Stability of Quadratic Functional Equation in Two Variables

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Abstract In this paper, we establish the general solution of a 2-variable quadratic functional equation
\[ f(x + y, 2z + w) - f(x - y, z - w) + 4f(x, z) + f(y, w) \]
and prove the generalized Hyers-Ulam stability of this functional equation.

Keywords: solution, stability, quadratic functional equation

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1. Introduction

One of the interesting questions in the theory of functional equations is the following (see [2]):

**When is it true that a function which approximately satisfies a functional equation \( F \) must be close to an exact solution of \( F \)?**

If there exists an affirmative answer we say that the equation \( F \) is stable. The stability problems of functional equations were raised by S. M. Ulam during his talk before a Mathematical Colloquium at the University of Wisconsin in 1940 [15]:

Given a group \( G_1 \), a metric group \( (G_2, d) \) and a positive number \( \varepsilon \), does there exist a number \( \delta > 0 \) such that if a function \( f : G_1 \rightarrow G_2 \) satisfies the inequality
\[ d(f(xy), f(x)f(y)) < \delta \quad \text{for all} \quad x, y \in G_1, \]
then there exists a homomorphism \( T : G_1 \rightarrow G_2 \) such that
\[ d(f(x), T(x)) < \varepsilon \quad \text{for all} \quad x \in G_1. \]

If the answer is affirmative, we would say that the equation of homomorphism \( T(xy) = T(x)T(y) \) is stable.

The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Subsequently, his result was extended and generalized in several ways (see e.g. [13]). Th.M. Rassias [15] extended Hyers’ theorem in the following form where Cauchy difference is allowed to be unbounded:

Let \( X \) and \( Y \) be real normed spaces with \( Y \) complete, for a mapping \( f : X \times X \rightarrow Y \), consider the 2-variable quadratic functional equation:
\[ f(x + y, 2z + w) - f(x - y, z - w) + 4f(x, z) + f(y, w) \]
When \( X = \mathbb{R} \), we see the quadratic form given by
\[ f(x, y) = ax^2 + bxy + cy^2 \]
is a solution of (1.1). In fact, we can check that
\[
\begin{align*}
\|f(x) - T(x)\| &\leq \frac{\varepsilon}{1 - 2^p} \\
\text{for all} \quad x \in X.
\end{align*}
\]
In 1994, a generalization of Rassias’ theorem was obtained by Gavruta P. Gavruta [10] in the spirit of Th. M. Rassias’ approach.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem. A large list of references can be found, the reader is referred to [5,13,14] and references therein for further information on stability.

Let \( X \) and \( Y \) be vector spaces. For a mapping \( f : X \times X \rightarrow Y \), consider the 2-variable quadratic functional equation:
\[ f(x + y, 2z + w) = f(x + y, z + w) - f(x - y, z - w) + 4f(x, z) + f(y, w) \]
When \( X = Y = \mathbb{R} \), we see the quadratic form given by
\[ f(x, y) = ax^2 + bxy + cy^2 \]
is a solution of (1.1). In fact, we can check that
\[
\begin{align*}
f(x + y, 2z + w) &= a(x + y)^2 + b(x + y)(2z + w) + c(2z + w)^2 \\
&= a(x + y)^2 + b(x + y)(z + w) + c(z + w)^2 \\
&\quad - \left[ a(x - y)^2 + b(x - y)(z - w) + c(z - w)^2 \right] \\
&\quad + 4ax^2 + bxz + cz^2 + 4ay^2 + byw + cw^2 \\
&= f(x + y, z + w) - f(x - y, z - w) + 4f(x, z) + f(y, w).
\end{align*}
\]
For a mapping \( g : X \rightarrow Y \), Now, we consider the quadratic functional equation:
One can easily verify that \( g(2x + y) = g(x + y) - g(x - y) + 4g(x) + g(y) \) \((1.2)\)

In a one paper, by using the fixed point theorem method, C. Park \([3]\) proved the generalized Hyers-Ulam stability of the quadratic functional equation \((1.2)\).

In this paper, we investigate the relation between \((1.1)\) and \((1.2)\). And we find out the general solution and the generalized Hyers-Ulam stability of \((1.1)\).

2. The Relation between \((1.1)\) and \((1.2)\)

**Theorem 2.1.** Let \( f : X \times X \to Y \) be a mapping satisfying \((1.1)\) and let \( g : X \to Y \) be the mapping given by

\[
g(x) = f(x, x) \tag{2.1}
\]

for all \( x \in X \), then \( g \) satisfies \((1.2)\).

**Proof.** By \((1.1)\) and \((2.1)\), we can show that

\[
g(2x + y) = f(2x + y, 2x + y)
= f(x + y, x + y) + f(x - y, x - y)
+ 4f(x, y) - f(y, y)
= g(x + y) - g(x - y) + 4g(x) + g(y)
\]

for all \( x, y \in X \).

**Theorem 2.2.** Let \( a, b, c \in \mathbb{R} \) and \( g : X \to Y \) be a mapping satisfying \((1.2)\). If \( f : X \times X \to Y \) is the mapping given by

\[
f(x, y) := ag(x) + b\left[ g(x + y) - g(x - y) \right] + cg(x) \tag{2.2}
\]

for all \( x, y \in X \), then \( f \) satisfies \((1.1)\).

**Proof.** By \((1.2)\) and \((2.2)\), we can show that

\[
f(2x + y, 2z + w)
= ag(2x + y) + b\left[ g(2x + y + 2z + w) \right]
= 4ag(x) + 4ag(y) + 4ag(x + y) - 4ag(x - y)
+ b\left[ 4g(x + z) - g(y + w) \right]
+ g(x + z + y + w) - g(x + z - y - w)
- b\left[ 4g(x - z) - g(y + w) \right]
+ g(x - z + y - w) - g(x - z - y + w)
+ 4cg(z) + 4cg(w) + cg(z + w) - cg(z + w)
= ag(x + y) + b\left[ g(x + z + y + w) \right]
- g(x - y + z - w) + cg(z + w)
- ag(x - y) + b\left[ g(x - y + z - w) - g(x - y + z - w) \right]
+ cg(z - w) + 4ag(x) + b\left[ g(x + z) - g(x - z) \right]
+ cg(z) + 4cg(w) + b\left[ g(x + z) - g(x - z) \right] + cg(w)
\]

for all \( x, y, z, w \in X \). This completes the proof.

3. Solution and Stability Results

In the following theorem, we find out the general solution of the main functional equation \((1.1)\).

**Theorem 3.1.** A mapping \( f : X \times X \to Y \) satisfies \((1.1)\) if and only if there exist two symmetric bi-additive mappings \( S_1, S_2 : X \times X \to Y \) and a bi-additive mapping \( B : X \times X \to Y \) such that

\[
f(x, y) = S_1(x, x) + B(x, y) + S_2(x, y)
\]

for all \( x, y \in X \).

**Proof.** We first assume that there exist two symmetric bi-additive mappings

\[
S_1, S_2 : X \times X \to Y
\]

and a bi-additive mapping

\[
B : X \times X \to Y
\]

such that

\[
f(x, y) = S_1(x, x) + B(x, y) + S_2(x, y)
\]

for all \( x, y \in X \). Then we have

\[
f(2x + y, 2z + w) - f(x + y, z + w) - f(x - y, z - w)
= S_1(2x + y, 2x + y) + B(2x + y, 2z + w)
+ S_2(2z + w, 2z + w)
-[S_1(x, x, x, y) + B(x + y, z + w) + S_2(z + w, z + w)]
+[S_1(x, y, x, y) + B(x - y, z - w) + S_2(z - w, z - w)]
= 4S_1(x, x) + S_1(y, y) + 4B(x, z)
+ B(y, w) + 4S_2(z, z) + S_2(w, w)
= 4S_1(x, x) + B(x, z) + S_2(z, z)
+[S_1(y, y) + B(y, w) + S_2(w, w)]
= 4f(x, z) + f(y, w)
\]

for all \( x, y, z, w \in X \).

Conversely, we assume that \( f \) is a solution of \((1.1)\). Define \( f_1, f_2 : X \to Y \) by \( f_1(x) := f(x, 0) \) and \( f_2(x) = f(0, x) \) for all \( x \in X \). One can easily verify that \( f_1, f_2 \) are quadratic. By \([16]\), there exist two symmetric bi-additive mappings

\[
S_1, S_2 : X \times X \to Y
\]

such that \( f_1 = S_1(x, x) \) and \( f_2 = S_2(x, x) \) for all \( x \in X \).

Define \( B : X \times X \to Y \) by

\[
B(x, y) := f(x, y) - \left[ f(x, 0) + f(0, y) \right]
\]

for all \( x, y \in X \). Then, it is easy to investigate that \( B \) is bi-additive. This completes the proof.
In the following theorem, let $X$ be a vector space and $Y$ be a Banach space. Given a function $f : X \times X \to Y$, we set
\[
Df(x,y,z,w) = f(2x+y, z+w) - f(x+y, z+w) + f(x, z-w) - 4f(x, z) - f(y, w)
\]
for all $x, y, z, w \in X$.

**Theorem 3.2.** Let $f : X \times X \to Y$ be a mapping for which there exists a function $\phi : X^4 \to [0, \infty)$, such that
\[
\left\|Df(x,y,z,w)\right\| \leq \phi(x, y, z, w) \tag{3.1}
\]
for all $x, y, z, w \in X$. Then there exists a unique 2-variable quadratic mapping $A : X \times X \to Y$ such that
\[
\|f(x,y) - A(x,y)\| \leq \frac{1}{4} \phi(x,0,0,0) \tag{3.3}
\]
for all $x, y \in X$. The mapping $A$ is given by
\[
A(x,y) = \lim_{n \to \infty} \frac{1}{4^n} f(2^n x, 2^n y)
\]
for all $x,y \in X$.

**Proof.** Letting $y = 0$ and $w = 0$ in (3.2), we get
\[
\left\|\frac{1}{4} f(2x,2z) - f(x,0) + \frac{1}{4} f(0,0)\right\| \leq \frac{1}{4} \phi(x,0,0,0)
\]
for all $x, z \in X$. Thus we obtain
\[
\left\|\frac{1}{4^{j+1}} f(2^{j+1} x, 2^{j+1} z) - \frac{1}{4^j} f(2^j x, 2^j z) + \frac{1}{4^{j+1}} f(0,0)\right\| \leq \frac{1}{4^{j+1}} \phi(2^j x, 0, 2^j z, 0)
\]
for all $x, z \in X$ and all $j$. Replacing $z$ by $y$ in the above inequality, we see that
\[
\left\|\frac{1}{4^{j+1}} f(2^{j+1} x, 2^{j+1} y) - \frac{1}{4^j} f(2^j x, 2^j y) + \frac{1}{4^{j+1}} f(0,0)\right\| \leq \frac{1}{4^{j+1}} \phi(2^j x, 0, 2^j y, 0)
\]
for all $x, y \in X$ and all $j$. For given integers $l, m (0 \leq l < m)$, we get
\[
\left\|\frac{1}{4^m} f(2^m x, 2^m y) - \frac{1}{4^l} f(2^l x, 2^l y) + \frac{1}{4^m} f(0,0)\right\| \leq \frac{1}{4^m} \phi(2^l x, 0, 2^l y, 0)
\]
for all $m > 1$ and all $x, y \in X$. It follows from (3.1) and (3.4) that the sequence $\left\{\frac{1}{4^n} f(2^n x, 2^n y)\right\}$ is Cauchy. Due to the completeness of $Y$, this sequence is convergent. So we can define the mapping $A : X \times X \to Y$ by
\[
A(x,y) = \lim_{n \to \infty} \frac{1}{4^n} f(2^n x, 2^n y)
\]
for all $x, y \in X$. By (3.2) and (3.1), we have
\[
\left\|DA(x,y,z,w)\right\| = \lim_{n \to \infty} \frac{1}{4^n} \left\|Df(2^n x, 2^n y, z, w)\right\| = \lim_{n \to \infty} \frac{1}{4^n} \phi(2^n x, 2^n y, z, w)
\]
for all $x, y, z, w \in X$. So $DA(x,y,z,w) = 0$. Moreover, letting $I = 0$ and passing the limit $m \to \infty$ in (3.4), we get (3.3).

Now let $A : X \times X \to Y$ be another 2-variable quadratic mapping satisfying (3.3). Then we have
\[
\left\|A(x,y) - A'(x,y)\right\| = \frac{1}{4^n} \left\|A(2^n x, 2^n y) - A'(2^n x, 2^n y)\right\| \leq \frac{1}{4^n} \left\|A(2^n x, 2^n y) - f(2^n x, 2^n y)\right\| + \frac{1}{4^n} \left\|A'(2^n x, 2^n y) - f(2^n x, 2^n y)\right\| \leq \frac{1}{2^n} \phi(2^n x, 2^n y, 2^n x, 2^n y)
\]
which tends to zero as $n \to \infty$ for all $x, y \in X$. So we can conclude that $A(x, y) = A'(x, y)$ for all $x, y \in X$. This proves the uniqueness of $A$. This completes the proof.

**Remark 3.3.** Let $f : X \times X \to Y$ be a mapping for which there exists a function $\phi : X^4 \to [0, \infty)$ satisfying (3.2) such that
\[
\phi(x, y, z, w) = \sum_{j=1}^{\infty} 4^j \phi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}, \frac{w}{2^j}\right) < \infty
\]
for all $x, y, z, w \in X$. By a similar method to the proof of Theorem 3.2, one can show that there exists a unique 2-variable quadratic mapping $A : X \times X \to Y$ such that
\[
\|f(x,y) - A(x,y)\| \leq \frac{1}{4} \phi(x,0,y,0)
\]
for all $x, y \in X$. The mapping $A$ is given by
\[
A(x,y) = \lim_{n \to \infty} 4^n f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)
\]
for all $x, y \in X$.

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