Research Article

Comparative Study of Some Fixed-Point Methods in the Generation of Julia and Mandelbrot Sets

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Fractal is a geometrical shape with property that each point of the shape represents the whole. Having this property, fractals procured the attention in computer graphics, engineering, biology, mathematics, physics, art, and design. The fractals generated on highest priorities are the Julia and Mandelbrot sets. So, in this paper, we develop some necessary conditions for the convergence of sequences established for the orbits of \( M, M^*, \) and \( K \)-iterative methods to generate these fractals. We adjust algorithms according to the develop conditions and draw some attractive Julia and Mandelbrot set with sequences of iterates from proposed fixed-point iterative methods. Moreover, we discuss the self-similarities with input parameters in each graph and present the comparison of images with proposed methods.

1. Introduction

The Latin word fractal (means fractured, divided, or broken) is commonly used for an image having the property of self-similarity in complex graphics [1]. Fractals have many applications in social sciences and engineering. In computer engineering, fractals are used to establish the security system, computer networking, image encryption, image compression, and cryptography [2]. In biology, fractals are used to study the culture of microorganisms, nerve system, etc. [3]. In physics, fractals are used in fluid mechanics to understand the nature of fluids and their properties. Fractals are used in electrical and electronics engineering (i.e., in the fabricating of antennae, radar system, capacitors, security control system, radio, and antennae for wireless system) [4, 5]. Moreover, architectural patterns and designs are also fractals [6]. Fractals have application in many other emerging fields [7–9].

Before the invention of computer, the researchers sketched aesthetic patterns, images, graphs, and geometries manually. The graph of cantor set, Koch snowflake, and Sierpinski’s triangles are the patterns that can be generated manually. In 1918, Gaston Julia and Pierre Fatou defined two complementary sets (i.e., Julia set and Fatou set). But they could not sketch the graphs of Julia set and Fatou set. After the invention of computers, Mandelbrot made it possible to draw the graphs of Julia set with help of computers in 1970. He studied the Julia set for a polynomial \( Q_{a_0}(z_m) = z_m^2 + a_0 \), where \( z \) is a complex variable and \( a_0 \) is a complex parameter. Mandelbrot presented the characteristics of Julia set in [10] and explained that Julia set had great diversity of aesthetic designs [11]. The Mandelbrot set for \( Q_{a_0}(z_m) = z_m^2 + a_0 \), where \( z \) is a complex variable and \( a_0 \) is a complex parameter, was discussed in [12]. The images resembled with Julia and Mandelbrot sets for rational and transcendental complex functions were visualized in [13]. Some 4D and 3D fractals for quaternions and bicomplex and tricomplex functions were studied in [14, 15] and [16]. To generalize Julia and Mandelbrot sets, initially Rani et al. used fixed-point theory in the generation of fractals (refer in [17, 18]). Some generalized fractals via explicit fixed-point iterative methods were studied in [19–24]. The implicit iterative methods were used to develop convergence criterion for fractals in [25–30]. There are many fixed-point methods that can be used for fractal generation [31–36].
There are some well-known criterions to generate the fractals such as distance estimator [37], potential function algorithms [38], and escape criteria [39]. In this paper, we use escape criterion conditions to sketch some bewitching Julia and Mandelbrot sets. In this paper, we develop some necessary conditions for the convergence of $|Q_m^n|$ to generate fractals (i.e., especially for Julia and Mandelbrot sets) via some fixed-point iterative methods. We use proposed algorithms [38], and escape criteria [39]. In this paper, we fractals such as distance estimator [37], potential function [40].

Definition 1 (Julia set [40]). Let $Q_m(z_m) = z_m^p + a_0$ be a complex polynomial with $p \geq 2$. Then, the set of points $J_{Q_m}$ in $C$ is named as the filled Julia set, when the orbits of the points in $J_{Q_m}$ does not move to $\infty$ as $m \rightarrow \infty$, i.e.,

$$J_{Q_m} = \{ z \in C : \left\{ \left\{ Q_m^m \right\}_{m=0}^{\infty} \right\}_{m=0}^{\infty} \text{ is bounded} \}, \tag{1}$$

where $Q_m^m$ is the $m$-th iterate of $z$. The set of boundary points of $J_{Q_m}$ is called the simple Julia set.

Definition 2 (Mandelbrot set [41]). The collection of all connected Julia sets is defined as the Mandelbrot set $M$, i.e.,

$$M = \left\{ a_0 \in C : J_{Q_m} \text{ is connected} \right\}. \tag{2}$$

Equivalently, the Mandelbrot set is defined as [42]

$$M = \left\{ a_0 \in C : \left\{ Q_m^m (0) \right\} \rightarrow \infty \text{ as } m \rightarrow \infty \right\}. \tag{3}$$

Since the critical point of $Q_m$ is $0$, so the authors set $z_0 = 0$ as an initial guess. There are many fixed-point iterative methods in literature that can be used to generate fractals. For each method, the authors prove escape criterion to generate fractals. In this paper, we use $M$, $M^*$, and $K$-iterative methods to visualize Julia and Mandelbrot sets. The proposed fixed-point iterative methods are defined as follows.

Definition 3 ($M^*$-iterative method [43]). Let $Q : C \rightarrow C$ be a complex polynomial with $p \geq 2$. For any $z_0 \in C$, the $M^*$-iterative method is defined as

$$\begin{align*}
z_{m+1} &= Q(w_k), \\
w_m &= Q(u_m), \\
\quad u_m &= (1 - a)z_m + aQ(z_m),
\end{align*} \tag{4}$$

where $a \in (0, 1]$ and $m = 0, 1, 2, \ldots$

Definition 4 ($M^*$-iterative method [43]). Let $Q : C \rightarrow C$ be a complex polynomial with $p \geq 2$. For any $z_0 \in C$, the $M^*$-iterative method is defined as

$$\begin{align*}
z_{m+1} &= Q(w_k), \\
w_m &= Q(v_m), \\
\quad u_m &= (1 - a)z_m + aQ(z_m),
\end{align*} \tag{5}$$

where $v_m = (1 - b)z_m + bQ(u_m), a, b \in (0, 1)$, and $m = 0, 1, 2, \ldots$

Definition 5 ($K$-iterative method [43]). Let $Q : C \rightarrow C$ be a complex polynomial with $p \geq 2$. For any $z_0 \in C$, the $M^*$-iterative method is defined as

$$\begin{align*}
z_{m+1} &= Q(w_k), \\
w_m &= Q(v_m), \\
\quad u_m &= (1 - a)z_m + aQ(z_m),
\end{align*} \tag{6}$$

where $v_m = (1 - b)Q(z_m) + bQ(u_m), a, b \in (0, 1)$, and $m = 0, 1, 2, \ldots$

The sequence of iterates $\{z_m\}_{m \in \mathbb{N}}$ defined by (4)–(6) is called the orbit.

3. Convergence Analysis

Here, we prove some convergence conditions (i.e., escape criterion) for complex polynomial $Q_{a_0}(z) = z^p + a_0$, where $p \geq 2$ and $a_0 \in C$ via $M$, $M^*$, and $K$-iterative methods, respectively. Without necessary conditions, we cannot generate fractal because the convergence condition is the basic key to run the algorithm. Throughout this section, we use $Q(z)$ as $Q_{a_0}(z)$ and $z_0 = z, u_0 = u, v_0 = v$, and $w_0 = w$ in the following way.

Theorem 1. Let $Q(z) = z^p + a_0$ be a complex polynomial with $|z| \geq |a_0| > (2/a)^{(p-1)/p}$ and $|z| \geq |a_0| > (2/b)^{(p-1)/p}$, where $p \geq 2, a, b \in (0, 1)$, and $a_0 \in C$. The sequence of iterates $\{z_m\}_{m \in \mathbb{N}}$ for the $K$-iterative method is defined as follows:

$$\begin{align*}
z_{m+1} &= Q(w_k), \\
w_m &= Q(v_m), \\
\quad u_m &= (1 - a)z_m + aQ(z_m),
\end{align*} \tag{7}$$

where $v_m = (1 - b)Q(z_m) + bQ(u_m), a, b \in (0, 1)$, and $m = 0, 1, 2, \ldots$ Then, $|z_m| \rightarrow \infty$ as $m \rightarrow \infty$.

Proof. Because $Q(z) = z^p + a_0$, where $a_0 \in C$, $z_0 = z$, $u_0 = u$, $v_0 = v$, and $w_0 = w$, then, for first step of the $K$-iterative method, we have

$$u_m = (1 - a)z_m + aQ(z_m). \tag{8}$$

For $m = 0$, we have
\[|u_0| = |(1 - a)z + aQ(z)| = |(1 - a)z + a(z^p + a_0)|\]
\[|u_0| \geq a|z^p + a_0| - (1 - a)|z|\]
\[\geq a|x^2| - a|a_0| - |z| + |z|\]
\[\geq a|z^2| - |z|, \quad \therefore |z| \geq |a_0|\]
\[|u_0| = |z|(a|z| - 1).\]

Since \(|z| \geq |a_0| > (2/a)^{(1/p-1)}\), this yields \(|z|^p(a|z|^{p-1} - 1)^p \geq |a|^p|z|^p|\). Thus, \(|u|^p \geq |a|^p|z|^p|\).

For second step of K-iteration, we have
\[w_m = Q(v_m)\]
where \(v_m = (1 - b)Q(z_m) + bQ(u_m)\). For \(m = 0\), we get
\[|w_0| = |Q(v_0)| = |v_0 + a_0|.\]

Thus,
\[|w_0| \geq |v_0^p| - |a_0|.\]

Since \(v_m = (1 - b)Q(z_m) + bQ(u_m)\), then
\[|v_0| = |(1 - b)Q(z_0)| + bQ(u_0)\]
\[|v_0| = |(1 - b)(z^p + a_0) + b(u^p + a_0)|\]
\[\geq b|u|^p - b|a_0| - (1 - b)|a_0|\]
\[\geq ab|z|^{p-1} - (1 - b)|z| - |a_0|, \quad \therefore |v_0^p| \geq |a|^p|z|^p|\]
\[\geq (ab - b + 1)|z|^p - |z|, \quad \therefore ab + (1 - b) > ab\]
\[|v_0| = |z|(|a|^p|z|^{p-1} - 1).\]

Since \(|z| > (2/a)^{(1/p-1)}\) and \(|z| > (2/b)^{(1/p-1)}\), this yields \(|z| > (2/ab)^{(1/p-1)}\). Following this, we get \(ab|z|^{p-1} - 1 > 1\) and \(|z|^p(ab|z|^{p-1} - 1)^p \geq |z|^p > ab|z|^p|\). Therefore,
\[|v_0^p| > ab|z|^p|\]

From (12) and (14), we have
\[|w_0^p| \geq ab|z|^p - |a_0|\]
\[\geq |z|(|ab|z|^{p-1} - 1), \quad \therefore |z| > |a_0|.\]

It follows that
\[|w_0^p| > ab|z|^p|\]

The last step of the K-iterative method is
\[z_{m+1} = Q(w_m).\]

For \(m = 0\), we have
\[|z_1| = |Q(w_0)| = |w_0^p + a_0|\]
\[\geq |w_0^p| - |a_0|\]
\[z_1 \geq ab|z|^p - |z|, \quad \therefore |z| > |a_0|.\]

From (17),
\[|z_1| \geq |z|(|ab|z|^{p-1} - 1)|.\]

Since \(|z| > (2/a)^{(1/p-1)}\) and \(|z| > (2/b)^{(1/p-1)}\), then \(|z| > (2/ab)^{(1/p-1)}\), this implies \(ab|z|^{p-1} - 1 > 1\). Thus, there exists positive number \(\eta > 0\) such that \(ab|z|^{p-1} - 1 > 1 + \eta\), which yields \(|z_1| > (1 + \eta)|z|\). Particularly, \(|z_1| > |z|\). Subsequently, \(|z_m| > (1 + \eta)^m|z|\). Hence, \(|z_m| \rightarrow \infty\) as \(m \rightarrow \infty\).

**Corollary 1.** Suppose that
\[|a_0| > \left(\frac{2}{a}\right)^{(1/p-1)},\]
\[|a_0| > \left(\frac{2}{b}\right)^{(1/p-1)},\]

then the orbit of the K-iterative method escapes to infinity.

**Corollary 2.** Suppose that \(a, b \in (0, 1)\) and
\[|z| > \max\left[\left|a_0\right|, \left(\frac{2}{a}\right)^{(1/p-1)}, \left(\frac{2}{b}\right)^{(1/p-1)}\right],\]

therefore, there exists \(\eta > 0\) such that \(|z_m| > (1 + \lambda)^m|z|\) and \(|z_m| \rightarrow \infty\) as \(m \rightarrow \infty\).

**Corollary 3.** Assume that
\[|z_k| > |z| > \max\left[\left|a_0\right|, \left(\frac{2}{a}\right)^{(1/p-1)}, \left(\frac{2}{b}\right)^{(1/p-1)}\right]\]

for some \(k \geq 0\). Thus, there exists \(\eta > 0\) such that \(|z_k| > (1 + \lambda)^m|z_m|\) and \(|z_m| \rightarrow \infty\) as \(m \rightarrow \infty\).

**Theorem 2.** Let \(Q(z) = z^p + a_0\) be a complex polynomial with \(|z| \geq |a_0| > (2/a)^{(1/p-1)}\) and \(|z| \geq |a_0| > (2/b)^{(1/p-1)}\), where \(p \geq 2, a, b \in (0, 1), \) and \(a_0 \in C\). The sequence of iterates \(\{z_m\}_{m\in\mathbb{N}}\) for the \(M^*-\)iterative method is defined as follows:
\[x_{m+1} = Q(x_m),\]
\[w_m = Q(v_m),\]
\[u_m = (1 - a)z_m + aQ(z_m),\]

where \(v_m = (1 - b)z_m + bQ(u_m), a, b \in (0, 1), \) and \(m = 0, 1, 2, \ldots\). Then, \(|z_m| \rightarrow \infty\) as \(m \rightarrow \infty\).

**Proof.** Because \(Q(z) = z^p + a_0\), where \(a_0 \in C, \) \(z_0 = z, \) \(u_0 = u, v_0 = v, \) and \(w_0 = w,\) then, for first step of the \(M^*-\)iterative method, we have
\[u_m = (1 - a)z_m + aQ(z_m).\]

For \(m = 0,\) we have
\[ |u_0| = |(1 - a)z + aQ(z)| \]
\[ = |(1 - a)z + a(z^p + a_0)| \]
\[ \geq |a|z^p + a_0 - (1 - a)|z| \]
\[ \geq a|z|^p - a|a_0| - |z| + |z| \]
\[ \geq a|z|^p - |z|, \quad \vdash |z| \geq |a_0| \]
\[ |u_0| = |z|(a|z|^{p-1} - 1). \] (25)

Since \(|z| \geq |a_0| > (2/a)^{(1/p-1)}\), this creates the situation \(|z|(a|z|^{p-1} - 1) > |z| \geq |a|z|.\) Thus, \(|u|^p \geq |a|z|^p\).

For the second step of the \(M^\ast\)-iterative method, we have
\[ w_m = Q(v_m), \] (26)
where \(v_m = (1 - b)z_m + bQ(u_m).\) For \(m = 0\), we get
\[ |w_0| = |Q(v_0)| = |v_0 + a_0|. \] (27)

Thus,
\[ |w_0| \geq |v_0| - |a_0|. \] (28)

Since \(v_m = (1 - b)z_m + bQ(u_m),\) then
\[ |v_0| = |(1 - b)z_0 + bQ(u_0)| \]
\[ |v_0| = |(1 - b)z + b(u^p + a_0)| \]
\[ \geq b|u|^p - b|a_0| - (1 - b)|z| \]
\[ \geq ab|z|^p - (1 - b)|z| - b|a_0|, \quad \vdash |u|^p \geq a|z|^p \] (29)
\[ \geq ab|z|^p - |z| + b|z| - b|z|, \quad \vdash |z| > |a_0| \]
\[ = ab|z|^p - |z||v_0| \]
\[ = |z|(a|z|^{p-1} - 1). \]

Since \(|z| > (2/a)^{(1/p-1)}\) and \(|z| > (2/b)^{(1/p-1)}\), this yields \(|z| > (2/a)^{(1/p-1)}\). Following this, we get \(ab|z|^{p-1} - 1 > 1\) and \(|z|^p(ab|z|^{p-1} - 1)^{1/p} > |z|^p > ab|z|^p\). Therefore,
\[ |v_0|^p > ab|z|^p. \] (30)

From (28) and (30), we have
\[ |w_0|^p \geq ab|z|^p - |a_0|^p \]
\[ \geq |z|(a|z|^{p-1} - 1), \quad \vdash |z| > |a_0|. \] (31)

It follows that
\[ |w_0|^p > ab|z|^p. \] (32)

The last step of the \(M^\ast\)-iterative method is
\[ z_{m+1} = Q(w_m). \] (33)

For \(m = 0\) and using (32), we have
\[ |z_1| = |Q(w_0)| \]
\[ = |w^p + a_0| \]
\[ \geq |w|^p - |a_0| \]
\[ z_1 \geq ab|z|^p - |z|, \quad \vdash |z| > |a_0|. \] (34)

Therefore,
\[ |z_1| \geq |z|(ab|z|^{p-1} - 1). \] (35)

Since \(|z| > (2/a)^{(1/p-1)}\) and \(|z| > (2/b)^{(1/p-1)}\), then \(|z| > (2/ab)^{(1/p-1)}\), this implies \(ab|z|^{p-1} - 1 > 1\). Thus, there exists positive number \(\eta > 0\) such that \(ab|z|^{p-1} - 1 > 1 + \eta,\) which yields \(|z_1| > (1 + \eta)|z|\). Particularly, \(|z_1| > |z|\). Subsequently, \(|z_m| > (1 + \eta)^m|z|\). Hence, \(|z_m| \to \infty\) as \(m \to \infty\).

**Corollary 4.** Suppose that
\[ |a_0| > \left(\frac{2}{a}\right)^{(1/p-1)}, \] (36)
\[ |a_0| > \left(\frac{2}{b}\right)^{(1/p-1)}, \]
then the orbit of the \(M^\ast\)-iterative method escapes to infinity.

**Corollary 5.** Suppose that \(a, b \in (0, 1)\) and
\[ |z| > \max\left[|a_0|, \left(\frac{2}{a}\right)^{(1/p-1)}, \left(\frac{2}{b}\right)^{(1/p-1)}\right], \] (37)
therefore, there exists \(\eta > 0\) such that \(|z_m| > (1 + \lambda)^m|z|\) and \(|z_m| \to \infty\) as \(m \to \infty\).

**Corollary 6.** Assume that
\[ z_k > |z| \geq \max\left[|a_0|, \left(\frac{2}{a}\right)^{(1/p-1)}, \left(\frac{2}{b}\right)^{(1/p-1)}\right], \] (38)
for some \(k \geq 0\). Thus, there exists \(\eta > 0\) such that \(|z_k| > (1 + \lambda)^m|z_k|\) and \(|z_k| \to \infty\) as \(m \to \infty\).

**Theorem 3.** Let \(Q(z) = z^n + a_0\) be a complex polynomial with \(|z| > |a_0| > (2/a)^{(1/p-1)}\), where \(p \geq 2, a, b \in (0, 1],\) and \(a_0 \in \mathbb{C}\).

The sequence of iterates \(\{z_m\}_{m=0}^\infty\) for the \(M^\ast\)-iterative method is defined as follows:
\[
\begin{align*}
x_{m+1} &= Q(x_m), \\
w_m &= Q(u_m), \\
u_m &= (1 - a)z_m + aQ(z_m),
\end{align*}
\] (39)
where \(a, b \in (0, 1]\) and \(m = 0, 1, 2, \ldots\) Then, \(|z_m| \to \infty\) as \(m \to \infty\).

**Proof.** Because \(Q(z) = z^n + a_0\), where \(a_0 \in \mathbb{C}, \ z_0 = z, \ u_0 = u, \ v_0 = v,\) and \(w_0 = w,\) then, for the first of \(M^\ast\)-iterative method, we have
\[ u_m = (1 - a)z_m + aQ(z_m). \] (40)
For \(m = 0\), we have
\[ |u_0| = |1 - a|z + aQ(z) |\]
\[ = |1 - a|z + a(z^p + a_0)| \]
\[ |u_0| \geq |a|z^p + |a|z - (1 - a)|z| \]
\[ \geq |a|z^p - |a||z| - |z|, \therefore |u_0| \geq |a|z^p - |z| \]
\[ |u_0| = |z|(|a|z^{p-1} - 1). \]

Since \( |z| \geq |a| > (2/a)^{(1/p-1)} \), this creates the situation \( |z|(|a|z^{p-1} - 1) \geq |a||z| \). Thus, \( |u_0| \geq |a|z^p \).

For the second step of \( M \)-iteration, we have
\[ w_m = Q(u_m). \] (42)

For \( m = 0 \), we get
\[ |w_0| = |Q(u_0)| = |u^p + a_0|. \] (43)

Thus,
\[ |w_0| = |Q(u_0)| = |u^p + a_0| \]
\[ \geq |u^p| - |a_0| \]
\[ \geq |a|z^p - |a||z|, \therefore |u^p| \geq |a|z^p - |z| \]
\[ |u_0| = |z|(|a|z^{p-1} - 1). \]

Since \( |z| > (2/a)^{(1/p-1)} \), it follows \( |a|z^{p-1} - 1 > 1 \) and \( |z^p|(|a|z^{p-1} - 1)^p > |z|^p |a|z^p \). Therefore,
\[ |w_0^p| > |bl|z^p|. \] (45)

In the last step of the \( M \)-iterative method, we have
\[ z_{m+1} = Q(w_m). \] (46)

For \( m = 0 \), we have
\[ |z_1| = |Q(w_0)| = |u^p + a_0| \]
\[ \geq |u^p| - |a_0| \]
\[ \geq |a|z^p - |a||z|, \therefore |z_1| > |a_0|. \] (47)

From (46),
\[ |z_1| \geq |z|(|a|z^{p-1} - 1). \] (48)

Since \( |z| > (2/a)^{(1/p-1)} \), this implies \( |a|z^{p-1} - 1 > 1 \). Thus, there exists positive number \( \eta > 0 \) such that \( |a|z^{p-1} - 1 > 1 + \eta, \) which yields \( |z_1| > (1 + \eta)|z| \). Particularly, \( |z_1| > |z| \). Subsequently, \( |z_{m+1}| > (1 + \eta)^m|z| \). Hence, \( |z_{m+1}| \longrightarrow \infty \) as \( m \longrightarrow \infty \).

\[ |a_0| > \left( \frac{2}{a} \right)^{(1/p-1)}, \] (49)

then the orbit of the \( M \)-iterative method escapes to infinity.

**Corollary 8.** Suppose that \( a, b \in (0, 1) \) and
\[ |z| > \max \left[ |a_0|, \left( \frac{2}{a} \right)^{(1/p-1)} \right], \] (50)

then, there exists \( \eta > 0 \) such that \( |z_m| > (1 + \lambda)^m|z| \) and \( |z_m| \longrightarrow \infty \) as \( m \longrightarrow \infty \).

**Corollary 9.** Assume that
\[ |z_k| > |z| > \max \left[ |a_0|, \left( \frac{2}{a} \right)^{(1/p-1)} \right] \] (51)

for some \( k \geq 0 \). Thus, there exists \( \eta > 0 \) such that \( |z_{km+1}| > (1 + \lambda)^m|z_m| \) and \( |z_m| \longrightarrow \infty \) as \( m \longrightarrow \infty \).

4. Applications of Fractals

To visualize the fractals, some convergence conditions are required, and actually, these are the main tools to execute the algorithm properly and sketch the desired type of fractals. In literature, the authors fixed maximum number of iterations up to hundred. To check self-similarity and get better results, we fixed the maximum number of iterations at 1000. In this section, we adjust two algorithms: one for the Julia set and other for the Mandelbrot set to generate fractals via proposed methods. We visualize some Julia and Mandelbrot sets for different involve parameters.

4.1. Julia Sets. Julia is known as the pioneer of complex fractals. In this subsection, we sketch some graphs of Julia set at different input parameters. We generate Julia sets for \( M, M^*, \) and \( K \)-iterative methods by using Algorithm 1 and compare the images of Julia set for proposed methods.

**Example 1.** In this example, we present the Julia sets for a polynomial \( Q(z) = z^2 + a_0 \) where \( a_0 \in \mathbb{C} \) in the orbits of \( M, M^*, \) and \( K \)-iterative methods, respectively. The graphs in Figures 1–3 for \( a_0 = -0.05 - 0.63i, \) \( a = 0.01, \) \( b = 0.9, \) and \( A = [-1.5, 1.5] \) are quadratic Julia sets in the orbits of \( M, M^*, \) and \( K \)-iterative methods, respectively. The images in Figures 1 and 3 are Julia sets resembling Chinese dragon having two repelling fixed points: one is at the right end spiral, and other is in spiral on the left side. The image in Figure 2 is a filled connected quadratic Julia set. The graphs in Figures 4–6 with \( a_0 = -0.8, \) \( a = 0.01, \) \( b = 0.9, \) and \( A = [-1.8, 1.8] \times [-1.3, 1.3] \) are in symmetry along the \( x \)-axis. Each image is a junction of two quadratic Mandelbrot sets having opposite directions but slightly different from each other in shape of bulbs on main body. Now for \( a_0 = 0.749i \) with \( a = 0.01, \) \( b = 0.9, \) and \( A = [-1.5, 1.5] \times [-1.3, 1.3] \), we notice the graphs have quite different shapes, as shown in Figures 7–9. The images in Figures 7 and 8 resemble the lighting in sky, while the image in Figure 8 has two big and many small spirals.
Example 2. In second example, we visualize some cubic Julia sets for a polynomial $Q(z) = z^3 + a_0$, where $a_0 \in \mathbb{C}$ in the orbits of $M, M^*$, and $K$-iterative methods, respectively. The images in Figures 10–12 are like the cubic Douady rabbits. We observe the image in Figure 10 is a smart Douady rabbit, in Figure 11 is a fat Douady rabbit, and in Figure 12 is a relatively weak but more attractive Douady rabbit for cubic complex polynomial. The main body of graphs in Figures 13–15 is like a circular saw having three teeth.
The curl shape top of each teeth of the main saw is joint with the teeth of small circular saw and so on. Each image in Figures 13–15 have a main circular saw-type body with three large saws and six small saws. The saws for $M$ and $K$ methods are sharp, while saw for $M^*$ is blunt. The input parameters are as follows:

(i) Figures 10–12 have input parameters $a_0 = 0.5 + 0.5i$, $a = 0.01$, $b = 0.9$, and $A = [-1.5, 1.5]^2$.
Figures 13–15 are $a = -0.8 - 0.3i, a = 0.7, b = 0.7,$ and $A = [-1.5, 1.5]^2$.

Example 3. This example presents some biquadratic Julia sets for a polynomial $Q(z) = z^4 + a_0$, where $a_0 \in \mathbb{C}$ in the orbits of $M, M^*$, and $K$-iterative methods, respectively. The image in Figure 16 has a variety of colours. The images in Figures 16–18 are disconnected Julia sets and different in shapes at the same inputs $a_0 = -0.8 - 0.3i, a = 0.7, b = 0.7$, and $A = [-1.5, 1.5]^2$.

4.2. Mandelbrot Sets. Mandelbrot examined the graph of complex polynomial $Q(z) = z^2 + a_0$ and observed that the main body of image is a cardioid having a large bulb symmetry along $x$-axis and two small bulbs symmetry along $y$-axis. The image of $Q(z)$ is usually called the classical Mandelbrot set, and it is also called God’s thumb. In this subsection, we sketch some graphs of Mandelbrot set at different input parameters for $M, M^*$, and $K$-iterative methods by using Algorithm 2 and compare the images of Mandelbrot set for proposed methods.

Example 4. In this example, we visualize some graphs of Mandelbrot sets for a polynomial $Q(z) = z^2 + a_0$, where $a_0 \in \mathbb{C}$ in the orbits of $M, M^*$, and $K$-iterative methods,
respectively. The input parameters for the graphs in Figures 19–21 are $a = 0.01$, $b = 0.9$, and $A = [-2, 0.55] \times [-1.5, 1.5]$. Figures 19–21 are quadratic Mandelbrot sets via $M$, $M^*$, and $K$-iterative methods, respectively. The shapes of bulbs in each image are different. The images in Figures 19 and 20 relatively resemble classical Mandelbrot set, but Figure 21 is slightly different in shape. The main body or primary part of the images contains a large
Input: $Q_{a_0} = z^p + a_0$ - a complex polynomial, $A$-covered area, $M = 1000$, $a, b \in (0, 1]$ - involved parameters, coloursmap$[0..h-1]$ with $h$ colours.

Output: sketched Mandelbrot set.

\begin{itemize}
\item[(1)] for $a_0 \in A$
do\hfill
\item[(2)] $R$-convergence condition for proposed method\hfill
\item[(3)] $m = 0$
\item[(4)] $z_0$ - initial guess for $Q_{a_0}$
\item[(5)] while $m \leq K$
do\hfill
\item[(6)] Proposed iterative method
\item[(7)] if $|z_{m+1}| > R$	hen\break\hfill
\item[(8)] $m = m + 1$
\item[(9)] $i = \lfloor (h - 1)m/M \rfloor$
\item[(10)] colour $a_0$ with coloursmap$[i]$
\end{itemize}

\textbf{Algorithm 2:} Pseudocode for Mandelbrot set.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{Mandelbrot_set_19.png}
\caption{Mandelbrot set for $Q(z)$ with $n = 2$ in $M$-orbit.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{Mandelbrot_set_20.png}
\caption{Mandelbrot set for $Q(z)$ with $n = 2$ in $M^*$-orbit.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{Mandelbrot_set_21.png}
\caption{Mandelbrot set for $Q(z)$ with $n = 2$ in $K$-orbit.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{Mandelbrot_set_22.png}
\caption{Mandelbrot set for $Q(z)$ with $n = 2$ in $M$-orbit.}
\end{figure}
number of bulbs in different sizes, but if we magnify any bulb of image, it reflects the shape of whole image. In the generation of graphs in Figures 22–24, we change input
parameters $a$ and $b$. We observe that, for every method, images are completely different in shapes. Also the graphs in Figures 22 and 24 have input parameters $a = 0.1$, $b = 0.5$, and $A = [-2, 0.55] \times [-1.5, 1.5]$, but the graph in Figure 23 has input parameters $a = 0.1$, $b = 0.5$, and $A = [-3, 0.85] \times [-2.5, 2.5]$. The large bulbs in Figure 23 are like the wings of fish, and also the image covered a large area compared to images in Figures 22 and 24. In Figures 25–27, we again change the parameters $a$ and $b$ as $a, b = 0.9$. All images have the same area as $A = [-2, 0.55] \times [-1.5, 1.5]$ but
different in shapes primary (i.e., main cardioid) and secondary (i.e., bulbs on cardioid) parts.

Example 5. In this example, we visualize some graphs of Mandelbrot sets for a polynomial $Q(z) = z^3 + a_0$, where $a_0 \in \mathbb{C}$ in the orbits of $M$, $M^*$, and $K$-iterative methods, respectively. From Figures 28–35, we perceive that each image has 2 cardioids, 2 large bulbs, and 4 small bulbs symmetry along $y$-axis. The shape of bulbs for each iterative method is different. The inputs for each cubic Mandelbrot set are as follows:

- Figures 28–30 input parameters are $a = 0.01$, $b = 0.9$, and $A = [-1.5, 1.5]^2$
- Figures 31–33 input parameters are $a = 0.1$, $b = 0.5$, and $A = [-1.5, 1.5]^2$
- Figures 34–35 input parameters are $a = 0.9$, $b = 0.9$, and $A = [-1.5, 1.5]^2$

Example 6. The last example demonstrates the ochto Mandelbrot sets for a polynomial $Q(z) = z^8 + a_0$, where $a_0 \in \mathbb{C}$ in the orbits of $M$, $M^*$, and $K$-iterative methods, respectively. All images for the graphs in Figures 36–38 have the same inputs as $a = 0.01$, $b = 0.9$, and $A = [-1.5, 1.5]^2$. We notice that 7 large bulbs appear on the main body of each ochto Mandelbrot set. The shape of bulbs for each method is also different in images.

5. Conclusions

We analyzed $M$, $M^*$, and $K$-iterative methods in the generation of Julia and Mandelbrot sets. We established some convergence conditions for the orbits of $M$, $M^*$, and $K$-iterative methods, respectively. We used the established convergence conditions in algorithms to sketch some Julia and Mandelbrot sets. Fascinating Julia and Mandelbrot sets were generated for different input parameters and compared the images. We observed that, for each proposed method, image is slightly different in shape from other two methods. Furthermore, we noticed that, for a very small change in any input parameter, the images drastically changed. Moreover, we concluded that the complex graphs of Julia and Mandelbrot sets generated in this research were the application of fractal geometry.
Data Availability
Data are included within this paper.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

Authors’ Contributions
All authors have contributed equally.

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