Generic Hopf Galois extensions

Christian Kassel

Abstract. In previous joint work with Eli Aljadeff we attached a generic Hopf
Galois extension $A^\alpha_H$ to each twisted algebra $\alpha H$ obtained from a Hopf alge-
bra $H$ by twisting its product with the help of a cocycle $\alpha$. The algebra $A^\alpha_H$
is a flat deformation of $\alpha H$ over a “big” central subalgebra $B^\alpha_H$ and can be viewed
as the noncommutative analogue of a versal torsor in the sense of Serre.

After surveying the results on $A^\alpha_H$ obtained with Aljadeff, we establish three
new results: we present a systematic method to construct elements of the com-
mutable algebra $B^\alpha_H$, we show that a certain important integrality condition
is satisfied by all finite-dimensional Hopf algebras generated by grouplike and
skew-primitive elements, and we compute $B^\alpha_H$ in the case where $H$ is the Hopf
algebra of a cyclic group.

Introduction

In this paper we deal with associative algebras $\alpha H$ obtained from a Hopf alge-
bra $H$ by twisting its product by a cocycle $\alpha$. This class of algebras, which for sim-
pleity we call twisted algebras, coincides with the class of so-called clef Hopf Galois
extensions of the ground field; classical Galois extensions and strongly group-graded
algebras belong to this class. As has been stressed many times (see, e.g., [22]), Hopf
Galois extensions can be viewed as noncommutative analogues of principal fiber
bundles (also known as $G$-torsors), for which the rôle of the structural group is
played by a Hopf algebra. Hopf Galois extensions abound in the world of quantum
groups and of noncommutative geometry. The problem of constructing systemat-
ically Hopf Galois extensions of a given algebra for a given Hopf algebra and of
classifying them up to isomorphism has been addressed in a number of papers over
the last fifteen years; let us mention [4, 5, 10, 12, 13, 14, 15, 16, 17, 19, 20, 21].
This list is far from being exhaustive, but gives a pretty good idea of the activity
on this subject.

A new approach to this problem was recently considered in [2]; this approach
mixes commutative algebra with techniques from noncommutative algebra such as
polynomial identities. In particular, in that paper Eli Aljadeff and the author

2000 Mathematics Subject Classification. Primary (16W30, 16S35, 16R50) Secondary
(13B05, 13B22, 16E99, 58B32, 58B34, 81R50).

Key words and phrases. Hopf algebra, Galois extension, twisted product, generic, cocycle,
integrality.

Partially funded by ANR Project BLAN07-3_ 183390.
attached two "universal algebras" $U^\alpha_H$, $A^\alpha_H$ to each twisted algebra $\alpha H$. The algebra $U^\alpha_H$, which was built out of polynomial identities satisfied by $\alpha H$, was the starting point of loc. cit. In the present paper we concentrate on the second algebra $A^\alpha_H$ and survey the results obtained in [2] from the point of view of this algebra. In addition, we present here two new results, namely Theorem 6.2 and Proposition 7.1 as well as a computation in Subsection 3.3.

The algebra $A^\alpha_H$ is a “generic” version of $\alpha H$ and can be seen as a kind of universal Hopf Galois extension. To construct $A^\alpha_H$ we introduce the generic cocycle cohomologous to the original cocycle $\alpha$ and we consider the commutative algebra $B^\alpha_H$ generated by the values of the generic cocycle and of its convolution inverse. Then $A^\alpha_H$ is a cleft $H$-Galois extension of $B^\alpha_H$. We call $A^\alpha_H$ the generic Galois extension and $B^\alpha_H$ the generic base space. They satisfy the following remarkable properties.

Any “form” of $\alpha H$ is obtained from $A^\alpha_H$ by a specialization of $B^\alpha_H$. Conversely, under an additional integrality condition, any central specialization of $A^\alpha_H$ is a form of $\alpha H$. Thus, the set of algebra morphisms $\text{Alg}(B^\alpha_H, K)$ parametrizes the isomorphism classes of $K$-forms of $\alpha H$ and $A^\alpha_H$ can be viewed as the noncommutative analogue of a versal deformation space or a versal torsor in the sense of Serre (see [11, Chap. I]). We believe that such versal deformation spaces are of interest and deserve to be computed for many Hopf Galois extensions. Even when the Hopf algebra $H$ is a group algebra, in which case our theory simplifies drastically, not many examples have been computed (see [1, 3] for results in this case).

Our approach also leads to the emergence of new interesting questions on Hopf algebras such as Question 6.1 below. We give a positive answer to this question for a class of Hopf algebras that includes the finite-dimensional ones that are generated by grouplike and skew-primitive elements.

Finally we present a new systematic way to construct elements of the generic base space $B^\alpha_H$. These elements are the images of certain universal noncommutative polynomials under a certain tautological map. In the language of polynomial identities, these noncommutative polynomials are central identities.

The paper is organized as follows. In Section 1 we recall the concept of a Hopf Galois extension and discuss the classification problem for such extensions. In Section 2 we define Hopf algebra cocycles and the twisted algebras $\alpha H$. We construct the generic cocycle and the generic base space $B^\alpha_H$ in Section 3; we also compute $B^\alpha_H$ when $H$ is the Hopf algebra of a cyclic group. In Section 4 we illustrate the theory with a nontrivial, still not too complicated example, namely with the four-dimensional Sweedler algebra. In Section 5 we define the generic Hopf Galois extension $A^\alpha_H$ and state its most important properties. Some results of Section 5 hold under a certain integrality condition; in Section 6 we prove that this condition is satisfied by a certain class of Hopf algebras. In Section 7 we present the above-mentioned general method to construct elements of $B^\alpha_H$. The contents of Subsection 3.3 and of Sections 6 and 7 are new.

We consistently work over a fixed field $k$, over which all our constructions will be defined. As usual, unadorned tensor symbols refer to the tensor product of $k$-vector spaces. All algebras are assumed to be associative and unital, and all algebra morphisms preserve the units. We denote the unit of an algebra $A$ by $1_A$, or simply by 1 if the context is clear. The set of algebra morphisms from an algebra $A$ to an algebra $B$ will be denoted by $\text{Alg}(A, B)$. 

1. Principal fiber bundles and Hopf Galois extensions

1.1. Hopf Galois extensions. A principal fiber bundle involves a group $G$ acting, say on the right, on a space $X$ such that the map $X \times G \to X \times_Y X ; (x, g) \mapsto (x, xg)$ is an isomorphism (in the category of spaces under consideration). Here $Y$ represents some version of the quotient space $X/G$ and $X \times Y$ the fiber product.

In a purely algebraic setting, the group $G$ is replaced by a Hopf algebra $H$ with coproduct $\Delta : H \to H \otimes H$, coeunit $\varepsilon : H \to k$, and antipode $S : H \to H$. In the sequel we shall make use of the Heyneman-Sweedler sigma notation (see [23, Sect. 1.2]): we write

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$$

for the coproduct of $x \in H$ and

$$\Delta^{(2)}(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$$

for the iterated coproduct $\Delta^{(2)} = (\Delta \otimes \text{id}_H) \circ \Delta = (\text{id}_H \otimes \Delta) \circ \Delta$, and so on.

The $G$-space $X$ is replaced by an algebra $A$ carrying the structure of an $H$-comodule algebra. Recall that an algebra $A$ is an $H$-comodule algebra if it has a right $H$-comodule structure whose coaction $\delta : A \to A \otimes H$ is an algebra morphism.

The space of coinvariants of an $H$-comodule algebra $A$ is the subspace $A^H$ of $A$ defined by

$$A^H = \{ a \in A \mid \delta(a) = a \otimes 1 \} .$$

The subspace $A^H$ is a subalgebra and a subcomodule of $A$. We then say that $A^H \subset A$ is an $H$-extension or that $A$ is an $H$-extension of $A^H$. An $H$-extension is called central if $A^H$ lies in the center of $A$.

An $H$-extension $B = A^H \subset A$ is said to be $H$-Galois if $A$ is faithfully flat as a left $B$-module and the linear map $\beta : A \otimes_B A \to A \otimes H$ defined for $a, b \in A$ by

$$a \otimes b \mapsto (a \otimes 1_H) \delta(b)$$

is bijective. For a survey of Hopf Galois extensions, see [18 Chap. 8].

Example 1.1. The group algebra $H = k[G]$ of a group $G$ is a Hopf algebra with coproduct, coeunit, and antipode respectively given for all $g \in G$ by

$$\Delta(g) = g \otimes g , \quad \varepsilon(g) = 1 , \quad S(g) = g^{-1} .$$

This is a pointed Hopf algebra. It is well known (see [7, Lemma 4.8]) that an $H$-comodule algebra $A$ is the same as a $G$-graded algebra

$$A = \bigoplus_{g \in G} A_g .$$

The coaction $\delta : A \to A \otimes H$ is given by $\delta(a) = a \otimes g$ for $a \in A_g$ and $g \in G$. We have $A^H = A_e$, where $e$ is the neutral element of $G$. Such a algebra is an $H$-Galois extension of $A_e$ if and only if the product induces isomorphisms

$$A_g \otimes_{A_e} A_h \cong A_{gh} \quad (g, h \in G) .$$
1.2. (Uni)versal extensions. An isomorphism $f : A \to A'$ of $H$-Galois extensions is an isomorphism of the underlying $H$-comodule algebras, i.e., an algebra morphism satisfying

$$\delta \circ f = (f \otimes \text{id}_H) \circ \delta.$$

Such an isomorphism necessarily sends $A^H$ onto $A'^H$.

For any Hopf algebra $H$ and any commutative algebra $B$, let $\text{CGal}_H(B)$ denote the set of isomorphism classes of central $H$-Galois extensions of $B$. It was shown in [13, Th. 1.4] (see also [14, Prop. 1.2]) that any morphism $f : B \to B'$ of commutative algebras induces a functorial map

$$f_* : \text{CGal}_H(B) \to \text{CGal}_H(B')$$

given by $f_*(A) = B' \otimes_B A$ for all $H$-Galois extensions $A$ of $B$. The set $\text{CGal}_H(B)$ is also a contravariant functor in $H$ (see [14, Prop. 1.3]), but we will not make use of this fact here.

Recall that principal fiber bundles are classified as follows: there is a principal $G$-bundle $EG \to BG$, called the universal $G$-bundle such that any principal $G$-bundle $X \to Y$ is obtained from pulling back the universal one along a continuous map $f : Y \to BG$, which is unique up to homotopy.

By analogy, a (uni)versal $H$-Galois extension would be a central $H$-Galois extension $B_H \subset A_H$ such that for any commutative algebra $B$ and any central $H$-Galois extension $A$ of $B$ there is a (unique) morphism of algebras $f : B_H \to B$ such that $f_*(A_H) \cong A$. In other words, the map

$$\text{Alg}(B_H, B) \to \text{CGal}_H(B) ; f \mapsto f_*(A_H)$$

would be surjective (bijective). We have no idea if such (uni)versal $H$-Galois extensions exist for general Hopf algebras.

In the sequel, we shall only consider the case where $B = k$ and the $H$-Galois extensions of $k$ are cleft. Such extensions coincide with the twisted algebras $^aH$ introduced in the next section. To such an $H$-Galois extension we shall associate a central $H$-Galois extension $B_H^a \subset A_H^a$, such that the functor $\text{Alg}(B_H^a, -)$ parametrizes the “forms” of $^aH$. In this way we obtain an $H$-Galois extension that is versal for a family of $H$-Galois extensions close to $^aH$ in some appropriate étale-like Grothendieck topology.

2. Twisted algebras

The definition of the twisted algebras $^aH$ uses the concept of a cocycle, which we now recall.

2.1. Cocycles. Let $H$ be a Hopf algebra and $B$ a commutative algebra. We use the following terminology. A bilinear map $\alpha : H \times H \to B$ is a cocycle of $H$ with values in $B$ if

$$\sum_{(x), (y)} \alpha(x_{(1)}, y_{(1)}) \alpha(x_{(2)}y_{(2)}, z) = \sum_{(y), (z)} \alpha(y_{(1)}, z_{(1)}) \alpha(x, y_{(2)}z_{(2)})$$

for all $x, y, z \in H$. In the literature, what we call a cocycle is often referred to as a “left 2-cocycle.”

A bilinear map $\alpha : H \times H \to B$ is said to be normalized if

$$\alpha(x, 1_H) = \alpha(1_H, x) = \varepsilon(x) \cdot 1_B$$

for all $x \in H$. 


Two cocycles $\alpha, \beta : H \times H \to B$ are said to be cohomologous if there is an invertible linear map $\lambda : H \to B$ such that
\begin{equation}
\beta(x, y) = \sum_{(x), (y)} \lambda(x_{(1)}) \lambda(y_{(1)}) \alpha(x_{(2)}, y_{(2)}) \lambda^{-1}(x_{(3)}y_{(3)})
\end{equation}
for all $x, y \in H$. Here “invertible” means invertible with respect to the convolution product and $\lambda^{-1} : H \to B$ denotes the inverse of $\lambda$. We write $\alpha \sim \beta$ if $\alpha, \beta$ are cohomologous cocycles. The relation $\sim$ is an equivalence relation on the set of cocycles of $H$ with values in $B$.

2.2. Twisted product. Let $H$ be a Hopf algebra, $B$ a commutative algebra, and $\alpha : H \times H \to B$ be a normalized cocycle with values in $B$. From now on, all cocycles are assumed to be invertible with respect to the convolution product.

Let $u_H$ be a copy of the underlying vector space of $H$. Denote the identity map from $H$ to $u_H$ by $x \mapsto u_x$ ($x \in H$).

We define the twisted algebra $B \otimes {}^\alpha H$ as the vector space $B \otimes u_H$ equipped with the associative product given by
\begin{equation}
(b \otimes u_x)(c \otimes u_y) = \sum_{(x), (y)} bc \alpha(x_{(1)}, y_{(1)}) \otimes u_{x_{(2)}y_{(2)}}
\end{equation}
for all $b, c \in B$ and $x, y \in H$. Since $\alpha$ is a normalized cocycle, $1_B \otimes u_1$ is the unit of $B \otimes {}^\alpha H$.

The algebra $A = B \otimes {}^\alpha H$ is an $H$-comodule algebra with coaction
\[ \delta = \text{id}_B \otimes \Delta : A = B \otimes H \to B \otimes H \otimes H = A \otimes H. \]
The subalgebra of coinvariants of $B \otimes {}^\alpha H$ coincides with $B \otimes u_1$. Using (2.1) and (2.3), it is easy to check that this subalgebra lies in the center of $B \otimes {}^\alpha H$.

It is well known that each twisted algebra $B \otimes {}^\alpha H$ is a central $H$-Galois extension of $B$. Actually, the class of twisted algebra coincides with the class of so-called central cleft $H$-Galois extensions; see [6, 9, 18 Prop. 7.2.3].

An important special case of this construction occurs when $B = k$ is the ground field and $\alpha : H \times H \to k$ is a cocycle of $H$ with values in $k$. In this case, we simply call $\alpha$ a cocycle of $H$. Then the twisted algebra $k \otimes {}^\alpha H$, which we henceforth denote by $^\alpha H$, coincides with $u_H$ equipped with the associative product
\[ u_x u_y = \sum_{(x), (y)} \alpha(x_{(1)}, y_{(1)}) u_{x_{(2)}y_{(2)}} \]
for all $x, y \in H$. The twisted algebras of the form $^\alpha H$ coincide with the so-called cleft $H$-Galois objects, which are the cleft $H$-Galois extensions of the ground field $k$.

We point out that for certain Hopf algebras $H$ all $H$-Galois objects are cleft, e.g., if $H$ is finite-dimensional or is a pointed Hopf algebra.

When $H = k[G]$ is the Hopf algebra of a group as in Example 1.1, then a $G$-graded algebra $A = \bigoplus_{g \in G} A_g$ is an $H$-Galois object if and only if $A_gA_h = A_{gh}$ for all $g, h \in G$ and $\dim A_g = 1$ for all $g \in G$. Such an $H$-extension is cleft and thus isomorphic to $^\alpha H$ for some normalized invertible cocycle $\alpha$.

2.3. Isomorphisms of twisted algebras. By [6, 8] there is an isomorphism of $H$-comodule algebras between the twisted algebras $^\alpha H$ and $^\beta H$ if and only if the cocycles $\alpha$ and $\beta$ are cohomologous in the sense of (2.2). It follows that the
set of isomorphism classes of cleft $H$-Galois objects is in bijection with the set of cohomology classes of invertible cocycles of $H$.

When the Hopf algebra $H$ is cocommutative, then the convolution product of two cocycles is a cocycle and the set of cohomology classes of invertible cocycles of $H$ is a group. This applies to the case $H = k[G]$; in this case the group of cohomology classes of invertible cocycles of $k[G]$ is isomorphic to the cohomology group $H^2(G, k^\times)$ of the group $G$ with values in the group $k^\times$ of invertible elements of $k$.

In general, the convolution product of two cocycles is not a cocycle and thus the set of cohomology classes of invertible cocycles is not a group. One of the raisons d'être of the constructions presented here and in [2] lies in the lack of a suitable cohomology group governing the situation. We come up instead with the generic Galois extension defined below.

3. The generic cocycle

Let $H$ be a Hopf algebra and $\alpha : H \times H \to k$ an invertible normalized cocycle.

3.1. The cocycle $\sigma$. Our first aim is to construct a “generic” cocycle of $H$ that is cohomologous to $\alpha$.

We start from the equation (2.2)

$$\beta(x, y) = \sum_{(x), (y)} \lambda(x(1)) \lambda(y(1)) \alpha(x(2), y(2)) \lambda^{-1}(x(3)y(3))$$

expressing that a cocycle $\beta$ is cohomologous to $\alpha$, and the equation

(3.1) $$\sum_{(x)} \lambda(x(1)) \lambda^{-1}(x(2)) = \sum_{(x)} \lambda^{-1}(x(1)) \lambda(x(2)) = \varepsilon(x) 1$$

expressing that the linear form $\lambda$ is invertible with inverse $\lambda^{-1}$. To obtain the generic cocycle, we proceed to mimic (2.2), replacing the scalars $\lambda(x)$, $\lambda^{-1}(x)$ respectively by symbols $t_x$, $t_x^{-1}$ satisfying (3.1).

Let us give a meaning to the symbols $t_x, t_x^{-1}$. To this end we pick another copy $t_H$ of the underlying vector space of $H$ and denote the identity map from $H$ to $t_H$ by $x \mapsto t_x$ ($x \in H$).

Let $S(t_H)$ be the symmetric algebra over the vector space $t_H$. If $\{x_i\}_{i \in I}$ is a basis of $H$, then $S(t_H)$ is isomorphic to the polynomial algebra over the indeterminates $\{t_{x_i}\}_{i \in I}$.

By [2] Lemma A.1 there is a unique linear map $x \mapsto t_x^{-1}$ from $H$ to the field of fractions $\text{Frac} S(t_H)$ of $S(t_H)$ such that for all $x \in H$,

(3.2) $$\sum_{(x)} t_{x(1)} t_{x(2)}^{-1} = \sum_{(x)} t_{x(1)}^{-1} t_{x(2)} = \varepsilon(x) 1.$$

Equation (3.2) is the symbolic counterpart of (3.1).

Mimicking (2.2), we define a bilinear map

$$\sigma : H \times H \to \text{Frac} S(t_H)$$

with values in the field of fractions $\text{Frac} S(t_H)$ by the formula

(3.3) $$\sigma(x, y) = \sum_{(x), (y)} t_{x(1)} t_{y(1)} \alpha(x(2), y(2)) t_{x(3)y(3)}^{-1}$$
for all \( x, y \in H \). The bilinear map \( \sigma \) is a cocycle of \( H \) with values in \( \text{Frac} S(t_H) \); by definition, it is cohomologous to \( \alpha \). We call \( \sigma \) the \textit{generic cocycle attached to} \( \alpha \).

The cocycle \( \alpha \) being invertible, so is \( \sigma \), with inverse \( \sigma^{-1} \) given for all \( x, y \in H \) by

\[
\sigma^{-1}(x, y) = \sum_{(x), (y)} t_{x(1)} y_{(1)} \alpha^{-1}(x(2), y(2)) t_{x(3)}^{-1} y_{(3)}^{-1},
\]

where \( \alpha^{-1} \) is the inverse of \( \alpha \).

In the case where \( H = k[G] \) is the Hopf algebra of a group, the generic cocycle and its inverse have the following simple expressions:

\[
\sigma(g, h) = \alpha(g, h) \frac{t_g t_h}{t_{gh}} \quad \text{and} \quad \sigma^{-1}(g, h) = \frac{1}{\alpha(g, h)} \frac{t_{gh}}{t_g t_h}
\]

for all \( g, h \in G \).

### 3.2. The generic base space.

Let \( B^H_\alpha \) be the subalgebra of \( \text{Frac} S(t_H) \) generated by the values of the generic cocycle \( \sigma \) and of its inverse \( \sigma^{-1} \). For reasons that will become clear in Section 5, we call \( B^H_\alpha \) the \textit{generic base space}.

Since \( B^H_\alpha \) is a subalgebra of the field \( \text{Frac} S(t_H) \), it is a domain and the transcendence degree of the field of fractions of \( B^H_\alpha \) cannot exceed the dimension of \( H \).

In the case where \( H \) is finite-dimensional, \( B^H_\alpha \) is a finitely generated algebra. One can obtain a presentation of \( B^H_\alpha \) by generators and relations using standard monomial order techniques of commutative algebra.

### 3.3. A computation.

Let \( H = k[\mathbb{Z}] \) be the Hopf algebra of the group \( \mathbb{Z} \) of integers. We write \( \mathbb{Z} \) multiplicatively and identify its elements with the powers \( x^m \) of a variable \( x \) (\( m \in \mathbb{Z} \)).

We take \( \alpha \) to be the trivial cocycle, i.e., \( \alpha(g, h) = 1 \) for all \( g, h \in \mathbb{Z} \) (this is no restriction since \( H^2(\mathbb{Z}, k^*) = 0 \)). In this case the symmetric algebra \( S(t_H) \) coincides with the polynomial algebra \( k[t_m \mid m \in \mathbb{Z}] \). Set \( y_m = t_m / t_1^m \) for each \( m \in \mathbb{Z} \). We have \( y_1 = 1 \) and \( y_0 = t_0 \).

By (3.5), the generic cocycle is given by

\[
\sigma(x^m, x^n) = \frac{t_m t_n}{t_{m+n}}
\]

for all \( m, n \in \mathbb{Z} \). This can be reformulated as

\[
\sigma(x^m, x^n) = \frac{y_m y_n}{y_{m+n}}.
\]

The inverse of \( \sigma \) is given by

\[
\sigma^{-1}(x^m, x^n) = \frac{1}{\sigma(x^m, x^n)} = \frac{y_{m+n}}{y_m y_n}.
\]

A simple computation yields the following expressions of \( y_m \) in the values of \( \sigma \) and \( \sigma^{-1} \):

\[
y_m = \begin{cases} 
\sigma^{-1}(x^{m-1}, x) \sigma^{-1}(x^{m-2}, x) \cdots \sigma^{-1}(x, x) & \text{if } m \geq 2, \\
1 & \text{if } m = 1, \\
\sigma(x^0, x^0) & \text{if } m = 0, \\
\sigma(x^m, x^{-m}) \sigma(x^{-m-1}, x) \sigma(x^{-m-2}, x) \cdots \sigma(x, x) \sigma(x^0, x^0) & \text{if } m \leq -1.
\end{cases}
\]
It follows that the elements $y_m^{\pm 1}$ belong to $\mathcal{B}_H^\alpha$ for all $m \in \mathbb{Z} - \{1\}$ and generate this algebra. It is easy to check that the family $(y_m)_{m \neq 1}$ is algebraically independent, so that $\mathcal{B}_H^\alpha$ is the Laurent polynomial algebra

$$\mathcal{B}_H^\alpha = k[y_m^{\pm 1} \mid m \in \mathbb{Z} - \{1\}] .$$

We deduce the algebra isomorphism

$$k[t_m^{\pm 1} \mid m \in \mathbb{Z}] \cong \mathcal{B}_H^\alpha .$$

If in the previous computations we replace $\mathbb{Z}$ by the cyclic group $\mathbb{Z}/N$, where $N$ is some integer $N \geq 2$, then the algebra $\mathcal{B}_H^\alpha$ is again a Laurent polynomial algebra:

$$\mathcal{B}_H^\alpha = k[y_0^{\pm 1}, y_2^{\pm 1}, \ldots, y_N^{\pm 1}]$$

where $y_0, y_2, \ldots, y_{N-1}$ are defined as above and $y_N = t_0/t_1^N$. In this case, the algebra $k[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_{N-1}^{\pm 1}]$ is an integral extension of $\mathcal{B}_H^\alpha$:

$$k[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_{N-1}^{\pm 1}] \cong \mathcal{B}_H^\alpha / (t_1^N - y_0/y_N).$$

4. The Sweedler algebra

We now illustrate the constructions of Section 3 on Sweedler’s four-dimensional Hopf algebra. We assume in this section that the characteristic of the ground field $k$ is different from 2.

The Sweedler algebra $H_4$ is the algebra generated by two elements $x, y$ subject to the relations

$$x^2 = 1, \quad xy + yx = 0, \quad y^2 = 0 .$$

It is four-dimensional. As a basis of $H_4$ we take the set $\{1, x, y, z\}$, where $z = xy$.

The algebra $H_4$ carries the structure of a Hopf algebra with coproduct, counit, and antipode given by

\[
\begin{align*}
\Delta(1) &= 1 \otimes 1, & \Delta(x) &= x \otimes x, \\
\Delta(y) &= 1 \otimes y + y \otimes 1, & \Delta(z) &= x \otimes z + z \otimes 1, \\
\varepsilon(1) &= \varepsilon(x) = 1, & \varepsilon(y) &= \varepsilon(z) = 0, \\
S(1) &= 1, & S(x) &= x, \\
S(y) &= z, & S(z) &= -y.
\end{align*}
\]

By definition, the symbols $t_x$ and $t_{x^{-1}}$ satisfy the equations

$$t_1 t_{x^{-1}} = 1, \quad t_x t_x^{-1} = 1, \quad t_x t_{x^{-1}} + t_{x^{-1}} t_x = 0 .$$

Hence,

$$t_{x^{-1}} = \frac{1}{t_x}, \quad t_y = \frac{1}{t_x}, \quad t_y^{-1} = -\frac{t_y}{t_{1x}}, \quad t_{x^{-1}} = -\frac{t_{x^{-1}}}{t_{1x}} .$$

Masuoka [15] showed that any cleft $H_4$-Galois object has, up to isomorphism, the following presentation:

$$a H_4 = k\langle u_x, u_y \mid u_x^2 = a, \quad u_x u_y + u_y u_x = b, \quad u_y^2 = c \rangle$$

for some scalars $a, b, c$ with $a \neq 0$. To indicate the dependence on the parameters $a, b, c$, we denote $a H_4$ by $A_{a,b,c}$.

It is easy to check that the center of $A_{a,b,c}$ is trivial for all values of $a, b, c$. Moreover, the algebra $A_{a,b,c}$ is simple if and only $b^2 - 4ac \neq 0$. If $b^2 - 4ac = 0$, then $A_{a,b,c}$ is isomorphic as an algebra to $H_4$; the latter is not semisimple since the two-sided ideal generated by $y$ is nilpotent.
The generic cocycle $\sigma$ attached to $\alpha$ has the following values:

\[
\begin{align*}
\sigma(1, 1) &= \sigma(1, x) = \sigma(x, 1) = t_1, \\
\sigma(1, y) &= \sigma(y, 1) = \sigma(1, z) = \sigma(z, 1) = 0, \\
\sigma(x, x) &= at^2_xt^{-1}, \\
\sigma(y, y) &= \sigma(z, y) = -\sigma(y, z) = (at^2_y + bt_y + ct^2_z)t^{-1}_1, \\
\sigma(x, y) &= -\sigma(x, z) = (at_xt_y - t_xt_z)t^{-1}_1, \\
\sigma(y, x) &= \sigma(z, x) = (bt_xt_x + at_xt_y + t_xt_z)t^{-1}_1, \\
\sigma(z, z) &= -(t^2_z + bt_xt_x + act^2_z)t^{-1}_1,
\end{align*}
\]

The values of the inverse $\sigma^{-1}$ are equal to the values of $\sigma$ possibly divided by positive powers of $t_1$ and of $\sigma(x, x) = at^2_xt^{-1}$.

By definition, $B_{H_4}^a$ is the subalgebra of $\text{Frac} S(t_{H_4})$ generated by the values of $\sigma$ and $\sigma^{-1}$. If we set

\[
E = t_1, \quad R = at^2_x, \quad S = at^2_y + bt_y + ct^2_z, \\
T = t_x(2at_y + bt_1), \quad U = at^2_x(2t_z + bt_x),
\]

then we can reformulate the above (nonzero) values of $\sigma$ as follows:

\[
\sigma(1, 1) = \sigma(1, x) = \sigma(x, 1) = E, \\
\sigma(x, x) = \frac{R}{E}, \\
\sigma(y, y) = \sigma(z, y) = -\sigma(y, z) = \frac{S}{E}; \\
\sigma(x, y) = -\sigma(x, z) = \frac{RT - EU}{2ER}, \\
\sigma(y, x) = \sigma(z, x) = \frac{RT + EU}{2ER}, \\
\sigma(z, z) = \frac{aU^2 - (b^2 - 4ac)R^3}{4aER^2}.
\]

From the previous equalities we conclude that $E^{\pm 1}$, $R^{\pm 1}$, $S$, $T$, $U$ belong to $B_{H_4}^a$ and that they generate it as an algebra.

In [2 Sect. 10] we obtained the following presentation of $B_{H_4}^a$ by generators and relations.

**Theorem 4.1.** We have

\[
B_{H_4}^a \cong k[E^{\pm 1}, R^{\pm 1}, S, T, U]/(P_{a,b,c}),
\]

where

\[
P_{a,b,c} = T^2 - 4RS - \frac{b^2 - 4ac}{a} E^2 R.
\]

It follows from the previous theorem that the algebra morphisms from $B_{H_4}^a$ to a field $K$ containing $k$ are in one-to-one correspondence with the quintuples $(e, r, s, t, u) \in K^5$ verifying $e \neq 0$, $r \neq 0$, and the equation

\[
t^2 - 4rs = \frac{b^2 - 4ac}{a} e^2 r.
\]

In other words, the set of $K$-points of $B_{H_4}^a$ is the hypersurface of equation (4.2) in $K^5 \times K^5 \times K \times K \times K$. 
5. The generic Galois extension

As in Section 3, we consider a Hopf algebra $H$ and an invertible normalized cocycle $\alpha : H \times H \rightarrow k$. Let $^o H$ be the corresponding twisted algebra.

5.1. The algebra $A^o_H$. By the definition of the commutative algebra $B^o_H$ given in Section 3.2, the generic cocycle $\sigma$ takes its values in $B^o_H$. Therefore we may apply the construction of Section 2.2 and consider the twisted algebra

$$A^o_H = B^o_H \otimes \sigma H.$$ 

The product of $A^o_H$ is given for all $b, c \in B^o_H$ and $x, y \in H$ by

$$(b \otimes u_x)(c \otimes u_y) = \sum_{(x),(y)} bc \sigma(x(1), y(1)) \otimes u_{x(2)y(2)}.$$ 

We call $A^o_H$ the generic Galois extension attached to the cocycle $\alpha$.

The subalgebra of coinvariants of $A^o_H$ is equal to $B^o_H \otimes u_1$; this subalgebra is central in $A^o_H$. Therefore, $A^o_H$ is a central cleft $H$-Galois extension of $B^o_H$.

By [2] Prop. 5.3, there is an algebra morphism $\chi_0 : B^o_H \rightarrow k$ such that

$$\chi_0(\sigma(x, y)) = \alpha(x, y) \quad \text{and} \quad \chi_0(\sigma^{-1}(x, y)) = \alpha^{-1}(x, y)$$

for all $x, y \in H$. Consider the maximal ideal $m_0 = \ker(\chi_0 : B^o_H \rightarrow k)$ of $B^o_H$. According to [2] Prop. 6.2, there is an isomorphism of $H$-comodule algebras

$$A^o_H/m_0 A^o_H \cong ^o H.$$ 

Thus, $A^o_H$ is a flat deformation of $^o H$ over the commutative algebra $B^o_H$.

Certain properties of $^o H$ lift to the generic Galois extension $A^o_H$ such as the one recorded in the following result of [2], where $\text{Frac} B^o_H$ stands for the field of fractions of $B^o_H$.

**THEOREM 5.1.** Assume that the ground field $k$ is of characteristic zero and the Hopf algebra $H$ is finite-dimensional. If the algebra $^o H$ is simple (resp. semisimple), then so is

$$\text{Frac} B^o_H \otimes_{B^o_H} A^o_H = \text{Frac} B^o_H \otimes \sigma H.$$ 

5.2. Forms. We have just observed that $^o H \cong A^o_H/m_0 A^o_H$ for some maximal ideal $m_0$ of $B^o_H$. We may now wonder what can be said of the other central specializations of $A^o_H$, that is of the quotients $A^o_H/m A^o_H$, where $m$ is an arbitrary maximal ideal of $B^o_H$. To answer this question, we need the following terminology.

Let $\beta : H \times H \rightarrow K$ be a normalized invertible cocycle with values in a field $K$ containing the ground field $k$. We say that the twisted $H$-comodule algebra $K \otimes^\beta H$ is a $K$-form of $^o H$ if there is a field $L$ containing $K$ and an $L$-linear isomorphism of $H$-comodule algebras

$$L \otimes_K (K \otimes^\beta H) \cong L \otimes_k ^o H.$$ 

We now state two theorems relating forms of $^o H$ to central specializations of the generic Galois extension $A^o_H$. For proofs, see [2] Sect. 7.

**THEOREM 5.2.** For any $K$-form $K \otimes^\beta H$ of $^o H$, where $\beta : H \times H \rightarrow K$ is a normalized invertible cocycle with values in an extension $K$ of $k$, there exist an algebra morphism $\chi : B^o_H \rightarrow K$ and a $K$-linear isomorphism of $H$-comodule algebras

$$K_\chi \otimes_{B^o_H} A^o_H \cong K \otimes^\beta H.$$
Here $K_\chi$ stands for $K$ equipped with the $B_H^\alpha$-module structure induced by the algebra morphism $\chi : B_H^\alpha \to K$. We have

$$K_\chi \otimes_{B_H^\alpha} A_H^\alpha \cong A_H^\alpha / \mathfrak{m}_\chi A_H^\alpha,$$

where $\mathfrak{m}_\chi = \text{Ker}(\chi : B_H^\alpha \to K)$.

There is a converse to Theorem 5.2; it requires an additional condition.

**Theorem 5.3.** If $\text{Frac} \ S(t_H)$ is integral over the subalgebra $B_H^\alpha$, then for any field $K$ containing $k$ and any algebra morphism $\chi : B_H^\alpha \to K$, the $H$-comodule $K$-algebra $K_\chi \otimes_{B_H^\alpha} A_H^\alpha = A_H^\alpha / \mathfrak{m}_\chi A_H^\alpha$ is a $K$-form of $^\alpha H$.

It follows that if $\text{Frac} \ S(t_H)$ is integral over $B_H^\alpha$, then the map

$$\text{Alg}(B_H^\alpha, K) \to K\text{-Forms}(^\alpha H)$$

$$\chi \mapsto K_\chi \otimes_{B_H^\alpha} A_H^\alpha = A_H^\alpha / \mathfrak{m}_\chi A_H^\alpha$$

is a surjection from the set of algebra morphisms $B_H^\alpha \to K$ to the set of isomorphism classes of $K$-forms of $^\alpha H$. Thus the set $\text{Alg}(B_H^\alpha, K)$ parametrizes the $K$-forms of $^\alpha H$. Using terminology of singularity theory, we say that the Galois extension $B_H^\alpha \subset A_H^\alpha$ is a versal deformation space for the forms of $^\alpha H$ (we would call this space universal if the above surjection was bijective).

By Theorem 5.2 the central localization $\text{Frac} B_H^\alpha \otimes_{B_H^\alpha} A_H^\alpha$ is a simple algebra if the algebra $^\alpha H$ is simple. Under the integrality condition above, we have the following related result (see [2 Th. 7.4]).

**Theorem 5.4.** If $\text{Frac} \ S(t_H)$ is integral over $B_H^\alpha$ and if the algebra $^\alpha H$ is simple, then $A_H^\alpha$ is an Azumaya algebra with center $B_H^\alpha$.

This means that $A_H^\alpha / \mathfrak{m} A_H^\alpha$ is a simple algebra for any maximal ideal $\mathfrak{m}$ of $B_H^\alpha$. For instance, any full matrix algebra with entries in a commutative algebra is Azumaya.

**Example 5.5.** For the Sweedler algebra $H_4$, we proved in [2 Sect. 10] that $A_{H_4}^\alpha$ is given as an algebra by

$$A_{H_4}^\alpha \cong B_{H_4}^\alpha (X, Y) / (X^2 - R, Y^2 - S, XY + YX - T),$$

where $B_{H_4}^\alpha$ is as in Theorem 4.1 and the elements $R, S, T$ of $B_{H_4}^\alpha$ are defined by (4.1). As an $B_{H_4}^\alpha$-module, $A_{H_4}^\alpha$ is free with basis $\{1, X, Y, XY\}$.

6. The integrality condition

In view of Theorem 5.3 it is natural to ask the following question.

**Question 6.1.** Under which condition on the pair $(H, \alpha)$ is $\text{Frac} \ S(t_H)$ integral over the subalgebra $B_H^\alpha$?

Question 6.1 has a negative answer in the case where $H = k[\mathbb{Z}]$ and $\alpha$ is the trivial cocycle. Indeed, it follows from (3.0) that $\text{Frac} \ S(t_H)$ is then a pure transcendental extension (of degree one) of the field of fractions of $B_H^\alpha$.

We give a positive answer in the following important case.

**Theorem 6.2.** Let $H$ be a Hopf algebra generated as an algebra by a set $\Sigma$ of grouplike and skew-primitive elements such that the grouplike elements of $\Sigma$ are of finite order and generate the group of grouplike elements of $H$ and such that each skew-primitive element of $\Sigma$ generates a finite-dimensional subalgebra of $H$. Then $\text{Frac} \ S(t_H)$ is integral over the subalgebra $B_H^\alpha$ for every cocycle $\alpha$ of $H$. 


Theorem 6.2 implies a positive answer to Question 6.1 for any finite-dimensional Hopf algebra generated by grouplike and skew-primitive elements. It is conjectured that all finite-dimensional pointed Hopf algebras are generated by grouplike and skew-primitive elements; if this conjecture holds, then Question 6.1 has a positive answer for any finite-dimensional Hopf algebra that is pointed.

Recall that \( g \in H \) is grouplike if \( \Delta(g) = g \otimes g \); it then follows that \( \varepsilon(g) = 1 \). The inverse of a grouplike element and the product of two grouplike elements are grouplike. An element \( x \in H \) is skew-primitive if

\[
\Delta(x) = g \otimes x + x \otimes h
\]

for some grouplike elements \( g, h \in H \); this implies \( \varepsilon(x) = 0 \). The product of a skew-primitive element by a grouplike element is skew-primitive.

In order to prove Theorem 6.2, we need the following lemma.

**Lemma 6.3.** If \( x^{[1]}, \ldots, x^{[n]} \) are elements of \( H \), then

\[
t_{x^{[1]},\ldots,x^{[n]}} = \sum_{x^{[1]},\ldots,x^{[n]}} \sigma^{-1}(x^{[1]}_{(1)} \cdots x^{[n-1]}_{(1)}, x^{[n]}_{(1)}) \times
\]

\[
\times \sigma^{-1}(x^{[1]}_{(2)} \cdots x^{[n-2]}_{(2)}, x^{[n-1]}_{(2)} \cdots \sigma^{-1}(x^{[1]}_{(n-1)}, x^{[n]}_{(n-1)}) \times
\]

\[
\times t^{[1]}_{(n)} \cdot t^{[2]}_{(n)} \cdots t^{[n]}_{(n)} \times
\]

\[
\alpha(x^{[1]}_{(n+1)}; x^{[2]}_{(n+1)}) \cdots \alpha(x^{[1]}_{(2n-2)}; x^{[2]}_{(2n-2)} \cdots x^{[1]}_{(6)}; x^{[2]}_{(4)} \cdots \alpha(x^{[1]}_{(2n-1)}; x^{[n]}_{(2n-1)} \cdots x^{[1]}_{(3)}).}
\]

**Proof.** We prove the formula by induction on \( n \). When \( n = 2 \) it reduces to

\[
t_{xy} = \sum_{(x), (y)} \sigma^{-1}(x_{(1)}, y_{(1)}) t_{x_{(2)}, y_{(2)}} \alpha(x_{(3)}, y_{(3)})
\]

for \( x, y \in H \). Let us first prove (6.2). By (3.4) the right-hand side of (6.2) is equal to

\[
\sum_{(x), (y)} t_{x_{(1)}y_{(1)}} \alpha^{-1}(x_{(2)}, y_{(2)}) t^{-1}_{x_{(3)}y_{(3)} t_{x_{(4)}y_{(4)}} t_{x_{(5)}y_{(5)}} \alpha(x_{(5)}, y_{(5)})
\]

\[
= \sum_{(x), (y)} t_{x_{(1)}y_{(1)}} \alpha^{-1}(x_{(2)}, y_{(2)}) t^{-1}_{x_{(3)}y_{(3)} t_{x_{(4)}y_{(4)}} \alpha(x_{(5)}, y_{(5)})
\]

\[
= \sum_{(x), (y)} t_{x_{(1)}y_{(1)}} \alpha^{-1}(x_{(2)}, y_{(2)}) \alpha(x_{(3)}, y_{(3)})
\]

\[
= \sum_{(x), (y)} t_{x_{(1)}y_{(1)}} \varepsilon(x_{(2)}) \varepsilon(y_{(2)})
\]

\[
= \sum_{(x), (y)} t_{x_{(1)}y_{(1)}} \varepsilon(x_{(2)}y_{(2)}) = t_{xy}.
\]

Let us assume that Lemma 6.3 holds for all \( n \)-tuples of \( H \) and consider a sequence \( (x^{[1]}, x^{[2]}, \ldots, x^{[n+1]}) \) of \( n+1 \) elements of \( H \). By the induction hypothesis
and by \([6.2]\), \(t_{x^1, x^2 \ldots x^{n+1}} = t_{x^{[1]} x^{[2]} \ldots x^{[n+1]}}\) is equal to
\[
\sum_{x^1, x^2 \ldots x^{n+1}} \sigma^{-1}(x^{[1]}_{(1)} x^{[2]}_{(1)} \ldots x^{[n]}_{(1)}) \times
\sigma^{-1}(x^{[1]}_{(n-1)} x^{[2]}_{(n-1)} \ldots x^{[n]}_{(n-1)}) \times
\cdot \cdot \cdot \times \sigma^{-1}(x^{[1]}_{(1)} x^{[2]}_{(1)} \ldots x^{[n]}_{(1)}) \times
\]
\[
\times t_{x^1} t_{x^2} \ldots t_{x^{n+1}} \times \cdot \cdot \cdot t_{x^1} t_{x^2} \ldots t_{x^{n+1}} \times
\times \alpha(x^{[1]}_{(n+1)} x^{[2]}_{(n+1)}) \times
\cdot \cdot \cdot \alpha(x^{[1]}_{(n+1)} x^{[2]}_{(n+1)}) \times
\]
\[
= \sum_{x^1, x^2 \ldots x^{n+1}} \sigma^{-1}(x^{[1]}_{(1)} x^{[2]}_{(1)} \ldots x^{[n]}_{(n+1)}) \sigma^{-1}(x^{[1]}_{(1)} x^{[2]}_{(1)} \ldots x^{[n]}_{(n+1)}) \times
\cdot \cdot \cdot \sigma^{-1}(x^{[1]}_{(n-1)} x^{[2]}_{(n-1)} x^{[3]}_{(n-1)}) \sigma^{-1}(x^{[1]}_{(n)} x^{[2]}_{(n)}) \times
\times t_{x^1} t_{x^2} \ldots t_{x^{n+1}} \times \cdot \cdot \cdot t_{x^1} t_{x^2} \ldots t_{x^{n+1}} \times
\times \alpha(x^{[1]}_{(n+1)} x^{[2]}_{(n+1)}) \alpha(x^{[1]}_{(n+1)} x^{[2]}_{(n+1)}) \times
\cdot \cdot \cdot \alpha(x^{[1]}_{(n+1)} x^{[2]}_{(n+1)}) \times
\times \alpha(x^{[1]}_{(n+1)} x^{[2]}_{(n+1)}) \alpha(x^{[1]}_{(n+1)} x^{[2]}_{(n+1)}) \times
\]
\[
which is the desired formula for \(n+1\) elements.
\]

**Proof of Theorem 6.2.** Let \(A\) be the integral closure of \(B_H^0\) in \(\text{Frac } S(t_H)\). To prove the theorem it suffices to establish that each generator \(t_z\) of \(S(t_H)\) belongs to \(A\).

We start with the unit of \(H\). By \([2\) Lemma 5.1], \(t_1 = \sigma(1, 1)\). Thus \(t_1\) belongs to \(B_H^0\), hence to \(A\).

Let \(g\) be a grouplike element of the generating set \(\Sigma\). By hypothesis, there is an integer \(n \geq 2\) such that \(g^n = 1\). We apply Lemma \([6.3\) to \(x^1 = \ldots = x^n = g\).

Since any iterated coproduct \(\Delta^{(p)}\) applied to \(g\) yields
\[
\Delta^{(p)}(g) = g \otimes g \otimes \cdots \otimes g,
\]
where the right-hand side is the tensor product of \(p\) copies of \(g\), we obtain
\[
t_{g^n} = \sigma^{-1}(g^{n-1}, g) \sigma^{-1}(g^{n-2}, g) \cdot \cdot \cdot \sigma^{-1}(g, g) t_g^n \times
\cdot \cdot \cdot \alpha(g^{n-2}, g) \alpha(g^{n-1}, g).
\]
Since the values of an invertible cocycle on grouplike elements are invertible elements, since \(t_{g^n} = t_1\), and since \(\sigma^{-1}(g, h) = 1/\sigma(g, h)\) for all grouplike elements \(g, h\), Formula \([6.3\) implies
\[
t_g^n = t_1 \frac{\sigma(g^{n-1}, g) \sigma(g^{n-2}, g) \cdot \cdot \cdot \sigma(g, g)}{\alpha(g, g) \cdot \cdot \cdot \alpha(g^{n-2}, g) \alpha(g^{n-1}, g)}.
\]
The right-hand side belongs to \(B_H^0\). It follows that \(t_g\) is in \(A\) for each grouplike element of \(\Sigma\).

Since the grouplike elements of \(\Sigma\) are of finite order and generate the group of grouplike elements of \(H\), any grouplike element \(g\) of \(H\) is a product \(g = g^{[1]} \ldots g^{[n]}\)
of grouplike elements of \( \Sigma \) for which we have just established that \( t_{g[1]}, \ldots, t_{g[n]} \) belong to \( A \). It then follows from Lemma 6.3 and 6.5 that

\[
t_{g[1] \ldots g[n]} = \kappa(g^{[1]}, \ldots, g^{[n]}) t_{g[1]} t_{g[2]} \cdots t_{g[n-1]} t_{g[n]},
\]

where \( \kappa(g^{[1]}, \ldots, g^{[n]}) \) is the invertible element of \( B_H^n \) given by

\[
\kappa(g^{[1]}, \ldots, g^{[n]}) = \frac{\alpha(g^{[1]}, g^{[2]}) \cdots \alpha(g^{[1]} g^{[2]} \cdots g^{[n-2]}, g^{[n-1]}) \alpha(g^{[1]} g^{[2]} \cdots g^{[n-1]}, g^{[n]})}{\sigma(g^{[1]} g^{[2]} \cdots g^{[n-1]}, g^{[n]}) \sigma(g^{[1]} g^{[2]} \cdots g^{[n-2]}, g^{[n-1]}) \cdots \sigma(g^{[1]}, g^{[2]})}.
\]

Therefore, \( t_g \in A \) for every grouplike element of \( H \).

We next show that \( t_x \) belongs to \( A \) for every skew-primitive element \( x \) of \( \Sigma \). It is easy to check that if \( x \) satisfies (6.1), then for all \( p \geq 2 \),

\[
\Delta^{(p)}(x) = g^\otimes p \otimes x + \sum_{i=1}^{p-1} g^\otimes(p-i) \otimes x \otimes h^\otimes i + x \otimes h^\otimes p.
\]

Thus the iterated coproduct of any skew-primitive element \( x \) is a sum of tensor product of elements, all of which are grouplike, except for exactly one, which is \( x \). It then follows from Lemma 6.3 and 6.5 that for each \( n \geq 1 \) the element \( t_{x^n} \) is a linear combination with coefficients in \( B_H^n \) of monomials of the form \( t_{g_1} t_{g_2} \cdots t_{g_{n-p}} t_x^p \), where \( 0 \leq p \leq n \) and \( g_1, \ldots, g_{n-p} \) are grouplike elements. It is easily checked that in this linear combination there is a unique monomial of the form \( t_x^n \) whose coefficient is the invertible element of \( B_H^n \)

\[
\sigma^{-1}(g^{n-1}, g) \sigma^{-1}(g^{n-2}, g) \cdots \sigma^{-1}(g, g) \alpha(h, h) \cdots \alpha(h^{n-2}, h) \alpha(h^{n-1}, h).
\]

Since \( t_g \) belongs to \( A \) for any grouplike element \( g \in H \), it follows that, for all \( n \geq 1 \), the element \( t_{x^n} \) is a polynomial of degree \( n \) in \( t_x \) with coefficients in \( A \). By hypothesis, there are scalars \( \lambda_1, \ldots, \lambda_{n-1}, \lambda_n \in k \) for some positive integer \( n \) such that

\[
x^n + \lambda_1 x^{n-1} + \cdots + \lambda_{n-1} x + \lambda_n = 0.
\]

Therefore, \( t_x \) satisfies a degree \( n \) polynomial equation with coefficients in the integral closure \( A \) and with highest-degree coefficient equal to 1. This proves that \( t_x \in A \).

To complete the proof, it suffices to check that \( t_z \) belongs to \( A \) for any product \( z \) of grouplike or skew-primitive elements \( x^{[1]}, \ldots, x^{[n]} \) such that \( t_{x^{[1]}}, \ldots, t_{x^{[n]}} \) belong to \( A \). It follows from Lemma 6.3, 6.3, and 6.5 that \( t_z \) is a linear combination with coefficients in \( B_H^n \) of products of the variables \( t_{x^{[1]}}, \ldots, t_{x^{[n]}} \) and of variables of the form \( t_g \), where \( g \) is grouplike. Since these monomials belong to \( A \), so does \( t_z \). \( \square \)

### 7. How to construct elements of \( B_H^n \)

In the example considered in Section 4, we reformulated the values of the generic cocycle in terms of certain rational fractions \( E, R, S, T, U \). The aim of this last section is to explain how we found these fractions by presenting a general systematic way of producing elements of \( B_H^n \) for an arbitrary Hopf algebra. To this end we introduce a new set of symbols.
7.1. The symbols $X_x$. Let $H$ be a Hopf algebra and $X_H$ a copy of the underlying vector space of $H$; we denote the identity map from $H$ to $X_H$ by $x \mapsto X_x$ for all $x \in H$.

Consider the tensor algebra $T(X_H)$ of the vector space $X_H$ over the ground field $k$:

$$T(X_H) = \bigoplus_{r \geq 0} X_H^r.$$ 

If $\{x_i\}_{i \in I}$ is a basis of $H$, then $T(X_H)$ is the free noncommutative algebra over the set of indeterminates $\{X_{x_i}\}_{i \in I}$.

The algebra $T(X_H)$ is an $H$-comodule algebra equipped with the coaction $\delta : T(X_H) \rightarrow T(X_H) \otimes H$ given for all $x \in H$ by

$$\delta(x) = \sum_{(x)} X_{x(1)} \otimes x_{(2)}.$$ (7.1)

7.2. Coinvariant elements of $T(X_H)$. Let us now present a general method to construct coinvariant elements of $T(X_H)$. We need the following terminology.

Given an integer $n \geq 1$, an ordered partition of $\{1, \ldots, n\}$ is a partition $I = (I_1, \ldots, I_r)$ of $\{1, \ldots, n\}$ into disjoint nonempty subsets $I_1, \ldots, I_r$ such that $i < j$ for all $i \in I_k$ and $j \in I_{k+1}$ $(1 \leq k \leq r - 1)$.

If $x[1], \ldots, x[n]$ are $n$ elements of $H$ and if $I = \{i_1 < \cdots < i_p\}$ is a subset of $\{1, \ldots, n\}$, we set $x[I] = x[i_1] \cdots x[i_p] \in H$. If $I = (I_1, \ldots, I_r)$ is an ordered partition of $\{1, \ldots, n\}$, then clearly $x[I_1] \cdots x[I_r] = x[1] \cdots x[n]$.

Now let $x[1], \ldots, x[n]$ be $n$ elements of $H$ and $I = (I_1, \ldots, I_r), J = (J_1, \ldots, J_s)$ be ordered partitions of $\{1, \ldots, n\}$. We consider the following element of $T(X_H)$:

$$P_{x[1],\ldots,x[n];I,J} = \sum_{(x[1]),\ldots,(x[n])} X_{x[I_1]} \cdots X_{x[I_r]} X_{S(x[J_1])} \cdots X_{S(x[J_s])}.$$ (7.2)

The element $P_{x[1],\ldots,x[n];I,J}$ is an homogeneous element of $T(X_H)$ of degree $r + s$. Observe that $P_{x[1],\ldots,x[n];I,J}$ is linear in each variable $x[1], \ldots, x[n]$.

We have the following generalization of [2] Lemma 2.1.

**Proposition 7.1.** Each element $P_{x[1],\ldots,x[n];I,J}$ of $T(X_H)$ is coinvariant.

**Proof.** By [7.1], $\delta(P_{x[1],\ldots,x[n];I,J})$ is equal to

$$\sum_{(x[1]),\ldots,(x[n])} X_{x[I_1]} \cdots X_{x[I_r]} X_{S(x[J_1])} \cdots X_{S(x[J_s])} \otimes x[I_1](2) \cdots x[I_r](2) S(x[J_1](3)) \cdots S(x[J_s](3))$$

$$= \sum_{(x[1]),\ldots,(x[n])} X_{x[I_1]} \cdots X_{x[I_r]} X_{S(x[J_1])} \cdots X_{S(x[J_s])} \otimes x[I_1](2) \cdots x[n](2) S(x[n](3)) \cdots S(x[1](3))$$

$$= \sum_{(x[1]),\ldots,(x[n])} X_{x[I_1]} \cdots X_{x[I_r]} X_{S(x[J_1])} \cdots X_{S(x[J_s])} \otimes \varepsilon(x[1](2)) \cdots \varepsilon(x[n](2))$$

$$= \sum_{(x[1]),\ldots,(x[n])} X_{x[I_1]} \cdots X_{x[I_r]} X_{S(x[J_1])} \cdots X_{S(x[J_s])} \otimes 1.$$
Therefore, \( \delta(P_{x[1],\ldots,x[n]}) = P_{x[1],\ldots,x[n]} \otimes 1 \) and the conclusion follows. \( \Box \)

As special cases of the previous proposition, the following elements of \( T(X_H) \) are coinvariant for all \( x, y \in H \):

\[
P_x = P_{x,((1),(1))} = \sum_{(x)} X_{x(1)} X S(x(2))
\]

and

\[
P_{x,y} = P_{x,y,((1),(2)),((1),(2))} = \sum_{(x), (y)} X_{x(1)} X_{y(1)} X S(x(2)y(2)).
\]

### 7.3. The generic evaluation map

As in Section 2 let \( H \) be a Hopf algebra, \( \alpha : H \times H \to k \) a normalized invertible cocycle, and \( ^\alpha H \) the corresponding twisted algebra.

Consider the algebra morphism \( \mu_\alpha : T(X_H) \to S(t_H) \otimes ^\alpha H \) defined for all \( x \in H \) by

\[
\mu_\alpha(x) = \sum_{(x)} t_{x(1)} \otimes u_{x(2)}.
\]

The morphism \( \mu_\alpha \) possesses the following properties (see Section 4).

**Proposition 7.2.** (a) The morphism \( \mu_\alpha : T(X_H) \to S(t_H) \otimes ^\alpha H \) is an \( H \)-comodule algebra morphism.

(b) If the ground field \( k \) is infinite, then for every \( H \)-comodule algebra morphism \( \mu : T(X_H) \to ^\alpha H \), there is a unique algebra morphism \( \chi : S(t_H) \to k \) such that

\[
\mu = (\chi \otimes \text{id}) \circ \mu_\alpha.
\]

In other words, any \( H \)-comodule algebra morphism \( \mu : T(X_H) \to ^\alpha H \) is obtained by specialization from \( \mu_\alpha \). For this reason we call \( \mu_\alpha \) the *generic evaluation map* for \( ^\alpha H \).

Now we have the following result (see Section 8).

**Proposition 7.3.** If \( P \in T(X_H) \) is coinvariant, then \( \mu_\alpha(P) \) belongs to \( B_H^n \).

It follows that the image \( \mu_\alpha(P_{x[1],\ldots,x[n]}) \) of all coinvariant elements defined by (7.2) belong to \( B_H^n \). This provides a systematic way to produce elements of \( B_H^n \).

**Example 7.4.** When \( H = H_4 \) is the Sweedler algebra, it is easy to check that the elements \( R, S, T, U \) of (4.1) are obtained in this way: we have

\[
R = \mu_\alpha(P_x), \quad T = \mu_\alpha(P_{y-z}), \quad U = \mu_\alpha(P_{x,z}), \quad ES = \mu_\alpha(P_{y,y})
\]

where \( \{1, x, y, z\} \) is the basis of \( H_4 \) defined in Section 4 and \( P_x, P_{y-z}, P_{x,z}, \) and \( P_{y,y} \) are special cases of the noncommutative polynomials defined by (7.3) and (7.4).

**Remark 7.5.** In Section 2 we developed a theory of polynomial identities for \( H \)-comodule algebras. This theory applies in particular to the twisted algebras \( ^\alpha H \). We established that the \( H \)-identities of \( ^\alpha H \), as defined in loc. cit., are exactly the elements of \( T(X_H) \) that lie in the kernel of the generic evaluation map \( \mu_\alpha \). Thus the \( H \)-comodule algebra \( U_H^\alpha = T(X_H)/\ker \mu_\alpha \) plays the rôle of a universal comodule algebra. We also constructed an \( H \)-comodule algebra morphism \( U_H^\alpha \to A_H^\alpha \); under certain conditions this map turns the generic Galois extension \( A_H^\alpha \) into a central localization of the universal comodule algebra \( U_H^\alpha \) (see Section 9 for details).
References

1. E. Aljadeff, D. Haile, M. Natapov, Graded identities of matrix algebras and the universal graded algebra, Trans. Amer. Math. Soc. (2009), in press.
2. E. Aljadeff, C. Kassel, Polynomial identities and noncommutative versal torsors, Adv. Math. 218 (2008), 1453–1495, doi:10.1016/j.aim.2008.03.014.
3. E. Aljadeff, M. Natapov, On the universal G-graded central simple algebra, Actas del XVI Coloquio Latinoamericano de Álgebra (eds. W. Ferrer Santos, G. Gonzalez-Sprinberg, A. Rittatore, A. Solotar), Biblioteca de la Revista Matemática Iberoamericana, Madrid 2007.
4. T. Aubriot, On the classification of Galois objects over the quantum group of a nondegenerate bilinear form, Manuscripta Math. 122 (2007), 119–135.
5. J. Bichon, Galois and bigalois objects over monomial non-semisimple Hopf algebras, J. Algebra Appl. 5 (2006), 653–680.
6. R. J. Blattner, M. Cohen, S. Montgomery, Crossed products and inner actions of Hopf algebras, Trans. Amer. Math. Soc. 298 (1986), 671–711.
7. R. J. Blattner, S. Montgomery, A duality theorem for Hopf module algebras, J. Algebra 95 (1985), 153–172.
8. Y. Doi, Equivalent crossed products for a Hopf algebra, Comm. Algebra 17 (1989), 3053–3085.
9. Y. Doi, M. Takeuchi, Cleft comodule algebras for a bialgebra, Comm. Algebra 14 (1986), 801–817.
10. Y. Doi, M. Takeuchi, Quaternion algebras and Hopf crossed products, Comm. Algebra 23 (1995), 3291–3325.
11. S. Garibaldi, A. Merkurjev, J.-P. Serre, Cohomological invariants in Galois cohomology, Univ. Lecture Ser. 28, Amer. Math. Soc., Providence, RI, 2003.
12. R. Güntner, Crossed products for pointed Hopf algebras, Comm. Algebra 27 (1999), 4389–4410.
13. C. Kassel, Quantum principal bundles up to homotopy equivalence, The Legacy of Niels Henrik Abel, The Abel Bicentennial, Oslo, 2002, O. A. Laudal, R. Piene (eds.), Springer-Verlag 2004, 737–748 (see also arXiv:math.QA/0507221).
14. C. Kassel, H.-J. Schneider, Homotopy theory of Hopf Galois extensions, Ann. Inst. Fourier (Grenoble) 55 (2005), 2521–2550.
15. A. Masuoka, Cleft extensions for a Hopf algebra generated by a nearly primitive element, Comm. Algebra 22 (1994), 4537–4559.
16. A. Masuoka, Cocycle deformations and Galois objects for some cosemisimple Hopf algebras of finite dimension, New trends in Hopf algebra theory (La Falda, 1999), 195–214, Contemp. Math., 267, Amer. Math. Soc., Providence, RI, 2000.
17. A. Masuoka, Abelian and non-abelian second cohomologies of quantized enveloping algebras, J. Algebra 320 (2008), 1–47.
18. S. Montgomery, Hopf algebras and their actions on rings, CBMS Conf. Series in Math., vol. 82, Amer. Math. Soc., Providence, RI, 1993.
19. F. Pacala, F. Van Oystaeyen, Clifford-type algebras as cleft extensions for some pointed Hopf algebras, Comm. Algebra 28 (2000), 585–600.
20. P. Schauenburg, Hopf bi-Galois extensions, Comm. Algebra 24 (1996), 3797–3825.
21. P. Schauenburg, Galois objects over generalized Drinfeld doubles, with an application to $u_q(sl_2)$, J. Algebra 217 (1999), 584–598.
22. H.-J. Schneider, Principal homogeneous spaces for arbitrary Hopf algebras, Israel J. Math. 72 (1990), 167–195.
23. M. Sweedler, Hopf algebras, W. A. Benjamin, Inc., New York, 1969.