On Energy Conversion in Port-Hamiltonian Systems

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Abstract—Port-Hamiltonian systems with two external ports are studied, together with the strategies and obstructions for conversion of energy from one port to the other. Apart from the cyclo-passivity properties, this turns out to be intimately related to the interconnection topology of the system. A prime source of motivation for energy conversion is thermodynamics, in particular the Carnot-Clausius heat engine theory about conversion of thermal into mechanical energy. This classical theory is extended to general port-Hamiltonian systems satisfying structural conditions on their topology. In particular, the operation of Carnot cycles is generalized. This is illustrated by the examples of a precursor to the Stirling engine and an electro-mechanical actuator. Finally, alternative energy conversion schemes for general port-Hamiltonian systems, such as energy-routers, are discussed from the same vantage point.

I. INTRODUCTION

Energy conversion and harvesting are among the most important current problems in engineering. They motivate a number of control questions such as the development of a general theory of energy conversion and efficiency, and of effective control strategies for achieving this. This paper aims at addressing some of these questions by making use of port-Hamiltonian systems theory (see e.g. [9], [11]). This theory offers a systematic framework for modeling and control of multiphysics systems. The current line of research was initiated in [12] by developing the notion of one-port cyclo-passivity, thereby identifying topological conditions which pose limitations to the energy transfer of one port of the system to the other. In the present paper, this is extended by defining and analyzing Carnot cycles for one-port cyclo-passive systems, and applying this to a gas-piston system and an electro-mechanical actuator. Furthermore, initial steps are made towards obviating the conditions for one-port cyclo-passivity by establishing direct topological connections by the use of feedback; thus relating to the Duindam-Stramigoli energy router [1], [8]. The paper closes by formulating a number of open research problems.

II. CYCLO-PASSIVITY

First recall passivity theory, and especially the less well-known theory of cyclo-passivity. Consider an input-state-output system

\[ \begin{aligned}
\dot{x} &= f(x) + G(x)u, \\
y &= h(x), \quad x \in \mathcal{X}, \quad u, y \in \mathbb{R}^{m},
\end{aligned} \tag{1} \]

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on an n-dimensional state space manifold \( \mathcal{X} \). Cyclo-passivity was coined in [15] generalizing [14], and was further developed in [3], [4] with recent extensions in [10].

Definition 1: \( \Sigma \) is statically passive if \( y^\top(t)u(t) \geq 0 \) for all input-output trajectories \( u(t), y(t) \). It is called cyclo-passive if

\[ \int_{0}^{\tau} y^\top(t)u(t)dt \geq 0 \tag{2} \]

for all \( \tau \geq 0 \) and all \( u : [0, \tau] \rightarrow \mathbb{R}^{m} \) such that \( x(\tau) = x(0) \). Furthermore, \( \Sigma \) is cyclo-passive with respect to \( x^* \) if (2) holds for all \( \tau \geq 0 \) and all \( u : [0, \tau] \rightarrow \mathbb{R}^{m} \) such that \( x(\tau) = x(0) = x^* \). In case (2) holds with equality, we speak about cyclo-losslessness (with respect to \( x^* \)).

Obviously, static passivity implies cyclo-passivity. With \( y^\top u \) denoting externally supplied power, static passivity means that at any moment of time the system consumes a nonnegative amount of power. Obvious example of a statically passive system is a damper. Cyclo-passivity means that cyclic motions always require a nonnegative net amount of externally supplied energy. While energy might be released during some sub-interval of the cyclic motion, over the whole time-interval this needs to be compensated by an amount of supplied energy which is at least as large. For the more general class of cyclo-passive systems energy may be internally stored and released. The characterization of the set of possible energy storage functions of a cyclo-passive system is done via the dissipation inequality [14].

Definition 2: A function \( S : \mathcal{X} \rightarrow (-\infty \cup \mathbb{R} \cup \infty) \) satisfies the dissipation inequality for system \( \Sigma \) if

\[ S(x(t_2)) \leq S(x(t_1)) + \int_{t_1}^{t_2} y^\top(t)u(t)dt \tag{3} \]

holds for all \( t_1 \leq t_2 \), all input functions \( u : [t_1, t_2] \rightarrow \mathbb{R}^{m} \), and all initial conditions \( x(t_1) \), where \( y(t) = h(x(t)) \), with \( x(t) \) denoting the solution of \( \dot{x} = f(x) + G(x)u \) for initial condition \( x(t_1) \) and input function \( u : [t_1, t_2] \rightarrow \mathbb{R}^{m} \). If \( S : \mathcal{X} \rightarrow (-\infty \cup \mathbb{R} \cup \infty) \), then \( S(x) < \infty \) for all \( x \), then \( S \) is called a storage function.

By assuming differentiability of \( S \), the dissipation inequality (3) is seen [9] to be equivalent to the differential dissipation inequality

\[ \left( \frac{\partial S}{\partial x}(x) \right)^\top (f(x) + G(x)u) \leq h^\top(x)u, \]

where \( f(x) \) and \( G(x)u \) denote the first and second arguments of the dissipation inequality.
for all $x, u$, or equivalently,
\[
h(x) = G^\top(x) \frac{\partial S}{\partial x}(x) \text{ and } \left[ \frac{\partial S}{\partial x}(x) \right]^\top f(x) \leq 0.
\]

Obviously, if there exists a storage function for $\Sigma$, then by substituting $x(\tau) = x(0)$ in (3), it follows that $\Sigma$ is cyclo-passive. A converse is given as follows. Assuming reachability from a ground-state $x^*$ and controllability to this same state $x^*$, define the functions $S_{ac} : \mathcal{X} \to \mathbb{R} \cup \mathbb{R}$ and $S_{rc} : \mathcal{X} \to \mathbb{R}$ as
\[
S_{ac}(x) := \sup_{u, \tau \geq 0} \int_0^\tau y^\top(t)u(t)dt,
\]
\[
S_{rc}(x) := \inf_{u, \tau \geq 0} \int_0^\tau y^\top(t)u(t)dt.
\]
The function $S_{ac}(x)$ is the maximal (in fact, supremal) energy that can be recovered from the system at state $x$ while returning the system to its ground-state $x^*$ (available energy under the constraint $x(\tau) = x^*$), and the function $S_{rc}(x)$ is the minimal (in fact, infimal) energy that is needed in order to transfer the system from the ground-state $x^*$ to $x$ (required energy under the constraint $x(-\tau) = x^*$).

It directly follows [14], [9], [10] from the properties of ‘supremum’ and ‘infimum’ that $S_{ac}$ and $S_{rc}$ satisfy the dissipation inequality. Hence, if either $S_{ac}$ or $S_{rc}$ are not taking values $\pm \infty$, then the system is cyclo-passive. Finally note that for a stastically passive system both $S_{ac}$ and $S_{rc}$ equal the zero function, and thus all storage functions are zero (up to a constant).

The following basic theorem was obtained in [10], extending the results of [3], [4].

**Theorem 1:** Assume $\Sigma$ is reachable from $x^*$ and controllable to $x^*$, then $\Sigma$ is cyclo-passive with respect to $x^*$ if and only if
\[
S_{ac}(x) \leq S_{rc}(x), \quad \text{for all } x \in \mathcal{X}.
\]

Furthermore, if $\Sigma$ is cyclo-passive with respect to $x^*$ then both $S_{ac}$ and $S_{rc}$ are storage functions for $\Sigma$, implying that $\Sigma$ is cyclo-passive. Moreover, if $\Sigma$ is cyclo-passive with respect to $x^*$, then $S_{ac}(x^*) = S_{rc}(x^*) = 0$, and any other storage function $S$ for $\Sigma$ satisfies
\[
S_{ac}(x) \leq S(x) - S(x^*) \leq S_{rc}(x).
\]

If the system is cyclo-lossless from $x^*$ then $S_{ac} = S_{rc}$, and the storage function is unique up to a constant. More generally, $\Sigma$ is cyclo-passive with unique (up to a constant) storage function if and only if for every $x \in \mathcal{X}$
\[
\inf_{u, \tau_1 \leq 0, \tau_2 \geq 0} \int_{x(\tau_1) = x(\tau_2) = x^*, x(0) = x}^{\tau_2} y^\top(t)u(t)dt = 0.
\]
Note that the choice of the ground-state $x^*$ in the above theorem is arbitrary. Indeed, if $\Sigma$ is reachable from and controllable to $x^*$, then so it is from any other ground-state. Furthermore, it follows from Theorem 1 that cyclo-passivity with respect to $x^*$ implies cyclo-passivity.

**Example 1:** A simple example of a passive system which is not cyclo-lossless, but still has unique storage function is the ubiquitous mass-spring-damper system
\[
\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix} =
\begin{bmatrix}
0 & -k \\
-k & -\frac{d}{m}
\end{bmatrix}
\begin{bmatrix}
q \\
p
\end{bmatrix} +
\begin{bmatrix}
0 \\
\frac{1}{m}
\end{bmatrix} u, \quad (u = \text{force})
\]
\[
y = \begin{bmatrix} 0 & \frac{1}{m} \end{bmatrix} q \quad (= \text{velocity})
\]

Here, $q$ is the extension of a linear spring with stiffness $k$, $p$ is the momentum of mass $m$, and $d > 0$ is the damping coefficient. Let $x = \text{col}(q, p)$. The dissipation inequality for quadratic storage functions $S(x) = \frac{1}{2}x^\top Qx$ with
\[
Q = \begin{bmatrix} Q_{11} & Q_{12} \\
Q_{12} & Q_{22} \end{bmatrix}
\]

reduces to the linear matrix inequality (LMI)
\[
\begin{bmatrix}
\frac{1}{m} & -k \\
-k & \frac{1}{m} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\
Q_{12} & Q_{22} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\
0 & 1 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\
Q_{12} & Q_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\
1 & \frac{1}{m} \end{bmatrix} \leq 0,
\]

The equality implies $Q_{12} = 0$ and $Q_{22} = \frac{1}{m}$. Substitution in the inequality yields the unique solution $Q_{11} = k$, corresponding to a unique quadratic storage function $S$ which equals the physical energy $H(q, p) = \frac{1}{2}m p^2 + \frac{1}{2}kq^2$. Thus, the mass-spring-damper system is such that for all $x$ we have
\[
\inf_{u, \tau_1 \leq 0, \tau_2 \geq 0} \int_{x(\tau_1) = x(\tau_2) = x^*, x(0) = x}^{\tau_2} \frac{p(t)}{m} u(t)dt = 0.
\]

Indeed, although all possible trajectories involve non-zero energy-dissipation due to the presence of the damper if $d > 0$, this dissipation can be made arbitrarily small.

The stronger notion of passivity corresponds to the existence of a nonnegative storage function. One considers instead of $S_{ac}$ the function $S_a : \mathcal{X} \to \mathbb{R} \cup \mathbb{R}$ given as
\[
S_a(x) := \sup_{u, \tau \geq 0} \int_0^\tau y^\top(t)u(t)dt,
\]
which is the supremal energy that can be extracted from the system at state $x$; whence the name available storage for $S_a$. Obviously, $S_a$ is nonnegative, and, again by the property of ‘supremum’, satisfies the dissipation inequality.

It follows [14], [9] that $\Sigma$ is passive if and only if $S_a(x) < \infty$ for every $x \in \mathcal{X}$. Furthermore, if $\Sigma$ is passive then $S_a$ is a nonnegative storage function satisfying $\inf_x S_a(x) = 0$, and all other nonnegative storage functions $S$ satisfy
\[
S_a(x) \leq S(x) - \inf_x S(x), \quad x \in \mathcal{X}.
\]
passivity implies (and in case Σ is reachable from $x^*$, is equivalent to)
$$
\int_0^\tau y(t)^\top u(t)dt \geq 0,
$$
(10)
for all $u : [0, \tau] \rightarrow \mathbb{R}^n, \tau \geq 0$, where $y(t)$ is the corresponding output for initial condition $x^*$. This is sometimes taken as the definition of passivity, and implies that the supplied energy is nonnegative for any motion from the state of minimal energy. However, in a nonlinear context there is often no natural ground-state $x^*$ with $S_a(x^*) = 0$, and there may even not exist such a ground-state, as demonstrated by the following example [10].

**Example 2:** Consider the scalar system
$$
\Sigma : \begin{cases}
    \dot{x} = u, \\
y = e^x.
\end{cases}
$$

The available storage (8) takes the form
$$
S_a(x) = \sup_{u, \tau \geq 0} \left\{ e^x - e^{x(\tau)} \right\} = e^x,
$$
and Σ is lossless with nonnegative storage function $e^x$, which is unique up to a constant. Note that $\inf S_a(x) = 0$, but there does not exist a finite $x^*$ such that $S_a(x^*) = 0$.

### III. ENERGY CONVERSION BY CARNOT-LIKE CYCLES

While the net total energy supplied to a cyclo-passive system with two ports during a cyclic motion is always nonnegative, energy conversion is concerned with the transfer of energy from one port to the other. As it will turn out, an important aspect in energy conversion is the interconnection topology of the physical system. This topology is clearly captured in its port-Hamiltonian formulation. An input-state-output port-Hamiltonian system is given as [9, 11]

$$
\begin{align*}
\dot{x} &= J(x)e - \mathcal{R}(x,e) + G(x)u, \\
y &= G(x)\dot{e}, \\
x &\in \mathcal{X},
\end{align*}
$$
(11)

with $n$-dimensional state space $\mathcal{X}$, Hamiltonian $H : \mathcal{X} \rightarrow \mathbb{R}$, skew-symmetric matrix $J(x) = -J^\top(x)$, and dissipation mapping $\mathcal{R}$ satisfying $e^\top \mathcal{R}(x,e) \geq 0$ for all $x, e$. By the properties of $J(x)$ and $\mathcal{R}(x,e)$, any port-Hamiltonian system (11) satisfies the dissipation inequality

$$
\frac{d}{dt}H(x) = e^\top J(x)e - e^\top \mathcal{R}(x,e) + e^\top G(x)u \leq y^\top u,
$$

and thus is cyclo-passive, with storage function $H$. Conversely, almost any cyclo-passive system can be formulated as a port-Hamiltonian system for some $J$ and $\mathcal{R}$ as above.\(^1\) Thus, by cyclo-passivity theory

$$
S_a(x) \leq H(x) - H(x^*) \leq S_{ac}(x).
$$

\(^1\)However, in systematic systems modeling one would start from a port-Hamiltonian description with $J$ and $\mathcal{R}$ dictated by the physics of the system.

Consider a port-Hamiltonian system (11) with two ports $(u_1, y_1)$ and $(u_2, y_2)$ as in Fig. 1. Consequently

$$
\frac{d}{dt}H(x) \leq y_1^\top u_1 + y_2^\top u_2.
$$
(12)

The central question addressed in this paper is how to convert energy which is flowing into the system at port 1 into energy which is flowing out of the system at port 2, and to identify the obstructions for doing so. It turns out that feasible strategies for energy conversion crucially depend on the interconnection topology given by $J(x), G(x), \mathcal{R}(x,e)$.

#### A. Topological Obstructions for Energy Conversion

Consider the subclass of port-Hamiltonian systems with two ports given as (for some splitting $x = (x_1, x_2)$)

$$
\begin{align*}
\dot{x}_1 &= J_1(x_1, x_2)e_1 - \mathcal{R}_1(x_1, x_2, e_1) + G_1(x_1, x_2)u_1, \\
\dot{x}_2 &= J_2(x_1, x_2)e_2 - \mathcal{R}_2(x_1, x_2, e_2) + G_2(x_1, x_2)u_2, \\
y_1 &= G_1(x_1, x_2)e_1, \\
y_2 &= G_2(x_1, x_2)e_2,
\end{align*}
$$
(13)

where $G_1(x_1, x_2)$ is an invertible matrix for all $x_1, x_2$. It follows that for any state there exists $u_1(\cdot)$ such that $\dot{x}_1 = 0$ (and thus $x_1 = \bar{x}_1$ is constant). This implies

$$
\frac{d}{dt}H(\bar{x}_1, x_2) = \left[ \frac{\partial H}{\partial x_2} (\bar{x}_1, x_2) \right]^\top \dot{x}_2 \leq y_2^\top u_2.
$$
(14)

Hence, for such input $u_1(\cdot)$ the system is cyclo-passive at the second port, with storage function $H(\bar{x}_1, x_2)$. Moreover, for any such $u_1$ and any $t$

$$
y_1^\top(t)u_1(t) = e_1^\top(t)G_1(\bar{x}_1, x_2(t))u_1(t) = e_1^\top(t)\left[ -J_1(\bar{x}_1, x_2(t))e_1(t) + \mathcal{R}_1(\bar{x}_1, x_2(t), e_1(t)) \right] = e_1^\top(t)\mathcal{R}_1(\bar{x}_1, x_2(t), e_1(t)) \geq 0.
$$

This implies the following theorem.

**Theorem 2:** Consider a port-Hamiltonian system of the form (13), with $G_1(x_1, x_2)$ invertible. Then for all cyclic motions with $x_1 = \bar{x}_1$ constant, $\int_0^\tau y_1^\top(t)u_1(t)dt \geq 0$, and thus the system is cyclo-passive at port 2 with storage function $H(\bar{x}_1, x_2)$. Also, for all motions along which $x_1 = \bar{x}_1$ is constant, $y_1^\top(t)u_1(t) \geq 0$ for all $t$, and thus the system...
is statically passive at port 1. Furthermore, the inequalities become equalities in case \( R_1(\bar{x}_1, x_2, e_2) = 0 \), respectively \( R_2(\bar{x}_1, x_2, e_2) = 0 \).

Hence, for constant \( x_1 = \bar{x}_1 \) not only the total net energy at the ports is flowing into the system, but in fact the power flow at port 1 is inwards at any time \( t \), while also the net energy flow at port 2 during a cyclic motion is inwards.

Additionally assume that \( G_1 \) is a constant matrix, and the partial Hessian matrix \( \frac{\partial^2 H}{\partial x_1^2} \) has full rank everywhere. Since

\[ \dot{e}_1 = \frac{\partial^2 H}{\partial x_1^2}(x_1, x_2) \dot{x}_1 + \text{other terms}, \]

it follows, after substituting the equation for \( \dot{x}_1 \) from (13), that \( u_1 \) in this case can be chosen such that \( e_1 \), or equivalently \( y_1 \), is constant. Furthermore, recall the partial Legendre transform \( H^*_1 \) of \( H \) with respect to \( x_1 \), given as

\[ H^*_1(e_1, x_2) = H(x_1, x_2) - e_1^T x_1, \quad (15) \]

where \( x_1 \) is expressed as a function of \( e_1, x_2 \) by means of the equation \( e_1 = \frac{\partial H}{\partial x_1}(x_1, x_2) \); as locally guaranteed by the full rank assumption on the partial Hessian. The following properties of the partial Legendre transform hold [6]

\[ \frac{\partial H^*_1}{\partial e_1}(e_1, x_2) = -x_1, \quad \frac{\partial H^*_1}{\partial x_2}(e_1, x_2) = \frac{\partial H}{\partial x_2}(x_1, x_2). \]

Hence, for \( u_1 \) such that \( y_1 = \bar{y}_1 \) and \( e_1 = \bar{e}_1 \) constant,

\[ \frac{d}{dt} H^*_1(e_1, x_2) = -x_1^T \dot{e}_1 + e_1^T \dot{x}_2 = e_2^T J_2(x_1, x_2) e_2 \]

\[ -e_2^T R_2(x_1, x_2, e_2) + e_2^T G_2(x_1, x_2, u_2) \leq y_2^T u_2. \]

Thus the system for constant \( y_1 = \bar{y}_1 \) and \( e_1 = \bar{e}_1 \) is cyclo-passive at the port \( (u_2, y_2) \), with respect to the storage function \( H^*_1(e_1, x_2) \). This property was called one-port cyclo-passivity in [12], and implies that we cannot convert energy from port 1 to port 2 while keeping \( y_1 \) constant.

Furthermore, in view of (13), all solutions with constant \( y_1 = \bar{y}_1 = G_1^T \bar{e}_1 \) satisfy (recall that \( \bar{e}_1 \) is the Carnot cycle)

\[ \int \bar{y}_1^T u_1(t) dt = \int \bar{e}_1^T G_1 u_1(t) dt \]

\[ = \int \bar{e}_1^T \left[ \dot{x}_1(t) + R_1(x_1(t), x_2(t), \bar{e}_1) \right] dt \]

\[ \geq \bar{e}_1^T (x_1(\tau) - x_1(0)). \]

This implies the following theorem, extending [12].

**Theorem 3:** Consider a port-Hamiltonian system of the form (13) with \( G_1 \) constant and invertible, and \( \frac{\partial^2 H}{\partial x_1^2}(x_1, x_2) \) full rank. Then, for all cyclic motions along which \( y_1 = \bar{y}_1 \) is constant:

\[ \int_0^\tau \bar{y}_1^T u_1(t) dt \geq 0 \quad \text{and} \quad \int_0^\tau \bar{y}_2^T u_2(t) dt \geq 0. \]

Furthermore, the inequalities become equalities in case \( R_1(\bar{x}_1, x_2, e_2) = 0 \), respectively \( R_2(\bar{x}_1, x_2, e_2) = 0 \).

Hence for constant \( y_1 = \bar{y}_1 \) the net energy flow at each port is inwards.

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**B. Carnot Cycles and Efficiency**

Within a thermodynamics context, with \((u_1, y_1)\) corresponding to the thermal port and \((u_2, y_2)\) to the mechanical port, the trajectories for which \( x_1 \) is constant are the isentropic curves (entropy \( S = x_1 \) constant), also called adiabatics, while the trajectories for which the temperature \( T = e_1 \) is constant are the isothermals.

In order to convert energy from port 1 to 2 we need by Theorem 3 more than one constant value of \( e_1 = \bar{e}_1 \). Within thermodynamics this amounts to the observation made by Carnot that at least two different constant temperatures \( T \) are needed in order to convert thermal energy into mechanical energy (‘work’) in a cyclic manner. This gives rise to the well-known *Carnot cycle*, consisting of an isothermal at high temperature, followed by an adiabatic leading to a lower temperature, another isothermal, and finally an adiabatic which brings the system back to its original state; see Fig. 2.

The notion of the Carnot cycle can be extended to general port-Hamiltonian systems of the form (13) as follows:

1. On the time-interval \([0, \tau_1]\) consider an ‘isothermal’ with respect to port 1, corresponding to a constant \( e_1 = e_1^0 \) (‘hot’ for ‘hot’). Then by (16)

\[ \frac{d}{dt} H^*_1(e_1, x_2) \leq y_2^T u_2. \]

2. On the time-interval \([\tau_1, \tau_2]\) consider an ‘adiabatic’ corresponding to a constant \( x_1 = \bar{x}_1 \). Then by (14)

\[ \frac{d}{dt} H(\bar{x}_1, x_2) \leq y_2^T u_2. \]

3. On the time-interval \([\tau_2, \tau_3]\) consider an ‘isothermal’ corresponding to a constant \( e_1 = e_1^0 \) (‘cold’). Then

\[ \frac{d}{dt} H^*_1(e_1, x_2) \leq y_2^T u_2. \]

4. Finally, on the time-interval \([\tau_3, \tau]\) consider an ‘adiabatic’ corresponding to a constant \( x_1 = \bar{x}_1 \). Then

\[ \frac{d}{dt} H(\bar{x}_1, x_2) \leq y_2^T u_2. \]

Note that by (15) on both isothermals:

\[ H(x_1, x_2) = H^*_1(e_1^0, x_2) + e_1^0 x_1, \]

\[ H(x_1, x_2) = H^*_1(e_1^0, x_2) + e_1^0 x_1, \]
respectively. Since the total process is a cycle, i.e., \(x(\tau) = x(0)\), addition of these inequalities yields
\[
0 = H(x(\tau)) - H(x(0)) = \int_0^\tau y_2(t)u_2(t)dt + e_1^h \Delta^h x_1 + e_1^c \Delta^c x_1,
\]
where \(\Delta^h x_1\) and \(\Delta^c x_1\) are the changes in \(x_1\) during the isothermals on \([0, \tau]\) and \([\tau, \tau_3]\), respectively. Thus, the total energy delivered (to the surroundings) via port 2 satisfies
\[
-\int_0^\tau y_2(t)u_2(t)dt \leq e_1^h \Delta^h x_1 + e_1^c \Delta^c x_1,
\]
with equality in case there is no dissipation, i.e., when \(R_1\) and \(R_2\) are zero. Note that by (17) the right-hand side of (19) is the total energy provided at port 1 to the system during the two isothermals if \(R_1 = 0\). Because of \(x(\tau) = x(0)\) necessarily \(\Delta^h x_1 + \Delta^c x_1 = 0\). Hence the right-hand side of (19) can be also written as \((e_1^h - e_1^c) \Delta^h x_1\).

Classically, the efficiency of the Carnot cycle (in case \(R_1\) and \(R_2\) are zero) is defined as the energy delivered via port 2 to the surroundings divided by the supplied energy via port 1 during the first isothermal (corresponding to 'hot' temperature \(e_1^h\)). Hence the efficiency is equal to
\[
\frac{e_1^h \Delta^h x_1 + e_1^c \Delta^c x_1}{e_1^h \Delta^h x_1} = 1 - \frac{e_1^c}{e_1^h},
\]
recovering the well-known expression from thermodynamics [2], [5].

**Example 3:** Consider a gas-piston system with incoming entropy flow (due to incoming heat) as in Fig. 3. The system is close to the original heat engine considered by Carnot and Clausius, and can be as well considered as a precursor to the Stirling engine [13]. Its port-Hamiltonian description is given by
\[
\begin{bmatrix}
\dot{S} \\
\dot{V} \\
\dot{\pi}
\end{bmatrix} = 
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & A \\
0 & -A & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H}{\partial S} \\
\frac{\partial H}{\partial V} \\
\frac{\partial H}{\partial \pi}
\end{bmatrix}
+ 
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix},
\]
with Hamiltonian \(H(S, V, \pi) = U(S, V) + \frac{1}{2m} \pi^2\) and outputs \(y_1 = \frac{\partial U}{\partial S} = (T), \ y_2 = \frac{\pi}{m}\).

**Example 4:** Consider an electro-mechanical actuator as depicted in Fig. 4. Such device converts electrical energy into mechanical motion (work).

The port-Hamiltonian description (with \(q\) and \(p\) the displacement and the momentum of the armature \(m\), respectively, and \(\varphi\) the magnetic flux of the coil) is given as
\[
\begin{bmatrix}
\dot{\varphi} \\
\dot{q} \\
\dot{p}
\end{bmatrix} = 
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H}{\partial \varphi} \\
\frac{\partial H}{\partial q} \\
\frac{\partial H}{\partial p}
\end{bmatrix}
+ 
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix},
\]
where \(u_1\) is the supplied voltage and \(y_1\) the associated current, while \(u_2\) is a mechanical force and \(y_2\) the velocity of the armature. The system (23) is in the form (13), with the mechanical and electromagnetic part coupled via the Hamiltonian
\[
H(\varphi, q, p) = \frac{\varphi^2}{2L(q)} + \frac{p^2}{2m},
\]
where the inductance \( L(q) > 0 \) depends inversely on the mechanical displacement \( q \).

Note that the system (23) has the same structure as the gas-piston system (21). Thus, (23) is one-port cyclo-passive at the mechanical port, and a similar Carnot cycle can be used in order to convert energy from the electrical port to the mechanical port. Let \( V_a > V_b \). The Carnot cycle described in Subsection III-B is implemented as follows:

1) Switch \( s_a \) is closed and \( s_b \) is open. The coil absorbs an amount of energy, say \( E_a \), from the upper source \( V_a \), while the current \( I = I_a \) remains constant. This results in a decreasing \( q \) and a certain amount of external work done by the armature (and port 2).

2) The coil is disconnected from the voltage source (both switches open) and undergoes an ‘adiabatic’ transformation in which the magnetic flux remains constant and the current \( I \) decreases from \( I_a \) to \( I_b \) as the armature recedes further to the coil.

3) Switch \( s_b \) is now closed and \( I = I_b \). The armature is turning back to its initial position with energy, say \( E_b \), being transferred to the lower voltage source \( V_b \).

4) Both switches open and \( I = I_b \). The armature displaces to change the inductance \( L(q) \) until the current equals \( I = I_a \). The magnetic flux remains constant.

Now, since the energy
\[
E_a = \int_{\varphi_a}^{\varphi_b} I_a d\varphi = I_a(\varphi_b - \varphi_a)
\]
and, likewise, \( E_b = -I_b(\varphi_b - \varphi_a) \), we get that
\[
\frac{E_b}{E_a} = -\frac{I_b}{I_a}.
\]
Hence, the efficiency of the Carnot cycle equals
\[
\frac{E_a + E_b}{E_a} = 1 + \frac{E_b}{E_a} = 1 - \frac{I_b}{I_a}.
\]

IV. ENERGY CONVERSION BY DIRECT INTERCONNECTION AND ENERGY-ROUTERS

In the previous section it was shown how port-Hamiltonian systems in the form (13) share well-known properties with standard thermodynamic systems, in the sense that no cyclic energy transfer is possible from port 1 to port 2 for constant \( y_1 \); thus leading to the use of Carnot-like cycles.

The essential assumption in (13) is the absence of off-diagonal blocks in the \( J \)-matrix, which implies that there is no direct topological connection between the dynamics of the sub-vectors \( x_1, x_2 \). As a result, the interaction is only via coupling of \( x_1, x_2 \) in the Hamiltonian \( H(x_1, x_2) \) (and to a lesser extent in \( J_1, J_2 \) and \( R_1, R_2 \)).

In this section, we start with an example where instead we do have off-diagonal terms; as a result of which there is easy energy transfer from one port to the other. Next we consider two examples where off-diagonal elements are intentionally introduced through the use of feedback, and how this leads to strategies for energy transfer.

A. Heat Exchanger

Consider a heat exchanger comprising two heat reservoirs connected via a conducting wall; see Fig. 5.

![Heat Exchanger Diagram]

The dynamics is given as
\[
\begin{bmatrix}
\dot{S}_1 \\
\dot{S}_2
\end{bmatrix} =
\begin{bmatrix}
0 & \lambda \frac{E_1' - E_2'}{E_1'E_2'} \\
\lambda \frac{E_1' - E_2'}{E_1'E_2'} & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial E}{\partial S_1} \\
\frac{\partial E}{\partial S_2}
\end{bmatrix} +
\begin{bmatrix}
1 & 0
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix},
\]
where \( \lambda \) is Fourier’s conduction coefficient, \( E_1(S_1) \) and \( E_2(S_2) \) are the (thermal) energies of heat reservoirs 1 and 2 with temperatures \( T_1 = E_1'(S_1) \) and \( T_2 = E_2'(S_2) \), and
\[
E(S_1, S_2) = E_1(S_1) + E_2(S_2)
\]
represents the total energy of the heat exchanger. Furthermore, \( u_1 \) and \( u_2 \) are the external entropy flows entering the reservoirs 1 and 2, so that \( y_1 u_1 \) and \( y_2 u_2 \) are the external heat flows entering the two reservoirs. The equations (24) constitute a quasi port-Hamiltonian system, where the total entropy \( S_1 + S_2 \) is always non-decreasing.

Although (24) is a thermodynamic system, it is not in the form (13), and, indeed, heat flow occurs from port 1 to port 2 for constant \( T_1 \) as long as \( T_1 > T_2 \).

B. Energy Transfer by Energy-Routing

Consider two cyclo-lossless port-Hamiltonian systems
\[
\Sigma_i : \begin{cases}
\dot{x}_i = J_i(x_i) \frac{\partial H_i}{\partial x_i}(x_i) + g_i(x_i)u_i, \\
y_i = \gamma_i(x_i) \frac{\partial H_i}{\partial x_i}(x_i),
\end{cases}
\]
with \( i \in \{1, 2\} \). The two systems may be coupled to each other by the output feedback (an elementary version of the Duindam-Stramigioli energy-router [1], [8])
\[
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} =
\begin{bmatrix}
0 & -y_1 y_2^\top \\
y_1 & 0
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} +
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix},
\]
with new inputs \( v_1 \) and \( v_2 \). Then, the closed-loop system \( \Sigma_1 \circ \Sigma_2 \) is a lossless port-Hamiltonian system with Hamiltonian
\[
H(x_1, x_2) = H_1(x_1) + H_2(x_2),
\]
but clearly not of the form (13). Due to the special form of the output feedback, we have that
\[
\frac{d}{dt} H_1 = -\|y_1\|^2 \|y_2\|^2 + y_1^\top v_1,
\]
\[
\frac{d}{dt} H_2 = \|y_1\|^2 \|y_2\|^2 + y_2^\top v_2.
\]

\(^2\)Quasi since the \( J \)-matrix does not directly depend on the state variables \((S_1, S_2)\), but through the temperatures \( T_1 = E_1'(S_1) \) and \( T_2 = E_2'(S_2) \).
Hence, if energy is pumped into $\Sigma_1 \circ \Sigma_2$ through port 1, then the stored energy in $\Sigma_2$ will increase—enabling its release through port 2.

C. Energy Transfer Using IDA-PBC

Consider again the electro-mechanical actuator (23). In order to establish an effective topological interconnection between the electrical and mechanical subsystems, we can try to enforce a coupling between the magnetic flux and the momentum; thus realizing a desired interconnection matrix of the form

$$J_d = \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 1 \\ -\alpha & -1 & 0 \end{bmatrix}, \quad (25)$$

where $\alpha$ is a constant or function to be defined.

The easiest way to accomplish this is when the second input is available for direct manipulation. Indeed, introducing the feedback

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

with new inputs $v_1$ and $v_2$, replaces the $J$-matrix in (23) by the interconnection matrix (25).

However, usually only the first input is available for control. On the other hand, invoking the IDA-PBC methodology [7], and selecting a state feedback $u_1 = \alpha(\varphi, q) \frac{\partial H}{\partial \varphi}(\varphi, q, p) + v_1$, with

$$\alpha(\varphi, q) = \frac{1}{4} L'(q) \varphi,$$

and $u_2 = v_2$, yields (see the Appendix for details)

$$\begin{bmatrix} \dot{\varphi} \\ \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \alpha(\varphi, q) \\ 0 & 0 & 1 \\ -\alpha(\varphi, q) & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_d}{\partial \varphi} \\ \frac{\partial H_d}{\partial q} \\ \frac{\partial H_d}{\partial p} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

in which the total energy is modified into

$$H_d(\varphi, q, p) = \frac{\varphi^2}{L(q)} + \frac{p^2}{2m}.$$  

As in the previous example, the system after the application of feedback is not in the form (13) anymore, and a direct interaction between the mechanical and magnetic subsystem is achieved through the lossless modulation $\alpha(\varphi, q)$. Moreover, the magnetic energy storage is doubled, enabling an increased energy conversion rate.

V. CONCLUSIONS AND OUTLOOK

Feasible strategies for energy conversion in physical systems are related to the structure of the $J$-matrix in their port-Hamiltonian formulation. In particular, if the off-diagonal blocks of $J$ are zero we generally have one-port cyclo-passivity [12], and need for more than one value of the output at port 1 in order to convert energy from port 1 to port 2. This may be achieved by a direct generalization of the classical Carnot cycle for heat engines. Another strategy is to use feedback in order to introduce non-zero off-diagonal blocks in the $J$-matrix.

Among the many open problems we mention:

- Consider a port-Hamiltonian system in the form (13). Can we generalize Carnot cycles to more general cyclic processes, while retaining a notion of efficiency?
- How to formulate optimal energy conversion from one port to another?
- The physical energy $H$ of any port-Hamiltonian system is somewhere "in between" $S_{\text{ac}}$ and $S_{\text{ca}}$. If we choose another storage function this will generally lead to a different $J$-matrix (as well as different dissipation structure $R$). How to exploit this for energy conversion?

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APPENDIX

The IDA-PBC methodology [7] entails to solving the so-called matching equation

$$J_d(x) \frac{\partial H_d}{\partial \varphi} = -J_a(x) \frac{\partial H}{\partial \varphi} + G(x) u,$$

with $J_a(x) = J_d(x) - J(x)$. For the system (23), desired interconnection structure (25), and $u_3 = \beta(\varphi, q, p) + v_1$, this amounts to solving

$$\alpha(\varphi, q, p) \frac{\partial H_d}{\partial \varphi}(\varphi, q, p) = -\alpha(\varphi, q, p) \frac{\partial H}{\partial \varphi}(\varphi, q, p) + \beta(\varphi, q, p),$$

$$\frac{\partial H_d}{\partial \varphi}(\varphi, q, p) = 0,$$

and

$$\alpha(\varphi, q, p) \frac{\partial H_d}{\partial q}(\varphi, q, p) = -\alpha(\varphi, q, p) \frac{\partial H}{\partial q}(\varphi, q, p) + \alpha(\varphi, q, p) \frac{\partial H}{\partial \varphi}(\varphi, q, p).$$

for $H_d$, $\alpha$ and $\beta$. Selecting $H_d(\varphi, q, p) = \frac{1}{L(q)} \varphi^2$, this leads to

$$\frac{1}{2} \frac{\partial H_d}{\partial \varphi} \frac{\partial \varphi}{\partial \varphi} = 2 \alpha(\varphi, q) \frac{\partial H_d}{\partial \varphi}(\varphi, q, p) \Rightarrow \alpha(\varphi, q) = \frac{1}{4} \frac{\partial H_d}{\partial \varphi}(\varphi, q, p),$$

together with the state feedback control

$$\beta(\varphi, q, p) = \alpha(\varphi, q) \frac{\partial H_d}{\partial \varphi}(\varphi, q, p),$$

and modified energy storage $H_d(\varphi, q, p) = H(\varphi, q, p) + H_a(\varphi, q)$. 

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