BERNSTEIN-ZELEVINSKY DERIVATIVES: A HECKE ALGEBRA APPROACH

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Abstract. Let $G$ be a general linear group over a $p$-adic field. It is well known that Bernstein components of the category of smooth representations of $G$ are described by Hecke algebras arising from Bushnell-Kutzko types. We describe the Bernstein components of the Gelfand-Graev representation of $G$ by explicit Hecke algebra modules. This result is used to translate the theory of Bernstein-Zelevinsky derivatives in the language of representations of Hecke algebras, where we develop a comprehensive theory.

1. Introduction

Bernstein-Zelevinsky derivatives were first introduced and studied in [BZ] and [Ze] and are an important tool in the representation theory of general linear groups over $p$-adic fields. One goal of this paper is to formulate functors for Hecke algebras that correspond to Bernstein-Zelevinsky derivatives and show that Bernstein-Zelevinsky derivatives can be determined from the corresponding Hecke algebra functors. An advantage of our approach is that the some representations, such as generalized Speh modules, have explicit description in terms of the corresponding Hecke algebra modules, rather than just being defined as Langlands quotients. Thus, as an application of our study, we compute the Bernstein-Zelevinsky derivatives of generalized Speh modules, by a method which does not use the determinantal formula of Tadić [Ta] and Lapid-Mínguez [LM] or Kazhdan-Lusztig polynomials [Ze2] [CG].

1.1. Main results. Let $F$ be a $p$-adic field. Let $G$ be a general linear group over $F$. The category $\mathcal{R}(G)$ of smooth representations of $G$ can be described by Hecke algebras arising from Bushnell-Kutzko types [BK]. In order to keep notation simple, we shall only discuss the simple types. This restriction will present no loss of generality, as far as the theory of Bernstein-Zelevinsky derivatives is concerned. So let $G$ (or $G_n$ if we need to distinguish between the general linear groups of different rank) be the group $GL_{nr}(F)$ where $r$ is a fixed integer. The group $G$ contains a Levi group $L = GL_r(F)^n$. Let $\delta$ be a supercuspidal representation of $GL_r(F)$. Then $\tau = \delta \boxtimes \ldots \boxtimes \delta$ is a supercuspidal representation of $L$. The pair $s = [L, \tau]$ (or $s_n$) determines a Bernstein component $\mathcal{R}^s(G)$ of $\mathcal{R}(G)$.

A type is a representation $\rho$ of an open compact subgroup $K$ of $G$. If $\pi$ is a smooth representation of $G$, then $\pi_{\rho} = (\pi \otimes \rho^\vee)^K$ is naturally a module for $\mathcal{H}(G, \rho)$, the Hecke algebra of $\text{End}(\rho^\vee)$-valued functions on $G$. A type $\rho$ is a Bushnell-Kutzko type if $\pi \mapsto \pi_{\rho}$ is an equivalence of $\mathcal{R}^s(G)$ and the category of $\mathcal{H}(G, \rho)$-modules. For $s_n$, described above, such type $\rho_n$ is constructed in [BK] and in [Wa] in the tame case. Moreover, it is proved
that \( \mathcal{H}(G, \rho_n) \) is isomorphic to \( \mathcal{H}_n \), the Iwahori Hecke algebra of \( GL_n(F') \), where \( F' \) is an extension of \( F \) depending on \( \rho_n \). The Weyl group of \( GL_n(F') \) is isomorphic to the group of permutation matrices \( S_n \), and \( \mathcal{H}_n \) has a finite-dimensional subalgebra \( \mathcal{H}_{S_n} \) with a basis \( T_w \) of characteristic functions of double cosets of \( w \in S_n \). The algebra \( \mathcal{H}_{S_n} \) has a one dimensional representation \( \text{sgn} \), \( T_w \mapsto (-1)^{l(w)} \), where \( l \) is the length function on \( S_n \).

Let \( U \) be the unipotent subgroup of all strictly upper triangular matrices in \( G \). Let \( \psi \) be a Whittaker character of \( U \). One of the main results of this paper is a description of the Bernstein components of the Gelfand-Graev representation \( \text{ind}_U^G \psi \) in terms of the Hecke algebra action:

**Theorem 1.1.** (Theorem [3.4]) The \( \mathcal{H}_n \)-module \( \text{ind}_U^G \psi \) is isomorphic to \( \mathcal{H}_n \otimes_{\mathcal{H}_{S_n}} \text{sgn} \).

This theorem is proved in [CS] for the component consisting of representations generated by their Iwahori-fixed vectors, by an explicit computation. Here we give a more abstract proof using projectivity of \( \text{ind}_U^G \psi \) and that the Bernstein components of \( \text{ind}_U^G \psi \) are finitely generated, a result of Bushnell and Henniart [BH]. Our result is therefore a refinement of theirs for the general linear group. Projectivity of \( \text{ind}_U^G \psi \) was proved by Prasad in [P] by an argument very specific to general linear groups. In the appendix we prove projectivity of the Gelfand-Graev representation in a very general setting.

Theorem 1.1 plays an important role in the formulation of the Bernstein-Zelevinsky derivatives in the language of Hecke algebras. To that end, let

\[
S_n = \left( \sum_{w \in S_n} \frac{1}{q^{l(w)}} \right) \sum_{w \in S_n} (-1/q)^{l(w)} T_w \in \mathcal{H}_n.
\]

If \( \sigma \) is an \( \mathcal{H}_n \)-module, then \( S_n(\sigma) \) is the \( \text{sgn} \)-isotypic subspace of \( \sigma \). For every \( i = 1, \ldots, n-1 \) we have an embedding of the Hecke algebra \( \mathcal{H}_{n-i} \otimes \mathcal{H}_i \) into \( \mathcal{H}_n \). In particular, the map \( h \mapsto h \otimes 1 \) realizes \( \mathcal{H}_{n-i} \) as a subalgebra of \( \mathcal{H}_n \). Let \( S^n_i \) be the image in \( \mathcal{H}_n \) of \( 1 \otimes S_i \), where \( S_i \) is the sign projector in \( \mathcal{H}_i \). For every \( \mathcal{H}_n \)-module \( \sigma \),

\[
S_i(\sigma) := \frac{S^n_i(\sigma)}{S_i(\sigma)}
\]

is naturally an \( \mathcal{H}_{n-i} \)-module. This is the \( i \)-th derivative of \( \sigma \). Let \( \pi \) be a smooth representation of \( G_n \), and \( \pi^{(l)} \) its \( l \)-th Bernstein-Zelevinsky derivative. If \( \pi \) is in \( \mathcal{R}^n(G_n) \), then \( \pi^{(l)} = 0 \) unless \( l \) is a multiple of \( r \), and then \( \pi^{(ir)} \) is an object in \( \mathcal{R}^{n-i}(G_{n-i}) \).

**Theorem 1.2.** (Theorem [3.2]) Let \( \pi \) be an admissible representation of \( G_n \) in \( \mathcal{R}^n(G_n) \). Let \( \mathcal{BZ}_i \) be the functor defined in [11]. There is a functorial isomorphism of \( \mathcal{H}_{n-i} \)-modules

\[
(\pi^{(ir)})_{\mathcal{H}_{n-i}} \cong \mathcal{BZ}_i(\pi_{\mathcal{H}_i}).
\]

One can similarly formulate Bernstein-Zelevinsky derivatives for graded Hecke algebras. We check in Sections [5] and [6] that Bernstein-Zelevinsky derivatives of affine Hecke algebras and graded Hecke algebras agree under the Lusztig’s reductions. A reason for formulating the Bernstein-Zelevinsky derivatives for graded Hecke algebras is that one can apply representation theory of symmetric groups, in particular the Littlewood-Richardson rule,
to compute the Bernstein-Zelevinsky derivatives of generalized Speh representations, see Section 7 for details.

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2. Affine Hecke algebra

2.1. Let \( \mathcal{H}_n \) be the Iwahori-Hecke algebra of \( GL(n) \) over the \( p \)-adic field \( F' \). As an abstract algebra, \( \mathcal{H}_n \) is generated by elements \( T_1, \ldots, T_{n-1} \) and by the algebra \( \mathcal{A}_n = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) of Laurent polynomials. The algebra \( \mathcal{A}_n \) is isomorphic to the group algebra of the lattice \( \mathbb{Z}^n \), as follows. The group algebra is spanned by the elements \( \theta_x \) where \( x \in \mathbb{Z}^n \) with the multiplication \( \theta_x \cdot \theta_y = \theta_{x+y} \). We can identify the two algebras by \( \theta_x = x_1^{m_1} \cdots x_n^{m_n} \) where \( x = (m_1, \ldots, m_n) \in \mathbb{Z}^n \). We shall use both notations for elements in \( \mathcal{A}_n \) at our convenience. The elements \( T_j \) satisfy the quadratic relation \( (T_j + 1)(T_j - q) = 0 \) (and braid relations) and the relationship between \( T_j \) and \( f \in \mathcal{A}_n \) is given by

\[
T_j f - f^{s_j} T_j = (q - 1)x_j \frac{f - f^{s_j}}{x_j - x_{j+1}}
\]

where \( s_j \) is the permutation \( (j, j+1) \) and \( f^{s_j} \) is obtained from \( f \) by permuting \( x_j \) and \( x_{j+1} \).

The Weyl group of \( GL(n) \) is isomorphic to the group of permutations \( S_n \), and the center \( Z_n \) of \( \mathcal{H}_n \) is equal to the subalgebra of \( S_n \)-invariant Laurent polynomials in \( \mathcal{A}_n \). We shall use the fact that \( \mathcal{A}_n \) is a free \( \mathbb{Z}_n \)-module of rank \( |S_n| \). Let \( \mathcal{H}_{S_n} \) be the subalgebra of \( \mathcal{H}_n \) generated by the elements \( T_j, j = 1, \ldots, n-1 \). It is a finite algebra spanned by elements \( T_w, w \in S_n \), where \( T_w \) is a product of \( T_j \) as given by a shortest expression of \( w \) as a product of simple reflections. In particular, the dimension of \( \mathcal{H}_{S_n} \) is \( |S_n| \). We shall also use the fact that the multiplication in \( \mathcal{H}_n \) of elements in \( \mathcal{A}_n \) and \( \mathcal{H}_{S_n} \) gives isomorphisms

\[
\mathcal{H}_n \cong \mathcal{A}_n \otimes \mathcal{H}_{S_n} \cong \mathcal{H}_{S_n} \otimes \mathcal{A}_n.
\]

The algebra \( \mathcal{H}_{S_n} \) has two one-dimensional representations: the trivial, where \( T_j = q \) for all \( j \), and the sign representation, where \( T_j = -1 \) for all \( j \). A twisted Steinberg representation is a one-dimensional representation of \( \mathcal{H}_n \) such that its restriction to \( \mathcal{H}_{S_n} \) is the sign representation. This section is devoted to the proof of the following theorem.

**Theorem 2.1.** Let \( \Pi \) be an \( \mathcal{H}_n \)-module such that:

- \( \Pi \) is projective and finitely generated.
- \( \dim \text{Hom}_{\mathcal{H}_n}(\Pi, \pi) \leq 1 \) for an irreducible principal series representation \( \pi \).
- A twisted Steinberg representation is a quotient of \( \Pi \).

Then \( \Pi \) is isomorphic to \( \mathcal{H}_n \otimes_{\mathcal{H}_{S_n}} \text{sgn} \).

**Lemma 2.2.** Let \( P \) be a projective and finitely-generated \( \mathcal{H}_n \)-module. Then \( P \) is free and finitely generated as an \( \mathcal{A}_n \)-module.
Proof. Let \( \sigma \) be an \( A_n \)-module. Recall that we have a natural isomorphism
\[
\text{Hom}_{A_n}(P|_{A_n}, \sigma) \cong \text{Hom}_{H_n}(P, \text{Hom}_{A_n}(H_n, \sigma)).
\]
Since \( H_n \) is a free \( A_n \)-module, \( \text{Hom}_{A_n}(H_n, \sigma) \) is exact in \( \sigma \). Since \( P \) is a projective \( H_n \)-module, the above isomorphism implies that \( P \), viewed as an \( A_n \)-module, is also projective. It is clear that \( P \) is finitely-generated \( A_n \)-module, hence \( P \) is a free \( A_n \)-module by a version of the Quillen-Suslin theorem for rings of Laurent polynomials due to Swan \([Sw]\). □

Lemma, combined with the first assumption on \( \Pi \), implies that \( \Pi \cong A'_n \) as an \( A_n \)-module.

**Lemma 2.3.** Let \( \pi \) be an irreducible principal series module of \( H_n \) i.e. an irreducible representation whose dimension is equal to the order of \( S_n \). Let \( J \) be the annihilator of \( \pi \) in the center \( Z_n \) of \( H_n \). Let
\[
0 \to \pi'' \to \pi' \xrightarrow{g} \pi \to 0
\]
be a non-split exact sequence \( H_n \)-modules. Then \( J\pi' \neq 0 \).

Proof. We abbreviate \( P_{\text{sgn}} = H_n \otimes_{H_{S_n}} \text{sgn} \). By the projectivity of \( P_{\text{sgn}} \) we have the following maps
\[
P_{\text{sgn}} \xrightarrow{f} \pi' \xrightarrow{g} \pi \to 0
\]
such that the composition \( g \circ f \) is non-trivial. Since \( J\pi = 0 \), the composition \( g \circ f \) descends to a map from \( P_{\text{sgn}}/JP_{\text{sgn}} \) to \( \pi \). Since
\[
\dim \mathbb{C}(P_{\text{sgn}}/JP_{\text{sgn}}) = \dim \mathbb{C}(A_n/JA_n) = |S_n| = \dim \mathbb{C}(\pi)
\]
the composition \( g \circ f \) descends to an isomorphism \( P_{\text{sgn}}/JP_{\text{sgn}} \cong \pi \). If \( J\pi' = 0 \) then \( f \) descends to a map from \( P_{\text{sgn}}/JP_{\text{sgn}} \cong \pi \) to \( \pi' \), contradicting the assumption on the exact sequence. □

Let \( J \) and \( \pi \) be as in the lemma. Then \( \Pi/J\Pi \) has a composition series such that any irreducible subquotient is isomorphic to \( \pi \). Since \( \Pi/J\Pi \) is annihilated by \( J \), by an easy application of Lemma 2.3, it is a direct sum of \( r \) copies of \( \pi \). Hence \( r = 1 \), by the second assumption on \( \Pi \).

**Lemma 2.4.** Let \( P \) be an \( H_n \)-module isomorphic to \( A_n \), as an \( A_n \)-module. Then \( P \) is isomorphic to \( H_n \otimes_{H_{S_n}} \text{sgn} \) or \( H_n \otimes_{H_{S_n}} 1 \).

A proof of this lemma is in the next section. Lemma, combined with the third assumption on \( \Pi \), implies that \( \Pi \) is isomorphic to \( H_n \otimes_{H_{S_n}} \text{sgn} \). This completes the proof of Theorem 2.1.

**Remark 2.5.** The authors would like to thank a referee for pointing out that the structure of finitely generated projective modules can be understood from \( K \)-theory of affine Hecke algebras \([So2, \text{Section 5.1}] \). Some explicit \( K \)-theoretic computations can be found in \([So, \text{Chapter 6}] \).
2.2. $\mathcal{H}_n$-module structure on $A_n$. The main goal of this section is to prove Lemma 2.4. This will be accomplished by an explicit calculation for $H_2$, from which we shall derive the general case. We work in a more general setting, and replace $A_2$ with $A[x_1^{\pm1}, x_2^{\pm1}]$ where $A$ is a $\mathbb{C}$-algebra. So assume we have an $\mathcal{H}_2$-structure on $A[x_1^{\pm1}, x_2^{\pm1}]$. In particular, if $g(x_1, x_2) \in A[x_1^{\pm1}, x_2^{\pm1}]$ is invertible, then

$$T_1(g(x_1, x_2)) = f(x_1, x_2)g(x_1, x_2)$$

for some $f(x_1, x_2) \in A[x_1^{\pm1}, x_2^{\pm1}]$, depending on $g(x_1, x_2)$. Using the relation (2.2), the relation $T_1^2 = (q-1)T_1 + q$ implies that $f(x_1, x_2)$ satisfies the following polynomial equation:

$$f(x_1, x_2)f(x_2, x_1) = (q-1)(\frac{x_1f(x_2, x_1) - x_2f(x_1, x_2)}{x_1 - x_2}) + q.$$ 

So our task is to solve this polynomial equation. To that end, we abbreviate

$$\tilde{f}(x_1, x_2) = \frac{x_1f(x_2, x_1) - x_2f(x_1, x_2)}{x_1 - x_2},$$

and derive some explicit formulae for $\tilde{f}$. Assume that $f(x_1, x_2) = x_1^nx_2^m$. If $m \geq n$ then

$$\tilde{f}(x_1, x_2) = x_1^nx_2^m + x_1^{n-1}x_2^{m+1} + \ldots + x_1^nx_2^m.$$

If $m < n$ then

$$\tilde{f}(x_1, x_2) = -x_1^{n-1}x_2^{m+1} + \ldots - x_1^{m+1}x_2^{n-1}.$$

Write $f(x_1, x_2) = \sum a_{n,m}x_1^nx_2^m$ and define

$$\maxdeg(f(x_1, x_2)) = \max \{n + m \in \mathbb{Z} : a_{n,m} \neq 0 \},$$

$$\mindeg(f(x_1, x_2)) = \min \{n + m \in \mathbb{Z} : a_{n,m} \neq 0 \}.$$

**Lemma 2.6.** Assume that $f(x_1, x_2) \in A[x_1^{\pm1}, x_2^{\pm1}]$ is a solution of the equation

$$f(x_2, x_1)f(x_1, x_2) = (q-1)f(x_1, x_2) + q.$$ 

Then $\maxdeg(f(x_1, x_2)) = \mindeg(f(x_1, x_2)) = 0$.

**Proof.** Let $f(x_1, x_2) \in A[x_1^{\pm1}, x_2^{\pm1}]$. If $\maxdeg(f(x_1, x_2)) = \mindeg(f(x_1, x_2)) = 0$ fails then $\maxdeg(f(x_1, x_2)) > 0$ or $\mindeg(f(x_1, x_2)) < 0$. Assume $\maxdeg(f(x_1, x_2)) > 0$. Then

$$\maxdeg(f(x_1, x_2)f(x_2, x_1)) > \maxdeg(f(x_1, x_2)) \geq \maxdeg((q-1)f(x_1, x_2) + q))$$

so $f(x_1, x_2)$ is not a solution. The case $\mindeg(f(x_1, x_2)) < 0$ is dealt with similarly. $\square$

Lemma implies that a solution of the polynomial equation is a Laurent polynomial $f(x)$ where $x = x_2/x_1$. We abbreviate

$$\tilde{f}(x) = \frac{x^{-1/2}f(x^{-1}) - x^{1/2}f(x)}{x^{-1/2} - x^{1/2}}.$$
Lemma 2.7. Let \( f(x) \in A[x^{\pm 1}] \) be a solution of
\[
f(x)f(x^{-1}) = (q-1)(\tilde{f}(x)) + q.
\]
Then there exists an integer \( m \) and \( \lambda = -1 \) or \( q \) such that \( f(x) = f_m^\lambda(x) \) where, if \( m \geq 0 \),
\[
f_m^\lambda = (q-1)(1 + x + \ldots + x^{m-1}) + \lambda x^m
\]
and, if \( m < 0 \),
\[
f_m^\lambda = -(q-1)(x^{-1} + \ldots + x^{m+1}) - \lambda x^m.
\]
Proof. It is trivial to check that a solution \( f(x) \) cannot have at the same time negative and positive powers of \( x \). So assume firstly that \( f(x) = a_m x^m + a_{m-1} x^{m-1} + \ldots + a_0 \), where \( m \geq 0 \) and \( a_m \neq 0 \). Since
\[
\tilde{f}(x) = a_m x^m + (a_m + a_{m-1}) x^{m-1} + \ldots + (a_m + a_{m-1} + \ldots + a_0) + \ldots + a_m x^{-m},
\]
equating coefficients of the two sides yields the following sequence of equations:
\[
a_m a_0 = (q-1)a_m
\]
\[
a_m a_1 + a_{m-1} a_0 = (q-1)(a_m + a_{m-1})
\]
etc and the last
\[
a_m^2 + a_{m-1}^2 + \ldots + a_0^2 = (q-1)(a_m + a_{m-1} + \ldots + a_0) + q.
\]
Since \( a_m \neq 0 \) the first equation implies \( a_0 = q - 1 \). Then the second implies that \( a_1 = q - 1 \) etc. Finally, the last implies that \( a_m^2 = (q-1)a_m + q \), and this has two solutions, \( -1 \) and \( q \). Now assume that \( f(x) = -a_0 - a_1 x^{-1} - \ldots - a_m x^m \), for \( m < 0 \) and \( a_m \neq 0 \). Then
\[
\tilde{f}(x) = a_m x^{m+1} + (a_m + a_{m+1}) x^{m+2} + \ldots + (a_m + a_{m+1} + \ldots + a_1) + \ldots + a_m x^{-m}.
\]
In particular, we do not have the \( x^m \) term. A comparison with the left hand side implies that \( a_0 = 0 \). The rest of the proof proceeds along the same lines as in the first case, giving \( a_{-1} = q - 1 \), \( a_{-2} = q - 1 \) ... and \( a_m \) a solution of \( a_m^2 = (q-1)a_m + q \).

Corollary 2.8. Assume we have an \( \mathcal{H}_2 \)-module structure on \( A[x_1^{\pm 1}, x_2^{\pm 1}] \). Then for every invertible \( g(x_1, x_2) \in A[x_1^{\pm 1}, x_2^{\pm 1}] \) there exists an integer \( m \) such that \( g(x_1, x_2)x_2^m \) is an eigenvector of \( T_1 \).

Proof. Two lemmas imply that \( T_1(g(x_1, x_2)) = f_m^\lambda(x_2/x_1) g(x_1, x_2) \). Now one checks that \( g(x_1, x_2)x_2^m \) is an eigenvector.

In view of the tensor product decomposition \( \mathcal{H}_n \cong A_\ast \otimes C \mathcal{H}_{S_n} \), the following corollary completes the proof of Lemma 2.4.

Corollary 2.9. Assume we have an \( \mathcal{H}_n \)-module structure on \( A_n = C[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \). Then there exists an invertible element in \( A_n \) that is an eigenvector for \( T_1, \ldots, T_{n-1} \).

Proof. We apply Corollary 2.8 to \( A[x_1^{\pm 1}, x_2^{\pm 1}] \) and \( g(x_1, x_2) = 1 \), where \( A = C[x_i^{\pm 1}] \) for \( i \neq 1, 2 \). Thus there is an integer \( m_2 \) such that \( x_2^{m_2} \) is an eigenvector of \( T_1 \). Next, we apply Corollary 2.8 to \( A[x_2^{\pm 1}, x_3^{\pm 1}] \) and \( g(x_1, x_2) = x_2^{m_2} \), where \( A = C[x_i^{\pm 1}] \) for \( i \neq 2, 3 \). Hence there exists an integer \( m_3 \) such that \( x_2^{m_2} x_3^{m_3} \) is an eigenvector of \( T_2 \). Since \( T_1 \) and
$x_3$ commute, $x_2^{m_2}x_3^{m_3}$ is still an eigenvector of $T_1$. Continuing in this fashion, we arrive to a monomial in $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ that is a joint eigenvector for all $T_j$. \hfill \Box

3. Gelfand-Graev representation

3.1. Hecke algebras. Let $G$ be a $p$-adic reductive group. Let $K$ be an open compact subgroup of $G$ and $(\rho, E)$ a smooth, finite-dimensional, representation of $K$. Let $\mathcal{H}(G, \rho)$ be the algebra of compactly supported $\text{End}(E^\vee)$-valued functions on $G$ such that $f(kgk') = \rho^\vee(k)f(g)\rho^\vee(k')$ for $k, k' \in K$. Let $\mathcal{S}(G)$ be the space of locally constant, compactly supported functions on $G$, and let $e_\rho \in \mathcal{S}(G)$ be defined by

$$e_\rho(x) = \frac{\dim(\rho)}{\text{vol}(K)} \int_E (x^{-1})$$

if $x \in K$ and 0 otherwise. Then $e_\rho \ast e_\rho = e_\rho$. Let $\mathcal{H}_\rho = e_\rho \ast \mathcal{S}(G) \ast e_\rho$. The two algebras are related by a canonical isomorphism $\mathcal{H}_\rho \cong \mathcal{H}(G, \rho) \otimes \text{End}(E)$, see [BK2]. If $(\pi, V)$ is a smooth representation of $G$, let

$$V_\rho = \text{Hom}_K(E, V) \cong (E^\vee \otimes V)^K.$$ 

Note that $f \in \mathcal{H}(G, \rho)$ naturally acts on $e_\rho \otimes v \in E^\vee \otimes V$ by the formula

$$\pi_\rho(f)(e_\rho \otimes v) = \int_G f(g)(e_\rho) \otimes \pi(g)(v) \, dg.$$ 

This action preserves the subspace $(E^\vee \otimes V)^K$, and defines a structure of $\mathcal{H}(G, \rho)$-module on $V_\rho$. On the other hand, $\pi(e_\rho) \cdot V \cong V_\rho \otimes E$ is naturally a $\mathcal{H}_\rho$-module. The two structures are compatible with respect to the isomorphism $\mathcal{H}_\rho \cong \mathcal{H}(G, \rho) \otimes \text{End}(E)$.

Lemma 3.1. Assume that a smooth representation $(\pi, V)$ of $G$ is finitely generated. Then $V_\rho$ is finitely generated $\mathcal{H}(G, \rho)$-module.

Proof. It suffices to show that $\pi(e_\rho) \cdot V$ is finitely generated $\mathcal{H}_\rho$-module. Let $V_0$ be a finite-dimensional subspace generating $V$. Let $J$ be an open compact subgroup of $K$ such that $V_0 \subseteq V^J$. Then $V = \pi(S(G/J)) \cdot V_0$. Assume, in addition, that $J$ is contained in the kernel of $\rho$, so $e_\rho \in S(J \backslash G/J)$. Then

$$\pi(e_\rho) \cdot V = \pi(e_\rho \ast S(J \backslash G/J)) \cdot V_0.$$ 

It is known that $S(J \backslash G/J)$ is finite over its center $Z_J$. Hence $\pi(e_\rho) \cdot V$ is finite over $e_\rho \ast Z_J \subseteq \mathcal{H}_\rho$. \hfill \Box

Let $f \in \mathcal{H}(G, \rho)$. Then $f(g) \in \text{End}(E)$. Let $\tilde{f}(g)$ be the image of $f(g)$ under the composite of the following isomorphisms:

$$\text{End}(E) \cong E \otimes E^\vee \cong \text{End}(E^\vee).$$
Let \( f^*(g) = \tilde{f}(g^{-1}) \). Then the map \( f \mapsto f^* \) is anti-isomorphism of \( \mathcal{H}(G, \rho) \) and \( \mathcal{H}(G, \rho^\vee) \).

Let \( (\pi^\vee, V^\vee) \) be the smooth dual of \( (\pi, V) \). Then \( V^\vee_{\rho^\vee} \) is an \( \mathcal{H}(G, \rho^\vee) \)-module. We have a natural isomorphism

\[
(V^\vee_{\rho^\vee})^* = ((E^\vee \otimes V)^K)^* \cong (E \otimes V^\vee)^K = V^\vee_{\rho^\vee}
\]

of vector spaces where \( (V^\vee_{\rho^\vee})^* \) is the linear dual of \( V^\vee_{\rho^\vee} \). On \( (V^\vee_{\rho^\vee})^* \) we have an anti-action \( \pi^\vee_{\rho^\vee} \) of \( \mathcal{H}(G, \rho) \). Via the isomorphism \( (V^\vee_{\rho^\vee})^* \cong V^\vee_{\rho^\vee} \) the two actions are related by the formula

\[
\pi^\vee_{\rho^\vee}(f) = \pi_{\rho^\vee}(f^*).
\]

### 3.2. Bernstein’s decomposition

Let \( \mathcal{R}(G) \) be the category of smooth representations of \( G \). We recall some notions and properties of Bernstein decomposition, and the Bushnell-Kutzko theory of types \([BK, BK2]\), mainly for the case of general linear groups.

Let \( \mathfrak{B}(G) \) be the set of \( G \)-inertial equivalence classes. For each \( s \in \mathfrak{B}(G) \), let \( \mathcal{R}^s(G) \) be the Bernstein component associated to \( s \). More precisely, an inertial equivalence class \( s \) consists of pairs \((L, \tau)\), where \( L \) is a Levi subgroup of \( G \) and \( \tau \) is a supercuspidal representation, and \( \mathcal{R}^s(G) \) is the full subcategory of \( \mathcal{R}(G) \) whose objects have the property that every irreducible subquotient appears as a composition factor of \( \text{Ind}_P^G(\tau \otimes \chi) \) for some unramified character \( \chi \) of \( L \) and \( P \) is a parabolic subgroup with the Levi part \( L \). Two pairs \((L_1, \tau_1)\) and \((L_2, \tau_2)\) are in the same equivalence class if and only if they determine the same subcategories in \( \mathcal{R}(G) \). The Bernstein decomposition asserts that there is an equivalence of categories:

\[
\mathcal{R}(G) \cong \prod_{s \in \mathfrak{B}(G)} \mathcal{R}^s(G).
\]

**Definition 3.2.** Fix an inertial equivalence class \( s \). Let \( K \) be an open compact subgroup of \( G \). Let \( \rho \) be a smooth finite-dimensional representation of \( K \). Then \( \rho \) is called an \( s \)-type if \( V \mapsto V_\rho \) is an equivalence of the category \( \mathcal{R}^s(G) \) and the category of \( \mathcal{H}(G, \rho) \)-modules.

We now look at the special case when \( G = GL_{nr}(F) \) and an inertial equivalence class \( s_n \) is given by

\[
L = GL_r(F) \times \ldots \times GL_{\nu}(F)
\]

and

\[
\tau = \delta \boxtimes \ldots \boxtimes \delta,
\]

where \( \delta \) is a supercuspidal representation of \( GL_r(F) \) and the number of factors is \( n \). Let \( P \) be the parabolic subgroup of \( GL_{nr}(F) \), with the Levi \( L \), consisting of block upper-triangular matrices. Let \( \text{St}_n(\delta) \) be the unique irreducible quotient of

\[
\text{Ind}_{P}^{G}(\nu^{\frac{1}{p-1}} \delta \boxtimes \nu^{\frac{1}{p-1}} \delta \boxtimes \ldots \boxtimes \nu^{\frac{1}{p-1}} \delta)
\]

as in \([BZ, \text{Sec. 9.1}]\). Then \( \text{St}_n(\delta) \) is an essentially square integrable representation, also known as the generalized Steinberg representation. We have the following result due to Bushnell and Kutzko (and Waldspurger \([Wa]\) in the tame case):
Theorem 3.3. Let $\mathfrak{s}_n$ be the inertial class of $G = GL_{nr}(F)$ as above. Then there exists an $\mathfrak{s}_n$-type $\rho_n$ and an isomorphism $\mathcal{H}(G, \rho_n) \cong \mathcal{H}_n$, where $\mathcal{H}_n$ is defined in Section 2 with $q$ equal to a power of the order of the residual field of $F$. Moreover, under the isomorphism $\mathcal{H}(G, \rho_n) \cong \mathcal{H}_n$, the generalized Steinberg representation $\text{St}_n(\delta)_{\rho_n}$ corresponds to the Steinberg module of $\mathcal{H}_n$.

Let $U$ be the unipotent group of upper-triangular matrices in $G$. Let $\psi : U \to \mathbb{C}^\times$ be a Whittaker functional. The Gelfand-Graev representation is the induced representation $\text{ind}_G^U(\psi)$, consisting of functions on $G$ with compact support modulo $U$.

Theorem 3.4. Let $G = GL_{nr}(F)$ and let $\rho_n$ be the $\mathfrak{s}_n$-type as in Theorem 3.3. Then

$$(\text{ind}_G^U(\psi))_{\rho_n} \cong \mathcal{H}_n \otimes \mathcal{S}_n \text{sgn}$$

as $\mathcal{H}(G, \rho_n) \cong \mathcal{H}_n$-modules.

Proof. We need to show that the conditions of Theorem 2.1 are satisfied. By a result of Bushnell and Henniart [BH], every Bernstein component of the Gelfand-Graev representation is finitely generated. Thus $\text{ind}_G^U(\psi)_{\rho_n}$ is finitely generated $\mathcal{H}(G, \rho_n)$-module by Lemma 3.1. Moreover, the Gelfand-Graev representation is projective by Corollary 8.6. Thus the first bullet in Theorem 2.1 holds. The second bullet holds since any Whittaker generic representation appears as a quotient, with multiplicity one, of the Gelfand-Graev representation. Finally, $\text{St}_n(\delta)_{\rho_n}$ is an essentially discrete series representation and therefore Whittaker generic. Hence the third bullet holds.

In addition to the isomorphism $\epsilon : \mathcal{H}_n \to \mathcal{H}(G, \rho_n)$, there is also an isomorphism $\epsilon^\vee : \mathcal{H}_n \to \mathcal{H}(G, \rho_n^\vee)$. Since $(\epsilon(T_j))^*$ is supported on the same double coset as $\epsilon^\vee(T_j)$, and satisfies the same quadratic equation, the two elements must be the same. Hence the following diagram commutes, here the left vertical arrow is the anti-involution of $\mathcal{S}_n$ defined by $T_j^* = T_j$ for all $j = 1, \ldots, n - 1$.

If $(\pi, V)$ is a smooth representation of $G$, let $V_{U, \psi}$ be the maximal quotient of $V$ such that $U$ acts on it by $\psi$. Recall that

$$\mathcal{S}_n = (\sum_{w \in \mathcal{S}_n} (1/q)^{l(w)})^{-1} \sum_{w \in \mathcal{S}_n} (-1/q)^{l(w)} T_w,$$

where $l$ is the length function on $\mathcal{S}_n$, is the sign projector.

Theorem 3.5. Let $G = GL_{nr}(F)$ and let $\rho_n$ be the $\mathfrak{s}_n$-type as in Theorem 3.3. Let $(\pi, V)$ be an admissible representation of $G$ in the component $\mathfrak{R}^\sharp(G)$. Then there exists a functorial isomorphism of vector spaces $\phi_V : \mathcal{S}_n(V_{\rho_n}) \to V_{U, \phi}$.

Proof. We need the following:
Lemma 3.6. For every smooth representation \( V \) in the component \( \mathfrak{H}^s(G) \) and every finite dimensional complex vector space \( X \), there is an isomorphism, functorial in \( V \) and \( X \),

\[
\Phi_X : \text{Hom}_\mathbb{C}(V_U, X) \to \text{Hom}_\mathbb{C}(\mathfrak{S}_n(V_{\rho_n}), X).
\]

Proof. We start by observing some facts that will be needed in the proof. Let \( Y \) and \( Z \) be two complex vector spaces, and \( Y^* \) and \( Z^* \) their linear duals. Then

\[
\text{Hom}_\mathbb{C}(Y, Z^*) \cong \text{Hom}_\mathbb{C}(Z, Y^*).
\]

If \( Y \) and \( Z \) are \( \mathcal{H}_{S_n} \)-modules, then \( Y^* \) and \( Z^* \) are \( \mathcal{H}_{S_n} \)-modules, where the action is the natural anti-action, precomposed with the anti-involution \( T_j^* = T_j \) for all \( j = 1, \ldots, n-1 \). Then

\[
\text{Hom}_{\mathcal{H}_{S_n}}(Y, Z^*) \cong \text{Hom}_{\mathcal{H}_{S_n}}(Z, Y^*).
\]

If \( Y \) and \( Z \) are smooth representations of \( G \), let \( Y^\vee \) and \( Z^\vee \) be the smooth duals of \( Y \) and \( Z \). Then

\[
\text{Hom}_G(Y, Z^\vee) \cong \text{Hom}_G(Z, Y^\vee).
\]

For every finite dimensional vector space \( X \) we have the following sequence of isomorphisms:

\[
\text{Hom}_\mathbb{C}(V_U, X) \cong \text{Hom}_\mathbb{C}(V, \text{Ind}_U^G(X \boxtimes \bar{\psi})) \quad \text{(by Frobenius reciprocity)}
\]

\[
\cong \text{Hom}_\mathbb{C}(\text{ind}_U^G(X^* \boxtimes \bar{\psi}), V^\vee) \quad \text{(since (ind}_U^G(X^* \boxtimes \bar{\psi})^\vee \cong \text{Ind}_U^G(X \boxtimes \bar{\psi}))}
\]

\[
\cong \text{Hom}_{\mathcal{H}_n}(\mathcal{H}_n \otimes_{\mathcal{H}_{S_n}} (X^* \boxtimes \text{sgn}), V_{\rho_n}^\vee) \quad \text{(by Theorem 3.4 for } \rho_n^\vee)
\]

\[
\cong \text{Hom}_{\mathcal{H}_{S_n}}(X^* \boxtimes \text{sgn}, V_{\rho_n}^\vee) \quad \text{(by Frobenius reciprocity)}
\]

\[
\cong \text{Hom}_{\mathcal{H}_{S_n}}(V_{\rho_n}, X \boxtimes \text{sgn}) \quad \text{(since } V_{\rho_n}^\vee \cong (V_{\rho_n})^*)
\]

\[
\cong \text{Hom}_\mathbb{C}(\mathfrak{S}_n(V_{\rho_n}), X).
\]

The map \( \Phi_X \) is the composite of the sequence of isomorphisms. \( \square \)

Now assume that \( V \) is admissible. Then \( V_U, \bar{\psi} \) and \( \mathfrak{S}_n(V_{\rho_n}) \) are finite-dimensional. By the Yoneda Lemma, Lemma 3.6 implies Theorem 3.5. \( \square \)

Let \( I \) be an Iwahori subgroup of \( G \). In the case when \( (\pi, V) \) belongs to the Bernstein component of representations generated by their \( I \)-fixed vectors Theorem 3.5 holds for all smooth representations, that is, without the admissibility assumption. This is Corollary 4.5 in [CS] which is proved using an explicit version of Theorem 3.4 available in the Iwahori case. In this case \( \mathfrak{S}_n(V_{\rho_n}) \) is simply \( \mathfrak{S}_n(V^I) \). The inclusion of \( \mathfrak{S}_n(V^I) \) into \( V \) followed with the projection on \( V_U, \bar{\psi} \) gives the map \( \phi_V \).

4. Bernstein-Zelevinsky derivatives

In this section we shall change notation slightly and write \( G_n = GL_{nr}(F) \). We shall also use \( \pi \) to denote the space of a smooth representation of \( G_n \). As previously, \( \rho_n \) is an \( s_n \)-type.
4.1. Jacquet functor. Let $P = MN$ be the minimal parabolic subgroup of $G_n$ of block-upper triangular matrices, with the Levi $M = G_{n-i} \times G_i$, and the unipotent radical $N$. The restriction of the $K$-type $\rho_n$ to $K_M = K \cap M$ is irreducible and isomorphic to $\rho_{n-i} \boxtimes \rho_i$.

We have the following commutative diagram, a consequence of Theorem (7.6.20) in [BK]:

$$
\begin{array}{ccc}
\mathcal{H}_{n-i} \otimes \mathcal{H}_i & \overset{\cong}{\longrightarrow} & \mathcal{H}(M, \rho_{n-i} \boxtimes \rho_i)
\\[5pt]
\downarrow m & & \downarrow m
\\[5pt]
\mathcal{H}_n & \cong & \mathcal{H}(G, \rho_n)
\end{array}
$$

where the vertical maps are injections. The left vertical map $m$ is explicitly described as follows: $m(T_j \otimes 1) \mapsto T_j$ and $m(x_j \otimes 1) \mapsto x_j$, for $j = 1, \ldots, n-i-1$; $m(1 \otimes T_j) \mapsto T_{j+n-i}$ and $m(1 \otimes x_j) \mapsto x_{j+n-i}$, for $j = 1, \ldots, i-1$.

Let $\pi$ be a smooth representation of $G_n$. Then $\pi_{\rho_n}$ is an $\mathcal{H}(M, \rho_{n-i} \boxtimes \rho_i)$-module by restriction from $\mathcal{H}(G_n, \rho_n)$. Let $\pi_N$ be the normalized Jacquet functor i.e. the maximal quotient of $\pi$ such that $N$ acts trivially. Then we have a natural map $\pi_{\rho_n} \mapsto (\pi_N)_{\rho_{n-i} \boxtimes \rho_i}$.

Proposition 4.1. ([BK2 Corollary 7.11]) As $\mathcal{H}(M, \rho_{n-i} \boxtimes \rho_i)$-modules, $\pi_{\rho_n} \cong (\pi_N)_{\rho_{n-i} \boxtimes \rho_i}$.

4.2. Bernstein-Zelevinsky derivatives. Let $U_i$ be the subgroup of $M$ consisting of matrices of the form

$$
\begin{pmatrix}
I_{r(n-i)} & 0 \\
0 & u
\end{pmatrix},
$$

where $u$ is a strictly upper-triangular matrix in $G_i$. The character $\overline{\psi}$ of conductor $p$ defines a Whittaker character $\psi$ of $U_i$

$$
\psi(u) = \sum_{j=r(n-i)+1}^{r_n-1} \overline{\psi}(u_{j,j+1})
$$

where $u_{j,j+1}$ refers to the matrix entries. Let $\omega$ be a smooth $M$-module. Let $\omega_{U_i, \psi}$ be the space of $\psi$-twisted $U_i$-coinvariants. It is naturally a $G_{n-i}$-module. The $ri$-th Bernstein-Zelevinsky derivative of a smooth $G_n$-module $\pi$ is defined by

$$
\pi^{(ri)} = (\pi_N)_{U_i, \psi}
$$

Thus the $ri$-th Bernstein-Zelevinsky derivative is a functor from the category of smooth $G_n$-modules to the category of smooth $G_{n-i}$-modules. We note that th $l$-th derivative $\pi^{(l)}$ is defined for any non-negative integer $l$, however, if $\pi$ is an object in $\mathcal{R}^*(G_n)$ then $\pi^{(l)} = 0$ unless $l$ is divisible by $r$.

4.3. Bernstein-Zelevinsky derivative for $\mathcal{H}_n$. Abusing notation, we shall identify $\mathcal{H}_{n-i}$ and $m(\mathcal{H}_{n-i} \otimes 1)$. Let $S_i \in \mathcal{H}_i$ be the sign projector. Let $S_i^p = m(1 \otimes S_i)$. Let $\sigma$ be an $\mathcal{H}_n$-module. The $i$-th Bernstein-Zelevinsky derivative of $\sigma$ is the natural $\mathcal{H}_{n-i}$-module

$$
\text{BZ}_i(\sigma) := S_i^p(\sigma).
$$

Theorem 4.2. Let $\pi$ be an admissible representation of $G_n$ in $\mathcal{R}^*(G_n)$. There is a functorial isomorphism $\text{BZ}_i(\pi_{\rho_n}) \cong (\pi^{(ri)})_{\rho_{n-i}}$ of $\mathcal{H}_{n-i}$-modules.
Proof. By Proposition 4.1, \( \pi_N \) belongs to the Bernstein component with the type \( \rho_{n-i} \boxtimes \rho_i \). It follows that \((\pi_N)_{\rho_{n-i}} \) is an admissible representation in \( \mathfrak{R}^s(G_i) \). Theorem 3.5, applied to \( G_i \), implies that there is an isomorphism

\[
S_i((\pi_N)_{\rho_{n-i}}) \cong ((\pi_N)_{\rho_{n-i}})_{U_i,\tilde{\psi}_i}.
\]

Since the isomorphism is functorial, it is also an isomorphism of vector spaces. \((\pi_N)_{\rho_{n-i}} \) is isomorphic to \( H \alpha \). Theorem 5.2 holds without the assumption that \( \pi \) is admissible.

As it is true for Theorem 3.3 in the case of the Bernstein component of representations generated by their Iwahori-fixed vectors, Theorem 4.2 holds without the assumption that \( \pi \) is admissible.

5. A Leibniz rule

5.1. Affine Hecke algebras. We shall state the definition of an affine Hecke algebra in a greater generality which will be needed in the following subsections.

Let \((X, R, X^\vee, R^\vee)\) be a root datum where \( R \) is a reduced root system and \( X \) a \( \mathbb{Z} \)-lattice containing \( R \). Let \( W \) be the Weyl group of \( R \). Fix a set of simple roots \( \Delta \). The choice of \( \Delta \) determines a set \( S \) of simple reflections in \( W \). Let \( l : W \to \mathbb{Z} \) be the length function such that \( l(s) = 1 \) for all \( s \in S \). Let \( \mathcal{A} \cong \mathbb{C}[X] \) be the group algebra of \( X \). In other words, \( \mathcal{A} \) has a basis of elements \( \theta_x, x \in X \), such that \( \theta_x \theta_y = \theta_{x+y} \), for all \( x, y \in X \).

Definition 5.1. The affine Hecke algebra \( \mathcal{H} := \mathcal{H}(X, R, \Delta, q) \) associated to the datum is defined to be the complex associative algebra generated by the elements \( T_w, w \in W \), and the algebra \( \mathcal{A} \), subject to the relations

1. \( T_w T_{w'} = T_{ww'} \) if \( l(ww') = l(w) + l(w') \),
2. \( (T_s + 1)(T_s - q) = 0 \) for \( s \in S \),
3. \( T_s \theta_x - \theta_s(x) T_s = (q - 1) \frac{\theta_x - \theta_s(x)}{1 - \theta_s(x)} \).

Denote by \( \mathcal{H}_W \) the finite-dimensional subalgebra of \( \mathcal{H} \) generated by \( T_w, w \in W \). We have an isomorphisms of vector spaces \( \mathcal{H} \cong \mathcal{A} \otimes \mathcal{H}_W \). Let \( \mathbb{T} = \text{Hom}(X, \mathbb{C}^\times) \). The center \( Z \) of \( \mathcal{H} \) is isomorphic to \( \mathbb{C}[X]^W \). Hence central characters of \( \mathcal{H} \) are parameterized by \( W \)-orbits in \( \mathbb{T} \). We shall denote by \( Wt \) the \( W \)-orbit of \( t \in \mathbb{T} \). Let \( \mathcal{J}_{Wt} \) be the corresponding maximal ideal in \( Z \). For a finite-dimensional \( \mathcal{H} \)-module \( \chi \), let \( \chi_{[Wt]} \) be the subspace of \( \chi \) annihilated by a power of \( \mathcal{J}_{Wt} \). Then

\[
\chi \cong \bigoplus_{Wt \in \mathbb{T}/W} \chi_{[Wt]}.
\]

Let \( X_n = X_n^{\vee} = \bigoplus_{k=1}^n \mathbb{Z} \epsilon_k \) be a \( \mathbb{Z} \)-lattice. Set \( \alpha_{kl} = \epsilon_k - \epsilon_l \) (\( k \neq l \)) and also set \( \alpha_k = \alpha_{k,k+1} \) (\( k = 1, \ldots, n \)). Let \( R_n = R_n^{\vee} = \{ \epsilon_k - \epsilon_l : l \neq k \} \) be a root system of type \( A_{n-1} \). Let \( \Delta_n = \{ \epsilon_i - \epsilon_{i+1} : i = 1, \ldots, n-1 \} \). The Iwahori-Hecke algebra \( \mathcal{H}_n \) of \( GL(n) \) (from Section 2) is isomorphic to \( \mathcal{H}(X_n, R_n, \Delta_n, q) \).
5.2. Lusztig’s first reduction theorem. We shall need a variation [OS Section 2] of Lusztig’s reduction theorem for the affine Hecke algebra \( \mathcal{H}_n \) [Lu Section 8]. Let \( T_n = \text{Hom}(X_n, \mathbb{C}^\times) \). Any \( t \in T_n \) is identified with an \( n \)-tuple \((z_1, \ldots, z_n)\) of non-zero complex numbers where \( z_i \) is the value of \( t \) at \( \epsilon_i \). Let \( T_r = \text{Hom}(X_n, \mathbb{R}_{>0}) \) and \( T_{un} = \text{Hom}(X_n, \mathbb{S}^1) \).

Any \( t \in T_n \) has a polar decomposition \( t = vu \) where \( v \in T_r \) and \( u \in T_{un} \). Write \( x(u) \) for the value of \( u \) at \( x \in X_n \). Hence \( u = (z_1, \ldots, z_m) \) where \( z_k = \epsilon_k(u) \). We can permute the entries of \( u \) such that, for a partition \( n = (n_1, \ldots, n_m) \) of \( n \), \( z_1 = \ldots = z_{n_1} \neq z_{n_1+1} = \ldots \) etc. Let

\[
R_n = \{ \alpha \in R_n : \alpha(u) = 1 \}.
\]

It is a root subsystem of \( R_n \) which, as the notation indicates, depends on the partition \( n \).

It is isomorphic to the product \( R_{n_1} \times \ldots \times R_{n_m} \). Let \( S_n \cong S_{n_1} \times \ldots \times S_{n_m} \) be its Weyl group. Let \( \Delta_n \) be the set of simple roots in \( R_n \) determined by \( R_n^+ = R_n^+ \cap R_n \).

\[
\mathcal{H}_n := \mathcal{H}(X_n, R_n, \Delta_n, q) \cong \mathcal{H}_{n_1} \otimes \ldots \otimes \mathcal{H}_{n_m}
\]

be the associated affine Hecke algebra (Definition [Lu 4.4]). Let \( Z_n = \mathcal{A}_n^{S_n} \) be the center of \( \mathcal{H}_n \). Let \( \mathcal{J}_{S_n,t} \) be an ideal in \( Z_n \) corresponding to the central character \( S_{n,t} \). Let \( \sigma \) be a finite-dimensional \( \mathcal{H}_n \)-module annihilated by a power of \( \mathcal{J}_{S_n,t} \). Then \( \iota(\sigma) \cong \mathcal{H}_n \otimes \mathcal{H}_n \sigma \) is annihilated by a power of \( \mathcal{J}_{S_n,t} \).

The following result and proof are a variation of [Lu Sections 8.16 and 10.9].

**Theorem 5.2.** The functor \( \iota \) defines an equivalence between the category of finite-dimensional \( \mathcal{H}_n \)-modules annihilated by a power of \( \mathcal{J}_{S_n,t} \) and the category of finite-dimensional \( \mathcal{H}_n \)-modules annihilated by a power of \( \mathcal{J}_{S_n,t} \).

5.3. First reduction for the Bernstein-Zelevinsky derivatives. We keep using notations from the previous subsection. In particular, we fixed \( t = vu \in T_n \), and we have a canonical isomorphism \( \mathcal{H}_n \cong \mathcal{H}_{n_1} \otimes \ldots \otimes \mathcal{H}_{n_m} \), where \( n = (n_1, \ldots, n_m) \) is a partition of \( n \), arising from \( u \).

Fix an integer \( i \leq n \). For each \( m \)-tuple \( i = (i_1, \ldots, i_m) \) of integers, such that \( i_1 + \ldots + i_m = i \) and \( 0 \leq i_k \leq n_k \) \((k = 1, \ldots, m)\), define another \( m \)-tuple \( n - i = (n_1 - i_1, \ldots, n_m - i_m) \).

Each pair \((n_k - i_k, i_k)\) gives rise to an embedding \( \mathcal{H}_{n_k - i_k} \otimes \mathcal{H}_{i_k} \subseteq \mathcal{H}_{n_k} \), as in Section 4.1, and these combine to give an embedding

\[
\mathcal{H}_{n-1} \otimes \mathcal{H}_i \subseteq \mathcal{H}_n
\]

where \( \mathcal{H}_i \cong \mathcal{H}_{i_1} \otimes \ldots \otimes \mathcal{H}_{i_m} \) etc. (Note, if \( i_k = 0 \), then the corresponding factor is the trivial algebra \( \mathbb{C} \).) Abusing notation, we shall identify \( \mathcal{H}_{n-1} \) with its image in \( \mathcal{H}_n \) via the map \( h \mapsto h \otimes 1 \). Let \( S_i \in \mathcal{H}_i \) be the sign projector in \( \mathcal{H}_i \), and let \( \mathcal{S}_i^n \) be the image of \( 1 \otimes S_i \) in \( \mathcal{H}_n \). Let \( \sigma \) be an \( \mathcal{H}_n \)-module. Then \( \mathcal{S}_i^n(\sigma) \) is naturally an \( \mathcal{H}_{n-1} \)-module. Thus we have a functor

\[
\mathbf{BZ}_i^n(\sigma) := \mathcal{S}_i^n(\sigma)
\]

from the category of \( \mathcal{H}_n \)-modules to the category of \( \mathcal{H}_{n-1} \)-modules.

Observe that \( \mathcal{H}_{n-1} \) is a Levi subalgebra of \( \mathcal{H}_n \) and \( \mathcal{H}_i \) is a Levi subalgebra of \( \mathcal{H}_i \). We are now ready to state the first reduction result.
Theorem 5.3. Let \( \pi \) be a finite-dimensional \( H_n \)-module annihilated by a power of \( J_{S_n} \). Let \( \sigma \) be a finite-dimensional \( H_n \)-module annihilated by a power of \( J_{S_n} \) such that \( \pi \cong \iota(\sigma) \) (see Theorem 5.2). Then there is an isomorphism
\[
BZ_i(\pi) \cong \bigoplus_i H_{n-i} \otimes H_{n-i} BZ_i(\sigma)
\]
where the sum is taken over all \( m \)-tuple of integers \( i = (i_1, \ldots, i_m) \) satisfying \( i_1 + \cdots + i_m = i \) and \( 0 \leq i_k \leq n_k \) \((k = 1, \ldots, m)\).

Proof. Using the Mackey Lemma for affine Hecke algebras (see [M], Section 2] and [K]),
\[
\text{res} \mathcal{H}_n \otimes H_i (H_n \otimes H_n \sigma) \cong \bigoplus_i (H_{n-i} \otimes H_i) \otimes (H_{n-i} \otimes H_i) \left( \text{res} \mathcal{H}_n \otimes H_i \sigma \right)
\]
where the sum is over \( i \) as in the statement of the theorem. We remark that the Mackey Lemma asserts that the composition factors of \( \text{res} \mathcal{H}_{n-i} \otimes H_i (H_n \otimes H_n \sigma) \) are precisely those on the right hand side of the above isomorphism. The composition factors are indeed direct summands since their \( H_{n-i} \otimes H_i \)-central characters are distinct. Furthermore, using the Frobenius reciprocity, we have
\[
S^n_i (H_{n-i} \otimes H_i) \otimes (H_{n-i} \otimes H_i) \sigma) \cong H_{n-i} \otimes H_{n-i} S^n_i (\sigma).
\]
Combining (5.7) and (5.6), we obtain (5.8). \( \square \)

Remark 5.4. When \( \sigma \) is an irreducible \( H_n \)-module, then \( \sigma \cong \sigma_1 \boxtimes \cdots \boxtimes \sigma_m \) for some irreducible \( H_n \)-modules \( \sigma_k \). In this case,
\[
BZ^n_i (\sigma) \cong BZ_{i_1} (\sigma_1) \boxtimes \cdots \boxtimes BZ_{i_m} (\sigma_m).
\]

From this viewpoint, Theorem 5.3 can be seen as a Leibniz rule.

6. Reduction to graded Hecke algebras

6.1. Graded affine Hecke algebras. We shall now need the graded affine Hecke algebra attached to the root datum \((X, R, X^\vee, R^\vee)\). Let \( V = X \otimes_{\mathbb{Z}} \mathbb{C} \).

Definition 6.1. [Lu] Section 4] The graded affine Hecke algebra \( \mathbb{H} = \mathbb{H}(V, R, \Delta, \log q) \) is an associative algebra with the unit over \( \mathbb{C} \) generated by the symbols \( \{ t_w : w \in W \} \) and \( \{ f_v : v \in V \} \) satisfying the following relations:

1. The map \( w \mapsto t_w \) from \( \mathbb{C}[W] = \bigoplus_{w \in W} \mathbb{C}w \rightarrow \mathbb{H} \) is an algebra injection,
2. The map \( v \mapsto f_v \) from \( S(V) \rightarrow \mathbb{H} \) is an algebra injection, where \( S(V) \) is the symmetric algebra of \( V \),
3. writing \( v \) for \( f_v \) from now on, for \( \alpha \in \Delta \) and \( v \in V \),
\[
vt_s_\alpha - t_s_\alpha s_\alpha(v) = \log q \cdot (v, \alpha^\vee).
\]

In particular, \( \mathbb{H} \cong S(V) \otimes \mathbb{C}[W] \) as vector spaces. We also set \( \mathbb{A} = S(V) \), the graded algebra analogue of \( \mathbb{A} \). Let \( \mathbb{Z} = \mathbb{A}^W \) be the center of \( \mathbb{H} \) [Lu, Sec. 4]. Let \( V^* = \text{Hom}(X, \mathbb{C}) \). The central characters of irreducible representations are parameterized by \( W \)-orbits in \( V^* \). If \( \zeta \in V^* \), let \( W_\zeta \) denote the corresponding orbit an the central character. Let \( \mathbb{J}_{W_\zeta} \subset \mathbb{Z} \) be the corresponding maximal ideal.
6.2. Lusztig’s second reduction theorem. Let $H = H(X, R, \Delta, q)$ be the affine Hecke algebra defined in Section 6.1. $A \cong \mathbb{C}[X]$ the commutative subalgebra, and $Z \cong \mathbb{C}[X]^W$ be the center of $H$. Let $F$ be the quotient field of $A$. Let $H_F \cong H_W \otimes_C F$ with the algebra structure naturally extended from $H$.

Following Lusztig [Lu] Section 5], for $\alpha \in \Delta$, define $\tau_{s_\alpha} \in H_F$ by

$$\tau_{s_\alpha} + 1 = (T_{s_\alpha} + 1)G(\alpha)^{-1},$$

where

$$G(\alpha) = \frac{\theta_\alpha q - 1}{\theta_\alpha - 1} \in F.$$ 

It is shown in [Lu] Section 5] that the map from $W$ to the units of $H_F$ defined by $s_\alpha \mapsto \tau_{s_\alpha}$ is an injective group homomorphism.

On the graded Hecke algebra side, let $H = H(V, R, \Delta, \log q)$ be as in Definition 6.1. Let $F$ be the quotient field of $A$ and let $Z$ be the center of $H$. Let $H_F \cong H_W \otimes_C F$ with the algebra structure naturally extended from $H$. For $\alpha \in \Delta$, define $\tau_{s_\alpha} \in H_F$ by

$$\tau_{s_\alpha} + 1 = (t_{s_\alpha} + 1)g(\alpha)^{-1},$$

where

$$g(\alpha) = \frac{\alpha + \log q}{\alpha} \in F.$$ 

As in the affine case, the map from $W$ to the units of $H_F$ defined by $s_\alpha \mapsto \tau_{s_\alpha}$ is an injective group homomorphism.

Any $\zeta \in V^*$ defines $t \in T = \text{Hom}(X, \mathbb{C})^\times$ by $x(t) = e^{\zeta(x)}$, for all $x \in X$. We shall express this relationship by $t = \exp(\zeta)$. We shall say that $\zeta$ is real for the root system $R$ if $\alpha(\zeta) \in \mathbb{R}$ for all $\alpha \in R$. Then $t = \exp(\zeta)$ satisfies $\alpha(t) > 0$, for all $\alpha \in R$. Conversely, every such $t$ arises in this fashion, from a real $\zeta$. Let $\widehat{Z}$ be the $J_{W_{\zeta}}$-adic completion of $Z$ and let $\widehat{Z}$ be the $J_{W_{\zeta}}$-adic completion of $Z$. Let $\widehat{H} = \widehat{Z} \otimes_Z H$ and let $\widehat{H}_F = \widehat{Z} \otimes_Z H_F$ and let $\widehat{H}_F = \widehat{Z} \otimes_Z H_F$. Let $\widehat{A} = \widehat{Z} \otimes_Z A$ and let $\widehat{A} = \widehat{Z} \otimes_Z A$. Let $\widehat{J}_{W_{\zeta}} = \widehat{Z} \otimes_Z J_{W_{\zeta}}$ and let $\widehat{J}_{W_{\zeta}} = \widehat{Z} \otimes_Z J_{W_{\zeta}}$.

**Theorem 6.2.** [Lu] Theorem 9.3, Section 9.6] Recall that we are assuming that $\zeta \in V^*$ is real for the root system $R$.

1. There is an isomorphism denoted $j$ between $\widehat{H}_F$ and $\widehat{H}_F$ determined by

$$j(\tau_{s_\alpha}) = \tau_{s_\alpha}, \quad j(\theta_{\zeta}) = e^\zeta.$$

2. The above map also induces isomorphisms between $\widehat{Z}$ and $\widehat{Z}$, between $\widehat{A}$ and $\widehat{A}$ and between $\widehat{H}$ and $\widehat{H}$.

A crucial point for the proof of (2) is the fact that

$$\frac{e^\alpha q - 1}{e^\alpha - 1} : \frac{\alpha}{\alpha + \log q} \in F$$

is holomorphic and non-vanishing at any $\zeta' \in W^\zeta$, and hence is an invertible element in $\widehat{A}$.

Now (2) gives the following isomorphisms:

$$\frac{H}{J_{W_{\zeta}} \cdot H} \cong \widehat{H}/\widehat{J}_{W_{\zeta}} \cdot \widehat{H} \cong \widehat{H}/\widehat{J}_{W_{\zeta}} \cdot \widehat{H} \cong \widehat{H}/\widehat{J}_{W_{\zeta}} \cdot \widehat{H}$$
and hence:

**Theorem 6.3.** [LR, Section 10] Assume that $\zeta \in V^*$ is real. There is an equivalence of categories between the category of finite-dimensional $\mathbb{H}$-modules annihilated by a power of $J_{W\zeta}$ and the category of finite-dimensional $\mathcal{H}$-modules annihilated by a power of $J_{W}$, where $t = \exp(\zeta)$.

Let $\Lambda$ be the functor in Theorem 6.3. Explicitly, for a finite-dimensional $\mathbb{H}$-module annihilated by a power of $J_{W\zeta}$, $\Lambda(\pi)$ is equal to $\pi$, as vector spaces, but the $\mathcal{H}$-action on $\pi$ is given by

$$h \cdot _{\mathcal{H}} x = j(h) \cdot _{\mathbb{H}} x,$$

where $h \in \mathcal{H}$ and $x \in \pi$. Note that the functor extends to the category of finite-dimensional $\mathbb{H}$-modules that are sums of $\mathbb{H}$-modules, where each summand is annihilated by a power of $J_{W\zeta}$ for some real $\zeta$.

**Proposition 6.4.** Recall the sign projector $S = (\sum_{w \in W} (1/q)^{l(w)} - 1) \sum_{w \in W} (-1/q)^{l(w)} T_w$ in $\mathcal{H}$ and let $s = |W|^{-1} \sum_{w \in W} (-1)^{l(w)} t_w$ be the sign projector in $\mathbb{H}$. Then $j(S) = a \cdot s$, where $a$ is an invertible element in $\mathbb{H}$.

**Proof.** Let $\alpha \in \Pi$. Note that $S(T_{s_{n+1}} + 1)G(\alpha)^{-1} = 0$. Applying $j$ to this equation gives

$$j(S)(t_{s_{n+1}} + 1)g(\alpha)^{-1} = 0,$$

hence $j(S)(t_{s_{n+1}} + 1) = 0$ for $\alpha \in \Pi$. This shows that $j(S) \in \mathbb{H} \cdot s$. Since $\mathbb{H} \cdot s = \mathbb{H} \mathbb{H} \cdot s$, we have $j(S) = a \cdot s$, for some $a \in \mathbb{H}$. Using the same argument for $j^{-1}$, we obtain $j^{-1}(s) = b \cdot S$ for some $b \in \mathbb{H}$. Hence $j(b)a = 1$ and $a$ is invertible. \hfill $\Box$

We have the following corollary to Proposition 6.4.

**Corollary 6.5.** Let $\pi$ be a finite-dimensional $\mathbb{H}$-module annihilated by a power of $J_{W\zeta}$, where $\zeta \in V^*$ is real. Identify $\pi$ and $\Lambda(\pi)$ as linear spaces. The multiplication by $a \in \mathbb{H}$ (from Proposition 6.4) provides a natural isomorphism between the vector spaces $s(\pi)$ and $S(\pi)$.

6.3. Bernstein-Zelevinsky derivatives for graded algebras. Let $V_n = X_n \otimes \mathbb{C}$, and $H_n := H(V_n, R_n, \Delta_n, \log q)$. For every $i = 0, \ldots, n$, we have a Levi subalgebra $H_{n-i} \otimes H_i$. Let $s_i \in H_i$ be the sign projector, and let $s_i^H \in H_n$ be the image of $1 \otimes s_i$ under the inclusion $H_{n-i} \otimes H_i \subseteq H_n$.

Let $\pi$ be a finite-dimensional representation of $H_n$. The $i$-th Bernstein-Zelevinsky derivative of $\pi$ is the natural $H_{n-i}$-module

$$gBZ_i(\pi) := s_i^H(\pi).$$

Write any $\zeta \in V_n^* = \text{Hom}(X_n, \mathbb{C})$ as an $n$-tuple $(\zeta_1, \ldots, \zeta_n)$ where $\zeta_i$ is the value of $\zeta$ on the standard basis element $e_i \in X_n$. In this case $\zeta$ is real for $R_n$ if and only if $\zeta_k - \zeta_l \in \mathbb{R}$ for all $1 \leq k, l \leq n$. 
**Theorem 6.6.** Assume that $\zeta \in V_n^*$ is real for the root system $R_n$, and $\pi$ is a finite-dimensional $\mathbb{H}_n$-module annihilated by a power of $\mathbb{J}_{S_n,\zeta}$. There is a natural isomorphism of $\mathcal{H}_{n-i}$-modules $\mathbf{BZ}_i(\Lambda(\pi))$ and $\Lambda(\mathbf{gBZ}_i(\pi))$.

**Proof.** Note that the functor $\Lambda$ commutes with the restriction to Levi subalgebras, that is, we can either restrict to $\mathbb{H}_{n-i} \otimes \mathbb{H}_i$ and then apply $\Lambda$, or apply $\Lambda$ and then restrict to $\mathcal{H}_{n-i} \otimes \mathcal{H}_i$. Decompose $\pi$ under the action of $\mathbb{H}_i$ $\pi = \oplus \pi_{[i,\zeta]}$ where $\pi_{[i,\zeta]}$ is the summand annihilated by a power of $\mathbb{J}_{S_i,\zeta}$. Concretely, the sum runs over $S_i$-orbits of the $i$-tuples $\zeta'$ that appear as the tail end of the $n$-tuples in the $S_n$-orbit of $\zeta$. We have the corresponding decomposition for the action of $\mathcal{H}_i$,

$$\Lambda(\pi) = \oplus \Lambda(\pi_{[i,\zeta']})$$

where $t' = \exp(\zeta')$. (The underlying vector spaces of $\pi_{[i,\zeta]}$ and $\Lambda(\pi_{[i,\zeta']})$ are the same.) It follows that $\Lambda(\pi_{[i,\zeta']})$ and $\Lambda(\pi_{[i,\zeta]})$ are isomorphic $\mathcal{H}_{n-i} \otimes \mathcal{H}_i$-modules. Recall that $S^n_i = 1 \otimes S_i$ and $S^n_i = 1 \otimes s_i$, where $S_i$ and $s_i$ are the sign projectors in $\mathcal{H}_i$ and $\mathbb{H}_i$, respectively. Now we have the following isomorphisms of $\mathcal{H}_{n-i}$-modules

$$S^n_i(\Lambda(\pi_{[i,\zeta']})) \cong S^n_i(\Lambda(\pi_{[i,\zeta]})) \cong \Lambda(s^n_i(\pi_{[i,\zeta]}))$$

where the second is furnished by Corollary 6.5. This isomorphism is given by the action of an invertible element in $\mathbb{H}_i$ and therefore intertwines $\mathcal{H}_{n-i}$-action. 

## 6.4. Second reduction for Bernstein-Zelevinsky derivatives

In this section, we transfer the problem of computing Bernstein-Zelevinsky derivatives $\mathbf{BZ}_i^n$ in Theorem 5.3 to the corresponding problem for graded Hecke algebras. We retain the notation from Sections 5.2 and 5.3. In particular, $n = (n_1, \ldots, n_m)$ is a partition of $n$, and we have fixed $t \in T_n$ such that $\alpha(t) > 0$ for all $\alpha \in R_n$. Then there exists $\zeta \in V_n^*$, real for the root system $R_n$, such that $t = \exp(\zeta)$.

Let $\mathbb{H}_n := \mathbb{H}(V_n, R_n, \Delta_n, \log q) \cong \mathbb{H}_{n_1} \otimes \ldots \otimes \mathbb{H}_{n_m}$.

Let $i = (i_1, \ldots, i_m)$ be an $m$-tuple of integers such that $0 \leq i_k \leq n_k$ for all $k$ and $n - i = (n_1 - i_1, \ldots, n_m - i_m)$. Each pair $(n_k - i_k, i_k)$ gives rise to an embedding $\mathbb{H}_{n_k - i_k} \otimes H_{i_k} \subseteq \mathbb{H}_{n_k}$, and these combine to give an embedding

$$\mathbb{H}_{n-i} \otimes H_i \subseteq \mathbb{H}_n$$

where $\mathbb{H}_i \cong \mathbb{H}_{i_1} \otimes \ldots \otimes \mathbb{H}_{i_m}$ etc. Abusing notation, we shall identify $\mathbb{H}_{n-i}$ with its image in $\mathbb{H}_n$ via the map $h \mapsto h \otimes 1$. Let $s_i \in \mathbb{H}_i$ be the sign projector in $\mathbb{H}_i$, and let $s^n_i$ be the image of $1 \otimes s_i$ in $\mathbb{H}_n$. Let $\sigma$ be an $\mathbb{H}_n$-module. Then $s^n_i(\sigma)$ is naturally an $\mathbb{H}_{n-i}$-module. Thus we have a functor

$$\mathbf{gBZ}_i^n(\sigma) := s^n_i(\sigma)$$

from the category of $\mathbb{H}_n$-modules to the category of $\mathbb{H}_{n-i}$-modules. The following is proved in the same way as Theorem 6.6.
Lemma 7.1. \( \text{irreducible} \ C \) \( gBZ \) Then under the Borel-Casselman equivalence and the Lusztig equivalence in Theorem 6.3.

Then \( \pi \) the fact that the category of \( C \) This follows from the construction of generalized Speh modules (see e.g. (7.9)) and

Proof. \( \text{constant} \) module pulled back from \( \sigma \) JM

Corollary 7.2. Let \( \pi \) be a generalized Speh representation of \( GL(n, F) \) associated to \( (\bar{n}, \rho) \). Then \( \pi^{(i)} \) is the direct sum of generalized Speh modules associated to \( (\bar{n}', \rho) \), where \( \bar{n}' \) runs
through all partitions obtained by removing \( i \) boxes from \( \bar{n} \) with at most one in each row such that the resulting diagram is still a Young diagram.

Proof. Since \( \Lambda(\sigma_{\bar{n},\kappa}) = \pi^{\bar{n}} \), it suffices to compute \( g_{BZ_i}(\sigma_{\bar{n},\kappa}) \) by Theorem 6.6. From the observation in Lemma 7.1, it suffices to determine the \( S_n \) of \( \pi \) satisfying the conditions of the Littlewood-Richardson rule (or the Pieri's formula).

Generalized Speh modules form a subclass of ladder representations defined by Lapid-Mínguez [LM]. Bernstein-Zelevinsky derivatives of ladder representations are computed there using a determinantal formula of Tadić.

8. Appendix: Projectivity of Gelfand-Graev representation

In this appendix we shall prove that the Gelfand-Graev representation of a quasi-split reductive group is projective. Roughly speaking, this follows from two facts: its Bernstein components are finitely generated, and its dual is injective.

8.1. Some algebra. Let \( H \) be a \( \mathbb{C} \)-algebra with 1, such that its center \( Z \) is a noetherian algebra, and \( H \) is a finitely generated \( Z \)-module. In particular, every finitely generated \( H \)-module \( \pi \) is also finitely generated \( Z \)-module. Hence any ascending chain of submodules of \( \pi \) stabilizes. It follows that any finitely generated \( H \)-module has irreducible quotients. Assume also that \( H \) is countably dimensional, as a vector space over \( \mathbb{C} \). Then any finitely generated \( H \)-module \( \pi \) is countably dimensional. In this situation the Schur lemma holds, that is, if \( \pi \) is irreducible then \( \pi \) is annihilated by a maximal ideal \( J \) in \( Z \). It follows that any irreducible \( H \)-module is finite dimensional.

Fix a maximal ideal \( J \) in \( Z \). Let \( \hat{Z} \) be the \( J \)-adic completion of \( Z \). It is known that \( \hat{Z} \) is a flat \( Z \)-module [AM]. If \( \pi \) is a \( Z \)-module, let \( \hat{\pi} \) be the \( J \)-adic completion of \( \pi \). If \( \pi \) is finitely generated then \( \hat{\pi} \cong \hat{Z} \otimes_Z \pi \). In particular, the \( J \)-adic completion gives an exact functor from the category of finitely generated \( H \)-modules to the category of finitely generated \( \hat{H} \cong \hat{Z} \otimes_Z H \)-modules.

Theorem 8.1. Assume that \( H \) satisfies all conditions spelled out above. Let \( \pi \) be a finitely-generated \( H \)-module. Suppose

\[
\text{Ext}_H^i(\pi, \sigma) = 0, \quad i \geq 1,
\]

for all finite-dimensional \( H \)-modules \( \sigma \), or \( \text{Hom}_H(\pi, \cdot) \) is an exact functor on the category of finite-dimensional \( H \)-modules. Then \( \pi \) is a projective \( H \)-module.

Proof. Suppose we have two \( H \)-modules \( \sigma, \tau \) with a surjection \( f : \sigma \to \tau \). To show \( \pi \) is projective, we have to show that the map \( f_* : \text{Hom}_H(\pi, \sigma) \to \text{Hom}_H(\pi, \tau) \) is surjective. Note that, since \( \pi \) is finitely generated, we can assume, without loss of generality, that \( \sigma \) and \( \tau \) are finitely generated. We shall prove firstly a local version of the desired result.

Lemma 8.2. For every maximal ideal \( J \) in \( Z \), the map \( \hat{f}_* : \text{Hom}_{\hat{H}}(\hat{\pi}, \hat{\sigma}) \to \text{Hom}_{\hat{H}}(\hat{\pi}, \hat{\tau}) \) is surjective.
Proof. Let $\psi \in \text{Hom}_{\mathcal{H}}(\widehat{\pi}, \widehat{\tau})$. Since $\widehat{\tau}$ is naturally isomorphic to the inverse limit of $\tau/J^i\tau$, the map $\psi$ is given by a system of $\psi_i \in \text{Hom}_{\mathcal{H}}(\pi, \tau/J^i\tau)$, commuting with the canonical projections $\tau/J^i+1\tau \to \tau/J^i\tau$. Thus, in order to find $\phi \in \text{Hom}_{\mathcal{H}}(\pi, \sigma/J^i\sigma)$ such that $\widehat{f}_i(\phi) = \psi$, it suffices to find a system of $\phi_i \in \text{Hom}_{\mathcal{H}}(\pi, \sigma/J^i\sigma)$, commuting with the canonical projections $\sigma/J^i+1\sigma \to \sigma/J^i\sigma$, such that $f_i^*(\phi_i) = \psi_i$, for all $i$,

$$f_i^*: \text{Hom}_{\mathcal{H}}(\pi, \sigma/J^i\sigma) \to \text{Hom}_{\mathcal{H}}(\pi, \tau/J^i\tau),$$

are naturally arising from $f$. Observe that the quotients $\sigma/J^i\sigma$ and $\tau/J^i\tau$ are finite dimensional, since $\sigma$ and $\tau$ are finitely generated. Consider the following commutative diagram:

$$
\begin{array}{ccc}
0 & \longrightarrow & \text{Hom}_{\mathcal{H}}(\pi, J^i\sigma/J^{i+1}\sigma) \\
& & \downarrow f_i^{i+1} \\
0 & \longrightarrow & \text{Hom}_{\mathcal{H}}(\pi, \sigma/J^{i+1}\sigma)
\end{array}
\begin{array}{ccc}
& & \longrightarrow \text{Hom}_{\mathcal{H}}(\pi, \sigma/J^i\sigma) \\
& & \downarrow f_i^* \\
& & \longrightarrow 0
\end{array}
\begin{array}{ccc}
0 & \longrightarrow & \text{Hom}_{\mathcal{H}}(\pi, J^i\tau/J^{i+1}\tau) \\
& & \downarrow f_i^{i+1} \\
0 & \longrightarrow & \text{Hom}_{\mathcal{H}}(\pi, \tau/J^{i+1}\tau)
\end{array}
\begin{array}{ccc}
& & \longrightarrow \text{Hom}_{\mathcal{H}}(\pi, \tau/J^i\tau) \\
& & \downarrow f_i^* \\
& & \longrightarrow 0
\end{array}
$$

The assumption on $\pi$ implies that the vertical maps are surjective and the horizontal sequences are exact. We can now construct the sequence $\phi_i$ by induction. Assume that $\phi_i$ has been constructed. Let $\phi_{i+1}'$ be any element in $\text{Hom}_{\mathcal{H}}(\pi, \sigma/J^{i+1}\sigma)$ in the fiber of $\phi_i$. Then $f_i^{i+1}(\phi_{i+1}')$ may not be equal to $\psi_{i+1}$, however, a simple diagram chase shows that there exists $\epsilon \in \text{Hom}_{\mathcal{H}}(\pi, \sigma/J^{i+1}\sigma)$, in the image of $\text{Hom}_{\mathcal{H}}(\pi, J^i\sigma/J^{i+1}\sigma)$, such that $\phi_{i+1} = \phi_{i+1}' + \epsilon$ satisfies all requirements.

We now pass to a global version of the above result.

Lemma 8.3. Let $\pi_1, \pi_2$ be finitely-generated $\mathcal{H}$-modules. Then $\text{Hom}_{\mathcal{H}}(\pi_1, \pi_2)$ is a finitely-generated $\mathcal{Z}$-module.

Proof. Let $x_1, \ldots, x_r$ be a finite set of generators for $\pi$ as $\mathcal{Z}$-module. Now we consider a $\mathcal{Z}$-submodule of $\oplus_{k=1}^r \pi$ given by

$$\{f(x_1), \ldots, f(x_r) : f \in \text{Hom}_{\mathcal{H}}(\pi_1, \pi_2)\}$$

By the Noetherian property, a submodule of a finitely-generated module is still finitely-generated. This implies the lemma.

Lemma 8.4. Let $\pi_1, \pi_2$ be finitely-generated $\mathcal{Z}$-modules, and $g : \pi_1 \to \pi_2$ a morphism of $\mathcal{Z}$-modules. Then the following statements are equivalent:

1. $g$ is surjective;
2. with respect to every maximal ideal $J$ in $\mathcal{Z}$, the map $\widehat{g} : \widehat{\pi}_1 \to \widehat{\pi}_2$, arising naturally from $g$, is surjective.

Proof. (1) implies (2) by the exactness of tensoring with $\widehat{\mathcal{Z}}$. For (2) implying (1), we proceed by contraposition. So assume that (1) fails. Let $\sigma$ be an irreducible quotient of $\pi_2/f(\pi_1)$, and let $J$ be the annihilator of $\sigma$. Then (2) fails for the $J$-adic completion.
**Lemma 8.5.** Let $\pi_1, \pi_2$ be finitely-generated $\mathcal{H}$-modules. Then the natural map
\[ \theta : \hat{\mathbb{Z}} \otimes \mathbb{Z} \operatorname{Hom}_\mathcal{H}(\pi_1, \pi_2) \to \operatorname{Hom}_{\hat{\mathcal{H}}}(\hat{\pi}_1, \hat{\pi}_2) \]
is an isomorphism.

**Proof.** Let 
\[ \ldots \to \mathcal{H}^{r_2} \to \mathcal{H}^{r_1} \to \pi_1 \to 0 \]
be a finitely generated free resolution for $\pi_1$. Then we have the following commutative diagram:
\[
\begin{array}{c}
0 \rightarrow \hat{\mathbb{Z}} \otimes \mathbb{Z} \operatorname{Hom}_\mathcal{H}(\pi_1, \pi_2) \rightarrow \hat{\mathbb{Z}} \otimes \mathbb{Z} \operatorname{Hom}_\mathcal{H}(\mathcal{H}^{r_1}, \pi_2) \rightarrow \hat{\mathbb{Z}} \otimes \mathbb{Z} \operatorname{Hom}_\mathcal{H}(\mathcal{H}^{r_2}, \pi_2) \\
\downarrow \theta \quad \downarrow \quad \downarrow \theta \\
0 \rightarrow \operatorname{Hom}_{\hat{\mathcal{H}}}(\hat{\pi}_1, \hat{\pi}_2) \rightarrow \operatorname{Hom}_{\hat{\mathcal{H}}}(\hat{\mathcal{H}}^{r_1}, \hat{\pi}_2) \rightarrow \operatorname{Hom}_{\hat{\mathcal{H}}}(\hat{\mathcal{H}}^{r_2}, \hat{\pi}_2).
\end{array}
\]
The first sequence is exact since $\operatorname{Hom}_\mathcal{H}(\cdot, \pi_2)$ is left exact and tensoring with $\hat{\mathbb{Z}}$ is exact. The second sequence is exact since tensoring with $\hat{\mathbb{Z}}$ is exact and $\operatorname{Hom}_{\hat{\mathcal{H}}}(\cdot, \hat{\pi}_2)$ is left exact. Since the last two vertical arrows are isomorphisms, $\theta$ is an isomorphism, because kernels of isomorphic maps are isomorphic.

We can now prove that $f_*$ is surjective. By the previous two lemmas, it suffices to show that $\hat{f}_*$ is surjective, for every completion. But this is true by Lemma 8.2. This completes the proof of the theorem. □

### 8.2. Gelfand-Graev representation

Let $G$ be a quasi-split reductive group over a $p$-adic field. Let $K$ be a good open compact subgroup of $G$, as in Corollaire 3.9 in [BD]. Let $\mathcal{H}$ be the Hecke algebra of compactly supported $K$-biinvariant functions on $G$. Then, by [BD], in particular, Corollaire 3.4 there, the algebra $\mathcal{H}$ satisfies the conditions spelled out at the beginning of this section. Let $\pi$ be a smooth $G$-module. In order to prove that $\pi$ is projective, by Theorem 8.1, it suffices to show the following two bullets:

- For every $K$, the summand of $\pi$ generated by $K$-fixed vectors is finitely generated.
- The functor $\operatorname{Hom}_G(\pi, \cdot)$ is exact on the category of finite length modules.

Let $\pi^*$ denote the smooth dual of $G$. Since $\operatorname{Hom}_G(\pi, \sigma^*) \cong \operatorname{Hom}_G(\sigma, \pi^*)$, the second bullet holds if $\pi^*$ is an injective $G$-module. This is true if $\pi$ is a Gelfand-Graev representation, by exactness of the Jacquet functor. The first bullet is also true for the Gelfand-Graev representation, by [BH]. Thus we have obtained the following corollary:

**Corollary 8.6.** The Gelfand-Graev representation is projective.

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