Dirac method and symplectic submanifolds in the cotangent bundle of a factorizable Lie group

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Abstract

In this work we study some symplectic submanifolds in the cotangent bundle of a factorizable Lie group defined by second class constraints. By applying the Dirac method, we study many issues of these spaces as fundamental Dirac brackets, symmetries, and collective dynamics. This last item allows to study integrability as inherited from a system on the whole cotangent bundle, leading in a natural way to the AKS theory for integrable systems.

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1 Introduction

Cotangent bundles of Lie groups are one of the most important symplectic models for the phase spaces of dynamical systems. They correspond to systems with configuration spaces being Lie groups, as it is the case for the rigid body and its generalizations (as the main finite dimensional examples [1]) and with the sigma and WZNW models in field theory as some of the infinite dimensional examples. They are symplectic manifolds and enjoy many nice properties related to the symmetry issues [2], [3], and they have close relationship with integrable systems, in particular for those groups where some kind of factorization holds [4].

Many interesting phase spaces arise as constrained submanifolds in a larger phase space. The Dirac method [5] is a successful algorithm for dealing with second class constraints, producing a Poisson bracket for a constrained submanifold, and thus allowing a consistent quantization of this kind of systems. It provides a way to obtain Lie derivatives of functions on the whole phase space along the projection of the hamiltonian vector fields on the tangent space of the constrained submanifold, and so it is a representation of these vectors fields in terms of the geometrical data in the total phase space.

In this work, we combine both the above ingredients to study a family of symplectic submanifolds in the cotangent bundle of a factorizable Lie group $G = G_+ \times G_-$, defined as the level set of a map which projects the cotangent bundle $T^*G$ onto the cotangent bundle of one of its factors. To deal with this kind of constraints we develop, in a general framework, a geometric approach to Dirac’s brackets addressed to describe immersed symplectic submanifold $\mathcal{N}, \iota: \mathcal{N} \hookrightarrow \mathcal{M}$, as the level sets of some submersive map from $\mathcal{M}$ into another manifold $\mathcal{P}$. This approach allows for a straightforward application to the projection maps $\Psi: G \times g^* \rightarrow G_- \times g_-^*$ and $\Upsilon: G \times g^* \rightarrow G_+ \times g_+^*$, from where the symplectic submanifolds of $T^*G$ are obtained, equipped with the corresponding
Poisson-Dirac structure, turning them in symplectic fibrations. This framework allows to study many aspects of theses spaces and related dynamical systems such as symmetries and integrability. In fact, as by product, we work out the application of the AKS ideas \[6\] to some collective models on these spaces exploiting the tools constructed previously, showing how to make contact with the AKS results from the Dirac bracket approach and offering an insight on the geometric aspects underlying these issues.

To put in practice these constructions, we apply them on the group $SL(2, \mathbb{C})$ which factorize as $SU(2) \times B$, where $B$ is the solvable group of $2 \times 2$ complex upper triangular matrices with real positive diagonal elements and determinant 1, coming to generalize the results of \[7\]. We work out a collective hamiltonian models on a generic fiber of $\Psi : SL(2, \mathbb{C}) \times \mathfrak{sl}_2 \to B \times b^*$, and construct the corresponding Lagrangian version.

This work is organized as follows: in the second Section we carry out a brief review of the Dirac procedure addressed to deal with level set of submersive maps from the phase space to another manifolds; in the third Sections we define, by means of the Dirac procedure developed in the previous Section, a class of phase spaces in the cotangent bundle of a factorizable Lie group, working out many issues as fundamental brackets, symmetries and Hamilton equations; in the fourth Section, the connection with integrable models and factorization problem is studied from the Dirac procedure and symplectic reduction points of view. Finally, in the fifth Section we present the example built on $SL(2, \mathbb{C}) = SU(2) \times B$, and in the sixth Section some conclusions are summarized.

2 A brief review of Dirac procedure

Let $(\mathcal{M}, \omega)$ be a symplectic manifold and $(\mathcal{N}, i^*\omega)$ a symplectic submanifold with the immersion $i : \mathcal{N} \hookrightarrow \mathcal{M}$.

**Lemma:** The tangent space at a point $i(n) \in \mathcal{M}$ can be decomposed in a direct sum as
\[
T_{i(n)}\mathcal{M} = i_* (T_n\mathcal{N}) \oplus [i_* (T_n\mathcal{N})]^\omega_{\perp}. \tag{1}
\]

**Proof:** Here $[V]^\omega_{\perp}$ means the symplectic orthogonal of $V$ by the 2-form $\omega$. This result stems from the non degeneracy of $\omega$. \qed

**Lemma:** Let be $F \in C^\infty(\mathcal{N})$, $V_F \in X(\mathcal{M})$ its Hamiltonian vector field and $\pi_{\mathcal{N}}|_{i(n)} : T_{i(n)}\mathcal{M} \to i_* (T_n\mathcal{N})$ the projection. Then
\[
i_* V_{i^* F}|_{i(n)} = \pi_{\mathcal{N}}(V_F)|_{i(n)}. \tag{2}
\]

The Dirac procedure arises from considering a submanifold defined as the level surfaces of some submersive map $\Psi$ from $\mathcal{M}$ into another manifold $\mathcal{P}$. So, for some regular value $p_0 \in \mathcal{P}$, we get a submanifold $\mathcal{N}$ defined as
\[i(\mathcal{N}) = \Psi^{-1}(p_0)\]

The following theorem is in truly realm of Dirac’s idea.
Theorem: Let $\Psi : M \to P$ be a surjective map, and $p_0 \in P$ a regular value of $\Psi$. Then, the cotangent space at a point $i(n) \in M$ is decomposed as

$$T_{i(n)}^* M = \omega(T_n N) \oplus \Psi^* T_{p_0}^* P. \quad (3)$$

Proof: For $n \in \Psi^{-1}(p_0)$, we have the exact short sequence

$$0 \to T_n N \xrightarrow{i_*} T_{i(n)} M \xrightarrow{\Psi^*} T_{p_0} \xrightarrow{i^*} T_n N \to 0$$

and by duality

$$0 \to T_{p_0}^* \xrightarrow{\Psi^*} T_{i(n)} M \xrightarrow{i^*} T_n^* N \to 0$$

so that

$$T_{i(n)}^* M = i^* \left[ \frac{T_{i(n)} M}{\Psi^* (T_{p_0}^* P)} \right]$$

from where we conclude that

$$(i_* (T_n N))^\perp = \Psi^* (T_{p_0}^* P) \quad (4)$$

Now, since $\omega|_{T_n N}$ is nondegenerate, $\omega(i_* (T_n N))$ is complementary to $(i_* (T_n N))^\perp$ in $T_{i(n)}^* M$, then

$$T_{i(n)}^* M = \omega(i_* (T_n N)) \oplus \Psi^* (T_{p_0}^* P)$$

Proposition: The induced bijection $\omega : T M \to T^* M$ provides the isomorphism

$$\Psi^*_{i(n)} (T_{p_0}^* P) \cong [i_*(T_n N)]^{\omega \perp}, \quad (5)$$

for all $n \in N$.

Proof: First, observe that $(\Psi^* \alpha)_{i(n)} \in [\Psi^* (T_{p_0}^* P)]_{i(n)}$, $\alpha \in T_{p_0}^* P$, the bijection $\omega$ assigns a $V_\alpha \in T_{i(n)}^* M$ such that $i_{V_\alpha} \omega = d\Psi^* \alpha$. Then, $\forall W \in i_* (T_n N) = \ker \Psi_{st(n)}$ we have

$$\langle \omega, V_\alpha \otimes W \rangle = \langle d\Psi^* \alpha, W \rangle = \langle d\alpha, \Psi_* W \rangle = 0$$

then $V_\alpha \in [i_*(T_n N)]^{\omega \perp}$.

Reciprocally, $\forall W \in [i_*(T_n N)]^{\omega \perp}$, $\exists \alpha_W T_{i(n)}^* M / i_W \omega = \alpha_W$. Moreover, $\forall V \in i_* (T_n N) = \ker \Psi_{st(n)}$,

$$\langle \alpha_W, V \rangle = \langle \omega, W \otimes V \rangle = 0$$

then $\alpha_W \in (i_* (T_n N))^\perp = \Psi^*_{i(n)} (T_{p_0}^* P)$. ■

Before to going on, it is worth to distinguish the two following situations which apply into the most important physical systems:
1. First class $\iff [i_*(T_nN)]^{\omega \perp} \subset i_*(T_nN) \iff \mathcal{N}$ is a coisotropic submanifold.

2. Second class $\iff [i_*(T_nN)]^{\omega \perp} \cap i_*(T_nN) = \{0\}$

From now on, we shall be concerned with the second class case.

**Theorem:** $\mathcal{N} = \Psi^{-1}(p_0)$ is symplectic if and only if

\[ \omega \left( \ker \Psi_{\ast(n)} \right) \cap \Psi^\ast(T_{p_0} \mathcal{P}) \big|_{i_{\ast(n)}} = \{0\} \]

**Proof:** Since $p_0$ is a regular value of $\Psi$ and $\mathcal{N}$ is a level surface of $\Psi$, $\ker \Psi_{\ast(n)} = i_*(T_n\mathcal{N})$. Then, because $\omega : T_n\mathcal{N} \rightarrow T^*_n\mathcal{N}$ is a bijection, there is a one to one correspondence between tangent vectors to $\mathcal{N}$ and linear forms. So that seeking for null vectors of $i^*\omega$ in $T_n\mathcal{N}$ is equivalent to look for $\alpha \in \omega(\ker \Psi_{\ast(n)})$ vanishing on $T_n\mathcal{N}$, that is $\alpha \in \omega(\ker \Psi_{\ast(n)}) \cap (i_*(T_n\mathcal{N}))^\circ$. Regularity of $i^*\omega$ implies no nontrivial such an $\alpha$ does exist, so $\omega(\ker \Psi_{\ast(n)}) \cap \Psi^\ast(T_{p_0} \mathcal{P}) \big|_{i_{\ast(n)}} = \{0\}$. ■

**Corollary:** $\Psi$ is a second class constraint if and only if

\[ \omega \left( \ker \Psi_{\ast(n)} \right) \cap \Psi^\ast(T_{p_0} \mathcal{P}) \big|_{i_{\ast(n)}} = \{0\} \]

**Proof:** $\Psi$ is a second class if and only if $[i_*(T_n\mathcal{N})]^{\omega \perp} \cap i_*(T_n\mathcal{N}) = \{0\}$, that is equivalent to $(i_*(T_n\mathcal{N}))^\circ \cap \ker \Psi_{\ast(n)} = \{0\}$. And it happens if and only if $\omega(\ker \Psi_{\ast(n)}) \cap \Psi^\ast(T_{p_0} \mathcal{P}) \big|_{i_{\ast(n)}} = \{0\}$. ■

Let us now introduce a suitable set of functions $\{f_1, \cdots, f_r\} \subset C^\infty(\mathcal{P})$, $r = \dim \mathcal{P}$, such that $\{df_1|_{p_0}, \cdots, df_r|_{p_0}\}$ is a basis of $T_{p_0} \mathcal{P}$. Then, $\{f_1, \cdots, f_r\}$ is local coordinate system for $\mathcal{P}$. Let us consider then the set of local hamiltonian vector fields associated with the pullback of these functions

$$\Psi^\ast f_j \longrightarrow V_{\Psi^\ast f_j} \big|_{\Psi^\ast f_j} \omega = d \left( \Psi^\ast f_j \right)$$

**Proposition:** Let be $\Psi$ a second class constraint, then the set $\{V_{\Psi^\ast f_j}\}_{i=1}^r$ is a basis for $[i_*(T_n\mathcal{N})]^{\omega \perp}$.

**Proof:** Let be $W \in i_*(T_n\mathcal{N}) = \ker \Psi_{\ast(n)}$, then

$$\langle \omega, V_{\Psi^\ast f_j} \otimes W \rangle_{i_{\ast(n)}} = \langle d \left( \Psi^\ast f_j \right), W \rangle_{i_{\ast(n)}} = \langle df_j, \Psi^\ast W \rangle_{\Psi_{\ast(n)}} = 0$$

from where we conclude that each $V_{\Psi^\ast f_j} \in [i_*(T_n\mathcal{N})]^{\omega \perp}$

One may see that the functions $\Psi^\ast f_1, \cdots, \Psi^\ast f_r$ are constant on $\mathcal{N} = \Psi^{-1}(p_0)$.

Also, we know that $\{df_1|_{p_0}, \cdots, df_r|_{p_0}\}$ is a linearly independent set, then so is $\{d\Psi^\ast f_1|_{i_{\ast(n)}}, \cdots, d\Psi^\ast f_r|_{i_{\ast(n)}}\}$ because $\Psi$ is a surjective map and $\Psi^\ast$ is an injective one. Then, the set $\{V_{\Psi^\ast f_j}\}_{i=1}^r$ is linearly independent too.
From theorem \(3\) we know that \(T^*_\iota(n)M = \omega (\iota_* (T_n \mathcal{N})) \oplus \Psi^*_\iota(n) (T^*_p \mathcal{P})\), and because \(\omega\) is nondegenerate on \(\iota_* (T_n \mathcal{N})\) we can apply \(\omega^{-1}\) on this direct sum to map it to the tangent space
\[
T^*_\iota(n)M = \iota_* (T_n \mathcal{N}) \oplus \omega^{-1} \left( \Psi^*_\iota(n) (T^*_p \mathcal{P}) \right)
\]
and this decomposition is symplectically orthogonal. This shows that
\[
\omega^{-1} \left( \Psi^*_\iota(n) (T^*_p \mathcal{P}) \right) = [\iota_* (T_n \mathcal{N})]^{\omega^\perp}
\]
This finish the proof. ■

**Corollary:** The set
\[
\left\{ d\Psi^* f_1|_{\iota(n)}, \ldots, d\Psi^* f_r|_{\iota(n)} \right\} \subset T^*_\iota(n)M
\]
is a basis of \([\iota_* (T_n \mathcal{N})]^0 = \Psi^*_\iota(n) (T^*_p \mathcal{P})\).

Then, from the last four assertions, we conclude that in the second class case any vector \(V \in T^*_\iota(n)M\) can be written as
\[
V_{\iota(n)} = (\iota_* v)_{\iota(n)} + \sum_{i=1}^{r} a^i(n) V_{\Psi^* f_i}
\]
where \(v \in T_n \mathcal{N}\). Let us introduce the Dirac matrix
\[
C_{jk} (\iota(n)) = \langle d\Psi^* f_j, V_{\Psi^* f_k} \rangle_{\iota(n)} = \{ \Psi^* f_j, \Psi^* f_k \} (\iota(n))
\]
Observe that, in the case of \(\Psi\) being second class, \(\omega (\iota_* (T_n \mathcal{N})) \cap \Psi^* (T^*_p \mathcal{P})|_{\iota(n)} = \{0\}\), there are no hamiltonian forms associated to vector of \(\iota_* (T_n \mathcal{N})\) contained in \(\Psi^* (T^*_p \mathcal{P})|_{\iota(n)}\). Then, in this case the vector space \(\Psi^*_\iota(n) (T^*_p \mathcal{P}) = [\iota_* (T_n \mathcal{N})]^0\) is the dual of \([\iota_* (T_n \mathcal{N})]^{\omega^\perp}\)
\[
\Psi^*_\iota(n) (T^*_p \mathcal{P}) = \left([\iota_* (T_n \mathcal{N})]^{\omega^\perp}\right)^*
\]
so that the matrix obtained from the contraction of both the basis \(\left\{ d\Psi^* f_j|_{\iota(n)} \right\}\) and \(\left\{ V_{\Psi^* f_j} \right\}_{j=1}^{r}\), namely \(C_{jk} (\iota(n))\), is an invertible one.

Contracting the vector \(V_{\iota(n)}\) with \(d\Psi^* f_j\), and because \(\langle d\Psi^* f_k, \iota_* v \rangle_{\iota(n)} = \langle df_k, \Psi^* \iota_* v \rangle_{\iota(n)} = 0\) (remember that \(\iota_* (T_n \mathcal{N}) = \ker \Psi^*_{\iota(n)}\)), we get
\[
a^j(n) = \sum_{k=1}^{r} C_{jk} (\iota(n)) \langle d\Psi^* f_k, V \rangle_{\iota(n)}
\]
Writing the vector \(V, W\) in terms of the expression \(3\), including the coefficients given in the last equation, we get from the contraction \(\langle \omega, V \otimes W \rangle_{\iota(n)}\) the following relation
\[
\langle \iota_* \omega, v \otimes w \rangle_n = \langle \omega, V \otimes W \rangle_{\iota(n)} + \left( \sum_{k,l=1}^{r} C_{lk} (\iota(n)) \Psi^* (df_l \wedge df_k), V \otimes W \right)_{\iota(n)}
\]
Lemma: Let be \( \alpha \in \Omega^1(M) \) and let be \( V_\alpha \in \mathfrak{X}(M) \) the associated hamiltonian vector field. Let the surjection \( \Psi : M \rightarrow \mathcal{P} \) be a second class constraint, and \( \pi_N|_n : T_{(n)}M \rightarrow T_n\mathcal{N} \) the projection on the first factor in the direct sum in \( \mathfrak{g} \). Then

\[
i_*v_{i*}\alpha = \pi_NV_\alpha.
\]

So, let us now specialize the expression of \( (i^*\omega, v \otimes w)_n \) given above to the hamiltonian vector fields like in the last Lemma, and defining as usual

\[
\{f, g\}_N(n) = \langle i^*\omega, v_f \otimes v_g \rangle_n
\]

\( \forall f, g \in C^\infty(\mathcal{N}) \), we obtain the celebrated Dirac formula relating the Poisson bracket on \( \mathcal{N} \) with the one defined on \( M \):

\[
\{i^*F, i^*H\}_N(n) = \{F, H\}_M(i(n)) - \sum_{l,k=1}^n \{F, \Psi^*f_l\}_M(i(n)) \cdot C^{lk}(i(n)) \cdot \{\Psi^*f_k, H\}_M(i(n))
\]

### 3 Symplectic submanifolds in cotangent bundle of double Lie group as constrained system

In this section we shall consider tangent and cotangent bundles of Lie groups trivialized by left translation, a procedure equivalent to take global body coordinates.

Given three Lie groups \((G, G_+, G_-)\) where \(G_+\) and \(G_-\) are both closed Lie subgroups of \(G\), we say they form a **double Lie group** if there exist a diffeomorphism \(\alpha : G_+ \times G_- \rightarrow G\) defined as \((g_+, g_-) \rightarrow g_+g_-\). In this case, the Lie algebras \((\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)\) form a **double Lie algebra** if \(\mathfrak{g}_+\) and \(\mathfrak{g}_-\) are Lie subalgebras of \(\mathfrak{g}\), and \(\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-\) as vector spaces. By duality, this decomposition gives rise also to the factorization of \(\mathfrak{g}^* = \mathfrak{g}_+^* \oplus \mathfrak{g}_-^*\), where

\[
\mathfrak{g}^*_\pm = \{ \eta \in \mathfrak{g}^* / \langle \eta, X_\pm \rangle = 0, \ \forall X_\pm \in \mathfrak{g}_\pm^* \}
\]

that allows to make the identifications \(\mathfrak{g}^*_\pm \cong \mathfrak{g}_\pm^*\). In particular, if \(\mathfrak{g}\) is supplied with a \(Ad^G\)-invariant nondegenerate symmetric bilinear form turning \(\mathfrak{g}_+^*, \mathfrak{g}_-^*\) into isotropic subspaces, the previous identifications implies \(\mathfrak{g}_\pm^* \cong \mathfrak{g}_{\mp}\). Let us denote the corresponding projectors \(\Pi_{\mathfrak{g}_\pm} : \mathfrak{g} \rightarrow \mathfrak{g}_\pm\) and \(\Pi_{\mathfrak{g}_\mp} : \mathfrak{g}^* \rightarrow \mathfrak{g}_{\pm}^*\). We also denote \(\eta_{\pm} = \Pi_{\mathfrak{g}_\pm} \eta\).

Writing out every element \(g \in G\) as \(g = g_+g_-\), with \(g_+ \in G_+\) and \(g_- \in G_-\), the product \(g_+g_-\) in \(G\) can be expressed as \(g_-g_+ = g_-g_+g_+g_- = g_+g_-g_+\), with \(g_+g_- \in G_+\) and \(g_-g_+ \in G_-\). The dressing action of \(G_-\) on \(G_+\) is defined to be

\[
\text{Dr} : G_- \times G_+ \rightarrow G_+ / \text{Dr} (h_-, g_+) := \Pi_{G_+} h_-g_+ = g_+^{h_-} \quad (9)
\]

The infinitesimal generator of this action at the point \(g_+ \in G_+\) is, for \(X_- \in \mathfrak{g}_-, \)

\[
X_- \rightarrow g_+^{X_-} = -\frac{d}{dt} \text{Dr}(e^{tX_-}, g_+) \bigg|_{t=0}
\]
such that, for \( X_-, Y_- \in g_- \), we have \([g_+^{-1}X_-, g_+^{-1}Y_-] = -g_+^{-1}[X_-, Y_-]g_-\). It satisfies the relation
\[
Ad_{g_+^{-1}}^G X_- = g_+^{-1}g_+X_- + Ad_{g_+}^* X_- \tag{10}
\]
where \( Ad_{g_+}^* X_- \in g_- \) is the adjoint action of \( G_- \), and \( Ad_{g_+}^* X_- \in g_- \) is the coadjoint action of \( G_+ \) on the dual of its Lie algebra \( g_+^* \cong g_-^* \). Then, we may write \( g_+ X_- = g_+ \Pi_{g_+} Ad_{g_+^{-1}}^G X_- \).

Let us introduce the fibrations
\[
\Psi : G \times g^* \longrightarrow G_- \times g_-^* \rightarrow \Psi (g, \eta) \rightarrow (\Pi_{G_-} (g), \Pi_{g_-} (\eta)) \tag{11}
\]
\[
\Upsilon : G \times g^* \longrightarrow G_+ \times g_+^* \rightarrow \Upsilon (g, \eta) \rightarrow (\Pi_{G_+} (g), \Pi_{g_+} (\eta))
\]
with fibers
\[
\mathcal{N} (g_-, \eta_-) = \{(g, \eta) \in G \times g^* / \Psi (g, \eta) = (g_-, \eta_-) \}
\]
\[
\mathcal{M} (g_+, \eta_+) = \{(g, \eta) \in G \times g^* / \Upsilon (g, \eta) = (g_+, \eta_+) \}
\]
Observe that the corresponding fibers on \((e, 0) \in G_- \times g_-^*\) reduce to the trivialized cotangent bundles of the factors \( G_+, G_- \), namely \( \mathcal{N} (e, 0) = G_+ \times g_+^* \cong T^*G_+ \) and \( \mathcal{M} (e, 0) = G_- \times g_-^* \cong T^*G_- \).

The differential of the maps \( \Psi \) and \( \Upsilon \) involve the following digression: a vector \( v \in T_g G \) with \( g = g_+g_- \) can be written as
\[
v = v_+g_- + g_+v_-
\]
that in terms of left translated vectors on \( G_+ \times g_+^* \) and \( G_- \times g_-^* \) reads
\[
\dot{g} = g_+ (g_+^{-1}v_+) g_- + g_+g_- (g_-^{-1}v_-)
\]
Hence, the relation with the left trivialization of \( TG \cong G \times g \) arises from
\[
g^{-1}v = Ad_{g_+^{-1}}^G (g_+^{-1}v_+) + (g_-^{-1}v_-)
\]
from where we define
\[
X_+ = \Pi_{g_+} g_+^{-1}v = Ad_{g_-}^* (g_+^{-1}v_+) \tag{12}
\]
\[
X_- = \Pi_{g_-} g_-^{-1}v = (g_+^{-1})g_+^{-1}v_+ g_- + g_-^{-1}v_-
\]
Here we can see that there are no null vectors of the canonical 2-form $\omega_T$ shall be regarded as phase spaces.

In fact, since the cotangent bundle $T^*G \cong G \times \mathfrak{g}^*$ is a symplectic manifold with the canonical 2-form $\omega$, which has associated the nondegenerate Poisson bracket

$$\{ \mathcal{F}, \mathcal{G} \} (g, \eta) \equiv \langle d\mathcal{F}, \delta \mathcal{G} \rangle_{(g,\eta)} - \langle d\mathcal{G}, \delta \mathcal{F} \rangle_{(g,\eta)} = \left[ \eta, \frac{d\mathcal{F}_{\mid (g,\eta)}}{d\mathcal{G}_{\mid (g,\eta)}} \right]$$

where we wrote $d\mathcal{F}_{\mid (g,\eta)} = (d\mathcal{F}, \delta \mathcal{G})_{\mid (g,\eta)} \in T^*G \oplus \mathfrak{g}^*$, we shall study how to apply the Dirac method to supply the submanifolds $\mathcal{N}(g_-, \eta_-)$ and $\mathcal{M}(g_+, \eta_+)$ with a Poisson-Dirac structure.

3.1 The phase spaces $\mathcal{N}(g_-, \eta_-)$

Let us consider the restriction of the canonical Poisson bracket of $T^*G \cong G \times \mathfrak{g}^*$ to the fibers $\mathcal{N}(g_-, \eta_-) := \Psi^{-1}(g_-, \eta_-)$. Then the differential map $\Psi_* : T(G \times \mathfrak{g}^*) \rightarrow T(G_- \times \mathfrak{g}^+)$ is

$$\Psi_* (gX, \xi)_{(g,\eta)} = \frac{d}{dt} \Pi_{G_-}(ge^{tX}) \bigg|_{t=0} = \left( g_- X_+ + g_- X_- , \xi_- \right)_{(g_-, \eta_-)}$$

Therefore, the kernel of $\Psi_*$, that coincides with $TN(g_-, \eta_-)$, is

$$\ker \Psi_*_{\mid (g,\eta)} = \left\{ \begin{array}{l} g_+ \left( Ad_{g_-}^* X_+ \right) g_- , \xi_+ \end{array} / (X_+ , \xi_+ ) \in \mathfrak{g}_+^c \oplus \mathfrak{g}_+^* \right\}$$

At each point $(g, \eta) \in \mathcal{N}(g_-, \eta_-)$, the intersection of $T_{(g,\eta)} \mathcal{N}(g_-, \eta_-)$ with its symplectic orthogonal is $\{ 0 \}$, so we have the following result.

**Proposition:** $(\mathcal{N}(g_-, \eta_-), \tilde{\omega}_o)$, where $\tilde{\omega}_o$ is the restriction to $\mathcal{N}(g_-, \eta_-)$ of the canonical symplectic form $\omega_o$ on $G \times \mathfrak{g}^*$, is a symplectic manifold.

**Proof:** The restriction of the symplectic form to $TN(g_-, \eta_-)$ reduces to

$$\left( \omega_o , \left( -g_+ \left( Ad_{g_-}^* X_+ \right) g_- , \xi_+ \right) \otimes \left( -g_+ \left( Ad_{g_-}^* Y_+ \right) g_- , \lambda_+ \right) \right)_{(g,\eta)}$$

$$= - \langle \xi_+ , Y_+ \rangle + \langle \lambda_+ , X_+ \rangle + \left( Ad_{g_-} \eta_+ , \left[ Ad_{g_-}^* X_+ , Ad_{g_-}^* Y_+ \right] \right)_{(g,\eta)}$$

$$+ \left( \eta_- , g_- \left[ Ad_{g_-}^* X_+ , Ad_{g_-}^* Y_+ \right] \right)_{(g,\eta)}$$

(16)

Here we can see that there are no null vectors of $\omega_o$ on $T\Psi^{-1}(g_-, \eta_-) = \ker \Psi_*_{\mid (g,\eta)}$.
Corollary: \((T_{(g,\eta)}N (g_-,\eta_-))^\perp \cap T_{(g,\eta)}N (g_-,\eta_-) = \{0\},\) then \(N (g_-,\eta_-)\) is a second class constraint.

3.1.1 Dirac brackets on \(N (g_-,\eta_-)\)

In order to built up the Dirac brackets, we choose a basis for \(T^\ast (g_-,\eta_-)\)
\(\cong g^\ast_+ \oplus g^\ast_-\). In doing so, we introduce the basis \(\{T_a\}_{a=1}^n\) of \(g^\ast_+ \cong g^\ast_+\) and the basis \(\{T^a\}_{a=1}^n\) of \(g^\ast_- = g^\ast_-\), which provide a set of linearly independent 1-forms on \(G_- \times g^\ast_-\):

\[
\alpha_a = \left( T_{g_-^{-1}}^\ast T_a, 0 \right) \in T^\ast_{(g_-,\eta_-)} G_- \times g^\ast_+
\]

\[
\beta_a = (0, T^a) \in T^\ast_{(g_-,\eta_-)} G_- \times g^\ast_-
\]

for \(a = 1, 2, \ldots, n\). Their pullback amount a set \(\{\Psi^\ast \alpha_a\}_{a=1}^n \cup \{\Psi^\ast \beta^a\}_{a=1}^n\) of linearly independent 1-forms on \(G \times g^\ast\), whose null distribution is precisely the tangent space of \(N (g_-,\eta_-)\).

Hence, the hamiltonian vector fields associated with these forms through \(\omega\) are

\[
V_{\Psi \alpha_a} = \left( 0, g^{-1} g T_a - T_a \right)_{(g,\eta)}
\]

\[
V_{\Psi \beta^a} = \left( g T^a, ad_{T^a} \eta \right)
\]

So, we calculate the entries of the Dirac matrix:

\[
C_{\Psi^\ast \alpha_a, \Psi^\ast \alpha_b} (g,\eta) = \langle \Psi^\ast \alpha_a, V_{\Psi^\ast \alpha_b} \rangle_{(g,\eta)} = 0
\]

\[
C_{\Psi^\ast \beta^a, \Psi^\ast \beta^b} (g,\eta) = \langle \Psi^\ast \beta^a, V_{\Psi^\ast \beta^b} \rangle_{(g,\eta)} = -\langle \eta_-, [T^a, T^b] \rangle
\]

\[
C_{\Psi^\ast \alpha_a, \Psi^\ast \beta^b} (g,\eta) = \langle \Psi^\ast \alpha_a, V_{\Psi^\ast \beta^b} \rangle_{(g,\eta)} = \delta^b_a
\]

that finally produces the matrix

\[
C (g,\eta) = \begin{bmatrix} 0_{n\times n} & -I_{n\times n} \\ I_{n\times n} & \Omega (\eta) \end{bmatrix}
\]

where \(\Omega (\eta)\) stands for the \(n \times n\) matrix of entries

\[
\Omega_{ab} (\eta) = -\langle \eta_-, [T^a, T^b] \rangle
\]

Now, we are ready to introduce the Dirac brackets: carrying these results in the expression \([\mathcal{F}, \mathcal{G}] \in C^\infty (G \times g^\ast)\), the Dirac
bracket gives the restriction of Poisson bracket on $G \times \mathfrak{g}^*$ to constraint submanifold $\mathcal{N} (g_-, \eta_-)$ and it is defined as

$$\{ F, G \}^N (g, \eta) = \{ F, G \} (g, \eta) - \{ \alpha_\eta, G \} \Omega_{ab} (\eta) \{ \alpha_\alpha, F \} (g, \eta)$$

that has the explicit form

$$\{ F, G \}^N (g, \eta) = \left\{ \begin{array}{c}
g dF, \delta G_g \delta \eta_l - \delta \delta G_g \delta F_l \\
g dG, \delta F_g \delta \eta_l - \delta \delta F_g \delta G_l \\
\eta, \left[ \delta F_g \delta \eta_l - \delta \delta F_g \delta \eta_l \right] \end{array} \right\}$$

Observe that, for $\mathcal{N} (e, 0)$, it turns in

$$\{ F, G \}^N (g, \eta) = \left\{ \begin{array}{c}
g dF, \delta \delta F_g \delta \eta_l - \delta \delta \delta F_g \delta \eta_l \\
g dG, \delta \delta F_g \delta \eta_l - \delta \delta \delta F_g \delta \eta_l \\
\eta, \left[ \delta \delta F_g \delta \eta_l - \delta \delta \delta \eta_l \right] \end{array} \right\}$$

giving the usual canonical Poisson structure on $T^* G_+ = G_+ \times \mathfrak{g}_+^*$.

3.1.2 The fundamental brackets

Let $T : G \to GL (n, \mathbb{C})$ a representation of the group $G$, with the associated set of functions $T^i_A : G \to \mathbb{C}$ such that $T = E^j_i T^i_j$, being $E^j_i$, $i, j = 1, ... , n$, the elementary $n \times n$ matrices with entries $[ E^j_i ]^l_k = \delta_{ik} \delta_{kl}$. For $g \in G$

$$T^i_j (g) = g^i_j \implies T (g) = \sum_{i,j=1}^n g^i_j E^j_i$$

Also, we consider the coordinates $\{ \xi_A \}^N_{A=1}$ for $\mathfrak{g}^*$, associated with the basis $\{ T_A \}^N_{A=1}$ of $\mathfrak{g}$, such that

$$\xi_A : \mathfrak{g}^* \to \mathbb{C} / \xi_A = (\xi, T_A)$$

The representation map $T$ induces the map $dT : \mathfrak{g} \to gl (n, \mathbb{C})$ such that

$$(dT)_* X = \sum_{i,j=1}^n X^j_i E^j_i.$$ 

The corresponding fundamental Poisson bracket on $G \times \mathfrak{g}^*$, associated with the canonical Poisson bracket, are:

$$\{ T^i_j, T^k_l \} (g, \xi) = 0$$

$$\{ \xi_A, T^i_j \} (g, \xi) = - \sum_{k=1}^n g^k_j [ T_A ]^j_k$$

Let us now calculate the Dirac brackets for these functions. First observe that

$$\delta T^i_j = 0 \quad , \quad d \xi_A = 0$$
the fundamental Dirac brackets can be written as

\[ \{ T_i^j, T_k^l \}^D_{g, \eta} = 0 \]

\[ \{ \xi_a, T_i^j \}^D_{g, \eta} = \langle g d T_i^j, g^{-1} g T_a - T_a \rangle \]

\[ \{ \xi^a, T_i^j \}^D_{g, \eta} = 0 \]

\[ \{ \xi_a, \xi_b \}^D_{g, \eta} = -\langle \eta, [T_a - g^{-1} g T_a, T_b - g^{-1} g T_b] \rangle \]

\[ \{ \xi^a, \xi^b \}^D_{g, \eta} = 0 \]

Since

\[ \langle g d T_i^j, g^{-1} g T_a - T_a \rangle = T^k_i (g+) \left[ g_{T_a}^j \right]_{k} - T^k_i (g) [T_a]^j_k \]

and from the definition of the Lie bracket in \( g = g^a_+ \oplus g^a_- \),

\[ [T_a - g^{-1} g T_a, T_b - g^{-1} g T_b] = [T_a, T_b] + a d^*_{g^{-1} g T_a} T_b - a d^*_{g^{-1} g T_b} T_a \]

\[ + [g^{-1} g T_a, g^{-1} g T_b] + a d^*_{T_a} g^{-1} g T_b - a d^*_{T_b} g^{-1} g T_a \]

the fundamental Dirac brackets can be written as

\[ \{ T_i^j, T_k^l \}^D_{g, \eta} = 0 \]

\[ \{ \xi_a, T_i^j \}^D_{g, \eta} = T^k_i (g+) \left[ g_{T_a}^j \right]_{k} - T^k_i (g) [T_a]^j_k \]

\[ \{ \xi^a, T_i^j \}^D_{g, \eta} = 0 \]

\[ \{ \xi_a, \xi_b \}^D_{g, \eta} = -f^{c}_{a b} \xi^c (\eta) + m^{c}_{a b} (g-) \xi^c (\eta) + n^{c}_{a b} (g-) \xi^c (\eta) \]

\[ \{ \xi^a, \xi^b \}^D_{g, \eta} = 0 \]

where

\[ m^{c}_{a b} (g-) = \langle \left[ g^{-1} g T_a, T_c \right], T_a \rangle - \langle \left[ g^{-1} g T_a, T_a \right], T_b \rangle \]

\[ n^{c}_{a b} (g-) = \langle [T_b, T_a], g^{-1} g T_a \rangle - \langle [T_a, T_c], g^{-1} g T_a \rangle - \langle T_c, \left[ g^{-1} g T_a, g^{-1} g T_b \right] \rangle \]

are constant coefficients on each submanifold \( N (g_-, \eta_-) \) and, in particular, they vanish for \( g_- = e \), \( m^{c}_{a b} (e) = n^{c}_{a b} (e) = 0 \).

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3.1.3 The action of $G$ on $N (g_-, \eta_-)$

The group $G$ acts on itself by left translations, $L : G \times G \rightarrow G$ as

$$L_ag = ag = a+ag$$

for $a_+, g_+ \in G_+$ and $a_- g_- \in G_-$. It can be easily lifted to the cotangent bundle $T^*G \cong G \times g^*$, in body coordinates, as the hamiltonian action

$$\rho : G \times (G \times g^*) \rightarrow G \times g^* \ / \ \rho_h(g, \eta) = (hg, \eta)$$

with associated $Ad$-equivariant momentum map $\Phi^L : G \times g^* \rightarrow g^*$

$$\Phi^L (g, \eta) = Ad_{\rho^{-1}(g)}^* \eta$$  \hspace{1cm} (22)

So, the map

$$\phi_X (g, \eta) = \left\langle \eta, Ad_{\rho^{-1}(g)}^* X \right\rangle$$

is the hamiltonian function associated with the infinitesimal generator $X_{G \times g^*}$ corresponding to $X \in g$, namely $i_{X_{G \times g^*}} \omega_0 = d\phi_X$.

We now use the Dirac bracket (20) to get the hamiltonian vector field $X_N$ on $N (g_-, \eta_-)$, associated with $X \in g$, such that

$$\{dF, X_N\}_{(g, \eta)} := \{F, \phi_X\}_{D}^D (g, \eta)$$

Writing $d\phi_X = (d\phi_X, \delta \phi_X) \in T^*G \oplus T^*g^*$,

$$(gd\phi_X, \delta \phi_X) = \left( \left[ \eta, Ad_{\rho^{-1}(g)}^* X \right], Ad_{\rho^{-1}(g)}^* X \right)$$

the Poisson-Dirac bracket turns into

$$\{ F, \phi_X \}_{D}^D (g, \eta) = \left\langle gdF, Ad_{\rho^{-1}(g)}^* \Pi_{\rho^+} Ad_{\rho^+}^* X \right\rangle$$

$$- \left\langle \delta F, \left[ \eta, Ad_{\rho^{-1}(g)}^* \Pi_{\rho^-} Ad_{\rho^-}^* X \right] \right\rangle$$

$$+ \left\langle \delta F, Ad_{\rho^{-1}(g)}^* \Pi_{\rho^+} Ad_{\rho^-}^* \left[ \eta, Ad_{\rho^{-1}(g)}^* \Pi_{\rho^-} Ad_{\rho^-}^* X \right] \right\rangle$$

and from it we get the hamiltonian vector field associated with $\phi_X$,

$$X_N (g, \eta) = V_{\phi_X}^{N (g_-, \eta_-)} (g, \eta)$$

$$= \left( gAd_{\rho^{-1}(g)}^* \Pi_{\rho^+} Ad_{\rho^-}^* X, Ad_{\rho^{-1}(g)}^* \Pi_{\rho^-} \left[ \Pi_{\rho^+} Ad_{\rho^-}^* X, Ad_{\rho^-}^\eta \right] \right)$$  \hspace{1cm} (23)

These vector fields are the projection on $TN (g_-, \eta_-)$ of the infinitesimal generators $X_{G \times g^*}$ of the action of $G$ on $G \times g^*$ associated with $X \in g$. However, it is not clear at this point whether these vector fields are infinitesimal generators for an action of $G$ on $N (g_-, \eta_-)$. This question is addressed in the following proposition.
Proposition: The assignment $X \in \mathfrak{g} \mapsto X_N \in \mathfrak{X}(\mathcal{N}(g_-,\eta_-))$ defines a Hamiltonian action of the Lie algebra $\mathfrak{g}$ on $\mathcal{N}(g_-,\eta_-)$ provided $\eta_-$ is a character of $\mathfrak{g}_{\eta}^\circ$.

Proof: The Dirac bracket of the Hamiltonian functions $\phi_X,\phi_Y$ is

$$\{\phi_X,\phi_Y\}_D(g,\eta) = \phi_{[X,Y]}(g,\eta) - \left\langle Ad_{g_-}^*\eta_-, \left[ \Pi_{\eta^-} Ad_{g_-}^* Y, \Pi_{\eta^-} Ad_{g_-}^* X \right] \right\rangle$$

where it is obvious that the second term in the rhs vanish for every $X$ and $Y$ in $\mathfrak{g}$ only if $\eta_-$ is a character of $\mathfrak{g}_{\eta}^\circ$. In this case, it is easy to see using the Jacobi identity that, for an arbitrary function $f$ on $G \times \mathfrak{g}^*$,

$$(L_{Y_N} L_{X_N} - L_{X_N} L_{Y_N}) f (g,\eta) = \left\{ \{f, \phi_X\}_D, \phi_Y \right\}_D (g,\eta) = \left\{ \phi_Y, \{f, \phi_X\}_D \right\}_D (g,\eta) = L_{[X,Y]_N} f (g,\eta)$$

that is equivalent to say that the assignment $X \in \mathfrak{g} \mapsto X_N \in \mathfrak{X}(\mathcal{N}(g_-,\eta_-))$ is an antihomomorphism of Lie algebras

$$[X_N,Y_N] = -[X,Y]_N$$

Therefore, it defines a left action of $\mathfrak{g}$ on $\mathcal{N}(g_-,\eta_-)$.

In particular, at $\eta_- = 0$ and $g_- = e$, we get

$$X_N(g_+,\eta_+) = \left( g_+ \left( \Pi_{\eta^+_e} Ad_{g_+}^* X \right), -\Pi_{\eta^+_e} \left[ \eta_+, \Pi_{\eta^+_e} Ad_{g_+}^* X \right] \right)$$

Observe that, for $X \in \{T^a\}_{a=1}^n \subset \mathfrak{g}_{\eta}^\circ$, the momentum functions $\phi^a := \phi_{T^a}$ generate the infinitesimal dressing action of $\mathfrak{g}_{\eta}^\circ$ on $\mathcal{N}(g_-,\eta_-)$.

This infinitesimal action of $\mathfrak{g}$ on $\mathcal{N}(g_-,\eta_-)$ corresponds to the following action of $G$ on $\mathcal{N}(g_-,\eta_-)$.

Proposition: The vector field $X_N \in \mathfrak{X}(\mathcal{N}(g_-,\eta_-))$, for $X \in \mathfrak{g}$ and $\eta_-$ a character of $\mathfrak{g}_{\eta}^\circ$, is the infinitesimal generator associated with the action $G \times \mathcal{N}(g_-,\eta_-) \rightarrow \mathcal{N}(g_-,\eta_-)$ defined as

$$d (h, (g,\eta)) = \left( gAd_{g_-}^* \Pi_{\eta^+_e} (g_+^{-1}hg_+), Ad_{g_-}^* \Pi_{\eta^-} Ad_{g_-}^* \Pi_{\eta^-} (g_+^{-1}hg_+) \right)$$

$$\forall \ (g,\eta) = (g_+g_-,\eta_+ + \eta_-) \in \mathcal{N}(g_-,\eta_-).$$

Proof: It follows by straightforward calculation of the differential of this map.

Observe that it can be written as

$$d (h, (g_+g_-,\eta_+ + \eta_-))$$

$$= \left( h_+g_+^{-1}g_-, Ad_{g_-}^* \left[ (h_+g_+^{-1})^{\eta_-} (h_+g_+^{-1})^{-1} + Ad_{g_-}^* g_+^{-1} \right] + Ad_{g_-}^* g_+^{-1} \right)$$
so that for \( g = e \) and \( \eta = 0 \) it turns into

\[
\text{d} (h, (g_+\eta_+)) = (h_+g_+^{\eta_-}, \text{Ad}_{h_+^{\eta_-}}\eta_+)
\]

This action was introduced in [10] as the fundamental ingredient underlying the Poisson Lie T-duality scheme.

**Note:** The submanifolds we refer above are particular members of a bigger family of symplectic submanifolds in \( G \times \mathfrak{g}^* \) which are \( G \)-spaces and can be constructed by means of reduction theory [12]. In this framework, the procedure shows in particular that \( N (g_-, \eta_-) \) and \( M (g_+, \eta_+) \) are symplectic submanifolds of \( G \times \mathfrak{g}^* \) if \( \eta_\pm \) are characters of the coadjoint action of \( G \) on \( \mathfrak{g}^* \), respectively. Additionally, it gives an interpretation for the actions of the factorizable Lie group \( G = G_+G_- \) on \( N (g_-, \eta_-) \) and \( M (g_+, \eta_+) \), providing us an explanation to the symplecticity of these actions. For instance, in the case of \( N (g_-, \eta_-) \) we can think of it as follows: the right action of \( G_- \) on \( G \) induces on \( G \times \mathfrak{g}^* \) a symplectic action by lifting and, by applying the Marsden-Weinstein reduction via its momentum map \( J_- : G \times \mathfrak{g}^* \to \mathfrak{g}^*_\eta_- \simeq \mathfrak{g}^+_\eta_- \), for \( \eta_- \in \mathfrak{g}^+_\eta_- \), we get the quotient map

\[
\pi_{\eta_-} : J_-^{-1} (\eta_-) \to J_-^{-1} (\eta_-) / (G_-)_{\eta_-}
\]

such that the symplectic form \( \omega_{\eta_-} \) on \( J_-^{-1} (\eta_-) / (G_-)_{\eta_-} \) is defined by the condition

\[
\pi_{\eta_-}^* \omega_{\eta_-} = \omega_0 |(J_-)^{-1} (\eta_-)|
\]

where \( \omega_0 \) indicates the canonical 2-form on \( G \times \mathfrak{g}^* \). In this context we have the following remarkable facts:

- for \( \eta_- \in \mathfrak{g}^+_\eta_- \), a character, we have that \( (G_-)_{\eta_-} = G_- \) for its isotropy group,
- any submanifold \( S \subset J_-^{-1} (\eta_-) \) transverse to the \( G_- \)-orbits and such that \( \dim S = \dim [J_-^{-1} (\eta_-) / G_-] \) (i.e. \( S \) is a cross-section for the \( G_- \)-action on \( J_-^{-1} (\eta_-) \)), is a symplectic submanifold of \( G \times \mathfrak{g}^* \) via the restriction of the canonical 2-form, and
- \( \pi_{\eta_-} \big| S \) is a symplectomorphism.

By viewing \( N (g_-, \eta_-) \) (\( \eta_- \) a character!) as submanifolds of \( J_-^{-1} (\eta_-) \) transverse to the \( G_- \)-orbits, we find that they are symplectic submanifolds of \( G \times \mathfrak{g}^* \) and symplectomorphic to the Marsden-Weinstein reduced space \( J_-^{-1} (\eta_-) / G_- \).

On the other side, it can be proved that the Marsden-Weinstein reduced space obtained above is symplectomorphic to the cotangent bundle of a (reductive) homogeneous space \( M \) of \( G \), and this fact has two consequences:

- any slice in \( J_-^{-1} (\eta_-) \) is symplectomorphic to \( T^* M \), and
the lifted canonical action of $G$ on $T^*M$ induces a symplectic action on these slices.

We see that the action of $G$ on $N(\mathfrak{g}, \eta)$ defined above (24) comes from this construction.

Summarizing, we can fit all the relevant structures in the following diagram:

\[
\begin{array}{c}
 N(\mathfrak{g}, \eta) \xrightarrow{\pi} J_{\eta_-}^{-1}(\eta_-) \\
 T^*M \xrightarrow{\pi} J_{\eta_-}^{-1}(\eta_-)/G-
\end{array}
\]

The dotted arrow can be viewed as a consequence of the rest of the structures in the diagram, inducing the action while keeping it symplectic. The particular case $(\mathfrak{g}, \eta) = (e, 0)$ gives the $G$-action on $G^+$ considered in [10].

3.1.4 The Hamilton equations on $N(\mathfrak{g}, \eta)$

Let us consider a generic Hamiltonian function $H$ on $G \times \mathfrak{g}^*$, so the associated Hamiltonian vector field by the Poisson-Dirac structure (20) is

\[
V_{H}^{N}(g, \eta) = \left( g \ \text{Ad}_{g_-}^{G}, \Pi_{\mathfrak{g}^+}^{G} \text{Ad}_{g_-}^{G} \delta H, \text{Ad}_{g_-}^{G}, \Pi_{\mathfrak{g}^+}^{G} \text{Ad}_{g_-}^{G} \left( [\eta, \text{Ad}_{g_-}^{G}, \Pi_{\mathfrak{g}^+}^{G} \text{Ad}_{g_-}^{G} \delta H] - g dH \right) \right)
\]

and the Hamilton equation are

\[
\begin{align*}
\dot{g}^{-1} \dot{g} &= \left( \text{Ad}_{g_-}^{G}, \Pi_{\mathfrak{g}^+}^{G} \text{Ad}_{g_-}^{G} \right) \delta H \\
\dot{\eta} &= \left( \text{Ad}_{g_-}^{G}, \Pi_{\mathfrak{g}^+}^{G} \text{Ad}_{g_-}^{G} \right) \left( [\eta, \text{Ad}_{g_-}^{G}, \Pi_{\mathfrak{g}^+}^{G} \text{Ad}_{g_-}^{G} \delta H] - g dH \right)
\end{align*}
\]

In terms of the factors, $g = g_+ g_-$ and $\eta = \eta_+ + \eta_-$, they are equivalent to

\[
\begin{align*}
g_+^{-1} \dot{g}_+ &= \Pi_{\mathfrak{g}^+}^{G} \text{Ad}_{g_-}^{G} \delta H \\
\dot{\eta}_+ &= \text{Ad}_{g_-}^{G}, \Pi_{\mathfrak{g}^+}^{G} \text{Ad}_{g_-}^{G} \left( [\eta, \text{Ad}_{g_-}^{G}, \Pi_{\mathfrak{g}^+}^{G} \text{Ad}_{g_-}^{G} \delta H] - g dH \right) \\
\dot{g}_-^{-1} g_- &= 0 \\
\dot{\eta}_- &= 0
\end{align*}
\]
By using the fundamental coordinates, the Hamilton equation on \( N (g_-, \eta_-) \) are
\[
\begin{align*}
\dot{T}_k^j &= \{ T_k^j, \mathcal{H} \}^D \\
\dot{\xi}_a &= \{ \xi_a, \mathcal{H} \}^D
\end{align*}
\]
that turn out to be
\[
\begin{align*}
\dot{T}_k^j &= \frac{\partial \mathcal{H}}{\partial \xi_b} \{ T_k^j, \xi_b \}^D \\
\dot{\xi}_a &= \frac{\partial \mathcal{H}}{\partial T^j_i} \left\{ \xi_a, T^j_i \right\}^D + \frac{\partial \mathcal{H}}{\partial \xi_b} \left\{ \xi_a, \xi_b \right\}^D
\end{align*}
\]

3.2 The phase spaces \( \mathcal{M} (g_+, \eta_+) \)

Let us consider the fibration \( \Upsilon : G \times g^* \rightarrow G_+ \times g^*_+ \) with fiber \( \mathcal{M} (g_+, \eta_-) \) at \( (g_+, \eta_-) \in G_+ \times g^*_+ \). The differential of the projection of a vector \((v, \xi) \in G \times g^* \) on \( G_+ \times g^*_+ \) is
\[
\Pi^* (v, \xi) = \left( g_+ \text{Ad}_{g_+}^{-1} X_+, \xi_+ \right)_{(g_+, \eta_+)}
\]
where \( X_+ = \Pi_{g_+} g_+^{-1} v \) and \( \xi = \xi_+ + \xi_- \). Hence, coming back to the expression for \( \Upsilon_* \), we have
\[
\Upsilon_* (g X, \xi)(g, \eta) = \left( g_+ \text{Ad}_{g_+}^{-1} X_+, \xi_+ \right)_{(g, \eta_+)}
\]

Let us find out \( \ker \Upsilon_* |_{(g, \eta)} = T_{(g, \eta)} \mathcal{M} (g_+, \eta_+) \), using (13) we get
\[
\ker \Upsilon_* |_{(g, \eta)} = \left\{ \left( g X_-, \xi_- \right) / \left( X_-, \xi_- \right) \in g_-^* \oplus g_+^* \right\}
\]
We may use this result to analyze the intersection of \( \ker \Upsilon_* \) with the symplectic orthogonal of \( T \Upsilon^{-1} (g_+, \eta_+) \). In order to do this we evaluate
\[
\langle \omega, (g X_-, \xi_-) \otimes (g Y_-, \lambda_-) \rangle_{(g, \eta)} = - \langle \xi_-, Y_- \rangle + \langle \lambda_-, X_- \rangle + \langle \eta_-, [X_-, Y_-] \rangle
\]
that is regular on \( T \mathcal{M} (g_+, \eta_+) \), so that \( \left[ T_{(g, \eta)} \mathcal{M} (g_+, \eta_+) \right]_{\perp \omega} \cap \ker \Upsilon_* |_{(g, \eta)} = \{ 0 \} \).

**Proposition:** \( (\mathcal{M} (g_+, \eta_+), \tilde{\omega}_o) \), where \( \tilde{\omega}_o \) stands for the restriction of the canonical symplectic form \( \omega_o \) on \( G \times g^* \) to \( \Upsilon^{-1} (g_+, \eta_+) \), is a symplectic manifold. Moreover \( \tilde{\omega}_o \) coincides with the canonical symplectic form of \( G_- \times g_+^* \).

Consequently, the restriction to \( \Upsilon^{-1} (g_+, \eta_+) \) is a second class constraint, as expected.
In order to built up the Dirac brackets, we take a set of linearly independent 1-forms on $G_+ \times g^*_+$:

$$\theta_a = \left(L_{g_+}^* T^a, 0 \right) \in T_{(g_+ , \eta_+)}^* (G_+ \times g^*_+)$$

$$\gamma_a = (0, T_a) \in T_{(g_+ , \eta_+)}^* (G_+ \times g^*_+)$$

Then their pullback $\{ \Upsilon^* \theta^a , \Upsilon^* \gamma_a \}_{a=1}^n$ are a set of linearly independent 1-forms on $G \times g^*$, such that $\{ \{ \Upsilon^* \theta^a , \Upsilon^* \gamma_a \}_{a=1}^n \} = \mathcal{T} \mathcal{M} (g_+ , \eta_+)$. The hamiltonian vector fields of these 1-forms on $G \times g^*$ are

$$V_{\Upsilon^* \theta^a} \big|_{(g, \eta)} = \left( 0, -\text{Ad}_{g_+} T^a \right)_{(g, \eta)}$$

$$V_{\Upsilon^* \gamma_a} \big|_{(g, \eta)} = (g T_a , a \partial_{\delta^a} \eta)$$

and the Dirac matrix

$$C = \left( \begin{array}{cc}
\langle \Upsilon^* \theta^a , V_{\Upsilon^* \theta^b} \rangle_{(g, \eta)} & \langle \Upsilon^* \theta^a , V_{\Upsilon^* \gamma_b} \rangle_{(g, \eta)} \\
\langle \Upsilon^* \gamma_a , V_{\Upsilon^* \theta^b} \rangle_{(g, \eta)} & \langle \Upsilon^* \gamma_a , V_{\Upsilon^* \gamma_b} \rangle_{(g, \eta)}
\end{array} \right)$$

is

$$C (g, \eta) = \left[ \begin{array}{cc}
0_{n \times n} & F (g_-) \\
-F^\dagger (g_-) & \Theta (\eta_+)
\end{array} \right]$$

where $\Theta (\eta_+)$ is a $n \times n$ matrix with entries

$$\Theta_{ab} (\eta_+) = \{ \Upsilon^* \gamma_a , \Upsilon^* \gamma_b \} (\eta_+) = - \langle \eta_+ , [T_a , T_b] \rangle$$

and

$$F^a_b (g_-) = \{ \Upsilon^* \theta^a , \Upsilon^* \gamma_b \} (g_-) = \langle \text{Ad}_{g_-} T^a , T_b \rangle$$

The inverse of the Dirac matrix is

$$C^{-1} (g, \eta) = \left( \begin{array}{cc}
F^a_{\theta} (g^{-1}) \Theta_{cd} (\eta_+) F^d_b (g^{-1}) & -F^a_b (g^{-1}) \\
F^a_{\gamma} (g^{-1}) & 0_{n \times n}
\end{array} \right)$$

that reduces to

$$C^{-1} (g, \eta) = \left( \begin{array}{cc}
-\langle \eta_+ , \left[ \text{Ad}_{g_-} T_a , \text{Ad}_{g_-} T_b \right] \rangle & \langle \text{Ad}_{g_-} T^a , T_b \rangle \\
\langle \text{Ad}_{g_-} T^a , T_b \rangle & 0_{n \times n}
\end{array} \right)$$

We are now ready to construct the Dirac bracket on $\mathcal{M} (g_+ , \eta_+)$. Introducing this result into (25), we get

$$\{ \mathcal{F} , \mathcal{H} \}^{\mathcal{M}} (g, \eta) = \left\langle g \mathcal{d} \mathcal{F} , \Pi g^\delta \mathcal{H} \right\rangle - \left\langle g \mathcal{d} \mathcal{H} , \Pi g^\delta \mathcal{F} \right\rangle$$

$$- \left\langle \eta_- , \left[ \Pi g^\delta \mathcal{F} , \Pi g^\delta \mathcal{H} \right] \right\rangle$$

that coincides with the canonical Poisson bracket on $G_- \times g_-^*$.
3.2.1 The left action of $G$ on $G \times g^*$ and its restriction to $\mathcal{M}(g_+, \eta_+)$

As described at the beginning of the section (3.1.3), we want to study how the left action of $G$ on its cotangent bundle project on the fibers $\mathcal{M}(g_+, \eta_+)$. In doing so, we consider the Dirac bracket (25) involving the momentum function $\phi_X$, which for an arbitrary function $F \in C^\infty(G \times g^*)$ gives

$$\{F, \phi_X\}_{\mathcal{M}}(g, \eta) = \left\langle gdF, \Pi_{g^*} \circ \text{Ad}_{g^{-1}} \right\rangle_{\mathcal{M}} = \left\langle \eta, \Pi_{g^*} \circ \text{Ad}_{g^{-1}} \right\rangle_{\mathcal{M}}$$

therefore the hamiltonian vector field of $\phi_X$ is

$$V_{\phi_X}^\mathcal{M}(g_+, \eta_+) = \left\langle g\Pi_{g^*} \circ \text{Ad}_{g^{-1}} X, \Pi_{g^*} \circ \text{Ad}_{g^{-1}} X, \eta \right\rangle$$

(26)

On the other side, the Dirac bracket between two moment functions reduces to

$$\{\phi_X, \phi_Y\}_{\mathcal{M}}(g, \eta) = \phi_{[X,Y]}(g, \eta) - \left\langle \eta, \left[ \Pi_{g^*} \circ \text{Ad}_{g^{-1}} X, \Pi_{g^*} \circ \text{Ad}_{g^{-1}} Y \right] \right\rangle$$

so, as it happens on $\mathcal{N}(g_-, \eta_-)$, it closes an algebra provided $\eta_+$ is a character of $\mathfrak{g}_+^*$.

From the expression of the hamiltonian vector field $V_{\phi_X}^\mathcal{M}(g_+, \eta_+)$ given in eq. (26), we retrieve the action of $G$ on $\mathcal{M}(g_+, \eta_+)$, $b : G \times \mathcal{M}(g_+, \eta_+) \rightarrow \mathcal{M}(g_+, \eta_+)$, in the case $\eta_+$ is a character of $\mathfrak{g}_+^*$,

$$b(h, (g_+ g_-, \eta_+ + \eta_-)) = \left( g\Pi_{-} (g^{-1} h g), \Pi_{g^*} \circ \text{Ad}_{g^{-1}} \right)$$

3.2.2 The Hamilton equations

The Hamilton equations on $\mathcal{M}(g_+, \eta_+)$ are defined by the Dirac bracket (25)

$$\{F, \mathcal{H}\}_{\mathcal{M}}(g, \eta) = \left\langle gdF, \Pi_{g^*} \circ \text{Ad}_{g^{-1}} \right\rangle - \left\langle gd\mathcal{H}, \Pi_{g^*} \circ \text{Ad}_{g^{-1}} \right\rangle - \left\langle \eta, \left[ \Pi_{g^*} \circ \text{Ad}_{g^{-1}} \right] \right\rangle$$

from where we get the hamiltonian vector field for the Hamilton function $\mathcal{H}$:

$$V_{\mathcal{H}}^\mathcal{M}(g, \eta) = \left\langle g\Pi_{g^*} \circ \text{Ad}_{g^{-1}} X, \Pi_{g^*} \circ \text{Ad}_{g^{-1}} X, \eta \right\rangle$$

The reduced Hamilton equations are then

$$\begin{cases} g^{-1} \dot{g} = \Pi_{g^*} \circ \mathcal{H} \\ \dot{\eta} = \Pi_{g^*} \left( gd\mathcal{H} - \left[ \eta, \Pi_{g^*} \circ \mathcal{H} \right] \right) \end{cases}$$
that, when expressed in terms of the factors, \( g = \varphi \varphi^{-1} \) and \( \eta = \eta_+ + \eta_- \), becomes in

\[
\begin{cases}
\varphi^{-1} \dot{\varphi} = \Pi \varphi_+ \delta \mathcal{H} \\
\dot{\eta}_- = \Pi \varphi_+ \left( g \delta \mathcal{H} - \left[ \eta_-, \Pi \varphi_+ \delta \mathcal{H} \right] \right) \\
\varphi^+_1 \dot{\varphi}_+ = 0 \\
\dot{\eta}_+ = 0
\end{cases}
\]

4 The Dirac method and integrable systems

4.1 Involutive function algebra in \( \mathcal{N} \left( \varphi, \eta_- \right) \)

The vector space \( \mathfrak{g}^* \) turns into a Poisson manifold provided we equip the set \( C^\infty \left( \mathfrak{g}^* \right) \) with one of the Kirillov-Kostant bracket \( \{ , \} : C^\infty \left( \mathfrak{g}^* \right) \times C^\infty \left( \mathfrak{g}^* \right) \rightarrow C^\infty \left( \mathfrak{g}^* \right) \) defined as

\[
\{ f, g \} \left( \eta \right) = - \langle \eta, [ \mathcal{L}_f \left( \eta \right), \mathcal{L}_g \left( \eta \right) ] \rangle
\]

where \( \mathcal{L}_h : \mathfrak{g}^* \rightarrow \mathfrak{g} \) stands for the Legendre transformation of a function \( h : \mathfrak{g}^* \rightarrow \mathbb{R} \) such that for any \( \xi \in \mathfrak{g}^* \)

\[
\langle \xi, \mathcal{L}_h (\eta) \rangle = \langle \frac{d}{dt} h|_{\eta} + \xi, \eta \rangle = \left. \frac{d}{dt} (h|_{\eta} + t \xi) \right|_{t=0}
\]

The canonical Poisson bracket on \( G \times \mathfrak{g}^* \)

\[
\{ \mathcal{F}, \mathcal{G} \} \left( g, \eta \right) = \langle \mathcal{d} \mathcal{F}, g \delta \mathcal{G} \rangle - \langle \mathcal{d} \mathcal{G}, g \delta \mathcal{F} \rangle - \langle \eta, [ \delta \mathcal{F}, \delta \mathcal{G} ] \rangle
\]

that, for \( \mathcal{F}, \mathcal{G} = f, g C^\infty \left( \mathfrak{g}^* \right) \), such that \( df|_{\eta} = (0, \delta f)|_{\eta} = (0, \mathcal{L}_f (\eta)) \), it reduces to

\[
\{ f, g \} \left( g, \eta \right) = - \langle \eta, [ \mathcal{L}_f (\eta), \mathcal{L}_g (\eta) ] \rangle
\]

A remarkable fact here is that the symplectic leaves of this Poisson structure coincides with the orbits of the coadjoint action of \( G \) on \( \mathfrak{g}^* \).

Let us now study the constraint submanifold \( \mathcal{N} \left( \varphi, \eta_- \right) \), equipped with Poisson-Dirac structure derived from the Dirac bracket \( \{ , \} \)

\[
\{ \mathcal{F}, \mathcal{G} \}^{\mathcal{N}} \left( g_+, \eta \right) = \langle g \mathcal{d} \mathcal{F}, \Pi \varphi_+ \delta \mathcal{G} \rangle - \langle g \mathcal{d} \mathcal{G}, \Pi \varphi_+ \delta \mathcal{F} \rangle - \langle \eta, [ \Pi \varphi_+ \delta \mathcal{F}, \Pi \varphi_- \delta \mathcal{G} ] \rangle
\]

which, when applied to \( f, g C^\infty \left( \mathfrak{g}^* \right) \), gives

\[
\{ f, g \}^{\mathcal{N}} \left( g_+, \eta \right) = - \langle \eta, [ \Pi \varphi_+ \mathcal{L}_f (\eta), \Pi \varphi_- \mathcal{L}_g (\eta) ] \rangle
\]

meaning that \( \mathcal{N} \left( \varphi, \eta_- \right) \) is equipped with a kind of Kirillov-Kostant bracket just as \( \mathfrak{g}^* \), where \( \eta_- \) enters as a parameter in the Legendre transform of the functions \( f, g C^\infty \left( \mathfrak{g}^* \right) \). Then, for \( f, g \) being \( Ad^G \)-invariant functions we have the relation

\[
\mathcal{L}_h (Ad^*_g \eta) = Ad^*_{g^{-1}} \mathcal{L}_h (\eta)
\]
that infinitesimally is \([\mathcal{L}_t(\eta), \eta] = 0\), therefore we may also write
\[
\{f, g\}^N (g_+, \eta) = \left\langle \eta, \left[ \Pi_{g^+} \mathcal{L}_g(\eta), \Pi_{g^+} \mathcal{L}_h(\eta) \right] \right\rangle
\]
Hence, provided \(\eta_-\) is a character of \(g^\circ\), we get
\[
\{f, g\}^N (g_+, \eta) = 0
\]
meaning that \(f, g\) are involutive in relation to the Dirac bracket which restricts them to the submanifold \(N (e, \eta_-)\). For the special value \(\eta_- = 0\), it is just the AKS result \([6]\).

### 4.2 Solving a system in \(N (g_-, \eta_-)\) by factorization

Let us now consider the collective hamiltonian \(h \circ \Phi^L : G \times g^* \rightarrow \mathbb{R}\), with \(h\) being \(Ad^G\)-invariant as above, and \(\Phi^L\) the momentum map associated with the left translation symmetry given in eq. \([22]\). In this way, the hamiltonian function is bi-invariant. Since
\[
d(h \circ \Phi^L) = dh \circ \Phi^L = \delta h \circ \Phi^L
\]
Let \(\mathcal{L}_h : g^* \rightarrow g\) be the Legendre transformation of \(h\) so, the differential of the Hamilton function \(h \circ \Phi^L\) reduces to
\[
d(h \circ \Phi^L) |_{(g, \eta)} = (0, \mathcal{L}_h(Ad^*_g \Phi^L (g, \eta))) = (0, \mathcal{L}_h(\eta))
\]
and the associated hamiltonian vector field on \(G \times g^*\) is
\[
V_{h \circ \Phi^L} = (g \mathcal{L}_h(\eta), 0)
\]
With this result we evaluate the hamiltonian vector field using the Dirac bracket \(\{F, h \circ \Phi^L\}_D (g, \eta)\) defined in \([??]\), obtaining
\[
V_{h \circ \Phi^L}^D = \left( gAd^G_{g_{-1}} \Pi_{g^+} Ad^G_{g_{-1}} \mathcal{L}_h(\eta), Ad^*_g \Pi_{g^+} Ad^G_{g_{-1}} \left[ \eta, Ad^G_{g_{-1}} \Pi_{g^+} Ad^G_{g_{-1}} \mathcal{L}_h(\eta) \right] \right)
\]
Therefore, the Hamilton eqs. motion are
\[
\begin{align*}
g^{-1} \dot{g} &= Ad^G_{g_{-1}} \Pi_{g^+} Ad^G_{g_{-1}} \mathcal{L}_h(\eta) \\
\dot{\eta} &= Ad_{g_{-1}} \Pi_{g^+} Ad^G_{g_{-1}} \left[ \eta, Ad^G_{g_{-1}} \Pi_{g^+} Ad^G_{g_{-1}} \mathcal{L}_h(\eta) \right]
\end{align*}
\]
that in terms of \(g = g_+ g_-\) and \(\eta = \eta_+ + \eta_-\) gives rise to the
\[
\begin{align*}
g_+^{-1} \dot{g}_+ &= \Pi_{g^+} Ad^G_{g_{-1}} \mathcal{L}_h(\eta) \\
\dot{\eta}_+ &= -Ad_{g_{-1}} \Pi_{g^+} Ad^G_{g_{-1}} \left[ \Pi_{g^+} Ad^G_{g_{-1}} \mathcal{L}_h(\eta), \Pi_{g^+} Ad^G_{g_{-1}} \eta \right] \\
\dot{g}_-^{-1} &= 0 \\
\dot{\eta}_- &= 0
\end{align*}
\]
Introducing $\lambda = \text{Ad}_{g_-}^G \eta$ so, if $\eta_-$ is a character of $g_-^\circ$, 
\begin{align*}
\lambda_- = \Pi_{g_-^\circ} \text{Ad}_{g_-^\circ}^G \eta_+ + \Pi_{g_-^\circ} \text{Ad}_{g_-^\circ}^G \eta_- = \text{Ad}_{g_-^\circ}^G \eta_- = \eta_-
\end{align*}

$\lambda_-$ is also a character of $g_-^\circ$, the first couple of equation turns in
\begin{align*}
\begin{cases}
g_+^{-1} \dot{g}_+ = \Pi_{g_-^\circ} \mathcal{L}_h(\lambda) \\
\dot{\lambda}_+ = -\Pi_{g_-^\circ} \left[ \Pi_{g_-^\circ} \mathcal{L}_h(\lambda), \lambda \right]
\end{cases}
\end{align*}

Because the Ad-invariance of $h$ and having in mind that $\lambda_-$ is a character of $g_-^\circ$, the second equation is equivalent to
\begin{align*}
\dot{\lambda}_+ = a d_{\Pi_{g_-^\circ}}^g \mathcal{L}_h(\lambda) \lambda
\end{align*}

and because $\lambda_- = \eta_- = \text{cte}$, we may write
\begin{align*}
\dot{\lambda} = a d_{\Pi_{g_-^\circ}}^g \mathcal{L}_h(\lambda) \lambda \tag{32}
\end{align*}

Introducing the curve $h_- (t) \subset G_-$ satisfying the differential equation
\begin{align*}
\dot{h}_- h_-^{-1} = \Pi_{g_-^\circ} \mathcal{L}_h(\lambda) \tag{33}
\end{align*}

we see that
\begin{align*}
g_+^{-1} \dot{g}_+ + \dot{h}_- h_-^{-1} = \mathcal{L}_h(\lambda)
\end{align*}

Let us now write $\xi = \text{Ad}_{h_-^\circ} \lambda$ so, the last equation is
\begin{align*}
\text{Ad}_{h_-^\circ} \left( g_+^{-1} \dot{g}_+ + \dot{h}_- h_-^{-1} \right) = \mathcal{L}_h (\text{Ad}_{h_-^\circ}^G \xi)
\end{align*}

that can be written as
\begin{align*}
-1 (t) \dot{k} (t) = \mathcal{L}_h (\xi (t)) \tag{34}
\end{align*}

where $k (t) := g_+ (t) h_- (t)$.

Having in mind eqs \(33\) and \(32\), we get
\begin{align*}
\dot{\xi} = 0
\end{align*}

Thus, the couple of equations
\begin{align*}
\begin{cases}
-1 \dot{k} = \mathcal{L}_h (\xi) \\
\dot{\xi} = 0
\end{cases}
\end{align*}

are solved by the curves
\begin{align*}
\begin{cases}
k (t) = e^{\mathcal{L}_h (\xi)} \\
\xi = \xi_o
\end{cases}
\end{align*}
Therefore, we have shown that the Hamilton equations on $\mathcal{N}(g_-, \eta_-)$ have the solutions
\[
\begin{align*}
g_+(t) &= \Pi_{G_+} e^{\xi_0(t)} \\
\eta(t) &= \text{Ad}^{G_{g_+}}_{\Pi_g} e^{\xi_0(t)} \xi_0
\end{align*}
\]
encountering the Adler-Kostant-Symes result [6], [11], relating a system of differential equations on a coadjoint orbit with the factorization problem of an exponential curve in $G$. This issue deserves a deeper insight which is addressed in the next subsection.

4.3 AKS theory

In order to understand the above results, let us digress on the meaning of the Adler-Kostant-Symes approach to integrability [6], [11]. An AKS systems can be characterized as a reduced spaces derived from a dynamical systems defined on the cotangent bundle of a Lie group, so our main concern in this section will be to provide a connection between this kind of systems and the dynamical systems studied above on the phase spaces $\mathcal{N}(g_-, \eta_-)$ (analogous considerations can be made for the spaces $\mathcal{M}(g_+, \eta_+)$).

Let us first stress the role played by some symmetries of a factorizable Lie group. The space $G \times g^*$ can be considered as a $G_+ \times G_-$-space, if as above $G \times g^* \simeq T^*G$ via left trivialization and we lift the $G_+ \times G_-$-action on $G$ given by
\[
G_+ \times G_- \times G \rightarrow G : (a_+, a_-; g) \mapsto a_+ g a_-^{-1}.
\]
By using the facts that the action is lifted and the symplectic form on $G \times g^*$ is exact, we can determine the momentum map associated to this action; then we obtain that
\[
J : G \times g^* \rightarrow g_- \times g_+^*
\]
\[
(g, \xi) \mapsto \left( \Pi_{g_-} (\text{Ad}_g^G \xi), \Pi_{g_+} (\xi) \right)
\]
where $\text{Ad}_g^G$ indicates the coadjoint action of $G$ on $g^*$. Let us now define the submanifold
\[
\Lambda_{\eta_+ \eta_-} := \left\{ (g, \xi) \in G \times g^* / \Pi_{g_-} (\text{Ad}_g^G \xi) = \eta_+, \ \Pi_{g_+} (\xi) = \eta_- \right\}
\]
for each pair $\eta_+ \in g_-^*, \eta_- \in g_+^*$. We have the following lemma.

**Lemma** Let $\xi_\pm \in g_\pm^*$, $a_+ \in G_+, \ a_- \in G_-$ be arbitrary elements. Then the formulas
\[
a_+ \cdot \xi_+ := \Pi_{g_+} \left( \text{Ad}_a^G \xi_+ \right)
\]
\[
a_- \cdot \xi_- := \Pi_{g_-} \left( \text{Ad}_a^G \xi_- \right)
\]
defines an action of \( G_\pm \) on \( \mathfrak{g}_\pm^* \); in fact, under the identification \( \mathfrak{g}_\pm^* \simeq \mathfrak{g}_\pm^* \) induced by the decomposition \( \mathfrak{g} = \mathfrak{g}_+^* \oplus \mathfrak{g}_-^* \), these actions are just the coadjoint actions of each factor on the dual of its Lie algebras.

**Note:** The symbol \( \mathcal{O}^{G_\pm}_{\xi_\pm} \) will denote the orbit in \( \mathfrak{g}_\pm^* \) under the actions defined in the previous lemma. Additionally, for each \( \xi \in \mathfrak{g}^* \), the form \( \xi^g \in \mathfrak{g}^* \) is given by \( \xi^g := B(\xi, \cdot) \), where \( B(\cdot, \cdot) \) is the invariant bilinear form on \( \mathfrak{g}^* \) induced by the Killing form.

Therefore \( \Lambda_{\eta_+, \eta_-} = J^{-1}(\eta_+, \eta_-) \) and, taking into account the Marsden-Weinstein reduction (see [2]), the projection of \( \Lambda_{\eta_+, \eta_-} \) on \( \Lambda_{\eta_+, \eta_-} / \{(G_+)^{\eta_+} \times (G_-)^{\eta_-} \} \) is presymplectic, and the solution curves for the dynamical system defined there by the invariant Hamiltonian \( H(g, \xi) := \frac{1}{2} \xi^g(\xi^g) \) are closely related with the solution curves of the system induced in the quotient. To work out these equations, let us introduce some convenient coordinates. The map \( L_{\eta_+, \eta_-} : \Lambda_{\eta_+, \eta_-} \rightarrow \mathcal{O}^{G_+}_{\eta_+} \times \mathcal{O}^{G_-}_{\eta_-} \) defined as

\[
L_{\eta_+, \eta_-}(g, \xi) = \left( \Pi_{g_\pm} Ad_{g_\pm}^G \eta_\pm, \Pi_{g_\pm} Ad_{g_\pm}^G \eta_- \right) = \left( \Pi_{g_\pm} Ad_{g_\pm}^G \xi, \Pi_{g_\pm} Ad_{g_\pm}^G \xi \right)
\]

where \( g = g_+ g_- \), induces a diffeomorphism on \( \Lambda_{\eta_+, \eta_-} / \{(G_+)^{\eta_+} \times (G_-)^{\eta_-} \} \). If \( (g, \xi; X, \lambda) \) is a tangent vector to \( G \times \mathfrak{g}^* \) at \( (g, \xi) \) (all the relevant bundles are left trivialized) then the derivative of \( L_{\eta_+, \eta_-} \) is

\[
\left. \left( L_{\eta_+, \eta_-} \right)_* \right|_{(g, \xi)} (g, \xi; X, \lambda) = \left( -\Pi_{g_\pm} ad_{X, g_\pm}^G \eta_\pm, \Pi_{g_\pm} ad_{X, g_\pm}^G \eta_- \right) = \left( \Pi_{g_\pm} ad_{X, g_\pm}^G \xi + \Pi_{g_\pm} ad_{X, g_\pm}^G \lambda \right)
\]

if and only if \( g = g_+ g_- \), \( X = \Pi_{g_\pm} \left( Ad_{g_\pm}^G X \right) \). So the following remarkable result is true.

**Proposition** Let \( \mathcal{O}^{G_+}_{\eta_+} \times \mathcal{O}^{G_-}_{\eta_-} \) be the phase space whose symplectic structure is \( \omega_{\eta_+, \eta_-} = \omega_{\eta_+} - \omega_{\eta_-} \), where \( \omega_{\eta_\pm} \) are the corresponding Kirillov-Kostant symplectic structures on each orbit. If \( i_{\eta_+, \eta_-} : \Lambda_{\eta_+, \eta_-} \rightarrow G \times \mathfrak{g}^* \) is the inclusion map, then we have that

\[
i^*_{\eta_+, \eta_-} \omega = L^*_{\eta_+, \eta_-} \omega_{\eta_+, \eta_-}.
\]

**Proof:** Let \( (\varsigma_+, \varsigma_-) = \left( \Pi_{g_\pm} \left( Ad_{a_+}^G \eta_\pm \right), \Pi_{g_\pm} \left( Ad_{a_-}^G \eta_- \right) \right) \) be an arbitrary element of \( \mathcal{O}^{G_+}_{\eta_+} \times \mathcal{O}^{G_-}_{\eta_-} \), then the tangent space at this point is given by

\[
T_{(\varsigma_+, \varsigma_-)} \left( \mathcal{O}^{G_+}_{\eta_+} \times \mathcal{O}^{G_-}_{\eta_-} \right) = \left\{ \left( \Pi_{g_\pm} ad_{X, g_\pm}^G \varsigma_+, \Pi_{g_\pm} ad_{X, g_\pm}^G \varsigma_- \right) / X \in \mathfrak{g}_\pm \right\}
\]

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The symplectic structure $\omega_{\eta_+,\eta_-}$ is given in these terms as
\[
\langle \omega_{\eta_+,\eta_-}, \left( \Pi_{g^+_+} ad^g_{X^+} \eta_+, \Pi_{g^+_-} ad^g_{X^-} \eta_- \right) \rangle \otimes \left( \Pi_{g^+_+} ad^g_{Y^+} \eta_+, \Pi_{g^-_-} ad^g_{Y^-} \eta_- \right) \rangle \rangle_{(\eta_+, \eta_-)}
\]
\[
= \langle \eta_+, [X_+, Y_+] \rangle - \langle \eta_-, [X_-, Y_-] \rangle
\]

Let us take now an element $(g, \xi, X, \lambda)$ tangent to $\Lambda_{\eta_+,\eta_-}$ at $(g, \xi)$; then it is true that
\[
\begin{cases}
\Pi_{g^+} (Ad^g (ad^g X \xi + \lambda)) = 0 \\
\Pi_{g^-} (\lambda) = 0
\end{cases}
\tag{35}
\]
Because of $g = g_+ g_-$, the first condition can be written as
\[
\Pi_{g^+} \left( Ad^g (ad^g X \xi + \lambda) \right) = 0
\]
and because of the nondegeneracy condition\footnote{That is, such that $Ad^g g^+_+ \cap g^-_+ = 0$ for all $g \in G.$}, it is equivalent to
\[
\Pi_{g^+} \left( ad^g_{Ad^g_- X} Ad^g_- \xi + Ad^g_- \lambda \right) = 0
\tag{36}
\]
Let $(g, \xi, X, \lambda), (g, \xi, Y, \mu) \in T(g, \xi) \Lambda_{\eta_+,\eta_-}$; then by evaluating on the canonical form we have that
\[
\langle \omega, (g, \xi, X, \lambda) \otimes (g, \xi, Y, \mu) \rangle_{(g, \xi)}
\]
\[
= \langle \lambda, Y \rangle - \langle \mu, X \rangle - \langle \xi, [X, Y] \rangle
\]
\[
= \left\langle \Pi_{g^+} Ad^g_- \lambda, Y_+ \right\rangle + \left\langle \Pi_{g^+} Ad^g_- \lambda, Y_- \right\rangle
- \left\langle \Pi_{g^+} Ad^g_- \mu, X_+ \right\rangle - \left\langle \Pi_{g^+} Ad^g_- \mu, X_- \right\rangle
- \left\langle Ad^g_+ \xi, [X_+, Y_+] \right\rangle - \left\langle Ad^g_+ \xi, [X_-, Y_-] \right\rangle - \left\langle Ad^g_- \xi, [X_+, Y_-] \right\rangle
\]
where we have used the notation according to which $(X)_{\pm} = \Pi_{g^\pm} \left( Ad^g_- X \right)$ and $(Y)_{\pm} = \Pi_{g^\pm} \left( Ad^g_- Y \right).$ The first term in this expression annihilates because of the eq. \footnote{That is, such that $Ad^g g^+_+ \cap g^-_+ = 0$ for all $g \in G.$}; additionally, the second and fourth vanishes as a consequence of the second eq. in \footnote{That is, such that $Ad^g g^+_+ \cap g^-_+ = 0$ for all $g \in G.$}, which implies that $\lambda, \mu \in g^-_+$, and because this subspace is invariant for the $G_-$. -action through the coadjoint action. Moreover, for the third term in the second hand side, we use the eq. \footnote{That is, such that $Ad^g g^+_+ \cap g^-_+ = 0$ for all $g \in G.$} again, and therefore we can write
\[
\Pi_{g^+} \left( Ad^g_- \mu + ad^g_+ Ad^g_- \xi \right) (X_+) = - \left\langle Ad^g_- \xi, [X_+, Y_+] \right\rangle
\]
From eq. (determined by $H$) reverse direction: to this end let us note that the dynamical system on the solution of the reduced system. In case of AKS systems, we proceed in the degrees of freedom, and the solution of the original system is found by lifting the reduced system to be easier to solve than the original because it involves less.

In general, in dealing with a dynamical systems via reduction it is expected the

and the hamiltonian vector field is then $H$.

Therefore

Let us now address the dynamical data related to an AKS system. We have shown that the symplectic manifold $(\mathcal{O}_{\mathbb{R}^+}^{G^+} \times \mathcal{O}_{\mathbb{R}^-}^{G^-}, \omega_{\eta, \eta_-})$ is symplectomorphic to the reduced space associated to the $G^+ \times G^-$-action defined above on $G \times \mathfrak{g}^*$. As it was pointed out above, the Hamilton function $H \ (g, \xi) = \frac{1}{2} \langle \xi, \xi^b \rangle$ is invariant for this action, and it implies that the solutions of the dynamical system defined by such a hamiltonian on $G \times \mathfrak{g}^*$ are in one-to-one correspondence with those of the dynamical system induced on $\mathcal{O}_{\mathbb{R}^+}^{G^+} \times \mathcal{O}_{\mathbb{R}^-}^{G^-}$ by the hamiltonian $H_{_{\eta, \eta^-}}$ defined according to the formula

Let us note now that if $L_{_{\eta, \eta^-}} (g, \xi) = (\xi_+, \xi_-)$, then $Ad_{_{g^1}}^{G^+} \xi = \xi_+ + \xi_-$, and so

Therefore

and the hamiltonian vector field is then

In general, in dealing with a dynamical systems via reduction it is expected the reduced system to be easier to solve than the original because it involves less degrees of freedom, and the solution of the original system is found by lifting the solution of the reduced system. In case of AKS systems, we proceed in the reverse direction: to this end let us note that the dynamical system on $G \times \mathfrak{g}^*$ determined by $H$ has hamiltonian vector field given by

\[ V_{_{(g, \xi)}} = \left( \xi^b, -ad_{\xi}^{\mathfrak{g}} \xi \right) = \left( \xi^b, 0 \right) \]
because of the invariance condition $ad_X^{g} X^g = 0$ for all $X \in g$. Then the solution in the original phase space passing through $(g, \xi)$ at the initial time is

$$t \mapsto \left(g \exp t \xi^g, \xi\right)$$

and if this initial data verifies $\Pi_{g^\pm} \left(Ad^g_\xi\right) = \eta_+, \Pi_{g_+}(\xi) = \eta_-$, then this curve belongs to $\Lambda_{\eta_+ \eta_-}$ for all $t$. Therefore the map

$$t \mapsto \left(\Pi_{g^\pm} Ad^G_{(g_+,(t))}^{-1} \eta_+, \Pi_{g^\pm} Ad^G_{g_-,(t)} \eta_-ight)$$

where $g^\pm : \mathbb{R} \to G^\pm$ are the curves defined by the factorization problem $g_+ (t) g_- (t) = g \exp t \xi^g$, is solution for the dynamical system associated to the vectorial field $\mathcal{L}$, yielding to a differential equation which is harder to solve than the original system.

4.4 AKS dynamics on the spaces $\mathcal{N} \left(g_-, \eta_-\right)$

Let $\eta_- \in g_+^\circ$ be a character. As it was shown above, the spaces $\mathcal{N} \left(g_-, \eta_-\right)$ are $G$-spaces, and in particular $G_+ \times G_- \times G^\dag$-spaces; therefore there exists a momentum map $J_- : \mathcal{N} \left(g_-, \eta_-\right) \to g_+^\circ$. The connection with the dynamics on the phase spaces $\mathcal{N} \left(g_-, \eta_-\right)$ is given by the following theorem, which is a consequence of the hamiltonian reduction by stages [13].

**Theorem** Let $\eta_- \in g_+^\circ$ be an arbitrary element. The reduced space

$$J_-^{-1}(\eta_-)/ (G_+)_{\eta_-}$$

is therefore symplectomorphic to the orbit space $\mathcal{O}_{\eta_+}^{G_+} \times \mathcal{O}_{\eta_-}^{G_-}$.

**Proof:** Simply note that $G_+$ and $G_-$ commutes each other as subgroups in $G_+ \times G_-$, and the same is true for their actions on $G$; then by performing the reduction into $J_-^{-1}(\eta_+, \eta_-)/ [(G_+)_{\eta_+} \times (G_-)_{\eta_-}]$ by stages, first using the right $G_-$-action and then the left $G_+\times$-action, we prove the theorem.

The following diagram can be useful to explain the contents of the previous statement:

![Diagram](image-url)
where the $\pi$-maps denotes the canonical projections onto the corresponding quotients. Accordingly, the picture can be described as follows: the hamiltonian $H(g,\xi) = \frac{1}{2}\xi(\xi^\flat)$ defines on $G \times g^*$ a dynamical system whose solutions are easier to be found; they are given by the formula

$$t \mapsto (g \exp t\xi^\flat, \xi)$$

for any initial data $(g_0, \xi_0) \in G \times g^*$. Since $G$ is a factorizable Lie group, $G = G_+G_-$ for some subgroups $G_{\pm} \subset G$ such that $G_+ \cap G_- = \{e\}$; on $G$ there exists a $G_+ \times G_-$-action defined as

$$(a_+, a_-) \cdot g := a_+ga_-^{-1}$$

for all $a_+ \in G_+, a_- \in G_-$ and $g \in G$. This action lift to a symplectic action on $G \times g^*$ (left trivialized), and $H$ turns out to be an invariant hamiltonian; therefore there exists a $G_+ \times G_-$-action defined on $J^{-1}(\eta_+, \eta_-)/[(G_+)_{\eta_+} \times (G_-)_{\eta_-}]$ is an AKS system.

Let us now suppose that $\eta_- \in g_{\otimes}^0$ is a character for the coadjoint action. It was shown above that the spaces $\mathcal{N}(g_-, \eta_-)$ are slices for the right $G_-$-action on $G \times (g_{\otimes}^2 + \eta_-)$, and therefore they are symplectic submanifolds of $G \times g^*$. Further, it was shown that these spaces are symplectomorphic to the space $G \times G_- (g_{\otimes}^2 + \eta_-)$, the reduced space for the right $G_-$-action on $G \times g^*$. Additionally, this reduced space is a $G$-space via the lift of the canonical action of $G$ on its homogeneous space $M = G/G_-; in particular, it is a $G_+$-space, and we can apply the reduction scheme once more. The reduction by stages theorem guarantees that the resulting space is symplectomorphic to the AKS system as obtained in section 4.3, and the solution curves for the system induced on $J^{-1}(\eta_+, \eta_-)/[(G_+)_{\eta_+} \times (G_-)_{\eta_-}]$ is an AKS system.

Let us now address the lifting the dynamics to $\mathcal{N}(g_-, \eta_-)$. As before, here $\eta_-$ is a character for the $Ad^*$-action and $g_-$ an arbitrary element in $G_-$. We want to describe the dynamical system on $\mathcal{N}(g_-, \eta_-)$ defined through the reduction procedure described above. In order to achieve this, let us recall that the map

$$\tilde{\pi}_{\eta_-} := \pi_{\eta_-} \big| \mathcal{N}(g_-, \eta_-) : \mathcal{N}(g_-, \eta_-) \rightarrow J^{-1}(\eta_-)/G_- =: M_{\eta_-}$$

between this symplectic submanifold and the reduced space $M_{\eta_-}$ is a symplectomorphism (i.e. the note at the end of section 3.1.3♣). Let us denote by $\omega_{\mathcal{N}}$ the symplectic structure on $\mathcal{N}(g_-, \eta_-)$; now the reduction procedure defines on $M_{\eta_-}$ both the symplectic structure $\omega_{\eta_-}$ and the hamiltonian function $H_{\eta_-}$.
through the formulas
\[ \pi^*_\eta_\omega \eta_\pi^* \omega \eta_- = \omega_0 \mid J_-^{-1} (\eta_-) \]
\[ \pi^*_\eta_\omega \eta_- = \mathcal{H} \mid J_-^{-1} (\eta_-) \]
so that if \( i_{\eta_-} : \mathcal{N} (g_-, \eta_-) \hookrightarrow G \times \mathfrak{g}^* \) is the inclusion, then
\[ i_{\eta_-}^* \omega = i_{\eta_-}^* \left[ \omega_0 \mid J_-^{-1} (\eta_-) \right] \]
\[ = i_{\eta_-}^* \pi^*_\eta_- \omega \eta_- \]
\[ = \left( \pi^*_\eta_- \circ i_{\eta_-} \right)^* \omega \eta_- \]
\[ = \pi^*_\eta_- \omega \eta_- \]
\[ = \omega \mathcal{N} \]
and accordingly \( \mathcal{H}_\mathcal{N} := i_{\eta_-}^* \mathcal{H} \) verifies \( \pi^*_\eta_- \mathcal{H} \eta_- = \mathcal{H}_{\mathcal{N}} \). Therefore the dynamics defined on \( \mathcal{N} (g_-, \eta_-) \) via the identification with the reduced space \( M_{\eta_-} \) is equivalent to the dynamics defined on the same space through the restriction of the dynamical data \((\omega_0, \mathcal{H})\). These considerations enables to lift the AKS dynamics from the full reduced space \( J_-^{-1} (\eta_-, \eta_-) / \left( (G_+)_{\eta_+} \times G_- \right) \) to \( M_{\eta_-} \); that is, if \((h, \xi)\) is an arbitrary point at \( J_-^{-1} (\eta_-) \), then the solution \( \gamma : t \mapsto \left( h \exp t \xi, \xi \right) \) for the dynamical system defined on \( G \times \mathfrak{g}^* \) by the data \((\omega_0, \mathcal{H})\) remains there for all \( t \), and \( t \mapsto \pi_{\eta_-} (\gamma (t)) \) is the solution for the dynamical system defined on \( M_{\eta_-} \) via \( (\omega_{\eta_-}, \mathcal{H}_{\eta_-}) \) passing through \( \pi_{\eta_-} (h, \xi) \). Finally, \( t \mapsto \left[ \pi_{\eta_-}^{-1} \circ \pi_{\eta_-} \right] (\gamma (t)) \) is the solution to the Hamilton equations on \( \mathcal{N} (g_-, \eta_-) \) associated to the dynamical data \((\omega_{\mathcal{N}}, \mathcal{H}_{\mathcal{N}})\), with initial data \( \gamma (0) = \pi_{\eta_-}^{-1} \circ \pi_{\eta_-} (h, \xi) \). An explicit formula for this solution curve can be given, because
\[ \left[ \pi_{\eta_-}^{-1} \circ \pi_{\eta_-} \right] (h_+ h_-, \xi_+ + \eta_-) = \pi_{\eta_-}^{-1} \left[ [h_+ h_-, \xi_+ + \eta_-] \right] \]
\[ = \pi_{\eta_-}^{-1} \left[ [h_+ g_-, \operatorname{Ad}_{g_+^{-1} h_-} \xi_+ + \eta_-] \right] \]
\[ = (h_+ g_-, \operatorname{Ad}_{g_+^{-1} h_-} \xi_+ + \eta_-) \]
This means that the solution curve has the following form
\[ t \mapsto (h_+ (t) g_-, \operatorname{Ad}_{g_+^{-1} h_- (t)} \xi_+ + \eta_-) \]
where \( h_\pm : \mathbb{R} \rightarrow G_\pm \) are the solutions of the factorization problem \( h_+ (t) h_- (t) = h \exp t \xi_\pm \).

In the same vein, the previous setting can be applied in order to find the hamiltonian vector field associated to the hamiltonian \( \mathcal{H}_{\mathcal{N}} \) on \( \mathcal{N} (g_-, \eta_-) \). First of all, we need the following proposition.
**Proposition:** By left trivializing $T_{(h,\xi)} (G \times \mathfrak{g}^*) = T_hG \times T_\xi \mathfrak{g}^* = \mathfrak{g} \times \mathfrak{g}^*$, the derivative of the map $\Xi_{\eta_-} := \hat{\pi}_{\eta_-} \circ \pi_\nu$ is given by

$$
\Xi_{\eta_-}(h,\xi)(X,\lambda_+) = \left( Ad_{g_-^1} \Pi_{g_+} Ad_{h_-^G} X, Ad_{h_-^G} \xi_+ + Ad_{g_-^1 h_-^G} \lambda_+ \right)
$$

for all $(X,\lambda_+) \in T_{(h,\xi)} J_{\eta_-}^{-1}(\eta_-) \subset \mathfrak{g} \times \mathfrak{g}^*$.

**Proof:** Let $h_\pm : \mathbb{R} \to G_\pm$ be the solution curves for the factorization problem

$$
h_\pm(t) h_- (t) = h \exp tX
$$

then if

$$
X_\pm := L_{g_\pm^{-1}} \left\{ \frac{d}{dt} \bigg|_{t=0} [h_\pm(t)] \right\}
$$

we see that

$$
X = Ad_{h_-^G} X_+ + X_-
$$

It is immediate from here to conclude that

$$
X_+ = \Pi_{g_+} \left( Ad_{h_-^G} X \right)
$$

and additionally

$$
Ad_{h_-^G} X_- = \Pi_{g_-} Ad_{h_-^G} X
$$

By using the previous formula for map $\pi_{\eta_-}$ and the identity

$$
\frac{d}{dt} \bigg|_{t=0} \left[ Ad_{h_-^G(t)} \xi_+ \right] = ad_{Ad_{h_-^G} X_-} Ad_{h_-^G} \xi_+
$$

we obtain the proposition. $\blacksquare$

Finally it remains to take into account that

$$
\mathcal{V}_\mathcal{H}|_{(g_+ g_-, \eta_+ + \eta_-)} = \Xi_{\eta_-}|_{(h,\xi)} \left( \mathcal{V}_\mathcal{H}|_{(h,\xi)} \right)
$$

for any $(h,\xi) \in \Xi_{\eta_-}^{-1}(g_+ g_-, \eta_+ + \eta_-)$. From the expression

$$
\mathcal{V}_\mathcal{H}|_{(h,\xi)} = \left( \xi_+,0 \right)
$$

in the left trivialization, and by realizing that $(h,\xi) \in \Xi_{\eta_-}^{-1}(g_+ g_-, \eta_+ + \eta_-)$ if and only if

\[
\begin{cases}
    h_+ = g_+ \\
    \xi_+ = Ad_{h_-^G} g_- \eta_+
\end{cases}
\]

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we can conclude that

\[ V_{\mathcal{L}^*}(g, \eta) = \left( Ad_{g_-}^G \Pi_{\mathfrak{g}_+^*} Ad_{g_+} \eta_+, Ad_{g_-}^G \Pi_{\mathfrak{g}_+^*} Ad_{g_+} \eta_+ \right) \] (39)

for \((g, \eta) = (g_+ g_-, \eta_+ + \eta_-)\). It was used here that the \(Ad\)-invariance of the Killing form implies the equivariance formula

\[ Ad_{g_-}^G \xi_+ = \left( Ad_{g_-}^G \xi_+ \right)^b \]

for all \(g_- \in G_-, \xi_+ \in \mathfrak{g}_+^*\).

It is interesting to relate this vector field with the vector field obtained in eq. (29). If \(h(\xi) := \frac{1}{2} \langle \xi, \xi^b \rangle\), then \(\mathcal{L}_h(\eta) = \eta^b\) and the vector field (29) (evaluated at \((h_+ h_-, \eta_+ + \eta_-)\)) can be written as

\[
V_{\mathcal{L}_h} = \left( g Ad_{h_-}^G \Pi_{\mathfrak{g}_+^*} Ad_{h_+} \left( \eta_+ + \eta_- \right)^b, Ad_{h_-}^G \Pi_{\mathfrak{g}_+^*} Ad_{h_+} \left( \eta_+ + \eta_- \right)^b \right)
\]

where it was used that, under the performed identifications, \(\eta^b_\pm = \eta_\pm\) for all \(\eta_\pm \in \mathfrak{g}_\pm^*\), and moreover, that \(\eta_- \in \mathfrak{g}_-^*\) is a character. Finally

\[
V_{\mathcal{L}_h} = \left( g Ad_{h_-}^G \Pi_{\mathfrak{g}_+^*} Ad_{h_+} \left( \eta_+ \right)^b, Ad_{h_-}^G \Pi_{\mathfrak{g}_+^*} Ad_{h_+} \left( \eta_+ \right)^b \right)
\]

gives us the desired expression, to be compared with eq. 29.

5 Example: \(SL(2, \mathbb{C}) = SU(2) \times B\)

We now specialize the above abstract structure to \(G = SL(2, \mathbb{C})\) and its Iwasawa decomposition \(SL(2, \mathbb{C}) \cong SU(2) \times B\), where \(B\) is the group of \(2 \times 2\) complex upper triangular matrices, with real diagonal and determinant equal to 1, and we identify \(G_+ = SU(2)\) and \(G_- = B\).

The Killing form for \(\mathfrak{sl}_2(\mathbb{C})\) is \(\kappa(X, Y) := tr(\text{ad}(X) \text{ad}(Y)) = 4 \text{tr}(XY)\), the restrictions to \(\mathfrak{su}_2\), \(a\), and \(n\) are negative defined; positive defined, and 0, respectively. We consider the non degenerate symmetric bilinear form on \(\mathfrak{sl}_2(\mathbb{C})\)

\[ (X, Y)_{\mathfrak{sl}_2} = -\frac{1}{4} \text{Im} \kappa(X, Y) \] (40)
which turns $\mathfrak{b}$ and $\mathfrak{su}_2$ into isotropic subspaces. Also, we take the basis

$$T_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad T_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

for $\mathfrak{su}_2$, and

$$T^1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad T^3 = -\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in $\mathfrak{b}$. Then, the crossed product are

$$(T_1, T^1) = 1 \quad (T_2, T^1) = 0 \quad (T_3, T^1) = 0$$  \hspace{1cm} (41)

$$(T_1, T^2) = 0 \quad (T_2, T^2) = 1 \quad (T, T^2) = 0$$

$$(T_1, T^3) = 0 \quad (T_2, T^3) = 0 \quad (T_3, T^3) = 1$$

allowing for the identification

$$\psi : \mathfrak{su}_2 \to \mathfrak{b}^*$$ \hspace{1cm} (42)

$$\psi(T_1) = t^1, \quad \psi(T_2) = t^2, \quad \psi(T_3) = t^3$$

where $\{t^1, t^2, t^3\} \subset \mathfrak{b}^*$ is the dual basis to $\{T^1, T^2, T^3\} \subset \mathfrak{b}$.

Let us come back to the expression (??)

$$\{F, G\}^D(g, \eta) = \left\langle gdF, Ad_{g_{-1}}^G \Pi_{\mathfrak{g}^+} Ad_{g_{-1}}^G \delta G \right\rangle$$

$$- \left\langle gdG, Ad_{g_{-1}}^G \Pi_{\mathfrak{g}^+} Ad_{g_{-1}}^G \delta F \right\rangle$$

$$- \left\langle \eta, \left[ Ad_{g_{-1}}^G \Pi_{\mathfrak{g}^+} Ad_{g_{-1}}^G \delta F, Ad_{g_{-1}}^G \Pi_{\mathfrak{g}^+} Ad_{g_{-1}}^G \delta G \right] \right\rangle$$

and apply it to the functions

$$T^j_i : G \to \mathbb{C} / T^j_i(g) = g^i_j$$

$$\xi_A : \mathfrak{g}^* \to \mathbb{C} / \xi_A = \langle \xi, T_A \rangle$$

with

$$\delta T^j_i = 0$$

$$d\xi_A = 0$$

So, it is necessary to calculate the expressions

$$Ad_{g_{-1}}^G \Pi_{\mathfrak{g}^+} Ad_{g_{-1}}^G \delta \xi_A$$

The differential $d\xi_A$ coincides with the generator $T_A$ of the Lie algebra $\mathfrak{g}$, being $T_A \in \{T_a, T^a\}_{a=1}^n$. This relations can be written in terms of the coordinates for $\eta_+ = \langle \eta_+, T_a \rangle t_a = \xi_a(\eta) t_a$ and $\eta_- = \langle \eta_-, T^a \rangle t^a = \xi^a(\eta) t^a$. Therefore, for $A$
running on the superindex \( \delta \xi_A \equiv \delta \xi^a = T^a \) we have \( \delta \xi_A = \delta \xi_{A-} \) and \( \delta \xi_{A+} = 0 \).

On the other side, for \( A \) running on the subindex \( \delta \xi_a \equiv \delta \xi_a = T_a \), we have \( \delta \xi_A = \delta \xi_{A-} = \delta \xi_a \) and \( \delta \xi_{A+} = 0 \). Therefore, since the only non vanishing Dirac brackets just involve \( \xi_a \), we evaluate

\[
\text{Ad}_{g^-}^{G} \Pi_{\theta^+}^{G} \text{Ad}_{g^-}^{G} \delta \xi_a = \text{Ad}_{g^-}^{G} \Pi_{\theta^+}^{G} \text{Ad}_{g^-}^{G} T_a
\]

Writing a generic element

\[
g_+ = \left( \begin{array}{cc} \alpha & \beta \\ -\beta & \alpha \end{array} \right) \in G_+ \cong SU(2)
\]

and

\[
g_- = \left( \begin{array}{cc} a & b + ic \\ 0 & a^{-1} \end{array} \right) \left( \begin{array}{cc} a & z \\ 0 & a^{-1} \end{array} \right) \in G_- \cong B
\]

with \( a \in \mathbb{R}_{>0}, \ b, c \in \mathbb{R} \), we have that

\[
\text{Ad}_{g^-_1}^{G} \Pi_{\theta^+}^{G} \text{Ad}_{g^-_1}^{G} T_1 = T_1 - \left( 1 - \frac{b^2}{a^2} - \frac{c^2}{a^2} - \frac{1}{a^4} \right) T^2 + 2 \frac{c}{a} T^3
\]

\[
\text{Ad}_{g^-_1}^{G} \Pi_{\theta^+}^{G} \text{Ad}_{g^-_1}^{G} T_2 = T_2 + \left( 1 - \frac{b^2}{a^2} - \frac{c^2}{a^2} - \frac{1}{a^4} \right) T^1 + 2 \frac{b}{a} T^3
\]

\[
\text{Ad}_{g^-_1}^{G} \Pi_{\theta^+}^{G} \text{Ad}_{g^-_1}^{G} T_3 = T_3 + 2 \frac{c}{a} T^1 + 2 \frac{b}{a} T^2
\]

Explicit calculation for the non trivial Dirac brackets gives

\[
\{ \xi_1, T_i \}^D (g, \eta) = -T_i^k (g) \left[ T_1 - \left( 1 - \frac{b^2}{a^2} - \frac{c^2}{a^2} - \frac{1}{a^4} \right) T^2 + 2 \frac{c}{a} T^3 \right]_k
\]

\[
\{ \xi_2, T_i \}^D (g, \eta) = -T_i^k (g) \left[ T_2 + \left( 1 - \frac{b^2}{a^2} - \frac{c^2}{a^2} - \frac{1}{a^4} \right) T^1 + 2 \frac{b}{a} T^3 \right]_k
\]

\[
\{ \xi_3, T_i \}^D (g, \eta) = -T_i^k (g) \left[ T_3 + 2 \frac{c}{a} T^1 + 2 \frac{b}{a} T^2 \right]_k
\]
\{\xi_1, \xi_2\}^D (g, \eta) = 2\frac{b}{a} \xi_1 (\eta) - 2\frac{c}{a} \xi_2 (\eta) + 2 \left( \frac{b^2}{a^2} + \frac{c^2}{a^2} + \frac{1}{a^4} \right) \xi_3 (\eta)

- 2\frac{c}{a} \left( 1 + \frac{b^2}{a^2} + \frac{c^2}{a^2} + \frac{1}{a^4} \right) \xi^1 (\eta)

- 2\frac{b}{a} \left( 1 + \frac{b^2}{a^2} + \frac{c^2}{a^2} + \frac{1}{a^4} \right) \xi^2 (\eta)

\{\xi_1, \xi_3\}^D (g, \eta) = -2\xi_2 (\eta) - 2\frac{c}{a} \xi_3 (\eta)

+ 2 \left( \frac{b^2}{a^2} + 3\frac{c^2}{a^2} + \frac{1}{a^4} - 1 \right) \xi^1 (\eta)

+ 4\frac{bc}{a^2} \xi^2 (\eta) + 4\frac{b}{a} \xi^3 (\eta)

\{\xi_2, \xi_3\}^D (g, \eta) = 2\xi_1 (\eta) - 2\frac{b}{a} \xi_3 (\eta) + 4\frac{bc}{a^2} \xi^1 (\eta)

+ 2 \left( 3\frac{b^2}{a^2} + \frac{c^2}{a^2} + \frac{1}{a^4} - 1 \right) \xi^2 (\eta) - 4\frac{c}{a} \xi^3 (\eta)

Setting \eta_\pm = 0, the only character in \mathfrak{b}^*, they reduce to

\{\xi_1, T^j_i\}^D (g, \eta) = -T^k_i (g) [T^j_k]_k

+ T^k_i (g) \left[ \left( 1 - \frac{b^2}{a^2} - \frac{c^2}{a^2} - \frac{1}{a^4} \right) T^2 + 2\frac{c}{a} T^3 \right]_k

\{\xi_2, T^j_i\}^D (g, \eta) = -T^k_i (g) [T^j_k]_k

- T^k_i (g) \left[ \left( 1 - \frac{b^2}{a^2} - \frac{c^2}{a^2} - \frac{1}{a^4} \right) T^1 + 2\frac{b}{a} T^3 \right]_k

\{\xi_3, T^j_i\}^D (g, \eta) = -T^k_i (g) [T^j_k]_k - T^k_i (g) \left[ \frac{c}{a} T^1 + 2\frac{b}{a} T^2 \right]_k

\{\xi_1, \xi_2\}^D (g, \eta) = 2\xi_3 (\eta) + 2\frac{b}{a} \xi_1 (\eta)

- 2\frac{c}{a} \xi_2 (\eta) - 2 \left( 1 - \frac{b^2}{a^2} - \frac{c^2}{a^2} - \frac{1}{a^4} \right) \xi_3 (\eta)

\{\xi_3, \xi_1\}^D (g, \eta) = 2\xi_2 (\eta) + 2\frac{c}{a} \xi_3 (\eta)

\{\xi_2, \xi_3\}^D (g, \eta) = 2\xi_1 (\eta) - 2\frac{b}{a} \xi_3 (\eta)

Let us now address some dynamical model on the phase space \mathcal{N} (g, 0), taking a collective hamiltonian function

\mathcal{H} (g, \eta) = -\frac{1}{16} \kappa (\Pi_{g^k} Ad_g^C \tilde{\psi} (\eta_+) , \Pi_{g^k} Ad_g^C \tilde{\psi} (\eta_+))

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We may write then
\[ \mathcal{H}(g, \eta) = \frac{1}{2} \sum_{a=1}^{3} \phi^a(g, \eta)\phi^a(g, \eta) \]

The Hamilton equation of motion are then
\[ \dot{T}_j^k = \{ T^k_j, \mathcal{H} \}^D \]
\[ \dot{\xi}_a = \{ \xi_a, \mathcal{H} \}^D \]

We may use the properties of the Poisson-Dirac bracket to get the Hamilton equation of motion
\[ \begin{cases} \dot{T}_v^u = \sum_{a=1}^{3} \phi^a \{ T_v^u, \phi^a \}^D \\ \dot{\xi}_c = \sum_{a=1}^{3} \phi^a \{ \xi_c, \phi^a \}^D \end{cases} \]

A more explicit expression can be obtained in a compact matrix notation and considering the Hamilton equations in the form (\ref{eq:47}). Therefore, for the Hamiltonian (\ref{eq:47}), and introducing the identification \( \hat{\kappa} g^\circ \rightarrow g^\ast \), such that
\[ \langle \hat{\kappa} g^\circ (X_+), Y_+ \rangle = \kappa(X_+, Y_+), \] with the relations \( \hat{\kappa} g^\circ (T_i) = -8t_i \), and using the bijection (\ref{eq:47}), \( \bar{\psi}^\ast (\hat{\kappa} g^\circ (T_i)) = -8T^i \), where \( \Psi : g^\ast \rightarrow g \) stands for the identification provided by the non degenerate scalar product (,\( _g \)) which, when restricted to \( g^\circ \), coincides with \( \psi \). So that differential is
\[ d\mathcal{H} = (d\mathcal{H}, \delta \mathcal{H}) \]
\[ = \left( -\frac{1}{2} \lambda R^*_{g^{-1}} \left( a d^B_{\Psi^g (\eta)} \hat{\kappa} g^\circ \left( \Pi_{g^\circ} A d^C g^\ast \psi (\eta) \right) \right) \right) \]
\[ + \frac{1}{2} \lambda A d^C_{g^{-1}} \bar{\psi}^\ast \left( \hat{\kappa} g^\circ \left( \Pi_{g^\circ} A d^C g^\ast \psi (\eta) \right) \right) \]

and remembering that \( \eta_- = 0 \), the Hamilton equations reads, for \( \eta_- = 0 \),
\[ \begin{cases} g_+^{-1} g_+ = -\frac{1}{8} \Pi_{g^\circ} A d^C_{g^\circ} \bar{\psi}^\ast \left( \hat{\kappa} g^\circ \left( \Pi_{g^\circ} A d^C g^\ast \psi (\eta) \right) \right) \\ A d^C_{g^\circ} \eta_- = -\frac{1}{8} \Pi_{g^\circ} a d^B_{\Pi_{g^\circ} A d^C g^\circ} \bar{\psi}^\ast \left( \hat{\kappa} g^\circ \left( \Pi_{g^\circ} A d^C g^\ast (\eta) \right) \right) A d^C_{g^\circ} \eta_+ \end{cases} \] (\ref{eq:47})
\[ \begin{cases} g_- g_-^{-1} = 0 \\ \eta_- = 0 \end{cases} \]

To get some insight on the meaning physical content of this model, let us built up the Lagrangian function associated with this hamiltonian system. Since
\[ \bar{\psi}^\ast (\eta) = \eta_1 T^1 + \eta_2 T^2 + \eta_3 T^3 \]
and

\[ g_+^{-1} g_+^T = \frac{1}{2} i (\alpha \beta - \bar{\alpha} \bar{\beta}) T_1 + \frac{1}{2} (\alpha \bar{\beta} + \bar{\alpha} \beta) T_2 \]
\[ g_+^{-1} g_+^{T2} = -\frac{1}{2} \left( \beta^2 + \bar{\beta}^2 \right) T_1 - \frac{1}{2} i \left( \beta^2 - \bar{\beta}^2 \right) T_2 - \frac{1}{2} (\alpha \bar{\beta} + \bar{\alpha} \beta) T_3 \]
\[ g_+^{-1} g_+^{T1} = \frac{1}{2} i \left( \beta^2 - \bar{\beta}^2 \right) T_1 - \frac{1}{2} \left( \beta^2 + \bar{\beta}^2 \right) T_2 + \frac{1}{2} i (\alpha \beta - \bar{\alpha} \bar{\beta}) T_3 \]

where it was used that we get

\[ \bar{\psi}^* (t_1) = T^1 \quad \bar{\psi}^* (t_2) = T^2 \quad \bar{\psi}^* (t_3) = T^3 \] (48)

we get

\[ Ad_{g_-}^G \bar{\psi} (\eta_+) = a (a \eta_{+1} - b \eta_{+3}) T^1 + a (a \eta_{+2} + c \eta_{+3}) T^2 + \eta_{+3} T^3 \]

Let us recall that \( \phi_X (g, \eta) = \left< Ad_{g_-}^G \eta, X \right> \), so that \( \phi^a (g, \eta) = \left< Ad_{g_-}^G \eta, T^a \right> \) with

\[ \phi^1 (g, \eta) = \frac{1}{2} i a^2 \left( \beta^2 - \bar{\beta}^2 \right) \eta_{+1} - \frac{1}{2} a^2 \left( \beta^2 + \bar{\beta}^2 \right) \eta_{+2} + \frac{1}{2} i \left( \alpha \beta - \bar{\alpha} \bar{\beta} - a z \beta^2 + a z \bar{\beta}^2 \right) \eta_{+3} \]
\[ \phi^2 (g, \eta) = -\frac{1}{2} a^2 \left( \beta^2 + \bar{\beta}^2 \right) \eta_{+1} - \frac{1}{2} i a^2 \left( \beta^2 - \bar{\beta}^2 \right) \eta_{+2} + \frac{1}{2} \left( a z \beta^2 + a z \bar{\beta}^2 - \bar{\alpha} \beta - \alpha \bar{\beta} \right) \eta_{+3} \]
\[ \phi^3 (g, \eta) = -\frac{1}{2} a^2 \left( \bar{\alpha} \beta - \alpha \bar{\beta} \right) \eta_{+1} + \frac{1}{2} a^2 \left( \alpha \bar{\beta} + \bar{\alpha} \beta \right) \eta_{+2} + \frac{1}{2} i a \left( z \bar{\alpha} \beta - z \alpha \bar{\beta} \right) \eta_{+3} \]

Written in components, the first Hamilton equation in (47) gives rise to the differential equations

\[ \text{Im} \left( \bar{\alpha} \beta - \bar{\beta} \bar{\alpha} \right) = -\frac{1}{2} \left( \left( \beta^2 + \bar{\beta}^2 \right) \phi^2 - i \left( \beta^2 - \bar{\beta}^2 \right) \phi^1 - i \left( \alpha \bar{\beta} - \bar{\alpha} \beta \right) \phi^3 \right) \]
\[ \text{Re} \left( \bar{\alpha} \beta - \bar{\beta} \bar{\alpha} \right) = -\frac{1}{2} \left( \left( \beta^2 + \bar{\beta}^2 \right) \phi^1 + i \left( \beta^2 - \bar{\beta}^2 \right) \phi^2 - (\alpha \bar{\beta} + \bar{\alpha} \beta) \phi^3 \right) \]
\[ \left( \bar{\alpha} \dot{\alpha} + \beta \dot{\beta} \right) = -\frac{1}{2} \left( (\bar{\alpha} \bar{\beta} + \alpha \beta) \phi^2 - i \left( \alpha \beta - \bar{\alpha} \bar{\beta} \right) \phi^1 \right) \]

We use these equations to invert the Legendre transformation in order to retrieve the Lagrange function. From these equations we write \( \eta_{+1} \) and \( \eta_{+2} \).
in terms of the velocities $\dot{\alpha} - \dot{\beta} \bar{\alpha}$ and $\bar{\alpha} \dot{\alpha} + \beta \dot{\beta}$. Thus, the expression of the momentum map associated with the dressing action are

$$\Pi_{\eta_{+}} A\tilde{g}_{\tilde{g}} \tilde{\psi} (\eta_{+})_1 = -\frac{1}{\alpha \beta} (\alpha \beta + \gamma \delta) \left( \dot{\alpha} \dot{\beta} - \alpha \dot{\beta} \right)$$

$$+ \frac{1}{2 \alpha |\beta|^4} \left( |\beta|^4 + |\beta|^2 - \gamma^2 \delta^2 \right) \left( \bar{\alpha} \dot{\alpha} + \bar{\beta} \dot{\beta} \right)$$

$$\Pi_{\eta_{+}} A\tilde{g}_{\tilde{g}} \tilde{\psi} (\eta_{+})_2 = -i \frac{1}{2 \alpha} (\alpha \beta - \gamma \delta) \left( \dot{\alpha} \dot{\beta} - \alpha \dot{\beta} \right)$$

$$+ \frac{1}{2 \alpha |\beta|^4} \left( \gamma^2 \delta^2 + |\beta|^4 + |\beta|^2 \right) \left( \bar{\alpha} \dot{\alpha} + \bar{\beta} \dot{\beta} \right)$$

$$\Pi_{\eta_{+}} A\tilde{g}_{\tilde{g}} \tilde{\psi} (\eta_{+})_3 = \frac{\alpha}{\beta} \left( \dot{\alpha} \dot{\beta} - \alpha \dot{\beta} \right) - \left( \bar{\alpha} \dot{\alpha} + \bar{\beta} \dot{\beta} \right)$$

where we may observe that there are no trace of $g_-$. So, the expression for the Hamilton function in terms of the velocities is the same in each phase space $N (g_-, 0)$,

$$H(\eta, \tilde{g}) = \frac{1}{2 |\beta|^2} \left( \frac{\alpha \gamma}{\alpha \beta} \left( \dot{\alpha} \dot{\beta} - \alpha \dot{\beta} \right)^2 - 2 \frac{\gamma}{\delta} \left( \bar{\alpha} \dot{\alpha} + \bar{\beta} \dot{\beta} \right) \left( \dot{\alpha} \dot{\beta} - \alpha \dot{\beta} \right) - \left( \bar{\alpha} \dot{\alpha} + \bar{\beta} \dot{\beta} \right)^2 \right)$$

Let us then build up the Lagrangian function describing the dynamics of these systems. It is obtained as usual

$$L_{N(g_{-}, 0)} (g, \tilde{g}) = \langle \eta_{+}, g_{+}^{-1} \tilde{g}_{+} \rangle - H(g_{+}, \eta_{+})$$

where

$$\langle \eta_{+}, g_{+}^{-1} \tilde{g}_{+} \rangle = \eta_{+1} \operatorname{Im} \left( \dot{\alpha} \dot{\beta} - \alpha \dot{\beta} \right) + \eta_{+2} \operatorname{Re} \left( \dot{\alpha} \dot{\beta} - \alpha \dot{\beta} \right) - i \eta_{+3} \left( \bar{\alpha} \dot{\alpha} + \bar{\beta} \dot{\beta} \right)$$

After some tedious calculations, we get

$$L_{N(g_{-}, 0)} (g_{+}, \tilde{g}_{+}) = \frac{1}{2 a^2 |\beta|^2} \left| \dot{\alpha} \dot{\beta} - \alpha \dot{\beta} \right|^2 + \frac{1}{2 |\beta|^2} \frac{\alpha \beta}{\alpha \beta} \left( \frac{1}{a^2} - 1 \right) \left( \dot{\alpha} \dot{\beta} - \alpha \dot{\beta} \right)^2$$

$$+ \frac{1}{2 a^2 |\beta|^2} \bar{\beta} \left( 2 a^2 - 1 - \frac{1}{|\beta|^2} \right) \left( \dot{\alpha} \dot{\beta} - \alpha \dot{\beta} \right) \left( \bar{\alpha} \dot{\alpha} + \bar{\beta} \dot{\beta} \right)$$

$$+ \frac{1}{2 a^2 |\beta|^2} |\beta|^2 \bar{\alpha} \dot{\alpha} + \bar{\beta} \dot{\beta} \left( \dot{\alpha} \dot{\beta} - \alpha \dot{\beta} \right) + \frac{1}{2 |\beta|^2} \left( \bar{\alpha} \dot{\alpha} + \bar{\beta} \dot{\beta} \right)^2$$

$$+ \eta_{+3} \frac{1}{2 a^2 |\beta|^2} \left( 2 \operatorname{Im} \left( \bar{\beta} (a \bar{\beta} - \alpha) \left( \dot{\alpha} \dot{\beta} - \alpha \dot{\beta} \right) \right) - 2 a^2 |\beta|^2 \left( \bar{\alpha} \dot{\alpha} + \bar{\beta} \dot{\beta} \right) \right)$$

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Observe that the last term contains the Lagrange multiplier $\eta$ realizing the constraint

$$\Omega (g_+, g_-) = i\beta (az\bar{\beta} - \bar{a}) (\bar{a}\beta - \beta a) - 2ia^2 |\beta|^2 (\bar{a}\alpha + \beta \bar{\beta})$$

Despite its rather complicated expression, this Lagrange function can be written in the compact form

$$L_N (g_+, g_-) = \frac{1}{8}\kappa (g_+^{-1} \dot{g}_+, \mathbb{K} (g_+, g_-) g_+^{-1} \dot{g}_+) - \lambda \Omega (g_+, g_-)$$

where $\lambda$ is a redefinition of the Lagrange multiplier, and $\mathbb{K} (g_+, g_-) : \mathfrak{su}_2 \rightarrow \mathfrak{su}_2$ is a bijection that in the basis $\{T_1, T_2, T_3\}$ is represented by the symmetric matrix, a metric tensor,

$$\mathbb{K} (g_+ g_-) = \frac{1}{2|\beta|^2} \begin{pmatrix} 1 & 0 & m \\ 0 & 1 & n \\ m & n & 1 \end{pmatrix}$$

where

$$m = \frac{(a^2 - 1) (\beta \bar{a} + a \bar{\beta}) + a (z + \bar{z}) \beta \bar{\beta})}{2a^2 \beta \bar{\beta}}$$

$$n = \frac{i(1 - a^2)(\alpha \bar{\beta} - \beta \bar{a}) + a (z - \bar{z}) \beta \bar{\beta})}{2a^2 \beta \bar{\beta}}$$

It is worth to remark that $g_- = e$, it turns in

$$\mathbb{K} = \frac{1}{2|\beta|^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

thus recovering the Lagrangian function described in ref [7].

The constraint $\Omega$ can be written as

$$\Omega = \kappa (\mathcal{A} (g_+ g_-), g_+^{-1} \dot{g}_+)$$

for

$$\mathcal{A} (g_+, g_-) = \frac{1}{8} i (\beta (az\gamma - \bar{a}) + \gamma (a\bar{z}\beta - \alpha)) T_1 - \frac{1}{8} (\beta (az\gamma - \bar{a}) - \gamma (a\bar{z}\beta - \alpha)) T_2 + \frac{1}{4} a^2 \beta \gamma T_3$$

(49)

so that the velocities must be confined in the orthogonal complement of the vector $\mathcal{A} (gb)$. The Lagrangian function is now written as

$$L_{N(g_-, g_+)} (g_+, \dot{g}_+) = -\frac{1}{8}\kappa (g_+^{-1} \dot{g}_+, \mathbb{K} (g_+ g_-) g_+^{-1} \dot{g}_+) - \frac{1}{8}\lambda \kappa (\mathcal{A} (g_+ g_-), g_+^{-1} \dot{g}_+) g_+$$

where $\mathcal{A} (gb) \in \mathfrak{g}$ was given in (49). It is equivalent to

$$L_{N(g_-, g_+)} (g_+, \dot{g}_+) = -\frac{1}{8}\kappa (g_+^{-1} \dot{g}_+, \mathbb{K} (g_+ g_-) g_+^{-1} \dot{g}_+ + \lambda \mathcal{A} (g_+ g_-))$$

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6 Conclusions

Using the Dirac method we have studied a class of homogenous submanifolds in the cotangent bundle of a factorizable Lie group. They appear as generalizations of the cotangent bundles of the factors, in fact, they are retrieved when the fibers at \((e,0) \in G \times g^*\) are considered. These fibrations amounts to be symplectic: any horizontal curve is a symplectomorphism between fibers.

By using the Dirac brackets, we described the restriction of the Lie algebra of functions on \(G \times g^*\) to the submanifolds \(\mathcal{N}(g_-, \eta_-)\) and \(\mathcal{M}(g_+, \eta_+)\) in a natural fashion, showing how symmetries and dynamics projects out from the big phase space to the smaller ones. Dirac brackets of the momentum functions associated with the left translations in \(G \times g^*\) give rise to an action of \(G\) on \(\mathcal{N}(g_-, \eta_-)\) and \(\mathcal{M}(g_+, \eta_+)\) provided \(\eta_-\) and \(\eta_+\) are characters of the corresponding coadjoint actions of \(G_-\) and \(G_+\). These actions play a central role in the connection with integrable systems: collective dynamics built with these momentum functions on \(G \times g^*\), give rise to Dirac hamiltonian vector fields which are of the AKS type, being integrable in this sense. Thus, Dirac brackets nicely reduces a somehow trivial systems on \(G \times g^*\) into a systems with plenty of non trivial dynamics.

A deeper geometric insight, besides a description of AKS systems suitable for our purposes, allowed us to explain this (apparently) unexpected connection between Dirac constraints and integrable AKS system, as arising from the description of the latter as reduced space of \(T^*G\), and by the existence of immersions of the cotangent bundle of some homogeneous spaces of \(G\) into \(T^*G\).

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