A Quasi 3D Modified Strain Gradient Formulation for Static Bending of Functionally Graded Micro beams Resting on Winkler-Pasternak Elastic Foundation

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Abstract

This paper presents the bending analysis of simply supported functionally graded (FG) size dependent beams based on modified strain gradient theory. The shear and normal deformations are considered in displacement field according to hyperbolic shear deformation theory. Governing equations and corresponding boundary conditions for FG micro beam are derived utilizing principle of minimum total potential energy. Mori–Tanaka homogenization scheme and the classical rule of mixture are used for prediction of material properties through the thickness. Effects of Winkler-Pasternak elastic foundation parameters are studied for different side to thickness ratios. Effects of different aspect ratios, elastic foundation parameters, power law gradient indexes and different loading conditions are investigated. The efficiency and accuracy of present model is demonstrated by comparing to the existing results in especial cases.

Key words: Modified strain gradient theory, quasi-3D theory, two-parameter elastic foundation, size dependent behavior, functionally graded materials
Nomenclature

\[ x, y, z \]  
Co-ordinates system

\[ b, h, L \]  
Width, thickness and length of the beam, respectively

\[ E_m, E_c \]  
Young modulus of metal and ceramic constituents, respectively

\[ \nu_m, \nu_c \]  
Poisson’s ratio of metal and ceramic constituents, respectively

\[ K_m, K_c \]  
Bulk modulus of metal and ceramic constituents, respectively

\[ G_m, G_c \]  
Shear modulus of metal and ceramic constituents, respectively

\[ K_e, G_e \]  
Effective bulk and shear modulus, respectively

\[ V_m, V_c \]  
Volume fraction of metal and ceramic constituents, respectively

\[ \lambda, \mu \]  
Lame constants

\[ u, w \]  
Displacement of micro beam in x and z directions, respectively

\[ u_0 \]  
Middle surface displacements in the x-direction

\[ w_b, w_s, w_z \]  
Bending, shear and normal part of transverse displacement, respectively

\[ u_0, w_{bm}, w_{sm}, w_{zm} \]  
Undetermined Fourier coefficient

\[ \bar{w}, \bar{w}^\wedge \]  
Dimensionless transverse deflection

\[ \bar{\sigma}_{xx}, \bar{\sigma}_{xz} \]  
Dimensionless axial and transverse stress, respectively

\[ q \]  
Distributed applied load

\[ f, g \]  
Shear deformation shape functions

\[ p \]  
Gradient power law index in z direction

\[ k_w, k_p \]  
Winkler and Pasternak stiffness parameters of the elastic foundation, respectively

\[ \bar{k}_w, \bar{k}_p \]  
Dimensionless Winkler and Pasternak stiffness parameters, respectively

\[ U, U_f, W \]  
Strain energy, strain energy of elastic foundation and work of external applied forces,

\[ \delta_{ij}, \epsilon_{ipq} \]  
Kronecker delta and permutation symbol, respectively

\[ \sigma_{ij}, \tau_{ijk}^{(1)}, \iota_{ij}^s, \iota_{ij}^p \]  
Classical and higher order stresses tensors

\[ \varepsilon_{ij}, \gamma_i, \eta_{ijk}^{(1)}, \xi_{ij}^s \]  
Classical strain, dilatation gradient, deviatoric stretch gradient and symmetric rotation

\[ \eta_{ijk} \]  
Symmetric part of second order deformation gradient tensor

\[ \eta_{ijk} \]  
Second order deformation gradient tensor

\[ l_0, l_1, l_2 \]  
Length scale parameters

\[ M, N, R, S \]  
Classical and non-classical forces and moment resultants

\[ P \]  
Dilatation resultants and moments

\[ T \]  
Deviatoric stretch resultants and moments
1. Introduction

Beams on the order of microns and sub-microns are unavoidable parts of modern world and widely used as sensors [1], actuators [2], atomic force microscopes[3] and in micro/nano electro-mechanical systems (NEMS/MEMS). A number of experiments proved that when the size decreases, results of classical continuum approach are not acceptable and lead to deficiency of classical theory for capturing size effects [4], which can account the size dependencies of micro and nano structures.

Developing higher order theories date back to 19\textsuperscript{th} century by works of Piola [5] and Cosserat and Cosserat [6]. Several approaches have been introduced in order to consider micro/nano scale effects; Among them, continuum mechanics approach provides more simplicity and efficiency in predicting size effect behavior in comparison to molecular dynamic (M.D) approach[7]. Mindlin [8] considered second order gradients of deformation and introduced general higher order theory with five length scale parameters. Couple stress theory (CST) presented by works of Toupin [9], Mindlin, Tiersten [10] and Koiter [11]. In this theory, higher order rotation gradients were incorporated. Yang et.al [12] modified the classical CST and developed modified couple stress theory (MCST) by enforcing the couple stress tensor to be symmetric, in which only one length scale parameter was included.

Subsequently, Fleck and Hutchinson [13, 14] extended and reformulated the first version of Mindlin theory and renamed it to the strain gradient theory (SGT), in which the deformation gradient tensor was composed of one rotation gradient and two independent stretch gradient tensors. Lam et al.[15] utilized the higher-order equilibrium equation suggested by Yang et al. [12] and presented modified strain gradient theory(MSGT). The presented theory was included three material length scale parameters to characterize the dilatation, deviatoric and symmetric rotation
gradient tensors. MCST can be achieved as a special case of MSGT by including only rotation tensor.

MSGT formulation has been gained more attention recently and has been widely employed by researchers. Dal [16] analyzed Euler-Bernoulli micro gold beams and demonstrated the accuracy of the results by comparing to the existing experiment tests. Ashoori and Mahmoodi [17] presented a geometric nonlinear formulation for analysis of thick plates. Chu et.al [18] elaborated general MSGT for static bending and natural frequency analysis of functionally graded material (FGM) Euler-Bernoulli piezoelectric nanobeams. They used volume fraction function for FGM nanobeams and concluded that material distribution function, flexoelectric coefficient ratio and span-to-depth ratio have been had considerable effects on electromechanical response of the nanobeams. Tai et.al [19] investigated free vibration of functionally graded (FG) hexagonal beryllium crystal micro plates by using Isogeometric analysis (IGA). They employed MSGT in conjunction with higher order shear deformation theory (HSDT) in order to consider shear effects. Another study on micro plates using IGA was performed by Farzam and Hassani [20]. They analyzed bending and buckling responses of FGM micro plates under mechanical and thermal loads. They assumed materials with temperature-dependent properties and several rise patterns were explored. They also presented the margins for material length scale ratio in which scale effects were negligible. Cornacchia et.al [21] solved the static bending of laminated Kirchhoff nano plates and different stacking sequences and loading profiles were examined. Some recent studies have been performed using IGA analysis and based on the nonlocal and HSDT for free vibration and bending analysis of FG plates [22], geometrically nonlinear transient analysis of FGM nanoplates [23, 24] and static and free vibration analysis of porous FG nanoplates [25].
Farzam and Hassani [26] examined the bending, buckling and free vibration behaviors of in-plane FG porous microplates. They also investigated thermal and mechanical buckling analysis of FG carbon nanotube reinforced composite nanoplates based on MCST and IGA [27] and investigated the accuracy and efficiency of the proposed model for different dimensional and power indexes. Effective computational optimization approaches based on the Eringen’s nonlocal elasticity and four variables refined plate theory were introduced for optimal design [28] and porosity dependent analysis of FG sandwich nanoplates [29]. Zhao et.al [30] proposed a nonlinear size-dependent formulation for bending and vibration analysis of nanobeams. They used MSGT and generalized differential quadrature method to derive and discretize nonlinear governing equations. They found that both strain gradient and flexoelectric coupling have been had considerable impact on nonlinear behavior, moreover inclusion of surface effects have been diminished the flexoelectric response. Pasha Zanoosi [31] argued the free vibration of porous FG micro beams under thermo-mechanical loading by using MSGT. He discussed the effects of different parameters such as thermal loading, slender ratio and gradient index for different beam theories.

The classical beam theory (CBT) is the simplest theory for beam analysis, but the results are limited to thin beams. Rotary inertia and shear effects were first reported by Timoshenko [32] with some improvements over CBT and several studies performed [33] using first order shear deformation theory (FSDT). Higher order shear deformation theories introduced to cover shortcomings of the latter theory, i.e. stress free surface condition. These theories assume power series expansion in thickness coordinate and offer acceptable precision compared to existing theories. In HSDT’s, the additional higher order terms account shear effects and compensate the above mentioned drawbacks of FSDTs. Reddy[34] considered a third order polynomial expansion for displacement field and studied the bending analysis of isotropic and anisotropic beams.
Afterwards, various types of HSDTs were implemented [35, 36]. Ninh and Bich analyzed the nonlinear vibration [37] and nonlinear torsional buckling [38] of eccentrically stiffened (ES) FG toroidal shell segments in thermal environment with the geometrical nonlinearity and surrounded by an elastic medium based on the classical shell theory.

Quasi-3D theories presented as another extension of shear deformation theories by introducing thickness stretching effects in transverse deflection function. The quasi-3D theories can be used by a unified formulation presented by Carrera [39] and consequently developed by Demasi [40]. Karamanli and Vo [41] investigated the flexural behavior of FG micro beams by employing quasi-3D formulation and MCST. They used finite element method and studied different boundary conditions and length scale parameters. Benahmed et.al [42] used a hyperbolic quasi-3D theory for bending and free vibration analysis of FG plates. They also studied the effects of elastic foundation parameters. Nguyen et.al [43] investigated the free vibration and buckling analysis of FG sandwich beams for various boundary conditions by employing Ritz type quasi-3D solution. Farzam and Hassani [44] developed a new HSDT for static, free vibration, and buckling analysis of FG plates with in-plane and through-thickness stiffness variations. Farzam-Rad et.al [45] studied static and free vibration of FG and sandwich plates based on IGA and quasi-3D theory. A quasi-3D shear deformation plate theory combined with MCST theory was employed by Thai et.al [46]. They studied the influence of length-to-thickness ratios, weight fraction values and material length scale-to-thickness ratios on free vibration and buckling behavior of multilayer FG graphene platelet-reinforced composite microplates.

In order to simulate the interaction between beams and elastic foundations, various models have been introduced. The simplest model is Winkler [47] elastic foundation that assumes a series of vertical independent springs. Pasternak [48] offered a more general model which also accounts the
shear interaction between Winkler springs. Atmane et.al [49] studied the effects of porosity and thickness stretching on the static and dynamic response of FG micro beams resting on elastic foundation. Lee et.al [50] studied the static bending response of simply supported plates. The material properties assumed to vary according to exponential power-law and effects of two-parameter Pasternak elastic foundation were investigated. Qingya et.al [51] employed MSGT combined with HSDT in order to study buckling behavior of organic solar cell. The model was rested on Winkler-Pasternak elastic foundation and Galerkin procedure was used to determine critical buckling loads. They also considered thermal effects and stated that mechanical buckling of the organic solar cell was more critical than thermal buckling. Ninh et.al [52] Investigated nonlinear vibration of W-Cu sandwich shell containing heavy water under thermo-mechanical loads. They concluded that the nonlinear response of sandwich shells have been significantly influenced by the geometrical parameters, material, temperature and elastic foundation.

Zeighampour et.al [53] studied the wave propagation in viscoelastic single walled carbon nanotubes resting on a viscoelastic Pasternak foundation. They employed Hamilton’s principle for deriving governing equations and Kelvin–Voigt model for expressing the viscoelastic property. Several studies have been performed for responses of beams and plates resting on elastic foundation and under moving loads [54], micro beams conveying fluid [55], and interaction with viscoelastic foundations [56].

Bich and Ninh performed several studies on static and dynamic analysis of FGM toroidal shell segments, including nonlinear dynamic buckling [57], nonlinear vibration in external thermal environment containing fluid [58] and nonlinear buckling and post-buckling behavior of shells surrounded by elastic foundation [59]. They examined effects of imperfection, fluid, geometrical and material parameters, on the nonlinear behavior of shell segments.
In the present study, a simply supported size dependent beam resting on elastic foundation is considered. The displacement field is based on quasi-3D theory. Static bending responses for different geometrical and foundation parameters were investigated using MSGT. To the best of the authors’ knowledge, effects of foundation parameters on bending behavior of FGM size dependent beams based on quasi-3D approach (accounting for $\varepsilon_{zz} \neq 0$) in conjunction with MSGT have not been studied before. Quasi 3-D theories offer less computational costs compared to 3-D theories; moreover, they offer acceptable accuracies compared to existing approaches. Possible applications of the present model might include curvature sensors, structural health monitoring and implantable Bio-MEMS devices.

2. Problem Formulation

2.1 Quasi-3D displacement field

The beam under study is assumed to have rectangular cross section with length $L$ along $x$ direction and is rested on two-parameter elastic foundation as depicted in Fig.1. Right-handed Cartesian coordinate system is adopted. Width $b$ and thickness $h$ lie along $y$ and $z$ directions, respectively. The FG size dependent beams are generally composed of two different materials at top and bottom surfaces. Gradual changes in material properties for beam from bottom to top surface were estimated according to power law index. Here the Young modulus and Poisson’s ratio are assumed to vary through thickness according to classical rule of mixture and Mori-Tanaka scheme.

The displacement field for quasi-3D HSDTs is commonly considered as follows:

\[
\begin{align*}
    u(x,z) &= u_0(x) - z \frac{d w_b(x)}{dx} - f(z) \frac{d w_s(x)}{dx} \\
    w(x,z) &= w_b(x) + w_s(x) + g(z) w_z(x)
\end{align*}
\]  

(1)
where \( u_o, w_b \) and \( w_s \) are axial displacement, bending and shear parts of transverse displacements, respectively. The contribution of normal strain into the displacement field is accounted by \( g(z)w_z(x,t) \) term. In the present research, \( f(z) \) and \( g(z) \) functions in Equation 1 are assumed as [60]:

\[
f(z) = \frac{h}{\pi} \frac{\sinh(\frac{\pi}{h}z) - z}{\cosh(\frac{\pi}{2}z) - 1} \\
g(z) = 1 - \frac{df(z)}{dz}
\]

Based upon the assumed displacement field in Equation 1, the non-zero strain-displacement terms can be achieved by differentiating respect to the variables \( x \) and \( z \) as below:

\[
\varepsilon_x = \frac{du_0}{dx} - z - \frac{d^2w_b}{dx^2} + f(z) \frac{d^3w_x}{dx^3} \\
\varepsilon_z = \frac{dg(z)}{dz} w_z \\
2\varepsilon_{xz} = g(z)(\frac{dw_z}{dx} + \frac{dw_s}{dx})
\]

2.2 Functionally graded materials

2.2.1 Classical rule of mixture

The effective material properties of the FG beam using classical rule of mixture will be as below:
where subscripts $m$ and $c$ represent metal and ceramic constituents, respectively and $p$ is the gradient power law index in $z$ direction.

### 2.2.2 Mori–Tanaka homogenization scheme

Based on the Mori–Tanaka homogenization scheme, the effective bulk modulus ($K_e$) and the effective shear modulus ($G_e$) of the FG beam are given by [61]:

\[
\begin{align*}
\frac{K_e - K_m}{K_e - K_m} &= \frac{V_c}{1 + V_m (K_c - K_m) / (K_m + 4G_m / 3)} \\
\frac{G_e - G_m}{G_e - G_m} &= \frac{V_c}{1 + V_m (G_c - G_m) / [G_m + G_m (9K_m + 8G_m) / (6(K_m + 2G_m))]]}
\end{align*}
\]

where $V$ is the volume fraction of the phase materials. The volume fraction of metal and ceramic constituents of the FG microbeam can be related as below:

\[V_m + V_c = 1\]

where:

\[
\begin{align*}
V_c(z) &= \left(\frac{z}{h} + \frac{1}{2}\right)^p \\
V_m(z) &= 1 - \left(\frac{z}{h} + \frac{1}{2}\right)^p
\end{align*}
\]
Consequently according to Mori–Tanaka homogenization scheme, the effective Young’s modulus and Poisson’s ratio can be expressed as below:

\[ E(z) = \frac{9K_e G_e}{3K_e + G_e} \]

\[ \nu(z) = \frac{3K_e - 2G_e}{6K_e + 2G_e} \]

### 2.3 The modified strain gradient theory

According to the MSGT presented by Lam et al. [15] stored strain energy in a linear elastic continuum, including higher order terms, can be expressed as below:

\[ U = \frac{1}{2} \int_\Omega (\sigma_{ij} \varepsilon_{ij} + p_i \gamma_i + \tau^{(1)}_{ijk} \eta^{(1)}_{ijk} + m^{s}_{ij} \chi^{s}_{ij}) d\Omega \]

where \( \varepsilon_{ij}, \gamma_i, \eta^{(1)}_{ijk} \) and \( \chi^{s}_{ij} \) are strain tensor, dilatation gradient tensor, deviatoric stretch gradient tensor and symmetric rotation gradient tensor, respectively. where:

\[ \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \]

\[ \gamma_i = \varepsilon_{mm,i} \]

\[ \eta_{ijk} = \eta^{s}_{ijk} - \frac{1}{5}(\delta_{ij} \eta^{s}_{mmk} + \delta_{jk} \eta^{s}_{imi} + \delta_{ik} \eta^{s}_{mmj}) \]

\[ \chi^{s}_{ij} = \frac{1}{4}(e_{ipq} \varepsilon_{aj,p} + e_{jpq} \varepsilon_{ai,p}) \]

in which :
\[
\eta_{ijk} = \frac{1}{3}(u_{i,jk} + u_{j,ki} + u_{k,ij})
\]

According to the assumed displacement field, the non-zero coefficients of Equation 10 will be as below:

\[
\begin{align*}
\varepsilon_{11} &= \frac{du}{dx} - z \frac{d^2 w_b}{dx^2} - f \frac{d^3 w_s}{dx^3} \\
\varepsilon_{13} &= \frac{1}{2} g \left( \frac{dw_s}{dx} + \frac{dw_z}{dz} \right) \\
\varepsilon_{33} &= \frac{1}{2} \frac{dg}{dz} w_z \\
\gamma_1 &= \frac{d^2 u}{dx^2} - z \frac{d^3 w_b}{dx^3} - f \frac{d^4 w_s}{dx^4} + \frac{dg}{dx} \frac{dw_z}{dz} \\
\gamma_3 &= \frac{d^3 w_b}{dx^3} - \frac{df}{dz} \frac{d^2 w_z}{dz^2} + \frac{d^3 g}{dz^2} w_z \\
\eta_{111} &= \frac{2}{5} \frac{d^2 u}{dx^2} - \frac{2}{5} z \frac{d^3 w_b}{dx^3} - \frac{2}{5} \frac{f}{dx} \frac{d^3 w_s}{dx^3} - \frac{2}{5} \frac{dg}{dx} \frac{dw_z}{dz} + \frac{1}{5} \frac{df}{dx} \frac{dw_z}{dz} \\
\eta_{113} = \eta_{311} = \eta_{131} &= -8 \frac{df}{dz} \frac{d^3 w_s}{dx^3} + \frac{4}{15} \frac{d^3 w_s}{dx^3} - \frac{4}{15} \frac{d^3 w_s}{dx^3} - \frac{4}{15} \frac{d^3 w_b}{dx^3} - \frac{1}{15} \frac{d^3 w_b}{dx^3} \\
\eta_{212} = \eta_{221} &= -\frac{1}{5} \frac{d^2 u}{dx^2} + \frac{1}{5} z \frac{d^3 w_b}{dx^3} + \frac{1}{5} \frac{f}{dx} \frac{d^3 w_s}{dx^3} - \frac{2}{5} \frac{dg}{dx} \frac{dw_z}{dz} + \frac{1}{5} \frac{df}{dx} \frac{dw_z}{dz} \\
\eta_{223} = \eta_{322} = \eta_{323} &= \frac{1}{15} \frac{d^3 w_b}{dx^3} + \frac{2}{15} \frac{d^3 w_s}{dx^3} - \frac{1}{15} \frac{d^3 g}{dx} \frac{dw_z}{dz} - \frac{1}{15} \frac{d^3 w_s}{dx^3} - \frac{1}{15} \frac{d^3 w_s}{dx^3} \\
\eta_{333} &= \frac{2}{5} \frac{d^3 g}{dz^2} w_z + \frac{1}{5} \frac{d^3 w_b}{dx^3} + \frac{2}{5} \frac{d^3 w_z}{dx^3} - \frac{1}{5} \frac{d^3 w_s}{dx^3} - \frac{1}{5} \frac{d^3 w_s}{dx^3} \\
\chi_{ij} &= \frac{1}{4} (e_{pq} e_{ij,p} + e_{pq} e_{qi,p})
\end{align*}
\]

The corresponding classical stress field associated with the above strain terms will be as below [62]:

\[\text{[Equation]}\]
And higher order stresses can be expressed as:

\[
\begin{align*}
& p_i = 2\mu l_0^2 \gamma_i, \\
& \tau_{ij} = 2\mu l_1^2 \eta_{ij}, \\
& m_{ij} = 2\mu l_2^2 \chi_{ij},
\end{align*}
\]

where \( l_0, l_1, l_2 \) are three length scale parameters in MSGT. This theory can be converted to MCST by letting \( l_0 = l_1 = 0 \) and CBT by letting \( l_0 = l_1 = l_2 = 0 \).

### 2.4 Governing Equations

In order to obtain the governing equations, the minimum total potential energy principle is used as below:

\[
\delta \int \left( U + U_f - W \right) dt = 0
\]

where \( U \) is the strain energy, \( U_f \) is the strain energy of elastic foundation and \( W \) is the work done by the external applied forces. The strain energy induced by elastic foundation will be:

\[
U_f = \frac{1}{2} \int_A k_w \left( w_b^2 + w_s^2 \right) dA + \frac{1}{2} \int_A k_p \left( \frac{d(w_b + w_s)}{dx} \right)^2 dA
\]

where \( k_w \) and \( k_p \) are Winkler and Pasternak stiffness parameters of the elastic foundation, respectively. Work done by external force can be obtained as:
Substituting Equations 9, 16 & 17 in Equation 15, using some mathematical process and employing integration by part technique, the following governing equations and corresponding boundary conditions can be achieved:

\[
\delta u : -\frac{dN_{11}}{dx} + \frac{d^2P_1}{dx^2} + \frac{2 d^2T_{111}}{5 dx^2} - \frac{3 d^2T_{133}}{5 dx^2} - \frac{3 d^2T_{122}}{5 dx^2} = 0
\]  

(18a)

\[
\delta w_b : -\frac{d^2N_{11}^b}{dx^2} - \frac{d^2M_{12}}{dx^2} - \frac{d^2P_{1}^a}{dx^2} + \frac{2 d^2T_{111}^a}{5 dx^3} - \frac{4 d^2T_{133}^a}{5 dx^3} - \frac{3 d^2T_{133}^a}{5 dx^3} - \frac{3 d^2T_{122}^a}{5 dx^3} \\
+ \frac{1 d^2T_{223}}{5 dx^2} + \frac{1 d^2T_{333}}{5 dx^2} + k_w (w_b + w_s) - k_p (\frac{d^3w_b}{dx^2} + \frac{d^3w_s}{dx^2}) - q = 0
\]  

(18b)

\[
\delta w_s : -\frac{d^2N_{11}^b}{dx^2} + 2 \frac{dS_{13}}{dx} - \frac{d^2M_{12}}{dx^2} + \frac{1 d^2M_{12}^a}{2 dx^2} - \frac{1 dM_{23}}{2 dx} - \frac{d^2P_{1}^b}{dx^2} + \frac{d^2P_{3}^b}{dx^2} + \frac{2 d^2T_{111}^b}{5 dx^3} \\
- \frac{1 dT_{111}^d}{5 dx^2} - \frac{8 d^2T_{133}^b}{5 dx^2} + \frac{4 d^2T_{133}^d}{5 dx^2} + \frac{3 d^2T_{133}^d}{5 dx^2} + \frac{3 d^2T_{133}^d}{5 dx^2} - \frac{1 dT_{122}^d}{5 dx^2} + \frac{2 d^2T_{223}^b}{5 dx^2} \\
- \frac{1 dT_{223}^b}{5 dx^2} + \frac{2 d^2T_{333}^b}{5 dx^2} + \frac{1 d^2T_{333}^b}{5 dx^2} + k_w (w_b + w_s) - k_p (\frac{d^3w_b}{dx^2} + \frac{d^3w_s}{dx^2}) - q = 0
\]  

(18c)

\[
\delta w_c : -\frac{2 dS_{13}}{dx} + R_{33} - \frac{1 d^2M_{12}^c}{2 dx^2} - \frac{1 dM_{23}}{2 dx} - \frac{dP_{1}^c}{dx} + \frac{P_{3}^c}{dx} + \frac{2 dT_{111}^c}{5 dx^2} + \frac{4 d^2T_{133}^c}{5 dx^2} - \frac{3 d^2T_{133}^c}{5 dx^2} \\
- \frac{8 dT_{133}^c}{5 dx^2} + \frac{2 d^2T_{122}^c}{5 dx^2} - \frac{3 dT_{223}^c}{5 dx^2} + \frac{1 d^2T_{223}^c}{5 dx^2} + \frac{2 d^2T_{333}^c}{5 dx^2} - \frac{1 d^2T_{333}^c}{5 dx^2} = 0
\]  

(18d)
where stress resultants in above equations can be calculated by integrating through the thickness as below:

\[
\begin{align*}
\{N_{11}, N_{11}^a, N_{11}^b\} &= \int_{-h/2}^{h/2} \{\sigma_{11}, \sigma_{11}^z, \sigma_{11} f\} \, dA \\
S_{13} &= \int_{-h/2}^{h/2} \sigma_{13} g \, dA \\
R_{33} &= \int_{-h/2}^{h/2} \sigma_{33} \frac{dg}{dz} \, dA \\
\{M_{12}, M_{12}^a, M_{23}\} &= \int_{-h/2}^{h/2} \{m_{12}, \sigma_{11} g, \sigma_{23}\} \, dA \\
\{P_1, P_1^a, P_1^b, P_1^c\} &= \int_{-h/2}^{h/2} \left\{p_1, p_1^z, p_1 f, p_1 \frac{dg}{dz}\right\} \, dA \\
\{P_3, P_3^a, P_3^b\} &= \int_{-h/2}^{h/2} \left\{p_3, p_3^f, p_3 \frac{dg}{dz}, p_3 \frac{d^2g}{dz^2}\right\} \, dA \\
\{T_{111}, T_{111}^a, T_{111}^b, T_{111}^c, T_{111}^d\} &= \int_{-h/2}^{h/2} \left\{\tau_{111}, \tau_{111}^z, \tau_{111} f, \tau_{111} \frac{dg}{dz}, \tau_{111} \frac{d^2f}{dz^2}\right\} \, dA \\
\{T_{113}, T_{113}^a, T_{113}^b, T_{113}^c, T_{113}^d\} &= \int_{-h/2}^{h/2} \left\{\tau_{113}, \tau_{113}^g, \tau_{113} f, \tau_{113} \frac{dg}{dz}, \tau_{113} \frac{d^2f}{dz^2}\right\} \, dA \\
\{T_{133}, T_{133}^a, T_{133}^b, T_{133}^c, T_{133}^d\} &= \int_{-h/2}^{h/2} \left\{\tau_{133}, \tau_{133}^z, \tau_{133} f, \tau_{133} \frac{dg}{dz}, \tau_{133} \frac{d^2f}{dz^2}\right\} \, dA \\
\{T_{122}, T_{122}^a, T_{122}^b, T_{122}^c, T_{122}^d\} &= \int_{-h/2}^{h/2} \left\{\tau_{122}, \tau_{122}^z, \tau_{122} f, \tau_{122} \frac{dg}{dz}, \tau_{122} \frac{d^2f}{dz^2}\right\} \, dA \\
\{T_{223}, T_{223}^a, T_{223}^b, T_{223}^c, T_{223}^d\} &= \int_{-h/2}^{h/2} \left\{\tau_{223}, \tau_{223}^g, \tau_{223} \frac{df}{dz}, \tau_{223} \frac{d^2g}{dz^2}\right\} \, dA \\
\{T_{333}, T_{333}^a, T_{333}^b, T_{333}^c\} &= \int_{-h/2}^{h/2} \left\{\tau_{333}, \tau_{333} g, \tau_{333} \frac{df}{dz}, \tau_{333} \frac{d^2g}{dz^2}\right\} \, dA
\end{align*}
\]

By Substituting Equations 12-14 into Equation 19, the components of classical stress resultants in terms of displacements can be obtained as follows:
\[
\begin{align*}
\{N_{11}\} &= \begin{bmatrix} A_1 \\
A_2 \\
A_3 \end{bmatrix} \frac{du}{dx} - \begin{bmatrix} A_4 \\
A_5 \\
A_6 \end{bmatrix} \frac{d^2 w}{dx^2} - \begin{bmatrix} A_7 \\
A_8 \\
A_9 \end{bmatrix} \frac{d^3 v}{dx^3} + \begin{bmatrix} B_1 \\
B_2 \\
B_3 \end{bmatrix} w_z \\
\{N'_{11}\} &= \begin{bmatrix} A_1 \\
A_2 \\
A_3 \end{bmatrix} \frac{du}{dx} - \begin{bmatrix} A_4 \\
A_5 \\
A_6 \end{bmatrix} \frac{d^2 w}{dx^2} - \begin{bmatrix} A_7 \\
A_8 \\
A_9 \end{bmatrix} \frac{d^3 v}{dx^3} + \begin{bmatrix} B_1 \\
B_2 \\
B_3 \end{bmatrix} w_z
\end{align*}
\]

\[
R_{33} = B_1 \frac{du}{dx} - B_2 \frac{d^2 w}{dx^2} - B_3 \frac{d^3 v}{dx^3} + A_3 \frac{d^3 v}{dx^3}
\]

\[
S_{13} = \frac{1}{2} D_8 \left( \frac{dw_x}{dx} + \frac{dw_y}{dy} \right)
\]

and the higher order stress resultants can be achieved as below:
\[
P_{1} = 2 I_0^2 (D_1 - D_3) \frac{d^2u}{dx^2} - \frac{D_4}{D_5} \frac{d^3w_b}{dx^3} - \frac{D_6}{D_7} \left( \frac{d^3w}{dx^3} + \frac{d^3w}{dx^3} \right) + \frac{D_8}{D_9} \left( \frac{d^3w}{dx^3} - \frac{d^3w}{dx^3} \right)
\]

\[
M_{23} = \frac{1}{2} \int (\frac{d^2w_s}{dx^2} - \frac{d^2w_z}{dx^2})
\]

\[
T_{113} = 2 I_1^2 \left( \begin{array}{c} D_1 \\ D_2 \\ D_3 \\ E_1 \\ E_2 \\ D_4 \\ E_3 \\ D_5 \\ D_6 \\ E_4 \\ D_7 \\ E_5 \\ D_8 \\ E_6 \\ D_9 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ \end{array} \right) \frac{dw_z}{dx}
\]

\[
T_{223} = 2 I_1^2 \left( \begin{array}{c} D_1 \\ D_2 \\ D_3 \\ E_1 \\ E_2 \\ D_4 \\ E_3 \\ D_5 \\ D_6 \\ E_4 \\ D_7 \\ E_5 \\ D_8 \\ E_6 \\ D_9 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ \end{array} \right) \frac{dw_z}{dx}
\]

\[
T_{333} = 2 I_1^2 \left( \begin{array}{c} D_1 \\ D_2 \\ D_3 \\ E_1 \\ E_2 \\ D_4 \\ E_3 \\ D_5 \\ D_6 \\ E_4 \\ D_7 \\ E_5 \\ D_8 \\ E_6 \\ D_9 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ \end{array} \right) \frac{dw_z}{dx}
\]
where:

\[
\{A_1, A_2, A_3, A_4, A_5, A_6, A_7\} = \int_A \frac{E(z)}{1 - v^2(z)} \left\{ 1, z, f, z f, z^2, f^2, \left( \frac{d g}{dz} \right)^2 \right\} dA
\]

\[
\{B_1, B_2, B_3\} = \int_A \frac{E(z) \nu(z)}{1 - v^2(z)} \left\{ \frac{d g}{dz} - \frac{d g}{dz}, z, \frac{f}{dz} \frac{d g}{dz} \right\} dA
\]

\[
\{C_1, C_2, C_3, C_4, C_5, C_6\} = \int_A \frac{E(z) \nu^2(z)}{1 - v^2(z)} \left\{ 1, g, g^2, \frac{df}{dz}, \frac{g}{dz} \frac{df}{dz}, \left( \frac{df}{dz} \right)^2 \right\} dA
\]

\[
\{D_1, D_2, D_3, D_4, D_5, D_6, D_7, D_8\} = \int_A \mu \left\{ 1, z, f, z f, z^2, f^2, g^2 \right\} dA
\]

\[
\{E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8\} = \int_A \mu \left\{ \frac{df}{dz}, \frac{d g}{dz}, \frac{d g}{dz}, \frac{df}{dz}, \frac{df}{dz}, \frac{df}{dz}, \left( \frac{df}{dz} \right)^2, \left( \frac{df}{dz} \right)^2 \right\} dA
\]

\[
\{F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8\} = \int_A \mu \left\{ \frac{d f}{dz}, \frac{d g}{dz}, \frac{d g}{dz}, \frac{d f}{dz}, \frac{d f}{dz}, \frac{d g}{dz}, \frac{d g}{dz}, \left( \frac{df}{dz} \right)^2, \left( \frac{df}{dz} \right)^2 \right\} dA
\]

and corresponding boundary conditions at the beam ends (x=0, L) are expressed as:

\[
\delta u = 0 \quad \text{or} \quad N_{11} = \frac{d P_1}{dx} - \frac{2}{5} \frac{d T_{111}}{dx} + \frac{3}{5} \frac{d T_{133}}{dx} + \frac{3}{5} \frac{d T_{122}}{dx} = 0
\] (22a)

\[
\frac{\delta (du)}{dx} = 0 \quad \text{or} \quad P_1 + \frac{2}{5} T_{111} - \frac{3}{5} T_{133} - \frac{3}{5} T_{122} = 0\] (22b)

\[
\delta w_b = 0 \quad \text{or} \quad \frac{d N_{11}^a}{dx} + \frac{d M_{12}}{dx} - \frac{2}{5} \frac{d T_{111}^a}{dx} + \frac{4}{5} \frac{d T_{113}^a}{dx} + \frac{3}{5} \frac{d T_{133}^a}{dx} + \frac{3}{5} \frac{d T_{122}^a}{dx} - \frac{1}{5} \frac{d T_{223}^a}{dx} - \frac{1}{5} \frac{d T_{333}^a}{dx} - \frac{d^2 P_1^a}{dx} = 0
\] (22c)
\[
\delta \left( \frac{dW_{lb}}{dx} \right) = 0 \quad \text{(22d)}
\]

or

\[
-N_{11}^a - M_{12}^a + \frac{2}{5} \frac{dT_{111}^a}{dx} - \frac{4}{5} \frac{T_{113}^a}{dx} - \frac{3}{5} \frac{dT_{133}^a}{dx} - \frac{3}{5} \frac{dT_{122}^a}{dx} + \frac{1}{5} \frac{T_{223}^a}{dx} + \frac{1}{5} \frac{T_{333}^a}{dx} + \frac{dP_1^a}{dx} = 0
\]

\[
\delta \left( \frac{d^2 W_{lb}}{dx^2} \right) = 0 \quad \text{or} \quad -\frac{2}{5} T_{111}^a + \frac{3}{5} T_{133}^a + \frac{3}{5} T_{122}^a + \frac{1}{5} T_{223} - P_1^a = 0 \quad \text{(22e)}
\]

\[
\delta w_z = 0 \quad \text{(22f)}
\]

or

\[
\frac{dN_{11}^b}{dx} - 2S_{13} + \frac{dM_{12}^a}{dx} - \frac{1}{2} \frac{dM_{12}^a}{dx} + \frac{1}{2} M_{23}^a + \frac{dP_1^b}{dx} - \frac{2}{5} \frac{dT_{111}^b}{dx} + \frac{1}{5} T_{111}^a + \frac{8}{5} \frac{dT_{133}^b}{dx} - \frac{4}{5} \frac{dT_{133}^b}{dx} - \frac{4}{5} \frac{dT_{133}^b}{dx} - \frac{3}{5} \frac{dT_{133}^b}{dx} - \frac{3}{5} \frac{dT_{223}^b}{dx} - \frac{2}{5} \frac{dT_{223}^b}{dx} - \frac{2}{5} \frac{dT_{333}^b}{dx} - \frac{1}{5} T_{333}^a = 0
\]

\[
\delta \left( \frac{dW_{lc}}{dx} \right) = 0 \quad \text{(22g)}
\]

or

\[
-N_{11}^b - M_{12}^a + \frac{1}{2} M_{12}^a - P_3^a + \frac{2}{5} \frac{dT_{111}^b}{dx} - \frac{8}{5} \frac{T_{113}^b}{dx} + \frac{4}{5} \frac{T_{113}^b}{dx} + \frac{3}{5} \frac{T_{133}^b}{dx} + \frac{dP_1^b}{dx} - \frac{3}{5} \frac{dT_{122}^b}{dx} + \frac{2}{5} T_{223}^b - \frac{1}{5} T_{223}^b + \frac{2}{5} T_{333}^b = 0
\]

\[
\delta \left( \frac{d^2 W_{lc}}{dx^2} \right) = 0 \quad \text{or} \quad -\frac{2}{5} T_{111}^b - P_1^b + \frac{3}{5} T_{122}^b = 0 \quad \text{(22h)}
\]
\[ \delta w_z = 0 \] (22i)

or

\[ -2S_{13} + \frac{1}{2} \frac{dM_{12}^a}{dx} - \frac{1}{2} M_{23} + P_c^e - \frac{2}{5} T_{111}^e - \frac{4}{5} \frac{dT_{113}^a}{dx} + \frac{8}{5} T_{133}^e - \frac{2}{5} T_{122}^e + \frac{1}{5} \frac{dT_{223}^a}{dx} + \frac{1}{5} \frac{dT_{333}^a}{dx} = 0 \]

\[ \delta \left( \frac{dw_z}{dx} \right) = 0 \quad \text{or} \quad -\frac{1}{2} M_{12}^a + \frac{4}{5} T_{113}^a - \frac{1}{5} T_{223}^a - \frac{1}{5} T_{333}^a = 0 \] (22j)

### 3. Solution Methodology

In this section, the Navier exact closed form solution is employed in order to automatically satisfy the simply supported boundary conditions. In the latter method, the displacement terms are expressed as functions with undetermined coefficients. The beam is supposed to be simply supported at \( x = 0, L \). The expansion of displacement field is expressed as:

\[
\begin{align*}
\delta w_c &= 0 \\
\delta \left( \frac{dw_z}{dx} \right) &= 0 \\
\delta \left( \frac{dw_y}{dx} \right) &= 0 \\
\delta \left( \frac{dw_x}{dx} \right) &= 0
\end{align*}
\]

where \( u_m, w_{bm}, w_{sm}, \text{ and } w_{zm} \) are arbitrary parameters to be determined. Also the transverse applied load can be expanded in Fourier series as:

\[ q(x) = \sum_{n=1}^{N} Q_n \sin \left( \frac{n \pi}{L} x \right) \] (24)
where:

\[ Q_n = q_0 \quad \text{for single sine load} \]
\[ Q_n = \frac{4q_0}{n\pi} \quad n = 1, 3, 5, \ldots \quad \text{for uniformly distributed load} \]
\[ Q_n = \frac{2P}{L} \sin \frac{n\pi x}{L} \quad n = 1, 2, 3, \ldots \quad \text{for point load at the middle} \]

4. Results and discussion

In order to demonstrate the accuracy of the present formulation, results for bending of micro beams in special case \((l_0 = l_1 = 0)\) are compared to the results of Ref.[62]. The FG beam is composed of Sic/Al and associated material properties are expressed in Equation 26. The classical rule of mixture and Mori-Tanaka scheme are employed to estimate material properties through the thickness. The material length scale parameter in the present study is considered to be \(l = 15 \mu m\)[63]. Comparison of results for different aspect ratios and dimensionless size dependent parameters for different power indexes are presented in Table 1. It can be observed that there is good agreement between results for both classical rule of mixture and Mori-Tanaka scheme.

\[ E_c = 427 GPa \] \hspace{1cm} (26)
\[ \nu_c = 0.17 \]
\[ E_m = 70 GPa \]
\[ \nu_m = 0.3 \]

The dimensionless transverse deflections are calculated as below:
\[ w = \frac{100E_{m}h^{3}}{12q_{0}L^{4}}w (x,0) \]
\[ w^{n} = \frac{100E_{m}h^{3}}{12PL^{3}}w (x,0) \]
\[ \bar{\sigma}_{xx} = \frac{\sigma_{xx}h}{q_{0}L} (x,z) \]
\[ \bar{\sigma}_{xz} = \frac{\sigma_{xz}h}{q_{0}L} (x,z) \]
\[ \bar{k}_{w} = \frac{k_{w}L^{4}}{EI} \]
\[ \bar{k}_{p} = \frac{k_{p}L^{2}}{EI} \]

Table 2 lists dimensionless deflections of MSGT and MCST theories for different aspect ratios \((L/h)\), dimensionless size dependent parameters \((h/l)\) and power index numbers. The dimensionless deflections for both MSGT and MCST theories increase with the increase of power index numbers and dimensionless size dependent parameters \((h/l)\). For simplicity and in order to compare the results of different theories, it was assumed that \((l_{0}=l_{1}=l_{2}=l)\) for MSGT.

Table 3 contains effects of foundation parameters and power law indexes on dimensionless deflection for different theories under point load at the middle of the beam based on classical rule of mixture. Remarkable variations have been observed for results of CBT; whereas results of MSGT have been underwent relatively little changes. Also effects of dimensionless Pasternak shear parameter \((\bar{k}_{p})\) have been more noticeable than dimensionless Winkler parameter \((\bar{k}_{w})\).

Fig. 2 illustrates the dimensionless deflections for different theories. Results of MSGT and MCST theories are closer to each other compared to the results of CBT. Classical continuum theory is not capable of approximating size dependent behavior and assumes the beam under study more
flexible (less stiff) than higher order continuum theories (MSGT, MCST). For different theories (CBT, MSGT, MCST), Mori-Tanaka scheme offers higher results than classical rule of mixture for dimensionless deflections.

Fig. 3 and Fig. 4 show the changes of dimensionless axial and normal stresses through dimensionless thickness \((L=10h)\) based on MSGT theory and under single sine load \(q_0=10\text{ N/m}^2\) for different power indexes, respectively. An increase in power index number results in material constituents tend from ceramic to metal and both axial and normal stresses undergo more obvious changes through non-dimensional thickness. For the present dimensionless thickness \((L=10h)\), that thickness is comparable to the length of size dependent beam, rate of changes for normal stresses are more pronounced than axial stresses. The material properties of the size dependent FG beams are very important in the bending response, especially in higher gradient indexes \((p=10)\) the rate of variations from bottom to top surface are remarkable.

Fig. 5 and Fig. 6 depict the variation of non-dimensional axial and transverse shear stresses through dimensionless thickness based on MSGT theory for \(p=1\) and \(L=10h\) under single sine load \(q_0=10\text{ N/m}^2\), respectively. Effects of foundation parameters were investigated for both dimensionless axial and transverse shear stresses through non-dimensional thickness of the FG size dependent beams.

In Fig. 5 and Fig. 6, inclusion of foundation parameters, result in decrease in variation of non-dimensional axial and transverse shear stresses through dimensionless thickness, respectively. Also calculated dimensionless axial, normal and transverse shear stresses using Mori-Tanaka scheme follow the same trends as the classical rule of mixture, albeit mostly with slightly higher values. Pasternak parameters (shear layer) have been resulted in reduction of the dimensionless
axial stresses in magnitude compared to Winkler elastic foundation and the situation that the elastic foundation is neglected. Results of dimensionless transverse shear stresses (Fig.6) approach each other for Mori-Tanaka and classical rule of mixture schemes as the foundation parameters \((K_w, K_p)\) increase.

5. Conclusion

In the present study static bending of a functionally graded size dependent beam is performed by using modified strain gradient theory. Different types of loading, volume fraction indexes, material properties schemes and elastic foundation parameters were investigated for simply supported beams. Stretching effects are also considered in displacement field, which are noticeable in the case of thick beams. Parametric study showed that impact of the Pasternak coefficient is more prominent than the Winkler one for bending behavior. Results of the present paper compared to the existing ones in special cases (MCST and CBT). It can be seen that the beam behaves stiffer in MSGT than MCST theory, because the additional higher order terms are considered. The MCST only involves the rotation gradients as higher order terms in governing equations, while CBT is not capable of predicting small scale behavior. The obtained results showed that inclusion of foundation parameters, result in decrease in variation of non-dimensional axial and transverse shear stresses through dimensionless thickness.
6. References

[1] Patocka, F., Schneidhofer, C. and Dörr, N. et.al, “Novel resonant MEMS sensor for the detection of particles with dielectric properties in aged lubricating oils”, Sens. Actuators A Phys., 315, pp.112290 (2020).

[2] Ulkir, O. “Design and fabrication of an electrothermal MEMS micro-actuator with 3D printing technology”, Mater. Res. Express, 7, pp. 075015 (2020).

[3] Moutlana, MK. and Adali, S. “Fundamental frequencies of a nano beam used for atomic force microscopy (AFM) in tapping mode”, MRS Adv., 3, pp. 2617-26 (2018).

[4] Li, X-F., Wang, B-L. and Lee, KY. “Size Effect in the Mechanical Response of Nanobeams”, J. of Adv. Res. in Mech. Eng., 1(1), pp. 4-16 (2010).

[5] Dell’Isola, F., Andreaus, U. and Placidi, L. “At the origins and in the vanguard of peridynamics, non-local and higher-gradient continuum mechanics: An underestimated and still topical contribution of Gabrio Piola”, Math. Mech. Solids, 20(8), pp. 887-928 (2015).

[6] Cosserat, E. and Cosserat F. “Théorie des corps déformables”, (1909).

[7] Thai, H-T., Vo, TP. and Nguyen, T-K. et.al, “A review of continuum mechanics models for size-dependent analysis of beams and plates”, Compos. Struct., 177, pp. 196-219 (2017).

[8] Mindlin, RD. “Second gradient of strain and surface-tension in linear elasticity”, Int. J. Solids Struct., 1(4), pp. 417-38 (1965).

[9] Toupin, R. “Elastic materials with couple-stresses”, (1962).

[10] Mindlin, R. and Tiersten, H. “Effects of couple-stresses in linear elasticity”, Arch. Ration. Mech. Anal, 11, pp. 415-48 (1962).

[11] Koiter, W. T. “Couple-stresses in the theory of elasticity, I & II”, pp. 17-44 (1969).

[12] Yang, F., Chong, A. and Lam, DCC. et.al, “Couple stress based strain gradient theory for elasticity”, Int. J. Solids Struct., 39(10), pp. 2731-43 (2002).

[13] Fleck, N. and Hutchinson, J. “Strain gradient plasticity”, Adv. Appl. Mech., 33, (1997).

[14] Fleck, N. and Hutchinson, J. “A reformulation of strain gradient plasticity”, J. Mech. Phys. Solids, 49(10), pp. 2245-71 (2001).

[15] Lam, DC., Yang, F. and Chong, A. et.al, “Experiments and theory in strain gradient elasticity”, J. Mech. Phys. Solids., 51(8), pp. 1477-508 (2003).

[16] Dal, Hüsnü. “Analysis of Gold Micro-Beams with Modified Strain Gradient Theory”, Anadolu Üniversitesi Bilim Ve Teknoloji Dergisi A-Uygulamali Bilimler ve Mühendislik, 18(3), pp. 663-681 (2017).
[17] Ashoori, A. and Mahmoodi, M. “A nonlinear thick plate formulation based on the modified strain gradient theory”, *Mech. Adv. Mater. Struct.*, 25(10), pp. 813-9 (2018).

[18] Chu, L., Dui, G. and Ju, C. “Flexoelectric effect on the bending and vibration responses of functionally graded piezoelectric nanobeams based on general modified strain gradient theory”, *Compos. Struct.*, 186, pp. 39-49 (2018).

[19] Thai, CH., Ferreira, A. and Phung-Van, P. “Free vibration analysis of functionally graded anisotropic microplates using modified strain gradient theory”, *Eng. Anal. Bound. Elem.*, 117, pp. 284-98 (2020).

[20] Farzam, A. and Hassani, B. “Size-dependent analysis of FG microplates with temperature-dependent material properties using modified strain gradient theory and isogeometric approach”, *Compos. B. Eng.*, 161, pp. 150-68 (2019).

[21] Cornacchia, F., Fantuzzi, N. and Luciano, R. et.al, “Solution for cross-and angle-ply laminated Kirchhoff nano plates in bending using strain gradient theory”, *Compos. B. Eng.*, 173, pp. 107006 (2019).

[22] Thai, CH., Ferreira, A. and Phung-Van, P. “A nonlocal strain gradient isogeometric model for free vibration and bending analyses of functionally graded plates”, *Compos. Struct.*, 251, pp. 112634 (2020).

[23] Phung-Van, P., Ferreira, A. and Nguyen-Xuan, H. et.al, “An isogeometric approach for size-dependent geometrically nonlinear transient analysis of functionally graded nanoplates”, *Compos. B. Eng.*, 118, pp. 125-34 (2017).

[24] Phung-Van, P., Thai, CH. and Nguyen-Xuan, H. et.al, “Porosity-dependent nonlinear transient responses of functionally graded nanoplates using isogeometric analysis”, *Compos. B. Eng.*, 164, pp. 215-25 (2019).

[25] Phung-Van, P., Thai, CH. and Nguyen-Xuan, H. et.al, “An isogeometric approach of static and free vibration analyses for porous FG nanoplates”, *Eur J Mech A Solids*, 78, pp. 103851 (2019).

[26] Farzam, A. and Hassani, B. “Isogeometric analysis of in-plane functionally graded porous microplates using modified couple stress theory”, *Aerosp Sci Technol.*, 91, pp. 508-24 (2019).

[27] Farzam A. and Hassani, B. “Thermal and mechanical buckling analysis of FG carbon nanotube reinforced composite plates using modified couple stress theory and isogeometric approach”, *Compos. Struct.*, 206, pp. 774-90 (2018).

[28] Phung-Van, P., Thai, CH. and Wahab, MA. et.al, “Optimal design of FG sandwich nanoplates using size-dependent isogeometric analysis”, *Mech. Mater.*, 142, pp. 103277 (2020).

[29] Phung-Van, P., Ferreira, A. and Thai, CH. “Computational optimization for porosity-dependent isogeometric analysis of functionally graded sandwich nanoplates”, *Compos. Struct.*, 239, pp. 112029 (2020).

[30] Zhao, X., Zheng, S. and Li, Z. “Size-dependent nonlinear bending and vibration of flexoelectric nanobeam based on strain gradient theory”, *Smart Mater. Struct.*, 28, pp. 075027 (2019).
[31] Zanoosi, AAP. “Size dependent thermo mechanical free vibration analysis of functionally graded porous microbeams based on modified strain gradient theory”, *J. Braz. Soc. Mech. Sci.*, **42**(5), pp. 236 (2020).

[32] Timoshenko, S. P. “On the correction for shear of the differential equation for transverse vibrations of prismatic bars”, *Philos. Mag.*, **41**(245), pp. 744-746 (1921).

[33] Nam, VH., Vinh, PV. and Chinh, NV. et.al, “A New Beam Model for Simulation of the Mechanical Behaviour of Variable Thickness Functionally Graded Material Beams Based on Modified First Order Shear Deformation Theory”, *Materials*, **12**(3), pp. 404 (2019).

[34] Reddy, J. “A general non-linear third-order theory of plates with moderate thickness”, *Int. J. Nonlin. Mech.*, **25**(6), pp. 677-86 (1990).

[35] Ghugal, YM. and Sharma, R. “A hyperbolic shear deformation theory for flexure and vibration of thick isotropic beams”, *Int. J. Comput. Methods*, **6**(04), pp. 585-604 (2009).

[36] Soldatos, K. “A transverse shear deformation theory for homogeneous monoclinic plates”, *Acta Mech.*, **94**(3-4), pp. 195-220 (1992).

[37] Ninh, DG. and Bich, DH. “Nonlinear thermal vibration of eccentrically stiffened ceramic-FGM-metal layer toroidal shell segments surrounded by elastic foundation”, *Thin-Walled Struct.*, **104**, pp. 198-210 (2016).

[38] Ninh, DG. and Bich, DH. “Nonlinear buckling of eccentrically stiffened functionally graded toroidal shell segments under torsional load surrounded by elastic foundation in thermal environment”, *Mech. Res. Commun.*, **72**, pp. 1-15 (2016).

[39] Carrera, E. “Theories and finite elements for multilayered plates and shells: a unified compact formulation with numerical assessment and benchmarking”, *Arch. Comput. Methods Eng.* **10**(3), pp. 215-96 (2003).

[40] Demasi, L. “∞ 6 mixed plate theories based on the generalized unified formulation. Part I: governing equations”, *Compos. Struct.*, **87**(1), pp. 1-11 (2009).

[41] Karamanli, A. and Vo, TP. “Size dependent bending analysis of two directional functionally graded microbeams via a quasi-3D theory and finite element method”, *Compos. B. Eng.*, **144**, pp. 171-83 (2018).

[42] Benahmed, A., Houari, MSA. and Benyoucef, S. et.al, “A novel quasi-3D hyperbolic shear deformation theory for functionally graded thick rectangular plates on elastic foundation”, *Geomech. Eng.*, **12**(1), pp. 9-34 (2017).

[43] Nguyen, T-K., Vo, TP. and Nguyen, B-D. et.al, “An analytical solution for buckling and vibration analysis of functionally graded sandwich beams using a quasi-3D shear deformation theory”, *Compos. Struct.*, **156**, pp. 238-52 (2016).
[44] Farzam, A. and Hassani, B. “A new efficient shear deformation theory for FG plates with in-plane and through-thickness stiffness variations using isogeometric approach”, Mech. Adv. Mater. Struct., 26, pp. 512-25 (2019).

[45] Farzam-Rad, SA., Hassani, B. and Karamodin A. “Isogeometric analysis of functionally graded plates using a new quasi-3D shear deformation theory based on physical neutral surface”, Compos. B. Eng., 108, pp.174-89 (2017).

[46] Thai, CH., Ferreira, A. and Tran, T. et.al, “A size-dependent quasi-3D isogeometric model for functionally graded graphene platelet-reinforced composite microplates based on the modified couple stress theory”, Compos. Struct., 234, pp. 111695 (2020).

[47] Winkler, E. “Theory of elasticity and strength”, Dominicus Prague, Czechoslovakia, (1867).

[48] Pasternak, P. “On a New Method of Analysis of an Elastic Foundation by Means of Two Foundation Constants (in Russian), Gosudarstvennoe Izdatelstvo Literaturi po Stroitelstvu i Arkhitekture, USSR”, (1954).

[49] Atmane, HA., Tounsi, A. and Bernard, F. “Effect of thickness stretching and porosity on mechanical response of a functionally graded beams resting on elastic foundations”, Int. J. Mech. Mater. Des., 13(1), pp. 71-84 (2017).

[50] Lee, W-H., Han, S-C. and Park, W-T. “A refined higher order shear and normal deformation theory for E-, P-, and S-FGM plates on Pasternak elastic foundation”, Compos. Struct., 122, pp. 330-42 (2015).

[51] Li, Q., Wu, D. and Gao, W. et.al, “Size-dependent instability of organic solar cell resting on Winkler–Pasternak elastic foundation based on the modified strain gradient theory”, Int. J. Mech. Sci., 177, pp. 105306 (2020).

[52] Ninh, DG., Tien, ND. and Hoang, VNV. et.al, “Vibration of cylindrical shells made of three layers W-Cu composite containing heavy water using Flügge-Lur'e-Byrne theory”, Thin-Walled Struct., 146, pp. 106414 (2020).

[53] Zeighampour, H., Beni, YT. and Dehkordi, MB. “Wave propagation in viscoelastic thin cylindrical nanoshell resting on a visco-Pasternak foundation based on nonlocal strain gradient theory”, Thin-Walled Struct., 122, pp. 378-86 (2018).

[54] Cao, C-Y. and Zhong, Y. “Dynamic response of a beam on a Pasternak foundation and under a moving load”, J. of Chongqing Univ., 7(4), pp. 311-6 (2008).

[55] Kural, S. and Özkaya, E. “Size-dependent vibrations of a micro beam conveying fluid and resting on an elastic foundation”, J. Vib. Control, 23(7), pp. 1106-14 (2017).

[56] Eyebe, G., Betchewe, G. and Mohamadou, A. et.al, “Nonlinear vibration of a nonlocal nanobeam resting on fractional-order viscoelastic Pasternak foundations”, Fractal Fract., 2(3), pp. 21 (2018).

[57] Bich, DH. and Ninh, DG. “Research on dynamical buckling of imperfect stiffened three-layered toroidal shell segments containing fluid under mechanical loads”, Acta Mech., 228, pp. 711-30 (2017).
[58] Bich, DH. Ninh, DG. “An analytical approach: Nonlinear vibration of imperfect stiffened FGM sandwich toroidal shell segments containing fluid under external thermo-mechanical loads”, *Compos. Struct.*, 162, pp. 164-81 (2017).

[59] Bich, DH. and Ninh, DG. “Post-buckling of sigmoid-functionally graded material toroidal shell segment surrounded by an elastic foundation under thermo-mechanical loads”, *Compos. Struct.*, 138, pp. 253-63 (2016).

[60] El Meiche, N., Tounsi, A. and Ziane, N. et.al, “A new hyperbolic shear deformation theory for buckling and vibration of functionally graded sandwich plate”, *Int. J. Mech. Sci.*, 53(4), pp. 237-47 (2011).

[61] Mori, T. and Tanaka, K. “Average stress in matrix and average elastic energy of materials with misfitting inclusions”, *Acta Metall.*, 21(5), pp. 571-4 (1973).

[62] Trinh, LC., Nguyen, HX. and Vo, TP. et.al, “Size-dependent behavior of functionally graded microbeams using various shear deformation theories based on the modified couple stress theory”, *Compos. Struct.*, 154, pp. 556-72 (2016).

[63] Ke, L-L. and Wang, Y-S. “Size effect on dynamic stability of functionally graded microbeams based on a modified couple stress theory”, *Compos. Struct.*, 93(2), pp. 342-50 (2011).
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Table 2. Dimensionless deflection of functionally graded micro beam under uniformly distributed loading for different gradient power indexes ($q_0=10 \text{ N/m}^2$)

MCST= Modified couple stress theory, CBT= Classical beam theory

| $L/h$ | $h/l$ | Theory                  | $\bar{w}$ (Classical rule of mixture) | $\bar{w}$ (Mori-Tanaka scheme) |
|-------|-------|-------------------------|---------------------------------------|---------------------------------|
|       |       |                         | $p=0$       | $p=0.5$       | $p=1$       | $p=10$      | $p=0$       | $p=0.5$       | $p=1$       | $p=10$      |
| 1     | Ref.[62] | 0.0364 | 0.0527 | 0.0663 | 0.1569 | 0.0364 | 0.0713 | 0.0920 | 0.1870 |
|       | Present MCST | 0.0363 | 0.0525 | 0.0660 | 0.1550 | 0.0363 | 0.0708 | 0.0914 | 0.1845 |
| 5     | Ref.[62] | 0.0989 | 0.1461 | 0.1861 | 0.4034 | 0.0989 | 0.1975 | 0.2535 | 0.4771 |
|       | Present MCST | 0.0986 | 0.1454 | 0.1851 | 0.3980 | 0.0986 | 0.1964 | 0.2518 | 0.4707 |
| $\infty$ | Ref.[62] | 0.2313 | 0.3567 | 0.4670 | 0.8599 | 0.2313 | 0.4823 | 0.6109 | 0.9956 |
|       | Present CBT | 0.2310 | 0.3550 | 0.4647 | 0.8528 | 0.2306 | 0.4800 | 0.6079 | 0.9876 |
| 1     | Ref.[62] | 0.0352 | 0.0510 | 0.0643 | 0.1522 | 0.0352 | 0.0691 | 0.0894 | 0.1809 |
|       | Present MCST | 0.0352 | 0.0510 | 0.0643 | 0.1518 | 0.0352 | 0.0690 | 0.0893 | 0.1803 |
| 10    | Ref.[62] | 0.0949 | 0.1404 | 0.1792 | 0.3838 | 0.0949 | 0.1900 | 0.2437 | 0.4535 |
|       | Present MCST | 0.0949 | 0.1403 | 0.1791 | 0.3824 | 0.0949 | 0.1898 | 0.2434 | 0.4517 |
| $\infty$ | Ref.[62] | 0.2178 | 0.3380 | 0.4429 | 0.7818 | 0.2178 | 0.4554 | 0.5737 | 0.9129 |
|       | Present CBT | 0.2179 | 0.3379 | 0.4426 | 0.7789 | 0.2179 | 0.4552 | 0.5732 | 0.9096 |
Table 2. Dimensionless deflections of functionally graded size dependent beam under point load, $P=100\mu N$

MCST = Modified couple stress theory, MSGT = Modified strain gradient theory

| Theory | $h/l$ | $w$ (Classical rule of mixture) | $w$ (Mori-Tanaka scheme) |
|--------|-------|--------------------------------|--------------------------|
| MCST   |       | $L/h=10$ | $L/h=100$ | $L/h=10$ | $L/h=100$ | $L/h=10$ | $L/h=100$ | $L/h=10$ | $L/h=100$ | $L/h=10$ | $L/h=100$ |
|        |       | $p=0$ | $p=10$ | $p=0$ | $p=10$ | $p=0$ | $p=10$ | $p=0$ | $p=10$ | $p=0$ | $p=10$ |
| 1      |       | 0.0564 | 0.1029 | 0.2429 | 0.0557 | 0.1017 | 0.2409 | 0.0564 | 0.1429 | 0.2885 | 0.0557 | 0.1416 | 0.2859 |
| 2      |       | 0.1500 | 0.2538 | 0.6125 | 0.1497 | 0.2829 | 0.6025 | 0.1520 | 0.3898 | 0.7236 | 0.1497 | 0.3844 | 0.7115 |
| 4      |       | 0.2639 | 0.5186 | 0.9919 | 0.2586 | 0.5097 | 0.9643 | 0.2639 | 0.6865 | 1.1641 | 0.2586 | 0.6731 | 1.1332 |
| 8      |       | 0.3236 | 0.6502 | 1.1757 | 0.3162 | 0.6375 | 1.1347 | 0.3236 | 0.8481 | 1.3746 | 0.3162 | 0.8286 | 1.3304 |
| 1      |       | 0.0348 | 0.0623 | 0.1512 | 0.0346 | 0.0627 | 0.1525 | 0.0348 | 0.0871 | 0.1797 | 0.0346 | 0.0878 | 0.1812 |
| MSGT   |       | 0.1087 | 0.1995 | 0.4462 | 0.1079 | 0.2010 | 0.4494 | 0.1087 | 0.2744 | 0.5276 | 0.1079 | 0.2764 | 0.5313 |
| 4      |       | 0.2290 | 0.4370 | 0.8579 | 0.2259 | 0.4399 | 0.8626 | 0.2290 | 0.5831 | 1.0073 | 0.2259 | 0.5865 | 1.0138 |
| 8      |       | 0.3112 | 0.5971 | 1.0795 | 0.3050 | 0.6123 | 1.1025 | 0.3112 | 0.7846 | 1.2235 | 0.3050 | 0.7985 | 1.2922 |
Table 3. Effects of foundation parameters on bending of functionally graded size dependent beam under point load, $P=100\mu N$, $h/l=1$, $L=50h$

CBT = Classical beam theory, MCST = Modified couple stress theory, MSGT = Modified strain gradient theory

| Theory | $k_w$ | $w$ | $p=0$ | $p=1$ | $p=10$ |
|--------|-------|-----|-------|-------|--------|
| CBT    | $k_w = 0$ | $k_w = 10$ | $k_w = 10^2$ | $k_w = 10$ | $k_w = 10^2$ | $k_w = 10$ | $k_w = 10^2$ |
| 0      | 0.3418 | 0.2937 | 0.1308 | 0.6960 | 0.5223 | 0.1641 | 1.2072 | 0.7663 | 0.1846 |
| 10     | 0.3362 | 0.2896 | 0.1300 | 0.6733 | 0.5095 | 0.1629 | 1.1404 | 0.7392 | 0.1831 |
| $10^2$ | 0.2932 | 0.2573 | 0.1233 | 0.5209 | 0.4177 | 0.1527 | 0.7635 | 0.5619 | 0.1707 |
| $10^3$ | 0.1304 | 0.1230 | 0.0817 | 0.1647 | 0.1534 | 0.0953 | 0.1880 | 0.1738 | 0.1038 |
| 0      | 0.0558 | 0.0543 | 0.0440 | 0.1018 | 0.0971 | 0.0685 | 0.2410 | 0.2160 | 0.1125 |
| MCST   | $k_w = 0$ | $k_w = 10$ | $k_w = 10^2$ | $k_w = 10$ | $k_w = 10^2$ | $k_w = 10$ | $k_w = 10^2$ |
| 10     | 0.0556 | 0.0542 | 0.0439 | 0.1013 | 0.0966 | 0.0683 | 0.2382 | 0.2138 | 0.1119 |
| $10^2$ | 0.0543 | 0.0529 | 0.0431 | 0.0970 | 0.0927 | 0.0663 | 0.2158 | 0.1956 | 0.1068 |
| $10^3$ | 0.0439 | 0.0430 | 0.0363 | 0.0683 | 0.0661 | 0.0516 | 0.1120 | 0.1064 | 0.0738 |
| 0      | 0.0347 | 0.0341 | 0.0297 | 0.0628 | 0.0609 | 0.0483 | 0.1524 | 0.1420 | 0.0883 |
| MSGT   | $k_w = 0$ | $k_w = 10$ | $k_w = 10^2$ | $k_w = 10$ | $k_w = 10^2$ | $k_w = 10$ | $k_w = 10^2$ |
| 10     | 0.0346 | 0.0340 | 0.0297 | 0.0626 | 0.0607 | 0.0481 | 0.1513 | 0.1411 | 0.0879 |
| $10^2$ | 0.0341 | 0.0335 | 0.0293 | 0.0609 | 0.0592 | 0.0472 | 0.1419 | 0.1329 | 0.0847 |
| $10^3$ | 0.0297 | 0.0292 | 0.0260 | 0.0481 | 0.0471 | 0.0392 | 0.0880 | 0.0845 | 0.0623 |

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