SHORTEST LENGTH GEODESICS ON CLOSED HYPERBOLIC SURFACES

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ABSTRACT. Given a hyperbolic surface, the set of all closed geodesics whose length is minimal form a graph on the surface, in fact a so called fat graph, which we call the systolic graph. The central question that we study in this paper is: which fat graphs are systolic graphs for some surface - we call such graphs admissible. This is motivated in part by the observation that we can naturally decompose the moduli space of hyperbolic surfaces based on the associated systolic graphs.

A systolic graph has a metric on it, so that all cycles on the graph that correspond to geodesics are of the same length and all other cycles have length greater than these. This can be formulated as a simple condition in terms of equations and inequations for sums of lengths of edges which we call combinatorial admissibility.

Our first main result is that admissibility is equivalent to combinatorial admissibility. This is proved using properties of negative curvature, specifically that polygonal curves with long enough sides, in terms of a lower bound on the angles, are close to geodesics.

Using the above result, it is easy to see that a subgraph of an admissible graph is admissible. Hence it suffices to characterize minimal non-admissible fat graphs. Another major result of this paper is that there are infinitely many minimal non-admissible fat graphs (in contrast, for instance, to the classical result that there are only two minimal non-planar graphs).

1. Introduction

A closed hyperbolic surface $F$ of genus $g \geq 2$ can be decomposed into pairs of pants along $3g - 3$ pairwise disjoint simple closed geodesics. If the geodesics are sufficiently short then the geometry of the surface is essentially determined by the combinatorics of the pants decomposition. These combinatorics are determined by the trivalent graph associated with the pants decomposition.

Given a hyperbolic surface $F$, each homotopy class of closed curves has a unique geodesic representative $[5]$. The lengths of the closed geodesics form the so called length spectrum, and the minimum of these lengths is the systole.

We shall call the union of all closed geodesics whose length is the systole the systolic graph associated to a surface. This is in fact a so called fat graph, with all nodes of valence even and at least 4 together with a disjoint union of closed curves. Henceforth when we refer to fat graphs we always assume that this valence condition is satisfied.

The central question of this paper is the following:

Question. What fat graphs are systolic graphs of hyperbolic surfaces.
Besides its relation to the study of systolic geometry and length spectra, we are motivated to study this question as we get a natural decomposition of the moduli space of hyperbolic surfaces by associating to a surface its systolic graph.

We call a fat graph admissible if it is the systolic graph of a hyperbolic surface, so that no complementary region is a disc (i.e., the graph is essential). Thus, our central goal is to understand which fat graphs are admissible.

**Combinatorial formulation.** An essential systolic graph $\Gamma$ of a hyperbolic surface $F$ can be viewed as a metric graph, with distance obtained by measuring along paths in the graph using the metric from the surface. The minimal geodesics are loops in this graph, all of which have the same length, namely the systole. Further, suppose $\lambda$ is any other loop in $\Gamma$ which is freely homotopic to the essential closed geodesic $\lambda'$ in the surface $F$. Then length of $\lambda'$ greater than $\text{Sys}(F)$. It follows that the length of $\lambda$ is greater than that of the systole.

Note that which loops of $\Gamma$ correspond to minimal geodesics of $F$ can be determined from the fat graph $\Gamma$ (which satisfies the valence condition) – we call these the standard cycles of $\Gamma$. Thus, we can formulate a necessary condition for $\Gamma$ to be admissible in terms of metric graph structures on $\Gamma$. Namely, if $\Gamma$ is admissible then we can associate lengths to the edges of $\Gamma$ so that

1. All standard cycles have the same length, say $\sigma$.
2. All other cycles have length greater than $\sigma$.

We say that a graph is combinatorially admissible if we can associate lengths to edges satisfying the above condition. Our first main result says that this condition in fact characterizes admissibility.

**Theorem 1.1.** A fat graph $\Gamma$ is admissible if and only if it is combinatorially admissible.

The proof of this result is based on negative curvature of hyperbolic space. The crucial ingredient is that if the systolic length is very large, then the lengths of loops in a systolic graphs in the metric on the graph are very close to the lengths of the corresponding geodesics on a hyperbolic surface.

**Minimal obstructions.** Given a fat graph $\Gamma$, we can associate to it sub-graphs that are unions of some of the standard cycles of $\Gamma$. It is easy to see that if $\Gamma$ is admissible, each such subgraph is combinatorially admissible, hence is admissible. Thus, it suffices to understand which fat graphs are \textit{minimally non-admissible}, i.e., which are non-admissible but with all proper subgraphs admissible.

This is a common situation in graph theory – for instance planarity is similarly characterized by describing the minimally non-planar graphs, namely $K_{3,3}$ and $K_5$. However, in contrast to the simple answer in that case, we see that the complexity of the question we are studying in the following result.

**Theorem 1.2.** There are infinitely many minimally non-admissible fat graphs.

\section{Decorated fat graphs}

In this section we recall some definition on fat graphs. The following definition of an ordinary graph is not the standard one which is used for ordinary graphs. One can easily see that this definition is equivalent to the standard definition of a graph. This definition is a convenient starting point for describing a fat graph.
Definition 2.1. A graph is a quadruple $G = (V, H, s, i)$ where

(1) $V$ is a non-empty set, called the set of vertices or nodes.
(2) $H$ is a set (possibly empty), called the set of half edges.
(3) $s : H \to V$ is a function, thought of as sending each half edge to the node that is incident on.
(4) $i : H \to H$ is a fixed point free involution map, thought of as sending each half edge to its other half.

The set $E = H/i$ of all cycles of $i$ is the set of all full edges. An edge is a loop if its constituent half-edges are incident with the same vertex.

Definition 2.2. The girth $T(G)$ of a graph $G$ is the length of a shortest nontrivial cycle, namely

$$T(G) = \min\{l(C) | C \text{ is a cycle in } G\}.$$ 

If a graph does not contain any cycle (i.e., it is a tree), its girth is defined to be infinity.

2.1. Fat graphs. A fat graph is a finite graph (possibly disconnected) equipped with a cyclic ordering on the set of half edges incident to each node. For a fat graph it is also required that valency of each node is at least 3. To each fat graph, we associate an oriented surface with boundary by replacing edges by thin oriented rectangles and nodes by discs and pasting rectangles to discs according to the chosen cyclic ordering at the nodes.

Definition 2.3. A fat graph is a graph $(V, E, s, i)$ with a bijection $\sigma : H \to H$ whose cycles corresponds to the sets $s^{-1}(v)$ of half edges incident on nodes $v \in V$.

Definition 2.4. A decorated fat graph is a fat graph together with union of disjoint circles (possibly empty) such that the degree of each node is even and at least 4.

In this paper by a fat graph we always mean a decorated fat graph. Let us consider a simple path $p = e_1 * e_2 * \cdots * e_n$ in a decorated fat graph and a node $v$ which is the endpoint of $e_i$ and $e_{i+1}$. Suppose the edges incident at $v$ has the following order

$$e_i = e_{v,1} < e_{v,2} < \cdots < e_{v,2n}.$$ 

Then the path $p$ is called transversal at $v$ if $e_{i+1} = e_{v,n+1}$. Now, we define a standard cycle and non-standard cycle in a decorated fat graph.

Definition 2.5. A simple cycle $C = e_1 * e_2 * \cdots * e_n$ is called a standard cycle if the cycle is transversal at each of its node. If a cycle is not standard, we say the cycle is non-standard.

2.2. Deletion of a cycle. Let $G$ be a fat graph and $C = e_1 * e_2 * \cdots * e_n$ be a cycle in $G$. Consider the graph obtained by deleting the (interior of) the edges $e_i$. This is a fat graph except that it may have nodes of valency two.

If $v$ is a node of valency two and $f_1$, $f_2$ are the edges incident at $v$, we remove the node $v$ and the edges $f_1$, $f_2$ by introduce a simple edge $e$ between $u_1$ and $u_2$, where $u_1$, $u_2$ are the other ends of $f_1$, $f_2$ respectively. We repeat the above procedure till there is no node of valency two.

The resulting graph is a decorated fat graph and is denoted by $G - C$. 


3. Minimal non-admissible fat graph

The purpose of this section is to study non-admissible fat graphs. Let $G$ be a decorated fat graph and let $l$ be a positive real number. If there exist a metric on $G$ such that the length of each standard cycle in $G$ is $l$ and the length of each non-standard cycle is strictly greater than $l$ then the graph is called admissible. A decorated fat graph is called non-admissible if the graph is not admissible. Recall that, a non-admissible fat graph $G$ is called minimal non-admissible if any subgraph $G'$ obtained from $G$ by deleting a standard cycle is an admissible fat graph.

In graph theory, a graph is called planar if the graph can be embedded in the plane. A graph is non-planar if it is not planar. A graph is called minimal non-planar if any proper subgraph of the graph is planar. Kuratowski proved that there are only two minimal non-planar graphs. These are the complete bipartite graph $K_{3,3}$ and the complete graph with five vertices $K_5$. In contrast of this result we have the following question in the context of minimal non-admissible fat graphs.

**Question.** What are the minimal non-admissible fat graphs?

The main theorem of this section deals with this question.

**Theorem 3.1.** There are infinitely many minimal non-admissible fat graphs.

Suppose $G$ is a given fat graph such that the intersection graph $\Gamma'(G)$ is a planar graph. Consider a standard cycle $C = e_1 * e_2 * \cdots * e_n$ and let $v_1, v_2, \ldots, v_n$ be the nodes of $C$ labelled in a fixed orientation. The standard cycle in $G$ meeting with $C$ at the node $v_i$ is denoted by $C_i$. If the orientation induced from the orientation of plane gives the cyclic ordering $C_1 < C_2 < \cdots < C_n$ (clock-wise or anti-clock wise) in the set of nodes adjacent to the node $C$ in the intersection graph then we say that the node $C$ in the intersection graph respects the orientation of the fat graph. If each of the nodes of the planar graph respect an orientation of the fat graph then we say that the intersection graph respects the orientation of the fat graph.

3.1. A motivating Example. Here we give an explicit example of minimal non-admissible fat graph.

**Example 3.2.** Let us consider the fat graph $G = (V, H, i, s)$ as in the following (see Figure 1).

- $V = \{v_i|i = 1, 2, \ldots, 6\}$, the set of vertices.
- $H = \{e'_i, e''_i|i = 1, 2, \ldots, 12\}$, the set of half edges.
- The function $s : H \rightarrow V$ is given by
  
  \[s(e'_2) = s(e'_4) = s(e'_{12}) = s(e'_{11}) = v_1,\]
  \[s(e'_1) = s(e''_7) = s(e'_7) = s(e''_6) = v_2,\]
  \[s(e'_2) = s(e'_4) = s(e'_5) = s(e''_7) = v_3,\]
  \[s(e''_6) = s(e'_6) = s(e'_8) = s(e''_5) = v_4,\]
  \[s(e''_9) = s(e'_9) = s(e'_{10}) = s(e''_{11}) = v_5,\]
  \[s(e''_{12}) = s(e''_8) = s(e''_{10}) = v_6.\]

- The function $i : H \rightarrow H$ is given by
  \[i(e'_i) = e''_i, \forall i.\]

The set of all full edges is given by

\[E = H/i = \{e_i|i = 1, 2, \ldots, 12\}, e_i = \{e'_i, e''_i\}.\]
The permutation \( \sigma : H \to H \) is given by the following ordering or cycles,

\[
(\epsilon'_2, \epsilon'_3, \epsilon'_11), \\
(\epsilon'_1, \epsilon'_7, \epsilon'_9, \epsilon'_3), \\
(\epsilon''_1, \epsilon''_2, \epsilon'_4, \epsilon'_5), \\
(\epsilon''_5, \epsilon''_6, \epsilon''_8, \epsilon''_7), \\
(\epsilon''_4, \epsilon''_{11}, \epsilon''_{10}, \epsilon''_6), \\
(\epsilon''_8, \epsilon''_{10}, \epsilon''_{12}, \epsilon''_9).
\]

There are four standard cycles in \( G \) which are given by

\[
c_1 = \epsilon_1 * \epsilon_2 * \epsilon_3, \\
c_2 = \epsilon_4 * \epsilon_5 * \epsilon_6, \\
c_3 = \epsilon_7 * \epsilon_8 * \epsilon_9 \text{ and} \\
c_4 = \epsilon_{10} * \epsilon_{11} * \epsilon_{12}.
\]

First, we show that \( G \) is non-admissible. If possible assume that there exist a metric on \( G \) which makes it admissible and without loss of generality assume that the length of each standard cycle is 1. Now, each vertex \( c_i \) in the intersection graph corresponds to an equation – the length of the standard cycle. For example, the standard cycle \( c_1 \) gives

\[
e_1 + e_2 + e_3 = 1.
\]

Each face of the intersection graph is corresponds to a non-standard cycle and hence to an in-equation. For example, the outer face of the intersection graph corresponds to the non-standard cycle \( c = \epsilon_6 * \epsilon_8 * \epsilon_{10} \) and the corresponding inequation is

\[
e_6 + e_8 + e_{10} > 1.
\]

Adding the inequations corresponding to all four faces we get
\[ \sum_{i=1}^{12} e_i > 4. \]

On the other hand adding all four equations we get

\[ \sum_{i=1}^{12} e_i = 4. \]

Thus we arrive at a contradiction. Hence the graph \( G \) is not admissible.

To prove \( G \) is minimal non-admissible it remains to show that \( G \setminus c \) is admissible, where \( c \) is any standard cycle in \( G \). It is easy to see that \( G \setminus c_i \) and \( G \setminus c_j \) are isomorphic for any \( i, j \in \{1, 2, 3, 4\} \). Each standard cycle in \( G - c \) consists of exactly two edges and each non-standard cycle consists of at least three edges. We can define a metric by specifying that length of each edge \( l(e) := \frac{1}{2} \) for each edge \( e \) in the graph \( G \setminus c \). This metric shows that the graph \( G \setminus c \) is admissible.

3.2. Proof of the Theorem 3.1

We consider a fat graph \( G \) so that the intersection graph is a prism over a polygon which is a planar graph and respects an orientation. In a planar representation, such a prism has same number of faces as the number of nodes. Similar arguments to the above example shows that the fat graph is non-admissible.

Proof of the Theorem 3.1: To prove the Theorem 3.1 for each integer \( n \geq 3 \) we construct a fat graph \( \Gamma_n \) with \( n + 1 \) standard cycles and show that the fat graph \( \Gamma_n \) is a minimal non-admissible graph. The proof of the theorem is given in three steps. In the first step we construct the fat graph \( \Gamma_n \), in next step show that the graph in non-admissible and in the final step we prove the minimality.

Step 1. Construction of \( \Gamma_n \): The standard cycles \( C_i \), \( i = 0, 1, 2, \ldots, n \), are given by following (see Figure 2):

\[ C_0 = (v_{0,1}, v_{0,2}, v_{0,3}, \ldots, v_{0,n}) \]

and for all \( i > 0 \),

\[ C_i = (v_{i,0}, v_{i,1}, v_{i,2}). \]

![Figure 2. The cycles of \( \Gamma_n \)](image)

We identify the nodes on the cycle by following

\[ v_{0,i} = v_{i,0} \text{ and } v_{1,1} = v_{2,1}, v_{2,2} = v_{3,2}, \ldots, v_{n,2} = v_{1,2}. \]
Step 2. Non-admissibility: In this step we prove that $\Gamma_n$ is non-admissible. The intersection graph $\Gamma'(\Gamma_n) = (V,E)$ is given by following. The set $V$ of nodes is

$$V = \{v_i| i = 0,1,2,\ldots,n\}$$

where the node $v_i$ corresponds to the standard cycle $C_i$. The set $E$ of edges of $\Gamma'(\Gamma_n)$ is given by

$$E = E_1 \cup E_2.$$ 

The sets $E_i$'s are defined by the following:

$$E_1 := \{e_i| i = 1,2,\ldots,n\}$$

where $e_i$ is the simple edge between $v_i$ and $v_{i+1}$ for $1 \leq i \leq n-1$ and $e_n$ is the edge between $v_n$ and $v_1$ and

$$E_2 := \{f_i| i = 1,2,\ldots,n\}$$

where $f_i$ is the edge between $v_0$ and $v_i$. The intersection graph $\Gamma'(\Gamma_n)$ is a planar graph in which the number of nodes is the same as the number of faces. Hence the fat graph is non-admissible. For $n=6$, the Figure 4 describes the intersection graph $\Gamma'(\Gamma_6)$ of $\Gamma_6$.

Step 3. Minimality of $\Gamma_n$: Now, we show that, if we delete any cycle $C_i$ from $\Gamma_n$ then the resulting graph becomes admissible. Let us denote the fat graph obtained by removing the cycle $C_i$ from $\Gamma_n$ by $\Gamma_i$. Note that $\Gamma_i$ and $\Gamma_j$ are isomorphic for all $i,j \geq 1$. Therefore, it is enough to show that $\Gamma_0$ and $\Gamma_1$ are admissible.

Case 1. In $\Gamma_0$ every standard cycle consists of two edges and on the other hand every non-standard cycle consists of at least three edges. Hence we define the metric $l : E \rightarrow \mathbb{R}_+$ by

$$l(e) = \frac{1}{2}$$

for each edge $e$ in $E$. Therefore, by definition, the length of each standard cycle is 1 and the length of each non-standard cycle is at least $\frac{3}{2}$. 

Figure 3. The schematics for building the graph $\Gamma_n$
Case 2. Let us describe $\Gamma_n^1$. The cycles are given by,

$$
\begin{align*}
C_0 &= (v_{0,2}, v_{0,3}, \ldots, v_{0,n-1}), \\
C_2 &= (v_{2,0}, v_{2,1}), \\
C_i &= (v_{i,0}, v_{i,1}, v_{i,2}); \text{ where } 2 < i < n \text{ and} \\
C_n &= (v_{n,0}, v_{n,1}).
\end{align*}
$$

The nodes are identified by following relations, $v_{0,i} = v_{i,0}$, $v_{2,1} = v_{3,1}$, $v_{3,2} = v_{4,2}$, and $v_{4,2} = v_{5,2}, \ldots, v_{n-1,1} = v_{n,1}.$

![Diagram of the fat graph $\Gamma_n^1$](image)

**Figure 5.** The fat graph $\Gamma_n^1$.

We define $d : E(\Gamma_n^1) \rightarrow \mathbb{R}_+$ by following,
\[ d(v_{0,i}, v_{0,i+1}) := \frac{1}{n-1}, \quad 2 \leq i \leq n-1, \]
\[ d(v_{0,n}, v_{0,2}) := \frac{1}{n-1}, \]
\[ d(v_{j,0}, v_{j,2}) = \frac{1}{2} + \epsilon - \frac{1}{2(n-1)}; \forall j = 3, \ldots, n-1, \]
\[ d(v_{2,0}, v_{2,1}) = d(v_{n,0}, v_{n,1}) := \frac{1}{2}, \]
\[ d(v_{i,1}, v_{i,2}) := \frac{1}{n} - \epsilon. \]

We choose \( \epsilon \in \mathbb{R}_+ \) so that \( d \) is a positive function and \( d \) is an admissible metric, namely one can choose positive \( \epsilon \) strictly less than \( \frac{1}{n-1} \). Note that the length of outer non-standard cycle is \( \text{length}(\text{outer cycle}) = 2 + \frac{1}{2} - (n-1)\epsilon \) which is strictly greater than one. If \( C \) is any non-standard cycle other than the outer cycle then it must consists of at least three edges and at least two of them have length at least \( \frac{1}{2} + \frac{\epsilon}{2} - \frac{1}{2(n-1)} \) and at least one edge of length \( \frac{1}{n-1} \). Thus we have,
\[
\text{length}(C) \geq 2\left(\frac{1}{2} + \frac{\epsilon}{2} - \frac{1}{2(n-1)}\right) + \frac{1}{n-1} = 1 + \epsilon > 1.
\]

\[
\square
\]

3.3. Remark. In this situation we have following natural question.

**Question.** Does \( \{ \Gamma \mid n(\geq 3) \in \mathbb{N} \} \) exhaust the set of all minimal non-admissible fat graphs?

**Answer.** No. Consider the following example.

Let us consider a fat graph \( G \) presented by the following:
\[
\begin{align*}
 v_1 : [v_2, v_4, v_3, v_8], \\
v_2 : [v_1, v_4, v_3, v_6], \\
v_3 : [v_1, v_7, v_2, v_6], \\
v_4 : [v_1, v_7, v_2, v_8], \\
v_5 : [v_4, v_8, v_6, v_7], \\
v_6 : [v_2, v_3, v_5, v_7], \\
v_7 : [v_3, v_5, v_6, v_8], \\
v_8 : [v_1, v_7, v_4, v_5],
\end{align*}
\]

where, \( v_i, i = 1, 2, \ldots, 8 \) are the nodes of \( G \) and \( v_i : [u_1, \ldots, u_{m_i}] \) means \( u'_i, i = 1, 2, \ldots, m_i \) are nodes adjacent to \( v_i \) and the edges has the following order:
\[
(v_i, u_1) < (v_i, u_2) < \cdots < (v_i, u_{m_i}).
\]

Here \((u, v)\) denotes a simple edge between the nodes \( u \) and \( v \). It is easy to see that the intersection graph corresponding to the fat graph is planar and respects an orientation of the fat graph. In a planar representation of the intersection graph the number of nodes is 5 which is equal to the number of faces. So by the similar argument as in the Example 3.2 the graph \( G \) is non-admissible. We claim that \( G \) is minimal non-admissible.
We show that if we delete any standard cycle $c$ from $G$, then the resulting graph $G \setminus c$ is admissible.

First see that $G \setminus c_2$ where, $c_2 = [v_2, v_4] \ast [v_4, v_5] \ast [v_5, v_6] \ast [v_6, v_2]$ is admissible. The intersection graph of $G \setminus c_2$ is a rectangle. Each standard cycle consists of exactly two edges. On the other hand each non-standard cycle consists of at least four edges. So we assign length of each edge being $\frac{1}{2}$ which makes $G \setminus c_2$ admissible.

Now we define a metric on $G \setminus c_3$ where, $c_3 = [v_3, v_6] \ast [v_6, v_7] \ast [v_7, v_3]$. To find a metric on the fat graph we need to solve the following system:

\begin{equation}
\sum_{e \in c} l(e) = 1;
\end{equation}

where $c$ is a standard cycle and inequations each for a non-standard cycle $d$,

\begin{equation}
\sum_{e \in d} l(e) > 1
\end{equation}

and non-negativity, for each edge $e$ in the graph,

\begin{equation}
l(e) > 0;
\end{equation}

We use the Z3 SMT solver to solve the above system. After renaming the nodes, the graph $G \setminus c_3$ is given by,

\begin{align*}
v_1 & : [v_2, v_4, v_2, v_5] \\
v_2 & : [v_1, v_3, v_1, v_4] \\
v_3 & : [v_2, v_5, v_4, v_5] \\
v_4 & : [v_1, v_3, v_5, v_2] \\
v_5 & : [v_1, v_3, v_4, v_3]
\end{align*}

We define, $l([v_4, v_1]) = x_0, l([v_4, v_3]) = x_1, l([v_4, v_5]) = x_2, l([v_4, v_2]) = x_3, l([v_5, v_1]) = x_4, l([v_5, v_3]) = x_5, l([v_1, v_2]) = x_6, l([v_2, v_3]) = x_7$. The system of equation and inequation is given by following:

Equations:

\begin{align*}
x_5 + x_5 & = 1, \\
x_6 + x_6 & = 1, \\
x_2 + x_4 + x_0 & = 1, \\
x_3 + x_7 + x_1 & = 1.
\end{align*}

Inequations:
The author proved that, for all \( n \) of a shortest cycle of a graph \( G \) such that \( \|G\| \in \mathbb{N} \) and \( g \geq g_0 \) there exist a trivalent graph \( G \) with \( |G| = 2g - 2 \) nodes and girth \( T(G) \geq n_0 \), where for \( t \in \mathbb{N} \) the natural number \( g_t \) is given by,

\[
g_t = \begin{cases} 
2 & \text{if } t = 1, 2, \\
t + 1 & \text{if } t = 3, 4, 5, \\
2^t & \text{if } t \geq 6.
\end{cases}
\]

Non-negativity of variables:

\[ x_i > 0, \ i = 1, 2, \ldots, 7. \]

Each solution of the above system will give a metric on the fat graph. Using the Z3 SMT solver we have that the above system is satisfiable that is the graph is admissible and a solution is given by, \( x_1 = \frac{1}{7}, \ x_5 = \frac{1}{7}, \ x_6 = \frac{1}{7}, \ x_4 = \frac{3}{7}, \ x_0 = \frac{1}{2}, \ x_7 = \frac{1}{8}, \ x_2 = \frac{1}{8} \).

By symmetry, we have \( G \setminus c_1 = G \setminus c_3 \) where \( c_1 = [v_1, v_2] * [v_2, v_3] * [v_3, v_1] \). Also \( G \setminus c_4 = G \setminus c_5 \), where \( c_4 = [v_1, v_8] * [v_8, v_4] * [v_4, v_1], c_5 = [v_5, v_8] * [v_8, v_7] * [v_7, v_5] \).

It suffices to show that \( G \setminus c_4 \) is admissible. After a renaming the nodes the graph \( G \setminus c_4 \) is given by,

\[
\begin{align*}
\{v_1 : [v_2, v_5, v_4, v_3] \\
\{v_2 : [v_1, v_3, v_4, v_3] \\
\{v_3 : [v_2, v_5, v_2, v_4] \\
\{v_4 : [v_1, v_5, v_3, v_3] \\
\{v_5 : [v_1, v_4, v_1, v_3]
\end{align*}
\]

As above we define, \( l([v_4, v_1]) = x_0, \ l([v_4, v_5]) = x_1, \ l([v_4, v_2]) = x_2, \ l([v_4, v_3]) = x_3, \ l([v_5, v_1]) = x_4, \ l([v_3, v_3]) = x_5, \ l([v_1, v_2]) = x_6, \ l([v_2, v_3]) = x_7. \)

Using the Z3 SMT solver we see that the fat graph is admissible and a solution is given by, \( x_1 = 3/8, \ x_5 = 1/8, \ x_0 = 3/4, \ x_4 = 1/2, \ x_6 = 1/8, \ x_7 = 1/2, \ x_3 = 1/2, \ x_2 = 1/8. \)

4. Uni-trivalent graphs of large girth

A graph is called trivalent if the degree of each node of the graph is 3. The length of a shortest cycle of a graph \( G \) is called the girth of the graph. In (P. Buser [3]) the author proved that, for all \( n_0 \in \mathbb{N} \) and \( g \geq g_0 \) there exist a trivalent graph \( G \) with \( |G| = 2g - 2 \) nodes and girth \( T(G) \geq n_0 \), where for \( t \in \mathbb{N} \) the natural number \( g_t \) is given by,

\[
g_t = \begin{cases} 
2 & \text{if } t = 1, 2, \\
t + 1 & \text{if } t = 3, 4, 5, \\
2^t & \text{if } t \geq 6.
\end{cases}
\]
Recall a graph \( G = (V, E) \) is called a uni-trivalent graph if there is a node \( v_0 \in V \) of degree one and all other nodes has degree three. In this section we prove the following lemma.

**Lemma 4.1.** For given any \( n_0 \in \mathbb{N} \), there exist a uni-trivalent graph \( G \) with 
\[ 2(n_0^2 - 3n_0 + 1) \] nodes of girth 
\[ T(G) = n_0. \]

Before going to the proof of the Lemma 4.1, let us consider the following motivating example of an uni-trivalent graph of girth six which needs only 38 nodes.

**Example 4.2.** First, we construct a trivalent graph \( G'(3, 6) \) of girth 6. The graph \( G'(3, 6) = (V, E) \) (Figure 6) is given by following

- \( V = \{v_i, u_i|i = 1, 2, \ldots, 18\} \) is the set of nodes.
- \( E = E_0 \cup E_1 \cup E_2 \cup E_3 \) is the set of edges, where \( E_i;i \in \{0, 1, 2, 3\} \) are given by following:
  
  \[
  E_0 = \{(v_{18}, v_1), (v_i, v_{i+1})|i = 1, 2, \ldots, 17\} \cup \{(v_j, u_j)|j = 1, 2, \ldots, 18\},
  \]
  
  \[
  E_1 = \{(u_1, u_4), (u_4, u_7), (u_7, u_{10}), (u_{10}, u_{13}), (u_{13}, u_{16}), (u_{16}, u_1)\},
  \]
  
  \[
  E_2 = \{(u_2, u_5), (u_5, u_8), (u_8, u_{11}), (u_{11}, u_{14}), (u_{14}, u_{17}), (u_{17}, u_2)\},
  \]
  
  \[
  E_3 = \{(u_3, u_6), (u_6, u_9), (u_9, u_{12}), (u_{12}, u_{15}), (u_{15}, u_{18}), (u_{18}, u_3)\}.
  \]

Now we construct \( G = (V, E) \) by following:

- \( V = V' \cup \{u, v\} \).
- Choose an edge \( (x, y) \) in \( G' \). To construct the set of edges of \( G \), we delete \( (x, y) \) from \( G' \) and introduce three new edges \( (x, u), (y, u) \) and \( (u, v) \), i.e
  
  \[ E = (E' - \{(x, y)\}) \cup \{(x, u), (y, u), (u, v)\}. \]

In the Figure 6 we choose the edge \( (x, y) = (v_1, v_{18}) \).

**Figure 6.** Uni-trivalent graph of girth six.
In a similar way, we can also construct a trivalent graph of girth 7 with 56 nodes. In general, in the following theorem we show that for given \( n_0 \), we can construct a trivalent graph of girth \( n_0 \) with \( 2(n_0^2 - 3n_0) \) nodes.

**Theorem 4.3.** Let \( n_0 \in \mathbb{N} \). Then there is a trivalent graph \( G \) with \( |G| = 2(n_0^2 - 3n_0) \) nodes and girth

\[ T(G) = n_0. \]

**Proof.** Initially, we start with a graph \( G_0 = (V, E_0) \), where \( V \) and \( E_0 \) are given by following:

- \( V = \{v_i, u_i|i = 1, 2, \ldots, M\} \), where \( M = 2(n_0^2 - 3n_0) \).
- \( E_0 = \{(v_i, v_{i+1})|i = 1, 2, \ldots, M-1\} \cup \{(v_M, v_1)\} \cup \{(v_j, u_j)|j = 1, 2, \ldots, M\}. \)

Then the girth \( T(G_0) \) of \( G_0 \) is \( M \) which is greater than \( n_0 \). Now, we introduce edges to the initial graph \( G_0 \) to get a trivalent graph of girth \( n_0 \).

**Step 1.** The distance \( d(u_1, u_1+(n_0-3)) \) between the nodes \( u_1, u_1+(n_0-3) \) in \( G_0 \) is \( n_0 - 1 \), where the length of each edge is 1 and the distance between two nodes is the length of the shortest path joining them. So, the girth of the graph

\[ (V, E_0 \cup \{(u_1, u_1+(n_0-3))\}) \]

is \( n_0 \). Similarly, the distance \( d(u_1+(n_0-3), u_1+(2n_0-2-3)) \) in \( V, E_0 \cup \{(u_1, u_1+(n_0-3))\} \)

is \( n_0 - 1 \). Therefore, we have

\[ (V, E_0 \cup \{(u_1, u_1+(n_0-3)), (u_1+(n_0-3), u_1+(2n_0-2-3))\}) \]

a graph of girth \( n_0 \). Proceeding in this way at the end of step 1, we get a graph \( G_1 = (V, E_0 \cup E_1) \) where

\[ E_1 = \{(u_1, u_1+(n_0-3)), (u_1+(n_0-3), u_1+2(n_0-3)), \ldots (u_1+(n_0-1)(n_0-3), u_1)\} \]

which has girth \( n_0 \).

**Step 2.** As in step 1, we add the edges

\[ E_2 = \{(u_2, u_2+(n_0-3)), (u_2+(n_0-3), u_2+2(n_0-3)), \ldots (u_2+(n_0-1)(n_0-3), u_2)\} \]

to the graph \( G_1 \) to obtain the graph \( G_2 = (V, E_0 \cup E_1 \cup E_2) \) of girth \( n_0 \).

Proceeding in this way at the end of step \((n_0 - 3)\) we obtain a trivalent graph

\[ G = G_{n_0-3} = (V, E = \bigcup_{i=0}^{n_0-3} E_i) \]

of girth \( n_0 \) where

\[ E_i = \{(u_i, u_i+(n_0-3)), (u_i+(n_0-3), u_i+2(n_0-3)), \ldots (u_i+(n_0-1)(n_0-3), u_i)\} \]

for \( i = 1, 2, \ldots, n_0 - 3 \) with \( T(G) = n_0 \). Hence the proof.

**Proof of the Lemma 4.2** Here we proceed as in example 4.2 i.e., first we construct a trivalent graph of girth \( n_0 \) with \( 2(n_0^2 - 3n_0) \) nodes. Next, we delete one edge and then introduce two more nodes and three edges to obtain a uni-trivalent graph.

Consider the trivalent graph \( G' = (V', E') \) of girth \( n_0 \) which is given in the Theorem 4.3. Consider an edge \( e = (x, y) \) in \( G' \). Then the uni-trivalent graph \( G = (V, E) \) of girth \( n_0 \) is given by

\[ V = V' \cup \{u, v\}, \]

\[ E = (E' - \{e\}) \cup \{(u, x), (u, y), (u, v)\}. \]
Therefore, we get a uni-trivalent graph $G$ with $2(n_0^2 - 3n_0 + 1)$ nodes with girth $n_0$.

\[ \square \]

5. Hyperbolic pair of pants

In this section, we prove three lemmas on hyperbolic pair of pants which will be needed for subsequent sections. A pair of pants is a compact surface with genus zero and three boundary components. The Euler characteristic $\chi(P)$ of a pair of pants $P$ is $-1$. Hence it admits a hyperbolic structure with geodesic boundary. A pair of pants together with a hyperbolic structure is called a hyperbolic pair of pants. Topologically, a pair of pants is a sphere with three holes in it. Unless otherwise noted, we assume that pair of pants are hyperbolic pair of pants.

It is a fact of hyperbolic geometry that a Poincaré metric on a pair of pants is determined by the length of its boundary curves. For an arbitrary prescribed triple $(a, b, c)$ of positive real numbers there is up to isometry a unique right angled convex hyperbolic hexagon with three pairwise non-consecutive sides of lengths $a$, $b$ and $c$. In particular, gluing two such hexagons along the remaining sides yields a hyperbolic pair of pants with three boundary geodesics of length $2a$, $2b$ and $2c$ respectively. On the other way, suppose we are given a hyperbolic pair of pants $P$ with boundary geodesics $\gamma_i$, $i = 1, 2, 3$ of lengths $l_i$, $i = 1, 2, 3$ respectively. Then for given two distinct boundary geodesics $\gamma_i$ and $\gamma_j$ there is a unique embedded geodesic arc $\delta_{i,j}$ connecting $\gamma_i$ and $\gamma_j$ meeting both boundary components perpendicularly at its end points. Suppose $\delta_{i,j}$ denotes the distance realizing geodesic between $\gamma_i$ and $\gamma_j$ (we refer to such an arc as a seam). Cutting $P$ open along the three seams, we get two isometric right angled convex hyperbolic hexagon with non-consecutive sides of lengths $\frac{l_i}{2}$; $i = 1, 2, 3$.

**Proposition 5.1.** [5] For any triple $(a, b, c)$ of positive real numbers, there is up to isometry a unique hyperbolic pair of pants with boundary geodesics of lengths $a, b, c$.

**Definition 5.2.** Let $P$ be a hyperbolic pair of pants with boundary components $\gamma_1, \gamma_2, \gamma_3$. The length of the simple geodesic arc $\delta_1$ [Figure 7] with both end points at $\gamma_i$ meeting perpendicularly is called the height of the hyperbolic pair of pants $P$ with respect to the waist $\gamma_i$.

For three positive real numbers $l_1, l_2, l_3 \in \mathbb{R}_+$, the hyperbolic pair of pants with boundary geodesics $\gamma_1, \gamma_2, \gamma_3$ of lengths $l_1, l_2, l_3$ respectively, is denoted by $P(l_1, l_2, l_3)$.

![Figure 7. Hyperbolic pair of pants.](image-url)
**Question:** Given any positive real number $l$, does there exist a pair of pants $P(l, kl, kl)$ for some $k \geq 1$ such that

$$\text{length}(\delta_1) \geq l$$

where $\delta_1$ is the height of $P(l, kl, kl)$ with the waist $\gamma_1$.

The following lemma answers the above question.

**Lemma 5.3.** Given any positive number $l$, there exist a pair of pants $P = P(l, kl, kl)$ for some $k$ such that if $\delta_1$ is the height of $P$ with waist $\gamma_1$ then

$$\text{length}(\delta_1) \geq l.$$

**Proof.** For every positive $k$, a hyperbolic structure on $P$ is uniquely determined by the lengths $\text{length}(\gamma_1) = l$, $\text{length}(\gamma_2) = kl$, $\text{length}(\gamma_3) = kl$ of the boundary geodesics. For a fixed positive number $k$, consider the hyperbolic structure on $P$. Suppose $\delta_{2,3}, \delta_{1,3}$ and $\delta_{1,2}$ are the distance realizing geodesics between $\gamma_2, \gamma_3; \gamma_1, \gamma_3$ and $\gamma_1, \gamma_2$ respectively. Now, we cut $P$ along $\delta_{2,3}, \delta_{1,3}$ and $\delta_{1,2}$ and get two isometric hexagons as in the Figure 8.

![Figure 8. Hyperbolic right angled hexagon.](image)

Let $GG'$ be the distance realizing geodesic between the sides $FA$ and $CD$ which divide the hexagon into two isometric right angled pentagons. From the right angled hyperbolic pentagon $(GG'CBA)$, using hyperbolic trigonometry, we get

$$\cosh \frac{m}{2} = \sinh \frac{kl}{2} \sinh l'.$$

Now, from the right angled hyperbolic hexagon we get

$$\cosh l' = \frac{\cosh \frac{l}{2} \cosh \frac{kl}{2} + \cosh \frac{kl}{2}}{\sinh \frac{l}{2} \sinh \frac{kl}{2}}.$$
Now using hyperbolic trigonometric identities and the equations (7) and (8), we get
\[
\cosh m = 2 \cosh^2 \frac{m}{2} - 1 \\
= 2 \sinh^2 \frac{kl}{2} \sinh^2 l' - 1 \\
= 2 \sinh^2 \frac{kl}{2} (\cosh^2 l' - 1) - 1 \\
= 2 \sinh^2 \frac{kl}{2} \left( \frac{\cosh \frac{l}{2} \cosh \frac{kl}{2} + \cosh \frac{kl}{2}}{\sinh \frac{1}{2} \sinh \frac{kl}{2}} \right)^2 - 1 - 1 \\
= 2 \sinh^2 \frac{kl}{2} \left( \frac{\cosh^2 \frac{l}{2} \cosh^2 \frac{kl}{2} + \cosh^2 \frac{kl}{2} + 2 \cosh \frac{l}{2} \cosh^2 \frac{kl}{2}}{\sinh^2 \frac{1}{2} \sinh^2 \frac{kl}{2}} \right) - 1 - 1 \\
= 2 \sinh^2 \frac{kl}{2} \left( \frac{\cosh^2 \frac{l}{2} \cosh^2 \frac{kl}{2} + \cosh^2 \frac{kl}{2} + 2 \cosh \frac{l}{2} \cosh^2 \frac{kl}{2}}{\sinh^2 \frac{1}{2} \sinh^2 \frac{kl}{2}} \right) - 1 - 1 \\
= 2 \sinh^2 \frac{l}{2} \frac{\cosh^2 \frac{kl}{2} + \cosh^2 \frac{kl}{2} + 2 \cosh \frac{l}{2} \cosh^2 \frac{kl}{2}}{\sinh^2 \frac{1}{2} \sinh^2 \frac{kl}{2}} - 1 - 1 \\
= 2 \sinh^2 \frac{l}{2} \frac{1}{\sinh^2 \frac{kl}{2}} + 2 \cosh \frac{l}{2} \cosh^2 \frac{kl}{2} - 1 - 1 \\
= 4 \cosh^2 \frac{kl}{2} + 4 \cosh \frac{l}{2} \cosh^2 \frac{kl}{2} + 2 \cosh^2 \frac{l}{2} - 1 - 1 \\
= 4 \cosh^2 \frac{kl}{2} + 4 \cosh \frac{l}{2} \cosh^2 \frac{kl}{2} + 1.
\]

Now,
\[
\frac{4 \cosh^2 \frac{kl}{2}}{\sinh^2 \frac{1}{2}} = \frac{4(1 + \sinh^2 \frac{kl}{2})}{\sinh^2 \frac{l}{2}} \\
= \frac{4}{\sinh^2 \frac{1}{2}} + \frac{4 \sinh^2 \frac{kl}{2}}{\sinh^2 \frac{l}{2}} \\
= \frac{4}{\sinh^2 \frac{1}{2}} + \frac{4 \sinh^2 l}{\sinh^2 \frac{1}{2}} \quad (\text{for } k = 2) \\
= 16 \cosh^2 \frac{l}{2} \\
= 8(1 + \cosh l).
\]

Hence we have
\[
\cosh m = \frac{4}{\sinh^2 \frac{1}{2}} + 8(1 + \cosh l) + \frac{4 \cosh \frac{l}{2} \cosh^2 l}{\sinh^2 \frac{1}{2}} + 1 \\
\Rightarrow \cosh m > \cosh l \\
\Rightarrow m > l \quad \text{for } k = 2.
\]

Therefore, for any \( l > 0 \), the height \( \text{length}(\delta_1) \) of the pair of pants \( P(l, 2l, 2l) \) with waist \( \gamma_1 \) satisfies
\[
\text{length}(\delta_1) \geq l.
\]
\[\square\]
Lemma 5.4. If $P(l, l, l)$ is a hyperbolic pair of pants with $\gamma_i; i = 1, 2, 3$ boundary components for some positive real $l$ and $\delta$ is the simple geodesic with both end points on the same boundary component $\gamma_1$ meeting perpendicularly. Then

$$\text{length}(\delta) > \frac{l}{2}$$

Proof. To prove the lemma, we use the same idea as in the proof of the Lemma 5.3. Cut the pair of pants $Y(l) = P(l, l, l)$ along the distance realizing geodesics of each pair of boundary components of $Y(l)$ and get two isometric hyperbolic hexagons as in the following picture. We need to show that

$$m \geq \frac{l}{2}.$$  

The right angled pentagon $DEFGH$ gives

(9) \[ \cosh \frac{m}{2} = \sinh \frac{l}{2} \sinh l'. \]

From the hyperbolic hexagon, we have

(10) \[ \cosh l' = \frac{\cosh^2 \frac{l}{2} + \cosh \frac{l}{2}}{\sinh^2 \frac{l}{2}}. \]

Now, using the equations (9), (10) and hyperbolic trigonometric identities we have

$$\cosh m = 2 \cosh^2 \frac{m}{2} - 1$$

$$= 2 \sinh^2 \frac{l}{2} \sinh^2 l' - 1$$

$$= 2 \sinh^2 \frac{l}{2} (\cosh^2 l' - 1) - 1$$

$$= 2 \sinh^2 \frac{l}{2} \left\{ (\frac{\cosh^2 \frac{l}{2} + \cosh \frac{l}{2}}{\sinh^2 \frac{l}{2}})^2 - 1 \right\} - 1$$

$$= \frac{2}{\sinh^2 \frac{l}{2}} \left\{ (\cosh^2 \frac{l}{2} + \cosh \frac{l}{2})^2 - (\cosh^2 \frac{l}{2} - 1)^2 \right\} - 1$$

$$= (4 \coth^2 \frac{l}{2}) \cosh \frac{l}{2} + \frac{4}{\sinh^2 \frac{l}{2}} + 5.$$

$$\Rightarrow \cosh m > \cosh \frac{l}{2}$$

$$\Rightarrow m > \frac{l}{2}.$$  

Consider a pair of pants $Y(l)$ whose boundary geodesics are $\gamma_i, i = 1, 2, 3$ and each of length $l$. The following lemma gives the distance between two boundary components of $Y(l)$.

Lemma 5.5. The distance between any two distinct boundary components of $Y(l)$ is given by

(11) \[ \text{dist}(\gamma_i, \gamma_j) = \text{arc sinh} \left( \frac{1}{2 \sinh \frac{l}{4}} \right) \text{ for } i \neq j. \]
Proof. Let $\delta_{2,3}$ (respectively $\delta_{1,3}$ and $\delta_{1,2}$) be the distance realizing geodesic of the boundary components $\gamma_2, \gamma_3$ (respectively $\gamma_1, \gamma_3$ and $\gamma_1, \gamma_2$). Let $D$ be the one of the right angled hyperbolic convex hexagon obtained by cutting $Y(l)$ along $\delta_{i,j}; i, j = 1, 2, 3; i \neq j$.

![Hyperbolic right angled hexagon](image)

**Figure 9. Hyperbolic right angled hexagon**

The distance realizing geodesics of the opposite edges of the hexagon $D$ divide it into six sharp corners $pqp'q'$ (recall, the geodesic rectangle with three right angles and one acute angle $\phi$ is called a sharp corner with an angle $\phi$) where

$$\phi = \frac{\pi}{3}, \ q' = \frac{l}{4}$$

and

$$2q = d(\gamma_i, \gamma_j), i \neq j.$$

Using hyperbolic trigonometry, we get

$$\cos \phi = \sinh q \cdot \sinh q'$$

which gives

$$\cos \frac{\pi}{3} = \sinh \frac{l}{4} \sinh q.$$

Therefore, we have $q = \text{arcsinh} \left( \frac{1}{2 \sinh \frac{\pi}{4}} \right)$. Hence

$$l(\tau_i) = 2\text{arcsinh} \left( \frac{1}{2 \sinh \frac{\pi}{4}} \right).$$

\[ \square \]

6. **Polygonal Quasi-geodesics**

In this section we develop three lemmas which are used in the next section to prove the first main theorem.

The first lemma says that a piecewise geodesic path with interior angles at the vertices bounded below by some positive real number and with the lengths of the edges bounded from below by some positive real number is a quasi-geodesic which we define below.

The second lemma says that the ratio of the length of a geodesic segment in a corridor with end points in the geodesic sides and the length of the corridor is close to 1 if the length of the corridor is sufficiently large.
It is a fact in hyperbolic geometry that there exist a unique simple closed geodesic of minimal length in the free homotopy class of a simple closed curve in a hyperbolic surface. The third lemma deals with the length of a piecewise geodesic simple closed curve and the length of the simple closed geodesic in its free homotopy class.

Before proceeding any further, we state a theorem due to Cannon that is useful in this section. Cannon’s theorem says that any local quasi geodesic is a global geodesic. We apply this theorem to prove the first lemma and the third lemma. It also says that a quasi-geodesic contained in some finite neighbourhood of the axis of the quasi-geodesic.

**Theorem 6.1. (Cannon, [4]).** Let $\lambda \geq 1$, $\delta \geq 0$ and $c \geq 0$. There exist effectively computable constants $\epsilon = \epsilon(\lambda, c, \delta)$, $L = L(\lambda, c, \delta)$, $c' \geq 0$ and $\lambda' \geq 1$ such that the following hold for any $\delta$-hyperbolic space $X$.

1. If $\gamma : I \to X$ is a $(\lambda, c, L)$-local quasi-geodesic then for all $a, b \in I$, $\gamma([a, b])$ contained in the $\epsilon$-neighborhood of the geodesic segment $[\gamma(a), \gamma(b)]$.
2. Any $(\lambda, c, L)$-local quasi-geodesic is a global $(\lambda', c')$-quasi-geodesic.

**6.1. Quasi-geodesics.** Let $(X, d)$ be a metric space. For $\lambda \geq 1$ and $\epsilon \geq 0$, a $(\lambda, \epsilon)$-quasi-isometric embedding $f : I \to X$ is a map $f : I \to X$ satisfying

$$
\frac{1}{\lambda} |a - b| - \epsilon \leq d(f(a), f(b)) \leq \lambda |a - b| + \epsilon,
$$

for all $a, b \in I$. If the restriction of $f$ to any subsegment $[x, y] \subset I$ of length at most $L$ is a $(\lambda, c)$-quasi-isometric embedding then we call $f$ is a $(L, \lambda, c)$-local quasi-isometric embedding. Note that, a quasi-isometric embedding need not be continuous. Now, we are ready to define a quasi-geodesic.

**Definition 6.2.** A curve $\gamma : I \to X$ in a geodesic metric space $(X, d)$ is called a $(\lambda, \epsilon)$-quasi-geodesic for some $\lambda \geq 1$ and $\epsilon \geq 0$, if the following inequality

$$
\frac{1}{\lambda} l(\gamma_{|[t_1, t_2]}) - \epsilon \leq d(\gamma(t_1), \gamma(t_2))
$$

holds for all $t_1, t_2 \in I$.

**Remarks 6.3.**

1. In the Definition 6.2 if $\epsilon = 0$ then $\gamma$ is simply called a $\lambda$-quasi-geodesic.
2. If the restriction of $\gamma$ to any subsegment $[a, b] \subset I$ of length at most $L$ is a $(\lambda, c)$-quasi-geodesic then we call $\gamma$ is a $(L, \lambda, c)$-local quasi-geodesic.

The definition of a quasi-geodesic does not depend on the parametrization of the curve. This often allows us to identify quasi-geodesics with their image and forget about the map.

If $\gamma : I \to X$ is parameterized by arc length then $\gamma$ is a $(\lambda, c)$-quasi-geodesic if and only if $\gamma$ is a $(\lambda, c)$-quasi-isometric embedding as

$$
d(\gamma(a), \gamma(b)) \leq l(\gamma_{|[a, b]}) = |b - a| \leq \lambda |b - a| + c
$$

holds. As a consequence we get

$$
\frac{1}{\lambda} |b - a| - c \leq d(\gamma(a), \gamma(b)) \leq \lambda |b - a| + c
$$

which is same as the inequality (13).
Example 6.4. Suppose \( \gamma_i : [0, \infty) \rightarrow \mathbb{H}, i = 1, 2 \) are two geodesics parameterized by arc length such that \( \gamma_1(0) = \gamma_2(0) \) and \( \sphericalangle(-\gamma_1'(0), \gamma_2'(0)) = \frac{\pi}{2} \). Then the piecewise geodesic curve \( \gamma : \mathbb{R} \rightarrow \mathbb{H} \) defined by
\[
\gamma(t) = \begin{cases} 
\gamma_2(t) & \text{if } t \in [0, \infty) \\
\gamma_1(-t) & \text{if } t \in (-\infty, 0]
\end{cases}
\]
is a 2-quasi-geodesic. Namely, suppose \( \gamma(t_1) \) and \( \gamma(t_2) \) are two points on \( \gamma \) such that \( t_1 < t_2 \). If both \( t_1, t_2 \) are either positive or negative then
\[
\frac{1}{2} \text{length}_{\mathbb{H}}(\gamma|_{[t_1, t_2]}) \leq \text{length}_{\mathbb{H}}(\gamma|_{[t_1, t_2]}) = d_{\mathbb{H}}(\gamma(t_1), \gamma(t_2)).
\]

Now suppose \( t_1 < 0 \) and \( t_2 > 0 \) then
\[
\text{length}_{\mathbb{H}}(\gamma|_{[t_1, t_2]}) = \text{length}_{\mathbb{H}}(\gamma|_{[t_1, 0]}) + l(\gamma|_{[0, t_2]}) \\
\leq 2d_{\mathbb{H}}(\gamma(t_1), \gamma(t_2))
\]
which implies that \( \gamma \) is a 2-quasi-geodesic.

6.2. Toponogov comparison theorem. Consider a geodesic triangle \( \triangle = (\gamma_1, \gamma_2, \gamma_3) \) in a Riemannian manifold \( M \) which is a set of three geodesics segments \( \gamma_i : [0, l_i] \rightarrow M, i = 1, 2, 3 \) parameterized by arc length such that \( \gamma_1(l_1) = \gamma_2(0), \gamma_2(l_2) = \gamma_3(0), \gamma_3(l_3) = \gamma_1(0) \). These three points are called vertices of \( \triangle \). Suppose the angle between \( -\gamma_i'(l_i+1) \) and \( \gamma_{i+2}'(0) \) is \( \sphericalangle(-\gamma_i'(l_i+1), \gamma_{i+2}'(0)) = \alpha_i \) where \( i = 1, 2, 3 \) with \( 0 \leq \alpha_i \leq \pi \). We specify a geodesic triangle by giving its sides \( (\gamma_1, \gamma_2, \gamma_3) \). The Toponogov theorem says that a pair of geodesics emanating from a point \( p \) spared apart more slowly in the region of high curvature than they would in a region of low curvature. Now, we state the Toponogov’s theorem, though we use it in a particular case which is easier to prove.

**Theorem 6.5.** (Toponogov, [7]). Let \( M \) be a complete manifold with \( K_M \geq H \), where \( K_M \) denotes the curvature of \( M \).

(A) Let \( (\gamma_1, \gamma_2, \gamma_3) \) determine a geodesic triangle in \( M \). Suppose \( \gamma_1, \gamma_3 \) are minimal and if \( H > 0 \) suppose \( l(\gamma_2) \leq \frac{\pi}{\sqrt{H}} \). Then in \( M^H \), the simply connected 2-dimensional space of constant curvature \( H \), there exist a geodesic triangle \( (\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3) \) such that \( l(\tilde{\gamma}_1) = l(\gamma_1) \) and \( \tilde{\alpha}_1 \leq \alpha_1, \tilde{\alpha}_3 \leq \alpha_3 \). Except in case \( H < 0 \) and \( l(\gamma) = \frac{\pi}{\sqrt{H}} \) for some \( i \), the triangle in \( M^H \) is uniquely determined.

(B) Let \( \gamma_1 \) and \( \gamma_2 \) be two geodesic segments in \( M \) of lengths \( l_1 \) and \( l_2 \) respectively parameterized by arc length such that \( \gamma_1(l_1) = \gamma_2(0) \) and \( \sphericalangle(-\gamma_1'(l_1), \gamma_2'(0)) = \alpha \).
We call such a configuration a hinge \( l \) and denote it by \((\gamma_1, \gamma_2, \alpha)\). Let \( \gamma_1 \) be minimal and if \( H > 0 \) then \( l(\gamma_2) \leq \frac{1}{\sqrt{H}} \). Let \( \bar{\gamma}_1, \bar{\gamma}_2 \subset M^H \) be such that \( \bar{\gamma}_1(l_1) = \bar{\gamma}_2(0) \), \( l(\bar{\gamma}_1) = l(\bar{\gamma}_2) = l_i \), \( i = 1, 2 \) and \( \angle(-\bar{\gamma}_1'(l_1), \bar{\gamma}_2(0)) = \alpha \). Then

\[
\rho(\gamma_1(0), \gamma_2(l_2)) \leq \rho(\gamma_1(0), \gamma_2(l_1)).
\]

For details see Comparison theorem in Riemannian Geometry by Cheeger [7]

6.3. Technical lemmas.

Lemma 6.6. Let \( \gamma : \mathbb{R} \to \mathbb{H} \) be a polygonal curve such that the interior angles are bounded below by some \( \theta_0 \in \mathbb{R}_+ \) and the lengths of geodesic pieces bounded below by \( L(> 0) \). Then there exist \( K \geq 1 \) and \( \epsilon \geq 0 \) such that \( \gamma \) is a \((K, \epsilon)\)-quasi-geodesic.

Proof. Step 1. In this step we prove the fact which is described bellow to conclude that \( \gamma \) is a locally quasi-geodesic.

Suppose \( \beta_i : [0, l_i] \to \mathbb{H}, i = 1, 2 \) are geodesic segments parameterized by arc length such that

1. \( \beta_1(l_i) = \beta_2(0) \) and
2. the interior angle at the connecting point is \( \alpha \).

Then the piece-wise geodesic arc \( \beta = \beta_1 \ast \beta_2 : [0, l_1 + l_2] \to \mathbb{H} \) defined by

\[
\beta(t) = \begin{cases} 
\beta_1(t) & \text{if } t \in [0, l_1] \\
\beta_2(t - l_i) & \text{if } t \in [l_1, l_1 + l_2]
\end{cases}
\]

is a \( k(\alpha) \)-quasi-geodesic for some constant \( k(\alpha) \geq 1 \).

Proof. Suppose \( \Delta(\gamma_1, \gamma_2, \gamma_3) \) is a triangle in the Euclidean plane \( \mathbb{R}^2 \) with

\[
l(\gamma_1) = t_1, l(\gamma_2) = t_2 \quad \text{and} \quad \angle(-\gamma_1'(l), \gamma_2'(0)) = \alpha.
\]

Claim: There exist a real number \( k(\alpha) \geq 1 \) which depends only on \( \alpha \), such that

\[
\frac{1}{k(\alpha)}(t_1 + t_2) \leq l(\gamma_3).
\]

Suppose the above claim is true. We prove that \( \beta \) is a \( k(\alpha) \)-quasi-geodesic. Let \( P, Q \) be two points on \( \beta \) and \( P = \beta(t_1), Q = \beta(t_2) \). If \( t_1, t_2 \) satisfy \( 0 \leq t_1, t_2 \leq l_1 \) or \( 0 \leq t_1, t_2 \leq l_2 \) then the points \( P, Q \) are both on either \( \beta_1 \) or \( \beta_2 \). Therefore, we have

\[
\text{length}_{\mathbb{H}}(\beta|_{[t_1, t_2]}) = d_{\mathbb{H}}(\beta(t_1), \beta(t_2)) = d_{\mathbb{H}}(P, Q).
\]

Hence for any \( k(\alpha) \geq 1 \) we have

\[
\frac{1}{k(\alpha)}\text{length}_{\mathbb{H}}(\beta|_{[t_1, t_2]}) \leq d_{\mathbb{H}}(P, Q).
\]

Now let \( 0 \leq t_1 \leq l_1 \) and \( l_1 \leq t_2 \leq l_1 + l_2 \). We want to show that

\[
\frac{1}{k(\alpha)}\text{length}_{\mathbb{H}}(\beta|_{[t_1, t_2]}) \leq d_{\mathbb{H}}(P, Q).
\]

Consider the Euclidean triangle \( \Delta(ABC) \) in \( \mathbb{C} \) such that \( d(A, C) = t_1, d(A, B) = t_2 \) and the interior angle at the vertex \( A \) is \( \alpha \). Then we have

\[
\frac{1}{k(\alpha)}(t_1 + t_2) \leq d(B, C)
\]
which follows from the above claim. Also by the Toponogov’s theorem we have
\[ d(B, C) \leq d_H(P, Q). \]
The above two inequalities give
\[ \frac{1}{k(\alpha)}(t_1 + t_2) \leq d_H(P, Q) \]
which is the same as the following inequality
\[ \frac{1}{k(\alpha)}length_{H}(\beta\vert_{[t_1,t_2]}) \leq d_H(P, Q). \]
Hence \( \beta \) is a \( k(\alpha) \)-quasi-geodesic.

To complete the proof it remains to prove the inequality \( (15) \).

**Proof of the claim:**
There are two cases to be considered.

**Case 1.** \((\alpha \leq \frac{\pi}{2})\)
Suppose \( \delta \) is the perpendicular line from \( \gamma(0) \) to the line \( \beta_2 \) (or an extension of \( \gamma_2 \)) meeting at the point \( R \). Let us denote \( P := \gamma_1(0), O := \gamma_2(0), Q := \gamma_3(0). \) Then we have
\[ d(P, R)) \leq d(P, Q) \quad \text{and} \quad d(Q, R) \leq d(P, Q). \]
From the right angled triangle with vertices at \( P, O, R \), we have
\[ d(O, P) + d(O, R) = \left( \frac{1}{\sin \alpha} + \frac{1}{\tan \alpha} \right)d(P, R). \]
Combining the inequalities \( (16) \) and \( (17) \) we have,
\[ (d(O, P) + d(O, R)) \leq \left( \frac{1}{\sin \alpha} + \frac{1}{\tan \alpha} \right)d(P, Q). \]
Now,
\[ d(O, P) + d(O, Q) \leq d(O, P) + d(O, R) + d(R, Q) \]
\[ \leq \left( \frac{1}{\sin \alpha} + \frac{1}{\tan \alpha} \right)d(P, Q) + d(P, Q) \]
\[ = \left( \frac{1}{\sin \alpha} + \frac{1}{\tan \alpha} + 1 \right)d(P, Q) \]
which gives
\[ \frac{1}{k(\alpha)}(t_1 + t_2) \leq l(\gamma_3) \]
where
\[ k(\alpha) = \frac{1}{\sin \alpha} + \frac{1}{\tan \alpha} + 1. \]

Now, we consider the second case.

**Case 2.** \((\alpha > \frac{\pi}{2})\)
We draw the perpendicular line \( \delta \) from \( P \) on the extended line of \( \gamma_2 \) which meets at \( R \). Then we have
\[ d(P, R) \leq l(\gamma_3) \quad \text{and} \quad d(R, Q) \leq l(\gamma_3). \]
Therefore, we get
\[
\begin{align*}
d(P, O) + d(O, Q) & \leq \frac{d(P, Q)}{\sin \alpha} + d(R, Q) \\
& \leq \frac{d(P, Q)}{\sin \alpha} + d(P, Q) \\
& = \left( \frac{1}{\sin \alpha} + 1 \right) d(P, Q).
\end{align*}
\]

Hence we have
\[
\frac{1}{k(\alpha)} (t_1 + t_2) \leq l(\gamma_3)
\]
where
\[
k(\alpha) = \frac{1}{\sin \alpha} + 1.
\]

Hence our claim is proved.

**Step 2.** It follows from the **Step 1** that \(\gamma\) is a locally \(k(\theta_0)\)-quasi-geodesic. Now it follows from the **Theorem 6.1** there are \(k \geq 1\) and \(\varepsilon > 0\) such that \(\gamma\) is a quasi-geodesic.

Now we recall some basic definitions before stating the second lemma. Let us define an orthogonal projection in \(\mathbb{H}\). Let \(L\) be a hyperbolic complete geodesic in \(\mathbb{H}\) and \(p \in \mathbb{H}\). Then there exist a unique point \(p'\) in \(L\) which minimizes the distance of \(p\) from \(L\). We define a function \(\rho_L : \mathbb{H} \rightarrow L\) by
\[
(19) \quad \rho_L(p) = p', \quad \forall p \in \mathbb{H}.
\]
The function \(\rho_L\) is called an orthogonal projection map and it is the projection of \(\mathbb{H}\) onto \(L\).

**Definition 6.7.** Let \(L\) be a hyperbolic line and \(I \subset L\). For a positive real number \(W \in \mathbb{R}_+\), the \(W\)-corridor about \(I\) along \(L\) is the set
\[
(20) \quad \{ z \in H | d(z, L) \leq W, \rho_L(z) \in I \}.
\]

Let \(\delta\) be any geodesic segment in the corridor \(W(\gamma, \gamma')\) such that
- The end points of \(\delta\) are on the sides of \(W(\gamma, \gamma')\) of length \(2W\) [Figure 11].
- \(\delta\) lies on the closure of one of the components \(W(\gamma, \gamma')\) as in the following picture.

The Figure 11 describes \(W(\gamma, \gamma')\) and a \(\delta\) when \(\gamma'\) is positive imaginary axis. Now, we state the second lemma.

**Lemma 6.8.** In the above setting, for a fixed \(W > 0\), if the length of the corridor is sufficiently large then the ratio \(\frac{\text{length}(\delta)}{\text{length}(\gamma)}\) is close to 1.

**Proof.** If necessary after using an isometry we assume that \(\gamma\) is a segment of the positive imaginary axis in the upper half plane model and \(\gamma(0) = i\) and \(\gamma(1) = ri\) (Figure 11). Without loss of generality we assume that \(\delta\) is on the right side of the imaginary axis. Let \(\alpha\) and \(\beta\) are the geodesic sides of the right side region. The end points of \(\alpha\) (\(\beta\) respectively) are \(a, a'\) (\(b, b'\) respectively) (Figure 11). The equations of \(\alpha\) and \(\beta\) are given by
\[ \alpha(t) = e^{it}, \quad t \in \left[ \frac{\pi}{2} - \psi, \frac{\pi}{2} \right] \quad \text{and} \]
\[ \beta(t) = re^{it}, \quad t \in \left[ \frac{\pi}{2} - \psi, \frac{\pi}{2} \right] \]

where \( 0 < \psi < \frac{\pi}{2} \) is determined by the equation:

\[ \int_{\psi}^{\pi/2} \frac{1}{\sin t} dt = W. \]

Then we have

\[ l(r) = \max \{ d(\alpha(t), \beta(s)) \mid t, s \in \left[ \frac{\pi}{2} - \psi, \frac{\pi}{2} \right] \}. \]

Then the maximum distance is realized by the distance between two opposite corner points. Hence

\[ l(r) = d(a, b') = \log \left( \frac{|a - \bar{b'}| + |a - b'|}{|a - b'| - |a - b'|} \right) \]
\[ = \log \left( \frac{|i - re^{-i\psi}| + |i - re^{i\psi}|}{|i - re^{-i\psi}| - |i - re^{i\psi}|} \right) \]
\[ = \log \frac{\sqrt{r^2 + 1 + 2r \sin \psi} + \sqrt{r^2 + 1 - 2r \sin \psi}}{\sqrt{r^2 + 1 + 2r \sin \psi} - \sqrt{r^2 + 1 - 2r \sin \psi}} \]
\[ = \log \frac{r^2 + 1 + \sqrt{(r^2 + 1)^2 + 4r^2 \sin^2 \psi}}{2r \sin \psi} \]

Hence, we have the following inequality

\[ 1 \leq \frac{l(\gamma)}{l(\sigma)} \leq \frac{l(r)}{\log r} \leq \frac{\log(2r \sin \psi)}{\log r} \]

To prove the lemma it is enough to prove that,

\[ \lim_{r \to +\infty} \frac{\log(2r \sin \psi)}{\log r} = 1. \]
By a straight forward calculation it is easy to show that

$$\lim_{r \to +\infty} \frac{l(r)}{\log r} = 1$$

which completes the proof.

Before stating the third lemma let us consider a hyperbolic surface $S$ and a piecewise simple closed geodesic $\gamma$ on $S$ such that

1. $\gamma$ is homotopically non-trivial.
2. The interior angle at each corner of $\gamma$ is bounded below by some positive number $\theta_0$.
3. The interior angle at each corner is less than $\pi$.

The third condition implies that the lift of unique simple closed geodesic in the free homotopy class of $\gamma$ does not cross the lift of $\gamma$ in the universal cover $\mathbb{H}$ of the surface.

**Lemma 6.9.** Let $\gamma$ be a piecewise geodesic simple closed curve in a hyperbolic surface $S$ and $\tilde{\gamma}$ be a lift of $\gamma$ in the universal cover of $S$ then:

1. There exist $k \geq 1$ and $\epsilon > 0$ such that $\tilde{\gamma}$ is a $(k, \epsilon)$-quasi-geodesic.
2. If $\gamma'$ is the simple closed geodesic in the free homotopy class of $\gamma$ then $\frac{\text{length}(\gamma)}{\text{length}(\gamma')}$ is close to 1 as the length of the smallest geodesic segment of $\gamma$ is sufficiently large.

**Proof.** Let $\theta_1, \theta_2, \ldots, \theta_n$ be the interior angles at the corners of $\gamma$. We write

$$\gamma = \gamma_1 \star \gamma_2 \cdots \star \gamma_n$$

where $\gamma_i$'s are geodesic segments in $\gamma$. Then we define

$$t = \min\{\text{length}(\gamma_i) : i = 1, 2, \ldots, n\} \quad \text{and} \quad \theta_0 = \min\{\theta_i : i = 1, 2, \ldots, n\}.$$

Now consider a lift $\tilde{\gamma}$ of $\gamma$ in the universal cover of $S$. Hence it follows from the Lemma 1 that $\tilde{\gamma}$ is a $t$-locally $k(\theta_0)$-quasi geodesic. Moreover it follows from the Theorem 6.1 that there exist $k \geq 1$ and $\epsilon > 0$ such that $\tilde{\gamma}$ is a globally $(k, \epsilon)$-quasi geodesic.

Let $A, B$ be the end points of $\tilde{\gamma}$ in the boundary at infinity. The geodesic joining $A$ and $B$, denoted by $\tilde{\gamma}'$ is the axis of $\tilde{\gamma}$ which projects onto $\gamma'$. It follows from the Theorem 6.1 that there exist $W > 0$ such that

$$\tilde{\gamma} \subset \text{Nbd}(\tilde{\gamma}', W)$$

where $\text{Nbd}(\tilde{\gamma}', W)$ denotes the $W$ neighbourhood of $\tilde{\gamma}'$.

Let $P = \gamma_1 \star \gamma_2 \cdots \star \gamma_n$ be a continuous subsegment of $\tilde{\gamma}$ such that $\gamma_i$'s project onto $\gamma_i$, $i = 1, 2, \ldots, n$ and we have

$$\text{length}(\gamma) = \sum_{i=1}^{n} \text{length}_{\mathbb{H}}(\gamma_i).$$

Suppose $P_0, P_1, \ldots, P_n$ are the points on the path $P$ such that $\gamma_i$'s are the geodesic segments joining $P_{i-1}$ and $P_i$, $i = 1, 2, \ldots, n$. Also the orthogonal projection of the
Figure 12. A lift of $\gamma$ point $P_i$ is denoted by $P'_i$, $i = 1, 2, \ldots, n$ and the geodesic segment joining $P'_{i-1}$ and $P'_i$ is denoted by $\tilde{\gamma}_i'$. Then we have

$$\text{length}(\gamma') = \sum_{i=1}^{n} \text{length}_{\mathbb{H}}(\tilde{\gamma}_i').$$

The geodesic segment $\tilde{\gamma}_i$ lies in the $W$-corridor of $\tilde{\gamma}_i'$ along $[A,B]$ such the end points are on the geodesic sides of the corridor. Then it follows from the Lemma 2 the ratio $\frac{\text{length}_{\mathbb{H}}(\tilde{\gamma}_i)}{\text{length}_{\mathbb{H}}(\tilde{\gamma}_i')}$ is close to 1 if $t$ is sufficiently large, $i = 1, 2, \ldots, n$. Hence

$$\frac{\text{length}(\gamma)}{\text{length}(\gamma')} = \frac{\sum_{i=1}^{n} \text{length}_{\mathbb{H}}(\tilde{\gamma}_i)}{\sum_{i=1}^{n} \text{length}_{\mathbb{H}}(\tilde{\gamma}_i')}$$

is close to 1 when $t$ is sufficiently large. Hence the proof. \hfill $\square$

7. Realization using hyperbolic surfaces with boundary

In this section, we prove that any fat graph is realized by the systolic graph of some hyperbolic surface with geodesic boundary if it is combinatorially admissible. We know that any decorated fat graph is the union of edge disjoint standard cycles. We construct hyperbolic cylinder corresponding to each standard cycle. Next, plumb the hyperbolic cylinders according to the intersections of the standard cycles of the decorated fat graph to obtain a hyperbolic surface with boundary. The boundary components are not geodesics, in fact not even piecewise geodesic. We obtain a hyperbolic surface with totally geodesic boundary by cutting the surface along the unique geodesics in the free homotopy classes of the boundary components. We show that the systolic graph of the surface is isomorphic to the fat graph.

7.1. Hyperbolic Cylinder. Now we give a construction of hyperbolic cylinder with width $2\epsilon$ and the central simple closed geodesic of length $l$. Let us consider the geodesic $\delta$ in $\mathbb{H}$ given by

$$t \mapsto \delta(t) = i e^t, t \in \mathbb{R}.$$
Let $\gamma$ and $\gamma'$ be geodesics intersecting $\delta$ perpendicularly at $i$ and $ie^l$ (Figure 13) where $l > 0$. The length of the geodesic segment $[i, ie^l]_H$ is given by

$$\text{length}_{H}([i, ie^l]_H) = l.$$  

We parameterize $\gamma$ and $\gamma'$ with unit speed and opposite orientation such that $\gamma(0) = i$ and $\gamma'(0) = ie^l$. If $P = e^{i\theta}(0 < \theta < \pi)$ is the point on $\gamma$ so that $d_{H}(P, \delta) = \epsilon$, then

$$\epsilon = \int_{\theta}^{0} \frac{1}{\sin t} \, dt = -\frac{1}{2} \ln \frac{1 + \cos t}{1 - \cos t} |_{\theta}^{0} = \frac{1}{2} \ln \frac{1 + \cos \theta}{1 - \cos \theta} \Rightarrow 1 + \cos \theta = e^{2\epsilon}$$

$$\Rightarrow \cos \theta = \frac{e^{2\epsilon} - 1}{e^{2\epsilon} + 1} = \frac{e^{\epsilon} - e^{-\epsilon}}{e^{\epsilon} + e^{-\epsilon}}$$

$$\Rightarrow \theta = \cos^{-1}(\tanh \epsilon).$$

Let $R$ and $R'$ be the Euclidean rays from the origin passing through $P = e^{i\cos^{-1}(\tanh \epsilon)}$ and $P' = -e^{-i\cos^{-1}(\tanh \epsilon)}$ respectively. For $\epsilon > 0$, consider the region

$$S(\epsilon) = \{z = re^{i\theta} | 1 \leq r \leq e^l, d_{H}(g, z) \leq \epsilon\}$$

bounded by $\gamma$, $\gamma'$ and the Euclidean rays $R, R'$. Then $S(\epsilon)$ is a hyperbolic surface with atlas $\mathcal{A} = \{(S, id)\}$.

Consider the isometry $m : \mathbb{H} \rightarrow \mathbb{H}$ defined by $m(z) = e^l z$. Then the isometry $m$ maps $\delta$ to itself and satisfies the equation $m(\gamma(t)) = \gamma'(t)$. Now the pasting equation

$$\gamma(t) = \gamma'(t), -\epsilon \leq t \leq \epsilon,$$

yields the hyperbolic surface

$$C(l, \epsilon) = S(\epsilon) \mod \{\gamma(t) = \gamma'(t), -\epsilon \leq t \leq \epsilon\}.$$
The geodesic arc $\delta_{[0,l]}$ projects onto the closed geodesic $\delta'$ on $C(l,\epsilon)$ of length $l(\delta') = l$. We call the geodesic $\delta'$ as the central geodesic of $C(l,\epsilon)$. The distance between two boundary components of $C(l,\epsilon)$ is $2\epsilon$ which is called the width of the cylinder.

7.2. Plumbing. We construct a surface corresponding to a fat graph as described below. We take a closed disc corresponding to each vertex and a rectangle corresponding to each edge. Then we identify the sides of the rectangles with the segments of the boundary component of the discs according to the ordering of the edges incident to a vertex. The local picture at the vertex is the following (Figure 14):

In this way, we get a topological surface corresponding to a given fat graph. Thus we can talk about number of boundary components, genus of fat graph, Euler characteristic and many other topological notions.

Let $G$ be a given decorated fat graph. Then the fat graph is the edge disjoint union of standard cycles,

$$G = \bigcup_{i=1}^{k} C_i,$$

where $C_i, i = 1, 2, \ldots, k$ are the distinct standard cycles of $G$. Now we describe the construction of surface for a given decorated fat graph.

First we take hyperbolic cylinder $C_i(l,\epsilon)$ which are the copies of $C(l,\epsilon)$ corresponding to the standard cycle $C_i$, where $l, \epsilon$ are some arbitrary positive real numbers.

Then we plumb the cylinders according to the intersection of the standard cycles in the fat graph $G$ to obtain a hyperbolic surface which we denote by $\Sigma(G)$.

The surface $\Sigma(G)$ has a natural hyperbolic structure with nonempty boundary. The central geodesics of the cylinders form a fat graph embedded in $\Sigma(S)$ which is nothing but the given decorated fat graph (up to isomorphism). The boundary components of $\Sigma(G)$ are neither geodesics nor even piecewise geodesics. The union of the central geodesics is a spine of $\Sigma(G)$ which is isomorphic to the fat graph. It is easy to see that each boundary component is freely homotopic to a non-standard cycle of the spine.

Now as $G$ is a admissible fat graph with the metric $d$ then we choose the positive number $l$ which is the length of each standard cycle with respect to the metric $d$. Also we plumb the cylinders in such a way that the restriction of the hyperbolic metric on the spine is $d$. 

\[\text{Figure 14. Fat graph locally.}\]
7.3. Constructing surface with geodesic boundary. Let us consider the surface $\Sigma(G)$ for a given admissible decorated fat graph $G$. The surface $\Sigma(G)$ has non-empty boundary component. Now we state and prove the following lemma which shows that in the free homotopy class of a boundary component there is a unique simple closed geodesic which lies in between the boundary curve and the spine. Moreover, there exist a metric on the fat graph so that the length of that geodesic is strictly greater than the length of standard cycles.

**Lemma 7.1.** Let $\gamma'$ be a boundary component of $\Sigma(G)$, which is freely homotopic to the piecewise geodesic simple closed curve $\gamma''$ in the spine of $\Sigma(G)$. Then there exist a simple closed curve $\gamma$ in the free homotopy class of $\gamma'$ which lies in the (topological) cylinder formed by $\gamma'$ and $\gamma''$ for suitably chosen metric $d$ on $G$ and $\epsilon$, $\epsilon > 0$ is as in the previous section. Also, if the length of the standard cycles is chosen sufficiently large in our construction, the length of $\gamma$ is strictly greater than the length of a standard cycle.

**Proof.** Let the interior angles of the piecewise geodesic simple closed curve $\gamma''$ be $\theta_1, \theta_2, \ldots, \theta_k$. Set

$$\theta = \min\{\theta_i | i = 1, 2, \ldots, k\}.$$  

Then $\theta(> 0)$ is a lower bound of the interior angles of the lift $\tilde{\gamma}''$ of $\gamma''$ in the universal cover of $\Sigma(G)$. So it follows from Lemma 6.6 that $\tilde{\gamma}''$ is a $(K, \epsilon)$-quasi geodesic. Let us denote the axis of the quasi-geodesic $\tilde{\gamma}''$ by $\tilde{\gamma}$, which is the geodesic line in the hyperbolic plane joining the end points of $\tilde{\gamma}''$. By Theorem 6.1 there is a positive number $W \in \mathbb{R}$ such that $\tilde{\gamma}'' \subset \text{Nbd}(\tilde{\gamma}, W)$. Also each interior angle is strictly less than $\pi$, which follows that the lift $\tilde{\gamma}'$ of $\gamma'$ and the geodesic $\gamma$ lie on the same side of $\tilde{\gamma}''$. If we choose the positive number $\epsilon$ larger that $W$, then the geodesic $\tilde{\gamma}$ lies in the region bounded by $\tilde{\gamma}'$ and $\tilde{\gamma}''$.

The projection of $\tilde{\gamma}$ on $\Sigma(G)$ is a simple closed geodesic in the free homotopic class of $\gamma'$ and is denoted by $\gamma$. The geodesic $\gamma$ lies in between $\gamma'$ and $\gamma''$. Let $d$ be a metric on the fat graph $G$ which satisfy the following system of equations and in-equations:

- $\text{length}_d(C) = l$, for each standard cycle $C$ in $G$.
- $\text{length}_d(C') > l$, for each non-standard cycle $C'$ in $G$.

Then for any positive number $\lambda \in \mathbb{R}$ the metric $d_\lambda$ on $G$ defined by $d_\lambda(e) = \lambda d(e)$, for each edge $e$ in $G$ is also satisfy the above system. By the Lemma 6.9 it follows that there exist $\lambda > 0$ such that,

$$\frac{\text{length}_{d_\lambda}(\gamma'')}{\text{length}_{d_\lambda}(\gamma)} < r'.$$

Now notice that whatever $\lambda$ we choose, the ratio $\frac{\text{length}_{d_\lambda}(\gamma'')}{\text{length}_{d_\lambda}(C)} = r$, where $C$ is a standard cycle, is unchanged. If we choose $r' < r$ then from the above inequality we have,

$$\text{length}_{d_\lambda}(\gamma) > \frac{\text{length}(\gamma'')}{r'} = \frac{r}{r'} \text{length}_{d_\lambda}(C) > \text{length}_{d_\lambda}(C).$$

$\square$
Lemma 7.2. There exist a metric $d$ on $G$ such that for each boundary component $\gamma'$ of $\Sigma(G)$:

1. The unique simple closed geodesic $\gamma$ in the free homotopy class of $\gamma'$ lies in the topological cylinder formed by $\gamma'$ and $\gamma''$, the piecewise geodesic simple closed curve in the spine of $\Sigma(G)$ freely homotopic to $\gamma'$.
2. The length of each geodesic $\gamma$ in each free homotopy class of each boundary component is strictly greater than the length of a standard cycle of $G$.

Hence cutting off the surface $\Sigma(G)$ along these geodesics in the free homotopy classes of boundary components we get a hyperbolic surface with totally geodesic boundary.

Proof. Let $d$ be a metric on the fat graph as in the proof of the Lemma 7.1. Let $\gamma'_i, i = 1, 2, \ldots, b$ be the boundary components of the surface $\Sigma(G)$. The piecewise geodesic simple closed curve in the spine freely homotopic to the boundary component $\gamma'_i$ is denoted by $\gamma''_i$. For each $i$, suppose $\lambda_i$ and $\epsilon_i$ are the positive numbers such that:

- The simple closed geodesic $\gamma_i$ in the free homotopy class of $\gamma'_i$ lies in the topological cylinder formed by two simple closed curves $\gamma'_i$ and $\gamma''_i$.
- $\text{length}(\gamma_i) > \text{length}_{d_{\lambda_i}}(C)$, where $\text{length}_{d_{\lambda_i}}(C)$ is the length of a standard cycle $C$ in $G$ with respect to the metric $d_{\lambda_i}$.

We take,

$$\epsilon := \max\{\epsilon_i | i = 1, 2, \ldots, b\} \quad \text{and} \quad \lambda := \max\{\lambda_i | i = 1, 2, \ldots, b\}.$$

Then $d_{\lambda}$ is a metric on $G$ for which following are satisfied:

- $\gamma_i, i = 1, 2, \ldots, b$ are the boundary components of $\Sigma(G)$.
- $\gamma''_i, i = 1, 2, \ldots, b$ are piecewise geodesic simple closed curves in the spine homotopic to $\gamma'$.
- $\gamma_i, i = 1, 2, \ldots, b$ are the simple closed geodesics in the homotopy class of $\gamma'_i$ lies in the topological cylinder formed by the pair of simple closed curves $\gamma'_i, \gamma''_i$.
- The length of a standard cycle is $\lambda l$.
- For each $\gamma_i$, $\text{length}(\gamma_i) > \lambda l$.

Hence cutting off the boundary curves of the surface $\Sigma(G)$ along the geodesics $\gamma_i$ we get a hyperbolic surface $\Sigma_1(G)$ and the systolic graph of $\Sigma_1(G)$ is $G$. $\square$

8. CAPPING

In this section we cap a hyperbolic surface with boundary to obtain a closed hyperbolic surface which satisfies our desired conditions. Let $\Sigma$ be a hyperbolic surface with boundary. We embed the surface $\Sigma$ isometrically into a closed hyperbolic surface $S$ such that they have the same systole and all geodesics realizing the systole are contained in $\Sigma$. Thus the systolic graph of the closed hyperbolic surface is the same as the systolic graph of $\Sigma$, a hyperbolic surface with boundary. The main result in this section is the following:

Theorem 8.1. Let $\Sigma$ be a hyperbolic surface with totally geodesic boundary. Then there exist a closed hyperbolic surface $S$ and an isometric embedding $i : \Sigma \rightarrow S$ such that the following hold:
(1) \( \text{sys}(S) = \text{sys}(\Sigma) \).
(2) The systolic graph of \( S \) is isomorphic to the systolic graph of \( \Sigma \).

8.1. Cubic graphs and closed hyperbolic surfaces. In this subsection we give a correspondence between cubic graphs and the closed hyperbolic surfaces together with pair of pants decompositions. Let \( \Gamma \) be a cubic graph with \( 2g - 2 \) nodes where \( g \geq 2 \) is an integer. We construct a closed hyperbolic surface of genus \( g \) corresponding to the graph \( \Gamma \). We consider each edge of the graph as the composition of two semi-edges. Let us denote the nodes of the graph \( \Gamma \) by \( v_i; i = 1, 2, \ldots, 2g - 2 \). Let us consider the local picture of a trivalent graph at a node \( v_k \) with the semi-edges \( c_{k,1}, c_{k,2} \) and \( c_{k,3} \) incident at \( v_k \) (Figure 15).

![Figure 15. Local picture of a trivalent graph.](image)

For each node \( v_k \), we construct \( Y(l) \) completely identical copies \( Y_k \) where \( k = 1, 2, \ldots, 2g - 2 \). We denote the boundary components of \( Y_k \) by \( \gamma_{k,1}, \gamma_{k,2} \) and \( \gamma_{k,3} \) which are parameterized with unit speed. Moreover the boundary components of \( Y_k \) are defined periodically on \( \mathbb{R} \). Now we glue the pair of pants \( Y_k \) and \( Y_l \) along the boundary components \( \gamma_{k,i} \) and \( \gamma_{l,j} \) by the following identification

\[
\gamma_{k,i}(t) = \gamma_{l,j}(\alpha - t), \quad \forall t \in \mathbb{R}
\]

if the semi-edges \( c_{k,i} \) and \( c_{l,j} \) together forms an edge of the graph. In the equation \( 23 \) the constant \( \alpha \) is a twist parameter. In the above construction, we obtain a closed hyperbolic surface \( F(\Gamma) \) of genus \( g \). A simple closed geodesic \( \gamma \) in \( F(\Gamma) \) is called a rim if it is obtained by gluing \( \gamma_{k,i} \) and \( \gamma_{l,j} \). Thus we have a set of \( 3g - 3 \) rims which gives a pair of pants decomposition of \( F(\Gamma) \). This pants decomposition of \( F(\Gamma) \) is the associated pair of pants decomposition to the trivalent graph \( \Gamma \). Also note that if the girth of the graph is sufficiently large then the systole of \( F(\Gamma) \) is \( l \).

Conversely, let \( S_g \) be a closed hyperbolic surface of genus \( g \) and

\[
P = \{ \gamma_1, \gamma_2, \ldots, \gamma_{3g-3} \}
\]

be a pair of pants decomposition. We construct the trivalent graph by following: Let \( Y_1, Y_2, \ldots, Y_{2g-2} \) be the pair of pants obtained by decomposing the surface along the pair of pants decomposition \( P \). The trivalent graph \( \Gamma(S_g) = (V,E) \) is defined as follows:

- \( V = \{ Y_i | i = 1, 2, \ldots, 2g - 2 \} \) is the set of nodes, i.e. there is a node for each pair of pants \( Y_i \) in the pants decomposition.
- Two nodes \( Y_k \) and \( Y_l \) (possibly same) are joined by a simple edge if there are boundary geodesics \( \gamma_{k,i} \) and \( \gamma_{l,j} \) of \( Y_k \) and \( Y_l \) respectively which are obtained by cutting \( S_g \) along the geodesic \( \gamma_{i_0} \) for some \( i_0 \in \{ 1, 2, \ldots, 3g-3 \} \).
8.2. Uni-trivalent graphs and Hyperbolic surfaces. In this subsection we construct hyperbolic surfaces which have a single boundary component and with large systole. Suppose we are given a uni-trivalent graph $G = (V, E)$ where $V = \{v_0, v_1, \ldots, v_n\}$, $n = 2g - 2$ for some $g \geq 0$ and $l > 0$ some positive number. We label the nodes such that

$$deg(v_0) = 1 \quad \text{and} \quad deg(v_i) = 3, \quad \forall i \geq 1.$$ 

Without loss of generality we assume that $v_1$ is the adjacent node to $v_0$. We consider each edge of the graph as a composition of two semi-edges. For each node $v_k (k \geq 1)$ go to the semi edges $c_{k,i}, i = 1, 2, 3$. We construct a hyperbolic surface $\Sigma_l(G)$ as follows.

We construct $Y(2l)$ completely identical copies $Y_k$ with boundary geodesics $\gamma_{k,i}, i = 1, 2, 3$ for each $k \geq 2$. For $k = 1$, $Y_1$ is the pair of pants with boundary geodesics $\gamma_{1,1}, \gamma_{1,2}$ and $\gamma_{1,3}$ of length $l$, $2l$ and $2l$ respectively, where the boundary geodesic $\gamma_{1,1}$ is corresponding to the semi edge $c_{1,1}$ which is adjacent to the degree one node $v_0$. We glue the pair of pants $Y_k$ and $Y_l$ along the $\gamma_{k,i}$ and $\gamma_{l,j}$ by the following identification

$$\gamma_{k,i}(t) = \gamma_{l,j}(\alpha - t), \forall t \in \mathbb{R},$$

when the semi-edges $c_{k,i}$ and $c_{l,j}$ together form an edge of the graph $G$. Also we assume that the rims are defined periodically on $\mathbb{R}$. The constant $\alpha$ is twist parameter. In this way we get a hyperbolic surface of genus $g$ and single boundary component $\gamma_{1,1}$ of length $l$. We denote the hyperbolic surface obtained in the above construction by $\Sigma_l(G)$.

We define a continuous function $a(l), l \in \mathbb{R}_+$ by

$$a(l) = \min\{\frac{2 \arcsinh(\frac{1}{2 \sinh \frac{l}{2}})}{2 \sinh \frac{l}{2}}, \cosh(1 + \frac{1 + \cosh \frac{l}{2}}{\sinh^2 \frac{l}{2}})\}.$$ 

Also we define the natural number $t = t(l), l \in \mathbb{R}_+$ by

$$t(l) = \lfloor \frac{l}{a(l)} \rfloor + 1.$$ 

**Lemma 8.2.** For given positive constant $l \in \mathbb{R}$, there exist a uni-trivalent graph $G$ of girth $t(l)$ such that

1. The length of any closed geodesic in $\Sigma_l(G)$ is greater than or equal to $l$.
2. If $\delta$ is an essential geodesic arc with endpoints on the geodesic boundary then the length of $\delta$ is greater than or equal to $l$.

**Proof.** Let $l$ be a positive constant and $G$ be the uni-trivalent graph of girth $t(l)$ given in the Lemma 4.1. Now, consider the surface with boundary $\Sigma_l(G)$. Observe that the rims of $\Sigma_l(G)$ are of length $2l$ and the boundary geodesic has length $l$. So it is sufficient to demonstrate that no further geodesic $\sigma$ have length less than $l$.

Without loss of generality we assume that $\sigma$ is a simple closed geodesic. Then $\sigma$ cannot run into a single $Y$-piece otherwise $\sigma$ would equal to one of the rim which we have already excluded. Therefore there is a partition

$$0 = t_0 < t_1 < \cdots < t_n = 1$$
of $[0,1]$ and the rims $\gamma_0, \gamma_1, \ldots, \gamma_n = \gamma_0$, where individuals can occur more than once such that

$$\sigma(t_i) \in \gamma_i, i = 0, 1, \ldots, n.$$ 

Moreover no other rim is crossed over. Hence each segment $\sigma_i = \sigma|_{t_{i-1}, t_i}$ lies in a single $Y$-piece, denoted by $Y_i$. Again the individual $Y$-pieces $Y_i$ can occur multiple times. Therefore we have

$$\text{(27) } \text{length}(\sigma) = \sum_{i=1}^{n} \text{length}(\sigma_i).$$

Now there are following three cases to be considered.

Case 1. First, we consider the case where $\gamma_{i-1} \neq \gamma_i \forall i$.

It is easy to see that each rim corresponds to an edge of $G$. Hence $\sigma$ corresponds to a closed path in the graph $G$. We denote the closed path by $P(\sigma)$. The closed path $P(\sigma)$ contains a cycle (simple closed path) whose length is greater than or equal to the girth $T(G)$ of the graph $G$. Therefore we have $n \geq n_0$. For each $i$, the geodesic segment $\sigma_i$ is a geodesic in $Y_i$ joining two distinct boundary components. If $Y_i$ is the pair of pants with boundary geodesics of length $2l$ then it follows from Lemma 5.5 that,

$$\text{(28) } \text{length}(\sigma_i) \geq 2 \text{arcsinh}\left(\frac{1}{2 \sinh \frac{l}{2}}\right).$$

If $Y_i = Y(l, 2l, 2l)$ then $\sigma_i$ is a geodesic with endpoints on the boundaries of length $2l$. In this case we have

$$\text{(29) } \text{length}(\sigma_i) \geq \text{arcosh}\left(1 + \frac{1 + \cosh \frac{l}{2}}{\sinh^2 \frac{l}{2}}\right).$$

Now from the definition of the function $a(l)$ in equation (25) and the above two inequalities,

$$\text{(30) } \text{length}(\sigma_i) \geq a(l).$$

Therefore,

$$\text{length}(\sigma) = \sum_{i=1}^{n} \text{length}(\sigma_i) \geq n \cdot a(l) \geq n_0 \cdot a(l) \geq l.$$ 

Case 2. Suppose there exist $i_0$ such that $\gamma_{i_0-1} = \gamma_{i_0}$ and $Y_{i_0} = Y(2l, 2l, 2l)$. Then it follows from the Lemma 5.4 that

$$\text{length}(\sigma_{i_0}) \geq l$$

which implies that $\text{length}(\sigma) \geq l$.

Case 3. The remaining case is the following. There is $i_0$ such that $\gamma_{i_0-1} = \gamma_{i_0}$ where $Y_{i_0} = Y(l, 2l, 2l)$. Then the both end points of $\gamma_{i_0}$ is on the same boundary of length $2l$. In that case

$$\text{length}(\sigma_{i_0}) \geq \frac{l}{2}.$$
If there are two such different \(i_0\) and \(j_0\) then we have two distinct geodesic segments \(\sigma_{i_0}\) and \(\sigma_{j_0}\) each of length greater than or equal to \(\frac{l}{2}\). Therefore the length of \(\sigma\) is greater than \(l\).

So we assume that there is only one such \(i_0\). Hence the seams

\[
\gamma_1, \ldots, \gamma_{i_0-2}, \gamma_{i_0+1}, \ldots, \gamma_\nu
\]

are pairwise distinct and correspond to a closed path \(P\) in \(G\). The length of the closed path \(P\) is \((n-2)\) which is greater than or equal to \(T(G) = n_0\). Therefore using similar arguments to Case 1 we conclude that \(\text{length}(\sigma) \geq l\).

Now we prove the second part of the lemma. Consider an essential geodesic arc \(\delta\) in \(\Sigma_i(G)\) with the end points on the boundary geodesic. If \(\delta\) lies in the pair of pants \(Y_1 = Y(l, 2l, 2l)\) then the length of \(\delta\) is greater than or equal to the height of \(Y(l, 2l, 2l)\) which is greater than \(l\) [Lemma 5.4]. Hence we have \(\text{length}(\delta) \geq l\).

In the remaining cases, there is a sequence of rims \(\gamma_0, \gamma_1, \ldots, \gamma_n = \gamma_0\) and a partition \(0 = t_0 < t_1 < \cdots < t_n = 1\) such that \(\delta(t_i) \in \gamma_i\) and no other rim is crossed over. So, \(\delta_i = \delta|_{[t_{i-1}, t_i]}\) lies in a single pair of pants, denoted by \(Y_i\). If \(\gamma_i = \gamma_{i-1}\) for some \(i\) then the length of \(\delta_i\) is greater than or equal to \(l\). So the length of \(\delta\) is greater than or equal to \(l\).

Now we assume that \(\gamma_i's\) are distinct. The rims \(\gamma_i\) determines a closed path \(P\) in \(G\) which contains a cycle. Hence as in the proof of first part, we have \(n \geq n_0\) and length of each \(\delta_i\) is greater than or equal to \(a(l)\). The length of \(\delta\) satisfies the following:

\[
\text{length}(\delta) = \sum_{i=1}^{n} \text{length}(\delta_i) \geq n \cdot a(l) \geq n_0 \cdot a(l) \geq l.
\]

\(\Box\)

### 8.3. Proof of the Theorem 8.1

**Proof.** Let \(\Sigma\) be a hyperbolic surface with \(b\) boundary components. Suppose the boundary components are \(\gamma_1, \gamma_2, \ldots, \gamma_b\) of length \(l_1, l_2, \ldots, l_b\) respectively. We construct a closed hyperbolic surface by the following:

**Step 1.** For each boundary component \(\gamma_i\) of \(\Sigma\), we consider the hyperbolic surface \(\Sigma_i(G_i)\) with single boundary component as in the Lemma 8.2. We denote the boundary component of \(\Sigma_i(G_i)\) by \(\delta_i\).

**Step 2.** For each \(i = 1, 2, \ldots, b\), we attach the surface \(\Sigma_i(G_i)\) with \(\Sigma\) along the geodesic boundaries \(\gamma_i\) and \(\delta_i\) by the following identification map

\[
\gamma_i(t) = \delta_i(\alpha_i - t), t \in \mathbb{R}.
\]

For each \(i = 1, 2, \ldots, b\), the constant \(\alpha_i \in \mathbb{R}\) is a twist parameter.

We denote the closed surface obtained above by \(S\). Then the surface \(S\) has a natural hyperbolic atlas. Now it remains to show that \(S\) satisfies the conditions of Theorem 8.1.

Suppose \(\gamma\) is a shortest closed geodesic in \(S\). Then we show that \(\gamma \subset \Sigma\) and \(\gamma\) is a shortest geodesic in \(\Sigma\). If \(\gamma\) is not entirely contained in \(\Sigma\) then \(\gamma \cap \Sigma_i(G_i) \neq \phi\) for some \(i = 1, 2, \ldots, b\). Then either \(\Sigma_i(G_i)\) entirely contains \(\gamma\) or a essential sub arc of \(\gamma\) with end points on the boundary of \(\Sigma_i(G_i)\).

If \(\gamma\) is entirely contained in \(\Sigma_i(G_i)\) then it is a simple closed geodesic in \(\Sigma_i(G_i)\). Hence by Lemma 8.2 we have

\[
\text{length}(\gamma) \geq l_i \quad \text{and} \quad l_i > l.
\]
Hence $\text{length}(\gamma) > l$, contradicts that $\gamma$ is a shortest closed geodesic in $S$.

In the remaining case, suppose $\gamma'$ is a sub arc of $\gamma$. Then $\gamma'$ is an essential geodesic arc with end points on the boundary of $\Sigma_i (G_i)$. It follows from the Lemma 8.2 that

$$\text{length}(\gamma') \geq l_i \text{ and } l_i > l.$$ 

Therefore $\text{length}(\gamma) > \text{length}(\gamma') \geq l_i > l$, which contradicts that $\gamma$ is a shortest geodesic in $S$.

Thus we conclude that if $\gamma$ is a shortest closed geodesic in $S$ then it is a shortest closed geodesic in $\Sigma$ as well, which is what we wanted to prove. $\square$

9. Minimum genus for embedding systolic graphs

In this section we prove the following:

**Theorem 9.1.** For each $g \geq 9$, there exist a closed hyperbolic surface $S_g$ such that the systolic graph $SLG(S_g)$ cannot be realized as a systolic graph of a closed surface of genus less than $g$.

Our proof is based on the following result.

**Theorem 9.2** (Theorem 1.1, [1]). There exist a sequence $S_{g_k}$ of genus $g_k \to \infty$ with a filling set of systoles and with Bers constant $> \sqrt{g_k}$.

To prove the Theorem 9.2 [1], for each sufficiently large $g$ the authors constructed a closed hyperbolic surface of whose systolic graph fills the surface. First, arrange $mn$ copies of hyperbolic squares in $m \times n$ rectangular grid. Next, identify the opposite sides of the rectangle in the obvious way and obtain a torus $T(m, n)$ with $mn$ cone points such that each cone angle is $\pi$. It follows from the proof the Theorem 1.1 [1] that, if $mn$ is even, there exists a closed hyperbolic surface of genus $g = \frac{mn + 2}{2}$ such that $T(m, n) = S_g < \sigma >$

where $\sigma$ is a orientation preserving isometry with $2g - 2$ fixed points. Moreover, the systolic graph of $S_g$ fills the surface if $m, n \geq 4$. The condition $m, n \geq 4$ on $m$ and $n$ implies that the genus $g$ is at least 9.

**The proof of the Theorem 9.1.** Let $S_g$ be a closed hyperbolic surface of genus $g$ such that the systolic graph $SLG(S_g)$ fills $S_g$. Then we show that the graph $SLG(S_g)$ cannot be realized as a systolic graph of any hyperbolic surface of genus less than $g$.

Let $F$ be a closed hyperbolic such that the systolic graph $SLG(F)$ is isomorphic to $SLG(S_g)$. We have $S_g = SLG(S_g) \bigcup_{i=1}^{n} D_i$ where $D_i$’s are discs and $n$ is the number of components in $S_g - SLG(S_g)$, the complement of $SLG(S_g)$ in $S_g$. Therefore,

$$\chi(S_g) = \chi(SLG(S_g)) + n.$$ 

Let $F_i, i = 1, 2, \ldots, k$ be the connected components in $F - SLG(F)$ then ($k \leq n$) and we have,

$$\chi(F) = \chi(SLG(F)) + \sum_{i=1}^{k} \chi(F_i).$$
Each surface $F_i$ satisfies $\chi(F_i) \leq 1$. Thus we have

$$\sum_{i=1}^{k} \chi(F_i) \leq n$$

$\Rightarrow \chi(SLG(S_g)) \leq \chi(F)$.

Moreover, equality holds if and only if $n = k$ and each $F_i$ is a disc. In that case $F$ is isometric to $S_g$. Hence the result follows. \qed

We remark that the above result is based on a topological obstruction for a fat graph being embedded in a low genus surface. It would be interesting to know if there are geometric obstruction, i.e., admissible fat graphs that topologically embed in a surface but which cannot be the systolic graph of the surface. Such a result may be based on a lower bound on the shortening of lengths of cycles under rounding, given an upper bound on the injectivity radius (in contrast to our main result being based on an upper bound on shortening of length due to rounding), with admissibility shown using computational tools. We hope to address this question in the future.

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