We present a new approach to the problem of Bloch electrons in magnetic field, by making explicit a natural relation between magnetic translations and the quantum group $U_q(sl_2)$. The approach allows to express the spectrum and the Bloch function as solutions of the Bethe-Ansatz equations typical for completely integrable quantum systems.

I. INTRODUCTION

Several times a peculiar problem of Bloch electrons in magnetic field emerged with a new face to describe another physical application [1], [2], [3], [4], [5], [6], [7] (see also [8] for a review).

i) it resembles some properties of the integer Hall effect [5],

ii) its spectrum has an extremely rich structure of Cantor set, and exhibits a multifractal behaviour [4], [6], [7] (see also [8] for a review).

iii) it describes the localization phenomenon in quasiperiodic potential (see e.g. [8] and references therein).

iv) it has been recently conjectured that the symmetry of magnetic group may appear dynamically in strongly correlated electronic systems [9], [10].

In this paper we show that this problem (some times called Hofstadter problem) and a class of quasiperiodic equations are solvable by the Bethe-Ansatz. We made explicit a long time anticipated connection of the group of magnetic translations with the Quantum Group $U_q(sl_2)$ and with Quantum Integrable Systems. The result of the paper is the algebraic Bethe-Ansatz equations for the spectrum. Although we do not solve the Bethe-Ansatz equation...
here, we are confident that they provide a basis for analytical study of the multifractal properties of the spectrum.

II. MODEL AND RESULT

The Hamiltonian of a particle on a two dimensional square lattice in magnetic field is
\[
H = \sum_{<n,m>} e^{iA\vec{n},\vec{m}}c_{\vec{n}}^\dagger c_{\vec{m}}
\]
(1)
\[
\prod_{\text{plaquette}} e^{iA\vec{n},\vec{m}} = e^{i\Phi}
\]
(2)
where \(\Phi = 2\pi \frac{P}{Q}\) is a flux per plaquette, \(P\) and \(Q\) are mutually prime integers. In the most conventional Landau gauge \(A_x = A_{\vec{n},\vec{n}+\vec{1}_x} = 0, A_y = \Phi n_x\) the Bloch wave function is
\[
\psi(\vec{n}) = e^{-ik\vec{n}}\psi_n(k), \psi_n = \psi_{n+Q}
\]
(3)
where \(n_x \equiv n = 1...Q\) is a coordinate of the magnetic cell. With these substitution the Schrodinger equation turns into a famous one-dimensional quasiperiodic difference equation ("Harper's" equation):
\[
e^{ik_x}\psi_{n+1} + e^{-ik_x}\psi_{n-1} + 2\cos(k_y + n\Phi)\psi_n = E\psi_n
\]
(4)
The spectrum of this equation has \(Q\) bands and feels the difference between rational and irrational numbers - if the flux is irrational, the spectrum is singular continuum - uncountable but measure zero set of points (Cantor set). If the flux is rational, then the spectrum has \(Q\) bands.

To ease the reference we state the main result of the paper:
It is known that due to gauge invariance, the energy depends on a single parameter \(\lambda = \cos(Qk_x) + \cos(Qk_y)\). We find that the spectrum at \(\lambda = 0\) ("mid" band spectrum) is given by the sum of roots \(z_l\)
\[
E = iq^Q(q - q^{-1}) \sum_{l=1}^{Q-1} z_l
\]
(5)
of the Bethe-Ansatz equations for the Quantum Group \(U_q(sl_2)\)
\[
\frac{z_l^2 + q}{qz_l^2 + 1} = q^Q \prod_{m=1, m\neq l}^{Q-1} \frac{z_l - z_m}{z_l - qz_m}, \quad l = 1...Q - 1.
\]
(6)
with
\[
q = e^{\frac{\Phi}{2}},
\]
(7)
Another version of the Bethe-Ansatz equations is presented in the end of the paper. Solution of the model with anisotropic hopping (when the coefficient in front of \(\cos\) in Eq.(4) differs
from 2 is also available. It will be published elsewhere. The Quantum Group symmetry is
more transparent in another gauge \( A_x = -\frac{\Phi}{2}(n_x + n_y), A_y = \frac{\Phi}{2}(n_x + n_y + 1) \). In this gauge
a discrete coordinate of the Bloch function \( \psi(\vec{n}) = \exp(i\vec{p}\vec{n})\psi_n(\vec{p}) \), turns to \( n = n_x + n_y \).

It is defined in two magnetic cells: \( n = 1...2Q, p_\pm = (p_x \pm p_y)/2 \in [0, \Phi/2] \). An equivalent
form of the Harper Eq. (4) for \( \psi_n \) is

\[
2e^{i\Phi + p_+} \cos(\frac{1}{2}\Phi n + \frac{1}{4}\Phi - p_-)\psi_{n+1} + \\
2e^{-i\Phi - p_+} \cos(\frac{1}{2}\Phi n - \frac{1}{4}\Phi - p_-)\psi_{n-1} = E\psi_n
\]  

In the new gauge the "midband" \( \lambda = 0 \) corresponds to the point \( \vec{p} = (\frac{\pi}{Q}, \frac{\pi}{Q}) \). The
advantage of this gauge is that the wave function turns into the polynomial with roots \( z_m \)

\[
\Psi(z) = \prod_{m=1}^{Q-1} (z - z_m)
\]

at the points \( z = q^{2j} \)

\[
\psi_n = \Psi(q^n)
\]

### III. GROUP OF MAGNETIC TRANSLATIONS

The wave function of a particle in a magnetic field forms a representation of the group
of magnetic translations \[1\]: let generators of translations be

\[
T_{\vec{\mu}}(\vec{i}) = e^{iA_{\vec{i}, \vec{i} + \vec{\mu}}} | \vec{i} > < \vec{i} + \vec{\mu} |
\]

They form the algebra

\[
T_{\vec{\mu}} = T_{-\vec{\mu}}^{-1}, T_{\vec{\mu}}, T_{\vec{\mu}}T_{\vec{\nu}} = q^{-\vec{n} \times \vec{\nu}}T_{\vec{\mu} + \vec{\nu}}, \\
T_{\vec{y}}T_{\vec{x}} = q^2T_{\vec{x}}T_{\vec{y}}, T_{\vec{y}}T_{-\vec{x}} = q^{-2}T_{-\vec{x}}T_{\vec{y}}...
\]

with \( q \) given by the Eq.(7). The Hamiltonian \[2\] can be expressed

\[
H = T_x + T_{-x} + T_y + T_{-y}
\]

This group is also equivalent to the Heisenberg-Weyl group: \( [\hat{p}, \hat{q}] = i\Phi, T_x = \exp \hat{q}, T_y = \exp \hat{p} \).

---

\[1\] The doubling of period in comparison with original Harper’s equation is artificial: using a simple
transformation (multiplying \( \psi_n \) by \( e^{-i\Phi n^2/4} \)) one comes to an equation with coefficients of period
Q. We are indebted to Alexandre Abanov for clarifying connection between two gauges.
IV. QUANTUM GROUP

The algebra $U_q(sl_2)$ (a $q$-deformation of the universal enveloping of the $sl_2$) is generated by the elements $A, B, C, D$, with the commutation relations \[ AB = qBA, BD = qDB, \]
\[ DC = qCD, CA = qAC, \]
\[ AD = 1, [B, C] = \frac{A^2 - D^2}{q - q^{-1}} \] (14)

The center of this algebra is a $q$-analog of the Casimir operator
\[ c = \left( \frac{q^{-\frac{1}{2}}A - q^{\frac{1}{2}}D}{q - q^{-1}} \right)^2 + BC \] (15)

In the classical limit $q \to 1 + \frac{i}{2} \Phi$, the quantum group turns to the $sl_2$ algebra:
\[ (A - D)/(q - q^{-1}) \to S_3, B \to S_+^-, C \to S_-, c \to \tilde{S}^2 + 1/4. \]

The commutation relations (14) are simply another way to write the Yang-Baxter equation
\[ R_{a_1a_2}^{b_1b_2}(u/v)L_{b_1c_1}(u)L_{b_2c_2}(v) = L_{a_1b_1}(u)L_{a_2b_2}(v)R_{b_1b_2}^{c_1c_2}(u/v) \] (16)

where generators $A, B, C, D$ are matrix elements of the $L$-operator
\[ L(u) = \left[ \frac{uA - u^{-1}D}{uB} \quad \frac{u^{-1}C}{uD - u^{-1}A} \right] \] (17)

Here $u$ is a spectral parameter and $R$-matrix is the $L$-operator in the spin 1/2-representation. It is given by the same matrix (17) with elements: $A = q^{\frac{1}{2}\sigma_3}, D = q^{-\frac{1}{2}\sigma_3}, B = \sigma_+, C = \sigma_-$, where $\sigma$ are the Pauli matrices.

Finite dimensional representations (except some representations of dimension $Q$) of the $U_q(sl_2)$ can be expressed in the weight basis, where $A$ and $D$ are diagonal matrices: $A = diag(q^j, \ldots, q^{-j})$. An integer or halfinteger $j$ is the spin of the representation, and $2j + 1$ is its dimension. The value of the Casimir operator (15) in this representation is given by the $q$-analog of $(j + 1/2)^2$
\[ c = \left( \frac{q^{j+1/2} - q^{-j-1/2}}{q - q^{-1}} \right)^2 = [j + 1/2]^2_q \] (18)

Representations can be realized by polynomials $\Psi(z)$ of the degree $2j$:
\[ A\Psi(z) = q^{-j}\Psi(qz), D\Psi(z) = q^{j}\Psi(q^{-1}z), \]
\[ B\Psi(z) = z(q - q^{-1})^{-1} \left( q^{2j}\Psi(q^{-1}z) - q^{-2j}\Psi(qz) \right) \]
\[ C\Psi(z) = -z^{-1}(q - q^{-1})^{-1} \left( \Psi(q^{-1}z) - \Psi(qz) \right) \] (19)

This again the $q$-analog of the representation of the $sl_2$ algebra by a differential operator:
\[ S_3 = z \frac{d}{dz} - j, S_+ = z(2j - z \frac{d}{dz}), S_- = \frac{d}{dz} \] (20)
V. MAGNETIC TRANSLATIONS AS A SPECIAL REPRESENTATION OF THE QUANTUM GROUP

Dimension of our physical space of states is \(2j + 1 = Q\). This is a very special dimension when \(q^{2j+1} = \mp 1\) for \(P\) - odd (even). The Casimir operator \([18]\) in this case is

\[
c = -4(q - q^{-1})^{-2}, \text{ for } P - \text{odd} \\
c = 0, \text{ for } P - \text{even}
\]  

(21)

In this special case \([16], [17]\) representation of the quantum group can be naturally (but not unambiguously \([18]\)) expressed in terms of Magnetic Translations. Say one may choose

\[
T_x + T_y = \pm i(q - q^{-1})B, \\
T_x + T_y = i(q - q^{-1})C, \\
T_y T_x = \pm q^{-1} A^2, T_x T_y = \pm qD^2
\]

(22)

where the upper sign corresponds to an odd \(P\) and the lower to an even \(P\) (representation of \(U_q(sl_2)\) in terms of different but related Weyl basis can be found in Ref. \([16], [17], [18], [21]\)). It is straightforward to check that this representation obeys commutation relations \((12)\) and \((14)\) and gives correct values \((21)\) of the Casimir operator \((15)\).

The Hamiltonian \((1,13)\) now can be expressed in terms of the quantum group generators

\[
H = i(q - q^{-1})(C \pm B)
\]

(23)

whereas the Schrodinger equation becomes a difference functional equation

\[
i(z^{-1} + qz)\Psi(qz) - i(z^{-1} + q^{-1}z)\Psi(q^{-1}z) = E\Psi(z)
\]

(24)

The original Harper’s discrete equation in the form \((8)\) at the point \(\vec{p} = (\pi/2, \pi/2)\) can be obtained from the functional equation \((24)\), by setting \(z = q^l\) and \(\psi_l = \Psi(q^l)\). The advantage to use the extention of \(\psi_l\) to a complex plane \(z\) is that the representation theory of the quantum group garantees that in a proper gauge the extended wave function would be polynomial \((9)\).

In addition to representation \([19]\) having the the highest and the lowest weight , in the special dimension \(q^{2j+1} = \pm 1\) there is a parametric family of representations having in general no highest or lowest weights \([17], [21]\). The parameter describes the anisotropy of the hopping amplitude in the Hamiltonian \((1)\) or the strength of the potential (i.e.the coefficient in front of \(cos\)) in the Harper Equation \((4)\). In this case the wave function is not polynomial in any gauge. Nevertheless the Bethe Ansatz solution of the anisotropic problem is also possible but acqueres much heavier mathematical techique. We postpone it for a more extended paper.

VI. FUNCTIONAL BETHE-ANSATZ

Among various methods of the theory of quantum integrable systems the functional Bethe-Ansatz \([19]\) seems to be the most direct way to diagonalize the Hamiltonian \((23)\).
We know that the solution of Eq. (24) is polynomial
\[ \Psi(z) = Q^{-1} \prod_{m=1}^{Q-1} (z - z_m) \tag{25} \]
at the points \( z = q^{2l} \)
\[ \psi_n = \Psi(q^n) \tag{26} \]
Let us substitute it in the Eq. (24) and divide both sides by \( \Psi(z) \). We obtain
\[ i(z^{-1} + qz) \prod_{m=1,m\neq l}^{Q-1} \frac{qz - z_m}{z - z_m} - i(z^{-1} + q^{-1}z) \prod_{m=1,m\neq l}^{Q-1} \frac{q^{-1}z - z_m}{z - z_m} = E \tag{27} \]
The l.h.s. of this equation is a meromorphic function, whereas the r.h.s. is a constant. To make them equal we must null all residues of the l.h.s. They appear at \( z = 0 \), \( z = \infty \) and at \( z = z_m \). The residue at \( z = 0 \) vanishes automatically.

The residue at \( z = \infty \) is \( -iq^Q + iq^{-Q} \). Its null determines the degree of the polynomial.

Comparing the coefficients of \( z^{Q-1} \) in the both sides of Eq. (24), we obtain the energy given by Eq. (5).

Finally, annihilation of poles at \( z = z_m \) gives the Bethe-Ansatz equations (8) for roots of the polynomial (9). We write them here in a more conventional form by setting \( z_l = \exp (2\varphi_l) \)
\[ \frac{\cosh(2\varphi_l - i\frac{\Phi}{4})}{\cosh(2\varphi_l + i\frac{\Phi}{4})} = \pm \prod_{m=1,m\neq l}^{Q-1} \frac{\sinh(\varphi_l - \varphi_m + i\frac{\Phi}{4})}{\sinh(\varphi_l - \varphi_m - i\frac{\Phi}{4})} \tag{28} \]
Another form of the Bethe-Ansatz equations is given in the next section.

**VII. MISCELLANEOUS RESULTS**

1. Another Form of the Bethe-Ansatz Equations

As we already mentioned the representation of the quantum group by magnetic translations is not unique. This means that there is another gauge where the wave function is a polynomial. Consider the gauge \( A_x = -A_y = -\Phi n_x \). Then instead of Harper’s equation (4) we obtain an equivalent equation for the gauge transformed wave function \( \psi_n \to \phi_n = \exp(i\frac{\Phi}{4}n(n-1))\psi_n \) (for an odd \( Q \) it respects the periodic conditions (3)). At \( \lambda = 0 \), it has the form
\[ e^{-i\Phi n} \phi_{n+1} + e^{i\Phi(n-1)}\phi_{n-1} - 2\cos(n\Phi)\phi_n = E\phi_n \tag{29} \]
This choice of the gauge corresponds to another representation of the quantum group by magnetic translations. Say for an odd \( P \) we have
\[ T_{-x} + T_{-y} = -i(q - q^{-1})q^{-\frac{1}{2}}BD, \]
\[ T_x + T_y = -i(q - q^{-1})q^{-\frac{1}{2}}CA, \]
\[ T_{-y}T_x = q^{-1}A^2, T_{-x}T_y = qD^2 \quad (30) \]

Then the Hamiltonian (33) turns into quadratic form in the \( U_q(sl_2) \) generators

\[ H = -i(q - q^{-1})q^{-\frac{1}{2}}(CA + BD) \quad (31) \]

The representation (19) (for \( q^{2j+1} = -1 \)) now gives another functional equation

\[ z^{-1}\Psi(q^2z) + q^{-2}z\Psi(q^{-2}z) - (z + z^{-1})\Psi(z) = E\Psi(z) \quad (32) \]

which is identical to (29) on the set of points \( z = q^{2l} \), where \( l = 0...Q - 1 \). The Bethe-Ansatz may be obtained in the similar way:

\[ z_l^2 = q^Q \prod_{m=1, m \neq l}^{Q-1} \frac{q^2z_l - z_m}{z_l - q^2z_m}, \quad l = 1...Q - 1. \quad (33) \]

The energy is given again by the sum of roots

\[ E = -q(q - q^{-1}) \sum_{l=1}^{Q-1} z_l, \quad (34) \]

Inspite of the difference Eqs.(33,34) must be equivalent to the Eqs.(31,32).

2. Quadratic Form of Quantum Group Generators

A limited number of other interesting solvable discrete equations may be obtained from a general quadratic form of quantum group generators. Their "classical" version \( (q \to 1) \) would be differential equations generated by quadratic forms of \( sl_2 \) -generators (20). These differential equations are known in the literature as so-called "quasi exactly solvable" problems of quantum mechanics [20], [22]. A quadratic form can be considered as trace of a monodromy matrix of an integrable model with nonperiodic boundary conditions

\[ \tau = trK_+(u)L(su)K_-(u)\sigma_2L^l(su^{-1})\sigma_2 \quad (35) \]

where \( K_\pm \) are c-number matrices - solutions of "reflective" Yang-Baxter equations (RYB) [23], [24]. These matrices describe all boundary conditions consistent with integrability. There is a 3-parametric family of boundary matrices \( K_\pm \) which generates a general quadratic form of \( A, B, C, D \). Besides, there is a parameter \( s \) in (35) which one can introduce in the \( L \)-operator (17) preserving integrability. For a particular choice of boundary \( K \)-matrices and parameter \( s \tau \) is proportional to the hamiltonian (31). Therefore, one can say, that the Hofstadter problem is equivalent to an integrable magnet of spin \((Q - 1)/2\) on one site with a proper boundary condition.

3. q-Analog of Orthogonal Polynomials

There is an intriguing connection between a wave function (25) and q-generalization of orthogonal polynomials [25]. They satisfy the difference equation (q-analog of differential hypergeometrical equation)
\[
A(z)P_n(q^2 z) + A(z^{-1})P_n(q^{-2} z) - (A(z) + A(z^{-1}))P_n(z) = (q^{-2n} - 1)(1 - abcdq^{2n-2})P_n(z) \quad (36)
\]

where \( A(z) = (1 - az)(1 - bz)(1 - cz)(1 - dz)/(1 - z^2)(1 - q^2 z^2) \) and \( a, b, c, d \) are parameters and \( n \) is a degree of the polynomial. Choosing \( c = -d = q, a = -b = 0 \) we arrive at the equation for the \( q \)-Hermite polynomials \( H_n^{(q)} \)

\[
H_n^{(q)}(q^2 z) - z^2 H_n^{(q)}(q^{-2} z) = q^{-2n}(1 - z^2)H_n^{(q)}(z)
\]  
(37)

(these are polynomials in \( z + z^{-1} \) of degree \( n \)). For an odd \( Q \) at \( n = (Q - 1)/2 \) this yields a zero energy solution to eq.(32) \( \Psi^{(E=0)}(iz) = z^{(Q-1)/2}H_{(Q-1)/2}^{(q)}(z) \).

Another choice \( c = -d = q, a = -b = q \) and then the replacement \( q \) by \( q^{1/2} \) gives the \( q \)-Legendre equation

\[
\frac{1 - qz^2}{1 - z^2} P_n^{(q)}(qz) + \frac{q - z^2}{1 - z^2} P_n^{(q)}(q^{-1}z) = (q^{-n} + q^{n+1}) P_n^{(q)}(z)
\]  
(38)

Then, comparing with the Eq.(24) we conclude that the zero mode solution is given by the \( q \)-Legendre polynomial \( \Psi^{(E=0)}(iz) = z^{(Q-1)/2}P_{(Q-1)/2}^{(q)}(z) \).

The almost immediate and the most interesting task now is to solve the Bethe-Ansatz equations in the limit \( P, Q \to \infty \) when the flux \( \Phi/2\pi \) is irrational. In all previous examples of integrable systems it was always possible to derive an integral equation for a distribution function of roots \( z_l \). We hope that it would be also possible for this problem and after all allows to obtain fractal properties of the spectrum analytically.

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