ON SMOOTH 4-DIMENSIONAL POINCARÉ CONJECTURE

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Abstract. A proof of 4-dimensional smooth Poincaré Conjecture.

0. Statement and a proof

It is known that homotopy equivalent simply connected closed smooth 4-manifolds are $h$-cobordant to each other, and they differ from each other by corks [A4], [M]. For the background material, and the techniques used in this paper we refer reader to [A1] and [A2].

Theorem 1. Every smooth homotopy 4-sphere $\Sigma^4$ (i.e. closed smooth 4-manifold homotopy equivalent to $S^4$) is diffeomorphic to $S^4$.

Proof. We know that every such $\Sigma^4$ is smoothly $h$-cobordant to $S^4$, through a cobordism $Z^5$ consisting of only 2- and 3-handles. Let $Z^5$ be such an $h$-cobordism from $S^4$ to $\Sigma^4$, obtained by attaching 2-handles $\{h^2_j\}$ to $S^4$, then attaching 3-handles top of them. The 3-handles upside down are duals to the 2-handles $\{k^2_j\}$, attached to $\Sigma^4$. This decomposes $Z^5$ into union of two parts, meeting in the middle 4-manifold $X^4$.

$$Z^5 = [S^4 \cup h^2_j] \sim_X [\Sigma^4 \cup k^2_j]$$

\[\text{Figure 1. } h\text{-cobordism}\]

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Since $S^4$ is simply connected, $X$ is diffeomorphic to $\#_n S^2 \times S^2$. By looking $Z$ from the other side, since $\Sigma^4$ is simply connected, we see $X$ is diffeomorphic to $(\#_n S^2 \times S^2) \# \Sigma$. Therefore $X$ contains two families of disjointly imbedded 2-spheres $A = \{ \sqcup A_i \}_{i=1}^n$ and $B = \{ \sqcup B_j \}_{j=1}^n$, which are the belt spheres of 2-handles $\{ h^2_i \}$ and $\{ k^2_j \}$, respectively. Since $Z$ is homotopy product, they are imbedded with trivial normal bundles, and we can arrange algebraic intersection numbers $A_i B_j = \delta_{ij}$.

![Diagram](image)

**Figure 2. Various protocorks**

Let us denote the union of the tubular neighborhoods of these spheres in $X$ by $V \subset X$. Clearly surgering $X$ along the $A$-spheres gives $S^4$, and surgering $X$ along the $B$-spheres gives $\Sigma^4$ (because these surgeries undo the affect of the 2-handles). Now denote the manifold obtained from $V$ by surgering $A$-spheres by $V(A)$; and denote the manifold obtained from $V$ by surgering $B$-spheres by $V(B)$. Clearly we have the inclusions $V(A) \subset S^4$ and $V(B) \subset \Sigma^4$. Note that $\partial V(A) = \partial V(B)$, and we have a diffeomorphism $X - V(A) \approx X - V(B)$. We call $V(A)$ a protocork, and call the following operation “protocork twisting” of $S^4$ along $V(A)$.

$$f : S^4 \rightsquigarrow (S^4 - V(A)) \sim V(B) = \Sigma^4$$

This corresponds to “zero-dot exchange operation” of corks. Let us denote the protocork shown in the left picture of Figure 2 by $P_n$ ($2n+1$ intersection points). Handlebody of $P_n$ can be described as in Figure 3. The first cork appeared in [A3] then generalized in [M]. Protocorks are simpler objects than corks, which are basic building blocks of corks. Recently they were used by Ladu as tools in geometric analysis [L].
Next, starting with any imbedding of a protocork $V(\mathcal{A}) \subset S^4$, we will describe the handlebody of the protocork twisting of $S^4$ along $V(\mathcal{A})$. For simplicity, we will first take $V(\mathcal{A}) = \mathbb{P}_n$, then indicate how to modify the argument for the general case. We proceed by two steps.

1. Draw the handlebody picture of the complement $C := S^4 - \mathbb{P}_n$.
2. Glue $\mathbb{P}_n$ to $C$ by the involution $f : \partial \mathbb{P}_n \rightarrow \partial \mathbb{P}_n$.

To simplify this process we first carve $\mathbb{P}_n$ across the meridianal 2-disk $D$ of its 2-handle (which $f(\gamma)$ bounds). Then notice that, up to 3-handles this gives the identification $S^4 = C \sim h^2_{f(\gamma)}$, because $N(D)$ upside-down is just the 2-handle $h^2_{f(\gamma)}$ attached to $C$, and $\mathbb{P}_n - N(D) \approx \#_k B^3 \times S^1$ can be viewed as 3-handles attached to $C \sim h^2_{f(\gamma)}$. Hence, up to 3-handles, gluing $\mathbb{P}_n$ to $C$ by $f$ can be described by $C \sim h^2_{\gamma}$.

From any imbedding $\mathbb{P}_n \subset S^4$, we can construct a handlebody of $C = S^4 - \mathbb{P}_n$ in two steps, which amounts to turning $S^4$ upside down:

(a) Isotope the 1-handles of $\mathbb{P}_n$ into standard position $\#_k B^3 \times S^1$, then take its complement in $S^4$, which is $\#_k S^2 \times D^2$.

(b) Carve this $\#_k S^2 \times D^2$ along the core $\mathbb{D}$ of the 2-handle of $\mathbb{P}_n$. 
Remark 1. 1-handles $\#_k B^3 \times S^1$ of $\mathbb{P}_n$ are just carved out 2-discs from the 4-ball $B^4_+$ (Figure 5), hence their complement in $S^4$ is the standard $\#_k S^2 \times B^2 \subset S^4$. Here $B^4_+ = S^4 - B^4_-$, and $D$ is the core of the 2-handle of $\mathbb{P}_n$ attached to carved $B^4_+$. Note that even though $D$ is just the core of the 2-handle, its complement in $S^4$ (viewed upside down) could be complicated looking slice disk complement. That is, 2-handle $N(D)$ of $\mathbb{P}_n$, is attached to $\#_k B^3 \times S^1$, and $S^4 - \mathbb{P}_n \subset S^4$ will be the complement of the slice disk $D$ in $\#_k S^2 \times B^2$. So all possible embeddings $\mathbb{P}_n \subset S^4$ are determined by the choice of this slice disk $D$.

Even though the slice disk $D$ is not unique, from the linking pattern of the circles $A$ and $B$ in Figure 3, we can determine intersections of $D$ with the 2-handles $\#_k S^2 \times B^2$ of $S^4 - \mathbb{P}_n$. Figure 6 summarizes what we have done so far. The doted circle labeled by $f(\gamma)$ indicates carving of $\mathbb{P}_n$ along the meridianal disk $D$ (looking from outside its just a 2-handle $h^2_{f(\gamma)}$). Then going from the second to the last picture of Figure 6 describes the protocork twisting, i.e.

$$(S^4 - \mathbb{P}_n) \sim h^2_{f(\gamma)} \rightarrow (S^4 - \mathbb{P}_n) \sim h^2_{(\gamma)}$$

![Figure 6. Protocork twisting of $S^4$ along $\mathbb{P}_n$, $n = 1$](image_url)
Then by the isotopy in Figure 7, followed by the indicated handle slide, we arrive to the last picture of Figure 7. After cancelling the two isolated 0-framed unknots (i.e. $S^2 \times B^2$’s) with 3-handles, we are now left with a chain of three linked circles: $A$, $B$, and $C$; where $A$ is the boundary of some carved 2-disk in $S^2 \times B^2$ (bounding a meridian $\partial B^2$).

Note that, by construction, this 2-disk is now slid over to the other side of $C$, i.e. it doesn’t go over the 2-handle $C$ (compare this to the sliding operation in Section 1.4 of [A1]). Hence by [G] the red dotted circle, with a 0-framed circle $B$ linking it, gives $B^4$. More specifically, any imbedded 2-disk in $S^2 \times B^2$ whose boundary is $p \times \partial B^2$ is isotopic to $p \times B^2$. Therefore the last picture of Figure 7 is just $B^4$ with a 2-handle $C$ attached to it (with 0-framing); and its boundary is diffeomorphic to $S^1 \times S^2$. Hence by the Property $R$ it must be $S^2 \times B^2$, so this final manifold must be $S^4$, up to 3-handles. Therefore Theorem 1 in the case of the protocork $\nabla_n(A) = \mathbb{P}_n$ has now been proven.

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![Figure 7. Protocork twist continued](image)

1. **General case of $\mathbb{P}_n$ and further remarks**

The protocork $\mathbb{P}_n$ can be identified with manifold called $L^0_n$ in [AY]. Namely there is the identification of Figure 8. We leave checking this as an exercise to the reader (Hint: Cancel the large dotted circle with the large 0-framed circle of $\mathbb{P}_n$, along the way slide dotted circles over each other if you have to). $L^0_n$ is the right picture of Figure 8, which is the manifold obtained by carving $B^4$ along the indicated slice disks. Figure 9 indicates a proof in the case $n = 1$. 

![Figure 8. Identification of protocork $\nabla_n(A) = \mathbb{P}_n$](image)
In general a protocork $\mathbb{V}(\mathcal{A})$ is a union of $\mathbb{P}_n$’s where you are allowed to link the dotted circles with the 0-framed circles algebraically zero times. Protocork twisting map $f$ is just the “zero-dot exchanging map” between $\mathcal{A}$ and $\mathcal{B}$ spheres (swapping 1- and 2-handles of $\mathcal{A}$ and $\mathcal{B}$).

With this, we can identify protocork twistings of $S^4$ along the protocorks $\mathbb{P}_n \subset S^4$, and along $L_n^0 \subset S^4$, which by [AY] is diffeomorphic to $S^4$. This gives another proof for Theorem 1 in the case of $\mathbb{V}(\mathcal{A}) = \mathbb{P}_n$. So to prove 4-dim smooth Poincaré conjecture we need to show that protocork twisting of $S^4$ along any protocork in $S^4$ gives back $S^4$. 

FIGURE 8. The equivalence $\mathbb{P}_n \approx L_n^0$, when $n = 2$

FIGURE 9. Describing $\mathbb{P}_1 \approx L_1^0$
For this we consider “excess intersections” between different $\mathbb{P}_n$'s of a protocork $\mathbb{V}(A)$, i.e. the case of $A$ spheres of $\mathbb{V}$ intersecting its $B$ spheres (algebraically zero times). The protocol described in the right picture of Figure 2 is a good example to understand the general case; it corresponds to the handlebody in the first picture of Figure 10. Now we will proceed to show how to modify the proof in case $\mathbb{V}(A) = \mathbb{P}_n$ to this general case.

Now denote the protocork complement in $S^4$ by $C = S^4 - \mathbb{P}$, and let $f(\gamma_1), ..., f(\gamma_n)$ be the linking circles of the 2-handles $A_1, ..., A_n$ of the protocork $\mathbb{P}$. Clearly, carving $\mathbb{P}$ along its 2-handles (that is carving it along the meridional discs which $f(\gamma_i)$'s bound) will turn $\mathbb{P}$ into disjoint union of thickened circles $#S^1 \times B^3 \hookrightarrow S^4$, which is the second picture of Figure 10; hence its complement in $S^4$ is $C^* := C \smallsetminus \{2$-handles along the curves $f(\gamma_i)$\} is a disjoint union of thickened 2-spheres $#B^2 \times S^2$ in $S^4$. Hence $S^4 = C^* \smallsetminus 3$-handles attached at top (in Figure 4).
Now assume the first handlebody of Figure 10 lies in \( S^4 \), then by an isotopy and carving across its 2-handles we obtain the second picture. Then by taking its complement in \( S^4 \) we obtain the third picture (the thick dots indicate carving along "some" disk bounding the corresponding circles). Then by applying protocork twist to the third picture and proceeding as in Figure 7 we obtain the last picture which is diffeomorphic to \( S^4 \) by the same argument (after cancelling its 2-handles with 3-handles ). □

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