Closed string disk amplitudes in the pure spinor formalism

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Overview

● Motivation

● Review of pure spinor formalism

● Adaption to purely closed string disk amplitudes

● Application to 2- and 1-point functions
Why disk amplitudes?

- First quantum correction to sphere level

- Higher derivative corrections to DBI action?
  
  \[ \sim e^{-\Phi} \epsilon_{10} \epsilon_{10} R^4 \]
  
  leads to correction to 4D EH-term

- Heterotic / Type I duality seems to predict such a term

  [Antoniadis, Ferrara, Minasian, Narain (1997)]

  [Green, Rudra (2016)]
• Type II:

\[
(\zeta(3)e^{-2\Phi} + \frac{\pi^2}{6})t_8 t_8 R^4 - (\zeta(3)e^{-2\Phi} \pm \frac{\pi^2}{6})\epsilon_{10}\epsilon_{10} R^4
\]

leads to correction to 4D kinetic terms of scalars

leads to correction to 4D EH-term

IIA

IIB (inhereted by I)

• Heterotic:

\[
(\zeta(3)e^{-2\Phi} + \frac{\pi^2}{6})t_8 t_8 R^4 - \zeta(3)e^{-2\Phi}\epsilon_{10}\epsilon_{10} R^4
\]

[Antoniadis, Ferrara, Minasian, Narain (1997)]
How is this compatible with heterotic / type I duality?

\[ g_{\mu \nu}^{(I)} = e^{-\Phi_{\text{het}}} g_{\mu \nu}^{(\text{het})} , \quad \Phi_I = -\Phi_{\text{het}} \]

Possible answer: \( S^{(\text{het})} \) in 10D contains

\[ \sqrt{g^{(\text{het})}} e^{-\Phi_{\text{het}}/2} J_0 E_{3/2}(e^{-\Phi_{\text{het}}}) - \sqrt{g^{(\text{het})}} \frac{\pi^2}{6} \mathcal{I}_2 \]

\[ \rightarrow \sqrt{g^{(I)}} e^{-\Phi_I/2} J_0 E_{3/2}(e^{-\Phi_I}) \]

\[ J_0 = t_8 t_8 R^4 - \frac{1}{8} \epsilon_1 \epsilon_{10} R^4 \]

\[ \mathcal{I}_2 = -\frac{1}{8} \epsilon_{10} \epsilon_{10} R^4 + \ldots \]

\[ E_{3/2} = \zeta(3) e^{-3/2\Phi_{\text{het}}} + \frac{\pi^2}{6} e^{\Phi_{\text{het}}/2} + \text{non-pert.} \]

Disk level!

S-duality invariant
• Origin: Disk and/or projective plane

• Direct check requires 5-point graviton amplitude!

• Alternatively: Check for EH-term correction from disk / projective plane in type I on Calabi-Yau $Y_3$

$$\sim e^{-\Phi} \chi_{Y_3} R$$

• How can disk / projective plane amplitude of gravitons depend on Euler number $\chi_{Y_3}$ of Calabi-Yau $Y_3$?
Pure spinor formalism

- Manifestly super-Poincare covariant
- Loop amplitudes calculable via non-minimal formulation
- Equivalence to RNS formalism shown in many examples (mainly amplitudes of massless states; for examples with massive states cf. [Chakrabarti, Kashyap, Verma (2018)])
Can lead to simplified calculations of amplitudes

- Complete quartic effective action of type II at tree level  
  [Policastro, Tsimpis (2006)]

- Closed string 4-point 3-loop amplitude in type II at low energy  
  [Gomez, Mafra (2013)]

- Arbitrary $n$-point amplitude of massless open strings on the disk  
  [Mafra, Schlotterer, Stieberger (2011)]
**CFT**

- **Type IIB-action for 10D flat space-time:**

  \[
  S = \frac{1}{2\pi} \int d^2 z \left( \frac{1}{2} \partial X^m \partial X_m + p_\alpha \partial \theta^\alpha + \bar{p}_\alpha \partial \bar{\theta}^\alpha + w_\alpha \partial \lambda^\alpha + \bar{w}_\alpha \partial \bar{\lambda}^\alpha \right) 
  \]

  \[
  (\lambda \gamma^m \lambda) = 0 
  \]

  i.e. \( \lambda \) pure spinor

- \( \lambda^\alpha, w_\alpha \) commuting

- \( h(\theta^\alpha) = h(\lambda^\alpha) = 0, \ h(p_\alpha) = h(w_\alpha) = 1 \)
• Supersymmetric fields (relevant for vertex operators):

\[ \Pi^m = \partial X^m + \frac{1}{2} (\theta \gamma^m \partial \theta), \]

\[ d_\alpha = p_\alpha - \frac{1}{2} \left( \partial X^m + \frac{1}{4} (\theta \gamma^m \partial \theta) \right) (\gamma_m \theta)_\alpha \]

(antiholomorphic analogs)

• OPEs (needed to perform contractions):

\[ X^m(z, \bar{z}) X^n(w, \bar{w}) = -\eta^{mn} \ln |z - w|^2 \]

\[ p_\alpha(z) \theta^\beta(w) = \frac{\delta_\alpha^\beta}{z - w}, \quad w_\alpha(z) \lambda^\beta(w) = -\frac{\delta_\alpha^\beta}{z - w} \]
• Nilpotent BRST operator:

\[ Q = \oint \frac{dz}{2\pi i} \lambda^\alpha(z) d_\alpha(z) \]

\[(\lambda \gamma^m \lambda) = 0 \implies Q^2 = 0\]

\[ d_\alpha(z) d_\beta(w) = -\frac{\gamma^m_{\alpha\beta} \Pi_m(w)}{z - w} \]

• Cohomology of \( Q \) coincides with superstring spectrum

[Berkovits (2000)]
Massless vertex operators

- Open string
  - Unintegrated
    \[ V^{(0)}(z) = [\lambda^\alpha A_{\alpha}(X, \theta)](z) \]
  - Integrated
    \[ V^{(1)}(z) = [\partial\theta^\alpha A_{\alpha}(X, \theta) + \Pi^m A_m(X, \theta) + d_{\alpha} W^\alpha(X, \theta) + \frac{1}{2} N^{mn} F_{mn}(X, \theta)](z) \]

- \( QV^{(0)} = 0 \), \( QV^{(1)} = \partial V^{(0)} \) (cf. RNS)
E.g.: Gauge field with polarisation $\xi_m$:

$$A_\alpha(X, \theta) = e^{ik \cdot X} \left\{ \frac{\xi_m}{2} (\gamma^m \theta)_\alpha - \frac{1}{16} (\gamma_p \theta)_\alpha (\theta \gamma^{mnp} \theta) i k_{[m} \xi_{n]} + \mathcal{O}(\theta^5) \right\}$$

$$A_m(X, \theta) = e^{ik \cdot X} \left\{ \xi_m - \frac{1}{4} i k_p (\theta \gamma_m{}^{pq} \theta) \xi_q + \mathcal{O}(\theta^4) \right\}$$

$$W^\alpha(X, \theta) = e^{ik \cdot X} \left\{ -\frac{1}{2} i k_{[m} \xi_{n]} (\gamma^{mn} \theta)^\alpha + \mathcal{O}(\theta^3) \right\}$$

$$F_{mn}(X, \theta) = e^{ik \cdot X} \left\{ 2i k_{[m} \xi_{n]} - \frac{1}{2} i k_{[p} \xi_{q]} i k_{[m} (\theta \gamma_{n]}{}^{pq} \theta) + \mathcal{O}(\theta^4) \right\}$$
• Closed string \((G_{mn}, B_{mn}, \Phi, \text{i.e. NS-NS-fields})\)

\[ \epsilon_{mn} = \xi_m \otimes \bar{\xi}_n \]

\[ V^{(a,b)}(z, \bar{z}) = V^{(a)}(z) \otimes \bar{V}^{(b)}(\bar{z}) \,, \quad a, b \in \{0, 1\} \]

\[ V^{(a,b)} \text{ contains factor} \]

\[ e^{ik \cdot X(z)} e^{ik \cdot \bar{X}(\bar{z})} = e^{ik \cdot [X(z) + \bar{X}(\bar{z})]} = e^{ik \cdot X(z, \bar{z})} \]
Tree level correlators

• After fixing conformal Killing group:

\[ A_{S^2}^{\text{closed}}(1, 2, \ldots, n) = \left\langle V_{1}^{(0,0)}(z_1, \bar{z}_1) \prod_{i=2}^{n-2} \int d^2 z_i V_{i}^{(1,1)}(z_i, \bar{z}_i)V_{n-1}^{(0,0)}(z_{n-1}, \bar{z}_{n-1}) V_{n}^{(0,0)}(z_n, \bar{z}_n) \right\rangle \]

\[ A_{D_2}^{\text{open}}(1, 2, \ldots, n) = \left\langle V_{1}^{(0)}(z_1) \prod_{i=2}^{n-2} \int_{z_{i-1}}^{z_{n-1}} dz_i V_{i}^{(1)}(z_i)V_{n-1}^{(0)}(z_{n-1}) V_{n}^{(0)}(z_n) \right\rangle \]

take this as example

• Integrate out non-zero modes via Wick's theorem, using

\[ \left\langle X^m(z) X^n(w) \right\rangle = -\eta^{mn} \ln(z - w) \]

\[ \left\langle p_\alpha(z) \theta^\beta(w) \right\rangle = \frac{\delta_\alpha^\beta}{z - w} \quad , \quad \left\langle w_\alpha(z) \lambda^\beta(w) \right\rangle = -\frac{\delta_\alpha^\beta}{z - w} \]
• Tree level: only $h = 0$ fields $X^m, \theta^\alpha, \lambda^\alpha$ have zero modes

$\implies$ All $h = 1$ fields $\partial\theta^\alpha, \Pi^m, d_\alpha, N^{mn}$ have to be integrated out via Wick's theorem

$\implies$ After Wick contractions and integrating out $X^m$ zero modes, one ends up with:

$$
\left\langle V_1^{(0)}(z_1) \prod_{i=2}^{n-2} V_i^{(1)}(z_i) V_{n-1}^{(0)}(z_{n-1}) V_n^{(0)}(z_n) \right\rangle = \delta(\sum_{i=1}^n k_i) \left\langle \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta; z_i) \right\rangle_0
$$

zero mode integration of $X^m$

zero mode integration of $\theta^\alpha, \lambda^\alpha$
Zero mode prescription:

\[ \langle (\lambda \gamma^m \theta)(\lambda \gamma^n \theta)(\lambda \gamma^p \theta)(\theta \gamma_{mnp} \theta) \rangle_0 = 1 \]

unique element of $Q$-cohomology with 3 factors of $\lambda$

Projects out coefficients of $\lambda^3 \theta^5$ -terms of

\[ \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha \beta \gamma}(\theta; z_i) \]

Need to integrate over $z_i$, $i = 2, \ldots, n - 2$
Closed string disk amplitudes

[Bischof, M.H.]

- Earlier results on 3-pt fct of 1 closed and 2 open states in type I
  [Alencar (2009); Alencar, Tahim, Landim, Costa Filho (2011)]

- Type IIB with Dp-brane along $X^1, \ldots, X^p$

- $V^{(a,b)}(z, \bar{z}) = V^{(a)}(z)\overline{V^{(b)}(\bar{z})}$, $z \in \mathbb{H}_+$

- Employ doubling trick:
  $$\overline{X}^m(\bar{z}) = D^m_n X^n(\bar{z})$$

  $D^{mn} = \begin{cases} 
  \eta^{mn} & m, n \in \{0, 1, \ldots, p\} \\
  -\eta^{mn} & m, n \in \{p + 1, \ldots, 9\} \\
  0 & \text{otherwise}
  \end{cases}$
Doubling trick for spinors

$$\bar{\Psi}^\alpha(\bar{z}) = M^\alpha_\beta \Psi^\beta(\bar{z}), \quad \bar{\Psi}_\alpha(\bar{z}) = N^\beta_\alpha \Psi_\beta(\bar{z})$$

Relations between $D, M, N$ (e.g. $N = (M^T)^{-1}$) allow rewriting:

$$V^{(0)}(\bar{z}) = \left( \bar{\lambda}^\alpha \bar{A}_\alpha[\bar{\xi}, k](\bar{X}, \bar{\theta}) \right)(\bar{z}) = \left( \lambda^\alpha A_\alpha[D\cdot\bar{\xi}, D\cdot k](X, \theta) \right)(\bar{z})$$

$$V^{(1)}(\bar{z}) = \left( \bar{\partial}\theta^\alpha A_\alpha[D\cdot\bar{\xi}, D\cdot k](X, \theta) + \Pi^m A_m[D\cdot\bar{\xi}, D\cdot k](X, \theta) + d_\alpha W^\alpha[D\cdot\bar{\xi}, D\cdot k](X, \theta) + \frac{1}{2}N^{mn}\mathcal{F}_{mn}[D\cdot\bar{\xi}, D\cdot k](X, \theta) \right)(\bar{z})$$

Can use same contractions as above, but allow both $z$ and $\bar{z}$
Conformal Killing group $PSL(2, \mathbb{R})$ only allows to fix one and a half closed string vertex operators.

$$\implies A_{D_2}^{\text{closed}}(1, \ldots, n) =$$

$$= 2 i g_c^n \tau_p \int_0^1 dy \left\langle V_1^{(0)}(iy) \overline{V}_1^{(1)}(-iy) \prod_{j=2}^{n-1} \int_{\mathbb{H}_+} d^2 z_j V_j^{(1,1)}(z_j, \overline{z}_j) V_n^{(0)}(i) \overline{V}_n^{(0)}(-i) \right\rangle$$

D-brane tension

cf. also [Grassi, Tamassia (2004); Alencar, Tahim, Landim, Costa Filho (2011)]

following [Hoogeveen, Skenderis (2007)]
\[ \mathcal{A}_{D_2}^{\text{closed}}(1, 2) = 2ig_c^2 \tau_p \int_0^1 dy \left< \left( V_1^{(0)}(iy) \bar{V}_1^{(1)}(-iy) V_2^{(0)}(i) \bar{V}_2^{(0)}(-i) \right) \right> \]

\[ = 2ig_c^2 \tau_p \int_0^1 dy \left< (\lambda A_1[\xi_1, k_1])(iy) \left( \bar{\partial} \theta^\alpha A_{1\alpha}[D \cdot \bar{\xi}_1, D \cdot k_1] + \Pi^m A_{1m}[D \cdot \bar{\xi}_1, D \cdot k_1] 
+ d_\alpha W_1^\alpha[D \cdot \bar{\xi}_1, D \cdot k_1] + \frac{1}{2} N^{mn} F_{1mn}[D \cdot \bar{\xi}_1, D \cdot k_1] \right) (-iy)(\lambda A_2[\xi_2, k_2])(i)(\lambda A_2[D \cdot \bar{\xi}_2, D \cdot k_2])(-i) \right> \]

\[ \sim g_c^2 \tau_p \int_0^1 dy \left( \frac{4y}{(1 + y)^2} \right)^{k_1 \cdot D \cdot k_1} \left( \frac{(1 - y)^2}{(1 + y)^2} \right)^{k_1 \cdot k_2} \left( \frac{d_1}{2y} + \frac{d_2}{1 + y} + \frac{d_3}{1 - y} \right) \]

**Koba-Nielsen factor**

**kinematic factors**
\[ d_1 = \langle i(\lambda A_1[\xi_1, k_1])k_1 \cdot A_1[D \cdot \bar{\xi}_1, D \cdot k_1](\lambda A_2[\xi_2, k_2])(\lambda A_2[D \cdot \bar{\xi}_2, D \cdot k_2]) + A_{1m}[\xi_1, k_1](\lambda \gamma^m W_1[D \cdot \bar{\xi}_1, D \cdot k_1])(\lambda A_2[\xi_2, k_2])(\lambda A_2[D \cdot \bar{\xi}_2, D \cdot k_2]) \rangle_0 \]

we applied zero mode prescription using Cadabra [Peeters]

\[ \mathcal{A}_{D_2}^{\text{closed}}(1, 2) \sim g_c^2 \tau_p \frac{\Gamma(-t/2)\Gamma(2q^2)}{\Gamma(1 - t/2 + 2q^2)} \left( 2q^2 a_1 + \frac{t}{2} a_2 \right) \]

\[ a_1 = \text{Tr}(\epsilon_1 \cdot D)k_1 \cdot \epsilon_2 \cdot k_1 - k_1 \cdot \epsilon_2 \cdot D \epsilon_1 \cdot k_2 - k_1 \cdot \epsilon_2 \cdot \epsilon_1^T \cdot D \cdot k_1 - k_1 \cdot \epsilon_2 \cdot \epsilon_1^T \cdot D \cdot k_1 + q^2 \text{Tr}(\epsilon_1 \cdot \epsilon_2^T) + \{1 \leftrightarrow 2\} \]

\[ a_2 = \text{Tr}(\epsilon_1 \cdot D)(k_2 \cdot D \cdot \epsilon_2 \cdot D \cdot k_2 + k_1 \cdot \epsilon_2 \cdot D \cdot k_2 + k_2 \cdot D \cdot \epsilon_2 \cdot k_1) + k_1 \cdot \epsilon_1 \cdot \epsilon_2 \cdot D \cdot k_2 - k_2 \cdot D \cdot \epsilon_2 \cdot \epsilon_1^T \cdot D \cdot k_1 + q^2 \text{Tr}(\epsilon_1 \cdot D \cdot \epsilon_2 \cdot D) - q^2 \text{Tr}(\epsilon_1 \cdot \epsilon_2^T) - (q^2 - \frac{t}{4})\text{Tr}(\epsilon_1 \cdot D)\text{Tr}(\epsilon_2 \cdot D) + \{1 \leftrightarrow 2\} \]

Same as RNS [Hashimoto, Klebanov; Garousi, Myers (1996)]
1-pt fct

- Earlier work in bosonic theory and RNS
  [Douglas, Grinstein (1987); Liu, Polchinski (1988); Ohta (1987)]

- Pure spinor: At most 2 factors of $\lambda^\alpha$ (in $V^{(0,0)}$) !?

- Alternative zero mode prescription (equivalent for higher-point tree amplitudes):
  [Berkovits (2016)]

$$\langle 1 \rangle_0 = 1$$

This corresponds to only alternative scalar element of $Q$-cohomology
\[ \mathcal{A}_{D_2}^{\text{closed}}(1) = g_c \tau_p \int_{\mathbb{H}^+} \frac{d^2 z}{V_{\text{CKG}}} \langle V^{(1)}(z, \bar{z}) \rangle \]

\[ \mathcal{A}_{D_2}^{\text{closed}}(1) = g_c \tau_p \Tr(\epsilon \cdot D) \]

For graviton this corresponds to linearisation of

\[ \tau_p \int d^p x \sqrt{-G} \]

\[ \Longrightarrow \text{ Fix position of } V^{(1)} \text{ to } z = i \text{ and divide by volume of } K \subset PSL(2, \mathbb{R}) \text{ leaving } z = i \text{ invariant} \]

\[ K = \left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \bigg| \theta \in [0, 2\pi] \right\}, \quad \frac{\cos \theta \ i + \sin \theta}{-\sin \theta \ i + \cos \theta} = i \]

volume: \(2\pi\)

\[ \Longrightarrow \mathcal{A}_{D_2}^{\text{closed}}(1) \sim g_c \tau_p \Tr(\epsilon \cdot D) \]

Alternative approach \[ \text{[Kashyap (2020)]} \]
Outlook

- RR-fields & fermions
- Projective plane  
  cf. [Garousi (2006)] for RNS
- Higher $n$-points
- Higher derivative corrections to DBI
Thank you!