We investigate a quasi-local energy naturally introduced by Kodama’s prescription for a spherically symmetric space-time with a positive cosmological constant $\Lambda$. We find that this quasi-local energy is well behaved inside a cosmological horizon. However, when there is a scalar field with a long enough Compton wavelength, the quasi-local energy diverges in the course of its time evolution outside the cosmological horizon. This means that the quasi-local energy has a meaning only inside the cosmological horizon.

In a spherically symmetric space-time, an energy (or equivalently a mass in Einstein’s theory) is well defined in a quasi-local manner since there are no degrees of gravitational radiations. Its reasonable expression was derived first by Misner and Sharp and later Kodama obtained the same one but by a different procedure in which a conserved current is introduced. This Misner-Sharp energy is deeply related to the local structure (e.g., the existence and location of the trapped region and the type of singularities, etc) and the dynamics of the space-time. Further, the Misner-Sharp energy agrees with the well defined global energy, i.e., the Bondi-Sachs energy at the null infinity and with the Arnowitt-Deser-Misner (ADM) energy at the spatial infinity in the case of the asymptotically flat space-time. Recently Hayward has extensively discussed it and shown its geometrical and physical properties and meaning.

In this paper, we shall investigate the behavior of the quasi-local energy associated with matter fields in a spherically symmetric space-time with a positive cosmological constant $\Lambda$. The quasi-local energy in the spherically symmetric space-time with $\Lambda$ is naturally introduced by Kodama’s prescription. As will be shown, this quasi-local energy agrees with the Misner-Sharp one in the case of $\Lambda = 0$ and further agrees with the Abbott-Deser (AD) energy at the “spacelike infinity” for the asymptotically de Sitter space-time, which is a conserved Killing energy in a global sense. One of differences between the asymptotically flat space-time and asymptotically de Sitter one is the existence of the cosmological horizon defined as an outermost spacelike closed two-surface enclosing the symmetric center (we assume the spherical symmetry) such that the expansion of the future directed ingoing null orthogonal to the surface vanishes. The quasi-local energy in the asymptotically de Sitter space-time is well behaved inside the cosmological horizon like as the Misner-Sharp one. However, here it is worthy to note that the AD energy is not bounded below in contrast with the ADM energy. The negative contribution to the AD energy comes from the distribution of matter fields outside the cosmological horizon and correspondingly, when we consider a scalar field with a long enough Compton wavelength, the quasi-local energy diverges negatively outside the cosmological horizon in the course of its time evolution. Hence, the quasi-local energy has a meaning only inside the cosmological horizon in such situations. In this paper, we follow the notation of Misner, Thorne and Wheeler for the sign convention of metric, etc and adopt the geometrical unit, i.e., $c = \hbar = G = 1$.

The space-time with a cosmological constant $\Lambda > 0$ which we consider here is governed by the Einstein equation,

$$G_{\mu\nu} = 8\pi \left( T_{\mu\nu} - \frac{\Lambda}{8\pi} g_{\mu\nu} \right).$$

As already mentioned, we focus on the spherically symmetric space-time and without loss of generality, the metric of it is written as
$$ds^2 = -\alpha^2(\eta, r)d\eta^2 + A^2(\eta, r)dr^2 + R^2(\eta, r)(d\theta^2 + \sin^2 \theta d\phi^2).$$  \hspace{1cm} (2)

Following Kodama [2], we shall first introduce a preferred “time” vector $K^\mu$. This Kodama’s time vector is given by

$$K^\eta = -\frac{1}{\alpha A} \frac{\partial R}{\partial r}, \quad K^r = +\frac{1}{\alpha A} \frac{\partial R}{\partial \eta}, \quad K^\theta = 0 = K^\phi,$$  \hspace{1cm} (3)

Then the current $S^\mu$ is defined by contracting $K^\mu$ with the right hand side of Eq.(1) as

$$S^\mu = S^\mu_{(M)} + S^\mu_{(\Lambda)},$$  \hspace{1cm} (4)

where

$$S^\mu_{(M)} = \alpha AR^2 T^\mu_\nu K^\nu, \quad \text{and} \quad S^\mu_{(\Lambda)} = -\frac{1}{8\pi} \alpha AR^2 \Lambda K^\mu.$$  \hspace{1cm} (5)

As shown by Kodama, the current $S^\mu$ satisfies the conservation law $\partial S^\mu/\partial x^\mu = 0$. Here it should be noted that $S^\mu_{(M)}$ and $S^\mu_{(\Lambda)}$ also satisfy the conservation law separately, i.e.,

$$\frac{\partial}{\partial x^\mu} S^\mu_{(M)} = 0 \quad \text{and} \quad \frac{\partial}{\partial x^\mu} S^\mu_{(\Lambda)} = 0.$$  \hspace{1cm} (6)

Since the conservation law (6) is defined by the ordinary derivative, we can rewrite those into integral forms separately. We shall focus on the integral form only of the first one in Eq.(6),

$$\frac{d}{d\eta} M(r, \eta) = -4\pi S^\eta_{(M)}(r, \eta),$$  \hspace{1cm} (7)

where

$$M(r, \eta) \equiv 4\pi \int_0^r S^\eta_{(M)}(x, \eta)dx.$$  \hspace{1cm} (8)

Eqs.(7) and (8) mean that $M$ can be regarded as a kind of a quasi-local energy associated with matter fields.

From the Einstein equation, we obtain

$$S^\eta_{(M)} = \frac{1}{8\pi} \alpha AR^2 \left( G^\eta_\eta + \Lambda \delta^\eta_\eta \right) K^\nu.$$  \hspace{1cm} (9)

Using the above equation, we get another useful expression for $M$ as

$$M = \frac{1}{2} R \left[ 1 + \frac{1}{2} R^2 \left( \rho_+ - \rho_- - \frac{2}{3} \Lambda \right) \right],$$  \hspace{1cm} (10)

where $\rho_+$ and $\rho_-$ is the expansions of future directed outgoing and ingoing null, respectively and are given by

$$\rho_\pm = -\frac{1}{\sqrt{2}} \left( K_T \mp \frac{2}{\alpha AR} \frac{\partial R}{\partial r} \right),$$  \hspace{1cm} (11)

where

$$K_T \equiv -\frac{2}{\alpha} \frac{\partial}{\partial \eta} \ln R.$$  \hspace{1cm} (12)

Hayward has shown that in the case of $\Lambda = 0$, the above quasi-local energy $M$ in the untrapped region, $\rho_+ > 0$ and $\rho_- < 0$, is non-decreasing in any outgoing spatial or null direction if the matter fields satisfy the dominant energy condition [6]. Following Hayward’s proof completely, we can see that same is true in the case of $\Lambda > 0$, if the energy-momentum tensor except for $\Lambda$, i.e., $T_{\mu\nu}$ in Eq.(1), satisfies the dominant energy condition (see Appendix A). This means that the quasi-local energy $M$ is well behaved quantity within the untrapped region. If the region including the origin $r = 0$ is untrapped, the quasi-local energy $M$ is non-negative within this region since a regular matter distribution implies $M(\eta, r = 0) = 0$.

Further, we should note that on a marginal surface $\rho_+ = 0$ or $\rho_- = 0$, Eq.(10) becomes
\[ 1 - \frac{2M}{R} - \frac{\Lambda}{3} R^2 = 0. \]  

(13)

The above relation is same as that satisfied on event horizons in the Schwarzschild-de Sitter space-time if the quasi-local energy \( M \) is identified with the mass parameter of that space-time [10]. From Eq. (13), we see that the quasi-local energy is related to the existence and location of the trapped region, marginal surfaces (apparent horizon and cosmological horizon) by an expected manner.

Here let’s consider the asymptotically de Sitter space-time which is here naively defined as, for \( r \to \infty \),

\[ \alpha, A \text{ and } R/r \to a(\eta) + \mathcal{O}(r^{-1}), \]  

(14)

and

\[ K_T \to -2H + \mathcal{O}(r^{-3}) \]  

(15)

where \( a(\eta) \) is the scale factor of the de Sitter space-time,

\[ a(\eta) = -\frac{1}{H\eta} \text{ with } -\infty < \eta < 0, \]  

(16)

and \( H = \sqrt{\Lambda/3} \) is the inverse of the de Sitter radius. In this case, for \( r \to \infty \), the quasi-local energy \( M \) agrees with the AD energy \( M_{AD} \) which is the conserved Killing energy in the asymptotically de Sitter space-time and defined by

\[ M_{AD} = -\lim_{r \to \infty} \left[ r \frac{\partial R}{\partial r} - rA + \frac{H}{2} R^3 (K_T - 2H) \right]. \]  

(17)

This fact shows that our definition of the quasi-local energy is reasonable.

However it should be noted that in the case of \( \Lambda > 0 \) there is a cosmological horizon. Outside of the cosmological horizon is not untrapped and therefore the quasi-local energy \( M \) need not be non-decreasing in outgoing null or spatial direction. Here we shall consider a scalar field and show that the quasi-local energy \( M \) negatively diverges outside the cosmological horizon. We assume that the back reaction of the scalar field to the space-time geometry is negligible and hence we can treat a scalar field in the de Sitter space-time, \( \alpha = A = R/r = a(\eta) \).

Then the equation of motion for a scalar field is given by

\[ \frac{\partial}{\partial \eta} k_\phi = -\frac{H\eta}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \phi \right) + \frac{3}{\eta} k_\phi + \frac{1}{H\eta} (m^2 + 12H^2 \xi) \phi, \]  

(18)

\[ \frac{\partial}{\partial \eta} \phi = -\frac{1}{H\eta} k_\phi, \]  

(19)

where \( m \) is a mass of the scalar field and \( \xi \) is a constant which determines the coupling between the scalar field and the gravity [11].

First, for simplicity, we consider the case of \( m = \xi = 0 \). Differentiating Eq. (18) with respect to \( \eta \) and using Eq. (19), we obtain

\[ \left( \frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial r^2} \right) \left( \frac{r}{\eta^2} k_\phi \right) = 0, \]  

(20)

and this can be easily solved. The solution is obtained as

\[ k_\phi = \frac{\eta^2}{r} [Q(\eta + r) - Q(\eta - r)], \]  

(21)

\[ \phi = -\frac{1}{Hr} \int_{\eta_0}^{\eta} d\omega [Q(\omega + r) - Q(\omega - r)] + \phi_0(r), \]  

(22)

where \( \eta_0 \) corresponds to an initial conformal time, \( Q(x) \) is an arbitrary functions and \( \phi_0(r) \) is also an arbitrary function but should satisfy

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi_0}{dr} \right) = -\frac{1}{H\eta} \left( \frac{\partial k_\phi}{\partial \eta} - \frac{3}{\eta} k_\phi \right) |_{\eta = \eta_0}, \]  

(23)
Here it should be noted that as $\eta \to 0$,

$$k_\phi \propto a^{-2} \to 0,$$

$$\phi \to -\frac{1}{H r} \int_{r_0}^{0} d\omega \omega [Q(\omega + r) - Q(\omega - r)] + \phi_0(r),$$

and, in general, the right hand side of Eq.(25) does not vanish. The above behavior is well-known fact that the massless scalar field does not decay but is frozen in the de Sitter space-time [11]. Of course, if $\phi$ and $k_\phi$ are small enough initially, these remain small for all time and therefore our assumption of no back reaction is justified.

The time evolution of the quasi-local energy $M$ is determined by Eq.(7). From this equation, the rate of change of the quasi-local energy $M$ is proportional to the flux $S_{(M)}$ which is given by

$$S_{(M)} = \frac{1}{2} a^2 r^2 \left[ \left( k_\phi - \frac{1}{a} \frac{\partial \phi}{\partial r} \right)^2 + (Ha - 1) \left\{ k_\phi^2 + \frac{1}{a^2} \left( \frac{\partial \phi}{\partial r} \right)^2 \right\} \right].$$

The cosmological horizon of the de Sitter space-time is located at $ar = H^{-1}$. From Eq.(26), we find that always $S_{(M)} \geq 0$ outside the cosmological horizon and hence from Eq.(7) the quasi-local energy $M$ monotonically decreases with time outside the cosmological horizon. Furthermore, for $\eta \to 0$ at $r =$constant,

$$S_{(M)} \to \frac{r^3}{2} H a^2 \left( \frac{\partial \phi}{\partial r} \right)^2 \propto a^2,$$

and hence form Eq.(7), we obtain

$$M \to -2\pi ar^3 \left( \frac{\partial \phi}{\partial r} \right)^2 \propto a.$$

The above equation shows that the quasi-local energy negatively diverges as long as $\partial \phi/\partial r|_{\eta=0} \neq 0$.

The energy divergence outside the cosmological horizon comes from the non-decaying outgoing flux $S_{(M)}$ due to the asymptotic fall-off behavior of the massless scalar field $\phi \to 0$. Even though the scalar field has a mass, if the mass is small enough, the quasi-local energy $M$ outside the cosmological horizon diverges. When $m \neq 0$ and $\xi \neq 0$, the asymptotic behavior of the scalar field is given by [11]

$$\phi \to f(r)a^{-1/4 + \sqrt{4 - \frac{m^2}{a^2} - 12\xi}},$$

where $f(r)$ is some function determined by the initial configuration of the scalar field. Hence if the scalar field minimally couples with gravity, i.e., $\xi = 0$, and the Compton wavelength of the scalar field $m^{-1}$ is larger than $2H^{-1}/\sqrt{5}$, the quasi-local energy $M$ diverges for $\eta \to 0$ since $S_{(M)} \propto a^{-1-\epsilon}$, where $\epsilon \geq 0$.

The energy divergence outside the cosmological horizon means that the information inside the cosmological horizon cannot be obtained by the quasi-local energy $M$ defined outside the cosmological horizon. Therefore, for the asymptotically de Sitter space-time, the quasi-local energy $M$ within the cosmological horizon seems to be more important to specify the structure of the space-time than the global conserved quantity like as the AD energy in such situations.

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APPENDIX: A

We shall see that the quasi-local energy $M$ is non-decreasing in the untrapped region. The proof is completely same as Hayward [1] and we shall write the metric in the double-null form as

$$ds^2 = -2 e^{-f}d\zeta_+ \zeta_- + R^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

The above form is obtained by taking $\alpha = A = e^{-f/2}$ and $\zeta_\pm = (\eta \pm r)/\sqrt{2}$ in Eq.(2). Here we shall write the quasi-local energy as

$$M(r, \eta) = \frac{1}{2} R \left[ 1 + \frac{1}{2} R^2 \left( e^{-f} \frac{\partial}{\partial r} \frac{\partial}{\partial r} - \frac{2}{3} \Lambda \right) \right].$$
where \( \vartheta_\pm = 2R^{-1} \partial_\pm R = e^{f/2} \varrho_\pm \) and \( \partial_\pm \) means the derivative with respect to \( \zeta_\pm \). Without loss of generality, the components of the energy-momentum tensor in the spherically symmetric space-time is written as

\[
T_{\pm\pm} = \psi_\pm, \quad T_{+-} = 2\varepsilon \quad \text{and} \quad T_{\theta\theta} = T_{\varphi\varphi}/\sin^2 \theta = R^2 P. \tag{A3}
\]

The Einstein equations Eq.(1) then become

\[
\partial_\pm \partial_\pm R + \partial_\pm f \partial_\pm R = -4\pi R \psi_\pm, \tag{A4}
\]

\[
R \partial_\pm \partial_- R + \partial_+ R \partial_- R + \frac{1}{2} e^{-f} = 4\pi R^2 \left( \varepsilon + \frac{1}{8\pi} e^{-f} \Lambda \right), \tag{A5}
\]

\[
R \partial_\pm \partial_- f + 2 \partial_+ R \partial_- R + e^{-f} = 8\pi R^2 (\varepsilon + e^{-f} P). \tag{A6}
\]

By using the above equations, the derivatives of \( M \) with respect to \( \zeta_\pm \) are obtained as

\[
\partial_\pm M = 4\pi e^{f} R^3 \left( \varepsilon \vartheta_\pm - \psi_\pm \vartheta_\mp \right). \tag{A7}
\]

Here we assume that the matter satisfies the dominant energy condition, i.e., \( \varepsilon \geq 0 \) and \( \psi_\pm \geq 0 \). Then, from Eq.(A7), we find that in the untrapped region \( \vartheta_+ > 0 \) and \( \vartheta_- < 0 \), the quasi-local energy satisfies \( \partial_+ M \geq 0 \) and \( \partial_- M \leq 0 \). If \( z^\mu \) is outgoing spatial vector, the derivative of \( M \) along this direction is given by

\[
z^\mu \partial_\mu M = \beta_+ \partial_+ M - \beta_- \partial_- M, \tag{A8}
\]

where \( \beta_\pm \) are positive. Hence, in the untrapped region, \( z^\mu \partial_\mu M \geq 0 \) and this means that \( M \) is non-decreasing in any outgoing null or spatial direction in this region.

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