Spinning Black Holes in $(2+1)$-dimensional String and Dilaton Gravity

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ABSTRACT

We present a new class of spinning black hole solutions in $(2+1)$-dimensional general relativity minimally coupled to a dilaton with potential $e^{b\phi}\Lambda$. When $b = 4$, the corresponding spinning black hole is a solution of low energy $(2+1)$-dimensional string gravity. Apart from the limiting case of the BTZ black hole, these spinning black holes have no inner horizon and a curvature singularity only at the origin. We compute the mass and angular momentum parameters of the solutions at spatial infinity, as well as their temperature and entropy.

Keywords: String; Dilaton; Black Holes; $2+1$

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1 Introduction

Lower dimensional gravity is a source of fascination for theorists, primarily because of the potential insights into quantum gravity that it offers. The lower-dimensional setting affords a significant amount of technical simplification of the gravitational field equations, bringing into sharper focus conceptual issues that are often obscured in the more complicated $(3+1)$-dimensional case. In particular, there has been great interest in recent years in $(2+1)$-dimensional general relativity ever since the discovery of the BTZ black hole solution [1]. This BTZ black hole has so far attracted much interest in its classical, thermodynamic, statistical and quantum properties (see [2] for a review). It is also a solution to the low energy string theory [3].

It is interesting and important to investigate how the properties of black holes are modified when a dilaton is present. The first such modification of the non-spinning charged BTZ black hole was investigated in [4] with an exponential potential and an asymptotically non-constant dilaton. In addition, by considering several asymptotically constant dilatons and different forms of potential functions, some interesting modifications of the BTZ black hole can be found [5]. The action considered in [4] is given by

$$ S = \int d^3 x \sqrt{-g} (\mathcal{R} - 4(\nabla \phi)^2 + 2e^{b\phi}\Lambda - e^{-4a\phi} F^2), $$

with arbitrary couplings $\Lambda$, $a$ and $b$, where $\mathcal{R}$ is the Ricci scalar, $\phi$ is the dilaton field and $F^2$ is the usual Maxwell contribution. The constants $a$ and $b$ govern the coupling of $\phi$ to $F^2$ and $\Lambda$ respectively. For a constant $\phi$, (1) admits the usual BTZ case, where $\Lambda$ is the cosmological constant [note that in the presence of a non-trivial dilaton, the space does not behave as either de Sitter ($\Lambda < 0$) or anti-de Sitter space ($\Lambda > 0$)]. When $a = 1$ and $b = 4$, the action (1) is that of low energy string theory in terms of the Einstein metric. The corresponding action in terms of the string metric can be obtained by performing the conformal transformation

$$ g^S_{\mu\nu} = e^{4\phi} g_{\mu\nu}, $$

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where $S$ and $E$ denote the string and Einstein metrics respectively. The corresponding string action is

$$S = \int d^3x \sqrt{-g^S} e^{-2\phi}(R[g^S] + 4(\nabla \phi)^2 - F^2 + 2\Lambda).$$

By taking the product of the $(1+1)$-dimensional string-theoretic black hole of Mandal, Sengupta and Wadia \[7\] with $S^1$, a static charged black hole solution to the field equations of string action \[3\] with $\phi = -\frac{1}{2}\ln(r)$ may be obtained (hereafter referred as 2+1 MSW black hole) – if a product with $R$ is taken instead, one gets a black string \[8\].

The black hole solutions obtained in \[4\] belong to \[1\] with the choice $\phi \propto \ln(r)$ and they contain the non-spinning $BTZ$ and 2+1 MSW black holes as special cases. In fact, \[1\] contains many other dilaton solutions (without black holes) which were previously investigated by a number of authors (see references in \[4\]). Note that \[1\] can be viewed as general relativity with an unusual matter Lagrangian. For example, one can easily see that the local energy density in the perfect fluid form is negative when $\phi = \text{constant}$ and $\Lambda > 0$.

In this paper, by adopting the same choice of dilaton as in the static black holes in \[4\], we obtain a family of spinning black holes as solutions to the field equations which follow from action \[1\]; in particular, the $b = 4$ spinning black hole belongs to the string theory \[3\] with the conformal transformation \[2\]. For simplicity, we ignore the Maxwell fields in this paper; the possibilities of including Maxwell fields will be investigated in a future work. The solutions we derive in this paper are not vacuum solutions in a strict sense since the action \[1\] contains a static “dilaton fluid” whose energy momentum tensor is nowhere vanishing. Here we adopt special dilaton fluid models in which the metric approaches neither the Minkowski nor anti-de Sitter metrics at spatial infinity similar to the previous black holes obtained in \[4\] and \[9\]. It is worthwhile mentioning that in recent a naive argument \[3\] demonstrates that there exist no asymptotically flat positive mass black hole solutions in 2+1 dimensions. Although the dilaton field diverges for large $r$ ($\phi \propto \ln r$), the (quasilocal) mass and angular momentum are finite for all values of $r$ outside the event horizon in every black hole solution. The Ricci and Kretschmann scalars are everywhere finite except at $r = 0$ but those singularities are hidden by the event horizon. It can also be checked that all terms in the action \[1\] are finite for $\infty > r > r_h$ (where $r_h$ denotes the event horizon), and vanish as $r \to \infty$ in all the black hole solutions we obtain. As a consequence we believe that our spinning black hole solutions are of some interest.

The organization of this paper is as follows. In section 2 we quickly review the static and circularly symmetric black hole solutions obtained in \[4\]. The quasilocal formalism for calculating energy, mass and angular momentum will also be briefly presented in this section. In section 3, we present the spinning black hole solutions. Various properties of the black hole metrics are discussed and their temperatures are also computed. We will see how a non-trivial dilaton modifies the spinning BTZ black hole. We summarize our work in a concluding section.

Our conventions are as in Wald [10]; we set the gravitational coupling constant $G$ equal to unity. The signature of the metric is $(-+++)$.

## 2 Review of Static Dilaton Black Holes and Quasilocal Formalism

We now consider the $2+1$-dimensional action \[1\] and review its uncharged dilaton black hole solutions. We refer readers to \[4\] for the charged cases. Varying \[1\] with respect to the metric and dilaton fields, respectively, yields (after some manipulation)

$$R_{\mu\nu} = 4\nabla_{\mu}\phi\nabla_{\nu}\phi - 2g_{\mu\nu}e^{b\phi}\Lambda,$$

$$4\nabla^2\phi = -be^{b\phi}\Lambda.$$

A family of static, circularly symmetric black hole solutions to these field equations was obtained in \[3\]. They are given by

$$\phi = k\ln(r),$$
\[ ds^2 = -U(r)dt^2 + \beta^2 \frac{dr^2}{U(r)} + \beta^2 r^2 d\theta^2, \]

(7)

with \n
\[ U(r) = \left[ -\frac{2M}{N} r^{1-\frac{N}{2}} + \frac{8\Lambda\beta^2 r^N}{(3N-2)N} \right], \]

(8)

and where \n
\[ k = \pm \frac{1}{4} \sqrt{N(2-N)}, \]

(9)

and \n
\[ bk = N - 2. \]

(10)

\( M \) is the quasilocal mass identified at spatial infinity. Positive mass \((M > 0)\) black hole solutions only exist for \( 2 \geq N > \frac{2}{3} \) and \( \Lambda > 0 \). It is obvious that as \( r \to \infty, \phi \to \infty \) too. However, the kinetic term and the potential term in the action (1) all vanish in that limit when \( 2 \geq N > 0 \). When \( N = 1 \), (8) reduces to the \( 2 + 1 \) the uncharged MSW black hole; if \( N = 2 \), it becomes the uncharged BTZ black hole which is also a solution to string theory. In general, (7) and (8) are neither asymptotically flat nor anti-de Sitter. Note that the radial co-ordinate \( r \) is chosen to be dimensionless and \( \beta \) is a length scale with dimensions of length \([L]\).

We now present the formulas for calculating the quasilocal mass, angular momentum and energy of a stationary and axisymmetric solution in an asymptotically non-flat spacetime; for a detailed derivation of these, see [11]. For a \((2+1)\)-dimensional stationary and axisymmetric spacetime, the metric can be written as [12]

\[ ds^2 = -L^2 dt^2 + f^{-2} dR^2 + R^2 (N^\theta dt + d\theta)^2, \]

(11)

where \( L(R) \), \( f(R) \) and \( N^\theta(R) \) are functions of \( R \) only – under a co-ordinate transformation \( R = R(r) \), they can be functions of a new radial variable \( r \). \( L(R) \) is called the lapse function and \( N^\theta(R) \) is called the angular shift. Using (11) and the formalism adopted in [11], the quasilocal energy \( E(R) \) at a radial boundary \( R \) can be shown to be

\[ E = 2(f_\phi(R) - f(R)). \]

(12)

Here \( f_\phi(R) = g^R_\phi(R) \) is a background metric function which determines the zero of the energy. The background metric can be obtained simply by setting constants of integration of a particular solution to some special value that then specifies the reference spacetime. The quasilocal mass \( m(R) \) at a \( R \) is given by

\[ m = L(R)E - j(R)N^\theta(R), \]

(13)

where \( j(R) \) is the quasilocal angular momentum given by

\[ j = \frac{f(R)N^\theta(R)R^3}{L(R)}, \]

(14)

where the prime denotes the ordinary derivative with respect to \( R \). As \( R \to \infty \), the analogous ADM mass and angular momentum are defined to be \( M \equiv m(\infty) \) and \( J \equiv j(\infty) \) respectively. Note that \( E(R) \), not \( m(R) \), is the thermodynamic internal energy [11]. This distinction between mass and energy is important for spacetimes which are not asymptotically flat, since in those cases the magnitude of the timelike killing vector field diverges as it approaches spatial infinity. The thermodynamic temperature \( T(R) \) at a given value of \( R \) is defined as

\[ T = \frac{\partial E}{\partial S}, \]

(15)

where \( S \) is the entropy of the black hole. General arguments similar to those in \( 3 + 1 \) dimensions (see, e.g. [13]) show that

\[ S = 4\pi R_+. \]

(16)
Thus using (12), (13) and (10), the temperature can be calculated easily. The analogous Hawking temperature is obtained by taking the spatial infinity limit. It can be shown that the temperature calculated using (14) is the usual surface gravity divided by a red-shift factor. For asymptotically flat spacetimes, the red-shift factor tends to unity at spatial infinity.

3 Spinning Black Hole Solutions

We now present a family of spinning black hole solutions characterized by the mass $M$, angular momentum $J$ and dilaton coupling $N$ parameters. We use the same dilaton (1) and relationships among $b$, $K$ and $N$ in (1) and (8). In terms of the same radial co-ordinate $r$ as in the static case (7) and (8), the spinning solutions are

$$ds^2 = -\left(\frac{8\Lambda\beta^2 r^N}{(3N-2)N} + Ar^{1-N}\frac{\beta^2 dr^2}{8\Lambda\beta^2 r^N + \left(A - \frac{2\omega^2}{(3N-2)NA}\right)r^{1-N}}\right)dt^2 + \frac{\beta^2 dr^2}{8\Lambda\beta^2 r^N + \left(A - \frac{2\omega^2}{(3N-2)NA}\right)r^{1-N}},$$

$$M = \frac{N}{2}\left[\frac{2\Lambda\omega^2}{(3N-2)NA} \left(\frac{4}{N} - 3\right) - A\right],$$

$$J = \frac{3N-2}{4}\omega.$$

Using (11) and the $g^{rr}$ in (17), the lapse, $f(r)$, angular shift and radial functions are given by

$$L^2 = \frac{g^{rr}\beta^4}{(\beta^2 - \frac{\omega^2}{4\pi}r^{1-N})},$$

$$f^2 = g^{RR} = g^{rr}\left(\frac{dR}{dr}\right)^2,$$

$$N^\theta = -\frac{\omega r^{1-N/2}}{2R^2},$$

and

$$R^2 = g_{\theta\theta} = \left(\frac{\beta^2 r^N - \frac{\omega^2}{4A}r^{1-N}}{2}\right).$$

$A$ and $\omega$ are two integration constants which are related to the mass $M$ and angular momentum $J$ respectively in (18) and (19). It is easy to check (see below) that black holes exist if $\Lambda > 0$ and $2 \geq N > \frac{2}{3}$. Except for the $N = 1$ and $N = 2$ cases, we cannot explicitly express $r$ in terms of $R$. As a result, all metric functions in the above are not explicit functions of $r$. However, when $r \to \infty$, by using binomial expansion and reversion of the series, an approximation is given by $r^{-1} = \left(\frac{3}{\pi}\right)^\frac{N}{2} - \frac{\omega^2}{4\Lambda^2 r^N} \left(\frac{3}{\pi}\right)^3 + \cdots$ for the range $2 \geq N > \frac{2}{3}$. The radial transformation (23) is based on the assumption that $g_{\theta\theta}$ part in (17) must always be greater than zero. However, if $g_{\theta\theta} < 0$ is allowed, one must have $g_{\theta\theta}dt^2 = -R^2d\theta^2$. Hence $\theta$ is a timelike co-ordinate, and because it is periodic, it corresponds to a region with closed timelike curves. To avoid this, $A$ must be chosen to be negative in (18) so that $g_{\theta\theta} > 0$. Thus $2A = -\frac{2M}{N} - \sqrt{\frac{4M^2}{N^2} + \left(\frac{4}{N} - 3\right)} \frac{8\Lambda\omega^2}{(3N-2)NA}. M$ is identified using (13) and the expansion of $\frac{1}{4}$ above when the background metric $f_o = f(A = \omega = 0)$ is chosen. It is taken to be positive. $J$ is identified at spatial infinity ($r \to \infty$) as well. This choice of background is consistent with the fact that $M$ is the mass parameter at spatial infinity identified by Brown, Creighton and Mann (14) for the BTZ black hole when $N = 2$. (17) reduces to the
static solution \( \text{(8)} \) when \( J = 0 \). We can see that the presence of \( J \) modifies the static solutions by introducing an extra term in \( g_{\theta\theta} \) and by causing the coefficients of the \( r^{1-N} \) term in \( g_{tt} \) and \( g_{rr} \) to become unequal. When \( N = 2 \), using \( \text{(17)}, \text{(18)} \) and \( \text{(19)} \), and the radial co-ordinate transformation \( \text{(23)} \), it is easy to show that

\[
\begin{equation}
\alpha = (\Lambda R^2 - M) dt^2 + \left(\Lambda R^2 - M + \frac{J^2}{4R^2}\right)^{-1} \left( R^2 - J dt d\theta + R^2 d\theta, \right.
\end{equation}
\]

which is exactly the spinning \( BTZ \) black hole with mass \( M \) and angular momentum \( J \).

One can locate the event horizon(s) from the vanishing of the lapse in \( \text{(20)} \). It can be checked that \( \text{(17)} \) admits an event horizon if the following conditions are satisfied: \( M > 0, \Lambda > 0 \) and \( 2 \geq N > \frac{2}{3} \). In terms of \( r, \text{(21)} \) vanishes at \( r_+ \):

\[
\begin{equation}
\left(\frac{4}{N} - 3\right) \frac{4\Lambda \beta^2}{(3N - 2)N} \frac{dR}{r} = \frac{M}{N} \left(\frac{2}{N} - 1\right) + \sqrt{\frac{4M^2}{N^2} + \left(\frac{4}{N} - 3\right) \frac{8\Lambda \omega^2}{(3N - 2)N} \left(\frac{1}{N} - 1\right)}
\end{equation}
\]

for \( N \neq \frac{4}{3} \). If \( N = \frac{4}{3} \), then \( r_+ \) is given by

\[
\begin{equation}
3\Lambda \beta^2 r_+ = \frac{3M}{2} - \frac{\Lambda \omega^2}{2M}.
\end{equation}
\]

Using \( \text{(18)} \), \( \text{(19)} \), \( \text{(23)} \) and \( \text{(25)} \) with \( N = 2 \), it can be shown that \( R^2_+ = \frac{M+\sqrt{M^2-4\Lambda J^2}}{2\Lambda} \) which is the expression for the location of the outer event horizon of the \( BTZ \) black hole in terms of the \( R \) co-ordinate. Thus except for the \( BTZ (N = 2) \) case, the family of spinning black holes \( \text{(17)} \) has no inner horizon even if \( J \neq 0 \). In addition, except for the \( BTZ (N = 2) \) case, where there are no divergences in curvature invariants, it is lengthly but straightforward to show that for \( 2 \geq N > \frac{2}{3} \), the scalar and Kretschmann scalars only diverge at \( r = r_- = 0 \) \( (R = 0) \). Thus as long as \( r_+ > 0 \), the solutions do not have naked singularities. The extremal limit between \( M \) and \( \omega \) \( (J) \) can be defined as the vanishing of \( r_+ \) in \( \text{(25)} \) and \( \text{(26)} \) for non-vanishing \( M \) and \( J \). In \( \text{(25)} \), there is no extremal limit for \( 1 \geq N > \frac{2}{3} \). \( r_+ \) is always real and greater than zero. Previous examples of black holes without extremal limit between mass and charge can be found in \( \text{(8)} \). For the range \( 2 \geq N > 1 \), \( r_+ \) is real and greater than zero in \( \text{(25)} \) if

\[
\begin{equation}
M^2 \geq (N-1)^2 \frac{8\Lambda \omega^2}{(3N-2)N}
\end{equation}
\]

where the equality sign corresponds to the extremal limit. When \( N = 2 \), \( M^2 = \Lambda J^2 \) as expected. In \( \text{(26)} \), the extremal limit simply is

\[
\begin{equation}
3M^2 = \Lambda \omega^2.
\end{equation}
\]

Solutions \( \text{(17)-(19)} \) contain three particular interesting cases. First, \( N = 2 \) is the \( BTZ \) case as we have just shown. Second, the \( N = \frac{4}{3} \) solution is conformally related to the the black hole solution previously derived by Lemos in \( \text{(14)} \). It was shown in \( \text{(14)} \) that his black hole solution can be translated to the \( 3+1 \) spacetime as a cylindrical solution to a \( (3+1) \)-dimensional dilaton gravity.
The third case is the $N = 1$ solution which corresponds to the low energy $2 + 1$ string theory. In terms of the Einstein metric, the solution reads

$$ds^2 = - \left( 8 \Lambda \beta^2 r - \frac{2M + \sqrt{4M^2 + 8\Lambda \omega^2}}{2} \sqrt{r} \right) dt^2 + \frac{\beta^2 dr^2}{[8 \Lambda \beta^2 r - 2M \sqrt{r}]} - \omega \sqrt{r} dtd\theta + \left( \beta^2 r + \frac{-2M + \sqrt{4M^2 + 8\Lambda \omega^2}}{16 \Lambda} \right) d\theta^2,$$  \hspace{1cm} (30)

$$J = \frac{1}{4} \omega.$$  \hspace{1cm} (31)

This $N = 1$ spinning black hole is a modification of the $2 + 1$ MSW static black hole. Using the conformal transformation (2), one can express the $N = 1$ black hole solution in terms of the string metric. It is the second example of a non-trivial black hole solution in $2 + 1$ string theory (the first one is the BTZ case). Note that in terms of the string action (3), one can see that the string coupling $e^{2\phi} = \frac{1}{\sqrt{r}}$ vanishes as $r \to \infty$. Due to the presence of a non-vanishing $J$, the dimensional reduction of the $N = 1$ string solution is no longer a solution to $1 + 1$ string theory, since one expects that $J$ will introduce an extra potential term in the original $1 + 1$ string action.

We can deduce the causal structures of the spinning dilaton black holes (except for the BTZ case) as follows. The causal structures of the static uncharged black holes in (8) were drawn in [4]. They all have a spacelike singularity at $r = 0$. Furthermore they either have a timelike or null-like null infinity, depending on the value of $N$. In the former case the causal structure is the same as the non-spinning BTZ one, while in the latter case it is the same as the Schwarzschild one. In the present spinning solutions they all asymptotically approach the static uncharged black hole solutions (8), and they have no inner horizons. There is a spacelike curvature singularity at $r = 0$. Thus one can deduce that the spinning dilaton black holes would have similar causal structures either to the non-spinning BTZ or Schwarzschild case. We will not repeat the drawing here.

Finally, we briefly discuss the thermodynamic properties of the spinning solutions. An important thermodynamic quantity in a stationary black hole is the temperature $T$ defined in (15). Using the $r$ co-ordinate, one can show that (15) at a radial boundary generally yields the surface gravity ($\kappa$) term

$$2\pi T = \kappa = \frac{N}{2R_+} \left( \frac{3N - 2}{4 - 3N} \right) \left[ \frac{M}{N} \left( \frac{2}{N} - 1 \right) + \sqrt{\frac{4M^2}{N^2} + \left( \frac{4}{N} - 3 \right) \frac{8\Lambda \omega^2}{(3N - 2)N} \left( \frac{1}{N} - 1 \right)} \right]$$  \hspace{1cm} (32)

along with a redshift factor which may be computed as in [11], and $R_+$ is defined in (23) with $r_+$ is given by (25). For (24), the temperature is

$$\kappa = \frac{1}{4R_+} \left( \frac{3M - \Lambda \omega^2}{M} \right).$$  \hspace{1cm} (33)

It is easy to check that when $N = 2$, $T$ in (32) reduces to the BTZ temperature obtained by Brown, Creighton and Mann in [11]. For the string case ($N = 1$) with $J = 0$, (32) is independent of the mass parameter $M$. In the extremal cases (28) and (29), the temperature vanishes in (32) and (33). The entropy can be trivially obtained using the entropy formula in (16). Other thermodynamic quantities such as heat capacity and chemical potential can be computed as in [11]. We do not discuss them here.

4 Conclusions

We have obtained a family of asymptotically non-flat spinning dilaton black hole solutions in $2 + 1$ with an exponential potential. The family contains the spinning BTZ metric as a special case. In addition, one member of this class of black holes is a solution to low energy string theory. All of
these black hole spacetimes have no inner horizons except the BTZ case. For the range $1 \geq N > \frac{2}{3}$ of the parameter $N$, the black holes have no extremal limit. $J$ can take any finite value for a given $M$ without causing the event horizon to disappear.

Although the static charged black hole solutions of (1) exist [4], at present we are unable to generalize our spinning solutions to charged cases. This endeavour is complicated by the fact that when one adds Maxwell fields to a spinning solution, both electric and magnetic fields must be present [12]. As a result, the field equations are considerably more complicated to solve. In order to simplify the differential equations involved, a (anti) self dual condition on electric and magnetic fields was recently imposed to get the spinning charged BTZ black hole solution [12]. However, by using (13) and (14) in their spinning solution, it is easy to check that the quasilocal mass and angular momentum, instead of being a constant, diverges logarithmically at spatial infinity [15]. Recently, a class of static magnetic solutions to the Einstein-Maxwell equations with $\Lambda > 0$ was obtained, and it is horizonless and free of curvature singularities [16]. It can be shown that the mass has a logarithmic divergence at spatial infinity as well [15]. The logarithmic divergence is due to the fact that the electric potential is a logarithmic function of $r$ in 2+1 dimensions. One way of curing the divergence is to add a topological Chern-Simons term to the gauge action [17]. The resultant solution in [17] is horizonless, regular and asymptotic to the extremal BTZ black hole. Another way is to couple a dilaton to the Maxwell term in the action, similar to (1). It was shown in [1] that the static electrically charged black hole solution has a constant electric term in $g_{tt}$ and $g_{rr}$ instead of a logarithmic one. Thus one of the roles of a dilaton in 2 + 1 dimensions is to remove the logarithmic divergence in the electric potential. It is an open question as to whether one can get a charged version of the family of spinning string and dilaton black holes obtained in this paper. We expect that the introduction of electromagnetic fields will yield inner horizons to the present spinning dilaton solutions, since the corresponding static charged solutions in [1] have inner horizons. One could then attempt, say, to derive exact mass inflation solutions to the charged spinning solutions. Early work on mass inflation for spinning BTZ black hole can be found in [18]. We intend to relate further details elsewhere.

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