ON THE CONTINUITY OF CENTER-OUTWARD DISTRIBUTION AND QUANTILE FUNCTIONS

ALESSIO FIGALLI

To Carlo Sbordone, for his 70th birthday.

Abstract. To generalize the notion of distribution function to dimension \( d \geq 2 \), in the recent papers [6, 12] the authors propose a concept of center-outward distribution function based on optimal transportation ideas, and study the inferential properties of the corresponding center-outward quantile function. A crucial tool needed in [12] to derive the desired inferential properties is the continuity and invertibility for the center-outward quantile function outside the origin, as this ensures the existence of closed and nested quantile contours. The aim of this paper is to prove such a continuity and invertibility result.

AMS 1980 subject classification: 62G35, 35J96.
Keywords: Measure transportation; multivariate distribution function; multivariate quantiles; gradient of convex functions.

1. Introduction

Starting with dimension \( d = 2 \), the traditional definition of a distribution function, based on marginal orderings, is unsatisfactory on many counts. Indeed, the ranks induced by its empirical counterpart do not enjoy the properties that make traditional (univariate) ranks a successful tool of inference, while the corresponding quantile function—in the Lebesgue-absolutely continuous case, the inverse of the distribution function—does not exhibit the equivariance behaviour one is expecting from a quantile; see [7, 12]. This fact, which results from the absence of a canonical ordering of \( \mathbb{R}^d \), has been recognized long ago, and a number of ingenious alternative definitions—all of them reducing, for dimension \( d = 1 \), to the traditional univariate definition—have been considered in the statistical literature. None of them, however, is preserving the inferential properties of their univariate counterparts; see [12] for a survey.

Motivated by this lack of a statistically sound definition, first in [6] and then in [12], the authors proposed a new concept of center-outward distribution function based on optimal transportation ideas. The starting point is the fact that, denoting by \( F \) the traditional distribution function associated with an absolutely continuous distribution \( P \) on the real line (namely, \( F(z) = P((\infty, z], z \in \mathbb{R}) \)), then \( 2F - 1 \) is pushing \( P \) forward to the uniform distribution \( U_1 \) over \([0,1]\), that one can interpret as the unit ball in \( \mathbb{R} \). As the map \( 2F - 1 \) is monotone increasing, a classical result in optimal transportation theory [13] implies that this map is the unique gradient of a convex function mapping \( P \) onto \( U_1 \). Note that, whereas \( F(z) = P((\infty, z]) \) yields the probability of nested halflines of the form \((\infty, z] \), the map \( 2F - 1 \) is related to intervals of the form \([z^{-}, z^{+}] \) with \( F(z^{-}) + F(z^{+}) = 1 \), whence the terminology center-outward distribution function.

The definition of a center-outward distribution function as the unique gradient of function (denoted as \( F_\pm \)) pushing \( P \) to the uniform measure over the unit ball readily extends to absolutely continuous distributions over \( \mathbb{R}^d \); here, with the name uniform measure over the unit ball, we mean the measure \( U_\mathbb{B} \) obtained by considering the product of the uniform measure over the unit sphere...
and the uniform over the unit interval \([0,1]\). In other words, by the change of variable formula,

\[
U_d = u_d(x) \, dx \quad \text{with} \quad u_d(x) = \frac{c_d}{|x|^{d-1}} \mathbf{1}_{B_1}(x),
\]

where \(c_d = 1/\mathcal{H}^{d-1}(S^{d-1})\) is a dimensional normalizing constant (here \(\mathcal{H}^{d-1}(S^{d-1})\) denotes the area of the \((d-1)\)-dimensional unit sphere). The corresponding **center-outward quantile function** is then defined as the inverse \(Q_\pm := F_\pm^{-1}\). The properties of \(F_\pm\) have been studied, under the assumption of \(P\) being compactly supported, in [6]; such assumption has then been relaxed in [12], where it is shown that \(F_\pm\) and \(Q_\pm\) (and their empirical counterparts), contrary to all previous concepts that have been proposed in the literature, do enjoy the inferential properties expected from distribution and quantile functions in \(\mathbb{R}^d\). We refer to [12] for more details.

It is important to observe that, in order to derive these inferential properties, a fundamental fact needed in [12] is the fact that \(Q_\pm\) is a homeomorphism from \(B_1 \setminus \{0\}\) onto its image. Indeed, this ensures the existence of closed and nested **quantile contours**, obtained as the images under \(Q_\pm\) of the nested hyperspheres \(\{\partial B_r\}_{0 < r < 1}\). The objective of this paper is to prove this continuity property needed in [12]. We note that, although several fundamental results have been obtained in the last 25 years on the regularity of optimal transport maps (see [8, 9] for a survey), the proof of the above-mentioned property is rather delicate, due to the fact that the density of \(U_d\) is singular at the origin whenever \(d \geq 2\).

We recall that, given two absolutely continuous probability densities on \(\mathbb{R}^d\), there exists a unique transport map that pushes forward one density onto the other and which coincides almost everywhere with the gradient of a convex function (see [13]). We shall refer to this map as the **optimal transport map**, being implicit that this is the optimal transport map for the quadratic Euclidean cost (see [8] for more details).

Here is our main result.\(^1\)

**Theorem 1.1.** Let \(U_d\) be the uniform measure on \(B_1\) (see (1.1)), and let \(P = p(y)dy\) be a probability measure on \(\mathbb{R}^d\) satisfying \(0 < \lambda_R \leq p \leq \Lambda_R\) inside \(B_R\) for all \(R < \infty\). Let \(Q_\pm = \nabla \varphi: B_1 \to \mathbb{R}^d\) be the unique optimal transport map from \(U_d\) to \(P\). Then \(Q_\pm\) is a homeomorphism from \(B_1 \setminus \{0\}\) onto \(\mathbb{R}^d \setminus K\), where \(K\) is a compact convex set of Lebesgue measure zero.

In addition:

(a) If \(p \in C^{k,\alpha}_{\text{loc}}(\mathbb{R}^d)\) for some \(k \geq 0\) and \(\alpha \in (0,1)\), then \(Q_\pm: B_1 \setminus \{0\} \to \mathbb{R}^d \setminus K\) is a diffeomorphism of class \(C^{k+1,\alpha}_{\text{loc}}\) inside \(B_1 \setminus \{0\}\), and

\[
\det(\nabla Q_\pm(x)) = \frac{u_d(x)}{p(Q_\pm(x))} \quad \forall \, x \in B_1 \setminus \{0\}.
\]

(b) If \(p\) is locally analytic, then \(Q_\pm: B_1 \setminus \{0\} \to \mathbb{R}^d \setminus K\) is locally an analytic map.

(c) If \(d = 2\) then \(K = \{Q_\pm(0)\}\) and \(Q_\pm\) is a homeomorphism from \(B_1\) onto \(\mathbb{R}^2\).

**Remark 1.1.** When \(P = p(|y|)dy\) has a radial density, then also the map \(Q_\pm\) is radial (this follows from the uniqueness of the optimal transport map) and the above result is elementary. Indeed, in this case one can explicitly write the optimal map in terms of distribution function, and the explicit formula is given by

\[
Q_\pm(x) = q_\pm(|x|) \frac{x}{|x|},
\]

\(^1\)Here and in the sequel, \(|E|\) stands for the Lebesgue measure of a Borel set \(E\). Also, given \(k \geq 0\) and \(\alpha \in (0,1)\), we say that a function \(f\) belongs to \(C^{k,\alpha}_{\text{loc}}(\mathbb{R}^d)\) if \(f \in C^k(\mathbb{R}^d)\) and its \(k\)-th derivative is locally \(\alpha\)-Hölder continuous, namely

\[
\forall \, R > 0, \quad \sup_{x \neq y, \, x, y \in B_R} \frac{|D^k f(x) - D^k f(y)|}{|x-y|^{\alpha}} < \infty.
\]
where \( q_{\pm} : [0, 1] \to [0, \infty) \) is implicitly defined via the identity
\[
s = H^{d-1}(S^{d-1}) \int_0^{q_{\pm}(s)} r^{d-1} p(r) \, dr \quad \forall s \in (0, 1).
\]
Hence, in this very particular case, the conclusions of Theorem 1.1 hold with \( K = \{0\} \), as can easily be checked by direct computations.

**Remark 1.2.** As shown in point (c) of Theorem 1.1, in the case \( d = 2 \) the map \( Q_{\pm} \) is a homeomorphism up to the origin. It is a well-known fact that the Monge-Ampère equation behaves better in dimension two than in higher dimensions (see for instance [9, Sections 2.5 and 3.2]), and we do not expect Theorem 1.1(c) to be true in dimension \( d \geq 3 \). However, finding a counterexample would not be relevant to the problem under investigation (namely, the existence of quantile contours as the images of the sphere \( \{\partial B_r\}_{0<r<1} \) under \( Q_{\pm} \)), so we shall not investigate this question here.

## 2. Proof of Theorem 1.1

To prove our main theorem, we first introduce some notation: given a convex function \( \psi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\} \), the Monge-Ampère measure \( \mu_\psi \) associated to \( \psi \) is defined by
\[
\mu_\psi(A) = |\partial \psi(A)| \quad \text{for every Borel set } A \subset \mathbb{R}^n,
\]
where
\[
\partial \psi(A) := \bigcup_{x \in A} \partial \psi(x)
\]
and \( \partial \psi(x) \) denotes the subdifferential of \( \psi \) at \( x \), that is
\[
\partial \psi(x) := \{ p \in \mathbb{R}^n : \psi(z) \geq \psi(x) + \langle p, z-x \rangle \quad \forall z \in \mathbb{R}^n \}.
\]
Note that \( \partial \psi(x) \) is a convex set for any \( x \in \mathbb{R}^d \). Also, when \( \psi \) is of class \( C^2 \), the Monge-Ampère measure of \( \psi \) is given by \( \det(D^2\psi)dx \) (see [9, Example 2.2]).

**Proof of Theorem 1.1.** First of all we note that, since the optimal map \( Q_{\pm} = \nabla \varphi \) is unique a.e. inside \( B_1 \), the function \( \varphi \) is unique inside \( B_1 \) up to an additive constant. In particular, with no loss of generality we can set \( \varphi(0) = 0 \).

Outside \( B_1 \) we simply extend \( \varphi \) to be identically equal to \( +\infty \) (note that this preserves the convexity of \( \varphi \) ), and we consider the Legendre transform of \( \varphi \), namely
\[
\varphi^*(y) := \sup_{x \in \mathbb{R}^n} \{ \langle y, x \rangle - \varphi(x) \} = \sup_{x \in B_1} \{ \langle y, x \rangle - \varphi(x) \} \quad \forall y \in \mathbb{R}^d
\]
(the second equality follows from the fact that \( \varphi = +\infty \) on \( \mathbb{R}^n \setminus B_1 \) ). It is well known that \( \nabla \varphi^* \) is the optimal transport map from \( P \) onto \( U_d \), and that \( \nabla \varphi^* = (\nabla \varphi)^{-1} \) a.e. (see for instance [1, Section 6.2.3 and Remark 6.2.11]). In particular,\n\[
|\nabla \varphi^*| \leq 1 \quad \text{a.e. in } \mathbb{R}^d.
\]
Note that, because \( U_d \) is supported in \( B_1 \) which is a convex set, it follows by [4] (see also Step 1 in the proof of [9, Theorem 4.23]) that \( \varphi^* \) is an Alexandrov solution to the Monge-Ampère equation inside \( \mathbb{R}^d \), namely (recall (1.1))
\[
\mu_{\varphi^*}(A) = \int_A \frac{p(y)}{u_d(\nabla \varphi^*(y))} \, dy = \frac{1}{cd} \int_A p(y)|\nabla \varphi^*(y)|^{d-1} \, dy \quad \text{for all } A \subset \mathbb{R}^d \text{ Borel.}
\]
Let \( K := \partial \varphi(0) \), and observe that \( K \) is a closed convex. Also, since \( \varphi \) is locally Lipschitz in a neighborhood of the origin (being a finite convex function inside \( B_1 \) ), it follows that \( K \) is bounded.
Consider an arbitrary compact set $C \subset \mathbb{R}^d \setminus K$. First of all, we note that $\partial \varphi^*(C)$ is a compact set (see [9, Lemma A.22]). Also, thanks to (2.4) it follows by [9, Corollary A.27] that
\begin{equation}
\partial \varphi^*(C) \subset \partial \varphi^*(\mathbb{R}^d) \subset \overline{B}_1.
\end{equation}
Furthermore, because $\partial \varphi$ and $\partial \varphi^*$ are inverse to each other (see [9, Equation (A.20)]), noticing that $C \cap \partial \varphi(0) = C \cap K = \emptyset$ we deduce that $\partial \varphi^*(C) \cap \{0\} = \emptyset$.

In conclusion, this proves that $\partial \varphi^*(C)$ is a compact set satisfying $\partial \varphi^*(C) \subset \overline{B}_1 \setminus B_\rho$, for some $\rho > 0$ depending on $C$. In particular, this implies that $\rho \leq |\nabla \varphi^*(y)| \leq 1$ for a.e. $y \in C$. Hence, recalling that $p$ is locally bounded away from zero and infinity, thanks to (2.5) we obtain
\begin{equation}
m_C|A| \leq \mu_{\varphi^*}(A) \leq M_C|A|
\end{equation}
for all $A \subset C \subset \subset \mathbb{R}^d \setminus K$ Borel, for some constants $0 < m_C \leq M_C$.

In order to apply the regularity theory for the Monge-Ampère equation from [9, Chapter 4], we first need to prove that $\varphi^*$ is strictly convex inside $\mathbb{R}^d \setminus K$. Assume this is false. Then there exists $\hat{y} \in \mathbb{R}^d \setminus K$ and $\hat{q} \in \partial \varphi^*(\hat{y}) \subset \overline{B}_1$ such that, if we consider the affine function $\ell(z) := \varphi^*(\hat{y}) + \langle \hat{q}, z - \hat{y} \rangle$, the convex set $\Sigma := \{\varphi^* = \ell\}$ is not a singleton.

Notice that, thanks to (2.7), [9, Theorem 4.10] applies inside any compact subset of $\mathbb{R}^n \setminus K$, so the convex set $\Sigma$ cannot have any exposed point in $\mathbb{R}^d \setminus K$. Hence, the only possibilities are the following:
(a) either $\Sigma$ contains an infinite half-line $L$ going from $K$ to infinity;
(b) or $\Sigma$ contains an infinite line.

In case (b), [9, Lemma A.25] yields that $\partial \varphi^*(\mathbb{R}^d)$ is contained inside a hyperplane, contradicting the fact that $\nabla \varphi^*$ transports $P$ onto the measure $U_\ell$ which is supported on the whole unit ball.

We now need to exclude that case (a) occurs. The argument in this case is inspired by [10].

With no loss of generality, up to a translation and rotation, we can assume that $0 \in K$ and that $L = \{te_1 : t \geq 0\}$. By the monotonicity of the subdifferential of convex functions it follows that, given two points $y_1$ and $y_2$,
\begin{equation}
\langle q_2 - q_1, y_2 - y_1 \rangle \geq 0 \quad \forall q_i \in \partial \varphi^*(y_i), \; i = 1, 2.
\end{equation}
Since $\varphi^* = \ell$ on $L$, it follows that $\hat{q} = \nabla \ell \in \partial \varphi^*(\hat{q})$ for all $q \in L$. Hence, applying (2.8) with $y_1 = te_1 \in L$, $q_1 = \hat{q}$, and $y_2 = y$ an arbitrary point in $\mathbb{R}^d$, we get
\begin{equation}
\langle q - \hat{q}, y - te_1 \rangle \geq 0 \quad \forall y \in \mathbb{R}^d, \; q \in \partial \varphi^*(y), \; t \geq 0.
\end{equation}
Letting $t \to +\infty$ in the above inequality we deduce that
\begin{equation}
\langle q - \hat{q}, e_1 \rangle \leq 0 \quad \forall y \in \mathbb{R}^d, \; q \in \partial \varphi^*(y).
\end{equation}
Thus, we proved that $\partial \varphi^*(\mathbb{R}^d)$ is contained inside the half-space
\begin{equation}
H := \{q : \langle q - \hat{q}, e_1 \rangle \leq 0\}.
\end{equation}
Recalling (2.4), this implies that $\nabla \varphi^*$ takes values inside $H \cap \overline{B}_1$ a.e. Since $(\nabla \varphi^*)_#P = U_\ell$ and $u_\ell$ is strictly positive inside $\overline{B}_1$, it follows that $H \cap \overline{B}_1 = \overline{B}_1$, which is possible if and only if $\hat{q} = e_1 \in \partial B_1$ (recall that $\hat{q} \in \partial \overline{B}_1$, see (2.6)).

Let $\theta > 0$ small, and consider the cone
\begin{equation}
\mathcal{C}_\theta := \{y \in B_1 : |y| \leq (1 + \theta)\langle y, \hat{q} \rangle\}.
\end{equation}
Since $0 \in L$ we have $\hat{q} = \nabla \ell \in \partial \varphi^*(0)$, so applying (2.8) with $y_1 = 0$ and $q_1 = \hat{q} \in \partial B_1$, we obtain
\begin{equation}
\langle \nabla \varphi^*(y) - \hat{q}, y \rangle \geq 0 \quad \forall y \text{ where } \varphi^* \text{ is differentiable}.
\end{equation}
Combining this inequality with the definition of $\mathcal{C}_\theta$ and (2.4), we deduce that
\begin{equation}
\nabla \varphi^*(y) \in \mathcal{D}_\theta := \{x \in B_1 : \langle x - \hat{q}, \hat{q} \rangle \geq -C_d\theta|x - \hat{q}|\}\quad \text{for a.e. } y \in \mathcal{C}_\theta,
\end{equation}
where $C_d > 0$ is a dimensional constant (cp. [10, Figure 1]). Because $p \geq \lambda_1$ inside $C_\theta \subset B_1$ and $u_d \leq 2c_d$ inside $\mathcal{D}_\theta$ for $\theta$ small enough, it follows by the transport condition $(\nabla \varphi^* )_\# P = U_d$ and by (2.9) that

$$2c_d |\mathcal{D}_\theta| \geq \int_{\mathcal{D}_\theta} u_d(x) \, dx = \int_{(\nabla \varphi^* )^{-1}(\mathcal{D}_\theta)} p(y) \, dy \geq \int_{\mathcal{D}_\theta} p(y) \, dy \geq \lambda_1 |\mathcal{E}_\theta|.$$ 

Since $|\mathcal{E}_\theta| \sim \theta^{n-1}$ and $|\mathcal{D}_\theta| \sim \theta^n + 1$, we obtain a contradiction for $\theta$ small enough. This proves that also case (a) is impossible, thus $\varphi^*$ is strictly convex inside $\mathbb{R}^d \setminus K$.

Since $\varphi^*$ is a strictly convex Alexandrov solution of (2.7), it follows by [2, 3, 4] (see also [9, Corollary 4.21]) that $\varphi^*$ is of class $C^{1,\alpha}$ inside $\mathbb{R}^d \setminus K$. In particular, $\nabla \varphi^*$ is continuous inside $\mathbb{R}^d \setminus K$. Since, by the strict convexity of $\varphi^*$ inside $\mathbb{R}^d \setminus K$, $\nabla \varphi^*$ is an injective continuous map from $\mathbb{R}^d \setminus K$ onto $B_1 \setminus \{0\}$, we deduce that $\nabla \varphi^* : \mathbb{R}^d \setminus K \to B_1 \setminus \{0\}$ is a homeomorphism by the theorem on the invariance of domain. Recalling that $Q_\pm = \nabla \varphi = (\nabla \varphi^*)^{-1}$, we conclude that $Q_\pm$ is a homeomorphism from $B_1 \setminus \{0\}$ onto $\mathbb{R}^d \setminus K$.

To prove that $K$ has Lebesgue measure zero it suffices to observe that $K = (\nabla \varphi^*)^{-1}(\{0\})$, so by the transport condition $(\nabla \varphi^* )_\# P = U_d$ we get

$$\int_K p(y) \, dy = \int_{\{0\}} u_d(x) \, dx = 0.$$ 

Since $p > 0$ we conclude that $|K| = 0$, as desired.

We now prove the additional statements in the theorem.

- **Proof of (a).** We note that if $p \in C^{k,\alpha}_{\text{loc}}(\mathbb{R}^d)$ for some $k \geq 0$ and $\alpha \in (0,1)$, then [9, Remark 4.25] implies that $\nabla \varphi^*$ is a $C_{\text{loc}}^{k+1,\alpha}$ diffeomorphism from $\mathbb{R}^d \setminus K$ onto $B_1 \setminus \{0\}$, hence $Q_\pm : B_1 \setminus \{0\} \to \mathbb{R}^d \setminus K$ is a diffeomorphism of class $C_{\text{loc}}^{k+1,\alpha}$.

Since $Q_\pm|_{B_1 \setminus \{0\}}$ is a $C^1$ diffeomorphism, the validity of (1.2) is classical, and we give here a short proof for completeness. Since $\nabla \varphi^*$ is of class $C^1$ outside $K$, it follows by [9, Example 2.2] and (2.5) that

$$\int_A \det(D^2 \varphi^*(y)) \, dy = \int_A \frac{p(y)}{u_d(\nabla \varphi^*(y))} \, dy \quad \text{for all } A \subset (\mathbb{R}^d \setminus K) \text{ Borel}.$$ 

By the arbitrariness of $A$, this yields

$$\det(D^2 \varphi^*(y)) = \frac{p(y)}{u_d(\nabla \varphi^*(y))} \quad \forall y \in \mathbb{R}^d \setminus K.$$ 

Since $Q_\pm = \nabla \varphi = (\nabla \varphi^*)^{-1}$, for any $x \in (\nabla \varphi)^{-1}(\mathbb{R}^d \setminus K) = B_1 \setminus \{0\}$ we obtain

$$\det(\nabla Q_\pm(x)) = \det(D^2 \varphi(x)) = \frac{1}{\det(D^2 \varphi^*(\nabla \varphi(x)))} = \frac{u_d(x)}{p(\nabla \varphi(x))} = \frac{u_d(x)}{p(Q_\pm(x)).}$$ 

This proves (1.2), concluding the proof of (a).

- **Proof of (b).** It follows by [9, Proposition A.43] that the Monge-Ampère equation is uniformly elliptic on $C^2$ solutions. Hence, by the classical analytic regularity of solution to uniformly elliptic PDEs with analytic data [15], if $p$ is locally analytic then so is $Q_\pm$.

- **Proof of (c).** We now focus on the case $d = 2$. In this part we shall use coordinates $x = (x_1, x_2)$ and $y = (y_1, y_2)$ to denote points in $\mathbb{R}^2$.

\[\text{Actually, since } \partial \varphi^*(K) = \{0\}, \text{ it follows by the continuity of the subdifferential (see [9, Equation (A.15)]) that } \nabla \varphi^* \text{ is continuous on the whole space } \mathbb{R}^d, \text{ with } \nabla \varphi^*(y) = 0 \text{ for all } y \in K.\]
Assume by contradiction that $K$ is not a point. Since $K$ is a compact convex set of Lebesgue measure zero it must be a segment, say $K = [−1, 1] \times \{0\}$. With no loss of generality, we can assume that $\varphi(0) = 0$. Recalling that $K = \partial \varphi(0)$, this implies that

$$
\varphi(x_1, x_2) \geq |x_1| \quad \forall x = (x_1, x_2) \in \mathbb{R}^2.
$$

(2.10)

Also, since $\varphi : B_1 \to \mathbb{R}$ is convex, there exists a constant $R > 0$ such that

$$
|\nabla \varphi(x)| \leq R \quad \forall x \in B_{1/2}.
$$

(2.11)

Now, given $\delta \in (0, 1/4]$, we define

$$
h_\delta := \varphi(0, \delta), \quad \ell_\delta := \frac{h_\delta}{\delta}.
$$

Note that, because $\varphi$ is differentiable in the $x_2$ variable, $\ell_\delta \to 0$ as $\delta \to 0$.

Given $R$ as in (2.11), we set

$$
\mathcal{R}_\delta := [-h_\delta, h_\delta] \times [0, (1 + R)\delta].
$$

With this definition, thanks to (2.10) and (2.11) we can apply [11, Lemma 2.3] to deduce that

$$
\partial \varphi(\mathcal{R}_\delta) \supset [-1/2, 1/2] \times \left[0, \frac{\ell_\delta}{2(1 + R)}\right].
$$

Since $\nabla \varphi = \mathbf{Q}_\pm$ is differentiable outside the origin and $\partial \varphi(0) = [-1, 1] \times \{0\}$, this implies that

$$
\nabla \varphi(\mathcal{R}_\delta) \supset [-1/2, 1/2] \times \left(0, \frac{\ell_\delta}{2(1 + R)}\right).
$$

(2.12)

Hence, by the transport condition $(\nabla \varphi)_\# U_2 = P$ and because $\nabla \varphi(\mathcal{R}_\delta) \subset \nabla \varphi(B_{1/2}) \subset B_R$ (see (2.11)), we obtain

$$
\int_{\mathcal{R}_\delta} u_2(x) \, dx \geq \int_{-1/2}^{1/2} \left(\int_0^{\ell_\delta/(2(1 + R))} p(y) \, dy_2\right) \, dy_1 \geq \frac{\lambda_R \ell_\delta}{2(1 + R)},
$$

(2.13)

where we used that $p \geq \lambda_R$ inside $B_R$. Noticing that

$$
u_2(x_1, x_2) = \frac{1}{2\pi} \frac{1}{\sqrt{x_1^2 + x_2^2}},
$$

it follows that (recall that $\frac{h_\delta}{\delta} = \ell_\delta \to 0$ as $\delta \to 0$)

$$
\int_{\mathcal{R}_\delta} u_2(x) \, dx \leq \frac{1}{2\pi} \int_{-h_\delta}^{h_\delta} \frac{dx_1}{\sqrt{x_1^2 + x_2^2}} = \frac{1}{2\pi} \int_{-h_\delta}^{h_\delta} \frac{dx_1}{\sqrt{x_1^2 + x_2^2}} = \frac{1}{2\pi} \int_{-h_\delta}^{h_\delta} \frac{dx_1}{\sqrt{1 + s^2}} \leq C_R \int_{-\ell_\delta}^{\ell_\delta} \log \left(\frac{\delta}{|x_1|}\right) \, dx_1 = 2C_R h_\delta (|\log \ell_\delta| + 1),
$$

for some constant $C_R$ depending on $R$. Combining this bound with (2.13), we get

$$
\ell_\delta \leq \hat{C}_R h_\delta |\log \ell_\delta| \quad \Rightarrow \quad \frac{1}{\delta} \leq \hat{C}_R |\log \ell_\delta| = \hat{C}_R \left|\log \left(\frac{h_\delta}{\delta}\right)\right|.
$$

Recalling that $h_\delta = \varphi(0, \delta) = o(\delta)$, this proves that

$$
\varphi(0, \delta) \leq \delta e^{-c_R/\delta} \quad \forall \delta \in [0, 1/4],
$$

where $c_R := 1/\hat{C}_R$. Analogously, repeating the above argument with $h_\delta = \varphi(0, -\delta)$ we obtain

$$
\varphi(0, -\delta) \leq \delta e^{-c_R/\delta},
$$

therefore

$$
\varphi(0, x_2) \leq |x_2|e^{-c_R/|x_2|} \quad \forall x_2 \in [-1/4, 1/4].
$$
By the definition of $\varphi^*$ (see (2.3)), this implies that
\begin{equation}
\varphi^*(y_1, y_2) \geq \sup_{|x_2| \leq 1/4} \{ x_2 y_2 - \varphi(0, x_2) \} \geq \sup_{|x_2| \leq 1/4} \{ x_2 y_2 - |x_2| e^{-c_R/|x_2|} \} 
(2.14)
\end{equation}
for some constant $c_R' > 0$. 

At this moment one may conclude as follows: by (2.5), the Monge-Ampère measure of $\varphi^*$ is bounded from above. So, thanks to (2.14), we can apply [14, Theorem 1.4] to conclude that $K = [-1, 1] \times \{0\}$ contains the infinite line $\mathbb{R} \times \{0\}$, thus providing the desired contradiction. For completeness we provide here an alternative self-contained proof, that we believe to have its own interest.

Consider the sets $U_k := [-1/2, 1/2] \times [2^{-k}, 2^{-k+1}] \subset \mathbb{R}^2$. By the transport condition $(\nabla \varphi^*)_# P = U_2$ one has (recall that $\mu \geq \lambda_1$ inside $B_1$)
\[ \det(D^2 \varphi^*(y)) = \frac{p(y)}{u_2(\nabla \varphi^*(y))} = 2 \pi p(y)|\nabla \varphi^*(y)| \geq 2 \pi \lambda_1 |\nabla \varphi^*(y)| \quad \text{for a.e. } y \in U_k. \]

Hence, arguing as in [5], it follows by the arithmetic-geometric inequality that
\[ 2 \cdot (2 \pi \lambda_1)^{1/2} \int_{U_k} |\nabla \varphi^*(y)|^{1/2} \, dy \leq 2 \int_{U_k} \det(D^2 \varphi^*(y))^{1/2} \, dy \leq \int_{U_k} \left(t \partial_{y_1} \varphi^*(y) + \frac{1}{t} \partial_{y_2} \varphi^*(y) \right) \, dy \leq t \int_{\partial U_k} \partial_{y_1} \varphi^*(y) \nu_1 + \frac{1}{t} \int_{\partial U_k} \partial_{y_2} \varphi^*(y) \nu_2 \quad \forall t > 0, \]
where $\nu = (\nu_1, \nu_2)$ is the outer unit normal to $\partial U_k$. Observe that, since $\varphi(0) = 0$ and $\partial \varphi(0) = K$, it follows that $\varphi^* \geq 0$ and $\varphi^*|_K = 0$. Thus, since $K = [-1, 1] \times \{0\}$ and $\varphi^*$ is 1-Lipschitz (see (2.4)),
\[ 0 \leq \varphi^*(y_1, y_2) \leq |y_2| \quad \forall y_1 \in [-1, 1], \]
and [9, Corollary A.23] applied to the convex function $\varphi^*(\cdot, y_2)$ yields
\[ |\partial_{y_1} \varphi^*(y_1, y_2)| \leq 2|y_2| \quad \forall y_1 \in [-1/2, 1/2]. \]

Thanks to this estimate, since $|y_2| \leq 2^{-k}$ on $\partial U_k$ we can bound
\[ \int_{\partial U_k} \partial_{y_1} \varphi^*(y) \nu_1 \leq 2 \cdot 2^{-k} \int_{\partial U_k} |\nu_1| = 2^{-2k}, \]
thus
\[ 2 \cdot (2 \pi \lambda_1)^{1/2} \int_{U_k} |\nabla \varphi^*(y)|^{1/2} \, dy \leq t \cdot 2^{-2k} + \frac{1}{t} \int_{\partial U_k} \partial_{y_2} \varphi^*(y) \nu_2 \quad \forall t > 0. \]

Note that, by the monotonicity of the gradient of convex functions,
\[ \partial_{y_2} \varphi^*(y_1, 2^{-k}) \geq \partial_{y_2} \varphi^*(y_1, 2^{-(k+1)}) \quad \forall y_1 \Rightarrow \int_{\partial U_k} \partial_{y_2} \varphi^*(y) \nu_2 \geq 0. \]

\[ \text{To rigorously justify the inequalities} \]
\[ \int_{U_k} \partial_{y_1} \varphi^*(y) \, dy \leq \int_{\partial U_k} \partial_{y_1} \varphi^*(y) \nu_1, \quad \int_{U_k} \partial_{y_2} \varphi^*(y) \, dy \leq \int_{\partial U_k} \partial_{y_2} \varphi^*(y) \nu_2, \]
one can either use that any pointwise pure second derivative of a convex function is bounded from above by the corresponding distributional derivative, or prove the inequalities for smooth functions and then argue by approximation.
Thus, choosing \( t := 2^k \left( \int_{\partial U_k} \partial y_2 \varphi^*(y) \nu_2 \right)^{1/2} \) we obtain
\[
(2\pi \lambda_1)^{1/2} \int_{U_k} |\nabla \varphi^*(y)|^{1/2} \, dy \leq 2^{-k} \left( \int_{\partial U_k} \partial y_2 \varphi^*(y) \nu_2 \right)^{1/2},
\]
or equivalently
\[
\frac{\pi}{2} \lambda_1 \left( \int_{U_k} |\nabla \varphi^*(y)|^{1/2} \, dy \right)^2 \leq \int_{\partial U_k} \partial y_2 \varphi^*(y) \nu_2.
\]
Summing over \( k \) the above inequalities and noticing that the boundary integrals appearing in the right hand side form a telescopic series, recalling (2.4) we conclude that
\[
(2.15) \quad \frac{\pi}{2} \lambda_1 \sum_{k \geq 1} \left( \int_{U_k} |\nabla \varphi^*(y)|^{1/2} \, dy \right)^2 \leq \int_{-1/2}^{1/2} \partial y_2 \varphi^*(y_1, 1/2) \, dy_1 \leq 1.
\]
We now want to obtain a contradiction by showing that the series in the left hand side diverges.

To this aim notice that, by the convexity of \( \varphi^* \) and because \( \varphi^*|_K = 0 \), (2.14) implies
\[
|\nabla \varphi^*(y_1, y_2)| \geq \partial y_2 \varphi^*(y_1, y_2) \geq \frac{\varphi^*(y_1, y_2)}{y_2} \geq \frac{\hat{c}_R}{|\log y_2|} \geq \frac{\hat{c}_R}{k} \quad \forall (y_1, y_2) \in U_k.
\]
Recalling (2.15), we conclude that
\[
1 \geq \frac{\pi}{2} \lambda_1 \hat{c}_R \sum_{k \geq 1} \frac{1}{k} = +\infty,
\]
a contradiction.

This proves that \( K = \partial \varphi(0) \) must be a point, and recalling [9, Lemmata A.21 and A.24] we obtain that both \( \nabla \varphi \) and \( \nabla \varphi^* \) are continuous, thus \( Q_\pm : B_1 \to \mathbb{R}^2 \) is a homeomorphism, as desired. \( \square \)

Acknowledgments: The author is extremely grateful to Marc Hallin, both for proposing this problem to him and for several stimulating discussions during the preparation of this manuscript. The author is thankful to Connor Mooney for useful comments on a preliminary version of the manuscript. The author is supported by the ERC Grant “Regularity and Stability in Partial Differential Equations (RSPDE)”

References

[1] L. Ambrosio, N. Gigli, and G. Savaré. Gradient flows in metric spaces and in the space of probability measures. Second edition. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2008. x+334 pp.

[2] L. A. Caffarelli. A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity. Ann. of Math. (2) 131 (1990), no. 1, 129–134.

[3] L. A. Caffarelli. Some regularity properties of solutions of Monge Ampère equation. Comm. Pure Appl. Math. 44 (1991), no. 8-9, 965–969.

[4] L. A. Caffarelli. The regularity of mappings with a convex potential. J. Amer. Math. Soc. 5 (1992), no. 1, 99–104.

[5] L. A. Caffarelli. A note on the degeneracy of convex solutions to Monge Ampère equation. Comm. Partial Differential Equations 18 (1993), no. 7-8, 1213–1217.

[6] V. Chernozhukov, A. Galichon, M. Hallin, and M. Henry. Monge-Kantorovich depth, quantiles, ranks, and signs. Ann. Statist. 45 (2017), no. 1, 223–256.

[7] C. Genest and L.-P. Rivest. On the multivariate probability integral transformation, Statist. Probab. Lett. 53 (2001), no. 4, 391–399.

[8] G. De Philippis and A. Figalli. The Monge-Ampère equation and its link to optimal transportation. Bull. Amer. Math. Soc. (N.S.) 51 (2014), no. 4, 527–580

[9] A. Figalli. The Monge-Ampère equation and its applications. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2017. x+200
[10] A. Figalli, Y.-H. Kim, and R. J. McCann. Hölder continuity and injectivity of optimal maps. *Arch. Ration. Mech. Anal.* 209 (2013), no. 3, 747–795

[11] A. Figalli and G. Loeper. $C^1$ regularity of solutions of the Monge-Ampère equation for optimal transport in dimension two. *Calc. Var. Partial Differential Equations* 35 (2009), no. 4, 537–550.

[12] M. Hallin. On distribution and quantile functions, ranks and signs in $\mathbb{R}^d$, a measure transportation approach. Preprint 2017, available at [https://ideas.repec.org/p/eca/wpaper/2013-258262.html](https://ideas.repec.org/p/eca/wpaper/2013-258262.html)

[13] R. J. McCann. Existence and uniqueness of monotone measure-preserving maps. *Duke Math. J.* 80 (1995), no. 2, 309–323.

[14] C. Mooney. Some counterexamples to Sobolev regularity for degenerate Monge-Ampère equations. *Anal. PDE* 9 (2016), no. 4, 881–891

[15] C. B. Morrey. On the analyticity of the solutions of analytic non-linear elliptic systems of partial differential equations. I. Analyticity in the interior. *Amer. J. Math.* 80 (1958), 198–218.

ETH Zürich, DEPT. MATHEMATICS, Rämistrasse 101, 8092 Zürich, SWITZERLAND.

*Email address: alessio.figalli@math.ethz.ch*