ANDERSON LOCALIZATION FOR ONE-FREQUENCY QUASI-PERIODIC BLOCK OPERATORS WITH LONG-RANGE INTERACTIONS

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Abstract. In this paper, we study the quasi-periodic operators $H_{\epsilon, \omega}(x)$:

$$(H_{\epsilon, \omega}(x)\tilde{\psi})_n = \epsilon \sum_{k \in \mathbb{Z}} W_k \tilde{\psi}_{n-k} + V(x+n\omega)\tilde{\psi}_n,$$

where $\tilde{\psi} = \{\psi_n\} \in \ell^2(\mathbb{Z}, \mathbb{C})$, $V(x) = \text{diag}(v_1(x), \ldots, v_l(x))$ with $v_i$ $(1 \leq i \leq l)$ being real analytic functions on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $W_k$ $(k \in \mathbb{Z})$ being $l \times l$ matrices satisfying $\|W_k\| \leq C_0 e^{-\rho|k|}$. Using techniques developed by Bourgain and Goldstein [Ann. of Math. 152(3):835–879, 2000], we show that for $|\epsilon| \leq \epsilon_0(V, \rho, l, C_0)$ (depending only on $V, \rho, l, C_0$) and $x \in \mathbb{R}/\mathbb{Z}$, there is some full Lebesgue measure subset $F$ of the Diophantine frequencies such that $H_{\epsilon, \omega}(x)$ exhibits Anderson localization if $\omega \in F$.

1. Introduction and main result

Quasi-periodic operators have been widely studied in both physics and mathematics literatures, and one of the most famous and typical operators of such type may be the almost Mathieu operator (AMO for short):

$$(H_{\lambda, \omega, x}u)_n = u_{n+1} + u_{n-1} + 2\lambda \cos 2\pi(x+n\omega)u_n,$$

where $u = \{u_n\} \in \ell^2(\mathbb{Z}, \mathbb{C}), x \in \mathbb{T}$ and $\omega \in \mathbb{R}\setminus\mathbb{Q}$. In recent years, more and more research efforts have focused on the nature of the spectrum and the behaviour of the eigenfunctions, particularly on phenomenon of Anderson localization (AL for short) which means the operator has pure point spectrum with exponentially decaying eigenfunctions. The methods for establishing the AL for a quasi-periodic operator include mainly the perturbative one and the non-perturbative one. The KAM technique is a typical perturbative method, which relies heavily on intricate multi-step procedures, eigenvalue (eigenfunction) parametrization, and perturbation arguments [10, 13–15, 28, 30]. Thus the perturbation may depend on the Diophantine condition. However, the non-perturbative method treats the Green’s function directly and only finite scales are involved. As a result, in many cases, the smallness (largeness) of the perturbation is independent of the Diophantine condition (this is called a non-perturbative AL). For an elegant and more complete exposition of (non) perturbative results (methods), we refer the reader to [22] by Jitomirskaya. Let us give a more exact introduction of the non-perturbative AL results. In 1999, Jitomirskaya [24] showed that the AMO $H_{\lambda, \omega, x}$ exhibits AL for almost every $x \in \mathbb{T}$ if $\omega$ is a Diophantine frequency and $|\lambda| > 1$, here $\omega$ is a Diophantine.
frequency means there is $t > 0$ such that $\omega \in \text{DC}_t$ with

$$\text{DC}_t = \left\{ \omega \in \mathbb{R} : ||k\omega||_T \geq \frac{t}{|k|^2}, \forall k \in \mathbb{Z} \setminus \{0\} \right\}.$$ 

Subsequently, Bourgain and Goldstein [6] proved that for a non-constant real analytic potential $v$ on $\mathbb{T}$, the general one-frequency quasi-periodic Schrödinger operators which are given by

$$(H_{\lambda,\omega,x} u)_n = u_{n+1} + u_{n-1} + \lambda v(x + n\omega) u_n,$$

satisfy AL with $\omega$ being in a full Lebesgue measure subset of $\text{DC}_t$ and $|\lambda| \geq \lambda_0(v) \gg 1$ (independent of $\text{DC}_t$). In Chapter 11 of Bourgain’s monograph [4], he extended their result of [6] to long-range operators: (actually a sketch of the proof):

$$(H_{\epsilon,\omega,x} u)_n = \epsilon \sum_{k \in \mathbb{Z}} w_k u_{n-k} + v(x + n\omega) u_n,$$

where $\{w_k\}_{k \in \mathbb{Z}}$ are the Fourier coefficients of some real analytic function $w$ on $\mathbb{T}$ and $|\epsilon| \leq \epsilon_0(w, v) \ll 1$. We then turn to the block operators case. In [8], Bourgain and Jitomirskaya extended the result of [6] to the band Schrödinger operators:

$$(H_{\lambda,\omega,x} \vec{\psi})_n := \vec{\psi}_{n-1} + \vec{\psi}_{n+1} + (\lambda V(x + n\omega) + W_0) \vec{\psi}_n,$$

with

$$W_0 = \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
\vdots & \vdots & \ddots & \ddots
\end{pmatrix}_{l \times l}$$

and $V(x) = \text{diag}(v_1(x), \cdots, v_l(x))$, $\vec{\psi} = \{\vec{\psi}_n\} \in \ell^2(\mathbb{Z}, \mathbb{C}^l)$. In a recent paper by Klein [27], he studied the quasi-periodic block Jacobi operators:

$$(H_{\lambda,\omega} \vec{\psi})_n := -(\triangle_W(\vec{\psi})_n + \lambda V(x + n\omega) \vec{\psi}_n$$

with the “weighted” Laplacian

$$(\triangle_W(\vec{\psi})_n := W(x + (n + 1)\omega) \vec{\psi}_{n+1} + W^\top(x + n\omega) \vec{\psi}_{n-1} + R(x + n\omega) \vec{\psi}_n.$$ 

Klein proved a non-perturbative AL and generalized the result of [8]. For recent AL results, we refer the reader to [1, 2, 3, 4, 19, 21, 22].

In this paper, we study the one-frequency quasi-periodic block operators with long-range interactions:

$$(H_{\epsilon,\omega} \vec{\psi})_n := \epsilon \sum_{k \in \mathbb{Z}} W_k \vec{\psi}_{n-k} + V(x + n\omega) \vec{\psi}_n,$$

where $\vec{\psi} = \{\vec{\psi}_n\} \in \ell^2(\mathbb{Z}, \mathbb{C}^l)$, $V(x) = \text{diag}(v_1(x), \cdots, v_l(x))$ with $v_i \geq 1$ being real analytic functions on $\mathbb{T}$ and $W_k \in \mathbb{Z}$ being exponential decay $l \times l$ matrices satisfying $W_{-k} = W_k^\star$ ($W_k^\star$ denotes the complex conjugate of $W_k$). In general, we call $x \in \mathbb{T}$ the phase, $\omega \in \mathbb{R} \setminus \mathbb{Q}$ the frequency, $\epsilon \in \mathbb{R}$ the perturbation and $V$ the potential. It is well-known that every $H_{\epsilon,\omega}(x)$ is a bounded self-adjoint operator on the Hilbert space $\ell^2(\mathbb{Z}, \mathbb{C}^l)$. This kind of operators was studied in some papers before, such as in [17, 18].

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1. Where $||x||_T := \min_{k \in \mathbb{Z}} |x - k|$. It is well-known that the Lebesgue measure of $\text{DC}_t$ is $1 - \mathcal{O}(t)$. Our definition here is a little different from that in [8, 22]. However, it is not essential.
The purpose of the present work is to show the operator $H_{\omega}(x)$ defined in (1.1) exhibits non-perturbative AL. This generalizes a result of Bourgain [4] as well as a result of Klein [27]. More precisely, we have

**Theorem 1.1.** Let $H_{\omega}(x)$ be given by (1.1) with $\|W_k\| \leq C_0 e^{-\rho|k|}$ and $v_i$ ($1 \leq i \leq l$) be nonconstant real analytic functions on $\mathbb{T}$, where the norm $\| \cdot \|$ is the standard matrix norm. Then there exists $c_0 = c_0(V, \rho, l, C_0) > 0$ (depending only on $V, \rho, l, C_0$) such that for $|\epsilon| \leq \epsilon_0$, $x \in \mathbb{T}$, there is some zero Lebesgue measure set $\mathcal{R}$ so that for $\omega \in \text{DC}_l \setminus \mathcal{R}$, $H_{\epsilon}(x)$ shows the Anderson localization.

The proof of our main theorem employs techniques developed by Bourgain and Goldstein in [3]. We also use some tools in [4, 9], and some convenient notations of Klein in [27].

The main difficulty here is to establish large deviation theorem (LDT for short) for the restricted Green’s function $G_N(x; E)$ (see §5) and this leads to exploring efficiently upper bounds on minors of the block matrix $H_N(x) - E1_N$ (see §3) as well as a lower bound on $\int_{\mathbb{T}} \log |\det[H_N(x) - E1_N]| \, dx$. Since our block operator $H_{\epsilon}(x)$ (see (1.1)) is with a long-range perturbation, it is much more complicated and skilful to obtain such upper and lower bounds.

The structure of the paper is as follows. In §2, we introduce some notions and basic concepts. In §3, we prove uniformly upper bounds on the minors of the Dirichlet matrix. In §4, we obtain a lower bound on the average of the Dirichlet determinant on torus. The Green’s function estimates are established in §5. In §6, we finish the proof of our main theorem. We include some useful lemmata in Appendix A.

## 2. Some basic concepts and notations

### 2.1. Some notations

We use convenient notations introduced by Klein in [27]. Let $\text{Mat}_m(\mathbb{C})$ be the set of all $m \times m$ complex matrices. Given a block matrix $M$, we use roman letters for the indices of its block-matrix entries, and Greek letters for the indices of its scalar entries. More precisely, we write $M = (M_{\gamma, \gamma'})_{1 \leq \gamma, \gamma' \leq N_1} \in \text{Mat}_{N_1}(\mathbb{C})$ which can be identified with a block matrix $M = (M_{n,n'})_{1 \leq n, n' \leq N}$ (i.e. $M_{n,n'} \in \text{Mat}_l(\mathbb{C})$ for any $1 \leq n, n' \leq N$). Moreover, given $\gamma \in [1, N_1]$, there is a unique $1 \leq n(\gamma) \leq N$ such that $\gamma = l \cdot (n(\gamma) - 1) + r$ with $1 \leq r \leq l$. Thus any scalar $M_{\gamma, \gamma'}$ belongs to the block $M_{n(\gamma), n(\gamma')}$.

Given any interval $[a, b] = N \subset \mathbb{Z}$ with length $|N| = b - a + 1$ and any infinite $l \times l$-block matrix $M = (M_{n,n'})_{n,n' \in \mathbb{Z}}$ (i.e. $M_{n,n'} \in \text{Mat}_l(\mathbb{C})$ for all $n, n' \in \mathbb{Z}$), we denote by $M_N = (M_{n,n'})_{n,n' \in N}$ by restricting $n \in N_1, n' \in \mathbb{Z}_2$. Especially, we write $M_N = M_N$ if $N = [1, N]$. Finally, by $I$, we mean the block identity matrix, that is $I = \text{diag}(I_n)_{n \in \mathbb{Z}}$ with $I_n$ being $l \times l$ identity matrix.

We define for $\rho \geq 0$ the strip $\Delta_\rho = \{z \in \mathbb{C}/\mathbb{Z} : |3z| \leq \rho\}$. For any continuous mapping $f$ from $\Delta_\rho$ to some Banach space $(B, \| \cdot \|)$, we define $\| f \|_\rho = \sup_{z \in \Delta_\rho} \| f(z) \|$. For any measurable $z \in \Delta_\rho$, we denote by $\text{Leb}(\mathcal{A})$ its Lebesgue measure. If a constant $C$ depends only on functions $v_i(x)$ ($1 \leq i \leq l$), we write $C = C(V)$ with $V(x) = \text{diag}(v_1(x), \ldots, v_l(x))$. We also use Euclidean norm for a vector and the standard operator norm for a matrix.

Note that every real analytic function $f$ on $\mathbb{T}$ can be analytically extended to the strip $\Delta_{\epsilon(f)}$ with $c(f) > 0$ depending only on $f$. Thus without loss of generality, we assume each $v_i(x)$ ($1 \leq i \leq l$) is analytic on $\Delta_\rho$. For simplicity, we also assume the perturbation $\epsilon \geq 0$ and $\| W_k \| \leq e^{-\rho|k|}$ (i.e., $C_0 = 1$).

### 2.2. Harmonic measure

For reader’s convenience, we introduce the basic properties of the harmonic measure which will be used in §5. The materials in this subsection are from [16].
Write \( \mathbb{H} = \{ z \in \mathbb{C} : \Re z > 0 \} \) for the upper half-plane and \( \partial \mathbb{H} = \mathbb{R} \) for its boundary. If \( U \subset \mathbb{R} \) is measurable, the harmonic measure of \( U \) at \( z = x + iy \in \mathbb{H} \) is

\[
\mu(z, U, \mathbb{H}) = \int_U \frac{y}{(t - x)^2 + y^2} \frac{dt}{\pi}
\]

We note that

- If \( U = (a, b) \), then

\[
\mu(z; (a, b), \mathbb{H}) = \frac{1}{\pi} \arg \left( \frac{z - b}{z - a} \right).
\]

- For any bounded Borel function \( f \) on \( \mathbb{R} \), we have for \( x + iy \in \mathbb{H} \)

\[
\int_{\mathbb{R}} f(t) d\mu(x + iy, t, \mathbb{H}) = \int_{\mathbb{R}} f(t + x) d\mu(iy, t, \mathbb{H}).
\]

Let \( \Omega \) be a simply connected domain in the extended plane \( \mathbb{C}^* = \mathbb{C} \cup \{ \infty \} \) with its boundary \( \partial \Omega \) being a Jordan curve in \( \mathbb{C}^* \). If \( \phi \) is a conformal mapping from \( \mathbb{H} \) onto \( \Omega \), then by Carathéodory’s theorem, \( \phi \) has a continuous extension (again denoted by \( \phi \)) to \( \overline{\mathbb{H}} \) and this extension is a continuous bijective mapping from \( \overline{\mathbb{H}} \) to \( \overline{\Omega} \). Then for any Borel set \( U \subset \partial \Omega \), we can define the harmonic measure of \( U \) relative to \( \Omega \) at \( z \in \Omega \) by

\[
\mu(z, U, \Omega) = \mu(\phi^{-1}(z), \phi^{-1}(U), \mathbb{H}).
\]

**Remark 2.1.** This definition is independent of the choices of conformal mappings.

## 3. Uniformly upper bounds on minors of the Dirichlet matrix

In this section, we will prove uniformly upper bounds on minors of the Dirichlet matrix and we use tools in [8] (see also Chapter 11 of [9]).

Let \( \mathcal{N} \subset \mathbb{Z} \) be an interval and \( H_N(x) \) be the restriction of \( H_{\alpha,\omega}(x) \) on \( \ell^2(\mathcal{N}, \mathbb{C}^l) \). We fix \( \epsilon \) and restrict \( E \) in a compact interval \( \mathcal{A} \subset \mathbb{R} \). Then \( H_N(x, E) := H_N(x) - EI_N \) can be represented by a \( Nl \times Nl \) matrix with complex entries, which we denote by \( H_{N,(\alpha,\alpha')}(x, E) \), where \( 1 \leq \alpha, \alpha' \leq Nl \). We let \( \mu_{N,(\alpha,\alpha')}(x, E) \) be the \( (\alpha,\alpha') \)-minor of the Dirichlet matrix \( H_N(x, E) \). That is,

\[
\mu_{N,(\alpha,\alpha')}(x, E) := \det[(H_N(x, E))_{\sim \alpha, \sim \alpha'}],
\]

where for any \( 1 \leq \gamma \leq Nl, \\gamma = \{1, \cdots, Nl\} \setminus \{\gamma\} \) is arranged in natural order.

The following lemma gives an expression for a minor of a matrix.

**Lemma 3.1** (Lemma 10 in [19]). *Let \( G = (G_{\gamma,\gamma'}) \in \text{Mat}_m(\mathbb{C}) \) and \( \alpha, \alpha' \in \{1, 2, \cdots, m\} \) with \( \alpha \neq \alpha' \). Then

\[
\det[G_{\sim \alpha, \sim \alpha'}] = \sum_{\Gamma} A_\Gamma \det[G_{\Gamma}] \prod_{i=1}^{s-1} G_{\gamma_i, \gamma_{i+1}},
\]

where the sum is taken over all ordered subsets \( \Gamma = (\gamma_1, \cdots, \gamma_s) \) of \( \{1, 2, \cdots, m\} \) with \( \gamma_1 = \alpha, \gamma_s = \alpha' \), and \( A_\Gamma \in \{-1, 1\} \). If \( \Gamma = \{1, \cdots, m\} \setminus \Gamma \) is arranged in natural order.*

**Lemma 3.2.** *Assume \( b = \sum_{j=1}^{s-1} |n(\gamma_j+1) - n(\gamma_j)| \). Then \( s \leq (b + 1)l \).

**Proof.** We let \( n(\gamma_1) = n(\gamma_2) = \cdots = n(\gamma_l) = n_1, n(\gamma_{l+1}) = n(\gamma_{l+2}) = \cdots = n(\gamma_2) = n_2, \cdots, n(\gamma_{m-1}+1) = n(\gamma_{m-1}+2) = \cdots = n(\gamma_m) = n_m \), where \( l_m = s \) and \( n_i \neq n_j \) for \( i \neq j \).
Since each block has order $l$, we must have $l_1 \leq l$, $l_j - l_{j-1} \leq l$ $(2 \leq j \leq m)$. As a result, $s \leq ml$. Finally,
\[
    b = \sum_{j=1}^{m-1} |n(\gamma_j) - n(\gamma_{j+1})| = \sum_{j=1}^{m-1} |n_j - n_{j+1}| \geq m - 1 \geq \frac{s}{l} - 1.
\]

The lemma follows.

Lemma 3.3 (Lemma 11.29 in [4]). Let $\Gamma = (\gamma_1, \ldots, \gamma_s)$ be an ordered subset of \{1, 2, \ldots, Nl\} and $v$ be a non-constant real analytic function on $\mathbb{T}$. Then there is some $\delta = \delta(v) > 0$ such that for $0 < \epsilon \ll 1, s > \epsilon^5 Nl$ and $E \in \mathbb{R}$,
\[
    \inf_{x \in \Gamma} \sum_{k \in \mathbb{Z}} \log(|v(x + k\omega) - E| + \epsilon) \geq \frac{3}{4} s \log \epsilon.
\]

We can now state our main result of this section.

Proposition 3.4. Let $\omega \in \text{DC}_1$. There are $\sigma = \sigma(V) > 0$ (depending only on $V$) and $e_0 = e_0(l, \rho) > 0$ (depending only on $l, \rho$) such that if $0 < \epsilon \leq e_0$, then for all $x \in \mathbb{T}$ and $N \gg 1$ (depending on $V, \epsilon, l, \rho, t$), we have
\[
    |\mu_{N, (\alpha, \alpha')} (x, E)| \leq e^{Nl \sum_{j=1}^{s} |\det[H_N(x, E)]_{\Gamma^0}|} |\mu_{N, (\alpha, \alpha')} (x, E)|
\]

where $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_s)$ is a path in $[1, Nl]$ with $\gamma_1 = \alpha, \gamma_s = \alpha'$. Recalling the properties of $\{W_k\}$, one has
\[
    \sum_{s} \sum_{\Gamma:|\Gamma|=s} |\det[H_N(x, E)]_{\Gamma^0}| e^{s-1} \prod_{j=1}^{s-1} ||H_N(x, E)_{\gamma_j, \gamma_{j+1}}||
\]

where $b = \sum_{j=1}^{s-1} |n(\gamma_{j+1}) - n(\gamma_j)|$, and a $(s, b)$-path $\Gamma(s, b)$ means the path $\Gamma$ with the restrictions $s = |\Gamma|$ and $b = \sum_{j=1}^{s-1} |n(\gamma_{j+1}) - n(\gamma_j)|$. In the last inequality, we use the fact that there are at most $2l^{s-1} \binom{b+s-1}{s-1}$ many $(s, b)$-paths.
Thus it needs to give the upper bound on $|\det[(H_N(x,E))_{\Gamma^c}]|$ first. By Hadamard's inequality,

$$|\det[(H_N(x,E))_{\Gamma^c}]| \leq \prod_{\beta \in \Gamma^c} |v_{j(\beta)}(x + n(\beta)\omega) - E + C\epsilon|,$$

where $\beta = (n(\beta) - 1)l + j(\beta)$ with $1 \leq \beta \leq NL$, $0 < j(\beta) \leq l$ and $C > 0$ is a constant depending only on $\rho$. Hence

$$\log |\det[(H_N(x,E))_{\Gamma^c}]| = \sum_{j=1}^{l} \sum_{n=1}^{N} \log |v_{j}(x + n\omega) - E + C\epsilon|$$

$$- \sum_{\beta \in \Gamma^c} \log \left[|v_{j(\beta)}(x + n(\beta)\omega) - E + C\epsilon|\right]$$

(3.1)

:= (I) - (II).

From Lemmata A.3 and A.5, there is some $\delta = \delta(V) > 0$ such that for $0 < \epsilon \leq \epsilon_0(\rho)$ and $N \gg 1$ (depending on $V, t, \delta$),

(3.2) $$(I) \leq N \sum_{j=1}^{l} \int_T \log |v_{j}(x) - E| dx + Nl\epsilon^\delta.$$

For (II), using Lemma 3.3, there is some $\delta_1 = \delta_1(V) > 0$ such that for $s \geq 1$,

(3.3) $$(II) > s \log \epsilon,$$

and for $s \geq e^{\delta_1 NL}$,

(3.4) $$(II) \geq \frac{3}{4} s \log \epsilon,$$

where $s = |\Gamma|$. From (3.1), (3.2), (3.3) and (3.4), we have

(3.5) $$\log |\det[(H_N(x,E))_{\Gamma^c}]| \leq N \sum_{j=1}^{l} \int_T \log |v_{j}(x) - E| dx + Nl\epsilon^\delta + s \log \frac{1}{\epsilon},$$

and for $s > e^{\delta_1 NL}$,

(3.6) $$\log |\det[(H_N(x,E))_{\Gamma^c}]| \leq N \sum_{j=1}^{l} \int_T \log |v_{j}(x) - E| dx + Nl\epsilon^\delta + \frac{3}{4} s \log \frac{1}{\epsilon}.$$

We then can finish our proof as follows. Let

$$d_0 = \sum_{j=1}^{l} \int_T \log |v_{j}(x) - E| dx + l\epsilon^\delta.$$

Recalling (3.5), (3.6) and Lemma 3.2 we obtain

$$|\mu_{N,(\alpha,\alpha')}(x,E)| \leq e^{-d_0} \sum_{b \geq |n(\alpha) - n(\alpha')|} \sum_{s \leq (b+1)l \leq \epsilon_1 NL} (2l)^{s-1} \binom{b+s-1}{s-1} e^{-\rho b}$$

$$+ e^{-\frac{3}{4}d_0} \sum_{b \geq |n(\alpha) - n(\alpha')|} \sum_{s \leq (b+1)l \leq \epsilon_1 NL} (2l\epsilon^\delta)^{s-1} \binom{b+s-1}{s-1} e^{-\rho b}$$

(3.7) $$:= (III) + (IV).$$
Regarding (IV), we have for $N \gg 1$ and $2(l + 1)\epsilon^{\frac{1}{3}} < \rho$

\[
(IV) \leq e^{-\frac{4}{5}e^{Nld_0}} \sum_{b \geq |n(\alpha) - n(\alpha')|} e^{-\rho b} \sum_{s \leq (b+1)!} (2l e^{\frac{1}{5}})^{s-1} \left( \frac{b(2l + 1)}{s - 1} \right)
\]

\[
\leq e^{-\frac{4}{5}e^{Nld_0}} \sum_{b \geq |n(\alpha) - n(\alpha')|} e^{-\rho b} (1 + 2l e^{\frac{1}{5}})^{b(2l + 1)}
\]

\[
\leq e^{-\frac{4}{5}e^{Nld_0}} \sum_{b \geq |n(\alpha) - n(\alpha')|} e^{-\rho b} e^{2l(2l+1)e^{\frac{1}{5}} b}
\]

\[
\leq Ce^{-\frac{4}{5}e^{Nld_0} e^{(-\rho + 2l(2l+1)\epsilon^{\frac{1}{5}})|n(\alpha) - n(\alpha')|}}.
\]

(3.8)

For (III), we distinguish the cases $|n(\alpha) - n(\alpha')| > \epsilon^{\frac{4}{5}} Nl$ and $|n(\alpha) - n(\alpha')| \leq \epsilon^{\frac{4}{5}} Nl$. If $|n(\alpha) - n(\alpha')| > \epsilon^{\frac{4}{5}} Nl$, we obtain

\[
(III) \leq e^{-1}(2l)^{\epsilon^{\frac{1}{5}} Nl} e^{Nld_0} \sum_{b \geq |n(\alpha) - n(\alpha')|} e^{-\rho b} \sum_{s \leq (b+1)!} (b + s - 1) \left( \frac{b}{s - 1} \right)
\]

\[
\leq e^{-1 + \epsilon^{\frac{4}{5}} Nl(2l)^{\epsilon^{\frac{1}{5}} Nl}} e^{Nld_0} \sum_{b \geq |n(\alpha) - n(\alpha')|} e^{-\rho b} \left( \frac{b(2l + 1)}{e^{\frac{1}{5}} b} \right)^2.
\]

By Stirling formula, it appears $\left( \frac{m}{r_m} \right) \leq Cme^{\phi(r)m}$, where

\[
\phi(r) = -r \log r - (1 - r) \log(1 - r), \quad 0 < r < 1.
\]

Therefore, when $N$ is large enough (s.t. $\frac{\log b}{b} < \epsilon$) and $-\rho + (2l + 1)\phi((2l + 1)^{-1}e^{\frac{1}{5}}) + \epsilon < 0$, we have

\[
(III) \leq Ce^{-1 + \epsilon^{\frac{4}{5}} Nl(2l)^{\epsilon^{\frac{1}{5}} Nl}} e^{Nld_0} \sum_{b \geq |n(\alpha) - n(\alpha')|} be^{-\rho b} e^{(2l + 1)\phi((2l + 1)^{-1}e^{\frac{1}{5}}) b}
\]

\[
\leq Ce^{-1 + \epsilon^{\frac{4}{5}} Nl^2(2l)^{\epsilon^{\frac{1}{5}} Nl}} e^{Nld_0} e^{(-\rho + \epsilon + (2l + 1)\phi((2l + 1)^{-1}e^{\frac{1}{5}}))|n(\alpha) - n(\alpha')|}.
\]

(3.9)

If $|n(\alpha) - n(\alpha')| \leq \epsilon^{\frac{4}{5}} Nl$, we write

\[
(III) = \sum_{b \geq \epsilon^{\frac{4}{5}} Nl} \sum_{s \leq \epsilon^{\frac{1}{5}} Nl} \cdots + \sum_{b \geq \epsilon^{\frac{4}{5}} Nl} \sum_{b \geq \epsilon^{\frac{4}{5}} Nl} \sum_{s \leq \epsilon^{\frac{1}{5}} Nl} \cdots
\]

\[
:= (III_1) + (III_2).
\]

Similarly to the proof of (3.9), one has

\[
(III_1) \leq e^{-1 + \epsilon^{\frac{4}{5}} Nl(2l)^{\epsilon^{\frac{1}{5}} Nl}} e^{Nld_0} \sum_{b \geq \epsilon^{\frac{4}{5}} Nl} e^{-\rho b} \left( \frac{b(2l + 1)}{e^{\frac{1}{5}} b} \right)^2
\]

\[
\leq Ce^{-1 + \epsilon^{\frac{4}{5}} Nl^2(2l)^{\epsilon^{\frac{1}{5}} Nl}} e^{Nld_0} e^{(-\rho + \epsilon + (2l + 1)\phi((2l + 1)^{-1}e^{\frac{1}{5}}))|n(\alpha) - n(\alpha')|}.
\]

(3.10)
Note also that
\[
(\text{III}_2) \leq e^{-1}e^{Nd_0} \sum_{b \geq |n(\alpha) - n(\alpha')|} e^{-\rho b(1 + 2l)^{b(2l+1)}}
\]
\[
\quad \leq Ce^{-1+\frac{0}{4} Nl(3l)}e^{\frac{1}{2}Nl(2l+1)} e^{Nd_0} e^{-\rho |n(\alpha) - n(\alpha')|}.
\]
(3.11)

Putting all above estimates (3.7)-(3.11) together, we obtain that there are
\[
\varnothing
\]
\[
\text{two symmetric (on the real axis) strips containing no zero in their interiors. We denote } \Omega
\]
\[
\text{this needs a small modification of the proof of Proposition 4.3 in [11].}
\]
\[
\text{More precisely,}
\]
\[
\text{Proof.}
\]
\[
\text{This needs a small modification of the proof of Proposition 4.3 in [11].}
\]
\[
\text{More precisely, we denote}
\]
\[
\Sigma = \sum_{j=1}^{l} \sup_{E \in \mathbb{R}} \Sigma_j(E),
\]
\[
\text{where } \Sigma_j(E) \text{ is the number (counting multiplicities) of zeros of function } v_j(z) - E \text{ on the strip } \Delta_{\rho}.
\]
\[
\text{From Proposition 4.2 of [11], } \Sigma < \infty. \text{ We let } z_{j,1}, \ldots, z_{j,k_j} \text{ be distinct zeros of } v_j(z) - E \text{ on } \Delta_{\rho} \text{ with multiplicities } n_{j,1}, \ldots, n_{j,k_j}. \text{ Then for any } E \in \mathbb{R},
\]
\[
\Sigma_j(E) = \sum_{j=1}^{l} \sum_{m=1}^{k_j} n_{j,m} \leq \Sigma < \infty.
\]
\[
\text{Write}
\]
\[
v_j(z) - E = g_{j,E}(z) \prod_{m=1}^{k_j} (z - z_{j,m})^{n_{j,m}},
\]
\[
\text{where } g_{j,E} \text{ is the zero-free part of } v_j - E \text{ on } \Delta_{\rho}. \text{ It was proved in [11] (see Proposition 4.2 of [11]) that}
\]
\[
q = \min_{1 \leq j \leq l} \inf_{E \in \mathbb{R}} \inf_{z \in \Delta_{\rho}} |g_{j,E}(z)| > 0.
\]

We now define \( \Omega_\xi := \{ z \in \mathbb{C}/\mathbb{Z} : \frac{\xi}{2} < |\Re z| < \xi \} \) and divide the region \( \Omega_\xi \) into \( 8\Sigma + 4 \) parallel strips (along the real axis) such that every strip has the width \( \frac{1}{2\Sigma + 4} \). Then there are at least two symmetric (on the real axis) strips containing no zero in their interiors. We denote \( \Omega_0 \) one of such strips with \( \Re z > 0 \) and \( \Omega_0' \) its symmetric strip. We then divide \( \Omega_0 \) (resp. \( \Omega_0' \)) into
three smaller strips such that each of them has the width \(\frac{\xi}{3(8\Sigma + 4)}\). Let \(\Omega_0\) (resp. \(\Omega'_0\)) be the middle \(\frac{1}{3}\) part of \(\Omega_0\) (resp. \(\Omega'_0\)). Thus for any \(x + iy \in \Omega_0\) (and of course \(x - iy \in \Omega'_0\))

\[
|v_j(x \pm iy) - E| = |g_{j,E}(x \pm iy)| \prod_{m=1}^{k_j} |v_j(x \pm iy) - z_{j,m}|^m \geq q \left( \frac{\xi}{24\Sigma + 12} \right) \Sigma,
\]

which completes the proof. \(\Box\)

We can now state our main result of this section.

**Proposition 4.2.** Let \(\omega \in DC_t\). There are \(\epsilon_0 = \epsilon_0(V, \rho, l) > 0\) (depending only on \(V, \rho, l\)) and \(C = C(V, \rho) > 0\) (depending only on \(V, \rho\)) such that if \(0 < \epsilon \leq \epsilon_0\), then for all \(E\) in a compact interval \(\mathcal{A}\) and \(N\) large enough (depending on \(V, l, \epsilon, \rho, t\)) , we have

\[
\frac{1}{N} \int_{T} \log |\det[H_N(x, E)]| dx \geq \sum_{j=1}^{l} \int_{T} \log |v_j(x) - E| dx - C \epsilon \Xi.
\]

where \(\Xi\) is given by Lemma 4.1.

**Remark 4.3.** From Lemma A.2, we have

\[
\min_{E} \sum_{j=1}^{l} \int_{0}^{1} \log |v_j(x) - E| dx > -C > -\infty,
\]

where \(C > 0\) depends on \(v_1, \ldots, v_l\) only.

**Proof.** We first consider the upper bound on determinant of the matrix \(H_N(z, E) = H_N(z) - EI_N\) on the strip \(\Delta_\rho\). By Hadamard’s inequality, one has

\[
|\det[H_N(z, E)]| \leq \prod_{1 \leq j \leq l} \prod_{1 \leq n \leq N} (|v_j(z + n\omega) - E| + C\epsilon),
\]

where \(C > 0\) is a constant depending only on \(\rho\). Thus when \(E\) is in a compact interval \(\mathcal{A}\), we have

\[
\frac{1}{N} \log |\det[H_N(z, E)]| \leq C(V, \rho)l.
\]

Next, we consider a lower bound on \(|\det[H_N(x \pm iy, E)]|\) for some \(y \in (0, \rho)\). We fix \(\xi = \epsilon \Xi\), where \(\Xi\) is given by Lemma 4.1. Then \(\xi \in (0, \frac{\rho}{2})\) for \(0 < \epsilon < \left(\frac{\rho}{2}\right)^{2\Sigma}\). From Lemma 4.1, there is \(y_0 \in (\frac{\rho}{2}, \xi)\) such that

\[
\min_{E} \min_{1 \leq j \leq l} \min_{x \in \mathbb{R}} |v_j(x \pm iy_0) - E| \geq c\xi^\Sigma.
\]

We write

\[
H_N(x \pm iy_0, E) = D_N(x \pm iy_0) + \epsilon B_N
\]

\[
= (I_N + \epsilon B_N D_N^{-1}(x \pm iy_0)) D_N(x \pm iy_0),
\]

where

\[
D_N(x \pm iy_0) = \text{diag}[V(x \pm iy_0 + \omega) - EI_N, \ldots, V(x \pm iy_0 + N\omega) - EI_N]
\]
and
\[
B_N = \begin{pmatrix}
W_0 & W_{-1} & \cdots & W_{-N+1} \\
W_1 & W_0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & W_{-1} \\
W_{N-1} & \cdots & W_1 & W_0
\end{pmatrix}.
\]

Then it follows that
\[
\frac{1}{N} \log | \det[H_N(x \pm iy_0, E)] | = \frac{1}{N} \log | \det D_N(x \pm iy_0) | \\
+ \frac{1}{N} \log | \det[I_N + \epsilon B_N D_N^{-1}(x \pm iy_0)] |.
\]

Hence when \( N \) is large enough, from Lemma A.5 and (4.3), it follows that
\[
\frac{1}{N} \log | \det D_N(x \pm iy_0) | = \frac{1}{N} \sum_{j=1}^{l} \sum_{n=1}^{N} \log \left( |v_j(x \pm iy_0 + n\omega) - E| + \epsilon \right)
\]
\[
- \frac{1}{N} \sum_{j=1}^{l} \sum_{n=1}^{N} \log \left( 1 + \frac{\epsilon}{|v_j(x \pm iy_0 + n\omega) - E|} \right)
\]
\[
\geq \sum_{j=1}^{l} \int_0^1 \log(|v_j(x \pm iy_0) - E| + \epsilon) dx - C l \epsilon^\frac{1}{2}.
\]

On the other hand, from (4.3), one has
\[
\| \epsilon B_N D_N^{-1}(x \pm iy_0) \| \leq C \epsilon^\frac{1}{2},
\]
and for \( 0 < \epsilon \ll 1 \) (depending only on \( v_1, \cdots, v_l, \rho \)), \( \| \epsilon B_N D_N^{-1}(x \pm iy_0) \| \ll \frac{1}{N} \). Therefore when \( 0 < \epsilon \ll 1 \), by Hadamard’s inequality and Neumann expansion technique, it follows that
\[
\log | \det[I_N + \epsilon B_N D_N^{-1}(x \pm iy_0)] | = - \log | \det[I_N + \sum_{s=1}^{N} (\epsilon B_N D_N^{-1}(x \pm iy_0))^s] |
\]
\[
\geq - \sum_{n=1}^{N} \log(1 + 2\epsilon \| B_N D_N^{-1}(x \pm iy_0) \delta_n \| )
\]
\[
> - CNl \epsilon^\frac{1}{2}.
\]

Combing (4.4)-(4.6), for \( 0 < \epsilon \leq \epsilon_0 \) (depending only on \( V, \rho \)), \( N \) large enough (depending on \( V, l, \epsilon, \rho, \sigma \)) and any \( x \in \mathbb{T} \), we have
\[
\frac{1}{N} \log | \det[H_N(x \pm iy_0, E)] | \geq \sum_{j=1}^{l} \int_0^1 \log(|v_j(x \pm iy_0) - E| + \epsilon) dx - C l \epsilon^\frac{1}{2}.
\]

Finally, in order to get a lower bound on \( \int_{\mathbb{T}} \log | \det[H_N(x, E)]| dx \), we exploit subharmonicity of the function
\[
u(z) = u_N(z) := \frac{1}{N} \log | \det[H_N(z, E)] |, \quad z \in \Delta_\rho.
\]

Fix \( x \in \mathbb{T} \) and denote \( y_1 := \frac{\rho}{2}, \quad \Omega_\rho := \{ z \in \mathbb{C} : 0 \leq \Im z \leq y_1 \} \). We use harmonic measure estimates of Bourgain and Goldstein [6] here and we include the basic properties of the harmonic
Consequently, using Fubini's theorem, one has

$$u(x + iy_0) \leq \int_{\{z: \Im z = 0\}} u(z) d\mu(x + iy_0, z, \Omega_\rho) + \int_{\{z: \Im z = y_1\}} u(z) d\mu(x + iy_0, z, \Omega_\rho)$$

$$= \int_{-\infty}^{+\infty} u(t) d\mu(x + iy_0, t, \Omega_\rho) + \int_{-\infty}^{+\infty} u(t + iy_1) d\mu(x + iy_0, t, \Omega_\rho)$$

$$= \int_{-\infty}^{+\infty} u(x + t) d\mu(iy_0, t, \Omega_\rho) + \int_{-\infty}^{+\infty} u(x + t + iy_1) d\mu(iy_0, t, \Omega_\rho),$$

where \(\mu\) is the harmonic measure defined in §2.2 and the last equality follows from (2.2).

Consequently, using Fubini's theorem, one has

$$\int_0^1 u(x + iy_0) dx \leq \mu(iy_0, \{z: \Im z = 0\}, \Omega_\rho) \int_0^1 u(x) dx$$

$$+ \mu(iy_0, \{z: \Im z = y_1\}, \Omega_\rho) \int_0^1 u(x + iy_1) dx.$$

Choose a conformal mapping \(\phi: \Omega_\rho \to \mathbb{H}, z \mapsto e^{\frac{2\pi}{y_0} z}\). From (2.1) and (2.3), it follows that

$$\int_0^1 u(x + iy_0) dx \leq \mu(e^{\frac{2\pi}{y_0} y_0}, [0, +\infty), \mathbb{H}) \int_0^1 u(x) dx$$

$$+ \mu(e^{\frac{2\pi}{y_0} y_0}, (-\infty, 0], \mathbb{H}) \int_0^1 u(x + iy_1) dx.$$

$$= \left(1 - \frac{y_0}{y_1}\right) \int_0^1 u(x) dx + \frac{y_0}{y_1} \int_0^1 u(x + iy_1) dx.$$

Recalling (4.2) and (4.7), we have

$$\sum_{j=1}^l \int_0^1 \log(|v_j(x + iy_0) - E| + \epsilon) dx - C\epsilon^{\frac{1}{2}} \leq \left(1 - \frac{y_0}{y_1}\right) \int_0^1 u(x) dx + C\frac{y_0}{y_1}. $$

Hence,

$$\int_0^1 u(x) dx \geq \frac{y_1}{y_1 - y_0} \left(\sum_{j=1}^l \int_0^1 \log |v_j(x + iy_0) - E| dx - C\epsilon^{\frac{1}{2}} - C\frac{y_0}{y_1}\right)$$

$$\geq \sum_{j=1}^l \int_0^1 \log |v_j(x + iy_0) - E| dx - C\frac{2\epsilon}{\rho - \xi} - C\frac{\rho}{\rho - \xi} \epsilon^{\frac{1}{2}}$$

$$\geq \sum_{j=1}^l \int_0^1 \log |v_j(x + iy_0) - E| dx - C\epsilon^{\frac{1}{2}}(\xi + \epsilon^{\frac{1}{2}}).$$

Repeating the process above, we have

$$\int_0^1 u(x) dx \geq \sum_{j=1}^l \int_0^1 \log |v_j(x - iy_0) - E| dx - C\epsilon^{\frac{1}{2}}(\xi + \epsilon^{\frac{1}{2}}).$$
From the convexity argument (see \cite{26} for details), we obtain
\[
\int_{0}^{1} \log |v_j(x) - E| \, dx \leq \frac{1}{2} \int_{0}^{1} \log |v_j(x + iy_0) - E| \, dx + \frac{1}{2} \int_{0}^{1} \log |v_j(x - iy_0) - E| \, dx.
\]
Noting \( \xi = \frac{\varepsilon}{12} \), we have
\[
\frac{1}{N} \int_{0}^{1} \log |\det[H_{N}(x, E)]| \, dx \geq \sum_{j=1}^{k} \int_{0}^{1} \log |v_j(x) - E| \, dx - C \varepsilon \frac{1}{\pi}.
\]
The proof of Proposition 4.2 is finished. \( \square \)

5. Green’s function estimates

In this section, we consider the Green’s function
\[
G_{N}(x; E) := \left( H_{N}(x) - EI_{N} \right)^{-1},
\]
whenever \( H_{N}(x) - EI_{N} \) is invertible, where \( N \subset \mathbb{Z} \) being an interval. A crucial ingredient in the proof of AL is the so called LDT for Green’s function. In this section, we will prove the LDT and this can be achieved using a quantitative Birkhoff ergodic theorem for the function \( u_{N}(x) \) defined in \( \text{4.3} \) as well as the uniformly upper and averaging lower bounds in Propositions 3.4 and 4.2.

**Definition 5.1.** We say that \( G_{N}(x; E) \) is a good Green’s function if for all \( n, n' \in \mathbb{N} \),
\[
\|G_{N, (n, n')} (x; E)\| \leq e^{-\left(\left| n - n' \right| - \frac{|N|}{100}\right)(\rho - \varepsilon)},
\]
where \( \delta > 0 \) is a constant.

**Remark 5.2.** It is easy to show \( G_{N}(x; E) \) is a good Green’s function if and only if there is some \( C = C(l) > 0 \) (depending only on \( l \)) such that
\[
|G_{N, (\alpha, \alpha')} (x; E)| \leq Ce^{-\left(\left| n(\alpha) - n(\alpha') \right| - \frac{|N|}{100}\right)(\rho - \varepsilon)},
\]
where \( (a - 1)l < \alpha, \alpha' < bl \) with \( N = [a, b] \subset \mathbb{Z} \) being an interval.

We first recall a useful lemma concerning the semi-algebraic set (for the basic knowledge of the semi-algebraic sets, see Chapter 9 of \cite{4} by Bourgain).

**Lemma 5.3** (Lemma 9.7 in \cite{4}). Let \( S \subset [0, 1] \) be a semi-algebraic set of degree \( B \) and \( \text{Leb}(S) \leq \eta \). Let \( \omega \in \text{DC}_{1} \) and \( K \) be a large integer satisfying
\[
\log B \ll \log K < \log \frac{1}{\eta}.
\]
Then for any \( x \in \mathbb{T} \),
\[
\# \{ k = 1, \cdots, K : x + k\omega \in S \ (\text{mod} \ 1) \} \leq K^{1 - \tau},
\]
where \( \tau = \tau(t) \in (0, 1) \) depends only on \( t \) and \#A denotes the number of elements of finite set \( A \).

One also has the following important quantitative Birkhoff ergodic theorem for some subharmonic function.
Lemma 5.4 (Theorem 6.5 in [12]). Let \( u : \Delta_\rho \rightarrow [-\infty, \infty) \) be a subharmonic function satisfying

\[
(5.3) \quad \sup_{z \in \Delta_\rho} u(z) + \left( \int_{\mathbb{T}} |u(x)|^2 dx \right)^{\frac{1}{2}} \leq C_*.
\]

Let \( \omega \in DC_1, M_0 = t^{-2} \). Then there are absolute constant \( a > 0 \) and \( C = C(\rho) > 0 \) (depending only on \( \rho \)) such that for \( M \geq M_0 \),

\[
(5.4) \quad \text{Leb} \left\{ x \in \mathbb{T} : \left| \frac{1}{M} \sum_{j=0}^{M-1} u(x+j\omega) - \int_{\mathbb{T}} u(x) dx \right| \geq \frac{CC_*}{\rho} M^{-a} \right\} \leq e^{-2\frac{20a}{C_*} M^a}. \]

In the following, we let \( u = u_N(x) \) with \( u_N(x) \) being given by (5.5). Recalling Proposition 4.2 we can define the set

\[
B^M_N = B^M_N(\omega, E)
\]

\[
(5.5) \quad = \left\{ x \in \mathbb{T} : \left| \frac{1}{M} \sum_{j=0}^{M-1} u(x+j\omega) \leq \sum_{j=1}^{M} \int_{\mathbb{T}} \log |v_j(x) - E| dx - C\rho^\delta \right| \right\},
\]

where \( \omega \in DC_1, E \) belongs to an interval \( A \subset \mathbb{R} \) and \( C > 0 \) depends only on \( V, \rho, l \). We assume \( \text{Leb}(A) \leq C(V, \rho) \). We have the following proposition.

Proposition 5.5. Let \( B^M_N \) be defined by (5.5) and \( 0 < \epsilon \ll 1 \) (depending only on \( V, \rho \)). There are absolute constants \( a > 0 \) and \( P \in \mathbb{N} \) such that if \( M \geq M_0(V, \rho, \epsilon, l, t) \) and \( N \geq N_0(V, \rho, \epsilon, l, t) \), then the following holds.

(i) \( \text{Leb}(B^M_N(\omega, E)) < e^{-cM^a} \),

where \( c = \frac{20a}{CC_*} > 0 \) is given by Lemma 5.4.

(ii) For every \( x \notin B^M_N(\omega, E) \) there is \( 0 \leq j < M \) such that \( G_N(x+j\omega; E) \) is a good Green’s function.

(iii) For any \( x \in \mathbb{T} \),

\[
\# \left\{ 0 \leq n < N^P : G_N(x+n\omega; E) \text{ is not a good Green’s function} \right\} \leq N^{(1-\tau)P},
\]

where \( \tau = \tau(t) \) is given by Lemma 5.3.

Proof. (i) We use Lemma 5.5 for \( u(x) = u_N(x) \) defined in (4.8). From Proposition 4.2 and Lemma A.2 we can take \( C_* = C(V, \rho) \) and if \( M \geq M_0 \), we have \( \text{Leb}(B_1) \leq e^{-cM^a} \) with

\[
B_1 = \left\{ x \in \mathbb{T} : \left| \frac{1}{M} \sum_{j=0}^{M-1} u(x+j\omega) - \int_{\mathbb{T}} u(x) dx \right| \geq \frac{C_*}{\rho} M^{-a} \right\}.
\]

Thus it suffices to show \( B^M_N \subset B_1 \) if \( M \gg 1 \). In fact, if \( x \notin B_1 \), then by Proposition 4.2 and for \( M \gg 1 \), we have

\[
\frac{1}{M} \sum_{j=0}^{M-1} u(x+j\omega) \geq \int_{\mathbb{T}} u(x) dx - \frac{C_*}{\rho} M^{-a} \geq \sum_{j=1}^{l} \int_{\mathbb{T}} \log |v_j(x) - E| dx - C_1 \rho^\delta,
\]
that is \( x \notin B^M_N \).

(ii) By Cramer’s rule, we have
\[
G_{N,(\alpha,\alpha')}(x; E) = \frac{\mu_{N,(\alpha,\alpha')}(x, E)}{\det[H_N(x) - EI]},
\]
and thus
\[
(5.6) \quad \frac{1}{N} \log |G_{N,(\alpha,\alpha')}(x; E)| = \frac{1}{N} \log |\mu_{N,(\alpha,\alpha')}(x, E)| - u_N(x).
\]

From Proposition 3.3, we have for any \( x \in T \) and \( N \geq N_0 \),
\[
(5.7) \quad \frac{1}{N} \log |\mu_{N,(\alpha,\alpha')}(x, E)| \leq \sum_{j=1}^l \int_T \log |v_j(x) - E| dx + l \epsilon^\sigma + \frac{|n(\alpha) - n(\alpha')|}{N} (-\rho + \epsilon^\sigma).
\]

If \( x \notin B^M_N \), by item (i), we have
\[
\frac{1}{M} \sum_{j=0}^{M-1} u(x + j\omega) \geq \sum_{j=1}^l \int_T \log |v_j(x) - E| dx - Cl\epsilon^\delta.
\]

Then there is \( 0 \leq j_0 < M \), such that
\[
(5.8) \quad u_N(x + j_0\omega) \geq \sum_{j=1}^l \int_T \log |v_j(x) - E| dx - 2Cl\epsilon^\delta.
\]

Thus combing (5.6)-5.8, we conclude for \( 0 < \epsilon \ll 1 \) (depending only on \( V, \rho, l \)),
\[
\frac{1}{N} \log |G_{N,(\alpha,\alpha')}(x + j_0\omega; E)| \leq - (\rho - \epsilon^\sigma) \frac{|n(\alpha) - n(\alpha')|}{N} + Cl\epsilon_{\min}(\sigma, \delta).
\]

This implies \( G_N(x + j_0\omega; E) \) is a good Green’s function (see 5.1).

(iii) Let \( 1 \ll M \sim N^2 \). Fixing \( \omega \in DC_t \) and \( E \in A \), we rewrite the inequality defining the set \( B^M_N \) as
\[
(5.9) \quad Q^\pm_T(x) := \pm \prod_{j=1}^{M-1} \det[H_N(x + j\omega) - EI_N] \leq e^{NM(\sum_{j=1}^l f_j \log |v_j(x) - E| dx - Cl\epsilon^\delta)}.
\]

We truncate each \( v_j(x) \) as \( v_{j,N}(x) = \sum_{|k| \leq N^2} \hat{v}_j(k) e^{2\pi kix} \) first, where \( \hat{v}_j(k) \) are the corresponding Fourier coefficients. As a result, \( \|v_j - v_{j,N}\|_0 \leq Ce^{-\rho N^2} \) \((1 \leq j \leq l)\). If we replace each \( v_j \) with \( v_{j,N} \) in \( Q^\pm_T \), we get the trigonometric polynomials \( Q^\pm_T(\cos 2\pi x, \sin 2\pi x) \) and the degrees of \( Q^\pm_T \) are bounded by \( lMN^3 \). Obviously, \( \|Q^\pm_T - Q^\pm_T\|_0 \leq Ce^{-\rho N^2} \). Furthermore, in \( Q^\pm_T \), if we replace \( \sin 2\pi x, \cos 2\pi x \) with polynomials \( (\sin x) \) of degree \( N \), we can obtain polynomials \( Q^\pm_T(x) \) of degrees being bounded by \( O(N^4M) \) such that \( \|Q^\pm_T - Q^\pm_T\|_0 \leq Ce^{-\rho N^2} \). Thus we have showed \( B^M_N(\omega, E) \) can be replaced by a semi-algebraic set
\[
S = \left\{ x \in [0, 1] : Q^\pm_T(x) \leq e^{NM(\sum_{j=1}^l f_j \log |v_j(x) - E| dx - Cl\epsilon^\delta)} \right\},
\]
and the statements of items (i) (ii) are still valid for \( S \) (since \( MN \ll N^2 \) and Lemma A.2).

In fact, we know the degree of \( S \) is bounded by \( O(N^5) \). Thus using Lemma 5.3 there is some absolute constant \( P \in \mathbb{N} \), such that
\[
\#\{k = 0, 1, \ldots , N^P - 1 : x + k\omega \in S\} \leq N^{(1-\tau)P}.
\]
Since $S$ has property (ii), then (iii) holds.

Remark 5.6. One should note that the set $B^{M}_{N}(\omega, E)$ relies heavily on $E$.

Remark 5.7. The statements similar to those in items (ii) (iii) also hold if we replace $G_{N}(x; E)$ with $G_{[-N,N]}(x; E)$.

6. The proof of localization: eliminating the energy

Based on the Green’s function estimates and semi-algebraic sets considerations in Proposition 5.5, we will finish the proof of Theorem 1.1. We use the techniques developed by Bourgain and Goldstein in [6] to establish non-perturbative AL for quasi-periodic Schrödinger operators.

From Sch’nol-Simon theorem (see [22] for details), to prove AL, it suffices to show that every extended state decays exponentially.

Definition 6.1. We call $E \in \mathbb{R}$ a generalized eigenvalue of $H_{\epsilon, \omega}(x)$ if there is some $\vec{\psi} = \{\vec{\psi}_{n}\}_{n \in \mathbb{Z}}$ with $\|\vec{\psi}_{n}\| \leq C(\vec{\psi})(1 + |n|^{2})$ such that $H_{\epsilon, \omega}(x)\vec{\psi} = E\vec{\psi}$. Moreover, the corresponding $\vec{\psi}$ is called the extended state.

We note that

• Let $E$ be a generalized eigenvalue of $H_{\epsilon, \omega}(x)$ and $\vec{\psi}$ be the corresponding extended state. Then for any $j \in N \subset \mathbb{Z}$,

$$\vec{\psi}_{j} = - \sum_{i \in \mathbb{N}, k \notin \mathbb{N}} G_{N, (j, i)}(x; E)W_{i-k}\vec{\psi}_{k}.$$  

• For any $N \subset \mathbb{Z}$,

$$G_{N}(x + j\omega; E) = G_{N+j}(x; E),$$  

where $N + j = \{n + j : n \in N\}$.

In fact, good Green’s function implies exponential decay of the corresponding extended state.

Lemma 6.2. Let $N \geq N_{0}, N = [\sqrt{N}, 2N]$. Assume $E$ is a generalized eigenvalue of $H_{\epsilon, \omega}(x)$ with $\vec{\psi} = \{\vec{\psi}_{n}\}$ being corresponding extended state. If $G_{N}(x; E)$ is a good Green’s function, then for $\frac{N}{2} \leq j \leq N$,

$$\|\vec{\psi}_{j}\| \leq e^{-\frac{\rho}{3}j}.$$  

Proof. It is easy to show

$$\sum_{k \geq 0} k^{p}e^{-\rho k} \leq Ca^{p}e^{-\rho a}, \quad p \geq 0,$$

where $C > 0$ depends only on $\rho$ and $p$. Then for any $i \in N$,

$$\sum_{k \notin N} e^{-\rho|i-k-2N|} \leq C e^{-\rho|i-k|}(e^{-\rho|2N|} + e^{-\rho|2N|}).$$

As a result, recalling (6.1), we have for any $\frac{N}{2} \leq j \leq N$,

$$\|\vec{\psi}_{j}\| \leq C N^{2} \sum_{i \in \mathbb{N}} e^{-\rho|2N|}e^{-(\rho - \epsilon)\frac{j}{2N}} + C N^{2} \sum_{i \in \mathbb{N}} e^{-\rho|2N|}e^{-(\rho - \epsilon)\frac{j}{2N}} \leq e^{-\frac{\rho}{3}j} \quad \text{for } N \gg 1.$$  

The proof is finished. \qed
The following lemma suggests that a good Green’s function at large scale can be obtained from paving good Green’s functions at small scale.

**Lemma 6.3** (Lemma 10.33 in [4]). Let $\mathcal{N} \subset \mathbb{Z}$ be an interval with length $N$ and $\{\mathcal{N}_a\}$ be subintervals with length $M \ll N$. Assume

(i) If $k \in \mathcal{N}$, then there is some $\alpha$ such that $[k - \frac{M}{N}, k + \frac{M}{N}] \cap \mathcal{N}_a \subset \mathcal{N}$,

(ii) For all $\alpha$, $G_{N_a}$ are good.

Then $G_\mathcal{N}$ is good.

**Remark 6.4.** This lemma follows from the resolvent identity and see §15 of [3] for details.

We also need the following lemma which is crucial in the eliminating energy process.

**Lemma 6.5.** Let $N \geq N_0$ be fixed. Then there are some absolute constant $P_1 > 0$ and $N_1 \in \mathbb{N}$ with $N_1 = \mathcal{O}(N^{P_1})$ such that if $E$ is a generalized eigenvalue of $H_{\epsilon, \omega}(x)$ and $\psi$ is the corresponding extended state with $\|\psi_0\| = 1$, then

$$\text{dist}(E, \sigma(H_{[-N_1, N_1]}(x))) \leq e^{-\frac{\epsilon}{2}N^2},$$

where $\sigma(H)$ denotes the spectrum of the operator $H$.

**Proof.** The proof is similar to that of (6.6) in [3]. Using (iii) of Proposition 5.5 and Remark 5.7 at scale $N^2$, we obtain that there is some interval $\mathcal{I} \subset [0, N^{2P}]$ with length $|\mathcal{I}| \sim N^{2P}$ such that for $n \in \mathcal{I} \cup (-\mathcal{I})$, the Green’s function $G([-N^2, N^2]^2+n(x; E) = G([-N^2, N^2]^2+ \omega; E)$ is good, where $-\mathcal{I} = \{-n : n \in \mathcal{I}\}$. As a result, one has for any $n \in \mathcal{I} \cup (-\mathcal{I})$,

$$\|\psi_n\| \leq C \sum_{|j_1-n| \leq N^2, |j_2-n| > N^2} e^{-(\rho-\epsilon)(|j_1-n|-\frac{\epsilon}{2N^2})} e^{-\rho j_1-j_2} |j_2|^2$$

$$= C \sum_{|j_1-n| \leq N^2, |j_2-n| > N^2} e^{\frac{\epsilon N^2}{2}} e^{-\rho |j_2-n|} |j_2|^2 \quad (\text{for } 0 < \epsilon \ll 1)$$

$$\leq e^{-\frac{\epsilon}{2}N^2}.$$

We now let $N_1$ be the center of interval $\mathcal{I}$. Then

$$1 = \|\psi_0\| \leq \sum_{|j_1| \leq N_1, |j_2| > N_1} \|G([-N_1, N_1](x; E)|e^{-\rho j_1-j_2}\|^2 \|\psi_2\|$$

$$\leq \|G([-N_1, N_1](x; E))\| \sum_{|j_1| \leq N_1, |j_2| > N_1 \mathcal{I} \cup (-\mathcal{I})} e^{-\frac{\epsilon}{2}N^2}$$

$$+ C \|G([-N_1, N_1]x; E))\| \sum_{|j_1| \leq N_1, |j_2| > N_1 + \frac{\epsilon}{2}} e^{-\rho |j_1-j_2|} |j_2|^2$$

$$\leq \|G([-N_1, N_1](x; E))\| e^{-\frac{\epsilon}{2}N^2}.$$

Thus

$$\text{dist}(E, \sigma(H_{[-N_1, N_1]}(x))) = \|G([-N_1, N_1](x; E))\|^{-1} \leq e^{-\frac{\epsilon}{2}N^2}.$$

Now we can finish the proof of our main theorem.

**Proof of Theorem 1.1** Fix $x_0 \in \mathcal{T}$ and let $N \in \mathbb{N}$ be any fixed large enough scale. Consider a much larger scale $N' = N^{P_2}$ with $P_2 \gg 1$ being an absolute constant.

We want to prove for any $E$ being a generalized eigenvalue of $H_{\epsilon, \omega}(x_0)$, the corresponding extended state $\psi$ with $\|\psi_0\| = 1$ decays exponentially on $[\frac{N'}{2}, N']$. From Lemmata 6.2 and 6.3
one only needs to show that for any \( n \in \mathbb{N} \), there is some \( 0 \leq j_n \leq \frac{N}{100} \) such that

\[
G_{[1,N]+j_n}(x_0 + n\omega; E) \text{ is good for all } E.
\]

Then from (ii) of Proposition 6.5, the statement (6.3) is equivalent to

\[
x_0 + n\omega \not\in \bigcup_E B_N(\omega, E) \text{ for all } n \in [\sqrt{N'}, 2N'],
\]

where \( B_N(\omega, E) = B_N^{\omega \cdot N}(\omega, E) \) (see (6.3)). Fortunately, from Lemma 6.5, to prove statement (6.3), one just needs

\[
x_0 + n\omega \not\in \bigcup_{E \in \sigma(H_{-N_1,N_1}(x_0))} B_N(\omega, E) \text{ for all } n \in [\sqrt{N'}, 2N'],
\]

where \( N_1 \) is given by Lemma 6.5 and \( N_1 \ll N' \). Thus we define sets

\[
S^\omega_N = \bigcup_{E \in \sigma(H_{-K,K}(x_0))} B_N(\omega, E),
\]

and

\[
S_N = \{(\omega, x) \in \mathbb{T}^2 : \omega \in DC_t(N), x \in S^\omega_N \},
\]

where \( DC_t(N) := \{\omega : \|k\omega\|_T \geq \frac{t}{|k|} \text{ for } 0 < |k| \leq N^2\} \). As a result, similarly to the proof of (iii) of Proposition 6.5, \( S_N \) can be regarded as a semi-algebraic set with sub-exponentially small Lebesgue measure in \( N \) and polynomially bounded degree in \( N \). Using Lemma 9.9 in [4], the projective set along the frequencies

\[
R_N = \{\omega : (\omega, \{x + n\omega\}) \in S_N \text{ for some } \sqrt{N'} \leq n \leq 2N'\}
\]

can also be regarded as a semi-algebraic set with

\[
\text{Leb}(R_N) \leq N^{-C},
\]

where \( \{x + n\omega\} = x + n\omega \pmod{1} \) and \( C > 1 \). The detail elegant analysis can be found in the proof of Theorem 10.1 in [4]. Consequently, if \( \omega \in DC_t \setminus R_N \), then \( \|\tilde{\psi}_j\| \leq e^{-\frac{\rho}{3}j} \) for \( \frac{N'}{2} \leq j \leq N' \).

Note that \( \sum_{L \geq N} \text{Leb}(R_L) < \infty \). Then the set

\[
\mathcal{R} = \bigcap_{K \geq N} \bigcup_{L \geq K} R_L
\]

has zero Lebesgue measure by Borel-Cantelli theorem. Thus if \( \omega \in DC_t \setminus \mathcal{R} \), then \( \|\tilde{\psi}_j\| \leq e^{-\frac{\rho}{3}j} \) for \( j \geq N'/2 \).

Similarly, we can show \( \|\tilde{\psi}_j\| \leq e^{-\frac{\rho}{3}|j|} \) for \( j \leq -N' \) and the proof is finished. \( \square \)

**Appendix A.**

**Lemma A.1** (Lojasiewicz inequality, Lemma 7.3 in [4]). Let \( v \) be a nonconstant analytic function on \( \mathbb{T} \). Then there is a constant \( \sigma_0 = \sigma_0(v) > 0 \) (depending only on \( v \)) such that for sufficiently small \( \epsilon > 0 \) and all \( E \in \mathbb{R} \),

\[
\text{Leb}\{x \in \mathbb{T} : |v(x) - E| < \epsilon\} < e^{\sigma_0}.
\]

From this lemma, one can obtain the following two useful estimates.
Lemma A.2 (Lemma 6.2 in [12]). Let $v$ be as in Lemma A.1. Then there exists $C = C(v) > 0$ (depending only on $v$) such that

$$\int_T \log |v(x) - E|\,dx \geq -C.$$ 

Lemma A.3. Let $v$ be given by Lemma A.1. Then there is some $0 < \sigma_1 = \sigma_1(v) < 1$ such that

$$\int_T \log(|v(x) - E| + \epsilon)\,dx < \int_T \log |v(x) - E|\,dx + \epsilon^{\sigma_1},$$

where $0 < \epsilon \ll 1$.

Proof. Let $\sigma_0 > 0$ be given by Lemma A.1 and $0 < \sigma_2 < \sigma_0 < 1$. It is easy to see there is some constant $C_1 > 0$ such that $\log(1+x) \leq x^{\sigma_2}$ if $x > C_1$. Define $J = \{x \in T : \log\left(\frac{x}{|v(x) - E|}\right) > C\}$ and  

$$J_n = \{x \in J : 2^{-n}C_1^{-1}\epsilon \leq |v(x) - E| < 2^{-n}C_1^{-1}\epsilon\}$$

for $n \in \mathbb{N}$. Then $J = \bigcup_{n=0}^{\infty} J_n$ and $\text{Leb}(J_n) \leq 2^{-n\sigma_0}C_1^{-\sigma_0}\epsilon^{\sigma_0}$ by Lemma A.1. Thus

$$\int_J \log\left(1 + \frac{\epsilon}{|v(x) - E|}\right)\,dx \leq \epsilon^{\sigma_2} \int_J |v(x) - E|^{-\sigma_2}\,dx$$

$$\leq \sum_{n \geq 0} C_1\epsilon^{\sigma_2}2^{(n+1)\sigma_2}\text{Leb}(J_n)$$

$$\leq \sum_{n \geq 0} C_1\epsilon^{2\sigma_2 - \sigma_0}2^{-(\sigma_0 - \sigma_2)n}\epsilon^{\sigma_0}$$

$$\leq C_2\epsilon^{\sigma_0},$$

and

$$\int_{T \setminus J} \log\left(1 + \frac{\epsilon}{|v(x) - E|}\right)\,dx \leq \int_{\{x \in T : \frac{C_1\epsilon^{\sigma_0}}{|v(x) - E|} \leq \epsilon\}} \log\left(1 + \frac{\epsilon}{|v(x) - E|}\right)\,dx$$

$$+ \int_{\{x \in T : |v(x) - E| \geq \epsilon\}} \log\left(1 + \frac{\epsilon}{|v(x) - E|}\right)\,dx$$

$$\leq \log(1 + C_1)\epsilon^{\sigma_0} + \epsilon^{1-\sigma_0}.$$  

(A.1)

In the inequality (A.1), we use Lemma A.1 and the fact $\log(1+x) \leq x$ for $x \geq 0$. Recalling Lemma A.2 and by letting $\sigma_1 = (1 - \sigma_0)\sigma_0^2$, one has for $0 < \epsilon \ll 1$, 

$$\int_T \log(|v(x) - E| + \epsilon)\,dx = \int_T \log |v(x) - E|\,dx + \int_T \log\left(1 + \frac{\epsilon}{|v(x) - E|}\right)\,dx$$

$$\leq \int_T \log |v(x) - E|\,dx + \epsilon^{\sigma_1},$$

which completes the proof. \hfill \Box

Remark A.4. Actually in the proof of Lemma A.3, we have showed that for any $0 < \sigma_2 < \sigma_0$, 

$$\int_T |v(x) - E|^{-\sigma_2}\,dx > -C(\sigma_2, v) > -\infty,$$

which implies Lemma A.2.

We also need the following Denjoy-Koksma inequality.
Lemma A.5 (Lemma 12 in [24]). For any continuous real function $u$ on $\mathbb{T}$ and any interval $I \subset \mathbb{Z}$ with length $N$, we have

$$\left| \sum_{j \in I} u(x + j\omega) - N \int_{\mathbb{T}} u(x) \, dx \right| \leq CN^{\frac{1}{2}} \log N,$$

where $\omega \in DC_t$ and $C$ depends only on $u$ and $t$.

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