceeding 1.

**Definition 2.4 ($H^{\phi,q}(\mathbb{R}^n)$).** Let $\phi : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$, $1 < q \leq \infty$ and $1/q + 1/q' = 1$. Assume that $L_{q',\phi}(\mathbb{R}^n) \neq \{0\}$. We define the space $H^{\phi,q}(\mathbb{R}^n) \subset (L_{q',\phi}(\mathbb{R}^n))^*$ as follows:
\[ f \in H^{[\phi, q]}(\mathbb{R}^n) \text{ if and only if there exist sequences } \{a_j\} \subset A[\phi, q] \text{ and positive numbers } \{\lambda_j\} \text{ such that } \]
\[ f = \sum_j \lambda_j a_j \in \mathcal{L}_{q', \phi}(\mathbb{R}^n)^* \text{ and } \sum_j \lambda_j < \infty. \quad (2.6) \]

In general, the expression (2.6) is not unique. We define
\[ k_f H^{[\phi, q]} = \inf \left\{ \sum_j \lambda_j : X_j \lambda_j a_j \in (L^{q'}, \phi(\mathbb{R}^n))^* \text{ and } X_j \lambda_j < 1 \right\}. \]

where the infimum is taken over all expressions as in (2.6). Then \( k_f H^{[\phi, q]} \) is a norm and \( H^{[\phi, q]}(\mathbb{R}^n) \) is a Banach space. For sufficient conditions of \( C^\infty_{\text{comp}}(\mathbb{R}^n) \subset L_{p, \phi}(\mathbb{R}^n) \), see Proposition 4.6.

**Theorem 2.1** ([22]). Let \( \phi \) be in \( G^{\infty} \) and satisfy (2.3). Then
\[ \mathcal{L}_{q', \phi}(\mathbb{R}^n) = \mathcal{L}_{1, \phi}(\mathbb{R}^n), \quad \text{for } 1 \leq q' < \infty, \]
\[ H^{[\phi, q]}(\mathbb{R}^n) = H^{[\phi, \infty]}(\mathbb{R}^n), \quad \text{for } 1 < q \leq \infty, \]
with equivalent norms, respectively.

**Definition 2.5.** Denote by \( H^{[\phi, q]}_0(\mathbb{R}^n) \) the space of all finite linear combinations of \([\phi, q]\)-atoms and fix \( b \in \mathcal{L}_{q', \phi}(\mathbb{R}^n) \). We define a linear functional
\[ L_b(g) = \int_{\mathbb{R}^n} g(x)b(x) \, dx, \quad g \in H^{[\phi, q]}_0(\mathbb{R}^n) \quad (2.7) \]
as an absolutely convergent integral.

Since \( H^{[\phi, q]}_0(\mathbb{R}^n) \) is dense in \( H^{[\phi, q]}(\mathbb{R}^n) \), the linear functional \( L_b \) can be extended on the entire \( H^{[\phi, q]}(\mathbb{R}^n) \) in the usual way. Moreover, we have the following theorem.

**Theorem 2.2** ([22]). Assume that \( \phi, q \) satisfy the conditions of Definition 2.4. Then
\[ \left( H^{[\phi, q]}(\mathbb{R}^n) \right)^* = \mathcal{L}_{q', \phi}(\mathbb{R}^n). \]
More precisely, given \( b \in \mathcal{L}_{q', \phi}(\mathbb{R}^n) \), the linear functional \( L_b \) defined by Definition 2.5 can be extended on the entire \( H^{[\phi, q]}(\mathbb{R}^n) \). Conversely, for every bounded linear functional \( L \) on \( H^{[\phi, q]}(\mathbb{R}^n) \) there exists \( b \in \mathcal{L}_{q', \phi}(\mathbb{R}^n) \) such that for all \( f \in H^{[\phi, q]}_0(\mathbb{R}^n) \) we have \( L(f) = L_b(f) \). The norm \( \|L_b\|_{\left( H^{[\phi, q]} \right)^*} \) is equivalent to \( \|b\|_{\mathcal{L}_{q', \phi}} \).

**Theorem 2.3** ([34]). Let \( \phi \) be in \( G^{\infty} \) and satisfy (2.3). Assume that, for each \( M > 0 \),
\[ \lim_{r \to 0} \inf_{x \in B(0, M)} \frac{\phi(x, r)}{r} = \infty, \quad (2.8) \]
\[ \lim_{r \to \infty} \inf_{x \in \mathbb{R}^n} r^n \phi(x, r) = \infty. \quad (2.9) \]
Then
\[
\left( C^\infty_{\text{comp}}(\mathbb{R}^n)^{\mathcal{L}_1,\rho}(\mathbb{R}^n) \right)^* = H^{[\rho,\infty]}(\mathbb{R}^n).
\]

More precisely, for \( f \in H^{[\rho,\infty]}(\mathbb{R}^n) \), the linear functional
\[
\langle f, \psi \rangle = \int_{\mathbb{R}^n} f(x)\psi(x) \, dx, \quad \psi \in C^\infty_{\text{comp}}(\mathbb{R}^n)
\]
(2.10)
can be extended on \( C^\infty_{\text{comp}}(\mathbb{R}^n)^{\mathcal{L}_1,\rho}(\mathbb{R}^n) \). Conversely, each bounded linear functional on \( C^\infty_{\text{comp}}(\mathbb{R}^n)^{\mathcal{L}_1,\rho}(\mathbb{R}^n) \) has the form (2.10), for some \( f \in H^{[\rho,\infty]}(\mathbb{R}^n) \). The linear functional norm is equivalent to \( \| f \|_{H^{[\rho,\infty]}} \).

3. Main results

To define the generalized fractional integral operator \( I_\rho \) on the Campanato space \( \mathcal{L}_{p,\rho}(\mathbb{R}^n) \), we define the modified version of \( I_\rho \) by
\[
\tilde{I}_\rho f(x) = \int_{\mathbb{R}^n} f(y) \left( \frac{\rho(|x-y|)}{|x-y|^n} = \frac{\rho(|y|)(1 - \chi_{B_0}(y))}{|y|^n} \right) \, dy,
\]
where \( B_0 = B(0,1) \) and \( \chi_{B_0} \) is the characteristic function of \( B_0 \). Then the integral above converges for each \( f \in \mathcal{L}_{p,\rho}(\mathbb{R}^n) \). Moreover, if \( f \) is the constant function 1, then \( \tilde{I}_\rho \) is a constant function. Therefore, \( \tilde{I}_\rho \) is well defined on \( \mathcal{L}_{p,\rho}(\mathbb{R}^n) \) which is a space modulo constant functions. Further, if both \( I_\rho f \) and \( \tilde{I}_\rho f \) are well defined, then \( I_\rho f - \tilde{I}_\rho f \) is a constant function. Then it is enough to consider only \( I_\rho f \) instead of \( \tilde{I}_\rho f \) for \( f \in C^\infty_{\text{comp}}(\mathbb{R}^n) \).

We need the following condition on \( \rho \) for the well definedness of \( \tilde{I}_\rho \): There exist positive constants \( C_1 \) and \( C_2 \) such that, for all \( r, s \in (0, \infty) \),
\[
\int_r^\infty \frac{\rho(t)}{t^2} \, dt \leq C_1 \frac{\rho(r)}{r}, \quad (3.1)
\]
\[
\left| \frac{\rho(r)}{r^n} - \frac{\rho(s)}{s^n} \right| \leq C_2 |r - s| \frac{\rho(r)}{r^{n+1}}, \quad \text{if } \frac{1}{2} \leq \frac{s}{r} \leq 2. \quad (3.2)
\]

Then the main result is the following.

**Theorem 3.1.** Let \( \phi, \psi \in \mathcal{G} \) and \( p \in [1, \infty) \). Assume that \( \rho \) satisfies (1.2), (1.3), (1.1) and (3.2).

(i) If there exists a positive constant \( A \) such that, for all \( x \in \mathbb{R}^n \) and \( r \in (0, \infty) \),
\[
\int_r^\infty \frac{\rho(t)}{t} \, dt \phi(x, r) + r \int_r^\infty \frac{\rho(t)\phi(x, t)}{t^2} \, dt \leq A\phi(x, r), \quad (3.3)
\]
then \( \tilde{I}_\rho \) is bounded from \( \mathcal{L}_{p,\psi}(\mathbb{R}^n) \) to \( \mathcal{L}_{p,\phi}(\mathbb{R}^n) \).

(ii) Moreover, if both \( \phi \) and \( \psi \) are in \( \mathcal{G}^{\text{inc}} \) and satisfy (2.3) and (2.5), then \( I_\rho \) is bounded from \( C^\infty_{\text{comp}}(\mathbb{R}^n)^{\mathcal{L}_1,\rho}(\mathbb{R}^n) \) to \( C^\infty_{\text{comp}}(\mathbb{R}^n)^{\mathcal{L}_1,\psi}(\mathbb{R}^n) \).
From Theorem 3.1 we have the following corollary immediately.

**Corollary 3.2.** Let $0 < \alpha < 1$. Then $I_\alpha$ is bounded from $C_\infty^\comp(\mathbb{R}^n)^{BMO(\mathbb{R}^n)}$ to $C_\infty^\comp(\mathbb{R}^n)^{\Lip_\alpha(\mathbb{R}^n)}$. Moreover, if $0 < \beta < \alpha + \beta < 1$, then $I_\alpha$ is bounded from $C_\infty^\comp(\mathbb{R}^n)^{\Lip_\beta(\mathbb{R}^n)}$ to $C_\infty^\comp(\mathbb{R}^n)^{\Lip_{\alpha+\beta}(\mathbb{R}^n)}$.

As another corollary, we consider the Lipschitz (Hölder) space with variable exponent. For $\alpha(\cdot) : \mathbb{R}^n \to [0, \infty)$ and $\alpha_* \in [0, \infty)$, let $\Lip_{\alpha(\cdot)}^\ast(\mathbb{R}^n)$ be the set of all functions $f$ such that the following functional is finite:

$$
\|f\|_{\Lip_{\alpha(\cdot)}^\ast} = \max \left\{ \sup_{0 < |x-y| < 1} \frac{2|f(x) - f(y)|}{|x-y|^{\alpha(x)} + |x-y|^{\alpha(y)}}, \sup_{|x-y| \geq 1} \frac{|f(x) - f(y)|}{|x-y|^{\alpha_*}} \right\},
$$

see [23, Definition 2.1 and Remark 2.2]. For these $\alpha(\cdot)$ and $\alpha_*$, let

$$
\phi(x, r) = \begin{cases} 
   r^{\alpha(x)}, & 0 < r < 1, \\
   r^{\alpha_*}, & 1 \leq r < \infty.
\end{cases}
$$

If

$$
0 \leq \inf_{x \in \mathbb{R}^n} \alpha(x) \leq \sup_{x \in \mathbb{R}^n} \alpha(x) < 1, \quad 0 \leq \alpha_* < 1,
$$

then $\phi$ is in $G^{inc}$ and satisfies (2.8) and (2.9). If $\alpha(\cdot)$ is log-Hölder continuous also, that is, there exists a positive constant $C$ such that, for all $x, y \in \mathbb{R}^n$,

$$
|\alpha(x) - \alpha(y)| \leq \frac{C}{\log(e/|x-y|)} \quad \text{if} \quad 0 < |x-y| < 1,
$$

then $\phi$ satisfies (2.3), see [23, Proposition 3.3]. Moreover, if $\inf_{x \in \mathbb{R}^n} \alpha(x) > 0$ and $\alpha_* > 0$, then $L_1,\phi(\mathbb{R}^n) = \Lip_{\alpha(\cdot)}^\ast(\mathbb{R}^n)$ with equivalent norms, see [23, Corollary 3.5]. Hence we have the following corollary.

**Corollary 3.3.** Let $0 < \alpha < 1$. Let $\beta(\cdot), \gamma(\cdot) : \mathbb{R}^n \to (0, 1)$ be log-Hölder continuous and $\beta_* \gamma_* \in (0, 1)$. If $\gamma(x) = \alpha + \beta(x)$, $0 < \beta_- < \gamma_+ < 1$ and $\gamma_* = \alpha + \beta_*$, then $I_\alpha$ is bounded from $C_\infty^\comp(\mathbb{R}^n)^{\Lip_{\beta(\cdot)}^\ast(\mathbb{R}^n)}$ to $C_\infty^\comp(\mathbb{R}^n)^{\Lip_{\gamma(\cdot)}^\ast(\mathbb{R}^n)}$.

In a similar way to [34] we can apply Theorem 3.1 to the dual and bidual operators of $I_\rho$. In general, if a linear operator $T$ is bounded from a normed linear space $X$ to a normed linear space $Y$, then the dual operator $T^*$ is bounded from $Y^*$ to $X^*$, where $X^*$ and $Y^*$ are the dual spaces of $X$ and $Y$, respectively, see [35, Theorem 2’ (p. 195)]. Hence, by the duality $(C_\infty^\comp(\mathbb{R}^n)^{C_1,\phi(\mathbb{R}^n)})^* = H^{[\rho, \infty]}(\mathbb{R}^n)$ and $(H^{[\rho, \infty]}(\mathbb{R}^n))^* = L_1,\phi(\mathbb{R}^n)$, we can consider the dual and bidual operators $(I_\rho)^*$ and $(I_\rho)^{**}$, which are bounded linear operators. This idea was used by [30, 31] for Morrey spaces.

**Theorem 3.4.** Let $\rho$ satisfy (1.2), (1.3), (3.1) and (3.2). Assume that both $\phi$ and $\psi$ are in $G^{inc}$ and satisfy (2.3), (2.5), (2.8) and (2.9). Assume also that $\phi$, $\psi$ and $\rho$ satisfy (3.3).
(i) The dual operator \((I_\rho)^*\) coincide with \(I_\rho\) from \(H^{[0, \infty]}(\mathbb{R}^n)\) to \(H^{[\rho, \infty]}(\mathbb{R}^n)\). Consequently, \(I_\rho\) is a bounded linear operator from \(H^{[0, \infty]}(\mathbb{R}^n)\) to \(H^{[\rho, \infty]}(\mathbb{R}^n)\).

(ii) The bidual operators \((I_\rho)^{**}\) coincide with \(I_\rho\) from \(L_{1, \psi}(\mathbb{R}^n)\) to \(L_{1, \psi}(\mathbb{R}^n)\). Consequently, \(I_\rho\) is a bounded linear operator from \(L_{1, \psi}(\mathbb{R}^n)\) to \(L_{1, \psi}(\mathbb{R}^n)\).

For the boundedness and continuity of \(I_\alpha\) on preduals of \(L_{1, \psi}(\mathbb{R}^n)\), see [24].

4. Proof

To prove Theorem 3.1 we need several lemmas. The first lemma is the same as [16, Lemma 4.2]. However, we can get the conclusion without the doubling condition of \(\rho\), while [16, Lemma 4.2] needed the doubling condition. Then we give the precise proof.

Lemma 4.1. If \(\rho\) satisfies (1.2), (3.1) and (3.2), then

\[
\rho(\frac{|x_1 - y|}{|x_1 - y|^n} - \frac{|x_2 - y|}{|x_2 - y|^n})
\]

is integrable on \(\mathbb{R}^n\) as a function of \(y\) and, for every choice of \(x_1\) and \(x_2\),

\[
\int_{\mathbb{R}^n} \left( \frac{\rho(|x_1 - y|)}{|x_1 - y|^n} - \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} \right) dy = 0.
\]

Proof. Let \(r = |x_1 - x_2|\). For large \(R > 0\), let

\[
J_1 = \int_{B(x_1, R)} \frac{\rho(|x_1 - y|)}{|x_1 - y|^n} dy - \int_{B(x_2, R)} \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} dy,
\]

\[
J_2 = \int_{B(x_1, R+r) \setminus B(x_1, R)} \frac{\rho(|x_1 - y|)}{|x_1 - y|^n} dy - \int_{B(x_1, R+r) \setminus B(x_2, R)} \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} dy,
\]

\[
J_3 = \int_{B(x_1, R+r)^c} \left( \frac{\rho(|x_1 - y|)}{|x_1 - y|^n} - \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} \right) dy.
\]

Then

\[
J_1 + J_2 + J_3 = \int_{\mathbb{R}^n} \left( \frac{\rho(|x_1 - y|)}{|x_1 - y|^n} - \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} \right) dy.
\]

For the parts \(J_1\) and \(J_3\), we use the same way as in [16, Proof of Lemma 4.2]. That is, (1.2) implies that \(\rho(|x_i - y|)/|x_i - y|^n\) \((i = 1, 2)\) are locally integrable and \(J_1 = 0\). Using (3.2) and (3.1) we see that (4.1) is integrable on \(B(x_1, R+r)^c\) and \(J_3 \to 0\) as \(R \to \infty\). Then we also see that (4.1) is integrable on \(\mathbb{R}^n\). Finally, for \(R > 2r\), we have

\[
|J_2| \leq \int_{B(x_1, R+r) \setminus B(x_1, R-r)} \frac{\rho(|x_1 - y|)}{|x_1 - y|^n} dy \sim \int_{R-r}^{R+r} \frac{\rho(t)}{t} dt \leq 2r \sup_{R/2 \leq t \leq 2R} \frac{\rho(t)}{t}.
\]

Since (3.1) implies \(\int_1^\infty \rho(t)/t^2 dt < \infty\), we see that \(\rho(t)/t \to 0\) as \(t \to \infty\). Then \(J_2 \to 0\) as \(R \to \infty\).
Let 
\[
\text{MO}(f, B) = \int_B |f(x) - f_B| \, dx.
\]

**Lemma 4.2** (see [13, Corollary 2.4]). There exists a positive constant \(c_n\) dependent only on \(n\) such that, for all \(x \in \mathbb{R}^n\) and \(r, s \in (0, \infty)\),
\[
|f_{B(x,r)} - f_{B(x,s)}| \leq c_n \int_r^{\max(2r, s)} \frac{\text{MO}(f, B(x, t))}{t} \, dt, \quad \text{if } r < s.
\]

**Lemma 4.3.** Let \(\ell \geq 2\). If \(\phi\) satisfies (1.3) and (3.1), then there exists a positive constant \(C\) such that, for \(r \in (0, \infty)\),
\[
\sup_{r \leq t \leq \ell r} \rho(t) \leq C \frac{\ell K_2}{K_1} \rho(K_1 r),
\]
where \(K_1\) and \(K_2\) are the constants in (1.3).

**Proof.** Take the integer \(k\) such that \(2^{k-1} \leq \ell < 2^k\). Using (1.3) and (3.1), we have
\[
\sup_{r \leq t \leq \ell r} \rho(t) = \sup_{j=1,2,\ldots,k} \left( \sup_{2^{j-1}r \leq t \leq \max(2^j r, \ell r)} \rho(t) \right)
\lesssim \sup_{j=1,2,\ldots,k} \int_{2^{j-1}r}^{\max(2^j r, \ell r)} \frac{\rho(t)}{t} \, dt \leq \int_{K_1 r}^{\ell K_2 r} \frac{\rho(t)}{t} \, dt
\lesssim \ell K_2 r \int_{K_1 r}^{\ell K_2 r} \frac{\rho(t)}{t^2} \, dt \lesssim \ell K_2 r \rho(K_1 r).
\]
This is the conclusion. \(\square\)

The next lemma is the same as [16, Lemma 4.3]. However, we use Lemma 4.3 instead of the doubling condition of \(\rho\), while [16, Lemma 4.3] needed the doubling condition. Then we give the precise proof.

**Lemma 4.4.** Under the assumption in Theorem 3.1, there exists a positive constant \(C\) such that, for all \(a \in \mathbb{R}^n\) and \(r \in (0, \infty)\),
\[
\int_{B(a, r)^c} \frac{\rho(|a - y|)}{|a - y|^{n+1}} |f(y) - f_{B(a, r)}| \, dy \leq C \frac{\rho(r) \phi(a, r)}{r} \|f\|_{C_{p, \phi}}.
\]

**Proof.** We may assume that \(\|f\|_{C_{p, \phi}} = 1\). By Lemma 4.2 and the inequality \(\text{MO}(f, B) \leq \phi(B)\|f\|_{C_{p, \phi}} \leq \phi(B)\) we have
\[
\int_{B(a, 2^j r)} |f(y) - f_{B(a, r)}| \, dy \leq \int_{B(a, 2^j r)} |f(y) - f_{B(a, 2^j r)}| \, dy + |f_{B(a, r)} - f_{B(a, 2^j r)}|
\lesssim \left( \phi(a, 2^j r) + \int_{2^j r}^{2^{j+1} r} \frac{\phi(a, s)}{s} \, ds \right)
\lesssim \int_{2^j r}^{2^{j+1} r} \frac{\phi(a, s)}{s} \, ds, \quad j = 1, 2, \ldots.
\]
If $2^jr \leq u \leq 2^{j+1}r$, then by Lemma 4.3 we have
\[
\int_{2^{j-1}r \leq |a-y| \leq 2^jr} \frac{\rho(|a-y|)}{|a-y|^{n+1}} |f(y) - f_{B(a,r)}| \, dy
\leq \sup_{2^{j-1}r \leq t \leq 2^jr} \frac{\rho(t)}{(2^jr)^{n+1}} \int_{B(a,2^jr)} |f(y) - f_{B(a,r)}| \, dy
\]
\[
\leq \sup_{u/4 \leq t \leq u} \frac{\rho(t)}{2^jr} \int_r^u \frac{\phi(a,s)}{s} \, ds
\leq \frac{\rho(K_1u/A)}{u} \int_r^u \frac{\phi(a,s)}{s} \, ds,
\]
which shows
\[
\int_{2^{j-1}r \leq |a-y| \leq 2^jr} \frac{\rho(|a-y|)}{|a-y|^{n+1}} |f(y) - f_{B(a,r)}| \, dy
\leq \int_{2^{j+1}r} \left( \frac{\rho(K_1u/A)}{u} \int_r^u \frac{\phi(a,s)}{s} \, ds \right) \frac{du}{u}.
\]
Hence, we have
\[
\int_{B(a,r)} \frac{\rho(|a-y|)}{|a-y|^{n+1}} |f(y) - f_{B(a,r)}| \, dy
= \sum_{j=1}^{\infty} \int_{2^{j-1}r \leq |a-y| \leq 2^jr} \frac{\rho(|a-y|)}{|a-y|^{n+1}} |f(y) - f_{B(a,r)}| \, dy
\leq \int_r^\infty \left( \frac{\rho(K_1u/A)}{u} \int_r^u \frac{\phi(a,s)}{s} \, ds \right) \frac{du}{u}
= \int_r^\infty \left( \int_s^\infty \frac{\rho(K_1s/4)}{s} \, ds \right) \frac{\phi(a,s)}{s} \, ds.
\]
Since (3.1) implies
\[
\int_s^\infty \frac{\rho(K_1s/4)}{u} \, du \sim \int_{K_1s/4}^\infty \frac{\rho(u)}{u} \, du \leq \frac{\rho(K_1s/4)}{s},
\]
we have
\[
\int_{B(a,r)} \frac{\rho(|a-y|)}{|a-y|^{n+1}} |f(y) - f_{B(a,r)}| \, dy
\leq \int_r^\infty \frac{\rho(K_1s/8)}{s^2} \phi(a,s) \, ds \sim \int_{K_1r/8}^\infty \frac{\rho(s)\phi(a,s)}{s^2} \, ds \leq \frac{\psi(a,r)}{r}.
\]
In the above we use (3.3) and the doubling condition of $\phi$ and $\psi$. The proof is complete. \qed

Now we prove Theorem 3.1 (i). We use the almost same method as in the proof of [16, Theorem 3.4]. Then we give only a sketch of the proof.
Proof of Theorem 3.1 (i). For any ball $B = B(a, r)$, let $\hat{B} = B(a, 2r)$ and

$$E_B(x) = \int_{\mathbb{R}^n} (f(y) - f_{\hat{B}}) \left( \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)(1 - \chi_{\hat{B}}(y))}{|a-y|^n} \right) dy,$$

$$C_B^1 = \int_{\mathbb{R}^n} (f(y) - f_{\hat{B}}) \left( \frac{\rho(|a-y|)(1 - \chi_{\hat{B}}(y))}{|a-y|^n} - \frac{\rho(|y|)(1 - \chi_{B_0}(y))}{|y|^n} \right) dy,$$

$$C_B^2 = \int_{\mathbb{R}^n} f_{\hat{B}} \left( \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)(1 - \chi_{B_0}(y))}{|a-y|^n} \right) dy,$$

$$E_B^{-1}(x) = \int_{\hat{B}^c} (f(y) - f_{\hat{B}}) \left( \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)}{|a-y|^n} \right) dy.$$

Then

$$I_p f(x) - (C_B^1 + C_B^2) = E_B(x) = E_B^{-1}(x) + E_B^2(x) \quad \text{for } x \in B.$$

By (3.2) and Lemma 4.4 we see that $C_B^1$ is a constant. By Lemma 4.1 and (1.2) we also see that $C_B^2$ is a constant. For $E_B^{-1}$ we use Minkowski’s integral inequality. Then

$$\left( \int_B |E_B^{-1}(x)|^p \, dx \right)^{1/p} \geq \left( \int_B \left| \int_{\hat{B}} (f(y) - f_{\hat{B}}) \frac{\rho(|x-y|)}{|x-y|^n} \, dy \right|^p \, dx \right)^{1/p}.$$

By (3.3) and the doubling condition of $\phi$ and $\psi$. For $E_B^2$ we use (3.2) and Lemma 4.4. Then we have

$$|E_2^2(x)| \preceq \psi(B) \| f \|_{L_\phi} \quad \text{for } x \in B.$$

Therefore, we have the conclusion. \qed

To prove Theorem 3.1 (ii) we need the following known results.

**Theorem 4.5 ([1]).** Let $\phi$ be in $G^{inc}$ and satisfy (2.3), (2.8) and (2.9). Let $f \in L_{1, \phi}({\mathbb{R}^n})$. Then $f \in C_{com}^{inc} L_{1, \phi}({\mathbb{R}^n})$ if and only if $f$ satisfies the following three conditions:

(i) \( \lim_{r \to +0} \sup_{x \in \mathbb{R}^n} \frac{\text{MO}(f, B(x, r))}{\phi(x, r)} = 0. \)

(ii) \( \lim_{r \to +\infty} \sup_{x \in \mathbb{R}^n} \frac{\text{MO}(f, B(x, r))}{\phi(x, r)} = 0. \)

(iii) \( \lim_{|x| \to \infty} \frac{\text{MO}(f, B(x, r))}{\phi(x, r)} = 0 \) for each \( r > 0. \)

In [1], the hypothesis \( \lim_{r \to +0} \inf_{x \in \mathbb{R}^n} \phi(x, r)/r = \infty \) was used instead of (2.8). However, we can relax the hypothesis to (2.8). Moreover, we do not need (2.8) and (2.9) to prove that, if \( f \) satisfies (i)–(iii), then \( f \in C^\infty_{\text{comp}}(\mathbb{R}^n)^{C_{1,\phi}(\mathbb{R}^n)}. \) We do not need (2.3) to prove that, if \( f \in C^\infty_{\text{comp}}(\mathbb{R}^n)^{C_{1,\phi}(\mathbb{R}^n)}, \) then \( f \) satisfies (i)–(iii). Theorem 4.5 is a generalization of [33, Lemma].

**Proposition 4.6** ([34, Proposition 6.4]). Let \( 1 \leq p < \infty \) and \( \phi \in \mathcal{G}. \) Assume that \( \phi \) satisfies (2.3).

(i) If \( r \mapsto r^{n/p}\phi(x, r) \) is almost increasing, then \( C^\infty_{\text{comp}}(\mathbb{R}^n) \subset \mathcal{L}_{p,\phi}(\mathbb{R}^n). \)

(ii) If \( \phi \) is almost increasing and \( \psi(x, r) = \phi(x, r) \min\{1, \frac{1}{|x|^{n/p}}\} \) for all \( x \in \mathbb{R}^n \) and \( r \in (0, \infty), \) then \( C^\infty_{\text{comp}}(\mathbb{R}^n) \subset \mathcal{L}_{p,\psi}(\mathbb{R}^n). \)

**Lemma 4.7** ([14, Lemma 2], [22, Lemma 7.1]). Let \( \phi : \mathbb{R}^n \times (0, \infty) \to (0, \infty). \) If \( \phi \) satisfies (2.5) for some constant \( C \) and for all \( r \in (0, \infty), \) then, for \( \epsilon \in (0, 1/C), \) the function \( \phi(x, r)^{r^\epsilon} \) satisfies (2.5) for the constant \( C/(1 - \epsilon C) \) and for all \( r \in (0, \infty). \)

Now we prove Theorem 3.1 (ii).

**Proof of Theorem 3.1** (ii). For \( f \in C^\infty_{\text{comp}}(\mathbb{R}^n), \) both \( \hat{I}_p f \) and \( I_p f \) are well defined, and then \( \hat{I}_p f - I_p f \) is a constant. Then, by Theorem 3.1 (i) we have \( \|I_p f\|_{L_{1,\phi}} \lesssim \|f\|_{L_{1,\phi}} \) for \( f \in C^\infty_{\text{comp}}(\mathbb{R}^n). \) We prove that \( I_p f \in C^\infty_{\text{comp}}(\mathbb{R}^n)^{C_{1,\phi}(\mathbb{R}^n)} \) by using Theorem 4.5. By the assumption and Lemma 4.7 we see that \( \phi(x, r)^{r^\epsilon} \) and \( \psi(x, r)^{r^\epsilon} \) also satisfy (2.5) for a small \( \epsilon > 0. \) Let

\[
\begin{align*}
\phi_1(x, r) &= \phi(x, r)^{r^\epsilon}, & \psi_1(x, r) &= \psi(x, r)^{r^\epsilon}, \\
\phi_2(x, r) &= \phi(x, r)^{r^{-\epsilon}}, & \psi_2(x, r) &= \psi(x, r)^{r^{-\epsilon}}, \\
\phi_3(x, r) &= \phi(x, r) \min\{1, |x|^{-n}\}, & \psi_3(x, r) &= \psi(x, r) \min\{1, |x|^{-n}\}
\end{align*}
\]

for \( x \in \mathbb{R}^n \) and \( r \in (0, \infty). \) From Proposition 4.6 it follows that \( C^\infty_{\text{comp}}(\mathbb{R}^n) \subset \mathcal{L}_{1,\phi_i}(\mathbb{R}^n) \) (\( i = 1, 2, 3 \)). Moreover, \( \phi_i, \psi_i \in \mathcal{G} \) (\( i = 1, 2, 3 \)) and (3.3) holds for each pair of \( (\phi_i, \psi_i) \), since, for \( i = 1, \)

\[
\rho(r) \phi(a, r)^{r^\epsilon} \lesssim \int_0^{K^r} \frac{\rho(t)}{t} \, dt \phi(a, r)^{r^\epsilon} \lesssim \psi(a, r)^{r^\epsilon},
\]

and

\[
\int_r^{\infty} \frac{\rho(t) \phi(x, t)^{r^\epsilon}}{t^2} \, dt \lesssim \int_r^{\infty} \frac{\psi(x, t)^{r^\epsilon}}{t^2} \, dt \lesssim \frac{\psi(x, r)^{r^\epsilon}}{r},
\]
and the other cases are clear. Then by Theorem 3.1 (i) we have the norm inequalities \( \|I_\rho f\|_{L^1,\psi_i} \lesssim \| f\|_{L^1,\psi_i} \) (i = 1, 2, 3). Hence, we get

\[
\frac{1}{\psi(a,r)} \int_{B(a,r)} |I_\rho f(x) - (I_\rho f)_{B(a,r)}| \, dx \leq \frac{\psi_1(a,r)}{\psi(a,r)} \|I_\rho f\|_{L^1,\psi_1} \lesssim r^\nu \| f\|_{L^1,\psi_1} \to 0 \quad \text{as} \quad r \to 0.
\]

Similarly,

\[
\frac{1}{\psi(a,r)} \int_{B(a,r)} |I_\rho f(x) - (I_\rho f)_{B(a,r)}| \, dx \lesssim r^\nu \| f\|_{L^1,\psi_2} \to 0 \quad \text{as} \quad r \to \infty,
\]

and

\[
\frac{1}{\psi(a,r)} \int_{B(a,r)} |I_\rho f(x) - (I_\rho f)_{B(a,r)}| \, dx \lesssim |a|^{-\omega} \| f\|_{L^1,\psi_3} \to 0 \quad \text{as} \quad |a| \to \infty.
\]

Thus, \( I_\rho f \) satisfies (i)–(iii) in Theorem 4.5. Hence, \( I_\rho f \in C_{\text{comp}}^{\infty} (\mathbb{R}^n) \). We get the conclusion.

To prove Theorem 3.4 we give a lemma.

**Lemma 4.8.** Assume that \( \rho \) satisfies (1.2), (3.1) and (3.2). If \( g \in L^1_{\text{comp}} (\mathbb{R}^n) \) and \( \int_{\mathbb{R}^n} g(x) \, dx = 0 \), then \( I_\rho g \in L^1 (\mathbb{R}^n) \) and \( \int_{\mathbb{R}^n} I_\rho g(x) \, dx = 0 \).

**Proof.** Let \( \text{supp} \ g \subset B(a,r) \) and \( B(0,1) \cup B(a,2r) \subset B(a,R) \), and let

\[
I_1 = \int_{B(a,R)} \left| \int_{\mathbb{R}^n} g(y) \frac{\rho(|x-y|)}{|x-y|^n} \, dy \right| \, dx,
\]

\[
I_2 = \int_{B(a,R)} \left( \int_{\mathbb{R}^n} g(y) \left( \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|x|)}{|x|^n} \right) \, dy \right) \, dx.
\]

Then

\[
I_1 \leq \int_{B(a,r)} |g(y)| \left( \int_{B(a,R)} \frac{\rho(|x-y|)}{|x-y|^n} \, dx \right) \, dy \lesssim \| g \|_{L^1} \int_0^{R+r} \frac{\rho(t)}{t} \, dt < \infty,
\]

and

\[
I_2 \leq \int_{B(a,r)} |g(y)| \left( \int_{B(a,R)} \frac{\rho(|x-y|)}{|x-y|^n} \, dx \right) \, dy,
\]

\[
\lesssim \int_{B(a,r)} |g(y)| \left( \int_{B(a,R)} \frac{\rho(|x-y|)}{|x-y|^{n+1}} \, dx \right) \, dy,
\]

\[
\lesssim (|a| + r) \| g \|_{L^1} \int_{R-r}^{\infty} \frac{\rho(t)}{t^2} \, dt < \infty.
\]
This shows the conclusion. □

Now we prove Theorem 3.4.

Proof of Theorem 3.4. First, we prove (i). Let \( f \in C_{\text{comp}}^{\infty}(\mathbb{R}^n) \subset L_{1,\phi}(\mathbb{R}^n) \) and \( g \in H_0^{[\psi,\infty]}(\mathbb{R}^n) \subset L_{\text{comp}}^{\infty}(\mathbb{R}^n) \). From Lemma 4.8 it follows that \( \int_{\mathbb{R}^n} I_{\rho} g(x) \, dx = 0 \). Then, even if \( f \) is modulo constant functions, the integral \( \int_{\mathbb{R}^n} I_{\rho} g(x) f(x) \, dx \) is well defined, that is, the integral is determined independently of the choice of the representative element \( f \). Moreover, we have

\[
\langle f, (I_{\rho})^* g \rangle = \langle I_{\rho} f, g \rangle = \int_{\mathbb{R}^n} I_{\rho} f(x) g(x) \, dx = \int_{\mathbb{R}^n} f(x) I_{\rho} g(x) \, dx,
\]

that is, \( (I_{\rho})^* = I_{\rho} \). Hence, \( I_{\rho} \) is bounded from \( H^{[\psi,\infty]}(\mathbb{R}^n) \) to \( H^{[\psi,\infty]}(\mathbb{R}^n) \), since \( (I_{\rho})^* \) is bounded.

Next, we prove (ii). Let \( g \in H_0^{[\psi,\infty]}(\mathbb{R}^n) \), \( \text{supp} \, g \subset B \) and \( f \in L_{1,\phi}(\mathbb{R}^n) \). We write \( f = f_1 + f_2 \), where \( f_1 = f \chi_{2B} \), \( f_2 = f(1 - \chi_{2B}) \). Then

\[
\langle g, (I_{\rho})^{**} f_1 \rangle = \langle (I_{\rho})^* g, f_1 \rangle = \langle I_{\rho} g, f_1 \rangle = \int_{\mathbb{R}^n} I_{\rho} g(x) f_1(x) \, dx
\]

\[
= \int_{\mathbb{R}^n} g(x) I_{\rho} f_1(x) \, dx = \int_{\mathbb{R}^n} g(x) \tilde{I}_{\rho} f_1(x) \, dx,
\]

and

\[
\langle g, (I_{\rho})^{**} f_2 \rangle = \langle (I_{\rho})^* g, f_2 \rangle = \langle I_{\rho} g, f_2 \rangle
\]

\[
= \int_{(2B)^c} \left( \int_B \frac{\rho(|x - y|)}{|x - y|^n} g(y) \, dy \right) f(x) \, dx
\]

\[
= \int_{(2B)^c} \left( \int_B \frac{\rho(|x - y|)}{|x - y|^n} \frac{\rho(|x|)}{|x|^n} (1 - \chi_{B(0,1)}(x)) \, g(y) \, dy \right) f(x) \, dx
\]

\[
= \int_B \left( \int_{(2B)^c} \frac{\rho(|x - y|)}{|x - y|^n} \frac{\rho(|x|)}{|x|^n} (1 - \chi_{B(0,1)}(x)) \, f(x) \, dx \right) g(y) \, dy
\]

\[
= \int_B \tilde{I}_{\rho} f_2(y) g(y) \, dy.
\]
Hence $\langle g, (I_\rho)^{**} f \rangle = \int_{\mathbb{R}^n} g(x) \tilde{I}_\rho f(x) \, dx$, that is, $(I_\rho)^{**} = \tilde{I}_\rho$ on $L_{1,\phi}(\mathbb{R}^n)$. Therefore, $\tilde{I}_\rho$ is bounded from $L_{1,\phi}(\mathbb{R}^n)$ to $L_{1,\psi}(\mathbb{R}^n)$, since $(I_\rho)^{**}$ is bounded.

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