On Pseudo-slant sub-manifolds of generalised sasakian space form

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Abstract.
In this article, we investigate pseudo-slant sub-manifold (PSSM) of generalised sasakian space form(GSSF). We obtain some integrability aspects of the distributions on the PSSM.

1. Introduction
Chen \cite{2} initiated the idea of slant immersions in 1990. Many researchers have achieved quality of result in this topic. Lotta \cite{1} commenced the approach of slant immersions in contact manifolds(CM). Cabrerizo et al achieved results on K-contact and sasakian manifolds under slant sub-manifold in \cite{4,3}. Lately Carriazo \cite{5} achieved quality of result in bi-slant immersions and notion of PSSM. Khan V. A. as well as Khan M. A. \cite{8} gave the result on contact versions of PSSM. Gupta et al \cite{6} studied TSM under slant sub-manifolds. U. C. De and Avijit Saskar studied integrability conditions of distributions of TSM under PSSM \cite{7}.

In this paper we deliberately had put an effort to procure the integrability aspects of generalised sasakian space forms under PSSM.

2. Preliminaries
If \((\overline{M}, g)\) is a manifold with dimension \((2n + 1)\) furnished with ACM structure \((\xi, \phi, g, \eta)\) having a vector field \(\xi\), a \((1,1)\) tensor field \(\phi\), a Riemannian metric \(g\) as well as a 1-form \(\eta\) then we have following equations satisfied,

\begin{align}
\phi^2 Y_1 &= -Y_1 + \eta(Y_1)\xi, \eta(\xi) = 1, \eta(Y_1) = g(Y_1, \xi), \\
g(\phi Y_1, \phi Y_2) &= g(Y_1, Y_2) - \eta(Y_1)\eta(Y_2), \\
g(\phi Y_1, Y_2) + g(Y_1, \phi Y_2) &= 0, \phi \xi = 0, \eta(\phi Y_1) = 0,
\end{align}

for any vector fields \(Y_1, Y_2 \in T\overline{M}\).

An ACM manifold \(\overline{M}\) is known as sasakian manifold if

\[ (\overline{\nabla}_{Y_1} \phi)Y_2 = g(Y_1, Y_2)\xi - \eta(Y_2)Y_1, \]
\[ \nabla_{Y_1}\xi = -\phi Y_1 \]
where \( \nabla \) stands for the Levi-Civita connection of \( g \). Also an ACM manifold \( \bar{M} \) with an ACM structure \((\xi, \phi, \eta, g)\) is known as a generalised sasakian space form (GSSF) if there exist three functions \( f_1, f_2, f_3 \) on \( \bar{M} \) such that the curvature tensor \( R \) is given by
\[
R(Y_1, Y_2)Y_3 = f_1[g(Y_2, Y_3)Y_1 - g(Y_1, Y_3)Y_2] + f_2[g(Y_1, \phi Y_3)\phi Y_2 - g(Y_2, \phi Y_3)\phi Y_1 \\
+ 2g(Y_1, \phi Y_2)\phi Y_3] + f_3[g(Y_1, Y_3)\eta(Y_2)\xi - g(Y_2, Y_3)\eta(Y_1)\xi \\
+ \eta(Y_1)\eta(Y_3)Y_2 - \eta(Y_2)\eta(Y_3)Y_1]
\]
for all vector fields \( Y_1, Y_2, Y_3 \) on \( \bar{M} \).
If \( f_1 = \frac{c + 3}{4}, f_2 = f_3 = \frac{c - 1}{4} \), then \( \bar{M} \) is a sasakian space form, where \( c \) is the constant sectional curvature.
In a \((2n + 1)\)-dimensional GSSF \( \bar{M}^{(2n+1)}(f_1, f_2, f_3) \), we have the following:
\[
(\nabla_{Y_1}\phi)Y_2 = (f_1 - f_3)[g(Y_1, Y_2)\xi - \eta(Y_2)Y_1], \tag{2}
\]
Also,
\[
\nabla_{Y_1}\xi = (f_3 - f_1)\phi Y_1, \tag{3}
\]
where \( \xi \) is the structure vector field and \( Y_1 \in T\bar{M} \).

If \( M \) is a sub-manifold of the contact manifold \( \bar{M} \) of dimension \((2n + 1)\) induced with metric \( g \) then \( M \) has the tangent bundle \( TM \) and the normal bundle \( T^\perp M \).
Weingarten and Gauss formula are given by,
\[
\nabla_{Y_1}N = \nabla_{Y_1}N - ANY_1, \tag{4}
\]
\[
\nabla_{Y_1}Y_2 = \nabla_{Y_1}Y_2 + h(Y_1, Y_2), \tag{5}
\]
\[ \forall \ Y_1, Y_2 \in TM \text{ as well as } N \in T^\perp M, \text{ where } \nabla^\perp \text{ stands for the connection in the normal bundle.} \]
Also the shape operator \( A_N \) and the second fundamental form \( h \) are connected by the relation,
\[
g(A_N Y_1, Y_2) = g(h(Y_1, Y_2), N) \tag{6}
\]
\[ \forall \ N \in T^\perp M, Y_1 \in TM \text{ we can write}
\]
\[ \phi Y_1 = TY_1 + NY_1, \quad (TY_1 \in TM, NY_1 \in T^\perp M) \tag{7}
\]
Similarly,
\[ \phi N = tN + nN, \quad (tN \in TM, nN \in T^\perp M) \tag{8}
\]
The covariant derivative of \( T \) and \( N \) are given by
\[
(\nabla_{Y_1}T)Y_2 = \nabla_{Y_1}TY_2 - T\nabla_{Y_1}Y_2 \tag{9}
\]
\[
(\nabla_{Y_1}N)Y_2 = \nabla_{Y_1}NY_2 - N\nabla_{Y_1}Y_2 \tag{10}
\]
The sub-manifold \( M \) is invariant if \( N \) is identically zero. If \( T \) is identically zero, then \( M \) is called anti-invariant.
From (1) and (7) we write,
\[
g(Y_1, TY_2) = -g(TY_1, Y_2) \tag{11}
\]
for any \( Y_1, Y_2 \in TM \) From (3), (5), as well as (7) we get
\[
\nabla_{Y_1}\xi = (f_3 - f_1)TY_1, \tag{12}
\]
\[ h(Y_1, \xi) = (f_3 - f_1)NY_1. \tag{13}
\]
3. Pseudo-slant sub-manifolds (PSSM) of Generalised Sasakian Space Forms (GSSF)

**Definition:** $M$ is called pseudo-slant sub-manifold (PSSM) of generalised Sasakian space form (GSSF) $\overline{M}$ if on $M$ there exists two orthogonal distributions $D_1$ and $D_2$ such that:

i. $TM$ admits the orthogonal direct decomposition.

$$TM = D_1 \oplus D_2 \oplus \langle \xi \rangle.$$ 

ii. $D_1$ is an anti-invariant distribution, that is

$$\phi D_1 \subseteq T^\perp M.$$ 

iii. $D_2$ is the slant distribution with slant angle $\theta \neq \pi/2$, the angle between $D_2$ as well as $\phi(D_2)$ is a fixed constant $\theta$.

If $\theta = 0$, then the pseudo-slant sub-manifold is a semi-invariant sub-manifold. If the dimensions of $D_i$ is denoted by $d_i$, for $i = 1, 2$, then we have the following cases:

a. $M$ is known as an anti-invariant sub-manifold, if $d_2 = 0$.

b. $M$ is said to be an invariant sub-manifold, if $d_1 = 0$ as well as $\theta = 0$.

c. $M$ is called a proper slant sub-manifold having the slant angle $\theta \neq 0$, if $d_1 = 0$ as well as $\theta \neq 0$.

A pseudo-slant sub-manifold is said to be proper if $d_1d_2 \neq 0$ and $\theta \neq 0$.

4. Integrability of the distributions

**Theorem 1.** Let $M$ be a PSSM of a GSSF of $\overline{M}$. Then $A_{\phi Y_2}Y_1 = A_{\phi Y_1}Y_2$ provided,

(i) $Y_1, Y_2 \in D_1$ or

(ii) $f_1 = f_3$ 

**Proof.** Inview of (6)

$$g(A_{\phi Y_2}Y_1, Y_3) = g(h(Y_1, Y_3), \phi Y_2) = -g(\phi h(Y_1, Y_3), Y_2) \tag{14}$$

By view of (5), (14) simplifies to

$$g(A_{\phi Y_2}Y_1, Y_3) = -g(\phi \nabla Y_3 Y_1, Y_2) + g(\phi \nabla Y_3 Y_1, Y_2)$$

Since $\phi \nabla Y_3 Y_1 \in T^\perp M$, $Y_2 \in D_1$, $g(\phi \nabla Y_3 Y_1, Y_2) = 0$.

$$g(A_{\phi Y_2}Y_1, Y_3) = g((\nabla Y_3)\phi Y_1, Y_2) - g(\nabla Y_3\phi Y_1, Y_2) \tag{15}$$

From (4) we have,

$$\nabla Y_3\phi Y_1 = -A_{\phi Y_1}Y_3 + \nabla Y_3\phi Y_1 \tag{16}$$

Combining (15) and (16) we get,

$$g(A_{\phi Y_2}Y_1, Y_3) = g((\nabla Y_3)\phi Y_1, Y_2) + g(A_{\phi Y_2}Y_3, Y_2) \tag{17}$$

Since $h(Y_1, Y_2) = h(Y_2, Y_1)$, it follows that

$$g(A_{\phi Y_1}Y_3, Y_2) = g(A_{\phi Y_1}Y_2, Y_3)$$
Therefore from (17) and (2) we obtain,
\[ g(A_\phi Y_2 Y_1, Y_3) - g(A_\phi Y_1 Y_2, Y_3) = (f_1 - f_3)[g(Y_3, Y_1)\eta(Y_2) - \eta(Y_1)g(Y_3, Y_2)] \]

Thus the above equation yields,
\[ A_\phi Y_2 Y_1 - A_\phi Y_1 Y_2 = (f_1 - f_3)[Y_1 \eta(Y_2) - \eta(Y_1)Y_2] \] (18)

(i) Since \( Y_1, Y_2, Y_3 \in D_1 \) an orthonormal distribution \( < \xi > \), it follows that \( \eta(Y_1) = \eta(Y_2) = 0 \)

Therefore (18) reduces to
\[ A_\phi Y_2 Y_1 = A_\phi Y_1 Y_2 \]

(ii) If \( f_1 = f_3 \) then \( A_\phi Y_2 Y_1 = A_\phi Y_1 Y_2 \)

Conversely: If \( A_\phi Y_2 Y_1 = A_\phi Y_1 Y_2 \), then from (18)
\[ (f_1 - f_3)[Y_1 \eta(Y_2) - \eta(Y_1)Y_2] = 0. \]
implies \( f_1 = f_3 \) or \( \eta(Y_2) = \eta(Y_1) = 0 \) implies \( Y_1, Y_2 \in D_1 \)

\[ \square \]

**Theorem 2.** Let \( M \) be a PSSM of GSSF of \( \overline{M} \), then
(i) \( [Y_1, \xi] \in D_1 \), for all \( Y_1 \in D_1 \).
(ii) The distribution \( D_1 \oplus < \xi > \) is integrable.

**Proof.** Since \( h(Y_1, Y_2) = h(Y_2, Y_1) \), we have from (5)
\[ \nabla Y_1 Y_2 - \nabla Y_2 Y_1 = \nabla Y_1 Y_2 - \nabla Y_2 Y_1 \]

For any \( Y_1 \in D_1 \) and \( Y_2 \in D_2 \)
\[ g([Y_1, \xi], TY_3) = g(\nabla Y_1 \xi, TY_3) - g(\nabla \xi Y_1, TY_3) \] (19)

Also \( Y_1 \in D_1 \), \( Y_2 \in D_2 \), we have
\[ (\nabla Y_1 g)(Y_2, Y_3) = \nabla Y_1 g(Y_2, Y_3) - g(\nabla Y_1 Y_2, Y_3) - g(Y_2, \nabla Y_1 Y_3) \]
\[ g(\nabla Y_1 Y_2, Y_3) = -g(Y_2, \nabla Y_1 Y_3) \] (20)

Since \( D_1 \) and \( D_2 \) are two orthogonal distributions as well as \( D_1 \) is anti-invariant, Using (3), (20) in (19) we obtain,
\[ g([Y_1, \xi], TY_3) = g(\nabla \xi TY_3, Y_1) \] (21)

In view of (2) we get
\[ (\nabla \xi \phi)Y_2 = 0. \] (22)

In virtue of (7) and (22),(21) yields
\[ g([Y_1, \xi], TY_3) = 0 \]

Hence \([Y_1, \xi] \in D_1 \) for \( Y_1 \in D_1 \)
Therefore the distribution \( D_1 \oplus < \xi > \) is integrable.

\[ \square \]
**Theorem 3.** If $M$ is a PSSM of GSSF of $\overline{M}$. Then for any $Y_1, Y_2 \in D_1 \oplus D_2$
\[ g([Y_1, Y_2], \xi) = 2(f_3 - f_1)g(TY_2, Y_1) \]  
(23)

**Proof.**
\[ g([Y_1, Y_2], \xi) = g(\nabla Y_1 Y_2, \xi) - g(\nabla Y_2 Y_1, \xi) \]
In view of (20). The above equation becomes,
\[ g([Y_1, Y_2], \xi) = g(Y_1, \nabla Y_2 \xi) - g(Y_2, \nabla Y_1 \xi) \]
Using (12), we have
\[ g([Y_1, Y_2], \xi) = -(f_3 - f_1)g(Y_2, TY_1) + (f_3 - f_1)g(Y_1, TY_2) \]
Hence the proof (23).

**Theorem 4.** If $M$ is a PSSM of GSSF of $\overline{M}$. Then the anti-invariant distribution $D_1$ is integrable.

**Proof.** For any $Y_1 \in TM$, Let
\[ Y_1 = P_1 Y_1 + P_2 Y_1 + \eta(Y_1) \xi \]  
(24)
where $P_1$ and $P_2$ are projections on the distribution $D_1$ and $D_2$. From (24) it follows that,
\[ \phi Y_1 = NP_1 Y_1 + TP_2 Y_1 + NP_2 Y_1 \]
From (7) on equating tangential and normal parts, we get
\[ TY_1 = TP_2 Y_1, \quad NY_1 = NP_1 Y_1 + NP_2 Y_1 \]
Now for any $Y_1, Y_2 \in D_1$ and $Y_3 \in D_2$
\[ g([Y_1, Y_2], TY_3) = g([Y_1, Y_2], TP_2 Y_3) = -g(\phi[Y_1, Y_2], P_2 Y_3) \]  
(25)
Now
\[ \phi[Y_1, Y_2] = \phi \nabla Y_1 Y_2 - \phi \nabla Y_2 Y_1 = \phi \nabla Y_1 Y_2 - \phi \nabla Y_2 Y_1 \]
\[ \phi[Y_1, Y_2] = \nabla Y_1 \phi Y_2 - (\nabla Y_1 \phi) Y_2 - \nabla Y_2 \phi Y_1 + (\nabla Y_2 \phi) Y_1 \]
In view of (2) and (4).
Also by the fact that $g(Y_1, Y_2) = 0$ for $Y_1 \in D_1$ and $Y_2 \in D_2$. We obtain from (25)
\[ g([Y_1, Y_2], TP_2 Y_3) = \eta(Y_2)g(Y_1, P_2 Y_3) + \eta(Y_1)g(Y_2, P_2 Y_3) \]
For any $Y_1, Y_2 \in D_1$ we get $\eta(Y_1) = \eta(Y_2) = 0$.
Therefore by Theorem [1] above equation yields
\[ g([Y_1, Y_2], TY_3) = 0 \]
That is $[Y_1, Y_2] \in D_1$ for $Y_1, Y_2 \in D_1$.
Therefore the distribution $D_1$ is integrable.

**Theorem 5.** If $M$ is a (PSSM) of (GSSF) of $\overline{M}$. Then the slant distribution $D_2$ is not integrable.
Proof. From Theorem [2] we have,
\[ g([Y_1, Y_2], \xi) = 2(f_3 - f_1)g(TY_2, Y_1). \]
Also by the definition of pseudo-slant sub-manifold the result follows. ☐

**Theorem 6.** If \( M \) is a PSSM of GSSF of \( \overline{M} \). Then
\[
(\nabla_{Y_1}T)Y_2 = A_{NY_2}Y_1 + th(Y_1, Y_2) + (f_1 - f_3)(g(Y_1, Y_2)\xi - \eta(Y_2)Y_1)
\]

Proof. For any \( Y_1, Y_2 \in TM \) we have
\[
\nabla_{Y_1}Y_2 = (\nabla_{Y_1}\phi)Y_2 + \phi \nabla_{Y_1}Y_2
\]
By (5) and (7) we have,
\[
\nabla_{Y_1}TY_2 + \nabla_{Y_1}NY_2 = (\nabla_{Y_1}\phi)Y_2 + \phi (\nabla_{Y_1}Y_2 + h(Y_1, Y_2))
\]
Again using (2),(5),(7) and (8) we get
\[
\nabla_{Y_1}TY_2 + h(Y_1, TY_2) - A_{NY_2}Y_1 + \nabla_{Y_1}^{T}NY_2 = (f_1 - f_3)[g(Y_1, Y_2)\xi - \eta(Y_2)Y_1] + T\nabla_{Y_1}Y_2 + N\nabla_{Y_1}Y_2 + th(Y_1, Y_2) + nh(Y_1, Y_2)
\]
Comparing the tangential and normal components, we have
\[
\nabla_{Y_1}TY_2 - A_{NY_2}Y_1 = (f_1 - f_3)[g(Y_1, Y_2)\xi - \eta(Y_2)Y_1] + T\nabla_{Y_1}Y_2 + th(Y_1, Y_2).
\]
Hence
\[
(\nabla_{Y_1}T)Y_2 = A_{NY_2}Y_1 + th(Y_1, Y_2) + (f_1 - f_3)(g(Y_1, Y_2)\xi - \eta(Y_2)Y_1)
\]

Definition: If \( M \) is a PSSM of GSSF \( \overline{M} \) then by direct calculation using (2),(5),(4),(7) and (8) we get
\[
(\nabla_{Y_1}T)Y_2 = \lambda (f_1 - f_3)(g(Y_1, Y_2)\xi - \eta(Y_2)Y_1) \tag{26}
\]
where \( \lambda = \cos^2 \theta \) [3] and \( Y_1, Y_2 \in TM \).
The above equation provides a sufficient condition for a sub-manifold to be a proper slant submanifold of a GSSF where \( \theta \in (0, \frac{\pi}{2}) \) which is the slant angle of the slant distributions \( D_2 \). Then we have,
\[
(\nabla_{Y_1}T)Y_2 = \cos^2 \theta (f_1 - f_3)(g(P_2Y_1, Y_2)\xi - \eta(Y_2)P_2Y_1) \tag{27}
\]
for any \( Y_1, Y_2 \in TM \)
If \( Y_1 = P_2Y_1 + \eta(Y_1)\xi \) and \( Y_2 = P_2Y_2 + \eta(Y_2)\xi \), then (27) implies that
\[
(\nabla_{Y_1}T)Y_2 = \cos^2 \theta (f_1 - f_3)(g(Y_1, Y_2)\xi - \eta(Y_2)Y_1)
\]
Thus in view of (26) which provides a sufficient condition for the distribution \( D_2 \oplus < \xi > \) is a tangent bundle of proper slant sub-manifold.
On the otherhand if \( Y_1, Y_2 \in D_1 \oplus < \xi > \) then (27) implies
\[
(\nabla_{Y_1}T)Y_2 = 0
\]
Theorem 7. If $M$ is a proper PSSM with angle $\theta$ of GSSF of $\overline{M}$. Then for any $Y_1, Y_2 \in TM$

$$\nabla_{Y_1}T Y_2 = A_{N P_1 Y_1} Y_1 + A_{N P_2 Y_2} Y_1 + th(Y_1, Y_2) + (f_1 - f_3)(g(Y_1, Y_2)\xi - \eta(Y_2)Y_1)$$  \hspace{1cm} (28)

Hence $M$ satisfies (27) if and only if

$$A_{N Y_1} Y_1 = A_{N Y_2} Y_2 + (f_1 - f_3)(\eta(Y_2)P_1 Y_1 - \eta(Y_1)P_1 Y_2) - (f_1 - f_3)\sin^2(\eta(Y_1)P_2 Y_2 - \eta(Y_2)P_2 Y_1)$$  \hspace{1cm} (29)

where $NY_1 = NP_1 Y_1 + NP_2 Y_1$

Proof. For any $Y_1, Y_2 \in TM$ we have,

$$\nabla_{Y_1} \phi Y_2 = (\nabla_{Y_1} \phi) Y_2 + \phi(\nabla_{Y_1} Y_2)$$

Using (2), (5), (4), (7) and (9) and comparing tangential components we obtain (28).

Suppose $M$ is a proper pseudo-slant sub-manifold satisfying (27), then by (28)

$$\cos^2 (f_1 - f_3)|g(P_2 Y_1, Y_2)\xi - \eta(Y_2)P_2 Y_1| = A_{N P_1 Y_1} Y_1 + A_{N P_2 Y_2} Y_1 + th(Y_1, Y_2)$$

$$+ (f_1 - f_3)(g(P_1 Y_1, Y_2)\xi + g(P_2 Y_1, Y_2)\xi - \eta(Y_2)P_1 Y_1 - \eta(Y_2)P_2 Y_1)$$

$$A_{N P_1 Y_1} Y_1 + A_{N P_2 Y_2} Y_1 =$$

$$- th(Y_1, Y_2) - (f_1 - f_3)(g(P_1 Y_1, Y_2)\xi - \eta(Y_2)P_1 Y_1) - \sin^2 (f_1 - f_3)(g(P_2 Y_1, Y_2)\xi - \eta(Y_2)P_2 Y_1)$$  \hspace{1cm} (30)

On interchanging $Y_1$ to $Y_2$ and $Y_2$ to $Y_1$

$$A_{N P_1 Y_1} Y_2 + A_{N P_2 Y_2} Y_1 = - th(Y_1, Y_2) - (f_1 - f_3)(g(P_1 Y_2, Y_1)\xi - \eta(Y_1)P_1 Y_2)$$

$$- \sin^2 (f_1 - f_3)(g(P_2 Y_2, Y_1)\xi - \eta(Y_1)P_2 Y_2)$$  \hspace{1cm} (31)

From (30) and (31) and also from the condition $NY_1 = NP_1 Y_1 + NP_2 Y_1$ we get (29)  \hspace{1cm} \Box

Conversely:

Suppose (29) holds, then for any $Y_3 \in TM$

$$g(A_{N Y_1} Y_1, Y_3) = g(A_{N Y_2} Y_1, Y_3) + (f_1 - f_3)[\eta(Y_2)g(P_1 Y_1, Y_3) - \eta(Y_1)g(P_1 Y_2, Y_3)]$$

$$- (f_1 - f_3)\sin^2(\eta(Y_1)g(P_2 Y_2, Y_3) - \eta(Y_2)g(P_2 Y_1, Y_3)]$$

Using (6)

$$g(A_{N Y_2} Y_1, Y_3) = -g(th(Y_2, Y_3), Y_1) + (f_1 - f_3)[\eta(Y_2)g(P_1 Y_1, Y_3) - \eta(Y_1)g(P_1 Y_2, Y_3)]$$

$$- (f_1 - f_3)\sin^2(\eta(Y_1)g(P_2 Y_2, Y_3) - \eta(Y_2)g(P_2 Y_1, Y_3)]$$

Interchange $Y_1$ to $Y_2$ and using the fact that $g(P_1 Y_1, Y_2) = g(Y_1, P_1 Y_2)$ for each $Y_1, Y_2 \in TM$

also $A_{N Y_2} Y_1 = A_{N Y_1} Y_2$, we get

$$g(A_{N Y_2} Y_1, Y_3) = -g(th(Y_2, Y_1), Y_3) + (f_1 - f_3)[\eta(Y_2)g(P_1 Y_1, Y_3) - \eta(Y_3)g(P_1 Y_2, Y_2)]$$

$$- (f_1 - f_3)\sin^2(\eta(Y_3)g(P_2 Y_2, Y_1) - \eta(Y_2)g(P_2 Y_1, Y_3)]$$

Since $Y_1 = P_1 Y_1 + P_2 Y_1$ we have,

$$g(A_{N Y_2} Y_1, Y_3) = -g(th(Y_1, Y_2), Y_3) + (f_1 - f_3)[\eta(Y_2)g(Y_1, Y_3) - \eta(Y_3)g(Y_1, Y_2)]$$

$$- (f_1 - f_3)\cos^2(\eta(Y_3)g(Y_2, Y_1) - \eta(Y_2)g(Y_2, Y_3)]$$
Implies that,
\[ A_{NY_2}Y_1 = -th(Y_1,Y_2) + (f_1 - f_3)[\eta(Y_2)Y_1 - g(Y_1,Y_2)\xi] \\
+ (f_1 - f_3)\cos^2\theta [g(P_2Y_1,Y_2)\xi - \eta(Y_2)P_2Y_1] \]

using (28) in the above expression we get
\[ (\nabla_{Y_1}T)Y_2 = \cos^2\theta(f_1 - f_3)(g(P_2Y_1,Y_2)\xi - \eta(Y_2)P_2Y_1). \]

Which proves the assertion.

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