The width difference of rho vector mesons

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We compute the difference in decay widths of charged and neutral $\rho(770)$ vector mesons. The isospin breaking arising from mass differences of neutral and charged $\pi$ and $\rho$ mesons, radiative corrections to $\rho \to \pi\pi$, and the $\rho \to \pi\pi\gamma$ decays are taken into account. It is found that the width difference $\Delta\Gamma_\rho$ is very sensitive to the isospin breaking in the $\rho$ meson mass $\Delta m_\rho$. This result can be useful to test the correlations observed between the values of these parameters extracted from experimental data.

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I. INTRODUCTION.

Lowest-lying vector mesons undergo predominantly strong interaction decays. The masses and decay widths of members of the same isomultiplet will therefore look very similar \[1\], with small differences induced by the breaking of isospin symmetry. The isospin breaking effects in the $\rho$ meson parameters have raised an interest recently, due to both experimental and theoretical reasons \[2,3,4,5,6\]. According to the PDG \[1\] the weighted averages of available measurements are:

$$\Delta m_\rho \equiv m_{\rho^0} - m_{\rho^\pm} = (-0.7 \pm 0.8) \text{ MeV},$$

$$\Delta\Gamma_\rho \equiv \Gamma_{\rho^0} - \Gamma_{\rho^\pm} = (0.3 \pm 1.3) \text{ MeV}.$$

These results are consistent with the absence of isospin breaking in the $\rho^0 - \rho^\pm$ system. Note however that the scale factors associated with the above averages are, respectively, 1.5 and 1.4 \[1\], which reflects an important spread in the yields from different experiments.

Some recent theoretical calculations of $\Delta m_\rho$ seem to confirm the above result. Using a
vector-meson dominance model to parametrize the $\gamma\rho\rho$ vertex, the authors of Ref. [4] have obtained $\Delta m_{\rho} = (-0.02 \pm 0.02)$ MeV. Also, using $1/N_c$ expansion techniques, the authors of Ref. [3] have obtained $-0.4 \text{ MeV} \leq \Delta m_{\rho} \leq 0.7 \text{ MeV}$. On another hand, it has been found that the width difference of $\rho$ mesons is of great importance to understand the current discrepancy between the hadronic vacuum polarization contributions to the muon anomalous magnetic moment obtained from $\tau$ decay and $e^+e^-$ annihilation data [2, 5, 6].

In this paper we provide an estimate of $\Delta \Gamma_{\rho}$ by considering the isospin breaking corrections in the exclusive modes that contribute to the decay widths of $\rho^{0,\pm}$ vector mesons. A previous estimate of this effect was done in reference [2] taking into account several sources of isospin breaking as mass differences and other subleading $\rho$ meson decays. Their result $\Delta \Gamma_{\rho} \approx (-0.42 \pm 0.59)$ MeV [2] is consistent with the world average given in Eq. (2). Additional contributions to isospin breaking in $\Delta \Gamma_{\rho}$, including the radiative corrections to the dominant $\rho \to \pi\pi$ decays, are considered in this paper.

II. SOURCES OF ISOSPIN BREAKING

At a fundamental level, isospin symmetry is broken by the different masses of $u$ and $d$ quarks and by the effects of electromagnetic interactions. At the hadronic level all manifestations of isospin breaking can be traced back to such fundamental sources. In the absence of isospin breaking, the $\rho^{0,\pm}$ mesons must have equal masses and decay widths, thus $\Delta m_{\rho} = \Delta \Gamma_{\rho} = 0$.

The dominant decay modes of $\rho$ mesons that are common to charged and neutral $\rho$’s are the $\pi\pi$ decay and its radiative mode. The branching fraction of other modes contributing only to the $\rho^0$ meson adds up to [1]:

$$B_{\text{rest}}^0 = B(\pi^0\pi^0\gamma) + B(\eta\gamma) + B(\mu^+\mu^-) + B(e^+e^-) + B(\pi^+\pi^-\pi^0) \approx 5.3 \times 10^{-4}.$$  \hspace{1cm} (3)

There is also a dipole transition $\rho \to \pi\gamma$ which is common to $\rho^{\pm,0}$ vector mesons with branching fractions of a few times $10^{-4}$ [1]. Since the $\rho$ meson widths are of order 150 MeV, all these subleading decay modes will contribute to the width difference at the tiny level of:

$$\Delta \Gamma_{\rho}^{\text{sub}} \approx 0.08 \text{ MeV}.$$  \hspace{1cm} (4)
Thus, any sizable difference in the decay widths can only originate from the dominant decay modes. To be more precise, we will define explicitly the contributions to the width difference as follows:

\[
\Delta \Gamma_\rho = \Gamma(\rho^0 \rightarrow \pi^+\pi^- (\gamma), \omega \leq \omega_0) - \Gamma(\rho^+ \rightarrow \pi^+\pi^0 (\gamma), \omega \leq \omega_0) \\
+ \Gamma(\rho^0 \rightarrow \pi^+\pi^- (\gamma), \omega \geq \omega_0) - \Gamma(\rho^+ \rightarrow \pi^+\pi^0 (\gamma), \omega \geq \omega_0) + \Delta \Gamma^{\text{sub}}_\rho. \tag{5}
\]

The first two terms in Eq. (5) denote the \( \pi\pi \) decay rates that include virtual and soft-photon corrections and the next two terms are related to hard-photon in \( \pi\pi\gamma \) decays. In the above equation, \( \omega_0 \) is an arbitrary value of the photon energy that separates the decay rates of soft- and hard-photon bremsstrahlung. It is expected that the \( \omega_0 \) dependence will cancel in the sum of the first (second) and third (fourth) terms of Eq. (5).

In the following section we will consider in more detail the isospin breaking corrections to \( \rho \rightarrow \pi\pi \) and their radiative decays. Later, we will evaluate the contributions of such corrections to the r.h.s. of Eq. (5).

### III. Radiative Corrections to \( \rho \rightarrow \pi\pi \) Decays

At the lowest order (indicated with superscript 0), the rates of \( \rho \rightarrow \pi\pi \) decays are given by:

\[
\Gamma_0^0 \equiv \Gamma^0(\rho^0 \rightarrow \pi^+\pi^-) = \frac{g_+^2}{48\pi}m_{\rho^0}v_0^3, \tag{6}
\]
\[
\Gamma_+^0 \equiv \Gamma^0(\rho^+ \rightarrow \pi^+\pi^0) = \frac{g_+^2}{48\pi}m_{\rho^+}v_+^3, \tag{7}
\]

where

\[
v_0 \equiv \sqrt{1 - \frac{4m_{\pi^+}^2}{m_{\rho^0}^2}}, \quad v_+ \equiv \sqrt{\left(1 - \frac{(m_{\pi^+} - m_{\pi^0})^2}{m_{\rho^+}^2}\right)\left(1 - \frac{(m_{\pi^+} + m_{\pi^0})^2}{m_{\rho^+}^2}\right)}, \tag{8}
\]

are the pion velocities in the rest frame of the \( \rho^i \) meson and \( g_{ij} \) are the \( \rho^i\pi^j \) coupling constants such that \( g_{+-} = g_{+0} \) owing to the isospin symmetry of strong interactions. In the limit of isospin symmetry, the masses of neutral and charged pions (\( \rho^i \)’s) are the same, \( v_0 = v_+ \), and consequently the three-level decay rates are equal (\( \Gamma_0^0 = \Gamma_+^0 \)). In the following subsections we will consider the different ingredients to get the \( O(\alpha) \) radiative corrections to the decay rates given above.
FIG. 1: Feynman graphs for $\rho^0 \rightarrow \pi^+\pi^-\gamma$ decays.

A. Real soft-photon corrections

In order to get an infrared safe result, the radiative corrections of order $\alpha$ to Eqs. (6,7) must include the sum of virtual corrections and soft-photon bremsstrahlung. In order to define the soft-photon contribution, let us consider the radiative $\rho \rightarrow \pi\pi\gamma$ decays shown in Figures 1 and 2. In each of Figures 1 and 2, the first three diagrams correspond to the model-independent contributions and the other graphs denote the model-dependent terms.

The decay amplitude of the process $\rho^0 (d, \eta) \rightarrow \pi^+(p)\pi^- (p')\gamma (k, \epsilon^*)$ (four-momenta and polarization four-vectors are indicated within parenthesis), see Figure 1, is given by

$$\mathcal{M}(\rho^0 \rightarrow \pi^+\pi^-\gamma) = i e g_\pi \left\{(p - p') \cdot \eta \left(p' \cdot \epsilon^* - \frac{p \cdot \epsilon^*}{p' \cdot \epsilon^*} \right) + 2 \left[\epsilon^* \cdot \eta - \frac{p \cdot \epsilon^* k \cdot \eta}{p \cdot k} \right] \right\} + \mathcal{M}_{d,e,f} (9)$$

where the model-independent pieces of the amplitude are given explicitly, and the remaining terms $\mathcal{M}_{d,e,f}$ correspond to the model-dependent amplitudes arising from Figures 1e)-1f). Similarly, the decay amplitude for the decay $\rho^+ (d, \eta) \rightarrow \pi^+(p)\pi^0 (p')\gamma (k, \epsilon^*)$, see Figure 2, is given
by the following expression \[7, 9, 10, 11\]:

\[
\mathcal{M}(\rho^+ \to \pi^+\pi^0\gamma) = ieg_0 \left\{ \left( \frac{p.e^*}{p.k} - \frac{d.e^*}{d.k} \right) (p - p') \cdot \eta + \left( \frac{p.e^*}{p.k} - \frac{d.e^*}{d.k} \right) k \cdot \eta \right.
\]

\[
+ \left[ 2 + \frac{\Delta\kappa}{2} \left( 1 + \frac{\Delta^2}{m^2_{\rho^+}} \right) \right] \left( \frac{d \cdot e^*}{d \cdot k} k \cdot \eta - e^* \cdot \eta \right)
\]

\[
- (2 + \Delta\kappa) \left( \frac{p \cdot e^*}{p \cdot k} k \cdot \eta - e^* \cdot \eta \right) \left( \frac{p \cdot k}{d \cdot k} \right) \right\} + \mathcal{M}_{d,e}, \quad (10)
\]

where $\Delta\kappa$ denotes the anomalous magnetic dipole moment of the $\rho^+$ vector meson in units of $e/2m_{\rho^+}$ (in our numerical evaluations we will set $\Delta\kappa = 0$), and we have defined $\Delta^2 \equiv m^2_{\pi^+} - m^2_{\pi^0}$. Once again, only the model-independent amplitudes for Figures 2a)-2c) have been written explicitly.

The soft-photon bremsstrahlung is the divergent piece of the amplitudes \[9,10\] in the infrared region ($k \to 0$). The diagrams contributing to the soft-photon amplitude are shown in Figs. 1a)-1b) for $\rho^0 \to \pi^+\pi^-\gamma$ and in Figs. 2a)-2b) for $\rho^+ \to \pi^+\pi^0\gamma$ decays. As usual, only photons of energy smaller than $\omega_0$ must be considered in the real-photon emission rate that is included.
in radiative corrections. Thus, the radiative corrected rates of $\rho^i \rightarrow \pi\pi$ decays ($i = 0, +$) can be written in the following general form:

$$
\Gamma(\rho^i \rightarrow \pi\pi(\gamma), \omega \leq \omega_0) = \Gamma^0_i + \Gamma^{1v}_i + \Gamma^{\text{soft}}_i(\omega_0) \\
= \Gamma^0_i \left[ 1 + \frac{\Gamma^{1v}_i}{\Gamma^0_i} + \frac{\Gamma^{\text{soft}}_i(\omega_0)}{\Gamma^0_i} \right] \\
= \Gamma^0_i \left[ 1 + \delta_{\rho^i} \right]. \quad (11)
$$

where $\Gamma^{1v}_i$ denotes the virtual corrected rate at order $\alpha$ for $\rho^i \rightarrow \pi\pi$ decay and $\Gamma^{\text{soft}}_i(\omega_0)$ is the soft-photon rate of $\rho^i \rightarrow \pi\pi\gamma$ obtained from the first term in Eqs. (9, 10).

The soft-photon rates can be computed in analytical form by integrating over photon energies smaller than $\omega_0$, with the following results, respectively for $\rho^0$ and $\rho^+$ radiative decays:

$$
\frac{\Gamma^{\text{soft}}_0(\omega_0)}{\Gamma^0_0} = \frac{\alpha}{\pi} \left\{ 2 \ln \left( \frac{\lambda}{2\omega_0} \right) \left( 1 + \frac{1 + v_0^2}{2v_0} \ln \left[ \frac{1 - v_0}{1 + v_0} \right] \right) - \frac{1}{v_0} \ln \left( \frac{1 - v_0}{1 + v_0} \right) \\
+ \frac{1 + v_0^2}{2v_0} \left[ \text{Li}_2 \left( \frac{1 - v_0}{1 + v_0} \right) - \text{Li}_2 \left( \frac{1 + v_0}{1 - v_0} \right) + i\pi \ln \left( \frac{1 - v_0}{1 + v_0} \right) \right] \\
+ \ln \left( \frac{1 - v_0}{1 + v_0} \right) \ln \left( \frac{4v_0^2}{1 - v_0^2} \right) \right\}, \quad (12)
$$

and

$$
\frac{\Gamma^{\text{soft}}_+}{\Gamma^0_+} = \frac{\alpha}{\pi} \left\{ 1 - 2\ln 2 - 2 \ln \left[ \frac{\lambda}{\omega_0} \right] \left( 1 + \frac{1}{2v_+^2} \ln \left[ \frac{1 - v_+^2}{1 + v_+^2} \right] \right) \\
+ \frac{1}{2v_+^2} \ln \left( \frac{1 + v_+^2}{1 - v_+^2} \right) - \frac{1}{2v_+^2} \left[ \text{Li}_2 \left( \frac{1 + v_+^2}{1 - v_+^2} \right) - \text{Li}_2 \left( \frac{1 - v_+^2}{1 + v_+^2} \right) \right] \\
+ \ln \left( \frac{v_+^2}{1 - v_+^2} \right) \ln \left( \frac{1 + v_+^2}{1 - v_+^2} \right) + i\pi \ln \left( \frac{1 + v_+^2}{1 - v_+^2} \right) \right\}, \quad (13)
$$

where we have defined $v_+ = v_+/\xi$, with $\xi = 1 + \Delta^2 / m^2_{\rho^+}$, and $\text{Li}(x)$ denotes the Spence function.

Observe that the soft-photon rates depend logarithmically upon the infrared cutoff $\lambda$ (the photon mass regulator) and the photon energy parameter $\omega_0$. We have checked that Eq. (12) coincides numerically with the result reported long ago by Cremmer and Gourdin \[12\] in the case of $\phi \rightarrow K^+K^-\gamma$ decays, although our corresponding expressions are written in a different form. Finally, let us point out that a very small contribution arising from the regular part (finite when the photon energy goes to zero) of the radiative decay rate must be added to the r.h.s. of Eq. (11).
The Feynman diagrams of virtual corrections of $O(\alpha)$ to the $\rho \rightarrow \pi\pi$ amplitudes are shown in Figures 3 and 4. As in the real photon case, we will use scalar QED for the electromagnetic couplings of charged pions and assume that the $\rho^\pm$ electromagnetic vertex (see for example [9, 10]) is similar to the $WW\gamma$ gauge boson vertex.

The sum of virtual corrections to the $\rho^0 \rightarrow \pi^+\pi^-$ amplitude is finite in the ultraviolet region owing to a Ward identity satisfied by the vertex and self-energy corrections [12]. In the case of the $\rho^+ \rightarrow \pi^+\pi^0$ decay, the sum of virtual corrections diverges. One way of getting rid of such divergences is by absorbing them into a redefinition of the strong coupling constant $g_{+0}$. Instead of dealing with divergent amplitudes, in the case of $\rho^+ \rightarrow \pi^+\pi^0$ decays we will follow a procedure introduced long ago by Yennie [13]. According to this method we pick only the convection terms from virtual corrections in Figures 4a)-c). The sum of such terms gives rise to a one-loop amplitude which is finite in the ultraviolet region and is also gauge-invariant and model-independent [13]. Furthermore, such an amplitude contains all the infrared divergent terms of virtual corrections [13].
FIG. 4: Feynman graphs for the virtual photonic corrections to $\rho^+ \to \pi^+ \pi^0$ decays.

Just to illustrate the cancellation of infrared divergencies in the sum of virtual and real photon corrections, we reproduce here the expression for the virtual corrections in the case of $\rho^0 \to \pi^+ \pi^-$ decays [12]:

$$\Gamma_{1V}^{\rho^0} = \frac{\alpha}{\pi} \left\{ -1 - 2 \ln \left( \frac{\lambda}{m_{\rho^+}} \right) \right\} \left\{ 1 + \frac{1}{2v'_+} \ln \left( \frac{1-v'_+}{1+v'_+} \right) \right\} + \frac{3}{4} \ln \left( \frac{m_{\pi^+}^2}{m_{\rho^+}^2} \right)$$

$$+ \frac{m_{\rho^+}^2}{4m_{\pi^0}^2} \left[ \ln \left( \frac{m_{\pi^+}^2}{m_{\rho^+}^2} \right) \left( 1 - \frac{\Delta_{\pi}^2}{m_{\rho^+}^2} \right) - v_+ \left[ \ln \left( \frac{1-v_+}{1+v_+} \right) - \frac{\Delta_{\pi}^2}{m_{\rho^+}^2} \right] + \ln \left( \frac{1+v_+ - \frac{v_{\rho}^2}{m_{\rho^+}^2}}{1-v_+ - \frac{\Delta_{\pi}^2}{m_{\rho^+}^2}} \right) \right]$$

$$+ \frac{1}{2v'_+} \left[ \ln \left( \frac{1-v'_+}{1+v'_+} \right) - \frac{1}{4} \ln \left( \frac{1-v'_+}{1+v'_+} \right) + 2 \ln \left( \frac{2v'_+}{1+v'_+} \right) - \ln \left( \frac{m_{\rho^+}}{m_{\pi^+}} \right) \right]$$

$$- \frac{\pi^2}{3} + 2 \text{Li}_2 \left( \frac{1-v'_+}{1+v'_+} \right) + \ln^2 \left( \frac{m_{\pi^+}}{m_{\rho^+}} \right).$$

The corresponding result in the case of $\rho^+ \to \pi^+ \pi^0$ decays can be computed using the procedure described above [13]. We obtain:

$$\frac{\Gamma_{1V}^{\rho^+}}{\Gamma_0^{\rho^+}} = \frac{\alpha}{\pi} \left\{ -1 - 2 \ln \left( \frac{\lambda}{m_{\rho^+}} \right) \right\} \left\{ 1 + \frac{1}{2v'_+} \ln \left( \frac{1-v'_+}{1+v'_+} \right) \right\} + \frac{3}{4} \ln \left( \frac{m_{\pi^+}^2}{m_{\rho^+}^2} \right)$$

$$+ \frac{m_{\rho^+}^2}{4m_{\pi^0}^2} \left[ \ln \left( \frac{m_{\pi^+}^2}{m_{\rho^+}^2} \right) \left( 1 - \frac{\Delta_{\pi}^2}{m_{\rho^+}^2} \right) - v_+ \left[ \ln \left( \frac{1-v_+}{1+v_+} \right) - \frac{\Delta_{\pi}^2}{m_{\rho^+}^2} \right] + \ln \left( \frac{1+v_+ - \frac{v_{\rho}^2}{m_{\rho^+}^2}}{1-v_+ - \frac{\Delta_{\pi}^2}{m_{\rho^+}^2}} \right) \right]$$

$$+ \frac{1}{2v'_+} \left[ \ln \left( \frac{1-v'_+}{1+v'_+} \right) - \frac{1}{4} \ln \left( \frac{1-v'_+}{1+v'_+} \right) + 2 \ln \left( \frac{2v'_+}{1+v'_+} \right) - \ln \left( \frac{m_{\rho^+}}{m_{\pi^+}} \right) \right]$$

$$- \frac{\pi^2}{3} + 2 \text{Li}_2 \left( \frac{1-v'_+}{1+v'_+} \right) + \ln^2 \left( \frac{m_{\pi^+}}{m_{\rho^+}} \right) \right\}. \quad (14)$$
\[ +2\text{Li}_2\left(\frac{1 + v'_+ - 2/\xi}{1 + v'_+}\right) + 2\text{Li}_2\left(\frac{-1 + v'_+ + 2/\xi}{2/\xi}\right) \],

where we have defined \( \Sigma_\pi^2 = m_{\pi^0}^2 + m_{\pi^+}^2 \). As pointed out before, this result is free from ultraviolet divergencies, and it depends logarithmically on the infrared mass regulator \( \lambda \).

As we can easily check, the sum of the virtual corrections, Eqs. (14,15) and the corresponding soft-photon bremsstrahlung, Eqs. (12,13), are free from the infrared regulator \( \lambda \) as it must be according to the Bloch-Nordsieck theorem. Note however, that this sum depends on the photon energy cut \( \omega_0 \). We can expect that this \( \omega_0 \) dependence will be largely cancelled in the photon-inclusive \( \rho \rightarrow \pi\pi \) decay rate in such a way that \( \Delta\Gamma_\rho \) defined in Eq. (5) does not depend on this parameter. Since the soft- and hard-photon decay rates have a different dependence upon \( \omega_0 \), we need to choose values of \( \omega_0 \) which are sufficiently small (typically much smaller than the masses of hadronic particles) [14]. In the following subsection we briefly comment about the decay rates for hard photons.

**C. Hard-photon emission**

In order to estimate the contributions to \( \Delta\Gamma_\rho \) due to photons of energy larger than \( \omega_0 \), the third and fourth terms in Eq. (5), we require the complete expressions of the radiative decay amplitudes corresponding to the Feynman diagrams in Figures 1 and 2. These amplitudes include also the model-dependent contributions shown in Figures 1d)-1f) and 2d)-2e). The explicit contributions of the model-dependent amplitudes can be found in Refs. [7,8,11].

When we evaluate the decay rates of radiative decays, we note that the contributions of the model-dependent terms (intermediate states with \( \omega, a_1, \sigma \) and \( f_0 \) mesons) are typically two orders of magnitude below the contributions due to model-independent terms [8,9,11]. Therefore, such model-dependent terms would affect the width difference \( \Delta\Gamma_\rho \) at a completely negligible level. Note however that their effects have been included in our numerical evaluations given below.
IV. RESULTS AND DISCUSSION

In this section we present the main numerical results of our calculations. The values of the radiative corrections $\delta_{\rho^i}$ ($i = +, 0$) defined in Eq. (11) are shown in Tables I and II for $\rho^0 \to \pi^+\pi^-(\gamma)$ and $\rho^+ \to \pi^+\pi^0(\gamma)$ decays, respectively. The values of $\delta_{\rho^i}$ are tabulated as a function of the photon energy cut $\omega_0$ for three different values of the $\rho^i$ meson mass which are consistent with a small isospin breaking. As expected, these radiative corrections depend only slightly on the specific value of the $\rho$ meson mass. It is interesting to observe that in $\phi \to K^+K^-$ decays (and other analogous quarkonia decays that occur close to threshold) the radiative corrections are dominated by the coulombic interaction in the final state [12], while in $\rho^0 \to \pi^+\pi^-$ this is not the case.

Table III shows the radiative decay rates of $\rho^i \to \pi\pi\gamma$ normalized to the corresponding tree-level rates $\Gamma^0_i$. The tabulated values are defined as follows ($i = 0, +$):

$$\Delta_{\rho^i} \equiv \frac{\Gamma(\rho^i \to \pi\pi\gamma, \omega \geq \omega_0)}{\Gamma^0_i}.$$  \hspace{1cm} (16)

These values were calculated for photon energies larger than $\omega_0$ and for three different values of the $\rho$ meson mass. The values of $\Delta_{\rho^i}$ are always positive and exhibit the typical decreasing behavior as the lower energy photon cut $\omega_0$ increases. We observe that these normalized rates are in good agreement with the corresponding results reported long ago in Ref. [7] using slightly different values for the mass and width of the $\rho$ meson.

The predicted branching fraction for the neutral radiative mode:

$$B(\rho^0 \to \pi^+\pi^-\gamma, \omega \geq 50 \text{ MeV}) = 11.5 \times 10^{-3},$$  \hspace{1cm} (17)

compares very well (they are in agreement within 1$\sigma$) with the experimental measurement $(9.9 \pm 1.6) \times 10^{-3}$ reported in [1], which was obtained for the same value of $\omega_0$. For the same value of $\omega_0$, we obtain the isospin breaking in the radiative modes to be (assuming $\Gamma_{\rho^0} \approx \Gamma_{\rho^i} = 150$ MeV):

$$\Gamma(\rho^0 \to \pi^+\pi^-\gamma, \omega_0 = 50 \text{ MeV}) - \Gamma(\rho^+ \to \pi^+\pi^0\gamma, \omega_0 = 50 \text{ MeV}) \approx 1.1 \text{ MeV},$$  \hspace{1cm} (18)

which is larger than the estimate $(0.45 \pm 0.45)$ MeV considered in Ref. [5] and essentially independent of the $\rho$ meson mass.
In Table IV we display the sum of the radiative corrections $\delta \rho_i$ and the radiative branching ratios $\Delta \rho_i$ for a common value (775 MeV) of the neutral and charged $\rho$ meson mass. According to the definitions given in the previous sections we have:

$$\frac{\Gamma(\rho^i \rightarrow \pi\pi(\gamma), \omega \leq \omega_0) + \Gamma(\rho^i \rightarrow \pi\pi\gamma, \omega \geq \omega_0)}{\Gamma^0_i} = 1 + \delta \rho_i + \Delta \rho_i$$

$$\equiv 1 + \sigma \rho_i .$$  \hspace{1cm} (19)

As we have pointed out in section II, we expect that the $\sigma \rho_i$ correction term will be independent of the photon energy cut $\omega_0$ as far as Eq. (19) describes a photon inclusive rate and $\omega_0$ is an arbitrary reference value used to separate soft- and hard-photon emission. In practice this cancellation is better for values of $\omega_0$ (typically below a few MeV’s, see Table IV) because radiative corrections includes only the logarithmic dependence in $\omega_0$, while the hard-photon radiative rate contains also linear and other $\omega_0$ dependent terms.

Finally, we can evaluate the difference in decay widths arising from different corrections. Based on previous definitions, we can write the width difference of rho mesons, Eq. (5), as follows:

$$\Delta \Gamma \rho = \Gamma^0_0 \left[ 1 + \sigma \rho \left( \frac{m_{\rho^0}v_0^2}{m_{\rho^0}v^2} \right) [1 + \sigma \rho^+] \right] + \Delta \Gamma_{\text{sub}} .$$  \hspace{1cm} (20)

Once the isospin breaking in $\Delta m_\rho$ is known, the values of $\Delta \Gamma \rho$ can be easily evaluated from Tables I-III, using the above expression. For illustrative purposes, we provide the values of $\Delta \Gamma \rho$ for two interesting cases (we have used $\Gamma^0_0 = 150$ MeV and the value of $\sigma \rho_i$ at $\omega_0 = 10$ MeV):

$$\Delta \Gamma \rho = \begin{cases} 
0.86 \text{ MeV}, & \text{if } \Delta m_\rho = 0 \\
0.02 \text{ MeV}, & \text{if } \Delta m_\rho = -3 \text{ MeV} \\
1.70 \text{ MeV}, & \text{if } \Delta m_\rho = +3 \text{ MeV}
\end{cases} .$$  \hspace{1cm} (21)

Note that the width difference is very sensitive to the size and sign of $\Delta m_\rho$. This is very interesting because it can be useful as a test of $(\Delta m_\rho(\text{exp}), \Delta \Gamma \rho(\text{exp}))$ correlated values extracted from fits to experimental data. In particular, the above results are in good agreement with the values ($-2.4 \pm 0.8$, $-0.2 \pm 1.0$) MeV extracted from a combined fit to $\tau$ and $e^+e^-$ data \cite{13}, but the agreement is less good with the central values ($-3.1 \pm 0.9$, $-2.3 \pm 1.6$) MeV reported in \cite{6}.
V. CONCLUSIONS

The isospin breaking in the \( \rho^0 - \rho^\pm \) system is a very important ingredient to understand the current discrepancy in predictions of the hadronic vacuum polarization contributions to the muon magnetic moment based in \( \tau \) decay and \( e^+e^- \) annihilation data. In this paper we have evaluated the difference in decay widths (\( \Delta \Gamma_{\rho} \)) of \( \rho(770) \) vector mesons. We have considered in our calculation the isospin breaking in the exclusive decay modes of charged and neutral \( \rho \) mesons. In particular, we have carried out a calculation of the radiative corrections to the dominant \( \rho \rightarrow \pi\pi \) decay modes and have done a careful reevaluation of the differences in their radiative \( \rho \rightarrow \pi\pi\gamma \) decays.

We found that \( \Delta \Gamma_{\rho} \) is sensitive to the isospin breaking in the \( \rho^0 - \rho^\pm \) mass difference. This provides a useful tool to test the (model-dependent) values of these parameters extracted from experimental data. In particular we have found that positive values of \( \Delta \Gamma_{\rho} \) are favored by \( |\Delta m_{\rho}| \leq 3 \text{ MeV} \).

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\[ m_{\rho^0} = 772 \text{ MeV} \quad m_{\rho^0} = 775 \text{ MeV} \quad m_{\rho^0} = 778 \text{ MeV} \]

| \( \omega_0 \) (MeV) | \( \delta_{\rho^0} \) | \( \delta_{\rho^0} \) | \( \delta_{\rho^0} \) |
|-----------------|-----------------|-----------------|-----------------|
| 2               | -0.03670        | -0.03692        | -0.03714        |
| 4               | -0.02910        | -0.02930        | -0.02949        |
| 6               | -0.02465        | -0.02483        | -0.02501        |
| 8               | -0.02150        | -0.02167        | -0.02183        |
| 10              | -0.01905        | -0.01921        | -0.01937        |
| 12              | -0.01705        | -0.01720        | -0.01736        |
| 14              | -0.01536        | -0.01550        | -0.01565        |
| 16              | -0.01389        | -0.01403        | -0.01477        |
| 18              | -0.01260        | -0.01273        | -0.01287        |
| 20              | -0.01144        | -0.01157        | -0.01170        |
| 30              | -0.00697        | -0.00708        | -0.00720        |
| 40              | -0.00378        | -0.00388        | -0.00399        |
| 50              | -0.00130        | -0.00139        | -0.00150        |
| 60              | 0.00074         | -0.00065        | 0.00056         |
| 70              | 0.00249         | 0.00240         | 0.00232         |
| 80              | 0.00401         | 0.00393         | 0.00384         |
| 90              | 0.00536         | 0.00529         | 0.00521         |
| 100             | 0.00659         | 0.00651         | 0.00643         |

TABLE I: Radiative correction \( \delta_{\rho^0} \) to the \( \rho^0 \to \pi^+\pi^- (\gamma) \) decay rate (see definition in Eq. (5)) as a function of \( \omega_0 \) and for three different values of \( m_{\rho^0} \).
| $\omega_0$ (MeV) | $m_{\rho^+} = 772$ MeV | $m_{\rho^+} = 775$ MeV | $m_{\rho^+} = 778$ MeV |
|----------------|------------------------|------------------------|------------------------|
|                | $\delta_{\rho^+}$      | $\delta_{\rho^+}$      | $\delta_{\rho^+}$      |
| 2              | -0.01959               | -0.01968               | -0.01970               |
| 4              | -0.01701               | -0.01710               | -0.01718               |
| 6              | -0.01551               | -0.01558               | -0.01566               |
| 8              | -0.01444               | -0.01451               | -0.01459               |
| 10             | -0.01361               | -0.01368               | -0.01375               |
| 12             | -0.01293               | -0.01300               | -0.01307               |
| 14             | -0.01236               | -0.01242               | -0.01249               |
| 16             | -0.01186               | -0.01192               | -0.01199               |
| 18             | -0.01142               | -0.01149               | -0.01155               |
| 20             | -0.01103               | -0.01109               | -0.01115               |
| 30             | -0.00953               | -0.00958               | -0.00963               |
| 40             | -0.00844               | -0.00849               | -0.00854               |
| 50             | -0.00761               | -0.00765               | -0.00769               |
| 60             | -0.00692               | -0.00696               | -0.00700               |
| 70             | -0.00633               | -0.00637               | -0.00639               |
| 80             | -0.00582               | -0.00584               | -0.00589               |
| 90             | -0.00536               | -0.00540               | -0.00544               |
| 100            | -0.00495               | -0.00499               | -0.00502               |

TABLE II: Radiative corrections $\delta_{\rho^+}$ to the $\rho^+ \to \pi^+\pi^0(\gamma)$ decay rate (see definition in Eq. (5)) as a function of $\omega_0$ and for three different values of $m_{\rho^+}$. 
| \( \omega_0 \) (MeV) | \( \Delta_{\rho^+} \) | \( \Delta_{\rho^0} \) | \( \Delta_{\rho^+} \) | \( \Delta_{\rho^0} \) | \( \Delta_{\rho^+} \) | \( \Delta_{\rho^0} \) |
|-----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 2               | 0.01544        | 0.04475        | 0.01553        | 0.04497        | 0.01561        | 0.04518        |
| 4               | 0.01290        | 0.03724        | 0.01297        | 0.03742        | 0.01305        | 0.03761        |
| 6               | 0.01143        | 0.03288        | 0.01149        | 0.03305        | 0.011556       | 0.03322        |
| 8               | 0.01039        | 0.02981        | 0.01045        | 0.02997        | 0.01051        | 0.03013        |
| 10              | 0.00959        | 0.02745        | 0.00965        | 0.02760        | 0.00970        | 0.02775        |
| 12              | 0.00894        | 0.02553        | 0.00900        | 0.02568        | 0.00905        | 0.02582        |
| 14              | 0.00840        | 0.02393        | 0.00845        | 0.02406        | 0.00850        | 0.02420        |
| 16              | 0.00793        | 0.02255        | 0.00798        | 0.02268        | 0.00803        | 0.02281        |
| 18              | 0.00753        | 0.02134        | 0.00758        | 0.02147        | 0.00762        | 0.02159        |
| 20              | 0.00717        | 0.02027        | 0.00721        | 0.02039        | 0.00726        | 0.02051        |
| 30              | 0.00581        | 0.01624        | 0.00585        | 0.01635        | 0.00589        | 0.01645        |
| 40              | 0.00488        | 0.01350        | 0.00492        | 0.01359        | 0.00495        | 0.01369        |
| 50              | 0.00420        | 0.01146        | 0.00423        | 0.01155        | 0.00426        | 0.01163        |
| 60              | 0.00366        | 0.00987        | 0.00369        | 0.00994        | 0.00372        | 0.01002        |
| 70              | 0.00322        | 0.00857        | 0.00325        | 0.00864        | 0.00327        | 0.00871        |
| 80              | 0.00286        | 0.00750        | 0.00288        | 0.00757        | 0.00291        | 0.00763        |
| 90              | 0.00255        | 0.00659        | 0.00257        | 0.00665        | 0.00259        | 0.00672        |
| 100             | 0.00228        | 0.00582        | 0.00230        | 0.00588        | 0.00232        | 0.00593        |

TABLE III: Decay rates of \( \rho^i \to \pi\pi\gamma \) (normalized to the tree-level rates \( \Gamma^0_i \), see definition in Eq. (16)) as a function of the low photon energy cut \( \omega_0 \) and for three different values of \( m_\rho \).
TABLE IV: Photon inclusive corrections for $\rho^i \rightarrow \pi \pi \gamma$ (see definition in Eq. (19)) as a function of the photon energy $\omega_0$. The masses of charged and neutral $\rho$ mesons were fixed to 775 MeV's.

| $\omega_0$ (MeV) | $\sigma_{\rho^0}$ | $\sigma_{\rho^+}$ | $\sigma_{\rho^0} - \sigma_{\rho^+}$ |
|------------------|-------------------|-------------------|-------------------------------|
| 2                | $8.05 \times 10^{-3}$ | $-4.15 \times 10^{-3}$ | $12.20 \times 10^{-3}$ |
| 4                | $8.12 \times 10^{-3}$ | $-4.13 \times 10^{-3}$ | $12.25 \times 10^{-3}$ |
| 6                | $8.22 \times 10^{-3}$ | $-4.09 \times 10^{-3}$ | $12.31 \times 10^{-3}$ |
| 8                | $8.30 \times 10^{-3}$ | $-4.06 \times 10^{-3}$ | $12.36 \times 10^{-3}$ |
| 10               | $8.39 \times 10^{-3}$ | $-4.03 \times 10^{-3}$ | $12.42 \times 10^{-3}$ |
| 12               | $8.48 \times 10^{-3}$ | $-4.00 \times 10^{-3}$ | $12.48 \times 10^{-3}$ |
| 14               | $8.56 \times 10^{-3}$ | $-3.97 \times 10^{-3}$ | $12.53 \times 10^{-3}$ |
| 16               | $8.65 \times 10^{-3}$ | $-3.94 \times 10^{-3}$ | $12.59 \times 10^{-3}$ |
| 18               | $8.74 \times 10^{-3}$ | $-3.91 \times 10^{-3}$ | $12.65 \times 10^{-3}$ |
| 20               | $8.82 \times 10^{-3}$ | $-3.88 \times 10^{-3}$ | $12.70 \times 10^{-3}$ |