Self-organized Boolean game on networks

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A model of Boolean game with only one free parameter $p$ that denotes the strength of herd behavior is proposed where each agent acts according to the information obtained from his neighbors in network and those in the minority are rewarded. The simulation results indicate that the dynamic system is sensitive to network topology, where the network of larger degree variance, i.e. the system of greater information heterogeneity, leads to less system profit. The system can self-organize to a stable state with more profit comparing with random choice game, although only the local information is available to the agents. In addition, in heterogeneity networks, the agents with more information gain more than those with less information for a wide extent of herd strength $p$.

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I. INTRODUCTION

Complex adaptive systems composed of agents under mutual influence have attracted considerable interest in recent years. It is not unexpected that the systems with globally shared information can be organized. A basic question in studies of complexity is how large systems with only local information available to the agents may become complex through a self-organized dynamical process\cite{1}.

The mutual influence can be properly described as the so-called information network, in which the nodes represent agents and the directed edge from $x$ to $y$ means the agent $y$ can obtain information from agent $x$. For simplicity, the undirected networks are considered in this paper. In this way, node degree $k$ is proportional to the quantity of information available to the corresponding agent. The two extensively studied information networks of ecosys-\cite{2} tem are regular (\cite{3,4}) and random (\cite{1,5}) networks, both of which have a characterized degree-the mean degree $\langle k \rangle$: for regular networks, all the node are of degree $\langle k \rangle$; and for random ones, the degree distribution decays quickly in a Possionian form when $k > \langle k \rangle$. The existence of characterized degree means every node has almost the same capacity of information. However, previous empirical studies have revealed that the information networks may be of scale-free property (\cite{6,7,8}), in which the giant heterogeneity of information exists. The nodes of larger degree predominate much more information than those of less degree thus the information heterogeneity can be measured by the degree variance $\langle k^2 \rangle$. The question is how the topology affects the system dynamic, will the greater information heterogeneity induce more profit for the system, or contrarily?

Another question being concerned of in this paper is about the herd behavior, which has been extensively studied in Behavioral Finance and is usually considered as one factor of the origins of complexity that may enhance the fluctuation and reduce the system profit\cite{9,10,11,12,13}. Here we argue that, to measure the strength of herd behavior, it is more proper to look at how far the agents’ actions are determined by others rather than how far the agents want to be in majority, since in many real-life cases, the agents would like to be in minority but the herd behavior still occurs. We wonder whether agents have different responses under a fixed herd strength, and whether the variation trends of system profit and individual profit are the same as the increase of herd strength.

In this paper, a model of Boolean game with only one free parameter $p$ that denotes the strength of herd behavior is proposed where each agent acts according to the information obtained from his neighbors in network and those in the minority are rewarded. Although the model may be too simple and rough, it offers a starting point aiming at those questions above. We have found that the topology of information network affects the system dynamic much and the system can self-organize to a stable state with more profit comparing with random choice game even only the local information is available.

II. MODEL

Boolean game is firstly proposed by Kauffman where each agent has only one binary choice such as either buying or selling a stock\cite{14}. The studies of Boolean game have attracted not only the physicists’ but also the ecol-
FIG. 1: The variance of the number of agent choosing +1 as a function of herd strength $p$. The four plots are the cases of star, regular, random and scale-free networks, respectively. The solid line represents the random choice game where $\sigma^2 = 0.25N$. It is clear that the system profit is more than random choice game when $p \in (0, 0.7)$, $p \in (0, 0.7)$ and $p \in (0, 0.4)$ in regular, random and scale-free networks, respectively. For any $p \in (0, 1)$, $\sigma^2$ of the four cases satisfy that $\sigma^2_{\text{regular}} < \sigma^2_{\text{random}} < \sigma^2_{\text{scale-free}} < \sigma^2_{\text{star}}$, that means the system profit $S$ satisfy that $S_{\text{regular}} > S_{\text{random}} > S_{\text{scale-free}} > S_{\text{star}}$.

FIG. 2: (Color online)The number of agent choosing +1 vs time. The simulation takes place on regular networks of size $N = 1001$. At the beginning, a large event with 701 agent choosing +1 happens. The red thick and black thin curve show the variety of $A_t$ after this large event for the two extreme cases $p = 0$ and $p = 1$, respectively. Clearly, in the case $p = 0$, $A_t$ slowly reverts to the equilibrium position $A \approx \frac{2m}{N}$, while in the case $p = 1$, the system displays obvious oscillation behavior. The inset exhibits the oscillation of $A_t$ in the case $p = 1$ for the first 30 time steps.

III. SIMULATIONS

In this paper, all the simulation results are the average of 100 realizations and for each realizations the time length is $T = 10^4$ unless a special statement is addressed. The number of agents $N = 1001$ and mutation probability $m = 0.01$ are fixed. Figure one shows the variance $\sigma^2 = \frac{1}{T} \sum_{t=1}^{T} (A_t - \frac{N}{2})^2$ as a function of $p$ in star, regular, random and scale-free networks, where $A_t$ is the num-

ogists’ and economists’ attention since it could explain many empirical data and might contribute to the understanding of the underlying mechanisms of the many-body ecosystems, although the dynamic rule is simple.

Inspired by the idea of minority game\cite{14}, which is a simple but rich model describing a population of selfish individuals fighting for a common resource, we propose the present Boolean game where each agent chooses between two opposing actions, simplified as $+1$ and $-1$, and the agents in the minority are rewarded. Each winner’s score increases by one thus the system profit equals to the number of winners\cite{17,18,19}. In our model, at each time step, each agent acts based on his neighbors at probability $p$, or acts all by himself at probability $1 - p$. In the former case, we assume each neighbor has the same opposite action at a small probability $s$, simplified as $+1$ and $-1$ in the last time step, respectively. In the latter case, since there is no information from others, the agent will simply inherits his action in the last time step or chooses the opposite action at a small probability $m$, named mutation probability. It is worthwhile to emphasize that, the agents do not know who are winners in previous steps since the global information is not available, which is also one of the main differences from the previous studies on minority game.

The real-life ecosystem often seems a black box to us: the outcome may be observed, but the underlying mechanism is not eyeable. If we see many agents display the same action, we say the herd behavior occurs, although those agents might prefer to be in the minority. In another point of view, if each agent acts all by himself, there is no preferential choice for $+1$ and $-1$ so as no herd behavior will occur. Therefore, if the herd behavior occurs, the agents’ actions must be at least partly based on the information obtained from others. In this paper, the strength of herd behavior is measured by how far the underlying possibility of the occurrence of herd behavior.

The number of agents $N = 1001$ and mutation probability $m = 0.01$ are fixed. Figure one shows the variance $\sigma^2 = \frac{1}{T} \sum_{t=1}^{T} (A_t - \frac{N}{2})^2$ as a function of $p$ in star, regular, random and scale-free networks, where $A_t$ is the num-
ber of agents who choose +1 at time step \( t \). Clearly, the smaller \( \sigma^2 \) corresponds to the more system profit, and for the completely random choice game, \( \sigma^2 = 0.25N \). The regular network is a one-dimension lattice with periodic boundary conditions and coordination number \( z = 3 \) \cite{21, 22}, the random network is the ER network of connecting probability \( 6 \times 10^{-3} \) \cite{21, 22}, and the scale-free network is the BA network of \( m_0 = m = 3 \) \cite{22}. Therefore, all the networks except the star networks are of average degree \( \langle k \rangle = 6 \). Since the number of edges \( \langle k \rangle N \) is proportional to the total quantity of information available to agents, the networks used for simulating except star networks have the same capacity of information. In star network, it is not unexpected that the system profit will be reduced when the herd strength increases. More interesting, in each of the latter three cases, the system preforms better than the random choice game when \( p \) is in a certain interval, indicating the self-organized process has taken place upon those networks.

Although having the same capacity of information, the dynamic of scale-free networks is obviously distinguishable from that in regular and random networks, indicating that the topology affects the dynamic behavior much. Note that, although the topology of regular and random networks are obviously different for they have completely different average distance and clustering coefficient and so on \cite{23}, the dynamic behaviors are almost the same in those two networks. The common ground is they have almost the same degree variance \( \langle k^2 \rangle \). According to the inequality

\[
\langle k^2 \rangle_{\text{star}} > \langle k^2 \rangle_{\text{scale-free}} > \langle k^2 \rangle_{\text{random}} > \langle k^2 \rangle_{\text{regular}}
\]

and the simulation results, we suspect that the larger degree variance, i.e. the greater information heterogeneity, will lead to less system profit.

In figure one, one can see clearly that for all the four cases, the variance \( \sigma^2 \) is remarkably greater than the random choice game at large \( p \). Consider the extreme case \( p = 1 \), if the agent choosing +1 and -1 are equally mixed up in the networks, and the number of agent choosing +1 at present time is \( A_t \), then in the next time step, the expectation of \( A_{t+1} \) is \( \langle A_{t+1} \rangle = N - A_t \), with departure \( \left| \langle A_{t+1} \rangle - \frac{N}{2} \right| = |A_t - \frac{N}{2}| \). If at present time \( A_t \) is larger than \( \frac{N}{2} \), then \( A_{t+1} \) will be smaller than \( \frac{N}{2} \) most probably, and the departure from \( \frac{N}{2} \) will not be reduced in average. Therefore, in the case of \( p = 1 \), when the "large event" happens, that is to say \( A_t \) is much larger or much smaller than \( \frac{N}{2} \) at some time \( t \), there will be a long duration of oscillation after \( t \), in which \( A \) skips between up-side \( A > \frac{N}{2} \) and down-side \( A < \frac{N}{2} \). The oscillation behavior of \( A \) is shown in figure two. At the beginning, a large event with \( A_0 = 701 \) is given, then the large oscillation goes on about 30 time steps. In \( p = 1 \) case, if \( A \) gets apart from \( \frac{N}{2} \), the influence (large oscillation behavior) will stand for long time, leading very large \( \sigma^2 \). However, in random choice game, whatever \( A_{t-1} \), the expectation of \( A_t \) is always \( \langle A_t \rangle = \frac{N}{2} \), and the distribution of \( A_t - \frac{N}{2} \) obeys Guassian form. That is the reason why the systems having poor performance at large \( p \) comparing with the random choice game.

![FIG. 3: The agent’s winning rate vs degree. Each point denotes one agent and the solid line represents the average winning rate over all the agents. In the cases of \( p = 0.0 \) and \( p = 1.0 \), no correlation is detected. In the cases of \( p = 0.03 \) and \( p = 0.4 \), the positive correlation between agent’s profit and degree is observed.](image1)

![FIG. 4: (Color online) The agent’s winning rate as a function of herd strength. The main plot is obtained by the simulation upon a BA network of size \( N = 1001 \), in which the black, red, green and blue curves from up to bottom represent the four agents of degree 105, 46, 6 and 3 respectively. The inset shows the case upon a BA network of size \( N = 2001 \), where the black, red, green and blue curves from up to bottom represent the four agents of degree 137, 77, 13 and 3 respectively. It is observed that the agents having more information gain more than those with less information.](image2)
is

A shows the agent’s winning rate as a function of

will perform better than those of less degree. Figure four

p

dividual during one time step.

winning rate is denoted by the average score

we report the agent’s winning rate vs degree, where the

lation between agent’s degree and profit. In figure three,

p

em performs best, and

cases respectively,

can see clearly that there exist the positive correlation

between agent’s profit and degree in the cases

1

G

As

is fixed as

such, for

0 < m < 1/2, we have

A

1

is very

large, the system profit will be equal to random

choice game, that means

σ2 = 0.25N.

This is strongly

supported by the simulation results shown in figure one. We

also have check that the value of m will not affect the

characters of these dynamic systems unless m is very

large. The red thick curve in figure two is an example for

the case

p = 0. At time

t = 0, a large event

A0 = 701

occurs, and then the curve

A

slowly reverts to

N

After

about 170 time steps, it arrives at the equilibrium

position

A ≈ \frac{N}{2}.

The two extreme cases also exhibit a clear picture why

the system profit can be maximized at a special value of

p. The herd mechanism (with probability

p) will bring

oscillation, while the independent mechanism (with proba-

bility

1 - p) will lead to a long reversion process. The

former mechanism makes

A

skipping from one side to another,

while the latter one keeps

A’s side. So, under a

proper value of

p, the system can quickly arrive at the

equilibrium position

A ≈ \frac{N}{2} after a large event occurs,

which leads to more system profit. The existence of opt-

imal

p has been demonstrated in figure one.

In succession, let’s focus on the scale-free since it

may be closer to reality. Firstly, we assume the agent

choosing +1 and -1 are equably mixed up in the net-

work. Since there is also no degree-degree correlation for

BA networks,

for arbitrary agent of degree

k (here we do not differentiate between node and the corre-

sponding agent), the probability at which he will choose +1 at time

t + 1 is

| $G_0$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-------|---|---|---|---|---|---|
| $k=105$ | 0.67 | 0.65 | 0.63 | 0.61 | 0.59 | 0.57 |
| $k=3$  | 0.67 | 0.65 | 0.63 | 0.61 | 0.59 | 0.57 |

\[ \eta_1(k, t + 1) = p \left( \sum_{i=0}^{k} \binom{k}{i} C_k^i \rho_1(t)(1 - \rho_1(t))^{k-i} \right) + (1 - p)(1 - \rho_1(t)) = 1 + 2p\rho_1(t) - p - \rho_1(t), \]

where

\rho_1(t)

denotes the density of agents choosing +1 at time step

t, and

C_k^i = \frac{k!}{i!(k-i)!}. Since the probability

\eta_1(k, t + 1)

is independent to

k, there must be no corre-

lation between agent’s degree and profit. In figure three,

we report the agent’s winning rate vs degree, where the

winning rate is denoted by the average score

\langle s \rangle

for individual during one time step. \( p = 0.0 \) and \( p = 1.0 \) corre-

spond to the completely independent and dependent cases

respectively. \( p = 0.03 \) is the point where the system

performs best, and \( p = 0.4 \) is another point where the

system profit is equal to the random choice game. One

can see clearly that there exist the positive correlation

between agent’s profit and degree in the cases \( p = 0.03 \)

and \( p = 0.4 \), which means the agents of larger degree

will perform better than those of less degree. Figure four

shows the agent’s winning rate as a function of \( p \) for dif-

ferent \( k \). It is clear that for a wide extent of \( p \), the agents

having more information will gain more. Therefore, the

assumption is not true, thus there must be some kind of

correlation, which is another evidence of the existence of

self-organized process.

A natural question is addressed: why the agents of

large degree will gain more than those of less degree? The

reason is the choice of a few hub nodes (i.e. the nodes of

very large degree) can strongly influence many other

small nodes (i.e. the nodes of very small degree) choice

in the next time step, and those hub nodes can clean up

from this influence. Denote

H

the set of those hub nodes

and

G_0(t)

the number of hub nodes choosing +1 at time

t. We assume at a certain time step

t,

G_0(t) > \frac{\mu_1}{\mu_2},

that means the number of hub nodes choosing +1 is more than

half. This departure will make some nodes connecting to

FIG. 5: The agents’ winning rate \( \langle s \rangle \) under different choice

patterns \( G_0 \) of the five hub nodes. The simulation takes place

on the BA networks of size \( N = 1001 \), and the herd strength

is fixed as \( p = 0.1 \). The hollow and solid histograms represent

the winning rates of a hub node (\( k = 105 \)) and a small node

(\( k = 3 \)), respectively. One can see clearly, the winning rates

of the small node under different patterns are almost the same

as \( \langle s \rangle_{k=3} \approx 0.49 \), which is obviously small than those of the

hub node especially in the case \( G_0 = 0 \) and \( G_0 = 5 \).
those hub nodes, especially the small nodes, choose -1 in time $t + 1$ with a greater probability. Because the majority of these small nodes’ hub neighbors choose +1 at present, this influence is remarkable and can not be neglected since the small nodes have only a few neighbors. The more departure $|G_0 - \frac{|H|}{2}|$ will lead to the greater influence.

Figure five exhibits an example on BA networks of size $N = 1001$, where $H$ contains only five hub nodes of the highest degree. In each time step, all the choice of these five nodes form a choice configuration. There are in total $2^5 = 32$ different configurations, which are classified into 6 patterns by identifying the number of agents choosing +1. For example, $G_0 = 2$ denotes the pattern that there are 2 agents choosing +1 and other 3 choosing -1. Under each choice pattern $0 \leq G_0 \leq 5$, since $|G_0 - \frac{|H|}{2}| = |G_0 - 2.5|$ is bigger than zero at all time, the hub node can always gain more than the small node. And clearly, under the choice pattern with larger departure, such as $G_0 = 0$ and $G_0 = 5$, the different of winning rates between the hub node and the small node under these patterns is much greater than the case of smaller departure.

**IV. CONCLUSION**

In summary, inspired by the minority game, we propose a model of Boolean game. The simulation results upon various networks is shown, which indicate the dynamic of system is sensitive to the topology of network, where the network of larger degree variance, i.e. the system of greater information heterogeneity, leads to less system profit. The system can perform better than the random choice game. That is a believable evidence of the existence of self-organized process taking place upon the networks although only local information is available to agents. We also have found that in heterogeneity networks, the agents with more information gain more than those with less information for a wide extent of herd strength $p$. In addition, it is clear that the trends of varying of system profit and individual profit are different as the increasing of herd strength, for example, in the scale-free network with $p = 0.5$, the system profit is less than random choice game but the profit of agent of large degree is much more than that in random choice game.

Although this model is rough, it offers a simple and intuitionistic paradigm of the many-body systems that can self-organize even when only local information is available. Since the self-organized process is considered as one of the key ingredients of the origins of complexity, it might contribute to the understanding of the underlying mechanism of the complex systems.

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