MOTION OF CHARGED PARTICLES IN HOMOGENEOUS FIBRATIONS

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Abstract. Let \((M = G/H, g)\) be a reductive homogeneous Riemannian manifold, where \(g\) is a \(G\)-invariant metric and let \(T_o M\) be the tangent space of \(M\) at \(o = eH\). We study the differential equation \(\nabla \dot{x} = kI(\dot{x})\) (\(k \in \mathbb{R}\) and \(I\) an endomorphism of \(T_o M\)), whose solution \(x(t)\) represents the motion of a charged particle in \(M\) under the electromagnetic field \(kI\). If \(k = 0\) then \(x(t)\) is a geodesic. We solve such an equation in a Riemannian fibration \(K/H \rightarrow G/H \rightarrow G/K\), where \(G\) is a Lie group with a bi-invariant Riemannian metric and \(H \subset K \subset G\) are closed connected subgroups of \(G\). To this end, we prove a more general result which has its own interest. Namely, we study the motion of a charged particle in a homogeneous space \(G/H\) with reductive decomposition \(g = h \oplus m\) (\(g\), \(h\) the Lie algebras of \(G\) and \(H\) respectively), under the following conditions: (i) the tangent space \(T_o M \cong m\) admits a decomposition \(m = m_1 \oplus \cdots \oplus m_s\) into \(\text{Ad}(H)\)-modules, orthogonal with respect to an \(\text{Ad}\)-invariant inner product of \(g\) and the \(G\)-invariant metric \(g\) is diagonal on \(m\), (ii) there exist subspaces \(m_a, m_b \subset m\) (\(a, b = 1, \ldots, s\)) such that \([m_a, m_b] \subset m_a\) and (iii) the restriction of the endomorphism \(I\) into \(m_a \oplus m_b\) is determined by an element in the center of \(h\).

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1. Introduction and statement of results

Let \((M = G/H, g)\) be a homogeneous Riemannian manifold, where \(g\) is a \(G\)-invariant metric with corresponding Levi-Civita connection \(\nabla\). The aim of the present paper is the study of the "charged particle" differential equation

\[\nabla \dot{x} = kI(\dot{x}),\]

where \(k\) is some real constant and \(I\) an endomorphism of \(T_o M\), the tangent space of \(M\) at \(o = eH\). Equation (1) appears in a more general context in general relativity as follows ([9]).

Let \((M, g)\) be a Riemannian manifold, \(F\) a closed 2-form, and \(X\) a vector field on \(M\). We denote by \(\iota_X : \Lambda^p(M) \rightarrow \Lambda^{p-1}(M)\) the interior product operator induced by \(X\), and by \(L : TM \rightarrow T^*M\) the Legendre transformation defined by \(u \mapsto L(u), L(u)(v) = g(u, v)\) \((v \in TM)\). A curve \(x(t)\) in \(M\) is called a motion of a charged particle under electromagnetic field \(F\) if it satisfies the differential equation

\[\nabla \dot{x} = -L^{-1}(\iota_X F),\]

where \(\nabla\) is the Levi-Civita connection of \(M\). When \(F = 0\) then \(x(t)\) is a geodesic in \(M\). In particular, if \(M\) is a Kähler manifold with complex structure \(J\) there is a natural choice of an electromagnetic field \(F\), namely a scalar multiple of the Kähler form \(\omega\), defined by \(\omega(X, Y) = g(X, JY)\). Since \(-L^{-1}(\iota_X \omega) = JX\), a curve \(x(t)\) is a motion of a charged particle
under electromagnetic field $k\omega$ if and only if $\nabla_\omega \dot{x} = k J(\dot{x})$. We also refer to [7] and [8] for other relevant applications in physics.

Differential equation (1) has been studied by O. Ikawa for various homogeneous spaces (cf. [3], [4], [5], [6]). A class of homogeneous spaces considered in [5] were generalized flag manifolds (or Kähler C-spaces) with two isotropy summands. These spaces were classified by the first author and I. Chrysikos in [1], hence obtaining a concrete class of homogeneous spaces where equation (1) can be solved.

In the present article we solve differential equation (1) for a large class of homogeneous spaces, which are described as follows.

Let $M = G/H$ be a homogeneous space with reductive decomposition $g = h \oplus m$ with respect to an Ad-invariant inner product $B$ of $g$ and assume that the Lie algebra $h$ has non trivial center $z(h)$. The tangent space $T_o M$ at $o = eH$ can be identified to $m$. Let $\pi : G \to G/H$ be the projection and for $p \in G$, let $\tau_p : G/H \to G/H$ be the left translation in $G/H$ by $p$. We assume that the following conditions are satisfied:

(i) The tangent space $m$ admits an Ad($H$)-invariant and $B$-orthogonal decomposition

\[ m = m_1 \oplus \cdots \oplus m_s. \]

(ii) There exist subspaces $m_a, m_b$, ($a, b = 1, \ldots, s$) of $m$ such that

\[ [m_a, m_b] \subset m_a. \]

(iii) For $W \in z(h)$ we define the endomorphism $I_o : m \to m$ such that

\[ I_o|_{m_a \oplus m_b} = \text{ad}(W)|_{m_a} + \frac{1}{\lambda} \text{ad}(W)|_{m_b}, \quad \lambda = \frac{\lambda_b}{\lambda_a}. \]

Since $\text{Ad}(h) I_o = I_o \text{Ad}(h)$ for all $h \in H$, we can extend $I_o$ to a $G$-invariant (1,1)-tensor $I$ on $G/H$ by defining

\[ I_{\pi(p)} V_{\pi(p)} := ((\tau_p)_* \circ I_o \circ (\tau_{p^{-1}})_*) V_{\pi(p)}, \quad V_{\pi(p)} \in T_{\pi(p)}(G/H). \]

We denote a homogeneous space satisfying conditions (i) – (iii) by $(G/H, g, I, \lambda)$.

**Definition 1.1.** For $k$ a real constant a curve $x(t)$ is called a motion of a charged particle under the electromagnetic field $kI$ if it is a solution of the differential equation

\[ \nabla_\omega \dot{x} = k J(\dot{x}). \]

Note that if $k = 0$ then $x(t)$ is a geodesic.

We prove the following:
Theorem 1.2. Let \((G/H, g, I, \lambda)\) be a Riemannian homogeneous space satisfying conditions (i), (ii) and (iii). Let \(x(t)\) be the motion of a charged particle given by (7) with initial conditions

\[
\begin{align*}
x(0) &= o \quad \text{and} \quad \dot{x}(0) = X_a + X_b, \\
\end{align*}
\]

where \(X_a \in m_a, X_b \in m_b\). Then the curve \(x(t)\) is given by

\[
\begin{align*}
x(t) &= \exp t(X_a + \lambda X_b + kW) \exp t(1 - \lambda)(X_b + \frac{k}{\lambda}W) \cdot o.
\end{align*}
\]

The above Theorem generalizes Ikawa’s result.

As a consequence, we obtain the following description of corresponding motion in homogeneous fibrations:

Theorem 1.3. Let \(G\) be a Lie group admitting a bi-invariant Riemannian metric and let \(B\) be the corresponding \(\text{Ad}\)-invariant positive definite inner product on \(g\). Let \(K, H\) be closed and connected subgroups of \(G\), such that \(H \subset K \subset G\) and such that the Lie algebra of \(H\) has non trivial center. We identify the tangent spaces \(T_o(G/H), T_o(G/K)\) and \(T_o(K/H)\) with corresponding subspaces \(m, m_1\) and \(m_2\) of \(g\), such that \(m = m_1 \oplus m_2\). We endow \(G/H\) with a \(G\)-invariant Riemannian metric \(g_\lambda\) corresponding to the \(\text{Ad}(H)\)-invariant positive definite inner product

\[
\langle \cdot, \cdot \rangle = B|_{m_1} + \lambda B|_{m_2}, \quad \lambda > 0
\]

on \(m\). Moreover, for \(W \in \mathfrak{z}(\mathfrak{h})\), let \(I^W\) be the \(G\)-invariant \((1,1)\)-tensor on \(G/H\) such that \(I^W_o = \text{ad}(W)|_{m_1} + \frac{1}{\lambda} \text{ad}(W)|_{m_2}\). Let \(X = X_1 + X_2 \in m\) with \(X_i \in m_i, i = 1, 2\). Then the motion of a charged particle in \(G/H\) under electromagnetic field \(kI\) with initial conditions \(x(0) = o\) and \(\dot{x}(0) = X_1 + X_2\) is the curve \(x : \mathbb{R} \rightarrow G/H\) given by

\[
x(t) = \exp t(X_1 + \lambda X_2 + kW) \exp t(1 - \lambda)(X_2 + \frac{k}{\lambda}W) \cdot o.
\]

We will prove Theorems 1.2 and 1.3 in Section 2. In Section 3 we will give examples of homogeneous spaces satisfying the conditions of Theorem 1.3.

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2. Proof of the main results

We need to show that

\[
g(\nabla_{\dot{x}} \ddot{x}, V) = g(kI(\dot{x}), V),
\]

for any vector field \(V\) in \(G/H\). By using Koszul’s formula the left-hand side of (11) is given by

\[
g(V, \nabla_{\dot{x}} \ddot{x}) = \dot{x} g(V, \dot{x}) + g(\dot{x}, [V, \dot{x}]) - \frac{1}{2} V g(\dot{x}, \dot{x}).
\]
We set \( X = X_a + \lambda X_b + kW, \) \( Y = (1 - \lambda)(X_b + \frac{k}{\lambda}W), \) and \( \alpha : \mathbb{R} \to G \) with \( \alpha(t) = \exp tX \exp tY \) so \( x = \pi \circ \alpha. \) We also consider the one-parameter family of automorphisms
\[
T : \mathbb{R} \to \text{Aut}(G) \quad \text{with} \quad T(t) = \text{Ad}(\exp(-tY)).
\] (13)

We will need the following:

**Lemma 2.1.** The following relations are satisfied:

1) \( T(t)X_a \in m_a, \ t \in \mathbb{R}, \)

2) \( \dot{x}(t) = (\tau_{\alpha(t)})_*(T(t)X_a + X_b). \)

**Proof.** To prove 1) we recall that
\[
T(t)X_a = \sum_{n=0}^{\infty} \frac{t^n}{n!} \text{ad}^n(-Y)X_a.
\] (14)

Moreover, by taking into account relation (4) and the \( \text{ad}(h) \)-invariance of the subspace \( m_a \) we deduce that \( [Y, m_a] \subset m_a. \) Therefore, for any \( N \in \mathbb{N}, \) it is \( \sum_{n=0}^{N} \sum_{\alpha} \text{ad}^n(-Y)X_a \in m_a, \) so by taking the limit \( N \to \infty \) in (14) we obtain that \( T(t)X_a \in m_a. \)

We now prove 2). For \( p \in G, \) let \( L_p, R_p \) be the left and right translations respectively in \( G \) by \( p. \) We compute:

\[
\dot{x}(t) = \pi_*(\dot{\alpha}(t)) = \pi_*(\frac{d}{ds}\bigg|_{s=0} \alpha(t + s)) = \pi_*(\frac{d}{ds}\bigg|_{s=0} \exp(t + s)X \exp(t + s)Y)
\]

\[
= \pi_*(\frac{d}{ds}\bigg|_{s=0} \exp(t + s)X \exp tY + \frac{d}{ds}\bigg|_{s=0} \exp tX \exp(t + s)Y)
\]

\[
= \pi_*(\frac{d}{ds}\bigg|_{s=0} \exp sX \exp tX \exp tY + \frac{d}{ds}\bigg|_{s=0} \exp tX \exp tY \exp sY)
\]

\[
= \pi_*(\frac{d}{ds}\bigg|_{s=0} \exp sX \alpha(t) + \frac{d}{ds}\bigg|_{s=0} \alpha(t) \exp sY) = \pi_*((\tau_{\alpha(t)})_*X + (L_{\alpha(t)})_*Y)
\]

\[
= (\pi \circ L_{\alpha(t)} \circ L_{\alpha(t)^{-1}})_*((R_{\alpha(t)})_*(X) + (L_{\alpha(t)})_*Y) = (\pi \circ L_{\alpha(t)})_*(\text{Ad}(\alpha(t)^{-1})X + Y)
\]

\[
= (\tau_{\alpha(t)} \circ \pi)_*(\text{Ad}(\alpha(t)^{-1})X + Y) = (\tau_{\alpha(t)} \circ \pi)_*(\text{Ad}(\exp(-tY) \exp(-tX))X + Y)
\]

\[
= (\tau_{\alpha(t)} \circ \pi)_*(\text{Ad}(\exp(-tY))X_a + \text{Ad}(\exp(-tY))(\lambda X_b + kW + Y)
\]

\[
= (\tau_{\alpha(t)} \circ \pi)_*(\text{Ad}(\exp(-tY))X_a + \text{Ad}(\exp(-tY))(\frac{\lambda}{1-\lambda}Y + Y)
\]

\[
= (\tau_{\alpha(t)} \circ \pi)_*(\text{Ad}(\exp(-tY))X_a + \frac{\lambda}{1-\lambda}Y + Y) = (\tau_{\alpha(t)} \circ \pi)_*(\text{Ad}(\exp(-tY))X_a + X_b + \frac{k}{\lambda}W)
\]

\[
= (\tau_{\alpha(t)})_*\pi_*(T(t)X_a + X_b) = (\tau_{\alpha(t)})_*((T(t)X_a + X_b).
\]

\[\square\]

For any \( p \in G \) we observe that the vector field
\[
\pi(p) \mapsto (\tau_p \circ \pi)_*(\text{Ad}(p^{-1})X + Y),
\] (15)
is a well defined local extension of $\dot{x}(t)$ in $G/H$, therefore, the vector field $\nabla_{\dot{x}}\dot{x}$ is well defined. Moreover, in view of equation (12), since $T_{\pi(p)}(G/K)$ admits a basis of left invariant vectors, it suffices to take $V$ as

$$V_{\pi(p)} = (\tau_p)_*Z, \quad Z \in \mathfrak{m}.$$  \hspace{1cm} (16)

**Proof of Theorem 1.2**

We will first simplify each term of the right-hand side of equation (12). By using Lemma 2.1, equations (16) and (3) as well as the $G$-invariance of the metric $g$, the first term of the right-hand side of (12) becomes

$$\dot{x}(t)g(V, \dot{x}) = \left| \frac{d}{ds} \right|_{s=0} g((\tau_{\alpha(t+s)})_*Z, (\tau_{\alpha(t+s)})_*(T(t+s)X_a + X_b))$$

$$= \left| \frac{d}{ds} \right|_{s=0} \langle Z, T(t+s)X_a + X_b \rangle = \left| \frac{d}{ds} \right|_{s=0} \langle Z, T(s)T(t)X_a + X_b \rangle$$

$$= \langle Z, [T(t)X_a, Y] \rangle = (1 - \lambda)\langle Z, [T(t)X_a, X_b + \frac{k}{\lambda}W] \rangle$$

$$= (\lambda_a - \lambda_b)B(Z, [T(t)X_a, X_b + \frac{k}{\lambda}W]).$$  \hspace{1cm} (17)

Similarly, and by using the $B$-orthogonality of $\mathfrak{h}$ and $\mathfrak{m}$ as well as the Ad-invariance of $B$, the second term of the right-hand side of (12) becomes

$$g(\dot{x}(t), [V, \dot{x}]_{\pi(t)}) = g((\tau_{\alpha(t)})_*[T(t)X_a + X_b], [(\tau_{\alpha(t)})_*Z, (\tau_{\alpha(t)})_*(T(t)X_a + X_b)]_m)$$

$$= g((\tau_{\alpha(t)})_*[T(t)X_a + X_b], (\tau_{\alpha(t)})_*[Z, T(t)X_a + X_b]_m)$$

$$= \langle T(t)X_a + X_b, [Z, T(t)X_a + X_b]_m \rangle$$

$$= \lambda_aB(T(t)X_a, [Z, T(t)X_a]_m) + \lambda_aB(T(t)X_a, [Z, X_b]_m)$$

$$+ \lambda_aB(X_b, [Z, T(t)X_a]_m) + \lambda_bB(X_b, [Z, X_b]_m)$$

$$= \lambda_aB(T(t)X_a, [Z, T(t)X_a]) + \lambda_aB(T(t)X_a, [Z, X_b])$$

$$+ \lambda_bB(X_b, [Z, T(t)X_a]) + \lambda_bB(X_b, [Z, X_b])$$

$$= (\lambda_a - \lambda_b)B(Z, [T(t)X_a, X_b]),$$  \hspace{1cm} (18)

where for the last equality we used the Ad-invariance of $B$.

Finally, for the third term of the right hand side of equation (12), we use the local extension (13) of $\dot{x}$ as well as (16). We have that

$$V_{\pi(t)}g(\dot{x}, \dot{x}) = (\tau_{\alpha(t)})_*Z)g(\dot{x}, \dot{x})$$

$$= \left| \frac{d}{ds} \right|_{s=0} g((\tau_{p(-1)} \circ \pi_*)(Ad(p^{-1})X + Y), ((\tau_{p(-1)} \circ \pi_*)(Ad(p^{-1})X + Y))$$

$$= \langle \pi_*(Ad(p^{-1})X + Y), \pi_*(Ad(p^{-1})X + Y) \rangle,$$  \hspace{1cm} (19)
where \( p = \alpha(t) \exp sZ \). Moreover, we have that

\[
\frac{d}{ds} \bigg|_{s=0} \pi_*(\text{Ad}(p^{-1})X) = \frac{d}{ds} \bigg|_{s=0} \pi_*(\text{Ad}(\exp(-sZ))T(t)X)
\]

\[
= \pi_* \left( \frac{d}{ds} \bigg|_{s=0} (\text{Ad}(\exp(-sZ))T(t)X) \right)
\]

\[
= [T(t)X, Z]_m = [T(t)X_a + \lambda X_b + \frac{k}{\lambda} W, Z]_m. \quad (20)
\]

By substituting equation (20) into equation (19) and by using Lemma 2.1 and the Ad-invariance of \( B \) we obtain that

\[
\frac{1}{2} V_{x(t)} g(\dot{x}, \dot{x}) = -\langle [T(t)X_a + \lambda X_b + \frac{k}{\lambda} W, Z]_m, T(t)X_a + X_b \rangle
\]

\[
= -\langle [T(t)X_a, Z]_m, T(t)X_a \rangle - \langle [T(t)X_a, Z]_m, X_b \rangle - \langle [X_b, Z]_m, T(t)X_a \rangle
\]

\[
- \lambda \langle [X_b, Z]_m, X_b \rangle - \frac{k}{\lambda} \langle [W, Z]_m, T(t)X_a \rangle - \frac{k}{\lambda} \lambda B([W, Z]_m, X_b)
\]

\[
= -\lambda_a B([T(t)X_a, Z], T(t)X_a) - \lambda_b B([T(t)X_a, Z], X_b) - \lambda \lambda_a B([X_b, Z]_m, T(t)X_a)
\]

\[
- \frac{k}{\lambda} \lambda_a B([W, Z], T(t)X_a) - \frac{k}{\lambda} \lambda B([W, Z], X_b)
\]

\[
= -\frac{k}{\lambda} \lambda_a B([W, Z], T(t)X_a) - \frac{k}{\lambda} \lambda_b B([W, Z], X_b)
\]

\[
= -\frac{k}{\lambda} \lambda_a B(Z, [T(t)X_a, W]) - \frac{k}{\lambda} \lambda_b B(Z, [X_b, W]). \quad (21)
\]

By substituting equations (17), (18) and (21) in equation (12) and by adding these we obtain that

\[
g(V, \nabla_{\dot{x}} \dot{x}) = -k \lambda_a B(Z, [TX_a + X_b, W]). \quad (22)
\]

Next, we simplify the right-hand side of equation (11). By using relations (3), (5) and (6), Lemma 2.1, the G-invariance of \( g \) and the Ad-invariance of \( B \), we obtain that

\[
g(kI(\dot{x}), V) = kg(((\tau_{\alpha(t)})_*) \circ I_0)(T(t)X_a + X_b), (\tau_{\alpha(t)})_* Z)
\]

\[
= k \langle I_0(T(t)X_a + X_b), Z \rangle = k \langle [W, T(t)X_a], Z \rangle + \frac{k}{\lambda} \langle [W, X_b], Z \rangle
\]

\[
= k \lambda_a B([W, T(t)X_a], Z) + \frac{k}{\lambda} \lambda_b B([W, X_b], Z)
\]

\[
= -k \lambda_a B(Z, [TX_a + X_b, W]),
\]

which, by equation (22) is equal to \( g(V, \nabla_{\dot{x}} \dot{x}) \). Therefore, relation (11) holds and the proof of Theorem 1.3 is completed. \qed

Proof of Theorem 1.3
Let \( h, \mathfrak{k}, \mathfrak{g} \) be the Lie algebras of the groups \( H, K, G \) respectively. The subspaces \( m_1 \) and \( m_2 \) can be obtained from the \( B \)-orthogonal decompositions

\[
\mathfrak{g} = \mathfrak{k} \oplus m_1 \quad \text{and} \quad \mathfrak{k} = \mathfrak{h} \oplus m_2,
\]

(23)

such that

\[
\text{Ad}(K)m_1 \subset m_1, \quad \text{Ad}(H)m_2 \subset m_2.
\]

(24)

Therefore, the decomposition \( m = m_1 \oplus m_2 \) is \( \text{Ad}(H) \)-invariant and \( B \)-orthogonal. Moreover, since \( \text{Ad}(K)m_1 \subset m_1 \), we have that \( [m_1, \mathfrak{k}] \subset m_1 \), therefore,

\[
[m_1, m_2] \subset [m_1, \mathfrak{t}] \subset m_1.
\]

(25)

By taking into account relation (25), it is straightforward to check that the space \( (G/H, g_\lambda, I^W, \lambda) \) satisfies the conditions of Theorem 1.2, and this completes the proof.

3. Examples

3.1. Lie groups. Let \( G \) be a Lie group admitting a bi-invariant Riemannian metric \( g \). Let \( K \) be a connected subgroup of \( G \) and let \( \mathfrak{g}, \mathfrak{k} \) be the Lie algebras of \( G, K \) respectively. The bi-invariant metric \( g \) corresponds to an \( \text{Ad} \)-invariant positive definite inner product \( B \) on \( \mathfrak{g} \), which induces an orthogonal decomposition \( \mathfrak{g} = \mathfrak{k} \oplus m \). This decomposition is \( \text{Ad}(K) \)-invariant. Indeed, for any \( X_k, Y_k \in \mathfrak{k} \) and \( X_m \in m \), we set \( Z_k = [X_k, Y_k] \in \mathfrak{k} \). Then

\[
B([X_k, X_m], Y_k) = -B(X_m, [X_k, Y_k]) = -B(X_m, Z_k) = 0.
\]

It follows that \( [X_k, X_m] \in m \), and since \( K \) is connected then \( \text{Ad}(K)m \subset m \), which in turn gives that \( \mathfrak{g} = \mathfrak{k} \oplus m \).

We view \( G \) as the homogeneous space \( G/\{e\} \) and we endow \( G \) with the left invariant metric \( g_\lambda \) corresponding to the positive definite inner product

\[
\langle \cdot, \cdot \rangle = B|_m + \lambda B|_\mathfrak{t}, \quad \lambda > 0.
\]

(26)

By taking \( m_1 = m \) and \( m_2 = \mathfrak{t} \) in Theorem 1.3 then we obtain the motion of a charged particle in a Lie group \( G \).

3.2. Ikawa’s result. Let \( M = G/H \) be a homogeneous space satisfying the conditions of Theorem 1.2 with \( m = m_1 \oplus m_2 \). Then we obtain Ikawa’s result [5, Theorem 1.1] under the only condition \( [m_1, m_2] \subset m_1 \).

3.3. Hopf bundles. Let \( G = U(n+1), H = U(n) \) and \( K = U(n) \times U(1) \). Then the fibration \( K/H \to G/H \to G/K \) is the homogeneous Hopf bundle

\[
\mathbb{S}^1 \to \mathbb{S}^{2n+1} \to \mathbb{C}P^n.
\]

(27)

Since \( U(n+1) \) is compact, it admits a bi-invariant metric corresponding to an \( \text{Ad}(U(n+1)) \)-invariant positive definite inner product \( B \) on \( u(n+1) \). We identify each of the spaces \( T_p\mathbb{S}^{2n+1} = T_p(G/H), T_p\mathbb{C}P^n = T_p(G/K) \), and \( T_p\mathbb{S}^1 = T_p(K/H) \) with corresponding subspaces \( \mathfrak{m}, m_1, \) and \( m_2 \) of \( u(n+1) \). Consider the one parameter family of metrics \( g_\lambda \) on \( \mathbb{S}^{2n+1} \) corresponding to the positive definite inner products.
\( \langle , \rangle = B|_{m_1} + \lambda B|_{m_2}, \quad \lambda > 0 \) \hspace{1cm} (28)

on \( m = m_1 \oplus m_2 \). Note that for \( \lambda = 1 \) the inner product (28) gives the standard metric \( g_1 \) on \( S^{2n+1} \). Then the hypotheses of Theorem 1.3 are satisfied, hence the curve (10) describes the motion of a charged particle in \((S^{2n+1}, g_\lambda)\). However, it is known ([Ya]) that \( G/H = U(n+1)/U(n) \) is a weakly symmetric space, hence in this case it can be shown that motion (10) reduces to \( x(t) = \exp(t(a + X_1 + X_2 + kW)) \cdot o, \ X_i \in m_i \ (i = 1, 2) \) and \( a \in \mathfrak{h} \), which depends on \( X_1, X_2 \).

3.4. **Twistor fibrations.** For \( G \) semisimple let \( F = G/H \) be a generalised flag manifold, i.e. an adjoint orbit of an element \( W \) in \( \mathfrak{g} \). It is known ([2]) that any flag manifold \( F = G/H \) can be fibered over a compact inner symmetric space \( G/K \) \((H \subset K)\) under the twistor fibration \( \pi : G/H \to G/K \). The normal metric of \( G/H \) is the \( G \)-invariant metric induced by the negative of the Killing form \( \mathfrak{g} \), denoted by \( B \). We endow \( G/H \) with the “deformation” of the normal metric along the fibers \( K/H \) of the twistor fibration, given by

\[ \langle , \rangle = B|_{T_o(G/K)} + \lambda B|_{T_o(K/H)}, \quad \lambda > 0. \]

Then, by virtue of Theorem 1.3, the equation of motion of a charged particle in \((F, \langle , \rangle)\) under electromagnetic field \( kI \), is given by (10).

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