STAR MEAN CURVATURE FLOW ON 3 MANIFOLDS AND ITS BÄCKLUND TRANSFORMATIONS

HSIAO-FAN LIU

Department of Mathematics, National Tsing Hua University, Taiwan
hfliu@math.nthu.edu.tw

ABSTRACT. The Hodge star mean curvature flow on a 3-dimensional Riemannian or pseudo-Riemannian manifold is a natural nonlinear dispersive curve flow in geometric analysis. A curve flow is integrable if the local differential invariants of a solution to the curve flow evolve according to a soliton equation. In this paper, we show that this flow on $S^3$ and $H^3$ are integrable, and describe algebraically explicit solutions to such curve flows. The Cauchy problem of the curve flows on $S^3$ and $H^3$ and its Bäcklund transformations follow from this construction.

1. Introduction

Suppose $g$ is a Riemannian or Lorentzian metric on a 3-dimensional manifold $N^3$. The hodge star mean curvature flow ($\ast$-MCF) on $(N^3, g)$ is the following curve evolution on the space of immersed curves in $N^3$,

$$\gamma_t = \ast_{\gamma}(H(\gamma(\cdot, t))),$$

(1.1)

where $\ast_{\gamma(x)}$ is the Hodge star operator on the normal plane $\nu(\gamma_x)$ and $H(\gamma(\cdot, t))$ is the mean curvature vector for $\gamma(\cdot, t)$. It can be checked that $\ast$-MCF preserves arc length parameter. As shown by Terng in [1], the $\ast$-MCF on $\mathbb{R}^3$ parametrized by arc length is the vortex filament equation, first modeled by Da Rios [2] for a self-induced motion of vortex lines in an incompressible fluid,

$$\gamma_t = \gamma_x \times \gamma_{xx},$$

(1.2)

which is directly linked to the famous nonlinear Schrödinger equation (NLS)

$$q_t = i(q_{xx} + 2|q|^2q).$$

(1.3)
Hasimoto transform [3] shows that the correspondence between the VFE and the NLS is given as follows. If $\gamma$ is a solution of the VFE, then there exists a function $\theta : \mathbb{R} \to \mathbb{R}$ such that

$$q(x, t) = k(x, t)e^{i\theta(t) + \int_0^x \tau(s, t) ds} \quad (1.4)$$

is a solution of the NLS, where $\tau(\cdot, t)$ is the torsion for $\gamma(\cdot, t)$ and $x$ is the arc-length parameter. Due to this transform, VFE is regarded as a completely integrable curve flow and has been studied widely (see [4]). In [5, 6], Terng and Uhlenbeck gave a systematic method to construct such a correspondence and they further gave a way to derive explicit Bäcklund transformations for curve flows.

A large literature has been developed about a more generalized NLS or the Gross-Pitaevskii equation in [7] given by

$$i\psi_t + \mu \psi = -\psi_{xx} + u(x)\psi + \alpha |\psi|^2 \quad (1.5)$$

with a trapping potential $u(x)$ and a chemical potential constant $\mu$. $\psi$ and $|\psi|^2$ represent a condensed wave-function and its local density of matter, respectively. $\alpha = +1$ or $-1$ is repulsive or attractive interactions between atoms. The Gross-Pitaevskii equation (1.5) is a fundamental model in nonlinear optics and low temperature physics, such as Bose-Einstein condensation and Superfluids [8, 9, 10, 11].

In the present article, we will show that the $*-\text{MCF}$ on 3-sphere and hyperbolic 3-space are respectively related to

$$q_t = i(q_{xx} \pm q + 2|q|^2 q), \quad (1.6)$$

the simplest case of Gross-Pitaevskii equation, which is more natural in the physical context [12]. This is a Schrödinger-type equation, however, the Gross-Pitaevskii equation is not integrable in general because of the external potential. For certain potentials, the Gross-Pitaevskii equation (1.1) admits special solutions. Fortunately, the equation (1.6) is completely integrable, since there exists a transform

$$q \mapsto e^{\pm it} q \quad (1.7)$$

between solutions of (1.6) and that of NLS. From now on, we refer (1.6) to (GP$^\pm$), where the superscript $\pm$ indicates the sign in front of the external potential $q$ in (1.6).

We aim to write down the explicit solutions of such a curve evolution on the three sphere and three dimensional hyperbolic space using the relation with (GP$^\pm$). This paper is organized as follows. In section 2, we review and modify moving frames along a curve in $\mathbb{R}^4$ from that in [11] and find periodic frames along a closed curve in order to investigate periodic Cauchy problems for $*-\text{MCF}$. We also give two examples related to soliton equations. We then show how the $*-\text{MCF}$ is related to the nonlinear Schrödinger flow in Section 3 and a Lax pair is given for the simple case of (1.1) in
Section 4. Section 5 and Section 6 are devoted to find solutions of (periodic) Cauchy problems of $\ast$-MCF on $N^3$. We give the Bäcklund transformations in the last section.

2. Moving Frames along a Curve

Let $\gamma(x) : \mathbb{R} \to \mathbb{R}^4$ be a curve parametrized by its arc-length parameter $x$, then there exists an orthonormal frame $g \in SO(4)$ such that $g^{-1}g_x$ is a $\mathfrak{so}(4)$-valued connection 1-form consisting of 6 local invariants. Since two orthonormal frames are differed by an element in $SO(4)$, one may choose a suitable frame that contains the least number of local invariants. Let

$$R(c) = \begin{pmatrix} \cos c & \sin c \\ -\sin c & \cos c \end{pmatrix}$$

denote the rotation of $\mathbb{R}^2$ by angle $c$, we consider the following types of frames.

2.1. Parallel Frame along curves on $S^3$. Given a curve $\gamma : \mathbb{R} \to S^3 \subset \mathbb{R}^4$ parametrized by arc length $x$. Since the position vector $\gamma$ is perpendicular to the tangent vector $\gamma_x$, we choose $e_0 = \gamma, e_1 = \gamma_x, n_2, n_3$ normal to $e_0$ and $e_1$ such that $\{e_0, e_1, n_2, n_3\}$ is an orthonormal basis in $\mathbb{R}^4$. Then

$$(e_0, e_1, n_2, n_3)_x = (e_0, e_1, n_2, n_3) \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -\xi_1 & -\xi_2 \\ 0 & \xi_1 & 0 & -\omega \\ 0 & \xi_2 & \omega & 0 \end{pmatrix}.$$  \hspace{1cm} (2.1)

Rotate $n_2, n_3$ by the angle $\theta$ for $\theta_x = -\omega$, i.e.,

$$(e_0, e_1, e_2, e_3) = (e_0, e_1, n_2, n_3) \begin{pmatrix} I_2 & 0 \\ 0 & R(\theta) \end{pmatrix}.$$  \hspace{1cm} (2.2)

One obtains an orthonormal frame $g = (e_0, e_1, e_2, e_3)$ satisfying

$$(e_0, e_1, e_2, e_3)_x = (e_0, e_1, e_2, e_3) \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -k_1 & -k_2 \\ 0 & k_1 & 0 & 0 \\ 0 & k_2 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (2.3)

where

$$\begin{cases} k_1 = \xi_1 \cos \theta + \xi_2 \sin \theta \\ k_2 = -\xi_1 \sin \theta + \xi_2 \cos \theta \end{cases}$$ \hspace{1cm} (2.4)
The frame $g$ is called a parallel frame and $k_i$ the principal curvature along $e_i$ for $i = 2, 3$ for $\gamma$. Moreover, $g^{-1}g_x \in o(4)$.

Next, we consider the Minkowski spacetime, denoted by $\mathbb{R}^{3,1}$, with Lorentzian metric $-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$, and the hyperbolic 3-space $\mathbb{H}^3$ is the hyperquadric defined by

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1.$$  

2.2. Parallel Frame along curves on $\mathbb{H}^3$.

Let $\gamma(x) \in \mathbb{H}^3 \subset \mathbb{R}^{3,1}$ be a curve parametrized by arc length $x$. A similar discussion to Example 2.1 shows that there exists a parallel frame $h = (e_0, e_1, e_2, e_3)$ with $e_0 = \gamma, e_1 = \gamma_x$ such that

$$h_x = h \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -\mu_1 & -\mu_2 \\ 0 & \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \end{pmatrix}. \tag{2.5}$$

Notice that $h^{-1}h_x \in o(1, 3)$.

From (2.2), we see that there are other choices for orthonormal base. Hence, given a periodic curve $\gamma(x)$, there exists a periodic frame along the curve $\gamma(x)$.

2.3. Periodic moving frames on $S^3$ and $\mathbb{H}^3$. (cf. [1])

Let $c_0 \in \mathbb{R}$ be a constant, and

$$M_{c_0} = \{ \gamma : S^1 \to N | ||\gamma_x|| = 1, \text{ the normal holonomy of } \gamma \text{ is } R(-2\pi c_0) \},$$

where $N = S^3, \mathbb{H}^3$. If $(e_0, e_1, e_2, e_3)$ is a parallel frame along $\gamma \in M_{c_0}$, then

$$(e_0, e_1, e_2, e_3)(2\pi) = (e_0, e_1, e_2, e_3)(0) \text{diag}(I_2, R(-2\pi c_0)).$$

Let $(v_2(x), v_3(x))$ be the orthonormal normal frame obtained by rotating $(e_2(x), e_3(x))$ by $c_0 x$. Then the new frame

$$\tilde{g}(x) = (e_0, e_1, v_2, v_3)(x) = (e_0, e_1, e_2, e_3)(x) \begin{pmatrix} I_2 & 0 \\ 0 & R(c_0 x) \end{pmatrix}$$

is periodic in $x$ (see Lemma 2.1). Moreover,

$$\tilde{g}^{-1}g_x = (e_{21} - \sigma e_{12}) + c_0(e_{43} - e_{34}) + \sum_{i=1}^{2} k_i(e_{i+2,2} - e_{2,i+2}), \tag{2.6}$$

where $\sigma = 1, -1$ when $N = S^3$ and $\mathbb{H}^3$, respectively, and $e_{ij}$ denote a matrix with 1 in $(i, j)$ entry and 0 elsewhere. Direct computations imply that $(\tilde{k}_1 + i\tilde{k}_2)(x, t) =
$e^{-ic_0}(k_1 + ik_2)(x, t)$ are periodic. We call $\tilde{g} = (e_0, e_1, v_2, v_3)$ a periodic $h$-frame along $\gamma$ on $N$.

**Lemma 2.1.** Let $c_0 \in \mathbb{R}$ and $\gamma \in M_{c_0}$. If $(e_0, e_1, e_2, e_3)$ is a parallel frame along $\gamma$ and for any non-negative integer $n$, define

$$(v_2^n, v_3^n)(x) = (e_2, e_3)(x)R(\gamma(c_0 + n)x). \quad (2.7)$$

Then $(v_2^n, v_3^n)(x)$ is periodic in $x$.

**Proof.** Since

$$v_2^n(2\pi) = \cos((c_0 + n)2\pi)e_2(2\pi) - \sin((c_0 + n)2\pi)e_3(2\pi)$$

$$= \cos(2\pi c_0)e_2(2\pi) - \sin(2\pi c_0)e_3(2\pi)$$

$$= v_2^0(2\pi) \quad (2.8)$$

and $v_2^0(0) = v_2^0(2\pi)$ followed from $\square$, $v_2^n(x)$ is periodic and, similarly, so is $v_3^n(x)$. $\square$

**Remark 2.1.** Notice that if a curve $\gamma(x) \in \mathbb{R}^4$ parametrized by arc length is given to be periodic in $x$, it is obvious that $e_1 = \gamma_x$ is periodic as well. Therefore, $(e_0, e_1, v_2^n, v_3^n)(x)$ is a periodic $h$-frame in $x$, where $e_0 = \gamma$.

### 3. $\ast$-MCF on 3-Sphere $S^3$ and Hyperbolic 3-space $\mathbb{H}^3$

In what follows, evolutions of invariants for the $\ast$-MCF on a 3-manifold $N = S^3, \mathbb{H}^3$ will be discussed. Denote $\sigma = 1$ and $-1$ for $N = S^3$ and $\mathbb{H}^3$, respectively. Now, let us evolve $\gamma : \mathbb{R}^2 \to N$ with the $\ast$-MCF flow (1.1), where $x$ is the arc-length parameter, $e_0 = \gamma$ and $e_1 = \gamma_x$.

Recall that the Hodge star operator on an oriented two dimensional inner product space is the rotation of $\pi$. So if $(u_1, u_2)$ is an oriented orthonormal basis then

$$\ast(u_1) = u_2, \ast(u_2) = -u_1.$$

Under this orthonormal frame $\{e_0, e_1, n_2, n_3\}$ showed in Section 2 the mean curvature vector $H$ is $\xi_1 n_2 + \xi_2 n_3$, and $\ast$-MCF (1.1) on $N$ is written as

$$\gamma_t = \xi_1 n_3 - \xi_2 n_2. \quad (3.1)$$

On one hand, a direct computation implies the following properties.

**Proposition 3.1.** For any $\gamma(x, t) \in N$ satisfying the $\ast$-MCF (3.1) parametrized by arc length, there exists a moving frame $h = \{e_0, e_1, n_2, n_3\}$, where $e_0 = \gamma, e_1 = \gamma_x$.
satisfying

\[
\begin{align*}
    h^{-1}h_x &= \begin{pmatrix} 0 & -\sigma & 0 & 0 \\ 1 & 0 & -\xi_1 & -\xi_2 \\ 0 & \xi_1 & 0 & -\omega \\ 0 & \xi_2 & \omega & 0 \end{pmatrix}, \\
    h^{-1}h_t &= \begin{pmatrix} 0 & 0 & \sigma \xi_2 & -\sigma \xi_1 \\ 0 & 0 & (\xi_2)_x + \xi_1 \omega & -(\xi_1)_x + \xi_2 \omega \\ -\xi_2 & -(\xi_2)_x - \xi_1 \omega & 0 & -u \\ \xi_1 & (\xi_1)_x - \xi_2 \omega & u & 0 \end{pmatrix},
\end{align*}
\]  

(3.2)

where

\[
\begin{align*}
    (\xi_1)_t &= -(\xi_2)_x - 2(\xi_1)_x \omega - \xi_1 \omega x + \xi_2(\omega^2 - \sigma + u), \\
    (\xi_2)_t &= (\xi_1)_x - 2(\xi_2)_x \omega - \xi_2 \omega x - \xi_1(\omega^2 - \sigma + u), \\
    \omega_t &= u_x + \frac{1}{2}(\xi_1^2 + \xi_2^2)_x.
\end{align*}
\]  

(3.3)

A similar result in [1] has been derived:

**Proposition 3.2.** Let \( \gamma(x, t) : \mathbb{R}^2 \to N \) be a closed curve parametrized by arc length and \( \gamma(0, t) = \gamma(2\pi, t) \) for all \( t \). Then

\[
\int_0^{2\pi} \omega(x, t) \, dx
\]  

is independent of \( t \).

**Proof.** It follows from Proposition 3.1 that there exists a moving \( h \) satisfying the system (3.2). The third equation of (3.3) implies

\[
\frac{d}{dt} \int_0^{2\pi} \omega(x, t) \, dx = \int_0^{2\pi} \omega_t(x, t) \, dx = \int_0^{2\pi} u_x + \frac{1}{2}(\xi_1^2 + \xi_2^2)_x \, dx = 0.
\]

\[\square\]

**Remark 3.1.** The normal holonomy of \( \gamma \) (cf. [1]) is then defined as

\[
\frac{1}{2\pi} \int_0^{2\pi} \omega(x, t) \, dx.
\]
On the other hand, if one considers parallel frames for curves, we have the following consequence.

**Proposition 3.3.** For any \( \gamma(x,t) \in N \) satisfying the \( \star\)-MCF (3.1) parametrized by arc length, there exists a parallel frame \( g = (e_0, e_1, e_2, e_3) \in G \) with \( e_0 = \gamma, e_1 = \gamma_x \), such that

\[
\begin{align*}
\left\{ \begin{array}{l}
g^{-1}g_x = \\
g^{-1}g_t = 
\end{array} \right. \\
\begin{pmatrix}
0 & -\sigma & 0 & 0 \\
1 & 0 & -k_1 & -k_2 \\
0 & k_1 & 0 & 0 \\
0 & k_2 & 0 & 0
\end{pmatrix}, \\
\begin{pmatrix}
0 & 0 & \sigma k_2 & -\sigma k_1 \\
0 & 0 & (k_2)_x & -(k_1)_x \\
-k_2 & -(k_2)_x & \frac{1}{2}(k_1^2 + k_2^2) & 0 \\
k_1 & (k_1)_x & -\frac{1}{2}(k_1^2 + k_2^2) & 0
\end{pmatrix}.
\end{align*}
\] (3.5)

Here, \( k_1, k_2 \) are principal curvatures along \( e_2, e_3 \) and \( (G, \sigma) = (SO(4), 1), (O(1, 3), -1) \) when \( N = \mathbb{S}^3, \mathbb{H}^3 \), respectively.

**Proof.** We give computations for the 3-sphere case below, and omit the similar calculations for the case \( \mathbb{H}^3 \). Using the curve evolution (3.1), we get

\[
\begin{align*}
(e_0)_t & = k_1 e_3 - k_2 e_2, \\
(e_1)_t & = (k_1)_x e_3 - (k_2)_x e_2.
\end{align*}
\] (3.6)

Therefore, we may assume

\[
(e_0, e_1, e_2, e_3)_t = (e_0, e_1, e_2, e_3) \begin{pmatrix}
0 & 0 & k_2 & -k_1 \\
0 & 0 & (k_2)_x & -(k_1)_x \\
-k_2 & -(k_2)_x & \frac{1}{2}(k_1^2 + k_2^2) & 0 \\
k_1 & (k_1)_x & -\frac{1}{2}(k_1^2 + k_2^2) & 0
\end{pmatrix}.
\]

Since \( (e_2)_x \cdot e_3 = (e_2)_x \cdot e_3 \), it is easy to see

\[
(e_2)_x \cdot e_3 = -k_1(k_1)_x \quad \text{and} \quad (e_2)_x \cdot e_3 = k_2(k_2)_x + \nu_x,
\]

which implies

\[
\nu_x = -k_1(k_1)_x - k_2(k_2)_x = -\frac{1}{2}(k_1^2 + k_2^2)_x.
\]
So,
\[ \nu = -\frac{1}{2}(k_1^2 + k_2^2) + c(t). \]

Changing frames again to make \( c(t) = 0 \), and then we have the equation (3.5) desired.

The compatibility condition of (3.5) leads to

\[
\begin{cases}
(k_1)_t = -(k_2)_{xx} - \sigma k_2 - k_2 \frac{k_1^2 + k_2^2}{2} \\
(k_2)_t = (k_1)_{xx} + \sigma k_1 + k_1 \frac{k_1^2 + k_2^2}{2}
\end{cases},
\]
(3.7)

therefore we have the following theorem.

**Theorem 3.4.** Let \( \gamma(x, t) \in \mathbb{N} \) be a solution to the curve evolution (3.1) parametrized by arc length with principal curvatures \( k_1, k_2 \) and \( g \in G \) a parallel frame along \( \gamma \) with \( G = SO(4), SO(1, 3) \) when \( \mathbb{N} = S^3, \mathbb{H}^3 \), respectively. Then \( k(\cdot, t) = (k_1 + ik_2)(\cdot, t) \) solves

\[ k_t = i(k_{xx} + \sigma k + \frac{1}{2}|k|^2 k). \]
(3.8)

**Proof.** The assertion follows directly from (3.7).

Let \( q = \frac{k}{2} \). Then it is obvious that

\[ q_t = i(q_{xx} + \sigma q + 2|q|^2 q). \]
(3.9)

We denoted the equation (3.9) by (GP\( ^\pm \)), where \( \pm \) indicates the sign of the external potential \( q \), i.e., (3.8) is called (GP\( ^+ \)) or (GP\( ^- \)) when \( \sigma = 1 \) or \( \sigma = -1 \), respectively.

4. Lax Pair of GP\( ^\pm \)

In the previous section, we see that the *-MCF is related to (3.9), which also can be transformed to the NLS. In this section, we will give Lax pairs of (GP\( ^\pm \)), and the relation between (GP\( ^\pm \)) and NLS, which implies that (GP\( ^\pm \)) is integrable. We first review some known properties of the SU(2)-hierarchy.

4.1. The SU(2)-hierarchy. (cf. [13, 14])

Let \( a = \text{diag}(i, -i) \). It can be checked that given \( u = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} : \mathbb{R} \to \mathfrak{su}(2) \), there is a unique

\[ Q(u, \lambda) = a\lambda + Q_0(u) + Q_{-1}(u)\lambda^{-1} + \cdots \]
satisfying \( Q_0(u) = u \),

\[
\begin{aligned}
\left\{ \begin{array}{l}
[\partial_x + a\lambda + u, Q(u, \lambda)] = 0, \\
Q(u, \lambda)^2 = -\lambda^2 I_2,
\end{array} \right.
\]

(4.1)

where \( I_2 \) is the \( 2 \times 2 \) identity matrix. Moreover, entries of \( Q_{-j}(u) \) are differential polynomials in \( q \) and its \( x \)-derivatives. It follows from (4.1) that we have the recursive formula

\[
(Q_{-j}(u))_x + [u, Q_{-j}(u)] = [Q_{-(j+1)}(u), a].
\]

(4.2)

In fact, the \( Q_j(u) \)'s can be computed directly from (4.1) and (4.2). For example,

\[
Q_{-1} = \frac{i}{2} \left( \begin{array}{cc}
-|q|^2 & q_x \\
\bar{q}_x & |q|^2
\end{array} \right).
\]

Let

\[
V = \left\{ \left( \begin{array}{cc}
0 & q \\
-\bar{q} & 0
\end{array} \right) \mid q \in \mathbb{C} \right\}.
\]

The \( j \)-th flow in \( SU(2) \)-hierarchy is the following flow on \( C^\infty(\mathbb{R}, V) \),

\[
u_t = [\partial_x + u, Q_{-(j-1)}(u)] = [Q_{-j}(u), a].
\]

(4.3)

The second flow in the \( SU(2) \)-hierarchy is the focusing NLS

\[
q_t = \frac{i}{2} \left( q_{xx} + 2|q|^2 q \right).
\]

(4.4)

It follows from (4.1) that \( u : \mathbb{R}^2 \to V \) is a solution of (4.3) if and only if

\[
\theta_j = (a\lambda + u)dx + \left( \sum_{-(j-1) \leq i \leq 1} Q_i(u)\lambda^{j-1+i} \right) dt
\]

(4.5)

is a flat \( \mathfrak{su}(2) \)-valued connection one form on the \((x, t)\)-plane for all parameter \( \lambda \in \mathbb{C} \), where \( Q_1(u) = a \) and \( Q_0(u) = u \). The connection 1-form \( \theta_j \) is the Lax pair of the solution \( u \) of the \( j \)-th flow (4.3). The Lax pair \( \theta_j \) (defined by (4.5)) for solution

\[
u = \left( \begin{array}{cc}
0 & q \\
-\bar{q} & 0
\end{array} \right)
\]

of the \( j \)-th flow (4.3) satisfies the \( \mathfrak{su}(2) \)-reality condition

\[
\theta_j(x, t, \bar{\lambda})^* + \theta_j(x, t, \lambda) = 0.
\]

(4.6)
We call a solution $E$ of
$$E^{-1}dE := E^{-1}E_x dx + E^{-1}E_t dt = \theta_j$$
a frame of the $j$-th flow in SU(2)-hierarchy if $E$ satisfies the SU(2)-reality condition,
$$E(x, t, \bar{\lambda})^*E(x, t, \lambda) = I.$$  

4.2. Lax pairs of $(GP^\pm)$. Due to the transformation, (1.7), between GP$^\pm$ and NLS, we are able to construct Lax pairs of GP$^\pm$.

**Proposition 4.1.** Let $a = \text{diag}(i, -i)$, 
$$u = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, Q_{-1} = \frac{i}{2} \begin{pmatrix} -|q|^2 & q_x \\ \bar{q}_x & |q|^2 \end{pmatrix}.$$  
(1) The equation $(GP^\pm)$
$$qt = \frac{i}{2}(q_{xx} + \sigma q + 2|q|^2q)$$
has a Lax pair
$$\tau = (a\lambda + u)dx + (a\lambda^2 + u\lambda + Q_{-1} - \frac{\sigma}{4}a)dt.$$  
(2) The above equation (4.9) is gauge equivalent to the Lax pair $\theta_2$ (defined by (4.5)) of the focusing NLS.
(3) If $v$ is a solution of the focusing NLS (4.4), then the transform
$$q = e^{\frac{\sigma}{2}t}v$$
gives a solution to $(GP^\pm)$.

**Proof.** (1) follows from the flatness of $\tau$ and direct calculations imply (3). For (2), let $g = e^{\frac{\sigma}{4}t}$. Then
$$(dg)g^{-1} = \frac{\sigma}{4}adt,$$
and hence $\tau$ is given by the gauge transform of $\theta_2$ via $g$.

Notice that two frames of the $j$-th flow in SU(2)-hierarchy are differed by a constant of SU(2). The following states the relation between frames of NLS and $(GP^\pm)$.

**Proposition 4.2.** Suppose $E$ and $F$ are frames of $\theta_2$ and $\tau$ given as in Proposition 4.1, respectively. Then $F = CEG^{-1}$ for some constant $C \in SU(2)$. 
Proof. Let $g = e^{\tau at}$. 

\[
F^{-1}dF = \tau = g\theta_2 g^{-1} - dgg^{-1} = gE^{-1}(dEg^{-1} - Eg^{-1}dgg^{-1}) = gE^{-1}d(Eg^{-1})
\]

\[\square\]

5. Solutions to $*$-MCF on $S^3$ and $H^3$

In this section, we construct solutions of the $*$-MCF on $S^3$ and $H^3$ under identifications of the ambient spaces $\mathbb{R}^4$ and $\mathbb{R}^{1,3}$. For later use, we write down the identifications in need as below.

$\mathbb{R}^4$ as the Quaternions.

Now we identify $\mathbb{R}^4$ as the quaternion matrices $\mathcal{H}$, where

\[
\mathcal{H} \equiv \{ \begin{pmatrix} \alpha & -\beta \\ \beta & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C} = \mathbb{R}^2 \}.
\]

As a real vector space, $\mathcal{H}$ has a standard basis consisting of the four elements

\[
I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, a = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ and } c = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad (5.1)
\]

with rules of multiplication

\[
a^2 = b^2 = c^2 = -I_2, \quad ab = -ba = c, \quad bc = -cb = a, \quad \text{and } ca = -ac = b.
\]

We identify $\mathcal{H}$ as the Euclidean $\mathbb{R}^4$ via

\[
xi_2 + ya + zb + wc = \begin{pmatrix} x + iy & z + iw \\ -z + iw & x - iy \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}. \quad (5.2)
\]

Let $SU(2) \times SU(2)$ act on $\mathbb{R}^4$ by

\[
(h_-, h_+) \cdot v = h_-vh_+^{-1},
\]

where $(h_-, h_+) \in SU(2) \times SU(2)$ and $v \in \mathbb{R}^4$. 

This gives an isomorphism $SO(4) \cong SU(2) \times SU(2)/\pm I_2$. Let $\delta = (I_2, a, b, c)$ be an orthonormal basis of $\mathcal{H}$, where $I_2, a, b$ and $c$ are defined as in (5.1). Denote $(h_-, h_+ \cdot \delta$ by

$$(h_-, h_+) \cdot \delta = (h_+^{-1}, h_- a h_+^{-1}, -h_- c h_+^{-1}, h_- b h_+^{-1}).$$

Then $(h_-, h_+) \cdot \delta$ is in $SO(4)$.

$\mathbb{R}^{1,3}$ as the collection of Hermitian matrices. (cf. [15])

We identify every point $x \in \mathbb{R}^{1,3}$ as a $2 \times 2$ self-adjoint matrix as follows:

$$x = \begin{pmatrix} x_0 - x_3 & x_1 - i x_2 \\ x_1 + i x_2 & x_0 + x_3 \end{pmatrix} \text{ represents } x = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}. \quad (5.4)$$

Notice that

$$x^* = \bar{x}^t = x \text{ and } \det x = \|x\|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2.$$

Let $\mathcal{M}$ be the collection of $2 \times 2$ self-adjoint matrices. Then $\mathcal{M}$ is a 4-dimensional real vector space, whose basis consists of the identity matrix $I_2$ and

$$a_M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b_M = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{ and } \quad c_M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (5.5)$$

We have identified the vector $x = (x_0, x_1, x_2, x_3)^t$ with the matrix $x = x_0 I_2 + x_1 c_M + x_2 b_M + x_3 a_M$. Let $\mathcal{M}$ be equipped with an inner product

$$<x, y> = \frac{1}{2} \text{tr}(xy),$$

then $\delta_M = (I_2, a_M, b_M, c_M)$ is an orthonormal basis of $\mathcal{M}$. Let $SL(2, \mathbb{C})$ act on $\mathcal{M}$ by

$$h \cdot v = hvh^t,$$

where $h \in SL(2, \mathbb{C})$ and $v \in \mathcal{M}$. This action preserves the determinant. Therefore, it maps to a subgroup of $O(1, 3)$. Indeed, the image is the identity component of $O(1, 3)$. Moreover, $h \cdot \delta_M$ is in $SO(1, 3)$. Note that

$$(a_M, b_M, c_M) = -i(a, b, c), \quad (5.6)$$

where $(a, b, c)$ is defined as in (5.1).
5.1. **Constructions of Solutions to $*$-MCF on $S^3$.** There are many solutions for the NLS constructed, and hence solutions to (GP$^\pm$) are obtained via $e^{it/4}$-transform. We have seen how the $*$-MCF relates to (GP$^\pm$) and their Lax pairs in the previous section. The natural question is can we construct a $*$-MCF corresponding to a given solution of (GP$^\pm$)? The answer is positive. We will use such a correspondence to establish solutions to $*$-MCF. That is, solutions to $*$-MCF can be written down explicitly.

**Theorem 5.1.** Let $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$ and $\lambda_1 \neq \lambda_2$. Suppose $E$ is a frame of a solution $q$ to (GP$^+$) and define

$$
\eta(x, t) = E(x, t, \lambda_1)E(x, t, \lambda_2)^{-1}.
$$

Then

$$
\gamma(x, t) = \eta\left(\frac{1}{\lambda_1 - \lambda_2} x - \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} t, \frac{1}{\lambda_1 - \lambda_2} t\right)
$$

is a solution of $*$-MCF (3.1) on $S^3$.

**Proof.** Note that for any arbitrary $\lambda$, $E(x, t, \lambda)$ satisfies

$$
E^{-1}E_x = a\lambda + u, E^{-1}E_t = a\lambda^2 + u\lambda + Q_{-1} - \frac{a}{4},
$$

where $a = \text{diag}(i, -i)$,

$$
u = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, Q_{-1} = \frac{i}{2} \begin{pmatrix} -|q|^2 & q_x \\ \bar{q}_x & |q|^2 \end{pmatrix}.
$$

We denote $E(x, t, \lambda_j)$ by $E_j$ and $E_{j,z}$ indicates the partial derivative of $E_j$ with respect to the variable $z$. Then it is easy to see

$$
\eta_x = E_{1,x}E_2^{-1} - E_1E_2^{-1}E_{2,x}E_2^{-1} = E_1(a\lambda_1 + u)E_2^{-1} - E_1(a\lambda_2 + u)E_2^{-1} = (\lambda_1 - \lambda_2)E_1aE_2^{-1},
$$

and similarly,

$$
\eta_t = (\lambda_1^2 - \lambda_2^2)E_1aE_2^{-1} + (\lambda_1 - \lambda_2)E_1uE_2^{-1}.
$$

Let

$$
e_0 = E_1E_2^{-1}(x, \tilde{t}), e_1 = E_1aE_2^{-1}(\tilde{x}, \tilde{t}), e_2 = -E_1cE_2^{-1}(\tilde{x}, \tilde{t}), e_3 = E_1bE_2^{-1}(\tilde{x}, \tilde{t}),
$$
where \( \tilde{x} = \frac{1}{\lambda_1 - \lambda_2} x - \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} t \), \( \tilde{t} = \frac{1}{\lambda_1 - \lambda_2} t \) and \( a, b, c \) are the same as stated in (5.1).

Since \( E_1, E_2 \in SU(2) \) and \((I_2, a, -c, b)\) is an orthonormal basis for \( \mathcal{H} \), \((e_0, e_1, e_2, e_3) \in SO(4) \). We write \( q = q_1 + iq_2 \) and hence \( u = q_1 b + q_2 c \). Use (5.9), (5.10) to obtain

\[
\gamma_x = \frac{1}{\lambda_1 - \lambda_2} \eta_x = e_1, \\
\gamma_t = -\frac{1}{\lambda_1 - \lambda_2} \eta_x + \frac{1}{\lambda_1 - \lambda_2} \eta_t = E_u E_2^{-1} = q_1 e_3 - q_2 e_2. \\
\]

We compute the following items:

\[
(e_0)_x = e_1, \\
(e_1)_x = \frac{1}{(\lambda_1 - \lambda_2)^2} \eta_{xx} = \frac{-(\lambda_1 - \lambda_2)^2 e_0 + (\lambda_1 - \lambda_2)(2q_1 e_2 + 2q_2 e_3)}{2q_0 + \lambda_1 - \lambda_2(2q_1 e_2 + 2q_2 e_3),} \\
(e_2)_x = \frac{-2q_0}{\lambda_1 - \lambda_2} e_1 + \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} e_3,
\]

namely, if \( g = (e_0, e_1, e_2, e_3) \), then

\[
g^{-1} g_x = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & -\frac{2q_1}{\lambda_1 - \lambda_2} & -\frac{2q_2}{\lambda_1 + \lambda_2} \\
0 & \frac{2q_1}{\lambda_1 - \lambda_2} & \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} & 0 \\
0 & \frac{2q_2}{\lambda_1 + \lambda_2} & \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} & 0
\end{pmatrix}
\]

Next, we compute the \( t \)-derivative of \( g \). (5.11) implies \( (e_0)_t = \gamma_t = q_1 e_3 - q_2 e_2 \). Note that \( (e_1)_t = (\gamma_x)_t = (\gamma_t)_x = (q_1 e_3 - q_2 e_2)_x \). Together with (5.13), we get

\[
(e_1)_t = \left(-\frac{1}{\lambda_1 - \lambda_2}(q_2)_x - \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} q_1 \right) e_2 + \left(\frac{1}{\lambda_1 - \lambda_2}(q_1)_x - \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} q_2 \right) e_3.
\]

Since \( (e_2)_t \) can be computed as

\[
q_2 e_0 + \left(\frac{1}{\lambda_1 - \lambda_2}(q_2)_x + \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} q_1 \right) e_1 + \left(-\frac{1}{2(\lambda_1 - \lambda_2)} - \frac{2\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \right) |q|^2 e_3,
\]
we obtain $g^{-1}g_t = A$, where $A$ is equal to

$$
\begin{pmatrix}
0 & 0 & q_2 & -q_1 \\
0 & -\gamma \frac{(q_2)_x}{\lambda_1-\lambda_2} + \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2} q_1 & 0 & \frac{-\gamma}{2(\lambda_1-\lambda_2)} + \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2} q_2 \\
-q_2 & \gamma \frac{(q_1)_x}{\lambda_1-\lambda_2} - \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2} q_2 & -\frac{1}{2(\lambda_1-\lambda_2)} - \frac{2\lambda_1\lambda_2+|q|^2}{\lambda_1-\lambda_2} & 0
\end{pmatrix}. 
$$

(5.14)

\[ \square \]

In particular, by choosing $\lambda_1 = 1, \lambda_2 = 0$, one obtains a neater solution. We state as follows:

**Corollary 5.2.** Let $E$ be a frame of a solution $q$ of $(GP^+)$ and $\eta(x, t)$ is defined as in (5.7). Then $\gamma(x, t) = \eta(x - t, t)$ solves the $\ast$-MCF on $S^3$ with principal curvatures $2q_1, 2q_2$. Moreover, let $\phi(x, t) = E(x, t, 1), \psi(x, t) = E(x, t, 0)$ and $g = (\phi, \psi) \cdot \delta(x - t, t)$. Then $g$ satisfies

$$
\begin{aligned}
g^{-1}g_x &= \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & -2q_1 & -2q_2 \\
0 & 2q_1 & 0 & -1 \\
0 & 2q_2 & 1 & 0
\end{pmatrix}, \\
g^{-1}g_t &= \begin{pmatrix}
0 & 0 & q_2 & -q_1 \\
0 & 0 & (q_2)_x + q_1 & -(q_1)_x + q_2 \\
-q_2 & -(q_2)_x - q_1 & 0 & \frac{1}{2} + |q|^2 \\
q_1 & (q_1)_x - q_2 & -\frac{1}{2} - |q|^2 & 0
\end{pmatrix},
\end{aligned}
$$

(5.15)

where $q = q_1 + iq_2$.

Notice that a frame $E$ is not unique. In other words, the solution constructed in this method is unique (up to the conjugation).

**Proposition 5.3.** Let $q$ be a solution of $(GP^+)$ and $E$ its frame. Let $F = CE$ for some constant $C \in SU(2)$ and $\gamma(x, t)$ defined as in Corollary 5.2. Define $\tilde{\eta}(x, t) = F(x, t, 1)F(x, t, 0)^{-1}$ and $\tilde{\gamma}(x, t) = \tilde{\eta}(x - t, t)$ Then $F$ is again a frame for $q$ and $\tilde{\gamma}(x, t)$ is also a solution of $\ast$-MCF (3.1), on $S^3$.

Proof. Denote $E_1, E_0$ by $E(x, t, 1), E(x, t, 0)$ and $F_1, F_0$ by $F(x, t, 1), F(x, t, 0)$, respectively. One sees that $F^{-1}dF = E^{-1}dE$ and $F_1 F_0^{-1} = C E_1 E_0^{-1} C^{-1}$, i.e., $\tilde{\gamma} = C \gamma C^{-1}$. Since $\gamma$ solves (3.1), we have

$$
\tilde{\gamma}_t = C \gamma_t C^{-1} = CE_1 u E_0^{-1} C^{-1} = F_1 u F_0^{-1}.
$$

(5.16)
According to (4.10), ∗-MCF on $H^3$ is closely related to $(GP^-)$ as well. And using the transform (1.7), the $(GP^-)$ can be linked to $(GP^+)$. Therefore, we derive a solution by means of frames for a solution $u$ to $(GP^+)$ and (5.6). A similar construction of solutions to ∗-MCF on $H^3$ can be found using Theorem 5.1. We state it as follows.

**Theorem 5.4 (Solutions on $H^3$).** Let $\lambda = \frac{1}{2} (1 - i)$ and suppose $E$ is a frame of a solution $q$ to $(GP^{+})$ and define

$$\eta(x,t) = E(x,t,\lambda)E(x,t,\bar{\lambda})^{-1}. \quad (5.17)$$

Then

$$\gamma(x,t) = \eta(x - t, t) \quad (5.18)$$

is a solution of ∗-MCF (3.1) on $H^3$.

**Proof.** Since $E$ satisfies the $SU(2)$-reality condition, it is clear that

$$\eta_x = -iE(x,t,\lambda)aE(x,t,\bar{\lambda})^{-1} = E(x,t,\lambda)a_ME(x,t,\lambda)^*, \quad \eta_t = -iE(x,t,\lambda)aE(x,t,\bar{\lambda})^{-1} = E(x,t,\lambda)a_ME(x,t,\lambda)^*,$$

where $a$ and $a_M$ are defined as in (5.2) and (5.5).

Using the fact that $E$ is a frame of $(GP^+)$ together with (5.2) and (5.5), it is easy to show

$$\eta_x = -iE(x,t,\lambda)aE(x,t,\bar{\lambda})^{-1} + (-i)E(x,t,\lambda)uE(x,t,\bar{\lambda})^{-1}, \quad (5.19)$$

and

$$\eta_{xx} = -i \left( E(x,t,\lambda)(\bar{\lambda} - \lambda + [u, a])E(x,t,\bar{\lambda})^{-1} \right) = E(x,t,\lambda)E(x,t,\bar{\lambda})^{-1} - iE(x,t,\lambda)[q_1 b + q_2 c, a]E(x,t,\bar{\lambda})^{-1} \quad \eta = -iE(x,t,\lambda)(2q_2 b - 2q_1 c)E(x,t,\bar{\lambda})^{-1} \quad \gamma = \eta + 2q_2 E(x,t,\lambda)b_mE(x,t,\bar{\lambda})^{-1} - 2q_1 E(x,t,\lambda)c_mE(x,t,\bar{\lambda})^{-1}.$$

Therefore, $\gamma_x = \eta_x$ and $\gamma_t = -\eta_x + \eta_t$ as desired. \hfill \Box

Similarly, choose any arbitrary $\lambda \in \mathbb{C} \setminus \mathbb{R}$, solutions to ∗-MCF on $H^3$ are constructed as follows.

**Theorem 5.5.** Suppose $\lambda = r - is$, where $r, s \in \mathbb{R}$ and $s > 0$. Let $E(x,t,\lambda)$ be a frame of a solution $q$ to $(GP^+)$. Define

$$\eta(x,t) = E(x,t,\lambda)E(x,t,\bar{\lambda})^{-1} \quad \text{and} \quad \gamma(x,t) = \eta(x - \frac{r}{s}, \frac{t}{2s}). \quad (5.21)$$

Then $\gamma(x,t)$ solves ∗-MCF on $H^3$. 

6. Cauchy Problems

Using the correspondence between Lax pairs of \((\text{GP}^+)\) and \(*\-\text{MCF}\), we are able to write down the explicit solutions to the curve evolution. In this section, we further investigate the Cauchy problem of the \(*\-\text{MCF}\) with an arbitrary initial curve or a periodic one. Without loss of generality, we assume the period is \(2\pi\).

**Theorem 6.1** (Cauchy problem for \(*\-\text{MCF}\) on \(\mathbb{S}^3\)). Given \(\gamma_0(x) : \mathbb{R} \to \mathbb{S}^3\) be an arc-length parametrized curve, \(g_0(x)\) a parallel frame along \(\gamma_0\), and \(q_1, q_2\) the corresponding principal curvatures. Given \(\phi_0, \psi_0 \in SU(2)\) such that \(g_0(0) = (\phi_0, \psi_0) \cdot \delta\). Suppose \(q = k_1 + i k_2\) is a solution of \((\text{GP}^+)\) with \(q(\cdot, 0) = q_1 + i q_2\). Let \(E, F\) be the frames of \(q\) satisfying \(E(0, 0, \lambda) = \phi_0, F(0, 0, \lambda) = \psi_0, \eta(x, t) = E(x, t, 1) F(x, t, 0)^{-1}\) and \(\alpha(x, t) = \eta(x - t, t)\). Then \(\gamma(x, t) = \alpha(x, t) - \eta(0, 0) + \gamma_0(0)\) is a solution of \((3.1)\) with \(\gamma(x, 0) = \gamma_0(x)\).

**Proof.** Note that \(\gamma_t = \alpha_t\) and Theorem 5.1 shows that \(\gamma\) satisfies the \(*\-\text{MCF}\) \((3.1)\). In particular, \(\alpha(x, 0) = \eta(x, 0)\). We claim that \(\eta(x, 0) = \gamma_0(x) + \eta(0, 0) - \gamma_0(0)\). In this case, one obtains \(\gamma(x, 0) = \gamma_0(x)\). Note that

\[
\eta_x(x, 0) = E(x, 0) a F(x, 0)^{-1} = \phi a \psi^{-1} = \gamma_0'(x),
\]

which implies

\[
\eta(x, 0) = \gamma_0(x) + c,
\]

for some constant \(c\). So \(c = \eta(0, 0) - \gamma_0(0)\). \(\square\)

Next, we turn our attention to construct \(x\)-periodic solutions to \(*\-\text{MCF}\) \((3.1)\) on \(\mathbb{S}^3\). By the construction of solutions in Theorem 5.1, the formula \((5.7)\) implies that it suffices to find periodic frames \(E\).

**Theorem 6.2.** Let \(\gamma(x, t)\) be a arc-length parametrized solution of the \(*\-\text{MCF}\) on \(\mathbb{S}^3\) and periodic in \(x\) with period \(2\pi\). Suppose \((e_0, e_1, \tilde{n}_2, \tilde{n}_3)\) is orthonormal along \(\gamma\) such that \(e_0 = \gamma, e_1 = \gamma_x\). Let \(\omega = (\tilde{n}_2)_x \cdot \tilde{n}_3\). Then \(c_0 = \frac{1}{2\pi} \int_0^{2\pi} \omega(x, t) \, dx\) is constant for all \(t\), and there exists \(g = (u_0, u_1, u_2, u_3)(x, t)\) such that

1. \(g(\cdot, t)\) is a periodic \(h\)-frame along \(\gamma(\cdot, t)\),
2. \(g^{-1} g_x = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & -\zeta_1 & -\zeta_2 \\
0 & \zeta_1 & 0 & -2 c_0 \\
0 & \zeta_2 & 2 c_0 & 0
\end{pmatrix}
\)
3. \(q = \frac{1}{2}(\zeta_1 + i \zeta_2)\) is a solution of the \((\text{GP}^+)\).
Lemma 6.3. Let $q$ be a $x$-periodic solution of $(\text{GP}^+)$ with period $2\pi$, $\lambda_0 \in \mathbb{R}$, and $E(x, t, \lambda)$ the extended frame of $q$. If $E(x, 0, \lambda_0)$ is periodic in $x$ with period $2\pi$, then so is $E(x, t, \lambda_0)$ for all $t$.

Proof. Recall that $E$ satisfies the following linear system

$$
\begin{cases}
E^{-1}E_x = a\lambda + u \\
E^{-1}E_t = a\lambda^2 + u\lambda + Q_{-1} - \frac{a}{4}
\end{cases}
$$

(6.2)

Let $y(t) = E(2\pi, t, \lambda_0) - E(0, t, \lambda_0)$ and $A(x, t) = a\lambda_0^2 + u\lambda_0 + Q_{-1} - \frac{a}{4}$. Note that $A(2\pi, t) = A(0, t)$ because of periodicity of $q$. Take the derivative with respect to $t$ to obtain

$$
y'(t) = E_t(2\pi, t, \lambda_0) - E_t(0, t, \lambda_0) = E(2\pi, t, \lambda_0)A(2\pi, t) - E(0, t, \lambda_0)A(0, t)
$$

(6.3)

Since $y(0) = 0$ solves $y'(t) = y(t)A(0, t)$, the uniqueness theorem of ODE implies that $y(t)$ is identically zero. □

As a consequence of Theorem 5.1 and Lemma 6.3, we have the following.

Theorem 6.4 (Periodic Cauchy problem for $\ast$-MCF on $S^3$).

Let $\gamma_0(x) : [0, 2\pi] \to S^3$ be a closed curve parametrized by arc length and $q^0_1, q^0_2$ principal curvatures. Let $(e^0_0, e^0_1, u^0_2, u^0_3)$ be a h-frame along $\gamma_0$ and $\phi, \psi \in SU(2)$ such that

$$(e^0_0, e^0_1, u^0_2, u^0_3) = (\phi \psi^{-1}, \phi a \psi^{-1}, -\phi c \psi^{-1}, \phi b \psi^{-1}),$$

where $a, b, c$ are defined as in (5.1). Suppose $q : \mathbb{R}^2 \to \mathbb{C}$ is a periodic solution of $(\text{GP}^+)$ with initial data $q(x, 0) = \frac{1}{2}(q_1^0 + iq_2^0)e^{-ic_0x}$, where $c_0$ is the normal holonomy of $\gamma_0$. Let $E$ and $F$ be frames with $E(0, 0, c_0 + 1) = \phi$ and $F(0, 0, c_0) = \psi$. Define

$$\eta(x, t) = EF^{-1}(x, t), \text{ and } \alpha(x, t) = \eta(x - (2c_0 + 1)t, t).$$

Then $\gamma(x, t) = \alpha(x, t) - \eta(0, 0) + \gamma_0(0)$ is a solution of periodic Cauchy problem of $\ast$-MCF on $S^3$ with initial data $\gamma_0(x)$.

Proof. Theorems 6.1 and 5.1 imply that $\gamma$ is a solution of $\ast$-MCF and the periodicity of $\gamma$ follows from Lemma 2.1 and Lemma 6.3. □
7. Bäcklund Transformation

From a conceptual point of view, once a solution \( q(x) \) of the NLS is given, and consequently, through formula (4.10), one obtains a solution \( Q(x) \) of GP. By solving the standard BT for the NLS [5], a new solution \( q(x) \) of NLS is derived and thus, using the transformation (4.10) again, one recovers a new solution of the GP equation. Of course, since we have established the correspondence between \( \ast \)-MCF and the GP equation, we shall try to construct a Bäcklund Transformation for \( \ast \)-MCF on a 3-sphere and \( \mathbb{H}^3 \). In this section, we first state the BT for NLS and give an example.

Given \( \alpha \in \mathbb{C} \backslash \mathbb{R} \), a Hermitian projection \( \pi \) of \( \mathbb{C}^2 \), and let
\[
g_{\alpha,\pi}(\lambda) = I + \frac{\alpha - \bar{\alpha}}{\lambda - \alpha} \pi^\perp,
\]
where \( \pi^\perp = I - \pi \). Then \( g_{\alpha,\pi}(\lambda)^{-1} = g_{\alpha,\pi}(\bar{\lambda})^* \).

**Theorem 7.1 (Algebraic BT for NLS).** [5]

Let \( E(x,t,\lambda) \) be a frame of a solution \( u = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} \) of the NLS, \( \pi \) the Hermitian projection of \( \mathbb{C}^2 \) onto \( \mathbb{C}v \), and \( \alpha \in \mathbb{C} \backslash \mathbb{R} \). Let \( \tilde{v} = E(x,t,\alpha)^{-1}(v) \), and \( \tilde{\pi} \) the Hermitian projection of \( \mathbb{C}^2 \) onto \( \mathbb{C}\tilde{v} \). Then
\[
\tilde{u} = u + (\bar{\alpha} - \alpha)[\tilde{\pi},a]
\]
is a solution of the NLS. Moreover, \( \tilde{E}(x,t,\lambda) = g_{\alpha,\pi}(\lambda)E(x,t,\lambda)g_{\alpha,\tilde{\pi}(x,t)}^{-1} \) is a new frame for \( \tilde{u} \).

Let
\[
W = e^{-\sigma at},
\]
where \( \sigma = 1, -1 \) for \( \mathbb{S}^3, \mathbb{H}^3 \), respectively. Since \( \tilde{E}(x,t,\lambda) = g_{\alpha,\pi}(\lambda)E(x,t,\lambda)g_{\alpha,\tilde{\pi}(x,t)}^{-1} \), described in Theorem 7.1, is a new frame for a solution \( \tilde{u} \) to NLS, so is \( E(x,t,\lambda)g_{\alpha,\tilde{\pi}(x,t)}^{-1} \); and the relation between frames the GP and NLS in Proposition 4.2 implies \( F = EW \) is a frame of the GP. It is obvious to see that \( \tilde{F} := E(x,t,\lambda)g_{\alpha,\tilde{\pi}(x,t)}^{-1}W \) is a new frame of the \( (GP^+) \).

**Lemma 7.2.** Suppose \( F(x,t,\lambda) = E(x,t,\lambda)W \) is a frame of the \( (GP^+) \) and \( \tilde{E} = E(x,t,\lambda)g_{\alpha,\tilde{\pi}(x,t)}^{-1} \), where \( \alpha, E(x,t,\lambda), \tilde{\pi}, g_{\alpha,\tilde{\pi}(x,t)} \) are defined as that in Theorem 7.1.
Then
\[ \tilde{F} = FW^{-1}g_{\alpha,\tilde{\pi}(x,t)}W \] (7.2)
is a new frame of the \((GP^+)\).

**Theorem 7.3** (Algebraic BT for \(*\)-MCF on \(\mathbb{S}^3\)). Let \(\gamma\) be a solution of \(*\)-MCF on \(\mathbb{S}^3\) and \(F\) the frame of a solution \(q\) of GP. Let \(\pi\) be the Hermitian projection of \(\mathbb{C}^2\) onto \(\mathbb{C}v\), and \(\alpha \in \mathbb{C} \setminus \mathbb{R}\). Let \(\tilde{v} = WF(x,t,\alpha)^{-1}(v)\), and \(\tilde{\pi}\) the Hermitian projection of \(\mathbb{C}^2\) onto \(\mathbb{C}\tilde{v}\).

\[ \tilde{\gamma} = \frac{\alpha(1-\tilde{\alpha})}{\tilde{\alpha}(1-\alpha)}\gamma + \frac{\tilde{\alpha} - \alpha}{\tilde{\alpha}(1-\alpha)}\phi_0W^{-1}\tilde{\pi}W\phi_1^{-1}, \] (7.3)
is a new solution of \(*\)-MCF. Here \(W = e^{-\sigma t}\), \(\phi_0 = F(x,t,0)\), \(\phi_1 = F(x,t,1)\)

**Proof.**

\[ \tilde{\gamma} = \tilde{F}(x,t,0)\tilde{F}(x,t,1)^{-1} \] (7.4)
\[ = F(x,t,0)W^{-1}(\tilde{g}_{\alpha,\tilde{\pi}(x,t)}(0)^{-1}\tilde{g}_{\alpha,\tilde{\pi}(x,t)}(1))WF(x,t,1)^{-1} \] (7.5)
\[ = F(x,t,0)W^{-1}(I + \frac{\tilde{\alpha} - \alpha}{\tilde{\alpha}(1-\alpha)}\tilde{\pi}^\perp)WF(x,t,1)^{-1} \] (7.6)

The equality (7.6) is obtained by the definition (7.1). Multiplying it out to have the desired result. \(\square\)

**Remark 7.1.** Based on the connection between constructions of solutions to \(*\)-MCF on \(\mathbb{S}^3\) and \(\mathbb{H}^3\) (see Theorems 5.1 6.2), the similar Bäcklund transformation for \(*\)-MCF on \(\mathbb{H}^3\) is omitted.

**References**

[1] Terng, C.L., *Dispersive Geometric Curve Flows*, Surveys in differential geometry 2014. Regularity and evolution of nonlinear equations, 179?229, Surv. Differ. Geom., 19, Int. Press, Somerville, MA, 2015.

[2] Sante Da Rios, L 1906 Sul moto d’un liquido indefinito con un filetto vorticoso di forma qualunque, Rend. Circolo Mat. Palermo 22, 117-135.

[3] Hasimoto, H., *Motion of a vortex filament and its relation to elastic*, J. Phys. Soc. Jap. 31 (1971), 293-295.

[4] Langer, J., Perline, Ron., *Geometric realizations of Fordy-Kulish nonlinear Schrödinger systems*, Pacific J. Math. 195 (2000) 157–178

[5] Terng, C.L., Uhlenbeck, K., *Bäcklund transformations and loop group actions*, Comm. Pure. Appl. Math. 53 (2000), 1-75.

[6] Terng, C.L., Uhlenbeck, K., *Schrödinger flows on Grassmannians*, Integrable systems, Geometry, and Topology, AMS/IP Stud. Adv. Math. 36 (2006), 235-256.
