Stochastic differential equations driven by $G$-Brownian motion and ordinary differential equations

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Abstract
In this paper, we show that the integration of a stochastic differential equation driven by $G$-Brownian motion ($G$-SDE for short) in $\mathbb{R}$ can be reduced to the integration of an ordinary differential equation (ODE for short) parametrized by a variable in $(\Omega, \mathcal{F})$. By this result, we obtain a comparison theorem for $G$-SDEs and its applications.

Keywords: $G$-Brownian motion, $G$-Itô’s formula, $G$-SDE, Comparison theorem.

Mathematics Subject Classification (2000). 60H30, 60H10.

1 Introduction
Motivated by uncertainty problems, risk measures and the superhedging in finance, Peng systemically established a time-consistent fully nonlinear expectation theory (see [13, 14, 15]). As a typical and important case, Peng introduced the $G$-expectation theory (see [16, 17] and the references therein) in 2006. In the $G$-expectation framework ($G$-framework for short), the notion of $G$-Brownian motion and the corresponding stochastic calculus of Itô’s type were established. On that basis, Gao [4] and Peng [16] studied the existence and uniqueness of the solution of $G$-SDE under a standard Lipschitz condition. Moreover, Lin [11] obtained the existence and uniqueness of the solution of $G$-SDE with reflecting boundary. For a recent account and development of this theory we refer the reader to [1, 7, 8, 9, 10, 12, 20].

Under the classical framework, Doss [3] and Huang, Xu and Hu [6] studied the sample solutions of stochastic differential equations, which enables us to transfer a stochastic differential equation into a set of ordinary differential equations for each sample path. Using the method of sample solutions to SDEs, Huang [5] established a comparison theorem of SDEs.

The aim of this paper is to study the sample solutions of $G$-SDEs by ODEs parameterized by a variable in basis probability space. Since $G$-SDE admits a unique solution in the space $M^2_G(0, T)$, the main difficulty is how to prove that the sample solution belongs to this space. We overcome this problem through some $G$-stochastic calculus techniques. Then we show that the solution of $G$-SDE can be represented as a function of both $G$-Brownian motion and a finite variation process. Since we can use the existing results in the theory of ordinary differential equations directly, this approach provides a powerful tool both in the theoretical analysis and in the practical computation of $G$-SDEs. In particular, we get a new kind of comparison theorem for $G$-SDEs. Moreover, a necessary and sufficient condition for comparison theorem of $G$-SDEs is also obtained.

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This paper is organized as follows: In the next section, we recall some notations and results that we will use in this paper. In section 3, we study the sample solution of $G$-SDE under some strong conditions, then, in section 4, we extend this result to a more general case. Finally in section 5, we establish a new kind of comparison theorem and give its applications.

2 Preliminaries

The main purpose of this section is to recall some preliminary results in $G$-framework which are needed in the sequel. More details can be found in Denis et al [2], Li and Peng [10], Lin [11 12] and Peng [10].

2.1 Sublinear expectation

**Definition 2.1** Given a set $\Omega$ and a linear space $\mathcal{H}$ of real valued functions defined on $\Omega$. Moreover, if $X_i \in \mathcal{H}, i = 1, \ldots, d$, then $\varphi(X_1, \ldots, X_d) \in \mathcal{H}$ for all $\varphi \in C_{b,Lip}(\mathbb{R}^d)$, where $C_{b,Lip}(\mathbb{R}^d)$ is the space of all bounded real-valued Lipschitz continuous functions. A sublinear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a functional $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$,

(a) **Monotonicity:** if $X \geq Y$, then $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$;

(b) **Constant preserving:** $\hat{\mathbb{E}}[c] = c$, $\forall$ $c \in \mathbb{R}$;

(c) **Sub-additivity:** $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$;

(d) **Positive homogeneity:** $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$, $\forall \lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space. $X \in \mathcal{H}$ is called a random variable in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. We often call $Y = (Y_1, \ldots, Y_d), Y_i \in \mathcal{H}$ a $d$-dimensional random vector in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$.

**Definition 2.2** In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a $d$-dimensional random vector $Y = (Y_1, \ldots, Y_d)$ is said to be independent from an $m$-dimensional random vector $X = (X_1, \ldots, X_m)$ under $\hat{\mathbb{E}}$ if for any test function $\varphi \in C_{b,Lip}(\mathbb{R}^{m+n})$

$$
\hat{\mathbb{E}}[\varphi(Y, X)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}]. 
$$

**Definition 2.3** Let $X_1$ and $X_2$ be two $n$-dimensional random vectors defined on sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$, respectively. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if

$$
\hat{\mathbb{E}}_1[\varphi(X_1)] = \hat{\mathbb{E}}_2[\varphi(X_2)], \forall \varphi \in C_{b,Lip}(\mathbb{R}^n).
$$

$X$ is said to be an independent copy of $X$ if $X \overset{d}{=} X$ and $X$ is independent from $X$.

**Definition 2.4** ($G$-normal distribution) A random variable $X$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called (centralized) $G$-normal distributed if for any $a, b \geq 0$

$$
aX + b\hat{X} \overset{d}{=} \sqrt{a^2 + b^2}X,
$$

where $\hat{X}$ is an independent copy of $X$. The letter $G$ denotes the function

$$
G(a) = \frac{1}{2}(\sigma^2 a^2 - \sigma^2 a^-)
$$

with $\sigma^2 := -\hat{\mathbb{E}}[-X^2] \leq \hat{\mathbb{E}}[X^2] =: \sigma^2$. 



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2.2 $G$-Brownian motion

**Definition 2.5 ($G$-Brownian motion)** A process $(B_t \in \mathcal{H})_{t \geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a $G$-Brownian motion if the following properties are satisfied:

(a) $B_0 = 0$.

(b) For each $t, s \geq 0$ the increment $B_{t+s} - B_t \overset{d}{=} \sqrt{s}X$ and independent from $(B_{t_1}, B_{t_2}, ..., B_{t_n})$ for each $n \in \mathbb{N}$, $0 \leq t_1 \leq t_2 \leq ... \leq t_n \leq t$, where $X$ is $G$-normal distributed.

Denote by $\Omega = C_0(\mathbb{R}^+)\,$ the space of all $\mathbb{R}$-valued continuous paths $(\omega_t)_{t \in \mathbb{R}^+}$, with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i}\max_{t \in [0,1]}|\omega^1_t - \omega^2_t| \wedge 1.$$  

$B(\Omega)$ is the Borel $\sigma$-algebra of $\Omega$.

For each $t \in [0, \infty)$, we introduce the following spaces.

- $\Omega_t := \{\omega(\cdot \wedge t) : \omega \in \Omega\}, F_t := \mathcal{B}(\Omega_t)$
- $L^0(\Omega)$: the space of all $\mathcal{B}(\Omega)$-measurable real functions,
- $L^0(\Omega_t)$: the space of all $F_t$-measurable real functions,
- $B^0_\Omega(\Omega) :=$ all bounded elements in $L^0(\Omega)$
- $C^0_\Omega(\Omega)$: all continuous elements in $B^0_\Omega(\Omega)$
- $B^0_\Omega(\Omega_t) := B^0_\Omega(\Omega) \cap L^0(\Omega_t)$
- $C^0_\Omega(\Omega_t) := C^0_\Omega(\Omega) \cap L^0(\Omega_t)$

In Peng [10], a $G$-Brownian motion is constructed on a sublinear expectation space $(\Omega, L^0_G, \hat{\mathbb{E}}, (\hat{\mathcal{F}}_t)_{t \geq 0})$, where $L^p_G(\Omega)$ is a Banach space under the natural norm $\|X\|_p := \hat{\mathbb{E}}[|X|^p]^{1/p}$. In this space the corresponding canonical process $B_t(\omega) = \omega_t$ is a $G$-Brownian motion. Denote by $L^G_2(\Omega)$ the completion of $B^0_\Omega(\Omega)$. Denis et al. [2] proved that $L^G_2(\Omega) \supset L^G_0(\Omega) \supset C^0_\Omega(\Omega)$, and there exists a weakly compact family $\mathcal{P}$ of probability measures defined on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E_P[X], \quad X \in L^G_2(\Omega).$$

**Remark 2.6** Denis et al. [2] gave a concrete set $\mathcal{P}_M$ that represents $\hat{\mathbb{E}}$. Consider a 1-dimensional Brownian motion $B_t$ on $(\Omega, F, P)$, then

$$\mathcal{P}_M := \{P_\theta : P_0 = P \circ X^{-1}, \quad X_t = \int_0^t \theta_s dB_s, \quad \theta \in L^G_2([0,T]; [\mathcal{F}^2_2, \mathcal{F}^2_\theta])\}$$

is a set that represents $\hat{\mathbb{E}}$, where $L^G_2([0,T]; [\mathcal{F}^2_2, \mathcal{F}^2_\theta])$ is the collection of all $\mathcal{F}$-adapted measurable processes with $\theta^2 \leq \theta(s)^2 \leq \mathcal{F}^2_\theta$.

Now we introduce the natural Choquet capacity

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

**Definition 2.7** A set $A \subset \mathcal{B}(\Omega)$ is polar if $c(A) = 0$. A property holds “quasi-surely" (q.s.) if it holds outside a polar set.

**Definition 2.8** A real function $X$ on $\Omega$ is said to be quasi-continuous if for each $\varepsilon > 0$, there exists an open set $O$ with $c(O) < \varepsilon$ such that $X|_O$ is continuous.

**Definition 2.9** We say that $X : \Omega \mapsto \mathbb{R}$ has a quasi-continuous version if there exists a quasi-continuous function $Y : \Omega \mapsto \mathbb{R}$ such that $X = Y$, q.s.
Then $L^p_b(\Omega)$ and $L^p_G(\Omega)$ can be characterized as follows:

$$L^p_b(\Omega) = \{ X \in L^0(\Omega) \mid \lim_{N \to \infty} \mathbb{E} [|X|^p I_{|X| \geq N}] = 0 \}$$

and

$$L^p_G(\Omega) = \{ X \in L^p_b(\Omega) \mid X \text{ has a quasi-continuous version} \}.$$  

2.3 G-stochastic calculus

Peng [16] also introduced the related stochastic calculus of Itô’s type with respect to $G$-Brownian motion (see Li and Peng [10], Lin [11] for more general and systematic research).

Let $T \in \mathbb{R}^+$ be fixed.

**Definition 2.10** For each $p \geq 1$, consider the following simple type of processes:

$$M^0_{G}(0,T) = \{ \eta := \eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j, t_{j+1})}(t) \}$$

$$\forall N > 0, 0 = t_0 < ... < t_N = T, \xi_j \in L^p_G(\Omega_{t_j}), j = 0, 1, 2, ..., N - 1 \}.$$ 

Denote by $M^p_G(0,T)$ the completion of $M^0_G(0,T)$ under the norm

$$||\eta||_{M^p_G(0,T)} = \left( \int_0^T \mathbb{E}[|\eta(t)|^p] dt \right)^{1/p}.$$ 

**Definition 2.11** For each $\eta \in M^0_{G}(0,T)$ with the form

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) I_{[t_k, t_{k+1})}(t),$$

define

$$I(\eta) = \int_0^T \eta_t dB_t := \sum_{k=0}^{N-1} \xi_k (B_{t_{k+1}} - B_{t_k}).$$

The mapping $I : M^0_{G}(0,T) \to L^2_G(\Omega_T)$ can be continuously extended to $I : M^2_G(0,T) \to L^2_G(\Omega_T)$. For each $\eta \in M^2_{G}(0,T)$, the stochastic integral is defined by

$$I(\eta) := \int_0^T \eta_t dB_t, \quad \eta \in M^2_{G}(0,T).$$ 

Unlike the classical theory, the quadratic variation process of $G$-Brownian motion $B$ is not always a deterministic process and it can be formulated in $L^2_G(\Omega_t)$ by

$$\langle B \rangle_t := \lim_{N \to \infty} \sum_{i=0}^{N-1} (B_{t_{i+1}}^N - B_{t_i}^N) = B^2_t - 2 \int_0^t B_s dB_s,$$

where $t_i^N = \frac{t_i}{N}$ for each integer $N \geq 1$. 

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Definition 2.12 Define a mapping $M_{1}^{0, 1}(0, T) \mapsto L_{G}^{1}(\Omega_{T})$:

$$Q(\eta) = \int_{0}^{T} \eta_{s} d(B)_{s} := \sum_{k=0}^{N-1} \xi_{k}[(B)_{t \xi_{k+1}} - (B)_{t \xi_{k}}].$$

Then $Q$ can be uniquely extended to $M_{1}^{1}(0, T) \mapsto L_{G}^{1}(\Omega_{T})$. We also denote this mapping by

$$Q(\eta) := \int_{0}^{T} \eta_{s} d(B)_{s}, \quad \eta \in M_{1}^{1}(0, T).$$

In view of the dual formulation of $G$-expectation as well as the properties of the quadratic variation process $(B)$ in $G$-framework, Gao [14] obtained the following BDG type inequalities.

Lemma 2.13 For each $p \geq 1$ and $\eta \in M_{1}^{p}(0, T)$,

$$\hat{\mathbb{E}}\left[ \sup_{0 \leq t \leq T} \left| \int_{0}^{t} \eta_{s} d(B)_{s} \right|^{p} \right] \leq \sigma^{2p} T^{p-1} \int_{0}^{T} \hat{\mathbb{E}}[|\eta_{s}|^{p}] ds.$$

Lemma 2.14 Let $p \geq 2$ and $\eta \in M_{1}^{p}(0, T)$. Then there exists some constant $C_{p}$ depending only on $p$ and $T$ such that

$$\hat{\mathbb{E}}\left[ \sup_{0 \leq t \leq T} \left| \int_{0}^{t} \eta_{s} dB_{s} \right|^{p} \right] \leq C_{p} \hat{\mathbb{E}}[|\eta_{0}|^{2}] T.$$  

3 G-Stochastic differential equation

Let us first recall some notations,

- $C^{n}(\mathbb{R}^{d})$: the space of all functions of class $C^{n}$ from $\mathbb{R}^{d}$ into $\mathbb{R}$,
- $C^{n}_{b, \text{lip}}(\mathbb{R}^{d})$: the space of all bounded functions of class $C^{n}(\mathbb{R}^{d})$ whose partial derivatives of order less than or equal to $n$ are bounded Lipschitz continuous functions,
- $C^{n}([0, T] \times \mathbb{R}^{d})$: the space of all functions of class $C^{n}$ from $[0, T] \times \mathbb{R}^{d}$ into $\mathbb{R}$,
- $C^{n}_{b, \text{lip}}([0, T] \times \mathbb{R}^{d})$: the space of all bounded functions of class $C^{n}([0, T] \times \mathbb{R}^{d})$ whose partial derivatives of order less than or equal to $n$ are bounded Lipschitz continuous functions.

Consider the following SDE driven by a 1-dimensional $G$-Brownian motion:

$$X_{t} = X_{0} + \int_{0}^{t} b(s, X_{s}) ds + \int_{0}^{t} h(s, X_{s}) d(B)_{s} + \int_{0}^{t} \sigma(s, X_{s}) dB_{s}, \quad t \in [0, T],$$

where the initial condition $X_{0} \in \mathbb{R}$ is a given constant.

We recall the following assumption.

(H) $b, h, \sigma : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ are given functions satisfying $b(\cdot, x), h(\cdot, x), \sigma(\cdot, x) \in M_{1}^{2}(0, T)$ for each $x \in \mathbb{R}$. Moreover, there exists some constant $K$ such that $|\varphi(t, x) - \varphi(t, y)| \leq K|x - y|$ for each $t \in [0, T], x, y \in \mathbb{R}, \varphi = b, h$ and $\sigma$, respectively.

From Peng [16],

Theorem 3.1 Under the assumption (H), there exists a unique solution $X \in M_{2}^{2}(0, T)$ to the stochastic differential equation (1).

Remark 3.2 We remark that there is a potential to extend our results to a much more general setting. However, in order to focus on the main ideas, in this paper we content ourselves with the case that the coefficients are 1-dimensional satisfying bounded condition. In particular, by slightly more involved estimates, we can extend our results to the multi-dimensional case without bounded condition.
3.1 A simple case

In order to explain the main ideas, we first consider a simple $G$-SDE,

\[ X_t = X_0 + \int_0^t b(X_s) ds + \frac{1}{2} \int_0^t \sigma(X_s)\partial_x \sigma(X_s) d(B)_s + \int_0^t \sigma(X_s) dB_s, \quad t \in [0, T], \]

(2)

where $\sigma(x) \in C^1_{b,lip}(\mathbb{R})$ and $b(x) \in C_{b,lip}(\mathbb{R})$. By Theorem 3.1 $G$-SDE (2) admits a unique solution $X \in \mathcal{M}_G^2(0, T)$.

Now consider the following ODE

\[ \frac{dy}{dx} = \sigma(y), \quad y(0) = v \in \mathbb{R}. \]

(3)

The above ODE has a unique solution $y = \varphi(x, v) \in C(\mathbb{R}^2)$. Then,

\[ \partial_x \varphi = \sigma(\varphi), \quad \varphi(0, v) = v. \]

Consequently,

\[ \partial_{\omega} \varphi(x, v) = \exp\left\{ \int_0^x \partial_x \sigma(\varphi(y, v)) dy \right\}, \quad \partial_{xx} \varphi(x, v) = (\partial_x \sigma)(\varphi(x, v)). \]

(4)

Next we introduce the following ODE with parameter $\omega$:

\[
\begin{cases}
  dV_t = \exp\{-\int_0^{B_t(\omega)} \partial_x \sigma(\varphi(y, V_t)) dy\} b(\varphi(B_t(\omega), V_t)) dt, \\
  V_0 = X_0.
\end{cases}
\]

(4)

For every fixed $\omega$, recalling Cauchy–Lipschitz theorem, the equation (4) has a unique solution $V_t = V_t(\omega)$ and $V_t$ is a continuous finite variation process. Moreover, $V_t(\omega)$ is a continuous function on $(\Omega, \rho)$.

The following result is important in our future discussion.

**Lemma 3.3** For any $p \geq 0$, there exists a constant $C_p$ depending only on $p$ such that,

\[ \hat{\mathbb{E}}[\sup_{0 \leq t \leq T} e^{p|B_t|}] \leq C_p. \]

**Proof.** For any $p \geq 0$, we have

\[ \hat{\mathbb{E}}[\sup_{0 \leq t \leq T} e^{p|B_t|}] \leq \sum_{n=0}^{\infty} \hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |pB_t|^n]. \]

Applying Doob’s maximal inequality yields that

\[ \hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |pB_t|^n] \leq (1 + \frac{1}{n-1})^n \hat{\mathbb{E}}[|pB_T|^n]. \]

By Exercise 1.7 in Chapter 3 of Peng [16], one can show that for some constant $C'_p$ depending only on $p$,

\[ \sum_{n=0}^{\infty} \frac{\hat{\mathbb{E}}[|pB_T|^n]}{n!} \leq C'_p. \]

Since $\lim_{n} (1 + \frac{1}{n-1})^n = e$, we can find some constant $C_p$ depending only on $p$ such that,

\[ \hat{\mathbb{E}}[\sup_{0 \leq t \leq T} e^{p|B_t|}] \leq C_p, \]

which is the desired result. ■
Lemma 3.4 For each $p \geq 1$, $V_t \in L^p_G(\Omega_t)$. Moreover, there exists some constant $C_p$ depending only on $p$ such that for each $s \leq t \in [0, T]$, 
\[
\hat{E}[[T^V_t - T^V_s]^p] \leq C_p|t - s|^p,
\]
where $T^V$ is the total variation process of $V$.

Proof. By equation (4),
\[
V_t = V_0 + \int_0^t \exp\{-\int_0^{B_u(\omega)} \partial_x \sigma(\varphi(y, V_u))dy\} b(\varphi(B_u(\omega), V_u))du.
\]
Denote by $C_p$ a constant depending only on $p$, which is allowed to change from line to line. Then applying Lemma 3.3 we conclude
\[
\hat{E}[\sup_{0 \leq t \leq T} |V_t|^p] \leq C_p \hat{E}[|V_0|^p] + \int_0^T \exp\{-p \int_0^{B_u(\omega)} \partial_x \sigma(\varphi(y, V_u))dy\}du]\]
\[
\leq C_p (|V_0|^p + \hat{E}[\sup_{0 \leq t \leq T} e^{Cp[B_t]}]) \leq C_p.
\]
Since $V_t(\omega)$ is a continuous function on $(\Omega, \rho)$, recalling the pathwise description of $L^p_G(\Omega_t)$, $V_t \in L^p_G(\Omega_t)$ for each $p \geq 1$.

Note that
\[
T^V_t = \int_0^t \exp\{-\int_0^{B_u} \partial_x \sigma(\varphi(y, V_u))dy\} b(\varphi(B_u, V_u))du,
\]
applying Lemma 3.3 again, we obtain for each $s \leq t \in [0, T]$,
\[
\hat{E}[[T^V_t - T^V_s]^p] \leq C_p |t - s|^p,
\]
which completes the proof. □

By Lemma 3.4, we deduce that $\varphi(B_t, V_t) \in M^2_G(0, T)$. Since $\varphi$ satisfies the conditions of Theorem 6.1, applying G-Itô formula, we get
\[
d\varphi(B_t, V_t) = \partial_x \varphi(B_t, V_t)dB_t + \partial_y \varphi(B_t, V_t)dV_t + \frac{1}{2} \partial_{xx} \varphi(B_t, V_t)d\langle B \rangle_t
\]
\[= b(\varphi(B_t, V_t))dt + \frac{1}{2} \partial_x \sigma(\varphi(B_t, V_t))\sigma(\varphi(B_t, V_t))d\langle B \rangle_t + \sigma(\varphi(B_t, V_t))dB_t.
\]
Consequently, $X_t = \varphi(B_t, V_t)$ is the unique $M^2_G(0, T)$-solution of G-SDE (2).

3.2 The general case

In this section, we will extend the above result to a more general case, where all the coefficients are functions in $t, B_t$ and $x$. Assume $b(t, x, y), h(t, x, y) \in C_{b, lip}([0, T] \times \mathbb{R}^2)$ and $\sigma(t, x, y) \in C_{b, lip}^1([0, T] \times \mathbb{R}^2)$. It is obvious G-SDE
\[
X_t = X_0 + \int_0^t b(s, B_s, X_s)ds + \int_0^t h(s, B_s, X_s)d\langle B \rangle_s + \int_0^t \sigma(s, B_s, X_s)dB_s, \ t \in [0, T]
\]
has a unique solution $X \in M^2_G(0, T)$.

Then the following ODE
\[
\frac{dy}{dx} = \sigma(t, x, y), \quad y(t, 0) = v \in \mathbb{R}
\]
(6)
admits a unique solution \( y = \varphi(t, x, v) \in C([0, T] \times \mathbb{R}^2) \). Moreover, we can get 
\[
\partial_v \varphi(t, x, v) = \exp\left\{ \int_0^t \partial_u \sigma(t, u, \varphi(t, u, v))du \right\}
\]
and 
\[
\partial_t \varphi(t, x, v) = \exp\left\{ \int_0^t \partial_z \sigma(t, z, \varphi(t, z, v))dz \right\}(\int_0^t \partial_t \sigma(t, u, \varphi(t, u, v))e^{-\int_0^t \partial_t \sigma(t, z, \varphi(t, z, v))dz}du)
\]
Set 
\[
g(t, x, v) := \partial_v \varphi^{-1}(t, x, v)(b(t, x, \varphi(t, x, v)) - \partial_t \varphi(t, x, v)),
\]
\[
f(t, x, v) := \partial_v \varphi^{-1}(t, x, v)(h(t, x, \varphi(t, x, v)) - \frac{1}{2}(\partial_x \sigma + \partial_y \sigma)(t, x, \varphi(t, x, v))).
\]
Then consider the following initial value problem with parameter \( \omega \):
\[
\begin{align*}
dV_t &= g(t, B_t(\omega), V_t)dt + f(t, B_t(\omega), V_t)d(B)_t(\omega), \\
V_0 &= X_0.
\end{align*}
\]
(7)

Note that \( \langle B \rangle_t \) is a continuous finite variation process, then the ODE (7) has a unique solution \( V = V_t(\omega) \) and \( V_t \) is a continuous finite variation process. Since \( \langle B \rangle_t(\omega) \) is not always a deterministic process, in general we can not get \( V_t(\omega) \) is a continuous function on \( (\Omega, \rho) \) as the above section. However, we also have the following result.

**Lemma 3.5** For each \( p \geq 1 \), there exists some constant \( C_p \) depending only on \( p \) such that, for each \( s \leq t \in [0, T] \),
\[
\mathbb{E}[|T^V_s - T^V_t|^p] \leq C_p|t - s|^p,
\]
where \( T^V \) is the total variation process of \( V \).

**Proof.** The proof is immediate in light of Lemma 3.3. \( \blacksquare \)

Now we shall give the main result of this section.

**Theorem 3.6** Assume \( b(t, x, y), h(t, x, y) \in C_{b, lip}([0, T] \times \mathbb{R}^2) \) and \( \sigma(t, x, y) \in C_{b, lip}([0, T] \times \mathbb{R}^2) \), then for each \( p \geq 1 \), \( V_t \in L^p_p(\Omega_t) \) and \( \varphi(t, B_t, V_t) \) is the unique \( M^p_{\mathcal{F}}(0, T) \)-solution of G-SDE (5).

**Proof.** It is obvious \( V_t \in L^p_p(\Omega_t) \). Then applying Theorem 6.1 we obtain q.s.
\[
\begin{align*}
d\varphi(t, B_t, V_t) &= \partial_{y} \varphi(t, B_t, V_t)dt + \partial_{z} \varphi(t, B_t, V_t)dB_t + \partial_{v} \varphi(t, B_t, V_t)dV_t + \frac{1}{2} \partial^2_{x z} \varphi(t, B_t, V_t)d\langle B \rangle_t \\
&= b(t, B_t, \varphi(t, B_t, V_t))dt + h(t, B_t, \varphi(t, B_t, V_t))d\langle B \rangle_t + \sigma(t, B_t, \varphi(t, B_t, V_t))d\langle B \rangle_t.
\end{align*}
\]
By a standard argument, there exists some constant \( C \) such that, 
\[
\mathbb{E}[|\varphi(t, B_t, V_t) - X_t|^2] \leq C \int_0^t \mathbb{E}[|\varphi(s, B_s, V_s) - X_s|^2]ds.
\]
Applying Gronwall’s lemma, we obtain \( \varphi(t, B_t, V_t) = X_t \), q.s.. By the uniqueness of solution of ODE (6),
\[
v = \varphi(t, -x, \varphi(t, x, v)),
\]
thus, \( V_t = \varphi(t, -B_t, X_t) \) q.s.. In particular, \( V_t \) has a quasi-continuous version and \( V_t \in L^p_{\mathcal{F}}(\Omega_t) \). The proof is completed. \( \blacksquare \)
4 G-diffusion process

The objective of this section is to remove the condition that $\sigma$ is continuously differentiable and to obtain a more general result on this topic. By an approximation approach, we can also represent the solution of $G$-SDE as a function of $B_t$ and a continuous finite variation process $V_t$ as the above section.

**Theorem 4.1** If $b, \sigma, h \in C_{b,\text{lip}}(\mathbb{R})$, then there exists a unique continuous finite variation process $V_t \in L^p_G(\Omega_t)$ for each $p \geq 1$ such that,

$$X_t = \varphi(B_t, V_t),$$

where $\varphi$ is the solution of the ODE:

$$\partial_x \varphi(x, v) = \sigma(\varphi(x, v)), \quad \varphi(0, v) = v.$$

Moreover if $\sigma \in C^1_{b,\text{lip}}(\mathbb{R})$, then for q.s. $\omega$, $V_t(\omega)$ is the solution of the following ODE:

$$\begin{align*}
dV_t &= \exp\{-\int_0^{B_t} \partial_x \sigma(\varphi(y, V_t))dy\}[b(\varphi(B_t, V_t))dt + (h(\varphi(B_t, V_t)), V_t)] - \frac{1}{2} \partial_x \sigma(\varphi(B_t, V_t))d(B_t)\|,
v_0 &= X_0.
\end{align*}$$

**Proof.** If $\sigma \in C^1_{b,\text{lip}}(\mathbb{R})$, then the theorem holds true. If $\sigma \in C_{b,\text{lip}}(\mathbb{R})$, one can define

$$\sigma_n(x) := \int_{\mathbb{R}} \sigma(y)\rho_n(y-x)dy = \int_{\mathbb{R}} \sigma(y+x)\rho_n(y)dy,$$

where $\rho_n$ is a nonnegative $C^\infty$ function defined on $\{x : |x| \leq \frac{1}{n}\}$ with $\int_{\mathbb{R}} \rho_n(y)dy = 1$. From this definition, we conclude that

$$|\sigma_n(x) - \sigma(x)| \leq \int_{\mathbb{R}} |\sigma(y + x) - \sigma(x)|\rho_n(y)dy \leq \int_{\mathbb{R}} K|y|\rho_n(y)dy \leq \frac{K}{n},$$

where $K$ is the Lipschitz coefficient of $b, h$ and $\sigma$.

For each $n$, it is obvious $\sigma_n \in C^1_{b,\text{lip}}(\mathbb{R})$. Thus, $X^n_t := \varphi^n(B_t, V^n_t)$ is the solution of $G$-SDE:

$$X^n_t = X_0 + \int_0^t b(X^n_s)ds + \int_0^t h(X^n_s)d(B)_s + \int_0^t \sigma^n(X^n_s)dB_s, \quad t \in [0, T],$$

where $\varphi^n$ satisfies

$$\partial_x \varphi^n(x, v) = \sigma^n(\varphi^n(x, v)), \quad \varphi^n(0, v) = v$$

and

$$\begin{align*}
dV^n_t &= \exp\{-\int_0^{B_t} \partial_x \sigma^n(\varphi^n(y, V^n_t))dy\}[b(\varphi^n(B_t, V^n_t))dt + (h(\varphi^n(B_t, V^n_t)), V^n_t)] - \frac{1}{2} \partial_x \sigma^n(\varphi^n(B_t, V^n_t))d(B_t)\|,
V^n_0 &= X_0.
\end{align*}$$

For each $n$, there exists some constant $C$ depending only on $T$ and $K$ such that,

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |X^n_t - X_t|^2 \right] \leq \frac{C}{n^2}.$$
Indeed, applying BDG inequalities, we obtain for some constant $C$, which is allowed to change from line to line,

$$
\hat{E}[\sup_{t \in [0,T]} |X^n_t - X_t|^2] \leq \hat{E}[\sup_{t \in [0,T]} |\int_0^t b(X^n_s) - b(X_s)ds + \int_0^t h(X^n_s) - h(X_s)d(B)_s + \int_0^t \sigma^n(X^n_s) - \sigma(X_s)dB_s|^2]
$$

$$
\leq C\hat{E}[(\int_0^T K|X^n_t - X_t|ds)^2 + (\int_0^T K|X^n_t - X_t|dt)^2 + (\int_0^T (K|X^n_t - X_t| + \frac{1}{n})dB_t)^2]
$$

$$
\leq C\left(\frac{1}{n^2} + \int_0^T \hat{E}[|X^n_t - X_t|^2]dt\right)
$$

$$
\leq C\left(\frac{1}{n^2} + \int_0^T \hat{E}[\sup_{s \in [0,t]} |X^n_s - X_s|^2]dt\right).
$$

By Gronwall’s lemma, we can get the desired result. Moreover, choosing a subsequence if necessary, $X^n \to X$ uniformly in $[0, T]$ q.s.

For each $v_1, v_2, x \in \mathbb{R}$,

$$
|\varphi^n(x, v_1) - \varphi(x, v_2)| \leq |\varphi^n(x, v_1) - \varphi^n(x, v_2)| + |\varphi^n(x, v_2) - \varphi(x, v_2)|.
$$

Applying Taylor formula yields that

$$
|\varphi^n(x, v_1) - \varphi^n(x, v_2)| \leq |\partial_x \varphi^n(x, v^*)||v_1 - v_2| \leq |v_1 - v_2|e^{C|x|}.
$$

By the definitions of $\varphi^n$ and $\varphi$, we obtain

$$
|\varphi^n(x, v_2) - \varphi(x, v_2)| \leq |\int_0^x \sigma^n(\varphi^n(s, v_2)) - \sigma(\varphi(s, v_2))ds| \leq \int_0^{|x|} K(|\varphi^n(s, v_2) - \varphi(s, v_2)| + \frac{1}{n})ds.
$$

From Gronwall’s lemma, we conclude for some constant $C$

$$
|\varphi^n(x, v_1) - \varphi(x, v_2)| \leq C(|v_1 - v_2| + \frac{|x|}{n})e^{C|x|}.
$$

Define $V_t := \varphi(-B_t, X_t)$, thus $X_t = \varphi(B_t, V_t)$ and $V_t$ has a quasi-continuous version. Moreover,

$$
\lim_{n \to \infty} \sup_{t \in [0,T]} |V^n_t - V_t| = \lim_{n \to \infty} \sup_{t \in [0,T]} |\varphi^n(-B_t, X^n_t) - \varphi(-B_t, X_t)|
$$

$$
\leq C \lim_{n \to \infty} (\sup_{t \in [0,T]} |X^n_t - X_t| + \sup_{t \in [0,T]} |B_t|) \frac{C}{n} = 0.
$$

Since for each $n$ and $t, s \in [0, T]$, there exists some constant $C$ such that

$$
|V^n_t - V^n_s| \leq C \sup_{t \in [0,T]} e^{C|B_t|}|t - s|.
$$

Thus

$$
|V_t - V_s| \leq C \sup_{t \in [0,T]} e^{C|B_t|}|t - s|.
$$

By the pathwise description of $L^p_G(\Omega_t)$, $V_t \in L^p_G(\Omega_t)$ for each $p \geq 1$ and the proof is completed. ■

In general, we can also get

**Theorem 4.2** If $b, \sigma, h \in C_{b, lip}([0, T] \times \mathbb{R}^2)$, then there exists a unique continuous finite variation process $V_t \in L^p_G(\Omega_t)$ for each $p \geq 1$ such that

$$
X_t = \varphi(t, B_t, V_t),
$$

where $\varphi$ is given by equation (12). Moreover if $\sigma \in C_{b, lip}([0, T] \times \mathbb{R}^2)$, then for q.s. $\omega$, $V_t$ is the solution of ODE (2).
5 Comparison Theorem for G-SDEs

In the above sections, we establish the relations between G-SDEs and ODEs. From these results, we shall study the comparison theorem for G-SDEs. We refer to Lin [9] for some sufficient condition under which a comparison theorem for G-SDEs is also obtained by virtue of a stochastic calculus approach.

We begin with a lemma, which is from [5].

**Lemma 5.1** Assume that two functions \( f(t, x) \) and \( \tilde{f}(t, x) \) are defined on \( \mathbb{R}^2 \), satisfying the Carathéodory conditions, that is, they are measurable in \( t \), continuous in \( x \) and dominated by a locally integrable function \( m_t \) in \( \mathbb{R}^2 \). Let \( (t_0, x_0), (\tilde{t}_0, \tilde{x}_0) \) be two points in \( \mathbb{R}^2 \) such that \( x_0 \leq \tilde{x}_0 \). Moreover, \( x_t \) is the solution to the initial value problem

\[
\frac{dx_t}{dt} = f(t, x_t)dt, \quad x_{t_0} = x_0,
\]

and \( \tilde{x}_t \) is the maximal solution to the problem

\[
\frac{d\tilde{x}_t}{dt} = \tilde{f}(t, \tilde{x}_t)dt, \quad \tilde{x}_{\tilde{t}_0} = \tilde{x}_0.
\]

If the inequality

\[
(t - t_0)f(t, x) \leq (t - \tilde{t}_0)\tilde{f}(t, \tilde{x})
\]

holds a.e. in \( \mathbb{R}^2 \), then \( x(t) \leq \tilde{x}(t) \) for every \( t \) in the common interval of existence of the solutions \( x_t \) and \( \tilde{x}_t \).

Then we have the following comparison theorem.

**Theorem 5.2** Let \( b(t, x, y), h(t, x, y) \in C_{b,\text{lip}}([0, T] \times \mathbb{R}^2) \) and \( \sigma(t, x, y) \in C_{b,\text{lip}}^1([0, T] \times \mathbb{R}^2) \) be given. If there exists three functions \( \sigma, \tilde{f}, \tilde{g} \) satisfying the Carathéodory conditions and the inequalities

\[
x\sigma(t, x, y) \leq x\tilde{\sigma}(t, x, y), \quad 2G(f(t, x, y) - \tilde{f}(t, x, y)) + g(t, x, y) - \tilde{g}(t, x, y) \leq 0 \quad \text{in} \quad [0, T] \times \mathbb{R}^2.
\]

Then for the unique solution \( X_t \) of SDE \( \tilde{X}_t \)

\[
X_t \leq \tilde{\varphi}(t, B_t, \tilde{V}_t)
\]

holds for q.s. \( \omega \) and every \( t \) in the common interval where both sides are defined. Here \( \tilde{\varphi} \) and \( \tilde{V} \) are the maximal solutions to the problems

\[
\frac{d\tilde{\varphi}}{dx} = \tilde{\sigma}(t, x, \tilde{\varphi}), \quad \tilde{\varphi}(t, 0, v) = v \in \mathbb{R},
\]

and

\[
\frac{d\tilde{V}}{dt} = \tilde{g}(t, B_t, \tilde{V}_t)dt + \tilde{f}(t, B_t(\omega), \tilde{V}_t)d(B)_t, \quad \tilde{V}_0 = \tilde{X}_0
\]

with \( X_0 \leq \tilde{X}_0 \), respectively.

**Proof.** According to the Lemma 5.1, we get

\[
\varphi(t, x, v) \leq \tilde{\varphi}(t, x, \tilde{v})
\]

provided \( v \leq \tilde{v} \). From \[2\] or \[19\], \( d(B)_t = \hat{\alpha}_t(\omega)dt \), where \( \hat{\alpha} \) is well defined for each \( \omega \) and q.s. takes value in \([\omega^2, \sigma^2] \). Since \( 2G(f(t, x, v) - \tilde{f}(t, x, v)) + g(t, x, v) - \tilde{g}(t, x, v) \leq 0 \), we obtain

\[
(f(t, x, v) - \tilde{f}(t, x, v))\hat{\alpha}_t + g(t, x, v) - \tilde{g}(t, x, v) \leq 0.
\]

Then applying Lemma 5.1 again, we also have \( V_t \leq \tilde{V}_t \) in the common interval where both sides are defined, which is the desired result. 

Now we consider some examples of its applications.
Example 5.3 Consider two G-SDEs with the same diffusion coefficient $\sigma$:

$$
\begin{align*}
\begin{cases}
  dX_i^t = b^i(t, B_t, X_i^t) dt + h^i(t, B_t, X_i^t) dB_t + \sigma(t, B_t, X_i^t) dB_t, \\
  X_0^i = X_i^0,
\end{cases} & (i = 1, 2),
\end{align*}
$$

where $\sigma$, $b^1$, $b^2$, $h^1$, $h^2$ satisfy the conditions in Theorem 5.2 and $X_0^1 \leq X_0^2$, $b^1 - b^2 + 2(\sigma^1 - \sigma^2) \leq 0$.

Denote:

$$
g^i(t, x, v) = \partial_x \varphi^{-1}_i(t, x, v)(b^i(t, x, \varphi(t, x, v))) - \partial_t \varphi(t, x, v),
$$

$$
f^i(t, x, v) = \partial_x \varphi^{-1}_i(t, x, v)(h^i(t, x, \varphi(t, x, v))) - \frac{1}{2}(\partial_x \varphi + \partial_y \varphi \varphi)(t, x, \varphi(t, x, v)), \quad (i = 1, 2)
$$

One can easily show that

$$
g^1 - g^2 + 2G(f^1 - f^2) \leq 0.
$$

Applying Theorem 5.2, we obtain $X_1^t \leq X_2^t$ q.s.

Remark 5.4 In Example 5.3, we can also assume that $\sigma \in C_{b, lip}([0, T] \times \mathbb{R}^2)$. Indeed, applying Theorem 4.1, there exists a sequence $X^{i,n} \to X^i$ uniformly in $[0, T]$. Then we conclude $X_1^t \leq X_2^t$ from $X_1^{i,n} \leq X_2^{i,n}$ for each $t \in [0, T]$.

In particular, we obtain a necessary and sufficient condition for comparison theorem of 1-dimensional G-SDEs.

Theorem 5.5 Consider two G-SDEs:

$$
\begin{align*}
\begin{cases}
  dX_{i,x}^t = b^i(X_{i,x}^t) dt + h^i(X_{i,x}^t) dB_t + \sigma^i(X_{i,x}^t) dB_t, \\
  X_{0,x}^i = x_i,
\end{cases} & (i = 1, 2),
\end{align*}
$$

where $\sigma^i, b^i, h^i \in C_{b, lip}(\mathbb{R})$ and $i \in \{1, 2\}$, then for each $x_1 \leq x_2$, $X_1^{1,x} \leq X_2^{2,x}$ if and only if

$$
b^1(x) - b^2(x) + 2G(h^1(x) - h^2(x)) \leq 0, \quad \sigma^1(x) = \sigma^2(x), \quad \forall x \in \mathbb{R}.
$$

Proof. We shall only have to prove that from $X_1^{1,x} \leq X_2^{2,x}$ for each $x \in \mathbb{R}$, we infer that

$$
b^1(x) - b^2(x) + 2G(h^1(x) - h^2(x)) \leq 0, \quad \sigma^1(x) = \sigma^2(x).
$$

By $X_1^{1,x} \leq X_2^{2,x}$, we get

$$
\int_0^t b^1(X_{s,x}^1) ds + \int_0^t h^1(X_{s,x}^1) dB_s + \int_0^t \sigma^1(X_{s,x}^1) dB_s \\
\leq \int_0^t b^2(X_{s,x}^2) ds + \int_0^t h^2(X_{s,x}^2) dB_s + \int_0^t \sigma^2(X_{s,x}^2) dB_s.
$$

Set $\alpha_s^i = b^i(X_{s,x}^i) - b^i(x)$, $\beta_s^i = h^i(X_{s,x}^i) - h^i(x)$ and $\gamma_s^i = \sigma^i(X_{s,x}^i) - \sigma^i(x)$. From Peng [16], there exists some constant $C$ such that,

$$
\mathbb{E} \left[ \sup_{s \in [0, t]} (|\alpha_s^1|^2 + |\beta_s^1|^2 + |\gamma_s^1|^2) \right] \leq Ct.
$$

For each $t \in [0, T]$, we have

$$
(b^1(x) - b^2(x))t + (h^1(x) - h^2(x))\langle B \rangle_t + (\sigma^1(x) - \sigma^2(x))B_t \\
\leq \int_0^t (\alpha_s^2 - \alpha_s^1) ds + \int_0^t (\beta_s^2 - \beta_s^1) dB_s + \int_0^t (\gamma_s^2 - \gamma_s^1) dB_s.
$$

(10)
Applying BDG inequalities, we can find some constant $C$ so that
\[\mathbb{E}[|\int_0^t \gamma_s^2 dB_s|^2] \leq C \int_0^t \mathbb{E}[\gamma_s^4] ds \leq Ct^2.\]
Thus q.s.
\[
\lim_{t \downarrow 0} \frac{1}{\sqrt{t}} \int_0^t \gamma_s dB_s = 0.
\]
In a similar way we can also obtain q.s.
\[
\lim_{t \downarrow 0} \frac{1}{\sqrt{t}} \int_0^t \alpha_s^2 ds = 0, \quad \lim_{t \downarrow 0} \frac{1}{\sqrt{t}} \int_0^t \beta_s^2 dB_s = 0.
\]
Recalling that
\[c(\limsup_{t \downarrow 0} \frac{B_t}{\sqrt{t}} = +\infty) = 1, \quad c(\liminf_{t \downarrow 0} \frac{B_t}{\sqrt{t}} = -\infty) = 1,
\]
then there exists a subset $\Omega \subset \Omega$ with $c(\Omega_0) = 1$, such that for each $\omega \in \Omega_0$, we can find a sequence $(r_n := r_n(\omega))$ so that $\lim_{r_n \downarrow 0} \frac{t}{r_n} = +\infty$. By equation (10), we derive that
\[\langle \sigma^1(x) - \sigma^2(x) \rangle \lim_{r_n \downarrow 0} \frac{B_{r_n}}{\sqrt{r_n}} \leq 0,
\]
Consequently, $\sigma^1(x) \leq \sigma^2(x)$. Similarly we can prove $\sigma^1(x) \geq \sigma^2(x)$, then,
\[\sigma^1(x) = \sigma^2(x).
\]
Finally, taking expectation on both sides of equation (10) yields
\[b^1(x) - b^2(x) + 2G(h^1(x) - h^2(x)) \leq \lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}[\int_0^t (\alpha^2_s - \alpha^1_s) ds + \int_0^t (\beta^2_s - \beta^1_s) dB_s] = 0,
\]
which completes the proof. $\blacksquare$

**Remark 5.6** Let $\sigma^1 = \sigma^2 = b^1 = h^2 = 0, b^2 = \sigma^2, h^1 = 1$, one can show that $b^1(x) - b^2(x) + 2G(h^1(x) - h^2(x)) \leq 0$ and $X^1_{1:x} = x + \langle B \rangle_t \leq x + \sigma^2 t = X^2_{1:x}$ q.s.. Thus $b^1(x) - b^2(x) + 2G(h^1(x) - h^2(x)) \leq 0$ does not imply $b^1(x) \leq b^2(x)$ and $h^1(x) \leq h^2(x)$.

**Example 5.7** Consider two $G$-SDEs with different diffusion coefficients:
\[
\begin{cases}
    dX^i_t = \sigma_i(X^i_t) dB_t + \frac{1}{2} \sigma_i(X^i_t) \sigma'_i(X^i_t) dB_t, \\
    X^i_0 = X^i_0 \quad (i = 1, 2),
\end{cases}
\]
where $\sigma_i > 0$ and $\sigma_i \in C^{1}_{b, lip}(\mathbb{R})$. Consider the following initial value problems:
\[\frac{d\varphi_i}{dx} = \sigma_i(\varphi_i), \quad \varphi_i(0) = v_i \quad (i = 1, 2).
\]
Clearly, the solutions $\varphi_i(x, v_i)$ satisfy the equalities
\[
\int_{v_1}^{\varphi_1(x, v_1)} ds = x = \int_{v_2}^{\varphi_2(x, v_2)} \frac{ds}{\sigma_2(s)}.
\]
Note that \( b_i \equiv 0 \), we can obtain \( V_i \equiv X_0^0 \) by equation (1). Hence, if for every \( x \in \mathbb{R} \) the inequality

\[
\int_{X_0^0}^x \frac{dy}{\sigma_1(y)} \geq \int_{X_0^0}^x \frac{dy}{\sigma_2(y)}
\]

holds, then \( \varphi_1(x, X_0^1) \leq \varphi_2(x, X_0^2) \) and therefore q.s.

\[
X_t^1 = \varphi_1(B_t, X_0^1) \leq \varphi_2(B_t, X_0^2) = X_t^2.
\]

**Example 5.8** Consider the following G-SDE:

\[
\begin{cases}
dX_t = b(X_t)dt + h(X_t)d(B)_t + \sigma(X_t)dB_t, \\
X_0 = X_0,
\end{cases}
\]

where \( b, h \in C^{b, \text{lip}}(\mathbb{R}) \) and \( \sigma \in C^{1, \text{lip}}(\mathbb{R}) \). Then we have for some constant \( C \)

\[
|\sigma(x)| \leq C, \quad |g(x, v)| \leq C|v|, \quad |f(x, v)| \leq C|v|.
\]

Let \( \hat{\sigma}(x) = C \text{sgn}(x) \), \( \hat{g}(x, v) = C|v| \) and \( \hat{f}(x, v) = C|v| \), combining these three inequalities and using Theorem 5.2, we obtain an asymptotic estimation for the paths of G-Itô diffusion process \( X_t \), for q.s. \( \omega \),

\[
X_t \leq C|B_t| + C \int_0^t e^{C|B_s|}ds + X_0.
\]

A symmetric argument shows that, for q.s. \( \omega \),

\[
X_t \geq -C|B_t| - C \int_0^t e^{C|B_s|}ds + X_0.
\]

### 6 Appendix

G-Itô formula for a G-Itô process was obtained by Peng \[10\] and improved by Gao \[3\], Zhang et al \[20\] in \( L^p_C(\Omega) \). Li and Peng \[10\] significantly improved the previous ones for a general \( C^{1,2} \)-function in a larger space \( L^p_C(\Omega) \) instead of \( L^p_C(\Omega) \). For reader’s convenience, we give the following G-Itô formula. Indeed, it can be viewed as a special case of Theorem 2.33 of Lin \[12\].

For each \( 0 \leq t \leq T \), consider a G-Itô process:

\[
X_t = X_0 + \int_0^t f_udu + \int_0^t h_ud(B)_u + \int_0^t g_udu.
\]

**Theorem 6.1** Suppose \( \varphi \in C([0, T] \times \mathbb{R}^2) \) satisfies for each \( t_1, t_2 \in [0, T], \ x_1, x_2, v_1, v_2 \in \mathbb{R}, \)

\[
|\psi(t_1, x_1, v_1) - \psi(t_2, x_2, v_2)| \leq C(1 + |x_1| + |x_2|)e^{C(|x_1| + |x_2|)}(|t_1 - t_2| + |x_1 - x_2| + |v_1 - v_2|),
\]

where \( \psi = \partial_t \varphi, \partial_x \varphi, \partial^2_{xx} \varphi \) and \( \partial_v \varphi \). Let \( f, h \) and \( g \) be bounded processes in \( M^2_C(0, T) \). If for each \( p \geq 1 \), the continuous finite variation process \( V_t \in L^p_C(\Omega_t) \) and there exists some constant \( C_p \), such that for each \( s \leq t \in [0, T] \):

\[
\mathbb{E}[|T^V_t - T^V_s|^p] \leq C_p|t - s|^p,
\]
where $T^V$ is the total variation process of $V$. Then in $L^2_b(\Omega_t)$,

$$
\varphi(t, X_t, V_t) = \varphi(0, X_0, V_0) + \int_0^t \partial_u \varphi(u, X_u, V_u) f_u du + \int_0^t \partial_x \varphi(u, X_u, V_u) g_u d\langle B \rangle_u + \int_0^t \partial_v \varphi(u, X_u, V_u) h_u dB_u + \frac{1}{2} \int_0^t \partial_{xx} \varphi(u, X_u, V_u) g^2_u d\langle B \rangle_u.
$$

**Proof.** The proof is immediate in light of Lemma 3.3, Theorem 5.4 of Li and Peng [10], and Theorem 2.33 of Lin [12].

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