HOMOTOPY DECOMPOSITION OF DIAGONAL ARRANGEMENTS

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Abstract. Given a space $X$ and a simplicial complex $K$ with $m$-vertices, the arrangement of partially diagonal subspaces of $X^m$, called the diagonal arrangement, is defined. We decompose the suspension of the diagonal arrangement when $2(\dim K + 1) < m$, which generalizes the result of Labassi [L]. As a corollary, we calculate the Euler characteristic of the complement $X^m - \Delta_K(X)$ when $X$ is a closed connected manifold.

1. Introduction and statement of results

A homotopy decomposition is a powerful tool in studying the topology of subspace arrangements and their complements. Ziegler and Živaljević [ZZ] give a homotopy decomposition of the one point compactification of affine subspace arrangements, from which one can deduce the well known Goresky-MacPherson formula [GM]. Bahri, Bendersky, Cohen, and Gitler [BBCG] give a homotopy decomposition of polyhedral products, a generalization of coordinate subspace arrangements and their complements, after a suspension, from which one can deduce Hochster’s formula on related Stanley-Reisner rings. A homotopy decomposition of polyedral products due to Grbić and Theriault [GT] and the authors [IK1, IK2] also implies the Golod property of several related simplicial complexes. In this paper, we consider a homotopy decomposition of diagonal arrangements which is defined as follows. Given a space $X$, we assign a partially diagonal subspace of $X^m$ corresponding to a subset $\sigma \subset [m] = \{1, \ldots, m\}$ as

$$\Delta_{\sigma}(X) = \{(x_1, \ldots, x_m) \in X^m \mid x_{i_1} = \cdots = x_{i_k} \text{ for } \{i_1, \ldots, i_k\} = [m] - \sigma\}. \tag{1}$$

Throughout the paper, let $K$ be a simplicial complex on the index set $[m]$, possibly with ghost vertices, where we always assume that the empty subset of $[m]$ is a simplex of $K$. We define the arrangement of partially diagonal subspaces of $X^m$ as

$$\Delta_K(X) = \bigcup_{\sigma \in K} \Delta_{\sigma}(X),$$

which is called the diagonal arrangement associated with $K$. Since $\Delta_K(X)$ is actually the union of the partially diagonal subspaces $\Delta_F(X)$ for facets $F$ of $K$, it is also called the hypergraph arrangement associated with the hypergraph whose edges are facets of $K$. Diagonal arrangements include many important subspace arrangements. For example, if $K$ is the $(m - 3)$-skeleton of $(m - 1)$-simplex, $\Delta_K(X)$ is the braid arrangement of $X$. Topology and combinatorics of diagonal arrangements have been studied in several directions. See [Ko, PRW, Ki, KS, L, MW, M] for

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example. We are particularly interested in the homotopy type of $\Delta_K(X)$. Labassi [L] showed that the suspension $\Sigma \Delta_K(X)$ decomposes into a certain wedge of smash products of copies of $X$ when $K$ is the $(m-d-1)$-skeleton of the $(m-1)$-simplex and $2d > m$, in which case $\Delta_K(X)$ consists of all $(x_1, \ldots, x_m) \in X^m$ such that at least $d$-tuple of $x_i$'s are identical. The proof for this decomposition in [L] heavily depends on the symmetry of the skeleta of simplices, and then it cannot apply to general $K$. The aim of this note is to generalize this result to arbitrary $K$ with $2(\dim K + 1) < m$ by a new method, where the result is best possible in the sense that if $2(\dim K + 1) \geq m$, the decomposition does not hold as is seen in [L].

**Theorem 1.1.** If $X$ is a connected CW-complex and $2(\dim K + 1) < m$, then

$$\Sigma \Delta_K(X) \simeq \Sigma \left( \bigvee_{\sigma \in K} \tilde{X}^{[\sigma]} \vee \tilde{X}^{[\sigma]+1} \right)$$

where $\tilde{X}^k$ is the smash product of $k$-copies of $X$ for $k > 0$ and $\tilde{X}^0$ is a point.

As a corollary, we calculate the Euler characteristic of the complement of the diagonal arrangement $\mathcal{M}_K(X) = X^m - \Delta_K(X)$.

**Corollary 1.2.** Let $X$ be a closed connected $n$-manifold. If $2(\dim K + 1) < m$, the Euler characteristic of $\mathcal{M}_K(X)$ is given by

$$\chi(\mathcal{M}_K(X)) = \chi(X)^m - (-1)^m \chi(X)(1 + \sum_{\emptyset \neq \sigma \in K} (\chi(X) - 1)^{|\sigma|}).$$

**Remark 1.3.** Corollary 1.2 does not hold without compactness of $X$. For example, if $X = \mathbb{R}$ (hence $n = 1$) and $K$ consists only of the empty subset of $[m]$, $\mathcal{M}_K(X)$ is the off-diagonal subset of $\mathbb{R}^m$ which has the homotopy type of $S^{m-2}$. Then $\chi(\mathcal{M}_K(X)) = 1 + (-1)^m$, which differs from Corollary 1.2.

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2. Proofs

Before considering the proof of Theorem 1.1, we prepare two lemmas on homotopy fibrations.

**Lemma 2.1** ([F, Proposition, pp.180]). Let $\{F_i \to E_i \to B\}_{i \in I}$ be an $I$-diagram of homotopy fibrations over a fixed connected base $B$. Then

$$\hocolim_i F_i \to \hocolim_i E_i \to B$$

is a homotopy fibration.

**Lemma 2.2.** Consider a homotopy fibration $F \overset{j}{\to} E \overset{s}{\to} B$ of connected CW-complexes. If $\Sigma j : \Sigma F \to \Sigma E$ has a homotopy retraction, then

$$\Sigma E \simeq \Sigma B \vee \Sigma F \vee \Sigma (B \wedge F).$$
Proof. Let \( r: \Sigma E \to \Sigma F \) be a homotopy retraction of \( \Sigma j \), and let \( \rho \) be the composite
\[
\Sigma E \to \Sigma E \vee \Sigma E \vee \Sigma E \xrightarrow{\Sigma (\pi \vee \nu \vee \Delta)} \Sigma B \vee \Sigma F \vee \Sigma (E \wedge E) \xrightarrow{1 \vee 1 \vee (\pi \wedge r)} \Sigma B
\]
where \( A = A \vee F \vee (A \wedge F) \) for a space \( A \). Since \( \Sigma E \) and \( \Sigma B \vee \Sigma F \vee \Sigma (B \wedge F) \) are simply connected CW-complexes, it is sufficient to show that \( \rho \) is an isomorphism in homology by the J.H.C. Whitehead theorem. We first observe the special case that there is a fiberwise homotopy equivalence \( \theta: B \times F \to E \) over \( B \). Then it is straightforward to see
\[
\rho_* \circ \theta_*(b \times f) = b \times \hat{\theta}_*(f) + \sum_{|b_i| < |b|} b_i \times f_i
\]
for singular chains \( b, b_i \) in \( B \) and \( f, f_i \) in \( F \), where we omit writing the suspension isomorphism of homology and \( \hat{\theta} \) is a self-homotopy equivalence of \( F \) given by the composite
\[
\Sigma F \xrightarrow{\rho} \Sigma (B \times F) \xrightarrow{\theta} \Sigma E \xrightarrow{\rho} \Sigma F.
\]
This readily implies that the map \( \rho \circ \theta \) is an isomorphism in homology, and then so is \( \rho \). For non-connected \( B \), the above is also true if we assume that \( r \) is a homotopy retraction of the suspension of the fiber inclusion on each component of \( B \). We next consider the general case. Let \( B_n \) be the \( n \)-skeleton of \( B \), and let \( E_n = \pi^{-1}(B_n) \). We prove that the restriction \( \rho|_{\Sigma E_n}: \Sigma E_n \to \Sigma \hat{B}_n \) is an isomorphism in homology by induction on \( n \). Since \( B \) is connected, \( j \) is homotopic to the composite
\[
F \simeq \pi^{-1}(b) \xrightarrow{\text{incl}} E
\]
for any \( b \in B \). Then \( \rho|_{\Sigma E_0}: \Sigma E_0 \to \Sigma \hat{B}_0 \) is an isomorphism in homology. Consider the following commutative diagram of homology exact sequences.
\[
\cdots \to H_*(E_{n-1}) \xrightarrow{(\rho|_{\Sigma E_{n-1}})_*} H_*(E_n) \xrightarrow{(\rho|_{\Sigma E_n})_*} H_*(E_n, E_{n-1}) \xrightarrow{(\rho|_{\Sigma E_n})_*} \cdots
\]
By the induction hypothesis, \( (\rho|_{\Sigma E_{n-1}})_* \) is an isomorphism. Since \( B_{n-1} \) is a subcomplex of \( B_n \), there is a neighborhood \( U \subset B_n \) of \( B_{n-1} \) which deforms onto \( B_{n-1} \). By the excision isomorphism, there is a commutative diagram of natural isomorphisms
\[
\begin{align*}
H_*(E_n, E_{n-1}) &\xrightarrow{\cong} H_*(E_n, \pi^{-1}(U)) \xrightarrow{\cong} H_*(E_n - E_{n-1}, \pi^{-1}(U) - E_{n-1}) \\
H_*(\hat{B}_n, \hat{B}_{n-1}) &\xrightarrow{\cong} H_*(\hat{B}_n, \hat{U}) \xrightarrow{\cong} H_*(\hat{B}_n - \hat{B}_{n-1}, \hat{U} - \hat{B}_{n-1})
\end{align*}
\]
where we may chose the basepoints of \( B_n \) and \( U \) in \( U - B_{n-1} \) since \( B \) is connected. Since each connected component of \( B_n - B_{n-1} \) is contractible, \( E_n - E_{n-1} \) is fiberwise homotopy equivalent to \( (B_n - B_{n-1}) \times F \) over \( B_n - B_{n-1} \), and then so is also \( \pi^{-1}(U) - E_{n-1} \) to \( (U - B_{n-1}) \times F \) over \( U - B_{n-1} \). As in the 0-skeleton case, we see that \( \Sigma r \) restricts to a homotopy retraction of
the suspension of the fiber inclusion on each component of $\Sigma(B_n - B_{n-1})$. Then by the above trivial fibration case, we obtain that the map

$$(\rho|_{\Sigma(B_n - B_{n-1})})_* : H_*(E_n - E_{n-1}, \pi^{-1}(U) - E_{n-1}) \to H_*(\tilde{B}_n - \tilde{B}_{n-1}, \tilde{U} - \tilde{B}_{n-1})$$

is an isomorphism, hence so is the right $(\rho|_{\Sigma E_n})_*$ in (2.1). Thus by the five lemma, the middle $(\rho|_{\Sigma E_n})_*$ in (2.1) is an isomorphism. We finally take the colimit to get that the map $\rho$ is an isomorphism in homology as desired, completing the proof. □

Remark 2.3. If we assume further that $F$ is of finite type, it immediately follows from the Leray-Hirsch theorem that the map $\rho$ is an isomorphism in cohomology with any field coefficient, implying that $\rho$ is an isomorphism in the integral homology by [H, Corollary 3A.7].

We now consider the diagonal arrangement $\Delta_K(X)$. Suppose that $2(\dim K + 1) < m$, or equivalently, $2|\sigma| < m$ for any $\sigma \in K$. Then for $(x_1, \ldots, x_m) \in \Delta_K(X)$, there is unique $x \in X$ such that $x_{i_1} = \cdots = x_{i_k} = x$ with $i_1 < \cdots < i_k$ and $2k > m$. Then by assigning such $x$ to $(x_1, \ldots, x_m) \in \Delta_K(X)$, we get a continuous map

$$\pi : \Delta_K(X) \to X.$$

For $\tau \subseteq [m]$, let $X^\tau = \{(x_1, \ldots, x_m) \in X^m \mid x_i = \ast \text{ for } i \in [m] - \tau\}$, and we put

$$X^K = \bigcup_{\sigma \in K} X^\sigma$$

which is called the polyhedral product or the generalized moment-angle complex associated with the pair $(X, \ast)$ and $K$. Observe that for $2(\dim K + 1) < m$, we have $\pi^{-1}(\ast) = X^K$.

Proposition 2.4. If $X$ is a CW-complex and $2(\dim K + 1) < m$, then $X^K \to \Delta_K(X) \xrightarrow{\pi} X$ is a homotopy fibration.

Proof. For each $\sigma \in K$, the map $\pi|_{\sigma} : \Delta_\sigma(X) \to X$ is identified with the projection from the product of copies of $X$. Then it follows from Lemma 2.1 that

$$\operatorname{hocolim}_{\sigma \in K} X^\sigma \to \operatorname{hocolim}_{\sigma \in K} \Delta_\sigma(X) \to X$$

is a homotopy fibration. Since the inclusions $X^\sigma \to X^\tau$ and $\Delta_\sigma(X) \to \Delta_\tau(X)$ for any $\sigma \subseteq \tau \subset [m]$ are cofibrations, we have

$$\operatorname{hocolim}_{\sigma \in K} X^\sigma \simeq \operatorname{colim}_{\sigma \in K} X^\sigma = X^K \quad \text{and} \quad \operatorname{hocolim}_{\sigma \in K} \Delta_\sigma(X) \simeq \operatorname{colim}_{\sigma \in K} \Delta_\sigma(X) = \Delta_K(X),$$

completing the proof. □

Put $\hat{X}^K = \bigvee_{\emptyset \neq \sigma \in K} \hat{X}^{[\sigma]}$. In [BBCG], it is proved that there is a homotopy equivalence

$$(2.2) \quad \epsilon_X : \Sigma X^K \xrightarrow{\simeq} \Sigma \hat{X}^K$$
which is natural with respect to \( X \), i.e. for a map \( f : X \rightarrow Y \), the square diagram

\[
\begin{array}{ccc}
\Sigma X^K & \xrightarrow{\epsilon} & \Sigma \hat{X}^K \\
\Sigma f^K & \downarrow & \Sigma j^K \\
\Sigma Y^K & \xrightarrow{\epsilon} & \Sigma \hat{Y}^K
\end{array}
\]

is homotopy commutative, where the vertical arrows are induced from \( f \).

**Proposition 2.5.** If \( X \) is a CW-complex and \( 2(\dim K + 1) < m \), the inclusion \( j : X^K \rightarrow \Delta_K(X) \) has a homotopy retraction after a suspension.

**Proof.** Let \( E : X \rightarrow \Omega \Sigma X \) be the suspension map. Since \( \Sigma E \) has a retraction, we easily see that the induced map \( \Sigma \hat{E}^K : \Sigma \hat{X}^K \rightarrow \Sigma \hat{\Omega \Sigma X}^K \) has a retraction, say \( r \). If \( Y \) is an H-space, the map

\[
Y \times Y^K \rightarrow \Delta_K(Y), \quad (y, (y_1, \ldots, y_m)) \mapsto (y \cdot y_1, \ldots, y \cdot y_m)
\]

is a map between homotopy fibrations with common base and fiber, and then is a weak homotopy equivalence. Hence if \( Y \) has the homotopy type of a CW-complex, the map is a homotopy equivalence, implying that there is a homotopy retraction \( r' : \Delta_K(Y) \rightarrow Y^K \) of the inclusion \( j : Y^K \rightarrow \Delta_K(Y) \). Combining the above maps, we get a homotopy commutative diagram

\[
\begin{array}{ccc}
\Sigma \hat{X}^K & \xrightarrow{\epsilon^{-1}} & \Sigma X^K \\
\Sigma \hat{E}^K & \downarrow & \Sigma \Omega \Sigma X^K \\
\Sigma \hat{X}^K & \xrightarrow{\epsilon} & \Sigma (\Omega \Sigma X)^K
\end{array}
\begin{array}{ccc}
\Sigma \Delta_K(X) & \xrightarrow{\Sigma j} & \Sigma \Delta_K(E) \\
\Sigma E^K & \downarrow & \Sigma j \\
\Sigma \Omega \Sigma X^K & \xrightarrow{\epsilon} & \Sigma \Delta_K(\Omega \Sigma X)
\end{array}
\]

where \( \Delta_K(E) : \Delta_K(X) \rightarrow \Delta_K(\Omega \Sigma X) \) is induced from \( E \). Thus the composite

\[
\Sigma \Delta_K(X) \xrightarrow{\Sigma \Delta_K(E)} \Sigma \Delta_K(\Omega \Sigma X) \xrightarrow{\Sigma r'} \Sigma (\Omega \Sigma X)^K \xrightarrow{\epsilon} \Sigma \hat{\Omega \Sigma X}^K \xrightarrow{r} \Sigma \hat{X}^K \xrightarrow{\epsilon^{-1}} \Sigma X^K
\]

is the desired homotopy retraction. \( \square \)

**Proof of Theorem 1.1.** If \( 2(\dim K + 1) < m \), there is a homotopy fibration \( X^K \rightarrow \Delta_K(X) \rightarrow X \), where the fiber inclusion has a homotopy retraction after a suspension by Proposition 2.5. Then by Lemma 2.2, we get a homotopy equivalence

\[
\Sigma \Delta_K(X) \simeq \Sigma X \vee \Sigma X^K \vee \Sigma (X \wedge X^K).
\]

Therefore the proof is completed by (2.2). \( \square \)

**Proof of Corollary 1.2.** Since \( X \) is a compact manifold, \( \Delta_K(X) \) is a compact, locally contractible subset of an \( mn \)-manifold \( X^m \). Then by the Poincaré-Alexander duality [H, Proposition 3.46], there is an isomorphism

\[
H_i(X^m, \mathcal{M}_K(X); \mathbb{Z}/2) \cong H^{mn-i}(\Delta_K(X); \mathbb{Z}/2),
\]
implying that $\chi(X^m, \mathcal{M}_K(X)) = (-1)^{mn}\chi(\Delta_K(X))$. Thus since $\chi(\Delta^k) = (\chi(X) - 1)^k + 1$ for $k \geq 1$, it follows from Theorem 1.1 that

$$\chi(X^m, \mathcal{M}_K(X)) = (-1)^{mn}\chi(X)(1 + \sum_{\emptyset \neq \sigma \in K} (\chi(X) - 1)^{||\sigma||}).$$

Therefore the proof is completed by the equality $\chi(X^m) = \chi(X^m, \mathcal{M}_K(X)) + \chi(\mathcal{M}_K(X))$. □

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