Li–He’s modified homotopy perturbation method coupled with the energy method for the dropping shock response of a tangent nonlinear packaging system

Qiu-Ping Ji¹, Jun Wang¹, Li-Xin Lu¹ and Chang-Feng Ge²

Abstract
This paper couples Li–He’s homotopy perturbation method with the energy method to obtain an approximate solution of a tangent nonlinear packaging system. A higher order homotopy equation is constructed by adopting the basic idea of the Li–He’s homotopy perturbation method. The energy method is used to improve the maximal displacement and the frequency of the system to an ever higher accuracy. Comparison with the numerical solution obtained by the Runge–Kutta method shows that the shock responses of the system solved by the new method are more effective with a relative error of 0.15%.

Keywords
Tangent nonlinear differential equation, Li–He’s homotopy perturbation method, energy method, approximate solution

Introduction
The homotopy perturbation method (HPM),¹,² which couples the homotopy technique with the perturbation method (PM), is one of the most popular methods to obtain approximate solutions of nonlinear problems.³–¹² Recently, some effective modifications of HPM and PM have been developed. Filobello-Nino et al.¹³ first introduced differential operators into HPM and proposed the differential operators HPM. Filobello-Nino et al.¹⁴ also proposed the enhanced perturbation method, which expanded the value range of the perturbation parameter by improving the order of equations. Li and He¹⁵ suggested an effective modification of the HPM combined with the enhanced perturbation method, and an extremely high accuracy of an approximate solution could be obtained for a nonlinear oscillator. Except for the above methods, other effective methods for nonlinear solutions have been found, like the variational iteration method¹⁶,¹⁷ and the max–min approach.¹⁸,¹⁹

The dropping shock responses of a nonlinear packaging system are very important for packaging protection. Chen²⁰ and Li and Wang²¹ respectively combined the variational iteration method and the max–min approach with the energy method to solve the nonlinear problems of different kinds of packaging systems. The energy method mentioned is similar to the variational approach²²–²⁶ to nonlinear oscillators. In this paper, a tangent nonlinear packaging system is considered and a new modification of Li–He’s homotopy perturbation method (LHHPM)¹⁵ is developed.
Basic idea of Li-He’s homotopy perturbation method

LHHPM uses the enhanced perturbation method to construct a homotopy equation with a higher order. To introduce the enhanced perturbation method, we take the linear vibration equation as example

\[ \ddot{u} + \omega^2 u = 0 \]  \hspace{1cm} (1)

Equation (1) can be written in the form

\[ (D^2 + \omega^2)u = 0 \]  \hspace{1cm} (2)

where \( D \) is a simple differential operator \( D = \frac{d}{dt} \).

According to the enhanced perturbation method, the so-called annihilator operator \( D^2 + \omega^2 \) is added to equation (2). Then we can obtain the following higher order differential equation

\[ (D^2 + \omega^2)(D^2 + \omega^2)u = 0 \]  \hspace{1cm} (3)

To introduce the HPM, we consider the following nonlinear equation

\[ A(x) = 0 \]  \hspace{1cm} (4)

Introducing an embedded variable \( p \in [0,1] \) and choosing linear operator \( L(u) \) and nonlinear operator \( N(u) \), equation (4) can be transferred as follows

\[ L(u) + N(u) = 0 \]  \hspace{1cm} (5)

To construct a homotopy equation \( H(u,q) : \Omega \times [0,1] \rightarrow R \), it should meet the conditions

\[
\begin{cases}
H(u,p) = (1-p)[L(u) - L(x_0)] + p[L(u) + N(u)] = 0 \\
H(u,p) = [L(u) - L(x_0)] + p[L(u) + N(u)] = 0
\end{cases}
\]  \hspace{1cm} (6)

where \( p \in [0,1] \) and \( x_0 \) is the initial solutions for equation (4).

When \( p = 0 \), equation (6) can be transferred to

\[ H(u,p) = L(u) - L(x_0) = 0 \]  \hspace{1cm} (7)

When \( p = 1 \), equation (6) can be transferred to

\[ H(u,p) = N(u) + L(u) = 0 \]  \hspace{1cm} (8)

Putting \( p \) as a small parameter, the solution can be expressed as a power series of \( p \)

\[ x = x_0 + px_1 + p^2x_2 + \ldots \]  \hspace{1cm} (9)

Basic idea of the energy method

For a nonlinear packaging system without damping, the gravitational potential energy of the system will all be ideally converted into elastic potential energy when the deformation of the buffer reaches its maximum value \( x_m \). Assuming that the drop height of the packaging system is \( h \) and the weight of the product is \( W \), the gravitational potential energy of the system can be expressed as follows

\[ U = Wh \]  \hspace{1cm} (10)
For nonlinear packaging systems, \( f(x) \) refers to restoring force, which is only a nonlinear function of \( x \). According to the energy method, there is

\[
Wh = \int_0^{x_m} f(x) dx
\]  

(11)

The maximal displacement \( x_m \) can be calculated by equation (11).

**A tangent nonlinear packaging system**

The dynamic equation of dropping shock response of a tangent nonlinear packaging system is shown in equation (12)

\[
m\ddot{x} + 2k_0 \frac{d_b}{\pi} \tan \frac{\pi x}{2d_b} = 0
\]  

(12)

where \( m \) is the mass of the product, the dot denotes the differentiation with respect to the time \( t \), \( x \) is the compressive deformation, \( k_0 \) and \( d_b \) are the material constant coefficients, respectively. \( 2k_0 \frac{d_b}{\pi} \) can be obtained in accordance to the mechanical properties of tangent materials.

The initial conditions of equation (12) are as follows

\[
x(0) = 0 \quad \dot{x}(0) = \sqrt{2gh}
\]  

(13)

where \( g \) is the gravitational acceleration and \( h \) is the drop height.

To solve equation (12), first we carry out the Taylor expansion of \( \tan \) to obtain

\[
\tan \frac{\pi x}{2d_b} = \frac{\pi x}{2d_b} + \frac{\pi^3}{3 (2d_b)^3} x^3 + \frac{2}{15 (2d_b)^5} x^5 + \ldots
\]  

(14)

Substituting equation (14) into equation (12), we have

\[
m\ddot{x} + k_0x + k_0 \frac{\pi^2}{12d_b^2} x^3 + k_0 \frac{\pi^4}{120d_b^4} x^5 = 0
\]  

(15)

By introducing parameters \( r = \frac{k_0 \pi^2}{12d_b} \) and \( q = \frac{k_0 \pi^4}{120d_b^4} \), and setting \( \omega_0^2 = \frac{k_0}{m} \), \( k = \frac{k_0}{m} \) and \( a = \frac{q}{m} \), one can obtain the equation as

\[
\ddot{x} + \omega_0^2 x + kx^3 + ax^5 = 0
\]  

(16)

in terms of differential operators

\[
(D^2 + \omega_0^2 + kx^2) x = 0
\]  

(17)

where \( D \) denotes differentiation respect to time \( t \).

Applying the annihilator operator \( D^2 \) to equation (17), we have

\[
(D^4 + \omega_0^2 D^2 + kD^2x^2 + aD^2 x^4) x = 0
\]  

(18)

or

\[
x^{(4)} + (\omega_0^2 + 3kx^2 + 5ax^4) \ddot{x} + (6k + 20ax^2)x \dot{x}^2 = 0
\]  

(19)
Substituting $\ddot{x}$ by $-(\omega_0^2x + kx^3 + ax^5)$ into equation (19), one can obtain
\begin{equation}
\dddot{x} - (\omega_0^2 + 3kx^2 + 5ax^4)(\omega_0^2x + kx^3 + ax^5) + (6k + 20ax^2)\ddot{x} = 0
\end{equation}

or
\begin{equation}
\dddot{x} - \omega_0^4x - 4k\omega_0^2x^3 - (3k^2 + 6a\omega_0^2)x^5 - 8kax^7 - 5a^2x^9 + (6k + 20ax^2)\ddot{x} = 0
\end{equation}

According to the HPM, we construct the equation as follows
\begin{equation}
\dddot{x} - \omega_0^4x + p\left[-4k\omega_0^2x^3 - (3k^2 + 6a\omega_0^2)x^5 - 8kax^7 - 5a^2x^9 + (6k + 20ax^2)\ddot{x}\right] = 0
\end{equation}

Assuming that the solution can be expanded into a series of $p$, we have
\begin{equation}
x = x_0 + px_1 + p^2x_2 + \ldots
\end{equation}

The parameter expansion technology can be used to expand the coefficient of the linear term $\omega_0^4$ into a series of $p$
\begin{equation}
\omega_0^4 = \omega_0^4 + p\omega_1 + p^2\omega_2 + \ldots
\end{equation}

Substituting equations (23) and (24) into equation (22) leads to
\begin{equation}
\left(x_0 + px_1 + p^2x_2 + \ldots\right)^{(4)} - (\omega_0^4 + p\omega_1 + p^2\omega_2 + \ldots)(x_0 + px_1 + p^2x_2 + \ldots) + \\
-4k\omega_0^2\left(x_0 + px_1 + p^2x_2 + \ldots\right)^3 - \\
(3k^2 + 6a\omega_0^2)(x_0 + px_1 + p^2x_2 + \ldots)^5 - \\
8kax_0 + px_1 + p^2x_2 + \ldots)^7 - 5a^2\left(x_0 + px_1 + p^2x_2 + \ldots\right)^9 + \\
\left(6k + 20a\left(x_0 + px_1 + p^2x_2 + \ldots\right)\right)^2 \\
\left(x_0 + px_1 + p^2x_2 + \ldots\right)(\left(x_0 + px_1 + p^2x_2 + \ldots\right)^{(1)})^2
\end{equation}

Arranging the same powers of $p$ from equation (25), we have
\begin{equation}
p^0: x_0^{(4)} - \omega_0^4x_0 = 0
\end{equation}

\begin{equation}
p^1: x_1^{(4)} - (\omega_1x_0 + \omega_0^4x_1) + \left[-4k\omega_0^2x_0^3 - (3k^2 + 6a\omega_0^2)x_0^5 - 8kax_0^7 - 5a^2x_0^9 + (6k + 20ax_0^2)x_0x_0^2\right] = 0
\end{equation}

Using the initial conditions, equation (13), $x_0$ can be solved as
\begin{equation}
x_0 = \frac{\sqrt{2gh}}{\omega} \sin\omega t
\end{equation}

By substituting $x_0$ into equation (27), one can obtain
\begin{equation}
\dddot{x}_1 - \omega_0^4x_1 + R_1\sin\omega t + R_2\sin2\omega t + R_3\sin3\omega t + R_4\sin5\omega t + R_5\sin7\omega t + R_6\sin9\omega t = 0
\end{equation}
where

\[ R_1 = -\omega_1 \sqrt{\frac{2gh}{\omega}} \]

\[ + \left( -3k\omega_0^2 + \frac{3}{2}k\omega^2 \right) \left( \sqrt{\frac{2gh}{\omega}} \right)^3 \left[ -\frac{5}{8}(3k^2 + 6a\omega_0^2) + \frac{5}{4}a\omega^2 \right] \left( \sqrt{\frac{2gh}{\omega}} \right)^5 - \frac{35}{8}ka \left( \sqrt{\frac{2gh}{\omega}} \right)^7 - \frac{315}{128}a^2 \left( \sqrt{\frac{2gh}{\omega}} \right)^9 \] (30)

\[ R_2 = \left( k\omega_0^2 + \frac{3}{2}k\omega^2 \right) \left( \sqrt{\frac{2gh}{\omega}} \right)^3 \left[ \frac{5}{16}(3k^2 + 6a\omega_0^2) + \frac{5}{4}a\omega^2 \right] \left( \sqrt{\frac{2gh}{\omega}} \right)^5 + \frac{21}{8}ka \left( \sqrt{\frac{2gh}{\omega}} \right)^7 + \frac{105}{64}a^2 \left( \sqrt{\frac{2gh}{\omega}} \right)^9 \] (31)

\[ R_3 = -\left[ \frac{1}{16}(3k^2 + 6a\omega_0^2) + \frac{5}{4}a\omega^2 \right] \left( \sqrt{\frac{2gh}{\omega}} \right)^5 - \frac{7}{8}ka \left( \sqrt{\frac{2gh}{\omega}} \right)^7 - \frac{45}{64}a^2 \left( \sqrt{\frac{2gh}{\omega}} \right)^9 \] (32)

\[ R_4 = \frac{1}{8}ka \left( \sqrt{\frac{2gh}{\omega}} \right)^7 + \frac{45}{256}a^2 \left( \sqrt{\frac{2gh}{\omega}} \right)^9 \] (33)

\[ R_5 = -\frac{5}{256}a^2 \left( \sqrt{\frac{2gh}{\omega}} \right)^9 \] (34)

where \( \omega \) is the frequency of the original system.

In order to avoid the secular terms in the solution of \( x_1 \), this requires

\[ R_1 = 0 \] (35)

Combining with \( w_1 = \omega_0^4 - \omega^4 \), then one can obtain

\[ \omega^4 + (\omega_0^4 + 3kgh)\omega^8 + (6kgh\omega_0^2 + 10a\omega^2h^2)\omega^6 + \frac{5}{2}g^2h^2(3k^2 + 6a\omega_0^2)\omega^4 - \frac{35}{8}ka\left( \sqrt{\frac{2gh}{\omega}} \right)^6 \omega^2 - \frac{315}{128} \left( \sqrt{\frac{2gh}{\omega}} \right)^8 a^2 = 0 \] (36)

After solving equation (36) with the parameters provided in Table 1, we can obtain \( = 108.21776s^{-1} \). Then, by solving equation (29), \( x_1 \) can be expressed as

\[ x_1 = -\frac{R_2}{80\omega^4} \sin 3\omega t - \frac{R_3}{624\omega^2} \sin 5\omega t - \frac{R_4}{2400\omega^6} \sin 7\omega t - \frac{R_5}{6560\omega^8} \sin 9\omega t \] (37)

| Method          | The maximal displacement (m) | Relative error (%) | The maximal acceleration (g) | Relative error (%) |
|-----------------|------------------------------|--------------------|-------------------------------|--------------------|
| R-K             | 0.033995                     |                    | 49.8099                       |                    |
| He²⁷            | 0.030823                     | 9.20               | 44.8980                       | 9.86               |
| LHHPM           | 0.032317                     | 4.94               | 44.6170                       | 10.43              |
| MLHHPM          | 0.033945                     | 0.15               | 49.7355                       | 0.15               |

R-K: Runge-Kutta; LHHPM: Li–He's homotopy perturbation method; MLHHPM: modified Li–He's homotopy perturbation method.
If we only focus on the first-order approximate solution, we set \( p = 1 \) and have
\[
x = x_0 + x_1 = \frac{\sqrt{2gh}}{\omega} \sin \omega t - \frac{R_2}{80\omega^4} \sin 3\omega t - \frac{R_3}{624\omega^4} \sin 5\omega t - \frac{R_4}{2400\omega^4} \sin 7\omega t - \frac{R_5}{6560\omega^4} \sin 9\omega t
\] (38)

According to the energy method, we have
\[
mgh = \int_0^{x_m} (k_0 x + rx^3 + qx^5) \, dx
\] (39)
or
\[
6k_0 x_m^2 + 3r x_m^4 + 2q x_m^6 - 12mgh = 0
\] (40)

We can obtain the maximal displacement \( x_m \) by solving equation (40). Assuming the modified zero-order solution as follows
\[
x_{0m} = A_m \sin \omega_m t
\] (41)
Then replacing $\frac{\sqrt{2gh}}{\omega}$ and $\omega$ by $A_m$ and $\omega_m$ in equations (31) to (34), respectively. The modified first-order solution can be obtained

$$x_m = A_m \sin \omega_m t - \frac{R_{2m}}{80 \omega_m^4} \sin 3 \omega_m t - \frac{R_{3m}}{624 \omega_m^4} \sin 5 \omega_m t - \frac{R_{4m}}{2400 \omega_m^4} \sin 7 \omega_m t - \frac{R_{5m}}{6560 \omega_m^4} \sin 9 \omega_m t \quad (42)$$

To set $\omega_m t = \frac{\pi}{2}$, then the maximal displacement $x_m$ and the maximal acceleration $x''_m$ can be obtained as follows

$$x_m = A_m - \frac{R_{2m}}{80 \omega_m^4} + \frac{R_{3m}}{624 \omega_m^4} - \frac{R_{4m}}{2400 \omega_m^4} + \frac{R_{5m}}{6560 \omega_m^4} \quad (43)$$

$$x''_m = -A_m \omega_m^2 + \frac{R_{2m}}{80 \omega_m^4} (3 \omega_m)^2 - \frac{R_{3m}}{624 \omega_m^4} (5 \omega_m)^2 + \frac{R_{4m}}{2400 \omega_m^4} (7 \omega_m)^2 - \frac{R_{5m}}{6560 \omega_m^4} (9 \omega_m)^2 \quad (44)$$

Combining equations (40), (43), and (44), we can obtain the modified frequency $\omega_m = 110.7254$ s$^{-1}$ and the modified maximal displacement $A_m = 0.03330418$ m. Then the modified displacement $x_m$ is obtained.

To evaluate the dropping shock responses of a tangent packaging system, we are highly concerned about the maximal displacement response and the maximal acceleration response of the system. Therefore, in order to illustrate the validity of the new method for the tangent nonlinear packaging system, we compare the solutions obtained by literature, LHHPM, and the modified Li–He’s homotopy perturbation method (MLHHPM) with the exact solution obtained by the R-K method, respectively. The results are shown in Table 1, Figure 1, and Figure. 2. It is obvious that the new method MLHHPM gives more accurate results than others for this nonlinear equation both in the maximal displacement and the maximal acceleration, and the relative error of the solutions obtained by MLHHPM is less than 0.15%.

Conclusions

In this paper, a novel method (MLHHPM) coupled the LHHPM with the energy method is proposed and applied to study the dropping shock response of a tangent packaging system. The results show that the maximal displacement and acceleration responses obtained by the new method are very close to the numerical ones obtained by the Runge–Kutta method. This new method can also have wide applications to other nonlinear packaging systems, jet vibration in electrospinning, and nonlinear transverse vibration of a nanofiber-reinforced concrete pillar.

This paper sheds a bright light on the nonlinear vibration theory, the method is especially effective for nonlinear packaging systems, jet vibration in electrospinning, and nonlinear transverse vibration of a nanofiber-reinforced concrete pillar.

Declaration of conflicting interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: This work was supported by Natural Science Foundation of Jiangsu Province, China (Grant No. BK20151128), National first-class discipline program of Light Industry Technology and Engineering (LITE2018-29), the 111 Project (No. B18027), and Natural Science Foundation of Zhejiang Province (Grant number: LY16A020004).

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