Dynamic cumulative residual entropy generating function and its properties

S. Smitha\textsuperscript{a}, Sudheesh K. Kattumannil\textsuperscript{b}, and E. P. Sreedevi\textsuperscript{c}

\textsuperscript{a}K E College Mannanam, Kerala, India; \textsuperscript{b}Indian Statistical Institute, Chennai, India; \textsuperscript{c}Cochin University of Science and Technology, Kochi, India

\textbf{ABSTRACT}
In this work, we study the properties of the cumulative residual entropy generating function (CREGF). We also discuss the non parametric estimation of CREGF. We introduce dynamic cumulative residual entropy generating function (DCREGF). It is shown that the DCREGF determines the distribution uniquely. We study some characterization results using the relationship between DCREGF, hazard rate, and mean residual life function. A new class of life distributions based on decreasing DCREGF is introduced. Finally, we develop a test for decreasing DCREGF and study its performance.

\textbf{ARTICLE HISTORY}
Received 11 January 2023
Accepted 06 July 2023

\textbf{KEYWORDS}
Entropy; entropy generating function; U-statistics.

\section{1. Introduction}
Entropy is an important concept in the field of information theory and Shannon (1948) was the first who formally introduced it. To measure the uncertainty contained in a random variable \(X\), entropy is defined as

\[H(X) = -\int_0^\infty f(x) \log f(x)\,dx = E(- \log f(X)),\]

where “\(\log\)” denotes the natural logarithm. Several measures of entropy have been introduced in the literature, each one suitable for some specific situations. Cumulative residual entropy (CRE) is given by Rao et al. (2004)

\[C\mathcal{RE}(X) = -\int_0^\infty \bar{F}(x) \log \bar{F}(x)\,dx,\]

where \(\bar{F}(x) = 1 - F(x)\) is the survival function of \(X\). Di Crescenzo and Longobardi (2009) introduced cumulative entropy (CE) for estimating the uncertainty in the past lifetime of a system as

\[C\mathcal{E}(X) = -\int_0^\infty F(x) \log F(x)\,dx.

The weighted versions of \(C\mathcal{RE}(X)\) and \(C\mathcal{E}(X)\) have been studied in the literature as well. These are given by Mirali, Baratpour, and Fakoor (2017)

\[C\mathcal{RE}^w(X) = -\int_0^\infty x\bar{F}(x) \log \bar{F}(x)\,dx\]
and Mirali and Baratpour (2017)

\[ CE^w(x) = - \int_0^\infty xF(x) \log F(x) \, dx. \]

For recent development in this area, we refer to Rajesh and Sunoj (2019), Kharazmi and Balakrishnan (2022, 2021a, 2021b), Ahmadi (2021), and Sudheesh, Sreedevi, and Balakrishnan (2022). Among these, Sudheesh, Sreedevi, and Balakrishnan (2022) defined a generalized cumulative residual entropy and studied its properties. They showed that the cumulative residual entropy and the weighted cumulative residual entropy are special cases of the proposed measure.

The moment generating function (m.g.f) of a probability distribution is a convenient tool for evaluating mean, variance, and other moments of a probability distribution. The successive derivatives of the m.g.f at a point zero give the successive moments of the probability distribution, provided these moments exists. In information theory, generating functions have been defined for probability density function to determine information quantities such as Shannon information, entropy, and Kullback-Leibler divergence. Golomb (1966) introduced entropy generating function of a probability distribution which is given by

\[ B(s) = \int f^s(x) \, dx, \quad s \geq 1. \]

It may be noted that the first derivative of this generating function at \( s = 1 \), gives the negative of Shannon's entropy in (1).

The generating function approach to describe different entropy measures is widely discussed in the literature. Guiasu and Reischer (1985) made a significant contribution to the relative information generating function and they observed that its derivatives provide well-known statistical measures like the Kullback and Leibler (1951) divergence between two probability distributions. Information-improving generating function was proposed by Hooda and Singh (1990). The first derivative of their measure at 1 gives Theils measures of information improvement. Jha, Dewangan, and Verma (2012) studied a few generating functions for probabilistic entropy and directed divergence measures. Papaioannou, Ferentinos, and Tsairidis (2007) introduced Fisher information and divergence generating functions. Kharazmi and Balakrishnan (2021) proposed a divergence generating function, which yields some well-known measures of entropy. Zamani, Kharazmi, and Balakrishnan (2022) studied the information generating function of record values and provided some main properties of it. They examined the relative information generating measure between the distribution of records values and the corresponding underlying distribution and present some results in this regard.

Asadi and Zohrevand (2007) have modified the definition of CRE in order to accommodate the current age of the system by

\[ CRE(X; t) = - \int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)} \log \left( \frac{\bar{F}(x)}{\bar{F}(t)} \right) \, dx. \]

The measure \( CRE(X; t) \) is known as dynamic cumulative residual entropy function. Obviously \( CRE(X; 0) = CRE(X) \). For more properties and applications of (2) and (3), one may refer to Rao et al. (2004), Asadi and Zohrevand (2007), Navarro, Del Aguila, and Asadi (2010), and the references therein.

The rest of the article is organized as follows. In Section 2, we study some properties of the cumulative residual entropy generating function (CREGF). We also discuss the non
parametric estimation of CREGF. In Section 3, we introduce dynamic cumulative residual entropy generating function. We prove that the proposed measure determines the distribution uniquely. We also study some characterization results using the relationship of DCREGF with hazard rate and mean residual life function. In Section 4, we develop a test for testing decreasing DCREGF. In Section 5, we conduct a Monte Carlo simulation study to assess the finite sample performance of the proposed test. Some concluding remarks with some open problems are given in Section 6.

2. Cumulative residual entropy generating function

Let $X$ be non negative continuous random variable having distribution function $F(x)$. Let $\bar{F}(x)$ be the survival function of $X$. The survival function is more useful in lifetime studies. This motivate us to study the properties of CREGF.

**Definition 1.** Let $X$ be a non negative random variable having an absolutely continuous survival function $\bar{F}(x)$, CREGF denoted by $C_s(X)$ is defined as

$$C_s(X) = \int_0^\infty (\bar{F}(x))^s dx, \ s > 0.$$

We noted that Kharazmi and Balakrishnan (2021) discussed the measure in (4) and obtained some relationship with Gini mean difference. They also discussed relative cumulative residual information generating measure to study the closeness between two survival function. We further study some properties of $C_s(X)$. We also discuss the non parametric estimation of $C_s(X)$.

It may be noted that negative of the first derivative of $C_s(X)$ at $s = 1$ gives the cumulative residual entropy function given in (2). Also note that (prime denote the derivative)

$$C_s(X)\big|_{s=2} = \int_0^\infty (\bar{F}(x))^2 \log \bar{F}(x) dx.$$ 

Table 1 gives the expression of $C_s(X)$ for some well-known distributions.

Next, we prove some properties of $C_s(X)$. In the following property, we show that $C_s(X)$ is a shift independent measure.

**Theorem 1.** Let $X$ be continuous non negative random variable and $Y = aX + b$, with $a > 0$ and $b \geq 0$, then we have

$$C_s(Y) = a \cdot C_s(X).$$

| Distribution     | $\bar{F}(x)$ | $C_s(X)$                  | $CE(X)$ |
|------------------|--------------|---------------------------|---------|
| (i) $U(0, a)$    | $1 - \frac{x}{a}, x > 0$ | $\frac{a}{s-1}$ | $\frac{a}{4}$ |
| (ii) GPD         | $(1 + \frac{ax}{b})^{-\left(1 + \frac{1}{b}\right)}, x > 0, a > -1, b > 0$ | $\frac{b}{(a+1)s-a}$ | $b(a+1)$ |
| (iii) Pareto $(k; \alpha)$ | $(\frac{k}{x})^{\alpha}, x \geq k$ | $\frac{1}{\alpha} \left(\frac{1}{x}\right)^{\alpha-1}$ | $\frac{k \alpha}{(\alpha-1)^2}$ |
| (iv) $\exp(\lambda)$ | $\exp(-\lambda x), x \geq 0$ | $\frac{1}{\lambda} \left(\frac{1}{x}\right)^{\frac{1}{\alpha}}$ | $\frac{1}{\lambda} \left(\frac{1}{x}\right)^{\frac{1}{\alpha}}$ |
| (v) Pareto II    | $(1 + \frac{x}{a})^{-b}, x \geq 0$ | $\frac{1}{bs-1}$ | $\frac{1}{ab} (b-1)^2$ |
Proof. The result follows by noting that \( \bar{F}_{ax+b}(x) = \bar{F}_X \left( \frac{x-b}{a} \right) \) for all \( x > b \).

**Theorem 2.** Suppose \( X \) and \( Y_\theta \) are two random variable admitting the proportional hazard model given by
\[
\bar{F}^*_\theta(x) = (\bar{F}(x))^\theta, \quad \theta > 0, \quad x > 0.
\]

Then the following statements hold;
(a) \( C_s(Y_\theta) = C_s(\theta X) \)
(b) \( C_s(Y_\theta) = \theta \cdot C_s(\theta X) \).

**Corollary 1.** Let \( X \) be non negative random variable having absolutely continuous distribution function \( F \). Let \( X_{1:n} \) be the first order statistic based on a random sample \( X_1, X_2, \ldots, X_n \) from \( F \). We have \( \bar{F}_{X_{1:n}}(x) = (\bar{F}(x))^n \) and so \( C_s(X_{1:n}) = C_{ns}(X) \).

**Example 1.** Suppose \( X \sim \exp(\lambda) \), then \( C_s(Y_\theta) = \frac{1}{\lambda^s}, C_s(\theta X) = \frac{1}{\lambda^s} \) and \( C_s(X) = \frac{1}{\lambda^s} \). So we have \( C_s(Y_\theta) = C_s(\theta X) \) and \( C_s(X) = \theta \cdot C_s(\theta X) \).

In the next theorem, we give bound of CREGF based on the mean of \( X \).

**Theorem 3.** Let \( X \) be a non negative continuous random variable with finite mean \( \mu \) and CREGF \( C_s(X) \). Then, \( C_s(X) < \mu \).

Proof. We have,
\[
(\bar{F}(x))^s < \bar{F}(x), \quad s > 1.
\]

Integrating both sides of the above equation with respect to \( x \),
\[
\int_0^\infty (\bar{F}(x))^s dx < \int_0^\infty \bar{F}(x) dx.
\]
That is,
\[
C_s(X) < \mu.
\]

Next, we find an estimator for \( C_s(X) \). Here we assume that \( s \) is a positive integer. Let \( X_{1:n} \) be the first-order statistic based on a random sample \( X_1, \ldots, X_n \) from \( F \). Then we have \( \bar{F}_{X_{1:n}}(x) = (\bar{F}(x))^n \). For a non negative random variable \( X \), we have \( \mu = E(X) = \int_0^\infty \bar{F}(x) dx \). Hence, we can write
\[
C_s(X) = \int_0^\infty (\bar{F}(x))^s dx = E(X_{1:s}). \tag{5}
\]

Hence a U-statistic-based estimator of \( C_s(X) \) is given by
\[
\hat{C}_s(X) = \frac{1}{C_{s,n}} \sum_{C_{i,n}} \min(X_{i_1}, X_{i_2}, \ldots, X_{i_s}), \tag{6}
\]
where the summations is over the set \( C_{s,n} \) of all combinations of \( s \) distinct elements \( \{i_1, i_2, \ldots, i_s\} \) chosen from \( \{1, 2, \ldots, n\} \). Now, we express \( \hat{C}_s(X) \) in a simple form. Let \( X_{i,n} \)
be the \(i\)-th order statistics based on \(n\) random sample \(X_1, \ldots, X_n\) from \(F\). In terms of the order statistics, we have the following equivalent expressions
\[
\sum_{i=1}^{n} \sum_{j=1, j<i}^{n} \min\{X_1, X_2\} = \sum_{i=1}^{n} (n - i)X_{1:n}
\]
and
\[
\sum_{i=1}^{n} \sum_{j=1, j<i}^{n} \sum_{k=1, k<j}^{n} \min\{X_1, X_2, X_3\} = \sum_{i=1}^{n} \frac{(n - i - 1)(n - i)}{2}X_{1:n}
\]
\[
= \sum_{i=1}^{n} \binom{n - i}{2}X_{1:n}.
\]
Therefore, the estimator given in (6) can be expressed as
\[
\widehat{C}_s(X) = \frac{1}{C_{5,n}} \sum_{i=1}^{n} \binom{n - i}{2}X_{1:n}.
\] (7)

Next we study the asymptotic properties of \(\widehat{C}_s(X)\). Clearly \(\widehat{C}_s(X)\) is an unbiased and consistent estimator of \(C_s(X)\) (Lehmann 1951). In the following result, we establish the asymptotic distribution of \(\widehat{C}_s(X)\).

**Theorem 4.** As \(n \to \infty\), \(\sqrt{n}(\widehat{C}_s(X) - C_s(X))\) converges in distribution to a normal random variable with mean zero and variance \(s^2\sigma^2\), where \(\sigma^2\) is given by
\[
\sigma^2 = \text{Var}\left(X\bar{F}^{s-1}(X) + (s - 1) \int_0^{X} y\bar{F}^{s-2}(y)dF(y)\right).
\] (8)

**Proof.** Using the central limit theorem for U-statistics, we have the asymptotic normality of \(\widehat{C}_s(X)\). The asymptotic variance is \(s^2\sigma^2_1\) where \(\sigma^2_1\) is given by Lee (2019)
\[
\sigma^2_1 = \text{Var}\left[E\left(\min(X_1, \ldots, X_s)\mid X_1\right)\right].
\] (9)

Denote \(Z = \min(X_2, X_3, \ldots, X_s)\), then the survival function of \(Z\) is given by \(\bar{F}^{s-1}(x)\). Consider
\[
E\left[\min(x, X_2, X_3, \ldots, X_s)\right] = E\left[xI(Z > x)\right] + E\left[ZI(Z \leq x)\right]
\]
\[
= x\bar{F}^{s-1}(x) + (s - 1) \int_0^{x} y\bar{F}^{s-2}(y)dF(y).
\]

Therefore, from (9) we obtain the variance expression specified in Equation (8). This completes the proof of the theorem. \(\square\)

The finite sample performance of the estimator given in (7) is evaluated through Monte Carlo simulation and the results of the same are reported in Section 5.

### 3. Dynamic cumulative residual entropy generating function

In this section, we define dynamic cumulative residual entropy generating function (DCREGF). We establish some characterization results for some well-known distributions in terms of DCREGF. Let \(X\) be the lifetime of a component or system under the condition
that the system has survived up to an age $t$. In such cases, we are interested in studying dynamic or time-dependent random variable $X_t = (X - t \mid X > t)$ having survival function

$$
\tilde{F}_t(x) = \begin{cases} 
\frac{F(x)}{F(t)}, & \text{where } x > t \\
1, & \text{otherwise.}
\end{cases}
$$

Then the mean residual life function $m(t)$ is defined as $m(t) = E(X_t)$. Using the survival function of $X_t$, we define DCREGF of $X$.

**Definition 2.** Let $X$ be a continuous random variable with distribution function $F(x)$. Then DCREGF of $X$ is defined as

$$
C_s(X; t) = \int_t^{\infty} \left( \frac{\tilde{F}(x)}{\tilde{F}(t)} \right)^s dx, \quad s \geq 1. \tag{10}
$$

From Definition 2, we readily observe the following properties of $C_s(X)$.

1. For $t = 0$, $C_s(X; t) = C_s(X)$;
2. When $s = 1$, $C_1(X; t) = m(t)$;
3. Differentiating $C_s(X; t)$ with respect to $s$ and taking $s = 1$, we obtain the negative of the dynamic cumulative residual entropy given in (3).

**Theorem 5.** For a non negative random variable $X$, let $Y = aX + b$, where $a > 0$ and $b \geq 0$. Then we have

$$
C_s(Y; t) = a \cdot C_s \left( X; \frac{t - b}{a} \right), \quad t \geq b.
$$

**Proof.** Consider

$$
C_s(Y; t) = \int_t^{\infty} \left( \frac{\tilde{F}(x)}{\tilde{F}(t)} \right)^s dx
= \int_{\frac{t - b}{a}}^{\infty} \left( \frac{\tilde{F}(u)}{\tilde{F}(\frac{t - b}{a})} \right)^s du
= a \cdot C_s \left( X; \frac{t - b}{a} \right), \quad t \geq b.
$$

From the above theorem, the following property can be easily obtained for particular choices of $a$ and $b$.

**Corollary 2.** We have

(a) $C_s(aX; t) = a \cdot C_s \left( X; \frac{t}{a} \right)$.
(b) $C_s(X + b; t) = C_s(X; t - b)$.

**Theorem 6.** Let $X$ be a continuous random variable with distribution function $F(x)$ and hazard rate function $h(t) = \frac{f(t)}{\tilde{F}(t)}$. The relationship between hazard rate and DCREGF is given by
\[ h(t) = \frac{1 + C_s'(X; t)}{s \cdot C_s(X; t)}. \]  

**Proof.** From the definition in (10), we have \[
(\tilde{F}(t))^s C_s(X; t) = \int_{t}^{\infty} (\tilde{F}(x))^s dx.
\]
Differentiating both sides of the above equation with respect to \(t\), we obtain \[
(\tilde{F}(t))^s C_s'(X; t) - C_s(X; t) \cdot s \cdot (\tilde{F}(t))^{s-1} f(t) = - (\tilde{F}(t))^s,
\]
where prime denotes the derivative. In terms of hazard rate, the above equation can be written as \[
C_s'(X; t) - s \cdot h(t) \cdot C_s(X; t) + 1 = 0.
\]
Hence, we obtain \[
h(t) = \frac{1 + C_s'(X; t)}{s \cdot C_s(X; t)}.
\]

In the following theorem, we show that the dynamic cumulative residual entropy generating function determines the distribution of \(X\) uniquely.

**Theorem 7.** Let \(X\) be a non negative random variable with density function \(f(x)\), survival function \(\tilde{F}(x)\) and hazard rate function \(h(x)\). Then \(C_s(F; t)\) uniquely determines the survival function \(\tilde{F}(t)\).

**Proof.** From the relationship between the hazard rate and the DCREGF given in (11), we have \[
- \frac{d}{dt} \log \tilde{F}(t) = \frac{1 + C_s'(X; t)}{s \cdot C_s(X; t)}.
\]
Integrating over the interval \((0, t)\), we obtain \[
- \log \tilde{F}(t) = \int_{0}^{t} \frac{1 + C_s'(X; u)}{s \cdot C_s(X; u)} du.
\]
That is \[
\tilde{F}(t) = \exp \left[ - \int_{0}^{t} \left( \frac{1 + C_s'(X; u)}{s \cdot C_s(X; u)} \right) du \right].
\]
This shows that the knowledge of \(C_s(X; t)\) enables us to determine the distribution.

Now, suppose that \(F(x)\) and \(G(x)\) are two distribution functions such that \[
C_s(F; t) = C_s(G; t).
\]
That is, \[
\frac{1}{(\tilde{F}(t))^s} \int_{t}^{\infty} (\tilde{F}(x))^s dx = \frac{1}{(\tilde{G}(t))^s} \int_{t}^{\infty} (\tilde{G}(x))^s dx.
\]
Differentiating with respect to $t$, we have

$$(\tilde{F}(t))^{-s} \cdot (- (\tilde{F}(t))^s) + \left( \int_t^\infty (\tilde{F}(x))^s \, dx \right) \cdot s (\tilde{F}(t))^{-s-1} f(t)$$

$$= (\tilde{G}(t))^{-s} \cdot (- (\tilde{G}(t))^s) + \left( \int_t^\infty (\tilde{G}(x))^s \, dx \right) \cdot s (\tilde{G}(t))^{-s-1} g(t).$$

In terms of hazard functions $h_1 = f(t)/\tilde{F}(t)$ and $h_2 = g(t)/\tilde{G}(t)$, we have

$$h_1(t) \cdot \frac{1}{(\tilde{F}(t))^s} \int_t^\infty (\tilde{F}(x))^s \, dx = h_2(t) \cdot \frac{1}{(\tilde{G}(t))^s} \int_t^\infty (\tilde{G}(x))^s \, dx,$$

which gives

$$h_1(t) C_s(F; t) = h_2(t) C_s(G; t).$$

In view of (12), we obtain

$$h_1(t) = h_2(t).$$

As the hazard rate function characterizes the distribution, this implies that $C_s(F; t)$ determines the distribution of $X$ uniquely.

Next theorem shows that the dynamic cumulative residual entropy generating function is independent of $t$ if and only if $X$ is a exponential random variable.

**Theorem 8.** Let $X$ be a non negative random variable admitting an absolutely continuous distribution function $F(x)$. The dynamic cumulative residual entropy generating function is independent of $t$ if and only if $X$ has exponential distribution.

**Proof.** Let $C_s(F; t) = k$, where $k$ is a positive constant. Hence

$$C_s'(F; t) = 0.$$

From (11), we obtain

$$s \cdot k \cdot h(t) = 1.$$

Hence, $h(t) = \frac{1}{s \cdot k} = \beta$, a constant. Constant hazard rate characterizes the exponential distribution. Therefore, $X$ is distributed as an exponential random variable with parameter $\beta$.

Conversely, assume that $X \sim \text{exp}(\beta)$ where the survival function is given by $\tilde{F}(x) = \exp(-\beta x)$. Therefore,

$$C_s(F; t) = \frac{1}{e^{-\beta ts}} \int_t^\infty e^{-\beta sx} \, dx$$

$$= \frac{1}{\beta s},$$

which is a constant. That is, DCREGF is independent of $t$ if and only if $X$ has exponential distribution.

The next theorem provides a characterization result for the generalized Pareto distribution based on a functional form for the dynamic cumulative residual entropy generating function.
Theorem 9. Let $X$ be a non-negative continuous random variable with survival function $\bar{F}(x)$. The dynamic cumulative residual entropy generating function is a linear function of $t$ if and only if $X$ follows generalized Pareto distribution.

Proof. Assume that, $C_s(F; t) = a + bt; b \neq 0$. Then $C_s'(F; t) = b$. From (11),

$$b - s \cdot h(t)(a + bt) + 1 = 0.$$ 

Or

$$h(t) = \frac{b + 1}{s} \frac{1}{(bt + a)}.$$

This can be rewritten as

$$h(t) = \frac{1}{ct + d},$$

where $c = \frac{bs}{b+1}$ and $d = \frac{as}{b+1}$. This is the hazard rate of GPD. Since the distribution function is uniquely determined by the hazard rate, $X \sim GPD$.

Conversely assume that, $X \sim GPD$, where the survival function of $X$ is given by

$$\bar{F}(x) = \left(1 + \frac{ax}{b}\right)^{-\left(1+\frac{1}{a}\right)}.$$ 

Using (10) we obtain

$$C_s(t) = k(b + at), \quad k = \frac{1}{(a + 1)s - a}.$$ 

That is, $C_s(t)$ is linear function in $t$. \qed

Now, we provide some characterization results in terms of relationship between the DCREGF and the hazard rate function $h(t)$.

Corollary 3. Let $X$ be a non-negative random variable with survival function $\bar{F}(t)$, hazard rate function $h(t)$, and DCREGF $C_s(t)$. Then the following relationship

$$C_s(t) = k(h(t))^{-1},$$

where $k$ is a positive constant, holds if and only if $F(x)$ is the GPD with survival function

$$\bar{F}(x) = \left(1 + \frac{ax}{b}\right)^{-\left(1+\frac{1}{a}\right)}, \quad a > -1, b > 0.$$ 

Proof. Under the assumption that (13) holds, we obtain

$$C_s'(t) = -k(h(t))^{-2}h'(t).$$ 

Using (11), we have

$$(sk - 1) = -k\frac{h'(t)}{(h(t))^2}.$$ 

That is

$$\frac{d}{dt} \left(\frac{1}{h(t)}\right) = \left(s - \frac{1}{k}\right).$$
Integrating on both sides of the above equation, we have

\[ \frac{1}{h(t)} = \left( \frac{sk - 1}{k} \right) t + d_2. \]

That is

\[ \frac{1}{h(t)} = d_1 t + d_2. \]

Therefore

\[ h(t) = \frac{1}{d_1 t + d_2}, \quad (14) \]

where \( d_1 = \frac{sk - 1}{k} \) and \( d_2^{-1} = h(0) \). Hall and Wellner (1981) have shown that (14) is a characteristic property of GPD. Conversely, assume that the random variable \( X \) follows GPD. By direct calculation, we obtain

\[ C_s(t) = b + at \quad \frac{1}{(a + 1)s - a} \]

\[ = \left( b + at \right) \left( \frac{a + 1}{(a + 1)s - a} \right) \]

\[ = k \cdot \frac{1}{h(t)} = k(h(t))^{-1}. \]

where \( k = \frac{a + 1}{(a + 1)s - a} \). Hence we have the proof of the theorem. \( \Box \)

In the following corollary, we characterize the distribution of \( X \) using the relationship between the DCREGF and the mean residual life function (MRL).

**Corollary 4.** Let \( X \) be a non negative random variable admitting an absolutely continuous distribution function \( F(x) \). Define MRL of \( X \) as \( m(t) = E(X - t|X > t) \). Then the relationship, \( C_s(t) = k \cdot m(t) \) holds for every \( t > 0 \), if and only if \( X \sim \text{GPD} \).

**Proof.** Assume

\[ C_s(t) = k \cdot m(t). \quad (15) \]

Differentiating with respect to \( t \), we have

\[ C'_s(t) = k \cdot m'(t). \]

From (11), we have

\[ s \cdot h(t) \cdot C_s(t) - 1 = k \cdot m'(t). \]

Using (15), we obtain

\[ k \cdot s \cdot h(t) \cdot m(t) - 1 = k \cdot m'(t) \]

Now, we have the relationship between the hazard rate and the mean residual life given by

\[ \frac{1 + m'(t)}{m(t)} = h(t). \]
Therefore, we have
\[ m'(t) = \frac{1 - ks}{ks - k}, \]
a constant, which implies that \( m(t) \) is a linear function in \( t \). The linear mean residual life function is a characteristic property of GPD.

Conversely, assume that \( X \sim GPD \). By direct calculation, we obtain
\[ C_s(X; t) = k(b + at) = k \cdot m(t), \quad (16) \]
where \( k = \frac{1}{(a+1)s-a} \). This proves the if part of the theorem.

**Remark 1.** Differentiating both sides of (16) with respect to \( s \) and setting \( s = 1 \) and evaluating it with a negative sign, we obtain Theorem 4.8 of Asadi and Zohrevand (2007).

Next, using DCREGF we introduce two new classes of lifetime distributions.

**Definition 3.** A random variable \( X \) is said to have increasing (decreasing) DCREGF, denoted by IDCREF (DDCREGF) if \( C_s(X; t) \) is increasing (decreasing) function in \( t \), \( \forall t \geq 0 \).

The following theorem gives the bounds for \( C_s(X; t) \) in terms of hazard rate function.

**Theorem 10.** The distribution function \( F \) is increasing (decreasing) DCREGF if and only if for all \( t > 0 \)
\[ C_s(F; t) \geq (\leq) \frac{1}{s \cdot h(t)}, \quad s \geq 1. \]

**Proof.** From the definition of new classes of lifetime distributions, the distribution function \( F \) is said to be increasing (decreasing) if \( C_s(F; t) \) is increasing (decreasing) in \( t \). That is
\[ C'_s(F; t) \geq (\leq) 0. \]
Hence
\[ s \cdot h(t) \cdot C_s(F; t) - 1 \geq (\leq) 0. \]
Or
\[ C_s(F; t) \geq (\leq) \frac{1}{s \cdot h(t)}. \]

**Theorem 11.** Let \( X \) and \( Y \) be two non negative absolutely continuous random variables with survival functions \( \tilde{F}(t) \) and \( \tilde{G}(t) \) and hazard rate functions \( h_1(t) \) and \( h_2(t) \), respectively. If \( X \geq_{hr} Y \), that is, \( h_1(t) \leq h_2(t) \) \( \forall t \geq 0 \), then \( C_s(F; t) \geq C_s(G; t) \).

**Proof.** Let \( X \geq_{hr} Y \), then we have
\[ h_1(t) \leq h_2(t) \implies \tilde{F}_X(t) \geq \tilde{G}_Y(t) \]
\[ \implies \frac{\tilde{F}(x)}{\tilde{F}(t)} \geq \frac{\tilde{G}(x)}{\tilde{G}(t)} \]
\[ \int_t^\infty \left( \frac{\bar{F}(x)}{F(t)} \right)^s \, dx \geq \int_t^\infty \left( \frac{\bar{G}(x)}{G(t)} \right)^s \, dx \]
\[ \implies C_s(F; t) \geq C_s(G; t). \]

The following example gives an application of Theorem 11 in order statistics.

**Example 2.** Let \( X_1, \ldots, X_n \) be independent and identical non-negative random variables with survival function \( \bar{F}(x) \). If \( X_{i:n} \) denotes the \( i \)-th order statistic based on a random sample \( X_1, \ldots, X_n \) from \( F \), then the following results hold:

(i) \( C_s(X_{i:n}; t) \leq C_s(X_{i+1:n}; t) \)

(ii) \( C_s(X_{i:n}; t) \leq C_s(X_{1:n-1}; t) \)

(iii) \( C_s(X_{n:n}; t) \geq C_s(X_{n-1:n-1}; t) \).

The following theorem shows that the exponential distribution is the unique distribution which is both IDCREGF and DDCREGF.

**Theorem 12.** Let \( X \) be a non-negative random variable having IDCREGF and DDCREGF, then \( X \) has exponential distribution.

**Proof.** As \( X \) has both IDCREGF and DDCREGF properties, \( C_s(X; t) \) is a constant. Hence, from Theorem 8, \( X \) has exponential distribution.

**Corollary 5.** Let \( X \) be DDCREGF (IDCREGF), then

\[ \bar{F}(t) \geq (\leq) \exp \left[ - \int_0^t \frac{1}{s \cdot C_s(F; t)} \, dx \right]. \]

**Proof.** Let \( X \) be DDCREGF, then

\[ h(t) \leq \frac{1}{s \cdot C_s(F; t)}. \]

Again, using the relationship between \( \bar{F}(t) \) and \( h(t) \), we have

\[ \bar{F}(t) = \exp \left[ - \int_0^t h(u) \, du \right] \geq \exp \left[ - \int_0^t \frac{1}{s \cdot C_s(F; t)} \, dx \right]. \]

Similarly, we can show that if \( X \) has IDCREGF, then

\[ \bar{F}(t) \leq \exp \left[ - \int_0^t \frac{1}{s \cdot C_s(F; t)} \, dx \right]. \]

**4. Test for decreasing DCREGF**

In the previous section, we proved that the DCREGF characterizes the distribution of \( X \). As constant DCREGF is a characterization of exponential random variable, we develop a test for testing exponentiality against the decreasing DCREGF class.
Let \( X_1, \ldots, X_n \) be a random sample of size \( n \) from \( F \). We are interested in testing the null hypothesis

\[
H_0 : X \text{ has exponential distribution}
\]

against the alternative

\[
H_1 : X \text{ has decreasing DCREGF and not exponential}.
\]

For testing the above hypothesis, first we define a departure measure which discriminates between null and alternative hypothesis. Note that \( C_s(X; t) \) is decreasing in \( t \) if \( C'_s(X; t) \leq 0 \). That is,

\[
\bar{F}^s + 1(t) - s f(t) \int_t^\infty \bar{F}^s(x) dx \geq 0.
\]

Changing the order of integration, from (18) we have

\[
\Delta(F) = E(\min(X_1, \ldots, X_{s+1})) - s \int_0^\infty f(t) \int_t^\infty \bar{F}^s(x) dx dt.
\]
where the summations is over the set \( C_{m,n} \) of all combinations of \((s + 1)\) distinct elements \( \{i_1, i_2, \ldots, i_{s+1}\} \) chosen from \( \{1, 2, \ldots, n\} \). We reject the null hypothesis \( H_0 \) against the alternative \( H_1 \) for large value of \( \Delta \).

**Remark 2.** When \( s = 1 \), the testing problem reduces to testing decreasing mean residual life, a important problem in the lifetime data analysis.

We find a critical region of the test using the asymptotic distribution of \( \hat{\Delta} \). Next we find the asymptotic distribution of \( \hat{\Delta} \).

**Theorem 13.** As \( n \to \infty, \sqrt{n}(\hat{\Delta} - \Delta(F)) \) converges in distribution to normal random variable with mean zero and variance \((s + 1)^2\sigma^2\), where \( \sigma^2 \) is given by

\[
\sigma^2 = \text{Var} \left( (s + 1)X\bar{F}^s(X) + s(s + 1) \int_0^X y\bar{F}^{s-1}(y) dF(y) \right) - \frac{s^2}{(s + 1)} X\bar{F}^{s-1}(X) - \frac{(s - 1)s^2}{(s + 1)} \int_0^X y\bar{F}^{s-2}(y) dF(y) \right). \tag{19}
\]

**Proof.** By using the central limit theorem for U-statistics, we have the asymptotic normality of \( \hat{\Delta}^* \). The asymptotic variance is \((s + 1)^2\sigma^2_1\) where \( \sigma^2_1 \) is given by Lee (2019)

\[
\sigma^2_1 = \text{Var} \left[ E(h(X_1, \ldots, X_{(s+1)})) \right]. \tag{20}
\]

Denote \( Z = \min(X_2, X_3, \ldots, X_{s}) \), then the distribution of \( Z \) is given by \( 1 - \bar{F}^{s-1}(x) \), where \( \bar{F}(x) = 1 - F(x) \). Consider

\[
E[\min(x, X_2, X_3, \ldots, X_{s})] = E[xI(Z > x)] + E[ZI(Z \leq x)]
= x\bar{F}^{s-1}(x) + (s - 1) \int_0^x y\bar{F}^{s-2}(y) dF(y).
\]

Similarly, we obtain

\[
E[\min(x, X_2, X_3, \ldots, X_{s+1})] = x\bar{F}^{s}(x) + s \int_0^x y\bar{F}^{s-1}(y) dF(y).
\]

Hence

\[
E(h(X_1, X_2, \ldots, X_{s+1}) = x\bar{F}^{s}(x) + s \int_0^x y\bar{F}^{s-1}(y) dF(y)
= (s + 1)x\bar{F}^s(x) + s + 1 \int_0^x y\bar{F}^{s-1}(y) dF(y)
- \frac{s^2}{(s + 1)} x\bar{F}^{s-1}(x) - \frac{(s - 1)s^2}{(s + 1)} \int_0^x y\bar{F}^{s-2}(y) dF(y) + \frac{sk}{(s + 1)},
\]

where \( k = E(\min(X_2, \ldots, X_{s+1})) \), a constant. Therefore, from (20) we obtain the variance expression specified in the statement of the theorem. \( \square \)

Under the null hypothesis \( H_0 \), \( \Delta(F) = 0 \). Hence we have the following corollary.

**Corollary 6.** Under \( H_0 \), as \( n \to \infty, \sqrt{n}\hat{\Delta} \) converges in distribution to a Gaussian random variable with mean zero and variance \( \sigma^2_0 \), where \( \sigma^2_0 \) is given by

\[
\sigma^2_0 = \frac{s}{(4s^2 - 1)\lambda^2}. \tag{21}
\]
Therefore, the variance expression in (19) reduces to

\[ \sigma_1^2 = Var \left( \frac{s^2}{(s + 1)(s - 1)\lambda} \bar{F}^{s-1}(X) - \frac{(s + 1)\bar{F}^s(X)}{s\lambda} \right) \]

Also

\[ \int_0^\infty (s - 1)y\bar{F}^{s-2}(y)dF(y) = \frac{1}{(s - 1)\lambda} - \bar{F}^{s-1}(x) \left( x + \frac{1}{(s - 1)\lambda} \right). \]

Therefore, the variance expression in (19) reduces to

\[ \sigma_1^2 = \frac{1}{(s + 1)^2\lambda^2} \left\{ Var \left( \frac{s^2}{(s - 1)} \bar{F}^{s-1}(X) - \frac{(s + 1)^2}{s} \bar{F}^s(X) \right) \right\} \]

Proof. For the exponential distribution, the mean is equal to the mean residual life function. Hence

\[ \int_0^\infty sy\bar{F}^{s-1}(y)dF(y) = \frac{1}{s\lambda} - \bar{F}^s(x) \left( x + \frac{1}{s\lambda} \right). \]

Also

\[ \int_0^\infty (s - 1)y\bar{F}^{s-2}(y)dF(y) = \frac{1}{(s - 1)\lambda} - \bar{F}^{s-1}(x) \left( x + \frac{1}{(s - 1)\lambda} \right). \]

Table 2. Bias and MSE of the proposed estimator for various distributions.

| n | Bias | MSE | Bias | MSE | Bias | MSE | Bias | MSE | Bias | MSE |
|---|------|-----|------|-----|------|-----|------|-----|------|-----|
| exp(1) | gamma(2, 2) | Weibull(3, 1) | lognormal(0.5, 0.5) | Makeham(1, 1) |
| s = 1 | | | | | |
| 10 | 0.0026 | 0.1014 | 0.0015 | 0.0495 | 0.0009 | 0.0106 | 0.0043 | 0.0988 | 0.0038 | 0.0180 |
| 20 | 0.0024 | 0.0501 | 0.0011 | 0.0249 | 0.0007 | 0.0053 | 0.0021 | 0.0494 | 0.0018 | 0.0088 |
| 30 | 0.0018 | 0.0336 | 0.0006 | 0.0171 | 0.0004 | 0.0035 | 0.0014 | 0.0327 | 0.0010 | 0.0059 |
| 40 | 0.0018 | 0.0257 | 0.0002 | 0.0127 | 0.0002 | 0.0026 | 0.0013 | 0.0246 | 0.0009 | 0.0044 |
| 50 | 0.0006 | 0.0196 | 0.0001 | 0.0098 | 0.0000 | 0.0021 | 0.0010 | 0.0201 | 0.0008 | 0.0035 |
| s = 2 | | | | | |
| 10 | 0.0017 | 0.0351 | 0.0007 | 0.0255 | 0.0007 | 0.0116 | 0.0037 | 0.0490 | 0.0026 | 0.0124 |
| 20 | 0.0016 | 0.0167 | 0.0009 | 0.0127 | 0.0005 | 0.0056 | 0.0014 | 0.0240 | 0.0005 | 0.0061 |
| 30 | 0.0008 | 0.0115 | 0.0005 | 0.0084 | 0.0004 | 0.0037 | 0.0012 | 0.0159 | 0.0004 | 0.0041 |
| 40 | 0.0005 | 0.0082 | 0.0002 | 0.0063 | 0.0003 | 0.0027 | 0.0011 | 0.0122 | 0.0002 | 0.0030 |
| 50 | 0.0003 | 0.0068 | 0.0001 | 0.0049 | 0.0001 | 0.0022 | 0.0004 | 0.0098 | 0.0001 | 0.0024 |
| s = 3 | | | | | |
| 10 | 0.0010 | 0.0217 | 0.0007 | 0.0215 | 0.0004 | 0.0125 | 0.0021 | 0.0417 | 0.0009 | 0.0108 |
| 20 | 0.0007 | 0.0106 | 0.0007 | 0.0101 | 0.0004 | 0.0061 | 0.0021 | 0.0204 | 0.0003 | 0.0049 |
| 30 | 0.0005 | 0.0068 | 0.0005 | 0.0068 | 0.0003 | 0.0041 | 0.0008 | 0.0137 | 0.0002 | 0.0032 |
| 40 | 0.0004 | 0.0053 | 0.0003 | 0.0049 | 0.0001 | 0.0031 | 0.0002 | 0.0101 | 0.0001 | 0.0024 |
| 50 | 0.0002 | 0.0041 | 0.0002 | 0.0040 | 0.0001 | 0.0024 | 0.0001 | 0.0078 | 0.0001 | 0.0020 |
| s = 4 | | | | | |
| 10 | 0.0015 | 0.0170 | 0.0011 | 0.0189 | 0.0011 | 0.0141 | 0.0016 | 0.0403 | 0.0005 | 0.0092 |
| 20 | 0.0005 | 0.0077 | 0.0006 | 0.0092 | 0.0011 | 0.0067 | 0.0013 | 0.0191 | 0.0004 | 0.0042 |
| 30 | 0.0005 | 0.0050 | 0.0004 | 0.0058 | 0.0007 | 0.0044 | 0.0011 | 0.0126 | 0.0002 | 0.0028 |
| 40 | 0.0002 | 0.0038 | 0.0003 | 0.0044 | 0.0003 | 0.0032 | 0.0006 | 0.0093 | 0.0001 | 0.0020 |
| 50 | 0.0001 | 0.0030 | 0.0001 | 0.0034 | 0.0001 | 0.0027 | 0.0001 | 0.0075 | 0.0001 | 0.0016 |
| s = 5 | | | | | |
| 10 | 0.0010 | 0.0136 | 0.0009 | 0.0177 | 0.0011 | 0.0153 | 0.0026 | 0.0411 | 0.0007 | 0.0081 |
| 20 | 0.0004 | 0.0062 | 0.0005 | 0.0082 | 0.0008 | 0.0073 | 0.0022 | 0.0193 | 0.0006 | 0.0037 |
| 30 | 0.0004 | 0.0039 | 0.0004 | 0.0054 | 0.0004 | 0.0048 | 0.0021 | 0.0126 | 0.0004 | 0.0025 |
| 40 | 0.0004 | 0.0029 | 0.0002 | 0.0040 | 0.0004 | 0.0035 | 0.0013 | 0.0094 | 0.0004 | 0.0017 |
| 50 | 0.0002 | 0.0022 | 0.0001 | 0.0032 | 0.0001 | 0.0029 | 0.0003 | 0.0074 | 0.0001 | 0.0014 |
where

\[ \frac{\Delta^*}{\bar{X}} = \frac{\Delta}{X}, \]

where \( \bar{X} \) is the sample mean. Using Slutsky's theorem, we have the following result.

In view of the null variance specified in (21), we consider a scale invariant test given by

\[ X \exp \left\{ -\frac{2s(s+1)^2}{(s-1)} \text{Cov}(\bar{F}^s(X), \bar{F}^{s-1}(X)) \right\}. \]

= \frac{1}{(s+1)^2 \lambda^2} \left\{ \frac{s^2}{2s-1} + \frac{(s+1)^2}{2s+1} - (s+1) \right\} = \frac{s}{(s+1)^2(4s^2-1)\lambda^2}. \]

Therefore, the asymptotic null variance is equal to \( s/(4s^2-1)\lambda^2 \), which proves the statement. \[ \square \]
Table 4. Power of the proposed test for various distributions-II.

| \( n/\alpha \) | \( s = 0.01 \) | \( s = 0.05 \) | \( s = 0.01 \) | \( s = 0.05 \) | \( s = 0.01 \) | \( s = 0.05 \) | \( s = 0.01 \) | \( s = 0.05 \) | \( s = 0.01 \) | \( s = 0.05 \) |
|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| \( \exp(1) \) | \( \gamma(2, 1) \) | \( \text{Weibull}(2, 1) \) | \( \lognormal(1, 0.5) \) | \( \text{Makeham}(1, 1) \) | \( \text{LFR}(1) \) | \( \exp(2) \) | \( \gamma(2, 2) \) | \( \text{Weibull}(2, 2) \) | \( \lognormal(2, 0.5) \) | \( \text{Makeham}(2, 2) \) | \( \text{LFR}(2) \) |
| 10            | 0.0161        | 0.0622        | 0.2253        | 0.3816        | 0.5486        | 0.7181        | 0.8424        | 0.9416        | 0.1034        | 0.2068        | 0.5271        | 0.7142        |
| 20            | 0.0138        | 0.0616        | 0.4275        | 0.6289        | 0.8419        | 0.9485        | 0.9958        | 1.0000        | 0.1672        | 0.2995        | 0.8735        | 0.9556        |
| 30            | 0.0132        | 0.0552        | 0.6114        | 0.7952        | 0.9697        | 0.9905        | 1.0000        | 1.0000        | 0.2107        | 0.3736        | 0.9683        | 0.9921        |
| 40            | 0.0123        | 0.0523        | 0.7596        | 0.8943        | 0.9968        | 0.9996        | 1.0000        | 1.0000        | 0.2562        | 0.4295        | 0.9951        | 0.9994        |
| 50            | 0.0110        | 0.0511        | 0.8580        | 0.9476        | 0.9999        | 1.0000        | 1.0000        | 1.0000        | 0.3023        | 0.4808        | 1.0000        | 1.0000        |
| \( \exp(0.5) \) | \( \gamma(3, 1) \) | \( \text{Weibull}(3, 1) \) | \( \lognormal(3, 0.5) \) | \( \text{Makeham}(2, 1) \) | \( \text{LFR}(0.5) \) | \( \exp(0.2) \) | \( \gamma(3, 2) \) | \( \text{Weibull}(3, 2) \) | \( \lognormal(0.5, 0.5) \) | \( \text{Makeham}(1, 0.5) \) | \( \text{LFR}(0.2) \) |
| 10            | 0.0162        | 0.0574        | 0.5345        | 0.7213        | 0.9357        | 0.9713        | 0.8492        | 0.9510        | 0.1618        | 0.3228        | 0.5209        | 0.7196        |
| 20            | 0.0124        | 0.0565        | 0.8747        | 0.9572        | 0.9995        | 1.0000        | 0.9991        | 1.0000        | 0.3069        | 0.4757        | 0.8593        | 0.9395        |
| 30            | 0.0123        | 0.0524        | 0.9692        | 0.9917        | 1.0000        | 1.0000        | 1.0000        | 1.0000        | 0.4467        | 0.6263        | 0.9773        | 0.9962        |
| 40            | 0.0114        | 0.0489        | 0.9982        | 0.9997        | 1.0000        | 1.0000        | 1.0000        | 1.0000        | 0.5628        | 0.7464        | 0.9971        | 1.0000        |
| 50            | 0.0099        | 0.0495        | 0.9961        | 1.0000        | 1.0000        | 1.0000        | 1.0000        | 1.0000        | 0.6674        | 0.8025        | 0.9998        | 1.0000        |
| \( \exp(5) \) | \( \gamma(2, 1.5) \) | \( \text{Weibull}(2, 3) \) | \( \lognormal(3, 0.8) \) | \( \text{Makeham}(2, 0.5) \) | \( \text{LFR}(3) \) | \( \exp(2) \) | \( \gamma(3, 2) \) | \( \text{Weibull}(3, 2) \) | \( \lognormal(0.5, 0.5) \) | \( \text{Makeham}(1, 0.5) \) | \( \text{LFR}(0.2) \) |
| 10            | 0.0156        | 0.0579        | 0.2235        | 0.3976        | 0.5348        | 0.7182        | 0.1946        | 0.3712        | 0.3158        | 0.4616        | 0.5127        | 0.7083        |
| 20            | 0.0138        | 0.0565        | 0.4156        | 0.6254        | 0.8655        | 0.9557        | 0.3258        | 0.5154        | 0.5368        | 0.6856        | 0.8427        | 0.9337        |
| 30            | 0.0128        | 0.0526        | 0.6146        | 0.8028        | 0.9695        | 0.9958        | 0.5331        | 0.7390        | 0.7113        | 0.8475        | 0.9778        | 0.9948        |
| 40            | 0.0112        | 0.0516        | 0.7571        | 0.8913        | 0.9981        | 0.9992        | 0.6317        | 0.8309        | 0.8424        | 0.9235        | 0.9938        | 0.9993        |
| 50            | 0.0109        | 0.0509        | 0.8542        | 0.9489        | 0.9997        | 1.0000        | 0.7954        | 0.9156        | 0.9413        | 0.9786        | 1.0000        | 1.0000        |

**Corollary 7.** Under \( H_0 \), as \( n \to \infty \), \( \sqrt{n} \Delta^* \) converges in distribution to a Gaussian random variable with mean zero and variance \( \sigma_0^2 \), where \( \sigma_0^2 \) is given by

\[
\sigma_0^2 = \frac{s}{(4s^2 - 1)}. 
\]

An asymptotic critical region of the test can be obtained using Corollary 7. We reject the null hypothesis \( H_0 \) against the alternative hypothesis \( H_1 \) at a significance level \( \alpha \), if

\[
\frac{\sqrt{n}(4s^2 - 1)|\Delta^*|}{\sqrt{s}} > Z_{\alpha/2},
\]

where \( Z_{\alpha} \) is the upper \( \alpha \)-percentile point of the standard normal distribution.
Table 5. Power of the proposed test for various distributions-III.

| n/α | exp(1) | gamma(2, 1) | Weibull(2, 1) | lognormal(1, 0.5) | Makeham(1, 1) | LFR(1) |
|-----|--------|-------------|---------------|-------------------|---------------|--------|
| 10  | 0.0151 | 0.0564      | 0.3275        | 0.4856            | 0.5824        | 0.7276 |
| 20  | 0.0127 | 0.0548      | 0.4716        | 0.641             | 0.8477        | 0.9182 |
| 30  | 0.0125 | 0.0526      | 0.6425        | 0.8165            | 0.9557        | 0.9836 |
| 40  | 0.0117 | 0.0517      | 0.7672        | 0.8876            | 0.9881        | 0.9942 |
| 50  | 0.0112 | 0.0515      | 0.8638        | 0.9423            | 0.9986        | 1.0000 |
|     | exp(1) | gamma(2, 2) | Weibull(2, 2) | lognormal(0.5, 0.5) | Makeham(2, 2) | LFR(2) |
| 10  | 0.0151 | 0.0599      | 0.2955        | 0.4286            | 0.5985        | 0.7395 |
| 20  | 0.0142 | 0.0561      | 0.5095        | 0.6693            | 0.8481        | 0.9216 |
| 30  | 0.0123 | 0.0547      | 0.6535        | 0.8124            | 0.9526        | 0.9845 |
| 40  | 0.0117 | 0.0523      | 0.7735        | 0.8863            | 0.9854        | 0.9968 |
| 50  | 0.0112 | 0.0516      | 0.8714        | 0.9584            | 0.9976        | 1.0000 |
|     | exp(0.5) | gamma(3, 1) | Weibull(3, 1) | lognormal(0.5, 0.5) | Makeham(2, 1) | LFR(0.5) |
| 10  | 0.0152 | 0.0593      | 0.6438        | 0.7785            | 0.9287        | 0.9645 |
| 20  | 0.0136 | 0.0574      | 0.8925        | 0.9623            | 0.9981        | 1.0000 |
| 30  | 0.0121 | 0.0562      | 0.9823        | 0.9932            | 1.0000        | 1.0000 |
| 40  | 0.0118 | 0.0548      | 0.9991        | 1.0000            | 1.0000        | 1.0000 |
| 50  | 0.0112 | 0.0567      | 0.9974        | 1.0000            | 1.0000        | 1.0000 |
|     | exp(0.2) | gamma(2, 3) | Weibull(2, 3) | lognormal(0.5, 0.5) | Makeham(1, 0.5) | LFR(0.2) |
| 10  | 0.0144 | 0.0612      | 0.6566        | 0.7862            | 0.9178        | 0.9616 |
| 20  | 0.0129 | 0.0572      | 0.8953        | 0.9675            | 1.0000        | 1.0000 |
| 30  | 0.0124 | 0.0563      | 0.9751        | 0.9942            | 1.0000        | 1.0000 |
| 40  | 0.0118 | 0.0544      | 0.9968        | 0.9999            | 1.0000        | 1.0000 |
| 50  | 0.0112 | 0.0527      | 0.9998        | 1.0000            | 1.0000        | 1.0000 |
|     | exp(5) | gamma(2, 1.5) | Weibull(2, 3) | lognormal(0.8, 0.8) | Makeham(2, 0.5) | LFR(3) |
| 10  | 0.0153 | 0.0558      | 0.2871        | 0.4569            | 0.5438        | 0.6799 |
| 20  | 0.0139 | 0.0547      | 0.4555        | 0.6488            | 0.8455        | 0.9232 |
| 30  | 0.0129 | 0.0536      | 0.6038        | 0.7872            | 0.9557        | 0.9856 |
| 40  | 0.0116 | 0.0527      | 0.7759        | 0.9091            | 0.9862        | 0.9988 |
| 50  | 0.0115 | 0.0511      | 0.8460        | 0.9345            | 0.9989        | 1.0000 |

5. Simulation and data analysis

We conduct Monte Carlo simulation studies using R software to evaluate the finite sample performance of the estimator $\hat{C}_s(X)$ and the proposed test for DDCREGF. The simulation is repeated 10,000 times.

First, we evaluate the performance of the estimator $\hat{C}_s(X)$ in terms of bias and MSE. In the simulation study, we generate observations from various lifetime distributions including exponential, gamma, Weibull, lognormal, and Makeham distributions. Various parameters are chosen for generating these observations. The MSE and the absolute bias of the estimators based on samples of sizes $n = 10, 20, 30, 40,$ and 50 are calculated. Different choices of $s$ are used in the simulation study. The results of the simulation study are presented in Table 2. From Table 2, we can observe that both the MSE and the absolute bias are negligible for all distributions and both decrease as $n$ increases.

Next, we conduct an extensive simulation study to assess the performance of the proposed test for decreasing DCREGF. The exponential distribution with different choices of parameters is used to find the empirical type I error of the test. For finding the empirical power of the test, lifetime distributions including, gamma, Weibull, lognormal, Makeham, and linear
failure rate, which are members of the DDCREGF class are used. Random samples of sizes \( n = 10, 20, 30, 40, \) and 50 are generated from these distributions where the parameters are chosen in such a way that the distribution belongs to decreasing DCREGF family. Different choices of \( s \) are considered in the study. The results of the simulation study are presented in Tables 3–5. In Tables 3–5 we reported the results for \( s = 1, s = 2 \) and \( s = 3 \), respectively.

From Tables 3–5, we can see that empirical type I error of the test approaches chosen significance level for all choices of parameters of the exponential distribution. For all choices of alternatives, the test yields good power also. When the samples are generated from a gamma distribution with parameters such that, the distribution is approaching exponential, we observe small power. The random samples from Makeham distribution also show low power, compared to the other distributions. The proposed test yields very good power for all the other distributions with various choices of parameters we considered in the simulation.

Next, we use two real-life data sets to illustrate the proposed testing procedure.

**Illustration 1.** The following data set represents the failure times (in minutes) for a sample of 15 electronic components in an accelerated life test, Lawless (2011). Data: 1.4, 5.1, 6.3, 10.8, 12.1, 18.5, 19.7, 22.2, 23.0, 30.6, 37.3, 46.3, 53.9, 59.8, 66.2. After applying the above testing procedure, we obtain the test statistic value (for \( s = 1 \)) as 0.8409. Hence, we accept the null hypothesis that above data set is exponentially distributed. Our conclusion is same as previous studies on this data.

**Illustration 2.** The second data set is also taken from Lawless (2011). We considered the data which arose in tests on endurance of deep groove ball bearings. The data are the number of million revolutions before failure for each of the 23 ball bearings in the life tests and the complete data is given below. Data: 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.40, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40. We obtain the test statistic value (for \( s = 1 \)) as 3.5099. This concludes that for this data set, we reject the null hypothesis and that it is DCREGF. We use the the function “fitdist” in R-package “fitdistrplus” to fit the ball bearing data. Weibull and lognormal distributions are two possible models which fit the data and belong to DCREGF class of distributions.

### 6. Conclusions

In this article, we studied the properties of the cumulative residual entropy generating function (CREGF). We obtain the non parametric estimator of CREGF and evaluated its performance. We then introduced a dynamic version of the cumulative residual entropy generating function (DCREGF). We have shown that DCREGF determines the distribution uniquely. Further, we studied the relationships between DCREGF, hazard rate, and mean residual life function. This led to the development of a non parametric test for decreasing DCREGF. We evaluated the finite sample behavior of the proposed test through an extensive Monte Carlo simulation study. The empirical type I of the test is well maintained. The practical applications of the test are illustrated using real-life examples. The proposed test can be used for testing decreasing mean residual life function.

Different extensions of measures of entropy and extropy are studied in the literature. We can extend the generating function approach to study these measures. One can consider developing the empirical likelihood and jackknife empirical likelihood inference for these measures.
References

Ahmadi, J. 2021. Characterization of continuous symmetric distributions using information measures of records. *Statistical Papers* 62 (6):2603–26. 10.1007/s00362-020-01206-z.

Asadi, M., and Y. Zohrevand. 2007. On the dynamic cumulative residual entropy. *Journal of Statistical Planning and Inference* 137 (6):1931–41. 10.1016/j.jspi.2006.06.035.

Di Crescenzo, A., and M. Longobardi. 2009. On cumulative entropies. *Journal of Statistical Planning and Inference* 139 (12):4072–87. 10.1016/j.jspi.2009.05.038.

Golomb, S. 1966. The information generating function of a probability distribution (Corresp.). *IEEE Transactions on Information Theory* 12 (1):75–7. 10.1109/TIT.1966.1053843.

Guasu, S., and C. Reischer. 1985. The relative information generating function. *Information Sciences* 35 (3):235–41. 10.1016/0020-0255(85)90053-2.

Hall, W. J., and J. Wellner. 1980. Mean residual life. In *Statistics and related topics*, 169–84. Ottawa, ON: North-Holland, Amsterdam.

Hooda, D. S., and U. Singh. 1990. An information improvement generating function. *Communications in Statistics- Theory and Methods* 19 (3):1037–46. 10.1080/03610929008830245.

Jha, D. V. R. K., C. L. Dewangan, and R. K. Verma. 2012. Some generating function for measures of probabilistic entropy and directed divergence. *International Journal of Pure and Applied Mathematics* 74 (1):21–32.

Kharazmi, O., and N. Balakrishnan. 2021a. Jensen-information generating function and its connections to some well-known information measures. *Statistics & Probability Letters* 170:108995. 10.1016/j.spl.2021.108995.

Kharazmi, O., and N. Balakrishnan. 2021b. Cumulative and relative cumulative residual information generating measures and associated properties. *Communications in Statistics-Theory and Methods* 52 (15):5260–73.

Kharazmi, O., and N. Balakrishnan. 2022. Cumulative residual and relative cumulative residual Fisher information and their properties. *IEEE Transactions on Information Theory* 67 (10):6306–12. 10.1109/TIT.2021.3073789.

Kullback, S., and R. A. Leibler. 1951. On Information and Sufficiency. *The Annals of Mathematical Statistics* 22 (1):79–86. 10.1214/aoms/1177729694.

Lawless, J. F. 2011. *Statistical models and methods for lifetime data*. Hoboken, NJ: John Wiley and Sons.

Lee, A. J. 2019. *U-statistics: Theory and practice*. New York, NY: Routledge.

Lehmann, E. L. 1951. Consistency and unbiasedness of certain nonparametric tests. *The annals of mathematical statistics*, 165–179.

Mirali, M., and S. Baratpour. 2017. Some results on weighted cumulative entropy. *Journal of the Iranian Statistical Society* 16 (2):21–32.

Mirali, M., S. Baratpour, and V. Fakoor. 2017. On weighted cumulative residual entropy. *Communications in Statistics- Theory and Methods* 46 (6):2857–69. 10.1080/03610929.2015.1053932.

Navarro, J., Y. Del Aguila, and M. Asadi. 2010. Some new results on the cumulative residual entropy. *Journal of Statistical Planning and Inference* 140 (1):310–22. 10.1016/j.jspi.2009.07.015.

Papaoannou, T., K. Ferentinos, and C. Tsairidis. 2007. Some information theoretic ideas useful in statistical inference. *Methodology and Computing in Applied Probability* 9 (2):307–23. 10.1007/s11009-007-9017-7.

Rajesh, G., and S. M. Sunoj. 2019. Some properties of cumulative Tsallis entropy of order α. *Statistical Papers* 60 (3):933–43. 10.1007/s00362-016-0855-7.

Rao, M., Y. Chen, B. C. Vemuri, and F. Wang. 2004. Cumulative residual entropy: A new measure of information. *IEEE Transactions on Information Theory* 50 (6):1220–8. 10.1109/TIT.2004.828057.

Shannon, C. E. 1948. A mathematical theory of communication. *Bell System Technical Journal* 27 (3):379–423. 10.1002/j.1538-7305.1948.tb01338.x.

Sudheesh, K. K., E. P. Sreedevi, and N. Balakrishnan. 2022. A generalized measure of cumulative residual entropy. *Entropy* 24 (4):444. 10.3390/e24040444.

Zamani, Z., O. Kharazmi, and N. Balakrishnan. 2022. Information generating function of record values. *Mathematical Methods of Statistics* 31 (3):120–33. 10.3103/S1066530722030036.