Equilibria and precession in a uniaxial antiferromagnet driven by the spin Hall effect

Qiao-Hua Li1, Peng-Bin He1, *, Meng-Qiu Cai1 and Zai-Dong Li2,3,4,5

1 School of Physics and Electronics, Hunan University, Changsha 410082, People’s Republic of China
2 Department of Applied Physics, Hebei University of Technology, Tianjin 300401, People’s Republic of China
3 State Key Laboratory of Quantum Optics and Quantum Optics Devices, Shandong University, Taiyuan 030006, People’s Republic of China
4 School of science, Tianjin University of Technology, Tianjin 300384, People’s Republic of China
5 Key Laboratory for Magnetism and Magnetic Materials of the Ministry of Education, Lanzhou University, Lanzhou 730000, People’s Republic of China

* Author to whom any correspondence should be addressed.
E-mail: hepengbin@hnu.edu.cn

Keywords: antiferromagnet, spin–orbit torque, magnetic precession, Landau–Lifshitz–Gilbert equation

Abstract

We systematically study the stationary and precessional states in a uniaxial antiferromagnet under the dampinglike spin–orbit torque (SOT). The stable regions for all equilibria are defined by the linear stability analysis. In the regions without any stable equilibrium, we confirm that a stable conical precession may exist. Invoking symmetry arguments, we rigorously reduce the coupled Landau–Lifshitz–Gilbert equations to a single-vector one, which allows us to derive the analytic expressions of the lower and upper thresholds, the frequency, and the amplitude of precession for a weak uniaxial anisotropy. Its frequency is of the order of THz and increases almost linearly with the current. Further analysis reveals that the precession is mainly propelled by the exchange interaction in the promise that the SOT balances the damping one in average. Moreover, the investigation uncovers that a weak anisotropy can improve the frequency tunability, and a small damping benefits lowering the exciting current. Finally, the analytic expressions are verified by comparing with numerical simulations of the original Landau–Lifshitz–Gilbert equations.

1. Introduction

Recent years, antiferromagnetic (AFM) spintronics has spurred intensive research because of the potential to serve as active components in spintronic applications [1]. One of the major topics in this field is the AFM oscillator [2–15], which frequency is in the THz regime due to the strong exchange interaction. Generally, this AFM oscillator is based on the stable precession of magnetization. It appears particularly attractive to excite and adjust this precession by the dampinglike spin–orbit torque (SOT) [7–14], which can be generated in the AFM/heavy metal heterostructure through the spin Hall effect [16].

In order to elucidate the key physical features of the AFM oscillator, such as the threshold, frequency, and amplitude, an analytic solution is indispensable. However, the AFM Landau–Lifshitz–Gilbert (LLG) equations are too complicated to be dealt with analytically. A usual approximate analytic method is the \( l \)-\( m \) scheme [6–9, 11–15, 17], In which, the coupled LLG equations of the sublattice magnetization are firstly transformed into another coupled equations about the Néel vector \( l \) and the average magnetization \( m \). Then, taking the strong exchange limit and the approximation \( |m| \ll |l| \), the coupled \( l \)-\( m \) equations are reduced to a single-vector equation of \( l \) with \( m \) being a slave vector. Because the reduced \( l \)-equation is decoupled with \( m \), it is easily treated analytically. According to this scheme, the AFM dynamics under spin torques have been reduced to a nonlinear forced vibration [6, 8, 14], which has been used to investigate the precession and switching more succinctly. In these cases, the sublattice moments remain nearly antiparallel, so that the approximation \( |m| \ll |l| \) is satisfied. However, for general cases, such as the conical precession, \( m \) may be sizable comparing with \( l \). For example, the numeric results of this paper (figures 2–4) reveal that,
unless the sublattice moments are approximately antiparallel, \(|\mathbf{m}| \ll |\mathbf{I}|\) is unsatisfied. Therefore, it is not appropriate to adopt the \(1-\mathbf{m}\) scheme in spite of its validity. A concrete analysis should be made for each specific stationary or dynamic state in the AFM system driven by different forces.

Although extensive works have already been performed on the AFM oscillators, no analytic description is available in the whole current region, and the thresholds of stable precession is yet to be fully understood, which are the aim of this work. Firstly, by linearizing the AFM LLG equations around the equilibria, we obtain four coupled ordinary differential equations about the linear excitations. Furthermore, the stability of every equilibrium is judged by the Routh–Hurwitz criterion \([18, 19]\), which apply to the high-order secular equation. This method has been used to study the stability condition of the synthetic AFM \([20]\) and the AFM domain wall \([21]\). By reference to regions without any stable equilibria, we will determine an approximate regime that a stable precession may emerge.

Secondly, while the linear analysis helps to roughly determine the thresholds of stable precession, it cannot predict the details of this dynamic state in the nonlinear regime. Hence, we carry out the analysis of stable precession by Melnikov’s method \([22]\), which is suitable for studying the limit circles in the weakly disturbed dynamic systems with periodic trajectories. As we known, Melnikov method (or named as Poincaré–Melnikov method) has been applied to study the self-oscillation of ferromagnet driven by different spin torques from the 2000s \([23–27]\). In these literatures, the method was named after Melnikov’s method or theory explicitly. In some literatures about the ferromagnetic precession, this method was not known as ‘Melnikov’s method’ apparently. But the main idea originates from the Melnikov theory \([22]\). So, we call it Melnikov’s method following references \([26, 27]\). Henceforth, many related investigations have been performed for different configurations and driving forces, such as the biaxial anisotropy with an easy axis parallel to the spin polarization \([23–27]\), the perpendicular anisotropy and in-plane spin polarization \([28–31]\), the in-plane anisotropy and vertical spin polarization \([32, 33]\), the biaxial layer driven by the spin torques with a tilted polarization and thermal fluctuations \([34, 35]\), and the perpendicularly magnetized layer driven by the spin Hall effect \([36]\). These studies presented a good agreement between the analytical theory and the numerical simulation. It is also worth mentioning that, in order to realize a stable precession in the ferromagnet driven by spin torques, a magnetic field is generally essential. This is undesired for application.

So far as we know, Melnikov’s method has not been applied on the AFM system. For the considered model, owing to the strong exchange interaction, both the intrinsic damping and the SOT can be treated as perturbations. So, it is suitable to apply the Melnikov’s method here. Through the linear stability analysis of equilibria and solving the precession solution by Melnikov’s method, we will derive the analytic expressions of the thresholds, the frequency, and the amplitude of stable precession. It will be found that the stable precession can be sustained under the dampinglike SOT and in the absence of magnetic field.

This paper is structured as follows. In section 2, we introduce the model. In section 3, we formulate a general method for the linear stability analysis of the AFM LLG equations. The equilibria and precession are analyzed in section 4 for the case with the spin polarization normal to the easy axis, and in section 5 for the case with the spin polarization along the easy axis. Then, several remarks are provided in section 6. Finally, section 7 concludes the results.

2. Model

We consider an AFM thin film attached with a heavymetal (HM) layer, as shown in figure 1. With a longitudinal electric current flowing through the HM layer, the dampinglike SOT is generated via the spin Hall effect \([16]\). Under the SOT, the magnetic dynamics in AFM layer is governed by two coupled sublattice LLG equations,

\[
\frac{d\mathbf{m}_i}{dt} = \mathbf{m}_i \times \frac{d\mathcal{E}}{d\mathbf{m}_i} + \alpha \mathbf{m}_i \times \frac{d\mathbf{m}_i}{dt} + \tau_i, \tag{1}
\]

where \(\mathbf{m}_i\) is the unit vectors of magnetization in two sublattices marked by \(i = 1, 2\). \(\alpha\) is the Gilbert constant of damping. Here, the inhomogeneous exchange contribution is ignored. So, we focus on the magnetic dynamics within the framework of a two-macrospin model \([15]\), which is a reasonably good approximation for the small-size sample. Then, including the exchange and uniaxial anisotropy terms, the magnetic energy reads,

\[
\mathcal{E} = \omega_\alpha \mathbf{m}_1 \cdot \mathbf{m}_2 - \frac{1}{2} \omega_{\text{an}} \left[ (\mathbf{m}_1 \cdot \mathbf{e}_a)^2 + (\mathbf{m}_2 \cdot \mathbf{e}_a)^2 \right], \tag{2}
\]

with \(\mathbf{e}_a\) being the unit vector along the easy axis. Here, the demagnetization energy is not considered. This can be justified by the fact that the demagnetization along the film normal is dominant. In the following sections, we find three states under the SOT, including the tilted-AFM, the precession, and the \(\mathbf{e}_p\)-FM, for
matrix, torques have been scaled with frequency. Analysis is somewhat different from the ferromagnetic counterpart. In this section, we present a general parameterization of the spin transparency of the interface [39].

Because the AFM LLG equations are coupled ones about four independent variables, the linear stability stabilities can be judged by the linear stability analysis. Table 1 lists all the equilibria of Equation (2) which the components of \( \mathbf{m} \) along the film normal are all zero. So, the demagnetization effect (shape anisotropy) is negligible.

The dampinglike SOT is expressed as

\[
\tau_i = -\omega_{\text{sot}} \mathbf{m}_i \times (\mathbf{m}_i \times \mathbf{e}_p),
\]

with \( \mathbf{e}_p \) being the unit vector along the spin polarization. All parameters related with the strengths of torques have been scaled with frequency, \( \omega_{\text{ex}} = \gamma H_{\text{ex}} \) with \( \gamma \) being the gyromagnetic ratio and \( H_{\text{ex}} \) the inter-sublattice exchange field. \( \omega_{\text{an}} = \gamma H_{\text{an}} \) with \( H_{\text{an}} \) being the anisotropy field. \( \omega_{\text{tot}} = \omega/d \) with \( d \) being the thickness of AFM layer. \( u \) has the dimension of velocity, \( u = \mu_0/(eM_s) \xi_b \), with \( \mu_0 \) being the Bohr magneton, \( e \) the element charge, \( M_s \) the sublattice saturation magnetization, and \( \xi_b \) the electric current density. \( \xi \) is the SOT efficiency which equals to \( \xi_{\text{sh}} \theta_{\text{sh}} [37, 38] \), with \( \theta_{\text{sh}} \) being the spin Hall angle, and \( \xi_{\text{tot}} \) the spin transparency of the interface [39].

Without the SOT, from the static equation \( \mathbf{m}_i \times \partial \mathbf{E} / \partial \mathbf{m}_i = 0 \), it is easy to get the equilibria. Their stabilities can be judged by the linear stability analysis. Table 1 lists all the equilibria of \( \mathbf{m}_{1,2} \) and their stability types. When applying the SOT, do these equilibria remain their stability? Do new equilibria and dynamic states emerge? In the following, we will investigate these issues for two configurations: the easy axis perpendicular and parallel to the spin polarization.

3. Linear stability analysis

Because the AFM LLG equations are coupled ones about four independent variables, the linear stability analysis is somewhat different from the ferromagnetic counterpart. In this section, we present a general method to judge the stability of equilibria for bipartite AFM systems. Due to \( \mathbf{m}_1 = 1 \), it is convenient to parameterize \( \mathbf{m}_2 \) in terms of the polar angle \( \theta_1 \) and the azimuthal one \( \phi_1 \) according to \( \mathbf{m}_2 = (\sin \theta_1 \cos \phi_1, \sin \theta_1 \sin \phi_1, \cos \theta_1) \). Then, the coupled LLG equations (equation (1)) can be written in the form of a \( 4 \times 4 \) matrix,

\[
\begin{align*}
\mathbf{A} \frac{\mathbf{d} \mathbf{x}}{\mathbf{d} t} &= \partial \mathbf{E} + \mathbf{B},
\end{align*}
\]

where \( \mathbf{x} = (\theta_1, \phi_1, \theta_2, \phi_2)^T \), with T denoting the matrix transpose, and

\[
\partial = \left( \partial / \partial \theta_1, \partial / \partial \phi_1, \partial / \partial \theta_2, \partial / \partial \phi_2 \right)^T.
\]

In addition, \( \mathbf{A} = \text{diag}(A_1, A_2) \), with

\[
A_2 = \begin{pmatrix}
-\alpha & \sin \theta_1 \\
-\sin \theta_1 & -\alpha \sin^2 \theta_1
\end{pmatrix}.
\]

The SOT term is expressed as \( \mathbf{B} = \omega_{\text{sot}} (b_1, b_2)^T \) with \( b_1 = \mathbf{\tau}_1 \cdot \mathbf{e}_{\phi_1} - \sin \theta_1 \mathbf{\tau}_1 \cdot \mathbf{e}_\theta \), where the spherical unit vectors \( \mathbf{e}_\theta = (\cos \theta_1 \cos \phi_1, \cos \theta_1 \sin \phi_1, -\sin \theta_1) \) and \( \mathbf{e}_{\phi_1} = (-\sin \phi_1, \cos \phi_1, 0) \).

The equilibrium directions of \( \mathbf{m}_{1,2} \) can be obtained by setting \( \mathbf{d} \mathbf{x} / \mathbf{d} t = 0 \). The solutions are denoted by \( \mathbf{x}_0 = (\theta_0^1, \phi_0^1, \theta_0^2, \phi_0^2)^T \). Applying small perturbations, the system deviates from the equilibrium state slightly.

Table 1. Equilibria and their stability types in the absence of SOT.

| Equilibria | AFM state \((\mathbf{m}_{1,2} \parallel \mathbf{e}_a)\) | FM state \((\mathbf{m}_{1,2} \parallel \mathbf{e}_p)\) | AFM state \((\mathbf{m}_{1,2} \perp \mathbf{e}_a)\) | FM state \((\mathbf{m}_{1,2} \perp \mathbf{e}_p)\) |
|------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| Stability  | Stable focus or node            | Unstable focus                  | Stable focus-node               | Unstable focus-node             |
Namely, $x = x_0 + x'$, where $x' = (\theta'_1, \phi'_1, \theta'_2, \phi'_2)^T$ is regarded as a response to the perturbations. Inserting this ansatz into equation (4), and linearizing it in the vicinity of equilibria, one can get

$$\mathcal{A}^0 \frac{dx'}{dt} = \mathcal{H} x' + \mathcal{T} x',$$

where $\mathcal{A}^0 = \mathcal{A}|_{\dot{\theta}_i = \theta_i^0, \phi_i = \phi_i^0}$. The Hessian matrix

$$\mathcal{H} = \begin{pmatrix} H_1 & H_{12} \\ H_{12}^T & H_2 \end{pmatrix},$$

where

$$H_i = \begin{pmatrix} \frac{\partial^2 E}{\partial \theta_i^2} & \frac{\partial^2 E}{\partial \theta_i \partial \phi_i} \\ \frac{\partial^2 E}{\partial \phi_i \partial \theta_i} & \frac{\partial^2 E}{\partial \phi_i^2} \end{pmatrix} \bigg|_{\theta_i = \theta_i^0, \phi_i = \phi_i^0},$$

and

$$H_0 = \begin{pmatrix} \frac{\partial^2 E}{\partial \theta_1 \partial \theta_2} & \frac{\partial^2 E}{\partial \theta_1 \partial \phi_2} \\ \frac{\partial^2 E}{\partial \phi_1 \partial \theta_2} & \frac{\partial^2 E}{\partial \phi_1 \partial \phi_2} \end{pmatrix} \bigg|_{\theta_i = \theta_i^0, \phi_i = \phi_i^0}.$$

In addition, the matrix $\mathcal{T}$ is related with the SOT, which is written as $\mathcal{T} = \text{diag}(T_1, T_2)$, where

$$T_i = \begin{pmatrix} \frac{\partial}{\partial \theta_i} (\tau_i \cdot e'_i) & \frac{\partial}{\partial \phi_i} (\tau_i \cdot e'_i) \\ -\frac{\partial}{\partial \theta_i} (\tau_i \cdot e'_i \sin \theta_i) & -\frac{\partial}{\partial \phi_i} (\tau_i \cdot e'_i \sin \theta_i) \end{pmatrix} \bigg|_{\theta_i = \theta_i^0, \phi_i = \phi_i^0}.$$

It should be mentioned that this $4 \times 4$ matrix description of the linearized AFM dynamics has been used to rewrite the spin wave Hamiltonian of AFM [40]. This treatment avoid taking the approximation $|\mathbf{m}| \ll |\mathbf{I}|$. Our formalism includes the nonconservative terms, such as the SOT and the damping.

Usually, the solutions of equation (6) take the form $x' \propto e^{\lambda t}$. To ensure the existence of nontrivial solutions, $\lambda$ satisfies the secular equation,

$$|\lambda \mathcal{A}^0 - \mathcal{H} - \mathcal{T}| = 0.$$  

Generally, this secular equation can be expanded as a quartic one,

$$a_0 \lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4 = 0,$$  

where the parameters are determined by the equilibrium solutions $(\theta_i^0, \phi_i^0)$. If all the roots of $\lambda$ have a negative real part, the corresponding equilibrium state is stable. According to Routh–Hurwitz criterion [18–21], one can define a series of determinants,

$$\Delta_1 = a_1,$$  

$$\Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix},$$  

$$\Delta_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{vmatrix},$$  

$$\Delta_4 = a_4 \Delta_3.$$  

If all $\Delta$ are positive, the real parts of all roots of $\lambda$ are negative. Namely, one gets a stable equilibrium state. In the following sections, we will use this formalism to seek the stable equilibria of AFM system driven by the damping-like SOT.
4. Case that $\mathbf{e}_p \perp \mathbf{e}_a$

In this section, we obtain all the equilibria for the case with the spin polarization ($\mathbf{e}_p$) perpendicular to the easy axis ($\mathbf{e}_a$) and analyze their stability. By Melnikov’s method, the thresholds, the frequency, and the amplitude of stable precession are also calculated analytically.

4.1. Equilibria and their stabilities

The equilibria are defined if the SOT balances out the precession torques. So, setting $d\mathbf{x}/dt = 0$ in equation (4), and taking $\mathbf{e}_a = \mathbf{e}_1$ and $\mathbf{e}_p = \mathbf{e}_2$ (as shown in figure 1), the equilibrium equations can be obtained,

$$\omega_e \sin \theta_{3-i} \sin (\phi^0_i - \phi^0_{3-i}) = \sin \theta_i^0 (\omega_{an} - \omega_{ex} \sin \phi^0_i \cos \phi^0_i),$$  

$$\omega_e \sin \theta_{3-i} \cos \theta_i^0 \cos (\phi^0_i - \phi^0_{3-i}) = \sin \theta_i^0 (\omega_{ex} \cos \theta^0_{3-i} + \omega_{an} \sin^2 \phi^0_i \cos \theta_i^0),$$

where $i = 1, 2$, denoting the two sublattices.

Firstly, it is easy to observe that $\sin \theta_i^0 = 0$ solve equations (17) and (18). This generates two FM states that $\theta^0_1 = \theta^0_2 = 0$ or $\pi$, and two AFM ones that $\theta^0_1 = 0$ ($\pi$) and $\theta^0_2 = \pi$ ($0$). For these two kinds of equilibria, because $\mathbf{m}_1$ and $\mathbf{m}_2$ are collinear and along $\mathbf{e}_p$ ($z$ axis), the exchange torque $\propto \mathbf{m}_1 \times \mathbf{m}_2$, the SOT $\propto (\mathbf{m}_1 \times \mathbf{m}_2)$, and the anisotropy torque $\propto (\mathbf{m}_1 \times \mathbf{e}_p)$, are all zero. Considering $\mathbf{m}_{1,2}$ point along the spin polarization ($\mathbf{e}_p$), these equilibria are named after $\mathbf{e}_p$-FM and $\mathbf{e}_p$-AFM states. Utilizing the linear stability analysis formulated in section 3, we find that the $\mathbf{e}_p$-AFM states are unstable (see appendix A2 for details of the derivation). The $\mathbf{e}_p$-FM states are stable under the condition that $|\omega_{an}| > \omega_{ex}^u$ (see appendix A1 for details of the derivation), with

$$\omega_{an}^u = \frac{\alpha}{2} (4\omega_{ex} + \omega_{an}).$$  

Secondly, for $\sin \theta_i^0 \neq 0$, from equation (17) one has

$$\sin 2\phi^0_i = \frac{2\omega_{an}}{\omega_{ex}},$$  

$$\sin (\phi^0_i - \phi^0_{3-i}) = 0.$$  

Equation (21) allows $\phi^0_i = \phi^0_{3-i}$ or $\phi^0_i = \pm \pi + \phi^0_{3-i}$. Inserting these two relations into equation (18), one has the solution $\theta_1^0 = \theta_2^0 = \pi/2$. From equation (20), we can get $\phi^0_1 = \phi^0_2 = \phi^0$ (FM) or $\phi^0_1 = \pm \pi + \phi^0_2 = \phi^0$ (AFM), where

$$\phi^0 = \frac{1}{2} \left[ P - 1 \right] \pi - (-1)^P \arcsin \left( \frac{2\omega_{an}}{\omega_{ex}} \right).$$  

with $P = 1, 2, 3, 4$. These two kinds of equilibria indicate that $\mathbf{m}_{1,2}$ remain parallel (FM) or antiparallel (AFM) in the $x$–$y$ plane. Based on this characteristic, we name them after tilted FM and tilted AFM states. Stability analysis reveals that the tilted FM states are unstable (see appendix A4 for the detailed derivations). The tilted AFM states are unstable for $P = 1, 3$, while stable for $P = 2, 4$ under the condition $|\omega_{an}| < \omega_{ex}/2$ (see appendix A3 for the detailed derivations). The formation of the stable tilted AFM state can be illustrated by a simple argument. The SOT pushes $\mathbf{m}_{1,2}$ deviate from the easy axis ($y$ axis), then the exchange and anisotropy torques rotate $\mathbf{m}_{1,2}$ around the easy axis. When $\mathbf{m}_1$ and $\mathbf{m}_2$ rotate to the $x$–$y$ plane, they restore antiparallel. In this scene, the exchange torques vanish. In a proper current region, the SOT is balanced by the anisotropy torque, and the tilted AFM state emerges.

For clarity, we list all the equilibria and their stabilities in table 2. Comparing with the equilibria listed in table 1, one can find that, under the SOT with $\mathbf{e}_p \perp \mathbf{e}_a$, the AFM and FM states along the easy axis do not exist anymore, to be replaced by the novel tilted AFM and FM states. The AFM and FM states normal to the easy axis are also not equilibria except that $\mathbf{m}$ lie along the spin-polarization direction.

For the uniaxial AFM with a weak anisotropy, such as MnF$_2$, $\omega_{an}^u > \omega_{an}/2$. Therefore, by linearization method, there is no equilibrium in the regions that $\omega_{an}/2 < |\omega_{an}| < \omega_{an}^u$. By integrating the coupled LLG equations (equation (1)), it can be found that the precession may emerge in this region (see figure 3). In the next subsection, the precession state is analyzed by combining the analytic and numeric methods.

4.2. Precession

4.2.1. Reduced equation

In general, the two coupled LLG equations are too complex to be solved analytically. Then, in many works [6–9, 11–15, 17], equation (1) is transformed into the coupled equations in terms of the uncompensated
the N\textsubscript{x} along the easy axis, as listed in table 1. Applying the SOT, they are deviated from the easy axis, but remain in magnetization components of \( \mathbf{m} \). Without the SOT, the AFM state is stable with \( |\omega_{\text{sot}}| > \omega_{\text{an}} \), and \( |\omega_{\text{an}}| < \omega_{\text{sot}}/2 \) for the tilted AFM state, as shown in figure 4.

Before dealing with equation (1) analytically, we solve them numerically for different strengths of SOT. Figures 2–4 (and figure 7 in next section) are plotted by this data directly. In these figures, we summarize the evolutions of components of \( \mathbf{m}_{1,2} \) (the magnetization of two sublattices), \( \mathbf{m} \) (the uncompensated magnetization), and \( \mathbf{l} \) (the Néel vector), as well as the magnitudes of \( \mathbf{m} \) and \( \mathbf{l} \). Without the SOT, the AFM state is stable with \( \mathbf{m}_{1,2} \) along the easy axis, as listed in table 1. Applying the SOT, they are deviated from the easy axis, but remain in the \( x-y \) plane and antiparallel for the weak SOT, as shown in figure 2. Increasing the strength of SOT, \( \mathbf{m}_{1,2} \) are tilted out of the \( x-y \) plane and precess out of phase around the spin polarization (z axis), as shown in figure 3. Increasing the strength of SOT further, \( \mathbf{m}_{1,2} \) are oriented along the spin polarization and the system enters \( \mathbf{e}_\gamma \)-FM state, as shown in figure 4.

From these numeric results, several important conclusions can be obtained. Firstly, comparing figures 2 and 3, one can find that the threshold of \( \omega_{\text{sot}} \) is about 0.392\( \omega_{\text{an}} \), for the switching from the tilted AFM state to the precession. This is less than \( 1/2\omega_{\text{an}} \), the critical value to destabilize the tilted AFM state. Secondly, observing figures 3(f) and 4(f), the conditions \( |\mathbf{m}| \ll 1 \) and \( |\mathbf{l}| \approx 1 \) are unsatisfied for the precession and the \( \mathbf{e}_\gamma \)-FM state. The deviations of \( |\mathbf{m}| \) from 0 and \( |\mathbf{l}| \) from 1 increase with \( \omega_{\text{sot}} \) increasing. So, it is impossible to decouple the Néel vector \( \mathbf{l} \) and the uncompensated magnetization \( \mathbf{m} \). The usual \( \mathbf{l} \)-\( \mathbf{m} \) scheme

| Equilibria | \( \mathbf{e}_\gamma \)-FM | \( \mathbf{e}_\gamma \)-AFM | Tilted FM | Tilted AFM I | Tilted AFM II |
|-----------|-----------------|-----------------|-----------|-------------|-------------|
| \( \theta_1^* \) and \( \phi_1^* \) | \( \theta_1^* = 0 \) or \( \pi \) | \( \theta_1^* = 0 \) (0) | \( \theta_1^* = \frac{\pi}{4} \) | \( \theta_1^* = \frac{\pi}{4} \) | \( \theta_1^* = \frac{\pi}{4} \) |
| \( \theta_2^* \) | \( \theta_2^* = \pi \) | \( \phi_2^* = \phi_0 \) | \( \phi_2^*(\phi_0) = \frac{\pi}{4} \phi_0 \) | \( \phi_2^*(\phi_0) = \frac{\pi}{4} \phi_0 \) | \( \phi_2^*(\phi_0) = \frac{\pi}{4} \phi_0 \) |
| Stable region | \( |\omega_{\text{sot}}| > \omega_{\text{an}} \) | Unstable | Unstable | Unstable | \( |\omega_{\text{an}}| < \omega_{\text{sot}}/2 \) |

**Table 2.** Equilibria and their stable regions in the presence of SOT. Here, \( \Psi = \arcsin(2\omega_{\text{sot}}/\omega_{\text{an}}) \), and \( \omega_{\text{sot}} \) and \( \phi_0^* \) are expressed by equations (19) and (22), respectively.
Figure 3. Time-evolutions of x-components (a), y-components (b), and z-components (c) of $m_1$ and $m_2$ for the precessional state of $e_p \perp e_a$. (d) and (e) show the evolutions of components of the average magnetization $m$ and Néel vector $l$, respectively. (f) shows the evolutions of the magnitudes of $m$ and $l$. Here, $\omega_{an} = 0.392\omega_{so}$. The rest parameters are the same as those in figure 2.

Figure 4. Time-evolutions of x-components (a), y-components (b), and z-components (c) of $m_1$ and $m_2$ for the $e_p$-FM state of $e_p \perp e_a$. (d) and (e) show the evolutions of components of the average magnetization $m$ and Néel vector $l$, respectively. (f) shows the evolutions of the magnitudes of $m$ and $l$. Here, $\omega_{an} = 3\omega_{so}$. The rest parameters are the same as those in figure 2.

fails here. Thirdly, these numeric results provide some clues on how to deal with this model analytically in a simple way. From figures 2(d), 3(d), and 4(d), one can find that $m_{1x} + m_{2x} = 0$ and $m_{1y} + m_{2y} = 0$.

Furthermore, from figures 2(c), 3(c), and 4(c), $m_{1z} = m_{2z}$. This can also be illustrated by the symmetry analysis. For the model considered, the two sublattices are equivalent. So, the system (equations (1)–(3)) is invariant with respect to reflections across the x–z and y–z planes. Namely, if taking a substitution $m_{1x} \rightarrow -m_{2x}$, $m_{1y} \rightarrow -m_{2y}$, and $m_{1z} \rightarrow m_{2z}$, the two sublattice equations are equivalent.

Based on these numerical analysis and symmetry argument, we can simplify equation (1) by setting $m_{1x} = -m_{2x} = n_x$, $m_{1y} = -m_{2y} = n_y$ and $m_{1z} = m_{2z} = n_z$. Then, the coupled LLG equations (equation (1))
are reduced as
\[
\frac{dn}{dt} = n \times \frac{dE_n}{dn} + \alpha n \times \frac{dn}{dt} + \tau_n, \tag{23}
\]
where the reduced magnetic energy
\[
E_n = -\frac{1}{2} \omega_{\text{ex}} (n \cdot e_y)^2 + \omega_{\text{ex}} (n \cdot e_z)^2 - \frac{1}{2} \omega_{\text{ex}}, \tag{24}
\]
and the SOT
\[
\tau_n = -\omega_{\text{sot}} n \times (n \times e_z). \tag{25}
\]
So, a bipartite AFM with a uniaxial anisotropy is translated into a single-order-vector magnetic system with a biaxial anisotropy. Apart from the easy axis along \( e_y \), there exists an easy plane normal to \( e_y \) and determined by the AFM exchange interaction.

Because this reduced dynamic system is two-dimensional, it can be determined that there exist non-decaying precessional states in the region without stable equilibria. This is a logical consequence of Poincaré–Bendixson theorem [22, 43]. From figures 2 and 3, we have find that the lower threshold of precession is \( \omega_{\text{ex}} \approx 0.39 \omega_{\text{in}} \), which is less than the critical value \( (1/2 \omega_{\text{in}}) \) to destabilize the tilted AFM state. Thus, the linear analysis cannot predict the details of this precessional state. In the following, based on equations (23)–(25), we will use the Melnikov’s method to redefine the lower and upper thresholds of precession, and to calculate its frequency and amplitude.

Melnikov’s method is suited to the weakly perturbed conservative systems [22] and has been successfully applied on the ferromagnetic systems driven by spin torques [24]. Here, due to the strong exchange interaction, the damping and the SOT can be treated as small perturbations. When the energy influx from the SOT balances the damping dissipation, a stable precession can be excited, which is approximately along some constant-energy trajectories derived from the energy landscape defined by equation (24).

4.2.2. Constant-energy trajectories

Considering the conservation of modulus, the vector \( n \) evolves on the surface of the unit sphere \( |n| = 1 \). So, the unperturbed trajectories can be defined as the intersection of the elliptic cylinders determined by equation (24) with this unit sphere. These trajectories are identified by \( E_n \). In addition, there are three equilibria defined by equation (24). The first one is that \( n \) is oriented along the easy axis (\( y \)-axis). The corresponding energy is minimal and defined as \( E_{\text{min}} = -1/2 (\omega_{\text{ex}} + \omega_{\text{in}}) \). The second one is a saddle point, for which \( n \) is perpendicular to the easy axis and the spin polarization, as marked in figure 5(b). The corresponding energy \( E_{\text{saddle}} = -1/2 \omega_{\text{ex}} \). The third one is that \( n \) points along the spin polarization (\( z \)-axis). The corresponding energy is maximal and defined as \( E_{\text{max}} = 1/2 \omega_{\text{ex}} \).

The trajectories through the saddle points act as separatrixes, which are defined as the locus of intersection of two planes (determined by \( n_y = \pm \sqrt{2 \omega_{\text{ex}} / \omega_{\text{in}}^2} \)) with the unit sphere, as shown by the red solid curves in figure 5(a). For a weak anisotropy \( (\omega_{\text{in}} \ll \omega_{\text{ex}}) \), these trajectories are very close to the \( x-y \) plane.

When \( E_{\text{min}} < E_n < E_{\text{saddle}} \), \( n \) rotates around the easy axis (\( y \)-axis), as represented by the green dotted curves in figure 5(a). These trajectories are named after low-energy ones. Here, the spin polarization \( e_p \)
points outside of the trajectory. With that in mind, projecting the SOT along the direction of damping torque, it can be find that the component of SOT is parallel to the damping torque in a half trajectory, whereas antiparallel in another half \([44, 45]\). Therefore, it is impossible to balance the energy gain from the SOT and the energy dissipation from the damping during a whole precession. There cannot exist stable precessions on the low-energy trajectories.

When \(E_{\text{saddle}} < E_n < E_{\text{max}}\) rotates around the spin polarization \((z\text{-axis})\), as represented by the blue dashed curves in figure 5(a). For these high-energy trajectories, the SOT may oppose the damping torque during the entire precession. Thus, a stable precession is possible. For these trajectories, it is convenient to parameterize equation (24) as

\[
\begin{align*}
  n_x &= a \cos \eta, \\
  n_y &= b \sin \eta,
\end{align*}
\]

where \(\eta\) varies from 0 to 2\(\pi\), and

\[
\begin{align*}
  a &= \sqrt{\frac{\omega_{\text{ex}} - 2E_n}{2\omega_{\text{ex}}}}, \\
  b &= \sqrt{\frac{\omega_{\text{ex}} - 2E_n}{2\omega_{\text{ex}} + \omega_{\text{an}}}}
\end{align*}
\]

4.2.3. Balance between SOT and damping

For a high-energy trajectory (denoted by \(\Gamma\)), the dissipative energy by the damping torque during an entire precession is

\[
W_{\text{damp}} = \alpha \oint \frac{\text{d}n}{\text{d}t} \cdot \text{dn}.
\]

Inserting the components of equation (23) into equation (30), and by use of the parametrization of trajectory (equations (26) and (27)), the integral can be calculated analytically. Keeping only the linear terms of \(\alpha\) and \(\omega_{\text{sot}}\), the dissipative energy reads

\[
W_{\text{damp}} = 4\alpha \sqrt{\frac{\omega_{\text{ex}} + \omega_{\text{an}} + 2E_n}{2\omega_{\text{ex}}}} \left[ 2\omega_{\text{ex}} E(k) - (\omega_{\text{ex}} + 2E_n) K(k) \right],
\]

where \(E\) and \(K\) denote the complete elliptic integral of the second and first kinds, and the modulus

\[
k = \frac{\omega_{\text{an}} (\omega_{\text{ex}} - 2E_n)}{2\omega_{\text{ex}} (\omega_{\text{ex}} + \omega_{\text{an}} + 2E_n)}.
\]

In addition, around the same trajectory, the pumped energy by the SOT during a whole precession is calculated as

\[
W_{\text{sot}} = \omega_{\text{sot}} \oint (n \times e_z) \cdot \text{dn} = \frac{2\pi (\omega_{\text{ex}} - 2E_n) \omega_{\text{sot}}}{\sqrt{2\omega_{\text{ex}} (2\omega_{\text{ex}} + \omega_{\text{an})}}}
\]

Equating \(W_{\text{damp}}\) and \(W_{\text{sot}}\), the SOT strength \(\omega_{\text{sot}}\) necessary to excite a stable precession on a constant-energy trajectory marked by \(E_{\text{\textbf{n}}}\) can be expressed as,

\[
\omega_{\text{sot}} = \frac{2}{\pi} \alpha \sqrt{\frac{(\omega_{\text{ex}} + \omega_{\text{an}} + 2E_n) (2\omega_{\text{ex}} + \omega_{\text{an}})}{\omega_{\text{ex}} - 2E_n}} \left[ 2\omega_{\text{ex}} E(k) - (\omega_{\text{ex}} + 2E_n) K(k) \right].
\]

It is worth noting that, for \(\omega_{\text{sot}} > 0\), the trajectories are on the half sphere with \(m_{1z}(m_{2z}) > 0\). \(\mathbf{m}_{1,2}\) rotate left-handedly around the \(z\) axis, as labeled by the blue arrow in figure 5(a). This results in that the damping torque points away from the rotational axis. The SOT is just the reverse. So, the balance between both of them may be realized during a whole precession. Moreover, the rotation is mainly propelled by the exchange interaction \([6]\). If \(\omega_{\text{sot}} < 0\), the trajectories are on the hemisphere with \(m_{1z}(m_{2z}) < 0\) and the precessional direction is reversed.

Setting \(E_{\text{n}} = E_{\text{max}}\) in equation (34), one can get the upper threshold for precession, which is just \(\omega_{\text{max}}\) (equation (19)). This is consistent with the result of linear stability analysis of \(\mathbf{e}_r\)-FM state. Taking the limit that \(E_{\text{n}} \to E_{\text{saddle}}\), one has another lower threshold

\[
\omega_{\text{sot}}^{\text{th}} = \frac{2}{\pi} \alpha \sqrt{\frac{(2\omega_{\text{ex}} + \omega_{\text{an}})\omega_{\text{an}}}{\omega_{\text{ex}}}}.
\]

Taking the parameters in the caption of figure 2, the value of this lower threshold is about 0.072\(\omega_{\text{an}}\), which is much less than the numeric result 0.392\(\omega_{\text{an}}\). This indicates that, in common with the linearization
method, the Melnikov method also fails in the transitional region between the tilted AFM state and the precession.

4.2.4. Lower threshold of precession
To derive the lower threshold analytically, we use a method developed by Taniguchi [33], which gives an analytic result consistent with the numeric calculation very well. In figure 5(b), we plot the evolution of $n$ at the lower threshold $\omega_{\text{tot}} = 0.392\omega_{\text{an}}$. Initially, $n$ is oriented along the positive $y$ direction (point $F$ on the sphere) for $\omega_{\text{tot}} = 0$. The plot indicates that before entering the high-energy trajectory, $n$ must evolve from the initial stable state to the saddle point (point $S$ on the sphere). This part is not a constant-energy trajectory. To reach the saddle point, $n$ needs to climb over an energy barrier $E_{\text{saddle}} = E_{\text{min}} = 1/2\omega_{\text{an}}$. Namely, apart from balancing the damping torque, the SOT should also do work to overcome this barrier. This work-energy relation can be expressed as,

$$\omega_{\text{tot}} \int_F^S (n \times e_i) \cdot \mathrm{d}n = \alpha \int_F^S \frac{\mathrm{d}n}{\mathrm{d}t} \cdot \mathrm{d}n + \frac{1}{2} \omega_{\text{an}}^2.$$  \hspace{1cm} (36)

It is difficult to solve this equation analytically, because there is no explicit expression for the trajectory from the stable focus ($F$) to the saddle point ($S$). However, due to $\omega_{\text{an}} \ll \omega_{\text{ex}}$, the trajectory of $E_{\text{saddle}}$ is very close to the $x$-$y$ plane. Therefore, as show in figure 5(b), the integral trajectory can be approximated by the constant-energy one of $E_{\text{saddle}}$ from $F_\pm$ to $S$. $F_\pm$ situate in the $y$-$z$ plane with $n_z = \pm \sqrt{\omega_{\text{an}}/(2\omega_{\text{ex}} + \omega_{\text{an}})}$ and $n_y = \sqrt{2\omega_{\text{ex}}/(2\omega_{\text{ex}} + \omega_{\text{an}})}$.

Then, by the similar method used in obtaining equations (34) and (35), integrating equation (36) (after replacing the lower limit $F$ by $F_+$ or $F_-$), a lower threshold of precession can be derived,

$$\omega_{\text{tot}} = \frac{1}{\pi} \left(2\pi + \frac{\omega_{\text{an}}}{\omega_{\text{ex}}} \right) \sqrt{\frac{2\omega_{\text{ex}} + \omega_{\text{an}}}{\omega_{\text{ex}}}}.$$  \hspace{1cm} (37)

Taking the parameters in the caption of figure 2, $\omega_{\text{tot}} \approx 0.3918\omega_{\text{an}}$, which well reproduces the value of the lower threshold of precession estimated from the numeric simulation.

4.2.5. Frequency
For the high-energy trajectories, the SOT balances with the damping torque in average during a precession. So, the period (frequency) is mainly determined by the unperturbed parts of equation (23), which are written in the Cartesian coordinate system as

$$\frac{\mathrm{d}n_x}{\mathrm{d}t} = 2\omega_{\text{ex}}n_y n_z,$$  \hspace{1cm} (38)

$$\frac{\mathrm{d}n_y}{\mathrm{d}t} = -(2\omega_{\text{ex}} + \omega_{\text{an}})n_x n_y,$$  \hspace{1cm} (39)

$$\frac{\mathrm{d}n_z}{\mathrm{d}t} = \omega_{\text{an}} n_x n_y.$$  \hspace{1cm} (40)

Substituting equations (26) and (27) into one of above equations, and integrating over the whole trajectory, the period can be calculated as a function of $E_n$,

$$T = \int_0^{2\pi} \sqrt{\frac{2\omega_{\text{ex}}(2\omega_{\text{ex}} + \omega_{\text{an}})}{1 - a^2 \cos^2 \eta - b^2 \sin^2 \eta}} \mathrm{d}n.$$  \hspace{1cm} (41)

The final result (frequency) is expressed by the elliptical integral,

$$f = \frac{1}{T} = \frac{\sqrt{2}}{4} \frac{\sqrt{\omega_{\text{ex}} + \omega_{\text{an}} + 2E_n}}{K(k)},$$  \hspace{1cm} (42)

where $K$ denotes the complete elliptic integral of the first kind, and the modulus $k$ is still equation (32). When $E_n = E_{\text{saddle}}, f = 0$. When $E_n = E_{\text{max}}, f = 1/(2\pi)\sqrt{2\omega_{\text{ex}}(2\omega_{\text{ex}} + \omega_{\text{an}})}$, which is just the resonance frequency of $e_y$-FM states.

Removing $E_n$ from equations (34) and (42), one can get the dependence on frequency $\omega_{\text{tot}}$. But, there is no explicit expression. We plot $f$-$\omega_{\text{tot}}$ curve in figure 6(a). There is a good agreement between the numerical simulation (red stars) and the analytical theory (blue solid line) in the region between $\omega_{\text{tot}}$ and $\omega_{\text{an}}$. Below $\omega_{\text{tot}}$, the numerical simulation indicates that the tilted AFM state is stable, as shown in figure 2. Obviously, the frequency increases almost linearly with $\omega_{\text{tot}}$. In the absence of uniaxial anisotropy ($\omega_{\text{an}} = 0$), $f = 1/(2\pi)\omega_{\text{an}}/\alpha$. 

---

10
It is interesting to discuss the relation between the material parameters and the frequency tunability. From equations (34) and (42), it is easy to observe that the frequency is a function of $\omega_{\text{sot}}/\alpha$. So, to excite a precession with a certain frequency, small damping can decrease the required current density. Additionally, from equations (19) and (37), it can be proved that $d(\omega^u_{\text{sot}} - \omega^l_{\text{sot}})/d\omega_{\text{an}} < 0$. Namely, the current window gets narrow with $\omega_{\text{an}}$ increasing, so does the frequency window. In the limit of weak anisotropy, from equation (37) the lowest frequency is about $1/\pi(1 + 2\alpha\sqrt{2\omega_{\text{ex}}/\omega_{\text{an}}})\omega_{\text{an}}/\alpha$. There is no simple relation with the material parameters. On the other hand, equation (19) indicates that the upper threshold of $\omega_{\text{sot}}$ is proportional to $\alpha$. Then, the highest frequency, which is about $2\omega_{\text{ex}}$, mainly depends on the exchange interaction.

4.2.6. Amplitude

Removing $E_n$ from equations (28), (29) and (34), one can obtain the dependence of amplitudes of $n_x$ and $n_y$ on $\omega_{\text{sot}}$. In figure 6(b), the result from the analytic formula is displayed as the solid lines. The validity of the derived formula can be confirmed by comparison with a numeric integration of the coupled LLG equations (equation (1)). With $\omega_{\text{sot}}$ increasing, the amplitudes decrease and $n$ is directed along the spin polarization at the upper threshold. This means that the conical angle of precession decreases with $\omega_{\text{sot}}$ increasing.

5. Case that $e_p \parallel e_a$

For this configuration, the system has the axial symmetry which makes the analysis easier than the case that $e_p \perp e_a$.

5.1. Equilibria and their stabilities

Here, $e_p = e_a = e_z$, as shown in figure 1. The equilibria are determined by the balance between the SOT and the precession torques produced by the exchange and anisotropy fields. Taking $dx/dt = 0$ in equation (4), the equilibrium equations read,

$$\omega_{\text{ex}} \sin \theta^0_{3-i} \sin(\varphi^0_{i} - \varphi^0_{3-i}) - \omega_{\text{sot}} \sin \theta^0_i = 0,$$

$$\omega_{\text{ex}} [\cos \theta^0_{i} \sin \theta^0_{3-i} \cos(\varphi^0_{i} - \varphi^0_{3-i}) - \cos \theta^0_{3-i} \sin \theta^0_i] + \omega_{\text{an}} \cos \theta^0_i \sin \theta^0_0 = 0,$$

where $i = 1, 2$, denoting the two sublattices. From equation (43), one can derive that $\sin^2 \theta^0_{3-i} + \sin^2 \theta^0_i = 0$. This means that $\sin \theta^0_{3-i} = 0$ which also satisfy equation (44). So, there are four equilibria: $\theta^0_i = 0$ and $\theta^0_2 = \pi$, $\theta^0_i = \pi$ and $\theta^0_2 = 0$, as well as $\theta^0_{3-i} = \pi$. The first two correspond to degenerate AFM states. The last two are FM states. The stability can be determined by the linearization method and the Routh–Hurwitz criterion which have been formulated in section 3. The detailed derivations are presented in appendix B.
The AFM states are stable under the condition that $|\omega_{\text{tot}}| < \omega_{\text{tot}}^{\mu}$ with

$$\omega_{\text{tot}}^{\mu} = \alpha \sqrt{\omega_{\text{an}}(2 \omega_{\text{ex}} + \omega_{\text{an}})}.$$  \hfill (45)$$

The FM state with $\theta_{1,2} = 0 (\pi)$ is stable under the condition that $\omega_{\text{tot}} > \omega_{\text{tot}}^{\mu} (\omega_{\text{tot}} < -\omega_{\text{tot}}^{\mu})$, where

$$\omega_{\text{tot}}^{\mu} = \alpha (2 \omega_{\text{ex}} - \omega_{\text{an}}).$$  \hfill (46)$$

All the equilibria have been found. So, in the region that $\omega_{\text{tot}}^{\mu} < |\omega_{\text{tot}}| < \omega_{\text{tot}}^{\mu}$, there is no stable equilibrium and a precession possibly emerges.

5.2. Precession

Analogous to section 4.2.1, in view of the symmetry and the numeric results (for example, figure 7), a reduced equation like equation (23) is obtained from equation (1) by setting $m_{1x} = -m_{2x} = n_x$, $m_{1y} = -m_{2y} = n_y$ and $m_{1z} = m_{2z} = n_z$. These relations can also be observed in figure 7. Only the reduced magnetic energy is different from the case of $\mathbf{e}_y \parallel \mathbf{e}_x$, which reads,

$$E_n = \frac{1}{2} \frac{1}{2} (2 \omega_{\text{ex}} - \omega_{\text{an}}) (\mathbf{n} \cdot \mathbf{e}_x)^2 - \frac{1}{2} \omega_{\text{an}}.$$  \hfill (47)$$

Considering $\omega_{\text{ex}} > \omega_{\text{an}}$ generally, a bipartite AFM with $\mathbf{e}_y \parallel \mathbf{e}_x$ is translated into a single-order-vector magnetic system with an easy plane perpendicular to the spin polarization ($\mathbf{e}_y$).

Due to the axial symmetry, the constant-energy trajectories are just the latitude lines on the unit sphere and there exists an analytic solution of precession. It is not necessary to resort to Melnikov's method. In order to solve equation (23) for the case of $\mathbf{e}_y \parallel \mathbf{e}_x$, the vector $\mathbf{n}$ is conveniently parametrized in spherical coordinates as $\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. Then, equation (23) with the magnetic energy equation (47) is transformed into

$$\frac{d\theta}{dt} + \alpha \sin \theta \frac{d\phi}{dt} = -\omega_{\text{tot}} \sin \theta,$$  \hfill (48)$$

$$-\alpha \frac{d\theta}{dt} + \sin \theta \frac{d\phi}{dt} = -(2 \omega_{\text{ex}} - \omega_{\text{an}}) \sin \theta \cos \theta.$$  \hfill (49)$$

The functions on the right of equations (48) and (49) are independent of $\phi$. In order to look for the solutions, it is convenient to remove $d\phi/dt$ from equations (48) and (49). Then, one has

$$(1 + \alpha^2) \frac{d\theta}{dt} = \sin \theta [\alpha (2 \omega_{\text{ex}} - \omega_{\text{an}}) \cos \theta - \omega_{\text{tot}}].$$  \hfill (50)$$
For this \( \theta \)-equation, there exist two kinds of equilibria. The first one is \( \theta_0 = 0 (\pi) \), which is stable for \( \omega_{\text{tot}} > \omega_{\text{tot}}' \) (\( \omega_{\text{tot}} < -\omega_{\text{tot}}' \)). These stable conditions are derived in appendix C by linear stability analysis of equation (50). These two equilibria are exactly the FM states illustrated in section 5.1.

The other equilibrium of equation (50) is

\[
\theta_0 = \arccos \frac{\omega_{\text{tot}}}{\omega_{\text{tot}}'},
\]

which is stable for \( |\omega_{\text{tot}}| < \omega_{\text{tot}}' \) (see appendix C). For this stable equilibrium of \( \theta \), one can obtain the velocity of \( \phi \) from equation (48),

\[
\frac{d\phi}{dt} = -\frac{\omega_{\text{tot}}}{\alpha}.
\]

This is also the circular frequency of precession. Based on these solutions, the precessions of \( m_1,2 \) can be described by

\[
m_{ix} = (-1)^{-1} \sqrt{1 - \left( \frac{\omega_{\text{tot}}}{\omega_{\text{tot}}'} \cos \left( \frac{\omega_{\text{tot}}}{\alpha} t \right) \right)^2},
\]

\[
m_{iy} = (-1)^{-1} \sqrt{1 - \left( \frac{\omega_{\text{tot}}}{\omega_{\text{tot}}'} \sin \left( \frac{\omega_{\text{tot}}}{\alpha} t \right) \right)^2}.
\]

These analytic expressions are consistent with the numerical results shown in figure 7, supporting the validity of above argument.

Unlike the case of \( e_\perp e_a \), here the frequency window can be given analytically, which ranges from \( \sqrt{\omega_{\text{an}}(2\omega_{\text{ex}} + \omega_{\text{an}})} \) to \( 2\omega_{\text{ex}} + \omega_{\text{an}} \). This is easily obtained from equations (45), (46) and (52). The frequency window is mainly determined by the exchange interaction and the magnetic anisotropy, independent of the damping. But, a small damping favors decreasing the exciting current. The frequency window is shrunk if increasing \( \omega_{\text{an}} \), because \( d(\omega_{\text{tot}}' - \omega_{\text{tot}}')/d\omega_{\text{an}} < 0 \). When \( \omega_{\text{an}} > 2/3\omega_{\text{ex}} \), the window closes, i.e. there is no precession. Therefore, a small damping and a weak anisotropy benefit the frequency tunability.

Furthermore, the characteristics of precession are summarized in figure 8. For \( \omega_{\text{tot}} > 0 \), equation (51) means that \( \theta_0 < \pi / 2 \). Namely, the precession trajectories are above the equator. From equation (52), \( d\phi/dt < 0 \). So, the precession direction obeys the left-hand thumb rule when the thumb points along the positive \( z \)-direction, as marked by the arrowhead on the upper trajectory (dotted red line) in figure 8. For this precession direction, the damping torque makes \( m_1 \) deviate from the easy axis. But, the SOT drives \( m_1 \) approaching the easy axis. The balance between these two torques enables the stable precession. On the other hand, along the tangential of this trajectory, there exist an exchange torque \( \omega_{\text{ex}} m_1 \times m_2 \) and an anisotropy torque \( -\omega_{\text{an}}(m_1 \cdot e_a)(m_1 \times e_a) \). As indicated by the tangential arrows at the endpoint of \( m_1 \), the exchange torque makes \( m_1 \) rotate left-handedly around the axis. While, the anisotropy torque makes \( m_1 \) rotate right-handedly around the axis. Because the exchange is stronger than the anisotropy, \( m_1 \) rotates
left-handedly. This is very different from the ferromagnet, for which the anisotropy field determines the sense of rotation and the left-hand rotation is not allowed.

6. Discussions

In this section, several instructive remarks are in order. Firstly, it is helpful for application to estimate the thresholds and the frequency of precession. By the definition of $\omega_{\text{sot}}$, the corresponding current density can be calculated from $j = (eM_{s}d)/(\xi \mu_{B} \omega_{\text{sot}})$. The saturation magnetization $M_{s} \approx 47.7 \text{ kA m}^{-1}$ [42].

Considering the experimentally feasible parameter, the SOT efficiency $\xi = 0.32$ [46], and the thickness of AFM layer $d = 4 \text{ nm}$. Then, by use of the magnetic parameters in the caption of figure 2 and equations (19), (37), (45) and (46), the lower and upper thresholds of current density can be calculated. For the case $e_{p} \parallel e_{s}$, the precession happens with the current varying from $5.82 \times 10^{7} \text{ A cm}^{-2}$ to $1.92 \times 10^{8} \text{ A cm}^{-2}$. The corresponding frequency varies from $0.90 \text{ THz}$ to $2.97 \text{ THz}$. For the case $e_{p} \perp e_{s}$, the precession happens with the current varying from $1.69 \times 10^{7} \text{ A cm}^{-2}$ to $1.89 \times 10^{8} \text{ A cm}^{-2}$. The corresponding frequency varies from $0.26 \text{ THz}$ to $2.92 \text{ THz}$. These frequencies lie in the range of terahertz radiation. To decrease the exciting current, one can use the material with a smaller damping. The adjustable range for $e_{p} \perp e_{s}$ is slightly smaller than that for $e_{p} \parallel e_{s}$. This is because the SOT must overcome the energy barrier of uniaxial anisotropy before the stable precession happens for $e_{p} \perp e_{s}$. MnF$_{2}$ is a nearly ideal uniaxial antiferromagnet. But, the low Néel temperature is unfavorable for application. So, a uniaxial antiferromagnet with higher critical temperature is anticipated.

Secondly, it should be emphasized that no approximation is taken when reducing the coupled LLG equations. This is different from the usual $1-\mathbf{m}$ scheme, in which, to decouple the stagger magnetization $\mathbf{l}$ and the average magnetization $\mathbf{m}$, too many terms related with the damping and spin torques were discarded. This results in omission of some dynamic properties. Therefore, entering the nonlinear regime, one has to be a bit careful when applying the $1-\mathbf{m}$ scheme on a spin torque-driven antiferromagnet. Moreover, if taking $\omega_{\text{an}} \rightarrow 0$, the two cases considered above become identical. Both the lower threshold of $\omega_{\text{sot}}$ (equation (37)) for $e_{p} \perp e_{s}$ and the one (equation (45)) for $e_{p} \parallel e_{s}$ become zero. Both the upper threshold of $\omega_{\text{sot}}$ (equation (19)) for $e_{p} \perp e_{s}$ and the one (equation (46)) for $e_{p} \parallel e_{s}$ become $2\alpha \omega_{\text{ex}}$. In this limit, the frequencies are also the same, taking $1/[(2\pi)\omega_{\text{sot}}/\alpha]$. These facts also justify our analytical treatment.

Thirdly, our results indicate that a stable AFM precession is free from the magnetic field. A relatively strong dampinglike SOT tilts the sublattice magnetization $\mathbf{m}_{1,2}$ deviating from the antiparallel state. Then, $\mathbf{m}_{1,2}$ are exposed to the action of exchange torque, which propels them precessing. When the SOT balances the intrinsic damping torque, the precession can exist stably. Unlike the ferromagnet, it is the exchange interaction that drives the precession for AFM.

Fourthly, it should be pointed out that the fieldlike SOT is ignored. Including this fieldlike term $-\beta \omega_{\text{sot}} \mathbf{m}_{1,2} \times \mathbf{e}_{p}$ with $\beta$ being the relative strength of fieldlike SOT to the dampinglike one, the calculations indicate that $\beta$ enters as a factor $1 + \alpha\beta$ before $\omega_{\text{sot}}$ for the threshold values and the frequency-current relations. For typical experimental parameters, $\alpha\beta \ll 1$. In addition, for the case of $e_{p} \perp e_{s}$, the fieldlike SOT changes the tilted AFM states slightly by cocking $\mathbf{m}_{1,2}$ up from the $x-y$ plane with the tilted angle $\pi/2 - \arccos[2\beta \omega_{\text{sot}}/(2\omega_{\text{ex}} + \omega_{\text{an}} + \sqrt{\omega_{\text{an}}^{2} - 4\omega_{\text{sot}}^{2}})]$. In the large-exchange limit, this angle is vanishingly small. Beyond these two points, although the field-like SOT make the calculations more lengthy, it nearly has no influence on the final results. Moreover, for the SOTs induced by the spin Hall effect, the dominant component is dampinglike [16]. Therefore, we neglect the field-like SOT.

Finally, it is worthwhile to compare with two similar previous works [8, 15]. In reference [15], a uniaxial AFM is studied under the spin torques with arbitrary spin polarizations. During the precession, two sublattice magnetic moments are assumed to remain antiparallel. Moreover, their precessions are restricted in a plane normal to the spin polarization. However, one can find that, by observing the evolutions of magnetization (for example, figure 7), this is just a special case near the lower threshold. In reference [8], several similar results, such as the lower threshold and the frequency-SOT relation, were presented for the case of $e_{p} \perp e_{s}$. The lower threshold of reference [8] can be obtained by taking the limit $\omega_{\text{an}}/\omega_{\text{ex}} \rightarrow 0$ in equation (37). Meanwhile, the frequency-SOT relation in reference [8] can be gotten from equations (34) and (42) by taking $\omega_{\text{an}} \rightarrow 0$. The upper threshold of precession is not concerned in this work. In addition, this work only consider a special case that the precession was confined in the plane normal to the spin polarization by a strong easy-plane anisotropy. More universally, our work demonstrates that the precessional magnetic moments form a conical surface with an adjustable cone angle.
7. Conclusion

We have investigated the stationary and dynamic states of the AFM films with a uniaxial magnetic anisotropy driven by the dampinglike SOT with the spin polarization perpendicular and parallel to the easy axis. The stabilities of all equilibria are analyzed by the linearization method and Routh–Hurwitz criterion. Based on a reduced single-vector equation, we obtain the analytic expressions of the thresholds, the frequency, and the amplitude of the stable precessions. These results are verified by numerically integrating the AFM LLG equations.

For the case with the spin polarization perpendicular to the easy axis, if $|\omega_{\text{sot}}| < \omega_{\text{sot}}^l$, the system is in a state of tilted AFM. If $|\omega_{\text{sot}}| > \omega_{\text{sot}}^l$, the system is in a ferromagnetic state. The stable precession occurs in the region that $\omega_{\text{sot}}^d < |\omega_{\text{sot}}| < \omega_{\text{sot}}^l$. The analytic formula of the lower threshold $\omega_{\text{sot}}^d$ (equation (37)) and the upper one $\omega_{\text{sot}}^l$ (equation (19)) have been derived.

For the case with the spin polarization parallel to the easy axis, the system remains AFM state provided $|\omega_{\text{sot}}| < \omega_{\text{sot}}^l$, while enters the ferromagnetic state when $|\omega_{\text{sot}}| > \omega_{\text{sot}}^l$. In the middle region that $\omega_{\text{sot}}^d < |\omega_{\text{sot}}| < \omega_{\text{sot}}^l$, a stable precession emerges. Also, we derive the analytic formula for the lower threshold $\omega_{\text{sot}}^d$ (equation (45)) and the upper one $\omega_{\text{sot}}^l$ (equation (46)).

For both cases, it is mainly the exchange interaction to propel the precession on the promise of an average balance between the SOT and the damping. The precessional frequency is of the order of a few THz, and increases almost linearly with the strength of SOT. Moreover, a weak anisotropy favors improving the average balance between the SOT and the damping. The precessional frequency is tunable through enlarging the frequency window. A small damping favors reducing the current to excite a precession. Our results not only enrich the study of current-driven AFM dynamics, but also provide a solid clue for designing the AFM-oscillator with a uniaxial magnetic anisotropy.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant No. 51972103, No. 21938002, and No. 61774001), the Natural Science Foundation of Hebei Province of China under Grant No. F2019202207, the Program of State Key Laboratory of Quantum Optics and Quantum Optics Devices, Shanxi University, Taiyuan, 030006, China (No. KF201906), and the basic scientific research business expenses of the central university and Open Project of Key Laboratory for Magnetism and Magnetic Materials of the Ministry of Education, Lanzhou University.

Appendix A. Stability analysis of the equilibria for the case that $e_p \perp e_a$

In this appendix, using the linear stability analysis method illustrated in section 3, we derive the expressions for the stable regions of all equilibria presented in section 4.1.

A1. Stability analysis of $e_p$-FM states

For these states, $\theta_i^0 = 0$ or $\pi$. $\phi_i^0$ is ill defined. Thus, to avoid the difficulties in studying the stability of those equilibria, we chose another coordinate system with $x$ axis along the easy axis ($e_x$), $y$ axis along the spin-polarization direction ($e_y$), and $z$ axis normal to the film. In this reference, the two $e_p$-FM states are described as $\theta^0_{1,2} = \pi/2$ and $\phi^0_1 = \phi^0_2 = \pm \pi/2$. The energy density reads $E = \omega_{\text{ex}} \mathbf{m}_1 \cdot \mathbf{m}_2 - \lambda (\mathbf{m}_1 \cdot \mathbf{e}_x)^2 + [\lambda (\mathbf{m}_1 \cdot \mathbf{e}_y)^2 + (\mathbf{m}_2 \cdot \mathbf{e}_y)^2]$, and the SOT $\mathbf{r}_i = -\omega_{\text{sot}} \mathbf{m}_i \times (\mathbf{m}_i \times \mathbf{e}_y)$. Expressing $E$ and $\mathbf{r}_i$ by the spherical coordinates $\theta_i$ and $\phi_i$, and inserting them into equations (7)–(11), we can obtain the secular equation for the $e_p$-FM states. Here, this quartic equation can be factorized into the product of two quadratic ones. For $\mathbf{m}_{1,2}$ along the spin-polarization direction ($\phi^0_1 = \phi^0_2 = \pi/2$), the secular equation reads,

$$ (a_0 \lambda^2 + a_1 \lambda + a_2) (b_0 \lambda^2 + b_1 \lambda + b_2) = 0, \quad (A1) $$

where

$$ a_0 = b_0 = 1 + \alpha^2, \quad (A2) $$

$$ a_1 = 2\alpha \omega_{\text{sot}} - \alpha (4\omega_{\text{ex}} + \omega_{\text{an}}), \quad (A3) $$

$$ a_2 = \omega_{\text{sot}}^2 + 2\alpha \omega_{\text{ex}} (2\omega_{\text{ex}} + \omega_{\text{an}}), \quad (A4) $$

$$ b_1 = 2\alpha \omega_{\text{sot}} - \alpha \omega_{\text{an}}, \quad (A5) $$

$$ b_2 = \omega_{\text{sot}}^2. \quad (A6) $$
According to the Routh–Hurwitz criterion, if \( a_1, a_2, b_1, \) and \( b_2 \) are all positive, the equilibrium state is stable. For a realistic AFM, the exchange coupling is generally stronger than the anisotropy, i.e. \( \omega_{\text{ex}} > \omega_{\text{an}} > 0 \). So, it is easy to conclude that the stability condition is \( \omega_{\text{tot}} > \omega_{\text{tot}}^u \), with

\[
\omega_{\text{tot}}^u = \frac{\alpha}{2} (4\omega_{\text{ex}} + \omega_{\text{an}}).
\]  

(A7)

Similarly, the stability condition for the \( e_p \)-FM state with \( m_{1,2} \) contrary to the spin-polarization direction \( (\phi_1^0 = \phi_2^0 = -\pi/2) \) can also be deduced as \( \omega_{\text{tot}} < -\omega_{\text{tot}}^u \).

A2. Stability analysis of \( e_p \)-AFM states

In the reference introduced in appendix A1, the \( e_p \)-AFM states are described as \( \theta_{1,2}^p = \pi/2 \) and \( \phi_1^p = \pm \pi/2, \phi_2^p = \mp \pi/2 \). By use of the same procedure, the secular equation (equation (12)) is calculated with the parameters listed as:

\[
a_0 = (1 + \alpha^2)^2,
\]  

(A8)

\[
a_1 = 2\alpha (1 + \alpha^2)(2\omega_{\text{ex}} - \omega_{\text{an}}),
\]  

(A9)

\[
a_2 = \alpha^2 \omega_{\text{ex}} - \omega_{\text{an}}^2 - 2(1 + \alpha^2)\omega_{\text{ex}}\omega_{\text{an}} - 2(1 - \alpha^2)\omega_{\text{tot}}^u,
\]  

(A10)

\[
a_3 = -2\alpha(2\omega_{\text{ex}} - \omega_{\text{an}})(\omega_{\text{ex}}\omega_{\text{an}} - \omega_{\text{tot}}^u),
\]  

(A11)

\[
a_4 = \omega_{\text{tot}}^u(2\omega_{\text{ex}} - \omega_{\text{an}} + \omega_{\text{tot}}^u).
\]  

(A12)

Then, the Routh–Hurwitz determinants (equations (13)–(16)) are calculated as

\[
\Delta_1 = 2\alpha (1 + \alpha^2)(2\omega_{\text{ex}} - \omega_{\text{an}}),
\]  

(A13)

\[
\Delta_2 = \Delta_1 [A_0 - (3 - \alpha^2)\omega_{\text{tot}}^u]
\]  

(A14)

\[
\Delta_3 = \frac{-\Delta_1^2}{1 + \alpha^2} \left[ \omega_{\text{ex}}\omega_{\text{an}}A_0 + B_2\omega_{\text{tot}}^u + 4\omega_{\text{tot}}^u \right],
\]  

(A15)

\[
\Delta_4 = a_4 \Delta_3,
\]  

(A16)

where

\[
A_0 = \alpha^2 (2\omega_{\text{ex}} - \omega_{\text{an}})^2 - (1 + \alpha^2)\omega_{\text{ex}}\omega_{\text{an}},
\]  

(A17)

\[
B_2 = 4\omega_{\text{ex}}(\omega_{\text{ex}} - \omega_{\text{an}}) + \alpha^2 (4\omega_{\text{ex}} - \omega_{\text{an}})\omega_{\text{an}}.
\]  

(A18)

The two \( e_p \)-AFM states are degenerate and have the same secular equation and Routh–Hurwitz determinants. In view of \( \omega_{\text{ex}} > \omega_{\text{an}} \), it can be inferred that \( \Delta_2 > 0 \) and \( \Delta_4 > 0 \). So, the stability demands that \( \Delta_2 > 0 \) and \( \Delta_4 > 0 \). Considering \( B_2 > 0 \), from equation (A15), it can be derived that \( A_0 < 0 \) is a necessary condition for \( \Delta_1 > 0 \). Furthermore, from equation (A14), it is easy to infer that \( \Delta_2 < 0 \) for \( A_0 < 0 \) because \( 3 - \alpha^2 > 0 \) generally. Hence, it is impossible that all \( \Delta \) are positive. The \( e_p \)-AFM states are unstable.

A3. Stability analysis of tilted AFM states

For these kinds of states, the energy density reads \( \mathcal{E} = \omega_{\text{ex}} \mathbf{m}_1 \cdot \mathbf{m}_2 - \frac{1}{2} \omega_{\text{an}} \left[ (\mathbf{m}_1 \cdot \mathbf{e})^2 + (\mathbf{m}_2 \cdot \mathbf{e})^2 \right] \), and the SOT \( \tau_i = -\omega_{\text{tot}} \mathbf{m}_i \times (\mathbf{m}_i \times \mathbf{e}) \). Inserting \( \mathcal{E} \) and \( \tau_i \) into equations (7)–(11), expressing \( \mathbf{m}_i \) by the spherical coordinates \( \theta_i \) and \( \phi_i \), and taking \( \theta_{1,2}^p = \pi/2, \phi_1^p = \phi^p \) and \( \phi_2^p = \pm \pi + \phi^p \) with \( \phi^p \) being equation (22), we can obtain the secular equation as equation (A1), where the parameters are listed as

\[
a_1 = \frac{\alpha}{2} \left( 4\omega_{\text{ex}} + \omega_{\text{an}} + 3(-1)^P \sqrt{\omega_{\text{an}}^2 - 4\omega_{\text{tot}}^u} \right),
\]  

(A19)

\[
a_2 = \frac{(-1)^P}{2} \sqrt{\omega_{\text{an}}^2 - 4\omega_{\text{tot}}^u} \left( 4\omega_{\text{ex}} + \omega_{\text{an}} + (-1)^P \sqrt{\omega_{\text{an}}^2 - 4\omega_{\text{tot}}^u} \right),
\]  

(A20)

\[
b_1 = a_1,
\]  

(A21)

\[
b_2 = \frac{1}{2} \left( 2\omega_{\text{ex}} + (-1)^P \sqrt{\omega_{\text{an}}^2 - 4\omega_{\text{tot}}^u} \right) \left( \omega_{\text{an}} + (-1)^P \sqrt{\omega_{\text{an}}^2 - 4\omega_{\text{tot}}^u} \right).
\]  

(A22)

Because \( \omega_{\text{ex}} > \omega_{\text{an}} \), it can be concluded that \( a_1 > 0, a_2 < 0, b_1 > 0, \) and \( b_2 > 0 \) for \( P = 1, 3 \). For \( P = 2, 4 \), the parameters are all positive. Thus, according to the Routh–Hurwitz criterion, the tilted AFM states are
stable for \( P = 2, 4 \), and unstable for \( P = 1, 3 \). For \( P = 2(4) \),

\[
\phi_1^0(\phi_2^0) = \frac{\pi}{2} - \frac{1}{2} \arcsin \frac{2\omega_{\text{sot}}}{\omega_{\text{an}}},
\]

(A23)

\[
\phi_2^0(\phi_1^0) = \frac{3\pi}{2} - \frac{1}{2} \arcsin \frac{2\omega_{\text{sot}}}{\omega_{\text{an}}},
\]

(A24)

Considering the equivalence of two sublattices, the solutions for \( P = 2 \) and 4 are equivalent.

**A4. Stability analysis of tilted FM states**

By use of the same procedure as appendix A3, and taking \( \theta_{1,2}^0 = \pi/2 \), and \( \phi_{1,2}^0 = \phi^0 \) with \( \phi^0 \) being equation (22), we can obtain the secular equation as equation (A1), where the parameters are listed as

\[
a_1 = -2\alpha \left[ (2\omega_{\text{ex}} - \omega_{\text{an}}) + \frac{\omega_{\text{sot}}^2}{\omega_{\text{an}}} + (-1)^p \sqrt{\omega_{\text{an}}^2 - 4\omega_{\text{sot}}^2} \right],
\]

(A25)

\[
a_2 = \left[ 2\omega_{\text{ex}} - (-1)^p \sqrt{\omega_{\text{an}}^2 - 4\omega_{\text{sot}}^2} \right] \left( 2\omega_{\text{ex}} - \omega_{\text{an}} + 2\frac{\omega_{\text{sot}}^2}{\omega_{\text{an}}} + (-1)^p \sqrt{\omega_{\text{an}}^2 - 4\omega_{\text{sot}}^2} \right),
\]

(A26)

\[
b_1 = 2\alpha \left( \omega_{\text{an}} - 3\frac{\omega_{\text{sot}}^2}{\omega_{\text{an}}} + (-1)^p \sqrt{\omega_{\text{an}}^2 - 4\omega_{\text{sot}}^2} \right),
\]

(A27)

\[
b_2 = (-1)^p \sqrt{\omega_{\text{an}}^2 - 4\omega_{\text{sot}}^2} \left( \omega_{\text{an}} - 2\frac{\omega_{\text{sot}}^2}{\omega_{\text{an}}} + (-1)^p \sqrt{\omega_{\text{an}}^2 - 4\omega_{\text{sot}}^2} \right).
\]

(A28)

Considering \( \omega_{\text{ex}} > \omega_{\text{an}} \), it can be inferred that \( a_1 < 0 \). Therefore, the tilted FM states are unstable according to the Routh–Hurwitz criterion.

**Appendix B. Stability analysis of the equilibria for the case that \( e_p \parallel e_a \)**

In this appendix, using the linear stability analysis method illustrated in section 3, we derive the expressions for the stable regions of AFM and FM states presented in section 5.1. Similar to appendix A1, to avoid the illness of equilibria such as \( \theta = 0 \) or \( \pi \), we chose the coordinate system with \( x \) axis along the easy axis (\( e_a \)), \( y \) axis along the spin-polarization direction (\( e_p \)), and \( z \) axis normal to the film.

**B1. Stability analysis of AFM states**

In the chosen coordinate system, the two AFM states are described as \( \theta_{1,2}^0 = \pi/2 \), and \( \phi_{1,2}^0 = \pm \pi/2 \). The energy density reads \( E = \omega_{\text{ex}} m_1 \cdot m_2 - \frac{1}{2} \omega_{\text{an}} (m_1 \cdot e_p)^2 + (m_1 \cdot e_a)^2 \), and the SOT \( \tau_1 = -\omega_{\text{an}} m_1 \times (m_1 \times e_p) \). Expressing \( E \) and \( \tau_1 \) by the spherical coordinates \( \theta \) and \( \phi \), and inserting them into equations (7)–(11), we can obtain the secular equation (12), in which, the parameters are identical for two AFM states and given as

\[
a_0 = (1 + \alpha^2)^2,
\]

(B1)

\[
a_1 = 4\alpha (1 + \alpha^2) (\omega_{\text{ex}} + \omega_{\text{an}}),
\]

(B2)

\[
a_2 = 2 \left[ 2\omega_{\text{ex}} \omega_{\text{an}} + \omega_{\text{an}}^2 - \omega_{\text{sot}}^2 + 2\alpha^2 \left( 2\omega_{\text{ex}}^2 + 6\omega_{\text{ex}} \omega_{\text{an}} + 3\omega_{\text{an}}^2 + \omega_{\text{sot}}^2 \right) \right],
\]

(B3)

\[
a_3 = 4\alpha (\omega_{\text{ex}} + \omega_{\text{an}}) \left( 2\omega_{\text{ex}} \omega_{\text{an}} + \omega_{\text{an}}^2 + \omega_{\text{sot}}^2 \right),
\]

(B4)

\[
a_4 = \left( \omega_{\text{an}}^2 + \omega_{\text{sot}}^2 \right) \left[ (2\omega_{\text{ex}} + \omega_{\text{an}})^2 + \omega_{\text{sot}}^2 \right].
\]

(B5)

Then, the Routh–Hurwitz determinants (equations (13)–(16)) are calculated as

\[
\Delta_1 = 4\alpha (1 + \alpha^2) (\omega_{\text{ex}} + \omega_{\text{an}}),
\]

(B6)

\[
\Delta_2 = \Delta_1 \left[ 2\omega_{\text{ex}} \omega_{\text{an}} + \omega_{\text{an}}^2 - 3\omega_{\text{sot}}^2 + \alpha^2 \left( 4\omega_{\text{ex}}^2 + 10\omega_{\text{ex}} \omega_{\text{an}} + 5\omega_{\text{an}}^2 + \omega_{\text{sot}}^2 \right) \right],
\]

(B7)

\[
\Delta_3 = 16\alpha \Delta_1 (\omega_{\text{ex}} + \omega_{\text{an}}) \left[ (\omega_{\text{ex}} + \omega_{\text{an}})^2 + \omega_{\text{sot}}^2 \right] \left[ \alpha^2 \omega_{\text{an}} (2\omega_{\text{ex}} + \omega_{\text{an}}) - \omega_{\text{sot}}^2 \right],
\]

(B8)

\[
\Delta_4 = a_4 \Delta_1.
\]

(B9)

Obviously, \( \Delta_1 > 0 \). Solving \( \Delta_3 > 0 \), one has \( |\omega_{\text{sot}}| < \omega_{\text{an}} \) with

\[
\omega_{\text{sot}}^\delta = \alpha \sqrt{\omega_{\text{an}} (2\omega_{\text{ex}} + \omega_{\text{an}})}.
\]

(B10)
Inserting this inequation into $\Delta_2$, one can get

$$\Delta_2 > \Delta_1 \left[ 4 \alpha^2 \omega_{\text{ex}}^2 + \left( 1 + \alpha^2 \right)^2 \omega_{\text{in}} (2 \omega_{\text{ex}} + \omega_{\text{in}}) \right].$$

(B11)

Apparently, $\Delta_2 > 0$. Because $\alpha > 0$, all $\Delta$’s are positive under the condition $|\omega_{\text{tot}}| < \omega_{\text{tot}}'$. In accordance with the Routh–Hurwitz criterion, the inequation $|\omega_{\text{tot}}| < \omega_{\text{tot}}'$ defines the stable region of the AFM states.

**B2. Stability analysis of FM states**

In the chosen coordinate system, the two FM states are described as $\theta_{1,2}^0 = \pi/2$ and $\phi_{1,2}^0 = \pm \pi/2$. Following the same procedure as appendix B1, the secular equation as equation (A1) for $\phi_{1,2}^0 = \pi/2$ can be obtained with the parameters listed as,

$$a_1 = 2(\alpha \omega_{\text{in}} + \omega_{\text{tot}}'),$$

(B12)

$$a_2 = \omega_{\text{in}}^2 + \omega_{\text{tot}}'^2,$$

(B13)

$$b_1 = 2 [\omega_{\text{tot}} - \alpha (2 \omega_{\text{ex}} - \omega_{\text{in}})],$$

(B14)

$$b_2 = (2 \omega_{\text{ex}} - \omega_{\text{in}})^2 + \omega_{\text{tot}}'^2.$$  

(B15)

It is evident that $a_{1,2} > 0$ and $b_{1,2} > 0$ if $\omega_{\text{tot}} > \omega_{\text{tot}}'$ with

$$\omega_{\text{tot}}' = \alpha (2 \omega_{\text{ex}} - \omega_{\text{in}}).$$

(B16)

According to the Routh–Hurwitz criterion, $\omega_{\text{tot}} > \omega_{\text{tot}}'$ is the stable condition of the FM state $\phi_{1,2}^0 = \pi/2$. For another FM state $\phi_{1,2}^0 = -\pi/2$, the stable condition is $\omega_{\text{tot}} < -\omega_{\text{tot}}'$ by use of the similar derivation.

**Appendix C. Linear stability analysis on equation (50)**

Assuming $\theta = \theta_0 + \theta'$, with $\theta'$ being the response to a small perturbation. Inserting this ansatz into equation (50) and keeping linear term of $\theta'$, we have

$$\frac{d\theta'}{dr} = \mathcal{A}(\theta_0) \theta',$$

(C1)

with

$$\mathcal{A}(\theta_0) = \frac{\alpha (2 \omega_{\text{ex}} - \omega_{\text{in}}) \cos 2 \theta_0 - \omega_{\text{tot}} \cos \theta_0}{1 + \alpha^2}.$$  

(C2)

If $\mathcal{A}(\theta_0) < 0$, the corresponding solution is stable. For $\theta_0 = 0$, $\mathcal{A} = \alpha (2 \omega_{\text{ex}} - \omega_{\text{in}}) - \omega_{\text{tot}}$. So, the stable region is $\omega_{\text{tot}} > \alpha (2 \omega_{\text{ex}} - \omega_{\text{in}})$. While, for $\theta_0 = \pi$, $\mathcal{A} = \alpha (2 \omega_{\text{ex}} - \omega_{\text{in}}) + \omega_{\text{tot}}$ and the stable region is $\omega_{\text{tot}} < -\alpha (2 \omega_{\text{ex}} - \omega_{\text{in}})$.

For the equilibrium

$$\cos \theta_0 = \frac{\omega_{\text{tot}}}{\alpha (2 \omega_{\text{ex}} - \omega_{\text{in}})},$$

(C3)

$$\mathcal{A} = \frac{\omega_{\text{tot}}^2}{\alpha (2 \omega_{\text{ex}} - \omega_{\text{in}})} - \alpha (2 \omega_{\text{ex}} - \omega_{\text{in}}).$$

(C4)

From $\mathcal{A} < 0$, one can infer that this solution is stable for $|\omega_{\text{tot}}| < \alpha (2 \omega_{\text{ex}} - \omega_{\text{in}})$.

**ORCID iDs**

Peng-Bin He  [https://orcid.org/0000-0002-0658-2860]
Meng-Qiu Cai  [https://orcid.org/0000-0002-5364-725X]

**References**

[1] Baltz V, Mancho A, Tsoi M, Moriyama T, Ono T and Tserkovnyak Y 2018 Rev. Mod. Phys. 90 015005
[2] Firastrau I, Buda–Prejbeanu L D, Dieny B and Ebels U 2013 J. Appl. Phys. 113 113908
[3] Zhou Y, Xiao J, Bauer G E W and Zhang F C 2013 Phys. Rev. B 87 020409(R)
[4] Johansen Ø and Linder J 2016 Sci. Rep. 6 33845
[5] Zhong H, Qiao S, Yan S, Liang L, Zhao Y and Kang S 2020 J. Magn. Magn. Mater. 497 166070
[6] Cheng R, Daniels M W, Zhou J-G and Xiao D 2015 Phys. Rev. B 91 064423
[7] Cheng R, Xiao D and Brataas A 2016 Phys. Rev. Lett. 116 207603
[8] Khymyn R, Liskenov I, Tiberkevich V, Ivanov B A and Slavin A 2017 Sci. Rep. 7 43705
[9] Zarzuela R and Tserkovnyak Y 2017 Phys. Rev. B 95 180402(R)
[10] Sulymenko O R, Prokopenko O V, Tiberkevich V S, Slavin A N, Ivanov B A and Khymyn R S 2017 Phys. Rev. Appl. 8 064007
[11] Chęciński J, Frankowski M and Stobiecki T 2017 Phys. Rev. B 96 174438
[12] Chen X Z et al 2018 Phys. Rev. Lett. 120 207204
[13] Puliafito V, Khymyn R, Carpentieri M, Azzerboni B, Tiberkevich V, Slavin A and Finocchio G 2019 Phys. Rev. B 99 024405
[14] Troncoso R E, Rode K, Stamenov P, Coey J M D and Brataas A 2019 Phys. Rev. B 99 054433
[15] Lee D-K, Park B-G and Lee K-J 2019 Phys. Rev. Appl. 11 054048
[16] Manchon A, Železný J, Miron I M, Jungwirth T, Sinova J, Thiaville A, Garello K and Gambardella P 2019 Rev. Mod. Phys. 91 035004
[17] Gomonay H V and Loktev V M 2010 Phys. Rev. B 81 144427
[18] Routh E J 1877 A Treatise on the Stability of a Given State of Motion: Particularly Steady Motion (London: Macmillan)
[19] Hurwitz A 1895 Math. Ann. 46 273
[20] Baláž P and Barnaš J 2013 Phys. Rev. B 88 014406
[21] Liang Y-C, He P-B, Cai M-Q and Li Z-D 2019 J. Magn. Magn. Mater. 479 291
[22] Perko L 1996 Differential Equations and Dynamical Systems (Berlin: Springer) pp 415–31
[23] Serpico C, d’Aquino M, Bertotti G and Mayergoyz I D 2005 Phys. Rev. B 94 127206
[24] Bertotti G, Mayergoyz I D and Serpico C 2009 Nonlinear Magnetization Dynamics in Nanosystems (Amsterdam: Elsevier) pp 241–60
[25] Thiaville A and Nakatani Y 2006 Spin Dynamics in Confined Magnetic Structures III ed B Hillebrands and A Thiaville (New York: Springer) pp 269–74
[26] Taniguchi T, Arai H, Tsunegi S, Tamaru S, Kubota H and Imamura H 2013 Appl. Phys. Express 6 123003
[27] Taniguchi T, Arai H, Kubota H and Imamura H 2014 IEEE Trans. Magn. 50 140040
[28] Taniguchi T, Ito T, Tsunegi S and Kubota H 2015 J. Appl. Phys. 118 053903
[29] Taniguchi T, Ito T, Tsunegi S, Kubota H and Utsumi Y 2017 Phys. Rev. B 96 024406
[30] Ebels U, Housameddine D, Firastrau I, Gusakova D, Thirion C, Diény B and Buda-Prejbeanu L D 2008 Phys. Rev. B 78 024436
[31] Taniguchi T and Kubota H 2016 Phys. Rev. B 93 174401
[32] Pinna D, Kent A D and Stein D I 2013 Phys. Rev. B 88 104405
[33] Pinna D, Stein D I and Kent A D 2014 Phys. Rev. B 90 174405
[34] Taniguchi T 2015 Phys. Rev. B 91 104406
[35] Zhu L, Ralph D C and Buhrman R A 2018 Phys. Rev. Appl. 10 031001(R)
[36] Pai C-F, Ou Y X, Vilela-Leão L H, Ralph D C and Buhrman R A 2015 Phys. Rev. B 92 064426
[37] Wang Z et al 2020 Nanoscale 12 15246
[38] Daniels M W, Cheng R, Yu W, Xiao J and Xiao D 2018 Phys. Rev. B 98 134450
[39] Rezende S M, Azevedo A and Rodríguez-Suárez R L 2019 J. Appl. Phys. 126 151101
[40] Vaidya P, Morley S A, van Tol J, Liu Y, Cheng R, Brataas A, Lederman D and del Barco E 2020 Science 368 160
[41] Hubbard J H and West B H 1995 Differential Equations: A Dynamical Systems Approach: Higher-Dimensional Systems (New York: Springer) pp 162–7
[42] Sun J 2008 Physics 1 33
[43] Taniguchi T, Tsunegi S, Kubota H and Imamura H 2014 Appl. Phys. Lett. 104 152411
[44] Liu Y-T, Chen T-Y, Lo T-H, Tsai T-Y, Yang S-Y, Chang Y-J, Wei J-H and Pai C-F 2020 Phys. Rev. Appl. 13 044032