IMPROVED LOWER BOUND FOR THE NUMBER OF UNIMODULAR ZEROS OF SELF-RECIPROCAL POLYNOMIALS WITH COEFFICIENTS IN A FINITE SET

Tamás Erdélyi

February 10, 2017

Abstract. Let \( n_1 < n_2 < \cdots < n_N \) be non-negative integers. In a private communication Brian Conrey asked how fast the number of real zeros of the trigonometric polynomials \( T_N(\theta) = \sum_{j=1}^{N} \cos(n_j \theta) \) tends to \( \infty \) as a function \( N \). Conrey’s question in general does not appear to be easy. Let \( \mathcal{P}_n(S) \) be the set of all algebraic polynomials of degree at most \( n \) with each of their coefficients in \( S \). For a finite set \( S \subset \mathbb{C} \) let \( M = M(S) := \max\{|z| : z \in S\} \).

It has been shown recently that if \( S \subset \mathbb{R} \) is a finite set and \( (P_n) \) is a sequence of self-reciprocal polynomials \( P_n \in \mathcal{P}_n(S) \) with \( |P_n(1)| \) tending to \( \infty \), then the number of zeros of \( P_n \) on the unit circle also tends to \( \infty \). In this paper we show that if \( S \subset \mathbb{Z} \) is a finite set, then every self-reciprocal polynomial \( P \in \mathcal{P}_n(S) \) has at least

\[
c(\log \log \log |P(1)|)^{1-\varepsilon} - 1
\]

zeros on the unit circle of \( \mathbb{C} \) with a constant \( c > 0 \) depending only on \( \varepsilon > 0 \) and \( M = M(S) \).

Our result improves the exponent \( 1/2 - \varepsilon \) in a recent result by Julian Sahasrabudhe to \( 1 - \varepsilon \). We note that in both Sahasrabudhe’s paper and our paper the assumption that the finite set \( S \) contains only integers is deeply exploited.

Preliminary Version

1. Introduction and Notation

Research on the distribution of the zeros of algebraic polynomials has a long and rich history. In fact all the papers [1–48] in our list of references are just some of the papers devoted to this topic. The study of the number of real zeros trigonometric polynomials and the number of unimodular zeros (that is, zeros lying on the unit circle of the complex plane) of algebraic polynomials with various constraints on their coefficients are the subject of quite a few of these. We do not try to survey these in our introduction.

Key words and phrases. self-reciprocal polynomials, trigonometric polynomials, restricted coefficients, number of zeros on the unit circle, number of real zeros in a period, Conrey’s question.

2010 Mathematics Subject Classifications. 11C08, 41A17, 26C10, 30C15

Typeset by \textsc{AMS-\LaTeX}
Let $S \subset \mathbb{C}$. Let $\mathcal{P}_n(S)$ be the set of all algebraic polynomials of degree at most $n$ with each of their coefficients in $S$. A polynomial $P$ of the form

\begin{equation}
(1.1) \quad P(z) = \sum_{j=0}^{n} a_j z^j, \quad a_j \in \mathbb{C},
\end{equation}

is called conjugate-reciprocal if

\begin{equation}
(1.2) \quad \bar{a}_j = a_{n-j}, \quad j = 0, 1, \ldots, n.
\end{equation}

A polynomial $P$ of the form (1.1) is called plain-reciprocal or self-reciprocal if

\begin{equation}
(1.3) \quad a_j = a_{n-j}, \quad j = 0, 1, \ldots, n.
\end{equation}

If a conjugate reciprocal polynomial $P$ has only real coefficients, then it is obviously plain-reciprocal. We note also that if

\begin{equation}
P(z) = \sum_{j=0}^{2n} a_j z^j, \quad a_j \in \mathbb{C},
\end{equation}

is conjugate-reciprocal, then there are $\theta_j \in \mathbb{R}$, $j = 1, 2, \ldots, n$, such that

\begin{equation}
T(t) := P(e^{it})e^{-int} = a_n + \sum_{j=1}^{n} 2|a_j + n| \cos(jt + \theta_j).
\end{equation}

If the polynomial $P$ above is plain-reciprocal, then

\begin{equation}
T(t) := P(e^{it})e^{-int} = a_n + \sum_{j=1}^{n} 2a_j + n \cos(jt).
\end{equation}

In this paper, whenever we write “$P \in \mathcal{P}_n(S)$ is conjugate-reciprocal” we mean that $P$ is of the form (1.1) with each $a_j \in S$ satisfying (1.2). Similarly, whenever we write “$P \in \mathcal{P}_n(S)$ is self-reciprocal” we mean that $P$ is of the form (1.1) with each $a_j \in S$ satisfying (1.3). This is going to be our understanding even if the degree of $P \in \mathcal{P}_n(S)$ is less than $n$. It is easy to see that $P \in \mathcal{P}_n(S)$ is self-reciprocal and $n$ is odd, then $P(-1) = 0$. Associated with an algebraic polynomial $P$ of the form (1.1) we introduce the numbers

\begin{equation}
\text{NC}(P) := |\{j \in \{0, 1, \ldots, n\} : a_j \neq 0\}|.
\end{equation}

Here, and in what follows $|A|$ denotes the number of elements of a finite set $A$. Let NZ($P$) denote the number of real zeros (by counting multiplicities) of an algebraic polynomial $P$ on the unit circle. Associated with a trigonometric polynomial

\begin{equation}
T(t) = \sum_{j=0}^{n} a_j \cos(jt)
\end{equation}


we introduce the numbers
\[ \text{NC}(T) := |\{ j \in \{0, 1, \ldots, n\} : a_j \neq 0\}|. \]

Let \( \text{NZ}(T) \) denote the number of real zeros (by counting multiplicities) of a trigonometric polynomial \( T \) in a period (of length \( 2\pi \)). Let \( \text{NZ}^*(T) \) denote the number of sign changes of a trigonometric polynomial \( T \) in a period (of length \( 2\pi \)). The quotation below is from [7]. “Let \( 0 \leq n_1 < n_2 < \cdots < n_N \) be integers. A cosine polynomial of the form \( T(\theta) = \sum_{j=1}^{N} \cos(n_j \theta) \) must have at least one real zero in a period. This is obvious if \( n_1 \neq 0 \), since then the integral of the sum on a period is 0. The above statement is less obvious if \( n_1 = 0 \), but for sufficiently large \( N \) it follows from Littlewood’s Conjecture simply. Here we mean the Littlewood’s Conjecture proved by S. Konyagin [27] and independently by McGehee, Pigno, and Smith [35] in 1981. See also [14, pages 285-288] for a book proof. It is not difficult to prove the statement in general even in the case \( n_1 = 0 \) without using Littlewood’s Conjecture. One possible way is to use the identity
\[ \sum_{j=1}^{n_N} T \left( \frac{(2j-1)\pi}{n_N} \right) = 0. \]

See [28], for example. Another way is to use Theorem 2 of [36]. So there is certainly no shortage of possible approaches to prove the starting observation of this paper even in the case \( n_1 = 0 \).

It seems likely that the number of zeros of the above sums in a period must tend to \( \infty \) with \( N \). In a private communication B. Conrey asked how fast the number of real zeros of the above sums in a period tends to \( \infty \) as a function \( N \). In [12] the authors observed that for an odd prime \( p \) the Fekete polynomial
\[ f_p(z) = \sum_{k=0}^{p-1} \left( \begin{array}{c} k \\ p \end{array} \right) z^k \]
(the coefficients are Legendre symbols) has \( \sim \kappa_0 p \) zeros on the unit circle, where \( 0.500813 > \kappa_0 > 0.500668 \). Conrey’s question in general does not appear to be easy.

Littlewood in his 1968 monograph ‘Some Problems in Real and Complex Analysis [33, problem 22] poses the following research problem, which appears to still be open: ‘If the \( n_m \) are integral and all different, what is the lower bound on the number of real zeros of \( \sum_{m=1}^{N} \cos(n_m \theta) \)? Possibly \( N - 1 \), or not much less. Here real zeros are counted in a period. In fact no progress appears to have been made on this in the last half century. In a recent paper [7] we showed that this is false. There exist cosine polynomials \( \sum_{m=1}^{N} \cos(n_m \theta) \) with the \( n_m \) integral and all different so that the number of its real zeros in a period is \( O(N^{9/10} \log N)^{1/5} \) (here the frequencies \( n_m = n_m(N) \) may vary with \( N \). However, there are reasons to believe that a cosine polynomial \( \sum_{m=1}^{N} \cos(n_m \theta) \) always has many zeros in a period.”

Let
\[ \mathcal{L}_n := \left\{ P : P(z) = \sum_{j=0}^{n} a_j z^j, \ a_j \in \{-1, 1\} \right\}. \]
Elements of $\mathcal{L}_n$ are often called Littlewood polynomials of degree $n$. Let

$$K_n := \left\{ P : P(z) = \sum_{j=0}^{n} a_j z^j, \ a_j \in \mathbb{C}, \ |a_0| = |a_n| = 1, \ |a_j| \leq 1 \right\} .$$

Observe that $\mathcal{L}_n \subset K_n$. In [10] we proved that any polynomial $P \in K_n$ has at least $8n^{1/2} \log n$ zeros in any open disk centered at a point on the unit circle with radius $33n^{-1/2} \log n$. Thus polynomials in $K_n$ have quite a few zeros near the unit circle. One may naturally ask how many unimodular roots a polynomial in $K_n$ can have. Mercer [36] proved that if a Littlewood polynomial $P \in L_n$ of the form (1.1) is skew reciprocal, that is, $a_j = (-1)^j a_{n-j}$ for each $j = 0, 1, \ldots, n$, then it has no zeros on the unit circle. However, by using different elementary methods it was observed in both [17] and [37] that if a Littlewood polynomial $P$ of the form (1.1) is self-reciprocal, that is, $a_j = a_{n-j}$ for each $j = 0, 1, \ldots, n$, $n \geq 1$, then it has at least one zero on the unit circle. Mukunda [37] improved this result by showing that every self-reciprocal Littlewood polynomial of odd degree has at least 3 zeros on the unit circle. Drungilas [15] proved that every self-reciprocal Littlewood polynomial of odd degree $n \geq 7$ has at least 5 zeros on the unit circle and every self-reciprocal Littlewood polynomial of even degree $n \geq 14$ has at least 4 zeros on the unit circle. In [4] two types of Littlewood polynomials are considered: Littlewood polynomials with one sign change in the sequence of coefficients and Littlewood polynomials with one negative coefficient, and the numbers of the zeros such Littlewood polynomials have on the unit circle and inside the unit disk, respectively, are investigated. Note that the Littlewood polynomials studied in [4] are very special. In [7] we proved that the average number of zeros of self-reciprocal Littlewood polynomials of degree $n$ is at least $n/4$. However, it is much harder to give decent lower bounds for the quantities

$$NZ_n := \min_P NZ(P),$$

where $NZ(P)$ denotes the number of zeros of a polynomial $P$ lying on the unit circle and the minimum is taken for all self-reciprocal Littlewood polynomials $P \in \mathcal{L}_n$. It has been conjectured for a long time that $\lim_{n \to \infty} NZ_n = \infty$. In [21] we showed that $\lim_{n \to \infty} NZ(P_n) = \infty$ whenever $P_n \in \mathcal{L}_n$ is self-reciprocal and $\lim_{n \to \infty} |P_n(1)| = \infty$. This follows as a consequence of a more general result, see Corollary 2.3 in [21], stated as Corollary 1.5 here, in which the coefficients of the self-reciprocal polynomials $P_n$ of degree at most $n$ belong to a fixed finite set of real numbers. In [7] we proved the following result.

**Theorem 1.1.** If the set $\{a_j : j \in \mathbb{N}\} \subset \mathbb{R}$ is finite, the set $\{j \in \mathbb{N} : a_j \neq 0\}$ is infinite, the sequence $(a_j)$ is not eventually periodic, and

$$T_n(t) = \sum_{j=0}^{n} a_j \cos(jt),$$

then $\lim_{n \to \infty} NZ(T_n) = \infty$. 

4
In [6] Theorem 1.1 is stated without the assumption that the sequence \((a_j)\) is not eventually periodic. However, as the following example shows, Lemma 3.4 in [7], dealing with the case of eventually periodic sequences \((a_j)\), is incorrect. Let

\[
T_n(t) := \cos t + \cos((4n + 1)t) + \sum_{k=0}^{n-1} (\cos((4k + 1)t) - \cos((4k + 3)t))
\]

\[
= \frac{1 + \cos((4n + 2)t)}{2 \cos t} + \cos t.
\]

It is easy to see that \(T_n(t) \neq 0\) on \([-\pi, \pi] \setminus \{-\pi/2, \pi/2\}\) and the zeros of \(T_n\) at \(-\pi/2\) and \(\pi/2\) are simple. Hence \(T_n\) has only two (simple) zeros in a period. So the conclusion of Theorem 1.1 above is false for the sequence \((a_j)\) with \(a_0 := 0, a_1 := 2, a_3 := -1, a_{2k} := 0, a_{4k+1} := 1, a_{4k+3} := -1\) for every \(k = 1, 2, \ldots\). Nevertheless, Theorem 1.1 can be saved even in the case of eventually periodic sequences \((a_j)\) if we assume that \(a_j \neq 0\) for all sufficiently large \(j\). See Lemma 3.11 in [22] where Theorem 1 in [7] is corrected as

**Theorem 1.2.** If the set \(\{a_j : j \in \mathbb{N}\} \subset \mathbb{R}\) is finite, \(a_j \neq 0\) for all sufficiently large \(j\), and

\[
T_n(t) = \sum_{j=0}^{n} a_j \cos(jt),
\]

then \(\lim_{n \to \infty} \text{NZ}(T_n) = \infty\).

It was expected that the conclusion of the above theorem remains true even if the coefficients of \(T_n\) do not come from the same sequence, that is,

\[
T_n(t) = \sum_{j=0}^{n} a_{j,n} \cos(jt),
\]

where the set

\[
S := \{a_{j,n} : j \in \{0, 1, \ldots, n\}, n \in \mathbb{N}\} \subset \mathbb{R}
\]

is finite and

\[
\lim_{n \to \infty} |\{j \in \{0, 1, \ldots, n\}, a_{j,n} \neq 0\}| = \infty.
\]

Associated with an algebraic polynomial

\[
P(z) = \sum_{j=0}^{n} p_j z^j, \quad p_j \in \mathbb{C},
\]

let

\[
\text{NC}_k(P) := |\{u : 0 \leq u \leq n - k + 1, a_u + a_{u+1} + \cdots + a_{u+k-1} \neq 0\}|.
\]

In [15] we proved the following results.
Theorem 1.3. If $S \subset \mathbb{R}$ is a finite set, $P_{2n} \in P_{2n}(S)$ are self-reciprocal polynomials,

$$T_n(t) := P_{2n}(e^{it})e^{-int},$$

and

$$\lim_{n \to \infty} NC_k(P_{2n}) = \infty$$

for every $k \in \mathbb{N}$, then

$$\lim_{n \to \infty} NZ(P_{2n}) = \lim_{n \to \infty} NZ(T_n) = \infty.$$

Some of the most important consequences of the above theorem obtained in [15] are stated below.

Corollary 1.4. If $S \subset \mathbb{R}$ is a finite set, $P_n \in P_n(S)$ are self-reciprocal polynomials, and

$$\lim_{n \to \infty} |P_n(1)| = \infty,$$

then

$$\lim_{n \to \infty} NZ(P_n) = \infty.$$

Corollary 1.5. Suppose the finite set $S \subset \mathbb{R}$ has the property that

$$s_1 + s_2 + \cdots + s_k = 0, \ s_1, s_2, \ldots, s_k \in S, \ \text{implies} \ s_1 = s_2 = \cdots = s_k = 0,$$

that is, any sum of nonzero elements of $S$ is different from 0. If $P_n \in P_n(S)$ are self-reciprocal polynomials and

$$\lim_{n \to \infty} NC(P_n) = \infty,$$

then

$$\lim_{n \to \infty} NZ(P_n) = \infty.$$

J. Sahasrabudhe [43] examined the case when $S \subset \mathbb{Z}$ is finite. Exploiting the assumption that the coefficients are integer he proved that for any finite set $S \subset \mathbb{Z}$ a self-reciprocal polynomial $P \in P_{2n}(S)$ has at least

$$c (\log \log \log |P(1)|)^{1/2-\varepsilon} - 1$$

zeros on the unit circle of $\mathbb{C}$ with a constant $c > 0$ depending only on $M = M(S) := \max\{|z| : z \in S\}$ and $\varepsilon > 0$.

Let $\phi(n)$ denote the Euler’s totient function defined as the number of integers $1 \leq k \leq n$ that are relative prime to $n$. In an earlier version of his paper Sahasrabudhe [43] used the trivial estimate $\phi(n) \neq \sqrt{n}$ for $n \geq 3$ and he proved his result with the exponent $1/4 - \varepsilon$ rather than $1/2 - \varepsilon$. Using the nontrivial estimate $\phi(n) \geq n/8(\log \log n)$ [40] for all $n > 3$ allowed him to prove his result with $1/2 - \varepsilon$.

In the papers [6], [21], [42] the already mentioned Littlewood Conjecture, proved by Konyagin [27] and independently by McGehee, Pigno, and B. Smith [35], plays a key role, and we rely on it heavily in the proof of the main results of this paper as well. This states the following.
Theorem 1.6. There is an absolute constant $c > 0$ such that
$$\int_0^{2\pi} \left| \sum_{j=1}^m a_j e^{i\lambda_j t} \right| dt \geq c \gamma \log m$$
whenever $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct integers and $a_1, a_2, \ldots, a_m$ are complex numbers of modulus at least $\gamma > 0$. Here $c = 1/30$ is a suitable choice.

This is an obvious consequence of the following result a book proof of which has been worked out by Lorentz and DeVore in [13, pages 285-288].

Theorem 1.7. If $\lambda_1 < \lambda_2 < \cdots < \lambda_m$ are integers and $a_1, a_2, \ldots, a_m$ are complex numbers, then
$$\int_0^{2\pi} \left| \sum_{j=1}^m a_j e^{i\lambda_j t} \right| dt \geq \frac{1}{30} \sum_{j=1}^m |a_j|.$$  

Associated with a finite set $\{A \subset \mathbb{C} \}$ we will use the notation $M(A) := \max \{|z| : z \in A\}$ throughout the paper.

2. New Results

Theorem 2.1. If $S \subset \mathbb{Z}$ is a finite set, $P \in \mathcal{P}_n(S)$ is a self-reciprocal polynomial,
$$T(t) := P(e^{it})e^{-int},$$
and $M = M(S) := \max\{|z| : z \in S\}$, then
$$\text{NZ}^*(T_n) \geq c (\log \log \log |P(1)|)^{1-\varepsilon} - 1$$
with a constant $c > 0$ depending only on $\varepsilon > 0$ and $M = M(S)$.

Corollary 2.2. If $S \subset \mathbb{Z}$ is a finite set, $P \in \mathcal{P}_n(S)$ is a self-reciprocal polynomial, and $M = M(S) := \max\{|z| : z \in S\}$, then
$$\text{NZ}(P) \geq c (\log \log \log |P(1)|)^{1-\varepsilon} - 1$$
with a constant $c > 0$ depending only on $\varepsilon > 0$ and $M = M(S)$.

This improves the exponent $1/2 - \varepsilon$ to $1 - \varepsilon$ in a recent breakthrough result [41] by Julian Sahasrabudhe. We note that in both Sahasrabudhe’s paper and this paper the assumption that the finite set $S$ contains only integers is deeply exploited. Our next result is an obvious consequence of Corollary 2.2.

Corollary 2.3. If the set $S \subset \mathbb{Z}$ is finite
$$T(t) = \sum_{j=0}^n a_j \cos(jt), \quad a_j \in S,$$
then
$$\text{NZ}^*(T) \geq c (\log \log \log |T(0)|)^{1-\varepsilon} - 1$$
with a constant $c > 0$ depending only on $\varepsilon > 0$ and $M = M(S)$.
3. Lemmas

Let $\mathcal{P}_n^c$ denote the set of all algebraic polynomials of degree at most $n$ with complex coefficients. Our first four lemmas are quite similar to some of the lemmas used in [21], but some modifications in the formulation of these lemmas and their proofs are needed. It will be convenient for us to define $\log 0 := -\infty$.

**Lemma 3.1.** If $S \subset \mathbb{C}$ is a finite set,

$$S_k := \{s_1 + s_2 + \cdots + s_k : s_j \in S \cup \{0\}\},$$

$$\gamma := \min_{z \in S_k \setminus \{0\}} |z|, \quad M := M(S) := \max\{|z| : z \in S\},$$

$P \in \mathcal{P}_{2n}(S)$, $H(z) := z^k - 1$, and

$$P \in \mathcal{P}_{2n}(S), \quad \text{then}$$

$$\int_{-\delta}^{\delta} |P(e^{it})| dt \geq \frac{\gamma}{30} \log(\text{NC}_k(P)) - \frac{\pi^2 \mu M}{\delta}$$

for every $\delta \in (0, \pi)$.

**Proof.** We define

$$G(z) := \sum_{j=0}^{k-1} z^j$$

so that $H(z) = G(z)(z - 1)$. As $P \in \mathcal{P}_{2n}(S)$ and the set $S$ is finite, the set $S^*$ is also finite. By Theorem 1.3 there is an absolute constant $c > 0$ such that

$$\int_0^{2\pi} |(PG)(e^{it})| dt \geq c \gamma \log(\text{NC}(PG)) \geq c_1 \gamma \log(\text{NC}_k(P)).$$

We define

$$M_k := M(S_k) = \max\{|z| : z \in S_k\} \leq k \max\{|z| : z \in S\} \leq kM.$$ 

Observe that

$$|(PG)(e^{it})| \leq \frac{1}{|e^{it} - 1|} |(PH)(e^{it})| \leq \frac{\mu M}{|e^{it} - 1|} \leq \frac{\mu M_k}{|2 \sin(t/2)|} \leq \frac{\mu M_k}{|2t|}, \quad t \in (-\pi, \pi).$$

It follows that

$$\int_{[-\pi, \pi] \setminus [-\delta, \delta]} |(PG)(e^{it})| dt \leq 2\pi \frac{\pi \mu M_k}{2\delta} = \frac{\pi^2 \mu kM}{\delta}.$$
Now (3.2) and (3.3) give
\[
\int_{-\delta}^{\delta} |P(e^{it})| \, dt \geq \frac{1}{k} \int_{-\delta}^{\delta} |(PG)(e^{it})| \, dt
\]
\[
= \frac{1}{k} \left( \int_{0}^{2\pi} |(PG(e^{it}))| \, dt - \int_{[-\pi,\pi] \setminus [-\delta,\delta]} |(PG)(e^{it})| \, dt \right)
\]
\[
\geq \frac{1}{k} c_{\gamma} \log(NC_k(P)) - \frac{\pi^2 \mu k M}{k\delta}
\]
\[
\geq \frac{1}{k} c_{\gamma} \log(NC_k(P)) - \frac{\pi^2 \mu M}{\delta}
\]

□

**Lemma 3.2.** If \( S \subset \mathbb{R} \) is a finite set, \( P \in \mathcal{P}_{2n}(S) \) is self-reciprocal, \( H(z) := z^k - 1 \),

(3.1) \[ NC(P_{2n}H) \leq \mu, \]

\( T(t) := P(e^{it})e^{-int} \), \( R(x) := \int_{0}^{x} T(t) \, dt \),

\( M := M(S) = \max\{|z| : z \in S\} \), and \( 0 < \delta \leq (2k)^{-1} \), then

\[ \max_{x \in [-\delta,\delta]} |R_n(x)| < 42k(\mu + 1)M. \]

**Proof.** Let

\[ T(t) = a_0 + \sum_{j=1}^{n} 2a_j \cos(jt), \quad a_j \in S. \]

Observe that (3.1) implies that

(3.4) \[ \left| \{ j : k \leq j \leq n, a_{j-k} \neq a_j \} \right| \leq \mu. \]

We have

\[ R(x) = a_0 x + \sum_{j=1}^{n} \frac{2a_j}{j} \sin(jx). \]

Now (3.4) implies that

\[ R_n(x) = a_0 x + \sum_{m=1}^{u} F_{m,k}(x), \]

where

\[ F_{m,k}(x) := \sum_{j=0}^{n} \frac{2A_{m,k} \sin((j_m + jk)x)}{j_m + jk} \]
with some $A_{m,k} \in S$, $m = 1, 2, \ldots, u$, $j_m \in \mathbb{N}$, and $n_m \in \mathbb{N}$, where $u \leq k\mu$ (we do not know much about $j_m$ and $n_m$). Since the set $S \subset [-M, M]$ is finite, it is sufficient to prove that

$$\max_{x \in [-\delta, \delta]} |F_{m,k}(x)| \leq 21, \quad m = 1, 2, \ldots, u,$$

that is, it is sufficient to prove that if

$$F(x) := \nu \sum_{j=0}^{\nu} \frac{\sin((j_0 + jk)x)}{j_0 + jk},$$

then

$$\max_{x \in [-\delta, \delta]} |F(x)| = \max_{x \in [0, \delta]} |F(x)| \leq 21,$$

Note that the equality in (3.5) holds as $F$ is odd. To prove the inequality in (3.5) let $x \in (0, \delta]$, where $0 < \delta \leq (2k)^{-1}$. We break the sum as

$$F = R + S,$$

where

$$R(x) := \sum_{j=0}^{\nu} \frac{\sin((j_0 + jk)x)}{j_0 + jk}$$

and

$$S(x) := \sum_{j=0}^{\nu} \frac{\sin((j_0 + jk)x)}{j_0 + jk}.$$

Here

$$|R(x)| \leq \sum_{j=0}^{\nu} \left| \frac{\sin((j_0 + jk)x)}{j_0 + jk} \right| \leq (x^{-1} + 1)|x|$$

$$\leq 1 + |x| \leq 1 + \delta = 1 + (2k)^{-1} \leq \frac{3}{2},$$

where each term in the sum in the middle is estimated by

$$\left| \frac{\sin((j_0 + jk)x)}{j_0 + jk} \right| \leq \left| \frac{(j_0 + jk)x}{j_0 + jk} \right| = |x|,$$

and the number of terms in the sum in the middle is clearly at most $x^{-1} + 1$. Further, using Abel rearrangement, we have

$$S(x) = -\frac{B_v(x)}{j_0 + vk} + \frac{B_u(x)}{j_0 + uk} + \sum_{j=0}^{\nu} B_j(x) \left( \frac{1}{j_0 + jk} - \frac{1}{j_0 + (j+1)k} \right)$$
with
\[ B_j(x) := B_{j,k}(x) := \sum_{h=0}^{j} \sin((j_0 + hk)x) \]
and with some \( u, v \in \mathbb{N}_0 \) for which \( x^{-1} < j_0 + (u + 1)k \) and \( x^{-1} < j_0 + (v + 1)k \). Hence,
\[
(3.8) \quad |S(x)| \leq \left| \frac{B_v(x)}{j_0 + vk} \right| + \left| \frac{B_u(x)}{j_0 + uk} \right| + \sum_{j=0}^{j_0} |B_j(x)| \left( \frac{1}{j_0 + jk} - \frac{1}{j_0 + (j + 1)k} \right).
\]
Observe that \( x \in (0, \delta] \), \( 0 < \delta \leq (2k)^{-1} \), \( x^{-1} < j_0 + (w + 1)k \), and \( w \in \mathbb{N}_0 \) imply
\[
x^{-1} < j_0 + (w + 1)k < 2(j_0 + wk) \quad \text{if} \quad w \geq 1,
\]
and
\[
2k \leq \delta^{-1} \leq x^{-1} < j_0 + k \quad \text{if} \quad w = 0,
\]
and hence
\[
(3.9) \quad \frac{1}{j_0 + wk} \leq 2x, \quad w \in \mathbb{N}_0.
\]
Observe also that \( x \in (0, \delta] \) and \( 0 < \delta \leq (2k)^{-1} \) imply that \( 0 < x < \pi k^{-1} \). Hence, with \( z = e^{ix} \) we have
\[
(3.10) \quad |B_j(x)| = \left| \frac{1}{2} \text{Im} \left( \sum_{h=0}^{j} z^{j_0 + hk} \right) \right| \leq \left| \frac{1}{2} \sum_{h=0}^{j} z^{j_0 + hk} \right| = \left| \frac{1}{2} \sum_{h=0}^{j} z^{hk} \right| \leq \left| \frac{1}{2} \frac{1 - z^{(j+1)k}}{1 - z^k} \right| \leq \frac{1}{2} \left| 1 - z^{(j+1)k} \right| \frac{1}{|1 - z^k|} \leq \frac{1}{1 - z^k} \leq \frac{1}{\sin(kx/2)} \leq \frac{\pi}{kx}.
\]
Combining (3.8), (3.9), and (3.10), we conclude
\[
(3.11) \quad |S(x)| \leq \frac{\pi}{kx} 2x + \frac{\pi}{kx} 2x + \frac{\pi}{kx} 2x \leq \frac{6\pi}{k}.
\]
Now (3.6), (3.7), and (3.11) give the inequality in (3.5) as \( 3/2 + 6\pi/k \leq 21. \) □

**Lemma 3.3.** If \( R \) is a continuously differentiable real-valued function on the interval \([-\delta, \delta], \delta > 0\),
\[
L := \int_{-\delta}^{\delta} |R'(x)| \, dx \quad \text{and} \quad N := \max_{x \in [-\delta, \delta]} |R(x)|,
\]
then there is an \( \eta \in [-N, N] \) such that \( R - \eta \) has at least \( L(2N)^{-1} \) distinct zeros in \([-\delta, \delta]\).
Lemma 3.4. If $S \subset \mathbb{R}$ is a finite set,

$$S_k := \{ s_1 + s_2 + \cdots + s_k : s_j \in S \cup \{0\} \} ,$$

$$\gamma := \min_{z \in S_k \setminus \{0\}} |z| , \quad M := M(S) := \max\{|z| : z \in S\} ,$$

$P \in P_{2n}(S)$ is self-reciprocal, $T(t) := P(e^{it})e^{-int}$, $H(z) := z^k - 1$, and



(3.1) \quad NC(PH) \leq \mu ,

then

$$NZ^*(T) \geq \left( \frac{\gamma}{30} \log(NC_k(P)) - 2k\pi^2 \mu M \right) \left( 84k(\mu + 1)M \right)^{-1} .$$

Proof. Let $0 < \delta := (2k)^{-1}$. Let $R$ be defined by

$$R(x) := \int_0^x T(t) \, dt .$$

Observe that $|T(x)| = |P(e^{ix})|$ for all $x \in \mathbb{R}$, and hence Lemma 3.1 yields that

$$\int_{-\delta}^\delta |R'(x)| \, dx = \int_{-\delta}^\delta |T(x)| \, dx = \int_{-\delta}^\delta |P(e^{ix})| \, dx$$

$$> \frac{\gamma}{30} \log(NC_k(P)) - \frac{\pi^2 \mu M}{\delta}$$

$$> \frac{\gamma}{30} \log(NC_k(P)) - 2k\pi^2 \mu M ,$$

while by Lemma 3.2 we have

$$\max_{x \in [-\delta, \delta]} |R(x)| < 42k(\mu + 1)M .$$

Therefore, by Lemma 3.3 there is an $\eta \in \mathbb{R}$ such that $R - \eta$ has at least

$$\left( \frac{\gamma}{30} \log(NC_k(P)) - 2k\pi^2 \mu M \right) \left( 84k(\mu + 1)M \right)^{-1} .$$

distinct zeros in $[-\delta, \delta]$. However, $T_n(x) = (R_n - c_n)'(x)$ for all $x \in \mathbb{R}$, and hence

$$NZ^*(T) \geq \left( \frac{\gamma}{30} \log(NC_k(P)) - 2k\pi^2 \mu M \right) \left( 84k(\mu + 1)M \right)^{-1}$$

follows by Rolle’s Theorem. \hfill \Box

The following lemma, in which the assumption $S \subset \mathbb{Z}$ is crucial, is simple to prove. It is stated as Lemma 8 in [42]. Its straightforward proof given in [42] is reduced to the fact that a determinant of integer entries is an integer, and hence if it is not 0, then its modulus is at least 1.
Lemma 3.5. For \( d \in \mathbb{N} \) let \( A \) be an \( d \times d \) invertible matrix with entries from \( S \subset \mathbb{Z} \). If \( Ax = b \) with
\[
\mathbf{x} = (x(1), x(2), \ldots, x(d)) \in \mathbb{C}^d \quad \text{and} \quad \mathbf{b} = (b(1), b(2), \ldots, b(d)) \in \mathbb{C}^d,
\]
then
\[
\max\{|x(1)|, |x(2)|, \ldots, |x(d)|\} \leq M^{d-1}d^{d/2} \max\{|b(1)|, |b(2)|, \ldots, |b(d)|\},
\]
where \( M := M(S) := \max\{|z| : z \in S\} \).

For integers \( 1 \leq D \leq N \) we call
\[
(x(1 + r), x(2 + r), \ldots, x(D + r)) \in \mathbb{C}^N, \quad r = 0, 1, \ldots, N - D,
\]
the \( D \)-tuples of
\[
(x(1), x(2), \ldots, x(N)) \in \mathbb{C}^N.
\]

The following lemma is Lemma 9 in [42].

Lemma 3.6. For \( u, v, D, t \in \mathbb{N} \), let \( S \subset \mathbb{Z} \) be a finite set such that \( v - u > |S|^D + 3D \), and let
\[
(x(u + 1), x(u + 2), \ldots, x(v)) \in S^{v - u}.
\]
Let \( V \) denote the linear space spanned by the \( D \)-tuples
\[
(x(r + 1), x(r + 2), \ldots, x(r + D)) \in S^D, \quad r = u, u + 1, \ldots, v - D,
\]
over \( \mathbb{R} \). If \( \dim(V) = t < D \), then there are
\[
(x_j(u + D), x_j(u + D + 1), \ldots, x_j(v - D)) \in \mathbb{C}^{v - u - 2D + 1}, \quad j = 1, 2, \ldots, t,
\]
such that
\[
x(r) = x_1(r) + x_2(r) + \cdots + x_t(r), \quad r \in [u + D, v - D],
\]
where
\[
(x_j(u + D), x_j(u + D + 1), \ldots, x_j(v - D)) \in \mathbb{C}^{v - u - 2D + 1}
\]
are periodic with period \( \alpha_j \leq 16t \log \log(t + 3) \) for each \( j = 1, 2, \ldots, t \).

Let \( f \) be a continuous, even, real-valued function on \( K := \mathbb{R} \) (mod \( 2\pi \)) which changes sign on \( (0, \pi) \) exactly at \( t_1 < t_2 < \cdots < t_d \). We define the companion polynomial \( Q \) of \( f \) by
\[
e^{-idt}P(e^{it}) := (-1)^p \prod_{j=1}^d (\cos t - \cos t_j),
\]
where \( p \in \{0, 1\} \) is chosen so that \( f(t)e^{-idt}P(e^{it}) \geq 0 \) for all \( t \in (-\pi, \pi) \). Observe that
\[
Q(z) := (-1)^p \prod_{j=1}^d (z - e^{it_j})(z - e^{-it_j})
\]
is a monic self-reciprocal algebraic polynomial of degree \( 2d \) with real coefficients and with constant term 1.

Associated with \( m \in \mathbb{N} \) let \( d_m := \text{LCM}(1, 2, \ldots, m) \). It is shown in [40] that \( d_m < 3^m \) holds for all \( m \in \mathbb{N} \), and this upper bound will be useful for us later in this paper. We remark though that \( \lim_{m \to \infty} d_m/e^m = 1 \) holds and it is equivalent to the Prime Number Theorem, see [47].
Lemma 3.7. Suppose $S \subset \mathbb{Z}$ is a finite set, $P \in P_{2n}(S)$ is self-reciprocal, and $T(t) := P(e^{it})e^{-int}$ has exactly $2d$ sign changes in $(-\pi, \pi)$. Let $Q$ be the companion polynomial of $T$ and let

$$F(z) := P(z)(z^{d_m} - 1)^2 Q(z) = \sum_{j=0}^{2n+2d_m+2d} a_j z^j,$$

where $m := \lfloor 32d \log \log(2d + 3) \rfloor$, and let

$$\{j \in [0, 2n + 2d_m + 2d] : a_j \neq 0\} = \{j_1 < j_2 < \cdots < j_q\}.$$

If for the integers $1 \leq u \leq v \leq q$

$$|a_{j_k}| < (4M)^{-2d}(2d + 1)^{-d-1/2}, \quad k \in [u, v],$$

where $M := M(S) := \max\{|z| : z \in S\}$, then

$$v - u < (|S| + 2)^{4m+2} + 6d + 3,$$

where $|S|$ denotes the number of elements in the set $S$.

Proof. Let

$$S^* := \{s_1 - s_2 : s_1, s_2 \in S \cup \{0\}\} \subset \mathbb{Z}$$

and

$$S^{**} := \{s_1 - s_2 : s_1, s_2 \in S^*\} \subset \mathbb{Z}.$$

Obviously

$$|S^*| \leq (|S| + 1)^2, \quad |S^{**}| \leq |S^*|^2 \leq (|S| + 1)^4,$$

and

$$M(S^{**}) \leq 2M(S^*) \leq 4M(S).$$

Put $D := 2d + 1$. Suppose to the contrary that

$$v - u \geq (|S| + 2)^{4m+2} + 3D.$$

Let

$$G(z) := P(z)(z^{d_m} - 1) := \sum_{j=0}^{2n+d_m} \alpha_j z^j, \quad \alpha_j \in S^*,$$

and

$$H(z) := G(z)(z^{d_m} - 1) = P(z)(z^{d_m} - 1)^2 = \sum_{j=0}^{2n+2d_m} \beta_j z^j,$$
where

\[(3.18) \quad \beta_j := \alpha_{j-d_m} - \alpha_j \in S^{**}, \quad j = 0, 1, \ldots, 2n + 2d_m,\]

with

\[\alpha_{j-d_m} := 0, \quad j = 0, 1, \ldots, d_m - 1.\]

Observe that \(H \in \mathcal{P}_{2n+d_m}(S^{**})\). Let \(V\) denote the linear space spanned by the \(D\)-tuples

\[(\beta_j, \beta_{j+1}, \ldots, \beta_{j+D-1}), \quad j \in [j_u, j_v - D + 1],\]

over \(\mathbb{R}\). Lemma 3.6 together with (3.13) and (3.15) implies that \(\dim(V) = t < D\). It follows from (3.13) and (3.15) that

\[j_v - j_u \geq v - u \geq (|S| + 2)^{4m+2} + 3D > |S^{**}|^m + 3D.\]

As \(t < D = 2d + 1\), we have \([16t \log \log(t + 3)] \leq [32d \log \log(2d + 3)] = m\). Applying Lemma 3.7 we obtain that

\[(\beta_{j_u + D}, \beta_{j_u + D + 1}, \ldots, \beta_{j_v - D + 1})\]

is periodic with period \(d_m\), that is,

\[(3.19) \quad \beta_{r+d_m} = \beta_r, \quad r \in [j_u + D, j_v - d_m - D].\]

We claim that

\[(3.20) \quad \beta_r = 0, \quad r \in [j_u + D, j_v - D - |S^*|d_m].\]

Indeed, if \(\beta_r \neq 0\) for some \(r \in [j_u + D, j_v - D - |S^*|d_m]\), then (3.18) and (3.19) give

\[\alpha_{r+hd_m} = \alpha_r + \sum_{j=1}^{h} (\alpha_{r+jd_m} - \alpha_{r+(j-1)d_m}) = \alpha_r - \sum_{j=1}^{h} \beta_{r+jd_m}\]

\[= \alpha_r - h\beta_r \in S^*, \quad h = 0, 1, \ldots, |S^*|,\]

exhibiting \(|S^*| + 1\) distinct elements of \(S^*\), which is impossible. It follows from (3.12), (3.17), and (3.20) that

\[a_j = 0, \quad j \in [j_u + 2D, j_v - D - |S^*|d_m],\]

hence, recalling \(d_m < 3^m\) and (3.14), we obtain

\[v - u \leq |S^*|d_m + 3D < |S^*|3^m + 3D < (|S| + 1)^2(|S| + 2)^{4m} + 3D\]

\[\leq (|S| + 2)^{4m+2} + 3D,\]

which contradicts (3.16). In conclusion

\[v - u < (|S| + 2)^{4m+2} + 3D = (|S| + 2)^{4m+2} + 6d + 3.\]

\(\square\)
**Lemma 3.8.** Under the assumptions of Lemma 3.7 we have
\[
\log q \leq 60(4M)^{2d+1}(2d + 1)^{d+3/2}(|S| + 2)^{4m+2} + 6d + 3.
\]

**Proof.** Let \( L := (|S| + 2)^{4m+2} + 6d + 3 \), and \( r := \lfloor q/L \rfloor \). Observe that
\[
P(e^{it})e^{-int}(e^{id_m t} - 1)^2e^{-id_m t}Q(e^{it})e^{-idt}
\]
is real and nonnegative for all \( t \in \mathbb{R} \). Combining this with Theorem 1.7 and Lemma 3.7 we obtain
\[
\int_0^{2\pi} P(e^{it})e^{-int}(e^{id_m t} - 1)^2e^{-id_m t}Q(e^{it})e^{-idt} dt
\]
\[
= \int_0^{2\pi} |P_{2n}(e^{it})(e^{id_m t} - 1)^2Q(e^{it})| dt
\]
\[
\geq \frac{1}{30} \sum_{k=1}^{q} \frac{|a_{jk}|}{k} \geq \frac{1}{30} \sum_{j=1}^{r} \sum_{k=(j-1)L+1}^{jL} \frac{|a_{jk}|}{k}
\]
\[
\geq \frac{1}{30} (4M)^{-2d}(2d + 1)^{-d-1/2} \sum_{j=1}^{r} \frac{1}{jL}
\]
\[
\geq \frac{1}{30} (4M)^{-2d}(2d + 1)^{-d-1/2}L^{-1} \log r
\]
On the other hand, using orthogonality, we have
\[
\int_0^{2\pi} P(e^{it})e^{-int}(e^{id_m t} - 1)^2e^{-id_m t}Q(e^{it})e^{-idt} dt \leq 4M(2d + 1).
\]
Combining (3.21) and (3.22) we conclude
\[
\frac{1}{30} (4M)^{-2d}(2d + 1)^{-d-1/2}L^{-1} \log r \leq 4M(2d + 1),
\]
and hence
\[
\log q = \log r + \log L \leq 60(4M)^{2d+1}(2d + 1)^{d+3/2}L
\]
\[
\leq 60(4M)^{2d+1}(2d + 1)^{d+3/2}(|S| + 2)^{4m+2} + 6d + 3.
\]
□

Our final lemma follows easily from Lemma 3.6.

**Lemma 3.9.** If \( S \subset \mathbb{Z} \) is a finite set, \( P \in \mathcal{P}_{2n}(S) \), \( 0 \neq R \in \mathcal{P}_u^c \),
\[
\text{NC}(PR) \leq \nu,
\]
\( \nu := [16u \log \log (u + 3)] \), \( k := d_u \), and \( H(z) = z^k - 1 \), then
\[
\text{NC}(PH) \leq \mu := (\nu + 1)(k + |S|^{u+1} + 3(u + 1) + 2).
\]
4. Proof of the Theorems

Proof of Theorem 2.1. Suppose $S \subseteq \mathbb{Z}$ is a finite set, $P \in \mathcal{P}_{2n}(S)$ is self-reciprocal, and $T(t) := P(e^{it})e^{-int}$ has exactly $2d$ sign changes in $(-\pi, \pi)$. Let $Q$ be the companion polynomial of $T$. Let

$$F(z) := P(z)(z^{d_m} - 1)^2 Q(z) = \sum_{j=0}^{2n+2d_m+2d} a_j z^j,$$

where $m := \lceil 32d \log \log(2d + 3) \rceil$. Let

$$\{ j \in [0, 2n + 2d_m + 2d] : a_j \neq 0 \} =: \{ j_1 < j_2 < \cdots < j_q \}.$$  

Lemma 3.8 together with $|S| + 2 \leq 2M(S) + 3 = 2M + 3 \leq 5M$ and $4m + 2d + 3 \leq 5m$ implies

(4.1) \[ \log q \leq 60(5M)^5(2d + 1)^{d + 3/2}, \]

and hence applying Lemma 3.9 with $u := 2d_m + 2d$, we have

(4.2) \[ \text{NC}(PH) \leq \mu := (q + 1)(d_v + |S|^{u+1} + 3(u + 1) + 2). \]

with $v := [16u \log \log(u + 3)]$, $k := d_v$, and $H(z) = z^k - 1$. Observe that if $S \subseteq \mathbb{Z}$ and

$$S_k := \{ s_1 + s_2 + \cdots + s_k : s_j \in S \cup \{ 0 \} \},$$

then

$$1 \leq \gamma := \min \{ |z| : z \in S_k \setminus \{ 0 \} \}.$$

Lemma 3.4 gives

$$84k(\mu + 1)kM(2d) + 2k\pi^2 \mu M \geq \log(\text{NC}_k(P)).$$

Combining this with $m := \lceil 32d \log \log(2d + 3) \rceil$, $u := 2d_m + 2d$, $v := [16u \log \log(u + 3)]$, $k = d_v < 3^v$, $d_m < 3^m$, (4.1), (4.2), $|S| \leq 2M(S) + 1 = 2M + 1$, and the inequality $a + b \leq ab$ valid for all $a \geq 1$ and $b \geq 1$, we obtain

$$\log(84k(\mu + 1)M2d) + 2k\pi^2 \mu M) \leq \log(84k(\mu + 1)M(2d + 1),$$

$$\leq 6 + \log k + \log \nu + \log M + \log(2d + 1)$$

$$\leq 6 + \log k + \log(q + 1) + \log d_v + \log(|S|^{u+1} + \log(3u + 5)) + \log M + \log(2d + 1)$$

$$\leq 6 + (\log 3)v + \log 2 + 60(5M)^5(2d + 1)^{d + 3/2} + (\log 3)v + (u + 1)\log(2M + 1)$$

$$+ \log(3u + 5) + \log M + \log(2d + 1)$$

$$\leq 7 + 2(\log 3)16u \log \log(u + 3) + 60(5M)^5(2d + 1)^{d + 3/2} + (2 \cdot 3^m + 2d + 1) \log(2M + 1)$$

$$+ (\log 2 + (\log 3)m) \log(2d + 1) \log(2M + 1) + \log M + \log(2d + 1)$$

17
and hence
\[
\log(84k(\mu + 1)M2d) + 2k\pi^2\mu M \\
\leq 7 + 2(\log 3)16(2 \cdot 3^m + 2d) \log \log(2 \cdot 3^m + 2d) + 60(5M)^{5m}(2d + 1)^{d+3/2} + \\
(2 \cdot 3^m + 2d + 1) \log(2M + 1) + (\log 2 + (\log 3)m) \log(2d + 1) \log(2M + 1) \\
+ \log M + \log(2d + 1).
\]

and it follows that there is an absolute constant \(c_1 > 0\) such that
\[
(4.3) \quad c_1((d + 1) \log(d + 2) + (d + 1) \log(2d + 3)(1 + \log M)) \geq \log \log(\text{NC}_k(P)).
\]

It is easy to see that
\[
\text{NC}_k(P) \geq \frac{k|P(1)| - 2k^2M}{kM} = \frac{|P(1)|}{M} - k.
\]

Therefore if \(|P(1)| \geq 2kM\), then \(\text{NC}_k(P) \geq \frac{1}{2} |P(1)|\), and the theorem follows from (4.3) after a straightforward calculus. If \(|P(1)| < 2kM\), then it follows from (4.3) that
\[
\log \log(|P(1)|) < \log \log(2kM) \leq c_2 \log(d + 2) \leq c_2(2d + 1)
\]

with an absolute constant \(c_2 > 0\), and the theorem follows. \(\square\)

**Proof of Corollary 2.2.** Let \(S \subset \mathbb{Z}\) be a finite set. If \(P \in \mathcal{P}_{2\nu}(S)\) is self-reciprocal, then the corollary follows from Theorem 2.1. If \(P \in \mathcal{P}_{2\nu+1}(S)\) is self-reciprocal, then \(\widetilde{P} \in \mathcal{P}_{2\nu+2}(S^*)\) defined by
\[
\widetilde{P}(z) := (z + 1)P(z) \in \mathcal{P}_{2\nu+2}(S^*)
\]
is also self-reciprocal, where the fact that \(S \subset \mathbb{Z}\) is finite implies that the set
\[
S^* := \{s_1 + s_2 : s_1, s_2 \in S \cup \{0\}\} \subset \mathbb{Z},
\]
is also finite. Observe also that
\[
M(S^*) = \max\{|z| : z \in S^*\} = 2 \max\{|z| : z \in S\} = 2M(S)
\]
and
\[
\widetilde{P}(1) = 2P(1).
\]

Hence applying Theorem 2.1 to \(\widetilde{P} \in \mathcal{P}_{2\nu+2}(S^*)\), we obtain the statement of the corollary for \(P \in \mathcal{P}_{2\nu+1}(S)\) from Theorem 2.1 again. \(\square\)

**Proof of Corollary 2.3.** The corollary follows from Theorem 2.1 and the fact that for every trigonometric polynomial \(T\) of the form
\[
T(t) := a_0 + \sum_{j=1}^{n} a_j \cos(jt), \quad a_j \in \mathbb{Z},
\]
there is a self-reciprocal algebraic polynomial \(P\) of the form
\[
P(z) = 2a_0 z^n + \sum_{j=1}^{n} a_j (z^{n+j} + z^{n-j}), \quad a_j \in \mathbb{Z},
\]
such that
\[
2T(t) := P(e^{it})e^{-int}.
\]
\(\square\)
1. V.V. Andrievskii and H-P. Blatt, *Discrepancy of Signed Measures and Polynomial Approximation*, Springer, New York, 2002.

2. A. Bloch and G. Pólya,, *On the roots of certain algebraic equations*, Proc. London Math. Soc. 33 (1932), 102–114.

3. P. Borwein, *Computational Excursions in Analysis and Number Theory*, Springer, New York, 2002.

4. P. Borwein, S. Choi, R. Ferguson, and J. Jankauskas, *On Littlewood polynomials with prescribed number of zeros inside the unit disk*, Canad. J. of Math. 67 (2015), 507–526.

5. P. Borwein and T. Erdélyi, *On the zeros of polynomials with restricted coefficients*, Illinois J. Math. 41 (1997), no. 4, 667–675.

6. P. Borwein and T. Erdélyi, *Lower bounds for the number of zeros of cosine polynomials in the period: a problem of Littlewood*, Acta Arith. 128 (2007), no. 4, 377–384.

7. P. Borwein, T. Erdélyi, R. Ferguson, and R. Lockhart, *On the zeros of cosine polynomials: solution to a problem of Littlewood*, Ann. Math. Ann. (2) 167 (2008), no. 3, 1109–1117.

8. P. Borwein, T. Erdélyi, and G. Kós, *Littlewood-type problems on [0, 1]*, Proc. London Math. Soc. 79 (1999), 22–46.

9. P. Borwein, T. Erdélyi, and G. Kós, *The multiplicity of the zero at 1 of polynomials with constrained coefficients*, Acta Arith. 159 (2013), no. 4, 387–395.

10. P. Borwein, T. Erdélyi, and F. Littmann, *Zeros of polynomials with finitely many different coefficients*, Trans. Amer. Math. Soc. 360 (2008), 5145–5154.

11. D. Boyd, *On a problem of Byrne’s concerning polynomials with restricted coefficients*, Math. Comput. 66 (1997), 1697–1703.

12. B. Conrey, A. Granville, B. Poonen, and K. Soundararajan, *Zeros of Fekete polynomials*, Ann. Inst. Fourier (Grenoble) 50 (2000), 865–889.

13. R.A. DeVore and G.G. Lorentz, *Constructive Approximation*, Springer-Verlag, Berlin, 1993.

14. Y. Do, H. Nguyen, and V. Vu, *Real roots of random polynomials: expectation and repulsion* (to appear).

15. P. Drungilas, *Unimodular roots of reciprocal Littlewood polynomials*, J. Korean Math. Soc. 45 (2008), no. 3, 835–840.

16. A. Edelman and E. Kostlan, *How many zeros of a random polynomial are real?*, Bull. Amer. Math. Soc. (N.S.) 32 (1995), 1–37; *Erratum*, Bull. Amer. Math. Soc. (N.S.) 33 (1996), 325.

17. T. Erdélyi, *On the zeros of polynomials with Littlewood-type coefficient constraints*, Michigan Math. J. 49 (2001), 97–111.

18. T. Erdélyi, *An improvement of the Erdős-Turán theorem on the distribution of zeros of polynomials*, C. R. Acad. Sci. Paris, Ser. I 346 (2008), no. 5, 267–270.

19. T. Erdélyi, *Extensions of the Bloch-Pólya theorem on the number of real zeros of polynomial*, J. Théor. Nombres Bordeaux 20 (2008), no. 2, 281-287.

20. T. Erdélyi, *Coppersmith-Rivlin type inequalities and the order of vanishing of polynomials at 1*, Acta Arith. 172 (2016), no. 3, 271284.

21. T. Erdélyi, *The number of unimodular zeros of self-reciprocal polynomials with coefficients in a finite set*, Acta Arith. 176 (2016), no. 2, 177–200.
22. P. Erdős and A. C. Offord, *On the number of real roots of a random algebraic equation*, Proc. London Math. Soc. **6** (1956), 139–160.

23. P. Erdős and P. Turán, *On the distribution of roots of polynomials*, Ann. Math. **57** (1950), 105–119.

24. D. Hanson, *On the product of the primes*, Canad.Math. Bull. **15** (1972), 33–37.

25. M. Kac, *On the average number of real roots of a random algebraic equation*, Bull. Amer. Math. Soc. **49** (1943), 314–320.

26. M. Kac, *On the average number of real roots of a random algebraic equation. II*, Proc. London Math. Soc. **50** (1949), 390–408.

27. S.V. Konyagin, *On a problem of Littlewood*, Mathematics of the USSR, Izvestia **18** (1981), 205–225.

28. S.V. Konyagin and V.F. Lev, *Character sums in complex half planes*, J. Theor. Nombres Bordeaux **16** (2004), no. 3, 587–606.

29. J.E. Littlewood, *On the mean values of certain trigonometrical polynomials*, J. London Math. Soc. **36** (1961), 307–334.

30. J.E. Littlewood, *On the real roots of real trigonometrical polynomials (II)*, J. London Math. Soc. **39** (1964), 511–552.

31. J.E. Littlewood, *On polynomials \( \sum \pm z^m \) and \( \sum e^{\alpha m} z^m \), \( z = e^{\theta i} \)*, J. London Math. Soc. **41** (1966), 367–376.

32. J.E. Littlewood, *Some Problems in Real and Complex Analysis*, Heath Mathematical Monographs, Lexington, Massachusetts, 1968.

33. J.E. Littlewood and A.C. Offord, *On the number of real roots of a random algebraic equation. II*, Proc. Cambridge Philos. Soc. **35** (1939), 133–148.

34. J.E. Littlewood and A.C. Offord, *On the number of real roots of a random algebraic equation. III*, Rec. Math. [Mat. Sbornik] N.S. **54** (1943), 277–286.

35. O.C. McGehee, L. Pigno, and B. Smith, *Hardy’s inequality and the \( L_1 \) norm of exponential sums*, Ann. Math. **113** (1981), 613–618.

36. I.D. Mercer, *Unimodular roots of special Littlewood polynomials*, Canad. Math. Bull. **49** (2006), no. 3, 438–447.

37. K. Mukunda,, *Littlewood Pisot numbers*, J. Number Theory **117** (2006), no. 1, 106–121.

38. H. Nguyen, O. Nguyen, and V. Vu, *On the number of real roots of random polynomials* (to appear).

39. I.E. Pritsker and A.A. Sola, *Expected discrepancy for zeros of random algebraic polynomials*, Proc. Amer. Math. Soc. **142** (2014), 4251-4263.

40. J.B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962), 64–94.

41. E. Schmidt, *Über algebraische Gleichungen vom Pólya-Bloch-Typus*, Sitz. Preuss. Akad. Wiss., Phys.-Math. Kl. (1932), 321.

42. I. Schur, *Untersuchungen über algebraische Gleichungen*, Sitz. Preuss. Akad. Wiss., Phys.-Math. Kl. (1933), 403–428.

43. J. Sahasrabudhe, *Counting zeros of cosine polynomials: on a problem of Littlewood* (to appear arXiv:1610.07680 [math.CA]).
44. B. Solomyak, *On the random series* $\sum \pm \lambda^n$ (*an Erdős problem*), Ann. Math. **142** (1995), 611–625.

45. G. Szegő, *Bemerkungen zu einem Satz von E. Schmidt über algebraische Gleichungen*, Sitz. Preuss. Akad. Wiss., Phys.-Math. Kl. (1934), 86–98.

46. T. Tao and V. Vu, *Local universality of zeros of random polynomials*, IMRN (2015) (to appear).

47. G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge University Press, Cambridge, England, 1995.

48. V. Totik and P. Varjú, *Polynomials with prescribed zeros and small norm*, Acta Sci. Math. (Szeged) **73** (2007), 593–612.

Department of Mathematics, Texas A&M University, College Station, Texas 77843, College Station, Texas 77843 (T. Erdélyi)