The geometry of bifurcation surfaces in parameter space. I. A walk through the pitchfork

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Abstract

The classical pitchfork of singularity theory is a twice-degenerate bifurcation that typically occurs in dynamical system models exhibiting $\mathbb{Z}_2$ symmetry. Non-classical pitchfork singularities also occur in many non-symmetric systems, where the total bifurcation environment is usually more complex. In this paper three-dimensional manifolds of critical points, or limit-point shells, are introduced by examining several bifurcation problems that contain a pitchfork as an organizing centre. Comparison of these surfaces shows that notionally equivalent problems can have significant positional differences in their bifurcation behaviour. As a consequence, the parameter range of jump, hysteresis, or phase transition phenomena in dynamical models (and the physical systems they purport to represent) is determined by other singularities that shape the limit-point shell.

Keywords: pitchfork; singularity theory; bifurcation theory; limit-point shell

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1 Introduction

Discontinuous behaviour in dissipative dynamical systems is often ascribed to the propinquity of a pitchfork, a codimension 2 singularity which requires two auxiliary parameters for a universal unfolding. In a 3-dimensional space labelled by the system parameters a universal unfolding forms a critical surface, or limit-point shell. *A priori* knowledge of the topology of critical surfaces, and how they change with variation of other parameters (i.e., animation) is a powerful aid in the design and optimization of dynamical systems and the control of jump, hysteresis, or oscillatory phenomena. In this work cartoons of computed bifurcation surfaces are used to visualize the pitchfork and its surroundings in

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parameter space. It is shown how these surfaces are shaped by the degree of symmetry and by the lower codimension antecedents of the pitchfork as the organizing centre, or highest order singularity, in particular bifurcation problems.

A mathematical description of the pitchfork in singularity theory terms will be given in §2, for now the name itself will suffice as a working definition. The classic trident representation is simply a graph depicting how the number of something (such as solutions of an equation, or states of an ideal ensemble) changes from one to three as a control variable is made to cross a critical value. Because of both its heuristic value in singularity theory and its importance in applied bifurcation problems, the pitchfork has been described in many texts dealing with dynamics and bifurcations. A recent exposition with this flavour can be found in [1].

Pitchforks have been most often reported as bifurcations of the steady states of idealized systems possessing $\mathbb{Z}_2$ or reflectional symmetry. Simple examples in this category include studies of the driven pendulum [2], [3], [4], and variations on the buckling beam problem [5], [6]. In these systems, which are also called imperfect bifurcation problems, symmetry-breaking perturbations dissolve the pitchfork leaving behind persistent limit points where discontinuous action may occur. In more complicated problems the symmetric pitchfork occurs as a bifurcation of periodic solutions [4], or may be embedded in a higher-order degeneracy [3].

Although the presence of a pitchfork in a mathematical model can sometimes be interpreted as *prima facie* evidence or diagnostic of $\mathbb{Z}_2$ invariance, there are also many bifurcation problems containing non-symmetric pitchforks. In some problems of this type a fully unfolded or perturbed pitchfork is intrinsic to a minimal description of the associated physical system. Thermokinetic and isothermal chemical systems are the most widely studied in this category; a good introduction to these is found in [9]. A partially unfolded pitchfork occurs in other non-symmetric problems, with a single static perturbation required to complete a realistic description of the system. Two recent examples are a model for L(low)–H(high) confinement state transitions in plasmas [10, 11] and experimental and numerical studies of an electronic Van der Pol oscillator [12].

With recent advances in inertial manifold theory [13], it is likely that the pitchfork in general will also appear in the dynamics of infinite-dimensional systems that exhibit low-dimensional behaviour on a long time-scale.

In this paper several bifurcation problems will be discussed and compared to exemplify the qualities of pitchfork manifolds in the different categories: (1) the prototypic universal unfolding of the pitchfork, (2) a well-known thermokinetic system, (3) the simplest universal unfolding of a non-symmetric pitchfork, and (4) the L–H transition problem mentioned above.

Bifurcation surfaces can also illuminate rather dramatically the truism that what you see depends on where you view the object from. Originally the pitchfork was described

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1“Bifurcation” simply means “fork”, so it seems unnecessary to refer to a pitchfork fork.
from an orthogonal point of view as the generic cusp, and in §2 we work around to the prototypic pitchfork beginning from the more familiar cusp manifold. Moving from the non-generic surface of equilibrium points or steady states, up a dimension to the invariant surface of fold points, the limit-point shell \( L_p \) of the prototypic pitchfork is presented as a fundamental, generic object. In §3 the limit-point shell of the CSTR problem, \( L_c \) (first described in [14] and [15]), and that of the simplest non-symmetric pitchfork, \( L_{TI} \), are compared. Some curious features of the extraordinary limit-point shell of the L–H problem, \( L_{LH} \), are also described in §3. §4 concludes with a brief discussion of what these result imply for the design and control of experimental dynamical systems.

2 From cusp manifold to limit-point shell

In 1955 Hassler Whitney published the first exposition of singularity theory [16], in which he derived conditions for a regular point \( p \) of a smooth mapping \( f \) from \( \mathbb{R}^2 \) into \( \mathbb{R}^2 \) to be a cusp point. In coordinates \((u, v), (x, y)\) these are

\[
\begin{align*}
\text{a.} & \quad u_x = u_y = v_x = 0, \quad v_y = 1 \quad \text{(singular condition)}, \\
\text{b.} & \quad u_{xx} = 0, \quad u_{xy} \neq 0, \quad u_{xxx} - 3u_{xy}v_{xx} \neq 0, \quad \text{(cusp condition)}
\end{align*}
\]

(1)

at \( p \). Whitney proved that any mapping containing a point at which the conditions (1) are satisfied can be transformed by coordinate changes into the following normal form for the cusp:

\[
\begin{align*}
u &= xy - x^3 \\
v &= y
\end{align*}
\]

(2)

Subsequently René Thom listed the cusp, which he called the Riemann-Hugoniot catastrophe, as the second of the famous (I mean, of course, infamous) seven elementary catastrophes [17]. In catastrophe theory the normal form (2) becomes the universal unfolding \( G(x, u, v) \) of the germ \( g(x) = x^3 \):

\[
G(x, u, v) = x^3 - vx + u,
\]

(3)

where \( G \) is the gradient of a corresponding governing potential \( V \). The familiar cusp surface, \( G(x, u, v) = 0 \), and the projection of the folds on the \( u, v \) plane are shown in figure 1. The surface may be viewed as the lateral unfurling of a path in the \( x, u \) plane into \( v \). Three qualitatively different paths — slices of the cusp surface at constant \( v \) or bifurcation diagrams — are sketched in figure 2.

Although the cusp is generic, in the sense that all other singularities may be perturbed to either a fold or a cusp [13], the surface in figure 1 is not a unique manifold of the cusp catastrophe. Since all paths through the cusp unfolding (3) are equally valid, we may choose a path in the \( x, v \) plane. Any such path unfurls laterally into \( u \) to form another
surface, shown from two points of view in figure 3 together with the projection of the folds on the \( u, v \) plane. Three qualitatively different bifurcation diagrams are shown in figure 4, whence we finally arrive at the classical pitchfork.

Returning to Whitney’s original defining conditions for the cusp, equations (\( \text{I} \)), we see now that they also define the pitchfork.

The surface in figure 3 is non-generic, because it does not represent all qualitative information about the pitchfork. Golubitsky and Schaeffer [19] proved that a universal unfolding of the pitchfork must include a fourth variable. Their classification of singularities by codimension and derivation of universal unfoldings has the contextual setting of bifurcations of steady states in autonomous dynamical systems dependent on parameters:

\[
\frac{dx}{dt} = G(x, \lambda, \alpha_i) = 0.
\]

Here \( x \) is the dynamical state variable, \( \lambda \) is the principal bifurcation parameter and the \( \alpha_i \) are auxiliary or unfolding parameters. Assuming henceforth this context and notation, the pitchfork conditions \( P \) are given in table 1, column 2. A bifurcation problem which satisfies \( P \) is said to be locally equivalent to the normal form

\[
g(x, \lambda) = \pm x^3 \pm \lambda x. \tag{4}
\]

The prototypic universal unfolding of the pitchfork, \( P_p \), is given as

\[
G(x, \lambda, \alpha, \beta) = x^3 - \lambda x + \alpha + \beta x^2. \tag{P_p}
\]

A naïve interpretation of the concept of a universal unfolding is more useful here than a rigorous definition or derivation. We apply the conditions \( P \) to the unfolding

\[
G(x, \lambda, \alpha) = x^3 - \lambda x + \alpha. \tag{5}
\]

(This is a partial unfolding of (\( \text{I} \)); compare equation (\( \text{I} \)).) The result comprises an over-determined system of four equations in three variables. If we perturb (\( \text{I} \)) at \( \alpha = 0 \) by adding the term \( \beta x^2 \), the bifurcation diagram acquires qualitatively different characteristics, as indicated in figure 3. However, more perturbations, or different perturbations, do not introduce any more qualitative differences to the bifurcation diagram. (Note: it can be guessed that universal unfoldings are not necessarily unique.)

For any value of \( \beta \neq 0 \) the bifurcation surface of \( P_p \) formed by unfolding a path in the \( x, \lambda \) plane into \( \alpha \) does not, therefore, include the pitchfork. Two views of this surface for \( \beta = 5 \) are shown in figure 3 along with the projection of the folds on the \( \lambda, \alpha \) plane. The famous five bifurcation diagrams for the universal unfolding \( P_p \) are given in figure 4.

There are two singularities on the fold lines of figure 3(b), a hysteresis point \( H \) and a transcritical point \( T \), definitions of which are given in in table 3 columns 2 and 3. It is evident from these definitions and from the surface for the partial unfolding in figure 3 that the pitchfork is the limiting degeneracy of \( H \) and \( T \) for \( \beta = 0 \). A unique, continuous
manifold $L_p$ around $P_p$ is obtained by unfurling the fold lines in the $\lambda, \alpha$ plane into $\beta$. This forms a limit-point shell, the surface of fold or limit points of an unfolding. $L_p$ is shown from four vantage points in figure 8. It is a self-contained and unique manifold around $P_p$; it cannot be translated, rotated, deformed or punctured. Let us inspect it closely.

A slice of the surface at constant $\beta \neq 0$ is simply the fold lines in figure 8(b). There is a distinct seam of hysteresis points that runs from positive to negative $\beta$ and negative to positive $\alpha$ through the pitchfork at $(0, 0, 0)$. Less obvious is the line of transcritical points, which lies along $\lambda = 0$ in the $\lambda, \beta$ plane. The shell is symmetric about two reflections: reflect it in a vertical mirror plane through $\beta = 0$, then reflect it again in a horizontal mirror plane through $\alpha = 0$.

3 The limit-point shell in applications

In dynamical models the limit-point shell outlines the boundary and extent of steady-state multiplicity over the parameter space. Therefore, the shape of a limit point shell is an important consideration in predictive work, because it is a guide in the selection of design criteria and operating conditions for an experimental system. However, $L_p$ is a poor metaphor for many models which contain a pitchfork. A specific limit-point shell is shaped by the presence and location of other bifurcations and by the symmetry of the problem. In this section these ideas are illustrated by comparing the limit point shells of three bifurcation problems which have pitchforks as organizing centres.

3.1 The CSTR problem

Interest has been maintained in this thermokinetic system since the 1950s, because it possesses a combination of unusual dynamical properties and real-world experimental accessibility. The first singularity theory study of the CSTR problem can be found in [20] and a recent alternative treatment is given in [14].

The simplest CSTR model describes an exothermal chemical reaction occurring in a well-stirred bounded medium. As a bifurcation problem it may be written as follows:

$$ G(u, f, \theta, \varepsilon, \ell) = \frac{fe^{-1/u}}{e^{-1/u} + f} + (\varepsilon f + \ell)(\theta - u). \quad (P_c) $$

The state variable is the temperature $u$ which depends on a number of parameters: $f$, an input rate; $\theta$, the coupled temperature of the thermostat and the input; $\ell$, the thermal dissipation rate; and $\varepsilon$, the intrinsic properties of the medium. For $u \neq 1/2$ it can be shown that an organizing centre is the pitchfork $P_c$, although the system has no symmetries. The limit point shell $L_c$ at a fixed value of $\ell$ is shown in figure 4. (Note: A detailed discussion of this structure may be found in [14]. The dynamical CSTR problem also contains Hopf bifurcations, the manifold of which will be discussed elsewhere.)
Contrasted with $L_p$, the prototypic limit point shell of figure 8, the overall qualities of $L_c$ appear to be asymmetry and convexity.

**The role of co-existing bifurcations**

Table 1 shows that embedded in the pitchfork are the codimension 1 bifurcations $H$ and $T$ or $I$, or $H$ and $T$ and $I$, since $T$ and $I$ are distinguished only by the sign of the non-degeneracy condition $\det d^2G \neq 0$.

For the prototypic unfolding, $P_p$, we find that $G = G_x = G_\lambda = 0$ and $\det (d^2G) = -1$, $G_{xx} = -6$ at $x = \lambda = \alpha = 0$, thus $T$ but not $I$ is embedded in $P_p$. For the CSTR problem it is shown in Appendix A that both $T$ and $I$ are embedded in $P_c$.

The last column in table 1 gives conditions for another codimension 2 bifurcation, the asymmetric cusp $A$. In Appendix A it is shown that this singularity is also present in the CSTR problem. $A$ is too inconspicuous to be pinpointed visually on the portion of $L_c$ shown in figure 9, but it is easily approximated numerically as $(u, f, \theta, \varepsilon) \approx (0.216, 0.016, 0.141, 1.91)$ at $\ell = 0.05$. Nevertheless, the form of $L_c$ is strongly sculpted by the presence of $A$.

**Is the normal form for the CSTR problem an adequate proxy?**

One of the greatest strengths of singularity theory is that it defines criteria by which two algebraic systems have equivalent solution sets. Singularity theory tells us that the CSTR problem is qualitatively equivalent to the simplest universal unfolding of a bifurcation problem containing $P$, $H$, $T$, $I$, and $A$. This is designated $P_{TI}$:

$$G(x, \lambda, \alpha, \beta) = x^3 + \lambda(x - \alpha + \beta x). \quad (P_{TI})$$

Evaluation of $T$ and $I$ embedded in $P_{TI}$ and $A$ (Appendix B) indicates that the bifurcation behaviour of the CSTR problem is completely encapsulated in the simpler problem $P_{TI}$. The limit point shell $L_{TI}$ of $P_{TI}$ is viewed from two vantage points in figure 11, and it appears to tell a different story.

All of the qualitatively distinct bifurcation diagrams of the CSTR can indeed be recovered from various slices of $L_{TI}$ — for example, figure 11 shows the isolated branch of solutions that also exists in the CSTR. However, the qualitative differences between the two limit-point shells are quite striking. We could not use $P_{TI}$ to predict the boundaries of multiplicity in the CSTR.

**3.2 A tale of three pitchforks**

An important issue in the physics of magnetically confined plasmas is the dramatic jump to an improved confinement régime, known as the L–H transition, that occurs at critical values of tunable parameters or internal system properties. A phenomenological model that described this critical behaviour was analysed by Ball and Dewar and
found to contain a partially unfolded pitchfork. The simplest universal unfolding for this bifurcation problem was derived as:

\[ G(u, q, d, \alpha) = \frac{(dq - u^2)(b + au^{5/2})}{u^{5/2}} + \frac{q(u^2 - dq)}{u^2} + \alpha, \]  

(P_{LH})

where \( u \) is the state variable, \( q, d \), and \( \alpha \) are parameters and \( a \) and \( b \) are numerical factors having the values 0.05 and 0.95 respectively. In the model, \( u, q, \) and \( d > 0 \). (Note: The dynamical L-H model also contains Hopf bifurcations, which will be discussed elsewhere.)

Unusually, although there is a degenerate pitchfork in the original partially unfolded (or overdetermined) problem with \( \alpha = 0 \), the universal unfolding \( P_{LH} \) contains two pitchforks in the real, physical space. They are found exactly by applying the conditions \( P \) in table 1 to \( P_{LH} \):

\[
\begin{align*}
P_{1LH} : & \quad (u, q, d, \alpha) = \left( \frac{b^{2/5}}{24^{5/5}a^{2/5}}, \frac{5b}{4\sqrt{u}}, \frac{4u^{5/2}}{5b}, 0 \right) \\
P_{2LH} : & \quad (u, q, d, \alpha) = \\
& \quad \left( \frac{(173 + 70\sqrt{6})^{1/5}}{22^{5/5}a^{2/5}}, \frac{b(2 + \sqrt{6})}{16\sqrt{u}}, \frac{8u^{5/2}(2 + \sqrt{6})}{5b}, -\frac{5b(2 + \sqrt{6})}{2\sqrt{u}} \right)
\end{align*}
\]

This introduces a formidable global aspect to what hitherto has been a purely local focus on the structure of the limit-point shell around a unique pitchfork. A bifurcation analysis of \( P_{LH} \) and construction of the limit-point shell is clearly not for the faint-hearted, because it is difficult to imagine the topological nightmares that must exist to connect \( P_{1LH} \) and \( P_{2LH} \). One can begin by searching for lower-order and same-order bifurcations; the conditions for \( T \) and \( I \) yield:

\[
\begin{align*}
(u, q, d) = u_{TI} : & \quad \left( \frac{5b(5b + 4\alpha\sqrt{u})}{16\alpha u}, \frac{-16\alpha u^3}{25b^2} \right), \\
\det d^2G = & \quad -\frac{96\alpha^2}{25b^2u}, \quad G_{uu} = \frac{25b^2/\alpha - 20b\sqrt{u} - 2\alpha u}{2u^3},
\end{align*}
\]

with \( u_{TI} \) given by real roots of \( 16\alpha u^3a - b(-25b + 24\alpha\sqrt{u}) = 0 \). From the condition \( \det d^2G \neq 0 \) we can infer that the bifurcation is transcritical, and since \( u, q, \) and \( d > 0 \) it is required that \( \alpha < 0 \) at this point. The conditions for \( H \) evaluate to expressions too complicated to be useful. There does not appear to be an asymmetric cusp in the problem.

Another, related complication is the existence of a third pitchfork in the unphysical region \( q < 0, d < 0 \). It is given by

\[
\begin{align*}
P_{3LH} : & \quad (u, q, d, \alpha) = \\
& \quad \left( \frac{(173 - 70\sqrt{6})^{1/5}}{22^{5/5}a^{2/5}}, \frac{5b(-2 + \sqrt{6})}{16\sqrt{u}}, \frac{-8u^{5/2}(-2 + \sqrt{6})}{5b}, \frac{5b(-2 + \sqrt{6})}{2\sqrt{u}} \right).
\end{align*}
\]
$P_{3,LH}$ is important because part of its limit-point manifold intrudes into the physical parameter region and is connected to the limit-point manifold of $P_{1,LH}$ by a seam of hysteresis points. This curved seam can be seen in figure 12, a fragment of the limit-point shell of $P_{LH}$ around the connection. In figure 13 a series of slices taken in the $q, \alpha$ plane illustrate how the connection occurs.

The putative condition for épée à épée contact of the two hysteresis points in figure 13, a singularity designated $E_2$, appears to be

$$G = G_u = G_{uq} = G_{uu} = 0, \quad G_q > 0, \quad G_{uuu} < 0.$$  \hspace{1cm} (6)

This condition yields a single exact $E_2$ point:

$$E_2 : (u, q, d, \alpha) = \left( \frac{b^{2/5}}{2^{2/5}a^{2/5}}, \frac{5b}{8\sqrt{u}}, \frac{8u^{5/2}}{25b}, \frac{2b}{5\sqrt{u}} \right),$$

at which $G_q = 24/25$ and $G_{uuu} = -3b/u^{7/2}$. The conditions (3) for the $E_2$ singularity seem pathological, but they can be understood by referring to the surface in figure 12 and considering the the singularities defined in table 1. At the hysteresis points, $H$, $G_q \neq 0$, and this condition holds at $E_2$ (since it is not a pitchfork). However, another derivative must be zero at the union of two degenerate points, this is the condition $G_{uq} = 0$.

4 Discussion and conclusion

The above analysis highlights the pitfalls of accepting qualitative equivalence of a bifurcation problem to a normal form as, in some sense, a “solution”. In the case of the CSTR problem and its normal form, the boundaries of multiplicity are profoundly different. The analysis of the L–H problem also hints at the bizarre and interesting features that a limit-point shell can have while still remaining continuous. Although $L_{LH}$ is locally equivalent to $L_p$ around each of the three pitchforks, the global definition of $L_{LH}$ involves at least one new singularity, $E_2$.

This is largely an interpretive and exploratory work, investigating the local and global environment of the pitchfork through the limit-point shell and using prototypic normal forms and real-world bifurcation problems as examples. It turns out that bifurcation problems containing an organizing centre as simple as the pitchfork can have rather complex boundaries of multiplicity. For this reason, and given the increasing availability of 3-dimensional computer visualization techniques, the limit-point shell has enormous potential as a design and control aid for experimental dynamical systems.

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Appendix

A.

The conditions for $T$, $I$, and $A$ in table I are applied to $P_c$. The condition for $P$ has been evaluated in \cite{14}. With $\ell$ fixed and nonzero, $T$ or $I$ points exist at

\[(f, \theta, \varepsilon) = \left( \frac{\ell^{1/3}u^{2/3}}{e^{-2/3u} - e^{-1/u}\ell^{1/3}u^{2/3}}, u - \frac{e^{-1/3u}u^{4/3}}{\ell^{1/3}}, \frac{e^{-1/3u}\ell^{1/3}(-1 + e^{-1/3u}\ell^{1/3}u^{2/3})^2}{u^{2/3}} \right).\]

(The implicit function theorem ensures that $u$ can in principle be given by these expressions.) The nondegeneracy conditions at $T$ or $I$ points evaluate to

\[G_{uu} = u^{-8/3}e^{-1/3u}\ell^{2/3}(-1 + 2e^{1/3u}\ell^{1/3}u^{2/3} - 2u),\]

\[\det d^2G = u^{-8/3}e^{-2/3u}\ell^{2/3}(-1 + e^{1/3u}\ell^{1/3}u^{2/3})^3(-1 + 3e^{1/3u}\ell^{1/3}u^{2/3} - 4u).\]

These expressions may be positive, negative, or zero. Where $\det d^2G = 0$ the conditions $A$ in table I give an asymmetric cusp for $0 < u < 5.4$, at

\[(f, \theta, \varepsilon, \ell) = \left( \frac{e^{-1/u}(-1 - 4u)}{-2 + 4u}, \frac{u(1 + u)}{1 + 4u}, \frac{4}{27} \left(16u + \frac{1}{u^2} - 12\right), \frac{e^{-1/u}(1 + 4u)^3}{27u^2} \right).\]

Since an asymmetric cusp is the point of coincidence of a transcritical point and an isola point, it may be inferred that at least one isola and one transcritical bifurcation exists in the CSTR problem.

B.

The conditions in table I are applied to $P_{ri}$. The pitchfork occurs at $(x, \lambda, \alpha, \beta) = (0, 0, 0, 0)$, at which $G_{xxx} = 6$, $G_{x\lambda} = -1$. $I$ occurs for

\[(x, \lambda, \alpha) = \left( \frac{1}{12} (1 + B), \frac{1}{24} (1 + B), \frac{1}{864} (1 + B - 24\beta (3 + 2B)) \right),\]

\[\det d^2G = B, \quad G_{xx} = \frac{1}{2} (1 + B);\]

$T$ occurs for

\[(x, \lambda, \alpha) = \left( \frac{1}{12} (1 - B), \frac{1}{24} (1 - B), \frac{1}{864} (1 - B - 24\beta (3 - 2B)) \right),\]

\[\det d^2G = -B, \quad G_{xx} = \frac{1}{2} (1 - B);\]

where $B = \sqrt{1 - 48\beta}$. The asymmetric cusp occurs at $(x, \lambda, \alpha, \beta) = (1/12, 1/24, -1/1728, 1/48)$. 

9
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