On intertwining implies conjugacy for classical groups

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Abstract

Let $G$ be a unitary group of an $\epsilon$-hermitian form $h$ given over a non-Archimedean local field $k$ of residue characteristic not two. Let $V$ be the vector space on which $h$ is defined. We consider minimal skew-strata, more precisely pairs $(\beta, a)$ consisting of a Lie algebra element $\beta$ and a hereditary order $a$ stable under the adjoint involution of $h$, such that $\beta$ generates a field whose multiplicative group is a subset of the normalizer of $a$, and some more conditions, see [BK93]. We prove that if two minimal skew-strata $(\beta_i, a_i), i = 1, 2$ interwine by an element of $G$, i.e.

$$g(\beta_1 + a_{\nu}(\beta_1))g^{-1} \cap (\beta_2 + a_{\nu}(\beta_2)) \neq \emptyset,$$

for some $g \in G$, then they are conjugate under $G$, i.e. there is a $g \in G$ such that

$$g(\beta_1 + a_{\nu}(\beta_1))g^{-1} = \beta_2 + a_{\nu}(\beta_2).$$

1 Introduction

For this introduction let $k$ be a non-Archimedean local field of residue characteristic not two. In the field local representation theory of classical groups on complex vector spaces many researches, e.g. C. Bushnell, P. Kutzko, V. Secherre, S. Stevens and P. Broussous, to mention some of them, have made big progress in classifying supercuspidal representations. For example the cases of $\text{GL}_m(D)$ where $D$ is a central finite skew-field over $k$ has been studied completely. The case of a unitary group $U(h)$ of a signed hermitian form

$$h : V \times V \rightarrow k$$
is also not far of being completely studied as Shaun Stevens anounced recently. What is missing is to understand how far two simple types are related if both represent a given supercuspidal representation of $U(h)$. In the case of $GL_m(D)$ the authors mentioned above and in addition M. Grabitz have considered rigid objects which they call simple strata. Essentially for $k = D$ a simple stratum is a coset $\beta + a_{\nu(a(\beta))}$ consisting of a hereditary order $a$ of $End_k(V)$, which has a filtration $a_\nu$ of $a$-lattices in $End_k(V)$, and an element $\beta \in End_k(V)$ generating a field over $k$ whose non-zero elements normalize $a$ and some more conditions which we skip because of clearness reasons. In [BK93] C. Bushnell and P. Kutzko proved that two simple strata $\beta_i + a_{\nu(a(\beta_i))}$, $i = 1, 2$, of $End_k(V)$ which intertwine under $\text{Aut}_k(V)$, i.e.

$$g\beta_1 + a_{\nu(a(\beta_1))}g^{-1} \cap (\beta_2 + a_{\nu(a(\beta_2))}) \neq \emptyset,$$

for some $g \in \text{Aut}_k(V)$, are conjugate under $\text{Aut}_k(V)$. P. Broussous and M. Grabinetz proved it for the case of $GL_m(D)$ under the assumption that both strata have the same embedding type [BG00]. For the case of the group $U(h)$ such a statement is still missing. Here we do further assume the simple strata to be skew, i.e. that $\beta$ is skew-symmetric and $a$ is stable under the action of the adjoint involution $\sigma$ of $h$. We recall that the involution $\sigma|_k$ is part of the data given by $h$ by the definition of a signed hermitian form. S. Stevens anounced a reduction to all simple strata if one can solve the case of a minimal simple skew-stratum. This article is devoted to the latter, more precisely we prove that two minimal simple skew-strata with the same hereditary order are conjugate under $U(h)$ if they intertwine under $U(h)$.

Let us be more precisely to explain the steps for the proof. Let $(\beta_1, a)$ and $(\beta_2, a)$ be two intertwining minimal skew-strata and let $E_i$ be the field generated by $\beta_i$ and $E'_i$ its maximal tamely ramified subextension. Let us denote $U(h) \cap a^\times$ by $U(a)$. We have two main steps:

1. Firstly we show that there is an element $g \in U(a)$ such that $gE'_1g^{-1}$ is equal to $E'_2$. This step uses that the strata intertwne. See Proposition 4.

2. Step 1 allows to assume that $E_1|k$ and $E_2|k$ are purely wildly ramified. And we show in section 4.3 that the strata are conjugate.

Step 1: For the sake of simplicity we assume $E_i = E'_i$. The minimality of $\beta_i$, $i = 1, 2$, and the intertwining of the strata imply that the field extensions $E'_1|k$ and $E'_2|k$ are $\sigma$-equivariantly $k$-algebra isomorphic, see Proposition 2. Thus we have two structures on $V$ as an $E_1$-vector space. It is a purely functorial property under a fixed non-zero $\sigma$-equivariant $k$-linear map $\lambda$ from
$E_1$ to $k$, that one can find unique signed hermitian forms $\tilde{h}_i$, $i = 1, 2$, with respect to the $E_1$-structures on $V$ and $(E_1, \sigma|_{E_1})$ such that

$$\lambda \circ \tilde{h}_i = h, \ i = 1, 2.$$

To prove the assertion of the first step it is enough to show that the hermitian spaces $(V, \tilde{h}_1)$ and $(V, \tilde{h}_2)$ are isomorphic, see Proposition 1. The motivation of the latter Proposition is taken from [BH96]. The signed hermitian spaces above are isomorphic if and only if the signed hermitian forms $\tilde{h}_1(\ast, \beta_1\ast)$ and $\tilde{h}_2(\ast, \beta_2\ast)$ denoted by $\tilde{h}_{\beta_1}$ and $\tilde{h}_{\beta_2}$ give $E_1$-isomorphic hermitian spaces. To establish the last isomorphism we consider sequences of sesqui-linear forms.

This is an abuse of notation, more precisely the self-dual lattice chain $L$ corresponding to $a$ defines two sequences

$$\tilde{h}_{\beta_i}^{j,L} : L_{j+1}^{\#h_{\beta_i}/L_j^{\#h_{\beta_i}}} \times L_j/L_{j-1} \to \kappa_{E_1},$$

for $i = 1, 2$, and $j \in \mathbb{Z}$. Here $M^{\#h_{\beta_i}}$ denotes the dual of an $o_{E_1}$-lattice $M$ in $V$, i.e.

$$M^{\#h_{\beta_i}} := \{v \in V | \tilde{h}_{\beta_i}(v, M) \subseteq p_{E_1}\}.$$

The intertwining implies precisely that the sesqui-linear forms $\tilde{h}_{\beta_1}^{j,L}$ and $\tilde{h}_{\beta_2}^{j,L}$ are in a simultaneous way $\kappa_{E}$-isomorphic. Comparing Gram matrices of $\tilde{h}_{\beta_1}$ and $\tilde{h}_{\beta_2}$ we get in the proof of Proposition 3 that the hermitian spaces are $E_1$-isomorphic.

**Step 2:** The whole section 4.3 is devoted to that step. Here let us assume that $E_i|k$ is purely wildly ramified. The fields $E_i$ need not to be isomorphic, and thus the strategy of step 1 is not working. We consider both fields as valued $k$-vector spaces and we construct a continuous linear and $\sigma$-equivariant isomorphism $\phi$ from $E_1$ to $E_2$ and a map $\lambda'$ from $E_2$ to $k$ such that

$$(\lambda' \circ \phi)(xy) = \lambda'(\phi(x)\phi(y)),$$

for all $x, y \in E_1$. After normalizing $\lambda'$ and $\lambda' \circ \phi$ to $\lambda_1$ and $\lambda_2$ we consider the lifts $h_i$. The sequences of sesqui-linear forms $(\tilde{h}_1^{j,L})_j$ and $(\tilde{h}_2^{j,L})_j$ are equal because the field extensions $E_i|k$ are purely ramified. This allows to consider Gram matrices with entries in $k$ and the construction of an element of $U(a)$ which conjugates the stratum $\beta_1)$ to $\beta_2(a)$.

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3
2 Notation

Let $k$ be a non-Archimedean local field of residue characteristic not two. We use usual notation, i.e. $o_k$, $p_k$, $\kappa_k$, $\nu$ for the valuation ring, the valuation ideal, the residue field and the valuation. We also adapt the above notation for all other fields $E$, but the valuation $\nu_E$ is assumed to be normalized. Let $h$ be an $\epsilon$-hermitian form on a finite dimensional $k$-vector $V$ corresponding to an involution $\rho$ on $k$, and let $G = U(h)$ be the unitary group of $h$. The set of fixed points of $\rho$ in $k$ is denoted by $k_0$. The adjoint involution of $h$ is denoted by $\sigma$. Let $[\beta_i, n-1, n, a]$, $i = 1, 2$, be two simple skew-strata with minimal elements $\beta_i$, i.e.

1. $a$ is a hereditary order of $\text{End}_k(V)$ and $\beta_i$ generates a field $E_i := k[\beta_i]$ such that $E_i^\times$ is a subset of the normalizer $a$ in $\text{End}_k(V)$,

2. $\nu_a(\beta_i) = n$ and $n$ is prime to the ramification index $e(E_i|k)$ where $\nu_a$ is defined via

\[
\nu_a(\gamma) := \sup\{l \in \mathbb{Z} \mid \gamma \in a_i\}, \; \gamma \in \text{End}_k(V).
\]

3. $\beta_i^{e(E_i|k)} \pi \omega + p_{E_i}$ generates the residue field extension $\kappa_{E_i}|\kappa_k$ for $\omega := \nu_{E_i}(\beta_i)$.

We denote $k[\beta_i]$ by $E_i$.

Remark 1 The lattice sequence $(L_i)_i$ in $V$ associated to the hereditary order $a$ defines again a lattice sequence $(a_i)_i$ in $\text{End}_k(V)$, via

\[
a_i := \{a \in \text{End}_k(V) \mid aL_i \subseteq L_{i+l}, \; \forall l \in \mathbb{Z}\}.
\]

We refer to [BL02] for more details about lattice chains.

The goal of this article is the following theorem.

Theorem 1 If both strata are intertwining over $G$, i.e.

\[
\exists \; g \in G : \; (\beta_1 + a_{1-n})^g \cap (\beta_1 + a_{1-n}) \neq \emptyset,
\]

then there are conjugate under $G$, i.e. there is a $g' \in G$ such that

\[
g'(\beta_1 + a_{1-n})g'^{-1} = (\beta_i + a_{1-n}).
\]

From now on we assume that the both given strata interwine.
3 Conjugacy of field extensions

Let \( E \) be a field extension of \( k \). For this section let us assume that there are two \( k \)-algebra isomorphisms

\[
\phi_i : (E, \sigma') \to (E_i, \sigma|_{E_i}), \quad i = 1, 2
\]

which are \( \sigma' \)-\( \sigma \)-equivariant and let \( E_0 \) be the set of \( \sigma' \)-fixed points of \( E \). In the manner of Broussous and Stevens given in [BS09] we fix a non-zero \( k \)-linear \( \sigma' \)-\( \sigma \)-equivariant map \( \lambda \) from \( E \) to \( k \) such that

\[
p_{E_0} = \{ e \in E_0 \mid \lambda(eo_{E_0}) \subseteq p_k \}. \quad (1)
\]

We attach to \( \phi_i \) an \( \epsilon \)-hermitian form \( \tilde{h}_i : V \times V \to E \) with respect to \( \sigma' \) such that

\[
h = \lambda \circ \tilde{h}_i. \quad (2)
\]

For the proof see [BS09]. The \( \epsilon \)-hermitian forms \( \tilde{h}_i \) differ because we have different \( E \)-actions on \( V \).

**Proposition 1** If \( (V, \tilde{h}_1) \) is isomorphic to \( (V, \tilde{h}_2) \) as hermitian \( E \)-spaces. Then there is an element \( g \) of \( U(a) \) such that

\[
\phi^g_1(x) := g\phi_1(x)g^{-1} = \phi_2(x),
\]

for all \( x \in E \).

We follow the proof given in [BH96] 1.6. Given a lattice chain \( L \) the sequence of natural numbers

\[
d_i(L) := \dim_{k_D}(L_i/L_{i+1})
\]

is called the **invariant** of \( L \). Analogously we define \( d_i \) for lattice sequences and lattice functions. We recall that \( h \) defines the notion of the dual of a lattice. More precisely if \( M \) is an \( o_k \)-lattice in \( V \) the dual of \( M \) with respect to \( h \) is defined to be

\[
M^\# := \{ v \in V \mid h(v, M) \subseteq p_k \}.
\]

We call a lattice chain \( L \) on \( V \) self-dual with respect to \( h \) if \( L \) equals either \( (L^\#)_{i \in \mathbb{Z}} \) (type I) or \( (L^\#_{i+1})_{i \in \mathbb{Z}} \) (type II). In particular we have

\[
L^\#_0 = L_0 \ (\text{I}) \quad \text{or} \quad L^\#_1 = L_0 \ (\text{II}).
\]
Lemma 1  If two self-dual lattice chains $L$ and $L'$ on $V$ have the same type and the same invariants, then there is an element $g$ of $G$ such that the lattice chains $gL$ and $L'$ equal.

Proof: There is a Witt decomposition $\{W_i \mid i \in I\}$ of $V$ with respect to $h$ which splits both lattice chains. W.l.o.g. we can assume that the anisotropic part $W_0$ of the Witt decomposition is trivial, because $L$ and $L'$ equal on $W_0$ by [BT87, 2.9]. Let $r$ be the period of $L$. We choose a decomposition of $I$ into two disjoint sets $I^+$ and $I^-$ such that

$$\sigma(I^+) = I^-.$$ 

Further we define

$$W^+ := \oplus_{i \in I^+} W_i, \quad W^- := \oplus_{i \in I^-} W_i$$

and

$$L^+ := L \cap W^+, \quad L^- := L \cap W^-.$$ 

Let $\mu(L, j)$ be the set of indexes $i \in I$ for which $W_i \cap L_j$ differs from $W_i \cap L_{j+1}$.

Analogously we define $\mu(L^+, j)$ and $\mu(L^-, j)$, but one of them can be empty.

Case 1: We assume that $L$ is of type (I) and $r$ is even. We choose, for $0 \leq j < \frac{r}{2}$, injective maps

$$\phi_j^+: \mu(L^+, j) \rightarrow \mu(L', j), \quad \phi_j^-: \mu(L^-, j) \rightarrow \mu(L', j) \setminus \text{im}(\phi_j^+).$$ 

Such a choice is possible because $d_j(L)$ equals $d_j(L')$. We define

$$I^+ := \bigcup_{0 \leq j < \frac{r}{2}} (\text{im}(\phi_j^+) \cup \sigma(\text{im}(\phi_j^-))),$$

and we put $I^-$ to be the complement of $I^+$ in $I$. Because of

$$i \in \mu(L', j) \text{ if and only if } \sigma(i) \in \mu(L', -j - 1), \quad (3)$$

for all $i \in I$, we have that $I^+ \cap \sigma(I^+)$ is empty, and by symmetry $I^- \cap \sigma(I^-)$ is empty too. Thus

$$\sigma(I^+) = I^-.$$ 

This new decomposition of $I$ defines

$$W^+ := \oplus_{i \in I^+} W_i, \quad W^- := \oplus_{i \in I^-} W_i, \quad L'^+ := L' \cap W^+, \quad L'^- := L' \cap W^-.$$ 

By construction $L'^+$ and $L^+$ are lattice sequences with the same invariants, and there is an isomorphism $u$ of $k$-vector spaces from $W^+$ to $W'^+$ such that $uL'^+$ equals $L'^+$. The map

$$g := (u, 0) + \sigma((u^{-1}, 0)) : W^+ \oplus W^- \rightarrow W'^+ \oplus W'^-$$
is an element of $G$ and $gL$ equals $L'$.

Case 2: The type of $L$ is (I) and $r$ is odd. We can construct $W'^+$ and $W'^-$ as in case 1, but the only thing we have to change is the definition of $\phi^+_{r-1}$. Because of (3) the set $\mu(L', \frac{r-1}{2})$ is invariant under the action of $\sigma$, i.e. since $L$ and $L'$ have the same invariants we can choose $\phi^+_{r-1}$ such that
$$\sigma(\text{im}(\phi^+_{r-1})) \cap \text{im}(\phi^+_{r-1}) = \emptyset.$$ We now conclude as in case 1.

Case 3: The type of $L$ is (II). Different to the cases before we have $i \in \mu(L')$ if and only if $\sigma(i) \in \mu(L', -j)$, (4)
We follow the proof of the cases 1 and 2, but with the following differences:

1. We consider $0 \leq j \leq \frac{r}{2}$, i.e. if $r$ is even the index $\frac{r}{2}$ is considered too in all formulas.

2. The set $\mu(L', 0)$ is $\sigma$-equivariant.

3. If $r$ is even the set $\mu(L', \frac{r}{2})$ is $\sigma$-equivariant.

For the $\sigma$-equivariant sets we apply the procedure of case 2 for the choice of the map $\phi^+_j$. After these preparations we conclude as in case one to finish the proof. q.e.d.

Proof: [of proposition 1] We only need to consider the self-dual lattice chain $L$ whose hereditary order is $a$.

As in [BH96, 1.6], we consider $V$ as a hermitian $E$-vector space $V_i$ via $\phi_i$ and $\tilde{h}_i$. By [BS09, 5.5] we have
$$L_i^{\#h} = L_i^{\#\tilde{h}_1} = L_i^{\#\tilde{h}_2},$$
for all integers $i$. Thus $L$ seen as an $o_E$-lattice chain in $V_1$ has the same type and the same invariants as it has in $V_2$. Thus, since $V_1$ and $V_2$ are isomorphic hermitian spaces and because of Lemma 1 there is an isomorphism $u$ of hermitian spaces from $V_1$ to $V_2$ such that
$$u(L_i) = L_i$$
for all integers $i$. By (2) the map $u$ is an element of $G$, and being $E$-linear implies
$$\phi^+_1 = \phi^+_2.$$ q.e.d.
4 Intertwining implies conjugacy for simple stata

We do not consider at all simple strata. Shaun Stevens proposed a reduction to the case given in the notation section. Recall that the intertwining of both strata implies

1. that the elements \( \beta_i^{E_i|k}, \pi \omega + p_{E_i}, i = 1, 2 \), have the same characteristic and the same minimal polynomial over \( \kappa_k \).

2. The field extensions \( E_i|k, i = 1, 2 \), have the same ramification index and the same inertia degree.

4.1 Decent

We consider the maximal tamely ramified subextension \( E'_i \) and the maximal unramified subextension \( E''_i \) of \( E_i|k \). We want to find proper elements \( \beta'_i \) and \( \beta''_i \) generating \( E'_i \) and \( E''_i \) respectively, such that the stata \( (\beta'_i, a) \) are still simple with minimal elements. Moreover we show that the fields \( E'_1 \) and \( E'_2 \) are \( \sigma \)-equivariantly isomorphic.

Notation 1

\[ e' := [E'_1 : E''_1], \quad p' := [E_i : E'_i], \quad e := p'e' \]

Lemma 2 We can choose \( \pi \) in a way such that

1. \( \sigma(\pi) \in \{\pi, -\pi\} \) and

2. \( \pi \) has an \( \epsilon' \)th root in \( E'_1 \).

**Proof:** Indeed if \( k|k_0 \) is unramified then \( E'_1|(E'_1)_0 \) is too and there is an \( \epsilon' \)th root in \( (E'_1)_0 \) of some \( \pi' \) in \( k \). If \( k|k_0 \) is ramified then there is a skew-symmetric prime-element \( \pi' \) in \( k \) and there is unit \( u \) of \( \alpha_k \) such that \( u\pi' \) has an \( \epsilon' \)th root in \( E'_1 \). To be ramified implies that there is a unit \( u_0 \) of \( \alpha_k \) such that \( \frac{u_0}{u} \) is one in \( \kappa_k \), i.e. has an \( \epsilon' \)th root by Hensel’s lemma. Thus \( u_0\pi' \) fulfills the above conditions. q.e.d.

We take an element \( \gamma_i \) of \( \mathcal{O}_{E''_1} \) representing \( \beta_i^{E''_1} \), such that \( \sigma(\gamma_i) \in \{\gamma_i, -\gamma_i\} \). We define the element \( \beta''_i \) of \( E''_1 \) to be \( \gamma_i\pi^{-\omega} \).

Lemma 3 The strata \( (\beta''_1, a) \) and \( (\beta''_2, a) \) are simple strata with minimal elements, such that

\[ \sigma(\beta_i'') \in \{\beta''_1, -\beta''_2\} \]

They intertwine and there is a \( \sigma \)-equivariant field isomorphism from \( E''_1 \) to \( E''_2 \) which sends \( \beta'_1 \) to an element congruent to \( \beta''_2 \).
Proof: The intertwining follows from taking \( e \)th powers and using the intertwining of \((\beta_i, a), \ i = 1, 2\). Let \( \mu_i \) be the minimal polynomial of \( \beta_i'' \) over \( k \). Then \( Q_i := \pi^{\deg(\mu_i)} \mu_i(X\pi^\omega) \) is the minimal polynomial of \( \beta_i''\pi^\omega \). By intertwining the reductions of \( Q_i \) to \( \kappa_k \) equal. Thus there is a skew-symmetric root \( \beta_i'' \in E_i'' \) of \( \mu_i \) which is congruent to \( \beta_2'' \). We map \( \beta_1'' \) to \( \beta_2'' \), q.e.d.

Note by the above lemma that the element \( \pi \) has an \( e' \)th root in \( E_2'' \). Thus the polynomial \( P_i := X^{e'} - \gamma_i\pi^{-\omega} \) has a skew-symmetric root \( \beta_i' \in E_1' \) congruent to \( \beta_i'' \), \( i = 1, 2 \). Then \( (\beta_i', a) \) are intertwining simple strata with minimal elements. The element \( \beta_i' \) generates \( E_1' \) because \( e' \) is prime to \( \omega = -\nu E_1'(\beta_i') \), i.e. \( P_i \) is the minimal polynomial of \( \beta_i' \).

Proposition 2 The field \( E_1' \) is \( \sigma \)-equivariantly \( k \)-algebras isomorphic to \( E_2' \), such that \( \beta_1' \) is mapped to a skew-symmetric element of \( E_2' \) congruent to \( \beta_2' \).

Proof: Because of Lemma 3 we can assume that \( E_1'' = E_2'' =: E'' \) and \( \beta_1'' \) is congruent to \( \beta_2'' \), i.e. we consider \( E_1' \) embedded in an algebraic closure of \( E_2 \) via an extension of the isomorphism of Lemma 3. Let \( \pi_1 \) be an \( e' \)th root of \( \pi \) in \( E_1' \). Now we apply Hensel’s lemma. Since \( \beta_1'' \pi_1^{-\omega} \) is a root of

\[
Q := X^{e'} - \gamma_1, 
\]

the polynomial must have a root of the form \( \tilde{\beta}_2' \pi_2'' \) in \( E_2' \), which is a lift of the residue class of \( \beta_1'' \pi_1^{-\omega} \).

The polynomial \( Q \) has the property that if two roots are congruent then they equal. Thus \( \sigma(P_1) \) also has this property. That implies that \( \tilde{\beta}_2' \) is skew-symmetric. We map \( \beta_1' \) to \( \tilde{\beta}_2' \), i.e. we can assume \( E_1' = E_2' = E' \). The fact

\[
\left( \frac{-\beta_1'}{-\beta_2'} \right)^{e'} - 1 \in p_{E_1'} 
\]

implies that there is an \( e' \)th root \( \zeta \) of unity in \( E'' \) congruent to \( \frac{-\beta_1'}{-\beta_2'} \). The elements \( \zeta \) and \( \sigma(\zeta) \) are congruent and the separability of

\[
X^{e'} - 1
\]

over \( \kappa_{E''} \) implies that \( \zeta \) is symmetric. We map \( \beta_1' \) to \( \beta_2' \zeta \). This proves the proposition, q.e.d.

Notation 2 To summarize: We have given

\[
\beta_1, \beta_2, \beta_1', \beta_2', \beta_1'', \beta_2'', E_1, E_2, E_1', E_2', E_1'', E_2''.
\]
4.2 Hermitian forms

We attach to the $\varepsilon$-hermitian form $h$ and the to a corresponding selfdual lattice chain $L$ a sequence of sesquilinear forms. For that we abbreviate $(L_i)_{-k} := L_{i-k}$. The map $h$ induces maps

$$h^{j,L} : ((L_j)_{-1}/(L_j)\#) \times (L_j/L_{j+1}) \to \kappa_k.$$  

**Definition 1** Let $(E, \sigma_E)|k$ be a finite field extension with an involution $\sigma'$ on $E$ which is an extension of $\sigma$. We also assume that we have given a $\sigma_E$-equivariant $k$-linear non-zero map

$$\lambda : E \to k,$$

such that $\lambda$ satisfies (1). Let $V'$ and $V''$ be two $\varepsilon$-hermitian forms on finite dimensional $k$-vector spaces $V'$ and $V''$ respectively, such that $(E, \sigma')|k$ can be embedded into $(\text{End}_k(V'), \sigma_{V'})$ and into $(\text{End}_k(V''), \sigma_{V''})$. We fix for each pair one embedding. Let $L'$ and $L''$ be two self-dual $\sigma_E$-lattice chains in $V'$ and $V''$ respectively both of period $r$. The sequences $(h^{j,L'}_j)_{j}$ and $(h^{j,L''}_j)_{j}$ are said to be $E|k$-isomorphic if there is a $k$-linear isomorphism $f$ from $V'$ to $V''$ such that

1. $f(L'_i) = L''_i$ and $f((L'_i)\#) = (L''_i)\#$ for all $i \in \mathbb{Z}$ and
2. for all $x \in E$ and all integers $j$ the $\kappa_k$-linear map

$$\bar{f}_j : (L'_j / L'_{j+1}) \to (L''_j / L''_{j+1})$$

satisfies

$$\bar{f}_j(xv') = x\bar{f}(v') \in (L''_{j+r_E(x)} / L''_{j+1+r_E(x)}),$$

in particular $\bar{f}_j$ is $\kappa_E$-linear,
3. and such that for all $j$ we have

$$h^{j,L''} \circ (\bar{f}_j^{-1} \times \bar{f}_j) = h^{j,L'},$$

if $(L'_j)\# = L''_j$.

**Proposition 3** Under the notation of Definition 1 assume that $(h^{j,L'}_j)_{j}$ and $(h^{j,L''}_j)_{j}$ are $E|k$- isomorphic. Then the lifts $(V', h')$ and $(V'', h'')$ of $(V', h')$ and $(V'', h'')$ to $E$ are isomorphic $\varepsilon$-hermitian spaces over $E$.

To prove the above proposition we analize how $\hat{h'}$ depends on $h^{j,L}$. 

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Definition 2 Let \( A \) be a subset of \( k \). A Witt basis of \( V \) with respect to \( h \) with values in \( A \) adapted to \( L \) is a basis \( (v_i)_{i \in I_0 \cup I_+ \cup I_-} \) such that there are bijections
\[-: I_+ \to I_- , - : I_- \to I_+ , - \circ - = \text{id},\]
such that
1. \( h(v_i, v_{-i}) = 1 \), \( i \in I_+ \), and \( h(v_i, v_j) = 0 \) for all \( i, j \in I_+ \cup I_- \) with \( i \neq -j \),
2. \( h(v_i, v_j) = 0 \) and \( h(v_l, v_l) \in A \) for all \( i, l \in I_0 \) and all \( j \in I_0 \cup I_+ \cup I_- \) with \( i \neq j \),
3. The vector space spanned by the \( v_i \) with \( i \in I_0 \) is anisotropic, and
4. \( kv_i \cap L_0 = o_k v_i \) for all \( i \in I_+ \) and \( (v_i)_i \) is a splitting basis for \( L \).

Remark 2 If in the above definition \( A \) only consists of elements of valuation \( 0 \) or \( 1 \) then \( kv_i \cap L_0 = o_k v_i \) for all \( i \in I_0 \).

Proof: [of Prop. 3] We fix a non-square unit \( z \) of \( o_E \) and a prime element \( \pi_E \) of \( E \) such that \( \pi_E \) is symmetric or skew-symmetric. We take a Witt basis \( (v'_i)_{i \in I_0 \cup I_+ \cup I_-} \) of \( V' \) with respect to \( \tilde{h}' \{1, z, \pi_1, z\pi_1\} \) adapted to \( L \). Let \( r \) be the period of the lattice chain \( L \) over \( E \). The type and the period of \( L \) depend only on \( E \). The map \( \lambda \) can be reduced to a map
\[\lambda : \kappa_E \to \kappa_k,\]
because of (1), more precisely we have
1. \( \lambda(o_E) \subset o_k \) and \( \lambda(p_E) \subset p_k \) and
2. \( \tilde{\lambda} \) is non-zero equivariant and \( \kappa_k \)-linear.

One proves that in considering the cases \( E = E_0 \), \( E \neq E_0 \) and \( E|E_0 \) is ramified or unramified. By uniqueness of the lift from \( \kappa_k \) to \( \kappa_E \) the lift of \( (h^{j,L})_j \) to \( \kappa_E \) equals \( (\tilde{h}'^{j,L})_j \) and the same is true for \( h'' \).

We only have to prove that the entries of the anisotropic part of \( \tilde{h}' \) are detected by \( (\tilde{h}'^{j,L})_j \). Analogous to the proof of Lemma 1 we define \( \mu(L,j) \) to be the set of all indexes \( i \) such that \( Ev'_i \cap L_j \) is different from \( Ev'_i \cap L_{j+1} \), i.e. we have the identity
\[L_j/L_{j+1} = \bigoplus_{i \in (I_0 \cup I_+ \cup I_-) \cap \mu(L,j)} (Ev'_i \cap L_j)/(Ev'_i \cap L_{j+1}) \quad (5)\]
Let $i$ be an element of $I_0$. We want to analyze for which $j$ it is possible that $i \in \mu(L, j)$. Say that $i \in \mu(L, j)$ and $0 \leq j \leq r - 1$. Then
\[ 0 \leq j \leq \frac{r}{2}. \]
Let $j'$ be the index such that $L'_j = L_{j'}$. Then $r$ is a divisor of $(j' - 1 - j)$, i.e. we only have the possibilities
1. $j = 0 = j' - 1$, or
2. $j - j' + 1 = r$.

- Case 1: $L$ is of type (I) and $r$ is even: Then $j' = -j$ and thus $I_0$ is empty.
- Case 2: $L$ is of type (I) and $r$ is odd: Then $j' = -j$ and $j = \frac{r-1}{2}$ and $\tilde{h}_1(v_i, v_i)$ equals $\pi_1$ or $z\pi_1$. The form $\tilde{h}_1^\epsilon_{\pi_1, L}$ induces a signed hermitian form on $L_j/L_{j+1}$ via
\[ (\bar{v}, \bar{w}) \in (L_j/L_{j+1}) \times (L_j/L_{j+1}) \mapsto \tilde{h}_1^\epsilon_{\pi_1, L}(\pi_{E}^{-1}v, \bar{w}). \]
As an abuse of notation we still denote $\tilde{h}_1^\epsilon_{\pi_1, L}$ as an $\epsilon$-hermitian form. The direct sum over $I_0$ in (5) gives the anisotropic part of $\tilde{h}_1^\epsilon_{\pi_1, L}$ and one gets the Gram matrix for the vectors $(v'_l)_{l \in I_0}$ after multiplying Gram matrix of the anisotropic part of $\tilde{h}'$ with $\pi_{E}^{-1}$ and taking residues modulo $p_E$. Thus we recover the Gram matrix of $\tilde{h}'$ from a Gram matrix of $\tilde{h}_1^\epsilon_{\pi_1, L}$.

- Case 3: $L$ is of type (II) and $r$ is odd: Then $j' - 1 = j = 0$ and $\tilde{h}_1(v'_i, v'_i)$ equals $1$ or $z$. We now argue analogously to Case 2.
- Case 4: $L$ is of type (II) and $r$ is even: Here we have a mixture of Case 2 and Case 3. The form $\tilde{h}_1^0_{0, L}$ detects the entries $1$ and $z$, and $\tilde{h}_1^{\pm, L}$ detects the entries $\pi_E$ and $z\pi_E$.

Now we could have assumed that each entry $z$ and $\pi_1z$ accurs at most one time. Then by the argument above we can deduce from $(\tilde{h}_1^{\epsilon, L})_j$ the size and the entries of the anisotropic part of the given Gram matrix of $\tilde{h}'$. This proves Proposition 3. q.e.d.
Definition 3 Let $\gamma$ be a skew-symmetric element of $\text{Aut}_k(V)$. We define the signed hermitian form

$$h_\gamma : V \times V \rightarrow k$$

via

$$h_\gamma(v, w) := \tilde{h}_i(v, \gamma w), \ v, w \in V.$$  

Corollary 1 Let $\gamma$ be a skew-symmetric element such that

$$\gamma + a_1 + \nu E_1(\beta_1) = \beta_1 + a_1 + \nu E_1(\beta_1),$$

then the lifts of $(V, h_\gamma)$ and $(V, h_{\beta_1})$ to $E_1$ are $E_1$-isomorphic signed hermitian spaces.

Proof: One takes for $f$ the identity and for $L'$ and $L''$ the lattice chain $L$, up to a shift. q.e.d.

For the rest of this section we assume that $E_i|k$ is tamely ramified.

Proposition 4 The strata $(\beta_1, a)$ and $(\beta_2, a)$ are conjugate under an element of $G$, i.e. under an element of $U(a)$.

Proof: Indeed the condition on $\beta_i$ to be minimal and the intertwining of the strata imply that $E_1$ is $\sigma$-equivariantly $k$-algebra isomorphic to $E_2$, in the way such that the image of $\beta_1$ is congruent to $\beta_2$ (see Proposition 2). W.l.o.g. we assume that $\beta_1$ is mapped to $\beta_2$. We want to apply Proposition 1, i.e. we have to prove that the $\epsilon$-hermitian spaces $(V, h_1)$ and $(V, h_2)$ are $E_1$-isomorphic. The latter is equivalent to the existence of an $E_1$-isomorphism from $(V, h_{\beta_1})$ to $(V, h_{\beta_2})$. The rest of the proof is devoted to prove the latter statement:

Let $g$ be an element of $G$ which intertwines $(\beta_1, a)$ and $(\beta_2, a)$ That implies that there are skew-symmetric elements $b_1 \in \beta_1 + a_1 + \nu E_1(\beta_1)$ and $b_2 \in \beta_2 + a_1 + \nu E_1(\beta_2)$ such that $gb_1g^{-1} = b_2$. Now [BH96, 1.6] provides an element $t \in a^\times$ such that

$$t\beta_1t^{-1} = \beta_2,$$

because $E_1^\times$ and $E_2^\times$ normalize $a$. Thus $gt^{-1}$ intertwines $(\beta_2, a)$ with itself, i.e. by [BK93, 1.5.8] there are elements $u$ and $u'$ of $1 + a_1$ and an element $b$ of $C_{\text{Aut}_k(V)}(E_2)$ such that

$$gt^{-1} = ubu'$$

and we define

$$f := bu't = u^{-1}g.$$  

By Corollary 1 it suffices to consider $h_{b_1}$ and $h_{\sigma(u)b_2}$. The map $f$ is a $k$-isomorphism between the two signed hermitian spaces. We want to apply
Proposition 3 and we consider $L' := L$ and $L'' := bL$ up to a shift. Both are $o_{E_1}$-lattice chains but for the different $E_1$-actions. It is enough to show property 2 of Definition 1. We have

$$f_{j-n} \beta_1 v = b_{j-n} u' \beta_2 tv = b_j \beta_2 u'tv = b \beta_2 tv = \beta_2 f(v),$$

for all $\tilde{v} \in L'_j/L'_{j+1}$. The last equality is true, because $b$ commutes with $\beta_2$. Note that $\beta_1 v, u' \beta_2 tv \in L'_{j-n}/L'_{j-n+1}$ and $\beta_2 f(v) \in L''_{j-n}/L''_{j-n+1}$.

From the minimality of $\beta_1$ we get

$$f_j x v = x \ast f(v),$$

for all $x \in E_1$ all $j \in \mathbb{Z}$ and all $v \in L'_j$. More precisely: It holds for every element of $k$ and for $\beta_1$, thus by minimality of $\beta_1$ it is true for every element of $o_{E_1}^k$ and for a prime element of $E_1$, i.e. for every element of $E_1$. Thus the lifts of $h_{b_1}$ and $h_{\sigma(u)b_2}$ to $E_1$ are $E_1$-isomorphic signed hermitian forms.

q.e.d.

In general in the non-tamely ramified case the fields $E_i$ are not isomorphic. To solve this case we use that they are isomorphic $k$-vector spaces.

4.3 Conjugacy in the general case

We could have defined $\lambda$ by taking

$$\lambda_{tr} : E'_1 \to k \text{ and } \lambda_{wr} : E_1 \to E'_1$$

and composing $\lambda := \lambda_{wr} \circ \lambda_{tr}$. We now can consider extensions of $h$ to $E'_1$ and $E_1$. W.l.o.g. we can assume that $E'_1 = E'_2$ By Proposition 3. We call them $E'$. Further we can assume and $\beta'_1$ and $\beta'_2$ to be congruent, in particular the classes $\beta'_1 \pi^{\omega} + \kappa_{E'}$ and $\beta'_2 \pi^{\omega} + \kappa_{E'}$ equal.

We are going to prove the following:

**Lemma 4** There is an element $g \in \text{Aut}_{E'}(V) \cap G$ such that

$$(\beta_1 + a_{1-n})^g = \beta_2 + a_{1-n},$$

where $n := -\nu_a(\beta_1)$.

**Proof:** W.l.o.g. we assume that $E'$ equals $k$. The first step is to find a map $\phi$ from $E_1$ to $E_2$ and a $\sigma|_{E_2}$-equivariant and $k$-linear non-zero map $\lambda'$ such that

$$\lambda'((\phi(ab)) = \lambda'((\phi(a)\phi(b)),$$

(6)
for all $a, b \in E_1$. This is a problem of linear algebra. We can choose a prime element $\pi$ of $k$, such that $\pi$ is symmetric if $k|k_0$ is unramified, and $\pi$ is skew-symmetric if $k|k_0$ is ramified. Note that

$$[k : k_0] = [E_1 : (E_1)_0] = 2.$$  

By Bezout’s Theorem there are integers $z$ and $z'$ such that

$$ze + z'\omega = 1.$$  

i.e.

$$\pi_i := \beta_i z' \pi^2, i = 1, 2,$$

are prime elements for $E_1$ and $E_2$ respectively. Note that both are symmetric or skew-symmetric. We define $\phi$ to be the $k$-linear homomorphism which maps $\pi_j$ to $\pi_j$, for $j = 0, \ldots, e - 1$. To get (7) we solve a linear equation system. It is enough to find values for $\lambda_j^*$ for $j = 0, \ldots, e - 1$. We are done, if we find $\lambda^*$ such that

$$\lambda_j^*(\phi(\pi^j_1)) = \lambda_j^*(\pi^j_2), j = e, \ldots, 2e - 2.$$  

Let

$$f(X) = \sum_{i=0}^{e} a_i X^i, \quad g(X) = \sum_{i=0}^{e} b_i X^i,$$

be the minimal polynomial of $\pi_1, \pi_2$, respectively. The equation system (8) is equivalent to

$$\sum_{i=0}^{e-1} (a_i - b_i) \lambda^* \lambda^*(\pi^j_2 + i) = 0, j = 0, \ldots, e - 2.$$  

We have the following restriction on $\lambda$. Let $\tau$ be a skew-symmetric element of $k$. Then $\lambda_j \in k_0$ if $\sigma(\pi^j_2) = \pi^j_2$ and $\lambda_j \in k_0 \tau$ if $\sigma(\pi^j_2) = -\pi^j_2$.

From (8) we get $e - 1$ equations with coefficients in $k_0$ on $e$ variables, more precisely on the $k_0$ part of $\lambda_j \in k_0 \cup k_0 \tau$. Thus the set of all solutions is a one dimensional $k_0$-vector space. Let us take one non-zero solution $\lambda^*$. There is an even exponent $\alpha$ and an exponent $\gamma$, such that $t_1 := \pi_1^\alpha \pi^\gamma$ and $t_2 := \pi_2^\alpha \pi^\gamma$ are symmetric and

$$\lambda_2(*) := \lambda^*(t_2*), \quad \lambda_1(*) := \lambda^*(\phi(t_1*)),$$

fulfill (7). We choose $\gamma = 0$ if $k|k_0$ is ramified, and we choose $\gamma = -1$ if $k|k_0$ is unramified. We now use the $\lambda_i$ to get the lifts $\tilde{h}_i$. Because of

$$\tilde{h}_1^{j,L} = h_1^{j,L} = \tilde{h}_2^{j,L}, j \in \mathbb{Z},$$

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we can choose Witt basis \((v_i)\) and \((w_i)\) adapted to \(L\) with entries in 
\[\{1, z, π, zπ\},\]
where \(z\) is a symmetric or skew-symmetric fixed non-square in \(o_k^\times\) such that each \(z\) and \(zπ\) occur at most one time, and such that:

1. \(\mu(L, j)^{\tilde{h}_1} = \mu(L, j)^{h_2}\), for all \(j \in \mathbb{Z}\). We mean the sets \(\mu(L, j)\) but with respect to the considered signed hermitian forms.

2. For all indexes \(i, i’\) we have \(\tilde{h}_1(v_i, v_{i'}) = \tilde{h}_1(w_i, w_{i'})\).

We the sake of completeness we want to recall where the signed hermitian family \((\tilde{h}_1^j)\) detects the values \(zπ\) and \(π\). If \(L\) is of type (I) and the period \(r\) is odd then the signed hermitian form \(\tilde{h}_1^{r-1/2} + \frac{1}{2} r, L\) detects these values, and if \(L\) is of type (II) and the period \(r\) is even then \(\tilde{h}_1^{r+1/2}, L\) detects them.

We now change both basis simultaneously to \((\tilde{v}_i)\) and \((\tilde{w}_i)\), by multiplication with appropriate powers of \(π_1\) and \(π_2\) respectively, such that the entries of the Gram matrices are elements of \(t_1^{-1} k\) and \(t_2^{-1} k\) respectively. Indeed we put \(\tilde{v}_i := t_1^{-1} v_i\), for \(i \in I_+\), \(\tilde{v}_i := v_i\), for \(i \in I_-\), and \(\tilde{v}_i := π_1^{-\frac{1}{2}} v_i\). Analogously we define \(\tilde{w}_i\).

We define a \(k\)-linear map \(g\) from \(V\) to \(V\) by
\[\pi_1^j \tilde{v}_i \mapsto \pi_2^j \tilde{w}_i.\]
We have \(g \in G\). Indeed, if
\[\tilde{h}_1(\tilde{v}_i, \tilde{v}_{i'}) =: t_1^{-1} a \in t_1^{-1} k\]
then
\[\tilde{h}_2(\tilde{w}_i, \tilde{w}_{i'}) =: t_2^{-1} a\]
and
\[
\lambda_1(\tilde{h}_1(\tilde{v}_i π_1^q, \tilde{v}_{i'} π_1^{q'})) = \lambda_1(t_1^{-1} a σ(π_1)^q π_1^{q'})) \\
= \lambda'(φ(aσ(π_1)^q π_1^{q'})) \\
= aλ'(σ(π_2)^q π_2^{q'}) \\
= \lambda'(t_2 t_2^{-1} aσ(π_2)^q π_2^{q'}) \\
= \lambda'(t_2 \tilde{h}_2(\tilde{w}_i π_2^q, \tilde{w}_{i'} π_2^{q'})) \\
= \lambda_2(\tilde{h}_2(\tilde{w}_i π_2^q, \tilde{w}_{i'} π_2^{q'})).
\]
for \( q, q' \in \{0, \ldots, e - 1\} \). Note that \( g \) is an element of \( U(a) \) by definition.

As the next step we prove that
\[
g\pi_1 g^{-1} - \pi_2 \in a_{\tilde{e}+1}
\] (9)
where
\[
\tilde{e} := e(a|o_{E_1}) = e(a|o_{E_2}) = \nu_a(\pi_1) = \nu_a(\pi_2).
\]
to get (9) it is enough to show
\[
(g\pi_1 g^{-1} - \pi_2)(\pi_2^{e-1}\tilde{w}_j) \in \pi_2^{e+1}o_{E_2}\tilde{w}_j.
\]
Note that for vectors \( \pi_2^i\tilde{w}_j \) the difference is zero for \( i \) from 0 to \( e - 2 \).

Now to the equation we have
\[
g(\pi_1^e\tilde{v}_j) - \pi_2^e\tilde{w}_j = g\left(\sum_{c=0}^{e-1}(-a_c\pi_1^c)\tilde{v}_j\right) - \pi_2^e\tilde{w}_j
\]
\[
= \sum_{c=0}^{e-1}(-a_c\pi_2^c)\tilde{w}_j - \pi_2^e\tilde{w}_j
\]
\[
= (\sum_{c=0}^{e-1}(b_c - a_c)\pi_2^c)\tilde{w}_j
\]
The last coefficient is an element of \( \pi_2^{e+1}o_{E_2} \) because the minimal polynomials are Eisenstein polynomials and the classes of \( \pi_1^{e}\pi^{-1} \) and \( \pi_2^{e}\pi^{-1} \) equal in \( \kappa_k \), i.e. \( \nu(a_0 - b_0) > \nu(\pi) \).

To finish the proof we need to show:
\[
g\beta_1 g^{-1} - \beta_2 \in a_{-\nu_a(\beta_1)+1}.
\] (10)

We see that from the definition of \( \pi_1 \) follows:
\[
\beta_1 = \pi_1^{-\omega}(\beta_1^{e}\pi^{-\omega})^x
\]
\[
= \pi_1^{-\omega}((\beta_1^{e}\pi^{-\omega})^x - x) + x\pi_1^{-\omega},
\]
for an \( x \in o_k \) congruent to \( (\beta_1^{e}\pi^{-\omega})^x \) in \( \kappa_k \). In particular the first summand lies in \( a_{-\nu_a(\beta_1)+1} \). We now take the analogous equation for \( \beta_2 \) and the observations above imply (10). q.e.d.

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