Some structural results on the non-abelian tensor square of groups

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Abstract

We study the non-abelian tensor square $G \otimes G$ for the class of groups $G$ that are finitely generated modulo their derived subgroup. In particular, we find conditions on $G/G'$ so that $G \otimes G$ is isomorphic to the direct product of $\nabla(G)$ and the non-abelian exterior square $G \wedge G$. For any group $G$, we characterize the non-abelian exterior square $G \wedge G$ in terms of a presentation of $G$. Finally, we apply our results to some classes of groups, such as the classes of free soluble and free nilpotent groups of finite rank, and some classes of finite $p$-groups.

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Introduction

The non-abelian tensor square $G \otimes G$ of a group $G$ is a special case of the non-abelian tensor product $G \otimes H$ of two arbitrary groups $G$ and $H$, that was introduced by Brown and Loday in [6, 7] and arises from applications of a generalized Van Kampen theorem in homotopy theory.

For all $g, h \in G$ let $^g h = ghg^{-1}$ and $[g, h] = ghg^{-1}h^{-1}$. Then $G \otimes G$ is defined as the group generated by the symbols $g \otimes h$, for $g, h \in G$, subject to the relations

$$gh \otimes k = (^g h \otimes ^g k)(g \otimes k) \quad \text{and} \quad g \otimes hk = (g \otimes h)(^h g \otimes ^h k).$$

The definition guarantees the existence of an epimorphism $\kappa : G \otimes G \longrightarrow G'$, defined on the generators by $\kappa(g \otimes h) = [g, h]$ for all $g, h \in G$. Let $J(G)$ be
the kernel of the map $\kappa$, and let $\nabla(G)$ be the normal subgroup generated by the elements $g \otimes g$, for all $g \in G$. The group $(G \otimes G)/\nabla(G)$ is called the nonabelian exterior square of $G$, and denoted by $G \wedge G$. The map $\kappa$ factorizes modulo $\nabla(G)$, thus inducing an epimorphism $\kappa': G \wedge G \to G'$. By results in [6, 7] the kernel of the map $\kappa'$ is isomorphic to the Schur multiplicator $M(G)$ of $G$. Let $\Gamma(G/G')$ be Whitehead’s quadratic functor, as defined in [19]. Then results in [6, 7] give a commutative diagram with exact rows and central extensions as columns:

\[
\begin{array}{cccccc}
1 & \rightarrow & \Gamma(G/G') & \rightarrow & J(G) & \rightarrow & M(G) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \nabla(G) & \rightarrow & G \otimes G & \rightarrow & G \wedge G & \rightarrow & 1 \\
\downarrow & & \kappa & & \downarrow & & \kappa' & & \downarrow \\
1 & \rightarrow & G' & \rightarrow & 1 & \rightarrow & G' \\
\end{array}
\]

We are interested in the case when the middle row of the above diagram splits. Our main result in this context is the following.

**Proposition 1** Let $G$ be a group such that $G/G'$ is finitely generated. If $G/G'$ has no elements of order two or if $G'$ has a complement in $G$ then $G \otimes G \simeq \nabla(G) \times (G \wedge G)$.

We will see that, under the hypotheses of Proposition 1, the structure of the tensor square $G \otimes G$ is completely determined once the structures of $G/G'$ and of $G \wedge G$ are known. In [5] Brown, Johnson and Robertson proved that if $M(G)$ is finitely generated then $G \wedge G$ is isomorphic to the derived subgroup of any covering group $\hat{G}$ of $G$ (the notion of a covering group is well known if $G$ is finite, see [13], and in the general case the authors of [5] adopted a similar definition).

Our contribution is the following.
Proposition 2 Let $G$ be a group and let $F$ be a free group such that $G \simeq F/R$ for some normal subgroup $R$ of $F$. Then

$$G \wedge G \simeq \frac{F'}{[F,R]}.$$ 

As corollaries of Propositions 1 and 2 we deduce the structures of non-abelian tensor squares of finitely generated groups that are free in some variety, for example, the free $n$-generated nilpotent groups of fixed class (see Corollary 3.3) or the free $n$-generated soluble groups of fixed derived length (see Corollary 3.4).

We mention here that a wide list of references on the non-abelian tensor square of a group $G$ can be found in [15] and that an effective algorithm for computing it in the case when $G$ is polycyclic has been recently developed by Eick and Nickel in [12] and is implemented in the computing program GAP.

The paper is organized as follows. In the first section we collect some background material and prove some new basic results on the tensor square of an arbitrary group $G$. Proposition 1 is proved in Section 2, while in Section 3 we prove Proposition 2 and derive several consequences. Section 4 deals with finite $p$-groups $G$; in particular some upper bounds on the orders of $G \otimes G$ and $M(G)$ are found.

The notation used in this paper is standard (the reader is referred for example to [13]), with the only exception that conjugation and commutation are as defined in the second paragraph of this Introduction.

1 Background

Let $G$ be an arbitrary group. In order to investigate the structure of $G \otimes G$, it is sometimes more convenient to consider the following construction, which was introduced in [10].

Let $G^\varphi$ be a group isomorphic to $G$ via the isomorphism $\varphi : G \longrightarrow G^\varphi$, and consider the group

$$\nu(G) := \left\langle G, G^\varphi | R, R^\varphi, g_3[g_1, g_2^\varphi] = [g_3 g_1, (g_3 g_2)^\varphi] = g_3^x [g_1, g_2^\varphi], \forall g_1, g_2, g_3 \in G \right\rangle,$$
where $\mathcal{R}, \mathcal{R}^\varphi$ are the defining relations of $G$ and $G^\varphi$ respectively (that is, $\nu(G)$ is the quotient of the free product $G \ast G^\varphi$ by its normal subgroup generated by all the words $g_1 g_2 \ldots g_n$ with any element of $G$, $G^\varphi$ is proved to be isomorphic to the commutator subgroup by all the words $g_1 g_2 \ldots g_n$). In [17] (Proposition 2.6), the non-abelian tensor square $G \otimes G$ is proved to be isomorphic to the commutator subgroup $[G, G^\varphi]$ inside $\nu(G)$.

From now on we identify $G$ with $G^\varphi$ and, unless differently specified, we write $[g, h^\varphi]$ in place of $g \otimes h$ (for $g, h \in G$). For the reader’s clarity we report here some results that we will often use.

**Lemma 1.1 (Lemma 2.1 in [17], Lemma 2.1 in [4])** Let $G$ be any group. The following relations hold in $\nu(G)$.

(i) $[g_1 g_2 \varphi][g_3 g_4 \varphi] = [g_1 g_3 \varphi][g_2 g_4 \varphi] = [g_1 \varphi g_3][g_2 \varphi g_4]$, for all $g_1, g_2, g_3, g_4 \in G$.

(ii) $[g_1 \varphi g_2, g_3] = [g_1, g_2 \varphi g_3] = [g_1, g_2, g_3] = [g_1 \varphi g_2, g_3] = [g_1, g_2, g_3 \varphi] = [g_1, g_2 \varphi, g_3 \varphi], \text{ for all } g_1, g_2, g_3 \in G$.

(iii) $[g_1, g_2, g_3 \varphi] = [g_2, g_3, g_1 \varphi] = [g_3, g_1, g_2 \varphi]$, for all $g_1, g_2, g_3 \in G$.

(iv) $[g, g^\varphi]$ is central in $\nu(G)$ for all $g \in G$.

(v) $[g_1, g_2 \varphi][g_3, g_4 \varphi]$ is central in $\nu(G)$ for all $g_1, g_2 \in G$.

(vi) $[g, g^\varphi] = 1$ for all $g \in G'$.

**Corollary 1.2** Let $G$ be any group. Then the following hold.

(i) If $a, b \in G$ commute, then $[a, b^\varphi]$ and $[b, a^\varphi]$ are central elements of $[G, G^\varphi]$.

(ii) If $g_1 \in G'$ or $g_2 \in G'$, then $[g_1, g_2 \varphi] = [g_1, g_2 \varphi] = [g_2, g_1 \varphi]$.

(iii) If $A$ and $B$ are two subgroups of $G$ with $B \leq G'$, then $[A, B^\varphi] = [B, A^\varphi]$.

In particular, $[G, G^\varphi] = [G', G^\varphi]$.

(iv) $[G', Z(G)^\varphi] = 1$.

**Proof.** (i) By Lemma 1.1 (ii) it follows that both $[a, b^\varphi]$ and $[b, a^\varphi]$ commute with any element of $G$ and $G^\varphi$, so they are indeed central elements of $\nu(G)$.

(ii) By Lemma 1.1 (iii) the result holds if either $g_1$ or $g_2$ is a commutator. An
easy calculation shows that the result holds if \( g_1 \) or \( g_2 \) are arbitrary elements of \( G' \). (See also [18], Lemma 3.1 (iii)).

(iii) Assume that \( A = \langle a_i | i \in I \rangle \) and \( B = \langle b_j | j \in J \rangle \leq G' \). Then \( [A, B^\varphi] \) is generated by \( [a_i, b_j^\varphi] \) \((i \in I, j \in J)\), which by (ii) is equal to \( [b_j, a_i^\varphi]^{-1} \). Hence \( [A, B^\varphi] \leq [B, A^\varphi] \), and, by a symmetric argument, we have \( [A, B^\varphi] = [B, A^\varphi] \).

(iv) See [17], Lemma 2.7.

□

These results permit a description to be provided for the derived and the lower central series of \( G \otimes G \) in terms of those of \( G \).

**Proposition 1.3** Let \( G \) be any group. Then the following hold:

(i) For every \( n \geq 0 \), \( [G, G^\varphi]^{(n)} = [G'^{(n)}, (G'^{(n)})^\varphi] \).

(ii) For every \( n \geq 1 \), \( \gamma_{n+1}([G, G^\varphi]) = [\gamma_n(G'), G'^\varphi] = [G', \gamma_n(G')^\varphi] \).

**Proof.**

(i) We use induction on \( n \). The result being trivial for \( n = 0 \), assume \( n = 1 \). For \( g_i \in G \) \((i = 1, \ldots, 4)\) we have the following identity (which is stated without proof in Lemma 11 of [3]):

\[
(*) \ [g_1, g_2^\varphi, [g_3, g_4^\varphi]] = [g_1, g_2^\varphi]([g_3 g_4^\varphi][g_1, g_2^\varphi])^{-1}
\]
\[
= [g_1, g_2^\varphi]([g_3 g_4][g_1, g_2^\varphi])^{-1} \quad \text{by Lemma 1.1(i)},
\]
\[
= [g_1, g_2^\varphi, [g_3, g_4]^\varphi] \quad \text{by the defining properties of } \nu(G),
\]
\[
= [g_1, g_2, [g_3, g_4]^\varphi] \quad \text{by Lemma 1.1(ii)}.
\]

We note that both \( [G, G^\varphi]' \) and \( [G', G'^\varphi] \) are normal in \( \nu(G) \). By 5.1.7 of [16] we have that \( [G, G^\varphi]' = [G, G^\varphi, [G, G^\varphi]] \) is the normal closure in \( \nu(G) \) of the subgroup generated by the elements of the form

\[
[g_1, g_2^\varphi, [g_3, g_4^\varphi]],
\]

and \( [G', G'^\varphi] \) is the normal closure in \( \nu(G) \) of the subgroup generated by the elements of the form

\[
[g_1, g_2, [g_3, g_4]^\varphi].
\]

Therefore (*) shows that \( [G, G^\varphi]' = [G', (G')^\varphi] \).

We now assume that the result is true for \( n \) and we prove it for \( n + 1 \).
By the inductive hypothesis and the argument above applied to the group $G^{(n)}$, we have that
\[ [G, G^{\varphi}]^{(n+1)} = ( [G, G^{\varphi}]^{(n)} )' = ( [G^{(n)}, (G^{(n)})^{\varphi}] )' = [G^{(n+1)}, (G^{(n+1)})^{\varphi}] . \]

(ii) The case $n = 1$ has already been proved in (i). Thus we assume that the result is true for $n$ and we prove it for $n + 1$. By the inductive hypothesis, we have that
\[ \gamma_{n+2}([G, G^{\varphi}]) = \gamma_{n+1}([G, G^{\varphi}]), \]
\[ [G, G^{\varphi}] = [\gamma_{n}(G'), G'^{\varphi}, [G, G^{\varphi}]] . \]

By 5.1.7 of [16], $[\gamma_{n}(G'), G'^{\varphi}, [G, G^{\varphi}]]$ is the normal closure in $\nu(G)$ of the group generated by the elements of the form
\[ [g_1, g_2, [g_3, g_4]], \text{ with } g_1 \in \gamma_{n}(G'), g_2 \in G' \text{ and } g_3, g_4 \in G . \]

Similarly, $[\gamma_{n+1}(G'), G'^{\varphi}]$ is the normal closure in $\nu(G)$ of the group generated by the elements of the form
\[ [[g_1, g_2], [g_3, g_4]^{\varphi}], \text{ with } g_1 \in \gamma_{n}(G'), g_2 \in G' \text{ and } g_3, g_4 \in G . \]

By $(*)$, we have that $[g_1, g_2^{\varphi}, [g_3, g_4]^{\varphi}] = [[g_1, g_2], [g_3, g_4]^{\varphi}]$, so that $\gamma_{n+2}([G, G^{\varphi}]) = [\gamma_{n+1}(G'), G'^{\varphi}]$. This completes the induction. Finally, by Corollary 1.2 (iii), we have
\[ [[\gamma_{n+1}(G'), G'^{\varphi}]] = [G', \gamma_{n+1}(G')^{\varphi}] , \]
for all $n \geq 1$.

We stress the fact that Proposition 1.3 does not say that in general $(G \otimes G)^{(n)}$ and $G^{(n)} \otimes G^{(n)}$ are isomorphic groups. Consider, for example, the case $G = S_3$, where $G \otimes G$ is elementary abelian of order 4, while $A_3 \otimes A_3$ has order order 3). Indeed, the computation of $(G \otimes G)^{(n)}$ as $[G, G^{\varphi}]^{(n)} = [G^{(n)}, (G^{(n)})^{\varphi}]$ occurs within the group $\nu(G)$, whereas the calculation of $G^{(n)} \otimes G^{(n)}$ occurs as $[G^{(n)}, (G^{(n)})^{\varphi}]$ within the group $\nu(G^{(n)})$.

The following facts (given in [5]) are seen to be consequences of Proposition 1.3 and Lemma 1.2.

**Corollary 1.4** If $G$ is a solvable group of derived length $d$, then $G \otimes G$ is solvable of derived length $d - 1$ or $d$.

If $G$ is a nilpotent group of class $c$, then $G \otimes G$ is nilpotent of class $\leq \lfloor \frac{c+1}{2} \rfloor$. 

6
2 The structure of the non-abelian tensor square

In this section we prove some fundamental facts about the structure of the non-abelian tensor square of any group $G$ such that $G^{ab} = G/G'$ is finitely generated. We conjecture that our results remain true under the weaker assumption that $G/G'$ is a restricted direct product of cyclic groups.

For a finitely generated abelian group $A$, its non-abelian tensor square is simply the ordinary tensor product of two copies of $A$. In particular, if $A = \{a_1, \ldots, a_s\}$ is a set of generators of $A$ such that $A$ is the direct product of the cyclic groups $\langle a_i \rangle$, $i = 1, \ldots, s$, then we can write

$$A \otimes A = \nabla(A) \times E_A(A),$$

where

$$\nabla(A) = \langle [a_i, a_j^r], [a_i, a_j^r][a_j, a_i^r]|1 \leq i < j \leq s \rangle$$

and

$$E_A(A) = \langle [a_i, a_j^r]|1 \leq i < j \leq s \rangle.$$

We observe that $\nabla(A)$ is independent of the set of generators $A$ of $A$, since in fact $\nabla(A) = \langle [a, a^r]|a \in A \rangle$, while $E_A(A)$ is a complement of $\nabla(A)$ in $A \otimes A$ that does depend on the choice of $A$.

It turns out that for any group $G$ such that $G^{ab}$ is finitely generated (in particular, for any finitely generated group $G$), the structure of $\nabla(G)$ essentially depends on $G^{ab}$. The following Lemma, which improves Proposition 3.3 of Rocco [18], makes this observation precise.

Lemma 2.1 Let $G$ be a group such that $G^{ab}$ is finitely generated by the elements $\{x_iG^r|i=1,\ldots,s\}$. Set $E(G)$ to be $\langle [x_i, x_j^r]|i < j \rangle [G^r, G^r]$. Then the following hold:

(i) $\nabla(G)$ is generated by the elements of the set

$$\{[x_i, x_j^r], [x_i, x_j^r][x_j, x_i^r]|1 \leq i < j \leq s\};$$

(ii) $[G, G^r] = \nabla(G)E(G)$.

Proof.

(i) Let $Y = \{y_\alpha\}_{\alpha \in I}$ be a set of generators for $G^r$ and let $X = \{x_i\}_{i=1}^s$. Then
\( G = X \cup Y \) generates \( G \). By Lemma 17 in [3] (or Proposition 3.3 in [18]) \( \nabla(G) \) is generated by
\[
\{[a, a^\varphi], [a, b^\varphi]|b, a^\varphi]|a, b \in G \}.
\]

Note that \([a, a^\varphi] = 1\) if \( a \in Y \) (by Lemma 1.1(vi)) and similarly \([a, b^\varphi][b, a^\varphi] = 1\) if at least one among \( a \) and \( b \) lies in \( Y \) (Corollary 1.2 (ii)).

(ii) The proof follows by a direct expansion of the factors \([x_i g_1, (x_j g_2)^\varphi] \)
\((g_1, g_2 \in G')\). Alternatively, consider the map \( f : [G, G^\varphi] \rightarrow [G^{ab}, (G^{ab})^\varphi] \)
induced by the projection onto \( G^{ab} \). Then \( \text{Im } f = \langle \nabla(G) \langle [x_i, x_j^\varphi]|i < j \rangle \rangle \) and
\( \text{Ker } f = [G', G^\varphi] = [G, (G')^\varphi] \) (see [17], Remark 3), so \([G, G^\varphi] = \nabla(G)E(G)\).

We are now able to describe the structure of the non-abelian tensor square \( G \otimes G \) in terms of \( \nabla(G) \) and the non-abelian exterior square \( G \wedge G \). Our result generalizes Proposition 8 in [5] and Proposition 3.1 in [4].

**Proposition 2.2** Assume that \( G^{ab} \) is finitely generated. Then, with the notation of Lemma 2.1, the following hold.

(i) The map \( f_1 \) defined to be the restriction \( f|_{\nabla(G)} : \nabla(G) \rightarrow \nabla(G^{ab}) \) of
the projection onto \( G^{ab} \), has kernel \( N = E(G) \cap \nabla(G) \). Moreover, \( N \) is
a central elementary abelian \( 2 \)-subgroup of \([G, G^\varphi]\) of rank at most the
\( 2 \)-rank \( \text{rk}_2(G^{ab}) \) of \( G^{ab} \).

(ii) \([G, G^\varphi]/N \simeq \nabla(G^{ab}) \times (G \wedge G)\).

(iii) Suppose either that \( G^{ab} \) has no elements of order two or that \( G' \) has a
complement in \( G \). Then \( \nabla(G) \simeq \nabla(G^{ab}) \) and \( G \otimes G \simeq \nabla(G) \times (G \wedge G) \).

**Proof.**

(i) Let \( w \in \nabla(G) \cap E(G) \). Then
\[
f_1(w) = f(w) \in \nabla(G^{ab}) \cap E(G^{ab}) = 1,
\]
and so \( N \leq \text{Ker } (f_1) \). Conversely, \( \text{Ker } (f_1) = \text{Ker } (f) \cap \nabla(G) = (G' \otimes G) \cap \nabla(G) \leq N \).

It is obvious that \( N \) is a central subgroup of \( G \otimes G \), since \( N \) is contained in \( \nabla(G) \). In order to show that \( N \) is an elementary abelian \( 2 \)-group, we recall
that there is a sequence of epimorphisms between finitely generated abelian
groups
\[ \Gamma(G^{ab}) \xrightarrow{\psi} \nabla(G) \xrightarrow{f_1} \nabla(G^{ab}), \]
where \( \Gamma(G^{ab}) \) is the Whitehead functor on \( G \) (see [5]). In particular, if
\( N_2 = \text{Ker}(\psi f_1) \) and \( N_1 = \text{Ker}(\psi) \), then \( N \simeq N_2/N_1 \). From [19] (II. 7), we
recall some basic facts about the functor \( \Gamma \). First, if \( A \) is a finitely generated
abelian group such that \( A = \prod_i A_i \), then
\[ \Gamma(A) = \prod_i \Gamma(A_i) \times \prod_{i<j} (A_i \otimes A_j). \]
Moreover, \( \Gamma \) acts in the following way on cyclic groups:
\[ \Gamma(\mathbb{Z}) \simeq \mathbb{Z} \quad \text{and} \quad \Gamma(\mathbb{Z}_n) \simeq \begin{cases} \mathbb{Z}_n & \text{if } n \text{ is odd,} \\ \mathbb{Z}_{2n} & \text{if } n \text{ is even.} \end{cases} \]
If the 2-Sylow subgroup of \( G^{ab} \) is \( \prod_{i=1}^r \langle x_i G' \rangle \), it follows that \( N_2 \leq \prod_{i=1}^r \Gamma(\langle x_i G' \rangle) \)
and hence that \( N_2 \) is an elementary abelian 2-group whose rank is \( r = \text{rk}_2(G^{ab}) \). Since \( N \simeq N_2/N_1 \), the result follows.

(ii) By Lemma 2.1 and (i), we have
\[ \frac{G \otimes G}{N} \simeq \frac{\nabla(G)/N}{\nabla(G)^{ab} / N} \times \frac{E(G)}{N}. \]
Note that \( \nabla(G)/N \simeq \nabla(G^{ab}) \) and
\[ E(G)/N \simeq E(G)\nabla(G)/\nabla(G) = (G \otimes G)/\nabla(G) = G \wedge G. \]

(iii) If \( G^{ab} \) has no elements of order two, then 2 does not divide the order
of the torsion part of \( \Gamma(G^{ab}) \), and so \( \Gamma(G^{ab}) \simeq \nabla(G) \simeq \nabla(G^{ab}) \), forcing the
result.
Assume now that \( G' \) has a complement \( A \) in \( G \). If we write \( g \in G \) as \( g = xa \),
with \( x \in G' \) and \( a \in A \), by Lemma 3.1 (iv) in [18] we have that
\[ [g, g^a] = [a, a^x], \]
forcing
\[ \nabla(G) = \langle [g, g^\varphi] | g \in G \rangle = \langle [a, a^\varphi] | a \in A \rangle \simeq \nabla(G^{ab}), \]
and \( N = 1. \)

**Observation.** In the proof of Proposition 2.2 (i) if \( |x| = |x_i G'|, \) for \( i = 1, \ldots, r, \) then \( N_1 \) has rank \( r, \) so \( N \cong N_2/N_1 = 1, \) \( \nabla(G) \cong \nabla(G^{ab}) \) and \( G \otimes G \cong \nabla(G) \times (G \wedge G). \)

**Corollary 2.3** Let \( G \) be a group such that \( G^{ab} \) is a finitely generated abelian group with no elements of order two. Then \( J(G) \cong \Gamma(G^{ab}) \times M(G). \)

**Proof.** Note that \( J(G) \) is by definition the kernel of the commutator map \( \kappa : G \otimes G \to G'. \) In particular, \( J(G) \) is a central subgroup of \( G \) containing \( \nabla(G). \) By Proposition 2.2 we have that \( G \otimes G = \nabla(G) \times H, \) where \( H \) is a subgroup isomorphic to \( G \wedge G. \) Therefore, applying Dedekind’s modular law, we have

\[ J(G) = \nabla(G) \times (H \cap J(G)) \cong \Gamma(G^{ab}) \times M(G), \]

since \( \nabla(G) \cong \Gamma(G^{ab}) \) and \( J(G)/\nabla(G) \cong M(G). \)

We recall the notions of **non-abelian tensor center** \( Z^\otimes(G) \) and **non-abelian exterior center** \( Z^\wedge(G) \) of a group \( G. \) These groups are defined in \([8]\) as

\[
Z^\otimes(G) = \{ g \in G | [g, x^\varphi] = 1, \forall x \in G \}
\]
\[
Z^\wedge(G) = \{ g \in G | [g, x^\varphi] \in \nabla(G), \forall x \in G \}.
\]

As Ellis showed in \([8]\) and \([9]\), \( Z^\otimes(G) \) is a characteristic central subgroup of \( G \) and is the largest normal subgroup \( L \) of \( G \) such that \( G \otimes G \cong G/L \otimes G/L. \) The non-abelian exterior center \( Z^\wedge(G) \) is a central subgroup of \( G \) and is equal to the epicenter \( Z^*(G) \) of \( G. \) In particular, a group \( G \) is capable (that is, is isomorphic to a central quotient \( E/Z(E) \) for some group \( E \)) if and only if \( Z^\wedge(G) = 1. \)

**Corollary 2.4** Let \( G \) be any group such that \( G^{ab} \) is finitely generated. With the notation of Proposition 2.2 if \( N = 1 \) then \( Z^\otimes(G) = Z^\wedge(G) \cap G'. \) In particular, the conclusion holds if \( G \) is a finite group of odd order.
Proof. By the definition of exterior center we have that
\[ [Z^\wedge(G) \cap G', G^e] \leq N = 1. \]
Therefore \( Z^\wedge(G) \cap G' \leq Z^\circ(G) \). Conversely, we trivially have \( Z^\circ(G) \leq Z^\wedge(G) \). Moreover, \( Z^\circ(G) \leq G' \), as if \( x \in Z^\circ(G) \) then \( [G'x, (G'x)^e] \) should be the trivial element of the tensor product \( G^{ab} \otimes G^{ab} \), being the image of \( 1 = [x, x^e] \) under the natural map from \( G \otimes G \) to \( G^{ab} \otimes G^{ab} \). This of course forces \( G'x \) being the identity element of \( G^{ab} \), so \( x \in G' \). \( \square \)

Question 1 With the notation of Proposition 2.2, is it always true that \( N = [Z^\wedge(G) \cap G', G^e] \)?

Note that a positive answer to the previous question will imply, by Proposition 9 in [5], that
\[ \frac{G \otimes G}{N} \simeq \frac{G}{H} \otimes \frac{G}{H}, \]
where \( H \) is defined to be \( Z^\wedge(G) \cap G' \). This is precisely the case for generalized quaternion groups and semi-dihedral groups, as the reader may check by some easy calculations and the aid of [5], Proposition 13, and [8], Proposition 16.

3 Structure of the non-abelian exterior square

We will now describe the structure of the non-abelian exterior square \( G \wedge G \) of a group \( G \). Throughout this section we view the non-abelian tensor square \( G \otimes G \) as defined at the beginning of the paper, with generators \( g_1 \otimes g_2 \), rather than via the isomorphic subgroup \( [G, G^e] \) of \( \nu(G) \). We denote with \( g_1 \wedge g_2 \) the coset of \( G \wedge G \) containing \( g_1 \otimes g_2 \).

Let \( G \) be a group and let \( R \xrightarrow{i} F \xrightarrow{\pi} G \) be a presentation for \( G \), where \( F \) is a free group. Set \( F^\circ \) to be the quotient \( F/[F, R] \) and set \( R^\circ \) to be \( R/[F, R] \), so that
\[ 1 \longrightarrow R^\circ \xrightarrow{i} F^\circ \xrightarrow{\eta} G \longrightarrow 1 \]
is a central exact sequence. From the sequence (1) and by Proposition 7 in [3], there exists a homomorphism
\[ \xi : G \otimes G \longrightarrow (F^\circ)' \]
such that \( \eta \xi \) is the commutator map \( \kappa : G \otimes G \to G' \). In particular, \( \xi \) operates as follows on the generators \( g_1 \otimes g_2 \) of \( G \otimes G \):

\[
\xi(g_1 \otimes g_2) = [f_1, f_2][F, R],
\]

where \( f_1 \) and \( f_2 \) are any two preimages of \( g_1 \) and \( g_2 \) in \( F \), respectively. Of course, \( \xi \) is trivial on the central subgroup \( \nabla(G) \), and so it induces a homomorphism

\[
\overline{\xi} : G \wedge G \to (F^\circ)'\quad (2)
\]

The following Proposition is the main result of this section. The proof uses an argument similar to that of Theorem 2 in [14].

**Proposition 3.1** Let \( G \) be a group and let \( F \) be a free group such that \( G \simeq F/R \) for some normal subgroup \( R \) of \( F \). Then

\[
G \wedge G \simeq \frac{F'}{[F, R]}.
\]

**Proof.** We will show that the map \( \overline{\xi} \) defined in (2) is an isomorphism. The surjectivity of \( \overline{\xi} \) is immediate. Let \( \pi : F \to G \) be the projection with kernel \( R \). An arbitrary generator \( [f_1, f_2][F, R] \) of \( F'/[F, R] \) lies in the image of \( \overline{\xi} \), since \( \overline{\xi}(\pi(f_1) \wedge \pi(f_2)) = [f_1, f_2][F, R] \).

We now prove that \( \overline{\xi} \) is injective. Using the same notation as in the introduction, for any group \( X \) we set \( J(X) \) to be \( \ker(\kappa) \) and \( M(X) \) to be \( \ker(\kappa') \), where \( \kappa \) and \( \kappa' \) are the commutator maps \( \kappa : X \otimes X \to X' \) and \( \kappa' : X \wedge X \to X' \) respectively, so that \( J(X)/\nabla(X) \simeq M(X) \).

Let \( \phi \) be the map \( \overline{\xi} \) restricted to \( M(G) \). We want to show that \( \phi \) is injective. Note that \( \phi \) is a map

\[
\phi : M(G) \to R^\circ \cap (F^\circ)' = \frac{F' \cap R}{[F, R]}.
\]

Now the quotient map \( \eta : F^\circ \to G \) induces a homomorphism

\[
\eta_* : M(F^\circ) \to M(G)
\]

(which is the restriction to \( M(F^\circ) \) of the map sending \( f_1[F, R] \wedge f_2[F, R] \) to \( \eta(f_1) \wedge \eta(f_2) \)). It is easy to notice that the following is an exact sequence

\[
M(F^\circ) \xrightarrow{\eta_*} M(G) \xrightarrow{\partial} R^\circ \cap (F^\circ)'.
\]
Therefore, in order to show that $\phi$ is injective, we will prove that $\eta_*$ is the trivial map. If $\alpha : J(F^\circ) \longrightarrow M(F^\circ)$ is the quotient map, we show that $\eta_*(\alpha(J(F^\circ))) = \eta_*(\alpha(\nabla(F^\circ)))$.

Let $\overline{w} = \prod (\overline{x}_i \otimes \overline{y}_i) \in J(F^\circ)$, with $\overline{x}_i, \overline{y}_i \in F^\circ$. Then there exist $x_i, y_i \in F$ and $w = \prod (x_i \otimes y_i) \in F \otimes F$ such that $\lambda_\otimes(w) = \overline{w}$ and $\lambda(\kappa(w)) = \kappa(\overline{w}) = 1$ (here $\lambda_\otimes : F \otimes F \longrightarrow F^\circ \otimes F^\circ$ is the map induced by the projection $\lambda : F \longrightarrow F^\circ$).

Thus $\kappa(w) = \prod [x_i, y_i] \in \text{Ker}(\lambda) = [F, R]$. As $F$ is a free group its Schur multiplier is trivial, so $M(F) = 1$, that is, $J(F) = \nabla(F)$. In particular, modulo $\nabla(F)$, the product $w$ is equivalent to $\prod (f_j \otimes r_j)$, for some $f_j \in F$ and $r_j \in R$. So $w = \prod (f_j \otimes r_j)z$, for some $z \in \nabla(F)$. We note that $(\alpha(\lambda_\otimes(z))) = 1$. Now

\[
\eta_*(\alpha(\overline{w})) = \eta_*(\alpha(\lambda(z))) = \eta_*(\alpha(\lambda_\otimes(\prod (f_j \otimes r_j)z)))
\]

\[
= \prod (\eta_*(\alpha(\overline{f}_j \otimes 1))) = 1,
\]

therefore $\eta_*$ is the trivial map and $\phi$ is injective.

Finally in order to show that $\overline{\xi}$ is an isomorphism between $G \wedge G$ and $F'/[F, R]$, we apply the Short Five Lemma ([1], Proposition 2.10) to the following commutative diagram.

\[
\begin{array}{ccc}
1 & \longrightarrow & M(G) & \xrightarrow{i} & G \wedge G & \xrightarrow{\kappa'} & G' & \longrightarrow & 1 \\
\downarrow{\phi} & & \downarrow{\overline{\xi}} & & \downarrow{1_{G'}} \ & & \ & & \\
1 & \longrightarrow & \frac{R\cap F'}{[F, R]} & \xrightarrow{i} & \frac{F'}{[F, R]} & \xrightarrow{\eta} & G & \longrightarrow & 1
\end{array}
\]

As both $\phi$ and the identity map of $G'$ are injective, it follows that also $\overline{\xi}$ is injective. This concludes the proof that $\overline{\xi}$ is an isomorphism.

As consequences of the results above we now describe the structures of the non-abelian tensor squares of some groups that are “universal” in the sense that they are free in suitable varieties.

**Corollary 3.2 ([5], Proposition 6)** Let $F_n$ be a free group of rank $n$. Then

\[F_n \otimes F_n \simeq \mathbb{Z}^{n(n+1)/2} \times (F_n)'.\]

**Proof.** Since $F_n^{ab}$ is a free abelian group of rank $n$, by Proposition 2.2 we have $\nabla(F_n) \simeq \nabla(F_n^{ab}) \simeq \mathbb{Z}^{n(n+1)/2}$ and $F_n \otimes F_n \simeq \nabla(F_n^{ab}) \times F_n \wedge F_n$. Finally Proposition 3.1 (with $F = F_n$ and $R = 1$) gives $F_n \wedge F_n \simeq (F_n)'$. □
Proposition 3.1 gives a nice description in the case of free nilpotent groups of finite rank. For a more detailed description of this case we refer to the paper [4].

Corollary 3.3 ([4], Corollary 1.7) Let $G = N_{n,c}$ be the free nilpotent group of rank $n > 1$ and class $c \geq 1$. Then

$$G \otimes G \simeq \mathbb{Z}^{n(n+1)/2} \times (N_{n,c+1})'$$. 

Proof. As before, $G^{ab}$ is a free abelian group of rank $n$, and so we have $\nabla(G) \simeq \nabla(G^{ab}) \times \mathbb{Z}^{n(n+1)/2}$ and $G \otimes G \simeq \nabla(G^{ab}) \times G \wedge G$. Finally apply Proposition 3.1 with $F$ free group of rank $n$ and $R = \gamma_{c+1}(F)$. \hfill \Box

Corollary 3.4 Let $F$ be the free group of finite rank $n > 1$, let $d$ be a natural number, and let $G = F/F^{(d)}$ be the free solvable group $S_{n,d}$ of derived length $d$ and rank $n > 1$. Then

$$G \otimes G \simeq \mathbb{Z}^{n(n+1)/2} \times F'/[F, F^{(d)}]$$

is an extension of a nilpotent group of class $\leq 3$ by a free solvable group of derived length $d - 2$ and infinite rank. In particular, if $d = 2$, then $G \otimes G$ is a nilpotent group.

Proof. Once again, Propositions 2.2 and 3.1 imply that $G \otimes G$ has the described factorization. Note that $F^{(d-1)}/[F, F^{(d)}]$ is a normal subgroup of the group $F'/[F, F^{(d)}]$ and that $F^{(d-1)}/[F, F^{(d)}]$ is nilpotent of class at most 3, as it is a quotient of $F^{(d-1)}/\gamma_3(F^{(d-1)})$. So $M = \mathbb{Z}^{n(n+1)/2} \times F^{(d-1)}/[F, F^{(d)}]$ is also nilpotent of class at most 3 and $G \otimes G/M$ is isomorphic to $F'/F^{(d-1)}$, so it is free solvable of derived length $d - 2$. The fact that $F'/F^{(d-1)}$ is of infinite rank follows from the well-known fact that $F'$ is not finitely generated. \hfill \Box

We recall, in view of Theorem A in [2], that the Schur multiplier of $S_{n,d}$ is not finitely generated. In particular, $S_{n,d} \wedge S_{n,d}$ can also be viewed as an extension of a central abelian group of infinite rank by a free solvable group of derived length $d - 1$.

We end this section by applying our results to a particular finite $p$-group. Let $d$ be an integer and, as before, denote by $F_d$ the free group on $d$ generators. We recall that for every integer $i$ the group $\gamma_i(F_d)/\gamma_{i+1}(F_d)$ is free.
abelian of rank

\[ m_d(i) := \frac{1}{i} \sum_{t \mid i} \mu(t)d^{i/t}, \]

where \( \mu \) is the Mobius function (see [13] Chapter 3).

We also recall for a fixed prime number \( p \) the notion of lower central \( p \)-series of a group \( G \). The terms of this series are \( \{ \lambda_i(G) \}_{i \geq 1} \), where

\[ \lambda_1(G) = G \]
\[ \lambda_{k+1}(G) = [\lambda_k(G), G]\lambda_k(G)^p, \quad \text{for } k \geq 1. \]

We note that this series is the most rapidly descending central series of \( G \) whose factors have exponent \( p \) (see [13], Chapter 3). The lower central \( p \)-series will be used in the next section to find some bounds on the orders of the non-abelian tensor and exterior squares of finite \( p \)-groups. Now we exhibit an explicit calculation of these objects in a particular case.

For every pair of positive integers \( d \) and \( c \) define \( G_{d,c} \) to be the quotient \( F_d/\lambda_{c+1}(F_d) \). According to [13] (Theorem 3.2.10), \( G_{d,c} \) is a finite \( p \)-group of class \( c \) and order \( p^m \), where \( m = \sum_{j=1}^{c} (c + 1 - j)m_d(j) \).

**Corollary 3.5** With the above notation, we have that \( G_{d,c} \wedge G_{d,c} \simeq (G_{d,c+1})' \) and

\[ G_{d,c} \otimes G_{d,c} \simeq (\mathbb{Z}_{p^c})^{(d+1)/2} \times (G_{d,c+1})'. \]

**Proof.** Let \( G := G_{d,c} \). We first prove that

\[ G \otimes G \simeq \nabla(G) \times (G \land G). \quad (3) \]

For \( p \) odd (3) follows from Proposition 2.2, while for the case \( p = 2 \) a little more care is needed. More precisely, we observe that if \( F_d = \langle f_1, \ldots, f_d \rangle \), then the image \( x_i \) in \( G = F_d/\lambda_{c+1}(F_d) \) of the generator \( f_i \) of \( F_d \) has order \( p^c \) for each \( i = 1, \ldots, d \). Moreover, by Theorem 3.2.10 in [13], \( G^{ab} \) is isomorphic to a direct product of \( d = m_d(1) \) cyclic groups \( \mathbb{Z}_{p^c} \) of order \( p^c \). So now our result follows from the observation following Proposition 2.2.

We have that \( \nabla(G) \simeq (\mathbb{Z}_{p^c})^{(d+1)/2} \). We claim that the derived subgroup of a covering group for \( G \) is isomorphic to \( (G_{d,c+1})' \). In the following, let \( L_i \) denote \( \lambda_i(F_d), i \geq 1 \). We note that the group \( G_{d,c+1} = F_d/L_{c+2} \) has \( L_{c+1}/L_{c+2} \) as a central elementary abelian subgroup. Moreover, the subgroup

\[ \frac{M}{L_{c+2}} \text{ defined to be } (G_{d,c+1})' \cap \frac{L_{c+1}}{L_{c+2}} = \frac{\gamma_2(F_d)L_{c+2} \cap L_{c+1}}{L_{c+2}} \]
is isomorphic to
\[ \frac{L_{c+1} \cap \gamma_2(F_d)}{L_{c+2} \cap \gamma_2(F_d)}, \]
which is isomorphic to \( M(G) \), by Theorem 3.2.10 in [13]. Now let \( H/L_{c+2} \) be a complement of \( M/L_{c+2} \) in \( L_{c+1}/L_{c+2} \) and consider the factor group
\[ \overline{G}_{d,c+1} = \frac{G_{d,c+1} \cap H}{L_{c+2}}. \]
If \( N \leq G_{d,c+1} \) we denote with \( \overline{N} \) the image of \( N \) in \( \overline{G}_{d,c+1} \) under the canonical projection; it follows that
\[
M(G) \simeq \overline{M} \leq Z(\overline{G}_{d,c+1}) \cap (\overline{G}_{d,c+1})'.
\]
Moreover, \( \overline{G}_{d,c+1} / \overline{M} \simeq F_d/L_{c+1} = G \), so \( \overline{G}_{d,c+1} \) is a covering group for \( G \). Finally, note that
\[
(\overline{G}_{d,c+1})' = \frac{(F_d)' H}{H} \simeq \frac{(F_d)' \cap H}{(F_d)' \cap L_{c+2}} = (G_{d,c+1})'.
\]
\[\square\]

4 Non-abelian tensor squares of finite \( p \)-groups

Throughout this section \( G \) is a finite \( p \)-group, for some prime \( p \). We start with a lemma concerning the lower central \( p \)-series of \( G \). We again identify the group \( G \otimes G \) with its isomorphic image \( [G, G^p] \) in the group \( \nu(G) \) defined in Section 2.

Lemma 4.1 Let \( G \) be a finite \( p \)-group. Then for every \( k \geq 1 \),
\[
[\lambda_k(G), G^p] = [G, (\lambda_k(G))^p].
\]

Proof. We prove the result by induction on \( k \). Since the result is trivial for \( k = 1 \), we assume \([\lambda_k(G), G^p] = [G, (\lambda_k(G))^p]\) and show that \([\lambda_{k+1}(G), G^p] = [G, (\lambda_{k+1}(G))^p]\].

First note that, since \([\lambda_k(G), G, G^p] \) and \([\lambda_k(G)^p, G^p]\) are both normal in
\(\nu(G)\), we have, using \([xy, a^\varphi] = x [y, a^\varphi] x, a^\varphi\) with \(x \in [\lambda_k(G), G]\), \(y \in \lambda_k(G)^p\) and \(a \in G\), that

\[ [\lambda_{k+1}(G), G^\varphi] = [\lambda_k(G), G, G^\varphi][\lambda_k(G)^p, G^\varphi]. \]

(As \([xy, a^\varphi] = x [y, a^\varphi] x, a^\varphi\), with \(x \in [\lambda_k(G), G]\), \(y \in \lambda_k(G)^p\) and \(a \in G\).

Using Lemma 1.1(ii), we have that

\[ [\lambda_k(G), G, G^\varphi] = [\lambda_k(G)^p, G^\varphi] \subseteq [G, [\lambda_k(G), G]^{\varphi}] \]

Thus our proof will be complete if we show that \(\lambda_k(G)^{p, G^\varphi} = \lambda_k(G)^{p, G^\varphi} \subseteq [G, \lambda_k(G), G]^{\varphi}\).

Define \(R\) to be \([\lambda_k(G), G, G^\varphi] = [G, [\lambda_k(G), G]^{\varphi}]\).

Note that \(R\) contains the derived subgroup of \([\lambda_k(G), G^\varphi]\). To see this, we observe that \([\lambda_k(G), G^\varphi]^p\) is generated by the elements

\[ [[x, a^\varphi], [y, b^\varphi]], \quad \text{where} \ x, y \in \lambda_k(G) \text{ and } a^\varphi, b^\varphi \in G^\varphi, \]

and, by Lemma 1.1(i) and the defining properties of \(\nu(G)\), we have that

\[ [[x, a^\varphi], [y, b^\varphi]] = [[x, a], [y, b]] \in R. \]

We claim that the following hold:

\[ [x^m, a^\varphi] \in [x, a^\varphi]^m R \quad \text{for all} \ x \in \lambda_k(G), a^\varphi \in G^\varphi, m \in \mathbb{N} \tag{4} \]

\[ [y, (b^m)^\varphi] \in [y, b^\varphi]^m R \quad \text{for all} \ y \in G, b^\varphi \in (\lambda_k(G))^\varphi, m \in \mathbb{N}. \tag{5} \]

We prove (4) by induction on \(m\). The proof of (5) is similar.

If \(m = 1\) then (4) is trivially true. Let \(m \geq 2\). Then

\[ [x^m, a^\varphi] = [x \cdot x^{m-1}, a^\varphi] = x [x^{m-1}, a^\varphi] [x, a^\varphi] = [x^{m-1}, (x a)^\varphi] [x, a^\varphi]. \]

Now the claim is proved since, by induction on \(m\), the term \([x^{m-1}, (x a)^\varphi]\) lies in the coset

\[ [x, (x a)^\varphi]^{m-1} R = [x, [x, a]^{\varphi} a^\varphi]^{m-1} R = ([x, [x, a]^{\varphi}] \cdot [x, a]^{\varphi} [x, a^\varphi])^{m-1} R = ([x, [x, a]^{\varphi}] \cdot [x, a^\varphi])^{m-1} R = ([x, a^\varphi])^{m-1} R, \]

by a repeated use of Lemma 1.1 and the fact that \([x, [x, a]^{\varphi}] = [x, a, x^\varphi]^{-1} \in R\). Therefore our claims (4) and (5) are true, and we now complete the
proof of the lemma as follows. We have that \([\lambda_k(G)^p, G^p]\) is generated by elements of the form \([x^p, a^p]\) with \(x \in \lambda_k(G)\) and \(a^p \in G^p\). By (4), \([x^p, a^p] \in ([x, a^p])^p R\). Now \([x, a^p] \in [\lambda_k(G), G^p] = [G, (\lambda_k(G))^p]\) by the inductive hypothesis, so we may write

\[ [x, a^p] = w_1 \cdot \ldots \cdot w_l, \]

where \(w_i = [y_i, b_i^p]\), \(y_i \in G\) and \(b_i^p \in \lambda_k(G)^p\) for \(i = 1, \ldots, l\). In particular, since \([\lambda_k(G), G^p]/R\) is abelian we have that \([(x, a^p)]^p R = w_1^p \ldots w_l^p R\). Finally, by (5) \(w_i^p R = [y_i, (b_i^p)] R\) for all \(i = 1, \ldots, l\), forcing

\[ [x^p, a^p] \in R[G, (\lambda_k(G)^p)^p] = [G, (\lambda_{k+1}(G))^p]. \]

The following result is an improvement of Corollary 3.12 in [17]. In his PhD thesis A. McDermott proves this result using arguments different from ours (see [11]).

**Proposition 4.2** Let \(G\) be a finite group of order \(p^n\) (\(p\) a prime) and let \(d = d(G)\) be the minimum number of generators of \(G\). Then \(p^d \leq |[G, G^p]| \leq p^{nd}\).

**Proof.** Of course \(|[G, G^p]| \geq p^d\), as \(G \otimes G\) admits \(G/\Phi(G) \otimes G/\Phi(G)\) as a quotient, and \(G/\Phi(G) \otimes G/\Phi(G)\) is elementary abelian of order \(p^d\), since it is an ordinary tensor product.

Let \(\lambda_k(G)\) be the last non-trivial term of the series \(\{\lambda_i(G)\}_i\), and let \(\pi : G \longrightarrow \overline{G} = G/\lambda_k(G)\) be the quotient map. \(\pi\) induces a natural epimorphism, say, \(\overline{\pi} : [G, G^p] \longrightarrow [\overline{G}, \overline{G}^p]\). According to [17, Remark 3] and using the previous Lemma, the kernel of \(\overline{\pi}\) consists in the subgroup

\[ \text{Ker}(\overline{\pi}) = [\lambda_k(G), G^p][G, \lambda_k(G)^p] = [\lambda_k(G), G^p]. \]

Since \(\lambda_k(G)\) is a central elementary abelian subgroup of \(G\), by Lemma [1.1(ii)], we have that \(\text{Ker}(\overline{\pi})\) is an elementary abelian \(p\)-subgroup lying in the center of \(\nu(G)\). Thus the map

\[ \theta : \lambda_k(G) \times G \longrightarrow [\lambda_k(G), G^p] \\
(a, g) \longmapsto [a, g^p], \]

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is bilinear. Let $\lambda_k(G)$ be generated by the set $\{a_i|i=1,\ldots,d_k\}$ and let $G$ be generated by $\{g_i|i=1,\ldots,d\}$. Therefore $\text{Ker}(\bar{\pi})$ is generated by the set

$$\{[a_i, g_j^\varphi]|i=1,\ldots,d_k, j=1,\ldots,d\},$$

forcing $|\text{Ker}(\bar{\pi})| \leq p^{d-d_k}$, and $|[G, G^\varphi]| \leq p^{d-d_k} |[\overline{G}, G^\varphi]|$. By induction we obtain that $|[G, G^\varphi]| \leq p^{d-d_k} \cdots p^{d^2} = p^d \sum_{i=1}^{d_k} d_i = p^{nd}$.

\[\blacksquare\]

**Remark 1** Homocyclic abelian groups show that the upper bound in the Proposition 4.2 is best possible. Another example in which the upper bound is reached is when $G$ is the group $G_{2,2} := F_2/\lambda_3(F_2)$.

As a consequence of our results we have the following bound for the order of the Schur multiplicator of finite $p$-groups.

**Corollary 4.3** Let $G$ be a finite $p$-group of order $p^n$ with $d = d(G)$ generators. If $p$ is odd, the order of the Schur multiplicator $M(G)$ of $G$ is at most $p^{d(n-(d+1)/2)}$. If $p = 2$, then $|M(G)| \leq 2^{d(n-(d+3)/2)}$.

**Proof.** By Proposition 3.1 and the definition of the exterior square

$$|M(G)| |G'| = |G \wedge G| = \frac{|G \otimes G|}{|\nabla(G)|}.$$

If $p$ is odd, by Proposition 2.2, $\nabla(G) \simeq \nabla(G^{ab})$, and so $|\nabla(G)| \geq p^{d(d+1)/2}$. If $p = 2$, then $|\nabla(G)| \geq p^{d(d+3)/2}$. The proof is now completed by using the bounds given in Proposition 4.2. \[\blacksquare\]

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