On the Manev spatial isosceles three-body problem
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Abstract
We study the isosceles three-body problem with Manev interaction. Using a McGehee-type technique, we blow up the triple collision singularity into an invariant manifold, called the collision manifold, pasted into the phase space for all energy levels. We find that orbits tending to/ejecting from total collision are present for a large set of angular momenta. We also find that as the angular momentum is increased, the collision manifold changes its topology. We discuss the flow near-by the collision manifold, study equilibria and homographic motions, and prove some statements on the global flow.

Keywords: spatial isosceles three-body problem, Manev interaction, topology of the collision manifold

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1 Introduction
In 1930, the Bulgarian physicist Georgy Manev proposed gravitational law of the form

$$U(r) = -\frac{\mu}{r} - \left(\frac{3\mu^2}{2c^2}\right) \frac{1}{r^2}$$  (1)

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where $r$ is the distance between the bodies, $\mu$ the gravitational parameter, and $c$ the speed of light. He showed that by applying a general action-reaction principle to classical mechanics, one is naturally led to the aforementioned law [Manev 1925; Manev 1930]. Provided the constants are chosen appropriately, the Manev model can be used in calculations involving the perihelion advance of Mercury and the other inner planets.

The $N$-body problem with Manev interaction was brought into focus in the early 90’s by Diacu [Diacu 1993]. Due to its rich and interesting dynamics, it became subject to many studies [Diacu & al. 1995], [Diacu & al. 2000], [Szenkovits & al. 1999], [Stoica 2000], [Diacu & Santoprete 2001], [Santoprete 2002], [Puta & Hedrea 2005], [Kyuldjiev 2007], [Balsas & al. 2009], [Llibre & Makhlouf 2012], [Lemou & al. 2012], [Alberti 2015], [Barrabès & al. 2017]. For instance, in contrast to its Newtonian counterpart, the Manev problem displays binary collisions for non-zero angular momenta: when approaching collision, two mass points spin infinitely many times around each other [Diacu & al. 1995; Diacu & al. 2000]. In celestial mechanics community, this dynamical behaviour is known as a black-hole, somehow in analogy with the black-hole gravitational effect [Diacu & al. 1995] in relativity.

In the relative two-body problem, the Manev interactions delineates two distinct type of near-collision dynamics. Let us consider the class of potentials of the form $-1/r - B/r^{\alpha}$, with $\alpha > 0$, and $B > 0$ small so that the term $-B/r^{\alpha}$ may thought as a corrective augmentation to the Newtonian potential. It can be shown that for all $\alpha > 0$ the collision manifold is a torus. For all $0 < \alpha < 2$, the collision is possible only for zero angular momentum. The dynamics on the collision manifold is similar to the Newtonian case, with a gradient-like flow matching to circles of equilibria. Moreover, when $\alpha = 2(1 - 1/n)$, $n \geq 2$, $n \in \mathbb{N}$, the flow is regularizable in the sense of Levi-Civita [Stoica 2000]. For $\alpha = 2$, the Manev case, the collision is possible for angular momenta $C$ with $|C| \leq b$, $b > 0$ being some constant depending on masses; on the collision manifold the dynamics is trivial displaying two circles of degenerate equilibria [Diacu & al. 2000]. For $\alpha > 2$ the collision manifold is reached for all angular momenta. Its the flow is gradient-like, matching two circles of equilibria as well, but is not regularizable [Stoica 1997]. An intuitive and physically reasonable explanation for the above is that the Manev corrective term $(-B/r^{2})$ adds to the rotational inertial term $C^{2}/r^{2}$ (the latter being a consequence of the angular momentum conservation).

We believe that, similar to the case of two bodies, in the generalized stands as the threshold between two distinct type of near-collision dynamics. This is suggested by the studies of the isosceles three-body problem with Newtonian [Devaney 1980; Shibayama & al. 2009], Manev [Diacu 1993] and Schwarzschild “$-1/r - B/r^{3}$” interaction [Arredondo & al. 2014]. The present paper is a further step aiming to this problem clarification.

In this paper we investigate the dynamics near total collapse in a three-body problem with Manev binary interaction. Considering two of the masses equal, we study the dynamics on the invariant manifold of isosceles configurations. Using a McGehee technique similar to that in [Devaney 1980], we blow up the collision singularity and replace it by an invariant collision manifold pasted to the phase space for all energy levels. While fictitious, due to the continuity of ODE solutions with respect to the initial data, the collision manifold provides information about orbits passing close to collision. Its flow is rendered by the evolution of 3 variables, $v, \theta$ and $w$, describing the (fictitious) rate of change of the size of the system, the shape of it configuration and the rate of change of the latter, respectively.

The Manev isosceles three-body problem, and in particular the near-collision dynamics, was studied by Diacu [Diacu 1993], but only for zero total angular momentum. The bodies were confined to a fixed plane, with the middle body oscillating above and below the line joining the other two. One of the open problems stated in Diacu’s paper concerns the existence of non-zero angular momenta orbits ejecting/tending asymptotically to triple collision. Are such orbits possible? In the present work we find that orbits tending to/ejecting from total collision are present for a large set of non-zero momenta.

We also detect an interesting feature of the Manev three-body problem: as the size $C$ of the total angular momentum increases from zero, the collision manifold changes its topology from a sphere with 4
points removed, as in the Newtonian [Shibayama & al. 2009] and Schwarzschild [Arredondo & al. 2014] cases, to the union of a sphere with two lines, to the union a point with two lines, and finally to two lines. To our knowledge, this phenomenon was not observed anywhere else. The lines that persist for all momenta correspond to (fictitious) double collisions.

On the collision manifold $\mathcal{C}$, for all angular momenta, the double collisions lines are filled with equilibria. For low momenta, we find six more equilibria, similar to the Newtonian case [Shibayama & al. 2009]. This points correspond to two distinct total collision limit configurations: one linear (with one of the body fixed on the midpoint between the other two) and one spatial (modulo a reflection symmetry), with the ratio of the triangle sides depending on the bodies’ masses. As $C$ is increased, the spatial limit configurations disappear. For high $C$, the linear limit configurations disappear as well and triple collision is reached (asymptotically) only by solutions with double collision as limit configuration. The flow on $\mathcal{C}$ is constant in the $v$ coordinate: for low $C$, the orbits connect the double collision manifolds, whereas when $\mathcal{C}$ is diffeomorphic to the union of a sphere with the double collision lines, all orbits are either periodic or equilibria. We prove that none of these periodic orbits is an attractor for the global flow. We also prove that homographic motions, that is motions for with self-similar configurations, have linear configurations only.

The paper is organized as follows: in Section 2 we introduce the isosceles Manev three-body problem and reduce the dynamics to a two degrees of freedom using the angular momentum conservation. In Section 3 we regularize the equations of motion. In Section 4 we define the collision manifold, and classify its topology and investigate the associated dynamics. In Section 5 we discuss the flow near-by the collision manifold, study equilibria and homographic motions, and prove some statements on the global flow.

2 Dynamics

In cylindrical coordinates $(R, \phi, Z, p_R, p_\phi, p_Z)$ (see Figure 1) the Hamiltonian is

$$H(R, \phi, Z, P_R, P_\phi, P_Z) = \frac{1}{M} \left( \frac{P_R^2}{R^2} + \frac{P_\phi^2}{R^2} \right) + \frac{2M + m}{4Mm} P_Z^2 + U(R, Z),$$

with a Manev-type potential given by

$$U(R, Z) = -\frac{GM^2}{R} \left( 1 + \frac{\gamma_0}{R} \right) - \frac{4GMm}{\sqrt{R^2 + 4Z^2}} \left( 1 + \frac{4\gamma}{\sqrt{R^2 + 4Z^2}} \right).$$

(2)

where $\gamma_0, \gamma > 0$ and $\gamma_0 \neq \gamma$. For reason to be discussed later, we assume that

$$16\gamma > \gamma_0$$

(3)

Using the angular momentum conservation $P_\phi(t) = \text{const.} =: C$ we reduced the dynamics to a two degree of freedom Hamiltonian system determined by

$$H_{\text{red}}(R, Z, P_R, P_Z; C)$$

$$= \frac{1}{2} (p_R \ p_Z) \left( \begin{array}{cc} \frac{2M}{R^2} & 0 \\ 0 & \frac{2M+m}{2Mm} \end{array} \right) \left( \begin{array}{c} P_R \\ P_Z \end{array} \right) + U_{\text{eff}}(R, Z; C)$$

(4)

where $U_{\text{eff}}(R, Z; C)$ the effective (or amended) potential

$$U_{\text{eff}}(R, Z; C) := \frac{C^2}{MR^2} + U(R, Z).$$

(5)
and $C \in \mathbb{R}$ is a parameter. The equations of motion are

\[
\begin{align*}
\dot{R} &= \frac{2PR}{M}, \\
\dot{Z} &= \frac{2M + m}{2Mm} P_Z, \\
\dot{P}_R &= \frac{2C^2}{MR^3} - \frac{\partial U(R, Z; C)}{\partial R}, \\
\dot{P}_Z &= -\frac{\partial U(R, Z; C)}{\partial Z}.
\end{align*}
\]

Since the Hamiltonian is time-independent, along any solution the energy is conserved:

\[
H_{\text{red}}(R(t), Z(t), P_R(t), P_Z(t); c) = \text{const.} = h. \tag{6}
\]

![Figure 1: The spatial isosceles three-body problem](image)

3 The regularized dynamics

We now regularize the equations of motion of the isosceles Manev three body problem. We follow closely the McGehee technique as used in the Newtonian isosceles problem by Devaney [Devaney 1980]. Denoting

\[
x := \begin{pmatrix} R \\ Z \end{pmatrix}, \quad p := \begin{pmatrix} p_R \\ p_Z \end{pmatrix}, \quad \mathbb{K} = \begin{pmatrix} M & 0 \\ 0 & \frac{2Mm}{2M+m} \end{pmatrix},
\]

we introduce the coordinates $(r, v, s, u)$ defined by

\[
\begin{align*}
r &= \sqrt{x^t \mathbb{K} x}, & v &= r(s \cdot p), \\
s &= \frac{x}{r}, & u &= r(\mathbb{K}^{-1} p - (s \cdot p)s).
\end{align*} \tag{7}
\]

Notice that $r = 0$ corresponds to $R = Z = 0$, i.e., to the triple collision of the bodies. The coordinate $v$ describes the rate of change of the size of the system as given by $r$, whereas the vector $s$ describes $R$ and $Z$ separately. One may verify that in the new coordinates we have that $s^t \mathbb{K} s = 1$ and $s^t \mathbb{K} u = 0.$
The equations of motion are
\[ \dot{r} = r^{-1}v, \]
\[ \dot{v} = r^{-2}v^2 + r^{-2}u^\top K u + r^{-2} \frac{2C^2}{Ms^2_1} \left( \frac{V(s)}{r} + \frac{2W(s)}{r^2} \right), \]
\[ \dot{s} = r^{-2}u, \]
\[ \dot{u} = \left[ -r^{-2}u^\top K u - r^{-2} \frac{2C^2}{Ms^2_1} + r^{-2} \left( \frac{V(s)}{r} + \frac{2W(s)}{r^2} \right) \right] s \]
\[ + r^{-1} \left( \frac{2}{M} \frac{\partial V}{\partial s_1} + \frac{4GMm}{(s_1^2 + 4s_2^2)^{\frac{3}{2}}} \right), \]
\[ + r^{-2} \left( \frac{2}{M} \frac{\partial W}{\partial s_1} + \frac{8GMm\gamma}{(s_1^2 + 4s_2^2)} \right), \]
with
\[ V(s) = \frac{GM^2}{s_1} + \frac{4GMm}{(s_1^2 + 4s_2^2)^{\frac{3}{2}}} \quad \text{and} \quad \]
\[ W(s) = \frac{GM^2\gamma_0}{s_1^2} + \frac{8GMm\gamma}{(s_1^2 + 4s_2^2)}. \]

We further introduce the change of coordinates
\[ s = \sqrt{(K^{-1})}(\cos \theta, \sin \theta)^t, \quad u = u\sqrt{(K^{-1})}(-\sin \theta, \cos \theta)^t \]
where \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\) so that the boundaries \(\theta = \pm \frac{\pi}{2}\) correspond in the original coordinates to \(R = 0\), that is, to double collisions of the masses \(M\). More precisely, at \(\theta = \pi/2\) we have \(R = 0\) and \(z > 0\), whereas at \(\theta = -\pi/2\), \(R = 0\) and \(z < 0\). Also, the \(\theta\) varies, the ratio between \(R\) and \(Z\) varies as well; a direct calculation also shows that
\[ Z \cos \theta = \frac{\sqrt{\mu}}{2} R \sin \theta. \]
Thus, for instance, \(Z = 0\) at \(\theta = 0\), and \(R = 0\) at \(\pm \theta = \pi/2\). One may also verify that \(u^\top T u = u^2\) and \(\dot{u} = (u'/u) u - u \dot{\theta} s\). Denoting
\[ \mu := \frac{2M + m}{m} \quad \text{(9)} \]
and applying the time re-parametrization \(dt = r^2 dr\), we obtain the system
\[ r' = rv, \quad \text{(10)} \]
\[ v' = v^2 + u^2 + \frac{C^2}{\cos^2 \theta} - rV(\theta) - 2W(\theta), \quad \text{(11)} \]
\[ \theta' = u, \quad \text{(12)} \]
\[ u' = -C^2 \frac{\sin \theta}{\cos^3 \theta} + r \frac{\partial V(\theta)}{\partial \theta} + \frac{\partial W(\theta)}{\partial \theta}, \quad \text{(13)} \]
where
\[ V(\theta) = GM \left( \frac{M}{2} \right)^{\frac{1}{2}} \left( \frac{M}{\cos \theta} + \frac{4m}{(\cos^2 \theta + \mu \sin^2 \theta)^{\frac{3}{2}}} \right), \quad \text{(14)} \]
\[ W(\theta) = GM \left( \frac{M}{2} \right)^{\frac{1}{2}} \left( \frac{M\gamma_0}{\cos^2 \theta} + \frac{8m\gamma}{(\cos^2 \theta + \mu \sin^2 \theta)} \right). \quad \text{(15)} \]
In the new coordinates the energy integral is given by

\[ h r^2 = \frac{1}{2} (u^2 + v^2) - r V(\theta) - W(\theta). \]  

(16)

### 3.1 Potential functions \( V(\theta) \) and \( W(\theta) \)

First we notice that \( V(\theta) \) and \( W(\theta) \) are positive on their domain \( \theta \in (-\pi/2, \pi/2) \). A direct calculation shows that, \( V(\theta) \) has three critical points at \( \theta_0 = 0 \) and \( \theta = \pm \theta_v \), where

\[ \cos \theta_v = \sqrt{\frac{\mu}{\mu + 3}}. \]  

(17)

Similarly, provided the conditions (3) is satisfied, \( W(\theta) \) displays three critical points at \( \theta_0 = 0 \) and \( \theta = \pm \theta_w \), where

\[ \cos \theta_w = \sqrt{\frac{\mu}{\mu + 4 \sqrt{\frac{\gamma}{\gamma_0}} - 1}}. \]  

(18)

We leave for future work the case when the parameters \( \gamma_0 \) and \( \gamma \) do not obey (3) (that is when \( \gamma_0 \geq 16 \gamma \)). It is immediate that the nonzero critical points of \( V(\theta) \) and \( W(\theta) \) coincide only if \( \gamma = \gamma_0 \), case already excluded in our model; see equation (2).

### 3.2 Regularized Equations of Motion

In the system (10)-(12) and the energy integral (16) we make the substitutions

\[ U(\theta) = W(\theta) \cos^2 \theta, \quad w = \frac{\cos^2 \theta}{\sqrt{U(\theta)}} u, \]  

(19)

and introduce a new time parametrization given by \( \frac{d\tau}{d\sigma} = \frac{\cos^2 \theta}{\sqrt{U(\theta)}} \) to obtain

\[ r' = \frac{\cos^2 \theta}{\sqrt{U(\theta)}} rv, \]

\[ v' = \left( v^2 + \frac{U(\theta)}{\cos^4 \theta} w^2 + \frac{C^2}{\cos^2 \theta} - r V(\theta) - 2\frac{U(\theta)}{\cos^2 \theta} \right) \frac{\cos^2 \theta}{\sqrt{U(\theta)}}, \]

(20)

\[ \theta' = w, \]

\[ w' = -C^2 \sin 2\theta + r \cos^4 \theta \frac{V''(\theta)}{U(\theta)} + \cos^2 \theta \frac{U''(\theta)}{U(\theta)} + \sin 2\theta, \]
and

\[ 2hr^2 \cos^4 \theta = w^2 U(\theta) + v^2 \cos^4 \theta + C^2 \cos^2 \theta - 2r^2 \cos^4 \theta V(\theta) - 2 \cos^2 \theta U(\theta). \] \tag{21} \]

Notice that \( U(\theta) \) is smooth and \( U(\theta) > 0 \) for all \( \theta \in (-\pi/2, \pi/2) \); see its sketch in Figure 3. Finally, using the energy relation, we substitute the term containing the angular momentum \( C \) into the \( v' \) equation and obtain

\[ r' = \frac{\cos^2 \theta}{\sqrt{U(\theta)}} rv, \] \tag{22} \]

\[ v' = r(2hr + V(\theta)) \frac{\cos^2 \theta}{\sqrt{U(\theta)}}, \] \tag{23} \]

\[ \theta' = w, \] \tag{24} \]

\[ w' = \frac{\cos \theta}{U(\theta)} (rV'(\theta) \cos^3 \theta + U'(\theta) \cos \theta - (C^2 - 2U(\theta)) \sin \theta). \] \tag{25} \]

\section{The Triple Collision Manifold}

The vector field (22)-(25) is analytic on \([0, \infty) \times \mathbb{R} \times (-\pi/2, \pi/2) \times \mathbb{R}\), and thus the flow is well defined everywhere on its domain, including the points corresponding to triple collision \((r = 0)\). The restriction of the energy relation (21) to \(r = 0\)

\[ \mathcal{C} := \{(v, \theta, w) \in \mathbb{R} \times \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \times \mathbb{R} \mid w^2 + v^2 \cos^4 \theta \frac{U(\theta)}{U(\theta)} + (C^2 - 2U(\theta)) \cos^2 \theta \frac{U(\theta)}{U(\theta)} = 0 \}, \] \tag{26} \]

is a (fictitious) invariant set, called the \textit{triple collision manifold}, pasted into the phase space for any level of energy. By continuity with respect to the initial data, the flow on the smooth subsets of \( \mathcal{C} \) provides information about the orbits that pass close to collision.

\subsection{Topology}

Let \( U_m \) denote by \( U_m \) the minimum and maximum values of \( U(\theta) \) (see Figure 3). We calculate

\[ U_m = U \left( \pm \frac{\pi}{2} \right) = \frac{GM^3 \gamma_0}{2}. \] \tag{27} \]
We also observe that the maximum value of \( U(\theta) \) occurs at \( \theta = 0 \) and it is given by

\[
U(0) = \frac{GM^2}{2}(M\gamma_0 + 8m\gamma).
\] (28)

The collision manifold is non-void if \((C^2 - 2U(\theta)) \leq 0\). Considering the graph of \(2U(\theta)\) and the sign of \((C^2 - 2U(\theta))\) as \( C^2 \) is increasing from zero, we distinguish the following cases:

1. If \(0 \leq |C| < \sqrt{2U_m}\) the collision manifold \( C \) is homeomorphic to a sphere with 4 points removed; see Figure 4. \( C \) is a smooth manifold everywhere, except at the (fictitious) double collision boundaries

\[
B_{l,r} := \{(v, \theta, w) | v = v_0 \in \mathbb{R}, \theta = \pm \frac{\pi}{2} w = 0\}.
\] (29)

![Figure 4: The collision manifold \( C \) for angular momenta \(0 \leq |C| \leq \sqrt{2U_m}\).](image)

2. If \(|C| \in \left(\sqrt{2U_m}, \sqrt{2U(0)}\right)\) then \( C \) consists in the union of a sphere with the lines \( B_{l,r} \); see Figure 5.

3. If \(|C| = \sqrt{2U(0)}\) then \( C \) is the union of one point, the origin, with \( B_{l,r} \).

4. If \(|C| > \sqrt{2U(0)}\) then \( C \) consists of the lines \( B_{l,r} \).

Thus we have proved:

**Proposition 4.1** As the momentum \(|C|\) is increased, the triple collision manifold changes its topology, from a sphere with 4 points removed, to the union of a sphere with two lines, to the union of a point with two lines and finally, to two lines.

### 4.2 Dynamics on the collision manifold

The vector field on the collision manifold is obtained by setting \( r = 0 \) in system (22) and it is given by

\[
v' = 0,
\]

\[
\theta' = w,
\]

\[
w' = \frac{\cos \theta}{U(\theta)} (U'(\theta) \cos \theta - (C^2 - 2U(\theta)) \sin \theta).
\] (32)
Figure 5: The collision manifold $\mathcal{C}$ for angular momenta $|C| \in \left(\sqrt{2U_m}, \sqrt{2U(0)}\right)$. The compact part $\mathcal{C} \setminus \mathcal{B}_{l,r}$ of the collision manifold shrinks as the total angular momentum $|C|$ is increasing, and it disappears for $|C| > \sqrt{2U(0)}$.

It is immediate that $v$ is constant along the orbits, the flow being degenerate in this direction. For every $v = \text{const.} = v_0$, the restriction of collision manifold $\mathcal{C}$ to a level $v = \text{const.} = v_0$ is

$$\mathcal{V}_{v_0} := \left\{ (\theta, w) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R} \bigg| w^2 + v_0^2 \cos^4 \theta + \left(C^2 - 2U(\theta)\right) \frac{\cos^2 \theta}{U(\theta)} = 0 \right\}. \quad (33)$$

When connected to $\mathcal{C}$, the double collision lines $\mathcal{B}$ consist in degenerate equilibria. All orbits are horizontal.

For all momentum values for $\mathcal{C}$ exists, that is for

$$|C| \leq \sqrt{2U(0)} = \sqrt{GM^3\gamma_0} + 8GM^2\gamma$$

we have two equilibria located at

$$P_\pm := (\pm \sqrt{2U(0)} - C^2, 0, 0). \quad (35)$$

For momenta

$$|C| = \sqrt{2U_m} = \sqrt{GM^3\gamma_0}$$

the equilibria $P_\pm$ coalesce.

For lower momenta

$$|C| \leq \sqrt{2U_m} = \sqrt{GM^3\gamma_0}$$

we also have four more equilibria located at

$$E^1_\pm = (\pm v_0, -\theta_0, 0) \quad \text{and} \quad E^2_\pm = (\pm v_0, \theta_0, 0) \quad (38)$$

where

$$v_0 = \frac{1}{\mu} \left[ \sqrt{8GM^2m\gamma} + \sqrt{\frac{2M}{m} (GM^3\gamma_0 - C^2)} \right] \quad (39)$$

and $\theta_0 \in (0, \pi/2)$ so that

$$\tan^2 \theta_0 = \frac{1}{\mu} \left( \sqrt{\frac{16GM^3\gamma}{GM^3\gamma_0 - C^2}} - 1 \right). \quad (40)$$
Consequently, for \(|C| \leq \sqrt{GM^3 \gamma_0}\) we have the following type of orbits (see Figure [4]):
- homoclinic connections joining a double collision equilibrium;
- heteroclinic connections joining a double collision equilibrium to one of the “E” points;
- homoclinic connections between two “E” points;
- heteroclinic connections joining two double collision equilibria.

On the edges \(B_{l,r}\) the system (30)-(32) may lose uniqueness of solutions. The double collisions are not regularizable (and thus they cannot be equivalent to elastic bounces, as in the Newtonian case), as it is known from the \([\text{Diacu \\ al. 2000, Stoica 2000}]\).

For \(\sqrt{2U_m} < |C| < \sqrt{2U(0)}\), that is for
\[
\sqrt{GM^3 \gamma_0} < |C| < \sqrt{GM^3 \gamma_0 + 8GM^2 m \gamma}
\]
the flow wraps around \(C\) (see Figure [5]).

5 The Near-Collision Flow

5.1 Equilibria and their stability

We now discuss the equilibria on the collision manifold as embedded in the full \((r, v, \theta, w)\) regularized phase-space, and calculate their stability. We have
- for all momenta \(|C| \leq \sqrt{2U(0)}\), we find a pair equilibria on \(C\) at
  \[
  P_{\pm} := (0, \pm \sqrt{2U(0) - C^2}, 0, 0).  \tag{41}
  \]
- for momenta such \(|C| \leq \sqrt{2U_m}\) the flow also displays four fixed points
  \[
  E^1_{\pm} = (0, \pm v_0, -\theta_0, 0) \quad E^2_{\pm} = (0, \pm v_0, \theta_0, 0) \tag{42}
  \]
with \(v_0\) and \(\theta_0\) given by (39) and (40), respectively. Also, we find an infinite number of equilibria on the edges \(B_{l,r}\).

To determine the stability of \(P_{\pm}\) we start by writing the energy relation (21) as a level set:
\[
\mathcal{E} := \{(r, v, \theta, w) \mid F(r, v, \theta, w) = 0\}  \tag{43}
\]
where
\[
F(r, v, \theta, w) := 2hr^2 \cos^4 \theta - w^2 U(\theta) - v^2 \cos^4 \theta - C^2 \cos^2 \theta + 2rV(\theta) \cos^4 \theta + 2U(\theta) \cos^2 \theta. \tag{44}
\]

Next we calculate the spectrum of the linearization of system (22) at an equilibrium and then we restrict it to the tangent space of the collision manifold \(\mathcal{E}\). We denote by \(J\) the linearization of (22) and \(\bar{J}\) its restriction to a tangent space.

At \(P_{\pm} = (0, \pm \sqrt{2U(0) - C^2}, 0, 0)\) we find
\[
J = \begin{pmatrix}
\pm \sqrt{2 - \frac{C^2}{U(0)}} & 0 & 0 & 0 \\
0 & \frac{V(0)}{\sqrt{U(0)}} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 2 + \frac{U''(0) - C^2}{U(0)} & 0 & 0
\end{pmatrix}.  \tag{45}
\]
The tangent space to $E$ at an equilibrium point $P_{\pm} = (0, \pm \sqrt{2U(0) - C^2}, 0, 0)$ is 

$$T_{P_{\pm}}E = \{(\rho_1, \rho_2, \rho_3, \rho_4) \mid \nabla F|_{P_{\pm}} \cdot (\rho_1, \rho_2, \rho_3, \rho_4) = 0\} = \{(\rho_1, \rho_2, \rho_3, \rho_4) \mid V(0)\rho_1 \pm \sqrt{2U(0) - C^2}\rho_2 = 0\}.$$ 

For angular momenta $|C| < \sqrt{2U(0)} = GM^2(M\gamma_0 + 8m\gamma)/2,$ 
a basis for $T_{P_{\pm}}E$ is given by 

$$\xi_1 = (\pm \sqrt{2U(0)} - C^2, -V(0), 0, 0),$$ 

$\xi_3 = (0, 0, 1, 0)$ and $\xi_4 = (0, 0, 0, 1)$ and a representative of $\vec{J}$ in this basis is 

$$\begin{pmatrix} \pm \sqrt{2 - \frac{C^2}{U(0)}} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 + \frac{U''(0) - C^2}{U(0)} & 0 \end{pmatrix}.$$ (46) 

The eigenvalues of $\vec{J}$ are given by 

$$\lambda_1 = \pm \sqrt{2 - \frac{C^2}{U(0)}} = \pm \sqrt{\frac{GM^3\gamma_0 - C^2}{GM^3\gamma_0}} \in \mathbb{R}$$ 

and 

$$\lambda_{2,3} = \pm i \sqrt{\frac{(-2)(GM^3(\gamma_0 - 16\gamma) - C^2)}{GM^2(M\gamma_0 + 8m\gamma)}}$$ (47) 

where the quantity under square root is positive given that condition (3) is satisfied. 

If $|C| = \pm \sqrt{2U(0)},$ the collision manifold collapses to a point, the origin $O,$ which is also an equilibrium. 

We have $T_OE = \{(\rho_1, \rho_2, \rho_3, \rho_4) \mid \rho_1 = 0\}$. The linear part of the vector field (22) restricted to the tangent space is given by 

$$\vec{J} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{U''(0)}{U(0)} & 0 \end{pmatrix},$$ (48) 

and so a basis for $T_{P_{\pm}}E$ is given by $\xi_2 = (0, 1, 0, 0), \xi_3 = (0, 0, 1, 0)$ and $\xi_4 = (0, 0, 0, 1).$ A representative of $\vec{J}$ in this basis is 

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{U''(0)}{U(0)} & 0 \end{pmatrix}.$$ (49) 

The eigenvalues are given by $\lambda_1 = 0$ and 

$$\lambda_{2,3} = \pm 4i \sqrt{m\gamma\mu/(M\gamma_0 + 8m\gamma)}.$$
Now we study the behaviour near the points $E_{1,2}^\pm$. We calculate the Jacobian matrix of system (22) evaluated at this points and find:

$$J = \begin{pmatrix}
\frac{\pm v_0 \cos^2 \theta_0}{\sqrt{U(\theta_0)}} & 0 & 0 & 0 \\
\frac{V(\theta_0) \cos^2 \theta_0}{\sqrt{U(\theta_0)}} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\frac{V'(\theta_0) \cos^4 \theta_0}{U(\theta_0)} & 0 & a & 0
\end{pmatrix} \quad (50)$$

where

$$a = \frac{16M m^2 (2M + m) \gamma \sin^2 \theta_0 \cos^4 \theta_0}{(M \cos^2 \theta_0 - M - \frac{m}{2})^2} \frac{1}{(M^2 \gamma_0 - 4m^2 \gamma) \cos^2 \theta_0 - M \left( M + \frac{m}{2} \right) \gamma_0}. \quad (51)$$

The sign of the term $a$ is decided by the sign of the expression

$$T := (M^2 \gamma_0 - 4m^2 \gamma) \cos^2 \theta_0 - M \left( M + \frac{m}{2} \right) \gamma_0. \quad (52)$$

For this we calculate $\cos^2 \theta_0 = 1/(1 + \tan^2 \theta_0)$ using (40) that we then substitute into (52). We obtain

$$T = -\frac{m(2M + m)}{2M + m} \left( 8m \gamma + M \gamma_0 \sqrt{\frac{16GM^4 \gamma}{GM^4 \gamma_0 - C^2}} \right). \quad (53)$$

Thus the sign of $a$ is negative. The tangent space to the energy level manifold (43) at an equilibrium point $E_{1,2}^\pm$ is

$$T_{E_{1,2}^\pm} = \{(\rho_1, \rho_2, \rho_3, \rho_4) \mid \cos^3 \theta_0 V(\theta_0) \rho_1 - v_0 \cos^3 \theta_0 \rho_2 + [\sin \theta_0 (2v_0^2 \cos^2 \theta_0 + C^2 - 2U(\theta_0)) + \cos \theta_0 U'(\theta_0)] \rho_3 = 0 \}. \quad (54)$$

Then a basis for $T_{E_{1,2}^\pm}$ is given by $\xi_1 = (1, 0, 0, 0)$, $\xi_3 = (0, 0, 1, 0)$ and $\xi_4 = (0, 0, 0, 1)$. A representative of $\bar{J}$ in this basis is

$$\bar{J} = \begin{pmatrix}
\frac{\pm v_0 \cos^2 \theta_0}{\sqrt{U(\theta_0)}} & 0 & 0 \\
0 & 0 & 0 & 1 \\
\frac{V'(\theta_0) \cos^4 \theta_0}{U(\theta_0)} & a & 0
\end{pmatrix}. \quad (55)$$

The eigenvalues are

$$\lambda_1 = \frac{v_0 \cos^2 \theta_0}{\sqrt{U(\theta_0)}} \quad \text{for } E_{1,2}^\pm,$$

$$\lambda_2 = -\frac{v_0 \cos^2 \theta_0}{\sqrt{U(\theta_0)}} \quad \text{for } E_{1,2}^\pm,$$

and $\lambda_{2,3} = \pm \sqrt{-a}$. Thus we have proven:
Proposition 5.1 For every fixed energy level $h$ and any fixed angular momentum $|C| \in [0, \sqrt{2U(0)}]$, the equilibria $P_+$ ($P_-$) have a one-dimensional unstable (stable) manifold and a two-dimensional centre manifold.

Proposition 5.2 For every fixed energy level $h$ and any fixed angular momentum $|C| \in [0, \sqrt{2U_m}]$, the equilibria $E^{1,2}_+$ ($E^{1,2}_-$) have a one-dimensional unstable (stable) manifold and a two-dimensional centre manifold.

Proposition 5.3 For every fixed energy level $h$ and any fixed angular momentum $|C| > \sqrt{U(0)}$, the triple collision manifold is reached (asymptotically) by solutions with double collision as limit configuration (i.e., the limit configuration has $R = 0$).

Remark 5.4 When $\gamma_0 \geq 16\gamma$, the functions $V(\theta)$ and $W(\theta)$ lose their critical points at $\theta \neq 0$, and consequently, the collision manifold does not display a “hump”. The only equilibria on $C \setminus B_{l,r}$ are those at $P_{\pm}$.

5.2 Homographic motions

Using similar arguments as in [Arreldondo & al. 2014], one may prove that motions ejecting/ending from/to the equilibria $P_{\pm}$ are homographic, i.e., they maintain the a self-similar shape of the triangle formed by the three bodies. In the Manev isosceles problem, homographic motions form the invariant manifold

$$\mathcal{H} := \{ (r,v,\theta,w) \mid \theta = 0, w = 0 \}$$

(56)

of the system (22)-(25), and the dynamics on $\mathcal{H}$ are given by

$$r' = \frac{\cos^2 \theta}{\sqrt{U(\theta)}} rv,$$

(57)

$$v' = r(2hr + V(\theta)) \frac{\cos^2 \theta}{\sqrt{U(\theta)}}.$$

(58)

with the energy integral

$$v^2 + 2(-h)r^2 - 2rV(0) + C^2 - 2U(0) = 0.$$ (59)

Since on $\mathcal{H}$ we have $\theta(t) = 0$ for all $t$, physically homographic motions have a linear configurations, with body $m$ positioned midway between the other two. For $h < 0$ we re-write the energy relation (59) as

$$\frac{v^2}{2(-h)} + \left( r - \frac{V(0)}{2(-h)} \right)^2 + \frac{1}{2(-h)} \left( C^2 - 2U(0) - \frac{V^2(0)}{2(-h)} \right) = 0.$$ (60)

and notice that the motion is possible only for momenta $C$ such that

$$|C| < \sqrt{2U(0) + \frac{V^2(0)}{2(-h)}}.$$ (61)
We also observe that for
\[ \sqrt{2U(0)} < |C| < \sqrt{2U(0) + \frac{V^2(0)}{2(-h)}} \] (62)
all orbits are periodic and non-collisional, and surround the equilibrium located at
\[ S = \left( \frac{V(0)}{2(-h)}, 0 \right). \] (63)

As mentioned, in physical space, homographic motions correspond to motions with linear configuration. The homographic equilibrium is a rotating steady state with the outer bodies rotating at a fixed distance from the central body. The homographic periodic orbits are motions in which the outer bodies rotate and “pulsate” between a maximum and minimum distance from the central body. For \( h > 0 \) all homographic orbits are unbounded. They either eject/fall into the collision manifold or come from infinity, attain a configuration minimal size, and return to infinity. A sketch of the phase portrait of homographic motions is given in Figure 6.

![Figure 6: Homographic motions. Orbits with \( h < 0 \) and \( h > 0 \) are represented with solid lines and dashlines, respectively.](image)

### 5.3 Other aspects of the global flow

**Proposition 5.5** For every fixed \( h < 0 \) and \( |C| \in \left( \sqrt{2U_m}, \sqrt{2U(0)} \right) \) the set \( C \setminus (\mathcal{B}_{l,r} \cup P_{\pm}) \) is not an attractor.

**Proof:** Let \( h < 0 \) and \( |C| \in \left( \sqrt{2U_m}, \sqrt{2U(0)} \right) \) be fixed. In this case the collision manifold and its flow are depicted in Figure 5. The evolution of the \( r \) and \( v \) variables is driven by the equations (22) and (23); for reader’s convenience we re-write these equations below

\[ r' = \frac{\cos^2 \theta}{\sqrt{U(\theta)}} rv, \] (64)

\[ v' = 2h \left( \frac{\cos^2 \theta}{\sqrt{U(\theta)}} \right) r^2 + 2V(\theta) \left( \frac{\cos^2 \theta}{\sqrt{U(\theta)}} \right). \] (65)

We will show that for the given \( h \) and \( C \) no orbit can tend to \( C \setminus (\mathcal{B}_{l,r} \cup P_{\pm}) \). Assume that there is an orbit that approaches asymptotically \( C \setminus (\mathcal{B}_{l,r} \cup P_{\pm}) \). This means that from some \( t_0 \) the function \( r(t) \) is monotone decreasing for all \( t > t_0 \). Looking at (64), this implies that \( v(t) < 0 \) for all \( t > t_0 \). Since \( h \) is
finite, the term $\frac{\cos^2 \theta}{\sqrt{U(\theta)}}$ bounded and $V(\theta) > 0$ for all $\theta$, for $r$ small enough the right hand side of the
(65) becomes positive, so making $v' > 0$. Then $v$ starts increasing, becoming positive again for some
$t_1 > t_0$, and thus implying that $r$ is increasing for $t > t_1$. But this contradicts the assumption that $r(t)$
is decreasing for all $t > t_0$. □

Corollary 5.6 The triple collision manifold is reached (asymptotically) by solutions for which the limit
configuration have zero area, i.e., by solutions with limit configurations that are either linear ($Z = 0$),
or vertical, with the equal mass bodies in double collision ($R = 0$).

Using Propositions 5.1 and 5.2 we also deduce

Proposition 5.7 For any $h < 0$ fixed and low angular momenta $|C| < \sqrt{2U_m}$ the triple collision is
attainable (either as a ejection or collision) by solutions with spatial and linear limit configurations.

A direct analysis of the system (22) also implies that

Proposition 5.8 For $h > 0$, all orbits are unbounded.

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