Algebraic structures in quantum gravity

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Abstract
Starting from a recently introduced algebraic structure on spin foam models, we define a Hopf algebra by dividing with an appropriate quotient. The structure, thus defined, naturally allows for a mirror analysis of spin foam models with quantum field theory, from a combinatorial point of view. A grafting operator is introduced allowing for the equivalent of a Dyson–Schwinger equation to be written. Non-trivial examples are explicitly worked out. Finally, the physical significance of the results is discussed.

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1. Introduction and motivation

Formulating a renormalizable quantum theory for gravity is perhaps the most important open question of contemporary fundamental physics. Noncommutative geometry [1] can be requested when quantum mechanics and gravity meet at some energy scale [2]. String theory, loop quantum gravity, dynamical triangulations, etc have made, in the last few decades, different propositions for new physics such that this crucial task of unification can be achieved.

When considering loop quantum gravity (see for example [3]), the historic way to approach the quantification is to use the spin foam (SF) formalism. Lately it has also been indicated that this formalism can be equivalent to a new type of formulation, the group field theoretical one (see for example [4]).

In this paper, we investigate some algebraic properties of the SF formulation of loop quantum gravity. The starting point is the algebraic structure of SF models introduced in [5] (following previous work in [6]), structure related to the Connes–Kreimer algebra. Note that in commutative or resp. noncommutative quantum field theory (QFT) it was proved that the Connes–Kreimer Hopf algebra of Feynman graphs gives rise to the celebrated BPHZ forest formula ([7] and resp. [8, 9]). Nevertheless, a key ingredient of renormalizability is the power
counting theorem, which tells us which are the primitive divergent graphs to sum on in the
definition of the Connes–Kreimer coproduct.

This power counting result for SF models is not known and therefore is not present in the
construction proposed in [5]. Moreover, in commutative QFT the notion of locality is essential
for the renormalizability of the model (see for example [10]). This notion generalizes in the
case of noncommutative QFT to the notion of ‘Moyality’ (one has non-local counterterms of
the same form as the original non-local ones in the initial, noncommutative action, see for
example [11] or [12] for details).

The Connes–Kreimer Hopf algebra underlying renormalization has recently been
extended to a more general Hopf algebra, the core Hopf algebra [13], where the coproduct
sums over any subgraph. This core Hopf algebra was also introduced within the framework
of noncommutative QFT [9]. This implies that the only Hopf primitives (that is the graphs
which have a trivial coproduct) are the 1-loop graphs.

In this paper, we start from the construction proposed in [5] and we quotient out a Hopf
coidal in order to obtain a new algebraic structure whose properties are more naturally
interconnected to the algebraic properties one is familiar with in (non)commutative QFT.
This new construction is easily proved to be a Hopf algebra; its graduation structure will be
explained here. Furthermore, we note that this algebra can be interpreted as the core Hopf
algebra of SFs, since in the coproduct (just as in the one introduced in [5]) one sums over all
sub-SFs.

To further support this idea comes the remark that, when dealing with perturbative gravity,
the core Hopf algebra is the pertinent Hopf algebra structure, because the one-loop graphs are
the Hopf primitives (that is, the graphs which have a trivial coproduct) [15].

Let us also emphasize that in a commutative or noncommutative QFT, once one has a
Hopf algebra structure, one can define some grafting operator $B_+$. In the language of Feynman
diagrams of (non)commutative QFT, this corresponds to the operator of insertions of subgraphs
into graphs. To any primitively divergent graph in a (non)commutative QFT model one can
associate such an operator. Any relevant graph in perturbation theory is then in the image of
such an operator $B_+$. This property is intimately related to the physical principle of locality
in commutative QFT [16] or to the one of ‘Moyality’ in noncommutative QFT [9]. One
can then write down the combinatorial Dyson–Schwinger equation in a recursive way, as a
power series written in terms of these insertion operators $B_+$. When applying the renormalized
Feynman rules to the combinatorial Dyson–Schwinger equations in QFT, one deals with the
usual analytic Dyson–Schwinger equations [16, 17].

Recently, within the QFT core Hopf algebra setting, the role of the same operator $B_+$ has
been thoroughly investigated [15]. Moreover, the structure of Dyson–Schwinger equations
in the perturbative quantum field theory of gravity has been recently studied in [14] and
it was suggested that gravity, regarded as a probability conserving but perturbatively non-
renormalizable theory, is renormalizable after all.

In this paper, we define an appropriate grafting operator $B_+$ and we perform this type of
analysis for SFs in 2D, 3D and 4D. We propose a way of adapting all of these notions of
(non)commutative QFT for this completely different setting. The physical meaning of these
results is however related to a possible generalization of the locality (or ‘Moyality’) notions
mentioned above. We will argue further on that in section 6.

This paper is structured as follows. In the next section, we recall from [5] the algebraic
construction proposed there. We then define, in the third section, for 2D SFs the Hopf
algebra $\mathbb{T}$ obtained from the construction of [5] by taking some appropriate quotient. The
gradation of $\mathbb{T}$ is presented and the grafting operator $B_+$ is defined. We then give a list of
the algebraic properties existing in $\mathbb{T}$, properties which are in perfect analogy with the ones
existing in (non)commutative QFT. We also explicitly work out some non-trivial examples which illustrate these properties. In the following section, the generalizations of these results to 3D and 4D SFs are presented. The last section is dedicated to the conclusions and to a final discussion.

2. SFs: partitioned SFs and parenthesized weights

A SF is a combinatorial object which can be seen as the world-surface swept by a spin network. The spin networks are graphs labelled by the representations of some group (edges are labelled by representations and nodes are labelled by intertwiners). This implies that the faces of the SF are labelled by representations, and the edges by the intertwiners; the vertices carry the evolution amplitudes.

Consider now the following partition function, defined as the sum:

\[ Z(s_i, s_f) = \sum_{\Gamma} N(\Gamma) \prod_{f \in \Gamma} \dim j_f \prod_{v \in \Gamma} A_v(j). \]  

We have denoted by \( s_i \) and \( s_f \) the initial and the final spin networks, respectively, between which SFs \( \Gamma \) interpolate. A face of the SF is denoted by \( f \) and the dimension of the group representation \( j \) by \( \dim j_f \). The function \( A_v \) is the vertex amplitude and is associated with any vertex \( v \) of the SF. Finally, \( N \) is a weight factor depending only on the SF itself.

We also denote by

\[ \omega_\Gamma = \prod_{f \in \Gamma} \dim j_f \prod_{v \in \Gamma} A_v(j) \]  

the weight of the respective SF. It is this weight which encodes the physical content of the SF.

Choosing the set of SFs \( \Gamma \), associated factors \( N(\Gamma) \), the set of representations and intertwiners as well as the amplitudes \( A_v \) defines the respective SF model. For a general review of SFs, the interested reader may refer himself, for example, to [18]. The EPRL [19] and Freidel–Krasnov [20] models are the current SF models candidate to describe a microscopic structure of spacetime and to have a good low energy limit (which contains the known theories).

We now follow [5] to define partitioned SFs and parenthesized SFs. A sub-SF \( \gamma \) of a SF \( \Gamma \) is a subset of faces of \( \Gamma \), together with any vertices and edges that are boundaries of these faces.

A sub-SF \( \gamma_1 \) is nested into a sub-SF \( \gamma_2 \), \( \gamma_1 \subset \gamma_2 \), if the set of faces of \( \gamma_1 \) is a proper subset of faces of \( \gamma_2 \).

Two sub-SFs \( \gamma_i \) (\( i = 1, 2 \)) are disjoint sub-SFs, \( \gamma_1 \cap \gamma_2 = \emptyset \), if and only if they have no faces, edges or vertices in common.

One says that two sub-SFs are not overlapping if the respective sub-SFs are either nested or disjoint. Furthermore, an allowed partition into sub-SFs of a SF is a partition for which any two sub-SFs are not overlapping.

A partitioned SF is a SF marked with an allowed partitioned into sub-SFs. We denote by \( \Gamma / \gamma \) a co-SF, that is the SF obtained from shrinking the sub-SF \( \gamma \) of the SF \( \Gamma \) into a single vertex.

One says that two sub-SFs are not overlapping if the respective sub-SFs are either nested or disjoint. Furthermore, an allowed partition into sub-SFs of a SF is a partition for which any two sub-SFs are not overlapping.

As in [5], we will work out in this paper with partitioned SFs, referred however to as SFs. The weight of a given SF is represented, as explained in [5] by a parenthesized weight. For example, for the SF \( \Gamma \) of figure 1 one has

\[ \omega_\Gamma = \left( \left( \omega_{\gamma_1} \right) \omega_{\gamma_2} \omega_{\gamma_3} \right) \omega_{\Gamma / \gamma} \omega_{\Gamma / (\gamma_1 \cup \gamma_2 \cup \gamma_3)}, \]

\[ = \left( \left( d_{\gamma_1} A_{\gamma_1} A_{\gamma_2} A_{\gamma_3} \right) \left( d_{\gamma_2} A_{\gamma_2} A_{\gamma_3} A_{\gamma_1} \right) \left( d_{\gamma_3} A_{\gamma_3} A_{\gamma_1} A_{\gamma_2} \right) d_p \right) d_t d_s d_e. \]  

(2.3)
Figure 1. An example of a two-dimensional SF $\Gamma$ and the way to represent its parenthesized weight.

where $\gamma_1$, $\gamma_2$ and $\gamma_3$ are the sub-SFs with faces $l$, $m$ and $n$, respectively, and $\Gamma'$ is the sub-SF containing $\gamma_i$ ($i = 1, 2, 3$), see again Figure 1.

In [5], on the space of these SFs, a coproduct was defined:

$$\Delta M \Gamma = \Gamma \otimes 1_M + 1_M \otimes \Gamma + \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma, \quad (2.4)$$

where $\Gamma$ is some SF and $\gamma$ any of its SFs. Finally, we have denoted by $1_M$ the empty SF.

3. The core Hopf algebra and the grafting operator—definition and mirror analysis with QFT

In this section we focus on the 2D SFs. In [5], when applying the coproduct $\Delta_M$ one has (see example 1 of [5])

$$\Delta'_M \left( \begin{array}{c} \parbox{1cm}{\includegraphics{triangle}} \end{array} \right) = \begin{array}{c} \parbox{1cm}{\includegraphics{triangle}} \end{array} \otimes \begin{array}{c} \parbox{1cm}{\includegraphics{triangle}} \end{array} \quad + \quad \begin{array}{c} \parbox{1cm}{\includegraphics{triangle}} \end{array} \otimes \begin{array}{c} \parbox{1cm}{\includegraphics{triangle}} \end{array} \quad + \quad \begin{array}{c} \parbox{1cm}{\includegraphics{triangle}} \end{array} \otimes \begin{array}{c} \parbox{1cm}{\includegraphics{triangle}} \end{array} \quad . \quad (3.1)$$

Note that, unlike [5], we do not use a white vertex (or any other graphical object) to remind where the shrinking of the sub-SF was done in the SF. A crucial observation is that in the...
algebraic construction of \[5\] one allows ‘tree’-like elements such as

\[
\begin{array}{c}
\text{ } \\
\text{ } \\
\end{array}
\]

Nevertheless, this type of SFs cannot be obtained on the LHS when acting with the coproduct, unless some supplementary notion (like some kind of ‘color’ of the SFs) is defined. In order to obtain the same properties as in QFT, we need for this to be satisfied also. This will become clear in the following. We propose here to \textit{quotient out} this sector. Note that these ‘tree’-like SFs form a trivial Hopf coideal. For the sake of completeness let us also remark that they do not form a Hopf ideal.

We denote the quotiented structure by \(T\) and we refer to it as the \textit{core Hopf algebra of SFs} for the reasons explained above. To check that \(T\) is a Hopf algebra one has just made the correspondence with the Hopf algebra of rooted trees \[21, 22\]. This correspondence is immediate. The graduation of \(T\) is given naturally by the number of triangles of the respective SF. In the language of rooted trees, this corresponds to the weight of the tree (the number of vertices of the respective tree).

The empty SF is denoted by \(1_T\) and is the only element of the algebra of graduation 0. For graduation 1 one has the SF

\[
\begin{array}{c}
\text{ } \\
\text{ } \\
\end{array}
\]

For graduation 2 one has the SFs

\[
\begin{array}{c}
\text{ } \\
\text{ } \\
\end{array}, \quad \begin{array}{c}
\text{ } \\
\text{ } \\
\end{array}, \quad \begin{array}{c}
\text{ } \\
\text{ } \\
\end{array}
\]

The graduation 3 ones are

\[
\begin{array}{c}
\text{ } \\
\text{ } \\
\end{array}, \quad \begin{array}{c}
\text{ } \\
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\text{ } \\
\text{ } \\
\end{array}, \quad \begin{array}{c}
\text{ } \\
\text{ } \\
\end{array}
\]

and so on.

We denote the coproduct by \(\Delta_T\):

\[
\Delta_T \Gamma = \Gamma \otimes 1_T + 1_T \otimes \Gamma + \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma,
\]

where, as in (2.4), \(\Gamma\) is some SF and \(\gamma\) any of its SFs. We denote the non-trivial part of this coproduct by \(\Delta'_T\).

The multiplication is, as in [5], the disjoint union. The rest of the operations are also defined as in [5].

We now define a \textit{grafting operator} \(B_+ : T \rightarrow T\) which increases the graduation by one unit by inserting the respective SF into a bigger triangle. Note that one has three distinct insertion places, corresponding to the three corners of the triangle. One has

\[
B_+ \left( \begin{array}{c}
\text{ } \\
\text{ } \\
\end{array} \right) = \frac{1}{3} \left( \begin{array}{c}
\text{ } \\
\text{ } \\
\end{array} + \begin{array}{c}
\text{ } \\
\text{ } \\
\end{array} + \begin{array}{c}
\text{ } \\
\text{ } \\
\end{array} \right).
\]

Note that the internal structure (i.e. internal triangles) does not play when acting with the
grafting operator. Furthermore, one has

\[
B_+ \left( \bigtriangleup \bigtriangleup \bigtriangleup \right) = \frac{1}{3} \left( \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \right),
\]

(3.5)

\[
B_+ \left( \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \right) = \bigtriangleup \bigtriangleup,
\]

\[
B_+ \left( \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \ldots \right) = 0,
\]

where, by \ldots in the last line above, we mean any number (0 included) of \bigtriangleup \bigtriangleup. This comes from the fact that we work with three maximal insertion places. This is related, in the rooted tree language, to the fertility of a vertex of a tree, that is, the number of outgoing edges (see for example [16]). To complete the definition, one has

\[
B_+ (1_T) \bigtriangleup \bigtriangleup .
\]

(3.6)

The naturality of these equations will become clear in the following (see equations (3.9)). The operator \( B_+ \) is, from a mathematical point of view, a Hochschild 1-cocycle [22].

Let us now write down the following equation in \( \mathbb{T}[[t]] \):

\[
X = 1_T + t B_+ (X^2),
\]

(3.7)

where \( t \) is a parameter which counts the number of triangles (this is the equivalent of the parameter counting the number of loops in the Feynman graph Connes–Kreimer algebra of renormalization). Using the ansatz

\[
X = \sum_{n=0}^{\infty} t^n c_n,
\]

(3.8)

one can determine \( X \) by induction. In the QFT language, this equation is nothing but a cubic combinatorial Dyson–Schwinger equation. Nevertheless, there are some differences with the combinatorial Dyson–Schwinger equations generally used in the rooted tree framework; this implies important differences in the results, which are to be obtained in the rest of this section (see the discussion at the end of the following section). Here we deal with such a cubic equation, because the maximal number of insertion places is three, as already stated above.

Equations (3.7) and (3.8) allow one to obtain the following results (at the first four orders in the development in the constant \( t \)):

\[
c_0 = 1_T,
\]

\[
c_1 = B_+ (c_0^2) = B_+ (c_0) = B_+ (1_T) = \bigtriangleup \bigtriangleup .
\]

\[
c_2 = 3 B_+ (c_0^2 c_1) = 3 B_+ (c_1) = 3 B_+ \left( \bigtriangleup \bigtriangleup \right) = \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup ,
\]

\[
c_3 = 3 B_+ (c_0 c_1^2 + c_0^2 c_2) = 3 B_+ (c_1^2 + c_2) = 3 B_+ \left( \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \right)
\]

\[
= \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup + \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup .
\]

(3.9)

where we have used (3.4) and (3.5).
In all generality, using the Newton binomial formula, one proves
\[ c_{n+1} = \sum_{k_1+k_2+k_3=n} B_s(c_{k_1}c_{k_2}c_{k_3}). \] (3.10)

All this allows one to state the following results:
\[ \Delta_T(B_s) = B_s \otimes 1_T + (\text{id}_T \otimes B_s)\Delta_T \] (3.11)
and
\[ \Delta_T(c_n) = \sum_{k=0}^n P^n_k \otimes c_k, \] (3.12)
where \( P^n_k \) is a polynomial in the variables \( c_\ell, \ell \leq n \) of total degree \( n-k \).

The proof of these identities is straightforward, being a consequence of the fact that, as stated above, one has a direct correspondence between the Hopf algebra \( T \) and the Hopf algebra of rooted trees. Thus, identity (3.11) is in mathematical language the translation of the fact that the operator \( B_s \) is a Hochschild 1-cocycle [16]. Identity (3.12) is a consequence of (3.11) and can be proved by induction. In [16] such a proof was given for the combinatorial Dyson–Schwinger equation (4.7). Let us give a proof for our case.

For \( n = 0 \), identity (3.12) is trivially satisfied. We now start our induction. Using (3.10), one writes
\[ \Delta_T c_n = \Delta_T \sum_{k_1+k_2+k_3=n-1} B_s(c_{k_1}c_{k_2}c_{k_3}). \] (3.13)

We now make use of (3.11) to obtain
\[ \Delta_T c_n = \sum_{k_1+k_2+k_3=n-1} B_s \left( c_{k_1}c_{k_2}c_{k_3} \right) \otimes 1_T + (\text{id}_T \otimes B_s)\Delta_T \left( \sum_{k_1+k_2+k_3=n-1} c_{k_1}c_{k_2}c_{k_3} \right). \] (3.14)

Using again (3.10), equation (3.14) becomes
\[ \Delta_T c_n = c_n \otimes 1_T + (\text{id}_T \otimes B_s)\Delta_T \left( \sum_{k_1+k_2+k_3=n-1} c_{k_1}c_{k_2}c_{k_3} \right). \] (3.15)

Making now use of the induction hypothesis, one has
\[ \Delta_T c_n = c_n \otimes 1_T + (\text{id}_T \otimes B_s) \sum_{k_1+k_2+k_3=n-1} \sum_{\ell_1,\ell_2,\ell_3} P^{k_1}_{\ell_1} P^{k_2}_{\ell_2} P^{k_3}_{\ell_3} \otimes c_{\ell_1}c_{\ell_2}c_{\ell_3}, \] (3.16)
which further can be written as
\[ \Delta_T c_n = c_n \otimes 1_T + \sum_{k_1+k_2+k_3=n-1} \sum_{\ell_1,\ell_2,\ell_3} P^{k_1}_{\ell_1} P^{k_2}_{\ell_2} P^{k_3}_{\ell_3} \otimes B_s \left( c_{\ell_1}c_{\ell_2}c_{\ell_3} \right). \] (3.17)

By rearranging the indices of the last term above the left-hand tensor factor gives \( P^n_k \) and the right-hand tensor factor, once again using (3.10), gives \( c_q (q = 1, \ldots, n) \):
\[ \Delta_T c_n = c_n \otimes 1_T + \sum_{q=1}^n P^n_q \otimes c_q. \] (3.18)

By a direct inspection, one can see that \( P^n_q \) is nothing more than a homogeneous polynomial in the variables \( c_\ell (\ell \leq n) \) of total degree \( n-q \). Furthermore, let us recall that \( P^n_0 = c_n \).

Identity (3.12) thus shows that the elements \( c_\ell \) form Hopf subalgebras in \( T \). In QFT, this type of result is of fundamental importance for finding some exact solutions of the Dyson–Schwinger equations [25].
To end this section, let us illustrate identities (3.11) and (3.12) on some non-trivial particular cases of small graduation SFs. The LHS of (3.11) applied for $c_1$ gives

$$\Delta B_* \left( \begin{array}{c} \triangle \end{array} \right) = \Delta \left( \frac{1}{3} \left( \begin{array}{c} \triangle + \triangle + \triangle \end{array} \right) \right) \times 1_\mathbb{T} + 1_\mathbb{T}$$

$$\otimes 1_\mathbb{T} \left( \begin{array}{c} \triangle + \triangle + \triangle \end{array} \right) + \triangle \otimes \triangle .$$

On the RHS, one has

$$B_* \left( \begin{array}{c} \triangle \end{array} \right) \otimes 1_\mathbb{T} + (\text{id}_\mathbb{T} \otimes B_*) \Delta \left( \begin{array}{c} \triangle \end{array} \right) = \frac{1}{3} \left( \begin{array}{c} \triangle + \triangle + \triangle \end{array} \right) \otimes 1_\mathbb{T} + 1_\mathbb{T} \otimes \frac{1}{3} \left( \begin{array}{c} \triangle + \triangle + \triangle \end{array} \right) . \tag{3.20}$$

which is identical to (3.19), as expected. The polynomials given by formula (3.12) are trivial.

Let us now go further and verify the identity (3.11) for $c_2$ given by (3.9). The non-trivial part of the LHS is written as

$$\frac{1}{3} \Delta'_\mathbb{T} \left( \begin{array}{c} \triangle + \triangle + \triangle + \triangle + \triangle + \triangle + \triangle + \triangle + \triangle \end{array} \right) . \tag{3.21}$$

This further gives the following six terms:

$$\triangle \otimes \left( \begin{array}{c} \triangle + \triangle + \triangle \end{array} \right) . \tag{3.22}$$

On the RHS, the non-trivial terms are obtained from

$$(\text{id}_\mathbb{T} \otimes B_*) \Delta'_\mathbb{T} \left( \begin{array}{c} \triangle + \triangle + \triangle \end{array} \right) \tag{3.23}$$

The non-trivial terms of (3.23) are given by

$$(\text{id}_\mathbb{T} \otimes B_*) \left( \begin{array}{c} \triangle + \triangle + \triangle \end{array} \right) \otimes 1_\mathbb{T} + 3 \triangle \otimes \triangle . \tag{3.24}$$

This gives

$$\triangle \otimes \left( \begin{array}{c} \triangle + \triangle + \triangle \end{array} \right) \left( \begin{array}{c} \triangle + \triangle + \triangle \end{array} \right) \otimes \triangle . \tag{3.25}$$

which are, as expected, the same six terms as in (3.22).
Table 1. First values of the polynomial $P_n^k$ for the 2D case.

| $P_n^k$ | $n = 0$ | $n = 1$ | $n = 2$ | $n = 3$ |
|---------|---------|---------|---------|---------|
| $k = 0$ | $1_\tau$ | $c_1$   | $c_2$   | $c_3$   |
| $k = 1$ | $1_\tau$ | $3c_1$  | $3c_1^2$ | $3c_2$  |
| $k = 2$ | $1_\tau$ | $5c_1$  | $c_1$   |         |
| $k = 3$ | $1_\tau$ |         |         |         |

Let us now explicitly verify identity (3.12) at the graduation 2-level. Considering only the non-trivial part of the coproduct, one has

$$\Delta_\tau' c_2 = 3 \bigtriangleup \otimes \bigtriangleup = 3 \bigtriangleup \otimes c_1,$$

and thus

$$P_1^2 = 3c_1^2.$$  \hspace{1cm} (3.26)

Finally, let us now explicitly verify identity (3.12) at the graduation 3-level. Considering the non-trivial part of the coproduct, one has

$$\Delta_\tau' = 5 \bigtriangleup \otimes \left( \bigtriangleup + \bigtriangleup + \bigtriangleup \right) + \left( 3 \bigtriangleup \bigtriangleup \bigtriangleup + 3 \left( \bigtriangleup + \bigtriangleup + \bigtriangleup \right) \right) \otimes \bigtriangleup$$

$$= 5 \bigtriangleup \otimes c_2 + \left( 3 \bigtriangleup \bigtriangleup \bigtriangleup + 3 \left( \bigtriangleup + \bigtriangleup + \bigtriangleup \right) \right) \otimes c_1,$$  \hspace{1cm} (3.28)

which leads to

$$P_1^3 = 3c_1^2 + 3c_2,$$

$$P_2^3 = 5c_1.$$  \hspace{1cm} (3.29)

We list these results in table 1.

Let us end this section by stating that, as in [5], one can define an analogous algebraic structure on the vector space of parenthesized SFs over $\mathbb{C}$.

4. Generalization to 3D and 4D

In this section, we generalize the previous results to the case of 3D and resp. 4D SFs. This generalization is rather natural, since the number of maximal insertion places goes from three (in the 2D case) to four (in the 3D case) and resp. five (in the 4D case).

4.1. The 3D case

In the 3D case, the building block which replaces the triangle is the tetrahedron of figure 2. These tetrahedrons are related to SFs, as shown in figure 3.

One naturally generalizes (3.4), (3.5) and (3.6) to define the grafting operator $B_\tau$. The equation corresponding to (3.7) is now

$$X = 1_\tau + t B_\tau(X^4).$$  \hspace{1cm} (4.1)
Figure 2. A tetrahedron is the building block of the core Hopf algebra of 3D SFs. It plays the same role as the triangle in the 2D construction, being the graduation one generator in the algebra.

Figure 3. An example of a 3D SF.

Proceeding as in the previous section, one writes down (at the first four orders in the development in the constant $t$)

$$c_0 = 1_T,$$

$$c_1 = B_+(c^3_0) = B_+(1_T) = \begin{array}{c}
\end{array},$$

$$c_2 = 4B_+(c^3_0c_1) = 4B_+(c_1)$$

$$= 4B_+\left(\begin{array}{c}
\end{array}\right) = \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array},$$

$$c_3 = B_+(6c^3_0c_1^2 + 4c_0^3c_2) = B_+(6c^2_1 + 4c_2)$$

$$= B_+\left(6\begin{array}{c}
\end{array} + 4\left(\begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array}\right)\right). \tag{4.2}$$

We have left the last set of figures in the last equation above for the interested reader.
Figure 4. A 4-simplex is the building block of the core Hopf algebra of 4D SFs. It plays the same role as the triangle in the 2D construction or the tetrahedron in 3D, being the graduation one generator in the algebra.

Table 2. First values of the polynomial $P^n_k$ for the 3D case.

| $n$ | $P^n_0$ | $P^n_1$ | $P^n_2$ | $P^n_3$ |
|-----|---------|---------|---------|---------|
| 0   | 1       | $c_1$   | $c_2$   | $c_3$   |
| 1   | $1_T$   | $4c_1$  | $6c_1^2 + 4c_2$ |
| 2   | $1_T$   | $7c_1$  |          |
| 3   | $1_T$   |          |          |

The general recursive solution for equation (4.1), is given again by the Newton binomial formula

$$c_{n+1} = \sum_{k_1 + \ldots + k_4 = n} B_4(c_{k_1} \ldots c_{k_4}). \quad (4.3)$$

Identities (3.11) and (3.12) are also respected, the proof being analogous to the one of the previous section. Let us end this subsection by listing in table 2 the polynomials $P^n_k$, which are obtained analogously by applying the coproduct on the elements given in (4.2).

4.2. The 4D case

The 4D case is treated along the same lines. The building brick is now the 4-simplex of figure 4. The appropriate equation to investigate (generalizing (4.1)) is

$$X = 1_T + t B_4(X^5), \quad (4.4)$$

since the maximal number of insertion places is now five.

The solution of this equation (in the first four orders of the development in the constant $t$) is

$$c_0 = 1_T,$$
$$c_1 = B_4(c_0^5) = B_4(1_T),$$
$$c_2 = 5B_4(c_0^4c_1) = 5B_4(c_1) = 4B_4(1_T),$$
$$c_3 = B_4(10c_0^3c_2^2 + 5c_0^2c_2) = B_4(10c_1^3 + 5c_2). \quad (4.5)$$
The general solution can be written as

$$c_{n+1} = \sum_{k_1 + \cdots + k_s = n} B_s(c_{k_1} \ldots c_{k_s}).$$  \hspace{1cm} (4.6)$$

As in the previous cases, identities (3.11) and (3.12) hold in the same manner. Let us list here the set of the first polynomials $P_n^k$.

We end this section with the following comparison. In the rooted tree Hopf algebra literature, general combinatorial Dyson–Schwinger equations can be considered (see for example [26]). Nevertheless, let us note that, as already announced in the previous section, combinatorial Dyson–Schwinger equations of the following particular form are generally used (see for example [16, 27, 28]):

$$X = 1\tau + \sum_{n=1}^{\infty} t^n \omega_n B_d^v(X^{n+1}),$$

where $\omega_n$ are scalars and $(B_d^v)$ is a collection of Hochschild 1-cocycles on the algebra (see again [16] for details). The main difference with equation (3.7) (or (4.1) or (4.4) that we use here) is in the power of the constant $t$. This leads to crucial differences in the calculus of the polynomials $P_n^k$. For example, when considering the equation

$$X = 1\tau + t^2 \omega_2 B_d^v(X^3),$$

one obtains

$$c_0 = 1\tau,$$
$$c_1 = 0,$$
$$c_2 = B_d^v(c_0) = B_d^v(1\tau),$$
$$c_3 = 3B_d^v(c_0 c_1) = 0,$$
$$c_4 = 3B_d^v(c_0 c_1^2 + c_0^2 c_2) = 3B_d^v(c_2)$$

and so on. One can directly see that this is different from equation (3.9) (or (4.2) or (4.5)). This further leads to a different set of polynomials than the ones listed in table 1. To end this discussion, let us also remark that the polynomials $P_n^k$ associated with the combinatorial Dyson–Schwinger equation (4.7) do not depend on the scalars $\omega_n$ or on $B_d^v$. This is not the case for the polynomials exhibited in this paper, which are different for the 2D, 3D or 4D cases (see for example tables 1–3).

5. Comments on the physical relevance of the approach: example

As already stated in the introduction, a general power counting theorem for SF models is not known today; in [5], an algebraic structure was introduced where the coproduct $\Delta_M$ sums over all sub-SFs. In the Hopf algebra defined in this paper, the same definition of the coproduct is
kept, i.e. one sums over all sub-SFs. These constructions, both the one in [5] and the one here, can be seen as a first attempt towards better understanding the renormalizability properties of SF models.

Furthermore, we have also argued above that this type of algebraic structure can be well suited to deal with quantum gravity because of the following argument. Hopf primitives (i.e. the elements of the Hopf algebra which have a trivial coproduct) of perturbative quantum gravity are one-loop graphs. Hopf primitives are directly related to the primitive divergent graph of a field theory. Therefore, it appears natural, from this point of view, to consider core Hopf algebra (i.e. Hopf algebra in which the coproduct sums on all respective sub-SFs) as an interesting structure to investigate.

Finally, let me give one additional argument, using this time the group field theoretical approach. We focus on the 3D case (the 2D one being trivial). One can associate the SF of figure 3—the divergent quantity and also the Hopf primitive here—to the graph of figure 5. One can easily identify a bubble (a closed three-dimensional region of the graph or a closed bi-circuit) in this graph. This topological notion of bubble (see for example [29]) is the natural generalization of the notion of face (closed circuits in the graph). Let us also emphasize that in [29] an algorithm for identifying the bubbles of a generic three-dimensional group field theory was given. The Feynman amplitude associated with the graph of figure 5 is divergent [30]. This fact is thus a further indication for choosing the coproduct used here and in [5].

Let us now comment further on the significance of the combinatorial Dyson–Schwinger equation (3.7) (and its generalization to higher dimensions). As already mentioned above, this is the analogue of a cubic combinatorial Dyson–Schwinger equation from the QFT framework. The field action related to this equation is the group field theoretical one, which in the 3D case is written as

$$S[\phi] = \frac{1}{2} \int dg_1 \, dg_2 \, dg_3 \phi(g_1, g_2, g_3)\phi(g_3, g_2, g_1),$$

$$+ \frac{\lambda}{4} \int \phi(g_1, g_2, g_3)\phi(g_3, g_4, g_5)\phi(g_5, g_6, g_1)\phi(g_6, g_4, g_2).$$

(5.1)
The integrations over the group (left here implicit in the interaction term) are performed as usually with the invariant de Haar measure.

This equation is the combinatorial backbone of non-perturbative QFT. The analytic Dyson–Schwinger equation (the one used in physics) is obtained by applying the renormalized Feynman rules to the combinatorial one. Let us recall here that Dyson–Schwinger equations are quantum equations of motion for the Green (or Schwinger) functions, being thus a crucial tool of any QFT. We argue that it is thusly justified to analyse (here from a combinatorial point of view) such an equation in our efforts towards a better understanding of a quantum formulation of gravity.

As we will also comment on in the following section, it would be interesting to adapt such tools to the group field theoretical approach also (which is naturally suited for such a study) and then to compare the results with the one obtained in this paper.

6. Conclusions and perspectives

We have thus explicitly exhibited, in the framework of the SF formalism, some combinatorial notions which naturally appear in QFT. The 2D, 3D and 4D cases have been analysed and some non-trivial examples have been worked out as an illustration of our results.

The correspondence of the Hopf algebra $T$ defined here is done with the Hopf algebra of rooted trees and not with the Connes–Kreimer Hopf algebra of Feynman graphs. This comes from the fact that we do not deal with overlapping SFs. In QFT, dealing with overlapping divergences by rooted trees is also more involved (see [31, 32]). A $1PI$ Feynman graph can be uniquely represented by a rooted tree (with labels on each vertex corresponding to the associated subgraph) iff all subdivergences are nested and not overlapping and if there is only one way to make each insertion. It would be interesting to investigate whether or not a correspondence between overlapping SFs and the Connes–Kreimer algebra of the Feynman graph can be obtained.

Nevertheless, let us stress on the following issue. As already mentioned in the introduction, in commutative (resp. noncommutative) QFT, behind the combinatorial properties investigated here lies the physical principle of locality (or resp. ‘Moyal’). The renormalizability of local theories (or of non-local scalars models on the Moyal space—see [33]) is by now well understood. One cannot say today that this is also the case for quantum gravity models. It appears to us of crucial importance to investigate whether or not a generalization of the principles of locality (or ‘Moyal’) can exist. This new type of principle could be related, from a combinatorial point of view, to the fact that the triangular character of SFs reproduces itself when inserting SFs into SFs, having thus some kind of ‘triangularity’ (or similarly ‘simplexality’ for higher dimensions).

A promising way of approaching the renormalizability of quantum gravity can be a thorough study of group field theoretical models (see for example [4]). These models were developed as a generalization of 2D matrix models to 3D or 4D. Thus, group field theoretical models are duals to the Ponzano–Regge model, when considering the 3D gravity, or to the Ouguri model, when considering the 4D one.

These models can be seen nowadays not only as a technical tool but as a proposition for a quantum formulation of gravitation. Behind this lies the idea that group field theories are theories of spacetime, while QFT are theories on spacetime. Feynman graphs of these models are tensor graphs, a natural generalization of the matrix graphs of noncommutative QFT. Recently, insights on the renormalizability of 3D models have been given [29, 34].

A perspective to be mentioned here is the investigation of the combinatorial properties studied in this paper within this new context of group field theory. Moreover, a comparison
of the results obtained from this program with the results of this paper could offer a better understanding of the physical properties of these gravitational models.

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