Analytical results for the confinement mechanism in QCD$_3$

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Abstract

We present analytical methods for investigating the interaction of two heavy quarks in QCD$_3$ using the effective action approach. Our findings result in explicit expressions for the static potentials in QCD$_3$ for long and short distances. With regard to confinement, our conclusion reflects many features found in the more realistic world of QCD$_4$.

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I. INTRODUCTION

Gauge theory models in other than 3 + 1 space–time dimensions have been a center of interest for many years. In the present article we want to explore QCD$_3$ and work out its similarities and differences in comparison to QCD$_4$, the presumably correct theory of strong interaction. Our investigation follows in many respects the work of Adler and Piran [1] in QCD$_4$. Their starting point, the renormalization group approach [2], is, however, bound to fail; there is simply no renormalization group in QCD$_3$ nor is there a Callan–Symanzik $\beta$–function, etc., well–known attributes of QCD$_4$; our theory, QCD$_3$, is super–renormalizable. However, the infrared problem, which we are about to analyze, is shared by both theories. Now, since the incompletely understood gluonic vacuum structure is at the heart of both QCD$_4$ and QCD$_3$, we must rely on some more or less reasonable effective action models. In Adler’s case it is the leading–log model while ours might be termed the leading–root model. The latter will be represented by the one–loop effective action with constant color magnetic background field. Among many interesting features revealed in our work is the QCD$_3$ vacuum acting like a dielectric medium, the elliptic shape of the confinement region and its scaling properties and, finally, the behavior of the static potential between two massive color test charges for large and small distances. In Section II we present the essentials for the calculation of the one–loop effective action. In Section III we focus on the large distance (confinement) problem and give an expression for the linearly rising potential plus correction terms. In Section IV we treat the short distance domain and derive the classical formula for the interaction of two static charges augmented, again, by correction terms. Section V summarizes our findings.

II. COMPUTATION OF THE ONE–LOOP EFFECTIVE LAGRANGIAN

Since much work has been invested in the calculation of effective Lagrangians, we will use some short cuts to quickly reach our present goal. The Lagrangian for a pure SU(N)
The gauge field theory is

\[ \mathcal{L} = -\frac{1}{4} F_{\mu \nu}^a F^{a \mu \nu} \]  

(1)

where

\[ F^{a \mu \nu} = \partial^{\mu} A^{a \nu} - \partial^{\nu} A^{a \mu} + g f^{abc} A^{b \mu} A^{c \nu}. \]  

(2)

Working in \( d = 3 \) dimensions we list the following dimensions of various quantities, some of them to be introduced at a later stage: \([g] = [Q] = [A^0] = m^{1/2}, [J^0] = m^{5/2}.\)

Next, the gauge field is decomposed into

\[ A^{a \mu} = A_{B}^{a \mu} + b^{a \mu}, \]  

(3)

where \( b^{a \mu} \) represents the fluctuating Yang–Mills field. The external field configuration \( A_{B}^{a \mu} \) that probes our QCD vacuum is the commonly chosen color magnetic background field

\[ A_{B}^{\mu} = -\frac{1}{2} F_{B}^{\mu \nu} x_{\nu}, \quad F_{B}^{\mu \nu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & B \\ 0 & -B & 0 \end{pmatrix}, \]  

(4)

where \( F_{B}^{a \mu \nu} = F_{B}^{\mu \nu} V^a, \quad A_{B}^{a \mu} = A_{B}^{\mu} V^a, \quad (V^a)^2 = 1. \)

(5)

The magnetic field points along a fixed unit direction \( V^a \) in color space. Inserting the parametrization (3) into the Lagrangian (1) we obtain

\[ \mathcal{L} = -\frac{1}{4} F_{B}^{a \mu \nu} F_{B}^{a \mu \nu} + \frac{1}{2} b^{a \mu} \left[ g_{\mu \nu} (D_B^{ab} D_b^{bc}) \right. \]

\[ - (D_{B \nu}^{ab} D_{B \mu}^{bc}) + g f^{abc} F_{B \mu \nu}^{b} \big] b^{c \nu} + O(b^3), \]  

(6)

where \( D_{B \mu}^{ab} = (\partial_{\mu} \delta^{ab} + g f^{acb} A_{B \mu}^{c}). \)

(7)

The Lagrangian (6) together with the background gauge fixing term,

\[ \mathcal{L}_{GF} = \frac{1}{\alpha} b^{a \mu} (D_{B \mu}^{ab} D_{B \nu}^{bc}) b^{c \nu}, \]  

(8)

connects us with the effective Lagrangian via the functional integral.
\[ N' \exp \left[ i \int d^3x \mathcal{L}_{\text{eff}}^\prime \right] = N \int \mathcal{D}b \Delta_{\text{FP}} \exp \left[ i \int d^3x (\mathcal{L} + \mathcal{L}_{\text{GF}}) \right]. \] 

Using the Feynman gauge, \( \alpha = 1 \), we obtain

\[ \mathcal{L} + \mathcal{L}_{\text{GF}} = -\frac{1}{2} B^2 + \frac{1}{2} b^{\mu \nu} [g_{\mu \nu} (D_B^{ab} D_B^{bc})] + 2g f^{abc} F^a_{\mu \nu} b^c. \] 

To compute the two terms in the square brackets of (10) we follow the author of ref. [3] and so reproduce the result (pay attention to signs!)

\[ \mathcal{L} + \mathcal{L}_{\text{GF}} = -\frac{1}{2} B^2 + \sum_{T^b \in C_V} \frac{1}{2} b^{\mu \nu} (\partial^2 g_{\mu \nu} - \partial_\mu \partial_\nu) b^{\mu \nu} + \sum_\alpha W^{\mu \nu}_\alpha (D^2 g_{\mu \nu} + 2igQ_\alpha F_{\mu \nu}) W^{\nu}_\alpha, \] 

where \( D^\mu = \partial^\mu + igQ_\alpha A^\mu_B = \partial^\mu - \frac{1}{2} gQ_\alpha F^{\mu \nu} x_\nu, \)

\[ F_{\mu \nu} \equiv F_{B \mu \nu}. \] 

and \( C_V = \{ T^b | [V, T^b] = 0 \} \), \( V = V^a T^a \),

where the \( T^a \)'s denote the standard SU(N)–generators: \( [T^a, T^b] = if^{abc} T^c \), \( \text{Tr} (T^a T^b) = \frac{1}{2} \delta^{ab}. \) The remaining generators \( \notin C_V \) can be expressed in terms of eigenvectors of \( V \) with eigenvalues \( Q_\alpha \). Their fluctuating Yang–Mills field components \( b^{\mu \nu} \) form certain complex linear combinations \( W^{\mu}_\alpha \) depending on the choice of \( V \) in color space.

At this point we perform a Wick–rotation so that all subsequent calculations are done in Euclidean space:

\[ N'' \exp \left[ \int d^3x \mathcal{L}_{\text{E}}^\prime \right] = N'' \int \mathcal{D}b \Delta_{\text{FP}} \exp \left[ \int d^3x (\mathcal{L}_E + \mathcal{L}_{\text{EGF}}) \right] = N''' \int \mathcal{D}W^* \mathcal{D}W \Delta_{\text{FP}} \exp \left[ \int d^3x \left( -\frac{1}{2} B^2 + \sum_\alpha W^{\mu \nu}_\alpha (D^2 \delta_{\mu \nu} + 2igQ_\alpha F_{\mu \nu}) W^{\nu}_\alpha \right) \right]. \]
Due to the fact that the fields $W^*$ and $W$ have to be treated independently the Faddeev–Popov Determinant appears in the form

$$\Delta_{FP} = \Delta_{FP}^W \Delta_{FP}^{W^*} = \text{Det}(-D^2)\text{Det}(-D^2)$$

$$= \exp[2 \ln \text{Det}(-D^2)] = \exp[2\text{Tr} \ln(-D^2)]$$

$$= \exp \left[ \int d^3x \, 2\text{Tr} \ln(-D^2) \right],$$

Now we can write (13) as

$$N' \exp \left[ \int d^3x \, L_{E}^{\text{eff}} \right] = N'' \exp \left[ \int d^3x \left( -\frac{1}{2} B^2 + 2\text{tr} \ln(-D^2) \right) \right]$$

$$\times \int D W^* D W \exp \left[ \int d^3x \sum_{\alpha} W_{\alpha \mu}^* (D^2 \delta_{\mu \nu} + 2i g Q_{\alpha} F_{\mu \nu}) W_{\alpha \nu} \right].$$

Here we meet the standard functional integral

$$\int D W^* D W \exp \left[ - \int d^3x \, W_{\mu \nu}^* M_{\mu \nu} W_{\nu} \right]$$

$$= \exp[-\text{Tr} \ln M_{\mu \nu}].$$

Hence eq. (14) takes the form

$$N' \exp \left[ \int d^3x \, L_{E}^{\text{eff}} \right] = N'' \exp \left[ \int d^3x \left( -\frac{1}{2} B^2 + 2\text{tr} \ln(-D^2) \right) \right.$$$$- \text{tr} \ln(-D^2 \delta_{\mu \nu} - 2i g Q_{\alpha} F_{\mu \nu})) \right],$$

from which we read off the expression for $L_{E}^{\text{eff}}$:

$$L_{E}^{\text{eff}} = -\frac{1}{2} B^2 + 2\text{tr} \ln(-D^2) - \text{tr} \ln(-D^2 \delta_{\mu \nu} - 2i g Q_{\alpha} F_{\mu \nu}).$$

There are several ways to compute the various traces in this expression. We prefer the $\zeta$–function regularization method [5]. The result is
\[ L_{\text{eff}} = -\frac{1}{2}B^2 + |gB|^{3/2} \times \frac{1}{2\pi} \left[ 1 - \frac{(\sqrt{2} - 1)}{4\pi} \zeta\left(\frac{3}{2}\right) \right] \sum_{\alpha} |Q_\alpha|^{3/2}. \]  

(17)

In ref. [3] and [6] we found a proper–time calculation of expression (16). We agree with the result contained in ref. [3]. The numerical value of \( \zeta\left(\frac{3}{2}\right) \) is \( \approx 2.61238 \) so that the sign in front of \( |gB|^{3/2} \) is indeed positive. Hence \( L_{\text{eff}} \) takes a maximum (\( V_{\text{eff}} \) a minimum) at a nonzero value of the background field:

\[ |B_{\text{ex}}|^{1/2} = g^{3/2} \frac{3}{4\pi} \left[ 1 + \frac{2}{\sqrt{2} - 1} \|F\| \zeta\left(\frac{3}{2}\right) \right] \sum_{\alpha} |Q_\alpha|^2. \]  

(18)

The gauge invariant generalization of (17) can be obtained by the replacement \( B^2 \rightarrow -\frac{1}{2}F_{\mu\nu}F_{\mu\nu}^a = (E^a)^2 - (B^a)^2 = F^a, \)

\[ L_{\text{eff}} = \frac{1}{2}F^a \left[ 1 - \frac{4}{3} \left( \frac{F^a}{\kappa^2} \right)^{1/4} \right], \]  

(19)

where \( \kappa^{1/2} = |B_{\text{ex}}|^{1/2} = \text{eq.}(18). \)

(20)

So we obtain a gauge field condensate \( F = \kappa^2 \) due to radiative corrections, just like in four dimensions, which determines the interesting features of the model. If we choose SU(N=3) as our gauge group and the unit color vector pointing along the three-direction we find \( \kappa^{1/2} = 0.37245 \ldots g^{3/2}. \)

At this stage it is important to point out that we are only dealing with the lowest (first) order loop approximation. Higher order loop calculations will certainly modify the position and the shape of the extremum, i.e., the value of \( \kappa. \) However, concordant with our own approach there are other strong indications for gauge field condensation as implied, e.g., by the so called average action method advocated by the authors of ref. [4]. Thus, we expect our leading–root model to describe the exact QCD3–vacuum structure at least qualitatively accurate.

Furthermore, we assume that the form of (19) also holds for static fields which are slowly varying in space.
III. FLUX CONFINEMENT AND THE HEAVY QUARK STATIC POTENTIAL

In this section we study the statics of two massive test charges at large distances. The approximation in which leading QCD radiative corrections are retained is given by the effective Lagrangian (19). Following Adler [1] we write for the potential of static, infinitely massive test charges \[ J^a_\mu = (J^a_0, 0) \]

\[ V_{\text{static}} = -\text{extr}_{A^a_\mu, \varphi} \left\{ \int d^2 x \left( \mathcal{L}_{\text{eff}}(A^a_\mu) - A^a_0 J^a_0 \right) \right\}, \]  

(21)

Limiting ourselves to the case of quark–antiquark source charges we set \( R = 2a \)

\[ J^a_0 = Q \hat{q}^a [\delta^2(r - a\hat{x}) - \delta^2(r + a\hat{x})] = \hat{q}^a J_0, \]  

(22)

where \( \hat{q}^a \) is the internal unit vector in color space. Similarly we introduce scalar and vector potentials \( \varphi \) and \( A \) by writing

\[ A^a_0 = \hat{q}^a \varphi, \quad A^a_j = \hat{q}^a A_j. \]  

(23)

Our variational problem can now be restated in the form

\[ V_{\text{static}} = -\text{extr}_{A, \varphi} \left\{ \int d^2 x \left( \mathcal{L}_{\text{eff}}(\mathcal{F}) - \varphi J_0 \right) \right\}, \]  

(24)

where

\[ \mathcal{F} = E^2 - B^2, \quad E = -\nabla \varphi, \quad B = \epsilon_{ij} \partial_i A_j. \]  

(25)

With \( \mathcal{L}_{\text{eff}}(\mathcal{F}) \) given by (19) we write more explicitly

\[ V_{\text{static}} = -\text{extr}_{A, \varphi} \left\{ \int d^2 x \left( \frac{1}{2} \mathcal{F} \left( 1 - \frac{4}{3} \left| \mathcal{F} \right|^{-\frac{1}{3}} \right) - \varphi J_0 \right) \right\}, \]  

(26)

\[ =: \text{extr}_{A, \varphi} \left\{ \int d^2 x \mathcal{L}_{\text{stat}} \right\}. \]  

(27)

Given \( \mathcal{L}_{\text{stat}} \) we now can apply the Euler–Lagrange equations which imply the field equations.
\[ \nabla \cdot \mathbf{D} = J_0 \quad \epsilon_{ij} \partial_i E_j = 0 = \epsilon_{ij} \partial_i H, \]  

(28)

where \( \mathbf{D} = \epsilon \mathbf{E} \), \( H = \epsilon B \), \( \epsilon = 1 - \left| \frac{F}{\kappa^2} \right|^{\frac{1}{4}} \).

(29)

The source–free equation (28) for the magnetic field can be satisfied by

\[ \epsilon B^2 = 0. \]

(30)

Here we have to distinguish three cases:

(Ia) \( B = 0 \), \( \mathbf{E}^2 > \kappa^2 \)

(Ib) \( B = 0 \), \( \mathbf{E}^2 < \kappa^2 \)

(II) \( \epsilon = 0 \), \( B^2 = \mathbf{E}^2 - \kappa^2 \).

(31)

For short distances we expect Coulomb–like field configurations with \( \mathbf{E} \) large and \( B \) vanishing. Hence there should exist a finite region containing the source charges for which (Ia) is satisfied. In this domain we have reduced our original variational problem to a problem in nonlinear electrostatics with a field strength dependent dielectric constant:

\[ \nabla \cdot \mathbf{D} = J_0 \quad \epsilon_{ij} \partial_i E_j = 0, \]

(32)

\[ \mathbf{D} = \epsilon(E) \mathbf{E}, \quad \epsilon(E) = 1 - \sqrt{\frac{\kappa}{E}}, \quad E = |\mathbf{E}| \in \mathbb{R}. \]

(33)

Now, in Adler’s leading–log model of QCD, it proved very effective to work with a manifestly flux conserving quantity. So let us likewise parametrize \( \mathbf{D} \) by introducing a scalar flux function \( \mathbf{D} = f(\Phi) \). Without going into all the details (and subtleties) of how to arrive at the explicit relation between \( \mathbf{D} \) and \( \Phi \), we just state the result:

\[ \mathbf{D} = \left( \frac{1}{2} \frac{\partial \Phi}{\partial y}, -\frac{1}{2} \frac{\partial \Phi}{\partial x} \right). \]

(34)

(The authors of ref. [7] missed the important factor \( \frac{1}{2} \).) The boundary conditions imposed on the flux function \( \Phi(x, y) \) are
\[ \Phi(|x| < a, y \to 0) = Q, \]
\[ \Phi(x > a, y \to 0) = 0, \quad (35) \]
\[ \Phi \to 0 \quad \text{for} \quad x^2 + y^2 \to \infty. \]

For future calculations it is useful to derive some relations between the fields \( E \) and \( D \). From eqs. (33) and still considering the branch (Ia), where \( E > \kappa \), we obtain
\[ D = |D| = |\epsilon||E| = \epsilon E, \quad \epsilon = \sqrt{1 - \frac{\kappa}{E}} > 0. \quad (36) \]
This equation implies \( D = -\sqrt{\kappa} E^{1/2} + E \) or, setting \( e := \sqrt{E} \),
\[ e_{1,2} = \frac{\sqrt{\kappa}}{2} \pm \sqrt{\frac{\kappa^2}{4} + \kappa D}. \quad (37) \]
We need \( e_{1,2}^2 = \frac{\kappa}{2} \pm \sqrt{\frac{\kappa^2}{4} + \kappa D} \)
and select the positive sign to guarantee a single–valued potential. So we have the relation
\[ E = E(D) = \frac{\kappa}{2} + \sqrt{\frac{\kappa^2}{4} + \kappa D + D}. \quad (38) \]

At last we turn to the solution of eq. (32). In order to obtain more insight into the behavior of the solution, we begin by rewriting the field equation for \( \Phi(x, y) \) in its characteristic form. So let us start with
\[ \epsilon_{ij} \partial_i E_j = 0 \]
\[ 0 = \partial_x E_y - \partial_y E_x = \partial_x \left( \frac{D_y}{\epsilon} \right) - \partial_y \left( \frac{D_x}{\epsilon} \right) \]
\[ = \partial_x \left( \frac{D_y \sqrt{E}}{\sqrt{E - \sqrt{\kappa}}} \right) - \partial_y \left( \frac{D_x \sqrt{E}}{\sqrt{E - \sqrt{\kappa}}} \right). \quad (39) \]

Here we employ eq. (37) and introduce the flux function via relation (34). The result of a rather lengthy chain of partial derivatives is given by the exact field equation
\[ \left( \partial_x^2 + \partial_y^2 + (\alpha - 1) \partial_n^2 \right) \Phi = 0, \quad (40) \]
where \( \partial_n = \hat{n} \cdot \nabla \) and \( \hat{n} = \frac{\nabla \Phi}{|\nabla \Phi|} \)

\[
\alpha = - \frac{1 + \frac{1}{2} \frac{\kappa}{\sqrt{\frac{\kappa^2}{4} + \kappa D}}}{- \frac{\kappa}{2} D - 1 - \frac{\sqrt{\frac{\kappa^2}{4} + \kappa D}}{D}}
\]

(41)

Letting \( \partial_t \) be the tangential derivative, we can replace the coordinate derivatives \( (\partial_x, \partial_y) \) in (40) by

\[
\partial_x^2 + \partial_y^2 = \partial_t^2 + \partial_n^2 + \text{first derivative terms}
\]

and so the field equation (40) takes the form

\[
(\partial_t^2 + \alpha \partial_n^2) \Phi + \mathcal{O}(\partial_t, \partial_n) = 0.
\]

(42)

Now, for weak fields, \( D \to 0 \), i.e., away from the charges, we have

\[
\lim_{D \to 0} \alpha = \lim_{D \to 0} \frac{\kappa}{2} D + D = 0,
\]

(43)

and for strong fields, \( D \to \infty \), i.e., close to the charges, we obtain

\[
\lim_{D \to \infty} \alpha = 1.
\]

(44)

The flux equation (40) is quite similar to the one found in QCD\(_4\); it is of degenerating elliptic type and has a real characteristic at a surface of constant \( \Phi \), where \( \nabla \Phi = 0 \). Using the same arguments as in Adler’s leading–log model, we have here the first indication of confinement in QCD\(_3\). Next we want to show quantitatively that, in fact, the total flux between two massive color charges is confined to a domain with a characteristic as boundary on which \( \Phi \) vanishes. To do so it is useful to reformulate our problem in still another form for the equation for \( \Phi \). For this reason let us go back to eq. (32). We also know that \( E \) depends on \( D \) in a way stated in (38):

\[
f(D) := E(D) = \frac{\kappa}{2} + \sqrt{\frac{\kappa^2}{4} + \kappa D + D}.
\]
Then we obtain

\[ 0 = \partial_x E_y - \partial_y E_x = \partial_x \left( \frac{E_y}{E} f(D) \right) - \partial_y \left( \frac{E_x}{E} f(D) \right) \]

\[ = \partial_x \left( \frac{D_y}{D} f(D) \right) - \partial_y \left( \frac{D_x}{D} f(D) \right). \]

Recalling relation (34) we get

\[ \partial_x \left[ \frac{\partial_x \Phi f(D)}{((\partial_x \Phi)^2 + (\partial_y \Phi)^2)^{1/2}} \right] + \partial_y \left[ \frac{\partial_y \Phi f(D)}{((\partial_x \Phi)^2 + (\partial_y \Phi)^2)^{1/2}} \right] = 0, \tag{45} \]

where \( ((\partial_x \Phi)^2 + (\partial_y \Phi)^2)^{1/2} = 2D \).

After performing the various partial derivatives we end up with the following, still exact, differential equation for \( \Phi \):

\[ 0 = \left( (\partial_x \Phi)^2 + \left( \frac{g(D)}{D} + 1 \right) (\partial_y \Phi)^2 \right) \partial_x^2 \Phi \]

\[ + \left( \frac{g(D)}{D} + 1 \right) (\partial_x \Phi)^2 + (\partial_y \Phi)^2 \partial_y^2 \Phi \]

\[ - 2 \frac{g(D)}{D} \partial_x \Phi \partial_y \Phi \partial_{xy} \Phi, \tag{46} \]

where \( g(D) := \frac{f(D)}{f'(D)} - D. \tag{47} \)

At this stage we make contact with calculations contained in ref. [8]. Needless to say, a solution of (46) is not easily available. However, being interested in the far–field approximation, we now study the limiting case of weak fields:

\[ f(D) = \frac{\kappa}{2} + D + \sqrt{\frac{\kappa^2}{4} + \kappa D} \]

\[ = \frac{\kappa}{2} + D + \frac{\kappa}{2} \left( 1 + \frac{2D}{\kappa} - \frac{2D^2}{\kappa^2} + \mathcal{O}(3) \right), \]

so that to first order in \( D \): \( f(D) = \kappa + 2D \). From here we obtain for (47)

\[ g(D) = \frac{\kappa}{2}, \tag{48} \]

meaning \( g(D) \sim \mathcal{O}(D^0) \).
Using the value (18) in eq. (46) we arrive at

\[
0 = \left[ (\partial_x \Phi)^2 + (1 + \frac{\kappa}{2D}) (\partial_y \Phi)^2 \right] \partial_x^2 \Phi \\
+ \left[ (1 + \frac{\kappa}{2D}) (\partial_x \Phi)^2 + (\partial_y \Phi)^2 \right] \partial_y^2 \Phi \\
- \frac{\kappa}{D} \partial_x \Phi \partial_y \Phi \partial_{xy} \Phi. 
\] (49)

Following the strategies in QCD it is convenient to rescale \( x \) and \( y \) in terms of dimensionless parameters:

\[
x = R \bar{x}, \quad y = R^\alpha \bar{y}. 
\] (50)

The authors of ref. [8] supply arguments as to why the transverse coordinate scales with \( \alpha = \frac{2}{3} \). Now, we try the following ansatz:

\[
\Phi = \Phi^{(0)} + \frac{1}{R} \Phi^{(1)} + \frac{1}{R^2} \Phi^{(2)} + \ldots 
\] (51)

Earlier we found \((2D) = ((\partial_x \Phi)^2 + (\partial_y \Phi)^2)^{1/2}\), so that

\[
\frac{1}{2D} = \left[ \frac{1}{R^2} (\partial_x (\Phi^{(0)} + \frac{1}{R} \Phi^{(1)} + \ldots))^2 \\
+ \frac{1}{R^{4/3}} (\partial_y (\Phi^{(0)} + \frac{1}{R} \Phi^{(1)} + \ldots))^2 \right]^{1/2} \\
= -\frac{R^{2/3}}{\partial_y \Phi^{(0)}} + \frac{1}{2} \frac{\partial_y(\Phi^{(0)})^2}{(\partial_y \Phi^{(0)})^3} + \mathcal{O}(R^{-1/3}), 
\] (52)

with \( \partial_y \Phi^{(0)} < 0 \). (53)

Again, we skip the details of the calculation for the various partial derivatives in (49). Then we obtain the following differential equation for the flux function in zeroth order \((\bar{x}, \bar{y} \equiv x, y)\):

\[
0 = \kappa (\partial_x^2 \Phi^{(0)})(\partial_y \Phi^{(0)}) + \kappa (\partial_y^2 \Phi^{(0)}) (\partial_x \Phi^{(0)})^2 \\
- (\partial_y^2 \Phi^{(0)})(\partial_y \Phi^{(0)})^2 - 2\kappa (\partial_x \Phi^{(0)})(\partial_{xy} \Phi^{(0)}). 
\] (54)

Another useful form of this equation can be obtained by multiplication with \((\partial_y \Phi^{(0)})^2\):

\[
0 = \partial_y \left[ \frac{\kappa}{2} \left( \frac{\partial_x \Phi^{(0)}}{\partial_y \Phi^{(0)}} \right)^2 + \partial_y \Phi^{(0)} \right] - \kappa \partial_x \left[ \frac{\partial_x \Phi^{(0)}}{\partial_y \Phi^{(0)}} \right]. 
\] (55)
We do not expect eqs. (54) or (55) to be soluble for \( \Phi(0)(x, y) \) by the separation of variables method since there is still the boundary condition (35) to be taken into account. Yet, as in QCD$_4$, there is hope for the existence of a separable solution \( y(x, \Phi) \). Hence our next step is to rewrite (54) into a differential equation for \( y \). Here are the rules governing how to achieve this (\( \Phi \equiv \Phi(0) \)):

\[
\begin{align*}
\Phi_y &= y^{-1}_\Phi, & \Phi_x &= -\frac{y_x}{y\Phi}, & \Phi_{yy} &= -\frac{y\Phi y}{y^3\Phi}, \\
\Phi_{xx} &= -\frac{y_{xx}}{y\Phi} - \frac{y\Phi y^2_x}{y^2\Phi} + 2 \frac{y_{x\Phi} y_x}{y^2\Phi}, \\
\Phi_{xy} &= -\frac{y_{x\Phi}}{y^2\Phi} + \frac{y_{\Phi\Phi} y_x}{y^3\Phi}.
\end{align*}
\]

This set of partial derivatives enables us to rewrite (54) in the form

\[
\frac{y\Phi y}{y^3\Phi} - \kappa \frac{y_{xx}}{y^2\Phi} = 0.
\]

(57)

Here we can try the ansatz

\[ y(x, \Phi) = X(x) F(\Phi) \]

(58)

and so obtain for (57) instead

\[
\frac{F''}{F F'\Phi} = c, \\
\kappa X'' X^2 = c.
\]

(59) (60)

where \( c \) is the separation constant. Integrating (59) yields

\[
\Phi = -\frac{1}{6} c \frac{y^3}{X^3(x)} + k_1 \frac{y}{X(x)} + k_2.
\]

(61)

With the aid of (55) we find for \( k_2 \):

\[ \Phi(y = 0) = Q : k_2 = Q. \]

Since it is easier to work with \( X(x) \) in the numerator of (61), we redefine \( X \rightarrow \frac{1}{X} \) and so obtain
Φ = Q(1 − a₁yX(x) + a₃y³X³(x)) \tag{62}

with new constants \( a₁ = \frac{k₁}{Q}, \quad a₃ = \frac{1}{6} \frac{c}{Q} \). \tag{63}

Now recall that (i) we have to satisfy \( \partial_y \Phi \leq 0 \) in the confining domain and (ii) we also want \( \Phi \) to approach the boundary \( \Phi = 0 \) continuously: \( \partial_y \Phi|_{y = y_b} = 0 \). The first condition tells us that \( a₁ > 0 \) while the second implies \( a₃ > 0 \). These considerations lead to the following two equations:

\[
\begin{align*}
0 &= \Phi(x, y_b(x)) \\
&= Q(1 − a₁ y_b(x)X(x) + a₃ y_b³(x)X³(x)), \tag{64} \\
0 &= \partial_y \Phi|_{y = y_b} \\
&= Q(−a₁ X(x) + 3a₃ y_b²X³(x)). \tag{65}
\end{align*}
\]

The last equation yields the explicit expression

\[
y_b(x) = \pm \sqrt{\frac{a₁}{3a₃ X(x)}} \tag{66}.
\]

The two signs reflect the symmetry with respect to the \( x \)-axis. Substituting (66) into (64) we obtain

\[
a₃ = \frac{4}{27} a₁³. \tag{67}
\]

Thus, there is only one free parameter left. So far we have

\[
\Phi(x, y) = Q(1 − \frac{3}{2} b yX(x) + \frac{1}{2} b³ y³X³(x)) \\
y_b(x) = \pm \frac{1}{b X(x)}, \tag{68} \\
\text{where} \quad b = \frac{2}{3} a₁, \quad \text{or} \quad a₃ = \frac{1}{2} b³. \tag{69}
\]

What remains is an explicit solution for \( X(x) \). Because of our redefinition \( X \rightarrow \frac{1}{X} \) the equation following from (61) is

\[
κ \left( \frac{1}{X} \right)'' X^{-2} = c. \tag{70}
\]
Introducing $X' = \sqrt{p}$ we can cast (71) into the form

$$\frac{dp}{dX} = \frac{4p}{X} + \frac{6Q}{\kappa}b^3X^4,$$

which can be solved by

$$p(x) = \frac{6Qb^3}{\kappa}X^4 \left[ X - X_0 + \frac{\kappa p_0}{6QX_0^4b^3} \right],$$

where $X_0 = X(x = 0)$ and $p_0 = p(X_0)$. Among the three free parameters $b, X_0, p_0$ we find, via equation (69),

$$y_b(x)|_{x=0} = \frac{1}{bX_0},$$

that $b$ and $X_0$ merely rescale the size of the confinement domain. Hence solution (73) is essentially determined by $p_0$. Because of

$$p_0 = p(X_0) = p(X(x = 0)) = \left( \frac{dX}{dx} \right)^2|_{x=0}$$

we see that $p_0$ is (a) related to the tangent of $\Phi$ in $x$–direction at $x = 0$,

$$\partial_x \Phi|_{x=0} = Q( -\frac{a}{2}b y X' + \frac{a}{2}b^3y^3X^2X')|_{x=0}$$

and (b) related to the shape of the confinement boundary at $x = 0$:

$$\partial_x y_b(x)|_{x=0} = -\frac{1}{bX^2}X'|_{x=0}.$$

Now there are three different cases to be distinguished:

(1) $p_{01} = \frac{6QX_0^5b^3}{\kappa}$

(2) $p_{02} = 0$

(3) $p_{03} \neq p_{01}, p_{02}$, arbitrary.

Case (1) has been treated in the literature [7] and has the advantage of being analytically soluble. We regard this solution as unphysical, since, looking at (76), $y_b(x)$ does not behave smoothly at $x = 0$; this solution never yields an extremum with regard to, e.g., the energy.
density, assuming an underlying variational principle. On the other hand, case (2) contains a physical, smooth boundary, $\partial_x y_b|_{x=0} = 0$, and is our preferred solution. Its disadvantage is that it cannot be solved analytically. Case (3) is neither analytically soluble nor is it physical and so will not be considered any further.

Without going into further details we now present our solution of eqs. (54,55) for the case (1):

$$\Phi^{(0)}(x, y) = Q \left[ 1 - 2^{-2/3} \left( \frac{\kappa}{Q} \right)^{1/3} \frac{y}{(a-x)^{2/3}} + \frac{\kappa}{27Q} \frac{y^3}{(a-x)^2} \right]$$

$$y_b(x) = \pm \frac{3}{2^{1/3}} \left( \frac{Q}{\kappa} \right)^{1/3} \frac{(a-x)^{2/3}}{79}$$

(where we have switched back to our original coordinates $x, y \rightarrow R^{-1} x, \frac{1}{R^{2/3}} y$). We have assumed $x, y > 0$; otherwise we would have to use moduli. Note the scaling behavior of $y$ with a $2/3$ power.

Finally we come to the calculation of the static potential. This is achieved with the aid of the formula

$$V_{\text{static}} = \int d^2x \int_0^D E(D') dD'$$

$$= \int d^2x \int_0^D dD' \left[ \frac{\kappa}{2} + \sqrt{\frac{\kappa^2}{4} + \kappa D'} + D' \right]$$

$$= \int d^2x \left( \frac{\kappa^2}{2} D + 2 \frac{\kappa^2}{3\kappa} + \frac{\kappa^3}{2} - \frac{\kappa^2}{12} + \frac{1}{2} D^2 \right)$$

$$= \int d^2x \left[ \% \right].$$

The surface integration has to be performed over the confinement region defined by (79):

$$V_{\text{static}} = \int d^2x \left[ \% \right] = \int_{-a+\epsilon}^{a-\epsilon} dx \int_{-y_b(x)}^{y_b(x)} dy \left[ \% \right]$$

$$= 4 \int_0^{a-\epsilon} dx \int_0^{y_b(x)} dy \left[ \% \right].$$
Restricting ourselves to weak fields, i.e., to large distances (confinement or infrared domain), we approximate the integrand in (80) by

\[ \[\kappa D + D^2 + \ldots \] \]

where \( D = \frac{1}{2}[(\partial_x \Phi)^2 + (\partial_y \Phi)^2]^{1/2} \) in which we substitute the perturbative ansatz (51) and rescale according to (50). An intermediate step on our way to \( V_{\text{static}} \) is

\[ V_{\text{static}} = I_1 + I_2, \quad (81) \]

where

\[
I_1 = -2\kappa \int_0^{a-\epsilon} dx \int_0^{y_0(x)} dy \left[ \partial_y \Phi^{(0)} + \frac{1}{\kappa} \partial_y \Phi^{(1)} \right] \\
= -2\kappa \int_0^{a-\epsilon} dx \int_0^{y_0(x)} dy \partial_y \Phi \\
= -2\kappa \int_0^{a-\epsilon} dx \left[ \Phi|_{y_0(x)} - \Phi|_{y=0} \right] \\
= \lim_{\epsilon \to 0} 2\kappa Q(a - \epsilon) = \kappa QR. \quad (82)\]

So indeed we have linear confinement!

The correction term \( I_2 \) turns out to be

\[
I_2 = \frac{12}{5} \sqrt[3]{\kappa Q^5} R^{1/3} - \frac{6}{35} \sqrt[3]{\frac{Q^7}{\kappa}} R^{-1/3}. \quad (83)\]

Here, then, is our final expression [case(1)] for the static potential of two oppositely charged massive test sources at large distances:

\[
V_{\text{static}} = \kappa QR + \frac{12}{5} \sqrt[3]{\kappa Q^5} R^{1/3} - \frac{6}{35} \sqrt[3]{\frac{Q^7}{\kappa}} R^{-1/3} + \ldots. \quad (84)\]

Note the linear rising with distance as it is familiar from QCD.

Now we turn to case (2) with \( p_0 = p_{02} = 0 \), which guarantees a smooth boundary, \( X'(x)|_{x=0} = 0 \). The equation to be integrated follows from (73):

\[
\frac{dX}{dx} = \pm \sqrt{\frac{6b^2 Q}{\kappa} X^2 X - X_0}. \quad (85)\]
The solution is easily found:

\[
x = \pm \sqrt{\frac{\kappa}{6Q(bX_0)^3}} \left[ \frac{X(x)}{X_0} - 1 + \arctan \sqrt{\frac{X(x)}{X_0}} - 1 \right].
\]

This equation is transcendental with respect to \(X(x)\), i.e., neither \(X(x)\) nor \(\Phi(x, y)\) can be written in an explicit analytical form. It is, however, possible to find an approximate solution which is sufficiently close to an exact solution and which maintains the smooth boundary \(y_b(x)\). Defining \(t = X(x)/X_0\) it turns out that \(\sqrt{t-1/t} + \arctan \sqrt{t-1}\) can be excellently approximated by \(\arctan \sqrt{t-q} - 1\) with \(q \approx 3.301\ldots\), whereby \(q = 3\) already represents a fairly good approximation. So we can write instead of (86):

\[
x = \pm \sqrt{\frac{\kappa}{6Q(bX_0)^3}} \arctan \left( \frac{(X(x))^q}{X_0} - 1 \right) - q
\]

or \(X(x) = X_0 \left( 1 + \tan^2 \left[ \sqrt{\frac{6Q(bX_0)^3}{\kappa}} x \right] \right)^{1/q}\).

With the aid of eq. (69), i.e., employing the condition

\[
0 = y_b(x = \frac{1}{2}) = \pm \frac{1}{bX_0} \left( 1 + \tan^2 \left[ \sqrt{\frac{6Q(bX_0)^3}{\kappa}} \frac{1}{2} \right] \right)^{-1/q}
\]

we obtain \(bX_0 = \left( \frac{\kappa \pi^2}{6Q} \right)^{1/3}\).

Hence our approximate solution for the flux function is given by \((x, y \rightarrow x/R, y/R^{2/3})\):

\[
\Phi^{(0)}(x, y) = Q \left[ 1 - \left( \frac{9\kappa \pi^2}{16Q} \right)^{1/3} \frac{y}{R^{2/3}} \left( 1 + \tan^2 \frac{\pi x}{2a} \right)^{1/q} + \frac{\kappa \pi^2 y^3}{12Q R^2} \left( 1 + \tan^2 \frac{\pi x}{2a} \right)^{3/q} \right]
\]

with the confining boundary

\[
y_b(x) = 2 \left( \frac{3Q}{\kappa \pi^2} \right)^{1/3} \frac{a^{2/3}}{1 + \tan^2 \frac{\pi x}{2a}^{1/q}}.
\]

Again, the scaling behavior \(x \propto a\) and \(y \propto a^{2/3}\) is visible.
With the aid of the flux function we now turn to the calculation of the static potential. According to eqs. (81,82) there is no change in first (linear) order $\propto \kappa QR$; for the calculation of this term the explicit form of $\Phi$ is not needed. The correction term yields

$$I_2 = -\int_0^{a-\epsilon} dx \int_0^{y_n(x)} dy \left[ \kappa \frac{(\partial_x \Phi(0))^2}{\partial_y \Phi(0)} - (\partial_y \Phi(0))^2 \right] - \frac{\kappa (\partial_x \Phi(0))^4}{4 (\partial_y \Phi(0))^3} - (\partial_x \Phi(0))^2 \right]$$

(91)

and after performing the various integrals we end up with

$$V_{\text{static}} = \kappa QR + 2.17 \cdot \sqrt[3]{\kappa Q^5 R^{1/3}} - 0.36 \cdot \sqrt[3]{\frac{Q^2}{\kappa}} R^{-1/3}.$$ 

(92)

IV. SHORT DISTANCE BEHAVIOR

Let us recall that in four dimensional space the static potential can be evaluated by \[\quad V_{\text{static}} = -(\text{extr}_\varphi W - \Delta V_{\text{Coulomb}}), \quad (93)\]

where $W = \int d^3 x \left[ \frac{1}{8} b_0 (\nabla \varphi)^2 \ln \left( \frac{(\nabla \varphi)^2}{e\kappa^2} \right) - \varphi J_0 \right].$ 

(94)

By means of a cleverly chosen rescaling of coordinates, fields and charges, Adler succeeds in splitting up the effective Lagrangian into a classical part and extra terms due to quantum corrections containing a dimensionless *running coupling* $\zeta(R)$. This function goes to zero as $R$ approaches zero. Hence limiting oneself to short distance behavior, $\varphi$ and $V_{\text{static}}$ can be expanded in a perturbation series around the classical Coulomb solution with $\zeta(R)$ as the small parameter.

In QCD$_3$ we found for the effective action

$$W = \int d^2 [L_{\text{eff}}(E) - \varphi J_0]$$

$$= \int d^2 \left[ \frac{1}{2} (\nabla \varphi)^2 \left( 1 - \frac{4}{3} \sqrt[3]{\frac{\kappa}{E}} \right) - \varphi J_0 \right].$$

(95)
Here it is obvious that the classical part $\propto (\nabla \varphi)^2$ is already separated from the part containing the QCD corrections. Hence, rescaling the various parameters does in no way improve the situation. In fact the static potential approaches its classical limit in a natural manner, since close to the test charges we have $E \gg \kappa$. Thus, for short distances, $R \to 0$, our zeroth order approximation is sufficiently described by electrostatics:

$$V_{\text{static}} = -\text{extr}\varphi \int d^2x \left[\frac{1}{2}(\nabla \varphi)^2 - \varphi J_0\right] + \triangle V_c. \quad (96)$$

From (96) we obtain the Poisson equation,

$$\nabla^2 \varphi = -J_0 = -Q[\delta(r - a\hat{x}) - \delta(r + a\hat{x})], \quad (97)$$

which has the well–known solution

$$\varphi = -\frac{Q}{2\pi} \ln \frac{|r - a\hat{x}|}{|r + a\hat{x}|} = -\frac{Q}{2\pi} \ln \frac{\sqrt{(x - a)^2 + y^2}}{\sqrt{(x + a)^2 + y^2}}. \quad (98)$$

The Coulomb counter term becomes

$$\triangle V_c = \text{extr}\varphi \left\{ \int d^2x \left[\frac{1}{2}(\nabla \varphi)^2 - \varphi J_1\right] + \int d^2x \left[\frac{1}{2}(\nabla \varphi)^2 - \varphi J_2\right] \right\}$$

$$J_1 = Q\delta(r - a\hat{x}), \quad J_2 = -Q\delta(r + a\hat{x})$$

and the solutions of the corresponding field equations are

$$\varphi_1 = -\frac{Q}{2\pi} \ln \lambda |r - a\hat{x}|, \quad \varphi_2 = \frac{Q}{2\pi} \ln \lambda |r + a\hat{x}|; \quad (99)$$

where the arbitrary parameter $\lambda$ has dimension $[L]^{-1}$. Using the Poisson equation (97) in the action we find for $W$ and similarly for $\triangle V_c$

$$W = \int d^2x \left[\frac{1}{2}(\nabla \varphi)^2 - \varphi J_0\right]$$

$$= \int d^2x \left[-\frac{1}{2}\varphi \nabla^2 \varphi - \varphi J_0\right] = -\frac{1}{2} \int d^2x \varphi J_0.$$
\[
V_{\text{static}} = \frac{1}{2} \int d^2 x \varphi J_0 - \frac{1}{2} \int d^2 x \varphi_1 J_1 - \frac{1}{2} \int d^2 x \varphi_2 J_2
= \frac{Q^2}{2\pi} \ln \lambda R. \tag{100}
\]

This, then, is the leading short distance behavior of the static potential in QCD. In the sequel we will demonstrate that expression (100) also shows up when we now compute \(V_{\text{static}}\) by means of the formalism we developed for large distances. To do so we have to return to the exact quasilinear, second order differential equation (46). Expanding the coefficient function \(g(D)\) in terms of \(D\) near the sources where \(D \gg \kappa\), we obtain

\[
g(D) = \frac{1}{2} \sqrt{\kappa D} + \frac{\kappa}{4} + \mathcal{O}(D^{-1/2}). \tag{101}
\]

The relevant ratio \(g(D)/D\) is thus of order \(\mathcal{O}(D^{-1/2})\); hence we are permitted to omit it for short distances, obtaining, not surprisingly, Laplace’s equation as approximation of eq. (46):

\[
(\partial_x^2 + \partial_y^2) \Phi = 0 \tag{102}.
\]

The well-known solutions satisfying the boundary conditions (35) are

\[
\Phi(x, y) = \Phi_{\text{cl}} = \frac{Q}{\pi} \left[ \arctan \frac{y}{x - a} - \arctan \frac{y}{x + a} \right]. \tag{103}
\]

This expression for our flux function can be used to find

\[
D = \frac{1}{2} \begin{pmatrix}
\partial_y \Phi \\
- \partial_x \Phi
\end{pmatrix}
= \frac{Q}{2\pi} \begin{pmatrix}
\hat{r}_1 \\
- \hat{r}_2
\end{pmatrix}. \tag{104}
\]

Again, we come to the hardly surprising conclusion that the classical linearization is sufficient for treating the short distance behavior of two static color test sources.

At last we turn to the calculation of the static potential, which can be achieved with the aid of formula (80):

\[
V_{\text{static}} = \int d^2 x \left[ \frac{1}{2} D^2 + \frac{2}{3} \sqrt{\kappa} D^{3/2} + \frac{\kappa}{2} D + \mathcal{O}(D^{1/2}) \right]
=: \nu_{\text{static}}^D + V_{\text{static}}^{3/2} + V_{\text{static}}^D + \ldots. \tag{105}
\]
Of course, $V_{\text{static}}^{D^2}$ yields the classical potential and, with due consideration of the Coulomb counter terms, reads, as before,

$$V_{\text{static}}^{D^2} = \frac{Q^2}{2\pi} \ln \lambda R .$$  \hspace{1cm} (106)

The other two remaining terms in (105) allow us to augment the classical potential by correction terms. Needless to say, the respective integrations have to be performed with great care \[9\]. The leading order contribution to the classical potential comes from $V_{\text{static}}^{D^{3/2}}$, while $V_{\text{static}}^D$ provides us with a correction term $\propto R$. Our findings for the potential of two static color charges at short distances can be summarized in

$$V_{\text{static}} = \frac{Q^2}{2\pi} \ln \lambda R + \frac{\sqrt{2}}{3} \left( \frac{\pi^2}{2} - \frac{\Psi'(1/4)}{4} \right) \sqrt{\kappa Q^3} R^{1/2} + \frac{2G}{\kappa Q R} ,$$  \hspace{1cm} (107)

where $\Psi'(x)$ denotes the derivative of the psi function with $\Psi'(1/4) \simeq 17.1973$ and $G$ is Catalan’s constant, $G = 0.915965 \ldots$. With these numbers we can rewrite (107) in the final form

$$V_{\text{static}} = \frac{Q^2}{2\pi} \ln \lambda R + 0.054\ldots \sqrt{\kappa Q^3} R^{1/2} + 0.583\ldots \kappa Q R .$$  \hspace{1cm} (108)

Hence in addition to the dominant classical potential, there exist subdominant contributions behaving like $R^{1/2}, R, \ldots$ vanishing as $R$ approaches zero. Equation (108) should be read side by side with Adler’s formula (40) of ref. \[10\].

V. CONCLUSION

The results in this paper point to great similarities between QCD$_3$ and QCD$_4$. Despite major differences in physical and analytical details we find that in both theories radiative corrections to 1-loop order spontaneously generate a gauge field vacuum condensate leading directly to a confining theory. We find it interesting that quantum contributions arise
from the employment of purely classical differential equations. In this way large distance as well as short distance correction terms to the classical potentials were found. Without overestimating the importance of low–dimensional field theories, our calculation may lend some further insight into the mechanisms of classical approximations of QCD$_4$.

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