Parafermions, ternary algebras and their associated superspace

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Abstract
Parafermions of order two are shown to be the fundamental tool to construct ternary superspaces related to cubic extensions of the Poincaré algebra.

1 Introduction
Ternary (and in general $n$-ary) algebras have been considered for some time in the literature from the purely mathematical point of view, and only recently there has been some revival of these structures in connection with physics, like in the field theory of multiple $M$-2-branes, where certain metric 3-algebras appear naturally [1] (see also [2] for a formal description). In a different context it has been observed that some cubic extensions of the Poincaré algebra can be implemented into the Quantum Field Theory frame [3] (the underlying algebraic structure, called Lie algebras of order three, having been introduced in [4]).

It should however be taken into account that ternary (and higher order ones) structures are quite different from their quadratic analogue, the Lie algebras and superalgebras. In particular, for the extensions considered in [4], the various cubic brackets (see below) do not allow us to order a given monomial in a definite order. This means specifically that finite dimensional representations are automatically non-faithful (see the second paper of [4] or [9]). The latter obstruction is certainly only one among the various reasons that justify the formal difficulties encountered in order to construct an appropriate ternary superspace. However, despite these difficulties, it has recently been realised that Lie algebras of order three share some similarities with Lie superalgebras. Indeed, a formal study of Lie algebras of order three enables us to identify the parameters of the corresponding transformation and then to define groups associated to Lie algebras of order three. It turns out that the fundamental variables which naturally describe the parameters of the transformation correspond to the genuine cubic
extension of the Grassmann algebra, called the three-exterior algebra (see [4] below) introduced by Roby [10]. These similarities allow the construction of linear representations of groups associated to Lie algebras of order three, in terms of matrices, the entries of which belong to the three-exterior (or Roby) algebra [6], in straight analogy with Lie supergroups.

It is then natural to construct a ternary superspace using the Roby variables. In this paper, we give a step-by-step construction of the ternary superspace associated to the cubic extension of the Poincaré algebra (3) (see below). At the very end, the fundamental variables needed to define a ternary superspace turn out to be the order two parafermions. Parafermions and more generally parastatistics were introduced a long time ago as an exotic possibility extending the Bose and Fermi statistics [11, 12]. In particular, order two parafermions satisfy cubic relations, the latter allowing us to generate a ternary algebra. It is very interesting to notice that two different structures, which have a priori no relation, can be unified by this ansatz. The question whether these two structures (parafermions and Lie algebras of order three) have some further hidden relations arises at once.

Let us we mention that parafermions and parabosons have also been considered in a rather different context [13–15]. It is also important to mention that ternary superspaces were defined by several authors in one or two space-time dimensions, where the situation is somewhat exceptional (the Lorentz algebra being either trivial or abelian) [16, 17].

The contents of this paper is the following. In section 2, the basic definitions of Lie algebras of order three, together with the specific cubic extension of the Poincaré algebra relevant for the sequel are given. Section 3 is devoted to the explicit construction of the ternary superspace. Finally, some conclusions and perspectives are given in Section 4.

2 Lie algebra of order three and cubic extensions of the Poincaré algebra

In this section, we recall the basic properties of Lie algebras of order three. We also recall how a cubic extension of the Poincaré algebra arises in this context. Higher order algebraic structures (in fact F-ary), called Lie algebras of order F and generalising Lie (super)algebras, were introduced in [4]. In this note, we are mostly interested in elementary real Lie algebras of order three. An elementary (real) Lie algebra of order three is given by \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \), where \( \mathfrak{g}_0 \) is a real Lie algebra and \( \mathfrak{g}_1 \) is a real representation of \( \mathfrak{g}_0 \), satisfying the following brackets

\[
\begin{align*}
[&X_i;X_j] = f_{ij}^k X_k; \\
[&X_i;Y_a] = R^a_{\phantom{a}b} Y_b; \\
\sum Y_a Y_b Y_c g &= Y_a Y_b Y_c + Y_b Y_c Y_a + Y_c Y_a Y_b \\
&+ Y_a Y_c Y_b + Y_b Y_a Y_c + Y_c Y_b Y_a = Q_{abc} X_i;
\end{align*}
\]

and fulfilling the following fundamental identities for any \( X_i; Y_b; Y_c; Y_d \) in \( \mathfrak{g}_1 \).
Looking at the various brackets, one immediately observes that a Lie algebra of order three is endowed with two different products: one binary given by the usual commutator, and a ternary given by a fully symmetric product. Furthermore, a direct inspection of (1) and (2) shows that Lie algebras of order three are a ternary extension of Lie superalgebras, where the anticommutator is replaced by a fully symmetric cubic bracket. Moreover, the second equation of (1) is just a consequence of our assumption that $g_1$ is a representation of $g_0$ which is specified by the matrices $R_i$. This representation is denoted with $D$. Finally, the last equations just assume that $g_0 = S^3(g_1)$ (where $S^3(g_1)$ is the three-fold symmetric tensor product of $g_1$). Many examples of Lie algebras of order $F$ were given in [4], and a formal study of this algebraic structures was initiated [5–7].

Having introduced the ternary algebra (1), one immediately may wonder if it could be applied in physics. In fact, among various possibilities, the cubic extension of the Poincaré algebra $\mathfrak{so}(1;3) = g_0 \oplus g_1$, with $g_0 = hL = L ; P ; 0$; $3i$ generating the Poincaré algebra and $g_1 = hV ; 0$; $3i$ being the vector representation, and brackets

$$
\begin{align*}
[L ; L ] & = L + L ; \\
[L ; P ] & = P ; \\
[L ; V ] & = V ; \\
[P ; V ] & = 0; \\
\mathcal{F}V ; V ; V g & = P + P + P ;
\end{align*}
$$

(3)

where $\mathcal{F} = \text{diag}(1; 1; 1; 1)$ is the Minkowski metric was intensively studied in the framework of Quantum Field Theory [3, 8, 9]. This ternary extension of the Poincaré algebra is non-trivial in the sense that a space-time translation is generated by a cubic composition of three elements of $g_1$. Since, $g_1$ is in the vector representation of the Lorentz algebra, differently from supersymmetry, a multiplet contains states of different spin, but underlying the same statistics [3].

## 3 Ternary superspace

The representation theory of (3) was analysed in [3], and invariant Lagrangians were explicitly constructed. However, since our basic algebra is cubic instead of quadratic, the construction of a ternary superspace is more involved. For instance, looking the the fifth relation in (3), one observes that we cannot order a monomial in $V$ in a definite order. And, in particular, defining the universal enveloping algebra $U(g)$ a Poincaré-Birkhoff-Witt was established, and it was
shown [6] that $U(g_1)$ is isomorphic (as a vector space) to the three-exterior algebra (see (3)), which in turn is infinitely generated [10]. This means that if we consider a finite dimensional representation of (3), it cannot be faithful (see e.g. [4, 9]). This observation is certainly one of the reasons for the difficulty to construct a ternary superspace. In this section we show one possibility of associating an appropriate ternary superspace to the algebra (3). In the following construction, the various assumptions leading to the ternary superspace are introduced step-by-step. The result of this section was given in [18].

3.1 The natural variables

Starting from a Lie algebra of order three $g = g_0 \oplus g_1 = hX_i; i = 1; \ldots; \dim hX_i; a = 1; \ldots; \dim g_1$ and using the Hopf algebra techniques, it can be shown that [6]

1. to each generator of $g_0$, one associates a commuting parameter $X_i$;
2. to each generator of $g_1$, one associates a variable $Y_a$,

such that the variables $X_i$; $Y_a$; $i \in g_1$ generate the three exterior algebra

$$a b c + b c a + c a b + a c b + b a c + c b a = 0; \quad (4)$$

Furthermore, the variables $Y_a$ are in the dual representation of $D$. It has to be mentioned that the algebra generated by the $X_i$'s can either be real or complex. From now on, we are considering only real Roby algebras.

It is then natural in the case of the cubic extension (3), to postulate that the ternary superspace is generated by

$$X = (x; \ )$$

where $x$ are the space-time coordinates and are their ternary analogues which are in the vector representation of the Lorentz group and which satisfy the algebra (4).

3.2 Realisation of the Poincaré algebra

The next step in our construction is to define differential operators which act on the ternary superspace (5) and which realise the Poincaré algebra. It is then necessary to introduce some variables conjugate the the variables $X$. Denote $P$ (resp. $\partial$) the conjugate variables of $x$ (resp. $\partial$). The action of $P$ on $x$ is straightforward ($[P;x] = 0$). Following Green [11, 12], the more general quantisation which ensures that are vectors of the Lorentz algebra is given by the parafermions. We thus assume the parafermionic relations

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1 Or parabosons, but the parabosonic algebra turns out to be incompatible with requirement (4).
\[ [ ; ]; ] = 0; \quad [ ; ]; @ ] = + ; \quad [ ; ]; @ ] = + ; \quad [ ; ]; @ ] = @ ; \quad [ @ ; ]; ] = @ + @ \quad [ @ ; ]; @ ] = 0 : \] (6)

As a consequence, if we define
\[ J = [ ; @ ] [ ; @ ]; \] (7)
the relations (6) ensure that (7) act correctly on \( @ \) and \( @ \):
\[ [ J ; ] = : \] (8)

This means that \( P \) and
\[ L = \times P \times P + J ; \] (9)
generate the Poincaré transformations upon the ternary superspace (5).

### 3.3 Order two parafermions

Since we are considering ternary algebras involving fully symmetric products, putting (4) and (6) together shows explicitly that we are considering parafermions of order two. This means that the relations (4) have to be supplemented by the additional conditions [12]:

\[ f ; ; g = 0 ; \quad f ; ; @ g = 2 + 2 ; \quad f ; @ ; g = 2 @ + 2 @ ; \quad f @ ; @ ; g = 0 ; \] (10)

It is interesting to observe that the construction leading to (10) and (6) goes in reverse order to that of parafermionic algebras. Historically, parafermions were defined by means of equation (6), in order to realise the Lorentz algebra. After all the order of paraquantisation (here two, but in general \( p \)) is specified by assuming on which representation of the Lorentz algebra the parafermionic algebra acts. Order \( p \) parafermionic algebras involved fully symmetric brackets of order \( p + 1 \) and, in particular, order two parafermionic algebra give rise to the brackets (10). However, in our construction, the cubic brackets (4) are obtained from the very beginning, by our superspace assumption. Finally, notice that the order two parafermionic algebra (5), (10) is a non-faithful representation of the algebra (4) since with respect to the Roby algebra we have one more relation \([ [ ; ] ; ] = 0 \). (In particular the Roby algebra is infinitely many generated [6, 10] although the order three parafermionic algebra is finite dimensional).

As a final observation of this section, let us mention that if we define
\[ @ \] we have
which is very similar to the fifth equation of (3). This is certainly not a coincidence. Indeed, as we have seen $U(g)$ can be endowed with a Hopf algebra structure \cite{6}, and in particular with a coproduct. This is precisely this coproduct, that makes that three-exterior algebra (11) emerges naturally on $U(g_1)$ (the dual of $U(g_1)$). The introduction of the conjugate momenta of $\theta$ lead to (10) and finally to (11).

3.4 Realisation of the ternary part of the algebra

Now, we would like to construct a differential realisation of the purely ternary part of the algebra. The relations (6) shows that the natural relations upon the $\theta$’s and the $\theta$’s involve double commutators. This means in particular that we cannot expect to construct a differential operator $V$ from $\theta$ and acting on and satisfying the cubic relations (3). For instance, if we assume that $V = \theta + \theta + \theta$, we obtain that $[V;\theta]=\theta+\theta+\theta$. But since the relations (6) and (10) are cubic, there is no bilinear relations upon $\theta$ and $\theta$ and consequently $\theta$ emerges as a new object.

This situation is very similar to the implementation of the Noether theorem within the framework of ternary symmetries, where the conserved charges generate the symmetry through quadratic relations using the usual quantisation procedure (e.g. the equal-time (anti)-commutation relations). Let us briefly recall how it works. In [3] we have obtained some representations of the algebra $\hat{P}$ and $\hat{L}$ and $\hat{V}$ acting on a multiplet (in fact various multiplets were obtained). Next, we have constructed an invariant Lagrangian $\hat{\mathcal{L}}(\cdot)$ and obtained the associated conserved charges \cite{3,8}$\hat{\mathcal{L}};\hat{\theta};\hat{\theta}$ and $\hat{\mathcal{L}};\hat{\theta};\hat{\theta}$. For instance, $\hat{\mathcal{L}}$ is given by

$$\hat{\mathcal{L}} = \int d^4x \hat{\theta}\hat{L} \hat{\theta}\hat{\theta}$$

(standard expressions were obtained for $\hat{\theta}$ and $\hat{\theta}$), and is such that after quantisation

$$\mathcal{V} = [\hat{\mathcal{L}};\hat{\theta};\hat{\theta}]$$

(12)

(in the usual way the Poincaré transformations of the multiplet are given by $[\hat{\mathcal{L}};\hat{\theta};\hat{\theta}]$). It is important to emphasise that at this point we are dealing with usual bosonic and fermionic fields satisfying the standard (anti-)commutation relations i.e. there is no need to introduce some fields with exotic behaviour in order to obtain (12). Next, we have shown that the algebra is realised through multiple-commutators

$$[\hat{\mathcal{L}};\hat{\theta};\hat{\theta}] + \text{perm.} = [\hat{\theta};\hat{\theta};\hat{\theta}] + \text{perm.}$$

(13)
This procedure is standard in the implementation of Lie (super)algebra in Quantum Field Theory, but the equation corresponding to (13) in this case is not the end of the story since the Jacobi identities allow to obtain a relation which is independent of the fields \( \beta \). But here, in the context of ternary symmetries, the situation is very different, since the fundamental identities (2) do not allow to write the algebra in an independent form. This weaker realisation of the algebra has the interesting consequence that it enables us to consider algebraic structure (in Quantum Field Theories), different from Lie superalgebras, without contradicting the spin-statistics theorem (see [9] for a discussion). Finally, it is a matter of calculation to check that the fundamental identities (2) are satisfied by the realisation (13).

We thus see that the implementation of Noether theorem in ternary algebras presents some similarities with the natural action defined on parafermions. Indeed, in both cases the natural objects are the commutators (or the anticommutators for fermions). This suggests to try to realise (4) on the superspace \( \mathbb{C} \) in the form of (13). We introduce the parameters of the transformations \( \beta \) (of the same nature of and as such satisfying (4), (10) and (6)) such that we have the transformation

\[
! \overset{0}{=} + \beta ;
\]

under (3). Now we define the generator

\[
\mathcal{V} = [\beta ; \beta ] + [ ; ] [\beta ; ] [\beta ] ; (14)
\]

such that

\[
= [\mathcal{V}; ] = \beta ; \times = [\mathcal{V}; \times ] = [ ; ] [\beta ; ] ; : (15)
\]

It is important to realize that the \( \times \)'s are commuting real variables. Two observations are in order here. Firstly, in the realisation of the algebra, due to the nature of the para-commutation relations (6), it is not possible to dissociate the generators \( \mathcal{V} \) and the parameters \( \beta \). Secondly, in order to have the appropriate transformations properties for \( \beta \), we are forced to introduce one more parafermionic variable in the scalar representation of the Lorentz algebra. This new variable can be seen, together with \( \beta \), to be some parafermion associated to \( \mathfrak{so}(1,4) \).

3.5 Closure of the algebra

Now we have to check the closure of the algebra in the form (13). In particular, if we compute

\[
[V_1; V_2 ; V_3 ; 1 \quad 2 \quad 3 ] ] = \frac{1}{1 \quad 2 \quad 3} + \frac{1}{2 \quad 3 \quad 1} + \frac{1}{3 \quad 1 \quad 2} + \frac{1}{1 \quad 3 \quad 2} + \frac{1}{2 \quad 1 \quad 3} + \frac{1}{3 \quad 2 \quad 1} \quad (16)
\]
we observe that it is fully symmetric with respect to the indices 1, 2, 3. This means that \([V_1; V_2; V_3] + \text{perm.} \neq 0\). From now, in order to simplify the notations, we denote \([V_1; V_2; V_3] + \text{perm.} = 1\).

If one proceeds along these lines in the case of supersymmetry, one obtains the same kind of results. However, in this case the closure of the algebra is guaranteed by the introduction of the Grassmann (anti-commuting) variables and consequently, the anti-commutators get replaced by the commutators. There is some analogous substitution in the context of ternary algebras but here the situation is more involved because the variables do not satisfy quadratic relations.

Indeed, we have shown in [6] that ternary algebras of order three inherit of a \(Z_3\) graded structure, or more precisely of a \(Z_3\) twisted tensor product. This means in particular, that if we consider three successive transformations specified by \(\cdots\), we get a \(Z_3\) \(Z_3\) \(Z_3\) graded structure. This \(Z_3\) structure implies that, taking the parameters of the transformation, the bracket of order three is no longer fully symmetric, but as to be defined with the cubic primitive root of unity that we denote by \(q\). This is a kind of Jordan-Wigner transformation adapted to ternary algebras. In fact these types of structures, where the brackets are neither symmetric nor antisymmetric have been considered before in the literature, even for quadratic algebras (as a possible generalisation of Lie (super)algebras) and have been called colour algebras [19]. The basic tool to define colour Lie (super)algebras is a grading determined by an Abelian group. The latter, besides defining the underlying grading in the structure, moreover provides a new object known as commutation factor associated to an Abelian group (here \(Z_3\) \(Z_3\) \(Z_3\)). A commutation factor \(N\) is a map \(N : \mathbb{C} \times \mathbb{G}\) satisfying the following constraints:

1. \(N(a; b)N(b; a) = 1\) for all \(a, b \in \mathbb{G}\);
2. \(N(a; b + c) = N(a; b)N(a; c)\) for all \(a, b, c \in \mathbb{G}\);
3. \(N(a + b; c) = N(a; c)N(b; c)\) for all \(a, b, c \in \mathbb{G}\).

In our case, the commutation factor is given by

\[N(a; b) = q^{a_1(b_2 + b_3) + a_2 b_3 + (a_2 + a_1) b_2 a_3};\]

where \(q = e^{\frac{2\pi}{3}}\) and \(a, b \in Z_3\). Then in the same vain of the colour algebras, colour algebras of order three may be defined [7]. In particular, defining
\[ f \mathcal{Y}_1 ; \mathcal{V}_2 ; \mathcal{V}_3 \mathcal{X}_0 = V_1 V_2 V_3 + N \text{gr}("_1 ); \text{gr}("_2 ) + \text{gr}("_3 ) \ V_2 V_3 V_1 \]
\[ + N \text{gr}("_1 ) + \text{gr}("_2 ); \text{gr}("_3 ) \ V_3 V_1 V_2 \]
\[ + N \text{gr}("_2 ); \text{gr}("_3 ) \ V_1 V_3 V_2 + N \text{gr}("_1 ); \text{gr}("_2 ) \ V_2 V_1 V_3 \]
\[ + N \text{gr}("_1 ); \text{gr}("_2 ) N \text{gr}("_1 ); \text{gr}("_3 ) \ V_3 V_2 V_1 ; \]

with \(\text{gr}("_1 ) = (1;0;0); \text{gr}("_2 ) = (0;1;0); \text{gr}("_3 ) = (0;0;1)\), the cubic brackets \((18)\) adopt the following form (there is also corresponding fundamental identities, but there are not relevant for our purpose \([7]\))

\[ f \mathcal{Y}_1 ; \mathcal{V}_2 ; \mathcal{V}_3 \mathcal{X}_0 = V_1 V_2 V_3 + \mathcal{q}^2 V_2 V_3 V_1 + \mathcal{q}^2 V_3 V_1 V_2 \quad (19) \]
\[ + \mathcal{q}V_1 V_3 V_2 + \mathcal{q}V_2 V_1 V_3 + V_3 V_2 V_1 ; \]

In particular, since the constraint \(1 + \mathcal{q} + \mathcal{q}^2 = 0\) is satisfied and \(V_1 V_2 V_3 : ( \ 1 \ 2 \ 3 \ )\) is fully symmetric in the subindices \(1;2;3\), we automatically have that

\[ f \mathcal{Y}_1 ; \mathcal{V}_2 ; \mathcal{V}_3 \mathcal{X}_0 : ( \ 1 \ 2 \ 3 \ ) = 0 ; \]

Performing a similar computation for the space-time coordinates, we obtain the identities

\[ f \mathcal{Y}_1 ; \mathcal{V}_2 ; \mathcal{V}_3 \mathcal{X}_0 : ( \ 1 \ 2 \ 3 \ ) = \mathcal{q}^2 \mathcal{q}[ ; "_2 ];"_3 ;"_1 ] + \mathcal{q}^2 \mathcal{q}[ ; "_1 ];"_3 ;"_2 ] \]
\[ [ ;"_2 ];"_3 ;"_1 ] [ ;"_3 ];"_1 ;"_2 ] \quad (20) \]

It is important to notice that the \(\mathcal{a}\) are complex numbers. This means that the “coloration” of the algebra \((3)\), coming from our adapted Jordan-Wigner transformation gives rise to the algebra \((19)\), which is manifestly a complex algebra since the structure constants are complex. This deserves some explanation. The "\ are real parafermions, therefore the transformation properties \((15)\) ensure that \(\mathcal{x}\) and \(\mathcal{y}\) are both real. However, since \(\mathcal{a}\) is complex, this means that the cubic algebra \((3)\) is realised in a complexification of the superspace \((\mathcal{x};\mathcal{y})\). In other words, the algebra cannot be realised on a real vector space. This is the best possible result in this direction using this ansatz.

4 Conclusion and perspectives

We have shown that two different cubic algebras (the cubic extension of the Poincaré algebra \((3)\) and the order two parafermions) “unify” in the sense that the latter become the relevant variables for the construction of an adapted ternary...
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superspace for the former. In particular this means that we were able to construct a differential realisation of the algebras \( \mathfrak{f} \). Having such differential operators, the next step would be to define some appropriate superfields (depending on \( x \) and \( y \)) and to define certain operators which could be interpreted as a covariant derivative. This possibilities were analysed in [18] (together with the study of specific quaternary extensions of the Poincaré algebra). This construction opens the possibility, using the standard techniques, for the proposal of interesting physical model constructions based on cubic (and in general higher order) extensions of the Poincaré algebra. One step to be carefully analyzed under this perspective is the explicit construction of Lagrangians and other invariant quantities that provide the experimental confirmation of the model and fixes to which extent the considered parameters and variables can be actually identified with well known physical observables. Further work in this direction is currently in progress.

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