Lorentz violation in solar-neutrino oscillations

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Abstract

Neutrino oscillations are conventionally accounted for using a three-flavor massive model. This agrees with most experimental data, but fails to accommodate some notable results. A possible explanation is Lorentz violation, which could affect oscillatory behavior. Previous models have used Lorentz violation to describe oscillations with and without neutrino mass. These models have been developed for oscillations in a vacuum or a uniform medium, such as the Earth. However, variable media, such as the Sun, have not been considered.

Solar-neutrino oscillations are studied using a massive model with perturbative Lorentz violation and an approximate model for the composition of the Sun. The adiabatic approximation is used to eliminate high-frequency oscillations and hence obtain analytic expressions for the average probability of detecting each flavor. Lorentz violation is incorporated into the adiabatic results, and changes in energy and directional dependence are considered. Results are more compact than previous work, and are found to be very accurate when compared to exact numerical calculations. The analysis is first carried out for two neutrino flavors and then generalized to include three.

Fits to time-dependent experimental data from the solar-neutrino experiments Super-Kamiokande and the Sudbury Neutrino Observatory are demonstrated, providing approximate values for several combinations of Lorentz-violating coefficients. Most of these results are consistent with zero, but some are not. There is no obvious explanation for this behavior, but it is likely due to uncorrected effects from the eccentricity of Earth’s orbit or some other periodic effect.
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Chapter 1

Introduction

Neutrinos are neutrally charged elementary particles, first proposed by Wolfgang Pauli in 1930 as a means of conserving energy in beta decay [1]. They travel at essentially the speed of light, interact very weakly, and pass through matter virtually unobstructed.

In the Standard Model of particle physics, neutrinos are the neutral leptons. There are three known flavors, or types: electron, muon, and tau neutrinos (the particle postulated by Pauli was in fact the electron neutrino). For many years, neutrinos were assumed to be massless, but data from a variety of experiments have indicated otherwise. Neutrino mass was one of the first examples of physics beyond the Standard Model.

Neutrinos are produced—among other sources—as a byproduct of fusion in the Sun’s core. These solar neutrinos (all of which are electron neutrinos) quickly propagate to the surface and then through space to Earth, where they can be detected. The first solar-neutrino experiment was conducted in 1968 at the Homestake mine in South Dakota [2]. The Homestake results showed about one-third as many neutrinos as had been predicted [3], a discrepancy that became known as the solar-neutrino problem.

Later in 1968, a solution to the solar-neutrino problem was proposed: neutrinos oscillate between flavors as they propagate [4]. The Homestake experiment was only sensitive to electron neutrinos, so if about two-thirds of solar neutrinos had changed flavors, the results could be justified. This model requires neutrinos to have mass—which would indicate new physics beyond the Standard Model—but the Homestake data were insufficient to confirm neutrino oscillations.

It wasn’t until the 1990–2000s that compelling evidence for neutrino oscillations was observed. Data from the solar-neutrino experiments Super-Kamiokande (Super-K) [5] and the Sudbury Neutrino Observatory (SNO) [6] showed that the total solar-neutrino flux agreed with predictions, and that about a third were electron neutrinos, in agreement with the massive oscillation model. The mass model successfully fits most other experiments, but fails in some notable cases. Around the same time as Super-K and SNO, the Liquid Scintillator Neutrino Detector (LSND), an accelerator experiment, reported results that the mass model could not accommodate along with other data. LSND proposed a sterile neutrino—another kind of neutrino that could mix with the known flavors but could not be detected directly—as an explanation [7]. Though there is some evidence for the sterile neutrino, there is another possible resolution: Lorentz violation.

Lorentz violation is the idea that Lorentz symmetry is not exact, meaning the laws of physics are not perfectly invariant under rotations and boosts. If Lorentz symmetry is violated, it could
have a number of important implications, including modification of neutrino oscillations. Indeed, oscillations could be induced by mass, Lorentz violation, or a combination of the two. Lorentz-violating oscillations introduce strange energy dependence, directional dependence, and neutrino-antineutrino oscillations. This makes Lorentz violation an interesting theoretical possibility and a potential explanation of neutrino results.

In this work, solar-neutrino oscillations will be considered using the mass model with perturbative Lorentz violation. Previous work, such as reference [8], have used similar models to study oscillations in a vacuum or uniform matter. However, very little has been done with nonuniform media, such as the Sun. Here, we extend the analysis of [8] to accommodate the Sun’s variable composition. Results will provide a method for analyzing possible Lorentz violation in solar-neutrino data.

The organization of this paper is as follows. Background information necessary for the development and understanding of neutrino oscillations and Lorentz violation is provided in chapter 2. In chapter 3, a complete analysis of oscillations including perturbative Lorentz violation is carried out using a two-flavor model. These results are generalized to include three flavors in chapter 4. In chapter 5 data from Super-K and SNO are fit to results from chapters 3 and 4 and bounds on several combinations of Lorentz-violating coefficients are obtained.
Chapter 2

Background

2.1 Neutrinos and the Standard Model

The Standard Model of particle physics is a theory that describes the fundamental particles and interactions of matter. Matter is made up of elementary particles, divided into two classes called quarks and leptons. Neutrinos are the uncharged leptons. The four fundamental forces—strong, electromagnetic, weak, and gravitational—are attributed to the exchange of mediator particles. This section provides an overview of the Standard Model; for a more comprehensive treatment, see references [9] and [10].

The Standard Model provides for six leptons, three charged and three uncharged. The charged leptons are the electron ($e$), muon ($\mu$), and tau ($\tau$). Neutrinos, the uncharged leptons, come in three flavors: electron neutrinos ($\nu_e$), muon neutrinos ($\nu_\mu$), and tau neutrinos ($\nu_\tau$). They get their names from the charged leptons they interact with. For example, in $\beta^{-}$ decay, a neutron decays to a proton, electron, and electron antineutrino:\footnote{An antiparticle has the same mass as its corresponding normal particle, but the signs of its charge and other quantum numbers are reversed. An antiparticle is denoted by a bar on top, i.e. an electron antineutrino is written $\bar{\nu}_e$.}

$$n \rightarrow p + e^- + \bar{\nu}_e.$$

(2.1)

Similarly, muon neutrinos are observed in muon decay, and tau neutrinos in tau decay:

$$\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu, \quad \tau^- \rightarrow e^- + \bar{\nu}_e + \nu_\tau.$$

(2.2)

All these decays—and in fact all neutrino interactions—are weak force interactions. The weak force is mediated by the intermediate vector bosons, $W^\pm$ and $Z$. The $W^\pm$ carries a charge of ±1, while the $Z$ is neutral. Both are massive and highly unstable, so they decay very quickly. This causes the weak force to have a very short range and to be very weak (hence its name).
Figure 2.1: The Feynman diagram for $\beta^-$ decay. A neutron is converted to a proton via emission of a $W$, which then decays into an electron and electron antineutrino.

We can represent interactions graphically using Feynman diagrams. These diagrams are actually visual representations of terms in an infinite perturbation series, but they are intuitive to understand without addressing the underlying mathematics. The rules of Feynman diagrams are

- Time progresses left to right.
- Fermions are represented by straight lines. Normal particles point left to right; antiparticles point right to left.
- Bosons (mediators) are represented by wavy lines.
- When three lines meet, they form a vertex. Every vertex must conserve energy, momentum, and charge.

For example, figure 2.1 shows the Feynman diagram for $\beta^-$ decay. The diagrams for muon and tau decay are similar.

There are three other neutrino interactions that play an important role in solar-neutrino experiments. The first is elastic scattering (ES), in which a neutrino of any flavor scatters off an electron:

$$\nu_x + e^- \rightarrow \nu_x + e^-, \quad x = e, \mu, \tau.$$ \hspace{1cm} (2.3)

There are two possible Feynman diagrams for ES. One is mediated by a $Z$, and all neutrino flavors can participate equally well. The other is mediated by a $W$, and only $\nu_e$ can participate. The ES cross section is thus greater for $\nu_e$ than for $\nu_{\mu,\tau}$: $\sigma(\nu_{\mu,\tau}e) \approx 0.16 \sigma(\nu_e e)$ \[11\]. Figure 2.2 shows the two Feynman diagrams for ES.

Figure 2.2: Feynman diagrams for the elastic scattering of a neutrino and an electron.
The other two processes involve the scattering of a neutrino off a deuteron (a proton and a neutron). In the charged current process (CC), a $\nu_e$ and a deuteron become two protons and an electron via $W$ exchange:

$$\nu_e + d \rightarrow p + p + e^-.$$  

(2.4)

In the neutral current process (NC), a neutrino of any flavor scatters off a deuteron via $Z$ exchange:

$$\nu_x + d \rightarrow p + n + \nu_x.$$  

(2.5)

Figure 2.3 shows the Feynman diagrams for CC and NC.

The Standard Model describes two other forces: electromagnetic and strong. The electromagnetic force acts on electrically charged particles and is hence responsible for holding atoms together. Its mediator is the photon ($\gamma$), which is massless and uncharged. Photons are stable, so the electromagnetic force is much stronger than the weak force and has infinite range, although its strength decreases with distance squared.

The electromagnetic and weak forces in fact have a unified theory called the electroweak interaction. It states that above energies of about 100 GeV (temperatures of $\sim 10^{15}$ Kelvin), the two forces are unified into a single indistinguishable force—the electroweak force—and all particles are massless. Below this energy, the two forces appear different and particles acquire masses, due to a spontaneous breaking of symmetry\(^2\) called the Higgs mechanism. The Standard Model predicts that the Higgs mechanism should produce a massive spin-zero particle called the Higgs boson. This is the only particle predicted by the Standard Model that has not yet been observed.

Electroweak theory defines a complex Higgs field $\phi$ that is present in all of space and can interact with particles and other fields. The Higgs field has a corresponding potential

$$V = -\frac{1}{2} \mu^2 |\phi|^2 + \frac{1}{4} \lambda |\phi|^4,$$  

(2.6)

where $\mu, \lambda$ are constants. The minima of $V$ are thus the points on a circle of radius $\mu/\lambda$. Potentials like this are commonly known as Mexican hat potentials, as shown in figure 2.4. Above the electroweak unification energy, the bulge at the bottom is negligible. Particles have enough energy to stay at $\phi = 0$, and symmetry remains unbroken. As energy decreases, they fall to some point on the minimum circle, and symmetry is spontaneously broken. When this happens, the $W$, $Z$, and the massive quarks and leptons acquire a mass term.

\(^2\)Spontaneous symmetry breaking is a general term that describes processes in which a symmetric system becomes nonsymmetric. For example, a ball sitting on top of a symmetrical hill has an equal chance of rolling any direction. Once it starts rolling, symmetry is said to be spontaneously broken.
The other Standard Model force is the strong force. It is mediated by gluons ($g$), which don’t have mass or electric charge. They do, however, have a different kind of charge called color. The strong force acts only on colored particles—quarks and gluons. Therefore—in contrast to the other mediators—gluons can couple with each other, which causes the strong force to increase with distance. As a result, quarks do not exist as free particles, but as color-neutral composites, collectively known as hadrons. There are two ways to make a colorless particle: three quarks, called a baryon, or a quark and an antiquark, called a meson.

Quarks come in six flavors: up ($u$), down ($d$), strange ($s$), charm ($c$), bottom ($b$), and top ($t$). Two up quarks and a down quark, for example, make a proton, i.e. $p = uud$. A neutron is $udd$.

The strong force is therefore not only responsible for making protons and neutrons stable, but also for holding them together to form nuclei. It’s stronger than the electromagnetic force, as it must be to overcome electromagnetic repulsion in nuclei.

It’s possible that the strong force could merge with the electromagnetic and weak forces at high enough energies. Theories that propose this kind of unification are called Grand Unified Theories (GUTs), and predict an energy scale of about $10^{16}$ GeV.

There is a fourth force, gravity, but it does not yet have a satisfactory quantum theory. On the other hand, it’s so much weaker than the other forces that this shortcoming is of limited significance. Table 2.1 summarizes the forces and their mediators.

It’s important to note that neutrinos are neutral leptons, and are therefore unaffected by both the electromagnetic and strong forces. They are influenced only by the weak force (and gravity, but this can be neglected), a unique feature among all Standard Model particles. Since the weak force is so much weaker than the other forces, neutrinos interact much less than any of other the particles. This makes them notoriously difficult to detect.

Table 2.1: The four fundamental forces and their mediators [9, 10, 12].

| Force           | Mediator | $m$ / GeV | $Q$  | Affected Particles          |
|-----------------|----------|-----------|------|-----------------------------|
| Strong          | $g$      | 0         | 0    | color charged               |
| Electromagnetic | $\gamma$ | 0         | 0    | electrically charged        |
| Weak            | $W^\pm$  | 80.399    | $\pm 1$ | all                         |
|                 | $Z$      | 91.1876   | 0    | all                         |
| Gravitational   | Graviton | unknown   | unknown | all                         |
2.2 Solar Neutrinos

Electron neutrinos are produced in vast numbers in the Sun’s core via a fusion reaction called the proton-proton chain. This process has a number of steps, cumulatively resulting in protons combining to form \( \alpha \) particles, with neutrinos as byproducts [12]:

\[
4p \rightarrow ^4\text{He} + 2e^+ + 2\nu_e. \tag{2.7}
\]

Neutrinos are produced during several steps, with most coming from the initial combining of two protons to form a deuteron:

\[
p + p \rightarrow d + e^+ + \nu_e. \tag{2.8}
\]

However, these neutrinos have too low energy for most detectors (less than 0.5 MeV). The boron-8 process,

\[
^8\text{B} \rightarrow ^8\text{Be}^* + e^+ + \nu_e, \tag{2.9}
\]

produces far fewer neutrinos, but they have higher energies (up to about 10 MeV), making them much more useful experimentally [9].

John Bahcall et al. calculated solar neutrino abundances and fluxes from several different solar models [13]. Figure 2.5 shows solar neutrino flux from various processes, determined from the BS05(OP) Solar Standard Model (SSM).

The proton-proton chain produces photons as well as neutrinos, but photons interact significantly with matter in the Sun, which means they can take thousands of years to reach the surface. The weakly interacting neutrinos can traverse the same distance at essentially the speed of light, making them ideal for studying the Sun’s interior.
The first solar neutrino experiment was conducted in 1968 using a tank of tetrachloroethylene (C\(_2\)Cl\(_4\)) located in the Homestake mine in South Dakota \([2]\). The detector observed the process
\[
{}^{37}\text{Cl} + \nu_e \rightarrow {}^{37}\text{Ar} + e^-,
\]
which is essentially the charged current scattering (a neutron and \(\nu_e\) are converted to a proton and electron). It requires about 5.15 MeV, so \(^8\text{B}\) neutrinos are the only ones capable of inducing it. At the conclusion of the experiment, about one-third of the predicted number of argon atoms had accumulated \([3]\). This significant discrepancy became known as the solar neutrino problem.

### 2.3 Oscillation Models

#### 2.3.1 The Mass Model

In 1968, Bruno Pontecorvo proposed a possible explanation to the solar neutrino problem: neutrinos oscillate between flavors as they propagate \([4]\). Since the Homestake experiment was only sensitive to electron neutrinos, two-thirds of the total flux could have been undetected muon and tau neutrinos. This model states that the flavors \((\nu_e, \nu_\mu, \nu_\tau)\) are not the eigenstates of the Hamiltonian; instead, there are three energy eigenstates \(\nu_1, \nu_2, \nu_3\). The flavor states are superpositions of the energy states, and vice versa. Oscillations result from mass differences between the energy states.

To see why mass differences cause oscillations, we begin by writing the energy of the \(j\)th energy eigenstate \((\nu_j)\) using the relativistic energy-momentum equation:
\[
E_j = \sqrt{p^2 + m_j^2},
\]
using natural units where \(c = \hbar = 1\). Since neutrinos travel at almost the speed of light, we invoke the ultrarelativistic limit
\[
p \gg m, \quad p \simeq E,
\]
where \(E\) (without a subscript) is the energy of the particle, not of a certain eigenstate. The energy of the \(j\)th state is then
\[
E_j \simeq p + \frac{m_j^2}{2p} \simeq E + \frac{m_j^2}{2E}.
\]
Note that if the masses \(m_j\) are not equal, than the energies \(E_j\) aren’t either. Now, the energy states evolve in time as usual:
\[
\nu_j(t) = \nu_j(0) e^{-iE_j t}.
\]
Provided that the masses—and hence the energies—are not all identical, the states will evolve at different rates. So if a neutrino is in some linear combination of energy states at time \(t = 0\), it will be in a different combination at a later time \(t\). Since the flavor states are linear combinations of the energy eigenstates, the flavor composition will change with time. This is called flavor mixing.

Because oscillations only depend on mass differences, the eigenstates are usually simplified further. We subtract a factor of \(E + m_1^2/2E\) from each energy—this is just an overall phase of no physical significance—so that the energies become
\[
E_1 = 0, \quad E_2 = (m_2^2 - m_1^2)/2E, \quad E_3 = (m_3^2 - m_1^2)/2E.
\]

\(^3\)Neutrinos had been assumed to be massless, but it is possible to introduce mass without breaking the gauge invariance of the Standard Model \([14]\)—see the next section, 2.3.2.
We then define the mass-squared splittings as
\[ \Delta m_{jk}^2 \equiv m_k^2 - m_j^2. \]

Note that although there are three splittings (\(\Delta m_{12}^2, \Delta m_{23}^2, \Delta m_{13}^2\)), only two are independent since \(\Delta m_{13}^2 = \Delta m_{12}^2 + \Delta m_{23}^2\). Figure 2.6 shows an energy-level diagram. In terms of the splittings, the energies are simply:
\[ E_1 = 0, \quad E_2 = \Delta m_{12}^2/2E, \quad E_3 = \Delta m_{13}^2/2E. \]

As it turns out, the splittings that neutrino oscillation experiments can currently measure are \(\Delta m_{12}^2\) and \(\Delta m_{23}^2\). The most recent experimental values for the measurable splittings are \(^{12}\)
\[ \Delta m_{12}^2 = (7.59 \pm 0.20) \times 10^{-5} \text{eV}^2, \quad \Delta m_{23}^2 = (2.43 \pm 0.13) \times 10^{-3} \text{eV}^2. \] (2.15)

Since \(\Delta m_{23}^2 \sim 32 \Delta m_{12}^2\), we usually assume \(\Delta m_{23}^2 \sim \Delta m_{13}^2\). The absolute neutrino masses do not affect oscillations, so they have to be measured some other way. Currently, only upper bounds exist. They are \(^{12}\)
\[ m_{\nu_e} < 2 \text{ eV}, \quad m_{\nu_{\mu}} < 0.19 \text{ MeV}, \quad m_{\nu_{\tau}} < 18.2 \text{ MeV}. \] (2.16)

As we’ll see in chapters 3 and 4, the oscillation frequencies depend on several parameters: the mass splittings; the distance between the neutrino source and detector, \(L\), usually called baseline; and the neutrino energy, \(E\). Therefore, in order to effectively test the model and determine \(\Delta m_{jk}^2\), experiments must observe different combinations of \(L\) and \(E\).

The oscillation amplitudes are described by a different set of parameters, the mixing angles \(\theta_{12}, \theta_{23}, \theta_{13}\). A mixing angle of 45° implies maximal mixing, while 0° means no mixing. The most recent experimental values are \(\theta_{12} \simeq 34^\circ, \theta_{23} \simeq 45^\circ, \theta_{13} \simeq 0^\circ\) \(^{12}\). The mixing angles quantify how much of each flavor makes up each energy state (and vice versa). This requires some more information about exactly how the flavors mix, which we’ll see chapters 3 and 4. For now, the results are shown in figure 2.7.

There is a complication for solar neutrinos: they have to propagate through the Sun. Although they interact very weakly, they sometimes interact with electrons, protons, and neutrons in the

\(^{4}\)This is simply because oscillations due to \(\Delta m_{12}^2\) and \(\Delta m_{23}^2\) dominate those due to \(\Delta m_{13}^2\). Future experiments could be sensitive enough to measure \(\Delta m_{13}^2\) separately.
Figure 2.7: Composition of the energy and flavor states using experimental values for the mixing angles. In the energy states, \( \nu_e \) content is blue, \( \nu_\mu \) is green, and \( \nu_\tau \) is red; in the flavor states, \( \nu_1 \) is black, \( \nu_2 \) is dark grey, and \( \nu_3 \) is light grey.

Sun via the ES, CC, and NC scatterings. This effectively changes the mixing angles, leading to modified oscillations. In addition, the Sun changes composition with radius, so the effective mixing angles change as neutrinos propagate. Matter effects like these are known as the MSW effect, for the physicists who first studied it (Mikheev, Smirnov, and Wolfenstein). Wolfenstein published an early paper on the MSW effect in 1978 \[15\]. In 1986, Mikheev and Smirnov published a paper specifically about oscillations in variable media \[16\].

On the other hand, it turns out that \( \nu_\mu \) and \( \nu_\tau \) behave almost identically for solar neutrinos. It’s often an excellent approximation to combine \( \nu_\mu \) and \( \nu_\tau \) into a single flavor, and consider only two flavors, which significantly simplifies much of the theory. The two-flavor case will be considered in chapter 3 and a three-flavor analysis will be carried out in chapter 4.

### 2.3.2 Sterile Neutrinos and the Seesaw Mechanism

The mass model agrees with most neutrino data, but fails to accommodate some important results—most notably the LSND anomaly (see section 2.4.3). This has led to various alternatives and modifications to the mass model, including Lorentz violation (see section 2.5) and sterile neutrinos.

Before sterile neutrinos can be discussed, we must address the question of neutrino mass for a moment. Though the Standard Model originally assumed neutrinos were massless, it is possible to introduce mass without causing any problems. The question is whether neutrinos are Dirac or Majorana particles. Dirac particles are different from their antiparticles, and their mass originates from couplings between left- and right-handed particles. Majorana particles are identical to their antiparticles, and their mass is due to either left-left or right-right couplings.

Since neutrinos are neutral, they could be Dirac or Majorana fermions. But current neutrino data does not allow us to determine which \[12\], so we must assume they have both kinds of mass. The interesting thing is that all observed neutrinos are left-handed \[17\], while all observed antineutrinos are right-handed \[18\]. This has led to the notion of right-handed sterile neutrinos—meaning they could mix with the left-handed flavors but can’t be detected. Sterile neutrinos could be light (around the same mass as known neutrinos) or heavy.

There is a conveniently simple model, the seesaw mechanism, that incorporates both kinds of neutrinos and could account for the small masses of left-handed neutrinos. Using the basis \( \{ \nu_L, \nu_R \} \) for left- and right-handed neutrinos, the mass matrix for all neutrino types is

\[
m = \begin{pmatrix} 0 & D \\ D & R \end{pmatrix}, \tag{2.17}\]

\[A\ right-handed\ particle\ has\ positive\ spin\ along\ its\ direction\ of\ propagation,\ while\ a\ left-handed\ particle\ has\ negative\ spin.\]
where $D$ and $R$ are the Dirac and right-handed Majorana mass matrices $[12, 19, 20]$. Left-handed Majorana coupling is neglected because it is not compatible with Standard Model electroweak-gauge invariance. Dirac and right-handed Majorana couplings, on the other hand, do not cause any such problems $[14]$. The eigenvalues of the mass matrix are

$$\lambda_{\pm} = \frac{R \pm \sqrt{R^2 + 4D^2}}{2}.$$  \hfill (2.18)

The larger eigenvalue is the sterile neutrino mass, and the smaller eigenvalue is the left-handed neutrino mass. If we take $D$ to be electroweak scale (100 GeV) and $R$ to be GUT scale ($10^{16}$ GeV), we have $R \gg D$, and the eigenvalues simplify to

$$m_{\nu_R} = |\lambda_+| \simeq R, \quad m_{\nu_L} = |\lambda_-| \simeq D^2/R.$$  \hfill (2.19)

This gives a right-handed neutrino mass of approximately $10^{16}$ GeV and a far smaller left-handed mass, about $10^{-3}$ eV. Also note that if one mass increases, the other must decrease, hence the name seesaw mechanism.

The sterile neutrino has been used to explain some experimental results, and there is some evidence for its existence. In this work, sterile neutrinos will not be considered; instead, Lorentz violation will be used to modify oscillations.
2.4 Neutrino Experiments

2.4.1 Types of Experiments

Since the Homestake result, an array of experiments have attempted to confirm the original discrepancy, test the mass model, and measure the mass splittings and mixing angles. These experiments complement each other by detecting neutrinos produced by a variety of sources, which have different combinations of $L$ and $E$. The most common types of experiments are solar, atmospheric, reactor, and accelerator.

Solar-neutrino experiments detect neutrinos from the Sun. The detectors are large, underground structures, often located at the bottom of mines. By placing them far underground, cosmic rays and other background particles are filtered out simply by interacting with the ground, while neutrinos pass through. The detectors must also be large enough to observe a significant number of events. Solar neutrinos have low energies and a very long baseline. As we’ll see later, this means the smaller mass difference $\Delta m_{12}^2$ dominates oscillations, so it is often called $\Delta m_{\odot}^2$.

Atmospheric neutrinos are produced by cosmic rays colliding with air molecules in the upper atmosphere. The resulting decays yield $\nu_e, \bar{\nu}_e, \nu_\mu$, and $\bar{\nu}_\mu$. These neutrinos have high energies and long baselines. In contrast to solar neutrinos, the larger mass difference dominates here, so $\Delta m_{23}^2$ is often called $\Delta m_{\text{atm}}^2$.

Reactor neutrinos are produced in nuclear reactors from $\beta^-$ decay. Energies are low. Baseline depends on how far away the detector is from the reactor, which varies across experiments. Regardless, baselines are much shorter than solar or atmospheric experiments.

Accelerator experiments generate their own neutrinos by inducing muon decays to produce $\nu_\mu$ or $\bar{\nu}_\mu$. A neutrino beam is then formed and detected some distance away. Accelerator neutrinos have high energies and, like reactor experiments, long or short baselines. Table 2.2 summarizes the properties of each class of experiment.

Table 2.2: General properties of neutrino experiments [12]. Energies and baselines are approximate.

| Experiment               | Type of $\nu$ | $\langle E \rangle$ / MeV | $L$ / km |
|--------------------------|---------------|---------------------------|----------|
| Reactor (short baseline) | $\bar{\nu}_e$ | 1                         | 1        |
| Reactor (long baseline)  | $\bar{\nu}_e$ | 1                         | 100      |
| Accelerator (short baseline) | $\nu_\mu, \bar{\nu}_\mu$ | $10^3$                  | 1        |
| Accelerator (long baseline) | $\nu_\mu, \bar{\nu}_\mu$ | $10^3$                  | $10^3$   |
| Atmospheric              | $\nu_{e,\mu}, \bar{\nu}_{e,\mu}$ | $10^3$                  | $10^4$   |
| Solar                    | $\nu_e$       | 1                         | $1.5 \times 10^8$ |
2.4.2 Solar-Neutrino Experiments

Two of the most important solar-neutrino experiments are Super-Kamiokande and the Sudbury Neutrino Observatory. Super-Kamiokande (Super-K), located in Hida-city, Japan, provided some of the first evidence for neutrino oscillations. Unlike the Homestake experiment, Super-K uses water in its detector in order to detect neutrinos via elastic scattering. Since ES is more sensitive to $\nu_e$ than $\nu_{\mu,\tau}$, the ES flux has the form

$$\phi_{ES} = \phi_e + \epsilon \phi_{\mu\tau}, \quad \epsilon \simeq 0.16.$$  \hspace{1cm} (2.20)

Super-K collected data for 1258 days from 1996 to 2000, and detected about 45% of the predicted neutrino flux from the SSM [5]. These data provided some evidence for the oscillation model, since they suggested that some $\nu_e$ had converted to $\nu_\mu$ or $\nu_\tau$. However, without knowing how many, the oscillation model could not be confirmed.

In 2002, results from the Sudbury Neutrino Observatory (SNO) in Sudbury, Ontario, Canada provided direct evidence for neutrino flavor change. SNO uses heavy water ($D_2O$) in its detector, which allows charged current and neutral current to be used in addition to elastic scattering. SNO was thus able to detect all neutrino flavors both separately and together. Their data showed that the total neutrino flux agreed with the SSM prediction, and that only about 35% was $\nu_e$ [6]. This explained Homestake and Super-K’s results and confirmed the oscillation model. Figure 2.8 shows SNO and Super-K’s results for the fluxes $\phi_e$ and $\phi_{\mu\tau}$ along with the prediction of the SSM.

![Figure 2.8: Solar neutrino fluxes from SNO, Super-K, and the SSM. The bands are 1σ error bars. The dot represents the calculated value for $\phi_e$ and $\phi_{\mu\tau}$, and the contours around it represent 1, 2, and 3σ errors. Figure taken from [12].]
2.4.3 The LSND Anomaly

The Liquid Scintillator Neutrino Detector (LSND) was a short-baseline accelerator neutrino experiment that was operational from 1993–1998. It produced neutrinos via the decays \( \pi^+ \rightarrow \mu^+ \nu_\mu \) and \( \mu^+ \rightarrow e^+ \nu_e \bar{\nu}_\mu \). A small excess of \( \bar{\nu}_e \) was detected via the process \( \bar{\nu}_e p \rightarrow e^+ n \). This was attributed to \( \bar{\nu}_\mu \rightarrow \bar{\nu}_e \) oscillations with an oscillation probability of about 0.26%, which suggests a mass difference in the range 0.2–10 eV\(^2\) and a neutrino mass greater than 0.4 eV\(^2\) [7]. A mass difference of this size is incompatible with existing results, as it is larger than the sum of the other two mass differences, \( \Delta m_{12}^2 \sim 10^{-5} \) eV\(^2\) and \( \Delta m_{23}^2 \sim 10^{-3} \) eV\(^2\).

Assuming the LSND result is correct, the conventional three-flavor massive model cannot accommodate all neutrino oscillation data. LSND therefore proposed a light sterile neutrino as a justification for their results [7]. However, data from various cosmological experiments bound the mass of the sterile neutrino to \( m_s < 0.26 \) eV (0.44 eV) at 95% (99%) confidence level, which is incompatible with the LSND result [21].

MiniBooNE (Mini Booster Neutrino Experiment) is an experiment at Fermilab that was designed to unambiguously confirm or refute LSND results. Initial experiments ran in neutrino mode, searching for \( \nu_\mu \rightarrow \nu_e \) mixing at the \( \Delta m^2 \sim 1 \) eV\(^2\) scale. No excess \( \nu_e \) events were observed. If neutrino oscillations are the same as antineutrino oscillations, this result excludes the \( \bar{\nu}_\mu \rightarrow \bar{\nu}_e \) explanation of the LSND anomaly at 98% CL [22].

MiniBooNE then switched to antineutrino mode to search for \( \bar{\nu}_\mu \rightarrow \bar{\nu}_e \) oscillations at the \( \Delta m^2 \sim 1 \) eV\(^2\) scale. No significant excess of \( \bar{\nu}_e \) events were observed. These results were inconclusive with respect to LSND [23].

Recent results, published on 26 October 2010, show an excess of \( \bar{\nu}_e \) results. These data give a mass difference of 0.1–1.0 eV\(^2\) for \( \bar{\nu}_\mu \rightarrow \bar{\nu}_e \) oscillations, consistent with LSND. MiniBooNE plans to continue running in antineutrino mode, and expects to double the current amount of data [24].
2.5 Lorentz and CPT Violation

Lorentz and CPT symmetry are fundamental physical laws of nature. A system is Lorentz invariant if its physical laws are unaffected by a Lorentz transformation, which can either be a boost (change in velocity) or a rotation. Similarly, CPT symmetry means a CPT transformation does not affect a system’s physics. CPT is an abbreviation for charge conjugation (C), which converts a particle into its antiparticle; parity (P), which transforms a system to its mirror image; and time reversal (T), which reverses the direction of time. Only CPT is a symmetry; C, P, T, and CP can all be broken independently [9]. Lorentz and CPT violation are the ideas that these symmetries may not be exact. The CPT theorem states that CPT violation implies Lorentz violation, but the reverse doesn’t necessarily hold [25].

Lorentz and CPT violation are interesting theoretical possibilities, and could provide explanations of observations such as the LSND anomaly. However, Lorentz and CPT symmetry are integral to most physical theories. A theory that allows violations of these symmetries should do so while remaining compatible with known theories, such as the Standard Model.

2.5.1 The Standard-Model Extension

The Standard-Model Extension (SME) was developed by Alan Kostelecky et al. as a theoretical framework for quantifying Lorentz and CPT violation. It is an effective field theory that couples the Standard Model and General Relativity to allow for violation of Lorentz and CPT symmetry. The SME provides for a number of coefficients which must be determined by experiment [26]. In recent years, a number of searches for Lorentz and CPT violation have been carried out; their results are used to place upper bounds on various SME coefficients. Experiments include studies of

- neutrino oscillations
- neutral-meson oscillations
- clock comparisons
- spin-polarized torsion pendulums
- astrophysical tests such as cosmological birefringence
- optical and microwave resonators

Results and publications are compiled in the Data Tables for Lorentz and CPT violation [27].

The SME works by adding coefficients to existing equations. For example, consider the Dirac equation for a single fermion:

\[(i \gamma^\beta \partial_\beta - m) \psi = 0, \tag{2.21}\]

where \( \gamma^\beta \) are 4 \( \times \) 4 matrices that generalize the Pauli matrices, and \( m \) is mass. In the SME, this equation becomes

\[(i \Gamma^\beta \partial_\beta - M) \psi = 0, \tag{2.22}\]

where \( \Gamma^\beta \) and \( M \) are Lorentz and CPT-violating generalizations of \( \gamma^\beta \) and \( m \):

\[\Gamma^\beta \equiv \gamma^\beta + c^{\alpha\beta} \gamma_\alpha + d^{\alpha\beta\gamma} \gamma_\alpha, \quad M \equiv m + a^\alpha \gamma_\alpha + b^\alpha \gamma_\beta \gamma_\alpha. \tag{2.23}\]

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There are additional terms in $\Gamma$ and $M$, but they are not relevant to this work. The matrix $\gamma_5$ is the product of the other four $\gamma$ matrices, and the coefficients $a, b, c, d$ are SME coefficients. For neutrinos, $a, b$ are combined to form a coefficient $a_L$, while $c, d$ combine to form $c_L$:

$$a_L^\alpha \equiv (a + b)^\alpha, \quad c_L^{\alpha\beta} \equiv (c + d)^{\alpha\beta}.$$  

(2.24)

Here, the subscript $L$ means left-handed; there are different coefficients for right-handed neutrinos.

### 2.5.2 Lorentz Violation in Neutrino Oscillations

Both neutrino mass and Lorentz violation can induce oscillations. Lorentz-violating oscillations may exhibit strange energy dependence, directional dependence, and neutrino-antineutrino oscillations. Previous models include massless Lorentz-violating models, hybrid models where some neutrinos have mass, and massive models where all neutrinos have mass.

Reference [8] uses a massive model with perturbative Lorentz violation to study neutrino oscillations in a vacuum or uniform medium. In this work, a similar model is used for solar neutrinos. This requires a somewhat modified approach, as the Sun’s composition is nonuniform. Lorentz-violating energy and directional dependence will be considered to first order. Neutrino-antineutrino oscillations are a second-order effect for solar neutrinos, and will not be addressed here. Directional dependence is particularly noteworthy for several reasons:

- If Lorentz symmetry were not violated, oscillations would still have energy dependence, but would not have directional dependence. Any directional dependence at all would imply Lorentz violation.

- It’s difficult to obtain precise energy-dependent data for solar neutrinos, but trivial to measure directional dependence. The solar-neutrino propagation direction naturally changes with the time of year due to Earth’s orbit, so for solar-neutrino experiments, time-dependent data is effectively equivalent to direction-dependent data.

In chapter 5, time-dependent data from Super-K and SNO will be used to look for direction-dependent Lorentz violation.

Lorentz violation is another possible resolution—instead of the sterile neutrino—to the LSND anomaly. LSND’s results could have been affected by Lorentz-violating energy dependence, directional dependence, or neutrino-antineutrino mixing. In particular, the $\bar{\nu}_e$ appearance could have been due to $\nu_e \rightarrow \bar{\nu}_e$ mixing rather than $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ [14].
Chapter 3

Two-Flavor

This chapter begins the analysis of solar-neutrino oscillations using a two-flavor model. This is an excellent approximation for low energy, long baseline neutrinos, such as solar neutrinos. Muon and tau neutrinos behave very similarly, so they are grouped into a single flavor. This significantly simplifies much of the theory and makes the results easier to understand. Results will be generalized to include three flavors in chapter 4.

Section 3.1 develops the basic theory of massive neutrino oscillations in a vacuum. Matter effects due to the Sun are considered in section 3.2. Lorentz violation is added as a small perturbative correction to the massive model in section 3.3.

3.1 Basic Theory

Neutrinos oscillate because their flavor states are not eigenstates of the Hamiltonian, they are superpositions of mass eigenstates. We can write a neutrino’s state in either the flavor or mass basis, and a given state in either basis can be written as a linear combination of the other’s eigenstates. As neutrinos propagate, the mass states evolve in time at different rates, so their flavor composition changes. This is known as flavor mixing.

To transform between the two bases, we use the unitary mixing matrix $U$. We’ll define $U$ to be the matrix that transforms from flavor to mass, and its conjugate $U^\dagger$ to be the matrix that transforms from mass to flavor. Let $\nu$ be the two-dimensional neutrino state vector, and let a prime denote the mass basis. We can then write

$$\nu' = U\nu, \quad \nu = U^\dagger\nu'$$  \hspace{1cm} (3.1)

or as sums,

$$\nu'_j = \sum_a U_{ja} \nu_a, \quad \nu_a = \sum_j U^*_{ja} \nu'_j.$$  \hspace{1cm} (3.2)

In the two-flavor case, $U$ is a standard rotation matrix, parameterized by the mixing angle $\theta$:

$$U = \begin{pmatrix} c & -s \\ s & c \end{pmatrix},$$  \hspace{1cm} (3.3)
where \( s \equiv \sin \theta, \ c \equiv \cos \theta \). Writing the mixing matrix in terms of sines and cosines ensures that the neutrino states \( \nu, \nu' \) are always normalized. In general, \( U \) also has a phase, but it can be absorbed into the neutrino states.

The mass basis Hamiltonian is the diagonal matrix of the eigenvalues, \( E_1 \) and \( E_2 \):

\[
H'_{jk} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}_{jk}.
\]  

(3.4)

Assuming the ultrarelativistic limit, we have

\[
t \simeq L, \quad p \gg m, \quad p \simeq E,
\]  

(3.5)

where \( E \) (without a subscript) is the neutrino energy, not an eigenvalue. Putting this approximation into the relativistic energy-momentum equation, the energy of the \( j \)th eigenstate is

\[
E_j = \sqrt{p^2 + m_j^2} \simeq p + \frac{m_j^2}{2p} \simeq E + \frac{m_j^2}{2E}.
\]  

(3.6)

The Hamiltonian is then

\[
H'_{jk} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}_{jk} + \frac{1}{2E} \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}_{jk}.
\]  

(3.7)

We can subtract a factor of \( E + m_j^2/2E \) off the diagonal—it’s just an overall phase of no physical significance. This gives

\[
H'_{jk} = \frac{1}{2E} \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}_{jk} - \frac{1}{2E} \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}_{jk},
\]  

(3.8)

where \( \Delta m^2 = m_1^2 - m_2^2 \) is the mass splitting. The quantity \( \Delta m^2/2E \) will occur frequently, so we’ll define the abbreviation

\[
\Lambda \equiv \frac{\Delta m^2}{2E}.
\]  

(3.9)

The Hamiltonian can be transformed to the flavor basis by surrounding it with the mixing matrix:

\[
H_{ab} = (U^\dagger H'U)_{ab} = \sum_{j,k} U^*_{ja} U_{kb} H'_{jk} = \Lambda \begin{pmatrix} s^2 & sc \\ sc & c^2 \end{pmatrix}_{ab}.
\]  

(3.10)

Note the indices used so far. \( a, b \) are flavor indices (summed over \( e, \mu \)); specifically, \( a \) is the initial flavor (always \( e \) for solar neutrinos) and \( b \) is the final flavor. \( j, k \) are energy indices (summed over 1, 2). This convention will be used for the remainder of this work.

The mass splitting and mixing angle are the only parameters in the two-flavor case. Their approximate experimental values are \([12]\)

\[
\Delta m^2 \simeq 7.6 \times 10^{-17} \text{ MeV}^2, \quad \theta \simeq 34^\circ.
\]

Note that \( \Delta m^2 \) is given in MeV\(^2\), rather than eV\(^2\) or GeV\(^2\). Megaelectronvolts are the most convenient unit of energy for solar neutrinos, and will be used as the natural unit for the remainder of this work.
We’ll now derive the probability of detecting each neutrino flavor. Say a $\nu_e$ is produced at $t = L = 0$, so the initial state in the flavor basis is $\nu_0 = (1 \ 0)$. To add time dependence, we transform to the mass basis and multiply by $e^{-iH't}$:

$$\nu' = \nu'(t) = U\nu_0 e^{-iH't} \approx \begin{pmatrix} \cos \theta \\ \sin \theta e^{-i\Delta m^2 L/2E} \end{pmatrix}, \quad (3.11)$$

where we’ve again used the ultrarelativistic limit to approximate $t \approx L$. To obtain the time-dependent probability of detecting each flavor, we transform back to the flavor basis and take the absolute square of each component of the flavor state. So the probability of a $\nu_e$ oscillating into a $\nu_\mu$ is

$$P_{\nu_e \rightarrow \nu_\mu} = |\nu_\mu|^2 = |(U^\dagger \nu')_\mu|^2 = \sin^2(2\theta) \sin^2 \left( \frac{\Delta m^2 L}{4E} \right), \quad (3.12)$$

which is a simple sinusoidal oscillation with amplitude $\sin^2(2\theta)$ and frequency $\Delta m^2 L/4E$. Note that both the mixing angle and the mass splitting must be nonzero for oscillation to occur. The electron neutrino survival probability is just $P_{\nu_e \rightarrow \nu_e} = 1 - P_{\nu_e \rightarrow \nu_\mu}$. Figure 3.1 shows the probabilities as functions of $L/E$.

![Figure 3.1: Probabilities of detecting $\nu_e$ (blue) and $\nu_\mu$ (green) as functions of $L/E$. Note that though oscillation does occur, the average probabilities remain constant.](image)

### 3.2 Matter Effects

We now consider neutrinos in the Sun. This complicates matters, because neutrinos interact with other particles as they propagate from the core to the surface, which changes the Hamiltonian and hence affects oscillations. Further, the composition of the Sun varies with distance from the core, so the Hamiltonian isn’t constant and exact analytic solutions can’t be found. The method we used for vacuum oscillations isn’t going to work—we’ll need a different approach.

It’s worth taking a moment to consider what the goal is here. Solar neutrinos are not produced at a single point, but throughout a large volume in the Sun’s core. This means every neutrino detected on Earth will have traveled a different distance through the Sun. The relevant experimental quantities are therefore the average probabilities—the oscillation amplitude and frequency aren’t detectable and hence unnecessary for data analysis. It doesn’t matter if that information is buried
in the solution, so long as we get the averages out in the end. And, as we saw in figure 3.1, the averages conveniently don’t change in a vacuum, so we just need the averages at the Sun’s surface. They should be functions of energy, since the Hamiltonian is energy dependent and the Sun produces neutrinos over a range of energies.

There are a couple of possible methods for obtaining the desired result:

- Solve the Schrödinger equation numerically at many different energies, then calculate moving averages of the results. This would give exact solutions, but no analytic expressions. It would also be very computationally expensive.

- Use an approximation. The adiabatic approximation assumes that the Hamiltonian changes sufficiently slowly that neutrinos do not change energy eigenstates as they propagate \[28\]. If this is valid, we could calculate simple analytic expressions for the averages, which would certainly be preferable to numerical results.

We’ll do both and compare them. First, we need to be more specific about the Sun’s effect on oscillations.

The relevant interactions are the charged current (CC) and neutral current (NC) scatterings. NC affects all three flavors equally, so it just introduces an overall phase shift, which we can disregard. CC only affects electron neutrinos, so we must consider its effect \[15\]. The potential due to CC is \[28\]

\[
V \equiv \sqrt{2} G_F n_\nu,
\]

where \(G_F\) is the Fermi coupling constant and \(n_\nu\) is the number density of electrons in the Sun. The idea here is that the temperature and density of the Sun—which depend on radius—affect the concentration of electrons, which in turn affects neutrino oscillations. A position-dependent approximation of the electron number density is given by the decaying exponential

\[
n_\nu/N_A = 245 e^{-10.54 R/R_\odot} \text{ cm}^{-3},
\]

where \(N_A\) is Avogadro’s number, \(R\) is the distance from the Sun’s core, and \(R_\odot\) is the Sun’s radius \[29\]. The ratio \(R/R_\odot\) will appear often, so it’s convenient to define the dimensionless variable

\[
r \equiv R/R_\odot.
\]

In natural units, the matter potential is

\[
V = V(r) = 1.87 \times 10^{-17} e^{-10.54 r} \text{ MeV}.
\]

Adding the matter potential, the Hamiltonian becomes

\[
H = H(r) = \Lambda \begin{pmatrix} s^2 & sc \\ sc & c^2 \end{pmatrix} + \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} = \Lambda \begin{pmatrix} s^2 + V/\Lambda & sc \\ sc & c^2 \end{pmatrix}.
\]

Since the Hamiltonian is now a function of \(r\), the matrices that diagonalize it must be as well. The mass basis Hamiltonian is the diagonal matrix of the \(r\)-dependent eigenvalues of \(3.17\):

\[
H' = H'(r) = \begin{pmatrix} E_-(r) & 0 \\ 0 & E_+(r) \end{pmatrix},
\]

(3.18)
where

$$E_{\pm} = \frac{1}{2} \left( \Lambda + V \pm \sqrt{\Lambda^2 - 2\Lambda V \cos 2\theta + V^2} \right).$$

These expressions are the matter-dependent analogs of the vacuum eigenvalues, 0 and $\Delta m^2/2E$. When $V = 0$, $E_-$ reduces to zero and $E_+$ to $\Lambda = \Delta m^2/2E$, as they must. The difference between the eigenvalues will appear frequently, so we’ll also define

$$\Delta E \equiv E_+ - E_- = \sqrt{\Lambda^2 - 2\Lambda V \cos 2\theta + V^2},$$

which reduces to $\Lambda$ when $V = 0$. Figure 3.2 shows the eigenvalues at $E = 5$ and 50 MeV.

The mixing matrix $U$ is also a function of $r$—no longer parameterized by the constant mixing angle $\theta$, but by an effective matter mixing angle $\theta_m(r)$:

$$U = U(r) = \begin{pmatrix} c_m(r) & -s_m(r) \\ s_m(r) & c_m(r) \end{pmatrix},$$

where $s_m \equiv \sin \theta_m(r)$, $c_m \equiv \cos \theta_m(r)$ are the sine and cosine of the effective mixing angle. The simplest way to find $s_m, c_m$ is to write the Hamiltonian in two ways and solve the resulting equations. We already know one form—it’s just the matter potential added to the vacuum Hamiltonian:

$$H = \Lambda \begin{pmatrix} s^2 + V/\Lambda & sc \\ sc & c^2 \end{pmatrix}.$$  

The second expression is obtained by surrounding the mass basis Hamiltonian with the mixing matrix:

$$H = U^\dagger H' U = \begin{pmatrix} c_m & s_m \\ -s_m & c_m \end{pmatrix} \begin{pmatrix} E_- & 0 \\ 0 & E_+ \end{pmatrix} \begin{pmatrix} c_m & -s_m \\ s_m & c_m \end{pmatrix}$$

$$= \begin{pmatrix} s_m^2 E_+ + c_m^2 E_- & s_m c_m \Delta E \\ s_m c_m \Delta E & s_m^2 E_+ + c_m^2 E_- \end{pmatrix}.$$  

Figure 3.2: The matter-dependent eigenvalues $E_+$ (blue) and $E_-$ (green) and the vacuum eigenvalues (dashed) at $E = 5$ MeV (left) and 50 MeV (right). Note the different $y$-axes on the two plots. As energy increases, the matter potential becomes large compared to the vacuum Hamiltonian, hence the eigenvalues deviate from the vacuum values more. The effect diminishes at large $r$, where the matter potential becomes negligible.
Equating (3.23) with (3.22) gives three equations,

\[ c_m^2 E_+ - s_m^2 E_- = \Lambda s^2 + V, \quad s_m^2 E_+ + c_m^2 E_- = \Lambda c^2, \quad s_m c_m \Delta E = \Lambda s c, \]  

(3.24)

which are solved to give

\[ s_m^2 = \frac{\Lambda (s^2 - c^2) + V + \Delta E}{2 \Delta E}, \quad c_m^2 = \frac{\Lambda (c^2 - s^2) - V + \Delta E}{2 \Delta E}, \quad s_m c_m = \frac{\Lambda}{\Delta E} s c. \]  

(3.25)

Note that in a vacuum, \( V = 0 \) and \( \Delta E = \Lambda \), so all these expressions reduce to their vacuum values, as they must. Figure 3.3 shows \( s_m^2 \) and \( c_m^2 \) at \( E = 5 \) and 50 MeV.

### 3.2.1 Exact Solution

Exact solutions are obtained by numerically solving the Schrödinger equation. Since neutrinos interact so weakly, they propagate at essentially the speed of light, even in the Sun. We can thus approximate \( t \approx R = r R_\odot \), so the differential equation becomes

\[ \frac{d\nu}{dt} = -iH\nu \implies \frac{d\nu}{dr} = -iR_\odot H\nu. \]  

(3.26)

We encountered precision problems attempting to solve (3.26) directly, so we used an alternate method, outlined below. Later, this will also provide a convenient way of invoking the adiabatic approximation.

If \( H \) is constant, we already know the solution from equation (3.11):

\[ \nu = U^\dagger \nu' = U^\dagger e^{-iH' t} U \nu_0 \approx U^\dagger e^{-iR_\odot H' r} U \nu_0. \]  

(3.27)

This can be generalized for \( r \)-dependent \( H' \) and \( U \) by replacing the quantity in the exponential, \( R_\odot H' r \), with the integral

\[ \phi = R_\odot \int_0^r H' dr, \]  

(3.28)
where $H'$ is the $2 \times 2$ diagonal matrix of the $r$-dependent eigenvalues. If $H'$ is not $r$-dependent, the integral $\phi$ reduces to the old expression. We also need to replace $U\nu_0$ with an arbitrary $r$-dependent two-component vector $A$. The solution then takes the form

$$\nu = U^\dagger e^{-i\phi}A. \quad (3.29)$$

The integral $\phi$ is easily calculated, so our task is reduced to finding $A$. We can obtain a differential equation for $A$ by differentiating both sides of equation (3.29)

$$\frac{d\nu}{dr} = \frac{d}{dr}(U^\dagger e^{-i\phi}A) = \dot{U}^\dagger e^{-i\phi}A - iU^\dagger \dot{\phi} e^{-i\phi}A + U^\dagger e^{-i\phi}\dot{A}. \quad (3.30)$$

The left-hand side of this equation can be replaced by the Schrödinger equation (3.26). We still need to introduce $A$ somehow. First, rewrite $H$ as its diagonalization, $U^\dagger H'U$, then note that (3.29) can be rearranged to give $U\nu = e^{-\phi}A$. So the left-hand side becomes

$$\frac{d\nu}{dr} = -iR \circ H\nu = -iR \circ U^\dagger H'U\nu = -iR \circ U^\dagger H' e^{-i\phi}A. \quad (3.31)$$

Meanwhile, on the right-hand side, $\dot{\phi}$ can be obtained from (3.28),

$$\dot{\phi} = R \circ H'. \quad (3.32)$$

Putting (3.31) and (3.32) into (3.30), we obtain

$$-iR \circ U^\dagger H' e^{-i\phi}A = \dot{U}^\dagger e^{-i\phi}A - iR \circ U^\dagger H' e^{-i\phi}A + U^\dagger e^{-i\phi}\dot{A}. \quad (3.33)$$

Cancelling the term common to both sides and rearranging, we obtain the desired differential equation for $A$,

$$\dot{A} = -e^{i\phi}U\dot{U}^\dagger e^{-i\phi}A. \quad (3.34)$$

This can be solved numerically for $A$, and the result substituted into (3.29). Note that if $\dot{U}$ is small, $\dot{A}$ is correspondingly small. Since $U$ only changes with a decaying exponential, it should evolve slowly, hence $A$ should as well.

Numerical solutions were calculated at 10000 logarithmically spaced energies from 0.1 to 100 MeV; figure 3.4 shows a pair of example results at $E = 5$ and 50 MeV. The value of each solution at $r = 1$ gives the detection probabilities as functions of energy, as shown in figure 3.5. These plots are much easier to understand in the context of the adiabatic approximation, so they will be explained further in the next section.
The energy of the Hamiltonian during one period of oscillation must be much smaller than the energy difference between the initial and final states. Mathematically, this condition can be written \[30\]

\[
\left| \frac{1}{E_f - E_i} \frac{\partial}{\partial t} \langle \nu_f | H | \nu_i \rangle \right| \ll |E_f - E_i|,
\]

(3.35)

3.2.2 Adiabatic Approximation

The numerical method was successful, but has a few important shortcomings. It does not provide any analytic results, it’s very computationally expensive, and it gives highly oscillatory solutions. What we really want are the average probabilities, without the high-frequency oscillations, quickly computed from an analytic expression. We can calculate such an expression using the adiabatic approximation.

The adiabatic approximation assumes that the energy eigenstates change sufficiently slowly that neutrinos do not transition between them as they propagate. More precisely, the change in the energy of the Hamiltonian during one period of oscillation must be much smaller than the energy difference between the initial and final states. Mathematically, this condition can be written \[30\]
where \(i,f\) indicate the initial and final states. Consider the transition between the two energy eigenstates, \(\nu_1' \leftrightarrow \nu_2'\). Written the flavor basis, they are

\[
\nu_1 = \begin{pmatrix} c \\ -s \end{pmatrix}, \quad \nu_2 = \begin{pmatrix} s \\ c \end{pmatrix}.
\] (3.36)

Putting these into equation (3.35), and using that the only time-dependent part of \(H\) is \(V\), we get

\[
\left| sc \frac{\partial V}{\partial t} \right| \approx \left| \frac{sc}{R_\odot} \frac{\partial V}{\partial r} \right| \ll (\Delta E)^2.
\] (3.37)

The easiest way to check if this condition holds is to plot both sides of the equation, shown in figure 3.6 for a few different energies. Based on the results, the adiabatic approximation should be accurate.

We’ll now calculate adiabatic probabilities and compare them to the exact results from the previous section. Variables at \(r = 0\) will be written frequently, so it’s convenient to introduce a compact notation: we’ll put a ring on top. So, for example, the mixing matrix at \(r = 0\) is

\[
U(0) = \hat{U} = \begin{pmatrix} \hat{c}_m & -\hat{s}_m \\ \hat{s}_m & \hat{c}_m \end{pmatrix}.
\] (3.38)

When the condition (3.37) holds, the matter potential changes sufficiently slowly that we can assume \(\dot{V} \approx 0\). This implies the mixing matrix also changes slowly enough that \(\dot{U} \approx 0\). This is somewhat misleading, as \(U\) and \(V\) clearly aren’t constant. The point is that if \(r\) changes a little, they corresponding change in \(U\) and \(V\) is so small that we can approximate it as zero. In any case, this approximation lets us simplify the differential equation for \(A\) (3.34), which previously had to be solved numerically. Using \(\dot{U} \approx 0\), we get

\[
\dot{A} = -e^{i\phi} U \hat{U}^\dagger e^{-i\phi} A \approx 0,
\] (3.39)
hence $A$ remains at its initial value:

$$A \simeq \hat{A} = \hat{U}\nu.$$  

This is the fundamental assumption of the adiabatic approximation. By assuming $A \approx \text{constant}$, numerical methods are no longer required. Indeed, substituting (3.40) into the solution (3.29), we immediately obtain

$$\nu = U^{\dagger}e^{-i\phi}A \simeq U^{\dagger}e^{-i\phi}\hat{U}\nu,$$  

(3.41)

This expression says take the initial flavor state at the Sun’s core, transform to the mass basis, propagate to some new value of $r$, then transform back to the flavor basis. Evidently, the adiabatic approximation only needs to know the mixing matrix at the beginning and the end—it takes care of everything in the middle. Expanding this expression, we get

$$\nu = (c_{m} s_{m} - s_{m} c_{m} - s_{m} c_{m} s_{m} \cos(\phi_{1} - \phi_{2})),$$

(3.42)

and a muon neutrino

$$\langle P_{\nu_{e} \rightarrow \nu_{\mu}} \rangle = \langle |\nu_{\mu}|^2 \rangle = \langle (s_{m} c_{m})^2 + (s_{m} c_{m})^2 - 2 s_{m} s_{m} c_{m} c_{m} \cos(\phi_{1} - \phi_{2}) \rangle = (s_{m} c_{m})^2 + (s_{m} c_{m})^2.$$  

(3.43)

The $\cos(\phi_{1} - \phi_{2})$ is highly oscillatory and hence averages to zero. But its amplitude, $2s_{m}c_{m}s_{m}c_{m}$, is the envelope function. This isn’t of much physical interest, but it’ll be a nice additional check of the validity of the solution. Note that the integral $\phi$ need not be known to calculate any of these results.

Figure 3.7 shows the probabilities as functions of $r$ superimposed on the exact results from figure 3.4. As expected, the adiabatic approximation is excellent. Moving averages of the exact solutions were calculated, and the adiabatic averages only differ by $\sim 10^{-3} - 10^{-4} \%$. This is such a small difference that it could be due to precision errors in the moving average calculation, rather than the approximation. To justify this, we return to the fundamental assumption, $A \simeq \hat{A} = \hat{U}\nu$. Figure 3.8 shows the first components of both sides of this relation plotted together. $A$ actually does oscillate, but its average value remains constant, predicted perfectly by the approximation. This means that the adiabatic approximation is essentially exact for calculating averages.
The energy-dependent behavior is simple to explain now. At high energies, the Hamiltonian is almost diagonal, so oscillations are significantly damped. The diagonal Hamiltonian also means the flavor and energy states are almost the same. An initial $\nu_e$ is therefore also a $\nu_1$, and by the adiabatic approximation, it will stay a $\nu_1$. However, as the matter potential decreases, the $(\nu_e, \nu_e)$ component of the Hamiltonian becomes smaller. Therefore, the flavor content of the $\nu_1$ becomes less $\nu_e$ and more $\nu_\mu$. When the $\nu_1$ reaches the surface, it has become mostly $\nu_\mu$. As energy decreases, the Hamiltonian becomes less diagonal, so oscillations proceed more freely.

If we fix $r = 1$, we can use equations (3.42) and (3.43) to calculate energy-dependent probabilities. The matter potential is negligibly small at the surface, so the sine and cosine can be approximated as their vacuum values: $s_m(1) \simeq s$, $c_m(1) \simeq c$. The probabilities are shown in figure 3.9, superimposed on the exact results from figure 3.5.
\[ H = H + \delta H, \quad \delta H_{\alpha\beta} = \frac{1}{E} [(a_L)^{\alpha\beta} p_\alpha - (c_L)^{\alpha\beta} p_\alpha p_\beta] _{\beta}, \quad (3.44) \]

where \( p_\alpha \) is the neutrino energy-momentum four-vector,

\[ p_\alpha = (E, -\vec{p}) \simeq E(1, -\vec{p}), \quad (3.45) \]

and \((a_L)^{\alpha\beta}_{\alpha\beta}, (c_L)^{\alpha\beta}_{\alpha\beta}\) are complex Lorentz-violating SME coefficients. These coefficients represent a set of matrices whose elements depend on flavor. For example, \((a_L)^{\epsilon}_{\epsilon}\) is the \((\nu_\epsilon, \nu_\epsilon)\) component of the matrix \((a_L)^T\). \(a_L\) have dimensions of energy and hence produce energy-independent effects; \(c_L\) are dimensionless and their effects scale with energy. \(\delta H\) is a Hamiltonian and hence Hermitian, so

\[ \delta H_{\alpha\beta} = \delta H^{\alpha\beta}_{\beta}, \quad (a_L)^{\alpha}_{\alpha} = (a_L)^{\alpha*}_{\alpha}, \quad (c_L)^{\alpha}_{\alpha} = (c_L)^{\alpha*}_{\alpha} \quad (3.46) \]

for all \(c, d\). Also note that the coupling of \(c_L\) with momentum gives terms of the form

\[ (c_L)^{\alpha\beta} p_\alpha p_\beta = \frac{1}{2} [(c_L)^{\alpha\beta} + (c_L)^{\beta\alpha}] p_\alpha p_\beta, \quad (3.47) \]
so the antisymmetric parts of $c_L$ will always cancel. We therefore may as well assume $c_L$ are symmetric, i.e. $(c_L)^{\alpha \beta} = (c_L)^{\beta \alpha}$.

Coefficients with $\alpha, \beta = X, Y, Z$ couple with the neutrino propagation direction $\hat{p}$ and therefore cause anisotropic effects. For solar neutrinos, the propagation direction naturally changes with the time of year, which leads to annual variations in oscillatory behavior. The analysis of anisotropic Lorentz violation is not simplified by the two-generation case, so it will be carried out along with the three-generation analysis in section 4.3.

Coefficients with $\alpha, \beta = T$ cause isotropic effects, leading to changes in energy-dependent behavior called spectral anomalies. The remainder of this section generalizes the results of the previous section to include isotropic Lorentz violation. The exact probabilities will be far too cumbersome to work with, so the goal is to obtain leading order expressions of the form

$$\langle P_{\nu_a \rightarrow \nu_b} \rangle \simeq \langle P_{\nu_a \rightarrow \nu_b}^{(0)} \rangle + \langle P_{\nu_a \rightarrow \nu_b}^{(1)} \rangle,$$

(3.48)

where a superscript $(n)$ indicates the $n$th-order term. The zeroth-order term is just the Lorentz-invariant probability, and the leading order term is something linear in $\delta H$:

$$\langle P_{\nu_a \rightarrow \nu_b}^{(1)} \rangle = \frac{1}{\Lambda} M_{cd}^{\text{eff}} \delta H_{cd},$$

(3.49)

where $M_{cd}^{\text{eff}}$ is an effective scale factor for Lorentz violation and the factor of $\Lambda$ is present to make $\mathcal{M}$ dimensionless and order one. This form is similar to results from reference [8]. Our task is now reduced to finding $M_{cd}^{\text{eff}}$. We’ll use a similar method as the previous section: start with the Lorentz-violating eigenvalues, use them to write the sine and cosine, then calculate the probabilities.

The Hamiltonian can be written

$$\mathcal{H} = U^\dagger H' U + \delta \mathcal{H} = \begin{pmatrix} s_m^2 E_+ + c_m^2 E_- + \delta H_{ee} & s_m c_m \Delta E + \delta H_{e\mu} \\ s_m c_m \Delta E + \delta H_{e\mu} & s_m^2 E_- + c_m^2 E_+ + \delta H_{\mu\mu} \end{pmatrix}.\quad (3.50)$$

Using this matrix, the Lorentz-violating eigenvalues are

$$\mathcal{E}_\pm = \frac{1}{2} \left\{ E_+ + E_- + \delta H_{ee} + \delta H_{\mu\mu} \pm \left[ (\Delta E)^2 + (\delta H_{ee} - \delta H_{\mu\mu})^2 + 4|\delta H_{e\mu}|^2 \right]^{1/2} \right\},$$

(3.51)

Note that these Lorentz-violating eigenvalues reduce to the Lorentz-invariant values when $\delta H = 0$, as they must. To find the Lorentz-violating sine and cosine, $s_{\text{LV}}$ and $c_{\text{LV}}$, we write the Hamiltonian another way:

$$\mathcal{H} = U^\dagger \mathcal{H}' U = \begin{pmatrix} |s_{\text{LV}}|^2 \mathcal{E}_+ + |c_{\text{LV}}|^2 \mathcal{E}_- & s_{\text{LV}}^* c_{\text{LV}} \mathcal{E}_+ - s_{\text{LV}} c_{\text{LV}}^* \mathcal{E}_- \\ s_{\text{LV}} c_{\text{LV}}^* \mathcal{E}_+ - s_{\text{LV}}^* c_{\text{LV}} \mathcal{E}_- & |s_{\text{LV}}|^2 \mathcal{E}_- + |c_{\text{LV}}|^2 \mathcal{E}_+ \end{pmatrix}.\quad (3.52)$$

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Equating (3.52) and (3.50) gives three equations which can be solved to give

\[ |s_{LV}|^2 = s_m^2 \frac{\Delta E}{\Delta E} + \frac{(E_+ + \delta H_{ee})E_+ - (E_+ + \delta H_{\mu\mu})E_-}{\epsilon_+^2 - \epsilon_-^2} \quad (3.53) \]

\[ |c_{LV}|^2 = c_m^2 \frac{\Delta E}{\Delta E} + \frac{(E_+ + \delta H_{\mu\mu})E_+ - (E_+ + \delta H_{ee})E_-}{\epsilon_+^2 - \epsilon_-^2} \quad (3.54) \]

\[ s_{LV}^* c_{LV} = s_m c_m \Delta E + \Re (\delta H_{e\mu}) \Delta E + i \Im (\delta H_{e\mu}) \epsilon_+ + \epsilon_- \quad (3.55) \]

where \( \Delta E \equiv E_+ - E_- \). Again, these functions clearly reduce to the Lorentz-invariant case when \( \delta H = 0 \).

We can now find the Lorentz-violating probabilities. In analogy to the Lorentz-invariant case, they are

\[ \langle P_{\nu_e \rightarrow \nu_e} \rangle \simeq |s_{LV}|^2 |s_{LV}|^2 + |c_{LV}|^2 |c_{LV}|^2, \quad \langle P_{\nu_{\mu} \rightarrow \nu_{\mu}} \rangle = |s_{LV}|^2 |c_{LV}|^2 + |s_{LV}|^2 |c_{LV}|^2, \quad (3.56) \]

where the ring indicates \( r = 0 \), as before. To leading order in \( \delta H \), these become

\[ \langle P_{\nu_e \rightarrow \nu_e} \rangle \simeq \langle P_{\nu_{e} \rightarrow \nu_{e}}^{(0)} \rangle + \frac{1}{\Lambda} M_{ee}^{cd} \delta H_{cd}, \quad \langle P_{\nu_{\mu} \rightarrow \nu_{\mu}} \rangle \simeq \langle P_{\nu_{\mu} \rightarrow \nu_{\mu}}^{(0)} \rangle + \frac{1}{\Lambda} M_{e\mu}^{cd} \delta H_{cd} \quad (3.57) \]

where

\[ \frac{1}{\Lambda} M_{ee}^{cd} = - \frac{1}{\Lambda} M_{e\mu}^{cd} = 2 \left( \frac{s_m c_m (2 s_m^2 - 1)}{\Delta E} + \frac{s_m c_m (2 s_m^2 - 1)}{\Delta E} \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{cd} - (2 s_m^2 - 1) \begin{pmatrix} s_m c_m (s_m c_m) & s_m c_m (s_m c_m) \\ \Delta E & \Delta E \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{cd}. \quad (3.58) \]

Note that the first term scales the diagonal terms \( \delta H_{ee}, \delta H_{\mu\mu} \) by the same magnitude, but different signs. The second term scales the off-diagonal terms \( \delta H_{e\mu}, \delta H_{\mu e} \) by the same factor, including the sign. Since \( \delta H_{e\mu} = \delta H_{\mu e}^* \) by Hermicity, their imaginary parts will always cancel. Evidently, the imaginary parts cause second-order effects.

The following two pages show some example Lorentz-violating effects and provide interpretations of the results.
Figure 3.10 shows the effects of isotropic Lorentz-violation on energy-dependent probabilities. In each plot, a single coefficient is given the arbitrary value $5 \times 10^{-19}$. This number is not chosen for accuracy; it is an intentionally large number so that Lorentz-violating effects are easily visible. In fact, effects should be very pronounced, because $10^{-19}$ is not very small compared to either the vacuum Hamiltonian ($10^{-17}$–$10^{-19}$ depending on energy) or the matter potential ($10^{-17}$ at its largest). This also means that the first-order approximation (3.57) won’t be very accurate. Still, the approximation should be tested, so figure 3.10 compares it to exact numerical results. The following trends are observed:

**Energy dependence:** Lorentz-violating effects increase with energy. This is because the vacuum Hamiltonian is inversely proportional to energy, so the Lorentz-violating Hamiltonian becomes more significant as energy increases. The effect is much more dramatic for $(c_L)_{T\ell}^{TT}$ than $(a_L)_{T\ell}^T$ due to the extra factor of energy.

**Effects of individual coefficients:**

$(a_L)_{T\ell}^T$: Adds a constant term to the $(\nu_e, \nu_e)$ term of the Hamiltonian, which damps oscillations (because the Hamiltonian is closer to diagonal) and causes the energy eigenstates to have more $\nu_e$ content. Initial $\nu_e$ are more likely to stay $\nu_e$.

$(a_L)_{T\mu}^T$, $(a_L)_{T\ell}^T$: Adds to the off-diagonal terms. This enhances oscillations, so the probabilities converge to $1/2$ each.

$(a_L)_{\mu\mu}^T$: Adds a constant term to the $(\nu_\mu, \nu_\mu)$ component of the Hamiltonian, which damps oscillations and causes the energy eigenstates to have more $\nu_\mu$ content. Initial $\nu_e$ are more likely to become $\nu_\mu$.

$(c_L)_{T\ell}^{TT}$: Subtracts a term proportional to energy from the $(\nu_e, \nu_e)$ component. At low energies, this just decreases the $\nu_e$ content of the energy eigenstates, so the $\nu_\mu$ probability increases. As energy increases, this term becomes much larger and makes the Hamiltonian close to diagonal. This essentially eliminates oscillations, so initial $\nu_e$ always stay $\nu_e$.

$(c_L)_{T\ell}^{TT}$, $(c_L)_{T\mu}^{TT}$: Subtracts from the off-diagonal terms. At about 6 MeV, the off-diagonal is exactly cancelled. This means the flavor states are identical to the energy states, so an initial $\nu_e$ is exactly a $\nu_1$. However, when neutrinos reach the surface of the Sun, the matter potential has vanished and the $\nu_1$ state has become almost completely $\nu_\mu$. So, at 6 MeV, all neutrinos oscillate into $\nu_\mu$. As energy increases, the Hamiltonian becomes more off-diagonal and mixing is enhanced, so the probabilities converge to $1/2$ each.

$(c_L)_{T\mu}^{TT}$: Subtracts from the $(\nu_\mu, \nu_\mu)$ component, damping oscillations and decreasing $\nu_\mu$ content of the energy eigenstates. Initial $\nu_e$ are more likely to stay $\nu_e$.

**Accuracy of approximation:** The approximation is reasonably accurate when energy is low and the net Lorentz-violating effect is small. Fortunately, all solar-neutrino energies are less than ~$11$ MeV. Also, the values chosen for the SME coefficients are intentionally large—the approximation will perform better with smaller coefficients.

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1 The exact results are not calculated from the Schrödinger equation, they are numerical results from the adiabatic approximation (3.56). However, previous results have shown that the adiabatic approximation is essentially exact, so it is safe to treat it this way.
Figure 3.10: Comparison of Lorentz-violating energy-dependent probabilities at $r = 1$. In each plot, a single isotropic coefficient is made nonzero. Blue lines are $\nu_e$; green lines are $\nu_\mu$. The solid lines are calculated from the approximation (3.57), and the dashed are calculated numerically from the adiabatic expressions (3.56). The grey lines are the Lorentz-invariant probabilities, shown for comparison.
Chapter 4

Three-Flavor

We now consider three neutrino flavors: $\nu_e, \nu_\mu, \nu_\tau$. The following analysis will generalize the two-flavor results from the previous chapter and reduce to them as a special case. We’ll see that $\nu_\mu$ and $\nu_\tau$ are close to identical for solar neutrinos, which is why it’s often adequate to use a two-flavor model.

The method will be similar to the two-flavor chapter. Section 4.1 develops the basic theory of vacuum oscillations, section 4.2 adds matter effects, and section 4.3 adds Lorentz violation.

4.1 Basic Theory

The mass basis Hamiltonian now has two mass differences:

$$H'_{jk} = \frac{1}{2E} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta m^2_\odot & 0 \\ 0 & 0 & \Delta m^2_{\text{atm}} \end{pmatrix}_{jk},$$

(4.1)

where $\Delta m^2_\odot$ and $\Delta m^2_{\text{atm}}$ are the solar- and atmospheric-neutrino mass splittings. The smaller of the two, $\Delta m^2_\odot$, is the same as the two-flavor splitting. As we’ll see shortly, $\Delta m^2_\odot$ dominates solar-neutrino oscillations, while $\Delta m^2_{\text{atm}}$ dominates in the atmospheric regime—hence their names.

As before, the flavor basis Hamiltonian is

$$H_{ab} = (U^\dagger H'U)_{ab} = \sum_{j,k} U^*_{ja} U_{kb} H'_{jk}.$$  

(4.2)

The mixing matrix is parameterized by three mixing angles ($\theta_{12}, \theta_{23}, \theta_{13}$) and a CP-violating phase ($\delta$). It can be written as the product of three rotation matrices

$$U = U_{12} U_{13} U_{23} = \begin{pmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{13} & 0 & -s_{13} e^{-i\delta} \\ 0 & 1 & 0 \\ s_{13} e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & -s_{23} \\ 0 & s_{23} & c_{23} \end{pmatrix},$$

(4.3)

where $s_{jk} \equiv \sin \theta_{jk}$, $c_{jk} \equiv \cos \theta_{jk}$. The mixing angle $\theta_{12}$ is the same as the two-flavor angle, so
$U_{12}$ is just the two-flavor matrix with an extra dimension. Roughly speaking, $U_{12}$ accounts for $\nu_e \leftrightarrow \nu_\mu$ mixing. Similarly, $U_{13}$ is responsible for $\nu_e \leftrightarrow \nu_\tau$ mixing, and $U_{23}$ for $\nu_\mu \leftrightarrow \nu_\tau$.

Unless otherwise noted, the following approximate experimental values will be used [12]:

$$\Delta m^2_{\odot} = 7.6 \times 10^{-17} \text{ MeV}^2, \quad \Delta m^2_{\text{atm}} = 2.4 \times 10^{-15} \text{ MeV}^2,$$

$$\theta_{12} = 34^\circ, \quad \theta_{23} = 45^\circ, \quad \theta_{13} = 0^\circ, \quad \delta = 0.$$ 

A few things are worth noting. $\theta_{23}$ is maximal, which implies full $\nu_\mu \leftrightarrow \nu_\tau$ mixing, and $\theta_{13}$ is minimal, so there shouldn’t be any $\nu_e \leftrightarrow \nu_\tau$ mixing. It’s possible that $\theta_{13}$ is not quite zero, so we’ll spend some time examining the effects of a small but nonzero $\theta_{13}$. Finally, the phase $\delta$ only appears with $\sin \theta_{13}$, so it will have very little to no effect.

We’ll now derive oscillation probabilities. Since there are three mixing angles, it’s usually impractical to write out expressions in terms of sines and cosines, as we did in the two-flavor case. When possible, we’ll write sums of the mixing matrix instead.

The flavor state is

$$\nu = U^\dagger e^{-im^2 L/2E} U \nu_0,$$  \hspace{1cm} (4.4)

where $\nu_0$ is the initial flavor state and the ultrarelativistic limit is assumed. This can be written as a sum,

$$\nu_b = \sum_{a,j,k} (U^*)_{jb} (e^{-im^2 L/2E})_{jk} (U)_{ka} (\nu_0)_a,$$  \hspace{1cm} (4.5)

where $a$ is the initial flavor (always $e$ for solar neutrinos) and $b$ the final flavor, as usual. Since $e^{-im^2 L/2E}$ is diagonal, $j = k$, and since $\nu_0$ is a standard basis vector, $(\nu_0)_a$ is always unity. So the sum reduces to

$$\nu_b = \sum_j U_{ja} U^*_{jb} e^{-im^2 L/2E},$$  \hspace{1cm} (4.6)

and the oscillation probabilities are the absolute squares

$$P_{\nu_a \rightarrow \nu_b} = |\nu_b|^2.$$  \hspace{1cm} (4.7)

Using the unitarity of $U$, this can be reexpressed in terms of the mass splittings as

$$P_{\nu_a \rightarrow \nu_b} = \delta_{ab} - 4 \sum_{j>k} \Re (U^*_{ja} U_{ka} U^*_{kb}) \sin^2 \left( \frac{\Delta m^2_{jk} L}{4E} \right)$$

$$+ 2 \sum_{j>k} \Im (U^*_{ja} U_{ka} U^*_{kb}) \sin \left( \frac{\Delta m^2_{jk} L}{2E} \right).$$  \hspace{1cm} (4.8)

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1This is only rough because the oscillations are not independent of each other, so we can’t exactly say that each rotation matrix affects an individual type of oscillation.
These are certainly more complicated expressions than the two-flavor probabilities. But with $\theta_{13} = 0^\circ$, they reduce to

$$P_{\nu_e \to \nu_e} = 1 - \sin^2(2\theta_{12}) \sin^2\left(\frac{\Delta m^2_{\odot} L}{4E}\right),$$

(4.9)

$$P_{\nu_e \to \nu_\mu} = \cos^2(\theta_{23}) \sin^2(2\theta_{12}) \sin^2\left(\frac{\Delta m^2_{\odot} L}{4E}\right),$$

(4.10)

$$P_{\nu_e \to \nu_\tau} = \sin^2(\theta_{23}) \sin^2(2\theta_{12}) \sin^2\left(\frac{\Delta m^2_{\odot} L}{4E}\right).$$

(4.11)

Compared to the two-flavor case, $P_{\nu_e \to \nu_e}$ is identical and $P_{\nu_e \to \nu_\mu}$ is just multiplied by a factor of $\cos^2(\theta_{23}) = 1/2$. And of course $\sin^2(\theta_{23}) = 1/2$ too, so $P_{\nu_e \to \nu_{\tau}}$ is the same as $P_{\nu_e \to \nu_{\mu}}$. Evidently, $\nu_\mu$ from the two-flavor case has simply split into two identical flavors. If $\theta_{23}$ were zero, we would recover the two-flavor probabilities exactly, as we should.

Figure 4.1 compares oscillations with $\theta_{13} = 0$ and $\theta_{13} \neq 0$. In the latter case, probabilities become more complicated—in particular, there are some new low-amplitude, high-frequency oscillations. The amplitude is low because $\theta_{13}$ is small, and the frequency is high because $\Delta m^2_{\odot}$ is large. Obviously, $\nu_\mu$ and $\nu_\tau$ aren’t identical, but the oscillations between them are fast enough that they become irrelevant over long length scales. This is why the smaller mass difference, $\Delta m^2_{\odot}$, dominates oscillations in the long-baseline, low-energy regime. On the other hand, $\Delta m^2_{\odot}$ oscillations are too slow to have much effect over short length scales. $\Delta m^2_{\odot}$ thus dominates with short baselines and high energies.

![Figure 4.1: Probabilities of detecting $\nu_e$ (blue), $\nu_\mu$ (green), and $\nu_\tau$ (red) as functions of $L/E$. The left plot uses the values $\theta_{13} = 0^\circ$, $\delta = 0$. This causes the $\nu_\mu$ probability to be the same as $\nu_e$, which is why $\nu_\mu$ appears to missing. The right plot uses the values $\theta_{13} = 5^\circ$, $\delta = \pi/4$. The high-frequency oscillations are due to $\theta_{13}$, and the phase shift is due to $\delta$.](image-url)
4.2 Matter Effects

We now add the matter potential and carry out a similar analysis as the two-flavor case. The numerical solution method is identical, but the adiabatic solution needs to be generalized for three flavors. Unfortunately, the eigenvalues of the matter-dependent Hamiltonian are roots of the cubic equation, and are far too cumbersome to work with. This leaves two choices:

- Assume \( \theta_{13} = 0 \) from the beginning. This may not be exactly true, but it’s at least close, and it significantly simplifies the results. In fact, they essentially reduce to two-flavor results.
- Use numerically calculated eigenvalues and eigenvectors. This would allow any value of \( \theta_{13} \), but wouldn’t provide completely analytic expressions.

We’ll do both and compare them.

4.2.1 Special Case: \( \theta_{13} = 0 \)

When \( \theta_{13} = 0 \), \( U_{13} \) reduces to the identity, so \( U = U_{12}U_{23} \). The matter-dependent Hamiltonian becomes

\[
H = H(r) = \frac{1}{2E}U_{23}^\dagger U_{12}^\dagger \Delta m^2 U_{12} U_{23} + V,
\]

where \( V \) is the matter potential. Since the only nonzero term in \( V \) is the upper-left, \( U_{23} \) has no effect on it. Equation (4.12) can thus be rewritten

\[
U_{23} H U_{23}^\dagger = \frac{1}{2E}U_{12}^\dagger \Delta m^2 U_{12} + V = \Lambda \begin{pmatrix} s_{12}^2 + V & s_{12} c_{12} & 0 \\ s_{12} c_{12} & c_{12}^2 & 0 \\ 0 & 0 & \Delta m_{\text{atm}}^2 / \Delta m_{\odot}^2 \end{pmatrix},
\]

where \( \Lambda \equiv \Delta m_{\odot}^2 / 2E \). This matrix is just the two-flavor Hamiltonian with an extra dimension. And, since \( U_{23} \) is unitary, the eigenvalues of \( U_{23}^\dagger H U_{23} \) are the same as \( H \). This lets us reuse most of the results from the two-flavor case.

We’ll label the eigenvalues \( E_1, E_2, E_3 \). The first two are simply \( E_{\pm} \), equation (3.19). The third can be read off the diagonal:

\[
E_3 = \Lambda (\Delta m_{\text{atm}}^2 / \Delta m_{\odot}^2) = \Delta m_{\text{atm}}^2 / 2E.
\]

The matter-dependent sine and cosine, \( s_{12,m} \) and \( c_{12,m} \), are the same as \( s_m \) and \( c_m \), equation (3.26).

Finally, since \( U_{23} \) is unaffected by \( V \), \( s_{23} \) and \( c_{23} \) are unchanged: \( s_{23,m} = s_{23} \) and \( c_{23,m} = c_{23} \).

We can now use the adiabatic approximation to find probabilities. The flavor state is

\[
\nu = U^\dagger e^{-i\phi} \tilde{\nu} \nu = \begin{pmatrix} s_{12,m} \hat{s}_{12,m} e^{-i\phi_3} + c_{12,m} \hat{c}_{12,m} e^{-i\phi_1} \\ c_{23} (\hat{s}_{12,m} \hat{c}_{12,m} e^{-i\phi_2} - \hat{s}_{12,m} \hat{c}_{12,m} e^{-i\phi_1}) \\ s_{23} (\hat{s}_{12,m} \hat{c}_{12,m} e^{-i\phi_1} - \hat{s}_{12,m} \hat{c}_{12,m} e^{-i\phi_2}) \end{pmatrix},
\]

where the ring indicates \( r = 0 \). The average probabilities are

\[
\langle P_{\nu_e \rightarrow \nu_e} \rangle = (s_{12,m} \hat{s}_{12,m})^2 + (c_{12,m} \hat{c}_{12,m})^2,
\]

\[
\langle P_{\nu_e \rightarrow \nu_{\mu}} \rangle = c_{23}^2 [(s_{12,m} \hat{c}_{12,m})^2 + (\hat{s}_{12,m} c_{12,m})^2],
\]

\[
\langle P_{\nu_e \rightarrow \nu_{\tau}} \rangle = s_{23}^2 [(s_{12,m} \hat{c}_{12,m})^2 + (\hat{s}_{12,m} c_{12,m})^2].
\]
These is very similar to what we observed for vacuum oscillations with $\theta_{13} = 0$. $\nu_e$ are the same as the two-flavor case and $\nu_\mu$ simply splits into two identical flavors. The envelope functions do the same:

\begin{align*}
\nu_e \text{ envelope} &= 2 s_{12,m} \bar{s}_{12,m} c_{12,m} \bar{c}_{12,m}, \\
\nu_\mu \text{ envelope} &= c_{23}^2 (\nu_e \text{ envelope}), \\
\nu_\tau \text{ envelope} &= s_{23}^2 (\nu_e \text{ envelope}).
\end{align*} \hfill (4.19) \hfill (4.20) \hfill (4.21)

Figure 4.2 shows the adiabatic probabilities and envelope functions compared to the exact solution.

Figure 4.2: Probabilities of detecting $\nu_e$ (blue), $\nu_\mu$ (green), and $\nu_\tau$ (red) as functions of energy with $r$ fixed at 1. The solid black lines are the adiabatic averages, and the dashed lines are the envelopes.
4.2.2 General Case: $\theta_{13} \neq 0$

If $\theta_{13}$ is not assumed to be zero, the $r$-dependent eigenvalues are roots of the cubic equation, which means they’re too complicated to really be useful. However, if we skip this step and calculate the mixing matrix numerically, then the probabilities are straightforward to derive. $U$ can be found numerically by calculating the eigenvalues of $H$. If we write the eigenvectors as the column vectors $v_1, v_2, v_3$, then they are the columns of $U^\dagger$:

$$U^\dagger = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}.$$  \hspace{1cm} (4.22)

Assuming this has been done, we’ll find probabilities in terms of $U$. The following results will of course work equally well for a numerically calculated or analytic $U$.

The adiabatic approximation says

$$\nu = U^\dagger e^{-i\phi} \tilde{\nu}, \quad \text{or as a sum: } \nu_b = \sum_j \hat{U}_{ja} U^*_j b e^{-i\phi_j}. \hspace{1cm} (4.23)$$

This sum is analogous to vector addition in the complex plane. Each vector has a time-dependent magnitude $|\hat{U}_{ja} U^*_j b|$ and a phase. The magnitude $|\nu_b|$ is the distance of the vector sum from the origin. The average probabilities are the average magnitudes squared:

$$\langle P_{\nu_a \rightarrow \nu_b} \rangle = \langle |\nu_b|^2 \rangle = \left\langle \sum_{j,k} \hat{U}_{ka} \hat{U}_{ja} U^*_k b U^*_j b e^{i(\phi_k - \phi_j)} \right\rangle. \hspace{1cm} (4.24)$$

The phase $e^{i(\phi_k - \phi_j)}$ is highly oscillatory, so its average is simply

$$\langle e^{i(\phi_k - \phi_j)} \rangle = \delta_{jk}. \hspace{1cm} (4.25)$$

In other words, all the cross terms average away. Equation (4.24) simplifies to

$$\langle P_{\nu_a \rightarrow \nu_b} \rangle = \sum_j |\hat{U}_{ja} U^*_j b|^2, \hspace{1cm} (4.26)$$

so the average is the sum of the squares of the magnitudes. The maximum occurs when all three vectors are in phase, as in figure 4.3b, so it’s the square of the sum of the magnitudes:

$$P_{\nu_a \rightarrow \nu_b}^{\max} = \left( \sum_j |\hat{U}_{ja} U^*_j b| \right)^2. \hspace{1cm} (4.27)$$

![Graphical representations of the vector sum](image.png)

Figure 4.3: Graphical representations of the vector sum (4.23). In each plot, the origin is the lower left corner. (a) shows the vectors at an arbitrary time. (b) shows the three vectors with the same phase, which gives the maximum magnitude. (c) and (d) show the minimum magnitude in two scenarios. In the first case, the largest vector is larger than the sum of the other two, so the minimum magnitude is greater than zero. In the second case, the opposite is true, so the minimum is zero.
The minimum has two possibilities, depending on the relative sizes of the vectors. Without loss of
generality, assume that the vector with \( j = 1 \) is the largest:

\[
|\hat{U}_{1a}U_{1b}| > |\hat{U}_{2a}U_{2b}|, |\hat{U}_{3a}U_{3b}|. 
\] (4.28)

If vector 1 is larger than the sum of 2 and 3, then the minimum occurs when 1 is exactly out of
phase with 2 and 3, as in figure 4.3c. In this case, the minimum is greater than zero. If the sum of
vectors 2 and 3 is greater than 1, then the minimum is zero, as in figure 4.3d. Mathematically, this
is written

\[
P_{\text{min}}^{\nu_a \rightarrow \nu_b} = \begin{cases} 
\left(|\hat{U}_{1a}U_{1b}| - |\hat{U}_{2a}U_{2b}| - |\hat{U}_{3a}U_{3b}|\right)^2 & \text{if } |\hat{U}_{1a}U_{1b}| - |\hat{U}_{2a}U_{2b}| - |\hat{U}_{3a}U_{3b}| > 0 \\
0 & \text{if } |\hat{U}_{1a}U_{1b}| - |\hat{U}_{2a}U_{2b}| - |\hat{U}_{3a}U_{3b}| \leq 0.
\end{cases} 
\] (4.29)

Figure 4.4 shows the averages, maxima, and minima with \( \theta_{13} = 5^\circ \). Exact solutions were also
calculated. Figure 4.5 shows the averages when \( \theta_{13} \) is zero and nonzero. There are two notable
changes when it’s nonzero. First, there is a rapid change in probability for all three flavors at 50–60
MeV. But this is well above solar-neutrino energies, so this effect is of limited relevance. Second,
\( \nu_\mu \) and \( \nu_\tau \) aren’t identical anymore, particularly in solar-neutrino energy ranges. So if \( \theta_{13} \) is not
exactly zero, we expect to see differences between the two flavors.
Figure 4.4: Average (solid), maximum (dashed), and minimum (dashed) probabilities with $\theta_{13} = 5^\circ$. The exact solutions are also shown for comparison. $\nu_e$ is blue, $\nu_\mu$ green, and $\nu_\tau$ red.

Figure 4.5: Comparison of average probabilities for $\nu_e$ (blue), $\nu_\mu$ (green), and $\nu_\tau$ (red) with $\theta_{13} = 0^\circ$ (left) and $\theta_{13} = 5^\circ$ (right). $\nu_\mu$ appears to be missing in the left plot because it is identical to $\nu_e$. 

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4.3 Lorentz Violation

We now consider Lorentz violation as a small perturbative effect. The approach will be similar to the two-flavor case, though we’ll use sums of mixing matrix components rather than sines and cosines. Both isotropic (time-independent) and anisotropic (time-dependent) effects will be considered.

4.3.1 General Case

The Lorentz-violating Hamiltonian is

\[ H = H + \delta H, \tag{4.30} \]

where

\[ \delta H_{cd} = \frac{1}{E} \left[ (a_L)^\alpha p_\alpha - (c_L)^{\alpha\beta} p_\alpha p_\beta \right]_{cd}, \tag{4.31} \]

as before. The Hamiltonian is diagonalized by the Lorentz-violating mixing matrix \( U \):

\[ H' = U H U^\dagger. \tag{4.32} \]

Since \( \delta H \) is small, we’ll find \( U \) in the form

\[ U \approx U^{(0)} + U^{(1)} = U + \delta U, \tag{4.33} \]

where \( U \) is the Lorentz-invariant matrix and \( \delta U \) is a leading order correction.

If we surround \( H \) with the Lorentz-invariant mixing matrix, we get

\[ U H U^\dagger = H' + U \delta H U^\dagger, \tag{4.34} \]

which is not diagonal, but it’s close, because \( H' \) is diagonal and \( \delta H \) is small. We can diagonalize it to first order using a small matrix \( \Delta \) to cancel the off-diagonal terms of \( U \delta H U^\dagger \). This gives the mass basis Lorentz-violating Hamiltonian,

\[ H' = (1 + \Delta)(H' + U \delta H U^\dagger)(1 + \Delta^\dagger). \tag{4.35} \]

Comparing this to (4.32), we immediately have

\[ U = (1 + \Delta)U \implies \delta U = (\Delta)(U). \tag{4.36} \]

We can find \( \Delta \) by using standard time-independent perturbation theory to calculate the first-order eigenvectors of (4.34). Since \( H' \) is diagonal, the zeroth-order eigenvectors are just the standard basis vectors, \( \hat{e}_j \). The \( j \)th row of \( \Delta \) is the first-order term of the \( j \)th perturbed eigenvector:

\[ \Delta_j = \sum_{k \neq j} \frac{\hat{e}_k (U \delta H U^\dagger) \hat{e}_j^\dagger}{E_j - E_k}, \tag{4.37} \]

where \( E_j \) are the Lorentz-invariant eigenvalues. This simplifies to

\[ \Delta_{jk} = \frac{(U \delta H U^\dagger)_{kj}}{E_j - E_k} = \sum_{c,d} \frac{U_{jc} U_{kd}^*}{E_j - E_k} \delta H_{cd}. \tag{4.38} \]
Putting this into (4.36), we obtain
\[ \delta U_{ja} = [\Delta(U)]_{ja} = \sum_{k \neq j, c, d} \frac{U_{ka}U_{jc}^*U_{kd}^*}{E_j - E_k} \delta \mathcal{H}_{cd}. \] (4.39)

The average probabilities are given by equation (4.26),
\[ \langle P_{\nu_a \rightarrow \nu_b} \rangle = \sum_j |\hat{U}_{ja}|^2. \] (4.40)

To leading order in \( \delta U \), this becomes
\[ \langle P_{\nu_a \rightarrow \nu_b} \rangle \simeq \sum_j \left[ |\hat{U}_{ja}|^2 + |U_{jb}|^2 2\Re(\hat{U}_{ja}^* \delta \mathcal{H}_{ja}) + |\hat{U}_{ja}|^2 2\Re(U_{ja}^* \delta \mathcal{H}_{ja}) \right]. \] (4.41)

The first term is just the zeroth-order (Lorentz-invariant) probability, while the second and third terms are first-order and account for Lorentz violation. Replacing \( \delta U \) with (4.39), (4.41) can be written in the usual form
\[ \langle P_{\nu_a \rightarrow \nu_b} \rangle \simeq \langle P_{\nu_a \rightarrow \nu_b}^{(0)} \rangle + \frac{1}{\Lambda} \Re(M_{ab}^d \delta \mathcal{H}_{cd}), \] (4.42)

where summation over \( c, d \) is implied. The scale factor \( M \) is
\[ M_{ab}^d = 2\Lambda \sum_{j, k, j \neq k} \left[ \frac{|U_{ja}|^2 U_{ka}^* \hat{U}_{jc} \hat{U}_{kd}^*}{E_j - E_k} \right]. \] (4.43)

This expression will in fact work for any number of flavors, provided there are no degeneracies. In the two-generation case, it reduces to equation (3.58).

The effects of SME coefficients are very similar to the two-flavor case. The most notable new feature is that it’s now possible to make \( \nu_\mu \) and \( \nu_\tau \) nonidentical without a nonzero \( \theta_{13} \). An example is shown in figure 4.6.

Figure 4.6: An example of Lorentz violation with three flavors. The solid lines are calculated from the approximation (4.42), the dashed lines are exact, and the grey lines are the Lorentz-invariant probabilities. Blue is \( \nu_e \), green is \( \nu_\mu \), and red is \( \nu_\tau \). Arbitrary values were chosen for SME coefficients: \( (aL)_{ee}^T = 10^{-18} \) MeV and \( (cL)_{\tau\tau}^T = -5 \times 10^{-20} \). \( \nu_\mu \) and \( \nu_\tau \) are nonidentical due to Lorentz violation, not \( \theta_{13} \).
4.3.2 Time-Dependent Effects

We’ll now consider time-dependent anisotropic Lorentz violation. These effects are caused by the coefficients \((a_L)_{\alpha}^{cd}\) with \(\alpha \neq T\) and \((c_L)^{\alpha\beta}_{cd}\) with \(\alpha\beta \neq TT\). If any of these are nonzero, they will couple with the momentum vector, \(\vec{p} \simeq E \hat{p}\), which changes direction as the year progresses due to Earth’s orbit. This would produce annual variations in oscillatory behavior.

\(\delta\mathcal{H}\) needs to be expressed as a function of time in order to calculate these annual variations. This requires that the momentum vector be parameterized as a function of time. We’ll use the standard Sun-centered frame as our coordinate system, shown in figure 4.7. In this frame, the Z-axis is parallel to the Earth’s rotational axis, the X-axis points from the sun to the vernal equinox, and the Y-axis completes a right-handed system. The time \(T\) is zero at the 2000 vernal equinox [27]. The angle \(\eta\) is well-known: it’s the amount the Earth tilts during its orbit, 23.5°. Using this coordinate system, the momentum vector is

\[
\vec{p}(T) = -E \left( \cos \omega T \hat{X} + \sin \omega T \cos \eta \hat{Y} + \sin \omega T \sin \eta \hat{Z} \right),
\]

where \(\omega\) is the annual frequency.

We can now expand the Lorentz-violating Hamiltonian,

\[
\delta\mathcal{H}_{cd} = \frac{1}{E} \left[ (a_L)_{\alpha}^{cd} p_{\alpha} - (c_L)^{\alpha\beta}_{cd} p_{\alpha} p_{\beta} \right],
\]

using (4.44). We’ll decompose the result in the form

\[
\delta\mathcal{H}_{cd} = (C)_{cd} + (A_S)_{cd} \sin \omega T + (A_C)_{cd} \cos \omega T + (B_S)_{cd} \sin 2\omega T + (B_C)_{cd} \cos 2\omega T,
\]

similarly to reference [5]. In this form, \(C\) accounts for isotropic effects, \(A_S\) and \(A_C\) account for anisotropic effects whose period is one year, and \(B_S\) and \(B_C\) account for anisotropic effects whose period is half a year. The subscripts \(S, C\) mean “sine” and “cosine,” respectively. In terms of SME
coefficients, energy, and the angle $\eta$, we get

\[
(C)_{cd} = (a_L)_{cd}^T - \frac{E}{2} \left[ 2(c_L)_{cd}^{TT} + (c_L)_{cd}^{XX} + (c_L)_{cd}^{YY} \cos^2 \eta + (c_L)_{cd}^{ZZ} \sin^2 \eta + (c_L)_{cd}^{YZ} \sin 2\eta \right] \tag{4.47}
\]

\[
(A_S)_{cd} = (a_L)_{cd}^Y \cos \eta + (a_L)_{cd}^Z \sin \eta - 2E \left[ (c_L)_{cd}^{TY} \cos \eta + (c_L)_{cd}^{TZ} \sin \eta \right] \tag{4.48}
\]

\[
(A_C)_{cd} = (a_L)_{cd}^X - 2E (c_L)_{cd}^{TX} \tag{4.49}
\]

\[
(B_S)_{cd} = -E \left[ (c_L)_{cd}^{XY} \cos \eta + (c_L)_{cd}^{YZ} \sin \eta \right] \tag{4.50}
\]

\[
(B_C)_{cd} = -\frac{E}{2} \left[ (c_L)_{cd}^{XX} - (c_L)_{cd}^{YY} \cos^2 \eta - (c_L)_{cd}^{ZZ} \sin^2 \eta - (c_L)_{cd}^{YZ} \sin 2\eta \right]. \tag{4.51}
\]

The relation $(c_L)^{\alpha \beta} = (c_L)^{\beta \alpha}$ has been used to simplify these expressions. Note that the isotropic coefficients $(a_L)_{cd}^T, (c_L)_{cd}^{TT}$ only appear in $C$. If all the anisotropic coefficients are zero, then $\delta \mathcal{H}$ reduces to

\[
\delta \mathcal{H}_{cd} = (C)_{cd} = (a_L)_{cd}^T - E (c_L)_{cd}^{TT}, \tag{4.52}
\]

as it must.

Time-dependent probabilities are obtained by inserting the decomposed version of $\delta \mathcal{H}$ into the probability $\langle P \rangle_{\nu_a \rightarrow \nu_b}$. Again, it’s convenient to express the result in the form

\[
\langle P \rangle_{\nu_a \rightarrow \nu_b} = \langle P \rangle_{\nu_a \rightarrow \nu_b}^{(0)} + \langle P \rangle_{\nu_a \rightarrow \nu_b}^{(1)} \mathcal{M}
\]

\[
+ \langle P \rangle_{\nu_a \rightarrow \nu_b}^{(1)} A_S \sin \omega T + \langle P \rangle_{\nu_a \rightarrow \nu_b}^{(1)} A_C \cos \omega T
\]

\[
+ \langle P \rangle_{\nu_a \rightarrow \nu_b}^{(1)} B_S \sin 2\omega T + \langle P \rangle_{\nu_a \rightarrow \nu_b}^{(1)} B_C \cos 2\omega T. \tag{4.53}
\]

The first-order probabilities are calculated by taking one of $C, A_S, A_C, B_S, B_C$, contracting it with $\mathcal{M}$, taking the real part, and dividing through by $\Lambda$:

\[
\langle P \rangle_{\nu_a \rightarrow \nu_b}^{(1)} \mathcal{X} = \frac{1}{\Lambda} \Re \left[ \mathcal{M}^{cd}_{ab} \mathcal{X}_{cd} \right], \quad \mathcal{X} = C, A_S, A_C, B_S, B_C. \tag{4.54}
\]

We’ll write SME coefficients that have been contracted with $\mathcal{M}$ with a tilde over them:

\[
(\tilde{a}_L)^{\alpha}_{ab} = \mathcal{M}^{cd}_{ab} (a_L)^{\alpha}_{cd}, \quad (\tilde{c}_L)^{\alpha \beta}_{ab} = \mathcal{M}^{cd}_{ab} (c_L)^{\alpha \beta}_{cd}. \tag{4.55}
\]

These tilde coefficients are the effective SME coefficients for initial flavor $a$ and final flavor $b$ at a given energy. Or, put a different way, they are the linear combinations of the non-tilde $a_L$ and $c_L$ that affect solar neutrinos. The first-order probabilities can be written in terms of the tilde coefficients; for example

\[
\langle P \rangle_{\nu_a \rightarrow \nu_b}^{(1)} A_C = \frac{1}{\Lambda} \Re \left[ \mathcal{M}^{cd}_{ab} (A_C)_{cd} \right] = \frac{1}{\Lambda} \Re \left[ (\tilde{a}_L)^{X}_{ab} - 2E (\tilde{c}_L)^{TX}_{ab} \right], \tag{4.56}
\]
We can now examine the time-dependent effects of anisotropic coefficients. Energy will need to be fixed, so we’ll choose the energy of maximum $^8$B solar-neutrino flux, 6.4 MeV [31]. The effective scale factors for an initial $\nu_e$ at 6.4 MeV are

\[
\begin{align*}
M_{ee}^{cd} &= \begin{pmatrix}
0.401 & -0.236 & 0.236 \\
0.496 & -0.201 & 0.201 \\
-0.496 & 0.201 & -0.201
\end{pmatrix}_{cd}, \\
M_{e\mu}^{cd} &= \begin{pmatrix}
-0.201 & 0.069 & -0.090 \\
-0.210 & 0.089 & -0.111 \\
0.210 & -0.089 & 0.111
\end{pmatrix}_{cd}, \\
M_{e\tau}^{cd} &= \begin{pmatrix}
-0.201 & 0.090 & -0.069 \\
-0.210 & 0.111 & -0.089 \\
0.210 & -0.111 & 0.089
\end{pmatrix}_{cd}.
\end{align*}
\] (4.57)

There are far too many coefficients to plot each one, so a couple representative examples are chosen. For example, we can arbitrarily let

\[
(a_L)^X_{cd} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -2
\end{pmatrix}_{cd} \times 10^{-19} \text{ MeV.}
\] (4.58)

This is contracted with $\mathcal{M}$, as in equation (4.55), to obtain the tilde coefficients

\[
\begin{align*}
(\tilde{a}_L)^X_{ee} &= 0.401 \times 10^{-19} \text{ MeV,} \\
(\tilde{a}_L)^X_{e\mu} &= -0.245 \times 10^{-19} \text{ MeV,} \\
(\tilde{a}_L)^X_{e\tau} &= -0.156 \times 10^{-19} \text{ MeV.}
\end{align*}
\] (4.59)

Since $(a_L)^X$ is the only nonzero coefficient, the probabilities (4.53) are

\[
\langle P_{\nu_a \rightarrow \nu_b} \rangle = \langle P_{\nu_a \rightarrow \nu_b}^{(0)} \rangle + \frac{1}{\Lambda} (a_L)^X_{ab} \cos \omega T,
\] (4.60)

which evaluates to

\[
\begin{align*}
\langle P_{\nu_e \rightarrow \nu_e} \rangle &= \langle P_{\nu_e \rightarrow \nu_e}^{(0)} \rangle + (6.76 \times 10^{-3}) \cos \omega T, \\
\langle P_{\nu_e \rightarrow \nu_\mu} \rangle &= \langle P_{\nu_e \rightarrow \nu_\mu}^{(0)} \rangle - (4.12 \times 10^{-3}) \cos \omega T, \\
\langle P_{\nu_e \rightarrow \nu_\tau} \rangle &= \langle P_{\nu_e \rightarrow \nu_\tau}^{(0)} \rangle - (2.63 \times 10^{-3}) \cos \omega T.
\end{align*}
\] (4.61)

This is a simple annual variation centered around the Lorentz-invariant probability with a period of one year. Figure 4.8 shows the results.
Figure 4.8: Lorentz-violating probabilities with nonzero \((a_L)_{cd}^X\). Blue is \(\nu_e\), green is \(\nu_\mu\), red is \(\nu_\tau\), and the straight grey lines are the Lorentz-invariant probabilities. Both the approximate (solid lines) and exact (thick dotted lines) probabilities are plotted. Evidently, the approximation is very accurate under these conditions.

As a second example, let

\[
(c_L)_{X}^{XX} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}_{cd} \times 10^{-19},
\]

which is the same value we just used for \((a_L)_{cd}^X\). Following the same procedure, we get the tilde coefficients

\[
(\tilde{c}_L)_{ee}^{XX} = 0.401 \times 10^{-19}, \quad (\tilde{c}_L)_{e\mu}^{XX} = -0.245 \times 10^{-19}, \quad (\tilde{c}_L)_{e\tau}^{XX} = -0.156 \times 10^{-19},
\]

and the probabilities

\[
\langle P_{\nu_a \rightarrow \nu_b} \rangle = \langle P_{\nu_a \rightarrow \nu_b}^{(0)} \rangle - \frac{E}{2\Lambda} (\tilde{c}_L)_{ab}^{XX} (1 + \cos 2\omega T),
\]

which evaluates to

\[
\langle P_{\nu_e \rightarrow \nu_e} \rangle = \langle P_{\nu_e \rightarrow \nu_e}^{(0)} \rangle - (2.16 \times 10^{-2})(1 + \cos 2\omega T), \\
\langle P_{\nu_e \rightarrow \nu_\mu} \rangle = \langle P_{\nu_e \rightarrow \nu_\mu}^{(0)} \rangle + (1.32 \times 10^{-2})(1 + \cos 2\omega T), \\
\langle P_{\nu_e \rightarrow \nu_\tau} \rangle = \langle P_{\nu_e \rightarrow \nu_\tau}^{(0)} \rangle + (0.842 \times 10^{-2})(1 + \cos 2\omega T).
\]

This will produce an offset from the Lorentz-invariant probability and annual variations with a period of half a year. Figure 4.9 shows the results.
The first-order approximation appears to be quite accurate in these cases. Indeed, we’ve previously observed that it performs reasonably well at energies below 10 MeV, so it should be adequate at normal solar-neutrino energies. It will of course be even better for smaller SME coefficients.
Chapter 5
Experiments

The time-dependent probability (4.53) is ideal for fitting experimental data. In this chapter, data from the solar-neutrino experiments Super-K and SNO will be analyzed and bounds on several linear combinations of SME coefficients will be calculated.

First, time-dependent data should be fit to equation (4.53):

\[
\langle P_{\nu_a \rightarrow \nu_b} \rangle = \langle P_{\nu_a \rightarrow \nu_b}^{(0)} \rangle + \langle P_{\nu_a \rightarrow \nu_b}^{(1)} \rangle C + \langle P_{\nu_a \rightarrow \nu_b}^{(1)} \rangle A_S \sin \omega T + \langle P_{\nu_a \rightarrow \nu_b}^{(1)} \rangle A_C \cos \omega T + \langle P_{\nu_a \rightarrow \nu_b}^{(1)} \rangle B_S \sin 2\omega T + \langle P_{\nu_a \rightarrow \nu_b}^{(1)} \rangle B_C \cos 2\omega T.
\]

(5.1)

A standard least-squares fit will yield values for the first-order probabilities. Referring to equation (4.54), these are defined by

\[
\langle P_{\nu_a \rightarrow \nu_b}^{(1)} \rangle X = \frac{1}{\Lambda} \Re [\mathcal{M}_{ab}^{cd} (X)_{cd}] , \quad X = C, A_S, A_C, B_S, B_C.
\]

(5.2)

As before, we let energy be 6.4 MeV, the energy of maximum $^8B$ solar-neutrino flux. The scale factors $\mathcal{M}_{ab}^{cd}$ at this energy are given in equation (4.57), and $\eta$ is 23.5°. Using these numbers, the first-order probabilities can be evaluated numerically in terms of the tilde coefficients. We get the linear combinations

\[
\Lambda \langle P_{\nu_a \rightarrow \nu_b}^{(1)} \rangle C = (\tilde{a}_L)^T_{ab} - 6.4 (\tilde{c}_L)^T_{ab} - 3.2 (\tilde{c}_L)^X_{ab}
- 2.7 (\tilde{c}_L)^Y_{ab} - 2.3 (\tilde{c}_L)^Z_{ab} - 0.51 (\tilde{c}_L)^{ZZ}_{ab},
\]

(5.3)

\[
\Lambda \langle P_{\nu_a \rightarrow \nu_b}^{(1)} \rangle A_S = 0.91 (\tilde{a}_L)^X_{ab} + 0.40 (\tilde{a}_L)^Z_{ab} - 12 (\tilde{c}_L)^T_{ab} - 5.1 (\tilde{c}_L)^T_{ab},
\]

(5.4)

\[
\Lambda \langle P_{\nu_a \rightarrow \nu_b}^{(1)} \rangle A_C = (\tilde{a}_L)^X_{ab} - 13 (\tilde{c}_L)^T_{ab},
\]

(5.5)

\[
\Lambda \langle P_{\nu_a \rightarrow \nu_b}^{(1)} \rangle B_S = -5.9 (\tilde{c}_L)^X_{ab} - 2.6 (\tilde{c}_L)^Z_{ab},
\]

(5.6)

\[
\Lambda \langle P_{\nu_a \rightarrow \nu_b}^{(1)} \rangle B_C = -3.2 (\tilde{c}_L)^X_{ab} + 2.7 (\tilde{c}_L)^Y_{ab} + 2.3 (\tilde{c}_L)^Z_{ab} + 0.51 (\tilde{c}_L)^{ZZ}_{ab}.
\]

(5.7)
In practice, we can’t obtain a bound for the constant term \( \langle \mathcal{P}_{\nu_a \rightarrow \nu_b} \rangle_C \) because it will be too small compared to the Lorentz-invariant probability. Any slight error in the mass splittings or mixing angles would have a very significant effect on this bound, so its accuracy can’t be trusted. Also note that \((\tilde{a}_L)^a_{ab}\) have units of MeV, while \((\tilde{c}_L)^{ab}\) are dimensionless. In these expressions, numbers multiplying \((\tilde{c}_L)^{ab}\) implicitly have units of MeV.

5.1 Super-K

Reference [32] presents results from 1496 days of data collection at Super-K. We’ll fit to the flux data binned in 10-day samples (Table 1 in [32]). First, flux needs to be converted to probability. Recall that elastic scattering is much more sensitive to \(\nu_e\) than \(\nu_\mu\) or \(\nu_\tau\), so the flux is

\[
\phi_{ES} = \phi_e + \epsilon \phi_{\mu\tau}, \quad \epsilon \simeq 0.16.
\]

This can be rewritten as

\[
\phi_{ES} = (1 - \epsilon)\phi_e + \epsilon \phi_{\text{total}},
\]

where \(\phi_{\text{total}} \simeq 5.69 \times 10^6 \text{ cm}^{-2} \text{ s}^{-1}\) is the total \(^8\text{B}\) solar-neutrino flux from the solar standard model [13]. The \(\nu_e\) survival probability is then the ratio of the \(\nu_e\) flux to the total flux, which is easily solved for using (5.9):

\[
\langle \mathcal{P}_{\nu_e \rightarrow \nu_e} \rangle = \frac{\phi_e}{\phi_{\text{total}}} = \frac{\phi_{ES}/\phi_{\text{total}} - \epsilon}{1 - \epsilon}.
\]

The raw data in [32] gives the following parameters for each 10-day bin: start time, end time, mean time, measured neutrino flux with errors, and \(R^2\) correction for the eccentricity of Earth’s orbit. \(T\) was taken to be the mean time. \(\phi_{ES}\) was calculated by multiplying the measured flux by \(1/R^2\), and probability was obtained from that using equation (5.10). These data were then fit to (4.53). The data and the fit are shown in figure 5.1, and the best-fit parameters with 1\(\sigma\) errors are

\[
\langle \mathcal{P}_{\nu_e \rightarrow \nu_e} \rangle_A = -0.031 \pm 0.007, \quad \langle \mathcal{P}_{\nu_e \rightarrow \nu_e} \rangle_B = -0.005 \pm 0.007,
\]

\[
\langle \mathcal{P}_{\nu_e \rightarrow \nu_e} \rangle_A = 0.014 \pm 0.007, \quad \langle \mathcal{P}_{\nu_e \rightarrow \nu_e} \rangle_B = -0.006 \pm 0.007.
\]

These are multiplied by \(A\) to obtain bounds on the corresponding combinations of SME coefficients. Results are presented next section in table 5.1 along with the SNO results.
5.2 SNO

SNO detected neutrinos during two phases. Phase I used pure D$_2$O in the detector. It began on November 2, 1999 and ran for 572.2 calendar days with an actual livetime of 312.9 days [33], detecting 1967.71 ± 61.36 CC events, 263.64 ± 25.68 ES, and 576.53 ± 48.82 NC [34]. During phase II, 2 tons of NaCl were added to the D$_2$O to increase the efficiency of the NC reaction. This phase started on July 26, 2001 and ran for 762.7 calendar days with an actual livetime of 398.6 days [33]. The event totals were 2176 ± 78 CC, 279 ± 26 ES, and 2010 ± 85 NC [35].

Reference [33] (figure 7) provides combined results from D$_2$O and salt phases as relative event rates in 12 one-month bins. The relative event rate is the ratio of the total number of events in a given bin to the average number of events, i.e.

\[
\text{relative event rate} = \frac{n_{\text{total}}}{\langle n_{\text{total}} \rangle} = \frac{n_{\text{total}}}{N_{\text{total}}/12},
\]

(5.11)

where a lowercase $n$ indicates the count for a specific month and a capital $N$ is the combined count for the year. So we need an expression for probability that includes the relative event rate.

The $\nu_e$ probability is proportional to the number of $\nu_e$ events:

\[
\langle P_{\nu_e \rightarrow \nu_e} \rangle \propto n_e.
\]

(5.12)

Similarly, the constant part of the probability is proportional to the average number of $\nu_e$ events. This includes both the zeroth-order probability $\langle P_{\nu_e \rightarrow \nu_e}^{(0)} \rangle$ and the constant first-order term $\langle P_{\nu_e \rightarrow \nu_e}^{(1)} \rangle_c$, but we’ll assume that the first-order term is much smaller and ignore it:

\[
\langle P_{\nu_e \rightarrow \nu_e}^{(1)} \rangle_c \ll \langle P_{\nu_e \rightarrow \nu_e}^{(0)} \rangle \implies \langle P_{\nu_e \rightarrow \nu_e}^{(0)} \rangle + \langle P_{\nu_e \rightarrow \nu_e}^{(1)} \rangle_c \simeq \langle P_{\nu_e \rightarrow \nu_e}^{(0)} \rangle.
\]

(5.13)

We then have

\[
\langle P_{\nu_e \rightarrow \nu_e}^{(0)} \rangle \propto \langle n_e \rangle.
\]

(5.14)
The number of $\nu_e$ events, $n_e$, is the sum of CC and ES events, assuming that all ES events are $\nu_e$. We must make this assumption because we don’t know the event breakdown for each month, i.e. we don’t know how many were ES, CC, and NC individually, only the total. In any case, ES is the smallest contributor to SNO’s data, so this approximation is of limited significance. Dividing (5.12) by (5.14) and putting in $n_e = n_{CC} + n_{ES}$, we get

$$\frac{\langle P_{\nu_e \to \nu_e} \rangle}{\langle P_{\nu_e \to \nu_e}^0 \rangle} = \frac{n_e}{\langle n_e \rangle} = \frac{n_{CC} + n_{ES}}{\langle n_{CC} + n_{ES} \rangle} = \frac{n_{CC} + n_{ES}}{(N_{CC} + N_{ES})/12}. \quad (5.15)$$

Replacing $n_{CC} + n_{ES}$ with $n_{total} - n_{NC}$ and dividing everything by $\langle n_{total} \rangle$, we get

$$\langle P_{\nu_e \to \nu_e} \rangle = \frac{n_{total}/\langle n_{total} \rangle - n_{NC}/\langle n_{total} \rangle}{(N_{CC} + N_{ES})/N_{total}} \langle P_{\nu_e \to \nu_e}^0 \rangle, \quad (5.16)$$

which depends on the relative event rate. The final step is to correct the event counts for the eccentricity of Earth’s orbit. SNO gives the correction in (5.17):

$$\text{ecc}(T) = \left[1 + 0.0167 \cos \omega(T - T_0)\right]^2, \quad (5.17)$$

where $T_0$ is the time of perihelion. The total event counts should be divided by this correction. Also, since NC detects all flavors equally, its event counts should only vary due to the eccentricity, so $n_{NC} = \langle n_{NC} \rangle / \text{ecc}(T)$. The corrected probability is then

$$\langle P_{\nu_e \to \nu_e} \rangle = \frac{1}{\text{ecc}(T)} \frac{n_{total}/\langle n_{total} \rangle - N_{NC}/N_{total}}{(N_{CC} + N_{ES})/N_{total}} \langle P_{\nu_e \to \nu_e}^0 \rangle. \quad (5.18)$$

Relative rates were obtained from figure 7 in reference [33]. The probabilities were then calculated using equation (5.18) and fit to equation (4.53). The fit is shown in figure 5.2, and the best-fit parameters with $1\sigma$ errors are

$$\langle P_{\nu_e \to \nu_e}^{(1)} \rangle_{A_e} = -0.001 \pm 0.005, \quad \langle P_{\nu_e \to \nu_e}^{(1)} \rangle_{B_e} = 0.002 \pm 0.005,$$

$$\langle P_{\nu_e \to \nu_e}^{(1)} \rangle_{A_c} = 0.012 \pm 0.005, \quad \langle P_{\nu_e \to \nu_e}^{(1)} \rangle_{B_c} = 0.004 \pm 0.005.$$  

These are multiplied by $\Lambda$ to obtain bounds on SME coefficients. Results are presented in table 5.1 along with the Super-K results.
Although the uncertainties from the SNO fits are smaller, we still consider the Super-K results to be more reliable. Super-K provided far more data points, and their data required fewer approximations and assumptions to convert to probabilities. Most results are consistent with zero within $1\sigma$, but several are not. This is likely due to some uncorrected effect from Earth’s orbit or another unknown periodic effect.

Table 5.1: Best-fit parameters from fits to Super-K and SNO data with $1\sigma$ errors. Results are in units of $10^{-20}$ MeV. As before, the numbers multiplying $\tilde{c}_L$ implicitly have units of MeV.

| Coefficients | Super-K | SNO |
|--------------|---------|-----|
| $0.91 (\tilde{a}_L)^{\nu}_{ee} + 0.40 (\tilde{a}_L)^{Z}_{ee} - 12 (\tilde{c}_L)^{TY}_{ee} - 5.1 (\tilde{c}_L)^{TZ}_{ee}$ | $-18 \pm 4.3$ | $-0.56 \pm 3.1$ |
| $(\tilde{a}_L)^{X}_{ee} - 13 (\tilde{c}_L)^{TX}_{ee}$ | $8.4 \pm 4.4$ | $7.0 \pm 3.2$ |
| $-5.9 (\tilde{c}_L)^{XY}_{ee} - 2.6 (\tilde{c}_L)^{XZ}_{ee}$ | $-3.1 \pm 4.4$ | $1.3 \pm 3.1$ |
| $-3.2 (\tilde{c}_L)^{XX}_{ee} + 2.7 (\tilde{c}_L)^{YY}_{ee} + 2.3 (\tilde{c}_L)^{YZ}_{ee} + 0.51 (\tilde{c}_L)^{ZZ}_{ee}$ | $-3.5 \pm 4.3$ | $2.5 \pm 3.1$ |
Chapter 6

Conclusion

Solar-neutrino oscillations are studied using a massive model with perturbative Lorentz violation. Results extend previous works that use similar models in a vacuum or constant medium, such as reference [8].

First, the Lorentz-invariant case is considered. The adiabatic approximation is used to obtain simple analytic expressions for the probabilities of detecting solar neutrinos. The approximation allows high-frequency oscillations to be removed, leaving average probabilities. It proves to be accurate, agreeing with exact results to a very high precision.

Lorentz violation is then incorporated into the adiabatic expressions by adding a small Lorentz-violating Hamiltonian. The average probabilities are calculated as first-order approximations by finding effective leading order scale factors for Lorentz violation. These scale factors are expressed compactly in terms of fundamental parameters. Various Lorentz-violating coefficients are made nonzero and their effects on energy and directional dependence are examined. The first-order approximation is accurate provided the Lorentz-violating effects are sufficiently small.

The complete analysis is first carried out using two neutrino flavors, and is then generalized for three generations.

Fits to time-dependent data from Super-K and SNO are demonstrated. Standard least-squares fits immediately give bounds on several linear combinations of SME coefficients. Most bounds are consistent with zero within $1\sigma$, but several are not. This is likely due to some uncorrected effect from Earth’s orbit or another unknown periodic effect.

It’s possible that better data could yield improved results for Lorentz-violating coefficients. These fits will have to be carried out by those with better access to solar-neutrino data.

Other future work could include Lorentz-violating neutrino-antineutrino oscillations. The analysis will be somewhat different—and more complicated—as this is a second-order effect for solar neutrinos, and the addition of three antineutrino flavors makes the Hamiltonian a $6 \times 6$ matrix. There are also entirely new SME coefficients: $a_R$ and $c_R$, for antineutrino-antineutrino mixing; and $\tilde{g}$ and $\tilde{H}$, for neutrino-antineutrino mixing [14].
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