MEAN CURVATURE FLOW WITH CONVEX GAUSS IMAGE

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ABSTRACT. We study the mean curvature flow of complete space-like submanifolds in pseudo-Euclidean space with bounded Gauss image, as well as that of complete submanifolds in Euclidean space with convex Gauss image. By using the confinable property of the Gauss image under the mean curvature flow we prove the long time existence results in both cases. We also study the asymptotic behavior of these solutions when $t \to \infty$.

1. INTRODUCTION

There are many works on the mean curvature flow of hypersurfaces in Riemannian manifolds (see [15], [16], [11], [12] for example). The impressive features of mean curvature flow for codimension one are as follows.

1. If the initial hypersurface $M_0 \subset \mathbb{R}^{m+1}$ is uniformly convex, then the hypersurfaces under the mean curvature contract smoothly to a single point in finite time and the shape of the hypersurfaces becomes spherical at the end of the contraction. If the ambient manifold is a general Riemannian manifold, such a contraction is still working.

2. If the initial hypersurface $M_0 \subset \mathbb{R}^{m+1}$ is an entire graph with linear growth, then there is long time existence for the mean curvature flow and the shape of the hypersurfaces becomes flat.

We know that J. Moser [21] proved that an entire minimal graph in $\mathbb{R}^{m+1}$ given by $x^{m+1} = f(x^1, \ldots, x^m)$ with bounded gradient $|\nabla f| < c < \infty$ has to be hyperplane. This is closely related to the result of Ecker-Huisken [11], which reveals the second feature of the mean curvature flow of hypersurfaces mentioned above. On the other hand, Moser’s result [21] has been generalized to higher codimension in [14] [10], and in author’s joint work with J. Jost [18]. Those are the underline motivation of the present work.

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It is natural to study the mean curvature flow of higher codimension. In recent years some works have been done in [3], [4], [5], [24], [25], [26], [28] and [29]. In the present paper we show the second feature in higher codimension. The terminology of linear growth in [11] can be interpreted as the image under the Gauss map of the hypersurface lies in an open hemisphere. We investigate the mean curvature flow of submanifolds with convex Gauss image naturally.

The target manifold of the Gauss map is the Grassmannian manifold in this situation. It is a symmetric space of compact type. It has non-negative sectional curvature. The mean curvature flow is closely related to its harmonic Gauss heat flow [29]. The knowledge of the harmonic map theory inspires us to investigate the dual situation firstly: mean curvature flow of a space-like $m$-submanifold in pseudo-Euclidean space $\mathbb{R}^{m+n}_n$ with index $n$. Now, the target manifold of the Gauss map is a pseudo-Grassmannian manifold. It has non-positive sectional curvature. In the literature, rather few papers studied mean curvature flow in an ambient Lorentzian manifold. Among them Ecker-Huisken [13] studied the mean curvature flow of a compact space-like hypersurface in a Lorentzian manifold.

Whereas, there is a plenty of works on the Bernstein problem for complete space-like submanifolds. E. Calabi raised the Bernstein problem for complete space-like extremal hypersurfaces in Minkowski space $\mathbb{R}^{m+1}_m$. He proved that such hypersurfaces have to be hyperplanes when $m \leq 4$ [1]. Cheng-Yau solved the problem for all $m$, in sharp contrast to the situation of Euclidean space [8].

In [27] and [9], H. I. Choi and A. E. Triebergs constructed many complete space-like hypersurfaces with nonzero constant mean curvature by prescribing boundary data at infinity for the Gauss map.

On the other hand, we proved [31][32] that for any complete space-like hypersurface $M$ with constant mean curvature in Minkowski space $\mathbb{R}^{m+1}_n$, if the image under the Gauss map $\gamma : M \rightarrow \mathbb{H}^m(-1)$ is bounded, then $M$ has to be an $m$-plane.

Cheng-Yau’s result was generalized to the higher codimension in [17][19]. We proved in [19] a higher codimensional generalization of the above mentioned result in [31].

In this paper we investigate the mean curvature deformation of a complete submanifold both in ambient pseudo-Euclidean space and Euclidean space. The paper is divided by two part for two cases. The contents are organized as in the following table.
We will prove the following main theorems in this paper.

**Theorem 1.1.** Let \( F : M \to \mathbb{R}^{m+n} \) be a space-like complete \( m \)-submanifold which has bounded curvature and bounded Gauss image. Then the evolution equation of mean curvature flow has long time smooth solution.

**Remark 1.1.** This theorem was announced in Geometric Analysis Meetings in Changsha (June) and San Diego (July) in this summer.

**Remark 1.2.** Our result in [31] has been refined by Xin-Ye [34], independently by Cao-Shen-Zhu [2], as follows.

Let \( M \) be a complete space-like hypersurface of constant mean curvature in Minkowski space \( \mathbb{R}^{m+1}_1 \). If the image of the Gauss map \( \gamma : M \to \mathbb{H}^m(-1) \) lies in a horoball in \( \mathbb{H}^m(-1) \), then \( M \) has to be a hyperplane.

This is the best possible result. It implies that we may have better result than Theorem 1.1 in codimension one case.
Theorem 1.2. Let \( F : M \rightarrow \mathbb{R}^{m+n} \) be a complete \( m \)-submanifold which has bounded curvature. Suppose that the image under the Gauss map from \( M \) into \( G_{m,n} \) lies in a geodesic ball of radius \( R_0 < \frac{\sqrt{2}}{12} \pi \). Then the evolution equation of mean curvature flow has long time smooth solution.

Remark 1.3. Compare the above theorem with our Bernstein type result in [18], better results would be expected. It suffices to improve curvature estimates in §3.2.

Remark 1.4. From Theorem 3.1 and the proof of Theorem 1.2 we see that when the image under the Gauss map from \( M \) into \( G_{m,n} \) lies in a geodesic ball of radius \( R_0 < \frac{\sqrt{2}}{4} \pi \). The equation of the mean curvature is uniformly parabolic and has smooth solution on some short time interval.

We use the same idea to prove Theorem 1.1 and Theorem 1.2. Consider the image of the Gauss map under the mean curvature flow. If an initial submanifold has convex Gauss image, then deforming submanifolds under the mean curvature flow have the ”confinable property” (Theorem 2.3 and Theorem 3.1). This is an adequate higher codimensional generalization of the ”linear growth preserving property” in [11]. In the case of the ambient Euclidean space we have more technical issues, because of the nonnegative curvature of the target manifold of the Gauss map.

We also study the asymptotic behavior of these solutions when \( t \rightarrow \infty \), namely we study the rescaled mean curvature flow in both cases in §2.5 and §3.3, respectively. The corresponding results as in [11] can be obtained similarly.

2. Space-like Submanifolds

Let \( \mathbb{R}^{m+n} \) be an \((m+n)\)-dimensional pseudo-Euclidean space with the index \( n \). The indefinite metric is defined by

\[
\text{d} s^2 = \sum_{i=1}^{m} (\text{d}x^i)^2 - \sum_{\alpha=m+1}^{m+n} (\text{d}x^\alpha)^2
\]

Let \( F : M \rightarrow \mathbb{R}^{m+n} \) be a space-like \( m \)-submanifold in \( \mathbb{R}^{m+n} \) with the second fundamental form \( B \) defined by

\[
B_{XY} \overset{\text{def}}{=} (\nabla_X Y)^N
\]

for \( X, Y \in \Gamma(TM) \). We denote \((\cdots)^T\) and \((\cdots)^N\) for the orthogonal projections into the tangent bundle \( TM \) and the normal bundle \( NM \), respectively. For \( \nu \in \Gamma(NM) \) we define the shape operator \( A^\nu : TM \rightarrow TM \) by

\[
A^\nu (V) = -(\nabla_V \nu)^T.
\]
Taking the trace of $B$ gives the mean curvature vector $H$ of $M$ in $\mathbb{R}^{m+n}$ and

$$H \overset{\text{def}}{=} \text{trace}(B) = B_{e_i e_i},$$

where $\{e_i\}$ is a local orthonormal frame field of $M$. Here and in the sequel we use the summation convention. The mean curvature vector is a cross-section of the normal bundle.

Choose a local Lorentzian frame field $\{e_i, e_\alpha\}$ along $M$ with dual frame field $\{\omega_i, \omega_\alpha\}$, such that $e_i$ are tangent vectors to $M$. We agree with the following range of indices

$$A, B, C, \ldots = 1, \ldots, m+n;$$

$$i, j, k \ldots = 1, \ldots, m; s, t = 1, \ldots, n; \alpha, \beta, \ldots = m+1, \ldots, m+n.$$

The induced Riemannian metric of $M$ is given by $ds_M^2 = \sum_i \omega_i^2$ and the induced structure equations of $M$ are

$$d\omega_i = \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \omega_{ik} \wedge \omega_{kj} - \omega_{i\alpha} \wedge \omega_{\alpha j},$$

$$\Omega_{ij} = d\omega_{ij} - \omega_{ik} \wedge \omega_{kj} = -\frac{1}{2} R_{ijkl} \omega_k \wedge \omega_l.$$

By Cartan’s lemma we have

$$\omega_{\alpha i} = h_{\alpha ij} \omega_j.$$

### 2.1. Bochner Type Formula

We now derive the following Bochner type formula.

**Proposition 2.1.**

$$\langle \nabla^2 B \rangle_{XY} = \nabla_X \nabla_Y H + \langle B_{X e_i}, H \rangle B_{Y e_i} - \langle B_{X e_i}, B_{e_i e_j} \rangle B_{Y e_j}$$

$$+ 2 \langle B_{X e_i}, B_{Y e_i} \rangle B_{e_i e_j} - \langle B_{Y e_i}, B_{e_i e_j} \rangle B_{X e_j} - \langle B_{X e_i}, B_{e_i e_j} \rangle B_{Y e_j},$$

(2.1)

$$\Delta |B|^2 = 2 |\nabla B|^2 + 2 \langle \nabla_i \nabla_j H, B_{ij} \rangle + 2 \langle B_{ij}, H \rangle \langle B_{ik}, B_{jk} \rangle + 2 |R^\perp|^2 - 2 \sum_{\alpha, \beta} S_{\alpha \beta}^2,$$

(2.2)

where $R^\perp$ denotes the curvature of the normal bundle and $S_{\alpha \beta} = h_{\alpha ij} h_{\beta ij}$.

**Proof.** Choose a local orthonormal tangent frame field $\{e_i\}$ of $M$ near $x \in M$. Let $X, Y, \ldots$ be tangent vector fields and $\mu, \nu$ normal vector fields to $M$ near $x$ with

$$\nabla e_i |_x = \nabla_{e_i} X |_x = \nabla_{e_i} Y |_x = \cdots = \nabla_{e_i} \mu |_x = \nabla_{e_i} \nu |_x = \cdots = 0.$$

Thus,

$$\nabla_X Y |_x = \nabla_X Y |_x - \nabla_X Y |_x = (\nabla_X Y) |^N_x = B_{XY},$$
\[ \nabla_X \mu |_x = \nabla_X \mu |_x - \nabla_X \mu |_x = (\nabla_X \mu |_x)^T = -A^\mu(X), \]

\[ \nabla_{XY}|_x \overset{def}{=} \nabla_X \nabla_Y|_x - \nabla_{\nabla X Y}|_x = \nabla_X \nabla_Y|_x. \]

By the Codazzi equations we have at \( x \)

\[ (\nabla^2 B)_{XY} = (\nabla_{e_i} \nabla_{e_i} B)_{XY} \]

\[ = \nabla_{e_i} (\nabla_{e_i} B)_{XY} - (\nabla_{e_i} B)\nabla_{e_i} X Y - (\nabla_{e_i} B)_X \nabla_{e_i} Y \]

\[ = \nabla_{e_i} (\nabla_{e_i} B)_{e_i Y} \]

\[ = (\nabla_X \nabla_{e_i} B)_{e_i Y} + (R_{X e_i} B)_{e_i Y} \]

\[ = \nabla_X (\nabla_{e_i} B)_{e_i Y} + (R_{X e_i} B)_{e_i Y} \]

\[ = \nabla_X \nabla_Y H + (R_{X e_i} B)_{e_i Y} \]

\[ (2.3) \]

Noting the Gauss equations and the Ricci equations

\[ \langle R_{XY} Z, W \rangle + \langle Q^T_{XY Z}, W \rangle = 0, \]

\[ \langle R_{XY} \mu, \nu \rangle + \langle Q^N_{XY} \mu, \nu \rangle = 0, \]

where

\[ \langle Q^T_{XY Z}, W \rangle = \langle B_{X W}, B_{Y Z} \rangle - \langle B_{X Z}, B_{Y W} \rangle, \]

\[ \langle Q^N_{XY} \mu, \nu \rangle = \langle B_{X e_i} \mu \rangle \langle B_{Y e_i} \nu \rangle - \langle B_{X e_i} \mu \rangle \langle B_{Y e_i} \nu \rangle. \]

Hence,

\[ (R_{X e_i} B)_{e_i Y} = -Q^N_{X e_i} B_{e_i Y} + B_{Q^T_{X e_i e_i Y}} + B_{e_i Q^T_{X e_i Y}}. \]

Substituting the above equality into (2.3) gives

\[ (\nabla^2 B)_{XY} = \nabla_X \nabla_Y H + A_0 + B_0 + C_0, \]

where

\[ A_0 = -Q^N_{X e_i} B_{e_i Y}, \quad B_0 = B_{Q^T_{X e_i e_i Y}}, \quad C_0 = B_{e_i Q^T_{X e_i Y}}. \]
Now we calculate $A_0$, $B_0$ and $C_0$ in (2.4).

\[
\langle A_0, \mu \rangle = -\langle Q_X^e, B_{e_i} e_j, \mu \rangle = \langle B_{Xe_i}, B_{e_i e_j}, \mu \rangle - \langle B_{Ye_i}, B_{e_i e_j} \rangle \langle B_{Xe_j}, \mu \rangle,
\]

\[
\langle B_0, \mu \rangle = \langle B_{Q_X^e e_i Y}, \mu \rangle = \langle A^n(Y), Q_X^e e_i \rangle
\]

\[
= \langle B_X A^n(Y), H \rangle - \langle B_{X e_i}, B_{e_i A^n(Y)} \rangle
\]

\[
= \langle B_{X e_i}, H \rangle \langle B_{Y e_i}, \mu \rangle - \langle B_{X e_i}, B_{e_i e_j} \rangle \langle B_{Y e_j}, \mu \rangle,
\]

\[
\langle C_0, \mu \rangle = \langle B_{e_i Q_X^e Y}, \mu \rangle = \langle A^n(e_i), Q_X^e Y \rangle
\]

\[
= \langle B_{e_i e_j}, \mu \rangle \langle Q_X^e Y, e_j \rangle
\]

\[
= \langle B_{e_i e_j}, \mu \rangle \langle B_{X e_j}, B_{Y e_i} \rangle - \langle B_{e_i e_j}, \mu \rangle \langle B_{X Y}, B_{e_i e_j} \rangle.
\]

Hence,

\[
(A_0 + B_0 + C_0, \mu)
\]

\[
= \langle B_{X e_i}, H \rangle \langle B_{Y e_i}, \mu \rangle - \langle B_{X Y}, B_{e_i e_j} \rangle \langle B_{e_i e_j}, \mu \rangle + 2 \langle B_{X e_j}, B_{Y e_i} \rangle \langle B_{e_i e_j}, \mu \rangle - \langle B_{X e_i}, B_{e_i e_j} \rangle \langle B_{Y e_j}, \mu \rangle.
\]

Substituting (2.5) into (2.4) gives (3.1).

Denote

\[
B_{ij} = B_{e_i e_j} = (\nabla_{e_i e_j})^N = h_{aij} e_\alpha,
\]

where \(\{e_\alpha\}\) is a local orthonormal frame field of the normal bundle near \(x \in M\). It follows that

\[
|B|^2 = \sum_{i,j} \langle B_{ij}, B_{ij} \rangle = -\sum_{\alpha, \beta, j} h_{aij}^2 \leq 0.
\]

We denote the absolute value of \(|B|^2\) by \(|B|\|^2\), which is nonnegative. The same notation for other time-like quantities. Then \(|B|^2 = \sum_{\alpha} S_{\alpha}^2\). It is easy to see that

\[
\sum_{\alpha, \beta} S_{\alpha \beta}^2 \geq \frac{1}{n} (\sum_{\alpha} S_{\alpha\alpha}^2) = -\frac{1}{n} ||B||^4
\]

Noting

\[
- \langle B_{kl}, B_{ij} \rangle \langle B_{ij}, B_{kl} \rangle = -(h_{aij} h_{bij})(-h_{bij} h_{bij})
\]

\[
= -h_{aij} h_{bij} h_{bij} h_{bij} = -\sum_{\alpha, \beta} S_{\alpha \beta}^2
\]
\[
|R^+|^2 = \left< R_{e_i e_j} \nu_\alpha, R_{e_i e_j} \nu_\alpha \right>
\]
\[
= \left< Q_{e_i e_j} \nu_\alpha, Q_{e_i e_j} \nu_\alpha \right>
\]
\[
= \left< B_{ik}, Q_{ij}^N \nu_\alpha \right> \left< B_{jk}, \nu_\alpha \right> - \left< B_{ik}, \nu_\alpha \right> \left< B_{ij}, Q_{jk}^N \nu_\alpha \right>
\]
\[
= \left< B_{ik}, \nu_\alpha \right> (\left< B_{il}, B_{jk} \right> \left< B_{jl}, \nu_\alpha \right> - \left< B_{il}, \nu_\alpha \right> \left< B_{jl}, B_{jk} \right>)
\]
\[
= 2 \left< B_{il}, B_{jk} \right> \left< B_{jl}, B_{ik} \right> - 2 \left< B_{il}, B_{jk} \right> \left< B_{jk}, B_{il} \right>
\]
we have
\[
\left< \nabla^2 B, B \right> = \left< \nabla, \nabla_j H, B_{ij} \right> + \left< B_{ik}, H \right> \left< B_{il}, B_{kl} \right> + |R^+|^2 - \sum_{\alpha,\beta} S_{\alpha\beta}^2.
\]
It gives (2.2). \hfill \square

2.2. Evolution Equations. We now consider the deformation of a submanifold under the mean curvature flow (abbreviated by MCF). Namely, consider a one-parameter family \( F_t = F(\cdot,t) \) of immersions \( F_t : M \to \mathbb{R}^{m+n} \) with corresponding images \( M_t = F_t(M) \) such that
\[
\begin{align*}
\frac{d}{dt} F(x, t) &= H(x, t), \quad x \in M \\
F(x, 0) &= F(x)
\end{align*}
\]
is satisfied, where \( H(x, t) \) is the mean curvature vector of \( M_t \) at \( F(x, t) \). Denote \( e_i(t) = F_* e_i \) which is abbreviated to \( e_i \) if there is no ambiguity.

\[
\begin{align*}
\frac{d g_{ij}}{dt} &= 2 \left< \frac{d F_* e_i}{dt}, F_* e_j \right> \\
&= 2 \left< \frac{d \nabla_{e_i} F}{dt}, F_* e_j \right> = 2 \langle \nabla_{e_i} H, F_* e_j \rangle \\
&= 2 \langle \nabla_{e_i} (H, F_* e_j) - \langle H, \nabla_{e_i} F_* e_j \rangle \rangle \\
&= -2 \langle H, B_{ij} \rangle.
\end{align*}
\]
It follows that
\[
\frac{d g^{ij}}{dt} = 2 g^{ik} g^{jl} \langle H, B_{kl} \rangle
\]
and
\begin{equation}
\frac{dg}{dt} = -2 |H|^2 g, \label{eq:10}
\end{equation}
where \( g = \det(g_{ij}) \). For a space-like submanifold the mean curvature vector field is a normal vector field. It is time-like vector field.

(2.10) shows that if the initial submanifold is space-like, then it will remain space-like for any \( t \) under the mean curvature flow.

**Lemma 2.1.** The second fundamental form and the mean curvature satisfy
\begin{align}
\left( \frac{d}{dt} - \Delta \right) ||B||^2 &\leq -\frac{2}{n}||B||^4, \label{eq:11} \\
\left( \frac{d}{dt} - \Delta \right) ||H||^2 &\leq -\frac{2}{n}||H||^4. \label{eq:12}
\end{align}

**Proof.** For fixed \( x_0 \in M \), and \( t_0 \) choose orthonormal frame field \( \{e_i\} \) of \( M_{t_0} \) near \( x_0 \) which is normal at \( x_0 \), and orthonormal normal field \( \{e_\alpha\} \). Then we evaluate at \( x_0 \) and \( t_0 \) in the following calculation.
\begin{align}
\frac{d h_{\alpha ij}}{dt} &= -\nabla_{\frac{d}{dt}} \left( \nabla_{e_i} e_j, e_\alpha \right) \\
&= -\left( \nabla_{H} \nabla_{e_i} e_j, e_\alpha \right) - \left( \nabla_{e_i} e_j, \nabla_{H} e_\alpha \right) \\
&= -\left( \nabla_{e_i} \nabla_{e_j} H, e_\alpha \right) - \left( B_{ij}, \nabla_{H} e_\alpha \right) \\
&= -\left( \nabla_{e_i} \nabla_{e_j} H, e_\alpha \right) + \left( B_{ik} e_k, e_\alpha \right) \left( B_{jk}, H \right) - \left( B_{ij}, \nabla_{H} e_\alpha \right) \\
&= -\left( \nabla_{e_i} \nabla_{e_j} H, e_\alpha \right) + h_{\alpha ik} h_{\beta jk} H_\beta - h_{\alpha ij} \left( \nabla_{H} e_\alpha, e_\beta \right). \label{eq:13}
\end{align}

Since in a non-orthonormal frame field \( g_{ij} = \langle F_* e_i, F_* e_j \rangle \) is not a unit matrix (except at \( (x_0, t_0) \)),
\[ |B|^2 = -g^{ik} g^{jl} h_{\alpha ij} h_{\alpha kl}. \]
We have at \( (x_0, t_0) \)
\begin{align}
\frac{d |B|^2}{dt} &= -g^{ik} g^{jl} h_{\alpha ij} h_{\alpha kl} - g^{ij} \frac{dh_{\alpha ij}}{dt} h_{\alpha kl} \\
&\quad - g^{ik} g^{jl} \frac{dh_{\alpha ij}}{dt} h_{\alpha kl} - g^{ik} g^{jl} h_{\alpha ij} \frac{dh_{\alpha kl}}{dt} \\
&= -2 g^{ik} \frac{dh_{\alpha ij}}{dt} h_{\alpha kj} - 2 \frac{dh_{\alpha ij}}{dt} h_{\alpha ij}, \label{eq:14}
\end{align}
From (2.13) we have

\[
\frac{dh_{\alpha ij}}{dt} = h_{\alpha ij} \left( -\langle \nabla e_i \nabla e_j H, e_{\alpha} \rangle + h_{\alpha ik}h_{\beta jk}H_\beta - h_{\alpha ij} \langle \nabla H e_{\alpha}, e_{\beta} \rangle \right)
\]

(2.15)

\[
\begin{align*}
&= -h_{\alpha ij} \langle \nabla e_i \nabla e_j H, e_{\alpha} \rangle + h_{\alpha ij}h_{\alpha ik}h_{\beta jk}H_\beta \\
&\quad - h_{\alpha ij} \langle \nabla H e_{\alpha}, e_{\beta} \rangle \\
&= -h_{\alpha ij} \langle \nabla e_i \nabla e_j H, e_{\alpha} \rangle + h_{\alpha ij}h_{\alpha ik}h_{\beta jk}H_\beta.
\end{align*}
\]

The third term of the second equality vanishes, because of the symmetric and antisymmetric properties in \(\alpha\) and \(\beta\) simultaneously. Noting (2.9), we have

\[
\frac{dg_{ik}}{dt}h_{\alpha ij}h_{\alpha kj} = 2h_{\alpha ij}h_{\alpha kj} \langle H, B_{ik} \rangle = -2h_{\alpha ij}h_{\alpha kj}h_{\beta ik}H_\beta
\]

(2.16)

Substituting (2.15) and (2.16) into (2.14) gives

\[
\frac{1}{2} \left( \frac{d}{dt} \right) |B|^2 = \langle \nabla_i \nabla_j H, B_{ij} \rangle + \langle B_{ij}, H \rangle \langle B_{ik}, B_{jk} \rangle.
\]

(2.17)

From (2.2) and (2.17) we obtain the evolution equation for the norm of the second fundamental form

\[
\frac{1}{2} \left( \frac{d}{dt} - \Delta \right) |B|^2 = -|\nabla B|^2 - |R^\perp|^2 + \sum_{\alpha,\beta} S^2_{\alpha\beta}.
\]

(2.18)

From (2.6) and (2.18) it follows that

\[
\frac{1}{2} \left( \frac{d}{dt} - \Delta \right) ||B||^2 = -||\nabla B||^2 - ||R^\perp||^2 - \sum_{\alpha,\beta} S^2_{\alpha\beta}
\]

(2.19)

\[
\leq -\frac{1}{n} ||B||^4.
\]

\[
|H|^2 = \langle g^{ij}h_{\alpha ij}e_{\alpha}g^{kl}h_{\beta kl}e_{\beta} \rangle = -g^{ij}g^{kl}h_{\alpha ij}h_{\alpha kl}.
\]

At \((x_0, t_0)\)

\[
\frac{d|H|^2}{dt} = -2 \frac{d}{dt}g^{ij}g^{kl}h_{\alpha ij}h_{\alpha kl} - 2g^{ij}g^{kl} \frac{dh_{\alpha ij}}{dt}h_{\alpha kl}
\]

\[
= -2 \frac{d}{dt}g^{ij}h_{\alpha ij}H_\alpha - 2g^{ij} \frac{dh_{\alpha ij}}{dt}H_\alpha.
\]

Noting (2.9) and (2.13)

\[
\frac{d|H|^2}{dt} = 2h_{\alpha ij}h_{\beta ij}H_\alpha H_\beta + 2 \langle \nabla^2 H, H \rangle
\]

\[
= 2S_{\alpha\beta}H_\alpha H_\beta + \Delta |H|^2 - 2|\nabla e_i H|^2.
\]
It follows that
\[
\left( \frac{d}{dt} - \Delta \right) |H|^2 = -2 |\nabla H|^2 + 2 S_{\alpha \beta} H_\alpha H_\beta.
\]
By using the Cauchy inequality
\[
||H||^2 = \sum_\alpha H_\alpha^2 \leq \sqrt{n} \sqrt{H_\alpha h_{\alpha ij} H_\beta h_{\beta ij}} = \sqrt{n} \sqrt{S_{\alpha \beta} H_\alpha H_\beta}
\]
we obtain
\[
\frac{1}{2} \left( \frac{d}{dt} - \Delta \right) ||H||^2 = -||\nabla H||^2 - S_{\alpha \beta} H_\alpha H_\beta
\leq -\frac{1}{n} ||H||^4.
\]
(2.20)

2.3. Maximum Principle and Curvature Estimates. For a complete space-like submanifold \( M \) \( \in \mathbb{R}^{m+n} \) with bounded curvature, the Gauss equation implies its Ricci curvature is bounded from below. We can use the well-known Omori-Yau maximum principle:

Let \( u \) be a \( C^2 \) function bounded from above on a complete manifold \( M \) with Ricci curvature bounded from below. Then for any \( \varepsilon > 0 \), there exists a sequence of points \( \{x_k\} \in M \), such that
\[
\lim_{k \to \infty} u(x_k) = \sup u,
\]
and when \( k \) is sufficiently large
\[
|\nabla u|(x_k) < \varepsilon,
\]
\[
\Delta u(x_k) < \varepsilon.
\]

Remark 2.1. When the Ricci curvature of \( M \) is bounded below by \(-C \log(1 + r^2 \log^2(r + 2))\), where \( r \) is the distance function from a fixed point \( x_0 \in M \), then the maximum principle ia also applicable (see [6]).

Now, we can use Omori-Yau maximum principle to do curvature estimate.

**Theorem 2.1.** If \( M_t \) is a smooth solution of (2.7) in \([0, T)\). If \( M_0 \) has bounded curvature, then there is estimate
\[
\sup_{M_t} ||B||^2 \leq \sup_{M_0} ||B||^2
\]
for \( t \in [0, T) \).
Proof. Let \( t_0 \in [0, T) \) be any time such that \( \sup_{M_{t_0}} \| B \|^2 \) is bounded. Let \( x_k(t_0) \) be a sequence of points on \( M_{t_0} \) such that when \( k \to \infty \)
\[ \| B \|^2(x_k(t_0)) \to \sup_{M_0} \| B \|^2 > 0 \]
and when \( k \) is sufficiently large
\[ |\nabla \| B \|^2(x_k(t_0))| < \varepsilon, \]
\[ \Delta \| B \|^2(x_k(t_0)) < \varepsilon \]
for any \( \varepsilon > 0 \). When \( k \) is large enough from (2.11)
\[ \frac{d\| B \|^2}{dt}
\bigg|_{x_k(t_0)} \leq \left( \Delta \| B \|^2 - \frac{1}{n} \| B \|^4 \right)
\bigg|_{x_k(t_0)} < 0, \]
which is valid on \( U \times [t_0, t_0 + \delta] \) as well, where \( U \) is an open neighborhood of \( x_k(t_0) \) in \( M_{t_0} \). For \( t_1 \) is sufficiently close to \( t_0 \), there exists \( x_k(t_1) \in U \times [t_0, t_0 + \delta) \), such that \( x_k(t_1) \) is evolved from some point \( y \) on \( M_{t_0} \). Therefore,
\[ \| B \|^2(x_k(t_1)) < \| B \|^2(y) < \sup_{M_{t_0}} \| B \|^2. \]
Let \( k \to \infty \), we have the desired estimate in \([t_0, t_0 + \delta']\), hence in \([0, T)\). □

2.4. Gauss Maps under the Evolution. For any \( p \in M \) let \( \{e_1, \cdots, e_m\} \) be a local orthonormal frame field near \( p \). Define the Gauss map \( \gamma : p \to \gamma(p) \) which is obtained by parallel translation of \( T_pM \) to the origin in the ambient space \( R^{m+n}_n \). The image of the Gauss map lies in a pseudo-Grassmannian \( G_{m,n}^n \) - the totality of all the space-like \( m \)-planes in \( R^{m+n}_n \). It is a specific Cartan-Hadamard manifold.

For any \( P \in G_{m,n}^n \), there are \( m \) vectors \( v_1, \cdots, v_m \) spanning \( P \). Then we have Plücker coordinates \( v_m \wedge \cdots \wedge v_m \) for \( P \) up to a constants. The Gauss map \( \gamma \) can be described by \( p \to e_1 \wedge \cdots \wedge e_m \). Since
\[ d(e_1 \wedge \cdots \wedge e_m) = de_1 \wedge \cdots \wedge e_m + \cdots + e_1 \wedge \cdots \wedge de_m \]
\[ = \omega_{\alpha_1e_\alpha} \wedge e_2 \wedge \cdots \wedge e_m + \cdots + e_1 \wedge \cdots \wedge e_{m-1} \wedge \omega_{e_m}e_\alpha \]
\[ = \omega_{e_\alpha}e_{\alpha i} \]
and the canonical metric on \( G_{m,n}^n \) is defined by
\[ ds^2 = \sum_{\alpha,i} \omega_{\alpha i}^2, \]
where \( e_{\alpha i} = e_1 \wedge \cdots \wedge e_{i-1} \wedge e_\alpha \wedge e_{i+1} \wedge \cdots \wedge e_m \) are orthonomal basis for \( TG_{m,n}^n \). It follows that
\[ \gamma^*\omega_{\alpha i} = h_{\alpha ij}\omega_j \]
which means that the relation

\[ e(\gamma) = \frac{1}{2}||B||^2. \]

The tension field of the Gauss map

\[
\tau(\gamma) = h_{aij}e_{ai} = h_{aij}e_{ai}
\]

(2.21)

\[
= h_{aij}e_1 \wedge \cdots \wedge e_{i-1} \wedge e_\alpha \wedge e_{i+1} \wedge \cdots \wedge e_m
\]

\[
= \sum_i e_1 \wedge \cdots \wedge e_{i-1} \wedge \nabla_{e_i}H \wedge e_{i+1} \wedge \cdots \wedge e_m,
\]

where we use the Codazzi equation.

Now, we compute the evolution of the Gauss map under the mean curvature flow. From

\[
\gamma(t) = \frac{1}{\sqrt{g}} e_1(t) \wedge \cdots \wedge e_m(t)
\]

and (2.10), we have

\[
\frac{d\gamma}{dt} = -\frac{1}{g} \frac{d\sqrt{g}}{dt} (e_1(t) \wedge \cdots \wedge e_m(t))
\]

\[
+ \frac{1}{\sqrt{g}} \left( \frac{d e_1(t)}{dt} \wedge \cdots \wedge e_m(t) + \cdots + e_1(t) \wedge \cdots \wedge \frac{d e_m(t)}{dt} \right)
\]

\[
= \frac{|H|^2}{\sqrt{g}} (e_1(t) \wedge \cdots \wedge e_m(t))
\]

\[
+ \frac{1}{\sqrt{g}} (\nabla_{e_1}H \wedge \cdots \wedge e_m(t) + \cdots + e_1(t) \wedge \cdots \wedge \nabla e_m H)
\]

(2.22)

\[
= \frac{|H|^2}{\sqrt{g}} (e_1(t) \wedge \cdots \wedge e_m(t))
\]

\[
+ \frac{1}{\sqrt{g}} (e_1(t) \wedge \cdots \wedge \langle \nabla_{e_1}H, e_j(t) \rangle e_k(t) g^{jk} \wedge e_2(t) \wedge \cdots \wedge e_m(t) + \cdots)
\]

\[
+ \frac{1}{\sqrt{g}} e_1(t) \wedge \cdots \wedge \langle \nabla_{e_m}H, e_j(t) \rangle e_k(t) g^{ik}
\]

\[
+ \frac{1}{\sqrt{g}} (\nabla_{e_1}H \wedge \cdots \wedge e_m(t) + \cdots + e_1(t) \wedge \cdots \wedge \nabla e_m H).
\]
Since
\begin{equation}
\langle \nabla_{e_1} H, e_j(t) \rangle e_k(t) g^{jk} \wedge e_2(t) \wedge \cdots \wedge e_m(t) + \cdots e_1(t) \wedge \cdots \wedge \langle \nabla_{e_m} H, e_j(t) \rangle e_k(t) g^{jk} \\
= - \left( \langle H, \nabla_{e_1} e_j(t) \rangle g^{1j} + \langle H, \nabla_{e_m} e_j(t) \rangle g^{jm} \right) e_1(t) \wedge \cdots \wedge e_m(t) \\
= - |H|^2 (e_1(t) \wedge \cdots \wedge e_m(t)),
\end{equation}
we have
\begin{equation}
\frac{d\gamma}{dt} = \frac{1}{\sqrt{g}} (\nabla_{e_1} H \wedge \cdots \wedge e_m(t) + \cdots + e_1(t) \wedge \cdots \wedge \nabla_{e_m} H).
\end{equation}

We may assume \{e_1(t_0), \ldots, e_m(t_0)\} form an orthonormal basis of \(F(M)\) at \((p, t_0)\). Then, from (2.21) and (2.24) we obtain the following equation which is the Lorentzian version of a result in [29].

**Theorem 2.2.**
\begin{equation}
\frac{d\gamma}{dt} = \tau(\gamma(t)).
\end{equation}

2.5. **Proof of the First Main Theorem.** We study the Gauss image under the flow. The relevant Bernstein type theorem inspires us to consider the bounded Gauss image of the initial submanifold. In fact, any geodesic ball in any Cartan-Hadamard manifold is convex. Precisely, we have the following ”confinable property” of the Gauss image under the mean curvature flow.

**Theorem 2.3.** Let \(M\) be a complete space-like \(m\)-submanifold in \(\mathbb{R}^{m+n}\) with bounded curvature. If the image under Gauss map is contained in a bounded geodesic ball in \(G_{m,n}\), then the images of all the submanifolds under the MCF are also contained in the same geodesic ball.

**Proof.** Let \(h\) be any function on \(G_{m,n}\). The composition function \(h \circ \gamma\) of \(h\) with the Gauss map \(\gamma\) defines a function on \(M_t = F(M, t)\). We have
\[ \frac{d}{dt}(h \circ \gamma) = dh \left( \frac{d\gamma}{dt} \right) = dh(\tau(\gamma)). \]

By the composition formula (see [32], p.28)
\[ \Delta(h \circ \gamma) = \text{Hess}(h)(\gamma_* e_i, \gamma_* e_i) + dh(\tau(\gamma)), \]
where \(\{e_i\}\) is a local orthonormal frame field on \(M_t\). It follows that
\begin{equation}
\left( \frac{d}{dt} - \Delta \right) h \circ \gamma = -\text{Hess}(h)(\gamma_* e_i, \gamma_* e_i).
\end{equation}
Noting $G^m_{n,m,n}$ has non-positive sectional curvature, the standard Hessian comparison theorem implies
\[
\text{Hess}(\tilde{r}) \geq \frac{1}{\tilde{r}}(\tilde{g} - d\tilde{r} \otimes d\tilde{r}),
\]
where $\tilde{r}$ is the distance function from a fixed point in $G^m_{n,m,n}$, $\tilde{g}$ is the metric tensor on $G^m_{n,m,n}$. Choose $h = \tilde{r}^2$ and we have
\[
\text{Hess}(h) \geq 2\tilde{g}.
\]
Hence,
\[
\text{Hess}(h)(\gamma \ast e_i, \gamma \ast e_i) \geq 4 e(\gamma) = 2 ||B||^2.
\]

On the other hand,
\[
|\nabla (h \circ \gamma)|^2 = \langle \nabla h, \gamma \ast e_i \rangle \langle \nabla h, \gamma \ast e_i \rangle
\]
\[
= \langle 2\tilde{r} \nabla \tilde{r}, \gamma \ast e_i \rangle \langle 2\tilde{r} \nabla \tilde{r}, \gamma \ast e_i \rangle
\]
\[
\leq 8\tilde{r}^2 e(\gamma) = 4\tilde{r}^2 ||B||^2.
\]
Therefore
\[
\left( \frac{d}{dt} - \Delta \right) h \circ \gamma \leq -2 ||B||^2 \leq -\frac{1}{2 h \circ \gamma} |\nabla (h \circ \gamma)|^2.
\]

First of all, by Theorem 2.1 we always have bounded curvature for the smooth solution of (2.7). Denote $u = h \circ \gamma$. Then, by (2.28)
\[
\left( \frac{d}{dt} - \Delta \right) u \leq 0.
\]
Let $u_k(t_0)$ be a sequence of points on $M_{t_0}$ such that when $k \to \infty$
\[
u(x_k(t_0)) \to \sup_{M_{t_0}} u > 0
\]
and when $k$ is sufficiently large
\[
|\nabla u|(x_k(t_0)) < \varepsilon,
\]
\[
\Delta u(x_k(t_0)) < \varepsilon
\]
for any $\varepsilon > 0$. Define
\[
u_1 = (u - \sup_{M_{t_0}} u) - \delta (t - t_0) - \delta
\]
for any $\delta > 0$ and when $t = t_0$
\[
u_1 \leq -\delta < 0.
\]
Thus,
\[
\left. \frac{d}{dt} u_1 \right|_{x_k(t_0)} = \left. \frac{d}{dt} u \right|_{x_k(t_0)} - \delta \leq \Delta u|_{x_k(t_0)} - \delta < 0.
\]
The similar argument as that in the proof of Theorem 2.1 gives
\[ u_1(x_k(t_1)) < u_1(y) \leq -\delta \]
for \( t_1 \) close to \( t_0 \), namely
\[ u(x_k(t_1)) \leq \sup_{M_{t_1}} u. \]
Let \( k \to \infty \) gives
\[ \sup_{M_{t_1}} u \leq \sup_{M_{t_0}} u. \]
□

Choose a Lorentzian base \( \{\varepsilon_i, \varepsilon_\alpha\} \) in \( \mathbb{R}^{m+n}_n \) with space-like \( \{\varepsilon_i\} \) and time-like \( \{\varepsilon_\alpha\} \). For a space-like submanifold \( F : M \to \mathbb{R}^{m+n}_n \) we assume \( 0 \in M \) and define coordinate functions
\[ x^i = \langle F, \varepsilon_i \rangle, \quad y^\alpha = \langle F, \varepsilon_\alpha \rangle. \]
Denote
\[ x = \sqrt{\sum_{i=1}^{m}(x^i)^2}, \quad y = \sqrt{\sum_{\alpha=m+1}^{m+n}(y^\alpha)^2}, \]
such that \( |F|^2 = x^2 - y^2 \). It is non-negative. The function \( y \) is called the height function of \( M \). The growth
\[ y^2 \leq x^2 \]
is always satisfied for any space-like submanifold.

**Theorem 2.4.** Let \( F : M \to \mathbb{R}^{m+n}_n \) be a space-like complete \( m \)-submanifold which has bounded curvature and bounded Gauss image. Then the mean curvature flow equation (2.7) has long time smooth solution.

**Proof.** Let \( p : \mathbb{R}^{m+n}_n \to \mathbb{R}^m \) be the natural projection defined by
\[ p(x^1, \ldots, x^m, x^{m+1}, \ldots, x^{m+n}) = (x^1, \ldots, x^m), \]
which induces a map from \( M \) into \( \mathbb{R}^m \). It is a smooth map from a complete manifold to \( \mathbb{R}^m \). For any vector \( v = (v^1, \ldots, v^m, v^{m+1}, \ldots, v^{m+n}) \) tangent to \( M \subset \mathbb{R}^{m+n}_n \), we define
\[ \langle p_*v, p_*v \rangle_{\mathbb{R}^m} = \langle v, v \rangle + \sum_{\alpha} (v^{m+\alpha})^2 \geq \langle v, v \rangle. \]
This means that \( p \) increases the distance. It follows that \( p \) is a covering map, and a diffeomorphism, since \( \mathbb{R}^m \) is simply connected. Hence, the induced Riemannian metric on \( M \) can be expressed as \( (\mathbb{R}^m, ds^2) \) with
\[ ds^2 = g_{ij}dx^i dx^j. \]
In this chart, the identity map \((\mathbb{R}^m, ds^2) \to (\mathbb{R}^m, ds^2_0)\) is a distance increasing map, where \(ds^2_0\) is the Euclidean metric on \(\mathbb{R}^m\). It follows that any eigenvalue of \((g_{ij})\) is not big than 1.

Choose \(P_0\) as an \(m\)-plane spanned by \(\varepsilon_1 \wedge \cdots \wedge \varepsilon_m\). At each point in \(M\) its image \(m\)-plane \(P\) under the Gauss map is spanned by \(f_i = \varepsilon_i + c_i \varepsilon_{m+s}\) and \(\sqrt{g} = |f_1 \wedge \cdots \wedge f_m|\).

We then have
\[
\langle P, P_0 \rangle = \frac{1}{\sqrt{g}} \det \left( \langle f_i, \varepsilon_j \rangle \right) = g^{-\frac{1}{2}}.
\]

Now, drawing a minimal geodesic \(C(r)\) between \(P_0\) and \(P\) parameterized by arc length \(r\). By a result in [30], \(C(r)\) can be represented by \(P(r)\) which is spanned by \(h_i = \varepsilon_i + z_is(r)\varepsilon_{m+s}\),

where
\[
z_is(r) = \begin{pmatrix} \tanh(\lambda_1 r) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & \tanh(\lambda_m r) & 0 \end{pmatrix}
\]

for \(\sum_i \lambda_i^2 = 1\). Let \(\tilde{h}_1 = \cosh(\lambda_1 r)h_1, \cdots, \tilde{h}_m = \cosh(\lambda_m r)h_m\).

Since
\[
|\tilde{h}_i|^2 = \langle \varepsilon_i + z_is(r)\varepsilon_{m+s}, \varepsilon_i + z_is(r)\varepsilon_{m+s} \rangle = 1 - \tanh^2(\lambda_i r) = \frac{1}{\cosh^2(\lambda_i r)},
\]

the vectors \(\tilde{h}_1, \cdots, \tilde{h}_m\) are orthonormal. Therefore, we can compute the inner product \(\langle P_0, P \rangle\) again by
\[
\langle P_0, P \rangle = \det \left( \langle \varepsilon_i, \tilde{h}_j \rangle \right) = \prod_{i=1}^{m} \cosh(\lambda_i r).
\]
If the distance $r$ between $P_0$ and $P$ is bounded by a finite number $R$, then combine the above formula and (2.29) yields

$$\sqrt{g} \geq \left( \prod_{i=1}^{m} \cosh(\lambda_i R) \right)^{-1}.$$

Thus, we prove that any eigenvalue of the induced metric tensor of a complete space-like $m-$submanifold in $\mathbb{R}^{m+n}$ with bounded Gauss image is uniformly bounded.

Noting Theorem 2.3, we know that the equation (2.7) is uniformly parabolic and has a unique smooth solution on some short time interval. By the curvature estimate (see Theorem 2.1), we have uniform estimate on $||B||$. Then we can proceed as in [16] (Prop. 2.3) to estimate all derivatives of $B$ in terms of their initial data

$$\sup_{M_t} ||\nabla^a B|| \leq C(m),$$

where $C(m)$ only depends on $q, m$ and $\sup_{M_0} ||\nabla^j B||$ for $0 \leq j \leq q$. It follows that this solution can be extended to all $t > 0$. \qed

It is easy to verified that

$$\left( \frac{d}{dt} - \Delta \right) y^a = 0$$

and

$$\left( \frac{d}{dt} - \Delta \right) y^2 = -2 \sum |\nabla y^a|^2 \leq 0.$$  

Omori-Yau maximum principle implies that if $M_0$ has finite curvature and finite height function, then the height function of $M_t$ is also finite under the evolution.

By (2.11) and (2.28) we have

$$\left( \frac{d}{dt} - \Delta \right) (2t||B||^2 + h \circ \gamma) \leq -\frac{4t}{n}||B||^4,$$

where $h$ denotes the square of the distance function on $G_{m,n}$, the target manifold of the Gauss map. By using maximum principle again we have an estimate for $t > 0$

$$\sup_{M_t} ||B||^2 \leq \frac{c}{t},$$

where $c$ is a constant depending on the bound of the Gauss image of the initial submanifold.

If the height function is going to infinity, we can consider rescaled mean curvature flow as done by Ecker-Huisken [11]. Define

$$\tilde{F}(\tilde{t}) = \frac{1}{\sqrt{2t+1}} F(t),$$
where
\[ \tilde{t} = \log(2t + 1). \]

Hence
\[ \frac{\partial}{\partial \tilde{t}} \tilde{F} = \tilde{H} - \tilde{F}. \]

It is not hard to verify that the Gauss map \( \tilde{\gamma} \) of the rescaled mean curvature flow is as same as the original \( \gamma \). Furthermore, the previous estimates translate to
\[ |\tilde{A}|^2 \leq (2t + 1)|A|^2 \leq C \]
which is dependent on the initial bound on \( M \).

Choose a Lorentzian frame field near \( p \in M \) along \( M \) in \( R^{m+n}_n \), such that \( e_i \in TM \) and \( e_\alpha \in NM \) with \( \nabla_{e_j} e_i|_p = \nabla_{e_i} e_\alpha|_p = 0 \). We have

**Lemma 2.2.**

\[ \left( \frac{d}{dt} - \Delta \right) \langle F, e_\alpha \rangle = 2 \langle H, e_\alpha \rangle - S_{\alpha\beta} \langle F, e_\beta \rangle + C_{\alpha\beta} \langle F, e_\beta \rangle \]

with anti-symmetric \( C_{\alpha\beta} \) in \( \alpha \) and \( \beta \), and

\[ \left( \frac{d}{dt} - \Delta \right) \sum_\alpha \langle F, e_\alpha \rangle^2 = 4 \langle H, e_\alpha \rangle \langle F, e_\alpha \rangle - 2 \sum_\alpha |\nabla \langle F, e_\alpha \rangle|^2 - 2 S_{\alpha\beta} \langle F, e_\alpha \rangle \langle F, e_\beta \rangle \]
\[ \leq C \left( \sum_\alpha \langle F, e_\alpha \rangle^2 + 1 \right) - 2 \sum_\alpha |\nabla \langle F, e_\alpha \rangle|^2. \]

**Proof.** Since at the point \( p \)

\[ \nabla_{e_i} A^\alpha(e_i) = \nabla_{e_i} \langle B_{ij}, e_\alpha \rangle e_j \]
\[ = \nabla_{e_i} \langle \nabla_{e_j} e_i, e_\alpha \rangle e_j \]
\[ = \langle \nabla_{e_j} \nabla_{e_i} e_i, e_\alpha \rangle e_j \]
\[ = \langle \nabla_{e_j} (\nabla_{e_i} e_i + B_{ii}), e_\alpha \rangle e_j \]
\[ = \langle B_{ij} e_i, e_\alpha \rangle + \langle \nabla_{e_j} H, e_\alpha \rangle e_j \]
\[ = \langle \nabla_{e_j} H, e_\alpha \rangle e_j, \]

(2.31)
\[ \Delta \langle F, e_\alpha \rangle = e_i e_i \langle F, e_\alpha \rangle = -e_i \langle F, A^{e_\alpha} (e_i) \rangle = -\langle e_i, A^{e_\alpha} (e_i) \rangle - \langle F, \nabla e_i A^{e_\alpha} (e_i) \rangle \]

(2.32)

\[ = -\langle H, e_\alpha \rangle - \langle F, \nabla e_i A^{e_\alpha} (e_i) \rangle - \langle F, B^{e_\alpha} (e_i) \rangle \]

\[ = -\langle H, e_\alpha \rangle - \langle \nabla e_i H, e_\alpha \rangle \langle F, e_i \rangle - \langle F, B_{ij} \rangle \langle B_{ij}, e_\alpha \rangle \]

\[ = -\langle H, e_\alpha \rangle - \langle \nabla e_i H, e_\alpha \rangle \langle F, e_i \rangle + S_{\alpha \beta} \langle F, e_\beta \rangle. \]

On the other hand,

\[ \left\langle \frac{d e_\alpha}{d t}, e_i \right\rangle = -\left\langle e_\alpha, \frac{d e_i}{d t} \right\rangle = -\langle e_\alpha, \nabla e_i H \rangle, \]

\[ \frac{d e_\alpha}{d t} = -\langle e_\alpha, \nabla e_i H \rangle e_i + C_{\alpha \beta} e_\beta \]

with anti-symmetric \( C_{\alpha \beta} \) in \( \alpha, \beta \). It follows that

\[ \frac{d}{d t} \langle F, e_\alpha \rangle = \langle H, e_\alpha \rangle + \left\langle F, \frac{d e_\alpha}{d t} \right\rangle \]

\[ = \langle H, e_\alpha \rangle - \langle e_\alpha, \nabla e_i H \rangle \langle F, e_i \rangle + C_{\alpha \beta} \langle F, e_\beta \rangle. \]

Furthermore, we have

\[ \frac{d}{d t} \sum_\alpha \langle F, e_\alpha \rangle^2 = 2 \sum_\alpha \langle F, e_\alpha \rangle \frac{d}{d t} \langle F, e_\alpha \rangle \]

(2.34)

\[ = 2 \sum_\alpha \langle H, e_\alpha \rangle \langle F, e_\alpha \rangle - 2 \sum_\alpha \langle \nabla e_i H, e_\alpha \rangle \langle F, e_i \rangle \langle F, e_\alpha \rangle \]

\[ + 2 C_{\alpha \beta} \langle F, e_\beta \rangle \langle F, e_\alpha \rangle \]

\[ = 2 \sum_\alpha \langle H, e_\alpha \rangle \langle F, e_\alpha \rangle - 2 \sum_\alpha \langle \nabla e_i H, e_\alpha \rangle \langle F, e_i \rangle \langle F, e_\alpha \rangle, \]

and

\[ \Delta \langle F, e_\alpha \rangle^2 = 2 |\nabla \langle F, e_\alpha \rangle|^2 + 2 \langle F, e_\alpha \rangle \Delta \langle F, e_\alpha \rangle \]

(2.35)

\[ = 2 |\nabla \langle F, e_\alpha \rangle|^2 + 2 \langle F, e_\alpha \rangle (-\langle H, e_\alpha \rangle - \langle \nabla e_i H, e_\alpha \rangle \langle F, e_i \rangle + S_{\alpha \beta} \langle F, e_\beta \rangle). \]

Hence,

\[ \left( \frac{d}{d t} - \Delta \right) \sum_\alpha \langle F, e_\alpha \rangle^2 = 4 \langle H, e_\alpha \rangle \langle F, e_\alpha \rangle - 2 \sum_\alpha |\nabla \langle F, e_\alpha \rangle|^2 - 2 S_{\alpha \beta} \langle F, e_\alpha \rangle \langle F, e_\beta \rangle. \]

(2.36)

Noting the estimates of \( \|H\|^2 \), \( \|B\|^2 \) and \( S_{\alpha \beta} \), we obtain the desired estimate. \( \square \)
The subsequent estimates in [11] can be carried out in the same way to derive

**Theorem 2.5.** Suppose that $F : M \rightarrow \mathbb{R}^{m+n}$ is a space-like complete $m$-submanifold with bounded curvature and bounded Gauss image. If in addition assume that

$$\sum_{\alpha} \langle F, e_{\alpha} \rangle^2 \leq C'(1 + |F|^2)^{1-\delta}$$

is valid on $M$ for some constants $C' < \infty, \delta > 0$, then the solution $\tilde{M}_t$ of the rescaled equation converges for $t \rightarrow \infty$ to a limiting submanifold $M_\infty$ satisfying the equation

$$F^\perp = H.$$

### 3. Submanifolds in Euclidean space

Let $F : M \rightarrow \mathbb{R}^{m+n}$ be an $m$−submanifold in $(m + n)$−dimensional Euclidean space with the second fundamental form $B$ which can be viewed as a cross-section of the vector bundle Hom$(\otimes^2 TM, NM)$ over $M$, where $TM$ and $NM$ denote the tangent bundle and the normal bundle along $M$, respectively. A connection on Hom$(\otimes^2 TM, NM)$ can be induced from those of $TM$ and $NM$ naturally. We investigate the higher codimension $n \geq 2$ situation in this section.

For $\nu \in \Gamma(NM)$ the shape operator $A^\nu : TM \rightarrow TM$ satisfies

$$\langle B_{XY}, \nu \rangle = \langle A^\nu(X), Y \rangle.$$

The second fundamental form, curvature tensors of the submanifold, curvature tensor of the normal bundle and that of the ambient manifold satisfy the Gauss equations, the Codazzi equations and the Ricci equations.

Taking the trace of $B$ gives the mean curvature vector $H$ of $M$ in $\mathbb{R}^{m+n}$, a cross-section of the normal bundle.

Choose a local orthonormal frame field $\{e_i, e_{\alpha}\}$ along $M$ with dual frame field $\{\omega_i, \omega_{\alpha}\}$, such that $e_i$ are tangent vectors to $M$. The induced Riemannian metric of $M$ is given by $ds_M^2 = \sum_i \omega_i^2$ and the induced structure equations of $M$ are

$$d\omega_i = \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \omega_{ik} \wedge \omega_{kj} + \omega_{i\alpha} \wedge \omega_{\alpha j},$$

$$\Omega_{ij} = d\omega_{ij} - \omega_{ik} \wedge \omega_{kj} = -\frac{1}{2} R_{ijkl} \omega_k \wedge \omega_l.$$

By Cartan’s lemma we have

$$\omega_{\alpha i} = h_{\alpha ij} \omega_j.$$
3.1. Evolution Equation under the Mean Curvature Flow. As the same
derivation as in the space-like $m-$submanifold in pseudo-Euclidean space, we have
the following formula.

**Proposition 3.1.**

\[(\nabla^2 B)_{XY} = \nabla_X \nabla_Y H + \langle B_{Xe_i}, H \rangle B_{Ye_i} - \langle B_{XY}, B_{ei, ej} \rangle B_{ei, ej} + 2 \langle B_{Xe_i}, B_{Ye_i} \rangle (B_{ei, ej} - \langle B_{Ye_i}, B_{ei, ej} \rangle B_{Ye_j} - \langle B_{Xe_i}, B_{ei, ej} \rangle B_{Ye_j}).\]

Denote

\[B_{ij} = B_{ei, ej} = (\nabla_{e_i} e_j)^N = h_{\alpha ij}e_\alpha,\]

where \(\{e_\alpha\}\) is a local orthonormal frame field of the normal bundle near \(x \in M\). Let

\[S_{\alpha\beta} = h_{\alpha ij}h_{\beta ij}.\]

Then \(|B|^2 = \sum_\alpha S_{\alpha\alpha} \).

Noting

\[- \langle B_{kl}, B_{ij} \rangle \langle B_{ij}, B_{kl} \rangle = -h_{\alpha kl}h_{\alpha ij}h_{\beta ij}h_{\beta kl} = -\sum_{\alpha, \beta} S_{\alpha\beta}^2,\]

\[2 \langle B_{ik}, B_{jk} \rangle \langle B_{kl}, B_{ij} \rangle - 2 \langle B_{jk}, B_{kl} \rangle \langle B_{il}, B_{ij} \rangle\]

\[= 2 \sum_{\alpha \neq \beta} \langle [A^{e_\alpha} A^{e_\beta}, A^{e_\alpha} A^{e_\beta}] - 2 \langle A^{e_\beta} A^{e_\alpha}, A^{e_\alpha} A^{e_\beta} \rangle \rangle\]

\[= -\sum_{\alpha \neq \beta} \|[A^{e_\alpha}, A^{e_\beta}]\|^2,\]

we then have

\[\langle \nabla^2 B, B \rangle = \langle \nabla_i \nabla_j H, B_{ij} \rangle + \langle B_{ik}, H \rangle \langle B_{il}, B_{kl} \rangle - \sum_{\alpha \neq \beta} \|[A^{e_\alpha}, A^{e_\beta}]\|^2 - \sum_{\alpha, \beta} S_{\alpha\beta}^2.\]

The following expression follows immediately.

**Proposition 3.2.**

\[\Delta |B|^2 = 2 |\nabla B|^2 + 2 \langle \nabla_i \nabla_j H, B_{ij} \rangle + 2 \langle B_{ij}, H \rangle \langle B_{ik}, B_{jk} \rangle - 2 \sum_{\alpha \neq \beta} \|[A^{e_\alpha}, A^{e_\beta}]\|^2 - 2 \sum_{\alpha, \beta} S_{\alpha\beta}^2.\]

We now consider the MCF for a submanifold in \(\mathbb{R}^{m+n}\). Namely, consider a one-
parameter family \(F_t = F(\cdot, t)\) of immersions \(F_t : M \rightarrow \mathbb{R}^{m+n}\) with corresponding
images \(M_t = F_t(M)\) such that

\[\frac{d}{dt} F(x, t) = H(x, t), \quad x \in M\]

\[F(x, 0) = F(x)\]

\[(3.3)\]
is satisfied, where $H(x, t)$ is the mean curvature vector of $M_t$ at $F(x, t)$ in $\mathbb{R}^{m+n}$. We also have

$$\frac{d g_{ij}}{dt} = -2 \langle H, B_{ij} \rangle,$$

(3.4)

$$\frac{d g^{ij}}{dt} = 2 g^{ik} g^{jl} \langle H, B_{kl} \rangle,$$

(3.5)

and

$$\frac{d g}{dt} = -2 |H|^2 g,$$

(3.6)

where $g = \det(g_{ij})$. We now derive the evolution equation for the squared norm of the second fundamental form.

**Lemma 3.1.** The second fundamental form satisfies

$$\left( \frac{d}{dt} - \Delta \right) |B|^2 \leq -2 |\nabla B|^2 + 3 |B|^4,$$

(3.7)

**Remark 3.1.** Compare (3.7) with (2.11), we see that now the curvature estimates is not directly as that in the last section.

**Proof.** For fixed $x_0$, $t_0$ choose a local orthonormal frame $\{e_i\}$ of $M_{t_0}$ near $x_0$ which is normal at $x_0$. By the immersion $F_{t_0}$ we have $\{e_i\}$ on $M$, which is not orthonormal in general. Then by $F_t$ we obtain $\{F_t e_i\}$ which is denoted by $\{e_i\}$ for simplicity. We also choose a local orthonormal frame field $\{e_\alpha\}$ of the normal bundle of $M_t$ near $x_0$. Then at $(x_0, t_0)$

$$\frac{d h_{\alpha ij}}{dt} = \nabla_{\frac{d}{dt}} \langle \nabla_{e_i} e_j, e_\alpha \rangle$$

$$= \langle \nabla_{e_i} e_j, e_\alpha \rangle + \langle \nabla_{e_i} H, e_\alpha \rangle$$

$$= \langle \nabla_{e_i} e_j, e_\alpha \rangle + \langle B_{ij}, \nabla_{e_i} H \rangle$$

$$= \langle \nabla_{e_i} (\nabla_{e_j} H + (\nabla_{e_j} H)^T), e_\alpha \rangle + \langle B_{ij}, \nabla_{e_i} H \rangle$$

$$= \langle \nabla_{e_i} \nabla_{e_j} H, e_\alpha \rangle - h_{\alpha ik} h_{\beta jk} H_\beta + h_{\beta ij} \langle \nabla_{e_i} H e_\alpha, e_\beta \rangle.$$

(3.8)

Since in a non-orthonormal frame field $g_{ij} = \langle F_i e_i, F_j e_j \rangle$ (except at $t_0$) is not a unit matrix,

$$|B|^2 = g^{ik} g^{jl} h_{\alpha ij} h_{\alpha kl}.$$

We have at $(x_0, t_0)$

$$\frac{d |B|^2}{dt} = 2 \frac{d g^{ik}}{dt} h_{\alpha ij} h_{\alpha kl} + 2 \frac{d h_{\alpha ij}}{dt} h_{\alpha ij}.$$
From (3.8) we have
\[ \frac{d h_{\alpha ij}}{dt} = h_{\alpha ij} \left( \nabla_{e_i} \nabla_{e_j} H, e_\alpha \right) - h_{\alpha ij} h_{\alpha ik} h_{\beta jk} H_\beta \]  
Noting (3.5), we have
\[ \frac{d g_{ik}}{dt} h_{\alpha ij} h_{\alpha kj} = 2 h_{\alpha ij} h_{\alpha kj} \left( H, B_{ik} \right) = 2 h_{\alpha ij} h_{\alpha kj} h_{\beta ik} H_\beta \]
Substituting (3.10) and (3.11) into (3.9) gives
\[ \frac{1}{2} \left( \frac{d}{dt} - \Delta \right) |B|^2 = \left( \nabla_i \nabla_j H, B_{ij} \right) + \left( B_{ij}, H \right) \left( B_{ik}, B_{jk} \right) \]
From (3.2) and (3.12) we obtain the evolution equation for the squared norm of the second fundamental form
\[ \frac{1}{2} \left( \frac{d}{dt} - \Delta \right) |B|^2 = -|\nabla B|^2 + \sum_{\alpha \neq \beta} |[A^{\alpha \epsilon}, A^{\epsilon \beta}]|^2 + \sum_{\alpha, \beta} S_{\alpha \beta}^2. \]
We know from [23] in general
\[ \sum_{\alpha \neq \beta} |[A^{\alpha \epsilon}, A^{\epsilon \beta}]|^2 + \sum_{\alpha, \beta} S_{\alpha \beta}^2 \leq \left( 2 - \frac{1}{n} \right) |B|^4. \]
When the codimension \( n \geq 2 \) the above estimate was refined \[20][7]\]
\[ \sum_{\alpha \neq \beta} |[A^{\alpha \epsilon}, A^{\epsilon \beta}]|^2 + \sum_{\alpha, \beta} S_{\alpha \beta}^2 \leq \frac{3}{2} |B|^4. \]
On the other hand, by the Schwartz inequality
\[ |\nabla |B| | \leq |\nabla B|. \]
Therefore, the inequality (3.7) is obtained. \( \square \)

3.2. Main Estimates. For any \( p \in M \) let \( \{ e_1, \ldots, e_m \} \) be a local orthonormal frame field near \( p \). Define the Gauss map \( \gamma : p \mapsto \gamma(p) \) which is obtained by parallel translation of \( T_p M \) to the origin in the ambient space \( \mathbb{R}^{m+n} \). The image of the Gauss map lies in a Grassmannian \( G_{m,n} \). It is a symmetric space of compact type.

For any \( P \in G_{m,n} \), there are \( m \) vectors \( v_1, \ldots, v_m \) spanning \( P \). Then we have Plücker coordinates \( v_m \wedge \cdots \wedge v_1 \) for \( P \) up to a constants. The Gauss map \( \gamma \) can be described by \( p \mapsto e_1 \wedge \cdots \wedge e_m \). Since
\[
\begin{align*}
    d(e_1 \wedge \cdots \wedge e_m) &= de_1 \wedge \cdots \wedge e_m + \cdots + e_1 \wedge \cdots \wedge de_m \\
    &= \omega_{a1} e_a \wedge e_2 \wedge \cdots \wedge e_m + \cdots + e_1 \wedge \cdots \wedge e_{m-1} \wedge \omega_{am} e_a \\
    &= \omega_{ai} e_{ai}
\end{align*}
\]
and the canonical metric on $G_{m,n}$ is defined by

$$ds^2 = \sum_{\alpha,i} \omega^2_{\alpha i},$$

where $e_{\alpha i} = e_1 \wedge \cdots \wedge e_{i-1} \wedge e_\alpha \wedge e_{i+1} \wedge \cdots \wedge e_m$ are orthonomal basis for $TG_{m,n}$ (see [33], pp. 188-194). It follows that

$$\gamma^* \omega_{\alpha i} = h_{\alpha ij} \omega_j$$

and the tension field of the Gauss map

$$\tau(\gamma) = h_{\alpha ij} e_\alpha = h_{\alpha ji} e_\alpha$$

$$= h_{\alpha ji} e_1 \wedge \cdots \wedge e_{i-1} \wedge e_\alpha \wedge e_{i+1} \wedge \cdots \wedge e_m$$

$$= \sum_i e_1 \wedge \cdots \wedge e_{i-1} \wedge \nabla e_i H \wedge e_{i+1} \wedge \cdots \wedge e_m,$$

where we use the Codazzi equation. In [29], there is the following relation.

**Proposition 3.3.**

$$(3.15) \quad \frac{d\gamma}{dt} = \tau(\gamma(t)).$$

We consider the mean curvature flow of a complete manifold. We will assume that integration by parts is permitted and all integrals are finite for the submanifolds and functions we will consider in the sequel. We have the following maximum principle for parabolic equations on complete manifolds.

Define the backward heat kernel $\rho = \rho(x, t)$ by

$$\rho(x, t) = \frac{1}{(4\pi(t_0 - t))^\frac{m+n}{2}} \exp\left(-\frac{|x|^2}{4(t_0 - t)}\right), \quad t_0 > t, \quad x \in \mathbb{R}^{m+n}.$$  

We have the following formula. It is derived in the mean curvature flow in Euclidean space. Since (3.6), the formula is unchanged in higher codimension.

**Proposition 3.4.** (Huisken [16]) For a function $f(x, t)$ on $M$ we have

$$(3.16) \quad \frac{d}{dt} \int_M f \rho d\mu_t = \int_M \left(\frac{d}{dt} f - \Delta f\right) \rho d\mu_t - \int_M f \rho \left|H + \frac{F^\perp}{2(t_0 - t)}\right|^2 d\mu_t,$$

where $d\mu_t$ is the volume form of $M_t$.

**Corollary 3.1.** (Ecker and Huisken [11]) Suppose the function $f = f(x, t)$ satisfies the inequality

$$\left(\frac{d}{dt} - \Delta\right) f \leq \langle a, \nabla f \rangle$$
for some vector field \( a \) with uniformly bounded norm on \( M \times [0,t_1] \) for some \( t_1 > 0 \), then

\[
\sup_{M_t} f \leq \sup_{M_0} f
\]

for all \( t \in [0,t_1] \).

Now, we consider the convex Gauss image situation which is preserved under the flow, as shown in the following theorem.

**Theorem 3.1.** (confinable property) If the Gauss image of the initial submanifold \( M \) is contained in a geodesic ball of the radius \( \rho_0 < \sqrt{\frac{2}{4}} \pi \) in \( G_{m,n} \), then the Gauss images of all the submanifolds under the MCF are also contained in the same geodesic ball.

**Proof.** We consider a smooth bounded function on \( G_{m,n} \)

\[
h = 1 + \varepsilon - \cos(\sqrt{2} \rho),
\]

where \( \rho \) is the distance function from a point in \( G_{m,n} \), \( \varepsilon > 0 \) is a fixed constant. When \( \rho < \sqrt{\frac{2}{4}} \pi \), \( h \) is convex. By the Hessian comparison theorem we have

\[
\text{Hess}(h) \geq 2 \cos(\sqrt{2} \rho) g,
\]

where \( g \) is the metric tensor on \( G_{m,n} \). Hence,

\[
\text{Hess}(h)(\gamma_\ast e_i, \gamma_\ast e_i) \geq 2 \cos(\sqrt{2} \rho) |B|^2
\]

The composition function \( h \circ \gamma \) of \( h \) with the Gauss map \( \gamma \) defines a function on \( M_t = F(M,t) \). We have

\[
\frac{d}{dt}(h \circ \gamma) = \frac{d}{dt} h(\frac{d}{dt} \gamma) = \frac{d}{dt} \tau(\gamma)).
\]

By the composition formula (see [32], p.28)

\[
\Delta(h \circ \gamma) = \text{Hess}(h)(\gamma_\ast e_i, \gamma_\ast e_i) + \frac{d}{dt} \tau(\gamma)),
\]

where \( \{e_i\} \) is a local orthonormal frame field on \( M_t \).

It follows that

\[
(3.17) \quad \left( \frac{d}{dt} - \Delta \right) h \circ \gamma \leq -2 \cos(\sqrt{2} \rho \circ \gamma) |B|^2.
\]

Thus, we can use Corollary 3.1 to get conclusion. \( \square \)
For simplicity $h \circ \gamma$ is denoted by $h_1$ in the sequel. On the other hand,

$$(3.18) \quad |\nabla h_1|^2 = |\langle \nabla h, \gamma_* e_i \rangle \langle \nabla h, \gamma_* e_i \rangle| \leq 2 \sin^2(\sqrt{2} \rho \circ \gamma)|B|^2.$$ 

From (3.17) and (3.18) we have

$$(3.19) \quad \left(\frac{d}{dt} - \Delta\right) h_1 \leq -\cos(\sqrt{2} \rho \circ \gamma)|B|^2 - \frac{\cos(\sqrt{2} \rho \circ \gamma)}{2 \sin^2(\sqrt{2} \rho \circ \gamma)}|\nabla h_1|^2.$$ 

For any $q > 0$,

$$(3.20) \quad \left(\frac{d}{dt} - \Delta\right) h_1^q = q h_1^{q-1} \left(\frac{d}{dt} - \Delta\right) h_1 - q(q - 1) h_1^{q-2} |\nabla h_1|^2 \leq -q h_1^{q-1} \cos(\sqrt{2} \rho \circ \gamma)|B|^2 - \left(q(q - 1) h_1^{q-2} + q h_1^{q-1} \frac{\cos(\sqrt{2} \rho \circ \gamma)}{2 \sin^2(\sqrt{2} \rho \circ \gamma)}\right) |\nabla h_1|^2.$$ 

From (3.7) and (3.20), we have

$$(3.21) \quad \left(\frac{d}{dt} - \Delta\right) (|B|^2 h_1^q) = |B|^2 \left(\frac{d}{dt} - \Delta\right) h_1^q + h_1^q \left(\frac{d}{dt} - \Delta\right) |B|^2 - 2 |\nabla |B|^2| \cdot \nabla h_1^q \leq \left(-q \cos(\sqrt{2} \rho \circ \gamma) + 3 h_1\right) |B|^4 h_1^{q-1} - \left[q(q - 1) h_1^{q-2} + q \frac{\cos(\sqrt{2} \rho \circ \gamma)}{2 \sin^2(\sqrt{2} \rho \circ \gamma)} h_1^{q-1}\right] |B|^2 |\nabla h_1|^2 - 2 h_1^q |\nabla |B|^2| - 2 |\nabla |B|^2| \cdot \nabla h_1^q = [3(1 + \varepsilon) - (3 + q) \cos(\sqrt{2} \rho \circ \gamma)] h_1^{q-1} |B|^4 - \left[q(q - 1) + q \frac{\cos(\sqrt{2} \rho \circ \gamma)}{2 \sin^2(\sqrt{2} \rho \circ \gamma)} h_1\right] h_1^{q-2} |B|^2 |\nabla h_1|^2 - 2 h_1^q |\nabla |B|^2| - 2 |\nabla |B|^2| \cdot \nabla h_1^q.
By using the Young inequality we have
\[
-2\nabla |B|^2 \cdot \nabla h_1^q = -(h_1^{-1} \nabla h_1^q) \cdot \nabla (|B|^2 h_1^q) + |B|^2 h_1^{-2} |\nabla h_1^q|^2 \\
- |\nabla |B|^2 \cdot \nabla h_1^q |
\leq -q(h_1^{-1} \nabla h_1) \cdot \nabla (|B|^2 h_1^q) + \frac{1}{2} q^2 h_1^{q-2} |B|^2 |\nabla h_1|^2 + 2 h_1^q |\nabla |B|^2 h_1^q |^2 \\
+ q(h_1^{-1} \nabla h_1) \cdot \nabla (|B|^2 h_1^q)
\leq -q(h_1^{-1} \nabla h_1) \cdot \nabla (|B|^2 h_1^q) + \\
+ \frac{3}{2} q^2 h_1^{q-2} |B|^2 |\nabla h_1|^2 + 2 h_1^q |\nabla |B|^2 h_1^q |^2.
\]
(3.22)

Thus, (3.21) becomes
\[
\left( \frac{d}{dt} - \Delta \right) (|B|^2 h_1^q) \leq \left[ 3(1 + \varepsilon) - (3 + q) \cos(\sqrt{2} \rho \cos(\gamma)) \right] |B|^4 h_1^{q-1} \\
+ \left( \frac{1}{2} q + 1 - \frac{\cos(\sqrt{2} \rho \cos(\gamma))}{2 \sin^2(\sqrt{2} \rho \cos(\gamma))} \right) q h_1^{q-2} |B|^2 |\nabla h_1|^2 \\
- q(h_1^{-1} \nabla h_1) \cdot \nabla (|B|^2 h_1^q).
\]
(3.23)

We now give the following result.

**Theorem 3.2.** Let $M$ be a complete $m$-submanifold in $\mathbb{R}^{m+n}$ with bounded curvature. Suppose that the image under the Gauss map from $M$ into $G_{m,n}$ lies in a geodesic ball of radius $R_0 < \sqrt{\frac{2}{2\pi}}$. If $M_t$ is a smooth solution of (3.3), then there is the following estimate
\[
\sup_{M_t} |B|^2 h_1^q \leq \sup_{M_0} |B|^2 h_1^q,
\]
(3.24)

where $q$ is a fixed constant depending on $R_0$.

**Proof.** Let $r_0 = \cos(\sqrt{2} R_0)$. Then $r_0 > \sqrt{\frac{2}{2\pi}}$. It follows that
\[
\frac{3}{2r_0} - \frac{r_0}{2(1 - r_0^2)} < 0.
\]

It is possible to choose $\varepsilon > 0$ satisfying
\[
\left( \frac{3}{2r_0} - \frac{r_0}{2(1 - r_0^2)} \right) \varepsilon + \frac{3}{2r_0} - \frac{1}{2} \frac{r_0}{2(1 + r_0)} \leq 0.
\]
(3.25)

Set
\[
q = 3 \left( \frac{1 + \varepsilon}{r_0} - 1 \right).
\]
Then for $r = \cos(\sqrt{2}\rho \circ \gamma) \geq r_0$,

$$3(1 + \varepsilon) - (3 + q)r = 3(1 + \varepsilon) - 3(1 + \varepsilon) \frac{r}{r_0} \leq 0,$$

which implies the first term of the right hand side of (3.23) is non-positive. Note

$$\frac{1}{2}q + 1 - \frac{r}{2(1 - r^2)}(1 + \varepsilon - r)$$

(3.26)

$$= \frac{3}{2} \left( \frac{1 + \varepsilon}{r_0} - 1 \right) + 1 - \frac{r}{2(1 - r^2)}(1 + \varepsilon - r)
= \left( \frac{3}{2r_0} - \frac{r}{2(1 - r^2)} \right) \varepsilon + \frac{3}{2r_0} \frac{1}{2} - \frac{r}{2(1 + r)},$$

which is non-increasing in $r$. Since (3.25), (3.26) is non-positive when $r \geq r_0$. It follows that under the conditions of the theorem, (3.23) becomes

$$\left( \frac{d}{dt} - \Delta \right) (|B|^2 h_1^q) \leq -q(h_1^{-1}\nabla h_1) \cdot \nabla(|B|^2 h_1^q).$$

From (3.18) we have

$$|h_1^{-1}\nabla h_1| \leq \frac{\sqrt{2} \sin(\sqrt{2}\rho \circ \gamma)}{1 + \varepsilon - \cos(\sqrt{2}\rho \circ \gamma)} |B|$$

Let

$$f(\theta) = \frac{\sin \theta}{1 + \varepsilon - \cos \theta}.$$ 

Since $f''(\theta)|_{f'(\theta)=0} \leq 0$,

$$f(\theta) \leq f(\theta)|_{f'(\theta)=0} = \frac{\sqrt{1 - \frac{1}{(1+\varepsilon)^2}}}{1 + \varepsilon - \frac{1}{1+\varepsilon}}
= \frac{\sqrt{(1 + \varepsilon)^2 - 1}}{(1 + \varepsilon)^2 - 1} = \frac{\sqrt{\varepsilon(\varepsilon + 2)}}{\varepsilon(\varepsilon + 2)}.$$

(3.28)

It follows that

$$|h_1^{-1}\nabla h_1| \leq \frac{\sqrt{2} \varepsilon(\varepsilon + 2)}{\varepsilon(\varepsilon + 2)} |B|$$

Thus, we can use Corollary 3.1 and the estimate (3.24) has been obtained. \hfill \Box

**Corollary 3.2.** Suppose that the image under the Gauss map from $M$ into $G_{m,n}$ lies in a geodesic ball of radius $R_0 < \sqrt{\frac{7}{12}} \pi$. If $M_t$ is a smooth solution of (3.3), then there is the following estimate

$$\sup_{M_t} |B|^2 \leq \frac{c}{t},$$

(3.29)
where $c$ is depends only on the bound of the Gauss image of its initial manifold.

**Proof.** From (3.17)

$$
\left( \frac{d}{dt} - \Delta \right) h_1^q = qh_1^{q-1} \left( \frac{d}{dt} - \Delta \right) h_1 - q(q - 1)h_1^{q-2} |\nabla h_1|^2
\leq -2qh_1^{q-1} \cos(\sqrt{2}\rho \circ \gamma) |B|^2 - q(q - 1)h_1^{q-2} |\nabla h_1|^2.
$$

It follows that

$$
\left( \frac{d}{dt} - \Delta \right) (t|B|^2 h_1^q + h_1^q) \leq -q(h_1^{-1} \nabla h_1) \cdot \nabla (t|B|^2 h_1^q)
+ |B|^2 h_1^q - 2qh_1^{q-1} \cos(\sqrt{2}\rho \circ \gamma) |B|^2 - q^2 h_1^{q-2} |\nabla h_1|^2 + qh_1^{q-2} |\nabla h_1|^2.
$$

Since

$$
q(h_1^{-1} \nabla h_1) \cdot \nabla h_1^q = q^2 h_1^{q-2} |\nabla h_1|^2,
$$

(3.30) becomes

$$
\left( \frac{d}{dt} - \Delta \right) (t|B|^2 h_1^q + h_1^q) \leq -q(h_1^{-1} \nabla h_1) \cdot \nabla (t|B|^2 h_1^q + h_1^q)
+ |B|^2 h_1^q - 2qh_1^{q-1} \cos(\sqrt{2}\rho \circ \gamma) |B|^2 + qh_1^{q-2} |\nabla h_1|^2.
$$

Noting (3.18), the above inequality becomes

$$
\left( \frac{d}{dt} - \Delta \right) (t|B|^2 h_1^q + h_1^q) \leq -q(h_1^{-1} \nabla h_1) \cdot \nabla (t|B|^2 h_1^q + h_1^q)
+ |B|^2 h_1^{q-2} (h_1^2 - 2qh_1 \cos(\sqrt{2}\rho \circ \gamma)) + 2q \sin^2(\sqrt{2}\rho \circ \gamma)).
$$

Let

$$
A(r) = (h_1^2 - 2qh_1 \cos(\sqrt{2}\rho \circ \gamma)) + 2q \sin^2(\sqrt{2}\rho \circ \gamma))
= (1 + \varepsilon - r)^2 - 2q(1 + \varepsilon - r)r + 2q(1 - r^2) = (1 + \varepsilon - r)^2 - 2q(r + \varepsilon r - 1),
$$

where $r = \cos(\sqrt{2}\rho \circ \gamma)$. Since $A'(r) < 0$ and $q = 3 \left( \frac{1 + \varepsilon}{r_0} - 1 \right)$, then for $r \geq r_0$

$$
A \leq (1 + \varepsilon - r_0)^2 - 2q(r_0 + \varepsilon r_0 - 1) = (1 + \varepsilon - r_0) \left( \frac{6}{r_0} - r_0 - 5 - 5\varepsilon \right).
$$

We know that $\varepsilon$ is chosen by (3.25). If necessary we choose $\varepsilon$ larger such that $A \leq 0$. Therefore, from (3.31) we have

$$
\left( \frac{d}{dt} - \Delta \right) (t|B|^2 h_1^q + h_1^q) \leq -q(h_1^{-1} \nabla h_1) \cdot \nabla (t|B|^2 h_1^q + h_1^q)
$$
and by Corollary 3.1 again we have the desired estimate □

3.3. Proof of the Second Main Theorem. We are now in a position to prove the following theorem.

Theorem 3.3. Let \( F : M \to \mathbb{R}^{m+n} \) be a complete \( m \)-submanifold which has bounded curvature. Suppose that the image under the Gauss map from \( M \) into \( \mathbb{G}_{m,n} \) lies in a geodesic ball of radius \( R_0 < \frac{\sqrt{2}}{12} \pi \). Then the mean curvature flow equation (3.3) has long time smooth solution.

Proof. Let \( P_0 \in \mathbb{G}_{m,n} \) be a fixed point which is described by
\[
P_0 = \varepsilon_1 \wedge \cdots \wedge \varepsilon_m,
\]
where \( \varepsilon_1, \ldots, \varepsilon_m \) are orthonormal vectors in \( \mathbb{R}^{m+n} \). Choose complementary orthonormal vectors \( \varepsilon_{m+1}, \ldots, \varepsilon_{m+n} \), such that \( \{\varepsilon_1, \ldots, \varepsilon_m, \varepsilon_{m+1}, \ldots, \varepsilon_{m+n}\} \) is an orthonormal base in \( \mathbb{R}^{m+n} \).

Let \( p : \mathbb{R}^{m+n} \to \mathbb{R}^m \) be the natural projection defined by
\[
p(x^1, \ldots, x^m; x^{m+1}, \ldots, x^{m+n}) = (x^1, \ldots, x^m),
\]
which induces a map from \( M \) to \( \mathbb{R}^m \). It is a smooth map from a complete manifold to \( \mathbb{R}^m \).

For any point \( x \in M \) choose a local orthonormal tangent frame field \( \{e_1, \ldots, e_m\} \) near \( x \). Let \( v = v_i e_i \in TM \). Its projection
\[
p_*v = \langle v_i e_i, e_j \rangle e_j = v_i \langle e_i, e_j \rangle e_j.
\]

Now, we consider the case of the image under the Gauss map \( \gamma \) containing in a geodesic ball of radius \( R_0 < \frac{\sqrt{2}}{12} \pi \) and centered at \( P_0 \). For any \( P \in \gamma(M) \),
\[
w \text{ def.} = \langle P, P_0 \rangle = \langle e_1 \wedge \cdots \wedge e_m, \varepsilon_1 \wedge \cdots \wedge \varepsilon_m \rangle = \det W,
\]
where \( W = ((e_i, \varepsilon_j)) \). The Jordan angles between \( P \) and \( P_0 \) are
\[
\theta_i = \cos^{-1}(\lambda_i),
\]
where \( \lambda_i^2 \) are eigenvalues of the symmetric matrix \( W^T W \). It is well known that
\[
W^T W = O^T \Lambda O,
\]
where \( O \) is an orthogonal matrix and
\[
\Lambda = \begin{pmatrix}
\lambda_1^2 & 0 \\
0 & \ddots \\
0 & \lambda_m^2
\end{pmatrix},
\]
where each $0 < \lambda_i^2 < 1$. We know that
\[
w = \prod \cos \theta_i.
\]
On the other hand, the distance between $P_0$ and $P$ (see [33], pp. 188-194)
\[
d(P_0, P) = \sqrt{\sum \theta_i^2}
\]
which is less than $\sqrt{\pi} \frac{7}{12}$ by the assumption. It follows that
\[
w > w_0 = \left( \cos \frac{\sqrt{2} \pi}{12} \right)^m.
\]
We now compare the length of any tangent vector $v$ to $M$ with its projection $p_* v$.
\[
|p_* v|^2 = \sum_{j=1}^{m} (v_i \langle e_i, \xi_j \rangle)^2 = (W V)^T W V,
\]
where $V = (v^1, \cdots, v^m)^T$. Hence,
\[
(3.33) \quad |p_* v|^2 \geq (\lambda')^2 |v|^2 > w^2 |v|^2 > w_0^2 |v|^2,
\]
where $\lambda' = \min \{\lambda_i\}$. The induced metric $ds^2$ on $\mathbf{R}^{m+n}$ is complete, so is the homothetic metric $\tilde{d}s^2 = w_0^2 ds^2$. (3.33) implies
\[
p : (M, \tilde{d}s^2) \to (\mathbf{R}^m, \text{canonical metric})
\]
increases the distance. It follows that $p$ is a covering map from a complete manifold into $\mathbf{R}^m$, and a diffeomorphism, since $\mathbf{R}^m$ is simply connected. Hence, the induced Riemannian metric on $M$ can be expressed as $(\mathbf{R}^m, ds^2)$ with
\[
ds^2 = g_{ij} dx^i dx^j.
\]
Furthermore, the immersion $F : M \to \mathbf{R}^{m+n}$ is realized by a graph $(x, f(x))$ with $f : \mathbf{R}^m \to \mathbf{R}^n$ and
\[
g_{ij} = \delta_{ij} + \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j}
\]
It follows that any eigenvalue of $(g_{ij})$ is not less than 1.

At each point in $M$ its image $m$-plane $P$ under the Gauss map is spanned by
\[
f_i = e_i + \frac{\partial f^\alpha}{\partial x^i} e_\alpha.
\]
It follows that
\[
|f_1 \wedge \cdots \wedge f_m|^2 = \det \left( \delta_{ij} + \sum_{\alpha} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j} \right)
\]
and
\[
\sqrt{g} = |f_1 \wedge \cdots \wedge f_m|.
The $m$-plane $P$ is also spanned by

$$p_i = g^{-\frac{1}{2m}} f_i,$$

furthermore,

$$|p_1 \wedge \cdots \wedge p_m| = 1.$$

We then have

$$\langle P, P_0 \rangle = \det(\langle \varepsilon_i, p_j \rangle)$$

$$= \begin{pmatrix} g^{-\frac{1}{2m}} & 0 \\ \vdots & \ddots \\ 0 & g^{-\frac{1}{2m}} \end{pmatrix}$$

$$= \frac{1}{\sqrt{g}} > w_0$$

and

$$\sqrt{g} \leq \frac{1}{w_0}$$

Thus, we prove that any eigenvalue of $(g_{ij}) \leq \frac{1}{w_0^2}$. Noting Theorem 3.1, we know that the equation (3.3) is uniformly parabolic and has a unique smooth solution on some short time interval. By the curvature estimate (see Theorem 3.1 and Theorem 3.2), we have uniform estimate on $|B|$. Then we can proceed as in [16] (Prop. 2.3) to estimate all derivatives of $B$ in terms of their initial data

$$\sup_{M_t} |\nabla^q B| \leq C(m),$$

where $C(m)$ only depends on $q, m$ and $\sup_{M_0} |\nabla^j B|$ for $0 \leq j \leq q$. It follows that this solution can be extended to all $t > 0$. \hfill \Box

We assume $0 \in M$ and define coordinate functions

$$x^i = \langle F, \varepsilon_i \rangle, \quad y^\alpha = \langle F, \varepsilon_\alpha \rangle.$$

Denote

$$x = \sqrt{\sum_{i=1}^{m} (x^i)^2}, \quad y = \sqrt{\sum_{\alpha=m+1}^{m+n} (y^\alpha)^2},$$

It is easy to verified that

$$\left( \frac{d}{dt} - \Delta \right) y^\alpha = 0$$

and

$$\left( \frac{d}{dt} - \Delta \right) y^2 = -2 \sum |\nabla y^\alpha|^2 \leq 0.$$
Corollary 3.1 implies that if the height function of \(M_0\) is finite, then the height function of \(M_t\) is also finite under the evolution.

If the height function is going to infinity, we can consider rescaled mean curvature flow as done in [11]. Define

\[
\tilde{F}(\tilde{t}) = \frac{1}{\sqrt{2t + 1}} F(t),
\]

where

\[
\tilde{t} = \log(2t + 1).
\]

Hence

\[
\frac{\partial}{\partial \tilde{t}} \tilde{F} = \tilde{H} - \tilde{F}.
\]

It is not hard to verify that the Gauss map \(\tilde{\gamma}\) of the rescaled mean curvature flow is as same as the original \(\gamma\). Furthermore, the previous estimates (3.29) translate to

\[
|\tilde{A}|^2 \leq (2t + 1)|A|^2 \leq C
\]

which is dependent on the initial bound on \(M\).

We can carried out in the same way as in [11] to derive

**Theorem 3.4.** Let \(F : M \to \mathbb{R}^{m+n}\) be a complete \(m\)-submanifold with bounded curvature. Suppose that the image under the Gauss map from \(M\) into \(G_{m,n}\) lies in a geodesic ball of radius \(R_0 < \sqrt{\frac{2}{12}} \pi\). If in addition assume that

\[
\sum \alpha (F, e_\alpha)^2 \leq C'(1 + |F|^2)^{1-\delta}
\]

is valid on \(M\) for some constants \(C' < \infty, \delta > 0\), then the solution \(\tilde{M}_t\) of the rescaled equation converges for \(\tilde{t} \to \infty\) to a limiting submanifold \(\tilde{M}_\infty\) satisfying the equation

\[
F^\perp = H.
\]

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