WAVE PARTICLE DUALITY IN GENERAL RELATIVITY.

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Abstract: In this paper a one to one correspondence is established between space-time metrics of general relativity and the wave equations of quantum mechanics. This is done by first taking the square root of the metric associated with a space and from there, passing directly to a corresponding expression in the dual space. It is shown that in the case of a massless particle, Maxwell’s equation for a photon follows while in the case of a particle with mass, Dirac’s equation results as a first approximation. Moreover, this one to one correspondence suggests a natural explanation of wave-particle duality. As a consequence, the distinction between quantum mechanics and classical relativistic mechanics is more clearly understood and the key role of initial conditions is emphasized.

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I Introduction

In this paper a one to one correspondence is established between space-time metrics of general relativity and the wave equations of quantum mechanics. This is done by first taking the square root of the metric associated with a space. In this way, spin is introduced in a natural way into the space-time metric and would seem to be equivalent to the approach of Cordero, Tabensky and Teitelboim [1, 2, 3] in their formulation of a theory of supergravity. They introduce the notion of spin “into general relativity by taking the square root, a la Dirac, of the Hamiltonian constraints of the theory”[4]. The paper differs in that we take the square roots of metrics (not Hamiltonians) and focus primarily on the relationship between metrics and particle-wave equations. We also emphasize the probability aspects of the problem and the key role of initial conditions. As a consequence, the distinction between quantum mechanics and classical relativistic mechanics is more clearly understood.

Once the square root of the metric is taken, it is easy to pass directly to a corresponding expression in the dual space. This “corresponding expression” in the dual space will oftentimes be referred to as a “wave equation”. It is shown that in the case of a massless particle, the wave equation for a photon follows, while in the case of a particle with mass, Dirac’s equation results as a first approximation. Finally, we focus on the fact that the factoring technique which gives rise to the square root of the metric is not unique and allows us an alternative way of interpreting the negative energy levels associated with the solutions of the Dirac equation.

II Gravity and Spin

Goudsmit and Uhlenbeck were the first to introduce the notion of spin into quantum physics. Their work was eventually developed further by Pauli who was able to formulate a matrix representation of the spin operator. Later with the introduction
of the Dirac equation it was found that spin was a relativistic effect obtained by linearizing the Hamiltonian of special relativity. The corresponding spin matrices which resulted from this linearization also helped to explain the peculiar electric and magnetic moments associated with the motion of an electron in an electromagnetic field. However, in the light of the geometrical and isotropical nature of spin [5], it also suggests that spin is intrinsically linked to the geometrical properties of space-time and as a consequence gravity. This is even more evident when we linearize not the Hamiltonian of relativity theory as did Dirac, but rather the space-time metric itself. In fact, given the usual metric of Minkowski’s space

\[ ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2 \]  

(2-1)

where \( dx_0 = cdt \) and linearizing it in the same way as the Hamiltonian, we find that

\[ ds = \alpha_0 dx_0 + i\alpha_1 dx_1 + i\alpha_2 dx_2 + i\alpha_3 dx_3. \]  

(2-2)

On squaring this out and equating it to 1-1, gives for each \( \mu, \nu \in \{0, 1, 2, 3\} \)

\[ \alpha^2_\mu = 1 \quad \text{and} \quad \alpha_\mu \alpha_\nu + \alpha_\nu \alpha_\mu = 0. \]

These operators, therefore, are identical to the operators that are obtained from the Dirac equation and so spin can be defined in the usual way by putting

\[ \sigma_1 = -i\alpha_2 \alpha_3, \quad \sigma_2 = -i\alpha_3 \alpha_1 \quad \sigma_3 = -i\alpha_1 \alpha_2. \]

This relationship between the metric structure and the special relativistic Hamiltonian should come as no surprise. Both the metric and the Hamiltonian are covariant under the Lorentz transformation and as such reflect the same geometrical properties. However, if this is to be true in general, then we should expect the wave equations associated with a given space to reflect the underlying geometrical structure. In other words the space-time structure should be an effective cause [6] of the wave equation, in the sense that once the wave equation is given the metric can be immediately written down and once the metric is given the wave equation can be written down. Moreover, the resulting wave equation will be relativistically covariant by definition. The objective of the next section is to write down these globally covariant wave equations. The essential idea lies in the fact that if \( hdx \), where \( h \) is a constant associated with a curvilinear coordinate, lies in some dual vector space then this can be associated in a unique way with the differential operator \( \partial / h \partial x \) in the original space. For example, if the euclidean metric in curvilinear coordinates is of the form

\[ \tilde{ds} = h_1 dx_1 \tilde{e}_1 + h_2 dx_2 \tilde{e}_2 + h_3 dx_3 \tilde{e}_3 \]

where \( h_1, h_2, h_3 \) are the curvilinear coefficients, \( \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \) are unit vectors, then the corresponding “particle-wave” equation can be expressed as

\[ \nabla \psi = \frac{1}{h_1} \frac{\partial \psi}{\partial x_1} \tilde{e}_1 + \frac{1}{h_2} \frac{\partial \psi}{\partial x_2} \tilde{e}_2 + \alpha_3 \frac{1}{h_3} \frac{\partial \psi}{\partial x_3} \tilde{e}_3, \]

with the actual form of the \( \nabla \) operator being determined from the physics of the situation (see below). Note, also, that squaring out this latter equation gives the Laplace operator.

In the next section, we establish the formal relationship between the metric and the “wave” equation, for the general relativistic case. We then proceed in subsequent sections to explore this rapport for some concrete examples.
III From Particles to Waves

We shall take as our starting point the natural canonical correspondence that exists between the differential 1-forms of the type $dx$, used in defining the metric, and the covariant tensors over a manifold of the type $\frac{\partial}{\partial x}$. It is precisely this canonical 1-1 correspondence that allows us to define the wave equations we seek.

Once the wave equations are obtained we will then show that in the special relativistic case this gives rise to the wave equation for spin 0 massless particles and to the usual Dirac equation for neutrinos. In the case of particles with mass, however, both the Klein-Gordan and the general Dirac equation will be seen to be only first approximations of the more general wave equations.

Let $C$ be the set of complex numbers and $\mathcal{F}(M)$ be the set of twice differentiable functions over a manifold $M$. Define a vector field $v$ as a map $v : \mathcal{F}(M) \rightarrow \mathcal{F}(M) : f \mapsto vf.$

with properties

$\begin{align*}
    v(af + bg) &= avf + bv g \\
v(fg) &= f(vg) + g(vf)
\end{align*}$

where $f \in \mathcal{F}(M)$, $g \in \mathcal{F}(M)$ and $a$, $b \in C$. Denote the set of vector fields $\{v\}$ by $T^1_0(M)$. For the purpose of this paper we will work with a 4-dimensional pseudo-Riemannian manifold, with $x_0 = ct$ where $t$ is the local time and $c$ is the velocity of light. We will also use Einsteinian notation throughout for the indices; in other words, $a^\mu b_\mu = \sum_\mu a^\mu b_\mu$, $\mu \in \{0, 1, 2, 3\}$. This means that we can represent $v \in T^1_0(M)$ in terms of a local coordinate system as $v = v^\mu \frac{\partial}{\partial x^\mu}$ where $v^\mu \in \mathcal{F}(M)$.

We likewise, define the differential one forms as a map $\omega : T^1_0 \rightarrow \mathcal{F}(M) : v \mapsto \omega(v)$

with the properties

$\begin{align*}
    \omega(u + v) &= \omega(u) + \omega(v) \\
\omega(fu) &= f\omega(u) \quad f \in (M).
\end{align*}$

We denote the space of 1-forms by $T^0_1$. In terms of a local coordinate system, if $df \in T^0_1$ then we can write this as $df = f_\mu dx^\mu = \frac{\partial f}{\partial x^\mu} dx^\mu$.

Given a metric tensor we can define a 1-1 canonical correspondence between elements $T^1_0(M)$ and $T^0_1(M)$ by the map

$\wedge : T^1_0 \rightarrow T^0_1 : u \mapsto \hat{u}$

where $\hat{u}(v) = <u, v>$, $\forall v \in T^1_0(M)$. In particular, if $\hat{u} = \hat{u}_\mu dx^\mu$ then we can identify $\hat{u}$ and $u$ and write $u_\mu = g_{\mu\nu} u^\nu$. Similarly, if $\vee$ is the inverse of $\wedge$, i.e.

$\vee : T^0_1 \rightarrow T^1_0 : \omega \mapsto \check{\omega}$

where $<\check{\omega}, v> = \omega(v)$, $\forall v \in T^0_1(M)$. In particular, if $\check{\omega} = \check{\omega}_\mu \partial^\mu$ then we can identify $\check{\omega}$ and $\omega$ and write $\omega_\mu = g^{\mu\nu} \omega_\nu$.

By means of these canonical transformations we can now easily pass from particles to waves and vice-versa, or more precisely we can pass from metrics associated with the particles to the corresponding wave equation for the particle. In general, the form of the metric associated with a particle is

$$\begin{align*}
ds^2 &= g_{\mu\nu} dx^\mu dx^\nu.
\end{align*}$$
Since $g_{\mu\nu}$ is a symmetric matrix, its square root exists. The square root is symmetric but not unique and the significance of this non-uniqueness is discussed in the last section of this paper. For the moment we denote this square root matrix by $h_{\mu\nu}$ and note that $h_{\mu\nu} = h_{\nu\mu}$.

The linearized metric can now be written as a spinor: $ds = h_{\mu\nu} \alpha_\mu dx^\nu$, where $(\alpha^\mu)^2 = 1$ and $\alpha^\mu \alpha_\nu + \alpha_\nu \alpha^\mu = 0$. Note also that if $\gamma_\mu = h_{\mu\nu} \alpha_\nu$ then $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}$. It follows by the canonical correspondence that the associated particle wave equation will be given by:

$$\frac{\partial \psi}{\partial s} = h^{\mu\nu} \alpha_\mu \frac{\partial \psi}{\partial x^\nu}. \quad (3-4)$$

Note that $\alpha_\mu = \alpha^\mu$ for each $\mu$ and that $ds$ and hence $\frac{\partial}{\partial s}$ are covariant under coordinate transformations, according to the rules for a tetrad formalism.[7] Equation 3-4 will be seen as the most general form of a particle-wave equation. In effect, it describes the motion of a particle from a reference frame within the field of the particle. The wave function $\psi$ could represent a function describing the distribution of matter within a classical object, such as a star, as seen from the reference frame. Similarly, if the initial conditions are unknown, then $\psi$ could represent the probability distribution associated with the initial position of the object. This is the case with elementary particles where the initial positions, on account of the uncertainty principle, are in principle unknowable. Finally we note that if the rest mass $m$ of the particle is a constant, $h$ is Planck’s constant and $i = \sqrt{-1}$, then if we seek solutions of the form $\frac{\partial \psi}{\partial s} = k\psi$ where $k = \frac{imc}{h}$, equation 3-4 can be reduced to

$$mc^2 \psi = -i\frac{hc}{2\pi} \alpha_\nu h^{\mu\nu} \frac{\partial \psi}{\partial x^\nu}. \quad (3-5)$$

We will now use equation 3-4 (3-5) to investigate specific types of equations for specific types of metrics.

IV Photon and Neutrino Equations

The linearized metric for a massless particle is given by

$$0 = \alpha^0 c dt - \alpha^1 dx^1 - \alpha^2 dx^2 - \alpha^3 dx^3$$

from which it follows by the canonical correspondence established above that the associated wave equation for the particle is given by:

$$0 = \alpha_0 \frac{\partial \psi}{\partial t} - \alpha_1 \frac{\partial \psi}{\partial x^1} - \alpha_2 \frac{\partial \psi}{\partial x^2} - \alpha_3 \frac{\partial \psi}{\partial x^3}.$$  

This is the Dirac equation for a massless particle. Squaring this out we get Maxwells equation (or the Klein-Gordan equation for a massless particle) namely:

$$\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = \sum_{i=1}^3 \frac{\partial^2 \psi}{\partial x_i^2}.$$  

Note that, in this formulation, solutions of the massless Dirac equation are also solutions of the usual massless Klein-Gordan equation (Maxwell’s equation).

\[1\] It has been noted in a previous paper [8] that Fermi-Dirac statistics is a consequence of particle coupling and Bose-Einstein statistics is a consequence of decoupled particles. Moreover, it follows as a trivial consequence of that result that bosons cannot be second quantized as fermions and fermions cannot be second quantized as bosons. In other words, particles which at time $t$ are coupled with probability one cannot at the same time, $t$, be coupled with probability less than 1 (decoupled). They are either in one state or another. However, it is possible to make and break couplings. Transposed into the context of quantum field theory this means that the wave function of coupled particles will have the anti-commutator $=0$ (the singlet state being a case in point) while the wave function of the decoupled particles will have the commutator $=0$.[9]
Since the wave equation emerges from the structure of space-time itself, the question arises as to how to distinguish classical mechanics from quantum mechanics. We investigate this by analyzing the motion of a photon in a Minkowski space, subject to different sets of boundary conditions. In the first case we consider the motion of a photon moving on the x-axis with uniform velocity \( c \), but constrained by two mirrors placed at \( x = 0 \) and \( x = \xi \) to move uniformly on the interval \([0, \xi]\). We will assume that perfect reflection takes place at the mirrors and that no energy is exchanged. In this case, if the photon were a strictly classical particle with position \( x = 0 \) at \( t = 0 \) then its equation of motion would be of the form:

\[
x = \begin{cases} 
ct - 2n\xi, & \text{for } t \in \left[\frac{2n\xi}{c}, \frac{(2n+1)\xi}{c}\right] \\
2(n+1)\xi - ct & \text{for } t \in \left[\frac{(2n+1)\xi}{c}, \frac{(2n+2)\xi}{c}\right]
\end{cases}
\]

and its wave function \( \psi(x, t) \) would be of the form

\[
\psi(x, t) = \begin{cases} 
\delta[k(x - ct)] & \text{for } x - ct = -2n\xi \\
\delta[k(x + ct)] & \text{for } x + ct = 2(n+1)\xi \\
0 & \text{otherwise.}
\end{cases}
\]

The wave function in this case pinpoints the position of the particle with probability 1. Moreover, there is no restriction on the energy of the photon in this case. Theoretically, it may have values ranging from 0 to \( \infty \).

However, the classical particle is an idealized situation. In reality, the position of the photon constrained to move on the line is unknown and any attempt to know its exact position will be subject to Heisenberg’s uncertainty relations. In other words, its exact position cannot be known in principle, because any attempt to pinpoint it will scuttle the position and defeat the whole purpose of the experiment. The best we can do is to describe the position by means of a uniform probability density \( f(x - ct) = 1/\xi \) for \( x \in [0, \xi] \) which means \( \psi(x, t) = e^{\pm ik(x - ct)}/\sqrt{\xi} \). This does not mean that causality is violated nor that the particle does not have an exact position. It simply affirms that our initial conditions have to be defined statistically and as a consequence the future evolution of the system is best interpreted in a statistical way. Finally, note that in this model the energy of the particle can once again vary from 0 to \( \infty \) in a continuous manner.

Thirdly, the particle may be constrained to move in a potential well in such a way that the wave function is continuous (= 0) at the boundaries. In the case of the above problem, this would mean that the wave function would have stationary solutions of the form \( \psi(x) = c \sin\left(\frac{2n\pi}{\xi}x\right) \) where \( c \) is a constant and the photon energy would be quantized and of the form \( E = h\nu \).

The purpose of the above three examples is to highlight the importance of the boundary conditions when distinguishing between a classical type problem and a quantum mechanical problem, a point also stressed by Lindsey and Margenau [10]. Classical and quantum laws are not in opposition to each other. There is not one set of laws on the microscopic level and another on the macroscopic. On the contrary, classical and statistical methodologies are complimentary to each other and are in principle, applicable at all levels. However, on the microscopic level, statistical fluctuations will be more pronounced because of the uncertainty principle and in this case, the effects associated with quantum physics will become more apparent.

V Particle Wave Equations for Particles with Mass

We now turn our attention to particles with mass. Our approach will contrast with classical relativistic quantum theory in that we are no longer dealing with a particle existing within a given space-time manifold (for example an electron in a Minkowski space) rather our space-time structure is inherently related to the nature of the particle. The effect of mass on the wave equation and the boundary conditions is now a dominant factor. The purpose of the class of problems we will consider in this section is to understand how the nature of the particle’s wave function is influenced by the presence of mass and how this affects the behavior of the particle in a given potential field. We begin by considering the wave equation for a particle of mass \( m \) in a given potential field \( V(x) \) and show how the solutions of this equation are related to the particle’s energy and momentum. We then turn our attention to the boundary conditions and how they affect the particle’s wave function and the evolution of the system. Finally, we discuss the significance of the results obtained in this section and how they relate to the broader context of particle physics and quantum mechanics.
presence of the mass contained within it. In other words, the presence of the mass is an effective cause of the curvature of the space time [11] and not just an incidental presence within the space time manifold. In this regard, the Minkowski metric of special relativity

$$ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$$

cannot be an appropriate metric for any particle with mass. Moreover, the Dirac equation which comes from linearizing the Hamiltonian of special relativity, cannot be the proper wave equation for any elementary particle with mass. This follows from the fact that such a particle by definition curves space-time and hence cannot be imbedded in the flat space-time which underlies the Dirac equation. We begin our analysis with massive particles without charge.

To determine the wave equation of an neutral particle such as a neutron, it is assumed that a free neutron is spherical. Hence, the metric associated with it will be the usual Schwarzschild one:

$$ds^2 = \left(1 - \frac{2Gm}{c^2r}\right)dx_0^2 - \left(1 - \frac{2Gm}{c^2r}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Linearizing this we get

$$ds = \alpha_0 \left(1 - \frac{2Gm}{c^2r}\right)^{\frac{1}{2}}dx_0 + i\alpha_1 \left(1 - \frac{2Gm}{c^2r}\right)^{-\frac{1}{2}}dr + ir(\alpha_2 d\theta + \alpha_3 \sin\theta d\phi). \quad (5-1)$$

It now follows from the canonical correspondence discussed above that the generalized Dirac equation for the Schwarzschild metric is given by

$$\frac{\partial \psi}{\partial s} = \alpha_0 \left(1 - \frac{2Gm}{c^2r}\right)^{\frac{1}{2}}\frac{\partial \psi}{\partial x_0} - i\alpha_1 \left(1 - \frac{2Gm}{c^2r}\right)^{-\frac{1}{2}}\frac{\partial \psi}{\partial r} - i\frac{1}{r} \left(\alpha_2 \frac{\partial \psi}{\partial \theta} + \alpha_3 \frac{1}{\sin\theta} \frac{\partial \psi}{\partial \phi}\right). \quad (5-2)$$

Note that no distinction has been made between a general-relativistic-classical-type problem and a quantum-mechanical problem. For example, solutions to the above equation could be obtained by analyzing the reflection of a massless particle, confined to move on a straight line segment with endpoints \((r = r_0, \theta = 0, \phi = 0)\) and \((r = r_1, \theta = 0, \phi = 0)\), within the gravitational field of the particle, in a way analogous to the motion of a photon discussed in the previous section. However, we will not pursue this discussion here. Instead, we will focus on the conditions necessary to reduce the first approximation of equation 5-2 to the Dirac equation. This can be done by seeking eigenvalue solutions to the equation. We will refer to them as equilibrium solutions. For example, in the case of the photon problem discussed above, these solutions occurred when continuity of the wave function was required.

Returning to the problem of the Dirac equation, note that the set of equilibrium solutions \(\{k\}\) are given by the equation \(\frac{\partial \psi}{\partial s} = k\psi\). In other words \(k\) represents the set of eigenvalues of the system. If we now put \(k = -\frac{\hbar c}{2m}i_k\alpha_0\) and denote \(-i\alpha_0 \alpha_i\) by \(\alpha_i\) then for \(2m/r < 1\), we obtain as a first order approximation for equation 5-2:

$$\frac{i\hbar}{2\pi} \frac{\partial \psi}{\partial t} = -\frac{i\hbar c}{2\pi} \alpha_i \frac{\partial \psi}{\partial r} - i\frac{\hbar c}{2\pi} \frac{1}{r} \left(\alpha_i' \frac{\partial \psi}{\partial \theta} + \frac{\alpha_i'}{\sin\theta} \frac{\partial \psi}{\partial \phi}\right) + \alpha_0 mc^2 \psi.$$
\[ \frac{i\hbar}{2\pi} \frac{\partial \psi}{\partial t} = E\psi \] then Dirac’s equation follows. At this stage a couple of interesting observations arise based on the above analysis. It would appear that the Dirac equation is just an approximation of a 4-dimensional gradient times the \( \alpha \) matrix. From this perspective the Dirac equation could also be interpreted in a classical or non-quantum-mechanical way. For example, it could represent the movement of a large planetary size free mass with spin. How then do we distinguish quantum mechanics from a classical general relativistic theory? It is comparable to the theory of the photon already discussed above. The difference rests in assigning the probability interpretation to the wave function \[ 12 \], making full use of the uncertainty principle, and recognizing the fundamental role of Planck’s constant as a unit of measurement in physics. It is the choice of \( \hbar \) as a non-zero constant that causes the quantization procedure to come about. In the photon problem discussed earlier, the imposition of continuity on the wave function forced \( \hbar \) to be a non-zero constant.

A second question that arises is in the interpretation of equation 5-2 from the perspective of general relativity; to quote Marie Antoinette Tonnelat in this regard[13]: “in quantum mechanics, curved space remains a permissible framework; according to general relativity it becomes an effective cause”. I claim that the above approach to the wave equation resolves this in a natural way.

In the approach given in this paper general relativity is the effective cause of the form of the quantum mechanical wave equations. For example, consider as a frame of reference, a tetrad with origin at the particle’s center of mass. Let \( \psi(r) \) represent the wave function of a massless particle in the field of the massive particle. The corresponding wave equation can now be written as

\[
\alpha_0 \left( 1 - \frac{2Gm}{c^2 r} \right)^{-\frac{1}{2}} \frac{\partial \psi}{\partial x_0} - i\alpha_1 \left( 1 - \frac{2Gm}{c^2 r} \right)^{\frac{1}{2}} \frac{\partial \psi}{\partial r} - \frac{i}{r} \left( \alpha_2 \frac{\partial \psi}{\partial \theta} + \alpha_3 \sin \theta \frac{\partial \psi}{\partial \phi} \right) = 0. \tag{5-3}
\]

From this perspective the above equation can be taken as describing the motion of a massless fluid in a Schwarzschild space of the particle. Similarly, it may describe the motion of a probability density function for a massless particle within the same space. The distinction between the two cases depends on the boundary conditions being considered. As a final example, we choose an arbitrary point within the space-time containing the particle and let \( (t, r, \theta, \phi) \) be the position of the particle with respect to this arbitrary point. Denote its wave function by \( \psi(r, t) \), with probability density \( |\psi(r, t)|^2 \). The potential positions of the free particle are isotropic with respect to our chosen fixed point and as such are spherically symmetrical in a Schwarzschild space. The wave equation once again takes the form 5-2. If solutions of the form \( \frac{\partial \psi}{\partial r} = k\psi \) are sought and the limit as \( r \to \infty \) is taken as the special relativistic limit, then this will reduce to the Dirac equation if \( k = \pm 2\pi i\frac{mc}{\hbar} \). Hence, from this perspective we can take the equation:

\[
\pm \frac{imc}{\hbar} \psi = \alpha_0 \left( 1 - \frac{2Gm_p}{c^2 r} + \frac{Ge^2}{r^2} \right)^{-\frac{1}{2}} r \left( \alpha_2 \frac{\partial \psi}{\partial \theta} + \alpha_3 \sin \theta \frac{\partial \psi}{\partial \phi} \right) \tag{5-4}
\]

as describing the motion of a free electron in a “Schwarzschild space”.

**VI The Hydrogen Atom** We now apply the techniques of this paper to describe the hydrogen atom. More specifically, we describe the motion of the electron lying within the Reissner-Nordstrom metric [14] of the proton. Linearizing this metric gives:

\[
ds = \alpha_1 \left( 1 - \frac{2Gm_p}{c^2 r} + \frac{Ge^2}{r^2} \right)^{-\frac{1}{2}} dr + r \left( \alpha_2 d\theta + \alpha_3 \sin \theta d\phi \right) + \alpha_0 \left( 1 - \frac{2Gm_p}{c^2 r} + \frac{Ge^2}{r^2} \right)^{\frac{1}{2}} c dt.
\]
Denoting the rest masses of the electron and proton by \( m \) and \( m_p \) respectively, then the rest energy of the electron relative to the proton will be given by \( mc^2 - eA_0 \) where \( A_0 = \frac{e}{c} \) and \( r \) is the distance between the proton and electron. It follows from the usual 1-1 correspondence rule (c.f. equation 3-5) that the particle wave equation for this metric becomes, on multiplying across by \( \alpha_0^2 \):

\[
-\frac{2\pi i}{\hbar}(mc - eA_0)\psi(r, t) = \left\{ \alpha_0^2 \left( 1 - \frac{2Gm_p}{c^2r} + \frac{Ge^2}{c^4r^2} \right) \frac{\partial}{\partial r} + \frac{1}{r} \left( \alpha_2' \frac{\partial}{\partial \theta} + \alpha_3' \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right) \right. \\
\left. + \left( 1 - \frac{2Gm_p}{c^2r} + \frac{Ge^2}{c^4r^2} \right) \frac{\pi i}{c} \frac{\partial}{\partial t} \right\} \psi(r, t) \tag{6-1}
\]

where \( \alpha_i' = -i\alpha_0 \alpha_i \). Retaining only first order stationary terms gives:

\[
\left\{ \alpha_0 \frac{2\pi i}{\hbar c} (mc^2 - eA_0) + \alpha_1' \frac{\partial}{\partial r} + \frac{1}{r} \left( \alpha_2' \frac{\partial}{\partial \theta} + \alpha_3' \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right) \right\} \psi(r) = \frac{2\pi i}{\hbar c} E \psi(r) \tag{6-2}
\]

which is essentially the same as the equation of Dirac for the hydrogen atom. The only difference is the presence of the \( \alpha_0 eA_0 \) term, instead of \( eA_0 \). However, as will be pointed out in the next section, there is a lot of arbitrariness associated with the choice of the \( \alpha \) spinors. Secondly, if we “square” out equation 6-2 and denote \( \frac{\hbar}{2\pi i} \nabla \) by \( p \), we obtain the equation:

\[
(c^2p^2 + m^2c^4 + \frac{\hbar ec}{2\pi} \alpha' \mathcal{G})\psi = (E + \alpha_0 eA_0)^2 \psi \tag{6-3}
\]

The only difference between this and Dirac’s equation is that we have written \( \alpha_0 eA_0 \) instead of \( eA_0 \) on the right hand side of the equation while on the left hand side we have written \( \frac{\hbar ec}{2\pi} \alpha' \mathcal{G} \) instead of \( \frac{\hbar ec}{2\pi} \alpha' \mathcal{G} \). In other words, we have absorbed the imaginary component into the spin matrix. Moreover, this absorbing of the \( i \) into \( \alpha' \) is more than just a handy notation. Earlier, we defined \( \alpha_i' \) by \( \alpha_i' = -i\alpha_0 \alpha_i \) and pointed out that \( \{ \alpha_i' \} \) has all the same properties associated with the set \( \{ \alpha_i \} \). Moreover, if we had written the rest energy of the electron as \( mc^2 - \alpha_0 eA_0 \) instead of \( mc^2 - eA_0 \), then the regular Dirac equation would follow. It can be argued that both are valid: the regular Dirac case could represent coupled states of an electron-positron combination while the other case as given by equation 6-3 could represent the coupled states of a single electron. Finally, we point out that in both cases, the Schrödinger equation also follows as a first approximation.

**VII Negative Energy Levels Reinterpreted**

The negative energy solutions of the Dirac equation are usually interpreted in terms of virtual electrons occupying negative energy levels. This may be seen as a metaphor but it also prescinds from what is really taking place. The first thing to notice is that neither the representation of the square root matrix, \( h^{\mu \nu} \) nor the representation of \( \alpha_0, \alpha_1, \alpha_2, \alpha_3 \) are unique. We could equally work with any representation of the form \( \epsilon_0 \alpha_0, \epsilon_1 \alpha_1, \epsilon_2 \alpha_2, \epsilon_3 \alpha_3 \) where \( \epsilon_\mu \) can take on the values \( \pm 1 \). In particular, if a free particle with spin satisfies the Dirac equation in the form:

\[
\left\{ c \left[ \alpha_1 \frac{h}{2\pi i} \frac{\partial}{\partial x} + \alpha_2 \frac{h}{2\pi i} \frac{\partial}{\partial y} + \alpha_3 \frac{h}{2\pi i} \frac{\partial}{\partial z} \right] + \alpha_0 mc^2 \right\} \psi = E \psi,
\]

then the corresponding negative energy equation for a free particle with spin satisfies the Dirac equation in the form:

\[
\left\{ c \left[ \alpha_1 \frac{h}{2\pi i} \frac{\partial}{\partial x} + \alpha_2 \frac{h}{2\pi i} \frac{\partial}{\partial y} + \alpha_3 \frac{h}{2\pi i} \frac{\partial}{\partial z} \right] + \alpha_0 mc^2 \right\} \psi = -E \psi \tag{7-1}
\]
which is the same as

\[
\left\{ c \left[ -\alpha_1 \frac{h}{2\pi i} \frac{\partial}{\partial x} - \alpha_2 \frac{h}{2\pi i} \frac{\partial}{\partial y} - \alpha_3 \frac{h}{2\pi i} \frac{\partial}{\partial z} \right] - \alpha_0 mc^2 \right\} \psi = E\psi. \tag{7-2}
\]

Written in this last form we can see immediately that this too is the equation of a free particle of POSITIVE energy but of a spin state equal and opposite to the other particle. In particular, if the particles are coupled [15, 16] then they are in mutually opposite states with respect to the spin operator, in which case the Pauli exclusion principle applies. It follows that as long as the coupling lasts, the particles are not free to be in the same state. This coupling is implicit in the solutions of the equations. However, if the particles are not coupled then the particles can be in either one of the spin states and are free to switch from one to another in accordance with the usual probability laws but only after an interaction takes place. It follows that the Dirac equation gives the solution for a pair of coupled particles and implicitly contains a proof of the Pauli exclusion principle.

VIII Conclusion We have given a heuristic approach to unifying the theory of general relativity and quantum mechanics. As a consequence of this unification, we have highlighted the fact that the difference between classical mechanics and quantum mechanics rests primarily on the choice of initial conditions. It is the imposition of probability conditions on a sample space that leads to quantum mechanics. Classical mechanics on the other hand finds its intelligibility in its deterministic approach.

Finally, we note that to the extent that our approach is heuristic it remains relatively invariant. In other words, any modifications to the theory will be to the substance of either relativity theory or quantum mechanics. However, the general heuristic approach should remain unchanged.

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