intuitionistic fuzzy pseudo ideals in Q-algebra

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Abstract
Present several types in this paper of intuitionistic fuzzy ideal in Q-algebra, called (intuitionistic fuzzy pseudo ideal, intuitionistic fuzzy k-pseudo ideal, intuitionistic fuzzy c-pseudo ideal, intuitionistic fuzzy complete-k-pseudo ideal). We have introduced and illustrated several ideas that evaluate their relationship in a Q-algebra.

1 Introduction
In 1966, K.Iseki and Y.Imai([7], [14]) introduced BCK-and BCI-algebras. In 2001 H.S.Kim([6]) introduced a new notion, known as Q-algebra, which is BCH / BCI / BCK-algebra generalization. At the same time, A.Iorgulescu and G.Georgescu ([3]) introduced pseudo BCK-algebras as an exemption from bck-algebras. In 2016, Y.B.jun, H.S.Kim and S.S Ahn([13])introduced pseudo Q-algebra as ageneralization of Q-algebra the concept of fuzzy set was introduced in 1969 by L. A. Zadeh ([10]). In 2005, J.Meng, X.Guo([5]) studied fuzzy ideals of BCK / BCI-algebras. W.A.Dudek and Y.B.Jun ([15]) in 2008, introduced pseudo-BCI-algebras as a natural generalization of BCI-algebras and pseudo-BCI-algebras. At the same time, K. J. Lee([8]) established the fuzzy ideals in pseudo BCI-algebras. In([4]) H. K. Jawad introduced the notion of fuzzy pseudo Ideals of pseudo Q-algebra. In K. ([9]) Intuitionistic Fuzzy Sets(1986) was introduced by T. Atanassov. In 2012 S.M. Abdelnaby and O.R.Elgendy applied the concept of Intuitionistic fuzzy sets on Q-algebra. In this article, we will describe some of the new types of I F pseudo ideal, called (I F pseudo ideal, I F K-pseudo ideal, I F complete ?k-pseudo ideal). Also, we introduced and illustrated the proposition that defines the relationship among them in Q-algebra.

2 Basic concept and notations
In this section, We define Q-algebra, pseudo Q-algebra, bounded, involutory, and some properties.

Definition (2.1) [11]
A Q-algebra is a set M with a binary operation * and constant 0 that fulfilled the following axioms:
1. \( m * m = 0 \quad \forall m \in M \)
2. \( m * 0 = m \quad \forall m \in M \)
3. \((m \ast b) \ast d = (m \ast d) \ast b, \quad \forall m, b, d \in M\)

Remark (2.2) [11]
In a Q-algebra \(M\), we can define a binary relation \(\leq\) on \(M\) by \(m \geq b\) if and only if \(m \ast b = 0\), \(\forall m, b \in M\).

Definition (2.3) [1]
A Q-algebra \((M, \ast, 0)\) is called bounded if there is an element \(e \in M\) that satisfies \(m \leq e\), \(\forall m \in M\), then \(e\) is said to be an unit. We denoted \(e \ast m\) by \(m^*\), for each \(m \in M\) in bounded Q-algebra.

Example (2.4)
Let \(M = \{0, \eta, \theta, \beta\}\) be a set with the following table:

|   | 0 | \(\eta\) | \(\theta\) | \(\beta\) |
|---|---|-----|-----|-----|
| 0 | 0 | 0   | 0   | 0   |
| \(\eta\) | \(\eta\) | 0   | 0   | 0   |
| \(\theta\) | \(\theta\) | 0   | 0   | 0   |
| \(\beta\) | \(\beta\) | \(\beta\) | \(\beta\) | 0   |

Thereafter \((M, \ast, 0)\) be a Q-algebra. Note that \(M\) is bounded by unit \(\beta\).

Remark (2.5) [1]
As stated in the following example, the unit in bounded Q-algebra is not unique in general.

Example (2.6)
A binary operation \(\ast\) with \(M = \{0, \eta, \theta\}\) can be shown in the table:

|   | 0 | \(\eta\) | \(\theta\) |
|---|---|-----|-----|
| 0 | 0   | 0   | 0   |
| \(\eta\) | \(\eta\) | 0   | 0   |
| \(\theta\) | \(\theta\) | 0   | 0   |

Note that \(M\) is bounded with two units \(\eta, \theta\).

Proposition (2.7) [4]
In a bounded Q-algebra \(M\), for any \(m, b \in M\), the following are hold:
1. \(e^* = 0\), \(0^* = e\)
2. \(m^* \ast b = b^* \ast m\)
3. \(0 \ast b = 0\)
4. \(e^* \ast m = 0\)
5. \(m^{**} \leq m\)

Definition (2.8) [1]
For a bounded Q-algebra \(M\), If element \(m\) of \(M\) satisfies \(m^{**} = m\), then \(m\) is called an involution. If every element of \(M\) is an involution, we call \(M\) is an involutory Q-algebra.

Example (2.9)
Let \(M = \{0, \eta, \theta, \beta, \psi\}\), can be shown in table:
An intuitionistic fuzzy set (IFS for short) \( A \) in a set \( M \) is an object having the form
\[
A = \langle m, \mu_A(m), \nu_A(m) : m \in M \rangle
\]
satisfy the following axioms:

**Definition (2.10)** [9]

A pseudo Q-algebra is a non-empty set of \( M \) with constant 0 and two binary operations \( * \) and \( \# \) that
satisfy the following axioms:

1. \( A \subseteq B \) if and only if for all \( m \in M \) \( \mu_A(m) \geq \mu_B(m) \) and \( \mu_A(m) \geq \mu_B(m) \)
2. \( A = B \) if and only if for all \( m \in M \) \( \mu_A(m) = \mu_B(m) \) and \( \mu_A(m) = \mu_B(m) \)
3. \( A \cap B = \langle m, (\mu_A \cap \mu_B)(m), (\nu_A \cap \nu_B)(m) : m \in M \rangle \)
   where \( (\mu_A \cap \mu_B)(m) = \min\{\mu_A(m), \mu_B(m)\} \)
   and \( (\nu_A \cap \nu_B)(m) = \max\{\nu_A(m), \nu_B(m)\} \)
4. \( A \cup B = \langle m, (\mu_A \cup \mu_B)(m), (\nu_A \cup \nu_B)(m) : m \in M \rangle \)
   where \( (\mu_A \cup \mu_B)(m) = \max\{\mu_A(m), \mu_B(m)\} \)
   and \( (\nu_A \cup \nu_B)(m) = \min\{\nu_A(m), \nu_B(m)\} \)

**Definition (2.11)** [2]

If \( A = \langle m, \mu_A(m), \nu_A(m) : m \in M \rangle \) and \( B = \langle m, \mu_B(m), \nu_B(m) : m \in M \rangle \)
be any two IFS of a set \( M \) then

1. \( A \subseteq B \) if and only if for all \( m \in M \) \( \mu_A(m) \geq \mu_B(m) \) and \( \mu_A(m) \geq \mu_B(m) \)
2. \( A = B \) if and only if for all \( m \in M \) \( \mu_A(m) = \mu_B(m) \) and \( \mu_A(m) = \mu_B(m) \)
3. \( A \cap B = \langle m, (\mu_A \cap \mu_B)(m), (\nu_A \cap \nu_B)(m) : m \in M \rangle \)
   where \( (\mu_A \cap \mu_B)(m) = \min\{\mu_A(m), \mu_B(m)\} \)
   and \( (\nu_A \cap \nu_B)(m) = \max\{\nu_A(m), \nu_B(m)\} \)
4. \( A \cup B = \langle m, (\mu_A \cup \mu_B)(m), (\nu_A \cup \nu_B)(m) : m \in M \rangle \)
   where \( (\mu_A \cup \mu_B)(m) = \max\{\mu_A(m), \mu_B(m)\} \)
   and \( (\nu_A \cup \nu_B)(m) = \min\{\nu_A(m), \nu_B(m)\} \)

**Definition (2.12)**

An intuitionistic fuzzy set \( A = \langle m, \mu_A(m), \nu_A(m) : m \in M \rangle \) in a Q-algebra \( M \) is called an intuitionistic fuzzy ideal if

1. \( \mu_A(0) \geq \mu_A(m) \) \( \forall m \in M \)
2. \( \nu_A(0) \leq \nu_A(m) \) \( \forall m \in M \)
3. \( \mu_A(m) \geq \min\{\mu_A(m * b), \mu_A(b)\} \) \( \forall b, m \in M \)
4. \( \nu_A(m) \leq \max\{\nu_A(m * b), \nu_A(b)\} \) \( \forall b, m \in M \)

**Definition (2.13)** [13]

A pseudo Q-algebra is a non-empty set of \( M \) with constant 0 and two binary operations \( * \) and \( \# \) that
satisfy the following axioms:

1. \( m \# m = m * m = 0 \) \( \forall m \in M \)
2. \( m \# 0 = m * 0 = 0 \) \( \forall m \in M \)
3. \( (m \# b) \# c = (m * c) \# b \) \( \forall m, b, c \in M \)
Remark (2.14) [13]
In pseudo Q-algebra M, we can define a binary relation \( \leq \) by
\[
\begin{align*}
  m \leq b & \text{ if and only if } m \# b = 0 \\
  & \text{ and } m \ast b = 0 \quad \forall m, b \in M
\end{align*}
\]

Remark (2.15) [13]
That Q-algebra is a pseudo Q-algebra but the converse is not true as shown in the example below

Example (2.16) Let \( M = \{0, \eta, \theta, \beta\} \)

Table 4: pseudo Q-algebra but not Q-algebra

| * | 0 | \( \eta \) | \( \theta \) | \( \beta \) |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| \( \eta \) | \( \eta \) | 0 | 0 | 0 |
| \( \theta \) | \( \theta \) | \( \theta \) | 0 | \( \eta \) |
| \( \beta \) | \( \beta \) | \( \beta \) | 0 | 0 |

Then \((M, *, 0)\) and \((M, \#, 0)\) are not Q-algebra, since \((\theta \ast \eta) \ast \beta \neq \eta \neq 0 = (\theta \ast \beta) \ast \eta\) and \((\theta \# \eta) \# \beta = 0 \neq \beta = (\theta \# \beta) \# \eta\), but \((M, *, 0)\) is pseudo Q-algebra.

Proposition (2.17) [12]
Let \((M, *, \#, 0)\) be a pseudo Q-algebra. Then the following hold:

1. \((m \ast (m \# b)) \# b = (m \# (m \ast b)) \ast b = 0 \quad \forall m, b \in M\)

Definition (2.18) [4]
A pseudo -Q-algebra M it is said to be bounded if there is an element \( n \in M \) satisfying \( m \leq n \quad \forall m \in M \) i.e. \( m \leq n \iff m \ast n = 0 \) and \( m \# n = 0 \) then \( n \) is called pseudo unit of M.

A pseudo-Q-algebra with a pseudo unit is called bounded.

Proposition (2.19) [4]
Let \((M, *, \#, 0)\) be a bounded pseudo Q-algebra. Then the following hold:

1. \( e \ast 0 = 0 = e \# \)
2. \( m \ast \# b = b \# \ast m \quad \forall m, b \in M \)
3. \( m \ast \# b \ast = (b \ast) \# \ast m \quad \forall m, b \in M \)
4. \( m \# \ast b \# = (b \# \ast \# m \quad \forall m, b \in M \)

Definition (2.20) [13]
Let \((M, *, \#, 0)\) be a bounded pseudo Q-algebra. A subset I of M is called the pseudo -ideal of M if it satisfies:

1. \( 0 \in I \)
2. \( m \ast b, m \# b \in I \) and \( b \in I \) imply \( m \in I \) \( \forall m, b \in I \) whenever \( m, b \in I \)

Definition (2.21) [9]
Let \((M, *, \#, 0)\) be a bounded pseudo Q-algebra and let \( \phi \neq I \subseteq M \). I is called a pseudo subalgebra of M if \( m \ast b, m \# b \in I \) wenever \( m, b \in I \)

Definition (2.22) [4]
Let M be a pseudo Q-algebra. A fuzzy set \( \mu \) in M is called a fuzzy pseudo ideal of M if it satisfies:
1. \( \mu(0) \geq \mu(m) \), \( \forall m \in M \)

2. \( \mu(m) \geq \text{Min}\{\mu(m \ast b), \mu(m \# b), \mu(b)\} \) \( \forall m, b \in M \)

**Example (2.23)**

In Example (2.17), define the fuzzy set \( \mu \) by

\[
\mu(m) = \begin{cases} 
0.8 & \text{if } m = 0, \eta \\
0.6 & \text{if } m = \theta, \beta 
\end{cases}
\]

Then \( \mu \) is fuzzy pseudo ideal, since \( \mu(0) \geq \mu(m) \), \( \forall m \in M \) and

\[
\mu(m) = 0.6 \geq \text{Min}\{\mu(m \ast b), \mu(m \# b), \mu(b)\} = 0.6 \quad \forall m \in M \setminus \{\eta, 0\} \quad \text{and} \quad \forall b \in M
\]

While \( \varphi(m) = \begin{cases} 
0.7 & \text{if } m = 0, \eta, \theta \\
0.5 & \text{if } m = \beta 
\end{cases} \]

is not fuzzy pseudo ideal of \( M \), since \( \varphi(\beta) = 0.5 \not\geq \text{Min}\{\varphi(\beta \ast \theta), \varphi(\beta \# \theta), \varphi(\theta)\} = 0.7 \)

**Definition (2.24)**

A nonempty subset \( I \) of a pseudo Q-algebra \( (M, \ast, \# , 0) \) is called complete pseudo ideal (briefly, c-pseudo ideal), if

1. \( 0 \in I \)

2. \( m \ast b, m \# b \in I, \forall b \in I \) such that \( b \neq 0 \) implies \( m \in I \)

**Definition (2.25)**

A nonempty subset \( I \) of a bounded pseudo Q-algebra \( (M, \ast, \# , 0) \) is called complete k-pseudo ideal (briefly, c-k-pseudo ideal), if

1. \( 0 \in I \)

2. \( m \ast \ast b, b \# \ast m \in I \) (resp. \( m \# \ast b, b \# \# m \in I \)), \( \forall b \in I \) such that \( b \neq 0 \) imply \( m \ast \in I \) (resp. \( m \# \in M \)), \( \forall m \in M \)

Note that in bounded pseudo Q-algebra \( M \) there are trivial c-k-pseudo ideals, \( \{0\} \) and \( M \)

**Proposition (2.26)**

Any c-pseudo ideal from bounded pseudo Q-algebra is c-k-pseudo ideal.

**Definition (2.27)**

Let \( M \) be a bounded pseudo Q-algebra. An element \( m \in M \) satisfies \( m \ast = m = m \# \) then \( m \) is called pseudo involution (i.e. \( m \) is \( \ast \)-involution and \( \# \)-involution). If every element \( m \in M \) is pseudo involution, we call \( M \) is a pseudo Q-algebra.

**Example (2.28)**

Let \( M = \{0, \eta, \theta, \beta, \psi\} \) be a set with tables below

| \* \ 0 \ \eta \ \theta \ \beta \ \psi | \# \ 0 \ \eta \ \theta \ \beta \ \psi |
|----------------|----------------|
| 0 \ 0 \ 0 \ 0 \ 0 | 0 \ 0 \ 0 \ 0 \ 0 |
| \eta \ \eta \ 0 \ 0 \ 0 | \eta \ \eta \ 0 \ 0 \ 0 |
| \theta \ \theta \ \psi \ 0 \ 0 | \theta \ \theta \ 0 \ 0 \ 0 |
| \beta \ \beta \ \eta \ \psi \ 0 \ | \beta \ \beta \ \psi \ \theta \ \eta |
| \psi \ \psi \ 0 \ 0 \ 0 | \psi \ \psi \ 0 \ 0 \ 0 |

Then \( (M, \ast, \#, 0) \) is bounded pseudo Q-algebra with unit \( \beta \). Notice that \( M \) is a pseudo involution.

**Proposition (2.29)**

If \( I \) be a c-k-pseudo-ideal in a pseudo-involutory pseudo-Q-algebra \( M \), then \( I \) is c-pseudo-ideal.
Proposition (2.30) [4]
Let $\mu$ be a fuzzy pseudo ideal of a pseudo Q-algebra $M$ if $m \leq b$, then $\mu(m) \geq \mu(b), \forall m, b \in M$

Definition (2.31)
Let $M$ be a pseudo Q-algebra. A fuzzy set $\mu$ in $M$ is called a fuzzy pseudo subalgebra of $M$ if it satisfies:

1. $\mu(m \ast b) \geq \min\{\mu(m), \mu(b)\} \quad \forall m, b \in M$
2. $\mu(m \# b) \geq \min\{\mu(m), \mu(b)\} \quad \forall m, b \in M$

3 Some types of intuitionistic fuzzy pseudo ideal

In this section, we define IF pseudo ideal and IF complete pseudo ideal, IF k-pseudo ideal, IF c-pseudo ideal and some properties among them.

Definition (3.1)
Let $M$ be a pseudo Q-algebra. An intuitionistic fuzzy set $A$ of $M$ is called an intuitionistic fuzzy pseudo ideal if it satisfies:

1. $\mu_A(0) \geq \mu_A(m) \quad \forall m \in M$
2. $\nu_A(0) \leq \nu_A(m) \quad \forall m \in M$
3. $\mu_A(m) \geq \min\{\mu_A(m \ast b), \mu_A(m \# b), \mu_A(b)\} \quad \forall m, b \in M$
4. $\nu_A(m) \leq \max\{\nu_A(m \ast b), \nu_A(m \# b), \nu_A(b)\} \quad \forall m, b \in M$

Example (3.2)
In Example (2.23) define the intuitionistic fuzzy set $A$ by

$\mu_A(m) = \begin{cases} 
0.8 : & \text{if } m = 0, \eta \\
0.6 : & \text{if } m = \theta, \beta
\end{cases}$

$\nu_A(m) = \begin{cases} 
0.2 : & \text{if } m = 0, \eta \\
0.4 : & \text{if } m = \theta, \beta
\end{cases}$

Then $A$ is intuitionistic fuzzy pseudo ideal since,

$\mu_A(0) \geq \mu_A(m) \quad \text{and} \quad \nu_A(0) \leq \nu_A(m) \quad \forall m \in M,$

$\mu(b) = 0.6 \geq \min\{\mu_A(b \ast m), \mu_A(b \# m), \mu_A(m)\} = 0.6,$

$\nu_A(b) = 0.4 \leq \max\{\nu_A(b \ast m), \nu_A(b \# m), \nu_A(b)\} = 0.4 \quad \forall m \in M \quad \text{and} \quad \forall b \in M \setminus \{0, \eta\}$

Definition (3.3)
Let $I$ be a c-pseudo ideal of a pseudo Q-algebra $(M, \ast, \#)$. An intuitionistic fuzzy set $A$ is called intuitionistic fuzzy complete pseudo ideal at $I$ (briefly, IF c-pseudo ideal), if

1. $\mu_A(0) \geq \mu_A(m) \quad \forall m \in M$
2. $\nu_A(0) \leq \nu_A(m) \quad \forall m \in M$
3. $\mu_A(m) \geq \min\{\mu_A(m \ast b), \mu_A(m \# b), \mu_A(b)\} \quad \forall m, b \in M, b \in I$
4. $\nu_A(m) \leq \max\{\nu_A(m \ast b), \nu_A(m \# b), \nu_A(b)\} \quad \forall b \in I, \forall m \in M$

Example (3.4)
Let $M = \{0, \eta, \theta, \beta\}$ be a set with the tables below
Then $(M, \ast, \#, 0)$ is pseudo Q-algebra, a subset $I = \{0, \eta, \theta\}$ is a c-pseudo ideal of $M$. Let $A$ is the intuitionistic fuzzy c-ideal at $I$ in $M$, then by definitin (2.22) we have,

$$\mu_A = \begin{cases} 
0.5 & \text{if } m = 0, \eta, \beta \\
0.4 & \text{if } m = \theta 
\end{cases} \quad \nu_A(m) = \begin{cases} 
0.5 & \text{if } m = 0, \eta, \beta \\
0.6 & \text{if } m = \theta 
\end{cases}$$

Then $A$ is the intuitionistic fuzzy c-ideal at $I$ in $M$, because

$\mu_A(0) \geq \mu_A(m) \quad \nu_A(0) \leq \nu_A(m) \quad \forall m \in M$

$\nu_A(\theta) = 0.4 \geq \min\{\mu_A(\theta \ast b), \mu_A(\theta \# b), \mu_A(b)\} = 0.4 \quad \forall b \in I$

$\nu_A(\theta) = 0.6 \leq \max\{\nu_A(\theta \ast b), \nu_A(\theta \# b), \mu_A(b)\} = 0.6 \quad \forall b \in I$

**Proposition (3.5)**

Every intuitionistic fuzzy pseudo ideal of a pseudo Q-algebra is an intuitionistic fuzzy c- pseudo ideal.

**Proof**

suppose that I be a c-pseudo ideal and A is intuitionistic fuzzy pseudo ideal of a pseudo Q-algebra M then by definitin (2.22) we have ,

1. $\mu_A(0) \geq \mu_A(m) \quad \forall m \in M$
2. $\nu_A(0) \leq \nu_A(m) \quad \forall m \in M$
3. $\mu_A(m) \geq \min\{\mu_A(m \ast b), \mu_A(m \# b), \mu_A(b)\} \quad \forall m, b \in M$
4. $\nu_A(m) \leq \{\nu_A(m \ast b), \nu_A(m \# b), \nu_A(b)\} \quad \forall m, b \in M$

since $I \subseteq M$, then $\mu_A(m) \geq \min\{\mu_A(m \ast b), \mu_A(m \# b), \mu_A(b)\}$ and

$\nu_A(m) \leq \max\{\nu_A(m \ast b), \nu_A(m \# b), \nu_A(b)\} \quad \forall b \in I$

Thus $A$ is intuitionistic fuzzy c-pseudo ideal of $M$.

**Remark (3.6)**

The following example shows that the converse of proposition (3.5) is not true in general.

**Example (3.7)**

In example (3.2), notice that $A$ is intuitionistic fuzzy c-pseudo ideal at $I$ in $M$ (When $I = \{0, \eta, \theta\}$, but its not is intuitionistic fuzzy pseudo ideal because

$\mu_A(\theta) = 0.4 \not\geq \min\{\mu_A(\theta \ast \beta), \mu_A(\theta \# \beta), \mu_A(\beta)\} = 0.5$

**Proposition (3.8)**

Let $I$ be a c-pseudo ideal of a pseudo involutory pseudo Q-algebra $M$. An intuitionistic fuzzy set $A$ is intuitionistic fuzzy c-pseudo ideal if and only if satisfies:

1. $\mu_A(0) \geq \mu_A(m) \quad \forall m \in M$
2. $\nu_A(0) \leq \nu_A(m) \quad \forall m \in M$
3. $\mu_A(m) \geq \min\{\mu_A(m \ast b), \mu_A(b \# m \ast), \mu_A(b)\}$
   $= \mu_A(m) \geq \min\{\mu_A(b \# m \# b), \mu_A(m \# b), \mu_A(b)\} \quad \forall m, b \in M$
4. \( \nu_A(m) \leq \text{Max}\{\nu_A(m^* \ast b), \nu_A(b^\# \ast m^*), \nu_A(b)\} \) and 
\( \nu_A(m^\#) \leq \text{Max}\{\nu_A(m^\# b), \nu_A(b^\# m), \nu_A(b)\}. \ \forall m, b \in M \)

**Proof**
by definition(2.27) and definition(3.3)

**Definition(3.9)**
An intuitionistic fuzzy set \( A \) in bounded pseudo Q-algebra \((M, \#, *, 0)\) is called intuitionistic fuzzy k-pseudo ideal, if

1. \( \mu_A(0) \geq \mu_A(m) \ \forall m \in M \)
2. \( \nu_A(0) \leq \nu_A(m) \ \forall m \in M \)
3. \( \mu_A(m^*) \geq \text{Min}\{\mu_A(m^* b), \mu_A(b^* \ast m), \mu_A(b)\} \) and 
\( \mu_A(m^\#) \geq \text{Min}\{\mu_A(m^\# b), \mu_A(b^\# m), \mu_A(b)\}. \ \forall m, b \in M \)
4. \( \nu_A(m^*) \leq \text{Max}\{\nu_A(m^* b), \nu_A(b^* \ast m), \nu_A(b)\} \) and 
\( \nu_A(m^\#) \leq \text{Max}\{\nu_A(m^\# b), \nu_A(b^\# m), \nu_A(b)\}. \ \forall m, b \in M \)

**Example(3.10)**
1. Every intuitionistic fuzzy constant in bounded pseudo Q-algebra \( M \) is intuitionistic fuzzy k-pseudo ideal .
2. Let \( M = \{0, \eta, \theta, \beta, \psi\} \) be a set with the tables below

|   | 0 | \eta | \theta | \beta | \psi |
|---|---|------|-------|------|-----|
| 0 | 0 | 0    | 0     | 0    | 0   |
| \eta | \eta | 0 | 0 | 0     | \eta |
| \theta | \theta | 0 | 0 | \theta | \theta |
| \beta | \beta | 0 | \beta | \beta | \psi |
| \psi | \psi | 0 | \psi | \psi | 0   |

then \((M, *, \#, 0)\) is bounded pseudo Q-algebra with unit \( \eta \) and define an intuitionistic fuzzy A by
\( \mu_A(m) = \begin{cases} 0.9 & \text{if } m = 0, \eta \\ 0.3 & \text{if } m = \theta, \beta, \psi \end{cases} \) & \( \nu_A(m) = \begin{cases} 0.1 & \text{if } m = 0, \eta \\ 0.7 & \text{if } m = \theta, \beta, \psi \end{cases} \)

then \( A \) is intuitionistic fuzzy k-pseudo ideal of \( M \), because
\( \mu_A(0) \geq \mu_A(m) \) and \( \nu_A(0) \leq \nu_A(m), \ \forall m \in M \)
\( \mu_A(m^*) = 0.9 \geq \text{Min}\{\mu_A(m^* b), \mu_A(b^* \ast m), \mu_A(b)\} \) is hold \( \forall m, b \in M \).
also \( \mu_A(m^\#) = 0.9 \geq \text{Min}\{\mu_A(m^\# b), \mu_A(b^\# m), \mu_A(b)\} \) is hold \( \forall m, b \in M \) also
\( \nu_A(m^*) = 0.1 \leq \text{Max}\{\nu_A(m^* b), \nu_A(b^* \ast m), \nu_A(b)\} \) and \( \nu_A(m^\#) = 0.1 \leq \text{Max}\{\nu_A(m^\# b), \nu_A(b^\# m), \nu_A(b)\} \)

**Proposition(3.11)**
Every intuitionistic fuzzy pseudo ideal of a bounded pseudo Q-algebra is an intuitionistic fuzzy k-pseudo ideal

**Proof**
Let \( A \) is an intuitionistic fuzzy pseudo ideal of a bounded pseudo Q-algebra then by definition (3.1) we have

1. \( \mu_A(0) \geq \mu_A(m) \ \forall m \in M \)
2. \( \nu_A(0) \leq \nu_A(m) \quad \forall m \in M \)

3. \( \mu_A(m) \geq \text{Min} \{ \mu_A(m \ast b), \mu_A(m \# b), \mu_A(b) \} \) then 
   \[ \mu_A(m^*) \geq \text{Min} \{ \mu_A(m^* \ast b), \mu_A(m^* \# b), \mu_A(b) \} \]
   \[ = \text{Min} \{ \mu_A(m^* \ast b), \mu_A(b^* \# m), \mu_A(b) \} \quad \forall m, b \in M \]
   Also 
   \[ \mu_A(m^\#) \geq \text{Min} \{ \mu_A(m^\# \ast b), \mu_A(m^\# \# m), \mu_A(b) \} \]
   \[ = \text{Min} \{ \mu_A(m^\# \# b), \mu_A(b^* \# m), \mu_A(b) \} \quad \forall m, b \in M \]

4. \( \nu_A(m) \leq \text{Max} \{ \nu_A(m \ast b), \nu_A(m \# b), \nu_A(b) \} \) then 
   \[ \nu_A(m^*) \leq \text{Max} \{ \nu_A(m^* \ast b), \nu_A(m^* \# b), \nu_A(b) \} \]
   \[ = \text{Max} \{ \nu_A(m^* \ast b), \nu_A(b^* \# m), \nu_A(b) \} \quad \forall m, b \in M \]

Thus A is intuitionistic fuzzy K-pseudo ideal of M .

Remark(3.12)
In general, the converse of Proposition (3.11) needs not true as shown in the following example .

Example(3.13)
in Example (3.10 -2) A is intuitionistic fuzzy k-pseudo ideal in M , but not intuitionistic fuzzy pseudo ideal in M , because \( \mu_A(\theta) = 0.3 \geq \text{Min} \{ \mu_A(\theta \ast \eta), \mu_A(\theta \# \eta), \mu_A(\eta) \} = 0.9 \)

Proposition(3.14)
Every intuitionistic fuzzy k-pseudo ideal in a pseudo involutory pseudo Q-algebra M is intuitionistic fuzzy pseudo ideal .

Proof
Assume that A be an intuitionistic fuzzy k-pseudo ideal of M since M is pseudo involutory pseudo Q-algebra , then 
\[ \mu_A(m) = \mu_A(m^*) \geq \text{Min} \{ \mu_A(m^* \ast b), \mu_A(b^* \# m), \mu_A(b) \} \]
\[ = \text{Min} \{ \nu_A(m \ast b), \nu_A(m \# b), \nu_A(b) \} \quad \forall m, b \in M \]

Proposition(3.15)
Let A be intuitionistic fuzzy k-pseudo ideal of a bounded pseudo Q-algebra M , then

1. \( \mu_A(m^*) \geq \mu_A(e) \quad \text{and} \quad \mu_A(m^\#) \geq \mu_A(e) \quad \forall m \in M \)
2. \( \nu_A(m^*) \leq \nu_A(e) \quad \nu_A(m^\#) \leq \nu_A(e) \quad \forall m \in M \)
3. \( m^* \leq b \), then \( \mu_A(b) \geq \mu_A(m^*) \) also \( \nu_A(b) \leq \nu_A(m^*) \)
4. \( m^\# \leq b \), then \( \mu_A(b) \geq \mu_A(m^\#) \) also \( \nu_A(b) \leq \nu_A(m^\#) \)

Proof
1. Since A is intuitionistic fuzzy k-pseudo ideal, we have 
   \[ \mu_A(m^*) \geq \text{Min} \{ \mu_A(m^* \ast e), \mu_A(e \ast m), \mu_A(e) \} \]
   \[ = \text{Min} \{ \mu_A(0), \mu_A(e) \} = \mu_A(e) \quad \text{and} \quad \mu_A(m^\#) \geq \text{Min} \{ \mu_A(m^\# \ast e), \mu_A(e \# m), \mu_A(e) \} \]
   \[ = \text{Min} \{ \mu_A(0), \mu_A(e) \} = \mu_A(e) \quad \forall m \in M \]

2. Since A is intuitionistic fuzzy k-pseudo ideal, we have 
   \[ \nu_A(m^*) \leq \text{Max} \{ \nu_A(m^* \ast e), \nu_A(e \ast m), \nu_A(e) \} \]
   \[ = \text{Max} \{ \nu_A(0), \nu_A(e) \} = \nu_A(e) \quad \text{and} \quad \nu_A(m^\#) \leq \text{Max} \{ \nu_A(m^\# \ast e), \nu_A(e \# m), \nu_A(e) \} \]
   \[ = \text{Max} \{ \nu_A(0), \nu_A(e) \} = \nu_A(e) \quad \forall m \in M \]
3. if $m^* \leq b$ i.e. $m^* * b = 0$ and $m^* \# b = 0$, then
\[
\mu_A(m^*) \geq \min \{ \mu_A(m^* * b), \mu_A(b^\# * m), \mu_A(b) \} \quad \forall m, b \in M
\]
(proof as intuitionistic fuzzy k-pseudo ideal)
\[
= \min \{ \mu_A(0_\#), \mu_A(b) \} = \mu_A(b)
\]
\[
\nu_A(m^*) \leq \max \{ \nu_A(m^* * b), \nu_A(b^\# * m), \nu_A(b) \} \quad \forall m, b \in M
\]
(proof as intuitionistic fuzzy k-pseudo ideal)
\[
= \max \{ \nu_A(0_\#), \nu_A(b) \} = \nu_A(b)
\]

4. is similar to the proof of (3)

**Definition (3.16)**

Let $I$ be a c-k-pseudo ideal of a bounded pseudo Q-algebra $(M, *, \#, 0)$. An intuitionistic fuzzy set $A$ is called intuitionistic fuzzy complete k-pseudo ideal (briefly, intuitionistic fuzzy c-k-pseudo ideal), if

1. $\mu_A(0) \geq \mu_A(m) \quad \forall m \in M$
2. $\nu_A(0) \leq \nu_A(m) \quad \forall m \in M$
3. $\mu_A(m^*) \geq \min \{ \mu_A(m^* * b), \mu_A(b^\# * m), \mu_A(b) \}$
   and $\mu_A(m^\#) \geq \min \{ \mu_A(m^\# * b), \mu_A(b^\# * m), \mu_A(b) \} \quad \forall m, b \in M, b \in I$
4. $\nu_A(m^*) \leq \max \{ \nu_A(m^* * b), \nu_A(b^\# * m), \nu_A(b) \}$
   and $\nu_A(m^\#) \leq \max \{ \nu_A(m^\# * b), \nu_A(b^\# * m), \nu_A(b) \} \quad \forall m, b \in M, b \in I$

**Example (3.17)**

In Example (2.16) let $A$ be intuitionistic fuzzy set of $M$ where $I = \{0, \eta, \theta\}$ be c-k-pseudo ideal defined by

\[
\mu_A(m) = \begin{cases} 
0.6 & \text{if } m = 0, \eta, \beta \\
0.2 & \text{if } m = \theta
\end{cases} \quad \eta, \beta \quad \& \quad \nu_A(m) = \begin{cases} 
0.4 & \text{if } m = 0, \eta, \beta \\
0.8 & \text{if } m = \theta
\end{cases}
\]

Then $A$ is intuitionistic fuzzy complete k-pseudo ideal of $M$ because

\[
\mu_A(0) \geq \mu_A(m) \text{ and } \nu_A(0) \leq \nu_A(m) \quad \forall m \in M,
\]
\[
\mu_A(0^*) = 0.2 \geq \min \{ \mu_A(0^* * b), \mu_A(b^\# * 0), \mu_A(b) \} = 0.2 \quad \forall b \in I
\]
\[
\mu_A(0^\#) = 0.2 \geq \min \{ \mu_A(0^\# * b), \mu_A(b^\# * 0), \mu_A(b) \} = 0.2 \quad \forall b \in I
\]
\[
\nu_A(0^*) = 0.8 \leq \max \{ \nu_A(0^* * b), \nu_A(b^\# * 0), \nu_A(b) \} = 0.8 \quad \forall b \in I
\]
\[
\nu_A(0^\#) = 0.8 \leq \max \{ \nu_A(0^\# * b), \nu_A(b^\# * 0), \nu_A(b) \} = 0.8 \quad \forall b \in I
\]

**Proposition (3.18)**

Every intuitionistic fuzzy k-pseudo ideal of a bounded pseudo Q-algebra is an intuitionistic fuzzy c-k-pseudo ideal

**Proof**

Let $I$ be a c-k-pseudo ideal in bounded pseudo Q-algebra $M$ and $A$ be an intuitionistic fuzzy k-pseudo ideal of $M$, then

\[
\mu_A(m^*) \geq \min \{ \mu_A(m^* * b), \mu_A(b^\# * m), \mu_A(b) \} \quad \text{and}
\]
\[
\nu_A(m^*) \leq \max \{ \nu_A(m^* * b), \nu_A(b^\# * m), \nu_A(b) \} \quad \forall m, b \in M
\]

Since $I \subseteq M$ we have

\[
\nu_A(m^*) \leq \min \{ \nu_A(m^* * b), \nu_A(b^\# * m), \nu_A(b) \} \quad \forall b \in I
\]

Also

\[
\mu_A(m^\#) \geq \min \{ \mu_A(m^\# * b), \mu_A(b^\# * m), \mu_A(b) \} \quad \text{and}
\]
\[
\nu_A(m^\#) \leq \max \{ \nu_A(m^\# * b), \nu_A(b^\# * m), \nu_A(b) \} \quad \forall b \in I
\]

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\[ \nu_A(m^b) \leq \max \{ \nu_A(m^b \# b), \nu_A(b^\# m), \nu_A(b) \} \quad \forall m, b \in M \]

Since \( I \subseteq M \) we have
\[ \mu_A(m^b) \geq \min \{ \mu_A(m^b \# b), \mu_A(m^b \# b), \mu_A(b) \} \quad \text{and} \]
\[ \nu_A(m^b) \leq \max \{ \nu_A(m^b \# b), \nu_A(b^\# m), \nu_A(b) \} \quad \forall b \in I \]

**Remark (3.19)**

The converse of proposition (3.18) may not be true and the following example explainwd that.

**Example (3.20)**

In example (3.17) if \( A \) be intuitionistic fuzzy c-k-pseudo ideal in \( M \)

\( \text{(Where } I = \{0, \eta, \theta\} \text{ is c-k pseudo ideal) , since } \mu_A(0^#) = 0.2 \leq \min \{ \mu_A(0^# \# \beta), \mu_A(\beta^# 0), \mu_A(\beta) \} = 0.6 \)

**Corollary (3.21)**

Every intuitionistic fuzzy pseudo ideal of bounded pseudo Q-algebra is intuitionistic fuzzy c-k-pseudo ideal.

**Proof**

by proposition (3.11) and proposition (3.18).

**Proposition (3.22)**

Any intuitionistic fuzzy c-pseudo ideal from bounded pseudo Q-algebra is intuitionistic fuzzy c-k-pseudo ideal.

**Proof**

Let \( A \) be an intuitionistic fuzzy c-pseudo ideal from bounded pseudo Q-algebra \( M \) and \( I \) be c-pseudo ideal of \( M \).

then \( I \) is c-k pseudo ideal of \( M \) by proposition (2.26).

since \( A \) intuitionistic fuzzy c-pseudo ideal of \( M \), from definition (3.3) we have:

1. \( \mu_A(0) \geq \mu_A(m) \quad \forall m \in M \)
2. \( \nu_A(0) \leq \nu_A(m) \quad \forall m \in M \)
3. \( \mu_A(m) \geq \min \{ \mu_A(m \# b), \mu_A(m^b \#), \mu_A(b) \} \quad \forall b \in I \) thus
   \[ \mu_A(m) \geq \min \{ \mu_A(m \# b), \mu_A(m^b \# b), \mu_A(b) \} \]
   \[ = \min \{ \mu_A(m \# b), \mu_A(b^# m), \mu_A(b) \} \quad \forall b \in I \]
   also \( \mu_A(m^#) \geq \min \{ \mu_A(m^# \# b), \mu_A(m^b \#), \mu_A(b) \} \)
   \[ = \min \{ \mu_A(m^# \# b), \mu_A(b^# m), \mu_A(b) \} \quad \forall b \in I \]
4. \( \nu_A(m) \leq \max \{ \nu_A(m \# b), \nu_A(m^b \#), \nu_A(b) \} \quad \forall b \in I \) thus
   \[ \nu_A(m^b) \leq \max \{ \nu_A(m \# b), \nu_A(m^b \# b), \nu_A(b) \} \]
   \[ = \max \{ \nu_A(m \# b), \nu_A(b^# m), \nu_A(b) \} \quad \forall b \in I \]
   also \( \nu_A(m^#) \leq \max \{ \nu_A(m^# \# b), \nu_A(m^b \#), \nu_A(b) \} \)
   \[ = \max \{ \nu_A(m^# \# b), \nu_A(b^# m), \nu_A(b) \} \quad \forall b \in I \]

Hance \( A \) is intuitionistic fuzzy c-pseudo ideal of \( M \)

**Example (3.23)**

In example (3.10) if \( I = \{0, \beta, \psi\} \), then \( i \) is a c-k pseudo ideal and c-pseudo ideal of a bounded Q-algebra \( M \)

**Example (3.23)**

define the intuitionistic fuzzy set \( A \) by:

\[ \mu_A(m) = \begin{cases} 0.9 & : \text{if } m = 0, \eta, \beta \\ 0.6 & : \text{if } m = \theta, \psi \\ \end{cases} \quad \& \nu_A(m) = \begin{cases} 0.1 & : \text{if } m = 0, \eta, \beta \\ 0.4 & : \text{if } m = \theta, \psi \\ \end{cases} \]

then \( A \) is intuitionistic fuzzy c-k pseudo ideal because

\( \mu_A(0) \geq \mu_A(m) \) and \( \nu_A(0) \leq \nu_A(m) \quad \forall m \in M \)
Also $\mu_A(m^*) = 0.9 \geq \min\{\mu_A(m^* \ast b), \mu_A(b^\# \ast m), \mu_A(b)\}$ is hold $\forall b \in I$, $\forall m \in M$ and $\mu_A(m^\#) = 0.9 \geq \min\{\mu_A(m^\#b), \mu_A(b^\#m), \mu_A(b)\}$ is hold too $\forall b \in I$, $\forall m \in M$ and $\nu_A(m^*) = 0.1 \leq \max\{\nu_A(m^* \ast b), \nu_A(b^\# \ast m), \nu_A(b)\}$ is hold $\forall b \in I$, $\forall m \in M$ and $\nu_A(m^\#) = 0.1 \leq \max\{\nu_A(m^\#b), \nu_A(b^\#m), \nu_A(b)\}$ is hold too $\forall b \in I$, $\forall m \in M$ but $A$ is not intuitionistic fuzzy c-pseudo ideal because $\mu_A(\psi) = 0.6 \not\geq \min\{\mu_A(\psi \ast \beta), \mu_A(\psi^\#\beta), \mu_A(\beta)\} = 0.9$

**Proposition (3.24)**

Every intuitionistic fuzzy c-k-pseudo ideal in a pseudo involutory pseudo Q-algebra $M$ is intuitionistic fuzzy c-pseudo ideal.

**Proof**

suppose that $A$ is intuitionistic fuzzy c-k-pseudo ideal of $M$. Then $I$ is c-pseudo ideal of $M$ by (proposition (2.29)) since $M$ is pseudo involutory, then

$\mu_A(m) = \mu_A(m^\#) \geq \min\{\mu_A(m^\#b), \mu_A(b^\#m), \mu_A(b)\} = \min\{\mu_A(m^\#b), \mu_A(m^\#b), \mu_A(b)\}$

$\forall b \in I$

Thus $A$ is intuitionistic fuzzy c-pseudo ideal of $M$.

**Remark (3.25):**

The following diagram shows the relation among intuitionistic fuzzy pseudo ideal, intuitionistic fuzzy k-pseudo ideal, intuitionistic fuzzy c-pseudo ideal, intuitionistic fuzzy c-k-pseudo ideal in bounded Q-algebra:
In involutory Q-algebra

IF pseudo ideal

IF k- pseudo ideal

IF c-pseudo ideal

IF c-K-pseudo ideal
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