Scale-Free Percolation in Continuum Space

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Abstract The study of real-life network modeling has become very popular in recent years. An attractive model is the scale-free percolation model on the lattice $\mathbb{Z}^d$, $d \geq 1$, because it fulfills several stylized facts observed in large real-life networks. We adopt this model to continuum space which leads to a heterogeneous random-connection model on $\mathbb{R}^d$: Particles are generated by a homogeneous marked Poisson point process on $\mathbb{R}^d$, and the probability of an edge between two particles is determined by their marks and their distance. In this model we study several properties such as the degree distributions, percolation properties and graph distances.

Keywords Scale-free percolation · Continuum percolation · Random-connection model · Degree distribution · Phase transition · Graph distance

Mathematics Subject Classification 60K35 · 05C80

1 Introduction

The study of real-life networks such as virtual social networks or financial networks has become very popular in recent years, see, for example, [1,8,19]. Such networks can be seen as sets of particles that are possibly linked to each other. Several stylized facts of large real-life networks have been observed using large empirical data sets (see [19] and Section 1.3 in [12] for further details):

– The minimal number of links that connect two particles, called the graph distance, is typically small for distant particles. This is called the “small-world effect.”
is the observation that most particles in many real-life networks are connected by at most six links, see [23].

- Particles that are linked tend to have common friends, which is called the “clustering property.”
- The number of links of a given particle, called the degree, has a heavy-tailed distribution with (power law) tail parameter \( \tau > 0 \). The tail parameter is often observed to be between 1 and 2, i.e., the degree distribution has finite mean and infinite variance. We refer to [12] for explicit examples.

Since it is too complicated to model large real-life networks particle by particle, many theoretical random graph models have been developed and their geometrical properties studied. One of these models is the homogeneous long-range percolation model on \( \mathbb{Z}^d, d \geq 1 \), first introduced in [25] for \( d = 1 \). The set of particles is the lattice \( \mathbb{Z}^d \), and any two particles \( x, y \in \mathbb{Z}^d \) are independently linked with probability \( p_{xy} \) which behaves as \( \lambda |x - y|^{-\alpha} \) for \( |x - y| \to \infty \), with fixed constants \( \lambda, \alpha > 0 \). Since close particles are likely linked, this model has a local clustering property. Moreover, depending on \( \alpha \), the graph distance of two connected particles is roughly of logarithmic order as their separation tends to infinity, see [6]. This is a version of the small-world effect. However, this model does not fulfill the stylized fact of having heavy-tailed degree distributions. Therefore, [10] extended the homogeneous long-range percolation model to a scale-free percolation model on \( \mathbb{Z}^d \) (also known as inhomogeneous long-range percolation model). In their model they consider a collection \( (W_x)_{x \in \mathbb{Z}^d} \) of i.i.d. positive weights that are heavy-tailed with tail parameter \( \beta > 0 \), and they assign to each particle \( x \in \mathbb{Z}^d \) the random weight \( W_x \). Given these weights, any two particles \( x, y \in \mathbb{Z}^d \) are independently linked with probability \( p_{xy} \) which is approximately \( \lambda W_x W_y |x - y|^{-\alpha} \) for large \( |x - y| \) and given constants \( \lambda, \alpha > 0 \). Note that \( p_{xy} \) is increasing in the weights \( W_x \) and \( W_y \), and decreasing in the distance between \( x \) and \( y \). This means that the weights make particles more or less attractive, i.e., particles with large weights play the role of hubs in this network. This extension of the homogeneous model is very natural since the existence of hubs is often observed in real-life networks. Again, this model has a local clustering property. Depending on \( \alpha \) and \( \beta \), [10] showed that the degree distribution is heavy-tailed, i.e., this model fulfills the stylized fact of having heavy-tailed degree distributions. Moreover, they showed that whenever the degree distribution has finite mean but infinite variance, the graph distance of two particles behaves doubly logarithmically as their separation tends to infinity. This is again a version of the small-world effect, sometimes called the ultra-small-world effect.

In this article we adopt the scale-free percolation model on \( \mathbb{Z}^d \) to the continuum space \( \mathbb{R}^d \) as proposed in [10], which leads to a heterogeneous random-connection model (RCM) on \( \mathbb{R}^d \) where particles are no longer restricted to a lattice. Instead of taking the particles to be the vertices of \( \mathbb{Z}^d \) with assigned weights, we distribute particles randomly in space according to a homogeneous Poisson point process on \( \mathbb{R}^d \), and to each particle \( x \) we attach (independently of its location) a positive random weight \( W_x \) whose distribution is heavy-tailed with tail parameter \( \beta > 0 \). Given the Poisson cloud and the weights, two particles \( x \) and \( y \) are linked with probability \( p_{xy}(\lambda, \alpha) \) as in the scale-free percolation model on \( \mathbb{Z}^d \). This heterogeneous RCM can be seen as an extension of the homogeneous RCM on \( \mathbb{R}^d \), which was introduced and
studied in [20], while an applied version already appeared in [14]. The main reference for the homogeneous RCM and other continuum percolation models is [17]. The goal of this article is to prove similar results as in [10,11] for the heterogeneous RCM. In particular, depending on $\alpha$ and $\beta$, we show that in our heterogeneous RCM the degree distribution is heavy-tailed with tail parameter $\tau(\alpha, \beta) > 0$. Assuming that the weights follow a Pareto distribution with tail parameter $\beta$, we give an explicit expression of the degree distribution as well as the expected degree of a given particle in terms of the model parameters $\beta$, $\alpha$, $\lambda$ and the intensity of the Poisson point process. This result improves the bounds given in Proposition 2.3 in [10] for this particular choice of weight distribution. This explicit expression is also helpful to calibrate the model parameters for real-life network applications. Moreover, we show that there is a non-trivial phase transition depending on $\alpha$ and $\beta$, where $\lambda$ plays the role of the percolation parameter. Above criticality there is a unique infinite connected component. For real-life network applications the interesting case is $\tau(\alpha, \beta) \in (1,2)$, and we observe that in this case the model percolates for all $\lambda > 0$. In other words, in this latter case the network contains infinitely many particles that are all connected through links. We furthermore study graph distances between particles that lie in the same connected component. Similar to [10,11] we prove the existence of different asymptotic regimes, which are characterized by $\alpha$ and the power law constant $\beta$ of the marks. As key step in that proof, we show that the size of the largest connected component restricted to a finite box is of the same order as the total number of particles in that box. This result is of independent interest and states that the number of particles belonging to the largest connected network in a finite box $[0, m]^d$ is of order $m^d$. Moreover, we show that there is no percolation at criticality whenever $p_{xy}(\lambda, \alpha)$ does not decrease too fast in the distance of two particles.

Compared to inhomogeneous long-range percolation on $\mathbb{Z}^d$, the heterogeneous RCM has the advantage that some proofs of the results are more easy to handle since we can use standard integration in $\mathbb{R}^d$. This also allows us for calculating several graph properties explicitly which is of central interest for calibrating model parameters. On the other hand, some proofs are more involved because one needs to make sure that the Poisson cloud is sufficiently regular. We also mention that the continuum space model, as an extension of the lattice model, has the advantage that it can be extended to Poisson point processes with space-dependent (random) intensity functions. This can be used to model networks that have more densely populated areas than other areas.

The paper is organized as follows. In the next section we introduce the model. In Sect. 3 we state the main results on the degree distributions, the percolation properties, the absence of percolation at criticality, the size of largest connected components in finite boxes and the graph distances in the random graph. Section 4 gives the proofs of the results on the degree distributions, and in Sect. 5 we prove the percolation properties. In Sect. 6 we prove the absence of percolation at criticality and the results on the size of largest connected components in finite boxes are given. Finally, Sect. 7 contains the proofs of the results on graph distances.
2 The Model

We introduce a heterogeneous RCM which modifies the homogeneous RCM defined in [20] and which is a continuum space analogue to the inhomogeneous long-range percolation model presented by Deijfen et al. [10]. The tuple \((X, \nu, \beta, \lambda, \alpha)\) denotes a heterogeneous RCM on \(\mathbb{R}^d, d \geq 1\), where we make the following assumptions:

1. \((X, \nu, \beta)\) is a homogeneous marked Poisson point process, where \(X\) denotes the spatially homogeneous Poisson point process on \(\mathbb{R}^d\) with fixed intensity \(\nu > 0\), and \(\beta > 0\) denotes the power law tail parameter of the distribution of the i.i.d. marks \(W_x, x \in X\). We assume that \(W_x, x \in X\), has Pareto distribution with scale parameter 1, i.e.,
   \[ P[W_x > w] = w^{-\beta}, \quad \text{for } w \geq 1. \]

2. Given \(X\) and \((W_x)_{x \in X}\), we have an edge (link) between two distinct particles \(x \neq y \in X\), write \(x \leftrightarrow y\), independently of all other possible edges, with probability
   \[ p_{xy} = p_{xy}(\lambda, \alpha) = 1 - \exp\left\{ -\lambda W_x W_y |x - y|^{-\alpha} \right\}, \]
   with constants \(\lambda > 0\) and \(\alpha > 0\), and \(|\cdot|\) denoting the Euclidean norm on \(\mathbb{R}^d\).

By replacing \(\lambda\) by \(\theta^2 \lambda\) we can extend the results to Pareto distributions with arbitrary scale parameter \(\theta > 0\), but we use \(\theta = 1\) as normalization of the model. In [10] there is a more general version for the choice of the distribution of the marks \((W_x)_{x}\), but since eventually only the choice of \(\beta > 0\) of regular variation at infinity is relevant, the Pareto distribution provides the full flavor of the asymptotic results. Moreover, for our results only the tail behavior of \(p_{xy}\) is relevant, which is of order \(\lambda W_x W_y |x - y|^{-\alpha}\), but we make the particular choice of \(p_{xy}\) to simplify calculations. We call \(X\) the Poisson cloud with particles \(x \in X\). The marks \((W_x)_{x \in X}\) are the weights in the particles \(x \in X\) that determine the edge probabilities \(p_{xy}\) between the corresponding particles \(x\) and \(y\) of \(X\). It follows from [22] that the model is shift invariant and ergodic.

3 Main Results

3.1 Degree Distribution

We define the degree \(D_x\) of particle \(x \in X\) to be the number of particles \(y \in X\) such that \(x\) and \(y\) are linked, i.e., \(x \leftrightarrow y\). Observe that the distribution of \(D_x\) is translation invariant in the sense that we may start at every particle \(x\) of the Poisson cloud \(X\). Since \(D_0\) is only defined if the origin belongs to the Poisson cloud \(X\), we consider \(D_0\) under the conditional probability \(\mathbb{P}_0\), conditionally given that the Poisson cloud has a particle at the origin. The probability \(\mathbb{P}_0\) is the Palm measure of \(\mathbb{P}\), and the conditioning on the event of having a particle at the origin does not influence the rest of the Poisson process, see, for instance, Chapter 12 in [9]. The first result describes the distribution of the degree \(D_0\) under \(\mathbb{P}_0\).
Theorem 3.1 We obtain the following cases.

(i) For \( \min\{\alpha, \beta \alpha\} \leq d \) we obtain \( \mathbb{P}_0[D_0 = \infty] = 1 \).

(ii) For \( \min\{\alpha, \beta \alpha\} > d \) we obtain that \( D_0 \) has (under \( \mathbb{P}_0 \)) a mixed Poisson distribution with mixing distribution being the Pareto distribution with shape parameter \( \tau = \beta \alpha / d > 1 \) and scale parameter \( c_1^{1/\tau} \), where

\[
\begin{align*}
 c_1 &= c_1(d, \beta, \alpha, \lambda, \nu) = \left( \nu v_d \Gamma(1 - d/\alpha) \frac{\tau}{\tau - 1} \right)^\tau \lambda^\beta,
\end{align*}
\]

and where \( v_d \) denotes the volume of the unit ball in \( \mathbb{R}^d \). That is, for \( k \geq 0 \),

\[
\mathbb{P}_0[D_0 = k] = \frac{\tau c_1}{k!} \int_{c_1}^{\infty} t^{k-\tau-1} e^{-t} dt.
\]

Moreover, the survival probability of this distribution fulfills

\[
\lim_{n \to \infty} \frac{\mathbb{P}_0[D_0 > n]}{n^{-\tau}} = c_1,
\]

and hence the degree distribution is heavy-tailed with tail parameter \( \tau = \beta \alpha / d > 1 \). The first moment of this distribution is given by

\[
\mathbb{E}_0[D_0] = \nu v_d \Gamma(1 - d/\alpha) \left( \frac{\tau}{\tau - 1} \right)^2 \lambda^{d/\alpha}.
\]

This theorem is the continuum space analogue to Theorems 2.1 and 2.2 in [10]; the explicit expression of \( \mathbb{E}_0[D_0] \) improves the bounds given in Proposition 2.3 in [10] for our choice of the distribution of the weights \( (W_x)_{x \in X} \). We observe that if \( \beta \alpha / d \leq 1 \) or if the decay \( |x|^{-\alpha} \) is too slow for \( |x| \to \infty \), namely if \( \alpha \leq d \), then any given particle shares edges with infinitely many other particles, a.s. This trivial case is, of course, not of interest for real-life network modeling. In the non-trivial case \( \min\{\alpha, \beta \alpha\} > d \) the distribution of the degree of a given particle is heavy-tailed with tail parameter \( \tau = \beta \alpha / d > 1 \). Hence, in this latter case, the continuum space model fulfills the stylized fact of having heavy-tailed degree distributions. This differs from the homogeneous RCM, where the degree distribution always is light-tailed, see formula (6.1) in [17]. According to the stylized facts the interesting case for real-life applications is \( \tau = \beta \alpha / d \in (1, 2) \) with \( \alpha > d \), see also Section 1.4 in [12]. Note that, even if \( \alpha > d \), weight distributions having an infinite variance \( (\beta < 2) \) do not immediately imply degree distributions having an infinite variance \( (\tau < 2) \). On the other hand, under the assumption \( \alpha > d \), if the weight distributions have a finite variance \( (\beta > 2) \), the degree distributions have a finite variance \( (\tau > 2) \) as well.

3.2 Phase Transition

In order to study the percolation properties of the heterogeneous RCM, denote the (maximal) connected component of \( x \in X \) by

\[
\mathcal{C}(x) = \{ y \in X \mid \text{there is a finite path of edges connecting } x \text{ and } y \},
\]
which is the set of all particles that can be reached from \( x \) within the network. The percolation probability is defined by

\[
\theta(\lambda) = \mathbb{P}_0[|C(0)| = \infty],
\]

where \( |C(0)| \) denotes the number of particles in the connected component of the origin. The critical percolation value is defined by

\[
\lambda_c = \inf \{ \lambda > 0 \mid \theta(\lambda) > 0 \}.
\]

By ergodicity it follows that there are only finite connected components, a.s., whenever \( \lambda < \lambda_c \), and there exists an infinite connected component, a.s., if \( \lambda > \lambda_c \). By the uniqueness theorem for the homogeneous RCM, see Theorem 6.3 of \cite{17}, and the fact that \( p_{xy} \in (0, 1) \) for all particles \( x \) and \( y \), a.s., such an infinite connected component is unique, a.s.; we denote it by \( C_\infty \).

We refer to \cite{7, 15} for a general introduction to percolation theory. For \( \min\{\alpha, \beta\alpha\} \leq d \), it follows from Theorem 3.1 \((i)\) that \( \theta(\lambda) = 1 \) for all \( \lambda > 0 \), hence \( \lambda_c = 0 \). The next theorem gives the percolation properties in the non-trivial case \( \min\{\alpha, \beta\alpha\} > d \), see also Fig. 1.

**Theorem 3.2** Assume \( \min\{\alpha, \beta\alpha\} > d \).

(a) In the case \( d \geq 2 \) we obtain:

(a1) if \( \beta\alpha < 2d \), then \( \lambda_c = 0 \);

(a2) if \( \beta\alpha > 2d \), then \( \lambda_c \in (0, \infty) \).

(b) In the case \( d = 1 \) we obtain:

(b1) if \( \beta\alpha < 2 \), then \( \lambda_c = 0 \);
(b2) if \( \beta \alpha > 2 \) and \( \alpha \in (1, 2] \), then \( \lambda_c \in (0, \infty) \);
(b3) if \( \min\{\alpha, \beta \alpha\} > 2 \), then \( \lambda_c = \infty \).

This result also holds true in the discrete space model, see [10]. It shows the existence of a non-trivial phase transition if the degree distribution has finite variance (\( \tau = \beta \alpha/d > 2 \)) and \( \alpha > d \) (\( \alpha \in (1, 2] \) in \( d = 1 \)). Note that in the interesting case for real-life network applications (\( \tau = \beta \alpha/d \in (1, 2) \) with \( \alpha > d \)) there is a unique infinite connected component \( C_\infty \) for all \( \lambda > 0 \). In the one-dimensional case, similar results to Theorem 3.2 hold true for the homogeneous long-range percolation model on \( \mathbb{Z} \), where the probability of an edge between two sites \( x \) and \( y \) is given by \( 1 - \exp\{-\lambda|x-y|^{-\alpha}\} \), see [18,21]. It is shown that for \( \alpha \leq 1 \) percolation occurs for any \( \lambda > 0 \), for \( \alpha \in (1, 2] \) percolation occurs only for \( \lambda \) sufficiently large, and for \( \alpha > 2 \) there does not exist an infinite connected component, a.s.

Note that \( \lambda_c = 0 \) whenever \( \tau = \beta \alpha/d \in (1, 2) \) and, therefore, there is trivially no infinite connected component at criticality \( \lambda_c \). The next theorem states that there is no infinite connected component at criticality \( \lambda_c > 0 \) also in the case \( \alpha \in (d, 2d) \) and \( \tau = \beta \alpha/d > 2 \). This corresponds to Theorem 1.5 of [3] for homogeneous long-range percolation and to Corollary 4 of [11] for inhomogeneous long-range percolation on the lattice. The case \( \alpha > 2d \) and \( \tau = \beta \alpha/d > 2 \) is still open, except if \( d = 1 \) where there is never an infinite connected component.

**Theorem 3.3** Assume \( \alpha \in (d, 2d) \) and \( \tau = \beta \alpha/d > 2 \). There is no infinite connected component at criticality \( \lambda_c > 0 \), a.s.

### 3.3 Percolation on Finite Boxes

For \( n \in (0, \infty) \) we define the box \( \Lambda_n = [-n,n]^d \) and we denote by \( C_n \) the largest connected component in \( \Lambda_n \) (with a deterministic rule if there is more than one largest connected component). The next result shows that in case of percolation and \( \alpha \in (d, 2d) \), the number of particles in \( C_n \) presents with high probability at least a positive fraction of the Lebesgue measure of box \( \Lambda_n \). This result is the continuum space analogue to Theorem 6 of [11].

**Theorem 3.4** Assume \( \alpha \in (d, 2d) \) and \( \tau = \beta \alpha/d > 1 \). Choose \( \lambda \in (0, \infty) \) such that \( \theta(\lambda, \alpha) > 0 \). Then, for all \( \alpha' \in (\alpha, 2d) \) there exist \( \rho > 0 \) and \( n_0 < \infty \) such that for all \( n \geq n_0 \),

\[
\mathbb{P}\left[|C_n| \geq \rho n^d\right] \geq 1 - \exp\{-\rho n^{2d-\alpha'}\},
\]

where \( |C_n| \) denotes the number of particles of the largest connected component in \( \Lambda_n \).

For \( x \in \mathbb{R}^d \) and \( n \in (0, \infty) \) we write \( \Lambda_n(x) = x + [-n,n]^d \) for the box of side length \( 2n \) centered at \( x \). For \( x \in X \) we write \( C_n(x) \) for the set of particles in \( \Lambda_n(x) \cap X \) that are connected to \( x \) within \( \Lambda_n(x) \). For \( \ell > 0 \) and \( \rho > 0 \), we call \( x \in X \) a \((\rho, \ell)\)-dense particle if the number of particles in \( X \) belonging to \( C_\ell(x) \), denoted by \( |C_\ell(x)| \), is at least \( \rho (2\ell)^d \), see also Definition 4.1 of [6]. The set of \((\rho, \ell)\)-dense particles in \( \Lambda_n = \Lambda_n(0) \) is denoted by

\[
D_n^{(\rho, \ell)} = \left\{ x \in \Lambda_n \cap X : |C_\ell(x)| \geq \rho (2\ell)^d \right\}.
\]
Corollary 3.5 below shows that whenever a particle \( x \in X \) belongs to the infinite connected component \( C_\infty \), the probability that it is \((\rho, \ell)\)-dense converges to 1 as \( \ell \to \infty \) for some \( \rho > 0 \). This result can be interpreted as a local clustering property in the sense that a particle in the infinite connected component is surrounded by many other particles that are connected to it. Moreover, Corollary 3.5 shows that the number of dense particles in box \( \Lambda_n \) presents with high probability at least a positive fraction of the Lebesgue measure of that box. Corollary 3.5 is the analogue to Corollaries 3.3 and 3.4 of [6]. We use it to prove an estimate on the graph distance in the infinite connected component, below.

**Corollary 3.5** Assume \( \alpha \in (d, 2d) \) and \( \tau = \beta\alpha/d > 1 \). Choose \( \lambda \in (0, \infty) \) such that \( \theta(\lambda, \alpha) > 0 \).

(i) There exists \( \rho > 0 \) such that for \( x \in \mathbb{R}^d \),

\[
\lim_{\ell \to \infty} \mathbb{P} \left[ |C_\ell(x)| \geq \rho(2\ell)^d \mid x \in C_\infty \right] = 1.
\]

(ii) For all \( \alpha' \in (\alpha, 2d) \) there exist \( \rho > 0 \) and \( \ell_0 > 0 \) such that for all \( n > \ell_0^2 \) and \( \ell \in (\ell_0, n/\ell_0) \),

\[
\mathbb{P} \left[ |D_n^{(\rho, \ell)}| \geq \rho(2n)^d \right] \geq 1 - \exp\{-\rho n^{2d-\alpha'}\},
\]

where \( |D_n^{(\rho, \ell)}| \) denotes the number of \((\rho, \ell)\)-dense particles in \( \Lambda_n \).

### 3.4 Graph Distances

For \( x, y \in X \), we write \( d(x, y) \) for the graph distance or chemical distance between \( x \) and \( y \), i.e.,

\[
d(x, y) = \inf \{ n \in \mathbb{N} \mid \exists x_1, \ldots, x_n \in X : x \leftrightarrow x_1 \leftrightarrow \cdots \leftrightarrow x_{n-1} \leftrightarrow x_n = y \},
\]

where we use the convention that \( d(x, y) = \infty \) if \( x \) and \( y \) are not in the same connected component. In order to measure events involving \( d(x, y) \) for \( x, y \in \mathbb{R}^d \), one needs to make sure that the particles \( x \) and \( y \) lie in the Poisson cloud \( X \). We therefore consider the twofold Palm measure \( \mathbb{P}_{x,y} \) of \( \mathbb{P} \) which can be interpreted as the conditional distribution of the marked Poisson point process under the condition that there are particles of the process in \( x \) and \( y \), i.e., \( \mathbb{P}_{x,y}[\cdot] = \mathbb{P}[\cdot \mid x, y \in X] \). Note that we have \( \mathbb{P}[\cdot \mid x, y \in C_\infty] = \mathbb{P}_{x,y}[\cdot \mid x, y \in C_\infty] \). The next theorem states bounds on the graph distance in the case \( \min\{\alpha, \beta\alpha\} > d \), see also Fig. 1 for an illustration.

**Theorem 3.6** Assume \( \min\{\alpha, \beta\alpha\} > d \).

(a) Assume \( \tau = \beta\alpha/d \in (1, 2) \) and choose \( \lambda > \lambda_c = 0 \). There exists \( \eta_1 > 0 \) such that for all \( \varepsilon > 0 \),

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\[
\lim_{|x| \to \infty} \mathbb{P} \left[ \eta_1 \frac{2}{\log(\alpha (\beta \wedge 1)/d - 1)} \leq \frac{d(0, x)}{\log |x|} \leq (1 + \epsilon) \frac{2}{\log(\beta \alpha/d - 1)} \right] = 1.
\]

(b1) Assume \( \alpha \in (d, 2d) \), \( \tau = \beta \alpha/d > 2 \) and choose \( \lambda > \lambda_c \). For all \( \epsilon > 0 \),

\[
\lim_{|x| \to \infty} \mathbb{P} \left[ 1 - \epsilon \leq \frac{\log d(0, x)}{\log \log |x|} \leq (1 + \epsilon) \frac{\log 2}{\log(2d/\alpha)} \right] = 1.
\]

(b2) Assume \( \min\{\alpha, \beta \alpha\} > 2d \). There exists \( \eta_2 > 0 \), such that

\[
\lim_{|x| \to \infty} \mathbb{P} \left[ \eta_2 \leq \frac{d(0, x)}{|x|} \right] = 1.
\]

This theorem is the continuum space analogue to the results in Section 5 of [10] and Theorem 8 of [11]. We note that in homogeneous long-range percolation on \( \mathbb{Z}^d \) the picture about graph distances in the case \( \alpha > d \) (where the degree distribution has finite mean) has two different regimes: The graph distances behave roughly logarithmically if \( \alpha \in (d, 2d) \), while if \( \alpha > 2d \), there is a linear lower bound on the graph distances, see Theorem 1.1 of [6] and Theorem 1 of [4], respectively. In our model we observe the same behavior if in addition the degree distribution has finite variance \( (\tau = \beta \alpha/d > 2) \). But if the degree distribution has infinite variance, we get an additional regime where the graph distances behave doubly logarithmically. According to the stylized facts this latter case is interesting for real-life network applications. In particular, if \( d < \min\{\alpha, \beta \alpha\} < 2d \), we see that distant particles are connected by very short paths of edges which is a version of the (ultra-) small-world effect.

An upper bound on the graph distances in the case \( \min\{\alpha, \beta \alpha\} > 2d \) is still open. As in homogeneous long-range percolation with \( \alpha > 2d \) it is believed that a linear upper bound should hold, see Conjecture 1 of [4]. For independent nearest neighbor bond percolation on \( \mathbb{Z}^d, d \geq 2 \), this result was proved in [2]. Note that (b1) states that \( d(0, x) \) is roughly \( (\log |x|)^\Delta \) for large \( |x| \) and some constant \( \Delta > 0 \). The existence of \( \Delta \) is still unknown, even in homogeneous long-range percolation. Moreover, the optimal constants in all asymptotic behaviors are still open.

\section{4 Degree Distribution}

In this section we prove Theorem 3.1. We start with the following observation.

\textbf{Lemma 4.1} The distribution of degree \( D_0 \), conditionally given \( W_0 \), is given by

\[
P_0 \left[ D_0 = k \mid W_0 \right] = \exp \left\{ -v \int_{\mathbb{R}^d} \mathbb{E}_0 \left[ p_{0x} \mid W_0 \right] \, dx \right\} \left( v \int_{\mathbb{R}^d} \mathbb{E}_0 \left[ p_{0x} \mid W_0 \right] \, dx \right)^k / k!
\]
for $k \in \mathbb{N}_0$. Note that this distribution is trivial if the integral appearing twice on the right-hand side does not exist.

**Proof of Lemma 4.1** Let $X$ be a Poisson cloud with $0 \in X$ and denote by $X(A)$ the number of particles in $X \cap A$ for $A \subset \mathbb{R}^d$. Conditionally given $W_0$, every particle $x \in X \setminus \{0\}$ is now independently of the others removed from the Poisson cloud with probability $1 - p_{0x}$. By Proposition 1.3 of [17], the resulting process $X$ is a thinned Poisson cloud, conditionally given $W_0$, with intensity function $x \mapsto \nu \mathbb{E}_0 \{ p_{0x} \mid W_0 \}$. Since $D_0 = \tilde{X}(\mathbb{R}^d \setminus \{0\})$ in distribution, it follows that, conditionally given $W_0$, $D_0$ has a Poisson distribution with parameter $\nu \int_{\mathbb{R}^d} \mathbb{E}_0 \{ p_{0x} \mid W_0 \} \, dx$. $\square$

We now provide a necessary and sufficient condition for the existence of $\int_{\mathbb{R}^d} \mathbb{E}_0 \{ p_{0x} \mid W_0 \} \, dx$ in terms of $\alpha$ and $\beta$.

**Proposition 4.2** The following two statements are equivalent:

(i) $\min\{ \alpha, \beta \alpha \} > d$;

(ii) $\int_{\mathbb{R}^d} \mathbb{E}_0 \{ p_{0x} \mid W_0 \} \, dx < \infty$.

Lemma 4.1 and Proposition 4.2 imply that the distribution of degree $D_0$, conditionally given $W_0$, has a Poisson distribution whenever $\min\{ \alpha, \beta \alpha \} > d$, and that $D_0$ is infinite, a.s., otherwise. (i) of Theorem 3.1 is therefore a direct consequence of Lemma 4.1 and Proposition 4.2. The proof of Proposition 4.2 is based on integral calculations.

**Proof of Proposition 4.2** We obtain, using integration by parts in the second step,

$$
\mathbb{E}_0 \{ p_{0x} \mid W_0 \} = \int_1^\infty \beta w^{-\beta - 1} \left( 1 - \exp \{-\lambda W_0|x|^{-\alpha}w \} \right) \, dw \\
= 1 - \exp \{-\lambda W_0|x|^{-\alpha} \} + \lambda W_0|x|^{-\alpha} \int_1^\infty w^{-\beta} \exp \{-\lambda W_0|x|^{-\alpha} w \} \, dw \\
= 1 - \exp \{-\lambda W_0|x|^{-\alpha} \} + (\lambda W_0|x|^{-\alpha})^\beta \int_{\lambda W_0|x|^{-\alpha}}^\infty z^{-\beta} e^{-z} \, dz.
$$

Note that, given $W_0$, $1 - \exp \{-\lambda W_0|x|^{-\alpha} \}$ is integrable over $\mathbb{R}^d$ if and only if $\alpha > d$. It therefore remains to consider the integrability of $|x|^{-\beta \alpha} \int_{|x|^{-\alpha}}^\infty z^{-\beta} e^{-z} \, dz$. For $|x|^\alpha \geq 1$ we obtain lower bound

$$
|x|^{-\beta \alpha} \int_{|x|^{-\alpha}}^\infty z^{-\beta} e^{-z} \, dz \geq |x|^{-\beta \alpha} \int_1^\infty z^{-\beta} e^{-z} \, dz,
$$

which is not integrable over $\mathbb{R}^d$ for $\beta \alpha \leq d$. This finally shows that (ii) implies (i). For $|x|^\alpha \geq 1$ we obtain upper bound, assume $\beta \neq 1$,

$$
|x|^{-\beta \alpha} \int_{|x|^{-\alpha}}^\infty z^{-\beta} e^{-z} \, dz \leq |x|^{-\beta \alpha} \left[ \int_{|x|^{-\alpha}}^1 z^{-\beta} \, dz + \int_1^\infty e^{-z} \, dz \right] = |x|^{-\beta \alpha} \left[ \frac{1 - |x|^{\beta \alpha - \alpha}}{1 - \beta} + e^{-1} \right],
$$

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which is integrable over \( \mathbb{R}^d \) for \( \min\{\alpha, \beta \alpha\} > d \). Similarly, if \( \beta = 1 \), this finally shows that \((i)\) implies \((ii)\). \( \square \)

In order to prove part \((ii)\) of Theorem 3.1 we first calculate the integral \( \nu \int_{\mathbb{R}^d} \mathbb{E}_0 [ p_{0x} | W_0 ] \, dx \), which is finite for \( \min\{\alpha, \beta \alpha\} > d \).

**Proposition 4.3** Assume \( \min\{\alpha, \beta \alpha\} > d \), and set \( \tau = \beta \alpha / d > 1 \). We obtain

\[
\nu \int_{\mathbb{R}^d} \mathbb{E}_0 [ p_{0x} | W_0 ] \, dx = \frac{c_1}{\tau} W_0^{d/\alpha},
\]

where \( c_1 \) is defined in Theorem 3.1, which has a Pareto distribution with scale parameter \( c_1/\tau \) and shape parameter \( \tau \).

**Proof of Proposition 4.3** From Proposition 4.2 we obtain that we can apply Fubini’s theorem which provides

\[
\nu \int_{\mathbb{R}^d} \mathbb{E}_0 [ p_{0x} | W_0 ] \, dx = \nu \int_{\mathbb{R}^d} \left( \int_1^{\infty} \beta w^{-\beta-1} \left( 1 - \exp \left\{ -\lambda W_0 |x|^{-\alpha} w \right\} \right) \, dw \right) \, dx
\]

\[
= \nu \int_1^{\infty} \beta w^{-\beta-1} \left( \int_{\mathbb{R}^d} \left( 1 - \exp \left\{ -\lambda W_0 |x|^{-\alpha} w \right\} \right) \, dx \right) \, dw.
\]

We first calculate the inner integral. Using polar coordinates and integration by parts, we obtain for \( w \geq 1 \) and for \( v_d \) denoting the volume of the unit ball in \( \mathbb{R}^d \),

\[
\int_{\mathbb{R}^d} 1 - \exp \left\{ -\lambda W_0 |x|^{-\alpha} w \right\} \, dx = dv_d \int_0^{\infty} \left( 1 - \exp \left\{ -\lambda W_0 w^{-\alpha} t \right\} \right) r^{d-1} \, dr
\]

\[
= \frac{dv_d}{\alpha} \int_0^{\infty} \left( 1 - \exp \left\{ -\lambda W_0 w t \right\} \right) t^{-d/\alpha-1} \, dt
\]

\[
= - v_d \left( 1 - \exp \left\{ -\lambda W_0 w t \right\} \right) t^{-d/\alpha} \bigg|_0^{\infty}
\]

\[
+ v_d \lambda W_0 w \int_0^{\infty} \exp \left\{ -\lambda W_0 w t \right\} t^{-d/\alpha} \, dt
\]

\[
= v_d \lambda W_0 w \frac{\Gamma(1 - d/\alpha)}{(\lambda W_0 w)^{1-d/\alpha}} \int_0^{\infty} \frac{(\lambda W_0 w)^{1-d/\alpha} t^{1-d/\alpha-1} \exp \left\{ -\lambda W_0 w t \right\} \, dt}{\Gamma(1 - d/\alpha)}
\]

The latter is an integral over a gamma density for \( 1 - d/\alpha > 0 \). Therefore, we obtain

\[
\int_{\mathbb{R}^d} 1 - \exp \left\{ -\lambda W_0 |x|^{-\alpha} \right\} \, dx = v_d \Gamma(1 - d/\alpha) (\lambda W_0 w)^{d/\alpha}.
\]
For $\beta \alpha > d$, this implies that
\[
\nu \int_{\mathbb{R}^d} \mathbb{E}_0[\rho_{0x} W_0] \, dx = \nu v_d \Gamma(1 - d/\alpha) (\lambda W_0)^{d/\alpha} \int_1^\infty \beta w^{-\beta - 1} w^{d/\alpha} \, dw
\]
\[= \nu v_d \Gamma(1 - d/\alpha) \frac{\beta}{\beta - d/\alpha} (\lambda W_0)^{d/\alpha}.\]

Since $\beta/(\beta - d/\alpha) = \tau/(\tau - 1)$, the claim follows. $\square$

Proof of Theorem 3.1 The proof of part (i) is a direct consequence of Lemma 4.1 and Proposition 4.2. In order to prove part (ii), assume $\min\{\alpha, \beta \alpha\} > d$ and set $\tau = \beta \alpha/d > 1$. Then, by Lemma 4.1 and Proposition 4.3, $D_0$ has (under $\mathbb{P}_0$) a mixed Poisson distribution with mixing distribution being the Pareto distribution with scale parameter $c_1^{1/\tau}$ and shape parameter $\tau$. From this we obtain, since $\mathbb{E}_0[W_0^{d/\alpha}] = \beta/(\beta - d/\alpha) = \tau/(\tau - 1)$,
\[
\mathbb{E}_0[D_0] = \mathbb{E}_0[c_1^{1/\tau} W_0^{d/\alpha}] = \nu v_d \Gamma(1 - d/\alpha) \left(\frac{\tau}{\tau - 1}\right)^2 \lambda^{d/\alpha} < \infty,
\]
and for $n \geq 0$, see, for instance, Lemma 3.1.1 of [24],
\[
\mathbb{P}_0[D_0 > n] = \frac{1}{n!} \int_0^\infty x^n e^{-x} \mathbb{P}_0[c_1^{1/\tau} W_0^{d/\alpha} > x] \, dx
\]
\[= \frac{1}{n!} \int_0^{c_1^{1/\tau}} x^n e^{-x} \, dx + \frac{c_1}{n!} \int_{c_1^{1/\tau}}^\infty x^{n-\tau} e^{-x} \, dx
\]
\[= e^{-c_1^{1/\tau}} \sum_{j=n+1}^{\infty} \frac{c_1^{j/\tau}}{j!} + \frac{c_1}{\Gamma(n + 1)} \int_{c_1^{1/\tau}}^\infty x^{n-\tau} e^{-x} \, dx.
\]
Choose $n \geq 0$ with $n - \tau + 1 > 0$. Then,
\[
\mathbb{P}_0[D_0 > n] = \mathbb{P}_0[Z > n] + c_1 \frac{\Gamma(n + 1 - \tau)}{\Gamma(n + 1)} \mathbb{P}_0[Y_n > c_1^{1/\tau}],
\]
where $Z$ has a Poisson distribution with parameter $c_1^{1/\tau}$, and $Y_n$ has a gamma distribution with shape parameter $n - \tau + 1 > 0$ and scale parameter 1. Markov’s inequality provides
\[
n^n \mathbb{P}_0[Z \geq n] = n^n \mathbb{P}_0[e^{Z-n} \geq 1] \leq n^n \mathbb{P}_0[e^{Z}] e^{-n},
\]
which converges to 0 as $n \to \infty$. Moreover, by Stirling’s formula, $n^n \Gamma(n + 1 - \tau)/\Gamma(n + 1)$ tends to 1 as $n \to \infty$, and so does $\mathbb{P}_0[Y_n > s]$ for any $s > 0$. Therefore,
\[
\lim_{n \to \infty} n^n \mathbb{P}_0[D_0 > n] = c_1.
\]
\[\square\]
5 Phase Transition

In this section we prove Theorem 3.2 which gives the phase transition picture of the heterogeneous RCM in the case \( \min\{\alpha, \beta \alpha\} > d \). We first prove (a1) and (b1), namely that in any dimension \( d \geq 1 \) the critical percolation value \( \lambda_c \) equals 0 whenever \( \alpha > d \) and \( \tau = \beta \alpha/d \in (1, 2) \).

**Proof of (a2) and (b2)** The idea of the proof is similar to the proof of Theorem 4.4 in [10]. Assume \( \alpha > d \) and \( \tau = \beta \alpha/d \in (1, 2) \). The goal is to prove that \( \theta(\lambda) > 0 \) for all \( \lambda > 0 \). Choose \( \lambda > 0 \) and \( 0 < \varepsilon < \min\{d/\beta, \alpha(2/\tau - 1)\} \).

Define for \( k \geq 0 \) boxes \( A_{2^k} = [-2^k, 2^k]^d \) and for \( k \geq 1 \) define disjoint annuli \( R_k = A_{2^k} \setminus A_{2^{k-1}} \). For \( k \geq 1 \) denote by \( z_k \) the particle with maximal weight in \( R_k \) (if it exists). Using that for \( k \geq 1 \), the number of particles in \( R_k \), denoted by \( X(R_k) \), has a Poisson distribution with parameter of order \( \nu 2^{dk} \), one derives that the event \( \{X(R_k) \geq 1 \text{ and } W_{z_k} \geq 2^{k(d/\beta - 1)} \text{ for all } k \geq 1\} \) has positive probability. Given this event, we obtain that \( z_{k-1} \leftrightarrow z_k \) for all \( k \geq 1 \) with positive probability, where we set \( z_0 = 0 \). This implies \( \theta(\lambda) > 0 \). We refer to [10] for the details.

Next, we prove (a2) and (b2), which provide the non-trivial phase transition for appropriate choices of \( \alpha \) and \( \beta \).

**Proof of (a2) and (b2)** We first show that for any dimension \( d \geq 1, \lambda_c > 0 \) whenever \( \alpha > d \) and \( \tau = \beta \alpha/d > 2 \). We mimic the proof of Theorem 4.2 of [10]. Assume \( \alpha > d \) and \( \tau = \beta \alpha/d > 2 \). Set \( x_0 = 0 \). We say that \((x_1, \ldots, x_n) \in X^n\) is a self-avoiding path in \( X \) of length \( n \in \mathbb{N} \) starting from the origin, write \((x_1, \ldots, x_n)\) s.a., if for all \( i = 1, \ldots, n \) there is an edge between \( x_{i-1} \) and \( x_i \), and every particle \( x_i \) in that path occurs at most once. Since the degree distribution has finite mean, see Theorem 3.1, we obtain that the degree of each particle in such a path is bounded, a.s. Therefore, the event that the origin lies in an infinite connected component implies that for each \( n \in \mathbb{N} \) there is a self-avoiding path in \( X \) of length \( n \) starting from the origin. Therefore, using \( 1 - e^{-x} \leq x \wedge 1 \),

\[
\theta(\lambda) \leq \mathbb{E}_0 \left[ \sum_{\text{s.a.}} \prod_{i=1}^n p_{x_{i-1}x_i} \right] \leq \mathbb{E}_0 \left[ \sum_{\text{s.a.}} \mathbb{E}_0 \left[ \prod_{i=1}^n \left( \frac{\lambda W_{x_{i-1}} W_{x_i}}{|x_{i-1} - x_i|^\alpha} \wedge 1 \right) \right] \right],
\]

where the sums are over all self-avoiding paths \((x_1, \ldots, x_n) \in X^n\). For \( n \) even and distinct \( x_1, \ldots, x_n \in \mathbb{R}^d \), Cauchy–Schwarz inequality implies that

\[
\mathbb{E}_0 \left[ \prod_{i=1}^n \left( \frac{\lambda W_{x_{i-1}} W_{x_i}}{|x_{i-1} - x_i|^\alpha} \wedge 1 \right) \right] \leq \prod_{i=1}^n \mathbb{E}_0 \left[ \left( \frac{\lambda W_{x_{i-1}} W_{x_i}}{|x_{i-1} - x_i|^\alpha} \wedge 1 \right)^2 \right]^{1/2},
\]

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and similarly for \( n \) odd, it follows that for all \( n \in \mathbb{N} \),

\[
\theta(\lambda) \leq \mathbb{E}_0 \left[ \sum_{(x_1, \ldots, x_n) \text{ s.a.}} \prod_{i=1}^n \mathbb{E}_0 \left[ \left( \frac{\lambda W_{x_{i-1}} W_{x_i}}{|x_{i-1} - x_i|^\alpha} \right)^2 \right] \right].
\]

Similar to Lemma 4.3 of [10] we get for \( u \geq 1 \) and two i.i.d. random variables \( W_1 \) and \( W_2 \) having a Pareto distribution with scale parameter 1 and shape parameter \( \beta \), using integration by parts in the first step,

\[
\mathbb{E} \left[ (W_1 W_2/u)^2 \wedge 1 \right] = \frac{1}{u^2} + \frac{2}{u^3} \int_1^u v \mathbb{P} \left[ W_1 W_2 > v \right] dv
\]

\[
= \frac{1}{u^2} + \frac{2}{u^3} \int_1^u v^{1-\beta} (1 + \beta \log v) dv
\]

\[
\leq (1 + \beta \log u) \left( u^{-(\beta \wedge 2)} + \frac{2}{u^2} \int_1^u v^{1-\beta} dv \right)
\]

\[
\leq (1 + 1_{(\beta \neq 2)}/|\beta - 2|) (1 + \max\{2, \beta\} \log u)^2 u^{-(\beta \wedge 2)},
\]

where the last step follows by considering the cases \( \beta < 2, \beta = 2 \) and \( \beta > 2 \) separately. We finally have for any \( u \geq 0 \), set \( c_2 = c_2(\beta) = (1 + 1_{(\beta \neq 2)}/|\beta - 2|)^{1/2} \),

\[
\mathbb{E} \left[ (W_1 W_2/u)^2 \wedge 1 \right]^{1/2} \leq 1_{[u < 1]} + 1_{[u \geq 1]} \lambda (\beta^{(1/4)})(1/2) (1 + \max\{2, \beta\} \log u) u^{-(\beta \wedge 1)} = g(u), \quad (5.1)
\]

where the equality defines the function \( g \). Using this bound, we obtain

\[
\theta(\lambda) \leq \mathbb{E}_0 \left[ \sum_{(x_1, \ldots, x_n) \text{ s.a.}} \prod_{i=1}^n g\left(\lambda^{-1}|x_i - x_{i-1}|^\alpha\right) \right]
\]

\[
\leq v^n \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{i=1}^n g\left(\lambda^{-1}|x_i - x_{i-1}|^\alpha\right) dx_1 \cdots dx_n = \left( v \int_{\mathbb{R}^d} g\left(\lambda^{-1}|x|^\alpha\right) dx \right)^n.
\]

(5.2)

Choose \( \lambda < 1 \) such that \((1 + \max\{2, \beta\} \log(\lambda^{-1})) \lambda^{(\beta/4) \wedge (1/2)} \leq 1 \). This implies for all \( u \geq 0 \),

\[
g\left(\lambda^{-1}u\right) = 1_{[u < \lambda]} + 1_{[u \geq \lambda]} c_2 \left(1 + \max\{2, \beta\} \log(\lambda^{-1}u)\right) (\lambda^{-1}u)^{-(\beta \wedge 1)}
\]

\[
\leq 1_{[u < \lambda]} + 1_{[u \geq \lambda]} \lambda^{(\beta/4) \wedge (1/2)} c_2 \left(1 + \max\{2, \beta\} \log u\right) u^{-(\beta \wedge 1)}.
\]
This provides upper bound
\[
\int_{\mathbb{R}^d} g(\lambda^{-1}|x|^\alpha)\,dx 
\leq v_d \lambda^{d/\alpha} + \lambda^{(\beta/4) \land (1/2)} c_2 \int_{|x| \geq \lambda^{1/\alpha}} \left(1 + \max\{2, \beta\} \log((|x|^\alpha))\right)|x|^{-\alpha(\beta/2 \land 1)}\,dx.
\]

Note that the latter integral is finite because \(\alpha > d\) and \(\beta \alpha / 2 > d\). Therefore, we can choose \(\lambda > 0\) so small that the right-hand side is less than \(1/(2v)\). We finally obtain for all \(\lambda > 0\) sufficiently small,
\[
\theta(\lambda) \leq 2^{-n} \to 0, \quad \text{as } n \to \infty.
\]

This finally implies that for any dimension \(d \geq 1\), \(\lambda_c > 0\) whenever \(\alpha > d\) and \(\tau = \beta \alpha / d > 2\).

To finish the proof of (a2) and (b2) it remains to show that the critical percolation value \(\lambda_c\) is finite in the case \(\tau = \beta \alpha / d > 2\) and \(\alpha > d\) (\(\alpha \in (1, 2]\) if \(d = 1\)). We adapt the proof of Theorem 3.1 of [10].

Partition the space \(\mathbb{R}^d\) into cubes of side length \(n\), and let \(r = r(n, d)\) be the maximal possible distance between two particles in neighboring cubes. We call a cube \(\Lambda\) in the partition of \(\mathbb{R}^d\) good if \(X(\Lambda) \geq 1\), i.e., at least one particle \(x\) of the Poisson cloud \(X\) falls into \(\Lambda\). Note that a cube \(\Lambda\) is good with probability \(1 - \exp(-\nu n^d)\).

If two neighboring cubes are both good, then the probability that the two particles with maximal weight in the respective cubes are connected is bounded below by \(1 - \exp(-\lambda r^{-\alpha})\), note that \(W_0 \geq 1\), a.s. We now consider the site-bond percolation model on \(\mathbb{Z}^d\) where sites are alive independently with probability \(1 - \exp(-\nu n^d)\) and edges are added independently between alive nearest neighbor sites with probability \(1 - \exp(-\lambda r^{-\alpha})\). Note that \(1 - \exp(-\nu n^d)\) can be chosen arbitrarily close to 1 by taking \(n\) large and then the nearest neighbor edges can have probabilities arbitrarily close to 1 by choosing \(\lambda\) large. Therefore, in dimensions \(d \geq 2\), the site-bond percolation model on \(\mathbb{Z}^d\) percolates for sufficiently large \(\lambda\), see [16] and Theorem 3.2 of [10]. In the case \(d = 1\) and \(\alpha \in (1, 2]\) it follows from Theorem 1.2 of [18] that the described site-bond percolation model percolates. But this immediately implies that there exists an infinite connected component, a.s., in our model for sufficiently large \(\lambda\).

In order to complete the proof of Theorem 3.2 it remains to prove (b3), namely that the critical percolation value \(\lambda_c\) is infinite in dimension \(d = 1\) whenever \(\min\{\alpha, \beta \alpha\} > 2\). The proof is similar to the one of part (c) in Theorem 3.1 of [10]. Since the Poisson cloud induces an additional level of complexity, we prove (b3) in detail.

**Proof of (b3)** Assume \(d = 1\) and \(\min\{\alpha, \beta \alpha\} > 2\). We describe the particles of a Poisson cloud \(X = (x_i)_{i \in \mathbb{Z}}\) containing the origin as follows.

\[
\begin{align*}
x_0 &= 0; \\
x_i &= \inf\{x \in X | x > x_{i-1}\}, \quad &\text{for } i \in \mathbb{N}; \\
x_{-i} &= \sup\{x \in X | x < x_{-i+1}\}, \quad &\text{for } i \in \mathbb{N}.
\end{align*}
\]
Note that \( x_i - x_{i-1}, i \in \mathbb{Z} \), are i.i.d. having an exponential distribution with parameter \( \nu \). The proof of (b3) is simpler in the case where the weights \((W_x)_{x \in X}\) have finite mean. We start with this case.

**Case 1** Assume \( \beta > 1 \) so that we obtain \( \mathbb{E}_0[W_0] < \infty \). Choose \( x \in \mathbb{R} \) and define the event
\[
A_x = \{ \text{no particle } y \leq x \text{ shares an edge with any particle } z > x \}.
\]
The aim is to prove \( \mathbb{P}_0[A_0] > 0 \). Stationarity and ergodicity then imply that \( A_x \) occurs for infinitely many \( x \in \mathbb{R} \), a.s., hence \( \lambda_c = \infty \). We define for \( n \in \mathbb{N} \) the event
\[
A_0^{(n)} = \{ x_{-n+k} \leftrightarrow x_k \text{ for all } k = 1, \ldots, n \}.
\]

We get, using independence of edges,
\[
\mathbb{P}_0[A_0] = \mathbb{P}_0 \left[ \bigcap_{n \in \mathbb{N}} A_0^{(n)} \right] = \mathbb{E}_0 \left[ \prod_{n \in \mathbb{N}} \prod_{k=1}^{n} \exp \left\{ -\lambda W_{x_{-n+k}} W_{x_k} (x_k - x_{-n+k})^{-\alpha} \right\} \right].
\]

Note that the weights \((W_x)_{x \in X}\) are independent for different particles and also independent of their locations. Using Jensen’s inequality, we therefore get
\[
\mathbb{P}_0[A_0] \geq \exp \left\{ -\lambda \sum_{n \in \mathbb{N}} \sum_{k=1}^{n} \mathbb{E}_0[W_{x_{-n+k}}] \mathbb{E}_0[W_{x_k}] \mathbb{E}_0[(x_k - x_{-n+k})^{-\alpha}] \right\}.
\]

Since for each \( n \in \mathbb{N} \) and \( k = 1, \ldots, n \), the difference \( x_k - x_{-n+k} \) is the sum of \( n \) i.i.d. random variables having exponential distributions with parameter \( \nu \), \( x_k - x_{-n+k} \) has a gamma distribution with shape parameter \( n \) and scale parameter \( \nu \). Therefore,
\[
\mathbb{P}_0[A_0] \geq \exp \left\{ -\lambda \sum_{n \in \mathbb{N}} \sum_{k=1}^{n} v^{\alpha} \frac{\Gamma(n-\alpha)}{\Gamma(n)} \right\}
\]
\[
= \exp \left\{ -\lambda \mathbb{E}_0[W_0]^2 v^{\alpha} \sum_{n \in \mathbb{N}} n^2 \frac{\Gamma(n-\alpha)}{n!} \right\}.
\]

By Stirling’s approximation and since \( \alpha > 2 \), the latter sum is finite, which finishes the proof in the case \( \beta > 1 \).

**Case 2** Assume \( \beta \leq 1 \). Using independence of edges, we obtain
\[
\mathbb{P}_0[A_0] = \mathbb{E}_0 \left[ \exp \left\{ -\lambda \sum_{i,j \geq 0, (i,j) \neq (0,0)} W_{x-i} W_{x_j} (x_j - x_i)^{-\alpha} \right\} \right].
\]
Since we condition on having a particle at the origin, we obtain \( x_j - x_i = x_j + |x_i| \) for all \( i, j \geq 0 \) and \((x_j + |x_i|)^2 \geq x_j |x_i| > 0 \) for all \( i, j \geq 1 \). This implies

\[
P_0[A_0] = \mathbb{E}_0 \left[ \exp \left\{ -\lambda W_0 \sum_{j \geq 1} W_{x_j} x_j^{-\alpha} - \lambda \sum_{i \geq 1} W_{x_i} (x_j - x_i)^{-\alpha} \right\} \right] 
\geq \mathbb{E}_0 \left[ \exp \left\{ -\lambda W_0 \sum_{j \geq 1} W_{x_j} x_j^{-\alpha} - \lambda \sum_{i \geq 1} W_{x_i} |x_i|^{-\alpha/2} \sum_{j \geq 1} W_{x_j} x_j^{-\alpha/2} \right\} \right].
\]

In order to prove that all sums on the right-hand side are finite, a.s., it suffices to check that \( \mathbb{E}_0 \left[ \sum_{j \geq 1} W_{x_j} x_j^{-\alpha/2} \right] < \infty \). Since \( \beta \leq 1 \) and \( \beta \alpha > 2 \) we can choose \( \epsilon > 0 \) such that \(-\alpha/2 + (1 - \beta)(1 - \epsilon)/\beta < -1\), and we set \( a_j = j^{(1+\epsilon)/\beta} \) for \( j \geq 1 \). Write

\[
\sum_{j \geq 1} W_{x_j} x_j^{-\alpha/2} = \sum_{j \geq 1} \frac{W_{x_j} \wedge a_j}{x_j^{\alpha/2}} + \sum_{j \geq 1} \frac{(W_{x_j} - a_j)^+}{x_j^{\alpha/2}}. \tag{5.3}
\]

Since \( \mathbb{P} \left[ W_0 > a_j \right] = a_j^{-\beta} = j^{-(1+\epsilon)} \), the Borel–Cantelli lemma implies that \((W_{x_j} - a_j)^+\) is positive for only finitely many \( j \geq 1 \), a.s. Hence, the second sum in (5.3) is finite, a.s. For the first sum in (5.3) note that

\[
\mathbb{E}_0 \left[ \frac{W_{x_j} \wedge a_j}{x_j^{\alpha/2}} \right] = \mathbb{E}_0 \left[ x_j^{-\alpha/2} \right] \mathbb{E}_0 \left[ W_{x_j} \wedge a_j \right] 
\leq \nu^{\alpha/2} \frac{\Gamma(j - \alpha/2)}{\Gamma(j)} \sum_{1 \leq k \leq a_j} \mathbb{P} [W_0 > k] 
= \nu^{\alpha/2} \frac{j \Gamma(j - \alpha/2)}{j!} \sum_{1 \leq k \leq a_j} k^{-\beta}.
\]

For \( \beta < 1 \) this implies for an appropriate constant \( c_3 > 0 \), using Stirling’s approximation,

\[
\mathbb{E}_0 \left[ \frac{W_{x_j} \wedge a_j}{x_j^{\alpha/2}} \right] \leq c_3 \frac{j \Gamma(j - \alpha/2)}{j!} a_j^{1-\beta} 
= c_3 \frac{\Gamma(j - \alpha/2)}{j!} j^{1+(1-\beta)(1+\epsilon)/\beta} 
= c_3 j^{-\alpha/2+(1-\beta)(1+\epsilon)/\beta} (1 + o(1)),
\]
as \( j \to \infty \). By the choice of \( \varepsilon > 0 \), the right-hand side is summable in \( j \). The same conclusion holds true in the case \( \beta = 1 \). This completes the proof of (b3). \( \square \)

6 Percolation on Finite Boxes

In this section we prove Theorems 3.3 and 3.4 and Corollary 3.5. The key result is Lemma 6.1 below which corresponds to Lemma 2.3 of [3] in homogeneous long-range percolation on \( \mathbb{Z}^d \).

**Lemma 6.1** Assume \( \alpha \in (d, 2d) \) and \( \tau = \beta \alpha / d > 1 \), and choose \( \lambda \in (0, \infty) \) with \( \theta(\lambda, \alpha) > 0 \). For every \( \varepsilon \in (0, 1) \), \( \rho > 0 \) and \( \alpha' < 2d \) there exists \( m' \geq 1 \) such that for all \( m \geq m' \),

\[
\mathbb{P} \left[ |C_m| \geq \rho m^{\alpha'/2} \right] \geq 1 - \varepsilon,
\]

where \( |C_m| \) denotes the number of particles of the largest connected component in \( \Lambda_m \).

The proof of Lemma 6.1 is based on a renormalization technique introduced in [3]. We explain this renormalization in detail because the Poisson cloud induces an additional level of complexity. For integers \( m \geq 1 \), \( k \geq 0 \) and \( x \in m\mathbb{Z}^d \) we define the box with corner \( x \) and side length \( m \) and its \( k \)-enlargement by

\[
B_m(x) = x + [0, m)^d \quad \text{and} \quad B_m^{(k)}(x) = x + [-k, m+k)^d,
\]

respectively. We write \( B_m \) and \( B_m^{(k)} \) if \( x = 0 \). We call a set of at least \( \ell \geq 1 \) particles in \( B_m(x) \cap X \) an \( \ell \)-semi-cluster if these particles are connected within its \( k \)-enlargement \( B_m^{(k)}(x) \).

For an integer-valued sequence \( (a_n)_{n \in \mathbb{N}_0} \) with \( a_n > 1 \), \( n \geq 0 \), we define for \( n \in \mathbb{N} \) the cube lengths

\[
m_n = a_n m_{n-1} = m_0 \prod_{i=1}^{n} a_i = \prod_{i=0}^{n} a_i, \quad \text{with} \ m_0 = a_0.
\]

For \( x \in m_n \mathbb{Z}^d \) we call \( B_{m_n}(x) \) an \( n \)-stage box. Note that each \( n \)-stage box contains \( a_n^d \) of \( (n-1) \)-stage boxes \( B_{m_{n-1}}(z) \subset B_{m_n}(x) \) with \( z \in m_{n-1} \mathbb{Z}^d \), which we call children of \( B_{m_n}(x) \). In the following we recursively define the aliveness of \( n \)-stage boxes.

**Definition 6.2** Let \( (a_n)_{n \in \mathbb{N}_0} \) be an integer-valued sequence with \( a_n > 1 \), \( n \geq 0 \), and define \( (m_n)_{n \in \mathbb{N}_0} \) as above. Choose \( k \geq 0 \), and let \( (\theta_n)_{n \in \mathbb{N}_0} \) be a real-valued sequence with \( \theta_n \in (0, 1) \) for \( n \geq 0 \).

- For \( x \in m_0 \mathbb{Z}^d \), we say that \( 0 \)-stage box \( B_{m_0}(x) \) is alive if it contains a \( (\theta_0 a_0^d) \)-semi-cluster, i.e., it contains at least \( \theta_0 a_0^d \) particles that are connected within \( B_{m_0}^{(k)}(x) \).
- For \( n \in \mathbb{N} \) and \( x \in m_n \mathbb{Z}^d \), we say that \( n \)-stage box \( B_{m_n}(x) \) is alive if the event \( A_{n,x} = A_{n,x}^{(a)} \cap A_{n,x}^{(b)} \) occurs, where

\[
A_{n,x}^{(a)} = \{ \text{at least} \ \theta_n a_n^d \ \text{children of} \ B_{m_n}(x) \ \text{are alive} \}.
\]
\[ A_{n,x}^{(b)} = \{ \text{all}(\prod_{i=0}^{n-1} \theta_i a_i^d)\text{-semi-clusters of all alive children of } B_{m_n}(x) \} \].

For \( n \in \mathbb{N}_0 \), define \( u_n = \prod_{i=0}^{n} \theta_i \) and note that every alive \( n \text{-stage box } B_{m_n}(x) \) contains at least \( \prod_{i=0}^{n} \theta_i a_i^d = m_n^d u_n \) particles that are connected within \( k \)-enlargement \( B_{m_n}^{(k)}(x) \). The next lemma provides a recursive lower bound for \( p_n = \mathbb{P}[A_{n,x}], n \in \mathbb{N} \) and \( x \in m_n \mathbb{Z}^d \).

**Lemma 6.3** Assume \( \alpha \in (d, 2d) \), and choose \( \lambda > 0 \). Let \( \xi \in (\alpha/d, 2) \) and \( \gamma \in (0, 1) \) such that \( 18\gamma > 16 + \xi \). Choose a real-valued sequence \( (\theta_n)_{n \in \mathbb{N}_0} \) with \( \theta_n \in (0, 1), n \in \mathbb{N}_0 \), and an integer-valued sequence \( (a_n)_{n \in \mathbb{N}} \) with \( a_n > 1, n \in \mathbb{N} \). Assume that there exists \( m'_0 \geq 1 \) such that for all \( a_0 = m_0 \geq m'_0 \), all \( n \in \mathbb{N} \) and all \( s \in (2e^{-1} v m_{n-1}^d, 2e v m_{n-1}^d) \),

\[
s'^{−} < m_{n-1}^d \prod_{i=0}^{n-1} \theta_i = u_{n-1} m_{n-1}^d \quad \text{and} \quad s'^{−} < \lambda \left( \sqrt{d} m_n \right)^{−\alpha}. \tag{6.1}
\]

There exist \( \varphi > 0 \) and \( m''_0 \geq m'_0 \) such that for every \( a_0 = m_0 \geq m''_0, k \geq 0, \) and for all \( n \in \mathbb{N} \) and \( x \in m_n \mathbb{Z}^d \),

\[
p_n = \mathbb{P}[A_{n,x}] \geq 1 - \frac{1 - p_{n-1}}{1 - \theta_n} - a_n^2 d \left( 2e^{-2v m_{n-1}^d(1-2/e)} + \left( 2e^{-1} v m_{n-1}^d \right)^{−\varphi} \right).
\]

**Proof of Lemma 6.3** Let \( n \in \mathbb{N} \) and \( x \in m_n \mathbb{Z}^d \). We obtain

\[
1 - p_n = \mathbb{P}[A_{n,x}^c] = \mathbb{P}\left[ (A_{n,x}^{(a)} \cap A_{n,x}^{(b)})^c \right] \leq \mathbb{P}\left[ (A_{n,x}^{(a)})^c \right] + \mathbb{P}\left[ (A_{n,x}^{(b)})^c \right]. \tag{6.2}
\]

For the first term in (6.2), Markov’s inequality and translation invariance provide

\[
\mathbb{P}\left[ (A_{n,x}^{(a)})^c \right] = \mathbb{P}\left[ \sum_{B_{m_{n-1}}(z) \subset B_{m_n}(x)} 1_{A_{n-1,z} < \theta_n a_n^d} \right] = \mathbb{P}\left[ \sum_{B_{m_{n-1}}(z) \subset B_{m_n}(x)} 1_{A_{n-1,z}^c > (1 - \theta_n) a_n^d} \right] \leq \frac{1}{1 - \theta_n} \mathbb{P}\left[ A_{n-1,z}^c \right] = \frac{1 - p_{n-1}}{1 - \theta_n}.
\]

The second term in (6.2) is more involved due to possible dependence in the \( k \)-enlargements, \( k \geq 0 \). For two children \( B^1 \) and \( B^2 \) of \( B_{m_n}(x) \), let \( E(B^1, B^2) \) be...
the event that at least two \((m_{n-1}^d)^d\)-semi-clusters in \(B^1 \cup B^2\) are not connected within \(B_{mn}^d(x)\). We obtain

\[
\Prob\left[ (A_{n,x}^{(b)})^c \right] \leq \left( \frac{a_n^d}{2} \right) \sup_{(B^1, B^2)} \Prob\left[ E(B^1, B^2) \right].
\]

(6.3)

where the supremum is taken over all possible choices of two distinct children \(B^1\) and \(B^2\) of \(B_{mn}^d(x)\). We fix two different children \(B^1\) and \(B^2\) of \(B_{mn}^d(x)\). We obtain

\[
\Prob\left[ E(B^1, B^2) \right] 
\leq \Prob\left[ E(B^1, B^2), 2e^{-1}vm_{n-1}^d < X(B^1 \cup B^2) < 2evm_{n-1}^d \right] 
+ \Prob\left[ X(B^1 \cup B^2) \geq 2evm_{n-1}^d \right] + \Prob\left[ X(B^1 \cup B^2) \leq 2e^{-1}vm_{n-1}^d \right],
\]

where \(X(A)\) denotes the number of particles of \(X\) in \(A \subset \mathbb{R}^d\). A random variable \(Y\) having a Poisson distribution with parameter \(\mu > 0\) satisfies, using Chernoff bound,

\[
\Prob\left[ Y \leq e^{-1}\mu \right] \leq e^{-\mu(1-2/e)} \quad \text{and} \quad \Prob\left[ Y \geq e\mu \right] \leq e^{-\mu},
\]

(6.4)

see, for instance, (A.12) of [13]. Using these bounds we obtain

\[
\Prob\left[ E(B^1, B^2) \right] \leq \Prob\left[ E(B^1, B^2), 2e^{-1}vm_{n-1}^d < X(B^1 \cup B^2) < 2evm_{n-1}^d \right] + 
+ 2e^{-2vm_{n-1}^d(1-2/e)}.
\]

To estimate the probability above we will condition on the Poisson cloud restricted to \(B_{mn}^d(x)\). We fix integers \(s \in (2e^{-1}vm_{n-1}^d, 2evm_{n-1}^d)\), \(t \geq 0\), and choose \(x_1, \ldots, x_{s+t} \in B_{mn}^d(x)\) with \(x_1, \ldots, x_s \in B^1 \cup B^2\). Assume that \(\{x_1, \ldots, x_s\} = (B^1 \cup B^2) \cap X\) and \(\{x_1, \ldots, x_t\} = B_{mn}^d(x) \cap X\). Consider the edge probabilities

\[
\tilde{p}_{x_i x_j} = 1 - \exp\{-\lambda|x_i - x_j|^{-\alpha}\} \leq 1 - \exp\{-\lambda W_{x_i} W_{x_j} |x_i - x_j|^{-\alpha}\} = p_{x_i x_j},
\]

a.s., for every \(i \neq j \in \{1, \ldots, s+t\}\). Denote by \(\tilde{P}_X\) the probability measure of the resulting edge configurations restricted to \(\{x_1, \ldots, x_{s+t}\}\) induced by \(\tilde{p}_{x_i x_j}, i \neq j \in \{1, \ldots, s+t\}\). Note that on \(B_{mn}^d(x) \cap X = \{x_1, \ldots, x_{s+t}\}\), \(E(B^1, B^2)\) is determined by edges with end points in \(\{x_1, \ldots, x_{s+t}\}\). We therefore get

\[
\Prob\left[ E(B^1, B^2) \mid X \cap (B^1 \cup B^2) = \{x_1, \ldots, x_s\}, X \cap B_{mn}^d(x) = \{x_1, \ldots, x_{s+t}\} \right] \leq \tilde{P}_X[E(B^1, B^2)].
\]
We can now argue as in the proof of Lemma 2.3 of [5], which we briefly recall. By an abuse of notation, assume that $B^1 \cup B^2 = \{x_1, \ldots, x_s\}$ and $B^{(k)}_{m_n}(x) = \{x_1, \ldots, x_{s+t}\}$.

Using that sequence $(a_n)_{n \in \mathbb{N}_0}$ satisfies (6.1) and $B^1 \cup B^2 \subseteq B_{m_n}(x)$, we obtain $\tilde{p}_{x_j} > 1 - \exp(-s^{-\xi}) = \nu_n > 0$ for all $x_i \neq x_j \in B^1 \cup B^2$, where the equality defines $\nu_n$. For $x_i \neq x_j \in B^1 \cup B^2$, we define $\tilde{q}_{x_i,x_j}$ by $\tilde{p}_{x_i,x_j} = \tilde{q}_{x_i,x_j} + \nu_n - \nu_n \tilde{q}_{x_i,x_j}$.

We now sample an edge configuration $\omega$ on $B^{(k)}_{m_n}(x)$ induced by the $\tilde{p}_{x_i,x_j}$’s in two steps. We sample $\omega'$ on $B^{(k)}_{m_n}(x)$ with edge probabilities $\tilde{q}_{x_i,x_j}$ if $x_i \neq x_j \in B^1 \cup B^2$ and with edge probabilities $\tilde{p}_{x_i,x_j}$ otherwise. $\omega''$ is then independently sampled on $B^1 \cup B^2$ with edge probabilities $\nu_n$. Note that $\omega = \omega' \lor \omega''$ in distribution. Let $S_1 \neq S_2 \subset B^1 \cup B^2$ be two disjoint maximal sets in $B^1 \cup B^2$ that are each connected within $B^{(k)}_{m_n}(x)$ by $\omega'$-edges. Here, the maximality of set $S_1$ means that there is no particle in $B^1 \cup B^2$ which does not belong to $S_1$, but is connected to $S_1$ within $B^{(k)}_{m_n}(x)$ by $\omega'$-edges. Given $\omega'$, the probability that there is an $\omega$-edge between $S_1$ and $S_2$ is equal to the probability that there is an $\omega''$-edge between $S_1$ and $S_2$, given $\omega'$, which follows by maximality of $S_1$ and $S_2$. The latter probability is by definition of $\omega''$ and $\nu_n$ given by $1 - \exp(-|S_1||S_2|s^{-\xi})$, where $|S_i|$ denotes the number of particles of $S_i$, $i = 1, 2$. Given $\omega'$, denoting by $S_1, \ldots, S_l$ all disjoint maximal sets in $B^1 \cup B^2$ that are connected within $B^{(k)}_{m_n}(x)$ by $\omega'$-edges, the indices $\{1, \ldots, l\}$ form an inhomogeneous random graph of size $\sum_{i=1}^l |S_i| = s$ and parameter $\xi$, as defined in [5]. Lemma 2.5 of [5] shows that there exists $\varphi > 0$ such that for all $M$ sufficiently large and all inhomogeneous random graphs of size $M$ and parameter $\xi$, the probability that such a graph contains more than one cluster of size at least $M^{\varphi}$ is at most $M^{-\varphi}$. This implies that there exist $\varphi > 0$ and $m''_0 \geq m'_0$ such that for all $m_0 \geq m''_0$, given $\omega'$, there is at most one cluster of size at least $s^{\varphi}$ formed by $S_1, \ldots, S_l$ that are connected by $\omega$-edges, with probability at most $s^{-\varphi}$. Note that the existence of $\varphi$ is uniform in $s \in (2e^{-1}vm_{n-1}^d, 2evm_{n-1}^d)$, $t \geq 0$, the locations of $\{x_1, \ldots, x_{s+t}\}$, $\omega'$, $n \geq 1$, $k \geq 0$ and $m_0 \geq m''_0$. Using that sequences $(a_n)_{n \in \mathbb{N}_0}$ and $(\theta_n)_{n \in \mathbb{N}_0}$ satisfy (6.1), i.e., $s^{\varphi} < u_{n-1}m_{n-1}^d$, we conclude that there exist $\varphi > 0$ and $m''_0 \geq m'_0$ such that for every $m_0 \geq m''_0$, $k \geq 0$, $n \in \mathbb{N}$, $s \in (2e^{-1}vm_{n-1}^d, 2evm_{n-1}^d)$, $t \geq 0$, and $x_1, \ldots, x_{s+t} \in B^{(k)}_{m_n}(x)$ with $x_1, \ldots, x_s \in B^1 \cup B^2$,

$$\mathbb{P}[X(E(B^1, B^2)) \leq s^{-\varphi}].$$

Integrating over the particles in $B^{(k)}_{m_n}(x) \setminus (B^1 \cup B^2)$ and $B^1 \cup B^2$ we then get for all $m_0 \geq m''_0$,

$$\mathbb{P}[E(B^1, B^2)] \leq 2e^{-2vm_{n-1}^d(1-2/e)} + \sum_{s=1}^{[2evm_{n-1}^d-1]} \mathbb{P}[X(B^1 \cup B^2) = s] s^{-\varphi} \leq 2e^{-2vm_{n-1}^d(1-2/e)}$$

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\[ + \left( 2e^{-1}vn_{n-1}^d \right)^{-\varphi} \sum_{s=\lfloor 2e^{-1}vn_{n-1}^d \rfloor +1}^{\lfloor 2e\nu v_{n-1}^{d} \rfloor -1} \mathbb{P} \left[ X(B_1 \cup B^2) = s \right] \]
\[ \leq 2e^{-2\nu v_{n-1}^{d}(1-2/\varepsilon)} + \left( 2e^{-1}vn_{n-1}^d \right)^{-\varphi}, \]

which together with (6.3) implies for all \( m_0 \geq m_0'' \), \( k \geq 0 \) and \( n \in \mathbb{N} \),
\[ \mathbb{P} \left[ (A_{n,\lambda}^{(b)})^c \right] \leq a_n^{2d} \left( 2e^{-2\nu v_{n-1}^{d}(1-2/\varepsilon)} + \left( 2e^{-1}vn_{n-1}^d \right)^{-\varphi} \right). \]

This finishes the proof of Lemma 6.3. \( \square \)

Using the recursion in Lemma 6.3 and sequences \( (a_n)_{n \in \mathbb{N}} = (n^d)_{n \in \mathbb{N}} \) and \( (\theta_n)_{n \in \mathbb{N}} = (n^{-b})_{n \in \mathbb{N}} \) for appropriate constants \( 1 < a < b \), for explicit choices see (5) in [5], \( 1 - p_n \) can be bounded by a multiple (independent of \( n \)) of \( 1 - p_0 \). In order to prove that \( p_n \) is arbitrarily close to 1, it remains to show that 0-stage boxes are alive with arbitrarily high probability for sufficiently large \( a_0 = m_0 \) and well-chosen \( \theta_0 \in (0, 1) \).

**Proof of Lemma 6.1** Note that \( \alpha \in (d, 2d) \) and \( \tau = \beta \alpha / d > 1 \) imply \( \lambda_c < \infty \), see Theorem 3.2. Hence, there exists \( \lambda \in (0, \infty) \) with \( \theta = \theta(\lambda, \alpha) > 0 \) and for these parameters we have a unique infinite connected component \( C_\infty \subset X \), a.s. In order to prove Lemma 6.1 it suffices to check that for any \( \varepsilon' \in (0, 1) \) there exists \( \theta_0 \in (0, 1) \) such that \( p_0 = \mathbb{P} \left[ A_{0,0} \right] > 1 - \varepsilon' \) for all \( m_0 \) sufficiently large. The remainder of the proof then follows from the one of Lemma 2.3 of [5] using Lemma 6.3 above with appropriate sequences \( (a_n)_{n \in \mathbb{N}} \) and \( (\theta_n)_{n \in \mathbb{N}} \). Choose \( \varepsilon' \in (0, 1) \), and set \( \theta_0 = \upsilon \upsilon_d 2^{-d} \theta / 2 \), where \( \upsilon_d \) denotes the Lebesgue measure of the unit ball in \( \mathbb{R}^d \). For \( m \geq 1 \) denote by \( \left| C_\infty \cap B_m \right| \) the number of Lebesgue measure of the unit ball in \( \mathbb{R}^d \). For \( m \geq 1 \) it holds that
\[ \mathbb{P} \left[ C_\infty \cap B_m \geq \theta_0 m^d \right] \geq \mathbb{P} \left[ \sum_{x \in X \cap B(m/2)} 1_{\{x \in C_\infty \}} \geq \theta_0 m^d \right], \]

where \( B(m/2) \) denotes the ball of radius \( m/2 \) around the origin. Using ergodicity and (12.4.3) of [9] it follows that
\[ \frac{1}{\upsilon \upsilon_d (m/2)^d} \sum_{x \in X \cap B(m/2)} 1_{\{x \in C_\infty \}} \rightarrow \mathbb{P}_0 \left[ 0 \in C_\infty \right] = \theta, \quad \text{asm} \rightarrow \infty, \text{ a.s.} \]
Hence, for all \( m_0 \) sufficiently large we obtain
\[ \mathbb{P} \left[ \left| C_\infty \cap B_{m_0} \right| \geq \theta_0 m_0^d \right] \geq 1 - \varepsilon'/2. \]

Since the infinite connected component \( C_\infty \) is unique, a.s., there exists \( k = k(m_0) \geq 0 \) such that \( C_\infty \cap B_{m_0} \) is connected within \( k \)-enlargement \( B_{m_0}^{(k)} \). Choose \( k = k(m_0) \geq 0 \).
such that
\[ P[C_\infty \cap B_{m_0} \text{ is connected within } B_{m_0}^{(k)}] > 1 - \varepsilon'/2. \]

This implies that for all \( m_0 \) sufficiently large,
\[ p_0 = P[A_{0,0}] \geq P[|C_\infty \cap B_{m_0}| \geq \theta_0 m_0^d \text{ and } C_\infty \cap B_{m_0} \text{ is connected within } B_{m_0}^{(k)}] \geq 1 - \varepsilon'. \]

**Proof of Theorems 3.3 and 3.4** Note that by Lemma 6.1, the number of particles of the largest connected component in \( \Lambda_m \) is at least \( \rho m^{\alpha'} \) for \( \rho > 0 \) and \( \alpha' < d \), with high probability. In order to prove that this number is proportional to \( m^d \), with high probability, we apply a second renormalization based on site-bond percolation, as in the proof of Theorem 6 in [11], where the bound on the probability that a site is alive is given by Lemma 6.1 above. Theorem 3.4 then follows directly from the results in [11]. This is immediately clear because the bounds on attachedness of alive sites derived in Lemma 10 (a) in [11] also apply to the Poisson case, see in particular estimate (2) in [11]. In the same way, using a site-bond percolation model, Theorem 3.3 follows from Theorem 3 in [11].

**Proof of Corollary 3.5** Using Theorem 3.4, the statements \((i)\) and \((ii)\) of Corollary 3.5 follow from Corollaries 3.3 and 3.4 of [6], respectively. Note that Lemma 3.5 therein corresponds to Lemma 10 (a) in [11]. We refer to [6] for the details.

**7 Graph Distances**

**7.1 Infinite Variance of Degree Distribution, Upper Bound**

In order to prove Theorem 3.6 we first prove the upper bound of statement \((a)\) which we recall in the next proposition.

**Proposition 7.1** Assume \( \alpha > d \) and \( \tau = \beta \alpha / d \in (1, 2) \). For every \( \lambda > \lambda_c = 0 \) and \( \varepsilon > 0 \) we obtain
\[ \lim_{|x| \to \infty} P[d(0, x) \leq (1 + \varepsilon) \frac{2 \log \log |x|}{\log(\beta \alpha / d - 1)}] \text{ for } 0, x \in C_\infty = 1. \]

For the proof of this proposition, we use the following technical lemma.

**Lemma 7.2** Assume \( \alpha > d \) and \( \tau = \beta \alpha / d \in (1, 2) \). If there exists a particle in \( \Lambda_r = [-r, r]^d \), \( r > 0 \), let \( z_0 \in X \cap \Lambda_r \) be the particle with maximal weight \( W_{z_0} \) in \( \Lambda_r \). We obtain for any constant \( c \geq 1 \),
\[ \lim_{w \to \infty} P[|C(z_0)| = \infty \mid W_{z_0} \geq w, X(\Lambda_r) \geq c] = 1. \]
where $|C(z_0)|$ denotes the number of particles of the connected component of $z_0$, and where $X(\Lambda_r)$ denotes the number of particles in box $\Lambda_r$.

Note that this result corresponds to Lemma 5.2 of [10]. The only difference lies in the additional condition that $\Lambda_r$ contains at least $c \geq 1$ particles, which ensures that particle $z_0$ with maximal weight in $\Lambda_r$ exists.

Proof of Lemma 7.2 Let $\alpha > d$ and $\tau = \beta \alpha / d \in (1, 2)$. Choose $b > 1$ such that $2d / \beta - b \alpha > 0$, and choose $\epsilon \in (\alpha / 2, d / \beta)$. Choose $r > 0$, and for $w \geq 1$ sufficiently large we define disjoint annuli $R_1 = \Lambda_{w_{1/\alpha}} \setminus \Lambda_r$, $R_2 = \Lambda_{w^{1/\alpha}} \setminus \Lambda_{w^{1/\alpha}}$ and $R_k = \Lambda_{w^{1/\alpha}} \setminus \Lambda_{w^{1/\alpha}}$ for $k \geq 3$. If $X(\Lambda_r) \geq c$, let $z_0 \in X \cap \Lambda_r$ be the particle with maximal weight $W_{z_0}$ in $\Lambda_r$. For $k \geq 1$, if $X(R_k) \geq 1$, let $z_k \in X \cap R_k$ be the particle with maximal weight $W_{z_k}$ in $R_k$. By the choices of $b$ and $\epsilon$, and since $X(R_k)$ is Poisson distributed, we obtain that the event $\{X(R_k) \geq 1\}$ and $W_{z_k} \geq 2^{k \epsilon} w^{d / \beta}$ for all $k \geq 1$ has probability arbitrarily close to 1 for $w$ sufficiently large. Given this event and the event $\{X(\Lambda_r) \geq c\}$, we obtain that $z_0 \leftrightarrow z_1$ and $z_{k+1} \leftrightarrow z_k$ for all $k \geq 1$ with probability arbitrarily close to 1 for $w$ sufficiently large, which implies the claim. We refer to the corresponding proof in [10] for the details. □

Proof of Proposition 7.1 Let $\alpha > d$ and $\tau = \beta \alpha / d \in (1, 2)$. Fix $\epsilon > 0$ and choose $b \in (0, 1)$ such that

$$d(1 + b) / \beta > \alpha \quad \text{and} \quad \frac{1 + \epsilon / 2}{|\log b|} \leq \frac{1 + \epsilon}{|\log(\beta \alpha / d - 1)|}.$$ 

Fix $m > \max\{e, 3^{1/(1-b)} , c_4^{-1/d}, m_0, m_1\}$, with $c_4 = (1 - 2/e) v_2 - d$ and where $m_0 = m_0(b) \geq 0$ and $m_1 = m_1(b) \geq 0$ are defined below. Choose $x \in \mathbb{R}^d$ with $|x| \geq e^{(\log m) / b}$, and set

$$k = k(x) = \left[ \frac{\log |x| - \log \log m}{\log |b|} \right] \geq 1.$$ 

Note that $m \leq |x|^{b_k} \leq m^{1 / b}$ for all $x \in \mathbb{R}^d$. For $i = 0, 1, \ldots, k$, write $\Lambda(x, b^i)$ for the box with side length $|x|^{b^i} / 2$ centered at the point at distance $|x|^{b^i} / 2$ from the origin on the segment with end points 0 and $x$. If there is a particle in $\Lambda(x, b^i)$, let $z_i \in \Lambda(x, b^i) \cap X$ be the particle with maximal weight $W_{z_i}$ in $\Lambda(x, b^i)$. Moreover, write $\Lambda'(x, b^i)$ for the box with side length $|x|^{b^i} / 2$ centered at the point at distance $|x|^{b^i} / 2$ from $x$ (instead of the origin) on the segment with end points 0 and $x$. If there is a particle in $\Lambda'(x, b^i)$, let $z'_i \in \Lambda'(x, b^i) \cap X$ be the particle with maximal weight $W_{z'_i}$ in $\Lambda'(x, b^i)$, and this choice is similar to the one in the proof of Theorem 5.1 of [10]. In order to make sure that particles $z_i$ and $z'_i$ exist and that boxes $\Lambda(x, b^i)$ and $\Lambda'(x, b^i)$ contain sufficiently many particles for all $i = 0, \ldots, k$, we consider the probability measure

\[ Springer \]
\[ \mathbb{P}^k[.] = \mathbb{P} \left[ \cdot \mid X(A(x, b^i)) \geq c_4|x|^{db^i} \text{ and } X(A'(x, b^i)) \geq c_4|x|^{db^i} \text{ for all } i = 0, \ldots, k \right]. \]

which is the conditional probability given that there are at least \( c_4|x|^{db^i} \geq c_4m^d \geq 1 \) particles in each of boxes \( A(x, b^i) \) and \( A'(x, b^i) \) for \( i = 0, \ldots, k \). Using Chernoff bound, see (6.4), we obtain for each \( i = 0, \ldots, k \), since \( X(A(x, b^i)) \) has a Poisson distribution with parameter \( \nu 2^{-d}|x|^{db^i} \),

\[ \mathbb{P} \left[ X(A(x, b^i)) < c_4|x|^{db^i} \right] \leq \mathbb{P} \left[ X(A(x, b^i)) \leq e^{-1}\nu 2^{-d}|x|^{db^i} \right] \leq e^{-c_4|x|^{db^i}}, \]

for each \( i = 0, \ldots, k \). Note that \( m > 3^{1/(1-b)} \) and \( m \leq |x|^{b^k} \) imply \( |x|^{b^i} - 3|x|^{b^{i+1}} > 0 \) for all \( i = 0, \ldots, k - 1 \). This implies that all boxes \( A(x, b^i) \) are disjoint for \( i = 0, \ldots, k \). It follows that

\[ \mathbb{P} \left[ X(A(x, b^i)) \geq c_4|x|^{db^i} \text{ and } X(A'(x, b^i)) \geq c_4|x|^{db^i} \text{ for all } i = 0, \ldots, k \right] \geq \prod_{i=0}^{k} \left( 1 - e^{-c_4|x|^{db^i}} \right)^2, \]

where the inequality comes from the fact that \( A(x, 1) = A'(x, 1) \). We write \( k = k(x) = \lfloor N(x) - M \rfloor \) with \( N = N(x) = (\log \log |x|)/| \log b | \) and \( M = (\log \log m)/| \log b | \), and we obtain, note that \( | \log b | = - \log b \) (because \( b \in (0, 1) \)),

\[ \lim_{|x| \to \infty} \sum_{i=0}^{k} \exp \left\{ -c_4|x|^{db^i} \right\} = \lim_{N \to \infty} \sum_{i=0}^{\lfloor N-M \rfloor} \exp \left\{ -c_4e^{b-N} db^i \right\} \leq \lim_{N \to \infty} \sum_{i=0}^{\lfloor N-M \rfloor} \exp \left\{ -c_4e^{db^{-(\lfloor N-M \rfloor)-i}-M} \right\} = \lim_{N \to \infty} \sum_{j=0}^{\lfloor N-M \rfloor} \exp \left\{ -c_4e^{db^{-j-M}} \right\} = \sum_{j \geq 0} \exp \left\{ -c_4e^{db^{-j-M}} \right\} \in (0, \infty). \quad (7.1) \]

For fixed \( \varepsilon' > 0 \), we therefore can choose \( m_0 > 0 \) (used for the choice of \( m > m_0 \)) so large that for any sufficiently large \( |x| \), the first equality defines event \( N_k = N_k(x, b) \),

\[ \mathbb{P}[N_k] = \mathbb{P} \left[ X(A(x, b^i)) \geq c_4|x|^{db^i} \text{ and } X(A'(x, b^i)) \geq c_4|x|^{db^i} \text{ for all } i = 0, \ldots, k \right] \geq 1 - \varepsilon'. \quad (7.2) \]
Note that for every $\delta \in (0, 1)$ and $i = 0, \ldots, k$,
\[ \mathbb{P}^k \left[ W_{z_i} \leq \left( c_4 |x|^{db_j} \right)^{(1-\delta)/\beta} \right] \leq \mathbb{P} \left[ W_{z_i} \leq X(A(x, b_j))^{(1-\delta)/\beta} \right] \geq c_4 |x|^{db_j} \]
\[ = \mathbb{P} \left[ \max_{z \in A(x, b_j) \cap X} W_z \leq X(A(x, b_j))^{(1-\delta)/\beta} \right] \geq c_4 |x|^{db_j} \]
\[ = \mathbb{E} \left[ (1 - X(A(x, b_j))^{\delta-1}) X(A(x, b_j)) \right] \geq c_4 |x|^{db_j}. \]

Using $1 - x \leq e^{-x}$, we obtain
\[ \mathbb{P}^k \left[ W_{z_i} \leq \left( c_4 |x|^{db_j} \right)^{(1-\delta)/\beta} \right] \leq \mathbb{E} \left[ e^{-X(A(x, b_j))^{\delta}} \right] \geq c_4 |x|^{db_j}. \]
\[ \leq e^{-c_4 |x|^{db_j}}. \] (7.3)

Since $|z_i - z_{i+1}| < d|x|^{b_j}$ for each $i = 0, \ldots, k - 1$, we therefore obtain
\[ \mathbb{P}^k \left[ \bigcup_{i=0}^{k-1} \{ z_i \neq z_{i+1} \} \right] \leq \sum_{i=0}^{k-1} \mathbb{E}^k \left[ e^{-\lambda d^{-\alpha} W_{z_i} W_{z_{i+1}} |x|^{-\alpha b_j}} \right] \]
\[ \leq \sum_{i=0}^{k-1} \mathbb{E}^k \left[ e^{-\lambda d^{-\alpha} c_4 |x|^{db_j}} \mathbb{1}_{W_{z_j} > \left( c_4 |x|^{db_j} \right)^{(1-\delta)/\beta}} \right] + 2e^{-c_4 |x|^{db_j}} \]
\[ \leq \sum_{i=0}^{k-1} \exp \left\{ -\lambda d^{-\alpha} c_4 2^{(1-\delta)/\beta} |x|^{b_j d(1-\delta)(1+b)/\beta} |x|^{-\alpha b_j} \right\} + 2e^{-c_4 |x|^{db_j}} \]
\[ = \sum_{i=0}^{k-1} \exp \left\{ -\lambda d^{-\alpha} c_4 2^{(1-\delta)/\beta} |x|^{b_j d(1-\delta)\beta - \alpha} \right\} + 2e^{-c_4 |x|^{db_j}}. \]

Since $b$ was chosen such that $d(1+b)/\beta > \alpha$, we can choose $\delta = \delta(b) \in (0, 1)$ so small that $d(1-\delta)(1+b)/\beta - \alpha > 0$. We therefore proceed as in (7.1) to see that there exists $m_1 \geq 0$ (used for the choice of $m > m_1$) so large that for any sufficiently large $|x|$, $\mathbb{P}^k \left[ \bigcup_{i=0}^{k-1} \{ z_i \neq z_{i+1} \} \right] \leq \varepsilon'$. Using symmetry we therefore obtain for sufficiently large $|x|$,
\[ \mathbb{P}^k \left[ z_i \leftrightarrow z_{i+1} \text{ and } z'_i \leftrightarrow z'_{i+1} \text{ for all } i = 0, \ldots, k - 1 \right] \geq 1 - 2\varepsilon'. \] (7.4)

Recall that $z_0$ is the Poisson particle with maximal weight $W_{z_0}$ in box $A(x, 1)$ with side length $|x|/2$ centered at the midpoint of the segment with end points 0 and $x$. For every $w \geq 1$ it holds that for sufficiently large $|x|$, see also (7.3) with $i = 0$, 

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\[ \mathbb{P}^k [ W_{z_0} \leq w ] \leq \varepsilon'. \]

Moreover, using that box \( A(x, 1) \) does not intersect boxes \( A(x, b^i) \) and \( A'(x, b^i) \) for all \( i = 1, \ldots, k \), we obtain for \(|x| \) sufficiently large,

\[
\mathbb{P}^k \left[ |C(z_0)| < \infty \mid W_{z_0} \geq w \right] \leq \mathbb{P} \left[ |C(z_0)| < \infty \mid \begin{array}{l}
W_{z_0} \geq w, X(A(x, 1)) \geq c_4 |x|^d \\
X(A(x, b^i)) \geq c_4 |x|^{db^i}, X(A'(x, b^i)) \geq c_4 |x|^{db^i} \forall i = 1, \ldots, k
\end{array} \right] \\
\leq \frac{\mathbb{P} \left[ X(A(x, 1)) \geq c_4 |x|^d \wedge X(A'(x, b^i)) \geq c_4 |x|^{db^i} \forall i = 1, \ldots, k \right]}{1 - \varepsilon'},
\]

where we used (7.2) for the second inequality. Using Lemma 7.2 with \( r = |x|/4 \) we see that the numerator is less than \( \varepsilon' \) for sufficiently large \( w \). (Note that convergence in Lemma 7.2 is uniform in \( r \) and \( c \geq 1 \).) We therefore obtain for sufficiently large \(|x|\),

\[
\mathbb{P}^k [ |C(z_0)| = \infty ] \geq \mathbb{P}^k [ |C(z_0)| = \infty \mid W_{z_0} \geq w ] \mathbb{P}^k [ W_{z_0} \geq w ] \\
\geq \frac{1 - 2\varepsilon'}{1 - \varepsilon'} (1 - \varepsilon') = 1 - 2\varepsilon'.
\]

Together with (7.4), this implies that for sufficiently large \(|x|\), the event

\[ E = \{ z_i \leftrightarrow z_{i+1} \text{ and } z'_i \leftrightarrow z'_{i+1} \text{ for all } i = 0, \ldots, k - 1 \} \cap \{|C(z_0)| = \infty\} \]

satisfies \( \mathbb{P} [ E ] > 1 - 4\varepsilon' \). It follows that, using (7.2),

\[
\mathbb{P}^k [ E \mid 0, x \in C_\infty ] \geq 1 - 4\varepsilon'/\mathbb{P}^k_{0,x} [0, x \in C_\infty ] \\
= 1 - \frac{4\varepsilon' \mathbb{P}[N_k]}{\mathbb{P}[N_k, 0, x \in C_\infty ] \mathbb{P}_{0,x} [0, x \in C_\infty ]} \\
\geq 1 - \frac{4\varepsilon' \mathbb{P}[N_k]}{(1 - \varepsilon'/\mathbb{P}_{0,x} [0, x \in C_\infty ]) \mathbb{P}_{0,x} [0, x \in C_\infty ]} \\
= 1 - \frac{4\varepsilon' \mathbb{P}[N_k]}{\mathbb{P}_{0,x} [0, x \in C_\infty ] - \varepsilon'} = 1 - \varepsilon'').
\]

where the last equality defines \( \varepsilon'' > 0 \). Note that on event \( E \), because of the choice of \( k \) and since \( z_0 = z'_0 \),

\[
d(0, x) \leq d(0, z_k) + \sum_{i=1}^k (d(z_i, z_{i-1}) + d(z'_{i-1}, z'_i)) + d(z'_k, x) \\
= d(0, z_k) + 2k + d(z'_k, x) \leq d(0, z_k) + 2\frac{\log \log |x|}{|\log b|} + d(z'_k, x).
\]
Moreover, on event $E$, because $z_0 = z'_0$, we have that $z_k$ and $z'_k$ are both in the infinite connected component $C_\infty$. If, in addition, we assume that $0 \in C_\infty$, then $0$ and $z_k$ are in the same component $C_\infty$. Since $|z_k| \leq 3|x|^b/4 \leq m^{1/b}$ and the infinite connected component $C_\infty$ is unique, a.s., it follows that on $E \cap \{0 \in C_\infty\}$, $0$ and $z_k$ are connected within box $\Lambda_{\tilde{m}}$ with probability arbitrarily close to 1, for some $\tilde{m} = m(m) < \infty$. Note that $\tilde{m}$ is independent of $x$. This implies for any $\kappa > 0$ and sufficiently large $|x|$, 

$$
\mathbb{P}^k \left[ d(0, z_k) \leq \kappa \log \log |x| \left| 0 \in C_\infty, \ E \right. \right] \geq 1 - \epsilon'.
$$

By symmetry we therefore obtain for $|x|$ sufficiently large, 

$$
\mathbb{P}^k \left[ d(0, z_k) + d(z'_k, x) \leq 2\kappa \log \log |x| \left| 0, x \in C_\infty, \ E \right. \right] \geq 1 - 2\epsilon' .
$$

Therefore, if we choose $\kappa = \epsilon/(2|\log b|)$ and $|x|$ sufficiently large, 

$$
\mathbb{P}^k \left[ d(0, x) \leq \frac{2(1 + \epsilon/2) \log \log |x|}{|\log b|} \left| 0, x \in C_\infty \right. \right] \geq \mathbb{P}^k \left[ d(0, x) \leq \frac{2(1 + \epsilon/2) \log \log |x|}{|\log b|} \left| 0, x \in C_\infty, \ E \right. \right] \geq \mathbb{P}^k \left[ d(0, x) \leq \frac{\epsilon \log \log |x|}{|\log b|} \left| 0, x \in C_\infty, \ E \right. \right] (1 - \epsilon'') 

\geq (1 - 2\epsilon')(1 - \epsilon'').
$$

It follows that for sufficiently large $|x|$, using (7.2) in the last step, 

$$
\mathbb{P} \left[ d(0, x) \leq \frac{2(1 + \epsilon/2) \log \log |x|}{|\log b|} \left| 0, x \in C_\infty \right. \right] \geq (1 - 2\epsilon')(1 - \epsilon'') \mathbb{P} \left[ N_k | 0, x \in C_\infty \right. \right] \geq (1 - 2\epsilon')(1 - \epsilon'')(1 - \epsilon'/\mathbb{P}_{0,x} | 0, x \in C_\infty \right. )).
$$

This finishes the proof of Proposition 7.1. 

\[ \square \]

### 7.2 Infinite Variance of Degree Distribution, Lower Bound

Next, we give the proof of the lower bound of statement (a) of Theorem 3.6 which we recall in the following proposition. Note that this proposition differs from the corresponding Theorem 5.3 of [10] in the discrete space model.

**Proposition 7.3** Assume $\alpha > d$ and $\tau = \beta \alpha/d \in (1, 2)$. For every $\lambda > \lambda_c = 0$ there exists $\eta_1 > 0$ such that

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\[ \lim_{|x| \to \infty} \mathbb{P}_{0,x} \left[ d(0, x) \geq \eta_1 \frac{2 \log \log |x|}{\log \kappa} \right] = 1, \]

with \( \kappa = \alpha (\beta \wedge 1)/d - 1 \in (0, 1) \).

**Proof of Proposition 7.3** We modify the proof of Theorem 5.3 of [10] to our situation. Choose \( \vartheta > 1 \) and \( \mu > 0 \) such that

\[ d/\vartheta - d \kappa + \mu < 0 \quad \text{and} \quad \mu < d \kappa. \]

Note that this choice is possible since the above constraints require \( \vartheta > 1/\kappa \) and \( \mu \in (0, d(\kappa - 1/\vartheta)) \). For \( x \in X \) and \( n \in \mathbb{N} \) we define the random variable

\[ S_n(x) = \sup_{y \in X : d(x, y) \leq n} |x - y|, \]

which represents the Euclidean distance between \( x \) and the furthest particle that can be reached from \( x \) using at most \( n \) edges. For \( r > 0 \) we denote by \( B(r) \) the ball of (Euclidean) radius \( r \) around the origin. For \( t > 1 \) we obtain, using \( 1 - e^{-x} \leq x \wedge 1 \),

\[ \mathbb{P}_{0,x} \left[ S_{n-1}(0) < t^{1/\vartheta}, \ S_n(0) \geq t \right] \leq \mathbb{P}_{0,x} \left[ \exists z \in B(t^{1/\vartheta}) \cap X, \ z' \in B(t) \cap X : z \leftrightarrow z' \right] \leq \mathbb{E}_{0,x} \left[ \sum_{z \in B(t^{1/\vartheta}) \cap X} \sum_{z' \in B(t) \cap X} \mathbb{E} \left[ \lambda W_{Z,W_{Z'}}^\alpha \left| z - z' \right|^{\beta \wedge 1} \left| X \right. \right] \right]. \]

For two i.i.d. random variables \( W_1 \) and \( W_2 \) having a Pareto distribution with scale parameter 1 and shape parameter \( \beta \), we obtain for \( u \geq 1 \), using integration by parts in the first step,

\[
\mathbb{E} \left[ \frac{W_1 W_2}{u} \left| X \right. \right] = \frac{1}{u} + \frac{1}{u} \int_1^u \mathbb{P}[W_1 W_2 > v]dv = \frac{1}{u} + \frac{1}{u} \int_1^u v^{-\beta} (1 + \beta \log v)dv \\
\leq (1 + \beta \log u) \left( u^{-(\beta \wedge 1)} + \frac{1}{u} \int_1^u v^{-\beta} dv \right) \\
\leq \max \{1 + \log u, 1 + 1_{\{\beta \neq 1\}/|\beta - 1|} \} (1 + \beta \log u) u^{-(\beta \wedge 1)},
\]

where the last step follows by distinguishing between the cases \( \beta = 1, \beta > 1 \) and \( \beta < 1 \). This provides for \( u \geq 1 \),

\[
\mathbb{E} \left[ \frac{W_1 W_2}{u} \left| X \right. \right] \leq (1 + 1_{\{\beta \neq 1\}/|\beta - 1|}) (1 + \max \{1, \beta \} \log u)^2 u^{-(\beta \wedge 1)}. \quad (7.5)
\]

Choose \( t \) so large that \( \lambda^{-1}(t - t^{1/\vartheta})^\alpha \geq 1 \) which together with (7.5) implies that

\[ \mathbb{P}_{0,x} \left[ S_{n-1}(0) < t^{1/\vartheta}, \ S_n(0) \geq t \right] \leq \max \{1 + \log u, 1 + 1_{\{\beta \neq 1\}/|\beta - 1|} \} (1 + \beta \log u) u^{-(\beta \wedge 1)} \]

(7.6)
We therefore obtain for an appropriate constant $c > 0$

\[ 
\begin{aligned}
\sum_{z' \in B(t)^c \cap X} \mathbb{E}
\left[ \frac{W_z W_{z'}}{\lambda^{-1}|z - z'|^\alpha} \wedge 1 \mid X \right]
\end{aligned}
\]

\[ 
\leq (1 + 1_{\{\beta \neq 1\}/|\beta - 1|}) \mathbb{E}_0, x \left[ \sum_{z \in B(t^{1/\beta}) \cap X} \sum_{z' \in B(t)^c \cap X} (1 + \max\{1, \beta\} \log (\lambda^{-1}|z - z'|^\alpha)^2 \lambda^{\beta \wedge 1} \leq |z - z'|^{\mu} \text{ for all } z \in B(t^{1/\beta}) \text{ and } z' \in B(t)^c. \right]
\]

Choose $t$ so large that $(1 + 1_{\{\beta \neq 1\}/|\beta - 1|}) (1 + \max\{1, \beta\} \log (\lambda^{-1}|z - z'|^\alpha)^2 \lambda^{\beta \wedge 1} \leq |z - z'|^{\mu}$ for all $z \in B(t^{1/\beta})$ and $z' \in B(t)^c$. It follows that for sufficiently large $t$, note that $d(\kappa + 1) = \alpha(\beta \wedge 1)$,

\[ 
\begin{aligned}
P_{0, x} \left[ S_{n-1}(0) < t^{1/\beta}, S_n(0) \geq t \right] & \leq \mathbb{E}_0, x \left[ \sum_{z \in B(t^{1/\beta}) \cap X} \sum_{z' \in B(t)^c \cap X} |z - z'|^{-\alpha(\beta \wedge 1) + \mu} \right] \\
& = \mathbb{E}_0, x \left[ \sum_{z \in B(t^{1/\beta}) \cap X} \mathbb{E}_{0, x} \left[ \sum_{z' \in B(t)^c \cap X} |z - z'|^{-d(\kappa + 1) + \mu} \right] \right].
\end{aligned}
\]

We estimate the right-hand side under the unconditional measure $\mathbb{P}$ instead of $\mathbb{P}_{0, x}$. Note that the tail behavior is the same under both measures. We obtain

\[ 
\begin{aligned}
& \mathbb{E} \left[ \sum_{z \in B(t^{1/\beta}) \cap X} \mathbb{E} \left[ \sum_{z' \in B(t)^c \cap X} |z - z'|^{-d(\kappa + 1) + \mu} \right] \right] \\
& = \mathbb{E} \left[ \sum_{z \in B(t^{1/\beta}) \cap X} \nu \int_{z' \in B(t)^c} |z - z'|^{-d(\kappa + 1) + \mu} dz' \right] \\
& \leq \mathbb{E} \left[ \sum_{z \in B(t^{1/\beta}) \cap X} \nu \int_{|z'| \geq t^{-1/\beta}} |z'|^{-d(\kappa + 1) + \mu} dz' \right] \\
& = \nu^2 \nu \int_{|z'| \geq t^{-1/\beta}} |z'|^{-d(\kappa + 1) + \mu} dz'.
\end{aligned}
\]

We therefore obtain for an appropriate constant $c_5 > 0$ and for $t$ sufficiently large,

\[ 
\begin{aligned}
P_{0, x} \left[ S_{n-1}(0) < t^{1/\beta}, S_n(0) \geq t \right] \leq c_5 t^{d/\beta - d\kappa + \mu}. \tag{7.6}
\end{aligned}
\]

Define $f : \mathbb{N}_0^2 \to (0, \infty)$ by $f(m, n) = m^{\beta n}$ for all $m, n \in \mathbb{N}_0$. Observe
(1) $\sum_{k=2}^{\infty} f(2, k)^{d/\vartheta - d\kappa + \mu} < \infty$ because $\vartheta > 1$ and $d/\vartheta - d\kappa + \mu < 0$;

(2) for all $m \geq 2$ and sufficiently small $\eta_1 = \eta_1(m) > 0$: $f \left( m, \left\lceil \eta_1 \frac{\log \log |x|}{|\log \kappa|} \right\rceil \right) \leq |x|/2$ for all sufficiently large $|x|$.

We choose $m_0 \geq 2$ so large that (7.6) holds true for all $t = f(m, n)$ with $m \geq m_0$ and $n \geq 2$. Using (7.6) and induction we obtain for each $n \geq 2$ and $m \geq m_0$, note that $f^{1/\vartheta} = f(m, n - 1)$,

$$
\mathbb{P}_{0,x} [S_n(0) \geq f(m, n)] \leq \mathbb{P}_{0,x} [S_{n-1}(0) \geq f(m, n-1)]
+ \mathbb{P}_{0,x} [S_{n-1}(0) < f(m, n-1), S_n(0) \geq f(m, n)]
\leq \mathbb{P}_{0,x} [S_{n-1}(0) \geq f(m, n-1)] + c_5 f(m, n)^{d/\vartheta - d\kappa + \mu}
\leq \mathbb{P}_{0,x} [S_1(0) \geq f(m, 1)] + c_5 \sum_{k=2}^{n} f(m, k)^{d/\vartheta - d\kappa + \mu}
\leq \mathbb{P}_{0,x} [\exists \ y \in X \text{ with } |y| \geq f(m, 1) \text{ and } 0 \leftrightarrow y]
+ c_5 \sum_{k=2}^{\infty} f(m, k)^{d/\vartheta - d\kappa + \mu}.
$$

Note that the right-hand side is independent of $n \geq 2$ and is finite for any $m \geq 2$. Since $f(m, k)^{d/\vartheta - d\kappa + \mu}$ is decreasing in $m \geq 2$, there exists $m \geq m_0$ such that the right-hand side is less than $\varepsilon$ for fixed $\varepsilon > 0$. We finally obtain for sufficiently small $\eta_1 = \eta_1(m) > 0$ and for all sufficiently large $|x|$, set $n(x) = \left\lceil \eta_1 \frac{\log \log |x|}{|\log \kappa|} \right\rceil \geq 2$,

$$
\mathbb{P}_{0,x} \left[ d(0, x) \leq \eta_1 \frac{2 \log \log |x|}{|\log \kappa|} \right]
\leq \mathbb{P}_{0,x} [d(0, x) \leq 2n(x)]
\leq \mathbb{P}_{0,x} [S_{n(x)}(x) \geq |x|/2] + \mathbb{P}_{0,x} [S_{n(x)}(0) \geq |x|/2]
= 2\mathbb{P}_{0,x} [S_{n(x)}(0) \geq |x|/2]
\leq 2\mathbb{P}_{0,x} [S_{n(x)}(0) \geq f \left( m, \left\lceil \eta_1 \frac{\log \log |x|}{|\log \kappa|} \right\rceil \right)] \leq 2\varepsilon,
$$

which finishes the proof of Proposition 7.3. \hfill \Box

### 7.3 Finite Variance of Degree Distribution, Case 1, Lower Bound

In the following, we give the proof of part (b1) of Theorem 3.6. We first prove the lower bound which follows from the following proposition.

**Proposition 7.4** Assume $\alpha > d$ and $\tau = \beta \alpha / d > 2$. For every $\lambda > \lambda_c$ there exists $\eta' > 0$ such that

$$
\mathbb{P}_{0,x} [d(0, x) \geq \eta' \log |x|] = 1.
$$
Proof of Proposition 7.4 Choose \( n \in \mathbb{N}, 0, x \in X \), and set \( x_0 = 0 \) and \( x_n = x \). As in (5.2) we obtain, the first sum is over all self-avoiding paths of length \( n \) starting from 0, note that \( x_n = x \) is now fixed,

\[
\mathbb{P}_{0,x} [d(0, x) = n] \leq \mathbb{E}_{0,x} \left[ \sum_{(x_1, \ldots, x_n) \text{ s.a.}} \prod_{i=1}^{n} p_{x_{i-1}x_i} \right] 
\leq v^{n-1} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \left( \prod_{i=1}^{n} h(x_i) \right) h \left( x - \sum_{i=1}^{n-1} x_i \right) dx_1 \ldots dx_{n-1},
\]

where for \( y \in \mathbb{R}^d \) we define function \( h \) by, recall function \( g \) defined in (5.1),

\[
h(y) = g(\lambda^{-1}|y|^{\alpha}) = 1_{\{|y|<\lambda^{1/\alpha}\}} + 1_{\{|y|\geq\lambda^{1/\alpha}\}} c_2 \lambda^{(\beta/2\wedge 1)} \left( 1 + \max\{2, \beta\} \log(\lambda^{-1}|y|^{\alpha}) \right)|y|^{-\alpha(\beta/2\wedge 1)}.
\]

Note that \( h \) is integrable because \( \alpha > d \) and \( \tau = \beta \alpha/d > 2 \). Using \( x_0 = 0 \) and \( x_n = x \), and substituting inductively \( x_i \) by \( x_i - \sum_{j=1}^{i-1} x_j \) for \( i = 1, \ldots, n-1 \), it follows that

\[
\mathbb{P}_{0,x} [d(0, x) = n] \leq v^{n-1} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \left( \prod_{i=1}^{n-1} h(x_i) \right) h \left( x - \sum_{i=1}^{n-1} x_i \right) dx_1 \ldots dx_{n-1}.
\]

We condition on \( 1_{\{|x_i|<|x|/n\}} \) and \( 1_{\{|x_i|\geq|x|/n\}} \) for all \( i = 1, \ldots, n-1 \). Note that if \( |x_i| < |x|/n \) for all \( i = 1, \ldots, n-1 \) we have \( |x - \sum_{i=1}^{n-1} x_i| \geq |x|/n \), and we bound the corresponding factor in the integral by \( \sup_{y \in \mathbb{R}^d: |y|\geq|x|/n} h(y) \). Otherwise, by exchangeability of the \( x_i \)'s, there are \( n-1 \) different cases where at least one of the \( x_i \)'s satisfies \( |x_i| \geq |x|/n \). In each of these \( n-1 \) cases we bound one corresponding factor in the integral by \( \sup_{y \in \mathbb{R}^d: |y|\geq|x|/n} h(y) \). Note that the restriction on \( x - \sum_{i=1}^{n-1} x_i \) then drops and we obtain

\[
\mathbb{P}_{0,x} [d(0, x) = n] \leq n \left( \sup_{y \in \mathbb{R}^d: |y|\geq|x|/n} h(y) \right) \left( v \int_{\mathbb{R}^d} h(y)dy \right)^{n-1}, \tag{7.7}
\]

where \( v \int_{\mathbb{R}^d} h(y)dy < \infty \) since \( h \) is integrable. Next, we bound the supremum on the right-hand side of (7.7). Choose \( \eta > 0 \), and let \( |x| \) be so large that \( \eta \log |x| \geq 1 \). Choose \( n \in \mathbb{N} \) with \( n \leq \eta \log |x| \). Let \( \mu \in (0, \alpha(\beta/2 \wedge 1)) \), and choose \( |x| \) so large that any \( y \in \mathbb{R}^d \) with \( |y| \geq |x|/n \) satisfies

\[
c_2 \lambda^{(\beta/2\wedge 1)} \left( 1 + \max\{2, \beta\} \log(\lambda^{-1}|y|^{\alpha}) \right) \leq |y|^{\mu}.
\]

If, in addition, \( |x| \) is so large that \( |x|/n > \lambda^{1/\alpha} \), then for any \( y \in \mathbb{R}^d \) with \( |y| \geq |x|/n > \lambda^{1/\alpha} \),
We finally obtain for all $\eta > 0$ and $1 \leq n \leq \eta \log |x|$ with $|x|$ sufficiently large,

$$
\sup_{y \in \mathbb{R}^d: |y| \geq |x|/n} h(y) \leq \eta^{(\beta/2 \land 1) - \mu} \left( \log |x| \right)^{\alpha(\beta/2 \land 1) - \mu} |x|^{-\alpha(\beta/2 \land 1) + \mu}.
$$

Together with (7.7) we obtain for any $\eta > 0$ and $1 \leq n \leq \eta \log |x|$ with $|x|$ sufficiently large,

$$
P_{0,x} \left[ d(0, x) = n \right] \leq n \left( v \int_{\mathbb{R}^d} h(y)dy \right)^{n-1} \eta^{(\beta/2 \land 1) - \mu} \left( \log |x| \right)^{\alpha(\beta/2 \land 1) - \mu} |x|^{-\alpha(\beta/2 \land 1) + \mu}
$$

$$
\leq \left( 1 + v \int_{\mathbb{R}^d} h(y)dy \right)^{\eta \log |x|} \eta^{(\beta/2 \land 1) - \mu + 1} \left( \log |x| \right)^{\alpha(\beta/2 \land 1) - \mu + 1} |x|^{-\alpha(\beta/2 \land 1) + \mu + \eta \log (1 + v \int_{\mathbb{R}^d} h(y)dy)} \leq |x|^{-\delta},
$$

where the last inequality holds for some $\delta > 0$ whenever $|x|$ is sufficiently large and $\eta > 0$ is chosen so small that $-\alpha(\beta/2 \land 1) + \mu + \eta \log (1 + v \int_{\mathbb{R}^d} h(y)dy) < 0$. We conclude that there exist $\eta' > 0$ and $\delta > 0$ such that for all $|x|$ sufficiently large,

$$
P_{0,x} \left[ d(0, x) \leq \eta' \log |x| \right] = \sum_{1 \leq n \leq \eta' \log |x|} P_{0,x} \left[ d(0, x) = n \right] \leq \eta' \left( \log |x| \right) |x|^{-\delta},
$$

which converges to 0 as $|x| \to \infty$. \hfill \Box

### 7.4 Finite Variance of Degree Distribution, Case 1, Upper Bound

In order to finish the proof of statement (b1) of Theorem 3.6 it remains to show the corresponding upper bound on the graph distances. The result follows from the following proposition, see also Proposition 4.1 of [6].

**Proposition 7.5** Assume $\alpha \in (d, 2d)$ and $\tau = \beta \alpha/d > 2$, and choose $\lambda > \lambda_c$. For each $\varepsilon > 0$ and $\Delta' > \Delta = \log(2)/\log(2d/\alpha)$ there exists $N_0 < \infty$ such that for all $x, y \in \mathbb{R}^d$ with $|x - y| \geq N_0$,

$$
P_{x,y} \left[ d(x, y) \geq (\log |x - y|)^{\Delta'}, x, y \in C_\infty \right] \leq \varepsilon.
$$

Note that the latter statement implies

$$
P \left[ d(x, y) \leq (\log |x - y|)^{\Delta'} \bigg| x, y \in C_\infty \right]
$$
To prove Proposition 7.5 we use the concept of hierarchies of particles. For \( k \geq 1 \) we call an element \( \sigma \in \{0, 1\}^k \), such as \( \sigma = 01110001 \), a hierarchical index. If \( k = 0 \), \( \sigma \in \{0, 1\}^k \) denotes the empty string. For \( \sigma_1 \in \{0, 1\}^k \) and \( \sigma_2 \in \{0, 1\}^l \), \( k, l \geq 1 \), we denote by \( \sigma_1 \sigma_2 \) the concatenation of \( \sigma_1 \) and \( \sigma_2 \). Then [6] provides the following definition of a hierarchy.

**Definition 7.6** For \( m \geq 1 \) and two distinct particles \( x, y \in X \) we say that the set of particles

\[
\mathcal{H}_m(x, y) = \left\{ z_{\sigma} \in X \mid \sigma \in \{0, 1\}^k, k = 1, \ldots, m \right\} \subset X
\]

is a hierarchy of depth \( m \) connecting \( x \) and \( y \) if

1. \( z_0 = x \) and \( z_1 = y \);
2. \( z_{\sigma_{00}} = z_{\sigma_0} \) and \( z_{\sigma_{11}} = z_{\sigma_1} \) for all \( \sigma \in \{0, 1\}^k \) and \( k = 0, \ldots, m - 2 \);
3. for all \( \sigma \in \{0, 1\}^k \) and \( k = 0, \ldots, m - 2 \), there is an edge between \( z_{\sigma_{01}} \) and \( z_{\sigma_{10}} \) as long as \( z_{\sigma_{01}} \neq z_{\sigma_{10}} \);
4. each edge as in 3 appears only once in \( \mathcal{H}_m(x, y) \).

For \( \sigma \in \{0, 1\}^{m-2} \), we call the pairs of particles \((z_{\sigma_{00}}, z_{\sigma_{01}})\) and \((z_{\sigma_{11}}, z_{\sigma_{10}})\) “gaps.”

Recall that for \( x \in \mathbb{R}^d \) and \( n \in (0, \infty) \) we write \( \Lambda_n(x) = x + [-n, n]^d \) for the box with center \( x \) and side length \( 2n \), and for \( x \in X \) we write \( C_n(x) \) for the set of particles in \( \Lambda_n(x) \cap X \) that are connected to \( x \) within \( \Lambda_n(x) \). For any \( L > 0 \) and \( x \in \mathbb{R}^d \) we set annulus \( R_L(x) = \Lambda_L(x) \setminus \Lambda_{L/2}(x) \). Moreover, for \( \ell \in (0, L) \) and \( \rho \in (0, 1) \) we define by

\[
\mathcal{D}_L^{(\rho, \ell)}(x) = \left\{ z \in R_L(x) \cap X \mid |C_\ell(z)| \geq \rho(2\ell)^d \right\}
\]

the set of \((\rho, \ell)\)-dense particles in \( R_L(x) \). Note that this definition differs from \( \mathcal{D}_{[L]}^{(\rho, \ell)} \) defined in Sect. 3.3 because we now consider the particles in annulus \( R_L(x) \) instead of the particles in box \( \Lambda_L \). For \( x, y \in \mathbb{R}^d \), \( \alpha \in (d, 2d) \) and \( \gamma \in (\alpha/(2d), 1) \) we define for \( m \geq 1 \),

\[
N_m = N^\gamma^m \text{ with } N = |x - y|.
\]

Denote by \( \mathcal{B}_m = \mathcal{B}_m^{(\rho, \ell)}(x, y) \) the event that there exists a hierarchy \( \mathcal{H}_m(x, y) \) of depth \( m \) connecting \( x \) and \( y \) such that for all \( k = 0, \ldots, m - 2 \) and all \( \sigma \in \{0, 1\}^k \),

\[
z_{\sigma_{01}} \in \mathcal{D}_{N_{k+1}}^{(\rho, \ell)}(z_{\sigma_0}) \quad \text{and} \quad z_{\sigma_{10}} \in \mathcal{D}_{N_{k+1}}^{(\rho, \ell)}(z_{\sigma_1}),
\]

see also (4.5) of [6] and Fig. 2 for an illustration. Moreover, we denote by \( \mathcal{T} = \mathcal{T}^{(\rho, \ell)}(x, y) \) the event that \( x \) and \( y \) are \((\rho, \ell)\)-dense. Note that the event \( \mathcal{B}_m \cap \mathcal{T} \) ensures that there is a hierarchy \( \mathcal{H}_m(x, y) \) of depth \( m \) connecting \( x \) and \( y \) such that all particles in this hierarchy are \((\rho, \ell)\)-dense and lie in the corresponding annulus.
Fig. 2 Illustration of a hierarchy in $B_m$. Assume we are given $z_{\sigma 0}$ and $z_{\sigma 1}$ for some $\sigma \in \{0, 1\}^k$ and $k = 0, \ldots, m - 3$. We consider an edge between $R_{N_k+1}(z_{\sigma 0})$ and $R_{N_k+1}(z_{\sigma 1})$ and call the end points $z_{\sigma 0 1}$ and $z_{\sigma 1 0}$, respectively. Then, set $z_{00} = z_{\sigma 0}$ and consider an edge between $R_{N_k+2}(z_{00})$ and $R_{N_k+2}(z_{01})$ and call the end points $z_{00 1}$ and $z_{01 0}$, respectively. Similarly, we define $z_{11 0}$ and $z_{11 1}$. To get a path of edges from $z_{00}$ to $z_{11}$ it remains to connect the two particles of each gap $(z, z')$. (If we assume $m = 3$, the pairs $(z_{00}, z_{00 1}), (z_{01}, z_{01 0}), (z_{11}, z_{11 0})$, and $(z_{01}, z_{11 0})$ in the figure are the gaps of the illustrated hierarchy in $B_3$.) Such particles are likely connected by a short path of edges because they are $(\rho, \ell)$-dense and close to each other if $m$ is sufficiently large, conditional on $B_m$

$R_{N_k}(z_\sigma)$. Finally, given $B_m$, we denote by $S = S^{(\ell)}$ the event that all gaps $(z, z')$ in a hierarchy in $B_m$ satisfy

$$X(A_\ell(z)) \leq e v(2 \ell)^d \quad \text{and} \quad X(A_\ell(z')) \leq e v(2 \ell)^d.$$  

(7.8)

The event $S$ ensures that we do not have too many particles in boxes $A_\ell(z)$ and $A_\ell(z')$; in particular, the graph distance of connected paths within such boxes is bounded by $e v(2 \ell)^d$. For the proof of Proposition 7.5 we use Lemma 7.7 below. Part (i) of Lemma 7.7 corresponds to Lemma 4.2 of [6]. The only difference lies in the additional event $S$ defined by (7.8). Part (ii) of Lemma 7.7 is the continuum space analogue to Lemma 4.3 of [6], and it shows that the event $B_m$ occurs with sufficiently high probability.

Lemma 7.7 Choose $\alpha \in (d, 2d)$. For all $\varepsilon \in (0, 1)$, $\gamma \in (\alpha/(2d), 1)$, $\Delta' > \log(2)/\log(1/\gamma)$ and $\alpha' \in (\alpha, 2d \gamma)$ there exist $N' = N'(\varepsilon, \gamma, \Delta') < \infty$, $\rho_0 \in (0, 1)$ and a constant $c_6 > 0$ such that the following holds true: For all $x, y \in \mathbb{R}^d$ with $N = |x - y| \geq N'$ let $m \in \mathbb{N}$ be the maximal integer such that

$$m \log(1/\gamma) \leq \log \log N - \varepsilon \log \log \log N.$$  

(7.9)

For all $\rho \in (0, \rho_0)$ and $\ell \in (N_m, 2N_m)$ in the definitions of $B_m, T$ and $S$ we obtain

(i) $P_{x,y}[d(x, y) \geq (\log N)^{\Delta'} \cap B_m \cap T \cap S] \leq \varepsilon$;

(ii) $P_{x,y}[B_m^c] \leq 2^{m+1} e^{-c_6 N_m^{2d \gamma - \alpha'}}$;

(iii) $P_{x,y}[B_m \cap S^c] \leq \varepsilon$.
Note that choice (7.9) implies, see also (4.9) of [6],

\[ 2^m \leq (\log N)^{\log 2/(\log(1/\gamma))} \quad \text{and} \quad e^{(\log \log N)^{\varepsilon}} \leq N_m \leq e^{(1/\gamma)(\log \log N)^{\varepsilon}}. \]  

(7.10)

**Proof of Lemma 7.7** We first prove (i). Let \( \rho \in (0, 1) \). Using the additional event \( \mathcal{S} \) we argue as in the proof of Lemma 4.2 of [6] to see that it remains to estimate the probability of the event \( \mathcal{A}_m \cap \mathcal{B}_m \cap \mathcal{T} \cap \mathcal{S} \subset \mathcal{A}_m \cap \mathcal{B}_m \cap \mathcal{T} \) for \( N \) sufficiently large, where \( \mathcal{A}_m \) is the event that for any hierarchy in \( \mathcal{B}_m \) there exists a gap \((z, z')\) such that there is no edge between the sets \( C_{\ell}(z) \) and \( C_{\ell}(z') \). Let \( z \in \mathbb{R}^d \) be such that the event \( \mathcal{A}_m \cap \mathcal{B}_m \cap \mathcal{T} \) only depends on the particles in \( \mathcal{A}_N(z) \) and edges with end points in \( \mathcal{A}_N(z) \) (which exists if \( N \) is sufficiently large). For \( l \geq 1 \) and \( x = x_0, y = x_1, x_2, \ldots, x_l \in \mathcal{A}_N(z) \), we consider the edge probabilities

\[ \tilde{p}_{x_i x_j} = 1 - \exp\{-\lambda|x_i - x_j|^{-\alpha}\} \leq 1 - \exp\{-\lambda W_{x_i} W_{x_j}|x_i - x_j|^{-\alpha}\} = p_{x_i x_j}, \]

a.s., for every \( i \neq j \in \{0, \ldots, l\} \), and denote by \( \tilde{P}_X \) the probability measure of the resulting edge configurations. We obtain

\[ \tilde{P}_X[\mathcal{A}_m \cap \mathcal{B}_m \cap \mathcal{T}] \leq \tilde{P}_X[\mathcal{A}_m \cap \mathcal{B}_m \cap \mathcal{T}] \leq \varepsilon; \]

(7.11)

note that the left-hand side is zero for \( l \) too small. This proves (i) by additionally integrating over the Poisson cloud restricted to \( \mathcal{A}_N(z) \). (Note that the right-hand side of (7.11) does not depend on the Poisson cloud.)

To prove (ii) we note that

\[ \tilde{P}_X[\mathcal{B}_c^m] \leq \sum_{k=1}^{m-1} \tilde{P}_X[\mathcal{B}_k \cap \mathcal{B}_c^{k+1}] \]

\[ + \tilde{P}_X[\mathcal{B}_m^{-1}] \]

Hence, it is sufficient to show that there exist \( \rho_0 \in (0, 1) \) and a constant \( c_6 > 0 \) such that for all \( \rho \in (0, \rho_0) \) and for \( \ell \in (N_m, 2N_m) \) in the definition of \( \mathcal{B}_m \) we have for sufficiently large \( N \),

\[ \tilde{P}_X[\mathcal{B}_k \cap \mathcal{B}_c^{k+1}] \leq 2^{k+1} e^{-c_6 N_m^{2d\gamma - \alpha'}} \]

for all \( k = 1, \ldots, m - 1 \), with \( m \) as in (7.9). For \( k = 1, \ldots, m - 1 \) we denote by \( \mathcal{B}_k' \) the event that there is a hierarchy \( \mathcal{H}_k(x, y) \) of depth \( k \) connecting \( x \) and \( y \) such that for each \( j = 0, \ldots, k - 2 \) and each \( \sigma \in \{0, 1\}^j \),

\[ z_{\sigma 01} \in R_{N_{j+1}}(z_{\sigma 0}) \quad \text{and} \quad z_{\sigma 10} \in R_{N_{j+1}}(z_{\sigma 1}). \]
By definition we obtain \( B_k \subset B'_k \). For all \( \ell \in (N_m, 2N_m) \) and \( \rho \in (0, \rho_0) \), where \( \rho_0 \) will be chosen below, we define the events

\[
A_1 = \{ \text{for any hierarchy in } B'_k \text{ there exists } \sigma \in \{0, 1\}^k \text{ such that } |D_{N_k}^{(\rho, \ell)}(z_\sigma)| \leq \rho_0(2N_k)^d \},
\]

\[
A_2 = \{ \text{for any hierarchy in } B'_k \text{ there exists a gap } (z, z') \text{ such that there is no edge between the sets } D_{N_k}^{(\rho, \ell)}(z) \text{ and } D_{N_k}^{(\rho, \ell)}(z') \}. \]

\( B_k \cap B'_{k+1} \) implies that there is a hierarchy in \( B'_k \) but, by 3 of Definition 7.6, there is no edge between \( D_{N_k}^{(\rho, \ell)}(z) \) and \( D_{N_k}^{(\rho, \ell)}(z') \) for any pair \((z, z')\) as in the definition of \( A_2 \).

Hence, \( B_k \cap B'_{k+1} \subset B'_k \cap A_2 \), and therefore we obtain

\[
\mathbb{P}_{x,y}[B_k \cap B'_{k+1}] \leq \mathbb{P}_{x,y}[B'_k \cap A_2] \leq \mathbb{P}_{x,y}[B'_k \cap A_1] + \mathbb{P}_{x,y}[B'_k \cap A_1^c \cap A_2],
\]

and it remains to bound the two terms on the right-hand side. To bound the first term we note that for \( N \) sufficiently large it holds that for any hierarchy in \( B'_k \) and \( \sigma, \sigma' \in \{0, 1\}^k \) with \( z_\sigma \neq z_{\sigma'} \), \( R_{N_k+\ell}(z_\sigma) \cap R_{N_k+\ell}(z_{\sigma'}) = \emptyset \) for all \( \ell \in (N_m, 2N_m) \). The latter implies that the events \( \{|D_{N_k}^{(\rho, \ell)}(z_\sigma)| \leq \rho_0(2N_k)^d \} \) are independent for different \( \sigma \in \{0, 1\}^k \).

It follows that for \( N \) sufficiently large,

\[
\mathbb{P}_{x,y}[B'_k \cap A_1] \leq 2^k \mathbb{P}

\[ |D_{N_k}^{(\rho, \ell)}(0)| \leq \rho_0(2N_k)^d \].
\]

By Corollary 3.5 (ii) there exist \( \rho_0 \in (0, 1) \) and \( \ell_0 < \infty \) such that for all \( \ell \in (\ell_0, N_k/\ell_0) \),

\[
\mathbb{P}

\[ |D_{N_k}^{(\rho, \ell)}(0)| \leq \rho_0(2N_k)^d \] \leq e^{-\rho_0N_k^{2d-\alpha'}}.
\]

(Although the definition of \( D_{N_k}^{(\rho, \ell)}(0) \) differs from \( D_{N_k}^{(\rho, \ell)} \) in Corollary 3.5 (ii) we can still use the result because \( R_{N_k}(0) \) contains a box of side length \( N_k/3 \).) If we choose \( N \) so large that \( N_m \geq \ell_0 \) and \( N_k/\ell_0 \geq 2N_m \) for all \( k = 1, \ldots, m-1 \), we finally obtain for all \( N \) sufficiently large, \( \rho \in (0, \rho_0) \) and \( \ell \in (N_m, 2N_m) \),

\[
\mathbb{P}_{x,y}[B'_k \cap A_1] \leq 2^k e^{-\rho_0N_k^{2d-\alpha'}} \leq 2^k e^{-\rho_0N_m^{2d-\alpha'}} \leq 2^k e^{-\rho_0N_{m-1}^{2d\alpha'-\alpha'}}. \quad (7.13)
\]

We proceed as in the derivation of (7.11), using the same arguments as in [6], to see that the second term in (7.12) satisfies

\[
\mathbb{P}_{x,y}[B'_k \cap A_1^c \cap A_2] \leq 2^k \exp

\[ -\lambda\rho_0^2 2^{2d}N_k^{2d}(5dN_m-1)^{-\alpha} \]

\[
\leq 2^k \exp

\[ -\lambda\rho_0^2 2^{2d}(5d)^{-\alpha}N_m^{2d\alpha'-\alpha} \],
\]

\( Springer \)
for all $N$ sufficiently large. Using (7.12), (7.13) and that $N_m^{2dγ−α} ≥ N_m^{2dγ−α'}$, it follows that for all $N$ sufficiently large,

$$\mathbb{P}_{x,y} [B_k \cap B_{k+1}^c] \leq 2^k e^{-\rho_0 N_m^{2dγ−α'}} + 2^k \exp \left\{-\lambda \rho_0^2 2d (5d)^α N_m^{2dγ−α} \right\} \leq 2^{k+1} e^{-c_6 N_m^{2dγ−α'}},$$

with $c_6 = \min\{\rho_0, \lambda \rho_0^2 2d (5d)^α\}$.

To prove (iii) we note that any hierarchy in $B_m$ has $2^{m−1}$ gaps whose $2^m$ particles \{v_1 = x, v_2 = y, v_3, \ldots, v_{2^m}\} satisfy $Λ_\ell(v_i) \cap Λ_\ell(v_j) = \emptyset$ for all $i = j$. We therefore obtain for $N$ sufficiently large,

$$\mathbb{P}_{x,y} [B_m \cap S^c] \leq 2^m \mathbb{P}_x \left[ X(Λ_\ell(x)) > ev(2\ell)^d \right].$$

Using Chernoff bound, see (6.4), and (7.10) we obtain for $\ell ≥ N_m$,

$$2^m \mathbb{P} \left[ X(Λ_\ell(0)) > ev(2\ell)^d \right] \leq 2^m e^{-v(2\ell)^d} \leq (\log N)^{\log 2/\log(1/γ)} \exp \left\{-\nu 2^d d^{(\log \log N)^ε} \right\},$$

which converges to 0 as $N → \infty$.

\textbf{Proof of Proposition 7.5} Using part (i) of Corollary 3.5 (to bound the probability of event $T^c$) and Lemma 7.7, Proposition 7.5 is proven by using the same arguments as in [6].

\textbf{7.5 Finite Variance of Degree Distribution, Case 2}

In order to finish the proof of Theorem 3.6 it remains to show statement (b2). We start with the following lemma.

\textbf{Lemma 7.8} Assume $\min\{α, βα\} > d$. For all $δ ∈ (0, α(β ∧ 1) − d)$ there exist $t_0 ≥ 1$ and a constant $c_7 > 0$ such that for all $s ≥ 1$ and $t ≥ t_0$,

$$\mathbb{P} \left[ \text{there is an edge in } Λ_s \text{ with size at least } t \right] \leq c_7 s^d t^{d−α(β ∧ 1)+δ}.$$

\textbf{Proof of Lemma 7.8} Using (7.5) we obtain for all $|x − y| ≥ t_0$ with $t_0$ sufficiently large,

$$\mathbb{E} \left[ \left( \frac{W_x W_y}{|x − y|^{α'}} \right)^\wedge 1 \right] \leq (1 + 1_{[β \neq 1]} / |β − 1|) \lambda^{β ∧ 1} \times \left( 1 + \max\{1, β\} \log(\lambda − 1 |x − y|^{α'}) \right)^2 |x − y|^{−α(β ∧ 1)} \leq |x − y|^{−α(β ∧ 1)+δ}.$$
It follows that for all $t \geq t_0$, using $1 - e^{-x} \leq x \wedge 1$,
\[
P\left[ \text{there is an edge in } \Lambda_s \text{ with size at least } t \right] 
\leq \mathbb{E} \left[ \sum_{x, y \in X \cap \Lambda_s} 1_{\{|x-y|>t\}} |x - y|^{-\alpha(\beta \wedge 1) + \delta} \right]
\]
\[
= \sum_{k \geq 1} \frac{\mathbb{P}[X(\Lambda_s) = k]}{(2s)^d} \int_{\Lambda_s} \cdots \int_{\Lambda_s} \sum_{i=1}^{k} \sum_{j=1}^{k} 1_{\{|x_i-x_j|>t\}} |x_i - x_j|^{-\alpha(\beta \wedge 1) + \delta} \, dx_1 \ldots dx_k
\]
\[
= \sum_{k \geq 1} \frac{\mathbb{P}[X(\Lambda_s) = k]}{(2s)^d} (k-1) \int_{\Lambda_s} \int_{\Lambda_s} 1_{\{|x-y|>t\}} |x - y|^{-\alpha(\beta \wedge 1) + \delta} \, dx \, dy
\]
\[
\leq \sum_{k \geq 1} \frac{\mathbb{P}[X(\Lambda_s) = k]}{(2s)^d} (k-1) \int_{|y|>t} |y|^{-\alpha(\beta \wedge 1) + \delta} \, dy
\]
\[
= \nu^2 (2s)^d \int_{|y|>t} |y|^{-\alpha(\beta \wedge 1) + \delta} \, dy,
\]
where in the last step we used that $X(\Lambda_s)$ has a Poisson distribution with parameter $\nu(2s)^d$. It follows that for an appropriate constant $c_7 > 0$ and for all $t \geq t_0$ with $t_0$ sufficiently large,
\[
P\left[ \text{there is an edge in } \Lambda_s \text{ with size at least } t \right] \leq c_7 s^d t^{d-\alpha(\beta \wedge 1) + \delta},
\]
which finishes the proof of Lemma 7.8.

Proof of (b2) of Theorem 3.6 Once the proof of Lemma 7.8 is established, the proof of (b2) of Theorem 3.6 follows one-to-one from the proof of Theorem 1 of [4] and Theorem 8 (b2) of [11]. There, a renormalization is applied to see that we have a linear lower bound on the graph distances within “good” finite boxes, see also Definition 2 and Lemma 2 of [4]. Lemma 7.8 is then used to prove that all centered boxes of sufficiently large side lengths are “good,” a.s., see Lemma 14 of [11], which then implies (b2) of Theorem 3.6. We refer to [11] for the details of the proof.

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