Stochastic maximum principle for optimal control problem with a stopping time cost functional

Shuzhen Yang

Shandong University-Zhong Tai Securities Institute for Financial Studies, Shandong University, Shandong, People’s Republic of China

ABSTRACT

In this study, we consider an optimal control problem driven by a stochastic differential system with a stopping time terminal cost functional. To solve the optimal control problem under the stopping time terminal cost functional, we introduce a multi-time state optimal control systems, and prove that the multi-time state optimal control systems is a near-optimal control problem. We use the near-optimal control problem to take place the original optimal control problem under the cost functional with a stopping time terminal. Then, we establish the stochastic maximum principle for this kind of optimal control problem by introducing a discrete terminal system. Finally, we provide an example to describe the main results of this study.

1. Introduction

In the real world, we usually use controlled stochastic differential equation (1) to describe systems, for example, the optimal investment problem and production planning problem in the market:

\[ X^u(s) = x + \int_0^s b(X^u(t), u(t)) \, dt + \int_0^s \sigma(X^u(t), u(t)) \, dW(t), \]

where \( X^u(\cdot) \) denotes the controlled process under the given control process \( u(\cdot) \). Generally, the cost functional is given as follows:

\[ J(u(\cdot)) = E \left[ \int_0^T f(X^u(t), u(t)) \, dt + \Psi(X^u(T)) \right]. \]

where \( f(\cdot) \) is the running cost and \( \Phi(\cdot) \) is the terminal cost.

In the financial market, for a given terminal time \( T \), an investor wants to minimise the variance of the portfolio asset for a given mean level. The investor can stop the investment plan at an uncertain horizon time \( \tau \) before the terminal time \( T \). Martellini and Urošević (2006) considered the static mean–variance analysis with an uncertain time horizon. Yi et al. (2008) studied the mean–variance model of a multi-period asset-liability management problem under uncertain exit time. Furthermore, see Wu et al. (2011), Yao and Ma (2010) and Yu (2013) for additional studies in this vein. In the literature of the mean–variance model under uncertain or random exit time, we always suppose that the uncertain horizon time \( \tau \) satisfies a distribution (or a conditional distribution) and investigate the related mean–variance model at time \( \tau \).

At the moment, we haven’t found the work on the stochastic maximum principle of a general cost functional with a stopping time terminal. Thus, in this study, we want to construct the maximum principle for a general cost functional with a stopping time terminal, in which the terminal cost \( \Psi(X(\cdot)) \) can be dependent on a given stopping time \( \tau \leq T \) but not only a constant \( T \). In the optimal investment problem, \( \tau \) could be used to describe the action of the investor. For example, \( \tau \) could be used to represent a signal of the market, such as bear market or bull market (for more details, see Example 3.3).

Over the past decades, there has been much work concerning the optimality of state process (1) with cost functional (2). Bensoussan (1981) and Bismut (1978) presented a local maximum principle with a convex control set, while Peng (1990) presented a global maximum principle with a general control domain that may not be convex. Dynamic programming with related HJB (Hamilton–Jacobi–Bellman) equations and the maximum principle were powerful approaches for solving optimal control problems (see Yong & Zhou, 1999). Initially, Zhou (1995, 1996, 1998) introduced the concept of near-optimisation. Subsequently, many authors used the near-optimisation method to study the switching LQ-problem, stochastic recursive problem, linear forward–backward stochastic systems, etc. (Huang et al., 2010; Hui et al., 2011; Liu et al., 2005). For the discrete state constraints problem, Dmitruk and Kaganovich (2008, 2011) established the standard problem of Pontryagin-type maximum principle by a simple change of variables. More general versions of the maximum principle on time scales were given in Bourdin and Trélat (2013, 2016, 2017). In particular, Bourdin and Trélat (2017) studied a linear-quadratic optimal control problem. Yang (2016) investigated a stochastic maximum principle for a stochastic differential system with a general cost functional. In particular, a varying ter-
minal time structure for stochastic optimal control under con-
strained conditions was introduced and the maximum principle
was established in Yang (2020).

In this study, we find it difficult to investigate the stochastic
maximum principle for state process (1) under the cost func-
tional $J(u(\cdot))$ with the terminal cost $\Psi(X^u(\tau))$ by directly
applying the spike variation technique. This is because for a given
$\varepsilon > 0$ and $E_\varepsilon = [\nu, \nu + \varepsilon] \subset [0, T]$, we cannot verify whether $E_\varepsilon \subset [0, \tau]$. Thus, we establish a near-optimal control problem
for this study by applying the following multi-time state cost func-
tional:

$$J^\varepsilon(u(\cdot)) = E \left\{ \int_0^T f(X^u(t), u(t)) \, dt + \sum_{i=1}^{n-1} \Psi(X^u(t_i)) 1_{\{t_{i-1} \leq \tau < t_i\}} + \Psi(X^u(t_n)) 1_{\{t_{n-1} \leq \tau \leq t_n\}} \right\},$$

where $0 = t_0 < t_1 < t_2 \cdots < t_n = T$. Based on cost func-
tional (3), we investigate an optimal control problem for state
process (1) under a general control domain.

The main contributions of this study are given as follows:

(i) Different from the traditional cost functional in a stochas-
tic optimal control system, we introduce a stopping time
terminal in the cost functional, which can deal with the
financial problem. To establish the related maximum prin-
ciple, we consider a near-optimal control problem. Then,
we investigate the maximum principle for this near-optimal
control problem.

(ii) We use the near-optimal control problem to take place the
original optimal control problem under the cost functional
with a stopping time terminal. We can show the neces-
sary condition, which is called the maximum principle,
for the near-optimal control problem. The necessary condition
can help us to find the optimal control and state. Fur-
thermore, see Examples 3.3 and 4.3, in which we describe
how to find the optimal solution based on the maximum
principle results (Theorem 3.1).

This paper is organised as follows: In Section 2, we present
the stochastic optimal control problem with a stopping time
cost functional and investigate a related near-optimal con-
control problem. The stochastic maximum principle for the near-
optimal control problem is given in Section 3. Then, we
solve a financial example via this new maximum principle. In
Section 4, we prove the stochastic maximum principle for the
optimal control problem under state constraints with a stop-
ning time. Subsequently, we provide an example to illustrate
the main results of this study. In Section 5, we conclude this
study.

2. The optimal control problem

Let $W$ be a $d$-dimensional standard Brownian motion defined
on a complete filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}(t)\}_{t \geq 0})$,
given control. We introduce a discrete version of the stopping time $\tau$ as

$$
\tau^n = \sum_{i=1}^{n-1} t_i \mathbb{1}_{\{t_{i-1} \leq \tau < t_i\}} + t_n \mathbb{1}_{\{t_{n-1} \leq \tau \leq t_n\}}
$$

and consider the following multi-time state cost functional:

$$
J^n(u(\cdot)) = E \left[ \int_0^T f(X^u(t), u(t)) \, dt + \Psi(X^u(\tau^n)) \right]
$$

Let Assumptions 2.1, 2.2 hold. Then, for any given $\varepsilon > 0$, there exists a large value of $n$ such that

$$
|J(\bar{u}(\cdot)) - J^n(\bar{u}^n(\cdot))| < \varepsilon,
$$

where $(\bar{u}(\cdot), \bar{X}(\cdot))$ is an optimal solution of cost functional (5), while $(\bar{u}^n(\cdot), \bar{X}^n(\cdot))$ is an optimal solution of cost functional (7).

**Proof:** Based on Assumptions 2.1 and 2.2, for any given solution $(u(\cdot), X^u(\cdot))$, we have

$$
E[|X^u(t) - X^u(s)|^2] \leq C|t - s|,
$$

where $C$ is a positive constant and $t, s \in [0, T]$. Based on the above inequality, we get

$$
E \left| X^u(t) - \sum_{i=1}^{n-1} X^u(t_i) \mathbb{1}_{\{t_{i-1} \leq \tau < t_i\}} - X^u(t_n) \mathbb{1}_{\{t_{n-1} \leq \tau \leq t_n\}} \right|
$$

$$
= E \left| \sum_{i=1}^{n-1} (X^u(\tau) - X^u(t_i)) \mathbb{1}_{\{t_{i-1} \leq \tau < t_i\}} + (X^u(\tau) - X^u(t_n)) \mathbb{1}_{\{t_{n-1} \leq \tau \leq t_n\}} \right|
$$

$$
\leq C \sqrt{\frac{T}{n}}.
$$

Thus, for any given $\varepsilon > 0$, we set $n = \lceil \frac{C^2T}{\varepsilon^2} \rceil + 1$ and have

$$
|J(\bar{u}(\cdot)) - J^n(\bar{u}^n(\cdot))| < \varepsilon.
$$

Hence, we complete the proof. □

**Remark 2.5:** For a large value of $n$, Lemma 2.4 shows that the minimum value of cost functional (7) is close to the minimum value of cost functional (5). Thus, we call that state process (1) under cost functional (7) is a near-optimal control problem of state process (1) under cost functional (5). In the following, we will consider the optimal control problem with cost functional (7).

### 3. Stochastic maximum principle

For notation simplicity, we denote

$$
A_i = \{ \omega : t_{i-1} \leq \tau(\omega) \leq t_i, \omega \in \Omega \}, \quad i = 1, 2, \ldots, n - 1,

A_n = \{ \omega : t_{n-1} \leq \tau(\omega) \leq t_n, \omega \in \Omega \}.
$$

In this section, we study the stochastic maximum principle for the following cost functional:

$$
J^n(u(\cdot)) = E \left[ \int_0^T f(X^u(t), u(t)) \, dt + \sum_{i=1}^{n} \Psi(X^u(t_i))1_{A_i} \right],
$$

with state equation (4). Notice that we consider a general control domain $U$ that need not be a convex set. The main difficulty is to investigate the variational and adjoint equations. We introduce the first-order and second-order adjoint equations as follows:

The first-order adjoint equations are

$$
-dp(t) = \left[ b_x(\tilde{X}(t), \tilde{u}(t))^\top p(t) + \sum_{j=1}^{d} \sigma_j(\tilde{X}(t), \tilde{u}(t))^\top q_j(t) \right. 
$$

$$
\left. - f_x(\tilde{X}(t), \tilde{u}(t)) \right] \, dt - q(t) \, dW(t), \quad t \in (t_{i-1}, t_i),
$$

where $'\top'$ means the transpose of a vector or matrix, $p(t_i^+)$ is the right limit of $p(\cdot)$ at $t_i$, and $p(t_i^-) = 0$.

We define the following Hamiltonian function:

$$
H(x, u, p, q) = b(x, u)^\top p + \sum_{j=1}^{d} \sigma_j(x, u)^\top q_j - f(x, u),
$$

where $(x, u, p, q) \in \mathbb{R}^m \times U \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$. 
The second-order adjoint equations are
\[ -dP(t) = \left[ b_x(\bar{X}(t), \bar{u}(t))^T P(t) + P(t) b_x(\bar{X}(t), \bar{u}(t)) \right. \]
\[ + \sum_{j=1}^{d} \sigma_j^x(\bar{X}(t), \bar{u}(t))^T P(t) \sigma_j^x(\bar{X}(t), \bar{u}(t)) \]
\[ + \sum_{j=1}^{d} \left[ \sigma_j^x(\bar{X}(t), \bar{u}(t))^T Q_j(t) \right. \]
\[ + Q_j(t) \sigma_j^x(\bar{X}(t), \bar{u}(t)) \left. \right] dt - Q(t) dW(t), \]
\[ t \in (t_{i-1}, t_i), \]
\[ -P(t_i) = \Psi_{xx}(\bar{X}(t_i))1_{A_i} - P(t_i^-), \quad i = 1, \ldots, n, \quad (14) \]
where \( P(t_i^+) = 0. \)

The main result of this section is the following theorem:

**Theorem 3.1:** Let Assumptions 2.1, 2.2 and 2.3 hold, and \((\bar{u}(\cdot), \bar{X}(\cdot))\) be an optimal pair of (12). Then, there exists \((p(\cdot), q(\cdot))\) and \((P(\cdot), Q(\cdot))\) satisfying the series of first-order adjoint equation (13) and second-order adjoint equation (14) such that
\[ H(\bar{X}(t), \bar{u}(t), p(t), q(t)) - H(\bar{X}(t), u, p(t), q(t)) \]
\[ \geq \frac{1}{2} \sum_{j=1}^{d} \left[ \sigma_j^x(\bar{X}(t), \bar{u}(t)) - \sigma_j^x(\bar{X}(t), u) \right]^T \]
\[ \times P(t) \left[ \sigma_j^x(\bar{X}(t), \bar{u}(t)) - \sigma_j^x(\bar{X}(t), u) \right], \quad (15) \]
for any \( u \in U \) and \( t \in (t_{i-1}, t_i), \) \( i = 1, 2, \ldots, n. \)

Let \((\bar{u}(\cdot), \bar{X}(\cdot))\) be the given optimal pair of cost functional (12). Let \( \epsilon > 0, \) and \( E_\epsilon = [v, v + \epsilon] \subset (t_{i-1}, t_i), \) for some \( i \in \{1, 2, \ldots, n\}. \) Let \( u^\epsilon(\cdot) \in U[0, T] \) be any given control. By applying the spike variation technique, we define the following control:
\[ u^\epsilon(t) = \begin{cases} \bar{u}(t), & \text{if } t \in [0, T] \setminus E_\epsilon, \\ \bar{u}(t), & \text{if } t \in E_\epsilon, \end{cases} \]
where \( u^\epsilon(\cdot) \in U[0, T]. \) The following lemma is useful for proving Theorem 3.1.

**Lemma 3.2:** Let Assumptions 2.1, 2.2 and 2.3 hold, and \( X^\epsilon(\cdot) \) be the solution of Equation (4) under the control \( u^\epsilon(\cdot), \) and \( y(\cdot), z(\cdot) \) be the solutions of the following equations:
\[ dy(t) = \left[ b_x(\bar{X}(t), \bar{u}(t))^T y(t) dt + \sum_{j=1}^{d} \left[ \sigma_j^x(\bar{X}(t), \bar{u}(t)) y(t) \right. \right. \]
\[ \left. + \sigma_j^x(\bar{X}(t), u^\epsilon(t)) - \sigma_j^x(\bar{X}(t), \bar{u}(t)) \right] dW^j(t), \]
\[ y(0) = 0, \quad t \in (0, T], \quad (16) \]
and
\[ dz(t) = \left[ b_x(\bar{X}(t), \bar{u}(t))^T z(t) + \frac{1}{2} b_{xx}(\bar{X}(t), \bar{u}(t))(y(t))^2 \right. \]
\[ + b(\bar{X}(t), u^\epsilon(t)) - b(\bar{X}(t), \bar{u}(t)) \right] dt \]
\[ + \sum_{j=1}^{d} \left[ \sigma_j^x(\bar{X}(t), \bar{u}(t)) z(t) + \frac{1}{2} \sigma_j^{xx}(\bar{X}(t), \bar{u}(t))(y(t))^2 \right. \]
\[ \left. + (\sigma_j^x(\bar{X}(t), u^\epsilon(t)) - \sigma_j^x(\bar{X}(t), \bar{u}(t)))y(t) \right] dW^j(t), \]
\[ z(0) = 0, \quad t \in (0, T]. \quad (17) \]

Then,
\[ \max_{t \in [0, T]} E |y(t)| = O(\epsilon^{1/2}), \]
\[ \max_{t \in [0, T]} E |z(t)| = O(\epsilon), \]
\[ \max_{t \in [0, T]} E |X^\epsilon(t) - \bar{X}(t) - y(t)| = O(\epsilon), \]
\[ \max_{t \in [0, T]} E |X^\epsilon(t) - \bar{X}(t) - y(t) - z(t)| = o(\epsilon), \quad (18) \]

**Proof:** Applying the technique in Lemma 1 of Peng (1990), we can obtain Equation (18).

Note that
\[ J(u^\epsilon(\cdot)) - J(\bar{u}(\cdot)) = \sum_{i=1}^{n} E \left[ \Psi_x(\bar{X}(t_i))1_{A_i} y(t_i) \right. \]
\[ + z(t_i)) + y(t_i)^T \Psi_{xx}(\bar{X}(t_i))1_{A_i} y(t_i) \]
\[ + E \int_{0}^{T} \left[ f_x(\bar{X}(t), \bar{u}(t))(y(t) + z(t)) \right. \]
\[ \left. + \frac{1}{2} (y(t)^T f_{xx}(\bar{X}(t), \bar{u}(t)) y(t) \right. \]
\[ \left. + f(\bar{X}(t), \bar{u}(t)) - f(\bar{X}(t), \bar{u}(t)) \right] dt + o(\epsilon). \quad (19) \]

Applying Equation (18), it follows
\[ J(u^\epsilon(\cdot)) - J(\bar{u}(\cdot)) = \sum_{i=1}^{n} E \left[ \Psi_x(\bar{X}(t_i))1_{A_i} (y(t_i) + z(t_i)) \right. \]
\[ \left. + \int_{0}^{T} (f(\bar{X}(t), \bar{u}(t)) - f(\bar{X}(t), \bar{u}(t))) dt \right]. \quad (20) \]
Adding the total derivative rule to Equation (21), we have

\[ + y(t_i)\top \Psi_x (\bar{X}(t_i)) \varepsilon(t_i) \]

\[ + E \int_0^T \left[ f_x (\bar{X}(t), \bar{u}(t)) (y(t) + z(t)) \right. \]

\[ + \frac{1}{2} y(t)\top f_{xx} (\bar{X}(t), \bar{u}(t)) y(t) \]

\[ + f (\bar{X}(t), u^e (t)) - f (\bar{X}(t), \bar{u}(t)) \, dt + o(\varepsilon). \]

(21)

This completes the proof. \[ \blacksquare \]

Based on the above lemma, we carry out the proof for Theorem 3.1.

**Proof of Theorem 3.1:** For \( t \in (t_{i-1}, t_i) \), applying the differential chain rule to \( p(t)\top (y(t) + z(t)) \), and by Assumption 2.3, we have

\[
E \left[ -\Psi_x (\bar{X}(t_i)) 1_{A_x} (y(t_i) + z(t_i)) + p(t_i)\top (y(t_i) + z(t_i)) \right. \\
- p(t_i)\top (y(t_{i-1}) + z(t_{i-1})) \\
= E \left[ p(t_i)\top (y(t_i) + z(t_i)) - p(t_i)\top (y(t_{i-1}) + z(t_{i-1})) \right] \\
= E \int_{t_{i-1}}^{t_i} \left[ f_x (\bar{X}(t), \bar{u}(t)) (y(t) + z(t)) \right. \\
+ \frac{1}{2} p(t)\top b_{xx} (\bar{X}(t), \bar{u}(t)) (y(t))^2 \\
+ p(t)\top (b(\bar{X}(t), u^e (t)) - b(\bar{X}(t), \bar{u}(t))) \\
+ \sum_{j=1}^d \left[ \frac{1}{2} q_j(t)\top \sigma_{xj} (\bar{X}(t), \bar{u}(t)) (y(t))^2 \\
+ q_j(t)\top (\sigma_j (\bar{X}(t), u^e (t)) - \sigma_j (\bar{X}(t), \bar{u}(t))) \right] \, dt. \] (22)

Adding \( i \) on both sides of Equation (22), we have

\[
E \left[ -\sum_{i=1}^n \Psi_x (\bar{X}(t_i)) 1_{A_x} (y(t_i) + z(t_i)) \right.
\]

\[
= E \sum_{i=1}^n \left[ -\Psi_x (\bar{X}(t_i)) 1_{A_x} (y(t_i) + z(t_i)) \\
+ p(t_i)\top (y(t_i) + z(t_i)) - p(t_i)\top (y(t_{i-1}) + z(t_{i-1})) \right] \\
= E \sum_{i=1}^n \left[ p(t_i)\top (y(t_i) + z(t_i)) \\
- p(t_i)\top (y(t_{i-1}) + z(t_{i-1})) \right] \\
= E \int_0^T \left[ f_x (\bar{X}(t), \bar{u}(t)) (y(t) + z(t)) \right. \\
+ \frac{1}{2} p(t)\top b_{xx} (\bar{X}(t), \bar{u}(t)) (y(t))^2 \\
+ p(t)\top (b(\bar{X}(t), u^e (t)) - b(\bar{X}(t), \bar{u}(t))) \\
+ \sum_{j=1}^d \left[ \frac{1}{2} q_j(t)\top \sigma_{xj} (\bar{X}(t), \bar{u}(t)) (y(t))^2 \\
+ q_j(t)\top (\sigma_j (\bar{X}(t), u^e (t)) - \sigma_j (\bar{X}(t), \bar{u}(t))) \right] \, dt. \] (23)

Now, let \( u(t) = u \) be a constant, and \( E_t = [v, v + \varepsilon] \subset [0, T] \). Combining Equation (19) with (23) and noting the optimality of \( \bar{u} (\cdot) \), we obtain

\[ 0 \leq f(u^e (\cdot)) - f(\bar{u}(\cdot)) \]

\[ = \sum_{i=1}^n \left[ \Psi_x (\bar{X}(t_i)) 1_{A_x} (y(t_i) + z(t_i)) \\
+ y(t_i)\top \Psi_x (\bar{X}(t_i)) 1_{A_x} (y(t_i)) \right] \\
+ E \int_0^T \left[ f_x (\bar{X}(t), \bar{u}(t)) (y(t) + z(t)) \right. \\
+ \frac{1}{2} p(t)\top b_{xx} (\bar{X}(t), \bar{u}(t)) y(t) \\
+ f(\bar{X}(t), u^e (t)) - f(\bar{X}(t), \bar{u}(t)) \right] \, dt + o(\varepsilon) \]
\[
= E \sum_{i=1}^{n} y(t_i)^{\top} \Psi_{xx}(\tilde{X}(t_i)) 1_{A_i} y(t_i) \\
+ E \int_{0}^{T} \left[ f_x(\tilde{X}(t), \tilde{u}(t))(y(t) + z(t)) \\
+ \frac{1}{2} y(t)^{\top} f_{xx}(\tilde{X}(t), \tilde{u}(t)) y(t) \\
+ f(\tilde{X}(t), u^\varepsilon(t)) - f(\tilde{X}(t), \tilde{u}(t)) \right] dt + o(\varepsilon) \\
- E \int_{0}^{T} \left[ f_x(\tilde{X}(t), \tilde{u}(t))(y(t) + z(t)) \\
+ \frac{1}{2} p(t)^{\top} b_{xx}(\tilde{X}(t), \tilde{u}(t))(y(t))^2 \\
+ p(t)^{\top} (b(\tilde{X}(t), u^\varepsilon(t)) - b(\tilde{X}(t), \tilde{u}(t))) \\
+ \sum_{j=1}^{d} \left[ \frac{1}{2} q(t)^{\top} \sigma_{xx}^{j}(\tilde{X}(t), \tilde{u}(t))(y(t))^2 \\
+ q(t)^{\top} (\sigma^{j}(\tilde{X}(t), u^\varepsilon(t)) - \sigma^{j}(\tilde{X}(t), \tilde{u}(t))) \right] ight] dt \\
= E \sum_{i=1}^{n} y(t_i)^{\top} \Psi_{xx}(\tilde{X}(t_i)) 1_{A_i} y(t_i) \\
+ E \int_{0}^{T} \left[ \frac{1}{2} y(t)^{\top} f_{xx}(\tilde{X}(t), \tilde{u}(t)) y(t) \\
+ f(\tilde{X}(t), u^\varepsilon(t)) - f(\tilde{X}(t), \tilde{u}(t)) \\
- \frac{1}{2} p(t)^{\top} b_{xx}(\tilde{X}(t), \tilde{u}(t))(y(t))^2 \\
- p(t)^{\top} (b(\tilde{X}(t), u^\varepsilon(t)) - b(\tilde{X}(t), \tilde{u}(t))) \\
+ \sum_{j=1}^{d} \left[ \frac{1}{2} q(t)^{\top} \sigma_{xx}^{j}(\tilde{X}(t), \tilde{u}(t))(y(t))^2 \\
+ q(t)^{\top} (\sigma^{j}(\tilde{X}(t), u^\varepsilon(t)) - \sigma^{j}(\tilde{X}(t), \tilde{u}(t))) \right] ight] dt + o(\varepsilon).
\]

We recall that

\[
H(x, u, p, q) = b(x, u)^{\top} p + \sum_{j=1}^{d} \sigma^{j}(x, u)^{\top} q^{j} - f(x, u),
\]

\[(x, u, p, q) \in \mathbb{R}^{m} \times U \times \mathbb{R}^{m} \times \mathbb{R}^{m \times d}.
\]

Denote \(\tilde{H}(t) := H(\tilde{X}(t), \tilde{u}(t), p(t), q(t))\) and \(H^f(t) := H(\tilde{X}(t), u^\varepsilon(t), p(t), q(t))\). Thus,

\[
f(u^\varepsilon(\cdot)) - f(\tilde{u}(\cdot))
= E \sum_{i=1}^{n} y(t_i)^{\top} \Psi_{xx}(\tilde{X}(t_i)) 1_{A_i} y(t_i) \\
- E \int_{0}^{T} \left[ \frac{1}{2} y(t)^{\top} \tilde{H}_{xx}(t)y(t) + H^f(t) - \tilde{H}(t) \right] dt + o(\varepsilon).
\]

Similar to the proof in Lemma 3.2, by applying the Itô formula to \(y(t)^{\top} P(t)y(t)\) over \((t_{i-1}, t_i)\), we have

\[
E \sum_{i=1}^{n} y(t_i)^{\top} \Psi_{xx}(\tilde{X}(t_i)) 1_{A_i} y(t_i) \\
= -E \int_{0}^{T} \frac{1}{2} \left[ \sum_{j=1}^{d} (\sigma^{j}(\tilde{X}(t), u^\varepsilon(t)) - \sigma^{j}(\tilde{X}(t), \tilde{u}(t)))^{\top} \\
\times P(t)(\sigma^{j}(\tilde{X}(t), u^\varepsilon(t)) - \sigma^{j}(\tilde{X}(t), \tilde{u}(t))) \\
+ y(t)^{\top} \tilde{H}_{xx}(t)y(t) \right] + o(\varepsilon).
\]

Then, we obtain

\[
o(\varepsilon) \geq E \int_{0}^{T} \left[ H^f(t) - \tilde{H}(t) + \frac{1}{2} \sum_{j=1}^{d} (\sigma^{j}(\tilde{X}(t), u^\varepsilon(t)) - \sigma^{j}(\tilde{X}(t), \tilde{u}(t)))^{\top} \\
\times P(t)(\sigma^{j}(\tilde{X}(t), u^\varepsilon(t)) - \sigma^{j}(\tilde{X}(t), \tilde{u}(t))) \\
- \sigma^{j}(\tilde{X}(t), \tilde{u}(t))) \right] dt.
\]

Note that \(E_{\varepsilon} = [\nu, \nu + \varepsilon] \subset (t_{i-1}, t_i)\), for \(i = 1, 2, \ldots, n\), and \(\tau^n \not\in (t_{i-1}, t_i)\), where

\[
\tau^n = \sum_{i=1}^{n-1} t_i 1_{\{t_{i-1} \leq t < t_i\}} + t_n 1_{\{t_{n-1} \leq t \leq t_n\}},
\]

which deduces that

\[
H(\tilde{X}(t), \tilde{u}(t), p(t), q(t)) - H(\tilde{X}(t), u, p(t), q(t)) \\
\geq \frac{1}{2} \sum_{j=1}^{d} [\sigma^{j}(\tilde{X}(t), \tilde{u}(t)) - \sigma^{j}(\tilde{X}(t), u)]^{\top} \\
\times P(t) [\sigma^{j}(\tilde{X}(t), \tilde{u}(t)) - \sigma^{j}(\tilde{X}(t), u)],
\]

for any \(u \in U\) and \(t \in (t_{i-1}, t_i)\), \(i = 1, 2, \ldots, n\).

This completes the proof. \(\blacksquare\)

In the following, we use the main result of Theorem 3.1 to solve a financial example.

**Example 3.3:** Let the risk-free return rate is 0. We consider one asset which is traded in the market. The stock asset is described by

\[
\left\{ \begin{array}{l}
\frac{dS(t)}{S(t)} = b(t)S(t) dt + \sigma S(t) dW(t), \ t \in (0, 1], \\
S(0) = 1.
\end{array} \right.
\]

We denote \(\tau\) as the signal of the bear market, which can be defined by the return rate of the benchmark \(\gamma(t), \ t \in [0, 1]\) (the benchmark could be some kind of stock index),

\[
\tau = \inf \{t : \gamma(t) \leq \alpha\} \wedge 1,
\]

where \(\alpha < 0\). Here, \(\tau\) takes values 0.5, 1 and \(P(\tau = 0.5) = P(\tau = 1) = 0.5\). We suppose that the return \(b(\cdot)\) of the stock \(S(\cdot)\)
dependents on $\tau$:

$$b(t) = \begin{cases} 
\beta, & t \in [0, \tau), \\
-\beta, & t \in [\tau, T], 
\end{cases}$$

where $\beta > 0$ and the volatility $\sigma > 0$ is a constant. Thus, the investor’s wealth $X^u(\cdot)$ satisfies

$$\begin{cases} 
\frac{dX^u(t)}{dt} = u(t) [b(t) \, dt + \sigma \, dW(t)], \\
X^u(0) = 0, 
\end{cases}$$

(24)

where $u(\cdot)$ is the capital invested in the risky asset and takes values in $U = [-1, 1]$.

It should be noted that the investor needs to minimise whose cost before stopping time $\tau$ (bear market). Thus, the cost functional is given as follows:

$$J(u(\cdot)) = E[-X^u(\tau)].$$

(25)

Based on Theorem 3.1, we introduce the following first-order adjoint equations:

$$dp(t) = q(t) \, dW(t), \quad t \in (0, 1),$$

$$p(1) = 1_{[\tau=1]},$$

and

$$dp(t) = q(t) \, dW(t), \quad t \in [0, 0.5),$$

$$p(0.5) = 1_{[\tau=0.5]} + p(0.5^+).$$

We can verify that the solution of the second-order adjoint equation is $(0, 0)$.

Denote

$$H(t, x, u, p, q) = ub(t)p + u\sigma q,$$

$$(t, x, u, p, q) \in [0, 1] \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R}.$$  

By Theorem 3.1, for any $u \in [-1, 1]$, we have

$$H(\bar{X}(t), \bar{u}(t), p(t), q(t)) - H(\tilde{X}(t), u, p(t), q(t))$$

$$= (\bar{u}(t) - u)(b(t)p(t) + \sigma q(t))$$

$$\geq 0.$$  

Denoting $A(t) = \{\omega : b(t)p(t) + \sigma q(t) \geq 0\}$, we have

$$\bar{u}(t) = 1_{A(t)} - 1_{\bar{A}(t)}, \quad t \in [0, 0.5) \cup (0.5, 1).$$

To solve the first-order equation, we suppose $\tau$ independents on Brownian motion $W(\cdot)$. Thus, we have

$$(p(t), q(t)) = \begin{cases} 
(0.5, 0), & t \in (0.5, 1), \\
(1.0, 0), & t \in [0, 0.5), 
\end{cases}$$

which deduces that

$$\bar{u}(t) = 1, \quad t \in [0, 0.5] \cup (0.5, 1).$$

To describe the control efficiency, we set

$$\beta = 1, \quad \sigma = 1, \quad \tau = 1.$$  

The optimal state trajectory under the optimal control $\bar{u}(t)$ is denoted as $\bar{X}(\cdot)$. We plot the figure of state trajectories of $E[\bar{X}(\cdot)]$ and $E[X^u(\cdot)]$ as follows:

![Figure 1. State trajectories of $E[\bar{X}(\cdot)]$ and $E[X^u(\cdot)]$ with $u = 0.2$ and $u = t$.](image-url)
In Figure 1, the topside line denotes the state trajectories $E[\bar{X}(\cdot)]$, the bottom line denotes the state trajectories $E[X_u(\cdot)]$ with $u = 0.2$, and the middle line denotes the state trajectories $E[X^u(\cdot)]$ with $u = t$. Note that the state $E[X^u(\cdot)]$ is the average wealth of the investment. Thus, from this example, we can see that the optimal control can be used to find a better return.

4. Maximum principle under constraints

Recalling that
$$ r^n = \sum_{i=1}^{n-1} t_i 1_{\{t_{i-1} \leq \tau < t_i\}} + t_n 1_{\{t_{n-1} \leq \tau \leq t_n\}}, $$
where $0 = t_0 < t_1 < t_2 \cdots < t_n = T$ and $t_i - t_{i-1} = \frac{T}{n}, \ i = 1, 2, \ldots, n$. Similar to the proof in Lemma 2.4, for any given $\varepsilon > 0$, there exists a sufficiently large $n$ such that
$$ E \left| \Psi(X^n_u(\tau^n)) - \Psi(X^n_u(\tau)) \right| < \varepsilon. $$
Thus, $E\Psi(X^n_u(\tau^n))$ is close to $E\Psi(X^n_u(\tau))$. In the following, we give the Pontryagin stochastic maximum principle under constrained conditions with stopping time $\tau^n$, i.e. the cost functional is given as follows:
$$ J(u(\cdot)) = E \int_0^T f(X^n_u(t), u(t)) \, dt, $$
where the state process $X(\cdot)$ satisfies the following constrained condition:
$$ 0 \leq E\Psi(X^n_u(\tau^n)). $$
Notice that
$$ \Psi(X^n_u(\tau^n)) = \sum_{i=1}^{n-1} \Psi(X^n_u(t_i)) 1_{\{t_{i-1} \leq \tau < t_i\}} + \Psi(X^n_u(t_n)) 1_{\{t_{n-1} \leq \tau \leq t_n\}}. $$
For notation simplicity, again, we set
$$ A_i = \{t_{i-1} \leq \tau < t_i\}, \ i = 1, 2, \ldots, n - 1, $$
$$ A_n = \{t_{n-1} \leq \tau \leq t_n\}. $$
Thus, we can rewrite constrained condition (27) as
$$ 0 \leq E \sum_{i=1}^{n} \Psi(X^n_u(t_i)) 1_{A_i}. $$
(29)
To prove the main result of this section, we introduce the following Ekeland variational principle, which comes from Corollary 6.3 in Yong and Zhou (1999).

**Lemma 4.1:** Let $F : V \to \mathbb{R}$ be a continuous function on a complete metric space $(V, \bar{d})$. Given $\theta > 0$ and $v_0 \in V$ such that
$$ F(v_0) \leq \inf_{v \in V} F(v) + \theta. $$
Then, there exists a $v_0 \in V$ such that
$$ F(v_0) \leq F(v), \ \bar{d}(v_0, v) \leq \sqrt{\theta}, $$
and for all $v \in V$,
$$ -\sqrt{\theta} \bar{d}(v_0, v) \leq F(v) - F(v_0). $$

The related Hamiltonian is given as follows:
$$ H(\beta^0, X, u, p, q) = b(x, u)^T p + \sum_{j=1}^{d} \sigma^j(x, u)^T q^j - \beta^0 f(x, u), $$
where $(\beta^0, x, u, p, q) \in \mathbb{R} \times \mathbb{R}^m \times U \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$. Next, we present the main results of this section.

**Theorem 4.2:** Let Assumptions 2.1, 2.2 and 2.3 hold, and $(\tilde{u}(\cdot), \tilde{X}(\cdot))$ be an optimal pair of (26). Then, there exists $(\beta^0, \beta^1, \ldots, \beta^n) \in \mathbb{R}^{n+1}$ satisfying
$$ \beta^0 \geq 0, \ \|\beta^0\|^2 + \sum_{j=1}^{n} |\beta^j|^2 = 1, $$
and
$$ \beta^j (\gamma - E[\Psi(\tilde{X}(t_j))])_{A_j} \geq 0, \ \gamma \leq 0, \ j = 1, 2, \ldots, n. $$
The adapted solution $(p(\cdot), q(\cdot))$ satisfies the following series of first-order adjoint equations:
$$ -dp(t) = \begin{bmatrix} b_x(\tilde{X}(t), \tilde{u}(t))^T p(t) + \sum_{j=1}^{d} \sigma^j_x(\tilde{X}(t), \tilde{u}(t))^T q^j(t) \\
- \beta^0 f_x(\tilde{X}(t), \tilde{u}(t)) \end{bmatrix} dt - q(t) \, dW(t), $$
$$ t \in (t_{i-1}, t_i), $$
$$ p(t_i) = -\beta^i \Psi_x(\tilde{X}(t_i))^T 1_{A_i} + p(t_i^+), \ i = 1, 2, \ldots, n, $$
and second-order adjoint equations,
$$ -dP(t) = \begin{bmatrix} b_x(\tilde{X}(t), \tilde{u}(t))^T P(t) + P(t) b_x(\tilde{X}(t), \tilde{u}(t)) \\
+ \sum_{j=1}^{d} \sigma^j_x(\tilde{X}(t), \tilde{u}(t))^T P(t) \sigma^j_x(\tilde{X}(t), \tilde{u}(t)) \\
+ \sum_{j=1}^{d} \left[ \sigma^j_x(\tilde{X}(t), \tilde{u}(t))^T Q^j(t) \\
+ Q^j(t) \sigma^j_x(\tilde{X}(t), \tilde{u}(t)) \right] \end{bmatrix} dt - Q(t) \, dW(t), $$
$$ t \in (t_{i-1}, t_i), $$
$$ -P(t_i) = \beta^i \Psi_{xx}(\tilde{X}(t_i))^T 1_{A_i} - P(t_i^+), $$
where $p(t^+_i) = 0, P(t^+_i) = 0$, and
$$ H(\beta^0, \tilde{X}(t), \tilde{u}(t), p(t), q(t)) - H(\beta^0, \tilde{X}(t), u, p(t), q(t)) \geq \frac{1}{\tau} \sum_{j=1}^{d} \left[ \sigma^j(\tilde{X}(t), \tilde{u}(t)) - \sigma^j(\tilde{X}(t), u) \right]^T \sigma^j(\tilde{X}(t), \tilde{u}(t)).
\[ \times P(t) \left[ \sigma^i(\tilde{x}(t), \tilde{u}(t)) - \sigma^j(\tilde{x}(t), u) \right], \]

for any \( u \in U \) and \( t \in (t_i, t_{i+1}) \), where \( i = 0, 1, \ldots, n - 1 \).

**Proof:** Without loss of generality, we assume that \( J(\tilde{u}(\cdot)) = 0 \), where \((\tilde{u}(\cdot), \tilde{X}(\cdot))\) is the optimal pair of problem (26) with constrained condition (29). For any \( \theta > 0 \), we set

\[ f^\theta(\cdot) = \left[ [J(u(\cdot)) + \theta]^+ \right]^2 + \sum_{i=1}^{n} \left[ (-E\Psi(X^u(t_i))1_A_j)^+ \right]^2. \]

In addition, one can verify that \( f^\theta : \mathcal{U}[0, T] \to \mathbb{R} \) is a continuous function and satisfies

\[ f^\theta(\tilde{u}(\cdot)) = \theta \leq \inf_{u \in \mathcal{U}[0, T]} f^\theta(u(\cdot)) + \theta. \]

Now, by Lemma 4.1, there exists a \( u^\theta(\cdot) \in \mathcal{U}[0, T] \) such that

\[ f^\theta(u^\theta(\cdot)) \leq f^\theta(\tilde{u}(\cdot)) = \theta, \]

where \( d(u^\theta(\cdot), u^\theta(\cdot)) = M((t, \omega) \in [0, T] \times \Omega : u^1(t, \omega) \neq u^2(t, \omega)), M \) is the product measure of the Lebesgue measure and probability on the set of \([0, T] \times \Omega\). We can verify that \((\mathcal{U}[0, T], d)\) is a complete metric space. Also, we have

\[ -\sqrt{\theta}d(u^\theta(\cdot), u(\cdot)) \leq f^\theta(u(\cdot)) - f^\theta(u^\theta(\cdot)), \quad \forall u(\cdot) \in \mathcal{U}[0, T], \]

which deduces

\[ f^\theta(u^\theta(\cdot)) + \sqrt{\theta}d(u^\theta(\cdot), u(\cdot)) \leq f^\theta(u^\theta(\cdot)) + \sqrt{\theta}d(u^\theta(\cdot), u(\cdot)), \quad \forall u(\cdot) \in \mathcal{U}[0, T]. \]

Thus, inequality (35) shows that \((u^\theta(\cdot), X^\theta(\cdot))\) is the optimal pair for the cost functional

\[ f^\theta(u(\cdot)) + \sqrt{\theta}d(u^\theta(\cdot), u(\cdot)), \]

without the state constraints.

Since \( U \) is a general control domain, let \( \rho > 0 \) and \( E_\rho = [v, v + \rho] \subset (t_{i-1}, t_i) \), for some \( i \in \{1, 2, \ldots, n\} \). Let \( u \in U \) be any given constant. We define the following:

\[ u^{\theta, \rho}(t) = \begin{cases} u^\theta(t), & \text{if } t \in [0, T] \setminus E_\rho, \\ u, & \text{if } t \in E_\rho, \end{cases} \]

which belongs to \( \mathcal{U}[0, T] \). It is easy to verify that

\[ d(u^{\theta, \rho}(\cdot), u^\rho(\cdot)) \leq \rho, \]

By applying Equation (35), it follows

\[ -\sqrt{\theta} \leq f^\theta(u^{\theta, \rho}(\cdot)) - f^\theta(u^\theta(\cdot)) \]

\[ \leq \left[ [J(u^{\theta, \rho}(\cdot)) + \theta]^+ \right]^2 - \left[ [J(u^\theta(\cdot)) + \theta]^+ \right]^2 \]

\[ + \sum_{j=1}^{n} \left[ (-E\Psi(X^{u^\theta}(t_j))1_A_j)^+ \right]^2 \]

\[ \leq \left[ [J(u^{\theta, \rho}(\cdot)) + \theta]^+ \right]^2 - \left[ [J(u^\theta(\cdot)) + \theta]^+ \right]^2 \]

\[ + \sum_{j=1}^{n} \left[ (-E\Psi(X^{u^\theta}(t_j))1_A_j)^+ \right]^2 \]

\[ \leq [J(u^\theta(\cdot)) + \theta]^+ f^\theta(u^\theta(\cdot)) + f^\theta(u^\theta(\cdot)) \]

where \( X^{\theta, \rho}(\cdot) \) and \( X^\theta(\cdot) \) are the related solutions of Equation (4) with controls \( u^{\theta, \rho}(\cdot) \) and \( u^\theta(\cdot) \). We set

\[ \beta^{0, \theta}_j = \frac{[f^\theta(u^\theta(\cdot)) - \theta]^+}{f^\theta(u^\theta(\cdot))}, \quad \beta^{1, \theta}_j = \frac{[-E\Psi(X^\theta(t_j))1_A_j]^+}{f^\theta(u^\theta(\cdot))}, \quad j = 1, 2, \ldots, n. \]

Then, by the continuity of \( f^\theta(\cdot) \), we have

\[ \beta^{0, \theta}(u^{\theta, \rho}(\cdot)) - \beta^{0, \theta}(u^\theta(\cdot)) \]

\[ = \beta^{0, \theta} \left[ J(u^{\theta, \rho}(\cdot)) - J(u^\theta(\cdot)) \right] \]

\[ + \sum_{j=1}^{n} \beta^{1, \theta}_j \left[ E\Psi(X^{\theta, \rho}(t_j)1_A_j - E\Psi(X^\theta(t_j))1_A_j \right] + o(1), \]

\[ = E \sum_{j=1}^{n} \beta^{1, \theta}_j \left[ \Psi(X^{\theta, \rho}(t_j))1_A_j - \Psi(X^\theta(t_j))1_A_j \right] \]

\[ + \beta^{0, \theta} \int_0^T \left( f(X^{\theta, \rho}(t), u^{\theta, \rho}(t)) - f(\tilde{X}^\theta(t), u^\theta(t))) \right) dt \]

\[ + o(\rho), \]

where \( o(1) \) converges to 0 when \( \rho \to 0 \).

Similar to the proof in Lemma 3.2, let \((\tilde{X}(\cdot), \tilde{u}(\cdot))\) be replaced by \((X^\theta(t), u^\theta(t)))\), \(y(\cdot)\) be replaced by \(\tilde{y}(\cdot)\) in Equation (16), and \(z(\cdot)\) be replaced by \(\tilde{z}(\cdot)\) in Equation (17). Thus, we obtain

\[ -dp^\theta(t) \leq f^\theta(u^{\theta, \rho}(\cdot)) - f^\theta(u^\theta(\cdot)) \]

\[ \leq E \sum_{j=1}^{n} \beta^{1, \theta}_j \Psi_\rho(X^{\theta, \rho}(t_j))1_A_j \tilde{y}(t_j) + \tilde{z}(t)) \]

\[ + \sum_{j=1}^{n} \beta^{1, \theta}_j \frac{1}{2} \tilde{y}(t_j)^T \Psi_\rho(X^{\theta, \rho}(t_j))1_A_j \tilde{y}(t_j) \]

\[ + \beta^{0, \theta} \int_0^T \left( f_{\rho}(X^\theta(t), u^\theta(t))\tilde{y}(t) + \tilde{z}(t) \right) \]

\[ + \frac{1}{2} \tilde{y}(t)^T f_{\rho}(X^\theta(t), u^\theta(t))\tilde{y}(t) \]

\[ + f(\tilde{X}^\theta(t), u^{\theta, \rho}(t)) - f(\tilde{X}^\theta(t), u^\theta(t))) \right) dt \]

\[ + o(\rho). \]

In addition, we introduce the following adjoint equations:

\[ -dp^\theta(t) = \left[ \sigma_\rho(X^\theta(t), u^\theta(t))^T p^\theta(t) \right] \]

\[ + \sum_{j=1}^{d} \sigma_\rho(X^\theta(t), u^\theta(t))^T q^\theta(t) \]

\[ - \beta^{0, \theta} f_{\rho}(X^\theta(t), u^\theta(t)) \right] dt \]
\[ -q^\theta(t) \, dW(t), \quad t \in (t_{i-1}, t_i), \]
\[ p^\theta(t_i) = -\beta^\theta \Psi_x(X^\theta(t_i)) \top 1_{A_i} + p^\theta(t_i^+), \quad i = 1, \ldots, n, \]
where \( q^\theta(\cdot) = (q^1(\cdot), q^2(\cdot), \ldots, q^n(\cdot)) \), and
\[
-dp^\theta(t) = \left[ b_x(X^\theta(t), u^\theta(t)) \top p^\theta(t) + p^\theta(t)b_x(X^\theta(t), u^\theta(t)) \\
+ \sum_{j=1}^d \sigma_j^x(X^\theta(t), u^\theta(t)) \top \sigma_j^x(X^\theta(t), u^\theta(t)) \\
+ \sum_{j=1}^d \left[ \sigma_j^x(X^\theta(t), u^\theta(t)) \top Q^\theta(t) \\
+ Q^\theta(t) \sigma_j^x(X^\theta(t), u^\theta(t)) \right] \\
+ H_{xx}(\beta^0, \bar{X}^\theta(t), u^\theta(t), p^\theta(t), q^\theta(t)) \right] \, dt \\
- \frac{1}{2} \sum_{j=1}^d \left[ \sigma_j^x(X^\theta(t), u^\theta(t)) - \sigma_j^x(X^\theta(t), u^\theta(t)) \right] \top \sigma_j^x(X^\theta(t), u^\theta(t)) \, dt,
\]
\[ -p^\theta(t_i) = \beta^\theta \Psi_x(\bar{X}(t_i))1_{A_i} - p^\theta(t_i^+), \]
where \( p^\theta(t_i^+) = 0 \) and \( Q^\theta(\cdot) = (Q^1(\cdot), Q^2(\cdot), \ldots, Q^n(\cdot)) \).

Now, by applying the duality relation as in the proof of Theorem 3.1, we get
\[
\sqrt{\theta} \geq E \int_0^T \left[ H^\theta(t, u^\theta(t)) - H^\theta(t, X^\theta(t)) \\
+ \frac{1}{2} \sum_{j=1}^d \left[ \sigma_j^x(X^\theta(t), u^\theta(t)) - \sigma_j^x(X^\theta(t), u^\theta(t)) \right] \top \right. \left. \sigma_j^x(X^\theta(t), u^\theta(t)) \right] \, dt,
\]
where
\[ H^\theta(t, u^\theta(t)) = H(\beta^0, X^\theta(t), u^\theta(t), p^\theta(t), q^\theta(t)), \]
and
\[ H^\theta(t, u^\theta(\cdot)) = H(\beta^0, X^\theta(t), u^\theta(\cdot), p^\theta(t), q^\theta(t)). \]

Notice that \( o(1) \to 0 \) when \( \rho \to 0 \). Thus, when \( \rho \to 0 \), we get
\[
\sqrt{\theta} \geq H^\theta(t, u) - H^\theta(t, u^\theta(t)) \\
+ \frac{1}{2} \sum_{j=1}^d \left[ \sigma_j^x(X^\theta(t), u) - \sigma_j^x(X^\theta(t), u^\theta(t)) \right] \top \sigma_j^x(X^\theta(t), u^\theta(t)) \]
\[ \times P^\theta(t) \left[ \sigma_j^x(X^\theta(t), u) - \sigma_j^x(X^\theta(t), u^\theta(t)) \right], \]
From inequality (34), we observe that \( u^\theta(\cdot) \) converges to \( \bar{u}(\cdot) \) under \( d \) as \( \theta \to 0 \). Then, by Assumptions 2.1, 2.2, and the basic theory of stochastic differential equation, we have
\[
\sup_{0 \leq t \leq T} E \left| X^\theta(t) - \bar{X}(t) \right| \to 0,
\]
as \( \theta \to 0 \). From Equation (38), we have
\[
\left| \beta^\theta(\cdot) \right|^2 + \frac{n}{2} \left| \beta^\theta(\cdot) \right|^2 = 1. \tag{44}
\]
Thus, we can choose a sequence \( \{\theta_k\}_{k=1}^\infty \) satisfying \( \lim_{k \to \infty} \theta_k = 0 \) such that the limitations of \( \beta^{\theta_k} \) and \( \beta^{\theta_k} \) exist, and we denote them by
\[
\beta^0 = \lim_{k \to \infty} \beta^{\theta_k}, \quad \beta^1 = \lim_{k \to \infty} \beta^{\theta_k}, \tag{45}
\]
respectively, with \( j = 1, 2, \ldots, n \). From Equation (44), we have
\[
\left| \beta^0 \right|^2 + \frac{n}{2} \left| \beta^1 \right|^2 = 1,
\]
and
\[
\beta^j \left[ y - E(\Psi(\bar{X}(t))1_{A_j}) \right] \geq 0, \quad y \geq 0, \quad j = 1, 2, \ldots, n.
\]
Similarly, we can obtain
\[
\sup_{0 \leq t \leq T} E \left[ |p^{\theta_k}(t) - p(t)|^2 + \int_0^T |q^{\theta_k}(t) - q(t)|^2 \, dt \right] \to 0,
\]
as \( k \to \infty \). When \( k \to \infty \), from Equation (43), we have
\[
H(\beta^0, \bar{X}(t), \bar{u}(t), p(t), q(t)) - H(\beta^0, \bar{X}(t), u, p(t), q(t))) \geq \frac{1}{2} \sum_{j=1}^d \left[ \sigma_j^x(\bar{X}(t), \bar{u}(t)) - \sigma_j^x(\bar{X}(t), u(t)) \right] \\
\times P(t) \left[ \sigma_j^x(\bar{X}(t), \bar{u}(t)) - \sigma_j^x(\bar{X}(t), u(t)) \right], \tag{46}
\]
for any \( u \in U \) and \( t \in (t_i, t_{i+1}) \), \( i = 0, 1, \ldots, n - 1 \).

Thus, we complete this proof.

In the following, we provide an example to illustrate the optimal production planning problem under stopping time state constraints.

**Example 4.3:** Let \( T = 1 \), and consider the following controlled stochastic differential equation:
\[
X^u(t) = \int_0^t [u(s) - y(s)] \, ds, \tag{47}
\]
where \( y(\cdot) \) denotes an uncertain demand
\[
y(t) = \frac{8}{3} t - W(t),
\]
and \( u(\cdot) = \{u(t), 0 \leq t \leq 1\} \) is a control process taking values in a compact set \( U = [0, 2] \). Thus, we minimise the following
cost functional:
\[ J(u(\cdot)) = E[X^u(1)], \]  
(48)

with the state constraints
\[ 0 \leq EX^u(\tau), \quad EX^u(1), \quad 0 \leq \tau \leq 0.5. \]  
(49)

For a given integer \( N > 0 \), let \( \tau \) takes the values \( \frac{i}{2N} \), \( i = 1, 2, \ldots, N \). Substituting \( X^u(\cdot) \) into Equation (48), we obtain

\[ J(u(\cdot)) = E \int_0^1 \left[ u(s) - \frac{8}{3} s \right] ds. \]

We can show that
\[
(\hat{u}(t), \hat{X}(t)) = \begin{cases} \left( \frac{2i - 1}{4N}, \int_0^t W(s) ds \right), & \frac{i - 1}{2N} \leq t < \frac{i}{2N}, \quad i = 1, 2, \ldots, N, \\ \left( 2, 2t - \frac{4}{3} t^2 - \frac{2}{3} + \int_0^t W(s) ds \right), & \frac{1}{2} \leq t < 1, \end{cases}
\]

(50)
is an optimal pair of system (48) under state constraints (49).

Next, we show the maximum principle for optimal control under multi-time state constraints (49). Notice that it is difficult to obtain the adjoint equations for state process (47) directly. To get the related adjoint equations, we rewrite Equation (47) as follows:

\[
X^u(t) - W(t)t = \int_0^t \left[ u(s) - \frac{8}{3} s \right] ds - \int_0^t s dW(s).
\]

Denoting \( \hat{X}^u(t) = X^u(t) - W(t)t \), we have
\[
d\hat{X}^u(t) = \left[ u(t) - \frac{8}{3} t \right] dt - t dW(t),
\]

and
\[
E[\hat{X}^u(1)] = E[X^u(1)],
\]

which shows that \((\hat{u}(t), \hat{X}(t)) : (\hat{u}(t), \hat{X}(t) - W(t)t)\) is the optimal pair of the following cost functional:

\[ J(u(\cdot)) = E[\hat{X}^u(1)], \]  
(51)

with the following constrained conditions:
\[ 0 \leq E[\hat{X}^u(\tau)], \quad E[\hat{X}^u(1)]. \]

Next, we introduce the following first-order adjoint equations for functional (51):

\[
dp(t) = q(t) dW(t), \quad \frac{1}{2} < t < 1,
\]

\[
p(1) = -(\beta^0 + \beta^{N+1})^T,
\]

and
\[
dp(t) = q(t) dW(t), \quad \frac{i-1}{2N} < t < \frac{i}{2N},
\]

\[
p(t_i) = -\beta^i + p(t_i^+),
\]

and second-order adjoint equations
\[
dP(t) = Q(t) dW(t),
\]

\[
P(T) = 0,
\]

(52)

as follows:
\[
(p(t), q(t)) = \begin{cases} \left( -\left( \beta^0 + \sum_{j=i}^{N+1} \beta^j \right), 0 \right), & \frac{i-1}{2N} < t < \frac{i}{2N}, \\ \left( -\left( \beta^0 + \beta^{N+1} \right), 0 \right), & \frac{1}{2} \leq t \leq 1, \end{cases}
\]

(53)

and
\[
(P(t), Q(t)) = (0, 0).
\]

(54)

Now, from Theorem 4.2, we have \( \beta^0 = -\frac{\sqrt{2}}{2}, \beta^{N+1} = \frac{\sqrt{2}}{2}, \beta^1 = 0, i = 1, 2, \ldots, N \). From \( \beta^0 + \beta^{N+1} \leq 0 \) and \( \beta^0 + \sum_{j=i}^{N+1} \beta^j = 0, i = 1, 2, \ldots, N \), we obtain
\[
H(\beta^0, X(t), \hat{u}(t), p(t), q(t)) = H(\beta^0, \hat{X}(t), u, p(t), q(t)) = \begin{cases} -\frac{1}{2} \sum_{j=1}^d \left( \theta_j(X(t), \hat{u}(t)) - \theta_j(\hat{X}(t), u) \right)^T \\ \times P(t) \theta_j(\hat{X}(t), \hat{u}(t) - \theta_j(X(t), u)) \end{cases}
\]

(55)

Finally,
\[
[\hat{u}(t) - u] p(t)
\]

(56)

where \( u \in [0, 2] \). Hence, the optimal control pair \((\hat{u}(\cdot), \hat{X}(\cdot))\) satisfies Theorem 4.2.

5. Conclusion

In this study, we considered the stochastic optimal control problem with a stopping time cost functional. We studied the classical constant terminal time as a stopping time in the cost functional. For example, in the financial market, the time when the investor leaves the market can be modelled as the stopping time. To solve this kind of problem, we introduced a near-optimal control problem for the stopping time cost functional. Based on a series of first- and second-order adjoint equations, we established a stochastic maximum principle for the near-optimal control problem and the near-optimal control problem under a stopping time state constraint. A closely related work (Yang, 2018) provides the necessary and sufficient conditions for stochastic differential systems under multi-time state cost functional with a convex control domain. In the future, certain related topics, such as to show the dynamic programming principle for this system, to solve the mean–variance problem, and to perform empirical analysis, should be considered for the stopping time cost functional.
Acknowledgements
This work was supported by the National Key R&D Program of China (No. 2018YFA0703900), the National Natural Science Foundation of China (Grant No. 11701330) and Young Scholars Program of Shandong University.

Disclosure statement
No potential conflict of interest was reported by the author(s).

Funding
This work was supported by the National Key Research and Development Project [No. 2018YFA0703900], the National Natural Science Foundation of China [Grant No. 11701330, 11871050] and Young Scholars Program of Shandong University.

ORCID
Shuzhen Yang
http://orcid.org/0000-0001-8501-7634

References
Bensoussan, A. (1981). Lecture on stochastic control. In Nonlinear filtering and stochastic control. Lecture Notes in Mathematics 972, Proc. Cortona. Springer-Verlag, Editors: Morel, Jean-Michel and Teissier, Bernard, 1-62. Bismut, J. (1978). An introductory approach to duality in optimal stochastic control. SIAM Review, 20, 62–78. https://doi.org/10.1137/1020004 Bourdin, L., & Trélat, E. (2013). Pontryagin maximum principle for finite dimensional nonlinear optimal control problems on time scales. SIAM Journal on Control and Optimization, 51(5), 3781–3813. https://doi.org/10.1137/130912219 Bourdin, L., & Trélat, E. (2016). Optimal sampled-data control, and generalizations on time scales. Mathematical Control & Related Fields, 6(1), 53–94. https://doi.org/10.3934/mcrf Bourdin, L., & Trélat, E. (2017). Linear-quadratic optimal sampled-data control problems: Convergence result and Riccati theory. Automatica Journal of IFAC, 79, 273–281. https://doi.org/10.1016/j.automatica.2017.02.013 Dmitruk, A. V., & Kaganovich, A. M. (2008). The hybrid maximum principle is a consequence of Pontryagin maximum principle. Systems & Control Letters, 57, 964–970. https://doi.org/10.1016/j.sysconle.2008.05.006 Dmitruk, A. V., & Kaganovich, A. M. (2011). Maximum principle for optimal control problems with intermediate constraints. Computational Mathematics and Modeling, 22(2), 180–215. https://doi.org/10.1007/s10598-011-9096-8 Huang, J., Li, X., & Wang, G. (2010). Near-optimal control problems for linear forward backward stochastic systems. Automatica, 46, 397–404. https://doi.org/10.1016/j.automatica.2009.11.016 Hui, E., Huang, J., Li, X., & Wang, G. (2011). Near-optimal control for stochastic recursive problems. Systems & Control Letters, 60, 161–168. https://doi.org/10.1016/j.sysconle.2010.10.010 Lipster, R. S., & Shiryaev, A. N. (1978). Statistics of random processes I. Springer. Liu, Y., Yin, G., & Zhou, X. Y. (2005). Near-optimal controls of random-switching LQ problems with indefinite control weight costs. Automatica, 41, 1063–1070. https://doi.org/10.1016/j.automatica.2005.01.002 Martellini, L., & Urošević, B. (2006). Static mean-variance analysis with uncertain time horizon. Management Science, 52, 955–964. https://doi.org/10.1287/mnsc.1060.0507 Peng, S. (1990). A general stochastic maximum principle for optimal control problem. SIAM Journal on Control and Optimization, 28(4), 966–979. https://doi.org/10.1137/0328054 Wu, H. L., Li, Z. F., & Li, D. (2011). Multi-period mean-variance portfolio selection with Markov regime switching and uncertain time horizon. Journal of Systems Science and Complexity, 24, 140–155. https://doi.org/10.1007/s11424-011-9184-z Yang, S. (2016). The maximum principle for stochastic differential systems with general cost functional. Systems & Control Letters, 90, 1–6. https://doi.org/10.1016/j.sysconle.2016.01.001 Yang, S. (2018). The necessary and sufficient conditions for stochastic differential systems with multi-time states cost functional. Systems & Control Letters, 114, 11–18. https://doi.org/10.1016/j.sysconle.2018.02.002 Yang, S. (2020). A varying terminal time structure for stochastic optimal control under constrained condition. International Journal of Robust and Nonlinear Control, 30, 5181–5204. https://doi.org/10.1002/rnc.v30.13 Yao, H., & Ma, Q. (2010). Continuous time mean-variance model with uncertain exit time. In International Conference on Management and Service Science. (MASS2010)-Wuhan, China, (pp. 1–4). Yi, L., Li, Z. F., & Li, D. (2008). Multi-period portfolio selection for asset-liability management with uncertain investment horizon. Journal of Industrial and Management Optimization, 4, 535–552. https://doi.org/10.3934/jimo.2008.4.535 Yong, J., & Zhou, X. (1999). Stochastic controls: Hamiltonian systems and HJB equations. In Stochastic differential equations: An introduction with applications. Springer Verlag, Editors: I. Karatzas and M. Yor, 1-438. Yu, Z. (2013). Continuous time mean-variance portfolio selection with random horizon. Applied Mathematics and Optimization, 68, 333–359. https://doi.org/10.1007/s00245-013-9209-1 Zhou, X. (1995). Deterministic near-optimal controls, part I: Necessary and sufficient conditions for near-optimality. Journal of Optimization Theory Applications, 85, 473–488. https://doi.org/10.1007/BF02192237 Zhou, X. (1996). Deterministic near-optimal controls, part II: Dynamic programming and viscosity solution approach. Mathematics of Operations Research, 21, 655–674. https://doi.org/10.1287/moor.21.3.655 Zhou, X. (1998). Stochastic near-optimal controls: Necessary and sufficient conditions for near-optimality. SIAM Journal on Control and Optimization, 39, 929–947. https://doi.org/10.1137/S0363012996302664