COSMOLOGICAL MODELS WITH FRACTIONAL DERIVATIVES AND FRACTIONAL ACTION FUNCTIONAL

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Abstract

Cosmological models of a scalar field with dynamical equations containing fractional derivatives or derived from the Einstein-Hilbert action of fractional order, are constructed. A number of exact solutions to those equations of fractional cosmological models in both cases is given.

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1 Introduction

As is observed [1], there has been in the last decade active interest to application of fractional derivatives in various fields of physics, where they play essential and sometimes leading role in understanding the complex classical and quantum systems. Surprisingly, the achievements of the fractional-differential calculus in the theory of gravitation and cosmology are very insufficient and probably no more then thirty or so publications are devoted to this subject (see, for example, [2]-[7]). At the same time, some essential problems in modern theoretical cosmology [8, 9] are known that lead to serious modification of the possible sources of the accelerated expansion of the Universe (or the gravitational theory itself). Under these circumstances, the interest to the fractional-differential modifications of cosmological models that appeared recently is fully justified.

In this paper, we provide a review of the main publications devoted to the fractional differential approach in cosmology (known to the author). We also offer a novel approach to obtaining the modified Friedmann equations consistent with main principles and allowing, in our opinion, to avoid the conflict between the integer dimension of (pseudo) Riemann space - time of GR and fractional order of derivatives in the modified equations. As is mentioned in [5], there are two different methods of approaching what fractional derivative cosmology could be. The simplest is the Last Step Modification (LSM) method, in which the Einstein’s field equations for a given configuration are replaced with analogous fractional field equations. In other words, \( \partial_t \rightarrow D_t^\alpha \) after the field equations for a specific geometry have been derived. The fundamentalist methodology is the First Step Modification (FSM), in which one starts by constructing fractional derivative geometry. The intensively developed approach to modification of the main cosmological equations and non-conservative systems of Lagrangian dynamics on the basis of a variational principle for the action of a fractional order (Fractional Action-Like Variational Approach FALVA) developed in [2]-[4] represents one of the possible version of intermediate modification (Intermediate Step Approach, ISA) mentioned in [5].

Our work is based in its main part on ISA and FALVA, because the equations of standard cosmology can be obtained from a variational principle for the Einstein-Hilbert action, in which the variation is made both over the scale factor and a laps function \( N \), or, as is shown in [10], the last can be replaced by a condition of a time scale invariancy, and in any case it is easy to generalize masters equations to the case of fractional derivatives. We obtain the fractional-differential analogue of the Friedmann equations including a scalar field as a source of gravitation in frameworks of ISA and on the basis of FALVA. Until now in cosmology with fractional derivatives, there exist more questions than answers to such of them as: How should such models be obtained? How should the basic equations of these models be written down? What is the meaning of these solutions for the modified models in aspect of the modern problems of cosmology? In our work, we try to answer some of these questions.

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2 Fractional integrals and derivatives

There are no less than two dozens of definitions of the fractional derivative [1]. In the physical applications of the fractional differential calculus, more often one deals with Riemann-Liouville derivative (RLD), Caputo derivative (CD), and some others.

Such derivatives are defined by means of analytical continuation of the Cauchy formula for the multiple integral of integer order as a single integral with a power-law core into the field of real order $\mu > 0$:

$$\mathcal{I}_x^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_c^x f(t)(x - t)^{\mu - 1} dt.$$ (1)

The Riemann-Liouville derivative of fractional order $\alpha \geq 0$ of function $f(x)$ is defined as the integer order derivative of the fractional-order integral (1):

$$D_x^\alpha f(x) \equiv \mathcal{D}_x^\alpha \mathcal{I}_x^{\alpha - \alpha} f(x) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dx^n} \int_c^x \frac{f(t)}{(x - t)^{\alpha - n + 1}} dt$$ (2)

where $D_x^n \equiv d^n/dx^n$, $n = [\alpha] + 1$. This definition corresponds to the so-called left derivative, frequently denoted as $f(x)$. For the limit $\alpha = 1$, this definition gives $df(x)/dx$. For example, the left RLD of $x^k$ for $\alpha \leq 1$, $c = 0$ equals:

$$D_x^\alpha x^k = \frac{\Gamma(k + 1)}{\Gamma(k + 1 - \alpha)} x^{k - \alpha}.$$ (3)

For $\alpha = 1$, one has the usual result: $D_x^1 x^k = kx^{k-1}$. The interesting feature of RLD is that RLD of non-zero constant $C_0$ does not equal zero, but for $\alpha \leq 1$ it equals $D_x^\alpha C_0 = C_0 x^{-\alpha}/\Gamma(1 - \alpha)$. The right RLD is defined similarly to (2) on the interval $[c, d]$:

$$x D_d^\alpha f(x) = \frac{1}{\Gamma(n - \alpha)} \left( -\frac{d}{dx} \right)^n \int_c^d \frac{f(t)}{(t - x)^{\alpha - n + 1}} dt$$ (4)

The Caputo derivative is the integral transform of a regular derivative, and it is defined by moving the integer-order derivative in the Riemann-Liouville definition (2) inside the integral to act on the function $f(t)$:

$$C D_x^\alpha f(x) \equiv \mathcal{D}_x^{\alpha - \alpha} \mathcal{I}_x^\alpha f(x) = \frac{1}{\Gamma(n - \alpha)} \int_c^x \frac{d^n f(t)}{dt^n} \frac{dt^n}{(x - t)^{\alpha - n + 1}} dt.$$ (5)

So, for $\alpha \leq 0$, $n = 1$ $c = 0$, we have the following expression:

$$C D_x^\alpha x^k = \frac{\Gamma(k)}{\Gamma(k + 1 - \alpha)} kx^{k - \alpha},$$

which coincides with RLD $D_x^\alpha x^k$. In general, RLD does not coincide with CD [1]:

$$C D_x^\alpha f(x) = D_x^\alpha f(x) + \sum_{j=0}^n \frac{1}{\Gamma(1 + j - \alpha)} \frac{f^{(j)}(c)}{(x - c)^{\alpha - j}}.$$ (6)

RLD and CD are not defined for power function $x^k$ with $k = -1$ due to divergency of the integrals in the upper limit. The derivative of $1/x$ can be obtained as the Weyl derivative defined as

$$D^\alpha f(x) = \frac{(-1)^{n-1}}{\Gamma(n - \alpha)} \int_x^\infty \frac{d^n f(t)}{dt^n} (t - x)^{n-\alpha} dt.$$
which provides $D^\alpha x^{-1} = -x^{-(1+\alpha)}\Gamma(1+\alpha)$ for $\alpha \leq 1 (n=1)$.

The left and the right CD are defined similarly to those for RLD, i.e., due to replacing
the limits $c \rightarrow x$ $x \rightarrow d$ in definition (5) consequently.

One needs to be aware that according to the formulas of addition of orders, the following
holds([1], p.161):

$$D^\alpha_x D^\beta_x f(x) = D^\alpha_x f(x) - \sum_{j=1}^{n} D^{\beta-j}_x f(c+) \frac{(x-c)^{-\alpha-j}}{\Gamma(1-\alpha-j)} ,$$

that is $D^\alpha_x D^\beta_x f(x) \neq D^{\alpha+\beta}_x f(x)$, if only not all derivatives $D^{\beta-j}_x f(c+)$ at the beginning of the
interval are equal to zero. That is why $D^\alpha_x D^\beta_x f(x) \neq D^{2\alpha}_x f(x)$ in the general case. Generalizing
the Laplace operator in the equation for Newtonian gravitational potential, the author of [5]
wrongly doubles the order of the repeated fractional derivative. The authors of [11] have avoided
this mistake, having written down the Laplacian $\Delta^\alpha$ as:

$$\Delta^\alpha u = \frac{1}{r^{2\alpha}} D^\alpha_\rho(r^{2\alpha} D^\rho_\rho u) + \frac{\Gamma(\alpha + 1)}{r^{2\alpha} \sin^\alpha \theta} \frac{\partial}{\partial \theta} (\sin^\alpha \theta \frac{\partial u}{\partial \theta}) + \frac{\Gamma(\alpha + 1)}{r^{2\alpha} \sin^\alpha \theta} \frac{\partial^2 u}{\partial \phi^2}.$$  

However their research did not concern cosmological problems.

Let us note one more property of the fractional derivative expressed in modification of the
Leibniz rule ([1],p.162):

$$D^\alpha_x [f(x)g(x)] = \sum_{k=0}^{\infty} \frac{\alpha + 1}{k!\Gamma(\alpha - k + 1)} D^\alpha_x D^{\alpha-k}_x f(x) D_x^k g(x) ,$$  

which becomes the usual rule as $\alpha = n$. It can be represented as the integral over the order of
fractional derivative:

$$D^\alpha_x [f(x)g(x)] = \int_{-\infty}^{\infty} \frac{\Gamma(\alpha + 1)}{\Gamma(\mu + 1)\Gamma(\alpha + 1 - \mu)} D^\alpha_x D^{\alpha-\mu}_x f(x) D_x^\mu g(x)d\mu .$$

Later these rules of fractional differentiation will give us an essential modification to the cosmo-
logical models with fractional derivatives.

At last we want to note, that RLD can be expressed with the help of the integer derivative at
the initial point of the interval and CD ([1], p. 163) due to definition (5):

$$D^{\alpha}_x f(x) = \sum_{k=0}^{n-1} \frac{(x-c)^{k-\alpha}}{\Gamma(1 + k - \alpha)} f^{(k)}(c+) + CD^\alpha_x f(x) .$$

### 3 Cosmological models with fractional derivatives

Presumably, for the first time such models within LSM were considered in [2]. To avoid a
conflict between the occurrence of fractional derivatives in the Friedmann equations and classical
definition of a tensor in the Einstein equation, based on the integer order derivative in tensor law
of transformation, the quoted author has repeated the well known derivation of the Friedmann
equations for a dust from the classical approach (see [3], [12]) and than has replaced all integer
derivatives with its fractional analog. For the spatially flat Universe, the Friedmann equations
with a cosmological term are written down in [2] as follows:

$$(D^\alpha_\tau a(t))^2 = (A_1(G\rho)^\alpha + B(\Lambda c^2)^\alpha)a^2,$$

$$D^\alpha_\tau D^\alpha_\tau a(t) = -(A_2(G\rho)^\alpha - B(\Lambda c^2)^\alpha)a ,$$

where $a(t)$ is a scale factor of the Friedmann-Robertson-Walker (FRW) line element,

$$ds^2 = dt^2 - a^2(t)(dr^2 + \xi^2(r)d\Omega^2) ,$$  

where $a(t)$ is a scale factor of the Friedmann-Robertson-Walker (FRW) line element,
where $\xi(r) = \sin r, r, \sinh r$ for the sign of space curvature $k = +1, 0, -1$, consequently. The occurrence of $\alpha$ - degrees in the right-hand side of equations (7) is caused, probably, by dimensional reasons. Then the author of the quoted work obtained the following solution to this set in a static case $a = a_0 = \text{const}$:

$$(G\rho)^{\alpha} = \frac{C_1}{t^{2\alpha}}, \quad (\Lambda c^2)^{\alpha} = \frac{C_2}{t^{2\alpha}}.$$  

The conclusion is made, that the density of matter and cosmological constant decrease as $1/t^2$ if $G$ and speed of light in vacuum $c$ remains constant, but if the density of matter and $\Lambda$ remain constant, then the following holds: $G \sim 1/t^2$ and $c \sim 1/t$. The solution of equations (7), mentioned as an illustration of the method of [2], in the form $a(t) = a_0E_1,\alpha(Ct^{\alpha})$, where

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}$$

is the so-called Mittag-Leffler two-parametric function [1], is not confirmed by any calculations.

It is interesting to note that for $k = 0$ and in the absence of matter and the cosmological constant, that is if $\rho = 0$ and $\Lambda = 0$, equations (7) are reduced to

$$D_\alpha^\mu a(t) = 0, \quad D_\alpha^\mu (D_\alpha^\mu a(t)) = 0,$$  

(9)

which for $\alpha = 1$ gives the obvious result: $a(t) = a_0 = \text{const}$, and interval (8) is reduced to those of the Minkowski space. If $\alpha \neq 1$, then the substitution of the zero constant (that is the derivative $D_\alpha^\mu a(t)$) from the first equation of (9) into the second one results in identity irrespectively to the definition of fractional derivative: RLD or CD. The solution of the first equation in (9) for CD, as well as in the case of integer derivative, is equal to constant, but for RLD its partial solution is zero, and the general solution depends on time: $a(t) \sim t^{\alpha-1}$ for $0 < \alpha \leq 1$ (see [1], p. 216). This circumstance and also the fact, that initial conditions for equations with CD should be expressed by means of integer derivative, instead of fractional one, as in the case of RLD, frequently decline a choice of definition for the benefit of CD. In general considerations, we shall not specify a type of fractional derivative so long as it will be possible.

Let us notice that in [5] the Universe dynamics equations are written down in other form, namely:

$$3[k + (D_\alpha^\mu a)^2] = \kappa \rho a^2, \quad a^3D_\alpha^\mu p = D_\alpha^\mu [a^3(\rho + p)],$$  

(10)

where we use $\rho$ for the energy density and $p$ for the pressure. The second equation represents the energy-momentum conservation law ($T_{ij}^\gamma = 0$) for the perfect fluid in space-time (8) with the integer-order derivatives replaced by its fractional analogues. Assuming further $k = 0$ and the power-law dependence of $a, p$ and $\rho$ on the time: $a = C t^n$, $p = A t^m$, $\rho = B t^\gamma$, with the help of formulas (3) and (10) we can find:

$$m = r = -2\alpha, \quad B = \frac{3}{\kappa} \frac{\Gamma(n + 1)^2}{\Gamma(n + 1 - \alpha)^2},$$

and then from the equation of state $p = (\gamma - 1)\rho$ and the second equation in (10), the following equation for degree $n$ in the law of evolution of the scale factor is obtained:

$$\frac{(\gamma - 1)\Gamma(1 - 2\alpha)}{\Gamma(1 - 3\alpha)} = \frac{\Gamma(3n - 2\alpha + 1)}{\Gamma(3n - 3\alpha + 1)}.$$  

In the same spirit of naive approach, as it was named by the author of [5], the article [6] is prepared. In it, the Riemann curvature tensor and, as a consequence, the Einstein tensor are defined by the unusual Christoffel symbols containing fractional (of order $0 < \alpha \leq 1$) derivatives of metrics coefficients,

$$\Gamma_{\mu\lambda}^\nu(\alpha) = \frac{1}{2} g^{\mu\nu}(\partial^\mu g_{\beta\lambda} + \partial^\mu g_{\mu\lambda} + \partial^\mu g_{\nu\lambda}),$$  

(11)

where $\partial^\mu g_{\nu\lambda}$ is a fractional derivative (2) or (5) with respect to $x^\nu$. The quoted author writes down the Einstein equation,

$$R_{\mu\nu}(\alpha) - \frac{1}{2} g_{\mu\nu} R(\alpha) = \frac{8\pi G}{c^4} T_{\mu\nu}(\alpha),$$
and the geodetic equation,
\[
\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}(\alpha) \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0,
\]
but does not give any solutions, and only demonstrates that linearized equations in the limit \( \alpha \rightarrow 1 \) are reduced to the usual equations with the integer derivatives for the gravitational waves and the Newtonian potential, that was expected from the very beginning due to the definition of a fractional derivative. Let us note, that an attempt to prove such fractional derivative replacement and the Newtonian potential, that was expected from the very beginning due to the definition of a fractional derivative, that is so far not clear what such geometry should be. However, in [5] an attempt of construction of fractional geometry with the help of fractional coordinate transformations \( dx^i = D^\alpha_i x^i dx^j \) is undertaken for the flat two-dimensional space.

There were much more advanced and proved results concerning FSM formalism and stated by S. Vacaru [13] (see the bibliography therein). In those works, the results of construction of the fractional theory of gravitation for the space-time of fractional (not integer) dimension are obtained. The author sees one of the simplest motivation for application of fractional differential calculus in the theory of gravitation in an opportunity to avoid singularities of the curvature tensor of physical meaning due to the completely different geometrical and physical solutions of the fundamental equations. Besides, it is noted that models of fractional order are more adapted to the description of processes with memory, branching and heredity, than those of integer order. The result of the application of the method developed by the author of nonholonomic deformations to cosmology was the construction of new classes of cosmological models [14]. However, the last does not concern to the models under consideration, and it will not be considered here.

Let us now mention papers [3, 4], where the approach to the dynamical field theories in general and to the theory of gravitation is developed on the basis of the variational principle, formulated by the author, for the action of fractional order (FALVA). In this approach concerning ISA, the integral of action \( S_L[q] \) for the Lagrangian \( L(\tau, q(\tau), \dot{q}(\tau)) \) is written down as the fractional integral (1):
\[
S_L[q] = \frac{1}{\Gamma(\alpha)} \int_0^t L(\tau, q(\tau), \dot{q}(\tau))(t - \tau)^{\alpha - 1} d\tau,
\]
being at fixed \( t \) the Stieltjes integral with integrating function \( g_t(\tau) = \frac{1}{\Gamma(1 + \alpha)}[t^{\alpha} - (t - \tau)^{\alpha}] \), having the following scale property:
\[
g_{\mu t}(\mu \tau) = \mu^\alpha g_{t}(\tau), \quad \mu > 0.
\]
Then \( q_i(\tau) \) satisfies the fractional (or modified) Euler-Lagrange equation:
\[
\frac{\partial L}{\partial q_i} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = \frac{1 - \alpha}{t - \tau} \frac{\partial L}{\partial \dot{q}_i} \equiv \mathcal{F}_i, \quad i = 1, 2, ..., n; \quad \tau \in (0, t),
\]
where a dot above the appropriate function stands for the first derivative with respect to time \( \tau \), \( \mathcal{F}_i \) is the modified decaying force of “friction”, that is the general expression for non-conservative force. In the article [15], time \( \tau \) is treated as the intrinsic (proper) time, and \( t \) is the observer time. The author of [3, 4], [16] states that at \( \tau \rightarrow \infty \) we have \( \mathcal{F}_i = 0 \), and provides some examples of application FALVA to the Riedmann geometry and perturbed cosmological models. Every time the quoted author does not apply FALVA directly to the gravitational action \( S_G[a(t)] \), where \( a(t) \) is the scale factor in the FRW model (8), but tries to take into account influence of the fractional order action (12) on the Friedmann equations through perturbed (and time-dependent) classical gravitational constant. Considering Lagrangian \( L = g_{ij}(x, \dot{x})\dot{x}^i\dot{x}^j \), the author obtains the modified geodesic equation:
\[
\ddot{x}^i + \frac{\alpha - 1}{T} \dot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0,
\]
where \( \Gamma^i_{jk} \) are the standard Christoffel symbols, and \( T = t - \tau \). The second term here is interpreted as a dissipative force, which infinitely increases as \( \tau \rightarrow t \) for \( \alpha \neq 1 \) and under condition of fixing
future time $t$. Considering equation (13) in the Newtonian approximation, the author established the time variation of Newton’s gravitational constant (i.e., tried to realize the Dirac hypothesis[17]) and introduced the perturbation of the gravitational constant $\Delta G = \frac{3(1 - \alpha) \dot{a}}{4\pi \rho T a}$. Only then, the effective gravitational constant $G_{\text{eff}} = G + \Delta G$ is substituted to the standard Friedmann equations:

$$\frac{1}{a^2} (\dot{a}^2 + k) = \frac{8\pi G_{\text{eff}}}{3} \rho + \frac{\Lambda}{3}, \quad (14)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_{\text{eff}}}{3} (\rho + 3p) + \frac{\Lambda}{3}, \quad (15)$$

where the cosmological constant $\Lambda$ equals zero or $\Lambda = (\beta/t) (\dot{a}/a)$, as it is made in [18], where $G_{\text{eff}} = G = \text{const}$. Then in [18], [19] the solutions of equations (14), (15) are investigated for various equations of state ($p = \gamma \rho$), which could make sense if these equations or the way one obtains them were enough justified. It would be natural to expect of the author of FALVA the direct applications of FALVA to construction of the modified Friedmann equations from the fractional functional $S_G[a(t)]$ that leads to other results, as it will be shown below, without necessity to use equation (13) for the mentioned above interpretation of the extra terms in modified equation. Moreover, with the help of canonical parameter $s = g_t(\tau)$, equation (13) is reduced to:

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0, \quad (16)$$

where the over dots stand for derivatives with respects to $s$. Actually, the last means that equations (13) could be obtained without applying FALVA but simply by replacement of the parameter $s = g_t(\tau)$ in equation (16). By the way, it is possible to address precisely the same remark to [20], where the modification of cosmology is undertaken on the basis of the periodic weight function $g_t(\tau)$ in action:

$$S_L[q_t] = \int_{t_0}^t L(\tau, q_t(\tau), \dot{q}_t(\tau)) \exp(-\chi \sin(\beta \tau)) d\tau, \quad (17)$$

which is named fractional in [20], though such name could be applied to functional (12) only, meaning its origination from fractional integral (1). As it follows from (17), in this case the weight function is defined by $\frac{dg_t(\tau)}{d\tau} = \exp(-\chi \sin(\beta \tau))$, and replacement $s = g_t(\tau)$ in (16) is immediately resulted in equation obtained in [20] by variation of (17):

$$\ddot{x}^i - \beta \chi \cos(\beta \tau) \dot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0, \quad (18)$$

again with treatment of the second term in (18) as the perturbation of gravitational constant $\Delta G = 3\chi \beta \cos(\beta \tau) \frac{H}{4\pi \rho}$, where the Hubble parameter $H = \frac{\dot{a}}{a}$. Certainly, it would be possible to experiment further with various weight functions to solve those or other problems of the gravitational theory and cosmology but it is reasonable before all to return to the framework stated at the beginning of our paper and to apply FALVA directly to the gravitational field of the Universe. By the way, it was possible to understand the purpose that El-Nabulsi declared while formulating FALVA (see, for example, [21], [22] just so).

### 4 Fractional derivative cosmology of scalar field

First we consider the naive (or LSM) approach to the fractional derivative cosmological models of a scalar field. For what follows, it is useful to reproduce the derivation of the gravitation and scalar field equations from the variational principle for the Einstein-Hilbert action inspired by ADM formalism in cosmology (see, e.g., [23]). The Einstein-Hilbert action-like functional for FRW model of the Universe

$$ds^2 = N(t)^2 dt^2 - a^2(t) (dr^2 + \xi^2(r) d\Omega^2), \quad (19)$$
potential

earlier in the frameworks of the naive approach, which are submitted by the modified equations required set of equations becomes as follows:

\[ \frac{\ddot{\phi}}{a} + 3\frac{\dot{a}}{a} \dot{\phi} + \frac{dV}{d\phi} = 0, \]  
\[ 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi G \left( \frac{\dot{\phi}^2}{2} - V(\phi) \right) + \Lambda, \]  
\[ \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = 8\pi G \left( \frac{\dot{\phi}^2}{2} + V(\phi) \right) + \frac{\Lambda}{3}. \]

On the other hand, equation (23) could be derived with preliminary gauge \( N = 1 \) in (19), that is with proper time and representation of the FRW space-time interval as (8), proceeding from time invariance of the action (20), as it was made in [10]. As a matter of fact, both approaches are equivalent, because (23) was derived from the Euler-Lagrange equation for invariancy of the action (20), as it was made in [10]. As a matter of fact, both approaches are equivalent, because (23) was derived from the Euler-Lagrange equation for \( N(t) \), that is \( \frac{\partial L}{\partial N} = 0 \), which just expresses noted time scale invariancy. Besides, instead of equation (22) one frequently uses the following equation:

\[ \frac{\ddot{a}}{a} = -8\pi G \left( \frac{\dot{\phi}^2}{2} - V(\phi) \right) + \frac{\Lambda}{3}, \]

which turns out from equations (22), (23).

Considering LSM, it is necessary to make substitution of fractional derivative of the scale factor and the scalar field instead of integer derivatives in equations (21), (23) and (24). Therefore the required set of equations becomes as follows:

\[ D^\alpha_\beta (D^\alpha_\beta \phi) + 3 \left( \frac{D_\alpha^\beta a}{a} \right) D^\alpha_\beta \phi + \frac{dV(\phi)}{d\phi} = 0, \]  
\[ (D^\alpha_\beta a)^2 + k = 8\pi G \frac{1}{3} \left( \frac{1}{2} (D^\alpha_\beta \dot{\phi})^2 + V(\phi) \right) a^2 + \frac{\Lambda}{3} a^2, \]  
\[ D^\beta_\alpha (D^\alpha_\beta a) = -8\pi G \frac{1}{3} \left( (D^\alpha_\beta \dot{\phi})^2 - V(\phi) \right) a + \frac{\Lambda}{3} a. \]

While among three equations (21) - (23) of standard cosmology only two equations are independent, and the third one can be derived as the differential consequence of two others, due to the modified Leibniz rule for the fractional derivative (6), all equations of (25) - (27) are generally independent. The last means that one has either to solve equations (25) - (27) for \( a(t), \phi(t) \) and \( V(\phi(t)) \) with constant \( G \) and \( \Lambda \) or to admit the dependence of \( G \) and/or \( \Lambda \) on time when the potential \( V(\phi) \) is given. This circumstance allows us to doubt the acceptability of models offered earlier in the frameworks of the naive approach, which are submitted by the modified equations (7) and (10).

The solution of nonlinear fractional equations (25) - (27) is rather problematic. Even their classical prototype (21) - (23) does not always have exact solutions. Unfortunately, the theory of equations in fractional derivatives is incomparably more difficult and less advanced in comparison with the theory of integer-order differential equations (see, e.g., [1], [24]). Nevertheless, we would like to give an example of an exact solution to (25) - (27) demonstrating the fact of existence of some solutions. Let us consider the spatially flat model of the Universe without cosmological constant \( k = \Lambda = 0 \). We assume the dependence of the scale factor, scalar field, and potential on time as follows: \( a = a_0 t^\alpha, \phi = \phi_0 t^m, V(\phi(t)) = V_0 t^r \). Due to the modified rules of differentiation (3) for the given functions, it follows from (25) - (27) that:
derivatives over time in the action (20) with its fractional (of order
this set undergoes at the intermediate approach (or ISA). For this purpose, we replace the
following basic equations of the model:

\[
q_t = \frac{\Gamma(n+1)}{\Gamma(n+2-\alpha)} \left( 2 + \frac{\Gamma(n+1-\alpha)}{\Gamma(n+2-\alpha)} \right).
\]

where we denote \[ D^\alpha_t \equiv D^\alpha, \] and \[ D^\infty_t \equiv D^\infty. \] If in the action \( S[q_j](t) \) the derivatives are CDs, then (34) remains practically without changes, with the only exception of the derivatives of Lagrangian in the brackets are CDs [26]. Hence if the derivatives \( D^\alpha_t a \) and \( D^\alpha_t \phi \) in (33) are determined as CDs they also should be understood as CDs in equations (35),(37) but the repeated right derivatives (see definition (4)) in the first terms of (35) and (37) do not undergo redefinition. Let us note also, that in the case of integer order derivative (\( \alpha = 1 \)) due to the obvious expressions \( \frac{\partial}{\partial t} = \frac{d}{dt} \), the set of the fractional equations (35 -37) coincides with the classical one (21 -23).
5 Cosmological models of scalar field with fractional action

We consider now the cosmological model of a scalar field, which follows from the variational principle for the fractional action (12). This approach is in the framework of the intermediate approach (ISA) and frequently, as it was mentioned above, is referred to FALVA. Here, it is necessary to pay attention to the following peculiar properties of computation to avoid mistakes in obtaining the master equations of the model under consideration. The first term in the action (20) is obtained as the result of integration by parts of the first term in $R\sqrt{-g}$, which depends on the second derivative $\ddot{a}$. Actually this integration removes the derivative of the laps function $N$ from the action. It is a well known procedure. But now, if one makes simply fractional, as (12), generalization of the Einstein-Hilbert action on the basis of expression (20), the result will be incorrect, and some terms will be omitted. Due to the stated above reason, such terms are absent in all quoted articles by El-Nabulsi, unfortunately also containing several other inaccuracies.

Therefore, we use the modified Einstein-Hilbert action $S_{EH}^0 \equiv \int_0^t L_{EH}^0 \, d\tau$ as the following fractional integral:

$$S_{EH}^0 = \frac{1}{\Gamma(\alpha)} \int_0^t N \left[ \frac{3}{8\pi G} \left( \frac{a^2 \ddot{a}}{N^2} + \frac{a \dot{a}^2}{N^2} - \frac{a^2 \dot{N}}{N^3} + ka - \frac{\Lambda a^3}{3} \right) + a^3 \left( \frac{\epsilon \phi^2}{2N^2} - V(\phi) \right) \right] (t-\tau)^{\alpha-1} d\tau ,$$

where all functions in $L_{EH}^0$ depend on the intrinsic time $\tau$, and $\epsilon = +1, -1$ for the usual and phantom scalar fields respectively. Varying the action (38) over $q_i = \phi, a$ and $N$ with the subsequent choice of the gauge $N = 1$, we obtain the following Euler-Poisson equations,

$$\frac{\partial L_{EH}^0}{\partial \dot{q}_i} - \frac{d}{d\tau} \left( \frac{\partial L_{EH}^0}{\partial \ddot{q}_i} \right) + \frac{d^2}{d\tau^2} \left( \frac{\partial L_{EH}^0}{\partial \dot{q}_i} \right) = 0 ,$$

for our model [27]:

$$\ddot{\phi} + 3 \left( \frac{\dot{a}}{a} + \frac{1-\alpha}{3t} \right) \dot{\phi} + \epsilon \frac{dV(\phi)}{d\phi} = 0 , \quad (39)$$

$$\frac{\ddot{a}}{a} - \frac{1-\alpha}{2t} \left( \frac{\dot{a}}{a} + \frac{(1-\alpha)(2-\alpha)}{2t^2} \right) = -\frac{8\pi G}{3} \left( \frac{\epsilon \phi^2}{2} - V(\phi) \right) + \frac{\Lambda}{3} , \quad (40)$$

$$\left( \frac{\dot{a}}{a} \right)^2 + \frac{1-\alpha}{t} \left( \frac{\dot{a}}{a} \right) + k \frac{\dot{a}}{a^2} = \frac{8\pi G}{3} \left( \frac{\phi^2}{2} + V(\phi) \right) + \frac{\Lambda}{3} , \quad (41)$$

where and below the dots above functions designate the derivatives of the appropriate order with respect to time $t$ obtained by $t-\tau = T \rightarrow t$. Precisely the same equations one can obtain from generalization of the Euler-Lagrange equations in FALVA for the Lagrangian with higher (second) derivative (see [28]) or from the previous integration by parts as mentioned above. In the last case, one has to take into account the weight function $(t-\tau)^{\alpha-1}$ in the action integral (38) requiring the limit $\lim_{t \rightarrow 0}(a^2 \ddot{a}/t)$ to be finite.

It is easy to verify that the set of equations (39) - (41) is represented by three independent equations, against two independent equations in the classical case (21) - (23), that is connected with violation of the energy - momentum conservation law $T_{\mu\nu}^{\text{fl}} = 0$. Therefore in the set of equations (39) - (41) for three unknown functions ($a(t), \phi(t)$ and $V(t)$), potential is not an arbitrary function in general. One can rewrite equations (39) - (41) in terms of effective energy density $\rho(t)$ and pressure $p(t)$, taking into account the well known expressions:

$$\rho = \frac{\epsilon}{2} \dot{\phi}^2 + V(\phi) , \quad p = \frac{1}{2} \dot{\phi}^2 - V(\phi) ,$$

As a result, it follows from (39)-(41) and (42) that:

$$\dot{\rho} + 3 \left( \frac{\dot{a}}{a} + \frac{1-\alpha}{3t} \right) (\rho + p) = 0 , \quad (43)$$
\[ \frac{\ddot{a}}{a} + \frac{1 - \alpha}{2t} \left( \frac{\dot{a}}{a} \right) + \frac{(1 - \alpha)(2 - \alpha)}{2t^2} = -\frac{4\pi G}{3} (\rho + 3p) + \frac{\Lambda}{3}, \]  
\[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{1 - \alpha}{t} \left( \frac{\dot{a}}{a} \right) + \frac{k a^2}{\alpha^2} = \frac{8\pi G}{3} \rho + \frac{\Lambda}{3}, \]  
(44)

It is easy to integrate equation (43) for the perfect fluid with equation of state \( p = \gamma \rho \):

\[ \rho = \rho_0 a^3 (1 + \gamma)(1 + \gamma)(1 - \alpha). \]  
(46)

The interesting fact is that the cosmological \( \Lambda \)-term in the action (38) can depend on time, but in equations (39) - (41), and also in (43) - (45), the only change appears in the dependence \( \Lambda = \Lambda(t) \). Therefore the expression (46) is also valid.

Let us provide an example of an exact solution for the flat model \( (k = 0) \) and for the quasi-vacuum state of matter: \( \gamma = -1 \). From (46) it follows that \( \rho(t) = \rho_0 = \text{constant} \), as well as in the standard model. Then, the remaining equations of (43) - (45) for the Hubble parameter and \( \Lambda \)-term can be copied as follows:

\[ \dot{H} - \frac{1 - \alpha}{2t} H + \frac{(1 - \alpha)(2 - \alpha)}{2t^2} = 0, \]  
(47)
\[ H^2 + \frac{1 - \alpha}{t} H = \frac{8\pi G}{3} \rho_0 + \frac{\Lambda}{3}. \]  
(48)

From equation (47), it is easy to find that the Hubble parameter varies with time as

\[ H = \frac{C_\alpha}{t} + H_0 t \frac{1 - \alpha}{2}, \]  
(49)

where \( C_\alpha = \frac{(1 - \alpha)(2 - \alpha)}{(3 - \alpha)} \), \( H_0 \) is a constant of integration. From (49), the following dependence of the scale factor on time \( t \) appears:

\[ a = a_0 C_\alpha \exp \left( \frac{3 - \alpha}{2} \frac{3 - \alpha}{H_0 t} \right), \]  
(50)

while the cosmological \( \Lambda \)-term changes with time as

\[ \Lambda = 3H_0^2 t^{1 - \alpha} + 3H_0 \frac{(1 - \alpha)(7 - 3\alpha)}{(3 - \alpha)} t \frac{(1 + \alpha)}{2} + \frac{3(1 - \alpha)^2(2 - \alpha)(5 - 2\alpha)}{(3 - \alpha)^2} \frac{1}{t^2} - 8\pi G \rho_0. \]  
(51)

It is obvious that at the proceeding to the standard model in the limit \( \alpha \to 1 \), the obtained solutions (49) - (51) reduce to the known exponential expansion of the Universe: \( a = a_0 e^{H_0 t} \), \( H = H_0 \), \( \Lambda = 3H_0^2 - 8\pi G \rho_0 \).

We consider now the dynamics of the flat model of the Universe \( (k = 0) \) filled by a scalar field. It is convenient to rewrite equations (39) - (41) in terms of the Hubble parameter in the following form:

\[ \ddot{\phi} + 3 \left( H + \frac{1 - \alpha}{3t} \right) \dot{\phi} + \epsilon \frac{dV(\phi)}{d\phi} = 0, \]  
(52)
\[ \dot{H} - \frac{1 - \alpha}{2t} H + \frac{(1 - \alpha)(2 - \alpha)}{2t^2} = -4\pi G \epsilon \dot{\phi}^2, \]  
(53)
\[ H^2 + \frac{1 - \alpha}{t} H = \frac{8\pi G}{3} \left( \frac{\dot{\phi}^2}{2} + V(\phi) \right) + \frac{\Lambda}{3}. \]  
(54)

It is easy to prove that the given set of the independent equations can contain some arbitrariness in a choice of unknown functions, for instance \( H(t) \) or \( V(t) \), only if the cosmological \( \Lambda \)-term depends
on time. However, it is possible to proceed from some dependence $\Lambda(t)$. Let us consider one example of an exact solution, assuming that the Hubble parameter and the scalar field depend on time as follows:

$$H = \frac{C}{t}, \quad \dot{\phi} = \frac{B}{t},$$

(55)

where $C$, $B$ are the constants to be defined. We take into account definitions (42) and the equation of state $p = \gamma \rho$ for convenience of representation of solutions in terms of physically clear parameters and fractional order of the action $\alpha$. Using substitution (55) in (52) - (54) and considering (42), we obtain the following result:

$$a = a_0 t^{\lambda}, \quad \phi = \frac{1}{\lambda} \ln t + \phi_0, \quad V(\phi) = \frac{\epsilon}{2\lambda^2} \left(1 - \gamma\right) e^{-2\lambda(\phi - \phi_0)},$$

(56)

where the following notation is used:

$$\lambda = \lambda(\alpha, \gamma) = \pm \sqrt{\frac{24\pi G}{3 - \alpha}} \left[\frac{1 - \gamma}{1 + \gamma} - 2\epsilon \left(\frac{3}{3 - \alpha} - 2\alpha\right)\right]^{-1/2},$$

$$C = \frac{\epsilon}{3} \left(\frac{1 - \gamma}{1 + \gamma}\right) + \frac{\alpha}{3}, \quad B = \lambda^{-1},$$

It is easy to see that the barotropic index $\gamma$ is not arbitrary but it should satisfy the following inequality:

$$\frac{1 - \gamma}{1 + \gamma} > 2\epsilon \left(\frac{3}{3 - \alpha} - 2\alpha\right),$$

and also the inequality which follows from a weak power condition $\rho > 0$: $\gamma > -1$. Thus the $\Lambda$-term depends on time $t$ as follows: $\Lambda = \Lambda_0/t^2$, where

$$\Lambda_0 = (1 - \alpha) \left[\frac{(1 - \gamma)^2}{6(1 + \gamma)^2} - \frac{\epsilon(1 - \gamma)}{6(1 + \gamma)(9 - 4\alpha)} - 1\right].$$

The above mentioned solutions demonstrate the existence of exact solutions to the model, which can be of interest in aspect of modelling of processes occurring in the Universe. Let us investigate more generally the properties of the masters equations for the spatially flat cosmological model (52) - (54). After simple manipulation, this set of equations can be reorganized as follows:

$$\dot{H} + 3H^2 - \frac{2(4 - \alpha)}{t} H - \frac{(1 - \alpha)(2 - \alpha)}{t^2} = \frac{t\dot{\Lambda}}{1 - \alpha},$$

(57)

$$4\pi G\epsilon \dot{\phi}^2 = 3H^2 - \frac{3(5 - \alpha)}{2t} H - \frac{3(1 - \alpha)(2 - \alpha)}{2t^2} - \frac{t\dot{\Lambda}}{1 - \alpha},$$

(58)

$$8\pi G V(t) + \Lambda = \frac{3(7 - 3\alpha)}{2t} H + \frac{3(1 - \alpha)(2 - \alpha)}{2t^2}. $$

(59)

The term $\frac{t\dot{\Lambda}}{1 - \alpha}$ in the right-hand side of equation (57) is obtained by dividing the both sides of this equality by $(1 - \alpha)$. Therefore equation (57) is valid for all $\alpha \in (0; 1)$ but in the classical limit $\alpha \to 1$ it simply means $\dot{\Lambda} = 0$, i.e. $\Lambda = constant$. The latter is the only consequence of $\alpha = 1$, and among three equations (57) - (59) it is necessary to solve only two of them with $\dot{\Lambda} = 0$, that allows us to define one of the function among $H(t), \phi(t)$ or $V(\phi)$ arbitrarily.

The situation is essentially different for $\alpha \neq 1$. Now we have three independent equations for four functions: $H(t), \phi(t), V(\phi)$ and $\Lambda(t)$, and specification of $\Lambda(t)$ allows us to find the dependence $H(t)$ from equation (57). After that, it is possible to find $\phi(t)$ and $V(t)$ from equations (58) and (59). It is important that the evolutionary, i.e. containing derivative over time, equation (57) for the Hubble parameter experiences the influence of the parameter $\Lambda(t)$ only. Thus, the behaviour of the field and its potential becomes secondary, not determining an expansion dynamics of the Universe directly, and can be simply found from the equations (58), (59). In some sense, the
scalar field here plays a role of a latent parameter. Indeed, the system behaviour is determined by the field behaviour but the Hubble parameter and the scale factor are determined not directly by the scalar field but only through the $\Lambda(t)$-term. Using some phenomenological expression for the cosmological parameter, it is possible to find the dependence of expansion on time, and then the scalar field that provided it. There is a certain sense in that, as we know practically nothing about a scalar field that evoked cosmological inflation and the present accelerated expansion of the Universe.

We consider one example of the mentioned approach, assuming $\Lambda = constant$. In this case, equation (57) contains the only free parameter: $\alpha$, and can be easily solved, that gives the Hubble parameter as:

$$H = \frac{1}{6t} \left( 9 - 2\alpha - w_\alpha \frac{1 - c_0 t^{w_\alpha}}{1 + c_0 t^{w_\alpha}} \right), \quad w_\alpha = \sqrt{16\alpha^2 - 72\alpha + 105}, \quad c_0 > 0. \quad (60)$$

One can notice, that $w_\alpha > 0$ for $\alpha \in (0; 1)$. Integrating equation (60), we obtain the following expression for the scale factor:

$$a = a_0 \left\{ t(9 - 2\alpha - w_\alpha)/2 \left( 1 + c_0 t^{w_\alpha} \right) \right\}^{1/3}. \quad (61)$$

Substitution of the Hubble parameter from equation (60) in (58), (59) gives us expressions for $\phi(t)$ and $V(t)$. Due to the definition (42) and equations (58), (59) the energy density and the pressure of the scalar field ($\Lambda = 0$) are as follows:

$$\rho = \frac{1}{96\pi G t^2} \left( 9 - 2\alpha - w_\alpha \frac{1 - c_0 t^{w_\alpha}}{1 + c_0 t^{w_\alpha}} \right) \left( 15 - 8\alpha - w_\alpha \frac{1 - c_0 t^{w_\alpha}}{1 + c_0 t^{w_\alpha}} \right),$$

$$p = \frac{1}{96\pi G t^2} \left[ \left( 9 - 2\alpha - w_\alpha \frac{1 - c_0 t^{w_\alpha}}{1 + c_0 t^{w_\alpha}} \right) \left( 10\alpha - 27 - w_\alpha \frac{1 - c_0 t^{w_\alpha}}{1 + c_0 t^{w_\alpha}} \right) - 36(1 - \alpha)(2 - \alpha) \right].$$

From these expressions, it follows that the barotropic index $\gamma = \rho/p$ is not constant, and the equation of state changes with time. It is possible to show that at the initial moment of time, $\gamma(0)$ strongly depends on the fractional order $\alpha$, but at $t \to \infty$ it practically loses this dependence, slowly approaching $-5/7$ as $\alpha \to 1$.

It is interesting that if we assume the cosmological term decays as $\Lambda = \frac{\beta}{t}H$, where $\beta$ is a positive constant, then equation (ref 56) is reduced into the following one:

$$(1 - \alpha - \beta) \dot{H} + 3(1 - \alpha)H^2 - \frac{2[4(\alpha)(1 - \alpha) - \beta]}{t}H - \frac{(1 - \alpha)^2(2 - \alpha)}{t^2} = 0,$$

differing from the same equation in the case $\beta = 0$, considered above, only by the constant multipliers. Therefore its solution is structurally similar to (60), (61).

6 Conclusion

In our article, we have given the critical analysis of the main results and approaches known by now that are directed towards the application in cosmology of the ideas of fractional differentiation and integration. At the same time, we offer the unified consecutive approach to such a problem on the basis of the variational principle for the Einstein-Hilbert action which either includes fractional derivatives of the scale factor and the scalar field in the Lagrangian, or becomes fractional one. The exact solutions of the modified Friedmann and scalar field equations essentially differ from the ones known earlier that could be useful in aspect of modifications undertaken in modern cosmology in view of new observational puzzles [29]. We hope that further investigation of the obtained equations and its solutions will allow us at last to answer the question: in what degree the ideas of the fractional differential calculus are productive in cosmology.

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