0. Introduction

Let $V$ be a 6-dimensional complex vector space. Let $L \mathbb{G}(\wedge^3 V) \subset \text{Gr}(10, \wedge^3 V)$ be the symplectic Grassmannian parametrizing subspaces which are lagrangian for the symplectic form given by wedge-product. Given $A \in L \mathbb{G}(\wedge^3 V)$ we let

$$Y_A := \{ [v] \in \mathbb{P}(V) | A \cap (v \wedge \wedge^2 V) \neq \{0\} \}. $$

Then $Y_A$ is a degeneracy locus and hence it is naturally a subscheme of $\mathbb{P}(V)$. For certain pathological choices of $A$ we have $Y_A = \mathbb{P}(V)$: barring those cases $Y_A$ is a sextic hypersurface named EPW-sextic. An EPW-sextic comes equipped with a double cover

$$f_A : X_A \rightarrow Y_A. \tag{0.0.1}$$

$X_A$ is what we call a double EPW-sextic. There is an open dense subset $L \mathbb{G}(\wedge^3 V)^0 \subset L \mathbb{G}(\wedge^3 V)$ parametrizing smooth double EPW-sextics - these 4-folds are hyperkähler (HK) deformations of the Hilbert square of a $K3$ (i.e. the blow-up of the diagonal in the symmetric product of a $K3$ surface), see [19]. By varying $A \in L \mathbb{G}(\wedge^3 V)^0$ one gets a locally versal family of HK varieties - one of the five known such families in dimensions greater than 2, see [3, 4, 9, 10, 11] for the construction of the other families. The complement of $L \mathbb{G}(\wedge^3 V)^0$ in $L \mathbb{G}(\wedge^3 V)$ is the union of two prime divisors, $\Sigma$ and $\Delta$: the former consists of those $A$ containing a non-zero decomposable tri-vector, the latter is defined in Subsection 1.5. If $A$ is generic in $\Sigma$ then $X_A$ is singular along a $K3$ surface, see Corollary 3.2, if $A$ is generic in $\Delta$ then $X_A$ is singular at a single point whose tangent cone is isomorphic to the contraction of the zero-section of the cotangent sheaf of $\mathbb{P}^2$, see Prop. 3.10 of [22]. By associating to $A \in L \mathbb{G}(\wedge^3 V)^0$ the Hodge structure on $H^2(X_A)$ one gets a regular map of quasi-projective varieties

$$\mathcal{P}^0 : L \mathbb{G}(\wedge^3 V)^0 \rightarrow \mathbb{D}_A \tag{0.0.2}$$

where $\mathbb{D}_A$ is a quasi-projective period domain, the quotient of a bounded symmetric domain of Type IV by the action of an arithmetic group, see Subsection 1.6. Let $\mathbb{D}^{BB}_A$ be the Baily-Borel compactification of $\mathbb{D}_A$ and

$$\mathcal{P} : L \mathbb{G}(\wedge^3 V) \rightarrow \mathbb{D}^{BB}_A \tag{0.0.3}$$

the rational map defined by (0.0.2). The map $\mathcal{P}$ descends to the GIT-quotient of $L \mathbb{G}(\wedge^3 V)$ for the natural action of $\text{PGL}(V)$. More precisely: the action of $\text{PGL}(V)$ on $L \mathbb{G}(\wedge^3 V)$ is uniquely linearized and hence there is an unambiguous GIT quotient

$$\mathcal{M} := L \mathbb{G}(\wedge^3 V) // \text{PGL}(V). \tag{0.0.4}$$

Let $L \mathbb{G}(\wedge^3 V)^{st}, L \mathbb{G}(\wedge^3 V)^{ss} \subset L \mathbb{G}(\wedge^3 V)$ be the loci of (GIT) stable and semistable points of $L \mathbb{G}(\wedge^3 V)$. By Cor. 2.5.1 of [23] the open $\text{PGL}(V)$-invariant subset $L \mathbb{G}(\wedge^3 V)^0$ is contained in $L \mathbb{G}(\wedge^3 V)^{st}$: we let

$$\mathcal{M}^0 := L \mathbb{G}(\wedge^3 V)^0 // \text{PGL}(V). \tag{0.0.5}$$

Then $\mathcal{P}$ descends to a rational map

$$p : \mathcal{M} \rightarrow \mathbb{D}^{BB}_A \tag{0.0.6}$$

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which is regular on $\mathcal{M}$. By Verbitzky’s global Torelli Theorem and Markman’s monodromy results the restriction of $p$ to $\mathcal{M}$ is injective, see Theorem 1.3 and Lemma 9.2 of [13]. Since domain and codomain of the period map have the same dimension it follows that $p$ is a birational map. In the present paper we will be mainly concerned with the following problem: what is the indeterminacy locus of $p$? In order to state our main results we will go through a few more definitions. Given $A \in \mathbb{L}G(\Lambda^3 V)$ we let
\[ \Theta_A := \{ W \in \text{Gr}(3, V) | \bigwedge^3 W \subseteq A \}. \] (0.0.7)

Thus $A \notin \Sigma$ if and only if $\Theta_A \neq \emptyset$. Suppose that $W \in \Theta_A$: there is a natural determinantal subscheme $C_{W,A} \subset \mathbb{P}(W)$, see Subsect. 3.2 of [23], with the property that
\[ \text{supp} C_{W,A} = \{ [v] \in \mathbb{P}(W) | \dim(A \cap (v \wedge \Lambda^2 V)) \geq 2 \}. \] (0.0.8)

$C_{W,A}$ is either a sextic curve or (in pathological cases) $\mathbb{P}(W)$. Let
\[ \mathcal{D}^{BB}_{p(W)}(6) \longrightarrow \mathcal{D}^{BB}_{\Phi} \] (0.0.9)

be the compactified period map where $\mathcal{D}^{BB}_{\Phi}$ is the Baily-Borel compactification of the period space for $K3$ surfaces of degree 2, see [23].

**Definition 0.1.** (1) Let $\mathcal{M}^{ADE} \subset \mathcal{M}$ be the subset of points represented by $A \in \mathbb{L}G(\Lambda^3 V)^{ss}$ for which the following hold:

(1a) The orbit $\text{PGL}(V)A$ is closed in $\mathbb{L}G(\Lambda^3 V)^{ss}$.

(1b) For all $W \in \Theta_A$ we have that $C_{W,A}$ is a sextic curve with simple singularities.

(2) Let $\mathcal{J} \subset \mathcal{M}$ be the subset of points represented by $A \in \mathbb{L}G(\Lambda^3 V)^{ss}$ for which the following hold:

(2a) The orbit $\text{PGL}(V)A$ is closed in $\mathbb{L}G(\Lambda^3 V)^{ss}$.

(2b) There exists $W \in \Theta_A$ such that $C_{W,A}$ is either $\mathbb{P}(W)$ or a sextic curve in the indeterminacy locus of $\Theta_A$.

Then $\mathcal{M}^{ADE}$, $\mathcal{J}$ are respectively open and closed subsets of $\mathcal{M}$, and since every point of $\mathcal{M}$ is represented by a single closed $\text{PGL}(V)$-orbit $\mathcal{J}$ is in the complement of $\mathcal{M}^{ADE}$. As is well-known the family of double EPW-sextics is analogous to the family of varieties of lines on a smooth cubic 4-fold, and the period map for double EPW-sextics is analogous to the period map for cubic 4-folds: the subset $\mathcal{M}^{ADE}$ is the analogue in our context of the moduli space of cubic 4-folds with simple singularities, see [13] [14]. Below is the main result of the present paper.

**Theorem 0.2.** The period map $p$ is regular away from $\mathcal{J}$. Let $x \in (\mathcal{M} \setminus \mathcal{J})$: then $p(x) \in \mathcal{D}_A$ if and only if $x \in \mathcal{M}^{ADE}$.

**Theorem 0.2** is the analogue of the result that the period map for cubic 4-folds extends regularly to the moduli space of cubic 4-folds with simple singularities, and it maps it into the interior of the relevant Baily-Borel compactification, see [13] [14]. The above result is a first step towards an understanding of the rational map $\mathcal{M} \longrightarrow \mathcal{D}^{BB}_{\Phi}$. Such an understanding will eventually include a characterization of the image of $(\mathbb{L}G(\Lambda^3 V) \setminus \Sigma)$. (Notice that if $A \in (\mathbb{L}G(\Lambda^3 V) \setminus \Sigma)$ then $A$ is a stable point (Cor. 2.5.1 of [23]) and hence $[A] \in \mathcal{M}^{ADE}$ because $\Theta_A$ is empty.) In the present paper we will give a preliminary result in that direction. In order to state our result we will to introduce more notation. Let
\[ \mathcal{M} := \Sigma//\text{PGL}(V). \] (0.0.10)

(The generic point of $\Sigma$ is $\text{PGL}(V)$-stable by Cor. 2.5.1 of [23], hence $\mathcal{M}$ is a prime divisor of $\mathcal{M}$.)

In Subsection 1.7 we will prove that the set of Hodge structures in $\mathcal{D}_A$ which have a $(1,1)$-root of negative square is the union of four prime divisors named $S_2^0, S_2, S_2^0, S_4$.

**Theorem 0.3.** The restriction of the period map $p$ to $(\mathcal{M} \setminus \mathcal{M})$ is an open embedding
\[ (\mathcal{M} \setminus \mathcal{M}) \hookrightarrow (\mathbb{D}_A \setminus (S_2^0 \cup S_2^0 \cup S_2^0 \cup S_4)). \] (0.0.11)

Let us briefly summarize the main intermediate results of the paper and the proofs of **Theorem 0.2** and **Theorem 0.3**. In Subsection 1.7 we will define the Noether-Lefschetz prime divisor $S_2^0$ of $\mathbb{D}_A$ and its closure $S_2$ in $\mathbb{D}^{BB}_{\Phi}$; later we will prove that the closure of $\mathcal{P}(\Sigma)$ is equal to $S_2$. In Subsection 1.7 we will show that the normalization of $S_2$ is equal to the Baily-Borel compactification $\mathcal{D}^{BB}_{\Gamma}$ of the quotient of a bounded symmetric domain of Type IV modulo an arithmetic group and we will define
a natural finite map \( \rho; \mathbb{D}_{BB} \to \mathbb{D}_{BB} \) where \( \mathbb{D}_{BB} \) is as in (0.0.9) - the map \( \rho \) will play a key rôle in the proof of Theorem 0.2. In Section 2 we will prove that \( \mathcal{P} \) is regular away from a certain closed subset of \( \Sigma \) which has codimension 4 in \( LG(\Lambda^3 V) \). The idea of the proof is the following. Suppose that \( X_A \) is smooth and \( L \subset P(V) \) is a 3-dimensional linear subspace such that \( f_A^{-1}(Y_A \cap L) \) is smooth:

by Lefschetz’ Hyperplane Theorem the periods of \( X_A \) inject into the periods of \( f_A^{-1}(Y_A \cap L) \). This together with Griffiths’ Removable Singularity Theorem gives that the period map extends regularly over the subset of \( LG(\Lambda^3 V) \) parametrizing those \( A \) for which \( f_A^{-1}(Y_A \cap L) \) has at most rational double points for generic \( L \subset P(V) \) as above. We will prove that the latter condition holds away from the union of the subsets of \( \Sigma \) denoted \( \Sigma[2] \) and \( \Sigma_{\infty} \), see Proposition 2.4. One gets the stated result because the codimensions in \( LG(\Lambda^3 V) \) of \( \Sigma[2] \) and \( \Sigma_{\infty} \) are 4 and 7 respectively, see (13.4.3) and (13.4.7).

Let \( LG(\Lambda^3 V) \subset LG(\Lambda^3 V) \times \mathbb{D}_{BB} \) be the closure of the graph of \( \mathcal{P} \). Since \( LG(\Lambda^3 V) \) is smooth the projection \( p; LG(\Lambda^3 V) \to LG(\Lambda^3 V) \) is identified with the blow-up of the indeterminacy locus of \( \mathcal{P} \) and hence the exceptional set of \( p \) is the support of the exceptional Cartier divisor of the blow-up. Let \( \tilde{\Sigma} \subset LG(\Lambda^3 V) \) be the strict transform of \( \Sigma \). The results of Section 2 described above give that if \( A \) is in the indeterminacy locus of \( \mathcal{P} \) (and hence \( A \in \Sigma \)) then

\[
\dim(p^{-1}(A) \cap \tilde{\Sigma}) \geq 2.
\]

Section 3 starts with an analysis of \( X_A \) for generic \( A \in \Sigma \): we will prove that it is singular along a K3 surface \( S_A \) which is a double cover of \( P(W) \) where \( W \) is the unique element of \( \Theta_A \) (unique because \( A \) is generic in \( \Sigma \)) and that the blow-up of \( X_A \) with center \( S_A \) - call it \( \tilde{X}_A \) - is a smooth HK variety deformation equivalent to smooth double EPW-sextics, see Corollary 3.2 and Corollary 3.6. It follows that \( \mathcal{P}(A) \) is identified with the weight-2 Hodge structure of \( \tilde{X}_A \). Let \( \zeta_A \) be the Poincaré dual of the exceptional divisor of the blow-up \( \tilde{X}_A \to X_A \). Then \( \zeta_A \) is a \((-2)\)-root of divisibility 1 perpendicular to the pull-back of \( c_1((\mathcal{O}_{Y_A}(1)) \): it follows that \( \mathcal{P}(A) \in \mathbb{S}_2 \) and that \( \mathcal{P}(\Sigma) \) is dense in \( \mathbb{S}_2 \), see Proposition 3.13. We will also define an index-2 inclusion of integral Hodge structures \( \zeta_A^* \to H^2(S_A; \mathbb{Z}) \), see (3.6.7), and we will show that the inclusion may be identified with the value at \( \mathcal{P}(A) \) of the finite map \( p; \mathbb{D}_{BB} \to \mathbb{D}_{BB} \) mentioned above (this makes sense because \( \rho \) is the map associated to an extension of lattices), see (3.6.6). Let

\[
\tilde{\Sigma} := \{(W, A) \in \text{Gr}(3, V) \times LG(\bigwedge^3 V) \mid \bigwedge^3 W \subset A \}.
\]

The natural forgetful map \( \tilde{\Sigma} \to \Sigma \) is birational (for generic \( A \in \Sigma \) there is a unique \( W \in \text{Gr}(3, V) \) such that \( \bigwedge^3 W \subset A \); since the period map is regular at the generic point of \( \Sigma \) it induces a rational map \( \tilde{\Sigma} \dashrightarrow \mathbb{S}_2 \) and hence a rational map to its normalization

\[
\tilde{\Sigma} \dashrightarrow \mathbb{D}_{BB}.
\]

Let \( (W, A) \in \tilde{\Sigma} \) and suppose that \( C_{W, A} \) is a sextic (i.e. \( C_{W, A} \neq P(W) \)) and the period map (0.0.9) is regular at \( C_{W, A} \); the relation described above between the Hodge structures on \( \zeta_A^* \) and \( H^2(S_A) \) gives that Map (0.0.11) is regular at \( (W, A) \). Now let \( x \in (\mathfrak{M} \setminus \mathfrak{J}) \) and suppose that \( x \) is in the indeterminacy locus of the rational period map \( p \). One reaches a contradiction arguing as follows. Let \( A \in LG(\Lambda^3 V) \) be semistable with orbit closed in \( LG(\Lambda^3 V)^{ss} \) and representing \( x \). Results of [23] and [21] give that \( \dim \Theta_A \leq 1 \); this result combined with the regularity of (0.0.11) at all \( (W, A) \) with \( W \in \Theta_A \) gives that \( \dim(p^{-1}(A) \cap \tilde{\Sigma}) \leq 1 \): that contradicts (0.0.12) and hence proves that \( p \) is regular at \( x \) (it proves also the last clause in the statement of Theorem 0.2). In Section 4 we will prove Theorem 0.3. The main ingredients of the proof of \( p \) maps \( (\mathfrak{M} \setminus \mathfrak{R}) \) into the right-hand side of (0.0.11) are Verbitsky’s Global Torelli Theorem and our knowledge of degenerate EPW-sextics whose periods fill out open dense subsets of the divisors \( \mathbb{S}_2', \mathbb{S}_2'' \) and \( \mathbb{S}_4 \). Here “degenerate” means that we have a hyperkähler deformation of the Hilbert square of a K3 and a map \( f: X \to \mathbb{P}^3 \): while \( X \) is not degenerate, the map is degenerate in the sense that it is not a double cover of its image, either it has (some) positive dimensional fibers (as in the case of \( \mathbb{S}_4 \) that we discussed above) or it has degree (onto its image) strictly higher than 2. Lastly we will prove that the restriction of \( p \) to \( (\mathfrak{M} \setminus \mathfrak{R}) \) is open. Since \( \mathbb{D}_A \) is \( \mathbb{Q} \)-factorial it will suffice to prove that the exceptional set of \( p|_{(\mathfrak{M} \setminus \mathfrak{R})} \) has codimension at least 2. If \( A \in LG(\Lambda^3 V)^p \) then \( p \) is open at \( [A] \) because \( X_A \) is smooth (injectivity and surjectivity of the local period map). On the other hand there is a regular involution of \( (\mathfrak{M} \setminus \mathfrak{R}) \) which lifts a regular involution of \( \mathbb{D}_A \) and \( (\Delta \setminus \Sigma)/\text{PGL}(V) \) is not sent to itself by the involution: this will allow us to conclude that the exceptional set of \( p|_{(\mathfrak{M} \setminus \mathfrak{R})} \) has codimension at least 2.
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1. Preliminaries

1.1. Local equation of EPW-sextics. We will recall notation and results from [21]. Let $A \in LG(\Lambda^3 V)$ and $[v_0] \in \mathbb{P}(V)$. Choose a direct-sum decomposition

$$V = [v_0] \oplus V_0. \quad (1.1.1)$$

We identify $V_0$ with the open affine $(\mathbb{P}(V) \setminus \mathbb{P}(V_0))$ via the isomorphism

$$V_0 \xrightarrow{\sim} \mathbb{P}(V) \setminus \mathbb{P}(V_0) \quad v \mapsto [v_0 + v]. \quad (1.1.2)$$

Thus $0 \in V_0$ corresponds to $[v_0]$. Then

$$Y_A \cap V_0 = V(f_0 + f_1 + \cdots + f_6), \quad f_1 \in S^4 V_0^\vee. \quad (1.1.3)$$

The following result collects together statements contained in Corollary 2.5 and Proposition 2.9 of [21].

**Proposition 1.1.** Keep assumptions and hypotheses as above. Let $k := \dim(\Lambda \cap (v_0 \wedge \Lambda^2 V))$.

1. Suppose that there is no $W \in \Theta_A$ containing $v_0$. Then the following hold:
   1a) $0 = f_0 = \cdots = f_{k-1}$ and $f_k \neq 0$.
   1b) Suppose that $k = 2$ and hence $[v_0] \in Y_A(2)$. Then $Y_A(2)$ is smooth two-dimensional at $[v_0]$.

2. Suppose that there exists $W \in \Theta_A$ containing $v_0$. Then $0 = f_0 = f_1$.

Next we recall how one describes $Y_A \cap V_0$ under the following assumption:

$$\bigwedge^3 V_0 \cap A = \{0\}. \quad (1.1.4)$$

Decomposition (1.1.1) determines a direct-sum decomposition $\Lambda^3 V = [v_0] \wedge \Lambda^2 V_0 \oplus \Lambda^3 V_0$. We will identify $\Lambda^2 V_0$ with $v_0 \wedge \Lambda^2 V_0$ via

$$\Lambda^2 V_0 \xrightarrow{\sim} v_0 \wedge \Lambda^2 V_0 \quad \beta \mapsto v_0 \wedge \beta \quad (1.1.5)$$

By (1.1.4) the subspace $A$ is the graph of a linear map

$$\tilde{q}_A : \bigwedge^2 V_0 \to \bigwedge^3 V_0. \quad (1.1.6)$$

Choose a volume-form

$$\text{vol}_0 : \bigwedge^5 V_0 \xrightarrow{\sim} \mathbb{C} \quad (1.1.7)$$

in order to identify $\Lambda^3 V_0$ with $\Lambda^2 V_0^\vee$. Then $\tilde{q}_A$ is symmetric because $A \in LG(\Lambda^3 V)$. Explicitly

$$\tilde{q}_A(\alpha) = \gamma \iff (v_0 \wedge \alpha + \gamma) \in A. \quad (1.1.8)$$

We let $q_A$ be the associated quadratic form on $\Lambda^2 V_0$. Notice that

$$\ker q_A = \left\{ \alpha \in \bigwedge^2 V_0 \mid v_0 \wedge \alpha \in A \cap (v_0 \wedge \bigwedge V) \right\} \quad (1.1.9)$$

is identified with $A \cap (v_0 \wedge \Lambda^2 V)$ via (1.1.5). Let $v \in V_0$ and $q_v$ be the Plücker quadratic form defined by

$$\Lambda^2 V_0 \xrightarrow{q_v} \mathbb{C} \quad \alpha \mapsto \text{vol}_0(v \wedge \alpha \wedge \alpha) \quad (1.1.10)$$

**Proposition 1.2** (Proposition 2.18 of [21]). Keep notation and hypotheses as above, in particular (1.1.3) holds. Then

$$Y_A \cap V_0 = V(\det(q_A + q_v)). \quad (1.1.11)$$
Notice that

$$Y_A \text{ Explicit description of double EPW-sextics.}$$

1.2. following is an immediate consequence of the definition of $v$

where $\xi$ (1.1.1)

In particular there exists a decomposition such that (1.1.4) holds.

We start by introducing some notation. Let

Choose a basis of $v$

$V$ for all $v$

$E$

Equation (2.82) of [21].

Next we will state a hypothesis which ensures existence of a decomposition such that (1.1.4) holds.

for this we need $v$

The EPW-sextic $Y_A$ holds if and only if $Y_A = P(V)$. The proposition below follows at once from Claim 2.11 and Equation (2.82) of [21].

**Proposition 1.3.** Let $A \in \mathbb{L}(\wedge^3 V)$ and suppose that $\dim \Theta_A \leq 2$. Then

$$Y_A \neq \mathbb{P}(V), \quad Y_A \neq \mathbb{P}(V').$$

In particular there exists a decomposition such that (1.1.3) holds.

Let $A \in \mathbb{L}(\wedge^3 V)$. We will need to consider higher degeneracy loci attached to $A$. Let

$$Y_A[k] = \{ [v] \in \mathbb{P}(V) \mid \dim (A \cap (v \wedge \wedge^2 V)) \geq k \}. \quad (1.1.14)$$

Notice that $Y_A[0] = \mathbb{P}(V)$ and $Y_A[1] = Y_A$. Moreover $A \in \Delta$ if and only if $Y_A[3]$ is not empty. We set

$$Y_A(k) := Y_A[k] \setminus Y_A[k + 1]. \quad (1.1.15)$$

1.2. Explicit description of double EPW-sextics. Throughout the present subsection we will assume that $A \in \mathbb{L}(\wedge^3 V)$ and $Y_A \neq \mathbb{P}(V)$. Let $f_A : X_A \to Y_A$ be the double cover of $[0, 1]$. The following is an immediate consequence of the definition of $f_A$, see [22]:

$$f_A \text{ is a topological covering of degree 2 over } Y_A(1). \quad (1.2.1)$$

Let $[v_0] \in Y_A$; we will give explicit equations for a neighborhood of $J_A^{-1}(v_0)$ in $X_A$. We will assume throughout the subsection that we are given a direct-sum decomposition such that (1.1.4) holds.

We start by introducing some notation. Let $K := \ker q_A$ and let $J \subset \wedge^2 V_0$ be a maximal subspace over which $q_A$ is non-degenerate; we have a direct-sum decomposition

$$\wedge^2 V_0 = J \oplus K. \quad (1.2.2)$$

Choose a basis of $\wedge^2 V_0$ adapted to Decomposition (1.2.2). Let $k := \dim K$. The Gram matrices of $q_A$ and $q_v$ (for $v \in V_0$) relative to the chosen basis are given by

$$M(q_A) = \begin{pmatrix} N_J & 0 \\ 0 & 0_k \end{pmatrix}, \quad M(q_v) = \begin{pmatrix} Q_J(v) & R_J(v)^t \\ R_J(v) & P_J(v) \end{pmatrix}. \quad (1.2.3)$$

(We let $0_k$ be the $k \times k$ zero matrix.) Notice that $N_J$ is invertible and $q_0 = 0$; thus there exist arbitrarily small open (in the classical topology) neighborhoods $V_0$ of 0 in $V_0$ such that $(N_J + Q_J(v))^{-1}$ exists for $v \in V_0$. We let

$$M_J(v) := P_J(v) - R_J(v) \cdot (N_J + Q_J(v))^{-1} \cdot R_J(v)^t, \quad v \in V_0'. \quad (1.2.4)$$

If $V_0'$ is sufficiently small we may write $(N_J + Q_J(v)) = S(v) \cdot S(v)^t$ for all $v \in V_0'$ where $S(v)$ is an analytic function of $v$ (for this we need $V_0'$ to be open in the classical topology) and $S(v)$ is invertible for all $v \in V_0'$. Let $j := \dim J$. For later use we record the following equality

$$\begin{pmatrix} 1_j & 0 \\ -R_J(v)S^{-1}(v)^t & 1_k \end{pmatrix} \begin{pmatrix} S(v)^{-t} & 0 \\ 0 & 1_k \end{pmatrix} \begin{pmatrix} N_J + Q_J(v) & R_J(v)^t \\ R_J(v) & P_J(v) \end{pmatrix} \begin{pmatrix} S^{-1}(v)^t & 0 \\ 0 & 1_k \end{pmatrix} \begin{pmatrix} 1_j & 0 \\ 0 & M_J(v) \end{pmatrix} = \begin{pmatrix} 1_j & 0 \\ 0 & M_J(v) \end{pmatrix} \quad (1.2.5)$$

Let $X_J \subset V_0' \times \mathbb{C}^k$ be the closed subscheme whose ideal is generated by the entries of the matrices

$$M_J(v) \cdot \xi, \quad \xi \cdot \xi^t - M_J(v)^t, \quad (1.2.6)$$

where $\xi \in \mathbb{C}^k$ is a column vector and $M_J(v)^t$ is the matrix of cofactors of $M_J(v)$. We identify $V_0'$ with an open neighborhood of $[v_0] \in \mathbb{P}(V)$ via (1.2.2). Projection defines a map $\phi : X_J \to \mathbb{P}(\det M_J)$. By (1.2.5) we have $\mathbb{P}(\det M_J) = V_0' \cap Y_A$. 


Proposition 1.4. Keep notation and assumptions as above. There exists a commutative diagram

\[ \frac{(X_A, f_A^{-1}([v_0]))}{\phi} \xrightarrow{f_A} \frac{(X_J, \phi^{-1}([v_0]))}{\phi} \]

where the germs are in the analytic topology. Furthermore \( \zeta \) is an isomorphism.

Proof. Let \( [v] \in \mathbb{P}(V) \): there is a canonical identification between \( v \wedge \Lambda^2 V \) and the fiber at \( [v] \) of \( \Omega^3_{\mathbb{P}(V)}(3) \), see for example Proposition 5.11 of [10]. Thus we have an injection \( \Omega^3_{\mathbb{P}(V)}(3) \hookrightarrow \Lambda^3 V \otimes O_{\mathbb{P}(V)} \). Choose \( B \in L \mathbb{G}(\Lambda^3 V) \) transversal to \( A \). The direct-sum decomposition \( \Lambda^3 V = A \oplus B \) gives rise to a commutative diagram with exact rows

\[
\begin{array}{ccl}
0 & \rightarrow & \Omega^3_{\mathbb{P}(V)}(3) \xrightarrow{\lambda_A} A^\vee \otimes O_{\mathbb{P}(V)} \xrightarrow{i_* \zeta_A} i_* \zeta_A & \rightarrow & 0 \\
0 & \rightarrow & A \otimes O_{\mathbb{P}(V)} \xrightarrow{\lambda_B} \Lambda^3 T_{\mathbb{P}(V)}(-3) \rightarrow Ext^1(i_* \zeta_A, O_{\mathbb{P}(V)}) \rightarrow 0
\end{array}
\]  

(1.2.7)

(As suggested by our notation the map \( \beta_A \) does not depend on the choice of \( B \).) Choosing \( B \) transverse to \( v_0 \wedge \Lambda^2 V \) we can assume that \( \mu_{A,B}(0) \) (recall that \( (\mathbb{P}(V) \setminus \mathbb{P}(V_0)) \) is identified with \( \mathbb{P}(V_0) \) and that \([v_0]\) corresponds to \( 0 \)) is an isomorphism. Then there exist arbitrarily small open (classical topology) neighborhoods \( \mathcal{U} \) of \( 0 \) such that \( \mu_{A,B}(v) \) is an isomorphism for all \( v \in \mathcal{U} \). The map \( \lambda_A \circ \mu_{A,B}^{-1} \) is symmetric because \( A \) is lagrangian. Choose a basis of \( A \) and let \( L(v) \) be the Gram matrix of \( \lambda_A \circ \mu_{A,B}^{-1}(v) \) with respect to the chosen basis. Let \( L(v)^c \) be the matrix of cofactors of \( L(v) \). Claim 1.3 of [22] gives an embedding

\[ f_A^{-1}(\mathcal{U} \cap Y_A) \hookrightarrow \mathcal{U} \times \mathbb{A}^{10} \]  

(1.2.8)

with image the closed subscheme whose ideal is generated by the entries of the matrices

\[ L(v) \cdot \xi - \xi \cdot L(v)^c. \]  

(1.2.9)

(Here \( \xi \) is a \( 10 \times 1 \)-matrix whose entries are coordinates on \( \mathbb{A}^{10} \).) We will denote the above subscheme by \( V(L(v) \cdot \xi, \xi \cdot \xi^t - L(v)^c) \). Under this embedding the restriction of \( f_A \) to \( f_A^{-1}(\mathcal{U} \cap Y_A) \) gets identified with the restriction of the projection \( \mathcal{U} \times \mathbb{A}^{10} \rightarrow \mathcal{U} \). Let \( G : \mathcal{U} \rightarrow GL_{10}(\mathbb{C}) \) be an analytic map and for \( v \in \mathcal{U} \) let \( H(v) := G(v) \cdot L(v) \cdot G(v) \). The automorphism of \( \mathcal{U} \times \mathbb{A}^{10} \) given by \( (v, \xi) \mapsto (v, G(v)^{-1} \xi) \) restricts to an isomorphism

\[ V(L(v) \cdot \xi, \xi \cdot \xi^t - L(v)^c) \xrightarrow{\sim} V(H(v) \cdot \xi, \xi \cdot \xi^t - H(v)^c). \]  

(1.2.10)

In other words we are free to replace \( L \) by an arbitrary congruent matrix function. Let

\[ \Lambda^2 V_0 \xrightarrow{\phi_{v_0+v}} (v_0 + v) \wedge \Lambda^2 V \xrightarrow{\alpha} (v_0 + v) \wedge \alpha \]  

(1.2.11)

A straightforward computation gives that

\[ \phi_{v_0+v} \circ \mu_{A,B}^l(v) \circ (\lambda_A(v) \circ \mu_{A,B}^{-1}(v)) \circ \mu_{A,B}(v) \circ \phi_{v_0+v} = \tilde{q}_A + \tilde{q}_v, \]

(1.2.12)

Thus the Gram matrix \( M(q_A + q_v) \) is congruent to \( L(v) \) and hence we have an embedding [1.2.8] with image \( V(M(q_A + q_v) \cdot \xi, \xi \cdot \xi^t - M(q_A + q_v)^c) \). On the other hand [1.2.5] shows that \( M(q_A + q_v) \) is congruent to the matrix

\[ E(v) := \begin{pmatrix} 1 & 0 \\ 0 & M_J(v) \end{pmatrix} \]  

(1.2.13)

Thus we have an embedding [1.2.8] with image \( V(E(v) \cdot \xi, \xi \cdot \xi^t - E(v)^c) \). A straightforward computation shows that the latter subscheme is isomorphic to \( X_J \cap (\mathcal{U} \times \mathbb{C}^k) \). \( \square \)
1.3. The subscheme \( C_{W,A} \). Let \((W,A)\) be an \( Y \) for the definition of the subscheme \( C_{W,A} \subset \mathbb{P}(W) \) we refer to Subs. 3.2 of [23].

**Definition 1.5.** Let \( \mathcal{R}(W,A) \subset \mathbb{P}(W) \) be the set of \([w]\) such that one of the following holds:

1. There exists \( W' \in (\Theta_A \setminus \{ W \}) \) containing \( w \).
2. \( \dim(A \cap (w \wedge 2V) \cap (\wedge^2 W \wedge V)) \geq 2 \).

The following result is obtained by pasting together Proposition 3.3.6 and Corollary 3.3.7 of [23].

**Proposition 1.6.** Let \((W,A)\) be an \( Y \). Then the following hold:

1. \( C_{W,A} \) is a smooth curve at \([v_0]\) if and only if \( \dim(A \cap (v_0 \wedge 2V)) = 2 \) and \([v_0] \notin \mathcal{R}(W,A) \).
2. \( C_{W,A} = \mathbb{P}(W) \) if and only if \( \mathcal{R}(W,A) = \mathbb{P}(W) \).

1.4. The divisor \( \Sigma \). Given \( d \geq 0 \) we let \( \tilde{\Sigma}[d] \subset \tilde{\Sigma} \) be

\[
\tilde{\Sigma}[d] := \{(W,A) \in \tilde{\Sigma} | \dim(A \cap (\wedge^2 W \wedge V)) \geq d + 1 \}.
\]  

(1.4.1)

Notice that \( \tilde{\Sigma}[0] \). Let

\[
\text{Gr}(3,V) \times \mathbb{L}G(\wedge^3 V) \xrightarrow{\pi} \mathbb{L}G(\wedge^3 V)
\]

be projection and \( \Sigma[d] := \pi(\tilde{\Sigma}[d]) \). Notice that \( \Sigma := \Sigma[0] \). Proposition 3.1 of [21] gives that

\[
\text{cod}(\Sigma[d], \mathbb{L}G(\wedge^3 V)) = (d^2 + d + 2)/2.
\]

(1.4.3)

Let

\[
\Sigma_+ := \{ A \in \Sigma | \text{Card}(\Theta_A) > 1 \}.
\]

(1.4.4)

Proposition 3.1 of [21] gives that \( \Sigma_+ \) is a constructible subset of \( \mathbb{L}G(\wedge^3 V) \) and

\[
\text{cod}(\Sigma_+, \mathbb{L}G(\wedge^3 V)) = 2.
\]

(1.4.5)

We claim that

\[
\text{sing} \Sigma = \Sigma_+ \cup \Sigma[1].
\]

(1.4.6)

In fact \( (\Sigma_+ \setminus \Sigma[1]) \subset \Sigma[1] \) by Equation (3.19) of [21] and hence (1.4.6) follows from Proposition 3.2 of [21]. We let \( \Sigma_{\infty} := \{ A \in \mathbb{L}G(\wedge^3 V) | \dim(\Theta_A) > 0 \} \).

(1.4.7)

Theorem 3.37 and Table 3 of [21] give the following:

\[
\text{cod}(\Sigma_{\infty}, \mathbb{L}G(\wedge^3 V)) = 7.
\]

(1.4.8)

1.5. The divisor \( \Delta \). Let

\[
\Delta := \{ A \in \mathbb{L}G(\wedge^3 V) | \exists [v] \in \mathbb{P}(V) \text{ such that } \dim(A \cap (v \wedge 2V)) \geq 3 \}.
\]

(1.5.1)

A dimension count gives that \( \Delta \) is a prime divisor in \( \mathbb{L}G(\wedge^3 V) \), see [22]. Let

\[
\hat{\Delta}(0) := \{( [v], A ) \in \mathbb{P}(V) \times \mathbb{L}G(\wedge^3 V) | \dim(A \cap (v \wedge 2V)) = 3 \}.
\]

(1.5.2)

The following result will be handy.

**Proposition 1.7.** Let \( A \in \mathbb{L}G(\wedge^3 V) \) and suppose that \( \dim Y_A[3] > 0 \). Then \( A \in (\Sigma_{\infty} \cup \Sigma[2]) \).

**Proof.** By contradiction. Thus we assume that \( \dim Y_A[3] > 0 \) and \( A \notin (\Sigma_{\infty} \cup \Sigma[2]) \). By hypothesis there exists an irreducible component \( C \) of \( Y_A[3] \) of strictly positive dimension. Let \([v] \in C \) be generic. We claim that one of the following holds:

(a) There exist distinct \( W_1([v]), W_2([v]) \in \Theta_A \) containing \( v \).

(b) There exists \( W([v]) \in \Theta_A \) containing \( v \) and such that

\[
\dim A \cap S_{W([v])} \cap (v \wedge 2V) \geq 2.
\]

(1.5.3)
In fact assume first that \( \dim(\mathcal{A} \cap (v \wedge \Lambda^2 V)) = 3 \) for \([v]\) in an open dense \( C^0 \subset C \). We may assume that \( C^0 \) is smooth; then we have an embedding \( \iota: C^0 \to \Delta(0) \) defined by mapping \([v]\) to \([v], A\). Let \([v]\) \( \in \mathcal{C}^0 \): the differential of the projection \( \Delta(0) \to \mathcal{L}/(\Lambda^3 V) \) at \([v], A\) is not injective because it vanishes on \( \text{Im}\, \delta(\mathcal{L}) \). By Corollary 3.4 and Proposition 3.5 of [22] we get that one of Items (a), (b) above holds. Now assume that \( \dim(\mathcal{A} \cap (v \wedge \Lambda^2 V)) > 3 \) for generic \([v]\) \( \in C \) (and hence for all \([v]\) \( \in C \)). Let notation be as in the proof of Proposition 3.5 of [22]: then \( \mathcal{K} \cap \mathcal{G}(2, V_0) \) is a zero-dimensional (if it has strictly positive dimension then \( \dim \Theta_A > 0 \) and hence \( A \in \Sigma_\infty \) against our assumption) scheme of length 5. It follows that either Item (a) holds (if \( \mathcal{K} \cap \mathcal{G}(2, V_0) \) is not a single point) or Item (b) holds (if \( \mathcal{K} \cap \mathcal{G}(2, V_0) \) is a single point \( p \) and hence the tangent space of \( \mathcal{K} \cap \mathcal{G}(2, V_0) \) at \( p \) has dimension at least 1). Now we are ready to reach a contradiction. First suppose that Item (a) holds. Since \( \Theta_A \) is finite there exist distinct \( W_1, W_2 \subset \Theta_A \) such that \( C \subset (\mathcal{P}(W_1) \cap \mathcal{P}(W_2)) \). Thus \( \dim(W_1 \cap W_2) = 2 \) and hence the line
\[
\{ W \in \mathcal{G}(3, V) \mid (W_1 \cap W_2) \subset W \subset (W_1 + W_2) \}
\] (1.5.4)
is contained in \( \Theta_A \), that is a contradiction. Now suppose that Item (b) holds. Since \( \Theta_A \) is finite there exists \( W \subset \Theta_A \) such that \( C \subset \mathcal{P}(W) \) and
\[
\dim A \cap S_W \cap (v \wedge \Lambda^2 V) \geq 2 \quad \forall [v] \in C. \tag{1.5.5}
\]
Since \( A \notin \Sigma[2] \) we have \( \dim(\mathcal{A} \cap (\Lambda^2 W \wedge V)) = 2 \). Let \( \{w_1, w_2, w_3\} \) be a basis of \( W \); then
\[
A \cap (\Lambda^2 W \wedge V) = \langle w_1 \wedge w_2 \wedge w_3, \beta \rangle. \tag{1.5.6}
\]
Let \( \bar{\beta} \) be the image of \( \beta \) under the quotient map \( (\Lambda^2 W \wedge V) \to (\Lambda^2 W \wedge V)/\Lambda^3 W \). Then
\[
\bar{\beta} \in \Lambda^2 W \wedge (V/W) \cong \text{Hom}(W, V/W). \tag{1.5.7}
\]
(We choose a volume form on \( W \) in order to define the isomorphism above.) By (1.5.5) the kernel of \( \bar{\beta} \) (viewed as a map \( W \to (V/W) \)) contains all \( v \) such that \([v]\) \( \in C \). Thus \( \bar{\beta} \) has rank 1. It follows that \( \beta \) is decomposable: \( \beta \in \Lambda^3 W' \) where \( W' \subset \Theta_A \) and \( \dim W \wedge W' = 2 \). Then \( \Theta_A \) contains the line in \( \mathcal{G}(3, V) \) joining \( W \) and \( W' \): that is a contradiction. \( \Box \)

1.6. Lattices and periods. Let \( L \) be an even lattice: we will denote by \( (,\, \cdot) \) the bilinear symmetric form on \( L \) and for \( v \subset L \) we let \( v^2 := (v, v) \). For a ring \( R \) we let \( L_R := L \otimes_{\mathbb{Z}} R \) and we let \( (,\, R) \) be the \( R \)-bilinear symmetric form on \( L_R \) obtained from \( (,\, \cdot) \) by extension of scalars. Let \( L^\vee := \text{Hom}(L, \mathbb{Z}) \). The bilinear form defines an embedding \( L \to L^\vee \): the quotient \( D(L) := L^\vee/L \) is the discriminant group of \( L \). Let \( 0 \neq v \subset L \) be primitive i.e. \( L/(v) \) is torsion-free. The divisibility of \( v \) is the positive generator of \( (v, L) \) and is denoted by \( \text{div}(v) \); we let \( v^* := v/\text{div}(v) \in D(L) \). The group \( O(L) \) of isometries of \( L \) acts naturally on \( D(L) \). The stable orthogonal group is equal to
\[
\tilde{O}(L) := \ker(O(L) \to D(L)). \tag{1.6.1}
\]
We let \( q_L : D(L) \to \mathbb{Q}/2\mathbb{Z} \) and \( b_L : D(L) \times D(L) \to \mathbb{Q}/\mathbb{Z} \) be the discriminant quadratic-form and symmetric bilinear form respectively, see [13]. The following criterion of Eichler will be handy.

Proposition 1.8 (Eichler’s Criterion, see Prop. 3.3 of [9]). Let \( L \) be an even lattice which contains \( U^2 \) (the direct sum of two hyperbolic planes). Let \( v_1, v_2 \subset L \) be non-zero and primitive. There exists \( g \subset \tilde{O}(L) \) such that \( gv_1 = v_2 \) if and only if \( v_1^2 = v_2^2 \) and \( v_1^* = v_2^* \).

Now suppose that \( L \) is an even lattice of signature \((2, n)\). Let
\[
\Omega_L := \{ [\sigma] \in \mathbb{P}(L) \mid (\sigma, \sigma)_C = 0, \ (\sigma, \sigma)_C > 0 \}, \tag{1.6.2}
\]
(Notice that the isomorphism class of \( \Omega_L \) depends on \( n \) only.) Then \( \Omega_L \) is the union of two disjoint bounded symmetric domains of Type IV on which \( O(L) \) acts. By Baily and Borel’s fundamental results the quotient
\[
\mathbb{D}_L := \tilde{O}(L)\backslash \Omega_L. \tag{1.6.3}
\]
is quasi-projective.
Remark 1.9. Suppose that \(v_0 \in L\) has square 2. The reflection

\[
L \xrightarrow{R_{v_0}} L \quad v \mapsto v - (v, v_0)v_0 \tag{1.6.4}
\]

belongs to the stable orthogonal group. We claim that \(R_{v_0}\) exchanges the two connected components of \(\Omega_L\). In fact let \(M \subset L_8\) be a positive definite subspace of maximal dimension (i.e. 2) containing \(v_0\). If \([\sigma]\in \Omega_L \cap (M_2)\) then \(R_{v_0}([\sigma]) = [\overline{\sigma}]\); this proves our claim because conjugation interchanges the two connected components of \(\Omega_L\). It follows that if \(L\) contains a vector of square 2 then \(\mathbb{D}_L\) is connected.

Let us examine the lattices of interest to us. Let \(O \in \mathbb{Z}\) be a single \(O\)-module deformation equivalent to the Hilbert square of a \(K_3\) then \(H^2(X; \mathbb{Z})\) equipped with the Beauville-Bogomolov quadratic form is isometric to \(\tilde{\Lambda}\). A vector in \(\tilde{\Lambda}\) of square 2 has divisibility 1: it follows from Proposition 1.8 that any two vectors in \(\tilde{\Lambda}\) of square 2 are \(O(\tilde{\Lambda})\)-equivalent and hence the isomorphism class of \(v^2\) for \(v^2 = 2\) is independent of \(v\). We choose \(v_1 \in J\) of square 2 and let \(\Lambda := v_1^+\). Then

\[
\Lambda \cong U^2 \oplus E_8(-1)^2 \oplus (-2)^2. \tag{1.6.5}
\]

If \(X\) is a HK manifold deformation equivalent to the Hilbert square of a \(K_3\) then \(H^2(X; \mathbb{Z})\) equipped with the Beauville-Bogomolov quadratic form is isometric to \(\tilde{\Lambda}\). A vector in \(\tilde{\Lambda}\) of square 2 has divisibility 1: it follows from Proposition 1.8 that any two vectors in \(\tilde{\Lambda}\) of square 2 are \(O(\tilde{\Lambda})\)-equivalent and hence the isomorphism class of \(v^2\) for \(v^2 = 2\) is independent of \(v\). We choose \(v_1 \in J\) of square 2 and let \(\Lambda := v_1^+\). Then

\[
\Lambda \cong U^2 \oplus E_8(-1)^2 \oplus (-2)^2. \tag{1.6.6}
\]

Such an isomorphism is a marking of \((X, h)\). If \(H\) is a divisor on \(X\) of square 2 a marking of \((X, H)\) is a marking of \((X, c_1(\mathcal{O}_X(H)))\). Let \(\psi_\mathbb{C} : H^2(X; \mathbb{C}) \to \tilde{\Lambda}_\mathbb{C}\) be the \(\mathbb{C}\)-linear extension of \(\psi\). Since \(h\) is of type \((1, 1)\) we have that \(\psi_\mathbb{C}(H(2,0)) \in v_1^+\). Well-known properties of the Beauville-Bogomolov quadratic form give that \(\psi_\mathbb{C}(H(2,0)) \in \Omega_\Lambda\). Any two markings of \((X, h)\) differ by the action of an element of \(O(\tilde{\Lambda})\). It follows that the equivalence class

\[
\Pi(X, h) := [\psi_\mathbb{C}(H(2,0))] \in \mathbb{D}_\Lambda \tag{1.6.8}
\]

is well-defined i.e. independent of the marking: that is the period point of \((X, h)\). Since the lattice \(\Lambda\) contains vectors of square 2 the quotient \(\mathbb{D}_\Lambda\) is irreducible by Remark 1.9. The discriminant group and discriminant quadratic form of \(\Lambda\) are described as follows. Let \(e_1\) be a generator of \(v_1^+ \cap J\) and let \(e_2\) be a generator of the last summand of \(\Lambda\). We have the following:

\[
\mathbb{D}_\Lambda = \mathbb{D}(\Lambda) / \mathbb{D}(\Lambda) \equiv -\frac{1}{2} x^2 - \frac{1}{2} y^2 \pmod{2\mathbb{Z}} \tag{1.6.10}
\]

In particular we get that

\[
[O(\Lambda) : \tilde{O}(\Lambda)] = 2. \tag{1.6.11}
\]

Let \(\iota \in O(\Lambda)\) be the involution characterized by

\[
\iota(e_1) = e_2, \quad \iota(e_2) = e_1, \quad \iota|_{\{e_1, e_2\}} = \text{Id}_{\{e_1, e_2\}}. \tag{1.6.12}
\]

Then \(\iota \notin \tilde{O}(\Lambda)\). Since \([O(\Lambda) : \tilde{O}(\Lambda)] = 2\) we get that \(\iota\) induces a non-trivial involution

\[
\mathbb{D}^{BB}_\Lambda \to \mathbb{D}^{BB}_\Lambda. \tag{1.6.13}
\]
The geometric counterpart of $\tau$ is given by the involution $\delta: \mathfrak{m} \to \mathfrak{m}$ induced by the map

$$LG(\Lambda^3 V) \quad \xrightarrow{\delta_V} \quad LG(\Lambda^3 V^\vee)$$

$$A \quad \mapsto \quad \delta_V(A) = \text{Ann} A.$$  \hfill (1.6.14)

(The geometric meaning of $\delta_V(A)$: for generic $A \in LG(\Lambda^3 V)$ the dual of $Y_A$ is equal to $Y_{\delta_V(A)}$.) In [20] we proved that

$$\tau \circ p = p \circ \delta.$$  \hfill (1.6.15)

1.7. Roots of $\Lambda$. Let $v_0 \in \Lambda$ be primitive and let $v_0^2 = -2d \neq 0$: then $v_0$ is a root if the reflection

$$\Lambda \quad \xrightarrow{v} \quad \Lambda, \quad \Lambda_0 \quad \mapsto \quad \Lambda_0 + \frac{\Lambda_0(v_0v_0)}{d}$$

is integral, i.e. $R(\Lambda) \subset \Lambda$. We record the square of $v_0$ by stating that $v_0$ is $(-2d)$-root. Notice that if $v_0^2 = \pm 2$ then $v_0$ is a root. In particular $e_1$ and $e_2$ are $(-2)$-roots of $\Lambda$. Let

$$e_3 \in M, \quad e_3^2 = -2.$$  \hfill (1.7.2)

Notice that $e_3 \in \Lambda$ and hence it is a $(-2)$-root of $\Lambda$. Since $(e_1 + e_2)^2 = -4$ and $\text{div}(e_1 + e_2) = 2$ we get that $(e_1 + e_2)$ is a $(-4)$-root of $\Lambda$.

**Proposition 1.10.** The set of negative roots of $\Lambda$ breaks up into 4 orbits for the action of $\tilde{O}(\Lambda)$, namely the orbits of $e_1, e_2, e_3$ and $(e_1 + e_2)$.

**Proof.** First let us prove that the orbits of $e_1$, $e_2$, $e_3$ and $(e_1 + e_2)$ are pairwise disjoint. Since $-2 = e_1^2 = e_2^2 = e_3^2$ and $(e_1 + e_2)^2 = -4$ the orbits of $e_1$, $e_2$ and $e_3$ are disjoint from that of $(e_1 + e_2)$. We have $\text{div}(e_3) = 1$ and hence $e_3^2 = 0$. Since $e_1$, $e_2$ and $e_3$ are pairwise distinct elements of $D(\Lambda)$ it follows that the orbits of $e_1$, $e_2$, $e_3$ are pairwise disjoint. Now let $v_0 \in \Lambda$ be a negative root. Since $D(\Lambda)$ is 2-torsion $\text{div}(v_0) \in \{1, 2\}$: it follows that $v_0$ is either a $(-2)$-root or a $(-4)$-root, and in the latter case $\text{div}(v_0) = 2$. Suppose first that $v_0$ is a $(-2)$-root. If $\text{div}(v_0) = 1$ then $v_0^2 = 0$ and hence $v_0$ is in the orbit of $e_3$ by **Proposition 1.8**. If $\text{div}(v_0) = 2$ then $v^* \in \{e_1^*, e_2^*\}$ because $qA(e_1^* + e_2^*) = -1 \equiv -1/2 \pmod{2\mathbb{Z}}$: it follows from **Proposition 1.8** that $v_0$ belongs either to the $\tilde{O}(\Lambda)$-orbit of $e_1$ or to that of $e_2$. Lastly suppose that $v_0$ is a $(-4)$-root. Since $\text{div}(v_0) = 2$ we have $qA(v_0^2) = -1$ and hence $v_0^4 = e_1/2 + e_2/2$: it follows from **Proposition 1.8** that $v_0$ belongs to the $\tilde{O}(\Lambda)$-orbit of $(e_1 + e_2)$. \hfill $\square$

Let $\kappa: \Omega \to \mathbb{D}_A$ be the quotient map. Let

$$S'_2 := \kappa(e_1^+ \cap \Omega), \quad S''_2 := \kappa(e_2^+ \cap \Omega), \quad S'_3 := \kappa(e_3^+ \cap \Omega), \quad S'_4 := \kappa((e_1 + e_2)^+ \cap \Omega).$$  \hfill (1.7.3)

**Remark 1.11.** Let $i = 1, 2, 3$: then $e_i^+ \cap \Omega$ has two connected components - see **Remark 1.9**. Let $v_0 \in N$ (we refer to (1.6.5) of square 2). Then $(v_0, e_i) = 0$ for $i = 1, 2, 3$ and hence Reflection (1.6.4) exchanges the two connected components of $e_i^+ \cap \Omega$ for $i = 1, 2, 3$ and also the two connected components of $(e_1 + e_2)^+ \cap \Omega$. It follows that each of $S'_2$, $S''_2$, $S'_3$ and $S'_4$ is a prime divisor in $\mathbb{D}_A$.

Let $\tau$ be the involution given by (1.6.13): then

$$\tau(S'_2) = S'_2, \quad \tau(S''_2) = S'_2, \quad \tau(S'_3) = S'_3, \quad \tau(S'_4) = S'_4.$$  \hfill (1.7.4)

We will describe the normalization of $S'_2$ and we will show that it is a finite cover of the period space for $K3$ surfaces of degree 2. Let $v_3$ be a generator of $e_3^+ \cap M$. Let

$$\tilde{\Gamma} := e_3^+ = J \oplus \mathbb{Z}v_3 \oplus N \oplus E_k(-1)^2 \oplus \mathbb{Z}e_2 \cong U \oplus (2) \oplus U \oplus E_k(-1)^2 \oplus (-2)$$

and

$$\Gamma := e_3^+ \cap \Lambda \cong \mathbb{Z}v_3 \oplus N \oplus E_k(-1)^2 \oplus \mathbb{Z}e_2 \cong (-2) \oplus (2) \oplus U \oplus E_k(-1)^2 \oplus (-2).$$  \hfill (1.7.5)

We have $\Omega_{\Gamma} = e_3^+ \cap \Omega$. Viewing $\tilde{O}(\Gamma)$ as the subgroup of $\tilde{O}(\Lambda)$ fixing $e_3$ we get a natural map

$$\nu: \mathbb{D}^{BB}_\Gamma \to \mathbb{S}'_2.$$  \hfill (1.7.7)

**Claim 1.12.** Map (1.7.7) is the normalization of $S'_2$. 


Proof. Since $\mathbb{D}^{BB}_1$ is normal and $\nu$ is finite it suffices to show that $\nu$ has degree 1. Let $[\sigma] \in e_1^\perp \cap \Omega$ be generic. Let $g \in \tilde{O}(\Lambda)$ and $[\tau] = g([\sigma])$. We must show that there exists $g' \in \tilde{O}(\Gamma)$ such that $[\tau] = g'( [\sigma])$. Since $[\sigma]$ is generic we have that

$$\sigma^\perp \cap \Lambda = \mathbb{Z} e_3. \quad (1.7.8)$$

It follows that $g(e_3) = \pm e_3$. If $g(e_3) = e_3$ then $g \in \tilde{O}(\Gamma)$ and we are done. Suppose that $g(e_3) = -e_3$. Let $g' := (-1)_{\Lambda} \circ g$. Since multiplication by 2 kills $D(\Lambda)$ we have that $(-1)_{\Lambda} \in \tilde{O}(\Lambda)$ and hence $g' \in \tilde{O}(\Lambda)$; in fact $g'(e_3) = e_3$. On the other hand $[\tau] = g'( [\sigma])$ because $(-1)_{\Lambda}$ acts trivially on $\Omega_\Lambda$. \Box

Our next task will be to define a finite map from $\mathbb{D}^{BB}_1$ to the Baily-Borel compactification of the period space for K3 surfaces with a polarization of degree 2. Let

$$\tilde{\Phi} := J \oplus (v_3, (v_3 + e_2)/2) \oplus N \oplus E_8(-1)^2 \cong U^3 \oplus E_8(-1)^2. \quad (1.7.9)$$

Then $\tilde{\Phi}$ is isometric to the K3 lattice i.e. $H^2(K3; \mathbb{Z})$ equipped with the intersection form. Let

$$\Phi := v_1^+ \cap \tilde{\Phi} := \mathbb{Z} e_1 \oplus (v_3, (v_3 + e_2)/2) \oplus N \oplus E_8(-1)^2 \cong (-2) \oplus U^2 \oplus E_8(-1)^2. \quad (1.7.10)$$

Then $\mathbb{D}_\Phi$ is the period space for K3 surfaces with a polarization of degree 2.

Claim 1.13. $\tilde{\Phi}$ is the unique lattice contained in $\Lambda_Q$ (with quadratic form equal to the restriction of the quadratic form on $\Lambda_Q$) and containing $\tilde{\Gamma}$ as a sublattice of index 2.

Proof. First it is clear that $\tilde{\Gamma}$ is contained in $\tilde{\Phi}$ as a sublattice of index 2. Now suppose that $L$ is a lattice contained in $\Lambda_Q$ and containing $\tilde{\Gamma}$ as a sublattice of index 2. Then $L$ must be generated by $\tilde{\Gamma}$ and an isotropic element of $D(\tilde{\Gamma})$: since there is a unique such element $L$ is unique. \Box

By Claim 1.13 every isometry of $\Lambda$ induces an isometry of $\tilde{\Phi}$. It follows that we have well-defined injection $O(\Lambda) < O(\tilde{\Phi})$. Since $\Omega_\Lambda = \Omega_\Phi$ there is an induced finite map

$$\rho: \mathbb{D}^{BB}_1 \longrightarrow \mathbb{D}^{BB}_\Phi. \quad (1.7.11)$$

Remark 1.14. Keep notation as above. Then $\deg \rho = 2^{20} - 1$.

1.8. Determinant of a variable quadratic form. Let $U$ be a complex vector-space of finite dimension $d$. We view $S^2 U^\vee$ as the vector-space of quadratic forms on $U$. Given $q \in S^2 U^\vee$ we let $\tilde{q}: U \to U^\vee$ be the associated symmetric map. Let $K := \ker q$; then $\tilde{q}$ may be viewed as a (symmetric) map $\tilde{q}: (U/K) \to \Ann K$. The dual quadratic form $q^\vee$ is the quadratic form associated to the symmetric map $\tilde{q}^{-1}: \Ann K \to (U/K)$; thus $q^\vee \in S^2(U/K)$. We will denote by $\wedge \ q$ the quadratic form induced by $q$ on $\wedge^i U$.

Remark 1.15. If $0 \neq \alpha = v_1 \wedge \cdots \wedge v_i$ is a decomposable vector of $\wedge^i U$ then $\wedge^i q(\alpha)$ is equal to the determinant of the Gram matrix of $q|_{\{v_1,\ldots,v_i\}}$ with respect to the basis $\{v_1,\ldots,v_i\}$.

The following exercise in linear algebra will be handy.

Lemma 1.16. Suppose that $q \in S^2 U^\vee$ is non-degenerate. Let $S \subset U$ be a subspace. Then

$$\cork(q|_S) = \cork(q^\vee|_{\Ann(S)}). \quad (1.8.1)$$

Let $q_* \in S^2 U^\vee$. Then

$$\det(q_* + q) = \Phi_0(q) + \Phi_1(q) + \cdots + \Phi_d(q), \quad \Phi_i \in S^i(S^2 U). \quad (1.8.2)$$

Of course $\det(q_* + q)$ is well-defined up to multiplication by a non-zero scalar and hence so are the $\Phi_i$’s. The result below is well-known (it follows from a straightforward computation).

Proposition 1.17. Let $q_* \in S^2 U^\vee$ and

$$K := \ker(q_*), \quad k := \dim K. \quad (1.8.3)$$

Let $\Phi_i$ be the polynomials appearing in (1.8.2). Then

1. $\Phi_i = 0$ for $i < k$, and
2. there exists $c \neq 0$ such that $\Phi_k(q) = c \det(q|_K)$.

Keep notation and hypotheses as in Proposition 1.17. Let $\mathcal{V}_K \subset S^2 U^\vee$ be the subspace of quadratic forms whose restriction to $K$ vanishes. Given $q \in \mathcal{V}_K$ we have $\tilde{q}(K) \subset \Ann K$ and hence it makes sense to consider the restriction of $q^\vee$ to $\tilde{q}(K)$.
Proposition 1.18. Keep notation and hypotheses as in Proposition 1.17. The restriction of $\Phi_1$ to $\mathcal{Y}_K$ vanishes for $I < 2k$. Moreover there exists $c \neq 0$ such that

$$\Phi_{2k}(q) = c \det(q^*_e | q|_K), \quad q \in \mathcal{Y}_K.$$  \hspace{1cm} (1.8.4)

Proof. Choose a basis $\{u_1, \ldots, u_d\}$ of $U$ such that $K = \langle u_1, \ldots, u_k \rangle$ and $\tilde{q}_*(u_i) = \tilde{u}_i$ for $k < i \leq d$. Let $M$ be the Gram matrix of $q$ in the chosen basis. Expanding $\det(q_* + tq)$ we get that

$$\det(q_* + tq) \equiv (-1)^i t^{2i} \sum_j (\det M_{k,j})^2 \pmod{t^{2k+1}}$$

where $M_{k,j}$ is the $k \times k$ submatrix of $M$ determined by the first $k$ rows and the columns indexed by $J = (j_1, j_2, \ldots, j_k)$. The claim follows from the equality

$$\sum_j (\det M_{k,j})^2 = \wedge^k q^*_e(\tilde{q}(u_1) \wedge \ldots \wedge \tilde{q}(u_k))$$

and Remark 1.15. \hfill \Box

Now suppose that

$$\text{cork} \tilde{q}_* = 1, \quad \ker \tilde{q}_* = \langle e \rangle.$$  \hspace{1cm} (1.8.5)

We let $\mathcal{T}_* \in S^2(U/\langle e \rangle)^\vee$ be the non-degenerate quadratic form induced by $q_*$ i.e. $\mathcal{T}_*(\mathcal{T}) := q_*(e)$ for $\mathcal{T} \in U/\langle e \rangle$. Let ..., $\Phi_i, \ldots$ be as in (1.8.2). In particular $\Phi_0 = 0$. Assume that

$$L \subset \ker \Phi_1 = \{q | q(e) = 0\}$$  \hspace{1cm} (1.8.6)

is a vector subspace. Thus

$$\det(q_* + q)|_{L} = \Phi_2|_{L} + \ldots + \Phi_d|_{L}.$$  \hspace{1cm} (1.8.7)

We will compute $\text{rk}(\Phi_2|_{L})$. Let $T \subset U$ be defined by

$$T := \text{Ann}(\tilde{q}(e))_{q \in L}.$$  \hspace{1cm} (1.8.8)

where $L$ and $e$ are as above. Geometrically: $P(T)$ is the projective tangent space at $[e]$ of the intersection of the projective quadrics parametrized by $P(L)$.

Proposition 1.19. Suppose that $L \subset S^2 U^\vee$ is a vector subspace such that (1.8.6) holds. Keep notation as above, in particular $T$ is given by (1.8.5). Then

$$\text{rk}(\Phi_2|_{L}) = \text{cod}(T, U) - \text{cork}(\mathcal{T}_*|_{T/\langle e \rangle}).$$  \hspace{1cm} (1.8.9)

(The last term on the right-side makes sense because $T \supset \langle e \rangle$.)

Proof. Let

$$L \xrightarrow{\alpha} (U/\langle e \rangle)^\vee \xrightarrow{\tilde{q}(e)} \mathcal{T}.$$  \hspace{1cm} (1.8.10)

By Proposition 1.18 we have

$$\text{rk}(\Phi_2|_{L}) = \text{rk}(\mathcal{T}_*|_{\text{Im}(\alpha)}).$$  \hspace{1cm} (1.8.11)

On the other hand Lemma 1.16 gives that

$$\text{rk}(\mathcal{T}_*|_{\text{Im}(\alpha)}) = \dim \text{Im}(\alpha) - \text{cork}(\mathcal{T}_*|_{\text{Ann}(\text{Im}(\alpha)))}).$$  \hspace{1cm} (1.8.12)

By definition $\text{Ann}(\text{Im}(\alpha)) = T/\langle e \rangle$. Since $\dim \text{Im}(\alpha) = \text{cod}(T, U)$ we get the proposition. \hfill \Box

2. First extension of the period map

2.1. Local structure of $Y_A$ along a singular plane. Let $(W, A) \in \tilde{\Sigma}$. Then $P(W) \subset Y_A$. In this section we will analyze the local structure of $Y_A$ at $v_0 \in (P(W) \setminus C_{W,A})$ under mild hypotheses on $A$. Let $[v_0] \in P(W)$ - for the moment being we do not require that $v_0 \notin C_{W,A}$. Let $V_0 \subset V$ be a subspace transversal to $[v_0]$. We identify $V_0$ with an open affine neighborhood of $[v_0]$ via (1.1.2); thus $0 < v_0$ corresponds to $[v_0]$. Let $f_i \in S^1 \overline{V_0}$ for $i = 0, \ldots, 6$ be the polynomials of (1.1.3). By Item (2) of Proposition 1.11 we have

$$Y_A \cap V_0 = V(f_2 + \ldots + f_6).$$  \hspace{1cm} (2.1.1)

Suppose that $Y_A \neq P(V)$. Then $[v_0] \in \text{sing} Y_A$; since $[v_0]$ is an arbitrary point of $P(W)$ we get that $P(W) \subset \text{sing} Y_A$. It follows that $\text{rk} f_2 \leq 3$. 

Proposition 2.1. Let \((W, A) \in \mathfrak{S}\) and suppose that \(Y_{(A)} \neq \mathbb{P}(V^\vee)\). Let \([v_0] \in (\mathbb{P}(W) \setminus C_{W,A})\) and \(f_2\) be the quadratic term of the Taylor expansion of a local equation of \(Y_A\) centered at \([v_0]\). Then

\[
\text{rk} f_2 = 4 - \dim (A \cap (\bigwedge^2 W \wedge V)).
\]  

(2.1.2)

Proof. By hypothesis there exists a subspace \(V_0 \subset V\) such that (1.1.1)-(1.1.3) hold. Let \(\tilde{q}_A\) be as in (1.1.10) and \(q_A\) be the associated quadratic form on \(\bigwedge^2 V_0\). Let \(Q_A := V(q_A) \subset \mathbb{P}(\bigwedge^2 V_0)\). By Proposition 1.19 we have

\[
V(Y_A)|_{V_0} = V(\det(q_A + q_v))
\]

(2.1.3)

where \(q_v\) is as in (1.1.10). Let \(W_0 := W \cap V_0\). Since \([v_0] \notin C_{W,A}\) we have \(A \cap (v_0 \wedge \bigwedge^2 V) = \bigwedge^3 W\). By (1.1.9) we get that \(\text{sing} Q_A = \{[\bigwedge^2 W_0]\}\). Thus

\[
\text{det}(q_A + q_v) = \Phi_2(v) + \Phi_3(v) + \ldots + \Phi_6(v), \quad \Phi_i \in \bigwedge^i V_0^\vee
\]

(2.1.4)

and the rank of \(\Phi_2\) is given by (1.8.9) with \(q_v = q_A\) and \(L = V_0\). Let us identify the subspace \(T \subset \bigwedge^2 V_0\) given by (1.8.8). Let \(U_0 \subset V_0\) be a subspace transversal to \(W_0\); since the Plücker quadrics generate the ideal of the Grassmannian we have

\[
T = 2 W_0 \oplus W_0 \oplus U_0.
\]

(2.1.5)

By Proposition 1.19 we get that

\[
\text{rk} \Phi_2 = 3 - \dim \ker(q_A|_{W_0 \wedge U}).
\]

(2.1.6)

We claim that

\[
\dim \ker(q_A|_{W_0 \wedge U}) = \dim (A \cap (\bigwedge^2 W \wedge V)).
\]

(2.1.7)

In fact let \(\alpha \in W_0 \wedge U\). Then \(\alpha \in \ker(q_A|_{W_0 \wedge U})\) if and only if

\[
\tilde{q}_A(\alpha) \in \text{Ann}(W_0 \wedge U_0) = \bigwedge^2 W_0 \wedge U_0 \oplus \bigwedge^3 U_0.
\]

(2.1.8)

Since \(A \subset (\bigwedge^3 W)^\perp\) it follows from (1.1.3) that necessarily \(\tilde{q}_A(\alpha) \in \bigwedge^2 W_0 \wedge U_0\). Equation (1.1.8) gives a linear map

\[
\ker(q_A|_{W_0 \wedge U_0}) \xrightarrow{\varphi} A \cap (\bigwedge^2 W \wedge U_0)
\]

(2.1.9)

The direct-sum decomposition

\[
\bigwedge^2 W \wedge U = [v_0] \wedge W_0 \wedge U_0 \oplus \bigwedge^2 W_0 \wedge U_0
\]

(2.1.10)

shows that \(\varphi\) is bijective. Since there is an obvious isomorphism \((A \cap (\bigwedge^2 W \wedge U_0)) \cong (A \cap (\bigwedge^2 W \wedge V))/\bigwedge^3 W\) we get that (2.1.7) holds.

\[\square\]

Remark 2.2. Suppose that \(\dim (A \cap (\bigwedge^2 W \wedge V)) > 4\). Then Equation (2.1.2) does not make sense. On the other hand \(C_{W,A} = \mathbb{P}(W)\) by Proposition 1.6 and hence there is no \([v_0] \in (\mathbb{P}(W) \setminus C_{W,A})\).

2.2. The extension.

Lemma 2.3. Let \(A_0 \in (L \mathcal{G}(\bigwedge^3 V) \setminus \Sigma_\infty \setminus \Sigma[2])\). Then \(Y_{A_0}[3]\) is finite and \(C_{W,A_0}\) is a sextic curve for every \(W \in \Theta_{A_0}\).

Proof. \(Y_{A_0}[3]\) is finite by Proposition 1.7. Let \(W \in \Theta_{A_0}\). Let us show that \(\mathcal{B}(W, A_0) \neq \mathbb{P}(W)\). Let \(W' \in (\Theta_{A_0} \setminus \{W\})\). Then \(\dim(W \cap W') = 1\) because otherwise \(\bigwedge^3 W\) and \(\bigwedge^3 W'\) span a line in \(\text{Gr}(3, V)\) which is contained in \(\Theta_{A_0}\) and that contradicts the assumption that \(\Theta_{A_0}\) is finite. This proves finiteness of the set of \([w] \in \mathbb{P}(W)\) such that Item (1) of Definition 1.5 holds. Since \(\dim(\bigwedge^2 W \wedge V) \leq 2\) a similar argument gives finiteness of the set of \([w] \in \mathbb{P}(W)\) such that Item (2) of Definition 1.5 holds. This proves that \(\mathcal{B}(W, A_0)\) is finite, in particular \(\mathcal{B}(W, A_0) \neq \mathbb{P}(W)\). By Proposition 1.6 it follows that \(C_{W,A_0}\) is a sextic curve. \(\square\)

Proposition 2.4. Let \(A_0 \in (L \mathcal{G}(\bigwedge^3 V) \setminus \Sigma_\infty \setminus \Sigma[2])\) and \(L \subset \mathbb{P}(V)\) be a generic 3-dimensional linear subspace. Let \(\mathcal{W} \subset (L \mathcal{G}(\bigwedge^3 V) \setminus \Sigma_\infty \setminus \Sigma[2])\) be a sufficiently small open set containing \(A_0\). Let \(A \in \mathcal{W}\). Then the following hold:

(a) The scheme-theoretic inverse image \(f_A^{-1}L\) is a reduced surface with DuVal singularities.

(b) If in addition \(A_0 \in \mathcal{L}(\bigwedge^3 V)^0\) then \(f_A^{-1}L\) is a smooth surface.
Proof. Let \( L \subset P(V) \) be a generic 3-dimensional linear subspace. Then

(1) \( L \cap Y_{A_0} [3] = \emptyset \).

(2) \( L \cap C_{W,A_0} = \emptyset \) for every \( W \in \Theta_{A_0} \).

In fact \( Y_{A_0} [3] \) is finite by Lemma 2.3 and hence (1) holds. Since \( \Theta_{A_0} \) is finite and \( C_{W,A_0} \) is a sextic curve for every \( W \in \Theta_{A_0} \). Item (2) holds as well. We will prove that \( f_{A_0}^{-1} L \) is reduced with DuVal singularities and that it is smooth if \( A_0 \in LG(\Lambda^3 V)^0 \). The result will follow because being smooth, reduced or having DuVal singularities is an open property. Write \( \Theta_{A_0} = \{ W_1, \ldots, W_d \} \). By Item (2) above the intersection \( L \cap P(W_i) \) is a single point \( p_i \) for \( i = 1, \ldots, d \). Since \( p_i \notin C_{W_i,A_0} \) the points \( p_1, \ldots, p_d \) are pairwise distinct. By Proposition 1.1 we know that away from \( \bigcup_{W \in \Theta_{A_0}} P(W) \) the locally closed sets \( Y_{A_1}(1) \) and \( Y_{A_2}(2) \) are smooth of dimensions 4 and 2 respectively. By Item (1) it follows that \( f_{A_0}^{-1} L \) is smooth away from

\[
\bigcup_{i=1}^{d} \{ p_1, \ldots, p_d \}.
\] (2.2.1)

It remains to show that \( f_{A_0}^{-1} L \) is DuVal at each point of (2.2.1). Since \( p_i \in Y_{A_0}(1) \) the map \( f_{A_0} \) is étale of degree 2 over \( p_i \), see (2.2.1). Thus \( f_{A_0}^{-1}(p_i) = \{ q_i^+, q_i^- \} \) and \( f_{A_0} \) defines an isomorphism between the germ \((X_{A_0}, q_i^+)\) (in the classical topology) and the germ \((Y_{A_0}, p_i)\). By Proposition 2.1 we get that the tangent cone of \( f_{A_0}^{-1} L \) at \( q_i^\pm \) is a quadric cone of rank 2 or 3; it follows that \( f_{A_0}^{-1} L \) has a singularity of type \( A_0 \) at \( q_i^\pm \).

\[ \square \]

Proposition 2.5. Let \( A_0 \in (LG(\Lambda^3 V) \setminus \Sigma_\infty \setminus \Sigma[2]) \). Then \( \mathcal{P} \) is regular at \( A_0 \) and \( \mathcal{P}(A_0) \in \mathbb{A}_A \).

Proof. Let \( \mathcal{U} \) and \( L \) be as in Proposition 2.4. Let \( U \subset \mathcal{U} \) be a subset containing \( A_0 \), open in the classical topology and contractible. Let \( U^0 := U \cap LG(\Lambda^3 V)^0 \). Let \( \overline{\mathcal{U}} \subset U^0 \); thus \( X_{\mathcal{U}} \) is smooth. By Lemma 2.3 we know that \( f_{A_0}^{-1} L \) is a smooth surface for every \( A \in U^0 \). Thus \( \pi_1(U^0, \overline{\mathcal{U}}) \) acts by monodromy on \( H^2(f_{A_0}^{-1} L) \) and by Item (a) of Proposition 2.4 the image of the monodromy representation is a finite group. On the other hand \( H_{\mathcal{U}} \) is an ample divisor on \( X_{\mathcal{U}} \); by the Lefschetz Hyperplane Theorem the homomorphism

\[
H^2(X_{\mathcal{U}}, \mathbb{Z}) \longrightarrow H^2(f_{A_0}^{-1} L; \mathbb{Z})
\] (2.2.2)

is injective. The image of (2.2.2) is a subgroup of \( H^2(f_{A_0}^{-1} L) \) invariant under the monodromy action of \( \pi_1(U^0, \overline{\mathcal{U}}) \). By injectivity of (2.2.2) the monodromy action of \( \pi_1(U^0, \overline{\mathcal{U}}) \) on \( H^2(X_{\mathcal{U}}) \) is finite as well. By Griffith’s Removable Singularity Theorem (see p. 41 of [3]) it follows that the restriction of \( \mathcal{P}^0 \) to \( U^0 \) extends to a holomorphic map \( U \to \mathbb{A}_A \). Hence \( \mathcal{P}^0 \) extends regularly in a neighborhood \( A_0 \) and it goes into \( \mathbb{A}_A \).

\[ \square \]

Definition 2.6. Let \( \overline{LG}(\Lambda^3 V) \subset LG(\Lambda^3 V) \times \mathbb{A}_A^{BB} \) be the closure of the graph of the restriction of \( \mathcal{P} \) to the set of its regular points and

\[
p : \overline{LG}(\Lambda^3 V) \longrightarrow LG(\Lambda^3 V)
\] (2.2.3)

the restriction of projection. Let \( \overline{\Sigma} \subset \overline{LG}(\Lambda^3 V) \) be the proper transform of \( \Sigma \).

Corollary 2.7. Keep notation as above. Let \( A \) be in the indeterminacy locus of \( \mathcal{P} \) and \( p \) be as in (2.2.3). Then \( \dim(p^{-1}(A) \cap \overline{\Sigma}) \) has dimension at least 2.

Proof. Let \( \text{Ind}(\mathcal{P}) \) be the indeterminacy locus of \( \mathcal{P} \). Since \( LG(\Lambda^3 V) \) is smooth the morphism \( p \) identifies \( \overline{LG}(\Lambda^3 V) \) with the blow-up of \( \text{Ind}(\mathcal{P}) \). Hence the exceptional set of \( p \) is the support of a Cartier divisor \( E \). By Proposition 2.5 the indeterminacy locus of \( \mathcal{P} \) is contained in \( \Sigma \) and thus \( A \in \Sigma \). It follows that \( p^{-1}(A) \cap \overline{\Sigma} \) is not empty. Since \( \overline{\Sigma} \) is a prime divisor in \( \overline{LG}(\Lambda^3 V) \) and \( E \) is a Cartier divisor every irreducible component of \( E \cap \overline{\Sigma} \) has codimension 2 in \( \overline{LG}(\Lambda^3 V) \). On the other hand Proposition 2.5 and (1.13) give that \( \text{cod}(\text{Ind}(\mathcal{P}), LG(\Lambda^3 V)) \geq 4 \) and hence every component of a fiber of \( E \cap \overline{\Sigma} \to \text{Ind}(\mathcal{P}) \) has dimension at least 2. Since \( p^{-1}(A) \cap \overline{\Sigma} \) is one such fiber we get the corollary.

\[ \square \]

3. Second extension of the period map

3.1. \( X_A \) for generic \( A \) in \( \Sigma \). Let \( A \in (\Sigma \setminus \Sigma[2]) \) and \( W \in \Theta_A \). Then \( \mathcal{P}(W,A) \neq P(W) \) because by Lemma 2.3 we know that \( C_{W,A} \neq P(W) \). By the same Lemma \( Y_A[3] \) is finite. In particular \((P(W) \setminus \mathcal{P}(W,A) \setminus Y_A[3]) \) is not empty.
Proposition 3.1. Let $A \in (\Sigma \setminus \Sigma[2])$ and $W \in \Theta_A$. Suppose in addition that $\dim(A \cap (\Lambda^2 W \wedge V)) = 1$. Let

$$x \in f_A^{-1}(\mathbb{P}(W) \setminus \mathcal{R}(W, A) \setminus Y_A[3]).$$

The germ $(X_A, x)$ of $X_A$ at $x$ in the classical topology is isomorphic to $(\mathbb{C}^2, 0) \times A$ and sing $X_A$ is equal to $f_A^{-1}(\mathbb{P}(W))$ in a neighborhood of $x$.

Proof. Suppose first that $f_A(x) \notin C_{W,A}$. Then $f_A(x) \in Y_A(1)$ and hence $f_A$ is etale over $f_A(x)$, see (1.2.4). Thus the germ $(X_A, x)$ is isomorphic to the germ $(Y_A, f_A(x))$ and the statement of the proposition follows from Proposition 2.1 because by hypothesis $B = 0$. It remains to examine the case

$$f_A(x) \in (C_{W,A} \setminus \mathcal{R}(W, A) \setminus Y_A[3]).$$

Let $f_A(x) = \{v_0\}$. Since $A \notin \Sigma_{\infty}$ there exists a subspace $V_0 \subset V$ transversal to $\{v_0\}$ and such that (1.1.7) holds - see Proposition 1.3. Thus we may apply Proposition 1.4. We will adopt the notation of that Proposition, in particular we will identify $V_0$ with $(\mathbb{P}(V) \setminus \mathbb{P}(V_0))$ via (1.1.2). Let $W_0 := W \cap V_0$; thus $\dim W_0 = 2$. Let $K \subset \Lambda^2 V_0$ be the subspace corresponding to $\langle v_0 \wedge \Lambda^2 V \rangle \cap A$ via (1.1.5). By (3.1.2) $\dim K = 2$. Let us prove that there exists a basis $\{w_1, w_2, u_1, u_2, u_3\}$ of $V_0$ such that $w_1, w_2 \in W_0$ and

$$K = \langle w_1 \wedge w_2, w_1 \wedge u_1 + w_2 \wedge u_3 \rangle.$$  

In fact since $[v_0] \notin \mathcal{R}(W, A)$ the following hold:

1. $\mathbb{P}(K) \cap \text{Gr}(2, V_0) = \{\Lambda^2 V_0\}$.
2. $\mathbb{P}(K)$ is not tangent to $\text{Gr}(2, V_0)$.

Now let $\{\alpha, \beta\}$ be a basis of $K$ such that $\Lambda^2 W_0 = \langle \alpha \rangle$. By (1) we have that $\beta \wedge \beta \neq 0$. Let $S := \text{supp}(\beta \wedge \beta)$; thus $\dim S = 4$. Let us prove that $W_0 \not\subset S$. In fact suppose that $W_0 \subset S$. Then $K \subset \Lambda^2 S$ and since $\text{Gr}(2, S)$ is a quadric hypersurface in $\mathbb{P}(\Lambda^2 S)$ it follows that either $\mathbb{P}(K)$ intersects $\text{Gr}(2, U)$ in two points or is tangent to it, that contradicts (1) or (2) above. Let $\{w_1, w_2\}$ be a basis of $W_0$ such that $w_1 \in W_0 \cap S$; it is clear that there exist $u_1, u_2, u_3 \in S$ linearly independent such that $\beta = w_1 \wedge u_1 + w_2 \wedge u_3$. This proves that 3.1.3 holds. Rescaling $u_1, u_3$ we may assume that

$$\text{vol}_0(\wedge w_1 \wedge w_2 \wedge u_1 \wedge u_2 \wedge u_3) = 1.$$  

where $\text{vol}_0$ is our chosen volume form, see (1.1.7). Let

$$J := \langle w_1 \wedge u_1, w_1 \wedge u_2, w_1 \wedge u_3, w_2 \wedge u_1, w_2 \wedge u_2, w_2 \wedge u_3, u_1 \wedge u_2, u_1 \wedge u_3 \rangle.$$  

Thus $J$ is transversal to $K$ by 3.1.3 and hence we have Decomposition 1.2.2. Given $v \in V_0$ we write

$$v = s_1w_1 + s_2w_2 + t_1u_1 + t_2u_2 + t_3u_3.$$  

(3.1.6)

Thus $(s_1, s_2, t_1, t_2, t_3)$ are affine coordinates on $V_0$ and hence by 1.1.2 they are also coordinates on an open neighborhood of $[v_0] \in V_0$. Let $N = N_J, P = P_J, Q = Q_J, R = R_J$ be the matrix functions appearing in 1.2.3. A straightforward computation gives that

$$P(v) = \begin{pmatrix} 0 & t_1 \\ t_1 & -2s_2 \end{pmatrix}, \quad R(v) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & t_3 & -t_2 \\ -s_2 & 0 & s_1 & -t_3 & t_2 & 0 & 0 \end{pmatrix}.$$  

(3.1.7)

The $8 \times 8$-matrix $(N + Q(v))$ is invertible for $(s,t)$ in a neighborhood of 0; we set

$$(c_{ij})_{1 \leq i,j \leq 8} = -(N + Q(v))^{-1}$$  

(3.1.8)

where $c_{ij} \in \mathcal{O}_{V_0,0}$. A straightforward computation gives that

$$P(v) - R(v) \cdot (N + Q(v))^{-1} \cdot R(v)^T = \begin{pmatrix} c_{7,7}t_3^2 - 2c_{7,8}t_3^3 + c_{8,7}t_2^2 & t_1 + \delta \\ -2s_2 + \epsilon \end{pmatrix}$$  

(3.1.9)

where $\delta, \epsilon \in \mathbb{m}_0$ (here $\mathbb{m}_0 \subset \mathbb{C}[s_1, s_2, t_1, t_2, t_3]$ is the maximal ideal of $(0,\ldots, 0)$). Let us prove that

$$\det \begin{pmatrix} c_{7,7}(0) & -c_{7,8}(0) \\ -c_{8,7}(0) & c_{8,8}(0) \end{pmatrix} \neq 0.$$  

(3.1.10)

Since $Q(0) = 0$ we have $c_{ij}(0) = -(\det N)^{-1} \cdot N^T$ where $N^c = (N^{ij})_{1 \leq i,j \leq 8}$ is the matrix of cofactors of $N$. Thus 3.1.10 is equivalent to

$$\det \begin{pmatrix} N^7,7 & N^7,8 \\ N^8,7 & N^8,8 \end{pmatrix} \neq 0.$$  

(3.1.11)
The quadratic form $q_A|J$ is non-degenerate and hence we have the dual quadratic form $(q_A|J)^\vee$ on $J^\vee$. Let $U := (u_1, u_2, u_3)$ where $u_1, u_2, u_3$ are as in (3.1.3). Applying Lemma 1.16 to $q_A|J$ and the subspace $W_0 \cup U \subset J$ we get that

$$\text{cork}(q_A|_{W_0 \cup U}) = \text{cork}((q_A|J)^\vee|_{\text{Ann}(W_0 \cup U)}).$$

(3.1.12)

By (2.1.7) $q_A|_{W_0 \cup U}$ is non-degenerate: it follows that $(q_A|J)^\vee|_{\text{Ann}(W_0 \cup U)}$ is non-degenerate as well. The annihilator of $W_0 \cup U$ in $J^\vee$ is given by

$$\text{Ann}(W_0 \cup U) = \langle u_1^2 \wedge u_2^2, u_1^3 \wedge u_3^2 \rangle$$

(3.1.13)

and the Gram-matrix of $(q_A|J)^\vee|_{\text{Ann}(W_0 \cup U)}$ with respect to the basis given by (3.1.3) is equal to $(\det N)^{-1}(N^\vee)^{12}_{7,12}$. Hence $\text{Ann}(W_0 \cup U)$ holds and this proves that (3.1.10) holds. By (3.1.9) and (3.1.10) there exist new analytic coordinates $(x_1, x_2, y_1, y_2)$ on an open neighborhood $U$ of $0 \in V_0$ - with $(0, \ldots, 0)$ corresponding to $0 \in V_0$ - such that

$$P(v) - R(v) \cdot (N + Q(v))^{-1} \cdot R(v)^t = \begin{pmatrix} x_1^2 + x_2^2 & y_1 \\ y_1 & y_2 \end{pmatrix}.$$  

(3.1.14)

(Recall that $\delta, \epsilon \in m_0^2$.) By Proposition 1.4 we get that

$$f_A^{-1}U = V(\xi_3^2 - y_2, \xi_1\xi_2 + y_1, \xi_3^2 - x_1^2 - x_2^2) \subset U \times C^2$$

(3.1.15)

where $(\xi_1, \xi_2)$ are coordinates on $C^2$ and our point $x \in X_A$ has coordinates $(0, \ldots, 0)$. (Notice that if $k = 2$ then the entries of the first matrix of (1.2.6) belong to the ideal generated by the second matrix of (1.2.5).) Let $B^3(0, r) \subset C^3$ be a small open ball centered at the origin and let $(x_1, x_2, y_1)$ be coordinates on $C^3$; there is an obvious isomorphism between an open neighborhood of $0 \in f_A^{-1}U$ and

$$V(\xi_3^2 - x_1^2 - x_2^2) \subset B^3(0, r) \times C^2$$

(3.1.16)

taking $(0, \ldots, 0)$ to $(0, \ldots, 0)$. This proves that $X_A$ is singular at $x$ with analytic germ as claimed. It follows that $f_A^{-1}(\mathbb{P}(W) \setminus \mathcal{S}(W, A) \setminus Y_A[3]) \cap \mathbb{P}(W)$. On the other hand an arbitrary point $x'$ in a sufficiently small neighborhood of $x$ is mapped to $Y_A(1)$ and if it does not belong to $f_A^{-1}(\mathbb{P}(W))$ the map $f_A$ is étale at $x'$: by Proposition 1.1 $Y_A$ is smooth at $f(x')$ and therefore $X_A$ is smooth at $x'$. □

Let $\Sigma^{sm}$ be the smooth locus of $\Sigma$.\n
**Corollary 3.2.** Let $A \in (\Sigma^{sm} \setminus \Delta)$ and $W$ be the unique element in $\Theta_A$ (unique by (1.4.6)). Then

1. $\text{sing } X_A = f_A^{-1}\mathbb{P}(W)$.
2. $x \in f_A^{-1}\mathbb{P}(W)$. The germ $(X_A, x)$ in the classical topology is isomorphic to $(C^2, 0) \times A_1$.
3. $C_{W,A}$ is a smooth sextic curve in $\mathbb{P}(W)$.
4. The map

$$f_A^{-1}\mathbb{P}(W) \rightarrow \mathbb{P}(W)$$

$$x \mapsto f_A(x)$$

(3.1.17)

is a double cover simply branched over $C_{W,A}$.

**Proof.** (1)-(2): By (1.4.6) $A \notin (\Sigma_\infty \cup \Sigma[2])$, $\dim(A \cap (A_1 \cap W \cap V)) = 1$ and $\mathcal{S}(W, A) = \emptyset$. Moreover $Y_A[3]$ is empty by definition. By Proposition 3.1 it follows that $f_A^{-1}\mathbb{P}(W) \subset X_A$ and that the analytic germ at $x \in f_A^{-1}\mathbb{P}(W)$ is as stated. It remains to prove that $X_A$ is smooth at $x \in f_A^{-1}\mathbb{P}(W)$. Since $A \notin \Delta$ we have that $f_A(x) \in (Y_A(1) \cup Y_A[2])$. If $f_A(x) \notin Y_A(1)$ then $f_A$ is étale over $f_A(x)$ (see (1.2.7)) and $Y_A$ is smooth at $f_A(x)$ by Proposition 1.1. If $f_A(x) \in Y_A(2)$ then $X_A$ is smooth at $x$ by Lemma 2.5 of (2.2.3). Immediate consequence of Proposition 1.6 (4): Map (3.1.17) is an étale cover away from $C_{W,A}$, see (1.2.1), while $f_A^{-1}(y)$ is a single point for $y \in C_{W,A}$ - see (3.1.15). Thus either $f_A^{-1}\mathbb{P}(W)$ is singular or else Map (3.1.17) is simply branched over $C_{W,A}$. Items (1), (2) show that $f_A^{-1}\mathbb{P}(W)$ is smooth: it follows that Item (4) holds. □

**Definition 3.3.** Suppose that $(W, A) \in \widetilde{\Sigma}$ and that $C_{W,A} \neq \mathbb{P}(W)$. We let

$$S_{W,A} \rightarrow \mathbb{P}(W)$$

(3.1.18)

be the double cover ramified over $C_{W,A}$. If $\Theta_A$ has a single element we let $S_A := S_{W,A}$.

**Remark 3.4.** Let $A \in (\Sigma^{sm} \setminus \Delta)$ and $W$ be the unique element of $\Theta_A$. By Item (4) of Corollary 3.2 $f_A^{-1}\mathbb{P}(W)$ is identified with $S_A$ and the restriction of $f_A$ to $f_A^{-1}\mathbb{P}(W)$ is identified with the double cover $S_A \rightarrow \mathbb{P}(W)$. In particular $f_A^{-1}\mathbb{P}(W)$ is a polarized K3 surface of degree 2.
3.2. Desingularization of $X_A$ for $A \in (\Sigma^m \setminus \Delta)$. Let $A \in (\Sigma^m \setminus \Delta)$ and $W$ be the unique element of $\Theta_A$. Let

$$\pi_A : \tilde{X}_A \to X_A$$

be the blow-up of sing $X_A$. Then $\tilde{X}_A$ is smooth by Corollary 3.2. Let

$$\tilde{H}_A := \pi_A^* H_A, \quad \tilde{h}_A := c_1(\mathcal{O}_{\tilde{X}_A}(\tilde{H}_A)).$$

Let

$$\mathcal{V} \subset \mathcal{H}(\mathcal{L}G(\mathcal{V}) \setminus \text{sing } \Sigma \setminus \Delta)$$

be an open (classical topology) contractible neighborhood of $A$. We may assume that there exists a tautological family of double EPW-sextics $\mathcal{F} \to \mathcal{W}$, see §2 of [22]. Let $\mathcal{H}$ be the tautological divisor class on $\mathcal{F}$; thus $\mathcal{H}|_{X_A} \sim H_A$. The holomorphic line-bundle $\mathcal{O}_{\mathcal{V}}(\Sigma)$ is trivial and hence there is a well-defined double cover $\phi : \mathcal{V} \to \mathcal{V}$ ramified over $\Sigma \cap \mathcal{W}$. Let $\mathcal{F}_2 := \mathcal{V} \times_\mathcal{V} \mathcal{F}$ be the base change:

$$\mathcal{F} \xrightarrow{\phi} \mathcal{V} \xrightarrow{\rho_2} \mathcal{F}_2 \xrightarrow{\pi} \mathcal{V}.$$  

(3.2.3)

Moreover $\pi$ is an isomorphism away from $g^{-1}(\phi^{-1}(\Sigma \cap \mathcal{W}))$ and

$$g^{-1}(A) \cong \tilde{X}_A, \quad \pi|_{g^{-1}(A)} = \pi_A, \quad \pi^* \mathcal{H}|_{g^{-1}(A)} \sim \tilde{H}_A.$$  

(3.2.6)

Proof. By Proposition 3.2 of [22], $\mathcal{F}$ is smooth and the map $\rho$ of (3.2.4) is a submersion of smooth manifolds away from points $x \in \mathcal{F}$ such that

$$\rho(x) := A' \in \Sigma \cap \mathcal{W}, \quad x \in S_{A'}.$$  

(3.2.7)

Let $(A',x)$ be as in (3.2.7). By Proposition 3.1 and smoothness of $\mathcal{F}$ we get that the map of analytic germs $(\mathcal{F},x) \to (\mathcal{W},A')$ is isomorphic to

$$\left(C^3_\mathfrak{a} \times C^2_\mathfrak{a} \times C^3_5, 0\right) \to \left(C^{54}_1, 0\right)$$

$$\left(\xi, \eta, t\right) \mapsto \left(\xi^3_1 + \xi^2_2 + \xi^3_3, t_2, \ldots, t_{54}\right).$$

(3.2.8)

Thus (3.2.5) is obtained by the classical process of simultaneous resolution of ordinary double points of surfaces. More precisely let $\tilde{\mathcal{F}}_2 \to \mathcal{F}_2$ be the blow-up of sing $\mathcal{F}_2$. Then $\tilde{\mathcal{F}}_2$ is smooth and the exceptional divisor is a fibration over sing $\mathcal{F}_2$ with fibers isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Since sing $\mathcal{F}_2$ is simply-connected we get that the exceptional divisor has two rulings by $\mathbb{P}^1$s. It follows that there are two small resolutions of $\mathcal{F}_2$ obtained by contracting the exceptional divisor along either one of the two rulings. Choose one small resolution and call it $\tilde{\mathcal{F}}_2$. Then (3.2.5) holds.

$$\square$$

**Corollary 3.6.** Let $A \in (\Sigma^m \setminus \Delta)$ and $A' \in \mathcal{L}(\mathcal{L}G(\mathcal{V}) \setminus \text{sing } \Sigma \setminus \Delta)$. Then $(\tilde{X}_A, \tilde{H}_A)$ is a HK variety deformation equivalent to $(X_A', H_A')$. Moreover $\mathcal{P}(A) = \Pi(\tilde{X}_A, \tilde{H}_A)$ where $\Pi(\tilde{X}_A, \tilde{H}_A)$ is given by (1.6.8).

Proof. Since $\pi_A : \tilde{X}_A \to X_A$ is a blow-up $\tilde{X}_A$ is projective. By Proposition 3.5 $\tilde{X}_A$ is a (smooth) deformation of $X_A$: it follows that $\tilde{X}_A$ is a HK variety. The remaining statements are obvious.

**Definition 3.7.** Let $A \in (\Sigma^m \setminus \Delta)$. We let $E_A \subset \tilde{X}_A$ be the exceptional divisor of $\pi_A : \tilde{X}_A \to X_A$ and $\zeta_A := c_1(\mathcal{O}_{\tilde{X}_A}(E_A))$.

Given $A \in (\Sigma^m \setminus \Delta)$ we have a smooth conic bundle $p : E_A \to S_A$.

(3.2.9)

$p$ is a smooth map and each fiber is isomorphic to $\mathbb{P}^1$. 

---

1. For clarity, $p$ is a smooth map and each fiber is isomorphic to $\mathbb{P}^1$. 

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(3.2.1)
Claim 3.8. Let \( (\cdot, \cdot) \) be the Beauville-Bogomolov quadratic form of \( \tilde{X}_A \). The following formulae hold:
\[
\begin{align*}
(\tilde{h}_A, \zeta_A) &= 0, \\
(\zeta_A, \zeta_A) &= -2.
\end{align*}
\] (3.2.10, 3.2.11)

Proof. We claim that
\[
6(\zeta_A, \tilde{h}_A) = \int_{\tilde{X}_A} \zeta_A \wedge \tilde{h}_A^3 = \int_{S_A} h_A^3 = 0.
\] (3.2.12)

In fact the first equality follows from Fujiki’s relation
\[
\int_X \alpha^4 = 3(\alpha, \alpha)^2, \quad \alpha \in H^2(X)
\] (3.2.13)
valid for any deformation of the Hilbert square of a \( K3 \) (together with the fact that \( (\tilde{h}_A, \tilde{h}_A) = 2 \)) and third equality in (3.2.12) holds because \( \dim S_A = 2 \). Equation (3.2.10) follows from (3.2.12). In order to prove (3.2.11) we notice that \( K_{E_A} \cong \Theta_{E}(E_A) \) by adjunction and hence
\[
\int_{p^{-1}(s)} \zeta_A = -2, \quad s \in S_A.
\] (3.2.14)

Using (3.2.13), (3.2.10) and (3.2.14) one gets that
\[
2(\zeta_A, \zeta_A) = (\tilde{h}_A, \tilde{h}_A) \cdot (\zeta_A, \zeta_A) = \int_{\tilde{X}_A} \tilde{h}_A^2 \wedge \zeta_A^2 = 2 \int_{p^{-1}(s)} \zeta_A = -4.
\] (3.2.15)

Equation (3.2.11) follows from the above equality. \( \Box \)

3.3. Conic bundles in HK fourfolds. We have shown that if \( A \in (\Sigma^m \setminus \Delta) \) then \( \tilde{X}_A \) contains a divisor which is a smooth conic bundle over a \( K3 \) surface. In the present section we will discuss HK fourfolds containing a smooth conic bundle over a \( K3 \) surface. (Notice that if a divisor in a HK fourfold is a smooth bundle over a smooth base then the base is a holomorphic symplectic surface.)

Proposition 3.9. Let \( X \) be a hyperkähler 4-fold. Suppose that \( X \) contains a prime divisor \( E \) which carries a conic fibration \( p: E \rightarrow S \) over a \( K3 \) surface \( S \). Let \( \zeta := c_1(\Theta_X(E)) \). Then:

1. \( h^0(\Theta_X(E)) = 1 \) and \( h^p(\Theta_X(E)) = 0 \) for \( p > 0 \).
2. \( q_X(\zeta) < 0 \) where \( q_X \) is the Beauville-Bogomolov quadratic form of \( X \).

Proof. By adjunction \( K_E \cong \Theta_{E}(E) \) and hence
\[
\int_{p^{-1}(s)} \zeta = -2, \quad s \in S.
\] (3.3.1)

Thus \( h^0(\Theta_{E}(E)) = 0 \) and hence \( h^0(\Theta_X(E)) = 1 \). Let us prove that the homomorphism
\[
H^q(\Theta_X) \rightarrow H^q(\Theta_E)
\] (3.3.2)
induced by restriction is an isomorphism for \( q < 4 \). It is an isomorphism for \( q = 0 \) because both \( X \) and \( E \) are connected. The spectral sequence with \( E_2 \) term \( H^i(R^j(p_E)\Theta_E) \) abutting to \( H^q(\Theta_E) \) gives an isomorphism \( H^q(\Theta_E) \cong H^q(\Theta_S) \). Since \( S \) is a \( K3 \) surface it follows that \( H^q(\Theta_E) = 0 \) for \( q = 1, 3 \). On the other hand \( H^q(\Theta_X) = 0 \) for odd \( q \) because \( X \) is a HK manifold. Thus \( H^q(\Theta_E) \) is an isomorphism for \( q = 1, 3 \). It remains to prove that (3.3.2) is an isomorphism for \( q = 2 \). By Serre duality it is equivalent to prove that the restriction homomorphism \( H^0(\Omega^1_X) \rightarrow H^0(\Omega^1_E) \) is an isomorphism. Since \( 1 = h^0(\Omega^1_X) = h^0(\Omega^1_E) \) it suffices to notice that a holomorphic symplectic form on \( X \) cannot vanish on \( E \) (the maximum dimension of an isotropic subspace for \( \sigma|_{T,E} \) is equal to 2). This finishes the proof that (3.3.2) is an isomorphism for \( q < 4 \). The long exact cohomology sequence associated to
\[
0 \rightarrow \Theta_X(-E) \rightarrow \Theta_X \rightarrow \Theta_E \rightarrow 0
\] (3.3.3)
gives that \( h^q(\Theta_X(-E)) = 0 \) for \( q < 4 \). By Serre duality we get that Item (1) holds. Let \( c_X \) be the Fujiki constant of \( X \); thus
\[
\int_X \alpha^4 = c_X q_X(\alpha)^2, \quad \alpha \in H^2(X).
\] (3.3.4)
Let \( \iota: E \hookrightarrow X \) be Inclusion. Let \( \sigma \) be a holomorphic symplectic form on \( X \). We proved above that there exists a holomorphic symplectic form \( \tau \) on \( S \) such that \( \iota^* \sigma = p^* \tau \). Thus we have
\[
\frac{c_X}{3} q_X(\zeta) q_X(\sigma + \tau) = \int_X \zeta^2 \wedge (\sigma + \tau)^2 = \int_E \iota^* \zeta \wedge p^* (\tau + \tau)^2 = -2 \int_S (\tau + \tau)^2.
\] (3.3.5)
(The first equality follows from \((\zeta, \sigma + \overline{\sigma}) = 0\), we used (3.3.11) to get the last equality.) On the other hand \(c_X > 0\) and \(q_X(\sigma + \overline{\sigma}) > 0\): thus \(q_X(\zeta) < 0\).

Let \(X\) and \(E\) be as in Proposition 3.9 Let \(\text{Def}_E(X) \subset \text{Def}(X)\) be the germ representing deformations for which \(E\) deforms and \(\text{Def}_E(X) \subset \text{Def}(X)\) be the germ representing deformations that keep \(\zeta\) of type \((1, 1)\). We have an inclusion of germs

\[
\text{Def}_E(X) \hookrightarrow \text{Def}_X(X).
\]

Corollary 3.10. Let \(X\) and \(E\) be as in Proposition 3.9 The following hold:

1. Inclusion (3.3.6) is an isomorphism.
2. Let \(C\) be a fiber of the conic vibration \(p: E \to S\). Then

\[
\{ \alpha \in H^2(X; \mathbb{C}) \mid (\alpha, \zeta) = 0 \} = \{ \alpha \in H^2(X; \mathbb{C}) \mid \int_C \alpha = 0 \}.
\]

3. The restriction map \(H^2(X; \mathbb{C}) \to H^2(E; \mathbb{C})\) is an isomorphism.

Proof. Item (1) follows at once from Item (1) of Proposition 3.9 and upper-semicontinuity of cohomology dimension. Let us prove Item (2). Let \(X_t\) be a very generic small deformation of \(X\) parametrized by a point of \(\text{Def}_E(X) \subset \text{Def}(X)\) and \(\zeta \in H^{2,1}_E\) be the class deforming \(\zeta\). A non-trivial rational Hodge sub-structure of \(H^2(X_t)\) is equal to \(\zeta^2\) to \(\zeta\mathbb{C}\). On the other hand (3.3.6) is an isomorphism: thus \(X_t\) contains a deformation \(E_t\) of \(E\) and hence also a deformation \(C_t\) of \(C\). Clearly \(\{ \alpha \in H^2(X_t; \mathbb{C}) \mid \int_{C_t} \alpha = 0 \}\) is a rational Hodge sub-structure of \(H^2(X_t)\) containing \(H^{2,0}(X_t)\) and non-trivial by (3.3.11): it follows that

\[
\{ \alpha \in H^2(X_t; \mathbb{C}) \mid (\alpha, \zeta) = 0 \} = \{ \alpha \in H^2(X_t; \mathbb{C}) \mid \int_{C_t} \alpha = 0 \}
\]

The kernel of the restriction map \(H^2(X_t; \mathbb{C}) \to H^2(E_t; \mathbb{C})\) is a rational Hodge sub-structure \(V_t \subset H^2(X_t)\). By (3.3.11) we know that \(\zeta \notin V_t\) and since (3.3.2) is an isomorphism for \(q = 2\) we know that \(H^{2,0}(X_t) \notin V_t\); thus \(V_t = 0\). Parallel transport by the Gauss-Manin connection gives Items (2) and (3).

Let \(\iota: E \hookrightarrow X\) be Inclusion. By Items (2) and (3) of Corollary 3.10 we have an isomorphism

\[
\zeta^\perp \xrightarrow{\sim} \{ \beta \in H^2(E; \mathbb{C}) \mid \int_C \beta = 0 \} \quad \text{and} \quad \alpha \mapsto \iota^* \alpha
\]

On the other hand \(p^*: H^2(S; \mathbb{C}) \to H^2(E; \mathbb{C})\) defines an isomorphism of \(H^2(S; \mathbb{C})\) onto the right-hand side of (3.3.9). Thus (3.3.9) gives an isomorphism

\[
r: \zeta^\perp \xrightarrow{\sim} H^2(S; \mathbb{C}).
\]

Claim 3.11. Let \(X, E\) be as in Proposition 3.9 and \(r\) be as in (3.3.10). Suppose in addition that the Fujiki constant \(c_X\) is equal to 3 and that \(q_X(\zeta) = -2\). Let \(\alpha \in \zeta^\perp\). Then

\[
q_X(\alpha) = \int_S r(\alpha)^2.
\]

Proof. Equality (3.3.11) gives that

\[
-2q_X(\alpha) = \frac{c_X}{3} q_X(\zeta) q_X(\alpha) = \int_X \zeta^2 \wedge \alpha^2 = \int_E \iota^* \zeta \wedge (\iota^* \alpha)^2 = -2 \int_S r(\alpha)^2.
\]

3.4. The period map on \((\Sigma^{2m}; D)\). Let \(A_0 \in (\Sigma^{2m}; D)\). By (1.4.6) and Cor. 2.5.1 of (33) \(A_0\) belongs to the GIT-stable locus of \(\mathbb{L}(\mathbb{A}^3 V)\). By Luna’s étale slice Theorem [15] it follows that there exists an analytic \(\mathbb{P}GL(V)\)-slice at \(A_0\), call it \(Z_{A_0}\), such that the natural map

\[
Z_{A_0}/\text{Stab}(A_0) \to \mathfrak{M}
\]

is an isomorphism onto an open (classical topology) neighborhood of \([A_0]\). We may assume that \(Z_{A_0} \subset \mathcal{W}\) where \(\mathcal{W}\) is as in (3.2.3). Let \(\tilde{Z}_{A_0} := \phi^{-1} Z_{A_0}\) where \(\phi: \mathcal{Y} \to \mathcal{W}\) is as in (3.2.4). Then \(\phi\) defines a double cover \(\tilde{Z}_{A_0} \to Z_{A_0}\) ramified over \(\Sigma \cap Z_{A_0}\); if \(A \in \Sigma \cap Z_{A_0}\) we will denote by the same letter the unique point in \(\phi^{-1}(A)\). By Proposition 3.5 points of \(\tilde{Z}_{A_0}\) parametrize deformations of \(X_A\) for \(A \in \mathbb{L}(\mathbb{A}^3 V)^0\). Since \(\Sigma\) is smooth at \(A_0\) also \(\Sigma \cap Z_{A_0}\) is smooth at \(A_0\). Thus \(\tilde{Z}_{A_0}\) is smooth at
\( A_0 \). Shrinking \( Z_{A_0} \) around \( A_0 \) if necessary we may assume that \( \tilde{Z}_{A_0} \) is contractible. Hence a marking \( \psi \) of \((\tilde{X}_{A_0}, \tilde{H}_{A_0})\) defines a marking of \((\tilde{X}_A, \tilde{H}_A)\) for all \( A \in Z_{A_0} \); we will denote it by the same letter \( \psi \). Thus we have a local period map

\[
\tilde{Z}_{A_0} \xrightarrow{\tilde{\mathcal{P}}} \Omega_A
\]

\[
t \mapsto \psi_C(H^{2,0}(g^{-1}t)).
\]

Claim 3.12. The local period map \( \tilde{\mathcal{P}} \) of (3.4.2) defines an isomorphism of a sufficiently small open neighborhood of \( A_0 \) in \( \tilde{Z}_{A_0} \) onto an open subset of \( \Omega_A \).

Proof. Since \( \tilde{Z}_{A_0} \) is smooth and \( \dim \tilde{Z}_{A_0} = \dim \Omega_A \) it suffices to prove that \( d\tilde{\mathcal{P}}(\tilde{A}_0) \) is injective. By Luna’s étale slice Theorem we have an isomorphism of germs

\[
(Z_{A_0}, A_0) \xrightarrow{\sim} \text{Def}(X_{A_0}, H_{A_0})
\]

induced by the local tautological family of double EPW-sextics parametrized by \( Z_{A_0} \). By Corollary 3.2 the points of \( Z_{A_0} \cap \Sigma \) parametrize deformations of \( X_{A_0} \) which are locally trivial at points of \( S_A \).

Let \( \Sigma_{A_0} \subset \tilde{Z}_{A_0} \) be the inverse image of \( \Sigma \cap Z_{A_0} \) with reduced structure. Let \( \text{Def}_\zeta_{A_0}(\tilde{X}_{A_0}, \tilde{H}_{A_0}) \subset \text{Def}(\tilde{X}_{A_0}, \tilde{H}_{A_0}) \) be the germ representing deformations that “leave \( \zeta_{A_0} \) of type \((1,1)\)”.

The natural map of germs

\[
(\Sigma_{A_0}, A_0) \rightarrow \text{Def}_\zeta_{A_0}(\tilde{X}_{A_0}, \tilde{H}_{A_0})
\]

is an inclusion because Map (3.4.3) is an isomorphism. Notice that \( \zeta_{A_0} \in \tilde{h}^1_{A_0} \) by (3.2.10); since \( \zeta_{A_0} \in \Omega_{\tilde{X}_{A_0}}^1(\tilde{X}_{A_0}) \) we have

\[
\tilde{\mathcal{P}}(\Sigma_{A_0}) \subset \psi(\zeta_{A_0})^\perp \cap \Omega_A.
\]

Notice that \( \zeta_{A_0}^\perp \cap \Omega_A \) has codimension 1 and is smooth because \( (\zeta_{A_0}, \zeta_{A_0}) = -2 \). By injectivity of the local period map we get injectivity of the period map restricted to \( \Sigma_{A_0} \):

\[
(\Sigma_{A_0}, A_0) \leadsto (\psi(\zeta_{A_0})^\perp \cap \Omega_A, \psi_C H^{2,0}(\tilde{X}_{A_0}))
\]

\[
t \mapsto \tilde{\mathcal{P}}(t)
\]

Since domain and codomain have equal dimensions the above map is a local isomorphism. In particular \( d\tilde{\mathcal{P}}(\tilde{A}_0) \) is injective when restricted to the tangent space to \( \Sigma_{A_0} \) at \( A_0 \). Thus it will suffice to exhibit a tangent vector \( v \in T_{A_0} \tilde{Z}_{A_0} \) such that \( d\tilde{\mathcal{P}}(v) \notin \psi(\zeta_{A_0})^\perp \). By Item (1) of Corollary 3.10 it suffices to prove that \( E_{A_0} \) does not lift to 1-st order in the direction \( v \). Let \( \Delta \) be the unit complex disc and \( \gamma: \Delta \rightarrow \tilde{Z}_{A_0} \) an inclusion with \( v := \gamma'(0) \notin \Sigma_{A_0} \).

Let \( \tilde{\mathcal{P}}_\Delta \rightarrow \Delta \) be obtained by base-change from \( g: \tilde{\mathcal{P}} \rightarrow \mathcal{Y} \). Let \( \mathbb{P}^1 \) be an arbitrary fiber of (3.2.9); then \( N_{\mathbb{P}^1, \tilde{\mathcal{P}}_\Delta} \cong \Theta^{\mathbb{P}'}(-1) \oplus \Theta^{\mathbb{P}'}(-1) \). It follows that \( E_{A_0} \) does not lift to 1-st order in the direction \( v \). This finishes the proof that \( d\tilde{\mathcal{P}}(\tilde{A}_0) \) is injective. \( \square \)

Proposition 3.13. The restriction of \( p \) to \((\Sigma^{\text{sm}} \setminus \Delta)/\text{PGL}(V)\) is a dominant map to \( S^\ast_2 \) with finite fibers. Let \( A \in (\Sigma^{\text{sm}} \setminus \Delta) \) and \( \psi \) be a marking of \((\tilde{X}_A, \tilde{H}_A)\): then \( \psi(\zeta_A) \) is a \((-2)\)-root of \( \Lambda \) and \( \text{div}(\psi(\zeta_A)) = 1 \).

Proof. Let \( A \in (\Sigma^{\text{sm}} \setminus \Delta) \). By Claim 3.12 we get that \( [A] \) is an isolated point in the fiber \( p^{-1}(p([A])) \).

In particular

\[
\text{cod}(p((\Sigma^{\text{sm}} \setminus \Delta)/\text{PGL}(V)), D_A) = 1.
\]

By (3.2.10) and (3.2.11) \( \psi(\zeta_A) \) is a \((-2)\)-root of \( \Lambda \). By (3.4.3) and Proposition 1.10 we get that

\[
p((\Sigma^{\text{sm}} \setminus \Delta)/\text{PGL}(V)) \subset S^\ast_2 \cup S^\ast_2' \cup S''_2.
\]

By (3.4.7) and irreducibility of \( \Sigma \) the left-hand side of (3.4.8) is dense in one of \( S^\ast_2, S^\ast_2', S''_2 \). Let \( \delta \) be as in (1.6.14) and \( \delta: \mathbb{M} \rightarrow \mathbb{M} \) be the induced involution, let \( \tau: \mathbb{D}^{BB}_A \rightarrow \mathbb{D}^{BB}_A \) be the involution given by (1.6.13). Then \((\Sigma/\text{PGL}(V))\) is mapped to itself by \( \delta \) and hence \((1.6.13)\) gives that its image under the period map \( p \) is mapped to itself by \( \tau \). By (1.7.4) it follows that \( p \) maps \((\Sigma^{\text{sm}} \setminus \Delta)/\text{PGL}(V) \) into \( S^\ast_2 \) and hence that \( \text{div}(\psi(\zeta_A)) = 1 \). \( \square \)
3.5. Periods of $K3$ surfaces of degree 2. Let $A \in (\Sigma^{sm} \setminus \Delta)$. We will recall results of Shah on the period map for double covers of $\mathbb{P}^2$ branched over a sextic curve. Let $\mathcal{C}_6 := |\sigma_{\mathbb{P}^2(6)}|/\text{PGL}_3$ and $\Phi$ be the lattice given by (4.4.4). There is a period map
\[
g : \mathcal{C}_6 \to \mathbb{D}_\Phi^{BB}
\] (3.5.1)
whose restriction to the open set parametrizing smooth sextics is defined as follows. Let $C$ be a smooth plane sextic and $f : S \to \mathbb{P}^2$ be the double cover branched over $C$. Then (3.5.1) maps the orbit of $C$ to the period point of the polarized $K3$ surface $(S, f^* \sigma_{\mathbb{P}^2(1)})$. Shah [25] determined the “boundary” and the indeterminacy locus of the above map. In order to state Shah’s results we recall a definition.

Definition 3.14. A curve $C \subset \mathbb{P}^2$ has a simple singularity at $p \in C$ if and only if the following hold:
(i) $C$ is reduced in a neighborhood of $p$.
(ii) $\text{mult}_p(C) \leq 3$ and if equality holds $C$ does not have a consecutive triple point at $p$.\[\text{Remark} 3.15.\] Let $C \subset \mathbb{P}^2$ be a sextic curve. Then $C$ has simple singularities if and only if the double cover $S \to \mathbb{P}^2$ branched over $C$ is a normal surface with DuVal singularities or equivalently the minimal desingularization $\widetilde{S}$ of $S$ is a $K3$ surface (with A-D-E curves lying over the singularities of $S$), see Theorem 7.1 of [1].

Let $C \subset \mathbb{P}^2$ be a sextic curve with simple singularities. Then $C$ is $\text{PGL}_3$-stable by [25]. We let
\[
\mathcal{C}_6^{ADE} := \{ C \in |\sigma_{\mathbb{P}^2(6)}| \mid C \text{ has simple singularities} \}/\text{PGL}_3.
\] (3.5.2)

Let $C$ be a plane sextic. If $C$ has simple singularities the period map (3.5.1) is regular at $C$ and takes value in $\mathbb{D}_\Phi$. - see Remark 3.15. More generally Shah [25] proved that (3.5.1) is regular at $C$ if and only if $C$ is $\text{PGL}_3$-semistable and the unique closed orbit in $\text{PGL}_3 C \cap |\sigma_{\mathbb{P}^2(6)}|$ is not that of triple (smooth) conics.

Definition 3.16. Let $\text{LG}(\Lambda^3 V)^{ADE} \subset \text{LG}(\Lambda^3 V)$ be the set of $A$ such that $C_{W,A}$ is a curve with simple singularities for every $W \in \Theta_A$. Let $\text{LG}(\Lambda^3 V)^{ILS} \subset \text{LG}(\Lambda^3 V)$ be the set of $A$ such that the period map (3.5.1) is regular at $C_{W,A}$ for every $W \in \Theta_A$.

Notice that both $\text{LG}(\Lambda^3 V)^{ADE}$ and $\text{LG}(\Lambda^3 V)^{ILS}$ are open. We have inclusions
\[
(\text{LG}(\Lambda^3 V) \setminus \Sigma) \subset \text{LG}(\Lambda^3 V)^{ADE} \subset \text{LG}(\Lambda^3 V)^{ILS}.
\] (3.5.3)
The reason for the superscript $ILS$ is the following: a curve $C \in |\sigma_{P(W)}(6)|$ is in the regular locus of the period map (4.1.3) if and only if the double cover of $P(W)$ branched over $C$ has Insignificant Limit Singularities in the terminology of Mumford, see [20].

Definition 3.17. Let $\Sigma^{ILS} := \Sigma \cap \text{LG}(\Lambda^3 V)^{ILS}$. Let $\hat{\Sigma}^{ILS} \subset \hat{\Sigma}$ be the inverse image of $\Sigma^{ILS}$ for the natural forgetful map $\hat{\Sigma} \to \Sigma$, and $\hat{\Sigma}^{ILS} \subset \hat{\Sigma}$
\[
\hat{\Sigma}^{ILS} := (p|_{\hat{\Sigma}})^{-1}(\Sigma^{ILS})
\] (3.5.4)
where $p : \text{LG}(\Lambda^3 V) \to \text{LG}(\Lambda^3 V)$ and $\hat{\Sigma}$ are as in Definition 2.6

3.6. The period map on $\Sigma$ and periods of $K3$ surfaces. Let
\[
\hat{\Sigma}^{ILS} (W, A) \mapsto \Sigma^{ILS} (W, A)
\] (3.6.1)
be the forgetful map. Let $A \in (\Sigma^{sm} \setminus \Delta)$: then $\Theta_A$ is a singleton by (4.4.4) and if $W$ is the unique element of $\Theta_A$ then $C_{W,A}$ is smooth sextic by Item (3) of Corollary 3.2. It follows that $(\Sigma^{sm} \setminus \Delta) \subset \text{LG}(\Lambda^3 V)^{ILS}$ and $\tau$ defines an isomorphism $\tau^{-1}(\Sigma^{sm} \setminus \Delta) \to (\Sigma^{sm} \setminus \Delta)$. Thus we may regard $(\Sigma^{sm} \setminus \Delta)$ as an (open dense) subset of $\hat{\Sigma}^{ILS}$:
\[
i : (\Sigma^{sm} \setminus \Delta) \hookrightarrow \hat{\Sigma}^{ILS}.
\] (3.6.2)
By definition of $\hat{\Sigma}^{ILS}$ we have a regular map
\[
\hat{\Sigma}^{ILS} (W, A) \mapsto \mathbb{D}_\Phi^{BB} (W, A) \mapsto \Pi(S_{W,A}, D_{W,A})
\] (3.6.3)

$^2C$ has a consecutive triple point at $p$ if the strict transform of $C$ in $\text{Bl}_p(\mathbb{P}^2)$ has a point of multiplicity 3 lying over $p$.\[\text{Remark} 3.18.\]
where $D_{W,A}$ is the pull-back to $S_{W,A}$ of $\Theta_{(W)}(1)$ and $\Pi(S_{W,A}, D_{W,A})$ is the (extended) period point of $(S_{W,A}, D_{W,A})$. Recall that we have defined a finite map $\rho : \mathbb{D}^B \to \mathbb{D}^B$, see (17.7.11), and that there is a natural map $\nu : \mathbb{D}_2^B \to \mathbb{D}_2$ which is identified with the normalization of $\mathbb{D}_2$, see (17.7.7).

**Proposition 3.18.** There exists a regular map

$$Q : \mathbb{H}_2^I \to \mathbb{D}_2^B$$

such that $\rho \circ Q = q$. Moreover the composition $\nu \circ (Q|_{\Sigma^m \setminus \Delta})$ is equal to the restriction of the period map $\mathcal{P}$ to $(\Sigma^m \setminus \Delta)$.

**Proof.** By Proposition 3.13 the restriction of the period map to $(\Sigma^m \setminus \Delta)$ is a dominant map to $\mathbb{D}_2^B$ and therefore it lifts to the normalization of $\mathbb{D}_2^B$:

$$\begin{array}{c}
\Sigma^m \setminus \Delta \\
\downarrow \mathcal{P} |_{\Sigma^m \setminus \Delta} \\
\mathbb{D}_2^B
\end{array}$$

We claim that

$$\rho \circ Q_0 = q|_{\Sigma^m \setminus \Delta}.$$  

(3.6.6)

In fact let $A \in (\Sigma^m \setminus \Delta)$. Let $r : \mathbb{H}_2^I \to H^2(S_A; \mathbb{C})$ be the isomorphism given by (3.3.10). Let's prove that

$$[H^2(S_A; \mathbb{C}) : r(\mathbb{H}_2^I \cap H^2(\tilde{X}_A; \mathbb{Z}))] = 2.$$  

(3.6.7)

In fact $r$ is a homomorphism of lattices by Claim 3.11. Since $H^2(S_A; \mathbb{Z})$ and $\mathbb{H}_2^I \cap H^2(\tilde{X}_A; \mathbb{Z})$ have the same rank it follows that $r(\mathbb{H}_2^I \cap H^2(\tilde{X}_A; \mathbb{Z}))$ is of finite index in $H^2(S_A; \mathbb{Z})$: let $d$ be the index. By the last clause of Proposition 3.13 the lattice $(\mathbb{H}_2^I \cap H^2(\tilde{X}_A; \mathbb{Z}))$ is isometric to $\Gamma$ - see (17.7). Hence we have

$$-4 = \text{discr} \Gamma = \text{discr}(\mathbb{H}_2^I \cap H^2(\tilde{X}_A; \mathbb{Z})) = d^2 \cdot \text{discr} H^2(S_A; \mathbb{Z}) = -d^2.$$  

(3.6.8)

Equation (3.6.7) follows at once. Next let $\psi : H^2(\tilde{X}_A; \mathbb{Z}) \sim \tilde{\Lambda}$ be a marking of $(\tilde{X}_A, \tilde{H}_A)$. By the last clause of Proposition 3.13 we know that $\psi(\mathbb{H}_2^I)$ is a $(\pm 2)$-root of $\Lambda$ of divisibility $1$. By Proposition 1.10 there exists $g \in O(A)$ such that $g \circ \psi(\mathbb{H}_2^I) = e_3$. Let $\phi := g \circ \psi$. Then $f$ is a marking of $(\tilde{X}_A, \tilde{H}_A)$ and $\phi(\mathbb{H}_2^I) = e_3$. It follows that $\phi(\mathbb{H}_2^I \cap H^2(\tilde{X}_A; \mathbb{Z})) = \tilde{\Lambda} = \tilde{\Lambda}$. Let $\phi_0 : H^2(\tilde{X}_A; \mathbb{Q}) \sim \tilde{\Lambda}_0$ be the $\mathbb{Q}$-linear extension of $\phi$. By (3.6.7) $H^2(S_A; \mathbb{Z})$ is an overlattice of $\mathbb{H}_2^I \cap H^2(\tilde{X}_A; \mathbb{Z})$ and hence it may be embebedded canonically into $H^2(\tilde{X}_A; \mathbb{Q})$: thus $\phi(\mathbb{H}_2^I(S_A; \mathbb{Z}))$ makes sense. By (3.6.7) we get that $\phi_0(H^2(S_A; \mathbb{Z}))$ is an overlattice of $\phi_1(\mathbb{H}_2^I \cap H^2(\tilde{X}_A; \mathbb{Z}))$ and that $\phi(\mathbb{H}_2^I \cap H^2(\tilde{X}_A; \mathbb{Z}))$ has index $2$ in $\phi_0(H^2(S_A; \mathbb{Z}))$. By Claim 1.11 it follows that $\phi_0(H^2(S_A; \mathbb{Z})) = \tilde{\Phi}$. Equation (3.6.6) follows at once from this. By (3.6.6) we have a commutative diagram

$$\begin{array}{cccc}
\mathbb{H}_2^I \times_{\mathbb{D}_2^B} \mathbb{D}_2^B & \to & \mathbb{D}_2^B & \\
\downarrow & & \downarrow \rho \\
\mathbb{H}_2^I & \to & \mathbb{D}_2^B
\end{array}$$

(3.6.9)

where $\iota$ is the inclusion map (3.6.2). Let $\mathcal{E}$ be the closure of $\text{Im}(\iota, Q_0)$. Then $\mathcal{E}$ is an irreducible component of $\Sigma^ILS \times_{\mathbb{D}_2^B} \mathbb{D}_2^B$ because $\iota$ is an open inclusion. The natural projection $\mathcal{E} \to \Sigma^ILS$ is a finite birational map and hence it is an isomorphism because $\Sigma^ILS$ is smooth. We define the map $Q : \Sigma^ILS \to \mathbb{D}_2^B$ as the composition of the inverse $\Sigma^ILS \to \mathcal{E}$ and the projection $\mathcal{E} \to \mathbb{D}_2^B$. The properties of $Q$ stated in the proposition hold by commutativity of (3.6.9).

**Corollary 3.19.** The image of the map $(\tau, \nu \circ Q) : \Sigma^ILS \to \Sigma^ILS \times_{\mathbb{D}_2^B} \mathbb{D}_2^B$ is equal to $\Sigma^ILS$.

**Proof.** Let $\rho : \mathbb{H}_2^I(\mathbb{Z}^2) \to \mathbb{H}_2^I(\mathbb{Z}^3)$ be as in Definition 2.6. Since $\mathcal{P}$ is regular on $(\Sigma^m \setminus \Delta)$ the map $p^{-1}(\Sigma^m \setminus \Delta) \to (\Sigma^m \setminus \Delta)$ is an isomorphism and $p^{-1}(\Sigma^m \setminus \Delta)$ is an open dense subset of $\Sigma^ILS$ (recall that $\Sigma$ is irreducible and hence so is $\Sigma$). By the second clause of Proposition 3.18 we have that

$$(\tau, \nu \circ Q)(\Sigma^m \setminus \Delta) = p^{-1}(\Sigma^m \setminus \Delta).$$  

(3.6.10)
Since $\Sigma$ is closed in $\overline{LG}(\Lambda^3 V) \times \mathbb{D}_A^{BB}$ it follows that $\text{Im}(\tau, \nu \circ Q) \subset \Sigma$. The commutative diagram

$$
\begin{array}{ccc}
\Sigma_{ILS} & \xrightarrow{(\tau, \nu \circ Q)} & \Sigma_{ILS} \\
\tau \downarrow & & \downarrow p|_{\Sigma_{ILS}} \\
\Sigma_{ILS} & \xrightarrow{\tau, \nu \circ Q} & \Sigma_{ILS}
\end{array}
$$

(3.6.11)
gives that $\text{Im}(\tau, \nu \circ Q) \subset \Sigma_{ILS}$. The right-hand side of (3.6.11) is dense in $\Sigma_{ILS}$: thus in order to finish the proof it suffices to show that $\text{Im}(\tau, \nu \circ Q)$ is closed in $\Sigma_{ILS}$. The equality $(p|_{\Sigma_{ILS}}) \circ (\tau, \nu \circ Q) = \tau$ and properness of $\tau$ give that $(\tau, \nu \circ Q)$ is proper (see Ch. II, Cor. 4.8, Item (e) of [18]) and hence closed: thus $\text{Im}(\tau, \nu \circ Q)$ is closed in $\Sigma_{ILS}$.

\[\Box\]

3.7. Extension of the period map.

Proposition 3.20. Let $A \in LG(\Lambda^3 V)^{ILS}$. If $\dim \Theta_A \leq 1$ the period map $\mathcal{P}$ is regular at $A$ and moreover $\mathcal{P}(A) \in \mathbb{D}_A$ if and only if $A \in \overline{LG}(\Lambda^3 V)^{ADE}$.

Proof. If $A \notin \Sigma$ then $\mathcal{P}$ is regular at $A$ by Proposition 2.5. Now assume that $A \in \Sigma_{ILS}$. Suppose that $\mathcal{P}$ is not regular at $A$: we will reach a contradiction. Let $p: \overline{LG}(\Lambda^3 V) \rightarrow \overline{LG}(\Lambda^3 V)$ be as in Definition 2.6. Then $p^{-1}(A) \cap \Sigma$ is a subset of $\{A\} \times \mathbb{D}_A^{BB}$ and hence we may identify it with its projection in $\mathbb{D}_A^{BB}$. This subset is equal to $\nu \circ Q(\tau^{-1}(A))$ by Corollary 3.19 and Commutative Diagram (3.6.11). On the other hand $\tau^{-1}(A) = \Theta_A$ and hence $\dim \tau^{-1}(A) \leq 1$ by hypothesis: it follows that $\dim p^{-1}(A) \leq 1$ and this contradicts Corollary 2.7. This proves that $\mathcal{P}$ is regular at $A$.

The last clause of the proposition follows from Corollary 3.19.

\[\Box\]

Proof of Theorem 0.2. Let $x \in (\mathfrak{M} \setminus \mathfrak{J})$. There exists a GIT-semistable $A \in \overline{LG}(\Lambda^3 V)$ representing $x$ with PGL(V)-orbit closed in the semistable locus $\overline{LG}(\Lambda^3 V)$, and such $A$ is determined up to the action of PGL(V). By Luna’s étale slice Theorem 15 it suffices to prove that the period map $\mathcal{P}$ is regular at $A$. If $A \notin \Sigma$ then $\mathcal{P}$ is regular at $A$ and $\mathcal{P}(A) \in \mathbb{D}_A$ by Proposition 2.5. Now suppose that $A \in \Sigma$. Then $A \in \Sigma_{ILS}$ because $x \notin \mathfrak{J}$. By Proposition 3.20 in order to prove that $\mathcal{P}$ is regular at $A$ it will suffice to show that $\dim \Theta_A \leq 1$. Suppose that $\dim \Theta_A \geq 2$, we will reach a contradiction. Theorem 3.26 and Theorem 3.36 of [21] give that $A$ belongs to certain subsets of $\overline{LG}(\Lambda^3 V)$, namely $X_{\mathscr{D}, X_{\mathscr{D}}, \ldots, X_{\mathscr{D}}}$ (notice the misprint in the statement of Theorem 3.36: $X_{\mathscr{D}}$ is to be replaced by $X_{\mathscr{D}, +}$). Thus, unless

$$A \in (X_{\mathscr{D}} \cup X_{\mathscr{W}} \cup X_h \cup X_h),
$$

(3.7.1)

(noting that $X_{\mathscr{D}} \subset X_{\mathscr{W}}$) the lagrangian $A$ belongs to one of the standard unstable strata listed in Table 2 of [23], and hence is PGL(V)-unstable. That is a contradiction because $A$ is semistable and hence we conclude that (3.7.1) holds. Proposition 4.3.7 of [23] gives that if $A \in X_{\mathscr{W}}$ then $A \in \text{PGL}_6 A_+$ i.e. $A \in X_{\mathscr{F}}$ (because $A$ has minimal PGL(V)-orbit), thus $A \in X_{\mathscr{F}} \cup X_h \cup X_h$). If $A \in X_{\mathscr{W}}$ then $A \notin \Sigma_{ILS}$ by Claim 4.4.5 of [23], if $A \in X_h$ then $A \notin \Sigma_{ILS}$ by (4.5.6) of [23], and $A \in X_h$ then $A \notin \Sigma_{ILS}$ by (4.5.5) of [23]: that is a contradiction. This shows that $\dim \Theta_A \leq 1$ and hence $p$ is regular at $A$.

The last clause of Proposition 3.20 gives that $p(x) \in \mathbb{D}_A$ if and only if $x \in \mathfrak{M}^{ADE}$.

\[\Box\]

4. On the image of the period map

We will prove Theorem 0.3. Most of the work goes into showing that $\mathcal{P}(\overline{LG}(\Lambda^3 V) \setminus \Sigma)$ does not intersect $S_2^2 \cup S_2^3 \cup S_3^2 \cup S_4$ i.e. that $p$ maps $(\mathfrak{M} \setminus \mathfrak{J})$ into the right-hand side of (0.10.11). First we will prove that result, then we will show that the restriction of $p$ to $(\mathfrak{M} \setminus \mathfrak{J})$ is an open embedding.

4.1. Proof that $\mathcal{P}(\overline{LG}(\Lambda^3 V) \setminus \Sigma) \cap S_2^2 = \emptyset$. Let $S$ be a K3 surface. We recall the description of $H^2(S^{[2]})$ and the Beauville-Bogomolov form $q_{S^{[2]}}$ in terms of $H^2(S)$. Let $\mu: H^2(S) \rightarrow H^2(S^{[2]})$ be the composition of the symmetrization map $H^2(S) \rightarrow H^2(S^{[2]})$ and the pull-back $H^2(S^{[2]}) \rightarrow H^2(S^{[2]})$. There is a direct sum decomposition

$$H^2(S^{[2]}) = \mu(H^2(S; \mathbb{Z})) \oplus 2\xi
$$

(4.1.1)

where $2\xi$ is represented by the locus parametrizing non-reduced subschemes. Moreover if $H^2(S)$ and $H^2(S^{[2]})$ are equipped with the intersection form and Beauville-Bogomolov quadratic form $q_{S^{[2]}}$ respectively, then $\mu$ is an isometric embedding. Decomposition (4.1.1) is orthogonal, and $q_{S^{[2]}}(\xi) = -2$. Recall that $\delta_V: LG(\Lambda^3 V) \xrightarrow{\sim} LG(\Lambda^3 V^\vee)$ is defined by $\delta_V(A) := \text{Ann} A$, see (1.6.14).
Lemma 4.1. $\mathcal{P}(\Delta \setminus \Sigma) \not\subset (S_2^2 \cup S_3^2 \cup S_4^2 \cup S_4)$ and $\mathcal{P}(\partial \nu(\Delta \setminus \Sigma)) \not\subset (S_2^2 \cup S_3^2 \cup S_4^2 \cup S_4)$.

Proof. Let $A \in (\Delta \setminus \Sigma)$ be generic. By Theorem 4.15 of [22] there exist a projective $K3$ surface $S_A$ of genus 6 and a small contraction $S_A^2 \to X_A$. Moreover the period point $\mathcal{P}(A)$ may be identified with the Hodge structure of $S_A^2$ as follows. The surface $S_A$ comes equipped with an ample divisor $D_A$ of genus 6 i.e. $D_A \cdot D_A = 10$, let $d_A$ be the Poincarè dual of $D_A$. Then $\mathcal{P}(A)$ is identified with the Hodge structure on $(\mu(d_A) - 2\xi)^{-1}$, where $\xi$ is as above. By Proposition 4.14 of [22] we may assume that $(S_A, D_A)$ is a general polarized $K3$ surface of genus 6. It follows that if $A$ is very general in $(\Delta \setminus \Sigma)$ then $H_2^1(S_A) = \mathbb{Z}d_A$. Thus for $A \in (\Delta \setminus \Sigma)$ very general we have that

$$H_2^1(S_A^2) \cap (\mu(d_A) - 2\xi)^{-1} = \mathbb{Z}(2\mu(d_A) - 5\xi).$$

(4.1.2)

Now suppose that $\mathcal{P}(A) \in (S_2^2 \cup S_3^2 \cup S_4^2 \cup S_4)$; by definition there exists $\alpha \in H_2^1(S_A^2) \cap (\mu(d_A) - 2\xi)^{-1}$ of square $(-2)$ or $(-4)$: since $q_{S_A^2}(2\mu(d_A) - 5\xi) = -10$ that contradicts (4.1.2). This proves that $\mathcal{P}(\Delta \setminus \Sigma) \not\subset (S_2^2 \cup S_3^2 \cup S_4^2 \cup S_3^2)$. Next let $\tau : \mathbb{P}^B_A \to \mathbb{P}^B_A$ be the involution defined by $\delta_V$, see (4.1.3). Then $\tau$ maps $(S_2^2 \cup S_3^2 \cup S_4^2 \cup S_3^2)$ to itself, see (4.1.4), and hence $\mathcal{P}(\Delta \setminus \Sigma) \not\subset (S_2^2 \cup S_3^2 \cup S_4^2 \cup S_4)$ because otherwise it would follow that $\mathcal{P}(\Delta \setminus \Sigma) \subset (S_2^2 \cup S_3^2 \cup S_4^2 \cup S_3^2)$.

Suppose that $\mathcal{P}(\mathbb{P}^2 \setminus (\Lambda^3 V) \setminus \Sigma) \cap S_4^2 \neq \emptyset$. Since $S_2^2$ is a $\mathbb{P}$-Cartier divisor of $\mathbb{P}^2$ it follows that $\mathcal{P}^{-1}(S_4^2) \cap (\mathbb{P}^2 \setminus (\Lambda^3 V) \setminus \Sigma)$ has pure codimension 1: let $C$ be one of its irreducible components. Then $C \neq \Delta$ by Lemma 4.1 and hence $C^0 := C \setminus \Delta$ is a codimension-1 PGL(V)-invariant closed subset of $(\mathbb{P}^2 \setminus (\Lambda^3 V) \setminus \Sigma)$. Since $C^0$ has pure codimension 1 and is contained in the stable locus of $\mathbb{P}^2 \setminus (\Lambda^3 V)$ (see Cor. 2.5.1 of [23]) the quotient $C^0//PGL(V)$ has codimension 1 in $\mathbb{P}^2$. If $A \in (C^0 \setminus \Delta)$ then $X_A$ is smooth and hence the period map $Def(X_A, H_A) \to \Omega_A$ is a (local) isomorphism (local Torelli for hyperkähler manifolds); it follows that the restriction of $p$ to $C^0//PGL(V)$ has finite fibers and hence $\mathcal{P}(C^0)$ is dense in $S_4$. Now consider the period map $p : (\mathbb{P} \setminus \Delta) \to D^B_A$; it is birational by Verbitsky’s Global Torelli and Markman’s monodromy results, see Theorem 1.3 and Lemma 9.2 of [16]. We have proved that there are (at least) two distinct components in $p^{-1}(S_4^2)$ which are mapped dominantly to $S_2^2$ by $p$, namely $(\mathbb{P}^2//PGL(V) \setminus \Delta)$ and the closure of $C^0//PGL(V)$: that is a contradiction because $D^B_A$ is normal.

4.2. Proof that $\mathcal{P}(\mathbb{P}^2 \setminus (\Lambda^3 V) \setminus \Sigma) \cap (S_4^2 \cup S_4^2) = \emptyset$. First we will prove that $\mathcal{P}(\mathbb{P}^2 \setminus (\Lambda^3 V) \setminus \Sigma) \cap S_4^2 = \emptyset$. Let $U$ be a 3-dimensional complex vector space and $\pi : S \to \mathbb{P}(U)$ a double cover branched over a smooth sextic curve; thus $S$ is a $K3$ surface. Let $D \in |\pi^*\mathcal{O}_S(1)|$ and $d \in H_2^1(S; \mathbb{Z})$ be its Poincarè dual. Since $S^2$ is simply connected there is a unique class $\mu(d) \in \text{Pic}(S^2)$ whose first Chern class is equal to $\mu(d)$. One easily checks the following facts. There is a natural isomorphism

$$\mathbb{P}(S^2 U^+) \cong |\mu(D)|$$

(4.2.1)

and the composition of the natural maps

$$S^2 \to S^2 \xrightarrow{\iota} \mathbb{P}(U)^{(2)} \to \mathbb{P}(S^2 U)$$

(4.2.2)

is identified with the natural map $f : S^2 \to |\mu(D)|$. The image of $f$ is the chordal variety $\mathcal{Y}_2$ of the Veronese surface $\{u^2 \mid 0 \neq u \in U\}$, and the map $S^2 \to \mathcal{Y}_2$ is of degree 4. Since $\mu(d)$ has square 2 we have a well-defined period point $\Pi(S^2, \mu(d)) \in \mathbb{P}^2$. The class $\xi \in H_2^1(S^2)$ is a (2)-root of divisibility 2 and it is orthogonal to $\mu(d)$: it follows that $\Pi(S^2, \mu(d)) \in (S_2^2 \cup S_3^2)$. Actually $\Pi(S^2, \mu(d)) \in S_2^2$ because the divisibility of $\xi$ as an element of $H^2(S^2; \mathbb{Z})$ is equal to 2 (and not only as element of $\mu(d)^{-1}$). The periods $\Pi(S^2, \mu(d))$ with $S$ as above fill-out an open dense subset of $S_2^4$. Now suppose that there exists $A \in (\mathbb{P}(\mathbb{A}^3 V) \setminus S)$ such that $\mathcal{P}(A) \in S_2^2$. Since $S_2^2$ is a $\mathbb{P}$-Cartier divisor of $\mathbb{P}^2$ it follows that $\mathcal{P}^{-1}(S_2^2) \cap (\mathbb{P}(\mathbb{A}^3 V) \setminus \Sigma)$ has pure codimension 1: let $C$ be one of its irreducible components. By Lemma 4.1 $C^0 := C \setminus \Delta$ is a codimension-1 subset of $\mathbb{P}(\mathbb{A}^3 V)$ and hence $\mathcal{P}(C^0)$ contains an open dense subset of $S_2^4$. It follows that there exist $A \in C^0$ and a double cover $\pi : S \to U$ as above with $\text{Pic}(S) = Z\mu(D)$ and such that $\mathcal{P}(A) = \Pi(S^2, \mu(d))$. By Verbitsky’s Global Torelli Theorem there exists a birational map $\varphi : S^2 \dashrightarrow X_A$. Now $\varphi^*h_A$ is a $(1,1)$-class of square 2: since $\text{Pic}(S) = Z\mu(D)$ it follows that $\varphi^*h_A = \pm \mu(d)$, and hence $\varphi^*H_A = \mu(D)$ because $| - \mu(D)|$ is empty. But that is a contradiction because the map $f_A : X_A \to |H_A|^*$ is 2-to-1 onto its image while the map $f : S^2 \to |\mu(D)|$ has degree 4 onto its image. This proves that

$$\mathcal{P}(\mathbb{P}(\mathbb{A}^3 V) \setminus \Sigma) \cap S_2^2 = \emptyset.$$  

(4.2.3)
It remains to prove that $\mathcal{P}(\mathbb{L}G(\Lambda^3 V) \setminus \Sigma) \cap S'_3 = \emptyset$. Suppose that $\mathcal{P}(\mathbb{L}G(\Lambda^3 V) \setminus \Sigma) \cap S'_3 \neq \emptyset$. Let $\Sigma(V^\vee)$ be the locus of $A \in \mathbb{L}G(\Lambda^3 V^\vee)$ containing a non-zero decomposable tri-vector. Since $\delta(V)(\Sigma) = \Sigma(V^\vee)$ we get that $\mathcal{P}(\mathbb{L}G(\Lambda^3 V^\vee \setminus \Sigma(V^\vee))) \cap S'_3 \neq \emptyset$ by \[\ref{1.6.15}\] and \[\ref{1.7.4}\] that contradicts \ref{1.2.3}. \hfill \Box

**Remark 4.2.** In the above proof we have noticed that the generic point of $S'_3$ is equal to $\Pi(S^{[2]}, \mu(d))$. One may also identify explicitly polarized hyperkähler varieties whose periods belong to $S'_3$. In fact let $\pi: S \to \mathbb{P}(U)$ and $D, d$ be as above. Let $v \in H^*(S; \mathbb{Z})$ be the Mukai vector $v := (0, d, 0)$ and let $\mathcal{M}_v$ be the corresponding moduli space of $D$-semistable sheaves on $S$ with Mukai vector $v$: the generic such sheaf is isomorphic to $\mathcal{E}_v$ where $\mathcal{E} \to S$ is the inclusion of a smooth $C \subset [D]$ and $\mathcal{E}$ is an invertible sheaf on $C$ of degree 1. As is well-known $\mathcal{M}_v$ is a hyperkähler variety deformation equivalent to $K^3[2]$. Moreover $H^2(\mathcal{M}_v)$ with its Hodge structure and B-B form is identified with $v^\perp$ with the Hodge structure it inherits from the Hodge structure of $H^*(S)$ and the quadratic form given by the Mukai pairing, see \[\ref{27}\]. Let $h \in H^2(\mathcal{M}_v)$ correspond to $\pm (1, 0, -1)$. Then $h$ has square 2 and, as is easily checked, the period point of $(\mathcal{M}_v, h)$ belongs to $S'_3$: more precisely $\Pi(\mathcal{M}_v, h) = \tau(\Pi(S^{[2]}, \mu(d))).$

4.3. **Proof that** $\mathcal{P}(\mathbb{L}G(\Lambda^3 V) \setminus \Sigma) \cap S_4 = \emptyset$. Let $S \subset \mathbb{P}^3$ be a smooth quartic surface, $D \subset |\mathcal{O}_S(1)|$ and $d$ be the Poincarè dual of $D$. We have a natural map

$$S^{[2]} \xrightarrow{\psi} \Gr(1, P^4) \subset \mathbb{P}^5$$

\[\d\] where $(Z)$ is the unique line containing the lenght-2 scheme $Z$. Let $H \in |f^*\mathcal{O}(1)|$ and $h$ be its Poincarè dual. One checks easily that $h = (\mu(d) - \xi)$, in particular $q_{S^{[2]}}(h) = 2$. Moreover pull-back gives an identification of $f$ with the natural map $S^{[2]} \to |H|^\vee$. The equalities

$$(h, \mu(d) - 2\xi)_{S^{[2]}} = 0, \quad q_{S^{[2]}}(\mu(d) - 2\xi) = 2$$

\[\d\] (here $h^+ \subset H^2(S^{[2]}; \mathbb{Z})$ is the subgroup of classes orthogonal to $h$) show that $(\mu(d) - 2\xi)$ is a $(-4)$-root of $h^+$ and hence $\Pi(S^{[2]}, h) \subset S_4$ by **Proposition 1.10**. Moreover the generic point of $S_4$ is equal to $\Pi(S^{[2]}, h)$ for some $(S, d)$ as above: in fact $S_4$ is irreducible, see **Remark 1.11** of dimension 19 i.e. the dimension of the set of periods $\Pi(S^{[2]}, h)$ for $(S, d)$ as above. Now assume that $\mathcal{P}(\mathbb{L}G(\Lambda^3 V) \setminus \Sigma) \cap S_4 \neq \emptyset$. Arguing as in the previous cases we get that there exists a closed $\text{PGL}(V)$-invariant codimension-1 subvariety $C^0 \subset (\mathbb{L}G(\Lambda^3 V) \setminus \Delta \setminus \Sigma)$ such that $\mathcal{P}(C^0) \subset S_4$. Thus $\mathcal{P}(C^0)$ contains an open dense subset of $S_4$ and therefore if $A \in C^0$ is very generic $h^{1,1}_Z(X_A) = 2$. By the discussion above we get that there exist $(S, d)$ as above such that $\Pi(S^{[2]}, h) = \Pi(X_A, h_A)$ with $h^{1,1}_Z(X_A) = 2$. By Verbitisky’s Global Torelli Theorem there exists a birational map $\varphi: S^{[2]} \dashrightarrow X_A$. Since the map $f_A: X_A \to |H_A|^\vee$ is of degree 2 onto its image, and since $\varphi$ defines an isomorphism between the complement of a codimension-2 subsets of $S^{[2]}$ and the complement of a codimension-2 subsets of $X_A$ (because both are varieties with trivial canonical bundle) we get that

$$q_{S^{[2]}}(\varphi^*h_A) = 2, \quad |\varphi^*H_A|^\vee$$

is of degree 2 onto its image. \[\d\] We will get a contradiction by showing that there exists no divisor of square 2 on $S^{[2]}$ such that \ref{4.3.3} holds. Notice that if $H$ is the divisor on $S^{[2]}$ defined above then the first two conditions of \ref{4.3.3} hold but not the third (the degree of the map is equal to 6). This does not finish the proof because the set of elements of $H^{1,1}_Z(S^{[2]})$ whose square is 2 is infinite.

**Lemma 4.3.** There exists $n \in \mathbb{Z}$ such that

$$\varphi^*h_A = x\mu(d) + y\xi, \quad y + x\sqrt{2} = (-1 + \sqrt{2})(3 + 2\sqrt{2})^n.$$ 

\[\d\]

**Proof.** Since $h^{1,1}_Z(X_A) = 2$ we have $h^{1,1}_Z(S^{[2]}) = 2$ and hence $H^{1,1}_Z(S^{[2]})$ is generated (over $\mathbb{Z}$) by $\mu(d)$ and $\xi$. Let

$$H^{1,1}_Z(S^{[2]}) \xrightarrow{\psi} \mathbb{Z}[\sqrt{2}]$$

\[\d\] Then

$$(\alpha, \beta) = -\text{Tr}(\psi(\alpha) \cdot \overline{\psi(\beta)}).$$

\[\d\] Since $\varphi^*h_A$ is an element of square 2 we will need to solve a (negative) Pell equation. Solving Pell’s equation $N(y + x\sqrt{2}) = 1$ (see for example Proposition 17.5.2 of \[\ref{12}\]) and noting that $N(-1 + \sqrt{2}) = -1$ we get that there exists $n \in \mathbb{Z}$ such that

$$\varphi^*h_A = x\mu(d) + y\xi, \quad y + x\sqrt{2} = \pm (1 + \sqrt{2})(3 + 2\sqrt{2})^n.$$ 

\[\d\]
Next notice that $S$ does not contain lines because $h^1_2(S) = 1$: it follows that the map $S^{[2]} \to \text{Gr}(1, \mathbb{P}^3)$ is finite and therefore $H$ is ample. Since $|\varphi^*H_A|$ is not empty and $\varphi^*H_A$ is not equivalent to 0 we get that
\[
0 < (\varphi^*H_A, h)_{S^{[2]}} = -\text{Tr} \left( \pm (\pm 1 + \sqrt{2})(3 + 2\sqrt{2})^n(\pm 1 - \sqrt{2}) \right).
\]
(4.3.8)

It follows that the $\pm$ is actually +. \hfill $\Box$

Next we will consider the analogue of nodal classes on $K3$ surfaces. For $n \in \mathbb{Z}$ we define $\alpha_n \in H^2_2(S^{[2]})$ by requiring that
\[
|\psi(\alpha_n)| = (3 - 2\sqrt{2})^n.
\]
(4.3.9)

Thus $q_{S^{[2]}}(\alpha_n) = -2$ for all $n$.

**Lemma 4.4.** If $n > 0$ then $2\alpha_n$ is effective, if $n \leq 0$ then $-2\alpha_n$ is effective.

**Proof.** By Theorem 1.11 of [17] either $2\alpha_n$ or $-2\alpha_n$ is effective (because $q_{S^{[2]}}(\alpha_n) = -2$). Since $(\mu(d) - \xi)$ is ample we decide which of $\pm 2\alpha_n$ is effective by requiring that the product with $(\mu(d) - \xi)$ is strictly positive. The result follows easily from (4.3.6). \hfill $\Box$

**Proposition 4.5.** Suppose that $\varphi^*H_A$ is given by (4.3.5) with $n \neq 0$. Then there exists an effective $\beta \in H^2_2(S^{[2]})$ such that $|\varphi^*H_A, \beta| < 0$.

**Proof.** Identify $H^2_2(S^{[2]})$ with $\mathbb{Z}[\sqrt{2}]$ via (4.3.6) and let $g: H^2_2(S^{[2]}) \to H^2_2(S^{[2]})$ correspond to multiplication by $(3 - 2\sqrt{2})$. Since $N(3 - 2\sqrt{2}) = 1$ the map $g$ is an isometry. Notice that $\alpha_2 = g^2(-\xi)$ and by **Lemma 4.4** we have that $\varphi^*H_A = g^{-n}(\mu(d) - \xi)$. Now suppose that $n > 0$. Then $-2\alpha_{n+1}$ is effective by **Lemma 4.4** and
\[
(\varphi^*H_A, -2\alpha_{n+1})_{S^{[2]}} = (\mu(d) - \xi, 2g^{-n-1}(\xi))_{S^{[2]}} = (\mu(d) - \xi, 2\mu(d) - 4\mu(d) + 6\xi)_{S^{[2]}} = -4 < 0.
\]
(4.3.10)

Lastly suppose that $n < 0$. Then $2\alpha_{n}$ is effective by **Lemma 4.4** and
\[
(\varphi^*H_A, 2\alpha_{n})_{S^{[2]}} = (g^{-n}(\mu(d) - \xi), 2g^{-n}(\mu(d) - \xi))_{S^{[2]}} = (\mu(d) - \xi, -2\xi)_{S^{[2]}} = -4 < 0.
\]
(4.3.11)

Now we are ready to prove that (4.3.3) cannot hold and hence reach a contradiction. By **Lemma 4.3** we know that $\varphi^*H_A$ is given by (4.3.6) for some $n \in \mathbb{Z}$. We have already noticed that (4.3.6) cannot hold if $n = 0$. Suppose that $n \neq 0$. By **Proposition 4.5** there exists an effective $\beta \in H^2_2(S^{[2]})$ such that
\[
(\varphi^*H_A, \beta)_{S^{[2]}} < 0.
\]
(4.12)

Let $B$ be an effective divisor representing $\beta$ and $C \in |\varphi^*H_A|$. Then $C \cap B$ does not have codimension 2 i.e. there exists at least one prime divisor $B_1$ which is both in the support of $B$ and in the support of $C$. In fact suppose the contrary. Let $c \in H^2_2(S^{[2]})$ be the Poincaré dual of $C$ and $\sigma$ be a symplectic form on $S^{[2]}$; then
\[
0 \leq \int_{B \cap C} \sigma \wedge \sigma = (\beta, c)_{S^{[2]}} (\sigma, \sigma)_{S^{[2]}}
\]
(4.13)

and since $(\sigma, \sigma)_{S^{[2]}} > 0$ we get that $(\beta, c) \geq 0$ i.e. $(\varphi^*H_A, \beta)_{S^{[2]}} \geq 0$, that contradicts (4.3.12). The conclusion is that there exists a prime divisor $B_1$ which is both in the support of $B$ and of any $C \in |\varphi^*H_A|$, i.e. $B_1$ is a base divisor of the linear system $|\varphi^*H_A|$; that shows that (4.3.3) does not hold. \hfill $\Box$

**4.4. Proof that $p$ restricted to $(\mathfrak{M} \setminus \mathfrak{R})$ is an open embedding.** Let $\mathfrak{X} := \Delta / \text{PGL}(V)$. Let $\delta: \mathfrak{M} \to \mathfrak{M}$ be the duality involution defined by (1.6.1). By [19], pp. 36-38, we have that
\[
\text{cod}(\mathfrak{X} \cap \delta(\mathfrak{X}), \mathfrak{M}) \geq 2.
\]
(4.4.1)

**Claim 4.6.** The restriction of $p$ to $(\mathfrak{M} \setminus \mathfrak{R}) \setminus (\mathfrak{X} \cap \delta(\mathfrak{X}))$ is open.

**Proof.** It suffices to prove that the restriction of $p$ to $(\mathfrak{M} \setminus \mathfrak{R}) \setminus (\mathfrak{X} \cap \delta(\mathfrak{X}))$ is open in the classical topology. Suppose first that $[A] \in (\mathfrak{M} \setminus \mathfrak{R} \setminus \mathfrak{X})$ i.e. $X_A$ is smooth. Then $A$ is stable by Corollary 2.5.1 of [23]. Let $Z_A \subset LG(A^3, V)$ be an analytic PGL(V)-slice at $A$, see [15]. We may and will assume that $Z_A$ is contractible; hence a marking of $(X_A, H_A)$ defines a lift of $\mathcal{P}|_{Z_A}$ to a regular map $\tilde{\mathcal{A}}_A: Z_A \to \Omega_\Lambda$. The family of double EPW-sextics parametrized by $Z_A$ is a representative of the universal deformation space of the polarized 4-fold $(X_A, H_A)$ and hence $\tilde{\mathcal{A}}_A$ is a local isomorphism $(Z_A, A) \to (\Omega_\Lambda, \mathfrak{P}(A))$ (injectivity and surjectivity of the local period map for compact hyperkähler manifolds). Now suppose
that $[A] \in \mathcal{S}$. By hypothesis $\delta([A]) \notin \mathcal{S}$, and since $\delta(\mathcal{R}) = \mathcal{R}$ it follows that $\delta([A]) \in (\mathcal{M} \setminus \mathcal{R} \setminus \mathcal{S})$; hence $p$ is open at $[A]$ by openness at $\delta([A])$ (which we have just proved) and \((4.4.15)\).

Next we notice that $D_\Lambda$ is $\mathbb{Q}$-factorial. In fact by Lemma 7.2 in Ch. 7 of \cite{24} there exists a torsion-free subgroup $G < \tilde{O}(\Lambda)$ of finite index. Thus the natural map $\pi_\ast: (G' \setminus \Omega_\Lambda) \to D_\Lambda$ is a finite map of quasi-projective varieties, and $(G' \setminus \Omega_\Lambda)$ is smooth: it follows that $D_\Lambda$ is $\mathbb{Q}$-factorial. (If $D$ is a divisor of $D_\Lambda$ then $(\deg \pi) D = \pi_\ast(\pi^\ast D)$ is the push-forward of a Cartier divisor, hence Cartier.) Since $D_\Lambda$ is $\mathbb{Q}$-factorial and the restriction of $p$ to $(\mathcal{M} \setminus \mathcal{R})$ is a birational map

$$(\mathcal{M} \setminus \mathcal{R}) \to D_\Lambda \quad (4.4.2)$$

the exceptional set of \((4.4.2)\) is either empty or of pure codimension 1. By \textbf{Claim 4.6} there are no components of codimension 1 in the the exceptional set of \((4.4.2)\), hence it is empty. This proves that \((4.4.2)\) is an open embedding.

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