A note on commutators in compact semisimple Lie algebras

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Dedicated to Jacques Tits

Abstract

Given any two elements \( A, B \) in a compact semisimple Lie algebra, we show that there exist elements \( X, Y, Z \) such that

\[
A = [X, Y] \quad \text{and} \quad B = [X, Z].
\]

The proof uses Cartan subalgebras and their root systems. We also review some related problems about Cartan subalgebras in compact semisimple Lie algebras.

Gotô’s Commutator Theorem \([4][9, 6.56]\) states that in a compact connected semisimple Lie group \( G \), every element is a commutator. There is an infinitesimal version of Gotô’s Theorem which says that every element in a compact semisimple Lie algebra \( \mathfrak{g} \) is a commutator, cp. \([8\ Thm. A3.2]\). The proof given in loc.cit, which uses Kostant’s Convexity Theorem, is attributed to K.-H. Neeb. Other proofs were given later by D’Andrea–Maffei and Malkoun–Nahlus \([3, 16, 17]\). We prove the following somewhat stronger result by elementary means.

**Theorem 1.** Let \( \mathfrak{g} \) be a semisimple compact Lie algebra and let \( A, B \in \mathfrak{g} \). Then there is a regular element \( X \in \mathfrak{g} \) with

\[
A, B \in [X, \mathfrak{g}] = \text{ad}(X)(\mathfrak{g}).
\]

Our Lemma \([6]\) which is the main step of the proof, uses a variant of Jacobi’s method, cp. \([14, 16\ App. B]\) and \([23]\). In the course of the proof we show in Corollary \([7]\) that every linear subspace \( W \subseteq \mathfrak{g} \) of codimension at most 2 contains a Cartan subalgebra.

**Definition 2.** A finite dimensional real semisimple Lie algebra \( \mathfrak{g} \) is called *compact* if its Killing form \( \langle -,- \rangle \) is negative definite. In this case its adjoint group

\[
G = \langle \exp(\text{ad}(X)) \mid X \in \mathfrak{g} \rangle
\]

\(^1\text{Note that }[16]\text{ and }[17]\text{ differ considerably.}\)
is compact and
\[ |X| = \sqrt{-\langle X, X \rangle} \]
is a $G$-invariant euclidean norm on $\mathfrak{g}$. In what follows, orthogonality in $\mathfrak{g}$ will always refer to the Killing form. The centralizer of $A \in \mathfrak{g}$ is the Lie subalgebra
\[ \text{Cen}_\mathfrak{g}(A) = \{ X \in \mathfrak{g} \mid [X, A] = 0 \}. \]

**Lemma 3.** Let $\mathfrak{g}$ be a compact semisimple Lie algebra and let $A \in \mathfrak{g}$. Then $\mathfrak{g}$ decomposes (as a $\text{Cen}_\mathfrak{g}(A)$-module) orthogonally as
\[ \mathfrak{g} = \text{Cen}_\mathfrak{g}(A) \oplus [A, \mathfrak{g}]. \]

**Proof.** Let $X, Y \in \mathfrak{g}$. If $X$ centralizes $A$, then
\[ \langle X, [A, Y] \rangle = \langle [X, A], Y \rangle = 0, \]
whence $X \in [A, \mathfrak{g}]^\perp$. Conversely, if $X \in [A, \mathfrak{g}]^\perp$, then
\[ 0 = \langle X, [A, Y] \rangle = \langle [X, A], Y \rangle \]
holds for all $Y$ and thus $[X, A] = 0$. The Jacobi identity shows that $[X, [A, \mathfrak{g}]] \subseteq [A, \mathfrak{g}]$ for $X \in \text{Cen}_\mathfrak{g}(A)$. \hfill $\square$

We recall some facts about the structure of compact semisimple Lie algebras, which can be found in [1, 2, 6, 7, 9].

**Facts 4.** Let $\mathfrak{g}$ be a compact semisimple Lie algebra. We call a maximal abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ a Cartan subalgebra or a CSA for short. All CSAs in $\mathfrak{g}$ are conjugate under the action of $G$, cp. [6, V.6.4] or [9, 6.27]. The dimension of $\mathfrak{h}$ is called the rank of $\mathfrak{g}$. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a CSA. Then
\[ T = \{ \exp(\text{ad}(H)) \mid H \in \mathfrak{h} \} \]
is a maximal torus in $G$. As a $T$-module, the Lie algebra $\mathfrak{g}$ decomposes as an orthogonal direct sum of irreducible $T$-modules
\[ \mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi^+} L_\alpha, \]
with $\Phi^+$ the positive real roots. The positive real roots $\alpha \in \Phi^+$ are certain nonzero linear forms $\alpha : \mathfrak{h} \rightarrow \mathbb{R}$. Each $T$-module $L_\alpha$ is 2-dimensional and carries a complex structure $i$ such that $L_\alpha \cong \mathbb{C}$ and
\[ \exp(\text{ad}(H))(X) = \exp(2\pi i \alpha(H))X \]
holds for all $H \in \mathfrak{h}$, $\alpha \in \Phi^+$ and $X \in L_\alpha$. Hence $H \in \mathfrak{h}$ acts on $L_\alpha$ as
\[ \text{ad}(H)(X) = [H, X] = 2\pi i \alpha(H)X. \]
The positive real roots separate the points in \( h \), i.e. \( \bigcap \{ \ker(\alpha) \mid \alpha \in \Phi^+ \} = \{ 0 \} \). The centralizer of an element \( H \in h \) is therefore

\[
\mathrm{Cen}_g(H) = h \oplus \sum_{\alpha \in \Phi^+} L_\alpha.
\]

Hence \( \mathrm{Cen}_g(H) = h \) holds if and only if \( \alpha(H) \neq 0 \) for all positive real roots \( \alpha \). Such elements \( H \) are called regular.

**Lemma 5.** Let \( g \) be a compact semisimple Lie algebra, with a CSA \( h \) and the corresponding decomposition

\[
g = h \oplus \sum_{\alpha \in \Phi^+} L_\alpha
\]

as above, and let \( \gamma \in \Phi^+ \) be a positive real root. Let \( H_\gamma \in h \) be a nonzero vector orthogonal to \( \ker(\gamma) \). Then

\[
m_\gamma = \mathbb{R}H_\gamma \oplus L_\gamma \cong \mathfrak{so}(3)
\]

is the Lie algebra generated by \( L_\gamma \).

**Proof.** We let \( m_\gamma \) denote the Lie algebra generated by \( L_\gamma \). The centralizer of \( \ker(\gamma) \) is \( h \oplus L_\gamma \), whence \( m_\gamma \subseteq h \oplus L_\gamma \). Let \( X \in L_\gamma \) be an element of norm \( |X| = 1 \). Then \( X, iX \) is an orthonormal basis for \( L_\gamma \), and we put \( Y = [X, iX] \). Then

\[
\langle X, Y \rangle = \langle [X, X], iX \rangle = 0 = \langle X, [iX, iX] \rangle = \langle Y, iX \rangle
\]

and thus \( Y \in h \). For \( H \in h \) we have

\[
\langle H, Y \rangle = \langle [H, X], iX \rangle = 2\pi\gamma(H)\langle iX, iX \rangle = -2\pi\gamma(H),
\]

hence \( Y \) is nonzero and orthogonal to \( \ker(\gamma) \). Moreover, \( \langle Y, Y \rangle = -2\pi\gamma(Y) < 0 \). If we put \( g = \frac{1}{\sqrt{2\pi\gamma(Y)}} \) and \( U = gX, V = g^2X, W = g^3Y \), then

\[
[U, V] = W, \ [V, W] = U, \ [W, U] = V
\]

and thus \( m_\gamma \cong \mathfrak{so}(3) \). \( \Box \)

**Key Lemma 6.** Let \( g \) be a compact semisimple Lie algebra and let \( A, B \in g \). Suppose that \( A \) is orthogonal to some CSA. Then there exists a CSA \( h \subseteq g \) which is orthogonal both to \( A \) and to \( B \).

**Proof.** Among all CSAs \( h \) orthogonal to \( A \), we choose one for which the orthogonal projection \( B_0 \) of \( B \) to \( h \) has minimal length \( r = |B_0| \). We claim that \( r = 0 \). Assume towards a contradiction that this is false. We decompose \( g \) orthogonally as

\[
g = h \oplus \sum_{\alpha \in \Phi^+} L_\alpha.
\]
Accordingly we have $A = \sum_\alpha A_\alpha$ and $B = B_0 + \sum_\alpha B_\alpha$, with $A_\alpha, B_\alpha \in L_\alpha$. By assumption, $B_0 \neq 0$. Hence there is a positive real root $\gamma \in \Phi^+$ with $\gamma(B_0) \neq 0$. We decompose $B_0$ further in $\mathfrak{h}$ as an orthogonal sum $B_0 = B_{00} + H_\gamma$, where $\gamma(B_{00}) = 0$ and $H_\gamma \neq 0$. Then

$$\mathfrak{h} = \mathbb{R}H_\gamma \oplus \ker(\gamma)$$

and

$$\mathfrak{m}_\gamma = \mathbb{R}H_\gamma \oplus L_\gamma \cong \mathfrak{so}(3)$$

by Lemma 5. In the 3-dimensional Lie algebra $\mathfrak{m}_\gamma \cong \mathfrak{so}(3)$ there is a 1-dimensional subspace $V \subseteq \mathfrak{m}_\gamma$ which is orthogonal to $H_\gamma$ and to $A_\gamma$. The adjoint representation of $SO(3)$ on its Lie algebra $\mathfrak{so}(3)$ is transitive on the 1-dimensional subspaces. Hence there is an element $g \in G$ of the form $g = \exp(\text{ad}(Z))$, for some $Z \in \mathfrak{m}_\gamma$, with $g(H_\gamma) \in V$. Moreover, $g$ fixes $\ker(\gamma)$ pointwise. The CSA $\mathfrak{h}' = g(\mathfrak{h}) = V \oplus \ker(\gamma)$ is then orthogonal to $A$. The projection of $B$ to $\mathfrak{h}'$ is $B_{00}$ and has therefore strictly smaller length than $B_0$. This is a contradiction. □

**Corollary 7.** Let $\mathfrak{g}$ be a compact semisimple Lie algebra and let $A, B \in \mathfrak{g}$. Then $A^\perp \cap B^\perp$ contains a CSA $\mathfrak{h}$.

**Proof.** We apply Lemma 6 to 0 and $A$ to obtain a CSA which is orthogonal to $A$. Another application of Lemma 6 to $A, B$ then yields a CSA $\mathfrak{h}$ which is orthogonal to both $A$ and $B$. □

**Proof of Theorem 1.** Let $\mathfrak{h}$ be a CSA which is orthogonal to $A$ and to $B$ and let $X \in \mathfrak{h}$ be a regular element. Then $\mathfrak{h} = \text{Cen}_\mathfrak{g}(X)$ and thus $A, B \in [X, \mathfrak{g}]$ by Lemma 3. □

**Some remarks and open problems.**

We close with some remarks and an open problem. Suppose that $\mathfrak{h}$ is a CSA in the compact Lie algebra $\mathfrak{g}$. If we pick nonzero elements $Z_\alpha \in L_\alpha$, for every positive root $\alpha$, and if we put $Z = \sum_{\alpha \in \Phi^+} Z_\alpha$, then $\mathfrak{h} \cap \text{Cen}_\mathfrak{g}(Z) = 0$. Since $\text{Cen}_\mathfrak{g}(Z)$ contains a CSA $\mathfrak{h}'$, this shows that there exists a CSA $\mathfrak{h}'$ which intersects $\mathfrak{h}$ trivially. However, one can do better. The following is shown in [17].

**Theorem 8** (Malkoun-Nahlus). Let $\mathfrak{h}$ be a CSA in a compact semisimple Lie algebra $\mathfrak{g}$. Then there exists a CSA $\mathfrak{h}' \subseteq \mathfrak{h}^\perp$.

We reproduce the beautiful proof from [17].

**Proof.** We may assume that $\mathfrak{g} \neq 0$. Let $w$ be a Coxeter element in the Weyl group $W = N/T$, where $T$ is the maximal torus corresponding to $\mathfrak{h}$, and $N \subseteq G$ is the normalizer of $T$. Then $W$ acts as a finite reflection group on $\mathfrak{h}$, and 1 is not an eigenvalue of $w$ in this action, cp. [10], 3.16. We choose $X \in \mathfrak{g}$ with $w = \exp(\text{ad}(X))T$ and we claim that every CSA $\mathfrak{h}'$ containing $X$ is orthogonal to $\mathfrak{h}$. The linear endomorphism $\exp(\text{ad}(X)) - \text{id}_\mathfrak{g}$ of $\mathfrak{g}$ maps $\mathfrak{h}$ onto $\mathfrak{h}$, and

$$\exp(\text{ad}(X)) - \text{id}_\mathfrak{g} = \sum_{k=1}^\infty \frac{1}{k!} \text{ad}(X)^k = \text{ad}(X) \sum_{k=1}^\infty \frac{1}{k!} \text{ad}(X)^{k-1}.$$

In particular, $\text{ad}(X)(\mathfrak{g}) \supseteq \mathfrak{h}$. Thus $\text{Cen}_\mathfrak{g}(X) \subseteq \mathfrak{h}^\perp$ by Lemma 3. □
Christoph Böhm has explained to me the following remarkable result.

**Theorem 9.** The orthogonal Lie algebras $\mathfrak{so}(m)$, for $m \geq 3$, can be decomposed as orthogonal direct sum of CSAs.

**Proof.** The rank of $\mathfrak{so}(m)$ is $r = \lfloor \frac{m}{2} \rfloor$, and the dimension of $\mathfrak{so}(m)$ is $n = \frac{m(m-1)}{2}$. We let $e_1, \ldots, e_m$ denote the standard basis of $\mathbb{R}^m$, and we put $X_{i,j} = e_i e_j^T - e_j e_i^T$. Then the $X_{i,j}$ with $i < j$ form an orthonormal basis of $\mathfrak{so}(m)$. Moreover, two distinct basis elements $X_{i,j}, X_{k,\ell}$ commute if and only if $\{i, j\} \cap \{k, \ell\} = \emptyset$. The standard CSA for $\mathfrak{so}(m)$ is spanned by $X_{1,2}, X_{3,4}, \ldots, X_{2r-1,2r}$. The claim follows if we can partition the set $\mathcal{T}_m$ of all two-element subsets of $\{1, \ldots, m\}$ into $\frac{n}{r}$ subsets consisting of $r$ pairwise disjoint 2-element subsets. The latter is possible by the scheduling algorithm for round robin tournaments.

An explicit construction of such a partition of $\mathcal{T}_m$ can be described as follows, cp. [22, Ex. 36.2]. For odd $m \geq 3$ put $M_k = \{\{i, j\} \mid i < j \text{ and } i + j \equiv 2k \pmod{m}\}$, for $k = 1, \ldots, m$. The $M_k$ partition $\mathcal{T}_m$ into $m$ subsets of cardinality $\frac{m-1}{2}$, each consisting of pairwise disjoint 2-element subsets. From this we obtain also such a partition of $\mathcal{T}_{m+1}$ by putting $M'_k = M_k \cup \{\{k, m+1\}\}$.

We cannot expect such a result for general compact semisimple Lie algebras. For example, the compact semisimple Lie algebra $\mathfrak{g} = \mathfrak{so}(5) \oplus \mathfrak{so}(3)$ has dimension 13, hence such a decomposition cannot exist. The following question is thus very natural.

**Problem 10.** Which compact semisimple Lie algebras $\mathfrak{g}$ can be decomposed as an orthogonal sum of CSAs?

The monograph [15] is devoted to the complex version of this problem.

For the Lie algebras $\mathfrak{su}(m)$, the problem can be rephrased as follows, using the Veronese embedding of $\mathbb{CP}^{m-1}$. To each unit vector $u \in \mathbb{C}^m$, we may assign the selfadjoint projector

$$P(u) = uu^*,$$

where $*$ denotes the conjugate-transpose, and its traceless part

$$P_0(u) = uu^* - \frac{1}{m} \text{id}_{\mathbb{C}^m}.$$ 

We note that $P(uz) = P(u)$ holds for all complex numbers $z$ with $|z| = 1$. Suppose that $u_1, \ldots, u_m$ is an orthonormal basis of $\mathbb{C}^m$. Then the projectors $P(u_1), \ldots, P(u_m)$ commute, and the matrices $iP_0(u_1), \ldots, iP_0(u_m)$ span a CSA $\mathfrak{h}$ in $\mathfrak{su}(m)$. Conversely, the CSA $\mathfrak{h}$ determines the set of subspaces $u_1 \mathbb{C}, \ldots, u_m \mathbb{C}$ uniquely, since these are the fixed points of the maximal torus $T \subseteq \text{PSU}(m)$ with Lie algebra $\mathfrak{h}$ in its action on complex projective space $\mathbb{CP}^{m-1}$. Hence $\mathfrak{h}$ determines the orthonormal basis $u_1, \ldots, u_m$ up to a permutation of vectors, and up to multiplication of the basis vectors by complex numbers of norm 1.
The Killing form for $\mathfrak{su}(m)$ is given by $\langle X, Y \rangle = 2m \text{tr}(XY)$. The CSAs $\mathfrak{h}$ and $\mathfrak{h}'$ provided by two orthonormal bases $u_1, \ldots, u_m, v_1, \ldots, v_m$ are thus orthogonal if and only if

$$|\langle u_k, v_\ell \rangle|^2 = \frac{1}{m}$$

holds for all $k, \ell$. In this case, the two bases are called mutually unbiased. Such bases were considered in quantum mechanics by J. Schwinger [18]. The construction of mutually unbiased bases has interesting connections to finite geometry, cp. [11], [12], [19], [20]. It is an open problem in which dimensions $m$ there exist $m + 1$ pairwise mutually unbiased orthonormal bases. They are known to exist if $m$ is a prime power [24], [13]. As we have seen, this question is equivalent to the existence of an orthogonal decomposition of $\mathfrak{su}(m)$ into CSAs. There is a related problem about MASAs in operator theory, cp. [5]. It is presently an open problem if $\mathfrak{su}(6)$ admits an orthogonal decomposition into 7 CSAs.

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