EVEN UNIVERSAL BINARY HERMITIAN LATTICES
OVER IMAGINARY QUADRATIC FIELDS

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Abstract. A positive definite even Hermitian lattice is called even universal if it represents all even positive integers. We introduce a method to get all even universal binary Hermitian lattices over imaginary quadratic fields \( \mathbb{Q}(\sqrt{-m}) \) for all positive square-free integers \( m \) and we list optimal criterions on even universality of Hermitian lattices over \( \mathbb{Q}(\sqrt{-m}) \) which admits even universal binary Hermitian lattices.

1. Introduction

Lagrange’s four square theorem says that every positive integer can be written as a sum of four squares of integers. A quadratic form, like sum of four squares, is called universal, if it represents all positive integers. In 1997, Conway, Schneeberger and Bhargava announced the fifteen theorem for classical universal quadratic forms, which characterizes the universality by representability of nine critical numbers, namely, 1, 2, 3, 5, 6, 7, 10, 14 and 15 (see [3], [1]). Recently, Bhargava and Hanke enunciated that they proved the 290-theorem characterizing the universality of (nonclassical) quadratic forms. That is, if a (nonclassical) quadratic form represents 29 numbers, say, 1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 34, 35, 37, 42, 58, 93, 110, 145, 203, 290, then it is universal (see [2]).

It is a natural attempt to extend these results to Hermitian lattices over imaginary quadratic fields. A positive definite Hermitian lattice over an imaginary quadratic field is called universal if it represents all positive integers. There are many contributions to list all universal binary Hermitian lattices over imaginary quadratic fields (see [4], [6], [11]). In the previous article [9], the authors provided a list of universal Hermitian lattices over imaginary quadratic field \( \mathbb{Q}(\sqrt{-m}) \) for all positive square-free integers \( m \), and the Conway-Schneeberger-Bhargava type criterion on universality, so called the fifteen theorem for universal Hermitian lattices.

The goal of this article is to prove the analogue of the previous paper [9] for even universal Hermitian lattices over imaginary quadratic fields \( \mathbb{Q}(\sqrt{-m}) \) for all possible positive square-free integers \( m \). First, we find all imaginary quadratic fields \( \mathbb{Q}(\sqrt{-m}) \) for positive square-free integers \( m \) that admit an even universal binary Hermitian lattice and classify all such lattices. A Hermitian lattice is even if its norm ideal is generated by 2. A primitive even Hermitian lattice is called even universal if it represents all positive even integers. Next, we determine the even

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critical set \( C[m] \) of an imaginary quadratic field \( \mathbb{Q}(\sqrt{-m}) \) which admits an even universal binary Hermitian lattice.

2. Review on Hermitian lattices over imaginary quadratic fields

In this chapter, we will review Hermitian lattices, in particular, we will review a matrix presentation of both free and non-free Hermitian lattices. That was suggested by the authors in [9].

Let \( E \) denote an imaginary quadratic field \( \mathbb{Q}(\sqrt{-m}) \) for a positive square-free integer \( m \) with nontrivial \( \mathbb{Q} \)-involution and let \( \mathfrak{o} \) be the ring of integers of \( E \). It is well-known that \( \mathfrak{o} = \mathbb{Z}[\omega] \), where \( \omega = \sqrt{-m} \) if \( m \equiv 1, 2 \pmod{4} \) or \( \omega = \frac{1 + \sqrt{-m}}{2} \) if \( m \equiv 3 \pmod{4} \). An \( \mathfrak{o} \)-lattice \( L \) means a finitely generated \( \mathfrak{o} \)-module in the Hermitian space \( (V, H) \), where \( V \) is an \( n \)-dimensional vector space over \( E \) with a nondegenerate Hermitian map \( H : V \times V \to E \). Without specific mention, we will assume that all lattices are positive definite and \( H(v_1, v_2) \in \mathfrak{o} \) for all \( v_1, v_2 \in L \). Since \( H(v_1, v_2) = \overline{H(v_2, v_1)} \), the (Hermitian) norm \( H(v) = H(v, v) \) is in \( \mathbb{Z} \) for every \( v \in L \). If \( a = H(v) \) for some \( v \in L \), we say that \( a \) is represented by \( L \).

The lattice \( L \) can be written as

\[
L = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n
\]

with ideals \( a_i \subset \mathfrak{o} \) and vectors \( v_i \in V \). The volume ideal \( \mathfrak{v} L \) of \( L \) is defined by

\[
\mathfrak{v} L = (a_1 \bar{a}_1)(a_2 \bar{a}_2)\cdots(a_n \bar{a}_n) \det(H(v_i, v_j))_{1 \leq i, j \leq n}.
\]

The volume of \( L \) is a principal ideal. The norm ideal \( \mathfrak{n} L \) of \( L \) is an \( \mathfrak{o} \)-ideal generated by the set \( \{H(v)|v \in L\} \). The scale ideal \( \mathfrak{s} L \) of \( L \) is an \( \mathfrak{o} \)-ideal generated by the set \( \{H(v, w)|v, w \in L\} \). It is clear that \( \mathfrak{n} L \subseteq \mathfrak{s} L \). If \( \mathfrak{n} L = \mathfrak{s} L \), then we call \( L \) normal. Otherwise, we call \( L \) subnormal. This paper focuses on the subnormal lattices with \( \mathfrak{n} L = 2\mathfrak{o} \).

If \( L \) is a free \( \mathfrak{o} \)-module, then we can write \( L = \mathfrak{o} v_1 + \cdots + \mathfrak{o} v_n \). The matrix \( M_L = (H(v_i, v_j))_{1 \leq i, j \leq n} \) is called the Gram matrix of \( L \) and is a matrix presentation of \( L \). If the matrix is diagonal, we denote it by \( \langle H(v_1), H(v_2), \ldots, H(v_n) \rangle \). But, if \( L \) is not a free \( \mathfrak{o} \)-module, then \( L = \mathfrak{o} v_1 + \cdots + \mathfrak{o} v_{n-1} + \alpha v_n \) for some ideal \( \mathfrak{a} \subset \mathfrak{o} \) [12, 81:5]. Since any ideal in \( \mathfrak{o} \) is generated by at most two elements, we can write \( L = \mathfrak{o} v_1 + \cdots + \mathfrak{o} v_{n-1} + (\alpha, \beta)\mathfrak{o} v_n \) for some \( \alpha, \beta \in \mathfrak{o} \). Therefore, we consider the following \((n+1) \times (n+1)\)-matrix as a formal Gram matrix for \( L \):

\[
M_L = \begin{pmatrix}
H(v_1, v_1) & \cdots & H(v_1, \alpha v_n) & H(v_1, \beta v_n) \\
\vdots & \ddots & \vdots & \vdots \\
H(\alpha v_n, v_1) & \cdots & H(\alpha v_n, \alpha v_n) & H(\alpha v_n, \beta v_n) \\
H(\beta v_n, v_1) & \cdots & H(\beta v_n, \alpha v_n) & H(\beta v_n, \beta v_n)
\end{pmatrix}.
\]

Note that this matrix \( M_L \) is positive semi-definite, but it represents an \( n \)-ary positive definite Hermitian lattice. A scaled lattice \( L^a \) is obtained from the Hermitian map \( H_{L^a}(\cdot, \cdot) = aH_L(\cdot, \cdot) \) with \( a \in \mathbb{Q} \). If \( M \) is a matrix presentation of a lattice \( L \), \( aM \) is the matrix presentation of a scaled lattice \( L^a \).
Considering a 2\textit{n}-dimensional vector space \(\tilde{V}\) over \(\mathbb{Q}\) corresponding to \(V\) as defined in [7], we can regard \((V, H)\) over \(E\) as a 2\textit{n}-dimensional quadratic space \((\tilde{V}, B_H)\) over \(\mathbb{Q}\) where \(B_H(x, y) = \frac{1}{2}[H(x, y) + H(y, x)] = \frac{1}{2}\text{Tr}_{E/\mathbb{Q}}(H(x, y))\). Analogously, by viewing \(L\) as a \(\mathbb{Z}\)-lattice in \((\tilde{V}, B_H)\) we can obtain a quadratic \(\mathbb{Z}\)-lattice \(\tilde{L}\) in \(\tilde{V}\) associated to a Hermitian \(\alpha\)-lattice \(L\) in \(V\). For example, the quadratic form associated to the Hermitian lattice \(\langle 1 \rangle\) over \(\mathbb{Q}(\sqrt{-m})\) is

\[
\begin{cases}
x_1^2 + my_1^2 & \text{if } m \equiv 1, 2 \pmod{4}, \\
x_1^2 + x_1y_1 + \frac{1+m}{2}y_1^2 & \text{if } m \equiv 3 \pmod{4}.
\end{cases}
\]

For unexplained terminologies, notations and basic facts about quadratic forms and Hermitian lattices, we refer the readers to [12], [7] and [9].

3. Even universal binary Hermitian lattices

A (even resp.) Hermitian lattice is said to be (even resp.) \textit{universal} if it represents every positive (even resp.) integers. If a Hermitian lattice \(L\) is not (even resp.) universal, we define the (even resp.) \textit{truant} of \(L\) to be the smallest positive (even resp.) integer not represented by \(L\). An \textit{even escalation} of a nonuniversal Hermitian lattice \(L\) is defined to be a process of getting any Hermitian lattice which generated by \(L\) and any vectors whose Hermitian norm is equal to the even truant of \(L\). An \textit{even escalation lattice} is a lattice that can be obtained from a finite sequence of successive even escalations starting from the trivial lattice \(\{0\}\). Then \(\langle 2 \rangle\) is the even escalation lattice of rank 1. A \textit{spurious escalation} of a Hermitian lattice \(L\) is a process of finding any lattice \(\tilde{L}\) which contains the lattice \(L\) and whose rank is the same as rank \(L\). Since \(vL \subseteq v\tilde{L}\), a spurious escalation terminates with finite steps. Important in the proof Theorem 1 is the notion of even escalations and spurious escalations. In particular, to find all candidates of even universal binary Hermitian lattices, the spurious escalations are imperative. In this article, if there is no confusion, we use the term escalation instead of the even escalation.

We set

\[
[a_1, \alpha, a_2] := \begin{pmatrix} a_1 & \alpha \\ \alpha & a_2 \end{pmatrix} \quad \text{and} \quad [a_1, a_2, a_3 ; \alpha, \beta, \gamma] := \begin{pmatrix} a_1 & \alpha & \beta \\ \alpha & a_2 & \gamma \\ \beta & \gamma & a_3 \end{pmatrix}
\]

for convenience.

\textbf{Lemma 1.} A \textit{primitive even Hermitian lattice} exists over the field \(\mathbb{Q}(\sqrt{-m})\) only when \(m \not\equiv 3 \pmod{4}\).

\textbf{Proof.} Suppose \(L\) is a primitive even universal binary Hermitian lattice over \(\mathbb{Q}(\sqrt{-m})\), where \(m \equiv 3 \pmod{4}\). Let \(u\) and \(v\) be vectors of \(L\) such that \(H(u) = 2a\), \(H(v) = 2b\), \(H(u, v) = c + d\omega\) for some integers \(a, b, c, d\). Since \(L\) is primitive, both \(c\) and \(d\) are not even. If \(d\) is odd, then \(H(u + v) = 2a + 2b + 2c + d\) is not even. If \(d\) is even, then \(c\) is odd and \(H(u + \omega v) = 2a + \frac{1+m}{2}b + c + \frac{1+m}{2}d\) is not even. Hence we get the result. \(\square\)
Remark 1. Let $L$ be an even binary Hermitian lattice over $\mathbb{Q}(\sqrt{-m})$ and let $f_L$ be a corresponding integral quadratic form of $L$. Since $\frac{1}{2}f_L$ is a nonclassical integral quadratic form, the universality of $\frac{1}{2}f_L$ can be determined by Bhargava and Hanke’s 290-theorem [2]. The theorem is that a nonclassical integral quadratic form over $\mathbb{Q}$ represents all $a \in \text{BH}$ if and only if it is universal, where

$$\text{BH} = \left\{ 1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 34, 35, 37, 42, 58, 93, 110, 145, 203, 290 \right\}.$$

Therefore $L$ is even universal if and only if $L$ represents all $2a$ for $a \in \text{BH}$.

Theorem 1. A primitive even universal binary Hermitian lattice exists over the field $\mathbb{Q}(\sqrt{-m})$ if and only if $m$ is

$$1, 2, 5, 6, 10, 13, 14, 21, 22, 29, 34, 37 \text{ or } 38.$$

Moreover, we have a complete list of fifty two primitive even universal binary Hermitian lattices (see Table[7]).

| Field | Primitive even universal binary Hermitian lattices (vol:a means the volume $a\omega$) |
|-------|----------------------------------------------------------------------------------|
| $\mathbb{Q}(\sqrt{-1})$ | $[2, -1 + \omega, 2]_{\text{vol:2}}$, $[2, 1, 4]_{\text{vol:7}}$, $[2, -1 + \omega, 6]_{\text{vol:10}}$, $[2, -1 + \omega, 4]_{\text{vol:6}}$, $[2, -1 + \omega, 4]_{\text{vol:6}}$ |
| $\mathbb{Q}(\sqrt{-2})$ | $[2, -1 + \omega, 2]_{\text{vol:1}}$, $[2, -1 + \omega, 4]_{\text{vol:5}}$, $[2, -1 + \omega, 4]_{\text{vol:9}}$, $[2, -1 + \omega, 4]_{\text{vol:14}}$, $[2, -1 + \omega, 10]_{\text{vol:17}}$, $[2, 1, 6]_{\text{vol:11}}$ |
| $\mathbb{Q}(\sqrt{-5})$ | $[2, 2, 4; -1, \omega, -1]_{\text{vol:1}}$, $[2, 2, 4; -1, 0, -1 + \omega]_{\text{vol:1}}$, $[2, -1 + \omega, 4]_{\text{vol:2}}$ |
| $\mathbb{Q}(\sqrt{-6})$ | $[2, -1 + \omega, 4]_{\text{vol:1}}$, $[2, 2, 4; -1, 0, \omega]_{\text{vol:1}}$, $[2, 4, 1; 0, \omega, -2 + 2\omega]_{\text{vol:2}}$, $[2, 2, 4; -1, 0, -1 + \omega]_{\text{vol:2}}$, $[2, -1 + \omega, 4]_{\text{vol:2}}$ |
| $\mathbb{Q}(\sqrt{-10})$ | $[2, -1 + \omega, 2]_{\text{vol:1}}$, $[2, 2, 4; -1, 0, -1 + \omega]_{\text{vol:1}}$, $[2, 4, 1; 0, \omega, -2 + 2\omega]_{\text{vol:2}}$, $[2, -1 + \omega, 4]_{\text{vol:2}}$, $[2, 1, 4]_{\text{vol:7}}$ |
| $\mathbb{Q}(\sqrt{-13})$ | $[2, 4, 4; 0, -1, -1 + \omega]_{\text{vol:1}}$, $[2, 4, 4; 0, -1, -1 + \omega]_{\text{vol:1}}$, $[2, 4, 4; 0, -1, -1 + \omega]_{\text{vol:1}}$, $[2, 4, 4; 0, -1, -1 + \omega]_{\text{vol:1}}$ |
| $\mathbb{Q}(\sqrt{-14})$ | $[2, 4, 4; -1, 0, \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, 0, \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, 0, \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, 0, \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, 0, \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, 0, \omega]_{\text{vol:1}}$ |
| $\mathbb{Q}(\sqrt{-17})$ | $[2, 4, 4; -1, -1, -1 + \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, -1, -1 + \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, -1, -1 + \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, -1, -1 + \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, -1, -1 + \omega]_{\text{vol:1}}$ |
| $\mathbb{Q}(\sqrt{-21})$ | $[2, 4, 4; -1, 0, \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, 0, \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, 0, \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, 0, \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, 0, \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, 0, \omega]_{\text{vol:1}}$ |
| $\mathbb{Q}(\sqrt{-29})$ | $[2, 4, 4; -1, 0, \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, 0, \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, 0, \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, 0, \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, 0, \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, 0, \omega]_{\text{vol:1}}$ |
| $\mathbb{Q}(\sqrt{-37})$ | $[2, 4, 4; -1, -1, -1 + \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, -1, -1 + \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, -1, -1 + \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, -1, -1 + \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, -1, -1 + \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, -1, -1 + \omega]_{\text{vol:1}}$ |
| $\mathbb{Q}(\sqrt{-41})$ | $[2, 4, 4; -1, -1, -1 + \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, -1, -1 + \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, -1, -1 + \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, -1, -1 + \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, -1, -1 + \omega]_{\text{vol:1}}$ |
| $\mathbb{Q}(\sqrt{-49})$ | $[2, 4, 4; -1, -1, -1 + \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, -1, -1 + \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, -1, -1 + \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, -1, -1 + \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, -1, -1 + \omega]_{\text{vol:1}}$, $[2, 4, 4; -1, -1, -1 + \omega]_{\text{vol:1}}$ |

**Table I.** Primitive even universal binary Hermitian lattices

Proof. Let $L$ be an even universal binary Hermitian lattice over $\mathbb{Q}(\sqrt{-m})$. By Lemma[1] $m \not\equiv 3 \pmod{4}$. To find all candidates of even universal binary Hermitian lattices, we do escalations and spurious escalations for all possible fields.
Table II. Spurious escalation when \( m = 10, 13, 14, 17, 21, 22, 29, 34, 37, 38 \)

Suppose \( m \geq 10 \). Since \( L \) represents all even positive integers, \( L \) contains a lattice \( \langle 2 \rangle \). Since a unary lattice \( \langle 2 \rangle \) fails to represent 4, \( L \) contains a lattice \( \sqrt{2} \alpha \) for some \( \alpha \in \mathfrak{o} \). From the positive definite condition, \( 8 - \alpha \overline{\alpha} \geq 0 \), hence \( \alpha = 0, \pm 1, \pm 2 \). So we obtain reduced escalation lattices \( \sqrt{2} \alpha \) which are contained in \( L \). Now we will do spurious escalations with these reduced binary lattices. Suppose \( L \) contains \( \ell = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \). Since \( \ell \) fails to represent 6, \( L \) contains

\[
\tilde{\ell} = \begin{pmatrix} 2 & 0 & \beta \\ 0 & 2 & \gamma \\ \beta & \gamma & 6 \end{pmatrix}
\]

for some \( \beta, \gamma \in \mathfrak{o} \) with \( \det \tilde{\ell} = 0 \). We have only one candidate
\[
\begin{pmatrix}
2 & 0 & -1 \\
0 & 2 & -1 + \omega \\
-1 & -1 + \sqrt{-1} & 6
\end{pmatrix} \cong \begin{pmatrix}
2 & -1 + \omega & -1 + \omega \\
-1 + \omega & 6
\end{pmatrix}
\]
over \(\mathbb{Q}(\sqrt{-10})\). Since \(\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\) fails to represent 10 and \(\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}\) fails to represent 6, spurious escalation lattices are as following Table III. Note that there is no escalation of \(\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\), \(\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}\) over \(\mathbb{Q}(\sqrt{-m})\) for all \(m \geq 41\) and \(m = 26, 30, 33\). All spurious escalation lattices in Table III represent all positive even integers except \(\begin{pmatrix} 2 & 0 \\ 0 & 4 \\ \pm 1 + \omega \\ \pm 1 + \omega \end{pmatrix}\) over \(\mathbb{Q}(\sqrt{-21})\). Since \(\begin{pmatrix} 2 & 0 \\ 0 & 4 \\ \pm 1 + \omega \\ \pm 1 + \omega \end{pmatrix}\) fails to represent 14 over \(\mathbb{Q}(\sqrt{-21})\), we spuriously escalate it with its truant 14 then we find a unimodular lattice \(\begin{pmatrix} 2 & 0 \\ 0 & 4 \\ \pm 1 + \omega \\ \pm 1 + \omega \end{pmatrix}\) which represents all positive even integers. But this lattice also can be spuriously escalated from \(\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}\) with the truant 6.

Suppose \(m = 1\). Since \(L\) contains a lattice \(\langle 2 \rangle\) and the unary lattice \(\langle 2 \rangle\) fails to represent 6 over \(\mathbb{Q}(\sqrt{-1})\), \(L\) contains a lattice \(\begin{pmatrix} 2 & \alpha \\ 0 & 6 \end{pmatrix}\), for some \(\alpha \in o\). From the positive definite condition, \(12 - \alpha \sqrt{-1} \geq 0\). So we obtain reduced lattices \(\begin{pmatrix} 2 & 0 \\ 0 & 2a \end{pmatrix}\), \(\begin{pmatrix} 2 & 1 \\ 1 & 2a \end{pmatrix}\), \(\begin{pmatrix} 2 & -1 + \omega \\ -1 + \omega & 2 \end{pmatrix}\) and \(\begin{pmatrix} 2 & -1 + \omega \\ -1 + \omega & 6 \end{pmatrix}\), where \(a = 1, 2, 3\). Even though these lattices are even universal, we should do spurious escalation with these lattices to find all binary even universal lattices. For \(\ell = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}\), a spuriously escalated lattice of \(\ell\) has a form \(\bar{\ell} = \begin{pmatrix} 2 & 0 \\ 0 & 6 \\ \alpha & \beta \end{pmatrix}\), where \(\alpha = \alpha_1 + \omega \alpha_2\), \(\beta = \beta_1 + \omega \beta_2 \in o\) such that \(|\alpha_i| \leq 1\), \(|\beta_i| \leq 3\) and \(b \in \mathbb{N}\). Since any sublattice of \(\bar{\ell}\) is positive definite, \(b\) is bounded. Moreover \(\bar{\ell}\) satisfies \(\text{det} \bar{\ell} = 0\) and the volume ideal of any sublattice of \(\bar{\ell}\) is contained in the volume ideal \(\mathfrak{v} \ell = 120o\). So we get spurious escalation lattices \(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\) and \(\begin{pmatrix} 2 & -1 + \omega \\ -1 + \omega & 4 \\ -3 + 3 \omega \end{pmatrix}\) \(\cong \begin{pmatrix} 2 & -1 + \omega \\ -1 + \omega & 4 \end{pmatrix}\). Note that \(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\) was already found at the previous step.

With almost identical arguments, we get the following Table III. Note that \(\mathbb{Q}(\sqrt{-1})\) and \(\mathbb{Q}(\sqrt{-2})\) are P.I.D.
| Fields | Reduced escalation lattices | Truant | Spurious escalation lattices | Truant |
|--------|-----------------------------|--------|-----------------------------|--------|
| \( \mathbb{Q}(\sqrt{-1}) \) | \([2, 0, 2]_{\text{vol:4}} \) none | No new lattice | \([2, -1 + \omega, 6]_{\text{vol:2}} \) none | No new lattice |
| | \([2, 0, 4]_{\text{vol:8}} \) none | No new lattice | \([2, -1 + \omega, 4]_{\text{vol:6}} \) none | No new lattice |
| | \([2, 0, 6]_{\text{vol:12}} \) none | No new lattice | \([2, 1, 4]_{\text{vol:11}} \) N.A. | No new lattice |
| | \([2, 1, 2]_{\text{vol:7}} \) none | No new lattice | \([2, -1 + \omega, 2]_{\text{vol:2}} \) none | No new lattice |
| | \([2, -1 + \omega, 6]_{\text{vol:10}} \) none | No new lattice | | |
| \( \mathbb{Q}(\sqrt{-2}) \) | \([2, 0, 2]_{\text{vol:4}} \) none | No new lattice | \([2, -1 + \omega, 4]_{\text{vol:5}} \) none | No new lattice |
| | \([2, 0, 4]_{\text{vol:8}} \) none | No new lattice | \([2, 2, 6 : 0, -1 + \omega, 1 + \omega]_{\text{vol:2}} \) none | No new lattice |
| | \([2, 0, 6]_{\text{vol:12}} \) none | No new lattice | \([2, 2, 4 : -1, 0, -1 + \omega]_{\text{vol:1}} \) none | No new lattice |
| | \([2, 0, 8]_{\text{vol:16}} \) none | No new lattice | \([2, 2, 4 : -1, -1, -1]_{\text{vol:1}} \) none | No new lattice |
| | \([2, 0, 10]_{\text{vol:20}} \) none | No new lattice | \([2, 2, 4 : -1, -1, 0]_{\text{vol:1}} \) none | No new lattice |
| | \([2, 2, 4 : -1, 0, -1 + \omega]_{\text{vol:1}} \) none | No new lattice | \([2, 2, 4 : -1, 0, -1]_{\text{vol:1}} \) none | No new lattice |
| | \([2, 0, 14]_{\text{vol:14}} \) none | No new lattice | \([2, 2, 4 : -1, -1, -1]_{\text{vol:1}} \) none | No new lattice |
| | \([2, 2, 4 : -1, 0, -1]_{\text{vol:1}} \) none | No new lattice | \([2, 2, 4 : -1, 0, -1 + \omega]_{\text{vol:1}} \) none | No new lattice |
| | \([2, 2, 4 : -1, 0, -1 + \omega]_{\text{vol:1}} \) none | No new lattice | \([2, 2, 4 : -1, 0, -1 + \omega]_{\text{vol:1}} \) none | No new lattice |
| \( \mathbb{Q}(\sqrt{-5}) \) | \([2, 0, 2]_{\text{vol:4}} \) none | \([2, 2, 6 : 0, -1 + \omega, -1 + \omega]_{\text{vol:2}} \) none | | |
| | \([2, 0, 4]_{\text{vol:8}} \) none | \([2, 2, 4 : -1, 0, -1 + \omega]_{\text{vol:1}} \) none | | |
| | \([2, 2, 4 : -1, 0, -1 + \omega]_{\text{vol:1}} \) none | \([2, 2, 4 : -1, -1, -1]_{\text{vol:1}} \) none | | |
| | | \([2, 2, 4 : -1, -1, 0]_{\text{vol:1}} \) none | | |
| | \([2, 2, 4 : -1, -1, -1]_{\text{vol:1}} \) none | \([2, 2, 4 : 1, -1, 0]_{\text{vol:1}} \) none | | |
| | \([2, 2, 4 : 1, -1, 0]_{\text{vol:1}} \) none | \([2, 2, 4 : 1, -1, -1]_{\text{vol:1}} \) none | | |
| \( \mathbb{Q}(\sqrt{-6}) \) | \([2, 0, 2]_{\text{vol:4}} \) none | \([2, 2, 6 : 0, -1 + \omega, -1 + \omega]_{\text{vol:2}} \) none | | |
| | \([2, 0, 4]_{\text{vol:8}} \) none | \([2, 2, 4 : -1, 0, -1 + \omega]_{\text{vol:1}} \) none | | |
| | | \([2, 2, 4 : -1, 0, -1 + \omega]_{\text{vol:1}} \) none | | |
| | \([2, 2, 4 : -1, 0, -1 + \omega]_{\text{vol:1}} \) none | \([2, 2, 4 : -1, 0, -1 + \omega]_{\text{vol:1}} \) none | | |
| | \([2, 2, 4 : -1, 0, -1]_{\text{vol:1}} \) none | \([2, 2, 4 : -1, 0, -1 + \omega]_{\text{vol:1}} \) none | | |

**Table III**: Spurious escalation when \( m = 1, 2, 5, 6 \)

Therefore, we have candidates of even universal Hermitian lattices over \( \mathbb{Q}(\sqrt{-m}) \) as listed in Table II and Table III. The proof of the even universalities of these lattices proceeds by Bhargava and Hanke’s 290-theorem (See Remark II). \( \square \)
4. FINITENESS THEOREMS

In this section, we will find a set $C[m]$ of even critical numbers for each imaginary quadratic field $\mathbb{Q}(\sqrt{-m})$ which admits even universal binary Hermitian lattices. From the Bhargava and Hanke’s 290-theorem, $L$ is even universal if and only if $L$ represents all $2a$ for $a \in \text{BH}$ (see Remark [1]). That implies $C[m] \subseteq \text{BH}^*$ where $\text{BH}^* = \{2a \mid a \in \text{BH}\}$. But these twenty nine positive integers in the set BH are redundant for some imaginary quadratic fields. So we will seek an optimal set $C[m]$ of even critical numbers for each imaginary quadratic field $\mathbb{Q}(\sqrt{-m})$ which admit even universal binary Hermitian lattices.

The following Lemma [2] says that an even truant of a Hermitian lattice is an even critical number, that is essential to the proof of the first part of Theorem 2.

**Lemma 2.** Let $L$ be an even Hermitian lattice over $\mathbb{Q}(\sqrt{-m})$. If $L$ represents all even positive integers less than $2a$ but does not represent $2a$, then

$$L' = L \perp \langle 2a + 2, 2a + 2, 2a + 2, 2a + 2 \rangle \perp \langle 4a + 2 \rangle$$

represents all even positive integers except $2a$.

**Proof.** By Lagrange’s four square theorem, it is clear. \qed

**Theorem 2.** If a primitive even Hermitian lattice over $\mathbb{Q}(\sqrt{-m})$ represents the following even critical numbers in $C[m]$ (see Table [IV]), then it is even universal.

| Field $\mathbb{Q}(\sqrt{-m})$ | The Set $C[m]$ of Even critical numbers |
|--------------------------------|----------------------------------------|
| $\mathbb{Q}(\sqrt{-1})$      | 2, 6,                                   |
| $\mathbb{Q}(\sqrt{-2})$      | 2, 10, 14,                              |
| $\mathbb{Q}(\sqrt{-5})$      | 2, 4, 6,                                |
| $\mathbb{Q}(\sqrt{-6})$      | 2, 4, 6, 10,                            |
| $\mathbb{Q}(\sqrt{-10})$     | 2, 4, 6, 10, 12, 14, 30,                |
| $\mathbb{Q}(\sqrt{-13})$     | 2, 4, 6, 10, 12, 14, 20,                |
| $\mathbb{Q}(\sqrt{-14})$     | 2, 4, 6, 10, 12, 14, 20, 26, 42,        |
| $\mathbb{Q}(\sqrt{-17})$     | 2, 4, 6, 10, 12, 14, 20, 26, 28, 30,    |
| $\mathbb{Q}(\sqrt{-21})$     | 2, 4, 6, 10, 12, 14, 20, 26, 28, 30, 34, 38, |
| $\mathbb{Q}(\sqrt{-22})$     | 2, 4, 6, 10, 12, 14, 20, 26, 28, 30, 34, 38, 42,  |
| $\mathbb{Q}(\sqrt{-26})$     | 2, 4, 6, 10, 12, 14, 20, 26, 28, 30, 34, 38, 42, 46, |
| $\mathbb{Q}(\sqrt{-29})$     | 2, 4, 6, 10, 12, 14, 20, 26, 28, 30, 34, 38, 42, 46, 52, |
| $\mathbb{Q}(\sqrt{-30})$     | 2, 4, 6, 10, 12, 14, 20, 26, 28, 30, 34, 38, 42, 46, 52, 58, |
| $\mathbb{Q}(\sqrt{-33})$     | 2, 4, 6, 10, 12, 14, 20, 26, 28, 30, 34, 38, 42, 46, 52, 58, 60, 62, |
| $\mathbb{Q}(\sqrt{-34})$     | 2, 4, 6, 10, 12, 14, 20, 26, 28, 30, 34, 38, 42, 46, 52, 58, 60, 62, |
| $\mathbb{Q}(\sqrt{-37})$     | 2, 4, 6, 10, 12, 14, 20, 26, 28, 30, 34, 38, 42, 46, 52, 58, 60, 62, 68, 70, |
| $\mathbb{Q}(\sqrt{-38})$     | 2, 4, 6, 10, 12, 14, 20, 26, 28, 30, 34, 38, 42, 46, 52, 58, 60, 62, 68, 70, 74 |

**Table IV.** The set $C[m]$ of even critical numbers for $\mathbb{Q}(\sqrt{-m})$

**Proof.** By Lemma [2], if $2a$ is a truant of a lattice $L$ over $\mathbb{Q}(\sqrt{-m})$, then $2a$ is a critical number of $\mathbb{Q}(\sqrt{-m})$. Most of all process of finding a set of critical numbers
$C[m]$ is based on the proof of Theorem 1 which indicates the truants or exceptional numbers of $L$.

If $m = 1$, the eight escalated lattices (not necessarily primitive) with truants 2 and 6 over $\mathbb{Q}(\sqrt{-1})$ are all even universal binary Hermitian lattices. So any lattice which represents both numbers 2 and 6 contains an even universal binary Hermitian lattice as a sublattice. Therefore

$$C[1] = \{2, 6\}.$$ 

If $m = 2$, the escalations by the truants 2 and 10 give 14 even lattices (which are not necessarily primitive) over $\mathbb{Q}(\sqrt{-2})$. These binary even lattices are all even universal except $[2, \omega, 10]$. The lattice $[2, \omega, 10]$ represents all positive even integers except 14 and escalated lattices of $[2, \omega, 10]$ by the truant 14 are even universal including the binary lattice $[2, \omega, 4]$. Therefore

$$C[2] = \{2, 10, 14\}.$$ 

With the identical arguments, we can show that

$$C[5] = \{2, 4, 6\} \quad \text{and} \quad C[6] = \{2, 4, 6, 10\}.$$ 

From now on, suppose $m = 10, 13, \ldots, 38$. We begin with listing lattices with their truant that is essential member of $C[m]$.

The escalations by truants 2 and 4 give lattices $\begin{pmatrix} 2 & 0 & \omega \\ 0 & 4 & 0 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 1 \end{pmatrix}$, and $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 4 & -1 \end{pmatrix}$, whose truants are 6, 10 and 6 respectively. So $2, 4, 6, 10 \in C[m]$.

If $m \geq 10$, then the truants of $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 6 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 4 & 1 \end{pmatrix}$ are 12 and 14, respectively. So $12, 14 \in C[m]$.

If $m = 10$, then the truant of $\begin{pmatrix} 2 & 0 & \omega \\ 0 & 4 & 0 \end{pmatrix}$ is 30. So $30 \in C[10]$.

If $m \geq 13$, then the truant of $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 4 \end{pmatrix}$ is 20. So $20 \in C[m]$.

If $m \geq 14$, then the truant of $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 10 \end{pmatrix}$ is 26. So $26 \in C[m]$.

If $m = 14$, then the truant of $\begin{pmatrix} 2 & 0 & \omega & 0 \\ 0 & 4 & -2 & -2 \\ -2 & 8 & 1 \\ 0 & -2 & 1 & 8 \end{pmatrix}$ is 42. So $42 \in C[14]$.

If $m \geq 17$, then the truants of $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 2 \\ 1 & 2 & 6 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 4 \end{pmatrix}$ are 28 and 30, respectively. So $28, 30 \in C[m]$. 

If \( m \geq 21 \), then the truants of 
\[
\begin{pmatrix}
2 & 1 & 1 \\
1 & 4 & -1 \\
1 & -1 & 6
\end{pmatrix}
\]
and 
\[
\begin{pmatrix}
2 & 1 & 1 \\
1 & 4 & 0 \\
1 & 0 & 6
\end{pmatrix}
\]
\( \perp \langle 18 \rangle \) are 34 and 38, respectively. So 34, 38 \( \in C[m] \).

If \( m \geq 22 \), then the truant of 
\[
\begin{pmatrix}
2 & 0 & 0 \\
0 & 2 & 1 \\
0 & 1 & 4
\end{pmatrix}
\]
is 42. So 42 \( \in C[m] \).

If \( m \geq 26 \), then the truant of 
\[
\begin{pmatrix}
2 & 0 & 0 \\
0 & 4 & 1 \\
0 & 1 & 6
\end{pmatrix}
\]
is 46. So 46 \( \in C[m] \).

If \( m \geq 29 \), then the truant of 
\[
\begin{pmatrix}
2 & 0 & 1 \\
0 & 4 & 0 \\
1 & 0 & 10
\end{pmatrix}
\]
is 58. So 58 \( \in C[m] \).

If \( m \geq 30 \), then the truant of 
\[
\begin{pmatrix}
2 & 0 & 0 \\
0 & 4 & 1 \\
1 & 0 & 10
\end{pmatrix}
\]
is 60 and 62, respectively. So 60, 62 \( \in C[m] \).

If \( m \geq 33 \), then the truants of 
\[
\begin{pmatrix}
2 & 0 & 1 & 1 \\
0 & 4 & 0 & 0 \\
1 & 0 & 0 & 58
\end{pmatrix}
\]
and 
\[
\begin{pmatrix}
2 & 1 & 0 \\
1 & 4 & 2 \\
0 & 2 & 6
\end{pmatrix}
\]
\( \perp \langle 34 \rangle \) are 60 and 62, respectively. So 60, 62 \( \in C[m] \).

If \( m \geq 37 \), then the truants of 
\[
\begin{pmatrix}
2 & 1 & 0 \\
1 & 4 & 2 \\
0 & 2 & 6
\end{pmatrix}
\]
\( \perp \langle 34 \rangle \) and 
\[
\begin{pmatrix}
2 & 0 & 0 \\
0 & 4 & 1 \\
0 & 1 & 8
\end{pmatrix}
\]
\( \perp \langle 2, 28 \rangle \) are 68 and 70, respectively. So 68, 70 \( \in C[m] \).

If \( m = 38 \), then the truant of 
\[
\begin{pmatrix}
2 & 0 & 1 & 0 \\
0 & 4 & 1 & 1 \\
1 & 1 & 10 & 0 \\
0 & 1 & 0 & 58
\end{pmatrix}
\]
is 74. So 74 \( \in C[38] \).

We showed that the numbers given in Table IV are elements of \( C[m] \). Now we prove that \( C[m] \) has no more numbers.

For \( m = 10 \), let a set \( D[10] = \{2, 4, 6, 10, 12, 14, 30\} \). It suffices to show that if \( L \) represents all numbers in \( D[10] \), \( L \) represents all numbers in \( BH^* \). By escalation we can find a finite set
\[
S = \{ M \mid \text{binary or ternary even Hermitian lattice such that } 2, 4, 6, 10 \rightarrow M \}.
\]

In the escalation process, \( L \) should contain some \( M \in S \) as a sublattice. By computer calculation, we confirmed that all \( M \in S \) represents all numbers \( n \in BH^* \setminus D[10] \). So \( D[10] \supseteq C[10] \). Similarly to \( C[10] \), we can check that the Table IV is optimal.

\( \square \)

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EVEN UNIVERSAL BINARY HERMITIAN LATTICES

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