A GENERALIZED QUANTUM RELATIVE ENTROPY

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ABSTRACT. We propose a generalization of the quantum relative entropy by considering the geodesic on a manifold formed by all the invertible density matrices $P$. This geodesic is defined from a deformed exponential function $\varphi$ which allows to work with a wider class of families of probability distributions. Such choice allows important flexibility in the statistical model. We show and discuss some properties of this proposed generalized quantum relative entropy.

1. Introduction

Quantum mechanics and quantum statistics are among the very important new theories of the 20th century [16]. In the last decades, information theory has been an important ally leading to important progress on quantum physics, and giving rise to quantum information theory [14]. In the quantum approach, a physical systems is described in a Hilbert space, where the observable corresponds to self-adjoint operators, and statistical operators are associated with the quantum states. Therefore, we have a similar framework to deal with uncertainty and information, as in the “classical” or non-quantum information theory.

The quantum relative entropy was introduced in the setting of Von Neumann algebras in 1962 [16]. In [7, 6], Tsallis relative entropy in quantum system was studied.

In the non-quantum case, Pistone and Sempi in [18] proposed a geometric structure on the space of all the probability distributions. This structure is given from the exponential families that were studied after in [4]. In [13] a deformed exponential was proposed, called $\phi$-exponential function and denoted by $\exp_\phi(\cdot)$. The exponential function, Tsallis $q$-exponential and $\kappa$-exponential [11, 17] are examples of $\phi$-exponential functions. The geometry of the deformed exponential family was later studied in [2].

Still in the non-quantum case, the authors in [22] proposed a generalization of the exponential family of probabilities by replacing the ordinary exponential function by a deformed exponential function $\varphi(\cdot)$, which is a function that respects some properties and presents a more flexible model for the probability density functions.

2010 Mathematics Subject Classification: Primary: 58F15, 58F17; Secondary: 53C35.
Key words and phrases: Quantum relative entropy, quantum information theory, deformed exponential function, $\varphi$-families of probability distributions.

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A deformed exponential function $\varphi(\cdot)$ is a $\phi$-exponential function if $\varphi(0) = 1$. Besides the proposition of the so-called $\varphi$-family of probability functions, the authors also provided a $\varphi$-divergence, that is obtained from the Bregman divergence based on the normalization function $\psi(\cdot)$. For $\varphi(\cdot) = \exp(\cdot)$, i.e., the ordinary exponential function, the $\varphi$-divergence reduces to the Kullback-Leibler divergence [12].

In [5], a generalization of the Rényi relative entropy was given considering a deformed exponential, and properties of such generalization were also discussed and evaluated from the information geometry point of view.

Although the generalization of the exponential family has been developed in a non parametric case, in this work we consider a finite-dimensional Hilbert space $H$ and the deformed exponential function $\varphi$ to provide a generalized quantum relative entropy by a broader statistical model and taking profit from the known generalizations already found in the literature for the classical information theory scenarios.

The rest of paper is organized as follows. In Section 2 we recall some important results about quantum relative entropy and about deformed exponential functions. Section 3 is devoted to find the generalized quantum relative entropy and where we prove some relevant properties. Finally, conclusions and perspectives are stated in Section 4.

2. Preliminary results

2.1. Quantum relative entropy and its properties. Let $\mathcal{H}$ be a finite-dimensional complex Hilbert space and $\mathcal{L}(\mathcal{H})$ be the set of all linear operators on $\mathcal{H}$. Mixed states of a quantum system are described by a density matrix (or statistical operator), which is a self-adjoint, positive definite matrix, whose trace equals 1. The space has a basis that can be expressed by the eigenvectors of the statistical operator and the sum of the corresponding eigenvalues is equal to 1. The pure states represented by unit vectors of the Hilbert space are among the density matrices under an appropriate identification. If $x = |x\rangle$ is a unit vector, then $|x\rangle\langle x|$ is a density matrix. Geometrically, the density matrix $|x\rangle\langle x|$ is the orthogonal projection onto the linear subspace generated by $x$ (see [16]).

Let $\rho$ and $\sigma$ be density matrices on the Hilbert Space $\mathcal{H}$. The quantum relative entropy between $\rho$ and $\sigma$ is defined as

$$S(\rho \parallel \sigma) = \text{Tr}[\rho(\log \rho - \log \sigma)].$$

The quantum relative entropy expresses statistical distinguishability. Observe that it is not a symmetric function. The relative entropy satisfies some important properties and here we discuss some of them:

1. Non-negativity:
$$S(\rho \parallel \sigma) \geq 0.$$

2. Monotonicity:
$$S(\rho \parallel \sigma) \geq S(\mathcal{E}(\rho) \parallel \mathcal{E}(\sigma)),$$

where $\mathcal{E} : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ is a state transformation.

3. Invariance under unitary conjugation
$$S(U\rho U^* \parallel U\sigma U^*) = S(\rho \parallel \sigma),$$

where the matrix $U$ satisfies $UU^* = U^*U = I$.

4. Joint Convexity:
$$S(\lambda \rho_1 + (1 - \lambda)\sigma_1 \parallel \lambda \rho_2 + (1 - \lambda)\sigma_2) \leq \lambda S(\rho_1 \parallel \rho_2) + (1 - \lambda)S(\sigma_1 \parallel \sigma_2),$$
where $\rho_i$ and $\sigma_i$, $i = 1, 2$, are density matrices and $0 \leq \lambda \leq 1$.

In the next section, we discuss the relationship between the quantum relative entropy and the normalizing function found in the definition of the $\varphi$-family of probability distributions. We show how we can derive the quantum relative entropy from the normalizing function that appears when two density matrices are connected by an arc.

2.2. Relationship between the quantum relative entropy and the normalizing function. Let $\mathcal{P}$ be the set of all invertible density matrices, and let $\mathcal{A}$ denote the subspace of self-adjoint operators in $\mathcal{L}(\mathcal{H})$. There exists an embedding of $\mathcal{P}$ into $\mathcal{A}$, defined as

$$l: \mathcal{P} \rightarrow \mathcal{A}$$

$$\rho \mapsto \log \rho.$$ 

Therefore $\mathcal{P}$ forms a manifold [8] which has the exponential function as a natural path in $\mathcal{P}$, and we can use the linear space structure of $\mathcal{A}$ to obtain a representation of the bundle of $\mathcal{P}$ in terms of operators in $\mathcal{A}$. At each point $\rho \in \mathcal{P}$, consider the subspace $\mathcal{A}_\rho = \{A \in \mathcal{A} : \text{Tr}(A\rho) = 0\}$ of $\mathcal{A}$. Let $H$ be a self-adjoint matrix such that $\rho = \exp(H)$, and let $\sigma$ be an invertible density matrix. The e-geodesic is the curve

$$\gamma_e(t) = \exp(H + tA - \log(\text{Tr} \exp(H + tA)))I,$$

for $t \in [0, 1]$, where $A = \log \sigma - \log \rho \in \mathcal{A}_\rho$. Then, $\gamma_e(0) = \rho$ and $\gamma_e(1) = \sigma$.

We can rewrite $\gamma_e(t)$ as

$$\gamma_e(t) = \exp(H + tA - \psi_H(tA))I,$$

where $\psi_H(tA) := \log(\text{Tr} \exp(H + tA))$, for $t \in [0, 1]$, and, as a consequence, the e-geodesic can be rewritten as:

$$\gamma_e(t) = \exp(H + tA - \psi_H(tA))I.$$ 

We want to verify whether $\gamma_e(t) \in \mathcal{P}$. Every differentiable convex function $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfies

$$f(y) \geq f(x) + (y - x)f'(x), \quad x, y \in \mathbb{R}.$$ 

From Klein’s inequality [15], for $X$ and $Y$ self-adjoint with the discrete spectrum, the inequality below follows:

$$\text{Tr} f(Y) \geq \text{Tr} f(X) + \text{Tr}(Y - X)f'(X).$$

Therefore, from the convexity of the exponential function and by the fact of $A$ and $H$ are self-adjoint, it follows that

$$\text{Tr} \exp(H + A) \geq \text{Tr} \exp(H) + \text{Tr} A \exp(H).$$

Consider the subspace of $\mathcal{A}$, defined as:

$$\mathcal{A}_\rho = \{A \in \mathcal{A} : \text{Tr}(A\rho) = 0\},$$

and the density matrix

$$\rho = \exp(H),$$
where \( H \in \mathcal{A} \). Thus, for any \( A \in \mathcal{A}_\rho \), we have \( H + A \in \mathcal{A} \) and

\[
\text{Tr} \exp(H + A) \geq \text{Tr} \exp(H) = \text{Tr} \rho = 1.
\]

Taking \( H + A \in \mathcal{A} \), from the Spectral Theorem \cite[Theorem 9, Chapter 9]{9} we can write \( H + A = \sum_i \lambda_i P_i \), where \( \lambda_i \) are eigenvalues. Thus, \( H + A - \lambda I = \sum_i (\lambda_i - \lambda) P_i \) and \( \exp(H + A - \lambda I) = \sum_i \exp(\lambda_i - \lambda) P_i \). The function \( V: \mathbb{R} \rightarrow \mathbb{R} \), given as

\[
V(\lambda) \mapsto \text{Tr} \exp(H + A - \lambda I),
\]

behaves in a way that, \( V(\lambda) \rightarrow 0 \) when \( \lambda \rightarrow \infty \). Thus, from the behavior of the function \( V(\cdot) \) and the continuity of the Trace function and the exponential function, we conclude that there exists only one \( \lambda_0 \geq 0 \) such that \( \text{Tr} \exp(H + A - \lambda_0 I) = 1 \).

Taking \( \lambda_0 = \psi_H(A) \), we have that \( \exp(H + A - \psi_H(A) I) \in \mathcal{P} \), for any \( A \in \mathcal{A}_\rho \) as we wanted to verify.

Given \( A \in \mathcal{A}_\rho \), a density matrix \( \sigma = \exp(H + A - \psi_H(A) I) \). Thus, \( \log \sigma = H + A - \psi_H(A) I \) and therefore

\[
\log \sigma = \log \rho + A - \psi_H(A) I.
\]

Multiplying (11) by the matrix \( \rho \), we obtain the matrix \( (\log \sigma)\rho = (\log \rho)\rho + A\rho - \psi_H(A)\rho \) and then

\[
\psi_H(A)\rho = (\log \rho - \log \sigma)\rho + A\rho.
\]

Applying the trace operator in (12) and using (8) it follows that

\[
\text{Tr} \psi_H(A)\rho = \text{Tr} \rho(\log \rho - \log \sigma) = S(\rho \mid \mid \sigma).
\]

Hence, we can infer that

\[
S(\rho \mid \mid \sigma) = \psi_H(A).
\]

In other words, the quantum relative entropy between the density matrices \( \rho \) and \( \sigma \) relates to the normalizing function defined in (4) which is the geodesic that connects the density matrix \( \rho \) to the density matrix \( \sigma \).

In the next section, we recall some results related to the deformed exponential function \( \varphi(\cdot) \).

2.3. Deformed exponential functions. A convex function \( \varphi: \mathbb{R} \rightarrow [0, \infty) \) is said to be a deformed exponential function if \( \lim_{x \rightarrow -\infty} \varphi(x) = 0 \) and \( \lim_{x \rightarrow \infty} \varphi(x) = \infty \) \cite{22}. The exponential function \( \exp(x) \) is an example of a deformed exponential. Another example is the Kaniadakis’ \( \kappa \)-deformed exponential \( \exp_{\kappa}: \mathbb{R} \rightarrow (0, \infty) \) for \( \kappa \in [-1,1] \) which is defined as \cite{22, 17, 10, 11}

\[
\exp_{\kappa}(u) = \begin{cases}
(ku + \sqrt{1 + \kappa^2u^2})^\frac{1}{\kappa}, & \text{if } \kappa \neq 0, \\
\exp(u), & \text{if } \kappa = 0,
\end{cases}
\]

whose inverse function defines the Kaniadakis’ \( \kappa \)-deformed logarithm, which is given by

\[
\ln_{\kappa}(u) = \begin{cases}
\frac{u^\kappa - u^{-\kappa}}{2\kappa}, & \text{if } \kappa \neq 0, \\
\ln(u), & \text{if } \kappa = 0.
\end{cases}
\]

Some algebraic properties of the ordinary exponential and logarithm functions are preserved, for example:

\[
\exp_{\kappa}(u) \exp_{\kappa}(-u) = 1, \quad \ln_{\kappa}(u) + \ln_{\kappa}(u^{-1}) = 0.
\]
The Tsallis $q$-exponential is also an example of a deformed exponential \cite{19, 20, 3}. This function, represented by $\exp_q: \mathbb{R} \to [0, \infty)$, for $q \in [0, \infty)$, is given by the following expression \cite{21}:

$$\exp_q(x) = [1 + (1 - q)x]^{rac{1}{1-q}},$$

where $[u]_+ = u$, if $u \geq 0$ and $[u]_+ = 0$, if $u \leq 0$. For $0 \leq q < 1$, the $q$-exponential function is injective for $u \in \mathbb{R}$ such that $\frac{1}{1-q} < u$ and for $1 \leq q$ the $q$-exponential is injective for all $u \in \mathbb{R}$ \cite{21}.

The Tsallis $q$-logarithm, denoted by $\ln_q: [0, \infty) \to \mathbb{R}$, for $q \in [0, \infty)$, is defined as the inverse of $\exp_q: X \to [0, \infty)$, where $X \subset \mathbb{R}$ is the set such that the function $\exp_q(\cdot)$ is injective, which is given by \cite{20}

$$\ln_q(x) = \frac{x^{1-q} - 1}{1-q}.$$ 

In next section, we will use a deformed exponential function $\varphi(\cdot)$ to define the generalized quantum relative entropy. We will define this new relative entropy using the same reasoning of Section 2.2. We relate the generalized quantum relative entropy with the normalizing function obtained when we connect two density matrices by an arc.

3. Generalized quantum relative entropy

Let $\mathcal{H}$ be a finite-dimensional Hilbert space. Let $\mathcal{A}$ be the subspace and the set $\mathcal{P}$ as in the Section 2.2. From now on we assume that $\varphi(\cdot)$ is a deformed exponential function continuously differentiable with its inverse continuously differentiable.

For any deformed exponential $\varphi(\cdot)$, expression (5) is satisfied. So, for $H$ and $A$ belonging to the subspace $\mathcal{A}$, from Klein’s inequality we may write

$$\text{Tr} \varphi(H + A) \geq \text{Tr} \varphi(H) + \text{Tr} A\varphi'(H),$$

where a density matrix $\rho$ is given by

$$\rho = \varphi(H).$$

We consider the subspace of $\mathcal{A}$, given as

$$\mathcal{A}_\rho^\sigma = \{A \in \mathcal{A} : \text{Tr} A\varphi'(H) = 0\},$$

that is a more general case of the space given in (8). One should note that in this case if $\varphi(x) = \exp(x)$ we have that $\mathcal{A}_\rho^\sigma = \mathcal{A}_\rho$. For $A \in \mathcal{A}_\rho^\sigma$, from (15) and (17), we obtain $\text{Tr} \varphi(H + A) \geq 1$. Similarly, there exists a unique $\psi_H(A) \geq 0$, such that $\text{Tr} \varphi(H + A - \psi_H(A)I) = 1$. Thus, taking

$$\sigma = \varphi(H + A - \psi_H(A)I),$$

the generalized arc, given as

$$\gamma_\varphi(t) = \varphi(H + tA - \psi_H(tA)I), \quad \text{for } t \in [0, 1],$$

connects the density matrix $\rho$ to the density matrix $\sigma$. Then, $\tilde{\varphi}(\sigma) = H + A - \psi_H(A)I$, from (16) we rewrite $\tilde{\varphi}(\sigma) = \varphi(\rho) + A - \psi_H(A)I$, where, in order to avoid confusion, we call $\tilde{\varphi}$ as the inverse function of $\varphi$.

Therefore,

$$\psi_H(A)I = \varphi(\rho) - \tilde{\varphi}(\sigma) + A,$$

and multiplying on both side by $\varphi'(H)$, we have

$$\psi_H(A)[\varphi'(H)] = [\varphi(\rho) - \tilde{\varphi}(\sigma)][\varphi'(H)] + A\varphi'(H),$$
that can be rewritten as
\begin{equation}
\psi_H(A)[\varphi'(H)] = [\tilde{\varphi}(\rho) - \tilde{\varphi}(\sigma)][\varphi'(\tilde{\varphi}(\rho))] + A\varphi'(H).
\end{equation}
Applying the trace operator in (19), using (17) and \(\varphi'(\tilde{\varphi}(x)) = [(\tilde{\varphi}')^{-1}(x)]^{-1}\), we obtain
\begin{equation}
\psi_H(A) = \frac{\text{Tr}[\tilde{\varphi}(\rho) - \tilde{\varphi}(\sigma)][((\tilde{\varphi}')(\rho))^{-1}]}{\text{Tr}[\varphi'(H)]},
\end{equation}
where we suppose from now on that the deformed exponential function is such that
\begin{equation}
\text{Tr}[\varphi'(H)] = 1.
\end{equation}
Thus, we obtain the generalized relative entropy which is given in the following form:
\begin{equation}
S_{\varphi}(\rho \mid \mid \sigma) = \text{Tr}[(\tilde{\varphi}')^{-1}[\tilde{\varphi}(\rho) - \tilde{\varphi}(\sigma)].
\end{equation}
As in Section 2.2, we relate the generalized quantum relative entropy with the normalizing function \(\psi_H(\cdot)\), the function that is obtained when we connect two density matrices by a generalized arc.

The non-negativity of \(S_{\varphi}(\rho \mid \mid \sigma) \geq 0\) is a consequence of the concavity of \(\tilde{\varphi}\). The concavity of \(\tilde{\varphi}\) implies that
\begin{equation}
(y - x)(\tilde{\varphi})'(y) \leq \tilde{\varphi}(y) - \tilde{\varphi}(x), \quad \text{for all } x, y > 0.
\end{equation}
Thus, we have \(\text{Tr}(\rho - \sigma) \leq \text{Tr}[\tilde{\varphi}(\rho) - \tilde{\varphi}(\sigma)][((\tilde{\varphi}')(\rho))^{-1}\) and
\begin{equation}
S_{\varphi}(\rho \mid \mid \sigma) = \text{Tr}[(\tilde{\varphi}')^{-1}[\tilde{\varphi}(\rho) - \tilde{\varphi}(\sigma)] \geq \text{Tr}(\rho - \sigma) = 0.
\end{equation}
It is clear that \(S_{\varphi}(\rho \mid \mid \sigma) = 0\) if \(\rho = \sigma\). The converse is true if we assume \(\tilde{\varphi}\) is strictly concave, since the equality in (22) occurs if and only if \(\rho = \sigma\). So, if \(\tilde{\varphi}\) is strictly concave, then \(S_{\varphi}(\rho \mid \mid \sigma) = 0\) if and only if \(\rho = \sigma\).

If \(\varphi(\cdot) = \exp(\cdot)\) the generalized quantum relative entropy (21) reduces to the quantum relative entropy presented in (1). In [22] the \(\kappa\)-relative entropy for the classical statistical was defined from the \(\varphi\)-divergence. Here, we will define the \(\kappa\)-quantum relative entropy, that is, the generalized quantum relative entropy obtained from the \(\kappa\)-logarithm as given in equation (14).

Using the derivative of the \(\kappa\)-deformed logarithm
\[\ln'_\kappa(y) = \frac{y^\kappa + y^{-\kappa}}{2} \frac{1}{y},\]
we can express the \(\kappa\)-quantum relative entropy as
\begin{equation}
S_\kappa(\rho \mid \mid \sigma) = \text{Tr}[\rho(\rho^\kappa + \rho^{-\kappa})^{-1}\left\{\frac{\rho^\kappa - \rho^{-\kappa} + \sigma^{-\kappa} - \sigma^\kappa}{2\kappa}\right\}].
\end{equation}
For \(\varphi(\cdot) = \exp_q(\cdot)\), and using the derivative of the \(q\)-logarithm \(\ln'_q(x) = x^{-q}\), for \(x > 0\), we obtain the Tsallis quantum relative entropy
\begin{equation}
S_q(\rho \mid \mid \sigma) = \text{Tr} \rho^q[\ln_q(\rho) - \ln_q(\sigma)],
\end{equation}
which was investigated in [1, 7, 6].

In the next section we discuss some properties of the generalized relative entropy.
3.1. Properties of the generalized quantum relative entropy.

**Proposition 1.** For any statistical operators $\rho$ and $\sigma$, the generalized relative entropy $S_\varphi(\rho \mid\mid \sigma)$ satisfies the following properties:

(a) Non-negativity: $S_\varphi(\rho \mid\mid \sigma) \geq 0$.
(b) $S_\varphi$ is invariant under the unitary transformation $U$:

$$S_\varphi(U\rho U^* \mid\mid U\sigma U^*) = S_\varphi(\rho \mid\mid \sigma).$$

(c) $S_\varphi(\rho \otimes \frac{I}{n} \mid\mid \sigma \otimes \frac{I}{n}) = nS_\varphi(\rho \mid\mid \sigma)$, where $n$ is the dimension of the Hilbert space $\mathcal{H}$.

**Proof.** By the concavity of $\tilde{\varphi}$, we have that $S_\varphi(\rho \mid\mid \sigma) \geq \text{Tr}[\rho - \sigma] = 0$, which implies in property (a). Let $U$ be a unitary operator, that is, $UU^* = U^*U = I$, then $U$ is diagonalizable and the eigenvalues of $U$ are of the form $\alpha = \exp(i\theta)$ where $|\alpha|^2 = 1$. To prove property (b) we consider the decomposition of density matrices $\rho = \sum_i \lambda_i P_i$ and $\sigma = \sum_i \mu_i Q_i$, where the matrices $P_i$ and $Q_i$ can be seen as $P_i = |x_i\rangle\langle x_i|$ and $Q_i = |y_i\rangle\langle y_i|$, with unitary vectors $|x_i\rangle$ and $|y_i\rangle$, that may be chosen pairwise orthogonal eigenvectors and $\lambda_i$ and $\mu_i$ are the corresponding eigenvalues, respectively. Using the fact that $\text{Tr}(AB) = \text{Tr}(BA)$, we have

$$S_\varphi(U\rho U^* \mid\mid U\sigma U^*) = \text{Tr}[(\tilde{\varphi}'(U\rho U^*))^{-1}][\tilde{\varphi}(U\rho U^*) - \tilde{\varphi}(U\sigma U^*)]$$

$$= \text{Tr}\left[\left(\tilde{\varphi}'\left(U\left(\sum_i \lambda_i P_i\right)U^*\right)\right)^{-1}\right]$$

$$= \text{Tr}\left[\tilde{\varphi}\left(U\left(\sum_i \lambda_i P_i\right)U^*\right) - \tilde{\varphi}\left(U\left(\sum_i \mu_i Q_i\right)U^*\right)\right]$$

$$= \text{Tr}\left[U\left(\sum_i (\tilde{\varphi}')(\lambda_i) P_i\right)U^*\right]$$

$$= \text{Tr}\left[U\sum_i \tilde{\varphi}(\lambda_i) P_i U^* - U\sum_i \tilde{\varphi}(\mu_i) Q_i U^*\right]$$

$$= \text{Tr}[(\tilde{\varphi}'(\rho))^{-1}][\tilde{\varphi}(\rho) - \tilde{\varphi}(\sigma)]$$

$$= S_\varphi(\rho \mid\mid \sigma),$$

and the proof of property (b) is concluded. Now, to prove property (c) we know that the trace of a Kronecker product is given by $\text{Tr}(A \otimes B) = \text{Tr}(A) \text{Tr}(B)$. We also know that the Kronecker product satisfies the following properties: $(A \otimes B)(C \otimes D) = (AC \otimes BD)$, the matrix $(A \otimes B)$ is an invertible matrix if and only if the matrices $A$ and $B$ are invertible matrices and we have $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$. The singular value decomposition of $\rho \otimes \frac{I}{n} = \sum_i \lambda_i P_i \otimes \frac{I}{n}$ and $\sigma \otimes \frac{I}{n} = \sum_i \mu_i Q_i \otimes \frac{I}{n}$, with $\sum_i \lambda_i = \sum_i \mu_i = 1$. Thus, we have

$$S_\varphi\left(\rho \otimes \frac{I}{n} \mid\mid \sigma \otimes \frac{I}{n}\right)$$
for all $\rho$

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Proposition 2. Fixed a deformed exponential $\phi$

Let $L(H_{AB}) = L(H_A) \otimes L(H_B)$ be a bipartite quantum system, that is, a quantum system which consists of two parts. In order to have the generalized quantum relative entropy as a statistical distance, it is interesting that the quantity $S_{\phi}(\cdot || \cdot)$ decreases if a part of the system is ignored. Supposing that the deformed exponential $\phi$ is such that the generalized quantum relative entropy satisfies the joint convexity we can prove that

$$S_{\phi}(\rho^A || \sigma^A) \leq S_{\phi}(\rho^{AB} || \sigma^{AB}),$$

where $\rho^A, \sigma^A \in L(H_A)$, and $\rho^{AB}, \sigma^{AB} \in L(H_{AB})$.

Proposition 2. Fixed a deformed exponential $\phi$, suppose that $S_{\phi}(\cdot || \cdot)$ satisfies the joint convexity. Then

$$S_{\phi}(\rho^A || \sigma^A) \leq S_{\phi}(\rho^{AB} || \sigma^{AB}).$$

Proof. We know that [14, Exercise 11.19]

$$\rho^A \otimes \frac{I}{n} = \sum_i \lambda_i U_i \rho^{AB} U_i^*,$$

for all $\rho^{AB}$. From the joint convexity we have

$$S_{\phi}(\rho^A \otimes \frac{I}{n} || \sigma^A \otimes \frac{I}{n}) \leq \sum_i \lambda_i S_{\phi}(U_i \rho^{AB} U_i^* || U_i \sigma^{AB} U_i^*).$$

From property (b) of Proposition 1, we obtain

$$S_{\phi}(\rho^A \otimes \frac{I}{n} || \sigma^A \otimes \frac{I}{n}) \leq \sum_i \lambda_i S_{\phi}(\rho^{AB} || \sigma^{AB}) = S_{\phi}(\rho^{AB} || \sigma^{AB}).$$
And, by the fact that
\[ S\varphi\left(\rho^A \parallel \sigma^A\right) \leq S\varphi\left(\rho^A \otimes \frac{1}{n} \parallel \sigma^A \otimes \frac{1}{n}\right), \]
the inequality follows.

4. Conclusions

We conclude that in the quantum e-geodesic, replacing the exponential function by a deformed exponential in the same way that was done in [22], we can obtain a more general form of the quantum relative entropy. This quantum relative entropy is invariant under unitary transformations and if it satisfies the joint convexity, then it satisfies the monotonicity.

The results provided in our work are of interest in the application of measurement of the entanglement of quantum states. The relative entropy is employed to verify if a given quantum state is more likely to be a entangled one or a disentangled (separable) one, which brings several information about the state. Moreover, quantum relative entropy can also be applied to the measure of the mutual quantum information which provides a quantification, for example, of how much the joint system is different from the individual parts combined. Those applications are of major interest and relevance in quantum information processing.

As a future work, we aim to study which conditions the deformed exponential must satisfy so that the generalized quantum relative entropy can be used for this quantification of entanglement.

Acknowledgments

This work was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001 and Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) (Procs. 309472/2017-2 and 408609/2016-8).

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Received December 2018; revised September 2019.

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