The regularity of the positive part of functions in $L^2(I; H^1(\Omega)) \cap H^1(I; H^1(\Omega)^*)$ with applications to parabolic equations

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Abstract. Let $u \in L^2(I; H^1(\Omega))$ with $\partial_t u \in L^2(I; H^1(\Omega)^*)$ be given. Then we show by means of a counter-example that the positive part $u^+$ of $u$ has less regularity, in particular it holds $\partial_t u^+ \notin L^1(I; H^1(\Omega)^*)$ in general. Nevertheless, $u^+$ satisfies an integration-by-parts formula, which can be used to prove non-negativity of weak solutions of parabolic equations.

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1 Introduction

In this note, we are concerned with the regularity of the positive part of functions from the function space

$$W := \{ u \in L^2(I; H^1(\Omega)) : \partial_t u \in L^2(I; H^1(\Omega)^*) \}$$

of Bochner integrable functions. Here, $I = (0, T)$, $T > 0$, is an open interval, and $H^1(\Omega)$ denotes the usual Sobolev space on the domain $\Omega \subset \mathbb{R}^n$; $\partial_t u$ denotes the weak derivative of $u$ with respect to the time variable $t \in I$. The underlying spaces form a so-called evolution triple (or Gelfand triple) $H^1(\Omega) \subset L^2(\Omega) = L^2(\Omega)^* \subset H^1(\Omega)^*$ with continuous and dense embeddings. In the sequel, we will use the commonly applied abbreviations

$$V := H^1(\Omega), \quad H := L^2(\Omega).$$

For an introduction to these kind of function spaces and their various properties, we refer to e.g. [1] Section IV.1, [2] Section 7.2, [4] Chapter 25.

Let $u \in W$ be given. Let us denote its positive part by $u^+$,

$$u^+(t, x) = \max(u(t, x), 0), \quad t \in I, \quad x \in \Omega.$$ 

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Due to the embedding \( W \hookrightarrow L^2(I \times \Omega) \), the positive part is well-defined. Moreover, since the mapping \( u \mapsto u^+ \) is bounded from \( H^1(\Omega) \) to \( H^1(\Omega) \), it follows that for \( u \in W \) also \( u^+ \in L^2(I; V) \) holds. Here, the question arises whether \( u \in W \) also implies \( u^+ \in W \). The aim of the short note is to provide an counter-example of this claim, see Theorem 2.7. Nevertheless, the following integration-by-parts formula holds true for all \( u \in W \)

\[
\int_I (u_t(s), u^+(s))_{V, V} \, ds = \frac{1}{2} \| u^+(T) \|_{H^1}^2 - \frac{1}{2} \| u^+(0) \|_{H^1}^2,
\]

which enables us to show positivity of weak solutions of linear parabolic equations, see Section 3.

2 The regularity of the positive part

In this section, we study the mapping properties of \( u \mapsto u^+ \). First, let us state the following well-known results:

**Proposition 2.1.** The mapping \( u \mapsto u^+ \) is Lipschitz continuous as mapping from \( H \) to \( H \). Furthermore it is bounded from \( V \) to \( V \), and for \( u \in V \) it holds

\[
\| \nabla u^+(x) \| = \begin{cases} \nabla u(x) & \text{if } u(x) > 0 \\
0 & \text{if } u(x) \leq 0 \end{cases} \quad x \in \Omega,
\]

which implies \( \| u^+ \|_V \leq \| u \|_V \).

The following result is an obvious consequence.

**Corollary 2.2.** Let \( u \in W \) be given. Then \( u^+ \in L^2(I; V) \cap C(\bar{I}; H) \), and it holds

\[
\| u^+ \|_{L^2(I; V)}, \| u^+ \|_{C(\bar{I}; H)} \leq \| u \|_W.
\]

With the same arguments that are classically used to proof Proposition 2.1 one can prove

**Corollary 2.3.** Let \( u \in W \) be given with \( u_t \in L^2(I; H) \). Then \( u^+ \in W \) with \( u^+_t \in L^2(I; H) \).

Moreover, in this case, we have \( \partial_t u^+ \in L^2(Q) \), and we can write for almost all \((t, x) \in Q\)

\[
\partial_t u^+(t, x) = \begin{cases} \partial_t u(t, x) & \text{if } u(t, x) > 0 \\
0 & \text{if } u(t, x) \leq 0 \end{cases} \quad (2)
\]

Now, if \( \partial_t u \) is in \( L^2(I; V^*) \) only, the representation (2) makes no sense, as \( \partial_t u(t, \cdot) \) is only in \( H^1(\Omega)^* \) for almost all \( t \).

In the following, we will construct a function \( u \in W \) with \( \partial_t u \not\in L^2(I; H) \) such that \( \partial_t u^+ \not\in L^2(I; V^*) \). The key idea is the observation that the mapping \( u \mapsto u^+ \) for \( u \in L^2(\Omega) \) is not bounded as mapping from \( H^1(\Omega)^* \) to \( H^1(\Omega)^* \).

To see this, set \( \Omega = (0, 1) \). Let us define \( \psi_n(x) = \sin(2\pi nx) \). Then it is well-known that \( \psi_n \) converges weakly to zero in \( L^2(\Omega) \), thus strongly to zero in \( H^1(\Omega)^* \). However, a short computation shows that

\[
\int_0^1 \psi_n^+(x) \, dx = \int_0^1 \psi_n^-(x) \, dx = \int_0^{1/2} \sin(2\pi x) \, dx = \frac{1}{\pi} \neq 0,
\]
which implies that $\psi_n^+$ converges weakly to the constant function $\check{\psi}(x) = 1/\pi$ in $L^2(\Omega)$. Hence, $\psi_n^+$ cannot converge to zero in $H^1(\Omega)^*$.

In the sequel, we will equip $V$ with the scalar product $(u, v)_V := \int_\Omega \nabla u \cdot \nabla v + u \cdot v \, dx$ and the associated norm. The space $H$ is equipped with the standard $L^2(\Omega)$ inner product and norm. We consider the family of functions $\psi_n(x) := \cos(n \pi x), \ x \in \Omega$ (3) for $n \in \mathbb{N}$. Now, we will derive quantitative estimates of the norm of $\psi_n$ in $V$, $H$, and $V^*$ for $n \to \infty$.

**Lemma 2.4.** Let $n \in \mathbb{N}$ be given. Then it holds

$$\|\psi_n\|_V = \left( \frac{n^2 \pi^2 + 1}{2} \right)^{1/2} \leq n \pi, \quad \|\psi_n\|_H = \frac{1}{\sqrt{2}}, \quad \|\psi_n\|_{V^*} \leq \frac{1}{\sqrt{2} n \pi}$$

**Proof.** The first two identities can be verified with elementary calculations. To prove the third, consider the solution $z \in V$ of $(z, v)_V = (\psi_n, v)_V$ for all $v \in V$. Then it follows $\|\psi_n\|_{V^*} = \|z\|_V$. The function $z$ is given by $z = \frac{1}{n \pi + 1} \psi_n$, and hence the third estimate follows from the first. \hfill \Box

Let us show that the $V^*$-norm of $\psi_n^+$ is bounded away from zero.

**Lemma 2.5.** There is $C > 0$ such that

$$\|\psi_n^+\|_{V^*} \geq C \quad \forall n.$$

**Proof.** Let $e \in H$ be defined by $e(x) = 1$. Then we have

$$\langle \psi_n^+, e \rangle_H = \int_0^1 \psi_n^+(x) \, dx = \int_0^1 (\cos(n \pi x))^+ \, dx = n \int_0^{1/2} \cos(n \pi x) \, dx = \frac{1}{\pi}.$$  

Let now $v_e \in V$ be defined by $v_e(x) = \min(4x, 1, 4(1-x))$. Then it holds $\|v_e - e\|_H^2 = 2 \int_0^{1/4} (4x)^2 \, dx = \frac{1}{2}$. Thus, we can estimate

$$\langle \psi_n^+, v_e \rangle_{V^*} \geq (\psi_n^+, e) - \|\psi_n^+\|_H \|v - e\|_H \geq \frac{1}{\pi} - \frac{1}{\sqrt{12}} = 0.0296 \cdots \geq \frac{1}{5}.$$  

Here, we used $\|\psi_n^+\|_H \leq \|\psi_n\|_H = 1/\sqrt{2}$. The lower bound implies that $\|\psi_n^+\|_{V^*} \geq \frac{1}{5} \|v_e\|^{-1}_{V^*}$, and the claim is proven. \hfill \Box

Let us now introduce a family of functions on small time intervals, which will be used to define the counterexample by means of an infinite series.

**Lemma 2.6.** Let $I := (0, 1)$. Let $\phi \in H^1(I)$ be given. Define

$$\phi_n(t) := n(n + 1) \cdot \phi(n(n + 1)t - n).$$

Then it holds $\text{supp} \phi_n \subset \left( \frac{1}{n+1}, \frac{1}{n} \right)$ and

$$\|\phi_n\|_{L^1(I)} = \|\phi\|_{L^1(I)}, \quad \|\partial \phi_n\|_{L^1(I)} \geq n^2 \|\partial \phi\|_{L^1(I)}, \quad \|\partial \phi_n\|_{L^2(I)} \leq \sqrt{2}n \|\phi\|_{L^2(I)}, \quad \|\partial \phi_n\|_{L^2(I)} \leq \sqrt{2}n^3 \|\partial \phi\|_{L^2(I)}.$$
Proof. This follows by elementary calculations. □

Let us now define the function
\[ u(x, t) = \sum_{n=1}^{\infty} n^{-3} \phi_n(t) \psi_n(x). \] (5)

**Theorem 2.7.** Let \( \phi \in H_0^1(I) \setminus \{0\} \) be given with \( \phi \geq 0 \). Then the function \( u \) defined in (5) with \( \psi_n \) and \( \phi_n \) from (3) and (4), respectively, belongs to \( W \). However, the time derivative of its positive part \( \partial_t u^+ \) does not belong to \( L^1(I; V^*) \).

Proof. Let us define the partial sum \( u_N := \sum_{n=1}^{\infty} \phi_n(t) \psi_n(x) \). We will exploit the fact that the supports of the functions \( \phi_n \) are distinct. From the Lemmas 2.4, 2.5 and 2.6 we have

\[ \|u_N\|_{L^2(I; V)}^2 = \sum_{n=1}^{N} n^{-6}\|\phi_n\|_{L^2(I)}^2\|\psi_n\|_V^2 \leq c \sum_{n=1}^{N} n^{-6} \cdot n^2 = c N^{-2}, \]

\[ \|\partial_t u_N\|_{L^2(I; V^*)}^2 = \sum_{n=1}^{N} n^{-6}\|\partial_t \phi_n\|_{L^2(I)}^2\|\psi_n\|_{V^*}^2 \leq c \sum_{n=1}^{N} n^{-6} \cdot n^2 = c N^{-2}, \]

\[ \|\partial_t u_N^+\|_{L^1(I; V^*)} = \sum_{n=1}^{N} n^{-3}\|\partial_t \phi_n\|_{L^1(I)}\|\psi_n\|_{V^*} \geq c \sum_{n=1}^{N} n^{-3} \cdot n^2 = c N^{-1}. \]

This proves that \( (u_N) \) strongly converges in \( W \) to \( u \). Since \( u = u_N \) on \( \left( \frac{1}{n+1}, 1 \right) \), the weak derivative \( \partial_t u^+ \) exists almost everywhere on \( I \), and belongs to the space \( L^1(I; V^*) \). Suppose that \( \partial_t u^+ \in L^1(I; V^*) \) holds. Then by the continuity of the integral it follows

\[ \|\partial_t u^+\|_{L^1(I; V^*)} = \lim_{N \to \infty} \int_{1/(N+1)}^{1} \|\partial_t u_N^+(t)\|_{V^*} dt = \lim_{N \to \infty} \|\partial_t u_N\|_{L^1(I; V^*)} \to \infty, \]

which is a contradiction, hence \( \partial_t u^+ \notin L^1(I; V^*) \). □

3 **Positivity of weak solutions to parabolic equations**

Let \( \Omega \subset \mathbb{R}^n \) be a domain. Again, we make use of the evolution triple \( V = H^1(\Omega), H = L^2(\Omega), V^* = (H^1(\Omega))^* \). Due to the counter-example in the previous section, we cannot apply the well-known integration-by-parts results for functions in \( W \) to \( u^+ \). In order to prove formula (1), we recall the following density result

**Proposition 3.1.** [3 Lemma 7.2] The space \( C^\infty([0, T], V) \) is dense in \( W \).

First, let us prove the integration-by-parts formula for smooth \( u \).

**Lemma 3.2.** Let \( u \in W \) with \( \partial_t u \in L^2(I; L^2(\Omega)) \) be given. Then it holds

\[ \int_0^T \langle \partial_t u(t), u^+(t) \rangle_{V^*, V} dt = \frac{1}{2} \int_0^T \partial_t \|u^+(t)\|_H^2 = \frac{1}{2} \left( \|u^+(0)\|_H^2 - \|u^+(0)\|_H^2 \right). \] (6)
Lemma 3.3. Let \( u \in W \) be given. Then it holds
\[
\int_0^T \langle \partial_t u(t), u^+(t) \rangle_{V^*, V} \, dt = \frac{1}{2} \int_0^T \partial_t \|u^+(t)\|^2_H \, dt.
\]
Proof. Since \( \partial_t u \in L^2(I; L^2(\Omega)) \), it holds \( \partial_t u^+ \in L^2(I; L^2(\Omega)) \). With the representation it follows
\[
\int_Q \partial_t u(x,t) u^+(x,t) \, dx \, dt = \int_Q \partial_t u^+(x,t) u^+(x,t) \, dx \, dt = \frac{1}{2} \int_0^T \partial_t \|u^+(t)\|^2_H \, dt,
\]
which proves the claim. \( \square \)

**Theorem 3.4.** Let \( f \in L^1(I; L^2(\Omega)) + L^2(I; V^*) \) be given, with \( f \geq 0 \), which is \( (f,v) \geq 0 \) for all \( v \in L^2(\Omega) \). Let \( u_0 \in H \) be given with \( u_0 \geq 0 \). Let \( u \) be a weak solution of the parabolic equation \( \partial_t u - \Delta u = f \) on \( Q \), \( \partial_n u = 0 \) on \( I \times \partial \Omega \). Then it holds \( u \geq 0 \).

Proof. Let us denote \( u^- = -(-u)^+ \in L^2(\Omega) \cap C(I; H) \). Testing the weak formulation with \( u^- \), integrating from 0 to \( t \), and using Proposition 2.1 and Lemma 3.3 yields
\[
0 \geq \int_0^t (f(s), u^-(s))_{V^*, V} \, ds
\]
\[
= \int_0^t \langle \partial_t u(s), u^-(s) \rangle_{V^*, V} \, ds + \int_0^t \int_\Omega \nabla u(x,s) \nabla u^-(x,s) \, dx \, ds
\]
\[
= \frac{1}{2} \left( \|u^-(t)\|^2_H - \|u^- (0)\|^2_H \right) + \|\nabla u^-\|^2_{L^2(0,t; L^2(\Omega))}
\]
\[
\geq \frac{1}{2} \|u^-(t)\|^2_H.
\]
Hence, it follows \( u^-(t) = 0 \) for almost all \( t \in I \), which implies \( u^- = 0 \) almost everywhere on \( Q \). \( \square \)
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