Decoherence free algebra

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Abstract

We consider the decoherence free subalgebra which satisfies the minimal condition introduced by Alicki \[1\]. We show the manifest form of it and relate the subalgebra with the Kraus representation. The arguments also provide a new proof for generalized Lüders theorem \[5\].

1 Introduction

Decoherence is a non-unitary dynamics that is a consequence of system-environment coupling. As a way to protect a quantum system against decoherence, a subalgebra whose dynamics is implementable by unitary operator has attracted considerable interests; that is, the decoherence free subalgebra.

Let \( M_n \) be a \( C^\ast \)-algebra of all \( n \times n \) matrices and \( \phi \) a unital trace preserving completely positive map from \( M_n \) into itself. It is known that \( \phi \) satisfies the Kadison inequality:

\[
\phi(x^*x) \geq \phi(x^*)\phi(x) \quad \text{for all } x \in M_n.
\]

The equality holds for all \( x \in M_n \) if and only if \( \phi \) is given by a unitary operator \( U \) as \( \phi(x) = U^*xU \). So \( \phi(x^*x) - \phi(x^*)\phi(x) \) can be interpreted as a dissipation function. Under these considerations, Alicki introduced the decoherence free subalgebra \( N_\phi \) of \( \phi \) as \[1\]

\[
N_\phi = \{ x_1 + ix_2 \in M_n; x_i = x_i^*, \phi(x_i^2) = \phi(x_i)^2, \ i = 1, 2 \}.
\]

It is a \( C^\ast \)-subalgebra of \( M_n \). Suppose that \( \phi \) can be joined to the identity map \( \text{id} \), in the sense that there exists a continuous map from interval \([0, 1]\) to completely positive map \( t \to \phi_t \) which satisfies \( \phi_0 = \text{id} \), and \( \phi_1 = \phi \). Then there exists a unitary operator \( U_\phi \) in \( M_n \) such that

\[
\phi(x) = U_\phi^*xU_\phi \quad \text{for all } x \in N_\phi.
\]

Hence it seems suitable to call \( N_\phi \) decoherence free algebra. However, how can we find the manifest form of this subalgebra? In particular, how is it related to the Kraus representation \( \phi(x) = \sum_{i=1}^k A_i^*xA_i \)? In this paper, we consider this problem and show the following proposition:
Proposition 1.1 The decoherence free subalgebra $\mathcal{N}_\phi$ is equal to the commutant of the algebra generated by $A_i A_j^*$;

$$\mathcal{N}_\phi = \{ A_i A_j^*, \text{ for all } 1 \leq i, j \leq k \},$$

which is independent of the choice of the Kraus representation.

Here and hereafter, we denote the commutant of $\mathcal{C}$ by $\mathcal{C}' \equiv \{ x \in M_n; xy = yx \text{ for all } y \in \mathcal{C} \}$. The arguments also provides a new proof for generalized Lüders theorem [5].

2 The decoherence free algebra

Let us first confirm the notations. We denote by $B(\mathcal{H})$ the set of all bounded operators on Hilbert space $\mathcal{H}$. A subset $\mathcal{S}$ of Hilbert space $\mathcal{H}$ is said to be total if it spans $\mathcal{H}$. We denote by $h_m$ the $m$-dimensional Hilbert space, and denote the orthonormal basis of it as $\{ e_i; 1 \leq i \leq m \}$. A linear map $\phi$ from $B(\mathcal{H}_2)$ to $B(\mathcal{H}_1)$ is called a unital completely positive map if it satisfies (i) $\phi(1) = 1$ and (ii) for every $n = 1, 2, \cdots$ the correspondence $(X_{ij}) \rightarrow (\phi(X_{ij}))$ $(1 \leq i, j \leq n)$ preserves positivity.

Before going into the proof, we introduce two famous Theorems. The first one is the Theorem of Choi [3]:

**Theorem 2.1** Suppose that $\phi$ is a unital completely positive map from $B(\mathcal{H}_2)$ to $B(\mathcal{H}_1)$. If $\phi(a^*a) = \phi(a)^*\phi(a)$ and $\phi(aa^*) = \phi(a)\phi(a)^*$ for some $a \in B(\mathcal{H}_2)$, then for all $b \in B(\mathcal{H}_2)$, we have $\phi(ba) = \phi(b)\phi(a), \phi(ab) = \phi(a)\phi(b)$.

The second one is the Theorem of Steinspring [3]:

**Theorem 2.2** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be finite dimensional Hilbert spaces and let $\phi$ be a unital completely positive map from $B(\mathcal{H}_2)$ to $B(\mathcal{H}_1)$. Then there exists a Hilbert space $\mathcal{K}$ and isometry $V$ from $\mathcal{H}_1$ to $\mathcal{H}_2 \otimes \mathcal{K}$ such that the following holds:

(a) $\phi(x) = V^* (x \otimes 1) V$ for all $x \in B(\mathcal{H}_2)$

(b) $\{ (x \otimes 1) Vu \mid x \in B(\mathcal{H}_2), u \in \mathcal{H}_1 \}$ is total in $\mathcal{H}_2 \otimes \mathcal{K}$.

(c) If $\mathcal{K}'$ is another Hilbert space and $V'$ is an isometry from $\mathcal{H}_1$ to $\mathcal{H}_2 \otimes \mathcal{K}'$ such that (a) and (b) hold, then there exists a unitary $W$ from $\mathcal{H}_2 \otimes \mathcal{K}$ to $\mathcal{H}_2 \otimes \mathcal{K}'$ such that $W(x \otimes 1_{\mathcal{K}}) V = (x \otimes 1_{\mathcal{K}'}) V'$.

What we consider is a unital completely positive map from $B(\mathcal{H})$ to $B(\mathcal{H})$, where $\mathcal{H}$ is $n$-dimensional Hilbert space and $B(\mathcal{H})$ is equal to $M_n$. Now let us proceed to the proof of the proposition.
Proof of Proposition 1.1: Let $A_i \in B(H), 1 \leq i \leq m$ be Kraus operators which give the Kraus representation of $\phi$ as

$$\phi(a) = \sum_{i=1}^{m} A_i^* a A_i.$$ 

Define a linear map $V$ from $H$ to $H \otimes h_m$ by

$$Vu = \sum_{i=1}^{m} A_i u \otimes e_i.$$ 

The adjoint of $V$ satisfies

$$V^* \left( \sum_{i=1}^{m} \psi_i \otimes e_i \right) = \sum_{i=1}^{m} A_i^* \psi_i,$$

and

$$V^*(a \otimes 1)V = \sum_{i=1}^{m} A_i^* a A_i = \phi(a).$$

Therefore, the condition (a) of the Steinspring’s Theorem holds for the pair $(K \equiv h_m, V)$.

First we consider the case that the condition (b) is also satisfied. Note that by the definition of $N_\phi$, and the Choi’s Theorem, any element $a$ of $N_\phi$ satisfies $\phi(a^* a) = \phi(a)^* \phi(a)$. Then again by Choi’s Theorem, we have

$$\phi(ba) = \phi(b) \phi(a), \phi(ab) = \phi(a) \phi(b)$$

for all $a \in N_\phi$ and all $b \in B(H)$. The relation (2) is represented in the Steinspring representation as

$$VV^*(a \otimes 1_K)VV^*(b \otimes 1_K)Vu = VV^*(a \otimes 1_K)(b \otimes 1_K)Vu$$

for all $u \in H, a \in N_\phi$ and $b \in B(H)$. As $\{(b \otimes 1_K)Vu | b \in B(H), u \in H\}$ is total in $H \otimes K$ by condition (b), this implies

$$VV^*(a \otimes 1_K)VV^* = VV^*(a \otimes 1_K),$$

on $H \otimes K$. The decoherence free algebra $N_\phi$ is involutive. So we also have

$$VV^*(a^* \otimes 1_K)VV^* = VV^*(a^* \otimes 1_K).$$

for all $a \in N_\phi$. Taking the adjoint of this, we have

$$VV^*(a \otimes 1_K)VV^* = (a \otimes 1_K)VV^*.$$ 

Combining these, we get

$$(a \otimes 1_K)VV^* = VV^*(a \otimes 1_K),$$
for all $a \in N_\phi$. As

$$VV^* = \sum_{i,j=1}^{m} A_i A_j^* \otimes |e_i)(e_j|,$$

we obtain

$$aA_i A_j^* = A_i A_j^* a \quad 1 \leq i, j \leq m \quad (3)$$

for all $a$ in $N_\phi$. Conversely, suppose that a self-adjoint element $a$ in $B(\mathcal{H})$ satisfies $\mathfrak{B}$. Then it follows that

$$\phi(a)^2 = \sum_{i,j=1}^{m} A_i^* a A_i^* a A_j = \sum_{i,j=1}^{m} A_i^* A_i A_j^* A_j a^2 A_j$$

$$= \phi(1)\phi(a^2) = \phi(a^2).$$

Here we used the fact that $\phi$ is unital. Hence $N_\phi$ is equal to the commutant

$$N_\phi = \{A_i A_j^* \mid 1 \leq i, j \leq k\}'.$$

Second we consider the case that condition (b) of the Steinspring’s Theorem fails. Let $\tilde{P}$ be a projection operator in $\mathcal{H} \otimes \mathfrak{h}_m$ onto the subspace spanned by $\{(x \otimes 1)Vu \mid x \in B(\mathcal{H}), u \in \mathcal{H}\}$. Note that $\tilde{P}$ satisfies

$$\tilde{P}(x \otimes 1)(y \otimes 1)Vu = (x \otimes 1)(y \otimes 1)Vu = (x \otimes 1)\tilde{P}(y \otimes 1)Vu.$$

This implies

$$\tilde{P}(x \otimes 1)\tilde{P} = (x \otimes 1)\tilde{P}, \quad \tilde{P}(x^* \otimes 1)\tilde{P} = (x^* \otimes 1)\tilde{P}, \quad x \in B(\mathcal{H}).$$

Taking the adjoint of the second equation, we obtain

$$(x \otimes 1)\tilde{P} = \tilde{P}(x \otimes 1), \quad x \in B(\mathcal{H}).$$

By Theorem of 5.9 of $\mathfrak{H}$, all the bounded operator which commutes with $B(\mathcal{H}) \otimes 1$ belongs to $1 \otimes M_m$. So $\tilde{P}$ is of the form $\tilde{P} = 1 \otimes P$, where $P$ is a projection in $\mathfrak{h}_m$. If (b) is not satisfied, $P \neq 1$. In this case, we define a new orthogonal basis $f_j$ of $\mathfrak{h}_j$ so that there exists $1 \leq l < m$, such that

$$P f_j = f_j \quad \text{for} \quad 1 \leq j \leq l$$

$$P f_j = 0 \quad \text{for} \quad l + 1 \leq j \leq m. \quad (4)$$

The orthonormal basis $\{e_i\}$ and $\{f_j\}$ are connected by a unitary transformation, as

$$e_i = \sum_{j=1}^{m} v_{i,j} f_j,$$
Define bounded operators $B_j$ for $1 \leq j \leq m$ as

$$B_j = \sum_{i=1}^{m} v_{i,j} A_i.$$  

Using these relations, we obtain

$$Vu = \sum_{i=1}^{m} A_i u \otimes e_i = \sum_{j=1}^{m} B_j u \otimes f_j.$$  

From $(1 \otimes (1 - P))Vu = 0$ and (4), we have $B_j = 0$ for $l + 1 \leq j \leq m$. We then see that condition (b) is satisfied if and only if $A_i$ are linear independent. Now as $(x \otimes 1)Vu | x \in B(H), u \in H}$ is total in $H \otimes P\mathfrak{h}_m = H \otimes \mathfrak{h}_l$, the pair $(\mathfrak{h}_l, V)$ satisfies conditions (a),(b) of the Stinespring’s Theorem. So by the preceding arguments, we have

$$N_\phi = \{B_i B_j^* \ 1 \leq i, j \leq l\}'.$$

On the other hands, as $v_{i,j}$ satisfies

$$\sum_{j=1}^{m} v_{i,j} v_{k,j} = \delta_{i,k},$$

we have

$$\sum_{j=1}^{m} \overline{v_{i,j}} B_j = \sum_{k,j=1}^{m} \overline{v_{i,j}} v_{k,j} A_k = A_i.$$  

Hence we have

$$\{B_i B_j^* \ 1 \leq i, j \leq l\}' \supset \{A_i A_j^* \ 1 \leq i, j \leq m\}' ,$$

and

$$\{B_i B_j^* \ 1 \leq i, j \leq l\}' \subset \{A_i A_j^* \ 1 \leq i, j \leq m\}' ,$$

Therefore, we obtain

$$N_\phi = \{B_i B_j^* \ 1 \leq i, j \leq l\}' = \{A_i A_j^* \ 1 \leq i, j \leq m\}' ,$$

which completes the proof. $\square$

Let us consider the relation between decoherence free algebra $N_\phi$ and the fixed points $M_\phi$ of $\phi$ defined by

$$M_\phi = \{x \in M_n|\phi(x) = x\}.$$  

We have the following Lemma:
Lemma 2.1 Let $\phi$ be a unital trace preserving completely positive map from $M_n$ into itself. Then $N_\phi$ includes $M_\phi$.

Proof
Any $x \in M_\phi$ is decomposed into $x = x_1 + ix_2$, where $x_1, x_2$ are self adjoint elements of $M_\phi$. So it suffices to show that any self-adjoint element $x$ of $M_\phi$ satisfies $\phi(x^2) = \phi(x)^2$. By the inequality (1), we have

$$\phi(x^2) - x^2 \geq 0.$$ 

On the other hands, we have

$$\text{Tr} \left( \phi(x^2) - x^2 \right) = 0,$$

because $\phi$ is trace preserving. As trace is faithful, we have

$$\phi(x^2) = x^2 = \phi(x)^2,$$

which completes the proof. □

Let $A$ be a $C^*$-algebra generated by $A_i$, and $B$, a $C^*$-algebra generated by $A_i A_i^*$. Clearly, $B \subset A$ and the following inclusion relation holds:

$$A' \subset M_\phi \subset N_\phi = B'.$$

If $A_i$ are positive, then $A_i = \sqrt{A_i A_i^*} \in B$, i.e., all the generators of $A$ is included in $B$. So we have $A = B$, which implies $A' = B'$. In this case, we have

$$A' = M_\phi = N_\phi = B'.$$

That is, all the decoherence free elements are fixed points if $A_i$ are positive. This is a new proof of a generalized Lüders theorem [5], [6].

References

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