Equivariant Higher Analytic Torsion and Equivariant Euler Characteristic

Ulrich Bunke*

November 7, 2018

Abstract

We show that J. Lott’s equivariant higher analytic torsion for compact group actions depends only on the equivariant Euler characteristic.

Contents

1 Introduction 1

2 Additivity of equivariant torsion 4

3 Products and coverings 10

4 Manifolds with corner singularities 12

5 Restriction to subgroups 14

6 The map $T_G$ 16

*Mathematisches Institut, Universität Göttingen, Bunsenstr. 3-5, 37073 Göttingen, Germany, bunke@uni-math.gwdg.de
1 Introduction

Let $G$ be a compact connected Lie group with Lie algebra $g$. Let $I(G)$ denote the ring of $\text{Ad}(G)$-invariant polynomials on $g$. Then $I(G)^1 := \{ f \in I(G) \mid f(0) = 0 \}$ is a maximal ideal of $I(G)$. By $\hat{I}(G)$ we denote the $I(G)^1$-adic completion of $I(G)$. We define $\tilde{I}(G) := \hat{I}(G)/\mathcal{C}1$.

Let $M$ be a closed oriented $G$-manifold. Then Lott $[5]$ defined equivariant higher analytic torsion $T(M)$ of $M$ (see Def. 2.1). To be precise, in $[5]$, Def. 2, he defined an element $T(M, g^M, F) \in \hat{I}(g)$, where $g^M$ is a $G$-equivariant Riemannian metric and $F$ is an equivariant flat hermitean vector bundle with trivial momentum $[5]$ (14). In our case for $F$ we take the trivial flat hermitean bundle $F := M \times \mathbb{C}$, where $G$ acts on the first factor. By $[5]$, Cor. 1, the class $T(M) := [T(M, g^M, M \times \mathbb{C})] \in \hat{I}(g)$ is independent of $g^M$. By definition $T(M)$ is a differential topological invariant of the $G$-manifold $M$. If $M$ is even-dimensional, then by $[5]$, Prop. 9, we have $T(M) = 0$.

Let $\text{Or}(G)$ denote the orbit category of $G$ (see Lück $[7]$, Def. 8.16), and let $U(G)$ be the Euler ring of $G$ ($[7]$, Def. 5.10). By $[7]$, Prop. 5.13, we can identify

$$U(G) = \prod_{[G/H] \in \text{Or}(G)} \mathbb{Z}[G/H],$$

where the product runs over all isomorphism classes of objects of $\text{Or}(G)$. If $X$ is a $G$-space of the $G$-homotopy type of a finite $G$-CW complex, then we can define its equivariant Euler characteristic $\chi_G(X) \in U(G)$. If $E_\alpha$ is the finite collection of $G$-cells of $X$, then

$$\chi_G(X) := \sum_\alpha (-1)^{\dim(E_\alpha)}[G/t(E_\alpha)],$$

where $t(E) = H$ is the type of the cell $E = G/H \times D^{\dim(E_\alpha)}$ (see $[7]$, Lemma 5.6). Any compact $G$-manifold has the $G$-homotopy type of a finite $G$-CW complex ($[7]$, 4.36), and thus $\chi_G(M) \in U(G)$ is well defined.

In the present note we define a homomorphism $T_G : U(G) \to \hat{I}(G)$ (Lemma 6.3), such that our main result can be formulated as follows.

**Theorem 1.1** Let $G$ be a compact connected Lie group. If $M$ is a closed oriented $G$-manifold, then

$$T(M) = T_G \chi_G(M).$$

This theorem answers essentially the question posed by Lott $[3]$, Note 4. As we will see below it can be employed to compute $T(M)$ effectively.

Let $H \subset G$ be a closed subgroup. Then by $[7]$, 7.25 and 7.27, there is a restriction map $\text{res}_H^G : U(G) \to U(H)$ such that $\text{res}_H^G \chi_G(M) = \chi_H(\text{res}_H^G M)$ for any compact $G$-manifold, where $\text{res}_H^G(M)$ denotes $M$ with the induced action of $H$. 


The inclusion $h \hookrightarrow g$ induces a map $\text{res}^G_H : \tilde{I}(G) \to \tilde{I}(H)$. It is an immediate consequence of the Definition 2.1 of $T(M)$, that

$$\text{res}^G_H T(M) := T(\text{res}^G_H M). \tag{1}$$

This is compatible with

$$\text{res}^G_H \circ T_G = T_H \circ \text{res}^G_H. \tag{2}$$

Let $S(G) \subset \text{Or}(G)$ be the full subcategory with objects $G/H$, where $H$ is isomorphic to $S^1$. By Corollary 5.2 the collection $\text{res}^G_H T(M), G/H \in S(G)$, determines $T(M)$. In order to compute $T(M)$ it is thus sufficient to define $T_{S^1} : U(S^1) \to \tilde{I}(S^1)$. If $H \subset G$ is isomorphic to $S^1$, then $T_H$ is defined, and we have

$$\text{res}^G_H T(M) = T(\text{res}^G_H M) = T_H \chi_H \text{res}^G_H(M) = T_H \text{res}^G_H \chi_G(M).$$

In order to give an explicit formula for $T(M)$ in terms of the $G$-homotopy type of $M$ it remains to give the formula for $T_{S^1}$.

Since $T_{S^1}$ has to satisfy Theorem 1.1, we are forced to put

$$T_{S^1}([S^1/S^1]) = T(*) = 0$$
$$T_{S^1}([S^1/H]) = T(S^1/H), \quad H \neq S^1. \tag{3}$$

For $n \in \mathbb{N}$ let $F_n : S^1 \to S^1$ be the $n$-fold covering. The derivative $F_{n*}$ of $F_n$ at $1 \in S^1$ is multiplication by $n$. By $\tilde{F}_n : \tilde{I}(S^1) \to \tilde{I}(S^1)$ we denote the induced map. If $H \subset S^1$ is different from $S^1$, then it is a cyclic subgroup of finite order $|H|$. It is again an easy consequence of the Definition 2.1 of $T(M)$, that

$$T(S^1/H) = \tilde{F}_{|H}|T(S^1). \tag{4}$$

Let $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$. We identify $s^1 \cong \mathbb{R}$ such that the exponential map is given by $\exp(y) := e^{iy}$. Then $I(S^1) = \mathbb{C}[y]$, and we identify $\tilde{I}(S^1) \cong y \mathbb{C}(y)$. By [3], Prop. 11, we then have

$$T(S^1) = 2 \sum_{k=1}^{\infty} \left( \frac{4k}{2k} \right) \text{Li}_{2k+1}(1) \left( \frac{y}{8\pi} \right)^{2k},$$

where

$$\text{Li}_j(z) := \sum_{m=1}^{\infty} \frac{z^m}{m^j}. \quad \text{It follows that}$$

$$T(S^1/H) = 2 \sum_{k=1}^{\infty} \left( \frac{4k}{2k} \right) \text{Li}_{2k+1}(1) \left( \frac{y|H|}{8\pi} \right)^{2k}.$$
Lemma 1.3 Let $T$ be a $k$-dimensional torus and $H \subset T$ be a closed subgroup. If $\dim(T/H) \geq 2$, then $T(T/H) = 0$, and if $\dim(T/H) = 1$, then $T(T/H) = \tilde{P}(T(S^1))$, where $\tilde{P} : \tilde{I}(S^1) \to \tilde{I}(T)$ is induced by the projection $P : T \to T/H \cong S^1$.

Proof. Let $R \in S(T)$. Then $\chi_R(\res^T_R T/H) = \chi((T/H)/R)[R/R \cap H]$. If $\dim(T/H) \geq 2$, then $(T/H)/R$ is a torus and $\chi((T/H)/R) = 0$. If $\dim(T/H) = 1$, then $\chi((T/H)/R) \neq 0$ iff $(T/H)/R$ is a point. Thus $\chi_R(\res^T_R T/H) = [R/R \cap H]$. The Lemma now follows from (2) and (4). \qed

Let $G/K$ be a compact symmetric space associated to the Cartan involution $\theta$ of $G$. We fix a $\theta$-stable maximal torus $T \subset G$. Then $T \cap K =: S$ is a maximal compact torus of $K$. The rank of $G/K$ is by definition $\rank(G/K) := \dim(T) - \dim(S)$. Let $W_G(T)$, and $W_K(T)$ be the Weyl groups of $(G, T)$ and $(K, T)$. If $\rank(G/K) = 1$, then for $w \in W_G(T)$ we have a projection $P_w : T \to T/S^w \cong S^1$, where $S^w = wSw^{-1}$. It induces a map $\tilde{P}_w : \tilde{I}(S^1) \to \tilde{I}(T)$. Since $\res^G_T : \tilde{I}(G) \to \tilde{I}(T)$ is injective, the following Lemma gives an explicit computation of $T(G/K)$.

Lemma 1.4 If $\rank(G/K) \geq 2$, then $T(G/K) = 0$, and if $\rank(G/K) = 1$, then $\res^G_T T(M) = \sum_{W_G(T)/W_K(T)} P_w(T(S^1))$.

Proof. Fix $S^1 \cong R \subset T$. If $H \subset T$ is a closed subgroup, then $\chi_R(\res^T_R T/H) = 0$ except if $\dim(T/H) = 1$. In [3] we have shown that

$$\chi_T(\res^G_T G/K) = \sum_{W_G(T)/W_K(T)} [T/S^w] + \text{higher dimensional staff}. $$

Hence if $\rank(G/K) \geq 2$, then by Lemma 1.3 $\res^G_T T(M) = 0$, and if $\rank(G/K) = 1$, then

$$\res^G_T T(G/K) = \sum_{W_G(T)/W_K(T)} T[T/S^w].$$

Applying 1.3 we obtain the desired result. \qed

We now briefly describe the contents of the remainder of the paper. In Section 3 we prove our main analytic result Theorem 2.2 saying that $T(M)$ is essentially additive. In Section 3
we study the behaviour of $T(M)$ under coverings and with respect to cartesian products. In Section 3 we extend the analytic results to manifolds with corner singularities using certain formal considerations. In Section 3 we show that $T(M)$ is determined by its restrictions to all subgroups $H \cong S^1$. In Section 4 we first prove Theorem 1.1 for $G = S^1$, and then we construct $T_G$ and finish the proof of Theorem 1.1 for general $G$.

2 Additivity of equivariant torsion

We first recall the definition of higher equivariant torsion [5], Def. 2. Let $G$ be a connected Lie group with Lie algebra $g$. Let $M$ be a closed oriented $G$-manifold. We write $\Omega(M) := C^\infty(M, \Lambda^\ast T^\ast M)$ and $d : \Omega(M) \to \Omega(M)$ for the differential of the de Rham complex.

For $X \in g$ let $X^* \in C^\infty(M, T^\ast M)$ denote the corresponding fundamental vector field. We set

$$I := \sum_\alpha X^\alpha \otimes i_{X^\alpha} \in S(g^\ast) \otimes \text{End}(\Omega(M)) ,$$

where $X^\alpha \in g$, $X^\ast \in g^\ast$ run over a base of $g$ or dual base of $g^\ast$, respectively, and $i_Y$ denotes interior multiplication by the vector field $Y$. We choose a $G$-invariant Riemannian metric $g_M$. It induces a pre Hilbert space structure on $\Omega(M)$, and we let $e_Y$ be the adjoint of $i_Y$. We set $E := \sum_\alpha X^\alpha \otimes e_{X^\ast}^\alpha$.

For $t > 0$ we define

$$d_t := \sqrt{t}d - \frac{1}{4\sqrt{t}}I, \quad \delta_t := \sqrt{t}d^\ast + \frac{1}{4\sqrt{t}}E .$$

Then we put

$$D_t := \delta_t - d_t \in S(g^\ast) \otimes \text{End}(\Omega(M)) .$$

Let $S(g^\ast)^1 := \{ f \in S(g^\ast) \mid f(0) = 0 \}$, and let $\hat{S}(g^\ast)$ be the $S(g^\ast)^1$-adic completion. Since

$$D_t^2 = -t\Delta \pmod{S^1(g^\ast) \otimes \text{End}(\Omega(M))}$$

we can form

$$e^{D_t^2} \in \hat{S}(g^\ast) \otimes \text{End}(\Omega(M)) .$$

Moreover we have

$$\text{Tr}_\ast N e^{D_t^2} \in \hat{I}(G) ,$$

where $N$ is the $\mathbb{Z}$-grading operator on $\Omega(M)$, and $\text{Tr}_\ast$ is the $\mathbb{Z}_2$-graded trace on $\text{End}(\Omega(M))$. Define $\chi'(M) := \sum_{p=0}^{\infty} p(-1)^p \dim H^s(M, \mathbb{R})$. Then the function

$$s \mapsto -\frac{1}{\Gamma(s)} \int_0^{\infty} (\text{Tr}_\ast N e^{D_t^2} - \chi'(M)) t^{s-1} dt$$

is holomorphic for $\text{Re}(s) >> 0$, and it has a meromorphic continuation to all of $\mathbb{C}$ which is regular at $s = 0$. 


**Definition 2.1** The equivariant higher torsion $T(M) \in \tilde{I}(G)$ of the $G$-manifold $M$ is represented by

$$
-\frac{d}{ds|_{s=0}} \frac{1}{\Gamma(s)} \int_0^\infty (\text{Tr}_sNe^{D_2^2} - \chi'(M)) t^{s-1} dt .
$$

If $M$ is odd-dimensional, then by [5], Cor. 1, $T(M)$ is independent of the choice of the $G$-invariant Riemannian metric $g^M$. If $M$ is even-dimensional, the by [5], Prop 9, we have $T(M) = 0$.

Let $M$ be a closed oriented $G$-manifold, and let $N$ be a $G$-invariant oriented hypersurface such that $M \setminus N$ has two components, i.e. there are compact manifolds $M_1$, $M_2$ with boundary $\partial M_i = N$, $i = 1, 2$ such that $M = M_1 \cup_N M_2$. We form the closed oriented $G$-manifolds $\tilde{M}_i := M_i \cup_N M_i$, the doubles of $M_i$.

**Theorem 2.2**

$$2T(M) = T(\tilde{M}_1) + T(\tilde{M}_2) .$$

**Proof.** We choose Riemannian metrics on $M$ and $\tilde{M}_i$, $i = 1, 2$. Then let $D_t$ and $D_{t,i}$, $i = 1, 2$ denote the operators (5) for $M$ and $\tilde{M}_i$, respectively. We define $\delta(t) \in \tilde{I}(G)$ by

$$\delta(t) := 2\text{Tr}_sNe^{D_2^2} - \text{Tr}_sNe^{D_{2,1}^2} - \text{Tr}_sNe^{D_{2,2}^2} - (2\chi'(M) - \chi'(\tilde{M}_1) - \chi'(\tilde{M}_2)) .$$

We have to show that

$$0 = \left[-\frac{d}{ds|_{s=0}} \frac{1}{\Gamma(s)} \int_0^\infty \delta(t)t^{s-1} dt \right] ,$$

where [-] denotes the class of "." in $\tilde{I}(G)$.

We now specialize the choice of Riemannian metrics. We choose a $G$-invariant collar neighbourhood $(-1, 1) \times N \hookrightarrow M$ such that $\{0\} \times N$ is mapped to $N$. Then we assume that $g^M$ is a product metric $dr^2 + g^N$ on the collar. The metric $g^M$ induces natural Riemannian metrics $g^{\bar{M}_i}$ on $\tilde{M}_i$.

For $R > 1$ let $g^M(R)$ be the Riemannian metric which coincides with $g^M$ outside the collar, and which is such that the collar is isometric to $(-R, R) \times N$. Similarly we obtain metrics $g^{\bar{M}_i}(R)$ on $\bar{M}_i$.

Let $\delta(t, R)$ be defined with respect to these choices of metrics. While $\delta(t, R)$ may depend on $R$, it is known that

$$\left[-\frac{d}{ds|_{s=0}} \frac{1}{\Gamma(s)} \int_0^\infty \delta(t, R)t^{s-1} dt \right] \in \tilde{I}(G)$$

is independent of $R$. The proof of the theorem is obtained by studying the behaviour of $\delta(t, R)$ as $R$ tends to infinity.

Note that $\tilde{I}(G)$ is a locally convex topological vector space.
Proposition 2.3 For any seminorm $\|\cdot\|$ on $\hat{I}(G)$ there are constants $C < \infty, c > 0$ such that for all $t > 0$, $R > 1$

$$|\delta(t, R)| < Ce^{-\frac{cR^2}{t}}.$$ 

Proof. This follows from a standard argument using the finite propagation speed method [4]. We leave the details to the interested reader. 

Let $I(G)_1 \subset \hat{I}(G)$ be the closed subspace of at most linear invariant polynomials on $g$ and put $\hat{I}(G) := \hat{I}(G)/I(G)_1$. By [[[]]] we denote classes in this topological quotient space.

Proposition 2.4 For any seminorm $\|\cdot\|$ on $\hat{I}(G)$ there is a constant $C < \infty$ such that for all $R > 1$, $t > 1$

$$\|[\delta(t, R)]\| < Ct^{-1}R.$$ 

Proof. This is a consequence of the more general estimate

$$\|[\text{Tr}_s \text{Ne}^{D_t(R)^2}]\| < Ct^{-1}R \tag{6}$$

which also holds for $M$ replaced by $\hat{M}_t$. Here $D_t(R)$ denotes the operator $\hat{M}_t$ associated to $g^M(R)$.

We can assume that $\|\cdot\|$ is the restriction to $\hat{I}(G)$ of a seminorm of $\hat{S}(g^*)/S_1(g^*)$, where $S_1(g^*)$ denotes the subspace $C \oplus g^*$. There is an $m > 0$ depending on $\|\cdot\|$ such that $\|[U]\| = 0$ for all $U \in \hat{S}(g^*)^m$. Let $\Delta(R)$ denote the Laplace operator on differential forms associated to the Riemannian metric $g^M(R)$. We have

$$D_t^2(R) = -t\Delta(R) + \mathcal{N} + \frac{1}{t}\mathcal{N}_1,$$

(to be precise we should write $\mathcal{N}(R), \mathcal{N}_1(R)$) where

$$\mathcal{N} := \frac{1}{4}[d^*(R) - d, E + I]$$

$$\mathcal{N}_1 := Q$$

$$Q := \frac{1}{16}[I, E],$$

(the commutators are understood in the graded sense) belong to $S(g^*)^1 \otimes \text{End}(\Omega(M))$.

As in [1], 9.46, we write

$$\text{Tr}_s \text{Ne}^{D_t(R)^2} = \sum_{k=0}^{\infty} \int_{\Delta_k} U_k(\sigma, R) d\sigma \tag{7},$$

$$U_k(\sigma, R) := \text{Tr}_s \text{Ne}^{-\frac{1}{t}\sigma_k D_t(R)^2} (\mathcal{N} + \frac{1}{t}\mathcal{N}_1) \ldots (\mathcal{N} + \frac{1}{t}\mathcal{N}_1)e^{-\frac{1}{t}\sigma_k D_t(R)^2},$$
We consider the first estimate. Note that \( \Delta_k \subset \mathbb{R}^{k+1} \) denotes the standard simplex such that \( \Delta_k \ni \sigma = (\sigma_0, \ldots, \sigma_k) \) satisfies \( \sum_{i=0}^{k} \sigma_i = 1 \).

The Riemannian metric \( g^M(R) \) induces a pre Hilbert space structure on \( \Omega(M) \). The trace (operator) norm \( \| \cdot \|_1 (\| \cdot \|) \) on \( \text{End}(\Omega(M)) \) and \( \| \cdot \| \) together induce norms on \( \mathcal{S}(g)/S_1(g) \otimes \text{End}(\Omega(M)) \) which we also denote by \( \| \cdot \|_1 (\| \cdot \|) \).

**Lemma 2.5** There is a constant \( C < \infty \) such that for all \( t > 1 \) and \( R > 1 \) we have
\[
\| e^{-t\Delta(R)} \|_1 < CR.
\]

**Proof.** The operator \( e^{-t\Delta(R)} \) is positive. Thus \( \| e^{-t\Delta(R)} \|_1 = \text{Tr} e^{-t\Delta(R)} \). Let \( W(t, x, y)(R) \) be the integral kernel of \( e^{-t\Delta(R)} \). The family \((M, g^M(R))\) of Riemannian manifolds has uniformly bounded geometry as \( R \) varies in \([1, \infty)\), i.e. there are uniform curvature bounds, and the injectivity radius is uniformly bounded from below. Standard heat kernel estimates (see e.g. [1]) imply that there is a constant \( C_1 < \infty \) such that for all \( x \in M, t > 1, R > 1 \) we have \( |W(t, x, x)(R)| < C_1 \). In particular, for some \( C, C_2 < \infty \) independent of \( R > 1, t > 1 \) we have
\[
\text{Tr} e^{-t\Delta(R)} = \int_M \text{Tr} W(t, x, x)(R) \text{vol}_{g^M(R)}(x) < C_2 \text{vol}_{g^M(R)}(M) < CR.
\]
This finishes the proof of the lemma. \( \square \)

**Lemma 2.6** There is a \( C < \infty \) such that for all \( R > 1 \) and \( t, s > 0 \) we have
\[
\| \left[ [e^{-t\Delta(R)} \mathcal{N} e^{-s\Delta(R)}] \right] \| < C(t^{-1/2} + s^{-1/2}) .
\]

**Proof.** Since \( \mathcal{N} = [d^*(R) - d, E + I] \) and \( \| E + I \| \) is uniformly bounded w.r.t. \( R \) it suffices to show that there exists \( C_1 < \infty \) such that for all \( R > 1 \) and \( t > 0 \) we have
\[
\| e^{-t\Delta(R)} d \| < C_1 t^{-1/2}, \quad \| e^{-t\Delta(R)} d^*(R) \| < C_1 t^{-1/2} .
\]
We consider the first estimate. Note that \( dd^*(R) + d^*(R) d = \Delta(R) \), and the ranges of \( dd^*(R) \) and \( d^*(R)d = \) are perpendicular. Thus
\[
\| e^{-t\Delta(R)} d \| = \| e^{-t\Delta(R)} dd^*(R)e^{-t\Delta(R)} \|^{1/2} \\
\leq \| e^{-t\Delta(R)} \Delta(R)e^{-t\Delta(R)} \|^{1/2} \\
= t^{-1/2} \| e^{-t\Delta(R)} t\Delta(R)e^{-t\Delta(R)} \|^{1/2} \\
\leq t^{-1/2} \sup_{x \geq 0} x e^{-x} \\
\leq C_1 t^{-1/2} .
\]
If $A$ is of trace class and $B$ is bounded, then we have $|\text{Tr} \ AB| \leq \|B\|\|A\|_1$. Note that $\|N_1\|$ is uniformly bounded w.r.t. $R$. Applying this and Lemmas 2.6 and 2.5 to $U_k$ we obtain $C, C_1 < \infty$ such that for all $R > 1$ and $t > 1$ we have

$$|[[U_k(\sigma, R)]]| < C_1 R t^{-k/2} \sum_{i=0}^{k} \sigma_i^{-1/2}$$

$$|[[\int_{\Delta_k} U_k(\sigma, R)d\sigma]]| < C t^{-k/2} R .$$

Note that $|[U_k]| = 0$ for $k > m$. In order to obtain (8) from (7) and (8) it remains to discuss $U_1$. Since $N \in S(g)^1$ there exists $C, C_1 < \infty$ such that for all $R > 1$ and $t > 1$

$$|[[U_1(\sigma, R)]]| = |[[\text{Tr}_s N e^{-t\sigma_0 \Delta(R)} 1^t N_1 e^{-t\sigma_1 \Delta(R)}]]|$$

$$= |[[\text{Tr}_s 1^t N_1 e^{-t\Delta(R)}]]|$$

$$< C_1 R t^{-1}$$

$$|[[\int_{\Delta_1} U_1(\sigma, R)d\sigma]]| < C R t^{-1}$$

This finishes the proof of the proposition. \qed

We now continue with the proof of the theorem. Let $|.|$ any seminorm on $\hat{I}(G)$ as in the proof of Proposition 2.4. By Propositions 2.3 and 2.4 we can write

$$\sigma(R) := - \frac{d}{ds}|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty |[\delta(t, R)]|^s t^{s-1} dt ,$$

and the integral converges at $t = 0$ and $t = \infty$ uniformly in $s \in (-1/2, 1/2)$. We can perform the derivative and obtain

$$\sigma(R) = - \int_0^\infty |[\delta(t, R)]| t^{-1} dt$$

$$= - \int_0^R |[\delta(t, R)]| t^{-1} dt + \int_R^\infty |[\delta(t, R)]| t^{-1} dt .$$

By Proposition 2.3 there are $C_1 < \infty, c_1 > 0$ such that for all $R > 1$ we have

$$|[\int_0^{R^3/2} \delta(t, R) t^{-1} dt]| \leq \int_0^{R^3/2} C e^{-c_1 R^3 t^{-1}} dt$$

$$\leq C_1 e^{-c_1 R^1/2} .$$

Moreover by Proposition 2.4 there is a $C < \infty$ such that for all $R > 1$

$$|[\int_{R^3/2}^{\infty} \delta(t, R) t^{-1} dt]| \leq \int_{R^3/2}^{\infty} C R t^{-2} dt$$

$$= CR^{-1/2} .$$
We now let $R$ tend to infinity and take into account that $\sigma(R)$ is independent of $R$ in order to conclude that

$$\sigma(R) = 0.$$  

(9)

We have shown that $[[T(M)]] = [[T(\tilde{M}_1)]] + [[T(\tilde{M}_2)]]$.

We now consider the remaining component $T_1(M) \in I_1(G)/C1$. Note that $\mathcal{N} = -\frac{1}{2}L + [d, E] + [d^*, I]$, where $L := \sum \alpha X^\alpha \otimes L_{X^\alpha}$ and $L_Y$ denotes the Lie derivative with respect to the vector field $Y$. Since $[d, E]$, and $[d^*, I]$ shift the form degree by $\pm 2$ we obtain

$$\text{Tr}_s NL^e_{-t\Delta} = -\frac{1}{2}\text{Tr}_s NL^e_{-t\Delta}.$$  

Let $\rho_{an}(M, g^M) : G \to \mathbb{C}$ denote the equivariant analytic torsion defined by

$$\rho_{an}(M, g^M)(g) := -\frac{d}{ds}_{|s=0} \frac{1}{\Gamma(s)} \int_0^\infty (\text{Tr}_s NL^e_{-t\Delta} - \chi'(M)) t^{s-1} dt.$$  

If we define

$$\delta(t, g) := 2\text{Tr}_s NL^e_{t\Delta} - \text{Tr}_s NL^e_{t\Delta_1} - \text{Tr}_s NL^e_{t\Delta_2} - (2\chi'(M) - \chi'(\tilde{M}_1) - \chi'(\tilde{M}_2)),$$

then there are $C < \infty$, $c > 0$ such that for all $g \in G$

$$|\delta(t, g)| \leq Ce^{-\frac{t}{2}} \quad \forall t \in (0, 1]$$

$$|\delta(t, g)| \leq Ce^{-ct} \quad \forall t \in [1, \infty).$$

The first estimate is again a consequence of the finite propagation speed method. Similar estimates hold for the derivative of $\delta(t, g)$ w.r.t. $g$. We have

$$\sigma_1(g) := -\int_0^\infty \delta(t, g)t^{-1} dt = 2\rho_{an}(M, g^M) - \rho_{an}(\tilde{M}_1, g^{\tilde{M}_1}) - \rho_{an}(\tilde{M}_2, g^{\tilde{M}_2}).$$

On the one hand in we have shown that on the dense subset of $G$ consisting of elements of finite order

$$2\rho_{an}(M, g^M) - \rho_{an}(\tilde{M}_1, g^{\tilde{M}_1}) - \rho_{an}(\tilde{M}_2, g^{\tilde{M}_2}) = \text{const}.$$  

On the other hand $\sigma_1$ is differentiable. We conclude

$$0 = d_{|g=1}\sigma_1$$

$$= -\int_0^\infty d_{|g=1}\delta(t, .)t^{-1} dt$$

$$= -\int_0^\infty (2\text{Tr}_s NL^e_{-t\Delta} - \text{Tr}_s NL^e_{-t\Delta_1} - \text{Tr}_s NL^e_{-t\Delta_2}) dt$$

$$= -2(T_1(M) - T_1(\tilde{M}_1) - T_1(\tilde{M}_2)).$$

This finishes the proof of the theorem. \qed
3 Products and coverings

Let $G$ be a compact connected Lie group and $\Gamma$ be a finite group. Let $C(\Gamma)$ denote the algebra of $C$-valued functions on $\Gamma$. We need the generalization of higher equivariant analytic torsion $T^\Gamma(M) \in \tilde{I}(G) \otimes C(\Gamma)$ mentioned in [5], Note 3. Let $M$ be a closed oriented $G \times \Gamma$-manifold equipped with a $G \times \Gamma$-invariant Riemannian metric $g^M$. Set

$$\chi'(M)(\gamma) := \sum_{p=0}^{\infty} p(-1)^p \text{Tr} H^p(\gamma) ,$$

where $H^p(\gamma)$ is the induced action of $\gamma \in \Gamma$ on $H^p(M, \mathbb{R})$. Then we define $T^\Gamma(M) \in \tilde{I}(G) \otimes C(\Gamma)$ to be the element represented by the function

$$\gamma \mapsto \frac{d}{ds}|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty (\text{Tr}_s N \gamma e^{D_t^2} t^{s-1} - \chi'(M)(\gamma)) dt .$$

Let $M$ be a closed oriented $G \times \Gamma$-manifold and $N$ be a closed oriented $\Gamma$-manifold. Then we form the closed oriented $G \times \Gamma$-manifold $M \times N$, where $\Gamma$ acts diagonally. We choose a $G \times \Gamma$-invariant Riemannian metric $g^M$, a $\Gamma$-invariant Riemannian metric $g^N$, and we let $g^{M \times N}$ be the product metric.

Define the $\Gamma$-equivariant Euler characteristic $\chi^\Gamma(N) \in C(\Gamma)$ of a closed $\Gamma$-manifold $N$ by

$$\chi^\Gamma(N)(\gamma) := \sum_{p=0}^{\infty} (-1)^p \text{Tr} H^p(\gamma) .$$

**Lemma 3.1** If $\chi^\Gamma(M) = 0$, then

$$T^\Gamma(M \times N) = T^\Gamma(M) \chi^\Gamma(N) .$$

**Proof.** We write $D_t(M), D_t(N), D_t(M \times N)$ for the operators on $M, N, M \times N$. Let $\Delta(N)$ be the Laplace operator on $\Omega(N)$. On the level of Hilbert space closures we have

$$\text{clo}_{L^2}\Omega(M \times N) = \text{clo}_{L^2}\Omega(M) \otimes \text{clo}_{L^2}\Omega(N) .$$

With respect to this splitting we can write

$$D_t(M \times N)^2 = D_t(M)^2 \otimes 1 - 1 \otimes t\Delta(N) .$$

If $\gamma \in \Gamma$, then

$$\begin{align*}
\text{Tr}_s N \gamma e^{D_t(M \times N)^2} &= \text{Tr}_s (N \otimes 1 + 1 \otimes N)(\gamma \otimes \gamma)(e^{D_t(M)^2} \otimes e^{-t\Delta(N)}) \\
&= \text{Tr}_s N \gamma e^{D_t(M)^2} \text{Tr}_s e^{-t\Delta(N)} + \text{Tr}_s N \gamma e^{D_t(M)^2} \text{Tr}_s N \gamma e^{-t\Delta(N)} .
\end{align*}$$
By the equivariant McKean-Singer formula [1], Thm. 6.3, we have $\text{Tr}_s \gamma e^{-t\Delta(N)} = \chi^\Gamma(N)(\gamma)$. Moreover we have

$$\frac{d}{dt} \text{Tr}_s \gamma e^{D_t(M)^2} = \text{Tr}_s \gamma \frac{d}{dt} D_t^2 e^{D_t(M)^2} = \text{Tr}_s \left[ \frac{d}{dt} D_t, \gamma D_t e^{D_t(M)^2} \right] = 0$$

$$\lim_{t \to \infty} \text{Tr}_s \gamma e^{D_t(M)^2} = \lim_{t \to \infty} \text{Tr}_s \gamma e^{-t\Delta(M)} = \chi^\Gamma(M)(\gamma) = 0.$$ 

It follows

$$\text{Tr}_s N \gamma e^{D_t(M \times N)^2} = \chi^\Gamma(N)(\gamma) \text{Tr}_s N \gamma e^{D_t(M)^2}.$$ 

This implies the assertion of the Lemma. \(\square\)

Let $N$ be a closed oriented $G \times \Gamma$-manifold such that $\Gamma$ acts freely on $N$. Let $M := \Gamma \setminus N$. Then $M$ is a closed oriented $G$-manifold. We equip $N$ with a $G \times \Gamma$-invariant Riemannian metric and define $g^M$ such that the projection $\pi : N \to M$ becomes a local isometry.

Let $\int_\Gamma : C(\Gamma) \to \mathcal{C}$ be the integral over $\Gamma$ with respect to the normalized Haar measure. We denote the induced map $\tilde{I}(G) \otimes C(\Gamma) \to \tilde{I}(G)$ by the same symbol.

**Lemma 3.2**

$$T(M) = \int_\Gamma T^\Gamma(N).$$

**Proof.** Note that $\Pi := \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma$ acts on $\Omega(N)$ as projection onto the subspace of $\Gamma$-invariant forms which can be identified with $\Omega(M)$ using the pull-back $\pi^*$. Moreover, $D_t(M)$ coincides with the restriction of $D_t(N)$ to the range of $\Pi$. We have

$$\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \text{Tr}_s N \gamma e^{D_t(N)^2} = \text{Tr}_s N \Pi e^{D_t(N)^2} = \text{Tr}_s N e^{D_t(M)^2}.$$ 

This implies the assertion of the Lemma. \(\square\)
4 Manifolds with corner singularities

In this section we extend the definition of $T(M)$, $T^\Gamma(M)$, and the results of Section 3 to manifolds with corner singularities.

A compact manifold with a corner singularity of codimension one is just a manifold with boundary. Corner singularities of codimension two arise if we admit that boundaries have itself boundaries. In general a corner singularity of codimension $m$ of a $n$-dimensional manifold is modelled on $(\mathbb{R}^+)^m \times \mathbb{R}^{n-m}$, where $\mathbb{R}^+ = [0, \infty)$.

Let $M$ be a compact manifold with corner singularities. Then the boundary of $M$ can be decomposed into pieces $\partial_1 M \cup \ldots \cup \partial_l M$. We do not require that the pieces $\partial_i M$ are connected.

For $i \in \{1, \ldots, l\}$ we can form the double $\tilde{M}_i := M \cup \partial_i M$ of $M$ along the piece $\partial_i M$. Then $\tilde{M}_i$ is again a compact manifold with corner singularities. In particular it has $l-1$ boundary pieces $\partial_j \tilde{M}_i = \partial_j M \cup \partial_j \partial_i M \partial_j M$, $j \neq i$.

The notion corner singularities and the construction of the double extends to compact oriented $G$-manifolds in the obvious way. We define $T(M)$ for compact oriented $G$-manifolds inductively with respect to the number $l(M)$ of boundary pieces.

If $l(M) = 0$, then $T(M)$ is already defined. Assume now that $l(M)$ is defined for all $M$ with $l(M) < l$. If $M$ is now a compact oriented $G$-manifold with $l(M) = l$. Then we set

$$T(M) := \frac{1}{2} T(\tilde{M}_1).$$

If $l > 1$, then we have to check that this definition is independent of the numbering of boundary components. It suffices to show that $T(\tilde{M}_1) = T(\tilde{M}_2)$. Note that $\tilde{M}_{12}$ and $\tilde{M}_{21}$ are $G$-diffeomorphic. Using the induction hypothesis

$$2T(\tilde{M}_1) = T(\tilde{M}_{12}) = T(\tilde{M}_{21}) = 2T(\tilde{M}_2).$$

Thus $T(M)$ is well defined.

The doubling trick was introduced by [6], Ch. IX. Instead of the formal definition above one could also employ absolute and relative boundary boundary conditions in order to define higher equivariant analytic torsion $T(M, \text{abs})$, $T(M, \text{rel})$ for $G$-manifolds with boundary. If the Riemannian metric is choosen to be product near the boundary, then $T(M) = \frac{1}{2} T(M, \text{abs}) + T(M, \text{rel})$.

The sum formula 2.2 has now the nice reformulation

$$T(M) = T(M_1) + T(M_2).$$

(10)

It has the following generalization:
Corollary 4.1 Let $M_i$, $i = 1, 2$, be compact oriented $G$-manifolds with corner singularities. If we are given a $G$-diffeomorphism $\partial_1 M_1 \cong \partial_1 M_2$, then we form the manifold with corner singularities $M := M_1 \cup_{\partial_1} M_2$, and we have

$$T(M) = T(M_1) + T(M_2).$$

Proof. We employ induction by the number of boundary pieces. The assertion is true if $M$ is closed. Assume that the corollary holds true for all $M$ with $l(M) < l$. Let $M := M_1 \cup_{\partial_1} M_2$ now be a manifold with $l(M) = l$ and $l \geq 1$. Then we can assume that $l(M_1) \geq 2$. Let $\partial_1 M$ be the piece corresponding to $\partial_2 M_1$. We distinguish the cases (a): $\partial_2 M_1 \cap \partial_1 M_1 = \emptyset$ and (b): $\partial_2 M_1 \cap \partial_1 M_1 \neq \emptyset$. In case (a) let $\partial_1 M$ be the piece corresponding to $\partial_2 M_1$. Then using the induction hypothesis

$$T(M) = \frac{1}{2} T(\tilde{M}_1) = \frac{1}{2} T(\tilde{M}_{12}) + T(M_2) = T(M_1) + T(M_2).$$

In case (b) there is a boundary piece $\partial_2 M_2$ meeting $\partial_1 M_2$. Then $M$ has a boundary piece $\partial_1 M := \partial_2 M_1 \cup_{\partial_1} \partial_1 M_1 \cap \partial_2 M_2$. Again using the induction hypothesis we have

$$T(M) = \frac{1}{2} T(\tilde{M}_1) = \frac{1}{2} T(\tilde{M}_{12}) + \frac{1}{2} T(\tilde{M}_{22}) = T(M_1) + T(M_2).$$

This proves the corollary. \qed

Let $\Gamma$ be an additional finite group. For a $G \times \Gamma$-manifold with corner singularities we require that the pieces $\partial_i M$ are compact $G \times \Gamma$-manifolds with corner singularities as well.

A Riemannian metric on a manifold with corner singularities is compatible if it is a product metric $g^{(R_+)^m} + g^{R_i - m}$ at a corner of codimension $m$. Then we can form the doubles $\tilde{M}_i$ metrically.

Let $M$ be a compact oriented $G \times \Gamma$-manifold with corner singularities equipped with a compatible $G \times \Gamma$-invariant Riemannian metric. Then we define $T^\Gamma(M)$ for $G \times \Gamma$-manifolds with corner singularities using the same formal procedure as for trivial $\Gamma$. We can generalize Lemma 3.2 to this case. Let $N$ be a compact oriented $G \times \Gamma$-manifold with corner singularities such that $\Gamma$ acts freely and form $M := \Gamma \backslash N$.

Corollary 4.2

$$T(M) = \int_\Gamma T^\Gamma(N).$$

Proof. We argue by induction with respect to the number of boundary pieces. If $l(N) = 0$, then this is just Lemma 3.2. Assume now that the corollary holds true for all $N$ with $l(N) < l$. Let now $N$ be a compact oriented $G \times \Gamma$-manifold with corner singularities such that $\Gamma$ acts freely
and \( l(N) = l \geq 1 \). Then consider the covering \( \tilde{N}_1 \rightarrow \tilde{M}_1 \). Applying the induction hypothesis we obtain
\[
T(M) = \frac{1}{2} T(\tilde{M}_1) = \frac{1}{2} \int_{\Gamma} T^\Gamma(\tilde{N}_1) = \int_{\Gamma} T^\Gamma(N).
\]
This proves the corollary.

Let \( M \) be a closed oriented \( G \times \Gamma \)-manifold and \( N \) be a compact oriented \( \Gamma \)-manifold with corner singularities. Then we form the compact oriented \( G \times \Gamma \)-manifold \( M \times N \) with corner singularities, where \( \Gamma \) acts diagonally. We choose a \( G \times \Gamma \)-invariant Riemannian metric \( g^M \), a \( \Gamma \)-invariant compatible Riemannian metric \( g^N \), and we let \( g^{M \times N} \) be the product metric which is again invariant and compatible.

We define the \( \Gamma \)-equivariant Euler characteristic \( \chi^\Gamma(N) \in C(\Gamma) \) of a \( \Gamma \)-manifold \( N \) with corner singularities with \( l(N) \geq 1 \) inductively with respect to the number of boundary pieces by
\[
\chi^\Gamma(N) := \frac{1}{2} \chi^\Gamma(\tilde{N}_1).
\]
We leave it to the interested reader to express \( \chi^\Gamma(N) \) in terms of equivariant Euler characteristics of the components of the filtration of \( N \). The main feature of this definition is that the equivariant Euler characteristic is additive under glueing along boundary pieces.

We have the following generalization of Lemma 3.1.

**Corollary 4.3** If \( \chi^\Gamma(M) = 0 \), then
\[
T^\Gamma(M \times N) = T^\Gamma(M) \chi^\Gamma(N).
\]

**Proof.** We argue by induction over the number of boundary pieces \( l(N) \). If \( l(N) = 0 \), then this is just Lemma 3.1. Assume that the corollary holds true if \( l(N) < l \). Let now \( N \) be such that \( l(N) = l \geq 1 \). Let \( \partial_1(M \times N) := M \times \partial_1 N \). Then using the induction hypothesis and the additivity of \( \chi^\Gamma \) we obtain
\[
T^\Gamma(M \times N) = \frac{1}{2} T^\Gamma((M \times N)_1) = \frac{1}{2} T^\Gamma(M) \chi^\Gamma(\tilde{N}_1) = T^\Gamma(M) \chi^\Gamma(N).
\]
This proves the corollary.

5 Restriction to subgroups

Let \( G, H \) be a connected compact Lie groups with Lie algebras \( g, h \). If \( f : H \rightarrow G \) is a homomorphism, then \( f_* : h \rightarrow g \) induces a map \( \tilde{f} : I(G) \rightarrow I(H) \). If \( H \subset G \) is a closed subgroup
and $i$ denotes the inclusion, then we set $\tilde{i} := \text{res}_H^G$. If $g \in G$, then we put $H^g := gHg^{-1}$. Let $\alpha_g : H \to H^g$ be given by $\alpha_g(h) := ghg^{-1}$.

Let $M$ be a closed oriented $G$-manifold with corner singularities. If $f : H \to G$ is a homomorphism, then we denote by $f^* M$ the $H$-manifold $M$ with action induced by $f$. If $H \subset G$ is a closed subgroup, then we put $\text{res}^G_H M := i^* M$. The following Lemma is an immediate consequence of the definition of $T(M)$.

**Lemma 5.1** (1) : If $f : H \to G$ is a homomorphism, then $\tilde{f} T(M) = T(f^* M)$. In particular, if $H \subset G$ is closed, then $\text{res}^G_H T(M) = T(\text{res}^G_H M)$.

(2) : If $H \subset G$ is closed, then for all $g \in G$ we have $\bar{\alpha}_g \text{res}^G_H T(M) = \text{res}^G_H T(M)$ for all $g \in G$.

The association $H \subset G \mapsto \tilde{I}(H) := \tilde{I}_G(H)$ assembles to give a contravariant functor $\tilde{I}_G : \text{Or}(G) \to \mathbf{C} - \text{vect}$. If $f : H \to G$ is a homomorphism, then it induces a natural functor $f_* : \text{Or}(H) \to \text{Or}(G)$ sending $H/K$ to $G/f(K)$. For $K \subset H$ let $f_K : K \to f(K)$ be the restriction of $f$ to $K$. The collection $\{f_K\}$, $K \subset H$, provides a natural transformation $\tilde{f} : \tilde{I}_G \circ f_* \to \tilde{I}_H$. Let $f^* : \lim_{\text{Or}(G)} \tilde{I}_G \to \lim_{\text{Or}(H)} \tilde{I}_H$ denote the induced map.

Lemma 5.1 says that $G/H \mapsto T(\text{res}^G_H M)$ is a section of $\tilde{I}_G$. Since $\text{Or}(G)$ has a final object $G/G$, we have an isomorphism

$$\lim_{\text{Or}(G)} \tilde{I}_G \cong \tilde{I}(G) \quad (11)$$

given by restriction to the final object.

By $S(G)$ we denote the full subcategory of $\text{Or}(G)$ of those objects $G/H$ with $H \cong S^1$. We denote the space of sections of $\tilde{I}_{G|S(G)}$ by $V(G)$, i.e.

$$V(G) := \lim_{S(G)} \tilde{I}_G .$$

There is a natural restriction map

$$R_G : \tilde{I}(G) \cong \lim_{\text{Or}(G)} \tilde{I} \to V(G) .$$

**Lemma 5.2** $R_G$ is injective.

*Proof.* Let $T \subset G$ be a maximal torus and denote by $j$ its inclusion. There is a functor $j_{*|S(T)} : S(T) \to S(G)$. Let $J^* : \lim_{S(G)} \tilde{I}_G \to \lim_{S(T)} \tilde{I}_T$ be induced by the natural transformation $j_{*|S(T)} : \tilde{I}_{G|S(G)} \circ j_{*|S(T)} \to \tilde{I}_{T|S(T)}$. Then $R_T \circ j^* = J^* \circ R_G$. In order to prove that $R_G$ is injective it is therefore sufficient to show that $j^*$ and $R_T$ are injective.

Now $j^*$ is injective since it coincides with $\text{res}^G_T : \tilde{I}(G) \to \tilde{I}(T)$ under the identification (11), and the latter map well known to be injective. Let $t$ be the Lie algebra of $T$. The kernel of
exp : t → T defines a $\mathbb{Z}$ structure on $t$. The set of subspaces $h \subset t$ corresponding to objects $T/H \in S(T)$ with $H \cong S^1$ is just the set of integral points of the projective space $P(t \otimes \mathbb{C})$. Injectivity of $R_T$ follows easily from the fact that the set of integral points of $P(t \otimes \mathbb{C})$ is Zariski dense. □

**Corollary 5.3** $T(M)$ is uniquely determined by the values of $T(\text{res}_H^G M)$ for all $H \subset G$ with $H \cong S^1$.

### 6 The map $T_G$

We need the following technical result.

**Lemma 6.1** Let $M$ be a closed manifold. Then there exists a Riemannian metric $g^M$ and a decomposition $M = \bigcup_i B_i$ of $M$ into manifolds with corner singularities such that the $B_i$ are contractible and the restriction of $g^M$ to $B_i$ is compatible for all $i$.

**Proof.** We choose a smooth triangulation of $M$. Then there is another smooth triangulation $\mathcal{T}$ which is dual to the first one. We choose small closed tubular neighbourhoods $U_\sigma$ of the simplices $\sigma$ of $\mathcal{T}$. We now proceed inductively. Assume that in the steps $0, \ldots, l - 1$ we have already defined $B_i, i = 1, \ldots r$. In the $l$'th step we let $B_{r+1}, \ldots$ be the intersections of $U_\sigma \cap (M \setminus \text{int}(\bigcup_{i=1}^r B_i))$, where $\sigma$ runs over all simplices of $\mathcal{T}$ of dimension $j$. By choosing the tubular neighbourhoods appropriately, this construction gives manifolds $B_i$ with corner singularities. Now one can construct an appropriate Riemannian metric. □

Recall that if $M$ is a manifold with corner singularities and $M$ has at least one boundary piece, then we define inductively $\chi(M) := \frac{1}{2} \chi(M_1)$. In particular, if $M = \bigcup_i B_i$ is a decomposition as in Lemma 6.1, then

$$\chi(M) = \sum_i \chi(B_i).$$

**Proposition 6.2** (1) : Let $M$ be a closed oriented $S^1$-manifold. Then $T(M) = T_{S^1} \chi_{S^1}(M)$. (2) : If in addition $M$ is even-dimensional, then $\chi_{S^1}(M) = a[S^1/S^1]$ for some $a \in \mathbb{Z}$.
Proof. A compact $S^1$-manifold has a finite number of orbit types $H_1, \ldots, H_l$. We employ induction by the number of orbit types $l(M)$. We first assume that $l(M) = 1$. If $H_1 = S^1$, then $T(M) = 0$ and $\chi_{S^1}(M) = \chi(M)[S^1/S^1]$. Thus $T_{S^1} \chi_{S^1}(M) = \chi(M)T_{S^1}[S^1/S^1] = 0$, too.

We now consider the case that $H_1 \neq S^1$. Then by [2], II.5.2., we have a smooth locally trivial fibre bundle $M \to M/S^1$ with fibre $S^1/H_1$. Let $M/S^1 = \bigcup_i B_i$ be a decomposition of $M/S^1$ into manifolds with corner singularities given by Lemma 6.1. Then $M/B_i \cong S^1/H_1 \times B_i$. Using Corollaries 4.1 and 4.3, (3), (12), and $\chi_{S^1}(M) = \chi(M/S^1)[S^1/H_1]$ we obtain

$$T(M) = \sum_i T(M|B_i)$$
$$= \sum_i T(S^1/H_1 \times B_i)$$
$$= T(S^1/H_1) \sum_i \chi(B_i)$$
$$= T_{S^1}[S^1/H_1]\chi(M/S^1)$$
$$= T_{S^1} \chi_{S^1}(M).$$

This proves assertion (1) for $l(M) = 1$. If $M$ is even-dimensional closed, then $M/S^1$ is odd-dimensional, and $\chi(M/S^1) = 0$ by Poincaré duality. Assertion (2) follows.

Now assume that the proposition holds true for all $M$ with $l(M) < l$. Let $M$ be a closed oriented $S^1$-manifold with $l(M) = l$. Without loss of generality we can assume that $H := H_1 \neq S^1$. By [2], VI 2.5., the fixed point set $M_H$ of $H$ is a smooth submanifold of $M$ with normal bundle $NM_H$, which we identify with an equivariant tubular neighbourhood of $M_H$ using the exponential map provided by a $S^1$-invariant Riemannian metric $g^M$.

Assume that $M$ is odd-dimensional. By Corollary 4.1 we have $T(M) = T(M \setminus NM_H) + T(NM_H)$. Let $N$ be the double of $M \setminus NM_H$. Then $l(N) \leq l-1$, and we can apply the induction hypothesis in order to obtain $T(M \setminus NM_H) = \frac{1}{2}T(N) = \frac{1}{2}T_{S^1} \chi_{S^1} T(N)$. Note that

$$\chi_{S^1}(N) = 2\chi_{S^1}(M \setminus NM_H) - \chi_{S^1}(\partial NM_H).$$

Note that $\partial NM_H$ is even-dimensional, closed and orientable. Since $l(\partial NM_H) < l$ we have by our induction hypothesis $\chi_{S^1}(\partial NM_H) = a[S^1/S^1]$ for some $a \in \mathbb{Z}$. This implies $T_{S^1} \chi_{S^1}(\partial NM_H) = 0$ and

$$T(M \setminus NM_H) = T_{S^1} \chi_{S^1}(M \setminus NM_H).$$

We now compute $T(NM_H)$. Since $l(M_H) = 1$ we have a smooth locally trivial fibre bundle $M_H \to M_H/S^1$ with fibre $S^1/H$. Let $M_H/S^1 = \bigcup_i B_i$ be a decomposition of $M_H/S^1$ into manifolds with corner singularities given by Lemma 6.1. Then $M_{H,i} := (M_H)|_{B_i} \cong S^1/H \times B_i$. Since $H$ acts orientation preserving, the bundle $NM_H$ admits an $H$-invariant complex structure. The restriction $(NM_H)|_{M_{H,i}}$ can be written as $S^1 \times V_i/H$, where $V_i \to B_i$ is a complex vector bundle on which $H$ acts fibrewise linear.

Since a complex linear action of a cyclic group $H$ can always be extended to the connected group $S^1$, we obtain $\chi^H(V_i)(\gamma) = \chi(V_i)$ for all $\gamma \in H$. Moreover we have $\chi^H(S^1) = 0$. Thus we
can apply Corollaries 4.3 and 4.2 in order to obtain
\[ T(S^1 \times V_i/H) = \int \Gamma T^H(S^1) \chi^H(V_i) = \int \Gamma T^H(S^1) \chi(V_i) = T(S^1/H) \chi(V_i). \]

Since \( \overline{NM}_H \) and \( M_H \) are \( S^1 \)-homotopy equivalent, we have \( \chi_{S^1}(\overline{NM}_H) = \chi_{S^1}(M_H) \). Moreover, \( \sum_i \chi(V_i) = \sum_i \chi(B_i) = \chi(M_H/S^1) \) and \( \chi_{S^1}(M_H) = \chi(M_H/S^1)[S^1/H] \). Thus we obtain by Corollary 4.1
\[ T(\overline{NM}_H) = \sum_i T(S^1 \times V_i/H) = \sum_i T(S^1/H) \chi(V_i) = T_{S^1}[S^1/H] \chi(M_H/S^1) = T_{S^1} \chi_{S^1}(\overline{NM}_H). \] (14)

We have
\[ \chi_{S^1}(M) = \chi_{S^1}(M \setminus NM_H) + \chi_{S^1}(NM_H) - \chi_{S^1}(\partial NM_H). \]

Since \( T_{S^1} \chi_{S^1}(\partial NM_H) = 0 \), combining (13) and (14) we obtain the desired formula 
\[ T(M) = T_{S^1} \chi_{S^1}(M) \] for \( M \) odd-dimensional.

Assume now that \( M \) is even-dimensional and that \( l(M) = l \). Then \( T(M) = 0 \), and (1) follows from (2). We now show (2). We have
\[ \chi_{S^1}(M) = \chi_{S^1}(M \setminus NM_H) + \chi_{S^1}(M_H). \]

We can apply the induction hypothesis to \( M_H \) and the double of \( M \setminus NM_H \). It follows that \( \chi_{S^1}(M \setminus NM_H) = \frac{1}{2} \chi_{S^1}(\partial NM_H) + a[S^1/S^1] \). The restriction \( \partial NM_H|_{M_H,i} \) is isomorphic to \( S^1 \times \partial SV_i/H \), where \( SV_i \) denotes the sphere bundle of \( V_i \).

Let \( U \) be the unit sphere in a fibre of \( NM_H \). Using that \( M_H/S^1 \) is closed, orientable, and odd-dimensional, we obtain
\[ \chi_{S^1}(\partial NM_H) = \sum_i \chi_{S^1}(S^1 \times U/H) \chi(B_i) = \chi_{S^1}(S^1 \times U/H) \chi(M_H/S^1) = 0. \]

This finishes the proof of (2). \qed

We now construct \( T_G \). The collection \( T_H : H \in S(G) \), forms a natural transformation from the functor \( H \mapsto U(H) \) to \( H \mapsto \tilde{I}(H) \). Thus we obtain a homomorphism
\[ \tilde{T} : \varprojlim_{S(G)} U \to V(G). \]
Let \( \tilde{\text{res}} : U(G) \to \lim S(G) \) be given by the collection \( \text{res}^G_H, \ H \in S(G) \). If \( M \) is a compact \( G \)-manifold, then we let \( \tilde{\chi}(M) \in \lim S(G) U \) be given by the section \( S(G) \ni H \mapsto \chi_H(M) \in U(H) \). Then \( \text{res}_G \chi_G(M) = \tilde{\chi}(M) \).

**Lemma 6.3** There is a unique homomorphism \( T_G : U(G) \to \tilde{I}(G) \) such that \( R_G \circ T_G = \tilde{T} \circ \text{res}_G \).

**Proof.** For \( G/K \in \text{Or}(G) \) we shall have

\[
R_G \circ T_G[G/K] = \tilde{T} \circ \text{res}_G(G/K)
= \tilde{T} \circ \tilde{\chi}(G/K)
= \{S(G) \ni H \mapsto T_H \circ \chi_H \circ \text{res}^G_H(G/K)\}
= \{S(G) \ni H \mapsto T(\text{res}^G_H G/K)\}
= \{S(G) \ni H \mapsto \text{res}^G_H T(G/K)\}
= R_G T(G/K).
\]

Hence by injectivity of \( R_G \) (Lemma 5.2) we are forced to define \( T_G[G/K] := T(G/K) \). \( \square \)

We now finish the proof of Theorem 1.1. Let \( M \) be a closed oriented \( G \)-manifold. Then we have

\[
R_G \circ T_G \chi_G(M) = \tilde{T} \circ \text{res}_G \chi_G(M)
= \tilde{T} \circ \tilde{\chi}(M)
= \{S(G) \ni H \mapsto T_H \circ \chi_H \circ \text{res}^G_H(M)\}
= \{S(G) \ni H \mapsto T(\text{res}^G_H M)\}
= \{S(G) \ni H \mapsto \text{res}^G_H T(M)\}
= R_G T(M).
\]

We conclude that \( T_G \chi_G(M) = T(M) \) by Lemma 5.2. \( \square \)

**References**

[1] N. Berline, E. Getzler, and M. Vergne. *Heat Kernels and Dirac Operators*. Springer-Verlag Berlin Heidelberg New York, 1992.

[2] G. E. Bredon. *Introduction to transformation groups*. Academic Press, 1972.

[3] U. Bunke. Equivariant torsion and \( G \)-CW complexes. Preprint, 1997.
[4] J. Cheeger, M. Gromov, and M. Taylor. Finite propagation speed, kernel estimates for functions of the Laplace operator and the geometry of complete Riemannian manifolds. *J.Diff.Geom.*, 17, 15–53, 1982.

[5] J. Lott. Equivariant analytic torsion for compact Lie group actions. *J. Funct. Anal.*, 125, 438–451, 1994.

[6] J. Lott and M. Rothenberg. Analytic torsion for group actions. *J.Diff.Geom*, 34, 431–481, 1991.

[7] W. Lück. *Transformation groups and algebraic $K$-theory*. LNM 1408. Springer Verlag, 1989.