Quantum evolution of Universe in the constrained quasi-Heisenberg picture: from quanta to classics?

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Abstract

The quasi-Heisenberg picture of minisuperspace model is considered. The suggested scheme consists in quantizing of the equation of motion and interprets all observables including the Universe scale factor as the time-dependent (quasi-Heisenberg) operators acting in the space of solutions of the Wheeler–DeWitt equation. The Klein-Gordon normalization of the wave function and corresponding to it quantization rules for the equation of motion allow a time-evolution of the mean values of operators even under constraint $H = 0$ on the physical states of Universe. Besides, the constraint $H = 0$ appears as the relation connecting initial values of the quasi-Heisenberg operators at $t = 0$. A stage of the inflation is considered numerically in the framework of the Wigner–Weyl phase-space formalism. For an inflationary model of the “chaotic inflation” type it is found that a dispersion of the Universe scale factor grows during inflation, and thus, does not vanish at the inflation end. It was found also, that the “by hand” introduced dependence of the cosmological constant from the scale factor in the model with a massless scalar field leads to the decrease of dispersion of the Universe scale factor. The measurement and interpretation problems arising in the framework of our approach are considered, as well.

Key words: quantization of the equations of motion, quantum stage of inflation, scale factor dispersion

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1 Introduction

COBE [1], WMAP [2] and other experiments on measurements of the cosmic microwave background anisotropy have inspired a lot of the works on the classical inflationary potential reconstruction (see, for example, Ref. [3]). However, it is generally accepted that the quantum effects have to be taken into account at initial stage of the cosmological evolution. The question arises, at what stage of the Universe evolution the classical description is applicable. A simplest possibility to clear up this is to build the Heisenberg picture of a minisuperspace model and to calculate the mean values and the dispersions of observables. If at some moment of cosmological time \( t \), we would reveal that for the operators \( \hat{A}(t) \) and \( \hat{B}(t) \) describing the Universe dynamics (they can be Universe scale factor, value of scalar field etc.) the relation \( \langle \hat{A}(t)\hat{B}(t) \rangle \approx \langle \hat{A}(t) \rangle \langle \hat{B}(t) \rangle \) is satisfied with a sufficient accuracy, then we would change the operators in the operator equations by their mean values and, hence, consider the Universe classically. Appearance of a classical world in quantum cosmology is widely discussed (see Refs. [4, 5], and citation therein). As a rule, the Wigner function served as a diagnostic tool for the problem. In quantum region the Wigner function is highly oscillating and has no classical limit in the general case. But under evaluation of the mean value its oscillations are averaged. Thus, the method considering the observable mean values and dispersions seems to be more straightforward for analyzing of transition to the classics than a direct analysis of the Wigner function.

The first step is choosing of an appropriate quantization scheme. A variety of the quantization schemes for a minisuperspace model can be roughly divided in two classes: imposing the constraints i) “before quantization” [6, 7] and ii) “after quantization” [8, 9] (see also reviews comparing both approaches [10, 11]).

In the former, the constraints are used to exclude “nonphysical” degrees of freedom. This allows then constructing Hamiltonian acting in the reduced “physical” phase space. In such models, Universe dynamics is introduced by the time-depended gauge. Gauge of this type should identify the Universe scale factor with the prescribed monotonic function of time [12, 13]. This results in a non-vanishing and generically non-stationary Hamiltonian of the system and, thus, in the equations of motion. Such a procedure cannot be wholly satisfactory, since it requires to introduce a priori arbitrary function and does not allow considering the Universe scale factor as quantum observ-
The alternative schemes prefer imposing the constraint “after quantization”. This leads to the Wheeler–DeWitt equation on quantum states of Universe \[8, 9\]. We believe that it is most correct description of the quantum Universe. Nevertheless, a problem of extraction of the information about the Universe evolution in time remains, because there is no an explicit “time” in the corresponding Wheeler–DeWitt equation. This inspires discussions about “time disappearance” and interpretation of the wave function of Universe \[14, 15\]. Possible solutions and interpretations of this problem like to introduce time along the quasi-classical trajectories, or subdivide Universe into classical and quantum parts have been offered \[16\].

Our point of view is that one can solve the "problem of time" radically, without appealing to the quasiclassics.

Let us note that i) for some observable \( A \) the commutators \([A, H]\) are non-trivial, i.e. the equation of motion remain in force even in ordinary Heisenberg picture, ii) absence of evolution of the mean values can be proved only in the Schrödinger normalization of the Universe wave function. Namely, for evolution of mean value of some Heisenberg operator \( \hat{A} \) we have

\[
\langle A(t) \rangle = \langle \psi | e^{i\hat{H}t} \hat{A} e^{-i\hat{H}t} | \psi \rangle .
\]  

Let \( H \) contains differential operators like \( \frac{\partial^2}{\partial a^2} \) or \( \frac{\partial^2}{\partial \phi^2} \). Assuming that the wave function \( \psi(a, \phi) \) obeys \( \hat{H}\psi(a, \phi) = 0 \) (i.e. it is “on shell”) and is normalized in the Schrödinger style, one can move \( \frac{\partial^2}{\partial a^2} \) to the left side by habitual operation \( \langle \psi | \frac{\partial^2}{\partial a^2} = \langle \frac{\partial^2}{\partial a^2} \psi \rangle \) through integration by parts (and do the same for \( \frac{\partial^2}{\partial \phi^2} \)). As a result, \( \langle \psi | \hat{H} = \langle \hat{H} \psi \rangle = 0 \) and one finds no an evolution of the mean values with the time. iii) However, the Universe wave function cannot be normalized in the Schrödinger style if the most natural Laplacian like operator ordering in the Wheeler-DeWitt equation is chosen (for closed Universe and unnatural operator ordering Schrödinger’s norm can be archived \[17\]). If the wave function is unbounded along one of the variables (e.g. \( a \) variable) its normalization differs from the Schrödinger one and absence of evolution of the operator mean values can not be proven. It gives a hope that some Heisenberg-like picture is possible for the Klein–Gordon normalization. Certainly, the ordinary Heisenberg operators are not suitable for this aim because they are not Hermite in the normalization above.
Also, it should be mentioned, there are the works where the Schrödinger normalization for the “off shell” states (i.e. not obeying the constraints) has been used [18, 19, 20, 21]. In Ref. [18] after evaluation of the mean values of the operators, the proceeding to limit of the “on shell” states (which satisfy the constraints) leads the time-dependence of the expectation values of some operators. Another procedure has been used in Refs. [19, 20], where the constraint is considered as an equation connecting expectation values of the operators.

The paper is organized as follows: in section 2, origin and description of our quantization scheme\(^1\) (i.e. quantization rules for the equations of motion and formula for evaluation of the mean values) are expounded. Quantization rules for the quasi-Heisenberg operators are defined consistently with choice of the hyperplane used for normalization of solutions of the Wheeler–DeWitt equation in the Klein–Gordon style.

In section 3 the approximate solution is obtained numerically for the quasi-Heisenberg operators and the corresponding mean values are evaluated and discussed. Transition to Universe having negligible dispersion of the scale factor is discussed in section 4. In section 5 the measurement and interpretation problems in the quantum Universe are discussed.

## 2 Quantization rules, operator equations of motion, mean values evaluation

Let us start from the Einstein action for a gravity and an one-component real scalar field:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \left[ \frac{1}{2}(\partial_{\mu}\phi)^2 - V(\phi) \right],$$  \hspace{1cm} (2)

where \(R\) is the scalar curvature and \(V\) is the matter potential which includes a possible cosmological constant effectively. We restrict our consideration to the homogeneous and isotropic metric:

$$ds^2 = N^2(t) dt^2 - a^2(t) d\sigma^2.$$  \hspace{1cm} (3)

\(^1\)This quantization scheme has many common features with a model of the relativistic-particle-clock (i.e. particle having its own clock, for instance, radioactive particle) [22].
Here the lapse function $N$ represents the general time coordinate transformation freedom. For the restricted metric the total action becomes

$$S = \Omega \int N(t) \left\{ \frac{3}{8\pi G} a \left( \mathcal{K} - \frac{\dot{a}^2}{N^2(t)} \right) + \frac{1}{2} a^3 \frac{\dot{\phi}^2}{N^2(t)} - a^3 V(\phi) \right\} dt, \quad (4)$$

where $\mathcal{K}$ is the signature of the spatial curvature, and $\Omega$ is the constant defining volume of the Universe. It is equal to $2\pi^2$ for the closed Universe and is infinite for the flat and open ones. For quantization of the flat and open Universes, $\Omega$ should be some properly fixed constant. It is suggested that some fluctuation, from which the Universe arises, can be approximately considered as isolated, having no local degrees of freedom and obeying dynamics of the uniform and isotropic Universe. Constant $\Omega$, corresponding to the "volume" occupied by this fluctuation is to be such that the value of $\Omega a^3_{\text{today}}$ is greater than the visible part of Universe, which is known to be isotropic and uniform. Further we set $\Omega$ to unity, i.e. approximately 1/18 part from the volume of the closed Universe.

The action (4) can be obtained from the following expression by varying on $p_a$ and $p_\phi$:

$$S = \int \left\{ p_\phi \dot{\phi} + p_a \dot{a} - N(t) \left( -\frac{3a\mathcal{K}}{8\pi G} - \frac{8\pi G p_a^2}{12a} + \frac{p_\phi^2}{2a^3} + a^3 V(\phi) \right) \right\} dt.$$

Varying on $N$ gives the constraint

$$H = -\frac{3a\mathcal{K}}{8\pi G} - \frac{8\pi G p_a^2}{12a} + \frac{p_\phi^2}{2a^3} + a^3 V(\phi) = 0. \quad (5)$$

This constraint turns into the Wheeler–DeWitt equation $\hat{H} \psi(a, \phi) = 0$ after quantization: $[\hat{a}, \hat{p}_a] = -i$, $[\hat{\phi}, \hat{p}_\phi] = i$.

Attempts to modernize or remove the constraint equation can be justified within the framework of theories implying existence of some preferred system of reference. For instance, the Logunov's relativistic theory of gravity $[23]$, which gives an adequate description of the Universe expansion $[24]$, allows omitting the constraint $[24]$. However, here we shall keep to the General Relativity.

Let us first consider the flat Universe ($\mathcal{K} = 0$) with $V(\phi) = 0$ (corresponding Hamiltonian is $H_0 = \frac{p_a^2}{2a} + \frac{p_\phi^2}{2a}$ in the units $4\pi G/3 = 1$).
Procedure, which is invariant under general coordinate transformations consists in postulating the quantum Hamiltonian \[26\]:

\[
\hat{H}_0 = \frac{1}{2}g^{-\frac{1}{4}}\hat{p}_\mu g^{\frac{1}{2}}g^{\mu\nu}\hat{p}_\nu g^{-\frac{1}{4}},
\]

where \(\hat{p}_\mu = -ig^{-\frac{1}{4}}\frac{\partial}{\partial x^\mu}g^{\frac{1}{4}}\). For our choice of variables \(x^\mu = \{a, \phi\}\), \(p_\mu = \{-p_a, p_\phi\}\), the metric has the form:

\[
g^{\mu\nu} = \begin{pmatrix}
-a_0 & 0 \\
0 & 1
\end{pmatrix}
\]

so that

\[
g = \det |g_{\mu\nu}| = a^4, \quad \hat{p}_a = i\left(\frac{\partial}{\partial a} + \frac{1}{a}\right).
\]

Then the Hamiltonian is

\[
\hat{H}_0 = -\frac{1}{4}\left(\hat{p}_a^2 + \frac{1}{a^2}\hat{p}_\phi^2\right) + \frac{\hat{p}_\phi^2}{2a^3} = \frac{1}{2a^2} a^2 \frac{\partial}{\partial a} a \frac{\partial}{\partial a} - \frac{1}{2a^3} \frac{\partial^2}{\partial \phi^2}.
\]

Explicit expression for the wave function satisfying \(\hat{H}_0\psi = 0\) is

\[
\psi_k(a, \phi) = a^{\pm i|k|} e^{ik\phi}.
\]

Exactly as in the case of the Klein–Gordon equation, we should choose only the positive frequency solutions \[16\]. Thus, the wave packet

\[
\psi(a, \phi) = \int c(k) \frac{a^{-i|k|}}{\sqrt{4\pi|k|}} e^{ik\phi} dk
\]

will be normalized by

\[
ia \int \left(\frac{\partial \psi}{\partial a} \psi^* - \frac{\partial \psi^*}{\partial a} \right) d\phi = \int c^*(k)c(k) dk = 1,
\]

where some hyperplane \(a = \text{const}\) is chosen.

Now we have to quantize the classical equations of motion

\[
p_\phi(t) = 0, \quad (a^3(t))^\cdot = 3p_a, \quad (p_a^a)^\cdot = -3H_0
\]

obtained from the classical hamiltonian \(H_0\) by taking Poisson brackets \(\hat{A} = \{H, A\}\), where

\[
\{A, B\} = \frac{\partial A}{\partial p_\mu} \frac{\partial B}{\partial x^\mu} - \frac{\partial B}{\partial x^\mu} \frac{\partial A}{\partial p_\mu} = \frac{\partial A}{\partial \phi} \frac{\partial B}{\partial p_\phi} - \frac{\partial B}{\partial \phi} \frac{\partial A}{\partial p_\phi} - \frac{\partial A}{\partial p_\phi} \frac{\partial B}{\partial a} - \frac{\partial B}{\partial p_\phi} \frac{\partial A}{\partial a}.
\]
For quantization it is sufficient to specify commutation relation for operators at initial moment of time \( t = 0 \). According to the Dirac quantization procedure [27], besides the hamiltonian constraint \( \Phi_1 = -\frac{p^2}{a} + \frac{p_a^2}{a^2} \) (see [5]), we have to set some additional gauge fixing constraint, which can be chosen in our case as \( \Phi_2 = a = \text{const} \), because the hyperplane \( a = \text{const} \) is chosen earlier for the normalization of the wave function in the Klein-Gordon style. Besides the ordinary Poisson brackets the Dirac brackets have to be introduced:

\[
\{A, B\}_D = \{A, B\} - \{A, \Phi_i\}(C^{-1})_{ij}\{\Phi_j, B\},
\]

where \( C \) is the nonsingular matrix with the elements \( C_{ij} = \{\Phi_i, \Phi_j\} \) and \( C^{-1} \) is the inverse matrix. Quantization consists in postulating the commutator relations to be equal to the Dirac brackets with the variables replaced by operators:

\[
[i\eta, i\eta'] = -i\{\eta, \eta'\}_D \bigg|_{\eta\rightarrow i\eta}.
\]

Here \( \eta \) implies set of the canonical variables \( p_\mu, x^\nu \). In contrast to the usual formalism of Refs. [6, 13, 28, 29], we postulate to impose constraints \( \Phi_1 = 0 \) and \( \Phi_2 = 0 \) at only hyperplane \( t = 0 \). Consequently the quasi-Heisenberg operators obey the commutation relations obtained from the Dirac quantization procedure at the initial moment \( t = 0 \). Direct evaluation gives

\[
[\hat{p}_a(0), \hat{a}(0)] = 0, \quad [\hat{p}_\phi(0), \hat{a}(0)] = 0,
\]

\[
[\hat{p}_\phi(0), \hat{\phi}(0)] = -i, \quad [\hat{a}(0), \hat{\phi}(0)] = -i\frac{\hat{p}_\phi(0)}{\hat{p}_a(0)\hat{a}^2(0)}.
\]

One has to solve Eqs. (10) with the given initial commutation relations. In contrast to the ordinary Heisenberg operators, the quasi-Heisenberg operators do not conserve their commutation relations during evolution. The commutation relations (14) can be satisfied through

\[
\dot{a}(0) = \text{const} = a, \quad \dot{p}_\phi(0) = \hat{p}_\phi, \quad \dot{p}_a(0) = |\hat{p}_\phi|/a, \quad \dot{\phi}(0) = \phi,
\]

where \( \dot{\phi} = -i\frac{\partial}{\partial \phi} \). Variable \( a = \dot{a}(0) \) is \( c \)-number now because it commutes with all operators [29]. Solutions of Eqs. (10) are

\[
\hat{p}_\phi(t) = \hat{p}_\phi, \quad \hat{a}^3(t) = a^3 + 3|\hat{p}_\phi|t, \quad \hat{p}_a(t) = \frac{|\hat{p}_\phi|}{(a^3 + 3|\hat{p}_\phi|t)^{1/3}},
\]

\[
\dot{\phi}(t) = \phi + \frac{\hat{p}_\phi}{3|\hat{p}_\phi|} \ln(a^3 + 3|\hat{p}_\phi|t) - \frac{\hat{p}_\phi}{|\hat{p}_\phi|} \ln a.
\]
We imply that these quasi-Heisenberg operators act in the Hilbert space with the Klein-Gordon scalar product. Expression for the mean value of an observable is

\[ < \hat{A}(t) > = i a \int \left( \psi^*(a, \phi) D^{\frac{1}{2}} \hat{A}(t) D^{-\frac{1}{2}} \hat{A}(t) D^{\frac{1}{2}} \psi(a, \phi) \right) d\phi \bigg|_{a \to 0} , \quad (17) \]

where operator \( D = -\frac{\partial^2}{\partial \phi^2} + 2a^3V(\phi) \) (since \( a \to 0 \) the \( V \)-term can be omitted in the expression for \( D \)). Eq. (17) is particular case of that suggested in Ref. [30], where an one-particle picture of the Klein-Gordon equation in the Foldy-Wouthausen representation has been considered. The adequacy of this definition can been seen in the momentum representation of the \( \phi \) variable, where \( \hat{p}_\phi = k \) and \( \hat{\phi} = i \frac{\partial}{\partial k} \). Then Eq. (17) gives

\[ < \hat{A}(t) > = \int a^{ij|k|} c^*(k) \hat{A}(t, \hat{\phi}, k, a) a^{-ij|k|} c(k) dk \bigg|_{a \to 0} , \quad (18) \]

which is similar to the ordinary quantum mechanical definition and certainly possesses hermicity.

Evaluation of the mean value \( \hat{a}^3(t) \) given by (16), (18) over the wave packet (8) reads

\[ < a^3(t) > = 3t \int |k||c(k)|^2 dk. \quad (19) \]

Next quantity is the mean value of the scalar field \( < \hat{\phi}(t) > : \)

\[ < \hat{\phi}(t) > = \int a^{ij|k|} c^*(k) \left( \frac{i}{a^3 + 3|k|} \ln a^3 + 3|k|t \right) a^{-ij|k|} c(k) dk \]

\[ = \int \left( c^*(k)i \frac{\partial}{\partial k} c(k) + \frac{k}{3|k|} \ln a^3 + 3|k|t \right) |c(k)|^2 dk, \quad (20) \]

Brackets \( < \ldots >_a \) with the index \( a \) in (20) mean that \( a \) is not equal to zero yet (compare with Eq. (18)). A remarkable property of Eq. (20) is that the term \( -\frac{k}{|k|} \ln a \) cancels the term arising from the differentiation:

\[ a^{ij|k|} \frac{\partial}{\partial k} a^{-ij|k|} = \frac{k}{|k|} \ln a. \]  
Thus, after \( a \to 0 \) one can obtain

\[ < \hat{\phi}(t) > = \int \left( \frac{k}{3|k|} \ln(3|k|t)|c(k)|^2 + c^*(k)i \frac{\partial}{\partial k} c(k) \right) dk. \quad (21) \]
Cancellation of the terms divergent under \( a \to 0 \) in the mean values of the quasi-Heisenberg operators is a general feature of the theory. As a result, it is possible to evaluate, for instance,

\[
< \hat{\phi}^2(t) > = \int \left( \frac{1}{9} \ln^2(3|k|t) |c(k)|^2 - c^*(k) \frac{\partial^2}{\partial k^2} c(k) \right) dk.
\]

(22)

One should not confuse the divergence at \( a \to 0 \) arising under evaluation of the mean values with the singularity at \( t \to 0 \). The mean values of operators, which are singular at \( t \to 0 \) in the classical theory remain singular also in the quantum case. According to (16), (17) the way to avoid a singularity is to guess, that the Universe evolution began from some “seed” scale factor \( a_0 \). Then in the expression for a mean value, one has to assume \( a \to a_0 \) instead of \( a \to 0 \). But, the mathematics is simplified greatly namely at \( a \to 0 \), because the use of asymptotical value of the wave function is possible in this case.

One more kind of the infinity can be found in Eqs. (21), (22): for \( c(k) \), which does not tend to zero at small \( k \), the mean values of \( \phi(t) \) and \( \phi^2(t) \) diverge. This is a manifestation of the well-known infrared divergence of scalar field minimally coupled with gravity. Thus, not all possible \( c(k) \) are suitable for construction of the wave packets.

Let us consider Hamiltonian, containing the cosmological constant \( V_0 \):

\[
H = H_0 + a^3 V_0.
\]

(23)

Explicit solution for the wave function \( \hat{H} \psi = 0 \) has the form

\[
\psi_k(a, \phi) = \left( \frac{18}{V_0} \right)^{\frac{i|k|}{6}} \Gamma(1 - \frac{i|k|}{3}) J_{\frac{3i|k|}{2}} \left( \frac{\sqrt{2V_0}}{3a^3} \right) e^{i k \phi},
\]

(24)

where \( \Gamma(z) \) is the Gamma function and \( J_{\mu}(z) \) is the Bessel function. The wave function (24) tends to \( a^{-i|k|} e^{ik\phi} (1+O(a^6)) \) asymptotically under \( a \to 0 \). Then for evaluation of the mean values according to (18), we can always build the wave packet \( a^{-i|k|} e^{ik\phi} \) from solutions of the free Wheeler–DeWitt equation and do not encounter with a problem of negative frequency solutions. The argumentation holds for any potential \( V(\phi) \), because it contributes into the Hamiltonian as a term multiplied by \( a^3 \).

Equations of motion obtained from the classical Hamiltonian (23) are

\[
(a^3(t))' = 3\hat{p}_a a, \quad (p_a a) = 3V_0 a^3 - 3H_0, \quad (V_0 a^3 - H_0)' = 6V_0 p_a a.
\]

(25)
The additional term $a^3V_0$ does not change relations (14) required for the quantization procedure. Only expression for $\hat{p}_a(0)$ changes in (15): $\hat{p}_a(0) = \sqrt{\frac{\hat{p}_a^2}{a^2} + 2V_0a^4}$.

Finally we arrive to

$$\dot{a}^3(t) = a^3 + 3|\hat{p}_\phi| \frac{\sinh(t\sqrt{18V_0})}{\sqrt{18V_0}} + a^3(\cosh(t\sqrt{18V_0}) - 1).$$  \tag{26}$$

Evaluation of the mean values according to Eq. (17) leads to

$$< \dot{a^3}(t) > = \frac{\sinh(3\sqrt{2V_0}t)}{\sqrt{2V_0}} \int |k||c(k)|^2dk,$$

$$< \dot{a^6}(t) > = \frac{(\sinh(3\sqrt{2V_0}t))^2}{2V_0} \int k^2|c(k)|^2dk.$$ 

One can see, that the dispersion $\sqrt{< \dot{a^6} > - < \dot{a^3} >^2} / < \dot{a^3} >$ does not depend on $t$. Thus, the evolution of Universe remains quantum during all time in the model with a cosmological constant.

This results from an absence of some scale length in the model with cosmological constant (besides the natural Plank length). Such a length can appear due to some mechanism reducing a cosmological constant during the Universe evolution. In the next section one of the possible mechanisms, namely an inflation derived by the quadratic potential of the scalar field, is considered.

### 3 Operator equations for the quadratic inflationary potential and Wigner-Weyl evolution of the minisuperspace

As it has been discussed, the quantization procedure consists in quantization of the equations of motion, i.e. considering them as the operator equations. These equations have to be solved with the initial conditions obeying to the constraint at $t = 0$. For the Hamiltonian

$$H = -\frac{\hat{p}_a^2}{2a} + \frac{\hat{p}_\phi^2}{2a^3} + a^3\frac{m^2\phi^2}{2}$$  \tag{27}$$
we have the equations:

\[
\begin{align*}
\ddot{a} &= -\frac{3}{2} a \dot{\phi}^2 - \frac{\dot{a}^2}{2a} + \frac{3}{2} a m^2 \phi^2, \\
\dot{\phi} &= -3 \frac{\ddot{a}}{a} \dot{\phi} - m^2 \phi
\end{align*}
\]  

and the constraint:

\[
- \dot{a}^2 a + \dot{\phi}^2 a^3 + a^3 m^2 \phi^2 = 0.
\]  

The point means the differentiation over \( t \). After quantization, Eqs. \( 28 \) lead to the equations for the quasi-Heisenberg operators, which have to be solved with the operator initial conditions:

\[
\begin{align*}
\hat{a}(0) &\equiv a, \quad \hat{\phi}(0) \equiv \phi, \quad \hat{p}_\phi(0) \equiv -i \frac{\partial}{\partial \phi}, \\
\dot{\hat{\phi}}(0) &= \frac{\hat{p}_\phi(0)}{\hat{a}^3(0)} = \frac{1}{a^3} \left(-i \frac{\partial}{\partial \phi}\right), \\
\dot{\hat{a}}(0) &= \hat{a}(0) \sqrt{\dot{\phi}(0)^2 + m^2 \phi^2(0)} = \sqrt{\frac{1}{a^4} \left(-i \frac{\partial}{\partial \phi}\right)^2 + m^2 a^2 \phi^2}.
\end{align*}
\]

According to our ideology, the operator constraint \( 29 \) is satisfied only at \( t = 0 \). The ordinary problem of the operator ordering arises, because the quasi-Heisenberg operators are noncommutative in the general case. The problem seems more transparent if we change the variable \( \hat{\alpha} = \ln \hat{a} \):

\[
\begin{align*}
\ddot{\hat{\alpha}} + \frac{3}{2} \ddot{\alpha}^2 - \frac{3}{2} m^2 \ddot{\phi}^2 + \frac{3}{2} \ddot{\phi}^2 &= 0, \\
\ddot{\hat{\phi}} + \frac{3}{2} \left( \ddot{\hat{\alpha}} \ddot{\phi} + \ddot{\alpha} \ddot{\phi} \right) + m^2 \ddot{\phi} &= 0,
\end{align*}
\]

where the symmetric ordering is used. The system \( 31 \) has to be solved with the initial conditions: \( \ddot{\phi}(0) = \phi, \ddot{\alpha}(0) = \ln a, \ddot{\phi}(0) = \frac{1}{a} \left(-i \frac{\partial}{\partial \phi}\right), \ddot{\alpha}(0) = \sqrt{m^2 \phi^2 + \frac{1}{a^4} \left(-i \frac{\partial}{\partial \phi}\right)^2} \).

The operator equations under consideration can be solved within the framework of the perturbation theory in the first order on interaction constant (i.e. on \( m^2 \)). The solution in analytical form can be found in [22]. The analytic solution is important because it allows ensuring that the divergent
Figure 1: Contour-plot of the Wigner function of Universe for $c(k) = e^2 \left( \frac{2}{\pi} \right)^{1/4} \exp(ik\phi_0 - k^2 - 1/k^2)$ at $a = 10^{-4}$.

terms at $a \to 0$ cancel each other under calculations of the mean values based on (18).

However, the most interesting is to consider an inflation at its late stages. This requires a numerical consideration of the operator equations and can be realized within the framework of the Weyl-Wigner phase-space formalism [31]. Let us remind that in this formalism every operator acting on $\phi$ variable has the Weyl symbol: $\mathcal{W}[\hat{A}] = A(k, \phi)$. For instance, the simplest Weyl symbols in our case are: $\mathcal{W}[-i\frac{\partial}{\partial \phi}] = k$, $\mathcal{W}[\phi] = \phi$. Weyl symbol of the symmetrized product of operators reads

$$W[\frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A})] = \cos\left(\hbar \frac{\partial}{2 \partial \phi_1} \frac{\partial}{\partial k_2} - \hbar \frac{\partial}{2 \partial \phi_2} \frac{\partial}{\partial k_1}\right)A(k_1, \phi_1)B(k_2, \phi_2) \bigg|_{k_1=k_2=k, \phi_1=\phi_2=\phi} \quad (32)$$

where the Planck constant is restored only to point the order of cosine
Figure 2: Contour-plot of the Wigner function of Universe for $c(k) = 2\sqrt{5} e^{20\sqrt{\pi} (\frac{3}{\pi})^{1/4}} \exp(-600k^2 - 1/k^2)$ at $a = 10^{-4}$.

expansion.

Let us consider the Weyl transformation of Eqs. (31) and expand the Weyl symmetrized product of operators up to second-order in $\hbar$. This results in:

$$
\partial_t^2 \alpha + \frac{3}{2} \left( (\partial_t \alpha)^2 + \frac{\hbar^2}{4} (\partial_k \partial_\phi \partial_t \alpha)^2 - \frac{\hbar^2}{4} (\partial_\phi \partial_t \alpha)(\partial_k^2 \partial_t \alpha) \right) + \frac{3}{2} \left( (\partial_t \varphi)^2 + \frac{\hbar^2}{4} (\partial_k \partial_\phi \partial_t \varphi)^2 
- \frac{\hbar^2}{4} (\partial_\phi \partial_t \varphi)(\partial_k^2 \partial_t \varphi) \right) - \frac{3}{2} m^2 \left( \varphi^2 + \frac{\hbar^2}{4} (\partial_k \partial_\phi \varphi)^2 - \frac{\hbar^2}{4} (\partial_\phi \varphi)(\partial_k^2 \varphi) \right) = 0,
$$

$$
\partial_t^2 \varphi + 3 \left( \partial_t \alpha \partial_t \varphi + \frac{\hbar^2}{4} (\partial_k \partial_\phi \partial_t \alpha)(\partial_k \partial_\phi \partial_t \varphi) - \frac{\hbar^2}{8} (\partial_\phi \partial_t \alpha)(\partial_k^2 \partial_t \varphi) 
- \frac{\hbar^2}{8} (\partial_\phi \partial_t \varphi)(\partial_k^2 \partial_t \varphi) \right) + m^2 \varphi = 0,
$$

(33)
where \( \alpha(k, \phi, t) \) and \( \varphi(k, \phi, t) \) are the Weyl symbols of the operators \( \hat{\alpha}(t) \) and \( \hat{\varphi}(t) \), respectively. These equations have to be solved with the initial conditions at \( t = 0 \):

\[
\alpha(k, \phi, 0) = \ln a, \quad \partial_t \alpha(k, \phi, 0) = W \left[ \sqrt{-\frac{1}{a^6} \frac{\partial^2}{\partial \phi^2} + m^2 \phi^2} \right],
\]

\[
\varphi(k, \phi, 0) = \phi, \quad \partial_t \varphi(k, \phi, 0) = \frac{k}{a^3}.
\] (34)

Weyl symbol of the square root can be expressed as [32]:

\[
W \left[ \sqrt{-\frac{1}{a^6} \frac{\partial^2}{\partial \phi^2} + m^2 \phi^2} \right] = \frac{m_1}{2 \pi^{1/2} a^{3/2}} \int_0^\infty t^{-1/2} \exp \left( -\frac{m^2 a^6 \phi^2 + k^2}{m a^3} \tanh(t) \right) 
\times \text{sech}(t) \left( \frac{m^2 a^6 \phi^2 + k^2}{m a^3} \text{sech}(t)^2 + \tanh(t) \right) dt. \] (35)

Since the mean values result from \( a \to 0 \), it is possible to take simply \( \partial_t \alpha(k, \phi, 0) = \frac{|k|}{a^3} \).

State of Universe is described by the Wigner function \( \wp(k, \phi) \), which is constructed on the basis of definition (17) and given by

\[
\wp(k, \phi) = ia \int \left[ -\frac{\partial^2}{\partial \phi^2} \right]^{-1/2} \psi^* \left( \phi + \frac{u}{2} \right) \left[ -\frac{\partial^2}{\partial \phi^2} \right]^{-1/2} \frac{\partial \psi \left( \phi - \frac{u}{2} \right)}{\partial a} e^{iku} du -
\]

\[
ia \int \left[ -\frac{\partial^2}{\partial \phi^2} \right]^{-1/2} \frac{\partial \psi^* \left( \phi + \frac{u}{2} \right)}{\partial a} \left[ -\frac{\partial^2}{\partial \phi^2} \right]^{-1/2} \psi \left( \phi - \frac{u}{2} \right) e^{iku} du. \] (36)

or in the momentum representation of the wave function corresponding to Eq. (8):

\[
\wp(k, \phi) = \frac{1}{\pi} \int c^*(2k - q)c(q)a^{-i|q|+i|2k-q|} e^{2i(q-k)\phi} dq. \] (37)

As a result of \( a \to 0 \), both Weyl symbols and Wigner function diverge. In particular, when \( a \to 0 \) the Wigner function becomes strongly oscillating. However, the divergences cancel each other in the expectation values, which can be constructed in an ordinary way. For instance, expectation values of \( \alpha \) and its square are:

\[
\langle \alpha(t) \rangle = \int dk d\phi \alpha(k, \phi, t) \wp(k, \phi) \bigg|_{a \to 0},
\]

\[
\langle \alpha^2(t) \rangle = \int dk d\phi (\alpha^2 + \frac{\hbar^2}{4} (\partial_k \partial_\phi \alpha)^2 - \frac{\hbar^2}{4} (\partial^2_\phi \alpha)(\partial^2_k \alpha)) \wp(k, \phi) \bigg|_{a \to 0}. \]
Figure 3: Evolution of $\langle \alpha \rangle$, $\langle \varphi \rangle$, dispersion $\sigma(\alpha) = \sqrt{\langle \alpha^2 \rangle - \langle \alpha \rangle^2}$ (solid curves); $\langle \varphi^2 \rangle$ and relative dispersion $\sigma(\alpha) / \langle \alpha \rangle$ (dashed curves) for $c(k) = e^2 \left( \frac{2}{\pi} \right)^{1/4} \exp(ik\phi_0 - k^2 - 1/k^2)$. $\bar{h} = 0$ (black curves), $\bar{h} = 1$ (gray curves).
Figure 4: Evolution of $\langle \alpha \rangle$, $\langle \varphi^2 \rangle$, dispersion $\sigma (\alpha) = \sqrt{\langle \alpha^2 \rangle - \langle \alpha \rangle^2}$ (solid curves) and relative dispersion $\sigma (\alpha) / \langle \alpha \rangle$ (dashed curve) for $c(k) = 2\sqrt{5} \epsilon^{20} \sqrt{\pi} \left( \frac{3}{\pi} \right)^{1/4} \exp(-600k^2 - 1/k^2)$. $\bar{h} = 0$ (black curves), $\bar{h} = 1$ (gray curves).
Let us discuss the parameters of inflationary model. Quadratic potential corresponds to the Linde’s “chaotic inflation” [33]. This model supposes that the value of potential at an initial stage of the inflation has to be an order of the Planck mass $M_p$ in fourth degree ($M_p = G^{-1/2} = \sqrt{\frac{4\pi}{3}}$). Hence, the corresponding value of the scalar field is $\phi_0 = \frac{\sqrt{2} M_p^2}{m}$. Constant $m^2$ dictated by the COBE data is $m^2 \sim 10^{-12} M_p^2 = 10^{-12} \frac{3}{4\pi}$. Still for the purposes of visuality of the numerical calculations we take $m^2 = 1.7 \times 10^{-3}$. This reflects the fact that $m^2 << M_p^2$ and the initial scalar field is sufficiently large to provide $V^{1/4} \sim M_p$. There are two possibility to create large scalar field: the first one is a wave packet with the non-zero mean field $\phi_0$, for instance, $c(k) = e^{\frac{1}{2} \left( \frac{2}{\pi} \right)^{1/4} \exp(ik\phi_0 - k^2 - 1/k^2)}$ (the corresponding Winger function is shown in Fig. 1). The second one is a “squeezed” packet having small uncertainty of $k$, but large square of the scalar field: for instance, $c(k) = 2\sqrt{5} e^{20\sqrt{6} \left( \frac{3}{2\pi} \right)^{1/4} \exp(-600k^2 - 1/k^2)}$ (the corresponding Winger function is shown in Fig. 2). Note that for ordinary systems, the decoherence principle forbids the highly “squeezed” packets because they should be “collapsed” due to interacting with environment. The minisuperspace model takes up a little number degrees of freedom and the other ones may serve as an environment. So, we cannot be fully sure that the “squeezed” packet is permitted.

For the both packets, the function $c(k)$ contains a multiplier $\exp(-1/k^2)$ suppressing the infrared divergence. As a result of the numerical solution of Eqs. (33), the evolution of the operators expectation values and their dispersions have be obtained. Results are shown in Figs. 3,4.

We consider two different cases for Eqs. (33): i) $\hbar = 0$ and ii) $\hbar = 1$ (that gives corrections to the equations of motion taking into account the operator noncommutativity).

The main conclusion is that the dispersion of logarithm of the Universe scale factor does not vanish during inflation. Even for the wave packet having a large mean value of the scalar field $\phi_0$ and a small dispersion, one can see (Fig. 3) the increasing dispersion of the scale factor logarithm during inflation without dispersion decay after the inflation end. In our particular case (small mass of the scalar field), corrections to the equations of motion due to noncommutativity of the quasi-Heisenberg operators do not change the picture qualitatively. Smallness of the corrections indicates that the contribution of the next terms in the cosine expansion (32) is negligible. Thus the accurate solutions of the operator equations of motion (31) for the
particular set of parameters are obtained.

4 Decrease of the Universe scale factor dispersion due to inflation dynamics.

Figure 5: Mean value of logarithm of the scale factor and its dispersion for the model with the cosmological constant, \( V_0 = 1, \beta = 0 \) (dashed curve), and for the model (33) with the decreasing cosmological constant, introduced “by hands”, \( V_0 = 1, \beta = 10^{-8} \).

As we seen in the previous section, the model with one scalar field provides Universe with growing dispersion of the scale factor. Nevertheless, how can Universe become classical? A possible answer is that this occurs due to decoherence. That is at some stage of the Universe evolution, it interacts with
the environment and such interaction suppresses the quantum properties of the system. As an “environment”, one can consider the remaining degrees of freedom (including matter \[33\]), which are not taken into account in the minisuperspace model. However, there exists one more possibility of the transition to the classics occurring only due to the Universe dynamics without appealing to the decoherence.

Let us consider the model with the massless scalar field but with the introduced “by hand” decrease of the cosmological constant. Hamiltonian of

Figure 6: Classical trajectories: (a) for the model with the decreasing cosmological constant: \( V_0 = 1, \beta = 10^{-8} \), and (b) for the model with the quadratic potential of the scalar field. Initial value of \( \phi \) is fixed, initial value of \( k \) is varying.
the model has the form:

\[ H = -\frac{p_\alpha^2}{2a} + \frac{p_\phi^2}{2a^3} + V_0 \frac{a^3}{1 + \beta a^3}, \]  

(38)

where \( \beta \) is some constant. The Hamiltonian suggests some modification of the theory with cosmological constant in a sense that the cosmological “constant” \( V_0/(1 + \beta a^3) \) is not zero at the small scale factors and decreases as \( a^{-3} \). We shall not discuss here what fundamental model is able to produce such a modification. Let us only remind that the first model of the inflation by Starobinsky \[34\] did not use a scalar field.

The corresponding equations of motion are

\[ \ddot{\alpha} + \frac{3}{2} \dot{\alpha}^2 + \frac{3}{2} \dot{\phi}^2 - V_0 \frac{3}{(1 + \beta e^{3\alpha})^2} = 0, \]

\[ \ddot{\phi} + 3\dot{\alpha} \dot{\phi} = 0. \]  

(39)

Initial conditions correspond to Eq. \[34\] apart from

\[ \partial_t \alpha(0) = \sqrt{-\frac{1}{a^6} \frac{\partial^2}{\partial \phi^2} + \frac{2V_0}{1 + \beta a^3}}. \]

The latter does not differ from \( \partial_t \alpha(0) = \frac{1}{a^3} \sqrt{-\frac{\partial^2}{\partial \phi^2}} \) due to limit \( a \to 0 \) under evaluation of the mean values.

Results of calculation are shown in Fig. 5. One can see that Universe becomes classical after the inflation end. A sufficiently quick decrease of the cosmological constant allows suppressing the dispersion of the scale factor logarithm.

The trajectories of \( \alpha(k, \phi, t) \) at the fixed \( \phi \) but for the different \( k \) are shown in Fig. 6. The trajectories are divergent for the inflationary model with quadratic potential, but are convergent for Eq. \[38\]. The last illustrates the transition to classics.

We did not investigate models with the multiple scalar fields, but considered different shapes of the potential both for the “small field” and “large field” inflation \[35\]. In both cases we had the divergent trajectories and the time-growing dispersion of the scale factor logarithm.

5 Measurement issue and interpretation

It is often stated that the quantum mechanics can not be applied to whole Universe, because one has the single Universe and is able to perform one mea-
Figure 7: Schematic picture: (a) Universe after measurements by the local observers. Scale factor is projected to defined value in the local regions. (b) A stage, when Universe becomes nonuniform, can be considered as a "self-measurement".

measurement. What is the sense of the scale factor dispersion in this case? Our suggestion is to consider some local system (Intermediate System for Measurements, IMS) inside Universe and imply that measurements are carried out under it. Denoting the IMS degrees of freedom as ξ = {ξ₁, ξ₂...}, one can write Lagrangian of the flat, uniform, and isotropic Universe including IMS:

\[
L(t) = \Omega \left\{ -\frac{3M_p^2}{8\pi} a \dot{a}^2 + \frac{1}{2} a^3 \dot{\phi}^2 - a^3 V(\phi) \right\} + L(\xi, \dot{\xi}, a),
\]

(40)

where \( M_p = 1/\sqrt{G} \) is the Plank mass, and \( \Omega \) is the "volume" over which the integration in the action (2) is doing. IMS is local, i.e. it fills some restricted region of the three-dimensional space which is much smaller than \( \Omega a^3 \) at the moment of time considered.

For simplicity, let us consider \( \Omega \) to be infinite. As a result, the equation of motion for Universe (including \( \dot{a} \)) becomes independent on the IMS variables \( \xi \), because the influence of the finite system to the infinite one is negligible (let us remind that we consider Universe with the "frozen" local degrees of freedom, implying that some change of Universe occurs in the hole space simultaneously). From the other hand, equations for IMS contain \( a(t) \) as the time-dependent parameter. After quantization this parameter becomes the operator. For convenience in the IMS description, we can turn to the Schrödinger picture and at the same time to consider the scale factor in the quasi-Heisenberg picture as before. Thus the IMS Hamiltonian \( H(p_\xi, \xi, \dot{a}(t)) \) contains the operator \( \dot{a}(t) \) as a parameter. State of the system is described by the density matrix

\[
ρ(\xi, \xi', t) = <a | \Psi(\xi', t, [\dot{a}])\Psi(\xi, t, [\hat{a}])|a >,
\]

(41)

where averaging over the Universe state is assumed. The wave function
\( \Psi(\xi', t, [\dot{a}]) \) is the solution of the Schrödinger equation and depends functionally on the operator of the Universe scale factor.

If an observable does not contain the scale factor explicitly, one can find its mean value from the density matrix. If an observable contains the scale factor, one has to use the wave function and to average over the Universe state at the last step. We can consider a number of IMSs: our consideration remains valid and the limited number of IMSs does not influence “large” Universe. In our model IMSs can be placed arbitrary close to each other and measurements can be proceeded during the infinitesimal time. This does not change the measurement results.

Let us consider the red shift from the distant sources in the fluctuating Universe. At first, let us note that the fluctuating scale factors does not change the spectral characteristics of the source itself. For example, the hydrogen atom hamiltonian is

\[
H = \frac{p_\xi^2}{2a^2 m} + \frac{e^2}{a |\xi|},
\]

where \( \xi \) is the IMS spatial coordinate. Here it is implied that \( \xi^i \gamma_{ij} \xi^j = a^2 \xi^2 \) and \( p_i \gamma^{ij} p_j = \frac{1}{\varepsilon^2} p^2 \), where \( \gamma_{ij} \) is the metric tensor of the three-space. Let us evaluate energies of a “pure” state, which is the eigenstate of Hamiltonian (42). The “pure” state depends on the operator of the Universe scale factor, which is implied to be time-independent (adiabatic approximation). To find energies we have to rescale the mass and the charge so that \( m \rightarrow a^2 m \) and \( e^2 \rightarrow e^2 / a \), respectively. Then using the ordinary formula

\[
E = -\frac{me^4}{2n^2}, \quad n = 1, 2, ...
\]

one can find that the energies do not change after rescaling.

When one observes the spectrum of an atom from a distant point, the red shifted frequencies are visible in agreement with the formula

\[
\omega = \omega_0 (1 + \frac{\dot{a}}{a} l),
\]

which is valid for not too large distances \( l \). We can consider the ray of light as a number of IMS situated along the ray trajectory. Each of these IMS shows different values of the scale factor. It is equivalent to the ray propagation
in the randomly nonuniform media. Since the scale factor is a fluctuating quantity, the additional level width

\[ \Gamma_a = \omega_0 l \sqrt{\left\langle \left( \frac{\dot{a}}{a} \right)^2 \right\rangle - \left\langle \frac{\dot{a}}{a} \right\rangle^2} \]  

appears besides the broadening of a line due to atom collisions.

Above we considered uniform Universe. In the general case of Universe with the local degrees of freedom, one cannot neglect a backreaction of the measuring system on Universe. The measurement process spoils suggested uniformity. Under the measurements carried out under IMS, the projection of the Universe state occurs (Fig. 7, a). The projection is local, i.e. the Universe state is “spoiled” only at the local region occupied by IMS. One does not need to consider the multiple Universes to build quantum mechanics: every measurements spoil Universe only in a local region, and far from this point Universe remains almost unchanged and ready for a next measurement.

Let us discuss one more sort of measurements. There exists an opinion [36], that the matter degrees of freedom in Universe serve as an environment leading to the decoherence of metric. In other words, we can assume that at some stage of the Universe evolution, a “self measurement” occurs due to a matter filling Universe. As a result, the scale factor is projected to the different values in the different spatial regions of Universe and the nonuniform structure of Universe appears (Fig. 7, b). Quantum dispersion of the scale factor turns to dispersion of the scale factor in the different spatial regions (i.e. the classical dispersion results from the set of measurements). Again, this results in an additional broadening of spectral lines. If a typical size of the scale factor nonuniformity is greater than the size of light source, it is not possible to see an additional line broadening. In this case, another effect like an observation of the Hubble constant dispersion in the different directions has to appear. In the case, when the size of the nonuniformity is greater than the observed range of Universe, we are not able to detect the spatial scale factor dispersion by the direct experiments.

6 Conclusion

Universe quantum evolution originated from the some fluctuation of the scalar field (wave packet) has been considered. No initial conditions for
inflation are needed because all information is contained in the quantum state.

Quantization procedure for the equation of motion has been introduced resulting in the quasi-Heisenberg operators, which are Hermite when the Universe wave function normalizes in the Klein-Gordon style.

For the quadratic inflationary potential, the numerical calculations have demonstrated that dispersion of logarithm of the scale factor grows during inflation and approaches some constant at the inflation end. Interpretation of this fact can be that the Universe scale factor is projected to the different spatial regions by the "self-measurement" process (associated with the decoherence) after inflation. This results in a highly nonuniform Universe. However, because there is no significant dispersion of the Hubble constant measured in a different directions, these spatial regions must have super-Hubble size.

In the model with massless scalar field but with the cosmological constant, which decreases with the scale factor growth, we obtain the decreasing dispersion of logarithm of the Universe scale factor. This causes the negligible dispersion after the inflation end without any need of the decoherence.

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