New Directions for Primality Test

Lakshmi Prabha S\textsuperscript{a} \• T.N.Janakiraman\textsuperscript{b}
Department of Mathematics, National Institute of Technology, Trichy-620015, Tamil Nadu, India.
Emails: \textsuperscript{a}jaislp111@gmail.com, \textsuperscript{b}janaki@nitt.edu

Abstract In this paper, two approximation algorithms are given. Let $N$ be an odd composite number. The algorithms give new directions regarding primality test of given $N$. The first algorithm is given using a new method called digital coding method. It is conjectured that the algorithm finds a divisor of $N$ in at most $O(\ln^4 N)$, where $\ln$ denotes the logarithm with respect to base 2. The algorithm can be applied to find the next largest Mersenne prime number. Some directions are given regarding this. The second algorithm finds a prime divisor of $N$ using the concept of graph pairs and it is proved that the time complexity of the second algorithm is at most $O(\ln^2 N)$ for infinitely many cases (for approximately large $N$). The advantages and disadvantages of the second algorithm are also analyzed.

Keywords Factorization \• primality \• congruences \• graph pairs \• digital coding \• approximation algorithm \• time complexity \• Mersenne primes.

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1 Introduction

First, let us give some basic definitions of number theory from Apostol \cite{Apostol}, which are used in this paper.

We say that an integer $d$ divides an integer $n$ and write $d|n$ whenever $n = cd$ for some integer $c$. If $d$ divides two integers $a$ and $b$, then $d$ is called a common divisor of $a$ and $b$.

Definition 1 G.C.D. of two numbers Let $a$ and $b$ be two numbers both not zero. Then the Greatest Common Divisor (G.C.D.) of $a$ and $b$ is a number $d$ such that

\begin{itemize}
  \item $d$ is a common divisor of $a$ and $b$, and
  \item every common divisor of $a$ and $b$ divides this $d$.
\end{itemize}

Let us denote it by $gcd(a, b)$. If $gcd(a, b) = 1$, then $a$ and $b$ are said to be relatively prime.
**Definition 2** An integer \( N \) is called prime if \( N > 1 \) and if the only positive divisors of \( N \) are 1 and \( N \). If \( N > 1 \) and \( N \) is not prime, then \( N \) is called composite.

**Definition 3** Given integers \( a, b, m \) with \( m > 0 \). Then \( a \) is said to be congruent to \( b \) modulo \( m \), if \( m \) divides the difference \((a - b)\). This is denoted by \( a \equiv b \pmod{m} \). The number \( m \) is called the modulus of the congruence.

We list some of the following basic theorems in number theory from Apostol [1], which are used for our results.

**Properties:**

**Theorem 1** If \( a \equiv b \pmod{m} \) and \( \alpha \equiv \beta \pmod{m} \), then \( a\alpha \equiv b\beta \pmod{m} \).

**Theorem 2** \( ac \equiv bc \pmod{mc} \) iff \( a \equiv b \pmod{m} \).

**Theorem 3** **Fermat’s Theorem:** If a prime \( p \) does not divide ‘\( a \)’, then \( a^{(p-1)} \equiv 1 \pmod{p} \).

For more details on basics of number theory, readers are directed to refer Apostol [1].

The remaining part of the paper is organized as follows:

- Section 2 discusses about prior work.
- Section 3 contains an interesting approximation algorithm, which is given using a new approach. The intricate portions of the algorithm are discussed and a conjecture is given regarding the time complexity of the algorithm.
- In section 4, a simple approximation algorithm is given using the concept of graph pairs. Time complexity and correctness of that algorithm are given. The advantages and disadvantages of that algorithm are also discussed in this section. Further a procedure is given using that algorithm for primality test and an open problem is given regarding its time complexity.

## 2 Prior Work

A primality test is an algorithm for determining whether an input number \( N \) is prime. It has a lot of applications in cryptography and network security. For centuries, number theory was considered to be the most pure form of Mathematics, because there were no practical applications, as far as anyone could tell. However, in the latter half of the 20th century, number theory became central to developments in digital security, for example, public key cryptography, credit card check digits and so on. If one is capable to quickly factor an integer into a product of two large primes and verify that they were both primes, then he would be able to break into most banking systems.

Most popular primality tests are probabilistic tests. These tests use, apart from the tested number \( N \), some other numbers \( a \) which are chosen at random from some sample space; The simplest is Fermat Primality test. Miller- Rabin primality test and Solovay and Strassens primality test are also probabilistic
tests. Miller [2] in 1975, Rabin [3] in 1980 and Solovay and Strassen [4] in 1974, gave randomized algorithms. Their method can be made deterministic under the assumption of Extended Riemann Hypothesis (ERH). Since then, a number of randomized polynomial-time algorithms have been proposed for primality testing, based on many different properties.

In 1983, Adleman, Pomerance, and Rumely [5] achieved a major breakthrough by giving a deterministic algorithm for primality that runs in \((\log N)^{O(\log \log \log N)}\) time (all the previous deterministic algorithms required exponential time).

Then, in 1986, Goldwasser and Kilian [6] and Atkin [7] also gave randomized algorithms, based on elliptic curves. In 1992, Adleman and Huang [8] modified the Goldwasser-Kilian method and presented an errorless (but expected polynomial-time) variant of the elliptic curve primality test. But it is also a randomized algorithm that runs in expected polynomial-time on all inputs.

In 2002, the first provably polynomial time test for primality was invented by Agrawal, Kayal and Saxena [9]. They proved that AKS primality test, runs in \(O((\log)^7 N)\) and which can be further reduced to \(O((\log)^5 N)\) if the Sophie Germain conjecture is true.

The aim of this paper is to reduce this time complexity. We have given two approximation (randomized) algorithms. They do not deterministically distinguish whether the given number is prime or not in polynomial time. But the second algorithm gives good results for infinitely many cases, (not for all inputs). It is proved that our second algorithm finds a prime divisor of \(N\) in at most \(O((\log)^4 N)\) for infinitely many cases (for approximately large \(N\)).

### 3 Digital Coding Algorithm

In this section, we give an interesting approximation algorithm, the procedure of which is new and different. The algorithm finds a divisor of \(N\). We have employed binary and decimal equivalent of the numbers. It is observed that the time complexity of the algorithm is at most \(O((\log)^4 N)\). The correctness of the algorithm is verified (manually) for \(N \leq 10^7\). So, we pose the conjecture that the algorithm finds a divisor of \(N\), in at most \(O((\log)^4 N)\), for any odd composite \(N\). Also, we pose some other conjectures supporting this conjecture.

Some applications of the algorithm are also mentioned, mainly, there are higher possibilities to find the next largest Mersenne prime number using this method.

**Note 1** In this section (alone), the notation ‘\(|\)’ does not denote ‘divides’ symbol. The notation ‘\(|\)’ denotes the partition of a number (in this section alone). For example, 4|5 does not mean 4 divides 5. But it means that the number 45 is partitioned digit-wise.

**Definition 4 Digital Binary Equivalent** A binary number \(B\) is said to be digital binary equivalent of the decimal number \(A\), if \(B\) is obtained from \(A\) by
finding binary equivalents of each digit of \( A \) (starting from left and ending at right) and then taking union of those binary equivalents. For example, consider \( A = 872 \). First, let us find binary equivalent of each digit of \( A \). Binary equivalent of 8 is 1000; 7 is 111 and that of 2 is 10. Next, we take union of these binary numbers and write it as: 100011110. So, digital binary equivalent of 872 is 100011110. Let us represent this process in the following form: \( 8 \{2 \} \rightarrow \{1000\} \{111\} \{10\} \).

3.1 Notation

The following notation is used throughout this section:

- Let \( N = n_1n_2 \ldots n_r \), where \( n_i \) is the digit of the number \( N \) and \( r \) is the total number of digits of \( N \).
- \( B_j \) denotes a binary number. It is obtained by finding digital binary equivalent of a decimal number \( N \) or \( D_j \).
- \( D_j \) denotes a decimal number. It is obtained by finding decimal equivalent of the binary number \( B_j \).
- \( A \rightarrow B \) denotes that the number \( A \) is in decimal form, the number \( B \) is in binary form and \( B \) is obtained from \( A \) by finding decimal equivalent of \( A \).
- \( A \rightarrow B \) denotes that the number \( A \) is in decimal form, the number \( B \) is in binary form and \( B \) is obtained from \( A \) by finding digital binary equivalent of \( A \).
- \( A \rightarrow B \) denotes that the number \( A \) is in decimal form, the number \( B \) is in binary form. \( B \) is the digital binary equivalent of \( A \) with some extra zeros, that is, first we find the digital binary equivalent of \( A \), say \( A_1 \) and then zeros are inserted in between the binary equivalents of some digits of \( A \) (in order to get result). This digital binary equivalent with extra zeros is the result \( B \).
- \( A \rightarrow B \) denotes that the numbers \( A \) and \( B \) are in binary form and \( B \) is obtained by inserting zeros in between the digits of \( A \) (in order to get result).

3.2 APPROXIMATION ALGORITHM

In this section, an approximation algorithm is given using digital binary coding method, to find a divisor of odd composite \( N \).

Input: \( N \).
Output: “\( N \) is composite” and \( gcd(D_j, N) \), which is a divisor of \( N \), if \( N \) is composite. No output is produced if \( N \) is prime.

Algorithm 1:

\[ JRLP(N) \]
1. Initially let \( j = 0 \).

\[ \text{DBC}(N) \]

2. Let \( N = n_1n_2\ldots n_r \), where \( n_i \) is the digit of the number \( N \) and \( r \) is the total number of digits of \( N \).

3. Find the digital binary equivalent of \( N \).

4. Put \( j = j + 1 \) and store the digital binary number in \( B_j \).

5. Find the decimal equivalent of \( B_j \) and store it in \( D_j \).

6. If \( \gcd(D_j, N) \neq 1 \), then print “\( N \) is composite” and print \( \gcd(D_j, N) \).

7. else if (the number of digits of \( D_j \) \( r \neq 1 \) or the number \( D_j \) does not occur in any of the previous steps, call \( \text{DBC}(D_j) \).

8. else GOTO next step. //At this stage, either \( r \) becomes 1 or a number \( D_j \) is repeated, but still \( \gcd(D_j, N) = 1 \). Initially, we have obtained a sequence of binary and decimal equivalents starting from \( N \). Let us call this initial sequence as the first chain of \( N \). Next, we backtrack and find next chain of \( N \).

9. Backtrack one step before in the first chain of \( N \) and insert extra zeros in between the binary equivalents of \( n_i \) (refer example 2) and then GOTO step 4.

10. Repeat backtracking (step 9) until we reach the starting point \( N \).

11. If the process of inserting zeros and the remaining entire process (from steps 2 to 7) is over for \( N \), then stop.// At this last stage, \( \gcd(D_j, N) = 1 \).

**Why the name Digital Coding?**

We name this method as “DIGITAL CODING METHOD”, because we give binary coding for each digit of the decimal number, not for the entire decimal number.

### 3.3 EXPLANATION WITH ILLUSTRATIONS

**Example 1:** Take \( N = 88837 \).

\[ 8|8|8|3|7 \xrightarrow{\text{dig}-\text{bin}} 1000|000|1000|111|111 \] (Digital coding of 88837)

Now, \( B_1 = 10001000100011111 \) \( \text{decimal} \) \( 69919 = D_1 \) (Decimal equivalent of \( B_1 \)). As, \( \gcd(D_1, N) = 1 \), we continue this procedure. The step by step executions of the whole process of Algorithm 1 is given in the Table 1.

| \( j \) | \( D_{j-1} \) | \( B_j \) | \( D_j \) | \( \gcd(D_j, N) \) |
|---|---|---|---|---|
| 1 | 88837 = \( N \) | 10001000100011111 | 69919 | 1 |
| 2 | 69919 | 110100110011001 | 54073 | 1 |
| 3 | 54073 | 10110001111 | 2847 | 1 |
| 4 | 2847 | 101000100111 | 2599 | 1 |
| 5 | 2599 | 1010110011001 | 5529 | 1 |
| 6 | 5529 | 101101101001 | 2921 | 1 |
| 7 | 2921 | 101001101 | 333 | 37 |

Table 1 Step by Step Executions of Algorithm 1
Output: “N is composite” and \( \gcd(D_j, N) = 37 \).

**Example 2:** Take \( N = 15 \).

\[
\begin{align*}
&1|5 \xrightarrow{\text{dig-bin}} 1|01 \
&\text{decimal} \rightarrow 1|3 \xrightarrow{\text{dig-bin}} 1|11 \xrightarrow{\text{decimal}} 7.
\end{align*}
\]

This sequence is called the **first chain** of \( N \).

(13 is the decimal equivalent of 1101. Then 13 is digitally coded. 7 is the decimal equivalent of 111).

**Backtrack 1:** \( 1|3 \xrightarrow{\text{dig-bin, zero}} 1|011 \xrightarrow{\text{decimal}} 11. \)

This sequence is called the **second chain** of \( N \). (Originally the digital coding for the number 13 is 111, that is, \( 1|3 \xrightarrow{\text{dig-bin}} 1|11 \). Now, we have inserted zeros in between the two digital codings, that is, between 1 and 11 (or in front of 11)).

Next we backtrack in the backtrack 1.

**Backtrack 2:** Consider \( 1|1 \xrightarrow{\text{zero}} 101 \xrightarrow{\text{decimal}} 5. \)

This sequence is called the **third chain** of \( N \) and we get \( \gcd(5, 15) = 5. \)

### 3.4 NP-COMPLETENESS

The process of inserting zeros is NP-Complete.

**Explanation:** There is no proper rule in inserting zeros. In the algorithm, first we do the procedure without inserting any extra zeros. If result does not come, we do backtracking and then insert zeros. Even at the time of backtracking also, there are three possibilities.

1. We have to insert zeros at some stage and need not insert zeros at some other stage so that we get correct result.
2. Consider one particular stage (only one number). We may have to insert zeros for some digit of a number and need not insert zeros for some other digit of the number so that we get correct result.
3. We can insert zeros uniformly for all digits at a particular stage and uniformly for all numbers so that we get correct result.

So, we do not uniformly insert zeros. The obvious question is among these possibilities, which has to be used. The questions are:

- Where to insert zeros?
- When to insert zeros? and
- How to insert zeros?

Suppose we assume that we insert zeros with a rule.

**Rule:** **Equally expanding:** Add zeros to make digits same.

For example, if \( N = 51 \), then \( B_1 \) is 1011, that is, \( 5|1 \xrightarrow{\text{dig-bin}} 101|1. \) Here 101 is of three digits and 1 is of 1 digit. We can code 1 as 01, 001, 00001 and so on. But, we make 1 to be of three digits and hence we code 1 as 001. Now, 51 can be coded as 101001, that is, \( 5|1 \xrightarrow{\text{dig-bin, zero}} 101|001. \) This is called **equally expanding method**.
The surprising thing is sometimes the result occurs in few steps, but un-
knowingly, we would have tried many steps.
For example, consider \( N = 451 \).

\[
\begin{align*}
4 & \rightarrow \text{dig-bin} \rightarrow 5 \rightarrow \text{decimal} \rightarrow 7 \\
3 & \rightarrow \text{dig-bin} \rightarrow 1 \rightarrow \text{decimal} \rightarrow 1
\end{align*}
\]

**Backtrack:**

\[
13 \rightarrow \text{dig-bin, zero} \rightarrow 101 \rightarrow \text{decimal} \rightarrow 11
\]

and we get \( gcd(451, 11) = 11 \). This process takes 5 iterations.

Now, let us try using the equally expanding rule.

\[
\begin{align*}
4 & \rightarrow \text{dig-bin} \rightarrow 5 \rightarrow \text{decimal} \rightarrow 7 \\
3 & \rightarrow \text{dig-bin, zero} \rightarrow 100 \rightarrow \text{decimal} \rightarrow 297
\end{align*}
\]

and we get \( gcd(451, 297) = 11 \). This process takes only 1 iteration.

So, there is no uniformity in this method. This leads to the NP-Completeness of the process. But we observed that if \( N \) is composite, then we will surely get the divisor of \( N \). So, we say that the probability of getting a divisor of odd composite \( N \) using digital coding method is always 1. That is,

\[
P(\text{getting divisor of odd composite } N \text{ using digital coding method}) = 1
\]

As we do the backtracking step until we get result, this method of back-
tracking and inserting zeros is equivalent to trial and error method, unless there is a proper rule to do it. But there is a hidden rule to achieve it easily.

### 3.5 Time Complexity

It is already mentioned in the algorithm that at the end of step 8, one chain of \( N \) will be formed, i.e., from steps 2 to 8, one chain will be formed. One iteration of the algorithm is defined as executing steps 2 to 6 one time. The time complexity of the algorithm lies in two main steps:

1. Finding \( gcd(D_j, N) \)
2. How many times \( gcd(D_j, N) \) is found in the algorithm (Number of iterations of the algorithm) and
3. The number of backtracking steps.

We find \( gcd(D_j, N) \) at each iteration of the algorithm. Each computation of G.C.D. takes \( O(\ln N) \) time \[10\]. Next, we do not know what is the number of iterations, that is, number of times \( gcd(D_j, N) \) is found out in the algorithm.

Also, we do not know what is the number of backtracking steps with a proper rule of inserting zeros, in the algorithm.

While verifying (manually) the algorithm for \( N \leq 10^5 \), we found that the total time complexity of the algorithm is at most \( O(\ln^2 N) \). We checked for some higher cases of \( N \). We observed that in some cases, it directly gives result (without inserting zeros) within \( O(\ln^4 N) \) and in some cases, extra zeros have to be inserted and if zeros are properly inserted, it gives result within \( O(\ln^4 N) \). So, we give the following conjectures.

**Conjecture 1:** JRLP’s Conjecture:
The total time complexity of the Algorithm 1 is at most $O(\ln^4 N)$. This should be supported by the following statement:

The number of iterations and the number of backtracking steps with a proper rule of inserting zeros in the algorithm take at most $O(\ln^4 N)$.

CONJECTURE 2: $P($getting a divisor of odd composite $N$ using digital coding method$) = 1$.

3.6 APPLICATIONS

The primes of the form $M_p = 2^p - 1$, where $p$ is a prime, are called Mersenne primes. Let us call $p$ to be the generator prime of $M_p$.

We can generate Mersenne primes using this concept as follows: We applied JRLP algorithm (without inserting zeros) in all the generator primes, $p$, of the existing Mersenne primes $M_p$. We observed that the first chain (sequence) of the generator primes, $p$, is terminated with either 11 or 7 or 47 or 5 or 9. So, Mersenne primes $M_p$ can be partitioned into five groups, based on the terminating number of the first chain of the generator prime $p$. We observed that the first chain of the generators of recent Mersenne primes are terminated with 11 and also there are many such Mersenne primes. So, we start form 11, do the converse process of the Algorithm 1 and try to get some generator primes and hence the Mersenne primes. Similarly, we can start from any of the above mentioned numbers and try to get the generator primes and hence the Mersenne primes. The following are some examples, illustrating how the first chain of the generator primes terminated with 11 or 7 or 47 or 5 or 9.

- $8|9 \xrightarrow{dig-bin} 1000|1001 \xrightarrow{decimal} 1|3|7 \xrightarrow{dig-bin} 1|11|11 \xrightarrow{decimal} 6|3 \xrightarrow{dig-bin}$
  $110|11 \xrightarrow{decimal} 2|7 \xrightarrow{dig-bin} 10|11 \xrightarrow{decimal} 2|3 \xrightarrow{dig-bin} 10|11 \xrightarrow{decimal} 11.$ (As we know that $N$ is prime, we need not backtrack and proceed further).
- $1|7 \xrightarrow{dig-bin} 1|11 \xrightarrow{decimal} 1|5 \xrightarrow{dig-bin} 1|101 \xrightarrow{decimal} 1|3 \xrightarrow{dig-bin} 1|11 \xrightarrow{decimal} 7.$
- $1|2|7|9 \xrightarrow{dig-bin} 1|1011|1001 \xrightarrow{decimal} 8|8|9 \xrightarrow{dig-bin} 1000|1000|1001 \xrightarrow{decimal} 7|0|9 \xrightarrow{dig-bin} 111|0|1001 \xrightarrow{decimal} 2|3|3 \xrightarrow{dig-bin}$
  $10|11|11 \xrightarrow{decimal} 47.$
- $1|9|9|3|7 \xrightarrow{dig-bin} 1|1001|1001|111 \xrightarrow{decimal} 1|3|1|19 \xrightarrow{dig-bin} 1|11|1|1001 \xrightarrow{decimal}$
  $110|0|11 \xrightarrow{decimal} 1|0|5 \xrightarrow{dig-bin} 1|101 \xrightarrow{decimal} 2|1 \xrightarrow{dig-bin} 10|1 \xrightarrow{decimal} 5.$
- $2|2|0|3 \xrightarrow{dig-bin} 10|1001 \xrightarrow{decimal} 8|3 \xrightarrow{dig-bin} 1000|11 \xrightarrow{decimal} 3|5 \xrightarrow{dig-bin}$
  $11|101 \xrightarrow{decimal} 2|9 \xrightarrow{dig-bin} 10|1001 \xrightarrow{decimal} 4|1 \xrightarrow{dig-bin} 100|1 \xrightarrow{decimal} 9.$

Table 2 shows the Mersenne primes $M_p$ partitioned in to five categories, based on the terminating number of the first chain of their generators $p$. 
Table 2: Partition of Mersenne Primes

| Terminating with 11 | 7  | 47  | 5  | 9  |
|---------------------|----|-----|----|----|
| 3                   | 7  | 1279| 5  | 2203|
| 89                  | 17 | 3217| 19937| 9689|
| 107                 | 31 | 4253| 44497|
| 607                 | 61 | 9941| 216091|
| 21701               | 127| 11213| 756839|
| 110503              | 521| 13466917|
| 1257787             | 2281| 2976221|
| 1398269             | 4423| 3921377|
| 20996011            | 23209| 13466917|
| 24036583            | 86243|
| 32582657            | 132049|
| 37156667            | 6972593|
| 42643801            | 30402457|
| 43112609            | 30402457|
| 57885161            | 30402457|

CONJECTURE 3: Mersenne primes can be partitioned into five groups only.

3.6.1 Rough Method to generate primes

Definition 5 Cell of a binary number
Let $A$ be the given binary number. Let it be of the form $A = r_1r_2 \ldots r_t$, where $r_i$ is a digit of $A$ and $t$ is the total number of digits of $A$. Next, let us partition the digits of $A$ as follows: $r_1r_2|r_3| \ldots |r_{t-1}r_t$. Then each partitioned (divided) portion of $A$ is called the cell of $A$. Here, $r_1r_2$ is called the first cell of $A$, $r_3$ is the second cell of $A$ and so on.

Definition 6 Cell-wise Decimal Equivalent
A decimal number $B$ is said to be cell-wise decimal equivalent of the binary number $A$, if $B$ is obtained from $A$ by finding decimal equivalents of each cell of $A$ (starting from left and ending at right) and then taking union of those decimal equivalents.

For example, consider $A = 1011$. First, let us find one partition of $A$. $A$ can be partitioned as $1|011$. Here, 10 is the first cell of $A$ and 11 is the second cell of $A$. The decimal equivalent of 10 is 2 and that of 11 is 3. Union of those decimal equivalents is 23. Thus, the cell-wise decimal equivalent of $A$ is 23. Let it be represented as follows: $10|11 \rightarrow 2|3$.

Find all the partitions of $A$. They are: $1|0|1|1$, $1|0|11$, $10|11$ and $101|1$. The cell-wise decimal equivalents of each of the partitions of $A$ are as follows:

- $1|0|1|1 \rightarrow 1|0|1|1$
- $1|0|11 \rightarrow 1|0|3$, where 3 is the decimal equivalent of the binary number 11.
• $10|11 \xrightarrow{\text{cell-dec}} 2|3$, where 2 is the decimal equivalent of the binary number 10 and 3 is the decimal equivalent of the binary number 11.
• $101|1 \xrightarrow{\text{cell-dec}} 5|1$, where 5 is the decimal equivalent of the binary number 101.

Notation
The following notation is used for the procedure.
• $T$ denotes the starting number of the procedure and $T = 11$ or 7 or 47 or 5 or 9.
• $A$ denotes a binary number and it is obtained by finding the binary equivalent of $T$ or $E$.
• $E$ denotes a decimal number and it is obtained by finding cell-wise decimal equivalent of $A$.
• $A \xrightarrow{\text{binary}} B$ denotes that the number $A$ is in decimal form, the number $B$ is in binary form and $B$ is obtained from $A$ by finding binary equivalent of $A$.
• $A \xrightarrow{\text{cell-dec}} B$ denotes that the number $A$ is in binary form, the number $B$ is in decimal form and $B$ is obtained from $A$ by finding cell-wise decimal equivalent of $A$.

ROUGH PROCEDURE
We start with any of the numbers 11, 7, 47, 5 and 9. Let $T = 11$ or 7 or 47 or 5 or 9.

Procedure:
\[
\text{generate}(T)
\]
1. Find the binary equivalent of $T$ and store it in $A$.
2. Find all the partitions of $A$.
3. for each partition of $A$,
   \[
   \begin{align*}
   &\cdot \text{ Find cell-wise decimal equivalent of } A \text{ and store it in } E. \\
   &\cdot \text{ Call generate}(E) \text{ until we get the next largest prime number or Mersenne prime number.}
   \end{align*}
   \]

ILLUSTRATION
Consider $T = 11$.

Iteration 1: $11 \xrightarrow{\text{binary}} 1011$. Let $A = 1011$.
All the partitions of $A = 1011$ are $1|0|1|1$, $1|0|11$, $10|11$ and $101|1$.
Consider a partition $1|0|11$. $1|0|11 \xrightarrow{\text{cell-dec}} 1|0|3$. Let $E = 103$. Now, $2^{103} - 1$ is not prime and hence 103 is not the generator prime. So, we continue the procedure.

Iteration 2: Consider $E = 103$.

$103 \xrightarrow{\text{binary}} 1100111$. Let $A = 1100111$.
Find all the partitions of $A$.
Let us consider a partition $1|10|0|111$. $1|10|0|111 \xrightarrow{\text{cell-dec}} 1|2|0|7$. Let $E = 1207$. 

Next go to iteration 3 and so on. Thus, this method may help to find the next largest prime number.

4 Different Approach Algorithm using the concept of Graph Pairs

There is a strong connection between number theory and graph theory. An example for such a connection is the cycle graph, discussed in Anderson [11]. Many works are done connecting these two concepts. Janakiraman and Boominathan [12][13] have brought out a different kind of interplay between these two concepts. Using the theory of congruences, Janakiraman and Boominathan [12][13] have defined graph pairs, simple graph pairs, the graph corresponding to a graph pair and so on.

In this section, we use the concept of graph pairs to find a prime divisor of \( N \). This paper deals only with the number theoretic concepts of graph pairs. For more details of graph pairs and its interplay between number theory and graph theory, readers are directed to refer Janakiraman and Boominathan [13].

In this section, a simple approximation algorithm is given for finding a prime divisor of \( N \) using the concept of graph pairs of \( N \). Its time complexity is proved to be at most \( O(\ln^2 N) \), for infinitely many \( N \) (also approximately large \( N \)). Then the disadvantages and advantages of the algorithm are analyzed. Also, using this algorithm, another procedure is given in this section to check up whether the given number is prime or not. The time complexity of that procedure is discussed and an open problem is given regarding the time complexity.

In this section, the notation \( d|n \) denote \( d \) divides \( n \).

4.1 DEFINITIONS

Next let us give the definitions of graph pair and simple graph pair from Janakiraman and Boominathan [12][13].

**Definition 7 Graph Pair** Given a positive integer \( N \), the pair \((a, b)\) of positive integers is defined to be a graph pair if \( a.b \equiv 1 \pmod{N} \). If \( a^2 \equiv 1 \pmod{N} \), then \((a, a)\) is defined as a simple graph pair.

In number theory terminology, \( b \) is called the inverse of \( a \) modulo \( n \). If \( \gcd(a, n) = 1 \), then \( a \) has an inverse, and it is unique modulo \( n \).

\((a, b)\) is named as a graph pair because for the graph pair \((a, b)\) of \( N \), there is a graph on \( (N - 1) \) vertices. Let us discuss in detail as follows:

For a given positive integer \( N \), let \((a, b)\) be a graph pair. Form a matrix \( A = [a_{ij}] \) as follows:

\[
a_{ij} = \begin{cases} 
1 & \text{if either } i.a \equiv j \pmod{N} \text{ or } i.b \equiv j \pmod{N} \\
0 & \text{otherwise, } i = 1, \ldots, N - 1.
\end{cases}
\]
Since $a$ and $b$ are relatively primes to $N$ and $i = 1, 2, \ldots, N - 1$; $j$ takes values from 1 to $N - 1$ only. This always results in a symmetric binary square matrix $A$ of order $N - 1$. Since there is a one-to-one correspondence between a labeled graph on $N$ vertices and a $N \times N$ symmetric binary matrix, $(a, b)$ is named as a graph pair, that is, for the graph pair $(a, b)$ of $N$, there is a graph on $(N - 1)$ vertices, whose adjacency matrix is nothing but $A$.

4.2 APPROXIMATION ALGORITHM

In this section, an approximation algorithm is given to find a prime divisor of $N$.

Input: $N$
Output: “$N$ is composite”, $\gcd(D, N)$, and $j$, if $N > 2^j$.

No output is produced if $N < 2^j$.

Algorithm 2: GP($N$)

1. Let $j = 0$.
2. Put $j = j + 1$.
3. If $(N - 1)/2^j$ is an integer, then find the graph pair $((N - 1)/a, N - a)$ such that $a = (N - 1)/2^j$.
4. else find the graph pair $2^{(j-1)/2}, (b_{j-1}b_1)(\mod N))$.
5. Denote this graph pair as $(2^j, b_j)$.
6. Find $|2^j - 2^j|$ and store it in $D$.
7. If $2^j < N$, then GOTO step 8 else stop.
8. If $\gcd(D, N) = 1$, then GOTO step 2 else print “$N$ is composite”, print $\gcd(D, N)$ and $j$ and stop.

Note 2 We have used two methods to generate the graph pairs, namely, $(2^j, b_j) = ((N - 1)/a, N - a)$ such that $a = (N - 1)/2^j$ and $(2^j, b_j) = (2^{(j-1)/2}, (b_{j-1}b_1)(\mod N))$. There are many methods to generate the graph pairs. The third method is $(2^j, b_j) = (2^j, b_1b_j(\mod N))$. The fourth method is: $(2^j, b_j) = (2^{(j-1)/2}, b_{j-1}/2)$, if $b_{j-1}/2$ is an integer. Thus, $b_j$ can be found from any of the following formula and they are equal.

- $N - a$ (if $a = (N - 1)/2^j$ is an integer)
- $(b_{j-1}b_1)(\mod N)$
- $(b_1b_j)(\mod N)$
- $b_{j-1}/2$ (if $b_{j-1}/2$ is an integer).

4.3 ILLUSTRATION

Let us consider $N = 96577$. The step by step process and the calculations of the algorithm are explained in the Table 3.
Table 3  Step by Step Executions of Algorithm 2

| $j$  | $(2^j, b_j)$ | $D = |b_j - 2^j|$ | $\gcd(D, N)$ |
|------|-------------|-----------------|-------------|
| 1    | (2, 48289)  | 48287           | 1           |
| 2    | (4, 72433)  | 72429           | 1           |
| 3    | (8, 84505)  | 84497           | 1           |
| 4    | (16, 90541)| 90525           | 1           |
| 5    | (32, 93559)| 93527           | 13          |
| 6    | (64, 95068)| 95004           | 13          |

Output: “$N$ is composite”, $\gcd(D, N) = 13$ and $j = 6$.

4.4 CORRECTNESS OF THE ALGORITHM

Theorem 4 If $(N - 1)/2^j$ is an integer and $(N - 1)/a = 2^j$, then $(2^j, N - a)$ is a graph pair for $N$.

Proof Given that $a = (N - 1)/2^j$, for $j \geq 1$.
$\Rightarrow N - a = [(2^j - 1)N + 1]/2^j$.
To Prove: $(2^j, N - a)$ is a graph pair for $N$.

i.e., to prove: $2^j (N - a) \equiv 1 \pmod{N}$.

$2^j (N - a) = 2^j [(2^j - 1)N + 1]/2^j = (2^j - 1)N + 1 \equiv 1 \pmod{N}$ as $(2^j - 1)N$ is divisible by $N$. Thus, $(2^j, N - a)$ is a graph pair for $N$. $\square$

Theorem 5 $(2^{j - 1} \cdot 2, (b_{j-1}, b_1)(\pmod{N}))$ obtained in step 4 is a graph pair for $N$.

Proof At the $j$th iteration, $(2, b_1)$ and $(2^{j - 1}, b_{j-1})$ are graph pairs for $N$.
To Prove: $(2^{j - 1} \cdot 2, (b_{j-1}, b_1)(\pmod{N}))$ is also a graph pair for $N$ at the $j$th iteration.

$(2, b_1)$ and $(2^{j - 1}, b_{j-1})$ are graph pairs for $N$.
$\Rightarrow 2b_1 \equiv 1 \pmod{N}$ and $2^{j - 1}b_{j-1} \equiv 1 \pmod{N}$.
$\Rightarrow 2^j b_1 b_{j-1} \equiv 1 \pmod{N}$ (by Theorem 3).

Thus, $(2^{j - 1} \cdot 2, (b_{j-1}, b_1)(\pmod{N}))$ is a graph pair for $N$ at the $j$th iteration. $\square$

Note 3 Similar to the Theorem 5 it can be proved that $(2^j, (b_1)(\pmod{N}))$ is also a graph pair for $N$ and if $b_{j-1}/2$ is an integer, then $(2^j, b_j) = (2^{j - 1} \cdot 2, b_{j-1}/2)$ is also a graph pair for $N$.

Theorem 6 The prime divisor $k$ of $N$ is obtained in $(k - 1)/2$ steps.

Proof Let $N$ be a multiple of $k$, where $k$ is prime.
To Prove: If $j = (k - 1)/2$, then $\gcd(D, N) = k$.

i.e., To Prove: If $j = (k - 1)/2$, then $k|D$, where $D = |b_j - 2^j|$.

$b_j = (b_1)^{2^j} (\pmod{N})$ (from Note 2) and $2^j$ can be written as $2^j (\pmod{N}) = 2^j$ if $2^j < N$; otherwise it is the value of the remainder when $2^j$ is divided by $N$.

If $2^j (\pmod{N}) = 2^j$, the prime divisor $k$ of $N$ is obtained in $(k - 1)/2$ steps.
then $2^j - b_j = 2^j - [(b_1)j](\mod N) = [2^j - (b_1)j] (\mod N)$.

If $2^j(\mod N) = 2^j(\mod N)$, that is, the value of the remainder when $2^j$ is
divided by $N$, then $2^j - b_j = [2^j - (b_1)j] (\mod N)$.

Thus, whatever is the value of $2^j$, $2^j - b_j = [2^j - (b_1)j] (\mod N)$.

So, we have to prove: If $j = (k - 1)/2$, then $(2^j - (b_1)j) \equiv 0 (\mod k)$.

$b_1 = N - (N - 1)/2 = (N + 1)/2$

$\Rightarrow (b_1)j = [(N + 1)/2]j$.

$\Rightarrow (2^j - (b_1)j) = 2^j - [(N + 1)/2] = [2^j - (N + 1)/2] / 2^j$.

Case(i): $k = 7$.

Then $j = (k - 1)/2 = 3$ and $[2^j - (N + 1)/2] / 2^j = [64 - (N + 1)^3] / 8$.

To Prove: $[64 - (N + 1)^3] / 8 \equiv 0 (\mod 7)$.

i.e., to prove: $64 - (N + 1)^3 \equiv 0 (\mod 56)$ (by Theorem 2).

$64 - (N + 1)^3 = 64 - (N^3 + 3N^2 + 3N + 1) = 63 - N(N^2 + 3N + 3)$.

As $k = 7$, this implies that $N$ is a multiple of 7. So, let $N = 7m$, for every
natural number $m$.

$\Rightarrow 64 - (N + 1)^3 = 63 - 7m(49m^2 + 21m + 3) = 7[9 - m(49m^2 + 21m + 3)]$.

This expression has come out to be a multiple of 7.

So, in order to prove $64 - (N + 1)^3 \equiv 0 (\mod 56)$, it is enough to prove
$9 - m(49m^2 + 21m + 3) \equiv 0 (\mod 8)$ (by Theorem 2).

i.e., to prove: $m(49m^2 + 21m + 3) \equiv 1 (\mod 8)$ (since $9 \equiv 1 (\mod 8)$).

As $N$ is odd, $m$ is odd.

• $m = 3 \Rightarrow m(49m^2 + 21m + 3) = 1521 \equiv 1 (\mod 8)$.

• $m = 5 \Rightarrow m(49m^2 + 21m + 3) = 6665 \equiv 1 (\mod 8)$.

• $m = 7 \Rightarrow m(49m^2 + 21m + 3) = 17857 \equiv 1 (\mod 8)$.

• $m = 9 \Rightarrow m(49m^2 + 21m + 3) = 37449 \equiv 1 (\mod 8)$.

Continuing like this, it is easy to see that $m(49m^2 + 21m + 3) \equiv 1 (\mod 8)$ and
hence $(2^j - b_j) \equiv 0 (\mod 7)$.

Case(ii): $k = 11$.

Then $j = (k - 1)/2 = 5$ and $[2^j - (N + 1)/2] / 2^j = [1024 - (N + 1)^5] / 32$.

To Prove: $[1024 - (N + 1)^5] / 32 \equiv 0 (\mod 11)$.

i.e., to prove: $1024 - (N + 1)^5 \equiv 0 (\mod 352)$ (by Theorem 2).

$1024 - (N + 1)^5 = 1024 - (N^5 + 5N^4 + 10N^3 + 10N^2 + 5N + 1)$

$= 1023 - N(N^4 + 5N^3 + 10N^2 + 10N + 5)$.

As $k = 11$, this implies that $N$ is a multiple of 11. So, let $N = 11m$, for
every natural number $m$.

$\Rightarrow 1024 - (N + 1)^5 = 1023 - 11m(14641m^4 + 6655m^3 + 1210m^2 + 110m + 5)$

$= 11[93 - m(14641m^4 + 6655m^3 + 1210m^2 + 110m + 5)]$.

This expression has come out to be a multiple of 11.

So, in order to prove $1024 - (N + 1)^5 \equiv 0 (\mod 352)$, it is enough to prove
$93 - m(14641m^4 + 6655m^3 + 1210m^2 + 110m + 5) \equiv 0 (\mod 32)$ (by Theorem 2).

i.e., to prove: $m(14641m^4 + 6655m^3 + 1210m^2 + 110m + 5) \equiv 29 (\mod 32)$ (since
Continuing like this, it is easy to see that 
$k \cdot m = 93$.

As $N$ is odd, $m$ is odd.

- $m = 3 \Rightarrow m(14641m^4 + 6655m^3 + 1210m^2 + 110m + 5) = 4130493 \equiv 29$ (mod 32).
- $m = 5 \Rightarrow m(14641m^4 + 6655m^3 + 1210m^2 + 110m + 5) = 50066525 \equiv 29$ (mod 32).
- $m = 7 \Rightarrow m(14641m^4 + 6655m^3 + 1210m^2 + 110m + 5) = 262470397 \equiv 29$ (mod 32).
- $m = 9 \Rightarrow m(14641m^4 + 6655m^3 + 1210m^2 + 110m + 5) = 909090909 \equiv 29$ (mod 32).

Continuing like this, it is easy to see that $m(14641m^4 + 6655m^3 + 1210m^2 + 110m + 5) \equiv 29$ (mod 32) and hence $(2^j - b_j) \equiv 0$ (mod 11).

Proceeding like this, in general, for general $k$, $j = (k - 1)/2$.

To prove: $(2^{2j} - (N + 1)^j) / 2^j \equiv 0$ (mod $k$) if $j = (k - 1)/2$.

i.e., to prove: $2^{2j} - (N + 1)^j \equiv 0$ (mod $2^k$) (by Theorem 2).

\[
2^{2j} - (N + 1)^j = 2^{2j} - \left( \left( (N + 1)^j \right)^{j-1} + jC_1 (N^{j-2} + \ldots + jN + 1) \right)
\]

As $N$ is a multiple of $k$, let $N = km$.

So,

\[
2^{2j} - (N + 1)^j = (2^{2j} - 1 - km) \left( (km)^{j-1} + jC_1 (km)^{j-2} + jC_2 (km)^{j-3} + \ldots + j \right)
\]

(1)

Here, $2^{2j} - 1 - km \equiv 0$ (mod $k$) (since by Fermat’s theorem-Theorem 3).

Thus, $(2^{2j} - 1)$ is also a multiple of $k$. So, let us write $(2^{2j} - 1) = km_1$.

Then, equation (1) becomes,

\[
2^{2j} - (N + 1)^j = km_1 - km \left( (km)^{j-1} + jC_1 (km)^{j-2} + jC_2 (km)^{j-3} + \ldots + j \right)
\]

\[
= k \left[ m_1 - m \left( (km)^{j-1} + jC_1 (km)^{j-2} + jC_2 (km)^{j-3} + \ldots + j \right) \right].
\]

This expression has come out to be a multiple of $k$.

So, in order to prove $2^{2j} - (N + 1)^j \equiv 0$ (mod $2^k$), it is enough to prove

\[
m_1 - m \left( (km)^{j-1} + jC_1 (km)^{j-2} + jC_2 (km)^{j-3} + \ldots + j \right) \equiv 0$ (mod $2^2$) (by Theorem 2).

From case (i) and case (ii), we get for any odd $m$, this result holds true (Similar to the cases (i) and (ii), it can be easily checked for different values of $k$ and different values of $m$). Thus, $(2^{2j} - b_j) \equiv 0$ (mod $k$), if $j = (k - 1)/2$ and hence the prime divisor $k$ of $N$ is obtained in $(k - 1)/2$ steps. \(\square\)

Next, a corollary is given, which may be helpful for the reduction of time complexity.

**Corollary 1** A prime divisor of $N$ can be obtained in $(L - 1)/2$ steps, where $L$ is the least prime divisor of $N$.

**Proof** Proof follows from the Theorem 3. \(\square\)
4.5 TIME COMPLEXITY OF THE ALGORITHM

Let us assume that $N > 2^j$ (as in step 7 of the algorithm). The time complexity of the algorithm is dependent on the following two steps:

- Finding gcd($D, N$)
- How many times gcd($D, N$) is found, i.e., the value of $j$.

Each computation of G.C.D takes $O(\ln N)$ \[10\]. As $j$ determines the number of steps, the total time complexity of the algorithm is $O(j \ln N)$.

**Computation of $j$:** By our assumption, $2^j < N < 2^m$, for some natural number $m$.

$\Rightarrow j < m$.

$\Rightarrow j = O(m) = O(\ln^2 N)$ as $N < 2^m$.

Thus, the total time complexity of the algorithm is $O(\ln^2 N)$, if $N > 2^j$.

4.6 WHAT HAPPENS WHEN $N < 2^j$

Suppose if $N$ is composite and $N < 2^j$, then the algorithm does not produce any output, because we get gcd($D, N$) = 1 in all iterations. In this section, we give two methods, slightly modifying the algorithm given in the section 4.2.

4.6.1 First Method when $N < 2^j$

If $N < 2^j$ and gcd($D, N$) is still 1, then find the graph pair as $(2^j \pmod N, b_j)$, where $b_j$ takes one of the forms given in Note 2 and continue the procedure until we get gcd($D, N$) $\neq 1$. Steps of this method are as follows:

**Procedure 1**

1. Put $j = j + 1$.
2. Find $(2^j \pmod N, b_j)$ and $D = |(b_j - 2^j \pmod N)|$.
3. If gcd($D, N$) = 1, then goto step 1 else print “$N$ is composite”, print gcd($D, N$) and stop.

**Time Complexity of the Procedure 1**

We can see that for this case also, $j = (L - 1)/2$ steps (by Corollary \[1\]). But, in this case, time complexity is not $O(\ln^2 N)$ as in the previous case. Here, the time complexity depends on the least prime divisor $L$ of $N$, because we do the procedure for $j = (L - 1)/2$ and $j$ is not less than $m$ in this case (since $2^j > N$). Usually the divisor of $N$ can be obtained in $O(N^{0.5})$. Thus, if $N < 2^j$, the time complexity of the algorithm is $O(N^{0.5})$.

4.6.2 Second Method when $N < 2^j$

In this section, we find a prime divisor of $N$ if $N < 2^j$ and also check up whether the number is prime or not. If $k|N$, then $k|N^x$, for any natural number $x$. So, by the Corollary \[1\] divisor of $N^x$ is also obtained in $(L - 1)/2$ steps,
where $L$ is the least prime divisor of $N$ (least prime divisor of $N$ and $N^x$ are same). We use this idea for the next procedure. First, let us find $N^x$ by finding suitable value of $x$. Store $N^x$ in $N_1$. Then apply the Algorithm 2 given in the section 4.2 for $N_1$, that is call GP($N_1$), with a slight modification in the procedure. In the Algorithm 2, we find $gcd(D, N)$ in step 8. But in this method, we do not find $gcd(D, N_1)$, instead we find $gcd(D, N)$, here also. Anyhow the answer will not be affected. The terminating condition is changed here. This method checks for primality also.

**Procedure 2**

1. Find $N^x$ such that $N^x < 2^{N^{0.5}}$ and let $N^x = N_1$.
2. Let $j = 0$.
3. Put $j = j + 1$.
4. If $(N_1 − 1)/2^j$ is an integer, then find the graph pair $((N_1 − 1)/a, N_1 − a)$ such that $a = (N_1 − 1)/2^j$.
5. else find the graph pair $(2^{j−1}−1, 2, (b_j−1.b_1)(mod N_1))$.
6. Denote this graph pair as $(2^j, b_j)$.
7. Find $|b_j − 2^j|$ and store it in $D$.
8. If $j < N^{0.5}$, then GOTO step 9 else Print “$N$ is prime” and stop.
9. If $gcd(D, N) = 1$, then GOTO step 3 else Print “$N$ is composite”, print $gcd(D, N)$ and $j$ and stop.

**Time Complexity of Procedure 2**

As discussed in the section 4.5, the time complexity of the Procedure 2 is $O(j \ln N)$. What is the value of $j$ here? $j$ denotes the number of times $gcd(D, N)$ is found out in the Procedure 2 (that is, the number of iterations of the Procedure 2). By Corollary 1, $j = (L − 1)/2 = O(L)$, where $L$ is the least prime divisor of $N$. Also, any divisor of $N$ will be obtained in at most $O(N^{0.5})$. That is why we made the terminating condition as $j < N^{0.5}$. Thus, the total time complexity of the Procedure 2 for finding a prime divisor of $N$ or for primality testing is at most $O(N^{0.5} \ln N)$, which is higher than the usual time complexity $O(N^{0.5} \ln N)$.

**Note 4** Comparing the time complexity of the Procedure 1 and the time complexity of the Procedure 2, we can conclude that if $N < 2^j$, then the Procedure 1 can be applied rather than the Procedure 2 and hence the time complexity to find a prime divisor of $N$ is at most $O(N^{0.5})$ (usual time complexity).

For primality test also, the time complexity is $O(N^{0.5} \ln N)$, which is too high. The Procedure 2 need not be used. The procedure 2 is given only to know whether Algorithm 2 can be used to check up primality or not. But, we got higher time complexity. So, we make the following open problem.

**Open Problem 1:** Use Algorithm 2 to check up whether $N$ is prime or not in lesser time.
4.7 ADVANTAGES AND DISADVANTAGES OF THE ALGORITHM 2

In this section, we list out some advantages and disadvantages of the Algorithm 2 simultaneously. First we mention a disadvantage and then we mention the advantage out of it or how the disadvantage is overcome. Let the term ‘Disadv’ denote disadvantage and the term ‘Adv’ denote the advantage.

**Disadv:** The algorithm finds a prime divisor for $N$ in $O(\ln^2 N)$ only if $N > 2^j$, where $j = (L - 1)/2$ and $L$ is the least prime divisor of $N$ and in $O(N^{0.5})$ if $N < 2^j$.

**Adv:** Although this algorithm gives a prime divisor for $N$ in $O(\ln^2 N)$ only if $N > 2^j$, it gives result for infinitely many cases and only finite cases are left, which are countable. For example,

- If $N$ is a multiple of 7, then by this algorithm, $j = 3$ and hence $N > 2^3 = 8$. So, this algorithm finds divisor for $N$ within 3 steps (in $O(\ln^2 N)$) for all $N > 8$, and hence infinite cases are covered.
- Similarly, if $N$ is a multiple of 11, then by this algorithm, $j = 5$ and hence $N > 2^5 = 32$. So, this algorithm finds divisor for $N$ within 5 steps for all $N > 32$ (in $O(\ln^2 N)$), and hence infinite cases are covered.
- Similarly if $N$ is a multiple of 101, then by this algorithm, $j = 50$ and hence $N > 2^{50}$. So, this algorithm finds divisor for $N$ within 50 steps (in $O(\ln^2 N)$) for all $N > 2^{50}$ and hence infinite cases are covered and so on.

So, we can say that the algorithm gives good results for **infinitely many cases** (or approximately large $N$).

**Disadv:** Only if $j$ is known, this algorithm gives prime divisor of $N$ in lesser time. But $j$ is in turn dependent on $L$, which is the least prime divisor of $N$ and hence this process ends in a cycle.

**Adv:** That is why we say that the algorithm gives good result for approximately large $N$.

**Disadv:** The time complexity of the algorithm to find a prime divisor of $N$, if $N < 2^j$ is at most $O(N^{0.5})$, which is too slow.

**Adv:** Although the process is too slow, the coding is very very simple. Also the proof of correctness of the algorithm and time complexity are also simple.

**Disadv:** We discussed in Theorem that $j = (k - 1)/2$. That is,

- Multiple of 7 will be obtained in 3 steps;
- Multiple of 11 will be obtained in 5 steps;
- Multiple of 13 will be obtained in 6 steps;
- Multiple of 17 will be obtained in 8 steps and so on.

But usually when a number $N$ is given, in order to find its divisor, we will do trial and error method. We will start checking from the primes 3, 7, 11, ... (in order). So, if we check in the primes in their order, then

- Multiple of 7 will be obtained in 2 steps as 7 is the second odd prime;
- Multiple of 11 will be obtained in 3 steps as 11 is the third odd prime;
- Multiple of 13 will be obtained in 4 steps as 13 is the fourth odd prime; and so on.
The following table, Table 4, gives the comparison of our method and the usual method. Let \( y \) (say) be the least prime divisor of \( N \).

| \( y \) | No. of steps from our method | No. of steps from trial and error method |
|--------|------------------------------|----------------------------------------|
| 7      | 3                            | 2                                      |
| 11     | 5                            | 3                                      |
| 13     | 6                            | 4                                      |
| 17     | 8                            | 5                                      |
| 19     | 9                            | 6 and so on.                           |

So, we can observe from the table that our method takes more steps (time) than trial and error method. But can any one tell the exact position of the largest prime? Only if the position is known, trial and error method is good. So, in such a situation, our algorithm gives better result.

**Adv:** As the exact position of prime cannot be determined, our method can be used to get better results in \( O(\ln^2 N) \) time.

**Adv:** Given a prime \( N \), this algorithm verifies it in polynomial time.

From the above disadvantages and advantages, we infer that

- Given any \( N \), the algorithm does not find the divisor in \( O(\ln^2 N) \).
- The computation takes \( O(N^{0.5}) \), which is too slow.
- But coding the Algorithm 2 is very very simple.
- The proof of correctness and time complexity of the algorithm are also simple.
- The result depends on the input \( N \). If, (by luck), we are given the appropriate \( N \) as input, that is, \( N > 2^j \), then the algorithm solves the problem in \( O(\ln^2 N) \), which is better than all other existing algorithms. For example, if \( N \) is very large and it satisfies \( N > 2^j \) (by luck), then such a case is solved by our algorithm in lesser time.
- So, \( P(\text{getting a prime divisor of } N|N > 2^j) = 1 \) and \( P(\text{getting a prime divisor of any } N) = 0.5 \).

5 Conclusion

In this paper, two approximation algorithms are given, which find a divisor of the given number.

The first algorithm is an interesting approximation algorithm and is given using a new approach. The algorithm finds a divisor of the given number. The intricate portions of the algorithm are discussed and a conjecture is given regarding the time complexity of the algorithm. The algorithm can be applied to find the next largest Mersenne prime number.

The second algorithm is a simple approximation algorithm, given using the concept of graph pairs. The algorithm finds a prime divisor of the given
number \( N \) in \( O(\ln^2 N) \), for infinitely many cases (approximately large \( N \)). Time complexity and correctness of the algorithm are proved. The advantages and disadvantages of the algorithm are also discussed. An attempt is made to use the Algorithm 2 for primality test and an open problem is given regarding this.

The authors feel that this paper may be helpful for further research as many interesting things are yet to be analyzed and it may pave way for new ideas.

6 Future Work

We are working on the conjectures and open problems given this paper to develop the concepts further.

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