Equations and first-order theory of one-relator and word-hyperbolic monoids

Albert Garreta∗, Robert D. Gray†

Abstract

We investigate systems of equations and the first-order theory of one-relator monoids and of word-hyperbolic monoids. We describe a family of one-relator monoids of the form $\langle A \mid w = 1 \rangle$ with decidable Diophantine problem (i.e. decidable systems of equations), and another family $\mathcal{F}$ of one-relator monoids $\langle A \mid w = 1 \rangle$ where for each monoid $M$ in $\mathcal{F}$, the longstanding open problem of decidability of word equations with length constraints reduces to the Diophantine problem in $M$. This is achieved by interpreting by systems of equations in $M$ a free monoid with a length relation. It follows that each monoid in $\mathcal{F}$ has undecidable positive AE-theory, hence in particular it has undecidable first-order theory. The family $\mathcal{F}$ includes many one-relator monoids with torsion $\langle A \mid w^n = 1 \rangle$ ($n > 1$), which have hyperbolic group of units and hyperbolic undirected Cayley graph. Contrastingly, all one-relator groups with torsion are hyperbolic, and all hyperbolic groups are known to have decidable Diophantine problem.

For word-hyperbolic monoids, we prove that the polycyclic monoid has decidable Diophantine problem but undecidable positive AE-theory. We shall also observe that there exist families of word-hyperbolic monoids such that the decidability problem of word equations with length constraints is reducible to the Diophantine problem in any of these monoids. We finish the paper with a list of open problems and questions.

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∗Albert Garreta, Department of Mathematics, University of the Basque Country, 48080 Bilbao, Spain, garreta.a@gmail.com,
†Robert D. Gray, School of Mathematics, University of East Anglia, Norwich NR4 7TJ, England, UK, Robert.D.Gray@uea.ac.uk

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1 Introduction

Equations in monoids and groups have been widely studied during the past few decades, being of interest in several areas, ranging from computer science to group and model theory. For a detailed account of the history, motivation and key results in this area we refer the reader to the survey articles [29, 48, 55]. By the Diophantine problem we mean the algorithmic problem of determining if any given system of equations has a solution or not. Two classical results due to Makanin show that the Diophantine problem is decidable in any free monoid [64] and in any free group [65]. Based on Makanin's algorithm, Razborov [81] provided a powerful description of the sets of solutions to systems of equations in free groups via what were later called Makanin-Razborov diagrams. This played a key part in the solution to the Tarski problems [54, 87] regarding groups elementary equivalent to a free group. In subsequent years new decidability algorithms and descriptions of solutions have appeared: in [79] Plandowski describes a polynomial space algorithm for deciding word equations based on a compression technique. In [46] Jez shows that word equations can be solved in non-deterministic linear space, and in [19] it is proved that the solution set of a word equation is an EDT0L language (in particular, it is an indexed language), furthermore this set can be computed in polynomial space [30]. More recently, in [88] Sela presents the first in a sequence of papers devoted to investigating the structure of sets of solutions to systems of equations over a free semigroup via a Makanin—Razborov diagram analogue.

Results regarding systems of equations in free monoids and groups have been extended in different directions. In [83] Rips and Sela proved that the Diophantine problem in any torsion-free hyperbolic group is reducible to the same problem in a free group, and hence that it is decidable. Later Dahmani and Guirardel [21] extended this result to arbitrary hyperbolic groups and, in particular, to virtually free groups. The latter is reducible to systems of twisted equations with rational constraints in free monoids with involution, which was studied in [21] and was later generalized in [30, 59]. In particular, in [30] it is proved that this problem is decidable in polynomial space with sets of solutions forming EDT0L languages. Additionally, systems of word equations are known to be decidable in free partially commutative monoids (i.e. trace monoids) [32], and free partially commutative groups (i.e. Right Angled Artin groups) [15, 28]. Closure properties of decidability of systems of equations have been established for some graph products of certain monoids and groups, including
free products \[16, 27\], and HNN or amalgamated products of groups over a finite subgroup \[59\]. In contrast, groups with “richer” structure, such as solvable groups, tend to exhibit a completely different behavior which leads or points to undecidability of their Diophantine problem \[38, 84\]. Equations in free inverse monoids are considered in \[25, 31, 85\] where, among other things, it is shown that the Diophantine problem for free inverse monoids is undecidable. Other interesting papers on Diophantine problems for monoids and groups include: \[53\] and \[26\].

The Diophantine problem of one-relator groups has also attracted some interest recently. In \[51\] it is proved that all solvable Baumslag-Solitar groups have decidable Diophantine problem. Also, one immediately has that one-relator groups with torsion have decidable Diophantine problem, since such groups are always hyperbolic. As yet, there is no known example of a one-relator group with undecidable Diophantine problem. Note that, since the conjugacy problem is still open for one-relator groups, if the Diophantine problem for arbitrary one relator groups is decidable then it is likely to be a difficult problem.

Since the Diophantine problem is decidable in free monoids by Makanin, but undecidable in general, it is reasonable to seek solutions to the Diophantine problem for classes of monoids which are close to being free in some sense. Motivated by the results for hyperbolic and one-relator groups above, a natural starting point in this direction is to investigate the Diophantine problem for one-relator monoids and word-hyperbolic monoids (in the sense of Duncan and Gilman \[33\]). The main topic of the present paper will be the study of the Diophantine problem for these classes of monoids and its relation to word equations with length constraints. While doing so we shall also investigate the first-order theory of such monoids (see Theorems \(C\) and \(E\) and the paragraph below Theorem \(E\) in this introduction).

The class of one-relator monoids is less well understood than the corresponding class of one-relator groups. Most notably, while the word problem for one-relator groups is known to be decidable by a classical result of Magnus \[62\], it is a longstanding open problem whether the word problem is decidable for monoids defined by a presentation \(\langle A \mid u = v \rangle\) with a single defining relation \(u = v\) where \(u, v \in A^*\). Of course, since the word problem is open, this indicates that if the Diophantine problem for all one-relator monoids is decidable then this is likely to be a difficult thing to prove. On the other hand, there are several natural classes of one-relator monoids for which the word problem has been shown to be decidable. Specifically, Adjan \[2\] showed that all one-relator monoids defined by presentations of the form \(\langle A \mid w = 1 \rangle\) have decidable word problem. Monoid presentations where all of the relations are of the form \(w = 1\) are commonly called special presentations. Adjan solved the word problem for special one-relator monoids by showing that the group of units of such a monoid is a one-relator group, and then reducing the word problem of such a monoid to the word problem of its group of units. Then decidability of the word problem for the special one-relator monoid follows from Magnus’s theorem. This result was generalised by Makanin in \[63\], who showed that for any finitely presented special monoid \(M = \langle A \mid w_i = 1 \ (i \in I) \rangle\) the group of units \(G\) of \(M\) is presentable by a finite presentation with \(|I|\) defining relations, and \(M\) has decidable word problem if and only if its group of units \(G\) does. A modern approach to finite special monoid presentations using techniques from the theory of string rewriting systems is given by Zhang in \[93\]. Zhang’s methods will play an important role in the results we prove in this paper for special one-relator monoids.

A number of natural decision problems in semigroup theory are special cases of the Dio-
Diophantine problem, in the sense that they ask about the existence of solutions to certain equations. For example the left divisibility problem asks for a solution to the equation \( u = vX \), and dually we also have the right divisibility problem. These problems are equivalent to the question of decidability of Green’s orders \( \preceq_R \) and \( \preceq_L \) in the semigroup. Here \( R \) and \( L \) denote Green’s equivalences in the semigroup; see [45]. Makanin showed that special one-relator monoids all have decidable left and right divisibility problem. Other interesting results on the decidability of Green’s orders, and Green’s relations, can be found in [77].

Several notions of conjugacy have been investigated for monoids, and all of them are expressible in terms of the existence of solutions to certain sets of equations. Results on decidability of conjugacy problems for monoids may be found in [3, 4, 13, 73, 76, 92, 93]. In particular, the decidability and complexity of conjugacy problems in polycyclic monoids is investigated in [3], and in [92] Zhang proves that the conjugacy problem is decidable in all one-relator monoids of the form \( \langle A \mid u^n = 1 \rangle \) with \( n > 1 \). Two natural definitions of conjugation for monoids that have been considered in the literature are the following: \( x \) is conjugate to \( y \) if there exist \( u, v \) such that \( x = uv \) and \( y = vu \); or \( x \) is conjugate to \( y \) if there exists \( w \) such that \( wx = yw \). In [92] Zhang proves that for finitely presented special monoids these definitions are equivalent and they both define an equivalence relation on the monoid. Zhang’s result, together with the fact that one-relator groups with torsion are known to have solvable Diophantine problem, leads naturally to the question of whether all one-relator monoids \( \langle A \mid u^n = 1 \rangle \) with \( n > 1 \) have decidable Diophantine problem. This is one of the questions that will be investigated in this paper.

Given that Adjan reduced the word problem for \( \langle A \mid w = 1 \rangle \) to the group of units, and Zhang reduces the conjugacy problem for \( \langle A \mid w = 1 \rangle \) to its groups of units, one result to hope to establish is that if \( M \) is a special one-relator monoid with group of units \( G \), and if \( G \) has decidable Diophantine problem, then so does \( M \). Whether or not this is true remains an open problem. However, the results we establish in this paper for special one-relator monoids with torsion indicate that it could be a difficult problem, especially if it has a positive solution; see Proposition 3.2 and Theorem 5.22. Note that since the group of units of a special one-relator monoid is a positive one-relator group in the sense of [8], this direction of research also motivates further investigation of the Diophantine problem for positive one-relator groups.

In the case that all of the generators that appear in the defining relator \( w \) represent invertible elements of the monoid, we shall observe that there is indeed a reduction of the Diophantine problem to the corresponding problem for the group of units of the monoid (which is a positive one-relator group by Adjan, as discussed above), see Theorem B in this introduction or 5.1. On the other hand, when not all of the generators appearing in the defining relator are invertible in the monoid, we shall see that in many of these cases one can encode solving word equations with length constraints in the monoid, creating an interesting link between equation solving in one-relator monoids and the important open problem of decidability of word equations with length constraints (see Theorem A or Theorems 5.19 and 5.22). This reduction is attained by interpreting by systems of equations a free monoid with a length relation.

The problem of determining whether word equations with length constraints (in short, WELCs — see Subsection 2.1) are decidable has been open for decades now and is of major interest in computer science. Some partial cases and variations have been successfully studied in [11, 23, 24, 36, 58]. WELCs are of interest in industry where they are applied in program
verifiers and debuggers [35, 86]. In this regard there exists a variety of fast solvers [1, 6, 7, 9, 35, 89, 91] for SMT formulae, which include in particular word equations with rational constraints and length constraints. These solvers are not complete, i.e. not all inputs are successfully solved. A further point of interest is that WELCs are reducible to the problem of solving systems of integer-coefficient polynomial equations in Z [71]. Thus a proof of undecidability of WELCs would provide a new solution to Hilbert’s 10th problem, which states that equations in the ring Z are undecidable [68].

We shall now explain the main results of the paper in more detail. Before doing so, we first need to give some background notions.

Given any one-relator monoid presentation of the form \( \langle A \mid r = 1 \rangle \), defining a monoid \( M \), there is a unique decomposition of the word \( r = r_1r_2 \ldots r_k \) such that each \( r_i \) belongs to \( A^+ = A^* \setminus \{1\} \), each of the words \( r_i \) represents an invertible element of \( M \), and no proper non-empty prefix of \( r_i \) is invertible, for all \( 1 \leq i \leq k \). The words \( r_i (1 \leq i \leq k) \) in this decomposition are called the minimal invertible pieces of \( r \). Adjan [2] gives an algorithm for computing this decomposition for any one-relator special monoid. Minimal invertible pieces are a key concept for relating a special monoid with its group of units.

Given a set \( S \) and a tuple of nonnegative integers \( \hat{\lambda} = (\lambda_s \mid s \in S) \), by \( | \cdot |_{\hat{\lambda}} \) we denote the \( \hat{\lambda} \)-weighted word-length in \( S^* \) defined as

\[
|w|_{\hat{\lambda}} = \text{def} \sum_{s \in S} \lambda_s |w|_s, \quad (w \in S^*),
\]

where \( |w|_s \) denotes the number of occurrences of the letter \( s \) in \( w \). By \( L_{\hat{\lambda}} \) we denote the \( \hat{\lambda} \)-length relation defined as \( L_{\hat{\lambda}}(w, u) \) if and only if \( |w|_{\hat{\lambda}} \leq |u|_{\hat{\lambda}} \). Note that if \( \lambda_s = 1 \) for all \( s \in S \) then \( | \cdot |_{\hat{\lambda}} \) and \( L_{\hat{\lambda}} \) are just the standard word length and the standard length relation, which we denote simply as \( | \cdot | \) and \( L \), respectively. Hence \( L(u, v) \) holds if and only if \( |u| \leq |v| \), for any two words \( u, v \in S^* \). The tuple \( (S^*, \cdot, 1, =, L_{\hat{\lambda}}) \) refers to the free monoid \( S^* \) equipped with the relation \( L_{\hat{\lambda}} \). This is the natural structure on which to write systems of word equations with (\( \hat{\lambda} \)-weighted) length constraints. See Subsection 2.1 for further details.

The main tool we use for reducing one problem to another is that of interpretability by systems of equations or by positive existential formulas (Definition 2.3). This is nothing more than the usual notion of interpretability [12, 67] restricting all formulas to be systems of equations or disjunctions of systems of equations, respectively.

Among other results, in this paper we prove the following.

**Theorem A** (Theorems 5.19 and 5.22). Let \( M \) be the one-relator monoid \( \langle A \mid r = 1 \rangle \). Write \( r = r_1r_2 \ldots r_k \) such that each \( r_i \in A^+ \), each of the words \( r_i \) represents an invertible element of \( M \), and no proper non-empty prefix of \( r_i \) is invertible, for all \( 1 \leq i \leq k \). Set \( \Delta = \{r_i \mid 1 \leq i \leq k\} \), so \( \Delta \) is the set of minimal invertible pieces of the relator \( r \). Suppose that:

\( (C1) \) no word from \( \Delta \) is a proper subword of any other word from \( \Delta \), and

\( (C2) \) there exist distinct words \( \gamma, \delta \in \Delta \) with a common first letter \( a \).

Then there exists a free monoid \( F \) of finite rank \( n \geq 2 \) and a tuple of positive integer weights \( \hat{\lambda} = (\lambda_1, \ldots, \lambda_n) \) such that the free monoid with weighted length relation \( (F, \cdot, 1, L_{\hat{\lambda}}) \) is interpretable in \( M \) by systems of equations. Consequently, the problem of solving systems of word
equations with weighted length constraints is reducible to the problem of solving systems of equations in \( M \).

If additionally to (C1) and (C2) we have:

(C3) no word in \( \Delta \) starts with \( a^2 \),

then the above result holds with \( L_{\vec{\lambda}} \) being the standard length relation \( L \), i.e. \( L(u,v) \) if and only if \( |u| \leqslant |v| \), for \( u,v \in F \). Consequently, in this case, the problem of solving systems of word equations with length constraints is reducible to the problem of solving systems of equations in \( M \).

Some examples of monoids satisfying conditions (C1), (C2) and (C3) are \( \langle a, b, c | (ab)(ac)(ab) = 1 \rangle \) and \( \langle a, b, c | ((ab)(ac)(ab))^n = 1 \rangle \) for \( n \geqslant 1 \), where we indicate the minimal invertible pieces with parentheses. In the two-generated case we have examples satisfying all of (C1), (C2) and (C3) such as \( \langle a, b | (ab abb)(aba abb)(ab abb) = 1 \rangle \) and \( \langle a, b | ((aba^n b^{n+1})(aba^{n+1} b^{n+1})(aba)^{n+1})^m = 1 \rangle \), for all \( n, m \geqslant 1 \). Dropping (C3) there are simpler two-generated examples which satisfy both (C1) and (C2) e.g. \( \langle a, b | ((aab)(abb)(aab)) = 1 \rangle \) (where \( n \geqslant 1 \). As seen in these examples, the family of one-relator monoids satisfying conditions (C1), (C2), and (C3) includes many one-relator monoids with torsion \( \langle A \mid w^n = 1 \rangle \), \( n > 1 \), which by Proposition \ref{prop:hyperbolic} have hyperbolic group of units and hyperbolic undirected Cayley graph. We stress that one-relator groups with torsion are hyperbolic and thus have decidable Diophantine problem \cite{21,83}.

In another direction we prove the following result, which can be used to obtain many examples of special one-relator monoids with decidable Diophantine problem, as described in Section \ref{sec:examples}.

**Theorem B (Theorem 5.1).** Let \( M = \langle A \mid w = 1 \rangle \) and suppose that every letter in \( w \) is invertible in \( M \). Let \( G = \langle B \mid w = 1 \rangle \) where \( B \subseteq A \) is the set of letters that appear in \( w \). Then \( G \) is a one-relator group, and if the Diophantine problem is decidable in \( G \) then it is decidable in \( M \).

In Section \ref{sec:examples} we provide some examples of monoids satisfying the hypotheses of Theorem \ref{thm:hyperbolic} as well as a list of questions and open problems.

The other main topic of the paper is the Diophantine problem in hyperbolic and word-hyperbolic monoids. By hyperbolic monoid we understand a monoid whose undirected Cayley graph is hyperbolic. This notion of hyperbolicity for monoids has been investigated e.g. in \cite{12} and \cite{17}. As explained in \cite{33}, in general the connection between the geometry of the Cayley graph of a semigroup and its algebraic properties is much weaker than for finitely generated groups. In particular there are geometrically hyperbolic monoids with undecidable word problem since, for example, adjoining a zero element to any monoid will result in a monoid with an undirected Cayley graph which is hyperbolic. This was part of the motivation for the introduction in \cite{33} of the notion of a word-hyperbolic monoid which is a monoid whose multiplication table with respect to a regular combing is a context-free language (see Subsection \ref{subsec:word_hyperbolic}). This definition is then strong enough to obtain a class of monoids with good algorithmic properties. In particular word hyperbolic monoids have decidable word problem. We note that if one works with the directed Cayley graph of a semigroup, and the corresponding class of semimetric spaces, then it is possible to develop a theory of directed hyperbolicity which is strong enough to imply good properties, like decidability of the word problem, in certain situations; see \cite{39} for more details.
As far as we are aware it is not known whether the Diophantine problem is decidable for word-hyperbolic monoids. Results in this direction include [14], where it is shown that Green’s relations $R$ and $L$ are decidable in word-hyperbolic monoids, that is, it is decidable whether the systems of equations $Xa = b$, $Yb = a$, has a solution (and the same for the system: $aX = b$, $bY = a$). It is also shown in that paper that the isomorphism problem is undecidable for word-hyperbolic monoids. In [72] the authors pose as an open problem whether cyclic conjugation is decidable in monoids defined by monadic complete rewriting systems. All such monoids are word-hyperbolic by Theorem 2.7. It is not known whether word-hyperbolic monoids have decidable conjugacy problem (for any of the standard notions of conjugacy in monoids).

One of the most fundamental examples of a monoid which is both one-relator and word-hyperbolic, and is not a group, is the bicyclic monoid $B = \langle a, b \mid ab = 1 \rangle$. In [27, Corollary 7] it is proved that the first-order theory of $B$ is decidable. This implies that many decision problems are decidable in $B$, such as the Diophantine problem and identity checking. Explicit algorithms for checking identities in $B$, and connections with identities in semigroups of tropical matrices, may be found in [22, 78]. Other interesting results on decidability of identity-checking in monoids can be found in [18].

Our study of word-hyperbolic monoids shall begin with the class of polycyclic monoids, which is closely related to the bicyclic monoid. The study of this class of monoids was introduced by Nivat and Perrot in [74]; see also [57]. Since their introduction, these monoids have received a great deal of attention in the literature, in part because of their connections with $C^*$-algebras and self-similarity, see [20] and [41]. Furthermore they are hyperbolic and word-hyperbolic due to a result of Cain [12] (see Theorem 2.7). Further motivation for the study of algorithmic problems in polycyclic monoids comes from [71].

**Theorem C** (Theorem 4.1). The Diophantine problem in the polycyclic monoid

$$P_n = \langle p_1, \ldots, p_n, p_1^{-1}, \ldots, p_n^{-1} \mid p_ip_i^{-1} = 1 (1 \leq i \leq n),
qquad p_ip_j^{-1} = 0 (1 \leq i \neq j \leq n) \rangle$$

is decidable, for all $n \geq 1$. On the other hand, the positive $AE$-theory of $P_n$ is undecidable for all $n \geq 2$. In particular, the first-order theory of $P_n$ ($n \geq 2$) is undecidable.

We refer to Subsection 2.1 for a definition of $AE$-theory. This theorem extends and improves on the result (see e.g. [3]) that polycyclic monoids have decidable conjugacy problem.

We shall also see that there exists a family of hyperbolic and word-hyperbolic monoids such that the decidability problem of word equations with length constraints is reducible to the Diophantine problem in any of these monoids (see Example 4.3).

The results above have several consequences, some of them are summarized in the following:

**Corollary D.** The following hold:

(i) If every finitely presented word-hyperbolic monoid has decidable Diophantine problem, then the problem of solving word equations with length constraints is decidable.

(ii) If every one-relator monoid of the form $\langle A \mid u = 1 \rangle$ with a hyperbolic group of units has decidable Diophantine problem, then the problem of solving word equations with length constraints is decidable.
(iii) If every one-relator monoid of the form $\langle A \mid w^n = 1 \rangle$, with $n > 1$, has decidable Diophantine problem, then the problem of solving word equations with length constraints is decidable.

This follows immediately from Examples 4.3, Theorems 2.7 and 5.19 and Proposition 3.2. We stress that the Diophantine problem is known to be decidable for the last two classes of groups, since it is decidable for hyperbolic groups, and one-relator groups with torsion are hyperbolic.

In addition to the Diophantine problem, we also obtain results about the decidability of the first-order theory of certain one-relator monoids. The first-order theory with coefficients of a free nonabelian semigroup was shown to be undecidable by Quine [80] (all free structures in this paragraph are implicitly assumed to be nonabelian). Quine’s result was strengthened in [34, 66] by proving that the positive AE-theory with coefficients of a free semigroup is undecidable. This contrasts with the aforementioned decidability result of Makanin for systems of equations, and also with the fact that the first-order theory of free groups is decidable as part of the solution to Tarski problems [54]. A consequence of Theorem A is the following

**Theorem E** (Theorem 5.21). Let $M$ be a monoid with presentation $\langle A \mid w = 1 \rangle$ for some set $A$ and some word $w \in A^*$ satisfying the conditions (C1) and (C2) of Theorem A. Then the positive AE-theory with coefficients of $M$ is undecidable. In particular, the first-order theory with coefficients of $M$ is undecidable.

To the best of our knowledge, Theorem E provides the first examples of one-relator monoids with undecidable positive AE-theory with coefficients, excluding the free monoid. Other examples of one-relator monoids (including special one-relator monoids) with undecidable first-order theory with coefficients can be found in [50, Theorem 1] (note that our examples include families of two-generated monoids which are different from the examples obtained in [51, Theorem 1]). Concerning groups, it is known that solvable Baumslag-Solitar groups have undecidable first-order theory [73]. Moreover, a recent result shows that the first-order theory is undecidable in any one-relator group containing a solvable Baumslag-Solitar subgroup [52] (because the subgroup is interpretable in the group).

2 Preliminaries

In this section we provide the necessary background definitions and results from model and semigroup theory that will be needed in this article. In Subsections 2.1 and 2.2 we shall state the model-theoretic definitions for general structures, although throughout the paper these will be used only on monoids, or on monoids with some extra function or relation such as a length relation. Further background on model theory can be found in [42, 67]. See [5] for notions of computational and complexity theory, [45] for semigroup and monoid theory background, and [61] for notions in combinatorial group theory.

2.1 Equations, first-order theory, and other problems

We follow Sections 1.1. and 1.3 from [42]. We fix $X$ and $A$ to denote a finite set of variables and a finite set of constants, respectively.
We describe structures by tuples \( S = (U, f_1, f_2, \ldots, r_1, r_2, \ldots, c_1, c_2, \ldots) \), where \( U \) is the domain of the structure, the \( f_i \) are function symbols, the \( r_i \) are relation symbols, and the \( c_i \) are constant symbols. The equality relation \( = \) is always assumed to be one of the relations of \( S \) and is usually omitted from the list \( r_1, \ldots \). The tuple \((f_1, f_2, \ldots, r_1, r_2, \ldots, c_1, c_2, \ldots)\) is the language (or signature) of \( S \). We make the convention that this tuple is implicitly enlarged with as many elements from \( U \) as needed. These extra elements are called coefficients (or parameters). Sometimes we identify the whole structure with its domain. For example, we denote the free monoid generated by \( A \) simply by \( A^* \), omitting any reference to the concatenation operation \( \cdot \) or the identity element 1 or the equality relation \( = \).

An equation in a structure \( S \) with language \( L \) is an atomic formula in the language \( L \) with coefficients. Recall that an atomic formula is one that makes no use of quantifiers, conjunctions, disjunctions, or negations. Thus an equation in \( S \) is a formula constructed using only variables, constant elements from \( U \) (because we allow the use of coefficients by convention), functions \( f_i \), and a single relation \( r_i \). For example, if \( S \) is a monoid generated by \( A \) then an equation in \( S \) is a formal expression of the form \( w_1(X, A) = w_2(X, A) \), where \( w_1(X, A) \) and \( w_2(X, A) \) are words in \((A \cup X)^*\). A solution to such equation is a map \( f : X \to S \) such that \( w_1(f(X), A) = w_2(f(X), A) \) is true in \( S \). By \( w_i(f(X), A) \) we refer to the word obtained from \( w_i \) after replacing each variable \( x \in X \) by the word \( f(x) \). A system of equations in \( S \) is a conjunction of equations in \( S \). Alternatively one can define equations as formulas of the form \( \exists x_1 \ldots \exists x_n \phi(x_1, \ldots, x_n) \) where \( \phi \) is an atomic formula as above on variables \( x_1, \ldots, x_n \). We use these two formulations interchangeably.

Equations in a free monoid \( A^* \) receive the special name of word equations. One can consider equations in more complicated structures, such as the structure \((A^*, \cdot, 1, =, L)\) obtained from the free monoid \( A^* \) (which we identify with the tuple \((A^*, \cdot, 1, =)\)) by adding the length relation \( L \) defined by the rule \( L(u, v) \) if and only if \(|u| \leq |v|\), for all \( u, v \in A^* \), where \(|\cdot|\) denotes length of words and 1 is the identity element. A system of equations in \((A^*, \cdot, 1, =, L)\) is called a system of word equations with length constraints. This is a system of word equations \( \Sigma \) together with a finite conjunction \( C \) of formal expressions of the form \( L(w_1, w_2) \), each called a length constraint, where \( w_1, w_2 \in (X \cup A)^* \). A map \( f : X \to A^* \) is a solution to such system if it is a solution to \( \Sigma \) and \(|w_1(f(X), A)| \leq |w_2(f(X), A)| \) for each length constraint \( L(w_1, w_2) \) appearing in \( C \).

Alternatively to the length constraint one can consider the more general notion of weighted length constraint, which we define now. Let \( \vec{k} = (k_a \mid a \in A) \) be a tuple of natural numbers, one for each constant \( a \in A \). Then by \(|\cdot|_{\vec{k}} \) we denote the map \( |\cdot|_{\vec{k}} : A^* \to \mathbb{N} \) defined by

\[
|h|_{\vec{k}} = \sum_{a \in A} k_a n_a(a),
\]

where \( n_a(h) \) is the number of times that the letter \( s \) appears in \( h \). We call \(|\cdot|_{\vec{k}} \) the \( \vec{k} \)-weighted length function of \( A^* \). We further let \( L_{\vec{k}} \) denote the relation in \( A^* \) defined by the rule \( L_{\vec{k}}(h, g) \) if and only if \(|h|_{\vec{k}} \leq |g|_{\vec{k}} \), and call \( L_{\vec{k}} \) the \( \vec{k} \)-weighted length relation in \( A^* \). Note that if \( \vec{k} \) consists solely of 1’s then \(|\cdot|_{\vec{k}} \) is the usual length of words \(|\cdot| \) and \( L_{\vec{k}} \) is the length relation \( L \).

The Diophantine problem in a structure \( S \), denoted \( D(S) \), refers to the algorithmic problem of determining if each given system of equations in \( S \) (with coefficients belonging to a fixed computable set) has a solution. One says that \( D(S) \) is decidable if there exists an algorithm (i.e., a Turing machine \( \vec{U} \)) that performs such task.
Given two algorithmic problems $P_1$ and $P_2$, we say that $P_1$ is reducible to $P_2$ if there exists an algorithm that solves $P_1$ using an oracle for the problem $P_2$ (i.e. a black-box algorithm that ‘magically’ solves $P_2$ —see Definition 3.4 in [5]). Thus in this case if $P_1$ is unsolvable then so is $P_2$: indeed, if $P_2$ was solvable then replacing the oracle in the definition above by an algorithm that solves $P_2$ would yield an algorithm that solves $P_1$, a contradiction. As an example, $D_pZ_q$ is undecidable for $Z$ the ring of integers (this is the answer to Hilbert’s 10th problem [69]), and hence $D_pM_q$ is undecidable for any structure $M$ such that $D_pZ_q$ is reducible to $D_pM_q$.

Let $L$ be some language. A positive AE-sentence in $L$ is a first-order sentence of the form

$$\forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_m \psi(x_1,\ldots,x_n,y_1,\ldots,y_m)$$

where $\psi$ is a quantifier-free formula without negations on the language $L$. The positive AE-theory of a structure $S$ is the set of all positive AE-sentences in the language of $S$ that are true in $S$. Analogously to the Diophantine problem, the positive AE-theory of $S$ is said to be decidable if there exists and algorithm that, given a positive AE-sentence, decides whether or not it holds in $S$.

One can generalize the notions in the paragraph above by replacing positive AE-sentences by any family of first-order sentences $\Phi$. In particular, if $\Phi$ is the set of all first-order sentences then one speaks of the first-order theory, or the elementary theory, of a structure. It is important to note that if the first-order theory is decidable then so is the Diophantine problem, the positive AE-theory, the positive universal theory (identity checking), etc.

Rational constraints and involutions A rational constraint consists in a tuple $(x,R)$ where $x \in X$ is a variable and $R$ is a rational subset of some monoid $A^*$, i.e. one that can be obtained from singletons and the empty set by successively applying the operations of union, product, or Kleene star. A map $f : X \to A^*$ satisfies the rational constraint $(x,R)$ if $f(x)$ belongs to $R$. Equations with rational constraints are known to be decidable in free monoids, in free monoids with involution, and in free groups [26], in fact they are PSPACE-complete.

An involution in a monoid $M$ is a bijection $\bar{\cdot} : M \to M$ such that $\bar{1} = 1$ and for all $x, y \in M$ one has $\bar{\bar{x}} = x$ and $\bar{xy} = \bar{y}\bar{x}$. A free monoid with involution generated by a set $A$ is the structure $((A \cup \bar{A})^*, 1, \bar{\cdot})$, where $\bar{A}$ is a disjoint copy of $A$ and $\bar{\cdot}$ is an involution that restricts to a bijection between $A$ and $\bar{A}$.

We remark that it is possible to define a multi-sorted structure in which one can write word equations with rational constraints. Such approach is unnecessarily formal for this paper and it will not be followed.

2.2 Reductions and interpretability

In this subsection we introduce the notion of interpretability with respect to some class of formulas. This is a powerful tool which in particular implies reducibility of the decision problem for such class of formulas. It is nothing else than the classical model-theoretical notion of interpretability [42, 67], with the modification that formulas are required to be of some specific form (such as systems of equations). We follow Section 1.3 of [67] (alternatively, see Sections 2.1 and 5.3 of [42]).
Definition 2.1. Let $M$ be a structure, $n$ a natural number, and $\Phi$ a set of formulas in the language of $M$. A subset $S \subseteq M^n$ is called definable in $M$ by formulas in $\Phi$ (in short, $\Phi$-definable) if there exists a formula

$$\Sigma_S(x_1, \ldots, x_n, y_1, \ldots, y_k) \in \Phi,$$

with free variables $(x_1, \ldots, x_n, y_1, \ldots, y_k) = (\vec{x}, \vec{y})$, such that for any $\vec{m} \in M^n$, one has that $\vec{m} \in S$ if and only if there exists $\vec{y}_0 \in M^k$ such that $\Sigma_S(\vec{m}, \vec{y}_0)$ is true in $M$. In this case $\Sigma_S$ is said to define $S$ in $M$.

We will make use of the following two classes of formulas $\Phi$:

1. Systems of equations. In this case we replace the prefix $\Phi$ by $e$-, speaking of $e$-definability.

2. Disjunctions of systems of equations. In this case we speak of $PE$-definability. See below for an explanation of this terminology.

Remark 2.2. It is well known that any disjunction of systems of equations is equivalent to a positive existential sentence with coefficients (hence the name $PE$-definability), i.e. formulas that can be constructed using only existential quantifiers, conjunctions, disjunctions, variables, and coefficients from the structure. To prove such an equivalence it suffices to use the distributive properties between conjunction $\land$ and disjunction $\lor$, together with the logical equivalences

$$(\exists \vec{x} \; \psi(\vec{x})) \land (\exists \vec{x} \; \phi(\vec{x})) \Rightarrow \exists \vec{x} \exists \vec{y} \; (\psi(\vec{x}) \land \phi(\vec{y})),
(\exists \vec{x} \; \psi(\vec{x})) \lor (\exists \vec{x} \; \phi(\vec{x})) \Rightarrow \exists \vec{x} \; (\psi(\vec{x}) \lor \phi(\vec{x})),
$$

which hold for any formulas $\psi$ and $\phi$ having $\vec{x}$ as free variables (and possibly other free variables).

We remark that, in the literature, positive existential formulas are sometimes referred to simply as existential formulas.

For example, the set of all elements that commute with a given element $m \in M$ is defined by the equation $xm = mx$. Likewise, the set of all elements of $M$ that are squares is defined by the equation $x = y^2$. The set of all elements that commute with $m$ or are a square is defined by the $PE$-formula $(xm = mx) \lor (x = y^2)$.

Observe that, by definition, $e$-interpretability and $PE$-interpretability allow the use of any coefficients in the domain of the structures at hand.

Definition 2.3. Let $A$ and $M$ be two structures and let $\Phi$ be a family of formulas in the language of $M$. Let further $A$ and $M$ be the domains of $A$ and of $M$, respectively. Then $A$ is called interpretable in $M$ by formulas $\Phi$ (in short, $\Phi$-interpretable) if there exists $n \in \mathbb{N}$, a subset $S \subseteq M^n$ and a bijective map, called interpreting map, $\phi : S \rightarrow A$, such that:

\[\text{The most general formulation of interpretability uses onto maps instead of bijective maps. Since only bijective maps appear in the interpretations of this paper, we have chosen to use this more restricted version of interpretability. This is similar to the approach followed in Section 1.3 of [67]. For the definition of interpretability with onto maps see Section 5.4 of [42] or Section 1.3 of [67].} \]
1. \( S \) is \( \Phi \)-definable in \( \mathcal{M} \).

2. For every function \( f = f(x_1, \ldots, x_n) \) in the language of \( \mathcal{A} \), the preimage by \( \phi \) of the graph of \( f \), i.e. the set \( \{(x_1, \ldots, x_k, x_{k+1}) \in M^{n(k+1)} \mid \phi(x_{k+1}) = f(\phi(x_1), \ldots, \phi(x_k))\} \), is \( \Phi \)-definable in \( \mathcal{M} \).

3. Similarly, for every relation \( r \) of \( \mathcal{A} \) (including the equality relation \( = \)), the preimage by \( \phi \) of the graph of \( r \) is \( \Phi \)-definable in \( \mathcal{M} \).

Similarly as before, if \( \Phi \) consists of all systems of equations in the language of \( \mathcal{M} \) then we speak of \( \text{e-interpretability} \), and if \( \Phi \) consists of all disjunctions of systems of equations we speak of \( \text{PE-interpretability} \). Note that a \( \text{PE-interpretation} \) is, in particular, an \( \text{e-interpretation} \).

The next two results are fundamental and they constitute the main reason we use interpretability in this paper. These are standard results whose proofs follow immediately from the Reduction Theorem 5.3.2 in [42] and Remark 3 after it (alternatively, see Lemma 2.7 of [37]).

**Proposition 2.4 (Interpretability is transitive).** Interpretability is a transitive relation. That is, given three structures \( M_1, M_2, \) and \( M_3 \), if \( M_1 \) is \( \text{e- or PE-} \) interpretable in \( M_2 \) and \( M_2 \) is \( \text{e or PE-} \) interpretable in \( M_3 \), then \( M_1 \) is \( \text{e- or PE-} \) interpretable in \( M_3 \), respectively.

**Proposition 2.5 (Reduction of problems).** Let \( M_1 \) and \( M_2 \) be two structures on languages \( L_1 \) and \( L_2 \), respectively. Assume \( M_1 \) is \( \text{e-interpret} \) or \( \text{PE-interpret} \) in \( M_2 \). Then the Diophantine problem in \( M_1 \) is reducible to the Diophantine problem in \( M_2 \). As a consequence, if the second problem is decidable, then so is the first.

Similarly, the problem of deciding if any given first-order formula in the language \( L_1 \) holds in \( M_1 \) is reducible to the problem of deciding if any given formula in the language \( L_2 \) holds in \( M_2 \). Consequently, if the first-order theory of \( M_2 \) is decidable then so is the first-order theory of \( M_1 \). The same statement holds when replacing first-order theory by positive \( \text{AE} \)-theory.

In order to illustrate how interpretability is nicely connected to the reduction of many algorithmic problems we sketch the proof of this last result.

**Sketch of the proof of Proposition 2.5 for monoids and \( \text{PE-interpretability} \).** Assume that a monoid \( M_1 \) is \( \text{PE-interpret} \) in another monoid \( M_2 \) through a map \( \psi : S \rightarrow M_2 \), where \( S \subseteq M_1^n \) for some \( n \). A reader may prefer to assume \( n = 1 \) throughout the proof, in which case the key ideas of the proof remain but some technical obfuscation disappears.

Let \( \Sigma(\vec{x}, \vec{q}) \) be a system of equations in \( M_1 \) on variables \( \vec{x} \) and coefficients \( \vec{q} \). We shall construct a disjunction of systems of equations \( \Sigma_1 \vee \cdots \vee \Sigma_k \) in \( M_2 \), with the property that \( \Sigma \) has a solution in \( M_1 \) if and only if there exists \( i = 1, \ldots, k \) such that \( \Sigma_i \) has a solution in \( M_2 \). This immediately gives a reduction of the Diophantine problem in \( M_1 \) to the Diophantine problem in \( M_2 \).

First, if necessary we add more variables and equations to \( \Sigma \), so that each equation \( e \) in \( \Sigma \) has the form \( z_{e1}z_{e2} = z_{e3} \), where each \( z_{ei} \) is either a coefficient, or a variable. For example, the equation \( x_1x_2q_2 = q_1x_1 \) is equivalent to the system of equations \( (y_1 = q_1x_1) \land (x_1x_2q_2 = y_1) \), which in turn is equivalent to \( (y_1 = q_1x_1) \land (x_1y_2 = y_1) \land (y_2 = x_2q_2) \). We denote still by \( \Sigma \) the resulting system of equations. We note that in a more general setting, this step corresponds to the construction of an equivalent formula where all atomic subformulas are unnested (see the proof of Theorem 5.3.2 in [42]).
For each coefficient or variable \( z_{ei} \) from \( \Sigma \) we introduce the notation \( \psi^{-1}(z_{ei}) \) with the following meaning: if \( z_{ei} \) is a coefficient then \( \psi^{-1}(z_{ei}) \) is just the preimage of such element by \( \psi \). On the other hand, if \( z_{ei} \) is a variable then \( \psi^{-1}(z_{ei}) \) is an \( n \)-tuple of variables each one taking values in \( M_1 \).

Since \( \psi \) is a PE-interpretation of \( M_1 \) in \( M_2 \), there exists a disjunction of systems of equations (i.e. a positive existential formula) in \( M_2 \), say \( \Psi_\cap(y_1,y_2,y_3,\bar{w}) \), that defines in \( M_2 \) the set
\[
\{(x_1,x_2,x_3) \in S^{3n} \mid \psi(x_1)\psi(x_2) =_{M_1} \psi(x_3)\} \subseteq M_2^{3n}.
\]
This set is nothing else than the preimage by \( \psi \) of the graph of monoid multiplication in \( M_1 \).

Now, for each equation \( z_{e1}z_{e2} = z_{e3} \) in \( \Sigma \), define the following disjunction of systems of equations in \( M_2 \):
\[
\Phi_e =_{def} \Psi_\cap(\psi^{-1}(z_{e1}),\psi^{-1}(z_{e2}),\psi^{-1}(z_{e3}),\bar{w}).
\]
By definition, the solutions to \( z_{e1}z_{e2} = z_{e3} \) in \( M_1 \) are in in bijective correspondence (under \( \psi \)) with the first \( 3n \) components of the solutions to \( \Phi_e \). Now let
\[
\Phi_\Sigma =_{def} \bigwedge_e \Phi_e,
\]
where the conjunction runs over all equations \( e \) of the system \( \Sigma \). Note that \( \Phi_\Sigma \) is a positive existential formula, and thus it is equivalent to a disjunction of systems of equations (see Remark 2.2). Moreover, it follows from the construction that \( \Sigma \) has a solution in \( M_2 \) if and only if \( \Phi_\Sigma \) has a solution in \( M_1 \), as required.

### 2.3 Hyperbolicity

Recall that the **undirected Cayley graph** of a monoid \( M \) with generating set \( A \) is a graph with vertex set \( M \) and undirected edges \{\( \{m,ma\} \mid m \in M, \ a \in A \}\).

**Definition 2.6.** Following [33], a monoid \( M = \langle A \rangle \) will be called **hyperbolic** if its undirected Cayley graph is hyperbolic as a metric space with the usual distance metric for graphs and it will be called **word-hyperbolic** if there is a regular language \( L \subseteq A^* \) such that
\[
\{(L) = \{u#v#w^r \mid u, v, w \in L, uv =_M w\}\}
\]
is a context-free language.

A more restrictive notion of hyperbolic monoid has been proposed in [43].

We follow Chapter 12 in [44] for the following definitions. Recall that a rewriting system \((A,R)\) consists in a (possibly infinite) collection of pairs \((u,v) \in A^* \times A^*\), denoted \( u \rightarrow v \) and called **rewriting rules**. One writes \( u \rightarrow^* v \) to indicate that \( v \) can be obtained from \( u \) by successively applying finitely many rewriting rules. The rewriting system \((A,R)\) is called **confluent** if whenever \( u \rightarrow^* v, u \rightarrow^* w \) for some words \( u, v, w \) there exists \( u' \in A^* \) such that \( v \rightarrow^* u' \) and \( w \rightarrow^* u' \). Moreover, \((A,R)\) is called **monadic** if all rewriting rules are of the form \( u \rightarrow v \) for some \( u \in A^* \), \( v \in A \cup \{1\} \), and \(|u| \geq |v|\). Finally, \((A,R)\) is said to be **regular** if the set of left-hand sides of all rewriting rules forms a regular language in \( A \). Note that in particular if \(|R| < \infty \) then \((A,R)\) is regular.

**Theorem 2.7** (Theorem 3.2 and Proposition 5.2 of [12]). Let \((A,R)\) be a regular confluent monadic rewriting system. Then the monoid \( \langle A \mid R \rangle \) is both word-hyperbolic and hyperbolic.
3 Hyperbolicity in one-relator monoids

In this section we will give some sufficient conditions for one-relator monoids to be hyperbolic and to have hyperbolic group of units. The following result gives a sufficient condition for the undirected Cayley graph of a special one-relator monoid to be hyperbolic. It follows from results in \cite{93} and \cite{40}. We include a proof for completeness.

**Proposition 3.1.** Let \( M = \langle A \mid w = 1 \rangle \) be a one-relator special monoid. Let \( G \) be the group of units of \( M \). If \( G \) is a hyperbolic group then the undirected Cayley graph of \( M \) is hyperbolic.

**Proof.** In this proof we make use of the definitions and notation from Zhang \cite{93}. More details on Zhang’s theory for the study of finitely presented special monoids will be given below at the beginning of Section 5. Following Zhang \cite{93}, let \( \Delta \) be the set of minimal invertible pieces of the relator \( w \). Let \( I \) be the set of all non-empty prefixes of the words from \( \Delta \), that is

\[
I = \{ x \in A^+ \mid xy \in \Delta \text{ for some } y \in A^* \}.
\]

Let \( Y = \{ [u] : u \in I \} \). Then, by Zhang \cite{93}, Lemma 3.3, \( Y \) is a finite generating set for the submonoid of right units \( R \) of \( M \). Note that \( R \) is the \( \mathcal{R} \)-class of the identity element of \( M \), where \( \mathcal{R} \) is Green’s \( \mathcal{R} \)-relation on \( M \) defined by saying \( mRn \) if and only if \( mM = nM \). Clearly \( \Delta \) is a subset of \( I \). Let \( \mathcal{G} \) be the underlying undirected graph of the right Cayley graph of the monoid \( R \), with respect to the generating set \( Y \). So \( \mathcal{G} \) has vertex set \( R \) and edges \( \{ [u], [ux] \} \) where \( u \in A^* \), \( x \in I \) and \( \{ [u], [ux] \} \) is a subset of \( R \). Note that, since \( R \) is a right cancellative monoid, \( \mathcal{G} \) is a connected infinite graph with vertices of bounded degree.

We use \( \mathcal{S} \) to denote the undirected Schützenberger graph of the \( \mathcal{R} \)-class \( R \). So \( \mathcal{S} \) also has vertex set \( R \) but has edges \( \{ [u], [ua] \} \) where \( u \in A^* \), \( a \in A \) and \( \{ [u], [ua] \} \) is a subset of \( R \).

We claim that the identity map on \( R \) defines a quasi-isometry between the graph \( \mathcal{G} \) and the graph \( \mathcal{S} \).

To prove this claim, let \( d_{\mathcal{G}} \) and \( d_{\mathcal{S}} \) denote the distances in each of these graphs. Consider an arbitrary edge \( \{ [u], [ux] \} \) in the graph \( \mathcal{G} \). Let \( D \) be the maximum length of a word in \( \Delta \). Then \( d_{\mathcal{S}}([u], [ux]) \leq D \). For the converse, let \( \{ [u], [ua] \} \) be an arbitrary edge in the graph \( \mathcal{S} \). We claim that \( d_{\mathcal{G}}([u], [ua]) \leq 2 \). We may assume without loss of generality that \( u \) is a reduced word. There are now two cases to consider.

First suppose that \( ua \) is a reduced word. It then follows from \cite{93}, Lemma 3.3] that \( ua \in I^* \) (i.e. is a graphical product of words from \( I \)). Note that \( ua \) may admit several different decompositions in \( I^* \). Write \( ua = u'\gamma \) where \( \gamma \in I \) and \( u' \in I^* \). If \( |\gamma| = 1 \) then \( a = \gamma \in I \) and so \( d_{\mathcal{G}}([u], [ua]) = 1 \). Now suppose that \( |\gamma| > 1 \). Write \( \gamma = \gamma' a \) with \( \gamma' \in I \).

Then we have \( u = u'\gamma' \) and both \( \{ [u'], [u'\gamma'] \} \) and \( \{ [u'], [u'\gamma] \} \) are edges in the graph \( \mathcal{G} \). It follows that \( d_{\mathcal{G}}([u], [ua]) = d_{\mathcal{G}}([u'\gamma'], [u'\gamma]) \leq 2 \).

Now suppose that \( ua \) is not a reduced word. Since \( u \) is reduced, it follows that we can write \( ua = u'\gamma \) where \( \gamma \in \Delta \) is a non-empty word. Then arguing as in the previous paragraph, either \( |\gamma| = 1 \) and \( d_{\mathcal{G}}([u], [ua]) = 1 \), or else \( |\gamma| > 1 \) and \( d_{\mathcal{G}}([u], [ua]) \leq 2 \). This completes the proof of the claim that the identity mapping on \( R \) induces a quasi-isometry between the graph \( \mathcal{G} \) and the graph \( \mathcal{S} \).

It follows from \cite{93}, Theorem 4.5] that the submonoid of right units \( R \) of \( M \) is isomorphic to a monoid free product \( T \ast G \) where \( T \) is a free monoid of finite rank, and \( G \) is the group of units of the monoid \( M \). Since the Cayley graph of a free monoid is a tree, it then follows that
the undirected Cayley graph $G$ of $R \cong T \ast G$ is hyperbolic. Since $S$ is quasi-isometric to $G$ we conclude that the undirected Schützenberger graph of the $\mathcal{R}$-class of the identity element is hyperbolic.

It follows from the results in [40, Section 3] that (i) the Schützenberger graphs of any pair of $\mathcal{R}$-classes of $M$ are isomorphic to each other, and (ii) for every $\mathcal{R}$-class $R'$ of $M$ there is at most one edge $\{m, ma\}$ in the Cayley graph of $M$ such that $m \in M$, $a \in A$, with $ma \in R'$ but $m \notin R'$, and (iii) the quotient graph with vertex set the $\mathcal{R}$-classes of $M$ and edges all edges $\{m, n\}$ from the Cayley graph of $M$ such that $(m, n) \notin \mathcal{R}$ is a rooted tree. Note that in general the vertices of this tree have infinite degree.

Combining these observations we see that the Cayley graph of $M$ has the structure of a “tree of copies of” the hyperbolic graph $S$. From this it quickly follows that the undirected Cayley graph of $M$ is hyperbolic.

Proposition 3.2. Let $M = \langle A \mid w^k = 1 \rangle$ be a one-relator monoid with torsion. Then the group of units of $M$ is a one-relator group with torsion. It follows that the group of units of $M$ is a hyperbolic group, and the undirected Cayley graph of $M$ is a hyperbolic metric space.

Proof. It follows from results of Adjan [2] that the group of units $G$ of $M$ is a one-relator group with torsion (see [40, Section 3] for a proof of this). By the Newman Spelling Theorem we have that $G$ is a hyperbolic group. This and Proposition 3.1 imply that the undirected Cayley graph of $M$ is hyperbolic.

We do not know if every one-relator monoid of the form $\langle A \mid w^n = 1 \rangle$ with $n > 1$ is word-hyperbolic.

4 Word-hyperbolic monoids

In this section we investigate the Diophantine problem of word-hyperbolic monoids (in the sense given in Subsection 2.3). Before anything else we comment on one of the simplest examples of such monoids, namely the bicyclic monoid $B = \langle a, b \mid ab = 1 \rangle$. This monoid is hyperbolic and word-hyperbolic due to Cain’s result (Theorem 2.7). Moreover, in [27, Corollary 7] it is proved that the first-order theory of $B$ is decidable. In particular, many decision problems are decidable in $B$, such as the Diophantine problem, the problem of identity checking, etc.

4.1 Polycyclic monoids

For $n \in \mathbb{N}$ the polycyclic monoid $P_n$ on $n$ generators is the monoid with zero defined by the presentation
Theorem 4.1. Without loss of generality that each equation
This follows from Cain’s result (Theorem 2.7) using the natural monoid presentation of
coefficients $\vec{q}$ and $i$
hold:
Hence for any two elements $g,h$
holds.

The following result improves on results in the literature showing that the conjugacy
problem is decidable in polycyclic monoids.

Theorem 4.1. The Diophantine problem in the polycyclic monoid $P_n$ is decidable, for all
On the other hand, the positive AE-theory of $P_n$ is undecidable for all $n \geq 2$. In
particular, the first-order theory of $P_n$ ($n \geq 2$) is undecidable.

Proof. We first prove that the Diophantine problem is decidable in $P_n$ for all $n \geq 1$. The
strategy is as follows: given a system of equations $\Sigma$ in $P_n$, we find finitely many systems of
equations with rational constraints in a free monoid with involution $(P^*, \cdot, 1, -1)$, for some
finite set $P$, such that $\Sigma$ has a solution in $P_n$ if and only if one of these systems has a solution
in $(P^*, \cdot, 1, -1)$ (see Subsection 2.1 for definitions). Then the result follows from the fact that
the latter problem is decidable due to 2.3.

Let $^{-1} : P_n \rightarrow P_n$ be the natural involution defined by sending $p_i$ to $p_i^{-1}$ and $p_i^{-1}$ to $p_i$
for all $i = 1, \ldots, n$: by sending every product $p_1 \ldots p_k$ to $p_k^{-1} \ldots p_1^{-1}$, and by letting $1^{-1} = 1$,
$0^{-1} = 0$.

Let $\Sigma(\vec{x}, \vec{q}) = 1$ be a system of equations in $P_n$ on variables $\vec{x} = (x_1, \ldots, x_m)$ and
coefficients $\vec{q} = (q_1, \ldots, q_k)$. By adding new variables and new equations to $\Sigma$ we can assume
without loss of generality that each equation $e$ in $\Sigma$ has the form $z_{e_1}z_{e_2} = z_{e_3}$ where for each $e$
and $i = 1, 2, 3$, $z_{e_i}$ is either a coefficient or a variable, for example the equation $x_1x_2x_3 = q_1x_1$
is equivalent to the system of equations $(y_1 = q_1x_1) \wedge (x_1x_2y_2 = y_1)$, which in turn is equiva-
lent to $(y_1 = q_1x_1) \wedge (x_1y_2 = y_1) \wedge (y_2 = x_2y_2)$ (this is the same example given in the sketch
of proof of Proposition 4.3).

Denote $P_n^+ = \{p_i \mid i = 1, \ldots, n\} \subseteq P_n$ and $P_n^- = \{p_i^{-1} \mid i = 1, \ldots, n\} \subseteq P_n$. Note that any
element $h \in P_n$ can be written uniquely as $h = 0$ or $h = h_-h_+$ where $h_- \in P_n^-$ and $h_+ \in P_n^+$.
Hence for any two elements $g,h \in P_n$ one and only one of the following three alternatives hold:

$$gh = g_--g_+h_-h_+ = \begin{cases} 0, & \text{or} \\ g_-h''_+h_+, & \text{and } h_- = h'_-h''_+, \text{ and } g_+^{-1} = h'_-, \text{ or} \\ g_-g''_+h_+, & \text{and } g_+ = g'_+g''_+, \text{ and } h_-^{-1} = g''_+, \end{cases}$$

for some elements $h'_-, h''_-$ if the second case holds, or some elements $g'_+, g''_+$ if the third case
holds.

Replace each occurrence of each variable $x \in X$ in $\Sigma$ by the expression $x_-x_+$, where $x_-$
and $x_+$ are new variables, and add formal constraints $x_- \in P_n^-$, $x_+ \in P_n^+$. Later we will
express these as suitable rational constraints in a free monoid. Also, replace each occurrence
of each coefficient $q_i$ by the product of the coefficient elements $q_{i,-} \in P_n^-$ and $q_{i,+} \in P_n^+$. 16
Let $e$ where in the first case obtained by making all possible substitutions of each equation and only if one of the following holds: there exists two indices the disjunction of the following systems of equations, where

\[ p \in \mathcal{P} \]

Denote the resulting system of equations by $\Sigma'$. Thus each equation $z_{e1}z_{e2} = z_{e3}$ in $\Sigma$ has been replaced in $\Sigma'$ by the equation with constraints

\[ z_{1-}z_{1+}z_{2-}z_{2+} = z_{3-}z_{3+}, \quad z_{i-} \in P_n^-, \quad z_{i+} \in P_n^+, \quad (i = 1, 2, 3), \]

where for readability we have removed the subscript $e$. Due to (2), the latter equation is equivalent to the disjunction of the following three systems of equations with constraints in $(P_n^+, \cdot^{-1}, 1)$, where $\cdot$ is the multiplication operation of $P_n$, 1 is the identity element, and $^{-1}$ is the involution defined above.

\[
\begin{align*}
  z_{2+} &= z_{3+}, \\
  z_{1-}z_{2-}^{-1} &= z_{3-}, \\
  z_{2-} &= z_{2-}^{-1}, \\
  z_{1-} &= z_{2-}^{-1}, \\
  z_{1+}, z_{2+}, z_{3+} &\in P_n^+,
\end{align*}
\]

\[
\begin{align*}
  z_{1-} &= z_{3-}, \\
  z_{1+}z_{2+} &= z_{3+}, \\
  z_{1+} = z_{1+}^{-1}, \\
  z_{2-} &= z_{1+}^{-1}, \\
  z_{1-}, z_{2-}, z_{3-} &\in P_n^-, \\
  z_{1+}, z_{2+}, z_{3+} &\in P_n^+.
\end{align*}
\]

(3)

where in the first case $z_{2-}^{-1}, z_{2-}^{-1}$ are fresh new variables, and in the second case $z_{1+}^{-1}, z_{1+}^{-1}$ are fresh new variables. Let $S_1$ be the first system of equations with constraints appearing in (3). Let $\mathcal{P}$ be the set $\{p_1, \ldots, p_n, p_1^{-1}, \ldots, p_n^{-1}\}$, and consider the free monoid with involution $(\mathcal{P}^*, \cdot, 1, -1)$ (note the slight abuse of notation: formally it would be more correct to use copies of the letters $\{p_i, p_i^{-1} \mid i = 1, \ldots, n\}$ as generators of $(\mathcal{P}^*, \cdot, 1, -1)$). We claim that $S_1$ is equivalent to a system of equations with rational constraints $S_1'$ in $(\mathcal{P}^*, \cdot, ^{-1}, 1)$, where by equivalent we mean that one system has a solution if and only if the other does. Indeed, to construct such $S_1'$ replace each constraint of the form $z_{i+} \in P_n^+$ in $S_1$ by a rational constraint requiring that $z_{i+}$ takes values in the rational set $\{p_1, \ldots, p_n\}^* \subseteq \mathcal{P}^*$. Similarly, replace each constraint $z_{i-} \in P_n^-$ by a rational constraint requiring that $z_{i-}$ takes values in the rational set $\{p_1^{-1}, \ldots, p_n^{-1}\}^* \subseteq \mathcal{P}^*$. All other equations of $S_1$ are left unchanged.

Similarly, the second system in (3) is equivalent to a system of equations with rational constraints in the free monoid with involution $(\mathcal{P}^*, \cdot, ^{-1}, 1)$. Regarding the third system of equations in (2), observe that two elements $z_{1+} \in P_n^+$ and $z_{2-} \in P_n^-$ satisfy $z_{1+}z_{2-} = 0$ if and only if one of the following holds: there exists two indices $1 \leq i \neq j \leq n$ such that $z_{1+} = z_{1+}^{-1}p_i$ and $z_{2-} = p_j^{-1}z_{2-}$. Hence the third system of equations in (3) is equivalent to the disjunction of the following systems of equations, where $1 \leq i \neq j \leq n$:

\[
\begin{align*}
  z_{1+} &= z_{1+}^{-1}p_i, \\
  z_{2-} &= p_j^{-1}z_{2-}^{-1}, \\
  z_{3} &= 0, \\
  z_{1-}, z_{2-}, z_{3-} &\in P_n^-, \\
  z_{1+}, z_{2+}, z_{3+} &\in P_n^+.
\end{align*}
\]

(4)

Let $\Sigma_1, \ldots, \Sigma_6$ be the list of all systems of equations with rational constraints in $(\mathcal{P}^*, \cdot, ^{-1}, 1)$ obtained by making all possible substitutions of each equation $z_{1-}z_{1+}z_{2-}z_{2+} = z_{3-}z_{3+}$ in $\Sigma'$ by either one of the first two systems of equations with rational constraints from (3) or a system of the form (4) for some $1 \leq i \neq j \leq n$. Then $\Sigma$ has a solution in $P_n$ if and only
if there exists $i = 1, \ldots, \ell$ such that $\Sigma_i$ has a solution in $(P^*, -^1, \ldots, 1)$. The first part of the theorem, namely that the Diophantine problem is decidable in $P_n$, now follows from the fact that systems of equations with rational constraints are decidable in any free monoid with involution $[20]$.

We next prove the second part of the theorem; namely that the positive $\mathcal{AE}$-theory (and thus the first-order theory) of $P_n$ is undecidable. Indeed, we have seen that the submonoid $P_n^+ = \langle p_1, \ldots, p_n \rangle$ of $P_n$ is a free monoid. Moreover, this submonoid is $e$-interpretable in $P_n$, because an element $g \in P_n$ belongs to $P_n^+$ if and only if there exists an element $h \in P_n$ such that $gh = 1$. Hence the equation $xy = 1$ serves as an $e$-interpretation of $P_n^+$ in $P_n$. Now by Proposition $2.5$, the positive $\mathcal{AE}$-theory of $P_n^+$ is reducible to the positive $\mathcal{AE}$-theory of $P_n$. Since the first is undecidable because $P_n^+$ is a free monoid $[80]$, so is the second. This completes the last part of the proof. $\square$

### 4.2 A general lemma and an example

The following lemma will be key later when studying systems of equations in some one-relator monoids (Section $5$). It is also used in Remark $4.3$ to obtain a simple example of word-hyperbolic monoid whose Diophantine problem is likely to be difficult. A definition of weighted length relation can be found in Subsection $2.1$.

**Lemma 4.2.** Let $M$ be a monoid, let $C = \langle c_0 \rangle$ be an infinite one-generated submonoid of $M$, and let $D$ be a free rank-$n$ submonoid of $M$ freely generated by a set $\{d_1, \ldots, d_n\} \subseteq M$. Assume that both monoids $C$ and $D$ are $e$-interpretable in $M$. Assume also that for each $i = 1, \ldots, n$ there exists $k_i \in \mathbb{N}$ such that $c_i^{k_i}d_i = 1$. Then the free monoid with weighted length relation $(D^*, 1, =, L_{\bar{k}})$ is $e$-interpretable in $M$, where $\bar{k} = (k_1, \ldots, k_n)$, and $\cdot$ is the usual concatenation operation.

**Proof.** Since the free monoid $D$ is $e$-interpretable in $M$, it suffices to show that so is the relation $L_{\bar{k}}$. Let $\Sigma_C(x, \bar{y})$ and $\Sigma_D(z, \bar{w})$ be two systems of equations $e$-interpreting $C$ and $D$ in $M$, so that an element $h \in M$ belongs to $C$ (respectively $D$) if and only if $\Sigma_C(h, \bar{y})$ (resp. $\Sigma_D(h, \bar{w})$) has a solution $\bar{y}_0$ (resp. $\bar{w}_0$) in $M$. Take arbitrary elements $c \in C$ and $d \in D$. Then $c = c_0^t$ for some $t \in \mathbb{N}$, and $d = d_{i_1} \cdots d_{i_r}$ for some $d_{i_j}$. Now

\[
    cd = \begin{cases} 
    t - |d\bar{k}|_0 & \text{if } t > |d\bar{k}|, \\
    1 & \text{if } t = |d\bar{k}|, \\
    d_{i_{t+1}} \cdots d_{i_r} & \text{if } t < |d\bar{k}|, \text{ and } cd \in D, \\
    c_0^s d_{i_{t+1}} \cdots d_{i_r} & \text{if } t < |d\bar{k}|, \text{ and } cd \notin D,
\end{cases}
\]  

(5)

where in the last two cases $\ell$ is the minimum number such that $|d_{i_1} \cdots d_{i_{t+1}}|_0 > t$ (we have $\ell < r$), and in the last case $s$ is some number such that $0 < s < k_{i_{t+1}}$. It follows that if $t \geq |d\bar{k}|$ then $cd \in C$. The other implication is true as well: if we had $cd \in C$ and $t < |d\bar{k}|$ then $cd = c_0^s d_{i_{t+1}} \cdots d_{i_r} = c_0^s$ for some $r, s \geq 0$ and some $0 < \ell < r$. Let $d' = d_{i_{t+1}} \cdots d_{i_r}$ and let $p = |d'\bar{k}|$. Note that $s < p$. Then $1 = c_0^s d' = c_0^{p-s} c_0^s d' = c_0^{p-s+r}$, contradicting the assumption that $\langle c_0 \rangle$ is infinite.

We have proved that $cd \in C$ if and only if $|c| = t \geq |d\bar{k}|$. Due to the e-definability of $C$, this in turn occurs if and only if $\Sigma_C(cd, \bar{y})$ has a solution $\bar{y}_0$. Moreover, the second case of
and the infiniteness of $\langle c_0 \rangle$ indicate that $t = |d|_k$ if and only if $cd = 1$. Hence given two elements $d_1, d_2 \in D$ we have that $|d_1|_k \leq |d_2|_k$ if and only if there exists an element $c \in C$ such that $cd_2 = 1$ (this ensures $|c| = |d_2|_k$) and $\Sigma_C(cd_1, \vec{y})$ has a solution $\vec{y}_0$ (this ensures $|d_1|_k \leq |c|$). Overall, $|d_1|_k \leq |d_2|_k$ if and only if the following system of equations has a solution $x_0, \vec{y}_0, \vec{z}_0$:

\[
\begin{cases}
\Sigma_C(x, \vec{y}), \\
\Sigma_C(xd_1, \vec{z}), \\
xd_2 = 1
\end{cases}
\] (6)

It follows that the $k$-weighted length relation $L_k$ is e-interpretable in $M$. \hfill $\square$

**Example 4.3.** The above result can be applied to the monoid with presentation

\[
\langle a, b_1, \ldots, b_n \mid ab_1 = 1, ab_2 = 1, \ldots, ab_n = 1 \rangle,
\] (7)

for any $n > 1$, thus we recover the reduction from Example 21 in [27]. We remark further that any such monoid is hyperbolic and word-hyperbolic due to Cain’s result (Theorem 2.7). Hence (7) constitutes a simple example of a word-hyperbolic monoid where the Diophantine problem is likely to be a difficult problem. Note also that the positive AE-theory with coefficients of (7) is undecidable (thus also its first-order theory).

## 5 One-relator monoids

Our interest in this section is in the Diophantine problem for one-relator monoids with presentation $\langle A \mid w = 1 \rangle$. The main theme of this section will be that the Diophantine problem for one-relator monoids of the form $\langle A \mid w = 1 \rangle$ is difficult. In more detail we shall see how this problem relates to other known difficult decidability problems. Before exploring those links we first observe one other situation where the Diophantine problem is decidable.

**Theorem 5.1.** Let $M = \langle A \mid w = 1 \rangle$ and suppose that every letter in $w$ is invertible in $M$. Let $G = \langle B \mid w = 1 \rangle$ where $B \subseteq A$ is the set of letters that appear in $w$. Then $G$ is a one-relator group, and if the Diophantine problem is decidable in $G$ then it is decidable in $M$.

**Proof.** The monoid $M$ is isomorphic to the monoid free product $G \ast C^*$ where $C = A \setminus B$. Both $G$ and $C^*$ satisfy Assumption 17 from [27] (a cancellativity condition which satisfied by any group and any free monoid) and Assumption 18 from [27] (decidability of the Diophantine problem). Hence applying [27, Theorem 19] (taking $C_\sigma$ to be just $\{U_\sigma, V_\sigma\}$) we obtain that the Diophantine problem of $M$ is decidable. \hfill $\square$

**Example 5.2.** As an easy example of an application of the previous theorem, we see that the Diophantine problem is decidable in the monoid $\langle a, b, c, d \mid aba = 1 \rangle$. Indeed, the monoid $\langle a, b \mid aba = 1 \rangle$ is the infinite cyclic group. Some more complicated examples to which this theorem applies will be discussed in Section 6.

Let $\Delta = \{ \alpha_i \ (i \in I) \} \subseteq A^+$ be the set of minimal invertible pieces of the relator $w$. So the word $w$ uniquely decomposes as

\[w \equiv \alpha_{i_1} \alpha_{i_2} \ldots \alpha_{i_k}\]
where each $\alpha_{i_j} \in \Delta$, and each of these words is invertible in $M$ and has no proper non-empty prefix which is invertible in $M$. As mentioned in the introduction, we call the $\alpha_{i_j}$ the minimal invertible pieces of $w$. In [2] Adjan gives an algorithm for computing the minimal invertible pieces of the defining relator of a one-relator special monoid. In particular, every letter appearing in the relator represents an invertible element of the monoid if and only if all the minimal invertible pieces have size one, and this can be decided using Adjan’s algorithm. Hence Adjan’s algorithm can be used to test whether the hypothesis of Theorem 5.1 above are satisfied. A good description of Adjan’s algorithm can be found in [56, Section 1].

Since each piece $\alpha_i$ is minimal invertible, none of them is a prefix of another piece $\alpha_j$, and so $\Delta$ is a prefix code. Hence the submonoid of $A^*$ generated by $\Delta$ is free. We shall denote it $\Delta^*$. Let $B = \{b_i \mid i \in I\}$ be an alphabet in bijective correspondence with $\Delta$. Let $\phi : \Delta^* \rightarrow B^*$ be the unique homomorphism extending $\alpha_i \mapsto b_i$ for $i \in I$. It follows from Adjan’s results [2] that the group of units $G$ of $M$ is isomorphic to the monoid defined by the monoid presentation

$$\langle B \mid \phi(w) = 1 \rangle = \langle B \mid b_i b_i \ldots b_i = 1 \rangle.$$  

**Theorem 5.3** ([93], Proposition 3.2). The infinite monoid presentation

$$\langle A \mid u = v : u, v \in \Delta^*, v \lessdot_{sh} u \& \phi(u) =_G \phi(v) \rangle$$

is an infinite complete rewriting system defining the monoid $M$.

In the above theorem $\lessdot_{sh}$ denotes shortlex ordering, and $\phi(u) =_G \phi(v)$ means that $\phi(u)$ and $\phi(v)$ both represent the same element in the group of units $G$. For the rest of this section, when we say a word $w$ is reduced we mean that it is reduced with respect to the above infinite complete rewriting system (8). Our aim is to show that for a wide class of special one-relator monoids, if we could solve equations for those monoids then that would imply a solution to equation solving with length constraints in free monoids—which is a longstanding open problem; see [11, 23, 24, 36, 58]. Of course, not every special one-relator monoid encodes equation solving with length constraints since, for instance, we have seen above that equations can be solved over the bicyclic monoid. So we will need some conditions on the monoid. We give conditions in terms of certain combinatorial properties on the set of minimal invertible pieces $\Delta$. We suppose that the following conditions are satisfied:

(C1) No word from $\Delta$ is a proper subword of any other word from $\Delta$.

(C2) There exist distinct words $\gamma, \delta \in \Delta$ with a common initial letter $a \in A$.

These conditions are easily satisfied and can be used to construct a wide variety of examples as we shall see in the next section. Note, for instance, if all the words from $\Delta$ have the same length, then condition (C1) will be satisfied. In particular there are one-relator monoids with torsion whose minimal invertible pieces satisfy these properties. A concrete example is given by the family monoids

$$\langle a, b, c \mid ((ab)(ac)(ab))^k = 1 \rangle,$$

for $k > 1$, where it is easily verified that every minimal invertible piece belongs to the set $\{ab, ac\}$. This gives many examples of special one-relator monoids with hyperbolic (right and left) Cayley graphs which satisfy the conditions (C1)-(C2). Applications to examples like this will be discussed below.
For the rest of this section let $M$ be the one-relator monoid defined by the monoid presentation

$$\langle A \mid r = 1 \rangle$$

where we suppose that conditions (C1)-(C2) are satisfied.

We use $=$ to denote equality in $M$ and $\equiv$ to denote graphical equality, that is, $w_1 \equiv w_2$ means $w_1$ and $w_2$ are equal as word in $A^\ast$. We denote the projection of a word $w$ onto $M$ by $[w]$, so $[w]$ is the element of $M$ represented by the word $w$.

We now give a series of important technical lemmas.

**Lemma 5.4.** Suppose that (C1) and (C2) are both satisfied. Then for every reduced word $w \in A^\ast$, and every positive integer $i > 0$, if $a^i w = 1$ then $w$ has no prefix in $\Delta$.

**Proof.** Since $a^i w = 1$ it follows that $a^i w$ is not reduced and since $w$ is assumed to be reduced it follows that we can write

$$a^i w \equiv a^j \alpha_1 \ldots \alpha_k w''$$

where $0 \leq j < i$, $w''$ is a suffix of $w$, $\alpha_1 \ldots \alpha_k$ is the left hand side of a rewrite rule from $\Delta$, and each $\alpha_i \in \Delta$. Since $a$ is not invertible and $w$ is reduced, we have $\alpha_1 \equiv a^k w'$ where $k = i - j > 0$, and $w'$ is a non-empty prefix of $w$. Suppose, seeking a contradiction, that $w = \beta w_2$ with $\beta \in \Delta$. Note that since $w'$ is a suffix of $\alpha_1$, where $\alpha_1$ is invertible, it follows that $w'$ is left invertible. Now, if $w'$ were a prefix of $\beta$ it would follow that $w'$ is also right invertible and hence invertible. But then since $\alpha_1$ and $w'$ are both invertible it would follow that $a^k$ is invertible and hence $a$ is invertible, which is a contradiction. Therefore we must have that $\beta$ is a prefix of $w'$, but then $\beta \in \Delta$ is a proper subword of $\alpha_1 \equiv a^k w' \in \Delta$, and this contradicts (C1). This completes the proof of the lemma. \[\square\]

**Lemma 5.5** ([93], Lemma 3.1 and Lemma 3.6). If $u_1, u_2 \in \Delta^\ast$ then $[u_1] = [u_2]$ in $M$ if and only if $[\phi(u_1)] = [\phi(u_2)]$ in the one-relator group $G$.

**Lemma 5.6.** Let $\delta$ and $\gamma$ be two distinct words in $\Delta$. Then $[\delta] \neq [\gamma]$ in $M$.

**Proof.** Since the words $\delta$ and $\gamma$ are distinct it follows that $\phi(\delta)$ and $\phi(\gamma)$ are distinct letters of $B$. This implies that $|B| \geq 2$. If $|B| \geq 3$ then it follows from Magnus’ Freiheitssatz that $\phi(\delta)$ and $\phi(\gamma)$ represent distinct elements of the group $G$ and hence $[\delta] \neq [\gamma]$ in $M$, by the previous Lemma 5.5.

Now suppose that $|B| = 2$. Set $c = \phi(\delta)$ and $d = \phi(\gamma)$. If $c = d$ in $G$ then $cd^{-1} = 1$ in $G$. Since $|B| = 2$ and $c = d$ it follows that the defining relator in the presentation of $G$ is a proper power. Then it follows from Newman’s spelling theorem that $cd^{-1}$ contains a subword of the defining relator (which uses no inverse of $c$ or $d$), or the inverse of such a subword, with length at least 2. This is clearly impossible and thus completes the proof. \[\square\]

**Lemma 5.7** ([93], Lemma 3.3). Let $u \in A^\ast$ be reduced. If $[u]$ is invertible then $u \in \Delta^\ast$.

We are interested in right inverses of powers of the element $a$. These elements clearly form a submonoid of $M$. The following result shows that the set of reduced words representing elements in this submonoid themselves form a submonoid of the free monoid $A^\ast$. 

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\textbf{Lemma 5.8.} Let $i,j \in \mathbb{N}$. Let $u, v \in A^*$ be reduced words such that $a^i u = 1$ and $a^j v = 1$. Then $uv$ is a reduced word such that $a^{i+j} uv = 1$.

\textit{Proof.} We just need to prove that $uv$ is a reduced word. By Lemma 5.4, the word $v$ does not have any prefix in $\Delta$. If $uv$ were reducible then it would follow that there is a non-empty suffix $u_1$ of $u$, and a non-empty prefix $v_1$ of $v$, such that $u_1v_1 \in \Delta$. But then $u_1$ is left invertible, since $u$ is left invertible, and right invertible, since $u_1v_1$ is right invertible. This contradicts $u_1v_1 \in \Delta$.

Let $F$ be the set of all reduced words $\beta$ such that $a^i \beta = 1$ for some $i \in \mathbb{N}$ with $i > 0$. By the previous lemma, $F$ is a submonoid of $A^*$. We shall now prove that $F$ is a free submonoid of $A^*$. For this it will be useful to recall some standard results about submonoids of free monoids. Recall from [60, Subsection 1.2] that given a submonoid $P$ of $A^*$ there is a unique set $B$ that generates $P$ and is minimal with respect to set-theoretic inclusion; it is the set 

$$(P \setminus \{1\}) \setminus ((P \setminus \{1\})^2).$$

The following nice characterisation of free subsemigroups of free semigroups, from Lothaire, will be useful for us; see [60, Proposition 1.2.3].

\textbf{Lemma 5.9.} A submonoid $P$ of $A^*$ is free if and only if for any word $w \in A^*$, one has $w \in P$ whenever there exist $p, q \in P$ such that

$$pw, wq \in P.$$ 

\textbf{Lemma 5.10.} $F$ is a free submonoid of $A^*$.

\textit{Proof.} Suppose that $p, q, pw, wq \in F$ where $p, q \in F$ and $w \in A^*$. We need to show that $w \in F$. Since $w$ is a subword of a reduced word (for example, $pw$), it is reduced. By assumption there are $i, j \geq 1$ such that $a^i p = 1$ and $a^j pw = 1$. If $i = j$ then $w = 1$ and since $w$ is reduced it is the empty word and this belongs to $F$. If $i < j$ then $a^{j-i} w = 1$ and so $w \in F$. Otherwise, if $i > j$ then it would follow that $w = a^k$ for some $k > 0$. But then $a^{j} pw = 1$ implies $a^j pw = 1$. This last equality implies that $a$ is invertible, contradicting (C2) and the definition of $\Delta$. In all cases $w \in F$ so this completes the proof of the lemma.

\textbf{Lemma 5.11.} Let $w \in A^*$ be arbitrary. Write $w \equiv w_1 w_2$ where $w_1$ is the longest prefix of $w$ which is invertible. Suppose that $w'$ may be obtained from $w$ by a single application of a relation from the presentation. Write $w' = w'_1 w'_2$ where $w'_1$ is the longest invertible prefix of $w'$. Then $w_1 = w'_1$ in $M$. This implies that for any pair of words $u, v$, if $u = v$ in $M$ then the longest invertible prefix of $u$ is equal to 1 in $M$ if and only if the longest invertible prefix of $v$ is equal to 1 in $M$.

\textit{Proof.} We consider where the relation is applied to the word $w \equiv w_1 w_2$. If the relation is applied within either $w_1$ or $w_2$ the result is immediate, so suppose otherwise. Let $\delta_1 \ldots \delta_m \in \Delta^*$ be the subword of $w$ to which the relation is being applied. If there is a non-empty suffix $w'_1$ of $w_1$, and non-empty prefix $w'_2$ of $w_2$ such that $w'_1 w'_2 \equiv \delta$, for some $r$, then since $w'_1$ is left invertible since it is a suffix of $w_1$, and $w'_2$ is right invertible since it is a prefix of $\delta$, it would follow that $w'_1$ is invertible, which would contradict the fact that $\delta$ has no proper prefix which is invertible. So we must have $w_1 \equiv a \delta r+1 \ldots \delta_m$, and $w_2 \equiv \delta r+1 \ldots \delta_m \beta$, but then $w_1 \delta_1 \ldots \delta_m$ is a prefix of $w$ which is invertible and is longer than $w_1$, contradicting the definition of $w_1$. \hfill \Box
Let $m \in \mathbb{N}$ be the maximum value $m$ such that there is a minimal invertible piece $\alpha \in \Delta$ such that $a^m$ is a prefix of $\alpha$. We define a finite set of words $X$ in the following way. For each $1 \leq j \leq m$ and for every piece $a^j \beta \in \Delta$ (where $\beta$ might begin with $a$) let $\eta$ be the reduced word representing the inverse of $a^j \beta$ and add the word $\beta \eta$ to the set $X$.

**Lemma 5.12.** Every word in the set $X$ is reduced.

*Proof.* Let $a^j \beta \in \Delta$ and let $\eta$ be a reduced word representing the inverse of $a^j \beta$. We claim that $\beta \eta$ is a reduced word as a consequence of assumption (C1). Indeed, suppose for a contradiction that $\beta \eta$ is not reduced. It follows from Lemma 5.7 that $\eta \in \Delta^*$. Then there is a rewrite rule from (8) which can be applied to the word $\beta \eta$. Let $\lambda$ be the left hand side of such a rule noting that $\lambda \in \Delta^+$. Since $\beta$ and $\eta$ are both reduced words we can write $\lambda = \beta_2 \eta_1$ where $\beta_2$ and $\eta_1$ are both non-empty, with $\beta = \beta_1 \beta_2$ and $\eta = \eta_1 \eta_2$. Let $\alpha_1 \in \Delta$ be the prefix of $\lambda$ which belongs to $\Delta$. Let $\alpha_2 \in \Delta$ be the prefix of $\eta$ which belongs to $\Delta$. Since $\alpha_1$ cannot be a subword of $\beta$ since by (C1) it is not a subword of $a^j \beta \in \Delta$ it follows that $\alpha_1 = \alpha_1^1 \alpha_1^2$ where $\alpha_1^2$ is a non-empty prefix of $\eta$. But since $\eta$ is invertible this would imply that $\alpha_1^2$ is invertible and thus $\alpha_1^1$ is invertible, contradicting the fact that $\alpha_1 \in \Delta$ is a minimal invertible piece. This is a contradiction, and we conclude that $\beta \eta$ is indeed a reduced word.

Thus $X$ is a finite set of reduced words, each of which is the right inverse of some $a^j$ with $1 \leq j \leq m$. Note also that $X$ is a finite subset of the free monoid $F$.

**Lemma 5.13.** Let $i \in \mathbb{N}$ and $w \in A^*$ be a reduced word such that $a^i w = 1$ in $M$. Then there is an integer $0 < j \leq i$, with $j \leq m$, and a non-empty prefix $w_1$ of $w$ such that $w_1 \in X$ and $a^j w_1 = 1$ in $M$. Moreover, with the same value of $j$, there is a decomposition

$$a^j w = a^k a^j w' w''$$

where $k + j = i$, $w = w' w''$ and $a^j w' \in \Delta$. In particular, if no word in $\Delta$ begins with $a^2$ then $w$ can be written as $w = \prod_{i=1}^{l} w_i$ such that $aw_i = 1$ for all $1 \leq l \leq i$.

*Proof.* Let $i \in \mathbb{N}$ and $w \in A^*$ be a reduced word such that $a^i w = 1$ in $M$. Since $a^i w$ is not reduced it follows that the left hand side $\lambda$ of one of the relations from (8) arises as a subword of $a^i w$. In particular $\lambda$ is a non-empty word with $\lambda \in \Delta^*$. Since $a$ is not invertible, no word from $\Delta$ is a subword of $a^i$, and since $w$ is reduced, $\lambda$ is not a subword of $w$. It follows that there is a prefix $\lambda'$ of $\lambda$ such that, $\lambda' = a^j w' \in \Delta$ with $j > 0$ and where $w'$ is a non-empty prefix of $w$. Thus we have the decomposition

$$a^j w = a^k a^j w' w''$$

where $k + j = i$, $w = w' w''$ and $a^j w' \in \Delta$.

If $k = 0$ then $i = j$ and $a^j w = a^i w = 1$. So we can write $a^j w = (a^j w')(w'')$ and since $(a^j w')(w'') = 1$ it follows that in $M$ we have $w = \beta \eta$ where $\beta = w'$, $\eta = w''$, where $\eta$ is equal to the inverse of $a^j \beta$ in $M$ (note $a^j \beta$ is invertible because it belongs to $\Delta$). Thus in this case the reduced word $w$ belongs to the set $X$, as required.

Now suppose that $k > 0$. Consider the longest invertible prefix of the word $a^j w$. It is certainly non-empty since $a^j w'$ is invertible. Set $v = \text{red}(a^j w)$. Then we have $a^k v = 1$ with $k > 0$ and $v$ a reduced word. It follows from Lemma 5.3 that $v$ cannot begin with a word
from $\Delta$. Hence $v$ has no invertible prefix. Now by the last part of Lemma 5.11 since $v = a^j w$ in $M$, it follows that the longest invertible prefix $p$ of $a^j w$ is equal to 1 in $M$. So now we can write
\[ a^i w \equiv a^k a^j w_1 w_2 \]
where $k + j = i$, $w \equiv w_1 w_2$ and $p \equiv a^j w_1 = 1$ in $M$, and $a^j w_1$ has prefix $a^j w' \in \Delta$. It then follows that in $M$ we have $w_1 = \beta \eta$ where $\beta \equiv w'$ and $\eta$ is equal to the inverse of $a^j w'$ in $M$. Also, $w_1$ is a reduced word because $w$ is reduced. It follows that $w_1 \in X$, as required. This completes the proof of the lemma.

Lemma 5.14. $X$ is a finite generating set for the monoid $F$.

Proof. Let $i \in \mathbb{N}$ and $w \in A^*$ be a reduced word such that $a^i w = 1$ in $M$. It follows from Lemma 5.13 that there is an integer $0 < j \leq m$, and a non-empty prefix $w_1$ of $w$ such that $w_1 \in X$ and $a^j w_1 = 1$ in $M$. The lemma now follows by induction.

Let $B$ be the unique subset of $F$ that generates $F$ and is minimal with respect to set-theoretic inclusion, that is $B$ is equal to the set
\[ (F \setminus \{1\}) \setminus (F \setminus \{1\})^2. \]

Since $X \subseteq F$ is a finite generating set for $F$ it follows that $B \subseteq X$.

Lemma 5.15. The basis $B$ has size at least two. Thus $F$ is a free monoid of rank at least two.

Proof. By assumption (C2) there are distinct words $\gamma, \delta \in \Delta$ with common initial letter $a \in A$. Write $\gamma = a\gamma'$ and $\delta = a\delta'$. Note that either $\gamma'$ or $\delta'$ can begin with the letter $a$. By Lemma 5.10 the words $\gamma$ and $\delta$ represent different elements of the monoid $M$. This in turn implies that $[\gamma'] \neq [\delta']$. Let $(a\gamma')^{-1}$ be a reduced word representing the inverse of $a\gamma'$ in $M$, and let $(a\delta')^{-1}$ be a reduced word representing the inverse of $a\delta'$ in $M$. In particular $(a\gamma')^{-1}, (a\delta')^{-1} \in \Delta^*$. Then by definition we have $\gamma_2 \equiv \gamma'(a\gamma')^{-1} \in X$ and $\delta_2 \equiv \delta'(a\delta')^{-1} \in X$, and both of these words are reduced words. Suppose, seeking a contradiction, that $F$ is a free monoid of rank 1. It follows that there is a word $\nu \in A^+$ such that each of $\gamma_2$ and $\delta_2$ is, in $A^+$, equal to some power of the word $\nu$. But this would imply that $\gamma'$ is a prefix of $\delta'$, or vice versa. Suppose without loss of generality $\gamma'$ is a proper prefix of $\delta'$. Then $a\gamma'$ is a proper prefix of $a\delta'$. But this contradicts condition (C1) since both of these words belong to $\Delta$. This completes the proof of the lemma.

Thus we have identified a free submonoid of $M$ of rank at least two.

Lemma 5.16. Let $w \in A^*$ be a word. If $a^i w = 1$ and $a^j w = 1$ then $i = j$.

Proof. Seeking a contradiction suppose that $a^i w = a^j w = 1$ with $j < i$. Then $a^{i-j} = a^{i-j} a^j w = a^i w = 1$. But this contradicts the fact that $a$ is not invertible.

Define a mapping $\omega : F \rightarrow \mathbb{Z}^{\geq 1}$ where $w \mapsto i$ if and only if $a^i w = 1$. This is a well-defined mapping by the previous lemma. Also, it is easy to see that $\omega$ is a homomorphism to $(\mathbb{Z}, +)$. The mapping $\omega$ assigns a weight to every element of the free monoid $F$.

The following result is now an immediate consequence of the previous results proved in this section.
Lemma 5.17. Let \( w \in A^* \) be a non-empty reduced word, and suppose that \( a^iw = 1 \) with \( i \geq 1 \). Then the word \( w \) can be written uniquely as

\[
w = w_1w_2 \ldots w_k
\]

where \( w_j \in B \) for all \( 1 \leq j \leq k \), and

\[
\omega(w_1) + \omega(w_2) + \ldots + \omega(w_k) = i.
\]

In the special case that \( \Delta \) contains no word beginning with \( a^2 \) then \( \omega(w_j) = 1 \) for all \( 1 \leq j \leq k \), i.e. the statement above holds with \( k = 1 \).

Note that in particular condition (C1) is satisfied if all the pieces have the same length. We note that Adjan [2] gives an algorithm for computing the set \( \Delta \) by analysing overlaps of the relator with itself.

The following lemma will allow us to express membership in \( \{a\}^* \) in terms of equations.

Lemma 5.18. Let \( u \in A^* \) be reduced. Then \( u \in \{a\}^* \) if and only if \([ua] = [au] \) in \( M \).

Proof. Clearly if \( u \in \{a\}^* \) then \([ua] = [au] \) in \( M \).

For the converse, suppose that \( u \) is right invertible and \([ua] = [au] \) in \( M \). Since \( a \) is right invertible and \( a \) is not invertible, it follows that for all \( \delta \in \Delta \) the last letter of \( \delta \) is not equal to \( a \). (Note this is true for all \( \delta \in \Delta \) including those \( \delta \) in \( \Delta \) where \( \delta \) does not begin with the letter \( a \).)

Seeking a contradiction, suppose that \( u \notin \{a\}^* \) and write \( u = u_1a^y \) where \( u_1 \in A^+ \) and the last letter of \( u_1 \) is not equal to \( a \), and \( y \geq 0 \). Consider \( \text{red}(ua) = \text{red}(u_1a^{y+1}) \). Since for every rewrite rule \( \alpha = \beta \) from \( S \) neither \( \alpha \) nor \( \beta \) ends in the letter \( a \), it follows that \( \text{red}(ua) = w_1a^{y+1} \) where \( w_1 \) does not end in the letter \( a \).

In contrast, consider \( \text{red}(au) = \text{red}(au_1a^y) \). Reasoning in the same way as in the previous paragraph \( \text{red}(au) = w_2a^y \) where \( w_2 \) does not end in the letter \( a \) (note it may start with the letter \( a \)). In particular this implies that \( \text{red}(ua) \neq \text{red}(au) \) which implies \([ua] \neq [au] \). This contradicts our original assumption, and completes the proof of the lemma.

The main result we shall prove in this section is the following.

Theorem 5.19. Let \( M = \langle A \mid r = 1 \rangle \) and let \( \Delta \subseteq A^* \) be the set of minimal invertible pieces of \( r \). Suppose that:

(C1) no word from \( \Delta \) is a proper subword of any other word from \( \Delta \), and

(C2) there exist distinct words \( \gamma, \delta \in \Delta \) with a common first letter.

Then there exists a free monoid \( F \) of finite rank \( n \geq 2 \) and a tuple of weights \( \vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \) such that the free monoid with weighted length relation \( (F, \cdot, 1, L_\vec{\lambda}) \) is interpretable in \( M \) by systems of equations and one coefficient.

Proof. Let \( F \) be the free submonoid of \( M \) defined in our previous arguments (see Lemma 5.10), and let \( \vec{\omega} \) be the tuple \( (\omega(w_1), \ldots, \omega(w_n)) \) where \( w_1, \ldots, w_n \) freely generate \( F \) and \( \omega : F \to \mathbb{Z}_{\geq 1} \) is the homomorphism defined after Lemma 5.16 so that \( a^{\omega(w_i)}w_i = 1 \) for all \( i \) (by Lemma 5.17).
We claim that $a$ generates an infinite submonoid of $M$. Indeed, if it did not, we would have $a^k = a^{k+\ell}$ for some $k, \ell \geq 0$. Since $a$ is right invertible (due to condition (C2)), this implies that $a^\ell = 1$, from where it follows that $a$ is invertible, a contradiction. This proves the claim.

By Lemma 5.18 the submonoid $\langle a \rangle$ is interpretable in $M$ by the equation $ax = xa$ (Lemma 5.18). Since $F = \{x \in M \mid a^t x = 1 \text{ for some } t \in \mathbb{N}\}$, it follows that $F$ is $e$-interpretable in $M$ by the system of two equations $ay = ya, yx = 1$. Hence the theorem follows from Lemma 4.2.

The following two results follow immediately from the above Theorem 5.19 and from Proposition 2.5 regarding reducibility of decision problems.

**Corollary 5.20.** Let $M$ be a monoid satisfying the hypothesis of Theorem 5.19. Then there exists a free monoid with a weighted length relation $(F', \cdot, 1, L_{\vec{\omega}})$ such that the Diophantine problem in $(F', \cdot, 1, L_{\vec{\omega}})$ is reducible to the Diophantine problem in $M$. In particular, if the latter is decidable, then systems of word equations with $\vec{\omega}$-weighted length constraints are decidable as well.

**Theorem 5.21.** Any one-relator monoid of the form $\langle A \mid w = 1 \rangle$ satisfying conditions (C1) and (C2) has undecidable positive AE-theory with coefficients. In particular, its first-order theory with coefficients is undecidable.

**Proof.** It is an immediate consequence of Theorem 5.19 of the fact that the AE-theory with coefficients of free monoids is undecidable [34, 66] and of reducibility of theories (Proposition 2.5).

If we add to Theorem 5.19 the extra condition that no word in $\Delta$ starts with $a^2$, then the same result holds with all weights being 1, i.e. $\vec{\lambda} = (1, \ldots, 1)$. In this case $L_{\vec{\lambda}}$ is the standard length relation $L$.

**Theorem 5.22.** Let $M = \langle A \mid r = 1 \rangle$ and let $\Delta \subseteq A^*$ be the set of minimal invertible pieces of $r$. Suppose that:

(C1) no word from $\Delta$ is a proper subword of any other word from $\Delta$,

(C2) there exist distinct words $\gamma, \delta \in \Delta$ with a common first letter, say $a$,

(C3) no word in $\Delta$ starts with $a^2$.

Then there exists a free monoid $F$ of finite rank $n \geq 2$ such that the free monoid with length relation $(F', \cdot, 1, =, L)$ is interpretable in $M$ by systems of equations.

**Proof.** The proof works in the same way as in Theorem 5.19 with the addition that the last part of Lemma 5.17 now ensures that $\omega(w_i) = 1$ for all $i = 1, \ldots, n$. Then $\vec{\omega} = (1, \ldots, 1)$ and $(F', 1, L_{\vec{\omega}}) = (F, 1, =, L)$. Hence $(F, 1, =, L)$ is interpretable in $M$ by systems of equations and one coefficient.

We obtain an analogue of Corollary 5.20.
Corollary 5.23. Let $M$ be a monoid satisfying the hypothesis of Theorem 5.22. Then there exists a free monoid with (non-weighted) length relation $(F, \cdot, 1, \cdot, L)$ such that the Diophantine problem in $(F, \cdot, 1, \cdot, L)$ is reducible to the Diophantine problem in $M$. In particular, if the latter is decidable, then systems of word equations with length constraints are decidable as well.

The following result illustrates how the results in the section can also be used to show that many one-relator monoids naturally embed the monoids from Example 4.3.

Theorem 5.24. Let $M = \langle A \mid r = 1 \rangle$ and let $\Delta \subseteq A^*$ be the set of minimal invertible pieces of $r$. Suppose conditions (C1), (C2), (C3) are satisfied, i.e.:

(C1) no word from $\Delta$ is a proper subword of any other word from $\Delta$,
(C2) there exist distinct words $\gamma, \delta \in \Delta$ with a common first letter, say $a$,
(C3) no word in $\Delta$ starts with $a^2$.

Let

$$\Sigma_a = \{ w \in A^* : w \text{ is reduced and } [aw] = 1 \}.$$

Then

(i) $\Sigma_a$ is a finite set with $|\Sigma_a| \geq 2$;

(ii) the submonoid of $M$ generated by $\Sigma_a$ is free with basis $\Sigma_a$.

Let $\Sigma_a = \{ \gamma_1, \ldots, \gamma_q \}$. Then the submonoid of $M$ generated by $\{ [a] \} \cup [\Sigma_a]$ is naturally isomorphic to the monoid defined by the presentation

$$\langle a, d_1, d_2, \ldots, d_q \mid ad_1 = 1, \ldots, ad_q = 1 \rangle.$$

Proof. We claim that $\Sigma_a$ is equal to the set $X = \{ \beta(a^j\beta)^{-1} \mid a^j\beta \in \Delta \}$ defined above; see Lemma 5.14. It is immediate from the definition of $X$ that $X \subseteq \Sigma_a$. For the converse, let $\gamma \in \Sigma_a$. This means that $\gamma$ is a reduced word and $[a\gamma] = 1$ in $M$. By Lemma 5.13, we can write $a\gamma \equiv a\gamma'\gamma''$ with $\gamma' \in X$ and $a\gamma' = 1$ in $M$. Then $\gamma'' = (a\gamma')\gamma'' = a\gamma = 1$ in $M$. Since $\gamma$ is a reduced word it follows that $\gamma'' \equiv \epsilon$ and thus $\gamma \equiv \gamma' \in X$. This completes the proof that $X = \Sigma_a$.

Since $X = \Sigma_a$, part (i) now follows from Lemmas 5.14 and 5.15.

To prove part (ii) it will suffice to prove that $\Sigma_a = \mathcal{B}$, where $\mathcal{B}$ is the unique basis of the free monoid generated by $\Sigma_a = X$. To prove this it will suffice to prove that no $\gamma \in \Sigma_a$ can be written as a product of other $\gamma$ from $\Sigma_a$. Suppose that

$$\gamma = \gamma_1 \gamma_2 \cdots \gamma_m$$

where $\gamma_i \in \Sigma_a$ for all $1 \leq i \leq m$. Then

$$1 = a\gamma = a\gamma_1 \gamma_2 \cdots \gamma_m = \gamma_2 \cdots \gamma_m$$

By Lemma 5.8, $\gamma_2 \cdots \gamma_m$ is a reduced word and hence it follows that it must equal the empty word. Hence $m = 1$ and $\gamma \equiv \gamma_1$ since they are both reduced words and they are equal in $M$. This completes the proof that $\Sigma_a = \mathcal{B}$, and hence completes the proof of (ii).
For the last part, let $w \in \{a\} \cup \Sigma_\alpha^* = \{a, \gamma_1, \ldots, \gamma_q\}^*$. Since $a\gamma_i = 1$ for all $i$, this word is equal in $M$ to a word $w'$ where $w'$ has the form $w' \equiv w_1a^j$, where $w_1 \in \{\gamma_1, \ldots, \gamma_q\}^*$. We claim that in fact $\text{red}(w) \equiv w_1a^j$. Indeed, since none of the words appearing in the rewrite rules in [8] ends in $a$ (because otherwise together with condition (C2) this would imply that $a$ is invertible) to show that $w_1a^j$ is reduced it suffices to prove that $w_1$ is reduced, and this was proved in Lemma 5.8. Therefore, each element of the submonoid of $M$ generated by $\{[a]\} \cup [\Sigma_\alpha]$ may be uniquely written in the form $\alpha a^j$ for some $j \geq 0$ and some word $\alpha \in \{\gamma_1, \ldots, \gamma_q\}^*$. Now consider the monoid $N$ defined by the presentation

$$\langle a, d_1, d_2, \ldots, d_q \mid ad_1 = 1, \ldots, ad_q = 1 \rangle$$

This is a finite complete presentation, and the reduced words are precisely those of the form $\beta a^j$ where $j \geq 0$ and $\beta \in \{d_1, \ldots, d_q\}^*$.

Let $\phi : \{a, d_1, \ldots, d_q\}^* \to A^*$ be the homomorphism induced by the map $a \mapsto a$, and $d_i \mapsto \gamma_i$ for $1 \leq i \leq q$. Since each relation in the presentation for $\langle a, d_1, \ldots, d_q \rangle$ is preserved by this homomorphism it follows that $\phi$ induces a homomorphism $\phi : \langle a, d_1, \ldots, d_q \rangle \to M$. Moreover, this homomorphism maps $\langle a, d_1, \ldots, d_q \rangle$ bijectively to the submonoid of $M$ generated by $\{[a]\} \cup [\Sigma_\alpha]$ since it clearly defines a bijection between the normal forms described above. This completes the proof of the theorem. \qed

**Remark 5.25.** We follow the notation of the previous Theorem 5.24. In the proof of Theorem 5.19 we showed that both $\langle a \rangle$ and $[\Sigma_\alpha]$ are $e$-interpretable in $M$. It is natural to ask whether the submonoid $\langle a, \Sigma_\alpha \rangle$, which by Theorem 5.24 is isomorphic to the monoid from Example 4.3, is itself $e$-interpretable in $M$. The answer to this question is not clear and we leave it open.

### 6 Applications, examples and open problems

In this section we list some examples, and classes of examples, of monoids to which the main results of this paper apply. We shall also collect together a selection of open problems, and possible future research directions, which naturally arise from our results. As part of this we will identify the simplest examples of one-relator monoids for which we do not yet know whether or not the Diophantine problem is decidable. In general, we do not know if there is an example of a one-relator monoid of the form $\langle A \mid r = 1 \rangle$ with undecidable Diophantine problem.

Let us begin by recording some examples of one-relator monoids of the form $\langle A \mid r = 1 \rangle$ where we have shown that the Diophantine problem is decidable. Consider, in particular the case of 2-generated one-relator monoids $\langle a, b \mid r = 1 \rangle$. Let $M$ denote the monoid defined by this presentation. Very often questions about one-relator monoids can be reduced to just considering the 2-generator case e.g. this is the case for the word problem.

By Makanin [64], the Diophantine problem is decidable for the free monoid $\langle a, b \rangle$, while in [27, Example 21] it is proved that it is decidable for the bicyclic monoid $\langle a, b \mid ab = 1 \rangle$. Now consider the general case $\langle a, b \mid r = 1 \rangle$ and let $r = r_1r_2 \ldots r_k$ be the decomposition of $r$ into minimal invertible pieces as described in Section 5. If $r \in \{a\}^*$ or $r \in \{b\}^*$ then the monoid is a free product of a free monoid of rank one and a finite cyclic group, and thus the Diophantine problem is decidable by [27]. Now suppose that both the letters $a$ and $b$ appear
in the defining relator \( r \). There are then two cases to consider. If there are minimal invertible pieces \( r_i \) and \( r_j \) such that the first letter of \( r_i \) equals the last letter of \( r_j \), then it follows that both \( a \) and \( b \) both represent invertible elements of \( M \) and hence \( M \) is a group. In this case, \( M \) is the group defined by the same one-relator group presentation, and hence \( M \) is a so-called positive one-relator group. Such groups have been studied e.g. by Baumslag [8] and Wise [90].

This motivates the question of whether the Diophantine problem is decidable for positive one-relator groups. Up to symmetry the case that remains is when all the invertible pieces \( r_i \) begin with the letter \( a \) and end with the letter \( b \). This case then divides into two subcases, either (i) all of the pieces \( r_i \) are equal to each other as words, or (ii) there is some pair of minimal invertible pieces \( r_i \) and \( r_j \) with \( r_i \neq r_j \). Note that subcase (i) includes in particular the case where there is a single invertible piece. This is precisely the case where the relator \( r \) is self-overlap free meaning that no proper non-empty prefix is equal to a proper non-empty suffix of \( r \). This in turn is equivalent to saying that the group of units of the monoid is the trivial group. Also note that many of the examples in (ii) will satisfy the conditions (C1) and (C2) (and (C3)) from Section 5, and thus the main theorems of that section, Theorem 5.19 and Theorem 5.22, will apply to them. Some examples of these are listed in the introduction after Theorem A.

A similar division into cases can also be done for one-relator monoids \( \langle A \mid r = 1 \rangle \) with more than two generators. For instance the monoid \( \langle a, b, c, d \mid aba = 1 \rangle \) has decidable Diophantine problem by Theorem 5.1 above, since all the letters in the relator are invertible, and the group of units is the infinite cyclic group which has decidable Diophantine problem. Similarly the monoid \( \langle a, b, c, d, e, f \mid abcddcbbaa = 1 \rangle \) has decidable Diophantine problem, again applying Theorem 5.1, where this time the group of units is isomorphic to the group defined by the group presentation

\[
Gp\langle a, b, c, d, e, f \mid cddc = b^{-1}a^{-1}a^{-1}b^{-1}b^{-1} \rangle.
\]

The words \( cddc \) and \( b^{-1}a^{-1}a^{-1}b^{-1}b^{-1} \) are non-primitive since the words \( cddc \) and \( bbbaab \) are not Christoffel words (see e.g. [22]), and neither are any of the conjugates of these words, since the first word have the same number of cs and ds, and similarly for the second word. It is known, see [10, 47, 49], that a cyclically pinched one-relator group defined by a presentation \( Gp\langle A \mid u = v \rangle \), where \( u \) and \( v \) are non-primitive words written over disjoint sets of letters, and it is not the case that both \( u \) and \( v \) are proper powers, is hyperbolic. Hence the group of units of \( \langle a, b, c, d, e, f \mid abedcba = 1 \rangle \) is a hyperbolic group and thus by Theorem 5.1 above this monoid has decidable Diophantine problem. Many other examples similar to this can be written down. This gives a reasonably rich source of examples of one-relator monoids \( \langle A \mid r = 1 \rangle \) which have solvable Diophantine problem as a consequence of the fact that their groups of units are hyperbolic. We do not know in general whether having a hyperbolic group of units is enough to imply that a one-relator monoid of the form \( \langle A \mid r = 1 \rangle \) has solvable Diophantine problem. As explained in the introduction, this was one of the original motivating questions for the work done in this paper. By Proposition 3.2 and Theorem 5.22 a positive answer to this questions implies decidability of word equations with length constraints.

In light of this discussion, it is sensible to identify the simplest examples of one-relator monoids of the form \( \langle A \mid r = 1 \rangle \) for which we neither know that the Diophantine problem is decidable, but we also do not know of a reduction theorem (like the theorems from Section 5)
above) of a known difficult open problem. Thus we ask whether either of the monoids 
\langle b, c \mid b^2c = 1 \rangle or \langle a, b, c \mid abc = 1 \rangle has decidable Diophantine problem? Initial investigations 
indicate that this might relate to solving word equations with a variation on the notion of 
twisting, in the sense of [30]. More generally we ask the following

**Question 6.1.** If the word \( w \in A^* \) has no self overlaps, i.e. there is no non-empty word 
which is both a proper prefix of \( w \) and a proper suffix of \( w \), then is the Diophantine problem 
for the one-relator monoid \( \langle A \mid w = 1 \rangle \) decidable?

Note that the condition that \( w \) has no self overlaps is equivalent to saying the group of 
units of this monoid is trivial. Also note that the bicyclic monoid is a basic example of a 
one-relator monoid satisfying this property.

The corresponding class of monoids with torsion are also not covered by any of the 
theorems in this paper. Thus we ask whether \( \langle b, c \mid bcbc = 1 \rangle \) has decidable Diophantine 
problem? More generally, of course, we can ask whether the Diophantine problem is decidable 
for monoids \( \langle A \mid w^n = 1 \rangle \) where \( w \) has no self overlaps.

Finally, we restate some natural questions which have arisen in this work. As already 
mentioned above, if any of these problems has a positive answer, then as a corollary this 
would give a positive solution to the open problem of solving word equations with length 
constraints.

**Question 6.2.** Is the Diophantine problem decidable for one-relator monoids of the form 
\( \langle A \mid w^n = 1 \rangle \) where \( n > 1 \)?

**Question 6.3.** Let \( M \) be the monoid defined by \( \langle A \mid w = 1 \rangle \) and let \( G \) be the group of units 
of \( M \). If the Diophantine problem is decidable in \( G \), then does it follow that it is decidable in 
\( M \)?

**Question 6.4.** Let \( M \) be the monoid defined by \( \langle A \mid w_1 = 1, \ldots, w_n = 1 \rangle \) and let \( G \) be the 
group of units of \( M \). If the Diophantine problem is decidable in \( G \), then does it follow that it 
is decidable in \( M \)?

**Question 6.5.** Is the Diophantine problem decidable for finitely presented word-hyperbolic 
monoids?

It follows from the results in the present paper that the positive AE-theory is in general 
undecidable in the classes of monoids from Questions 6.1 through Question 6.4 (due to 
Theorem 5.21, Proposition 3.2, and Remark 4.3). Note this does not include Question 6.5.

We finish the paper by stating a somewhat different kind of problem:

**Question 6.6.** Are special one-relator monoids with torsion \( \langle A \mid w^n = 1 \rangle \) \( (n > 1) \) word-
hyperbolic?

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