Exceptional boundary states at $c = 1$

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Abstract

We consider the CFT of a free boson compactified on a circle, such that the compactification radius $R$ is an irrational multiple of $R_{selfdual}$. Apart from the standard Dirichlet and Neumann boundary states, Friedan suggested that an additional 1-parameter family of boundary states exists. These states break $U(1)$ symmetry of the theory, but still preserve conformal invariance. In this paper we give an explicit construction of these states, show that they are uniquely determined by the Cardy-Lewellen sewing constraints, and we study the spectrum in the ‘open string channel’, which is given here by a continuous integral with a nonnegative measure on the space of conformal weights.

1 Introduction

Boundary states in Conformal Field Theories (CFT) correspond to boundary conditions that one may impose on the CFT that do not destroy conformal invariance. The study of these states (or equivalently boundary conditions) is interesting both from the point of view of statistical mechanics, where one can gain a lot of information on the

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behaviour of two-dimensional critical systems in confined geometries \[2\], and from the point of view of string theory where these states correspond to D-branes, nonperturbative objects in string theories \[3, 4\].

The theoretical description of these states and, more generally, boundary conformal field theory (BCFT), is very well developed in the context of rational theories (i.e. with a finite number of primary fields w.r.t Virasoro or an extended chiral algebra) \[5, 6, 7, 8, 9\]. However once we relax the requirement of rationality (by e.g. breaking the invariance w.r.t. an extended chiral algebra) new phenomena may appear such as moduli spaces for boundary states. In addition, continuous spectra of conformal weights may make the meaning of Cardy’s condition for boundary states unclear. It is thus very interesting to investigate BCFT in the nonrational case.

In this work we would like to consider boundary states in a (compactified) free boson theory. This theory has an affine $U(1)$ symmetry, and the boundary states preserving that are the classical Neumann and Dirichlet b.s. When the boundary states break $U(1)$ but still preserve conformal symmetry, Friedan \[1\] suggested that a continuous family of boundary states should appear parametrized by a real interval. The goal of this paper is to propose an explicit construction of these states and to study some of their properties like the ‘open string channel’ spectrum.

The interest in such symmetry breaking conditions lies also in the fact that usually a lot of symmetry which makes a theory rational leads to just a finite number of boundary states, like e.g. a finite number of D-brane states in the Gepner models of Calabi-Yau compactifications \[3\]. In order to explore D-branes with nontrivial moduli space in this context one has to impose much less stringent restrictions on the symmetry properties of the states.

The plan of this paper is as follows. In section 2 we will briefly recall basic properties of the $c = 1$ discrete states and some features of BCFT. In section 3 we state the proposal for the explicit form of Friedan’s boundary states. Then we show that the coefficients of these boundary states are in fact uniquely determined by the Cardy-Lewellen sewing conditions. In section 4 we show it for the lowest nontrivial case, while the general, more technical proof is relegated to the appendix. In section 6 we make the transformation into the open string channel. We close the paper with a discussion and a summary.

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1This is the case when the compactification radius is such that $R/R_{selfdual}$ is irrational.
2 $c = 1$ discrete states and BCFT

Let us consider a free boson compactified on a circle of radius $R$. The partition function of this theory is

$$Z = \frac{1}{\eta(\tau)\eta(\bar{\tau})} \sum_{e, m \in \mathbb{Z}} q^{(e/R+rnR/2)^2/2} \bar{q}^{(e/R-rmR/2)^2/2}$$ \hspace{1cm} (1)

This CFT has $c = 1$ and possesses a $U(1)$ current algebra. The terms in the sum are characters of $U(1)$ Kac-Moody representations labelled by $U(1)$ charges.

In this paper we will restrict ourself to the case when $R$ is an irrational multiple of $R_{\text{selfdual}}$. Then all the $U(1)$ representations for $e \neq 0$ or $m \neq 0$ are also irreducible w.r.t the Virasoro algebra. Only for $e = m = 0$ (the vacuum sector) the $U(1)$ representation decomposes into an infinite sum of Virasoro representations associated with the so-called discrete states $[10, 11]$: \hspace{1cm} (2)

$$\chi_{e=m=0}(q) = \frac{1}{\eta(q)} = \sum_{J=0}^{\infty} \chi_{J^2}^{Vir}(q)$$

where

$$\chi_{J^2}^{Vir}(q) = \frac{q^{J^2} - q^{(J+1)^2}}{\eta(q)}$$ \hspace{1cm} (3)

The corresponding (unnormalized) Virasoro primary fields are given by the expressions $[12]$: \hspace{1cm} (4)

$$V_J = \left( \int \frac{du}{2\pi} e^{-i\sqrt{2}X(u+z)} \right)^J e^{i\sqrt{2}X(z)}$$

The free boson $X(z)$ is normalized here so that the conformal weight $h_J = J^2$.

Boundary states are constructed from Ishibashi states $[13]$ labelled by Virasoro primaries. The standard boundary states corresponding to Neumann and Dirichlet boundary conditions are given by the expressions (see e.g. $[14]$):

$$|N(\theta)\rangle = \sum_{J=0}^{\infty} (-1)^J |J\rangle + \sum_{m \in \mathbb{Z}/\{0\}} e^{im\theta} |(0, m)\rangle$$ \hspace{1cm} (5)

$$|D(\theta)\rangle = \sum_{J=0}^{\infty} |J\rangle + \sum_{e \in \mathbb{Z}/\{0\}} e^{ie\theta} |(e, 0)\rangle$$ \hspace{1cm} (6)
where \(|(e, 0)\rangle\) (resp. \(|(0, m)\rangle\)) correspond to vertex operators with the same (resp. opposite) \(U(1)\) charges in the holomorphic and antiholomorphic sectors. It is well known [15, 16, 17] that for special values of the compactification radius additional boundary states may appear. However even for the generic case of irrational \(R/R_{selfdual}\) Friedan suggested [1] that a 1-parameter family of boundary states should appear. An explicit construction of this family is the main aim of this paper. Before we proceed let us first recall some features of BCFT.

In general we have an expansion

\[
|\alpha\rangle = \sum_{J=0}^{\infty} B^\alpha_J |J\rangle + \sum_{e \in \mathbb{Z}/\{0\}} B^\alpha_e |(e, 0)\rangle + \sum_{m \in \mathbb{Z}/\{0\}} B^\alpha_m |(0, m)\rangle
\]  

(7)

The coefficients \(B^\alpha_J, B^\alpha_e, B^\alpha_m\) are required to satisfy a number of constraints.

One fundamental property that one has to impose is that the spectrum of the theory between any two such states decomposes into characters of Virasoro representations (Cardy’s condition):

\[
\sum_i B_i^\alpha B_i^\beta \chi^{Vir}_i(q) = \sum_A n^A_{\alpha \beta} \chi^{Vir}_A(\tilde{q})
\]

(8)

where \(n^A_{\alpha \beta}\) are some coefficients, \(q = e^{2\pi i\tau}\), \(\tilde{q}\) is related to \(q\) by the modular transformation \(\tau \rightarrow -1/\tau\), and \(\chi^{Vir}_A\) are Virasoro characters at the same central charge.

When we are in the rational setting the sum is finite and the coefficients have to be integers (this is the main content of Cardy’s condition). In the nonrational case the summation is transformed into an integral and the meaning of Cardy’s condition then is not completely understood, in particular it is not obvious what conditions one should impose on the \(n^A_{\alpha \beta}\) which now is a ‘density function’ (measure) on the space of conformal weights \(A\).

The least that we may do is to require that for any choice of boundary states \(|\alpha\rangle, |\beta\rangle\) the density function \(n^A_{\alpha \beta}\) should be non-negative. In section 6 we will verify that this property holds for our construction of Friedan’s boundary states.

Apart from Cardy’s condition, boundary state coefficients have to satisfy a set of sewing constraints [1, 6, 8], which link them with other ingredients of a BCFT such as bulk-to-boundary operator product coefficients, OPE between boundary fields, 1-point functions etc.
We will show that these constraints determine unambiguously all
the coefficients of \( \Phi_i(z, \bar{z})_\alpha \). From this point of view, the most interesting
quantities in the BCFT are the 1-point functions \( A^\alpha_i \) defined through
\[
\langle \Phi_i(z, \bar{z}) \rangle_\alpha = \frac{A^\alpha_i}{|z - \bar{z}|^{2h_i}}
\] (9)
where the superscript \( \alpha \) denotes that the average is performed on the
upper half plane with the boundary condition \( |\alpha\rangle \) on the real axis.
There is a very close relation of these data to the coefficients of the
boundary state (7), first obtained in \([6]\):
\[
A^\alpha_i = \frac{B^\alpha_i}{B^\alpha_0}
\] (10)
where 0 denotes the identity. In the following we will normalize the
coefficient of the identity to be 1, which allows us to identify the
coefficients \( B^\alpha_i \) with the 1-point functions \( A^\alpha_i \).

Cardy and Lewellen derived a nonlinear constraint on the 1-point
functions \([6, 7, 9]\) which we will use in section 4 to determine the form
of the boundary states (7).

3 A proposal for Friedan’s boundary
states

We propose the following expression for the boundary states (7) con-
structed using the Ishibashi states corresponding to the fields (4):
\[
|x\rangle = \mathcal{N} \sum_{l=0}^{\infty} P_l(x) |l\rangle
\] (11)
where \( P_l(x) \) are the Legendre polynomials and \( x \in (-1, 1) \). In par-
ticular the \( U(1) \) charged states will have coefficient zero. The nor-
malization constant \( \mathcal{N} \) is undetermined and in the rest of the paper
we will set it equal to 1. We will use the coefficient of the \( |1\rangle \) state
\( (P_1(x) = x) \) as a label for the state.

This choice of coefficients has the most natural behaviour under
modular transformations to the open string channel, which was ini-
tially our main motivation for (11). In addition we will show that all
the coefficients of (11) are indeed determined \textit{uniquely} as a function
of \( x \) through the Cardy-Lewellen sewing conditions.
In the next section we will show it explicitly for the coefficient of \( |2\rangle \rangle \). The full proof is summarized in Appendix A.

Recently exactly this expression was reobtained through a certain limiting procedure \[18\] from theories with the compactification radius being \( R = \frac{N}{M} R_{selfdual} \).

### 4 Cardy-Lewellen sewing condition

From the point of view of determining the coefficients of the boundary states the most interesting condition is the sewing constraint for 1-point functions:

\[
B_i^\alpha B_j^\alpha = \sum_k C_{ij}^k B_k^\alpha F_{k0} \left[ \begin{array}{c} j \\ i \\ i \end{array} \right] \tag{12}
\]

From now on we will drop the superscript \( \alpha \) in \( B_i^\alpha \). This equation has been (implicitly or explicitly) used in recent studies of boundary states in Liouville and \( SL(2,\mathbb{R}) \) WZW theory \[13, 20, 21\]. The \( C_{ij}^k \) are the OPE coefficients for the bulk fields, while the fusing matrix \( F_{k0} \left[ \begin{array}{c} j \\ i \\ i \end{array} \right] \) relates conformal blocks of argument \( \eta \) to conformal blocks at \( 1 - \eta \):

\[
F_{ij,ij}(1 - \eta) = \sum_q F_{ij}^{pq} \left[ \begin{array}{c} j \\ i \\ i \end{array} \right] F_{ij,ii}^{q}(\eta) \tag{13}
\]

Here the conformal blocks are normalized as \( F_{ij}^{ql}(\eta) \sim 1 \cdot \eta^{b_q - h_j - h_i} \) for small \( \eta \).

Let us now set \( i = 1 \) (primary field of conformal weight \( h = 1^2 = 1 \)) in the CL sewing equation (12). Then it is easy to see that given the value of

\[
B_1 = x \tag{14}
\]

all the other coefficients are uniquely fixed. In order to verify that the proposal (11) is correct we have to verify that (12) becomes equivalent to the recursion relation for the Legendre polynomials:

\[
x P_j(x) = \frac{j}{2j+1} P_{j-1} + \frac{j+1}{2j+1} P_{j+1} \tag{15}
\]

We will now verify explicitly the lowest nontrivial case \( (j = 1) \), and state the general more technical argument in Appendix A.
From the explicit form of the primary fields (4) we see that the OPE coefficient $C_{ij}^k$ vanishes unless $k = j - 1$ or $k = j + 1$ (see also appendix A). Therefore (12) takes the form of a two term recursion relation similar to (13). It remains to show that the coefficients coincide.

**OPE coefficients**

We now have to compute the OPE’s. The fields should first be normalized so that the coefficient of the 2-point function on the plane is unity. The $C_{ij}^k$ can be calculated through 3-point functions, which follow from calculating the correlation function of the exponentials of the free field $X$ and then performing the integrals over the $u$’s by residues at $u = 0$ (cf (4)). The antiholomorphic sector contributes equally and multiplicatively since the relevant operators are diagonal. The results for the normalisations are

$$\langle V_1 V_1 V_0 \rangle = 2 \quad \langle V_2 V_2 V_0 \rangle = 24^2 \quad \langle V_3 V_3 V_0 \rangle = 720^2 \quad (16)$$

Here we suppressed the standard conformal prefactor. The normalised operators are therefore $O_0 = V_0$, $O_1 = \frac{1}{2} V_1$, $O_2 = \frac{1}{24} V_2$ and $O_3 = \frac{1}{720} V_3$. The lowest nonvanishing OPE’s are

$$\langle V_1 V_1 V_2 \rangle = 8^2 \quad \langle V_1 V_2 V_3 \rangle = 144^2 \quad (17)$$

Which lead to the OPE coefficients:

$$C_{11}^0 = 1 \quad C_{11}^2 = \frac{1}{4} \frac{1}{24} \cdot 8^2 = \frac{2}{3} \quad C_{12}^3 = \frac{1}{2} \frac{1}{24} \frac{1}{720} \cdot 144^2 = \frac{3}{5} \quad (18)$$

The CL equation (12) at $j = 1$ can then be rewritten as:

$$xB_1 = 1 \cdot F_{00} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} B_0 + \frac{2}{3} \cdot F_{20} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} B_2 \quad (19)$$

while the recursion relation for the Legendre polynomials takes the form $xP_1 = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)$. It remains to determine the fusing matrices.

**Fusing matrices**

We first have to determine the conformal blocks $\mathcal{F}^0_{11,11}(\eta)$, $\mathcal{F}^1_{11,11}(\eta)$ and $\mathcal{F}^2_{11,11}(\eta)$. To this end we note that there is a level-3 singular vector in the Virasoro representation with $h = 1$ and $c = 1$, namely

$$\langle L_{-3} - \frac{2}{3}L_{-1}L_{-2} + \frac{1}{6}L_{-1}^2 \rangle |1\rangle \quad (20)$$
This can be directly translated into a 3rd order differential equation for the conformal blocks (see Appendix B), which has the following three linear independent solutions

\[ f_0(\eta) = \frac{1}{\eta^2(1-\eta)^2} \sim \frac{1}{\eta^2} (1 + 2\eta + \ldots) \]  

\[ f_1(\eta) = \frac{1 - \frac{3}{2}\eta + \eta^2}{\eta(1-\eta)^2} \sim \frac{1}{\eta} (1 + \ldots) \]  

\[ f_2(\eta) = \frac{\eta^2}{(1-\eta)^2} \sim \eta^2 (1 + \ldots) \]  

From the behaviour of the conformal blocks \( F_{11,11}(\eta) \sim \eta^{h_{11}^p - 2}(1 + \ldots) \) we can identify \( F_{11,11}^0(\eta) = f_2(\eta) \). From Virasoro algebra one can show that \( F_{11,11}^0(\eta) = \eta^{-2}(1 + 0 \cdot \eta + 2\eta^2 + \ldots) \). Therefore

\[ F_{11,11}^0(\eta) = f_0(\eta) - 2f_1(\eta) + \gamma f_2(\eta) \]  

where \( \gamma \) is still unknown. From these considerations we may now determine the appropriate fusing matrices. Because \( f_2(1-\eta) = f_0(\eta) - 4f_1(\eta) + f_2(\eta) \) and the coefficient of \( f_0 \) in \( F_{11,11}^0(\eta) \) is 1, we may read off

\[ F_{20} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 1 \]  

Analogously from \( f_0(1-\eta) = f_0(\eta) \) and \( f_1(1-\eta) = \frac{1}{2}f_0(\eta) - f_1(\eta) \) we get

\[ F_{00} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \gamma \]  

It remains therefore to identify \( \gamma \). In principle it could be done by calculating the coefficient of \( \eta^2 \) in \( F_{11,11}^0(\eta) \) which follows just from conformal symmetry [22], but since such a calculation is very tedious we will determine \( \gamma \) indirectly.

Consider the 4-point function

\[ \lim_{z_1 \to \infty} z_1^2 \bar{z}_1^2 \langle O_1(z_1)O_1(1)(\eta)O_1(0) \rangle \]  

Since \( O_1 \) is proportional to \( \partial X \bar{\partial} X \) we may calculate the above correlation function using Wick contractions and express it in terms of the \( f_i(\eta) \) with the result

\[ \left( \frac{1}{\eta^2} + 1 + \frac{1}{(1-\eta)^2} \right) \cdot \text{c.c} \equiv (f_0(\eta) - 2f_1(\eta) + f_2(\eta)) \cdot \text{c.c} \]
where c.c. stands for complex conjugate. On the other hand we know how the same correlation function is expressed in terms of conformal blocks:

\[
\left( 2 \sum_{p=0}^{2} \sqrt{C_{11}^{p}} \sqrt{C_{11}^{p}} F_{11,11}^{p}(\eta) \right) \cdot \text{c.c.} = \left| F_{11,11}^{0}(\eta) + \frac{2}{3} F_{11,11}^{2}(\eta) \right|^2 (29)
\]

Keeping in mind that \( F_{11,11}^{2} = f_2 \) and \( F_{11,11}^{0} = f_0 - 2f_1 + \gamma f_2 \) we read off that \( \gamma = \frac{1}{3} \). Thus we have finally

\[
F_{00} \left[ \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right] = \frac{1}{3} (30)
\]

Note that this is exactly the value to make the CL equation (19) to be consistent with the recurrence relation for the Legendre orthogonal polynomials. From this we get that \( B_2 = \frac{1}{2}(3x^2 - 1) \equiv P_2(x) \).

In Appendix A we show that the CL equation for \( i = 1 \) and arbitrary \( j \) coincides with the recursion relation for the Legendre polynomials (15), thus fixing uniquely all the coefficients of the discrete states to the values given in the proposal (11).

## 5 Remarks

We will now show that in general the \( U(1) \) charged states in (11) have vanishing coefficients. The CL equation (22) for the fields \( i = 1 \) and \( j = (e,0) \) takes the form

\[
x B_e = C_{i e}^{e} B_e F_{e0} \left[ \begin{array}{cc} e & e \\ 1 & 1 \end{array} \right] (31)
\]

Here we used the fact that there is a unique Virasoro primary with charge \( (e,e) \) for the theory with irrational \( R/R_{selfdual} \). We see that \( B_e \) may be nonzero only for a single value of \( x \). By comparison with the formulas (5-6) we get that \( x = 1 \). In an analogous manner we get that the \( B_m \) coefficient may be nonzero only for \( x = -1 \).

Therefore in the case \( x = \pm 1 \) the boundary states (11) become superpositions of Dirichlet and Neumann boundary states i.e.

\[
|1\rangle = \frac{1}{2\pi} \int_{0}^{2\pi} d\theta |D(\theta)\rangle \quad |-1\rangle = \frac{1}{2\pi} \int_{0}^{2\pi} d\theta |N(\theta)\rangle (32)
\]

\(^2\)Recall that the theory is not diagonal.
A similar observation has been made in [18] for theories with $R = \frac{N}{M} R_{selfdual}$.

In this context we see that one cannot obtain the family of Friedan’s boundary states from the conventional (Dirichlet or Neumann) theory through deformation by some marginal boundary operator, i.e. by writing the partition function on the disk as

$$Z_{\text{disk}} = \int DX e^{-\frac{1}{8\pi} \int_D d^2z \partial X \partial X + (x+1) \int_{\partial D} d\phi O_{\text{boundary}}(e^{i\phi})}$$  \hspace{1cm} (33)$$

The undeformed theory (with $x = -1$), as can be seen from (32), should really be a superposition of theories with different boundary conditions. Therefore we cannot directly apply the methods of [15, 16, 17] which were used to construct exceptional boundary states at special values of the compactification radii.

In the remaining part of the paper we will calculate the spectrum of the theory on a strip.

6 Open string channel spectrum

Let us consider the partition function of the CFT on a cylinder on the ends of which we impose boundary conditions $|\cos \theta_1\rangle$ and $|\cos \theta_2\rangle$. In the ‘closed string channel’ we have

$$Z = \sum_{l=0}^{\infty} P_l(\cos \theta_1) P_l(\cos \theta_2) \chi^{Vir}_l(q)$$  \hspace{1cm} (34)$$

We will now show that in the ‘open string channel’ this partition function can be written as a continuous integral of $c = 1$ characters $\chi^{Vir}_h(\tilde{q})$ (see (8)) with a nonnegative measure on the space of conformal weights $h$.

We start with the formula

$$P_l(\cos \theta_1) P_l(\cos \theta_2) = \frac{1}{\pi} \int_0^\pi P_l(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \cos \phi_2) / \cos \theta d\phi_2$$ \hspace{1cm} (35)$$

This expresses the product of two Legendre polynomials in terms of a single one. Now it is convenient to express the resulting Legendre polynomial as an integral over the SU(2) group character through

$$P_l(\cos \theta) = \frac{1}{\pi} \int_0^\pi \frac{\sin \left( l + \frac{1}{2} \right) t}{\sin \frac{t}{2}} d\phi$$  \hspace{1cm} (36)$$
where \( \cos t/2 = \cos \theta/2 \cos \phi/2 \). It is this expression which facilitates performing the modular transformation since

\[
\sum_{l=0}^{\infty} \frac{\sin \left( \frac{l+1}{2} \right) t}{\sin \frac{l}{2}} \left( q^{l^2} - q^{(l+1)^2} \right) = 1 + 2 \sum_{n=1}^{\infty} q^n \cos nt
\]

(37)

Now we can use Poisson resummation to easily perform the modular transformation to obtain

\[
\frac{1}{\sqrt{-i2\tau}} \sum_{n=-\infty}^{+\infty} \hat{q}^{\frac{1}{4}(n+\frac{t}{2\pi})^2}
\]

(38)

The \( \sqrt{-i\tau} \) will be compensated by the transformation formula for the Dedkind eta function: \( \eta(-1/\tau) = \sqrt{-i\tau} \eta(\tau) \).

Finally we obtain for the partition function \([33]\) but now viewed in the ‘open string channel’:

\[
Z = \frac{1}{\sqrt{2\pi i}} \int_0^{\pi} d\phi_2 \int_0^{\pi} d\phi \sum_{n=-\infty}^{+\infty} \frac{\hat{q}^{\frac{1}{4}(n+\frac{t}{2\pi})^2}}{\eta(-1/\tau)}
\]

(39)

where \( t \) depends on \( \theta_1, \theta_2, \phi \) and \( \phi_2 \) through

\[
\begin{align*}
\cos \frac{t}{2} &= \cos \frac{\theta}{2} \cos \frac{\phi}{2} \\
\cos \theta &= \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \cos \phi_2
\end{align*}
\]

(40)

(41)

In the above formulas we thus get a combination of Virasoro/\( U(1) \) characters:

\[
Z = \frac{1}{\sqrt{2\pi i}} \int_0^{\pi} d\phi_2 \int_0^{\pi} d\phi \sum_{n=-\infty}^{+\infty} \chi_{\frac{1}{4}(n+t/2\pi)^2}(\hat{q})
\]

(42)

The conformal weights \( h = \frac{1}{4}(n+t/2\pi)^2 \) appear in bands of finite width parametrized by the angles \( \phi \) and \( \phi_2 \):

\[
t(\theta_1, \theta_2, \phi, \phi_2) = 2 \arccos \left( \cos \frac{\phi}{2} \sqrt{\frac{1 + \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \cos \phi_2}{2}} \right)
\]

(43)

We see that the measure is explicitly nonnegative for any choice of \( \theta_1 \) and \( \theta_2 \). This requirement limits, in a natural way, the range of the parameter \( x \) labelling the boundary states \( |x\rangle \) \([11]\) to lie between -1 and 1. If \( x \) would be outside this interval, complex conformal weights would be generated.
7 Discussion

In this paper we constructed explicitly the 1-parameter family of boundary states, mentioned in [1], in a $c = 1$ free boson compactified on a circle of radius $R$ with $R/R_{\text{selfdual}}$ irrational. We showed, in particular, that the coefficients of these boundary states are uniquely fixed by the Cardy-Lewellen sewing conditions. This implies that at these radii no additional boundary states may appear. Once we move away from these radii, additional $U(1)$ charged states may enter the expression for boundary states and there may appear additional parameters. Technically this arises due to the appearance of additional terms on the rhs of (31). This complementary case of $R/R_{\text{selfdual}}$ rational is studied in [18].

An interesting feature of the family of Friedan’s boundary states is that they may not be reached by a deformation of a single standard (i.e. Neumann or Dirichlet) theory by a marginal boundary operator.

This gives rise to an intriguing open question what is the Lagrangian formulation of a free boson theory with Friedan’s boundary conditions. In this paper these boundary conditions were determined algebraically by fixing the coefficients of appropriate Ishibashi states. A direct understanding, however, of the meaning of these conditions in terms of the field $X(z, \bar{z})$ would be very interesting.

The answer to these questions might also shed some light on another puzzling feature of these states, namely the continuous spectrum of conformal weights in the open string channel. Cardy’s condition is now much less restrictive than in the rational case. The multiplicities are no longer constrained to be positive integers, but become a measure on the space of conformal weights. It seems that the only requirement that we may impose is just the positivity of the measure. And indeed we checked that for any two states from this family the relevant measure turns out to be nonnegative.

The structure of the space of boundary states even for the simplest example of an irrational free boson CFT turns out to be quite complex. The irrational BCFT’s exhibit several new phenomena like continuous spectra and nontrivial structure of the moduli spaces of states. It would be very interesting, both by its own right and in view of potential applications to D-branes, to understand these new features more systematically for a wider class of irrational BCFT’s.
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Appendix A — recursion relations (general case)

Here we will show that the CL sewing conditions for any \( k \)

\[
x B_k = F_{k-10} \left[ \begin{array}{cc} k & k \\ 1 & 1 \end{array} \right] C_{1k}^{k-1} B_{k-1} + F_{k+10} \left[ \begin{array}{cc} k & k \\ 1 & 1 \end{array} \right] C_{1k}^{k+1} B_{k+1} \tag{44}
\]

coincides with the recursion relation for the Legendre polynomials, thus determining uniquely the coefficients of the boundary state \([11]\).

Step 1

We will first show that

\[
F_{k+10} \left[ \begin{array}{cc} k & k \\ 1 & 1 \end{array} \right] = 1
\] (45)

This fusing matrix is defined through the crossing property of conformal blocks:

\[
\mathcal{F}_{1k,1k}^{k+1}(1-\eta) = \sum_l F_{k+1l} \left[ \begin{array}{cc} k & k \\ 1 & 1 \end{array} \right] \mathcal{F}_{l,k,11}^l(\eta) \tag{46}
\]

We note that the identity \( l = 0 \) has the most singular behaviour \( \sim \eta^{-2} \) on the right hand side, so we can isolate the relevant fusing matrix through

\[
F_{k+10} \left[ \begin{array}{cc} k & k \\ 1 & 1 \end{array} \right] = \lim_{\eta \to 0} \eta^2 \mathcal{F}_{1k,1k}^{k+1}(1-\eta) \tag{47}
\]

The conformal block \( \mathcal{F}_{1k,1k}^{k+1}(\eta) \) can be determined from the appropriate differential equation (Appendix B). As in section 4 there is no ambiguity, as this conformal block behaves like \( \eta^{2k} \):

\[
\mathcal{F}_{1k,1k}^{k+1}(\eta) = \frac{\eta^{2k}}{(1-\eta)^2} \tag{48}
\]

Inserting (48) into (47) we get (45).
Step 2

We will show that the OPE coefficient $C_{1k}^{k+1}$ is

$$C_{1k}^{k+1} = \frac{k + 1}{2k + 1}$$

It is easy to see that the OPE coefficients between the discrete states $O_J$ do not depend on the radius of compactification. Therefore for the computation, we may choose the self dual radius. Then there appear additional Virasoro primaries $O_{J,m}$, which form multiplets of $SU(2)$

$$O_{J,m}(0) \propto \left( \int \frac{du}{2\pi i} e^{-i\sqrt{2}X(u)} \right)^{J-m} e^{i\sqrt{2}JX(0)} = (J_0^{-})^{J-m} e^{i\sqrt{2}JX(0)}.$$ 

Using $SU(2)$ symmetry, as was argued in [12], the OPE coefficients (just for the holomorphic parts) have the form

$$O_{J_1,m_1} O_{J_2,m_2} \sim \sum_{J,m} C(\{J_1, m_1\}, \{J_2, m_2\}, \{J, m\}) f(J_1, J_2, J) O_{J,m}$$

where $C(\{J_1, m_1\}, \{J_2, m_2\}, \{J, m\})$ is the $SU(2)$ Clebsch-Gordan coefficient and $f(J_1, J_2, J)$ is some unknown function which remains to be determined\footnote{This function is different from the one determined in [12], since there the $O_{J,m}$ have been ‘gravitationally dressed’.}. We can fix the function $f(1, k, k+1)$ by calculating directly

$$O_{1,1} O_{k,k} \sim 1 \cdot O_{k+1,k+1}$$

Here we used the fact that the $O_{J,J}$ operators are just equal to the vertex operators. The normalization of the $O_{J,m}$ operators follow from $SU(2)$ symmetry. Therefore using (51) and (52) we have

$$C_{1k}^{k+1} = \left[ \frac{C(\{1, 0\}, \{k, 0\}, \{k + 1, 0\})}{C(\{1, 1\}, \{k, k\}, \{k + 1, k + 1\})} \right]^2 = \frac{k + 1}{2k + 1}$$

which is the result we wanted to obtain.

Step 3

It remains to determine the remaining fusing matrix. Again for the same reason as discussed in section it is difficult to do it directly. 

$$14$$
Therefore we will use crossing symmetry for (the holomorphic part of) the correlation function \( \langle O_1 O_k O_k O_1 \rangle \):

\[
\sum_p \sqrt{C_{1k}^p C_{1k}^p F_{1k,1k}^p(\eta)} = \sum_q \sqrt{C_{11}^q C_{kk}^q F_{kk,11}^q(1-\eta)} \quad (54)
\]

We may now reexpress the conformal blocks \( F_{1k,1k}(\eta) \) in terms of \( F_{0k,0k}(1-\eta) \) using fusing matrices, and equate the coefficients of \( F_{0k,0k}(1-\eta) \). In this way we obtain

\[
\sum_p C_{1k}^p F_{p0} \begin{bmatrix} k & k \\ 1 & 1 \end{bmatrix} = \sqrt{C_{11}^0 C_{kk}^0} = 1 \quad (55)
\]

where the sum runs only over \( p = k - 1 \) and \( p = k + 1 \). From here we get the result

\[
F_{k-10} \begin{bmatrix} k & k \\ 1 & 1 \end{bmatrix} = \frac{1 - C_{1k}^{k+1}}{C_{1k}^{k-1}} = \frac{1 - C_{1k}^{k+1}}{C_{1k}^{k-1}} = \frac{1 - \frac{k+1}{2k+1}}{\frac{k}{2k-1}} = \frac{2k - 1}{2k + 1} \quad (56)
\]

**Final result**

When we insert the results (45), (49) and (56) into (44), and use \( C_{1k}^{k-1} \equiv C_{1k}^{k-1} = \frac{k}{2k-1} \), we obtain the recursion relation for the coefficients \( B_k \):

\[
x B_k = \frac{k}{2k + 1} B_{k-1} + \frac{k + 1}{2k + 1} B_{k+1} \quad (57)
\]

This allows us to unambiguously identify \( B_k = P_k(x) \).

**Appendix B — differential equations for conformal blocks**

The differential equation for the conformal blocks \( F_{11,11}^p \) is obtained by writing the differential equation for the 4-point correlation function:

\[
(\mathcal{L}_{-3} - \frac{2}{3} \mathcal{L}_{-1} \mathcal{L}_{-2} + \frac{1}{6} \mathcal{L}_{-1}^3) \left[ \prod_{i<j} (z_i - z_j)^{-\frac{4}{3}} G(x) \right] = 0 \quad (58)
\]

\((O_1(z_0) O_1(z_1) O_1(z_2) O_1(z_3))\)
and then taking the limit $z_1 \to 0$, $z_2 \to 1$, $z_3 \to \infty$ and $z_0 \to \eta$. When we substitute $G(x) = \eta^{\frac{3}{2}}(1 - \eta)^{\frac{3}{2}} F(\eta)$ we obtain the following equation for the conformal blocks $F_{11,11}(\eta)$:

$$
(4 - 9 \eta) F + 2 \eta (\eta - 1)(5 \eta^2 - 5 \eta - 1) F' + 4 \eta^2(\eta - 1)^2(2 \eta - 1) F'' + \eta^3 (\eta - 1)^3 F''' = 0
$$

(59)

The relevant equation for conformal blocks of $\langle O_1(z_0)O_k(z_1)O_1(z_2)O_k(z_3) \rangle$ used in Appendix A is

$$
-4(k^2(\eta - 1)^3 - \eta(1 - 3 \eta + \eta^2)) F - 2(\eta - 1) \eta (1 + 2k^2(\eta - 1) + 9 \eta - 7 \eta^2) F' + 4(\eta - 1)^2 \eta^2(2 \eta - 1) F'' + \eta^3 (\eta - 1)^3 F''' = 0
$$

(60)

The linear independent solutions of this equation are

$$
\frac{\eta^{-2k}}{(1 - \eta)^2}, \quad \frac{2k^2 - 4k^2 \eta + \eta + 2k^2 \eta^2}{\eta k^2(1 - \eta)^2}, \quad \frac{\eta^{2k}}{(1 - \eta)^2}
$$

(61)

The solution with the leading coefficient $\eta^{2k}$ may be unambiguously identified with the conformal block $F_{1k,1k}^{k+1}(\eta)$. 

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References

[1] D. Friedan, “The space of conformal boundary conditions for the c=1 Gaussian model”, unpublished note (1999).

[2] See e.g. the lectures:
   H. Saleur, “Lectures on non perturbative field theory and quantum impurity problems”, cond-mat/9812110.
   I. Affleck, “Conformal Field Theory Approach to the Kondo Effect”, Acta Phys. Polon. B 26 (1995) 1869 [cond-mat/9512099].

[3] A. Recknagel and V. Schomerus, “D-branes in Gepner models”, Nucl. Phys. B 531 (1998) 185 [hep-th/9712186].

[4] For a review see P. Di Vecchia and A. Liccardo, “D branes in string theory. I”, [hep-th/9912161].

[5] J. L. Cardy, “Boundary Conditions, Fusion Rules And The Verlinde Formula”, Nucl. Phys. B 324 (1989) 581.

[6] J. L. Cardy and D. C. Lewellen, “Bulk and boundary operators in conformal field theory”, Phys. Lett. B 259 (1991) 274.

[7] D. C. Lewellen, “Sewing constraints for conformal field theories on surfaces with boundaries”, Nucl. Phys. B 372 (1992) 654.

[8] G. Felder, J. Frohlich, J. Fuchs and C. Schweigert, “Conformal boundary conditions and three-dimensional topological field theory”, Phys. Rev. Lett. 84 (2000) 1659 [hep-th/9909140].

[9] R. E. Behrend, P. A. Pearce, V. B. Petkova and J. Zuber, “On the classification of bulk and boundary conformal field theories”, Phys. Lett. B 444 (1998) 163 [hep-th/9809097].
   R. E. Behrend, P. A. Pearce, V. B. Petkova and J. Zuber, “Boundary conditions in rational conformal field theories”, Nucl. Phys. B 570 (2000) 525 [Nucl. Phys. B 579 (2000) 707] [hep-th/9908036].

[10] V.G.Kac in Group Theoretical Methods in Physics, Lecture Notes in Physics, vol 94 (Springer-Verlag, 1979).

[11] G. Segal, “Unitarity Representations Of Some Infinite Dimensional Groups”, Commun. Math. Phys. 80 (1981) 301.

[12] I. R. Klebanov and A. M. Polyakov, “Interaction of discrete states in two-dimensional string theory”, Mod. Phys. Lett. A 6 (1991) 3273 [hep-th/9109032].

[13] N. Ishibashi, “The Boundary And Crosscap States In Conformal Field Theories”, Mod. Phys. Lett. A 4 (1989) 251.
[14] M. Oshikawa and I. Affleck, “Boundary conformal field theory approach to the critical two-dimensional Ising model with a defect line”, Nucl. Phys. B 495 (1997) 533 [cond-mat/9612187].

[15] C. G. Callan and I. R. Klebanov, “Exact C = 1 boundary conformal field theories”, Phys. Rev. Lett. 72 (1994) 1968 [hep-th/9311092];
C. G. Callan, I. R. Klebanov, A. W. Ludwig and J. M. Maldacena, “Exact solution of a boundary conformal field theory”, Nucl. Phys. B 422 (1994) 417 [hep-th/9402113].

[16] J. Polchinski and L. Thorlacius, “Free fermion representation of a boundary conformal field theory”, Phys. Rev. D 50 (1994) 622 [hep-th/9404008].

[17] A. Recknagel and V. Schomerus, “Boundary deformation theory and moduli spaces of D-branes”, Nucl. Phys. B 545 (1999) 233 [hep-th/9811237].

[18] M.R. Gaberdiel and A. Recknagel, “Conformal boundary states for free bosons and fermions”, [hep-th/0108238].

[19] J. Teschner, “Remarks on Liouville theory with boundary”, [hep-th/0009138].

[20] V. Fateev, A. Zamolodchikov and A. Zamolodchikov, “Boundary Liouville field theory. I: Boundary state and boundary two-point function”, [hep-th/0001012].
A. Zamolodchikov and A. Zamolodchikov, “Liouville field theory on a pseudosphere”, [hep-th/0101152].

[21] A. Giveon, D. Kutasov and A. Schwimmer, “Comments on D-branes in AdS(3)”, [hep-th/0106003].

[22] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, “Infinite Conformal Symmetry In Two-Dimensional Quantum Field Theory”, Nucl. Phys. B 241 (1984) 333.