INDUCED CONNECTIONS ON
SUBMANIFOLDS IN SPACES WITH
FUNDAMENTAL GROUPS

Maks A. Akivis, Vladislav V. Goldberg, and Arto V. Chakmazyan

Abstract. The authors establish a relation of the theory of varieties with degenerate Gauss maps in projective spaces with the theory of congruences and pseudocongruences of subspaces and show how these two theories can be applied to the construction of induced connections on submanifolds of projective spaces and other spaces endowed with a projective structure.

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0 Introduction

The theory of congruences and pseudocongruences of subspaces of a projective space is closely related to the theory of varieties with degenerate Gauss maps.

In a three-dimensional space \( \mathbb{P}^3 \) as well as in three-dimensional spaces endowed with a projective structure (such as an affine, Euclidean, and non-Euclidean space), the theory of congruences was studied by many geometers. The extensive monographs on this subject were published (see, for example, the monograph [Fi 50]).

In this paper, we establish a relation of the theory of varieties with degenerate Gauss maps in projective spaces with the theory of congruences and pseudocongruences of subspaces and show how these two theories can be applied to the construction of induced connections on submanifolds of projective spaces and other spaces endowed with a projective structure.

1 Basic Equations of a Variety with
a Degenerate Gauss Map.

A smooth \( n \)-dimensional variety \( X \) of a projective space \( \mathbb{P}^N \) is called tangentially degenerate or a variety with a degenerate Gauss map if the rank of its Gauss map

\[
\gamma : X \to \mathbb{G}(n, N)
\]

is less than \( n \), \( 0 \leq r = \text{rank} \gamma < n \). Here \( x \in X \), \( \gamma(x) = T_x(X) \), and \( T_x(X) \) is the tangent subspace to \( X \) at \( x \) considered as an \( n \)-dimensional projective space.
The number \( r \) is also called the rank of \( X \), \( r = \text{rank } X \). The case \( r = 0 \) is trivial: it gives just an \( n \)-plane.

Let \( X \subset \mathbb{P}^N \) be an almost everywhere smooth \( n \)-dimensional variety with a degenerate Gauss map. Suppose that \( 0 < \text{rank } \gamma = r < n \). Denote by \( L \) a leaf of this map, \( L = \gamma^{-1}(T_x) \subset X \); \( \dim L = n - r = l \).

As was proved in [AG 04] (see Theorem 3.1 on p. 95), a variety with a degenerate Gauss map of rank \( r \) foliates into its plane leaves \( L \) of dimension \( l \), along which the tangent subspace \( T_x(X) \) is fixed.

The tangent subspace \( T_x(X) \) is fixed when a point \( x \) moves along regular points of \( L \). This is the reason that we denote it by \( T_L, L \subset T_L \). A pair \((L, T_L)\) on \( X \) depends on \( r \) parameters.

The foliation on \( X \) defined as indicated above is called the Monge–Ampère foliation.

The varieties of rank \( r < n \) are multidimensional analogues of developable surfaces of a three-dimensional Euclidean space.

The main results on the geometry of varieties with degenerate Gauss maps and further references can be found in Chapter 4 of the book [AG 93] and in the recently published book [AG 04].

In this section, we find the basic equations of a variety \( X \) with a degenerate Gauss map of dimension \( n \) and rank \( r \) in a projective space \( \mathbb{P}^N \).

In what follows, we will use the following ranges of indices:

\[ a, b, c = 1, \ldots, l; \quad p, q = l + 1, \ldots, n; \quad \alpha, \beta = n + 1, \ldots, N. \]

A point \( x \in X \) is said to be a regular point of the map \( \gamma \) and of the variety \( X \) if \( \dim T_xX = \dim X = n \), and a point \( x \in X \) is called singular if \( \dim T_xX > \dim X = n \).

We associate a family of moving frames \( \{A_u\}, u = 0, 1, \ldots, N \), with \( X \) in such a way that the point \( A_0 = x \) is a regular point of \( X \); the points \( A_a \) belong to the leaf \( L \) of the Monge–Ampère foliation passing through the point \( A_0 \); the points \( A_a \) together with the points \( A_0, A_a \) define the tangent subspace \( T_LX \) to \( X \); and the points \( A_\alpha \) are located outside of the subspace \( T_LX \).

The equations of infinitesimal displacement of the moving frame \( \{A_u\} \) are

\[ dA_u = \omega_v^w A_v, \quad u, v = 0, 1, \ldots, N, \]

where \( \omega_v^w \) are 1-forms satisfying the structure equations of the projective space \( \mathbb{P}^N \):

\[ d\omega_v^w = \omega_v^w \wedge \omega_w^v, \quad u, v, w = 0, 1, \ldots, N. \]

As a result of the specialization of the moving frame mentioned above, we obtain the following equations of the variety \( X \) (see [AG 04], Section 3.1):

\[ \omega_0^\alpha = 0, \quad \omega_a^\alpha = 0, \]
\[ \omega^p = b^\alpha_{pq} \omega^q, \quad b^\alpha_{pq} = b^\alpha_{qp}, \quad (5) \]

\[ \omega^p = c^\alpha_{aq} \omega^q. \quad (6) \]

The 1-forms \( \omega^q := \omega^q_0 \) in these equations are basis forms of the Gauss image \( \gamma(X) \) of the variety \( X \), and the quantities \( b^\alpha_{pq} \) form the second fundamental tensor of the variety \( X \) at the point \( x = A_0 \). The quantities \( b^\alpha_{pq} \) and \( c^\alpha_{aq} \) are related by the following equations:

\[ b^\alpha_{sq} c^\alpha_{ap} = b^\alpha_{sp} c^\alpha_{aq}. \quad (7) \]

Equations (5) and (6) are called the basic equations of a variety \( X \) with a degenerate Gauss map (see [AG 04], Section 3.1).

Note that under transformations of the points \( A_p \), the quantities \( c^\alpha_{aq} \) are transformed as tensors with respect to the indices \( p \) and \( q \). As to the index \( a \), the quantities \( c^\alpha_{aq} \) do not form a tensor with respect to this index. Nevertheless, under transformations of the points \( A_0 \) and \( A_a \), the quantities \( c^\alpha_{aq} \) along with the unit tensor \( \delta^q_p \) are transformed as tensors. For this reason, the system of quantities \( c^\alpha_{aq} \) is called a quasitensor.

Denote by \( B^\alpha \) and \( C_a \) the \((r \times r)\)-matrices of coefficients occurring in equations (5) and (6):

\[ B^\alpha = (b^\alpha_{pq}), \quad C_a = (c^\alpha_{aq}). \]

Sometimes we will use the identity matrix \( C_0 = (\delta^p_q) \) and the index \( i = 0, 1, \ldots, l \), i.e., \( \{i\} = \{0, a\} \). Then equations (5) and (7) can be combined and written as follows:

\[ (B^\alpha C_i)^T = (B^\alpha C_i), \]

i.e., the matrices

\[ H^\alpha_i = B^\alpha C_i = (b^\alpha_{qs} c^\alpha_{sp}) \]

are symmetric.

The quadratic forms

\[ \Phi^\alpha = b^\alpha_{pq} \omega^p \omega^q \]

are the second fundamental forms of the variety \( X \) at the point \( x = A_0 \), and the forms

\[ \Phi^\alpha = b^\alpha_{pq} (\delta^p_q x^0 + c^\alpha_{aq} x^a) \omega^p \omega^q \]

are the second fundamental forms of the variety \( X \) at the point \( x = x^0 A_0 + x^a A_a \in L \).

Let \( \{\alpha^u\} \) be the dual coframe (or tangential frame) in the space \( (\mathbb{P}^N)^* \) to the frame \( \{A_u\} \). Then the hyperplanes \( \alpha^u \) of the frame \( \{\alpha^u\} \) are connected with the points of the frame \( \{A_u\} \) by the conditions

\[ (\alpha^u, A_v) = \delta^u_v. \quad (9) \]

Conditions (9) mean that the hyperplane \( \alpha^u \) contains all points \( A_v, v \neq u \), and that the condition of normalization \( (\alpha^u, A_u) = 1 \) holds.
The equations
\[ \xi_{\beta} \alpha^{\beta} = 0 \]
defines the system of tangent hyperplanes passing through the tangent subspace \( T_L X \), and this system of tangent hyperplanes defines the system of second fundamental forms
\[ II = \xi_{\beta} b_{pq} \omega^p \omega^q \] (10)
and the system of second fundamental tensors
\[ \xi_{\beta} b_{pq} \]
of the variety \( X \) at the point \( x = A_0 \).

A variety \( X \) with a degenerate Gauss map is \textit{dually nondegenerate} if the dimension of its dual variety \( X^* \) equals \( N - l - 1 \). By the generalized Griffiths–Harris theorem (see Theorem 3.2 and Corollary 3.3 in [AG 04], pp. 97–99), a variety \( X \) with a degenerate Gauss map is dually nondegenerate if and only if at any smooth point \( x \in X \) there is at least one nondegenerate second fundamental form in the system of second fundamental forms \( \xi_{\alpha} b_{pq} \omega^p \omega^q \) of \( X \).

In what follows, we will consider only dually nondegenerate varieties \( X \) with degenerate Gauss maps.

\section{Focal Images of a Variety with a Degenerate Gauss Map.}

Suppose that \( X \) is a variety with a degenerate Gauss map of dimension \( n \) and rank \( r \) in the space \( \mathbb{C}P^N \). As we noted earlier, such a variety carries an \( r \)-parameter family of \( l \)-dimensional plane generators \( L \) of dimension \( l = n - r \). Let \( x = x^0 A_0 + x^a A_a \) be an arbitrary point of a generator \( L \). For such a point, we have
\[ dx = (dx^0 + x^0 \omega^0_0 + x^a \omega^a_0) A_0 + (dx^a + x^0 \omega^a + x^b \omega^a_b) A_a + (x^0 \omega^p + x^a \omega^p_a) A_p. \]
By (6), it follows that
\[ dx \equiv (x^0 \delta^p_q + x^a e^p_{aq}) A_p \omega^q \text{ (mod } L). \] (11)

The matrix \( (J^p_q) = (x^0 \delta^p_q + x^a e^p_{aq}) \) is the \textit{Jacobi matrix} of the map \( \gamma : X \rightarrow \mathbb{G}(n, N) \), and the determinant
\[ J(x) = \det (J^p_q) = \det (x^0 \delta^p_q + x^a e^p_{aq}) \]
of this matrix is the \textit{Jacobian} of the map \( \gamma \).

It is easy to see that \( J(x) \neq 0 \) at regular points and \( J(x) = 0 \) at singular points.

By (6) and (11), the set of singular points of a generator \( L \) of the variety \( X \) is determined by the equation
\[ \det (\delta^p_q x^0 + e^p_{aq} x^a) = 0. \] (12)
Hence this set is an algebraic hypersurface of dimension \( l - 1 \) and degree \( r \) in the generator \( L \). This hypersurface (in \( L \)) is called the focus hypersurface and is denoted \( F_L \).

Because for \( x^a = 0 \) the left-hand side of equation (12) takes the form

\[
\det \left( x^0 \delta_{pq} \right) = (x^0)^r,
\]

it follows that the point \( A_0 \) is a regular point of the generator \( L \).

We call a tangent hyperplane \( \xi = (\xi_\alpha) \) singular (or a focus hyperplane) if

\[
\det (\xi_\alpha b_{\alpha pq}) = 0,
\]

i.e., if the rank of matrix \( (\xi_\alpha b_{\alpha pq}) \) is reduced. Condition (13) is an equation of degree \( r \) with respect to the tangential coordinates \( \xi_\alpha \) of the hyperplanes \( \xi \) containing the tangent subspace \( T_L(X) \).

Because we consider only dually nondegenerate varieties with degenerate Gauss maps, there exists at least one nondegenerate form in the system of second fundamental forms of \( X \) (see the end of Section 1). Hence in the space \( \mathbb{P}^N \), equation (13) defines an algebraic hypercone of degree \( r \), whose vertex is the tangent subspace \( T_L(X) \). This hypercone is called the focus hypercone and is denoted \( \Phi_L \) (see [AG 93], p. 119).

Note that if a variety \( X \) is dually degenerate, then on such a variety, equations (13) are satisfied identically, and the variety \( X \) does not have focus hypercones.

The focus hypersurface \( F_L \subset L \) and the focus hypercone \( \Phi_L \) with vertex \( T_L \) are called the focal images of the variety \( X \) with a degenerate Gauss map.

## 3 Congruences and Pseudocongruences in a Projective Space

In a projective space \( \mathbb{P}^n \), we consider a family \( Y \) of its \( l \)-dimensional subspaces \( L \), \( \dim L = l \), which depends on \( r = n - l \) parameters. We assume that not more than a finite number of subspaces \( L \) passes through every point \( x \in \mathbb{P}^n \).

If we restrict ourselves by a small neighborhood of a subspace \( L \), then we can assume that only one subspace \( L \subset Y \) passes through a generic point \( x \in L \).

Such families of the space \( \mathbb{P}^n \) are called the congruences.

The dual image for a congruence \( Y \) of \( l \)-dimensional subspaces in \( \mathbb{P}^n \) is a pseudocongruence \( Y^\ast \) which is an \( r \)-parameter family of subspaces of dimension \( r - 1 \). Every hyperplane \( \xi \subset \mathbb{P}^n \) contains not more than a finite number of subspaces \( L^\ast \subset Y^\ast \). However, if we consider an infinitesimally small neighborhood of the subspace \( L^\ast \) of the pseudocongruence \( Y^\ast \), then there is only a single subspace \( L^\ast \) in the hyperplane \( \xi \).

In what follows, we shall establish a relation of the theory of varieties with degenerate Gauss maps in projective spaces with the theory of congruences and pseudocongruences of subspaces and show how these two theories can be applied.
to the construction of induced connections on submanifolds of projective spaces and other spaces endowed with a projective structure.

So, consider in $\mathbb{P}^n$ a congruence $Y$ of $l$-dimensional subspaces $L$. We associate with its element $L$ a family of projective frames $\{A_0, A_1, \ldots, A_l\}$ chosen in such a way that the points $A_0, A_1, \ldots, A_l$ are located in $L$, and the points $A_{l+1}, \ldots, A_n$ are located outside of $L$. The equations of infinitesimal displacement of such frames have the form

\[
\begin{align*}
\text{d}A_i &= \omega^i_j A_j + \omega^i_p A_p, \\
\text{d}A_p &= \omega^p_i A_i + \omega^p_q A_q,
\end{align*}
\]

where $i, j = 0, 1, \ldots, l$; $p, q = l+1, \ldots, n$, and $L = A_0 \wedge A_1 \ldots \wedge A_l$ is a generator of the congruence $Y$ in question. Because this generator depends on $r$ parameters and is fixed, when $\omega^p_i = 0$, the forms $\omega^p_i$ are expressed linearly in terms of the differentials of these $r$ parameters or in terms of linearly independent 1-forms $\theta^p$—linear combinations of these differentials:

\[
\omega^p_i = c_{i q}^p \theta^q.
\]

Under admissible linear transformations of the basis forms $\theta^p$, the matrices $C_i = (c_{i q}^p)$ are transformed according to the tensor law with respect to the indices $p$ and $q$.

A point $F \in L \subset Y$ is called a focus of a generator $L$ if $\text{d}F \in L$ under some condition on the basis forms $\theta^p$. In order to find the foci, we represent them in the form $F = x^i A_i$. Then

\[
\text{d}F \equiv x^i \omega^p_i A_p \quad (\text{mod} \ L),
\]

and as a result, the foci are determined by the system of equations

\[
x^i \omega^p_i = 0.
\]

By (15), this system takes the form

\[
x^i c_{i q}^p \theta^q = 0.
\]

This system has a nontrivial solution with respect to the forms $\theta^q$ if and only if

\[
\det(x^i c_{i q}^p) = 0.
\]

Equation (17) determines on $L$ the focus hypersurface $F_L$, which is an algebraic hypersurface of degree $r$.

Suppose that the point $A_0$ of our moving frame does not belong to the hypersurface $F_L$. Then the 1-forms $\omega^p_0$ are linearly independent, and we can take these forms as basis forms of the congruence $Y$. As a result, equations (15) become

\[
\omega^p_a = c_{a q}^p \omega^q_0,
\]
where \( a = 1, \ldots, l, \) and \( c_{0q}^p = \delta_{0q}^p. \) Now equations (18) coincide with equations (6). As a result, equation (17) of the focus hypersurface \( F_L \) takes the form

\[
\det(x^0 \delta_{aq}^p + x^aq_{aq}) = 0. \tag{19}
\]

Equation (19) coincides with equation (12) defining the foci on a plane generator \( L \) of a variety \( X \) with a degenerate Gauss map of rank \( r. \) However, unlike in Section 1, the quantities \( c_{aq}^p \) are not connected by any relations of type (7), because now there is no matrices \( B^\alpha = (b_{pq}^\alpha). \) Thus, the focus hypersurfaces \( F_L \) determined by equation (19) are arbitrary determinant varieties on generators \( L \) of the congruence \( Y \) in question.

In particular, if \( l = 1 \) and \( n = r + 1, \) then \( Y \) becomes a rectilinear congruence. Equation (19) defining the focus hypersurfaces \( F_L \) of such a congruence becomes

\[
\det(x^0 \delta_{aq}^p + x^1 r_{aq}^p) = 0. \tag{20}
\]

Hence, each of the focus hypersurfaces \( F_L \) of \( Y \) decomposes into \( r \) real or complex points if each is counted as many times as its multiplicity. Each of these points describes a focal variety in \( \mathbb{P}^n \) tangent to the generators \( L \) of the congruence \( Y. \)

Next, we consider a pseudocongruence \( Y^* \) in the space \( \mathbb{P}^n. \) Its generator \( L^* \) is of dimension \( r - 1 \) and depends on \( r \) parameters. We place the points \( A_p, \)

\( p = l + 1, \ldots, n, l = n - r, \) of our moving frame into the generator \( L^* \subset Y^* \) and place the points \( A_i, i = 0, 1, \ldots, l, \) outside of \( L^*. \) The equations of infinitesimal displacement of such frames again have the form (14) but now the 1-forms \( \omega^i_p \) are linear combinations of the basis forms \( \theta^p \) defining a displacement of the generator \( L^* = A_{l+1} \wedge \ldots \wedge A_n. \) So now we have

\[
\omega^i_p = b^i_{pq} \theta^q \tag{21}
\]

and

\[
dA_p = \omega^i_p A_q + b^i_{pq} \theta^q A_i. \tag{22}
\]

Consider a hyperplane \( \xi \) passing through the generator \( L^* \subset Y^*. \) Relative to our moving frame, the equation of \( \xi \) is \( \xi x^i = 0, \) where \( \xi_i \) are tangential coordinates of the hyperplane \( \xi. \) The hyperplane \( \xi, \) which in addition to the generator \( L^* \) contains also a near generator \( 'L^* \) determined by the points \( A_p \) and \( dA_p, \) is called the focus hyperplane. By (22), the conditions defining the focus hyperplane are

\[
\xi_i b^i_{pq} \theta^q = 0. \tag{23}
\]

The system of equations (23) defines a displacement of the generator \( L^* \) if and only if this system has a nontrivial solution with respect to the forms \( \theta^q. \) The necessary and sufficient condition for existence of such a nontrivial solution is the vanishing of the determinant of system (23):

\[
\det(\xi_i b^i_{pq}) = 0. \tag{24}
\]

Equation (24) defines the family of focus hyperplanes passing through the generator \( L^* \subset Y^*. \) This family is an algebraic hypercone of degree \( r \) whose
vertex is the generator \( L^* \). Note that equation (24) is similar to equation (13) of the focus hypercone \( \Phi_L \) of a variety with a degenerate Gauss map.

4 Normalized Varieties in a Multidimensional Projective Space

1. Consider a smooth \( r \)-dimensional variety \( X \) in a projective space \( \mathbb{P}^n \), \( r < n \). The differential geometry on such a variety is rather poor. It is less rich than the differential geometry on varieties of the Euclidean space \( \mathbb{E}^n \) or the spaces of constant curvature \( \mathbb{S}^n \) and \( \mathbb{H}^n \), where by \( \mathbb{S}^n \) and \( \mathbb{H}^n \) we denote, respectively, an \( n \)-dimensional elliptic and hyperbolic space. With a first-order neighborhood of a point \( x \in X \subset \mathbb{P}^n \), only the tangent subspace \( T_x(X) \) is associated. For example, in Section 1.4 of the book [AG 04], the authors showed that in order to enrich the differential geometry of a curve in the projective plane \( \mathbb{P}^2 \), it is necessary to use differential prolongations of rather higher orders of the curve equations.

However, we can enrich the differential geometry of \( X \subset \mathbb{P}^n \) if we endow \( X \) with an additional construction consisting of a subspace \( N_x(X) \) of dimension \( n - r \) such that \( T_x(X) \cap N_x(X) = x \), and an \((r - 1)\)-dimensional subspace \( K_x(X) \), \( K_x(X) \subset T_x(X), x \notin K_x(X) \). We shall denote these subspaces simply by \( N_x \) and \( K_x \) and call the normals of the first and second kind (or simply the first and second normals) of the variety \( X \), respectively (see [N 76], p. 198).

The family of first normals forms a congruence \( N \), and the family of second normals forms a pseudocongruence \( K \) in the space \( \mathbb{P}^n \). If at any point of \( x \in X \), there are assigned a single first normal \( N_x \) and a single second normal \( K_x \), then the variety \( X \) is called normalized (cf. [N 76], p. 198, and [AG 93], Chapter 6).

As we will see below, on varieties of the Euclidean space \( \mathbb{E}^n \) and the non-Euclidean spaces \( \mathbb{S}^n \) and \( \mathbb{H}^n \), the first and second normals are determined by the geometry of these spaces while on varieties of the affine space \( \mathbb{A}^n \) and the projective space \( \mathbb{P}^n \), these normals should be assigned artificially, or to find them, one should use higher order neighborhoods of a point \( x \in X \). In this section, we shall apply the first method. Note that the second method is connected with great computational difficulties. One can find more details on this method and a related bibliography in the books [AG 93], Chapters 6, 7, and [N 76], Chapter 5.

Thus, we now consider a normalized variety \( X \) of dimension \( r \), \( r = \dim X \), in the projective space \( \mathbb{P}^n \). We associate with \( X \) a family of projective frames \( \{A_0, A_1, \ldots, A_n\} \) in such a way that \( A_0 = x; A_a \in N_x, a = 1, \ldots, l \), where \( l = n - r \); and \( A_p \in K_x, p = l + 1, \ldots, n \). The equations of infinitesimal
displacement of these frames have the form

\[
\begin{align*}
  dA_0 &= \omega_0^0 A_0 + \omega^p A_p, \\
  dA_a &= \omega_0^a A_0 + \omega_b^a A_b + \omega^p A_p, \\
  dA_p &= \omega_0^p A_0 + \omega_a^p A_a + \omega_q^p A_q,
\end{align*}
\]  

(25)

Equations (25) show that for the family of moving frames in question, the system of differential equations

\[
\omega^a = 0
\]  

(26)
is satisfied, and the 1-forms \(\omega^p\) are basis forms, because they determine a displacement of the point \(A_0 = x\) along the variety \(X\). Exterior differentiation of equations (26) and application of Cartan’s lemma lead to the following equations:

\[
\omega_p^a = b_{pq}^a \omega^q, \quad b_{pq}^a = b_{qp}^a
\]  

(27)
The quantities \(b_{pq}^a\) form a tensor and are coefficients of the second fundamental forms of the variety \(X\) at the point \(x\) (see [AG 04], Section 2.1):

\[
\Phi^a = b_{pq}^a \omega^p \omega^q.
\]  

(28)

2. The points \(A_p\) belong to the tangent subspace \(T_x(X)\). We assume that these points belong to the second normal \(K_x \subset T_x(X), K_x = A_{i+1} \land \ldots \land A_n\). Then, for \(\omega^p = 0\), the 1-forms \(\omega_0^p\) must also vanish, and as a result, we have

\[
\omega_0^p = l_{pq} \omega^q.
\]  

(29)

Next, we place the points \(A_0\) of our moving frame into the first normal \(N_x\) of \(X, N_x = A_0 \land A_1 \land \ldots \land A_l\). Then, for \(\omega^p = 0\), we obtain that \(\omega_0^p = 0\), and hence

\[
\omega_a^p = c_{aq}^p \omega^q.
\]  

(30)

Consider a point \(y \in N_x\) on the first normal. For this point, we have \(y = y^0 A_0 + y^a A_a\). Differentiating this point by means of (25), we find that

\[
dy = (dy^0 + y^0 \omega_0^0 + y^a \omega_a^0) A_0 + (y^0 \omega^p + y^a \omega_a^p) A_p + (dy^a + y^b \omega_b^a) A_a.
\]  

(31)

A point \(y\) is a focus of the first normal \(N_x\) if \(dy \in N_x\). By (31), this condition implies that

\[
y^0 \omega^p + y^a \omega_a^p = 0.
\]

Applying relations (30), we find that

\[
(y^0 \delta_q^p + y^a c_{aq}^p) \omega_q = 0.
\]

This system has a nontrivial solution with respect to the forms \(\omega_q\) if and only if

\[
det(y^0 \delta_q^p + y^a c_{aq}^p) = 0.
\]  

(32)

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Equation (32) differs from equation (19) only in notation, and it defines the focus hypersurface \( F_x \) in the generator \( N_x \) of the congruence of first normals associated with the variety \( X \). It follows from equation (32) that the point \( x \in X \), whose coordinates are \( y^0 = 1, y^a = 0 \), does not belong to the focus hypersurface \( F_x \).

Let us find the focus hypercones \( \Phi_x \) of the pseudocongruence \( K \) of second normals of \( X \). The hypercones \( \Phi_x \) are formed by the hyperplanes \( \xi \) of the space \( P^n \) containing the second normal \( K_x = A_{l+1} \wedge \ldots \wedge A_n \subset T_x(X) \) and its neighboring normal \( K_x + dK_x \), which contains not only the points \( A_p \) but also the points \( dA_p \equiv \omega_0^0 A_0 + \omega_0^a A_a \pmod{N_x} \).

As a result, tangential coordinates \( \xi_0 \) and \( \xi_a \) of such a hyperplane satisfy the equations

\[
\xi_0 \omega_0^0 + \xi_a \omega_0^a = 0.
\]

By (29) and (27), it follows from this equation that

\[
(\xi_0 l_{pq} + \xi_a b_{pq}) \omega^q = 0.
\]

This system has a nontrivial solution with respect to the forms \( \omega^q \) if and only if its determinant vanishes,

\[
\det(\xi_0 l_{pq} + \xi_a b_{pq}) = 0. \quad (33)
\]

Equation (33) determines an algebraic hypercone of order \( r \) whose vertex is the generator \( K_x \) of the pseudocongruence \( K \) of the second normals. This hypercone is called the \textit{focus hypercone} of the pseudocongruence \( K \).

3. Next, we consider the tangent and normal bundles associated with a normalized variety \( X \). The base of both bundles is the variety \( X \) itself, the fibers of the tangent bundle are the tangent subspaces \( T_x \), and the fibers of the normal bundle are the first normals \( N_x \).

Suppose that \( x = x + x^p A_p \) is an arbitrary point in the tangent subspace \( T_x \), and \( x = x - x = x^p A_p \) is a vector in the tangent bundle \( TX \). The differential of this vector has the form

\[
dx = (dx^p + x^q \omega_q^p)A_p + x^p(l_{pq} A_0 + b_{pq}^a A_a) \omega^q. \quad (34)
\]

The first term on the right-hand side of (34) belongs to the tangent subspace \( T_x \), and the second term belongs to \( N_x \). The 1-form

\[
Dx^p = dx^p + x^q \omega_q^p
\]

is called the \textit{covariant differential} of the vector \( x = (x^p) \) in the connection \( \gamma_t \), and the 1-forms \( \omega_q^p \) are the components of the \textit{connection form} \( \omega = \{ \omega_q^p \} \) of the affine connection \( \gamma_t \) induced on the variety \( X \) by a normalization of \( (N, K) \).

The vector field \( x \) is called \textit{parallel} in the connection \( \gamma_t \) if its covariant differential \( Dx^p \) vanishes, i.e., if

\[
Dx^p = dx^p + x^q \omega_q^p = 0. \quad (35)
\]
We find the exterior differentials of the components $\omega_{pq}$ of the connection form $\omega$. By (27), (29), and (30), these exterior differentials have the form

$$d\omega_{pq} = \omega_{qs} \wedge \omega_{sp} + (l_{qs} \delta_l^p + b_{qs}^a c_{al}^p) \omega^s \wedge \omega^l.$$  \hspace{1cm} (36)

The 2-form

$$\Omega_{pq} = d\omega_{pq} - \omega_{qs} \wedge \omega_{sp}$$

is said to be the curvature form of the affine connection $\gamma_t$ induced on the variety $X$. From equation (36) it follows that

$$\Omega_{pq} = \frac{1}{2} R_{qsrt} \omega^s \wedge \omega^r,$$  \hspace{1cm} (37)

where

$$R_{qsrt} = l_{qs} \delta_l^p + b_{qs}^a c_{al}^p - l_{qt} \delta_s^p - b_{qt}^a c_{as}^p$$  \hspace{1cm} (38)

is the curvature tensor of the affine connection $\gamma_t$ on $X$. Equations (38) allow us to compute the curvature tensor for different normalizations of the variety $X$.

If $R_{qsrt} = 0$ on the variety $X$, then the affine connection $\gamma_t$ on $X$ is flat, and a parallel translation of a vector $x$ does not depend on the path of integration (see, for example, [N 76], p. 118, or [KN 76], p. 70).

4. Further, we consider a vector field $y$ in the normal bundle $N(X)$. This vector is determined by the point $x$ and a point $y = y^0 A_0 + y^a A_a$ of the fiber $N_x \subset N(X)$. The differential of the point $y$ is defined by equation (31).

The 1-form

$$Dy^a = dy^a + y^b \omega_b^a$$  \hspace{1cm} (39)

is called the covariant differential of the vector field $y$ in the normal bundle $N(X)$, and the forms $\omega_b^a$ are the components of the connection form of the normal connection $\gamma_n$ on a normalized variety $X$ (see, for example, [Ca 01], p. 242; see more on the normal connection in [AG 95] and Section 6.3 of the book [AG 93]). The 2-form

$$\Omega^a_b = d\omega^a_b - \omega^a_c \wedge \omega^c_b$$

is called the curvature form of the normal connection $\gamma_n$. Note that Cartan in [Ca 01] called this form the Gaussian torsion of an embedded variety $X$.

Differentiating the forms $\omega_b^a$ and applying formulas (27) and (30), we find the expression of the curvature form $\Omega^a_b$:

$$\Omega^a_b = \frac{1}{2} R^a_{bst} \omega^s \wedge \omega^t,$$  \hspace{1cm} (40)

where

$$R^a_{bst} = c^a_{ba} p^b t^p - c^a_{bt} p^b s^p.$$  \hspace{1cm} (41)

The tensor $R^a_{bst}$ is called the tensor of normal curvature of the variety $X$.

The second normals $K_x$ associated with the variety $X$ allow us to find a distribution $\Delta_y$ of $r$-dimensional subspaces associated with $X$. The elements
of the distribution $\Delta_y$ are linear spans of the points $y \in N_x$ and the second normals $K_x$. By (31), the distribution $\Delta_y$ is determined by the system of equations
\[ dy^a + y^b \omega^b_a = 0. \] (42)
In the general case, the system of equations (42) is not completely integrable, and when a point $x$ moves along a closed contour $l \subset X$, the corresponding point $y$ does not describe a closed contour.

But the point $y$ describes a closed contour $l'$ if system (42) is completely integrable. The condition of complete integrability of (42) is the vanishing of the tensor of normal curvature (41) of the variety $X$. In this case, the distribution $\Delta_y$ defined by system (42) is completely integrable, and the closed contours $l'$ lie on integral varieties of this distribution. These integral varieties form an $(n-r)$-parameter family of $r$-dimensional subvarieties $X(y)$ which are “parallel” to the variety $X$ in the sense that the subspaces $T_x(X)$ and $T_x(X(y))$ pass through the same second normal $K_x$.

5. A normalization of a variety $X \subset P^n$ is called central if all its first normals $N_x$ form a bundle with an $(l-1)$-dimensional vertex $S$.

The following theorem gives necessary and sufficient conditions for a normalization of a variety $X$ to be central.

**Theorem 1.** A normalization of a normalized variety $X \subset P^n$ is central if and only if the quantities $l_{pq}$ and $c^p_{aq}$ in equations (29) and (30) vanish:
\[ l_{pq} = 0, \quad c^p_{aq} = 0. \]

**Proof.** Necessity: Suppose that a normalization of a variety $X \subset P^n$ is central with an $(l-1)$-dimensional vertex $S$. If we place the points $A_a$ into this vertex $S$, then we get
\[ dA_a = \omega^b_a A_b. \]
By (25), this implies that
\[ \omega^0_a = 0, \quad \omega^p_a = 0. \]
By (29) and (30), this means that
\[ l_{pq} = 0, \quad c^p_{aq} = 0. \]

Sufficiency: If $l_{pq} = 0, c^p_{aq} = 0$, then $\omega^0_a = 0$, $\omega^p_a = 0$, and $dA_a = \omega^b_a A_b$, (i.e., the subspace $S = A_1 \wedge \cdots \wedge A_l$ is fixed), then all first $l$-dimensional first normals $N_x$ pass through $S$, and a normalization of $X$ is central with the $(l-1)$-dimensional vertex $S$.

**Corollary 2.** The induced affine connection $\gamma_t$ and the normal connection $\gamma_n$ of a centrally normalized variety $X \subset P^n$ are flat.

**Proof.** Because for a centrally normalized variety $X$, we have $l_{pq} = 0, c^p_{aq} = 0$, and the curvature tensor of the induced affine connection $\gamma_t$ has the form (38), it follows that
\[ R^p_{qst} = 0, \]
i.e., the curvature tensor of the induced affine connection $\gamma_t$ of a centrally normalized variety vanishes.

In the same way, it follows from (41) that the tensor of the normal connection $\gamma_n$ of a centrally normalized variety also vanishes.

Note that both results also follow from the fact that a centrally normalized variety $X \subset \mathbb{P}^n$ can be bijectively projected onto an $r$-dimensional subspace $T$ that is complementary to the vertex $S$ of the bundle of first normals $N_x$, and the geometry on the variety $X$ induced by this central normalization is equivalent to the plane geometry in the subspace $T$.

Atanasyan [A 52] found necessary and sufficient conditions for normalization of a variety $X$ in an affine space $\mathbb{A}^N$ to be central and trivial. The trivial normalization of $X$ in $\mathbb{A}^N$ is a normalization for which all first normals $N_x$ of $X$ are parallel to some constant $l$-plane (i.e., they form a bundle of parallel $l$-planes).

In our notations, his conditions for a normalization to be central are

$$c_{aq}^p = \delta_q^p c_a,$$

where $\delta_q^p$ is the Kronecker delta, and $c_a$ are $(1,0)$-tensors, and the conditions for a normalization to be trivial are

$$c_{aq}^p = 0.$$

But for an affine space (and in particular, for a Euclidean space) we always have $l_{pq} = 0$. In addition, in the projective setting (as well as in the affine setting), a trivial normalization is a central normalization whose vertex is an $(l-1)$-plane at infinity. Therefore, Atanasyan’s results follow from Theorem 1.

6. Consider the normalization dual to a central normalization. For such a normalization all second normals $K_x$ belong to a fixed hyperplane $\alpha$. We will call such a normalization affine.

**Theorem 3.** A normalization of a variety $X \subset \mathbb{P}^n$ is affine if and only if the 1-forms $\omega_0^0$ and $\omega_0^a$ occurring in equations (25) vanish,

$$\omega_0^p = 0, \quad \omega_0^a = 0.$$

If a normalization of variety $X \subset \mathbb{P}^n$ is affine, then the space $\mathbb{P}^n$ carries an affine structure, i.e., $\mathbb{P}^n$ is an affine space $\mathbb{A}^n$.

**Proof.** We place the points $A_1, \ldots, A_n$ of our moving frame into the fixed hyperplane $\alpha$. Since for an affine normalization $K_x \subset \alpha$, and hence $dA_p \subset \alpha$, $p = l+1, \ldots, n$, it follows that

$$\omega_0^p = 0.$$

Moreover, the points $A_a, a = 1, \ldots, l$, of the first normal $N_x$ can be also placed into the hyperplane $\alpha$. Then $dA_a \subset \alpha$, and as a result, we have

$$\omega_0^a = 0.$$
Conversely, if $\omega^0_p = 0, \omega^0_a = 0$, then

$$dA_p \subset \alpha, \ dA_a \subset \alpha,$$

where $\alpha = A_1 \wedge \cdots \wedge A_n$. Hence the plane $\alpha$ is fixed, and the normalization of $X$ is affine. This proves the first part of Theorem 3.

To prove the second part of Theorem 3, note that we can take the hyperplane $\alpha$ as the hyperplane at infinity $\mathbb{P}_{\infty}^{n-1}$ of the space $\mathbb{P}^n$. Hence, this hyperplane defines an affine structure of the space $\mathbb{P}^n$. Thus, the space $\mathbb{P}^n$ become an affine space $\mathbb{A}^n$. \hfill \blackSquare

7. Now suppose that a normalized variety $X \subset \mathbb{P}^n$ has a flat normal connection $\gamma_n$, i.e., $R_{bst} = 0$. By (41), these conditions lead to the relation

$$b^p_{st} c^r_{bs} = b^p_{st} c^r_{bs}. \quad (43)$$

Relations (43) differ from relations (7) only in notation. If we introduce the matrix notations

$$B^a = (b^a_{pq}), \ C_b = (c^a_{bp}),$$

then relations (43) take the form

$$(B^a C_b) = (B^a C_b)^T. \quad (44)$$

We proved in Chapters 3 and 4 of [AG 04] that these relations imply that the matrices $B^a$ and $C_b$ can be simultaneously reduced to a diagonal form or a block diagonal form. Thus, we have proved the following result.

**Theorem 4.** The focus hypersurfaces $F_x \subset N_x$ of a normalized variety $X$ with a flat normal connection decompose into the plane generators of different dimensions.

This property of the varieties $X$ with a flat normal connection $\gamma_n$ allows us to construct a classification of such varieties in the same way as this was done for the varieties with degenerate Gauss maps in a projective space. For varieties in an affine space and a Euclidean space, such a classification was outlined in the papers [ACh 75, 76, 01].

## 5 Normalization of Varieties in Affine and Euclidean Spaces

1. An affine space $\mathbb{A}^n$ differs from a projective space $\mathbb{P}^n$ by the fact that in $\mathbb{A}^n$ a hyperplane at infinity $\mathbb{P}_{\infty}$ is fixed. If we place the points $A_i, i = 1, \ldots, n$, of our moving projective frame into this hyperplane, then the equations of infinitesimal displacement of the moving frame take the following form (see equations (1.81) in AG 04]):

$$\begin{align*}
    dA_0 &= \omega^0_a A_0 + \omega^i_j A_i, \\
    dA_i &= \omega^j_i A_j, \ i, j = 1, \ldots, n,
\end{align*} \quad (45)$$
and the structure equations of the affine space $\mathbb{A}^n$ take the form
\[
d\omega_0^0 = 0, \quad d\omega_i^0 = \omega_j^0 \wedge \omega_j^i, \quad d\omega_i^j = \omega_j^k \wedge \omega_k^i. \tag{46}
\]

Consider a variety $X$ of dimension $r$ in the affine space $\mathbb{A}^n$. The tangent space $T_x(X)$ intersects the hyperplane at infinity $\mathbb{P}_\infty$ in a subspace $K_x$ of dimension $r - 1$, $K_x = T_x \cap \mathbb{P}_\infty$. Thus, for a normalization of $X$, it is sufficient to assign only a family of first normals $N_x$. If we place the points $A_a, a = 1, \ldots, l$, of our moving frame into the subspace $N_x \cap \mathbb{P}_\infty$, and the points $A_p, p = l + 1, \ldots, n$, into the subspace $K_x$, then equations (45) take the form
\[
\begin{cases}
dA_0 = \omega_0^0 A_0 + \omega_0^p A_p, \\
dA_a = \omega_a^b A_b + \omega_a^p A_p, \\
dA_p = \omega_p^a A_a + \omega_p^q A_q \\
\end{cases} \tag{47}
\]
(cf. equations (25)).

As was in the projective space, we have equations (27),
\[
\omega^a_p = b^a_{pq} \omega^q, \quad b^a_{pq} = b^a_{qp}, \tag{48}
\]
where $b^a_{pq}$ is the second fundamental tensor of the variety $X$. Equations (30) also preserve their form:
\[
\omega^p_a = c^p_{aq} \omega^q, \tag{49}
\]
but equations (29) become
\[
\omega^0_p = 0. \tag{50}
\]

As a result, because $l_{pq} = 0$, the equation of the focus hypersurface $F_x \subset N_x$ preserves its form (32):
\[
det(y^0_q b^p_q + y^a c^p_a) = 0. \tag{51}
\]
As to equation (33) of the focus hypercone $\Phi_x$, by (50), this equation takes the form
\[
det(\xi_a b^a_{pq}) = 0. \tag{52}
\]

Expressions (38) for the components of the curvature tensor of the affine connection $\gamma_t$ induced on the normalized variety $X \subset \mathbb{A}^n$ take now the form
\[
R^p_{qst} = b^a_{qs} c^p_{at} - b^a_{qt} c^p_{as}, \tag{53}
\]
and the expression (41) for the components of the tensor of normal curvature of the variety $X$ preserves its form:
\[
R^a_{bst} = b^a_{pt} c^p_{bt} - b^a_{ps} c^p_{st}. \tag{54}
\]

Consider the tensor $R_{st}$ obtained from the curvature tensor $R^p_{qst}$ of the affine connection $\gamma_t$ with respect to the indices $p$ and $q$. This tensor is called the Ricci-type tensor of the connection $\gamma_t$. It follows from (53) that
\[
R_{st} = b^a_{ps} c^p_{st} - b^a_{pt} c^p_{st}. \tag{55}
\]
Similarly we can define the Ricci-type tensor of the normal connection $\gamma_n$. We denote it by $\tilde{R}_{st}$. It follows from (54) that

$$\tilde{R}_{st} = b^a_{pt} c^b_{as} - b^a_{ps} c^b_{at}.$$ 

Comparing the last two equations, we see that

$$R_{st} = -\tilde{R}_{st}.$$ 

Hence, the following theorem is valid.

**Theorem 5.** On a variety $X \subset \mathbb{A}^n$ endowed with an affine normalization, the Ricci-type tensors of the connections $\gamma_t$ and $\gamma_n$ are equal in absolute value but opposite in sign.

Note that if a normalized variety $X$ is a hypersurface in the space $\mathbb{A}^n$, then the following theorem is valid.

**Theorem 6.** If on a normalized hypersurface $X \subset \mathbb{A}^n$ the induced affine connection $\gamma_t$ is flat, then the normal connection $\gamma_n$ is also flat.

**Proof.** In fact, for a hypersurface $X$ we have the following ranges of the indices:

$$a, b = 1; \quad p, q, s, t = 2, \ldots, n.$$ 

Hence the curvature tensor of the normal connection $\gamma_n$ has the components $R^1_{1st}$. Now it follows from Theorem 5 that

$$R^1_{1st} = -R^0_{pst}.$$ 

But if the connection $\gamma_t$ is flat, then $R^0_{qst} = 0$ and hence $R^0_{pst} = 0$. As a result, we have $R^1_{1st} = 0$, and the connection $\gamma_n$ is also flat.

As was the case in the projective space, the vanishing of tensor of normal curvature $R^a_{bst}$ is equivalent to the complete integrability of the system defining the distribution $\Delta_y = y \wedge K_x$, where $y \in N_x$. But in the affine space, the elements $\Delta_y$ of this distribution are parallel to the subspace $T_x(X)$.

Thus, we have proved the following result.

**Theorem 7.** A variety $X \subset \mathbb{A}^n$ has a flat normal connection $\gamma_n$ if and only if this variety admits an $l$-parameter family of parallel varieties $X(y)$, where $y \in N_x$.

2. Another relation of the theory of varieties with degenerate Gauss maps and the theory of normalized varieties was established by Chakmazyan in [Cha 77] (see also Theorem 4 in [Cha 78] and Theorem 1 on p. 39 of his book [Cha 90]).

We will now present this theorem.

Suppose that at an arbitrary point $x$ of a normalized variety $X$, an $s$-dimensional direction $\nu^s(x)$ (i.e., an a $s$-dimensional plane passing through $x$)
belonging to the first normal $N_x(X)$ is given. This means we have a smooth field of $s$-dimensional directions $\nu^s(x)$ on $X$, where $s \leq l = n - r$. This field determines a normal subbundle $\nu^s(X)$, whose $s$-dimensional fibres are $s$-dimensional centroprojective spaces.

The plane $N^s(x)$ of this field corresponding to an arbitrary point $x \in X$, can be defined by the point $x$ and points $B_f$ given by

$$B_f = \xi_f^a A_a \in N_x,$$  \hspace{1cm} (55)

where $f, g, h = 1, \ldots, s$.

In addition, the plane $N^s(x)$ must be invariant with respect to admissible transformations of the moving frame in $N_x(X)$. The necessary and sufficient conditions for this invariance are

$$dB_f = \theta_f^g B_g + \theta_f^0 A_0 \quad (\mod \omega^p),$$  \hspace{1cm} (56)

where $\theta_f^g$ and $\theta_f^0$ are linearly independent 1-forms whose structure can be obtained by taking exterior derivatives of (56). Since we do not need these conditions, we will not derive them.

A field $\nu^s$ is called parallel with respect to the normal connection $\gamma_n$ if under any infinitesimal displacement of an arbitrary point $x \in X$, the displacement of the $s$-dimensional direction $\nu^s(x)$ remains in the $(r + s)$-dimensional plane passing through the tangent subspace $T_x(X)$, $x \in X$, and the direction $\nu^s(x)$.

Let us find analytic conditions for a field $\nu^s$ to be parallel. Any direction belonging to an $s$-dimensional element of the field $\nu^s$ is determined by the point $A_0 = x$ and

$$A = A_0 + \xi_f B_f,$$  \hspace{1cm} (57)

where $B_f$ are defined by (55).

Taking exterior derivative of (57) and applying (25), we find that

$$dA = (\omega_0^a + \xi_f^a \omega_0^a)A_0 + d\xi_f^a B_f + (\omega^p + \xi_f^a \omega^p)A_p + \xi_f (d\xi_f^a + \xi_f^b \omega^p)A_a.$$  \hspace{1cm} (58)

The field $\nu^s$ is parallel with respect to the normal connection $\gamma_n$ if and only if

$$(d\xi_f^a + \xi_f^b \omega^p)A_a = \theta_f^g B_g + \theta_f^0 A_0.$$  \hspace{1cm}

This and (55) implies that

$$d\xi_f^a + \xi_f^b \omega^p = \theta_f^g \xi_g.$$  \hspace{1cm} (59)

**Theorem 8.** A field $\nu^s$ of $s$-dimensional normal directions $\nu^s(x)$ on a normalized variety $X \subset \mathbb{P}^n$ is parallel with respect to the normal connection $\gamma_n$ if and only if all the planes $N^s(x)$, $x \in X$, form a variety $V_{r+s}^r$ with a degenerate Gauss map of rank $r$ with $s$-dimensional plane generators in the space $\mathbb{P}^n$.

**Proof.** See [Cha 77] or [Cha 90], pp. 39-40. \hfill \square
Theorem 8 indicates a method of construction of a variety $V_{r+s}$ with a degenerate Gauss map departing from a normalized variety $X$ of some special kind.

3. Further consider a variety $X$ of dimension $r$ in the Euclidean space $\mathbb{E}^n$. On $X$, both the second normal $K_x = T_x \cap \mathbb{P}_\infty$ and the first normal $N_x$ orthogonal to the tangent subspace $T_x(X)$ are naturally defined.

In the Euclidean space $\mathbb{E}^n$, there is defined a scalar product of vectors, and a scalar product of points in the hyperplane at infinity $\mathbb{P}_\infty$ is induced by the scalar product in $\mathbb{E}^n$. Because in our moving frame, we have $A_a \in N_x \cap \mathbb{P}_\infty; A_p \in T_x \cap \mathbb{P}_\infty = K_x, a = 1, \ldots, l; p = l + 1, \ldots, n$; and $T_x \perp N_x$, we find that
\[(A_a, A_p) = 0,\] where, as usually, the parentheses denote the scalar product of points in the hyperplane at infinity $\mathbb{P}_\infty$. In addition, we set
\[(A_a, A_b) = g_{ab}, \quad (A_p, A_q) = g_{pq},\] where $g_{ab}$ and $g_{pq}$ are nondegenerate symmetric tensors.

Differentiating equations (60) and using formulas (47), (60), and (61), we find that
\[g_{ab} \omega^b_p + g_{pq} \omega^q_a = 0.\] It follows that
\[\omega^p_a = -g^{pq} g_{ab} \omega^b_q.\] Equations (57) and (48) imply that
\[\omega^p_a = -g^{pq} g_{ac} b^c_{q} \omega^s.\] Comparing (63) and (49), we obtain
\[c^p_{as} = -g^{pq} g_{ac} b^c_{qs}.\]

Now we find the equation of the focus hypersurface $F_x$ of the variety $X \in \mathbb{E}^n$. By (51) and (64), we have the following equation for $F_x$: 
\[\det(y^0 \delta^p_q - y^a g^{ps} g_{ac} b^c_{qs}) = 0.\] The last equation is equivalent to the equation
\[\det(y^0 g_{pq} - y_a b^a_{pq}) = 0,\] where $y_a = g_{ab} y^b$.

In our moving frame, the hyperplane at infinity $\mathbb{P}_\infty$ is determined by the equation $y^0 = 0$. Hence by (65), the intersection $F_x \cap \mathbb{P}_\infty$ of the focus hypersurface $F_x$ with the hyperplane at infinity $\mathbb{P}_\infty$ is defined by the equation
\[\det(y_a b^a_{pq}) = 0.\] But this equation differs only in notation from equation (52) of the focus hypercone $\Phi_x$ of the variety $X$. Equations (52) and (66) coincide if $\xi_a = y_a = g_{ab} y^b$. Thus, we have proved the following result.
Theorem 9. The focus hypercone $\Phi_x$ of the variety $X \subset \mathbb{E}^n$ is formed by the hyperplanes $\xi$ containing the tangent subspace $T_x$ and orthogonal at the points $\tilde{y}$ of the hyperplane at infinity $\mathbb{P}_\infty$ lying in the intersection $F_x \cap \mathbb{P}_\infty$.

This result clarifies the geometric meaning of the focus hypercone $\Phi_x$ for the variety $X \subset \mathbb{E}^n$ and its relation with the focus hypersurface $F_x$ of $X$.

We also find the curvature tensor of the affine connection induced on the variety $X \subset \mathbb{E}^n$. Substituting the values of $c_{ap}^q$ from (64) into formula (53), we find that

$$R^p_{qst} = g^{pu} g_{ac} (b^a_{qt} b^c_{us} - b^a_{qs} b^c_{ut}).$$

(67)

Contracting equation (67) with the tensor $g_{pv}$ and changing the summation indices (if necessary), we find that

$$R_{pqst} = g_{ac} (b^a_{ps} b^c_{qt} - b^a_{pt} b^c_{qs}).$$

(68)

where $R_{pqst} = g_{pv} R^v_{qst}$. Formulas (67) and (68) give the usual expressions for the curvature tensor of the affine connection $\gamma^r_t$ induced on a normalized variety $X \subset \mathbb{E}^n$.

But in addition to the curvature tensor of the affine connection induced on a normalized variety $X \subset \mathbb{E}^n$, we considered also the tensor $R^a_{bst}$ of normal curvature defined by equation (54). Substituting the values of $c^a_{aq}$ from (64) into formula (54), we find that

$$R^a_{bst} = g^{pa} g_{bc} (b^c_{qt} b^a_{ps} - b^c_{qs} b^a_{pt}).$$

(69)

As we noted earlier, in the book [C 01], the exterior 2-form

$$\Omega^a_b = d\omega^a_b - \omega^a_c \wedge \omega^c_b = \frac{1}{2} R^a_{bst} \omega^s \wedge \omega^t$$

is called the Gaussian torsion of a variety $X \subset \mathbb{E}^n$.

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Authors’ addresses:
M. A. Akivis  
Department of Mathematics  
Jerusalem Institute of Technology—Mahon Lev  
Havaad Haleumi St., P. O. B. 16031  
Jerusalem 91160, Israel  
E-mail address: akivis@mail.jct.ac.il

V. V. Goldberg  
Department of Mathematical Sciences  
New Jersey Institute of Technology  
University Heights  
Newark, N.J. 07102, U.S.A.  
E-mail address: vlgold@m.njit.edu

A. V. Chakmazyan  
Department of Geometry  
Armyanskij Gosudarstvennyj Pedagogicheskij Institut  
Erevan, Armenia