STABLE PRINCIPAL BUNDLES AND REDUCTION OF STRUCTURE GROUP

INDRANIL BISWAS

Abstract. Let \( E_G \) be a stable principal \( G \)-bundle over a compact connected Kähler manifold, where \( G \) is a connected reductive linear algebraic group defined over \( \mathbb{C} \). Let \( H \subset G \) be a complex reductive subgroup which is not necessarily connected, and let \( E_H \subset E_G \) be a holomorphic reduction of structure group. We prove that \( E_H \) is preserved by the Einstein–Hermitian connection on \( E_G \). Using this we show that if \( E_H \) is a minimal reductive reduction in the sense that there is no complex reductive proper subgroup of \( H \) to which \( E_H \) admits a holomorphic reduction of structure group, then \( E_H \) is unique in the following sense: For any other minimal reductive reduction \((H', E_{H'})\) of \( E_G \), there is some element \( g \in G \) such that \( H' = g^{-1}Hg \) and \( E_{H'} = E_Hg \). As an application, we give an affirmative answer to a question posed in \( \text{[BK]} \).

1. Introduction

In \( \text{[BK]} \), the notion of algebraic holonomy for a stable vector bundle defined over a normal complex projective variety is introduced. Let \( X \) be a simply connected smooth complex projective variety equipped with a Kähler–Einstein metric. In \( \text{[BK]} \), Question 9 it is asked whether the algebraic holonomy of \( TM \) coincides with the complexification of the differential geometric holonomy of \( M \). The present work started by trying to answer it. We prove that for any stable vector bundle \( E \) defined over a complex projective manifold, the algebraic holonomy coincides with the complexification of the differential geometric holonomy of the Einstein–Hermitian connection on \( E \). Since a Kähler–Einstein connection is also an Einstein–Hermitian connection, this answers the above question affirmatively.

Let \( M \) be a compact connected Kähler manifold equipped with a Kähler form. Let \( G \) be a connected reductive linear algebraic group defined over \( \mathbb{C} \). It is known that any stable principal \( G \)-bundle over \( M \) admits a unique Einstein–Hermitian connection.

Our main result is the following (Theorem 2.3):

\textbf{Theorem 1.1.} Let \( E_G \) be a stable \( G \)-bundle over \( M \). Let \( H \) be a complex reductive subgroup of \( G \) which is not necessarily connected, and let \( E_H \subset E_G \) be a holomorphic reduction of structure group of \( E_G \) to \( H \). Then the Einstein–Hermitian connection on \( E_G \) is induced by a connection on \( E_H \).

Let \( H \) be a complex reductive subgroup of \( G \) which is not necessarily connected, and let \( E_H \subset E_G \) be a holomorphic reduction of structure group to \( H \) of a stable \( G \)-bundle \( E_G \) defined over \( M \). Assume that there is no complex reductive proper subgroup of \( H \) to which \( E_H \) admits a holomorphic reduction of structure group. Such reductions will
be called the minimal reductive ones. Theorem 1.1 says that $E_H$ is preserved by the Einstein–Hermitian connection on $E_G$.

Fix a point $x_0 \in M$, and also choose a point in the fiber $z \in (E_G)_z$. Taking parallel translations, for the Einstein–Hermitian connection on $E_G$, of $z$ along piecewise smooth paths in $M$ based at $x_0$ we get a subset of $E_G$. The topological closure, in $E_G$, of this subset gives a smooth reduction of structure group of $E_G$ to a compact subgroup $\overline{K}_E \subset G$. The corresponding smooth reduction of structure group $E_{z_0}^\overline{K}_E$ of $E_G$ to the Zariski closure $\overline{K}_E^\mathbb{C}$ of $\overline{K}_E$ in $G$ is actually holomorphic. We prove the following (see Theorem 3.1):

**Theorem 1.2.** There is a point $z_0 \in (E_G)_z$ such that the above minimal reductive reduction $E_H$ coincides with $E_{z_0}^\overline{K}_E$. In particular, $H$ coincides with $\overline{K}_E^\mathbb{C}$ for such a base point $z_0$.

Consequently, if $(H, E_H)$ and $(H', E_{H'})$ are two minimal reductive reductions of $E_G$, then there is an element $g \in G$ such that $H' = g^{-1}Hg$ and $E_{H'} = E_{H}g$. This gives us the following corollary (Corollary 3.2):

**Corollary 1.3.** Let $E_G$ be a stable principal $G$–bundle over $M$. Then there is a unique holomorphic sub–fiber bundle $\mathcal{G}_{E_G}$ of the adjoint bundle $\text{Ad}(E_G)$, with fibers being subgroups, that satisfies the following condition: For any minimal reductive reduction $E_H$ of $E_G$, the adjoint bundle $\text{Ad}(E_H)$, which is a sub–fiber bundle of $\text{Ad}(E_G)$, coincides with $\mathcal{G}_{E_G}$.

The sub–fiber bundle $\mathcal{G}_{E_G} \subset \text{Ad}(E_G)$ in Corollary 1.3 is the complexification of the holonomy of the Einstein–Hermitian connection on $E_G$. While the stability condition does not depend on the choice of a Kähler form in a given Kähler class, the Einstein–Hermitian connection on a stable bundle depends on the choice of the Kähler form. We note that the sub–fiber bundle $\mathcal{G}_{E_G}$ does not depend either on the Kähler form or on the Kähler class as long as $E_G$ remains stable. When $M$ is a complex projective manifold, $\mathcal{G}_{E_G}$ coincides with the algebraic monodromy of $E_G$ introduced in [BK].

2. Einstein–Hermitian connection and reduction of structure group

Let $M$ be a compact connected Kähler manifold equipped with a Kähler form $\omega$. The degree of a torsionfree coherent analytic sheaf $V$ defined over a dense open subset $U \subset M$, whose complement $M \setminus U$ is an analytic subset of $M$ of complex codimension at least two, is defined to be

\[
\text{degree}(V) := \int_M c_1(\iota_* V) \omega^{\dim M - 1} \in \mathbb{R},
\]

where $\iota : U \hookrightarrow M$ is the inclusion map; the codimension condition ensures that $\iota_* V$ is a coherent analytic sheaf.
Let $G$ be a connected reductive linear algebraic group defined over $\mathbb{C}$. Since the degree has been defined, we have the notions of a stable $G$–bundle and a polystable $G$–bundle over $M$. See [RS, AB, Ra] for definitions of stable and polystable principal $G$–bundles.

Fix a maximal torus $T \subset G$ and a Borel subgroup $B \subset G$ containing $T$. We also fix a maximal compact subgroup $K$ of $G$. By a parabolic subgroup of $G$ we will mean one containing $B$. Therefore, the Levi quotient $L(Q)$ of any parabolic subgroup $Q \subset G$ is also a subgroup of $Q$. Let $Z_0(G)$ denote the connected component of the center of $G$ that contains the identity element.

Take any polystable principal $G$–bundle $E_G$ over $M$. Therefore, there is a Levi subgroup $L(P)$ associated to some parabolic subgroup $P \subset G$ (the subgroup $P$ need not be proper) along with a holomorphic reduction of structure group $E_{L(P)} \subset E_G$ to $L(P)$, such that

- the principal $L(P)$–bundle $E_{L(P)}$ is stable, and
- the principal $P$–bundle $E_P := E_{L(P)}(P)$, obtained by extending the structure group of $E_{L(P)}$ using the inclusion $L(P)$ in $P$, is an admissible reduction of $E_G$.

Note that since $L(P) \subset P \subset G$, the $P$–bundle $E_P$ is a reduction of structure group of $E_G$ to $P$. The condition that $E_P$ is an admissible reduction of structure group of $E_G$ means that for each character $\chi$ of $P$, which is trivial on $Z_0(G)$, the associated line bundle $E_P(\chi)$ over $M$ is of degree zero. The $G$–bundle $E_G$ is stable if and only if $P = G$.

Since $E_G$ and $E_{L(P)}$ are polystable, they admit unique Einstein–Hermitian connections [AB p. 208, Theorem 0.1]. It should be clarified that the Einstein–Hermitian reduction of structure to a maximal compact subgroup depends on the choice of the maximal compact subgroup. Even for a fixed maximal compact subgroup, the Einstein–Hermitian reduction of structure group need not be unique. However, once the Kähler form on $M$ is fixed, the Einstein–Hermitian connection on a polystable principal bundle over $M$ is unique. Let $\nabla^{L(P)}$ be the Einstein–Hermitian connection on the stable $L(P)$–bundle $E_{L(P)}$. Let $\nabla^G$ be the connection on $E_G$ induced by $\nabla^{L(P)}$. Note that a connection on $L(P)$ induces a connection on any fiber bundle associated to $E_{L(P)}$. The principal $G$–bundle $E_G$ is associated to $E_{L(P)}$ for the left translation action of $L(P)$ on $G$, and $\nabla^G$ is the connection on it obtained from $\nabla^{L(P)}$.

**Proposition 2.1.** The induced connection $\nabla^G$ on $E_G$ coincides with the unique Einstein–Hermitian connection on $E_G$.

**Proof.** Since the inclusion map $L(P) \hookrightarrow G$ need not take the connected component, containing the identity element, of the center of $L(P)$ into $Z_0(G)$, the proposition does not follow immediately from [AB p. 208, Theorem 0.1]. The connection $\nabla^{L(P)}$ on $E_{L(P)}$ is induced by a connection on a smooth reduction of structure group of $E_{L(P)}$ to a maximal compact subgroup of $L(P)$ (this is a part of the definition of a Einstein–Hermitian connection). A maximal compact subgroup of $L(P)$ is contained in a maximal compact subgroup of $G$. Consequently, the connection $\nabla^G$ on $E_G$ is induced by a connection on a smooth
reduction of structure group of $E_G$ to a maximal compact subgroup of $G$. Therefore, to prove that $\nabla^G$ is the Einstein–Hermitian connection on $E_G$ it suffices to show that $\nabla^G$ satisfies the Einstein–Hermitian equation.

Let $\mathfrak{g}(L(P))$ be the Lie algebra of the center of $L(P)$. For any $\theta \in \mathfrak{z}(\mathfrak{g}(P))$, the holomorphic section of the adjoint bundle $\text{ad}(E_{L(P)})$ given by $\theta$ will be denoted by $\hat{\theta}$; the vector bundle $\text{ad}(E_{L(P)})$ is associated to $E_{L(P)}$ for the adjoint action of $L(P)$ on its Lie algebra. Let $\Lambda_\omega$ be the adjoint of multiplication by $\omega$ of differential form on $M$. That the connection $\nabla^L(P)$ is Einstein–Hermitian means that there is an element $\theta \in \mathfrak{z}(\mathfrak{g}(P))$ such that the Einstein–Hermitian equation

$$\Lambda_\omega K_{\nabla^L(P)} = \hat{\theta}$$

holds.

If $\theta$ in eqn. (1) is in the Lie algebra $\mathfrak{z}(G)$ of $Z_0(G)$, then $\nabla^G$ is a Einstein–Hermitian connection on $E_G$. Therefore, in that case the proposition is proved. Assume that

$$\theta \notin \mathfrak{z}(G).$$

Fix a character $\chi$ of $P$ which is trivial on $Z_0(G)$ but satisfies the following condition: the homomorphism of Lie algebras

$$d\chi : \mathfrak{p} \longrightarrow \mathbb{C}$$
given by $\chi$, where $\mathfrak{p}$ is the Lie algebra of $P$, is nonzero on $\theta$. (It is easy to check that the group of characters of $P$ coincides with the group of characters of $L(P)$.)

Consider the holomorphic line bundle $L_\chi := E_P(\chi)$ over $M$ associated to $E_P$ for $\chi$. Let $\nabla^\chi$ be the connection on $L_\chi$ induced by the connection on $E_P$ given by $\nabla^L(P)$. Since $\nabla^L(P)$ is a Einstein–Hermitian connection, the connection $\nabla^\chi$ is also Einstein–Hermitian. Indeed, if $K(\nabla^\chi)$ is the curvature of $\nabla^\chi$, then

$$\Lambda_\omega K(\nabla^\chi) = d\chi(\theta),$$

where $d\chi$ is the homomorphism in eqn. (3) and $\theta$ is the element in eqn. (1). Therefore,

$$\deg(L_\chi) = \frac{d\chi(\theta)\sqrt{-1}}{2\pi d} \int_M \omega^d,$$

where $d = \dim_{\mathbb{C}} M$; see [Ko] p. 103, Proposition 2.1).

The condition that $E_P \subset E_G$ is an admissible reduction of structure group says that

$$\deg(L_\chi) = 0.$$

This, in view of the assumption in eqn. (2) that $d\chi(\theta) \neq 0$, contradicts eqn. (4). Therefore, we conclude that $\theta \in \mathfrak{z}(G)$. This immediately implies that the connection $\nabla^G$ on $E_G$ is Einstein–Hermitian. This completes the proof of the proposition. □

Let $E_G$ be a holomorphic principal $G$–bundle over $M$ and $\nabla'$ a $C^\infty$ connection on $E_G$ compatible with the holomorphic structure of $E_G$. Such a connection is called a complex connection; see [AB] p. 230, Definition 3.1(1)] for the precise definition of a complex
connection. A connection \( \widetilde{\nabla} \) is complex if and only if the \((0,2)\)–Hodge type component of the curvature of \( \widetilde{\nabla} \) vanishes.

**Definition 2.2.** Let \( H \) be a closed complex subgroup of \( G \) and \( E_H \subset E_G \) a \( C^\infty \) reduction of structure group of \( E_G \) to \( H \). We will say that \( E_H \) is preserved by the connection \( \nabla' \) if \( \nabla' \) induces a connection on \( E_H \).

It follows immediately that \( E_H \) is preserved by \( \nabla' \) if and only if there is a smooth connection on \( E_H \) that induces the connection \( \nabla' \) on \( E_G \). It is easy to see that \( E_H \) is preserved by \( \nabla' \) if and only if for each point \( z \in E_H \), the horizontal subspace in \( T_zE_G \) for the connection \( \nabla' \) is contained in \( T_zE_H \). If \( E_H \) is preserved by \( \nabla' \), then \( E_H \) is a holomorphic reduction of structure group of \( E_G \).

**Theorem 2.3.** Let \( E_G \) be a stable principal \( G \)–bundle over \( M \) and \( \nabla^G \) the Einstein–Hermitian connection on \( E_G \). Let \( H \) be a complex reductive subgroup of \( G \) which is not necessarily connected, and let \( E_H \subset E_G \) be a holomorphic reduction of structure group of \( E_G \) to \( H \). Then \( E_H \) is preserved by the connection \( \nabla^G \).

**Proof.** Let \( H_0 \subset H \) be the connected component containing the identity element. Set

\[
X := \frac{E_H}{H_0},
\]

which is finite étale Galois cover of \( M \) with Galois group \( H/H_0 \). Let

\[
(5) \quad p : X \to M
\]

be the projection. We note that \( p^*E_H \) has a canonical reduction of structure group to the subgroup \( H_0 \subset H \). Set

\[
F_G := p^*E_G.
\]

Let \( F_{H_0} \subset F_G \) be the reduction of structure group to \( H_0 \) obtained from the canonical reduction of structure group of \( p^*E_H \) to \( H_0 \).

Equip \( X \) with the Kähler form \( p^*\omega \). The Einstein–Hermitian connection on \( E_G \) pulls back to a Einstein–Hermitian connection on \( F_G \) for the Kähler form \( p^*\omega \). Therefore, \( F_G \) is polystable with respect to \( p^*\omega \). Let \( \text{ad}(F_G) \) be the adjoint bundle over \( X \). We recall that \( \text{ad}(F_G) \) is associated to \( F_G \) for the adjoint action of \( G \) on its Lie algebra. The connection on the vector bundle \( \text{ad}(F_G) \) induced by the Einstein–Hermitian connection of \( F_G \) is clearly Einstein–Hermitian. Hence the adjoint vector bundle \( \text{ad}(F_G) \) is polystable.

Let \( Z(G) \) be the center of \( G \). Set

\[
H' := \frac{H_0}{(H_0 \cap Z(G))}.
\]

Therefore, \( H' \) is a complex reductive subgroup of the complex semisimple group \( G' := G/Z(G) \). Let \( \mathfrak{g} \) (respectively, \( \mathfrak{h}_0 \)) be the Lie algebra of \( G \) (respectively, \( H_0 \)). Note that the adjoint action makes \( \mathfrak{g} \) (respectively, \( \mathfrak{h}_0 \)) a \( G' \)–module (respectively, \( H' \)–module). We will also consider \( \mathfrak{g} \) as a \( H' \)–module using the inclusion of \( H' \) in \( G' \).
Since \( g \) is a faithful \( G' \)–module, and \( H' \) is a reductive subgroup of \( G' \), there is a positive integer \( N \) and nonnegative integers \( a_i, b_i \), \( i \in [1, N] \), such that the \( H' \)–module \( \mathfrak{h}_0 \) is a direct summand of the \( H' \)–module
\[
\bigoplus_{i=1}^{N} \mathfrak{g}^{\otimes a_i} \otimes (\mathfrak{g}^*)^{\otimes b_i} = \bigoplus_{i=1}^{N} \mathfrak{g}^{\otimes a_i} \otimes \mathfrak{g}^{\otimes b_i}
\]
\[\text{[De, p. 40, Proposition 3.1]}\]; since \( H' \) is complex reductive, any exact sequence of \( H' \)–modules splits; also, \( g = g^* \) as \( G \) is reductive. Therefore, the adjoint vector bundle \( \text{ad}(F_{H_0}) \) is a direct summand of the vector bundle
\[
\bigoplus_{i=1}^{N} \text{ad}(F_G)^{\otimes a_i} \otimes \text{ad}(F_G)^{\otimes b_i}.
\]
We note that the vector bundle in eqn. (7) is associated to \( F_{H_0} \) for the \( H_0 \)–module in eqn. (6).

Since the adjoint vector bundle \( \text{ad}(F_G) \) is polystable of degree zero (recall that \( \text{ad}(F_G) = \text{ad}(F_G)^* \)), the vector bundle
\[
\text{ad}(F_G)^{\otimes a} \otimes \text{ad}(F_G)^{\otimes b}
\]
is polystable of degree zero for all \( a, b \in \mathbb{N} \) \[\text{AB, p. 224, Theorem 3.9}\]. Therefore, the vector bundle in eqn. (7) is polystable of degree zero. Since \( \text{ad}(F_{H_0}) \) is a direct summand of it of degree zero, we conclude that \( \text{ad}(F_{H_0}) \) is also polystable. Consequently, the principal \( H_0 \)–bundle \( F_{H_0} \) is polystable \[\text{AB, p. 224, Corollary 3.8}\]. Hence \( F_{H_0} \) admits a unique Einstein–Hermitian connection \[\text{AB, p. 208, Theorem 0.1}\]. Let \( \nabla^{H_0} \) be the Einstein–Hermitian connection on \( F_{H_0} \). So, there is an element \( \nu \in \mathfrak{h} \) such that
\[
\Lambda_{p^*}\mathcal{K}(\nabla^{H_0}) = \hat{\nu},
\]
where \( \mathcal{K}(\nabla^{H_0}) \) is the curvature of \( \nabla^{H_0} \), and \( \hat{\nu} \) is the holomorphic section of \( \text{ad}(p^*E_H) = \text{ad}(F_{H_0}) \) given by \( \nu \).

Since \( H/H_0 \) is a finite group, giving a connection on a principal \( H_0 \)–bundle is equivalent to giving a connection on the principal \( H \)–bundle obtained from it by extension of structure group. From the uniqueness of the Einstein–Hermitian connection on \( F_{H_0} \) it follows that the corresponding connection on \( p^*E_H \) is left invariant by the action of the Galois group \( H/H_0 \). Consequently, the connection on \( p^*E_H \) given by \( \nabla^{H_0} \) descends to a connection on \( E_H \). Let \( \nabla^{H} \) denote the connection on \( E_H \) obtained this way. It is clear that \( \nabla^{H} \) is a complex connection.

Let \( \nabla \) be the complex connection on \( E_G \) induced by the above connection \( \nabla^{H} \) on \( E_H \). We will first show that \( \nabla \) is unitary, which means that \( \nabla \) is induced by connection on a smooth reduction of structure group of \( E_G \) to a maximal compact subgroup of \( G \). Then we will show that \( \nabla \) satisfies the Einstein–Hermitian equation.

To prove that \( \nabla \) is unitary, fix a point \( x_0 \in M \), and also fix a point \( z_0 \in (E_G)_{x_0} \) in the fiber of \( E_G \) over \( x_0 \). Taking parallel translations of \( z_0 \), with respect to the connection \( \nabla \), along piecewise smooth paths in \( M \) based at \( x_0 \) we get a subset \( \mathcal{S} \) of \( E_G \). Sending
any \( g \in G \) to the point \( z_0g \in (E_G)_{x_0} \) we get an isomorphism \( G \rightarrow (E_G)_{x_0} \). Using this isomorphism, the intersection \((E_G)_{x_0} \cap S\) is a subgroup \( K_E \) of \( G \). The condition that the connection \( \nabla \) is unitary is equivalent to the condition that \( K_E \) is contained in some compact subgroup of \( G \).

Fix a point \( x \in p^{-1}(x_0) \), and also fix a point \( z \in (p^*E_G)_{x_0} \) that projects to \( z \), where \( p \) is the covering map in eqn. (5). Consider parallel translations of \( z \), with respect to the connection \( p^*\nabla \) on \( p^*E_G =: F_G \), along piecewise smooth paths in \( X \) based at \( x \). As before, we get a subgroup \( K_F \subset G \) from the resulting subset of \( F_G \). It is easy to see that \( K_F \) is a finite index subgroup of the group \( K_E \) constructed above.

We note that the connection \( p^*\nabla \) on \( F_G \) is induced by the unitary connection \( \nabla^{H_0} \) on the reduction \( F_{H_0} \) of \( F_G \). Therefore, the connection \( p^*\nabla \) is unitary. Consequently, the subgroup \( K_F \subset G \) is contained in a compact subgroup of \( G \). Using this together with the observation that \( K_F \) is a finite index subgroup of the subgroup \( K_E \subset G \) we conclude that \( K_E \) is also contained in a compact subgroup of \( G \). Thus the connection \( \nabla \) is unitary.

We will now show that \( \nabla \) satisfies the Einstein–Hermitian equation.

Let \( K(\nabla) \) be the curvature of \( \nabla \). From eqn. (5) it follows immediately that \( \Lambda_p^*\omega K(\nabla^{H_0}) \) is a holomorphic section of \( \text{ad}(p^*E_H) \). Consequently, \( \Lambda_p^*\omega K(\nabla) \) is a holomorphic section of the adjoint vector bundle \( \text{ad}(E_G) \). Since \( E_G \) is stable, all holomorphic sections of \( \text{ad}(E_G) \) are given by the Lie algebra \( \mathfrak{z}(G) \) of \( Z_0(G) \) \cite[Proposition 3.3]{BC}. In other words, there is an element \( \theta \in \mathfrak{z}(G) \) such that \( \Lambda_p^*\omega K(\nabla) \) coincides with the section of \( \text{ad}(E_G) \) given by \( \theta \). Thus, we conclude that the connection \( \nabla \) is the unique Einstein–Hermitian connection on \( E_G \). Since \( \nabla \) is induced by a connection on \( E_H \), the proof of the theorem is complete. \( \square \)

Proposition 2.1 and Theorem 2.3 together have the following corollary.

**Corollary 2.4.** Let \( E_G \) be a polystable \( G \)-bundle over \( M \). Take \( E_{L(P)} \) as in Proposition 2.1. Let \( H \) be a complex reductive subgroup of \( L(P) \) (not necessarily connected), and let \( E_H \subset E_{L(P)} \) be a holomorphic reduction of structure group of \( E_{L(P)} \) to \( H \). Then \( E_H \) is preserved by the Einstein–Hermitian connection on \( E_G \).

3. **Properties of a smallest reduction**

Let \( E_G \) be a stable principal \( G \)-bundle over \( M \). Fix a point \( x_0 \in M \), and also fix a point \( z_0 \in (E_G)_{x_0} \) in the fiber over \( x_0 \). Let \( \nabla^G \) be the Einstein–Hermitian connection on \( E_G \).

Taking parallel translations of \( z_0 \), with respect to \( \nabla^G \), along piecewise smooth paths in \( M \) based at \( x_0 \) we get a subset \( S \) of \( E_G \). Let \( \overline{S} \) be the topological closure of \( S \) in \( E_G \). The map \( G \rightarrow (E_G)_{x_0} \) defined by \( g \mapsto z_0g \) is an isomorphism. Using this isomorphism, the intersection \((E_G)_{x_0} \cap \overline{S}\) gives a compact subgroup \( K_E \subset G \). The subset \( \overline{S} \subset E_G \) is a \( C^\infty \) reduction of structure group of \( E_G \) to the subgroup \( K_E \). Let \( \overline{K}_E \) be the complex
reductive subgroup of $G$ obtained by taking the Zariski closure of $\overline{K_E}$ in $G$. The subset
\begin{equation}
E_{K_E}^{z_0} := S\overline{K_E} \subset E_G
\end{equation}
is a holomorphic reduction of structure group of $E_G$ to $\overline{K_E}$; see [Bi, Section 3] for the
details. It is easy to see that the principal $K_E$–bundle $E_{z_0}^{K_E}$ is the extension of structure
group of the principal $K_E$–bundle $E_{z_0}$. We note that in [Bi], the point $z_0$ is taken to be in the subset of
$E_G$ given by an Einstein–Hermitian reduction of structure group (see [Bi, p. 71, (3.20)]). T his was only to make
$K_E$ lie inside a fixed maximal compact subgroup of $G$. If we replace the base point $z_0$ by $z_0g$, where $g$ is any point of $G$, then it is easy to see that the subset $S$ constructed above using $z_0$ gets replaced by $Sg$. Therefore, the subgroup $\overline{K_E}$ in eqn. (9) gets replaced by $g^{-1}\overline{K_E}g$, and the reduction $E_{z_0}^{K_E} \subset E_G$ in eqn. (9) gets replaced by $E_{z_0}^{gK_E}$. We note that $(\overline{K_E}, E_{z_0}^{K_E})$ in eqn. (9) is a minimal complex reduction of $E_G$ in the
following sense: There is no complex proper subgroup of $\overline{K_E}$ to which $E_{z_0}^{K_E}$ admits a
holomorphic reduction of structure group which is preserved by the connection $\nabla^G$. Indeed, this follows immediately from the construction of $E_{z_0}^{K_E}$. Furthermore, if $E_{H'} \subset E_G$
is a holomorphic reduction of structure group of $E_G$, to a complex subgroup $H' \subset G$, satisfying the two conditions:
\begin{itemize}
  \item $E_{H'}$ is preserved by $\nabla^G$, and
  \item there is no complex proper subgroup of $H'$ to which $E_{H'}$ admits a holomorphic
    reduction of structure group which is also preserved by $\nabla^G$,
\end{itemize}
then it follows immediately that there is a point $z_0 \in (E_G)_{x_0}$ such that $E_{H'} = E_{z_0}^{K_E}$. Indeed, $z_0$ can be taken to be any point of the fiber $(E_{H'})_{x_0}$.

Let $H$ be a complex reductive subgroup of $G$ which is not necessarily connected, and let $E_H \subset E_G$ be a holomorphic reduction of structure group to $H$ of the stable $G$–bundle $E_G$. This reduction $E_H$ will be called a *minimal reductive reduction* of $E_G$ if there is no complex reductive proper subgroup of $H$ to which $E_H$ admits a holomorphic reduction of structure group.

In view of the above observations, using Theorem 2.3 get the following theorem:

**Theorem 3.1.** Let $E_G$ be a stable principal $G$–bundle over a compact connected Kähler manifold $M$, where $G$ is a connected reductive complex linear algebraic group. Let $E_H \subset E_G$ be a minimal reductive reduction of $E_G$. Fix a point $x_0 \in M$. Then there is a point $z_0$ in the fiber $(E_G)_{x_0}$ such that $E_{z_0}^{K_E}$ (defined in eqn. (9)) coincides with $E_H$. In particular, the subgroup $H$ coincides with $\overline{K_E}$ for such a base point $z_0$.

Let $\text{Ad}(E_G)$ be the adjoint bundle for $E_G$. So $\text{Ad}(E_G)$ is the fiber bundle over $M$ associated to $E_G$ for the adjoint action of $G$ on itself. The fibers of $\text{Ad}(E_G)$ are groups
isomorphic to $G$. Let $\text{Ad}(E_{z_0}^{E_G})$ be the adjoint bundle of the principal $K_E^C$–bundle $E_{z_0}^{E_G}$ defined in eqn. \(\mathfrak{g}\). Since $E_{z_0}^{E_G}$ is a holomorphic reduction of structure group of $E_G$, the fiber bundle $\text{Ad}(E_{z_0}^{E_G})$ is a holomorphic sub–fiber bundle of $\text{Ad}(E_G)$ with the fibers of $\text{Ad}(E_{z_0}^{E_G})$ being subgroups of the fibers of $\text{Ad}(E_G)$.

We noted earlier that if $z_0$ is replaced by $z_0g$, where $g \in G$, then the subgroup $K_E^C$ gets replaced by $g^{-1}K_E^Cg$, and $E_{z_0}^{E_G}$ gets replaced by $E_{z_0}^{E_G}g$. From this it follows immediately that the subbundle $\text{Ad}(E_{z_0}^{E_G}) \subset \text{Ad}(E_G)$ is independent of the choice of the base point $z_0$.

Therefore, from Theorem 3.1 we have the following corollary:

**Corollary 3.2.** Let $E_G$ be a stable principal $G$–bundle over a compact connected Kähler manifold $M$, where $G$ is a connected reductive complex linear algebraic group. Then there exists a unique holomorphic sub–fiber bundle $G_E$ of the adjoint bundle $\text{Ad}(E_G)$, with fibers being subgroups, that satisfies the following condition: For any minimal reductive reduction $E_H$ of $E_G$, the adjoint bundle $\text{Ad}(E_H)$, which is a sub–fiber bundle of $\text{Ad}(E_G)$, coincides with $G_E$.

From Theorem 3.1 it follows that the sub–fiber bundle $G_E \subset \text{Ad}(E_G)$ in Corollary 3.2 is the complexification of the holonomy of the Einstein–Hermitian connection on $E_G$. We note that for defining stable bundles we need only the class $[\omega] \in H^2(M, \mathbb{R})$. In other words, the stability condition does not depend on the choice of the Kähler form in a given Kähler class. On the other hand, the Einstein–Hermitian connection on a stable bundle depends on the choice of the Kähler form. From Corollary 3.2 it follows that the sub–fiber bundle $G_E$ does not depend on the choice of the Kähler form in a given Kähler class. In fact, it does not even depend on the Kähler class as long as $E_G$ remains stable.

Using Corollary 3.2 and \[\text{BK}\] Theorem 20] it follows that the algebraic monodromy of $E_G$ coincides with the fiber $(G_E)_{z_0}$ when $M$ is a complex projective manifold. (See \[\text{BK}\] for algebraic monodromy where it is introduced.)

**References**

- [AB] B. Anchouche and I. Biswas, Einstein–Hermitian connections on polystable principal bundles over a compact Kähler manifold, Amer. Jour. Math. 123 (2001), 207–228.
- [BK] V. Balaji and J. Kollár, Holonomy groups of stable vector bundles, Preprint, \[\text{math.AG/0601120}\]
- [BG] I. Biswas and T. L. Gómez, Simplicity of stable principal sheaves, Bull. Lond. Math. Soc. (to appear).
- [Bi] I. Biswas, Stable bundles and extension of structure group, Diff. Geom. Appl. 23 (2005), 67–78.
- [De] P. Deligne, Hodge cycles on abelian varieties, (in: Hodge cycles, motives, and Shimura varieties, by P. Deligne, J. S. Milne, A. Ogus and K.-Y. Shih), Lecture Notes in Mathematics, 900, Springer-Verlag, Berlin-New York, 1982.
- [Ko] S. Kobayashi, Differential Geometry of Complex Vector Bundles, Publications of the Math. Society of Japan 15, Iwanami Shoten Publishers and Princeton University Press, 1987.
[RS] A. Ramanathan and S. Subramanian, Einstein–Hermitian connections on principal bundles and stability, Jour. Reine Angew. Math. 390 (1988), 21–31.

[Ra] A. Ramanathan, Moduli for principal bundles over algebraic curves: I, Proc. Ind. Acad. Sci. (Math. Sci.) 106 (1996), 301–328.

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

E-mail address: indranil@math.tifr.res.in