Sommerfeld Fine-Structure Formula for Two-Body Atoms

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For relativistic atomic two-body systems such as the hydrogen atom, positronium, and muon-proton bound states, a two-body generalisation of the single-particle Sommerfeld fine-structure formula for the relativistic bound-state energies is found. The two-body Sommerfeld bound-state energy formula is obtained from a two-body wave equation which is physically correct to order \((Z\alpha)^6\). The two-body Sommerfeld formula makes two predictions in order \((Z\alpha)^6\) for every bound state and every mass ratio. With \(N\) the Bohr quantum number: (a) The coefficient of the \((Z\alpha)^6/N^6\) energy term has a specified value which depends only on the masses of the bound particles, not on angular quantum numbers; (b) The coefficient of the \((Z\alpha)^2/N^2\) energy term is a specified multiple of the square of the coefficient of the \((Z\alpha)^4/N^4\) energy term. Both these predictions are verified in positronium by previous calculations to order \((Z\alpha)^6\) which used second-order perturbation theory. They are also correct in the Coulomb-Dirac limit. The effect of the two-body Sommerfeld formula on calculations of muon-proton bound-state energies is examined.

I. INTRODUCTION

The lack of an analytically solvable relativistic wave equation for two-body atomic systems has compelled physicists to use second-order perturbation theory in calculating energy levels to order \((Z\alpha)^6\) in systems such as positronium \([1, 2]\). Several two-body wave equations or prescriptions (reviewed in ref. \([3]\)), starting with the Breit equation \([4]\) in 1929, have correctly described the energy levels of two-body atomic systems to order \((Z\alpha)^4\) using first-order perturbation theory. But none has been solvable analytically, so none has been able to be a starting-point for the calculation of atomic energy levels at order \((Z\alpha)^6\) without the use of second-order perturbation theory.

For the rest of this paper the symbol \(\alpha\) will be a synonym for \(Z\alpha\). No radiative corrections are considered.

To explain what is found in this paper and what remains to be found, we remind the reader that the analytic solution \([5]\) to the Coulomb Dirac equation in 1928 gave the one-particle Sommerfeld fine-structure formula for the bound-state energies:

\[ E_{\text{Dirac}} = \frac{m}{\sqrt{1 + \frac{\alpha^2}{(n + \epsilon_{\text{Dirac}} + 1)^2}}}. \]  

(1)

Here \(m\) is the mass of the electron, \(\alpha\) is the fine-structure constant, \(n\) is the radial quantum number and \(\epsilon_{\text{Dirac}}\) is determined by the eigenvalue of a finite-dimensional angular equation

\[ (\epsilon_{\text{Dirac}} + 1)^2\xi = \epsilon^2 - \alpha^2\xi. \]  

(2)

Because equation (2) for the angular parameter \(\epsilon_{\text{Dirac}}\) is exactly solvable, the bound-state energies predicted by equation (1) can be evaluated by simple algebra not only to order \(\alpha^4\), but to order \(\alpha^6\) and even \(\alpha^8\). The Coulomb-Dirac problem does not need second- and third-order perturbation theory.

If a two-body wave equation accurate to order \((Z\alpha)^4\) were to be solvable analytically like the one-particle Coulomb Dirac equation, no second-order perturbation theory would be needed to obtain two-body energy levels to order \((Z\alpha)^6\). The other terms needed to complete the physical calculation to order \((Z\alpha)^6\) would only need to be evaluated in first-order perturbation theory.

II. MAIN RESULT

In this paper we will find a two-particle counterpart to the Sommerfeld fine-structure formula (1). It is

\[ E = \sqrt{m^2 + M^2 + \frac{2mM}{\sqrt{1 + \frac{n^2}{(n + \epsilon_{\text{Dirac}} + 1)^2}}}}. \]  

(3)

Here \(M\) is the mass of the other particle. It is easy to see that \(E - M\) reduces to the one-particle Sommerfeld fine-structure formula (1) when \(M \to \infty\). The formula (3) for the bound-state energies is obtained from a two-body relativistic atomic wave equation which was derived in ref. \([3]\) and is repeated below.

In this work we will not be able to derive an angular equation for the angular parameter \(\epsilon\) which is a two-particle counterpart to equation (2). Therefore we have only half the analytic solution.

Nevertheless, from the two-particle Sommerfeld formula (3) alone, assuming that \(\epsilon\) exists without knowing it, we have found two predictions for order \(\alpha^6\) bound-state energy terms for any masses, which are verified by previous calculations for positronium which had to use second-order perturbation theory. These results raise the hope that it may eventually be possible to find a full analytic solution to a relativistic atomic two-body wave
The eigenvalue is $\beta$, the particle of mass $m$, in ref. [3], contains an anomalous magnetic moment for the final term in the curly brackets, which was not included in the Blankenbecler-Sugar correction formalism. The derivation used a simple quasi-potential approximation [6, 7] with its associated Blankenbecler-Sugar correction series [8].

The masses of the bound particles are denoted by $m, M$. The bound-state energy $E$ is parametrised by a quantity $\beta$ as follows:

$$E = \sqrt{m^2 - \beta^2} + \sqrt{M^2 - \beta^2}$$  

(4)

The particles' individual bound-state energies also occur:

$$t = \sqrt{m^2 - \beta^2}, \quad T = \sqrt{M^2 - \beta^2}$$  

(5)

The Pauli matrices $\sigma$ and the Dirac matrices $\gamma, \gamma^0$ refer to the particle of mass $m$, while $\Sigma, \Gamma$ and $\Gamma^0$ refer to the particle of mass $M$. The operator $p$ is $-i\nabla$, where $\nabla$ refers to the relative position coordinate $r$. Also $r = |r|$ and $\hat{r} \equiv r/r$.

The relativistic bound-state wave equation in the centre-of-mass system is

$$[p^2 + \beta^2] \psi(r) = -\frac{1}{2E} \left[ m - \gamma \cdot p + \gamma^0 t \right] \left[ M + \Gamma \cdot p + \Gamma^0 T \right] \times$$

$$\times \left\{ -\gamma^0 \Gamma^0 \frac{\alpha}{r} + \frac{\gamma \cdot \Gamma + \gamma \cdot \hat{r} \Gamma \cdot \hat{r} \begin{array}{l} \frac{\alpha}{r} \\ \alpha \end{array}}{2} + \frac{1 + \gamma^0 1 + \Gamma^0}{2 \alpha} \left( \frac{\alpha}{r} \right)^2 - \frac{g - 2}{4M} \gamma \cdot \hat{r} \times \Sigma \frac{\alpha}{r^2} \right\} \psi(r)$$

(6)

The eigenvalue is $\beta^2$, not $E$. The constant $\beta^2$ is substituted into the square roots in equation (4) to obtain the energy $E$. There are no non-local operators of the form $\sqrt{p^2 + m^2}$, so no kinetic-energy correction terms $p^4, p^6$ appear in perturbation theory.

The first and second terms in the curly brackets are the standard binding potential and Breit interaction. The third term is an effective term equal to $1/2E$ times the square of the binding potential. This is a consequence of the Blankenbecler-Sugar correction formalism, which ensures that the relativistic energies are correct [9]. The final term in the curly brackets, which was not included in ref. [3], contains an anomalous magnetic moment for the particle of mass $M$. It is derived by adding a term $i(g - 2)(k \times \Sigma)/4M$ to the Dirac matrix $\Gamma$ in the vertex function of the particle of mass $M$ in the originating Bethe-Salpeter equation. The term is included here so that the particle of mass $M$ may represent a proton.

As discussed in ref. [3], the wave equation (6) is expected to be valid for $r \gtrsim \alpha/2E$, because the Blankenbecler-Sugar correction formalism results in a series in powers of $(-\alpha/r)/2E$. In addition, for the hydrogen atom and muonic hydrogen [10, 11], $\alpha/2E$ is about 0.001 fm. The mean-square charge radius of the proton is about 1 fm. Therefore except perhaps for positronium the validity of the two-body wave equation (6) is expected to end well before $r$ descends to $\alpha/2E$.

In the limit $M \to \infty$, dividing through by the first right-hand operator in square brackets, it is easy to see that equation (6) reduces to the one-body Coulomb Dirac equation

$$[m + \gamma \cdot p - \gamma^0 t] \psi(r) = \frac{1 + \Gamma^0}{2} \gamma^0 \frac{\alpha}{r} \psi(r)$$

(7)

which gives the energies (11) and the angular equation (2).

To verify the correctness of the wave equation (6) we give its bound-state energies to order $\alpha^4$. With $J$ (often written $F$) denoting the total spin of the bound state, notation used is:

$$\mu = \frac{mM}{m + M}, \quad x = \frac{\mu^2}{mM}, \quad \rho = \frac{\mu}{M} - \frac{\mu}{m}$$

(8)

$$a^2 = [\rho + (g - 2)x]^2 L(L + 1)$$

(9)

From first-order perturbation theory with $N$ the Bohr quantum number the bound-state energies of the wave equation (6) are

$$E = m + M - \frac{\alpha^2 \mu}{2N^2} + \frac{\alpha^3 \mu^3 - x}{8N^4} - \frac{\alpha^4 \mu}{2N^4} \left[ (1 - \delta_Lo)\Theta + \delta_Lo\Phi \right]$$

(10)

in which

$$\Theta = \begin{cases} \frac{1}{L + 1} & \text{if } L = J + 1 \\ \frac{L + 1}{(2L + 1)(2L + 3)} & \text{if } L = J - 1 \\ \frac{1}{L + 1} + \frac{1}{(2L + 1)(2L + 3)} & \text{if } L = J, S \approx 1 \\ \frac{1}{L + 1} - \frac{1}{(2L + 1)(2L + 3)} & \text{if } L = J, S \approx 0 \end{cases}$$

(11)

and

$$\Phi = \begin{cases} 1 - 2xg/3 & L = 0, S = 1 \\ 1 + 2xg & L = 0, S = 0 \end{cases}$$

(12)

The symbols $S \approx 1, S \approx 0$ for $L = J$ stand for the state in which $S$ is predominantly 1 or 0, respectively.

Examining the results above, we find that in the limit $M \to \infty$ the energies of the Dirac Coulomb equation are recovered. For $m \ll M$ it is easy to verify that the hyperfine splittings are correct to order $m/M$ (12), Sec. 22). With $m = M$ and $g = 2$ we find the standard positronium energies without the annihilation term (12), Sec. 23). These results verify that to order $\alpha^4$, the two-body wave equation (6) gives physically correct results.
IV. DERIVATION OF THE TWO-BODY ENERGY FORMULA

Following conventional treatments of the Coulomb Schrödinger equation and the Coulomb Dirac equation, we substitute

$$\psi(r) = e^{-\beta r} r^j \sum_j a_j r^j$$  \hspace{1cm} (13)$$

into the wave equation (6). Here the coefficients $a_j$ are 16-dimensional vectors. The expansion is expected to be valid for $r \gtrsim \alpha/2E$, or for $r$ greater than the proton radius if a proton is present. One obtains a four-term recurrence relation for the coefficients $a_j$.

When the dominant terms acting on the large component of the wave function in the wave equation (6) are examined, we see that they are the same as in the Coulomb Schrödinger equation. That means that if the series (13) does not terminate for some $j = n$, the wavefunction will diverge as $e^{+\beta r}$ for large $r$. So that the wavefunction (13) will go to zero at infinity, in this paper we will assume that the series terminates at $j = n$. Then it is easy to show that with $a_j = 0$ for $j > n$, the recurrence relation implies this equation for $a_n$:

$$2\beta (\epsilon + 1 + n) a_n = -\frac{\alpha}{2E} \tilde{m} \tilde{M} \left[ -\gamma^0 \Gamma^0 + \frac{\gamma \cdot \Gamma + \gamma \cdot \tilde{r} \Gamma \cdot \tilde{r}}{2} \right] a_n$$  \hspace{1cm} (14)$$

containing the abbreviations

$$\tilde{m} = m - i \beta \gamma \cdot \tilde{r} + \gamma^0 \alpha, \hspace{1cm} \tilde{M} = M + i \beta \Gamma \cdot \tilde{r} + \Gamma^0 \Gamma. \hspace{1cm} (15)$$

Equation (14) shows that $\alpha_n = \tilde{m} \tilde{M} \beta$. Substituting that back into (14) puts a projection operator $\tilde{m} \tilde{M}$ on each side of the Coulomb and Breit terms; multiplying them out gives a scalar multiple of $\tilde{m} \tilde{M}$ again. One thus finds

$$2\beta (\epsilon + 1 + n) a_n = -\frac{\alpha}{2E} [-4tT + 4\beta^2] a_n$$

which is to say

$$\frac{\beta E}{tT - \beta^2} = \frac{\alpha}{\epsilon + n + 1}$$  \hspace{1cm} (16)$$

Using equations (4) and (5), equation (16) gives the two-body Sommerfeld “fine-structure” bound-state energy formula (6).

The parameter $\epsilon$, which depends on the angular quantum numbers, cannot be found from the two-body wave equation (6) as easily as $\epsilon_{\text{Dirac}}$ is found from the Coulomb Dirac equation (7). The reason is that $\epsilon_{\text{Dirac}}$ is determined by the regular singularity of (7) as $r \to 0$. But for finite $m$ and $M$, the wave equation (6) is expected to be valid only for $r$ exceeding the small but finite value $\alpha/2E$, or exceeding the radius of a proton if a proton is one of the bound particles. While the two-body Sommerfeld energy formula (4) has come from the requirement that the wavefunction remain finite at large $r$, we cannot use the behaviour of eqn. (6) as $r \to 0$ to determine $\epsilon$.

However, the assumption that the quantity $\epsilon$ does exist will lead below to some new predictions which are verified in known physical systems such as the hydrogen atom and positronium. The predictions apply to muon-proton bound states as well.

V. EXPANSION OF $E$ IN POWERS OF $\alpha^2$

In the bound-state energy formula (3) the quantity $n + \epsilon + 1$ occurs. While in the Coulomb Schrödinger equation $\epsilon$ is always $L$, in the Coulomb Dirac case equation (2) sometimes gives $\epsilon_{\text{Dirac}} \approx L$, but sometimes $\epsilon_{\text{Dirac}} \approx L - 1$. In the latter case the ground state is known to be given by $n = 1$, not $n = 0$. Thus in either case, $n + \epsilon_{\text{Dirac}} + 1$ is still approximately the Bohr quantum number $N$. In fact it is easy to find from (2) that, with $\kappa = |L + \sigma/2| + 1/2$,

$$n + \epsilon_{\text{Dirac}} + 1 = N - \frac{\alpha^2}{2 \kappa} - \frac{\alpha^4}{8 \kappa^3} - \frac{\alpha^6}{16 \kappa^5} + \cdots.$$  \hspace{1cm} (17)$$

It is likely that the same thing happens for the two-body wave equation (6). Therefore we expand

$$n + \epsilon + 1 = n + \alpha^2 c_2 + \alpha^4 c_4 + \alpha^6 c_6 + \cdots$$  \hspace{1cm} (18)$$

When the expansion (18) is substituted into the two-body Sommerfeld formula (4), the energy levels $E$ can be expanded in powers of $\alpha^2$. Recalling that $x \equiv \mu^2/mM$, one finds

$$E = m + M + \alpha^2 \mu C_2 + \alpha^4 \mu C_4 + \alpha^6 \mu C_6 + \alpha^8 \mu C_8 + \cdots.$$  \hspace{1cm} (19)$$

in which

$$C_2 = -\frac{1}{2N^2}.$$  \hspace{1cm} (20)$$

$$C_4 = \frac{3 - x}{8N^4} + \frac{\epsilon_2}{N^3}.$$  \hspace{1cm} (21)$$

$$C_6 = -\frac{5 - 3x + x^2}{16N^6} - \frac{(3 - x)\epsilon_2}{2N^5} - \frac{3\epsilon_2^2}{2N^4} + \frac{\epsilon_4}{N^3}.$$  \hspace{1cm} (22)$$

$$C_8 = \frac{35 - 29x + 18x^2 - 5x^3}{128N^8} + $$

$$\frac{3 [5 - 3x + x^2] \epsilon_2}{8N^7} + \frac{5(3 - x)\epsilon_2^2}{4N^6} + $$

$$\frac{4\epsilon_2^3}{2N^5} - \frac{3\epsilon_2 \epsilon_4}{N^4} + \frac{\epsilon_6}{N^3}.$$  \hspace{1cm} (23)$$

In the Coulomb-Dirac limit, $\mu = m$ and $x = 0$. In this limit, using the expansion of $\epsilon_{\text{Dirac}}$ from eqn. (17), eqns. (19) to (23) reproduce the known expansion of $E_{\text{Dirac}}$ to order $\alpha^8$. For general masses $m, M$ we now examine the expansion (19) and the coefficients (20) to (23) to see whether first, they agree with the known results to order $\alpha^4$, and second, to see whether they predict anything not known yet.
VI. $\alpha^2$ AND $\alpha^4$ TERMS

The coefficient $C_2$ gives the Bohr levels correctly.

Next, referring to the energy levels $[10]$ given above, even the coefficient $C_4$ suggests the correctness of the two-body Sommerfeld formula $[3]$. Eqn. $[21]$ agrees that there are $\alpha^4/N^3$ and $\alpha^4/N^5$ terms only. Moreover, as eqn. $[10]$ shows, $C_4$ gives the $\alpha^4/N^3$ term correctly for all masses, verifying that it is independent of the unknown value of $\epsilon$. Finally, from the first-order perturbation theory result $[10]$ we can read off the value of $\epsilon_2$:

$$\epsilon_2 = -\frac{1}{2}[(1 - \delta_{L0})\Theta + \delta_{L0}\Phi]$$

(24)

with $\Theta$ and $\Phi$ given by $[11]$ and $[12]$.

VII. PREDICTIONS OF $\alpha^6$ TERMS

The $\alpha^6$ terms in $[22]$ will tell whether the two-body Sommerfeld energy formula $[3]$ contains any information previously obtainable only from using second-order perturbation theory, and whether it predicts anything as yet uncalculated. Note that the predictions described hold automatically in the Coulomb-Dirac limit $M \to \infty$.

A. $\alpha^6/N^6$ Terms

The coefficient of the $\alpha^6\mu/N^6$ term given by $[22]$ does not need knowledge of the angular eigenvalue $\epsilon$. It is predicted to have the same value $-5 - 3x + x^2)/16$ for every state, depending only on the masses. Therefore for positronium, for which $\mu = m/2$ and $x = 1/4$, eqn. $[22]$ predicts that the coefficient of $\alpha^6\mu/N^6$ is always $-69/512$. In fact, precisely this value has been found for all P-states $[13]$, for all S-states $[2]$, and in Zatorski’s recent calculations for all $L \geq 2$ states $[4]$. To contrast the methods, we note that in ref. $[1]$ the coefficient $-69/512$ is the sum of five terms, three from second-order perturbation theory (eqns. (159), (160) and (162) of ref. $[1]$) and two first-order terms, one of which contains the expectation value of the $p^6/m^5$ kinetic-energy correction which is absent in our formalism (ref. $[1]$ eqns. (91) and (122)).

In addition, Pachucki $[14]$ has found the same function $-5 - 3x + x^2)/16$ for the coefficient of the $\alpha^6\mu/N^6$ energy term as in $[22]$ for arbitrary masses — but only for S-states. Accordingly this coefficient remains to be tested for $L \geq 1$ when $m \neq M$.

B. $\alpha^6/N^4$ Terms

For $L \geq 1$, in ref. $[1]$ the $\alpha^6m/N^4$ terms of the positronium energy levels come from the sum of six second-order perturbation terms (eqns. (153), (158), (164), (171), (174) and (177)) and no first-order terms. Since the two-body Sommerfeld formula $[3]$ is a candidate to replace the results of second-order perturbation theory once $\epsilon$ is fully determined, these calculated $\alpha^6/N^4$ terms are a possible second test of the Sommerfeld formula $[3]$.

In positronium, from $[21]$ the $\alpha^4/N^3$ energy term is $(\epsilon_2/2)\alpha^4m/N^3$, and from $[22]$ the $\alpha^6/N^4$ energy term is $-3\alpha_3^2/4\alpha^6m/N^4$. Therefore it is predicted that the coefficient of the $\alpha^6m/N^4$ term of positronium should be $-3$ times the square of the coefficient of the $\alpha^4m/N^3$ term.

As an example for $L = J + 1$, with $m = M$ and $q = 2$, from $[10]$ and $[11]$ above the coefficient of $\alpha^4m/N^3$ in positronium is

$$-(4L^2 + L - 1)$$

$$4L(2L - 1)(2L + 1).$$

In eqn. (211) of ref. $[1]$ the calculated coefficient of the $\alpha^6m/N^4$ term for $L = J + 1$ is given from the sum of the six second-order terms as

$$\frac{3 - 6L - 21L^2 + 24L^3 + 48L^4}{16L^2(2L - 1)^2(2L + 1)^2}$$

It can be seen that the latter expression is indeed $-3$ times the square of the former. This verifies the prediction of the two-body Sommerfeld formula for positronium $L = J + 1$ states. The prediction also gives the other $\alpha^6m/N^4$ terms of positronium energies correctly (eqns. (207), (215) and (219) of ref. $[1]$).

For the $^1S_0$ and $^3S_1$ states of positronium the coefficients of the $\alpha^6m/N^4$ term contain other contributions (see ref. $[2]$) and no prediction can be made. For $L \geq 1$ and $m \neq M$ the prediction for the $\alpha^6/N^4$ term remains to be tested.

C. $\alpha^6/N^5$ AND $\alpha^6/N^3$ TERMS

While $[22]$ predicts the coefficient of $\alpha^6\mu/N^5$ to be $-3 - x)\epsilon_2/2$, in which $\epsilon_2$ is already known from $[24]$, we cannot tell whether this coefficient is confirmed by previous positronium calculations because many terms of order $\alpha^6$ contain $1/N^5$ $[1, 2]$.

The final coefficient $\epsilon_4$ of $\alpha^6\mu/N^3$ in $[22]$ cannot be evaluated and tested until an angular equation for $\epsilon$ analogous to $[2]$ is found from which $\epsilon_4$ can be calculated. Note, however, that since $L = J \pm 1$ mixing is prevented in order $\alpha^4$ because $(L = J - 1)|1/r^3)(L = J + 1) = 0$, this mixing does not occur in $\epsilon_2$ either, because of eqn. (24). Consequently, such mixing can only occur in the $\epsilon_4$ term. Therefore eqn. $[22]$ predicts that in order $\alpha^6$, $L = J \pm 1$ mixing should occur in the $\alpha^6/N^3$ term only.
VIII. OTHER REMARKS

A. \( m \ll M \)

For \( m \ll M \) and \( \epsilon = \epsilon_{\text{Dirac}} \) the two-body Sommerfeld formula reproduces a result in the review of hydrogen-like atoms by Eides, Groch and Shelyuto [13]. The two-body Sommerfeld formula (3) can be written, once again recalling that \( x = \mu/(m + M) \), as

\[
E = (m + M)\sqrt{1 + 2xh} \tag{25}
\]

with

\[
h = \frac{1}{\sqrt{1 + \frac{\alpha^2}{(n+\epsilon+1)^2}}} - 1 \tag{26}
\]

For \( m \ll M \), \( x \) is small. Equation (25) expanded in powers of \( x \) up to order \( x^2 \) is

\[
E = (m + M) + \mu h - \frac{1}{2} \frac{\mu^2}{m + M} h^2. \tag{27}
\]

With \( \epsilon = \epsilon_{\text{Dirac}} \) this is the same expression that appears in eqn. (38) of ref. [15]. Expansion of expression (27) to order \( \alpha^6 \) reproduces the expression (22) for \( C_6 \) except for the \( x^2 \) term, which is small for the hydrogen atom. This agreement represents another confirmation of the two-body Sommerfeld formula (3).

B. \( (Z\alpha)^6 \)

The expansion of the bound-state energy given by equations (19) to (23) is carried out to order \( \alpha^6 \), meaning \( (Z\alpha)^6 \). As a single example, in the unlikely event that any energy levels of positronium are calculated to order \( (Z\alpha)^6 \) for some angular quantum numbers and all Bohr quantum numbers \( N \), eqn. (23) predicts that the coefficient of \( (Z\alpha)^6 m/N^6 \) for every state of positronium will be 1843/16384.

C. Evaluation from Bethe-Salpeter Kernel

The wave equation (6) was derived from a Bethe-Salpeter equation whose kernel consists of the exchange of a single Coulomb-gauge photon. To evaluate physical energies to order \( (Z\alpha)^6 \) fully, the procedure would be (i) to evaluate the energies of this binding Bethe-Salpeter equation fully to order \( (Z\alpha)^6 \), then (ii) to evaluate the energy contributions of other relevant kernels, including crossed diagrams, all of which would need only first-order expectation values to obtain energies to order \( (Z\alpha)^6 \).

In part (i) the first task is evidently to find and solve a two-body angular equation similar to (2) for the unknown angular parameter \( \epsilon \), so that the two-body Sommerfeld formula (3) can be evaluated to any desired accuracy. Then the Blankenbecler-Sugar correction formalism [3] can be used to evaluate the remaining corrections.

Since for real two-body atoms the wave equation (6) has not revealed a way to calculate \( \epsilon \) yet, we report on a complete implementation of part (i) for a far simpler model.

Ref. [10] used a Bethe-Salpeter equation for scalar constituent particles of masses \( m, M \) bound by a scalar-Coulomb kernel \(-4mM\alpha/r\). Following through the formalism of ref. [3] gave a two-body wave equation with a solution which could be evaluated fully. The wave equation had a regular singularity as \( r \to 0 \) which gave an algebraic equation for \( \epsilon \) not unlike eqn. (2) of the present paper. The solution series terminated at \( j = n \) for appropriate energies. The resultant two-body Sommerfeld formula was quite similar to eqn. (3) of the present paper. It gave energies to order \( \alpha^0 \) since \( \epsilon \) was known. The Blankenbecler-Sugar correction formula was used to find the corrections to the approximations which led to the wave equation. Evaluation of their expectation values gave \( \alpha^6 \) terms for all \( L \), as well as \( \alpha^5 \) and \( \alpha^6 \log \alpha \) terms for \( L = 0 \). Thus the calculation of the bound-state energies of the model scalar Bethe-Salpeter equation was completed to order \( (Z\alpha)^6 \) without using second-order perturbation theory.

It is possible that a similar implementation for real two-body atoms using a Coulomb-gauge kernel would require a return to the derivation in ref. [3], perhaps obtaining a wave equation slightly different to (6) near the origin.

D. Muonic Hydrogen

For muon-proton bound states we have compared the energies calculated by first-order perturbation theory (10) to an evaluation using the exact two-body Sommerfeld formula (3) in which, given the value of \( n + \epsilon + 1 \), recoil effects appear to be exact.

Substituting the approximation

\[
n + \epsilon + 1 = N + \alpha^2 \epsilon_2
\]

into the two-body Sommerfeld formula (3) with \( \epsilon_2 \) given by the first-order perturbation value (24), it was found that the energy of the nominal 1S\(^{P=0}\) state evaluated by the two-body Sommerfeld formula was 0.005 meV lower than when evaluated by first-order perturbation theory. The other 1S state had a smaller difference. The energy of the nominal 2S\(^{P=0}\) state was 0.0008 meV lower when calculated by the Sommerfeld formula than by perturbation theory. The other \( N = 2 \) states had a smaller difference. Thus for muonic hydrogen the recoil correction to first-order perturbation theory given by the Sommerfeld formula appears to be small for the 2P and 2S states of current interest [10, 11].
IX. CONCLUSION

Verification of the two \((Z\alpha)^6\) predictions, especially for the \((Z\alpha)^6/N^4\) term which was quite unexpected, indicates two things. (i) The two-body Sommerfeld energy formula \(\Phi\) seems to be true. (ii) The angular eigenvalue \(\epsilon\) apparently exists. We do not know yet whether the two-body atomic wave equation \(\Delta\), valid as written for \(r > \alpha/2E\), determines a finite-dimensional angular equation for \(\epsilon\) analogous to the one-particle angular equation \(\Omega\) for \(\epsilon_{\text{Dirac}}\).

It is expected that any complete analytic solution to the relativistic two-body atomic bound-state problem will give a two-body Sommerfeld formula for the bound-state energies which would give the \((Z\alpha)^4\) levels completely, and which would also give the contribution of order \((Z\alpha)^6\) which previously has had to be calculated from second-order perturbation theory on the Breit interaction. To obtain complete physical results to order \((Z\alpha)^6\), as was done for the simple scalar Bethe-Salpeter equation in ref. [16], it is necessary first to go back to the originating Bethe-Salpeter equation and evaluate its energies to order \((Z\alpha)^6\). This is done by letting the formalism given in [3] determine what small additional terms are present beyond those contained in the Sommerfeld formula. The expectation values of these terms are evaluated.

More than 80 years have passed since the analytic solution to the Coulomb Dirac equation gave the Sommerfeld fine-structure formula \(\Phi\). The existence of the two-body Sommerfeld formula \(\Phi\), even though its angular eigenvalue \(\epsilon\) remains unknown for now, suggests that a wave equation for two-body atoms, perhaps somewhat similar to our equation \(\Omega\), may eventually be found and solved analytically.

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