ISOPHOTE CURVES ON SPACELIKE SURFACES IN
LORENTZ-MINKOWSKI SPACE $E^3_1$

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Abstract. Isophote curve consists of a locus of surface points whose normal vectors make a constant angle with a fixed vector (the axis). In this paper, we define an isophote curve on a spacelike surface in Lorentz-Minkowski space and then find its axis as timelike and spacelike vectors via the Darboux frame. Besides, we give some characterizations concerning isophote curve and its axis.

1. Introduction

Isophote is one of the characteristic curves on a surface such as parameter, geodesic and asymptotic curves or line of curvature. It comprises a locus of the surface points whose normal vectors make a constant angle with a given fixed vector.

Isophote on a surface can be regarded as a nice consequence of Lambert’s cosine law in optics branch of physics. Lambert’s law states that the intensity of illumination on a diffuse surface is proportional to the cosine of the angle generated between the surface normal vector $N$ and the light vector $d$. According to this law the intensity is irrespective of the actual viewpoint, hence the illumination is the same when viewed from any direction [20]. In other words, isophotes of a surface are curves with the property that their points have the same light intensity from a given source (a curve of constant illumination intensity). When the source light is at infinity, we may consider that the light flow consists in parallel lines. Hence, we can give a geometric description of isophotes on surfaces, namely they are curves such that the surface normal vectors in points of the curve make a constant angle with a fixed direction (which represents the light direction). These curves are successfully used in computer graphics but also it is interesting to study for geometry.

Then, to find an isophote on a surface we use the formula

$$\frac{\langle N(u,v),d \rangle}{\|N(u,v)\|} = \cos \theta, \quad 0 \leq \theta \leq \frac{\pi}{2},$$

where $\theta$ is the angle between the surface normal $N$ and the fixed vector $d$. In the special case, isophote works as a silhouette curve if

$$\frac{\langle N(u,v),d \rangle}{\|N(u,v)\|} = \cos \frac{\pi}{2} = 0.$$

Koenderink and van Doorn [10] studied the field of constant image brightness contours (isophotes). They showed that the spherical image (the Gaussian mapping) of an isophote is a latitude circle on the unit sphere $S^2$ and the problem was reduced to that of obtaining the inverse Gauss map of these circles. By means of

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this they defined two kind singularities of the Gaussian map: folds (curves) and simple cusps (apex, antapex points) and there are structural properties of the field of isophotes that bear an invariant relation to geometric features of the object.

Poeschl [17] used isophotes in car body construction via detecting irregularities along this curves on a free form surface. These irregularities (discontinuity of a surface or of the Gaussian curvature) emerge by taking differentiation of the equation \( (N(u, v), d) = \cos \theta = c \) (constant) as follows.

\[
\frac{dv}{du} = -\frac{N_u \cdot d}{N_v \cdot d}, \quad N_v \cdot d \neq 0
\]

Sara [19] researched local shading of a surface through isophotes properties. By using fundamental theory of surfaces, he focused on accurate estimation of surface normal tilt and on qualitatively correct Gaussian curvature recovery.

Kim and Lee [9] parameterized isophotes for surface of rotation and canal surface. They utilized both these surfaces decompose into a set of circles where the surface normal vectors at points on each circle construct a cone. Again the vectors that make a constant angle with given fixed vector construct another cone and thus tangential intersection of these cones give the parametric range of the connected component isophote. In the same way, these authors [8] parameterized the perspective silhouette of a canal surface by solving the problem that characteristic circles meet tangential each other.

Dillen et al. [2] studied the constant angle surfaces in the product space \( S^2 \times \mathbb{R} \) for which the unit normal makes a constant angle with the \( \mathbb{R} \)-direction first. Then Dillen and Munteanu [3] investigated the same problem in \( \mathbb{H}^2 \times \mathbb{R} \) where \( \mathbb{H}^2 \) is the hyperbolic plane. Again, Nistor [14] researched normal, binormal and tangent developable surfaces of the space curve stand of point constant angle surface. Recently, Munteanu and Nistor [13] studied the constant angle surfaces taking with a fixed vector direction being the tangent direction to \( \mathbb{R} \) in Euclidean 3-space. They gave a important characterization about constant angle surfaces. With this point of view, the curves on a constant angle surface are isophote curves.

Izumiya and Takeuchi [5] defined a slant helix as the space curve that the principal normal lines make a constant angle with a fixed direction. They displayed that a certain slant helix is also a geodesic on the tangent developable surface of a general helix. As an amazing consequence in our paper, we see that the curve which is both a geodesic and a slant helix on a spacelike surface is an isophote curve.

Recently, Ali and Lopez [1] looked into slant helices in Lorentz-Minkowski space \( E^3_1 \). They gave characterizations as to slant helix and its axis in \( E^3_1 \). In our paper, we see that there is a close relation between isophotes and slant helices on the spacelike surface.

More lately, we have investigated [4] isophote curves in the Euclidean space \( E^3 \). In that paper, we obtained various characterizations of isophote curve and its axis. This time, we study isophote curves on spacelike surfaces in Lorentz-Minkowski space \( E^3_1 \) and view that there are some differences between Minkowski case and Euclidean case. For example, if isophote is also asymptotic curve, then it is a general helix in \( E^3 \) but it is not possible in \( E^3_1 \) because there is not an asymptotic curve on spacelike surfaces. Again, contrary to Euclidean case, isophote with timelike axis can not be a line of curvature and apart from these it is not possible to define a silhouette curve on spacelike surfaces in \( E^3_1 \).
In this paper, we define isophote curves on spacelike surfaces in Lorentz-Minkowski space $E^3_1$ and find its axis by means of the Darboux frame. This paper is organized as follows. Section 2 brings in some basic facts and concepts concerning curves in $E^3_1$. In section 3, we concentrate on finding the axis of an isophote and also give some characterizations about it. Finally, in section 4 we conclude this paper.

2. Preliminaries

First of all, we begin to introduce Lorentz-Minkowski space. Later, we mention some fundamental concepts of curves and surfaces in the Minkowski 3-space $E^3_1$. The space $R^3_1$ is a three dimensional real vector space endowed with the inner product

$$\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3.$$ 

This space is called Lorentz-Minkowski space and denoted by $E^3_1$. A vector in this space is said to be spacelike, timelike and lightlike (null) if $\langle x, x \rangle > 0$ or $x = 0$, $\langle x, x \rangle < 0$ and $\langle x, x \rangle = 0$ or $x \neq 0$, respectively. Again, a regular curve $\alpha : I \rightarrow E^3_1$ is called spacelike, timelike and lightlike if the velocity vector $\dot{\alpha}$ is spacelike, timelike and lightlike, respectively [11].

The Lorentzian cross product of $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $R^3_1$ is defined as follows [6].

$$x \times y = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ e_1 & e_2 & e_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1),$$

where $\delta_{ij}$ is kronecker delta, $e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})$ and $e_1 \times e_2 = -e_3$, $e_2 \times e_3 = e_1$, $e_3 \times e_1 = -e_2$.

Let $\{t, n, b\}$ be the moving Frenet frame along the curve $\alpha$ with arclength parameter $s$. For a spacelike curve $\alpha$, the Frenet-Serret equations are

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\varepsilon \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix},$$

where $(t, t) = 1$, $(n, n) = \pm 1$, $(b, b) = -\varepsilon$, $(t, n) = (t, b) = (n, b) = 0$ and $\kappa$ is the curvature of $\alpha$ and again $\tau$ is the torsion of $\alpha$. Here, $\varepsilon$ determines the kind of spacelike curve $\alpha$. If $\varepsilon = 1$, then $\alpha(s)$ is a spacelike curve with spacelike principal normal $n$ and timelike binormal $b$. If $\varepsilon = -1$, then $\alpha(s)$ is a spacelike curve with timelike principal normal $n$ and spacelike binormal $b$.

**Definition 1** (Hyperbolic angle). Let $v$ and $w$ be in the same timecone of $R^3_1$. Then, there is a unique real number $\theta \geq 0$, called the hyperbolic angle between $v$ and $w$ such that [15]

$$\langle v, \omega \rangle = -\|v\| \|w\| \cosh \theta.$$  

**Definition 2**. Let $v$ be a spacelike vector and $w$ be a timelike vector in $R^3_1$. Then, there is a unique non-negative real number $\theta \geq 0$ such that [15]

$$\langle v, w \rangle = \|v\| \|w\| \sinh \theta.$$  

**Definition 3**. A surface in the Minkowski 3-space $E^3_1$ is called a spacelike surface if the induced metric on the surface is a positive definite Riemannian metric, i.e., the normal vector on the spacelike surface is a timelike vector.
Lemma 1. In the Minkowski 3-space $E^3_1$, the following properties are satisfied [6].

(i) Two timelike vectors are never orthogonal.
(ii) Two null vectors are orthogonal if and only if they are linearly dependent.
(iii) A timelike vector is never orthogonal to a null (lightlike) vector.

Let $M$ be a regular spacelike surface in $E^3_1$ and $\alpha : I \subset \mathbb{R} \rightarrow M$ be a unit speed spacelike curve on this surface. Then, Darboux frame $\{T, B = N \times T, N\}$ is well-defined and positively oriented along the curve $\alpha$ where $T$ is the tangent of $\alpha$ and $N$ is the unit normal of $M$. The Frenet-Serret equations which correspond to this frame are given by

\begin{align*}
T' &= k_g B + k_n N \\
B' &= -k_g T + \tau_g N \\
N' &= k_n T + \tau_g B,
\end{align*}

where $k_n$, $k_g$ and $\tau_g$ are the normal curvature, the geodesic curvature and the geodesic torsion of $\alpha$, respectively and $\langle T, T \rangle = \langle B, B \rangle = 1$, $\langle N, N \rangle = \langle n, n \rangle = -1$. Then, by using Eq (2.3) we reach,

\begin{align*}
\kappa^2 &= k_n^2 - k_g^2 \\
k_n &= \kappa \cosh \phi \\
k_g &= \kappa \sinh \phi \\
\tau_g &= \tau + \phi',
\end{align*}

where $\phi$ is the angle between the surface normal $N$ and the binormal $b$ to $\alpha$. If the surface $M$ is spacelike, then the tangent plane of $M$ has to be spacelike. Therefore, all curves lying on the spacelike surface $M$ are spacelike. In the rest of paper, all $M$ will be understood as a spacelike surface.

Proposition 1. Let $M$ be a spacelike surface and let $\alpha$ be a spacelike curve with timelike principal normal (\langle n, n \rangle = -1) on $M$. Then, there is not an asymptotic curve on $M$.

Proof. If $\alpha$ is a spacelike curve with timelike principal normal on $M$, by Eq (2.4) we have

\begin{align*}
\kappa^2 &= k_n^2 - k_g^2 \\
k_n &= \kappa \cosh \phi \\
k_g &= \kappa \sinh \phi \\
\tau_g &= \tau + \phi',
\end{align*}

Assume that $\alpha$ is an asymptotic curve on $M$. In this situation, the normal curvature $k_n = 0$. In the above equation for $k_n = 0$, the geodesic curvature $k_g$ does not have a solution in $R$. Then, our assumption is not true that is to say there is not an asymptotic curve on $M$. \hfill \Box

3. The Axis Of An Isophote Curve

In this section, we will obtain the axis (the fixed vector) of an isophote curve through its Darboux frame. Let $M$ be a regular spacelike surface and $\alpha : I \subset \mathbb{R} \rightarrow M$ be a unit speed isophote curve which is also spacelike. Then, from definition of the isophote curve

\[ \langle N(u, v), d \rangle = \text{constant}, \]

where $N(u, v)$ is the unit normal vector field of the surface $S(u, v)$ (a parameterization of $M$) and $d$ is the axis of isophote curve. Here, we examine two different cases of the axis $d$. Since the surface $M$ is spacelike, the surface normal $N$ is a
timelike vector.

**Case 1:** Let the surface normal $N$ and the axis $d$ be timelike vectors in the same timecone of $R^3$, then by Definition (1) $\langle N(u, v), d \rangle = -\cosh \theta$.

**Case 2:** Let the axis $d$ be a spacelike vector and the surface normal $N$ be a timelike vector, then by Definition (2) $\langle N(u, v), d \rangle = \sinh \theta$.

where $\theta$ is the angle between the surface normal $N$ and the axis $d$.

Now, we begin to find the axis $d$ for the case 1. Since $\alpha : I \subset \mathbb{R} \rightarrow M$ be a unit speed isophote curve, the Darboux frame can be defined as $\{T, B, N\}$ along the curve $\alpha$. Let the axis $d$ be a timelike vector. Then,

$$\langle N(u, v), d \rangle = -\cosh \theta. \tag{3.1}$$

If we derive the Eq (3.1) with respect to $s$ along the curve,

$$\langle N', d \rangle = 0. \tag{3.2}$$

From the last equation of (2.3), it follows

$$\langle k_n T + \tau_g B, d \rangle = 0$$

$$k_n \langle T, d \rangle + \tau_g \langle B, d \rangle = 0$$

$$\langle T, d \rangle = -\frac{\tau_g}{k_n} \langle B, d \rangle.$$

Because the Darboux frame $\{T, B, N\}$ is an orthonormal basis, if we say $\langle B, d \rangle = a$ in the last equation, then $d$ can be written as

$$d = -\frac{\tau_g}{k_n} a T + a B + \cosh \theta N.$$

As $\langle d, d \rangle = -1$, we get

$$a = \mp \frac{k_n}{\sqrt{k_n^2 + \tau_g^2}} \sinh \theta.$$

Thus, the timelike axis $d$ is obtained as

$$d = \pm \frac{\tau_g}{\sqrt{k_n^2 + \tau_g^2}} \sinh \theta T \mp \frac{k_n}{\sqrt{k_n^2 + \tau_g^2}} \sinh \theta B + \cosh \theta N. \tag{3.3}$$

If we derive $N'$ and Eq (3.2) with respect to $s$, we attain

$$N'' = (k'_n - k_g \tau_g) T + (k_n k_g + \tau'_g) B + (k_n^2 + \tau_g^2) N$$

$$\langle N'', d \rangle = 0$$

$$\langle N'', d \rangle = \frac{-\tau_g (k_n' k_n - k_g \tau_g) \mp k_g (k_n^2 + \tau_g^2)}{\sqrt{k_n^2 + \tau_g^2}} \sinh \theta - (k_n^2 + \tau_g^2) \cosh \theta = 0.$$

Therefore, we have

$$\tanh \theta = \mp \frac{(k_n^2 + \tau_g^2)^{3/2}}{k_g (k_n^2 + \tau_g^2) + (\tau_g k_n - k_g' \tau_g)} \tag{3.4}$$

$$\coth \theta = \mp \left[ \frac{k_n^2}{(k_n^2 + \tau_g^2)^{3/2}} \left( \frac{\tau_g}{k_n} \right) + \frac{k_g}{(k_n^2 + \tau_g^2)^{3/2}} \right].$$
From now on, we will prove that $d$ is a constant vector. In other words, $d' = 0$. By Eq (2.3) and Eq (3.3), the derivative of $d$ with respect to $s$ is that

$$
\dot{d} = \pm \sinh \theta \left[ \left( \frac{\tau_g}{\sqrt{k_n^2 + \tau_g^2}} \right) T + \frac{\tau_g}{\sqrt{k_n^2 + \tau_g^2}} (k_B + k_N) \right]
$$

$$
\mp \sinh \theta \left[ \left( \frac{k_n}{\sqrt{k_n^2 + \tau_g^2}} \right) B + \frac{k_n}{\sqrt{k_n^2 + \tau_g^2}} (-k_B + \tau_g N) \right] + \cosh \theta (k_B T + \tau_g B).
$$

If we arrange this equality, we obtain

$$
\dot{d} = \left( \pm \sinh \theta \left[ \left( \frac{\tau_g}{\sqrt{k_n^2 + \tau_g^2}} \right) + \frac{k_B k_N}{\sqrt{k_n^2 + \tau_g^2}} \right] + k_n \cosh \theta \right) T
$$

$$
+ \left( \pm \sinh \theta \left[ \left( \frac{k_n}{\sqrt{k_n^2 + \tau_g^2}} \right) + \frac{k_B \tau_g}{\sqrt{k_n^2 + \tau_g^2}} \right] + \tau_g \cosh \theta \right) B.
$$

From Eq (3.4), we have

$$
\cosh \theta = \mp \sinh \theta \frac{\tau_g' k_n - k_n' \tau_g + k_B (k_n^2 + \tau_g^2)}{(k_n^2 + \tau_g^2)^{3/2}}.
$$

If the last equality is replaced in the statement of $\dot{d}$, we get

$$
\dot{d} = \pm \sinh \theta \left( \frac{\tau_g' (k_n^2 + \tau_g^2) - \tau_g (k_n k_n' + \tau_g \tau_g') + k_B k_n (k_n^2 + \tau_g^2)}{(k_n^2 + \tau_g^2)^{3/2}} \right)
$$

$$
\pm \sinh \theta \left( \frac{-k_n' (k_n^2 + \tau_g^2) + k_n (k_n' k_n + \tau_g \tau_g') + k_B \tau_g (k_n^2 + \tau_g^2)}{(k_n^2 + \tau_g^2)^{3/2}} \right)
$$

$$
\pm \sinh \theta \left( \frac{\tau_g' (k_n^2 + \tau_g^2) - \tau_g (k_n k_n' + \tau_g \tau_g') + k_B k_n (k_n^2 + \tau_g^2)}{(k_n^2 + \tau_g^2)^{3/2}} \right)
$$

$$
\pm \sinh \theta \left( \frac{-k_n' (k_n^2 + \tau_g^2) + k_n (k_n' k_n + \tau_g \tau_g') + k_B \tau_g (k_n^2 + \tau_g^2)}{(k_n^2 + \tau_g^2)^{3/2}} \right)
$$

By the straight-forward calculation, it follows that $\dot{d} = 0$ namely $d$ is a constant vector.

**Theorem 1.** A unit speed curve $\alpha$ on the spacelike surface $M$ is an isophote curve with the timelike axis $d$ if and only if

$$
\psi(s) = \pm \left( \frac{k_n^2}{(k_n^2 + \tau_g^2)^{3/2}} \left( \frac{\tau_g}{k_n} \right)' + \frac{k_B}{(k_n^2 + \tau_g^2)^{3/2}} \right)(s)
$$

is a constant function.

**Proof.** We can show that $\alpha$ is an isophote curve on the spacelike surface if and only if the Gaussian mapping along the curve $\alpha$ is a latitude circle on the Lorentzian
unit sphere $S^2_1$. Hence, if we compute the Gaussian mapping $N_{|\alpha} : I \rightarrow S^2_1$ along the curve $\alpha$, the geodesic curvature of $N_{|\alpha}$ is $\psi(s)$ as shown below.

\[
N'_{|\alpha} = k_n T + \tau_g B \\
N''_{|\alpha} = (k_n' - k_n \tau_g) T + (k_n k_g + \tau_g') B + (k_n^2 + \tau_g^2) N \\
N'_{|\alpha} \times N''_{|\alpha} = \tau_g (k_n^2 + \tau_g^2) T + \left( k_g (k_n^2 + \tau_g^2) + k_n^2 \left( \frac{\tau_g}{k_n} \right) \right) N - k_n (k_n^2 + \tau_g^2) B.
\]

where $T \times B = N$, $B \times N = T$ and $N \times T = B$. Accordingly, we get

\[
\kappa = \sqrt{\left\langle N'_{|\alpha} \times N''_{|\alpha}, N'_{|\alpha} \times N''_{|\alpha} \right\rangle} = \frac{\sqrt{\tau_g^2 (k_n^2 + \tau_g^2)^2 - \left( k_g (k_n^2 + \tau_g^2) + k_n^2 \left( \frac{\tau_g}{k_n} \right) \right)^2 + k_n^2 (k_n^2 + \tau_g^2)^2}}{(k_n^2 + \tau_g^2)^3} = \sqrt{1 - \left( \frac{k_g (k_n^2 + \tau_g^2) + k_n^2 \left( \frac{\tau_g}{k_n} \right)'}{(k_n^2 + \tau_g^2)^3} \right)^2}.
\]

Let $\bar{k}_g$ and $\bar{k}_n$ be the geodesic curvature and the normal curvature of the Gaussian mapping $N_{|\alpha}$, respectively. Since the normal curvature $\bar{k}_n = 1$, if we substitute $\bar{k}_n$ and $\kappa$ in the following equation, we obtain the geodesic curvature $\bar{k}_g$ as follows.

\[
\kappa^2 = (\bar{k}_n)^2 - (\bar{k}_g)^2
\]

\[
\bar{k}_g(s) = \psi(s) = \coth \theta = \mp \left( \frac{k_n^2}{(k_n^2 + \tau_g^2)^2 \left( \frac{\tau_g}{k_n} \right)'} + \frac{k_g}{(k_n^2 + \tau_g^2)^2} \right) (s).
\]

where $\theta$ is the angle between the surface normal $N$ and the fixed vector $d$. Then, the spherical image of isophotes are latitude circles if and only if $\psi(s)$ is a constant function.

**Corollary 1.** If $\alpha$ is a unit speed isophote curve with the timelike axis $d$ on $M$, then $\alpha$ can not be a silhouette curve.

**Proof.** Let $\alpha$ be a unit speed isophote curve with timelike axis $d$ on $M$. Then,

\[
\langle N, d \rangle = -\cosh \theta \neq 0.
\]

Therefore, by the definition of silhouette curve $\alpha$ can not be a silhouette curve. \qed

**Proposition 2.** If $\alpha$ is a unit speed isophote curve the timelike axis $d$ on $M$, then $\alpha$ can not be a line of curvature.
Proof. Let $\alpha$ be a unit speed isophote curve on $M$. In that case, by Theorem (1) we have
\[
\coth \theta = \mp \left( \frac{k_n^2}{(k_n^2 + \tau^2/\kappa)^2} + \frac{k_g}{(k_n^2 + \tau^2/\kappa)^3/2} \right) (s) = \text{constant}.\]
Assume that $\alpha$ is a line of curvature. By applying $\tau_g = 0$ and Eq (2.4), we obtain
\[
\coth \theta = \mp \frac{k_n}{k_g} = \mp \frac{\kappa \sinh \phi}{\kappa \cosh \phi} = \mp \tanh \phi. \]
The last equation $\coth \theta = \mp \tanh \phi$ does not have a solution. For this reason, $\alpha$ cannot be a unit speed isophote curve on $M$. This is a contradiction with respect to our assertion. Hence, $\alpha$ cannot be a line of curvature.
\[\square\]
Corollary 2. Let $\alpha$ be a unit speed isophote curve on $M$. Then, we have the following.
(1) The axis $d$ cannot be perpendicular to the tangent line of $\alpha$.
(2) The axis $d$ cannot be perpendicular to the vector $B$.

Proof. (1) Suppose that $\alpha$ be a unit speed isophote curve on $M$. Then by Eq (3.3), it follows that
\[
\langle T, d \rangle = \pm \frac{\tau_g}{\sqrt{k_n^2 + \tau^2/\kappa}} \sinh \theta.
\]
By the definition of isophote curve, we must have $\sinh \theta \neq 0$ and again by Proposition (3) we have $\tau_g \neq 0$. Accordingly, it follows that $\langle T, d \rangle \neq 0$ in the above equation put differently the axis $d$ cannot be perpendicular to the tangent line of $\alpha$.
(2) Suppose that $\alpha$ be a unit speed isophote curve on $M$. Then, by Eq (3.3) we have
\[
\langle B, d \rangle = \mp \frac{k_n}{\sqrt{k_n^2 + \tau^2/\kappa}} \sinh \theta.
\]
From Proposition (1) and the definition of isophote curve, we have $k_n \neq 0$ and $\sinh \theta \neq 0$. Therefore, we conclude that $\langle B, d \rangle \neq 0$ that is the axis $d$ cannot be perpendicular to the vector $B$. \[\square\]

Lemma 2. Let $\alpha$ be a unit speed spacelike curve in $E^3_1$ with $\kappa(s) \neq 0$. Then, $\alpha$ is a slant helix with timelike principal normal if and only if $\sigma(s) = \left( \frac{\kappa^2}{(\kappa^2 + \tau^2)^3/2} \left( \frac{\tau}{\kappa} \right)' \right) (s)$ is a constant function [1].

Theorem 2. Let $\alpha$ be a unit speed isophote curve which is also spacelike on $M$. In that case, $\alpha$ is a geodesic on $M$ if and only if $\alpha$ is a slant helix with the timelike fixed vector
\[
d = \pm \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \sinh \theta T \pm \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \sinh \theta B + \cosh \theta N.
\]
Proof. Since $\alpha$ is a geodesic (i.e. surface normal $N$ concurs with the principal normal $n$ along the curve $\alpha$), we have $k_g = 0$ and therefore from Eq (2.4) $k_n = \kappa$
and \( \tau_g = \tau \). By substituting \( k_g \) and \( k_n \) in the expression of \( \psi(s) \), we follow that

\[
\psi(s) = \mp \left( \frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} (\tau_g') \right)(s)
\]

is a constant function. Then, by Lemma (1) \( \alpha \) is a slant helix. Using Eq (3.3), the timelike axis of slant helix is obtained as

\[
d = \pm \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \sinh \theta T \pm \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \sinh \theta B + \cosh \theta N.
\]

On the contrary, let \( \alpha \) be a slant helix with the timelike fixed vector \( d \). Then, from Eq (3.3) we have \( k_n = \kappa \) and \( \tau_g = \tau \). So, the geodesic curvature \( k_g \) must be zero that is to say \( \alpha \) is a geodesic on \( M \).

**Case 2:** From this time forth, we will obtain the spacelike fixed vector \( d \). If the axis \( d \) is spacelike, by the Definition (2) we possess

\[
\langle N(u,v), d \rangle = \sinh \theta.
\]

Just as the case of timelike fixed vector \( d \), we get

\[
\langle T, d \rangle = -\frac{\tau_g}{k_n} \langle B, d \rangle
\]

If we say \( \langle B, d \rangle = a \), because \( \langle d, d \rangle = 1 \) we gather that

\[
a = \mp \frac{k_n}{\sqrt{k_n^2 + \tau^2_g}} \cosh \theta
\]

and thus the spacelike axis \( d \) can be written as

\[
d = \pm \frac{\tau_g}{\sqrt{k_n^2 + \tau^2_g}} \cosh \theta T \mp \frac{k_n}{\sqrt{k_n^2 + \tau^2_g}} \cosh \theta B - \sinh \theta N.
\]

Seeing that \( \langle N', d \rangle = 0 \),

\[
\coth \theta = \pm \frac{(k_n^2 + \tau_g^2)^{3/2}}{k_g(k_n^2 + \tau^2_g) + (\tau_g'k_n - k'_n\tau_g)}
\]

\[
\tanh \theta = \pm \left[ \frac{k_n^2}{(k_n^2 + \tau^2_g)^{3/2}} \frac{(\tau_g')}{k_n^2} + \frac{k_g}{(k_n^2 + \tau^2_g)^{1/2}} \right].
\]

Here, applying Eq (3.7) it can be showed that \( d' = 0 \) in other words \( d \) is a constant vector.

**Theorem 3.** A unit speed spacelike curve \( \alpha \) on a spacelike surface is an isophote curve with the spacelike axis \( d \) if and only if

\[
\tanh \theta = \omega(s) = \pm \left( \frac{k_n^2}{(k_n^2 + \tau^2_g)^{3/2}} \frac{(\tau_g')}{k_n^2} + \frac{k_g}{(k_n^2 + \tau^2_g)^{1/2}} \right)(s)
\]

is a constant function.

The proof of this theorem is the same as Theorem (1).

**Corollary 3.** If \( \alpha \) is a unit speed isophote curve with the spacelike axis \( d \) on \( M \), then \( \alpha \) can not be a silhouette curve.
Proof. Let $\alpha$ be a unit speed isophote curve with spacelike axis $d$ on $M$. Then,

$$\langle N, d \rangle = \sinh \theta.$$ 

According to the definition of silhouette curve the surface normal must be orthogonal to the fixed vector. In the above equality when $\theta = 0$, $\langle N, d \rangle = 0$ in other words the surface normal $N$ coincides with the axis $d$ but it is not possible to define isophote curve like this. Hence, $\alpha$ can not be a silhouette curve. □

**Theorem 4.** Let $\alpha$ be a unit speed isophote curve on $M$. Then, $\alpha$ is a geodesic on $M$ if and only if $\alpha$ is a slant helix with the spacelike fixed vector

$$d = \pm \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \cosh \theta T \mp \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \cosh \theta B - \sinh \theta N.$$ 

The proof of this theorem like Theorem (2).

**Proposition 3.** Let $\alpha$ be a unit speed isophote curve with the spacelike axis $d$ on $M$. Then, $\alpha$ is a plane curve provided that $\alpha$ is a line of curvature on $M$.

Proof. Let $\alpha$ be a unit speed isophote curve on $M$. Then, by Theorem (4) we have

$$\tanh \theta = \pm \left( \frac{k_n}{(k_n^2 + \tau_g^2)^{\frac{3}{2}}} \left( \frac{\tau_g}{k_n} \right)' + \frac{\tau_g}{(k_n^2 + \tau_g^2)^{\frac{3}{2}}} \right) (s) = \text{constant}.$$ 

Assume that $\alpha$ is a line of curvature. By applying $\tau_g = 0$ and Eq (2.4), we obtain

$$\tanh \theta = \pm \frac{k_n}{k_n} = \pm \frac{\kappa \sinh \phi}{\kappa \cosh \phi} = \pm \tanh \phi.$$ 

In consequence, it traces that $\phi = \pm \theta$. Since $\phi$ is a constant, using $\tau_g = \tau + \phi' = 0$ we get $\tau = 0$ namely $\alpha$ is a plane curve. □

**Theorem 5.** Let $\alpha$ be a unit speed isophote curve with the spacelike axis $d$ on $M$. Then, we have the following.

1. The axis $d$ is orthogonal to the tangent line of $\alpha$ if and only if $\alpha$ is a line of curvature on $M$.
2. The axis $d$ can not be orthogonal to the vector $B$.

Proof. (1) Let the axis $d$ be orthogonal to the tangent line of $\alpha$, then from Eq (3.6)

$$\langle T, d \rangle = \pm \frac{\tau_g}{\sqrt{k_n^2 + \tau_g^2}} \cosh \theta = 0.$$ 

In that $\cosh \theta \neq 0$, $\tau_g = 0$ in the above equation. Consequently, the axis $d$ is orthogonal to the tangent line of $\alpha$ if and only if $\alpha$ is a line of curvature on $M$.

(2) According to Eq (3.6)

$$\langle B, d \rangle = \mp \frac{k_n}{\sqrt{k_n^2 + \tau_g^2}} \cosh \theta = 0.$$ 

On account of the fact that there is not asymptotic curve on a spacelike surface, $k_n \neq 0$. Furthermore, because $\cosh \theta \neq 0$ in the above equation, $\langle B, d \rangle \neq 0$ alternatively the axis $d$ can not be orthogonal to the vector $B$. □
4. Conclusions

In this paper, we found the axis (given fixed vector) of an isophote curve through its Darboux frame. Subsequently, we obtained some characterizations regarding this curves. By using these characterizations, it is investigated relation between special curves on a surface and an isophote. For instance, we viewed the curve which is both isophote and geodesic on the spacelike surface is a slant helix and also viewed that isophote curve with the timelike axis can not be a line of curvature while isophote curve with the spacelike axis can be a line of curvature. From now on, we will study isophote curves lying on timelike surfaces. We think that we will obtain important results here as in that paper.

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