New Facet Bell inequalities for multi-qubit states

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Non-trivial facet inequalities play important role in detecting and quantifying the nonlocality of a state – specially a pure state. Such inequalities are expected to be tight. Number of such inequalities depends on the Bell test scenario. With the increase in the number of parties, dimensionality of the Hilbert space, or/and the number of measurements, there are more nontrivial facet inequalities. By considering a specific measurement scenario, we find that for any multipartite qubit state, local polytope can have only one nontrivial facet. Therefore there exist a possibility that only one Bell inequality, and its permutations, would be able to detect the nonlocality of a pure state. The scenario involves two dichotomic measurement settings for two parties and one dichotomic measurement by other parties. This measurement scenario for a multipartite state may be considered as minimal scenario involving multipartite correlations that can detect nonlocality. We present detailed results for three-qubit states.

I. INTRODUCTION

Bell nonlocality [1], an intriguing feature of quantum mechanics has been studied extensively since the time of John S. Bell. From the time of the introduction of famous EPR paradox [2], entanglement has been known to be a source of many fascinating phenomena, including Bell nonlocality. However, entanglement in a state does not always guarantee Bell nonlocality; a simple example is the Werner state [3]. But the converse is true. The set of quantum correlations is convex but they do not form a polytope, whereas the set of local correlations is convex and also forms a polytope [4]. The nontrivial facets of this local polytope are known as tight Bell inequalities. The well known Clauser-Horner-Shimony-Holt (CHSH) inequality [6] is an example of facet Bell inequality for two parties, two measurement settings and two outcomes per setting. It is the only nontrivial facet inequality for this scenario giving the maximal quantum violation of $2\sqrt{2}$ which is also Tsirelson’s bound [7]. For three qubits Sliwa [8] constructed the local polytope for two dichotomic measurements per party, where Mermin inequality [9] is one of the facets. In our previous paper [10], we noted a particular limitation of Mermin-Ardehali-Belinskii-Klyshko (MABK) [9, 11] inequalities and constructed new Bell inequalities which removed this shortcoming. Particularly, the $n$-qubit state, $|\psi\rangle = \cos \alpha |0...0\rangle + \sin \alpha |1...1\rangle$ (generalized GHZ state) does not violate MABK inequalities [12] for $\sin 2\alpha \leq 1/\sqrt{2^{N-1}}$. In that paper, we constructed a set of six inequalities each of which is violated by generalized GHZ states for the whole parameter range. These six inequalities could be obtained from two inequalities after permutations of qubits. One important fact of those inequalities was the scenario we considered, i.e three parties, two dichotomic measurement settings for two parties and one dichotomic measurement for the remaining. But our inequalities were not facet inequalities for this particular scenario.

Question naturally arises what about the facet inequalities for this scenario. Will they also circumvent the obstacle posed by the MABK inequalities regarding the violation in the whole parameter range for generalized GHZ states and order them according to their entanglement? Besides, construction of facet Bell inequalities in this scenario is itself very interesting as it is the minimal scenario, where one can generate facet Bell inequalities. We need minimum two parties doing two dichotomic measurements, to have some non-trivial facet inequalities, also called facet Bell inequalities. To our knowledge, there is no work on constructing multiqubit facet Bell inequalities for this particular scenario. In this scenario, we find only one nontrivial facet inequality. With permutation of qubits, the number will be three. This shows that to uncover the nonlocality of a three-qubit, or multiqubit (as discussed below) system, one facet Bell inequality may be enough. This inequality involves multipartite correlations; so it explores multipartite nonlocality.

To go beyond three qubits, for the scenarios considered until now in the literature, it is computationally very difficult to construct facets of local polytope. Different generalization of facet Bell inequalities for three parties [8, 13] have been reported, but construction is hard. Only special situations are considered. In our new scenario, we have been able to construct the facet Bell inequalities for arbitrary number of qubits. In literature there are many multipartite Bell inequalities, each constructed for different purposes. Like in [14], authors constructed Bell inequalities to explore the nonlocality of cluster state. In [15, 16] Bell inequalities were devised to discriminate between multipartite entangled states etc. Our motivation here is two fold. Firstly, whether we can construct efficient facet Bell inequalities with the measurement settings of minimal requirement to explore the nonlocality of multiqubit states and secondly, whether for $n$-qubit generalized GHZ state we get violations for...
the whole parameter range like our previous inequalities [10].

In this paper, we have first constructed the facet Bell inequalities for three qubits and found only one nontrivial facet up to the relabeling of indices. We compare the results with that for other well known inequalities. We have also considered a few noisy mixed states. Next, we have constructed the facets for four- and five-qubit cases explicitly for the minimal measurement settings where, only two parties are doing two dichotomic measurements and the remaining parties are doing one dichotomic measurement each. It was very interesting to note that for each of these four- and five-qubit scenarios, there is again only one non-trivial facet, up to the relabeling of indices. Interestingly, we find that the structure of the facets are similar to three-qubit scenario, except for the addition of more parties. This observation enabled us to generalize our facet Bell inequality to n-qubit systems. We also show that generalized GHZ states of n qubits again violate the facet inequality for the whole parameter range.

The minimal scenario that we have considered can be thought of as two parties making measurements of two non-commuting dichotomic observables and other parties making measurements of commuting observables. We do not need to make measurement of noncommuting observables on all qubits. However, in this minimal scenario, the facet Bell inequality that we have obtained are not maximally violated by a maximally entangled state. The notion of a maximally entangled state for a multipartite system GHZ-state, for all practical purposes, can be considered to be maximally entangled. We find that the facet Bell inequality of our scenario is not maximally violated by the GHZ-state.

The paper is organized as follow. In the next section, we obtain facet Bell inequalities in the case of three qubits for our minimal scenario. In Sections III-VI, we discuss various aspects of these inequalities. In Section VII, we generalize the three-qubit facet Bell inequalities to multipartite case. In the last section, we present our conclusions.

II. FACET INEQUALITIES

Before stating our results, we will briefly review the polytope formed by local correlations and the significance of facet Bell inequalities. Polytope is a generalization of polygons to any dimension. Mathematically, there are two equivalent definitions [17] of a polytope: \( V \) representation and \( H \) representation. A \( V \)-polytope is the convex hull of a finite set of points \( \in \mathbb{R}^d \), which are called vertices. A \( H \) polytope is an intersection of a finite number of closed halfspaces in some \( \mathbb{R}^d \), which is bounded. So, a polytope is a set of finite number of points \( P \subseteq \mathbb{R}^d \), which can be represented as either a \( V \) or a \( H \) polytope. The dimension of a polytope is the dimension of its affine hull.

A Bell experiment can be described as follows. A source \( S \) distributes two particles (which may be entangled) to two spatially separated parties, Alice (A) and Bob (B). This situation can be easily generalized to multipartite scenarios, but for simplicity, we are discussing the preliminaries for two parties only. Now, Alice and Bob make local measurements labelled by the inputs \( x \) and \( y \) respectively. The outputs of their measurements are given by \( a \) and \( b \).

The joint probability distribution \( p = \{ p(ab|xy) \} \) characterizing the Bell experiment is called correlations or behaviour. We are interested only in these correlations, anything else is a black-box. Local Causality (LC) or Factorizability or Bell locality is defined as – \( p(ab|xy) = \int_{\lambda} d\lambda q(\lambda)p(a|x, \lambda)p(b|y, \lambda) \). Elements of \( p \), which satisfy the LC relation form the set of local correlations \( L \). This set is closed, bounded, convex and forms a polytope. Certain correlations in quantum mechanics are not compatible with local correlations; this is known as Bell nonlocality. The elements of \( p \) belong to the set of quantum correlations \( Q \) if, \( p(ab|xy) = Tr(\rho_{AB} M_{ax} \otimes M_{by}) \), where \( M_{ax} \) and \( M_{by} \) are POVM elements of corresponding measurements. Set of quantum correlations is closed, bounded and convex, but it is not a polytope as there are infinite number of extremal points. Any behaviour \( p \) is no-signalling \( NS \), if it satisfies the no-signalling constraints,

\[
\begin{align*}
\sum_b p(ab|xy) &= \sum_b p(ab|xy'), \forall a, x, y, y' \\
\sum_a p(ab|xy) &= \sum_a p(ab|x' y), \forall a, x, y, y' \tag{1}
\end{align*}
\]

No-signalling correlations also form a polytope, which consist of both local and nonlocal vertices. Both \( L \) and \( Q \) satisfy the no-signalling constraints, but there are \( NS \) correlations which do not satisfy locality and also do not belong to \( Q \). Any local behaviour admits a quantum description and hence belongs to \( Q \). But there are quantum correlations which do not belong to \( L \). So, finally we have, \( L \subset Q \subset NS \), which is shown in the Fig. (2).
From hyperplane separation theorem [18], for each behavior $p$ which is not the part of $L$ or $Q$ or $N'S$, there is a hyperplane that separates this $p$ from the corresponding set. If the set is $L$ then this is nothing but a facet Bell inequality. So, from the fig. (2), it is evident that facet Bell inequality is the tight or optimal Bell inequality for a set of local correlations. One can in principle construct Bell inequalities which are not facets of the local polytope, but these would not be optimal in the sense that there may be some quantum correlations which are nonlocal w.r.t a facet Bell inequality, but do not violate the non-optimal one. So, it is always desirable to find facet Bell inequalities for a set of local correlations. In literature, facet Bell inequalities have been constructed for many scenarios [4], like for higher dimensions, different measurement settings, multipartite settings etc. As we have seen, one of the important features of a local polytope is that only local correlations are inside it. Quantum correlations are outside it. Therefore, quantum correlations are expected to violate at least one of the facet inequalities of a given local polytope. From this point of view, it is of value to consider a local polytope with smallest number of nontrivial facet inequalities.

As stated in the introduction, we first construct facet Bell inequalities for three parties, two dichotomic measurements for two parties and one measurement for the rest. For this case we have a local polytope with 17 vertices in $V$ representation. By converting this $V$-representation to $H$-representation with the software cdd [5] we obtained total 48 facet inequalities. Detailed analysis of the local polytope and list of these 48 inequalities are given in the Appendix. Among 48 inequalities, 32 are just the positivity conditions for probabilities. Remaining 16 inequalities are the variations of four non-trivial facet inequalities. The four inequalities upto relabelling of indices are given below. In this list, the left-hand side should be thought of as the expectation value of the observables.

\[
\begin{align*}
(A_2B_2 - A_2B_1 - A_1B_2 - A_1B_1) \\
+ (A_2B_2 - A_2B_1 - A_1B_2 - A_1B_1)C_1 - 2C_1 &\leq 2 \quad (2)
\end{align*}
\]

\[
\begin{align*}
(A_2B_2 - A_2B_1 - A_1B_2 - A_1B_1) \\
+ (A_2B_2 - A_2B_1 - A_1B_2 - A_1B_1)C_1 - 2C_1 &\leq 2 \quad (3)
\end{align*}
\]

In terms of the well known CHSH inequality, these four can be written more simply as,

\[
\begin{align*}
-I_{\text{CHSH}} - I_{\text{CHSH}}C_1 - 2C_1 &\leq 2, \quad (6) \\
I_{\text{CHSH}} + I_{\text{CHSH}}C_1 - 2C_1 &\leq 2, \quad (7) \\
-I_{\text{CHSH}} + I_{\text{CHSH}}C_1 + 2C_1 &\leq 2, \quad (8) \\
I_{\text{CHSH}} - I_{\text{CHSH}}C_1 + 2C_1 &\leq 2. \quad (9)
\end{align*}
\]

But, these four inequalities are not inequivalent. We can see that if we make the interchange of the indices as, $A_1 \rightarrow A_2$, $A_2 \rightarrow -A_1$, $B_1 \rightarrow B_2$, $B_2 \rightarrow -B_1$ in the first inequality (Eqn.(2)), then it goes to the second inequality (Eqn.(3)). Similarly, one can see that with this type of interchange all the above inequalities are equivalent. So, finally we have only one inequality. We will choose the form of second inequality (if not mentioned) to do the rest of the analysis. Now other than Charlie, one can choose either Alice or Bob doing one measurement and rest are doing two dichotomic measurements. For each case we get one facet Bell inequality. In this way, there are three inequalities, where in our previous paper we had six inequalities. These three inequalities are,

\[
\begin{align*}
I_1 = I_{\text{CHSH}} + I_{\text{CHSH}}A_1 - 2A_1 &\leq 2 \quad (10) \\
I_2 = I_{\text{CHSH}} + I_{\text{CHSH}}B_1 - 2B_1 &\leq 2 \quad (11) \\
I_3 = I_{\text{CHSH}} + I_{\text{CHSH}}C_1 - 2C_1 &\leq 2. \quad (12)
\end{align*}
\]

In the following, we analyze these facet Bell inequalities for different purposes.

### III. Three-qubit generalized GHZ states

First, we will show that with the facet Bell inequalities, we can again have violation for all generalized GHZ states like our previous paper’s inequalities. Then we will show that amount of violation of the facet inequalities are in accordance with the amount of entanglement present in the generalized GHZ states, i.e more entangled a state is, more will be its violation. We will be using average von Neumann entropy over each bipartition as a measure of entanglement. One can take any other measure, and would get the same result. Let us consider the three-qubit generalized GHZ state,

\[
|GGHZ\rangle = \alpha |000\rangle + \beta |111\rangle. \quad (13)
\]

Without loss of generality, for simplicity, we take $\alpha$ and $\beta$ to be real and positive numbers, as the method will be same even if they are complex. Average von Neumann entropy for generalized GHZ state as defined above over

![FIG. 2. Schematic diagram of different type of correlations](image)
these bipartitions is \(-\alpha^2 \log_2 \alpha^2 - \beta^2 \log_2 \beta^2\), which is also the entropy for each bipartition for these states. Now to see the Bell violation by these states for the facet inequality, let’s take the facet inequality

$$I_B = I_{CHSH} + I_{CHSH} C_1 - 2 C_1 \leq 2,$$  \hspace{1cm} (14)

where \(I_{CHSH} = A_1 B_1 + A_1 B_2 + A_2 B_1 - A_2 B_2\). We choose \(A_1 = \sigma_z, A_2 = \sigma_x, B_1 = \cos \theta \sigma_x + \sin \theta \sigma_z, B_2 = - \cos \theta \sigma_x + \sin \sigma_z,\) and \(C_1 = \sigma_x\). For the generalised GHZ state \(|GGHZ\rangle = \alpha |000\rangle + \beta |111\rangle\), the expectation value of the operator \(I_B\) is

$$\langle GGHZ | I_B | GGHZ \rangle = 2 \sin \theta + 4 \alpha \beta \cos \theta$$ \hspace{1cm} (15)

As, \(a \sin \theta + b \cos \theta \leq \sqrt{a^2 + b^2}\), we have \(\langle I_B | GGHZ \rangle \leq 2 \sqrt{1 + 4 \alpha^2 \beta^2} = 2 \sqrt{1 + C}\), where \(C = 4 \alpha^2 \beta^2\) is nothing but the tangle [19] of the generalised GHZ state. The quantity \(C\) is also like concurrence for a two-qubit bipartite maximum is achieved when we choose \(\sin \theta = \frac{1}{\sqrt{1 + 4 \alpha^2 \beta^2}}\) and \(\cos \theta = \frac{2 \alpha \beta}{\sqrt{1 + 4 \alpha^2 \beta^2}}\).

Therefore, it is obvious that as long as the state is entangled i.e \(\alpha\) and \(\beta\) are not zero, the generalised GHZ states will violate the facet Bell inequality. This proves our first claim. Now, from this measurement setting, the maximum violation for the GHZ state is again \(2 \sqrt{2}\). Numerically, we have maximized the expectation value of the Bell operator for GHZ state and it is coming out to be \(2 \sqrt{2}\). So, this is the optimal measurement settings for GHZ state. Interesting fact is that there are many other states (not generalised GHZ states) which gives violation greater than \(2 \sqrt{2}\). We will discuss that later. Next, if we plot the entanglement (as calculated above) and the amount of optimal violation of the Bell inequality, we would get similar kind of plots such that they are monotonically related (see [10] for more details and the plots). So, more entangled a generalised GHZ state is more will be the violation of the facet Bell inequality.

One question may now arise that for this particular measurement settings, we are getting the expression of optimal violation which is a monotonic function of \(C\). If we choose other measurement settings, will this type of relation emerge? To answer this question, let us consider a general measurement settings as below,

\[
\begin{align*}
A_1 &= \sin \theta_{a1} \cos \phi_{a1} \sigma_x + \sin \theta_{a1} \sin \phi_{a1} \sigma_y + \cos \theta_{a1} \sigma_z \\
A_2 &= \sin \theta_{a2} \cos \phi_{a2} \sigma_x + \sin \theta_{a2} \sin \phi_{a2} \sigma_y + \cos \theta_{a2} \sigma_z \\
B_1 &= \sin \theta_{b1} \cos \phi_{b1} \sigma_x + \sin \theta_{b1} \sin \phi_{b1} \sigma_y + \cos \theta_{b1} \sigma_z \\
B_2 &= \sin \theta_{b2} \cos \phi_{b2} \sigma_x + \sin \theta_{b2} \sin \phi_{b2} \sigma_y + \cos \theta_{b2} \sigma_z \\
C_1 &= \cos \phi_{c1} \sigma_x + \sin \phi_{c1} \sigma_y
\end{align*}
\]

With these measurement settings we get

$$\langle I_B | GGHZ \rangle = X + CY,$$ \hspace{1cm} (16)

where \(X = \cos \theta_{a2}(\cos \theta_{b1} - \cos \theta_{b2}) + \cos \theta_{a1}(\cos \theta_{b1} + \cos \theta_{b2}), Y = \cos(\phi_{a1} + \phi_{b1} + \phi_{c1}) \sin \theta_{b1} \sin \theta_{b2} + \cos(\phi_{a2} + \phi_{b1} + \phi_{c1}) \sin \theta_{a1} \sin \theta_{b1} + \cos(\phi_{a1} + \phi_{b2} + \phi_{c1}) \sin \theta_{a1} \sin \theta_{b2} - \cos(\phi_{a2} + \phi_{b2} + \phi_{c1}) \sin \theta_{a2} \sin \theta_{b2} + \cos \phi_{c1} \sigma_x + \sin \phi_{c1} \sigma_y
\]

\(\phi_{a1}\) and \(\phi_{b1}\).

From the above relation, it is clear that for fixed values of \(X\) and \(Y\), the amount of violation is again monotonic in \(C\). So, no matter what the measurement settings, we will get more violation for a more entangled state, as long as we use same measurement settings for the states.

**A. Comparison with Mermin inequality**

Mermin inequality [9] can also track the entanglement, i.e violation of Mermin’s inequality will be more for more entangled generalized GHZ states. Mermin inequality is

$$I_M = A_1 B_1 C_2 + A_1 B_2 C_1 + A_2 B_1 C_1 - A_2 B_2 C_2 \leq 2.$$ \hspace{1cm} (17)

In this case if we choose the same general measurement settings as described above with \(C_2 = \cos \phi_{c2} \sigma_x + \sin \phi_{c2} \sigma_y\). The expectation value of the operator \(I_M^{MABK}\) for the generalized GHZ state is

$$\langle I_M | GGHZ \rangle = C \left( \cos(\phi_{a1} + \phi_{b1} + \phi_{c1}) \sin \theta_{a1} \sin \theta_{b1} + \cos(\phi_{a2} + \phi_{b1} + \phi_{c1}) \sin \theta_{a2} \sin \theta_{b1} + \cos(\phi_{a1} + \phi_{b2} + \phi_{c1}) \sin \theta_{a1} \sin \theta_{b2} - \cos(\phi_{a2} + \phi_{b2} + \phi_{c2}) \sin \theta_{a2} \sin \theta_{b2} \right).$$ \hspace{1cm} (18)

So, expectation value of the Bell-Mermin operator is again a monotonic function of \(C\). But the problem is that it does not show violation for the whole range of generalized GHZ states. So, for those states which do not violate Mermin inequality, this relation between entanglement and nonlocality has no meaning. But this relation can be used to measure the entanglement.

**B. Comparison with Svetlichny inequality**

Svetlichny first introduced [20] a definition of genuine tripartite nonlocality. Based on that definition he gave a inequality to detect genuine tripartite nonlocality. But this inequality is not violated [21, 22] by some tripartite genuinely entangled states, revealing that Svetlichny’s definition of genuine tripartite nonlocality is not equivalent to genuine tripartite entanglement, but a bit stronger. A strictly weaker definition of genuine tripartite nonlocality was given in [23], and the authors conjectured that every genuinely entangled tripartite pure state is also genuinely nonlocal according to their definition. But, if some state violates Svetlichny inequality, it must be a genuinely entangled state. This is not the case with Mermin inequality, as biseparable state also violate the Mermin inequality. This is also true for the facet and previous inequalities. They do not detect genuine entanglement, as biseparable states also violate them. But the class of states [21, 22] for which the Svetlichny inequality is not violated, our facet inequality and also the previous
inequalities get violated. There are two classes of states which do not Svetlichny inequality, which is,  
\[ I_S = A_1 D_1 C_1 + A_1 D_2 C_2 + A_2 D_2 C_1 - A_2 D_1 C_2 \leq 4, \tag{19} \]
where \( D_1 = B_1 + B_2 \) and \( D_2 = B_1 - B_2 \). One class is again the generalized GHZ class and another class is  
\[ |\psi_{gs} \rangle = \alpha |000 \rangle + \beta |111 \rangle (\cos \phi |0 \rangle + \sin \phi |1 \rangle) \tag{20} \]
For this class of states with the measurement settings chosen earlier, i.e \( A_1 = \sigma_x, A_2 = \sigma_x, B_1 = \cos \theta \sigma_x + \sin \theta \sigma_y, B_2 = -\cos \theta \sigma_x + \sin \theta \sigma_y \) and \( C_1 = \sigma_x \), the expectation value of our facet inequality operator is,  
\[ \langle I_B | \psi_{gs} \rangle = 4 \alpha \beta \cos \phi (\cos \theta + \sin \theta) + 2(1 + \beta^2 \sin 2\theta) \sin \phi - 2 \beta^2 \sin 2\theta \leq 2[\sqrt{\alpha^2 (\beta^2 + \beta^2 \sin 2\theta)} + (1 + \beta^2 \sin 2\theta)]^2 \tag{21} \]
It is clear from the expression that for any \( \alpha \) and \( \beta \), above expectation value is always greater than two. Therefore, our inequalities are also violated by those states, which do not Svetlichny inequality. Nevertheless our previous and facet inequalities can not be used to detect genuine tripartite entanglement just like Mermin inequality.

C. More Violation by a non-maximally entangled state

Unlike our previous inequalities, which are violated maximally by GHZ state by an amount \( 2\sqrt{2} \), our facet Bell inequalities are violated more by other genuinely entangled states. One very simple example is W state. Numerically we have found that W state gives maximum violation of 3.105 for the inequality, where Charlie makes one measurement. Obviously, there is no ordering of violation of the facet Bell inequality according to the entanglement within W class. Like the state \( \sqrt{1/6} |001 \rangle + \sqrt{3/6} |010 \rangle + \sqrt{2/6} |011 \rangle \) has average entropy 0.856 and violation of 3.33. And \( \sqrt{1/10} |001 \rangle + \sqrt{4/10} |010 \rangle + \sqrt{5/10} |011 \rangle \) has average entropy 0.813 and violation 3.475. Ordering is valid only for generalized GHZ states, not for whole GHZ class. Not only that, there are states within GHZ class, which violates the facet inequality more than the conventional GHZ state. Like the state \( |\psi \rangle = \sqrt{22/50} |000 \rangle + \sqrt{3/50} |100 \rangle + \sqrt{2/50} |011 \rangle + \sqrt{21/50} |110 \rangle + \sqrt{2/50} |111 \rangle \) has maximum expectation value 3.377 (found numerically) and also belongs to the GHZ class. For three-qubit systems, GHZ-state can be considered to be maximally entangled state. In this case, the subsystems are maximally mixed. Furthermore, for a number of communication protocols, the GHZ state is a task-oriented maximally entangled state [24]. But we see, that a facet Bell inequality is not maximally violated by this state. Non facet inequalities like in reference [10] and Mermin inequalities are violated maximally by the GHZ-state.

D. Three-qubit pure bi-separable states

The three facet Bell inequalities explore the entanglement of three types of bi-separable pure states like our previous inequalities. For example, the state which is separable in 1 – 23 bipartition will violate that facet inequality, which can explore the entanglement between the second and the third qubit. So in this case, the inequality \( I_{CHSH} + I_{CHSH} A_1 - 2A_1 \leq 2 \) will be violated. Similarly, other two types of biseparable states will violate other two inequalities. But we can not distinguish between bi-separable and genuinely entangled pure states like our previous set of inequalities. Because we had six inequalities for the previous paper and bi-separable state would violate exactly two inequalities from the state with same amount of optimal violation. But in the case of facet Bell inequalities bi-separable states will violate only one out of the three, and that optimal violation may be exhibited by some genuinely entangled state also. So, by a violation, we can not say whether it is for a bi-separable pure state or for a genuinely entangled pure state.

IV. Violation for three-qubit genuinely entangled states

In the previous subsection, we have shown that any biseparable pure state will violate one of our three facet Bell inequalities, depending upon in which bi-partition they are separable. In this section, we will investigate the case for genuinely entangled pure states. A genuinely entangled three-qubit pure state can be written in a canonical form [25] with six parameters as,  
\[ |\psi \rangle = \lambda_0 |0 \rangle |0 \rangle |0 \rangle + \lambda_1 e^{i\phi} |1 \rangle |0 \rangle |0 \rangle + \lambda_2 |1 \rangle |1 \rangle |0 \rangle + \lambda_3 |1 \rangle |1 \rangle |1 \rangle + \lambda_4 |1 \rangle |1 \rangle |1 \rangle, \tag{22} \]
where \( \lambda_i \geq 0, \sum_i \lambda_i^2 = 1, \lambda_0 \neq 0, \lambda_2 + \lambda_4 \neq 0, \lambda_3 + \lambda_4 \neq 0 \) and \( \phi \in [0, \pi] \). As there are many parameters involved (state parameters plus the parameters for the measurement operators), we don’t have any analytical claim for three-qubit genuinely entangled pure states. But we do have numerical evidence that all genuinely entangled pure states violate atleast one of the three inequalities listed above. We have generated 25000 random states and checked the expectation value of the facet-Bell operator. We numerically optimized the expectation value by considering all possible measurement settings and in each case we got a violation. In support of this we will provide some results for some special cases of pure three-qubit states. In section III, we have shown that all three
inequalities are violated for the whole range of generalized GHZ states. We consider another class of GHZ state, i.e., $|\psi\rangle_{GG} = \sin \alpha \cos \beta |000\rangle + \sin \alpha \sin \beta |010\rangle + \cos \alpha |111\rangle$. We find the expectation value of all three inequalities and then find out the maximum ($I_G = \max[I_1, I_2, I_3]$) among them. We plot $I_G$ with $\beta$ for some values of $\alpha$ in Fig.(3). Fig.(3) shows that $|\psi\rangle_{GG}$ violates three inequalities is violated by the generalized W state $|\psi\rangle_{GW}$ except when $\beta = 0; \alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{2}; \alpha = \frac{\pi}{2}$ as they are product states.

V. QUANTUM TO CLASSICAL RATIO

In this section, we study quantum to classical ratio and compare our inequalities with the well-known Mermin inequality. Quantum to classical ratio has a meaning in the sense that if quantum to classical ratio is large then the inequality is better suitable for an experiment.

In our case, we define the quantum to classical ratio as $I = \max[I_1, I_2, I_3]$. For generalized GHZ state ($|GGHZ\rangle = \sin \beta |000\rangle + \cos \beta |111\rangle$) our inequalities are not as good as Mermin. However, there is one drawback of Mermin inequality. It is not violated by the whole range of generalized GHZ state. In this sense our inequalities are better than Mermin inequalities. In Fig.(6), we compare our results for generalized W state of the form $|\psi\rangle_{GW} = \sin \alpha \cos \beta |001\rangle + \sin \alpha \sin \beta |010\rangle + \cos \alpha |100\rangle$, where we consider the case $\alpha = \frac{\pi}{4}$. From this figure it is clear that our facet Bell inequalities are better than of Mermin inequalities. Therefore, for experimental studies our inequalities are better.

VI. MIXED STATE SCENARIO

Mixed states present different challenges. There is a phenomenon of hidden nonlocality. We have the modest goal to examine where the facet Bell inequalities of this paper may be more useful. We consider a few noisy states, like noisy GHZ states, noisy W states with both white and colored noise, to see whether any advantages
are there for our facet Bell inequalities over the Mermin’s inequality for mixed states. First we will take a Werner like state for three qubits, which is GHZ state with white noise.

\[ |\text{NoisyGHZ} \rangle = p |\text{GHZ} \rangle \langle \text{GHZ} | + \frac{(1 - p)}{8} \mathbb{1}, \]  

(23)

where,

\[ 1 = |\psi_0^+ \rangle \langle \psi_0^+ | + |\psi_0^- \rangle \langle \psi_0^- | + |\psi_1^+ \rangle \langle \psi_1^+ | + |\psi_1^- \rangle \langle \psi_1^- | + |\psi_2^+ \rangle \langle \psi_2^+ | + |\psi_2^- \rangle \langle \psi_2^- | + |\psi_3^+ \rangle \langle \psi_3^+ | + |\psi_3^- \rangle \langle \psi_3^- | \]  

(24)

and,

\[ |\psi_0^+ \rangle = |\text{GHZ} \rangle = \sqrt{1/2}(|000 \rangle + |111 \rangle) \]  

(25)

\[ |\psi_0^- \rangle = \sqrt{1/2}(|000 \rangle - |111 \rangle) \]  

(26)

\[ |\psi_1^+ \rangle = \sqrt{1/2}(|010 \rangle + |101 \rangle) \]  

(27)

\[ |\psi_1^- \rangle = \sqrt{1/2}(|010 \rangle - |101 \rangle) \]  

(28)

\[ |\psi_2^+ \rangle = \sqrt{1/2}(|100 \rangle + |011 \rangle) \]  

(29)

\[ |\psi_2^- \rangle = \sqrt{1/2}(|100 \rangle - |011 \rangle) \]  

(30)

\[ |\psi_3^+ \rangle = \sqrt{1/2}(|110 \rangle + |001 \rangle) \]  

(31)

\[ |\psi_3^- \rangle = \sqrt{1/2}(|110 \rangle - |001 \rangle) \]  

(32)

For this noisy GHZ state, we have numerically obtained the optimal expectation value of the facet Bell operator for the whole range of \( p (0 \leq p \leq 1) \) and plotted them. The noisy GHZ states start violating our facet Bell inequality after \( p = 0.71 \). Now, let us see what is the scenario for Mermin inequality for the same noisy GHZ states.

![FIG. 7. Maximum expectation value of the Bell operator for a noisy GHZ states vs p plot.](image)

We see that for Mermin operator, the violation starts after \( p = 0.51 \). So, for this noisy GHZ states, our facet Bell inequality presents no advantage. One of the reasons for this is that, Mermin inequality is optimally constructed for GHZ states, giving a violation 4, whereas, our facet inequality gives only \( 2\sqrt{2} \) for GHZ states.

![FIG. 8. Maximum expectation value of the Mermin operator for a noisy GHZ states vs p plot.](image)

Let us now consider noisy \( W \) states to analyze the same thing. We take,

\[ |\text{NoisyW} \rangle = p |W \rangle \langle W | + \frac{(1 - p)}{8} \mathbb{1} \]  

(33)

For this case, we see that nosy \( W \) states start to violate our facet Bell inequality after \( p = 0.65 \).

![FIG. 9. Maximum expectation value of the Bell operator for a noisy \( W \) states vs p plot.](image)

For Mermin inequality violation starts after \( p = 0.66 \). So, in this case our inequality gives slight advantage over the Mermin inequality.

![FIG. 10. Maximum expectation value of the Mermin operator for a noisy \( W \) states vs p plot.](image)

Similarly, we can take colored noise and do the same analysis as before. In the following we give a table listing the results we have obtained numerically.

| Noise Type   | Maximum Expectation Value | Optimal Value |
|--------------|---------------------------|---------------|
| Noisy GHZ    | 0.0                       | 0.5           |
| Noisy W      | 0.5                       | 1.0           |
Our Inequalities

| States | Value of \( p \) to start violation | Our Inequalities | Mermin |
|--------|-----------------------------------|-----------------|--------|
| \( p |GHZ\rangle (|GHZ\rangle + \frac{(1-p)}{n} \mathbb{I} \) | 0.71 | 0.51 |
| \( p |GGHZ3\rangle (|GGHZ3\rangle + \frac{(1-p)}{3} \mathbb{I} \) | 0.80 | 0.69 |
| \( p |GGHZ2\rangle (|GGHZ2\rangle + \frac{(1-p)}{3} \mathbb{I} \) | 0.81 | 0.73 |
| \( p |GGHZ1\rangle (|GGHZ1\rangle + \frac{(1-p)}{3} \mathbb{I} \) | 0.83 | 0.97 |
| \( p |GHZ\rangle (|GHZ\rangle + \frac{(1-p)}{n} \mathbb{I} \) | 0.64 | 0.38 |
| \( p |W\rangle (|W\rangle + \frac{(1-p)}{n} \mathbb{I} \) | 0.65 | 0.66 |
| \( p |W1\rangle (|W1\rangle + \frac{(1-p)}{n} \mathbb{I} \) | 0.61 | 0.68 |

\[
|GHZ_1\rangle = \sqrt{\frac{8}{9}} |000\rangle + \sqrt{\frac{1}{9}} |111\rangle \tag{34}
\]

\[
|GGHZ_2\rangle = \sqrt{\frac{25}{29}} |000\rangle + \sqrt{\frac{4}{29}} |111\rangle \tag{35}
\]

\[
|GGHZ_3\rangle = \sqrt{\frac{21}{25}} |000\rangle + \sqrt{\frac{4}{25}} |111\rangle \tag{36}
\]

\[
col = |\psi_1^+\rangle \langle \psi_1^+| + |\psi_2^+\rangle \langle \psi_2^+| + |\psi_3^+\rangle \langle \psi_3^+| \tag{37}
\]

\[
|W\rangle = \sqrt{\frac{1}{3}} |001\rangle + \sqrt{\frac{1}{3}} |010\rangle + \sqrt{\frac{1}{3}} |100\rangle \tag{38}
\]

\[
|W1\rangle = \sqrt{\frac{1}{6}} |001\rangle + \sqrt{\frac{2}{6}} |010\rangle + \sqrt{\frac{3}{6}} |100\rangle \tag{39}
\]

In the above, we have taken \( col \) to be colored noise and \( |GGHZ_1\rangle, |GGHZ_2\rangle, |GGHZ_3\rangle \) are generalized GHZ states and \( |W1\rangle \) is a \( W \) class state.

Clearly, our inequality give advantages for noisy \( W \) states. For noisy GHZ states Mermin is better except for the cases starting from the close vicinity of the parameter range \( \theta = 15^\circ \) i.e. \( \sin \theta \sim 0.25 \), where Mermin does not get violated. From table it is evident that when \( \sin \theta = \sqrt{\frac{1}{9}} = 0.33 \), the noisy state violates Mermin when it is almost pure. Obviously in those regions our inequality is advantageous, because they are violated for all generalized \( GHZ \) states i.e \( GGHZ \) states. One can in principle check for other mixed states. We have analyzed the noisy ones, because they are experimentally relevant. Whenever one tries to prepare a \( GHZ \) or \( W \) state in lab, unavoidable noises add up, making the states noisy.

### VII. Extension to Multipartite Scenario

In this section, we will be extending the previous facet inequalities to more than parties. First, we will be dealing with four-qubit scenario and then with five qubits. After that results will be generalized for \( n \) qubits, where \( n \geq 3 \). In all these scenarios we will be restricting our calculations for the situations, where two parties are making two dichotomic measurements and rest are making only one dichotomic measurement. For this particular scenario, we will find nontrivial facets of the local polytope. Let’s start with four qubits.

#### A. Four-qubit scenario

For this case we have 35 vertices for the local polytope, where two parties are making two dichotomic measurements and remaining two parties are making one dichotomic measurement each. We again convert this \( V \)-representation of the polytope to the \( H \)-representation using the software cdd [5] and obtained a total of 96 facets. Out of which 64 facets are just the positivity conditions on the probabilities. So, we get 32 nontrivial facet inequalities. But, interestingly, these 32 inequalities are just the variants of one single inequality, upto the relabelling of indices. So, like the three-qubit scenario, we again get only one single inequality.

\[
(-2 + A_1(B_1 + B_2) + A_2(B_1 - B_2))(1 + C_1)(1 + D_1) \leq 0
\]

All the 32 facet inequalities are equivalent to this inequality up to the relabelling of indices. The form of this inequality is very similar to the inequality for the three-qubit case. Because one can write the inequality given by the Eqn. (12) as,

\[
(-2 + A_1(B_1 + B_2) + A_2(B_1 - B_2))(1 + C_1) \leq 0,
\]

which has the exactly similar structure like the four-qubit inequality except the extra party denoted by \( D \). Next we will explore whether five-qubit case has also the similar structure.

#### B. Five qubits or more

In this case, again we have two parties performing two dichotomic measurements and the remaining parties are performing only one dichotomic measurement. For this case, we have a total of 71 vertices. Converting from the \( V \) representation to \( H \) representation for this local polytope, we obtain total of 192 facets. Out of which, 128 inequalities are just the positivity conditions for the probabilities. Remaining 64 inequalities again give only one non-trivial inequality upto the relabelling of the indices.

\[
(-2 + A_1(B_1 + B_2) + A_2(B_1 - B_2))(1 + C_1)(1 + D_1)(1 + E_1) \leq 0
\]

Again for five-qubit case also, we have the same structure of the inequality like three- and four-qubit cases, with the addition of a new term for the party \( E \). So, after exploring these three cases extensively, we can generalize this structure to more qubits. For \( n \) number of qubits, we can generalize the structure as,

\[
(-2 + A_1(A_2 + A_3) + A'_1(A_2 - A'_2))(1 + A_3)(1 + A_4)...(1 + A_n) \leq 0,
\]

where \( A_1 \) and \( A'_1 \) are the two measurement choices for the party \( A_1 \) and similarly for \( A_2 \). If we just expand this
we will get,
\[
\begin{align*}
(A_1(A_2 + A_2') + A_1'(A_2 - A_2'))(1 + A_3) ... (1 + A_n) \\
- (2A_3 + 2A_4 + ... 2A_3A_4 + ... 2A_3A_4 ... A_n) \leq 2
\end{align*}
\]

So, our facet Bell inequalities have very simple and intuitive structure. Important point is that we have only one facet Bell inequality in our minimal scenario for any number of qubits. This Bell inequality involves multipartite correlations. We can permute the parties that make two dichotomic measurements to obtain the complete set. We now show that all n-qubit generalized GHZ state violate this n-qubit facet Bell inequality.

C. Violation by n-qubit GHZ state

Here, we will show that the n-qubit facet Bell inequality for n qubits will be violated by the generalized GHZ states for the whole parameter range. To show this, we take the n-qubit generalized GHZ state to be, $|GHZ_n\rangle = |00...0\rangle + |11...1\rangle$ and the similar measurement settings as the three-qubit scenario, i.e. we choose, $A_1 = \sigma_z, A_1' = \sigma_z, A_2 = \cos \theta \sigma_x + \sin \theta \sigma_z, A_2' = - \cos \theta \sigma_x + \sin \theta \sigma_z$ and all other measurement settings to be $\sigma_x$, i.e $A_3 = \sigma_x, A_4 = \sigma_x, ... A_n = \sigma_x$. Now for these measurement settings the expectation value of the Facet-Bell operator given by the Eqn.(41) is $(2\sin \theta + 4\alpha \beta \cos \theta)$, which is exactly equal to the previously obtained expectation value for the three-qubit scenario. So, the generalized GHZ state will violate the n-qubit facet inequality for all the range of parameters, giving the maximum violation of $2\sqrt{1 + 4\alpha^2 \beta^2}$ for the GHZ state for this measurement settings.

VIII. CONCLUSION

In this paper, we have considered a specific measurement scenario. This scenario may be thought of as minimal scenario that involves multipartite correlations. So one can explore multipartite nonlocality. In this scenario, there are two dichotomic measurement settings for two parties and one dichotomic measurement setting for each of the remaining parties. Interestingly, there is just one facet Bell inequality (up to permutation of parties) for n qubits. This is like the two-qubit scenario where only CHSH inequality is the facet Bell inequality. This suggests that we need only one facet Bell inequality that uses multipartite correlations to detect the nonlocality of a multipartite state. This gives significant advantage over the scenarios considered until now in the literature.

We first constructed facet Bell inequalities, in this scenario, for a three-qubit system. This was motivated by our previous work [10]. Then, we showed that the three facet inequalities give similar advantages like our previous inequalities [10]. However, the facet Bell inequalities are now not violated maximally by the GHZ states, which can be considered as maximally entangled three-qubit state. Then we computed the facets for four and five qubits in the minimal scenario. We found that each of these two cases again give only one non-trivial facet inequality up to the relabeling of indices. We then extended our results to n parties and shown that the n-qubit facet Bell inequality is violated by all n-qubit generalized GHZ states. We have compared our three-qubit inequalities with Mermin and Svetlichny inequalities and also analyzed some cases of mixed states, including noisy GHZ and W states. We have demarcated where our facet Bell inequalities present advantages. Inequalities in this paper can be tested experimentally as our previous ones [20].

[1] J. S. Bell, Physics 1, 195 (1964).
[2] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).
[3] R. F. Werner, Phys. Rev. A 40, 4277 (1989).
[4] N. Brunner et al., Rev. Mod. Phys., 86, 419 (2015).
[5] https://www.inf.ethz.ch/personal/fukudak/cdd/.
[6] J. F. Clauser et al., Phys. Rev. Lett. 23, 880 (1969).
[7] B. S. Tsirelson, Lett. Math. Phys. 4, 93 (1980); B. S. Tsirelson, J. Sov. Math. 36, 557 (1987).
[8] C. Siwa, Phys. Lett. A 317, 165 (2003).
[9] N. D. Mermin, Phys. Rev. Lett. 65, 1838 (1990).
[10] A. Das, C. Datta, P. Agrawal, Phys. Lett. A 381, 3928 (2017).
[11] M. Ardehali, Phys. Rev. A 46, 5375 (1992); A. V. Belinski and D. N. Klyshko, Phys. Usp. 36, 653 (1993).
[12] V. Scarani and N. Gisin, J. Phys. A 34, 6043 (2001).
[13] J. D. Bancal, N. Gisin and S. Pironio, J. Phys. A: Math. Theor. 43, 385303 (2010).
[14] V. Scarani et al., Phys. Rev. A 71, 042325 (2005).
[15] C. Schmid et al, Phys. Rev. Lett. 100, 200407 (2008).
[16] N. Brunner, J. Sharam and T. Vértesi, Phys. Rev. Lett. 108, 110501 (2012).
[17] G. M. Ziegler, Lectures notes on polytope, Springer (1994).
[18] S. P. Boyd and L. Vandenberghe, Convex Optimization, Cambridge University Press (2004).
[19] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A 61 052306 (2000).
[20] G. Svetlichny, Phys. Rev. D 35, 3066 (1987).
[21] S. Ghose et al, Phys. Rev. Lett. 102, 250404 (2009).
[22] A. Ajoy and P. Rungta, Phys. Rev. A 81, 052334 (2010).
[23] J. D. Bancal et al., Phys. Rev. A 88, 014102 (2013).
[24] P. Agrawal and B. Pradhan, J. Phys. A: Math. Theor. 43, 235302 (2010).
[25] A. Acín et al., Phys. Rev. Lett. 87, 040401 (2001).
[26] J- Q. Zhao et al., Phys. Lett. A 382, 1214 (2017).
APPENDIX

A. Facet

In our Bell test scenario, we have three parties with two dichotomic measurements for two parties and one dichotomic measurement for the other party. In a Bell test we usually measure the joint outcome probabilities i.e., $P(abc|xyz)$. Here $x, y, z \in \{0, 1\}$ are the measurement settings for Alice and Bob respectively and $a, b \in \{0, 1\}$ are the corresponding outcome for Alice and Bob respectively. As here Charlie is doing one measurement so $z = 0$ and $c \in \{0, 1\}$. Therefore, there are total of 32 joint probabilities. But all of them are not independent. No-signaling and normalization conditions constrain the number of independent joint probabilities and determine the dimension of the probability space $[13]$, which is $\left((m_1(d-1)+1)\cdot(m_2(d-1)+1)\cdot(m_3(d-1)+1)\right)-1$. In our case $m_1 = 2, m_2 = 2, m_3 = 1, \ d = 2$, which gives the dimension to be 17. Now the conditions on local correlations will determine the vertices. First, we have to choose the parametrization for this 17 dimensional space. We choose the following parametrization.

$$P = [P(a_0), P(a_1), P(b_0), P(b_1), P(c_0), P(a_0b_0), P(a_0b_1), P(a_1b_0), P(a_1b_1), P(a_0b_0c_0), P(a_0b_0c_1), P(a_0b_1c_0), P(a_0b_1c_1), P(a_1b_0c_0), P(a_1b_0c_1), P(a_1b_1c_0), P(a_1b_1c_1)], \quad (42)$$

where $P(a_x) = P(0|x), P(b_y) = P(0|y), P(c_z) = P(0|z), P(a_xb_y) = P(00|xyz)$ and $P(a_xb_yc_z) = P(000|xyz)$. So this polytope is 17 dimensional and consists of 32 extremal points or vertices. This polytope has been described using the V-representation. One can find the facets of this polytope using some standard algorithm. The number of facets for this polytope is 48 and these are as follows

$$P(a_xb_yc_z) \geq 0, \quad (43)$$
$$P(a_xb_y) - P(a_xb_yc_z) \geq 0, \quad (44)$$
$$P(b_yc_z) - P(a_xb_yc_z) \geq 0, \quad (45)$$
$$P(b_yc_z) - P(a_xb_yc_z) \geq 0, \quad (46)$$
$$P(a_x) - P(a_xb_y) + P(a_xb_yc_z) = P(a_xc_z) \geq 0, \quad (47)$$
$$P(b_y) - P(a_xb_y) + P(a_xb_yc_z) = P(b_yc_z) \geq 0, \quad (48)$$
$$P(c_z) - P(a_xb_y) + P(a_xb_yc_z) = P(b_yc_z) \geq 0, \quad (49)$$
$$P(a_x) - P(a_xb_y) + P(a_xb_yc_z) = P(a_xc_z) \geq 0, \quad (50)$$
$$P(a_0b_0c_0) - P(a_0b_1c_0) - P(a_1b_0c_0) - P(a_1b_1c_0) + P(a_0b_0c_0) + P(b_0c_0) - P(c_0) \leq 0, \quad (51)$$
$$P(a_0b_0c_1) - P(a_0b_1c_1) - P(a_1b_0c_0) - P(a_1b_1c_0) + P(a_0b_0c_0) + P(b_0c_0) - P(c_0) \leq 0, \quad (52)$$
$$P(a_0b_0c_1) - P(a_0b_1c_0) - P(a_1b_0c_1) - P(a_1b_1c_1) + P(a_0b_0c_0) + P(b_0c_0) - P(c_0) \leq 0, \quad (53)$$
$$P(a_1b_1c_0) - P(a_0b_0c_0) - P(a_0b_1c_1) - P(a_1b_1c_0) + P(a_0b_0c_0) + P(b_0c_0) - P(c_0) \leq 0, \quad (54)$$
$$P(a_0b_0c_0) - P(a_0b_1c_0) + P(a_1b_0c_0) + P(a_1b_1c_0) = P(a_2c_0) - P(b_0c_0) \leq 0, \quad (55)$$
$$P(a_0b_0c_0) + P(a_0b_1c_0) + P(a_1b_0c_0) - P(a_1b_1c_0) - P(a_0b_0c_0) - P(b_0c_0) \leq 0, \quad (56)$$
$$P(a_0b_0c_0) + P(a_0b_1c_0) + P(a_1b_0c_0) - P(a_1b_1c_0) - P(a_0b_0c_0) - P(b_0c_0) \leq 0, \quad (57)$$
$$P(a_0b_0c_0) + P(a_0b_1c_0) - P(a_1b_0c_0) + P(a_1b_1c_0) - P(a_0b_0c_0) - P(b_0c_0) \leq 0, \quad (58)$$
$$P(a_0b_0c_0) + P(a_0b_1c_0) + P(a_1b_0c_0) + P(a_1b_1c_0) + P(a_1b_1c_0) + P(a_1b_1c_0) - P(a_1b_1c_0) - P(b_0c_0) \leq 0, \quad (59)$$
$$P(a_0b_0c_0) + P(a_0b_1c_0) - P(a_1b_0c_0) + P(a_1b_1c_0) + P(a_1b_1c_0) - P(a_1b_1c_0) - P(b_0c_0) \leq 0, \quad (60)$$
$$P(a_0b_0c_0) + P(a_0b_1c_0) + P(a_1b_0c_0) + P(a_1b_1c_0) + P(a_1b_1c_0) - P(a_1b_1c_0) - P(b_0c_0) \leq 0, \quad (61)$$
$$P(a_0b_0c_0) + P(a_0b_1c_0) + P(a_1b_0c_0) + P(a_1b_1c_0) + P(a_1b_1c_0) - P(a_1b_1c_0) - P(b_0c_0) \leq 0, \quad (62)$$
$$P(a_0b_0c_0) + P(a_0b_1c_0) + P(a_1b_0c_0) + P(a_1b_1c_0) + P(a_1b_1c_0) - P(a_1b_1c_0) - P(b_0c_0) \leq 0, \quad (63)$$
$$P(a_0b_0c_0) + P(a_0b_1c_0) + P(a_1b_0c_0) + P(a_1b_1c_0) + P(a_1b_1c_0) - P(a_1b_1c_0) - P(b_0c_0) \leq 0, \quad (64)$$
$$P(a_0b_0c_0) + P(a_0b_1c_0) + P(a_1b_0c_0) + P(a_1b_1c_0) + P(a_1b_1c_0) - P(a_1b_1c_0) - P(b_0c_0) \leq 0, \quad (65)$$
$$P(a_0b_0c_0) + P(a_0b_1c_0) + P(a_1b_0c_0) + P(a_1b_1c_0) + P(a_1b_1c_0) - P(a_1b_1c_0) - P(b_0c_0) \leq 0, \quad (66)$$

Now, we can write the probabilities in terms of expectation values. Like $P(a_2) = \frac{1}{2}(1 + (a_x))$ and similarly for the joint probabilities. By this substitution of expectation values in place of probability distributions, we can write these inequalities as,

$$P(a_0b_0c_0) + P(a_0b_1c_0) - P(a_1b_0c_0) + P(a_1b_1c_0) - P(a_2c_0) - P(b_0c_0) \leq 0, \quad (67)$$
$$P(a_0b_0c_0) + P(a_0b_1c_0) - P(a_1b_0c_0) + P(a_1b_1c_0) - P(a_2c_0) - P(b_0c_0) \leq 0, \quad (68)$$
$$P(a_0b_0c_0) + P(a_0b_1c_0) - P(a_1b_0c_0) + P(a_1b_1c_0) - P(a_2c_0) - P(b_0c_0) \leq 0, \quad (69)$$
$$P(a_0b_0c_0) + P(a_0b_1c_0) - P(a_1b_0c_0) + P(a_1b_1c_0) - P(a_2c_0) - P(b_0c_0) \leq 0, \quad (70)$$
$$P(a_0b_0c_0) + P(a_0b_1c_0) - P(a_1b_0c_0) + P(a_1b_1c_0) - P(a_2c_0) - P(b_0c_0) \leq 0, \quad (71)$$
$$P(a_0b_0c_0) + P(a_0b_1c_0) - P(a_1b_0c_0) + P(a_1b_1c_0) - P(a_2c_0) - P(b_0c_0) \leq 0, \quad (72)$$
(1 − A_x)(1 − B_y)(1 + C_z) ≥ 0, \hspace{1cm} (73)
A_x(1 − B_y)(1 − C_z) + B_y(1 − C_z) + C_z ≤ 1, \hspace{1cm} (74)
[−2 + A_0(B_0 − B_1) − A_1(B_0 + B_1)](1 + C_0) ≤ 0, \hspace{1cm} (75)
[2 + A_1(−B_0 + B_1) + A_0(B_0 + B_1)](1 + C_0) ≥ 0, \hspace{1cm} (76)
[−2 + A_0(B_0 − B_1) + A_1(B_0 + B_1)](1 + C_0) ≥ 0, \hspace{1cm} (77)
[2 + A_1(−B_0 − B_1) + A_0(B_0 − B_1)](1 + C_0) ≤ 0, \hspace{1cm} (78)
[−2 + A_0(B_0 − B_1) + A_1(B_0 + B_1)](1 + C_0) ≤ 0, \hspace{1cm} (79)
[2 + A_1(−B_0 − B_1) + A_0(B_0 − B_1)](1 + C_0) ≤ 0, \hspace{1cm} (80)
[−2 + A_0(B_0 − B_1) − A_1(B_0 + B_1)](1 + C_0) ≥ 0, \hspace{1cm} (81)
[−2 + A_1(−B_0 − B_1) + A_0(B_0 − B_1)](1 + C_0) ≤ 0, \hspace{1cm} (82)
[A_0(B_0 − B_1) − A_1(B_0 + B_1)](1 − C_0) + 2C_0 ≤ 2, \hspace{1cm} (83)
[A_1(−B_0 + B_1) + A_0(B_0 + B_1)](−1 + C_0) + 2C_0 ≤ 2, \hspace{1cm} (84)

Similarly, we have computed the facets for four and five qubits and found that they have similar structure.