A note concerning a tidying procedure and contraction groups in non-metrizable, totally disconnected groups

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Abstract

We establish a technical result concerning automorphisms of not necessarily metrizable, totally disconnected, locally compact groups, which resolves an open problem formulated by George Willis in 2004. The result adds an important detail to a recent work of Wojciech Jaworski, who showed that the main technical tool from a 2004 article by Baumgartner and Willis on contraction groups remains valid in the non-metrizable case. Jaworski asserted without proof that, therefore, all results from that article remain valid. However, metrizability enters at a second point, and our result is designed to bypass this difficulty.

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1 Introduction and statement of the result

Consider a totally disconnected, locally compact topological group \( G \) and an automorphism \( \alpha: G \to G \). At the heart of the structure theory of totally disconnected groups is the discovery that there exist compact open subgroups of \( G \), the tidy subgroups, which are well-adapted to \( \alpha \) in a suitable sense (specified in [7] and [8]). Starting from an arbitrary compact open subgroup, subgroups tidy for \( \alpha \) can be constructed by various “tidying procedures.” Three different constructions can be found in [7], [8], and [9]. The tidying procedure from [9] is particularly important, and has been used in a recent article by U. Baumgartner and G. A. Willis to gain insight into the so-called contraction groups

\[
U_\alpha := \{ x \in G : \lim_{n \to \infty} \alpha^n(x) = 1 \}
\]

(where \( 1 \in G \) is the identity element), and into the role of contraction groups within the structure theory of totally disconnected groups [1]. In particular, twelve equivalent characterizations for closedness of \( U_\alpha \) were obtained, which provide links between contraction groups and other central concepts from the structure theory [1, Theorem 3.32]. The tidying procedure in [9] involves a certain compact subgroup \( K \subseteq G \). Before we can define \( K \), let us recall from [1] that \( \alpha \) goes along with a so-called parabolic subgroup \( P_\alpha \) and a Levi factor \( M_\alpha \), which consist of all \( x \in G \) whose forward orbit \( \{ \alpha^n(x) : n = 0, 1, 2, \ldots \} \) (resp., whose two-sided orbit \( \{ \alpha^n(x) : n = 0, \pm 1, \pm 2, \ldots \} \)) is relatively compact in \( G \). It is known that \( P_\alpha \) and \( M_\alpha = P_\alpha \cap P_{\alpha^{-1}} \) are closed subgroups of \( G \) (see [1, p. 224]). Given a compact, open subgroup \( \mathcal{O} \) of \( G \), let us write \( K_\mathcal{O} \) for the set of all \( x \in M_\alpha \) such that \( \alpha^n(x) \in \mathcal{O} \) for all sufficiently large \( n \). Moreover, let

\[
K_\mathcal{O} := \overline{\mathcal{L}_\mathcal{O}}
\]

(1)
be the closure of $K_\mathcal{O}$ in $G$. By [9, Lemma 2.2], a compact group is obtained via

$$K := \bigcap_{\mathcal{O} \in \mathcal{B}(G)} K_\mathcal{O}, \quad (2)$$

where $\mathcal{B}(G)$ denotes the set of all compact, open subgroups of $G$ (see Step 2a in [9, p. 4]). If $G$ is metrizable, the $K$ admits a simpler description, as the closure

$$K = \overline{U_\alpha \cap M_\alpha} \quad (3)$$

(see [9, lines preceding Proposition 2.1]). Note that $U_\alpha \cap M_\alpha = U_\alpha \cap P_\alpha$ here, because $U_\alpha \subseteq P_\alpha$ and $M_\alpha = P_\alpha \cap P_{\alpha^{-1}}$. Previously, it was not known whether (3) remains valid if $G$ is not metrizable (see [9, ibid.]). The result presented in this note clarifies this question. In Section 3, we prove:

**Theorem.** Let $G$ be a totally disconnected, locally compact topological group and $\alpha: G \to G$ be an automorphism. Then

$$\overline{U_\alpha \cap P_{\alpha^{-1}}} = \bigcap_{\mathcal{O} \in \mathcal{B}(G)} K_\mathcal{O}. \quad (4)$$

Our proof of this theorem is stimulated by Jaworski’s general strategy in [4]: We shall reduce to the case where $G$ is compactly generated as an $\langle \alpha \rangle$-group, and then use that any such group can be approximated by metrizable groups.

Recall that contraction groups $U_\alpha$ arise in representation theory in connection with the Mautner phenomenon [6, Chapter II, Lemma 3.2]. Moreover, they play an important role in the structure theory of totally disconnected groups [1], and in probability theory on locally compact groups (in connection with semi-stable convolution semigroups [3]). Also a certain generalization is needed in the latter areas: For $H \subseteq G$ a closed subgroup which is $\alpha$-stable (i.e., $\alpha(H) = H$), one considers the contraction set $U_{\alpha/H}$ of $\alpha$ modulo $H$, defined via

$$U_{\alpha/H} := \{ x \in G : \lim_{n \to \infty} \alpha^n(xH) = H \text{ in } G/H \}$$

(where $\alpha_H: G/H \to G/H, gH \mapsto \alpha(g)H$). If $H$ is compact or normal, then $U_{\alpha/H}$ is a subgroup of $G$. If $G$ is metrizable, Baumgartner and Willis were able to express $U_{\alpha/H}$ as the complex product

$$U_{\alpha/H} = U_\alpha H \quad (5)$$

(see [1, Theorem 3.8]), and this identity actually is their main technical tool for the discussion of contraction groups. Jaworski proved that (5) remains valid for general $G$, irrespective of metrizability [5, Theorem 1].

As already observed in [2, p. 331], Proposition 3.7–Corollary 3.30 from [1] remain valid for non-metrizable $G$ as soon as (5) is available for all $H$. In the introduction of [5], Jaworski asserts that, actually, all results from [1] (including
2 Preliminaries

Henceforth, we write \( \mathbb{Z} \) for the group of integers. By definition, automorphisms of a topological group \( G \) are, in particular, homeomorphisms. Surjective, open, continuous homomorphisms between topological groups are called quotient morphisms. As usual, \( G \) is said to be compactly generated if \( G = \langle K \rangle \) for a compact subset \( K \subseteq G \). If \( \alpha: G \to G \) is an automorphism, we say that \( G \) is compactly generated as an \( \langle \alpha \rangle \)-group if there exists a compact set \( K \subseteq G \) such that \( \bigcup_{n \in \mathbb{Z}} \alpha^n(K) \) generates \( G \). Various facts will help us to prove our theorem.

The first lemma varies an argument used in the proof of \([5\text{, Theorem 1]}\), where however \([4\text{, Theorem 8.7]}\) is misquoted\([1]\).

**Lemma 1.** Let \( G \) be a locally compact group and \( \alpha: G \to G \) be an automorphism. If \( G \) is compactly generated as an \( \langle \alpha \rangle \)-group, then every identity neighbourhood \( U \subseteq G \) contains a compact, normal, \( \alpha \)-stable subgroup \( N \) of \( G \) such that \( G/N \) is metrizable.

**Proof.** Let \( K \) be a compact subset of \( G \) which generates \( G \) as an \( \langle \alpha \rangle \)-group. Then the semidirect product \( H := G \rtimes_\alpha \mathbb{Z} \) is compactly generated (by \( K \), together with a generator of \( \mathbb{Z} \)), and hence has a compact normal subgroup \( N \subseteq U \) such that \( H/N \) is metrizable \([4\text{, Theorem 8.7]}\). Then \( N \) is as desired. □

The next observation is a special case of \([5\text{, Lemma 2]}\).

**Lemma 2.** Let \( q: G \to Q \) be a quotient morphism between topological groups. If \( \alpha: G \to G \) and \( \beta: Q \to Q \) are automorphisms such that \( \beta \circ q = q \circ \alpha \), then

\[
U_{\alpha/q^{-1}(H)} = q^{-1}(U_{\beta/H}),
\]

for each closed subgroup \( H \subseteq Q \). □

Finally, we shall need the following obvious fact concerning the compact subgroups defined in \([1]\) above: If \( \mathcal{O} \) and \( Q \) are compact, open subgroups of \( G \) such that \( \mathcal{O} \subseteq Q \), then \( K_{\mathcal{O}} \subseteq K_Q \) and hence

\[
K_{\mathcal{O}} \subseteq K_Q. \tag{6}
\]

\(^1\)As it stands, the theorem applies to compactly generated groups, not to \( \sigma \)-compact groups.
3 Proof of the theorem

Step 1. Let \( \beta \) be the restriction of \( \alpha \) to an automorphism of \( M_\alpha \). Given \( \mathcal{O} \in \mathcal{B}(M_\alpha) \), define \( K_\mathcal{O} \) as above, replacing \( G \) with \( M_\alpha \) and \( \alpha \) with \( \beta \). We have
\[
U_\alpha \cap P_{\alpha^{-1}} = U_\alpha \cap M_\alpha = U_\beta.
\]
Since \( M_\alpha \) is closed, taking the closure of \( U_\beta \) in \( G \) or in \( M_\alpha \) amounts to the same, and hence the left hand side of (4) remains unchanged if we replace \( G \) by \( M_\alpha \) and \( \alpha \) by \( \beta \). Now
\[
K_\mathcal{O} \cap M_\alpha = K_\mathcal{O} \cap M_\alpha
\]
for each \( \mathcal{O} \in \mathcal{B}(G) \). Since the sets \( \mathcal{O} \cap M_\alpha \) are cofinal in \( \mathcal{B}(M_\alpha) \) (when this set is directed via inverse inclusion), using (6) we see that the suggested replacements also leave the right hand side of (4) unchanged. After replacing \( G \) by \( M_\alpha \) and \( \alpha \) by \( \beta \), we may therefore assume henceforth that \( G = M_\alpha \).

Step 2. Every compact subset \( X \) of \( G \) is contained in an \( \alpha \)-stable open subgroup \( H \) of \( G \) which is compactly generated as an \( \langle \alpha \mid H \rangle \)-group.

[In fact, after replacing \( X \) by the union of \( X \) and a compact identity neighbourhood in \( G \), we may assume that \( X \) is an identity neighbourhood. Then the subgroup \( H \) of \( G \) generated by \( \bigcup_{n \in \mathbb{Z}} \alpha^n(X) \) is open, \( \alpha \)-stable, and generated by the compact set \( X \) as an \( \langle \alpha \mid H \rangle \)-group.] In particular, every point of \( G \) is contained in an \( \alpha \)-stable, open subgroup \( H \) of \( G \) which is compactly generated as an \( \langle \alpha \mid H \rangle \)-group, entailing that the validity of (4) can be checked by intersecting both sides with such \( H \). Consequently, we need only prove (4) for \( H \) instead of \( G \). Hence, after replacing \( G \) by \( H \), we may assume now that \( G \) is compactly generated as an \( \langle \alpha \rangle \)-group (and \( G = M_\alpha \)).

Step 3. As we assume \( G = M_\alpha \), we have \( G = P_{\alpha^{-1}} \). Thus (4) reads
\[
\bigcup \alpha = \bigcap_{\mathcal{O} \in \mathcal{B}(G)} K_\mathcal{O}.
\]
Let \( \mathcal{N}(G) \) be the set of all compact, normal, \( \alpha \)-stable subgroups \( N \subseteq G \) such that \( G/N \) is metrizable. As is well known, the closure of a subset \( S \subseteq G \) is given by \( \overline{S} = \bigcap_U SU \), for \( U \) ranging through the identity neighbourhoods in \( G \). Since any \( U \) contains some \( N \in \mathcal{N}(G) \) (by Lemma 1), we deduce that
\[
\overline{U_\alpha} = \bigcap_{N \in \mathcal{N}(G)} U_\alpha N.
\]
Given \( N \in \mathcal{N}(G) \), let \( q_N: G \to G/N \) be the canonical quotient map and \( \alpha_N \) the automorphism of \( G/N \) induced by \( \alpha \) (determined by \( \alpha_N \circ q_N = q_N \circ \alpha \)). Then
\[
\overline{U_\alpha} N = \overline{U_\alpha N} = \overline{U_\alpha /N} = \overline{q_N^{-1}(U_\alpha N)} = q_N^{-1} \left( \overline{U_\alpha N} \right),
\]
using the compactness of \( N \) for the first equality, then (3) (Jaworski’s main result), then Lemma 2, and finally that \( q_N \) is a closed map (since \( N \) is compact).
We now exploit that $G/N$ is metrizable, whence the conclusion of our theorem is available for $G/N$ (see [9, p. 4]). Using that $G/N = M_\alpha N$, we therefore have

$$U_{\alpha N} = \bigcap_{O \in B(G/N)} K_O. \quad (10)$$

Let $B(G)_N$ be the set of all compact, open subgroups $O$ of $G$ such that $N \subseteq O$. Then $B(G)_N = \{ q_N^{-1}(O) : O \in B(G/N) \}$. Moreover,

$$q_N^{-1}(K_O) = K_{q_N^{-1}(O)}$$

for each $O \in B(G/N)$, because $q_N^{-1}(K_O) = K_{q_N^{-1}(O)}$ and $q_N$ is a closed map. Hence (10) entails that

$$q_N^{-1}(U_{\alpha N}) = \bigcap_{O \in B(G/N)} q_N^{-1}(K_O) = \bigcap_{O \in B(G/N)} K_{q_N^{-1}(O)} = \bigcap_{O \in B(G)_N} K_O. \quad (11)$$

Since $B(G) = \bigcup_{N \in N(G)} B(G)_N$ by Lemma 1, combining (8), (9) and (11), we obtain that $U_{\alpha N} = \bigcap_{O \in B(G)} K_O$. This completes the proof. □

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