Transition to a supersolid phase in a two-dimensional dilute gas of electron-hole pairs

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Using coherent-state formalism (the Keldysh formalism), the article describes a transition from a homogeneous superfluid state to a supersolid state in a two-dimensional dilute gas of electron-hole pairs with spatially separated components. Such a transition is heralded by the appearance of a roton-type minimum in the collective excitation spectrum, which touches the abscissa axis as the distance between the layers or the pair density increases. This signals the instability of the system with respect to the appearance of a spatial modulation of the pair density. It has been found that a first-order transition to a hexagonal supersolid phase takes place a little earlier. A theory without phenomenological constants has been developed for an arbitrary relation between the effective masses of an electron and a hole. A phase diagram for the system has been plotted in the variables “the chemical potential of pairs - the distance between the layers”. It has been shown that there is a jump in the average density of the condensate during the phase transition. It has been established that with an increase in the chemical potential, the inhomogeneous phase breaks up into high-density regions surrounded by lines at which the density becomes zero, with these lines forming a continuous network.

I. INTRODUCTION

A supersolid phase is a state that combines superfluid properties with crystalline order. This possibility, as applied to quantum crystals of solid 4He, was predicted by A. F. Andreev and I. M. Lifshitz. There has been renewed interest in this phenomenon due to experiments that have revealed a decrease in the oscillation period of a torsion pendulum filled with solid helium, and which were repeatedly reproduced in different laboratories (see, for example, reviews in Refs. 2 and 3). The observed effect could be attributed to the appearance of a superfluid fraction, which is not involved in torsional oscillations, but additional experiments in Ref. 4 and the theory in Refs. 5, 7 suggest that this effect is caused not by the appearance of the supersolid phase, but rather by superplasticity.

A supersolid state can arise not only in quantum crystals. As was shown in Ref. 8 (see also Refs. 9, 10), Bose gases with dipole interaction between particles belong to systems in which superfluid properties and spatial periodicity can be expected to coexist. The supersolid state in dipole quantum gases was experimentally discovered recently in studies undertaken by three different groups15–17. The interaction between dipole particles comprises a long-range component. The Fourier component of the interaction potential $V_k$ is a function of the wave vector. In a Bose condensate state, the excitation spectrum $\omega(k)$ is described by an equation similar to the one for the Bogolyubov spectrum, with the difference being that the interaction constant is replaced with $V_k$: $\omega(k) = \sqrt{\varepsilon_k(\varepsilon_k + 2V_k n)}$, where $\varepsilon_k$ is the kinetic energy of the particles and $n$ is the condensate density (it is assumed that $V_k$ depends only on the modulus $k$). If in a certain range of wave vectors the Fourier component $V_k$ takes negative values and the inequality $\varepsilon_k + 2nV_k < 0$ is achieved, then a spatially uniform condensate is unstable.

The authors of Ref. 8 show that various inhomogeneous phases can arise in dipole quantum gases, depending on the value of the dimensionless parameter composed of the constants of two-particle and three-particle contact interactions, condensate density, and the dipole moment magnitude. Such phases include a one-dimensional supersolid phase (in the form of stripes), a two-dimensional supersolid phase with a triangular lattice, as well as a phase with a honeycomb-type lattice (similar to a graphene lattice). In Ref. 8, the discussion pertains to dipole molecules. Therefore, the temperature at which a transition from a homogeneous superfluid state to a supersolid state can occur is very low (no more than several tens of nK), which is consistent with the temperature (T=20 nK) at which the supersolid phase was observed experimentally15.

Due to the long-discussed possibility of superfluidity of coupled electron-hole pairs with spatially separated components (see Ref. 12), a transition to the supersolid phase in bilayer electron-hole systems has also been considered. A pioneering study on electron-hole pairing in quantum Hall systems16 showed that the collective excitation spectrum contains a roton-type minimum. It was found that when a critical distance between layers is reached, the minimum point touches the abscissa axis and a phase transition should be observed in the system. In Ref. 17, a phase diagram of a bilayer electron-hole sys-
tem in a zero magnetic field was analyzed qualitatively. Arguments were made that with increasing distance between the layers, a phase transition to a supersolid phase should first be observed, which is then followed by a transition to a Wigner crystal phase. A similar result was obtained in Ref. [18], based on the approach developed by the author of that study and referred to by him as the Ginzburg-Landau quantum theory. In Ref. [18] it was established that in a bilayer electron-hole system with a significant imbalance of electron and hole densities, the exciton gas condenses into a phase resembling the Fulde-Ferrel-Larkin-Ovchinnikov phase, which can be considered as a variation of the supersolid phase. In this case, the formation of such a phase is promoted by a considerable difference between the electron and hole effective masses.

Our previous study[19] developed an approach for describing the condensate of electron-hole pairs in a bilayer system, using the coherent-state formalism proposed by L. V. Keldysh in relation to a three-dimensional exciton condensate[20]. As part of the description[20], an analytical expression was obtained for the collective excitation spectrum. It turns out that it is important to take into account not only the dependence on the wave vector of the Fourier component of the direct Coulomb interaction between electron-hole pairs, but also a similar dependence for the exchange interaction. The spectrum obtained in Ref. [20] (much like the collective excitation spectrum in dipole Bose gases, as well as the spectrum obtained in Ref. [16], has a roton-type minimum, and with an increase in the distance between the layers or in the pair density, the state with a uniform Bose condensate becomes unstable.

In the present study, the approach developed in Ref. [20] is used to describe the supersolid phase. We have obtained expressions for the energy of inhomogeneous phases. The formalism used does not contain phenomenological parameters. The energy depends on the ratio of the distance between the layers to the effective Bohr radius of the pair $a_0$, on the chemical potential $\mu$, on the potential of the pairs, and the ratio of the effective masses of the electron ($m_e$) and the hole ($m_h$). The particular inhomogeneous phase that corresponds to the minimum energy has been established, and how the spatial distribution of the condensate in the inhomogeneous phase changes depending on the indicated parameters.

II. THE GENERAL EXPRESSION FOR THE EXCITON CONDENSATE ENERGY. DERIVATION WITHIN THE FRAMEWORK OF THE COHERENT-STATE FORMALISM

The coherent-state formalism[21] was further developed in Refs. [22] and [23], where it was used to describe electron-hole pairing in two-layer quantum Hall systems; in Refs. [24][25], where it was employed to analyze polarization phenomena in a three-dimensional superfluid gas of electron-hole pairs (without spatial separation of electrons and holes); as well as in Ref. [26], where this formalism was used to describe the superfluid state of a dilute gas formed by alkali metal atoms.

Let us outline the approach used. An exciton condensate is described by the many-particle wave function:

$$|\Phi\rangle = e^{D}(0),$$

where

$$\hat{D} = \int dr_1 dr_2 \Phi(r_1, r_2) \psi^+_e(r_1) \psi^+_h(r_2) - H.c.,$$

$$\psi^+_e$$ and $$\psi^+_h$$ are the electron and hole creation operators, $$\Phi(r_1, r_2)$$ is the pair wave function in the dilute condensate, the $$r_1$$ vectors are two-dimensional (i.e. they lie in the planes of the electron and hole layers), and the wave function $$|0\rangle$$ corresponds to a vacuum state (a state in which there are no electrons and holes). Here we consider a one-component condensate of pairs. As was shown in Ref. [20] (see also Ref. [23]), a two-component pair condensate (pairs differ in the projection of the spin of an electron or hole) is unstable with respect to spatial separation at $d/a_0 > 0.2$. At the same time, according to the estimates below, the supersolid phase can only occur at $d/a_0 > 1$.

The function $$\Phi(r_1, r_2)$$ is determined from the minimum condition for the energy of the system. The Hamiltonian of the system is taken as

$$H = - \sum_{\alpha=e,h} \int dr \frac{\hbar^2}{2m_\alpha} \nabla^2 \psi^+_\alpha(r) \psi^+_\alpha(r) + \frac{1}{2} \sum_{\alpha,\beta=e,h} \int dr dr' \psi^+_\alpha(r) \psi^+_\beta(r') V_{\alpha\beta}(|r - r'|) \psi^+_\beta(r') \psi^+_\alpha(r),$$

where $V_{\alpha\beta}(r)$ is the energy of the Coulomb interaction. Let us consider a system in a homogeneous dielectric matrix with the permittivity $\varepsilon$, which coincides with the dielectric constant of the insulator that separates the electron and hole layers. Then $V_{eh}(r) = V_{he}(r) = e^2/\varepsilon r$ and $V_{hh}(r) = -e^2/\varepsilon \sqrt{r^2 + d^2}$. The operators of the total number of electrons and holes are written as follows:

$$\hat{N}_e = \int dr \psi^+_e(r) \psi_e(r), \quad \hat{N}_h = \int dr \psi^+_h(r) \psi_h(r).$$

For further analysis it is convenient to move on to the operators $\tilde{\psi}_\alpha(r) = e^{-D} \psi_\alpha(r) e^D (\alpha = e, h)$, which are expressed in terms of the creation and annihilation operators for electrons and holes[22]:

$$\tilde{\psi}_e(r) = \int dr' [C_e(r, r') \psi_e(r') + S(r, r') \psi^+_e(r')],$$

$$\tilde{\psi}_h(r) = \int dr' [C_h(r', r) \psi_h(r') - S(r', r') \psi^+_h(r')],$$

where

$$C_e(r, r') = \delta(r - r') + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} (\Phi \cdot \Phi^+)^n,$$
\[
C_h(r, r') = \delta(r - r') + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \Phi^+ \Phi^n,
\]
\[
S(r, r') = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \Phi \cdot (\Phi^+ \Phi)^n.
\] (5)

The explicit form of the Hermitian-conjugate operators \( \tilde{\psi}_n(r) = e^{-D} \psi_n^+(r) e^D \) is determined by Hermitian-conjugate Eq. (4). Equation (6) uses the notation \( \Phi^+(r_1, r_2) \equiv \Phi^+(r_2, r_1) \) and the multiplication sign denotes convolution. Using Eq. (4) and (5), we can express the energy of the system \( E = \langle \Phi | H | \Phi \rangle = \langle 0 | H | 0 \rangle \), the total number of electrons \( N_e = \langle \Phi | \hat{N}_e | \Phi \rangle = \langle 0 | \hat{N}_e | 0 \rangle \) and the total number of holes \( N_h = \langle \Phi | \hat{N}_h | \Phi \rangle = \langle 0 | \hat{N}_h | 0 \rangle \) in terms of the operators \( \hat{N}_e \), \( \hat{N}_e \) and \( \hat{N}_h \). The latter are obtained from Hamiltonian (3) and the operators \( \hat{N}_e \) and \( \hat{N}_h \) by replacing \( \psi_{e(h)} \) with \( \tilde{\psi}_{e(h)} \), and by making a similar replacement for the Hermitian-conjugate operators. It is easy to show that in the state (1), the total number of electrons is equal to the total number of holes, \( N_e = N_h \).

Let us determine how the state of the system changes depending on the chemical potential of pairs \( \mu = (\mu_e + \mu_h)/2 \), which is considered as an external parameter. The desired state corresponds to \( \Phi(r_1, r_2) \), at which the minimum potential \( \Omega = E - \mu N \) is reached, where \( N = N_e + N_h \) is the number of pairs.

At low pair density (when the average distance between pairs is much larger than the pair size), the \( \Omega \) function can be written as a series in powers of \( \Phi \). We shall confine ourselves to taking into account the terms of the second and fourth order in \( \Phi \). In this approximation, we obtain the following expression:

\[
\frac{\Omega_u}{S} = -\mu \left( n_0 - \frac{1}{3} \pi^2 \int \frac{d^2q}{(2\pi)^2} |\phi_q|^4 \right) + \frac{\gamma_0 m \Omega}{2 n_0^2},
\] (10)

where \( S \) is the area of the system, \( \phi_q \) is the Fourier component of the function \( \phi_0(r) \) (to simplify, we write the Fourier component without index 0), \( \gamma_0 \) is the interaction constant. This constant is contributed to by direct and exchange interactions, \( \gamma_0 = \gamma^0 + \gamma^{(ex)} \), where

\[
\gamma^0 = \frac{4 \pi e^2 d}{\varepsilon},
\]
\[
\gamma^{(ex)} = -\frac{4 \pi e^2}{\varepsilon} \int \frac{d^2p}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \frac{1}{p} |\phi_q|^2 |\phi_{q+p}|^2 \frac{1}{2} \left( \phi^*_q + \phi_q \right) \left( \phi^*_p + \phi_p \right).
\] (12)

The minimum (10) corresponds to

\[
\frac{\Omega_u}{S} = \frac{\mu}{2 \gamma_0} \int \frac{d^2q}{(2\pi)^2} |\phi_q|^4.
\] (13)

As can be seen from Eq. (13), at small \( \mu \), the \( n_0 \) value depends almost linearly on \( \mu \). The inclusion of the second term in the denominator of Eq. (10) gives a correction to \( n_0 \) of the order of \( \mu^2 \). Substituting (12) into (10), we obtain the energy of the system, which is proportional (in the lowest approximation) to \( \mu^2 \):

\[
\frac{\Omega_u}{S} = \frac{1}{2} \frac{\mu^2}{\gamma_0}.
\] (14)
and the correction $\delta n_0 \propto \tilde{\mu}^2$ gives a correction to $\Omega_0$, which is proportional to $\tilde{\mu}^3$. Thus, the inclusion of the second term in the denominator of Eq. (14) is an excess of accuracy because in (6) we neglect the terms proportional to $\Phi^0$, which would also give a correction of the order of $\tilde{\mu}^3$. Since we restrict ourselves to an approximation which does not take account of the contribution to the energy from terms of the order of $n_0^2$, we can, from the outset, neglect the term $\Phi^+(r_2, r_3)\Psi(r_3, r_4)/3$ in the first round brackets in (6).

The quantity $\gamma_0$ is positive. If $\gamma_0$ were negative, there would be a collapse, but we do not consider this case here.

In a spatially homogeneous case, it is possible to find the exact relation between $n_0$ and the pair density $n_p$. In this case, Eq. (4) written in momentum representation are reduced to the usual $u-v$ transformation:

$$\tilde{\psi}_v(q) = u_q \psi_v(q) + v_q \tilde{\psi}_v(-q),$$
$$\psi_h(q) = u_q \psi_h(-q) - v_q \tilde{\psi}_v(q),$$

where

$$u_q = \cos |\Phi_q|, \quad v_q = \frac{\Phi_q}{|\Phi_q|} \sin |\Phi_q|,$$

and $\Phi_q = \sqrt{n_0} \phi_q$ is the Fourier component of the function $\Phi(r)$. The pair density is as follows:

$$n_p = \int \frac{d^2q}{(2\pi)^2} |v_q|^2 = \int \frac{d^2q}{(2\pi)^2} \left[ \sin(\sqrt{n_0}|\phi_q|) \right]^2.$$

As can be seen, in the limit of $\Phi_0 \ll 1$, $n_0$ approaches to the pair density, but in the general case $n_0 > n_p$. In the approximation being considered, the difference between $n_0$ and $n_p$ is neglected.

### III. ENERGY OF THE INHOMOGENEOUS PHASE

To describe the supersolid phase, we shall take $\Psi(R)$ as a spatially periodic function. We use the same functions as in Ref. 8.

A one-dimensional supersolid phase (wave) is set by the function

$$\Psi_h(R) = \sqrt{n_0} \left[ \cos \theta + \sqrt{2} \sin \theta \cos(kX) \right].$$

The parameter $\theta$ varies from $-\pi/2$ to $\pi/2$.

A hexagonal supersolid phase corresponds to the function

$$\Psi_h(R) = \sqrt{n_0} \left[ \cos \theta + \sqrt{2} \sin \frac{\theta}{3} \sum_{i=1}^{3} \cos(k_i R) \right],$$

where $k_1 = (k, 0)$, $k_2 = (-k/2, \sqrt{3}k/2)$, $k_3 = (-k/2, -\sqrt{3}k/2)$. At positive $\theta$ values ($\theta < \pi/2$), the main maxima of the $\Psi_h^2(R)$ function form a triangular lattice. At small negative $\theta$ values, a honeycomb-type lattice emerges. In the low density limit, $n_0$ corresponds to the average pair density with respect to both the phase (18) and the phase (19).

By substituting (18) and (19) into (6), we find the $n_0$ value corresponding to the minimum $\Omega$ at given $\theta$ and $k$:

$$n_0 = \frac{\tilde{\mu} - \epsilon_k \sin^2 \theta}{\gamma_{w,h}(k, \theta)} \cos \left( \frac{\tilde{\mu} - \epsilon_k \sin^2 \theta}{\gamma_{w,h}(k, \theta)} \right),$$

where $\epsilon_k = \hbar^2 k^2/2(m_e + m_h)$ is the kinetic energy of the pair, $\gamma_{w,h}(\theta, k)$ is the interaction constant that depends on the parameters $k$ and $\theta$ that determine the form of the function $\Psi(R)$, and $\Theta(x)$ is the Heaviside theta function. The appearance of the theta function is associated with the condition $n_0 \geq 0$. As a result, we obtain the following expression for the energy:

$$\frac{\Omega_{w,h}(k, \theta)}{S} = -\frac{1}{2} \left( \frac{\tilde{\mu} - \epsilon_k \sin^2 \theta}{\gamma_{w,h}(k, \theta)} \right)^2 \cos \left( \frac{\tilde{\mu} - \epsilon_k \sin^2 \theta}{\gamma_{w,h}(k, \theta)} \right).$$

The general structure of the functions $\gamma_w(k, \theta)$ and $\gamma_h(k, \theta)$ is given by the following equations:

$$\gamma_w(k, \theta) = \gamma_0 + 4\gamma_2(k) \cos^2 \theta \sin^2 \theta + \gamma_{4,w}(k) \sin^4 \theta,$$

$$\gamma_h(k, \theta) = \gamma_0 + 4\gamma_2(k) \cos^2 \theta \sin^2 \theta + \gamma_{3,h}(k) \cos \theta \sin^3 \theta + \gamma_{4,h}(k) \sin^4 \theta,$$

where $\gamma_0$ is the above introduced interaction constant for the homogeneous phase. Explicit expressions for the functions $\gamma_2(k)$ and $\gamma_{3,4,w,h}(k)$ are rather cumbersome, so they are given in the Appendix.

At $\theta = 0$ Eq. (21) turns into (14), i.e. (21) also describes the homogeneous phase. The expansion of (21) at small $\theta$ is as follows:

$$\frac{\Omega_{w,h}(k, \theta)}{S} = -\frac{1}{2} \frac{\tilde{\mu}^2}{\gamma_0} + \frac{\tilde{\mu}}{\gamma_0} \left( \epsilon_k + 2\frac{\tilde{\mu}}{\gamma_0} \right).$$

Given that for the homogeneous phase $n_0 = \tilde{\mu}/\gamma_0$, this phase has a higher energy compared to any of the inhomogeneous phases under consideration, if for some $k$ the inequality $\epsilon_k + 2\gamma_2(k)n_0 < 0$ is satisfied. The latter condition coincides exactly with the instability condition of the homogeneous state, which follows from the explicit expression for the collective excitation spectrum (22) (the condition under which the spectrum becomes imaginary). Even if this condition is not satisfied, i.e. $\epsilon_k + 2n_0 \gamma_2(k) > 0$ at all $k$, an inhomogeneous phase at some finite $\theta$ may have a lower energy compared to the homogeneous phase. Then a transition from the homogeneous phase to the supersolid phase will be a first-order phase transition.

To find an analytical expression for the energy (21), we approximate the function $\phi_0(r)$ by the wave function of the ground state of a two-dimensional harmonic oscillator. This approximation is justified for $d > a_0$, where $a_0 = \hbar^2 \varepsilon/m e^2$ is the effective Bohr radius of
the pair. In this case, the interaction potential in Eq. 14 can be replaced by its expansion near \( r = 0: V_{\text{eh}}(r) \approx -e^2/\varepsilon d + e^2r^2/2\varepsilon d^3 \). This results in \( \phi_\mathbf{q}(r) = (1/\sqrt{\pi}r_0) \exp(-r^2/\gamma_0^2), \) where \( \gamma_0 = \sqrt{\varepsilon_0 d^3} \) is the characteristic pair size. The Fourier component of this function is as follows: \( \phi_\mathbf{q} = \sqrt{4\pi}r_0 \exp(-q^2r_0^2/2). \) It is convenient to use the following as a unit of energy:

\[
\Xi_0 = \frac{e^2}{\varepsilon a_0} = \frac{me^4}{\varepsilon \hbar^2}
\]

(this is the doubled effective Rydberg). Then the energy \( \Omega \) related to the area \( a_0^2 \), is given by the expression

\[
\hat{\Omega}_{w,h}(k, \theta) = \frac{\Omega_{w,h}a_0^2}{8\pi r_0} \left[ \frac{\tilde{\mu}}{1} - \frac{k^2 a_0^2}{2} \left( 1 - x^2 \right) \sin^2 \theta \right]^2
\]

\[
= -\frac{\mathcal{E}_0 a_0}{8\pi r_0} \left[ \frac{\tilde{\mu}}{1} - \frac{k^2 a_0^2}{2} \left( 1 - x^2 \right) \sin^2 \theta \right]^2 \times \Theta \left[ \frac{\tilde{\mu}}{1} - \frac{k^2 a_0^2}{2} \left( 1 - x^2 \right) \sin^2 \theta \right],
\]

where \( \tilde{k} = k r_0 \) and to describe the electron-hole asymmetry, we introduce the parameter \( x = (m_e - m_h)/(m_e + m_h) \) (the case \( x = 0 \) corresponds to \( m_e = m_h \), and the limit \( x \to -1 \) corresponds to infinitely heavy holes). The functions \( \hat{\gamma}_{w,h}(\tilde{k}, \theta) \) have the same structure as expressions 22, 23 and depend on \( x \) and \( \tilde{d} = d/r_0 \), same as on parameters. The explicit form of these functions is given in the Appendix. It should be noted that \( \hat{\gamma}_{w,h}(\tilde{k}, \theta) \) do not change when \( x \) changes its sign.

By minimizing the energy 25 with respect to \( k \) and \( \theta \), we find the ground state of the system. If the minimum is reached at \( \theta = 0 \) (in this case, 25 does not depend on \( k \)), the ground state corresponds to a homogeneous condensate. This state is achieved at low \( \tilde{\mu} \) (\( \tilde{\mu} > 0 \)). As \( \tilde{\mu} \) increases, at certain critical value of \( \tilde{\mu} \) the global minimum jumps to the point with \( \theta_h \neq 0 \) and \( k \neq 0 \). This minimum corresponds to a hexagonal phase. Our analysis shows that \( \theta_h \) falls in the range \( (0, \pi/2) \), and the maxima of the condensate density form a triangular lattice. The lattice parameter is as follows: \( a_h = 4\pi/\sqrt{3}k \). In a transition from a homogeneous state to a hexagonal phase, the average condensate density increases abruptly.

### IV. RESULTS AND DISCUSSIONS

We will not analyze the situation with an arbitrary relation between \( m_e \) and \( m_h \), but will restrict ourselves to two cases. The first corresponds to the system MoS2-MoTe2 (System 1) for which \( m_e = 0.47m_0 \) \( \hat{m}_h = 0.62m_0 \), where \( m_0 \) is the free electron mass. In this case, \( |x| = 0.14 \). Considering this system in a hexagonal boron nitride matrix \( \varepsilon = 5 \), we obtain the effective Bohr radius \( a_0 \approx 1 \) nm. The second case corresponds to an AlGaAs-based heterostructure (System 2), for which \( m_e = 0.067m_0 \), \( m_h = 0.45m_0 \) and \( \varepsilon = 13 \). For this case, \( |x| = 0.74 \) and \( a_0 \approx 12 \) nm. Our analysis shows that over the entire range of parameters where the inhomogeneous phase is achieved, the energy of the phase 19 is lower than the energy of the phase 18. In this case, the transition from the homogeneous phase to phase 19 is accompanied by a jump in \( \theta \) and a jump in \( n_0 \), i.e. it is a first-order phase transition. Further, when referring to the inhomogeneous phase, we mean phase 19. Figure 4 shows a phase diagram of the system in variables \( (\mu/\Xi_0, d/a_0) \). It represents phase transition lines for Systems 1 and 2. With an increase in the parameter \( |x| \), the phase transition occurs at a lower value of \( d/a_0 \). In the limit of \( m_e/m_h \to 0 \), the inhomogeneous phase becomes energetically favorable at all \( d/a_0 \) (in the low pair density limit). However, as can be seen in Fig. 1, a wide variation in the ratio of the electron and hole masses has a weak effect on the position of the phase transition line.

Figure 4 shows phase boundary lines in the coordinates \( (n_0a_0^2, d/a_0) \), and an imaginary line that outlines the region of the phase diagram where the low-density approximation is applicable. This line is defined by the equation \( 4\pi n_0 a_0^2 = 1 \), which limits the region where \( \Phi_\mathbf{q} < 1 \) for all \( \mathbf{q} \) [see Eq. 17]. The lines dividing the region of parameters with a homogeneous and inhomogeneous condensate are doubled because there is a density jump associated with a jump in \( n_0 \) at the phase transition point.

Figure 3 shows the dependence of \( n_0 \) on the chemical potential at \( d = 5a_0 \) and \( |x| = 0.14 \). This dependence demonstrates that at the phase transition point, \( n_0 \) changes abruptly.

The lattice parameter \( a_h \) expressed in terms of \( a_0 \) increases with an increase in the ratio \( d/a_0 \) approximately according to the law \( d^{3/4} \). The same parameter expressed in terms of \( r_0 \) varies slightly and remains within \( 3.5r_0 < a_h < 5r_0 \) over the entire range of \( d \) and \( \tilde{\mu} \) values being considered. At a given \( d \), with an increase in the chemical potential and with a corresponding change in the condensate density, the average number of pairs per unit cell also changes. The dependence of the average number of pairs per unit cell on \( r_0 \) is shown in Fig. 3. It can be seen that in the case of the parameters considered here, this number is less than, or of the order of, one.

Figure 4 shows the dependence of \( \theta_h \) on \( n_0 \). At a given \( d \), the \( \theta_h \) value increases with an increase in \( n_0 \). The spatial distribution of density (calculated by the formula \( n(r) = |\Psi_\mathbf{R}(\mathbf{R})|^2 \)) at \( \theta_h \approx 0.48 \) is shown in Fig. 5. At this \( \theta_h \), a continuous network of regions with reduced condensate density is formed in the system, thus surrounding density maxima. In this case, the density of the condensate in the network remains quite high. In an inhomogeneous condensate, superfluid stiffness will be lower than in a homogeneous condensate with the same average density (see, for example, Ref. 29). Therefore, the temperature of transition from a supersolid phase to a normal state will be lower than the temperature of transition from a homogeneous superfluid phase to a normal state. With increasing \( \theta_h \), the density of the condensate in the
network decreases, and at $\theta_h = \arctan(\sqrt{3}/2) \approx 0.886$ there appears a continuous network of lines where the superfluid density becomes zero.

Figure 7 shows the density distribution at $\theta_h = 0.87$, which is achieved at $d = 10a_0$ and $4\pi n_0r_0^2 = 1$ (see Fig. 5). At $\theta_h = 0.87$, a zero-density network is already nearly formed. The formation of the network means that the condensate is divided into a system of weakly connected regions. Since the average number of pairs in each region will not be an integer (see Fig. 4), such a state should collapse already at a low temperature. It can also be assumed that the lattice parameter $a_h$ will adjust to the average density of the condensate so that the number of pairs in each of the weakly connected regions will become an integer. We leave this for further study.

Thus, within the coherent-state formalism, we have described the transition of a dilute gas of electron-hole pairs in a bilayer system to a supersolid state, have plotted a phase diagram for the system, and have demonstrated how the spatial distribution of the condensate in this phase changes with a change in the chemical potential.

**APPENDIX: DERIVATION OF EXPlicit EXPRESSIONS 389 FOR INTERACTION CONSTANTS**

The interaction constants $\gamma_2(k)$ and $\gamma_{3(4),w(h)}(k)$ can be written as the sum of the terms determined by direct and exchange interactions:

$$
\gamma_2(k) = \gamma_2^{(d)}(k) + \gamma_2^{(ex)}(k),
\gamma_{3(4),w(h)}(k) = \gamma_{3(4),w(h)}^{(d)}(k) + \gamma_{3(4),w(h)}^{(ex)}(k)
$$

(A.1)

($\gamma_{3,w}(k) \equiv 0$).

These terms are expressed in terms of the Fourier transform of the wave function of the bound state of the pair $\psi_0(r)$. In $\gamma_2(k)$, the term determined by direct interaction is as follows:

$$
\gamma_2^{(d)}(k) = V_{ex}(k) \int \frac{d^2p}{(2\pi)^2} \frac{d^2p'}{(2\pi)^2} \left[ \phi_{p+m_h,k} \phi_{p'-m_h,k} + \phi_{p-m_h,k} \phi_{p'+m_h,k} \right] \phi_p \phi_{p'}
$$

+ $V_{ch}(k) \int \frac{d^2p}{(2\pi)^2} \frac{d^2p'}{(2\pi)^2} \left[ \phi_{p+m_h,k} \phi_{p'+m_h,k} + \phi_{p-m_h,k} \phi_{p'-m_h,k} \right] \phi_p \phi_{p'}$

(A.2)

where $V_{ex}(k) = 2\pi e^2/(\varepsilon k)$, $V_{ch}(k) = -V_{ex}(k)e^{-kd}$ are the Fourier components of the Coulomb interaction, and the pair $\psi_0(r)$.
FIG. 4. Average number of pairs per unit cell versus $n_0$ (in units of $a_0^{-2}$) for $d/a_0 = 5$ (solid line) and $d/a_0 = 10$ (dashed line) in System 1.

FIG. 5. The parameter $\theta_h$ versus $n_0$ (in units of $a_0^{-2}$) for the same $d$ as in Fig. 4. The dash-dotted line represents the $\theta_h$ at which there appears a network of lines where the local condensate density is zero.

FIG. 6. Condensate density distribution at $\theta_h = 0.48$. Light (yellow-brown) shading indicates areas with high density; dark (grey-blue) shading indicates areas with low density. The numbers on the contours represent the values of local density in $n_0$. The size of the region shown is $4\pi/k \times 4\pi/k$.

FIG. 7. The same as in Fig. 6 for $\theta_h = 0.87$.

The following notation is introduced: $\tilde{m}_{e(h)} = m_{e(h)}/(m_e + m_h)$. In what follows, the function $\phi_p$ is considered real. The contributions of the direct interaction to $\gamma_{3(4),w(k)}(k)$ can be expressed in terms of the function $\gamma_{2}(k)$:

$$\gamma_{3,b}(k) = 4\sqrt{3} \gamma_{2}(k),$$

$$\gamma_{4,b}(k) = \frac{1}{2} \gamma_{2}(2k),$$

$$\gamma_{4,h}(k) = \frac{2}{3} \left[ \gamma_{2}(k) + \gamma_{2}(\sqrt{3}k) \right] + \frac{1}{6} \gamma_{2}(2k).$$

(A.3)

To take into account the contribution of the exchange interaction, we introduce the function

$$\Gamma(G, g) = -\int \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \left\{ V_{ee}(q) \right.$$  

$$\times \left[ \phi_{p+\hat{G}+\tilde{m}_h G} \phi_{p+q+g+\tilde{m}_h G} + \phi_{p+q+g+\tilde{m}_h G} \phi_{p+q+g+\tilde{m}_h G} \right] \phi_{p+G} \phi_{p} +$$

$$V_{eh}(q) \times \left[ \phi_{p+q+g+\tilde{m}_h G} \phi_{p+q+g+\tilde{m}_h G} + \phi_{p+q+g+\tilde{m}_h G} \phi_{p+q+g+\tilde{m}_h G} \right] \phi_{p+G} \phi_{p} +$$

$$+ \phi_{p+q+g+\tilde{m}_h G} \phi_{p+q-g+\tilde{m}_h G} \phi_{p+G} \phi_{p} +$$

$$+ \phi_{p+q+g+\tilde{m}_h G} \phi_{p+q-g+\tilde{m}_h G} \phi_{p+G} \phi_{p} \right\}.$$  

(A.4)

Expressions for the exchange interaction constants, as written in terms of this function, take the following form:

$$\gamma_0^{(ex)} = \Gamma(0, 0),$$

$$\gamma_2^{(ex)}(k) = \frac{1}{4} \Gamma(k_1, 0) + 2 \Gamma(\tilde{m}_h k_1, -\tilde{m}_e \tilde{m}_h k_1)$$

$$+ 2 \Gamma(\tilde{m}_e k_1, \tilde{m}_e \tilde{m}_h k_1)$$
\[ + \Gamma((\tilde{m}_e - \tilde{m}_h)k_1, 2\tilde{m}_e\tilde{m}_h k_1) - 2\Gamma(0,0) \] \tag{A.5}

\[
\gamma^{(ex)}_{3, h}(k) = \sqrt{\frac{2}{3}} \left\{ \Gamma(k_1 + \tilde{m}_e k_2, \tilde{m}_e \tilde{m}_h k_2) + \Gamma[\tilde{m}_h k_1 + m_k k_2, \tilde{m}_e \tilde{m}_h (k_1 - k_2)] + \Gamma[\tilde{m}_h k_1 + \tilde{m}_e k_2, -\tilde{m}_e \tilde{m}_h (k_1 - k_2)] \right\}, \tag{A.6}
\]

\[
\gamma^{(ex)}_{4, w}(k) = \frac{1}{2} \left[ \Gamma(2\tilde{m}_h k_1, -2\tilde{m}_e \tilde{m}_h k_1) + \Gamma(2\tilde{m}_h k_1, 2\tilde{m}_e \tilde{m}_h k_1) - \Gamma(0,0) \right], \tag{A.7}
\]

\[
\gamma^{(ex)}_{4, h}(k) = \frac{1}{6} \left[ \Gamma(2\tilde{m}_h k_1, -2\tilde{m}_e \tilde{m}_h k_1) + \Gamma(2\tilde{m}_h k_1, 2\tilde{m}_e \tilde{m}_h k_1) + 2\Gamma[\tilde{m}_h (k_1 - k_2), \tilde{m}_e \tilde{m}_h (k_1 - k_2)] + 2\Gamma[\tilde{m}_h (k_1 + k_2), \tilde{m}_e \tilde{m}_h (k_1 + k_2)] + 2\Gamma[(\tilde{m}_h - \tilde{m}_e) k_2, -2\tilde{m}_e \tilde{m}_h k_2] + 2\Gamma((\tilde{m}_e - \tilde{m}_h) k_2, 2\tilde{m}_e \tilde{m}_h k_2) - 5\Gamma(0,0) \right]. \tag{A.8}
\]

By substituting the function \( \phi_q = \sqrt{\frac{4\pi r_0}{\varepsilon}} \exp(-q^2 \varepsilon^2/2) \) into the general expressions above we obtain the following result:

\[
\gamma_{w, h}(k, \theta) = \frac{4\pi e^2 r_0}{\varepsilon} \tilde{\gamma}_{w, h}(\tilde{k}, \theta), \tag{A.9}
\]

where \( \tilde{k} = k r_0 \) and

\[
\tilde{\gamma}_{w, h}(\tilde{k}, \theta) = \tilde{\gamma}_0 + 4\tilde{\gamma}_2(\tilde{k}) \cos^2 \theta \sin^2 \theta + 4\tilde{\gamma}_{3, w(h)}(\tilde{k}) \cos \theta \sin^3 \theta + 4\tilde{\gamma}_{4, w(h)}(\tilde{k}) \sin^4 \theta \tag{A.10}
\]

\( \tilde{\gamma}_{3, w}(\tilde{k}) = 0 \). The \( \tilde{\gamma}(\tilde{k}) \) functions included in \( \tilde{\gamma}_{3, w(h)}(\tilde{k}) \) can be represented as the sum of the contributions of the direct and exchange interactions, similarly to \( \tilde{\gamma}_0 \). Explicit expressions for the direct interaction contributions are equal to \( \tilde{\gamma}_0(\tilde{d}) = \tilde{d} \).

\[
\tilde{\gamma}_2(\tilde{d}) = \frac{1}{2k} \left[ \exp \left( -\frac{\tilde{k}^2(1-x)^2}{8} \right) + \exp \left( -\frac{\tilde{k}^2(1+x)^2}{8} \right) - 2\exp \left( -\tilde{k}d - \frac{\tilde{k}^2(1+x^2)}{8} \right) \right], \tag{A.11}
\]

where \( \tilde{d} = d / r_0 \). We obtain expressions for \( \tilde{\gamma}^{(d)}_{3, w(h)}(\tilde{k}) \) by replacing, in \( \tilde{\gamma}_3 \), all \( \gamma \) with \( \tilde{\gamma} \), and \( k \) with \( \tilde{k} \). To keep a simple record of the exchange interaction contributions, we determine the functions as follows:

\[
\text{fd}(\tilde{k}, \tilde{d}) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \exp \left( -\frac{3p^2}{8} - pd \right) I_0(pk) dp.
\]

where \( I_0(y) \) is the modified Bessel function. At \( \tilde{k} = 0 \), the function \( \text{fd}(\tilde{k}, \tilde{d}) \) can be expressed in terms of the complementary error function:

\[
\text{fd}(0, \tilde{d}) = I_0(\tilde{d}) = \sqrt{\frac{4}{3}} \exp \left( \frac{2\tilde{d}^2}{3} \right) \text{erfc} \left( \sqrt{\frac{2}{3}} \tilde{d} \right). \tag{A.12}
\]

The \( \tilde{z}^{(ex)}(\tilde{k}) \) functions written in terms of the functions \( \tilde{\gamma}^{(ex)}_0 \) and \( \tilde{\gamma}^{(ex)}_2 \) are as follows:

\[
\tilde{z}^{(ex)}_0 = -\sqrt{\frac{\pi}{2}} \left[ 1 - f_0(\tilde{d}) \right], \tag{A.13}
\]

\[
\tilde{z}^{(ex)}_2(\tilde{k}) = -\frac{1}{4} \sqrt{\frac{\pi}{2}} A \left[ \frac{\tilde{k}^2(x - 1)^2}{16} \right] \times \left( \exp \left[ -\frac{\tilde{k}^2(x + 1)^2}{8} \right] + 1 \right)
\]

\[
+ A \left[ \frac{\tilde{k}^2(x + 1)^2}{16} \right] \exp \left[ -\frac{\tilde{k}^2(x - 1)^2}{8} \right] + 1 \right)
\]

\[
+ \exp \left[ -\frac{\tilde{k}^2(x + 1)^2}{8} \right] \left( 1 - 2fd \left[ \frac{\tilde{k}^2}{4}, \tilde{d} \right] \right)
\]

\[
+ \exp \left[ -\frac{\tilde{k}^2(x - 1)^2}{8} \right] \left( 1 - 2fd \left[ \frac{\tilde{k}^2}{4}, \tilde{d} \right] \right)
\]

\[
- \exp \left[ -\frac{\tilde{k}^2(x^2 + 1)}{4} \right] \left( \text{fd} \left[ \frac{\tilde{k}^2/2}{4}, \tilde{d} \right] \right) - 2 + 2I_0(\tilde{d}) \right), \tag{A.14}
\]

\[
\tilde{z}^{(ex)}_{3, h}(\tilde{k}) = -\sqrt{\frac{3\pi}{4}} \left\{ \exp \left[ -\frac{\tilde{k}^2(x + 1)^2}{8} \right] \right\}
\]

\[
+ \exp \left[ -\frac{\tilde{k}^2(x + 1)^2}{8} \right] \left[ A \left[ \frac{\tilde{k}^2(x - 1)^2}{16} \right] \right.
\]

\[
- \exp \left[ -\frac{\tilde{k}^2(x^2 + 1)}{4} \right] \left( \text{fd} \left[ \frac{\tilde{k}^2/2}{4}, \tilde{d} \right] \right)
\]

\[
+ \exp \left[ -\frac{\tilde{k}^2(x + 1)^2}{8} \right] \left( 1 - 2fd \left[ \frac{\tilde{k}^2}{4}, \tilde{d} \right] \right) \right\}, \tag{A.16}
\]

\[
\tilde{z}^{(ex)}_{4, w}(\tilde{k}) = -\frac{1}{4} \sqrt{\frac{\pi}{2}} A \left[ \frac{\tilde{k}^2(x + 1)^2}{4} \right] + A \left[ \frac{\tilde{k}^2(x - 1)^2}{4} \right]
\]

\[
+ \exp \left[ -\frac{\tilde{k}^2(x + 1)^2}{8} \right] \left( 1 - 2fd \left[ \frac{\tilde{k}^2}{4}, \tilde{d} \right] \right) \right\}. \tag{A.17}
\]
\[\frac{1}{6} \sqrt{\frac{\pi}{2}} \left\{ A \left[ \frac{3\tilde{k}^2(x+1)^2}{16} \left( 1 + \exp \left[ -\frac{\tilde{k}^2(x-1)^2}{8} \right] \right) \right] + A \left[ \frac{3\tilde{k}^2(x-1)^2}{16} \left( 1 + \exp \left[ -\frac{\tilde{k}^2(x+1)^2}{8} \right] \right) \right] + A \left[ \frac{\tilde{k}^2(x+1)^2}{16} \left( 1 + \exp \left[ -\frac{3\tilde{k}^2(x-1)^2}{8} \right] \right) \right] \right\} \]

(A.17)

\[\tilde{\gamma}_{4,h}(\tilde{k}) = \frac{1}{3} \tilde{\gamma}_{4,w}(\tilde{k}) - \frac{2}{3} \tilde{\gamma}_0 \]

\[-2 \exp \left[ -\frac{\tilde{k}^2(1-x+x^2)}{2} \right] \int \frac{d\tilde{k}}{\tilde{k}^\sqrt{1-x+x^2}} \]

(A.18)

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