A Potapov-type approach to a truncated matricial Stieltjes-type power moment problem

B. Fritzsche  B. Kirstein  C. Mädler  T. Makarevich

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The paper gives a parametrization of the solution set of a matricial Stieltjes-type truncated power moment problem in the non-degenerate and degenerate cases. The key role plays the solution of the corresponding system of Potapov’s fundamental matrix inequalities.

Keywords: Stieltjes moment problem, Potapov’s fundamental matrix inequalities, Herglotz–Nevanlinna functions, Stieltjes functions

1. Introduction and preliminaries

The starting point of studying power moment problems on semi-infinite intervals was the famous two part memoir of T. J. Stieltjes [59, 60]. A complete theory of the treatment of power moment problems on semi-infinite intervals in the scalar case was developed by M. G. Krein in collaboration with A. A. Nudelman (see [51, Section 10], [52, Chapter V]). What concerns an operator-theoretic treatment of the power moment problems named after Hamburger and Stieltjes and its interrelations, we refer the reader to Simon [58].

In the 1970’s, V. P. Potapov developed a special approach to discuss matrix versions of classical interpolation and moment problems. The main idea of his method is based on transforming such problems into equivalent matrix inequalities with respect to the Löwner semi-ordering. Using this strategy, several matricial interpolation and moment problems could successfully be handled (see, e.g. [8, 9, 16, 17, 20, 22, 23, 25, 27, 37, 38, 42, 49, 53, 61]). L. A. Sakhnovich enriched Potapov’s method by unifying the particular instances of Potapov’s procedure under the framework of one type of operator identities [11, 40, 56].

Matrix versions of the classical Stieltjes moment problem were studied by Adamyan/Tkachenko [1, 2], Andô [5], Bolotnikov [7, 8, 10], Bolotnikov/Sakhnovich [11], Chen/Hu [14], Chen/Li [15], Dyukarev [21, 22], Dyukarev/Katsnelson [26, 27], and Hu/Chen [39]. The considerations of this paper deal with the more general case of an arbitrary semi-infinite interval [α, ∞), where α is an arbitrarily given real number.

In order to formulate the moment problem, we are going to study, we first review some notation. Throughout this paper, let p and q be positive integers. Let C, R, N₀, and N be the set of all complex numbers, the set of all real numbers, the set of all non-negative integers, and the set of all positive integers, respectively. For every choice of υ, ω ∈ R ∪ {−∞, ∞}, let Z_{υ,ω} be the set of all integers k for which υ ≤ k ≤ ω holds. If X is a non-empty set, then X^{p×q} stands
for the set of all \( p \times q \) matrices each entry of which belongs to \( X \), and \( X^p \) is short for \( X^{p \times 1} \).

If \((\Omega, \mathcal{A})\) is a measurable space, then each countably additive mapping whose domain is \( \mathcal{A} \) and whose values belong to the set \( \mathbb{C}^{q \times q} \) of all non-negative Hermitian complex \( q \times q \) matrices is called a non-negative Hermitian \( q \times q \) measure on \((\Omega, \mathcal{A})\). By \( \mathcal{M}^q_2(\Omega, \mathcal{A}) \) we denote the set of all non-negative Hermitian \( q \times q \) measures on \((\Omega, \mathcal{A})\). For the integration theory for non-negative Hermitian measures, we refer to \cite{11,55}.

If \( \mu = \[\mu_{jk}\]_{j,k=0}^q \) is a non-negative Hermitian \( q \times q \) measure on a measurable space \((\Omega, \mathcal{A})\) and if \( \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\} \), then we use \( \mathcal{L}^1(\Omega, \mathcal{A}, \mu; \mathbb{K}) \) to denote the set of all Borel-measurable functions \( f : \Omega \to \mathbb{K} \) for which the integral exists, i.e., that \( \int_{\Omega} f(\omega) \mu(\mathrm{d}\omega) < \infty \) for every choice of \( j \) and \( k \) in \( \mathbb{Z}_{1,q} \), where \( \mu_{jk} \) is the variation of the complex measure \( \mu_{jk} \). If \( f \in \mathcal{L}^1(\Omega, \mathcal{A}, \mu; \mathbb{K}) \), then let \( \int_A f \mu = \int_{\Omega} 1_A f(\omega) \mu(\mathrm{d}\omega) \) for all \( A \in \mathcal{A} \) and we will also write \( \int_A f(\omega) \mu(\mathrm{d}\omega) \) for this integral.

Let \( \mathcal{B}_\mathbb{R} \) (resp. \( \mathcal{B}_\mathbb{C} \)) be the \( \sigma \)-algebra of all Borel subsets of \( \mathbb{R} \) (resp. \( \mathbb{C} \)). For all \( \Omega \in \mathcal{B}_\mathbb{R} \setminus \{\emptyset\} \), let \( \mathcal{B}_\Omega \) be the \( \sigma \)-algebra of all Borel subsets of \( \Omega \), let \( \mathcal{M}^q_2(\Omega) := \mathcal{M}^q_2(\Omega, \mathcal{B}_\Omega) \) and, for all \( \kappa \in \mathbb{N}_0 \cup \{\infty\} \), let \( \mathcal{M}^q_{\geq \kappa}(\Omega) \) be the set of all \( \sigma \in \mathcal{M}^q_2(\Omega) \) such that for all \( j \in \mathbb{Z}_{0,\kappa} \) the function \( f_j : \Omega \to \mathbb{C} \) defined by \( f_j(t) := t^j \) belongs to \( \mathcal{L}^1(\Omega, \mathcal{B}_\Omega, \sigma; \mathbb{C}) \). If \( \kappa \in \mathbb{N}_0 \cup \{\infty\} \) and if \( \sigma \in \mathcal{M}^q_{\geq \kappa}(\Omega) \), then we set

\[
\mathcal{M}^q_{\geq \kappa}(\Omega) := \{ \sigma \in \mathcal{M}^q_2(\Omega) : \sigma_j \text{ is non-negative Hermitian for which which, in the case } m > 0, \text{ moreover } \sigma^{[m]}_j = s_j \text{ is fulfilled for all } j \in \mathbb{Z}_{0,m-1} \}.
\]

The following matricial power moment problem lies in the background of our considerations:

**Problem** MP\([\Omega; (s_j)^{m}_{j=0}, \leq]\)

Let \( \Omega \in \mathcal{B}_\mathbb{R} \setminus \{\emptyset\} \), let \( m \in \mathbb{N}_0 \), and let \( (s_j)^{m}_{j=0} \) be a sequence of complex \( q \times q \) matrices. Describe the set \( \mathcal{M}^q_{\geq \kappa}(\Omega; (s_j)^{m}_{j=0}, \leq) \) of all \( \sigma \in \mathcal{M}^q_{\geq \kappa}(\Omega) \) for which the matrix \( s_m - s^{[\sigma]}_m \) is non-negative Hermitian and for which, in the case \( m > 0 \), moreover \( s_j^{[\sigma]} = s_j \) is fulfilled for all \( j \in \mathbb{Z}_{0,m-1} \).

Note that we also sometimes turn our attention to the following power moment problem:

**Problem** MP\([\Omega; (s_j)^{\infty}_{j=0}, =]\)

Let \( \Omega \in \mathcal{B}_\mathbb{R} \setminus \{\emptyset\} \), let \( \kappa \in \mathbb{N}_0 \cup \{\infty\} \), and let \( (s_j)^{\kappa}_{j=0} \) be a sequence of complex \( q \times q \) matrices. Describe the set \( \mathcal{M}^q_{\geq \kappa}(\Omega; (s_j)^{\infty}_{j=0}, =) \) of all \( \sigma \in \mathcal{M}^q_{\geq \kappa}(\Omega) \) for which \( s_j^{[\sigma]} = s_j \) is fulfilled for all \( j \in \mathbb{Z}_{0,\kappa} \).

The considerations of this paper are mostly concentrated on the case that the set \( \Omega \) is a one-sided bounded and closed infinite interval of the real axis. Such moment problems are called to be of Stieltjes type.

The key role for solving the moment problem MP\([\alpha, \infty); (s_j)^{m}_{j=0}, \leq]\), where \( \alpha \) is an arbitrarily given real number, is Theorem \ref{thm6.18} below. It will turn out that the solution set of the moment problem (obtained via Stieltjes transformation) coincides with the solution set of a certain system of Potapov’s fundamental matrix inequalities. The considerations in this paper are aimed to solve these inequalities. In Section \ref{sec12} we give a parametrization of the solution set \( \mathcal{M}^q_{\geq \kappa}([\alpha, \infty); (s_j)^{2n+1}_{j=0}, \leq] \) of Problem MP\([\alpha, \infty); (s_j)^{2n+1}_{j=0}, \leq]\), where \( \alpha \) is an arbitrarily given real number and where \( n \) is an arbitrarily given non-negative integer. Note that Problem MP\([\alpha, \infty); (s_j)^{2n}_{j=0}, \leq]\) can be discussed by similar methods. This will be done somewhere else.

In Section \ref{sec2} we recall necessary and sufficient conditions of solvability of the moment problems in question. In Section \ref{sec3} we reformulate these problems in the language of certain
matrix-valued functions. Section 4 is aimed at showing that every solution of the moment problem fulfills necessarily the corresponding system of Potapov’s fundamental matrix inequalities. Some integral estimates for the scalar case are given in Section 5. In Section 6 we will prove that each solution of the system of Potapov’s fundamental matrix is a solution of the moment problem as well. Section 7 is aimed to give some identities for block Hankel matrices. In Section 8 we study special subspaces of \( \mathbb{C}^q \), so-called Dubovoj subspaces. Particular matrix polynomials in connection with a signature matrix are considered in Section 9. In Sections 10 and 11 we study distinguished classes of meromorphic functions, which occur as parameters in our description of the solution set, which is stated in Section 12.

At the end of this section, let us introduce some further notations, which are useful for our considerations. We will write \( I_q \) for the identity matrix in \( \mathbb{C}^{q \times q} \), whereas \( 0_{p \times q} \) is the null matrix belonging to \( \mathbb{C}^{p \times q} \). If the size of the identity matrix or the null matrix is obvious, then we will also omit the indexes. The notations \( C_{H}^{p \times q} \) and \( C_{\mathbb{Z}}^{p \times q} \) stand for the set of all Hermitian complex \( q \times q \) matrices and the set of all non-negative Hermitian complex matrices, respectively. If \( A \) and \( B \) are complex \( q \times q \) matrices, then we will write \( A \leq B \) or \( B \geq A \) to indicate that \( A \) and \( B \) are Hermitian matrices such that the matrix \( B - A \) is non-negative Hermitian. For each \( A \in \mathbb{C}^{p \times q} \), let \( \mathcal{N}(A) \) be the null space of \( A \), let \( \mathcal{R}(A) \) be the column space of \( A \), and let rank \( A \) be the rank of \( A \). For each \( A \in \mathbb{C}^{p \times q} \), we will use \( \mathfrak{R} A \) and \( \mathcal{A} A \) to denote the real part of \( A \) and the imaginary part of \( A \), respectively: \( \mathfrak{R} A := \frac{1}{2}(A + A^*) \) and \( \mathcal{A} A := \frac{1}{2}(A - A^*) \). Furthermore, for each \( A \in \mathbb{C}^{p \times q} \), let \( \| A \|_F \) be the Frobenius norm of \( A \) and let \( \| A \|_S \) be the operator norm of \( A \). For each \( x \in \mathbb{C}^q \), we write \( \| x \|_E \) for the Euclidean norm of \( x \). If \( A \in \mathbb{C}^{q \times q} \), then \( \det A \) stands for the determinant of \( A \).

For each complex \( p \times q \) matrix \( A \), let \( A\{1\} := \{ X \in \mathbb{C}^{q \times p} : AXA = A \} \). Obviously, for each \( A \in \mathbb{C}^{p \times q} \), the Moore–Penrose inverse \( A^\dagger \) of \( A \) belongs to \( A\{1\} \).

If \( n \in \mathbb{N} \), if \( (p_j)_{j=1}^n \) is a sequence of positive integers, and if \( x_j \in \mathbb{C}^{p_j \times q} \) for each \( j \in \mathbb{Z}_{1,n} \), then let \( \text{col}(x_j)_{j=1}^n := \begin{bmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{nk} \end{bmatrix} \). If \( n \in \mathbb{N} \), if \( (q_k)_{k=1}^n \) is a sequence of positive integers, and if \( y_k \in \mathbb{C}^{p \times q_k} \) for each \( k \in \mathbb{Z}_{1,n} \), then let \( \text{row}(y_k)_{k=1}^n := [y_1, y_2, \ldots, y_n] \). If \( n \in \mathbb{N} \), if \( (p_j)_{j=1}^n \) and \( (q_j)_{j=1}^n \) are sequences of positive integers, and if \( A_j \in \mathbb{C}^{p_j \times q_j} \) for every choice of \( j \) in \( \mathbb{Z}_{1,n} \), then let \( \text{diag}(A_1, A_2, \ldots, A_n) := [\delta_{jk} A_{jk}]_{j,k=1}^n \), where \( \delta_{jk} \) is the Kronecker delta: \( \delta_{jk} := 1 \) in the case \( j = k \) and \( \delta_{jk} := 0 \) if \( j \neq k \). We also use the notation \( \text{diag}(A_j)_{j=1}^n \) instead of \( \text{diag}(A_1, A_2, \ldots, A_n) \). For each \( n \in \mathbb{N} \) and each \( A \in \mathbb{C}^{p \times q} \), we will also write \( I_n \otimes A \) for \( \text{diag}(A_j)_{j=1}^n \).

If \( \mathcal{M} \) is a non-empty subset of \( \mathbb{C}^q \), then let \( \mathcal{M}^\perp \) be the set of all vectors in \( \mathbb{C}^q \) which are orthogonal to \( \mathcal{M} \) (with respect to the Euclidean inner product). If \( \mathcal{X} \), \( \mathcal{Y} \), and \( \mathcal{Z} \) are non-empty sets with \( \mathcal{Z} \subseteq \mathcal{X} \) and if \( f : \mathcal{X} \to \mathcal{Y} \) is a mapping, then \( \text{Rstr}_\mathcal{Z} f \) stands for the restriction of \( f \) onto \( \mathcal{Z} \).

Furthermore, let \( \Pi_+ := \{ z \in \mathbb{C} : \exists z \in (0, \infty) \} \) and let \( \Pi_- := \{ z \in \mathbb{C} : \exists z \in (-\infty, 0) \} \).

2. On the solvability of matricial power moment problems

In this section, we recall necessary and sufficient conditions for the solvability of the Stieltjes moment problems \( \text{MP}[\alpha, \infty); (s_j)_{j=0}^m] \) and \( \text{MP}[\alpha, \infty); (s_j)_{j=0}^m] \), where \( \alpha \) is an arbitrarily given real number and where \( m \) is an arbitrarily given non-negative integer. First we introduce certain sets of sequences of complex \( q \times q \) matrices, which are determined by the properties
of particular block Hankel matrices built of them. For each \( n \in \mathbb{N}_0 \), let \( \mathcal{H}_{q,2n}^\geq \) be the set of all sequences \((s_j)_{j=0}^{2n}\) of complex \( q \times q \) matrices such that the block Hankel matrix \( H_n := [s_{j+k}]_{j,k=0}^{n} \) is non-negative Hermitian. Furthermore, let \( \mathcal{H}_{q,\infty}^\geq \) be the set of all sequences \((s_j)_{j=0}^\infty\) of complex \( q \times q \) matrices such that, for all \( n \in \mathbb{N}_0 \), the sequence \((s_j)_{j=0}^{2n}\) belongs to \( \mathcal{H}_{q,2n}^\geq \). The elements of the set \( \mathcal{H}_{q,2n}^\geq \), where \( \kappa \in \mathbb{N}_0 \cup \{ \infty \} \), are called **Hankel non-negative definite sequences**. For all \( n \in \mathbb{N}_0 \), let \( \mathcal{H}_{q,2n}^{\geq_{e}} \) be the set of all sequences \((s_j)_{j=0}^{2n}\) of complex \( q \times q \) matrices for which there are matrices \( s_{2n+1} \in \mathbb{C}^{q \times q} \) and \( s_{2n+2} \in \mathbb{C}^{q \times q} \) such that \((s_j)_{j=0}^{2(n+1)}\) belongs to \( \mathcal{H}_{q,2(n+1)}^\geq \). Furthermore, for all \( n \in \mathbb{N}_0 \), we will use \( \mathcal{H}_{q,2n+1}^{\geq_{e}} \) to denote the set of sequences \((s_j)_{j=0}^{2n+1}\) of complex \( q \times q \) matrices for which there is some \( s_{2n+2} \in \mathbb{C}^{q \times q} \) such that \((s_j)_{j=0}^{2(n+1)}\) belongs to \( \mathcal{H}_{q,2(n+1)}^\geq \). For all \( n \in \mathbb{N}_0 \), the elements of the set \( \mathcal{H}_{q,2n}^{\geq_{e}} \), are called **Hankel non-negative definite extendable sequences**. For technical reasons, we set \( \mathcal{H}_{q,\infty}^{\geq_{e}} := \mathcal{H}_{q,\infty}^\geq \). Observe that the solvability of the matricial Hamburger moment problems can be characterized by the introduced classes of sequences of complex \( q \times q \) matrices:

**Theorem 2.1** (see, e.g. [13] Theorem 3.2 or [25] Theorem 4.16). Let \( n \in \mathbb{N}_0 \) and let \((s_j)_{j=0}^{2n}\) be a sequence of complex \( q \times q \) matrices. Then \( \mathcal{M}_q^\geq [\mathbb{R}; (s_j)_{j=0}^{2n}, \leq] \neq \emptyset \) if and only if \((s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^\geq \).

**Theorem 2.2** (see [25] Theorem 4.17, [31] Theorem 6.6). Let \( \kappa \in \mathbb{N}_0 \cup \{ \infty \} \) and let \((s_j)_{j=0}^{\kappa}\) be a sequence of complex \( q \times q \) matrices. Then \( \mathcal{M}_q^\geq [\mathbb{R}; (s_j)_{j=0}^{\kappa}, \leq] \neq \emptyset \) if and only if \((s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\geq_{e}} \).

Let \( \alpha \in \mathbb{C} \), let \( \kappa \in \mathbb{N}_0 \cup \{ \infty \} \), and let \((s_j)_{j=0}^{\kappa}\) be a sequence of complex \( p \times q \) matrices. Then let the sequence \((s_{\alpha,j})_{j=0}^{\kappa-1}\) be defined by

\[
s_{\alpha,j} := -\alpha s_j + s_{j+1} \quad \text{for all } j \in \mathbb{Z}_{0,\kappa-1}.
\]

The sequence \((s_{\alpha,j})_{j=0}^{\kappa-1}\) is called the sequence generated from \((s_j)_{j=0}^{\kappa}\) by right-sided \( \alpha \)-shifting. (An analogous left-sided version is discussed in [32] Definition 2.1.) The sequence \((s_{\alpha,j})_{j=0}^{\kappa-1}\) is used to define further sets of sequences of complex matrices, which are useful to discuss the Stieltjes moment problems we consider. Let \( \mathcal{K}_{q,0,\alpha}^{\geq_{e}} := \mathcal{H}_{q,0}^{\geq_{e}} \). For every choice of \( n \in \mathbb{N} \), let \( \kappa_{q,2n,\alpha}^{\geq_{e}} := \{ (s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq_{e}} ; (s_{\alpha,j})_{j=0}^{2(n-1)} \in \mathcal{H}_{q,2(n-1)}^{\geq_{e}} \} \). For all \( m \in \mathbb{N}_0 \), by \( \mathcal{S}_m (\mathbb{C}^{q \times q}) \) we denote the set of all sequences \((s_j)_{j=0}^{m}\) of complex \( q \times q \) matrices. Then we set \( \kappa_{q,2n+1,\alpha}^{\geq_{e}} := \{ (s_j)_{j=0}^{2n+1} \in \mathcal{S}_{2n+1} (\mathbb{C}^{q \times q}) ; \{(s_j)_{j=0}^{2n}, (s_{\alpha,j})_{j=0}^{2n}\} \subseteq \mathcal{H}_{q,2n}^{\geq_{e}} \} \). For all \( m \in \mathbb{N}_0 \), let \( \mathcal{K}_{q,m,\alpha}^{\geq_{e}} \) be the set of all sequences \((s_j)_{j=0}^{m}\) of complex \( q \times q \) matrices for which there exists a complex \( q \times q \) matrix \( s_{m+1} \) such that \((s_j)_{j=0}^{m+1}\) belongs to \( \mathcal{K}_{q,m+1,\alpha}^{\geq_{e}} \). Obviously, we have \( \kappa_{q,2n,\alpha}^{\geq_{e}} = \{ (s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq_{e}} ; (s_{\alpha,j})_{j=0}^{2n-1} \in \mathcal{H}_{q,2n-1}^{\geq_{e}} \} \) for all \( n \in \mathbb{N} \) and \( \kappa_{q,2n+1,\alpha}^{\geq_{e}} = \{ (s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q,2n+1}^{\geq_{e}} ; (s_{\alpha,j})_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq_{e}} \} \) for all \( n \in \mathbb{N}_0 \).

**Remark 2.3.** Let \( \alpha \in \mathbb{R} \) and let \( m \in \mathbb{N}_0 \). Then \( \kappa_{q,m,\alpha}^{\geq_{e}} \subseteq \mathcal{K}_{q,m,\alpha}^{\geq_{e}} \). Furthermore, if \((s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\geq_{e}} \) (resp. \( \mathcal{K}_{q,m,\alpha}^{\geq_{e}} \)), then we easily see that \((s_j)_{j=0}^{\ell} \in \mathcal{K}_{q,\ell,\alpha}^{\geq_{e}} \) (resp. \((s_j)_{j=0}^{\ell} \in \mathcal{K}_{q,\ell,\alpha}^{\geq_{e}}\)) holds true for all \( \ell \in \mathbb{Z}_{0,m} \).

In view of Remark 2.3, for all \( \alpha \in \mathbb{R} \), \( \mathcal{K}_{q,\infty,\alpha}^{\geq_{e}} \) be the set of all sequences \((s_j)_{j=0}^{\infty}\) of complex \( q \times q \) matrices such that \((s_j)_{j=0}^{m} \in \kappa_{q,m,\alpha}^{\geq_{e}} \) for all \( m \in \mathbb{N}_0 \), and let \( \mathcal{K}_{q,\infty,\alpha}^{\geq_{e}} := \mathcal{K}_{q,\infty,\alpha}^{\geq_{e}} \). For all \( \kappa \in \mathbb{N}_0 \cup \{ \infty \} \), we call a sequence \((s_j)_{j=0}^{\infty} (\alpha, \infty)\)-Stieltjes right-sided non-negative definite.
Then there is a unique sequence \( \alpha, (\text{resp. } \alpha, q) \) complex matrices. Let \( \alpha, (\text{resp. } \alpha, q) \) be a sequence of complex \( q \times q \) matrices. Then \( \mathcal{M}_2^q[\alpha, \infty); (s_j)_{j=0}^m, \leq] \) and \( \mathcal{M}_2^q[\alpha, \infty); (s_j)_{j=0}^m, =: \)

**Theorem 2.4** ([24, Theorem 1.4]). Let \( \alpha, (\text{resp. } \alpha, q) \) be a sequence of complex \( q \times q \) matrices. Then \( \mathcal{M}_2^q[\alpha, \infty); (s_j)_{j=0}^m, \leq] \) if and only if \( (s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^\ep \).

**Theorem 2.5.** Let \( \alpha, \kappa, \mu, \nu, (\text{resp. } \alpha, \kappa, \mu, \nu) \) be a sequence of complex \( q \times q \) matrices. Then \( \mathcal{M}_2^q[\alpha, \infty); (s_j)_{j=0}^m, \leq] \) if and only if \( (s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^\ep \).

In the case \( \kappa, \mu, \nu, (\text{resp. } \kappa, \mu, \nu) \) of these notions are used in [32, Definition 1.3]. Note that left versions of these notions are used in [33, 36]. In particular, the functions \( \mu \) belong to \( \mathcal{M}_2^q \) and which are holomorphic in \( \Pi^+ \) and which satisfy \( \mathcal{G}(F(\Pi^+)) \subseteq C^q_{\geq} \). Detailed considerations of matrix-valued Herglotz–Nevanlinna functions in the upper half-plane \( \Pi^+ \) can be found in [33, 36]. In particular, the functions belonging to \( \mathcal{R}_q(\Pi^+) \) admit a well-known integral representation:

**Theorem 3.1.** (a) For each \( F \in \mathcal{R}_q(\Pi^+) \), there exist unique matrices \( A \in C^q_{H,q} \) and \( B \in C^q_{\geq} \) and a unique non-negative Hermitian measure \( \nu \in \mathcal{M}_2^q(\mathbb{R}) \) such that

\[
F(z) = A + zB + \int_{\mathbb{R}} \frac{1 + tz}{t - z} \nu(dt) \quad \text{for each } z \in \Pi^+. \tag{3.1}
\]

(b) If \( A \in C^q_{H,q} \), if \( B \in C^q_{\geq} \), and if \( \nu \in \mathcal{M}_2^q(\mathbb{R}) \), then \( F: \Pi^+ \to C^q_{\geq} \) defined by (3.1) belongs to \( \mathcal{R}_q(\Pi^+) \).

For each \( F \in \mathcal{R}_q(\Pi^+) \), the unique triple \( (A, B, \nu) \in C^q_{H,q} \times C^q_{\geq} \times \mathcal{M}_2^q(\mathbb{R}) \) for which the representation (3.1) holds true is called the Nevanlinna parametrization of \( F \) and we will also write \( (A_F, B_F, \nu_F) \) for \( (A, B, \nu) \). In particular, \( \nu_F \) is said to be the Nevanlinna measure of \( F \). If \( F \) belongs to \( \mathcal{R}_1(\Pi^+) \), then \( \mu_F: \mathcal{B}_\mathbb{R} \to [0, \infty] \) defined by

\[
\mu_F(B) := \int_B (1 + t^2) \nu_F(dt) \quad \text{for all } B \in \mathcal{B}_\mathbb{R} \tag{3.2}
\]
is a measure, which is called the spectral measure of $F$. By $R'_q(\Pi_+)$ we denote the set of all $F \in R_q(\Pi_+)$ for which $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) := 1 + t^2$ belongs to $L^1(\mathbb{R}, \mathcal{B}_R, \nu_F; \mathbb{R})$. Obviously, $R'_q(\Pi_+) = \{ F \in R_q(\Pi_+) : \nu_F \in M_{q2}^1(\mathbb{R}) \}$. If $F$ belongs to $R'_q(\Pi_+)$, then $\mu_F : \mathcal{B}_R \to \mathbb{C}_{\geq q}$ given by (3.2) is a well-defined non-negative Hermitian $q \times q$ measure belonging to $M_{q}^{2}(\mathbb{R})$, which is said to be the matricial spectral measure of $F$. Obviously, for functions which belong to $R'_1(\Pi_+)$, the notions spectral measure and matricial spectral measure coincide.

For our considerations, the class $R'_{0,q}(\Pi_+)$ of all $F \in R_q(\Pi_+)$ for which

$$\sup_{y \in [1, \infty]} y\|F(iy)\|_{S} < \infty$$

holds true plays an essential role. The class $R'_{0,q}(\Pi_+)$ is a subclass of $R'_q(\Pi_+)$ (see, e.g., [33, Lemma 6.1]). Furthermore, the functions belonging to $R'_{0,q}(\Pi_+)$ admit a particular integral representation:

**Theorem 3.2.** (a) For each $F \in R'_{0,q}(\Pi_+)$, there is a unique $\mu \in M_{q}^{2}(\mathbb{R})$ such that

$$F(z) = \int_{\mathbb{R}} \frac{1}{t - z} \mu(dt)$$

for each $z \in \Pi_+$,

namely the matricial spectral measure of $F$, and

$$\mu(\mathbb{R}) = \lim_{y \to \infty} (y\Im[F(iy)]) = -i \lim_{y \to \infty} [yF(iy)] = i \lim_{y \to \infty} [yF^*(iy)].$$

(b) If $F : \Pi_+ \to \mathbb{C}^{q \times q}$ is a matrix-valued function for which there exists a non-negative Hermitian measure $\mu \in M_{q}^{2}(\mathbb{R})$ such that (3.4) holds true, then $F$ belongs to $R'_{0,q}(\Pi_+)$. A proof of Theorem 3.2 is given, e.g., in [17, Theorem 8.7]. If $F \in R'_{0,q}(\Pi_+)$, then the unique $\mu \in M_{q}^{2}(\mathbb{R})$ for which (3.4) holds true is also called the Stieltjes measure of $F$. If a non-negative Hermitian $q \times q$ measure $\mu \in M_{q}^{2}(\mathbb{R})$ is given, then $F : \Pi_+ \to \mathbb{C}^{q \times q}$ defined by (3.4) is said to be the Stieltjes transform of $\mu$.

**Lemma 3.3.** Let $M \in \mathbb{C}^{q \times q}$ and let $F : \Pi_+ \to \mathbb{C}^{q \times q}$ be a matrix-valued function which is holomorphic in $\Pi_+$ and which satisfies the inequality

$$\begin{bmatrix} M & F(z) \\ F^*(z) & F(z) - F^*(z) \end{bmatrix} \geq 0$$

for each $z \in \Pi_+$. Then $F$ belongs to $R'_{0,q}(\Pi_+)$ and the inequality $\sup_{y \in (0, \infty)} y\|F(iy)\|_{S} \leq \|M\|_{S}$ holds true. Furthermore, the Stieltjes measure $\mu$ of $F$ fulfills $\mu(\mathbb{R}) \leq M$.

A proof of Lemma 3.3 is given, e.g., in [17, Lemma 8.9].

In view of the Stieltjes moment problem, a further class of matrix-valued functions plays a key role: For each $\alpha \in \mathbb{R}$, let $\mathcal{S}_{q;[\alpha, \infty)}$ be the set of all matrix-valued functions $S : \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q \times q}$ which are holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$ and which satisfy $\Im[S(\Pi_+)] \subseteq C_{\geq q}$ as well as $S([-\infty, \alpha)) \subseteq C_{\geq q}$. In [33, Theorems 3.1 and 3.6, Proposition 2.16], integral representations of functions belonging to $\mathcal{S}_{q;[\alpha, \infty)}$ are proved. Furthermore, several characterizations of the class $\mathcal{S}_{q;[\alpha, \infty)}$ are given in [33, Section 4].

For each $\alpha \in \mathbb{R}$, let $\mathcal{S}_{0,q;[\alpha, \infty)}$ be the class of all $F \in \mathcal{S}_{q;[\alpha, \infty)}$ which satisfy (3.3). The functions belonging to $\mathcal{S}_{0,q;[\alpha, \infty)}$ admit a particular integral representation. Before we state this, let us note the following:
Remark 3.4. For every choice of \( \alpha \in \mathbb{R} \) and \( z \in \mathbb{C} \setminus [\alpha, \infty) \), the function \( b_{\alpha,z} : [\alpha, \infty) \to \mathbb{C} \) given by \( b_{\alpha,z}(t) := 1/(t-z) \) is a bounded and continuous function which, in particular, belongs to \( \mathcal{L}_t^1([\alpha, \infty), \mathcal{B}_{[\alpha, \infty)}, \sigma; \mathbb{C}) \) for all \( \sigma \in \mathcal{M}_\leq^p([\alpha, \infty)) \).

Theorem 3.5 (\cite{35} Theorem 5.1). Let \( \alpha \in \mathbb{R} \).

(a) If \( S \in \mathcal{S}_{0,q;[\alpha,\infty)} \), then there is a unique \( \sigma \in \mathcal{M}_\leq^q([\alpha, \infty)) \) such that
\[
S(z) = \int_{[\alpha, \infty)} \frac{1}{t-z} \sigma(dt) \quad \text{for each } z \in \mathbb{C} \setminus [\alpha, \infty).
\]

(b) If \( \sigma \in \mathcal{M}_\leq^q([\alpha, \infty)) \) is such that \( S : \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C} \times q \) can be represented via (3.5), then \( S \) belongs to \( \mathcal{S}_{0,q;[\alpha,\infty)} \).

If \( F \in \mathcal{S}_{0,q;[\alpha,\infty)} \) is given, then the unique \( \sigma \in \mathcal{M}_\leq^q([\alpha, \infty)) \) which fulfills the representation (3.5) of \( F \) is called the \( F \). If \( \sigma \in \mathcal{M}_\leq^q([\alpha, \infty)) \) is given, then \( F : \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C} \times q \) defined by (3.5) is said to be the \([\alpha, \infty)\)-Stieltjes transform of \( \sigma \). In view of Theorem 3.5, the moment problems \( \mathcal{M}_p([\alpha, \infty); (s_j)_{j=0}^\infty, \leq] \) and \( \mathcal{M}_p([\alpha, \infty); (s_j)_{j=0}^\infty, =] \) admit reformulations in the language of \([\alpha, \infty)\)-Stieltjes transforms:

Problem \( S([\alpha, \infty); (s_j)_{j=0}^m, \leq] \)
Let \( \alpha \in \mathbb{R} \), let \( m \in \mathbb{N}_0 \), and let \( (s_j)_{j=0}^m \) be a sequence of complex \( q \times q \) matrices. Describe the set \( \mathcal{S}_{0,q;[\alpha,\infty)}((s_j)_{j=0}^m, \leq] \) of all \( F \in \mathcal{S}_{0,q;[\alpha,\infty)} \) the \([\alpha, \infty)\)-Stieltjes measure of which belongs to \( \mathcal{M}_\leq^q([\alpha, \infty); (s_j)_{j=0}^m, \leq] \).

Problem \( S([\alpha, \infty); (s_j)_{j=0}^\kappa, ] \)
Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N}_0 \cup \{ \infty \} \), and let \( (s_j)_{j=0}^\kappa \) be a sequence of complex \( q \times q \) matrices. Describe the set \( \mathcal{S}_{0,q;[\alpha,\infty)}((s_j)_{j=0}^\kappa, ] \) of all \( F \in \mathcal{S}_{0,q;[\alpha,\infty)} \) the \([\alpha, \infty)\)-Stieltjes measure of which belongs to \( \mathcal{M}_\leq^q([\alpha, \infty); (s_j)_{j=0}^\kappa, ] \).

Remark 3.6. Let \( \alpha \in \mathbb{R} \) and let \( F \in \mathcal{S}_{0,q;[\alpha,\infty)} \). Then \( \square := \text{Rstr}_{\Pi,+} F \) belongs to \( \mathcal{R}_{0,q}^p(\Pi_+) \), the matricial spectral measure \( \mu_\square \) of \( \square \) fulfills \( \mu_\square((-\infty, \alpha)) = 0 \), and \( \sigma := \text{Rstr}_{\Pi,+} \mu_\square \) is exactly the \([\alpha, \infty)\)-Stieltjes measure of \( F \) (see \cite{35} Proposition 2.16).

At the end of this section, we state two results, which are essential to discuss the so-called degenerate case.

Proposition 3.7 (cf. \cite{35} Proposition 5.3)). Let \( \alpha \in \mathbb{R} \) and let \( F \in \mathcal{S}_{0,q;[\alpha,\infty)} \). Then the \([\alpha, \infty)\)-Stieltjes measure \( \sigma \) of \( F \) fulfills \( \sigma([\alpha, \infty)) = -\lim_{y \to -\infty} y F(iy) \) and, for each \( z \in \mathbb{C} \setminus [\alpha, \infty) \), furthermore \( \mathcal{R}(F(z)) = \mathcal{R}(\sigma([\alpha, \infty])) \) and \( \mathcal{N}(F(z)) = \mathcal{N}(\sigma([\alpha, \infty])) \).

If \( \mathcal{G} \) is a region of \( \mathbb{C} \), i.e., a non-empty open connected subset of \( \mathbb{C} \), then a matrix-valued function \( S : \mathcal{G} \to \mathbb{C} \times q \) is called \( p \times q \) Schur function in \( \mathcal{G} \) if \( S \) is both holomorphic and contractive in \( \mathcal{G} \). The set of all \( p \times q \) Schur functions in a region \( \mathcal{G} \) of \( \mathbb{C} \) will be denoted by \( \mathcal{S}_{p,q}(\mathcal{G}) \).

Remark 3.8. Let \( \mathcal{G} \) be a region of \( \mathbb{C} \) and let \( S \in \mathcal{S}_{q \times p}(\mathcal{G}) \). Then \( \mathcal{G}^\vee := \{ z \in \mathbb{C} : \pi \in \mathcal{G} \} \) is a region of \( \mathbb{C} \) and \( \mathcal{S}^\vee : \mathcal{G}^\vee \to \mathbb{C} \times p \) given by \( \mathcal{S}^\vee(z) := [S(\pi)]^* \) belongs to \( \mathcal{S}_{q \times p}(\mathcal{G}^\vee) \).

Lemma 3.9. Let \( \mathcal{G} \) be a region of \( \mathbb{C} \) and let \( S \in \mathcal{S}_{q \times p}(\mathcal{G}) \). Then:

(a) If \( U \in \mathbb{C} \times q \) fulfills \( U^* U = I_q \), then \( \mathcal{N}(U + S(w)) = \mathcal{N}(U + S(z)) \) for every choice of \( w \) and \( z \) in \( \mathcal{G} \).
(b) If $V \in \mathbb{C}^{p \times q}$ fulfills $VV^* = I_p$, then $\mathcal{R}(V + S(w)) = \mathcal{R}(V + S(z))$ for every choice of $w$ and $z$ in $\mathcal{G}$.

**Proof.** Let $U \in \mathbb{C}^{p \times q}$ be such that $U^*U = I_p$, let $w \in \mathcal{G}$, and let $z \in \mathcal{G}$. We consider an arbitrary $v \in \mathcal{N}(U+S(w)) \setminus \{0\}$. Then $S(w)v = -Uv$. Consequently, $\|S(w)v\|_E^2 = \|Uv\|_E^2 = v^*U^*Uv = v^*v = \|v\|_E^2$. According to [20] Lemma 2.1.2, p. 61, this implies $S(w)v = S(z)v$. Hence, $[U + S(z)]v = Uv + S(w)v = 0_{p \times 1}$. Thus, $v \in \mathcal{N}(U + S(z))$. Therefore, $\mathcal{N}(U + S(w)) \subseteq \mathcal{N}(U + S(z))$ is proved. For reasons of symmetry, we also have $\mathcal{N}(U + S(z)) \subseteq \mathcal{N}(U + S(w))$. Part (b) is checked.

In order to prove part (a), we first apply Remark 3.3. Thus, we see that $S^\gamma$ belongs to $\mathcal{S}_{q \times p}(\mathbb{G}^\gamma)$. Consequently, part (a) yields

$$\mathcal{N}([V + S(w)]^*) = \mathcal{N}(V^* + S^\gamma(\overline{w})) = \mathcal{N}(V^* + S(\overline{v})) = \mathcal{N}([V + S(z)]^*)$$

for every choice of $w$ and $z$ in $\mathcal{G}$. In view of Remark 4.1, the proof is complete. $\square$

### 4. From the Stieltjes moment problem to the system of Potapov’s fundamental inequalities

In this section, we introduce the system of Potapov’s fundamental matrices corresponding to the matricial Stieltjes moment problem $\text{MP}[\alpha, \infty); (s_j)^n_{j=0}, \leq]$. We will see that each solution of this moment problem fulfills necessarily the system of Potapov’s fundamental matrix inequalities. First we are going to introduce further notations and, in particular, several block Hankel matrices which will play a key role in our considerations. For technical reason, let $s_{-1} := 0_{p \times q}$.

Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)^n_{j=0}$ be a sequence of complex $p \times q$ matrices. For each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let $H_n := [s_{j+k}]^n_{j,k=0}$, for each $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, let $K_n := [s_{j+k+1}]^n_{j,k=0}$, and, for each $n \in \mathbb{N}_0$ with $2n + 2 \leq \kappa$, let $G_n := [s_{j+k+2}]^n_{j,k=0}$. If $m$ and $n$ are integers such that $-1 \leq m \leq n \leq \kappa$, then we set

$$y_{m,n} := \text{col}(s_j)^n_{j=m} \quad \text{and} \quad z_{m,n} := \text{row}(s_k)^n_{k=m}. \quad (4.1)$$

Let $u_0 := 0_{p \times q}$, $w_0 := 0_{p \times q}$, and $w_0 := 0_{p \times q}$. For all $n \in \mathbb{N}$ with $n \leq \kappa + 1$, let $u_n := -y_{-1,n-1}$, and $w_n := z_{-1,n-1}$. Further, for each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let $u_n := [-y_{n+1,2n}]_{0 \times p q}$ and $w_n := [z_{n+1,2n}, 0_{p \times q}]$. If a real number $\alpha$ is additionally given, then we continue to use the notation given by (2.1), and we set $H_{\alpha n} := [s_{\alpha j+k}]^n_{j,k=0}$ for each $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$.

**Remark 4.1.** Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)^n_{j=0}$ be a sequence of complex $q \times q$ matrices. If $n \in \mathbb{N}_0$ is such that $2n \leq \kappa$, then $H_n \in \mathbb{C}^{(n+1)q \times (n+1)q}$ if and only if there is a $j \in \mathbb{Z}_{0,2n}$ such that $s_{j} \in \mathbb{C}^{q \times q}$. Furthermore, if $\alpha \in \mathbb{R}$, if $\kappa \geq 1$, and if $n \in \mathbb{N}_0$ is such that $2n + 1 \leq \kappa$, then $H_n, H_{\alpha n} \subseteq \mathbb{C}^{(n+1)q \times (n+1)q}$ if and only if there is a $j \in \mathbb{Z}_{0,2n+1}$ such that $s_j \in \mathbb{C}^{q \times q}$.

**Remark 4.2.** Let $n \in \mathbb{N}$ and let $(s_j)^{2n}_{j=0}$ be a sequence of complex $p \times q$ matrices. Then the block Hankel matrix $H_n$ admits the block representations

$$H_n = \begin{bmatrix} H_{n-1} & y_{n,2n-1} & 0_{2n-1} \\ z_{2n-1,n-1} & y_{n,2n-1} & 0_{2n-1} \\ s_{2n} & y_{n,2n-1} & 0_{2n-1} \end{bmatrix},$$

and

$$H_n = \begin{bmatrix} s_0 & y_{1,2n-1} & 0_{2n-1} \\ z_{0,2n-1} & y_{1,2n-1} & 0_{2n-1} \\ K_{n-1} & y_{1,2n-1} & 0_{2n-1} \end{bmatrix}, \quad (4.2)$$

and

$$H_n = \begin{bmatrix} s_0 & z_{1,n} \\ y_{1,n} & G_{n-1} \\ K_{n-1} & y_{1,n} \end{bmatrix}.$$
For each $n \in \mathbb{N}_0$, we set

$$T_{q,n} := \left[ \delta_{j,k+1} I_{q,j,k=0} \right]_n^0, \quad \nu_{q,n} := \text{col}(\delta_{j,0} I_{q})_j^0, \quad \text{and} \quad \nu_{q,n} := \text{col}(\delta_{n-j,0} I_{q})_j^0,$$

where $\delta_{j,k}$ is again the Kronecker delta. Obviously, $T_{q,n}^* = \left[ \delta_{j+1,k} I_{q,j,k=0} \right]_n^0$ for each $n \in \mathbb{N}_0$.

It seems to be useful to recall well-known Ljapunov identities for block Hankel matrices.

(These equations can be also easily proved by straightforward calculation.)

**Remark 4.3.** Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\infty$ be a sequence of complex $p \times q$ matrices.

(a) For each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, then $H_{p,n} T_{q,n} - T_{p,n} H_n = u_n v_{q,n}^* - v_{q,n} u_n$ and $H_{p,n} T_{q,n} - T_{p,n} H_n = u_n v_{q,n}^* - v_{q,n} u_n$. In particular, if $p = q$ and if $s_j = s_j$ for each $j \in \mathbb{Z}_0 \kappa$, then $H_{q,n} T_{q,n} - T_{q,n} H_n = u_n v_{q,n}^* - v_{q,n} u_n$ for each $n \in \mathbb{N}_0$ with $2n \leq \kappa$.

(b) For each $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, we have $H_{p,n} v_{q,n} = -\alpha H_n + K_n$, $v_{p,n} v_{p,n}^* H_n = \left[ R_{T_{p,n}}(\alpha) \right]^{-1} H_n - T_{p,n} H_{p,n}$, and, in the case that $p = q$ and if $s_j = s_j$ for each $j \in \mathbb{Z}_0 \kappa$, hold true, moreover $H_{\alpha,n} T_{q,n} - T_{q,n} H_{\alpha,n} = (\alpha u_n - y_0,n) v_{q,n}^* - v_{q,n} (-\alpha u_n - y_0,n) v_{q,n}^*$ for each $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$.

(c) For each $n \in \mathbb{N}_0$ with $2n + 2 \leq \kappa$, we have $H_{\alpha,n} T_{q,n} - T_{q,n} H_{\alpha,n} = (\alpha u_n - y_0,n) v_{q,n}^* - v_{q,n} (-\alpha u_n - y_0,n) v_{q,n}^*$ and, in particular, if $p = q$ and if $s_j = s_j$ for each $j \in \mathbb{Z}_0 \kappa$, then $H_{\alpha,n} T_{q,n} - T_{q,n} H_{\alpha,n} = (\alpha u_n - y_0,n) v_{q,n}^* - v_{q,n} (-\alpha u_n - y_0,n) v_{q,n}^*$ for each $n \in \mathbb{N}_0$ with $2n + 2 \leq \kappa$.

(d) The equations $H_n v_{q,n} = y_0,n$ and $-T_{p,n} H_n v_{q,n} = u_n$ hold true for each $n \in \mathbb{N}_0$ with $2n \leq \kappa$.

**Remark 4.4.** For each $n \in \mathbb{N}_0$, the matrix-valued functions $R_{T_{q,n}} : \mathbb{C} \to \mathbb{C}^{(n+1)q \times (n+1)q}$ and $R_{T_{q,n}}^* : \mathbb{C} \to \mathbb{C}^{(n+1)q \times (n+1)q}$ given by

$$R_{T_{q,n}}(z) := (I_{n+1})_q - z T_{q,n}^{-1} \quad \text{and} \quad R_{T_{q,n}}^*(z) := (I_{n+1})_q - z T_{q,n}^{-1}$$

are well-defined matrix polynomials of degree $n$, which can be represented, for each $z \in \mathbb{C}$, via $R_{T_{q,n}}(z) = \sum_{j=0}^n z^j T_{q,n}^j$ and $R_{T_{q,n}}^*(z) = \sum_{j=0}^n z^j (T_{q,n})^*$, respectively. In particular, $R_{T_{q,n}}^*(z) = [R_{T_{q,n}}(z)]^*$ for all $z \in \mathbb{C}$.

For each $n \in \mathbb{N}_0$, let $E_{q,n} : \mathbb{C} \to \mathbb{C}^{(n+1)q \times q}$ and $F_{q,n} : \mathbb{C} \to \mathbb{C}^{(n+1)q \times q}$ be defined by

$$E_{q,n}(z) := \text{col}(z^j I_{q})_j^0 \quad \text{and} \quad F_{q,n}(z) := z E_{q,n}(z),$$

respectively. Obviously, for each $n \in \mathbb{N}_0$ and each $z \in \mathbb{C}$, we have $R_{T_{q,n}}(z) v_{q,n} = E_{q,n}(z)$.

**Notation 4.5.** Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\infty$ be a sequence of complex $q \times q$ matrices. Further, let $\mathcal{G}$ be a subset of $\mathbb{C}$ with $\mathcal{G} \setminus \mathbb{R} \neq \emptyset$ and let $f : \mathcal{G} \to \mathbb{C}^{q \times q}$ be a matrix-valued function. Then, for each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let $P_{2n}^{[f]} : \mathcal{G} \setminus \mathbb{R} \to \mathbb{C}^{(n+2)q \times (n+2)q}$ be defined by

$$P_{2n}^{[f]}(z) := \left[ H_n (R_{T_{q,n}}(z) [v_{q,n} f(z) - u_n])^* \frac{R_{T_{q,n}}(z) [v_{q,n} f(z) - u_n]^*}{z-\alpha} (f(z) - \alpha^* f(z)) \right].$$
Remark 4.6. Then Remark A.4 shows that the following statements hold true:

If \( \kappa \geq 1 \), then, for each \( n \in \mathbb{N}_0 \) with \( 2n + 1 \leq \kappa \), let \( P^{[f]}_{2n+1} : \mathcal{G} \setminus \mathbb{R} \to \mathbb{C}^{(n+2)q \times (n+2)q} \) be given by

\[
P^{[f]}_{2n+1}(z) = \begin{bmatrix}
H_{\alpha \beta n} & R_{T_{q,n}}(z)(v_{q,n}([z - \alpha]f(z)] - (-\alpha u_n - y_{0,n})) \\
[R_{T_{q,n}}(z)(v_{q,n}([z - \alpha]f(z)] - (-\alpha u_n - y_{0,n}))]^* & \frac{\left(-\alpha u_n - y_{0,n}\right)(z - \alpha)f(z) - ([z - \alpha]f(z)]^*)}{z - \mathfrak{z}}
\end{bmatrix}.
\] (4.5)

Furthermore, let \( P^{[f]}_{-1} : \mathcal{G} \setminus \mathbb{R} \to \mathbb{C}^{q \times q} \) be defined by

\[
P^{[f]}_{-1}(z) := \frac{(z - \alpha)f(z) - ([z - \alpha]f(z)]^*)}{z - \mathfrak{z}}.
\]

With respect to the Stieltjes moment problem \( \text{MP}[[\alpha, \infty); (s_j)^\alpha_{j=0, \leq}] \) if \( \mathcal{G} = \mathbb{C} \), then the functions (4.4) and (4.5) are called the Potapov fundamental matrix-valued functions connected to the Stieltjes moment problem (generated by \( f \)). If these matrices are both non-negative Hermitian, then one says that the Potapov’s fundamental matrix inequalities for the function \( f \) are fulfilled.

Remark 4.6. Let \( \kappa \in \mathbb{N}_0 \cup \{\infty\} \) and let \( (s_j)_{j=0}^\alpha \) be a sequence from \( \mathbb{C}^{q \times q} \). Furthermore, let \( \mathcal{G} \) be a subset of \( \mathbb{C} \) with \( \mathcal{G} \setminus \mathbb{R} \neq \emptyset \), let \( f : \mathcal{G} \to \mathbb{C}^{q \times q} \) be a matrix-valued function, and let \( z \in \mathcal{G} \setminus \mathbb{R} \). Then Remark A.4 shows that the following statements hold true:

(a) Let \( n \in \mathbb{N}_0 \) with \( 2n \leq \kappa \). Then the matrix \( P^{[f]}_{2n}(z) \) is non-negative Hermitian if and only if the following three conditions are fulfilled:

(i) \( (s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq} \).

(ii) \( \mathcal{R}(R_{T_{q,n}}(z)[v_{q,n}f(z) - u_n]) \subseteq \mathcal{R}(H_n) \).

(iii) The matrix

\[
\Sigma^{[f]}_{2n}(z) := \frac{f(z) - f^*(z)}{z - \mathfrak{z}} - (R_{T_{q,n}}(z)[v_{q,n}f(z) - u_n])^*H_n^\dagger(R_{T_{q,n}}(z)[v_{q,n}f(z) - u_n])
\] (4.6)

is non-negative Hermitian.

If \( P^{[f]}_{2n}(z) \in \mathbb{C}^{(n+2)q \times (n+2)q} \geq \), then, for each \( H_n^{(1)} \in H_n \{1\} \), we have

\[
\Sigma^{[f]}_{2n}(z) = \frac{f(z) - f^*(z)}{z - \mathfrak{z}} - (R_{T_{q,n}}(z)[v_{q,n}f(z) - u_n])^*H_n^{(1)}(R_{T_{q,n}}(z)[v_{q,n}f(z) - u_n]).
\]

(b) Let \( \alpha \in \mathbb{R} \), let \( \kappa \geq 1 \), and let \( n \in \mathbb{N}_0 \) with \( 2n + 1 \leq \kappa \). Then the matrix \( P^{[f]}_{2n+1}(z) \) is non-negative Hermitian if and only if the following three conditions are valid:

(iv) \( (s_{\alpha \nu})_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq} \).

(v) \( \mathcal{R}(R_{T_{q,n}}(z)[v_{q,n}([z - \alpha]f(z)] - (-\alpha u_n - y_{0,n}))]) \subseteq \mathcal{R}(H_{\alpha \nu n}) \).

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The matrix

\[ \Sigma_{2n+1}(z) := \frac{(z - \alpha)f(z) - [(z - \alpha)f(z)]^*}{z - \overline{z}} - \left[R_{\alpha,n}(z)(v_{q,n}[(z - \alpha)f(z)] - (-\alpha u_n - y_{0,n})]\right]^* \times H_{\alpha,n}^1 \left[R_{\alpha,n}(z)(v_{q,n}[(z - \alpha)f(z)] - (-\alpha u_n - y_{0,n})]\right] \]  

(4.7)
is non-negative Hermitian.

If \( P_{2n+1}(z) \in \mathbb{C}^{(n+2)\times(n+2)} \), then for each \( H_{\alpha,n}^{(1)} \in H_{\alpha,n}\{1\} \), we have

\[ \Sigma_{2n+1}(z) = \frac{(z - \alpha)f(z) - [(z - \alpha)f(z)]^*}{z - \overline{z}} - \left[R_{\alpha,n}(z)(v_{q,n}[(z - \alpha)f(z)] - (-\alpha u_n - y_{0,n})]\right]^* \times H_{\alpha,n}^1 \left[R_{\alpha,n}(z)(v_{q,n}[(z - \alpha)f(z)] - (-\alpha u_n - y_{0,n})]\right]. \]

Remark 4.7. Let \( \kappa \in \mathbb{N}_0 \cup \{\infty\} \), let \( (s_j)^{\infty}_{j=0} \) be a sequence of complex \( q \times q \) matrices, let \( G \) be a subset of \( \mathbb{C} \) with \( G \setminus \mathbb{R} \neq \emptyset \), and let \( S : G \to \mathbb{C}^{q \times q} \) be a matrix-valued function. Straightforward calculations show then that the following statements hold true:

(a) For every choice of \( n \in \mathbb{N}_0 \) with \( 2n \leq \kappa \) and \( z \in G \setminus \mathbb{R} \), we have

\[ \left[ \begin{array}{c} s_0 \\ S^*(z) \\ S(z) \end{array} \right] = [v_{q,n+1}, v_{q,n+1}]^* P_{2n}^{[S]}(z) [v_{q,n+1}, v_{q,n+1}]. \]  

(4.8)

(b) If \( \kappa \geq 1 \), for each \( n \in \mathbb{N}_0 \) with \( 2n + 1 \leq \kappa \) and each \( z \in G \setminus \mathbb{R} \), then

\[ \left[ \begin{array}{c} -\alpha s_0 + s_1 \\ [(z - \alpha)S(z) + s_0]^* \\ [z - \overline{z}] \\ (z - \alpha)S(z) + s_0 \\ [z - \overline{z}]^{-1} \end{array} \right] = [v_{q,n+1}, v_{q,n+1}]^* P_{2n+1}^{[S]}(z) [v_{q,n+1}, v_{q,n+1}] \]  

(4.9)

In the following, for each \( k \in \mathbb{N}_0 \), let

\[ m_{2k} := k \quad \text{and} \quad m_{2k+1} := k. \]

(4.10)

Lemma 4.8. Let \( \kappa \in \mathbb{N}_0 \cup \{\infty\} \) and let \( (s_j)^{\infty}_{j=0} \) be a sequence of Hermitian complex \( q \times q \) matrices. Let \( G \) be a subset of \( \mathbb{C} \) with \( G \setminus \mathbb{R} \neq \emptyset \). Further, let \( f : G \to \mathbb{C}^{q \times q} \) be a matrix-valued function, let \( G' := \{ z \in \mathbb{C} : \overline{z} \in G \} \), and let \( f^* : G' \to \mathbb{C}^{q \times q} \) be defined by \( f^*(z) := f^*(\overline{z}) \). For each \( k \in \mathbb{Z}_{-1, \kappa} \) and each \( z \in G' \setminus \mathbb{R} \), then there is a complex \( (m_k + 2)q \times (m_k + 2)q \) matrix \( X_k(z) \) such that \( P_{k}^{[f^*]}(z) = X_k(z)P_{k}^{[f]}(\overline{z})X_k^*(z) \).

Proof. We give the proof which is stated with more details in \[54, \text{Lemma 3.6}\]. We consider an arbitrary \( z \in G' \setminus \mathbb{R} \). From Remark 4.3, for each \( n \in \mathbb{N}_0 \), we see that

\[ \left[R_{\alpha,n}(\overline{z})\right]^{-1}H_n \left[R_{\alpha,n}(\overline{z})\right]^{-*} + (\overline{z} - z)(v_{q,n}u_n^* - u_nv_{q,n}^*) \]

\[ = H_n - zH_n T_{q,n} - \overline{z}T_{q,n}H_n + z^2 T_{q,n}H_n T_{q,n}^* + (\overline{z} - z)(T_{q,n}H_n - H_n T_{q,n}^*) \]

\[ = H_n + z^2 T_{q,n}H_n T_{q,n}^* - \overline{z}T_{q,n}H_n - zT_{q,n}H_n \]  

(4.11)

\[ = (I + z^2 T_{q,n}H_n T_{q,n}^* - \overline{z}T_{q,n}H_n - zT_{q,n}H_n)^* = \left[R_{\alpha,n}(z)\right]^{-1}H_n \left[R_{\alpha,n}(z)\right]^{-*}. \]
Obviously, for each $n \in \mathbb{N}_0$, we also get
\[(z - \overline{z})v_{q,n}[f^*(\overline{z})v_{q,n}^* - u_n^*] - (z - \overline{z})[v_{q,n}f^*(\overline{z}) - u_n]v_{q,n}^* = (z - \overline{z})(u_nv_{q,n}^* - v_{q,n}u_n^*) \quad (4.12)\]
and
\[f^*(\overline{z})v_{q,n}^* - u_n^* - [f^*(\overline{z}) - f(\overline{z})]v_{q,n}^* = [v_{q,n}f^*(\overline{z}) - u_n]^* \quad (4.13)\]
For each $n \in \mathbb{N}_0$, let
\[A_{2n}(z) := \text{diag}\left( [R_{T_{q,n}}(\overline{z})]^{-1}, I_q \right), \quad B_{2n}(z) := \begin{bmatrix} I_{(n+1)q} & (z - \overline{z})v_{q,n} \\ 0_{q \times (n+1)q} & I_q \end{bmatrix},\]
\[C_{2n}(z) := \text{diag}(R_{T_{q,n}}(z), I_q), \quad A_{2n+1}(z) := A_{2n}(z), \quad B_{2n+1}(z) := B_{2n}(z), \quad C_{2n+1}(z) := C_{2n}(z).\]
For each $m \in \mathbb{N}_0$, let $X_m(z) := C_m(z)B_m(z)A_m(z)$.

First we observe now that $P_{-1}^{[f]}(z) = X_{-1}(z)P_{-1}^{[f]}(\overline{z})X_{-1}^*(z)$ is true with $X_{-1}(z) := I_q$.

Now we consider an arbitrary $n \in \mathbb{N}_0$ with $2n \leq \kappa$. Then we have
\[P_{2n}^{[f]}(\overline{z})A_{2n}^*(z) = \begin{bmatrix} H_n & R_{T_{q,n}}(\overline{z})[v_{q,n}f(\overline{z}) - u_n] \\ \overline{R_{T_{q,n}}(\overline{z})[v_{q,n}f(\overline{z}) - u_n]} & \text{diag}\left( [R_{T_{q,n}}(\overline{z})]^{-*}, I_q \right) \end{bmatrix} \cdot \begin{bmatrix} H_n[\overline{R_{T_{q,n}}(\overline{z})}]^{-*} & R_{T_{q,n}}(\overline{z})[v_{q,n}f(\overline{z}) - u_n] \\ \overline{R_{T_{q,n}}(\overline{z})[v_{q,n}f(\overline{z}) - u_n]} & \overline{[R_{T_{q,n}}(\overline{z})]^{-*}} \end{bmatrix} \cdot \begin{bmatrix} f^*(\overline{z})v_{q,n}^* - u_n^* \\ \overline{f^*(\overline{z})v_{q,n}^* - u_n^*} \end{bmatrix}.
\]
Consequently,
\[A_{2n}(z)P_{2n}^{[f]}(\overline{z})A_{2n}^*(z) = \begin{bmatrix} [R_{T_{q,n}}(\overline{z})]^{-*}H_n[\overline{R_{T_{q,n}}(\overline{z})}]^{-*} & \overline{v_{q,n}f(\overline{z}) - u_n} \\ f^*(\overline{z})v_{q,n}^* - u_n^* & \overline{f^*(\overline{z})v_{q,n}^* - u_n^*} \end{bmatrix} \begin{bmatrix} v_{q,n}f(\overline{z}) - u_n \\ \overline{f^*(\overline{z})v_{q,n}^* - u_n^*} \end{bmatrix}.
\]
Thus, we conclude
\[B_{2n}(z)A_{2n}(z)P_{2n}^{[f]}(\overline{z})A_{2n}(z) = \begin{bmatrix} I_{(n+1)q} & (z - \overline{z})v_{q,n} \\ 0_{q \times (n+1)q} & I_q \end{bmatrix} \begin{bmatrix} [R_{T_{q,n}}(\overline{z})]^{-*}H_n[\overline{R_{T_{q,n}}(\overline{z})}]^{-*} & \overline{v_{q,n}f(\overline{z}) - u_n} \\ f^*(\overline{z})v_{q,n}^* - u_n^* & \overline{f^*(\overline{z})v_{q,n}^* - u_n^*} \end{bmatrix} \begin{bmatrix} [R_{T_{q,n}}(\overline{z})]^{-*}H_n[\overline{R_{T_{q,n}}(\overline{z})}]^{-*} & \overline{v_{q,n}f(\overline{z}) - u_n} \\ f^*(\overline{z})v_{q,n}^* - u_n^* & \overline{f^*(\overline{z})v_{q,n}^* - u_n^*} \end{bmatrix}.
\]
Using (4.12), (4.13), and (4.11), we get then

\[
B_{2n}(z)A_{2n}(z)P_{2n}^{\dag}(\tau)A_{2n}^\ast(z)B_{2n}^\ast(z) = \begin{bmatrix}
[RT_{q,n}(\tau)]^{-1}H_n[RT_{q,n}(\tau)]^{-\ast} + (z - \tau)q_n[f^\ast(\tau)v_{q,n}^* - u_n] - (z - \tau)[v_{q,n}f^\ast(\tau) - u_n]v_{q,n}^* & v_{q,n}f^\ast(\tau) - u_n \\
(f^\ast(\tau)v_{q,n}^* - u_n) & (f^\ast(\tau) - f(\tau))v_{q,n}^* \end{bmatrix}
\]

\[
= \begin{bmatrix}
[RT_{q,n}(\tau)]^{-1}H_n[RT_{q,n}(\tau)]^{-\ast} + (z - \tau)u_nv_{q,n}^* - v_{q,n}u_n^* & v_{q,n}f^\ast(\tau) - u_n \\
(v_{q,n}f^\ast(\tau) - u_n) & f^\ast(\tau) - f(\tau) \end{bmatrix}.
\]

Hence, we obtain

\[
X_{2n}(z)P_{2n}^{\dag}(\tau)X_{2n}^\ast(z) = C_{2n}(z)B_{2n}(z)A_{2n}(z)P_{2n}^{\dag}(\tau)A_{2n}^\ast(z)B_{2n}^\ast(z)C_{2n}^\ast(z)
\]

\[
= \begin{bmatrix}
H_n & R_{T_{q,n}}(z)[v_{q,n}f^\ast(\tau) - u_n] \\
(v_{q,n}f^\ast(\tau) - u_n) & f^\ast(\tau) - f(\tau) \end{bmatrix} = P_{2n}^{\dag}(\tau).
\]

Now we consider the case that \( n \in \mathbb{N}_0 \) is such that \( 2n + 1 \leq \kappa \). Then Remark 4.3 yields

\[
[RT_{q,n}(\tau)]^{-1}H_{\alpha \vert n}[RT_{q,n}(\tau)]^{-\ast} + (\tau - z)\left[v_{q,n}(-\alpha u_n - y_{0,n})^* - (-\alpha u_n - y_{0,n})v_{q,n}^*\right]
\]

\[
= (I_{(n+1)q} - \tau T_{q,n})H_{\alpha \vert n}(I_{(n+1)q} - z T_{q,n}) + (\tau - z)\left[v_{q,n}(-\alpha u_n - y_{0,n})^* - (-\alpha u_n - y_{0,n})v_{q,n}^*\right]
\]

\[
= (I_{(n+1)q} - \tau T_{q,n})H_{\alpha \vert n}(I_{(n+1)q} - z T_{q,n}) + (\tau - z)(T_{q,n}H_{\alpha \vert n} - H_{\alpha \vert n}T_{q,n})
\]

\[
= H_{\alpha \vert n} + z I_{(n+1)q}T_{q,n}H_{\alpha \vert n} - \tau H_{\alpha \vert n}T_{q,n} - z T_{q,n}H_{\alpha \vert n} = (I_{(n+1)q} - z T_{q,n})H_{\alpha \vert n}(I_{(n+1)q} - \tau T_{q,n}) = [RT_{q,n}(\tau)]^{-1}H_{\alpha \vert n}[RT_{q,n}(\tau)]^{-\ast}.
\]

Obviously,

\[
v_{q,n}(\tau - \alpha)f(\tau) - (\alpha u_n - y_{0,n}) + (z - \tau)v_{q,n}(z - \alpha)f(\tau) - (\tau - \alpha)f(\tau)
\]

\[
= v_{q,n}(z - \alpha)f^\ast(\tau) - (-\alpha u_n - y_{0,n}),
\]

\[
(z - \tau)v_{q,n}((z - \alpha)f^\ast(\tau)v_{q,n}^* - (-\alpha u_n - y_{0,n})
\]

\[
- (z - \tau)[v_{q,n}(z - \alpha)f^\ast(\tau) - (-\alpha u_n - y_{0,n})]v_{q,n}^* = (z - \tau)(\tau - \alpha)f^\ast(\tau) - (\alpha u_n - y_{0,n})v_{q,n}^*,
\]

and

\[
(z - \alpha)f^\ast(\tau)v_{q,n}^* - (-\alpha u_n - y_{0,n}) - [(z - \alpha)f^\ast(\tau) - (\tau - \alpha)f(\tau)]v_{q,n}^* = [(z - \alpha)v_{q,n}f^\ast(\tau) - (-\alpha u_n - y_{0,n})]^*.
\]
Taking into account
\[ P_{2n+1}^{[f]}(\tau)A^{*}_{2n+1}(z) \]
\[
= \begin{bmatrix}
H_{\alpha \alpha}^{-1}[RT_{q,n}(\tau)] & -RT_{q,n}(\tau) \frac{v_{q,n}((\tau - \alpha) f(\tau) - (-\alpha u_n - y_0,n))}{(\tau - z)} \\
(z - \alpha)f^*(\tau)v_{q,n} - (-\alpha u_n - y_0,n) & (\tau - z) - (\alpha - f^*(\tau))v_{q,n}
\end{bmatrix},
\]
we conclude
\[ A_{2n+1}(z)P_{2n+1}^{[f]}(\tau)A^{*}_{2n+1}(z) \]
\[
= \begin{bmatrix}
\frac{[RT_{q,n}(\tau)]^{-1}H_{\alpha \alpha}[RT_{q,n}(\tau)]^{-*}}{(z - \alpha)f^*(\tau)v_{q,n} - (-\alpha u_n - y_0,n)} & v_{q,n}(\tau - z) - (\tau - z) f^*(\tau)
\end{bmatrix}
\]
In view of (4.1.1), (4.1.10), and (4.1.14), it follows
\[ B_{2n+1}(z)A_{2n+1}(z)P_{2n+1}^{[f]}(\tau)A^{*}_{2n+1}(z)B^{*}_{2n+1}(z) \]
\[
= \begin{bmatrix}
\frac{[RT_{q,n}(\tau)]^{-1}H_{\alpha \alpha}[RT_{q,n}(\tau)]^{-*}}{(z - \alpha)f^*(\tau)v_{q,n} - (-\alpha u_n - y_0,n)} & v_{q,n}(\tau - z) f^*(\tau)
\end{bmatrix}
\]
Consequently,
\[ X_{2n+1}(z)P_{2n+1}^{[f]}(\tau)X^{*}_{2n+1}(z) \]
\[
= \begin{bmatrix}
H_{\alpha \alpha} & RT_{q,n}(\tau) \frac{v_{q,n}(\tau - z) f^*(\tau) - (-\alpha u_n - y_0,n)}{(\tau - z)}
\end{bmatrix},
\]
In the following, we will use \( \mathfrak{B}_{p \times q} \) to denote the \( \sigma \)-algebra of all Borel subsets of \( \mathbb{C}^{p \times q} \). Let \( (\Omega, \mathfrak{A}) \) be a measurable space and let \( \mu \in M^+_c(\Omega, \mathfrak{A}) \). Then \( \mu \) is absolutely continuous with respect to its trace measure \( \tau := \text{tr} \mu \). Let \( \mu^* \) be a version of the Radon–Nikodym derivative of \( \mu \) with respect to \( \tau \). A pair \( \Phi, \Psi \) of an \( \mathfrak{A}-\mathfrak{B}_{p \times q} \)-measurable mapping \( \Phi : \Omega \to \mathbb{C}^{p \times q} \) and an \( \mathfrak{A}-\mathfrak{B}_{r \times q} \)-measurable mapping \( \Psi : \Omega \to \mathbb{C}^{r \times q} \) is called left-integrable with respect to \( \mu \).
if $\Phi \mu_n \Psi^*$ belongs to $[\mathcal{L}^1(\Omega, \mathfrak{A}, \tau; \mathbb{C})]^{p \times r}$. In this case, the corresponding integral is defined by $\int_\Omega \Phi \mu_n \Psi^* := \int_\Omega \Phi \mu_n \Psi^* d\tau$ and we also use the notation $\int_\Omega \Phi(\omega) \mu(\omega) \Psi^*(\omega)$ for it. In the following, when we write such an integral $\int_\Omega \Phi \mu_n \Psi^*$, then we also mean that the pair $[\Phi, \Psi]$ is left-integrable with respect to $\mu$. By $p \times q - \mathcal{L}^2(\Omega, \mathfrak{A}, \mu; \mathbb{C})$ we denote the set of all $\mathfrak{A}$-$\mathfrak{B}_{p \times q}$-measurable mappings for which the pair $[\Phi, \Phi]$ is left-integrable which respect to $\mu$.

Furthermore, for each subset $A$ of $\Omega$, we will use $1_A$ to denote the indicator function of the set $A$ (defined on $\Omega$).

**Remark 4.9.** Let $\Omega \in \mathfrak{B} \setminus \{\emptyset\}$, let $m \in \mathbb{N}_0$, and let $\sigma \in \mathcal{M}_{2m}^q(\Omega)$. In view of Lemma [C.3], it is readily checked that $\sigma$ belongs to $\mathcal{M}_{2m}^q(\Omega)$ if and only if $\text{Rstr}_\Omega E_{q,m}$ belongs to $(m+1)q \times q - \mathcal{L}^2(\Omega, \mathfrak{B}_0, \sigma; \mathbb{C})$, where $E_{q,m}$ is given by (1.3). If $\sigma \in \mathcal{M}_{2m}^q(\Omega)$, then Lemma [C.3] also shows that, for each $n \in \mathbb{N}_0$ with $n \leq m$, the block Hankel matrix $H_n^{[\sigma]} := [s_{q,n}^{[\sigma]}]_{j,k=0}^{n}$ admits the integral representation

$$H_n^{[\sigma]} = \int_\Omega E_{q,n}(t) \sigma(dt) E_{q,n}^*(t). \quad (4.17)$$

If $\alpha \in \mathbb{R}$, $\kappa \in \mathbb{N} \cup \{\infty\}$, and if $\sigma \in \mathcal{M}_{\kappa}^q((\alpha, \infty))$, then let $H_{\alpha \beta n}^{[\sigma]} := [s_{q,n}^{[\sigma]}]_{j,k=0}^{n}$ for each $n \in \mathbb{N}_0$ with $2n + 1 < \kappa$.

**Remark 4.10.** Let $\alpha \in \mathbb{R}$ and let $\sigma \in \mathcal{M}_{\kappa}^q((\alpha, \infty))$. Using Proposition [C.5] and Remark [C.4], it is readily checked that the following statements hold true (for details see [57], Lemma 5.7 and [53], Lemma 3.12):

(a) The function $\phi : [\alpha, \infty) \to \mathbb{C}^{q \times q}$ defined by $\phi(t) := \sqrt{t-\alpha}I_q$ belongs to $q \times q - \mathcal{L}^2([\alpha, \infty), \mathfrak{B}_{(\alpha, \infty)}, \sigma; \mathbb{C})$ and $\sigma^\#: \mathfrak{B}_{(\alpha, \infty)} \to \mathbb{C}^{q \times q}$ given by

$$\sigma^\#(B) := \int_B (\sqrt{t-\alpha}I_q) \sigma(dt) (\sqrt{t-\alpha}I_q)^* \quad (4.18)$$

belongs to $\mathcal{M}_{\kappa}^q((\alpha, \infty))$.

(b) If $n \in \mathbb{N}_0$ and if $\sigma \in \mathcal{M}_{2n+1}^q((\alpha, \infty))$, then

$$H_{\alpha \beta n}^{[\sigma]} = \int_{(\alpha, \infty)} [\sqrt{t-\alpha}E_{q,n}(t)] \sigma(dt) [\sqrt{t-\alpha}E_{q,n}(t)]^*. \quad (4.19)$$

(c) If $n \in \mathbb{N}_0$ and if $\sigma^\# \in \mathcal{M}_{2n}^q((\alpha, \infty))$, then $\sigma$ belongs to $\mathcal{M}_{2n+1}^q((\alpha, \infty))$ and furthermore $s_{q,n}^{[\sigma]} = s_{q+1,n}^{[\sigma]} - \alpha s_{q,n}^{[\sigma]}$ for all $j \in \mathbb{Z}_{0,2n}$ and $H_n^{[\sigma^\#]} = H_{\alpha \beta n}^{[\sigma]}$.

The next proposition shows that each solution of problem $\text{MP}([\alpha, \infty); (s_j)_{j=0}^m \leq \sigma]$ fulfills necessarily the system of the corresponding Potapov’s fundamental matrix inequalities.

**Proposition 4.11.** Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices such that $\mathcal{M}_{\kappa}^q([\alpha, \infty); (s_j)_{j=0}^m \leq \sigma]$ $\neq \emptyset$. Let $\sigma \in \mathcal{M}_{\kappa}^q([\alpha, \infty); (s_j)_{j=0}^m \leq \sigma]$ and let $S$ be the $[\alpha, \infty)$-Stieltjes transform of $\sigma$. For each $j \in \mathbb{Z}_{0,m}$, let $s_j^{[\sigma]}$ be given by (1.1). Then

$$P_{2n}^{[\sigma]}(z) = \int_{(\alpha, \infty)} [E_{q,n}(t)]_{q \times q} [E_{q,n}(t)]_{q \times q}^* + [v_{q,n}]_{q \times q} [s_{2n} - s_j^{[\sigma]}]_{q \times q}^*$$
for each \( n \in \mathbb{N}_0 \) with \( 2n \leq m \) and all \( z \in \mathbb{C} \setminus \mathbb{R} \), where \( E_{q,n} \) is given by (4.3), and
\[
P_{2n+1}(z) = \int_{[\alpha, \infty)} \left( \sqrt{t - \alpha} \left[ E_{q,n}(t) \right] \right) \sigma(dt) \left( \sqrt{t - \alpha} \left[ E_{q,n}(t) \right] \right)^* + \left[ \begin{array}{cc} v_{q,n} & 0_{q \times q} \\ 0_{q \times q} & 1 \end{array} \right] \left( s_{2n+1} - s_{2n+1}^{[\sigma]} \right) \left[ \begin{array}{c} v_{q,n} \\ 0_{q \times q} \end{array} \right]^* \]
for each \( n \in \mathbb{N}_0 \) with \( 2n + 1 \leq m \) and all \( z \in \mathbb{C} \setminus \mathbb{R} \). In particular, for every choice of \( k \in \mathbb{Z}_{0,m} \) and \( z \in \mathbb{C} \setminus \mathbb{R} \), the matrix \( P_k[S] \) is non-negative Hermitian.

Proposition 4.11 can be proved using standard arguments of integration theory of non-negative Hermitian measures (Lemma C.3 and Remark C.4). We omit the details.

5. Some integral estimates for the scalar case

In this section, we state some integral representations and integral estimates in the scalar case \( q = 1 \).

Lemma 5.1. Let \( \alpha \in \mathbb{R} \) and let \( F \in \mathcal{R}_1(\Pi_+) \) with Nevanlinna parametrization \((A, B, \nu)\) and spectral measure \( \mu \). Then:

(a) For each \( w \in \Pi_+ \), the integral \( \int_{\mathbb{R}} |t - w|^{-2} \mu(dt) \) is finite and
\[
\Im F(w) = \left( \Im \psi_w \left( t^2 \right) \right)
\]
(5.1)

(b) For each \( w \in \Pi_+ \), the integral \( \int_{\mathbb{R}} |t| |t - w|^{-2} - (1 + t^2)^{-1} - \alpha |t - w|^{-2} \mu(dt) \) is finite and \( F^\#: \Pi_+ \to \mathbb{C} \) defined by
\[
F^\#(w) := (w - \alpha)F(w) \tag{5.2}
\]
satisfies, for each \( w \in \Pi_+ \), the equation
\[
\Im F^\#(w) = (\Im \psi_w \left( t^2 \right) \right)
\]
(5.3)

Proof. In view of
\[
\int_{\mathbb{R}} \frac{1}{1 + t^2} \mu(dt) = \int_{\mathbb{R}} \frac{1}{1 + t^2} (1 + t^2)^{\nu}(dt) = \nu(\mathbb{R}) < \infty,
\]
we see that, for each \( w \in \Pi_+ \), the function \( \psi_w : \mathbb{R} \to \mathbb{C} \) given by the equation \( \psi_w(t) := (t - w)^{-1} - t(1 + t^2)^{-1} \) belongs to \( \mathcal{L}^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu; \mathbb{C}) \). By virtue of a result due to R. Nevanlinna (see, e.g. [52 Theorem A.2]), for each \( w \in \Pi_+ \), we have
\[
F(w) = A + Bw + \int_{\mathbb{R}} \left( \frac{1}{t - w} - \frac{t}{1 + t^2} \right) \mu(dt). \tag{5.4}
\]
(a) Let \( w \in \Pi_+ \). For each \( t \in \mathbb{R} \), then \( \Im \psi_w(t) = (\Im w)|t-w|^{-2} \). Thus,

\[
\int_{\mathbb{R}} \left| \frac{1}{|t-w|^2} \right| \mu(dt) = \frac{1}{3w} \int_{\mathbb{R}} \Im \psi_w(t) \mu(dt) = \frac{1}{3w} \Im \left[ \int_{\mathbb{R}} \psi_w(t) \mu(dt) \right] \\
\leq \frac{1}{3w} \int_{\mathbb{R}} |\psi_w(t)| \mu(dt) < \infty
\]

and

\[
\Im \left[ \int_{\mathbb{R}} \psi_w(t) \mu(dt) \right] = \int_{\mathbb{R}} \Im \psi_w(t) \mu(dt) = (\Im w) \int_{\mathbb{R}} \frac{1}{|t-w|^2} \mu(dt).
\]

Because of \( A \in \mathbb{R} \) and \( B \in [0, \infty) \), we have \( \Im A = 0 \) and \( \Im (wB) = (\Im w)B \). Consequently, from [5.4], and [5.5] we get then [5.1].

(b) Let \( w \in \Pi_+ \). In view of (5.2) and (5.4), we obtain

\[
F^\#(w) = A(w - \alpha) + Bw(w - \alpha) + \int_{\mathbb{R}} \left[ \frac{w - \alpha}{t-w} - \frac{t(w-\alpha)}{1+t^2} \right] \mu(dt).
\]

For each \( t \in \mathbb{R} \), we see that \((w - \alpha)\psi_w(t) = (w - \alpha)/(t - w) - t(w - \alpha)/(1 + t^2)\) holds true. Hence, \((w - \alpha)\psi_w \in L^1(\mathbb{R}, \mathfrak{B}_\mathbb{R}, \mu; \mathbb{C})\) and, for each \( t \in \mathbb{R} \), we have furthermore

\[
2i\Im [(w - \alpha)\psi_w(t)] = 2i(\Im w) \left[ t \left( \frac{1}{|t-w|^2} - \frac{1}{1+t^2} \right) - \frac{\alpha}{|t-w|^2} \right].
\]

This implies

\[
\int_{\mathbb{R}} \left| t \left( \frac{1}{|t-w|^2} - \frac{1}{1+t^2} \right) - \frac{\alpha}{|t-w|^2} \right| \mu(dt) = \frac{1}{3w} \int_{\mathbb{R}} |\Im [(w - \alpha)\psi_w(t)]| \mu(dt) \\
\leq \frac{1}{3w} \int_{\mathbb{R}} |(w - \alpha)\psi_w(t)| \mu(dt) < \infty
\]

and

\[
\Im \left[ \int_{\mathbb{R}} (w - \alpha)\psi_w(t) \mu(dt) \right] = \int_{\mathbb{R}} \Im [(w - \alpha)\psi_w(t)] \mu(dt) \\
= (\Im w) \int_{\mathbb{R}} \left[ t \left( \frac{1}{|t-w|^2} - \frac{1}{1+t^2} \right) - \frac{\alpha}{|t-w|^2} \right] \mu(dt).
\]

Obviously, \( \Im (w^2) = 2(\Re w)(\Im w) \). Consequently, \( \Im [w(w - \alpha)] = \Im (w^2) - \Im (w\alpha) = (\Im w)(2\Re w - \alpha) \). Thus, \( \Im [Bw(w - \alpha)] = B(\Im w)(2\Re w - \alpha) \). Then, by virtue of (5.6), and (5.7), we get

\[
\Im F^\#(w) = \Im A(w - \alpha) + \Im Bw(w - \alpha) + \int_{\mathbb{R}} \left[ \frac{w - \alpha}{t-w} - \frac{t(w-\alpha)}{1+t^2} \right] \mu(dt) \\
= \Im A(w - \alpha) + \Im [Bw(w - \alpha)] + \Im \left[ \int_{\mathbb{R}} \left( \frac{w - \alpha}{t-w} - \frac{t(w-\alpha)}{1+t^2} \right) \mu(dt) \right] \\
= A\Im w + B(\Im w)(2\Re w - \alpha) + (\Im w) \int_{\mathbb{R}} \left[ t \left( \frac{1}{|t-w|^2} - \frac{1}{1+t^2} \right) - \frac{\alpha}{|t-w|^2} \right] \mu(dt).
\]

Thus, (5.3) follows. \( \square \)
Remark 5.2. Let \( \alpha \in \mathbb{R} \) and let \( F \in \mathcal{R}_1(\Pi_+) \) with spectral measure \( \mu \). Further, let \( \ell_1, \ell_2 \in \mathbb{R} \) be such that \( \ell_1 < \ell_2 < \alpha \). Then it is readily checked that for every choice of \( a \in (-\infty, \ell_1) \) and \( b \in (\ell_2, \infty) \), there exists a \( K_{a,b} \in \mathbb{R} \) such that, for each \( x \in [\ell_1, \ell_2] \), the inequality \( \int_{r \leq x < s} (t-x)^{-2} \mu(dt) < K_{a,b} \) holds true.

Remark 5.3. Let \( r, s \in \mathbb{R} \). Then it is readily checked that the following statements hold true (for details, see also [57, Lemma 3.7]):

(a) If \( r < s \) and \( s \neq 0 \), then there exists a number \( a \in (-\infty, r) \cap (-\infty, 0) \) such that

\[
\left| t \left( \frac{1}{(t-x)^2 + y^2} - \frac{1}{1 + t^2} \right) \right| < \left( 2 + \frac{|r|}{s} \right) \cdot \left| t \left( \frac{1}{(t-s)^2 + 1} - \frac{1}{1 + t^2} \right) \right|
\]

(5.8)

is valid for every choice of \( x \in [r, s] \) and \( y \in (0, 1) \) and \( t \in (-\infty, a] \).

(b) If \( s < r \) and \( r \neq 0 \), then there exists a \( b \in (r, \infty) \cap (0, \infty) \) such that, for every choice of \( x \in [s, r] \) and \( y \in (0, 1) \) and \( t \in [b, \infty) \), inequality (5.8) holds true.

Lemma 5.4. Let \( \alpha \in \mathbb{R} \) and let \( F \in \mathcal{R}_1(\Pi_+) \) with spectral measure \( \mu \). Further, let \( \ell_1 \) and \( \ell_2 \) be real numbers with \( \ell_1 < \ell_2 < \alpha \). Then there are real numbers \( a, b, \) and \( C \) with \( a < \ell_1 \) and \( \ell_2 < b < \alpha \) such that

\[
\int_{r \leq x < s} \left| t \left( \frac{1}{(t-x)^2 + y^2} - \frac{1}{1 + t^2} \right) - \frac{\alpha}{(t-x)^2 + y^2} \right| \mu(dt) < C
\]

holds true for every choice of \( x \in [\ell_1, \ell_2] \) and \( y \in (0, 1) \).

Using Lemma [5.1] and Remarks [5.2] and [5.3], Lemma 5.4 can be proved analogous to the well-known special case \( \alpha = 0 \). However, in the general case of an arbitrary real number \( \alpha \), these straightforward calculations are very lengthy (see [57, Lemma 3.8] for details).

Lemma 5.5. Let \( \alpha \in \mathbb{R} \) and let \( F \in \mathcal{R}_1(\Pi_+) \) be such that \( F^\# : \Pi_+ \to \mathbb{C} \) defined by [6.2] belongs to \( \mathcal{R}_1(\Pi_+) \). Further, let \( \mu \) be the spectral measure of \( F \) and let \( \ell_1 \) and \( \ell_2 \) be real numbers with \( \ell_1 < \ell_2 < \alpha \). Then there are real numbers \( a, b, \) and \( C \) with \( a < \ell_1 \) and \( \ell_2 < b < \alpha \) such that

\[
\int_{r \leq x < s} \left| t - \frac{\alpha}{(t-x)^2 + y^2} \right| \sigma(dt) < C \quad \text{and} \quad \int_{r \leq x < s} \left| t \right| \frac{1}{(t-x)^2 + y^2} \sigma(dt) < C
\]

hold true for every choice of \( x \in [\ell_1, \ell_2] \) and \( y \in [0, \infty) \).

Lemma 5.5 can be proved, using Lemmata 5.1 and 5.4 and Beppo Levi’s Theorem of monotone convergence (for details, see [57, Lemma 3.12]).

Remark 5.6. Let \( \alpha \in \mathbb{R} \) and let \( F \in \mathcal{R}_1(\Pi_+) \) be such that \( F^\# : \Pi_+ \to \mathbb{C} \) defined by [5.2] belongs to \( \mathcal{R}_1(\Pi_+) \). Let \( \mu \) be the spectral measure of \( F \) and let \( \ell_1 \) and \( \ell_2 \) be real numbers with \( \ell_1 < \ell_2 < \alpha \). Then one can easily see from Remark 5.2 and Lemma 5.5 that there is a real number \( C \) such that \( \int_{r \leq x < s} (t-x)^{-2} \mu(dt) < C \) for all \( x \in [\ell_1, \ell_2] \).

Lemma 5.7. Let \( \alpha \in \mathbb{R} \) and let \( F \in \mathcal{R}_1(\Pi_+) \) be such that \( F^\# : \Pi_+ \to \mathbb{C} \) defined by [5.2] belongs to \( \mathcal{R}_1(\Pi_+) \). Then the Nevanlinna measure \( \nu \) of \( F \) and the spectral measure \( \mu \) of \( F \) fulfill \( \nu((-\infty, \alpha)) = 0 \) and \( \mu((-\infty, \alpha)) = 0 \).
Proof. We give the proof stated in [57, Lemmata 3.13 and 3.14].

(I) In the first step of the proof, we consider arbitrary real numbers \( \ell_1 \) and \( \ell_2 \) with \( \ell_1 < \ell_2 < \alpha \). Let \((A, B, \nu)\) be the Nevanilma parametrization of \( F \). Because of Remark 5.6, there is a \( C \in \mathbb{R} \) such that \( \int_{\mathbb{R}} (t-x)^{-2} \mu(dt) < C \) is true for all \( x \in [\ell_1, \ell_2] \). Since \( F \) belongs to \( \mathcal{R}_1(\Pi_+) \), for each \( x \in [\ell_1, \ell_2] \) and each \( \epsilon \in (0, \infty) \), from Lemma 5.11 we get then

\[
0 \leq \Im F(x + i\epsilon) = \epsilon \left[ B + \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{(t-x)^2 + \epsilon^2} \mu(dt) \right] < \epsilon(B + C)
\]

and, consequently,

\[
0 \leq \int_{[\ell_1, \ell_2]} \Im F(x + i\epsilon) \lambda^{(1)}(dx) \leq \epsilon(B + C)(\ell_2 - \ell_1), \tag{5.9}
\]

where \( \lambda^{(1)} \) is the Lebesgue measure defined on \( \mathbb{B}_\mathbb{R} \). In view of \( F \in \mathcal{R}_1(\Pi_+) \), the inversion formula of Stieltjes–Perron (see, e.g. [52, Appendix, p. 390]) yields

\[
\frac{1}{2} [\sigma(\{\ell_1\}) + \sigma(\{\ell_2\})] + \sigma(\{\ell_1, \ell_2\}) = \frac{1}{\pi} \lim_{\epsilon \to 0+0} \int_{[\ell_1, \ell_2]} \Im F(x + i\epsilon) \lambda^{(1)}(dx). \tag{5.10}
\]

Combining (5.10) and (5.9), we obtain \( \sigma(\{\ell_1, \ell_2\}) = 0 \), from

\[
0 \leq \sigma(\{\ell_1, \ell_2\}) \leq \frac{1}{2} [\sigma(\{\ell_1\}) + \sigma(\{\ell_2\})] + \sigma(\{\ell_1, \ell_2\})
\]

\[
= \frac{1}{\pi} \lim_{\epsilon \to 0+0} \int_{[\ell_1, \ell_2]} \Im F(x + i\epsilon) \lambda^{(1)}(dx) \leq \frac{1}{\pi} \lim_{\epsilon \to 0+0} [\epsilon(B + C)(\ell_2 - \ell_1)] = 0.
\]

(II) For each \( n \in \mathbb{N} \), the real numbers \( a_n := \alpha - (1 + n) \) and \( b_n := \alpha - \frac{1}{n} \) fulfill \( a_n < b_n < \alpha \). Consequently, part (I) of the proof provides us \( \mu((a_n, b_n)) = 0 \). Obviously, \( (a_n, b_n) \subseteq (a_{n+1}, b_{n+1}) \) for each \( n \in \mathbb{N} \) and \( \bigcup_{n=1}^{\infty} (a_n, b_n) = (-\infty, \alpha) \). Hence, \( \mu((-\infty, \alpha)) = \mu(\bigcup_{n=1}^{\infty} (a_n, b_n)) = \lim_{n \to \infty} \mu((a_n, b_n)) = 0 \). Thus, \( \nu((-\infty, \alpha)) = 0 \) follows from

\[
0 \leq \nu((-\infty, \alpha)) = \int_{(-\infty, \alpha)} 1 \nu(dt) \leq \int_{(-\infty, \alpha)} (1 + t^2) \nu(dt) = \mu((-\infty, \alpha)) = 0. \quad \square
\]

6. From the system of Potapov’s fundamental matrix inequalities to the moment problem

Proposition 5.11 showed that the Stieltjes transform of an arbitrary solution of problem \( MP[[\alpha, \infty); (s_j)_{j=0}^{\infty}, \le] \) fulfills necessarily the system of corresponding Potapov’s fundamental matrix inequalities. In this section, we are going to prove that the validity of the system of Potapov’s fundamental matrix inequalities for a holomorphic \( q \times q \) matrix-valued function defined on \( \mathbb{C} \setminus [\alpha, \infty) \) is also sufficient to be the Stieltjes transform of some solution of this matricial Stieltjes-type moment problem. For the convenience of the reader, first we state two well-known facts.

Remark 6.1. Let \( D \) be a discrete subset of \( \Pi_+ \) and let \( F: \Pi_+ \setminus D \to \mathbb{C}^{q \times q} \) be a matrix-valued function which is holomorphic in \( \Pi_+ \setminus D \) and which fulfills \( \Im F(z) \in \mathbb{C}^{q \times q} \) for all \( z \in \Pi_+ \setminus D \). Then one can easily see from [20, Lemma 2.1.9] that there is a function \( F^\Delta \in \mathcal{R}_q(\Pi_+) \) such that \( \text{Rstr}_{\Pi_+ \setminus D} F^\Delta = F \).
Remark 6.2. Let \( A, B \in \mathbb{C}^{q \times q} \), let \( M \) be an open subset of \( \mathbb{R} \), and let \( \nu \in \mathcal{M}_2^+(\mathbb{R} \setminus M) \). In view of a well-known result on integrals which depend on a complex parameter (see, e.g. [29, Satz 5.8]), it is readily checked that \( \phi: \Pi_+ \cup M \cup \Pi_- \to \mathbb{C}^{q \times q} \) given by

\[
\phi(z) := A + Bz + \int_{\mathbb{R} \setminus M} \frac{1 + tz}{t - z} \nu(dt)
\]

is holomorphic in \( \Pi_+ \cup M \cup \Pi_- \) (see also, e.g. [57, Lemma 3.17]).

In the following, for all \( \alpha \in \mathbb{R} \), let \( C_{\alpha,-} := \{z \in \mathbb{C}: \Re z \in (-\infty, \alpha)\} \).

Lemma 6.3. Let \( \alpha \in \mathbb{R} \) and let \( F \in \mathcal{R}_q(\Pi_+) \) be such that \( F^\#: \Pi_+ \to \mathbb{C}^{q \times q} \) defined by \( F^\#(w) := (w - \alpha)F(w) \) belongs to \( \mathcal{R}_q(\Pi_+) \). Further, let \( \nu \) be the Nevanlinna measure of \( F \). Then \( \nu((\infty, \alpha)) = 0 \) and the following two statements hold true:

(a) There is a function \( F_\alpha: \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q \times q} \) such that \( \text{Rstr}_{\Pi_+} F_\alpha = F \) and \( F_\alpha((-\infty, \alpha)) \subseteq \mathbb{C}^{q \times q}_\Pi \) are fulfilled.

(b) There exists a unique function \( S \in \mathcal{S}_{q:(\alpha, \infty)} \) with \( \text{Rstr}_{\Pi_+} S = F \).

Proof. Since \( F \) and \( F^\# \) belong to \( \mathcal{R}_q(\Pi_+) \), for all \( u \in \mathbb{C}^q \), we see that \( \{u^*F\alpha, u^*F^\#u\} \subseteq \mathcal{R}_1(\Pi_+) \) and that \( u^*\nu u \) is the Nevanlinna measure of \( u^*F \alpha \). Because of Lemma \([27]\) for all \( u \in \mathbb{C}^q \), we have \( u^*\nu((\infty, \alpha))u = (u^*\nu)u((-\infty, \alpha)) = 0 = u^*0_{q \times q}u \). Consequently, \( \nu((-\infty, \alpha)) = 0_{q \times q} \).

Obviously, \( \nu := \text{Rstr}_{\alpha,\alpha} \nu \) belongs to \( \mathcal{M}_2^+([\alpha, \infty)) \). By virtue of \( F \in \mathcal{R}_q(\Pi_+) \) and Theorem 6.1, there are matrices \( A \in \mathbb{C}^{q \times q}_\Pi \) and \( B \in \mathbb{C}^{q \times q}_\Pi \) such that \( (6.1) \) holds for each \( z \in \Pi_+ \). Remark 6.2 shows that \( F_\alpha: \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q \times q} \) given by

\[
F_\alpha(z) := A + Bz + \int_{[\alpha, \infty)} \frac{1 + tz}{t - z} \psi(dt)
\]

is holomorphic in \( \mathbb{C} \setminus [\alpha, \infty) \). Comparing \( (6.1) \) and \( (6.1) \), we get \( F_\alpha(z) = F(z) \) for each \( z \in \Pi_+ \). For every choice of \( x \in \mathbb{R} \), we have

\[
\left[ \int_{[\alpha, \infty)} \frac{1 + tx}{t - x} \psi(dt) \right]^* = \int_{[\alpha, \infty)} \left( \frac{1 + tx}{t - x} \right) \psi(dt) = \int_{[\alpha, \infty)} \frac{1 + tx}{t - x} \psi(dt).
\]

In view of \( (6.1) \), \( A \in \mathbb{C}_\Pi^{q \times q} \), and \( B \in \mathbb{C}_\Pi^{q \times q} \), then \( [F_\alpha(x)] = F_\alpha(x) \) follows for each \( x \in (-\infty, \alpha) \).

Because of part (a), there is a holomorphic function \( S: \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q \times q} \) such that

\[
\text{Rstr}_{\Pi_+} S = F \quad \text{and} \quad S((-\infty, \alpha)) \subseteq \mathbb{C}_\Pi^{q \times q}
\]

hold true. According to \( \{F, F^\#\} \subseteq \mathcal{R}_q(\Pi_+) \) and \( (6.2) \), for all \( z \in \Pi_+ \), then

\[
\Im S(z) = \Im F(z) \in \mathbb{C}_\Pi^{q \times q} \quad \text{and} \quad \Im(z - \alpha)S(z) = \Im F^\#(z) \in \mathbb{C}_\Pi^{q \times q}.
\]

For all \( z \in C_{\alpha,-} \cap \Pi_+ \), we have \( \Im((z - \alpha)S(z)) = [\Re(z - \alpha)]\Im S(z) + (\Im z)\Re S(z) \) and, by virtue of \( (6.3) \), consequently,

\[
\Re S(z) = \frac{\Im((z - \alpha)S(z))}{\Im z} + [-\Re(z - \alpha)] \frac{\Im S(z)}{\Im z} \in \mathbb{C}_\Pi^{q \times q}.
\]

(6.4)
Now we consider an arbitrary monotonically nondecreasing sequence \((y_n)_{n=1}^{\infty}\) of positive real numbers with \(\lim_{n \to \infty} y_n = 0\). Since the function \(S\) is holomorphic in \(\mathbb{C} \setminus [\alpha, \infty)\), the functions \(\Re S\) and \(\Im S\) are continuous in \(\mathbb{C} \setminus [\alpha, \infty)\). Thus, for each \(x \in (-\infty, \alpha)\), we have \(x + iy_n \in \mathbb{C}_{\alpha, \infty} \cap \Pi^+\) for all \(n \in \mathbb{N}\) and, hence, of (6.4), and (6.3), then

\[
\Re S(x) = \lim_{n \to \infty} \Re S(x + iy_n) \in \mathbb{C}_{\geq 0}^{q \times q} \quad \text{and} \quad \Im S(x) = \lim_{n \to \infty} \Im S(x + iy_n) \in \mathbb{C}_{\geq 0}^{q \times q}. \tag{6.5}
\]

Combining (6.2) and (6.5), for each \(x \in (-\infty, \alpha)\), we get \(\Re S(x) + i\Im S(x) = S(x) = |S(x)|^* = \Re S(x) - i\Im S(x)\) and, hence, \(\Im S(x) = 0\). From (6.5) then \(S(x) \in \mathbb{C}_{\geq 0}^{q \times q}\) follows for each \(x \in (-\infty, \alpha)\). Consequently, \(S \in \mathcal{S}_{q; [\alpha, \infty)}\).

Now we consider an arbitrary subset \(S \in \mathcal{S}_{q; [\alpha, \infty)}\) such that \(\text{Rstr}_{\Pi^+} S = F\). From (6.2) we get then \(S(z) = F(z) = S(z)\) for each \(z \in \Pi^+.\) Thus, the identity theorem for holomorphic functions provides us \(S = S\).

**Proposition 6.4.** Let \(\alpha \in \mathbb{R}\) and let \(D\) be a discrete subset of \(\Pi^+.\) Let \(F: \Pi^+ \setminus D \to \mathbb{C}_{\geq 0}^{q \times q}\) be a holomorphic matrix-valued function and let \(F^\#: \Pi^+ \to \mathbb{C}_{\geq 0}^{q \times q}\) be defined by \(F^\#(w) := (w - \alpha) F(w)\). Suppose \(\{\Im F(w), \Im F^\#(w)\} \subseteq \mathbb{C}_{\geq 0}\) for all \(w \in \Pi^+ \setminus D\). Then there is a unique \(S \in \mathcal{S}_{q; [\alpha, \infty)}\) such that \(\text{Rstr}_{\Pi^+ \setminus D} S = F\).

**Proposition 6.4** can be easily proved using Remark 6.1, Lemma 6.3, and the identity theorem for holomorphic functions (see also [57, Theorem 3.19]). We omit the details.

**Theorem 6.5.** Let \(\alpha \in \mathbb{R}\), let \(\kappa \in \mathbb{N}_0 \cup \{\infty\}\), let \((s_j)_{j=0}^{\kappa}\) be a sequence of complex \(q \times q\) matrices, and let \(m \in \mathbb{Z}_{0, \kappa}\). Further, let \(D\) be a discrete subset of \(\Pi^+\) and let \(F: \Pi^+ \setminus D \to \mathbb{C}_{\geq 0}^{q \times q}\) be a holomorphic matrix-valued function such that

\[
P_{m}^{[F]}(z) \geq 0 \quad \text{and} \quad P_{m-1}^{[F]}(z) \geq 0 \quad \text{for each } z \in \Pi^+ \setminus D. \tag{6.6}
\]

Then there exists a unique \(S \in \mathcal{S}_{0, q; [\alpha, \infty)}\) such that \(\text{Rstr}_{\Pi^+ \setminus D} S = F\). Moreover, the inequality \(P_{k}^{[S]}(z) \geq 0\) holds true for each \(k \in \mathbb{Z}_{-1, m}\) and each \(z \in \mathbb{C} \setminus \mathbb{R}\).

**Proof.** We give a version of the proof stated with more details in [57, Theorem 4.10]. From (6.6), Notation 4.5, and (1.2), we see that \(H_n \geq 0\) for each \(n \in \mathbb{N}_0\) with \(2n \leq m\), that \(H_{\text{even}} \geq 0\) for each \(n \in \mathbb{N}\) with \(2n + 1 \leq m\), that \(s_j^* = s_j\) for each \(j \in \mathbb{Z}_{0, m}\), and that \(\Im F(z) = (3z)^{\frac{F(z) - F^*(z)}{2}} \geq 0\) and \(\Im [(z - \alpha) F(z)] = (3z)^{\frac{(z - \alpha) F(z) - [(z - \alpha) F(z)]^*}{2}} \geq 0\) hold true for each \(z \in \Pi^+ \setminus D\). Thus, because of Proposition 6.4 there exists a unique \(S \in \mathcal{S}_{q; [\alpha, \infty)}\) such that \(\text{Rstr}_{\Pi^+ \setminus D} S = F\). By continuity arguments, from (6.6) we get then \(\{P_{m}^{[S]}(z), P_{m-1}^{[S]}(z)\} \subseteq \mathbb{C}_{\geq 0}\) for each \(z \in \Pi^+\) and, consequently,

\[
P_{k}^{[S]}(z) \geq 0 \quad \text{for each } k \in \mathbb{Z}_{-1, m}\) and each \(z \in \Pi^+. \tag{6.7}
\]

In particular, \(S := \text{Rstr}_{\Pi^+} S\) fulfills

\[
\begin{bmatrix}
 s_0 \\
 \hat{S}(z) \\
 \hat{S}^*(z)
\end{bmatrix}
= P_0^{[S]}(z) \geq 0 \quad \text{for each } z \in \Pi^+.
\]

Consequently, Lemma 3.3 provides us \(S \in \mathcal{R}_{q; [\Pi^+]}\) and sup_{y \in [1, \infty)} ||S(y)||_S < \infty. Hence, \(S\) belongs to \(\mathcal{S}_{q; [\alpha, \infty)}\). Then Theorem 3.3 shows that there is a \(\sigma \in \mathcal{M}_{[\alpha, \infty)}^{q}([\alpha, \infty))\) such that (3.3) holds true. Let \(S^\prime: \Pi^+ \to \mathbb{C}_{\geq 0}^{q \times q}\) be defined by \(S^\prime(z) := S^\prime(\hat{z})\). Thus, from (3.3), we get

\[
S^\prime(z) = \left[\int_{[\alpha, \infty)} \frac{1}{t - z} \sigma(dt)\right]^* = \int_{[\alpha, \infty)} \frac{1}{t - z} \sigma(dt) = S(z)
\]
for each $z \in \Pi_-$. Taking into account (6.7) and Lemma 4.8, we see then that, for each $k \in \mathbb{Z}_{-1,m}$ and each $z \in \Pi_-$, there exists a matrix $X_k(z)$ such that $P_k^{[S]}(z) = P_k^{[S]}(z) = X_k(z)P_k^{[S]}(z)X_k^*(z)$ is fulfilled for every choice of $k \in \mathbb{Z}_{-1,m}$ and $z \in \Pi_-$. In view of (6.7), this implies $P_k^{[S]}(z) \geq 0$ for each $k \in \mathbb{Z}_{-1,m}$ and each $z \in \Pi_-$. Because of $\mathbb{C} \setminus \mathbb{R} = \Pi_+ \cup \Pi_-$, the proof is complete.

**Remark 6.6.** For each $n \in \mathbb{N}_0$ and every choice of $w$ and $z \in \mathbb{C}$, it is readily checked that

$$(z - \overline{w}) T_{q,n}^*(w) \overline{T_{q,n}^* R_{q,n}(z) = R_{q,n}(z) - \overline{T_{q,n}^* R_{q,n}(z)}.$$  (6.8)

**Lemma 6.7.** Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of Hermitian complex $q \times q$ matrices. Then

$$H_n T_{q,n}^* R_{q,n}^*(z) - \overline{R_{q,n}^*(w)} T_{q,n}^* H_n + \overline{R_{q,n}^*(w)} \left( v_{q,n}^*u_n^* - u_n v_{q,n}^* \right) R_{q,n}^*(z)$$

$$= (z - \overline{w}) \left( \overline{T_{q,n}^* R_{q,n}^*(z)} + \overline{T_{q,n}^* R_{q,n}^*(z)} \right).$$

(6.8)

for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$ and every choice of $w$ and $z \in \mathbb{C}$. Furthermore,

$$(z - \overline{w}) T_{q,n}^* R_{q,n}^*(z) - \overline{R_{q,n}^*(w)} T_{q,n}^* H_n$$

$$= (z - \overline{w}) \left( \overline{T_{q,n}^* R_{q,n}^*(z)} + \overline{T_{q,n}^* R_{q,n}^*(z)} \right).$$

(6.9)

for all $\alpha \in \mathbb{R}$, all $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, and every choice of $w$ and $z \in \mathbb{C}$.

**Proof.** By virtue of Remark 4.3, we have

$$H_n T_{q,n}^* R_{q,n}^*(z) - \overline{R_{q,n}^*(w)} T_{q,n}^* H_n + \overline{R_{q,n}^*(w)} \left( v_{q,n}^*u_n^* - u_n v_{q,n}^* \right) R_{q,n}^*(z)$$

$$= \overline{R_{q,n}^*(w)} \left( \overline{T_{q,n}^* R_{q,n}^*(z)} + \overline{T_{q,n}^* R_{q,n}^*(z)} \right).$$

Using Remark 4.3, equation (6.9) can be proved analogous to (6.8).}

**Notation 6.8.** Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence from $\mathbb{C}^{q \times q}$. Let $G$ be a subset of $\mathbb{C}$ with $G \setminus \mathbb{R} \neq \emptyset$ and let $f: G \rightarrow \mathbb{C}^{q \times q}$ be a matrix-valued function. For each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let $F_{2n}: G \rightarrow \mathbb{C}^{(n+1)q \times (n+1)q}$ be given by

$$F_{2n}(z) := H_n T_{q,n}^* R_{q,n}^*(z) + \overline{R_{q,n}^*(w)} \left( v_{q,n}^*u_n^* - u_n v_{q,n}^* \right) R_{q,n}^*(z)$$

(6.10)

and let $Q_{2n}^f: G \setminus \mathbb{R} \rightarrow \mathbb{C}^{(2n+2)q \times (2n+2)q}$ be defined by

$$Q_{2n}^f(z) := \left[ \frac{H_n}{F_{2n}(z)} \frac{F_{2n}(z) - F_{2n}(z)}{z - \overline{w}} \right].$$

(6.11)
If $\kappa \geq 1$, then, for all $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, let $F_{2n+1} : \mathcal{G} \to \mathbb{C}^{(n+1)q \times (n+1)q}$ be given by

$$F_{2n+1}(z) := H_{\alpha n} T_{q,n}^* R_{T_{q,n}}^* (z)$$

$$+ R_{T_{q,n}}(z)[v_{q,n}(z - \alpha)f(z) - (-\alpha u_n - y_{0,n})]v_{q,n}^* R_{T_{q,n}}^* (z)$$  \hspace{1cm} (6.12)

and let $Q_{2n+1}^f : \mathcal{G} \setminus \mathbb{R} \to \mathbb{C}^{(2n+2)q \times (2n+2)q}$ be defined by

$$Q_{2n+1}^f(z) := \left[ \begin{array}{cc} H_{\alpha n} & F_{2n+1}(z) \\ F_{2n+1}^*(z) & F_{2n+1}(z) - F_{2n+1}^*(z) \end{array} \right].$$  \hspace{1cm} (6.13)

**Proposition 6.9.** Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^n$ be a sequence of Hermitian complex $q \times q$ matrices. Let $f : \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^q$ be a matrix-valued function. Further, for each $k \in \mathbb{N}_0$, let $m_k$ be given by (1.10) and let $F_k : \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{(m_k+1)q \times (m_k+1)q}$ be defined by Notation 6.8. For all $k \in \mathbb{Z}_{0,\kappa}$, then there are functions $\Gamma_k : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{(m_k+2)q \times (2m_k+2)q}$ and $\Delta_k : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{(2m_k+2)q \times (m_k+2)q}$ such that $P_k^f(z) = \Gamma_k(z) Q_k^f(z) \Gamma_k^*(z)$ and $Q_k^f(z) = \Delta_k(z) P_k^f(z) \Delta_k^*(z)$ hold true for each $z \in \mathbb{C} \setminus \mathbb{R}$.

**Proof.** (I) In the trivial case $k = 0$, choose $\Gamma_0 : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2q \times 2q}$ and $\Delta_0 : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2q \times 2q}$ given by $\Gamma_0(z) := I_{2q}$ and $\Delta_0(z) := I_{2q}$.

(II) Now we consider the case that $\kappa \geq 1$ and that $n \in \mathbb{N}_0$ is such that $2n + 1 \leq \kappa$. Let $\Delta_{2n+1} : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{(2n+2)q \times (n+2)q}$ be defined by

$$\Delta_{2n+1}(z) := \left[ \begin{array}{cc} I_{(n+1)q} & 0_{(n+1)q \times q} \\ [(R_{T_{q,n}}^* (z))^* T_{q,n}] & v_{q,n} \end{array} \right]$$  \hspace{1cm} (6.14)

and let $\Gamma_{2n+1} : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{(n+2)q \times (2n+2)q}$ be given by

$$\Gamma_{2n+1}(z) := \left[ \begin{array}{cc} I_{(n+1)q} & 0_{(n+1)q \times q} \\ -v_{q,n}^*[(R_{T_{q,n}}^* (z))^* T_{q,n}] & v_{q,n} \end{array} \right].$$  \hspace{1cm} (6.15)

Since $s_j = s_j$ holds true for each $j \in \mathbb{Z}_{0,\kappa}$, we have

$$H_{\alpha n}^* = H_{\alpha n}. \hspace{1cm} (6.16)$$

We consider an arbitrary $z \in \mathbb{C} \setminus \mathbb{R}$. Let

$$B_{2n+1}(z) := R_{T_{q,n}}(z)[v_{q,n}(z - \alpha)f(z) - (-\alpha u_n - y_{0,n})], \hspace{1cm} (6.17)$$

let

$$C_{2n+1}(z) := \frac{(z - \alpha)f(z) - [(z - \alpha)f(z)]^*}{z - \overline{z}} \hspace{1cm} (6.18)$$

and let

$$\Delta_{2n+1}(z) B_{2n+1}^f(z) \Delta_{2n+1}^*(z) = \left[ \begin{array}{cc} X_{2n+1}(z) & Y_{2n+1}(z) \\ Z_{2n+1}(z) & W_{2n+1}(z) \end{array} \right] \hspace{1cm} (6.19)$$

be the $(n+1)q \times (n+1)q$ block representation of $\Delta_{2n+1}(z) P_{2n+1}^f(z) \Delta_{2n+1}^*(z)$. Then

$$P_{2n+1}^f(z) = \left[ \begin{array}{cc} H_{\alpha n} & B_{2n+1}(z) \\ C_{2n+1}^*(z) & C_{2n+1}(z) \end{array} \right]. \hspace{1cm} (6.20)$$
Using (6.19), (6.14), and (6.20), straightforward calculations show that

\[
X_{2n+1}(z) = H_{n+1},
\]

\[
Y_{2n+1}(z) = H_{n+1} T_{q,n}^* R_{q,n}^* (z) + B_{2n+1}(z) v_{q,n}^* R_{q,n}^* (z),
\]

\[
Z_{2n+1}(z) = \left[ R_{q,n}^* (z) \right]^* T_{q,n} H_{n+1} + \left[ R_{q,n}^* (z) \right]^* v_{q,n}^* B_{2n+1}(z),
\]

and

\[
W_{2n+1}(z) = \left[ R_{q,n}^* (z) \right]^* T_{q,n} H_{n+1} T_{q,n}^* R_{q,n}^* (z) + \left[ R_{q,n}^* (z) \right]^* T_{q,n} B_{2n+1}(z) v_{q,n}^* R_{q,n}^* (z)
\]

\[
+ \left[ R_{q,n}^* (z) \right]^* v_{q,n}^* B_{2n+1}(z) T_{q,n}^* R_{q,n}^* (z) + \left[ R_{q,n}^* (z) \right]^* v_{q,n}^* C_{2n+1}(z) v_{q,n}^* R_{q,n}^* (z)
\]

hold true. Because of (6.23), (6.17), and (6.12), we see that

\[
Y_{2n+1}(z) = H_{n+1} T_{q,n}^* R_{q,n}^* (z) + R_{q,n}(z) [v_{q,n}(z - \alpha)f(z) - (-\alpha u_n - y_0,n)] v_{q,n}^* R_{q,n}^* (z)
\]

\[
= F_{2n+1}(z)
\]

is valid. From (6.29), (6.16), (6.22), and (6.25) we obtain then

\[
Z_{2n+1}(z) = Y_{2n+1}(z) = F_{2n+1}(z).
\]

Using (6.23), it follows

\[
W_{2n+1}(z) = \left[ R_{q,n}^* (z) \right]^* T_{q,n} H_{n+1} T_{q,n}^* R_{q,n}^* (z) + \left[ R_{q,n}^* (z) \right]^* T_{q,n} B_{2n+1}(z) v_{q,n}^* R_{q,n}^* (z)
\]

\[
+ \left[ R_{q,n}^* (z) \right]^* v_{q,n}^* B_{2n+1}(z) T_{q,n}^* R_{q,n}^* (z) + \left[ R_{q,n}^* (z) \right]^* v_{q,n}^* C_{2n+1}(z) v_{q,n}^* R_{q,n}^* (z).
\]

In view of Lemma 6.7, we have

\[
(z - \tau) \left[ R_{q,n}^* (z) \right]^* T_{q,n} H_{n+1} T_{q,n}^* R_{q,n}^* (z)
\]

\[
= H_{n+1} T_{q,n}^* R_{q,n}^* (z) - \left[ R_{q,n}^* (z) \right]^* T_{q,n} H_{n+1}
\]

\[
+ \left[ R_{q,n}^* (z) \right]^* \left[ v_{q,n}(-\alpha u_n - y_0,n) - (-\alpha u_n - y_0,n) v_{q,n}^* \right] R_{q,n}^* (z).
\]

By virtue of (6.17), Remark 6.6 and (6.14), we conclude

\[
(z - \tau) \left[ R_{q,n}^* (z) \right]^* T_{q,n} B_{2n+1}(z) v_{q,n}^* R_{q,n}^* (z)
\]

\[
= (z - \tau) \left[ R_{q,n}^* (z) \right]^* T_{q,n} R_{q,n}^* (z) [v_{q,n}(z - \alpha)f(z) - (-\alpha u_n - y_0,n)] v_{q,n}^* R_{q,n}^* (z)
\]

\[
= \left( R_{q,n}^* (z) \right)^* [v_{q,n}(z - \alpha)f(z) - (-\alpha u_n - y_0,n)] v_{q,n}^* R_{q,n}^* (z)
\]

\[
= R_{q,n}^* (z) [v_{q,n}(z - \alpha)f(z) - (-\alpha u_n - y_0,n)] v_{q,n}^* R_{q,n}^* (z)
\]

\[
- \left[ R_{q,n}^* (z) \right]^* \left[ v_{q,n}(z - \alpha)f(z) - (-\alpha u_n - y_0,n) \right] v_{q,n}^* R_{q,n}^* (z)
\]

\[
= F_{2n+1}(z) - H_{n+1} T_{q,n}^* R_{q,n}^* (z)
\]

\[
- \left[ R_{q,n}^* (z) \right]^* [v_{q,n}(z - \alpha)f(z) - (-\alpha u_n - y_0,n)] v_{q,n}^* R_{q,n}^* (z)
\]

\[
= F_{2n+1}(z) - H_{n+1} T_{q,n}^* R_{q,n}^* (z) - \left[ R_{q,n}^* (z) \right]^* v_{q,n}(z - \alpha)f(z) v_{q,n}^* R_{q,n}^* (z)
\]

\[
+ \left[ R_{q,n}^* (z) \right]^* (-\alpha u_n - y_0,n) v_{q,n}^* R_{q,n}^* (z),
\]

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which implies

\[(z - \overline{z})\left(\left[R_{q,n}^*(z)\right]^* T_{q,n} B_{2n+1(z)} v_{q,n}^* R_{q,n}^*(z)\right)^* \]

\[= -F_{2n+1}^*(z) + \left[R_{q,n}^*(z)\right]^* T_{q,n} H_{\alpha n}^* + \left[R_{q,n}^*(z)\right]^* v_{q,n}^* (z - \alpha f(z))^* v_{q,n}^* R_{q,n}^*(z) \]

\[= -\left[R_{q,n}^*(z)\right]^* v_{q,n}^* (z - \alpha f(z))^* v_{q,n}^* R_{q,n}^*(z). \quad (6.30)\]

Taking into account (6.18) we get

\[(z - \overline{z})\left[R_{q,n}^*(z)\right]^* v_{q,n} C_{2n+1(z)} v_{q,n}^* R_{q,n}^*(z) \]

\[= \left[R_{q,n}^*(z)\right]^* v_{q,n}^* (z - \alpha f(z))^* v_{q,n}^* R_{q,n}^*(z) \]

\[- \left[R_{q,n}^*(z)\right]^* v_{q,n}^* (z - \alpha f(z))^* v_{q,n}^* R_{q,n}^*(z). \quad (6.31)\]

and, taking (6.27), (6.28), (6.29), (6.30), (6.31), and (6.16) into account, furthermore

\[(z - \overline{z}) W_{2n+1(z)} \]

\[= (z - \overline{z})\left[R_{q,n}^*(z)\right]^* T_{q,n} H_{\alpha n}^* R_{q,n}^*(z) \]

\[= (z - \overline{z})\left[R_{q,n}^*(z)\right]^* T_{q,n} H_{\alpha n}^* R_{q,n}^*(z) \]

\[+ (z - \overline{z})\left[R_{q,n}^*(z)\right]^* T_{q,n} B_{2n+1(z)} v_{q,n}^* R_{q,n}^*(z) \]

\[+ (z - \overline{z})\left[R_{q,n}^*(z)\right]^* T_{q,n} B_{2n+1(z)} v_{q,n}^* R_{q,n}^*(z) \]

\[+ (z - \overline{z})\left[R_{q,n}^*(z)\right]^* v_{q,n} C_{2n+1(z)} v_{q,n}^* R_{q,n}^*(z) \]

\[= H_{\alpha n}^* R_{q,n}^*(z) - \left[R_{q,n}^*(z)\right]^* T_{q,n} H_{\alpha n}^* R_{q,n}^*(z) \]

\[+ \left[R_{q,n}^*(z)\right]^* v_{q,n}^* (-\alpha u_n - y_0, n)^* - (\alpha u_n - y_0, n) v_{q,n}^* R_{q,n}^*(z) \]

\[+ H_{\alpha n}^* R_{q,n}^*(z) - \left[R_{q,n}^*(z)\right]^* v_{q,n}^* (z - \alpha f(z))^* v_{q,n}^* R_{q,n}^*(z) \]

\[+ \left[R_{q,n}^*(z)\right]^* v_{q,n}^* (-\alpha u_n - y_0, n) v_{q,n}^* R_{q,n}^*(z) \]

\[+ \left[R_{q,n}^*(z)\right]^* T_{q,n} H_{\alpha n}^* + \left[R_{q,n}^*(z)\right]^* v_{q,n}^* (z - \alpha f(z))^* v_{q,n}^* R_{q,n}^*(z) \]

\[+ \left[R_{q,n}^*(z)\right]^* v_{q,n}^* (-\alpha u_n - y_0, n) R_{q,n}^*(z) \]

\[+ \left[R_{q,n}^*(z)\right]^* v_{q,n}^* (z - \alpha f(z))^* v_{q,n}^* R_{q,n}^*(z) \]

\[= F_{2n+1(z)} - F_{2n+1(z)}. \quad (6.32)\]

From (6.19), (6.21), (6.24), (6.26), (6.32), and (6.13) we infer

\[\Delta_{2n+1(z)} F_{2n+1(z)} = Q_{2n+1(z)}. \quad (6.33)\]

In view of \(v_{q,n}^* [R_{q,n}^*(z)] v_{q,n} = I_q\), we easily see that the matrices \(\Gamma_{2n+1(z)}\) and \(\Delta_{2n+1(z)}\) given by (6.15) and (6.14) obviously fulfill

\[\Gamma_{2n+1(z)} \Delta_{2n+1(z)} = I_q. \quad (6.34)\]
Thus, because of (6.33), we obtain
\[ P_{2n+1}^{(f)}(z) = I_{(n+2)q} P_{2n+1}^{(f)}(z) I_{(n+2)q}^* \]
\[ = \Gamma_{2n+1}(z) \Delta_{2n+1}(z) P_{2n+1}^{(f)}(z) \Gamma_{2n+1}^*(z) \Delta_{2n+1}^*(z) = \Gamma_{2n+1}(z) Q_{2n+1}^{(f)}(z) \Gamma_{2n+1}^*(z). \]

In this case \( k = 2n + 1 \) with some \( n \in \mathbb{N}_0 \), the proof is complete.

(III) Now we consider the case that \( \kappa \geq 2 \) and that there is an \( n \in \mathbb{N} \) such that \( k = 2n \). Let \( \Gamma_{2n} := \Gamma_{2n+1} \) and let \( \Delta_{2n} := \Delta_{2n+1} \). We consider again an arbitrary \( z \in \mathbb{C} \setminus \mathbb{R} \). Let
\[
\Delta_{2n}(z) P_{2n}^{(f)}(z) \Delta_{2n}^*(z) = \begin{bmatrix} X_{2n}(z) & Y_{2n}(z) \\ Z_{2n}(z) & W_{2n}(z) \end{bmatrix} \tag{6.35}
\]
be the \((n + 1)q \times (n + 1)q\) block representation of \( \Delta_{2n}(z) P_{2n}^{(f)}(z) \Delta_{2n}^*(z) \). Setting
\[
B_{2n}(z) := R_{T_{q,n}}(z)[v_{q,n}f(z) - u_n] \quad \text{and} \quad C_{2n}(z) := \frac{f(z) - f^*(z)}{z - \overline{z}}, \tag{6.36}
\]
we have
\[
P_{2n}^{(f)}(z) = \begin{bmatrix} H_n & B_{2n}(z) \\ B_{2n}^*(z) & C_{2n}(z) \end{bmatrix}. \tag{6.37}
\]
From (6.33) and (6.37) we easily see then that
\[
X_{2n}(z) = H_n, \quad Y_{2n}(z) = H_n T_{q,n}^* R_{T_{q,n}}^*(z) + B_{2n}(z) v_{q,n}^* R_{T_{q,n}}^*(z), \tag{6.38}
\]
\[
Z_{2n}(z) = \begin{bmatrix} R_{T_{q,n}}^*(z) \\ T_{q,n} H_n + \begin{bmatrix} R_{T_{q,n}}^*(z) \end{bmatrix}^* v_{q,n} B_{2n}(z) \end{bmatrix} \tag{6.39}
\]
and
\[
W_{2n}(z) = \begin{bmatrix} R_{T_{q,n}}^*(z) \\ T_{q,n} H_n + \begin{bmatrix} R_{T_{q,n}}^*(z) \end{bmatrix}^* v_{q,n} B_{2n}(z) \end{bmatrix} \begin{bmatrix} v_{q,n} B_{2n}(z) \\ \begin{bmatrix} v_{q,n}^* R_{T_{q,n}}^*(z) \end{bmatrix} \end{bmatrix} \tag{6.40}
\]
hold true. Because of (6.38), (6.39), and (6.41), we obtain
\[
Y_{2n}(z) = H_n T_{q,n}^* R_{T_{q,n}}^*(z) + R_{T_{q,n}}(z)[v_{q,n}f(z) - u_n] v_{q,n}^* R_{T_{q,n}}^*(z) = F_{2n}(z). \tag{6.41}
\]
Since \( s_j^* = s_j \) is supposed for each \( j \in Z_{0,e} \), we get \( H_n^* = H_n \). Consequently, in view of (6.39), (6.38), and (6.41), then
\[
Z_{2n}(z) = Y_{2n}^*(z) = F_{2n}^*(z) \tag{6.42}
\]
follows. By virtue of (6.40) and (6.39), we see that
\[
W_{2n}(z) = \begin{bmatrix} R_{T_{q,n}}^*(z) \\ T_{q,n} H_n T_{q,n}^* R_{T_{q,n}}^*(z) \end{bmatrix}^* v_{q,n} \begin{bmatrix} f(z) v_{q,n}^* - u_n^* \\ [R_{T_{q,n}}^*(z)]^* T_{q,n}^* R_{T_{q,n}}^*(z) \end{bmatrix} + \begin{bmatrix} R_{T_{q,n}}(z) \end{bmatrix}^* T_{q,n} R_{T_{q,n}}(z) [v_{q,n} f(z) - u_n] v_{q,n}^* R_{T_{q,n}}^*(z) \tag{6.43}
\]
\[ + \begin{bmatrix} R_{T_{q,n}}^*(z) \end{bmatrix}^* v_{q,n} \begin{bmatrix} f(z) - f^*(z) \end{bmatrix} v_{q,n} R_{T_{q,n}}^*(z). \]
holds true. Taking into account (6.43) and Remark 6.6, we conclude

\[ W_{2n}(z) = \left[ R_{\gamma_n}(z) \right]^* T_{q,n} H_n^* T_{q,n}^* R_{\gamma_n}^* (z) \]

\[ + \left[ R_{\gamma_n}(z) \right]^* v_{q,n} \left[ f^*(z) v_{q,n}^* - u_n^* \right] \left( \frac{1}{z - \bar{z}} \left( R_{\gamma_n}(z) - R_{\gamma_n}^*(z) \right) \right) \]

\[ + \left[ R_{\gamma_n}(z) \right]^* v_{q,n} \left[ \frac{f(z) - f^*(z)}{z - \bar{z}} \right] v_{q,n}^* R_{\gamma_n}^* (z). \] \tag{6.44}

Using Lemma 6.7, the equation \( H_n^* = H_n \), (6.10), and (6.44), we infer

\[ W_{2n}(z) = \frac{1}{z - \bar{z}} \left( H_n^* T_{q,n}^* R_{\gamma_n}^* (z) - R_{\gamma_n}^* (z) \right) T_{q,n} H_n^* \]

\[ + \left[ R_{\gamma_n}(z) \right]^* v_{q,n} \left[ f^*(z) v_{q,n}^* - u_n^* \right] \left( R_{\gamma_n}^* (z) - \left[ R_{\gamma_n}(z) \right]^* \right) \]

\[ + \left( R_{\gamma_n}(z) - \left[ R_{\gamma_n}(z) \right]^* \right) \left[ v_{q,n} f(z) - u_n \right] v_{q,n}^* R_{\gamma_n}^* (z) \]

\[ + \left[ R_{\gamma_n}(z) \right]^* v_{q,n} \left[ f(z) - f^*(z) \right] v_{q,n}^* R_{\gamma_n}^* (z) \] \tag{6.45}

\[ = \frac{1}{z - \bar{z}} \left( H_n^* T_{q,n}^* R_{\gamma_n}^* (z) + R_{\gamma_n}^* (z) \right) \left[ v_{q,n} f(z) - u_n \right] v_{q,n}^* R_{\gamma_n}^* (z) \]

\[ = \frac{1}{z - \bar{z}} \left[ F_{2n}(z) - F_{2n}(z) \right]. \]

Thus, (6.35), the first equation in (6.38), (6.41), (6.42), (6.45), and (6.11) show that

\[ \Delta_{2n}(z) P_{2n}^f(z) \Delta_{2n}^*(z) = \left[ H_n \quad F_{2n}(z) \right] \left[ F_{2n}(z) - F_{2n}(z) \right] = Q_{2n}^f(z) \] \tag{6.46}

is valid. Because of \( \Gamma_2 = \Gamma_{2n+1} \) and \( \Delta_{2n} = \Delta_{2n+1} \), equation (6.34) implies \( \Gamma_{2n}(z) \Delta_{2n}(z) = I_{(n+2)q} \). Consequently, from (6.46) we get

\[ P_{2n}^f(z) = I_{(n+2)q} P_{2n}^f(z) I_{(n+2)q} = \Gamma_{2n}(z) \Delta_{2n}(z) P_{2n}^f(z) \left[ \Gamma_{2n}(z) \Delta_{2n}(z) \right]^* \]

\[ = \Gamma_{2n}(z) \Delta_{2n}(z) P_{2n}^f(z) \Delta_{2n}^*(z) \Gamma_{2n}^*(z) = \Gamma_{2n}(z) Q_{2n}^f(z) \Gamma_{2n}^*(z). \]

**Lemma 6.10.** Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N}_0 \cup \{ \infty \} \), and let \( (s_j)_{j=0}^\kappa \) be a sequence from \( \mathbb{C}^{q \times q} \). Let \( f : \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q \times q} \) be a holomorphic matrix-valued function. Then:

(a) Let \( n \in \mathbb{N}_0 \) be such that \( 2n \leq \kappa \). If \( P_{2n}^f(z) \in \mathbb{C}^{(n+2)q \times (n+2)q} \) holds true for each \( z \in \mathbb{C} \setminus \mathbb{R} \), then \( F_{2n} : \Pi_+ \to \mathbb{C}^{(n+1)q \times (n+1)q} \) given by (6.10) belongs to \( \mathcal{R}_{0,(n+1)q}(\Pi_+) \) and the matricial spectral measure \( \mu_{2n} \) of \( F_{2n} \) fulfills \( \mu_{2n}(\mathbb{R}) \leq H_n \).

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(b) Let $n \in \mathbb{N}_0$ be such that $2n + 1 \leq \kappa$. If $P_{2n+1}^f(z) \in \mathbb{C}^{(n+2)\times(n+2)}$ for each $z \in \mathbb{C} \setminus \mathbb{R}$, then $F_{2n+1}: \Pi_+ \to \mathbb{C}^{(n+1)\times(n+1)}$ defined by (6.12) belongs to $\mathcal{R}^{(0,(n+1))}\Pi_+\) and the matricial spectral measure $\mu_{2n+1}$ of $F_{2n+1}$ fulfills $\mu_{2n+1}(\mathbb{R}) \leq H_{\alpha_0\kappa}$.

Proof. \(\Box\) Let $z \in \Pi_+$. Suppose that the matrix $P_{2n+1}^f(z)$ is non-negative Hermitian. From (4.4) we see then that $H_{\alpha_0\kappa}$ is non-negative Hermitian as well. In particular, $H_{\alpha}^* = H_{\alpha_0\kappa}$. This implies $s_j^* = s_j$ for each $j \in \mathbb{Z}_{0,2n}$. According to Proposition 6.9, there is a function $\Delta_{2n}: \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{(n+2)\times(n+2)}$ such that $\Delta_{2n}(z)P_{2n}^f(z)\Delta_{2n}^*(z) = Q_{2n}(z)$. Thus, since $P_{2n}^f(z)$ is non-negative Hermitian, the matrix $Q_{2n}^f(z)$ is non-negative Hermitian as well. Taking Notation 6.8 into account, this means that the block matrix on the right-hand side of (6.11) is non-negative Hermitian. Since $R_{T_{q,n}}$ and $R_{T_{q,n}}^*$ are matrix polynomials, we see from (6.10) that $F_{2n}$ is holomorphic in $\Pi_+$, and, thus, the application of Lemma 6.3 yields $F_{2n} \in \mathcal{R}^{(0,(n+1))}\Pi_+\) and $\mu_{2n}(\mathbb{R}) \leq H_{\alpha_0\kappa}$.

(b) Part (b) can be proved analogously. We omit the details. \(\square\)

Lemma 6.11. Let $\alpha \in \mathbb{R}$, let $f: \mathbb{C} \setminus \{\alpha, \infty\} \to \mathbb{C}^{q\times q}$ be a matrix-valued function, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j^*)_{j=0}^n$ be a sequence of Hermitian complex $q \times q$ matrices. Then:

(a) Let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$, let $F_{2n}: \Pi_+ \to \mathbb{C}^{(n+1)\times(n+1)}$ be defined by (6.10), and let $\Psi_{2n}: \mathbb{C} \to \mathbb{C}^{(n+1)\times(n+1)}$ be given by

$$\Psi_{2n}(z) := R_{T_{q,n}}(z)(H_{\alpha_0\kappa}T_{q,n}^* - u_n^*v_n^* - zT_{q,n}H_{\alpha_0\kappa}T_{q,n}^*)R_{T_{q,n}}^*(z).$$ (6.47)

Then $\Psi_{2n}$ is a continuous matrix-valued function such that $\Psi_{2n}(\mathbb{R}) \subseteq \mathbb{C}^{(n+1)\times(n+1)}$. In view of (6.3), furthermore,

$$F_{2n}(z) = \Psi_{2n}(z) + E_{q,n}(z)f(z)E_{q,n}^*(z) \quad \text{for each} \quad z \in \Pi_+. \quad (6.48)$$

(b) Let $n \in \mathbb{N}_0$ be such that $2n + 1 \leq \kappa$, let $F_{2n+1}: \Pi_+ \to \mathbb{C}^{(n+1)\times(n+1)}$ be defined by (6.12), and let $\Psi_{2n+1}: \mathbb{C} \to \mathbb{C}^{(n+1)\times(n+1)}$ be given by

$$\Psi_{2n+1}(z) := R_{T_{q,n}}(z)[H_{\alpha_0\kappa}T_{q,n}^* - (-\alpha u_n - y_{0,n})v_n^* - zT_{q,n}H_{\alpha_0\kappa}T_{q,n}^*]R_{T_{q,n}}^*(z). \quad (6.49)$$

Then $\Psi_{2n+1}$ is a continuous matrix-valued function such that $\Psi_{2n+1}(\mathbb{R}) \subseteq \mathbb{C}^{(n+1)\times(n+1)}$. Moreover,

$$F_{2n+1}(z) = \Psi_{2n+1}(z) + E_{q,n}(z)[(z - \alpha)f(z)]E_{q,n}^*(z) \quad \text{for each} \quad z \in \Pi_+. \quad (6.48)$$

Proof. \(\Box\) The case $n = 0$ is trivial. Suppose now $0 < 2n \leq \kappa$. Remark 6.4 shows that $\Psi_{2n}$ is a matrix polynomial. In particular, $\Psi_{2n}$ is continuous. By assumption, we have $s_j^* = s_j$ for each $j \in \mathbb{Z}_{0,2n}$. Thus, $H_{\alpha_0\kappa} = H_{\alpha_0\kappa}$. For each $x \in \mathbb{R}$, we have $R_{T_{q,n}}^*(x) = [R_{T_{q,n}}(x)]^*$ and, consequently,

$$[\Psi_{2n}(x)]^* = R_{T_{q,n}}(x)(T_{q,n}H_{\alpha_0\kappa} - v_n^*u_n^* - xT_{q,n}H_{\alpha_0\kappa}T_{q,n}^*)R_{T_{q,n}}^*(x),$$

which, in view of $H_{\alpha_0\kappa} = H_{\alpha_0\kappa}$, implies that

$$[\Psi_{2n}(x)]^* = R_{T_{q,n}}(x)(-H_{\alpha_0\kappa}T_{q,n}^* - T_{q,n}H_{\alpha_0\kappa}) + H_{\alpha_0\kappa}T_{q,n}^* - v_n^*u_n^* - xT_{q,n}H_{\alpha_0\kappa}T_{q,n}^*)R_{T_{q,n}}^*(x) = R_{T_{q,n}}(x)(-u_n^*v_n^* - v_n^*u_n^* + H_{\alpha_0\kappa}T_{q,n}^* - v_n^*u_n^* - xT_{q,n}H_{\alpha_0\kappa}T_{q,n}^*)R_{T_{q,n}}^*(x) = R_{T_{q,n}}(x)(H_{\alpha_0\kappa}T_{q,n}^* - u_n^*v_n^* - xT_{q,n}H_{\alpha_0\kappa}T_{q,n}^*)R_{T_{q,n}}^*(x) = \Psi_{2n}(x)$$

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hold true for all \( x \in \mathbb{R} \). Hence, \( \Psi_{2n}(\mathbb{R}) \subseteq \mathbb{C}^{(n+1)q \times (n+1)q} \). Taking into account (6.11), Remark 4.4 and (6.37), for all \( z \in \Pi_+ \), we see then that

\[
F_{2n}(z) = R_{T_{q,n}}(z) \left[ R_{T_{q,n}}(z) \right]^{-1} H_n T^*_{q,n} R_{T_{q,n}}(z) + R_{T_{q,n}}(z) v_{q,n} f(z) v^*_{q,n} R_{T_{q,n}}(z)
\]

\[
= R_{T_{q,n}}(z) \left[ (I_{(n+1)q} - z T_{q,n}) H_n T^*_{q,n} - u_n v^*_{q,n} \right] R_{T_{q,n}}(z) + R_{T_{q,n}}(z) v_{q,n} f(z) v^*_{q,n} [R_{T_{q,n}}(\overline{\tau})]^* 
\]

\[
= R_{T_{q,n}}(z) (H_n T^*_{q,n} - z T_{q,n} H_n T^*_{q,n} - u_n v^*_{q,n}) R_{T_{q,n}}(z) + R_{T_{q,n}}(z) v_{q,n} f(z) [R_{T_{q,n}}(\overline{\tau}) v_{q,n}]^*
\]

\[
= \Psi_{2n}(z) + E_{q,n}(z) f(z) E^*_{q,n}(\overline{\tau}).
\]

- Part (II) can be proved analogously.

**Lemma 6.12.** Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N} \cup \{\infty\} \), let \( (s_j)_{j=0}^{\infty} \) be a sequence from \( \mathbb{C}^{q \times q} \), and let \( n \in \mathbb{N}_0 \) be such that \( 2n + 1 \leq \kappa \). Further, let \( S \in \mathbb{S}_{0,\kappa};[\alpha,\infty) \) be such that

\[
P_{2n}^S(z) \in \mathbb{C}^{(n+2)q \times (n+2)q} \quad \text{and} \quad P_{2n+1}^S(z) \in \mathbb{C}^{(n+2)q \times (n+2)q} \quad \text{for all} \quad z \in \Pi_+.
\]

Then the \( [\alpha,\infty) \)-Stieltjes measure \( \sigma \) of \( S \) belongs to \( \mathcal{M}_{\geq 1}(][\alpha,\infty) \).

**Proof.** (I) For all \( z \in \Pi_+ \), from Remark 4.4 we see that (4.8) holds true and, in view of (6.50), hence, that the block matrix on the left-hand side of (4.8) is non-negative Hermitian. Consequently, since \( S \) is holomorphic in \( \mathbb{C} \setminus [\alpha,\infty) \), Lemma 6.11 yields that \( F := \text{Rstr}_{\Pi_+} S \) belongs to \( \mathcal{R}_{0,q}(\Pi_+) \) and that the matricial spectral measure \( \mu \) of \( F \) fulfills \( \mu(\mathbb{R}) \leq s_0 \). Thus, Remark 3.9 provides us

\[
\sigma([\alpha,\infty)) = \text{Rstr}_{[\alpha,\infty)} \mu([\alpha,\infty)) = \mu([\alpha,\infty)) \leq \mu(\mathbb{R}) \leq s_0.
\]

Because of (6.50) and (4.4), the matrix \( H_n \) is non-negative Hermitian. In particular, \( s_0 \in \mathbb{C}^{q \times q} \). Hence,

\[
s_0 = s_0 \quad \text{and} \quad \{u^* \sigma([\alpha,\infty)) u, u^* s_0 u\} \subseteq [0,\infty) \quad \text{for all} \quad u \in \mathbb{C}^q.
\]

(II) In the second part of the proof, we consider an arbitrary \( n \in \mathbb{N} \) and an arbitrary \( u \in \mathbb{C}^q \). From Remark 4.4 we see then that

\[
\int_{[\alpha,\infty)} \frac{\text{in}}{t - (\text{in} + \alpha)} (u^* \sigma u)(dt) = n \int_{[\alpha,\infty)} \frac{1}{t - (\text{in} + \alpha)} (u^* \sigma u)(dt) < \infty.
\]

In view of

\[
\frac{\text{in}}{t - (\text{in} + \alpha)} = -\frac{n^2}{|t - \alpha - \text{in}|^2} + \frac{(t - \alpha)n}{|t - \alpha - \text{in}|^2}
\]

and (6.53), we obtain

\[
\int_{[\alpha,\infty)} -\frac{n^2}{|t - \alpha - \text{in}|^2} (u^* \sigma u)(dt) = \int_{[\alpha,\infty)} \mathfrak{R} \left[ \frac{\text{in}}{t - (\text{in} + \alpha)} \right] (u^* \sigma u)(dt) < \infty
\]

and

\[
\int_{[\alpha,\infty)} \frac{(t - \alpha)n}{|t - \alpha - \text{in}|^2} (u^* \sigma u)(dt) = \int_{[\alpha,\infty)} \mathfrak{R} \left[ \frac{\text{in}}{t - (\text{in} + \alpha)} \right] (u^* \sigma u)(dt) < \infty.
\]

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For each \( t \in [\alpha, \infty) \), we have
\[
\left( \frac{\text{in}}{t - (\text{in} + \alpha)} + 1 \right) = \frac{(t - \alpha)n}{t - \alpha - \text{in}} = \frac{(t - \alpha)^2}{(t - \alpha)^2 + n^2} + i \frac{(t - \alpha)n}{(t - \alpha)^2 + n^2}.
\]  
(6.56)

Consequently, we get that
\[
|\Re \left( \frac{\text{in}}{t - (\text{in} + \alpha)} + 1 \right) | = \frac{(t - \alpha)^2}{(t - \alpha)^2 + n^2} \leq n \cdot 1_{[\alpha, \infty)}(t)
\]
and
\[
|\Im \left( \frac{\text{in}}{t - (\text{in} + \alpha)} + 1 \right) | = \frac{(t - \alpha)n^2}{(t - \alpha)^2 + n^2} \leq n \cdot \frac{2|t - \alpha|n}{(t - \alpha)^2 + n^2} \leq n = n \cdot 1_{[\alpha, \infty)}(t)
\]
hold true for each \( t \in [\alpha, \infty) \). This implies
\[
\int_{[\alpha, \infty)} |\Re \left( \frac{\text{in}}{t - (\text{in} + \alpha)} + 1 \right) | (u^*\sigma u)(dt) \leq \int_{[\alpha, \infty)} n \cdot 1_{[\alpha, \infty)} d(u^*\sigma u) = nu^*\sigma([\alpha, \infty))u < \infty
\]
and, analogously,
\[
\int_{[\alpha, \infty)} |\Im \left( \frac{\text{in}}{t - (\text{in} + \alpha)} + 1 \right) | (u^*\sigma u)(dt) \leq nu^*\sigma([\alpha, \infty))u < \infty.
\]

Thus, the function \( g_n : [\alpha, \infty) \rightarrow \mathbb{C} \) given by \( g_n(t) := n(\text{in}/[t - (\text{in} + \alpha)] + 1) \) fulfills
\[
g_n \in \mathcal{L}^1([\alpha, \infty), \mathcal{B}_{[\alpha, \infty)}, u^*\sigma u; \mathbb{C}).
\]
(6.57)

Using Theorem 3.5, Remark C.2, (6.51), and (6.55), we conclude
\[
u^*[\text{in} \cdot S(\text{in} + \alpha)]u = u^* \left[ \text{in} \int_{[\alpha, \infty)} \frac{1}{t - (\text{in} + \alpha)} \sigma(dt) \right] u
\]
\[
= \int_{[\alpha, \infty)} \frac{\text{in}}{t - (\text{in} + \alpha)} (u^*\sigma u)(dt)
\]
\[
= \int_{[\alpha, \infty)} \left[ -\frac{n^2}{|t - \alpha - \text{in}|^2} + i \frac{(t - \alpha)n}{|t - \alpha - \text{in}|^2} \right] (u^*\sigma u)(dt)
\]
\[
= -n^2 \int_{[\alpha, \infty)} \frac{1}{|t - \alpha - \text{in}|^2} (u^*\sigma u)(dt) + \int_{[\alpha, \infty)} \frac{t - \alpha}{|t - \alpha - \text{in}|^2} (u^*\sigma u)(dt)
\]
and, in particular,
\[
\Re(u^*[\text{in} \cdot S(\text{in} + \alpha)]u) = -n^2 \int_{[\alpha, \infty)} \frac{1}{|t - \alpha - \text{in}|^2} (u^*\sigma u)(dt).
\]
(6.58)

Taking into account (6.51), (6.58), and that \( 1 - n^2/|t - \alpha - \text{in}| = (t - \alpha)^2/[(t - \alpha)^2 + n^2] \) holds true, for each \( t \in [\alpha, \infty) \), we get
\[
\Re(u^*[\text{in} \cdot S(\text{in} + \alpha)]u) + u^*s_0u \geq \Re(u^*[\text{in} \cdot S(\text{in} + \alpha)]u) + u^*\sigma([\alpha, \infty))u
\]
\[
= -n^2 \int_{[\alpha, \infty)} \frac{1}{|t - \alpha - \text{in}|^2} (u^*\sigma u)(dt) + \int_{[\alpha, \infty)} 1_{[\alpha, \infty)} d(u^*\sigma u)
\]
\[
= \int_{[\alpha, \infty)} \left( 1 - \frac{n^2}{|t - \alpha - \text{in}|^2} \right) (u^*\sigma u)(dt) = \int_{[\alpha, \infty)} \frac{(t - \alpha)^2}{(t - \alpha)^2 + n^2} (u^*\sigma u)(dt) \geq 0
\]
and, consequently,
\[
| \Re(u^*[\im S(\im + \alpha)]u) + u^* s_0 u|^2 \geq \Re(u^*[\im S(\im + \alpha)]u) + u^* \sigma([\alpha, \infty))u|^2. \tag{6.59}
\]

Because of (6.52), (6.59), and again (6.52), it follows
\[
|nu^*[\im S(\im + \alpha) + s_0]u|^2 = n^2|u^*[\im S(\im + \alpha)]u + u^* s_0 u|^2
\]
\[
= n^2\left(\Re(u^*[\im S(\im + \alpha)]u) + u^* s_0 u^2 + |3(u^*[\im S(\im + \alpha)]u)|^2\right)
\]
\[
\geq n^2\left(\Re(u^*[\im S(\im + \alpha)]u) + u^* \sigma([\alpha, \infty))u|^2 + |3(u^*[\im S(\im + \alpha)]u)|^2\right)
\]
\[
= n^2|u^*[\im S(\im + \alpha)]u + u^* \sigma([\alpha, \infty))u|^2 = |nu^*[\im S(\im + \alpha) + \sigma([\alpha, \infty))u|^2
\]
and, therefore,
\[
|nu^*[\im S(\im + \alpha) + s_0]u| \geq |nu^*[\im S(\im + \alpha) + \sigma([\alpha, \infty))u| \tag{6.60}
\]

Since \(S\) belongs to \(S_{0,q|[\alpha, \infty)}\), the function \(G: \Pi_+ \to \mathbb{C}^{q \times q}\) given by \(G(w) := wS(\im + \alpha) + s_0\) is holomorphic in \(\Pi_+\). From Remark 4.4 we know that, for all \(z \in \mathbb{C} \setminus [\alpha, \infty)\), equation (4.9) is true. Hence, from (6.50) we see that the block matrix on the left-hand side of (4.9) is non-negative Hermitian. Consequently, we conclude
\[
\begin{bmatrix}
-\alpha s_0 + s_1 \\
G^*(w)
\end{bmatrix}
\begin{bmatrix}
G(w) \\
\frac{G(w) - G^*(w)}{w - \bar{w}}
\end{bmatrix}
= \begin{bmatrix}
-\alpha s_0 + s_1 \\
\Re(s_0) + s_0^2
\end{bmatrix}
\begin{bmatrix}
wS(\im + \alpha) + s_0 \\
\Re(s_0) + s_0^2
\end{bmatrix}
\]
\[
\geq \begin{bmatrix}
-\alpha s_0 + s_1 \\
\Re(s_0) + s_0^2
\end{bmatrix}
\begin{bmatrix}
(w + \alpha) - \alpha S(\im + \alpha) + s_0 \\
(w + \alpha) - \alpha S(\im + \alpha) + s_0
\end{bmatrix}
\in \mathbb{C}^{2 \times 2}. \tag{6.61}
\]

Since \(G\) is holomorphic, from (6.61) and Lemma 5.3 we infer sup\(_{y \in (0, \infty)} \|y\| \leq -\alpha s_0 + s_1 \|S\) and, hence, sup\(_{n \in \mathbb{N}} \|n \cdot S(\im + \alpha) + s_0\| \leq -\alpha s_0 + s_1 \|S\). Thus, the Bunjakowski–Cauchy–Schwarz inequality provides us
\[
|u^*[n \cdot S(\im + \alpha) + s_0]u| \leq \|n \cdot S(\im + \alpha) + s_0\| \cdot \|u\|_E \cdot \|u\|_E
\]
\[
\leq n \|S(\im + \alpha) + s_0\| + s_0 \|S\| \cdot \|u\|_E \leq -\alpha s_0 + s_1 \|S\| \cdot \|u\|_E^2. \tag{6.62}
\]

For each \(t \in [\alpha, \infty)\), we have \( |t - \alpha| = \liminf_{n \to \infty} (t - \alpha) \right)^n / \right)^n \). Fatou’s lemma yields then
\[
\int_{[\alpha, \infty)} |t - \alpha| (u^* \sigma u)(dt) = \int_{[\alpha, \infty)} \liminf_{n \to \infty} \frac{(t - \alpha)^n}{(t - \alpha)^2 + n^2} (u^* \sigma u)(dt)
\]
\[
\leq \liminf_{n \to \infty} \int_{[\alpha, \infty)} \frac{(t - \alpha)^n}{(t - \alpha)^2 + n^2} (u^* \sigma u)(dt). \tag{6.63}
\]

Obviously, from (6.57) and (6.58) it follows
\[
\int_{[\alpha, \infty)} 3 \left(\frac{\im}{t - \im + 1}\right) (u^* \sigma u)(dt) = \int_{[\alpha, \infty)} \frac{(t - \alpha)^n}{(t - \alpha)^2 + n^2} (u^* \sigma u)(dt) \tag{6.64}
\]

and
\[
\int_{[\alpha, \infty]} \Im \left( n \left[ \frac{\ln}{t - (\ln + \alpha)} + 1 \right] \right) (u^* \sigma u)(dt)
\]
\[
= \Im \left( \int_{[\alpha, \infty]} n \left[ \frac{\ln}{t - (\ln + \alpha)} + 1 \right] (u^* \sigma u)(dt) \right)
\]
\[
\leq \left| \int_{[\alpha, \infty]} n \left[ \frac{\ln}{t - (\ln + \alpha)} + 1 \right] (u^* \sigma u)(dt) \right|. \quad (6.65)
\]

(III) Since (6.57) holds true for every choice of \( u \in \mathbb{C}^q \) and \( n \in \mathbb{N} \), Remark C.2 yields
\[
g_n \in \mathcal{L}^{1}([\alpha, \infty), \mathcal{B}([\alpha, \infty), \sigma; \mathbb{C}).
\]
Hence, Remark C.2 shows that
\[
\int_{[\alpha, \infty]} n \left[ \frac{\ln}{t - (\ln + \alpha)} + 1 \right] (u^* \sigma u)(dt) = u^* \left( \int_{[\alpha, \infty]} n \left[ \frac{\ln}{t - (\ln + \alpha)} + 1 \right] \sigma(dt) \right) u \quad (6.67)
\]
is valid for each \( u \in \mathbb{C}^q \) and each \( n \in \mathbb{N} \). Combining (6.62), (6.65), and (6.67), we have
\[
0 \leq \int_{[\alpha, \infty]} \frac{(t - \alpha)n^2}{(t - \alpha)^2 + n^2} (u^* \sigma u)(dt) \leq \left| u^* \left( \int_{[\alpha, \infty]} n \left[ \frac{\ln}{t - (\ln + \alpha)} + 1 \right] \sigma(dt) \right) u \right| \quad (6.68)
\]
for each \( u \in \mathbb{C}^q \) and each \( n \in \mathbb{N} \). For all \( n \in \mathbb{N} \) and all \( t \in [\alpha, \infty) \), we see that \( g_n(t) = g_n(t) \) holds true, where \( g_n: [\alpha, \infty) \to \mathbb{C} \) is given by \( g_n(t) := \ln^2/[t - (\ln + \alpha)] \). Thus, for each \( n \in \mathbb{N} \), we get \( \tilde{g}_n = g_n - n \cdot 1_{[\alpha, \infty)} \), and, in view of (6.66), then \( \tilde{g}_n \in \mathcal{L}^{1}([\alpha, \infty), \mathcal{B}([\alpha, \infty), \sigma; \mathbb{C}) \) and
\[
\int_{[\alpha, \infty]} \tilde{g}_n d\sigma = \int_{[\alpha, \infty]} g_n d\sigma - n \int_{[\alpha, \infty]} 1_{[\alpha, \infty)} d\sigma = \int_{[\alpha, \infty]} g_n d\sigma - n\sigma([\alpha, \infty)).
\]
Consequently, for each \( n \in \mathbb{N} \), we conclude
\[
\int_{[\alpha, \infty]} n \left[ \frac{\ln}{t - (\ln + \alpha)} + 1 \right] \sigma(dt) = \int_{[\alpha, \infty]} \frac{\ln^2}{t - (\ln + \alpha)} \sigma(dt) + n\sigma([\alpha, \infty))
\]
\[
= \int_{[\alpha, \infty]} \frac{1}{t - (\ln + \alpha)} \sigma(dt) + n\sigma([\alpha, \infty))
\]
\[
= \ln \cdot S([\alpha, \infty) + n\sigma([\alpha, \infty)) = n[\ln \cdot S([\alpha, \infty) + \sigma([\alpha, \infty))].
\]
Thus, because of (6.60), for each \( u \in \mathbb{C}^q \) and each \( n \in \mathbb{N} \), we obtain
\[
\left| u^* \left( \int_{[\alpha, \infty]} n \left[ \frac{\ln}{t - (\ln + \alpha)} + 1 \right] \sigma(dt) \right) u \right| \leq |nu^*[\ln \cdot S([\alpha, \infty) + s_0]u|. \quad (6.69)
\]
Taking into account (6.63), (6.68), (6.69), and (6.62), for each \( u \in \mathbb{C}^q \), we get
\[
\int_{[\alpha, \infty]} |t - \alpha| (u^* \sigma u)(dt) \leq \liminf_{n \to \infty} \int_{[\alpha, \infty]} \frac{(t - \alpha)n^2}{(t - \alpha)^2 + n^2} (u^* \sigma u)(dt)
\]
\[
\leq \liminf_{n \to \infty} \left| u^* \left( \int_{[\alpha, \infty]} n \left[ \frac{\ln}{t - (\ln + \alpha)} + 1 \right] \sigma(dt) \right) u \right|
\]
\[
\leq \liminf_{n \to \infty} |nu^*[\ln \cdot S([\alpha, \infty) + s_0]u|
\]
\[
\leq \liminf_{n \to \infty} |\alpha s_0 + s_1|u| L^2 = \|\alpha s_0 + s_1\| L < \infty.
\]
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Therefore, from (6.70) we obtain that
\[
\int_{[\alpha,\infty)} |t|(u^*\sigma u)(dt) \leq \int_{[\alpha,\infty)} (|t| - |\alpha| + |\alpha|)(u^*\sigma u)(dt)
\]
\[
= \int_{[\alpha,\infty)} |t| - |\alpha|)(u^*\sigma u)(dt) + \int_{[\alpha,\infty)} |\alpha|(u^*\sigma u)(dt)
\]
\[
\leq \| - \alpha s_0 + s_1 \|_{S} |u|_{E}^2 + |\alpha|(u^*\sigma u)([\alpha,\infty)) < \infty
\]
is true for all \( u \in C^q \). Thus, Remark C.2 provides us \( \sigma \in \mathcal{M}_{\geq 1}([\alpha,\infty)) \).

**Lemma 6.13.** Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N} \cup \{\infty\} \), let \( (s_j)_{j=0}^{\kappa} \) be a sequence from \( C^{q \times q} \), and let \( n \in \mathbb{N}_0 \) be such that \( 2n + 1 \leq \kappa \). Further, let \( S \in S_{0,q}([\alpha,\infty)) \) be such that \( P_{2n}^[[S]](z) \in C_{\geq (n+2)q \times (n+2)q} \) and \( P_{2n+1}^[[S]](z) \in C_{\geq (n+2)q \times (n+2)q} \) hold true for all \( z \in \Pi_+ \). Then:

(a) The \([\alpha,\infty)\)-Stieltjes measure \( \sigma \) of \( S \) belongs to \( \mathcal{M}_{\geq 1}([\alpha,\infty)) \).

(b) The function \( \phi : [\alpha,\infty) \to C^{q \times q} \) given by \( \phi(t) := \sqrt{1 - \alpha}I_q \) belongs to \( q \times q - \mathcal{L}^2([\alpha,\infty), \mathfrak{B}_{[\alpha,\infty)}, \sigma; C) \) and \( \sigma^\#: \mathfrak{B}_{[\alpha,\infty)} \to C^{q \times q} \) defined by (4.18) belongs to \( \mathcal{M}_{\geq 1}([\alpha,\infty)) \).

(c) The function \( \tilde{S} : \mathbb{C} \setminus [\alpha,\infty) \to C^{q \times q} \) given by
\[
\tilde{S}(z) := (z - \alpha)S(z)
\]
and the \([\alpha,\infty)\)-Stieltjes transform \( S^[[\sigma^\#]] \) of \( \sigma^\# \) fulfill
\[
\tilde{S}(z) = S^[[\sigma^\#]](z) = \sigma([\alpha,\infty)) \text{ for each } z \in \mathbb{C} \setminus [\alpha,\infty). \text{ (6.72)}
\]

(d) The function \( (\tilde{S})_\square := \text{Rstr}_{\Pi_+} \tilde{S} \) belongs to \( \mathcal{R}'(\Pi_+) \) and \( (\tilde{S})_\square : \mathfrak{B}_{\mathbb{R}} \to C^{q \times q} \) given by \( (\tilde{S})_\square(B) := \sigma^\#(B \cap [\alpha,\infty)) \) is exactly the matricial spectral measure of \( (\tilde{S})_\square \).

**Proof.** (i) Part (i) is proved in Lemma 6.12.

(ii) In view of (ii), part (ii) follows immediately from Remark 4.10.

(iii) Let \( z \in \mathbb{C} \setminus [\alpha,\infty) \). According to Remark 4.4 and Theorem 3.3, the function \( g_{\alpha,z} : [\alpha,\infty) \to \mathbb{C} \) given by \( g_{\alpha,z}(t) := (z - \alpha)/(t - z) \) belongs to \( \mathcal{L}^1([\alpha,\infty), \mathfrak{B}_{[\alpha,\infty)}, \sigma; C) \) and
\[
(z - \alpha)S(z) = \int_{[\alpha,\infty)} \frac{z - \alpha}{t - z} \sigma(dt)
\]
is true. Consequently, in view of Lemma C.3, the pair \([g_{\alpha,z}I_q, 1_{[\alpha,\infty)}I_q]\) is left-integrable with respect to \( \sigma \) and
\[
\tilde{S}(z) = (z - \alpha)S(z) = \int_{[\alpha,\infty)} \left( \frac{z - \alpha}{t - z} I_q \right) \sigma(dt)I_q^*
\]
is valid. Thus, Remark C.4 shows that the pair \([g_{\alpha,z}I_q + 1_{[\alpha,\infty)}I_q, I_q]\) is left-integrable with respect to \( \sigma \) and that
\[
\int_{[\alpha,\infty)} \left( \frac{z - \alpha}{t - z} + 1 \right)I_q \sigma(dt)I_q^* = \int_{[\alpha,\infty)} \left( \frac{z - \alpha}{t - z} I_q \right) \sigma(dt)I_q^* + \int_{[\alpha,\infty)} I_q \sigma(dt)I_q^*
\]

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is fulfilled. Taking into account
\[ \sigma([\alpha, \infty)) = \int_{[\alpha, \infty)} 1_{[\alpha, \infty]} \, d\sigma = \int_{[\alpha, \infty)} (1_{[\alpha, \infty]} I_q) \, d\sigma(1_{[\alpha, \infty]} I_q)^* = \int_{[\alpha, \infty)} I_q \sigma(dt) I_q^* \]
and that \((z - \alpha)/(t - z) + 1 = (t - \alpha)/(t - z)\) holds true for each \(t \in [\alpha, \infty)\), we get then
\[ \tilde{S}(z) = \int_{[\alpha, \infty)} \left( \frac{t - \alpha}{t - z} I_q \right) \sigma(dt) I_q^* - \sigma([\alpha, \infty)). \] (6.73)

Because of Lemma C.3, Proposition C.5 and (4.18), we have
\[ \int_{[\alpha, \infty)} \left( \frac{t - \alpha}{t - z} I_q \right) \sigma(dt) I_q^* = \int_{[\alpha, \infty)} \left[ \left( \frac{1}{t - z} I_q \right) (\sqrt{t - \alpha} I_q) \right] \sigma(dt) \left[ I_q (\sqrt{t - \alpha} I_q) \right]^* \]
\[ = \int_{[\alpha, \infty)} \left( \frac{1}{t - z} I_q \right) \sigma^#(dt) I_q^* = \int_{[\alpha, \infty)} \frac{1}{t - z} \sigma^#(dt). \]
Thus, from (6.73) it follows
\[ \tilde{S}(z) = \int_{[\alpha, \infty)} \frac{1}{t - z} \sigma^#(dt) - \sigma([\alpha, \infty)) = S^{[\sigma^#]}(z) - \sigma([\alpha, \infty)). \]

[□] In view of Theorem 3.5, the function \(S^{[\sigma^#]}\) belongs to \(S_{0,q;[\alpha, \infty)}\). Thus, Remark 3.6 shows that \(\text{Rstr}_{\Pi^+} S^{[\sigma^#]} \in \mathcal{R}_q'(\Pi^+), \mathcal{R}_q'(\Pi^+)\), that the matricial spectral measure \(\mu^\#\) of \(\text{Rstr}_{\Pi^+} S^{[\sigma^#]}\) fulfills \(\sigma^# = \text{Rstr} S^{[\sigma^#]}\mu^\#\), and that \(\mu^\#(\mathbb{R} \setminus [\alpha, \infty)) = \mu^\#((-\infty, \alpha)) = 0_{q \times q}\). Consequently, \((\tilde{\sigma})_\square\) is the matricial spectral measure of \(\text{Rstr}_{\Pi^+} S^{[\sigma^#]}\). From Theorem 3.1 one can easily see that the function \(F: \Pi^+ \rightarrow \mathbb{C}^{q \times q}\) given by \(F(z) := -\sigma([\alpha, \infty))\) belongs to \(\mathcal{R}_q'(\Pi^+)\) and that the matricial spectral measure \(\theta\) of \(F\) fulfills \(\theta(B) = 0_{q \times q}\) for all \(B \in \mathcal{B}_\mathbb{R}\) (see also [16, Beispiel 1.2.1]). Because of (6.72), we have \((\tilde{S})_\square = \text{Rstr}_{\Pi^+} S^{[\sigma^#]} + F\). Since \(\text{Rstr}_{\Pi^+} S^{[\sigma^#]}\) and \(F\) both belong to \(\mathcal{R}_q'(\Pi^+)\), from [33, Remark 4.4] we see that \((\tilde{S})_\square \in \mathcal{R}_q'(\Pi^+)\) and that \((\tilde{\sigma})_\square + \theta\) is the matricial spectral measure of \((\tilde{S})_\square\). In view of \((\tilde{\sigma})_\square + \theta = (\tilde{\sigma})_\square\), the proof is complete. [□]

**Lemma 6.14.** Let \(\alpha \in \mathbb{R}\), let \(\kappa \in \mathbb{N}_0 \cup \{\infty\}\), and let \((s_j)_{j=0}^\infty\) be a sequence of complex \(q \times q\) matrices. Then:

(a) Let \(n \in \mathbb{N}_0\) be such that \(2n \leq \kappa\) and let \(S \in \mathcal{S}_{0,q;[\alpha, \infty)}\) be such that
\[ P_{2n}^S(z) \in \mathbb{C}^{(n+2)q \times (n+2)q} \quad \text{for all } z \in \mathbb{C} \setminus \mathbb{R}. \] (6.74)
Then the \(\{\alpha, \infty\}\)-Stieltjes measure \(\sigma\) of \(S\) belongs to \(\mathcal{M}_{2n}^\kappa([\alpha, \infty))\) and the inequality \(H_{\alpha}^{\kappa} \leq H_n\) holds true.

(b) Let \(n \in \mathbb{N}_0\) be such that \(2n + 1 \leq \kappa\) and let \(S \in \mathcal{S}_{0,q;[\alpha, \infty)}\) be such that
\[ \{P_{2n}^S(z), P_{2n+1}^S(z)\} \subseteq \mathbb{C}^{(n+2)q \times (n+2)q} \quad \text{for all } z \in \mathbb{C} \setminus \mathbb{R}. \] (6.75)
Then the \(\{\alpha, \infty\}\)-Stieltjes measure \(\sigma\) of \(S\) belongs to \(\mathcal{M}_{2n+1}^\kappa([\alpha, \infty))\) and the inequality \(H_{\alpha n}^{\kappa} \leq H_{\alpha n}^{\kappa}\) holds true.
Proof. Because of (6.73), we get $H_n \in C_\mathbb{H}^{(n+1)q \times (n+1)q} \subseteq C_{\mathbb{H}}^{(n+1)q \times (n+1)q}$ and, in particular, $s_j^2 = s_j$ for each $j \in \mathbb{Z}_{0,2n}$. In view of $S \in C_0, q, [\alpha, \infty)$, we see that the function $S$ is holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$ and, using additionally Propositions 8.9 and 8.8, we also obtain $\text{Rstr}_{\Pi^+} S \in C_0, q, (\Pi^+) \subseteq C_0, q, (\Pi^+)$. Let $f := S$ and let $F_n : \Pi^+ \to C_\mathbb{H}^{(n+1)q \times (n+1)q}$ be given by (6.10). Using Lemma 6.10 and Propositions 8.9 and 8.8, we conclude that $F_n \in C_0, q, (\Pi^+) \subseteq C_0, q, (\Pi^+)$ and that the matricial spectral measure $\mu_{2n}$ of $F_n$ fulfills $\mu_{2n}(\mathbb{R}) \leq H_n$. Let $\Psi_{2n} : \mathbb{C} \to C_\mathbb{H}^{(n+1)q \times (n+1)q}$ be given (6.47). Since $s_j^2 = s_j$ holds true for each $j \in \mathbb{Z}_{0,2n}$, from Lemma 6.11 we see that $\Psi_{2n}$ is a continuous matrix-valued function with $\Psi_{2n}(\mathbb{R}) \subseteq C_\mathbb{H}^{(n+1)q \times (n+1)q}$. Furthermore, Lemma 6.11 yields (6.48). According to Remark 8.6, the matricial spectral measure $\sigma_{\square}$ of $\text{Rstr}_{\Pi^+} S$ fulfills

$$\sigma = \text{Rstr}_{\square} \sigma_{\square} \quad \text{and} \quad \sigma_{\square}(\mathbb{R} \setminus [\alpha, \infty)) = 0. \quad (6.76)$$

Standard arguments of measure theory show that we can choose sequences $(a_k)_{k=1}^\infty$ and $(b_k)_{k=1}^\infty$ of real numbers such that

$$\sigma_{\square}(\{a_k\}) = 0, \quad \sigma_{\square}(\{b_k\}) = 0, \quad \mu_{2n}(\{a_k\}) = 0, \quad \mu_{2n}(\{b_k\}) = 0, \quad a_k < b_k, \quad \text{and} \quad (a_k, b_k) \subseteq (a_{k+1}, b_{k+1}) \quad (6.77)$$

hold true for each $k \in \mathbb{N}$ and that $\bigcup_{k=1}^\infty (a_k, b_k) = \mathbb{R}$. In view of $F_n \in C_0, q, (\Pi^+)$, a matricial version of Stieltjes’ inversion formula (see [17, Theorem 8.6]), and (6.77) provide us

$$\mu_{2n}((a_k, b_k)) = \frac{1}{\pi} \lim_{\epsilon \to 0+} \int_{[a_k, b_k]} \Im F_n(x + i\epsilon) \lambda^{(1)}(dx) \quad (6.79)$$

for all $k \in \mathbb{N}$, where $\lambda^{(1)}$ is the Lebesgue measure defined on $\mathbb{B}_\mathbb{R}$. The function $E_{q,n,2} : \mathbb{C} \to C_\mathbb{H}^{(n+1)q \times q}$ given by (1.3) is holomorphic in $\mathbb{C}$. Moreover, $\Psi_{2n}$ is continuous with $\Psi_{2n}(\mathbb{R}) \subseteq C_\mathbb{H}^{(n+1)q \times (n+1)q}$. Thus, for all $k \in \mathbb{N}$, we get from (6.48), a matricial version of Stieltjes’ inversion formula (see [17, Theorem 8.6]) and (6.77) that

$$\frac{1}{\pi} \lim_{\epsilon \to 0+} \int_{[a_k, b_k]} \Im F_n(x + i\epsilon) \lambda^{(1)}(dx)$$

$$= \frac{1}{2} E_{q,n}(a_k) \sigma_{\square}(\{a_k\}) [E_{q,n}(a_k)]^* + E_{q,n}(b_k) \sigma_{\square}(\{b_k\}) [E_{q,n}(b_k)]^*$$

$$+ \int_{(a_k, b_k)} E_{q,n}(t) \sigma_{\square}(dt) E_{q,n}^*(t)$$

$$= \int_{(a_k, b_k)} E_{q,n}(t) \sigma_{\square}(dt) E_{q,n}^*(t). \quad (6.80)$$

Combining (6.80) and (6.79), we obtain

$$\int_{(a_k, b_k)} E_{q,n}(t) \sigma_{\square}(dt) E_{q,n}^*(t) = \mu_{2n}((a_k, b_k)) \leq \mu_{2n}(\mathbb{R}) \quad \text{for all} \ k \in \mathbb{N} \quad (6.81)$$

and, consequently,

$$\text{tr} \int_{(a_k, b_k)} E_{q,n}(t) \sigma_{\square}(dt) E_{q,n}^*(t) \leq \text{tr} \mu_{2n}(\mathbb{R}) < \infty \quad \text{for all} \ k \in \mathbb{N}. \quad (6.82)$$
The trace measure $\tau := \text{tr} \sigma'$ of $\sigma'$ is a finite measure and $\sigma'$ is absolutely continuous with respect to $\tau$. We can choose a version $(\sigma')_t'$ of the matricial Radon–Nikodym derivative of $\sigma'$ with respect to $\tau$ such that $(\sigma')_t' \in C_q^{\sigma'}$ for all $t \in \mathbb{R}$. For all $k \in \mathbb{N}$, then $g_k := \|1_{(a_k, b_k)}(\text{Rstr}_R E_{q,n})\|_{\tau}^2 \in L^1(\mathbb{R}, \mathcal{B}_R, \tau; \mathbb{C})$ and $\text{tr} \{ \int_R (1_{(a_k, b_k)} \text{Rstr}_R E_{q,n}) d\sigma' (1_{(a_k, b_k)} \text{Rstr}_R E_{q,n})^* \} = \int_R g_k d\tau$. Thus, by virtue of (6.82), we get

$$\int_R g_k d\tau \leq \text{tr} \{ \mu_{2,n} (\mathbb{R}) \} < \infty$$

(6.83)

for all $k \in \mathbb{N}$. Obviously, $g: \mathbb{R} \to \mathbb{C}$ defined by $g(t) := \| E_{q,n}(t) \sigma' \|_{\tau}^2$ is an $\mathcal{B}_R \text{-} \mathcal{B}_C$-measurable function with $g(\mathbb{R}) \subseteq [0, \infty)$. For all $t \in \mathbb{R}$, we see that

$$g(t) = \left\| \left[ \lim_{k \to \infty} 1_{(a_k, b_k)}(t) \right] \cdot \left( \text{Rstr}_R E_{q,n}(t) \right)^2 \right\|_{\tau} = \lim_{k \to \infty} g_k(t) = \lim \inf_{k \to \infty} g_k(t).$$

(6.84)

In view of (6.84) and (6.83), Fatou’s lemma yields then

$$\int_R |g(t)| \tau(dt) = \int_R \lim \inf_{k \to \infty} g_k(t) \tau(dt) \leq \lim \inf_{k \to \infty} \int_R g_k(t) \tau(dt) \leq \text{tr} \{ \mu_{2,n} (\mathbb{R}) \} < \infty,$$

and, consequently, $g \in L^1(\mathbb{R}, \mathcal{B}_R, \tau; \mathbb{C})$. Because of Lemma [C.3] it follows

$$\text{Rstr}_R E_{q,n} \in (n + 1)q \times q - L^2(\mathbb{R}, \mathcal{B}_R, \sigma'; \mathbb{C}).$$

(6.85)

Hence, from (6.76) we obtain that $\text{Rstr}_{[0, \infty)} E_{q,n}$ belongs to $(n + 1)q \times q - L^2([0, \infty), \mathcal{B}_{[0, \infty)}, \sigma; \mathbb{C})$ and that

$$\int_{\mathbb{R}} E_{q,n}(t) \sigma'_B(dt) E_{q,n}^*(t) = \int_{[0, \infty)} E_{q,n}(t) \sigma(dt) E_{q,n}^*(t).$$

(6.86)

Furthermore, applying Remark 4.9 we get $\sigma \in \mathcal{M}^{2q}_{2n}([\alpha, \infty))$ and (4.17). Because of (6.85), we see that $\Theta_n: \mathcal{B}_R \to \mathbb{C}^{(n+1)q \times (n+1)q}$ defined by

$$\Theta_n(B) := \int_B E_{q,n}(t) \sigma'_B(dt) E_{q,n}^*(t)$$

(6.87)

is a well-defined non-negative Hermitian $(n + 1)q \times (n + 1)q$ measure on $(\mathbb{R}, \mathcal{B}_R)$. Using (6.87), $igcup_{k=1}^\infty (a_k, b_k) = \mathbb{R}$, (6.78), $\Theta_n \in \mathcal{M}^{(n+1)q}_{(n+1)q}(\mathbb{R}, \mathcal{B}_R)$, (6.81), and $\mu_{2,n} \in \mathcal{M}^{(n+1)q}_{(n+1)q}(\mathbb{R}, \mathcal{B}_R)$, we conclude

$$\int_{\mathbb{R}} E_{q,n}(t) \sigma'_B(dt) E_{q,n}^*(t) = \Theta_n(\mathbb{R}) = \Theta_n \left( \bigcup_{k=1}^\infty (a_k, b_k) \right) = \lim_{k \to \infty} \Theta_n((a_k, b_k))$$

$$= \lim_{k \to \infty} \int_{(a_k, b_k)} E_{q,n}(t) \sigma'_B(dt) E_{q,n}^*(t) = \lim_{k \to \infty} \mu_{2,n}((a_k, b_k))$$

(6.88)

$$= \mu_{2,n} \left( \bigcup_{k=1}^\infty (a_k, b_k) \right) = \mu_{2,n}(\mathbb{R}).$$

The combination of (4.17), (6.86), (6.88), and $\mu_{2,n}(\mathbb{R}) \leq H_n$ provides us then

$$H_n^{[\sigma]} = \int_{[\alpha, \infty)} E_{q,n}(t) \sigma(dt) E_{q,n}^*(t) = \int_{\mathbb{R}} \text{Rstr}_R E_{q,n} d\sigma'(\text{Rstr}_R E_{q,n})^* = \mu_{2,n}(\mathbb{R}) \leq H_n.$$

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Remark 6.16. We infer that $\sigma$ belongs to $M^\sigma_2([\alpha, \infty))$, that $\sigma^\#: \mathcal{B}_{[\alpha, \infty)} \to \mathcal{C}^{q \times q}$ defined by (4.18) belongs to $M^\sigma_2([\alpha, \infty))$, that $C_\beta \setminus [\alpha, \infty) \to \mathcal{C}^{q \times q}$ given by (6.71) is a function with $\text{Rstr}_{\Pi_j} \hat{S} \in \mathcal{R}^n_q(\Pi_j^+)$, and that $(\sigma^\#)_\square : \mathcal{B}_\mathbb{R} \to \mathcal{C}^{p \times \sigma}$ given by $(\sigma^\#)(B) := \sigma^\#(B \cap [\alpha, \infty))$ is the matricial spectral measure of $(\hat{S})_\square := \text{Rstr}_{\Pi_j} \hat{S}$. Observe that Remark 4.10(b) shows that (4.19) holds true.

Now part (b) can be proved analogously to part (a), where $F_{2n+1} : \Pi_+ \to \mathcal{C}^{(n+1)q \times (n+1)q}$ given by (6.12) and $\Psi_{2n+1} : \mathbb{C} \to \mathcal{C}^{(n+1)q \times (n+1)q}$ defined by (6.49) play the roles of $F_{2n}$ and $\Psi_{2n}$, respectively (for details, see [53, Lemma 7.9]).

Remark 6.15. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence from $\mathbb{C}^{q \times q}$. Using Remark A.6 and the definition of the class $\mathcal{S}_{0,q;[\alpha, \infty)}$, it is readily checked that the following statements hold true:

(a) If $n \in \mathbb{N}_0$ is such that $2n \leq \kappa$ and if $S \in \mathcal{S}_{0,q;[\alpha, \infty)}$ is such that $P_{2n}^S(iy) \in \mathcal{C}^{(n+2)q \times (n+2)q}$ for all $y \in (0, \infty)$, then
\[
\lim_{y \to \infty} R_{q,n}(iy)[v_{q,n}S(iy) - u_n] = 0. \tag{6.89}
\]

(b) If $\kappa \geq 1$ and $n \in \mathbb{N}_0$ are such that $2n + 1 \leq \kappa$ and if $S \in \mathcal{S}_{0,q;[\alpha, \infty)}$ is such that $P_{2n+1}^S(iy) \in \mathcal{C}^{(n+2)q \times (n+2)q}$ holds true for each $y \in (0, \infty)$, then
\[
\lim_{y \to \infty} R_{q,n}(iy)[v_{q,n}(iy - \alpha)S(iy) - (-\alpha u_n - y_0, n)] = 0.
\]

Remark 6.16. Let $n \in \mathbb{N}_0$ and let $y \in \mathbb{R}$. If $u \in \mathbb{C}^{(n+1)q \times p}$ is such that $\lim_{y \to \infty}[u^* R_{q,n}(iy)u] = 0$, then from Remark 4.14 one can easily see that $u = 0$.

Lemma 6.17. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices. Then:

(a) Let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$ and let $S \in \mathcal{S}_{0,q;[\alpha, \infty)}$ be such that $P_{2n}^S(z) \in \mathcal{C}^{(n+2)q \times (n+2)q}$ holds true for all $z \in \mathbb{C} \setminus \mathbb{R}$. Then the $[\alpha, \infty)$-Stieltjes measure $\sigma$ of $S$ belongs to $M^\sigma_{2,2n}([\alpha, \infty))$ and $S$ belongs to $\mathcal{S}_{0,q;[\alpha, \infty)}([s_j]_{j=0}^{2n}, \leq)$.

(b) Let $n \in \mathbb{N}_0$ be such that $2n + 1 \leq \kappa$ and let $S \in \mathcal{S}_{0,q;[\alpha, \infty)}$ be such that $\{P_{2n}^S(z), P_{2n+1}^S(z)\} \subseteq \mathcal{C}^{(n+2)q \times (n+2)q}$ holds true for each $z \in \mathbb{C} \setminus \mathbb{R}$. Then the $[\alpha, \infty)$-Stieltjes measure $\sigma$ of $S$ belongs to $M^\sigma_{2,2n+1}([\alpha, \infty))$ and $S$ belongs to $\mathcal{S}_{0,q;[\alpha, \infty)}([s_j]_{j=0}^{2n+1}, \leq)$.

Proof. We give a shortened version of the detailed proof stated in [57, Lemma 5.15].

Lemma 6.14 yields $\sigma \in M^\sigma_2([\alpha, \infty))$ and $H^\sigma_n \leq H_n$. If $n = 0$, then $\sigma \in M^\sigma_2([\alpha, \infty)); [s_j]_{j=0}^{2n}, \leq)$ follows. Suppose now $n \geq 1$. Remark 6.15 shows that (6.89) is valid.

Obviously, $\sigma \in M^\sigma_2([\alpha, \infty)); [s_j]_{j=0}^{2n}, \leq)$, where $(s_j)_{j=0}^{2n}$ is defined by (1.1). Thus, Proposition 4.11 and Remark 6.15 provide us
\[
\lim_{y \to \infty} R_{q,n}(iy)[v_{q,n}S(iy) - u_n^\sigma] = 0. \tag{6.90}
\]
where \( s_{-1}^{[\sigma]} := 0_{q \times q} \) and where \( u_n^{[\sigma]} := -\text{col}(s_{j-1}^{[\sigma]})_j^{n} \). Combining (6.89) and (6.90), we get

\[
\lim_{y \to \infty} (u_n^{[\sigma]} - u_n)^* R_{T_q,n}(iy)(u_n^{[\sigma]} - u_n) = 0.
\]

Consequently, Remark 6.16 yields \( u_n^{[\sigma]} = u_n \). Let \( d_j := s_j - s_j^{[\sigma]} \) for each \( j \in \mathbb{Z}_{0, 2n} \). Then \( u_n^{[\sigma]} = u_n \) and \( n \geq 1 \) imply \( d_k = 0_{q \times q} \). Furthermore, the inequality \( H_n^{[\sigma]} \leq H_n \) shows that the block Hankel matrix \( [d_{j+k}]_{j,k=0}^{n} \) is non-negative Hermitian. Thus, \( d_{2n} \in \mathbb{C}^{q \times q} \) and Remark A.5 yield \( d_j = 0_{q \times q} \) for each \( j \in \mathbb{Z}_{0, 2n-1} \). Hence, \( \sigma \) belongs to \( \mathcal{M}^q_{\leq}([\alpha, \infty); (s_j)_{j=0}^{2n}, \leq] \).

Part (ii) can be proved analogously.

Now we are able to prove that the solution set of the (reformulated) truncated Stieltjes-type moment problem and the solution set of the corresponding system of the fundamental Potapov’s matrix inequalities coincide.

**Theorem 6.18.** Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N}_0 \cup \{\infty\} \), and let \( (s_j)_{j=0}^{n} \) be a sequence of complex \( q \times q \) matrices. Let \( D \) be a discrete subset of \( \Pi_+ \) and let \( S : \mathbb{C} \setminus [\alpha, \infty] \to \mathbb{C}^{q \times q} \) be a holomorphic matrix-valued function. Then:

(a) Let \( n \in \mathbb{N}_0 \) be such that \( 2n \leq \kappa \). Then the following statements are equivalent:

(i) \( S \in S_{0; q; [\alpha, \infty]}((s_j)_{j=0}^{2n}, \leq] \).

(ii) \( P_{2n}^{[S]}(z) \in \mathbb{C}^{(n+2)q \times (n+2)q}_{\geq} \) and \( P_{2n}^{[S]}(z) \in \mathbb{C}^{(n+2)q \times (n+2)q}_{\leq} \) for all \( z \) in \( \Pi_+ \setminus D \).

(b) Let \( n \in \mathbb{N}_0 \) be such that \( 2n + 1 \leq \kappa \). Then the following statements are equivalent:

(iii) \( S \in S_{0; q; [\alpha, \infty]}((s_j)_{j=0}^{2n+1}, \leq] \).

(iv) \( \{P_{2n}^{[S]}(z); P_{2n+1}^{[S]}(z)\} \subseteq \mathbb{C}^{(n+2)q \times (n+2)q}_{\leq} \) for all \( z \) in \( \Pi_+ \setminus D \).

**Proof.** (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv)

Use Proposition 4.11.

Let \( m := 2n \). Observe that the function \( F := \text{Rstr}_{\Pi_+ \setminus D} S \) is holomorphic. Because of (iii), the inequalities \( P_{m-1}^{[F]}(z) \geq 0 \) and \( P_{m}^{[F]}(z) \geq 0 \) hold true for each \( z \) in \( \Pi_+ \setminus D \). From Theorem 6.5 we get then that there is a unique function \( \hat{S} \in S_{0; q; [\alpha, \infty]} \) such that \( \text{Rstr}_{\Pi_+ \setminus D} \hat{S} = F \), namely \( \hat{S} = S \), and that \( P_{k}^{[\hat{S}]}(z) \geq 0 \) are valid for all \( k \in \mathbb{Z}_{-1,m} \) and all \( z \in \mathbb{C} \setminus \mathbb{R} \). Applying Lemma 6.17 we get then (iv) \( \Rightarrow \) (iii) \( \Rightarrow \) (ii) \( \Rightarrow \) (i).

Let \( m := 2n+1 \) and use the same argumentation as in the proof of the implication (ii) \( \Rightarrow \) (iii).

7. Some considerations on block Hankel matrices

First we introduce some further block Hankel matrices. In Section 4.4 in particular in Remarks 4.14 and 4.3 we already discussed some aspects on such matrices.

Let \( \kappa \in \mathbb{N}_0 \cup \{\infty\} \) and let \( (s_j)_{j=0}^{n} \) be a sequence of complex \( p \times q \) matrices. Let \( H_n := [s_{j+k}]_{j,k=0}^{n} \) for each \( n \in \mathbb{N}_0 \) with \( 2n \leq \kappa \), let \( K_n := [s_{j+k}]_{j,k=0}^{n} \) for each \( n \in \mathbb{N}_0 \) with \( 2n+1 \leq \kappa \), and let \( G_n := [s_{j+k}]_{j,k=0}^{n} \) for each \( n \in \mathbb{N}_0 \) with \( 2n+2 \leq \kappa \). For every choice of integers \( m \) and \( n \) with \( 0 \leq m \leq n \leq \kappa \), let \( y_{m,n} \) and \( z_{m,n} \) be given by (4.1), let \( \hat{y}_{m,n} := \text{col}(s_{n-m})_{j=0}^{n-m} \), and let \( \hat{z}_{m,n} := \text{row}(s_{n-m})_{k=0}^{n-m} \).
We will see that certain Schur complements play an essential role for our considerations. Let \( L_0 := s_0 \) and, for each \( n \in \mathbb{N} \) with \( 2n \leq \kappa \), furthermore
\[
L_n := s_{2n} - z_{0,2n-1}H_n^\dagger y_{n,2n-1}.
\]
For every choice of integers \( m \) and \( n \) with \( 0 \leq m \leq n \leq \kappa - 1 \), let \( y_{ov,m,n} := \text{col}(s_{ov,m+k})_{j=0}^{m-n} \) and \( z_{ov,m,n} := \text{row}(s_{ov,m+k})_{j=0}^{m-n} \). Let \( L_{ov,0} := s_{ov,0} \) and, for each \( n \in \mathbb{N}_0 \) with \( 2n + 1 \leq \kappa \), moreover \( L_{ov,n} := s_{ov,2n} - z_{ov,2n-1}H_n^\dagger y_{ov,2n-1} \).

**Remark 7.1.** Let \( \kappa \in \mathbb{N}_0 \cup \{ \infty \} \) and let \( (s_j)_{j=0}^n \) be a sequence of Hermitian complex \( q \times q \) matrices. In view of Remark 4.13 it is readily checked that
\[
\begin{align*}
| R_{T_{q,n}}(w) |^* H_nT_{q,n}^* - T_{q,n}H_n [ R_{T_{q,n}}(z) ]^{-1} + (\overline{w} - z)T_{q,n}H_nT_{q,n}^* = v_{q,n}v_{q,n}^* H_nT_{q,n}^* - T_{q,n}H_nv_{q,n}v_{q,n}^* ,
\end{align*}
\]
and
\[
\begin{align*}
[ R_{T_{q,n}}(z) ]^{-1} H_nT_{q,n}^* - T_{q,n}H_n [ R_{T_{q,n}}(z) ]^{-*} = v_{q,n}v_{q,n}^* H_nT_{q,n}^* - T_{q,n}H_nv_{q,n}v_{q,n}^* \quad (7.1)
\end{align*}
\]
are fulfilled for every choice of \( n \in \mathbb{N}_0 \) with \( 2n \leq \kappa \) and \( w, z \in \mathbb{C} \).

**Remark 7.2.** Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N} \cup \{ \infty \} \), and let \( (s_j)_{j=0}^n \) be a sequence of Hermitian complex \( q \times q \) matrices. In view of Remarks 4.13 and 4.3 then
\[
\begin{align*}
[ R_{T_{q,n}}(z) ]^{-1} H_nT_{q,n}^* - T_{q,n}H_n [ R_{T_{q,n}}(z) ]^{-*} = v_{q,n}v_{q,n}^* H_nT_{q,n}^* - T_{q,n}H_nv_{q,n}v_{q,n}^* \quad (7.1)
\end{align*}
\]
is valid for each \( n \in \mathbb{N}_0 \) with \( 2n + 1 \leq \kappa \) and each \( z \in \mathbb{C} \). Remark 4.3 yields
\[
\begin{align*}
[ R_{T_{q,n}}(\alpha) ]^{-1} H_nv_{q,n} = [ R_{T_{q,n}}(\alpha) ]^{-1} y_{0,n}, \quad [ R_{T_{q,n}}(\alpha) ]^{-1} H_nv_{q,n} = \alpha w + y_{0,n} \quad (7.3)
\end{align*}
\]
and
\[
\begin{align*}
z_{0,n}[ R_{T_{q,n}}(\alpha) ]^{-*} = v_{q,n}^* H_n[ R_{T_{q,n}}(\alpha) ]^{-*}, \quad \alpha w + z_{0,n} = v_{q,n}^* H_n[ R_{T_{q,n}}(\alpha) ]^{-*}
\end{align*}
\]
for each \( n \in \mathbb{N}_0 \) with \( 2n \leq \kappa \).

**Remark 7.3 ( [25, Remark 2.1]).** Let \( n \in \mathbb{N} \) and let \( (s_j)_{j=0}^n \) be a sequence of complex \( q \times q \) matrices. In view of [12] and Lemma A.4 one can easily see that \( (s_j)_{j=0}^n \) belongs to \( \mathcal{H}_{q,2n}^Z \) if and only if the four conditions \( (s_j)_{j=0}^{2(n-1)} \in \mathcal{H}_{q,2(n-1)}^Z \), \( R(y_{n,2n-1}) \subseteq R(1-H_{n-1}) \), \( s_{2n-1} = s_{2n-1} \), and \( L_n \in \mathbb{C}^{q \times q} \) hold true. If \( (s_j)_{j=0}^n \) belongs to \( \mathcal{H}_{q,2n}^Z \), then rank \( H_n = \sum_{j=1}^n \text{rank} L_j \).

**Remark 7.4.** Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N} \cup \{ \infty \} \), and let \( (s_j)_{j=0}^n \in \mathcal{K}_{q,k}^Z \). By virtue of Remark 4.1 one can easily check then that \( s_j^* = s_j \) for each \( k \in \mathbb{Z}_0^\kappa \) and \( s_{\kappa \kappa}^* = s_{\kappa \kappa} \) for each \( k \in \mathbb{Z}_0^{\kappa - 1} \). Furthermore, for each \( n \in \mathbb{N}_0 \) with \( 2n \leq \kappa \), from Remark 7.3 one can see that the matrices \( s_{2n} \), \( H_n \), and \( L_n \) are non-negative Hermitian and, for each \( n \in \mathbb{N}_0 \) with \( 2n + 1 \leq \kappa \), the matrices \( s_{2n+2} \), \( H_{ov,n} \), and \( L_{ov,n} \) are non-negative Hermitian as well.

**Remark 7.5.** Let \( \alpha \in \mathbb{R} \) and let \( \kappa \in \mathbb{N} \cup \{ \infty \} \). According to the definition of \( \mathcal{K}_{q,k}^Z \) and [24, Lemma 4.7], one can easily check that \( \mathcal{K}_{q,k}^Z \subseteq \mathcal{K}_{q,k}^Z \cap \mathcal{H}_{q,k}^Z \). In particular, if \( (s_j)_{j=0}^n \) belongs to \( \mathcal{K}_{q,k}^Z \), then, in view of Remark 7.3 for each \( m \in \mathbb{Z}_0^\kappa \), the sequence \( (s_j)_{j=0}^m \) belongs to \( \mathcal{K}_{q,m}^Z \cap \mathcal{H}_{q,k}^Z \). Further, if \( (s_j)_{j=0}^m \in \mathcal{K}_{q,k}^Z \), then the definition of the sets \( \mathcal{K}_{q,k}^Z \) and \( \mathcal{H}_{q,k}^Z \) and [24, Proposition 4.8 and Lemma 4.11] show that, for each \( m \in \mathbb{Z}_0^{\kappa - 1} \), the sequence \( (s_{\kappa \kappa})_{j=0}^m \) belongs to \( \mathcal{H}_{q,m}^Z \).

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Remark 7.6. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa \in K_{q,\kappa,\alpha}^\geq$. For each $m \in \mathbb{Z}_{0,\kappa}$, we have $(s_j)_{j=0}^m \in K_{q,\kappa,\alpha}^\geq$. In view of Remarks 7.2 and 7.5 from [24, Lemmata 4.15 and 4.16] one can see that

$$\mathcal{N}(L_0) \subseteq \mathcal{N}(L_{\alpha \circ 0}) \subseteq \mathcal{N}(L_1) \subseteq \cdots \subseteq \mathcal{N}(L_n) \subseteq \mathcal{N}(L_{\alpha \circ n})$$

and that

$$\mathcal{R}(L_0) \supseteq \mathcal{R}(L_{\alpha \circ 0}) \supseteq \mathcal{R}(L_1) \supseteq \cdots \supseteq \mathcal{R}(L_n) \supseteq \mathcal{R}(L_{\alpha \circ n})$$

are valid for each $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$ and

$$\mathcal{N}(L_0) \subseteq \mathcal{N}(L_{\alpha \circ 0}) \subseteq \mathcal{N}(L_1) \subseteq \cdots \subseteq \mathcal{N}(L_{\alpha \circ n-1}) \subseteq \mathcal{N}(L_n)$$

and

$$\mathcal{R}(L_0) \supseteq \mathcal{R}(L_{\alpha \circ 0}) \supseteq \mathcal{R}(L_1) \supseteq \cdots \supseteq \mathcal{R}(L_{\alpha \circ n-1}) \supseteq \mathcal{R}(L_n)$$

hold true for each $n \in \mathbb{N}$ with $2n \leq \kappa$.

8. Dubovoj subspaces

If $\mathcal{U}$ and $\mathcal{W}$ are subspaces of $\mathbb{C}^p$, then we write $\mathcal{U} + \mathcal{W}$ for the sum of $\mathcal{U}$ and $\mathcal{W}$. To indicate that the sum $\mathcal{U} + \mathcal{W}$ is a direct sum, i.e., that $\mathcal{U} \cap \mathcal{W} = \{0_{q \times 1}\}$ is fulfilled, we use the notation $\mathcal{U} \oplus \mathcal{W}$. V. K. Dubovoj studied in [19] particular invariant subspaces to discuss the matricial Schur problem. Having in mind this, we give the following definition:

Definition 8.1. We call a subspace $\mathcal{D}$ of $\mathbb{C}^p$ a Dubovoj subspace corresponding to a given ordered pair $(H, T)$ of complex $p \times p$ matrices if $T^*(\mathcal{D}) \subseteq \mathcal{D}$ and $\mathcal{N}(H) \oplus \mathcal{D} = \mathbb{C}^p$ are fulfilled.

We are going to consider special Dubovoj subspaces.

Notation 8.2. Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. For each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let $\mathcal{D}_n := \mathcal{R}(\text{diag}(L_0, L_1, \ldots, L_n))$. Furthermore, if $\kappa \geq 1$, then, for every choice of $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, let $\mathcal{D}_{\alpha \circ n} := \mathcal{R}(\text{diag}(L_{\alpha \circ 0}, L_{\alpha \circ 1}, \ldots, L_{\alpha \circ n})).$

Using the Kronecker delta, we set

$$V_{q,n} := [\delta_{j,k} I_q]_{j=0,\ldots,n}^{k=0,\ldots,n-1} \quad \text{and} \quad \mathfrak{M}_{q,n} := [\delta_{j,k+1} I_q]_{j=0,\ldots,n}^{k=0,\ldots,n-1}.$$

Lemma 8.3. Let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa \in H_{q,\kappa}^\geq$. For each $n \in \mathbb{N}$ with $2n \leq \kappa$, then $T_{q,n}^*(\mathcal{D}_n) \subseteq \mathcal{D}_n$, $\mathfrak{M}_{q,n}^*(\mathcal{D}_n) \subseteq \mathcal{D}_{n-1}$, and $V_{q,n}^*(\mathcal{D}_n) \subseteq \mathcal{D}_{n-1}$.

Proof. Because of $T_{q,n}^* \cdot \text{diag}(L_0, L_1, \ldots, L_n) = \begin{bmatrix} 0_{q \times q} & \text{diag}(L_1, L_2, \ldots, L_n) \end{bmatrix}$, we have $T_{q,n}^*(\mathcal{D}_n) \subseteq \mathcal{R}(\text{diag}(L_1, L_2, \ldots, L_n, 0_{q \times q}))$. From Remark 7.3 we see that $\{L_0, L_1, \ldots, L_n\} \subseteq \mathbb{C}_{q \times q}^\geq$. Thus, using Remark 7.3 and [23, Proposition 2.13], we get

$$\mathcal{R}(L_j) = [\mathcal{N}(L_j)]^\perp \subseteq [\mathcal{N}(L_{j-1})]^\perp \subseteq \mathcal{R}(L_{j-1})$$

for each $j \in \mathbb{Z}_{1,n}$, (8.1)

which implies $\mathcal{R}(\text{diag}(L_1, L_2, \ldots, L_n, 0_{q \times q})) \subseteq \mathcal{R}(\text{diag}(L_0, L_1, \ldots, L_{n-1}, 0_{q \times q})) \subseteq \mathcal{D}_n$. Consequently, $T_{q,n}^*(\mathcal{D}_n) \subseteq \mathcal{D}_n$. Obviously, we have $\mathfrak{M}_{q,n}^* \cdot \text{diag}(L_0, L_1, \ldots, L_n) = [0_{q \times q}, \text{diag}(L_1, L_2, \ldots, L_n)]$ and $V_{q,n}^* \cdot \text{diag}(L_0, L_1, \ldots, L_n) = (\text{diag}(L_0, L_1, \ldots, L_{n-1}), 0_{q \times q})$. Because of (8.1), consequently, $\mathfrak{M}_{q,n}^*(\mathcal{D}_n) \subseteq \mathcal{R}(\text{diag}(L_1, L_2, \ldots, L_n)) \subseteq \mathcal{R}(\text{diag}(L_0, L_1, \ldots, L_{n-1})) = \mathcal{D}_{n-1}$ and $V_{q,n}^*(\mathcal{D}_n) \subseteq \mathcal{D}_{n-1}$.

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If \( n \in \mathbb{N}_0 \) and if \((s_j)_{j=0}^{2n} \in H_{q,2n}^\geq,e\), then the existence of a Dubovoj subspace corresponding to \((H_n, T_{q,n})\) was proved in [9] Lemma 3.2, [61] Satz 1.24, and [23]. An explicit construction of such a subspace gives the following result:

**Proposition 8.4.** Let \( \kappa \in \mathbb{N}_0 \cup \{\infty\} \) and let \((s_j)_{j=0}^{2n} \in H_{q,2n}^\geq,e\). For each \( n \in \mathbb{N}_0 \) with \( 2n \leq \kappa \), then \( D_n \) is a Dubovoj subspace for \((H_n, T_{q,n})\), where in particular \( \dim D_n = \text{rank } H_n \) and \( \dim D_n = \sum_{j=0}^n \text{rank } L_j \).

**Proof.** Let \( n \in \mathbb{N}_0 \) be such that \( 2n \leq \kappa \). Then \((s_j)_{j=0}^{2n} \in H_{q,2n}^\geq,e\). Furthermore, Lemma S.3 shows that \( T_{q,n}^*(D_n) \subseteq D_n \). Now we check that

\[
\dim D_k = \dim \mathcal{R}(H_k) \quad (8.2)
\]

holds true for each \( k \in \mathbb{Z}_{0,n} \). Because of \( L_0 = s_0 = H_0 \), equation (8.2) is valid for \( k = 0 \). Thus, there is an \( m \in \mathbb{Z}_{0,n} \) such (8.2) is fulfilled for each \( k \in \mathbb{Z}_{0,m} \). We consider the case that \( 2(m+1) \leq \kappa \). Then from Notation S.2 and (8.2) we obtain

\[
\dim D_{m+1} = \dim D_m + \dim \mathcal{R}(L_{m+1}) = \dim \mathcal{R}(H_m) + \dim \mathcal{R}(L_{m+1}). \quad (8.3)
\]

Since we know from Remark 7.3 that the right-hand side of (8.3) coincides with \( \dim \mathcal{R}(H_{m+1}) \), we see that (8.2) is true for \( k = m + 1 \) as well. Consequently, (8.2) holds for each \( k \in \mathbb{Z}_{0,n} \). This implies \( \dim D_n + \dim \mathcal{N}(H_n) = \dim \mathbb{C}^{(n+1)q} \). Furthermore, (8.2) and Remark 7.3 show that \( \dim D_n = \sum_{j=0}^n \text{rank } L_j \) holds true. It remains to prove that \( D_n \cap \mathcal{N}(H_n) \subseteq \{0_{(n+1)q \times 1}\} \). We consider an arbitrary \( x \in D_n \cap \mathcal{N}(H_n) \). Let \( x = \text{col}(x_j)_{j=0}^n \) be the \( q \times 1 \) block representation of \( x \). Because of \( x \in \mathcal{N}(H_n) \), from [25] Lemma A.2 we see that \( x_n \) belongs to \( \mathcal{N}(L_n) \). Since we know from Remark 7.3 that \( L_n \) is non-negative Hermitian, we conclude \( x_n \in \mathcal{R}(L_n)^\perp \). On the other hand, we have \( x \in D_n \), which implies \( \text{col}(x_j)_{j=0}^n \in \mathcal{R}(\text{diag}(L_0, L_1, \ldots, L_n)) \) and, consequently, \( x_n \in \mathcal{R}(L_n) \). Thus, \( x_n \in \mathcal{R}(L_n) \cap \mathcal{R}(L_n)^\perp = \{0_{q \times 1}\} \), i.e., \( x_n = 0_{q \times 1} \). Inductively, then \( x_{n-j} = 0_{q \times 1} \) follows for each \( j \in \mathbb{Z}_{0,n} \). Therefore, \( D_n \cap \mathcal{N}(H_n) \subseteq \{0_{(n+1)q \times 1}\} \).

For each \( n \in \mathbb{N}_0 \) and each \((s_j)_{j=0}^{2n} \in H_{q,2n}^\geq,e\), we will call \( D_n \) defined in Notation S.2 the canonical Dubovoj subspace corresponding to \((H_n, T_{q,n})\).

In [61] Abschnitt 1.4, H. C. Thiele showed that \((s_j)_{j=0}^{2n} \) given by \( s_0 := 0 \), \( s_1 := 0 \), and \( s_2 := 1 \) is a sequence belonging to \( H_{q,2}^\geq \) for which no Dubovoj subspace corresponding to \((H_1, T_{1,1})\) exists.

**Remark 8.5.** Let \( \kappa \in \mathbb{N}_0 \cup \{\infty\} \), let \((s_j)_{j=0}^\kappa \in H_{q,\kappa}^\geq,e\), and let \( n \in \mathbb{N}_0 \) be such that \( 2n \leq \kappa \). Let \( D_n \) be the canonical Dubovoj subspace corresponding to \((H_n, T_{q,n})\). In view of Proposition 8.4, one can easily see that \( \dim D_n \geq 1 \) if and only if \( s_0 \neq 0_{q \times q} \). Furthermore, it is readily checked that \( \dim D_n < (q+1)n \) if and only if \( H_n = 0 \).

**Remark 8.6.** Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N}_0 \cup \{\infty\} \), and let \((s_j)_{j=0}^\kappa \in K_{q,\kappa}^\geq,e,\alpha\). From Remark 7.3 and Proposition 8.4 one can see that the following statements hold true:

(a) For each \( n \in \mathbb{N}_0 \) with \( 2n \leq \kappa \), the subspace \( D_n \) of \( \mathbb{C}^{(n+1)q} \) is a Dubovoj subspace corresponding to \((H_n, T_{q,n})\).

(b) If \( \kappa \geq 1 \), then for each \( n \in \mathbb{N}_0 \) with \( 2n+1 \leq \kappa \), the subspace \( D_{\alpha n} \) of \( \mathbb{C}^{(n+1)q} \) is a Dubovoj subspace corresponding to \((H_{\alpha n}, T_{q,n})\).

**Proposition 8.7.** Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N} \cup \{\infty\} \), and let \((s_j)_{j=0}^\kappa \in K_{q,\kappa,\alpha}^\geq,e\). Then:
(a) For each \( n \in \mathbb{N}_0 \) with \( 2n + 1 \leq \kappa \), then \( T_{q,n}^*(\mathcal{D}_n) \subseteq \mathcal{D}_{\alpha n} \subseteq \mathcal{D}_n \),

\[
\mathcal{N}(H_n) + \mathcal{D}_n = \mathbb{C}^{(n+1)q}, \quad \text{and} \quad \mathcal{N}(H_{\alpha n}) + \mathcal{D}_{\alpha n} = \mathbb{C}^{(n+1)q}.
\] (8.4)

(b) For each \( n \in \mathbb{N} \) with \( 2n \leq \kappa \), furthermore \( \mathfrak{W}_{q,n}^*(\mathcal{D}_n) \subseteq \mathcal{D}_{\alpha n-1} \), \( V_{q,n}(\mathcal{D}_{\alpha n-1}) \subseteq \mathcal{D}_n \),

\[
\mathcal{N}(H_n) + \mathcal{D}_n = \mathbb{C}^{(n+1)q}, \quad \text{and} \quad \mathcal{N}(H_{\alpha n-1}) + \mathcal{D}_{\alpha n-1} = \mathbb{C}^{nq}.
\] (8.5)

Proof. According to Remark 7.5, we have \((s_j)_{j=0}^m \in K_{q,m}^{\geq e}\) for each \( m \in \mathbb{Z}_{0,\kappa} \). Consequently, Remark 7.6 yields

\[
\mathcal{R}(L_{j+1}) \subseteq \mathcal{R}(L_{\alpha j}) \quad \text{for each} \quad j \in \mathbb{N}_0 \text{ with } 2j + 2 \leq \kappa \quad \text{(8.6)}
\]

and

\[
\mathcal{R}(L_{\alpha j}) \subseteq \mathcal{R}(L_j) \quad \text{for each} \quad j \in \mathbb{N}_0 \text{ with } 2j + 1 \leq \kappa. \quad \text{(8.7)}
\]

(\[\square\]) Let \( n \in \mathbb{N}_0 \) be such that \( 2n + 1 \leq \kappa \). Because of Remark 8.6 and the definition of a Dubovoj subspace, we get \( \mathfrak{W}_{q,n}^* \). In view of (8.7), we have \( \mathcal{D}_{\alpha n} = \mathcal{R}(\text{diag}(L_{\alpha j}))_{j=0}^n \subseteq \mathcal{R}(\text{diag}(L_{j+1}))_{j=0}^n = \mathcal{D}_n \). If \( n = 0 \), then \( T_{q,n} = 0_{q \times q} \) and, consequently, \( T_{q,n}^*(\mathcal{D}_n) \subseteq \mathcal{D}_{\alpha n} \).

Now we assume that \( n \geq 1 \). In view of (8.6), then it is readily checked that

\[
T_{q,n}^*(\mathcal{D}_n) = T_{q,n}^* \left[ \mathcal{R} \left( \text{diag}(L_{j+1})_{j=0}^n \right) \right] \subseteq \mathcal{R} \left( \text{diag} \left( \text{diag}(L_{j+1})_{j=0}^{n-1}, 0_{q \times q} \right) \right)
\]

\[
\subseteq \mathcal{R} \left( \text{diag}(L_{\alpha j})_{j=0}^n = \mathcal{D}_{\alpha n} \right). \quad \text{(8.8)}
\]

(\[\square\]) Let \( \kappa \geq 2 \) and let \( n \in \mathbb{N} \) such that \( 2n \leq \kappa \). Because of Remark 8.6 and the definition of a Dubovoj subspace, we get \( \mathfrak{W}_{q,n}^* \). From \( \mathfrak{W}_{q,n}^* \), \( \text{diag}(L_{j+1})_{j=0}^n = \left[ 0_{q \times q}, \text{diag}(L_{j+1})_{j=0}^{n-1} \right] \) we conclude \( \mathfrak{W}_{q,n}^*(\mathcal{D}_n) \subseteq \mathcal{R}(\text{diag}(L_{j+1}))_{j=0}^n \). Using (8.6), we obtain \( \mathcal{R}(\text{diag}(L_{j+1}))_{j=0}^n \subseteq \mathcal{R}(\text{diag}(L_{\alpha j}))_{j=0}^n \) and, consequently, \( \mathfrak{W}_{q,n}^*(\mathcal{D}_n) \subseteq \mathcal{D}_{\alpha n-1} \). Obviously, \( V_{q,n} \cdot \text{diag}(L_{\alpha j})_{j=0}^{n-1} = \text{diag} \left( \text{diag}(L_{\alpha j})_{j=0}^{n-1}, 0_{q \times q} \right) \cdot V_{q,n} \) and, hence, \( V_{q,n}(\mathcal{D}_{\alpha n-1}) = \mathcal{R}(\text{diag}(L_{\alpha j}))_{j=0}^n = \mathcal{D}_n \), we get \( V_{q,n}(\mathcal{D}_{\alpha n-1}) \subseteq \mathcal{D}_n \). \( \square \)

Remark 8.8. If \( A \in \mathbb{C}^{p \times q} \) and if \( \mathcal{U} \) and \( \mathcal{V} \) are subspaces of \( \mathbb{C}^p \) and \( \mathbb{C}^p \), respectively, such that \( \mathcal{N}(A) + \mathcal{U} = \mathbb{C}^p \) and \( \mathcal{R}(A) + \mathcal{V} = \mathbb{C}^p \) are fulfilled, then there is a unique \( X \in \mathbb{C}^{p \times q} \) such that

\[
AXA = A, \quad XAX = X, \quad \mathcal{R}(X) = \mathcal{U}, \quad \text{and} \quad \mathcal{N}(X) = \mathcal{V},
\]

(see, e.g. [6] Chapter 2, Theorem 12(c)), and we will use \( A_{\mathcal{U},\mathcal{V}}^{(1,2)} \) to denote this matrix \( X \). In particular, if \( A \) is a Hermitian complex \( q \times q \) matrix and if \( \mathcal{U} \) is a subspace of \( \mathbb{C}^q \) with \( \mathcal{N}(A) + \mathcal{U} = \mathbb{C}^q \), then \( \mathcal{R}(A) + \mathcal{U}^\perp = \mathbb{C}^q \) and we will also write \( A_{\mathcal{U}} \) for \( A_{\mathcal{U},\mathcal{U}^\perp}^{(1,2)} \). (In Appendix B we turn our attention to the Hermitian case, in which special equations hold true.)

If \( \kappa \in \mathbb{N}_0 \cup \{ \infty \} \) and a sequence \((s_j)_{j=0}^\kappa \in \mathcal{H}_{q,\kappa}^{\geq e}\) is given, then, for each \( n \in \mathbb{N}_0 \) with \( 2n \leq \kappa \), let \( H_n := H_{D_n, D_n}^{(1,2)} \), where \( D_n \) is given by Notation 8.2. (Note that Remark 7.5 shows that \( K_{q,\kappa,\alpha}^{\geq e} \subseteq \mathcal{H}_{q,\kappa}^{\geq e} \) holds true for each \( \alpha \in \mathbb{R} \) and each \( \kappa \in \mathbb{N}_0 \cup \{ \infty \} \).) If \( \alpha \in \mathbb{R} \), \( \kappa \in \mathbb{N} \cup \{ \infty \} \), and \((s_j)_{j=0}^\kappa \in \mathcal{H}_{q,\kappa,\alpha}^{\geq e}\) are given, then, for each \( n \in \mathbb{N}_0 \) with \( 2n + 1 \leq \kappa \), let \( H_{\alpha n} := H_{D_{\alpha n}, D_{\alpha n}}^{(1,2)} \), where \( D_{\alpha n} \) is also given by Notation 8.2.
Remark 8.9. Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa \in \mathbb{H}_{q,\kappa}$. In view of Lemma 3.1 for each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, then it is readily checked that $H_n^* \in \mathbb{C}_{\geq n}^{(n+1)q \times (n+1)q}$,

$$H_n H_n^* = H_n^* = H_n,$$  \hspace{1cm} \text{(8.8)}

and

$$H_n^* H_n = H_n^* = H_n.$$

Lemma 8.10. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa \in \mathbb{K}_{q,\kappa}^{\geq e}$, and let $n \in \mathbb{N}_0$ be such that $2n + 1 \leq \kappa$. Then the matrices $H_n^*$ and $H_{\alpha n}^*$ are both non-negative Hermitian and fulfill

$$(H_n^*)^* = H_n^-,$$

and $$(H_{\alpha n}^*)^* = H_{\alpha n}^-.$$  \hspace{1cm} \text{(8.9)}

Furthermore, the equations in (8.8) as well as the following four identities hold true:

$$H_{\alpha n} H_{\alpha n}^* = H_{\alpha n}^* H_{\alpha n},$$

$$H_{\alpha n}^* H_{\alpha n} H_{\alpha n}^* = H_{\alpha n}^*,$$  \hspace{1cm} \text{(8.10)}

$$H_n^* H_n = H_n^* = H_n,$$  \hspace{1cm} \text{(8.11)}

Proof. The matrices $H_n$ and $H_{\alpha n}$ are both non-negative Hermitian. Lemma 3.1 yields then that $H_n^*$ and $H_{\alpha n}$ are both non-negative Hermitian and that the equations in (8.8), (8.9), and (8.10) hold true. In order to prove (8.11), we consider an arbitrary $n \in \mathbb{N}_0$ such that $2n + 1 \leq \kappa$. Taking into account $H_n^* = H_n$, Proposition 9.7, Lemma 3.1, and Remark 3.5, we conclude

$$\mathcal{R}(H_{\alpha n}^*) = \mathcal{R}(H_{\alpha n}^*) = \mathcal{D}_\alpha \subseteq \mathcal{D}_n = \mathcal{N}(I_{(n+1)q} - H_{\alpha n}^* H_n) = \mathcal{N}(I_{(n+1)q} - H_n^* H_n).$$

For every choice of $x \in \mathbb{C}_{(n+1)q}$, this implies $0 = (I_{(n+1)q} - H_n^* H_n)H_{\alpha n}^* x$ and, consequently, $H_n^* H_n H_{\alpha n}^* x = H_{\alpha n}^* x$. Thus, the first equation in (8.11) is verified. Hence $H_n^* = H_n$, as well.

Remark 8.11. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa \in \mathbb{K}_{q,\kappa}^{\geq e}$, and let $n \in \mathbb{N}_0$ be such that $2n + 1 \leq \kappa$. Then Remarks 7.3 and 7.4 yield $H_n^* = H_n$ and $H_{\alpha n}^* = H_{\alpha n}$. Thus,

$$H_n^* H_n = H_n^* H_n^\dagger$$

and

$$H_{\alpha n}^* H_{\alpha n} = H_{\alpha n}^* H_{\alpha n}^\dagger.$$  \hspace{1cm} \text{(8.12)}

In view of Lemma 8.10, consequently,

$$(I_{(n+1)q} - H_n^* H_n)H_n = 0,$$

$$(I_{(n+1)q} - H_{\alpha n}^* H_{\alpha n})H_{\alpha n} = 0,$$

$$(I_{(n+1)q} - H_n^* H_n)(I_{(n+1)q} - H_n^* H_n) = (I_{(n+1)q} - H_n^* H_n)(I_{(n+1)q} - H_n^* H_n)$$

$$= I_{(n+1)q} - H_n^* H_n H_n^* - H_n H_n^* H_n^* + H_n H_n^* H_n H_n^* = I_{(n+1)q} - H_n H_n^*,$$

and

$$(I_{(n+1)q} - H_n^* H_n)(I_{(n+1)q} - H_{\alpha n}^* H_{\alpha n})$$

$$= (I_{(n+1)q} - H_n^* H_n)(I_{(n+1)q} - H_{\alpha n}^* H_{\alpha n})$$

$$= I_{(n+1)q} - H_n^* H_n H_{\alpha n}^* - H_{\alpha n} H_{\alpha n}^* H_n H_{\alpha n} H_{\alpha n}^* = I_{(n+1)q} - H_n H_n^*.$$

Lemma 8.12. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa \in \mathbb{K}_{q,\kappa}^{\geq e}$. Further, let $n \in \mathbb{N}_0$ be such that $2n + 1 \leq \kappa$. For each $k \in \mathbb{N}_0$, then $H_n^* T_{q,n}^k (I_{(n+1)q} - H_n^* H_n) = 0$, $H_{\alpha n}^* T_{q,n}^k (I_{(n+1)q} - H_n^* H_n) = 0$, and $H_{\alpha n}^* T_{q,n}^k (I_{(n+1)q} - H_{\alpha n}^* H_{\alpha n}) = 0$ hold true. Furthermore, $H_n^* T_{q,n}^k (I_{(n+1)q} - H_{\alpha n}^* H_{\alpha n}) = 0$ for each $k \in \mathbb{N}$. 43
Proof. Use Lemma [8.10], Proposition [8.7] and Lemma [3.7]  

Lemma 8.13. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^e \in K_{q,\kappa,\alpha}$. For each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, then the following statements hold true:

(a) For every choice of $\zeta \in \mathbb{C}$ and $k \in \mathbb{N}_0$, 
\[
H_n R_{q,n}(\zeta) T_{q,n}^k (I_{(n+1)q} - H_n H_n^{-}) = 0, \\
(I_{(n+1)q} - H_n H_n^{-}) \left( T_{q,n}^\ast \right)^k \left[ R_{q,n}(\zeta) \right]^\ast H_n^- = 0, \\
H_{\alpha\beta n} R_{q,n}(\zeta) T_{q,n}^k (I_{(n+1)q} - H_{\alpha\beta n} H_{\alpha\beta n}^{-}) = 0, \\
\]
and 
\[
(I_{(n+1)q} - H_{\alpha\beta n} H_{\alpha\beta n}^{-})(T_{q,n}^\ast)^k \left[ R_{q,n}(\zeta) \right]^\ast H_{\alpha\beta n}^- = 0. \\
\]

(b) For each $\zeta \in \mathbb{C}$ and each $k \in \mathbb{N}$, 
\[
H_n R_{q,n}(\zeta) T_{q,n}^k (I_{(n+1)q} - H_{\alpha\beta n} H_{\alpha\beta n}^{-}) = 0, \\
\]
and 
\[
(I_{(n+1)q} - H_{\alpha\beta n} H_{\alpha\beta n}^{-})(T_{q,n}^\ast)^k \left[ R_{q,n}(\zeta) \right]^\ast H_n^- = 0. \\
\]

Proof. Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$ and let $\zeta \in \mathbb{C}$. Because of $K_{q,\kappa,\alpha} \subseteq K_{q,\kappa,\alpha}$, Remark [8.4] and Lemma [8.10] we get 
\[
H_n^\ast = H_n, \quad (H_n^-)^\ast = H_n^- , \quad (H_n^- H_n)^\ast = H_n^\ast (H_n^-)^\ast = H_n H_n^- , \\
H_{\alpha\beta n}^\ast = H_{\alpha\beta n}, \quad (H_{\alpha\beta n}^-)^\ast = H_{\alpha\beta n}^- , \quad \text{and} \quad (H_{\alpha\beta n}^- H_{\alpha\beta n}^-)^\ast = H_{\alpha\beta n}^- H_{\alpha\beta n}^- . \\
\]
In view of Remark [8.4] and Lemma [8.12] for each $k \in \mathbb{N}_0$, we obtain 
\[
H_n R_{q,n}(\zeta) T_{q,n}^k (I_{(n+1)q} - H_n H_n^{-}) = H_n^\ast \left( \sum_{j=0}^{n} \zeta^j T_{q,n}^j \right) T_{q,n}^k (I_{(n+1)q} - H_n H_n^{-}) \\
= \sum_{j=0}^{n} \zeta^j H_n^- T_{q,n}^{j+k} (I_{(n+1)q} - H_n H_n^{-}) = 0 \\
\]
and, analogously, [8.14]. Furthermore, the same arguments imply that [8.16] holds true for each $k \in \mathbb{N}$. For all $k \in \mathbb{N}_0$, equation [8.13] follows from [8.20], [8.18], and [8.19]. Moreover, for each $k \in \mathbb{N}_0$, equation [8.15] is a consequence of [8.18], [8.19], and [8.14]. Using [8.16], [8.18], and [8.19], we see that [8.17] holds true for each $k \in \mathbb{N}$.

Lemma 8.14. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^e \in K_{q,\kappa,\alpha}$. For each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$ and every choice of $\zeta \in \mathbb{C}$ and $\eta \in \mathbb{C}$, then 
\[
H_n R_{q,n}(\zeta) \left[ R_{q,n}^\ast (\eta) \right]^\ast (I_{(n+1)q} - H_n H_n^{-}) = 0, \\
(I_{(n+1)q} - H_n H_n^{-}) \left[ R_{q,n}^\ast (\eta) \right]^{-1} \left[ R_{q,n}(\zeta) \right]^\ast H_n^- = 0, \\
H_{\alpha\beta n} R_{q,n}(\zeta) \left[ R_{q,n}^\ast (\eta) \right]^\ast (I_{(n+1)q} - H_{\alpha\beta n} H_{\alpha\beta n}^{-}) = 0, \\
\]
and 
\[
(I_{(n+1)q} - H_{\alpha\beta n} H_{\alpha\beta n}^{-}) \left[ R_{q,n}^\ast (\eta) \right]^{-1} \left[ R_{q,n}(\zeta) \right]^\ast H_{\alpha\beta n}^- = 0. \\
\]
Proof. Let \( n \in \mathbb{N}_0 \) be such that \( 2n + 1 \leq \kappa \), and let \( \zeta, \eta \in \mathbb{C} \). Because of Remark 7.4 and Lemma 8.10, we obtain (8.18) and (8.19). From Remark 4.4 and Lemma 8.13[4], we conclude

\[
H_n^{-} T_{q,n}(\zeta) \left[ RT_{q,n}^{*}(\eta) \right]^{-1} (I_{n+1} q - H_n H_n^{-}) \\
= H_n^{-} T_{q,n}(\zeta) (I_{n+1} q - \eta T_{q,n}) (I_{n+1} q - H_n H_n^{-}) \\
= H_n^{-} T_{q,n}(\zeta) (I_{n+1} q - H_n H_n^{-}) - \eta H_n^{-} T_{q,n}(\zeta) T_{q,n} (I_{n+1} q - H_n H_n^{-}) = 0
\]  

(8.24)

and, analogously, (8.22). Obviously, (8.24) and (8.18) imply (8.21). Furthermore, (8.18), (8.19), and (8.22) show that (8.23) holds true as well. \( \square \)

Lemma 8.15. Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N} \cup \{ \infty \} \), and let \( (s_j)_{j=0}^{\kappa} \in K_{\mathbb{R},\mathbb{R},\alpha}^{>0} \). For each \( n \in \mathbb{N}_0 \) with \( 2n + 1 \leq \kappa \), then \( H_n^{-} T_{q,n}(\alpha)(v_{q,n} \varepsilon_{q,n} H_{\alpha n} + T_{q,n}) = H_{\alpha n}^{-} \)

\[
H_{\alpha n}^{-} [ (I_{n+1} q - H_n v_{q,n} v_{q,n}^{*} H_{\alpha n} + T_{q,n}) ] = T_{q,n}^{*} R_{q,n}^{*}(\alpha) H_n^{-}.
\]  

(8.25)

Proof. Let \( n \in \mathbb{N}_0 \) be such that \( 2n + 1 \leq \kappa \). Because of \( K_{\mathbb{R},\mathbb{R},\alpha}^{>0} \subseteq K_{\mathbb{R},\mathbb{R},\alpha}^{>0} \), Remark 7.4, and Lemma 8.10, we have (8.18) and (8.19). Remark 4.4, Lemma 8.10, and Lemma 8.13[4] yield

\[
H_n^{-} T_{q,n}(\alpha)(v_{q,n} \varepsilon_{q,n} H_{\alpha n} + T_{q,n}) \\
= H_n^{-} T_{q,n}(\alpha) \left[ [R_{q,n}(\alpha)]^{-1} H_n - T_{q,n} H_{\alpha n} \right] H_{\alpha n}^{-} + T_{q,n} \\
= H_n^{-} H_{\alpha n}^{-} - H_n^{-} R_{q,n}(\alpha) T_{q,n} H_{\alpha n}^{-} + H_n^{-} R_{q,n}(\alpha) T_{q,n} \\
= H_{\alpha n}^{-} + H_n^{-} R_{q,n}(\alpha) T_{q,n} (I_{n+1} q - H_{\alpha n} H_{\alpha n}^{-}) = H_{\alpha n}^{-}.
\]

This implies

\[
[I_{n+1} q - H_n^{-} R_{q,n}(\alpha) v_{q,n} v_{q,n}^{*} H_n] H_{\alpha n}^{-} = H_n^{-} R_{q,n}(\alpha) T_{q,n}.
\]  

(8.26)

In view of (8.18), (8.19), \([R_{q,n}(\alpha)]^{*} = R_{q,n}^{*}(\alpha)\), and (8.26), it follows (8.25). \( \square \)

9. Discussion of \( \tilde{J}_q \)-forms of particular matrix polynomials

In this section, we construct special matrix polynomials, which are useful to describe the solution set of the matricial truncated Stieltjes power moment problem \( MP[(\alpha, \infty); (s_j)_{j=0}^{\kappa}, \leq] \). Particular interest is focused to representations of \( \tilde{J}_q \)-forms, where

\[
\tilde{J}_q := \begin{bmatrix} 0_{q \times q} & -i I_q \\
I_q & 0_{q \times q}\end{bmatrix}.
\]

Obviously, \( \tilde{J}_q \) is a \( 2q \times 2q \) signature matrix, i.e., \( \tilde{J}_q^* = \tilde{J}_q \) and \( \tilde{J}_q^2 = I_{2q} \) hold true. We modify Bolotnikov’s [8] approach, who considered the particular case \( \alpha = 0 \). However, the calculations in the general case \( \alpha \in \mathbb{R} \) are much more complicated.

Remark 9.1. For every choice of \( A, B \in \mathbb{C}^{q \times q} \), we have \([A_B]^*(-\tilde{J}_q)[A_B]^* = -i(B^* A - A^* B)\). In particular, \([A_B]^*(-\tilde{J}_q)[A_B]^* = 23A\).

Remark 9.2. Let \( A \in \mathbb{C}_{n}^{q \times q} \). Then the matrices \( B := [I_q 0_{q \times q}] \) and \( C := [I_q A] \) fulfill \( B^* \tilde{J}_q B = \tilde{J}_q, C^* \tilde{J}_q C = \tilde{J}_q, B \tilde{J}_q B^* = \tilde{J}_q, \) and \( C \tilde{J}_q C^* = \tilde{J}_q \).
Remark 9.3. For each \( n \in \mathbb{N}_0 \) and each \( A \in \mathbb{C}^{(n+1)q \times (n+1)q} \), we have
\[
[I_{(n+1)q}, A](I_2 \otimes v_{q,n}) \tilde{J}_q = i[A, -I_{(n+1)q}](I_2 \otimes v_{q,n}),
\]
(9.1)
\[
[A, -I_{(n+1)q}](I_2 \otimes v_{q,n}) \tilde{J}_q = -i[I_{(n+1)q}, A](I_2 \otimes v_{q,n}),
\]
(9.2)
and
\[
[I_{(n+1)q}, A](I_2 \otimes v_{q,n}) \tilde{J}_q(I_2 \otimes v_{q,n})^*[A, -I_{(n+1)q}]^* = i(Av_{q,n}v_{q,n}^* - v_{q,n}v_{q,n}^* A)^*,
\]
(9.3)
\[
[A, -I_{(n+1)q}](I_2 \otimes v_{q,n}) \tilde{J}_q(I_2 \otimes v_{q,n})^*[A, -I_{(n+1)q}]^* = i(Av_{q,n}v_{q,n}^* - v_{q,n}v_{q,n}^* A)^*.
\]
(9.4)

Remark 9.4. Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N}_0 \cup \{ \infty \} \), and let \( (s_j)_{j=0}^\kappa \in \mathcal{H}_{q,\kappa}^\geq \). For each \( n \in \mathbb{N}_0 \) with \( 2n \leq \kappa \), Remark 9.4 shows that \( U_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q \times 2q} \) defined by
\[
U_{n,\alpha}(\zeta) := I_{2q} + (\zeta - \alpha)(I_2 \otimes v_{q,n})^*[T_{q,n}H_n, -I_{(n+1)q}]^* R_{T_{q,n}}(\zeta) R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n})
\]
is a matrix polynomial of degree not greater than \( n + 1 \), where \( H_n^* = H_n \) implies that, for each \( \zeta \in \mathbb{C} \), the matrix \( U_{n,\alpha}(\zeta) \) admits the block representation
\[
U_{n,\alpha}(\zeta) = \begin{bmatrix} A_n(\zeta) & B_n(\zeta) \\ C_n(\zeta) & D_n(\zeta) \end{bmatrix}
\]
(9.7)
with
\[
A_n(\zeta) := I_1 + (\zeta - \alpha)v_{q,n}H_nT_{q,n}^* R_{T_{q,n}}(\zeta)R_{T_{q,n}}(\alpha) v_{q,n},
\]
(9.8)
\[
B_n(\zeta) := - (\zeta - \alpha)v_{q,n}H_nT_{q,n}^* R_{T_{q,n}}(\zeta)R_{T_{q,n}}(\alpha) v_{q,n},
\]
(9.9)
\[
C_n(\zeta) := - (\zeta - \alpha)v_{q,n}^*R_{T_{q,n}}(\zeta)H_nR_{T_{q,n}}(\alpha) v_{q,n},
\]
(9.10)
\[
D_n(\zeta) := I_1 - (\zeta - \alpha)v_{q,n}^*R_{T_{q,n}}(\zeta)H_nR_{T_{q,n}}(\alpha)T_{q,n}H_n v_{q,n}.
\]
(9.11)

Lemma 9.5. Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N}_0 \cup \{ \infty \} \), and let \( (s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa}^\geq \). For all of \( n \in \mathbb{N}_0 \) with \( 2n \leq \kappa \) and all \( z, w \in \mathbb{C} \), the function \( U_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q \times 2q} \) given by \( (9.3) \) fulfills
\[
\tilde{J}_q - U_{n,\alpha}(z)\tilde{J}_q U_{n,\alpha}^*(w) = -i(z - w)(I_2 \otimes v_{q,n})^*[T_{q,n}H_n, -I_{(n+1)q}]^* R_{T_{q,n}}(z)H_n^-
\]
\[
\times \left[ R_{T_{q,n}}^*(w) \right]^*[I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n})
\]
(9.12)
\[
\tilde{J}_q - U_{n,\alpha}(z)\tilde{J}_q U_{n,\alpha}^*(w) = S_1(z) + S_2(w) + S_3(z, w)
\]
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where
\[
S_1(z) := -(z - \alpha)(I_2 \otimes v_{q,n})^*[T_{q,n}H_n, -I_{(n+1)q}]^* R_{T^*_{q,n}}(z) H_n^{-1}
\times R_{T^*_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n})\tilde{J}_q,
\]
(9.13)
\[
S_2(w) := - (\overline{w} - \alpha)\tilde{J}_q(I_2 \otimes v_{q,n})^*[I_{(n+1)q}, T_{q,n}H_n]^* [R_{T^*_{q,n}}(\alpha)]^* H_n^{-1}
\times \left[ R_{T^*_{q,n}}(w) \right]^* [T_{q,n}H_n, -I_{(n+1)q}]^* (I_2 \otimes v_{q,n}),
\]
(9.14)
and
\[
S_3(z, w) := -(z - \alpha)(\overline{w} - \alpha)(I_2 \otimes v_{q,n})^*[T_{q,n}H_n, -I_{(n+1)q}]^* R_{T^*_{q,n}}(z) H_n^{-1} R_{T^*_{q,n}}(\alpha)
\times [I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n})\tilde{J}_q(I_2 \otimes v_{q,n})^*[I_{(n+1)q}, T_{q,n}H_n]^*
\times \left[ R_{T^*_{q,n}}(\alpha) \right]^* H_n^{-1} \left[ R_{T^*_{q,n}}(w) \right]^* [T_{q,n}H_n, -I_{(n+1)q}]^* (I_2 \otimes v_{q,n}).
\]
(9.15)
Because of (9.13), (9.14), (9.1), (9.2), and Remark 1.4 we get then
\[
S_1(z) = -i(z - \alpha)(I_2 \otimes v_{q,n})^*[T_{q,n}H_n, -I_{(n+1)q}]^* R_{T^*_{q,n}}(z)
\times H_n^{-1} R_{T^*_{q,n}}(\alpha)\left[ R_{T^*_{q,n}}(w) \right]^* [T_{q,n}H_n, -I_{(n+1)q}] (I_2 \otimes v_{q,n}),
\]
(9.16)
\[
S_2(w) = i(\overline{w} - \alpha)(I_2 \otimes v_{q,n})^*[T_{q,n}H_n, -I_{(n+1)q}]^* R_{T^*_{q,n}}(z) \left[ R_{T^*_{q,n}}(w) \right]^{-1} [R_{T^*_{q,n}}(\alpha)]^*
\times H_n^{-1} \left[ R_{T^*_{q,n}}(w) \right]^* [T_{q,n}H_n, -I_{(n+1)q}] (I_2 \otimes v_{q,n}),
\]
(9.17)
and, according to (9.15), (9.3), and $H_n^* = H_n$, we have
\[
S_3(z, w) = -i(z - \alpha)(\overline{w} - \alpha)(I_2 \otimes v_{q,n})^*[T_{q,n}H_n, -I_{(n+1)q}]^* R_{T^*_{q,n}}(z) H_n^{-1}
\times R_{T^*_{q,n}}(\alpha)(T_{q,n}H_n v_{q,n}v_{q,n}^* - v_{q,n}v_{q,n}^* T_{n}^* T_{q,n}) [R_{T^*_{q,n}}(\alpha)]^*
\times H_n^{-1} \left[ R_{T^*_{q,n}}(w) \right]^* [T_{q,n}H_n, -I_{(n+1)q}] (I_2 \otimes v_{q,n}).
\]
(9.18)
In view of $H_n^* = H_n$, $R_{T^*_{q,n}}(\alpha) = [R_{T^*_{q,n}}(\alpha)]^*$, (7.1), (B.3), (B.3), and $R_{T^*_{q,n}}(\alpha) = [R_{T^*_{q,n}}(\alpha)]^*$, it follows
\[
(z - \alpha)(\overline{w} - \alpha)R_{T^*_{q,n}}(\alpha)(T_{q,n}H_n v_{q,n}v_{q,n}^* - v_{q,n}v_{q,n}^* T_{n}^* T_{q,n}) [R_{T^*_{q,n}}(\alpha)]^*
\times [R_{T^*_{q,n}}(w)]^{-1} (z - \alpha)(\overline{w} - \alpha)R_{T^*_{q,n}}(\alpha) - [R_{T^*_{q,n}}(\alpha)]^{-1} H_n^* [R_{T^*_{q,n}}(\alpha)]^*
\times [R_{T^*_{q,n}}(w)]^{-1} (z - \alpha)(\overline{w} - \alpha)R_{T^*_{q,n}}(\alpha) - [R_{T^*_{q,n}}(\alpha)]^{-1} H_n^* [R_{T^*_{q,n}}(\alpha)]^*
\times [R_{T^*_{q,n}}(w)]^{-1} (z - \alpha)(\overline{w} - \alpha)R_{T^*_{q,n}}(\alpha) - [R_{T^*_{q,n}}(\alpha)]^{-1} H_n^* [R_{T^*_{q,n}}(\alpha)]^*
\times [R_{T^*_{q,n}}(w)]^{-1} (z - \alpha)(\overline{w} - \alpha)R_{T^*_{q,n}}(\alpha) - [R_{T^*_{q,n}}(\alpha)]^{-1} H_n^* [R_{T^*_{q,n}}(\alpha)]^*
\times [R_{T^*_{q,n}}(w)]^{-1} (z - \alpha)(\overline{w} - \alpha)R_{T^*_{q,n}}(\alpha) - [R_{T^*_{q,n}}(\alpha)]^{-1} H_n^* [R_{T^*_{q,n}}(\alpha)]^*
\times [R_{T^*_{q,n}}(w)]^{-1} (z - \alpha)(\overline{w} - \alpha)R_{T^*_{q,n}}(\alpha) - [R_{T^*_{q,n}}(\alpha)]^{-1} H_n^* [R_{T^*_{q,n}}(\alpha)]^*$
\[
= -(z - \alpha)R_{T^*_{q,n}}(\alpha) [R_{T^*_{q,n}}(w)]^{-1} H_n + (\overline{w} - \alpha)H_n [R_{T^*_{q,n}}(z)]^{-1} [R_{T^*_{q,n}}(\alpha)]^* + (z - \overline{w})H_n.
\]
Consequently, from (9.18) we get then
\[
S_3(z, w) = -i(I_2 \otimes v_{q,n})^*[T_{q,n}H_n, -I_{(n+1)q}]^* R_{T^*_{q,n}}(z) H_n^{-1}
\times \left\{ -(z - \alpha)R_{T^*_{q,n}}(\alpha) [R_{T^*_{q,n}}(w)]^{-1} H_n + (\overline{w} - \alpha)H_n [R_{T^*_{q,n}}(z)]^{-1} [R_{T^*_{q,n}}(\alpha)]^*
\right.
\left. + (z - \overline{w})H_n \right\} T_{q,n}H_n, -I_{(n+1)q} (I_2 \otimes v_{q,n}).
\]
(9.19)
The combination of (9.12), (9.16), (9.17), and (9.19) yields

$$\tilde{J}_q - U_{n,\alpha}(z)\tilde{J}_q U_{n,\alpha}^*(w) = -i(I_2 \otimes v_{q,n})^*[T_{q,n}H_n, -I_{(n+1)q}]^*R_{T_q,n}(z)S(z, w)$$

$$\times \left[R_{T_q,n}(w)\right]^*[T_{q,n}H_n, -I_{(n+1)q}](I_2 \otimes v_{q,n}) \quad (9.20)$$

where

$$S(z, w) := (z - \alpha)H_n^- R_{T_q,n}(z)\left[R_{T_q,n}(w)\right]^{-*} - (\overline{w} - \alpha)\left[R_{T_q,n}(z)\right]^{-1}\left[R_{T_q,n}(\alpha)\right]^*H_n^-$$

$$- (z - \alpha)H_n^- R_{T_q,n}(\alpha)\left[R_{T_q,n}(w)\right]^{-*} H_nH_n^- + (\overline{w} - \alpha)H_n^- H_n\left[R_{T_q,n}(z)\right]^{-1}\left[R_{T_q,n}(\alpha)\right]^*H_n^-$$

$$+ (z - \overline{w})H_n^- H_nH_n^- \quad (9.21)$$

Using (9.21), Lemma 8.14 and Remark 8.9, we infer \( S(z, w) = (z - \overline{w})H_n^- \). Hence, because of (9.20), the proof is complete.

**Remark 9.6.** Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N} \cup \{\infty\} \), and let \( (s_j)_{j=0}^\kappa \in K_{q,\kappa,\alpha}^> \). For each \( n \in \mathbb{N}_0 \) with \( 2n+1 \leq \kappa \), Remark 4.4 shows then that \( \tilde{U}_{n,\alpha} : \mathbb{C} \rightarrow \mathbb{C}^{2q \times 2q} \) given by

$$\tilde{U}_{n,\alpha}(\zeta) := I_{2q} + (\zeta - \alpha)(I_2 \otimes v_{q,n})^*[R_{T_q,n}(\alpha)]^{-1}H_n, -I_{(n+1)q}]^*R_{T_q,n}(\zeta)H_{\alpha n}^-$$

$$\times R_{T_q,n}(\alpha)\left[I_{(n+1)q}]^*[R_{T_q,n}(\alpha)]^{-1}H_n\right](I_2 \otimes v_{q,n}) \quad (9.22)$$

is a matrix polynomial of degree not greater that \( n+1 \), where \( H_{\alpha n}^* = H_n \) shows that, for each \( \zeta \in \mathbb{C} \), the matrix \( \tilde{U}_{n,\alpha}(\zeta) \) admits the block representation

$$\tilde{U}_{n,\alpha}(\zeta) = \begin{bmatrix} \tilde{A}_n(\zeta) & \tilde{B}_n(\zeta) \\ \tilde{C}_n(\zeta) & \tilde{D}_n(\zeta) \end{bmatrix}$$

with

$$\tilde{A}_n(\zeta) := I_q + (\zeta - \alpha)v_{q,n}^*H_n\left[R_{T_q,n}(\alpha)\right]^{-1}R_{T_q,n}(\zeta)H_{\alpha n}^-R_{T_q,n}(\alpha)v_{q,n},$$

$$\tilde{B}_n(\zeta) := + (\zeta - \alpha)v_{q,n}^*H_n\left[R_{T_q,n}(\alpha)\right]^{-1}R_{T_q,n}(\zeta)H_{\alpha n}^-H_nv_{q,n},$$

$$\tilde{C}_n(\zeta) := - (\zeta - \alpha)v_{q,n}^*R_{T_q,n}(\zeta)H_{\alpha n}^-R_{T_q,n}(\alpha)v_{q,n},$$

$$\tilde{D}_n(\zeta) := I_q - (\zeta - \alpha)v_{q,n}^*R_{T_q,n}(\zeta)H_{\alpha n}^-H_nv_{q,n}.$$

**Lemma 9.7.** Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N} \cup \{\infty\} \), and let \( (s_j)_{j=0}^\kappa \in K_{q,\kappa,\alpha}^> \). Let \( \tilde{U}_{n,\alpha} : \mathbb{C} \rightarrow \mathbb{C}^{2q \times 2q} \) be given by (9.22). For all \( n \in \mathbb{N}_0 \) with \( 2n+1 \leq \kappa \) and all \( z, w \in \mathbb{C} \), then

$$\tilde{J}_q - \tilde{U}_{n,\alpha}(z)\tilde{J}_q \tilde{U}_{n,\alpha}^*(w) = -i(z - \overline{w})(I_2 \otimes v_{q,n})^*[R_{T_q,n}(\alpha)]^{-1}H_n, -I_{(n+1)q}]^*R_{T_q,n}(z)H_{\alpha n}^-$$

$$\times \left[R_{T_q,n}(w)\right]^*[R_{T_q,n}(\alpha)]^{-1}H_n, -I_{(n+1)q}]^*(I_2 \otimes v_{q,n}) \quad (9.23)$$

**Proof.** Let \( n \in \mathbb{N}_0 \) be such that \( 2n+1 \leq \kappa \). Because of \( (s_j)_{j=0}^\kappa \in K_{q,\kappa,\alpha}^> \subseteq K_{q,\kappa,\alpha}^> \) and Remark 7.21 we have \( H_{n}^* = H_n \) and \( H_{\alpha n}^- = H_{\alpha n}^- \). Lemma 8.10 provides us (8.9) and (8.10).
Let $z, w \in \mathbb{C}$. Taking into account (9.22), $\hat{J}_q^2 = I_{2q}$, and (8.31), we get then
\[
\hat{J}_q - \hat{U}_{n,\alpha}(z) \hat{J}_q \hat{U}_{n,\alpha}^*(w) = \hat{J}_q - \left\{ I_{2q} + (z - \alpha)(I_{2} \otimes v_{q,n})^* \left[ (RT_{q,n}(\alpha))^{-1} H_n, -I_{(n+1)q} \right]^* \right.
\]
\[
\times RT_{q,n}(z) H_{\alpha\circ n} RT_{q,n}^*(\alpha) \left[ I_{(n+1)q}, [RT_{q,n}(\alpha)]^{-1} H_n \right] (I_{2} \otimes v_{q,n}) \hat{J}_q
\]
\[
\left. \times \left\{ I_{2q} + (\bar{w} - \alpha)(I_{2} \otimes v_{q,n})^* \left[ I_{(n+1)q}, [RT_{q,n}(\alpha)]^{-1} H_n \right]^* \right. \right.
\]
\[
\times \left[ RT_{q,n}(\alpha) \right]^* H_{\alpha\circ n} \left[ RT_{q,n}(w) \right]^* \left[ [RT_{q,n}(\alpha)]^{-1} H_n, -I_{(n+1)q} \right] (I_{2} \otimes v_{q,n}) \right\} = \hat{S}_1(z) + \hat{S}_2(w) + \hat{S}_3(z, w)
\]

where
\[
\hat{S}_1(z) := -(z - \alpha)(I_{2} \otimes v_{q,n})^* \left[ [RT_{q,n}(\alpha)]^{-1} H_n, -I_{(n+1)q} \right]^* RT_{q,n}(z) H_{\alpha\circ n}
\]
\[
\times RT_{q,n}(\alpha) \left[ I_{(n+1)q}, [RT_{q,n}(\alpha)]^{-1} H_n \right] (I_{2} \otimes v_{q,n}) \hat{J}_q,
\]
\[
\hat{S}_2(w) := -(\bar{w} - \alpha) \hat{J}_q (I_{2} \otimes v_{q,n})^* \left[ I_{(n+1)q}, [RT_{q,n}(\alpha)]^{-1} H_n \right]^* \left[ RT_{q,n}(\alpha) \right]^* H_{\alpha\circ n}
\]
\[
\times \left[ RT_{q,n}(w) \right]^* \left[ [RT_{q,n}(\alpha)]^{-1} H_n, -I_{(n+1)q} \right] (I_{2} \otimes v_{q,n}),
\]
and
\[
\hat{S}_3(z, w) := -(z - \alpha)(\bar{w} - \alpha)(I_{2} \otimes v_{q,n})^* \left[ [RT_{q,n}(\alpha)]^{-1} H_n, -I_{(n+1)q} \right]^* RT_{q,n}(z) H_{\alpha\circ n}
\]
\[
\times RT_{q,n}(\alpha) \left[ I_{(n+1)q}, [RT_{q,n}(\alpha)]^{-1} H_n \right] (I_{2} \otimes v_{q,n}) \hat{J}_q
\]
\[
\times (I_{2} \otimes v_{q,n})^* \left[ I_{(n+1)q}, [RT_{q,n}(\alpha)]^{-1} H_n \right]^* \left[ RT_{q,n}(\alpha) \right]^* H_{\alpha\circ n}
\]
\[
\times \left[ RT_{q,n}^*(w) \right]^* \left[ [RT_{q,n}(\alpha)]^{-1} H_n, -I_{(n+1)q} \right] (I_{2} \otimes v_{q,n}).
\]

In view of (9.11), (9.12), and Remark 14 from (9.25) and (9.26) we obtain
\[
\hat{S}_1(z) = -i(z - \alpha)(I_{2} \otimes v_{q,n})^* \left[ [RT_{q,n}(\alpha)]^{-1} H_n, -I_{(n+1)q} \right]^* RT_{q,n}(z) H_{\alpha\circ n} RT_{q,n}^*(\alpha)
\]
\[
\times \left[ RT_{q,n}(w) \right]^{-*} \left[ RT_{q,n}^*(w) \right]^* \left[ [RT_{q,n}(\alpha)]^{-1} H_n, -I_{(n+1)q} \right] (I_{2} \otimes v_{q,n}),
\]
\[
\hat{S}_2(w) = i(\bar{w} - \alpha)(I_{2} \otimes v_{q,n})^* \left[ [RT_{q,n}(\alpha)]^{-1} H_n, -I_{(n+1)q} \right]^* RT_{q,n}(z) \left[ RT_{q,n}(z) \right]^{-1}
\]
\[
\times \left[ RT_{q,n}(\alpha) \right]^* H_{\alpha\circ n} \left[ RT_{q,n}^*(w) \right]^* \left[ [RT_{q,n}(\alpha)]^{-1} H_n, -I_{(n+1)q} \right] (I_{2} \otimes v_{q,n}),
\]
and, because of (9.27), (5.39), and $H_n^* = H_n$, furthermore,
\[
\hat{S}_3(z, w) = -i(z - \alpha)(\bar{w} - \alpha)(I_{2} \otimes v_{q,n})^* \left[ [RT_{q,n}(\alpha)]^{-1} H_n, -I_{(n+1)q} \right]^* RT_{q,n}(z) H_{\alpha\circ n}
\]
\[
\times RT_{q,n}(\alpha) \left[ [RT_{q,n}(\alpha)]^{-1} H_n v_{q,n} H_{\alpha\circ n} v_{q,n}^* - v_{q,n} v_{q,n}^* H_n [RT_{q,n}(\alpha)]^{-*} \right] \left[ RT_{q,n}(\alpha) \right]^*
\]
\[
\times H_{\alpha\circ n} \left[ RT_{q,n}^*(w) \right]^* \left[ [RT_{q,n}(\alpha)]^{-1} H_n, -I_{(n+1)q} \right] (I_{2} \otimes v_{q,n}).
\]
Using $H_n^* = H_n$, (7.2), (13.8), (13.9), and $R_{T,q,n}(\alpha) = [R_{T,q,n}(\alpha)]^*$, we infer

\[(z - \alpha)(\overline{w} - \alpha)R_{T,q,n}(\alpha)\left( [R_{T,q,n}(\alpha)]^{-1}H_n v_{q,n} v_{q,n}^* - v_{q,n} v_{q,n}^* H_n [R_{T,q,n}(\alpha)]^{-1} \right) [R_{T,q,n}(\alpha)]^*\]

\[= (z - \alpha)(\overline{w} - \alpha)R_{T,q,n}(\alpha)\left( T_{q,n} H_{\alpha n} [R_{T,q,n}(\alpha)]^{-1} - [R_{T,q,n}(\alpha)]^{-1} H_{\alpha n} T_{q,n}^* \right) [R_{T,q,n}(\alpha)]^*\]

\[= (z - \alpha)(\overline{w} - \alpha)R_{T,q,n}(\alpha)T_{q,n} H_{\alpha n} - (z - \alpha)(\overline{w} - \alpha)H_{\alpha n} T_{q,n}^* [R_{T,q,n}(\alpha)]^*\]

\[= -(z - \alpha)\left( R_{T,q,n}(\alpha) [R_{T,q,n}^*(w)]^{-1} - I_{(n+1)q} \right) H_{\alpha n}\]

\[+ (\overline{w} - \alpha)H_{\alpha n} \left( [R_{T,q,n}(\alpha)]^{-1} - I_{(n+1)q} \right)\]

\[= -(z - \alpha) R_{T,q,n}(\alpha) [R_{T,q,n}^*(w)]^{-1} H_{\alpha n} + (\overline{w} - \alpha)H_{\alpha n} [R_{T,q,n}^*(z)]^{-1} [R_{T,q,n}(\alpha)]^*\]

\[+ (z - \overline{w})H_{\alpha n}.\]

Consequently, from (9.30) we get then

\[\tilde{S}_3(z,w) = -i(I_2 \otimes v_{q,n})^* \left[ [R_{T,q,n}(\alpha)]^{-1} H_n, -I_{(n+1)q} \right]^* R_{T,q,n}^*(z) H_{\alpha n}\]

\[\times \left\{ -(z - \alpha) R_{T,q,n}(\alpha) [R_{T,q,n}^*(w)]^{-1} H_{\alpha n} + (\overline{w} - \alpha)H_{\alpha n} [R_{T,q,n}^*(z)]^{-1} [R_{T,q,n}(\alpha)]^*\right\}\]

\[\times [R_{T,q,n}(\alpha)]^{-1} H_n, -I_{(n+1)q} \right] (I_2 \otimes v_{q,n}). \quad (9.31)\]

The combination of (9.24), (9.28), (9.29), and (9.31) provides us

\[\tilde{J}_q - \tilde{U}_{n,\alpha}(z) \tilde{J}_q \tilde{U}_{n,\alpha}^*(w) = -i(I_2 \otimes v_{q,n})^* \left[ [R_{T,q,n}(\alpha)]^{-1} H_n, -I_{(n+1)q} \right]^* R_{T,q,n}^*(z) \tilde{S}(z,w)\]

\[\times \left[ R_{T,q,n}^*(w) \right]^* \left[ [R_{T,q,n}(\alpha)]^{-1} H_n, -I_{(n+1)q} \right] (I_2 \otimes v_{q,n}). \quad (9.32)\]

where

\[\tilde{S}(z,w) := (z - \alpha) H_{\alpha n} R_{T,q,n}(\alpha) [R_{T,q,n}^*(w)]^{-1} - (\overline{w} - \alpha) R_{T,q,n}^*(z) \left[ [R_{T,q,n}(\alpha)]^{-1} H_n \right] - (z - \alpha) H_{\alpha n} R_{T,q,n}(\alpha) [R_{T,q,n}^*(w)]^{-1} H_{\alpha n}\]

\[+ (\overline{w} - \alpha) H_{\alpha n} R_{T,q,n}(\alpha) [R_{T,q,n}^*(z)]^{-1} H_{\alpha n} + (z - \overline{w})H_{\alpha n} H_{\alpha n} H_{\alpha n}. \quad (9.33)\]

From (9.33) and the Lemmata 8.14 and 8.10 we obtain $\tilde{S}(z,w) = (z - \overline{w})H_{\alpha n}$. Hence, taking into account (9.32), we get (9.29).

**Remark 9.8.** Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa \in K_{q,n}^{\geq \kappa}$. For each $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, in view of the Remarks 7.2 and 7.4 and Lemma 8.10 it is readily checked that the matrices

\[B_{n,\alpha} := \begin{bmatrix} I_q & v_{q,n} H_{\alpha n} H_{\alpha n} H_{\alpha n} v_{q,n} \end{bmatrix}_{0 \times q} \quad (9.34)\]
and

\[ \tilde{B}_{n,\alpha} := \begin{bmatrix} I_q & 0_{q \times q} \\ -v_{q,\alpha}R_{T_{q,\alpha}}(\alpha)H_n R_{T_{q,\alpha}}(\alpha)v_{q,\alpha} & I_q \end{bmatrix} \]  \tag{9.35}

are $\tilde{J}_q$-unitary, i.e., that $B_{n,\alpha}\tilde{J}_qB_{n,\alpha}^* = \tilde{J}_q$, $B_{n,\alpha}^*\tilde{J}_qB_{n,\alpha} = \tilde{J}_q$, $\tilde{B}_{n,\alpha}\tilde{J}_q\tilde{B}_{n,\alpha}^* = \tilde{J}_q$, and $\tilde{B}_{n,\alpha}^*\tilde{J}_q\tilde{B}_{n,\alpha} = \tilde{J}_q$ hold true.

**Remark 9.9.** Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa \in K_{q,\kappa,\alpha}$. In view of (9.31) and (9.35), for each $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, then

\[ [I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n})B_{n,\alpha} = [I_{(n+1)q}, (v_{q,n}v_{q,n}^*H_nH_{\alpha n} + T_{q,n})H_n](I_2 \otimes v_{q,n}) \]

and

\[ [I_{(n+1)q}, [R_{T_{q,n}}(\alpha)]^{-1}H_n](I_2 \otimes v_{q,n})\tilde{B}_{n,\alpha} = [R_{T_{q,n}}(\alpha)]^{-1}[I_{(n+1)q} - H_nv_{q,n}v_{q,n}^*R_{T_{q,n}}(\alpha)H_n^{-1}R_{T_{q,n}}(\alpha), H_n](I_2 \otimes v_{q,n}). \]

**Remark 10.10.** Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa \in K_{q,\kappa,\alpha}$. In view of Remark 9.9 and Lemma 8.15, it is readily checked that, for each $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, then

\[ H_n^{-1}R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n})B_{n,\alpha} = [H_n^{-1}R_{T_{q,n}}(\alpha), H_{\alpha n}^{-1}H_n](I_2 \otimes v_{q,n}). \]

**Remark 11.11.** Let $\alpha \in \mathbb{R}$ and let $n \in \mathbb{N}_0$. According to Remark 11.1, the matrix-valued functions $\Omega_{q,n,\alpha} : \mathbb{C} \rightarrow \mathbb{C}^{2(n+1)q \times 2(n+1)q}$ and $\tilde{\Omega}_{q,n,\alpha} : \mathbb{C} \rightarrow \mathbb{C}^{2(n+1)q \times 2(n+1)q}$ given by

\[ \Omega_{q,n,\alpha}(\zeta) := \begin{bmatrix} (\zeta - \alpha)T_{q,n}^* & [R_{T_{q,n}}(\alpha)]^{-1} \\ -(\zeta - \alpha)I_{(n+1)q} & -(\zeta - \alpha)I_{(n+1)q} \end{bmatrix} [I_2 \otimes R_{T_{q,n}}^*(\zeta)] \]  \tag{9.36}

and

\[ \tilde{\Omega}_{q,n,\alpha}(\zeta) := \begin{bmatrix} (\zeta - \alpha)T_{q,n}^* & (\zeta - \alpha)[R_{T_{q,n}}(\alpha)]^{-1} \\ -I_{(n+1)q} & -(\zeta - \alpha)I_{(n+1)q} \end{bmatrix} [I_2 \otimes R_{T_{q,n}}^*(\zeta)] \]  \tag{9.37}

are both matrix polynomials of degree $n + 1$.

**Lemma 9.12.** Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa \in K_{q,\kappa,\alpha}$. Let $n \in \mathbb{N}_0$ be such that $2n + 1 \leq \kappa$. In view of (9.36) and (9.37), let $\Theta_{n,\alpha} : \mathbb{C} \rightarrow \mathbb{C}^{2q \times 2q}$ and $\tilde{\Theta}_{n,\alpha} : \mathbb{C} \rightarrow \mathbb{C}^{2q \times 2q}$ be given by

\[ \Theta_{n,\alpha}(\zeta) := I_{2q} + (I_2 \otimes v_{q,n})^* \cdot \text{diag}(H_n, I_{(n+1)q}) \cdot \Omega_{q,n,\alpha}(\zeta) \times \text{diag}(H_n^{-1}, H_{\alpha n}^{-1}) \cdot \text{diag}(R_{T_{q,n}}(\alpha), H_n) \cdot (I_2 \otimes v_{q,n}) \]  \tag{9.38}

and

\[ \tilde{\Theta}_{n,\alpha}(\zeta) := I_{2q} + (I_2 \otimes v_{q,n})^* \cdot \text{diag}(H_n, I_{(n+1)q}) \cdot \tilde{\Omega}_{q,n,\alpha}(\zeta) \times \text{diag}(H_n^{-1}, H_{\alpha n}^{-1}) \cdot \text{diag}(R_{T_{q,n}}(\alpha), H_n) \cdot (I_2 \otimes v_{q,n}). \]  \tag{9.39}
Then $\Theta_{n,\alpha}$ and $\tilde{\Theta}_{n,\alpha}$ are matrix polynomials of degree not greater than $n+1$ and, for each $\zeta \in \mathbb{C}$, the representations
\begin{equation}
\Theta_{n,\alpha}(\zeta) = U_{n,\alpha}(\zeta)B_{n,\alpha} \quad \text{and} \quad \tilde{\Theta}_{n,\alpha}(\zeta) = \tilde{U}_{n,\alpha}(\zeta)\tilde{B}_{n,\alpha} \tag{9.40}
\end{equation}
hold true, where $U_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q \times 2q}$ and $\tilde{U}_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q \times 2q}$ are defined by (9.10) and (9.22), and where $B_{n,\alpha}$ and $\tilde{B}_{n,\alpha}$ are given by (9.34) and (9.35), respectively. If
\begin{equation}
\Theta_{n,\alpha} = [\Theta_{n,\alpha}^{(j,k)}]_{j,k=1}^{q} \quad \text{and} \quad \tilde{\Theta}_{n,\alpha} = [\tilde{\Theta}_{n,\alpha}^{(j,k)}]_{j,k=1}^{q} \tag{9.41}
\end{equation}
are the $q \times q$ block representations of $\Theta_{n,\alpha}$ and $\tilde{\Theta}_{n,\alpha}$, respectively, for each $\zeta \in \mathbb{C}$, then
\begin{align*}
\Theta_{n,\alpha}^{(1,1)}(\zeta) &= I_q + (\zeta - \alpha)v_{q,n}^*H_nT_{q,n}^*T_{q,n}^-H_n(\zeta)H_nR_{T_{q,n}}^-R_{T_{q,n}}^-(\alpha)v_{q,n}, \\
\Theta_{n,\alpha}^{(1,2)}(\zeta) &= v_{q,n}^*H_n\left[R_{T_{q,n}}^-(\alpha)\right]^{-1}R_{T_{q,n}}^+(\zeta)H_{\alpha\beta\gamma}H_{\alpha\beta\gamma}v_{q,n}, \\
\Theta_{n,\alpha}^{(2,1)}(\zeta) &= -I_q - (\zeta - \alpha)v_{q,n}^*R_{T_{q,n}}^-(\zeta)H_nR_{T_{q,n}}^-H_n(\alpha)v_{q,n}, \\
\Theta_{n,\alpha}^{(2,2)}(\zeta) &= I_q + (\zeta - \alpha)v_{q,n}^*H_nT_{q,n}^*T_{q,n}^-H_n(\zeta)H_nR_{T_{q,n}}^-R_{T_{q,n}}^-(\alpha)v_{q,n},
\end{align*}
and
\begin{align*}
\tilde{\Theta}_{n,\alpha}^{(1,1)}(\zeta) &= I_q + (\zeta - \alpha)v_{q,n}^*H_nT_{q,n}^*T_{q,n}^-H_nv_{q,n}, \\
\tilde{\Theta}_{n,\alpha}^{(2,1)}(\zeta) &= -I_q - (\zeta - \alpha)v_{q,n}^*R_{T_{q,n}}^-(\zeta)H_nR_{T_{q,n}}^-H_nv_{q,n}, \\
\tilde{\Theta}_{n,\alpha}^{(2,2)}(\zeta) &= I_q + (\zeta - \alpha)v_{q,n}^*H_nT_{q,n}^*T_{q,n}^-H_nv_{q,n}.
\end{align*}

Proof. Remark 9.11 shows that $\Theta_{n,\alpha}$ and $\tilde{\Theta}_{n,\alpha}$ are matrix polynomials of degree not greater than $n+1$. Let $\zeta \in \mathbb{C}$. Because of the Remarks 7.4 and 7.4, we have $H_n^* = H_n$ and $H_{\alpha\beta\gamma}^* = H_{\alpha\beta\gamma}$. Using (9.38) and (9.39), one can easily check that (9.42), (9.43), (9.44), and (9.45) hold true. From (9.39), (9.37), and (9.41) we infer that (9.46), (9.47), (9.48), and (9.49) are valid. Let
\begin{equation}
\Phi_{n,\alpha} = [\Phi_{n,\alpha}^{(j,k)}]_{j,k=1}^{q} \quad \text{and} \quad \tilde{\Phi}_{n,\alpha} = [\tilde{\Phi}_{n,\alpha}^{(j,k)}]_{j,k=1}^{q} \tag{9.50}
\end{equation}
be the $q \times q$ block representations of
\begin{equation}
\Phi_{n,\alpha} := U_{n,\alpha}B_{n,\alpha} \quad \text{and} \quad \tilde{\Phi}_{n,\alpha} := \tilde{U}_{n,\alpha}\tilde{B}_{n,\alpha}. \tag{9.51}
\end{equation}
By virtue of (9.7), (9.11), (9.34), and (9.42), then
\begin{equation}
\Phi_{n,\alpha}^{(1,1)}(\zeta) = I_q + (\zeta - \alpha)v_{q,n}^*H_nT_{q,n}^*T_{q,n}^-H_n(\zeta)H_nR_{T_{q,n}}^-R_{T_{q,n}}^-(\alpha)v_{q,n} = \Theta_{n,\alpha}^{(1,1)}(\zeta), \tag{9.52}
\end{equation}
follows, whereas (9.7), (9.11), (9.34), and (9.44) show that
\begin{equation}
\Phi_{n,\alpha}^{(2,1)}(\zeta) = -I_q - (\zeta - \alpha)v_{q,n}^*R_{T_{q,n}}^-(\zeta)H_nR_{T_{q,n}}^-H_n(\alpha)v_{q,n} = \Theta_{n,\alpha}^{(2,1)}(\zeta). \tag{9.53}
\end{equation}
From (9.51), (9.7), (9.11), (9.34), Lemma 8.19, Remark 8.8, and (9.43) we conclude
\begin{align*}
\Phi_{n,\alpha}^{(1,2)}(\zeta) &= \left[I_q + (\zeta - \alpha)v_{q,n}^*H_nT_{q,n}^*T_{q,n}^-H_n(\zeta)H_nR_{T_{q,n}}^-R_{T_{q,n}}^-(\alpha)v_{q,n}\right]v_{q,n}^*H_nH_{\alpha\beta\gamma}H_nv_{q,n} \\
&\quad + (\zeta - \alpha)v_{q,n}^*H_nT_{q,n}^*T_{q,n}^-H_nR_{T_{q,n}}^-H_n(\alpha)v_{q,n} \\
&= v_{q,n}^*H_n\left[H_{\alpha\beta\gamma} + (\zeta - \alpha)T_{q,n}^*R_{T_{q,n}}^-H_nR_{T_{q,n}}^-H_n(\alpha)v_{q,n}^*H_nH_{\alpha\beta\gamma} + T_{q,n}\right]H_nv_{q,n} \\
&= v_{q,n}^*H_n\left[I_{(n+1)} + (\zeta - \alpha)T_{q,n}^*R_{T_{q,n}}^-\right]H_{\alpha\beta\gamma}H_nv_{q,n} \\
&= v_{q,n}^*H_n\left[R_{T_{q,n}}^-\right]^{-1}R_{T_{q,n}}^-H_{\alpha\beta\gamma}H_nv_{q,n} = \Theta_{n,\alpha}^{(1,2)}(\zeta)
\end{align*}
and, using additionally (9.45) instead of (9.43), furthermore
\[
\Phi_{n,\alpha}^{(2,2)}(\zeta) = - (\zeta - \alpha)v_q^n R_{T_{q,n}^*}(\zeta) H_{\alpha q}^- R_{T_{q,n}^*}(\alpha)v_q^n H_n H_{\alpha q}^- H_n v_q^n \\
+ I_q - (\zeta - \alpha)v_q^n R_{T_{q,n}^*}(\zeta) H_n R_{T_{q,n}^*}(\alpha)v_q^n \\
= I_q - (\zeta - \alpha)v_q^n R_{T_{q,n}^*}(\zeta) H_n R_{T_{q,n}^*}(\alpha)(v_q^n H_n H_{\alpha q}^- + T_{q,n})H_n v_q^n \\
= I_q - (\zeta - \alpha)v_q^n R_{T_{q,n}^*}(\zeta) H_{\alpha q}^- H_n v_q^n = \Theta_{n,\alpha}^{(2,2)}(\zeta)
\]
Consequently, taking additionally into account (9.52), (9.53), (9.41), (9.51), and (9.50), we obtain the first equation in (9.50). From (9.51), Remark 9.6 (9.35), (9.50), Lemma 8.15, Remark B.8, and (9.46) we get
\[
\Phi_{n,\alpha}^{(1,1)}(\zeta) = I_q + (\zeta - \alpha)v_q^n H_n R_{T_{q,n}^*}^{-1}(\zeta) H_{\alpha q}^- R_{T_{q,n}^*}^{-1}(\alpha)v_q^n \\
- (\zeta - \alpha)v_q^n H_n R_{T_{q,n}^*}^{-1}(\zeta) H_{\alpha q}^- H_n v_q^n v_q^n R_{T_{q,n}^*}^{-1}(\alpha)H_n R_{T_{q,n}^*}^{-1}(\alpha)v_q^n \\
= I_q + (\zeta - \alpha)v_q^n H_n R_{T_{q,n}^*}^{-1}(\zeta) H_{\alpha q}^- H_n v_q^n v_q^n R_{T_{q,n}^*}^{-1}(\alpha)H_n R_{T_{q,n}^*}^{-1}(\alpha)v_q^n \\
\times \left[ I_{(n+1)q} - H_n v_q^n v_q^n R_{T_{q,n}^*}^{-1}(\alpha)H_n R_{T_{q,n}^*}^{-1}(\alpha)v_q^n \right] \\
= I_q + (\zeta - \alpha)v_q^n H_n H_{\alpha q}^- R_{T_{q,n}^*}^{-1}(\zeta) H_{\alpha q}^- R_{T_{q,n}^*}^{-1}(\alpha)H_n R_{T_{q,n}^*}^{-1}(\alpha)v_q^n \\
= I_q + (\zeta - \alpha)v_q^n H_n H_{\alpha q}^- R_{T_{q,n}^*}^{-1}(\zeta) H_{\alpha q}^- R_{T_{q,n}^*}^{-1}(\alpha)H_n v_q^n = \Theta_{n,\alpha}^{(1,1)}(\zeta)
\]
whereas (9.51), Remark 9.6 (9.35), (9.50), and (9.47) show that
\[
\Phi_{n,\alpha}^{(1,2)}(\zeta) = (\zeta - \alpha)v_q^n H_n R_{T_{q,n}^*}^{-1}(\zeta) H_{\alpha q}^- H_n v_q^n = \Theta_{n,\alpha}^{(1,2)}(\zeta)
\]
holds true. Using (9.51), Remark 9.6 (9.35), (9.50), Lemma 8.15, Remark 4.4 and (9.48), we conclude
\[
\Phi_{n,\alpha}^{(2,1)}(\zeta) = - (\zeta - \alpha)v_q^n R_{T_{q,n}^*}(\zeta) H_{\alpha q}^- R_{T_{q,n}^*}(\alpha)v_q^n \\
- \left[ I_q - (\zeta - \alpha)v_q^n R_{T_{q,n}^*}(\zeta) H_{\alpha q}^- H_n v_q^n \right] v_q^n R_{T_{q,n}^*}(\alpha)H_n R_{T_{q,n}^*}(\alpha)v_q^n \\
= -v_q^n R_{T_{q,n}^*}(\zeta) \left\{ (\zeta - \alpha)H_{\alpha q}^- \left[ I_{(n+1)q} - H_n v_q^n v_q^n R_{T_{q,n}^*}(\alpha)H_n \right] \right\} \\
+ \left[ R_{T_{q,n}^*}(\zeta) \right]^{-1} R_{T_{q,n}^*}(\alpha)H_n \\
= -v_q^n R_{T_{q,n}^*}(\zeta) \left\{ (\zeta - \alpha)T_{q,n}^* R_{T_{q,n}^*}(\alpha)H_n + \left[ R_{T_{q,n}^*}(\zeta) \right]^{-1} R_{T_{q,n}^*}(\alpha)H_n \right\} R_{T_{q,n}^*}(\alpha)v_q^n \\
= -v_q^n R_{T_{q,n}^*}(\zeta) (I_{(n+1)q} - \alpha T_{q,n}^*) R_{T_{q,n}^*}(\alpha)H_n R_{T_{q,n}^*}(\alpha)v_q^n \\
= -v_q^n R_{T_{q,n}^*}(\zeta) H_n R_{T_{q,n}^*}(\alpha)H_n = \Theta_{n,\alpha}^{(2,1)}(\zeta)
\]
and, in view of (9.49), furthermore
\[
\Phi_{n,\alpha}^{(2,2)}(\zeta) = I_q - (\zeta - \alpha)v_q^n R_{T_{q,n}^*}(\zeta) H_{\alpha q}^- H_n v_q^n = \Theta_{n,\alpha}^{(2,2)}(\zeta).
\]
Lemma 9.13. Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N} \cup \{\infty\} \), and let \((s_j)_{j=0}^\infty \in K_{q,\kappa,\alpha}^e\). For each \( n \in \mathbb{N}_0 \) with \( 2n+1 \leq \kappa \), the functions \( \Theta_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q \times 2q} \) given by (9.38) and \( \tilde{\Theta}_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q \times 2q} \) given by (9.39) fulfill for each \( \zeta \in \mathbb{C} \setminus \{\alpha\} \) the identity

\[
\tilde{\Theta}_{n,\alpha}(\zeta) = \text{diag}(\zeta - \alpha)I_q, I_q) \cdot \Theta_{n,\alpha}(\zeta) \cdot \text{diag}\left((\zeta - \alpha)^{-1}I_q, I_q\right).
\]

Proof. Let \( n \in \mathbb{N}_0 \) with \( 2n+1 \leq \kappa \). For each \( \zeta \in \mathbb{C} \setminus \{\alpha\} \), we have

\[
\text{diag}\left((\zeta - \alpha)I_q, I_q\right) \cdot I_{2q} \cdot \text{diag}\left((\zeta - \alpha)^{-1}I_q, I_q\right) = I_{2q},
\]

and

\[
\text{diag}\left((\zeta - \alpha)I_q, I_q\right) \cdot (I_2 \otimes v_{q,n})^* \cdot \text{diag}(H_n, I_{(n+1)q}) = (I_2 \otimes v_{q,n})^* \cdot \text{diag}(H_n, I_{(n+1)q}),
\]

or

\[
\text{diag}(H_n^-, H_{\alpha n}^-) \cdot \text{diag}(RT_{q,n}(\alpha), H_n) \cdot (I_2 \otimes v_{q,n}) = \text{diag}(H_n^-, H_{\alpha n}^-) \cdot \text{diag}(RT_{q,n}(\alpha), H_n) \cdot (I_2 \otimes v_{q,n})
\]

and, in view of (9.36) and (9.37), furthermore,

\[
\text{diag}(\zeta - \alpha)I_q, I_q) \cdot (I_2 \otimes v_{q,n})^* \cdot \text{diag}(\zeta - \alpha)^{-1}I_q, I_q
\]

Thus, (9.38), (9.58), (9.59), (9.60), (9.61), and (9.39) imply

\[
\text{diag}(\zeta - \alpha)I_q, I_q) \cdot \Theta_{n,\alpha}(\zeta) \cdot \text{diag}\left((\zeta - \alpha)^{-1}I_q, I_q\right)
\]

Now we obtain a generalization of a result which is, for the special case \( \alpha = 0 \), due to Bolotnikov [3, Lemma 4.2].

Lemma 9.14. Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N} \cup \{\infty\} \), and let \((s_j)_{j=0}^\infty \in K_{q,\kappa,\alpha}^e\). For each \( n \in \mathbb{N}_0 \) with \( 2n+1 \leq \kappa \) and every choice of \( \zeta \in \mathbb{C} \) and \( w \in \mathbb{C} \), then

\[
\tilde{J}_q - \Theta_{n,\alpha}(z)\tilde{J}_q\Theta_{n,\alpha}^*(w) = -i(z - w)(I_2 \otimes v_{q,n})^*[T_{q,n}H_n, -I_{(n+1)q}]^*RT_{q,n}^*(z)H_n^-
\]

\[
\times \left[RT_{q,n}^*(w)\right]^*[T_{q,n}H_n, -I_{(n+1)q}]I_2 \otimes v_{q,n}
\]

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and

\[ \tilde{J}_q - \tilde{\Theta}_{n,\alpha}(z) \tilde{J}_q \tilde{\Theta}_{n,\alpha}(w) = -i(z - \overline{w})(I_2 \otimes v_{q,n})^* \left[ [RT_{q,n}(\alpha)]^{-1} H_n, -I_{(n+1)q} \right]^* R_{T_{q,n}(\alpha)}(z) H_{\overline{\alpha n}} \times \left[ [RT_{q,n}(\alpha)]^{-1} H_n, -I_{(n+1)q} \right] (I_2 \otimes v_{q,n}). \]

Proof. Let \( n \in \mathbb{N}_0 \) be such that \( 2n + 1 \leq \kappa \) and let \( z, w \in \mathbb{C} \). Using Lemma 9.12, Remark 9.8, and Lemma 9.5, we get

\[ \tilde{J}_q - \tilde{\Theta}_{n,\alpha}(z) \tilde{J}_q \tilde{\Theta}_{n,\alpha}(w) = \tilde{J}_q - U_{n,\alpha}(z) B_{n,\alpha} \tilde{J}_q U_{n,\alpha}(w) B_{n,\alpha} \]

\[ = -i(z - \overline{w})(I_2 \otimes v_{q,n})^* [T_{q,n} H_n, -I_{(n+1)q}]^* R_{T_{q,n}(\alpha)}(z) H_n \times \left[ [RT_{q,n}(\alpha)]^{-1} H_n, -I_{(n+1)q} \right] (I_2 \otimes v_{q,n}). \]

Analogously, from Lemma 9.12, Remark 9.8, and Lemma 9.7, we conclude

\[ \tilde{J}_q - \tilde{\Theta}_{n,\alpha}(z) \tilde{J}_q \tilde{\Theta}_{n,\alpha}(w) = \tilde{J}_q - U_{n,\alpha}(z) B_{n,\alpha} \tilde{J}_q B_{n,\alpha} U_{n,\alpha}(w) = \tilde{J}_q - U_{n,\alpha}(z) \tilde{J}_q U_{n,\alpha}(w) \]

\[ = -i(z - \overline{w})(I_2 \otimes v_{q,n})^* \left[ [RT_{q,n}(\alpha)]^{-1} H_n, -I_{(n+1)q} \right]^* \times R_{T_{q,n}(\alpha)}(z) H_{\overline{\alpha n}} \left[ [RT_{q,n}(\alpha)]^{-1} H_n, -I_{(n+1)q} \right] (I_2 \otimes v_{q,n}). \]

\[ \right]. \]

\[ \right]. \]

Lemma 9.15. Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N} \cup \{ \infty \} \), and let \( (s_j)_{j=0}^\infty \in K_{q,n,\alpha}^{e,c} \). For each \( n \in \mathbb{N}_0 \) with \( 2n + 1 \leq \kappa \) and each \( z \in \mathbb{C} \setminus \mathbb{R} \), then

\[ \frac{1}{23z} \left[ \tilde{J}_q - \tilde{\Theta}_{n,\alpha}(z) \tilde{J}_q \tilde{\Theta}_{n,\alpha}(z) \right] \geq 0 \quad \text{and} \quad \frac{1}{23z} \left[ \tilde{J}_q - \tilde{\Theta}_{n,\alpha}(z) \tilde{J}_q \tilde{\Theta}_{n,\alpha}(z) \right] \geq 0. \]

Proof. Let \( n \in \mathbb{N}_0 \) be such that \( 2n + 1 \leq \kappa \). From Lemma 9.10, we get \( \{ H_n^{-}, H_{\overline{\alpha n}}^{-} \} \subseteq C_{\geq(n+1)q \times (n+1)q} \). Applying Lemma 9.14 completes the proof.

Lemma 9.16. Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N} \cup \{ \infty \} \), and let \( (s_j)_{j=0}^\infty \in K_{q,n,\alpha}^{e,c} \). For each \( n \in \mathbb{N}_0 \) with \( 2n + 1 \leq \kappa \), then the following statements hold true:

(a) For each \( x \in \mathbb{R} \),

\[ \tilde{J}_q - \tilde{\Theta}_{n,\alpha}(x) \tilde{J}_q \tilde{\Theta}_{n,\alpha}(x) = 0 \quad \text{and} \quad \tilde{J}_q - \tilde{\Theta}_{n,\alpha}(x) \tilde{J}_q \tilde{\Theta}_{n,\alpha}(x) = 0. \]

(b) For all \( z \in \mathbb{C} \), the matrices \( \Theta_{n,\alpha}(z) \) and \( \tilde{\Theta}_{n,\alpha}(z) \) are both non-singular and fulfill

\[ \Theta_{n,\alpha}^{-1}(z) = \tilde{J}_q \tilde{\Theta}_{n,\alpha}(\overline{\tau}) \tilde{J}_q \quad \text{and} \quad \tilde{\Theta}_{n,\alpha}^{-1}(z) = \tilde{J}_q \tilde{\Theta}_{n,\alpha}(\overline{\tau}) \tilde{J}_q. \] (9.62)

(c) For every choice of \( z \) and \( w \) in \( \mathbb{C} \),

\[ \tilde{J}_q - \tilde{\Theta}_{n,\alpha}^{-1}(z) \tilde{J}_q \tilde{\Theta}_{n,\alpha}(w) = \tilde{J}_q \left[ \tilde{J}_q - \Theta_{n,\alpha}(\overline{\tau}) \tilde{J}_q \Theta_{n,\alpha}(\overline{w}) \right] \tilde{J}_q \]

and

\[ \tilde{J}_q - \tilde{\Theta}_{n,\alpha}^{-1}(z) \tilde{J}_q \tilde{\Theta}_{n,\alpha}(w) = \tilde{J}_q \left[ \tilde{J}_q - \tilde{\Theta}_{n,\alpha}(\overline{\tau}) \tilde{J}_q \tilde{\Theta}_{n,\alpha}(\overline{w}) \right] \tilde{J}_q. \]
Proof. We use Lemma 9.14. We know from Lemma 9.12 that $\Theta_{n,\alpha}$ and $\tilde{\Theta}_{n,\alpha}$ are matrix polynomials. Consequently, $\Theta_{n,\alpha}^\vee : \mathbb{C} \to \mathbb{C}^{2q \times 2q}$ and $\tilde{\Theta}_{n,\alpha}^\vee : \mathbb{C} \to \mathbb{C}^{2q \times 2q}$ defined by $\Theta_{n,\alpha}^\vee(\zeta) := \Theta_{n,\alpha}^*(\overline{\zeta})$ and $\tilde{\Theta}_{n,\alpha}^\vee(\zeta) := \tilde{\Theta}_{n,\alpha}^*(\overline{\zeta})$ are matrix polynomials as well. Thus, $F := \tilde{J}_q - \Theta_{n,\alpha} \tilde{J}_q \Theta_{n,\alpha}^\vee$ and $\tilde{F} := \tilde{J}_q - \Theta_{n,\alpha} \tilde{J}_q \tilde{\Theta}_{n,\alpha}^\vee$ are holomorphic in $\mathbb{C}$. For each $x \in \mathbb{R}$, we see from part (a) that

$$F(x) = \tilde{J}_q - \Theta_{n,\alpha}(x) \tilde{J}_q \Theta_{n,\alpha}^\vee(x) = \tilde{J}_q - \Theta_{n,\alpha}(x) \tilde{J}_q \Theta_{n,\alpha}^\vee(\overline{x}) = 0$$

and, analogously, that $\tilde{F}(x) = 0$. The identity theorem for holomorphic functions implies

$$\Theta_{n,\alpha}(z) \tilde{J}_q \Theta_{n,\alpha}^\vee(z) = \Theta_{n,\alpha}(z) \tilde{J}_q \Theta_{n,\alpha}^\vee(z) = \tilde{J}_q - F(z) = \tilde{J}_q$$

and, analogously, $\tilde{\Theta}_{n,\alpha}(z) \tilde{J}_q \tilde{\Theta}_{n,\alpha}^\vee(z) = \tilde{J}_q$ for all $z \in \mathbb{C}$. Because of $\tilde{J}_q^2 = I_{2q}$, we get $\Theta_{n,\alpha}(z) \tilde{J}_q \Theta_{n,\alpha}^\vee(z) \tilde{J}_q = \tilde{J}_q^2 = I_{2q}$ and $\tilde{\Theta}_{n,\alpha}(z) \tilde{J}_q \tilde{\Theta}_{n,\alpha}^\vee(z) \tilde{J}_q = \tilde{J}_q^2 = I_{2q}$ for each $z \in \mathbb{C}$. For all $z \in \mathbb{C}$, then det $\Theta_{n,\alpha}(z) \neq 0$ and det $\tilde{\Theta}_{n,\alpha}(z) \neq 0$ and both equations in (9.02) follow.

We use part (b) and $\tilde{J}_q^2 = I_{2q}$.

Lemma 9.17. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\kappa,\alpha}$. For each $n \in \mathbb{N}$ with $2n + 1 \leq \kappa$ and every choice of $z$ and $w \in \mathbb{C}$, then

$$\tilde{J}_q - \Theta_{n,\alpha}^{-1}(z) \tilde{J}_q \Theta_{n,\alpha}^{-1}(w) = -i(z - w)(I_{2q} \otimes \mathbb{C}^n)^*[I_{n+1,q}]^* \tilde{T}_{q,n} H_n \tilde{T}_{q,n}^*(\overline{z}) H_n$$

and

$$\tilde{J}_q - \tilde{\Theta}_{n,\alpha}^{-1}(z) \tilde{J}_q \tilde{\Theta}_{n,\alpha}^{-1}(w) = -i(z - w)(I_{2q} \otimes \mathbb{C}^n)^*[I_{n+1,q}]^* \tilde{T}_{q,n} H_n \tilde{T}_{q,n}^*(\overline{z}) H_n$$

Proof. Let $n \in \mathbb{N}$ be such that $2n + 1 \leq \kappa$ and let $z, w \in \mathbb{C}$. Using Lemma 9.10(c), Lemma 9.13 and Remark 9.3, we obtain

$$\tilde{J}_q - \Theta_{n,\alpha}^{-1}(z) \tilde{J}_q \Theta_{n,\alpha}^{-1}(w) = \tilde{J}_q \left[ \tilde{J}_q - \Theta_{n,\alpha}(\overline{z}) \tilde{J}_q \Theta_{n,\alpha}^*(w) \right] \tilde{J}_q$$

$$= \tilde{J}_q \left\{ -i(z - w)(I_{2q} \otimes \mathbb{C}^n)^*[T_{q,n} H_n, -I_{n+1,q}]^* \tilde{T}_{q,n}^*(\overline{z}) H_n \right\} \tilde{J}_q$$

$$= -i(z - w) \tilde{T}_{q,n}(w) \left( (-1) \cdot [I_{n+1,q}, T_{q,n} H_n]^* \tilde{T}_{q,n}^*(\overline{z}) H_n \right)$$

and

$$\tilde{J}_q - \tilde{\Theta}_{n,\alpha}^{-1}(z) \tilde{J}_q \tilde{\Theta}_{n,\alpha}^{-1}(w) = \tilde{J}_q \left[ \tilde{J}_q - \tilde{\Theta}_{n,\alpha}(\overline{z}) \tilde{J}_q \tilde{\Theta}_{n,\alpha}^*(w) \right] \tilde{J}_q$$

$$= \tilde{J}_q \left\{ -i(z - w)(I_{2q} \otimes \mathbb{C}^n)^*[I_{n+1,q}]^* \tilde{T}_{q,n}^*(\overline{z}) H_n \right\} \tilde{J}_q$$

$$= -i(z - w) \tilde{T}_{q,n}(w) \left( (-1) \cdot [I_{n+1,q}, T_{q,n} H_n]^* \tilde{T}_{q,n}^*(\overline{z}) H_n \right)$$

ordinary
and

\[ \hat{J}_q - \hat{\Theta}_{n,\alpha}^{-1}(z)\hat{J}_q \hat{\Theta}_{n,\alpha}^{-1}(w) = \hat{J}_q \left( \hat{J}_q - \hat{\Theta}_{n,\alpha}(z) \hat{J}_q \hat{\Theta}_{n,\alpha}(w) \right) \hat{J}_q \]

\[ = \hat{J}_q \left\{ -i(z - w)[I_2 \otimes v_{q,n}]^* \left[ [R_{T_q,n}(\alpha)]^{-1} H_n, -I_{(n+1)q} \right]^* R_{T_q,n}(\alpha)H_{\Omega,n} \right\} \hat{J}_q \]

\[ \times [R_{T_q,n}(\alpha)]^* \left[ [R_{T_q,n}(\alpha)]^{-1} H_n, -I_{(n+1)q} \right](I_2 \otimes v_{q,n}) \]

\[ = \left( i(z - w) \left( i[I_2 \otimes v_{q,n}]^* \left[ I_{(n+1)q}, [R_{T_q,n}(\alpha)]^{-1} H_n \right]^* R_{T_q,n}(\alpha) \right) \right) \hat{J}_q \]

\[ \times R_{T_q,n}(w) \left( (-i) \cdot \left[ I_{(n+1)q}, [R_{T_q,n}(\alpha)]^{-1} H_n \right](I_2 \otimes v_{q,n}) \right) \]

\[ = -i(z - w)[I_2 \otimes v_{q,n}]^* \left[ I_{(n+1)q}, [R_{T_q,n}(\alpha)]^{-1} H_n \right]^* R_{T_q,n}(\alpha)H_{\Omega,n} \]

\[ \times R_{T_q,n}(w) \left[ I_{(n+1)q}, [R_{T_q,n}(\alpha)]^{-1} H_n \right](I_2 \otimes v_{q,n}). \]

**Lemma 9.18.** Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N}_0 \cup \{ \infty \} \), and let \( (s_j)_{j=0}^\kappa \in K_{\kappa,\kappa,\alpha}^\infty \). Let \( U_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q \times 2q} \) be defined by (9.6). For each \( n \in \mathbb{N}_0 \) with \( 2n \leq \kappa \) and all \( z, w \in \mathbb{C} \), then

\[
\hat{J}_q - U_{n,\alpha}^*(w)\hat{J}_q U_{n,\alpha}(z) = i(z - w)(I_2 \otimes v_{q,n})^*[I_{(n+1)q}, T_{q,n}H_n]^* [R_{T_q,n}(\alpha)]^* H_n \]

\[ \times [R_{T_q,n}(\alpha)]^{-1} [R_{T_q,n}(\alpha)]^{-1} H_n \]

\[ \times R_{T_q,n}(w) \left[ I_{(n+1)q}, [R_{T_q,n}(\alpha)]^{-1} H_n \right](I_2 \otimes v_{q,n}). \]

**Proof.** Let \( n \in \mathbb{N}_0 \) be such that \( 2n \leq \kappa \). Since \( (s_j)_{j=0}^\kappa \in K_{\kappa,\kappa,\alpha}^\infty \subseteq K_{\kappa,\kappa,\alpha}^\infty \) holds true, Remark 7.4 yields \( H_n^* = H_n \) and from Lemma 8.10 we get (8.9). Now let \( z, w \in \mathbb{C} \). In view of (9.6), \( \hat{J}_q^2 = I_{2q} \), and \( H_{\Omega,n}^* = H_{\Omega,n} \), we conclude then

\[ \hat{J}_q - U_{n,\alpha}^*(w)\hat{J}_q U_{n,\alpha}(z) \]

\[ = \hat{J}_q - \left( I_{2q} + (z - w) \left( I_2 \otimes v_{q,n} \right)^*[I_{(n+1)q}, T_{q,n}H_n]^* [R_{T_q,n}(\alpha)]^* H_n \right) \]

\[ \times [R_{T_q,n}(\alpha)]^{-1} [R_{T_q,n}(\alpha)]^{-1} H_n \]

\[ \times R_{T_q,n}(w) \left[ I_{(n+1)q}, [R_{T_q,n}(\alpha)]^{-1} H_n \right](I_2 \otimes v_{q,n}) \]

\[ = S_1(w) + S_2(z) + S_3(z, w) \]

where

\[ S_1(w) := - (z - w) \left( I_2 \otimes v_{q,n} \right)^*[I_{(n+1)q}, T_{q,n}H_n]^* [R_{T_q,n}(\alpha)]^* H_n \]

\[ \times [R_{T_q,n}(\alpha)]^{-1} [R_{T_q,n}(\alpha)]^{-1} H_n \]

\[ \times R_{T_q,n}(w) \left[ I_{(n+1)q}, [R_{T_q,n}(\alpha)]^{-1} H_n \right](I_2 \otimes v_{q,n}) \]

\[ S_2(z) := - (z - w)(I_2 \otimes v_{q,n})^*[T_{q,n}H_n, -I_{(n+1)q}]^* R_{T_q,n}(z)H_n \]

\[ \times R_{T_q,n}(\alpha) \left[ I_{(n+1)q}, [R_{T_q,n}(\alpha)]^{-1} H_n \right](I_2 \otimes v_{q,n}) \]

\[ S_3(z, w) := - (z - w) \left( I_2 \otimes v_{q,n} \right)^*[I_{(n+1)q}, T_{q,n}H_n]^* [R_{T_q,n}(\alpha)]^* H_n \]

\[ \times [R_{T_q,n}(\alpha)]^{-1} [R_{T_q,n}(\alpha)]^{-1} H_n \]

\[ \times R_{T_q,n}(w) \left[ I_{(n+1)q}, [R_{T_q,n}(\alpha)]^{-1} H_n \right](I_2 \otimes v_{q,n}) \]
and

$$S_3(z, w) := -(\overline{w} - \alpha)(z - \alpha)(I_2 \otimes v_{q,n})^*[I_{(n+1)q}, T_{q,n}H_n]^*[R_{q,n}(\alpha)]^*H_n^-$$

$$\times \left[ R_{T_{q,n}}^*(w) \right]^*[R_{T_{q,n}}(\alpha)]^{-1}R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n})$$

$$\times [T_{q,n}H_n, -I_{(n+1)q}]^*RT_{q,n}^*(z)H_n^-RT_{q,n}(\alpha)[I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n}).$$

Keeping in mind the Remarks 9.3 and 4.4, we have

$$S_1(w) = i(\overline{w} - \alpha)(I_2 \otimes v_{q,n})^*[I_{(n+1)q}, T_{q,n}H_n]^*[R_{T_{q,n}}(\alpha)]^*H_n^-$$

$$\times \left[ R_{T_{q,n}}^*(w) \right]^*[R_{T_{q,n}}(\alpha)]^{-1}R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n}),$$

$$\text{(9.65)}$$

$$S_2(z) = -i(z - \alpha)(I_2 \otimes v_{q,n})^*[I_{(n+1)q}, T_{q,n}H_n]^*[R_{T_{q,n}}(\alpha)]^*[R_{T_{q,n}}(\alpha)]^{-*}RT_{q,n}^*(z)H_n^-$$

$$\times R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n}),$$

$$\text{(9.66)}$$

and, by virtue of (9.4) and $H_n^* = H_n$, furthermore

$$S_3(z, w) = -i(\overline{w} - \alpha)(z - \alpha)(I_2 \otimes v_{q,n})^*[I_{(n+1)q}, T_{q,n}H_n]^*[R_{T_{q,n}}(\alpha)]^*H_n^-$$

$$\times \left[ R_{T_{q,n}}^*(w) \right]^*[T_{q,n}H_n v_{q,n} v_{q,n}^* - v_{q,n} v_{q,n}^* H_n T_{q,n}^*]R_{T_{q,n}}^*(z)$$

$$\times H_n^- R_{T_{q,n}}(\alpha)[I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n}).$$

$$\text{(9.67)}$$

Remark B.8 shows that

$$(z - \alpha)T_{q,n}^*RT_{q,n}^*(z) = [R_{T_{q,n}}^*(\alpha)]^{-1}R_{T_{q,n}}^*(z) - I_{(n+1)q}$$

$$\text{(9.68)}$$

is valid and, because of $[R_{T_{q,n}}^*(w)]^* = R_{T_{q,n}}(\overline{w})$, that

$$(\overline{w} - \alpha)\left[ R_{T_{q,n}}(\overline{w}) \right]^*T_{q,n} = \left[ R_{T_{q,n}}(\overline{w}) \right]^*[R_{T_{q,n}}(\alpha)]^{-1} - I_{(n+1)q}$$

$$\text{(9.69)}$$
is also true. In view of (7.1), \([RT_{q,n}(\alpha)]^{-1} = [RT_{q,n}(\alpha)]^{-1}\), and \(9.68\), and \(9.69\), we obtain

\[
(w - \alpha)(z - \alpha)[RT_{q,n}(w)]^* (T_{q,n} H_n v_{q,n} v_{q,n} - v_{q,n} v_{q,n} H_n T_{q,n}^*) R_{q,n}(z)
\]

\[
= (w - \alpha)(z - \alpha)[RT_{q,n}(w)]^* T_{q,n} H_n [RT_{q,n}(\alpha)]^{-1} - [RT_{q,n}(\alpha)]^{-1} H_n T_{q,n}^* R_{q,n}(z)
\]

\[
= (w - \alpha)(z - \alpha)[RT_{q,n}(w)]^* T_{q,n} H_n [RT_{q,n}(\alpha)]^{-1} R_{q,n}(z)
\]

\[
- (w - \alpha)(z - \alpha)[RT_{q,n}(w)]^* [RT_{q,n}(\alpha)]^{-1} H_n T_{q,n}^* R_{q,n}(z)
\]

\[
= (z - \alpha)\left([RT_{q,n}(w)]^* [RT_{q,n}(\alpha)]^{-1} - I_{(n+1)q}\right) H_n [RT_{q,n}(\alpha)]^{-1} R_{q,n}(z)
\]

\[
- (w - \alpha)\left[R_{q,n}(w)]^* [RT_{q,n}(\alpha)]^{-1} H_n \right]\]

\[
(z - \alpha)\left([RT_{q,n}(w)]^* [RT_{q,n}(\alpha)]^{-1} - I_{(n+1)q}\right) H_n [RT_{q,n}(\alpha)]^{-1} R_{q,n}(z)
\]

\[
- (w - \alpha)\left[R_{q,n}(w)]^* [RT_{q,n}(\alpha)]^{-1} H_n \right]
\]

\[
(z - \alpha)\left([RT_{q,n}(w)]^* [RT_{q,n}(\alpha)]^{-1} - I_{(n+1)q}\right) H_n [RT_{q,n}(\alpha)]^{-1} R_{q,n}(z)
\]

which together with \(9.67\) implies

\[
S(z, w)
\]

\[
= -i(I_2 \otimes v_{q,n})^* [I_{(n+1)q}, T_{q,n} H_n]^* [RT_{q,n}(\alpha)]^* H_n
\]

\[
\times \left\{ (w - \alpha)[RT_{q,n}(w)]^* [RT_{q,n}(\alpha)]^{-1} H_n - (z - \alpha) H_n [RT_{q,n}(\alpha)]^{-1} R_{q,n}(z)\right\}
\]

\[
- (w - \alpha)[RT_{q,n}(w)]^* [RT_{q,n}(\alpha)]^{-1} H_n [RT_{q,n}(\alpha)]^{-1} R_{q,n}(z)
\]

\[
\times H_n R_{q,n}(\alpha) [I_{(n+1)q}, T_{q,n} H_n] (I_2 \otimes v_{q,n}).
\]

The combination of \(9.64\), \(9.65\), \(9.66\), and \(9.70\) provides us

\[
\tilde{J}_q - U_{n,\alpha}^*(w) \tilde{J}_q U_{n,\alpha}(z) = i(I_2 \otimes v_{q,n})^* [I_{(n+1)q}, T_{q,n} H_n]^* [RT_{q,n}(\alpha)]^* S(z, w)
\]

\[
\times R_{q,n}(\alpha) [I_{(n+1)q}, T_{q,n} H_n] (I_2 \otimes v_{q,n})
\]

\[
\]
Because of $R_{T,q,n}(\zeta) = [R_{T,q,n}(\zeta)]^*$, which is true for each $\zeta \in \mathbb{C}$, Lemma 8.13 shows that

$$H_n^* \left[ R_{T,q,n}(\zeta) \right]^* [R_{T,q,n}(\alpha)]^{-1} H_n^{-1} = H_n^* [R_{T,q,n}(\zeta)]^* [R_{T,q,n}(\alpha)]^{-1}$$  \quad (9.73)$$

and

$$H_n^{-1} H_n \left[ R_{T,q,n}(\alpha) \right]^{-1} R_{T,q,n}(\zeta) = [R_{T,q,n}(\alpha)]^{-*} R_{T,q,n}(\zeta) H_n^{-1}$$  \quad (9.74)$$

are valid for all $\zeta \in \mathbb{C}$. Thus, from (9.72), (9.73), and (9.74) we get

$$S(z,w) = (\overline{w} - z) H_n^* \left[ R_{T,q,n}(w) \right]^* [R_{T,q,n}(\alpha)]^{-1} H_n^* [R_{T,q,n}(\alpha)]^{-1} R_{T,q,n}(z) H_n^{-1}.$$  \quad (9.75)$$

Taking into account (9.71) and (9.75), we obtain (9.63). \qed

**Lemma 9.19.** Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^\infty$. For each $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$ and all $w, z \in \mathbb{C}$, then

$$\tilde{J}_q - \tilde{U}_{n,\alpha}^*(w) \tilde{J}_q \tilde{U}_{n,\alpha}(z)$$

\[= \frac{i(\overline{w} - z)(I_2 \otimes v_{q,n})^* \left[ I_{(n+1)q}, [R_{T,q,n}(\alpha)]^{-1} H_n \right]^* [R_{T,q,n}(\alpha)]^* H_{\alpha \cap n}^{-1}}{\tilde{J}_q} \times [R_{T,q,n}(w)]^* [R_{T,q,n}(\alpha)]^{-1} H_{\alpha \cap n}^{-1} [R_{T,q,n}(\alpha)]^{-1} \tilde{J}_q \]

\[\times \left[ I_2 + (z - \alpha)(I_2 \otimes v_{q,n})^* \left[ I_{(n+1)q}, [R_{T,q,n}(\alpha)]^{-1} H_n \right]^* [R_{T,q,n}(\alpha)]^* H_{\alpha \cap n}^{-1} \right] \tilde{J}_q \]

\[= \tilde{S}_1(w) + \tilde{S}_2(z) + \tilde{S}_3(z,w), \]

where

$$\tilde{S}_1(w) := -i(\overline{w} - \alpha)(I_2 \otimes v_{q,n})^* \left[ I_{(n+1)q}, [R_{T,q,n}(\alpha)]^{-1} H_n \right]^* [R_{T,q,n}(\alpha)]^* H_{\alpha \cap n}^{-1} \times [R_{T,q,n}(w)]^* [R_{T,q,n}(\alpha)]^{-1} H_n \times (I_2 \otimes v_{q,n}) \tilde{J}_q,$$

$$\tilde{S}_2(z) := -i(z - \alpha) \tilde{J}_q (I_2 \otimes v_{q,n})^* \left[ I_{(n+1)q}, [R_{T,q,n}(\alpha)]^{-1} H_n \right]^* [R_{T,q,n}(\alpha)]^{-1} \tilde{J}_q \times [R_{T,q,n}(w)]^* [R_{T,q,n}(\alpha)]^{-1} H_n \times (I_2 \otimes v_{q,n}),$$

and

$$\tilde{S}_3(z,w) := \ldots$$
From Remark 9.3 we obtain
\[ \tilde{S}_1(w) = (\bar{w} - \alpha) (z - \alpha) \left( I_{(n+1)q} \otimes v_{q,n} \right)^* \left[ [R_{T_{q,n}}(\alpha)]^{-1} H_n \right]^* \left[ R_{T_{q,n}}(\alpha) \right]^* H_{\alpha n} \]
\[ \times \left[ R_{T_{q,n}}(w) \right]^* \left[ [R_{T_{q,n}}(\alpha)]^{-1} H_n - I_{(n+1)q} \right] (I_{2} \otimes v_{q,n}) \]
\[ \times (I_{2} \otimes v_{q,n})^* \left[ [R_{T_{q,n}}(\alpha)]^{-1} H_n - I_{(n+1)q} \right]^* \left[ R_{T_{q,n}}(z) \right] H_{\alpha n} \]
\[ \times R_{T_{q,n}}(\alpha) \left[ I_{(n+1)q} \right]^* [R_{T_{q,n}}(\alpha)]^{-1} H_n \left( I_2 \otimes v_{q,n} \right). \]

(9.78)

and, in view of (9.1) and \( H_n^* = H_n \), furthermore
\[ \tilde{S}_3(z, w) \]
\[ = -i(\bar{w} - \alpha)(z - \alpha) \left( I_{(n+1)q} \otimes v_{q,n} \right)^* \left[ [R_{T_{q,n}}(\alpha)]^{-1} H_n \right]^* \left[ R_{T_{q,n}}(\alpha) \right]^* H_{\alpha n} \]
\[ \times \left[ R_{T_{q,n}}(w) \right]^* \left( [R_{T_{q,n}}(\alpha)]^{-1} H_n v_{q,n} v_{q,n}^* - v_{q,n}^* v_{q,n} H_n [R_{T_{q,n}}(\alpha)]^{-1} \right) \left[ R_{T_{q,n}}(z) \right] \]
\[ \times H_{\alpha n}^* R_{T_{q,n}}(\alpha) \left[ I_{(n+1)q} \right]^* [R_{T_{q,n}}(\alpha)]^{-1} H_n \left( I_2 \otimes v_{q,n} \right). \]

(9.80)

With the aid of Remark 3.8 and \( [R_{T_{q,n}}(\alpha)]^{-1} = [R_{T_{q,n}}(\alpha)]^{-*} \), we see that (9.68) and (9.69) hold true. Using equation (7.2) in Remark 7.2, we conclude
\[ (\bar{w} - \alpha)(z - \alpha) \left( I_{(n+1)q} \otimes v_{q,n} \right)^* \left[ [R_{T_{q,n}}(\alpha)]^{-1} H_n v_{q,n} v_{q,n}^* - v_{q,n}^* v_{q,n} H_n [R_{T_{q,n}}(\alpha)]^{-1} \right) \left[ R_{T_{q,n}}(z) \right] \]
\[ = (\bar{w} - \alpha)(z - \alpha) \left( I_{(n+1)q} \otimes v_{q,n} \right)^* \left( T_{q,n} H_{\alpha n} [R_{T_{q,n}}(\alpha)]^{-1} - [R_{T_{q,n}}(\alpha)]^{-1} H_{\alpha n} T_{q,n}^* \right) \left[ R_{T_{q,n}}(z) \right] \]
\[ = (\bar{w} - \alpha)(z - \alpha) \left( I_{(n+1)q} \otimes v_{q,n} \right)^* \left( T_{q,n} H_{\alpha n} [R_{T_{q,n}}(\alpha)]^{-1} - [R_{T_{q,n}}(\alpha)]^{-1} H_{\alpha n} T_{q,n}^* \right) \left[ R_{T_{q,n}}(z) \right] \]
\[ = (\bar{w} - \alpha)(z - \alpha) \left( I_{(n+1)q} \otimes v_{q,n} \right)^* \left( [R_{T_{q,n}}(\alpha)]^{-1} H_{\alpha n} T_{q,n}^* \right) \left[ R_{T_{q,n}}(z) \right] \]
\[ = (\bar{w} - \alpha)(z - \alpha) \left( I_{(n+1)q} \otimes v_{q,n} \right)^* \left( [R_{T_{q,n}}(\alpha)]^{-1} H_{\alpha n} T_{q,n}^* \right) \left[ R_{T_{q,n}}(z) \right] \]
\[ = (\bar{w} - \alpha)(z - \alpha) \left( I_{(n+1)q} \otimes v_{q,n} \right)^* \left( [R_{T_{q,n}}(\alpha)]^{-1} H_{\alpha n} T_{q,n}^* \right) \left[ R_{T_{q,n}}(z) \right] \]
\[ = (\bar{w} - \alpha)(z - \alpha) \left( I_{(n+1)q} \otimes v_{q,n} \right)^* \left( [R_{T_{q,n}}(\alpha)]^{-1} H_{\alpha n} T_{q,n}^* \right) \left[ R_{T_{q,n}}(z) \right] \]
\[ = (\bar{w} - \alpha)(z - \alpha) \left( I_{(n+1)q} \otimes v_{q,n} \right)^* \left( [R_{T_{q,n}}(\alpha)]^{-1} H_{\alpha n} T_{q,n}^* \right) \left[ R_{T_{q,n}}(z) \right] \]
\[ = (\bar{w} - \alpha)(z - \alpha) \left( I_{(n+1)q} \otimes v_{q,n} \right)^* \left( [R_{T_{q,n}}(\alpha)]^{-1} H_{\alpha n} T_{q,n}^* \right) \left[ R_{T_{q,n}}(z) \right] \]
\[ = (\bar{w} - \alpha)(z - \alpha) \left( I_{(n+1)q} \otimes v_{q,n} \right)^* \left( [R_{T_{q,n}}(\alpha)]^{-1} H_{\alpha n} T_{q,n}^* \right) \left[ R_{T_{q,n}}(z) \right] \]
\[ = (\bar{w} - \alpha)(z - \alpha) \left( I_{(n+1)q} \otimes v_{q,n} \right)^* \left( [R_{T_{q,n}}(\alpha)]^{-1} H_{\alpha n} T_{q,n}^* \right) \left[ R_{T_{q,n}}(z) \right] \]
\[ = (\bar{w} - \alpha)(z - \alpha) \left( I_{(n+1)q} \otimes v_{q,n} \right)^* \left( [R_{T_{q,n}}(\alpha)]^{-1} H_{\alpha n} T_{q,n}^* \right) \left[ R_{T_{q,n}}(z) \right] \]
\[ = (\bar{w} - \alpha)(z - \alpha) \left( I_{(n+1)q} \otimes v_{q,n} \right)^* \left( [R_{T_{q,n}}(\alpha)]^{-1} H_{\alpha n} T_{q,n}^* \right) \left[ R_{T_{q,n}}(z) \right] \]
\[ = (\bar{w} - \alpha)(z - \alpha) \left( I_{(n+1)q} \otimes v_{q,n} \right)^* \left( [R_{T_{q,n}}(\alpha)]^{-1} H_{\alpha n} T_{q,n}^* \right) \left[ R_{T_{q,n}}(z) \right] \]

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which, because of (9.80), implies

\[
\tilde{S}_3(z, w) = -i(I_2 \otimes v_{q,n})^*[I_{(n+1)q}, [RT_{q,n}(\alpha)]^{-1}H_n]^*[RT_{q,n}(\alpha)]^*H_{\alpha n}^-
\]

\[
\times \left\{ (\overline{w} - \alpha)[RT_{q,n}^* (w)]^*[RT_{q,n}(\alpha)]^{-1}H_{\alpha n}^-
\right. \\
\left. - (z - \alpha)H_{\alpha n}^-[RT_{q,n}(\alpha)]^{-*}RT_{q,n}^*(z)
\right. \\
\left. - (\overline{w} - z)[RT_{q,n}^*(w)]^*[RT_{q,n}(\alpha)]^{-1}H_{\alpha n}^-[RT_{q,n}(\alpha)]^{-*}RT_{q,n}^*(z) \right\}
\times H_{\alpha n}^-RT_{q,n}(\alpha)[I_{(n+1)q}, [RT_{q,n}(\alpha)]^{-1}H_n](I_2 \otimes v_{q,n}).
\]

Combining (9.77), (9.78), (9.79), and (9.81), we see that

\[
\tilde{J}_q - \tilde{U}_{n,\alpha}^*(w)\tilde{J}_q\tilde{U}_{n,\alpha}(z)
\]

\[
= i(I_2 \otimes v_{q,n})^*[I_{(n+1)q}, [RT_{q,n}(\alpha)]^{-1}H_n]^*[RT_{q,n}(\alpha)]^*\tilde{S}(z, w)
\]

\[
\times RT_{q,n}(\alpha)[I_{(n+1)q}, [RT_{q,n}(\alpha)]^{-1}H_n](I_2 \otimes v_{q,n})
\]

(9.82)

is valid, where

\[
\tilde{S}(z, w)
\]

\[
:= (\overline{w} - \alpha)H_{\alpha n}^-[RT_{q,n}^*(w)]^*[RT_{q,n}(\alpha)]^{-1} - (z - \alpha)[RT_{q,n}(\alpha)]^{-*}RT_{q,n}^*(z)H_{\alpha n}^-
\]

(9.83)

\[
+ (\overline{w} - \alpha)H_{\alpha n}^-H_{\alpha n}[RT_{q,n}(\alpha)]^{-*}RT_{q,n}^*(z)H_{\alpha n}^-
\]

\[
+ (\overline{w} - z)[RT_{q,n}^*(w)]^*[RT_{q,n}(\alpha)]^{-1}H_{\alpha n}^-[RT_{q,n}(\alpha)]^{-*}RT_{q,n}^*(z)H_{\alpha n}^-
\]

Since \([RT_{q,n}(\eta)]^* = RT_{q,n}(\overline{\eta})\) holds true for all \(\eta \in \mathbb{C}\), from Lemma 8.14 we infer that

\[
H_{\alpha n}^-[RT_{q,n}(\zeta)]^*[RT_{q,n}(\alpha)]^{-1}H_{\alpha n}^- = H_{\alpha n}^-[RT_{q,n}(\zeta)]^*[RT_{q,n}(\alpha)]^{-1}
\]

(9.84)

and

\[
H_{\alpha n}^-H_{\alpha n}[RT_{q,n}(\alpha)]^{-*}RT_{q,n}^*(\zeta)H_{\alpha n}^- = [RT_{q,n}(\alpha)]^{-*}RT_{q,n}^*(\zeta)H_{\alpha n}^-
\]

(9.85)

are fulfilled for each \(\zeta \in \mathbb{C}\). Using (9.84), (9.85), and (9.83), we get

\[
\tilde{S}(z, w) = (\overline{w} - z)H_{\alpha n}^-[RT_{q,n}^*(w)]^*[RT_{q,n}(\alpha)]^{-1}H_{\alpha n}^-[RT_{q,n}(\alpha)]^{-*}RT_{q,n}^*(z)H_{\alpha n}^-
\]

and, because of (9.82), then (9.77) follows.

\[
\Box
\]

Lemma 9.20. Let \(\alpha \in \mathbb{R}\), let \(\kappa \in \mathbb{N} \cup \{\infty\}\), and let \((s_j)^{\kappa}_{j=0} \in K_{q,\kappa,\alpha}^{\geq e}\). For each \(n \in \mathbb{N}_0\) with \(2n + 1 \leq \kappa\) and all \(w, z \in \mathbb{C}\), then

\[
\tilde{J}_q - \Theta_{n,\alpha}^*(w)\tilde{J}_q\Theta_{n,\alpha}(z) = i(\overline{w} - z)B_{n,\alpha}^*[I_{(n+1)q}, T_{q,n}H_n]^*[RT_{q,n}(\alpha)]^*H_n^-
\]

\[
\times [RT_{q,n}^*(w)]^*[RT_{q,n}(\alpha)]^{-1}H_n^-[RT_{q,n}(\alpha)]^{-*}RT_{q,n}^*(z)
\]

\[
\times H_n^-RT_{q,n}(\alpha)[I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n})B_{n,\alpha}
\]

(9.86)
\[ \tilde{J}_q - \Theta^*_{n,\alpha}(w) \tilde{J}_q \tilde{\Theta}_{n,\alpha}(z) \]
\[ = i(\bar{w} - z) \tilde{B}^*_{n,\alpha} (I_2 \otimes v_{q,n})^* \left[ I_{(n+1)q}, \left[ R_{T_{q,n}}(\alpha) \right]^{-1} H_n \right]^* \left[ R_{T_{q,n}}(\alpha) \right]^* H_{\alpha n} \]
\[ \times \left[ R_{T_{q,n}}(w) \right]^* \left[ R_{T_{q,n}}(\alpha) \right]^{-1} H_{\alpha n} \left[ R_{T_{q,n}}(\alpha) \right]^{-1} R_{T_{q,n}}(z) \]
\[ \times H_{\alpha n} R_{T_{q,n}}(\alpha) \left[ I_{(n+1)q}, \left[ R_{T_{q,n}}(\alpha) \right]^{-1} H_n \right] (I_2 \otimes v_{q,n}) \tilde{B}_{n,\alpha}. \] (9.87)

**Proof.** Let \( n \in \mathbb{N}_0 \) be such that \( 2n + 1 \leq \kappa \). From Remark 9.8 and Lemma 9.12 we get
\[ \tilde{J}_q - \Theta^*_{n,\alpha}(w) \tilde{J}_q \Theta_{n,\alpha}(z) = B^*_{n,\alpha} \tilde{J}_q B_{n,\alpha} - [U_{n,\alpha}(w) B_{n,\alpha}]^* \tilde{J}_q U_{n,\alpha}(z) B_{n,\alpha} \]
\[ = B^*_{n,\alpha} \left[ \tilde{J}_q - U^*_{n,\alpha}(w) \tilde{J}_q U_{n,\alpha}(z) \right] B_{n,\alpha} \] (9.88)
and, analogously,
\[ \tilde{J}_q - \tilde{\Theta}^*_{n,\alpha}(w) \tilde{J}_q \tilde{\Theta}_{n,\alpha}(z) = \tilde{B}^*_{n,\alpha} \left[ \tilde{J}_q - \tilde{U}^*_{n,\alpha}(w) \tilde{J}_q \tilde{U}_{n,\alpha}(z) \right] \tilde{B}_{n,\alpha} \] (9.89)
for every choice of \( z \) and \( w \) in \( \mathbb{C} \). Using (9.88) and Lemma 9.18 we obtain (9.89). Because of (9.89) and Lemma 9.19 then (9.87) follows. \( \square \)

**Lemma 9.21.** Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N} \cup \{ \infty \} \), and let \((s_j)_{j=0}^{\infty} \in K_{q,\kappa,\alpha}^{\geq e} \). For each \( n \in \mathbb{N}_0 \) with \( 2n + 1 \leq \kappa \) and each \( z \in \mathbb{C} \setminus \mathbb{R} \), then
\[ \frac{1}{23z} \left[ \tilde{J}_q - \Theta^*_{n,\alpha}(z) \tilde{J}_q \Theta_{n,\alpha}(z) \right] \geq 0 \quad \text{and} \quad \frac{1}{23z} \left[ \tilde{J}_q - \tilde{\Theta}^*_{n,\alpha}(z) \tilde{J}_q \tilde{\Theta}_{n,\alpha}(z) \right] \geq 0. \]

**Proof.** Let \( n \in \mathbb{N}_0 \) be such that \( 2n + 1 \leq \kappa \). Because of \((s_j)_{j=0}^{\infty} \in K_{q,\kappa,\alpha}^{\geq e} \subseteq K_{q,\kappa,\alpha}^{\geq} \) and Remark 7.4 we have \( H_n \in \mathbb{C}_{(n+1)q} \times (n+1)q \) and \( H_{\alpha n} \in \mathbb{C}_{(n+1)q} \times (n+1)q \). Using Lemma 9.20 and Lemma 8.10 for all \( z \in \mathbb{C} \setminus \mathbb{R} \), we get then
\[ \frac{1}{23z} \left[ \tilde{J}_q - \Theta^*_{n,\alpha}(z) \tilde{J}_q \Theta_{n,\alpha}(z) \right] \]
\[ = \frac{1}{23z} \left\{ i(\bar{z} - z) B^*_{n,\alpha} (I_2 \otimes v_{q,n})^* \left[ I_{(n+1)q}, T_{q,n} H_n \right]^* \left[ R_{T_{q,n}}(\alpha) \right]^* H_{\alpha n} \right. \]
\[ \times \left[ R_{T_{q,n}}(z) \right]^* \left[ R_{T_{q,n}}(\alpha) \right]^{-1} H_n \left[ R_{T_{q,n}}(\alpha) \right]^{-1} R_{T_{q,n}}(z) \]
\[ \times \left. H_{\alpha n} R_{T_{q,n}}(\alpha) \left[ I_{(n+1)q}, T_{q,n} H_n \right] (I_2 \otimes v_{q,n}) \right] \]
\[ \times \left[ \left[ R_{T_{q,n}}(\alpha) \right]^{-1} R_{T_{q,n}}(z) H_{\alpha n} R_{T_{q,n}}(\alpha) \left[ I_{(n+1)q}, T_{q,n} H_n \right] (I_2 \otimes v_{q,n}) \right]^* \]
\[ \times H_n \left( \left[ R_{T_{q,n}}(\alpha) \right]^{-1} R_{T_{q,n}}(z) H_{\alpha n} R_{T_{q,n}}(\alpha) \left[ I_{(n+1)q}, T_{q,n} H_n \right] (I_2 \otimes v_{q,n}) \right) \geq 0 \]
and
\[
\frac{1}{23z} \left[ \dot{j}_q - \Theta_n,\alpha(z) \dot{j}_q \Theta_n,\alpha(z) \right]
= \frac{1}{23z} \left\{ i(\sigma - z) \dot{B}_{n,\alpha}(I_{(n+1)q} \otimes v_{q,n}) \left[ R_{T_{q,n}}(\alpha) \right]^{-1} H_n \right\} \left[ R_{T_{q,n}}(\alpha) \right]^* H_{\alpha n}^-
\times \left[ R_{T_{q,n}}(z) \right]^* \left[ R_{T_{q,n}}(\alpha) \right]^{-1} H_{\alpha n} \left[ R_{T_{q,n}}(\alpha) \right]^{-1} R_{T_{q,n}}(z)
\times H_{\alpha n} R_{T_{q,n}}(\alpha) \left[ I_{(n+1)q} \otimes v_{q,n} \right] \dot{B}_{n,\alpha}
\right\} 
= \left( \left[ R_{T_{q,n}}(\alpha) \right]^{-1} R_{T_{q,n}}(z) H_{\alpha n} R_{T_{q,n}}(\alpha) \left[ I_{(n+1)q} \otimes v_{q,n} \right] \dot{B}_{n,\alpha} \right)^* H_{\alpha n}
\times \left( \left[ R_{T_{q,n}}(\alpha) \right]^{-1} R_{T_{q,n}}(z) H_{\alpha n} R_{T_{q,n}}(\alpha) \left[ I_{(n+1)q} \otimes v_{q,n} \right] \dot{B}_{n,\alpha} \right) \geq 0.
\]

Let \( \mathcal{G} \) be a non-empty open subset of \( \mathbb{C} \) and let \( f = [f_{jk}]_{j=1,...,p} \) be a \( p \times q \) matrix-valued function which is meromorphic in \( \mathcal{G} \). For every choice of \( j \) in \( \mathbb{Z}_{1,q} \) and \( k \) in \( \mathbb{Z}_{1,q} \), then let \( \mathbb{H}_{f_{jk}} \) be the set of all \( z \in \mathcal{G} \) in which \( f_{jk} \) is holomorphic and let \( \mathcal{P}_{f_{jk}} \) be the set of all poles of \( f_{jk} \). Furthermore, let \( \mathbb{H}_f := \bigcap_{j=1}^p \bigcap_{k=1}^q \mathbb{H}_{f_{jk}} \) and let \( \mathcal{P}_f := \bigcup_{j=1}^p \bigcup_{k=1}^q \mathcal{P}_{f_{jk}} \).

**Notation 9.22.** Let \( \alpha \in \mathbb{R} \). By \( \mathfrak{W}_{-\dot{j}_q,\alpha} \) we denote the set of all \( 2q \times 2q \) matrix-valued functions \( \Theta \) which are holomorphic in \( \mathbb{C} \setminus [\alpha, \infty) \) and for which there exists a discrete subset \( \mathcal{D} \) of \( \mathbb{C} \setminus [\alpha, \infty) \) such that the following three conditions are fulfilled:

1. \( \Theta \) is holomorphic in \( \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D}) \).
2. \( \Theta(z) \dot{j}_q \Theta^*(z) \leq \dot{j}_q \) for each \( z \in \Pi_+ \setminus \mathcal{D} \).
3. \( \Theta(x) \dot{j}_q \Theta^*(x) = \dot{j}_q \) for each \( x \in (-\infty, \alpha) \setminus \mathcal{D} \).

Observe that continuity arguments show that conditions **(II)** and **(III)** in Notation 9.22 can be replaced equivalently by the following conditions **(II)** and **(III)**, respectively:

1. \( \Theta(z) \dot{j}_q \Theta^*(z) \leq \dot{j}_q \) for each \( z \in \Pi_+ \cap \mathbb{H}_\Theta \).
2. \( (-\infty, \alpha) \subseteq \mathbb{H}_\Theta \) and \( \Theta(x) \dot{j}_q \Theta^*(x) = \dot{j}_q \) for each \( x \in (-\infty, \alpha) \).

**Remark 9.23.** Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N} \cup \{\infty\} \), and let \( (s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa}^{>0} \). From the Lemmas 9.12 and 9.13 we see that, for each \( n \in \mathbb{N}_0 \) with \( 2n + 1 \leq \kappa \), the functions \( \dot{\Theta}_{n,\alpha} := \text{Rstr}_{\mathcal{C}_{\mathbb{C}} \setminus [\alpha, \infty)} \Theta_{n,\alpha} \) and \( \dot{\Theta}_{n,\alpha} := \text{Rstr}_{\mathcal{C}_{\mathbb{C}} \setminus [\alpha, \infty)} \Theta_{n,\alpha} \) are holomorphic in \( \mathbb{C} \setminus [\alpha, \infty) \) and belong both to \( \mathfrak{W}_{-\dot{j}_q,\alpha} \).

**Notation 9.24.** Let \( \alpha \in \mathbb{R} \) and let the matrix-valued function \( P_{\alpha} : \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{2q \times 2q} \) be defined by \( P_{\alpha}(z) := \text{diag}((z - \alpha)I_q, I_q) \). Then let \( \mathfrak{W}_{-\dot{\Theta}_{n,\alpha}} \) be the set of all \( \Theta \in \mathfrak{W}_{-\dot{j}_q,\alpha} \) for which \( \dot{\Theta} := P_{\alpha} \Theta P_{\alpha}^{-1} \) belongs to \( \mathfrak{W}_{-\dot{j}_q,\alpha} \).

**Remark 9.25.** Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N} \cup \{\infty\} \), and let \( (s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa}^{>0} \). From Remark 9.23 and Lemma 9.13 one can easily see that, for each \( n \in \mathbb{N}_0 \) with \( 2n + 1 \leq \kappa \), the matrix-valued function \( \text{Rstr}_{\mathcal{C}_{\mathbb{C}} \setminus [\alpha, \infty)} \Theta_{n,\alpha} \) given by (9.38) is holomorphic in \( \mathbb{C} \setminus [\alpha, \infty) \) and belongs to \( \mathfrak{W}_{-\dot{j}_q,\alpha} \).
Lemma 9.26. Let $\alpha \in \mathbb{R}$ and let $\Theta \in \mathcal{W}_{-J,q,\alpha}$. Then there is a discrete subset $\mathcal{D}$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that $\Theta$ is holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$ and that $\det \Theta(z) \neq 0$ holds true for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$. Furthermore, $\Theta^{-1}$ is meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ with $\det \Theta^{-1} \geq \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$ and the identity $\Theta^{-1}(z) = J_q \Theta^*(\bar{z})\tilde{J}_q$ holds true for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$.

Proof. Let $\mathbb{H}_\Theta^\vee := \{ z \in \mathbb{C} \setminus [\alpha, \infty) : \tau \in \mathbb{H}_\Theta \}$. Then $\Theta^\vee : \mathbb{H}_\Theta^\vee \to \mathbb{C}^{2q \times 2q}$ given by $\Theta^\vee(z) := \Theta^*(\bar{z})$ is meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ with $\mathbb{H}_\Theta^\vee = (\mathbb{H}_\Theta)^\vee$. Thus, $\Omega := \tilde{J}_q - \Theta J_q \Theta^\vee$ is meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ with $\mathbb{H}_\Omega \supseteq \mathbb{H}_\Theta \cap \mathbb{H}_\Theta^\vee = \mathbb{H}_\Theta$. Because of $\Theta \in \mathcal{W}_{-J_q,\alpha}$, there is a discrete subset $\mathcal{D}$ of $\mathbb{C} \setminus [\alpha, \infty)$ with $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D}) \subseteq \mathbb{H}_\Theta$ such that $\Omega$ is holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$ and that

$$\Omega(x) = \tilde{J}_q - \Theta(x) J_q \Theta^\vee(x) = \tilde{J}_q - \Theta(x) J_q \Theta^\vee(\tau) = \tilde{J}_q - \Theta(x) \tilde{J}_q \Theta^\vee(x) = 0$$

holds true for each $x \in (-\infty, \alpha) \setminus \mathcal{D}$. Consequently, the identity theorem for holomorphic functions shows that $\Omega(z) \tilde{J}_q \Theta^\vee(z) = \tilde{J}_q$ is valid for each $z \in \mathbb{H}_\Theta \cap \mathbb{H}_\Theta^\vee$, which implies $\Theta(z) \tilde{J}_q \Theta^\vee(z) \tilde{J}_q = \Theta(z) \tilde{J}_q \Theta^\vee(z) \tilde{J}_q = \tilde{J}_q^2 = I_{2q}$ for each $z \in \mathbb{H}_\Theta \cap \mathbb{H}_\Theta^\vee$ and, in particular, for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$. The rest is plain. \hfill \Box

Lemma 9.27. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{ \infty \}$, and let $(s_j)_{j=0}^\kappa \in K_{q,k,\alpha}^{\geq e}$. Let $G$ be a subset of $\mathbb{C}$ with $G \setminus \mathbb{R} \neq \emptyset$ and let $f : G \to \mathbb{C}^{2q \times q}$ be a $q \times q$ matrix-valued function. For each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, then $\Sigma_{2n}^\kappa G \setminus \mathbb{R} \to \mathbb{C}^{2q \times q}$ and $\Sigma_{2n+1}^\kappa G \setminus \mathbb{R} \to \mathbb{C}^{2q \times q}$ given by (4.6) and (4.7) admit, for each $z \in G \setminus \mathbb{R}$, the representations

$$\Sigma_{2n}^\kappa f(z) = \begin{bmatrix} f(z) \\ I_q \end{bmatrix}^* \Theta_{n,\alpha}^{-1}(z) \begin{bmatrix} \frac{-\tilde{J}_q}{2z} \\ I_q \end{bmatrix} \Theta_{n,\alpha}^{-1}(z) \begin{bmatrix} f(z) \\ I_q \end{bmatrix}$$

and

$$\Sigma_{2n+1}^\kappa f(z) = \begin{bmatrix} (z - \alpha)f(z) \\ I_q \end{bmatrix}^* \Theta_{n,\alpha}^{-1}(z) \begin{bmatrix} \frac{-\tilde{J}_q}{2z} \\ I_q \end{bmatrix} \Theta_{n,\alpha}^{-1}(z) \begin{bmatrix} (z - \alpha)f(z) \\ I_q \end{bmatrix}.$$

Proof. Use Remarks 4.6 and 4.3 and Lemma 9.17. \hfill \Box

Lemma 9.28. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{ \infty \}$, and let $(s_j)_{j=0}^\kappa \in K_{q,k,\alpha}^{\geq e}$. For each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$ and all $z \in \mathbb{C}$, then

$$(I_{(n+1)q} - H_{n,H_n}^1)R_{T_{q,n}}(z)(I_{(n+1)q}, T_{q,n}H_n)(I_2 \otimes v_{q,n})\Theta_{n,\alpha}(z)$$

$$= \left[ I_{(n+1)q} + (z - \alpha)(I_{(n+1)q} - H_{n,H_n}^1)T_{q,n}R_{T_{q,n}}(z)(I_{(n+1)q} - H_{n,H_n}^1) \right]$$

$$\times (I_{(n+1)q} - H_{n,H_n}^1)R_{T_{q,n}}(x)(I_{(n+1)q}, T_{q,n}H_n)(I_2 \otimes v_{q,n})B_{n,\alpha}. \tag{9.90}$$

and

$$(I_{(n+1)q} - H_{\alpha,H_{\alpha,n}}^1)R_{T_{q,n}}(z)(I_{(n+1)q}, [R_{T_{q,n}}(x)]^{-1}H_n)(I_2 \otimes v_{q,n})\Theta_{n,\alpha}(z)$$

$$= \left[ I_{(n+1)q} + (z - \alpha)(I_{(n+1)q} - H_{\alpha,H_{\alpha,n}}^1)T_{q,n}R_{T_{q,n}}(z)(I_{(n+1)q} - H_{\alpha,H_{\alpha,n}}^1) \right]$$

$$\times (I_{(n+1)q} - H_{\alpha,H_{\alpha,n}}^1)R_{T_{q,n}}(x)(I_{(n+1)q}, [R_{T_{q,n}}(x)]^{-1}H_n)(I_2 \otimes v_{q,n})B_{n,\alpha}. \tag{9.91}$$
Proof. Let \( n \in \mathbb{N}_0 \) be such that \( 2n + 1 \leq \kappa \) and let \( z \in \mathbb{C} \). Remarks 7.25 and 7.23 yield \( H_n^* = H_n \) and \( H_{\alpha n}^* = H_{\alpha n} \). Using (9.40), (9.6), and \( R_{T_{q,n}}(z) = [R_{T_{q,n}}(z)]^* \), we have

\[
(I_{(n+1)q} - H_n^*H_n)R_{T_{q,n}}(z)[I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n}) \Theta_{n,\alpha}(z) = (I_{(n+1)q} - H_n^*H_n)R_{T_{q,n}}(z)[I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n}) \Phi(z) B_{n,\alpha}
\]

\[
= (I_{(n+1)q} - H_n^*H_n)R_{T_{q,n}}(z)[I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n}) B_{n,\alpha}
\]

\[
\times \left\{ I_{2q} + (z - \alpha)(I_2 \otimes v_{q,n})^*[T_{q,n}H_n, -I_{(n+1)q}] I_{2q} \right\} \]

\[
\times \left[ R_{T_{q,n}}(z) [R_{T_{q,n}}(z)]^* [R_{T_{q,n}}(z)]^* H_n^* \right].
\]

(9.92)

where

\[
\Phi(z) := R_{T_{q,n}}(z) [R_{T_{q,n}}(z)]^* + (z - \alpha) R_{T_{q,n}}(z) [I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n})
\]

\[
\times (I_2 \otimes v_{q,n})^*[T_{q,n}H_n, -I_{(n+1)q}] I_{2q} \right\} \]

\[
\times \left[ R_{T_{q,n}}(z) [R_{T_{q,n}}(z)]^* [R_{T_{q,n}}(z)]^* H_n^* \right].
\]

(9.93)

Taking into account equation (9.55) in Remark 9.3, \( H_n^* = H_n \), and (7.1), we obtain

\[
R_{T_{q,n}}(z)[I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n})(I_2 \otimes v_{q,n})^*[T_{q,n}H_n, -I_{(n+1)q}] I_{2q} \right\} \]

\[
\times \left[ R_{T_{q,n}}(z) [R_{T_{q,n}}(z)]^* H_n^* \right].
\]

(9.94)

From (9.44), (9.93), (11.7), and the identity \( R_{T_{q,n}}(z)T_{q,n} = T_{q,n}R_{T_{q,n}}(z) \) we get

\[
\Phi(z) = R_{T_{q,n}}(z) [R_{T_{q,n}}(z)]^* + (z - \alpha) \left( H_nT_{q,n}^*[R_{T_{q,n}}(z)]^* - R_{T_{q,n}}(z)T_{q,n}H_n \right) H_n^*
\]

\[
= I_{(n+1)q} + (z - \alpha)T_{q,n}R_{T_{q,n}}(z) + (z - \alpha)H_nT_{q,n}^*[R_{T_{q,n}}(z)]^* H_n^*
\]

\[
- (z - \alpha)T_{q,n}R_{T_{q,n}}(z)H_nH_n^*
\]

\[
= I_{(n+1)q} + (z - \alpha)T_{q,n}R_{T_{q,n}}(z)(I_{(n+1)q} - H_nH_n^*) + (z - \alpha)H_nT_{q,n}^*[R_{T_{q,n}}(z)]^* H_n^*.
\]

(9.95)

In view of Remark 8.11 consequently,

\[
(z - \alpha)(I_{(n+1)q} - H_n^*H_n)H_nT_{q,n}^*[R_{T_{q,n}}(z)]^* H_n^* = 0.
\]

(9.96)
By virtue of (9.95), Remark 8.11 and (9.96), we conclude

\[
(I_{(n+1)}q - H_n^1 H_n) \Phi(z) R_{T_{q,n}}(\alpha) \left[ I_{(n+1)}q, T_{q,n} H_n \right] (I_2 \otimes v_{q,n}) B_{n,\alpha} = (I_{(n+1)}q - H_n^1 H_n) \\
\times \left[ I_{(n+1)}q + (z - \alpha) T_{q,n} R_{T_{q,n}}(z) (I_{(n+1)}q - H_n H_n^-) + (z - \alpha) H_n T_{q,n}^* \left[ R_{T_{q,n}}(\zeta) \right]^* H_n^- \right] \\
\times R_{T_{q,n}}(\alpha) \left[ I_{(n+1)}q, T_{q,n} H_n \right] (I_2 \otimes v_{q,n}) B_{n,\alpha} \\
= \left\{ (I_{(n+1)}q - H_n^1 H_n) \\
+ (z - \alpha) (I_{(n+1)}q - H_n^1 H_n) T_{q,n} R_{T_{q,n}}(z) (I_{(n+1)}q - H_n H_n^-) (I_{(n+1)}q - H_n^1 H_n) \right\} \\
\times R_{T_{q,n}}(\alpha) \left[ I_{(n+1)}q, T_{q,n} H_n \right] (I_2 \otimes v_{q,n}) B_{n,\alpha} \\
= \left[ I_{(n+1)}q + (z - \alpha) (I_{(n+1)}q - H_n^1 H_n) T_{q,n} R_{T_{q,n}}(z) (I_{(n+1)}q - H_n H_n^-) \right] \\
\times (I_{(n+1)}q - H_n^1 H_n) R_{T_{q,n}}(\alpha) \left[ I_{(n+1)}q, T_{q,n} H_n \right] (I_2 \otimes v_{q,n}) B_{n,\alpha}.
\]

(9.97)

The combination of (9.92) and (9.97) yields (9.90). Furthermore, using (9.40), (9.22), and \( R_{T_{q,n}}(z) = [R_{T_{q,n}}(\zeta)]^* \), we infer

\[
(I_{(n+1)}q - H_n^1 H_n) R_{T_{q,n}}(\alpha) \left[ I_{(n+1)}q, [R_{T_{q,n}}(\alpha)]^{-1} H_n \right] (I_2 \otimes v_{q,n}) B_{n,\alpha} = \left\{ I_{(n+1)}q + (z - \alpha) [R_{T_{q,n}}(\alpha)]^{-1} H_n, \right\} \\
\times R_{T_{q,n}}(\alpha) \left[ I_{(n+1)}q, [R_{T_{q,n}}(\alpha)]^{-1} H_n \right] (I_2 \otimes v_{q,n}) B_{n,\alpha}.
\]

(9.98)

where

\[
\Phi(z) := R_{T_{q,n}}(z) [R_{T_{q,n}}(\alpha)]^{-1} + (z - \alpha) R_{T_{q,n}}(z) \left[ I_{(n+1)}q, [R_{T_{q,n}}(\alpha)]^{-1} H_n \right] (I_2 \otimes v_{q,n}) \\
\times (I_2 \otimes v_{q,n})^* \left[ [R_{T_{q,n}}(\alpha)]^{-1} H_n, -I_{(n+1)}q \right] \left[ R_{T_{q,n}}(\zeta) \right]^* H_{\alpha n}^-. \]

(9.99)

Because of identity (9.3) in Remark 9.3, \( H_n^* = H_n \), and equation (7.2) in Remark 7.2 we obtain

\[
R_{T_{q,n}}(z) \left[ I_{(n+1)}q, [R_{T_{q,n}}(\alpha)]^{-1} H_n \right] (I_2 \otimes v_{q,n}) \\
\times (I_2 \otimes v_{q,n})^* \left[ [R_{T_{q,n}}(\alpha)]^{-1} H_n, -I_{(n+1)}q \right] \left[ R_{T_{q,n}}(\zeta) \right]^* \\
= R_{T_{q,n}}(z) \left( v_{q,n} v_{q,n}^* H_n [R_{T_{q,n}}(\alpha)]^{-1} - [R_{T_{q,n}}(\alpha)]^{-1} H_n v_{q,n} v_{q,n}^* \right) \left[ R_{T_{q,n}}(\zeta) \right]^* \\
= R_{T_{q,n}}(z) \left( [R_{T_{q,n}}(\zeta)]^{-1} H_{\alpha n} T_{q,n}^* - T_{q,n} H_{\alpha n} \left[ R_{T_{q,n}}(\zeta) \right]^{-1} \right) \left[ R_{T_{q,n}}(\zeta) \right]^* \\
= H_{\alpha n} T_{q,n}^* \left[ R_{T_{q,n}}(\zeta) \right]^{-1} - R_{T_{q,n}}(z) T_{q,n} H_{\alpha n}.
\]

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which, in view of (9.99), (13.7), and the identity $R_{T_q,n}(z)T_{q,n} = T_{q,n}R_{T_q,n}(z)$, implies

$$
\Phi(z) = R_{T_q,n}(z)[R_{T_q,n}(\alpha)]^{-1} + (z - \alpha)\left[H_{\alpha}T_{q,n}[R_{T_q,n}(\alpha)] - R_{T_q,n}(z)T_{q,n}H_{\alpha}\right]H^{-1}_{\alpha}
$$

$$
= I_{(n+1)q} + (z - \alpha)T_{q,n}R_{T_q,n}(z) + (z - \alpha)H_{\alpha}T_{q,n}[R_{T_q,n}(\alpha)]H^{-1}_{\alpha}
$$

$$
- (z - \alpha)T_{q,n}R_{T_q,n}(z)H_{\alpha}H^{-1}_{\alpha}
$$

$$
= I_{(n+1)q} + (z - \alpha)T_{q,n}R_{T_q,n}(z)(I_{(n+1)q} - H_{\alpha}H^{-1}_{\alpha})
$$

$$
+ (z - \alpha)H_{\alpha}T_{q,n}[R_{T_q,n}(\alpha)]H^{-1}_{\alpha}.
$$

From Remark 8.11 we see that

$$
(z - \alpha)(I_{(n+1)q} - H^{-1}_{\alpha}H_{\alpha})H_{\alpha}T_{q,n}[R_{T_q,n}(\alpha)]H^{-1}_{\alpha} = 0
$$

is true. Using (9.100), Remark 8.11 and (9.101), we get

$$
(I_{(n+1)q} - H^{-1}_{\alpha}H_{\alpha})\Phi(z)R_{T_q,n}(\alpha)[I_{(n+1)q}, [R_{T_q,n}(\alpha)]^{-1}H_n] (I_2 \otimes v_{q,n})B_{n,\alpha}
$$

$$
= (I_{(n+1)q} - H^{-1}_{\alpha}H_{\alpha})\left( I_{(n+1)q} + (z - \alpha)T_{q,n}R_{T_q,n}(z)(I_{(n+1)q} - H_{\alpha}H^{-1}_{\alpha})
$$

$$
+ (z - \alpha)H_{\alpha}T_{q,n}[R_{T_q,n}(\alpha)]H^{-1}_{\alpha}\right)
$$

$$
\times R_{T_q,n}(\alpha)[I_{(n+1)q}, [R_{T_q,n}(\alpha)]^{-1}H_n] (I_2 \otimes v_{q,n})B_{n,\alpha}
$$

$$
= \left( I_{(n+1)q} + (z - \alpha)(I_{(n+1)q} - H^{-1}_{\alpha}H_{\alpha})T_{q,n}R_{T_q,n}(z)
$$

$$
+ (z - \alpha)(I_{(n+1)q} - H^{-1}_{\alpha}H_{\alpha})H_{\alpha}T_{q,n}[R_{T_q,n}(\alpha)]H^{-1}_{\alpha}\right)
$$

$$
\times R_{T_q,n}(\alpha)[I_{(n+1)q}, [R_{T_q,n}(\alpha)]^{-1}H_n] (I_2 \otimes v_{q,n})B_{n,\alpha}
$$

$$
= \left[ I_{(n+1)q} + (z - \alpha)(I_{(n+1)q} - H^{-1}_{\alpha}H_{\alpha})T_{q,n}R_{T_q,n}(z)(I_{(n+1)q} - H_{\alpha}H^{-1}_{\alpha})
$$

$$
\times (I_{(n+1)q} - H^{-1}_{\alpha}H_{\alpha})R_{T_q,n}(\alpha)[I_{(n+1)q}, [R_{T_q,n}(\alpha)]^{-1}H_n] (I_2 \otimes v_{q,n})B_{n,\alpha}.
$$

The combination of (9.98) and (9.102) provides us (9.99).

**Lemma 9.29.** Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\infty} \in K_{q,n,\alpha}^\kappa$. For each $n \in \mathbb{N}_0$ such that $2n + 1 \leq \kappa$, then

$$
(I_{(n+1)q} - H^{-1}_{\alpha}H_{\alpha})R_{T_q,n}(\alpha)[I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n})B_{n,\alpha}
$$

$$
= (I_{(n+1)q} - H^{-1}_{\alpha}H_{\alpha})R_{T_q,n}(\alpha)[I_{(n+1)q}, T_{q,n}(I_{(n+1)q} - H_{\alpha}H^{-1}_{\alpha})H_{\alpha}](I_2 \otimes v_{q,n})
$$

and

$$
(I_{(n+1)q} - H^{-1}_{\alpha}H_{\alpha})R_{T_q,n}(\alpha)[I_{(n+1)q}, [R_{T_q,n}(\alpha)]^{-1}H_n] (I_2 \otimes v_{q,n})B_{n,\alpha}
$$

$$
= (I_{(n+1)q} - H^{-1}_{\alpha}H_{\alpha})\left( I_{(n+1)q} - H_{\alpha}H^{-1}_{\alpha}\right)R_{T_q,n}(\alpha),H_{\alpha}(I_2 \otimes v_{q,n}).
$$

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Proof. Let \( n \in \mathbb{N}_0 \) be such that \( 2n + 1 \leq \kappa \). From Remarks 4.3 and 7.5 we get \( H_n^* = H_n \) and \( H^*_{\alpha \beta n} = H_{\alpha \beta n} \). Because of the Remarks 4.3 and 8.11 we have

\[
(I_{(n+1)q} - H_n^* H_n) R_{T,q,n}(\alpha)(v_{q,n} v_n^* H_n H^*_{\alpha \beta n} + T_{q,n})
= (I_{(n+1)q} - H_n^* H_n) R_{T,q,n}(\alpha) \left[ (R_{T,q,n}(\alpha))^{-1} H_n - T_{q,n} H^*_{\alpha \beta n} \right] H^-_{\alpha \beta n} + T_{q,n}
= (I_{(n+1)q} - H_n^* H_n) H_n H^-_{\alpha \beta n} - (I_{(n+1)q} - H_n^* H_n) R_{T,q,n}(\alpha) T_{q,n} H_{\alpha \beta n} H^-_{\alpha \beta n}
\]

(9.103)

Applying Remark 4.3 and 9.103, we conclude

\[
(I_{(n+1)q} - H_n^* H_n) R_{T,q,n}(\alpha) (I_{(n+1)q} T_{q,n} H_n)(I_2 \otimes v_{q,n}) B_{n,\alpha}
= (I_{(n+1)q} - H_n^* H_n) R_{T,q,n}(\alpha) \left[ (I_{(n+1)q} T_{q,n} H_n)(I_2 \otimes v_{q,n}) H_n \right] (I_2 \otimes v_{q,n})
= \left[ (I_{(n+1)q} - H_n^* H_n) R_{T,q,n}(\alpha), (I_{(n+1)q} - H_n^* H_n) R_{T,q,n}(\alpha) (v_{q,n} v_n^* H_n H^-_{\alpha \beta n} + T_{q,n}) H_n \right]
\times (I_2 \otimes v_{q,n})
\]

\[
= \left[ (I_{(n+1)q} - H_n^* H_n) R_{T,q,n}(\alpha), (I_{(n+1)q} - H_n^* H_n) R_{T,q,n}(\alpha) T_{q,n} (I_{(n+1)q} - H_{\alpha \beta n} H^-_{\alpha \beta n}) H_n \right]
\times (I_2 \otimes v_{q,n})
\]

(9.104)

Taking into account \( H_n^* = H_n \), \( H^*_{\alpha \beta n} = H_{\alpha \beta n} \), and Remark 4.3 we obtain \( H_n v_{q,n} v_n^* = H_n[R_{T,q,n}(\alpha)]^{-*} - H_{\alpha \beta n} T_{q,n}^* \) and, hence,

\[
I_{(n+1)q} - H_n v_{q,n} v_n^* [R_{T,q,n}(\alpha)]^{-*} H_n
= I_{(n+1)q} - \left( H_n[R_{T,q,n}(\alpha)]^{-*} - H_{\alpha \beta n} T_{q,n}^* \right) [R_{T,q,n}(\alpha)]^{-*} H_n
= I_{(n+1)q} - H_n H^-_{\alpha \beta n} + H_{\alpha \beta n} T_{q,n}^* [R_{T,q,n}(\alpha)]^{-*} H_n.
\]

(9.105)

Let \( P := I_{(n+1)q} - H_n^* H_{\alpha \beta n} \). From (9.104) and Remark 8.11 we see that

\[
P \left( I_{(n+1)q} - H_n v_{q,n} v_n^* [R_{T,q,n}(\alpha)]^{-*} H_n \right) R_{T,q,n}(\alpha)
= P \left( I_{(n+1)q} - H_n H^-_{\alpha \beta n} + H_{\alpha \beta n} T_{q,n}^* [R_{T,q,n}(\alpha)]^{-*} H_n \right) R_{T,q,n}(\alpha)
= P(I_{(n+1)q} - H_n H^-_{\alpha \beta n}) R_{T,q,n}(\alpha)
\]

holds true. Using Remark 4.3 and 9.105, we obtain finally

\[
PR_{T,q,n}(\alpha) \left[ I_{(n+1)q}, R_{T,q,n}(\alpha) \right]^{-1} H_n \] 

(9.106)
Lemma 9.30. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\kappa,e}_{q,\kappa,\alpha}$. Let $n \in \mathbb{N}_0$ be such that $2n + 1 \leq \kappa$ and let the matrix-valued functions $P_{n,\alpha}$, $Q_{n,\alpha}$, and $S_{n,\alpha}$ be defined on $\mathbb{C}$ and, for every choice of $z \in \mathbb{C}$, be given by

\[
P_{n,\alpha}(z) := I_{(n+1)q} + (z - \alpha)(I_{(n+1)q} - H_n^\dagger H_n)T_{q,n}RT_{q,n}(z)(I_{(n+1)q} - H_nH_n^-),
\]

(9.106)

\[
Q_{n,\alpha}(z) := I_{(n+1)q} + (z - \alpha)(I_{(n+1)q} - H_n^\dagger H_{\alpha\cap n})T_{q,n}RT_{q,n}(z)(I_{(n+1)q} - H_{\alpha\cap n}H_n^-),
\]

(9.107)

and

\[
S_{n,\alpha}(z) := I_{(n+1)q} - (z - \alpha)(I_{(n+1)q} - H_n^\dagger H_{\alpha\cap n})RT_{q,n}(\alpha)T_{q,n}(I_{(n+1)q} - H_{\alpha\cap n}H_n^-).\]

(9.108)

For each $z \in \mathbb{C}$, then

\[
\text{diag}[P_{n,\alpha}(z), Q_{n,\alpha}(z)]
\]

\[
\times \left[ \begin{array}{c|c}
I_{(n+1)q} & 0_{(n+1)q \times (n+1)q} \\
(z - \alpha)(I_{(n+1)q} - H_{\alpha\cap n}^\dagger H_{\alpha\cap n})(I_{(n+1)q} - H_nH_n^-) & S_{n,\alpha}(z) \\
\hline & \end{array} \right]
\]

\[
\times \left[ \begin{array}{c|c}
0_{(n+1)q \times (n+1)q} & (I_{(n+1)q} - H_n^\dagger H_n)T_{q,n}(\alpha)T_{q,n}(I_{(n+1)q} - H_{\alpha\cap n}H_n^-) \\
(9.109)
\end{array} \right]
\]

\[
\times \text{diag}(I_{(n+1)q} - H_n^\dagger H_n)RT_{q,n}(\alpha)q_n, (I_{(n+1)q} - H_{\alpha\cap n}^\dagger H_{\alpha\cap n})H_nq_n,
\]

\[
= \left[ \begin{array}{c|c}
(I_{(n+1)q} - H_n^\dagger H_n)RT_{q,n}(z)[I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes q_n)\Theta_{n,\alpha}(z) \\
(9.109)
\end{array} \right].
\]

Proof. Let $z \in \mathbb{C}$. Obviously, the matrix on the left-hand side of (9.109) coincides with

\[
R_{n,\alpha}(z) := \text{diag}(P_{n,\alpha}(z), Q_{n,\alpha}(z))
\]

\[
\begin{bmatrix}
\Psi_{n,\alpha}^{(1,1)}(z) & \Psi_{n,\alpha}^{(1,2)}(z) \\
\Psi_{n,\alpha}^{(2,1)}(z) & \Psi_{n,\alpha}^{(2,2)}(z)
\end{bmatrix}
\]

(9.110)

\[
(I_2 \otimes q_n),
\]

where

\[
\Psi_{n,\alpha}^{(1,1)}(z) := (I_{(n+1)q} - H_n^\dagger H_n)RT_{q,n}(\alpha),
\]

(9.110)

\[
\Psi_{n,\alpha}^{(1,2)}(z) := (I_{(n+1)q} - H_n^\dagger H_n)RT_{q,n}(\alpha)(I_{(n+1)q} - H_{\alpha\cap n}H_n^-),
\]

(9.111)

\[
\Psi_{n,\alpha}^{(2,1)}(z) := (z - \alpha)(I_{(n+1)q} - H_{\alpha\cap n}^\dagger H_{\alpha\cap n})(I_{(n+1)q} - H_{\alpha\cap n}H_n^-)RT_{q,n}(\alpha),
\]

(9.112)

and

\[
\Psi_{n,\alpha}^{(2,2)}(z) := (z - \alpha)(I_{(n+1)q} - H_n^\dagger H_n)(I_{(n+1)q} - H_nH_n^-)(I_{(n+1)q} - H_{\alpha\cap n}H_n^-)RT_{q,n}(\alpha),
\]

(9.113)

Because of (9.111) and Remark 8.11, we have

\[
\Psi_{n,\alpha}^{(1,2)}(z) = (I_{(n+1)q} - H_n^\dagger H_n)RT_{q,n}(\alpha)T_{q,n}(I_{(n+1)q} - H_{\alpha\cap n}H_n^-)H_n.
\]

(9.114)
Furthermore, \((9.112)\) and Remark \(8.11\) yield
\[
\Psi^{(2,1)}_{n,\alpha}(z) = (z - \alpha)(I_{(n+1)q} - H_{\alpha\beta n}^+H_{\alpha\beta n})(I_{(n+1)q} - H_n^+H_n)R_{\Gamma_{q,n}}(\alpha). \tag{9.115}
\]
From Remark \(8.11\), Lemma \(8.13\) and \((9.108)\) we conclude
\[
(z - \alpha)(I_{(n+1)q} - H_{\alpha\beta n}^+H_{\alpha\beta n})(I_{(n+1)q} - H_n^+H_n)
\times R_{\Gamma_{q,n}}(\alpha)T_{q,n}(I_{(n+1)q} - H_{\alpha\beta n}H_{\alpha\beta n})
= (z - \alpha)(I_{(n+1)q} - H_{\alpha\beta n}^+H_{\alpha\beta n})(I_{(n+1)q} - H_n^+H_n)
\times R_{\Gamma_{q,n}}(\alpha)T_{q,n}(I_{(n+1)q} - H_{\alpha\beta n}H_{\alpha\beta n})
\tag{9.116}
\]
Combining \((9.113)\) and \((9.116)\), we obtain
\[
\Psi^{(2,2)}_{n,\alpha}(z)
= (I_{(n+1)q} - S_{n,\alpha}(z))(I_{(n+1)q} - H_{\alpha\beta n}^+H_{\alpha\beta n})H_n + S_{n,\alpha}(z)(I_{(n+1)q} - H_{\alpha\beta n}^+H_{\alpha\beta n})H_n \tag{9.117}
\]
By virtue of Lemma \(9.28\), \((9.106)\), Lemma \(9.23\) \((9.110)\), and \((9.114)\), we get
\[
(I_{(n+1)q} - H_{\alpha\beta n}^+H_n)R_T(z)[I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n})\Theta_{n,\alpha}(z)
= \left[I_{(n+1)q} + (z - \alpha)(I_{(n+1)q} - H_{\alpha\beta n}^+H_n)T_{q,n}(z)(I_{(n+1)q} - H_n^+H_n)\right]
\times (I_{(n+1)q} - H_{\alpha\beta n}^+H_n)R_T(z)(I_{(n+1)q} - H_n^+H_n)\tag{9.118}
\]

Similarly, Lemma 9.28, 9.117, Lemma 9.29, 9.114, and 9.117 provide us

\[
(I_{n+1}q - H_{\alpha n}^\dag H_{\alpha n})R_T q_n(z)[I_{n+1}q, [R_{T q_n}(\alpha)]^{-1}H_n] \\
\times (I_2 \otimes v_{q,n})\tilde{\Theta}_{n,\alpha}(z) \text{diag}((z - \alpha)I_q, I_q)
\]

\[
= \left[I_{n+1}q + (z - \alpha)(I_{n+1}q - H_{\alpha n}^\dag H_{\alpha n})T q_nR_T q_n(z)(I_{n+1}q - H_{\alpha n}H_{\alpha n}) \right] \\
\times (I_{n+1}q - H_{\alpha n}^\dag H_{\alpha n})R_T q_n(\alpha)[I_{n+1}q, [R_{T q_n}(\alpha)]^{-1}H_n] \\
\times (I_2 \otimes v_{q,n})\tilde{B}_{n,\alpha} \text{diag}((z - \alpha)I_q, I_q)
\]

\[
= Q_{n,\alpha}(z)(I_{n+1}q - H_{\alpha n}^\dag H_{\alpha n})R_T q_n(\alpha)[I_{n+1}q, [R_{T q_n}(\alpha)]^{-1}H_n] \\
\times (I_2 \otimes v_{q,n})\tilde{B}_{n,\alpha} \text{diag}((z - \alpha)I_q, I_q)
\]

\[
= Q_{n,\alpha}(z)(I_{n+1}q - H_{\alpha n}^\dag H_{\alpha n})\left[(z - \alpha)(I_{n+1}q - H_nH_n^\dag)R_T q_n(\alpha), H_n\right](I_2 \otimes v_{q,n})
\]

\[
= Q_{n,\alpha}(z)\left[\Psi_{n,\alpha}^{(2,1)}(z), \Psi_{n,\alpha}^{(2,2)}(z)\right](I_2 \otimes v_{q,n}).
\]

Since \( R_{n,\alpha}(z) \) coincides with the matrix on the left-hand side of (9.109) from (9.118) and (9.119), equation (9.109) follows. \( \square \)

If \( \mathcal{G} \) is a non-empty subset of \( \mathbb{C} \) and if \( f: \mathcal{G} \rightarrow \mathbb{C} \) is a function, then let \( \mathcal{N}_f := \{ z \in \mathcal{G}: f(z) = 0 \} \).

**Lemma 9.31.** Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N} \cup \{ \infty \} \), let \( (s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\alpha}^{>\kappa} \), and let \( n \in \mathbb{N}_0 \) be such that \( 2n + 1 \leq \kappa \). Then:

(a) The set \( \mathcal{N}_{\det P_{n,\alpha}} \cup \mathcal{N}_{\det Q_{n,\alpha}} \cup \mathcal{N}_{\det S_{n,\alpha}} \) is finite.

(b) Let \( x \in \mathbb{C}^{q \times q} \) and let \( y \in \mathbb{C}^{q \times q} \). Then the following statements are equivalent:

(i) For each \( z \in \mathbb{C} \setminus (\mathcal{N}_{\det P_{n,\alpha}} \cup \mathcal{N}_{\det Q_{n,\alpha}} \cup \mathcal{N}_{\det S_{n,\alpha}}) \), the equations

\[
(I_{n+1}q - H_n^\dag H_n)R_T q_n(z)[I_{n+1}q, [R_{T q_n}(\alpha)]^{-1}H_n] \\
\times (I_2 \otimes v_{q,n})\tilde{\Theta}_{n,\alpha}(z) \begin{bmatrix} x \\ y \end{bmatrix} = 0
\]

and

\[
(I_{n+1}q - H_{\alpha n}^\dag H_{\alpha n})R_T q_n(z)[I_{n+1}q, [R_{T q_n}(\alpha)]^{-1}H_n] \\
\times (I_2 \otimes v_{q,n})\tilde{\Theta}_{n,\alpha}(z) \begin{bmatrix} x \\ y \end{bmatrix} = 0
\]

hold true.

(ii) There exists a number \( z \in \mathbb{C} \setminus (\mathcal{N}_{\det P_{n,\alpha}} \cup \mathcal{N}_{\det Q_{n,\alpha}} \cup \mathcal{N}_{\det S_{n,\alpha}}) \) such that (9.120) and (9.121) are valid.

(iii) The equations

\[
(I_{n+1}q - H_n^\dag H_n)R_T q_n(\alpha)v_{q,n}x = 0
\]

\[
(I_{n+1}q - H_{\alpha n}^\dag H_{\alpha n})R_T q_n(\alpha)v_{q,n}x = 0
\]

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are fulfilled.

Proof. By virtue of Remark 4.4, 9.106, 9.107, and 9.108, we see that \( P_{n,\alpha}, Q_{n,\alpha}, \) and \( S_{n,\alpha} \) are matrix polynomials with \( P_{n,\alpha}(\alpha) = I_{(n+1)q}, Q_{n,\alpha}(\alpha) = I_{(n+1)q}, \) and \( S_{n,\alpha}(\alpha) = I_{(n+1)q}. \) In particular, \( \det P_{n,\alpha}, \det Q_{n,\alpha}, \) and \( \det S_{n,\alpha} \) are polynomials which do not vanish identically. In view of the fundamental theorem of algebra, the proof of part (ii) is complete. For each \( z \in \mathbb{C}, \) Lemma 9.30 provides us

\[
(I_{(n+1)q} - H^*_{\alpha\beta}H_{\alpha\beta})H_{n,\alpha}v_ny = 0 \quad (9.123)
\]

This implication is trivial.

According to (ii), there exists a number

\[
z \in \mathbb{C} \setminus (\mathcal{N}_{\det P_{n,\alpha}} \cup \mathcal{N}_{\det Q_{n,\alpha}} \cup \mathcal{N}_{\det S_{n,\alpha}}) \quad (9.125)
\]

such that (9.120) and (9.121) hold true. Using (9.120), (9.121), and (9.124), we get

\[
\begin{bmatrix}
0_{(n+1)q \times q} \\
0_{(n+1)q \times q}
\end{bmatrix} = \text{diag}(P_{n,\alpha}(z), Q_{n,\alpha}(z))
\]

Using (9.120), (9.121), and (9.124), we get

\[
\begin{bmatrix}
0_{(n+1)q \times q} \\
0_{(n+1)q \times q}
\end{bmatrix} = \text{diag}(P_{n,\alpha}(z), Q_{n,\alpha}(z))
\]

\[
\begin{bmatrix}
I_{(n+1)q} \\
I_{(n+1)q} \\
I_{(n+1)q} \\
I_{(n+1)q}
\end{bmatrix} \begin{bmatrix}
0_{(n+1)q \times (n+1)q} \\
0_{(n+1)q \times (n+1)q} \\
0_{(n+1)q \times (n+1)q} \\
0_{(n+1)q \times (n+1)q}
\end{bmatrix}
\]

\[
\begin{bmatrix}
0_{(n+1)q \times q} \\
0_{(n+1)q \times q}
\end{bmatrix} = \text{diag}(P_{n,\alpha}(z), Q_{n,\alpha}(z))
\]

\[
\begin{bmatrix}
I_{(n+1)q} \\
I_{(n+1)q} \\
I_{(n+1)q} \\
I_{(n+1)q}
\end{bmatrix} \begin{bmatrix}
0_{(n+1)q \times (n+1)q} \\
0_{(n+1)q \times (n+1)q} \\
0_{(n+1)q \times (n+1)q} \\
0_{(n+1)q \times (n+1)q}
\end{bmatrix}
\]

\[
\begin{bmatrix}
0_{(n+1)q \times q} \\
0_{(n+1)q \times q}
\end{bmatrix} = \text{diag}(P_{n,\alpha}(z), Q_{n,\alpha}(z))
\]

\[
\begin{bmatrix}
I_{(n+1)q} \\
I_{(n+1)q} \\
I_{(n+1)q} \\
I_{(n+1)q}
\end{bmatrix} \begin{bmatrix}
0_{(n+1)q \times (n+1)q} \\
0_{(n+1)q \times (n+1)q} \\
0_{(n+1)q \times (n+1)q} \\
0_{(n+1)q \times (n+1)q}
\end{bmatrix}
\]

\[
\begin{bmatrix}
0_{(n+1)q \times q} \\
0_{(n+1)q \times q}
\end{bmatrix} = \text{diag}(P_{n,\alpha}(z), Q_{n,\alpha}(z))
\]

\[
\begin{bmatrix}
I_{(n+1)q} \\
I_{(n+1)q} \\
I_{(n+1)q} \\
I_{(n+1)q}
\end{bmatrix} \begin{bmatrix}
0_{(n+1)q \times (n+1)q} \\
0_{(n+1)q \times (n+1)q} \\
0_{(n+1)q \times (n+1)q} \\
0_{(n+1)q \times (n+1)q}
\end{bmatrix}
\]

\[
\begin{bmatrix}
0_{(n+1)q \times q} \\
0_{(n+1)q \times q}
\end{bmatrix} = \text{diag}(P_{n,\alpha}(z), Q_{n,\alpha}(z))
\]

\[
\begin{bmatrix}
I_{(n+1)q} \\
I_{(n+1)q} \\
I_{(n+1)q} \\
I_{(n+1)q}
\end{bmatrix} \begin{bmatrix}
0_{(n+1)q \times (n+1)q} \\
0_{(n+1)q \times (n+1)q} \\
0_{(n+1)q \times (n+1)q} \\
0_{(n+1)q \times (n+1)q}
\end{bmatrix}
\]

\[
\begin{bmatrix}
0_{(n+1)q \times q} \\
0_{(n+1)q \times q}
\end{bmatrix} = \text{diag}(P_{n,\alpha}(z), Q_{n,\alpha}(z))
\]

\[
\begin{bmatrix}
I_{(n+1)q} \\
I_{(n+1)q} \\
I_{(n+1)q} \\
I_{(n+1)q}
\end{bmatrix} \begin{bmatrix}
0_{(n+1)q \times (n+1)q} \\
0_{(n+1)q \times (n+1)q} \\
0_{(n+1)q \times (n+1)q} \\
0_{(n+1)q \times (n+1)q}
\end{bmatrix}
\]

\[
\begin{bmatrix}
0_{(n+1)q \times q} \\
0_{(n+1)q \times q}
\end{bmatrix} = \text{diag}(P_{n,\alpha}(z), Q_{n,\alpha}(z))
\]

\[
\begin{bmatrix}
I_{(n+1)q} \\
I_{(n+1)q} \\
I_{(n+1)q} \\
I_{(n+1)q}
\end{bmatrix} \begin{bmatrix}
0_{(n+1)q \times (n+1)q} \\
0_{(n+1)q \times (n+1)q} \\
0_{(n+1)q \times (n+1)q} \\
0_{(n+1)q \times (n+1)q}
\end{bmatrix}
\]

\[
\begin{bmatrix}
0_{(n+1)q \times q} \\
0_{(n+1)q \times q}
\end{bmatrix} = \text{diag}(P_{n,\alpha}(z), Q_{n,\alpha}(z))
\]
Because of (9.125), the first three factors of the matrix product on the right-hand side of equation (9.126) are non-singular matrices. Thus, (9.126) implies (9.122) and (9.123).

Taking into account (9.122), (9.123), and (9.124), we conclude that

\[
\begin{bmatrix}
0_{(n+1)q \times q} \\
0_{(n+1)q \times q}
\end{bmatrix}
\begin{bmatrix}
I_{(n+1)q} \\
(z - \alpha)(I_{(n+1)q} - H_n^\dagger H_n) + H_n H_n^{-1} \\
0_{(n+1)q \times (n+1)q}
\end{bmatrix}
\begin{bmatrix}
I_{(n+1)q} \\
(\alpha - \alpha)(I_{(n+1)q} - H_n^\dagger H_n) + H_n H_n^{-1} \\
\end{bmatrix}
\begin{bmatrix}
S_{n,\alpha}(z) \\
I_{(n+1)q}
\end{bmatrix}
\begin{bmatrix}
(z - \alpha)(I_{(n+1)q} - H_n^\dagger H_n)R_{q,n}(\alpha)T_{q,n}(I_{(n+1)q} - H_n H_n^{-1}) \\
0_{(n+1)q \times (n+1)q}
\end{bmatrix}
\begin{bmatrix}
I_{(n+1)q} \\
(I_{(n+1)q} - H_n^\dagger H_n)R_{q,n}(\alpha)H_n v_{q,n}
\end{bmatrix}
\]

\[
\begin{bmatrix}
(I_{(n+1)q} - H_n^\dagger H_n)T_{q,n}(z) \\
\end{bmatrix}
\begin{bmatrix}
(I_{(n+1)q} - H_n^\dagger H_n)R_{q,n}(\alpha)H_n v_{q,n} \\
0_{(n+1)q \times (n+1)q}
\end{bmatrix}
\begin{bmatrix}
I_{(n+1)q} \\
(I_{(n+1)q} - H_n^\dagger H_n)R_{q,n}(\alpha)H_n v_{q,n}
\end{bmatrix}
\begin{bmatrix}
\theta_{n,\alpha}(z) \\
\end{bmatrix}
\begin{bmatrix}
(I_{(n+1)q} - H_n^\dagger H_n)R_{q,n}(\alpha)H_n v_{q,n}
\end{bmatrix}
\]

and, consequently, (9.120) and (9.121) hold true for each \( z \in \mathbb{C} \).

If \( \mathcal{U} \) is a subspace of \( \mathbb{C}^q \), then by \( P_\mathcal{U} \) we denote the complex \( q \times q \) matrix which represents the orthogonal projection onto \( \mathcal{U} \), with respect to the standard basis of \( \mathbb{C}^q \) i.e., \( P_\mathcal{U} \) is the unique complex \( q \times q \) matrix which fulfills the three conditions \( P_\mathcal{U}^2 = P_\mathcal{U}, P_\mathcal{U}^\dagger = P_\mathcal{U}, \) and \( \mathcal{R}(P_\mathcal{U}) = \mathcal{U} \). In this case, we have \( \mathcal{N}(P_\mathcal{U}) = \mathcal{U}^\perp \).

**Lemma 9.32.** Let \( \alpha \in \mathbb{R}, \) let \( \kappa \in \mathbb{N} \cup \{\infty\} \), let \( (s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,p} \), and let \( n \in \mathbb{N}_0 \) be such that \( 2n + 1 \leq \kappa \). Then:

(a) The sets

\[
\mathcal{U}_{n,\alpha} := \mathcal{N}\left( (I_{(n+1)q} - H_n^\dagger H_n)R_{q,n}(\alpha)v_{q,n} \right) \quad (9.127)
\]

and

\[
\mathcal{V}_{n,\alpha} := \mathcal{N}\left( (I_{(n+1)q} - H_n^\dagger H_n)H_n v_{q,n} \right) \quad (9.128)
\]

are orthogonal subspaces of \( \mathbb{C}^q \) with

\[
\dim \mathcal{U}_{n,\alpha} = \text{rank} \left[ (I_{(n+1)q} - H_n^\dagger H_n)R_{q,n}(\alpha)v_{q,n} \right] \quad (9.129)
\]

and

\[
\dim \mathcal{V}_{n,\alpha} = \text{rank} \left[ (I_{(n+1)q} - H_n^\dagger H_n)H_n v_{q,n} \right] \quad (9.130)
\]

(b) Let \( A \in \mathbb{C}^{q \times p} \). Then \( (I_{(n+1)q} - H_n^\dagger H_n)R_{q,n}(\alpha)v_{q,n}A = 0 \) if and only if \( P_{\mathcal{U}_{n,\alpha}}A = 0 \). Moreover, \( (I_{(n+1)q} - H_n^\dagger H_n)H_n v_{q,n}A = 0 \) if and only if \( P_{\mathcal{V}_{n,\alpha}}A = 0 \).
This leads us to the set of parameters which occur in our description of the solution set of the
\( H_n \). Obviously, \( (H_n^* H_n)^* = H_n^* H_n \) and \( (H_n^* H_n)^* = H_n^* H_n \). Thus,

\[
\mathcal{U}_{n, \alpha} = \mathcal{R}\left(\left[ (I_{(n+1)q} - H_n^* H_n) R_{T_{q,n}}(\alpha)v_{q,n} \right]^* \right)
\]

and

\[
\mathcal{V}_{n, \alpha} = \mathcal{R}\left(\left[ (I_{(n+1)q} - H_n^* H_n) H_n v_{q,n} \right]^* \right) = \mathcal{R}\left( v_{q,n} H_n (I_{(n+1)q} - H_n^* H_n) \right).
\]

In particular, \( (9.129) \) and \( (9.130) \) hold true. Let \( f \in \mathcal{U}_{n, \alpha} \) and \( g \in \mathcal{V}_{n, \alpha} \) be arbitrary chosen.

According to \( (9.131) \) and \( (9.132) \), there are \( x, y \in C^{(n+1)q} \) such that \( f = [(I_{(n+1)q} - H_n^* H_n) R_{T_{q,n}}(\alpha)v_{q,n}]^* x \) and \( g = v_{q,n} H_n (I_{(n+1)q} - H_n^* H_n) y \). By virtue of the Remarks 4.3

and \( 8.11 \) we have

\[
f^* g = x^* (I_{(n+1)q} - H_n^* H_n) R_{T_{q,n}}(\alpha) v_{q,n} v_{q,n}^* H_n (I_{(n+1)q} - H_n^* H_n) y
\]

\[
= x^* (I_{(n+1)q} - H_n^* H_n) R_{T_{q,n}}(\alpha) \left( [R_{T_{q,n}}(\alpha)]^{-1} H_n - T_{q,n} H_n H_n \right) (I_{(n+1)q} - H_n^* H_n) y
\]

\[
= x^* (I_{(n+1)q} - H_n^* H_n) H_n (I_{(n+1)q} - H_n^* H_n) y = 0.
\]

Consequently, the subspaces \( \mathcal{U}_{n, \alpha} \) and \( \mathcal{V}_{n, \alpha} \) are orthogonal.

\( \square \) Use the equations \( \mathcal{N}((I_{(n+1)q} - H_n^* H_n) R_{T_{q,n}}(\alpha)v_{q,n}) = \mathcal{U}_{n, \alpha}^\perp = \mathcal{N}(P_{\mathcal{U}_{n, \alpha}}) \) and \( \mathcal{N}((I_{(n+1)q} - H_n^* H_n) H_n v_{q,n}) = \mathcal{V}_{n, \alpha}^\perp = \mathcal{N}(P_{\mathcal{V}_{n, \alpha}}) \).

10. Nevanlinna and Stieltjes pairs of meromorphic matrix-valued functions

In this section, we consider special classes of pairs of meromorphic matrix-valued functions.

This leads us to the set of parameters which occur in our description of the solution set of the
power moment problem in question.

Notation 10.1. A pair \( \begin{bmatrix} \phi & \psi \end{bmatrix} \) of \( q \times q \) matrix-valued functions \( \phi \) and \( \psi \) meromorphic in \( \Pi_+ \) is
called a \( q \times q \) Nevanlinna pair in \( \Pi_+ \) if there exists a discrete subset \( \mathcal{D} \) of \( \Pi_+ \) such that the following three conditions are fulfilled:

(i) \( \phi \) and \( \psi \) are holomorphic in \( \Pi_+ \setminus \mathcal{D} \).

(ii) \( \text{rank} [\phi(w)] = q \) for all \( w \in \Pi_+ \setminus \mathcal{D} \).

(iii) \( [\phi(w)]^* (-J_q) [\phi(w)] \geq 0_{q\times q} \) for each \( w \in \Pi_+ \setminus \mathcal{D} \).

The set of all \( q \times q \) Nevanlinna pairs in \( \Pi_+ \) will be denoted by \( \tilde{\mathcal{P}}_{-J_q}, q \geq 1 \).

Note the well-known fact that, for each \( S \in \mathcal{R}_q(\Pi_+) \), the pair \( \begin{bmatrix} S \\ \psi \end{bmatrix} \) belongs to \( \tilde{\mathcal{P}}_{-J_q}, q \geq 1 \).

Remark 10.2. If \( \begin{bmatrix} \phi & \psi \end{bmatrix} \in \tilde{\mathcal{P}}_{-J_q}, q \geq 1 \), then it is readily checked that, for each \( q \times q \) matrix-valued function \( g \) which is meromorphic in \( \Pi_+ \) and for which the function \( \det g \) does not
vanish identically, the pair \( \begin{bmatrix} \phi g & \psi g \end{bmatrix} \) belongs to \( \tilde{\mathcal{P}}_{-J_q}, q \geq 1 \) as well. Two \( q \times q \) Nevanlinna pairs \( \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix} \) and \( \begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix} \) in \( \Pi_+ \) are said to be equivalent if there are a \( q \times q \) matrix-valued function \( g \)
which is meromorphic in $\Pi_+$ and a discrete subset $D$ of $\Pi_+$ such that $\phi_1, \psi_1, \phi_2, \psi_2,$ and $g$ are holomorphic in $\Pi_+ \setminus D$ and that $\det(g(w)) \neq 0$ and $[\phi_2(w)] = [\psi_2(g(w))]$ hold true for all $w \in \Pi_+ \setminus D$. One can easily see that this implies an equivalence relation on $\tilde{P}_{q,q}^{(q,q)}(\Pi_+)$. For each $[\phi \psi] \in \tilde{P}_{q,q}^{(q,q)}(\Pi_+)$, let $\langle [\phi \psi] \rangle$ be the equivalence class generated by $[\phi \psi]$.

Let us recall a well-known interrelation between the classes $\tilde{P}_{q,q}^{(q,q)}(\Pi_+)$ and $S_{q\times q}(\Pi_+)$:

**Lemma 10.3.**  
(a) For each $[\phi \psi] \in \tilde{P}_{q,q}^{(q,q)}(\Pi_+)$, the function $\det(\psi - i\phi)$ does not vanish identically and $S := \{(\psi + i\phi)(\psi - i\phi)^{-1}\}$ belongs to the Schur class $S_{q\times q}(\Pi_+)$. 

(b) For each $S \in S_{q\times q}(\Pi_+)$, the pair $[\phi \psi]$ given by $\phi := i(I_q - S)$ and $\psi := I_q + S$ belongs to the class $\tilde{P}_{q,q}^{(q,q)}(\Pi_+)$, the functions $\phi$ and $\psi$ are holomorphic in $\Pi_+$, and $\det[\psi(w) - i\phi(w)] \neq 0$ and $S(w) = [\psi(w) + i\phi(w)][\psi(w) - i\phi(w)]^{-1}$ hold true for each $w \in \Pi_+$.

(c) Two $q \times q$ Nevanlinna pairs $[\phi_1 \psi_1]$ and $[\phi_2 \psi_2]$ in $\Pi_+$ are equivalent if and only if $(\psi_1 + i\phi_1)(\psi_1 - i\phi_1)^{-1} = (\psi_2 + i\phi_2)(\psi_2 - i\phi_2)^{-1}$.

A detailed proof of Lemma 10.3 can be found, e.g., in [61, Lemma 1.7].

**Proposition 10.4.** Let $[\phi \psi] \in \tilde{P}_{q,q}^{(q,q)}(\Pi_+)$. Then there exists a discrete subset $D$ of $\Pi_+$ such that $\phi$ and $\psi$ are holomorphic in $\Pi_+ \setminus D$ and that, for every choice of $w$ and $z$ in $\Pi_+ \setminus D$, the following four equations hold true:

$$
\mathcal{R}(\phi(w)) = \mathcal{R}(\phi(z)), \quad \mathcal{R}(\psi(w)) = \mathcal{R}(\psi(z)) \tag{10.1}
$$

$$
\psi(w)\mathcal{N}(\phi(w)) = \psi(z)\mathcal{N}(\phi(z)), \quad \text{and} \quad \phi(w)\mathcal{N}(\psi(w)) = \phi(z)\mathcal{N}(\psi(z)) \tag{10.2}
$$

**Proof.** Because of Lemma 10.3 the function $\det(\psi - i\phi)$ does not vanish identically and $S := \{(\psi + i\phi)(\psi - i\phi)^{-1}\}$ belongs to $S_{q\times q}(\Pi_+)$. Consequently, there is a discrete subset $D$ of $\Pi_+$ such that $\phi$ and $\psi$ are holomorphic in $\Pi_+ \setminus D$ and that $\det[\psi(w) - i\phi(w)] \neq 0$ holds true for each $w \in \Pi_+ \setminus D$. For each $w \in \Pi_+ \setminus D$, we obtain then

$$
\phi(w) = \frac{1}{2}[\psi(w) - i\phi(w) - [\psi(w) + i\phi(w)]] = \frac{1}{2}[I_q - S(w)][\psi(w) - i\phi(w)] \tag{10.3}
$$

and, analogously,

$$
\psi(w) = \frac{1}{2}[I_q + S(w)][\psi(w) - i\phi(w)]. \tag{10.4}
$$

In particular, for each $w \in \Pi_+ \setminus D$, we have

$$
\mathcal{R}(\phi(w)) = \mathcal{R}(I_q - S(w)) \quad \text{and} \quad \mathcal{R}(\psi(w)) = \mathcal{R}(I_q + S(w)). \tag{10.5}
$$

Thus, in view of (10.5), we see from Lemma 5.9 that (10.1) holds true for every choice of $w$ and $z$ in $\Pi_+ \setminus D$. For each $w \in \Pi_+ \setminus D$, from $\det[\psi(w) - i\phi(w)] \neq 0$, (10.3), (10.4), and Remark A.7 we get $\mathcal{N}(I_q - S(w)) = [\psi(w) - i\phi(w)]\mathcal{N}(\phi(w))$ and $\mathcal{N}(I_q + S(w)) = [\psi(w) - i\phi(w)]\mathcal{N}(\psi(w))$. Thus, Remark A.8 implies $\mathcal{N}(I_q - S(w)) = \psi(w)\mathcal{N}(\phi(w))$ and $\mathcal{N}(I_q + S(w)) = \phi(w)\mathcal{N}(\psi(w))$ for each $w \in \Pi_+ \setminus D$. Hence, $S \in S_{q\times q}(\Pi_+)$ and Lemma 5.9 show then that (10.2) holds true for every choice $w$ and $z$ in $\Pi_+ \setminus D$. \(\square\)
Now we introduce the class of pairs of meromorphic matrix-valued functions, which will be the set of parameters in the description of the solution set of the Stieltjes moment problem under consideration.

**Definition 10.5.** Let \( \alpha \in \mathbb{R} \). Let \( \phi \) and \( \psi \) be \( q \times q \) matrix-valued functions meromorphic in \( \mathbb{C} \setminus [\alpha, \infty) \). Then \( \left[ \frac{\phi}{\psi} \right] \) is called a \( q \times q \) Stieltjes pair in \( \mathbb{C} \setminus [\alpha, \infty) \) if there exists a discrete subset \( D \) of \( \mathbb{C} \setminus [\alpha, \infty) \) such that the following three conditions are fulfilled:

(i) \( \phi \) are \( \psi \) are holomorphic in \( \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D}) \).

(ii) \( \text{rank}[\phi(z)] = q \) for each \( z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D}) \).

(iii) For each \( z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D}) \),

\[
\left[ \frac{\phi(z)}{\psi(z)} \right]^* \left( \frac{-j_2}{23z} \right) \left[ \frac{\phi(z)}{\psi(z)} \right] \geq 0 \tag{10.6}
\]

and

\[
\left[ (z - \alpha)\phi(z) \right]^* \left( \frac{-j_2}{23z} \right) \left[ (z - \alpha)\phi(z) \right] \geq 0. \tag{10.7}
\]

The set of all \( q \times q \) Stieltjes pairs in \( \mathbb{C} \setminus [\alpha, \infty) \) will be denoted by \( \mathcal{P}^{(q,q)}_{-j_2} (\mathbb{C} \setminus [\alpha, \infty)) \).

A pair \( \left[ \frac{\phi}{\psi} \right] \in \mathcal{P}^{(q,q)}_{-j_2} (\mathbb{C} \setminus [\alpha, \infty)) \) is said to be a proper \( q \times q \) Stieltjes pair in \( \mathbb{C} \setminus [\alpha, \infty) \) if \( \det \psi \) does not vanish identically in \( \mathbb{C} \setminus [\alpha, \infty) \).

**Remark 10.6.** Let \( \alpha \in \mathbb{R} \) and let \( \left[ \frac{\phi}{\psi} \right] \in \mathcal{P}^{(q,q)}_{-j_2} (\mathbb{C} \setminus [\alpha, \infty)) \). Then one can easily see that \( \left[ \frac{\phi}{\psi} \right] \) given by \( \tilde{\phi} := \text{Rstr}_{\Pi_+ \cap H_{\tilde{\phi}}} \phi \) and \( \tilde{\psi} := \text{Rstr}_{\Pi_+ \cap H_{\tilde{\psi}}} \psi \) belongs to \( \mathcal{P}^{(q,q)}_{-j_2} (\Pi_+) \).

**Remark 10.7.** Let \( \alpha \in \mathbb{R} \), let \( \left[ \frac{\phi}{\psi} \right] \in \mathcal{P}^{(q,q)}_{-j_2} (\mathbb{C} \setminus [\alpha, \infty)) \), and let \( g \) be a \( q \times q \) matrix-valued function which is meromorphic in \( \mathbb{C} \setminus [\alpha, \infty) \) such that \( \det g \) does not vanish identically. Then it is readily checked that \( \left[ \frac{\phi g}{\psi g} \right] \in \mathcal{P}^{(q,q)}_{-j_2} (\mathbb{C} \setminus [\alpha, \infty)) \).

**Remark 10.8.** Two \( q \times q \) Stieltjes pairs \( \left[ \frac{\phi_1}{\psi_1} \right] \) and \( \left[ \frac{\phi_2}{\psi_2} \right] \) in \( \mathbb{C} \setminus [\alpha, \infty) \) are said to be equivalent if there exist a \( q \times q \) matrix-valued function \( g \) which is meromorphic in \( \mathbb{C} \setminus [\alpha, \infty) \) and a discrete subset \( D \) of \( \mathbb{C} \setminus [\alpha, \infty) \) such that \( \phi_1, \phi_2, \psi_1, \psi_2, \) and \( g \) are holomorphic in \( \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D}) \) and that \( \det g(z) \neq 0 \) and \( \left[ \frac{\phi_2}{\psi_2} \right] = \left[ \frac{\phi_1}{\psi_1} \right] g(z) \) hold true for each \( z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D}) \). It is readily checked that this implies an equivalence relation on \( \mathcal{P}^{(q,q)}_{-j_2} (\mathbb{C} \setminus [\alpha, \infty)) \). For each \( \left[ \frac{\phi}{\psi} \right] \in \mathcal{P}^{(q,q)}_{-j_2} (\mathbb{C} \setminus [\alpha, \infty)) \), by \( \left[ \frac{\phi}{\psi} \right] \) we denote the equivalence class generated by \( \left[ \frac{\phi}{\psi} \right] \).

**Remark 10.9.** Let \( \alpha \in \mathbb{R} \). For each \( j \in \{1, 2\} \), let \( \left[ \frac{\phi_j}{\psi_j} \right] \in \mathcal{P}^{(q,q)}_{-j_2} (\mathbb{C} \setminus [\alpha, \infty)) \), let \( \tilde{\phi}_j := \text{Rstr}_{\Pi_+} \phi_j \) and let \( \tilde{\psi}_j := \text{Rstr}_{\Pi_+} \psi_j \). Then it is readily checked that \( \left[ \frac{\phi_1}{\psi_1} \right] = \left[ \frac{\phi_2}{\psi_2} \right] \) if and only if \( \left[ \frac{\phi_1}{\psi_1} \right] = \left[ \frac{\phi_2}{\psi_2} \right] \).

**Example 10.10.** Let \( \alpha \in \mathbb{R} \) and let \( f \in S_q(\alpha, \infty) \). Using [34] Proposition 4.4 and Lemma 4.2, one can easily check then that \( \left[ \frac{f}{e} \right] \) belongs to \( \mathcal{P}^{(q,q)}_{-j_2} (\mathbb{C} \setminus [\alpha, \infty)) \). In particular, [34] Example 2.2 shows that \( \mathcal{P}^{(q,q)}_{-j_2} (\mathbb{C} \setminus [\alpha, \infty)) \neq \emptyset \). Furthermore, if \( f, g \in S_q(\alpha, \infty) \) are such that the pairs \( \left[ \frac{f}{e} \right] \) and \( \left[ \frac{g}{e} \right] \) are equivalent, then \( f = g \).
Remark 10.11. Let $\alpha \in \mathbb{R}$ and let $[\phi, \psi] \in \mathcal{P}^{(q,q)}_{-\mathbb{J}_{\alpha}}(\mathbb{C} \setminus [\alpha, \infty))$. Using a classical result of complex analysis (see, e.g., [12], Theorem 11.46, p. 395), one can prove that there is a $(\mathbb{C} \setminus \{0\})$ valued function $g$ holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$ such that $\phi := g\phi$ and $\psi := g\psi$ are holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$. In particular, $[\phi, \psi]$ belongs to $\mathcal{P}^{(q,q)}_{-\mathbb{J}_{\alpha}}(\mathbb{C} \setminus [\alpha, \infty))$ with $\langle [\phi, \psi] \rangle = \langle [\phi, \psi] \rangle$.

Remark 10.12. Let $\alpha \in \mathbb{R}$ and let $[\phi, \psi] \in \mathcal{P}^{(q,q)}_{-\mathbb{J}_{\alpha}}(\mathbb{C} \setminus [\alpha, \infty))$ be proper. Applying [33, Proposition 4.3] one can show then, that $S := \phi\phi^{-1}$ belongs to $\mathcal{S}_{\mathbb{R}[\alpha,\infty)}$ and that $[\frac{S}{\mathbb{J}_{\alpha}}]$ is a proper $q \times q$ Stieltjes pair in $\mathbb{C} \setminus [\alpha, \infty)$ which is equivalent to $[\phi, \psi]$. (Since we do not use this result in the following, we omit a detailed proof.)

Lemma 10.13. Let $\alpha \in \mathbb{R}$ and let $[\phi, \psi] \in \mathcal{P}^{(q,q)}_{-\mathbb{J}_{\alpha}}(\mathbb{C} \setminus [\alpha, \infty))$. Then there exists a discrete subset $\mathcal{D}$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that the conditions (i), (ii) and (iii) of Definition 10.5 hold true and that the following statements hold true:

(iv) $\frac{1}{3z} \Im[\psi^*(z)\phi(z)] \in \mathbb{C}^{q \times q}$ for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$.

(v) $\frac{1}{3z} \Im[(z-\alpha)\psi^*(z)\phi(z)] \in \mathbb{C}^{q \times q}$ for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$.

(vi) $\mathbb{R}[\psi^*(z)\phi(z)] \in \mathbb{C}^{q \times q}$ for each $z \in \mathbb{C}_{\alpha,-} \setminus \mathcal{D}$.

Proof. In view of Definition 10.5, there is a discrete subset $\mathcal{D}$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that (i), (ii) and (iii) of Definition 10.5 hold true. For each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$, from Remark 9.1 and condition (iii) of Definition 10.5 we get

$$\frac{1}{3z} \Im[\psi^*(z)\phi(z)] = \left[ \frac{\phi(z)}{\psi(z)} \phi(z) \right]^{*} - \left[ \frac{\psi(z)}{\phi(z)} \phi(z) \right] \in \mathbb{C}^{q \times q}$$

and, consequently, (iv) For each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$, Remark 9.1 and condition (iii) of Definition 10.5 yield

$$\frac{1}{3z} \Im[(z-\alpha)\psi^*(z)\phi(z)] = \left[\frac{(z-\alpha)\phi(z)}{\psi(z)}\right]^{*} - \left[\frac{\psi(z)}{\phi(z)}\phi(z)\right] \in \mathbb{C}^{q \times q}.$$ 

Hence, (v) is true. To verify (vi) we consider an arbitrary $z \in \mathbb{C}_{\alpha,-} \setminus (\mathbb{R} \cup \mathcal{D})$. Then $\alpha - \Re z > 0$ and (iv) implies $\frac{\alpha - \Re z}{3z} \Im[\psi^*(z)\phi(z)] \in \mathbb{C}^{q \times q}$. Because of Remark A.9 and (v) this implies

$$\mathbb{R}[\psi^*(z)\phi(z)] = \frac{1}{3z} \Im[(z-\alpha)\psi^*(z)\phi(z)] - \frac{\Re z}{3z} \Im[\psi^*(z)\phi(z)]$$

$$= \frac{1}{3z} \Im[(z-\alpha)\psi^*(z)\phi(z)] + \frac{\alpha - \Re z}{3z} \Im[\psi^*(z)\phi(z)] - \frac{\alpha}{3z} \Im[\psi^*(z)\phi(z)] \in \mathbb{C}^{q \times q}. (10.8)$$

Now we consider an arbitrary $z \in (\mathbb{C}_{\alpha,-} \setminus \mathcal{D}) \cap \mathbb{R}$. Since $\mathcal{D}$ is a discrete subset of $\mathbb{C} \setminus [\alpha, \infty)$, there is a sequence $(z_n)_{n=1}^{\infty}$ of numbers belonging to $\mathbb{C}_{\alpha,-} \setminus (\mathbb{R} \cup \mathcal{D})$ with $\lim_{n \to \infty} z_n = z$. For each $n \in \mathbb{N}$, in view of (10.8), then $\mathbb{R}[\psi^*(z_n)\phi(z_n)] \in \mathbb{C}^{q \times q}$. Thus, in view of condition (i) of Definition 10.5 then

$$\mathbb{R}[\psi^*(z)\phi(z)] = \mathbb{R}\left( \lim_{n \to \infty} [\psi^*(z_n)\phi(z_n)] \right) = \lim_{n \to \infty} \mathbb{R}[\psi^*(z_n)\phi(z_n)] \in \mathbb{C}^{q \times q} (10.9)$$

follows. Taking into account $\mathbb{C}_{\alpha,-} \setminus \mathcal{D} = [\mathbb{C}_{\alpha,-} \setminus (\mathbb{R} \cup \mathcal{D})] \cup [(\mathbb{C}_{\alpha,-} \setminus \mathcal{D}) \cap \mathbb{R}]$, (10.8), and (10.9), statement (vi) is proved as well. \qed
Lemma 10.14. Let $[\phi] \in \mathcal{P}^{(q,q)}_{-\mathbb{J}_q}(\mathbb{C} \setminus [\alpha, \infty))$. Then the function $\det(\psi - i\phi)$ does not vanish identically and the function $F := (\psi + i\phi)(\psi - i\phi)^{-1}$ is meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ and fulfills $\text{Rstr}_{\Pi_+} F \in \mathcal{S}_{q \times q}(\Pi_+)$. Furthermore, there exists a discrete subset $\mathcal{D}$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that $\phi$, $\psi$, $(\psi - i\phi)^{-1}$, and $F$ are holomorphic in $\Pi_+ \cup [\alpha, \infty) \cup \mathcal{D}$ and that $\det(\psi(z) - i\phi(z)) \neq 0$ and

$$F(z) = [(\psi(z) + i\phi(z))((\psi(z) - i\phi(z))^{-1}]$$

hold true for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$. Moreover, for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$, the matrix-valued functions $\phi$ and $\psi$ admit the representations

$$\phi(z) = \frac{i}{2}[I_q - F(z)][(\psi(z) - i\phi(z)) \quad \text{and} \quad \psi(z) = \frac{1}{2}[I_q + F(z)][(\psi(z) - i\phi(z)].$$

In view of Remark [10.6], Lemma [10.3] and Definition [10.5] one can easily prove Lemma [10.14]. We omit the details.

Proposition 10.15. Let $[\phi] \in \mathcal{P}^{(q,q)}_{-\mathbb{J}_q}(\mathbb{C} \setminus [\alpha, \infty))$. Then there exists a discrete subset $\mathcal{D}$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that $\phi$ and $\psi$ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$ and that (10.11) and (10.2) hold true for every choice of $z$ and $w$ in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$.

Proof. The proof is divided into five steps.

(I) Because of $[\phi] \in \mathcal{P}^{(q,q)}_{-\mathbb{J}_q}(\mathbb{C} \setminus [\alpha, \infty))$ and Lemma [10.13], there is a discrete subset $\mathcal{D}$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that the following four conditions hold true:

(i) $\phi$ and $\psi$ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$.

(ii) $\text{rank}[\phi(z)] = q$ for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$.

(iii) The inequalities (10.6) and (10.7) hold true for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$.

(iv) $\Re[\psi^*(z)\phi(z)] \in \mathbb{C}^{q \times q}$ for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$.

(II) Let $\Pi_1 := \Pi_+ \cap \mathbb{H}_\phi \cap \mathbb{H}_\psi$. Then $\mathcal{D}_1 := \mathcal{D} \cap \Pi_+$ is a discrete subset of $\Pi_+$ with $\Pi_1 \supseteq \Pi_+ \setminus \mathcal{D}_1$. Because of (11), the functions $\phi_1 := \text{Rstr}_{\Pi_1} \phi$ and $\psi_1 := \text{Rstr}_{\Pi_1} \psi$ are holomorphic in $\Pi_1$ as well as in $\Pi_+ \setminus \mathcal{D}_1$. If $k = 1$, then (II) and (III) imply

$$\text{rank} \begin{bmatrix} \phi_k(z) \\ \psi_k(z) \end{bmatrix} = q \quad (10.12)$$

and

$$\begin{bmatrix} \phi_k(z) \\ \psi_k(z) \end{bmatrix}^* \begin{pmatrix} -\mathbb{J}_q \\ 2sz \end{pmatrix} \begin{bmatrix} \phi_k(z) \\ \psi_k(z) \end{bmatrix} \geq 0_{q \times q} \quad (10.13)$$

for each $z \in \Pi_+ \setminus \mathcal{D}_1$. Thus, Notation [10.1] shows that $[\phi_1] \in \mathcal{P}^{(q,q)}_{-\mathbb{J}_q}([\Pi_1])$. From Proposition [10.4] we know that there is a discrete subset $\mathcal{D}_1$ of $\Pi_+$ such that $\phi_1$ and $\psi_1$ are holomorphic in $\Pi_+ \setminus \mathcal{D}_1$ and that

$$\mathcal{R}(\phi_k(w)) = \mathcal{R}(\phi_k(z)), \quad \mathcal{R}(\psi_k(w)) = \mathcal{R}(\psi_k(z)), \quad \mathcal{R}(\phi_k(w)) = \mathcal{R}(\phi_k(z)), \quad \mathcal{R}(\psi_k(w)) = \mathcal{R}(\psi_k(z)) \quad (10.14)$$

$$\psi_k(w)N(\phi_k(w)) = \psi_k(z)N(\phi_k(z)), \quad \text{and} \quad \phi_k(w)N(\psi_k(w)) = \phi_k(z)N(\psi_k(z)) \quad (10.15)$$

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hold true for \( k = 1 \) and for every choice of \( w \) and \( z \) in \( \Pi_+ \setminus D_1 \). Consequently, (10.11) and (10.12) are fulfilled for each \( w \in \Pi_+ \setminus D_1 \) and each \( z \in \Pi_+ \setminus D_1 \).

(III) Let \( \Pi_2 := \{ z \in \Pi_+ : -z \in \mathbb{H}_s \cap \mathbb{H}_q \} \) and let \( \tilde{D}_2 := \{ z \in \Pi_+ : -z \in \bar{D} \} \). Obviously, \( \bar{D}_2 \) is a discrete subset of \( \Pi_+ \) with \( \Pi_2 \supseteq \Pi_+ \setminus \bar{D}_2 \). The functions \( \phi_2 : \Pi_2 \rightarrow \mathbb{C}^{q \times q} \) defined by \( \phi_2(z) := -\phi(-z) \) and \( \psi_2 : \Pi_2 \rightarrow \mathbb{C}^{q \times q} \) defined by \( \psi_2(z) := \psi(-z) \) are holomorphic in \( \Pi_+ \setminus \bar{D}_2 \). For each \( z \in \Pi_+ \setminus \tilde{D}_2 \), we have \( -z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \bar{D}) \) and \( \left[ \phi_2(z) \right] = \text{diag}(-I_q, I_q) \cdot \left[ \phi(-z) \right] \). Consequently, (10.12) holds true for \( k = 2 \) and each \( z \in \Pi_+ \setminus \bar{D}_2 \) and, in view of (iii), furthermore,

\[
\left[ \phi_2(z) \right]^* \left[ \begin{array}{c}
\phi_2(z) \\
\psi_2(z)
\end{array} \right] = -23(-z) \left[ \begin{array}{c}
\phi(z) \\
\psi(z)
\end{array} \right]^* \left[ \begin{array}{c}
-\tilde{J}_q \\\n23(-z)
\end{array} \right] \left[ \begin{array}{c}
\phi(-z) \\
\psi(-z)
\end{array} \right] \in \mathbb{C}^{\geq q \times q}
\]

is fulfilled for each \( z \in \Pi_+ \setminus \bar{D}_2 \). Thus, according to Notation 10.1, the pair \( \left[ \phi_2 \psi_2 \right] \) belongs to \( \mathcal{P}_q \). Proposition 10.4 shows that there exists a discrete subset \( D_2 \) of \( \Pi_+ \) such that \( \phi_2 \) and \( \psi_2 \) are holomorphic in \( \Pi_+ \setminus D_2 \) and that (10.14) and (10.15) are valid for \( k = 2 \) and every choice of \( w \) and \( z \) in \( \Pi_+ \setminus D_2 \). Hence, \( \mathcal{R}(-\phi(-w)) = \mathcal{R}(-\phi(z)) \) and \( \mathcal{R}(\psi(-w)) = \mathcal{R}(\psi(z)) \), and \( \psi(-w)\mathcal{N}(-\phi(-w)) = \psi(-z)\mathcal{N}(-\phi(z)) \) and \( -\phi(w)\mathcal{N}(\psi(w)) = -\psi(w)\mathcal{N}(\psi(w)) \) for each \( w \in \Pi_+ \setminus D_2 \) and each \( z \in \Pi_+ \setminus D_2 \). Therefore, \( \bar{D}_2 := \{ z \in \Pi_+ : -z \in \bar{D}_2 \} \) is a discrete subset of \( \Pi_+ \) such that (10.11) and (10.12) are fulfilled for every choice of \( w \) and \( z \) in \( \Pi_+ \setminus \bar{D}_2 \).

(IV) Obviously, \( \bar{D}_3 := \{ z \in \Pi_+ : \alpha + iz \in \bar{D} \} \) is a discrete subset of \( \Pi_+ \) and \( \Pi_+ \setminus \bar{D}_3 \) is a subset of \( \Pi_3 := \{ z \in \Pi_+ : \alpha + iz \in \mathbb{H}_s \cap \mathbb{H}_q \} \). Because of (i), the functions \( \phi_3 : \Pi_3 \rightarrow \mathbb{C}^{q \times q} \) defined by \( \phi_3(z) := i\phi(\alpha + iz) \) and \( \psi_3 : \Pi_3 \rightarrow \mathbb{C}^{q \times q} \) defined by \( \psi_3(z) := \psi(\alpha + iz) \) are holomorphic in \( \Pi_+ \setminus \bar{D}_3 \). For each \( z \in \Pi_+ \setminus \bar{D}_3 \), we have \( \alpha + iz \in \mathbb{C} \setminus ([\alpha, \infty) \cup \bar{D}) \). Thus, for each \( z \in \Pi_+ \setminus \bar{D}_3 \), from \( \left[ \phi_3(z) \right] = \text{diag}(I_q, I_q) \cdot \left[ \phi(\alpha + iz) \right] \) and (ii) we see that (10.12) holds true for \( k = 3 \) and each \( z \in \Pi_+ \setminus \bar{D}_3 \). Obviously, \( \left[ \text{diag}(I_q, I_q) \right]^*(-\tilde{J}_q)\left[ \text{diag}(I_q, I_q) \right] = \left[ \phi_3 \psi_3 \right] I_q \). For each \( z \in \Pi_+ \setminus \bar{D}_3 \), then

\[
\left[ \begin{array}{c}
\phi_3(z) \\
\psi_3(z)
\end{array} \right]^* \left[ \begin{array}{c}
-\tilde{J}_q \\
23z
\end{array} \right] \left[ \begin{array}{c}
\phi_3(z) \\
\psi_3(z)
\end{array} \right] = \frac{1}{23z} \left[ \begin{array}{c}
\phi(\alpha + iz) \\
\psi(\alpha + iz)
\end{array} \right]^* \left[ \text{diag}(I_q, I_q) \right]^*(-\tilde{J}_q)\left[ \text{diag}(I_q, I_q) \right] \left[ \begin{array}{c}
\phi(\alpha + iz) \\
\psi(\alpha + iz)
\end{array} \right] = \frac{1}{23} \mathcal{R}[\psi^*(\alpha + iz)\phi(\alpha + iz)].
\]

For each \( z \in \Pi_+ \setminus \bar{D}_3 \), we have \( \alpha + iz \in \mathbb{C}_\alpha \setminus \bar{D} \) and, consequently, \( \mathcal{R}[\psi^*(\alpha + iz)\phi(\alpha + iz)] \in \mathbb{C}^{q \times q}_\geq \). Thus, (10.16) implies (10.13) for \( k = 3 \) and each \( z \in \Pi_+ \setminus \bar{D}_3 \). Hence, in view of Notation 10.1, the pair \( \left[ \phi_3 \psi_3 \right] \) belongs to \( \mathcal{P}_q \). Proposition 10.4 shows that there is a discrete subset \( D_3 \) of \( \Pi_+ \) such that \( \phi_3 \) and \( \psi_3 \) are holomorphic in \( \Pi_+ \setminus D_3 \) and that the equations in (10.14) and (10.15) hold true for \( k = 3 \) and every choice of \( z \) in \( \Pi_+ \setminus D_3 \). Therefore, for all \( w, z \in \Pi_+ \setminus D_3 \), we obtain

\[
\mathcal{R}(\phi(\alpha + iw)) = \mathcal{R}(i\phi(\alpha + iw)) = \mathcal{R}(\phi(\alpha + iz)) = \mathcal{R}(\phi(\alpha + iz)) = \mathcal{R}(\psi(\alpha + iz)),
\]

\[
\mathcal{R}(\psi(\alpha + iw)) = \mathcal{R}(\psi(\alpha + iz)),
\]

\[
\psi(\alpha + iw)\mathcal{N}(\phi(\alpha + iw)) = \psi(\alpha + i\omega)\mathcal{N}(i\phi(\alpha + iw)) = \psi(\alpha + iz)\mathcal{N}(i\phi(\alpha + iz)) = \psi(\alpha + iz)\mathcal{N}(\phi(\alpha + iz)),
\]

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and
\[
\phi(\alpha + iw)N(\psi(\alpha + iw)) = \phi(\alpha + iz)N(\psi(\alpha + iz)).
\] (10.18)

Since \( D_3 \) is a discrete subset of \( \Pi_+ \), we know that \( \hat{D}_3 := \{ \alpha + iz : z \in D_3 \} \) is a discrete subset of \( \mathbb{C}_\alpha^- \). Obviously, \( z \in \mathbb{C}_\alpha^- \setminus \hat{D}_3 \) if and only if \( \alpha + iz \) belongs to \( \Pi_+ \setminus D_3 \). Thus, (10.17) and (10.18) imply (10.1) and (10.2) for all \( w, z \in \mathbb{C}_\alpha^- \setminus \hat{D}_3 \).

(V) We easily see that \( D := D_1 \cup \hat{D}_2 \cup \hat{D}_3 \) is a discrete subset of \( \mathbb{C} \), that \( (\Pi_+ \setminus D_1) \cup (\Pi_- \setminus \hat{D}_2) \cup (\mathbb{C}_\alpha^- \setminus \hat{D}_3) = \mathbb{C} \setminus (\{ \alpha, \infty \} \cup D) \), that \( (\Pi_+ \setminus D_1) \cap (\mathbb{C}_\alpha^- \setminus \hat{D}_3) \neq \emptyset \), and that \( (\Pi_- \setminus \hat{D}_2) \cap (\mathbb{C}_\alpha^- \setminus \hat{D}_3) \neq \emptyset \). Hence, the equations in (10.1) and (10.2) hold true for every choice of \( w \) and \( z \) in \( \mathbb{C} \setminus (\{ \alpha, \infty \} \cup D) \).

Proposition 10.16. Let \( \alpha \in \mathbb{R} \), let \( \Theta \in \mathcal{M}_{-I_q,\alpha} \), and let
\[
\Theta = [\Theta_{jk}]_{j,k=1}^{2}
\] (10.19)
be the \( q \times q \) block representation of \( \Theta \). Then:

(a) The function \( \det \Theta \) does not vanish identically and the matrix-valued function \( \Theta^{-1} \) is meromorphic in \( \mathbb{C} \setminus [\alpha, \infty) \).

(b) Let \( f \) be a \( q \times q \) matrix-value function meromorphic in \( \mathbb{C} \setminus [\alpha, \infty) \). Suppose that there is a discrete subset \( D \) of \( \mathbb{C} \setminus [\alpha, \infty) \) such that \( f \) and \( \Theta \) are holomorphic in \( \mathbb{C} \setminus ([\alpha, \infty) \cup D) \), that \( \det \Theta(z) \neq 0 \) holds true for each \( z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D) \), and that
\[
\begin{bmatrix}
\left(\frac{-j}{23z}\right)
\end{bmatrix} \Theta^{-1}(z) \begin{bmatrix}
\left(\frac{-j}{23z}\right)
\end{bmatrix} \geq 0_{q \times q}
\] (10.20)
and
\[
\begin{bmatrix}
\left(\frac{-j}{23z}\right)
\end{bmatrix} \Theta^{-1}(z) \begin{bmatrix}
\left(\frac{-j}{23z}\right)
\end{bmatrix} \geq 0_{q \times q}
\] (10.21)
are fulfilled for each \( z \in \mathbb{C} \setminus (\mathbb{R} \cup D) \). For every such discrete subset \( D \) of \( \mathbb{C} \setminus [\alpha, \infty) \), there exists a pair \( \begin{bmatrix} \phi \\ \psi \end{bmatrix} \in \mathcal{P}_{-I_q,\alpha}^{q,q} \) such that \( \phi \) and \( \psi \) are holomorphic in \( \mathbb{C} \setminus ([\alpha, \infty) \cup D) \) and that
\[
\det[\Theta_{21}(z)\phi(z) + \Theta_{22}(z)\psi(z)] \neq 0
\] (10.22)
and
\[
f(z) = [\Theta_{11}(z)\phi(z) + \Theta_{12}(z)\psi(z)][\Theta_{21}(z)\phi(z) + \Theta_{22}(z)\psi(z)]^{-1}
\] (10.23)
hold true for each \( z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D) \).

(c) Let \( \begin{bmatrix} \phi \\ \psi \end{bmatrix} \in \mathcal{P}_{-I_q,\alpha}^{q,q} \) be such that \( \det(\Theta_{21}\phi + \Theta_{22}\psi) \) does not vanish identically. Then there exists a discrete subset \( D \) of \( \mathbb{C} \setminus [\alpha, \infty) \) such that the following three statements are valid:
(I) The the matrix-valued functions $\Theta$, $\phi$, and $\psi$ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$.

(II) The inequalities $\det \Theta(z) \neq 0$ and (10.22) hold true for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$.

(III) The function

$$f := (\Theta_{11} \phi + \Theta_{12} \psi)(\Theta_{21} \phi + \Theta_{22} \psi)^{-1}$$

(10.24)

is holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$, the inequalities (10.20) and (10.21) hold true for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$ and (10.23) is fulfilled for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$.

(d) For each $k \in \{1, 2\}$, let $[\phi_k \psi_k] \in P^{(q, q)}_{-j_{\psi}}(\mathbb{C} \setminus [\alpha, \infty))$ be such that $\det(\Theta_{21} \phi_k + \Theta_{22} \psi_k)$ does not vanish identically. Then $([\phi_1 \psi_1]) = ([\phi_2 \psi_2])$ if and only if

$$(\Theta_{11} \phi_1 + \Theta_{12} \psi_1)(\Theta_{21} \phi_1 + \Theta_{22} \psi_1)^{-1} = (\Theta_{11} \phi_2 + \Theta_{12} \psi_2)(\Theta_{21} \phi_2 + \Theta_{22} \psi_2)^{-1}.$$ 

Proof. Use Lemma 9.26.

[ ] Let $\mathcal{D}$ be a discrete subset of $\mathbb{C} \setminus [\alpha, \infty)$ such that $f$ and $\Theta$ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$, that $\det \Theta(z) \neq 0$ is valid for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$, and that (10.20) and (10.21) are fulfilled for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$. Then $\Theta^{-1}, \phi := [I_q, 0_{q \times q}] \Theta^{-1}[f_i], \text{ and } \psi := [0_{q \times q}, I_q] \Theta^{-1}[f_q]$ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$ and, for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$, we have

$$\begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} = \Theta^{-1}(z) \begin{bmatrix} f(z) \\ I_q \end{bmatrix},$$

(10.25)

consequently,

$$\Theta_{11}(z) \phi(z) + \Theta_{12}(z) \psi(z) = [I_q, 0_{q \times q}] \Theta(z) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} = f(z)$$

(10.26)

and, analogously,

$$\Theta_{21}(z) \phi(z) + \Theta_{22}(z) \psi(z) = I_q.$$ 

(10.27)

In particular, (10.27) implies (10.22) as well as

$$q \geq \text{rank} \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \geq \text{rank} \begin{bmatrix} \Theta_{21}(z), \Theta_{22}(z) \end{bmatrix} \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} = \text{rank} I_q = q$$

and, hence, $\text{rank} \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} = q$ for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$. In view of (10.25) and (10.24), we conclude that

$$\begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix}^\ast \begin{bmatrix} -j_{\psi} \\ 23z \end{bmatrix} \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \begin{bmatrix} f(z) \\ I_q \end{bmatrix}^\ast \Theta^{-1}(z) \begin{bmatrix} -j_q \\ 23z \end{bmatrix} \begin{bmatrix} f(z) \\ I_q \end{bmatrix} \geq 0$$

holds true for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$. From (10.26) we obtain the equation $\begin{bmatrix} (z - \alpha) \phi(z) \\ \psi(z) \end{bmatrix} = (\text{diag}([z - \alpha]I_q, I_q)) \Theta^{-1}(z) \begin{bmatrix} f(z) \\ I_q \end{bmatrix}$ for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$, and, according to (10.21), consequently, (10.7) for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$. Thus, we proved that $[\phi \psi]$ is a $q \times q$ Stieltjes pair in $\mathbb{C} \setminus ([\alpha, \infty)$.

Since $\Theta$ belongs to $\mathcal{W}_{-j_{\psi}}$, Lemma 9.26 shows that there is a discrete subset $\mathcal{D}_1$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that $\Theta$ is holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D}_1)$ and that $\det \Theta(z) \neq 0$ is valid for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D}_1)$. Because of $[\phi \psi] \in P^{(q, q)}_{-j_{\psi}}(\mathbb{C} \setminus [\alpha, \infty))$, there is a discrete subset $\mathcal{D}_2$
of \( C \setminus [0, \infty) \) such that \( \phi \) and \( \psi \) are holomorphic in \( C \setminus ([0, \infty) \cup \mathcal{D}_2) \) and that (10.13) and (10.17) hold true for each \( z \in C \setminus (\mathbb{R} \cup \mathcal{D}_2) \). Since the meromorphic function \( \det(\Theta_{21}\phi + \Theta_{22}\psi) \) does not vanish identically, there is a discrete subset \( \mathcal{D}_3 \subseteq C \setminus [0, \infty) \) such that \( \det(\Theta_{21}\phi + \Theta_{22}\psi) \) is holomorphic in \( C \setminus (\mathbb{R} \cup \mathcal{D}_3) \) and that \( \det(\Theta_{21}\phi + \Theta_{22}\psi)(z) \neq 0 \) holds true for all \( z \in C \setminus (\mathbb{R} \cup \mathcal{D}_3) \). Thus, the set \( \mathcal{D} := \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \) is a discrete subset of \( C \setminus [0, \infty) \) and we see that \( \Theta, \phi, \) and \( \psi \) are holomorphic in \( C \setminus (\mathbb{R} \cup \mathcal{D}) \) and that the inequalities (10.6), and (10.17) hold true for each \( z \in C \setminus (\mathbb{R} \cup \mathcal{D}) \). Furthermore, \( \det(\Theta(z)) \neq 0 \) and (10.22) are valid for all \( z \in C \setminus ([0, \infty) \cup \mathcal{D}) \). Consequently, \( f \) defined by (10.21) is holomorphic in \( C \setminus ([0, \infty) \cup \mathcal{D}) \) and (10.23) is valid for each \( z \in C \setminus ([0, \infty) \cup \mathcal{D}) \). Now we consider an arbitrary \( z \in C \setminus (\mathbb{R} \cup \mathcal{D}) \). Because of part (d), (10.22), (10.23), and (10.19), we have

\[
\Theta^{-1}(z) \begin{bmatrix} f(z) \\ I_q \end{bmatrix} = \Theta^{-1}(z) \begin{bmatrix} \Theta_{11}(z)\phi(z) + \Theta_{12}(z)\psi(z) \\ \Theta_{21}(z)\phi(z) + \Theta_{22}(z)\psi(z) \end{bmatrix} [\Theta_{21}(z)\phi(z) + \Theta_{22}(z)\psi(z)]^{-1} \\
= \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} [\Theta_{21}(z)\phi(z) + \Theta_{22}(z)\psi(z)]^{-1}
\]

(10.28)

and, consequently,

\[
\begin{bmatrix} f(z) \\ I_q \end{bmatrix}^* \Theta^{-*}(z)(-\tilde{J}_q)\Theta^{-1}(z) \begin{bmatrix} f(z) \\ I_q \end{bmatrix} = \begin{bmatrix} (z-\alpha)\phi(z) \\ \psi(z) \end{bmatrix} [\Theta_{21}(z)\phi(z) + \Theta_{22}(z)\psi(z)]^{-1},
\]

(10.29)

In view of (10.6), the matrix on the right-hand side of (10.29) is non-negative Hermitian. Thus, (10.20) holds true. Using (10.28), we get

\[
(\text{diag}([z-\alpha]I_q, I_q))\Theta^{-1}(z) \begin{bmatrix} f(z) \\ I_q \end{bmatrix} = \begin{bmatrix} (z-\alpha)\phi(z) \\ \psi(z) \end{bmatrix} [\Theta_{21}(z)\phi(z) + \Theta_{22}(z)\psi(z)]^{-1},
\]

which implies

\[
\begin{bmatrix} f(z) \\ I_q \end{bmatrix}^* \Theta^{-*}(z)(\text{diag}([z-\alpha]I_q, I_q))^* \begin{bmatrix} -\tilde{J}_q \\ 2\tilde{z} \end{bmatrix} \left( \begin{bmatrix} (z-\alpha)\phi(z) \\ \psi(z) \end{bmatrix} \right)^* \begin{bmatrix} (z-\alpha)\phi(z) \\ \psi(z) \end{bmatrix} [\Theta_{21}(z)\phi(z) + \Theta_{22}(z)\psi(z)]^{-1}.
\]

(10.30)

Because of (10.7), the matrix on the right-hand side of (10.30) is non-negative Hermitian. Consequently, (10.21) is proved as well.

In view of part (d) and Lemma 9.26, the proof of part (d) is straightforward.

**Lemma 10.17.** Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N} \cup \{\infty\} \), and let \((s_j)_{j=0}^\kappa \in \mathcal{K}_q^{\geq \kappa, \alpha} \). Let \( \phi \) and \( \psi \) be \( q \times q \) matrix-valued functions which are meromorphic in \( C \setminus [0, \infty) \). Let \( n \in \mathbb{N}_0 \) be such
that $2n + 1 \leq \kappa$ and let $\Theta_{n,\alpha} : \mathbb{C} \rightarrow \mathbb{C}^{2q \times 2q}$ be defined by (10.35). Let $\hat{\Theta}_{n,\alpha} := \text{Rstr}_{\mathbb{C}\setminus(\alpha,\infty)}(\Theta_{n,\alpha})$ and let

$$
\hat{\Theta}_{n,\alpha} = [\hat{\Theta}_{n,\alpha}^{(j,k)}]_{j,k=1}^{2q \times 2q}
$$

(10.31)

be the $q \times q$ block representation of $\hat{\Theta}_{n,\alpha}$. Let

$$
\hat{\phi} := \hat{\Theta}_{n,\alpha}^{(1,1)} \phi + \hat{\Theta}_{n,\alpha}^{(1,2)} \psi \quad \text{and} \quad \hat{\psi} := \hat{\Theta}_{n,\alpha}^{(2,1)} \phi + \hat{\Theta}_{n,\alpha}^{(2,2)} \psi.
$$

(10.32)

Furthermore, let $z \in (\mathbb{H}_\phi \cap \mathbb{H}_\psi) \setminus \mathbb{R}$ be such that

$$
\begin{bmatrix}
\phi(z) \\
\psi(z)
\end{bmatrix}
\begin{bmatrix}
\hat{J}_q \\
23z
\end{bmatrix}
\begin{bmatrix}
\phi(z) \\
\psi(z)
\end{bmatrix} \geq 0
$$

(10.33)

and

$$
(I_{(n+1)q} - H_n^* H_n) R(\alpha) v_{q,n}(\phi) = 0
$$

(10.34)

hold true. Then $\mathcal{N}(\hat{\psi}(z)) \subseteq \mathcal{N}(\hat{\phi}(z))$. Moreover, if

$$
\text{rank}\begin{bmatrix}
\phi(z) \\
\psi(z)
\end{bmatrix} = q
$$

(10.35)

holds true, then $\det\hat{\psi}(z) \neq 0$.

Proof. Because of $\mathcal{K}_{q,\kappa,\alpha} \subseteq \mathcal{K}_{q,\kappa,\alpha}^\infty$ and Lemma 8.10 the equations in (8.9) are true.

We consider an arbitrary $y \in \mathcal{N}(\hat{\psi}(z))$. Because of Remark 9.1 we have then

$$
y^* \begin{bmatrix}
\hat{\psi}(z) \\
\hat{\phi}(z)
\end{bmatrix}^* \hat{J}_q 
\begin{bmatrix}
\hat{\phi}(z) \\
\hat{\psi}(z)
\end{bmatrix} y = iy^* \left[ \hat{\psi}^*(z) \hat{\phi}(z) - \hat{\phi}^*(z) \hat{\psi}(z) \right] y = 0.
$$

(10.36)

Obviously, $\Theta_{n,\alpha}(z) = \hat{\Theta}_{n,\alpha}(z)$. From (10.31) and (10.32) we conclude

$$
\hat{\Theta}_{n,\alpha}(z) \begin{bmatrix}
\phi(z) \\
\psi(z)
\end{bmatrix} = \begin{bmatrix}
\hat{\phi}(z) \\
\hat{\psi}(z)
\end{bmatrix}.
$$

(10.37)

Using (10.37) and (10.36), we conclude

$$
y^* \begin{bmatrix}
\phi(z) \\
\psi(z)
\end{bmatrix}^* \begin{bmatrix}
\hat{J}_q - \Theta_{n,\alpha}^*(z) \hat{J}_q \Theta_{n,\alpha}(z) \\
\hat{J}_q - \Theta_{n,\alpha}^*(z) \hat{J}_q \Theta_{n,\alpha}(z)
\end{bmatrix} \begin{bmatrix}
\phi(z) \\
\psi(z)
\end{bmatrix} y = -y^* \begin{bmatrix}
\phi(z) \\
\psi(z)
\end{bmatrix}^* (-\hat{J}_q) \begin{bmatrix}
\phi(z) \\
\psi(z)
\end{bmatrix} y.
$$

(10.38)
Because of Lemma 8.10, Remark 4.4, Lemma 9.20, (10.38), and (10.33), we obtain

\[ 0 \leq \left\| \sqrt{H_n} \left[ R_{T, n}^* (\alpha) \right]^{-1} R_{T, n}^* (z) H_n^{-1} R_{T, n} (\alpha) [I_{n+1, q}, T_{\alpha, n} H_n] (I_2 \otimes v_{q, n}) B_{\alpha, n, \alpha} \phi (z) \psi (z) \right\|^2_{\mathcal{E}} \]

\[ = y^* \left[ \phi (z) \psi (z) \right]^* B_{\alpha, n, \alpha} (I_2 \otimes v_{q, n}) [I_{n+1, q}, T_{\alpha, n} H_n] [R_{T, n} (\alpha)]^* \times (H_n^{-1})^* \left[ R_{T, n}^* (z) \right]^* \left[ R_{T, n} (\alpha) \right]^{-1} H_n \left[ R_{T, n} (\alpha) \right]^{-1} R_{T, n}^* (z) H_n^{-1} \times R_{T, n} (\alpha) [I_{n+1, q}, T_{\alpha, n} H_n] (I_2 \otimes v_{q, n}) B_{\alpha, n, \alpha} \left[ \phi (z) \psi (z) \right] y \]

\[ = y^* \left[ \phi (z) \psi (z) \right]^* \frac{1}{i(z - \bar{z})} \left[ \tilde{J}_q - \Theta_{n, \alpha}^* (z) \tilde{J}_q \Theta_{n, \alpha} (z) \right] \left[ \phi (z) \psi (z) \right] y \leq 0 \]

and, consequently,

\[ \sqrt{H_n} \left[ R_{T, n}^* (\alpha) \right]^{-1} R_{T, n}^* (z) H_n^{-1} R_{T, n} (\alpha) [I_{n+1, q}, T_{\alpha, n} H_n] (I_2 \otimes v_{q, n}) B_{\alpha, n, \alpha} \left[ \phi (z) \psi (z) \right] y = 0. \]  

(10.39)

Multiplying equation (10.39) from the left by $\sqrt{H_n}$ and using Remark 9.10, we get

\[ H_n \left[ R_{T, n}^* (\alpha) \right]^{-1} R_{T, n}^* (z) [H_n^{-1} R_{T, n} (\alpha), H_{\alpha, n}^{-1} H_n] (I_2 \otimes v_{q, n}) \left[ \phi (z) \psi (z) \right] y = 0 \]

and, hence,

\[ \left[ H_n \left[ R_{T, n}^* (\alpha) \right]^{-1} R_{T, n}^* (z) H_n^{-1} R_{T, n} (\alpha), H_n \left[ R_{T, n} (\alpha) \right]^{-1} R_{T, n}^* (z) H_{\alpha, n}^{-1} H_n \right] \times (I_2 \otimes v_{q, n}) \left[ \phi (z) \psi (z) \right] y = 0. \]  

(10.40)

Let $X := H_n [R_{T, n}^* (\alpha)]^{-1} R_{T, n}^* (z) H_{\alpha, n}^{-1} H_n$. Because of (10.37), $\Theta_{n, \alpha} (z) = \tilde{\Theta}_{n, \alpha} (z)$, Lemma 9.12, and $v_{q, n}^* R_{T, n} (\alpha) v_{q, n} = I_q$, we have

\[ \tilde{\phi} (z) y = [I_q, 0_{q \times q}] \left[ \phi (z) \psi (z) \right] y = [I_q, 0_{q \times q}] \Theta_{n, \alpha} (z) \left[ \phi (z) \psi (z) \right] y \]

\[ = \left[ I_q + (z - \alpha) v_{q, n}^* H_n T_{\alpha, n} R_{T, n}^* (z) H_n^{-1} R_{T, n} (\alpha) \right] v_{q, n}^* X v_{q, n} \left[ \phi (z) \psi (z) \right] y \]

\[ = v_{q, n}^* \left[ R_{T, n} (\alpha) + (z - \alpha) H_n T_{\alpha, n} R_{T, n}^* (z) H_n^{-1} R_{T, n} (\alpha), X \right] (I_2 \otimes v_{q, n}) \left[ \phi (z) \psi (z) \right] y. \]  

(10.41)
Thus, using (10.43) and (10.34), we infer

\[(z - \alpha)H_nT_{q,n}^*RT_{q,n}(z)H_n^{-1}RT_{q,n}(\alpha) = H_n\left[RT_{q,n}^*(\alpha)]^{-1}RT_{q,n}(z) - I_{(n+1)q}\right]H_n^{-1}RT_{q,n}(\alpha)\]

\[= H_n[RT_{q,n}^*(\alpha)]^{-1}RT_{q,n}(z)H_n^{-1}RT_{q,n}(\alpha) - H_nH_n^{-1}RT_{q,n}(\alpha). \quad (10.42)\]

Taking (10.41), (10.42), and (10.40) into account, we obtain

\[\tilde{\phi}(z)y = v_{q,n}^*[RT_{q,n}(\alpha) - H_nH_n^{-1}RT_{q,n}(\alpha), 0_{(n+1)q \times (n+1)q}] (I_2 \otimes v_{q,n}) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y \quad (10.43)\]

\[+ v_{q,n}^*[H_n[RT_{q,n}^*(\alpha)]^{-1}RT_{q,n}(z)H_n^{-1}RT_{q,n}(\alpha), X] (I_2 \otimes v_{q,n}) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y \]

\[= v_{q,n}^*[RT_{q,n}(\alpha) - H_nH_n^{-1}RT_{q,n}(\alpha), 0_{(n+1)q \times (n+1)q}] (I_2 \otimes v_{q,n}) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y.\]

Thus, using (10.43), Remark 8.11 and (10.34), we infer

\[\tilde{\phi}(z)y = \begin{bmatrix} v_{q,n}^*[RT_{q,n}(\alpha)]v_{q,n} - v_{q,n}^*[H_nH_n^{-1}RT_{q,n}(\alpha)]v_{q,n}, 0_{q \times q} \end{bmatrix} \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y\]

\[= v_{q,n}^*(I_{(n+1)q} - H_nH_n^{-1})RT_{q,n}(\alpha)v_{q,n}\phi(z)y\]

\[= v_{q,n}^*(I_{(n+1)q} - H_nH_n^{-1})(I_{(n+1)q} - H_nH_n^{-1})RT_{q,n}(\alpha)v_{q,n}\phi(z)y = 0\]

and, consequently, \(y \in \mathcal{N}(\tilde{\phi}(z))\). Hence \(\mathcal{N}(\tilde{\psi}(z)) \subseteq \mathcal{N}(\tilde{\phi}(z))\) is proved.

Now we suppose (10.35). We consider again an arbitrary \(y \in \mathcal{N}(\tilde{\psi}(z))\). Then we already know that \(y \in \mathcal{N}(\tilde{\phi}(z))\). In view of Lemma 9.16(b) and (10.37), we get then

\[\begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y = [\Theta_{n,\alpha}(z)]^{-1}\Theta_{n,\alpha}(z) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y = [\Theta_{n,\alpha}(z)]^{-1} \begin{bmatrix} \tilde{\phi}(z)y \\ \tilde{\psi}(z)y \end{bmatrix} = 0.\]

Because of (10.35), this implies \(y = 0_{q \times 1}\), and hence, \(\det \tilde{\psi}(z) \neq 0\).

\[\text{Lemma 10.18. Let } \alpha \in \mathbb{R}, \text{ let } \kappa \in \mathbb{N} \cup \{\infty\}, \text{ let } (s_j)_{j=0}^\kappa \in K_{q,n,\alpha}^e, \text{ and let } [\Psi]_{s_j}^n \in \mathcal{P}_{-J_{q,\geq}}(\mathbb{C} \setminus [\alpha, \infty)) \text{ be such that } (I_{(n+1)q} - H_nH_n^{-1})RT_{q,n}(\alpha)v_{q,n}\phi = 0_{(n+1)q \times q}. \text{ Let } n \in \mathbb{N}_0 \text{ be such that } 2n + 1 \leq \kappa, \text{ let } \Theta_{n,\alpha}: \mathbb{C} \to \mathbb{C}^{2q \times 2q} \text{ be defined by } (9.38), \text{ and let } (10.31) \text{ be the } q \times q \text{ block partition of } Rstr_{\mathbb{C} \setminus [\alpha, \infty)} \Theta_{n,\alpha}. \text{ Then there is a discrete subset } \mathcal{D} \text{ of } \mathbb{C} \setminus [\alpha, \infty) \text{ such that } \phi \text{ and } \psi \text{ are holomorphic in } \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D}) \text{ and that}\]

\[\det[\hat{\Theta}_{n,\alpha}^{(2,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z)\psi(z)] \neq 0 \quad (10.44)\]

holds true for each \(z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})\).

\[\text{Proof. Use Definition } (10.5) \text{ and Lemma 10.17}.\]
11. A particular subclass of Stieltjes pairs

In this section, we study a particular subclass of Stieltjes pairs.

*Notation* 11.1. Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N} \cup \{ \infty \} \) and let \( (s_j)_{j=0}^{\infty} \) be a sequence of complex \( q \times q \) matrices. For each \( n \in \mathbb{N}_0 \) with \( 2n + 1 \leq \kappa \), let \( \mathcal{P}^{(q,q)}_{\infty n} [\mathbb{C} \setminus [\alpha, \infty) \}, (s_j)_{j=0}^{\infty} \] be the set of all \( \begin{bmatrix} \phi \\ \psi \end{bmatrix} \in \mathcal{P}^{(q,q)}_{\infty n} (\mathbb{C} \setminus [\alpha, \infty)) \) such that

\[
(I_{(n+1)q} - H_n^\dagger H_n) R_{T_n,n} (\alpha) v_{q,n} \phi = 0 \tag{11.1}
\]

and

\[
(I_{(n+1)q} - H_n^\dagger H_n \alpha v_{q,n} \psi = 0. \tag{11.2}
\]

*Remark* 11.2. Let \( \alpha \in \mathbb{R} \), let \( (s_j)_{j=0}^{\infty} \) be a sequence of complex \( q \times q \) matrices, let \( \begin{bmatrix} \phi \\ \psi \end{bmatrix} \in \mathcal{P}^{(q,q)}_{\infty n} (\mathbb{C} \setminus [\alpha, \infty)) \) such that \( \det g \) does not vanish identically. Then it is readily checked that \( \begin{bmatrix} \phi \\ \psi \end{bmatrix} \in \mathcal{P}^{(q,q)}_{\infty n} (\mathbb{C} \setminus [\alpha, \infty)) \) such that \( \det g \) does not vanish identically and that

\[
\begin{bmatrix} \phi \\ \psi \end{bmatrix} \in \mathcal{P}^{(q,q)}_{\infty n} (\mathbb{C} \setminus [\alpha, \infty)), (s_j)_{j=0}^{\infty} \] holds true, where \( \Theta_{n,\alpha} : \mathbb{C} \rightarrow \mathbb{C}^{2q \times 2q} \) be defined by (11.3), let \( \tilde{\Theta}_{n,\alpha} := \text{Rstr}_{\mathbb{C} \setminus [\alpha, \infty)} \Theta_{n,\alpha} \), let \( (10.31) \) be the \( q \times q \) block representation of \( \tilde{\Theta}_{n,\alpha} \), and let \( \hat{R}_{n,n} := \text{Rstr}_{\mathbb{C} \setminus [\alpha, \infty)} \tilde{\Theta}_{n,n} \). Let \( \begin{bmatrix} \phi \\ \psi \end{bmatrix} \in \mathcal{P}^{(q,q)}_{\infty n} (\mathbb{C} \setminus [\alpha, \infty)) \) such that \( \det(\hat{\Theta}^{(2,1)}_{n,\alpha} \phi + \hat{\Theta}^{(2,2)}_{n,\alpha} \psi) \) does not vanish identically and

\[
\hat{S}_{n,\alpha} := (\hat{\Theta}^{(2,1)}_{n,\alpha} \phi + \hat{\Theta}^{(2,2)}_{n,\alpha} \psi)/(\hat{\Theta}^{(2,1)}_{n,\alpha} \phi + \hat{\Theta}^{(2,2)}_{n,\alpha} \psi)^{-1}. \tag{11.3}
\]

Then the following statements (a) and (b) are equivalent:

(a) The equations

\[
(I_{(n+1)q} - H_n^\dagger H_n) \tilde{R}_{T_{n,n}} (I_{(n+1)q}, T_{n,n} H_n (I_2 \otimes v_{q,n}) \begin{bmatrix} \hat{S}_{n,\alpha} \\ I_q \end{bmatrix} = 0 \tag{11.4}
\]

and

\[
(I_{(n+1)q} - H_n^\dagger H_n \alpha v_{q,n} \psi = 0. \tag{11.2}
\]

hold true, where \( \hat{E} : \mathbb{C} \setminus [\alpha, \infty) \rightarrow \mathbb{C} \) is defined by \( \hat{E}(z) := z \).

(b) \( \begin{bmatrix} \phi \\ \psi \end{bmatrix} \in \mathcal{P}^{(q,q)}_{\infty n} (\mathbb{C} \setminus [\alpha, \infty)), (s_j)_{j=0}^{\infty} \]

*Proof.* The proof is partitioned into twelve steps.

(i) Since \( \det(\hat{\Theta}^{(2,1)}_{n,\alpha} \phi + \hat{\Theta}^{(2,2)}_{n,\alpha} \psi) \) does not vanish identically, there is a discrete subset \( D \) of \( \mathbb{C} \setminus [\alpha, \infty) \) such that the conditions (i), (ii), and (iii) of Definition 10.5 hold true and that (i) is fulfilled for each \( z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D) \).

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(II) In view of condition (i) of Definition \ref{def:10.1} \ref{def:11.3}, and \ref{thm:10.44}, the function $\hat{S}_{n,\alpha}$ admits the representation

$$
\hat{S}_{n,\alpha}(z) = \left[ \hat{\Theta}_{n,\alpha}^{(1,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(1,2)}(z)\psi(z) \right] \left[ \hat{\Theta}_{n,\alpha}^{(2,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z)\psi(z) \right]^{-1}
$$

(11.6)

for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$. Because of condition (i) of Definition \ref{def:10.1} \ref{def:11.3}, \ref{thm:10.44}, \ref{thm:10.31}, and \ref{thm:11.6}, we see that

$$
\hat{\Theta}_{n,\alpha}(z) \left[ \begin{array}{c} \phi(z) \\ \psi(z) \end{array} \right] \left[ \begin{array}{c} \hat{\Theta}_{n,\alpha}^{(2,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z)\psi(z) \end{array} \right]^{-1} = \left[ \begin{array}{c} \hat{\Theta}_{n,\alpha}^{(1,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(1,2)}(z)\psi(z) \\ \hat{\Theta}_{n,\alpha}^{(2,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z)\psi(z) \end{array} \right]^{-1}
$$

(11.7)

holds true for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$. Let $\hat{\Theta}_{n,\alpha} : \mathbb{C} \rightarrow \mathbb{C}^{2q \times 2q}$ given by \ref{thm:9.39}. Taking into account \ref{thm:10.44}, Lemma \ref{lem:9.13} and \ref{thm:11.7}, for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$, this implies

$$
\left[ \begin{array}{c} \text{Rstr}_{\mathbb{C}[\alpha, \infty]} \hat{\Theta}_{n,\alpha}(z) \\ \hat{\Theta}_{n,\alpha}(z) \end{array} \right] \left[ \begin{array}{c} \hat{\Theta}_{n,\alpha}^{(2,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z)\psi(z) \\ \hat{\Theta}_{n,\alpha}^{(2,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z)\psi(z) \end{array} \right]^{-1} = \left[ \begin{array}{c} \hat{\Theta}_{n,\alpha}^{(1,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(1,2)}(z)\psi(z) \\ \hat{\Theta}_{n,\alpha}^{(2,1)}(z)\phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z)\psi(z) \end{array} \right]^{-1}
$$

(11.8)

(III) Since the functions $\hat{\xi}$ and $\hat{R}_{q,n}$ are holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$, statement (iii) is equivalent to the following statement:

(c) There exists a discrete subset $\hat{\mathcal{D}}$ of $\mathbb{C} \setminus (\alpha, \infty)$ such that $\hat{S}_{n,\alpha}$ is holomorphic in $\mathbb{C} \setminus (\alpha, \infty) \cup \mathcal{D}$ and that

$$
(I_{(n+1)q} - H_{n}^t H_{n}) \hat{R}_{q,n}(z) [I_{(n+1)q}, T_{q,n} H_{n}] (I_2 \otimes v_{q,n}) \left[ \begin{array}{c} \hat{S}_{n,\alpha}(z) \\ I_q \end{array} \right] = 0
$$

(11.9)

and

$$
(I_{(n+1)q} - H_{\alpha \otimes n} H_{\alpha \otimes n}) \hat{R}_{q,n}(z) [I_{(n+1)q}, [R_{T_{q,n}}(\alpha)]^{-1} H_{n}] \\
\times (I_2 \otimes v_{q,n}) \left[ \begin{array}{c} \hat{\xi}(z) - \alpha I_q \hat{S}_{n,\alpha}(z) \\ I_q \end{array} \right] = 0
$$

(11.10)

hold true for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$.

(IV) In this step of the proof, we suppose (c). We are going to prove that the following statement holds true:
(d) There is a discrete subset $\hat{D}$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that $\phi$ and $\psi$ are holomorphic in $\mathbb{C} \setminus ((\alpha, \infty) \cup \hat{D})$ and that

$$
(I_{(n+1)q} - H_n^t H_n) \hat{R}_{T_{q,n}}(z) [I_{(n+1)q}, T_{q,n} H_n] \\
\times (I_2 \otimes v_{q,n}) \tilde{\Theta}_{n,\alpha}(z) \left[ \frac{\phi(z)}{\psi(z)} \right] \left[ \hat{\Theta}_{n,\alpha}^{(2,1)}(z) \phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z) \psi(z) \right]^{-1} = 0 \quad (11.11)
$$

and

$$
(I_{(n+1)q} - H_n^t H_n) \hat{R}_{T_{q,n}}(z) [I_{(n+1)q}, R_{T_{q,n}}(\alpha)]^{-1} H_n \\
\times (I_2 \otimes v_{q,n}) \left[ \text{Rstr}_{\mathbb{C} \setminus [\alpha, \infty)} \tilde{\Theta}_{n,\alpha}(z) \right] \left[ (\hat{\Theta}(z) - \alpha) \phi(z) \right] \\
\times \left[ \hat{\Theta}_{n,\alpha}^{(2,1)}(z) \phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z) \psi(z) \right]^{-1} = 0 \quad (11.12)
$$

are fulfilled for each $z \in \mathbb{C} \setminus ((\alpha, \infty) \cup \hat{D})$.

First we observe that $D_{\#} := D \cup \hat{D}$ is a discrete subset of $\mathbb{C} \setminus [\alpha, \infty)$. Since (11.9) and (11.10) are valid for each $z \in \mathbb{C} \setminus ((\alpha, \infty) \cup D_{\#})$ and since (II) shows that (11.7) and (11.8) are fulfilled for each $z \in \mathbb{C} \setminus ((\alpha, \infty) \cup \hat{D}_{\#})$, we get that (11.11) and (11.12) hold true for each $z \in \mathbb{C} \setminus ((\alpha, \infty) \cup D_{\#})$. Setting $\hat{D} = D_{\#}$, statement (d) is proved.

(V) In this step of the proof, we suppose (d). We are going to prove that (e) holds true. Obviously, $D_{\square} := D \cup \hat{D}$ is a discrete subset of $\mathbb{C} \setminus [\alpha, \infty)$. According to (I) and (II), we get (10.44), (11.7), and (11.8) for each $z \in \mathbb{C} \setminus ((\alpha, \infty) \cup D_{\#})$. Using these arguments and (11.11) and (11.12), we see that (11.9) and (11.10) are fulfilled for each $z \in \mathbb{C} \setminus ((\alpha, \infty) \cup D_{\#})$. Consequently, statement (e) holds true with $\hat{D} = D_{\square}$.

(VI) Now we verify that statement (d) implies the following statement:

(e) There is a discrete subset $\hat{D}_{\#}$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that the functions $\phi$ and $\psi$ are holomorphic in $\mathbb{C} \setminus ((\alpha, \infty) \cup \hat{D}_{\#})$ and that

$$
(I_{(n+1)q} - H_n^t H_n) \hat{R}_{T_{q,n}}(z) [I_{(n+1)q}, T_{q,n} H_n] [I_2 \otimes v_{q,n}] \tilde{\Theta}_{n,\alpha}(z) \left[ \frac{\phi(z)}{\psi(z)} \right] = 0 \quad (11.13)
$$

and

$$
(I_{(n+1)q} - H_n^t H_n) \hat{R}_{T_{q,n}}(z) [I_{(n+1)q}, R_{T_{q,n}}(\alpha)]^{-1} H_n \\
\times (I_2 \otimes v_{q,n}) \left[ \text{Rstr}_{\mathbb{C} \setminus [\alpha, \infty)} \tilde{\Theta}_{n,\alpha}(z) \right] \left[ (\hat{\Theta}(z) - \alpha) \phi(z) \right] \\
\times \left[ \hat{\Theta}_{n,\alpha}^{(2,1)}(z) \phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z) \psi(z) \right]^{-1} = 0 \quad (11.14)
$$

hold true for each $z \in \mathbb{C} \setminus ((\alpha, \infty) \cup \hat{D}_{\#})$.

Let us assume that (d) is fulfilled. Because of (I), we know that $\hat{D}_{\square} := D \cup \hat{D}$ is a discrete subset of $\mathbb{C} \setminus [\alpha, \infty)$. From (I) and (d) we see that (10.44), (11.11), and (11.12) are valid for each $\mathbb{C} \setminus ((\alpha, \infty) \cup \hat{D}_{\#})$, which implies (11.13) and (11.14) for each $z \in \mathbb{C} \setminus ((\alpha, \infty) \cup \hat{D}_{\#})$. Consequently, (e) holds true with $\hat{D}_{\#} = \hat{D}_{\square}$.

(VII) Now we show that (e) implies (d). Let (e) be fulfilled. Obviously, $\hat{D}_{\#} := \hat{D}_{\#} \cup D$ is a discrete subset of $\mathbb{C} \setminus [\alpha, \infty)$. Because of (I) and (e), we know that (10.44), (11.13), and (11.14)
are valid for each $z \in \mathbb{C} \setminus ([a, \infty) \cup \hat{D}_#)$. Consequently, (11.11) and (11.12) hold true for each $z \in \mathbb{C} \setminus ([a, \infty) \cup \hat{D}_#)$. Hence, (11) is fulfilled with $\hat{D} = \hat{D}_#$.

(VIII) Since $\hat{R}_{T_q,n}$ is the restriction of $R_{T_q,n}$ onto $\mathbb{C} \setminus [a, \infty)$, we see that (e) is equivalent to the following statement:

(f) There is a discrete subset $D'$ of $\mathbb{C} \setminus [a, \infty)$ such that $\phi$ and $\psi$ are holomorphic in $\mathbb{C} \setminus ([a, \infty) \cup D')$ and that

$$\left( I_{(n+1)q} - H_n^\dagger H_n \right) R_{T_q,n}(\alpha) [I_{(n+1)q}, H_n] (I_2 \otimes v_{q,n}) \Theta_{n,a}(z) \left[ \frac{\phi(z)}{\psi(z)} \right] = 0 \quad (11.15)$$

and

$$\left( I_{(n+1)q} - H_n^\dagger H_n \right) R_{T_q,n}(\alpha) [I_{(n+1)q}, [R_{T_q,n}(\alpha)]^{-1} H_n] \times (I_2 \otimes v_{q,n}) \Theta_{n,a}(z) [\text{diag}((z - \alpha) I_q, I_q)] \left[ \frac{\phi(z)}{\psi(z)} \right] = 0 \quad (11.16)$$

hold true for each $z \in \mathbb{C} \setminus ([a, \infty) \cup D')$.

(IX) Let $P_{n,a}$, $Q_{n,a}$, and $S_{n,a}$ be the matrix-valued functions defined (on $\mathbb{C}$) by (9.106), (9.107), and (9.108). According to Lemma 9.31(b), we see that $N := N_{\text{det} P_{n,a}} \cup N_{\text{det} Q_{n,a}} \cup N_{\text{det} S_{n,a}}$ is a finite and, in particular, discrete subset of $\mathbb{C}$.

(X) By virtue of (IX), we know that $N$ is a discrete subset of $\mathbb{C}$. We suppose now (f). Then $N' := N \cup D'$ is a discrete subset of $\mathbb{C}$, too. From Lemma 9.31(b) we see then that the following statement holds true:

(g) There is a discrete subset $D''$ of $\mathbb{C} \setminus [a, \infty)$ such that $\phi$ and $\psi$ are holomorphic in $\mathbb{C} \setminus ([a, \infty) \cup D'')$ and that

$$\left( I_{(n+1)q} - H_n^\dagger H_n \right) R_{T_q,n}(\alpha) v_{q,n} \phi(z) = 0 \quad (11.17)$$

and

$$\left( I_{(n+1)q} - H_n^\dagger H_n \right) R_{T_q,n}(\alpha) H_n v_{q,n} \psi(z) = 0 \quad (11.18)$$

are fulfilled for each $z \in \mathbb{C} \setminus ([a, \infty) \cup D'')$.

(XI) Conversely, now we suppose (g). We are going to prove (f). From (IX) we see that $N$ is a discrete subset of $\mathbb{C}$. Hence, $\hat{N} := N \cap (\mathbb{C} \setminus [a, \infty))$ and $D'_{\hat{c}} := D' \cup \hat{N}$ are discrete subsets of $\mathbb{C} \setminus [a, \infty)$. Because of (g), the functions $\phi$ and $\psi$ are holomorphic in $\mathbb{C} \setminus ([a, \infty) \cup D'_{\hat{c}})$ and (11.17) and (11.18) are valid for each $z \in \mathbb{C} \setminus ([a, \infty) \cup D'_{\hat{c}})$. Let us consider an arbitrary $z \in \mathbb{C} \setminus ([a, \infty) \cup D'_{\hat{c}})$. From (11.17) and (11.18) we get then that $x := \phi(z)$ and $y := \psi(z)$ fulfill (9.122) and (9.123). Consequently, Lemma 9.31 yields then that (9.120) and (9.121) hold true. Thus, we see that (11.15) and (11.16) are true. Hence, (e) is valid with $D' = D'_{\hat{c}}$.

(XII) In view of Notation 11.1 (g) and (h) are equivalent.

From (III)--(VIII) and (X)--(XII) we see that the statements (a) and (b) are equivalent. □

**Proposition 11.4.** Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{ \infty \}$, let $(s_j)_{j=0}^\kappa \in K_{q,r,a}^{2,\kappa}$, and let $n \in \mathbb{N}_0$ be such that $2n + 1 \leq \kappa$. Let (10.31) be the $q \times q$ block representation of $\hat{\Theta}_{n,a} := \text{Rstr}_{\mathbb{C} \setminus [a, \infty)} \Theta_{n,a}$.

Then:
(a) For each $[\phi_j] \in \mathcal{P}^{(q,q)}_{-J_q,\geq} [C \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1}]$, the function $\det(\hat{\Theta}_{n,\alpha}^{(1,1)} \phi + \hat{\Theta}_{n,\alpha}^{(2,2)} \psi)$ does not vanish identically in $C \setminus [\alpha, \infty)$ and

$$\hat{S}_{n,\alpha} := (\hat{\Theta}_{n,\alpha}^{(1,1)} \phi + \hat{\Theta}_{n,\alpha}^{(1,2)} \psi)(\hat{\Theta}_{n,\alpha}^{(2,1)} \phi + \hat{\Theta}_{n,\alpha}^{(2,2)} \psi)^{-1}$$  \hspace{1cm} (11.19)

belongs to the class $S_{0,q,\{\alpha,\infty\}}[(s_j)_{j=0}^{2n+1}, \leq]$.

(b) For each $S \in S_{0,q,\{\alpha,\infty\}}[(s_j)_{j=0}^{2n+1}, \leq]$, there exists a pair $[\phi_j \psi_j] \in \mathcal{P}^{(q,q)}_{-J_q,\geq} [C \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1}]$ consisting of two in $S$ belonging to the class $\mathbb{C}$ $q \times q$ matrix-valued functions $\phi$ and $\psi$ such that

$$\det(\hat{\Theta}_{n,\alpha}^{(2,1)} \phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)} \psi(z)) \neq 0$$  \hspace{1cm} (11.20)

and

$$S(z) = \left[\hat{\Theta}_{n,\alpha}^{(1,1)}(z) \phi(z) + \hat{\Theta}_{n,\alpha}^{(1,2)} \psi(z)\right]\left[\hat{\Theta}_{n,\alpha}^{(2,1)}(z) \phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)} \psi(z)\right]^{-1}$$  \hspace{1cm} (11.21)

hold true for each $z \in C \setminus [\alpha, \infty)$.

(c) Let $[\phi_1], [\phi_2] \in \mathcal{P}^{(q,q)}_{-J_q,\geq} [C \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1}]$. Then $\langle [\phi_1] \rangle = \langle [\phi_2] \rangle$ if and only if

$$\langle \hat{\Theta}_{n,\alpha}^{(1,1)} \phi_1 + \hat{\Theta}_{n,\alpha}^{(1,2)} \psi_1 \rangle \langle \hat{\Theta}_{n,\alpha}^{(2,1)} \phi_1 + \hat{\Theta}_{n,\alpha}^{(2,2)} \psi_1 \rangle^{-1} = \langle \hat{\Theta}_{n,\alpha}^{(1,1)} \phi_2 + \hat{\Theta}_{n,\alpha}^{(1,2)} \psi_2 \rangle \langle \hat{\Theta}_{n,\alpha}^{(2,1)} \phi_2 + \hat{\Theta}_{n,\alpha}^{(2,2)} \psi_2 \rangle^{-1}$$  \hspace{1cm} (11.22)

Proof. Since $(s_j)_{j=0}^{n}$ belongs to $\mathcal{K}_{q,e,\alpha}$, we have $s_j^\ast = s_j$ for all $j \in \mathbb{Z}_{\geq 0}$ and

$$\{H_n, H_{n+1}\} \subseteq C^{(n+1)q \times (n+1)q}_{\geq}.$$  \hspace{1cm} (11.23)

Remark 8.11 yields then 8.12. Remark 9.25 provides us $\hat{\Theta}_{n,\alpha} \in \mathcal{U}_{-J_q,\alpha}$. From Lemma 9.16 we get $\det(\hat{\Theta}_{n,\alpha}(z)) \neq 0$ and $\det(\Theta_{n,\alpha}(z)) \neq 0$ for each $z \in C$.

Let $[\phi_j] \in \mathcal{P}^{(q,q)}_{-J_q,\geq} [C \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1}]$. According to Notation 11.11 the equations (11.1) and (11.2) are fulfilled. Using Lemma 10.18 we see that there is a discrete subset $D$ of $C \setminus [\alpha, \infty)$ such that $\phi$ and $\psi$ are holomorphic in $C \setminus ([\alpha, \infty) \cup D)$ and that (10.44) holds true for each $z \in C \setminus (\mathbb{R} \cup D)$. Because of $\hat{\Theta}_{n,\alpha} \in \mathcal{U}_{-J_q,\alpha}$ and Proposition 10.16(c), the following three statements are valid:

(I) There is a discrete subset $D$ of $C \setminus [\alpha, \infty)$ such that $\hat{\Theta}_{n,\alpha}, \phi$, and $\psi$ are holomorphic in $C \setminus ([\alpha, \infty) \cup D)$.

(II) The matrix-valued function $\hat{S}_{n,\alpha}$ is holomorphic in $C \setminus ([\alpha, \infty) \cup D)$, and (10.44) as well as the representation

$$\hat{S}_{n,\alpha}(z) = \left[\hat{\Theta}_{n,\alpha}^{(1,1)}(z) \phi(z) + \hat{\Theta}_{n,\alpha}^{(1,2)} \psi(z)\right]\left[\hat{\Theta}_{n,\alpha}^{(2,1)}(z) \phi(z) + \hat{\Theta}_{n,\alpha}^{(2,2)} \psi(z)\right]^{-1}$$

of $S_{n,\alpha}$ are fulfilled for each $z \in C \setminus ([\alpha, \infty) \cup D)$.  

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(III) For each \( z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D}) \),

\[
\left[ \hat{S}_{n,\alpha}(z) \right]^* \Theta_{n,\alpha}^{-1}(z) \left( \frac{-\hat{J}_q}{23z} \right) \Theta_{n,\alpha}^{-1}(z) \left[ \hat{S}_{n,\alpha}(z) \right] \geq 0 \tag{11.24}
\]

and

\[
\left[ \hat{S}_{n,\alpha}(z) \right]^* \Theta_{n,\alpha}^{-1}(z) \left[ \text{diag}((z-\alpha)I_q, I_q) \right]^* \left( \frac{-\hat{J}_q}{23z} \right) \times \left[ \text{diag}((z-\alpha)I_q, I_q) \right] \Theta_{n,\alpha}^{-1}(z) \left[ \hat{S}_{n,\alpha}(z) \right] \geq 0. \tag{11.25}
\]

In view of (11.1) and (11.2), Lemma 11.3 provides us (11.4) and (11.5), where \( \hat{\alpha} \) and \( \hat{\beta} \) are.

\[
\hat{R}_{q,n}[I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n}) \left[ \hat{S}_{n,\alpha} \right] \left[ \hat{S}_{n,\alpha} \right]^* = H_n^\dagger H_n \hat{R}_{q,n}[I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n}) \left[ \hat{S}_{n,\alpha} \right] \left[ \hat{S}_{n,\alpha} \right]^* \tag{11.26}
\]

and

\[
\hat{R}_{q,n}[I_{(n+1)q}, [RT_{q,n}(\alpha)]^{-1}H_n](I_2 \otimes v_{q,n}) \left[ (\hat{\alpha} - \alpha)\hat{S}_{n,\alpha} \right] = H_n^\dagger H_n \alpha \otimes (RT_{q,n}(\alpha)]^{-1}H_n](I_2 \otimes v_{q,n}) \left[ (\hat{\alpha} - \alpha)\hat{S}_{n,\alpha} \right] \tag{11.27}
\]

Consequently, from (8.12) and (11) we get then

\[
\mathcal{R} \left( \hat{R}_{q,n}[z] | I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n}) \left[ \hat{S}_{n,\alpha}(z) \right] \right) \subseteq \mathcal{R}(H_n) \tag{11.28}
\]

and

\[
\mathcal{R} \left( \hat{R}_{q,n}[z] | I_{(n+1)q}, [RT_{q,n}(\alpha)]^{-1}H_n](I_2 \otimes v_{q,n}) \left[ (z - \alpha)\hat{S}_{n,\alpha}(z) \right] \right) \subseteq \mathcal{R}(H_n) \tag{11.29}
\]

for each \( z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D}) \). From Remark 11.3 we know that \( T_{q,n}H_n v_{q,n} = -u_n \). Therefore, for all \( z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D}) \), we get

\[
\hat{R}_{q,n}[z] | I_{(n+1)q}, T_{q,n}H_n](I_2 \otimes v_{q,n}) \left[ \hat{S}_{n,\alpha}(z) \right] = \hat{R}_{q,n}(z) \left[ v_{q,n,\alpha}(z) - u_n \right] \tag{11.30}
\]

and

\[
[RT_{q,n}(\alpha)]^{-1}H_n v_{q,n} = (I_{(n+1)q} - \alpha T_{q,n})H_n v_{q,n} = H_n v_{q,n} - \alpha T_{q,n}H_n v_{q,n} = y_0, u_n + \alpha u_n
\]
and, hence,

$$R_{T_{q,n}}(z) \left( I_{(n+1)q} [R_{T_{q,n}}(\alpha)]^{-1} H_n (I_2 \otimes v_{q,n}) \right) \left( z - \alpha \right) \tilde{S}_{n,\alpha}(z)$$

$$= R_{T_{q,n}}(z) \left( v_{q,n} \left[ (z - \alpha) \tilde{S}_{n,\alpha}(z) \right] + [R_{T_{q,n}}(\alpha)]^{-1} H_n v_{q,n} \right)$$

$$= R_{T_{q,n}}(z) \left( v_{q,n} \left[ (z - \alpha) \tilde{S}_{n,\alpha}(z) \right] - (-\alpha u_n - y_{0,n}) \right).$$

(11.29)

For all $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$, the equations (11.28) and (11.26) imply

$$\mathcal{R} \left( R_{T_{q,n}}(z) \left[ v_{q,n} \tilde{S}_{n,\alpha}(z) - u_n \right] \right) = \mathcal{R} \left( \tilde{R}_{T_{q,n}}(z) \left[ v_{q,n} \tilde{S}_{n,\alpha}(z) - u_n \right] \right) \subseteq \mathcal{R}(H_n)$$

(11.30)

and, in view of (11.29) and (11.27), furthermore,

$$\mathcal{R} \left( R_{T_{q,n}}(z) \left( v_{q,n} \left[ (z - \alpha) \tilde{S}_{n,\alpha}(z) \right] - (-\alpha u_n - y_{0,n}) \right) \right)$$

$$\mathcal{R} \left( \tilde{R}_{T_{q,n}}(z) \left( v_{q,n} \left[ (z - \alpha) \tilde{S}_{n,\alpha}(z) \right] - (-\alpha u_n - y_{0,n}) \right) \right) \subseteq \mathcal{R}(H_{\alpha,u}).$$

(11.31)

Lemma [9.27] shows that, for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$, the matrix $\Sigma_{2n}^\alpha(z)$ given by (4.6) admits the representation

$$\Sigma_{2n}^\alpha(z) = \left[ \tilde{S}_{n,\alpha}(z) \right] \hat{\Theta}_{n,\alpha}^{-1}(z) \left( \frac{-\tilde{f}_q}{23z} \right) \hat{\Theta}_{n,\alpha}(z) \left[ \tilde{S}_{n,\alpha}(z) \right].$$

(11.32)

In view of $\det \Theta_{n,\alpha}(z) \neq 0$, for each $z \in \mathbb{C} \setminus [\alpha, \infty)$, we have

$$\left[ \text{diag}((z - \alpha) I_q, I_q) \right] \hat{\Theta}_{n,\alpha}(z) \left[ \text{diag}((z - \alpha)^{-1} I_q, I_q) \right]^{-1}$$

$$= \left[ \text{diag}((z - \alpha) I_q, I_q) \right] \hat{\Theta}_{n,\alpha}^{-1}(z) \left[ \text{diag}((z - \alpha)^{-1} I_q, I_q) \right].$$

(11.33)

Taking into account Lemma [9.27], we see that, for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$, the matrix $\Sigma_{2n+1}^\alpha(z)$ given by (4.7) can be represented by

$$\Sigma_{2n+1}^\alpha(z) = \left[ \tilde{S}_{n,\alpha}(z) \right] \hat{\Theta}_{n,\alpha}^{-1}(z) \left( \frac{-\tilde{f}_q}{23z} \right) \hat{\Theta}_{n,\alpha}(z) \left[ \tilde{S}_{n,\alpha}(z) \right].$$

(11.34)

For each $z \in \mathbb{C} \setminus [\alpha, \infty)$, Lemma [9.13] and (11.33) yield

$$\hat{\Theta}_{n,\alpha}(z) \left[ (z - \alpha) \tilde{S}_{n,\alpha}(z) \right] = \left[ \text{diag}((z - \alpha) I_q, I_q) \right] \hat{\Theta}_{n,\alpha}(z) \left[ \tilde{S}_{n,\alpha}(z) \right],$$

which, because of (11.34), implies

$$\Sigma_{2n+1}^\alpha(z) = \left[ \tilde{S}_{n,\alpha}(z) \right] \hat{\Theta}_{n,\alpha}^{-1}(z) \left[ \text{diag}((z - \alpha) I_q, I_q) \right] \left( \frac{-\tilde{f}_q}{23z} \right)$$

$$\times \left[ \text{diag}((z - \alpha) I_q, I_q) \right] \hat{\Theta}_{n,\alpha}^{-1}(z) \left[ \tilde{S}_{n,\alpha}(z) \right].$$

(11.35)
for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$. From (11.32), (11.24), (11.35), and (11.25) it follows
\[
\left\{ \Sigma^{[\hat{S}_{n,\alpha}]}_{2n}(z), \Sigma^{[\hat{S}_{n,\alpha}]}_{2n+1}(z) \right\} \subseteq \mathbb{C}^{q \times q}
\] (11.36)
for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$. Thus, for all $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$, by virtue of \(13.7, 11.22, 11.30, 11.31, 11.36, \) and Remark 4.6, we conclude then
\[
\left\{ P^{[\hat{S}_{n,\alpha}]}_{2n}(z), P^{[\hat{S}_{n,\alpha}]}_{2n+1}(z) \right\} \subseteq \mathbb{C}^{(n+2)q \times (n+2)q}. 
\]
Obviously, $\mathcal{D} := \mathcal{D} \cap \Pi_+$ is a discrete subset of $\Pi_+$ and $f_{n,\alpha} := \text{Rstr}_{\Pi_+ \setminus \mathcal{D}} \hat{S}_{n,\alpha}$ is holomorphic in $\Pi_+ \setminus \mathcal{D}$. For each $z \in \Pi_+ \setminus \mathcal{D}$, then $\left\{ P^{[f_{n,\alpha}]}_{2n}(z), P^{[f_{n,\alpha}]}_{2n+1}(z) \right\} \subseteq \mathbb{C}^{(n+2)q \times (n+2)q}$. Thus, Theorem 6.5 provides us that there is a unique $S \in \mathcal{S}_{0,\alpha}((s_j)_{j=0}^{2n+1}, \leq]$ such that $\text{Rstr}_{\Pi_+ \setminus \mathcal{D}} S = f_{n,\alpha}$. Consequently, for each $z \in \Pi_+ \setminus \mathcal{D}$, we have $S(z) = f_{n,\alpha}(z) = \hat{S}_{n,\alpha}(z)$. Since $S$ is holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$, we get $S = \hat{S}_{n,\alpha}$.

Now we consider an arbitrary $S \in \mathcal{S}_{0,\alpha}((s_j)_{j=0}^{2n+1}, \leq]$. Then Proposition 4.11 yields $\{ P^{[S]}_{2n}(z), P^{[S]}_{2n+1}(z) \} \subseteq \mathbb{C}^{(n+2)q \times (n+2)q}$ for each $z \in \mathbb{C} \setminus \mathbb{R}$. Consequently, Remark 4.6 shows that (11.23) is valid and that, for each $z \in \mathbb{C} \setminus \mathbb{R}$, the following statements hold true:

(i) The inclusions $\mathcal{R}(R_{T_{\alpha,n}}(z)[v_{q,n}S(z) - u_n]) \subseteq \mathcal{R}(H_n)$ and
\[
\mathcal{R}(R_{T_{\alpha,n}}(z)(v_{q,n}((z - \alpha)S(z)) - (-\alpha u_n - y_0,n))) \subseteq \mathcal{R}(H_{\alpha,n})
\]
are valid.

(ii) The matrices $\Sigma^{[S]}_{2n}(z)$ and $\Sigma^{[S]}_{2n+1}(z)$ are both non-negative Hermitian.

For each $z \in \mathbb{C} \setminus [\alpha, \infty)$, from Remark 4.4 we get
\[
R_{T_{\alpha,n}}(z)[v_{q,n}S(z) - u_n] = \hat{R}_{T_{\alpha,n}}(z)[v_{q,n}, -u_n] \left[ \begin{array}{c} S(z) \\ I_q \end{array} \right] 
= \hat{R}_{T_{\alpha,n}}(z)[I_{(n+1)q}, T_{\alpha,n}H_n](I_2 \otimes v_{q,n}) \left[ \begin{array}{c} S(z) \\ I_q \end{array} \right]
\] (11.37)
and, because of the second equation in (7.3), furthermore
\[
R_{T_{\alpha,n}}(z)(v_{q,n}((z - \alpha)S(z)) - (-\alpha u_n - y_0,n)) \\
= R_{T_{\alpha,n}}(z)(v_{q,n}((z - \alpha)S(z))) + [R_{T_{\alpha,n}}(\alpha)]^{-1} H_n v_{q,n}
\]
\[
= \hat{R}_{T_{\alpha,n}}(z)[I_{(n+1)q}, [R_{T_{\alpha,n}}(\alpha)]^{-1} H_n](I_2 \otimes v_{q,n}) \left[ \begin{array}{c} (z - \alpha)S(z) \\ I_q \end{array} \right].
\] (11.38)
Using (i) and Lemmata 9.16 and 9.27, we see that
\[
\left[ \begin{array}{c} S(z) \\ I_q \end{array} \right] * \hat{\Theta}_{n,\alpha}(z) \left( \begin{array}{c} -\frac{i}{2} \\ 23z \end{array} \right) \hat{\Theta}_{n,\alpha}(z) \left[ \begin{array}{c} S(z) \\ I_q \end{array} \right] \geq 0
\] (11.39)
and
\[
\left[ \begin{array}{c} (z - \alpha)S(z) \\ I_q \end{array} \right] * \hat{\Theta}_{n,\alpha}(z) \left( \begin{array}{c} -\frac{i}{2} \\ 23z \end{array} \right) \hat{\Theta}_{n,\alpha}(z) \left[ \begin{array}{c} (z - \alpha)S(z) \\ I_q \end{array} \right] \geq 0
\] (11.40)
Applying again the identity theorem for holomorphic functions, it follows
\begin{equation}
\hat{\Theta}^{-1}_{n,\alpha}(z) \left[ (z - \alpha) S(z) \right] = \left[ \text{diag}((z - \alpha) I_q, I_q) \right] \hat{\Theta}^{-1}_{n,\alpha}(z) \left[ S(z) \right]
\end{equation}
for each \( z \in \mathbb{C} \setminus \mathbb{R} \). Consequently, from (11.40) it follows
\begin{equation}
\left[ S(z) \right]^* \hat{\Theta}^{-*}_{n,\alpha}(z) \left[ \text{diag}((z - \alpha) I_q, I_q) \right]^* \left[ \frac{-j_q}{2} z \right] \left[ \text{diag}((z - \alpha) I_q, I_q) \right] \hat{\Theta}^{-1}_{n,\alpha}(z) \left[ S(z) \right] \geq 0 \tag{11.41}
\end{equation}
for all \( z \in \mathbb{C} \setminus \mathbb{R} \). Since Lemma 9.16 shows that \( \det \hat{\Theta}_{n,\alpha}(z) \neq 0 \) holds true for each \( z \in \mathbb{C} \setminus [\alpha, \infty) \), we get from \( \hat{\Theta}_{n,\alpha} \in \mathbb{M}_{-J_q,\alpha} \) (10.31), Lemma 9.12 (11.39), (11.41), and Proposition 10.16 (with \( D = \emptyset \)) that there is a pair \( [\phi, \psi] \in \mathcal{D}^q_{\mathbb{R}^+(\alpha)}(\mathbb{C} \setminus [\alpha, \infty)) \) of in \( \mathbb{C} \setminus [\alpha, \infty) \) holomorphic matrix-valued functions \( \phi \) and \( \psi \) such that (11.20) and (11.21) hold true for each \( z \in \mathbb{C} \setminus [\alpha, \infty) \). Because of (11.37) and (i) we have
\begin{equation}
\mathcal{R} \left( \tilde{R}_{T_{q,n}}(z) \left[ I_{(n+1)q}, T_{q,n} H_n \right] (I_2 \otimes v_{q,n}) \left[ S(z) \right] \right) \subseteq \mathcal{R}(H_n)
\end{equation}
for each \( z \in \mathbb{C} \setminus \mathbb{R} \). Consequently, from Lemma A.2 and (8.12) it follows
\begin{equation}
(I_{(n+1)q} - H_n^1 H_n) \tilde{R}_{T_{q,n}}(z) \left[ I_{(n+1)q}, T_{q,n} H_n \right] (I_2 \otimes v_{q,n}) \left[ S(z) \right] = 0
\end{equation}
for each \( z \in \mathbb{C} \setminus \mathbb{R} \). Hence, the identity theorem for holomorphic functions yields
\begin{equation}
(I_{(n+1)q} - H_n^1 H_n) \tilde{R}_{T_{q,n}}[I_{(n+1)q}, T_{q,n} H_n] (I_2 \otimes v_{q,n}) \left[ S(z) \right] = 0. \tag{11.42}
\end{equation}
Because of (11.38) and (i) we obtain
\begin{equation}
\mathcal{R} \left( \tilde{R}_{T_{q,n}}(z) \left[ I_{(n+1)q}, [RT_{q,n}(\alpha)]^{-1} H_n \right] (I_2 \otimes v_{q,n}) \left[ (z - \alpha) S(z) \right] \right) \subseteq \mathcal{R}(H_{\alpha \cap n})
\end{equation}
for each \( z \in \mathbb{C} \setminus \mathbb{R} \). Thus, for each \( z \in \mathbb{C} \setminus \mathbb{R} \), Lemma A.2 and (8.12) imply
\begin{equation}
(I_{(n+1)q} - H_n^1 H_n) \tilde{R}_{T_{q,n}}(z) \left[ I_{(n+1)q}, [RT_{q,n}(\alpha)]^{-1} H_n \right] (I_2 \otimes v_{q,n}) \left[ (z - \alpha) S(z) \right] = 0.
\end{equation}
Applying again the identity theorem for holomorphic functions, it follows
\begin{equation}
(I_{(n+1)q} - H_n^1 H_n) \tilde{R}_{T_{q,n}}(z) \left[ I_{(n+1)q}, [RT_{q,n}(\alpha)]^{-1} H_n \right] (I_2 \otimes v_{q,n}) \left[ (z - \alpha) S(z) \right] = 0 \tag{11.43}
\end{equation}
for all \( z \in \mathbb{C} \setminus \mathbb{R} \). In view of (10.31), (11.20), (11.21), (11.32), and (11.43), then Lemma 11.3 shows that \( [\phi, \psi] \) belongs to \( \mathcal{D}^q_{\mathbb{R}^+(\alpha)}(\mathbb{C} \setminus [\alpha, \infty)), (s_j)_{j=0}^{2n+1} \].

(\( \Box \)) In view of part (ii), we know that, for each \( k \in \{1, 2\} \), the function \( \det(\hat{\Theta}^{(2,1)}_{n,\alpha} \phi_k + \hat{\Theta}^{(2,2)}_{n,\alpha} \psi_k) \) does not vanish identically in \( \mathbb{C} \setminus [\alpha, \infty) \). Because of \( \hat{\Theta}_{n,\alpha} \in \mathbb{M}_{-J_q,\alpha} \), the application of Proposition 10.16 (4) provides us the asserted equivalence. \( \Box \)
12. Parametrization of the solution set of the truncated matricial Stieltjes moment problem in the non-degenerate and degenerate cases

In this section, we state a parametrization of the solution set of the matricial truncated Stieltjes moment problem \( S[[\alpha, \infty); (s_j)_{j=0}^{2n+1}, \preceq] \) in the non-degenerate and degenerate cases. First we recall that, in view of Theorems 2.4 and 2.6, one can suppose that the given sequence \( (s_j)_{j=0}^{2n+1} \) of complex \( q \times q \) matrices belongs to the set \( K_{q,2n+1,\alpha}^e \).

Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N} \cup \{ \infty \} \), let \( (s_j)_{j=0}^{\infty} \in K_{q,\kappa,\alpha}^{e,\infty} \), and let \( n \) be a non-negative integer with \( 2n + 1 \leq \kappa \). According to Lemma 9.32, the non-negative integers

\[
m := \text{rank} \left[ (I_{(n+1)q} - H_n^\dagger H_n)RT_{T_\kappa,n}(\alpha)v_{q,n} \right]
\]

and

\[
\ell := \text{rank} \left[ (I_{(n+1)q} - H_{\alpha\infty,n}^\dagger H_{\alpha\infty,n})H_nv_{q,n} \right]
\]

fulfill \( m + \ell \leq q \). In particular, \( 0 \leq m \leq q \) and \( 0 \leq \ell \leq q \). We consider separately the following three cases:

(I) \( m + \ell = 0 \), i.e., \( m = 0 \) and \( \ell = 0 \).

(II) \( 1 \leq m + \ell \leq q - 1 \).

(III) \( m + \ell = q \).

Throughout this section, let \( \Theta_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q \times 2q} \) be defined by (9.38), let \( \hat{\Theta}_{n,\alpha} := R\text{str}_{\mathbb{C}[[\alpha, \infty]]} \Theta_{n,\alpha} \), and let \( (10.31) \) be the \( q \times q \) block partition of \( \hat{\Theta}_{n,\alpha} \).

12.1. The non-degenerate case

First we study the so-called non-degenerate case (I) i.e., we consider the situation that \( (I_{(n+1)q} - H_n^\dagger H_n)RT_{T_\kappa,n}(\alpha)v_{q,n} = 0 \) and \( (I_{(n+1)q} - H_{\alpha\infty,n}^\dagger H_{\alpha\infty,n})H_nv_{q,n} = 0 \).

Remark 12.1. Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N} \cup \{ \infty \} \), and let \( (s_j)_{j=0}^{\infty} \in K_{q,\kappa,\alpha}^{e,\infty} \). Let \( n \in \mathbb{N}_0 \) be such that \( 2n + 1 \leq \kappa \) and let \( m \) and \( \ell \) be given by (12.1) and (12.2), respectively. If \( m = 0 \) and \( \ell = 0 \), then Lemma 9.32 and Notation 11.1 show that \( U_{\kappa,\alpha} = \{ 0_{q \times 1} \} \), \( V_{\kappa,\alpha} = \{ 0_{q \times 1} \} \), and \( \mathcal{P}_{J_q \geq \kappa}[[\alpha, \infty), (s_j)_{j=0}^{2n+1}] = \mathcal{P}_{J_q \geq \kappa}((\alpha, \infty)) \) hold true.

Thus, in the non-degenerate case (I) we get immediately a parametrization of the set \( S_{0, \kappa; [\alpha, \infty)}[(s_j)_{j=0}^{2n+1}, \preceq] \):

Theorem 12.2. Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N} \cup \{ \infty \} \), let \( (s_j)_{j=0}^{\infty} \in K_{q,\kappa,\alpha}^{e,\infty} \), and let \( n \in \mathbb{N}_0 \) be such that \( 2n + 1 \leq \kappa \). Suppose that \( m \) and \( \ell \) are given by (12.1) and (12.2) fulfill \( m = 0 \) and \( \ell = 0 \). Then the following statements hold true:

(a) For each pair \( (\phi, \psi) \in \mathcal{P}_{J_q \geq \kappa}((\alpha, \infty)) \), the meromorphic function \( \det(\hat{\Theta}_{n,\alpha}^{(1,2)} \phi + \hat{\Theta}_{n,\alpha}^{(2,1)} \psi) \) does not vanish identically in \( \mathbb{C} \setminus [\alpha, \infty) \) and the matrix-valued function \( \hat{S}_{n,\alpha} := (\hat{\Theta}_{n,\alpha}^{(1,1)} \phi + \hat{\Theta}_{n,\alpha}^{(2,2)} \psi)^{-1} \) belongs to \( S_{0, q; [\alpha, \infty)}[(s_j)_{j=0}^{2n+1}, \preceq] \).
(b) For each $S \in S_{0,q;[\alpha,\infty]}(s_j)_{j=0}^{2n+1} \leq$, there is a pair $[\phi^\psi] \in P_{-J_{q,p}}^{(q,q)}(C \setminus [\alpha,\infty))$ of $q \times q$ matrix-valued functions $\phi$ and $\psi$ which are holomorphic in $C \setminus [\alpha,\infty)$ such that $[11.20]$ and $[11.21]$ hold true for each $z \in C \setminus [\alpha,\infty)$.

(c) Let $[\phi^1_{\psi_1}], [\phi^2_{\psi_2}] \in P_{-J_{q,p}}^{(q,q)}(C \setminus [\alpha,\infty))$. Then $[[\phi^1_{\psi_1}]] = [[\phi^2_{\psi_2}]]$ if and only if $[11.22]$ holds true.

Proof. Apply Proposition [11.4] and Remark [12.1].

12.2. The degenerate, but not completely degenerate case

Now we turn our attention to case $[11]$. Let $V$ and $W$ be complex $q \times q$ matrices with $W^*V = I_q$. Then it is readily checked that the following statements hold true:

(a) If $[\phi^\psi] \in P_{-J_{q,p}}^{(q,q)}(C \setminus [\alpha,\infty))$, then the pair $[\phi^\psi, \psi^\psi]$ given by $\phi^\psi := V\phi$ and $\psi^\psi := W\psi$ belongs to $P_{-J_{q,p}}^{(q,q)}(C \setminus [\alpha,\infty))$.

(b) Let $[\phi^1_{\psi_1}], [\phi^2_{\psi_2}] \in P_{-J_{q,p}}^{(q,q)}(C \setminus [\alpha,\infty))$ and let $\phi^\psi_1 := V\phi_1$, $\psi^\psi_1 := W\psi_1$, $\phi^\psi_2 := V\phi_2$, and $\psi^\psi_2 := W\psi_2$. Then $[[\phi^1_{\psi_1}]] = [[\phi^2_{\psi_2}]]$ if and only if $[[\phi^1_{\psi_1}]] = [[\phi^2_{\psi_2}]]$.

Lemma 12.4. Let $\alpha \in \mathbb{R}$ and let $r \in \mathbb{N}$ be such that $r < q$. Let $U$ and $V$ be complex $(q-r) \times (q-r)$ matrices with rank $[U] = q - r$ and $V^*U = 0_{(q-r)\times(q-r)}$. Let $U$ (resp. $V$) be the constant matrix-valued function (defined on $C \setminus [\alpha,\infty)$) with value $U$ (resp. $V$). Then:

(a) If $[\phi^\psi] \in P_{-J_{q,p}}^{(r,r)}(C \setminus [\alpha,\infty))$, then the pair $[\phi^\psi, \psi^\psi]$ given by $\phi^\psi := \text{diag}(\phi, U)$ and $\psi^\psi := \text{diag}(\psi, V) \in P_{-J_{q,p}}^{(q,q)}(C \setminus [\alpha,\infty))$.

(b) Let $[\phi^1_{\psi_1}], [\phi^2_{\psi_2}] \in P_{-J_{q,p}}^{(r,r)}(C \setminus [\alpha,\infty))$. For each $k \in \{1, 2\}$, let $\phi^\psi_k := \text{diag}(\phi_k, U)$ and $\psi^\psi_k := \text{diag}(\psi_k, V)$. Then $[[\phi^1_{\psi_1}]] = [[\phi^2_{\psi_2}]]$ if and only if $[[\phi^1_{\psi_1}]] = [[\phi^2_{\psi_2}]]$.

The proof of Lemma 12.4 is straightforward. We omit the details.

In the following, we will use again $P_{\mathcal{U}}$ to denote the complex $q \times q$ matrix which represents the orthogonal projection onto a given subspace $\mathcal{U}$ of $C^q$ with respect to the standard basis of $C^q$, i.e., for each subspace $\mathcal{U}$ of $C^q$, the matrix $P_{\mathcal{U}}$ is the unique complex $q \times q$ matrix $P$ which fulfills the three conditions $P^2 = P$, $P^* = P$, and $\mathcal{R}(P) = \mathcal{U}$.

Lemma 12.5. Let $m$ and $\ell$ be non-negative integers such that

$$r := q - (m + \ell) \quad (12.3)$$

fulfills $1 \leq r \leq q - 1$. Let $\mathcal{U}$ and $\mathcal{V}$ be orthogonal subspaces of $C^q$ with dim $\mathcal{U} = m$ and dim $\mathcal{V} = \ell$. Then:
(a) There exists a unitary complex $q \times q$ matrix $W$ such that
\[
W^* P_d W = \begin{cases} 
\text{diag}(0_{r \times r}, I_m, 0_{\ell \times \ell}), & \text{if } m \geq 1 \text{ and } \ell \geq 1 \\
\text{diag}(0_{r \times r}, I_m), & \text{if } m \geq 1 \text{ and } \ell = 0
\end{cases} \quad (12.4)
\]
and
\[
W^* P_v W = \begin{cases} 
\text{diag}(0_{r \times r}, 0_{m \times m}, I_\ell), & \text{if } m \geq 1 \text{ and } \ell \geq 1 \\
\text{diag}(0_{r \times r}, I_\ell), & \text{if } m = 0 \text{ and } \ell \geq 1
\end{cases} \quad (12.5)
\]

(b) Let $\alpha \in \mathbb{R}$ and let $W$ be a unitary complex $q \times q$ matrix such that (12.4) and (12.5) are fulfilled.

(b1) If $[\frac{\phi}{\psi}] \in \mathcal{P}_{-J_q \geq, (C \setminus [\alpha, \infty))}$ is such that
\[
P_d \tilde{\phi} = 0_{q \times q} \quad \text{and} \quad P_v \tilde{\psi} = 0_{q \times q}, \quad (12.6)
\]
then there exists a pair $[\frac{\phi}{\psi}] \in \mathcal{P}_{-J_q \geq, (C \setminus [\alpha, \infty))}$ such that $\phi$ and $\psi$ and the functions
\[
\phi^\Box := \begin{cases} 
W \cdot \text{diag}(\phi, 0_{m \times m}, I_\ell), & \text{if } m \geq 1 \text{ and } \ell \geq 1 \\
W \cdot \text{diag}(\phi, 0_{m \times m}), & \text{if } m \geq 1 \text{ and } \ell = 0 \\
W \cdot \text{diag}(\phi, I_\ell), & \text{if } m = 0 \text{ and } \ell \geq 1
\end{cases} \quad (12.7)
\]
and
\[
\psi^\Box := \begin{cases} 
W \cdot \text{diag}(\psi, I_m, 0_{\ell \times \ell}), & \text{if } m \geq 1 \text{ and } \ell \geq 1 \\
W \cdot \text{diag}(\psi, I_m), & \text{if } m \geq 1 \text{ and } \ell = 0 \\
W \cdot \text{diag}(\psi, 0_{\ell \times \ell}), & \text{if } m = 0 \text{ and } \ell \geq 1
\end{cases} \quad (12.8)
\]
fulfill the following three conditions:

(i) $\phi$, $\psi$, $\phi^\Box$, and $\psi^\Box$ are holomorphic in $\Pi_+$.  
(ii) $[\frac{\phi^\Box}{\psi^\Box}] \in \mathcal{P}_{-J_q \geq, (C \setminus [\alpha, \infty))}$.  
(iii) $\langle [\frac{\phi}{\psi}] \rangle = \langle [\frac{\phi^\Box}{\psi^\Box}] \rangle$.

(b2) For each pair $[\frac{\phi}{\psi}] \in \mathcal{P}_{-J_q \geq, (C \setminus [\alpha, \infty))}$, the functions $\phi^\Box$ and $\psi^\Box$ given by (12.7) and (12.8) fulfill (ii).

(b3) Let $[\frac{\phi}{\psi}] \in \mathcal{P}_{-J_q \geq, (C \setminus [\alpha, \infty))}$. Let $\phi^\Box$ and $\psi^\Box$ be defined by (12.7) and (12.8). Then every pair $[\frac{\phi}{\psi}] \in \mathcal{P}_{-J_q \geq, (C \setminus [\alpha, \infty))}$ for which (iii) holds true fulfills necessarily (12.6).

Lemma (12.6) is substantially proved in [53, Lemma 5.2, p. 459/460]. (A detailed proof for the case that $m \geq 1$ and $\ell \geq 1$ is also given in [53, Lemma 11.7].)

Lemma 12.6. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^\geq$, and let $n \in \mathbb{N}_0$ be such that $2n + 1 \leq \kappa$. Let $m$, $\ell$, and $r$ be given by (12.1), (12.2), and (12.3). Suppose $r \geq 1$. Let $\mathcal{U}_{n, \alpha}$ and $\mathcal{V}_{n, \alpha}$ be given by (12.12) and (12.128). Then:
(a) There exists a unitary complex $q \times q$ matrix $W$ such that
\[
W^* P_{U_{n,a}} W = \begin{cases}
\text{diag}(0, \ldots, 0_{x\times \ell}), & \text{if } m \geq 1 \text{ and } \ell \geq 1 \\
\text{diag}(0_{x\times \ell}, I_m), & \text{if } m \geq 1 \text{ and } \ell = 0
\end{cases}
\] (12.9)
and
\[
W^* P_{V_{n,a}} W = \begin{cases}
\text{diag}(0_{x\times \ell}, 0_{m \times m}, I_\ell), & \text{if } m \geq 1 \text{ and } \ell \geq 1 \\
\text{diag}(0_{x\times \ell}, I_\ell), & \text{if } m = 0 \text{ and } \ell \geq 1.
\end{cases}
\] (12.10)

(b) Let $W$ be a unitary complex $q \times q$ matrix such that (12.9) and (12.10) are valid.

(b1) Let $[\hat{\phi}_{\psi}] \in \mathcal{P}^{(q,r)}_{-J_\rho,\geq}(\mathbb{C} \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1})$. Then there exists a pair $[\hat{\phi}_{\psi}] \in \mathcal{P}^{(q,r)}_{-J_\rho,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ such that the conditions (i)–(iii) of Lemma 12.5 hold true with $\phi^\square$ and $\psi^\square$ given by (12.7) and (12.8).

(b2) If $[\hat{\phi}_{\psi}] \in \mathcal{P}^{(r,r)}_{-J_\rho,\geq}(\mathbb{C} \setminus [\alpha, \infty))$, then $\phi^\square$ and $\psi^\square$ be given by (12.7) and (12.8) fulfill condition (ii) of Lemma 12.5.

(b3) Let $[\hat{\phi}_{\psi}] \in \mathcal{P}^{(r,r)}_{-J_\rho,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ and let $\phi^\square$ and $\psi^\square$ be given by (12.7) and (12.8). If $[\hat{\phi}_{\psi}] \in \mathcal{P}^{(q,q)}_{-J_\rho,\geq}(\mathbb{C} \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1})$ fulfills condition (iii) of Lemma 12.5, then $[\hat{\phi}_{\psi}]$ belongs to $\mathcal{P}^{(q,q)}_{-J_\rho,\geq}(\mathbb{C} \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1})$.

Proof. Let us consider the case that $\ell \geq 1$ and $m \geq 1$ hold true. (If $m = 0$ or if $\ell = 0$, then the assertions can be checked analogously, so that we omit the details of the proof in these cases.) In view of (9.127), (9.128), Lemma 9.32 (12.1), and (12.2), we obtain that $U_{n,a}$ and $V_{n,a}$ are orthogonal subspaces of $\mathbb{C}^q$ with $\dim U_{n,a} = m \geq 1$ and $\dim V_{n,a} = \ell \geq 1$. Taking into account that $r = q - (m + \ell) \leq q - 2$, we see that we can apply Lemma 12.5 with $\mathcal{U} = U_{n,a}$ and $\mathcal{V} = V_{n,a}$.

(a1) Use Lemma 12.5(a).

(a2) Let $[\hat{\phi}_{\psi}] \in \mathcal{P}^{(q,q)}_{-J_\rho,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ be such that (11.1) and (11.2) hold true. Because of (11.1), (9.127), and Lemma 9.32 we have $P_{U_{n,a}} \hat{\phi} = 0$. Using (11.2), (9.128), and Lemma 9.32 we get $P_{V_{n,a}} \hat{\psi} = 0$. In view of Lemma 12.5(b) part (b1) is proved.

(a2) Apply Lemma 12.5(b2).

(b2) Use Lemma 12.5(b3) and Lemma 9.32.

Now we obtain a parametrization of the solution set of the matricial truncated Stieltjes moment problem in the so-called degenerate, but not completely degenerate case.

**Theorem 12.7.** Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, let $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{\varphi,\psi}^{\kappa}$, and let $n \in \mathbb{N}_0$ be such that $2n + 1 \leq \kappa$. Let the integers $m, \ell, r$ be given by (12.1), (12.2), and (12.3). Suppose $r \geq 1$. Let $U_{n,a}$ and $V_{n,a}$ be the subspaces of $\mathbb{C}^q$ which are defined in (9.127) and (9.128). Let $W$ be a unitary complex $q \times q$ matrix such that (12.9) and (12.10) hold true. Then:

(a) Let $[\hat{\phi}_{\psi}] \in \mathcal{P}^{(r,r)}_{-J_\rho,\geq}(\mathbb{C} \setminus [\alpha, \infty))$ and let $\phi^\square$ and $\psi^\square$ be defined by (12.7) and (12.8). Then the function $\det(\Theta_{n,a}^{(1,1)} \phi^\square + \Theta_{n,a}^{(2,2)} \psi^\square)$ does not vanish identically and

\[S := (\Theta_{n,a}^{(1,1)} \phi^\square + \Theta_{n,a}^{(2,2)} \psi^\square)(\Theta_{n,a}^{(2,1)} \phi^\square + \Theta_{n,a}^{(2,2)} \psi^\square)^{-1}
\]

belongs to the class $S_{(\kappa, q, [\alpha, \infty))}[(s_j)_{j=0}^{2n+1}; \leq]$. 

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(b) For each $S \in S_{0,q;\{\alpha,\infty\}}((s_j)_{j=0}^{2n+1}, \leq)$, there exists a pair $[\phi, \psi] \in \mathcal{P}(r,r)_{-J \geq}(\mathbb{C} \setminus [\alpha, \infty))$ such that the function $\det(\hat{\Theta}_{n,\alpha} \phi \overline{\phi} + \hat{\Theta}_{n,\alpha} \psi \overline{\psi})$ does not vanish identically and that $S$ admits the representation

$$S = (\hat{\Theta}_{n,\alpha} \phi \overline{\phi} + \hat{\Theta}_{n,\alpha} \psi \overline{\psi})(\hat{\Theta}_{n,\alpha} \phi \overline{\phi} + \hat{\Theta}_{n,\alpha} \psi \overline{\psi})^{-1} \quad (12.11)$$

where $\phi \overline{\phi}$ and $\psi \overline{\psi}$ are given by (12.7) and (12.8).

(c) Let $[\phi_1, \psi_1, [\phi_2, \psi_2] \in \mathcal{P}(r,r)_{-J \geq}(\mathbb{C} \setminus [\alpha, \infty))$. For each $k \in \{1, 2\}$, let $\phi_k$ be defined as in (12.7) where $\phi$ is replaced by $\phi_k$ and let $\psi_k$ be defined as in (12.8) where $\psi$ is replaced by $\psi_k$. Then the following statements are equivalent:

(i) $\langle [\phi_1, \psi_1] \rangle = \langle [\phi_2, \psi_2] \rangle$.

(ii) $\left( \hat{\Theta}_{n,\alpha} \phi_1 \overline{\phi_1} + \hat{\Theta}_{n,\alpha} \psi_1 \overline{\psi_1} \right) \left( \hat{\Theta}_{n,\alpha} \phi_1 \overline{\phi_1} + \hat{\Theta}_{n,\alpha} \psi_1 \overline{\psi_1} \right)^{-1}$

$$= \left( \hat{\Theta}_{n,\alpha} \phi_2 \overline{\phi_2} + \hat{\Theta}_{n,\alpha} \psi_2 \overline{\psi_2} \right) \left( \hat{\Theta}_{n,\alpha} \phi_2 \overline{\phi_2} + \hat{\Theta}_{n,\alpha} \psi_2 \overline{\psi_2} \right)^{-1}.$$

Proof. Let us consider the case that $m \geq 1$ and $\ell \geq 1$ hold true. (If $m = 0$ or if $\ell = 0$, then the assertions can be proved analogously.)

(a) Let $[\phi, \psi] \in \mathcal{P}(r,r)_{-J \geq}(\mathbb{C} \setminus [\alpha, \infty))$. Parts (b3) and (b2) of Lemma 12.6 and Notation 11.4 show that $[\phi \overline{\phi}, \psi \overline{\psi}]$ belongs to $\mathcal{P}(q,q)_{-J \geq}(\mathbb{C} \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1})$. Applying Proposition 11.4 to, completes the proof of part (a).

(b) Let $S \in S_{0,q;\{\alpha,\infty\}}((s_j)_{j=0}^{2n+1}, \leq)$. According to Proposition 11.4, then there is a pair $[\phi \#, \psi \#] \in \mathcal{P}(q,q)_{-J \geq}(\mathbb{C} \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1})$, where $\phi \#$ and $\psi \#$ are matrix-valued functions which are holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$ and which fulfill

$$\det \left[ \hat{\Theta}_{n,\alpha}^{(2,1)}(z) \phi \#(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z) \psi \#(z) \right] \neq 0 \quad (12.12)$$

and

$$S(z) = \left[ \hat{\Theta}_{n,\alpha}^{(1,1)}(z) \phi \#(z) + \hat{\Theta}_{n,\alpha}^{(1,2)}(z) \psi \#(z) \right] \left[ \hat{\Theta}_{n,\alpha}^{(2,1)}(z) \phi \#(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z) \psi \#(z) \right]^{-1} \quad (12.13)$$

for all $z \in \mathbb{C} \setminus [\alpha, \infty)$. In view of Notation 11.4 and Lemma 12.6, there is a pair $[\phi, \psi] \in \mathcal{P}(r,r)_{-J \geq}(\mathbb{C} \setminus [\alpha, \infty))$ such that $[\phi \overline{\phi}, \psi \overline{\psi}] \in \mathcal{P}(q,q)_{-J \geq}(\mathbb{C} \setminus [\alpha, \infty))$ and $\langle [\phi, \psi] \rangle = \langle [\phi \overline{\phi}, \psi \overline{\psi}] \rangle$ hold true. Consequently, there are a discrete subset $D$ of $\mathbb{C} \setminus [\alpha, \infty)$ and a $q \times q$ matrix-valued function $g$ which is meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ such that $\phi \#, \psi \#, \phi, \psi,$ and $g$ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup D)$ and that $\det g(z) \neq 0$ and

$$\begin{bmatrix} \phi \#(z) \\ \psi \#(z) \end{bmatrix} = \begin{bmatrix} \phi \overline{\phi}(z) & g(z) \\ \psi \overline{\psi}(z) & g(z) \end{bmatrix} \quad (12.14)$$

hold true for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D)$. Therefore, for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D)$, it follows from (12.12) that $0 \neq \det(\hat{\Theta}_{n,\alpha}^{(2,1)}(z) \psi \overline{\psi}(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z) \phi \overline{\phi}(z)) \cdot \det g(z)$. In particular, the function $\det(\hat{\Theta}_{n,\alpha}^{(2,1)}(z) \phi \overline{\phi}(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z) \psi \overline{\psi}(z))$ does not vanish identically in $\mathbb{C} \setminus [\alpha, \infty)$. Because of (12.13) and (12.14), for all $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D)$, we get furthermore

$$S(z) = \left[ \hat{\Theta}_{n,\alpha}^{(1,1)}(z) \phi \overline{\phi}(z) g(z) + \hat{\Theta}_{n,\alpha}^{(1,2)}(z) \psi \overline{\psi}(z) g(z) \right] \left[ \hat{\Theta}_{n,\alpha}^{(2,1)}(z) \phi \overline{\phi}(z) g(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z) \psi \overline{\psi}(z) g(z) \right]^{-1}$$

$$= \left[ \hat{\Theta}_{n,\alpha}^{(1,1)}(z) \phi \overline{\phi}(z) + \hat{\Theta}_{n,\alpha}^{(1,2)}(z) \psi \overline{\psi}(z) \right] \left[ \hat{\Theta}_{n,\alpha}^{(2,1)}(z) \phi \overline{\phi}(z) + \hat{\Theta}_{n,\alpha}^{(2,2)}(z) \psi \overline{\psi}(z) \right]^{-1}.$$
In particular, (12.11) holds true.

The matrices $U := \text{diag}(0_{m \times m}, I_\ell)$ and $V := \text{diag}(I_m, 0_{\ell \times \ell})$ fulfill rank $[V] = m + \ell = q - r$ and $V^*U = 0_{(q-r) \times (q-r)}$. For each $k \in \{1, 2\}$, let $\phi_k^\# = \text{diag}(\phi_k, U)$ and $\psi_k^\# = \text{diag}(\psi_k, V)$.

From Lemma 12.3 we see that $\begin{bmatrix} \phi_1^\# \\ \psi_1^\# \end{bmatrix}$ and $\begin{bmatrix} \phi_2^\# \\ \psi_2^\# \end{bmatrix}$ belong to $\mathcal{P}^{(q,q)}_{-J_{q,\geq}}(\mathbb{C} \setminus [\alpha, \infty))$ and that (1) is equivalent to $\langle \begin{bmatrix} \phi_1^\# \\ \psi_1^\# \end{bmatrix} \rangle = \langle \begin{bmatrix} \phi_2^\# \\ \psi_2^\# \end{bmatrix} \rangle$. Obviously, $\phi_k^\# := W\phi_k^\#$ and $\psi_k^\# := W\psi_k^\#$ for each $k \in \{1, 2\}$.

Taking into account $W^*W = I_q$ and Remark 12.3, we obtain that $\begin{bmatrix} \phi_1^\# \\ \psi_1^\# \end{bmatrix}$ and $\begin{bmatrix} \phi_2^\# \\ \psi_2^\# \end{bmatrix}$ belong to $\mathcal{P}^{(q,q)}_{-J_{q,\geq}}(\mathbb{C} \setminus [\alpha, \infty))$ and that $\langle \begin{bmatrix} \phi_1^\# \\ \psi_1^\# \end{bmatrix} \rangle = \langle \begin{bmatrix} \phi_2^\# \\ \psi_2^\# \end{bmatrix} \rangle$ is equivalent to $\langle \begin{bmatrix} \phi_1^\# \\ \psi_1^\# \end{bmatrix} \rangle = \langle \begin{bmatrix} \phi_2^\# \\ \psi_2^\# \end{bmatrix} \rangle$. Because of Lemma 12.5(b3), we obtain $\begin{bmatrix} \phi_k^\# \\ \psi_k^\# \end{bmatrix} \in \mathcal{P}^{(q,q)}_{-J_{q,\geq}}(\mathbb{C} \setminus [\alpha, \infty), (s_j)_{j=0}^{n+1})$ for each $k \in \{1, 2\}$. Using Proposition 11.4(c), we see that $\langle \begin{bmatrix} \phi_1^\# \\ \psi_1^\# \end{bmatrix} \rangle = \langle \begin{bmatrix} \phi_2^\# \\ \psi_2^\# \end{bmatrix} \rangle$ and (ii) are equivalent. Consequently, (i) holds true if and only if (iii) is fulfilled. □

### 12.3. The completely degenerate case

Now we consider the so-called completely degenerate case (III). We will see that, in this situation, the problem in question has a unique solution.

**Lemma 12.8.** Let $m, \ell \in \mathbb{N}$ be such that $m + \ell = q$. Let $\mathcal{U}$ and $\mathcal{V}$ be orthogonal subspaces of $\mathbb{C}^q$ with $\dim \mathcal{U} = m$ and $\dim \mathcal{V} = \ell$. Then:

(a) There exists a unitary complex $q \times q$ matrix $W$ such that

$$W^*P_uW = \text{diag}(I_m, 0_{\ell \times \ell}) \quad \text{and} \quad W^*P_vW = \text{diag}(0_{m \times m}, I_\ell). \quad (12.15)$$

(b) Let $W$ be a unitary complex $q \times q$ matrix such that (12.15) is fulfilled. Let $\phi^\#$ and $\psi^\#$ be the constant matrix-valued functions defined on $\mathbb{C} \setminus [\alpha, \infty)$ given by

$$\phi^\#(z) := W \cdot \text{diag}(0_{m \times m}, I_\ell) \quad \text{and} \quad \psi^\#(z) := W \cdot \text{diag}(I_m, 0_{\ell \times \ell}). \quad (12.16)$$

for all $z \in \mathbb{C} \setminus [\alpha, \infty)$. Then:

(b1) The pair $\begin{bmatrix} \phi^\# \\ \psi^\# \end{bmatrix}$ belongs to $\mathcal{P}^{(q,q)}_{-J_{q,\geq}}(\mathbb{C} \setminus [\alpha, \infty))$. Furthermore, $P_u\phi^\# = 0$ and $P_v\psi^\# = 0$.

(b2) If $\begin{bmatrix} \phi \\ \psi \end{bmatrix} \in \mathcal{P}^{(q,q)}_{-J_{q,\geq}}(\mathbb{C} \setminus [\alpha, \infty))$ fulfills

$$\begin{bmatrix} \langle \phi \\ \psi \rangle \end{bmatrix} = \begin{bmatrix} \langle \phi^\# \\ \psi^\# \rangle \end{bmatrix}, \quad (12.17)$$

then

$$P_u\phi = 0 \quad \text{and} \quad P_v\psi = 0. \quad (12.18)$$

(b3) If $\begin{bmatrix} \phi \\ \psi \end{bmatrix}$ belongs to $\mathcal{P}^{(q,q)}_{-J_{q,\geq}}(\mathbb{C} \setminus [\alpha, \infty))$ and fulfills (12.18), then (12.17) is valid.
Proof. Let \( \{u_1, u_2, \ldots, u_m\} \) be an orthonormal basis of \( U \) and let \( \{v_1, v_2, \ldots, v_\ell\} \) be an orthonormal basis of \( V \). Let \( U := \{u_1, u_2, \ldots, u_m\} \), let \( V := \{v_1, v_2, \ldots, v_\ell\} \), and let \( W := [U, V] \). Because of \( m + \ell = q \) and since \( U \) and \( V \) are orthogonal subspaces, the matrix \( W \) is unitary. Obviously, we have \( P_U U = U \), \( P_V V = V \), \( U^* U = I_m \), and \( V^* V = 0 \). Consequently, \( W^* P_W W = \text{diag}(I_m, 0_{\ell \times \ell}) \). Analogously, \( P_U U = 0 \), \( P_V V = V \), \( U^* V = 0 \), and \( V^* V = I_\ell \) imply the second equation in (12.15).

(1) Clearly, the constant matrix-valued functions \( \phi_\# \) and \( \psi_\# \) are holomorphic in \( \mathbb{C} \setminus [\alpha, \infty) \). Since the matrix \( W \) is non-singular, we have

\[
\text{rank} \begin{bmatrix} \phi_\#(z) \\ \psi_\#(z) \end{bmatrix} = \text{rank} \begin{bmatrix} \text{diag}(0_{m \times m}, I_\ell) \\ \text{diag}(I_m, 0_{\ell \times \ell}) \end{bmatrix} = m + \ell = q
\]

for each \( z \in \mathbb{C} \setminus [\alpha, \infty) \). For every choice of \( k \in \{0, 1\} \) and \( z \in \mathbb{C} \setminus \mathbb{R} \), from Remark 9.1 and \( W^* W = I_q \), we conclude

\[
\begin{align*}
\left[ (z - \alpha)^k \phi_\#(z) \right]^* & \left( -\frac{J_k}{25z^k} \right) \left[ (z - \alpha)^k \phi_\#(z) \right] \\
\psi_\#(z) & = -\frac{i}{23z} \left[ (z - \alpha)^k \phi_\#(z) \right] - \left( \bar{z} - \alpha \right)^k \phi_\#(z) \psi_\#(z) \\
& = -\frac{i}{23z} \left( (z - \alpha)^k \cdot \text{diag}(I_m, 0_{\ell \times \ell}) \cdot W^* W \cdot \text{diag}(0_{m \times m}, I_\ell) \right) \\
& \quad - \left( \bar{z} - \alpha \right)^k \cdot \text{diag}(0_{m \times m}, I_\ell) \cdot W^* W \cdot \text{diag}(I_m, 0_{\ell \times \ell}) \\
& = 0_{q \times q}.
\end{align*}
\]

In view of Definition 10.5 then \( [\phi_\#] \) belongs to \( \mathcal{P}_{-J_k}^{(q, q)}(\mathbb{C} \setminus [\alpha, \infty)) \). Further, from (12.15) we obtain \( P_U \phi_\# = I_q P_U W \cdot \text{diag}(0_{m \times m}, I_\ell) = W^* P_W W \cdot \text{diag}(0_{m \times m}, I_\ell) = W \cdot \text{diag}(I_m, 0_{\ell \times \ell}) \cdot \text{diag}(0_{m \times m}, I_\ell) = 0_{q \times q} \), and, analogously, \( P_V \psi_\# = 0_{q \times q} \).

(2) Let \( \left[ \phi \psi \right] \in \mathcal{P}_{-J_k}^{(q, q)}(\mathbb{C} \setminus [\alpha, \infty)) \) be such that (12.17) holds true. According to Remark 10.8 there are a discrete subset \( D \) of \( \mathbb{C} \setminus [\alpha, \infty) \) and a matrix-valued function \( g \) meromorphic in \( \mathbb{C} \setminus ([\alpha, \infty) \cup D) \) and that \( \det g(z) \neq 0 \) as well as \( \phi(z) = W \cdot \text{diag}(0_{m \times m}, I_\ell) \cdot g(z) \) and \( \psi(z) = W \cdot \text{diag}(I_m, 0_{\ell \times \ell}) \cdot g(z) \) hold true for each \( z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D) \). Taking into account (12.15) and \( W^* W = I_q \), for each \( z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D) \), we get

\[
P_U \phi(z) = W^* P_U \phi W \cdot \text{diag}(0_{m \times m}, I_\ell) \cdot g(z) = W \cdot \text{diag}(I_m, 0_{\ell \times \ell}) \cdot \text{diag}(0_{m \times m}, I_\ell) \cdot g(z) = 0_{q \times q}
\]

and, analogously \( P_V \psi(z) = 0_{q \times q} \). This implies (12.18).

(3) Let \( \left[ \phi \psi \right] \in \mathcal{P}_{-J_k}^{(q, q)}(\mathbb{C} \setminus [\alpha, \infty)) \) be such that (12.18) holds true. According to Lemma 10.14 we see that the function \( \det(\psi - i\phi) \) does not vanish identically. Let \( F := (\psi + i\phi)(\psi - i\phi)^{-1} \). Lemma 10.14 shows that there is a discrete subset \( D \) of \( \mathbb{C} \setminus [\alpha, \infty) \) such that the following three conditions are fulfilled:

(i) \( F \) is holomorphic in \( \Pi_+ \cup [\mathbb{C} \setminus ([\alpha, \infty) \cup D)] \).
(ii) The matrix-valued functions \( \phi, \psi \) and \( (\psi - i\phi)^{-1} \) are holomorphic in \( \Pi_+ \cup [\mathbb{C} \setminus ([\alpha, \infty) \cup D)] \).
(iii) For each \( z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D) \), the inequality \( \det[\psi(z) - i\phi(z)] \neq 0 \) and the equations in \((10.10)\) and \((10.11)\) hold true.

Obviously, because of \((\text{i})\) the functions \( \tilde{\phi} := \frac{1}{2}(I_q - F)W \) and \( \tilde{\psi} := \frac{1}{2}(I_q + F)W \) are meromorphic in \( \mathbb{C} \setminus [\alpha, \infty) \) and holomorphic in \( \Pi_+ \cup [\mathbb{C} \setminus ([\alpha, \infty) \cup D)] \). In view of \((\text{ii})\) the functions \( \phi, \psi, \tilde{\phi}, \tilde{\psi} \) and \( (\psi - i\phi)^{-1}W \) are holomorphic in \( \Pi_+ \cup [\mathbb{C} \setminus ([\alpha, \infty) \cup D)] \). From \((\text{iii})\) we see that

\[
\tilde{\phi}(z) = \phi(z)[\psi(z) - i\phi(z)]^{-1}W \quad \text{and} \quad \tilde{\psi}(z) = \psi(z)[\psi(z) - i\phi(z)]^{-1}W
\]

hold true for each \( z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D) \). In view of \((\text{ii})\) the matrix-valued functions \( (\psi + i\phi) \) and \( (\psi - i\phi)^{-1}W \) are holomorphic in \( \mathbb{C} \setminus [\alpha, \infty) \). Since the matrix \( W \) is unitary, for each \( z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D) \), we have \( \det[(\psi(z) - i\phi(z))^{-1}W] \neq 0 \) by \((\text{iii})\). Consequently, \((12.19)\) and Remark 10.7 imply that \( [\tilde{\phi}, \tilde{\psi}] \) belongs to \( \mathcal{P}(q,q_{\Xi}, \mathbb{C} \setminus [\alpha, \infty)) \). Furthermore, from Remark 10.8 we get

\[
\left\langle \begin{bmatrix} \tilde{\phi} \\ \tilde{\psi} \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} \phi \\ \psi \end{bmatrix} \right\rangle.
\]

(12.20)

By virtue of \( W^*W = I_q, (12.19), (12.15), \) and \((12.18)\), we conclude

\[
(I_m, 0_{m \times \ell})(I_q - W^*FW) = (I_m, 0_{m \times \ell})W^*(I_q - F)W = -2i(I_m, 0_{m \times \ell})W^*\tilde{\phi} = -2i(I_m, 0_{m \times \ell})W^*\phi(\psi - i\phi)^{-1}W = -2i(I_m, 0_{m \times \ell}) \cdot \text{diag}(I_m, 0_{\ell \times \ell}) \cdot W^*\phi(\psi - i\phi)^{-1}W = -2i(I_m, 0_{m \times \ell})W^*P_\ell \phi(\psi - i\phi)^{-1}W = -2i(I_m, 0_{m \times \ell})W^{0_{m \times \ell}}(\psi - i\phi)^{-1}W = 0_{m \times q}
\]

and, analogously,

\[
(0_{\ell \times m}, I_{\ell})(I_q + W^*FW) = 0_{\ell \times q}.
\]

(12.22)

Because of \((\text{i})\) we see that \( G := W^*FW \) is a matrix-valued function which is meromorphic in \( \mathbb{C} \setminus [\alpha, \infty) \) and holomorphic in \( \Pi_+ \cup [\mathbb{C} \setminus ([\alpha, \infty) \cup D)] \). From \((12.21)\) and \((12.22)\) we obtain \( G(w) = \text{diag}(I_m, -I_{\ell}) \) for each \( w \in \Pi_+ \). Hence, \( G = \text{diag}(I_m, -I_{\ell}) \) by the identity theorem for holomorphic functions. Thus, since the matrix \( W \) is unitary, this implies \( F = W \cdot \text{diag}(I_m, -I_{\ell}) \cdot W^* \). Then

\[
\tilde{\phi} = \frac{1}{2}(I_q - F)W = \frac{i}{2}[I_q - W \cdot \text{diag}(I_m, -I_{\ell}) \cdot W^*]W = \frac{i}{2}W[\text{diag}(I_m, I_{\ell}) - \text{diag}(I_m, -I_{\ell})] = W \cdot \text{diag}(0_{m \times m}, I_{\ell})
\]

(12.23)

and, analogously,

\[
\tilde{\psi} = W \cdot \text{diag}(I_m, 0_{\ell \times \ell}).
\]

(12.24)

Since \( \tilde{\phi} \) and \( \tilde{\psi} \) are holomorphic in \( \Pi_+ \cup [\mathbb{C} \setminus ([\alpha, \infty) \cup D)] \), the matrix-valued functions \( \phi_\square := \tilde{\phi} \cdot \text{diag}(I_m, -iI_{\ell}) \) and \( \psi_\square := \tilde{\psi} \cdot \text{diag}(I_m, -iI_{\ell}) \) are holomorphic in \( \Pi_+ \cup [\mathbb{C} \setminus ([\alpha, \infty) \cup D)] \). From \( \det(I_m, -iI_{\ell}) \neq 0 \), Remark 10.7, Remark 10.8, and \((12.20)\) we get

\[
\left\langle \begin{bmatrix} \phi_\square \\ \psi_\square \end{bmatrix} \right\rangle \in \mathcal{P}(q,q_{\Xi}, \mathbb{C} \setminus [\alpha, \infty)) \quad \text{and} \quad \left[ \begin{bmatrix} \phi_\square \\ \psi_\square \end{bmatrix} \right] = \left[ \begin{bmatrix} \tilde{\phi} \\ \tilde{\psi} \end{bmatrix} \right] = \left\langle \begin{bmatrix} \phi \\ \psi \end{bmatrix} \right\rangle.
\]

(12.25)

Because of \((12.23)\) and \((12.16)\), we have

\[
\phi_\square = \tilde{\phi} \cdot \text{diag}(I_m, -iI_{\ell}) = W \cdot \text{diag}(0_{m \times m}, I_{\ell}) \cdot \text{diag}(I_m, -iI_{\ell}) = \phi_{\#}.
\]

(12.26)

Analogously, \((12.24)\) and \((12.16)\) imply \( \psi_\square = \psi_{\#} \). Thus, \((12.17)\) follows from \((12.25)\).
Lemma 12.9. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)^{c_{j=0}} \in \mathcal{K}_{q,\kappa,\alpha}$. Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$. Suppose that the integers $m$ and $\ell$ given by (12.1) and (12.2) fulfill $m+\ell = q$, $m \geq 1$, and $\ell \geq 1$. Let $\mathcal{U}_{n,\alpha}$ and $\mathcal{V}_{n,\alpha}$ be given by (9.127) and (9.128). Then:

(a) There exists a unitary complex $q \times q$ matrix $W$ such that

$$W^*P_{\mathcal{U}_{n,\alpha}}W = \text{diag}(I_m, 0_{\ell \times \ell}) \quad \text{and} \quad W^*P_{\mathcal{V}_{n,\alpha}}W = \text{diag}(0_{m \times m}, I_\ell).$$  \hfill (12.26)

(b) Let $W$ be a unitary complex $q \times q$ matrix such that (12.26) holds true. Furthermore, let $\phi_\square$ and $\psi_\square$ be the constant matrix-valued functions defined on $\mathbb{C} \setminus [\alpha, \infty)$ given by $\phi_\square(z) := W \cdot \text{diag}(0_{m \times m}, I_\ell)$ and $\psi_\square(z) := W \cdot \text{diag}(I_m, 0_{\ell \times \ell})$ for all $z \in \mathbb{C} \setminus [\alpha, \infty)$. Then:

(b1) $\begin{bmatrix} \phi \cr \psi \end{bmatrix} \in \mathcal{P}^{(q,q)}_{-J_{n,\alpha} \geq}([\alpha, \infty))$.

(b2) Each pair $\begin{bmatrix} \phi \cr \psi \end{bmatrix} \in \mathcal{P}^{(q,q)}_{-J_{n,\alpha} \geq}([\alpha, \infty))$ with

$$\left\langle \begin{bmatrix} \phi \cr \psi \end{bmatrix} \right\rangle = \left\langle \phi_\square \right\rangle \left\langle \psi_\square \right\rangle \tag{12.27}$$

belongs to $\mathcal{P}^{(q,q)}_{-J_{n,\alpha} \geq}([\alpha, \infty), (s_j)_{j=0}^{2n+1})$.

(b3) Each $\begin{bmatrix} \phi \cr \psi \end{bmatrix} \in \mathcal{P}^{(q,q)}_{-J_{n,\alpha} \geq}([\alpha, \infty), (s_j)_{j=0}^{2n+1})$ fulfills (12.27).

Proof. From (9.127), (9.128), Lemma 9.32, (12.1) and (12.2) we see that $\mathcal{U}_{n,\alpha}$ and $\mathcal{V}_{n,\alpha}$ are orthogonal subspaces of $\mathcal{Q}$ with $\dim \mathcal{U}_{n,\alpha} = m$ and $\dim \mathcal{V}_{n,\alpha} = \ell$. Since $m$ and $\ell$ are positive integers with $m+\ell = q$, we can apply Lemma 12.8 with $\mathcal{U} = \mathcal{U}_{n,\alpha}$ and $\mathcal{V} = \mathcal{V}_{n,\alpha}$.

(a) Apply Lemma 12.8(a).

(b1) Suppose that $\begin{bmatrix} \phi \cr \psi \end{bmatrix} \in \mathcal{P}^{(q,q)}_{-J_{n,\alpha} \geq}([\alpha, \infty))$ is such that (12.27) holds true. Lemma 12.8(b2) shows then that $P_{\mathcal{U}_{n,\alpha}}\phi = 0$ and $P_{\mathcal{V}_{n,\alpha}}\psi = 0$. Thus, Lemma 9.32 implies (11.1) and (11.2). In view of Notation (11.1) part (12.2) is proved.

(b3) Let $\begin{bmatrix} \phi \cr \psi \end{bmatrix} \in \mathcal{P}^{(q,q)}_{-J_{n,\alpha} \geq}([\alpha, \infty), (s_j)_{j=0}^{2n+1})$. Then (11.1) and (11.2) hold true. Because of (9.127), (11.1) and Lemma 9.32 we have $P_{\mathcal{U}_{n,\alpha}}\phi = 0$, whereas (9.128), (11.2), and Lemma 9.32 yield $P_{\mathcal{V}_{n,\alpha}}\psi = 0$. Since $W$ is a unitary matrix which fulfills (12.26), Lemma 12.8(b3) implies (12.27). \hfill \Box

Remark 12.10. Let $W$ be a non-singular complex $q \times q$ matrix and let $\mathcal{W}$ be the constant function with value $W$ defined on $\mathcal{C} \setminus [\alpha, \infty)$. Then it is readily checked that the following statements hold true:

(a) The pairs $\left[ \begin{array}{c} 0_{q \times q} \\ W \end{array} \right]$ and $\left[ \begin{array}{c} W \\ 0_{q \times q} \end{array} \right]$ belong to $\mathcal{P}^{(q,q)}_{-J_{n,\alpha} \geq}([\alpha, \infty))$.

(b) Each pair $\begin{bmatrix} \phi \cr \psi \end{bmatrix} \in \mathcal{P}^{(q,q)}_{-J_{n,\alpha} \geq}([\alpha, \infty))$ with

$$\left\langle \begin{bmatrix} \phi \\ \psi \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 0_{q \times q} \\ W \end{bmatrix} \right\rangle \tag{12.28}$$

fulfills $\phi = 0_{q \times q}$. Conversely, if $\begin{bmatrix} \phi \cr \psi \end{bmatrix} \in \mathcal{P}^{(q,q)}_{-J_{n,\alpha} \geq}([\alpha, \infty))$ is such that $\phi = 0_{q \times q}$ holds true, then $\det \psi$ does not vanish identically and (12.28) is valid.

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(c) Each pair $[\phi \psi] \in \mathcal{P}^{(q,q)}_{-J_{q+1}}(C \setminus [\alpha, \infty))$ with

$$\left\langle \phi \psi \right\rangle = \left\langle W 0_{q \times q} \right\rangle$$  \hspace{1cm} (12.29)

fulfills $\psi = 0_{q \times q}$. Conversely, if $[\phi \psi] \in \mathcal{P}^{(q,q)}_{-J_{q+1}}(C \setminus [\alpha, \infty))$ is such that $\psi = 0_{q \times q}$ holds true, then $\det \phi$ does not vanish identically and (12.29) is valid.

**Lemma 12.11.** Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} = K_{q,\kappa,0}$. Let $n \in \mathbb{N}$ be such that $2n + 1 \leq \kappa$. Suppose that $\ell$ given by (12.22) fulfills $\ell = q$. Then $m$ given by (12.11) fulfills $m = 0$ and:

(a) $\mathcal{V}_{n,\alpha}$ defined by (12.28) fulfills $\mathcal{V}_{n,\alpha} = C^q$ and, in particular, $P_{\mathcal{V}_{n,\alpha}} = I_q$.

(b) Let $W$ be a non-singular complex $q \times q$ matrix and let $W$ be the constant function with value $W$ defined on $C \setminus [\alpha, \infty)$. Then $[W_{0 \times q}]$ belongs to $\mathcal{P}^{(q,q)}_{-J_q \geq \kappa}(C \setminus [\alpha, \infty))$ and each pair $[\phi \psi] \in \mathcal{P}_{-J_q \geq \kappa}(C \setminus [\alpha, \infty))$ with (12.29) belongs to the class $\mathcal{P}^{(q,q)}_{-J_q \geq \kappa}[C \setminus [\alpha, \infty),(s_j)_{j=0}^{2n+1}]$.

(c) If $[\phi \psi] \in \mathcal{P}_{-J_q \geq \kappa}(C \setminus [\alpha, \infty))$ fulfills (11.2), then (12.29) holds true.

**Proof.** Because of $0 \leq m \leq m + \ell \leq q = \ell$, we have $m = 0$.

(a) From (12.28), (12.22), $\ell = q$, and Lemma 9.32 we see that (a) is valid.

(b) Remark 12.10 shows that $[W_{0 \times q}]$ belongs to $\mathcal{P}^{(q,q)}_{-J_q \geq \kappa}(C \setminus [\alpha, \infty))$. Let $[\phi \psi] \in \mathcal{P}^{(q,q)}_{-J_q \geq \kappa}(C \setminus [\alpha, \infty))$ fulfill (12.29). Then part (a) and Remark 12.10(c) yield $P_{\mathcal{V}_{n,\alpha}} \psi = 0$ and, in view of Lemma 9.32 consequently (11.2). Since $m = 0$ holds, we get from (12.1) that (11.1) is true. In view of Notation 11.11 part (b) is proved.

(c) Let $[\phi \psi] \in \mathcal{P}_{-J_q \geq \kappa}(C \setminus [\alpha, \infty))$ be such that (11.2) holds true. Because of (11.2) and Lemma 9.32 we have $P_{\mathcal{V}_{n,\alpha}} \psi = 0$. Thus, part (a) implies $\psi = 0$. From Remark 12.10(c) it follows (12.29). \hfill $\Box$

**Lemma 12.12.** Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} = K_{q,\kappa,\alpha}$. Let $n \in \mathbb{N}$ be such that $2n + 1 \leq \kappa$. Suppose that $m$ given by (12.11) fulfills $m = q$. Let $\mathcal{U}_{n,\alpha}$ be defined by (9.127). Then $\ell$ given by (12.22) fulfills $\ell = 0$ and the following statements hold true:

(a) $\mathcal{U}_{n,\alpha} = C^q$ and, in particular, $P_{\mathcal{U}_{n,\alpha}} = I_q$.

(b) Let $W$ be a non-singular complex $q \times q$ matrix and let $W$ be the constant function with value $W$ defined on $C \setminus [\alpha, \infty)$. Then $[W_{0 \times q}]$ belongs to $\mathcal{P}_{-J_q \geq \kappa}(C \setminus [\alpha, \infty))$ and each pair $[\phi \psi] \in \mathcal{P}_{-J_q \geq \kappa}(C \setminus [\alpha, \infty))$ with (12.28) belongs to the class $\mathcal{P}^{(q,q)}_{-J_q \geq \kappa}[C \setminus [\alpha, \infty),(s_j)_{j=0}^{2n+1}]$.

(c) If $[\phi \psi] \in \mathcal{P}_{-J_q \geq \kappa}(C \setminus [\alpha, \infty))$ fulfills (11.1), then (12.28) holds true.

**Proof.** Using Lemma 9.32 and Remark 12.10, Lemma 12.12 can be proved analogous to Lemma 12.11. We omit the details \hfill $\Box$

**Theorem 12.13.** Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, let $(s_j)_{j=0}^{\kappa} = K_{q,\kappa,\alpha}$, and let $n \in \mathbb{N}$ be such that $2n + 1 \leq \kappa$. Suppose that the integers $m$ and $\ell$ given by (12.11) and (12.22) fulfill $m + \ell = q$. If $m \geq 1$ and $\ell \geq 1$, then let $W$ be a unitary complex $q \times q$ matrix such that the equations in (12.26) hold true where $\mathcal{U}_{n,\alpha}$ and $\mathcal{V}_{n,\alpha}$ are given by (9.127) and (12.12).
(a) If \( \phi \) and \( \psi \) are the matrix-valued functions defined on \( \mathbb{C} \setminus [\alpha, \infty) \) by

\[
\phi(z) := \begin{cases} 
W \cdot \text{diag}(0_{m \times m}, I_{\ell}), & \text{if } m \geq 1 \text{ and } \ell \geq 1 \\ 
I_q, & \text{if } m = 0 \\ 
0_{q \times q}, & \text{if } \ell = 0
\end{cases}
\]

and

\[
\psi(z) := \begin{cases} 
W \cdot \text{diag}(I_m, 0_{\ell \times \ell}), & \text{if } m \geq 1 \text{ and } \ell \geq 1 \\ 
0_{q \times q}, & \text{if } m = 0 \\ 
I_q, & \text{if } \ell = 0
\end{cases}
\]

then the function \( \det(\hat{\Theta}^{(1,1)}_{\alpha}(\phi) + \hat{\Theta}^{(2,2)}_{\alpha}(\psi)) \) does not vanish identically.

(b) The set \( S_{0,q;[\alpha,\infty)]}[(s_j)_{j=0}^{2n+1}], \leq \) consists of exactly one element, namely the matrix-valued function

\[
S := \begin{cases} 
(\hat{\Theta}^{(1,1)}_{n,\alpha}(\phi) + \hat{\Theta}^{(2,2)}_{n,\alpha}(\psi))^{-1}, & \text{if } m \geq 1 \text{ and } \ell \geq 1 \\ 
\hat{\Theta}^{(1,1)}_{n,\alpha}(\phi) \hat{\Theta}^{(2,2)}_{n,\alpha}(\psi)^{-1}, & \text{if } m = 0 \\ 
\hat{\Theta}^{(1,2)}_{n,\alpha}(\phi) \hat{\Theta}^{(2,1)}_{n,\alpha}(\psi)^{-1}, & \text{if } \ell = 0
\end{cases}
\]

where, in the case \( m \geq 1 \) and \( \ell \geq 1 \), the matrices \( U := [U, 0_{q \times \ell}] \) and \( V := [0_{q \times m}, V] \) are built with the \( q \times m \) block \( U \) and the \( q \times \ell \) block \( V \) from the block partition \( W = [U, V] \) of \( W \).

Proof. Let \( \phi \) and \( \psi \) be given by (12.30) and (12.31). Then Lemmas [12.9] and [12.11] yield \( [\hat{\Theta}^{(q,q)}_{n,\alpha}] \in \mathcal{P}^{(q,q)}_{J_{\alpha} \geq} [\mathbb{C} \setminus [\alpha, \infty), (s_j)_{j=0}^{2n+1}] \). Thus, Proposition [11.4] shows that \( \hat{\Theta}_{n,\alpha} \) is valid and that \( \hat{\Theta}_{n,\alpha} \) belongs to \( S_{0,q;[\alpha,\infty)]}[(s_j)_{j=0}^{2n+1}], \leq \). From (11.19), (12.30), (12.31), and (12.32) we get \( S = \hat{S}_{n,\alpha} \) and, consequently, \( \{S\} \subseteq S_{0,q;[\alpha,\infty)]}[(s_j)_{j=0}^{2n+1}], \leq \). Now we consider an arbitrary \( S_{0,q;[\alpha,\infty)]}[(s_j)_{j=0}^{2n+1}], \leq \). By virtue of Proposition [11.4], there exists a pair \( \{\phi_{0,q;[\alpha,\infty)]}, \psi_{0,q;[\alpha,\infty)]}\) of \( \mathbb{C} \setminus [\alpha, \infty) \) holomorphic \( q \times q \) matrix-valued functions \( \phi_{0,q;[\alpha,\infty)]} \) and \( \psi_{0,q;[\alpha,\infty)]} \) which fulfill (12.12) and

\[
S_{0,q;[\alpha,\infty)]}[(s_j)_{j=0}^{2n+1}], \leq \subseteq \{S\} \subseteq \{S\} \subseteq \{S\}.
\]

Remark 12.14. Under the assumptions of Theorem 12.13, we see from Theorem 12.13 [24] Theorems 6.5 and 6.4, [25] Definition 4.10, and [35] Theorem 5.1 that \( S \) given by (12.32) is exactly the \( [\alpha, \infty) \)-Stieltjes transform of the restriction onto \( \mathcal{B}_{[\alpha,\infty)]} \) of the completely degenerate non-negative Hermitian measure corresponding to \( (s_j)_{j=0}^{2n+1} \).

Observe that using Proposition 11.4, [24] Theorem 5.2, Example 10.10, Theorems 12.12, 12.2, 12.7, and 34, Theorem 5.1, one can obtain a self-contained proof of Theorem 2.4 in the case of a positive odd integer \( m \). We omit the details.
A. Particular results of matrix theory

Remark A.1. If $A \in \mathbb{C}^{p \times q}$, then $\mathcal{N}(A) = \mathcal{R}(A^*)^\perp$ and $\mathcal{R}(A) = \mathcal{N}(A^*)^\perp$.

Lemma A.2 ([18]). Let $A \in \mathbb{C}^{p \times q}$, let $B \in \mathbb{C}^{p \times r}$, and let $B^{(1)} \in B\{1\}$. Then the following statements are equivalent:

(i) $\mathcal{R}(A) \subseteq \mathcal{R}(B)$.

(ii) There is a matrix $X \in \mathbb{C}^{r \times q}$ such that $A = BX$.

(iii) $BB^{(1)}A = A$.

(iv) There is a positive real number $\beta$ such that $AA^* \leq \beta BB^*$.

(v) $\mathcal{N}(B^*) \subseteq \mathcal{N}(A^*)$.

It seems to be useful to state the following dual reformulation of Lemma A.2:

Lemma A.3. Let $A \in \mathbb{C}^{p \times q}$, let $C \in \mathbb{C}^{r \times q}$, and let $C^{(1)} \in C\{1\}$. Then the following statements are equivalent:

(i) $\mathcal{N}(C) \subseteq \mathcal{N}(A)$.

(ii) There is a matrix $Y \in \mathbb{C}^{p \times r}$ such that $A = YC$.

(iii) $AC^{(1)}C = A$.

(iv) There is a positive real number $\gamma$ such that $A^*A \leq \gamma C^*C$.

(v) $\mathcal{R}(A^*) \subseteq \mathcal{R}(C^*)$.

Lemma A.4 ([4, 28]). Let $E \in \mathbb{C}^{(p+q) \times (p+q)}$ and let $E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be the block partition of $E$ with $p \times p$ block $A$. Let $A^{(1)} \in A\{1\}$ and let $D^{(1)} \in D\{1\}$. Furthermore, let $L := D - CA^{(1)}B$ and let $R := A - BD^{(1)}C$. Then:

(a) The following statements are equivalent:

(i) $E \in \mathbb{C}^{(p+q) \times (p+q)}$.

(ii) $A \in \mathbb{C}^{p \times p}$, $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, $C = B^*$, and $L \in \mathbb{C}^{q \times q}$.

(iii) $D \in \mathbb{C}^{q \times q}$, $\mathcal{R}(C) \subseteq \mathcal{R}(D)$, $B = C^*$, and $R \in \mathbb{C}^{p \times p}$.

(b) If (i) is fulfilled, then $L = D - CA^1D$ and $R = A - BD^1C$.

(c) If (i) holds true, then $\text{rank } E = \text{rank } A + \text{rank } L$ and $\text{rank } E = \text{rank } D + \text{rank } R$.

A detailed proof of parts (ii) and (ii) of Lemma A.4 is given, e.g., in [20] Lemmata 1.1.9 and 1.1.7. Part (i) of Lemma A.4 is a consequence of the Lemmata A.2 and A.3.

Remark A.5. Let $n \in \mathbb{N}$ and let $(d_{j,k})_{j,k=0}^{n}$ be a sequence of complex $q \times q$ matrices. If $d_0 = 0_{q \times q}$ and if the block Hankel matrix $(d_{j+k})_{j,k=0}^{n}$ is non-negative Hermitian, then, in view of Lemma A.4, it is readily proved by induction that $d_j = 0_{q \times q}$ for all $j \in \mathbb{Z}_{0,2n-1}$.

Remark A.6. It is readily checked that if $E$ is non-negative Hermitian, then $\|B\|_S^2 \leq \|A\|_S \cdot \|D\|_S$ (see, e.g., [20] proof of Lemma 1.1.10)).
Remark A.7. Let $C$ be a non-singular complex $q \times q$ matrix, let $B \in \mathbb{C}^{q \times q}$ and let $A := BC$. Then $\mathcal{R}(B) = \mathcal{R}(A)$ and $\mathcal{N}(B) = C\mathcal{N}(A)$.

Remark A.8. Let $A, B \in \mathbb{C}^{q \times q}$ and let $\alpha, \beta \in \mathbb{C}$. Then $(\alpha A + \beta B)\mathcal{N}(B) \subseteq A\mathcal{N}(B)$. Furthermore, if $\alpha \neq 0$, then $(\alpha A + \beta B)\mathcal{N}(B) = A\mathcal{N}(B)$.

Remark A.9. Let $A \in \mathbb{C}^{q \times q}$. For all $z \in \mathbb{C}$, then $\mathfrak{R}(zA) = \mathfrak{R}(z)\mathfrak{R}(A) - \Im(z)\Im(A)$ and $\Im(zA) = \mathfrak{R}(z)\Im(A) + \Im(z)\mathfrak{R}(A)$.

### B. A particular generalized inverse of a complex matrix

In this section, we state some useful identities for the particular generalized inverse of a Hermitian complex matrix, which is introduced in Remark A.8.

**Lemma B.1.** Let $A$ be a Hermitian complex $q \times q$ matrix and let $U$ be a subspace of $\mathbb{C}^{q}$ such that $\mathcal{N}(A) + U = \mathbb{C}^{q}$. Then $(A^{-}_U)^* = A^{-}_U$, $\mathfrak{R}(A^{-}_U) = U$, $\mathcal{N}(A^{-}_U) = U^\perp$, $\dim(\mathfrak{R}(A^{-}_U)) = \text{rank } A$, and $\dim(\mathcal{N}(A^{-}_U)) = q - \text{rank } A$.

**Proof.** By definition of $A^{-}_U$, we have (B.1) as well as $\mathfrak{R}(A^{-}_U) = U$ and $\mathcal{N}(A^{-}_U) = U^\perp$. In view of $\mathcal{N}(A) + U = \mathbb{C}^{q}$, then (B.2) follows. Since $A^* = A$ is supposed, (B.1) implies $(A^{-}_U)^* A = A$ and $(A^{-}_U)^* (A^{-}_U)^* = (A^{-}_U)^*$. Moreover, $\mathcal{N}((A^{-}_U)^*) = \mathfrak{R}(A^{-}_U)^\perp = U^\perp$ and $\mathcal{N}((A^{-}_U)^*) = \mathcal{N}(A^{-}_U)^\perp = (U^\perp)^\perp = U$. Consequently, $(A^{-}_U)^* = A^{(1,2)}_{U^\perp U^\perp} = A^{-}_U$. $\square$

**Remark B.2.** Let $A \in \mathbb{C}^{q \times q}$ and let $U$ be a subspace of $\mathbb{C}^{q}$ such that $\mathcal{N}(A) + U = \mathbb{C}^{q}$. Then $AA^\dagger = A^\dagger A$ and, in view of Lemma B.1, one can easily check that

- $(A^{-}_U A)^* = AA^{-}_U A$,
- $(A^{-}_U A)^2 = A^{-}_U A$,
- $(AA^{-}_U)^2 = AA^{-}_U$,
- $\mathfrak{R}(A^{-}_U A) = U$,
- $\mathfrak{R}(AA^{-}_U) = \mathfrak{R}(A)$,
- $\mathcal{N}(AA^{-}_U) = U^\perp$,
- $\mathcal{N}(A^{-}_U A) = \mathcal{N}(A)$,
- $\dim(\mathfrak{R}(A^{-}_U A)) = \dim(\mathfrak{R}(AA^{-}_U)) = \dim(\mathcal{N}(AA^{-}_U)) = \dim(\mathcal{N}(A^{-}_U A)) = q - \text{rank } A$,
- $A^\dagger AA^{-}_U A = A^\dagger A = AA^\dagger A^{-}_U A$ and $AA^{-}_U A^\dagger A = A^\dagger A = AA^\dagger A^{-}_U A$.

Parts of the following result are already contained in [8, Lemma 2.3].

**Lemma B.3.** Let $A$ be a Hermitian complex $q \times q$ matrix and let $U$ be a subspace of $\mathbb{C}^{q}$ such that $\mathcal{N}(A) + U = \mathbb{C}^{q}$. Then

- $(I_q - AA^{-}_U)^* = I_q - A^{-}_U A$,
- $(I_q - AA^{-}_U)^2 = I_q - AA^{-}_U$,
- $\mathcal{N}(I_q - AA^{-}_U) = \mathfrak{R}(A)$,
- $\mathfrak{R}(I_q - AA^{-}_U) = U^\perp$,
- $(I_q - AA^{-}_U) A A^\dagger = 0$,
- $(I_q - AA^{-}_U) A^\dagger A = 0$,
- $(I_q - AA^{-}_U) (I_q - AA^{-}_U) = I_q - AA^{-}_U$ and $(I_q - AA^{-}_U) (I_q - A^\dagger A) = I_q - AA^{-}_U$.

**Proof.** The first two equations follow from Remark B.2. From Lemma B.1 we know that (B.1) is true. Using (B.1), the equation $\mathcal{N}(I_q - AA^{-}_U) = \mathfrak{R}(A)$ can be easily checked by straightforward calculations. In order to prove $\mathfrak{R}(I_q - AA^{-}_U) = U^\perp$, one shows that (B.1) implies $\mathfrak{R}(I_q - AA^{-}_U) = \mathcal{N}(AA^{-}_U)$ and one applies the equation $\mathcal{N}(AA^{-}_U) = U^\perp$ stated in Remark B.2. Because of $A^* = A$, we have $AA^\dagger = A^\dagger A$. Thus, from (B.1) we easily see that the remaining equations hold true. $\square$
Lemma B.4. Let $A$ be a Hermitian complex $q \times q$ matrix and let $\mathcal{U}$ be a subspace of $\mathbb{C}^q$ such that $\mathcal{N}(A) \perp \mathcal{U} = \mathbb{C}^q$. Then
\[
\mathcal{R}(I_q - A_{\mathcal{U}}^* A) = \mathcal{N}(A_{\mathcal{U}}^* A) = \mathcal{N}(A), \quad \mathcal{N}(I_q - A_{\mathcal{U}}^* A) = \mathcal{R}(A_{\mathcal{U}}^* A) = \mathcal{U},
\]
\[
dim \mathcal{R}(A_{\mathcal{U}}^* A) = \text{rank} A, \quad A^\dagger A A_{\mathcal{U}}^* A = A^\dagger A = A A^\dagger = A A_{\mathcal{U}}^* A,
\]
and $(I_q - A^\dagger A) A_{\mathcal{U}}^* A = A_{\mathcal{U}}^* A - A^\dagger A = A^\dagger A - A A^\dagger = (I_q - A A_{\mathcal{U}}^* A) A_{\mathcal{U}}^* A$.

Proof. Because of Lemma B.1, we get (B.1). From (B.1) we obtain $\mathcal{N}(A_{\mathcal{U}}^* A) = \mathcal{N}(A)$ and $\mathcal{R}(A_{\mathcal{U}}^* A) = \mathcal{R}(A_{\mathcal{U}}^* A) = \mathcal{U}$. In particular, $\dim \mathcal{R}(A_{\mathcal{U}}^* A) = \dim \mathcal{U} = \dim \mathbb{C}^q - \dim \mathcal{N}(A) = \text{rank} A$. Because of $A^\dagger A = A$, we have $AA^\dagger = A^\dagger A$. Therefore, (B.1) shows that $A^\dagger A A_{\mathcal{U}}^* A = A^\dagger A = AA^\dagger$ and $A^\dagger A A A_{\mathcal{U}}^* A = AA^\dagger$. Thus, the remaining equations immediately follow.

Remark B.5. Let $A$ be a Hermitian complex $q \times q$ matrix and let $\mathcal{U}$ be a subspace of $\mathbb{C}^q$ such that $\mathcal{N}(A) \perp \mathcal{U} = \mathbb{C}^q$. In view of the Lemmata B.3 and B.1 it is readily checked that
\[
(I_q - A_{\mathcal{U}}^* A)^* = I_q - A A_{\mathcal{U}}^*, \quad (I_q - A_{\mathcal{U}}^* A)^2 = I_q - A_{\mathcal{U}}^* A,
\]
\[
\mathcal{R}(I_q - A_{\mathcal{U}}^* A) = \mathcal{N}(A), \quad \mathcal{N}(I_q - A_{\mathcal{U}}^* A) = \mathcal{R}(A_{\mathcal{U}}^* A) = \mathcal{U},
\]
\[
A^\dagger A (I_q - A_{\mathcal{U}}^* A) = 0, \quad AA^\dagger (I_q - A_{\mathcal{U}}^* A) = 0,
\]
and $(I_q - A^\dagger A)(I_q - A_{\mathcal{U}}^* A) = I_q - A_{\mathcal{U}}^* A$, and $(I_q - A A^\dagger)(I_q - A_{\mathcal{U}}^* A) = I_q - A_{\mathcal{U}}^* A$.

Remark B.6. Let $T \in \mathbb{C}^{q \times q}$ and let $\mathcal{U}$ and $\mathcal{V}$ be subspaces of $\mathbb{C}^q$ with $T^\dagger(\mathcal{U}) \subseteq \mathcal{V} \subseteq \mathcal{U}$. Then it is readily checked that $(T^\dagger)^k(\mathcal{U}) \subseteq \mathcal{V} \subseteq \mathcal{U}$ and $T^k(\mathcal{V}^\perp) \subseteq \mathcal{U}^\perp \subseteq \mathcal{V}^\perp$ is valid for each $k \in \mathbb{N}$ and that $(T^\dagger)^k(\mathcal{V}) \subseteq \mathcal{U}$ and $T^k(\mathcal{U}^\perp) \subseteq \mathcal{V}^\perp$ for each $k \in \mathbb{N}_0$ hold true (see also [3, Corollary 3.3], where a special case is discussed).

The following lemma is a generalization of [3, Lemma 4.1], where special pairs of block Hankel matrices are considered.

Lemma B.7. Let $A$ and $B$ be Hermitian complex $q \times q$ matrices and let $T \in \mathbb{C}^{q \times q}$. Suppose that $\mathcal{U}$ and $\mathcal{V}$ are subspaces of $\mathbb{C}^q$ such that $\mathcal{N}(A) \perp \mathcal{U} = \mathbb{C}^q$ and $\mathcal{N}(B) \perp \mathcal{V} = \mathbb{C}^q$ and $T^\dagger(\mathcal{U}) \subseteq \mathcal{V} \subseteq \mathcal{U}$ hold true. Then
\[
A_{\mathcal{U}}^* T^\dagger (I_q - A A_{\mathcal{U}}^*) = 0 \quad \text{and} \quad B_{\mathcal{V}}^\dagger T^k (I_q - A A_{\mathcal{U}}^*) = 0 \quad (B.3)
\]
for each $\ell \in \mathbb{N}_0$ and, for each $k \in \mathbb{N}$, furthermore
\[
A_{\mathcal{U}}^* T^k (I_q - B B_{\mathcal{V}}) = 0 \quad \text{and} \quad B_{\mathcal{V}}^\dagger T^k (I_q - B B_{\mathcal{V}}) = 0. \quad (B.4)
\]

Proof. Lemma B.3 yields $\mathcal{R}(I_q - A A_{\mathcal{U}}^*) = \mathcal{U}^\perp$. Hence, from Remark B.6 we conclude
\[
T^\dagger \left( \mathcal{R}(I_q - A A_{\mathcal{U}}^*) \right) = T^\dagger (\mathcal{U}^\perp) \subseteq \mathcal{U}^\perp \subseteq \mathcal{V}^\perp. \quad (B.5)
\]
for each $\ell \in \mathbb{N}_0$. Since we know from Lemma B.1 that $\mathcal{N}(A_{\mathcal{U}}^*) = \mathbb{C}^q$ is valid, it follows $T^\dagger (I_q - A A_{\mathcal{U}}^*) x \in \mathcal{N}(A_{\mathcal{U}}^*)$ for each $\ell \in \mathbb{N}_0$ and each $x \in \mathbb{C}^q$. Consequently, the first equation in (B.3) is fulfilled for each $\ell \in \mathbb{N}_0$. Lemma B.1 yields $\mathcal{N}(B_{\mathcal{V}}^*) = \mathcal{V}^\perp$. Thus, we obtain from (B.3) that $T^\dagger (I_q - A A_{\mathcal{U}}^*) x \in \mathcal{N}(B_{\mathcal{V}}^*)$ is fulfilled for every choice of $\ell \in \mathbb{N}_0$ and $x \in \mathbb{C}^q$. Therefore, the second equation in (B.3) is proved for each $\ell \in \mathbb{N}_0$. Lemma B.3 provides us $\mathcal{R}(I_q - B B_{\mathcal{V}}) = \mathcal{V}^\perp$. Hence, Remark B.6 yields
\[
T^k \left( \mathcal{R}(I_q - B B_{\mathcal{V}}) \right) = T^k (\mathcal{V}^\perp) \subseteq \mathcal{U}^\perp \subseteq \mathcal{V}^\perp. \quad (B.6)
\]
for each \( k \in \mathbb{N} \). Since Lemma B.1 shows that \( \mathcal{N}(A_q^\perp) = U^\perp \) is true, we obtain then \( T^k(I_q - BB^\perp)x \in \mathcal{N}(A_q^\perp) \) for every choice of \( k \in \mathbb{N} \) and \( x \in \mathbb{C}^q \). Consequently, the first equation in (B.4) is true for each \( k \in \mathbb{N} \). Using (B.6) and the equation \( \mathcal{N}(B_q^\perp) = V^\perp \), which is proved in Lemma B.1, we get \( T^k(I_q - BB^\perp)x \in \mathcal{N}(B_q^\perp) \) for each \( k \in \mathbb{N} \) and each \( x \in \mathbb{C}^q \). Thus, the second equation in (B.4) is verified for each \( k \in \mathbb{N} \) as well.

Now we state some more or less known identities for the matrix-valued functions defined in Remark 4.3.

**Remark B.8.** Let \( n \in \mathbb{N}_0 \) and let \( w, z \in \mathbb{C} \). Then one can easily see that the equations

\[
RT_{q,n}(z)(I_{(n+1)q} - wT_{q,n}) = (I_{(n+1)q} - wT_{q,n})RT_{q,n}(z),
\]

\[
RT_{q,n}(z)(I_{(n+1)q} - wT^*_{q,n}) = (I_{(n+1)q} - wT^*_{q,n})RT^*_{q,n}(z),
\]

\[
RT_{q,n}(z) - RT_{q,n}(w) = (z - w)RT_{q,n}(w)T_{q,n}RT_{q,n}(z),
\]

\[
[RT_{q,n}(w)]^{-1} - [RT_{q,n}(z)]^{-1} = (z - w)T_{q,n},
\]

\[
RT_{q,n}(z) + (w - z)RT_{q,n}(z)T_{q,n}RT_{q,n}(w) = RT_{q,n}(w),
\]

\[
zRT_{q,n}(z) + (w - z)RT_{q,n}(z)RT_{q,n}(w) = wRT_{q,n}(w),
\]

\[
(z - \bar{w})[RT_{q,n}(w)] T_{q,n}RT_{q,n}(z) = RT_{q,n}(z) - [RT^*_{q,n}(w)],
\]

\[
(z - w)T_{q,n}RT_{q,n}(z) = RT_{q,n}(z)[RT_{q,n}(w)]^{-1} - I_{(n+1)q}, \tag{B.7}
\]

\[
(z - w)T^*_{q,n}RT^*_{q,n}(z) = [RT^*_q(w)]^{-1}RT^*_q(z) - I_{(n+1)q}, \tag{B.8}
\]

and

\[
(z - w)RT_{q,n}(z)T_{q,n} = RT_{q,n}(z)[RT_{q,n}(w)]^{-1} - I_{(n+1)q} \tag{B.9}
\]

hold true. Furthermore, for each \( \ell \in \mathbb{N}_0 \), it is readily checked that

\[
T^\ell_{q,n}RT_{q,n}(z) = RT_{q,n}(z)T^\ell_{q,n}, \quad RT_{q,n}(z)T^\ell_{q,n}RT_{q,n}(w) = RT_{q,n}(w)T^\ell_{q,n}RT_{q,n}(z),
\]

and

\[
(T^*_{q,n})^\ell RT^*_{q,n}(z) = RT^*_{q,n}(z)(T^*_{q,n})^\ell, \quad RT^*_{q,n}(z)(T^*_{q,n})^\ell RT^*_{q,n}(w) = RT^*_{q,n}(w)(T^*_{q,n})^\ell RT^*_{q,n}(z).\]

### C. Some considerations on non-negative Hermitian measures

In this appendix, we summarize some facts of the integration theory of non-negative Hermitian measures. We consider a measurable space \((\Omega, \mathfrak{A})\) and use the notation \( \mathcal{M}^b_\geq(\Omega, \mathfrak{A}) \) to denote the set of all non-negative Hermitian \( q \times q \) measures on \((\Omega, \mathfrak{A})\).

**Remark C.1.** Let \( \mu : \mathfrak{A} \to \mathbb{C}^{p \times p} \) be a mapping. Then \( \mu \in \mathcal{M}^b_\geq(\Omega, \mathfrak{A}) \) if and only if \( B^*\mu B : \mathfrak{A} \to \mathbb{C}^{p \times p} \) defined by \((B^*\mu B)(A) : = B^*\mu(A)B\) belongs to \( \mathcal{M}^b_\geq(\Omega, \mathfrak{A}) \) for all \( B \in \mathbb{C}^{q \times p} \).

**Remark C.2.** Let \( \mu \in \mathcal{M}^b_\geq(\Omega, \mathfrak{A}) \) and let \( f : \Omega \to \mathbb{C} \) be a function. Then it is readily checked by standard arguments of measure and integration theory that the following statements are equivalent:

\[(i) \quad f \in L^1(\Omega, \mathfrak{A}, \mu; \mathbb{C}).\]
(ii) $f \in L^1(\Omega, \mathcal{A}, B^* \mu; \mathbb{C})$ for all $B \in \mathbb{C}^{q \times p}$.

(iii) $f \in L^1(\Omega, \mathcal{A}, \tau; \mathbb{C})$ where $\tau := \text{tr } \mu$ is the trace measure of $\mu$.

If (i) holds true, then $\int_A f \, d(B^* \mu) = B^* (\int_A f \, d\mu) B$ for all $A \in \mathcal{A}$ and all $B \in \mathbb{C}^{q \times p}$.

Lemma C.3. Let $\mu \in M^{p}_\geq(\Omega, \mathcal{A})$ and let $\mu'_\tau$ be a version of the Radon–Nikodym derivative of $\mu$ with respect to the trace measure $\tau := \text{tr } \mu$. Let $f: \Omega \to \mathbb{C}$ and $g: \Omega \to \mathbb{C}$ be $\mathcal{A} \cdot \mathcal{B}_\mathbb{C}$-measurable functions. Then the following statements are equivalent:

(i) $f \mu \in L^1(\Omega, \mathcal{A}, \mu; \mathbb{C})$.

(ii) The pair $[fI_q, gI_q]$ is left-integrable with respect to $\mu$.

If (ii) is fulfilled, then $\int_\Omega f g d\mu = \int_\Omega (fI_q) d(\mu(gI_q)^*)$.

Lemma C.3 can be proved by standard methods of measure and integration theory.

Remark C.4. Let $\mu \in M^{p}_\geq(\Omega, \mathcal{A})$ and let $m, n \in \mathbb{N}$. For each $j \in \mathbb{Z}_{1,m}$, let $p_j \in \mathbb{N}$ and let $\Phi_j: \Omega \to C^{p_j \times q}$ be an $\mathcal{A} \cdot \mathcal{B}_{p_j \times q}$-measurable matrix-valued function. For each $k \in \mathbb{Z}_{1,n}$, let $r_k \in \mathbb{N}$ and let $\Psi_k: \Omega \to C^{r_k \times q}$ be an $\mathcal{A} \cdot \mathcal{B}_{r_k \times q}$-measurable matrix-valued function. Suppose that, for every choice of $j \in \mathbb{Z}_{1,m}$, and $k \in \mathbb{Z}_{1,n}$ the pair $[\Phi_j, \Psi_k]$ is left-integrable with respect to $\mu$. Let $s, t \in \mathbb{N}$. For each $j \in \mathbb{Z}_{1,m}$, let $A_j \in C^{s \times p_j}$, and, for each $k \in \mathbb{Z}_{1,n}$, let $B_k \in C^{t \times r_k}$. Then it is readily checked that the pair $[\sum_{j=1}^m A_j \Phi_j, \sum_{k=1}^n B_k \Psi_k]$ is left-integrable with respect to $\mu$ and that

$$\int_\Omega \left( \sum_{j=1}^m A_j \Phi_j \right) d\mu \left( \sum_{k=1}^n B_k \Psi_k \right)^* \geq \sum_{j=1}^m \sum_{k=1}^n A_j \left( \int_\Omega \Phi_k d\mu \Psi_k^* \right) B_k^*.$$

Proposition C.5. Let $\mu \in M^{p}_\geq(\Omega, \mathcal{A})$, let $\tau := \text{tr } \mu$ be the trace measure of $\mu$, and let $\mu'_\tau$ be a version of the Radon–Nikodym derivative of $\mu$ with respect to $\tau$. Furthermore, let $\Theta \in p \times q - L^2(\Omega, \mathcal{A}, \mu; \mathbb{C})$. Then:

(a) $\mu_\Theta: \mathcal{A} \to C^{p \times p}$ defined by $\mu_\Theta(A) := \int_\Omega \Theta d\mu \Theta^*$ belongs to $M^{p}_\geq(\Omega, \mathcal{A})$.

(b) The non-negative Hermitian measure $\mu_\Theta$ is absolutely continuous with respect to $\tau$ and $\Theta \mu'_\tau \Theta^*$ is a version of the Radon–Nikodym derivative of $\mu_\Theta$ with respect to $\tau$.

(c) Let $r, s \in \mathbb{N}$, let $\Phi: \Omega \to C^{r \times p}$ be an $\mathcal{A} \cdot \mathcal{B}_{r \times p}$-measurable function and let $\Psi: \Omega \to C^{s \times p}$ be an $\mathcal{A} \cdot \mathcal{B}_{s \times p}$-measurable function. Then the pair $[\Phi, \Psi]$ is left-integrable with respect to $\mu_\Theta$ if and only if the pair $[\Phi \Theta, \Psi \Theta]$ is left-integrable with respect to $\mu$. In this case, $\int_\Omega \Phi d\mu_\Theta \Psi^* = \int_\Omega (\Phi \Theta) d(\mu \Theta)^*$.

Proposition C.5 can be proved by standard arguments of measure and integration theory.

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