0. Introduction.

This is a second part of the treatment of complex analytic subvarieties of a holomorphically symplectic Kähler manifold. For the convenience of the reader, in the first two sections of this paper we recall the definitions and results of the first part ([V-pt I]).

By Calabi-Yau theorem, the holomorphically symplectic Kähler manifolds can be supplied with a Ricci-flat Riemannian metric. This implies that such manifolds are hyperkähler (Definition 1.1). Conversely, all hyperkähler manifolds are holomorphically symplectic (Proposition 2.1).

For a closed analytic subvariety $S$ of a holomorphically symplectic $M$, one can restrict the holomorphic symplectic form of $M$ to the Zarisky tangent sheaf of $S$. If this restriction is non-degenerate outside of singularities of $S$ and the same is true for the set of singular points of $S$, this subvariety is called non-degenerately symplectic (Definition 2.2). Of course, non-degenerately symplectic subvarieties are of even complex dimension.

The hyperkähler manifold is endowed with the canonical $SU(2)$-action in this cohomology space. Fix an induced complex structure on a hyperkähler manifold $M$. Let $N$ be a closed analytic subset of $M$. Then $N$ defines a cycle $[N]$ in cohomology of $M$. Denote the Poincare dual cocycle by $\langle N \rangle$. In [V-pt I] we proved that if $\langle N \rangle$ is $SU(2)$-invariant then $N$ is non-degenerately symplectic (Theorem 2.2 of [V-pt I]).

Take a generic element $M_0$ in a given deformation class of compact holomorphically symplectic Kähler manifolds. Then all integer $(p,p)$-cycles on $M_0$ (i.e., all elements of $H^{2p}(M, \mathbb{Z}) \cap H^{p,p}(M)$) are $SU(2)$-invariant (Proposition 2.2). According to Theorem 2.2 of [V-pt I], this immediately implies the following statement: All closed analytic subvarieties of $N$ are non-degenerately symplectic. If such subvariety is smooth, it is also a hyperkähler manifold (Proposition 2.1).

Let $M$ be a hyperkähler manifold with three complex structures $I$, $J$ and $K$. The closed subset $X \in M$ is called tri-analytic if $X$ is analytic with respect to $I$, $J$ and $K$.

In this paper we prove the following. Let $N$ be a closed analytic subset
of the compact holomorphic symplectic manifold $M$. Assume that $[N]$ is $SU(2)$-invariant. Then $N$ is trianalytic (Theorem 3.1). In particular, all closed analytic submanifolds of the compact holomorphic symplectic manifold of generic type are endowed with the hyperkähler structure which is preserved by the embedding (Corollary 3.1).

For the compact complex torus this statement implies a following corollary: if $N \subset M$ is a closed analytic subvariety of a generic compact torus, then $N$ is a point. However, this statement seems to be well known.

1. Hyperkähler manifolds.

**Definition 1.1:** ([Beau], [Bes]) A hyperkähler manifold is a Riemannian manifold $M$ endowed with three complex structures $I$, $J$ and $K$, such that the following holds.

(i) $M$ is Kähler with respect to these structures and
(ii) $I$, $J$ and $K$, considered as endomorphisms of a real tangent bundle, satisfy the relation $I \circ J = -J \circ I = K$.

This means that the hyperkähler manifold has the natural action of quaternions $\mathbb{H}$ in its real tangent bundle. Therefore its complex dimension is even.

Let $adI$, $adJ$ and $adK$ be the operators on the bundles of differential forms over a hyperkähler manifold $M$ which are defined as follows. Define $adI$. Let this operator act as a complex structure operator $I$ on the bundle of differential 1-forms. We extend it on $i$-forms for arbitrary $i$ using Leibnitz formula: $adI(\alpha \wedge \beta) = adI(\alpha) \wedge \beta + \alpha \wedge adI(\beta)$. Since Leibnitz formula is true for a commutator in a Lie algebras, one can immediately obtain the following identities, which are implied by the same identities in $\mathbb{H}$:

$$[adI, adJ] = 2adK; \quad [adJ, adK] = 2adI;$$

$$[adK, adI] = 2adJ$$

Therefore, the operators $adI$, $adJ$, $adK$ generate a Lie algebra $su(2)$ acting on the bundle of differential forms. We can integrate this Lie algebra action to the action of a Lie group $G_M = SU(2)$. In particular, operators $I$, $J$...
and $K$, which act on differential forms by the formula $I(\alpha \wedge \beta) = I(\alpha) \wedge I(\beta)$, belong to this group.

**Proposition 1.1:** There is an action of the Lie group $SU(2)$ and Lie algebra $\mathfrak{su}(2)$ on the bundle of differential forms over a hyperkähler manifold. This action is parallel, and therefore it commutes with Laplace operator. ■

If $M$ is compact, this implies that there is a canonical $SU(2)$-action on $H^i(M, \mathbb{R})$ (see [V-so(5)]).

Let $M$ be a hyperkähler manifold with a Riemannian form $\langle \cdot, \cdot \rangle$. Let the form $\omega_I := \langle I(\cdot), \cdot \rangle$ be the usual Kähler form which is closed and parallel (with respect to the connection). Analogously defined forms $\omega_J$ and $\omega_K$ are also closed and parallel.

The simple linear algebraic consideration ([Bes]) shows that $\omega_J + \sqrt{-1} \omega_K$ is of type $(2, 0)$ and, being closed, this form is also holomorphic. It is called the canonical holomorphic symplectic form of a manifold $M$. Conversely, if there is a parallel holomorphic symplectic form on a Kähler manifold $M$, then this manifold has a hyperkähler structure ([Bes]).

If some compact Kähler manifold $M$ admits non-degenerate holomorphic symplectic form $\Omega$, the Calabi-Yau ([Y]) theorem implies that $M$ is hyperkähler (Proposition 2.1). This follows from the existence of a Kähler metric on $M$ such that $\Omega$ is parallel for the Levi-Civită connection associated with this metric.

Let $M$ be a hyperkähler manifold with complex structures $I$, $J$ and $K$. For any real numbers $a$, $b$, $c$ such that $a^2 + b^2 + c^2 = 1$ the operator $L := aI + bJ + cK$ is also an almost complex structure: $L^2 = -1$. Clearly, $L$ is parallel with respect to connection. This implies that $L$ is a complex structure, and that $M$ is Kähler with respect to $L$.

**Definition 1.2:** If $M$ is a hyperkähler manifold, the complex structure $L$ is called **induced by a hyperkähler structure**, if $L = aI + bJ + cK$ for some real numbers $a, b, c \mid a^2 + b^2 + c^2 = 1$.

If $M$ is a hyperkähler manifold and $L$ is induced complex structure, we will denote $M$, considered as a complex manifold with respect to $L$, by $(M, L)$ or, sometimes, by $M_L$. 

3
Consider the Lie algebra $\mathfrak{g}_M$ generated by $adL$ for all $L$ induced by a hyperkähler structure on $M$. One can easily see that $\mathfrak{g}_M = su(2)$. The Lie algebra $\mathfrak{g}_M$ is called isotropy algebra of $M$, and corresponding Lie group $G_M$ is called an isotropy group of $M$. By Proposition 1.1, the action of the group is parallel, and therefore it commutes with Laplace operator in differential forms. In particular, this implies that the action of the isotropy group $G_M$ preserves harmonic forms, and therefore this group canonically acts on cohomology of $M$.

**Proposition 1.1:** Let $\omega$ be a differential form over a hyperkähler manifold $M$. The form $\omega$ is $G_M$-invariant if and only if it is of Hodge type $(p, p)$ with respect to all induced complex structures on $M$.

**Proof:** Assume that $\omega$ is $G_M$-invariant. This implies that all elements of $\mathfrak{g}_M$ act trivially on $\omega$ and, in particular, that $adL(\omega) = 0$ for any induced complex structure $L$. On the other hand, $adL(\omega) = (p - q)\sqrt{-1}$ if $\omega$ is of Hodge type $(p, q)$. Therefore $\omega$ is of Hodge type $(p, p)$ with respect to any induced complex structure $L$.

Conversely, assume that $\omega$ is of type $(p, p)$ with respect to all induced $L$. Then $adL(\omega) = 0$ for any induced $L$. By definition, $\mathfrak{g}_M$ is generated by such $adL(\omega) = 0$, and therefore $\mathfrak{g}_M$ and $G_M$ act trivially on $\omega$. ■

### 2. Holomorphic symplectic geometry.

**Definition 2.1:** The compact Kähler manifold $M$ is called holomorphically symplectic if there is a holomorphic 2-form $\Omega$ over $M$ such that $\Omega^n = \Omega \wedge \Omega \wedge ...$ is a nowhere degenerate section of a canonical class of $M$. There, $2n = \text{dim}_\mathbb{C}(M)$.

Note that we assumed compactness of $M$. One observes that the holomorphically symplectic manifold has a trivial canonical bundle. A hyperkähler manifold is holomorphically symplectic (see Section 1). There is a converse proposition:

**Proposition 2.1:** ([Beau], [Bes]) Let $M$ be a holomorphically symplectic Kähler manifold with the holomorphic symplectic form $\Omega$, a Kähler class $\omega$ and the holomorphic symplectic form $\Omega$. The operator $\nabla' : \Lambda^{p,0}(M) \to \Lambda^{p+1,0}(M)$ is a holomorphic differential defined on differential $(p, 0)$-forms ($[\mathbb{GH}]$).
\([\omega] \in H^{1,1}(M)\) and a complex structure \(I\). There is a unique hyperkähler structure \((I, J, K, (\cdot, \cdot))\) over \(M\) such that the cohomology class of the symplectic form \(\omega_I = (\cdot, I \cdot)\) is equal to \([\omega]\) and the canonical symplectic form \(\omega_I + \sqrt{-1} \omega_K\) is equal to \(\Omega\).

Proposition 2.1 immediately follows from the Calabi-Yau theorem ([Y]).

For each complex analytic variety \(X\) and a point \(x \in X\), we denote the Zariski tangent space to \(X\) in \(x\) by \(T_x X\).

**Definition 2.2:** Let \(M\) be a holomorphically symplectic manifold and \(S \subset M\) be its closed complex analytic subvariety. Assume that \(S\) is closed in \(M\) and reduced. It is called non-degenerately symplectic if for each point \(s \in S\) outside of the singularities of \(S\) the restriction of the holomorphic symplectic form \(\Omega\) to \(T_s M\) is nondegenerate on \(T_s S \subset T_s M\), and the set \(\text{Sing}(S)\) of the singular points of \(S\) is nondegenerately symplectic. This definition refers to itself, but since \(\dim \text{Sing}(S) < \dim S\), it is consistent.

Of course, the complex dimension of a non-degenerately symplectic variety is even.

Let \(M\) be a holomorphically symplectic Kähler manifold. By Proposition 2.1, \(M\) has a unique hyperkähler metric with the same Kähler class and holomorphic symplectic form. Therefore one can without ambiguity speak about the action of \(G_M\) on \(H^*(M, \mathbb{R})\) (see Proposition 1.1). Of course, this action essentially depends on the choice of Kähler class.

**Definition 2.3:** Let \(\omega \in H^{1,1}(M)\) be the Kähler of a Kähler metric defined on a holomorphically symplectic manifold \(M\). We say that \(\omega\) induces the \(SU(2)\)-action of general type when all elements of the group

\[
H^{pp}(M) \cap H^{2p}(M, \mathbb{Z})
\]

are \(G_M\)-invariant. The action of \(SU(2) \cong G_M\) is defined by Proposition 2.1. The holomorphically symplectic manifold \(M\) is called of general type if there exists a Kähler class on \(M\) which induces an \(SU(2)\)-action of general type.

As Theorem 2.2 implies, the holomorphically symplectic manifold of general type has no Weil divisors. Therefore these manifolds have connected Picard group. In particular, such manifolds are never algebraic.
**Proposition 2.2:** Let $M$ be a hyperkähler manifold. Let $S$ be the set of induced complex structures over $M$. Let $S_0 \subset S$ be the set of $R \in S$ such that the natural Kähler metric on $(M, R)$ induces the $SU(2)$ action of general type. Then $S_0$ is dense in $S$.

**Proof:** Let $A$ be the set of all $\alpha \in H^{2p}(M, \mathbb{Z})$ such that $\alpha$ is not $G_M$-invariant. The set $A$ is countable. For each $\alpha \in A$, let $S_\alpha$ be the set of all $R \in S$ such that $\alpha$ is of type $(p, p)$ with respect to $R$. The set $S_0$ of all induced complex structures of general type is equal to \( \bigcup_{\alpha \in A} S_\alpha \). Now, to prove Proposition 2.2 it is sufficient to show that $S_\alpha$ is a finite set for each $\alpha \in A$. This would imply that $S_0$ is a complement of a countable set to a 2-sphere $S$, and therefore dense in $S$.

As it follows from Section 1, $\alpha$ is of type $(p, p)$ with respect to $R$ if and only if $ad R(\alpha) = 0$. Now, let $V$ be a representation of $su(2)$, and $v \in V$ be a non-invariant vector. It is easy to see that the element $a \in su(2)$ such that $a(v) = 0$ is unique up to a constant, if it exists. This implies that if $\alpha$ is not $G_M$-invariant there are no more than two $R \in S$ such that $ad R(\alpha) = 0$. Of course, these two elements of $S$ are opposite to each other. $\blacksquare$

One can easily deduce from the results in [Tod] and from Proposition 2.2 that the set of points associated with holomorphically symplectic manifolds of general type is dense in the classifying space of holomorphically symplectic manifolds.

For a Kähler manifold $M$, $m = dim_\mathbb{C} M$ and a form $\alpha \in H^{2p}(M, \mathbb{C})$, define

$$deg(\alpha) := \int_M L^{m-p}(\alpha)$$

where $L$ is a Hodge operator of exterior multiplication by the Kähler form $\omega$ (see [GH]). Of course, the degree of forms of Hodge type $(p, q)$ with $p \neq q$ is equal zero, so only $(p, p)$-form can possibly have non-zero degree.

We recall that the real dimension of a holomorphically symplectic manifold is divisible by 4.

**Theorem 2.1:** (Theorem 2.1 of [V-pt1]). Let $M$ be a holomorphically symplectic Kähler manifold with a holomorphic symplectic form $\Omega$. Let $\alpha$ be a $G_M$-invariant form of non-zero degree. Then the dimension of $\alpha$ is divisible by 4.
divisible by 4. Moreover,

\[ \int_M \Omega^n \wedge \bar{\Omega}^n \wedge \alpha = 2^n \deg(\alpha), \]

where \( n = \frac{1}{4}(\dim \mathbb{R} M - \dim \alpha) \).

**Proposition 2.3:** (Theorem 2.2 of [V-pt I]). Let \( M \) be a holomorphic symplectic manifold of general type. All closed analytic subvarieties of \( M \) have even complex dimension.

**Proof:** Proposition 2.3 immediately follows from Theorem 2.1.

**Theorem 2.2:** (Theorem 2.3 of [V-pt I]). Let \( M \) be a holomorphic symplectic manifold of general type. All reduced closed analytic subvarieties of \( M \) are non-degenerately symplectic.

Combining this with Proposition 2.1, one obtains

**Theorem 2.3:** Let \( M \) be a holomorphically symplectic manifold of general type, and \( S \subset M \) be its smooth differentiable submanifold. If \( S \) is analytic in \( M \), it is a hyperkähler manifold.

### 3. When analytic implies tri-analytic

Let \( M \) be a compact hyperkähler manifold, \( \dim \mathbb{R} M = 2m \). Let \( I \) be an induced complex structure. As usually, \( (M, I) \) denotes \( M \) considered as a Kähler manifold with the complex structure defined by \( I \).

**Definition 3.1:** Let \( N \subset M \) be a closed subset of \( M \). Then \( N \) is called **tri-analytic** if \( N \) is an analytic subset of \( (M, I) \) for any induced complex structure \( I \).

Let \( N \subset (M, I) \) be a closed analytic subvariety of \( (M, I) \), \( \dim \mathbb{C} N = n \). Let \([N] \in H_{2n}(M)\) denote the homology class represented by \( N \). Let \( \langle N \rangle \in H^{2m-2n}(M) \) denote the Poincare dual cohomology class. Recall that the
hyperkähler structure induces the action of the group $G_M = SU(2)$ on the space $H^{2m-2n}(M)$. The main result of this section is following:

**Theorem 3.1:** Assume that $\langle N \rangle \in H^{2m-2n}(M)$ is invariant with respect to the action of $G_M$ on $H^{2m-2n}(M)$. Then $N$ is tri-analytic$^1$.

Theorem 3.1 has the following important corollary:

**Corollary 3.1:** Let $M$ be a holomorphically symplectic manifold of general type, and $S \subset M$ be its smooth complex submanifold. Let $\omega$ be the Kähler class which induces an $SU(2)$-action of general type. Pick a hyperkähler metric $s$ associated with $\omega$ by Proposition 2.1. Then the restriction of $s$ to $S$ is a hyperkähler metric.

The rest of this section is dedicated to the proof of Theorem 3.1.

**Proof of Theorem 3.1.** We use the following proposition of linear algebra. Let $V$ be a complex space and $V_\mathbb{R}$ be its underlying $\mathbb{R}$-linear space. Let $W \subset V_\mathbb{R}$ be an $\mathbb{R}$-linear subspace of $V_\mathbb{R}$, $dim W = 2n$. Denote the Hermitian form on $V$ by $\langle \cdot, \cdot \rangle$. The $\mathbb{R}$-linear space $V_\mathbb{R}$ is equipped with the positively defined symmetric scalar product $(u,v) := Re \langle u,v \rangle$. and the non-degenerated symplectic form $\langle u,v \rangle := Im \langle u,v \rangle$. Let $\langle \cdot, \cdot \rangle_W$ and $\omega = \langle \cdot, \cdot \rangle_W$ be the restrictions of these forms to $W \subset V_\mathbb{R}$. Clearly, $\langle \cdot, \cdot \rangle_W$ is a positively defined scalar product on $W$; therefore, $\langle \cdot, \cdot \rangle_W$ in non-degenerate. By a scalar product on an $R$-vector space, we define a **volume form** as follows.

**Definition 3.2:** Let $H$ be an $\mathbb{R}$-linear space equipped with a positively defined scalar product. Let $h = dim H$. The exterior form $Vol \in \Lambda^h(H)$ is called a **volume form** if the the standard hypercube with the side 1 has the volume 1 in the measure defined by $Vol$.

The proof of correctness of this definition can be found in any linear algebra textbook. Clearly, the volume form is defined up to a sign. This sign is determined by the choice of orientation on $H$. In the same manner we define the top degree differential form called a **volume form** $Vol$ on any oriented Riemannian manifold.

$^1$The number $n = dim_{\mathbb{C}} N$ is even by Theorem 2.1.
Let Vol be the volume form on W defined by the scalar product. Then Vol is a non-zero element of the 1-dimensional linear space
\[ \text{Vol} = \Lambda^{2n}(W). \]
The form \( \omega^n = \langle \cdot, \cdot \rangle_W^n \) is another element of Vol. The number \( \frac{\omega^n}{\text{Vol}} \in \mathbb{R} \) is defined up to a sign, because Vol is defined up to a sign. Let \( \eta_W := \left| \frac{\omega^n}{\text{Vol}} \right| \). This number is an invariant of W, V and \( \langle \cdot, \cdot \rangle \).

Consider the complex structure operator on V as the real endomorphism of \( V_\mathbb{R} \):
\[ I : V_\mathbb{R} \longrightarrow V_\mathbb{R}, \quad I^2 = -1. \]

**Proposition 3.1:** (Wirtinger’s inequality) Let \( 2n = \dim_\mathbb{R} W \). Then \( \eta_W \leq 2^n \). Moreover, if \( \eta_W = 2^n \), then \( I(W) = W \). In other words, if \( \eta_W = 2^n \), then W is a complex subspace of V.

**Proof:** [Sta] page 7.  

We return to the situation when M is a hyperkähler manifold and \( N \subset (M, I) \) is a closed analytic subvariety of \((M, I)\). Let J be an induced complex structure, and \( N^0 \subset M \) be the set of non-singular points of N. Denote the standard embedding \( N^0 \hookrightarrow M \) by \( \varphi \). Let \( \omega_J \in \Lambda^{1,1}(M) \) be the Kähler form induced by J. Denote the standard coupling of homology and cohomology by
\[ \langle \cdot, \cdot \rangle : H_i(M) \times H^j(M) \longrightarrow \mathbb{C}. \]

**Proposition 3.2:**
\[ \langle [N], \omega^n_J \rangle = \int_{N^n} \varphi^*(\omega^n_J). \]

**Proof:** Clear.  

Let \( \langle \cdot, \cdot \rangle_J \) denote the Hermitian form on M induced by the Riemannian metric and the complex structure J. Let \( \omega_J = \langle \cdot, \cdot \rangle_J \) and \( \langle \cdot, \cdot \rangle \) denote its
imaginary and real parts respectively. It is clear that the scalar product $(\cdot, \cdot)$ is the original Riemannian form on $M$. Thus, the real part of $(\cdot, \cdot)_J$ does not depend on the complex structure $J$.

Let $(\cdot, \cdot)_N := \varphi^*((\cdot, \cdot))$ be the Riemannian form on $N^0$. Since $N$ is oriented, $(\cdot, \cdot)_N$ defines a volume form $Vol \in \Lambda^{2n}(N^0)$. By definition, $Vol$ is a nowhere degenerate section of the 1-dimensional vector bundle $\Lambda^{2n}(N^0)$. Since $N$ is analytic in $(M, I)$, we have $Vol = 1/2^n \varphi^*(\omega_I)^n \in \Lambda^{2n}(N^0)$. Let $x \in N^0$. Consider $V = T_x M$ as a complex space with the complex structure induced by $J$. Let $W = T_x N^0$. Then $W \subset V$ defines a number $\eta_W$ as in [Proposition 3.1]. This number depends on $x$ and $J$. For any induced complex structure $J$, we define a function $\eta_J : N^0 \rightarrow \mathbb{R}_{\geq 0}$ which supplies the number $\eta_W \in \mathbb{R}_{\geq 0}$ by the point $x \in N^0$.

**Proposition 3.3:** Let $J$ be an induced complex structure. The closed set $N \subset M$ is analytic with respect to $J$ if and only if

$$\forall x \in N^0 \quad \eta_J(x) = 2^n.$$

**Proof:** The implication

$$(N \text{ is analytic w. r. to } J) \implies (\eta_J(x) \equiv 2^n)$$

is clear because if $N$ is analytic with respect to $J$, then $J(T_x N) = T_x N$ and $\eta_J(x) \equiv 2^n$ by [Proposition 3.1]. We proceed proving the converse implication. Assume that $\forall x \in N^0$ we have $\eta_J(x) = 2^n$. Then $J(T_x N) = T_x N$ by [Proposition 3.1]. Using Newlander-Nierenberg theorem, we see that $N^0$ is an analytic subset of $(M, J)$. Clearly, $N$ is a closure of $N^0$. Since the closure of an analytic set is also analytic, the set $N \subset M$ is also an analytic subset of $(M, J)$. ■

By definition of $\eta_J(x)$, we have

$$\int_{N^0} \varphi^*(\omega_J^{2n}) = \int_{N^0} Vol \cdot \eta_J(x). \quad (3.1)$$

By [Proposition 3.1], we have $\eta_J(x) \leq 2^n$. Clearly, the function

$$\eta_J(x) : N^0 \rightarrow \mathbb{R}_{\geq 0}$$

over $\mathbb{R}$
is continuous. Therefore

$$\int_{N^0} \text{Vol} \cdot \eta_J(x) = 2^n \int_{N^0} \text{Vol}$$

if and only if \( \eta_J(x) = 2^n \) for every \( x \in N^0 \). Combining this with (3.1) and Proposition 3.3, we obtain the following statement:

**Proposition 3.4:**

$$\int_{N^0} \varphi^*(\omega_J^{2n}) = 2^n \int_{N^0} \text{Vol}$$

if and only if \( N \) is analytic with respect to \( J \).

Since \( N \) is analytic with respect to \( J \), we have

$$2^n \int_{N^0} \text{Vol} = \int_{N^0} \varphi^*(\omega_J^{2n}).$$

Combining Proposition 3.4 with Proposition 3.2, we see that Theorem 3.1 is implied by the following statement.

**Proposition 3.5:** Let \( M \) be a compact hyperkähler manifold, \( \dim_C M = m \). Let \([N] \in H_{2n}(M)\) be a homology class of \( M \) such that its Poincare dual cocycle \( \langle N \rangle \in H^{2m-2n} \) is \( G_M \)-invariant. Let \( I \) and \( J \) be two induced complex structures on \( M \). Then \( \langle [N], \omega_I^{2n} \rangle = \langle [N], \omega_J^{2n} \rangle \).

**Proof:** By definition,

$$\langle [N], \omega_J^{2n} \rangle = \int_M \langle N \rangle \wedge \omega_J^{2n}.$$

Therefore all we need to show is that the number

$$\text{deg}_J \alpha := \int_M \alpha \wedge \omega_J^{2n}$$

is independent on the choice of \( J \) once the cohomology class \( \alpha \) is \( G_M \)-invariant.
Let \( L_J : \Lambda^i(M) \rightarrow \Lambda^{i+2}(M) \) denote the Hodge operator acting on differential forms over \( M \), \( L_J(\eta) = \omega_J \wedge \eta \). Let \( \Lambda_J : \Lambda^i(M) \rightarrow \Lambda^{i-2}(M) \) denote the adjoint operator. It is well known that \( L_J, \Lambda_J \) map harmonic form to harmonic ones. Therefore one can consider \( L_J, \Lambda_J \) as operators on the cohomology space \( H^*(M) \). Let \( a_M \subset \text{End}(H^*(M)) \) be the Lie algebra generated by \( L_J, \Lambda_J \) for all induced complex structures \( J \). Let \( g_M \cong \mathfrak{so}(3) \) be the Lie algebra of \( G_M \). Since \( G_M \) non-trivially acts on \( H^*(M) \), we may consider \( g_M \) as a subalgebra of \( \text{End}(H^*(M)) \). It is known that \( a_M \cong \mathfrak{so}(5) \) and that \( g_M \) considered as a Lie subalgebra of \( \text{End}(H^*(M)) \) lies in \( a_M \subset \text{End}(H^*(M)) \) (see [V-so(5)]).

Let \( H^*(M) = \oplus_{l \in \Pi} H_l \) be the isotypic decomposition of a \( a_M \)-module \( H^*(M) \). We recall that isotypic decomposition of a representation of an arbitrary reductive Lie algebra is defined as follows. For each \( l \in \Pi \), where \( \Pi \) is a weight lattice of \( a_M \), the module \( H_l \) is a union of all simple \( a_M \)-submodules of \( H^*(M) \) with a highest weight \( l \). One can easily see that the isotypic decomposition does not depend on a choice of a Cartan subalgebra of \( a_M \). This follows, for example, from Schur’s lemma.

Let \( H_0 \) be the \( a_M \)-submodule of \( H^*(M) \) generated by \( H^0(M) \cong \mathbb{C} \). Clearly, \( H_0 \) is an isotypic component of \( H^*(M) \) (see [V-pt I] for details). Let \( \alpha_o \) be the component of \( \alpha \) which corresponds to the summand \( H_0 \subset \oplus_{l \in \Pi} H_l \). Since \( g_M \subset a_M \), the isotypic component \( \alpha_o \) of \( \alpha \) is \( g_M \)-invariant. It was proven that \( \text{deg}_I(\alpha_o) = \text{deg}_J(\alpha) \) for all induced complex structures \( J \) (see the paragraph right after the proof of Lemma 3.1 in [V-pt I]).

Let \( I, J, K \) be a triple of induced complex structures on \( M \), such that

\[
I \circ J = -J \circ I = K.
\]

Lemma 3.2 of [V-pt I] implies that

\[
\alpha_o = c(I^2_J + J^2_J + K^2_K)^{m-n} \mathbb{I}
\]

where \( \mathbb{I} \) is a generator of \( H^0(M) \) and \( c \) is a constant. In this notation, the equality

\[
\text{deg}_I(\alpha_o) = \text{deg}_J(\alpha_o) = \text{deg}_K(\alpha_o)
\]

is obvious. We proved the following lemma
**Lemma 3.1:** If $I, J, K$ are induced complex structures on $M$, such that

$$I \circ J = -J \circ I = K,$$

then

$$deg_I(\alpha_o) = deg_J(\alpha_o) = deg_K(\alpha_o).$$

We deduce Proposition 3.5 from Lemma 3.1 as follows. Let $V$ be the space of purely imaginary quaternions. By definition, the space of purely imaginary quaternions $V \subset \mathbb{H}$ is a three-dimensional vector space over $\mathbb{R}$:

$$V = \{ t \in \mathbb{H} | \overline{t} = -t \},$$

$$V = \{ aI + bJ + cK, \ a, b, c \in \mathbb{R} \}.$$

The set of induced complex structures can be considered as a sphere $S$ of radius 1 in $V$. For $I, J$ in $S$, we have $I \circ J = -J \circ I$ if and only if the vector $(0, I) \in V$ is perpendicular to $(0, J) \in V$. On the other hand, if $(0, I) \perp (0, J)$, then $K := I \circ J$ belongs to $S$ and the vectors $(0, I), (0, J), (0, K)$ are pairwise orthogonal. We obtain that the triples $I, J, K$ of induced complex structures which satisfy

$$I \circ J = -J \circ I = K$$

are in one-to-one correspondence with the orthonormal repers in $V$. Therefore Lemma 3.1 implies the following statement:

**Lemma 3.2:** Let $I, J$ be the induced complex structures such that the vectors $(0, I) \in V$ and $(0, J) \in V$ are orthogonal. Then $deg_I(\alpha) = deg_J(\alpha)$.

**Lemma 3.2** trivially implies Proposition 3.5. Theorem 3.1 is proven.

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References

[Beau] Beauville, A. Varietes Kähleriennes dont la pere classe de Chern est nulle. // J. Diff. Geom. 18, p. 755-782 (1983).

[Bes] Besse, A., Einstein Manifolds. // Springer-Verlag, New York (1987)

[GH] Griffiths, Ph. and Harris, J. Principles of algebraic geometry. // Wiley-Interscience, New York (1978).

[Sto] Stonzenberg, G. Volumes, Limits and Extensions of Analytic Varieties. // Lect. Notes in Math. 19, 1966, Springer-Verlag.

[Tod] Todorov, A. Moduli of Hyper-Kählerian manifolds I,II. // Preprint MPI (1990)

[V-so(5)] Verbitsky, M. On the action of a Lie algebra SO(5) on the cohomology of a hyperkahler manifold. // Func. Analysis and Appl. 24(2) p. 70-71 (1990).

[V-pt I] Verbitsky M., Hyperkähler and holomorphic symplectic geometry I. // Electronic preprint alg-geom 9307009 (1993) 14 pp, LaTeX

[V-thesis] Verbitsky M., Coherent Perverse Sheaves // Ph. D. thesis, in preparation.

[Y] Yau, S. T. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I. // Comm. on Pure and Appl. Math. 31, 339-411 (1978).