Abstract

We introduce an associative algebra $M_k(x)$ whose dimension is the $2k$-th Motzkin number. The algebra $M_k(x)$ has a basis of “Motzkin diagrams,” which are analogous to Brauer and Temperley-Lieb diagrams, and it contains the Temperley-Lieb algebra $TL_k(x)$ as a subalgebra. We prove that for a particular value of $x$, the algebra $M_k(x)$ is the centralizer algebra of $U_q(gl_2)$ acting on the $k$-fold tensor power of the sum of the 1-dimensional and 2-dimensional irreducible $U_q(gl_2)$-modules. We show that $M_k(x)$ is generated by special diagrams $\ell_i, t_i, r_i$ ($1 \leq i < k$) and $p_j$ ($1 \leq j \leq k$), and that it has a factorization into three subalgebras $M_k(x) = RP_k TL_k(x) LP_k$, all of which have dimensions given by Catalan numbers. We define an action of $M_k(x)$ on Motzkin paths of rank $r$, and in this way, construct a set of indecomposable modules $C_k^{(r)}$, $0 \leq r \leq k$. We prove that $M_k(x)$ is cellular in the sense of Graham and Lehrer and that the $C_k^{(r)}$ are the left cell representations. We compute the determinant of the Gram matrix of a bilinear form on $C_k^{(r)}$ for each $r$ and use these determinants to show that $M_k(x)$ is semisimple exactly when $x$ is not the root of certain Chebyshev polynomials.

Keywords: Motzkin number, Catalan number, Motzkin algebra, Temperley-Lieb algebra, $gl_2$ quantum enveloping algebra, Schur-Weyl duality, cellular algebra

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1 Introduction

The Motzkin numbers $M_n$, $n = 0, 1, \ldots$, are defined by the generating function

\[
M(t) = \sum_{n \geq 0} M_n t^n = \frac{1 - t - \sqrt{1 - 2t - 3t^2}}{2t^2}
= 1 + t + 2t^2 + 4t^3 + 9t^4 + 21t^5 + 51t^6 + 127t^7 + 323t^8 + \cdots
\]

which satisfies

\[
M(t) = 1 + tM(t) + t^2M^2(t).
\]
Although not as ubiquitous as their relatives the Catalan numbers, the Motzkin numbers have appeared in a variety of combinatorial settings (see [M], [DS], [A], [SH 6.38], [E]). For example, $M_n$ counts

- the number of ways of drawing any number of nonintersecting chords among $n$ points on a circle;
- the number of lattice paths from $(0,0)$ to $(n,0)$ with steps $(1,-1)$, $(1,0)$, $(1,1)$, never going below the $x$-axis; or
- the number of walks on $\mathbb{N} = \{0,1,\ldots\}$ with $n$ steps from $\{-1,0,1\}$ starting and ending at 0;
- the number of standard tableaux of shape $\lambda$ with entries $\{1,\ldots,n\}$ over all partitions $\lambda$ of $n$ with no more than 3 parts.

In this work, we introduce a unital associative algebra $M_k(x)$, depending on the parameter $x$, whose dimension is the Motzkin number $M_{2k}$. The algebra $M_k(x)$ is defined over any commutative ring $\mathbb{K}$ with 1 and has a $\mathbb{K}$-basis of diagrams similar to those for the Temperley-Lieb, Brauer, and partition algebras.

When $\mathbb{K}$ is a field and $q \in \mathbb{K}$ is nonzero and not a root of unity, we show in Section 3 that the centralizer algebra of the action of the quantum enveloping algebra $U_q(\mathfrak{gl}_2)$ on the $k$-fold tensor power $V^{\otimes k}$, where $V = V(0) \oplus V(1)$, the sum of the trivial 1-dimensional module $V(0)$ and the natural 2-dimensional module $V(1)$ for $U_q(\mathfrak{gl}_2)$, is an algebra $M_k$ whose dimension is the Motzkin number $M_{2k}$. The algebra $M_k$ is semisimple and has irreducible modules $M_k^{(r)}$ indexed by the integers $r = 0, 1, \ldots, k$. The dimension of $M_k^{(r)}$ is $m_{k,r}$, the number of walks on $\mathbb{N}$ with $k$ steps from $\{-1,0,1\}$ starting at 0 and ending at $r$, and $M_{2k} = \sum_{r=0}^{k} m_{k,r}^2$.

By analogy, the centralizer algebra of the action of $U_q(\mathfrak{gl}_2)$ on $V(1)^{\otimes k}$ is the well-studied Temperley-Lieb algebra $TL_k(q+q^{-1})$, which first arose in statistical mechanics [TL] and has played a prominent role in Jones’ work ([Jo1], [Jo3]) on subfactors of von Neumann algebras and invariants of knots and links [Jo2]. Using facts about the Temperley-Lieb algebra and its irreducible modules, we derive a closed form expression for the numbers $m_{k,r}$.

We prove that the centralizer algebra $M_k$ is isomorphic to the Motzkin algebra $M_k(\zeta_q)$, where $\zeta_q = 1 - q - q^{-1}$. Using that result, we give an explicit decomposition of $V^{\otimes k}$ into irreducible $U_q(\mathfrak{gl}_2)$-modules. Finding this decomposition was one of our original motivations for introducing the Motzkin algebras. It follows that when $q \to 1$, the algebra $M_k(-1)$ is the centralizer algebra of the action of $\mathfrak{gl}_2$ on tensor powers of the sum of its natural 2-dimensional and trivial representations. We define an action of any Motzkin algebra $M_k(x)$ on certain paths, which we term Motzkin paths and which are related to the lattice paths and to the walks on $\mathbb{N}$ mentioned earlier. The Motzkin paths enable us to construct indecomposable $M_k(x)$-modules $C_k^{(r)}$ labeled by $r = 0, 1, \ldots, k$. The values of the characters of these modules on the basis diagrams are equal to the numbers $m_{k,r}$ for $0 \leq \ell \leq k$. The algebra $M_k(x)$ is cellular in the sense of Graham and Lehrer [GL], and the modules $C_k^{(r)}$ are the left cell representations. By computing the determinant of the Gram matrices for the modules $C_k^{(r)}$ explicitly, we determine conditions for the algebra $M_k(x)$ to be semisimple using expressions in the roots of the Chebyshev polynomials of the second kind. When $M_k(x)$ is semisimple, the modules $C_k^{(r)}$ are irreducible, and $\{C_k^{(r)} \mid r = 0, 1, \ldots, k\}$ is a complete set of representatives of isomorphism classes of irreducible $M_k(x)$-modules.
The q-rook monoid algebra $R_k(q)$ over $\mathbb{C}(q)$ specializes at $q = 1$ to the monoid algebra $R_k(1)$, which has as a basis the $k \times k$ complex matrices with entries 0,1 and at most one 1 in each row and column. The algebra $R_k(1)$ was first studied by Munn [Mu1, Mu2] and later by Solomon [So2], who gave a presentation for it and showed that there is an algebra epimorphism $R_k(1) \to \text{End}_{GL_n}(V^{\otimes k})$, where $V$ is the sum of the natural $n$-dimensional representation and the trivial representation of the general linear group $GL_n$. When $q$ is a prime power, $R_k(q) \cong H_C(M,B)$, the Iwahori-Hecke algebra of matrices $M = \text{Mat}_k(F_q)$ over a finite field of $q$ elements relative to the submonoid $B$ of upper-triangular matrices (see [So1, So2]). In [So3], Solomon showed that there is an algebra epimorphism $\psi_q : R_k(q) \to \text{End}_{U_q(\mathfrak{gl}_n)}(V^{\otimes k})$, where now $V$ is the sum of the natural $n$-dimensional representation and the trivial representation for the quantum enveloping algebra $U_q(\mathfrak{gl}_n)$. Halverson [Ha] used a different presentation for $R_k(q)$ to obtain $\psi_q$ via an $R$-matrix approach and to show it is an isomorphism for $n \geq k$. When $n = 2$, $\psi_q$ is an isomorphism only when $k = 1$ and 2.

One of our main results (Theorem 3.30) identifies $\text{End}_{U_q(\mathfrak{gl}_2)}(V^{\otimes k})$ with the Motzkin algebra $M_k(1-q-q^{-1})$. Thus, it follows from our work that $M_k(1-q-q^{-1})$ is a quotient of $R_k(q)$, hence also of the Iwahori-Hecke algebra $H_k(0,1;q)$ of type $B_k$ by [HR2]. Analogously, when $V$ is the natural $n$-dimensional representation of $U_q(\mathfrak{gl}_n)$, it is well known that there is an epimorphism $H_k(q) \to \text{End}_{U_q(\mathfrak{gl}_n)}(V^{\otimes k})$ of the Iwahori-Hecke algebra $H_k(q)$, which is an isomorphism for $n \geq k$. When $n = 2$, the centralizer algebra $\text{End}_{U_q(\mathfrak{gl}_2)}(V^{\otimes k})$ is the Temperley-Lieb algebra $\mathcal{T}_k(q)$.

In Section 2.3, we show that the Motzkin algebra $M_k(x)$ is generated by certain elements $\ell_i, r_i, t_i$ ($1 \leq i < k$), $p_j$ ($1 \leq j \leq k$). The algebra $M_k(x)$ contains several noteworthy subalgebras (all subalgebras considered are assumed to contain the identity element of $M_k(x)$). Indeed, the subalgebra generated by the elements $\ell_i, r_i$ ($1 \leq i < k$) and $p_j$ ($1 \leq j \leq k$) is the planar rook algebra (see [Ha], [FHIII]), and its irreducible modules have dimensions given by the binomial coefficients $\binom{k}{n}$, for $n = 0, 1, \ldots, k$. The subalgebra $R_k$ generated by the elements $r_i$ ($1 \leq i < k$) and $p_j$ ($1 \leq j \leq k$) has dimension the Catalan number $C_{k+1}$. The subalgebra $R_k$ generated by the elements $r_i$ ($1 \leq i < k$) has dimension equal to the $(k+1)$st term in the sequence 1, 1, 1, 3, 10, 31, 98, · · · (sequence #A114487 in [Sl]). This sequence has appeared in the work [STT], where the $n$th term in the sequence is shown to be the number of lattice paths (Dyck paths) from (0,0) to $(2n,0)$ with steps $u = (1,1)$ or $d = (1,-1)$, never falling below the $x$-axis and avoiding the pattern $uudd$. To keep the paper a reasonable length, we do not include the computations of the dimensions of $R_k$ and $R_k$ as they are not needed for any results here.

The Motzkin algebra has a factorization $M_k(x) = R_k TL_k(x) L_k$, where $L_k$ is the subalgebra generated by the elements $\ell_i$ ($1 \leq i < k$) and $p_j$ ($1 \leq j \leq k$). Every basis diagram of $M_k(x)$ can be expressed as a product of diagrams one from each of these three components. The subalgebra $L_k$ is anti-isomorphic to $R_k$ via an involution which interchanges $R_k$ and $L_k$ and restricts to an involution on $TL_k(x)$. Thus, $\dim(L_k) = \dim(R_k) = C_{k+1}$. Since $\dim(TL_k(x)) = C_k$, each of these three factors has dimension a Catalan number.

The results in this paper give representation-theoretic interpretations of known identities (2.2), (4.11) involving Motzkin numbers. Equations (2.16), (3.16), and (3.21) provide additional Motzkin identities which arise naturally in this setting.
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2 The Motzkin Algebra

2.1 Motzkin diagrams

A Motzkin $k$-diagram consists of two rows each with $k$ vertices such that each vertex is connected to at most one other vertex by an edge, and the edges are planar, which is to say they can be drawn without crossing while staying inside the rectangle formed by the vertices. For example

\[ d = \]

is a Motzkin 7-diagram, and

are the nine Motzkin 2-diagrams. The number of Motzkin $k$-diagrams is the same as the number of ways of drawing any number of nonintersecting chords among $2k$ points on a circle, which is known [SI, 6.38.a] to be the Motzkin number $M_{2k}$.

A Motzkin $k$-diagram with $k$ edges is a Temperley-Lieb diagram (see [TL] or [GHJ]). The number of Temperley-Lieb diagrams is the number of ways of drawing $k$ nonintersecting chords on $2k$ points on a circle, which is the Catalan number

\[ C_k = \frac{1}{k+1} \binom{2k}{k}. \]  

(2.1)

We can count the number of Motzkin $k$-diagrams according to the number $n$ of edges in the diagram. First choose, in $\binom{2k}{2n}$ ways, the $2n$ vertices to be connected by the $n$ edges. Then the number of planar ways to connect these edges is the Catalan number $C_n$. Thus,

\[ M_{2k} = \sum_{n=0}^{k} \binom{2k}{2n} C_n = \sum_{n=0}^{k} \frac{1}{n+1} \binom{2k}{2n} \binom{2n}{n}. \]  

(2.2)

2.2 The Motzkin algebra $M_k(x)$

Assume $\mathbb{K}$ is a commutative ring with 1 and $x$ is an element of $\mathbb{K}$. Set $M_0(x) = \mathbb{K}1$, and for $k \geq 1$, let $M_k(x)$ be the free $\mathbb{K}$-module with basis the Motzkin $k$-diagrams. We multiply two Motzkin $k$-diagrams $d_1$ and $d_2$ as follows. Place $d_1$ above $d_2$ and identify the bottom-row vertices in $d_1$ with the corresponding top-row vertices in $d_2$. The product is $d_1d_2 = x^{\kappa(d_1,d_2)}d_3$, where $d_3$ is the diagram consisting of the horizontal edges from the top row of $d_1$, the horizontal edges from the bottom row of $d_2$, and the vertical edges that propagate from the bottom of $d_2$ to the top of $d_1$, and $\kappa(d_1,d_2)$ is the number of loops that arise in the middle row. For example, if

\[ d_1 = \]

\[ d_2 = \]

then

\[ d_1d_2 = x \]

\[ = x \]
Diagram multiplication makes \( M_k(x) \) into an associative algebra with identity element

\[
1_k = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array}
\]

Note that under this multiplication, two vertical edges can become a single horizontal edge as in the example above, and a vertical edge can contract to a vertex. The \textit{rank} of a diagram \( \text{rank}(d) \) is the number of vertical edges in the diagram. Therefore,

\[
\text{rank}(d_1d_2) \leq \min(\text{rank}(d_1), \text{rank}(d_2)). \quad (2.3)
\]

For \( 0 \leq r \leq k \), we let \( J_r(x) \subseteq M_k(x) \) be the \( \mathbb{K} \)-span of the Motzkin \( k \)-diagrams of rank \textit{less than or equal} to \( r \). Then, by (2.3), \( J_r(x) \) is a two-sided ideal in \( M_k(x) \) and we have the tower of ideals,

\[
J_0(x) \subseteq J_1(x) \subseteq J_2(x) \subseteq \cdots \subseteq J_k(x) = M_k(x). \quad (2.4)
\]

The \( \mathbb{K} \)-span of the Temperley-Lieb diagrams in \( M_k(x) \) forms a subalgebra \( \text{TL}_k(x) \) known as the \textit{Temperley-Lieb algebra}, and \( \dim(\text{TL}_k(x)) = \mathcal{C}_k \).

\section{2.3 Generators for the Motzkin algebra \( M_k(x) \)}

For \( 1 \leq i \leq k - 1 \), consider the following diagrams in \( M_k(x) \),

\[
t_i = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array}
\]

\[
r_i = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array}
\]

\[
\ell_i = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array}
\]

\[
(2.5)
\]

It is well known that the \( t_i \)'s generate the Temperley-Lieb algebra \( \text{TL}_k(x) \) (see for example [3]). For \( 1 \leq i \leq k \), let

\[
p_i = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array}
\]

\[
(2.6)
\]

Diagram multiplication shows that

\[
p_1 = r_1 \ell_1, \quad p_i = r_i \ell_i = \ell_{i-1} r_{i-1}, \quad \text{for} \quad 1 < i < k, \quad \text{and} \quad p_k = \ell_{k-1} r_{k-1}. \quad (2.7)
\]

In this section, we prove that the diagrams \( t_i, \ell_i, r_i \) \((1 \leq i < k)\) generate \( M_k(x) \).

Let \( \text{RP}_k \subseteq M_k(x) \) denote the subalgebra with identity element \( 1_k \) spanned by the diagrams containing no horizontal edges and no vertical edges directed to the northeast; and analogously, let \( \text{LP}_k \subseteq M_k(x) \) denote the subalgebra with \( 1_k \) spanned by the diagrams containing no horizontal edges and no vertical edges directed to the southeast. For example,

\[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array} \in \text{RP}_{10} \quad \text{and} \quad \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array} \in \text{LP}_{10}.
\]

For \( 1 \leq j < i \leq k \), set

\[
r_{i,j} = r_{i-1} r_{i-2} \cdots r_j = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array}
\]

\[
(2.8)
\]

and let \( r_{i,i} = \ell_{i,i} = 1_k \) for \( 1 \leq i \leq k \).

Let \( d \) be a diagram in \( \text{RP}_k \). Thus, \( d \) has no horizontal edges. Let \( I = I(d) = \{i_1 < i_2 < \cdots < i_s\} \) denote the vertices on the top row of \( d \) which connect to vertices in the bottom
Thus, we have established the following result.

row of \( d \), and let \( J = J(d) = \{ j_1 < j_2 < \cdots < j_k \} \) denote the corresponding vertices in the bottom row of \( d \) so that \( i_q \) and \( j_q \) are connected by an edge for \( q = 1, \ldots, s \). In order for \( d \) to belong to \( \mathbb{R}P_k \), we must have \( i_q \geq j_q \) for each \( q = 1, \ldots, s \). It is easy to see (by multiplying diagrams) that

\[
d = \left( \prod_{i \notin I} p_i \right) r_{i_1,j_1} r_{i_2,j_2} \cdots r_{i_s,j_s} \left( \prod_{j \notin J} p_j \right).
\]

This shows that

\[
\mathbb{R}P_k = \langle 1_k, r_1, \ldots, r_{k-1}, p_1, \ldots, p_k \rangle.
\]

There is an involution “\( * \)” (a \( \mathbb{K} \)-linear anti-automorphism of order 2) on \( M_k(x) \) which interchanges the vertices in the top and bottom rows of a diagram while maintaining all the edge connections. Thus, for Motzkin diagrams \( d_1, d_2 \), the involution satisfies \((d_1 d_2)^* = d_2^* d_1^*\) and \((d^*)^* = d\). It has the following effect on the elements in (2.5):

\[
t_i^* = t_i, \quad \ell_i^* = r_i, \quad r_i^* = \ell_i \quad (1 \leq i < k), \quad p_j^* = p_j \quad (1 \leq j \leq k).
\]

As a result, the subalgebra \( \mathbb{L}P_k \) is anti-isomorphic to \( \mathbb{R}P_k \) and

\[
\mathbb{L}P_k = \langle 1_k, \ell_1, \ldots, \ell_{k-1}, p_1, \ldots, p_k \rangle.
\]

Similarly, the subalgebra \( \mathbb{L}_k \) with \( 1_k \) generated by the \( \ell_i \) \((1 \leq i < k)\) is anti-isomorphic to \( \mathbb{R}_k \), the subalgebra with \( 1_k \) generated by the \( r_i \) \((1 \leq i < k)\).

Every diagram in \( d \in M_k(x) \) can be factored as

\[
d = r t \ell, \quad \text{with } r \in \mathbb{R}P_k, \quad t \in \mathbb{L}T_k(x), \quad \text{and } \ell \in \mathbb{L}P_k.
\]

For example,

\[
d = \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array} \quad \text{with } r \in \mathbb{R}P_k, \quad t \in \mathbb{L}T_k(x), \quad \text{and } \ell \in \mathbb{L}P_k.
\]

This factorization can be done in the following way. To obtain \( t \), first shift the isolated vertices of \( d \) to the right of the diagram, preserving connections on the other vertices, to produce the diagram \( d_1 \). Second, fill in with enough non-nested horizontal edges so that the top and bottom rows have the same number of horizontal edges to produce the diagram \( d_2 \). Finally, finish off with identity edges. This produces the Temperley-Lieb diagram \( t \in TL_k(x) \) that is associated with \( d \). These steps are illustrated below.

Now, \( r \in \mathbb{R}P_k \) and \( \ell \in \mathbb{L}P_k \) are the unique diagrams that move the edges to their appropriate positions as seen in (2.11). This algorithm proves that

\[
M_k(x) = \mathbb{R}P_k TL_k(x) \mathbb{L}P_k.
\]

(2.12)

Thus, we have established the following result.
Proposition 2.13. The Motzkin algebra \( M_k(x) \) is generated by \( 1_k \) and the diagrams \( t_i, \ell_i, r_i \) for \( 1 \leq i < k \).

Remark 2.14. While these diagrams form a natural set of generators which obey nice relations, a problem not addressed here is to give a presentation of \( M_k(x) \) by generators and relations.

2.4 Motzkin paths and 1-factors

A Motzkin path of length \( k \) is a sequence \( p = (a_1, \ldots, a_k) \) with \( a_i \in \{-1,0,1\} \) such that \( a_1 + \cdots + a_s \geq 0 \) for all \( 1 \leq s \leq k \). Define \( \text{rank} ((a_1, \ldots, a_k)) = a_1 + \cdots + a_k \), so for example,

\[
p = (1, 1, 1, -1, 1, -1, 0, 1, 1, 0, 1, 1, 0, 1, -1, -1, 0, 1)
\]

is a Motzkin path of length 20 and rank 4. Let \( P_k \) denote the set of Motzkin paths of length \( k \), and

\[
P_k^r = \{ p \in P_k \mid \text{rank}(p) = r \} \quad \text{for} \quad r = 0, 1, \ldots, k. \tag{2.15}
\]

It follows from [St, Exercise 6.38 (d), p. 238] that the Motzkin number \( M_k \) is the number of lattice paths from \((0,0)\) to \((k,0)\) with steps \((1,-1), (1,0), (1,1)\), never going below the \(x\)-axis. These correspond precisely to our Motzkin paths in \( P_k \), as can be readily seen by taking the second coordinates of the steps. More generally, the Motzkin paths in \( P_k^r \) are in bijection with lattice paths from \((0,0)\) to \((k,r)\) with steps \((1,-1), (1,0), (1,1)\), never going below the \(x\)-axis. Assume \( m_{k,r} \) counts those paths so that \( |P_k^r| = m_{k,r} \). These paths are also in bijection with the walks with \( k \) steps from \{-1,0,1\} on \( \mathbb{N} \) starting at 0 and ending at \( r \).

We claim that

\[
\sum_{r=0}^{k} m_{k,r}^2 = M_{2k}. \tag{2.16}
\]

This follows from the simple observation that \( M_{2k} \) counts the number of Motzkin paths of length \( 2k \) and rank 0, hence also the number of lattice paths from \((0,0)\) to \((2k,0)\) with steps \((1,0), (1,1), (1,-1)\). The first \( k \) steps in that path form a lattice path from \((0,0)\) to \((k,r)\) for some \( r \in \{0,1,\ldots,k\} \) and hence correspond to a Motzkin path of length \( k \) and rank \( r \). The last \( k \) steps in the path form a lattice path from \((2k,0)\) to \((k,r)\) when it is read in reverse order and with a change of signs, hence also correspond to a Motzkin path of length \( k \) and rank \( r \). Equation (2.16) follows.

For each index \( i \) with \( a_i = 1 \) in the Motzkin path \( p = (a_1, \ldots, a_k) \in P_k \), let \( j \) be the smallest index (if it exists) such that \( i < j \leq k \) and \( a_i + a_{i+1} + \cdots + a_j = 0 \). Then \((a_i, a_j) = (1, -1)\) are said to be paired in \( p \); otherwise \( a_i \) is said to be unpaired in \( p \). For example, in the Motzkin path above, if we connect paired indices by an edge, and we associate unpaired 1s with \( \circ \), we have

\[
p = 1 \quad \circ \quad 1 \quad 1 \quad -1 \quad -1 \quad -1 \quad 0 \quad 1 \quad -1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad -1 \quad -1 \quad 0 \quad 1
\]

Then, since paired vertices cancel one another in the sum,

\[
\text{rank}(p) = \text{the number of white vertices in } p.
\]

Define a Motzkin 1-factor to be a diagram on a single row of \( k \) vertices, colored white or black, such that
1. black vertices are allowed to be connected by non-crossing horizontal edges drawn below the diagram, and

2. white vertices are not allowed between two vertices that are connected by an edge.

The pairing procedure above provides a map from Motzkin paths to Motzkin 1-factors. This process never allows white vertices between paired vertices or edge crossings. Furthermore, it is easy to find the labeling on a Motzkin 1-factor that produces its path $p$, so the procedure is reversible, and we have a bijection between paths and 1-factors. Therefore, in what follows, we use the terms (Motzkin) path and 1-factor interchangeably.

If $p$ and $q$ are Motzkin paths each of length $k$ and rank $r$, define a Motzkin diagram $d_p^q \in M_k(x)$ such that the horizontal edges in the bottom row of $d_p^q$ are from $p$ (but are reflected to be above the vertices), the horizontal edges in the top row of $d_p^q$ are from $q$, and the vertical edges in $d_p^q$ connect the $r$ white vertices in $p$ to the $r$ white vertices in $q$ in the one, and only one, planar way. For example,

\[
q = \begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

\[
p = \begin{array}{cccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\end{array}
\]

\[
d_p^q = \begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

In this way each Motzkin $k$-diagram of rank $r$ is uniquely identified with a pair $(p,q)$ of Motzkin 1-factors of rank $r$ and we have a second realization of identity (2.16).

In Section 3.3, we use Motzkin paths to count the multiplicity of irreducible quantum $\mathfrak{gl}_2$-modules in tensor space; in Section 3.5 we use Motzkin paths to explicitly decompose tensor space into irreducible $U_q(\mathfrak{gl}_2)$-modules; and in Section 4.1 we use Motzkin paths as a basis for the cell modules of $M_k(x)$.

2.5 Basic construction

The natural embedding of $M_k(x) \subseteq M_{k+1}(x)$, given by adding to a Motzkin diagram in $M_k(x)$ a vertical edge connecting the $(k+1)$st vertex in each row, allows one to study

\[
M_0(x) \subseteq M_1(x) \subseteq M_2(x) \subseteq \cdots
\]

collectively as a tower of algebras using the “Jones basic construction” (see [GHJ] or [HR1]) and the methods of recollement (see [CPS]). Define the idempotent $e_k = \frac{1}{2} t_k e_k \in M_k(x)$. Then, for $k \geq 2$, there is an algebra isomorphism

\[
M_{k-2}(x) \overset{\cong}{\to} e_k M_k(x) e_k
\]

given by sending a diagram $d \in M_{k-2}(x) \subseteq M_k(x)$ to $e_k d e_k$. This is easily confirmed by considering diagram multiplication. Note that under this map $1_{k-2} \mapsto e_k$, which is the unit in $e_k M_k(x) e_k$.

The algebras $M_k(x) e_k M_k(x)$ and $e_k M_k(x) e_k$ are full centralizers of each other on $M_k(x) e_k$ under left and right multiplication, respectively. Therefore, by double centralizer theory, the
irreducible modules for $M_k(x)e_kM_k(x)$ and $e_kM_k(x)e_k$ are in bijection. Thus, $M_k(x)e_kM_k(x)$ and $M_{k-2}(x)$ have the same irreducible modules, and they are labeled by the integers $0,1,2,\ldots,k-2$ as we shall see in Section 4. The ideal $M_k(x)e_kM_k(x)$ is referred to as the “basic construction” for $M_k(x)$ determined by the idempotent $e_k$.

It is easy to see by multiplying diagrams that $M_k(x)e_kM_k(x)$ is a non-cocommutative Hopf algebra with coproduct given by

$$d_q^p d_t^s \equiv \delta_{q,s} d_t^p \mod J_{k-2},$$

as the rank goes down when $q \neq s$; from this we see that the cosets $\{d_q^p + J_{k-2} \mid p, q \in \mathcal{P}_k\}$ form a basis of a $k \times k$ matrix algebra. Thus, when $\mathbb{K}$ is a field,

$$M_k(x)/M_k(x)e_kM_k(x) = M_k(x)/J_{k-2} \cong \mathbb{K}_1 \oplus \text{Mat}_k(\mathbb{K})$$

is semisimple. (2.21)

The basic construction tells us that when $x$ is chosen so that $M_k(x)$ is semisimple, the irreducible modules of $M_k(x)$ are in bijection with the union of the irreducible modules for $\mathbb{K}_1$, $\text{Mat}_k(\mathbb{K})$, and $M_{k-2}(x)$. This fact is explicitly realized when we construct the cell modules for $M_k(x)$ in Section 4.

3 Schur-Weyl duality

3.1 Quantum $\mathfrak{gl}_2$

We assume throughout this section that $\mathbb{K}$ is a field, and $q \in \mathbb{K}$ is nonzero and not a root of unity. The Lie algebra $\mathfrak{gl}_2$ of $2 \times 2$ matrices over a field $\mathbb{K}$ has standard basis elements

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

which satisfy the relations

$$[h_1, h_2] = 0, \quad [h_1, e] = e, \quad [h_1, f] = -f, \quad [h_2, e] = -e, \quad [h_2, f] = f, \quad \text{and} \quad [e, f] = h,$$

(3.1)

where $[a, b] = ab - ba$ and $h = h_1 + h_2$. Its universal enveloping algebra $U(\mathfrak{gl}_2)$ is the unital associative $\mathbb{K}$-algebra generated by $e, f, h_1, h_2$ with the defining relations in (3.1). The quantum enveloping algebra $U_q(\mathfrak{gl}_2)$ is the unital associative algebra generated by $E, F, K_i^{\pm 1}, i = 1, 2$, subject to the relations

$$K_1K_2 = K_2K_1, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1, \quad i = 1, 2,$$
$$K_1EK_1^{-1} = qE, \quad K_2EK_2^{-1} = q^{-1}E,$$
$$K_1FK_1^{-1} = q^{-1}F, \quad K_2FK_2^{-1} = qF,$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, \quad \text{where} \quad K = K_1K_2^{-1}.$$

(3.2)

In the classical limit $q \to 1$, the algebra $U_q(\mathfrak{gl}_2)$ specializes to $U(\mathfrak{gl}_2)$. Furthermore, $U_q(\mathfrak{gl}_2)$ is a non-cocommutative Hopf algebra with coproduct given by

$$\Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F,$$
$$\Delta(K_i) = K_i \otimes K_i, \quad i = 1, 2,$$

(3.3)
and counit \( u \) given by
\[
u(E) = u(F) = 0, \quad u(K_i) = 1, \quad i = 1, 2.\] (3.4)

The subalgebra of \( U_q(\mathfrak{gl}_2) \) generated by \( E, F, K^{\pm 1} \) is the quantum enveloping algebra \( U_q(\mathfrak{sI}_2) \) corresponding to the Lie algebra \( \mathfrak{sI}_2 \) of traceless matrices in \( \mathfrak{gl}_2 \).

For each \( r = 0, 1, \ldots \), the algebra \( U_q(\mathfrak{gl}_2) \) has an irreducible module \( V(r) \) of dimension \( r + 1 \) generated by a highest weight vector \( v_0 \) such that \( Ev_0 = 0, K_1 v_0 = q^r v_0 \), and \( K_2 v_0 = v_0 \). In particular, the 1-dimensional irreducible \( U_q(\mathfrak{gl}_2) \)-module \( V(0) = \text{span}_K \{ v_0 \} \) has \( U_q(\mathfrak{gl}_2) \)-action given by the counit,
\[
Ev_0 = 0, \quad Fv_0 = 0, \quad K_iv_0 = v_0, \quad i = 1, 2,
\] (3.5)

and the 2-dimensional irreducible \( U_q(\mathfrak{gl}_2) \)-module \( V(1) \) has action specified by
\[
Ev_1 = 0, \quad Fv_1 = v_{-1}, \quad K_1 v_1 = qv_1, \quad K_2 v_1 = v_1, \\
Ev_{-1} = v_1, \quad Fv_{-1} = 0, \quad K_1 v_{-1} = v_{-1}, \quad K_2 v_{-1} = qv_{-1},
\] (3.6)

so that the matrices of the representing transformations on \( V(1) \) relative to the basis \( \{ v_1, v_{-1} \} \) are given by
\[
E \rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F \rightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad K_1 \rightarrow \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}, \quad K_2 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}.
\] (3.7)

Set
\[
V = V(0) \oplus V(1) = \text{span}_K \{ v_{-1}, v_0, v_1 \}.
\] (3.8)

Then the \( k \)-fold tensor product space \( V^\otimes k \) has dimension \( 3^k \) and a basis of simple tensors,
\[
V^\otimes k = \text{span}_K \{ v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} \mid i_j \in \{-1, 0, 1\} \text{ for all } j \}.
\] (3.9)

The coproduct (3.3) affords an action of \( U_q(\mathfrak{gl}_2) \) on \( V^\otimes k \).

Under the above assumptions on \( K \) and \( q \), finite-dimensional modules for \( U_q(\mathfrak{gl}_2) \) are completely reducible. In particular, tensor products decompose according to the Clebsch-Gordan formulas
\[
V(r) \otimes V(0) = V(r),
\] (3.10)
\[
V(r) \otimes V(1) = V(r - 1) \oplus V(r + 1),
\] (3.11)

and thus,
\[
V(r) \otimes V = V(r - 1) \oplus V(r) \oplus V(r + 1),
\] (3.12)

where we set \( V(-1) = 0 \).

Iterating (3.12) creates the tower shown in Figure 1, which displays the decomposition of \( V^\otimes k \) into irreducible \( U_q(\mathfrak{gl}_2) \)-modules (where \( r \) stands for \( V(r) \)). By induction, the multiplicity of \( V(r) \) in \( V^\otimes k \) is equal to the number of paths of length \( k \) from the top of the diagram to \( (r) \). These multiplicities are shown as the subscripts. We record the steps in such a path, by writing \(-1\) for a northeast-to-southwest edge, \( 0 \) for a vertical edge, and \( 1 \) for a northwest-to-southeast edge. Thus a path from \((0)\) at the top to \((r)\) at the \( k \)th level
Then by the classical double-centralizer theory (see for example [CR, Secs. 3B and 68]), we know the following:

\[ M_k = \text{End}_{U_q(\mathfrak{gl}_2)}(V^{\otimes k}) \]

The centralizer algebra of the \( U_q(\mathfrak{gl}_2) \) tensor action

Let \( M_k = \text{End}_{U_q(\mathfrak{gl}_2)}(V^{\otimes k}) \) be the centralizer of \( U_q(\mathfrak{gl}_2) \) acting on \( V^{\otimes k} \), so that

\[ M_k = \left\{ \phi \in \text{End}(V^{\otimes k}) \mid \phi(yw) = y\phi(w) \text{ for all } y \in U_q(\mathfrak{gl}_2), w \in V^{\otimes k} \right\}. \]  

3.2 The centralizer algebra of the \( U_q(\mathfrak{gl}_2) \) tensor action

Let \( M_k = \text{End}_{U_q(\mathfrak{gl}_2)}(V^{\otimes k}) \) be the centralizer of \( U_q(\mathfrak{gl}_2) \) acting on \( V^{\otimes k} \), so that

\[ M_k = \left\{ \phi \in \text{End}(V^{\otimes k}) \mid \phi(yw) = y\phi(w) \text{ for all } y \in U_q(\mathfrak{gl}_2), w \in V^{\otimes k} \right\}. \]  

Then by the classical double-centralizer theory (see for example [CR, Secs. 3B and 68]), we know the following:

- \( M_k \) is a semisimple associative \( \mathbb{K} \)-algebra with irreducible representations labeled by \( 0, 1, \ldots, k \). We let \( \left\{ M_k^{(r)} \mid 0 \leq r \leq k \right\} \) denote the set of irreducible \( M_k \)-modules.

\[ \dim(M_k^{(r)}) = m_{k,r}. \]
We can naturally embed the algebras $M_0 \subseteq M_1 \subseteq M_2 \cdots$, and the edges from level $k + 1$ to level $k$ in Figure 1 represent the restriction rule for $M_k \subseteq M_{k+1}$. Therefore, Figure 1 gives the Bratteli diagram for the tower of semisimple algebras $M_k$.

The tensor space $V \otimes^k$ decomposes as
\[
V \otimes^k \cong \bigoplus_{r=0}^k m_{k,r} V(r) \quad \text{as a $U_q(\mathfrak{gl}_2)$-module,}
\]
\[
\cong \bigoplus_{r=0}^k (r + 1)M_k^{(r)} \quad \text{as an $M_k$-module,}
\]
\[
\cong \bigoplus_{r=0}^k \left(V(r) \otimes M_k^{(r)}\right) \quad \text{as a $(U_q(\mathfrak{gl}_2), M_k)$-bimodule.}
\]

Note that it follows from these expressions that
\[
3^k = \sum_{r=0}^k (r + 1)m_{k,r}.
\]

By general Wedderburn theory, the dimension of $M_k$ is the sum of the squares of the dimensions of its irreducible modules,
\[
\dim(M_k) = \sum_{r=0}^k m_{k,r}^2 = M_{2k}.
\]

### 3.3 The centralizer of the Temperley-Lieb action and a formula for Motzkin numbers

The centralizer algebra of $U_q(\mathfrak{gl}_2)$ acting on the $k$th tensor power $V(1) \otimes^k$ of the 2-dimensional irreducible module $V(1)$ is known to be the Temperley-Lieb algebra $TL_k(q + q^{-1})$. The irreducible modules for $TL_k(q + q^{-1})$ are labeled by the integers $k - 2\ell, \ell = 0, 1, \ldots, [k/2]$ (see for example [GHJ] or [W]), and if $T_k^{(k-2\ell)}$ is the corresponding irreducible $TL_k(q + q^{-1})$-module, then
\[
\dim \left(T_k^{(k-2\ell)}\right) = \binom{k}{\ell} := \binom{k}{\ell} - \binom{k}{\ell - 1}.
\]

Thus,
\[
V(1) \otimes^k \cong \bigoplus_{\ell=0}^{[k/2]} V(k - 2\ell) \otimes T_k^{(k-2\ell)}
\]

as an $(U_q(\mathfrak{gl}_2), TL_k(q + q^{-1}))$-bimodule. Now using the fact that $V(0)$ is a trivial 1-dimensional $U_q(\mathfrak{gl}_2)$-module isomorphic to the field $\mathbb{K}$, we have for $V = V(0) \oplus V(1)$,
\[
V \otimes^k = \bigoplus_{n=0}^k \binom{k}{n} V(1) \otimes^n
\]
\[
= \bigoplus_{n=0}^k \binom{k}{n} \left( \bigoplus_{\ell=0}^{[n/2]} V(n - 2\ell) \otimes T_n^{(n-2\ell)} \right).
\]
Let us fix \( r \), and as above, let \( m_{k,r} \) denote the multiplicity of \( V(r) \) in the decomposition of \( V^\otimes k \) as a \( U_q(\mathfrak{gl}_2) \)-module (which is also the dimension of the irreducible module \( M_k^{(r)} \) for \( M_k \)). Then by examining the above equation for when \( r = n - 2\ell \), we obtain

\[
m_{k,r} = \sum_{\ell=0}^{[(k-r)/2]} \binom{k}{r+2\ell} \dim(T_{r+2\ell}^{(r)})
\]

(3.21)

\[
= \sum_{\ell=0}^{[(k-r)/2]} \binom{k}{r+2\ell} \left\{ \frac{r+2\ell}{\ell} \right\}
\]

Example 3.22. When \( k = 4 \), this formula says the following:

\[
m_{4,0} = \left( \begin{array}{c} 4 \\ 0 \end{array} \right) \{0\} + \left( \begin{array}{c} 4 \\ 2 \end{array} \right) \{2\} + \left( \begin{array}{c} 4 \\ 4 \end{array} \right) \{4\} \times \frac{1}{2} = 1 + 6 + 2 = 9
\]

\[
m_{4,1} = \left( \begin{array}{c} 4 \\ 1 \end{array} \right) \{1\} + \left( \begin{array}{c} 4 \\ 3 \end{array} \right) \{3\} = 4 + 8 = 12
\]

\[
m_{4,2} = \left( \begin{array}{c} 4 \\ 2 \end{array} \right) \{2\} + \left( \begin{array}{c} 4 \\ 4 \end{array} \right) \{4\} = 6 + 3 = 9
\]

\[
m_{4,3} = \left( \begin{array}{c} 4 \\ 3 \end{array} \right) \{3\} = 4
\]

\[
m_{4,4} = \left( \begin{array}{c} 4 \\ 4 \end{array} \right) \{4\} = 1
\]

Therefore \( \dim(M_4) = \sum_{r=0}^{4} m_{k,r}^2 = 81 + 144 + 81 + 16 + 1 = 323 = M_8 \), as expected.

3.4 The action of the Motzkin algebra on tensor space

As the notation and equation (3.17) suggest, \( M_k \) is a Motzkin algebra \( M_k(x) \) for a specific choice of the parameter \( x \). In this section, we show that when \( x = 1 - q - q^{-1} \) there is an action of \( M_k(1 - q - q^{-1}) \) on the tensor space \( V^\otimes k \) such that \( M_k(1 - q - q^{-1}) \cong M_k \).

Define nondegenerate bilinear forms \( \langle \cdot, \cdot \rangle_t \) and \( \langle \cdot, \cdot \rangle_b \) on \( V \) by decreeing

\[
\langle v_{-1}, v_1 \rangle_t = q^{-1/2}, \quad \langle v_0, v_0 \rangle_t = 1, \quad \langle v_1, v_{-1} \rangle_t = -q^{1/2}, \quad \langle v_{-1}, v_1 \rangle_b = -q^{-1/2}, \quad \langle v_0, v_0 \rangle_b = 1, \quad \langle v_1, v_{-1} \rangle_b = q^{1/2},
\]

(3.23)

and

\[
\langle v_i, v_j \rangle_t = \langle v_i, v_j \rangle_b = 0, \quad \text{for all other } i, j.
\]

Notice that

\[
\sum_{i,j} \langle v_i, v_j \rangle_t \langle v_i, v_j \rangle_b = \langle v_{-1}, v_1 \rangle_t \langle v_1, v_{-1} \rangle_b + \langle v_0, v_0 \rangle_t \langle v_0, v_0 \rangle_b + \langle v_1, v_{-1} \rangle_t \langle v_{-1}, v_1 \rangle_b
\]

\[
= -q^{-1} + 1 - q.
\]

(3.24)

For \( d \) in the set \( \mathcal{M}_k \) of all Motzkin \( k \)-diagrams, we define an action of \( d \) on the basis of simple tensors in \( V^\otimes k \) by

\[
d(v_{i_1} \otimes \cdots \otimes v_{i_k}) = \sum_{j_1, \ldots, j_k} (d)_{j_1, \ldots, j_k}^{i_1, \ldots, i_k} v_{j_1} \otimes \cdots \otimes v_{j_k},
\]

(3.25)
where \((d)_{i_1,\ldots,i_k}^{j_1,\ldots,j_k}\) is computed by labeling the vertices in the bottom row of \(d\) with \(v_{i_1},\ldots,v_{i_k}\) and the vertices in the top row of \(d\) with \(v_{j_1},\ldots,v_{j_k}\). Then

\[
(d)_{i_1,\ldots,i_k}^{j_1,\ldots,j_k} = \prod_{\varepsilon \in d} (\varepsilon)_{i_1,\ldots,i_k}^{j_1,\ldots,j_k},
\]

where the product is over the weights of all connected components \(\varepsilon\) (edges and isolated vertices) in the diagram \(d\), where by the weight of \(\varepsilon\) we mean

\[
(\varepsilon)_{i_1,\ldots,i_k}^{j_1,\ldots,j_k} = \begin{cases} 
\delta_{a,0}, & \text{if } \varepsilon \text{ is an isolated vertex labeled by } v_a, \\
\delta_{a,b}, & \text{if } \varepsilon \text{ is a vertical edge connecting } v_a \text{ and } v_b, \\
\langle v_a, v_b \rangle_t, & \text{if } \varepsilon \text{ is a horizontal edge in the top row of } d \\
& \text{connecting } v_a \text{ (left) and } v_b \text{ (right)}, \\
\langle v_a, v_b \rangle_b & \text{if } \varepsilon \text{ is a horizontal edge in the bottom row of } d \\
& \text{connecting } v_a \text{ (left) and } v_b \text{ (right)}, 
\end{cases}
\]

where \(\delta_{a,b}\) is the Kronecker delta. For example, for this labeled diagram

![Diagram](image)

we have

\[
(d)_{i_1,\ldots,i_k}^{j_1,\ldots,j_k} = \langle v_{j_1}, v_{j_2} \rangle_t \langle v_{j_3}, v_{j_4} \rangle_t \langle v_{j_5}, v_{j_6} \rangle_t \langle v_{j_7}, v_{j_8} \rangle_t \langle v_{j_9}, v_{j_{10}} \rangle_t \langle v_{j_{11}} \rangle_t \times \\
\delta_{j_1,i_2} \delta_{j_6,i_4} \delta_{j_7,i_6} \delta_{j_{11},i_9} \delta_{j_{12},0} \delta_{j_4,0} \delta_{j_{10},0} \delta_{i_{11},0}.
\]

The representing transformation of \(d\) is obtained by extending this action linearly to all of \(V^\otimes k\).

Let \(t, l, r \in \mathcal{M}_2\) and \(id \in \mathcal{M}_1\) be the diagrams given by

\[
t = \begin{array}{c}
\bullet \\
\bullet \\
\end{array}, \quad 
 l = \begin{array}{c}
\bullet \\
\bullet \\
\end{array}, \quad 
 r = \begin{array}{c}
\bullet \\
\bullet \\
\end{array}, \quad 
 id = \begin{array}{c}
\bullet \\
\bullet \\
\end{array}.
\]

Then under the action defined in (3.25), \(id v_i = v_i\) for all \(i\), so \(id\) is represented by the identity map \(id\) on \(V\); the representing transformations \(R\) and \(L\) of \(r\) and \(l\) on \(V^\otimes 2\) move simple tensors right and left,

\[
R(v_i \otimes v_j) = \delta_{j,0} v_0 \otimes v_i, \quad L(v_i \otimes v_j) = \delta_{i,0} v_j \otimes v_0;
\]

and the representing transformation \(T\) of \(t\) on \(V^\otimes 2\) acts as a “contraction” map,

\[
T(v_i \otimes v_j) = \langle v_i, v_j \rangle_b (v_{i-1} \otimes v_1) + \langle v_0, v_0 \rangle_t (v_0 \otimes v_0) + \langle v_1, v_{i-1} \rangle_t (v_1 \otimes v_{i-1}) \\
= \langle v_i, v_j \rangle_b \left( q^{-1/2} v_{i-1} \otimes v_1 + v_0 \otimes v_0 - q^{1/2} v_1 \otimes v_{i-1} \right).
\]
Using these maps, we define the representing transformations of the generators $t_i$, $\ell_i$, $r_i$, $(1 \leq i < k)$ acting on $V^\otimes k$ to be

$$T_i = \text{id}_V^\otimes i-1 \otimes T \otimes \text{id}_V^\otimes k-(i+1), \quad 1 \leq i < k,$$

$$L_i = \text{id}_V^\otimes i-1 \otimes L \otimes \text{id}_V^\otimes k-(i+1), \quad 1 \leq i < k,$$

$$R_i = \text{id}_V^\otimes i-1 \otimes R \otimes \text{id}_V^\otimes k-(i+1), \quad 1 \leq i < k.$$  \hspace{1cm} (3.26)

Since $p_1 = \ell_1 r_1$, $p_i = \ell_i r_i = r_i \ell_i$ for $1 < i < k$, and $p_k = r_k \ell_k$, we see that the representing transformation $P_1$ of $p_1$ is the projection onto $V(0)$ in the $i$th tensor position.

Note that in order for this action to satisfy the relation $T_i^2 = xT_i$, the parameter for the Motzkin algebra must be $1 - q - q^{-1}$, since

$$T^2(v_a \otimes v_b) = \langle v_a, v_b \rangle_b \sum_{i,j} \langle v_i, v_j \rangle_t T(v_i \otimes v_j)$$

$$= \langle v_a, v_b \rangle_b \sum_{i,j} \langle v_i, v_j \rangle_t \langle v_i, v_j \rangle_b \sum_{\ell,m} \langle v_\ell, v_m \rangle_t (v_\ell \otimes v_m)$$

$$= \left( \sum_{i,j} \langle v_i, v_j \rangle_t \langle v_j, v_i \rangle_b \right) \langle v_a, v_b \rangle_b \sum_{\ell,m} \langle v_\ell, v_m \rangle_t (v_\ell \otimes v_m)$$

$$= (1 - q - q^{-1})T(v_a \otimes v_b),$$

by (3.24).

Set

$$\zeta_q = 1 - q - q^{-1}. \hspace{1cm} (3.27)$$

For example, $\zeta_1 = -1$ and $\zeta_{-1} = 3$.

**Proposition 3.28.** $\pi_k : M_k(\zeta_q) \to \text{End}(V^\otimes k)$ is an algebra representation.

**Proof.** For diagrams $d_1, d_2 \in M_k$ it suffices to show that

$$(d_1 d_2)^{j_1 \ldots j_k}_{\ell_1 \ldots \ell_k} = \sum_{j_1 \ldots j_k} (d_1)^{j_1 \ldots j_k}_{\ell_1 \ldots \ell_k} (d_2)^{\ell_1 \ldots \ell_k}_{j_1 \ldots j_k}.$$

We analyze the edges of $d_1 d_2$ case by case. Let $v^*_1 = v_1, v^*_0 = v_0$, and $v^*_1 = v_{-1}$.

**Case 1:** Vertical edges in $d_1 d_2$.

These arise from a vertical edge in $d_1$ connected to a vertical edge in $d_2$ via a chain of an even number (possibly 0) of horizontal edges in the middle of $d_1 d_2$. See the example below. The only way to achieve a nonzero matrix entry for $d_1 d_2$ is to have the labeling follow the pattern $v_a, v_a, v^*_a, v_a, v^*_a, \ldots, v^*_a, v_a, v_a$ as illustrated in the example below. Thus, the horizontal edge in the product $d_1 d_2$ will require that the top vertex have the same subscript as the bottom vertex.
We claim that the product of the edge weights corresponding to the $2h$ horizontal edges in the middle of $d_1d_2$ is 1. To see this, direct the edges in the path so that the path travels from the top of $d_1$ to the bottom of $d_2$ as we have done here in our example,

Observe that right arrows in the bottom of $d_1$ correspond to the weight $\langle v_a^*, v_a \rangle_b$, and left arrows correspond to the weight $\langle v_a, v_a^* \rangle_b$. Similarly, right arrows in the top of $d_2$ correspond to the weight $\langle v_a, v_a^* \rangle_t$, and left arrows correspond to the weight $\langle v_a^*, v_a \rangle_t$. Since our forms satisfy

$$\langle v_a^*, v_a \rangle_b \langle v_a, v_a^* \rangle_t = \langle v_a, v_a^* \rangle_b \langle v_a^*, v_a \rangle_t = 1,$$

the problem amounts to showing that there are the same number of right (and thus left) edges in $d_1$ as there are in $d_2$.

Our proof of this last statement is by induction on $h$. The case $h = 0$ is trivially true, so assume $h > 0$. Pick a horizontal edge $\{i, j\}$, with $i < j$, in the top row of $d_2$ with no edge nested above it. That is, there are no edges in $d_2$ connected to vertices in $\{i + 1, \ldots, j - 1\}$. Now consider the edges $\{s, i\}$ and $\{j, t\}$ in $d_1$. Then $s, t \notin \{i + 1, \ldots, j - 1\}$ and, by the planarity of $d_1$, we have $s < t$. This means that $s < i$ or $t > j$ (or both). Suppose that $t > j$ then the edges $\{i, j\}$ and $\{j, t\}$ have the same orientation in the path. We pair these two edges, delete them, and replace $\{s, i\}$ with $\{s, t\}$. Because there are no edges connecting with $\{i + 1, \ldots, j - 1\}$, the new diagram is planar and $\{s, t\}$ has the same direction as $\{s, i\}$. Now we apply induction to pair the rest of the edges in this new graph. The case $s < i$ is entirely similar, and the special case where $s$ or $t$ is in the top row is also easy. This proves that the edges can be paired, and thus the product of the weights is 1, resulting in a single vertical edge of weight 1 in $d_1d_2$.

**Case 2:** A loop in the middle row of $d_1d_2$.

When there is a loop, the middle row consists of a cycle of $h$ horizontal edges in the bottom row of $d_1$ and $h$ horizontal edges in the top row of $d_2$ as in the picture below, where only the edges in the loop are displayed. The only nonzero labeling will be a cycle $v_a, v_a^*, v_a, v_a^*, \ldots, v_a$ for $a = -1, 0, 1$ as illustrated here,

Removing the edges connected to the leftmost $v_a$ leaves a path from $v_a^*$ (starting from the top of $d_2$) to $v_a^*$ (ending in the bottom of $d_1$). The argument from the previous case shows that the product of the weights of this path is 1. The product of the weights on the
remaining two edges in the cycle is

\[
\langle v_a, v_a^* \rangle_t \langle v_a, v_a^* \rangle_b = \begin{cases} 
-q^{-1} & \text{if } a = -1, \\
1 & \text{if } a = 0, \\
-q & \text{if } a = 1.
\end{cases}
\]

Summing over \( a = -1, 0, 1 \), as in (3.24), gives the weight \( 1 - q - q^{-1} = \zeta_q \).

**Case 3**: Horizontal edges in the top row of \( d_1d_2 \).

A horizontal edge in the top row of \( d_1 \) will also appear in the top row of \( d_1d_2 \) with the appropriate weight. It is also possible to gain a horizontal edge in \( d_1d_2 \) through two vertical edges in \( d_1 \) connected by a path in the middle row of \( d_1d_2 \) as pictured in the example below. The path forces \( v_a \) to be connected to \( v_a^* \) in the top row. The path in the middle row from the first \( v_a \) to the last \( v_a^* \) will have a weight of 1 by the argument in Case 1. The remaining edge will be weighted \( \langle v_a, v_a^* \rangle_t \), as desired.

**Case 4**: Horizontal edges in the bottom row of \( d_1d_2 \).

This case is completely analogous to Case 3.

**Theorem 3.29.** Let \( \pi_k : M_k(\zeta_q) \rightarrow \text{End}(V^\otimes k) \) and \( \rho_{V^\otimes k} : U_q(\mathfrak{gl}_2) \rightarrow \text{End}(V^\otimes k) \) be the representations afforded by the action of \( M_k(\zeta_q) \) and \( U_q(\mathfrak{gl}_2) \) on \( V^\otimes k \). Then \( \pi_k(M_k(\zeta_q)) \) and \( \rho_{V^\otimes k}(U_q(\mathfrak{gl}_2)) \) commute. Thus, \( \pi_k(M_k(\zeta_q)) \subseteq \text{End}_{U_q(\mathfrak{gl}_2)}(V^\otimes k) \) and \( \rho_{V^\otimes k}(U_q(\mathfrak{gl}_2)) \subseteq \text{End}_{M_k(\zeta_q)}(V^\otimes k) \).

**Proof.** The elements \( t_i, r_i, \ell_i \) (\( 1 \leq i < k \)) generate \( M_k(\zeta_q) \). A straightforward matrix multiplication shows that their representing transformations, which are given in (3.26), commute with those of the generators \( E, F, K_i^{\pm 1}, i = 1, 2 \), of \( U_q(\mathfrak{gl}_2) \) on \( V^\otimes k \). We provide one illustrative calculation here. Consider \( TE \) acting on the simple tensor \( v_{-1} \otimes v_{-1} \in V \otimes V \). By (3.24) and (3.25) we have

\[
TE(v_{-1} \otimes v_{-1}) = T(Ev_{-1} \otimes Kv_{-1} + v_{-1} \otimes Ev_{-1})
= T(q^{-1}v_1 \otimes v_{-1} + v_{-1} \otimes v_1)
= (q^{-1}v_{-1}, v_1)_b + (v_1, v_{-1})_b)
\left(q^{-1/2}v_{-1} \otimes v_1 + v_0 \otimes v_0 - q^{1/2}v_1 \otimes v_{-1}\right).
\]

On the other hand \( ET(v_{-1} \otimes v_{-1}) = 0 \). Thus \( E \) and \( T \) commute as operators on \( v_{-1} \otimes v_{-1} \), since \( q^{-1}(v_{-1}, v_1)_b + (v_1, v_{-1})_b = 0 \).

Now we turn our attention to proving that the representation

\[
\pi_k : M_k(\zeta_q) \rightarrow \text{End}_{U_q(\mathfrak{gl}_2)}(V^\otimes k)
\]

is the maximal one satisfying the commutativity conditions imposed by the action of \( M_k(\zeta_q) \) and \( U_q(\mathfrak{gl}_2) \).
is faithful. The proof of this fact, as well as the proof of Theorem 3.32, will involve the following order \( \leq \). Suppose \( \underline{\alpha} = (\alpha_1, \ldots, \alpha_\ell) \) is a sequence of positive integers with \( 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_\ell \leq k \), and assume \( \underline{\alpha}' = (\alpha'_1, \ldots, \alpha'_\ell) \) is another such sequence. Then we say \( \underline{\alpha} < \underline{\alpha}' \) if \( \alpha_i = \alpha'_i \) for \( i < s \), but \( \alpha_s < \alpha'_s \). When the \( \alpha_i \)'s in such a sequence correspond to positions of the vertices which are the left ends of the horizontal edges in the bottom row of a \( k \)-diagram, then we say \( \underline{\alpha} \) is the left-end sequence of the bottom row.

Suppose there exists a nonzero \( y = \sum_{d \in \mathfrak{M}_k} a_{\underline{d}} d \in \ker \pi_k \). Choose a diagram \( d' \) so that

(i) \( a_{\underline{d}'} \neq 0 \);

(ii) among the diagrams satisfying (i), \( d' \) has a maximum number, say \( m \), of vertical edges;

(iii) among the diagrams satisfying (i) and (ii), \( d' \) has a maximum number, say \( \ell \), of horizontal edges in its bottom row;

(iv) among the diagrams satisfying (i), (ii), and (iii), \( d' \) has the minimal left-end sequence, say \( \underline{\alpha}' \), in its bottom row.

Let \( u \) in \( V^{\otimes k} \) be the simple tensor having \( v_{-1} \) as the factor in the positions in \( \underline{\alpha}' \). \( v_0 \) in the positions corresponding to isolated vertices in the bottom row of \( d' \); and \( v_1 \) in all the other tensor slots.

Now suppose \( d'' \) satisfies \( a_{\underline{d}''} \neq 0 \) and has \( d''u \neq 0 \). Consider the edges in the bottom row of \( d'' \) which match up with the \( \ell + m \) \( v_1 \)'s in \( u \). There can be at most \( \ell \) such horizontal edges in \( d'' \), as there are \( \ell \) factors \( v_{-1} \) in \( u \). There can be at most \( m \) vertical edges in \( d'' \) by (ii). But for \( d''u \) to be nonzero, there must be an edge in the bottom row of \( d'' \) in the same positions as the \( \ell + m \) \( v_1 \)'s in \( u \). So there must be exactly \( \ell \) horizontal and \( m \) vertical edges in the bottom row of \( d'' \) which meet the \( \ell + m \) factors \( v_1 \) in \( u \). This accounts for all the horizontal edges in the bottom row of \( d'' \) by (iii).

Assume \( \underline{\alpha}' = (\alpha'_1, \ldots, \alpha'_\ell) \), where \( \alpha'_1 < \cdots < \alpha'_\ell \), is the left-end sequence of the bottom row of \( d'' \). The first horizontal edge in \( d'' \) has one of its endpoints at \( \alpha'_1 \). Thus, \( \alpha'_1 \leq \alpha_1 \), and by the minimality of \( \underline{\alpha}' \), it must be that \( \alpha'_1 = \alpha_1 \). The next horizontal edge in \( d'' \) has one of its endpoints at \( \alpha_2 \), and again by minimality, \( \alpha'_2 = \alpha_2 \). Proceeding in this way, we determine that \( \underline{\alpha}' = \underline{\alpha} \).

For \( j = 1, \ldots, \ell \), adding the subscripts of the factors of \( u \) occurring in slots \( \alpha_j, \alpha_j + 1, \ldots, \), we see by the planar property of diagrams, that the subscript sum first becomes 0 at the right-hand endpoint of the horizontal edge in \( d' \) (and also \( d'' \)) with left endpoint at \( \alpha_j \). At the corresponding tensor slot in \( u \), there must be the vector \( v_1 \). The other \( v_1 \)'s in \( u \) occur where \( d' \) has a vertex in its bottom row with a vertical edge emanating from it. But since \( d''u \neq 0 \), \( d'' \) must have a vertical edge in its bottom row at those positions also. By the maximality of the number of vertical edges in the bottom row of \( d' \), this forces \( d'' \) to have exactly the same bottom row as \( d' \).

Now consider all the diagrams having a nonzero coefficient in \( y \) and having precisely the same bottom row as \( d' \). Look at the simple tensors produced by applying those diagrams to \( u \), and choose one of them, call it \( t \), with a maximal number, say \( n \), of \( v_{-1} \)'s and having them in positions \( \gamma_1 < \cdots < \gamma_n \), with \( \underline{\gamma} = (\gamma_1, \ldots, \gamma_n) \) minimal. The top row of any diagram which produces \( t \) when applied to \( u \) must have exactly \( m \) vertical edges, and \( t \) must have \( v_1 \)'s in the corresponding positions. Arguing as we have done with the bottom rows, we see that the top row of any such diagram producing \( t \) with nonzero coefficient when it is
applied to \( u \) is uniquely determined. Hence there is a unique diagram \( \tilde{d} \) with \( a_{\tilde{d}} \neq 0 \) which produces the simple tensor \( t \) when \( y \) is applied to \( u \). But since \( y \) is in \( \ker \pi_k \), and \( t \) occurs in \( yu = 0 \) with coefficient \( a_{\tilde{d}} \), we force \( a_{\tilde{d}} = 0 \), contrary to assumption. This shows that \( \ker \pi_k = 0 \). Thus, we have the following:

**Theorem 3.30.** The representation \( \pi_k : M_k(\zeta_q) \to \text{End}_{U_q(\mathfrak{gl}_2)}(V^{\otimes k}) \) is faithful. Hence \( \text{End}_{U_q(\mathfrak{gl}_2)}(V^{\otimes k}) \cong M_k(\zeta_q) \).

The second assertion in the theorem follows from the faithfulness of \( \pi_k \) and the fact that these two algebras have the same dimension.

**Remark 3.31.** It follows from the double centralizer theory that

\[
\text{End}_{M_k(\zeta_q)}(V^{\otimes k}) \cong \rho_{V^{\otimes k}}(U_q(\mathfrak{gl}_2)),
\]

which is the *Schur algebra* in this setting. For simplicity we denote the semisimple algebra \( \rho_{V^{\otimes k}}(U_q(\mathfrak{gl}_2)) \) by \( S(V^{\otimes k}) \) and observe that

\[
\dim \left( S(V^{\otimes k}) \right) = \sum_{r=0}^{k} (r + 1)^2 = \frac{1}{6}(k + 1)(k + 2)(2k + 3).
\]

If \( S(V(1)^{\otimes k}) = \rho_{V(1)^{\otimes k}}(U_q(\mathfrak{gl}_2)) \), the subalgebra of \( \text{End}(V(1)^{\otimes k}) \) generated by the representing transformations of \( U_q(\mathfrak{gl}_2) \) on \( V(1)^{\otimes k} \) (that is, the Schur algebra for \( V(1)^{\otimes k} \)), then it follows from the Clebsch-Gordan formula (3.11) that

\[
\dim \left( S(V^{\otimes k}) \right) = \dim \left( S(V(1)^{\otimes k}) \right) + \dim \left( S(V(1)^{\otimes k-1}) \right).
\]

### 3.5 An explicit decomposition of \( V^{\otimes k} \) into irreducible \( U_q(\mathfrak{gl}_2) \)-modules.

Recall the notion of a Motzkin path of length \( k \) as in Section 2.4. To each such path \( p \in \mathcal{P}_k \), we associate a simple tensor \( u_p \) as follows. Suppose that the pairs \( (\alpha_i, \beta_i) \) for \( 1 \leq i \leq \ell \) denote the left-hand and right-hand endpoints of the horizontal edges in \( p \), where \( \alpha_1 < \cdots < \alpha_\ell \). Construct the simple tensor \( u_p \) having \( v_{-1} \) at the positions \( \alpha_i \), \( v_1 \) at the positions \( \beta_i \), and also at the positions of the white vertices in \( p \), and \( v_0 \) at all other positions.

Now for each path \( p \), let \( d^p \) be the symmetric diagram in \( M_k(\zeta_q) \) determined by \( p \) as in Section 2.4 and set

\[
w_p = d^pu_p.
\]

We make a number of claims about the vectors \( w_p \).

**Claim 1.** \( w_p \neq 0 \).

If \( p \) has \( \ell \) horizontal edges, then in \( w_p \) the simple tensor \( u_p \) occurs with coefficient equal to

\[
\langle v_{-1}, v_1 \rangle^\ell \langle v_{-1}, v_1 \rangle^\ell = (-q^{-1/2})^\ell (q^{-1/2})^\ell = (-q^{-1})^\ell \neq 0.
\]

**Claim 2.** \( K_iu_p = q^{k_1}u_p \) and \( K_iw_p = q^{k_1}w_p \) for \( i = 1, 2 \), where \( k_1 \) is the number of \( v_1 \) in \( u_p \) and \( k_2 \) is the number of \( v_{-1} \) in \( u_p \). Hence, \( Ku_p = q^{\text{rank}(p)}u_p \) and \( Kw_p = q^{\text{rank}(p)}w_p \), where \( \text{rank}(p) \) is the number of white vertices in \( p \).
That \( w_p \) and \( u_p \) have the same weights relative to the \( K_i \) follows from the fact that the diagram \( \delta_p^p \) lives in the centralizer algebra of the \( \mathbb{U}_q(\mathfrak{g}_2) \)-action, hence commutes with \( K_i \). The rest is clear from the fact that \( K_i \) acts on \( V \otimes k \) by \( K_i \otimes \cdots \otimes K_i \).

**Claim 3.** \( \{ w_p \mid p \in \mathcal{P}_k \} \) is a linearly independent set.

Since we are assuming \( q \) is not a root of unity, and since vectors of different weights are linearly independent, we may assume that we have a nontrivial dependence relation
\[
\sum a_{p'} w_{p'} = 0,
\]
where the sum ranges over the paths \( p' \) of rank \( r \).

Now suppose among the terms occurring in this sum with nonzero coefficient, \( p \) is chosen so that it has a maximal number of horizontal edges, say \( \ell \), and minimal left-end sequence \( \underline{\alpha} = (\alpha_1, \ldots, \alpha_{\ell}) \), where \( \alpha_1 < \cdots < \alpha_{\ell} \). Suppose \( p' \) is another path with \( a_{p'} \neq 0 \) such that \( u_p \) occurs in \( w_{p'} \) with nonzero coefficient. In each tensor summand of \( w_{p'} \) the vector \( v_{-1} \) occurs only in positions which correspond to some horizontal edge of \( p' \). Let \( \underline{\alpha'} \) be the left-end sequence of \( p' \). The positions \( \alpha_j \) with the \( v_{-1} \) in \( u_p \) must correspond to some horizontal edge endpoint in \( p' \). Thus, \( p' \) also has \( \ell \) horizontal edges by the maximality of \( \ell \). Moreover, \( \alpha_1' \leq \alpha_1 \) to hold, and by the minimality of \( \underline{\alpha} \), we must have \( \alpha_1' = \alpha_1 \). Proceeding in order with the \( \alpha_j \), we determine that \( \underline{\alpha'} = \underline{\alpha} \).

Now \( w_{p'} = \delta_p^{p'} u_{p'} \) has \( 3^\ell \) summands, but only one of them has the vector \( v_{-1} \) in all the positions corresponding to \( \underline{\alpha} \), namely \( u_{\alpha'} \). Since we are assuming \( u_p \) occurs in \( w_{p'} \) with nonzero coefficient, and since it has the same property, \( u_{p'} = u_p \). We claim this forces \( p' = p \). Indeed, \( p \) has a horizontal edge with left-end position \( \alpha_j \). Starting at slot \( \alpha_j \) and summing the subscripts on the tensor factors of \( u_p = u_{\alpha'} \), the first place where the subscript sum is 0 must be right-hand end position \( \beta_j \) of the corresponding horizontal edge for each \( j = 1, \ldots, \ell \). The other positions where \( v_1 \) occurs correspond to white vertices of \( p \) and \( p' \), and the positions of the \( v_0 \) factors of \( u_p = u_{\alpha'} \) are the locations of the isolated black vertices in \( p \) and \( p' \). Hence \( p' = p \), and therefore \( u_p \) occurs in \( \sum a_{p'} w_{p'} \) exactly once, with coefficient \( (1 - q - q^{-1})^\ell a_p \), which implies \( a_p = 0 \). We have reached a contradiction, so no such nontrivial dependence relation can occur.

**Claim 4.** \( u_p \) is a highest weight vector for \( \mathbb{U}_q(\mathfrak{g}_2) \) of weight \( q^{k_i} \) relative to \( K_i \), where the values of \( k_i \) are as in Claim 2.

We know already that \( u_p \) is a weight vector for the \( K_i \) of the appropriate weight. We need to argue it is annihilated by \( E \). Using the expression for the coproduct, we see that \( E \) acts on \( V \otimes k \) by
\[
\sum_{j=0}^{k-1} \text{id}_V \otimes E \otimes K^{\otimes (k-1-j)}.
\]
Since \( Ew_p = Ed_p^{p}u_p = d_p^{p}Eu_p \), we first consider the vector \( Eu_p \). Now a term in the above sum has a nonzero action on \( u_p \) only when \( E \) acts on a tensor slot containing the vector \( v_{-1} \), which is a position in \( p \) that is a left-end node of a horizontal edge. It changes \( v_{-1} \) to \( v_1 \) at that position. But then when \( d_p^{p} \) acts on the changed tensor, it finds \( v_1 \) at both ends of a horizontal edge of \( p \), and so gives 0 since \( \langle v_1, v_1 \rangle_b = 0 \). Thus, \( Ew_p = 0 \).

**Claim 5.** \( \mathbb{U}_q(\mathfrak{g}_2)u_p \) is an irreducible \( \mathbb{U}_q(\mathfrak{g}_2) \)-module of dimension \( \text{rank}(p) + 1 \).

From Claims 1, 2, 4, we know that when \( \mathbb{U}_q(\mathfrak{g}_2)u_p \) is viewed as a module for the subalgebra \( \mathbb{U}_q(\mathfrak{sl}_2) \) it is a nonzero homomorphic image of the Verma module for \( \mathbb{U}_q(\mathfrak{sl}_2) \) with highest weight \( q^{\text{rank}(p)} \) relative to \( K \). But the only quotients of that Verma module are 0,
the whole Verma module, and the irreducible \( U_q(\mathfrak{sl}_2) \)-module of dimension \( \text{rank}(p) + 1 \) (see [Ja Prop. 2.5]). Since \( U_q(\mathfrak{gl}_2) w_p \subseteq V^\otimes k \) is finite dimensional and nonzero, it is the desired irreducible quotient.

**Claim 6.** The sum \( \sum_{p', \text{rank}(p')=r} U_q(\mathfrak{gl}_2) w_{p'} \) is direct.

If not, one of the summands, say \( U_q(\mathfrak{gl}_2) w_p \), intersects the sum of the others nontrivially, so by irreducibility, it must be contained in the sum of the others. The number of linearly independent highest weight vectors of weight \( q^r \) relative to \( K \) is exactly \( |P_k^r| \), the number of Motzkin paths of length \( k \) and rank \( r \). Since \( w_p \) is in \( \sum_{p' \neq p, \text{rank}(p')=r} U_q(\mathfrak{gl}_2) w_{p'} \), that would force it to be a linear combination of the \( w_{p'} \), \( p' \neq p \), which is impossible by Claim 3.

As a consequence of these claims we have the following result:

**Theorem 3.32.** \( V^\otimes k = \bigoplus_{p \in P_k} U_q(\mathfrak{gl}_2) w_p \) is a decomposition of \( V^\otimes k \) into irreducible \( U_q(\mathfrak{gl}_2) \)-modules.

**Corollary 3.33.** \( \bigoplus_{p \in P_k} U_q(\mathfrak{gl}_2) w_p \) is an irreducible \( (U_q(\mathfrak{gl}_2), M_k(\zeta_q)) \)-bimodule under the action, \( (a \times d). (b w_p) = ab w_{d,p} \) for \( a, b \in U_q(\mathfrak{gl}_2) \) and \( d \in M_k \), where \( d \cdot p \) is as in (4.6) below.

## 4 The Action on Motzkin Paths and Cellularity

In this section, we define a representation of the Motzkin algebra \( M_k(x) \) on Motzkin paths. This action is graded by the rank of the path, and so by taking quotients, we produce, in the case that \( x \) is chosen so that \( M_k(x) \) is a semisimple algebra, a complete set of irreducible modules for the Motzkin algebra. We show that the Motzkin algebra is cellular in the sense of Graham and Lehrer [GL] and that our Motzkin path modules are isomorphic to the (left) cell modules. We then compute the determinant of the Gram matrices of these cell modules inductively and determine for which values of the parameter \( x \) the algebra \( M_k(x) \) is semisimple. The parameter values where \( M_k(x) \) fails to be semisimple are roots of certain Chebyshev polynomials.

### 4.1 The action of \( M_k(x) \) on Motzkin paths

Let \( \mathbb{K} \) denote a commutative ring with 1, and let \( M_k(x) \) be the Motzkin algebra over \( \mathbb{K} \). Thus, \( M_k(x) \) is a free \( \mathbb{K} \)-module with basis the Motzkin \( k \)-diagrams in \( M_k \), and \( x \) is assumed to be an element of \( \mathbb{K} \).

Recall from Section 2.4 that \( P_k \) is the set of Motzkin paths, or 1-factors, of length \( k \) and that \( P_k^r \subseteq P_k \) is the subset of Motzkin paths of length \( k \) and rank \( r \). We define an action of a Motzkin diagram \( d \in M_k(x) \) on a Motzkin path \( p \in P_k \) as follows:

1. Place \( d \) above \( p \) and connect the vertices in the bottom row of \( d \) with the vertices in \( p \) to create a new Motzkin diagram \( dp \).

2. Color the vertices in the top row of \( dp \) so that vertices connected by an edge have the same color.

3. Let \( q \) be the 1-factor that is formed from the top row of \( dp \); the planarity of \( d \) ensures that \( q \) is a Motzkin 1-factor.
4. Let $\kappa(d,p)$ equal the number of closed loops formed in the bottom row of $dp$. Then set

$$dp = \kappa(d,p)q. \tag{4.1}$$

For example if $d$ and $p$ are given by

\[
\begin{align*}
    d &= \begin{array}{c}
            \includegraphics[width=0.5\textwidth]{d.png}
        \end{array}, \\
    p &= \begin{array}{c}
            \includegraphics[width=0.5\textwidth]{p.png}
        \end{array},
\end{align*}
\]

then

$$dp = \begin{array}{c}
            \includegraphics[width=0.5\textwidth]{dp.png}
        \end{array}.$$

There is one closed loop in the bottom row of $dp$, so we get $dp = xq$ where

$$q = \begin{array}{c}
            \includegraphics[width=0.5\textwidth]{q.png}
        \end{array}.$$

Let $W_k$ be the free $\mathbb{K}$-module with basis the Motzkin paths in $\mathcal{P}_k$. The action of diagrams on paths defined in (4.1) comes from the concatenation of diagrams, which is associative. Furthermore, vertex colors are simply pulled along the edges in the diagram, so we have $(d_1d_2)p = d_1(d_2p)$ for all Motzkin diagrams $d_1, d_2 \in M_k(x)$ and all paths $p \in \mathcal{P}_k$. This action extends linearly to an action of $M_k(x)$ on $W_k$ making $W_k$ a module for $M_k(x)$.

Recall from Section 2.4 the definition of the Motzkin diagram $d_p^r \in M_k(x)$ of rank $r$ associated with the pair $(p,q)$ of rank $r$ Motzkin paths. We have

$$d_p^r \cdot p = x^{\varepsilon(p)}q, \quad \text{where } \varepsilon(p) \text{ is the number of edges in } p. \tag{4.2}$$

In particular, $p$ is an eigenvector of $d_p^r$ with eigenvalue $x^{\varepsilon(p)}$. The action of a diagram $d \in M_k(x)$ on a path $p \in \mathcal{P}_k$ satisfies

$$\text{rank}(dp) \leq \min\left(\text{rank}(d), \text{rank}(p)\right), \tag{4.3}$$

since a white vertex in $p$ survives only if it passes along a vertical edge in $d$. This means that

$$W_k^{(r)} = \text{span}_\mathbb{K} \{ p \in \mathcal{P}_k \mid \text{rank}(p) \leq r \} \tag{4.4}$$

is a submodule of $W_k$, and $W_k^{(0)} \subseteq W_k^{(1)} \subseteq \cdots \subseteq W_k^{(r)} = W_k$ is a filtration of the $M_k(x)$-module $W_k$. Let

$$C_k^{(r)} = W_k^{(r)} / W_k^{(r-1)} \cong \text{span}_\mathbb{K} \{ p \mid p \in \mathcal{P}_k^r \}, \tag{4.5}$$

where $\mathcal{P}_k^r = \{ p \in \mathcal{P}_k \mid \text{rank}(p) = r \}$ as in (2.15). Under this isomorphism, if the Motzkin diagram $d \in M_k(x)$ acts on a path $p \in \mathcal{P}_k$ by $dp = x^{\kappa(d,p)}q$, then (abusing coset notation) we write the action in $C_k^{(r)}$ as follows:

$$d \cdot p = \begin{cases} x^{\kappa(d,p)}q, & \text{if } \text{rank}(q) = \text{rank}(p) \\ 0, & \text{if } \text{rank}(q) < \text{rank}(p). \end{cases} \tag{4.6}$$

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Let $C_0^0 = \mathbb{K}$ be the trivial module for $M_0(x) = \mathbb{K}1$.

The next theorem can also be obtained from general results on cellular algebras, once we establish, in Section 4.2, that the algebra $M_k(x)$ is cellular. Here we give an elementary proof of this result without evoking that machinery.

**Theorem 4.7.** Assume the Motzkin algebra $M_k(x)$ is defined over a field $\mathbb{K}$. For $k \geq 0$ and $0 \leq r \leq k$, the module $C_k^{(r)}$ is an indecomposable $M_k(x)$-module under the action given in (4.6). When $x$ is chosen so that $M_k(x)$ is a semisimple $\mathbb{K}$-algebra, then $\{ C_k^{(r)} \mid 0 \leq r \leq k \}$ is a complete set of pairwise nonisomorphic irreducible $M_k(x)$-modules.

**Proof.** First we prove that $C_k^{(r)}$ is indecomposable. Let $p$ be a Motzkin path in $P_k^r$ and let $e_p = x^{-\varepsilon(p)}d_p^r \in M_k(x)$. The image of $e_p$ on $C_k^{(r)}$ is $\mathbb{K}p$, so the kernel has codimension one. Thus, $e_p$ acts with eigenvalues 0 and 1 on $C_k^{(r)}$, and the eigenspace corresponding to the eigenvalue 1 is 1-dimensional with $p$ as a basis by (4.2).

Let $Z_k = \text{End}_{M_k(x)}(C_k^{(r)})$, and assume that $z \in Z_k$. Since $z$ and $e_p$ commute, $z$ maps the eigenspaces of $e_p$ into themselves. However, then $z(p) = \lambda p$ for some $\lambda \in \mathbb{K}$, since the eigenspace corresponding to the eigenvalue 1 is 1-dimensional. Now let $q$ be another path in $P_k^r$, and set $d = x^{-\varepsilon(q)}d_q^r$ so that $d \cdot p = q$. Then

$$z(q) = z(d \cdot p) = d \cdot z(p) = \lambda d \cdot p = \lambda q.$$ Consequently $z$ is a scalar multiple of the identity map. But $z$ is an arbitrary transformation in $Z_k$, which must then be 1-dimensional. This implies that $C_k^{(r)}$ is indecomposable.

Now when $x$ is chosen so that $M_k(x)$ is semisimple, then since modules for $M_k(x)$ are completely reducible, and since the module $C_k^{(r)}$ is indecomposable, $C_k^{(r)}$ must be irreducible in this case.

To prove that these modules are pairwise nonisomorphic, let

$$1_{\ell,k} = \begin{array}{c}
\ldots \\
\ell \\
k-k \\
\ldots
\end{array}, \quad 0 \leq \ell \leq k. \tag{4.8}$$

The diagram $1_{\ell,k}$ acts by 0 on all $C_k^{(r)}$ with $r > \ell$, since $1_{\ell,k}$ lowers the rank of any diagram of rank greater than $\ell$, and $1_{\ell,k}$ is nonzero on $C_k^{(\ell)}$. Thus, the diagrams $1_{\ell,k}, \ 0 \leq \ell \leq k,$ are sufficient to distinguish one $C_k^{(r)}$ from another.

To show that we have constructed all of the irreducible $M_k(x)$-modules, we sum the squares of the dimensions. Since $C_k^{(r)}$ has a basis consisting of the Motzkin paths of rank $r$, it has dimension $m_{k,r}$. But then by (2.16),

$$\sum_{r=0}^k \dim(C_k^{(r)})^2 = \sum_{r=0}^k m_{k,r}^2 = M_{2k},$$

which is the dimension of $M_k(x)$. Thus, by the general Wedderburn theory for semisimple algebras, we have found all of the irreducible $M_k(x)$-representations. $\square$

**Remark 4.9.** Let $T_k^r$ denote the set of all paths in $P_k^r$ which have no zeros, so $k - r$ is a nonnegative even integer. The definition of the action of diagrams on paths shows that the Temperley-Lieb diagrams (those Motzkin diagrams having no unconnected vertices) act on these paths. The same argument as in the proof of this theorem shows that the $\mathbb{K}$-span $T_k^{(r)}$ of $T_k^r$ is an indecomposable module for the Temperley-Lieb algebra $TL_k(x)$, and this module is irreducible when $TL_k(x)$ is semisimple. These modules give a complete set of nonisomorphic irreducible modules for the Temperley-Lieb algebra in the semisimple case.
As an immediate consequence of Theorem 4.7, we can explicitly derive the restriction rule for $M_{k-1}(x) \subseteq M_k(x)$, where this embedding is given by adding to a Motzkin $(k-1)$-diagram two vertices $k, k'$ on the right and connecting them with a vertical edge. For $p = (a_1, \ldots, a_k)$, set $p' = (a_1, \ldots, a_{k-1})$. Then

$$\text{rank}(p') = \begin{cases} 
\text{rank}(p) - 1 & \text{if } a_k = 1, \\
\text{rank}(p) & \text{if } a_k = 0, \\
\text{rank}(p) + 1 & \text{if } a_k = -1.
\end{cases}$$

Thus, the map $p \to p'$ realizes the decomposition,

$$\text{Res}_{M_{k-1}}^{M_k}(C^{(r)}_k) = C^{(r-1)}_{k-1} \oplus C^{(r)}_{k-1} \oplus C^{(r+1)}_{k-1},$$

(4.10)

where we define $C^{(\ell)}_{k-1} = 0$ if $\ell > k - 1$ or $\ell < 0$. This relation is to be expected from (3.12) and the Bratteli diagram (Figure 1), and it is a representation-theoretic interpretation of the identity

$$m_{k,r} = m_{k-1,r-1} + m_{k-1,r} + m_{k-1,r+1},$$

(4.11)

which appears in [DS].

### 4.2 Cellularity

In this section, we show that the Motzkin algebra $M_k(x)$ over any unital commutative ring $K$ is cellular in the sense of Graham and Lehrer [GL] and that the modules $C^{(r)}_k$, which have a basis the Motzkin paths in $P_k$, are its left cell modules.

First we verify that $M_k(x)$ has appropriate cell data (C1), (C2), (C3) as in [GL]:

(C1) The index set $\Lambda_k = \{0, 1, \ldots, k\}$ of indecomposable $M_k(x)$-modules is totally ordered under the usual ordering of integers. For $r \in \Lambda_k$ and $(p, q) \in P_k^r \times P_k^r$ the map $\prod_{r=0}^{k} P_k^r \times P_k^r \to M_k(x)$ which sends $(p, q) \to d^r_p$ (see (2.17)) is injective.

(C2) As observed in Section 2.3, the involution $\ast$ on $M_k(x)$, given by reflecting a diagram over its horizontal axis, satisfies $(d_1 d_2)^\ast = d_2^* d_1^*$ and $(d^r)^\ast = d$. In particular, $(d^r_p)^\ast = d^r_q$.

(C3) As in Section 2.2 we let $J_r = J_r(x) \subseteq M_k(x)$ be the $K$-span of the diagrams $d$ with $\text{rank}(d) \leq r$, satisfying

$$J_0 \subseteq J_1 \subseteq \cdots \subseteq J_k = M_k(x).$$

Let $p, q$ be Motzkin diagrams of rank $r$. Then for any diagram $d$, either $\text{rank}(dd^r_p) < \text{rank}(d^r_p)$, or $\text{rank}(dd^r_p) = \text{rank}(d^r_p)$ and $dd^r_p = x^\ell d^r_p$, for some nonnegative integer $\ell$. In this case, the bottom path $p$ is unchanged. It follows that for $a \in M_k(x)$,

$$ad^r_p \equiv \sum_{q' \in P_k^r} \mu_a(q', q)d^r_{p'} \mod J_{r-1}.$$  

(4.12)

where $\mu_a(q', q) \in K$ is independent of $p$.

Properties (C1), (C2), (C3) demonstrate that the Motzkin algebra $M_k(x)$ is cellular. Being cellular implies the existence of cell representations (Definition 2.1 in [GL]). For
each $r \in \Lambda_k$, the (left) cell representation $N^{(r)}_k$ is the free $K$-module with basis \{c_q \mid q \in \mathcal{P}^r_k\} such that for all $a \in \mathcal{M}_k(x)$ we have

$$ac_q = \sum_{q' \in \mathcal{P}^r_k} \mu_a(q', q)c_{q'},$$

where $\mu_a(q', q)$ is as in (4.12). The action of $a$ on $d^r_p \mod J_{r-1}$ is independent of the bottom 1-factor $p$, and when we replace the endpoints of the horizontal edges in $d^r_p$ with white vertices and remove the horizontal edges in the bottom row, then the action of $a$ on $d^r_p \mod J_{r-1}$ is precisely the same as the action $a$ on $q \mod \mathcal{W}^{(r-1)}_k$. Thus the action of $a$ on $q$ is the same as that of $a$ on $c_q$, so we have

$$N^{(r)}_k \cong C^{(r)}_k \text{ as } \mathcal{M}_k(x)-modules. \quad (4.13)$$

The cell representations $C^{(r)}_k$ come equipped with a $K$-bilinear form $\langle \cdot, \cdot \rangle : C^{(r)}_k \times C^{(r)}_k \to K$ defined for $p, q \in \mathcal{P}^r_k$ by the equation

$$d^p_pd^q_q \equiv \langle p, q \rangle d^p_pd^q_q \mod J_{r-1}. \quad (4.14)$$

Observe that

$$(d^p_pd^q_q)^* \equiv \langle p, q \rangle (d^p_pd^q_q)^* \mod J_{r-1},$$

which implies that $\langle p, q \rangle d^p_pd^q_q \equiv d^q_qd^p_p \equiv \langle q, p \rangle d^p_pd^q_q \mod J_{r-1}$, from which we may deduce that $\langle q, p \rangle = \langle p, q \rangle$ for all $p, q \in \mathcal{P}^r_k$. Moreover, since

$$\langle ap, q \rangle = \langle p, a^*q \rangle$$

for all $a \in \mathcal{M}_k(x), p, q \in \mathcal{P}^r_k$ (see [GL Prop. 2.4]), the radical $\text{rad}^{(r)}_k$ of the bilinear form is an $\mathcal{M}_k(x)$-submodule of $C^{(r)}_k$. As a consequence of [GL Prop. 3.2, Thm. 3.8], we know the following:

**Theorem 4.15.** Let $\mathcal{M}_k(x)$ be the Motzkin algebra over a field $K$.

(i) If $L^{(r)}_k := C^{(r)}_k / \text{rad}^{(r)}_k$ is nonzero, then it is an absolutely irreducible $\mathcal{M}_k(x)$-module; that is, it is irreducible over any extension field of $K$.

(ii) $\{L^{(r)}_k \neq (0) \mid r = 0, 1, \ldots, k\}$ is a complete set of representatives of isomorphism classes of absolutely irreducible $\mathcal{M}_k(x)$-modules.

(iii) The following are equivalent:

(a) the algebra $\mathcal{M}_k(x)$ is semisimple;

(b) for each $r = 0, 1, \ldots, k$, the cell module $C^{(r)}_k$ is absolutely irreducible;

(c) the bilinear form $\mathbb{K}$-bilinear form $\langle \cdot, \cdot \rangle : C^{(r)}_k \times C^{(r)}_k \to \mathbb{K}$ is nondegenerate for each $r = 0, 1, \ldots, k$.  

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4.3 Characters of $C_k^{(r)}$

Recall that a character $\chi$ of $M_k(x)$ on a module $W$ is the map $\chi : M_k(x) \to \mathbb{K}$ given by the trace on $W$. Thus $\chi$ is $\mathbb{K}$-linear, and for any two elements $a, b \in M_k(x)$, we have $\chi(ab) = \chi(ba)$. In this section, we compute the characters of the modules $C_k^{(r)}$.

**Proposition 4.16.** Any character $\chi$ of $M_k(x)$ is completely determined by its values on the diagrams $1_{\ell,k}$, $0 \leq \ell \leq k$, in $C_k$.

**Proof.** This follows by the Jones basic construction and [HR1] Lem. 2.8. However, we include the explicit calculation since it is useful to see the recursive algorithm for converting a general diagram to its representative of the form $1_{\ell,k}$. By the linearity of $\chi$, it is sufficient to compute the values of $\chi$ on the basis of diagrams. If the rank of a diagram $d$ is $k$, then $d = 1_{k,k}$. If the rank of $d$ is less than $k$, then there exist diagrams $d_1, d_2$ such that $d = d_1p_kd_2$; this can easily be seen by drawing diagrams. Thus $\chi(d) = \chi(d_1p_kd_2) = \chi(d_1p_kd_2) = \chi(p_kd_2d_1p_k) = x^{\kappa(d_1,d_2)}\chi(\delta(d_2d_1)p_k)$, where $\delta(d_2d_1)$ is a diagram in $M_{k-1}(x)$ and $\kappa(d_1, d_2)$ is the number of closed cycles produced in the product $d_1d_2$. Now, continue this argument on the diagram $\delta(d_2d_1) \in M_{k-1}(x)$ until we reach the identity diagram of rank $\ell$ in $M_{\ell}(x)$ for some $\ell$.

**Proposition 4.17.** If $\chi_k^{(r)}$ is the character of the module $C_k^{(r)}$, then

$$\chi_k^{(r)}(1_{\ell,k}) = \begin{cases} \dim(C_k^{(r)}) = m_{\ell,r}, & \text{if } r \leq \ell \\ 0, & \text{if } r > \ell. \end{cases}$$

**Proof.** We compute the trace of $d = 1_{\ell,k}$ on the basis of Motzkin paths $p \in P^r_k$. When $d = 1_{\ell,k}$ acts on the path $p$ (see [4,5]), we see that $d \cdot p = x^{\kappa(dp)}q$ modulo the span of paths of lower rank. To contribute to the trace, we must have $p = q$ (an entry on the main diagonal). If $dp$ has lower rank than $p$, then $d \cdot p = 0$, so we assume that $\text{rank}(dp) = \text{rank}(p)$.

By the structure of $d = 1_{\ell,k}$, the $1$-factor $p$ must have isolated vertices in the last $k - \ell$ positions, so for the product to be nonzero we need $\ell \geq r$. Letting $\tilde{p}$ be the diagram obtained by dropping the last $k - \ell$ isolated vertices of $p$, we see that $\tilde{p}$ is a $1$-factor in $C_k^{(r)}$ and that $d$ acts as identity on $C_k^{(r)}$, so its trace is the dimension $\dim(C_k^{(r)})$, which is $m_{\ell,r}$.

5 Gram matrices, Chebyshev Polynomials, and Semisimplicity

Throughout this section, we assume that $\mathbb{K}$ is a field, and we study the Gram matrix $G_k^{(r)}$ of the cell module $C_k^{(r)}$ with basis consisting of the Motzkin paths in $\mathcal{P}_k^r$. By explicitly changing the basis, we block diagonalize $G_k^{(r)}$ enabling us to give a precise formula for the determinant $\det(G_k^{(r)})$ in terms of certain Chebyshev polynomials $u_j$. We then show that the Motzkin algebra $M_k(x)$ is semisimple if and only if the parameter $x$ satisfies $u_j(x - 1) \neq 0$ for $1 \leq j \leq k - 1$. 

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5.1 Chebyshev polynomials

For \( k \geq 0 \), the Chebyshev polynomials of the second kind \( U_k(x) \) are defined by \( U_0(x) = 1, U_1(x) = 2x \), and the three-term recurrence relation

\[
U_k(x) = 2xU_{k-1}(x) - U_{k-2}(x), \quad \text{for} \ k \geq 2.
\]

Setting \( u_k(x) = U_k(x/2) \), we have the recurrence

\[
u_k(x) = xu_{k-1}(x) - u_{k-2}(x), \quad u_0(x) = 1, \ u_1(x) = x. \quad (5.1)
\]

The polynomials \( u_k(x-1) \) appear in the determinant of the Gram matrices in this section. The first few of them are

\[
\begin{align*}
u_0(x-1) &= 1, & \nu_3(x-1) &= (x-1)(x^2 - 2x - 1), \\
\nu_1(x-1) &= x - 1, & \nu_4(x-1) &= (x^2 - 3x + 1)(x^2 - x - 1), \\
\nu_2(x-1) &= x(x-2), & \nu_5(x-1) &= x(x-1)(x-2)(x^2 - 2x - 2).
\end{align*}
\]

The polynomial \( u_k(x-1) \) has degree \( k \) and its roots are given by (see [R], p. 229),

\[
\theta_m = e^{i\pi m/(k+1)} + e^{-i\pi m/(k+1)} + 1 = 2 \cos \left( \frac{\pi m}{k+1} \right) + 1, \quad m = 1, \ldots, k. \quad (5.2)
\]

5.2 Gram matrices

Let \( P_k^{r} \) denote the Motzkin paths (or, equivalently, 1-factors) of length \( k \) and rank \( r \). For \( p, q \in P_k^{r} \), let \( d^p_q \) be the unique Motzkin diagram whose top row is \( p \) and whose bottom row is \( q \). Let \( C_k^{(r)} \) be the cell module for \( M_k(x) \) with basis given by \( P_k^{r} \). Recall that for \( p, q \in P_k^{r} \), the bilinear form \( \langle \cdot, \cdot \rangle \) is defined on \( C_k^{(r)} \) by \( d^p_q \langle d^p_q \rangle \equiv \langle p, q \rangle d^p_q \mod J_{r-1} \). In particular, \( \langle p, q \rangle = 0 \) if \( \text{rank}(d^p_q) < r \) and, otherwise, \( \langle p, q \rangle = x^{\kappa(p,q)} \), where \( \kappa(p,q) \) is the number of loops removed from the middle row in the product \( d^p_q d^q_p \).

Let \( G_k^{(r)} \) be the Gram matrix of \( \langle \cdot, \cdot \rangle \) on \( C_k^{(r)} \) so that \( (G_k^{(r)})_{p,q} = \langle p, q \rangle \) for \( p, q \in P_k^{r} \). The module \( C_k^{(r)} \) has dimension 1 and is spanned by the 1-factor \( q \) consisting of all white vertices. The inner product is \( \langle q, q \rangle = 1 \). The module \( C_k^{(k-1)} \) is \( k \)-dimensional and is spanned by the 1-factors \( q_i \) having \( k-1 \) white vertices and 1 black vertex in position \( i \). It is easy to check by multiplying diagrams that \( \langle q_i, q_j \rangle = \delta_{i,j} \), the Kronecker delta. Thus,

\[
G_k^{(k)} = (1) \quad \text{and} \quad G_k^{(k-1)} = I_k, \quad \text{the} \ k \times k \text{ identity matrix}, \quad (5.3)
\]

and \( \det(G_k^{(k)}) = \det(G_k^{(k-1)}) = 1 \).

Order the Motzkin paths in \( P_k^{r} \) recursively as follows. Let \( p = (a_1, a_2, \ldots, a_k) \) and \( q = (b_1, b_2, \ldots, b_k) \) be two Motzkin paths of rank \( r \). Then \( p < q \) if and only if

\[
a_k > b_k \quad \text{or} \quad a_k = b_k \text{ and } (a_1, \ldots, a_{k-1}) < (b_1, \ldots, b_{k-1}). \quad (5.4)
\]

Let \( P_k^{r} = P_k^{r,1} \sqcup P_k^{r,0} \sqcup P_k^{r,-1} \), where \( P_k^{r,\ell} \) is the set of Motzkin paths (1-factors) of length \( k \) and rank \( r \) for which \( a_k = \ell \). If \( p \in P_k^{r,1}, q \in P_k^{r,0} \), then \( \langle p, q \rangle = 0 \), and so under this
ordering, the Gram matrix decomposes into the following symmetric block form,

\[
G_k^{(r)} = \begin{pmatrix}
G_{k-1}^{(r-1)} & 0 & AT \\
0 & G_{k-1}^{(r)} & B^T \\
A & B & H_{k-1}^{(r+1)}
\end{pmatrix},
\]  

(5.5)

where T denotes the transpose, and \(H_{k-1}^{(r+1)}\) is a matrix consisting of the inner products \(p, q\) where \(p, q\) are Motzkin paths ending in \(-1\). We will change the elements of \(P_{k}^{r-1}\) to get a basis for \(P_{k}^{r}\) which block diagonalizes the Gram matrix in (5.5).

### 5.3 Basis change

Let \(k \geq 2\) and \(0 \leq r \leq k - 2\). If \(p \in P_{k}^{r-1}\), then \(p\) has a horizontal edge connecting vertex \(k\) to some vertex \(i < k\). We refer to this horizontal edge as the pivot edge, and we denote it with a dashed line in the 1-factor \(p\) and in the diagram \(d_p^0\). If \(p \in P_{k}^{r-1}\), then we define \(p_+ \in P_{k}^{r,0}\) to be the 1-factor with the pivot edge deleted, we let \(p_+ \in P_{k}^{r,1}\) be the 1-factor such that the pivot edge is replaced with two white vertices, and we define \(p^{(1)} \in P_{k}^{r,1}\) to be the 1-factor created from \(p\) by removing the pivot edge and then adding a new horizontal edge connecting the \(i\)th vertex with the rightmost white vertex \(j < i\), thus making it black.

We refer to the added horizontal edge as the pivot edge of \(p^{(1)}\) and denote it with a dashed line. If there is no white vertex \(j < i\), then \(p^{(1)} = 0\). Here are four examples from \(P_6\):

\[
\begin{array}{c}
\begin{array}{c}
p = \circ \bullet \circ \circ \bullet \circ \\
p_+ = \bullet \circ \circ \circ \bullet \circ \\
p^{(1)} = \circ \bullet \circ \circ \bullet \circ \\
p^{(2)} = \bullet \circ \circ \circ \bullet \circ
\end{array}
\end{array}
\]

Now we define recursively

\[
[p] = p - p_+ - \frac{u_{r-1}(x-1)}{u_r(x-1)} [p^{(1)}]
\]

(5.6)

\[
[p^{(1)}] = p^{(1)} - (p^{(1)})_+ - \frac{u_{r-2}(x-1)}{u_{r-1}(x-1)} [p^{(2)}]
\]

(5.7)

where \(p^{(2)} = (p^{(1)})^{(1)}\). For example, applying these steps recursively to the Motzkin path \(p = \circ \bullet \bullet \circ \circ \bullet \circ \in P_5^2\), and omitting the argument \(x - 1\) to simplify the display, gives

\[
[p] = \circ \circ \bullet \circ \circ \bullet \circ - \frac{u_1}{u_2} (\circ \bullet \bullet \circ \circ \circ \circ) - \frac{u_0}{u_2} (\circ \bullet \circ \circ \circ \circ \circ \circ).
\]

The set \(\bar{P}_k := P_{k}^{r,1} \cup P_{k}^{r,0} \cup \{ [p] | p \in P_{k}^{r-1}\}\) is a basis for \(C_{k}^{r}\), since the change of basis matrix between it and \(P_{k}^r\) is unitriangular. The next lemma will help us compute the determinant.
respectively, the Gram matrix with respect to this new basis. Observe that the pivot edge of a diagram need not be connected to the \( k \)th vertex, as it moves to the left when the definition is applied recursively.

**Lemma 5.8.** Let \( p, q \in \mathbb{P}_k \) be such that \( p \) has a pivot edge and there are \( s \) vertical edges to the left of the pivot edge in \( d^p_{pq} \). Then \( \langle [p], q \rangle = \frac{u_{s+1}(x-1)}{u_s(x-1)} \langle p, q \rangle \), if the pivot edge is part of an inner loop in the product \( d^p_{pq}d^q_{pq} \). Otherwise, \( \langle [p], q \rangle = 0 \).

**Proof.** We prove this in four cases depending on the form of the path that contains the pivot edge in the product \( d^p_{pq}d^q_{pq} \).

**Case 1:** The pivot edge of \( p \) is part of a loop in the middle row of \( d^p_{pq}d^q_{pq} \).

First assume that there are \( s \geq 2 \) vertical edges to the left of the pivot edge. Then the products \( d^p_{pq}d^q_{pq} \) and \( d^p_{pq}d^q_{pq} \) take the following forms, respectively,

\[
d^p_{pq}d^q_{pq} = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\quad \text{and} \quad d^p_{pq}d^q_{pq} = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\]

so we have \( \langle [p], q \rangle = x \langle p, q \rangle \). The products \( d^p_{pq}d^q_{pq} \) and \( d^p_{pq}d^q_{pq} \) take the following forms, respectively,

\[
d^p_{pq}d^q_{pq} = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\quad \text{and} \quad d^p_{pq}d^q_{pq} = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\]

First, note that \( \langle [p^1], q \rangle = \langle p^1, q \rangle \). Second, observe that the rank goes down in the product \( d^p_{pq}d^q_{pq} \). In fact, since the pivot edges move to the left, this will be true for all \( d^p_{pq}d^q_{pq} \) for \( i \geq 2 \). Thus \( \langle [p^2], q \rangle = 0 \). Furthermore, \( \langle p^1, q \rangle = 0 \), since as can be seen in the product \( d^p_{pq}d^q_{pq} \), it is not possible for the rightmost edge of \( d^p_{pq} \) to propagate from bottom to top when the pivot edge of \( p^1 \) is removed. It follows that

\[
\langle [p], q \rangle = \langle p, q \rangle - \langle p, q \rangle - \frac{u_{s-1}}{u_s} \left[ \langle p^1, q \rangle - \langle p^1, q \rangle - \frac{u_{s-2}}{u_{s-1}} \langle p^2, q \rangle \right]
\]

\[
= (x-1) \langle p, q \rangle - \frac{u_{s-1}}{u_s} \langle p, q \rangle = \frac{u_{s+1}}{u_s} \langle p, q \rangle,
\]

using the recursion \((x-1)u_s - u_{s-1} = u_{s+1}\). Throughout this proof, we omit the argument \( x-1 \) from the polynomials \( u_s \) to simplify the display.

When \( s = 1 \), there is only one vertical edge to the left of the pivot, so \( p^2 = 0 \) and the proof follows as above, since we still have \( \langle p^2, q \rangle = 0 \). If \( s = 0 \), then \( p^1 = p^2 = 0 \), and the above calculation shows that \( \langle [p], q \rangle = (x-1) \langle p, q \rangle = \frac{u_{s+1}}{u_s} \langle p, q \rangle \), as desired.

**Case 2:** The pivot edge of \( p \) is part of a path that propagates from bottom to top in \( d^p_{pq}d^q_{pq} \).

In this case, we consider products of the form

\[
d^p_{pq}d^q_{pq} = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\quad \text{and} \quad d^p_{pq}d^q_{pq} = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\]
Now, \( \langle p_\bullet, q \rangle = 0 \), since the path no longer propagates when the pivot edge is removed. Furthermore, Case 1 tells us that \( \langle [p^{(1)}], q \rangle = \frac{u_{s-1}}{u_s} \langle [p^{(1)}_\bullet], q \rangle \), since \( d^{p^{(1)}_\bullet} q \) now has \( s-1 \) vertical edges to the left of the pivot edge. Also, by comparing the above products of diagrams (with the pivot edge removed in the second product), we see that \( \langle p, q \rangle = \langle p^{(1)}_\bullet, q \rangle \). Thus,

\[
\langle [p], q \rangle = \langle p, q \rangle - \langle p_\bullet, q \rangle - \frac{u_{s-1}}{u_s} \langle [p^{(1)}], q \rangle = \langle p, q \rangle - \frac{u_{s-1}}{u_s} \langle p^{(1)}_\bullet, q \rangle = \langle p, q \rangle - \langle p, q \rangle = 0.
\]

**Case 3:** In the product \( d^p p^q \), the pivot edge of \( p \) is not in a closed loop and is not in a path that contains a vertical edge of \( q \).

Consider products of the form shown below, where either of the dashed vertical edges may or may not be present,

\[
d^p p^q = \quad \text{and} \quad d^{p^{(1)}} (1) d^q = \]

Observe that \( \langle [p^{(1)}], q \rangle = 0 \), since the rank goes down in all products \( d^p p^{(1)} d^q \) for \( i \geq 1 \) (the pivot edge keeps moving left and so the rightmost vertical edge in \( d^p p^{(1)} \) never propagates). Furthermore, \( \langle p, q \rangle = \langle p_\bullet, q \rangle \), since the pivot edge does not form a loop, so \( \langle [p], q \rangle = \langle p, q \rangle - \langle p_\bullet, q \rangle - \frac{u_{s-1}}{u_s} \langle [p^{(1)}], q \rangle = 0 \).

**Case 4:** In the product \( d^p p^q \), the pivot edge of \( p \) is in a path that contains a vertical edge of \( d^q \) but does not contain a vertical edge of \( p \).

First we consider products of the form shown below, where when following the path starting from the vertical edge in \( d^q \), we first hit the left endpoint of the pivot edge,

\[
d^p p^q = \quad \text{and} \quad d^{p^{(1)}} (1) d^q = \]

We have \( \langle p^{(1)}, q \rangle = 0 \), since the rank will go down in all products as the pivot edge moves left. Furthermore, \( \langle p, q \rangle = \langle p_\bullet, q \rangle = 0 \), since no loop is removed, so \( \langle [p], q \rangle = \frac{u_{s-1}}{u_s} \langle [p^{(1)}], q \rangle = 0 \).

Second we consider products of the form shown below, where the path starting from the vertical edge in \( d^q \) first hits the right endpoint of the pivot edge. In this case, we will use induction on \( s \) (the number of vertical edges to the left of the pivot edge) to prove that \( \langle [p], q \rangle = 0 \). First, let \( s \geq 1 \),

\[
d^p p^q = \quad \text{and} \quad d^{p^{(1)}} (1) d^q = \]

Again, \( \langle p, q \rangle = \langle p_\bullet, q \rangle = 0 \), so \( \langle [p], q \rangle = \frac{u_{s-1}}{u_s} \langle [p^{(1)}], q \rangle \). If, in \( d^{p^{(1)}} (1) d^q \), the path containing the pivot edge does not reach the bottom of \( d^q \), then we are in Case 3, and \( \langle [p^{(1)}], q \rangle = 0 \).
If, in \( d_{p(1)}^p \cdot d_q^q \), the path containing the pivot edge reaches the bottom of \( d_q^q \), then since we have reduced the number of vertical edges that are to the left of the pivot edge, we apply induction to conclude that \( \langle p, q \rangle = \frac{u_{r+1}}{u_r} \langle p(1), q \rangle = 0 \). In the base case, when \( s = 0 \), we have \( p(1) = 0 \), so the result follows immediately.

**Proposition 5.9.** Let \( p, q \in \mathcal{P}_k^r \) with \( p \in \mathcal{P}_k^{r,-1} \). Then

\[
\begin{align*}
(i) & \quad \langle [p], q \rangle = 0, \text{ if } q \in \mathcal{P}_k^{r,0} \cup \mathcal{P}_k^{r,1}, \\
(ii) & \quad \langle [p], [q] \rangle = \frac{u_{r+1}(s-1)}{u_r(s-1)} \langle p+, q+ \rangle, \text{ if } q \in \mathcal{P}_k^{r,-1}.
\end{align*}
\]

**Proof.** (i) If \( p \in \mathcal{P}_k^{r,-1} \) and \( q \in \mathcal{P}_k^{r,0} \cup \mathcal{P}_k^{r,1} \), then the path that contains the pivot edge of \( p \) in \( d_{p(1)}^p \cdot d_q^q \) is not a loop, so by Lemma 5.8, we have \( \langle [p], q \rangle = 0 \).

(ii) If \( p, q \in \mathcal{P}_k^{r,-1} \) then \( q_\bullet \in \mathcal{P}_k^{r,0} \), so \( \langle [p], q_\bullet \rangle = 0 \), by part (a). Furthermore, each summand in \( q^{(1)} \) is in \( \mathcal{P}_k^{r,1} \), so \( \langle [p], [q^{(1)}] \rangle = 0 \), also by part (a). Thus,

\[
\langle [p], [q] \rangle = \langle [p], q \rangle - \langle [p], q_\bullet \rangle - \frac{u_{r+1}}{u_r} \langle [p], [q^{(1)}] \rangle = \langle [p], q \rangle.
\]

Let the pivot edge of \( p \) connect vertices \( i \) and \( k \), and let the rightmost horizontal edge of \( q \) connect vertex \( j \) and \( k \) (the right endpoint must be \( k \) in each case, since \( p, q \in \mathcal{P}_k^{r,-1} \)). Thus, the products \( d_{p(1)}^p \cdot d_q^q \) and \( d_{p+1}^p \cdot d_{q+1}^q \) look like,

\[
\begin{array}{c}
\begin{array}{c}
\cdots \\
i \quad j \quad k
\end{array}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\begin{array}{c}
\cdots \\
i \quad j \quad k
\end{array}
\end{array}
\]

If the pivot edge of \( p \) is contained in a loop in the middle row in the product \( d_{p(1)}^p \cdot d_q^q \), then there is a path in this middle row from vertex \( i \) to vertex \( j \). Furthermore, by Lemma 5.8, we have \( \langle [p], q \rangle = \frac{u_{r+1}}{u_r(s-1)} \langle p_\bullet, q_\bullet \rangle \). By comparing diagrams, we see \( \langle p_\bullet, q \rangle = \langle p+, q+ \rangle \), and so the result holds in this case.

If the pivot edge of \( p \) is not contained in a loop in the middle row in the product \( d_{p(1)}^p \cdot d_q^q \), then by Lemma 5.8, we have \( \langle [p], q \rangle = 0 \). There is no path from \( i \) to \( j \) in the middle row, and so it is not possible for the vertical edge at \( j \) in \( d_{q+1}^q \) to propagate through to the vertical edge at \( i \) in \( d_{p+1}^p \). Thus \( \langle p+, q+ \rangle = 0 = \langle [p], q \rangle \). \( \square \)

**Theorem 5.10.** With respect to the basis \( \mathcal{P}_k^r = \mathcal{P}_k^{r,1} \cup \mathcal{P}_k^{r,0} \} \{ [p] \mid p \in \mathcal{P}_k^{r,-1} \} \), the Gram matrix \( G_k^{(r)} \) takes the following block-diagonal form

\[
G_k^{(r)} = \begin{pmatrix}
G_{k-1}^{(r-1)} & 0 & 0 \\
0 & G_k^{(r)} & 0 \\
0 & 0 & \begin{pmatrix}
G_{k-1}^{(r-1)} \\
0 \\
0
\end{pmatrix}
\end{pmatrix},
\]

where \( G_j^{(i)} = \emptyset \) if \( j < 0 \) or \( i > j \).
Proof. For \( p \in \mathcal{P}_k \), let \( p' \) be the Motzkin path given by deleting the \( k \)th entry in \( p \). Then \( p \mapsto p' \) gives a bijection between \( \mathcal{P}^{r,1}_k \) and \( \mathcal{P}^{r-1,0}_{k-1} \) and a bijection between \( \mathcal{P}^{r,0}_k \) and \( \mathcal{P}^{r+1,1}_{k-1} \). If \( p \in \mathcal{P}^{r,1}_k \), then \( p' = (p_+)' \), so \( p \mapsto p_+ \) gives a bijection between \( \mathcal{P}^{r-1,0}_{k-1} \) and \( \mathcal{P}^{r+1,1}_{k-1} \). Since we are simply deleting the last entry, each bijection preserves the ordering of Motzkin paths defined in \((5.4)\). Now, if \( p, q \in \mathcal{P}^{r,1}_k \) or \( p, q \in \mathcal{P}^{r,0}_k \), then \( \langle p, q \rangle = \langle p', q' \rangle \). If \( p, q \in \mathcal{P}^{r-1,0}_{k-1} \), then by Proposition \( 5.9 \) (b), \( \langle p, q \rangle = \frac{u_{s+1}(x-1)}{u_{s+1}(x-1)} \langle p_+, q_+ \rangle = \frac{u_{s+1}(x-1)}{u_{s+1}(x-1)} \langle p', q' \rangle \). All other inner products are 0 by Proposition \( 5.9 \) (a) and the fact that if \( p \in \mathcal{P}^{r,1}_k \), \( q \in \mathcal{P}^{r,0}_k \), then \( \langle p, q \rangle = 0 \). \( \square \)

5.4 Determinant of the Gram matrix

Theorem 5.11. Recall from Section \( 2.4 \) that \( m_{k,r+2s} \) is the number of Motzkin paths of length \( k \) and rank \( r + 2s \). Then, for each \( k > 0 \) and \( 0 \leq r \leq k \), we have

\[
\det(G_k^{(r)}) = \prod_{s=1}^{k-r} \left( \frac{u_{s+r}(x-1)}{u_{s-1}(x-1)} \right)^{m_{k,r+2s}}. \tag{5.12}
\]

Proof. We use induction on \( k \), Theorem \( 5.10 \) and the fact that \( m_{k+1,r} = m_{k,r+1} + m_{k,r} + m_{k,r+1} \) as in \((4.11)\). The base case is when \( k = 1 \) and \( r = 0 \) or 1. We have \( G_1^{(1)} = G_1^{(0)} = (1) \), which trivially satisfy \((5.12)\).

From Theorem \( 5.10 \)

\[
\det(G_k^{(r)}) = \det(G_k^{(r-1)}) \det(G_k^{(r)}) \det(G_k^{(r+1)}) \left( \frac{u_{r+1}}{u_r} \right)^{m_{k,r+1}},
\]

and so by induction \( \det(G_k^{(r)}) \) equals

\[
\prod_{i=1}^{\lfloor k-(r-1) \rfloor} \left( \frac{u_{i+r-1}}{u_{i-1}} \right)^{m_{k,r-1+2i}} \prod_{j=1}^{\lfloor k-r \rfloor} \left( \frac{u_{j+r}}{u_{j-1}} \right)^{m_{k,r+2j}} \prod_{\ell=1}^{\lfloor k-(r+1) \rfloor} \left( \frac{u_{\ell+r+1}}{u_{\ell-1}} \right)^{m_{k,r+1+2\ell}} \left( \frac{u_{r+1}}{u_r} \right)^{m_{k,r+1}}.
\]

If \( s \neq r, r+1 \), then the exponent of \( u_{s-1} \) in the denominator of this product is

\[
m_{k,r-1+2s} + m_{k,r+2s} + m_{k,r+1+2s} = m_{k+1,r+2s},
\]

and the exponent of \( u_{s+r} \) in the numerator of this product is

\[
m_{k,r-1+2(s+1)} + m_{k,r+2(s+1)} + m_{k,r+1+2(s+1)} = m_{k,r+2s+1} + m_{k,r+2s} + m_{k,r+2s-1} = m_{k+1,r+2s}.
\]

The term \( u_s \) appears in the the numerator of the first product when \( i = 1 \) and its exponent is \( m_{k,r+1} \), which exactly cancels with the last term. The term \( u_{r+1} \) appears when \( i = 2 \), with exponent \( m_{k,r+3} \), and when \( j = 1 \) with exponent \( m_{k,r+2} \). It does not appear in the third term, but does appear in the fourth term with exponent \( m_{k,r+1} \). These exponents sum to \( m_{k,r+3} + m_{k,r+2} + m_{k,r+1} = m_{k+1,r+2} \). These are exactly the exponents expected in \((5.12)\) for \( \det(G_k^{(r)}) \). \( \square \)

Example 5.13. The following special cases are immediate consequences of \((5.12)\).

(i) \( \det(G_k^{(k)}) = \det(G_k^{(k-1)}) = 1 \),
(ii) \( \det(G_{k-2}^{(k-2)}) = u_{k-1}(x-1) \),

(iii) \( \det(G_{k}^{(k-3)}) = (u_{k-2}(x-1))^k \).

As a result of Theorem 5.11, we get the next theorem which gives the precise conditions for when \( M_k(x) \) is a semisimple algebra over a field \( \mathbb{K} \) in terms of the Chebyshev polynomials in Section 5.1.

**Theorem 5.14.** As an algebra over the field \( \mathbb{K} \), the Motzkin algebra \( M_k(x) \) is semisimple if and only if \( x \in \mathbb{K} \) satisfies \( u_j(x-1) \neq 0 \) for \( 1 \leq j \leq k-1 \).

**Proof.** By [GL, Thm. 3.8], \( M_k(x) \) is semisimple if and only if \( x \) avoids the zeros of \( \det(G_k^{(r-1)}) \), \( \det(G_k^{(r)}) \), and \( \frac{u_{r+1}(x-1)}{u_r(x-1)} \det(G_k^{(r+1)}) \). By induction, \( u_j(x-1) \neq 0 \) for \( 1 \leq j \leq k-2 \), and \( \frac{u_{r+1}(x-1)}{u_r(x-1)} u_j(x-1) \neq 0 \) for \( 1 \leq j \leq k-2 \). The maximum subscript appearing in these expressions is \( k-1 \), and it is attained when \( r = j = k-2 \). In that case, \( \frac{u_{k-1}(x-1)}{u_{k-2}(x-2)} u_{k-2}(x-2) = u_{k-1}(x-1). \)

**Remark 5.15.** One can verify that the Motzkin algebras \( M_0(x) \subseteq M_1(x) \subseteq \cdots \) satisfy the six axioms of [CMPX] to be a “tower of recollement.” (In fact, the properties of the Jones basic construction, verified in Section 2.5, constitute several of those axioms.) It then follows from the arguments used in [CMPX] Thm. 1.1 (ii), Cor. 5.1 that \( M_k(x) \) is semisimple if and only if \( x \in \mathbb{K} \) is chosen such that \( \det(G_j^{(i)}) \neq 0 \) for each \( 1 \leq j \leq k \) and each \( i \in \{ j-3, j-2, j-1, j \} \). Since by Example 5.13, \( \det(G_j^{(j-3)}) = (u_{j-2}(x-1))^j \), \( \det(G_j^{(j-2)}) = u_{j-1}(x-1) \), and \( \det(G_j^{(j-1)}) = \det(G_j^{(j)}) = 1 \), this approach generates exactly the same set of bad values for the parameter \( x \). However, the proof that \( \det(G_j^{(j-3)}) = (u_{j-2}(x-1))^j \) entails nearly the same amount of work as our general proof of Theorems 5.10 and 5.11 and the approach adopted here yields the general form of the determinant in Theorem 5.11 for all \( k \) and \( r \).

**Remark 5.16.** The change of basis in (5.6) should block-diagonalize the Gram matrix for the Temperley-Lieb algebra (compare [W]), except in the Temperley-Lieb case the diagram \( p_* \) does not exist and should be omitted.

**References**

[A] M. Aigner, Motzkin numbers, *European J. Combin.* 19 (1998), 663–675.

[CMPX] A. Cox, P. Martin, A. Parker, C. Xi, Representation theory of towers of recollement: Theory, notes, and examples, *J. Algebra* 302 (2006), 340–360.

[CPS] E. Cline, B. Parshall, L. Scott, Finite-dimensional algebras and highest weight categories, *J. Reine Angew. Math.* 391 (1988) 85–99.
[CR] C. Curtis and I. Reiner, *Methods of Representation Theory – With Applications to Finite Groups and Orders*, Pure and Applied Mathematics, vols. I and II, Wiley & Sons, Inc., New York, 1987.

[DS] R. Donaghey and L.W. Shapiro, Motzkin numbers, *J. Combin. Theory Ser. A* 23 (1977), 291-301.

[E] S.-P. Eu, Skew-standard tableaux with three rows, *Adv. in Appl. Math.* 45 (2010), no. 4, 463–469.

[FHH] D. Flath, T. Halverson, and K. Herbig, The planar rook algebra and Pascal’s triangle, *Enseign. Math.* (2) 54 (2008), 1-16.

[GHJ] F.M. Goodman, P. de la Harpe, and V.F.R. Jones, *Coxeter Graphs and Towers of Algebras*, Springer, New York, 1989.

[GL] J. Graham and G.I. Lehrer, Cellular algebras, *Invent. Math.* 123 (1996), 1–34.

[Ha] T. Halverson, Representations of the $q$-rook monoid, *J. Algebra* 273 (2004), 227–251.

[HR1] T. Halverson and A. Ram, Characters of algebras containing a Jones basic construction: the Temperley-Lieb, Okada, Brauer, and Birman-Wenzl algebras, *Adv. Math.* 116 (1995), 263–321.

[HR2] T. Halverson and A. Ram, $q$-rook monoid algebras, Hecke algebras, and Schur-Weyl duality, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov (POMI)* 283 (2001), 224–250.

[He] K. Herbig, *The Planar Rook Monoid*, Senior honor thesis, Macalester College 2006.

[Ja] J.C.J. Jantzen, *Lectures on Quantum Groups*, Grad. Studies in Math. 6 Amer. Math. Soc., Providence, RI, 1996.

[Jo1] V.F.R. Jones, Index for subfactors, *Invent. Math.* 72 (1983), 1–25.

[Jo2] V.F.R. Jones, A polynomial invariant for knots via von Neumann algebras, *Bull. Amer. Math. Soc.* 12 (1985), 103–111.

[Jo3] V.F.R. Jones, *Subfactors and Knots*, CBMS Regional Conference Series in Mathematics, 80. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1991.

[M] T. Motzkin, Relations between hypersurface cross ratios, and a combinatorial formul for partitions of a polygon, for permanent preponderance, and for non-associative products, *Bull. Amer. Math. Soc.* 54 (1948), 352–360.

[Mu1] W.D. Munn, Matrix representations of semigroups, *Proc. Cambridge Philos. Soc.* 53 (1957), 13–18.

[Mu2] W.D. Munn, The characters of the symmetric inverse semigroups, *Trans. Amer. Math. Soc.* 323 (1991), 563–581.
[R] T.J. Rivlin, Chebyshev polynomials, 2nd ed., Wiley-Interscience Pure and Applied Math., New York, 1990.

[STT] A. Sapounakis, I. Tasoulas, P. Tsikouras, Counting strings in Dyck paths, Discrete Math. 307 (2007), 2902–2924.

[Sl] N.J.A. Sloane, On-line Encyclopedia of Integer Sequences.

[So1] L. Solomon, The Bruhat decomposition, Tits system and Iwahori ring for the monoid of matrices over a finite field, Geom. Dedicata 36 (1990), 15–49.

[So2] L. Solomon, Representations of the rook monoid, J. Algebra 256 (2002), 309–342.

[So3] L. Solomon, The Iwahori algebra of $M_n(F_q)$. A presentation and a representation on tensor space, J. Algebra 273 (2004), 206–226.

[St] R. Stanley, Enumerative Combinatorics, Vol. 2, Cambridge University Press, Cambridge, 1999.

[TL] H.N.V. Temperley and E.H. Lieb, Relations between the “percolation” and the “colouring” problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the “percolation” problem, Proc. Roy. Soc. London Ser. A 322 (1971), 251–280.

[W] B.W. Westbury, The representation theory of the Temperley-Lieb algebras, Math. Z. 219 (1995), 539–565.