Topological aspects of the partition function of the NS5-brane

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Abstract

To study topological aspects of the partition function of the NS5-brane in type IIA string theory, we define a cohomology class whose vanishing is a necessary condition for this function to be well-defined. This leads to various topological conditions, including a twisted Fivebrane structure as well as secondary cohomology operations arising from a K-theoretic description. We explain how these operations also generate the topological part of the action as well as the phase of the partition function. Part of the discussion also applies to the M5-brane.

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1 Introduction

The NS5-brane in type IIA string theory is a magnetic brane [8] which enjoys many interesting physical and mathematical properties (see e.g. [12]). This brane is charged under the dual $\ast H_3 = H_7$ of the B-field $H_3$ and hence admits as a source the right hand side of the equation of motion of the B-field. The field $H_7$, unlike $H_3$, is hence not closed. This brane is obtained from the dimensional reduction of the M5-brane from eleven to ten dimensions, and hence shares many of the properties of the M5-brane. For example, the worldvolume theory carries a 2-form potential as part of the chiral $(0,2)$ supermultiplet. Associated with the M5-brane are several new geometric and topological structures that the author has been investigating [32] [33] [34] [36]. This note is in some sense a continuation of that work but for the closely related setting of the NS5-brane.

The action. An action for the NS5-brane is given in [3]. Another one, which is free of gravitational anomalies, for the coupled system of IIA $D = 10$ dynamical supergravity interacting with an NS5-brane is constructed in [9]. We will be mostly interested in the topological terms in the action, as described in [46] [48].

The partition function. The partition function of the NS5-brane is used in [46] [48] essentially as a tool to study the partition function of the M5-brane. This is studied further in [12] [13] in the special case when the worldvolume is a Calabi-Yau threefold. Instanton effects from Euclidean NS5-branes in type IIA string theory lead to quantum corrections to the hypermultiplet moduli space in Calabi-Yau compactifications of type IIA string theories and are governed by the wave function of the topological B-model [1] [2]. We will instead concentrate on the topological aspects of the partition function on general manifolds with some structure.

D-branes ending on NS5-branes. D-branes can end on NS5-branes [47] [29]. The tension of the NS5-brane is $(2\pi)^{-5}\alpha'^{-3}g_s^{-2}$ while that of a Dp-brane is $(2\pi)^{-p}\alpha'^{(-p-1)/2}g_s^{-1}$, so that NS5-branes are heavier than D-branes in the perturbative regime and the NS5-brane is a solitonic object at weak coupling (i.e. when $\alpha'$ is small). To study the dynamics of D-branes in the vicinity of NS5-branes in the weak string coupling regime, the NS5-brane is usually taken to be static [15] [24]. In particular, a D2-brane ending on an NS5-brane in type IIA string theory can be obtained from the

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dimensional reduction of a system consisting of an M2-brane ending on an M5-brane in M-theory. In fact, the NS5-brane behaves in many respects like the M5-brane, and so insights from one can be used to shed light on the other. We will analyze the first system in a way that is similar to the analysis of the latter in [11]. In particular, we will provide conditions for the NS5-brane partition function to be well-defined. The M-branes admit a description in terms of integral cohomology (however, more recent developments suggest generalized cohomology, see [31] [32]). For the NS5-brane, we have Ramond-Ramond fields on the right hand side of the equation of motion of the B-field. Thus we include in our analysis some effects arising from the K-theoretic description of these fields [18] [27].

Relating classes on the brane to classes on space-time. We will consider the action of the Steenrod square cohomology operations $Sq_i^z : H^m(X; \mathbb{Z}) \to H^{m+i}(X; \mathbb{Z})$ on cohomology classes on the target ten-manifold $X^{10}$ of type IIA string theory. We will also consider the NS5-brane with worldvolume $M^6$, a 6-dimensional oriented manifold, as well as its extension to a 7-manifold $Y^7$ and then to an 8-manifold $Z^8$. All three instances of the (extended) worldvolume admit maps to spacetime since they represent a brane living in the target space-time. Since the Steenrod square is natural, the following diagram is commutative

\[
\begin{array}{ccc}
H^*(W; \mathbb{Z}) & \xrightarrow{f^*} & H^*(X^{10}; \mathbb{Z}) \\
\downarrow{Sq^i} & & \downarrow{Sq^i} \\
H^*(W; \mathbb{Z}) & \xrightarrow{f^*} & H^*(X^{10}; \mathbb{Z})
\end{array}
\]  

(1.1)

where $W$ stands for any of the three manifolds $M^6$, $Y^7$, or $Z^8$. That is, if we have a class $x \in H^*(X^{10}; \mathbb{Z})$ then pulling back to $W$ and applying the Steenrod square gives the same result as applying the Steenrod square and then pulling back the result; this summarized by the relation $f^* Sq^i x = Sq^i f^* x$. Thus, in particular, if we take $x$ to be a degree four class and $i = 3$, then classes on $X^{10}$ that are annihilated by the first differential in the Atiyah-Hirzebruch spectral sequence (as in [11]) pull back to classes on $W$ with the same property.

We will apply the setting of Chern-Simons construction to the NS5-brane. That is, we consider the lift of the NS5-brane worldvolume $M^6$ via the circle to get the 7-dimensional circle bundle $Y^7$, which we take to be a boundary of an 8-manifold $Z^8$. As explained in [46] (mostly for the similar case of the M5-brane), the extension to $Z^8$ can be used to define the theory in six dimensions. Most applications of the fivebrane requires the six-manifold to be a Calabi-Yau space (see e.g. [12]). The spaces $Y^7$ and $Z^8$ should generally have compatible structures.

In section 2 we define a new class $\Pi$, which captures when the partition function is well-defined. The class $\Pi$ is integral for SU($n$) Chan-Paton bundles of even rank and even second Chern class, and its vanishing in cohomology is a necessary condition for the partition function of the NS5-brane to be nonzero. This is equivalent to saying that space-time admits a twisted Fivebrane structure in the sense of [38].

In section 3 we start considering the effect of the K-theoretic description of Ramond-Ramond fields. First, we show in section 3.1 that this requires the cohomology class corresponding to $F_4 \wedge F_4$ to be the mod 4 reduction of an integral class. This uses secondary cohomology operations. Then,
in section 3.2 we show that on an almost complex 8-dimensional manifold the cohomology classes of the fields $F_2$ and $F_6$ satisfy certain constraints arising from the Euler characteristic and the one-loop term.

Some consequences of the description of the fields and the corresponding constraints are considered in section 4. The first effect, described in section 4.1, is that the $F_4$-term in $\Pi$ when annihilated by the first differential in K-theory gives rise to a quadratic refinement of a bilinear form. The topological part of the action of the NS5-brane involving degree four classes is given by a secondary cohomology operation $\Omega$ on classes $z \in H^8(Z^8; \mathbb{Z})$ which are annihilated by the first differential in the Atiyah-Hirzebruch spectral sequence for K-theory. Finally, in section 4.2 we show that the phase of the partition function is captured by the signature via $\frac{1}{8} \sigma(Z^8)$, and is the $\mathbb{Z}_8$-valued phase of the Pontrjagin square operation on the degree four class. We do this in the setting of Gauss sums and cohomology operations.

Due to the similarities between the type IIA NS5-brane and the M-theory M5-brane, a good part of our discussion will apply to the latter as well.

2 The new class and its effect on the partition function

Defining the new class. The equation of motion for the B-field is given by (see [5] [17] [26])

$$d(e^{-2\phi} \ast H_3) = \ell_s^{-2} F_0 \wedge F_8 + \ell_s^2 F_2 \wedge F_6 - \frac{1}{2} \ell_s^4 F_4 \wedge F_4 + \ell_s^4 I_8 ,$$

(2.1)

where $\ell_s$ is the string length (a measure of coupling) and $I_8$ is the one-loop term [14] [44]. Since this expression involves distance (and hence energy) scales, we consider various limits as follows

1. $\ell_s$ is large: $d \ast H_3 \sim -\frac{1}{2} F_4 \wedge F_4 + I_8$.
2. $\ell_s$ is small: $d \ast H_3 \sim F_0 \wedge F_8$; and if $F_0 = 0$ then $d \ast H_3 \sim F_2 \wedge F_6$.

We assume a constant dilaton $\phi$.

The right hand side of the equation of motion (2.1) is the source for the NS5-brane. We define the new class corresponding to the right hand side by passing to de Rham cohomology

$$\Pi := \left[ \ell_s^{-2} F_0 \wedge F_8 + \ell_s^2 F_2 \wedge F_6 - \frac{1}{2} \ell_s^4 F_4 \wedge F_4 + \ell_s^4 I_8 \right]_{dR} .$$

(2.2)

We now explore properties of this class and their effect on corresponding properties of the NS5-brane partition function.

Integrality of the class. We consider whether the class $\Pi$ admits an integral lift, i.e. from de Rham cohomology to integral cohomology. Let us consider each one of the three terms separately. First, the one-loop term $I_8$ is always integral for Spin manifolds [45]. Then for the other two terms we use the K-theoretic quantization of the RR fields [18] [27]

$$F(E) = \sqrt{\hat{A}(X) \ ch(E)} ,$$

(2.3)
corresponding to a Chan-Paton bundle $E$. Considering $E$ to be an SU($n$) bundle, we have that $c_1(E) = 0$. This then implies that the cohomology class corresponding to the term $F_2 \wedge F_6$ will be zero, since $F_2 = c_1(E)$. Thus we still have the first term; consider the degree four RR class

$$F_4(E) = \left[ \sqrt{\hat{A}(X) \, \text{ch}(E)} \right]_4 = -\text{rank}(E) \frac{p_1(X)}{48} + c_2(E).$$

(2.4)

We now make use of the Atiyah-Singer index theorem, which gives that the index of the Dirac operator for Spin manifolds is an even integer in dimension four. As a consequence, $\frac{1}{48} p_1(X)$ is an integer. This, together with the fact that Chern classes are integral classes, implies that $F_4 \in H^4(X; \mathbb{Z})$. This results in the class corresponding to the wedge product $F_4 \wedge F_4$ being integral. However, this does not yet imply that the class of $\frac{1}{2} F_4 \wedge F_4$ is integral, as we are dividing by 2. This division is of no effect if we require the bundles $E$ and $E'$ corresponding to the first $F_4$ and second $F_4$, respectively, to have even rank and even second Chern class. Therefore,

**Proposition 1** The class $\Pi$ is integral for SU($n$) Chan-Paton bundles of even rank and even second Chern class.

**Explicit expression for $\Pi$.** The one-loop term is given by $I_8 = \frac{1}{48} [p_2(X) - \lambda(X)^2]$, where $\lambda(X) = \frac{1}{2} p_1(X)$ is the first Spin characteristic class. Together with above expressions for the quadratic RR fields, this leads to the explicit expression for $\Pi$

$$\Pi = \frac{1}{48} \left[ p_2(X) + \left( \frac{r^2}{12} - 1 \right) \lambda(X)^2 - 4r \lambda(X) c_2(E) + 48 c_2(E)^2 \right],$$

(2.5)

where $r = \text{rank}(E)$. This can be thought of as the analog of the one-loop term $I_8$, or more precisely an analog of the Green-Schwarz anomaly-cancellation term in heterotic and type I string theory [19]. The latter also contains terms that arise from Chern classes of a gauge bundle, there being an $E_8 \times E_8$ bundle, as opposed to an SU($n$) bundle. Note that in relating to eleven-dimensional M-theory, we can choose to retain an $E_8$ bundle– as in [30]– and not perform the breaking down to unitary groups as done in [11].

**Existence of a nonzero partition function.** In [10] a necessary condition for the existence of a nonzero partition function for the M5-brane in M-theory was derived. This decoupling from the bulk is essentially the condition that the class $\Theta_X(C)$ lifting the right hand side of the equation of motion of the C-field be zero. We will perform an analogous analysis for the NS5-brane, which is the result of the vertical dimensional reduction of the M5-brane down to type IIA string theory. Instead of satisfying a quadratic refinement as the class $\Theta$ did, the new class will enjoy index-theoretic properties arising from the K-theoretic quantization of the Ramond-Ramond fields [27 [18]. For the case of the NS5-brane, we require that the D-brane not end on it so that it is decoupled. Thus, in this sense, the D-branes ending on the NS5-branes are the analogs of the M2-brane ending on the M5-brane. We have

**Proposition 2** A necessary condition for the partition function of the NS5-brane to be nonzero is that the class $\Pi$ is zero in cohomology.

Now that we have a condition for a well-defined partition function, we ask when such a condition is satisfied. We provide one general such instance.
Relation to Fivebrane structures. When $\lambda(X) = 0$, that is when we have a String structure, expression (2.5) simplifies to

$$\Pi = \frac{1}{48}p_2(X) + c_2(E)^2. \quad (2.6)$$

We see that setting $\Pi = 0$ is equivalent to setting $\frac{1}{48}p_2(X) + c_2(E)^2$ to zero. The first term is essentially a Fivebrane structure [37] and since the second term is integral, the whole expression defines a twisted Fivebrane structure [38] with the twist given by the integral class $c_2(E)^2$. Note that having a Fivebrane structure implicitly also implies that we already have a String structure (this follows from the way such structures are defined via obstruction theory). Therefore,

**Proposition 3**

(i) The vanishing of the class $\Pi$ is equivalent to a twisted Fivebrane structure.

(ii) A necessary condition for the non-vanishing of the NS5-brane partition function is that space-time admits a twisted Fivebrane structure.

This is the analog for type IIA of the description in [37] [38] for M-theory and heterotic string theory.

3 Constraints on RR fields

We have seen that the expression (2.2) for the new class $\Pi$ involves a combination of Ramond-Ramond fields $F_2, F_4$, and $F_6$ of total degree eight. We will investigate constraints on such fields arising from K-theory and from geometric structures on the underlying space.

3.1 Conditions on $F_4$

We would like to consider some further K-theoretic aspects of the fields entering in the new class $\Pi$. We will first concentrate on the term quadratic in $F_4$. We will take our starting point that the classes representing the RR fields are annihilated by the first differential in the Atiyah-Hirzebruch spectral sequence for K-theory; that is we take the class $x_4$ representing $F_4$ to satisfy $Sq^3 Z x_4 = 0$.

One might ask whether the above condition is all what one should impose. Indeed, this is not the end of the story. In addition to the above primary operation, we will have a secondary operation, valid once the primary one is satisfied. This secondary operation is given by the Toda bracket $\langle Sq^3 Z, Sq^3 Z, x_4 \rangle$. This will in fact take us beyond the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ coefficients encountered in previous literature (see [11] and [16], respectively); we will be dealing with $\mathbb{Z}_4$ coefficients.

**Secondary cohomology operations.** 

If the cohomology class $x$ is such $Sq^3_2 x = 0$ then there is a secondary cohomology operation that can arise and that takes the form

$$\langle Sq^3_2, Sq^3_2, x \rangle = \beta_{Z_2}^2 Sq^4 x. \quad (3.1)$$

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2The author thanks Mike Hill for useful discussion on the material of this section.
This can be seen as follows. Consider the diagram which depicts the various cohomology operations and maps that we have

\[ \Sigma^2 HZ \xrightarrow{\Sigma^2 HZ} X_1 \xrightarrow{b} \Sigma^5 HZ. \]  

(3.2)

Here \( \Sigma^n HZ \) represents the \( n \)-suspension of integral cohomology, which is the same as integral cohomology with degree shifted up by \( n \). We start with a cohomology class \( x \) on \( M \). Applying \( Sq_3 \) to it gives a class \( Sq_3 x \in \Sigma^3 HZ \). The composition \( a \circ b \) is the secondary cohomology operation represented by the Toda bracket \( \langle Sq_3, Sq_3, x \rangle \).

We have the Adem relation \( Sq^2 Sq^3 = Sq^5 + Sq^4 Sq^1 \), which is equal to \( Sq^5 \), since we are working integrally. The spectral sequence is summarized as follows

\[ \begin{array}{c}
0 \\
1 \\
2 \quad Sq^2 \\
3 \quad Sq^3 \\
4 \quad Sq^4 \\
5 \quad Sq^5
\end{array} \xleftarrow{\downarrow} \begin{array}{c}
\iota \\
\iota_2 \\
\iota_3 \\
\iota_4 \\
\iota_5
\end{array} \]

(3.3)

Corresponding to the sequence of coefficients \( \mathbb{Z} \xrightarrow{x_4} \mathbb{Z} \xrightarrow{\rho_4} \mathbb{Z}_4 \), have the following relation between the corresponding cohomology groups

\[ \begin{array}{c}
HZ_4 \\
\Sigma HZ \\
\Sigma HZ_2
\end{array} \xleftarrow{\beta_2} \begin{array}{c}
\beta_2^Z \\
\beta_2^Z \\
\beta_2^Z
\end{array} \xrightarrow{\rho_2} \begin{array}{c}
\rho_2 \\
\rho_2 \\
\rho_2
\end{array} \]

(3.4)

so that \( \beta_2^Z \) is the Bockstein taking cohomology in \( \mathbb{Z}_4 \) coefficients to integral cohomology in one higher degree. The Toda bracket then takes the form

\[ \langle Sq_3^3, Sq_3^3, x_4 \rangle = \beta_2^Z Sq_4^4 x_4 = \beta_2^Z (x_4 \cup x_4). \]  

(3.5)

Therefore, we have
Proposition 4. The K-theoretic description of Ramond-Ramond fields requires the cohomology class corresponding to $F_4 \wedge F_4$ to be the mod 4 reduction of an integral class.

3.2 Conditions on $F_2$ and $F_6$.

We will apply the setting of the Chern-Simons construction to the NS5-brane. That is, we consider the lift of the NS5-brane worldvolume $M^6$ via the circle to get the 7-dimensional circle bundle $Y^7$, which we take to be a boundary of an 8-manifold $Z^8$. As explained in [16] (mostly for the similar case of the M5-brane), the extension to $Z^8$ can be used to define the theory in six dimensions. Most applications of the fivebrane requires the six-manifold to be a Calabi-Yau space (see e.g. [12]). The spaces $Y^7$ and $Z^8$ should generally have compatible structures. Thus we will work with 8-manifolds which admit an almost complex structure (and not necessarily Spin) and ask for constraints on $F_2$ and $F_6$ due to this. This discussion is in spirit (although not in techniques) analogous to the discussion of the one-loop term in [23] [4].

We will use the class of $F_2$ to define a complex line bundle over the NS5-brane worldvolume.

Almost complex structures. An almost complex structure (acs) on a manifold $M$ of dimension $2n$ is an endomorphism $J$ of the tangent bundle $TM$ satisfying $J^2 = -1$. This gives $TM$ the structure of a complex vector bundle. From obstruction theory, an almost complex structure can be viewed as a section of the $\text{SO}(2n)/\text{U}(n)$ bundle associated to $TM$ [39]. On the other hand, a stable almost complex structure (stable acs) on $M$ is a reduction of the structure group of the stable tangent bundle of $M$ from SO to U. The necessary conditions for the existence of a stable acs are the existence of integral lifts (for suitable $i$) $c_i \in H^{2i}(M;\mathbb{Z})$ of the even Stiefel-Whitney classes $w_{2i} \in H^{2i}(M;\mathbb{Z}_2)$, that is $w_{2i}(M) = \rho_2 c_i$, where $\rho_2$ is mod 2 reduction. This is equivalent to saying $W_{2i+1}(M) = 0 \in H^{2i+1}(M;\mathbb{Z})$. Such conditions are natural to impose on branes. In the 8-dimensional case, the two conditions are $W_3 = 0$ and $W_7 = 0$. The latter is automatically satisfied on our manifolds of dimension less than or equal to eight [25].

We will take $u \in H^2(Z^8;\mathbb{Z})$ and $v \in H^6(Z^8;\mathbb{Z})$ be the cohomology classes corresponding to the fields $F_2$ and $F_6$, respectively. Consider $Z^8$ to be an 8-manifold with a stable almost complex structure $\omega$. Let $(u, v)$ to be a pair of class in $H^2(Z^8;\mathbb{Z}) \times H^6(Z^8;\mathbb{Z})$ such that $\rho_2 u = w_2(Z^8)$ and $\rho_2 v = w_6(Z^8)$. From the expression of the Chern character $\chi = \frac{1}{8}(3c_3 - 3c_1 c_2 + c_1^3)$ we see that if $c_2(\omega) = 0 = p_1(Z^8)$ and $3c_3(\omega) = c_1(\omega)^3$, then $u = F_2 = c_1(\omega)$ and $v = F_3 = c_3(\omega)$. In this case, the Chern classes satisfy (see [21])

$$2c_4(\omega) + c_1(\omega) \cdot c_3(\omega) \equiv 0 \pmod{4},$$

$$2\chi(Z^8) + c_1(\omega) \cdot c_3(\omega) \equiv 0 \pmod{4}.$$ 

This follows from a more general formula, using $\rho_2 \chi(Z^8) = \rho_2 c_4(Z^8) = w_8(Z^8)$. Indeed, we will apply the more general main result of [20]: The manifold $Z^8$ has an almost complex structure if and only if [20]

1. $w_8(Z^8) \in Sq^2 H^6(Z^8;\mathbb{Z})$;

2. $\rho_2 u = w_2(Z^8)$ and $\rho_2 v = w_6(Z^8)$;
3. $2\chi(Z^8) + u \cdot v \equiv 0 \pmod{4}$;
4. $8\chi(Z^8) = 4p_2(Z^8) - p_1(Z^8)^2 + 8u \cdot v - u^4 + 2u^2p_1(Z^8)$

The manifold $Z^8$ has a stable acs $\omega$ if the tangent bundle $TZ^8$ is stably isomorphic to the underlying real bundle of some complex vector bundle over $Z^8$. This will be an actual acs if $c_4(\omega) = \chi(Z^8)$.

The above discussion immediately gives the following constraint on the cohomology classes $u$ and $v$ corresponding to the fields $F_2$ and $F_6$, respectively.

**Proposition 5** On an almost complex 8-dimensional manifold, the cohomology classes of the fields $F_2$ and $F_6$ satisfy

$$2u \cdot v - \frac{1}{8} u^4 + u^2 \cdot \lambda(Z^8) = 2\chi(Z^8) - 3I_8(Z^8).$$

In the particular case where the Pontrjagin classes vanish, and hence the one-loop term is zero in cohomology, the constraint simplifies to

$$u \cdot v - \frac{1}{8} u^4 = \chi(Z^8),$$

i.e. the two classes are constrained by the Euler characteristic.

## 4 Consequences for the action and partition function

In this section we consider some topological consequences of the classes we have been using. We will concentrate mostly on effects related to the degree four field $F_4$ and it corresponding cohomology class.

### 4.1 The action via cohomology operations

We will now show that the topological action involving the degree four field is generated by cohomology operations and leads to a quadratic refinement of a bilinear form. We will make use of some of the constructions in [7] and [43].

On $H^4(Z^8; \mathbb{Z})$ we have the relation among Steenrod squares

$$(Sq^2 \rho_2) \circ (\beta Sq^2 \rho_2) = 0,$$

where $\beta$ is the Bockstein. This can be seen as follows. For a degree four class $x$, we have $Sq^2 \rho_2(\beta Sq^2 \rho_2 u) = Sq^2 Sq^1 Sq^2 \rho_2 x$, which gives $Sq^2 Sq^3 \rho_2 x$ by virtue of the Adem relation $Sq^3 = Sq^1 Sq^2$. Another version of Adem relation leads to $Sq^1 Sq^4 \rho_2 x + Sq^4 Sq^1 \rho_2 u$, with the second factor being zero since $Sq^1 \rho_2 = 0$. Since $Sq^4$ acts as squaring on classes of degree $i$, this then gives $Sq^4 \rho_2 x^2 = 0$, again due to $Sq^4 \rho_2$ being identically zero.

Let $\Omega$ denote the secondary cohomology operation associated with the relation (4.1). The operation is defined on the subgroup

$$G(\Omega, Z^8) = \{ x \in H^4(Z^8; \mathbb{Z}) \mid \beta Sq^2 \rho_2 x = 0 \}$$

(4.2)
and takes values in the factor group $H^8(Z^8;\mathbb{Z}_2)/Sq^2\rho_2H^6(Z^8;\mathbb{Z})$, where the quotient is the indeterminacy. The operation $\Omega$ is not uniquely specified by the above relation: $\Omega' = \Omega + Sq^4$ is another secondary cohomology operation associated with the same relation. One can remove this ambiguity by normalizing, i.e. by taking $\rho_2x^2 \in \Omega(x)$ for $x \in H^4(\mathbb{HP}^2;\mathbb{Z})$. Let $z, z' \in H^4(Z^8;\mathbb{Z})$ be degree four classes in the domain of $\Omega$ that is annihilated by $Sq^3$. Then, from [43], the operation $\Omega$ leads to a quadratic refinement

$$\Omega(z + z') = \Omega(z) + \Omega(z') + \{z \cdot z'\},$$ (4.3)

where $\{z \cdot z'\}$ denotes the image of $\rho_2(z \cdot z')$ in $H^8(Z^8;\mathbb{Z}_2)/Sq^2\rho_2H^6(Z^8;\mathbb{Z})$.

The following is analogous to the corresponding statement in M-theory [31]

**Proposition 6** The $F_4$-term in $\Pi$ when annihilated by the first differential in $K$-theory gives rise to a quadratic refinement of a bilinear form.

**The case when $Z^8$ is Spin.** When $Z^8$ is taken to be Spin, then for all $z \in H^4(Z^8;\mathbb{Z})$ the action of mod 2 reduction on the expression appearing in the topological part of the action leads to

$$\rho_2(zq_1(Z^8) - z^2) = w_4(Z^8)\rho_2z - \rho_2z^2$$

$$= Sq^4\rho_2z - \rho_2z^2,$$

which gives zero. Since $H^8(Z^8;\mathbb{Z}) \cong \mathbb{Z}$, there is a unique $y \in H^8(Z^8;\mathbb{Z})$ such that $2y = zq_1(Z^8) - z^2$. Then it makes sense to denote $y = \frac{1}{2}(zq_1(Z^8) - z^2)$. In the Spin case, for every $z \in G(\Omega, Z^8)$ we have

$$\Omega(z) = \rho_2\frac{1}{2}(zq_1(Z^8) - z^2),$$ (4.4)

which is the topological part of the action, in the M5-brane case [46] 22. Therefore,\footnote{While this is done for the case when $Z^8$ has no boundary, we will consider the more general case in the next section.}

**Proposition 7** The topological part of the action of the NS5-brane involving degree four classes is given by the secondary cohomology operation $\Omega$ on classes $z \in H^8(Z^8;\mathbb{Z})$ which are annihilated by the first differential in the Atiyah-Hirzebruch spectral sequence for $K$-theory.

### 4.2 Quadratic refinements and the partition function

Having identified various structures associated with the topological action, we now turn to the partition function on which we consider the corresponding topological effects. Using this, we will show that the phase of the partition function, given by one eighth of the signature of the 8-manifold, is given by a secondary cohomology operation. The discussion in this section also applies to the M5-brane.

**Reduction of the Hirzebruch L-polynomial.** The topological action will involve the Hirzebruch L-polynomial of $Z^8$, via $\frac{1}{4}L_2(Z^8)$ as in [22]. We would like to characterize the division by 8. The division by 2 and 4 are easy to describe via reduction mod 2 and mod 4, respectively. Indeed, from the formulas in [42] we have

$$\rho_2L_2 = \rho_2(\mathcal{P}(v_4)) = v_4^2 \in H^8(\text{BSO};\mathbb{Z}_2).$$

$$\rho_4L_2 = \mathcal{P}(v_4) + i_4(w_8 + w_6w_2 + w_4^4) \in H^8(\text{BSO};\mathbb{Z}_4),$$
Here $P$ is the Pontrjagin square cohomology operation

$$P : H^4(Z^8; \mathbb{Z}_2) \to \mathbb{Z}_4$$

and $i : \mathbb{Z}_2 \to \mathbb{Z}_4$ is the inclusion. What we need is reduction modulo 8. We study this using the classic constructions in [6].

A partition function in our context is essentially an instance of a Gauss sum.

**Gauss sums.** Consider a finite group $T$ and a function $\psi : T \to \mathbb{R}/\mathbb{Z}$. The associated Gauss sum corresponding to $\psi$ is given by

$$G(\psi) = \sum_{x \in T} e^{2\pi i \psi(x)} .$$

This expression splits into a magnitude or norm, $N(\psi) = |G(\psi)|$, and a phase $\beta(\psi) \in \mathbb{R}/\mathbb{Z}$ (which can be defined only if the norm is nonzero) via

$$G(\psi) = N(\psi) \cdot e^{2\pi i \beta(\psi)} .$$

Let $K$ be a subgroup of $T$ the restriction of $\psi$ to which is zero, i.e. $\psi|_K = 0$. Then it can be shown (see [40]) that the Gauss sum is determined by the restriction to the complement

$$G(\psi) = |K| \cdot G(\psi|_{K^\perp/K}) .$$

Note that $\psi_{K^\perp/K} : K^\perp/K \to \mathbb{Q}/\mathbb{Z}$ is well-defined since $K \subset K^\perp$.

**The phase when $Z^8$ closed.** Consider the case when $Z^8$ is taken to be closed, in addition to being connected and oriented. The cup product pairing $H^4(Z^8; \mathbb{Z}) \times H^4(Z^8; \mathbb{Z}) \to \mathbb{Z}_2$ admits a quadratic refinement given by the Pontrjagin square $P$. By Morita’s proof [28] of Brown’s conjecture, the phase of the Pontrjagin square is given by

$$\beta(P) = \frac{\sigma(Z^8)}{8} \in \mathbb{Q}/\mathbb{Z} ,$$

where $\sigma(Z^8)$ is the signature of the 8-manifold $Z^8$. This can be shown as follows. Let $K$ denote the image in $H^4(Z^8; \mathbb{Z}) \otimes \mathbb{Z}_2$ of of the torsion in $H^4(Z^8; \mathbb{Z})$. Then $K^\perp/K = (H^4(Z^8; \mathbb{Z})/\text{torsion}) \otimes \mathbb{Z}_2$. Now since $H^4(Z^8; \mathbb{Z}) \otimes \mathbb{Z}_2 \subset H^4(Z^8; \mathbb{Z}_2)$ the Pontrjagin square is just the cup product square reduced mod 4. This implies that $P$ vanishes on $K$. Therefore, if $B$ is the bilinear form on $H^4(Z^8; \mathbb{Z})/\text{torsion}$ then the refinement induced by $P$ on $K^\perp/K$ is simply $B \otimes 1_2$.

**The phase when $Z^8$ has boundary.** We work with relative cohomology of the pair $(Z^8, Y^7)$, where $Y^7$ is the boundary of $Z^8$. In this case, the Pontrjagin square $P : H^4(Z^8, Y^7; \mathbb{Z}_2) \to \mathbb{Z}_4$ is still the quadratic refinement of the cup product pairing. However, in this case we distinguish two cases, depending on whether or not the cohomology of $Y^7$ has torsion in degree four.
1. **Torsion-free case.** If $H^4(Y^7; \mathbb{Z})$ is torsion-free, then \((4.5)\) holds. This can be seen as follows (see [40] for a general discussion). When there is a boundary, the cup product pairing has an annihilator: if $T = H^4(Z^8, Y^7; Z_2)$, then $T^\perp$ is the image of $H^3(Y^7; Z_2)$ in $H^4(Z^8, Y^7; Z_2)$. Let us start with a degree three cohomology class $x \in H^4(Y^7; Z_2)$ and get a $\mathbb{Z}_4$ quantity in two different ways. On the one hand, from the cohomology exact sequence corresponding to the relative pair $(Z^8, Y^7)$, we can map $x$ to a class in $H^4(Z^8, Y^7; Z_2)$, on which we can apply the Pontrjagin square to get

$$H^3(Y^7; Z_2) \to H^4(Z^8, Y^7; Z_2) \xrightarrow{\mathcal{P}} H^8(Z^8, Y^7; \mathbb{Z}_4) = \mathbb{Z}_4. \quad (4.10)$$

On the other hand, starting with $x$, we take its cup product with the degree four class $\text{Sq}^4 x$ to get a class in $H^7(Y^7; Z_2)$

$$H^3(Y^7; Z_2) \xrightarrow{x \cup \text{Sq}^4 x} H^7(Y^7; Z_2) = \mathbb{Z}_2 \subset \mathbb{Z}_4. \quad (4.11)$$

It follows from the general results of [41] that the two compositions, \((4.10)\) and \((4.11)\), are equal. If $H^4(Y^7; \mathbb{Z})$ is torsion-free then $\text{Sq}^4 x = 0$, so that $\mathcal{P}$ vanishes on $T^\perp$. We can identify $T/T^\perp$ with the image of $H^4(Z^8, Y^7; Z_2)$ in $H^4(Z^8; Z_2)$.

The cup product pairing on $H^4(Z^8, Y^7; Z_2)$ comes from a bilinear pairing

$$\lambda : H^4(Z^8, Y^7; Z_2) \otimes H^4(Z^8, Y^7; Z_2) \to \mathbb{Z}_2 \quad (4.12)$$

whose adjoint is an isomorphism. If $A \subset H^4(Z^8; Z_2)$ denotes the image of the torsion subgroup of $H^4(Z^8; \mathbb{Z})$ in $H^4(Z^8; Z_2)$, the $A^\perp \subset H^4(Z^8, Y^7; Z_2)$ is the image of $H^4(Z^8, Y^7; \mathbb{Z})$.

The enhancement $\mathcal{P}$ is related to the mod 2 reduction of the integral form. If there is no 2-torsion in $H^4(Y^7; \mathbb{Z})$ then the image of $H^4(Z^8, Y^7; \mathbb{Z})$ in $H^4(Z^8; \mathbb{Z})/\text{torsion}$ has a nonsingular bilinear pairing with a matrix $B_Z$ of determinant $\det B_Z = \pm 1$. This gives that $\mathcal{P}$ and $\psi_{B_Z}$ are equivalent.

2. **Torsion case.** We now consider the case when $H^4(Y^7; \mathbb{Z})$ has torsion, with torsion subgroup $T^4(Y^7)$. The various cosmology groups are related via the maps $i : T^4(Z^8) \to T^4(Y^7)$ and $j^* : H^4(Z^8, Y^7; \mathbb{Z}) \to H^4(Z^8; \mathbb{Z})$. Take $\psi : T^4(Y^7) \to \mathbb{Q}/\mathbb{Z}$ to be a quadratic function over the linking pairing $L : T^4(Y^7) \times T^4(Y^7) \to \mathbb{Q}/\mathbb{Z}$. Let $\hat{v}_4 \in H^4(Z^8, Y^7; Z_2)$ and $v_4 \in H^4(Z^8; \mathbb{Z})$ be liftings of the Wu class $v_4 \in H^4(Z^8; Z_2)$ to relative mod 2 cohomology and integral absolute cohomology, respectively, compatible with $\psi$. Compatibility with $\psi$ means that for all $x \in T^4(Y^7)$ such that $x = i^*(y)$ where $y \in H^4(Z^8, \mathbb{Z})$, we have

$$\psi(x) = \frac{1}{2} \langle y \cdot \hat{v}_4, [Z^8, Y^7]\rangle - \frac{1}{2} \langle y \cdot j^* y^{-1}, [Z^8, Y^7]\rangle \in \mathbb{Q}/\mathbb{Z}. \quad (4.13)$$

Such a lifting always exist [6]. As a consequence, we have $j^*(\hat{v}_4) = \rho_2(v_4^2)$; there is a class $b \in T^4(Y^7)$ such that $\psi i(x) = L(b, i(x))$ for $x \in T^4(Z^8)$, and a class $t \in T^4(Z^8)$ such that $i^*(t) = 2b$.

For $b \in H^4(Y^7; \mathbb{Z})$ and $z \in H^4(Z^8, Y^7; Z_2)$ have $\langle \mathcal{P}(z), [Z^8, Y^7]\rangle = L(b, b) \in \mathbb{Z}_4$. Now the index is given by $I(Z^8) = \langle v_4^2, [Z^8, Y^7]\rangle + L(b, b) - \langle G(T^4(Y^7), \psi)\rangle \in \mathbb{Z}_4$. Therefore, the signature mod 8 is given by the formula

$$\sigma(Z^8) = \langle (v_4^2, [Z^8, Y^7] + \psi(b)) - \langle G(T^4(Y^7), \psi)\rangle \rangle \in \mathbb{Z}_8. \quad (4.14)$$

We summarize our discussion above with the following
Proposition 8 The phase of the partition function, given by $\frac{1}{8} \sigma(Z^8)$, can be described as the phase of the Pontryagin square operation on the degree four class. In the case when $H^4(Y^7; \mathbb{Z})$ has torsion, the formula is given by (4.14).

The first part of the statement can be viewed as another way of viewing the phase vs. [46, 22], in the case when the Wu class can be set to zero (see [36] for a discussion of when this can occur). Other effects of the Pontryagin square in the case of the M5-brane is considered in [34]. The general index problem in the presence of a boundary can be described in a way analogous to the discussion in [35] for the case of M-theory.

Acknowledgements

The author would like to thank Mike Hill for useful discussions on secondary cohomology operations. He also acknowledges the kind hospitality of the American Institute of Mathematics, Palo Alto, and IHES, Bures-sur-Yvette, during the work on this project. This research is supported by NSF Grant PHY-1102218.

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