UNIFORM SHEAR FLOW VIA THE BOLTZMANN EQUATION WITH HARD POTENTIALS

RENJUN DUAN AND SHUANGQIAN LIU

ABSTRACT. The motion of rarefied gases for uniform shear flow at the kinetic level is governed by the spatially homogeneous Boltzmann equation with a deformation force. In the paper we study the corresponding Cauchy problem with initial data of finite mass and energy for the collision kernel in case of hard potentials $0 < \gamma \leq 1$ under the cutoff assumption. We prove the global existence and large time behavior of solutions provided that the force strength $\alpha > 0$ is small enough. In particular, when the initial perturbation is of order $\alpha^m$ for $m > 2$, we make a rigorous justification of the uniform-in-time asymptotic expansion of solutions up to order $\alpha^2$ under a homoeenergetic self-similar scaling.

1. Introduction

1.1. Problem. For a rarefied gas, the uniform shear flow (USF in short) is characterized at a macroscopic level as a state where the density is constant, the velocity at time $t$ and position $x$ and the temperature is uniform in space but may depend on time. For simplicity we always suppose $\text{tr} A = 0$. Due to the shearing motion and the associated viscous heating, the total energy and hence the temperature monotonically increase in time. It is more fundamental to understand the change of energy under the deformation force at the kinetic level (cf. [20]) where the gas motion is governed by the Boltzmann equation for a finite Knudsen number

$$\partial_t F + v \cdot \nabla_x F = Q(F, F).$$

Here $F = F(t, x, v) \geq 0$ stands for the density distribution function of gas particles with velocity $v$ at time $t$ and position $x$ and $Q$ is the collision operator to be specified later. In this context the kinetic USF state is then defined as the one that is spatially homogeneous when the velocities of particles are referred to a Lagrangian frame moving with the velocity field $\alpha Ax$. Consequently, the density distribution function has the form $F(t, x, v) = F(t, v - \alpha Ax)$. With this ansatz the Boltzmann equation above becomes

$$\partial_t F - \alpha \nabla_v \cdot (AvF) = Q(F, F),$$

for a spatially homogeneous unknown function $F = F(t, v) \geq 0$. Here, the bilinear collision operator $Q(\cdot, \cdot)$ is given as

$$Q(F_1, F_2) = \int_{\mathbb{R}^3} \int_{S^2} B(\omega, v - v_*) [F_1(v'_*) F_2(v) - F_1(v_*) F_2(v')] d\omega dv_*.$$

In the integral we have denoted $v'_* = v + (v_* - v) \cdot \omega$ and $v'_* = v_* - (v_* - v) \cdot \omega$ with $\omega \in S^2$ in terms of the conservation laws $v_* = v_* + v'$ and $|v_*|^2 + |v|^2 = |v'_*|^2 + |v'|^2$. Throughout this
paper, we define
\[ B(\omega, v - v_*) = |v - v_*|^\gamma B_0(\cos \theta), \quad \cos \theta = \frac{\omega \cdot (v - v_*)}{|v - v_*|}, \quad \omega \in S^2, \quad 0 \leq \gamma \leq 1, \quad 0 \leq B_0(\theta) \leq C|\cos \theta|. \tag{1.4} \]
This includes general hard potentials under the Grad’s angular cutoff assumption (cf. [16]).

In the paper, we are interested in studying the global existence and long time behavior of solutions to the spatially homogeneous Boltzmann equation (1.1) supplemented with initial data
\[ F(0, v) = F_0(v), \tag{1.5} \]
which has finite mass and energy. Since the case of \( \gamma = 0 \) has been considered in our previous work [20], we restrict our attentions in this paper to the only case of \( 0 < \gamma \leq 1 \). The problem in case of \( 0 < \gamma \leq 1 \) was addressed in [31, 32] by James, Nota and Velázquez; see also a recent survey [40]. More related results will be reviewed later on.

1.2. Normal solution. To solve the Cauchy problem (1.1) and (1.5), we consider the normal solution under a certain scaling such that the profile has conservative mass, momentum and energy for all nonnegative time. For the purpose, let us now define the mass \( \rho \), momentum \( u \) and temperature \( \theta \) associated with \( F(t, v) \) as follows
\[
\rho = \int_{\mathbb{R}^3} F(t, v) \, dv, \\
\rho u = \int_{\mathbb{R}^3} v F(t, v) \, dv, \\
\rho \theta = \int_{\mathbb{R}^3} \frac{1}{3} |v - u|^2 F(t, v) \, dv.
\]
From (1.1), it then follows
\[
\frac{d\rho}{dt} = 0, \tag{1.6} \\
\frac{d(\rho u)}{dt} + \alpha A\rho u = 0, \tag{1.7} \\
\frac{d\theta}{dt} + \frac{2\alpha}{3\rho} \int_{\mathbb{R}^3} (v - u) \cdot (A(v - u)) F(t, v) \, dv = 0. \tag{1.8}
\]
Without loss of generality we assume
\[
\int_{\mathbb{R}^3} F_0(v) \, dv = 1, \quad \int_{\mathbb{R}^3} v F_0(v) \, dv = 0, \tag{1.9}
\]
so that (1.6) and (1.7) give that \( \rho(t) \equiv 1 \) and \( u(t) \equiv 0 \) for all \( t \geq 0 \). To further get the conservative temperature, we introduce a scaled variable \( \xi = \frac{v}{\beta} \) with \( \beta = \beta(t) = \sqrt{\theta(t)} \) and set
\[
F(t, v) = \beta^{-3} G(t, \xi), \quad \xi = \frac{v}{\beta}, \tag{1.10}
\]
where \( \beta(t) \) is the so-called thermal speed measuring the temperature of particles. For simplicity, we assume \( \beta(0) = 1 \) or equivalently
\[
\int_{\mathbb{R}^3} |v|^2 F_0(v) \, dv = 3, \tag{1.11}
\]
so that \( G_0 \equiv F_0 \) at initial time. As a consequence, it holds
\[
\int_{\mathbb{R}^3} G(t, \xi) \, d\xi = 1, \quad \int_{\mathbb{R}^3} \xi G(t, \xi) \, d\xi = 0, \quad \int_{\mathbb{R}^3} |\xi|^2 G(t, \xi) \, d\xi = 3, \tag{1.12}
\]
for all \( t \geq 0 \). Substituting (1.10) into (1.11), we obtain
\[
\partial_t G - \frac{\beta'}{\beta} \nabla \cdot (\xi G) - \alpha \nabla \cdot (A\xi G) = \beta^7 Q(G, G), \tag{1.13}
\]
with the initial data
\[ G(0, \xi) = G_0(\xi) = F_0(\xi), \] (1.14)
where we have denoted \( \beta' = \frac{d\beta}{dt} \).

Note that under the above setting the time-dependent thermal speed \( \beta \) is given by the solution \( F(t, v) \) itself in the way that
\[ \beta(t) = \sqrt{\int_{\mathbb{R}^3} \frac{1}{3} |v|^2 F(t, v) \, dv}. \]
To determine \( \beta(t) \) in terms of \( G(t, \xi) \), it follows from the temperature equation (1.8) and the scaling (1.10) with \( \beta(t) = \sqrt{\theta(t)} \) that
\[ \frac{\beta'}{\beta} = -\frac{\alpha}{3} \int_{\mathbb{R}^3} \xi \cdot A\xi G(t, \xi) \, d\xi, \quad \beta(0) = 1. \] (1.15)

Therefore we conclude that under conditions (1.9) and (1.11) for the same initial data \( F_0 = G_0 \), to solve the Cauchy problem (1.1) and (1.5) is equivalent to solve the Cauchy problem (1.13) and (1.14) coupled with the first order ODE problem (1.15). Note that for the latter we have all the physical conservation laws as in (1.12).

1.3. Expansion. We solve the reformulated problem (1.13), (1.14) and (1.15) in the perturbation framework for any small enough deformation strength \( \alpha > 0 \). Note that for \( \alpha = 0 \) meaning that there is no deformation force, the solution to the problem exists globally in time and tends asymptotically toward the global Maxwellian determined by the conservation laws. Hence, in terms of initial conditions (1.9) and (1.11), we set the reference global Maxwellian to be
\[ \mu(\xi) = (2\pi)^{-3/2} e^{-|\xi|^2/2}. \]

For any \( \alpha > 0 \), we therefore define
\[ G = \mu + \sqrt{\mu} \{ \alpha G_1 + \alpha^2 G_2 + \alpha^m G_R \}, \] (1.16)
where \( G_1, G_2, G_R \) and the integer \( m > 2 \) are to be determined later. In order for (1.12) to be satisfied, we require
\[ (f, [1, 1, \frac{1}{2} |\xi|^2] \mu^{\frac{3}{2}}) = 0 \quad \text{for} \quad f = G_1, G_2, G_R. \] (1.17)
Here and in the sequel, \( (\cdot, \cdot) \) is used to denote the inner product of two functions on \( L^2(\mathbb{R}^3) \) for brevity. Further plugging (1.16) into (1.15) gives
\[ \frac{\beta'}{\beta} = \beta_0 \alpha^2 + \beta_1 \alpha^3, \] (1.18)
with
\[ \beta_0 = \beta_0(G) = -\frac{1}{3} \int_{\mathbb{R}^3} \xi \cdot A\xi \sqrt{\mu} G_1 \, d\xi, \] (1.19)
and
\[ \beta_1 = \beta_1(G) = -\frac{1}{3} \int_{\mathbb{R}^3} \xi \cdot A\xi \sqrt{\mu} G_2 \, d\xi - \frac{\alpha^{m-2}}{3} \int_{\mathbb{R}^3} \xi \cdot A\xi \sqrt{\mu} G_R \, d\xi. \] (1.20)
It should be noted that \( \beta, \beta_0 \) and \( \beta_1 \) all are functions of time \( t \) and depend on the solution itself. For simplicity, we will omit such dependence unless the explicit expressions are important for discussions at some places.

Plugging (1.16) into (1.13) and comparing the coefficients of terms with different powers of \( \alpha \), one has the equations for \( G_1 \) and \( G_2 \)
\[ -\nabla \xi \cdot (A\xi \mu)^{-\frac{1}{2}} + \beta^\gamma L G_1 = 0, \] (1.21)
\[ -\beta_0 \nabla \xi \cdot (\xi \mu)^{-\frac{1}{2}} - \nabla \xi \cdot (A\xi \sqrt{\mu} G_1)^{-\frac{1}{2}} + \beta^\gamma L G_2 = \beta^\gamma \Gamma(G_1, G_1), \] (1.22)
and the equation for the remainder \( G_R \)
\[
\partial_t G_R - \frac{\beta'}{\beta} \nabla_\xi \cdot (\xi \sqrt{\mu G_R}) \mu^{-\frac{1}{2}} - \nabla_\xi \cdot (A\xi \sqrt{\mu G_R}) \mu^{-\frac{1}{2}} + \beta^\gamma L G_R \\
= -\alpha^{1-m} \partial_t G_1 - \alpha^{2-m} \partial_t G_2 + \alpha^{3-m} \beta_1 \nabla_\xi \cdot (\xi \mu) \mu^{-\frac{1}{2}} + \alpha^{1-m} \frac{\beta'}{\beta} \nabla_\xi \cdot (\xi \sqrt{\mu G_1}) \mu^{-\frac{1}{2}} \\
+ \alpha^{2-m} \frac{\beta'}{\beta} \nabla_\xi \cdot (\xi \sqrt{\mu G_2}) \mu^{-\frac{1}{2}} + \alpha^{3-m} \nabla_\xi \cdot (A\xi \sqrt{\mu G_2}) \mu^{-\frac{1}{2}} + \alpha^{3-m} \beta^\gamma \{ \Gamma(G_1, G_2) + \Gamma(G_2, G_1) \} \\
+ \alpha^{4-m} \beta^\gamma \Gamma(G_2, G_2) + \alpha \beta^\gamma \{ \Gamma(G_1 + \alpha G_2, G_R) + \Gamma(G_R, G_1 + \alpha G_2) \} + \alpha^m \beta^\gamma \Gamma(G_R, G_R),
\]
with \( \sqrt{\mu G_R}(0, \xi) = G_{R,0} = \alpha^{-m} \{ G_0 - \mu - \alpha \sqrt{\mu G_1}(0, \xi) - \alpha^2 \sqrt{\mu G_2}(0, \xi) \} \).

Here, \( L \) is the linearized Boltzmann operator and \( \Gamma \) is the nonlinear collision Boltzmann operator, respectively given by
\[
Lg = -\mu^{-1/2} \{ Q(\mu, \sqrt{\mu g}) + Q(\sqrt{\mu g}, \mu) \},
\]
and
\[
\Gamma(f, g) = \mu^{-1/2} Q(\sqrt{\mu f}, \sqrt{\mu g}).
\]

For later use we first introduce some notations. Note that \( Lf = \nu f - Kf \)

\[
\begin{aligned}
\nu &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} B(\omega, \xi - \xi_*) (v_*) \mu \, d\omega \, d\xi_* \sim (1 + |\xi|)\gamma, \\
Kf &= \mu^{-\frac{1}{2}} \left\{ Q(\mu f, \mu) + Q_{\text{gain}}(\mu, \mu f) \right\},
\end{aligned}
\]

where \( Q_{\text{gain}} \) denotes the positive part of \( Q \) in \([12]\). Moreover, it holds that
\[
Kf = \int_{\mathbb{R}^3} k(\xi, \xi_*) f(\xi_*) \, d\xi_* = \int_{\mathbb{R}^3} (k_2 - k_1)(\xi, \xi_*) f(\xi_*) \, d\xi_*,
\]

with
\[
0 \leq k_1(\xi, \xi_*) \leq \tilde{c}_1 |\xi - \xi_*|^\gamma e^{-\frac{|\xi|^2}{2} \left( 1 + |\xi|^2 \right)}, \quad 0 \leq k_2(\xi, \xi_*) \leq \tilde{c}_2 |\xi - \xi_*|^{-2 + \gamma} e^{-\frac{|\xi|^2}{2} \left( 1 - |\xi|^2 \right)},
\]
where both \( \tilde{c}_1 \) and \( \tilde{c}_2 \) are positive constants. The kernel of \( L \), denoted as \( \ker L \), is a five-dimensional subspace of \( L^2(\mathbb{R}_+^3) \), spanned by
\[
\{1, \xi, |\xi|^2 - 3\} \sqrt{\mu} := \{\phi_i\}_{i=1}^5.
\]
We further define the projection from \( L^2 \) to \( \ker L \) by
\[
P_0 g = \{ a_g + b_g \cdot \xi + (|\xi|^2 - 3) c_g \} \sqrt{\mu}
\]
for \( g \in L^2 \), and correspondingly denote the operator \( P_1 \) by \( P_1 g = g - P_0 g \), which is orthogonal to \( P_0 \) in \( L^2 \). Traditionally, \( P_0 g \) is also called the macroscopic component, while \( P_1 g \) stands for the microscopic component. As in \([20]\), to treat the polynomial tail part of \( G_R \), it is also convenient to define
\[
L f = -\{ Q(f, \mu) + Q(\mu, f) \} = \nu f - Kf,
\]
with
\[
Kf = Q(f, \mu) + Q_{\text{gain}}(\mu, f) = \sqrt{\mu} K \left( \frac{f}{\sqrt{\mu}} \right).
\]

We now determine \( G_1 \) and \( G_2 \). Notice that one has \( \xi \cdot A \xi \mu^{\frac{3}{2}} \in (\ker L)^\perp \) due to \( \text{tr} A = 0 \). We then get from \([10]21 \) and \([11]17 \) that
\[
G_1 = -\beta^{-\gamma} L^{-1} \{ \xi \cdot A \xi \mu^{\frac{3}{2}} \}.
\]
As a consequence, (1.19) gives
\[ \beta_0 = -\frac{1}{3}(\xi \cdot A\xi \mu^\frac{1}{2}, G_1) = \frac{1}{3} \beta^{-\gamma} \left( \xi \cdot A\xi \mu^\frac{1}{2}, L^{-1}\{\xi \cdot A\xi \mu^\frac{1}{2}\} \right) = \beta^{-\gamma} \varrho_0, \] (1.28)
with
\[ \varrho_0 = \frac{1}{3} \left( \xi \cdot A\xi \mu^\frac{1}{2}, L^{-1}\{\xi \cdot A\xi \mu^\frac{1}{2}\} \right). \] (1.29)
Note that \( \varrho_0 > 0 \) is a constant independent of \( \alpha \) and \( t \). Moreover, (1.28) also implies
\[ (-\beta_0 \nabla \xi \cdot (\xi\mu)\mu^{-\frac{1}{2}} - \nabla \xi \cdot (A\xi\sqrt{\mu}G_1)\mu^{-\frac{1}{2}}, [1, \xi, \frac{1}{2}|\xi|^2]\mu^\frac{1}{2}) = 0, \]
and note that \( G_2 \) can be only microscopic as required in the condition (1.17). Then, it is valid to derive from (1.22) that
\[ G_2 = \beta^{-\gamma} L^{-1}\left\{ \beta^\gamma \Gamma(G_1, G_1) + \beta_0 \nabla \xi \cdot (\xi\mu)\mu^{-\frac{1}{2}} + \nabla \xi \cdot (A\xi\sqrt{\mu}G_1)\mu^{-\frac{1}{2}} \right\}. \] (1.30)
Using the expression of \( G_2 \) above, we define
\[ \varrho_1 = -\frac{\beta^{2\gamma}}{3} \int_{\mathbb{R}^3} \xi \cdot A\xi\sqrt{\mu}G_2 d\xi \]
\[ = -\frac{1}{3} \int_{\mathbb{R}^3} \xi \cdot A\xi\sqrt{\mu}\left\{ L^{-1}\left\{ \Gamma(L^{-1}\{\xi \cdot A\xi \mu^\frac{1}{2}\}, L^{-1}\{\xi \cdot A\xi \mu^\frac{1}{2}\}) + \varrho_0 \nabla \xi \cdot (\xi\mu)\mu^{-\frac{1}{2}} \right\} \right\} d\xi. \] (1.31)
Note again that \( \varrho_1 \) is a constant independent of \( \alpha \) and \( t \). Furthermore, by (1.30) and (1.27) as well as the definition (1.20), one gets from (1.22) that
\[ \left( G_R, [1, \xi, \frac{1}{2}|\xi|^2]\mu^\frac{1}{2} \right) = \left( G_{R,0}, [1, \xi, \frac{1}{2}|\xi|^2] \right), \] (1.32)
which coincides with our assumption (1.12) when further assuming
\[ \left( G_{R,0}, [1, \xi, \frac{1}{2}|\xi|^2] \right) = 0. \] (1.33)
Let us now briefly illustrate how to solve \( \beta \) from (1.18). By (1.28), (1.31) and (1.18), we rewrite
\[ \frac{\beta'}{\beta} = \varrho_0 \alpha^2 \beta^{-\gamma} + \varrho_1 \alpha^3 \beta^{-2\gamma} - \frac{\alpha^{m+1}}{3} \int_{\mathbb{R}^3} \xi \cdot A\xi\sqrt{\mu}G_R d\xi. \] (1.34)
It turns out that both constants \( \varrho_0 > 0 \) and \( \varrho_1 \) can be proved to be finite. Furthermore \( \beta^{2\gamma}\sqrt{\mu}G_R \) can be verified to be bounded in terms of \( G_1 \) and \( G_2 \). Therefore, if \( \alpha > 0 \) is suitably small and one takes \( 2 < m < 3 \), we formally have
\[ \frac{\beta'}{\beta} \sim \varrho_0 \beta^{-\gamma} \alpha^2, \]
which may give
\[ \beta \sim (1 + \varrho_0 \alpha^2 t)^{\frac{1}{2\gamma}}. \]
1.4. Main result. Define a polynomial velocity weighted function \( w^\ell(\xi) = (1 + |\xi|^2)^\ell \) for \( \ell \geq 0 \).
We now state the main result below for the Cauchy problem (1.1) and (1.5).

**Theorem 1.1.** Assume (1.3) and (1.4). Let \( 0 < \gamma \leq 1, 2 < m < 3 \) and an integer \( N \geq 1 \). Let \( \ell_{\text{max}} \gg 4 \) be a constant that can be arbitrarily large. Suppose \( \text{tr}A = 0 \). There are constants \( \alpha_0 > 0, M_0 > 0 \) and \( C > 0 \) such that for any \( \alpha \in (0, \alpha_0) \), if initial data \( F_0 = F_0(v) \geq 0 \) satisfies
\[ \int_{\mathbb{R}^3} F_0(v) dv = 1, \quad \int_{\mathbb{R}^3} v F_0(v) dv = 0, \quad \int_{\mathbb{R}^3} |v|^2 F_0(v) dv = 3, \]
and
\[ \sum_{|\alpha| \leq N} \| w^{\alpha} \partial_v^\alpha [F_0 - \mu - \alpha \sqrt{\theta} G_1(0, v) - \alpha^2 \sqrt{\theta} G_2(0, v)] \|_{L^\infty} \leq M_0 \alpha^m, \]

then the spatially homogeneous Cauchy problem \(^{(1.4)}\) and \(^{(1.5)}\) admits a unique global solution \(F = F(t, v) \geq 0\) satisfying
\[ \int_{\mathbb{R}^3} F(t, v) \, dv = 1, \quad \int_{\mathbb{R}^3} vF(t, v) \, dv = 0, \quad \int_{\mathbb{R}^3} |v|^2 F(t, v) \, dv = 3\beta^2(t), \quad \forall t \geq 0, \]

with estimates
\[ \beta(t) > 0, \quad \beta(0) = 1, \quad \lim_{t \to \infty} \frac{\beta(t)}{(1 + \gamma \varrho_0 \alpha^2 t)^{\frac{1}{2}}} = 1, \quad (1.35) \]
and
\[ \sum_{|\alpha| \leq N} \| w^{\alpha} \partial_v^\alpha [\beta^3(t) F(t, \beta(t)v) - \mu(v) - \alpha \sqrt{\theta} G_1(t, v) - \alpha^2 \sqrt{\theta} G_2(t, v)] \|_{L^\infty} \leq C(1 + \gamma \varrho_0 \alpha^2 t)^{-2}(M_0 \alpha^m + \alpha^3), \quad \forall t \geq 0. \quad (1.36) \]

Here \(\varrho_0 > 0\), \(G_1(t, \cdot)\) and \(G_2(t, \cdot)\) are respectively given by \(^{(1.29)}\), \(^{(1.27)}\) and \(^{(1.30)}\).

Theorem \(^{(1.4)}\) above shows the following uniform asymptotic expansion of the obtained global solution up to \(\alpha^2\) in the homoenergetic self-similar scaling
\[ \theta^{3/2}(t) F(t, \theta^{1/2}(t)v) = \mu + \alpha \sqrt{\mu} G_1(t, v) + \alpha^2 \sqrt{\mu} G_2(t, v) + O(1)\alpha^m(1 + \gamma \varrho_0 \alpha^2 t)^{-2}, \quad (1.37) \]
where the thermal speed \(\theta(t) = \beta^2(t)\) satisfies
\[ \theta(t) \sim (1 + \gamma \varrho_0 \alpha^2 t)^{\frac{3}{4}}. \quad (1.38) \]

By \(^{(1.27)}\) and \(^{(1.30)}\) one has
\[ G_1(t, v) \sim (1 + \gamma \varrho_0 \alpha^2 t)^{-1} \] and \(G_2(t, v) \sim (1 + \gamma \varrho_0 \alpha^2 t)^{-2}\) in large time. We remark that it is also interesting to carry out the higher order expansion of \(G\), for instance, up to the \(n\)th-order for an integer \(n > 0\), namely,
\[ G = \mu + \sqrt{\mu} \{ \alpha G_1 + \alpha^2 G_2 + \cdots + \alpha^n G_n + \alpha^m G_R \} , \quad m > n. \]

Then the corresponding decay rate in \(^{(1.36)}\) could be improved to be \((1 + \gamma \varrho_0 \alpha^2 t)^{-n}\). Furthermore, if one can obtain uniform estimates for any \(n > \frac{1}{2}\) as \(\gamma \to 0^+\), then the decay rate of the remainder \(G_R\) as \(\gamma \to 0^+\) should be recovered by taking the limit
\[ \lim_{\gamma \to 0^+} (1 + \gamma \varrho_0 \alpha^2 t)^{-\frac{1}{2}} = e^{-\varrho_0 \alpha^2 t}, \]

which is exponential in time with size of \(\alpha^2\) order, where \(\varrho_0 > 0\) is the limit of \(\varrho_0\) as \(\gamma \to 0^+\) in terms of \(^{(1.29)}\). This will coincide with the result proved in our previous work \(^{(20)}\) for the case of \(\gamma = 0\). The rigorous study of such issue is left for the future.

1.5. Literature. In what follows we mention some related literature. Readers may refer to our previous work \(^{(20–22)}\) for a detailed review. For instance, Galkin \(^{(25)}\) and Truesdell \(^{(43)}\) first independently introduced the concept of homoeenergetic solutions to the Boltzmann equation; see also an introduction to the topic in Truesdell-Muncaster \(^{(43)}\). The numerical investigation has been extensively made in the monograph Garzó-Santos \(^{(26)}\). Related to the USF state without any boundary, a physically more realistic topic is the planar Couette flow governed by the Boltzmann equation for a rarefied gas between two parallel infinite plates moving with opposite velocities and such topic was also discussed in many books on kinetic theory such as Kogan \(^{(36)}\) Chapter 4] and Sone \(^{(42)}\) Chapter 4]; see also \(^{(26)}\) Chapter 5].

When there is no deformation force, the topic on the self-similar solution and its asymptotic stability for the spatially homogeneous Boltzmann equation was investigated in early 2000s by Bobylev-Cercignani \(^{(9,11)}\) and later by Cannone-Karch \(^{(14,15)}\) and Morimoto-Yang-Zhao \(^{(39)}\) among many others.
When there is a deformation force, we mention many early results on the shear flow topic by Cercignani [17–19] and Bobylev-Caraffini-Spiga [12]. Recently, the significant progress was made by James-Nota-Velázquez [31–33] and later by Bobylev-Nota-Velázquez [13]. In particular, for the USF governed by the Boltzmann equation in case of the Maxwell molecule model $\gamma = 0$ under the cutoff assumption, [13] proved the existence (obtained also in [31]) and the uniqueness, non-negativity and stability (as well as the analysis of the moments and the exponential rate of convergence) of self-similar profiles in the class of measures for small enough deformation strength; see also Bobylev [7] for a further study to provide explicit estimates on smallness of the deformation matrix. Here, the approach used in [31] is based on the fixed point argument on the integral form of the problem over a set of non-negative Radon measures, while [13] gave a different proof by means of the Fourier transform method (cf. [7,8]) taking the full advantage of the Bobylev formula. An interesting result on self-similar profiles for the non-cutoff Maxwell molecule model was also obtained by Kepka [34]. Readers may refer to [40] for a thorough review to those and other related works. Moreover, following [31] and [13], in the case of Maxwell molecule with cutoff, we also constructed in [20] smooth self-similar profiles for the shear flow problem on the Boltzmann equation and proved the dynamical stability of the stationary solution via a perturbation approach.

In the current work, we are interested in the uniform shear flow governed by the Boltzmann equation in the case of hard potentials. The problem was addressed in [31, 32]; see also [10] as mentioned before. In fact, the formal Hilbert expansion similar to (1.16) as given in [32] implies that the temperature of gas particles increases in time with an algebraic rate in (1.38) and the self-similar asymptotics of the form (1.37) was conjectured. In particular, since $G_1$ and $G_2$ decay in time, the solution converges self-similarly in large time to the global Maxwellian in contrast to a non-equilibrium state with a polynomial large velocity tail obtained in case of the Maxwell molecule model (cf. [38]). To treat the issue we recently considered a closely related problem in [21] where an extra thermostated term is added to compensate the viscous heating energy such that the system of gas particles can be driven in large time to the non-equilibrium steady state under the interplay of both thermostated and sheared forces. Using the developed techniques in [20,21], we aim in this paper at making a rigorous justification of the expansion (1.37) with the temperature behavior (1.38). We remark that during the preparation of the current work, we have been aware of a preprint [35] for treating a similar issue which includes both cutoff and non-cutoff cases. The approach used in [35] is based on the construction of polynomial tail solutions with the help of the robust semigroup property, cf. [27].

In the end, we remark that there have been extensive studies for stability of shear flow in the context of fluid dynamic equations (cf. [41]). In particular, we mention major contributions [2–4] recently made by Bedrossian together with his collaborators; see the survey [5] for the subject. Regarding the shear flow with physical boundaries, we refer to recent progress by Ionescu-Jia [30] and Masmoudi-Zhao [37] which inspired us to study in [22] the kinetic planar Couette flow with boundaries as mentioned before. We point that it would be interesting to understand the relation of those fluid solutions and Boltzmann solutions through the rigorous justification of the hydrodynamic limit in case of small shear strength, cf. [23,24].

1.6. Strategies and ideas of the proof. We now outline some key ideas and methods used in the paper. One of typical features for the shear flow governed by the Boltzmann equation is the rapid increasing of the total energy of gas particles. A serious consequence of such a scenario is that the macroscopic component is out of control in $L^2$ setting. Therefore the self-similar structure of solutions need to be explored to look for the normal form. To do so, a suitable scaling should be introduced so that the energy of the self-similar profile can be conserved. In this paper, the scaling parameter $\beta$ of self-similar solutions is chosen to satisfy the ODE

$$\frac{\beta'}{\beta} = \frac{\alpha}{3} \int_{\mathbb{R}^3} \xi \cdot A\xi G \, d\xi.$$ 

In particular, $\beta = \beta(t)$ is an unknown function of time which depends on the solution itself. This leads to a nonlinear convection term $\frac{\alpha}{3} \nabla_{\xi} \cdot (\xi G)$ in the scaled equation. However, if the drift terms in the equation are of higher order when $t \to \infty$ compared with the Boltzmann collision...
operator, the solution should converge to Maxwellian equilibrium. Hence the following Hilbert type expansion is introduced

\[ G = \mu + \sqrt{\mu}\{\alpha G_1 + \alpha^2 G_2 + \alpha^m G_R\}, \]

with \( G_1, G_2 \) and \( G_R \) all belonging to \( (\ker L)^\perp \), provided that the shear strength \( \alpha > 0 \) is small enough. Unfortunately, unlike the case of Maxwell molecular, not only the remainder \( G_R \) but also both the correction terms \( G_1 \) and \( G_2 \) depend on \( \beta \). To overcome this difficulty, a continua argument is employed. More precisely, we first construct the local existence by the contraction mapping method, and then prove the \textit{a priori} estimates in \( L^\infty \) setting. To show the local existence, there are two difficulties: one is to determine the time-dependent scaling function \( \beta \) and the other is to justify that the solution operator is contractive in a short time. To overcome the first difficulty, an expansion in the form of

\[ \frac{\beta'}{\beta} = \beta_0 \alpha^2 + \beta_1 \alpha^3, \]

is considered, and then the problem is reduced to solve the ordinary differential inequalities

\[ \frac{1}{2} \varrho_0 \beta^{-\gamma} \alpha^2 \leq \frac{\beta'}{\beta} \leq \frac{3}{2} \varrho_0 \beta^{-\gamma} \alpha^2. \]

To treat the second difficulty, the stability of \( \beta \) with respect to the solution variable \( h \) is proved, namely we verify

\[ \|(\beta^\gamma(h) - \beta^\gamma(h))(t)\| \leq C(T_0) \|w^\gamma[h_1 - \bar{h}_1, h_2 - \bar{h}_2]\|_{L^\infty}. \]

Due to this, \( V(h) \), the velocity determined by the characteristic line as in [20, 21], and \( \mathcal{A}(h) \), the generator of the semigroup as given by [20, 21], both can be shown to be stable.

Another typical feature for the shear flow problem on the Boltzmann equation is the velocity growth caused by the shear force (or deformation force). To overcome this difficulty, we introduce the following Caflisch’s decomposition

\[ \partial_t G_{R,1} - \frac{\beta'}{\beta} \nabla_\xi \cdot (\xi G_{R,1}) - \alpha \nabla_\xi \cdot (A\xi G_{R,1}) + \beta^\gamma \nu G_{R,1} = \beta^\gamma \lambda M K G_{R,1} - \frac{\beta'}{2\beta} \|\xi\|^2 \sqrt{\mu} G_{R,2} - \frac{\alpha}{2} \xi \cdot (A\xi) \sqrt{\mu} G_{R,2} + \cdots, \]

and

\[ \partial_t G_{R,2} - \frac{\beta'}{\beta} \nabla_\xi \cdot (\xi G_{R,2}) - \alpha \nabla_\xi \cdot (A\xi G_{R,2}) + \beta^\gamma L G_{R,2} = \beta^\gamma (1 - \chi_M) \mu^{-\frac{1}{2}} K G_{R,1}. \]

The disadvantage of such a decomposition is that the large velocity behavior of the integration operator \( K \) is hard to be obtained. We have already settled this problem in \( L^\infty \) setting in our previous work [20] and [21] in the case of \( \gamma = 0 \) and \( \gamma \in (0, 1] \), respectively. Here the new difficulty stems from the time growth \( \beta^\gamma \) in \( \beta^\gamma (1 - \chi_M) \mu^{-\frac{1}{2}} K G_{R,1} \), which is caused by the hard potential kernel under the self-similar scaling \( \|L\|_{L^2} \). To overcome this difficulty, an \( L^2 \) estimate on the first component is developed, in particular, a new \( L^2 \) estimates on the collision operator \( K \) is proved with the aid of Riesz-Thorin’s interpolation inequality, cf. Lemma [21].

1.7. \textbf{Notations.} We list some notations and norms used in the paper. Throughout this paper, \( C \) denotes some generic positive (generally large) constant and \( \lambda \) denote some generic positive (generally small) constant. \( D \lesssim E \) means that there is a generic constant \( C > 0 \) such that \( D \leq CE \). \( D \sim E \) means \( D \lesssim E \) and \( E \lesssim D \). For multi-indices \( \vartheta = [\vartheta_1, \vartheta_2, \vartheta_3] \), we denote \( \partial_\vartheta = \partial_{\vartheta_1} \partial_{\vartheta_2} \partial_{\vartheta_3} \) and likewise for \( \partial_1 \vartheta \), and the length of \( \vartheta \) is denoted by \( |\vartheta| = \vartheta_1 + \vartheta_2 + \vartheta_3 \). \( \vartheta' \leq \vartheta \) means that no component of \( \vartheta' \) is greater than the component of \( \vartheta \), and \( \vartheta' < \vartheta \) means that \( \vartheta' \leq \vartheta \) and \( |\vartheta'| < |\vartheta| \). \( \langle \cdot, \cdot \rangle \) denotes the \( L^2 \) inner product in \( \mathbb{R}^3_\xi \), with the \( L^2 \) norm \( \| \cdot \| \).
1.8. Organization of the paper. The rest of this paper is arranged as follows. Section 2 is devoted to obtaining a crucial $L^2$ estimate for the integration operator $K$ given by (1.26). The local existence of Cauchy problem (1.23) and (1.24) is constructed in Section 3. The proof of Theorem 1.1 is given in Section 4. Finally, some basic estimates are collected in Appendix 5.

2. $L^2$ estimate for $K$ with large velocity

In this section we consider an $L^2$ estimate for $K$ defined by (1.26). The proof is based on the application of Riesz-Thorin’s interpolation inequality. For completeness, we first quote the following lemma, cf. [29, Theorem 1.3.4, pp.37].

**Lemma 2.1.** Let $(X_1, \mathcal{F}_1, \mathcal{M}_1)$ and $(X_2, \mathcal{F}_2, \mathcal{M}_2)$ be a finite measure spaces. Fix $1 \leq p_1, p_2, q_1, q_2 \leq \infty$ and $0 < \lambda < 1$. Define $p_\lambda$ and $q_\lambda$ by

$$p_\lambda = \frac{\lambda}{p_1} + \frac{1 - \lambda}{p_2}, \quad q_\lambda = \frac{\lambda}{q_1} + \frac{1 - \lambda}{q_2}.$$  

Let both $T_1$ and $T_2$ be continuous linear mappings such that

- $T_1 : L^{p_1}(\mathcal{M}_1) \to L^{q_1}(\mathcal{M}_2)$, with the operator norm $\|T_1\| = M_1$,
- $T_2 : L^{p_2}(\mathcal{M}_1) \to L^{q_2}(\mathcal{M}_2)$, with the operator norm $\|T_2\| = M_2$.

and $T_1 f = T_2 f$ for any $f \in L^{p_1}(\mathcal{M}_1) \cap L^{p_2}(\mathcal{M}_1)$. Here and below, we shorten $L^r(X, \mathcal{F}, \mathcal{M})$ to $L^r(\mathcal{M})$.

Thus we can define a mapping

$$T : L^{p_1}(\mathcal{M}_1) \cap L^{p_2}(\mathcal{M}_1) \to L^{q_1}(\mathcal{M}_2) \cap L^{q_2}(\mathcal{M}_2)$$

by

$$T f = T_1 f = T_2 f, \quad \text{for } f \in L^{p_1}(\mathcal{M}_1) \cap L^{p_2}(\mathcal{M}_1).$$

For each $f \in L^{p_1}(\mathcal{M}_1) \cap L^{p_2}(\mathcal{M}_1)$, it holds

$$\|T f\|_{L^{q_1}(\mathcal{M}_2) \cap L^{q_2}(\mathcal{M}_2)} \leq M_1^{1 - \lambda} M_2^{\lambda} \|f\|_{L^{p_1}(\mathcal{M}_1)}.$$  \hspace{1cm} (2.1)

Furthermore, $T$ has a unique continuous linear extension

$$T_\lambda : L^{p_\lambda}(\mathcal{M}_1) \to L^{q_\lambda}(\mathcal{M}_2)$$

with

$$\|T_\lambda\| \leq M_1^{1 - \lambda} M_2^\lambda.$$

With this lemma in hand, we now intend to prove the following $L^2$ estimate which plays an crucial role in the proof of global existence in Section 4. The estimate gives the smallness of $K$ at large velocities. We also consider the corresponding velocity derivative estimates.

**Proposition 2.1.** Let $0 < \gamma \leq 1$, then there is a constant $C > 0$ such that for suitably large $\ell_2 > 0$, there are sufficiently large $M = M(\ell_2) > 0$ and suitably small $\varsigma = \varsigma(\ell_2) > 0$ such that for any $\vartheta \geq 0$ it holds that

$$\|\nu^{-1/2} w^{1/2} \partial_\vartheta^\gamma (\chi_M K f)\| \leq C \{(1 + M)^{-\gamma} + \varsigma\}^{1/2} (1 + M)^{-\gamma/2} \sum_{\vartheta^j \leq \vartheta} \|\nu^{1/2} w^{1/2} \partial_\xi^\vartheta f\|, \hspace{1cm} (2.2)$$

where $\chi_M(\xi)$ is a non-negative smooth cutoff function such that

$$\chi_M(\xi) = \begin{cases} 1, & |\xi| \geq M + 1, \\ 0, & |\xi| \leq M. \end{cases}$$

**Proof.** We first consider the case that $\vartheta = 0$. From Lemma 5.6 it follows

$$\|\chi_M \nu^{-1} w^{2\ell_2} K f\|_{L^\infty} \leq C_1 (1 + M)^{-\gamma} \|w^{2\ell_2} f\|_{L^\infty}, \hspace{1cm} (2.3)$$

where $C_1 > 0$ and independent of $M$. Recalling Lemma 2.1 to prove (2.2) with $\vartheta = 0$, it suffices to prove the following

$$\|\nu^{-1} w^{2\ell_2} \chi_M K f\|_{L^1} \leq C_2 (1 + M)^{-\gamma} \|\nu w^{2\ell_2} f\|_{L^1}, \hspace{1cm} (2.4)$$

which we will do in the following section.
for some $C_2 > 0$ and independent of $M$. In view of (2.20), we have
\[
\int_{\mathbb{R}^3} \chi_M \nu^{-1} w^{2\ell_2} |f| \, d\xi \leq \int_{\mathbb{R}^3} \chi_M \nu^{-1} w^{2\ell_2} \int_{\mathbb{R}^3 \times S^2} B_0 |\xi - \xi'| \mu(\xi') |f(\xi')| \, d\xi, \, d\omega \, d\xi
\]
\[
+ \int_{\mathbb{R}^3} \chi_M \nu^{-1} w^{2\ell_2} \int_{\mathbb{R}^3 \times S^2} B_0 |\xi - \xi'| \mu(\xi') |f(\xi')| \, d\xi, \, d\omega \, d\xi
\]
\[
+ \int_{\mathbb{R}^3} \chi_M \nu^{-1} w^{2\ell_2} \int_{\mathbb{R}^3 \times S^2} B_0 |\xi - \xi'| \mu(\xi') |f(\xi')| \, d\xi, \, d\omega \, d\xi
\]
\[
= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,
\]
where $\mathcal{I}_i$ $(1 \leq i \leq 3)$ denote those terms on the right respectively. We now compute them individually. First of all, by choosing $M > 0$ sufficiently large, one sees that there exists a constant $C_0$ which is independent of $\ell_2$ such that
\[
\chi_M \nu w^{2\ell_2} \mu^{\frac{1}{2}} \leq C_0.
\]
(2.5)

For $\mathcal{I}_3$, by (2.20), we then have
\[
|\mathcal{I}_3| \leq C C_0 \nu \int_{\mathbb{R}^3} \chi_M \mu^{1/2}(\xi) d\xi \int_{\mathbb{R}^3} |\xi|^\gamma |(w^{-2\ell_2} w^{2\ell_2} f)(\xi)| d\xi \leq \frac{C}{(1 + M)^{\gamma}} \|\nu w^{2\ell_2} f\|_{L^1}.
\]
For $\mathcal{I}_1$, if $|\xi'| < \frac{1}{\sqrt{\ell_2}} |\xi|$, on the one hand, it follows $|\xi|^2 \geq |\xi|^2 - |\xi'|^2 \geq (1 - \ell_2^{-1}) |\xi|^2$, which gives
\[
\frac{w^{2\ell_2} (\xi)}{w^{2\ell_2} (\xi')} \leq \left( \frac{1}{1 - \ell_2^{-1}} \right)^{2\ell_2} \leq \epsilon^2.
\]
On the other hand, it follows
\[
|\xi - \xi'|^\gamma \leq (\xi'^\gamma)^{\gamma} (\xi)^{\gamma}.
\]
Thus, one has
\[
|\mathcal{I}_1| \leq C \int_{\mathbb{R}^3} \chi_M \nu^{-1} (\xi) \frac{w^{2\ell_2} (\xi)}{w^{2\ell_2} (\xi')} (\xi'^\gamma)^{\gamma} (\xi)^{\gamma} \mu(\xi') w^{2\ell_2} (\xi') |f(\xi')| d\omega d\xi d\xi d\omega \leq \frac{C}{(1 + M)^{\gamma}} \|\nu w^{2\ell_2} f\|_{L^1},
\]
where a change of variables $(\xi, \xi) \to (\xi', \xi')$ has been used.

If $|\xi'| \geq \frac{1}{\sqrt{\ell_2}} |\xi|$, one has by applying (2.5) and a change of variables $(\xi, \xi) \to (\xi', \xi')$ that
\[
|\mathcal{I}_1| \leq C \int_{\mathbb{R}^3} \chi_M \nu^{-1} (w^{2\ell_2} \mu^{\frac{1}{2}}) (\xi) d\xi \int_{\mathbb{R}^3} (\xi'^\gamma)^{\gamma} (\xi)^{\gamma} \mu^{\frac{1}{2}} (\xi') |f(\xi)| d\xi d\omega \leq \frac{C}{(1 + M)^{\gamma}} \|\nu w^{2\ell_2} f\|_{L^1}.
\]
Similarly, for $\mathcal{I}_2$, if $|\xi'| < \frac{1}{\sqrt{\ell_2}} |\xi|$, one gets $|\xi'|^2 \geq |\xi|^2 - |\xi'|^2 \geq (1 - \ell_2^{-1}) |\xi|^2$, which further implies
\[
\frac{w^{2\ell_2} (\xi)}{w^{2\ell_2} (\xi')} \leq \left( \frac{1}{1 - \ell_2^{-1}} \right)^{2\ell_2} \leq \epsilon^2.
\]
Thus, it follows
\[
|\mathcal{I}_2| \leq C \int_{\mathbb{R}^3} \chi_M \nu^{-1} (\xi) \frac{w^{2\ell_2} (\xi)}{w^{2\ell_2} (\xi')} (\xi'^\gamma)^{\gamma} (\xi)^{\gamma} \mu(\xi') w^{2\ell_2} (\xi') |f(\xi')| d\xi, d\omega \, d\xi \leq \frac{C}{(1 + M)^{\gamma}} \|\nu w^{2\ell_2} f\|_{L^1}.
\]
If $|\xi|^n \geq \frac{1}{\sqrt{2}} |\xi|$, similarly for obtaining (2.6), one has

$$|I_2| \leq C \int_{\mathbb{R}^3} \chi_M \nu^{-1} (\nu w^{2/2} \frac{1}{\sqrt{2}}) (\xi) d\xi \int_{\mathbb{R}^3 \times S^2} \langle \xi \rangle^{\gamma} \langle \xi \rangle^{\gamma} \mu^2 (\xi') |f(\xi)| d\xi_2 d\omega$$

$$\leq \frac{C}{(1 + M) \gamma} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \langle \xi \rangle^{\gamma} \langle \xi \rangle^{\gamma} \mu^2 (\xi) |f(\xi)| d\xi_2 d\xi \leq \frac{C}{(1 + M) \gamma} \|\nu w^{2/2} f\|_L^\infty.$$

Combining the above estimates together, we then see that (2.4) is true. Next, we define linear operators

$$T = T_1 = T_2 = \chi_M K : L^1(\nu w^{2/2}) \cap L^\infty(\nu w^{2/2}) \to L^1(\nu^{-1} w^{2/2}) \cap L^\infty(\nu^{-1} w^{2/2})$$

$$f \mapsto \chi_M K f,$$

where $L^p(W(\xi)) \ (p \in [1, +\infty])$ is a Lebesgue space with weighted measure $d\mathcal{M} = W d\xi$. Therefore (2.2) with $\vartheta = 0$ follows from (2.4) and (2.8), as well as (2.1).

We now turn to show that (2.2) is also true for $\vartheta > 0$. In this case, we first have by a change of variables $\xi = \xi - \xi \to u$ that

$$\partial^\vartheta_\xi (\chi_M K f) = \sum_{\vartheta^\prime + \vartheta^\prime \leq \vartheta} C^\vartheta_\vartheta (\partial^\vartheta_\xi \partial^\vartheta_\xi \chi_M) \int_{\mathbb{R}^3 \times S^2} \partial^\vartheta_\xi |\partial^\vartheta_\xi | \mu (\xi + u_\perp) (\partial^\vartheta_\xi f) (\xi + u_\|) d\omega d\xi$$

$$+ \sum_{\vartheta^\prime + \vartheta^\prime \leq \vartheta} C^\vartheta_\vartheta (\partial^\vartheta_\xi \partial^\vartheta_\xi \chi_M) \int_{\mathbb{R}^3 \times S^2} \partial^\vartheta_\xi |\partial^\vartheta_\xi | \mu (\xi + u_\perp) (\partial^\vartheta_\xi f) (\xi + u_\|) d\omega d\xi$$

$$+ \sum_{\vartheta^\prime + \vartheta^\prime \leq \vartheta} C^\vartheta_\vartheta (\partial^\vartheta_\xi \partial^\vartheta_\xi \chi_M) \int_{\mathbb{R}^3 \times S^2} \partial^\vartheta_\xi |\partial^\vartheta_\xi | \mu (\xi + u_\perp) (\partial^\vartheta_\xi f) (u + \xi) d\omega d\xi, \quad (2.7)$$

where we have also used the notations $u_\perp = (u \cdot \omega) \omega$ and $u_\perp = u - u_\perp$. Then, changing back to the original variables, one has

$$\partial^\vartheta_\xi (\chi_M K f) = \sum_{\vartheta^\prime + \vartheta^\prime \leq \vartheta} C^\vartheta_\vartheta (\partial^\vartheta_\xi \partial^\vartheta_\xi \chi_M) \int_{\mathbb{R}^3 \times S^2} \partial^\vartheta_\xi |\partial^\vartheta_\xi | \mu (\xi) (\partial^\vartheta_\xi f) (\xi) d\omega d\xi$$

$$- \sum_{\vartheta^\prime + \vartheta^\prime \leq \vartheta} C^\vartheta_\vartheta (\partial^\vartheta_\xi \partial^\vartheta_\xi \chi_M) \int_{\mathbb{R}^3 \times S^2} \partial^\vartheta_\xi |\partial^\vartheta_\xi | \mu (\xi) (\partial^\vartheta_\xi f) (\xi) d\omega d\xi$$

$$+ \sum_{\vartheta^\prime + \vartheta^\prime \leq \vartheta} C^\vartheta_\vartheta (\partial^\vartheta_\xi \partial^\vartheta_\xi \chi_M) \int_{\mathbb{R}^3 \times S^2} \partial^\vartheta_\xi |\partial^\vartheta_\xi | \mu (\xi) (\partial^\vartheta_\xi f) (\xi) d\omega d\xi. \quad (2.8)$$

We now define the following linear continuous operators

$$T_{\vartheta, 1} = T_{\vartheta, 2} : L^1(\nu w^{2/2}) \cap L^\infty(\nu w^{2/2}) \to L^1(\nu^{-1} w^{2/2}) \cap L^\infty(\nu^{-1} w^{2/2})$$

with

$$T_{\vartheta, 1} g = T_{\vartheta, 2} g = \sum_{\vartheta^\prime + \vartheta^\prime \leq \vartheta} C^\vartheta_\vartheta (\partial^\vartheta_\xi \partial^\vartheta_\xi \chi_M) \int_{\mathbb{R}^3 \times S^2} \partial^\vartheta_\xi |\partial^\vartheta_\xi | \mu (\xi) g(\xi) d\omega d\xi$$

$$+ \sum_{\vartheta^\prime + \vartheta^\prime \leq \vartheta} C^\vartheta_\vartheta (\partial^\vartheta_\xi \partial^\vartheta_\xi \chi_M) \int_{\mathbb{R}^3 \times S^2} \partial^\vartheta_\xi |\partial^\vartheta_\xi | \mu (\xi) g(\xi) d\omega d\xi$$

$$+ \sum_{\vartheta^\prime + \vartheta^\prime \leq \vartheta} C^\vartheta_\vartheta (\partial^\vartheta_\xi \partial^\vartheta_\xi \chi_M) \int_{\mathbb{R}^3 \times S^2} \partial^\vartheta_\xi |\partial^\vartheta_\xi | \mu (\xi) g(\xi) d\omega d\xi,$$

for any given $\vartheta \in \mathbb{Z}^d_+$. Because $|\partial^\vartheta_\xi \mu | \leq C_{\vartheta^\prime} \mu^\vartheta$, one can also deduce by performing the similar calculations as for obtaining (2.8) and (2.4) that

$$\|\nu^{-1} w^{2/2} T_{\vartheta, 2} (\partial^\vartheta_\xi f)\|_L^\infty \leq \tilde{C}_1 (1 + M)^{-\gamma} + \gamma \|\nu w^{2/2} \partial^\vartheta_\xi f\|_L^\infty, \quad (2.9)$$
and
\[ \| \nu^{-1} w^{2\ell_2} T_{\vartheta}(\partial^\rho f) \|_{L^1} \leq \frac{\hat{C}_2}{(1+M)^\gamma} \| \nu w^{2\ell_2} \partial^\rho f \|_{L^1}. \] (2.10)

Then, (2.3) together with (2.10) further gives
\[ \| \nu^{-1/2} w^{2\ell_2} T_{\vartheta}(\partial^\rho f) \| \leq \sqrt{\hat{C}_1 \hat{C}_2} \left\{ (1+M)^{-\gamma + \zeta} (1+M)^{-\gamma/2} \| \nu^{1/2} w^{2\ell_2} \partial^\rho f \| \right\}, \] (2.11)
according to Lemma 2.1. From (2.11) and (2.8), we see that (2.2) is also valid in the case \( \vartheta > 0 \). This ends the proof of Proposition 2.1. \( \square \)

3. Local existence

The goal of this section is to construct the local existence of the remainder problem (1.23) and (1.24) in the Sobolev space \( W^{N, \infty} \) for an arbitrary positive integer \( N \).

Since \( G_1 \) and \( G_2 \) are already given by (1.27) and (1.30), to solve (1.1) and (1.5) it suffices to determine \( G_R \) by the Cauchy problem (1.23) and (1.24). Here, we also recall that \( \beta = \beta(t) \) is determined by solving (1.19) or equivalently (1.18) in terms of \( G_R \). To do this, one crucial idea behind the proof is to split \( G_R \) as \( \sqrt{\mu} G_{R,1} + \sqrt{\mu} G_{R,2} \), where \( G_{R,1} \) and \( G_{R,2} \) satisfy
\[
\partial_t G_{R,1} - \frac{\beta'}{\beta} \nabla \cdot (\xi G_{R,1}) - \alpha \nabla \cdot (A\xi G_{R,1}) + \beta^\gamma \nu G_{R,1}
\]
\[
= \beta^\gamma \chi_M K G_{R,1} - \frac{\beta'}{2\beta} |\xi|^2 \sqrt{\mu} G_{R,2} - \alpha \frac{1}{2} \cdot (A\xi) \sqrt{\mu} G_{R,2} - \alpha^{1-m} \sqrt{\mu} \partial_t G_1 - \alpha^{2-m} \sqrt{\mu} \partial_t G_2
\]
\[
+ \alpha^{3-m} \beta_1 \nabla \cdot (\xi \mu) + \alpha^{1-m} \beta' \nabla \cdot (\xi \sqrt{\mu} G_1) + \alpha^{2-m} \beta' \nabla \cdot (\xi \sqrt{\mu} G_2) + \alpha^{3-m} \nabla \cdot (A\xi \sqrt{\mu} G_2) + \alpha^{3-m} \beta^\gamma Q(\sqrt{\mu} G_1, \sqrt{\mu} G_2)
\]
\[
+ \alpha \beta^\gamma Q(\sqrt{\mu} G_1 + \alpha \sqrt{\mu} G_2, \sqrt{\mu} G_1) + Q(\sqrt{\mu} G_R, \sqrt{\mu} G_1 + \alpha \sqrt{\mu} G_2)
\]
\[
+ \alpha^m \beta^\gamma Q(\sqrt{\mu} G_R, \sqrt{\mu} G_R),
\] (3.1)

\[ G_{R,1}(0, \xi) = G_{R,0}, \]

\[ \partial_t G_{R,2} - \frac{\beta'}{\beta} \nabla \cdot (\xi G_{R,2}) - \alpha \nabla \cdot (A\xi G_{R,2}) + \beta^\gamma L G_{R,2} = \beta^\gamma (1 - \chi_M) \mu^{-\frac{1}{2}} K G_{R,1}, \] (3.2)

and
\[ G_{R,2}(0, \xi) = 0, \]

respectively.

**Theorem 3.1** (Local existence). Let \( \gamma \in (0, 1], \ m \in (2, 3), \ \ell_\infty \gg 5 \) and assume \( \text{tr} A = 0 \). There exists a constant \( M_0 > 0 \) and a suitably small constant \( \alpha_0 > 0 \) such that if \( \alpha \in (0, \alpha_0) \) and
\[ \sum_{|\lambda| \leq N} \| w^{\ell_\infty} \partial^\lambda G_{R,0} \|_{L^\infty} \leq M_0, \] (3.3)

for an integer \( N \geq 1 \), then there exists \( T_0 > 0 \) which may depend on \( \alpha \) and \( M_0 \) such that (1.23) and (1.26) admits a unique local solution \( G_R(t, \xi) \) satisfying \( \sqrt{\mu} G_R = G_{R,1} + \sqrt{\mu} G_{R,2} \) with the estimate
\[
\sup_{0 \leq t \leq T_0} \sum_{|\lambda| \leq N} \left\{ \| w^{\ell_\infty} \beta^{2+\lambda} \partial^\lambda G_{R,1} \|_{L^\infty} + \| w^{\ell_\infty} \beta^{2+\lambda} \partial^\lambda G_{R,2} \|_{L^\infty} \right\} \leq 2M_0,
\]

where
\[ \beta^\gamma(t) \sim 1 + \gamma \beta_0 \alpha^2 t. \]
Proof. Our proof is based on the Duhamel’s principle and contraction mapping method. It is convenient to look for the weighted form \([g_1, g_2](t, \xi) = \beta^{2\gamma}(t)[G_{R,1}(t, \xi), G_{R,2}(t, \xi)].\) For the purpose, we consider the following linear inhomogeneous equations for the unknown \([g_1, g_2](t, \xi)\):

\[
\partial_t g_1 - \frac{(2\gamma + 3)\beta'}{\beta} \cdot \nabla g_1 = \frac{(2\gamma + 3)\beta'}{\beta} g_1 - \alpha A\xi \cdot \nabla g_1 + \nu \beta' g_1
\]

\[
= \beta' \chi_M K h_1 - \frac{(2\gamma + 3)\beta'}{\beta} \frac{\alpha}{2} (A\xi) \sqrt{\mu h_2} - \alpha^2 \frac{\beta}{2} \beta' (A\xi) \sqrt{\mu h_2} - \alpha - m \beta^2 \sqrt{\mu h_1} - \alpha^2 m \beta^2 \sqrt{\mu h_2}
\]

\[
+ \alpha^3 \beta^2 \beta_1 \nabla \xi \cdot (A\xi) + \alpha^3 \beta^2 \beta_1 \nabla \xi \cdot (A\xi) + \alpha^3 \beta^2 \beta_1 \nabla \xi \cdot (A\xi)
\]

\[
+ \frac{\alpha}{3} \int_{\mathbb{R}^3} \xi \cdot A\xi H d\xi, \quad \beta' = \frac{d\beta}{dt}, \quad \beta(0) = 1,
\]

\[
\beta_1 = \int_{\mathbb{R}^3} \xi \cdot A\xi \sqrt{\mu h_2} d\xi - \frac{\alpha}{3} \int_{\mathbb{R}^3} \xi \cdot A\xi \sqrt{\mu h_1} d\xi = \varrho_1 \beta^{-2\gamma} + \varrho_R \beta^{-2\gamma},
\]

with

\[
\varrho_1 = -\frac{\beta^{2\gamma}}{3} \int_{\mathbb{R}^3} \xi \cdot A\xi \sqrt{\mu h_2} d\xi
\]

\[
= -\frac{1}{3} \int_{\mathbb{R}^3} \xi \cdot A\xi \sqrt{\mu h_2} \left\{ L^{-1} \left\{ \Gamma(L^{-1} \{ \xi \cdot A\xi \mu^2 \}, L^{-1} \{ \xi \cdot A\xi \mu^2 \}) + \varrho_0 \nabla \xi \cdot (A\xi) \right\} - \nabla \xi \cdot (A\xi \mu L^{-1} \{ \xi \cdot A\xi \mu^2 \} \mu^2) \right\} d\xi,
\]

\[
\varrho_R = \beta_1 \beta^{-2\gamma}.
\]
and

\[ q_R = -\frac{\tilde{\beta}^2 \alpha^{m-2}}{3} \int_{\mathbb{R}^3} \xi \cdot A\xi \sqrt{\mu_R} d\xi. \]

Consequently, (3.11) and (3.14) give

\[ \frac{d\tilde{\gamma}}{dt} = \gamma \left\{ \phi_0 \alpha^2 + \tilde{\phi}_1 \alpha^3 \tilde{\beta}^{-\gamma} + q_R \alpha^3 \tilde{\beta}^{-\gamma} \right\}. \] (3.15)

It should be pointed out that both \( q_0 \) and \( q_1 \) are independent of \( \tilde{\beta} \).

Let \([g_1, g_2]\) be a solution of the coupled problems (3.4), (3.5) and (3.6), (3.7) with \([h_1, h_2]\) being given. Then the solution operator \( \mathcal{N} \) is formally defined as

\[ [g_1, g_2] = \mathcal{N}([h_1, h_2]). \]

We aim to prove that there exists a sufficiently small \( T_0 > 0 \) such that the solution mapping \( \mathcal{N}([\cdot, \cdot]) \) has a unique fixed point in some Banach space by adopting the contraction mapping method. In fact, from (3.3) and (3.7), one has

\[ \sum_{|\theta| \leq N} \| w^f \partial_\xi^\theta g_1(0, \xi) \|_{L^\infty} \leq M_0. \]

Thus we can define the following Banach space

\[ \mathbf{Y}_{\alpha,T} = \left\{ (\mathcal{G}_1, \mathcal{G}_2) \middle| \sup_{0 \leq t \leq T} \sum_{|\theta| \leq N} \left\{ \| w^f \partial_\xi^\theta \mathcal{G}_1(t) \|_{L^\infty} + \| w^f \partial_\xi^\theta \mathcal{G}_2(t) \|_{L^\infty} \right\} \leq 2M_0, \right\} \]

\[ \mathcal{G}_1(0) = G_{R,0}, \mathcal{G}_2(0) = 0, \langle \mathcal{G}_1, [1, \xi, \frac{1}{2} |\xi|^2] \rangle + \langle \mathcal{G}_2, [1, \xi, \frac{1}{2} |\xi|^2 \mu^2] \rangle = 0, \]

associated with the norm

\[ \| (\mathcal{G}_1, \mathcal{G}_2) \|_{\mathbf{Y}_{\alpha,T}} = \sup_{0 \leq t \leq T} \sum_{|\theta| \leq N} \left\{ \| w^f \partial_\xi^\theta \mathcal{G}_1(t) \|_{L^\infty} + \| w^f \partial_\xi^\theta \mathcal{G}_2(t) \|_{L^\infty} \right\}. \]

We now show that

\[ \mathcal{N} : \mathbf{Y}_{\alpha,T} \rightarrow \mathbf{Y}_{\alpha,T}, \]

is well-defined and \( \mathcal{N} \) is a contraction mapping for some \( T > 0 \). To do this, we start from the following approximation equations

\[ \partial_t (w^f \partial_\xi^\theta g_1) - \frac{\tilde{\beta}^2}{\tilde{\beta}} \xi \cdot \nabla \xi (w^f \partial_\xi^\theta g_1) + \frac{2\gamma^2 + 3}{3} \tilde{\beta} \frac{\alpha \xi}{1 + |\xi|^2} w^f \partial_\xi^\theta g_1 + \frac{2\gamma^2 + 3}{3} \tilde{\beta} \frac{\alpha \xi}{1 + |\xi|^2} w^f \partial_\xi^\theta g_1 \]

\[ = \sum_{\theta' \leq \theta} C_{\theta'}^\theta w^f \tilde{\beta} \gamma \partial_\xi^\theta (\chi M K) \partial_\xi^{\theta - \theta'} h_1 + \frac{\tilde{\beta}}{\beta} \sum_{|\theta'| = 1} C_{\theta'}^\theta w^f \partial_\xi^{\theta - \theta'} h_1 + \alpha 1_{|\theta'| > 0} \sum_{|\theta'| = 1} C_{\theta'}^\theta w^f \partial_\xi^{\theta - \theta'} h_1 \]

\[ + \alpha 1_{|\theta'| = 0} \sum_{|\theta'| = 1} C_{\theta'}^\theta w^f \partial_\xi^{\theta - \theta'} (A\xi) \cdot \nabla \xi \partial_\xi^{\theta - \theta'} h_1 - 1_{|\theta'| > 0} C_{\theta'}^\theta \tilde{\beta} \gamma \partial_\xi^{\theta - \theta'} \nu w^f \partial_\xi^{\theta - \theta'} h_1 - \frac{\tilde{\beta}}{\beta} \frac{\alpha \xi}{1 + |\xi|^2} w^f \partial_\xi^{\theta} \{ |\xi|^2 \sqrt{\mu h_2} \} \]

\[ - \frac{\alpha}{2} w^f \partial_\xi^{\theta} \{ (A\xi) \sqrt{\mu h_2} \} - \alpha^{-1} w^f \partial_\xi^{\theta} \{ \sqrt{\mu h_1} \partial_t H_1 \} - \alpha^{2-m} \tilde{\beta} \gamma w^f \partial_\xi^{\theta} \{ \sqrt{\mu} \partial_t H_2 \} \]

\[ + \alpha^{2-m} w^f \tilde{\beta} \gamma \partial_\xi^{\theta} \{ \nabla \xi \cdot \nabla (\xi \mu) \} + \alpha^{2-m} \tilde{\beta} \gamma w^f \partial_\xi^{\theta} \{ Q(\sqrt{\mu G_1}, \sqrt{\mu G_2}) + Q(\sqrt{G_2}, \sqrt{G_1}) \} \]

\[ + \alpha^{2-m} \tilde{\beta} \gamma w^f \partial_\xi^{\theta} \{ Q(\sqrt{G_1}, \sqrt{G_2}) + Q(\sqrt{\mu G_2}, \sqrt{G_1}) \}, \]

\[ g_1(0, \xi) = G_{R,0}. \] (3.16)
By this, we can write the solution of (3.16), (3.17), (3.18) and (3.19) as

\[ s, V \]

Next, we define the characteristic line \([s, V(s; t, \xi)]\) for equations (3.16) and (3.18) going through \((t, \xi)\) such that

\[
\begin{align*}
\frac{dV(s; t, \xi)}{ds} &= -\frac{\beta'(s)}{\beta(s)} V(s; t, \xi) - \alpha V(s; t, \xi), \\
V(t; t, \xi) &= \xi,
\end{align*}
\]

which can be solved as

\[
V(s) := V(s; t, \xi) = \frac{\tilde{\beta}(t)}{\beta(s)} e^{-(s-t)\alpha A \xi}.
\]

By this, we can write the solution of (3.16), (3.17), (3.18) and (3.19) as

\[
[w^t \omega_{\xi}^2 g_1, w^t \omega_{\xi}^2 g_2] = Q_{\theta}(h_1, h_2) = \{Q_{1, \theta}(h_1, h_2), Q_{2, \theta}(h_1, h_2)\},
\]

where

\[
Q_{1, \theta}(h_1, h_2) = \sum_{i=1}^{10} \mathcal{J}_i, \quad Q_{2, \theta}(h_1, h_2) = \sum_{i=11}^{15} \mathcal{J}_i,
\]

with

\[
\mathcal{J}_1 = e^{-\int_0^t \tilde{\alpha}(s) ds} w^t \omega_{\xi}^2 G_{R,0}(V(0)),
\]

\[
\mathcal{J}_2 = 1_{\theta > 0} \sum_{|\theta'| = 1} C_{\theta}^{\theta'} \int_0^t e^{-\int_0^t \tilde{\alpha}(r) dr} \left\{ \frac{\beta'}{\beta} w^t \omega_{\xi}^2 \xi \cdot \nabla \omega_{\xi}^{\theta' \theta} h_1 \right\} (s, V(s)) ds,
\]

\[
\mathcal{J}_3 = \alpha 1_{\theta > 0} \sum_{|\theta'| = 1} C_{\theta}^{\theta'} \int_0^t e^{-\int_0^t \tilde{\alpha}(r) dr} \left\{ w^t \omega_{\xi}^2 (A \xi) \cdot \nabla \omega_{\xi}^{\theta' \theta} h_1 \right\} (s, V(s)) ds,
\]

\[
\mathcal{J}_4 = -1_{\theta > 0} \sum_{0 < \theta' \leq \theta} C_{\theta}^{\theta'} \int_0^t e^{-\int_0^t \tilde{\alpha}(r) dr} \left\{ \tilde{\beta} w^t \omega_{\xi}^2 \nu \omega_{\xi}^{\theta' \theta} h_1 \right\} (s, V(s)) ds,
\]

\[
\mathcal{J}_5 = \sum_{\theta' \leq \theta} C_{\theta}^{\theta'} \int_0^t e^{-\int_0^t \tilde{\alpha}(r) dr} \left\{ \tilde{\beta} \nu w^t \omega_{\xi}^2 (A_M K) \omega_{\xi}^{\theta' \theta} h_1 \right\} (s, V(s)) ds,
\]

\[
\mathcal{J}_6 = -\int_0^t e^{\int_0^t \tilde{\alpha}(r) dr} \left\{ \frac{\beta'}{2\beta} w^t \omega_{\xi}^2 (|\xi|^2 \sqrt{\mu} H_2) + \frac{\alpha}{2} w^t \omega_{\xi}^2 (\xi \cdot A \xi \sqrt{\mu} h_2) \right\} (s, V(s)) ds,
\]

\[
\mathcal{J}_7 = -\int_0^t e^{\int_0^t \tilde{\alpha}(r) dr} \left\{ -\alpha^{-1-m} \tilde{\beta}^{2\gamma} w^t \omega_{\xi}^2 \{ \sqrt{\mu} \partial_t H_1 \} - \alpha^{2-m} \tilde{\beta}^{2\gamma} w^t \omega_{\xi}^2 \{ \sqrt{\mu} \partial_t H_2 \} + \alpha^{3-m} \tilde{\beta}^{2\gamma} \tilde{\beta} w^t \omega_{\xi}^2 \{ \nabla \xi \cdot (\xi h) \} + \alpha^{-1-m} \tilde{\beta}^{2\gamma-1} \tilde{\beta} w^t \omega_{\xi}^2 \{ \nabla \xi \cdot (\xi \sqrt{\mu} H_1) \} \right\} (s, V(s)) ds,
\]
\[
\mathcal{J}_8 = -\int_0^t e^{f_s^x A(\tau)} d\tau \left\{ \alpha^{2-m} \beta^{2\gamma-1} \tilde{\beta}^{\rho} w^\rho \partial_\xi^\rho \left\{ \nabla_\xi \cdot (\xi \sqrt{\mu} H_2) \right\} 
+ \alpha^{2-m} \beta^{2\gamma} w^\rho \partial_\xi^\rho \left\{ \nabla_\xi \cdot (A \xi \sqrt{\mu} H_2) \right\} \right\} (s, V(s)) ds,
\]

\[
\mathcal{J}_9 = \int_0^t e^{f_s^x A(\tau)} d\tau \left\{ \alpha^{3-m} \beta^{3\gamma} w^\rho \partial_\xi^\rho \left\{ Q(\sqrt{\mu} H_1, \sqrt{\mu} H_2) + Q(\sqrt{\mu} H_2, \sqrt{\mu} H_1) \right\} 
+ \alpha^{3-m} \beta^{3\gamma} w^\rho \partial_\xi^\rho Q(\sqrt{\mu} H_2, \sqrt{\mu} H_2) \right\} (s, V(s)) ds,
\]

\[
\mathcal{J}_{10} = \int_0^t e^{f_s^x A(\tau)} d\tau \left\{ \alpha^{m} \beta^{\gamma} w^\rho \partial_\xi^\rho \left\{ \beta^{\rho} w^\rho \partial_\xi^\rho (A \xi) \cdot \nabla_\xi \partial_\xi^\rho h_2 \right\} \right\} (s, V(s)) ds,
\]

\[
\mathcal{J}_{11} = 1_{|\vartheta| > 0} \sum_{|\varrho|=1} C_0^\varrho \int_0^t e^{f_s^x A(\tau)} d\tau \left\{ \beta^{\rho} w^\rho \partial_\xi^\rho \left\{ \beta^{\rho} w^\rho \partial_\xi^\rho h_2 \right\} \right\} (s, V(s)) ds,
\]

\[
\mathcal{J}_{12} = \alpha 1_{|\vartheta| > 0} \sum_{|\varrho|=1} C_0^\varrho \int_0^t e^{f_s^x A(\tau)} d\tau \left\{ \beta^{\rho} w^\rho \partial_\xi^\rho \left\{ \beta^{\rho} w^\rho \partial_\xi^\rho h_2 \right\} \right\} (s, V(s)) ds,
\]

\[
\mathcal{J}_{13} = -1_{|\varrho| > 0} \sum_{0 < w \leq \vartheta} \sum_{|\varrho|=1} C_0^\varrho \int_0^t e^{f_s^x A(\tau)} d\tau \left\{ \beta^{\rho} w^\rho \partial_\xi^\rho \left\{ \beta^{\rho} w^\rho \partial_\xi^\rho h_2 \right\} \right\} (s, V(s)) ds,
\]

\[
\mathcal{J}_{14} = \int_0^t e^{f_s^x A(\tau)} d\tau \left\{ \beta^{\gamma} w^\rho \partial_\xi^\rho (K h_2) \right\} (s, V(s)) ds,
\]

and

\[
\mathcal{J}_{15} = \int_0^t e^{f_s^x A(\tau)} d\tau \left\{ \beta^{\gamma} w^\rho \partial_\xi^\rho (1 - \chi_M) \mu^{-\delta} K h_1 \right\} (s, V(s)) ds,
\]

where we have denoted

\[
\tilde{A}(\tau, V(\tau)) = \tilde{\beta}^{\gamma} \nu(V(\tau)) - \frac{(2\gamma + 3)\beta^{\rho}}{\beta} + 2\epsilon \beta^{\rho} \frac{|V(\tau)|^2}{1 + |V(\tau)|^2} + 2\epsilon \alpha (V(\tau) \cdot (AV(\tau))).
\]

Before computing \( \mathcal{J}_i \) \((1 \leq i \leq 15)\), we first prove the following estimates.

**Lemma 3.1.** Let \([h_1, h_2] \in \mathbb{Y}_{\alpha, T}\) with \(0 \leq T \leq +\infty\). For any \(\ell \geq 0\), \(k \in \mathbb{Z}^+\), and \(N \in \mathbb{Z}^+\), it holds that

\[
\sum_{|\varrho| \leq N+1} \| w^\rho \partial_\xi^\rho H_1(t, \xi) \|_{L^\infty} \leq C_1 \alpha^{2k} \beta^{-\gamma-k\gamma},
\]

and

\[
\sum_{|\varrho| \leq N} \| w^\rho \partial_\xi^\rho H_2(t, \xi) \|_{L^\infty} \leq C_2 \alpha^{2k} \beta^{-2\gamma-k\gamma},
\]

where both \(C_1\) and \(C_2\) depend on \(\ell, k\) and \(N\). Moreover, if \(\tilde{\beta}(0) = 1\), it holds that

\[
\tilde{\beta}^{\gamma}(t) \sim 1 + \gamma_0 \alpha^2 t,
\]

for any \(0 \leq t \leq T\).
Proof. In light of (3.9) and \( \text{tr} A = 0 \), by using the same argument as for obtaining (3.28) below, one can show that for any \( \vartheta \in \mathbb{Z}_+^3 \) and \( \ell \geq 0 \)

\[
\|w^t \partial^\vartheta_\xi H_1(t, \xi)\|_{L^\infty} \leq C \tilde{\beta}^{-\gamma},
\]

where \( C > 0 \) depends on \( \vartheta \) and \( \ell \).

For \( H_2 \), we intend to prove

\[
\sum_{|\vartheta| \leq N} \|w^t \partial^\vartheta_\xi H_2(t, \xi)\|_{L^\infty} \leq C \tilde{\beta}^{-2\gamma}
\]

for any \( N \geq 0 \). To do this, we first get from (3.10) that

\[
w^t \partial^\vartheta_\xi H_2 = w^t K \partial^\vartheta_\xi H_2 + 1_{\vartheta > 0} \sum_{0 < \vartheta' + \vartheta'' \leq \vartheta} C^{\vartheta', \vartheta''} w^t \partial^\vartheta_\xi (\nu^{-1})(\partial^{\vartheta'}_\xi K) \partial^{\vartheta''} H_2 
\]

\[
+ \sum_{\vartheta' \leq \vartheta} C^{\vartheta', \vartheta''} w^t \tilde{\beta}^{-\gamma} \partial^\vartheta_\xi (\nu^{-1}) \partial^{\vartheta'}_\xi \left\{ \beta_0 \nabla \xi \cdot (\xi \mu) \mu^{-\frac{1}{2}} \right\} 
\]

\[
+ \sum_{\vartheta' \leq \vartheta} C^{\vartheta', \vartheta''} w^t \tilde{\beta}^{-\gamma} \partial^\vartheta_\xi (\nu^{-1}) \partial^{\vartheta'}_\xi \left\{ \nabla \xi \cdot (A \xi \sqrt{\tau} H_1) \mu^{-\frac{1}{2}} \right\}.
\]

By Lemma 5.2, (3.27) and (3.13), one further has

\[
|w^t \partial^\vartheta_\xi H_2| \leq C|w^t K \partial^\vartheta_\xi H_2| + C1_{\vartheta > 0} \sum_{\vartheta' \leq \vartheta} \|w^t \partial^\vartheta_\xi H_2\|_{L^\infty} + C \tilde{\beta}^{-2\gamma}.
\]

To handle the first term on the right hand side of (3.29), we rewrite

\[
w^t K \partial^\vartheta_\xi H_2 = \int_{\mathbb{R}^3} k_w(\xi, \xi_s) w^t \partial^\vartheta_\xi H_2(t, \xi_s) d\xi_s := \mathcal{I}_0,
\]

and then divide our computations in the following three cases.

**Case 1.** \( |\xi| \geq M \) with \( M > 0 \) suitably large. From Lemma 5.1 it follows that

\[
\int_{\mathbb{R}^3} k_w(\xi, \xi_s) d\xi_s \leq \frac{C}{1 + |\xi|} \leq \frac{C}{M}.
\]

Using this, one has

\[
|\mathcal{I}_0| \leq \int_{\mathbb{R}^3} k_w(\xi, \xi_s) d\xi_s \|w^t \partial^\vartheta_\xi H_2\|_{L^\infty} \leq \frac{C}{M} \|\partial^\vartheta_\xi H_2\|_{L^\infty}.
\]

**Case 2.** \( |\xi| \leq M \) and \( |\xi_s| \geq 2M \). In this situation, we have \( |\xi - \xi_s| \geq M \), then

\[
k_w(\xi, \xi_s) \leq Ce^{-\frac{M^2}{8}} k_w(\xi, \xi_s) e^{\frac{|\xi - \xi_s|^2}{8}}.
\]

In light of Lemma 5.1 one sees that \( \int k_w(\xi, \xi_s) e^{\frac{|\xi - \xi_s|^2}{8}} d\xi_s \) is still bounded. Thus, a similar computation as for obtaining (3.30) yields

\[
|\mathcal{I}_0| \leq Ce^{-\frac{M^2}{8}} \|w^t \partial^\vartheta_\xi H_2\|_{L^\infty}.
\]

**Case 3.** \( |\xi| \leq M \) and \( |\xi_s| \leq 2M \). In this case, we convert the bound in \( L^\infty \)-norm to the one in \( L^2 \)-norm which will be established later on. To do so, for any large \( M > 0 \), we choose a number \( p = p(M) \) to define

\[
k_{w,p}(\xi, \xi_s) \equiv 1_{|\xi - \xi_s| \geq M} k_w(\xi, \xi_s),
\]

(3.31)
such that \( \sup_{\xi} \int_{R^3} |k_{w,p}(\xi, \xi_*) - k_w(\xi, \xi_*)| \, d\xi_* \leq \frac{1}{M} \). Then it follows

\[
|\mathcal{I}_0| \leq C \int_{|\xi_*| \leq 2M} k_{w,p}(\xi, \xi_*) |\partial_\xi^\beta H_2| \, d\xi_* + \frac{1}{M} \|w' \partial_\xi^\beta H_2\|_{L^\infty}
\]

\[
\leq C_{p,M} \|\partial_\xi^\beta H_2\| + \frac{1}{M} \|w' \partial_\xi^\beta H_2\|_{L^\infty},
\]

according to Hölder’s inequality and the fact that \( \int_{R^3} k_{w,p}(\xi, \xi_*) \, d\xi_* < \infty \). Putting the calculations above together, we arrive at

\[
|\mathcal{I}_0| \leq C \|\partial_\xi^\beta H_2\| + \left\{ \frac{C}{M} + C e^{-\frac{M^2}{8}} \right\} \|w' \partial_\xi^\beta H_2\|_{L^\infty}.
\]

We now turn to deduce the \( L^2 \) estimate for \( H_2 \). In view of (3.10), one has

\[
-\tilde{\beta}^{-\gamma} \tilde{\beta}_0 \partial_\xi^\beta \left\{ \nabla_\xi \cdot (\xi \mu) \mu^{-\frac{1}{2}} \right\} - \partial_\xi^\beta \left\{ \tilde{\beta}^{-\gamma} \nabla_\xi \cdot (A\xi \sqrt{\mu} H_1) \mu^{-\frac{1}{2}} \right\} + \partial_\xi^\beta L H_2 = \partial_\xi^\beta \Gamma(H_1, H_1).
\]

Taking the inner product of (3.33) with \( \partial_\xi^\beta H_2 \) and applying (3.27), and Lemma 5.2 as well as Lemma 5.3 one has, if \( \vartheta = 0 \)

\[
\|H_2\|^2_{L^\infty} \leq C \tilde{\beta}^{-\gamma} \tilde{\beta}_0 \leq C \tilde{\beta}^{-2\gamma},
\]

and if \( \vartheta > 0 \)

\[
\|\partial_\xi^\beta H_2\|^2_{L^\infty} \leq C \|H_2\|^2 + \tilde{\beta}^{-2\gamma}.
\]

We now have by putting (3.34) and (3.35) into (3.32) that

\[
|\mathcal{I}_0| \leq C \tilde{\beta}^{-2\gamma} + \left\{ \frac{C}{M} + C e^{-\frac{M^2}{8}} \right\} \|w' \partial_\xi^\beta H_2\|_{L^\infty},
\]

which together with (3.29) gives

\[
\|w' \partial_\xi^\beta H_2\| \leq C \mathbf{1}_{\vartheta > 0} \sum_{\vartheta < \vartheta} \|w' \partial_\xi^\beta H_2\|_{L^\infty} + C \tilde{\beta}^{-2\gamma}.
\]

Consequently, (3.28) follows from a linear combination of (3.30) over \( \vartheta = 0, 1, 2, \ldots, N \). Once (3.27) and (3.28) are obtained, we now turn to determine \( \tilde{\beta} \). Since \( [h_1, h_2] \in \mathcal{Y}_{\alpha,T} \), from (3.8), it follows

\[
\|w' \partial_\xi^\beta H_{R,1}(t)\|_{L^\infty} + \|w' \partial_\xi^\beta H_{R,2}(t)\|_{L^\infty} \leq 2 M_0 \tilde{\beta}^{-2\gamma}(t),
\]

for any \( 0 \leq t \leq T \). Therefore, using (3.14), (3.28), and (3.37) and taking \( \alpha \) to be suitably small, we get

\[
\alpha |\tilde{\beta}_1| \leq \frac{1}{2} \vartheta_0 \tilde{\beta}^{-\gamma},
\]

where we also have used the fact that \( \tilde{\beta}^{-2\gamma} \leq C(T) \tilde{\beta}^{-2\gamma} \).

Next plugging (3.38) and (3.13) into (3.12), one has

\[
\frac{1}{2} \vartheta_0 \tilde{\beta}^{-\gamma} \alpha^2 \leq \frac{\tilde{\beta}'}{\tilde{\beta}} \leq \frac{3}{2} \vartheta_0 \tilde{\beta}^{-\gamma} \alpha^2,
\]

which further gives (3.20).

We now turn to prove (3.21) and (3.25) involving \( t \)-derivatives. As a matter of fact, since \( \tilde{\beta}(t) \) is given as (3.20), we see that

\[
\left| \frac{d^k}{dt^k} (\tilde{\beta}^{-\gamma}) \right| \leq C \alpha^{2k} \tilde{\beta}^{-\gamma-k\gamma}.
\]

On the other hand, it follows from (3.9) and (3.10) that

\[
w' \partial_\xi^\beta \partial_\xi^\beta H_1 = \frac{d^k}{dt^k} (\tilde{\beta}^{-\gamma}) w' \partial_\xi^\beta L^{-1}(\xi, A\xi \mu^{\frac{1}{2}}),
\]
and

\[ w^t \partial_t \partial^\beta H_2 = \nu^{-1} w^t K \partial_t \partial^\beta H_2 + 1_{\theta > 0} \sum_{0 < \theta' + \theta'' \leq \theta} C_{\theta}^{\theta', \theta''} w^t \partial^\theta' (\nu^{-1})(\partial^\theta'' K) \partial_t \partial^\beta \partial^\theta - \partial^\theta'' H_2 \]

\[ + \sum_{\theta' \leq \theta} C_{\theta}^{\theta'} \sum_{k' \leq k} C_{k'}^{k} w^t \partial^\theta' (\nu^{-1}) \partial^\theta'' \Gamma (\partial^{k'} H_1, \partial^k H_1) \]

\[ + \sum_{\theta' \leq \theta} C_{\theta}^{\theta'} w^t \frac{d}{dt} \left( \sum_{k \leq k} C_{k}^{k'} \nu \partial^{k'} \left( \sum_{\beta < \gamma} \partial^{\beta - \gamma} (\nu^{-1}) \partial^\theta - \partial^\theta'' \right) \right) \]

\[ + \sum_{\theta' \leq \theta} C_{\theta}^{\theta'} \sum_{k' \leq k} C_{k'}^{k} w^t \frac{d}{dt} \left( \sum_{\beta < \gamma} \partial^{\beta - \gamma} (\nu^{-1}) \partial^\theta - \partial^\theta'' \right) \right) \]

Thus we can perform similar calculations as for obtaining (3.27) and (3.28) and then see that both (3.24) and (3.25) with \( k > 0 \) also hold true. This ends the proof of Lemma 3.1.

With Lemma 3.1 in our hands, we now turn to estimate \( J_i (1 \leq i \leq 16) \) term by term. First of all, by taking \( \ell \alpha \ll 1 \), we get that

\[ \tilde{A}(t, V(t)) \geq \frac{1}{2} \nu \tilde{\beta}^\gamma (t) \geq c_0 \tilde{\beta}^\gamma (t), \]

for some \( c_0 > 0 \). Thus it follows

\[ \int_0^t e^{-s} f^\beta (\tilde{A}(t)) (s) ds \leq 2 \left( 1 - e^{-\frac{1}{2} \int_0^t (\tilde{\beta}^\gamma (t)) ds} \right) \leq 2. \]  

(3.39)

Moreover, one also has

\[ \int_0^t e^{-s} f^\beta (\tilde{A}(t)) (s) ds \leq 2 \left( 1 - e^{-c_0 \int_0^t \beta^\gamma (t) ds} \right) \leq \int_0^t \beta^\gamma (t) ds \leq t \beta^\gamma (t) \leq 2T_0, \]

(3.40)

provided that \( t \in [0, T_0] \) and \( T_0 \) is suitably small.

It is straightforward to see

\[ |J_1| \leq \| w^t \partial^\beta G_{R,0} (V(0)) \|_{L^\infty} \leq M_0. \]

Since

\[ \| \tilde{\beta}^t \|_{L^\infty} \leq C \alpha^2 \]

According to (3.26), one has

\[ |J_2| \leq C \alpha^{-1} \sum_{0 < \theta} \| w^t \partial^\theta h_1 \|_{L^\infty} \leq C \alpha^2 M_0, \]

\[ |J_1| \leq C \alpha^{-1} \sum_{0 < \theta} \| w^t \partial^\theta h_2 \|_{L^\infty} \leq C \alpha^2 M_0. \]

For \( J_3 \) and \( J_{12} \), it follows from (3.39) that

\[ |J_3| \leq C \alpha^{-1} \sum_{0 < \theta} \| w^t \partial^\theta h_1 \|_{L^\infty} \leq C \alpha M_0, \]

\[ |J_{12}| \leq C \alpha^{-1} \sum_{0 < \theta} \| w^t \partial^\theta h_2 \|_{L^\infty} \leq C \alpha M_0. \]

Similarly, for \( J_4 \) and \( J_{13} \), one has

\[ |J_4| \leq C T_0 \sum_{\theta < \theta} \| w^t \partial^\theta h_1 \|_{L^\infty} \leq C T_0 M_0, \]

\[ |J_{13}| \leq C T_0 \sum_{\theta < \theta} \| w^t \partial^\theta h_2 \|_{L^\infty} \leq C T_0 M_0. \]

For \( J_5 \), Lemma (3.6) and (3.39) give

\[ |J_5| \leq C (M^{-\gamma} + \zeta) \sum_{0 < \theta} \| w^t \partial^\theta h_1 \|_{L^\infty} \leq C M_0 (M^{-\gamma} + \zeta). \]

For \( J_6 \), by virtue of (3.41), (3.39), we get

\[ |J_6| \leq C \alpha \sum_{0 < \theta} \| w^t \partial^\theta h_2 \|_{L^\infty} \leq C \alpha M_0, \]
and

\[ |\mathcal{J}_{15}| \leq C\alpha \sum_{\theta' \leq \theta} \|w^{t}\phi_\theta^\theta h_2\|_{L^\infty} \leq C\alpha M_0. \]

For \(\mathcal{J}_7\) and \(\mathcal{J}_8\), Lemma 5.1, 3.31, and 3.39 imply

\[ |\mathcal{J}_7| + |\mathcal{J}_8| \leq C\alpha^{3-m} + C\alpha \sum_{\theta' \leq \theta} \|w^{t}\phi_\theta^\theta h_1, h_2\|_{L^\infty} \leq C\alpha^{3-m} + C\alpha M_0. \]

For the rest non-local terms, from Lemmas 5.7, 5.1 and 3.1 as well as 3.39 and 3.40, one has

\[ |\mathcal{J}_9| \leq C\alpha^{3-m} \sum_{\theta' \leq \theta} \|\tilde{\beta}\gamma w^{t}\phi_\theta^\theta [H_1, H_2]\|_{L^\infty} \leq C\alpha^{3-m}, \]

\[ |\mathcal{J}_{10}| \leq C\alpha^m \sum_{\theta' \leq \theta} \|w^{t}\phi_\theta^\theta h_1, h_2\|_{L^\infty}^2 + C\alpha \sum_{\theta' \leq \theta} \|w^{t}\phi_\theta^\theta [h_1, h_2]\|_{L^\infty} \leq C\alpha M_0, \]

\[ |\mathcal{J}_{14}| \leq CT_0 \sum_{\theta' \leq \theta} \|w^{t}\phi_\theta^\theta h_2\|_{L^\infty} \leq CT_0 M_0, \quad |\mathcal{J}_{15}| \leq CT_0 \sum_{\theta' \leq \theta} \|w^{t}\phi_\theta^\theta h_1\|_{L^\infty} \leq CT_0 M_0. \]

Putting the above estimates together, we arrive at

\[ \sum_{|\theta| \leq N} \left\{ \|w^{t}\phi_\theta^\theta g_1\|_{L^\infty} + \|w^{t}\phi_\theta^\theta g_2\|_{L^\infty} \right\} \]

\[ \leq M_0 + CN\alpha M_0 + CNT_0 M_0 + CN\alpha^{3-m} + CN\alpha^2 + CN M_0 (M^{-\gamma} + \varsigma). \]  \hspace{1cm} (3.42)

Choosing now both \(\alpha, \varsigma\) and \(T_0\) to be suitably small and letting \(M\) be sufficiently large such that

\[ \sigma_0 = \max \left\{ \frac{C N\alpha, CNT_0, CN (M^{-\gamma} + \varsigma), \frac{C N\alpha^{3-m}}{M_0}, \frac{C N\alpha^2}{M_0}}{M_0} \right\} \leq \frac{1}{16}, \]

we get from (3.42) that

\[ \sum_{|\theta| \leq N} \sup_{0 \leq t \leq T_0} \left\{ \|w^{t}\phi_\theta^\theta g_1(t)\|_{L^\infty} + \|w^{t}\phi_\theta^\theta g_2(t)\|_{L^\infty} \right\} \leq 2M_0. \]

Next, for \([h_1, h_2] \in \mathbf{Y}_{\alpha, T}\) and \([\bar{h}_1, \bar{h}_2] \in \mathbf{Y}_{\alpha, T}\), we aim to prove that

\[ \sum_{|\theta| \leq N-1} \sup_{0 \leq t \leq T_0} \left\{ \|Q_1(h_1, h_2) - Q_1(\bar{h}_1, \bar{h}_2)\|_{L^\infty} + \|Q_2(h_1, h_2) - Q_1(\bar{h}_1, \bar{h}_2)\|_{L^\infty} \right\} \]

\[ \leq C\{T_0 + \alpha + M^{-\gamma} + \varsigma\} \sum_{|\theta| \leq N-1} \sup_{0 \leq t \leq T_0} \left\{ \|w^{t}\phi_\theta^\theta [h_1 - \bar{h}_1]\|_{L^\infty} + \|w^{t}\phi_\theta^\theta [h_2 - \bar{h}_2]\|_{L^\infty} \right\}. \]  \hspace{1cm} (3.43)

To do this, as in the proof of Lemma 3.1, we first denote

\[ \sqrt{\mu} \hat{h} = \bar{h}_1 + \sqrt{\mu} \hat{h}_2 = \bar{\beta}^{2\gamma}(\hat{H}_{R,1} + \sqrt{\mu} \hat{H}_{R,2}), \quad \sqrt{\mu} \hat{H} = \hat{H}_{R,1} + \sqrt{\mu} \hat{H}_{R,2}, \]

\[ \hat{H} = \mu + \sqrt{\mu} (\alpha \hat{H}_1 + \alpha^2 \hat{H}_2 + \alpha^m \hat{H}_R), \]

\[ \hat{H}_1 = -\bar{\beta}^{-\gamma} L^{-1} \{ \xi \cdot A\xi \hat{H} \}, \]

\[ \hat{H}_2 = \bar{\beta}^{-\gamma} L^{-1} \left\{ \bar{\beta}^{-\gamma} \Gamma(H_1, H_1) + \beta_0 \nabla_\xi \cdot (\xi \mu) \mu^{-\frac{1}{2}} + \nabla_\xi \cdot (A\xi \sqrt{\mu} \hat{H}_1) \mu^{-\frac{1}{2}} \right\}, \]

\[ \frac{\bar{\beta}'}{\bar{\beta}} = -\frac{\alpha}{3} \int_{\mathbb{R}^3} \xi \cdot A\xi \hat{H} d\xi, \quad \bar{\beta}' = \frac{d\bar{\beta}}{dt}, \quad \bar{\beta}(0) = 1, \]

\[ \bar{\beta}_0 = -\frac{1}{3} \int_{\mathbb{R}^3} \xi \cdot A\xi \sqrt{\mu} \hat{H}_1 d\xi = \bar{\phi}_0 \bar{\beta}^{-\gamma}, \]  \hspace{1cm} (3.44)
\[
\bar{\beta}_1 = -\frac{1}{3} \int_{\mathbb{R}^3} \xi \cdot A\xi \sqrt{\mu H_2} d\xi - \frac{\alpha^{m-2}}{3} \int_{\mathbb{R}^3} \xi \cdot A\xi \sqrt{\mu H_R} d\xi = \varrho_1 \bar{\beta}^{-2\gamma} + \bar{\varrho}_R \bar{\beta}^{-2\gamma},
\]
(3.45)

with
\[
\bar{\varrho}_R = -\frac{\alpha^{m-2} \bar{\beta}^{-2\gamma}}{3} \int_{\mathbb{R}^3} \xi \cdot A\xi \sqrt{\mu H_R} d\xi,
\]

and similar to (3.15)
\[
d\bar{\beta}^{-\gamma} = \gamma \left\{ \varrho_0 \alpha^2 + \varrho_1 \alpha^3 \bar{\beta}^{-\gamma} + \bar{\varrho}_R \alpha^3 \bar{\beta}^{-\gamma} \right\}.
\]
(3.46)
Moreover, \([\bar{H}_1, \bar{H}_2, \bar{\beta}]\) also satisfies Lemma 3.1, Namely, we have the following lemma.

**Lemma 3.2.** Let \([\bar{h}_1, \bar{h}_2] \in Y_{\alpha,T}\) with \(0 \leq T \leq +\infty\). For any \(\ell \geq 0, k \in \mathbb{Z}^+, \) and \(N \in \mathbb{Z}^+\), it holds that
\[
\sum_{|\alpha| \leq N+1} \|w^\ell \partial^k \partial^\alpha \tilde{H}_1(t, \xi)\|_{L^\infty} \leq C_1 \alpha^{2k} \bar{\beta}^{-\gamma - k\gamma},
\]
and
\[
\sum_{|\alpha| \leq N} \|w^\ell \partial^k \partial^\alpha \tilde{H}_2(t, \xi)\|_{L^\infty} \leq C_2 \alpha^{2k} \bar{\beta}^{-2\gamma - k\gamma},
\]
where both \(C_1\) and \(C_2\) depend on \(\ell, k\) and \(N\). Moreover, if \(\bar{\beta}(0) = 1\), it holds that
\[
\bar{\beta}^\gamma(t) \sim 1 + \gamma \varrho_0 \alpha^2 t,
\]
(3.47)
for any \(0 \leq t \leq T\).

Using Lemmas 3.1 and 3.2 given \([\bar{h}_1, \bar{h}_2] \in Y_{\alpha,T_0}\) one gets from ODE equations (3.13) and (3.40) that
\[
|\langle \tilde{\beta}^\gamma - \bar{\beta}^\gamma \rangle(t)\rangle \leq C(T_0)\alpha|\varrho_R - \bar{\varrho}_R| \leq C(T_0)\alpha\|w^{\ell}\| \|h_1 - \bar{h}_1, h_2 - \bar{h}_2\|_{L^\infty},
\]
(3.48)
for \(\ell > 5/2\). This together with (3.11) and (3.14) further gives
\[
|\langle \tilde{\beta}^\gamma - \bar{\beta}^\gamma \rangle(t)\rangle \leq C(T_0)\alpha\|w^{\ell}\| \|h_1 - \bar{h}_1, h_2 - \bar{h}_2\|_{L^\infty}.
\]
(3.49)
In addition, (3.14) and (3.45) directly imply
\[
|\langle \tilde{\beta}^\gamma \bar{\beta}_1 - \bar{\beta}^\gamma \bar{\beta}_1 \rangle(t)\rangle \leq C(T_0)\alpha|\varrho_R - \bar{\varrho}_R| \leq C(T_0)\alpha\|w^{\ell}\| \|h_1 - \bar{h}_1, h_2 - \bar{h}_2\|_{L^\infty}.
\]
Next, as in (3.20), we define
\[
\tilde{A}(\tau, \tilde{V}(\tau)) = \tilde{\beta}^\gamma \nu(\tilde{V}(\tau)) - \frac{(2\gamma + 3)\tilde{\beta}^\gamma}{\beta} + 2\ell \bar{\beta}^\gamma \frac{1 + |\tilde{V}(\tau)|^2}{1 + |V(\tau)|^2},
\]
where \(\tilde{V}\) is given by
\[
\tilde{V}(s) = \tilde{V}(s; t, \xi) = \frac{\tilde{\beta}(t)}{\bar{\beta}(s)} e^{-\gamma(t-s)\alpha A \xi},
\]
(3.50)
according to (3.20). Note that \(\tilde{V}(0) = V(0)\), since \(\bar{\beta}(0) = \bar{\beta}(0)\). Furthermore, for \(0 \leq s \leq t \leq T_0\), it follows from (3.50), (3.21) and (3.48) that
\[
|\tilde{V}(s) - V(s)| \leq C(T_0)\alpha\|w^{\ell}\| \|h_1 - \bar{h}_1, h_2 - \bar{h}_2\|_{L^\infty} |\tilde{V}(s)|.
\]
(3.51)
Next, in view of (3.20), (3.47) and (3.45), one has
\[
|\tilde{A}(\tau, \tilde{V}(\tau)) - \tilde{A}(\tau, \tilde{V}(\tau))| \leq C(T_0)\alpha\|w^{\ell}\| |h_1 - \bar{h}_1, h_2 - \bar{h}_2\|_{L^\infty} \nu(\tilde{V}(\tau)),
\]
(3.52)
where the following fact has been used
\[
|\nu(\tilde{V}(\tau)) - \nu(\tilde{V}(\tau))| \leq C(T_0)\alpha\|w^{\ell}\| |h_1 - \bar{h}_1, h_2 - \bar{h}_2\|_{L^\infty} \nu(\tilde{V}(\tau))
\]
by mean value theorem. Consequently, by choosing both \(\alpha\) and \(\varepsilon_0\) are suitably small, one gets
\[
\frac{1}{C} |\tilde{V}(s)| \leq |V(s)| \leq C|\tilde{V}(s)|, \quad \frac{1}{C} |\nu(\tilde{V}(s))| \leq \nu(V(s)) \leq C \nu(\tilde{V}(s)),
\]
(3.53)
\[
\bar{A}(s, V(s)) \geq c_0 \nu(\bar{V}(\tau)), \quad \bar{A}(s, \bar{V}(s)) \geq c_0 \nu(\bar{V}(\tau)),
\]
for \(0 \leq s \leq T_0\) and some \(C > 0\) as well as \(c_0 > 0\).

Recalling (3.22), we have
\[
|Q_1,\check{\sigma}(h_1, h_2) - Q_1,\check{\sigma}(\check{h}_1, \check{h}_2)| \leq \sum_{1 \leq i \leq 10} |J_i - \bar{J}_i|, \quad |Q_2,\check{\sigma}(h_1, h_2) - Q_2,\check{\sigma}(\check{h}_1, \check{h}_2)| \leq \sum_{11 \leq i \leq 15} |J_i - \bar{J}_i|,
\]
where \(\bar{J}_i (1 \leq i \leq 15)\) is in the form of \(J_i\) with \([h_1, h_2, \hat{\beta}]\) replaced by \([\check{h}_1, \check{h}_2, \check{\beta}]\).

To prove (3.43), we now turn to compute \(J_i - \bar{J}_i (1 \leq i \leq 15)\) term by term. First of all, by mean value theorem, we have
\[
|e^{-\int_0^t \bar{A}(s)ds} - e^{-\int_0^t \bar{A}(s)ds}| \leq e^{-\Theta} \int_0^t |\bar{A}(s) - \bar{A}(s)|ds,
\]
where we have taken \(\Theta\) such that \(\min\{\int_0^t \bar{A}(s)ds, \int_0^t \bar{A}(s)ds\} \leq \Theta \leq \max\{\int_0^t \bar{A}(s)ds, \int_0^t \bar{A}(s)ds\} \).

Then, (3.52) and (3.54) further give
\[
|e^{-\int_0^t \bar{A}(s)ds} - e^{-\int_0^t \bar{A}(s)ds}| \leq C(T_0) \sup_{0 \leq t \leq T_0} \|w^{\ell_1} |h_1 - \bar{h}_1, h_2 - \bar{h}_2| (t)\|_{L^\infty}.
\]

Since \(\bar{V}(0) = V(0)\), one has by (3.57)
\[
|J_i - \bar{J}_i| = |e^{-\int_0^t \bar{A}(s)ds} - e^{-\int_0^t \bar{A}(s)ds}| \leq C(T_0) M_0 \sup_{0 \leq t \leq T_0} \|w^{\ell_1} |h_1 - \bar{h}_1, h_2 - \bar{h}_2| (t)\|_{L^\infty}.
\]

For \(J_2 - \bar{J}_2\), we first write
\[
J_2 - \bar{J}_2 = 1_{\eta > 0} \sum_{|\eta| = 1} C_0^{\eta} \int_0^t e^{-\int_0^t \bar{A}(s)ds} \left\{ \frac{\beta^{\eta}}{\beta} w^{\ell_1} \partial^{\eta_1}_x \cdot \nabla \partial^{\eta_2-\eta_1}_x h_1 \right\} (s, V(s)) ds
\]
\[
- 1_{|\eta| > 0} C_0^{\eta} \sum_{|\eta| = 1} \int_0^t e^{-\int_0^t \bar{A}(s)ds} \left\{ \frac{\beta^{\eta}}{\beta} w^{\ell_1} \partial^{\eta_1}_x \cdot \nabla \partial^{\eta_2-\eta_1}_x h_1 \right\} (s, V(s)) ds
\]
\[
+ 1_{|\eta| > 0} \sum_{|\eta| = 1} C_0^{\eta} \int_0^t e^{-\int_0^t \bar{A}(s)ds} \left\{ \frac{\beta^{\eta}}{\beta} \right\} \left\{ w^{\ell_1} \partial^{\eta_1}_x \cdot \nabla \partial^{\eta_2-\eta_1}_x h_1 \right\} (s, V(s)) ds
\]
\[
+ 1_{|\eta| > 0} \sum_{|\eta| = 1} C_0^{\eta} \int_0^t e^{-\int_0^t \bar{A}(s)ds} \left\{ \frac{\beta^{\eta}}{\beta} \right\} \left\{ w^{\ell_1} \partial^{\eta_1}_x \cdot \nabla \partial^{\eta_2-\eta_1}_x h_1 \right\} (s, V(s)) ds.
\]
On the other hand, it follows
\[ \{w^\infty \partial_\xi^\alpha \xi \cdot \nabla_\xi \partial_\xi^{\alpha-\delta'} h_1\}(s, V(s)) - \{w^\infty \partial_\xi^\alpha \xi \cdot \nabla_\xi \partial_\xi^{\alpha-\delta'} \bar{h}_1\}(s, \bar{V}(s)) \]
\[ = \{w^\infty \partial_\xi^\alpha \xi \cdot \nabla_\xi \partial_\xi^{\alpha-\delta'} h_1\}(s, V(s)) - \{w^\infty \partial_\xi^\alpha \xi \cdot \nabla_\xi \partial_\xi^{\alpha-\delta'} h_1\}(s, \bar{V}(s)) \]
\[ + \{w^\infty \partial_\xi^\alpha \xi \cdot \nabla_\xi \partial_\xi^{\alpha-\delta'} h_1\}(s, \bar{V}(s)) - \{w^\infty \partial_\xi^\alpha \xi \cdot \nabla_\xi \partial_\xi^{\alpha-\delta'} \bar{h}_1\}(s, \bar{V}(s)) \]
\[ = \left\{ \nabla_\xi \{w^\infty \partial_\xi^\alpha \xi \cdot \nabla_\xi \partial_\xi^{\alpha-\delta'} h_1\} \right\}(s, \Theta_1) \cdot (V(s) - \bar{V}(s)) \]
\[ + \{w^\infty \partial_\xi^\alpha \xi \cdot \nabla_\xi \partial_\xi^{\alpha-\delta'} h_1\}(s, \bar{V}(s)) - \{w^\infty \partial_\xi^\alpha \xi \cdot \nabla_\xi \partial_\xi^{\alpha-\delta'} \bar{h}_1\}(s, \bar{V}(s)), \]
where \( \Theta_1 = \bar{V}(s) + \sigma_1(V(s) - \bar{V}(s)) \) with \( \sigma_1 \in [0, 1] \). Since \([h_1, h_2] \in Y_{\alpha,T_0} \) and \(|\delta'| \leq N - 1 \) here, one has by (3.51) that
\[ \left| \left\{ \nabla_\xi \{w^\infty \partial_\xi^\alpha \xi \cdot \nabla_\xi \partial_\xi^{\alpha-\delta'} h_1\} \right\}(s, \Theta_1) \cdot (V(s) - \bar{V}(s)) \right| \leq C(T_0)M_0\alpha \|w^\infty\|_{L^\infty}[h_1 - \bar{h}_1, h_2 - \bar{h}_2](t)\|_{L^\infty}. \] (3.57)
Consequently, we get
\[ \left| \left\{ w^\infty \partial_\xi^\alpha \xi \cdot \nabla_\xi \partial_\xi^{\alpha-\delta'} h_1\right\}(s, V(s)) - \left\{ w^\infty \partial_\xi^\alpha \xi \cdot \nabla_\xi \partial_\xi^{\alpha-\delta'} \bar{h}_1\right\}(s, V(s)) \right| \leq C \sum_{\delta' \leq \delta} \|w^\infty \partial_\xi^\delta \partial_\xi^{\delta-\delta'} [h_1 - \bar{h}_1, h_2 - \bar{h}_2](t)\|_{L^\infty}. \] (3.58)
Therefore, (3.50), (3.55), (3.57) and (3.58) imply
\[ |J_2 - \bar{J}_2| \leq C_\alpha \sum_{\delta' \leq \delta} \|w^\infty \partial_\xi^\delta \partial_\xi^{\delta-\delta'} [h_1 - \bar{h}_1, h_2 - \bar{h}_2](t)\|_{L^\infty}. \]

In what follows, we only compute the nonlocal terms \( J_5 - \bar{J}_5 \) and \( J_{10} - \bar{J}_{10} \), since the other terms can be treated similarly. For \( J_5 - \bar{J}_5 \), as (3.50), we first write
\[ J_5 - \bar{J}_5 = \sum_{\delta' \leq \delta} C_{\delta'} \int_0^t e^{-J_{\alpha}^s} A(\tau) d\tau \left\{ \gamma^\infty \partial_\xi^\delta (\chi_\infty M) \partial_\xi^{\delta-\delta'} h_1\right\}(s, V(s)) ds \]
\[ + \sum_{\delta' \leq \delta} C_{\delta'} \int_0^t e^{-J_{\alpha}^s} A(\tau) d\tau \left\{ \gamma^\infty \partial_\xi^\delta (\chi_\infty M) \partial_\xi^{\delta-\delta'} h_1\right\}(s, V(s)) ds \]
\[ + \sum_{\delta' \leq \delta} C_{\delta'} \int_0^t e^{-J_{\alpha}^s} A(\tau) d\tau \left\{ \gamma^\infty \partial_\xi^\delta (\chi_\infty M) \partial_\xi^{\delta-\delta'} h_1\right\}(s, V(s)) ds, \] (3.59)
and furthermore
\[ \left\{ w^\infty \partial_\xi^\delta (\chi_\infty M) \partial_\xi^{\delta-\delta'} h_1\right\}(s, V(s)) - \left\{ w^\infty \partial_\xi^\delta (\chi_\infty M) \partial_\xi^{\delta-\delta'} \bar{h}_1\right\}(s, \bar{V}(s)) \]
\[ = \left\{ w^\infty \partial_\xi^\delta (\chi_\infty M) \partial_\xi^{\delta-\delta'} h_1\right\}(s, V(s)) - \left\{ w^\infty \partial_\xi^\delta (\chi_\infty M) \partial_\xi^{\delta-\delta'} h_1\right\}(s, \bar{V}(s)) \]
\[ + \left\{ w^\infty \partial_\xi^\delta (\chi_\infty M) \partial_\xi^{\delta-\delta'} h_1\right\}(s, \bar{V}(s)) - \left\{ w^\infty \partial_\xi^\delta (\chi_\infty M) \partial_\xi^{\delta-\delta'} \bar{h}_1\right\}(s, \bar{V}(s)). \] (3.59)
Next, by mean value theorem, Lemma 5.6 and 3.51 as well as 3.53, one has for \(|\delta'| \leq N - 1 \)
\[ \nu^{-1}(\bar{V}) \left| \left\{ w^\infty \partial_\xi^\delta (\chi_\infty M) \partial_\xi^{\delta-\delta'} h_1\right\}(s, V(s)) - \left\{ w^\infty \partial_\xi^\delta (\chi_\infty M) \partial_\xi^{\delta-\delta'} h_1\right\}(s, \bar{V}(s)) \right| \]
\[ = \nu^{-1}(V) \left| \nabla_\xi \left\{ w^\infty \partial_\xi^\delta (\chi_\infty M) \partial_\xi^{\delta-\delta'} h_1\right\}(s, \Theta_2) \cdot (V(s) - \bar{V}(s)) \right| \]
\[ \leq C_\alpha \|w^\infty\|_{L^\infty}[h_1 - \bar{h}_1, h_2 - \bar{h}_2](t)\|_{L^\infty}, \] (3.60)
Lemma 5.6 gives

\[ \nu^{-1}(\tilde{V}) \left\{ \left\{ w^\ell \partial^{\nu} (\chi M K) \partial_\xi^{\nu_1} h_1 \right\} (s, \tilde{V}(s)) - \left\{ w^\ell \partial^{\nu} (\chi M K) \partial_\xi^{\nu_1} h_1 \right\} (s, \tilde{V}(s)) \right\} \leq C((1 + M)^{-\gamma} + \varsigma) \sum_{\nu_2 \leq \nu} \| w^\ell \partial^{\nu_2} [h_1 - \bar{h}_1, h_2 - \bar{h}_2](t) \|_{L^\infty}. \]  

(3.61)

Moreover, it can be directly verified that

\[ \int_0^t e^{-\int_0^t \mathcal{A}(\tau)d\tau} \tilde{\mathcal{V}}(\tilde{V}(t))dt < +\infty. \]  

(3.62)

Consequently, when \(|\vartheta| \leq N - 1\), we get from (3.48), (3.55), (3.58), (3.59), (3.60), (3.61) and (3.62) that

\[ |\mathcal{J}_5 - \tilde{\mathcal{J}}_5| \leq C(\alpha + (1 + M)^{-\gamma} + \varsigma) \sum_{\nu_2 \leq \nu} \| w^\ell \partial^{\nu_2} [h_1 - \bar{h}_1, h_2 - \bar{h}_2](t) \|_{L^\infty}. \]

Likewise, for \(\mathcal{J}_{10} - \tilde{\mathcal{J}}_{10}\), we first have

\[ \mathcal{J}_{10} - \tilde{\mathcal{J}}_{10} = \int_0^t e^{-\int_0^t \mathcal{A}(\tau)d\tau} \left\{ m \tilde{\mathcal{A}}^{-\gamma} w^\ell \partial_\xi^\nu Q(\sqrt{\mu}h, \sqrt{\mu}h) \right\} \left( V(s) - \left\{ w^\ell \partial_\xi^\nu Q(\sqrt{\mu}h, \sqrt{\mu}h) \right\} (\tilde{V}(s)) \right\} ds, \]

Then, we write

\[ \{ w^\ell \partial_\xi^\nu Q(\sqrt{\mu}h, \sqrt{\mu}h) \}(V(s)) - \{ w^\ell \partial_\xi^\nu Q(\sqrt{\mu}h, \sqrt{\mu}h) \}(\tilde{V}(s)) \]

\[ + \{ w^\ell \partial_\xi^\nu Q(\sqrt{\mu}h, \sqrt{\mu}h) \}(V(s)) - \{ w^\ell \partial_\xi^\nu Q(\sqrt{\mu}h, \sqrt{\mu}h) \}(\tilde{V}(s)) \]

\[ \leq CM_0 \sum_{\nu_2 \leq \nu} \| w^\ell \partial^{\nu_2} [h_1 - \bar{h}_1, h_2 - \bar{h}_2](t) \|_{L^\infty}. \]

Next, Lemma 5.7 gives

\[ \nu^{-1}(\tilde{V}) \left\{ \left\{ w^\ell \partial_\xi^\nu Q(\sqrt{\mu}h, \sqrt{\mu}h) \right\} (V(s)) - \left\{ w^\ell \partial_\xi^\nu Q(\sqrt{\mu}h, \sqrt{\mu}h) \right\} (\tilde{V}(s)) \right\} \]

\[ \left. + \nu^{-1}(\tilde{V}) \right\{ \left\{ w^\ell \partial_\xi^\nu Q(\sqrt{\mu}h, \sqrt{\mu}h) \right\} (V(s)) - \left\{ w^\ell \partial_\xi^\nu Q(\sqrt{\mu}h, \sqrt{\mu}h) \right\} (\tilde{V}(s)) \right\} \]

\[ \leq CM_0 \sum_{\nu_2 \leq \nu} \| w^\ell \partial^{\nu_2} [h_1 - \bar{h}_1, h_2 - \bar{h}_2](t) \|_{L^\infty}. \]

Then Lemma 5.7 together with mean value theorem and (3.51) yields that for \(|\vartheta| \leq N - 1\),

\[ \nu^{-1}(\tilde{V}) \left\{ \left\{ w^\ell \partial_\xi^\nu Q(\sqrt{\mu}h, \sqrt{\mu}h) \right\} (V(t)) - \left\{ w^\ell \partial_\xi^\nu Q(\sqrt{\mu}h, \sqrt{\mu}h) \right\} (\tilde{V}(t)) \right\} \]

\[ \leq CM_0 \sum_{\nu_2 \leq \nu} \| w^\ell \partial^{\nu_2} [h_1 - \bar{h}_1, h_2 - \bar{h}_2](t) \|_{L^\infty}. \]

Hence it follows

\[ \sup_{0 \leq t \leq T_0} \nu^{-1}(\tilde{V}) \left\{ \left\{ w^\ell \partial_\xi^\nu Q(\sqrt{\mu}h, \sqrt{\mu}h) \right\} (V(t)) - \left\{ w^\ell \partial_\xi^\nu Q(\sqrt{\mu}h, \sqrt{\mu}h) \right\} (\tilde{V}(t)) \right\} \]

\[ \leq CM_0 \sup_{0 \leq t \leq T_0} \sum_{\nu_2 \leq \nu} \| w^\ell \partial^{\nu_2} [h_1 - \bar{h}_1, h_2 - \bar{h}_2](t) \|_{L^\infty}. \]
Moreover, one can directly show that
\[
\sup_{0 \leq s \leq T_0} \nu^{-1}(V) \left| w^{\ell} \partial_\xi^\alpha \{ Q(\sqrt{\mu}H_1 + \alpha \sqrt{\mu}H_2, \sqrt{\mu}h) + Q(\sqrt{\mu}h, \sqrt{\mu}H_1 + \alpha \sqrt{\mu}H_2) \} \right|(t, V(t)) \\
- w^{\ell} \partial_\xi^\alpha \{ Q(\sqrt{\mu}H_1 + \alpha \sqrt{\mu}H_2, \sqrt{\mu}h) + Q(\sqrt{\mu}h, \sqrt{\mu}H_1 + \alpha \sqrt{\mu}H_2) \} \right|(t, V(t)) \\
\leq C \sup_{0 \leq s \leq T_0} \sum_{|\theta| \leq \theta} \| w^{\ell} \partial_\xi^\alpha \{ h_1 - \tilde{h}_1, h_2 - \tilde{h}_2 \} \|_{L^\infty}.
\]

As a consequence, similarly for obtaining (3.58), we get
\[
|J_{10} - \tilde{J}_{10}| \leq C(\alpha + (1 + M)^{-\gamma} + \zeta) \sum_{|\theta| \leq \theta} \| w^{\ell} \partial_\xi^\alpha \{ h_1 - \tilde{h}_1, h_2 - \tilde{h}_2 \} \|_{L^\infty}.
\]

Therefore (3.43) holds true. Then, by taking $T_0$, $\alpha$ and $\zeta$ suitably small and $M$ sufficiently large, one has
\[
\| \mathcal{N}([h_1, h_2]) - \mathcal{N}([\tilde{h}_1, \tilde{h}_2]) \|_{\mathcal{Y}_{\alpha, T_0}} \leq \frac{1}{2} \| [h_1, h_2] - [\tilde{h}_1, \tilde{h}_2] \|_{\mathcal{Y}_{\alpha, T_0}}.
\]

Hence, $\mathcal{N}$ is a contraction mapping on $\mathcal{Y}_{\alpha, T_0}$. So, there exists a unique $[h_1, h_2] \in \mathcal{Y}_{\alpha, T_0}$ such that $[h_1, h_2] = \mathcal{N}([h_1, h_2])$.

This completes the proof of Theorem 3.1.

4. Global existence and large time behavior

In this section, we study the global existence and large time behavior of the remainder $G_R$ determined by (1.23) and (1.24) in order to complete the proof of Theorem 1.1.

In fact, the global existence of (1.23) and (1.24) follows from the local existence and a priori estimates as well as the continuum argument. Here, we only show the a priori estimates (4.1), because the local existence has been established in Theorem 3.1 in Section 3 and the nonnegativity can be justified in a similar way as that of [20]. The approach used in the following is based on Caflisch’s decomposition.

Now, our goal is to prove
\[
\sup_{0 \leq s \leq t} \sum_{|\theta| \leq N} \beta^{2\gamma}(s) \| w^{\ell} \partial_\xi^\alpha G_{R,1}(s) \|_{L^\infty} + \sup_{0 \leq s \leq t} \sum_{|\theta| \leq N} \beta^{2\gamma}(s) \| w^{\ell} \partial_\xi^\alpha G_{R,2}(s) \|_{L^\infty} \\
\leq C \sum_{|\theta| \leq N} \| w^{\ell} \partial_\xi^\alpha G_{R,0} \|_{L^\infty} + C\alpha^{3-m},
\]

for any $t \geq 0$ and for some constant $C > 0$, under the a priori assumption that
\[
\sup_{0 \leq s \leq t} \sum_{|\theta| \leq N} \beta^{2\gamma}(s) \| w^{\ell} \partial_\xi^\alpha G_{R,1}(s) \|_{L^\infty} + \sup_{0 \leq s \leq t} \sum_{|\theta| \leq N} \beta^{2\gamma}(s) \| w^{\ell} \partial_\xi^\alpha G_{R,2}(s) \|_{L^\infty} \leq 2M_0.
\]

The proof of (4.1) is proceeded in the following three subsections.

4.1. $L^\infty$ estimates. In this subsection, we deduce the $L^\infty$ estimates on $G_{R,1}$ and $G_{R,2}$. For this, we have the following result.

Lemma 4.1. Under the conditions (4.2), it holds that
\[
\sup_{0 \leq s \leq t} \sum_{|\theta| \leq N} \| w^{\ell} \partial_\xi^\alpha g_1(s) \|_{L^\infty} \\
\leq \sum_{|\theta| \leq N} \| w^{\ell} \partial_\xi^\alpha G_{R,0} \|_{L^\infty} + C\alpha \sup_{0 \leq s \leq t} \sum_{|\theta| \leq N} \| w^{\ell} \partial_\xi^\alpha g_2(s) \|_{L^\infty} + C\alpha^{3-m},
\]

and
\[
\sup_{0 \leq s \leq t} \sum_{|\theta| \leq N} \| w^{\ell} \partial_\xi^\alpha g_2(s) \|_{L^\infty} \leq C \sup_{0 \leq s \leq t} \sum_{|\theta| \leq N} \| w^{\ell} \partial_\xi^\alpha g_1(s) \|_{L^\infty} + C \sup_{0 \leq s \leq t} \sum_{|\theta| \leq N} \| \partial_\xi^\alpha g_2(s) \|.
\]
Proof. For brevity, we denote $$[g_1, g_2](s, \xi) = \beta^{2\gamma}(s)[G_{R,1}(s, \xi), G_{R,2}(s, \xi)].$$

In view of (4.1) and (4.2), one has
\[
\begin{align*}
\partial_t(w^t \partial^\beta g_1) & - \frac{\beta^7}{\beta} \xi \cdot \nabla_x (w^t \partial^\beta g_1) - \frac{(2\gamma + 3)\beta^7}{\beta} w^t \partial^\beta g_1 + \frac{2\beta^7}{\beta} \frac{|\xi|^2}{1 + |\xi|^2} w^t \partial^\beta g_1 \\
& - \alpha A\xi \cdot \nabla_x (w^t \partial^\beta g_1) + 2\ell A \xi \cdot \nabla_x (w^t \partial^\beta g_1) + \nu \beta \gamma w^t \partial^\beta g_1 \\
= & \beta^7 \chi_M w^t \partial^\beta \frac{K^\beta g_1 + 1_{\theta > 0}C^\beta \partial^\beta (\chi_M K)}{\partial^\beta g_1 + \frac{\beta^7}{\beta} \frac{|\xi|^2}{1 + |\xi|^2} w^t \partial^\beta g_1} \left(1 - \chi_M\right) \partial^\beta g_1,
\end{align*}
\]

with
\[
g_1(0, \xi) = G_{R,0},
\]
and
\[
\begin{align*}
\partial_t(w^t \partial^\beta g_2) & - \frac{\beta^7}{\beta} \xi \cdot \nabla_x (w^t \partial^\beta g_2) - \frac{(2\gamma + 3)\beta^7}{\beta} w^t \partial^\beta g_2 + \frac{2\beta^7}{\beta} \frac{|\xi|^2}{1 + |\xi|^2} w^t \partial^\beta g_2 \\
& - \alpha A\xi \cdot \nabla_x (w^t \partial^\beta g_2) + 2\ell A \xi \cdot \nabla_x (w^t \partial^\beta g_2) + \nu \beta \gamma w^t \partial^\beta g_2 \\
= & \beta^7 \chi_M w^t \partial^\beta \frac{K^\beta g_2 + 1_{\theta > 0}C^\beta \partial^\beta (\chi_M K)}{\partial^\beta g_2} \left(1 - \chi_M\right) \partial^\beta g_2 \\
& + \beta^7 \left(1_{\theta > 0}C^\beta \partial^\beta (\chi_M K) \partial^\beta g_2 \right) + \frac{\beta^7}{\beta} \frac{|\xi|^2}{1 + |\xi|^2} w^t \partial^\beta g_2 \\
& - \alpha A\xi \cdot \nabla_x (w^t \partial^\beta g_2) + 2\ell A \xi \cdot \nabla_x (w^t \partial^\beta g_2) + \nu \beta \gamma w^t \partial^\beta g_2 \\
& + \beta^7 \left(1_{\theta > 0}C^\beta \partial^\beta (\chi_M K) \partial^\beta g_2 \right) + \frac{\beta^7}{\beta} \frac{|\xi|^2}{1 + |\xi|^2} w^t \partial^\beta g_2 \\
& + \beta^7 \left(1_{\theta > 0}C^\beta \partial^\beta (\chi_M K) \partial^\beta g_2 \right) + \frac{\beta^7}{\beta} \frac{|\xi|^2}{1 + |\xi|^2} w^t \partial^\beta g_2 \\
& - \alpha A\xi \cdot \nabla_x (w^t \partial^\beta g_2) + 2\ell A \xi \cdot \nabla_x (w^t \partial^\beta g_2) + \nu \beta \gamma w^t \partial^\beta g_2 \\
& + \beta^7 \left(1_{\theta > 0}C^\beta \partial^\beta (\chi_M K) \partial^\beta g_2 \right) + \frac{\beta^7}{\beta} \frac{|\xi|^2}{1 + |\xi|^2} w^t \partial^\beta g_2 \\
& + \beta^7 \left(1_{\theta > 0}C^\beta \partial^\beta (\chi_M K) \partial^\beta g_2 \right) + \frac{\beta^7}{\beta} \frac{|\xi|^2}{1 + |\xi|^2} w^t \partial^\beta g_2 \\
& + \beta^7 \left(1_{\theta > 0}C^\beta \partial^\beta (\chi_M K) \partial^\beta g_2 \right) + \frac{\beta^7}{\beta} \frac{|\xi|^2}{1 + |\xi|^2} w^t \partial^\beta g_2
\end{align*}
\]

respectively, where $$\sqrt{\mu g} = g_1 + \sqrt{\mu g}_2.$$

Recalling Lemma 3.1 since $$[g_1, g_2] \in Y_{n,t}$$ for any $$t \geq 0$$, one can gets for any $$\ell \geq 0$$
\[
\sum_{|\theta| \leq N+1} \|w^t \partial^\beta g_1(t, \xi)\|_{L^\infty} \leq C \alpha^{2k} \beta^{-2 \gamma - k \gamma},
\]
and
\[
\sum_{|\theta| \leq N} \|w^t \partial^\beta g_2(t, \xi)\|_{L^\infty} \leq C \alpha^{2k} \beta^{-2 \gamma - k \gamma},
\]
where $$C > 0$$ depends on $$\ell, k$$ and $$N$$. Moreover, it holds that
\[
\beta^\gamma(t) \sim 1 + \gamma \theta_0 \alpha^2 t,
\]
for any $$t \geq 0$$. 

We now turn to estimate $g_1$ and $g_2$. Integrating along the backward trajectory with respect to $s \in [0, t]$, one can write the solutions of (4.3) and (4.6) as the following mild form

$$w^t \varphi_{t}^g g_1(t, x, v) = \sum_{i=1}^{11} \mathcal{H}_i,$$

(4.10)

with

$$\mathcal{H}_1 = e^{-\int_0^t (A(s))ds} w^t \varphi_{t}^g G_{R,0}(V(0)),$$

$$\mathcal{H}_2 = 1_{|\varphi|>0} \sum_{|\varphi|\neq 1} C_{\varphi}^g \int_0^t e^{-\int_0^\tau (A(\tau))d\tau} \left\{ \frac{\beta'}{\beta} w^t \varphi_{t}^g \xi \cdot \nabla \varphi_{t}^g g_1 \right\} (s, V(s))d\tau,$$

$$\mathcal{H}_3 = \alpha 1_{|\varphi|>0} \sum_{|\varphi|\neq 1} C_{\varphi}^g \int_0^t e^{-\int_0^\tau (A(\tau))d\tau} \left\{ w^t \varphi_{t}^g (A\xi) \cdot \nabla \varphi_{t}^g g_1 \right\} (s, V(s))d\tau,$$

$$\mathcal{H}_4 = -1_{|\varphi|>0} \sum_{0<|\varphi|\leq \vartheta} C_{\varphi}^g \int_0^t e^{-\int_0^\tau (A(\tau))d\tau} \left\{ \beta \gamma w^t \varphi_{t}^g \nu \partial_{\varphi_{t}^g}^{\varphi_{t}^g} g_1 \right\} (s, V(s))d\tau,$$

$$\mathcal{H}_5 = \int_0^t e^{-\int_0^\tau (A(\tau))d\tau} \left\{ \beta \gamma w^t \varphi_{t}^g \chi_{M} K \partial_{\varphi_{t}^g} g_1 \right\} (s, V(s))d\tau,$$

$$\mathcal{H}_6 = 1_{|\varphi|\geq 0} C_{\varphi}^g \int_0^t e^{-\int_0^\tau (A(\tau))d\tau} \left\{ \beta \gamma w^t \varphi_{t}^g \chi_{M} K \partial_{\varphi_{t}^g} g_1 \right\} (s, V(s))d\tau,$$

$$\mathcal{H}_7 = -\int_0^t e^{\int_0^\tau (A(\tau))d\tau} \left\{ \frac{\beta'}{2\beta} w^t \varphi_{t}^g (|\xi|^2 \sqrt{\mu} g_2) + \frac{\alpha}{2} w^t \varphi_{t}^g (\xi \cdot (A\xi) \sqrt{\mu} g_2) \right\} (s, V(s))d\tau,$$

$$\mathcal{H}_8 = -\int_0^t e^{\int_0^\tau (A(\tau))d\tau} \left\{ \alpha^{1-m} \beta^{2\gamma} w^t \varphi_{t}^g \partial_{\varphi_{t}^g} G_1 + \alpha^{2-m} \beta^{2\gamma} w^t \varphi_{t}^g \partial_{\varphi_{t}^g} G_2 \right\}$$

$$\quad - \alpha^{1-m} \beta^{2\gamma} w^t \varphi_{t}^g \partial_{\varphi_{t}^g} \left\{ \nabla \varphi_{t}^g \cdot (\xi \sqrt{\mu} G_1) \right\} (s, V(s))d\tau,$$

$$\mathcal{H}_9 = \int_0^t e^{\int_0^\tau (A(\tau))d\tau} \left\{ \alpha^{1-m} \beta^{2\gamma} \beta_1 w^t \varphi_{t}^g \nabla \varphi_{t}^g \cdot (\xi \mu) + \alpha^{2-m} \beta^{2\gamma-1} w^t \varphi_{t}^g \partial_{\varphi_{t}^g} \left\{ \nabla \varphi_{t}^g \cdot (\xi \sqrt{\mu} G_2) \right\}$$

$$\quad + \alpha^{3-m} \beta^{2\gamma} w^t \varphi_{t}^g \partial_{\varphi_{t}^g} \left\{ \nabla \varphi_{t}^g \cdot (A\xi \sqrt{\mu} G_2) \right\} \right\} (s, V(s))d\tau,$$

$$\mathcal{H}_{10} = \int_0^t e^{\int_0^\tau (A(\tau))d\tau} \left\{ \alpha^{3-m} \beta^{3\gamma} \partial_{\varphi_{t}^g} \left\{ Q(\sqrt{\mu} G_1, \sqrt{\mu} G_2) + Q(\sqrt{\mu} G_2, \sqrt{\mu} G_1) \right\}$$

$$\quad + \alpha^{1-m} \beta^{3\gamma} \partial_{\varphi_{t}^g} Q(\sqrt{\mu} G_2, \sqrt{\mu} G_2) \right\} (s, V(s))d\tau,$$

$$\mathcal{H}_{11} = \int_0^t e^{\int_0^\tau (A(\tau))d\tau} \left\{ \alpha^{m} \beta^{2\gamma} w^t \varphi_{t}^g \partial_{\varphi_{t}^g} \left\{ Q(\sqrt{\mu} g, \sqrt{\mu} g) \right\}$$

$$\quad + \alpha^{m} \beta^{2\gamma} \partial_{\varphi_{t}^g} \left\{ Q(\sqrt{\mu} G_1 + \alpha \sqrt{\mu} G_2, \sqrt{\mu} g) + Q(\sqrt{\mu} g, \sqrt{\mu} G_1 + \alpha \sqrt{\mu} G_2) \right\} \right\} (s, V(s))d\tau,$$

and

$$w^t \varphi_{t}^g g_2(t, x, v) = \sum_{i=1}^{17} \mathcal{H}_i,$$

(4.11)
with

\[ \mathcal{H}_{12} = \mathbf{1}_{|\vartheta| > 0} \sum_{|\vartheta'| = 1} C^{\vartheta'}_0 \int_0^t e^{- \int_0^r A(r)dr} \left\{ \frac{\beta'}{\beta} \frac{|V'(r)|}{1 + |V'(r)|^2} \right\} (s, V(s))ds, \]

\[ \mathcal{H}_{13} = \mathbf{a} \mathbf{1}_{|\vartheta| > 0} \sum_{|\vartheta'| = 1} C^{\vartheta'}_0 \int_0^t e^{- \int_0^r A(r)dr} \left\{ \frac{\beta'}{\beta} \frac{|V'(r)|}{1 + |V'(r)|^2} (A\xi) \cdot \nabla \xi \partial^{\vartheta - \vartheta'} g_2 \right\} (s, V(s))ds, \]

\[ \mathcal{H}_{14} = \mathbf{1}_{|\vartheta| > 0} \sum_{0 < \vartheta' \leq \vartheta} C^{\vartheta'}_0 \int_0^t e^{- \int_0^r A(r)dr} \left\{ \beta' \frac{|V'(r)|}{1 + |V'(r)|^2} \right\} (s, V(s))ds, \]

\[ \mathcal{H}_{15} = \int_0^t e^{- \int_0^r A(r)dr} \left\{ \beta' \frac{|V'(r)|}{1 + |V'(r)|^2} \right\} (s, V(s))ds, \]

\[ \mathcal{H}_{16} = \mathbf{1}_{\vartheta \geq \vartheta'} C^{\vartheta'}_0 \int_0^t e^{- \int_0^r A(r)dr} \left\{ \beta' \frac{|V'(r)|}{1 + |V'(r)|^2} \right\} (s, V(s))ds, \]

and

\[ \mathcal{H}_{17} = \int_0^t e^{- \int_0^r A(r)dr} \left\{ \beta' \frac{|V'(r)|}{1 + |V'(r)|^2} \right\} (s, V(s))ds. \]

Here, we have denoted

\[ A(r, V(r)) = \beta' \nu(V(r)) - \frac{(3 + 2\gamma)\beta'}{\beta} + 2\ell_\infty \frac{\beta'}{1 + |V(r)|^2} + 2\epsilon_\infty \frac{\nu(V(r) \cdot (AV(r)))}{1 + |V(r)|^2}. \]

One sees that, as long as \( \ell_\infty, \alpha > 0 \) and \( \alpha \) are suitably small,\n
\[ A(r, V(r)) \geq \frac{1}{2} \beta' \nu(V(r)) > \tilde{C}_0 \beta', \]

for some \( \tilde{C}_0 > 0 \). Moreover, it holds that

\[ \int_0^t e^{- \int_0^r A(r)dr} \beta'(s) \nu(V(s))ds < \infty. \]

We now turn to estimate \( \mathcal{H}_i \) (1 \( \leq i \leq 17 \)) separately. We start with the nonlocal terms \( \mathcal{H}_5, \mathcal{H}_6, \mathcal{H}_{10}, \mathcal{H}_{11}, \mathcal{H}_{15}, \mathcal{H}_{16} \) and \( \mathcal{H}_{17} \).

For \( \mathcal{H}_5 \), applying (4.12) and using (5.5) in Lemma 5.6, we obtain

\[ |\mathcal{H}_5| \leq C \int_0^t e^{- \int_0^r A(r)dr} \beta'(s) \nu(V(s))ds \sup_{0 \leq s \leq t} \left\| \nu^{-1} w^{\ell, \infty}(1 - M) g_1 \right\|_{L^\infty} \leq C \left( (1 + M)^{-\gamma} + \zeta \right) \sup_{0 \leq s \leq t} \left\| w^{\ell, \infty}(\partial^{\vartheta'} \xi g_1) \right\|_{L^\infty}. \]

Recalling (4.12), one gets from Lemma 5.7 that

\[ |\mathcal{H}_6| \leq C \mathbf{1}_{|\vartheta| > 0} \int_0^t e^{- \int_0^r A(r)dr} \beta'(s) \nu(V(s))ds \sum_{\vartheta' > 0} \sup_{0 \leq s \leq t} \left\| w^{\ell, \infty}(\partial^{\vartheta'} \xi g_1) \right\|_{L^\infty} \leq C \mathbf{1}_{|\vartheta| > 0} \sum_{\vartheta' > 0} \sup_{0 \leq s \leq t} \left\| w^{\ell, \infty}(\partial^{\vartheta'} \xi g_1) \right\|_{L^\infty}. \]

For \( \mathcal{H}_{10}, \mathcal{H}_{11}, \mathcal{H}_{12}, \mathcal{H}_{13} \) and Lemma 5.7, we give

\[ |\mathcal{H}_{10}| \leq C \alpha^{-m} \int_0^t e^{- \int_0^r A(r)dr} \beta'(s) \nu(V(s))ds \times \sum_{\vartheta', \vartheta'' \leq \alpha} \sup_{0 \leq s \leq t} \left\| \beta''(s) \right\|_{L^\infty} \left\| \nu^{-1} w^{\ell, \infty}(G_1) \right\|_{L^\infty} \leq C \alpha^{-m}. \]
Likewise, for $\mathcal{H}_{11}$, applying (4.7), (4.8) and Lemma 5.7 as well as the a priori assumption (4.2), one has
\[
|\mathcal{H}_{11}| \leq C \alpha \int_0^t e^{-\int_0^s \frac{\alpha'}{\alpha(s)} ds} \beta' \nu(V(s)) ds \sum_{\vartheta' + \vartheta'' \leq \vartheta} \sup_{0 \leq s \leq t} \left\| w^\infty \partial_\xi^{\vartheta'} [g_1, g_2](s) \right\|_{L^\infty} \left\| w^\infty \partial_\xi^{\vartheta''} [g_1, g_2](s) \right\|_{L^\infty} \\
+ C \alpha \int_0^t e^{-\int_0^s \frac{\alpha'}{\alpha(s)} ds} \beta' \nu(V(s)) ds \sum_{\vartheta' + \vartheta'' \leq \vartheta} \sup_{0 \leq s \leq t} \left\| w^\infty \partial_\xi^{\vartheta'} [G_1, G_2](s) \right\|_{L^\infty} \left\| w^\infty \partial_\xi^{\vartheta''} [g_1, g_2](s) \right\|_{L^\infty} \\
\leq C \alpha \sum_{\vartheta' + \vartheta'' \leq \vartheta} \sup_{0 \leq s \leq t} \left\| w^\infty \partial_\xi^{\vartheta'} [g_1, g_2](s) \right\|_{L^\infty}.
\]
For the delicate term $\mathcal{H}_{15}$, we first rewrite
\[
\mathcal{H}_{15} = \int_0^t e^{-\int_0^s \lambda'(r) dr} \beta'(s) \int_{\mathbb{R}^3} k_w(V(s), \xi_*) (w^\infty \partial_\xi^{\vartheta} g_2)(s, \xi_*) d\xi_* ds.
\]
As in the proof of Lemma 5.1, the computation for $\mathcal{H}_{15}$ is then divided in the following three cases.

**Case 1.** $|V(s)| > M$. In this case, we get from Lemma 5.1 that
\[
|\mathcal{H}_{15}| \leq \frac{C}{M} \sup_{0 \leq s \leq t} \left\| w^\infty \partial_\xi^{\vartheta} g_2(s) \right\|_{L^\infty}.
\]

**Case 2.** $|V(s)| \leq M$ and $|\xi_*| > 2M$. In this situation, one has $|V(s) - \xi_*| > M$, thus it follows
\[
k_w(V, \xi_*) \leq Ce^{-\frac{M^2}{4\xi^2}} k_w(V, \xi_*) e^{\frac{M^2}{\xi^2}},
\]
this together with Lemma 5.1 leads to
\[
|\mathcal{H}_{15}| \leq Ce^{-\frac{M^2}{4\xi^2}} \sup_{0 \leq s \leq t} \left\| w^\infty \partial_\xi^{\vartheta} g_2(s) \right\|_{L^\infty}.
\]

**Case 3.** $|V(s)| \leq M$ and $|\xi_*| \leq 2M$. At this stage, recalling $k_{w,p}(V, \xi_*)$ defined by (3.31), we write
\[
\mathcal{H}_{15} = \int_0^t e^{-\int_0^s \lambda'(r) dr} \int_{\mathbb{R}^3} \beta'(s) [k_w - k_{w,p} + k_{w,p}](V(s), \xi_*) (w^\infty \partial_\xi^{\vartheta} g_2)(s, \xi_*) d\xi_* ds,
\]
which further gives the bound
\[
|\mathcal{H}_{15}| \leq C \int_{|\xi_*| \leq 2M} k_{w,p}(\xi, \xi_*) \partial_\xi^{\vartheta} g_2|d\xi_* + \frac{1}{M} \left\| w^\infty \partial_\xi^{\vartheta} g_2 \right\|_{L^\infty} \\
\leq C_{p,M} \left\| \partial_\xi^{\vartheta} g_2 \right\| + \frac{C}{M} \left\| w^\infty \partial_\xi^{\vartheta} g_2 \right\|_{L^\infty}.
\]
To summarize, we arrive at
\[
|\mathcal{H}_{15}| \leq C \left\| \partial_\xi^{\vartheta} g_2 \right\| + C \left\{ \frac{1}{M} + e^{-\frac{M^2}{4\xi^2}} \right\} \left\| w^\infty \partial_\xi^{\vartheta} g_2 \right\|_{L^\infty}.
\]
For $\mathcal{H}_{16}$, from Lemma 5.2 it follows
\[
|\mathcal{H}_{16}| \leq C v > 0 \sum_{\vartheta'} \sup_{0 \leq s \leq t} \left\| w^\infty \partial_\xi^{\vartheta'} g_2(s) \right\|_{L^\infty}.
\]
For $\mathcal{H}_{17}$, one has by Lemma 5.7
\[
|\mathcal{H}_{17}| \leq C \sum_{\vartheta'} \sup_{0 \leq s \leq t} \left\| w^\infty \partial_\xi^{\vartheta'} g_1(s) \right\|_{L^\infty}.
\]
The remaining terms in (4.10) and (4.11) will be computed as follows
\[
|\mathcal{H}_3| \leq 1 v > 0 C \alpha \sup_{0 \leq s \leq t} \left\| w^\infty \partial_\xi^{\vartheta'} g_1(s) \right\|_{L^\infty}, \quad |\mathcal{H}_{13}| \leq 1 v > 0 C \alpha \sup_{0 \leq s \leq t} \left\| w^\infty \partial_\xi^{\vartheta'} g_2(s) \right\|_{L^\infty},
\]
\[
|\mathcal{H}_4| \leq 1 v > 0 C \gamma \sup_{0 \leq s \leq t} \left\| w^\infty \partial_\xi^{\vartheta'} g_1(s) \right\|_{L^\infty}, \quad |\mathcal{H}_{14}| \leq 1 v > 0 C \gamma \sup_{0 \leq s \leq t} \left\| w^\infty \partial_\xi^{\vartheta'} g_2(s) \right\|_{L^\infty}.
\]
Step 1. The estimates for $L^2$.

We have the following result.

Finally, using (4.7), (4.8) and (4.13), we obtain

$$|H_2| \leq 1_{\theta > 0} C \sum_{0 \leq s \leq t} \|w^t \partial^6_\xi g_1(s)\|_{L^\infty},$$

and

$$|H_7| \leq C \alpha \sup_{0 \leq s \leq t} \|w^t \partial^6_\xi g_2(s)\|_{L^\infty}.$$
with
\[ \sqrt{\mu}g(0, \xi) = G_{R,0}(\xi). \]

Next let us define
\[ P_{0g} = \{ a(t) + b(t) \cdot v + c(t)(|\xi|^2 - 3) \} \sqrt{\mu}, \]
\[ P_{0g_1} = \{ a_1(t) + b_1(t) \cdot \xi + c_1(t)(|\xi|^2 - 3) \} \mu, \]
and
\[ P_{0g_2} = \{ a_2(t) + b_2(t) \cdot v + c_2(t)(|v|^2 - 3) \} \sqrt{\mu}. \]

Then it follows
\[ a(t) = a_1(t) + a_2(t), \ b(t) = b_1(t) + b_2(t), \ c(t) = c_1(t) + c_2(t), \]
for any \( t \geq 0 \).

From (1.32) and (1.33), it is straightforward to check
\[ P_{0g} = 0, \]
hence
\[ a_1 = -a_2, \ b_1 = -b_2, \ c_1 = -c_2. \]

Thus, for \( \ell_2 \geq 2 \), it follows
\[ ||a_2, b_2, c_2(t)|| \leq C||w^{f_2}g_1(t)||. \]

Step 2. \( L^2 \) estimates for \( P_{1g_2} \). We now derive the \( L^2 \) estimate on \( P_{1g_2} \). Recall that \( g_2 \) satisfies
\[ \partial_t g_2 - \frac{\beta'}{\beta} \nabla_\xi \cdot (\xi g_2) - \frac{2\gamma}{\beta} g_2 + \alpha \nabla_\xi \cdot (A_\xi g_2) + \beta^\gamma Lg_2 = \beta^\gamma (1 - \chi_M) \mu^{-\frac{1}{2}} K_g. \]

Taking the inner product of (1.18) and \( P_{1g_2} \) over \( \mathbb{R}^3 \), applying Cauchy-Schwarz’s inequality and Lemmas 5.2 and 5.8 as well as (4.17), we obtain
\[ \frac{1}{2} \frac{d}{dt} ||P_{1g_2}||^2 + \lambda \beta^\gamma ||P_{1g_2}||^2 \leq C \alpha^2 \{ [a_2, b_2, c_2] \} + C \beta^\gamma ||w^{f_2}g_1||^2 \leq C \beta^\gamma ||w^{f_2}g_1||^2. \]

Step 3. Higher order estimates for \( P_{1g_2} \). Since \( ||\partial_\xi^2 g_2|| \leq C||[a_2, b_2, c_2]|| + C||\partial_\xi^2 P_{1g_2}|| \), to obtain the higher order \( L^2 \) estimates on \( g_2 \), it suffices to deduce the corresponding estimates on \( P_{1g_2} \).

For this, we first take \( P_1 \) projection of (4.18) to obtain
\[ \partial_t P_{1g_2} - \frac{(3 + 2\gamma)}{\beta} \partial_\xi^2 P_{1g_2} - \frac{\beta'}{\beta} \{ \nabla_\xi (P_{1g_2} + P_{0g_2}) - P_0 (\xi \cdot \nabla_\xi g_2) \} \\
- \alpha [A_\xi \cdot \nabla_\xi (P_{1g_2} + P_{0g_2}) - P_0 (A_\xi \cdot \nabla_\xi g_2)] + \beta^\gamma L P_{1g_2} = \beta^\gamma P_1 \{ (1 - \chi_M) \mu^{-\frac{1}{2}} K_g \}. \]

Then letting \( 1 \leq |\theta| \leq N \), taking inner product of \( \partial_\xi^2 (4.19) \) with \( \partial_\xi^2 P_{1g_2} \), and applying Lemmas 5.2, 5.3, and 5.8 as well as Cauchy-Schwarz’s inequality, one has
\[ \sum_{1 \leq |\theta| \leq N} \frac{d}{dt} ||\partial_\xi^2 P_{1g_2}||^2 + \lambda \sum_{1 \leq |\theta| \leq N} \beta^\gamma ||\partial_\xi^2 P_{1g_2}||^2 \\
\leq C \alpha^2 \beta^\gamma ||[a_2, b_2, c_2]||^2 + C \beta^\gamma ||P_{1g_2}||^2 + C \sum_{|\theta| \leq N} \beta^\gamma ||w^{f_2} \partial_\xi^2 g_1||^2 \\
\leq C \beta^\gamma ||P_{1g_2}||^2 + C \sum_{|\theta| \leq N} \beta^\gamma ||w^{f_2} \partial_\xi^2 g_1||^2, \]

where (4.17) has been used again.

Next, combing (4.19) and (4.21) gives
\[ \sum_{|\theta| \leq N} \frac{d}{dt} ||\partial_\xi^2 P_{1g_2}||^2 + \lambda \beta^\gamma \sum_{|\theta| \leq N} ||\partial_\xi^2 P_{1g_2}||^2 \leq C \beta^\gamma \sum_{|\theta| \leq N} ||w^{f_2} \partial_\xi^2 g_1||^2. \]
Step 4. Weighted $H^N_\xi$ estimates for $g_1$. In this step, we intend to obtain the estimates of $\| w^{e_2} \partial^\theta_\xi g_1 \|^2$ with $\ell_\infty \geq 2\ell_2 \gg 5$ and $|\theta| \leq N$. Recall the following equations for $g_1$

$$
\partial_\xi g_1 - \frac{\beta'}{\beta} \nabla_\xi \cdot (\xi g_1) - \alpha \nabla_\xi \cdot (A\xi g_1) - \frac{2\gamma\beta'}{\beta} g_1 + \beta^\gamma \nu g_1
$$

$$
= \beta^\gamma \chi_M K g_1 - \frac{\beta'}{\beta} \xi^2 \sqrt{\mu_2} - \frac{\alpha}{2} \xi \cdot (A\xi) \sqrt{\mu_2} - \alpha 1-m \sqrt{\mu_2}^2 \partial_1 G_1 - \alpha 2-m \sqrt{\mu_2}^2 \partial_1 G_2
$$

$$
+ \alpha 1-m \beta^2 \gamma^{-1} \nabla_\xi \cdot (\xi \sqrt{\mu_1} G_1) + \alpha 2-m \beta^2 \gamma^{-1} \nabla_\xi \cdot (\xi \sqrt{\mu_1} G_2) + \alpha 3-m \beta^2 \gamma \nabla_\xi \cdot (A\xi \sqrt{\mu_1} G_2)
$$

$$
+ \alpha 1-m \beta^2 \gamma \{ Q(\sqrt{\mu_1} G_1, \sqrt{\mu_2} G_2) + Q(\sqrt{\mu_2} G_1, \sqrt{\mu_1} G_2) \} + \alpha 3-m \beta^2 \gamma \{ Q(\sqrt{\mu_2} G_2, \sqrt{\mu_1} G_1) \}
$$

$$
+ \alpha 2-m \beta^2 \gamma \{ Q(\sqrt{\mu_1} G_1, 1) + \alpha \sqrt{\mu_2} G_2, 1 \} + Q(\sqrt{\mu_2} G_1, 1) + \alpha 2-m \beta^2 \gamma \{ Q(\sqrt{\mu_2} G_2, 1) \},
$$

(4.23)

$$
g_1(0, \xi) = G_{R,0}.
$$

Next, taking the inner product of $\partial^\theta_\xi g_1$ and $w^{e_2} \partial^\theta_\xi g_1$ and using Cauchy-Schwarz’s inequality, one has

$$
\frac{d}{dt} \| w^{e_2} \partial^\theta_\xi g_1 \|^2 + \lambda \beta^\gamma \| w^{e_2} \partial^\theta_\xi g_1 \|^2
$$

$$
\leq C \left( \alpha^2 + \left| \frac{\beta'}{\beta} \right|^2 \right) \beta^\gamma \sum_{\theta' \leq \theta} \| \partial^\theta_\xi g_2 \|^2 + C \alpha^{2-2m} \beta^\gamma \sum_{\theta' \leq \theta} \| \partial^\theta_\xi [\partial_1 G_1, \nabla_\xi G_1] \|^2
$$

$$
+ C \alpha^{2-2m} \beta^\gamma \sum_{\theta' \leq \theta} \| \partial^\theta_\xi [G_1, \nabla_\xi G_1, \nabla_\xi G_2] \|^2 + C \alpha^{6-2m} \beta^\gamma \sum_{\theta' \leq \theta} \| \partial^\theta_\xi [G_2, \nabla_\xi G_2] \|^2
$$

$$
+ \beta^\gamma \{ \partial^\theta_\xi (\chi_M K g_1), w^{e_2} \partial^\theta_\xi g_1 \} + \alpha 3-m \beta^2 \gamma \{ \partial^\theta_\xi Q(\sqrt{\mu_1} G_1, \sqrt{\mu_2} G_2) + Q(\sqrt{\mu_2} G_1, \sqrt{\mu_1} G_2) \}, w^{e_2} \partial^\theta_\xi g_1 \}
$$

$$
+ \alpha 3-m \beta^2 \gamma \{ \partial^\theta_\xi Q(\sqrt{\mu_2} G_2, \sqrt{\mu_1} G_1) \} + \alpha 2-m \beta^2 \gamma \{ \partial^\theta_\xi Q(g_1 + \sqrt{\mu_2} g_1, \sqrt{\mu_2} g_1, \sqrt{\mu_2} g_2) \}, w^{e_2} \partial^\theta_\xi g_1 \}
$$

$$
+ \alpha^2 \{ \partial^\theta_\xi Q(\sqrt{\mu_1} G_1 + \alpha \sqrt{\mu_2} G_2, g_1 + \sqrt{\mu_2} g_1, \sqrt{\mu_2} g_1) \}, w^{e_2} \partial^\theta_\xi g_1 \}.
$$

(4.24)

We now turn to compute the right hand side of term by term. First of all, in light of (4.13), one has

$$
\left( \alpha^2 + \left| \frac{\beta'}{\beta} \right|^2 \right) \beta^\gamma \sum_{\theta' \leq \theta} \| \partial^\theta_\xi g_2 \|^2 \leq C \alpha^2 \sum_{\theta' \leq \theta} \| \partial^\theta_\xi g_2 \|^2 \leq C \alpha^2 \sum_{\theta' \leq \theta} \| \partial^\theta_\xi P_1 g_2 \|^2 + C \alpha^2 \| w^{e_2} g_1 \|^2,
$$

where (4.7) has been used in the last inequality, and (4.13) together with (4.7) and (4.8) gives

$$
\alpha^{2-2m} \beta^\gamma \sum_{\theta' \leq \theta} \| \partial^\theta_\xi [G_1, \nabla_\xi G_1, \nabla_\xi G_2] \|^2 \leq C \alpha^{6-2m}.
$$

Moreover, (4.7) and (4.8) with $k = 1$ also imply

$$
\alpha^{2-2m} \beta^\gamma \sum_{\theta' \leq \theta} \| \partial^\theta_\xi [\partial_1 G_1, \nabla_\xi G_2] \|^2 \leq C \alpha^{6-2m},
$$

and

$$
\alpha^{6-2m} \beta^\gamma \sum_{\theta' \leq \theta} \| \partial^\theta_\xi [G_2, \nabla_\xi G_2] \|^2 \leq C \alpha^{6-2m}.
$$

Furthermore, employing Proposition 2.1 and Cauchy-Schwarz’s inequality, we have

$$
\| (\partial^\theta_\xi (\chi_M K g_1), w^{e_2} \partial^\theta_\xi g_1) \| \leq C \eta \{ (1 + M)^{-\gamma} + \varsigma \} \sum_{\theta' \leq \theta} \| w^{e_2} \partial^\theta_\xi g_1 \| + \eta \| w^{e_2} \partial^\theta_\xi g_1 \|^2.
$$
Next, Lemma 6.8, 4.7, 4.8 and 4.17 as well as the a priori assumption (4.2) give
\[
\begin{align*}
\alpha^3 - m \beta^2 |\{Q(\sqrt{\mu} G_1, \sqrt{\mu} G_2) + Q(\sqrt{\mu} G_2, \sqrt{\mu} G_1), w^{2f_2} \partial_{\xi}^\theta g_1)\} |^2 & \leq C \alpha^3 - m \beta^2 \| w^{f_\infty} \partial_{\xi}^\theta g_1 \|_{L^\infty}^2 \sum_{\theta' \leq \theta} \| w^{f_2} \partial_{\xi}^\theta ' g_1 \|_{L^\nu} \| w^{f_2} \partial_{\xi}^\theta g_2 \|_{L^\nu}^2 \\
& \leq C \alpha^3 - m \beta^2 \| w^{f_\infty} \partial_{\xi}^\theta g_1 \|_{L^\infty}^2,
\end{align*}
\]
\[
\begin{align*}
\alpha^4 - m \beta^2 |\{Q(\sqrt{\mu} G_1, \sqrt{\mu} G_2, w^{2f_2} \partial_{\xi}^\theta g_1)\} |^2 & \leq C \alpha^4 - m \beta^2 \| w^{f_\infty} \partial_{\xi}^\theta g_1 \|_{L^\infty}^2 \sum_{\theta' \leq \theta} \| w^{f_2} \partial_{\xi}^\theta ' g_1 \|_{L^\nu} \| w^{f_2} \partial_{\xi}^\theta g_2 \|_{L^\nu}^2 \\
& \leq C \alpha^4 - m \beta^2 \| w^{f_\infty} \partial_{\xi}^\theta g_1 \|_{L^\infty}^2,
\end{align*}
\]
\[
\begin{align*}
\alpha^m \beta^\gamma |\{Q(\sqrt{\mu} G_1 + \alpha \sqrt{\mu} G_2, g_1 + \sqrt{\mu} g_2 + Q(\sqrt{\mu} G_2, g_1 + \sqrt{\mu} G_1 + \alpha \sqrt{\mu} G_2), w^{2f_2} \partial_{\xi}^\theta g_1)\} |^2 & \leq C \alpha^m \| w^{f_\infty} \partial_{\xi}^\theta g_1 \|_{L^\infty}^2 \sum_{\theta' \leq \theta} \| w^{f_2} \partial_{\xi}^\theta ' g_1 \|_{L^\nu} \| w^{f_2} g_2 \|_{L^\nu}^2 + C \alpha^2 \| w^{f_\infty} \partial_{\xi}^\theta g_1 \|_{L^\infty}^2,
\end{align*}
\]
and
\[
\begin{align*}
\alpha^\beta |\{Q(\sqrt{\mu} G_1 + \alpha \sqrt{\mu} G_2, g_1 + \sqrt{\mu} g_2 + Q(\sqrt{\mu} G_2, g_1 + \sqrt{\mu} G_1 + \alpha \sqrt{\mu} G_2), w^{2f_2} \partial_{\xi}^\theta g_1)\} |^2 & \leq C \alpha^\beta \| w^{f_\infty} \partial_{\xi}^\theta g_1 \|_{L^\infty}^2 \sum_{\theta' \leq \theta} \| w^{f_2} \partial_{\xi}^\theta ' g_1 \|_{L^\nu} \| w^{f_2} g_2 \|_{L^\nu}^2 + C \alpha^2 \| w^{f_\infty} \partial_{\xi}^\theta g_1 \|_{L^\infty}^2.
\end{align*}
\]
Plugging the above estimates into (4.24) and adjusting constants, we obtain for \( m \in (2, 3) \)
\[
\frac{d}{dt} \sum_{\theta' \leq \theta} \| w^{f_2} \partial_{\xi}^\theta g_1 \|_{L^\nu}^2 + \lambda \beta^\gamma \sum_{\theta' \leq \theta} \| w^{f_2} \partial_{\xi}^\theta g_1 \|_{L^\nu}^2 & \leq C (\alpha^2 + \eta) \sum_{\theta' \leq \theta} \| w^{f_2} g_2 \|_{L^\nu}^2 + \| w^{f_\infty} \partial_{\xi}^\theta g_1 \|_{L^\infty}^2 + C \alpha^{6-2m}.
\]
Consequently, one gets from (4.22) and (4.25) that
\[
\frac{d}{dt} \sum_{\theta' \leq \theta} \| w^{f_2} \partial_{\xi}^\theta g_1 \|_{L^\nu}^2 + \lambda \beta^\gamma \sum_{\theta' \leq \theta} \| w^{f_2} \partial_{\xi}^\theta g_1 \|_{L^\nu}^2 & \leq \eta \| w^{f_\infty} \partial_{\xi}^\theta g_1 \|_{L^\infty}^2 + C \alpha^{6-2m},
\]
which together with (4.17) further implies (4.16). This then ends the proof of Lemma 4.2.
\[\square\]

4.3. Proof of Theorem 4.1. We are ready to complete the proof of Theorem 4.1. In fact, as explained at the beginning of this section, it suffices to prove (4.1) under the assumption (4.2). Indeed, combining (4.3), (4.4) and (4.16) together, we conclude that
\[
\sup_{0 \leq s \leq t} \sum_{\theta' \leq \theta} \| w^{f_\infty} \partial_{\xi}^\theta g_1(s) \|_{L^\infty} + \sup_{0 \leq s \leq t} \sum_{\theta' \leq \theta} \| w^{f_\infty} \partial_{\xi}^\theta g_2(s) \|_{L^\infty} & \leq C \sum_{|\theta| \leq N} \| w^{f_\infty} \partial_{\xi}^\theta G_{R,0} \|_{L^\infty} + C \alpha^{3-m},
\]
which in turn makes the assumption (4.22) close. The above estimate further gives
\[
\sum_{|\vartheta| \leq N} \| u^\vartheta \partial_\vartheta^\vartheta (\sqrt{\mu} G_R)(t) \|_{L^\infty}
\]
\[
\leq C \beta^{-2\gamma}(t) \left\{ \sum_{|\vartheta| \leq N} \| u^\vartheta \partial_\vartheta^\vartheta G_{R,0} \|_{L^\infty} + C \alpha^{3-m} \right\}
\]
\[
\leq C \beta^{-2\gamma}(t) \alpha^{-m} \sum_{|\vartheta| \leq N} \| u^\vartheta \partial_\vartheta^\vartheta [F_0(v) - (\mu + \sqrt{\mu} \{ \alpha G_1(0, v) + \alpha^2 G_2(0, v) \})] \|_{L^\infty}
\]
\[
+ C \beta^{-2\gamma}(t) \alpha^{3-m},
\]
for any \( t \geq 0 \), according to \( \sqrt{\mu} G_R = \beta^{-2\gamma}(g_1 + \sqrt{\mu} g_2) \) and
\[
G_{R,0}(\xi) = G_{R,0}(v) = \alpha^{-m} \left\{ F_0(v) - [\mu + \sqrt{\mu} \{ \alpha G_1(0, v) + \alpha^2 G_2(0, v) \}] \right\}.
\]
In addition, from (1.11), it follows
\[
G(t, \xi) = \beta^3 F(t, \beta \xi).
\]
Finally, (1.33) follows from (4.9) and (4.27) together with (4.26) by renaming the velocity variable. Moreover, since \( \beta(t) \to \infty \) as \( t \to \infty \), it follows from (1.34) and (4.26) that
\[
\lim_{t \to \infty} \frac{\beta^\gamma}{1 + \gamma g_0^2 \alpha^2 t} = \lim_{t \to \infty} \frac{\beta^{\gamma-1}}{\beta_0^2} = 1 + \lim_{t \to \infty} \frac{\alpha^m - \alpha^{m-1}}{3 \beta_0} \int_{\mathbb{R}^3} \xi \cdot A \xi \beta^\gamma(\sqrt{\mu} G_R) \, d\xi = 1,
\]
which proves (5.2). This ends the proof of Theorem 1.1.

5. Appendix

In this section, we provide those estimates that have been used in the previous sections. In particular, we give the basic estimates on the linearized operator \( L \) as well as the nonlinear operators \( \Gamma \) and \( Q \), and also present a key estimate for the operator \( K \) in the case of hard potentials.

The following lemma is concerned with the integral operator \( K \) given by (1.25), and its proof in case of the hard sphere model \( (\gamma = 1) \) has been given by [28, Lemma 3, pp. 727].

Lemma 5.1. Let \( K \) be defined as (1.25), then it holds that
\[
K f(\xi) = K_2 f(\xi) - K_1 f(\xi) = \int_{\mathbb{R}^3} (k_2(\xi, \xi_*) - k_1(\xi, \xi_*)) f(\xi_*) \, d\xi_*
\]
with
\[
k_1(\xi, \xi_*) = \tilde{C}_1 |\xi - \xi_*|^{\gamma} e^{-\frac{\xi_*^2 + |\xi_\bot|^2}{4}},
\]
and
\[
k_2(\xi, \xi_*) = \tilde{C}_2 |\xi - \xi_*|^{-2 \gamma} e^{-\frac{1}{8} |\xi - \xi_*|^2 - \frac{1}{8} \frac{|\xi_*^2 - |\xi_*\bot|^2|^2}{|\xi_* - \xi_\bot|^2}}.
\]
Here both \( \tilde{C}_1 \) and \( \tilde{C}_2 \) are positive constants.

In addition, let
\[
k(\xi, \xi_*) = k_2(\xi, \xi_*) - k_1(\xi, \xi_*), \quad k_w(\xi, \xi_*) = w^\ell(\xi) k(\xi, \xi_*) w^{-\ell}(\xi_*)
\]
with \( \ell \geq 0 \), then it also holds that
\[
\int_{\mathbb{R}^3} k_w(\xi, \xi_*) e^{\frac{\xi_*^2 + |\xi_\bot|^2}{8}} \, d\xi_* \leq \frac{C}{1 + |\xi|},
\]
for \( \varepsilon = 0 \) or any \( \varepsilon > 0 \) small enough.

Moreover, for any multi-indices \( \vartheta \) and any \( \ell \geq 0 \) it holds that
\[
|w^\ell \partial_\vartheta^\vartheta (K f)| \leq C \sum_{|\vartheta| \leq \vartheta} \| w^\ell \partial_\vartheta^\vartheta f \|_{L^\infty}.
\]
Proof. We prove \[5.3\] only, since the other statements in the Lemma is well known. By \[5.1\] and a change of variables $\xi_* - \xi \to u$, we have

$$
\partial_\xi^\gamma (K_1 f) = \tilde{C}_1 \sum_{\partial' \leq \partial} C_\partial^\gamma \int_{\mathbb{R}^3} |u| \gamma \partial_\xi^\gamma \left\{ e^{-\frac{|\xi|^2 + |u|^2 + |\xi_*|^2}{2}} \right\} \partial_\xi^{\partial - \partial'} f(u + \xi) du
$$

and

$$
\partial_\xi^\gamma (K_2 f) = \tilde{C}_1 \sum_{\partial' \leq \partial} C_\partial^\gamma \int_{\mathbb{R}^3} |u|^{-2+\gamma} \partial_\xi^\gamma \left\{ e^{-\frac{|\xi|^2 + |u|^2}{2}} \frac{|\xi|^2 - |u|^2}{|u|^2} \right\} \partial_\xi^{\partial - \partial'} f(u + \xi) du
$$

where

$$
\tilde{k}_1(\xi, \xi_*) = |\xi - \xi_*|^\gamma \left( \partial_\xi^\gamma \left\{ e^{-\frac{|\xi|^2 + |u|^2 + |\xi_*|^2}{2}} \right\} \right) \bigg|_{u = \xi - \xi_*},
$$

and

$$
\tilde{k}_2(\xi, \xi_*) = |\xi - \xi_*|^{-2+\gamma} \left( \partial_\xi^\gamma \left\{ e^{-\frac{|\xi|^2 + |u|^2}{2}} \frac{|\xi|^2 - |u|^2}{|u|^2} \right\} \right) \bigg|_{u = \xi - \xi_*}.
$$

Furthermore, it is direct to see

$$
|\tilde{k}_1(\xi, \xi_*)| \leq C(\theta)|\xi - \xi_*|\gamma e^{-\frac{|\xi|^2 + |\xi_*|^2}{2}},
$$

and

$$
|\tilde{k}_2(\xi, \xi_*)| \leq C(\theta)|\xi - \xi_*|^{-2+\gamma} e^{-\frac{|\xi|^2 + |\xi_*|^2}{2}}.
$$

Then performing the similar calculation as for obtaining \[5.2\], one sees that \[5.3\] is true. This completes the proof of Lemma \[5.1\].

For the weighted velocity derivative estimates on the nonlinear operator $\Gamma$, one has the following result.

Lemma 5.2. Let $0 \leq \gamma \leq 1$ and $\theta \in [0, 1]$. For any $p \in [1, +\infty]$ and any $\ell \geq 0$, it holds that

$$
\| v^\ell \nu^{-\theta} \partial_\xi^\gamma \Gamma(f, g) \|_{L^p} \leq C \sum_{\partial' \leq \partial} \left\{ \| v^\ell \nu^{1-\theta} \partial_\xi^\gamma f \|_{L^p} \| \partial_\xi^\gamma g \|_{L^p} + \| \partial_\xi^\gamma f \|_{L^p} \| v^\ell \nu^{1-\theta} \partial_\xi^\gamma g \|_{L^p} \right\}.
$$

The following lemma is concerned with coercivity estimates for the linear collision operator $L$.

Lemma 5.3. Let $0 \leq \gamma \leq 1$, then there is a constant $\delta_0 > 0$ such that

$$
\langle L f, f \rangle = \langle L P_1 f, P_1 f \rangle \geq \delta_0 \| P_1 f \|_{L^2}^2,
$$

where $\| \cdot \|_{L^2} = \| \nu^\frac{1}{2} \cdot \|$. Moreover, there are constants $\delta_1 > 0$ and $C > 0$ such that for $|\theta| > 0$

$$
\langle \partial_\xi^\gamma L f, \partial_\xi^\gamma f \rangle \geq \delta_1 \| \partial_\xi^\gamma f \|_{L^2}^2 - C \| f \|_{2}^2.
$$

Next, the following lemma which was proved in [20] Proposition 3.1, pp.13 gives the $L^\infty$ estimates of the solutions in the case of Maxwell molecule model.

Lemma 5.4. Let $\gamma = 0$ and $K$ be given by \[1.20\], then for any nonnegative integer $|\theta| \geq 0$, there is $C > 0$ such that for any arbitrarily large $\ell > 0$, there is $M = M(\ell) > 0$ such that it holds that

$$
\chi_M v^\ell |\partial_\xi^\gamma (Kf)| \leq \frac{C}{\ell} \sum_{0 \leq \partial' \leq \partial} \| v^\ell \partial_\xi^{\partial'} f \|_{L^\infty}.
$$

In particular, one can choose $M = \ell^2$. 
In the case of $0 < \gamma \leq 1$, the following lemma with $\theta = 0$ which can be found in \cite{{1}} Proposition 3.1, pp.397 enables us to gain the smallness property of $K$ at large velocity.

**Lemma 5.5.** Let $0 \leq \gamma \leq 1$, $\ell > 4$ and for any multi-indices $\theta \geq 0$, then there exists a function $\varsigma(\ell)$ which satisfies $\varsigma(\ell) \to 0$ as $\ell \to +\infty$ such that

$$
\begin{align*}
& w^\ell \{ |\partial^\theta_\xi Q_{\text{loss}}(f, g) | + |\partial^\theta_\xi Q_{\text{gain}}(f, g) | + |\partial^\theta_\xi Q_{\text{gain}}(g, f) | \} \\
& \leq \sum_{\theta' \leq \theta} \| w^\ell \partial^{\theta'}_\xi f \|_{L^\infty} \{ C(\ell) \| w^{\ell + \gamma/2} \partial^{\theta'}_\xi g \|_{L^\infty} + \varsigma(\ell) \| w^3 \partial^{\theta'}_\xi g \|_{L^\infty} (1 + |\xi|)^\gamma \},
\end{align*}
$$

(5.4)

where $Q_{\text{loss}}$ denotes the negative part of $Q$ in \cite{{1}}.

**Proof.** Since the case that $\theta = 0$ of (5.3) has been given in \cite{{1}} Proposition 3.1, pp.397], here we focus on the case of $\theta > 0$. As a matter of fact, for $\theta > 0$, as (2.7), by a change of variables $\xi - \xi \to u$, we have

$$
\begin{align*}
\partial^\theta_\xi Q(f, g) &= \sum_{\theta' \leq \theta} C^{\theta'}_\theta \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_0 |u|^{\gamma} (\partial^{\theta - \theta'}_\xi f)(\xi + u) (\partial^{\theta'}_\xi g)(\xi + u) d\omega du \\
& \quad - \sum_{\theta' \leq \theta} \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_0 |u|^{\gamma} (\partial^{\theta - \theta'}_\xi f)(u + \xi) (\partial^{\theta'}_\xi g)(\xi) d\omega du,
\end{align*}
$$

which further equals

$$
\begin{align*}
& \sum_{\theta' \leq \theta} C^{\theta'}_\theta \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_0 |\xi - \xi|^{\gamma} (\partial^{\theta - \theta'}_\xi f)(\xi')(\partial^{\theta'}_\xi g)(\xi') d\omega d\xi \\
& \quad - \sum_{\theta' \leq \theta} \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_0 |\xi - \xi|^{\gamma} (\partial^{\theta - \theta'}_\xi f)(\xi)(\partial^{\theta'}_\xi g)(\xi) d\omega d\xi,
\end{align*}
$$

by changing the variables back. Then performing the same calculations as for the case $\theta = 0$ in \cite{{1}} Proposition 3.1, pp.397], one sees that (5.3) holds true. This completes the proof of Lemma 5.5.

The following result is a direct consequence of Lemma 5.5.

**Lemma 5.6.** Let $0 < \gamma \leq 1$, then there is a constant $C > 0$ such that for any arbitrarily large $\ell > 4$ and any multi-indices $\theta \geq 0$, there are sufficiently large $M = M(\ell) > 0$ and suitably small $\varsigma = \varsigma(\ell) > 0$ such that it holds that

$$
\chi_M \nu^{-1} w^\ell |\partial^\theta_\xi K f | \leq C \sum_{\theta' \leq \theta} \{ (1 + M)^{-\gamma} + \varsigma \} \| w^\ell \partial^{\theta'}_\xi f \|_{L^\infty}.
$$

(5.5)

The following Lemma concerning the polynomial weighted estimates on the collision operator $Q$ can be verified by using a parallel argument as for obtaining \cite{{1}} Proposition 3.1, pp.397.

**Lemma 5.7.** Let $\ell > 4$ and $\gamma \geq 0$, then it holds that

$$
\begin{align*}
& \| w^\ell \nu^{-1} \partial^\theta_\xi Q_{\text{gain}}(F_1, F_2) \|, \| w^\ell \nu^{-1} \partial^\theta_\xi Q_{\text{loss}}(F_1, F_2) \| \leq C \sum_{\theta' \leq \theta} \| w^\ell \partial^{\theta'}_\xi F_1 \|_{L^\infty} \| w^\ell \partial^{\theta - \theta'}_\xi F_2 \|_{L^\infty}.
\end{align*}
$$

Finally, we give the following crucial estimates on the inner product involving $Q(f, g)$.

**Lemma 5.8.** Let $\ell_\infty > 2\ell_2 \gg 4$, then it holds that

$$
\begin{align*}
& \| (\partial^\theta_\xi Q(f, g), w^{2\ell_2} h) \| \leq C \sum_{\theta' + \theta'' \leq \theta} \| w^{\ell_\infty} h \|_{L^\infty} \| \nu^{1/2} w^{\ell_2} \partial^{\theta'}_\xi f \| \| \nu^{1/2} w^{\ell_2} \partial^{\theta''}_\xi g \|.
\end{align*}
$$

(5.6)

In particular, it holds that

$$
\begin{align*}
& \| (\partial^\theta_\xi [(1 - \chi_M)Q(f, g)], h) \| \leq C \sum_{\theta' + \theta'' \leq \theta} \| h \| \| w^{\ell_2} \partial^{\theta'}_\xi f \| \| w^{\ell_2} \partial^{\theta''}_\xi g \|.
\end{align*}
$$

(5.7)
We now turn to prove (5.7). As (5.8), it follows that
\[
\vartheta \text{ true in the case of } \end{proof}
\] Following the similar proof of Proposition 2.1, we first prove that both (5.6) and (5.7) are
\[
\text{true.}
\]
Proof. Notice
\[
\ell \geq C \int_\mathbb{R} (f^2) \int_\mathbb{S} \int_\mathbb{R} B_0 |\xi_\ast - \xi|^\gamma |f(\xi_\ast)g(\xi)| (w^{2\xi} h)(\xi) d\xi d\omega
\]
where (5.9) has been used again.

Since \( \ell_\infty > 2\ell_2 \gg 4 \), we have by Hölder’s inequality that
\[
Q_2 \leq C ||w^\ell h||_{L^\infty} \int_\mathbb{R} \langle \xi \rangle^\gamma w^{-2\ell_2} \int_\mathbb{S} \int_\mathbb{R} \|w^{2\xi} f\| ||w^{2\xi} g|| \leq C ||w^\ell h||_{L^\infty} \|w^{2\xi} f\| \|w^{2\xi} g\|
\]
where the fact that \( |\xi_\ast - \xi|^\gamma \leq (\langle \xi \rangle^\gamma (\xi))^\gamma \) has been used.

Likewise, for \( Q_1 \), Hölder’s inequality and a change of variables \((\xi, \xi_\ast) \rightarrow (\xi', \xi')\) give
\[
Q_1 \leq C ||w^\ell h||_{L^\infty} \int_\mathbb{R} \left( \int_\mathbb{S} \int_\mathbb{R} |\xi_\ast - \xi'|^\gamma w^{-2\ell_2} (\xi') \int_\mathbb{S} \int_\mathbb{R} \|w^{2\xi} f\| ||w^{2\xi} g|| \right) \]
\[
\leq C ||w^\ell h||_{L^\infty} \left( \int_\mathbb{R} \int_\mathbb{S} \int_\mathbb{R} |\xi_\ast - \xi|^\gamma w^{-2\ell_2} (\xi) \int_\mathbb{S} \int_\mathbb{R} \|w^{2\xi} f\| ||w^{2\xi} g|| \right)
\]
\[
\leq C ||w^\ell h||_{L^\infty} \|w^{2\xi} f\| \|w^{2\xi} g\|
\]
We now turn to prove (5.7). As (5.8), it follows that
\[
|\{1 - \chi_M\}Q(f, g, h)\| \leq \int_\mathbb{R} \int_\mathbb{S} \int_\mathbb{R} B_0 |\xi_\ast - \xi|^\gamma |f(\xi_\ast)\int_\mathbb{S} \int_\mathbb{R} \{1 - \chi_M\} h(\xi) d\omega d\xi
\]
\[
\leq \int_\mathbb{R} \int_\mathbb{S} \int_\mathbb{R} B_0 |\xi_\ast - \xi|^\gamma |f(\xi_\ast)\int_\mathbb{S} \int_\mathbb{R} \{1 - \chi_M\} h(\xi) d\omega d\xi =: \tilde{Q}_1 + \tilde{Q}_2.
\]
Notice \( \ell_2 \geq 2 \). It then follows
\[
\int_\mathbb{R} \int_\mathbb{S} \int_\mathbb{R} |\xi_\ast - \xi|^\gamma w^{-2\ell_2} (\xi_\ast) d\xi_\ast \leq C \langle \xi \rangle^\gamma
\]
which together with Hölder’s inequality gives
\[
\tilde{Q}_2 \leq C ||w^{2\xi} f\| \int_\mathbb{R} \left( \int_\mathbb{S} \int_\mathbb{R} |\xi_\ast - \xi|^\gamma w^{-2\ell_2} (\xi_\ast) d\xi_\ast \right) \frac{1}{2} \|g(1 - \chi_M) h(\xi)\| d\xi
\]
\[
\leq C ||w^{2\xi} f\| ||g|| ||h||.
\]
For \( \tilde{Q}_1 \), using Hölder’s inequality and a change of variables \((\xi, \xi_\ast) \rightarrow (\xi', \xi')\), one has
\[
\tilde{Q}_1 \leq C \int_\mathbb{R} \left( \int_\mathbb{S} \int_\mathbb{R} |\xi'|^\gamma w^{-2\ell_2} (\xi') \int_\mathbb{S} \int_\mathbb{R} \|w^{2\xi} f\| ||w^{2\xi} g|| \right) \]
\[
\leq C ||h|| \left( \int_\mathbb{R} \int_\mathbb{S} \int_\mathbb{R} |w^{2\xi} f (\xi) (w^{2\xi} g) (\xi')|^2 d\xi_\ast d\xi \right) \frac{1}{2} \|g(1 - \chi_M) h(\xi)\| d\xi
\]
\[
\leq C ||h|| \|w^{2\xi} f\| \|w^{2\xi} g\|.
\]
where (5.11) has been used again.
Next, if $\vartheta > 0$, one has as for obtaining (2.8) that
\[
\partial^\vartheta_\xi Q(f, g) = \sum_{\varrho' + \varrho'' \leq \vartheta} C^{\varrho'}_\vartheta (\varrho'' f) \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_0|\xi + \varrho f(\xi')| \omega d\xi
\]

Then, performing the similar calculations as in the case of $\vartheta = 0$, we see that both (5.8) and (5.7) are also valid for $\vartheta > 0$. The proof of Lemma 5.8 is finished.

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