On the stability of ground states in 4D and 5D nonlinear Schrödinger equation including subcritical cases

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Abstract

We consider a class of nonlinear Schrödinger equation in four and five space dimensions with an attractive potential. The nonlinearity is local but rather general encompassing for the first time both subcritical and supercritical (in $L^2$) nonlinearities. We show that the center manifold formed by localized in space periodic in time solutions (bound states) is an attractor for all solutions with a small initial data. The proof hinges on dispersive estimates that we obtain for the time dependent, Hamiltonian, linearized dynamics around a one parameter family of bound states that “shadows” the nonlinear evolution of the system. The methods we employ are an extension to higher dimensions, hence different linear dispersive behavior, and to rougher nonlinearities of our previous results [10, 11, 7].

1 Introduction

In this paper we study the long time behavior of solutions of the nonlinear Schrödinger equation (NLS)

\[ i\partial_t u(t,x) = [-\Delta_x + V(x)]u + g(u), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^N, \quad N = 4, 5 \]

where the local nonlinearity is constructed from the real valued, odd, $C^1$ function $g : \mathbb{R} \mapsto \mathbb{R}$ which is twice differentiable except maybe at zero and satisfies:

\[ g(0) = 0, \quad g'(0) = 0, \quad |g''(s)| \leq C(|s|^{\alpha_1-1} + |s|^{\alpha_2-1}), \quad \text{for } s \neq 0, \]

with

\[ \alpha_0(N) = \frac{2 - N + \sqrt{N^2 + 12N + 4}}{2N} < \alpha_1 < \frac{4}{N-2} \quad \text{for } N=4, 5. \]

$g$ is then extended to a complex function via the gauge symmetry:

\[ g(e^{i\theta}z) = e^{i\theta}g(z), \quad g(\bar{z}) = g(z). \]

Note that $g$ is not necessarily twice differentiable at 0, e.g. $g(z) = |z|^\frac{5}{2}z$.

We are going to show that the manifold of periodic solutions of (1.1)-(1.2) (center manifold) is a global attractor for all small initial data. More precisely, for $u_0 \in H^1 \cap L^{2\alpha_2+1}$ with sufficiently small norm the solution of (1.1)-(1.2) can be decomposed as follows:

\[ u(t, x) = e^{i\theta(t)} \psi_{E(t)}(x) + r(t, x) \]

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where for each fixed time $t_1 \in \mathbb{R}$, $E = E(t_1)$, we have that $u_E(t, x) = e^{-iE_t \psi_E(x)}$ is a periodic solution of (1.1) and, as $|t| \to \infty$, $r(t) \in H^1(\mathbb{R}^N)$ converges strongly to zero in $L^p(\mathbb{R}^N)$, $2 < p < 2N/(N - 2)$, spaces and weakly in $H^1(\mathbb{R}^N)$, see Section 3 for more details. We can also show that the full dynamics converges to a certain periodic solution, i.e. $\psi_E(t) \to \psi_{E_\infty}$, for $t \to \pm \infty$, provided we restrict the range of nonlinearities to the supercritical regime $\alpha > 4/N$, see Corollary 3.2. In this case we generalize the results in [19, 15].

Moreover, for the remaining range $\alpha_0(N) < \alpha_1 \leq 4/N$, we could show a type of asymptotic stability for periodic solutions (bound states) of (1.1). In other words, if $e^{-iE_t \psi_E(x)} \neq 0$ is a smooth periodic solution of (1.1) then there exists $\varepsilon > 0$ depending on $\psi_E$ such that for all initial data $u_0 \in H^1 \cap L^{p+4+2}$ satisfying

$$\inf_{0 \leq \theta < 2\pi} \| u_0 - e^{i\theta} \psi_E \|_{H^1 \cap L^{p+4+2}} < \varepsilon(\psi_E)$$

the solution of (1.1) converges to a periodic solution (close to $e^{-iE_t \psi_E(x)}$) strongly in $L^p$, $2 < p < 2N/(N - 2)$, spaces and weakly in $H^1$, as $t \to \pm \infty$. The proof of this result is left for another paper [6] because it involves a different decomposition of the dynamics and a more delicate way to obtain the linear estimates similar to the ones in Section 2. It has the advantage that it can be generalized to large periodic solutions and the disadvantage that it only describes the evolution of initial data in a conic like neighborhood (since $\varepsilon$ depends on $\psi_E$) of the set of periodic solutions (center manifold) with the zero solution removed. In fact the choice of $\varepsilon$ is such that the solution stays away from zero, the point where the center manifold fails to be $C^2$ smooth. Since the dynamics only sees the $C^2$ smooth part of the center manifold a better decomposition of the dynamics can be employed, see [7, 11], and convergence to a periodic solution follows.

The main contribution of our paper is to describe the long time evolution of all small initial data for rather general nonlinearities including for the first time the subcritical ones $\alpha_1 > 4/N$, see Section 3. We accomplish this by using a time-dependent projection of the solution of (1.1) onto the center manifold of periodic solutions described in Section 2 and (5.3). We first prove dispersive estimates for the propagator of the linearized equation at the time-dependent projection, see Section 4. We then estimate the error between the actual solution and its projection onto the center manifold via a Duhamel principle with respect to the linearized operator at the projection and a fixed point argument for the resulting integral equation, see (3.3). Since the operator on its right hand side contains no linear terms in the error, we are able to show that it is contractive in appropriately chosen Banach spaces for a large spectrum of nonlinearities $g$. This is in contrast with the approach in [15, 19] where linear and nonlinear terms had to be estimated at the same time, the linear ones requiring the use of $L^2$ weighted (localized) estimates which applied to the non-localized, nonlinear terms forced the assumption $\alpha_1 > 4/N$. In the current approach, as in our previous 3D and 2D results, see [7, 10, 11], we completely separate estimates for nonlinear terms from the ones for linear terms and use methods tailored for each of them.

The most difficult part is to obtain dispersive estimates for the propagator of the time-dependent linearized operator at the projections onto the center manifold, see Section 4. While estimates for the Schrödinger group of operators:

$$\| e^{-i(-\Delta + V(x))t} P_c \|_{L^{p'} \to L^p} \leq C_p |t|^{-N\left(\frac{1}{2p'} - \frac{1}{p'}\right)}, \quad 1/p' + 1/p = 1, \quad 2 \leq p \leq \infty$$

are well known, see [4] and references therein, they are almost non-existent when the potential $V$ depends on time $V = V(t, x)$. This is to be expected since the time-dependence of $V$ is the quantum mechanical analog of the parametric forcing in ordinary differential equations and, in principle, can lead to very different behavior compared to the time-independent case, see [21, 8, 12, 2, 3]. However, in the absence of resonant phenomena one might expect similarities between the two dynamics. Indeed, this is the case in [16] which cannot be generalized to our situation mostly because of the complex-valued potential, see (3.7) and (2.4). To overcome this issues we use smallness and localization of the time dependent terms (4.6)-(4.7) to first obtain dispersive estimates in weighted (localized) norms, see
Theorem 4.1. Then in Theorem 4.2 we rely on the integrability in time of the group of operators generated by the nearby time independent operator \(-i(-\Delta + V(x))\), see (1.9) with \(p > 2N/(N - 2)\) and \(t \geq 1\), to remove the weights and obtain dispersive estimates in non-localized \(L^p\) norms. But the integrability in time for \(t \geq 1\) comes at the cost of a non-integrable singularity at \(t = 0\) which we remove by using cancelations in highly oscillatory integrals. The method is similar to the one we employed in \([7]\), see also \([10, 11]\) for an alternate way of dealing with the singularity.

In a nutshell the results in this paper rely on shadowing the actual solution of (1.1)-(1.2) via a curve on the central manifold of periodic solutions for (1.1). Essential in showing that the distance between the solution and its shadow goes to zero are the new, apriori, dispersive estimates for the propagator of the linearized equation along the shadowing curve. In this regard the paper is an extension to higher dimensions, hence different linear dispersive behavior, and to rougher nonlinearities of our previous results in two respectively three space dimensions \([10, 11, 7]\).

Notations: \(H = -\Delta + V\);
\[L^p = \{f : \mathbb{R}^N \to \mathbb{C} | f \text{ measurable and } \int_{\mathbb{R}^N} |f(x)|^p dx < \infty\}, \|f\|_p = (\int_{\mathbb{R}^N} |f(x)|^p dx)^{1/p}\] denotes the standard norm in these spaces;
\(<x> = (1 + |x|^2)^{1/2}\), and for \(\sigma \in \mathbb{R}, L^2_\sigma\) denotes the \(L^2\) space with weight \(<x>^{\sigma}\), i.e. the space of functions \(f(x)\) such that \(<x>^\sigma f(x)\) are square integrable endowed with the norm \(\|f(x)\|_{L^2_\sigma} = \|<x>^\sigma f(x)\|_2\);
\[\langle f, g \rangle = \int_{\mathbb{R}^N} \overline{f}(x)g(x)dx\] is the scalar product in \(L^2\) where \(\overline{z}\) = the complex conjugate of the complex number \(z\);
\(P_c\) is the projection on the continuous spectrum of \(H\) in \(L^2\);
\(u\) denotes the Fourier transform of the temperate distribution \(u\);
\(H^s, s \in \mathbb{R}\) denote the Sobolev spaces of temperate distributions \(u\) such that \((1 + |\xi|^2)^{s/2} \hat{u}(\xi) \in L^2(\mathbb{R}^N)\) with norm \(\|u\|_{H^s} = \|(1 + |\xi|^2)^{s/2} \hat{u}(\xi)\|_{L^2}\). Note that for \(s = n, n\) a natural number, this spaces coincide with the space of measurable functions having all distributional partial derivatives up to order \(n\) in \(L^2\).

2 The Center Manifold

The center manifold is formed by the collection of periodic solutions for (1.1):
\[u_E(t, x) = e^{-iEt}\psi_E(x)\] (2.1)
where \(E \in \mathbb{R}\) and \(0 \neq \psi_E \in H^2(\mathbb{R}^N)\) satisfy the time independent equation:
\[-\Delta + V][\psi_E + g(\psi_E)] = E\psi_E\] (2.2)
Clearly the function constantly equal to zero is a solution of (2.2) but (iii) in the following hypotheses on the potential \(V\) allows for a bifurcation with a nontrivial, one parameter family of solutions:

(H1) Assume that
\(V(x)\) satisfies the following properties:
1. \(<x>^\rho V(x) : H^\eta \to H^\eta, \text{ for some } \rho > N + 4 \text{ and } \eta > 0;\)
2. \(\nabla V \in L^p(\mathbb{R}^N) \text{ for some } 2 \leq p \leq \infty \text{ and } |\nabla V(x)| \to 0 \text{ as } |x| \to \infty;\)
3. the Fourier transform of \(V\) is in \(L^1\).

(ii) \(0\) is a regular point\(^1\) of the spectrum of the linear operator \(H = -\Delta + V\) acting on \(L^2\).

\(^1\)see [18] Definition 7] or \(M_\nu = \{0\}\) in relation (3.1) in [13]
(iii) \(H\) acting on \(L^2\) has exactly one negative eigenvalue \(E_0 < 0\) with corresponding normalized eigenvector \(\psi_0\). It is well known that \(\psi_0(x)\) is exponentially decaying as \(|x| \to \infty\), and can be chosen strictly positive.

Conditions (i) and (ii) guarantee the applicability of dispersive estimates in [13] and [4] to the Schrödinger group \(e^{-iHt}P_c\). Condition (i)2. implies certain regularity of the nonlinear bound states while (i)3. allows us to use commutator type inequalities, see [14] and [11 Theorem 5.2]. All these are needed to obtain estimates for the semigroup of operators generated by our time dependent linearization, see Theorems 4.1 and 4.2 in section 4. In particular (i)1. implies the local well posedness in \(H^1\) of the initial value problem (1.1) - (1.2), see section 3.

Assume that for the positive solution of (2.2) we have initial value problem (1.1) - (1.2), see section 3.

By variational methods, see for example [17], one can show that the real valued solutions do not change sign. Then Harnack inequality for \(H^1(C(\mathbb{R}^N))\) solutions of (2.2) implies that these real solution cannot take the zero value. Hence \(\psi_E\) given by (2.3) for \(a \in \mathbb{R}\) is either strictly positive or strictly negative.

In section 4 we also need some smoothness for the effective (linear) potential induced by the nonlinearity which for the real valued bound states is:

\[
Dg|_{\psi_E}[u + iv] = g'(\psi_E)u + ig(\psi_E)v, \quad \psi_E > 0
\]

while for an arbitrary bound state \(\tilde{\psi}_E = e^{i\theta}\psi_E\), \(\psi_E > 0\) we have via the rotational symmetry of \(g\), see (1.3),

\[
Dg|_{\tilde{\psi}_E}[\beta] = e^{i\theta}Dg|_{\psi_E}[e^{-i\theta}\beta],
\]

(H2) Assume that for the positive solution of (2.2) we have \(\frac{g(\psi_E)}{\psi_E} \in L^1(\mathbb{R}^N)\) where \(\hat{f}\) stands for the Fourier transform of the function \(f\).

In concrete cases the hypothesis may be checked directly using the regularity of \(\psi_E\), the solution of an uniform elliptic e-value problem. In general we can prove the following result:
Proposition 2.2 If the following holds:

(H2') $g$ restricted to reals has four derivatives except at zero and

$$|g^{(m)}(s)| < C \left( \frac{1}{s^{m-1-\alpha_1}} + \frac{1}{s^{m-1-\alpha_2}} \right),$$
for $m = 3, 4$, and $s > 0$.

Then for the positive solution of (2.2), $\psi_E$, we have $g'(\psi_E) \in L^1$ and $\frac{g(\psi_E)}{\psi_E} \in L^1$.

Proof:

$$\|g'(\psi_E)\|_{L^1} = \left\| \frac{1}{(1 + |\xi|^2)^\frac{3}{2}} (1 + |\xi|^2) \frac{\partial g'(\psi_E)}{\partial \psi_E} \right\|_{L^1} \leq \left( \frac{1}{(1 + |\xi|^2)^\frac{3}{2}} \right) \|g'(\psi_E)\|_{L^2} \quad \text{for} \quad \xi \in \mathbb{R}^N \quad \text{and} \quad g'(\psi_E) \in H^3$$

So it suffices to show that $g'(\psi_E) \in H^3$ and $\frac{g(\psi_E)}{\psi_E} \in H^3$. We have:

$$\nabla g'(\psi_E) = g''(\psi_E) \nabla \psi_E$$

$$\Delta g'(\psi_E) = g'''(\psi_E) \Delta \psi_E + g''(\psi_E) \nabla \psi_E$$

$$\nabla^3 g'(\psi_E) = g^{(4)}(\psi_E) \nabla \psi_E^2 \nabla \psi_E + 3g'''(\psi_E) \nabla \psi_E \Delta \psi_E + g''(\psi_E) \nabla^2 \psi_E$$

and, using (1.3),

$$\|g'(\psi_E)\| \leq C(\psi_E^{\alpha_1} + |\psi_E|^{\alpha_2})$$

$$\|\nabla g'(\psi_E)\| \leq C \|\psi_E\| \left( \frac{1}{|\psi_E|^{1-\alpha_1}} + \frac{1}{|\psi_E|^{1-\alpha_2}} \right)$$

$$\|\Delta g'(\psi_E)\| \leq C \|\psi_E\| \left( \frac{1}{|\psi_E|^{2-\alpha_1}} + \frac{1}{|\psi_E|^{2-\alpha_2}} \right) + C(\psi_E^{\alpha_1} + |\psi_E|^{\alpha_2}) (|V| + |E|) + C(\psi_E^{2\alpha_1} + |\psi_E|^{2\alpha_2})$$

$$\|\nabla^3 g'(\psi_E)\| \leq C \|\nabla \psi_E\| \left( \frac{1}{|\psi_E|^{3-\alpha_1}} + \frac{1}{|\psi_E|^{3-\alpha_2}} \right) + 2C \|\nabla \psi_E\| (|V| + |E|) \left( \frac{1}{|\psi_E|^{1-\alpha_1}} + \frac{1}{|\psi_E|^{1-\alpha_2}} \right)$$

Similarly we get the estimates for $\frac{g(\psi_E)}{\psi_E}$ as follows:

$$\left| \frac{g(\psi_E)}{\psi_E} \right| \leq C(\psi_E^{\alpha_1} + |\psi_E|^{\alpha_2})$$

$$\left| \nabla \frac{g(\psi_E)}{\psi_E} \right| \leq C \|\psi_E\| \left( \frac{1}{|\psi_E|^{1-\alpha_1}} + \frac{1}{|\psi_E|^{1-\alpha_2}} \right)$$

$$\left| \Delta \frac{g(\psi_E)}{\psi_E} \right| \leq C \|\psi_E\| \left( \frac{1}{|\psi_E|^{2-\alpha_1}} + \frac{1}{|\psi_E|^{2-\alpha_2}} \right) + C(\psi_E^{\alpha_1} + |\psi_E|^{\alpha_2}) (|V| + |E|) + C(\psi_E^{2\alpha_1} + |\psi_E|^{2\alpha_2})$$

$$\left| \nabla^3 \frac{g(\psi_E)}{\psi_E} \right| \leq C \|\nabla \psi_E\| \left( \frac{1}{|\psi_E|^{3-\alpha_1}} + \frac{1}{|\psi_E|^{3-\alpha_2}} \right) + 2C \|\nabla \psi_E\| (|V| + |E|) \left( \frac{1}{|\psi_E|^{1-\alpha_1}} + \frac{1}{|\psi_E|^{1-\alpha_2}} \right)$$

Now, we will use the following bounds for $\psi_E$ and $\nabla \psi_E$, see [7, Section 5.2]. For any $0 < A < -E < A_2$ and any $0 < A_1 < -E$ there exist the constants $C_A$, $C_{A_1}$, $C_{A_2}$ > 0 such that:

$$C_{A_2} e^{-\sqrt{A_2}|x|} \leq \psi_E(x) \leq C_A e^{-\sqrt{A}|x|}, \quad \text{for} \quad x \in \mathbb{R}^N,$$
Choosing $A_1$ and $A_2$ such that $(3 - \alpha_1)^2 < 3\sqrt{A_2}$, we obtain $g'(\psi_E) \in H^3$ and $\frac{g(\psi_E)}{\psi_E} \in H^3$. The proposition is now completely proven. □

3 Main Results

**Theorem 3.1** Assume that the nonlinear term in (1.1) satisfies (1.3), (1.4) and (1.5). In addition assume that hypotheses (H1) and either (H2) or (H2') hold. Then there exists an $\varepsilon_0$ such that for all initial conditions $u_0(x)$ satisfying

$$\max\{\|u_0\|_{L^p}, \|u_0\|_{H^1}\} \leq \varepsilon_0, \quad p_2 = 2 + \alpha_2, \quad \frac{1}{p_2} + \frac{1}{p_2} = 1$$

the initial value problem (1.1)–(1.2) is globally well-posed in $H^1$ and the solution converges to the center manifold.

More precisely, there exist a $C^1$ function $a : \mathbb{R} \to \mathbb{C}$ such that, for all $t \in \mathbb{R}$ we have:

$$u(t, x) = a(t)\psi_0(x) + h(a(t)) + r(t, x)$$

where $\psi_{E(t)}$ is on the central manifold (i.e it is a ground state), see Proposition 2.1. Moreover, for all $t \in \mathbb{R}$ $r(t, x)$ satisfies the following decay estimates:

$$\|r(t)\|_{L^2} \leq C_0(\alpha_1, \alpha_2)\varepsilon_0$$

$$\|r(t)\|_{L^p_1} \leq C_1(\alpha_1, \alpha_2)\frac{\varepsilon_0}{(1 + |t|)^{N\left(\frac{4}{p_1} - \frac{1}{p_2}\right)}}$$

and, for $p_2 = 2 + \alpha_2$:

(i) if $\alpha_1 \geq \frac{4}{N}$ or $p_2 < \frac{2N}{2 + N - N\alpha_1}$ then $\|r(t)\|_{L^p_2} \leq C_2(\alpha_1, \alpha_2)\frac{\varepsilon_0}{(1 + |t|)^{N\left(\frac{4}{p_2} - \frac{1}{p_2}\right)}}$

(ii) if $\alpha_1 < \frac{4}{N}$ and $p_2 = \frac{2N}{2 + N - N\alpha_1}$ then $\|r(t)\|_{L^p_2} \leq C_2(\alpha_1, \alpha_2)\varepsilon_0\frac{\log(2 + |t|)}{(1 + |t|)^{N\left(\frac{4}{p_2} - \frac{1}{p_2}\right)}}$

(iii) if $\alpha_1 < \frac{4}{N}$ and $p_2 > \frac{2N}{2 + N - N\alpha_1}$ then $\|r(t)\|_{L^p_2} \leq C_2(\alpha_1, \alpha_2)\frac{\varepsilon_0}{(1 + |t|)^{N\left(\frac{4}{p_2} - \frac{1}{p_2}\right)}}$

where the constants $C_0$, $C_1$ and $C_2$ are independent of $\varepsilon_0$.

Before proving the theorem let us note that (3.1) decomposes the evolution of the solution of (1.1)–(1.2) into an evolution on the center manifold $\psi_{E(t)}$ and the “distance” from the center manifold $r(t)$. The estimates on the latter show collapse of solution onto the center manifold. A more precise decay of the ‘radiative’ part, $r(t)$, in different $L^p$ spaces is given in the following Corollary. It shows same decay as the solution of the free Schrödinger equation up to the threshold $p = \frac{2N}{2 + N - N\alpha_1}$ where it saturates:

**Corollary 3.1** Consider $2 \leq p < \frac{2N}{N-2}$, and $1/p' + 1/p = 1$. Under the hypotheses of Theorem 3.1 assuming also $u_0 \in L^p$, we have the following decay estimates:

if $\alpha_1 \geq 4/N$ then

$$\|r(t)\|_{L^p} \leq \frac{C(p)\max\{\|u_0\|_{L^{p'}}, \varepsilon_0\}}{(1 + |t|)^{N\left(\frac{4}{p} - \frac{1}{2}\right)}}$$

for all $2 \leq p < \frac{2N}{N-2}$. 

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otherwise
\[ \|r(t)\|_{L^p} \leq \frac{C(p) \max \{ \|u_0\|_{L^p}, \varepsilon_0 \}}{(1 + |t|)^{N(\frac{j}{2} - \frac{1}{p})}}, \quad \text{if } p_1 < p < \frac{2N}{2 + N - N \alpha_1} \]
\[ \|r(t)\|_{L^p} \leq \frac{C(p) \log(2 + |t|) \max \{ \|u_0\|_{L^p}, \varepsilon_0 \}}{(1 + |t|)^{N(\frac{j}{2} - \frac{1}{p})}}, \quad \text{if } p = \frac{2N}{2 + N - N \alpha_1} \]
\[ \|r(t)\|_{L^p} \leq \frac{C(p) \max \{ \|u_0\|_{L^p}, \varepsilon_0 \}}{(1 + |t|)^{\frac{N \alpha_1}{2} - 1}}, \quad \text{if } \frac{2N}{2 + N - N \alpha_1} < p < \frac{2N}{2 - 2N}. \]

The dynamics on the center manifold is determined by equation (3.4) below. In supercritical regimes \( \alpha_1 > 4/N \) we can actually show that it converges to a certain periodic orbit:

**Corollary 3.2** Under the hypotheses of Theorem 3.1 assuming also \( \alpha_2 \geq \alpha_1 > 4/N \) and \( u_0 \in L^p(\mathbb{R}^N) \) for some \( 1 \leq p' < 2N/(N + 2) \), we have in addition to the conclusion of Theorem 3.1 that there exists \( a_{\pm\infty} \in \mathbb{R} \) such that \( \lim_{t \to \pm\infty} |a(t)| = a_{\pm\infty} \). Moreover, if we denote by \( e^{itE_{\pm\infty}} \psi_{E_{\pm\infty}}(x) \) the periodic solutions of (1.1) corresponding to the parameters \( a_{\pm\infty} \) on the center manifold:

\[ \psi_{E_{\pm\infty}} = a_{\pm\infty} \psi_0 + h(a_{\pm\infty}), \quad E_{\pm\infty} = E(a_{\pm\infty}), \]

see Proposition 2.7, then there exists a \( C^1 \) function \( \theta : \mathbb{R} \to \mathbb{R} \) such that:

\[ \lim_{\imath \to \pm\infty} \| \psi_{E(t)} - e^{-itE_{\pm\infty}} \psi_{E_{\pm\infty}} \|_{H^2 \cap L^2_\sigma} = 0, \quad \lim_{|t| \to \infty} \theta(t) = 0, \]

where \( \psi_{E(t)} \) is the component on the central manifold of the actual solution of (1.1)-(2.2), see (3.1).

The corollary extends the results in [19, 15] to nonlinearities in more general form than pure power and to initial data that are not necessarily localized, i.e. in \( L^p(\mathbb{R}^N) \), \( \sigma > N \). As we shall see in Remark 3.2 the supercriticality restriction \( \alpha_1 > 4/N \) comes from the fact that the dispersive part \( r(t) \) appears linearly in the equation (3.4) for the central manifold parameter \( a \). Note that in [11, 7] we use an improved decomposition of the type (3.1) in which the equation on the central manifold corresponding to (3.4) contains only quadratic and higher order terms in \( r(t) \). While this decomposition allows us to show convergence to a periodic solution even for subcritical regimes \( \alpha_1 \leq 4/N \) in dimensions \( N = 2, 3 \), it requires the central manifold to be \( C^2 \), i.e. \( \alpha_1 \geq 1 \). However our central manifold is \( C^5 \) except at zero, so if the initial data is chosen in an appropriate manner such that the dynamics stays away from zero then Corollary 3.2 can be proven even in subcritical regimes, see [6].

**Remark 3.1** In conclusion the approach in [11, 7] would allow us to obtain Corollary 3.2 for the critical regime \( \alpha_1 = 1 \) in dimension \( N = 4 \), but it would require the stronger hypothesis \( \alpha_1 \geq 1 \) in dimension \( N = 5 \). However for initial data in a conic like neighborhood of the manifold of non-zero periodic solutions of (1.1) the conclusion of Corollary 3.2 is valid for all \( \alpha_0(N) < \alpha_1 \leq 2 < 4/(N - 2) \), see [9].

We now proceed with the proofs:

**Proof of Theorem 3.1** It is well known that under hypothesis (H1)(i) the initial value problem (1)-(2) is locally well posed in the energy space \( H^1 \) and its \( L^2 \) norm is conserved, see for example [11] Cor. 4.3.3 at p. 92. Global well posedness follows via energy estimates from \( \|u_0\|_{H^1} \) small, see [11] Remark 6.1.5 at p. 165.

In particular we can define

\[ a(t) = \langle \psi_0, u(t) \rangle, \quad \text{for } t \in \mathbb{R} \]

Cauchy-Schwarz inequality implies

\[ |a(t)| \leq \|u(t)\|_{L^2} \|\psi_0\|_{L^2} = \|u_0\|_{L^2} \leq \varepsilon_0, \quad \text{for all } t \in \mathbb{R}. \quad \text{(3.2)} \]
In order to apply a contraction mapping argument for (3.8) we use the following Banach spaces. Let $E$ and $F$ be Banach spaces. The latter is possible because (3.5) is: 

Hence, if we choose $\varepsilon_0 \leq \delta$ we can define $h(a(t))$, $t \in \mathbb{R}$, see Proposition 2.1. We then obtain (5.1) where 

$$r(t) = u(t) - a(t)\psi_0 - h(a(t)), \quad a(t) = \langle \psi_0, u(t) \rangle, \quad \langle \psi_0, r(t) \rangle = 0. \quad (3.3)$$

The solution is now described by the $C^1$ scalar function $a(t) \in \mathbb{C}$ and $r(t) \in C(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, H^{-1})$. To obtain their equations we plug in (3.1) into (1.1) and project onto $E$. Then using Duhamel’s principle (3.5) becomes 

$$i \frac{dz}{dt} = Hz(t) + P_c Dg|_{\psi_E(t)}[z(t)] + i Dh|_{a(t)} \varepsilon \langle \psi_0, Dg|_{\psi_E(t)}[\varepsilon r(t)] \rangle + F_2(\psi_E(t), r(t)) \quad (3.7)$$

Define $\Omega(t, s)v = z(t)$. Then using Duhamel’s principle (3.5) becomes 

$$r(t) = \Omega(t, 0)r(0) - \int_0^t \Omega(t, s)[P_c i F_2(\psi_E(s), r(s)) - Dh|_{a(s)} \varepsilon \langle \psi_0, i F_2(\psi_E(s), r(s)) \rangle] ds. \quad (3.8)$$

In order to apply the linear estimates of Section 4 to $\Omega(t, s)$, we fix $\sigma > N/2$ and $\frac{2N}{N-2} < q_2 < \frac{2N}{N-4}$, then we consider the $\varepsilon_1(q_2) > 0$ given by Theorem 4.1 and choose $\varepsilon_0 > 0$ in the hypotheses such that 

$$\|\langle x \rangle^{4\sigma} \psi_E(t)(x)\|_{L^\infty(\mathbb{R}^N)} \leq \varepsilon_1, \quad \text{for all } t \in \mathbb{R}. \quad (3.9)$$

The latter is possible because $E(t) = E(\|a(t)\|)$ and $E$ is a $C^1$ function from the compact interval $[-\delta, \delta]$ to the real numbers with $E(0) = E_0 < 0$. So, there exists $\varepsilon_0$ such that $E(\|a\|) \leq E_0/2 < 0$ for all $|a| \leq \varepsilon_0$. (3.9) now follows from the exponential decay estimates in Remark 2.1 and the observation $\|\psi_E(t)\|_{L^\infty} \leq C\varepsilon_0$, for some constant $C > 0$. This is a consequence of Sobolev imbeddings and $\|\psi_E(t)\|_{H^3} \leq C\varepsilon_0$ which follows from $\psi_E(t) = a(t)\psi_0 + h(a(t))$, the norm $\|\psi_0\|_{H^2}$ is fixed, $|a(t)| \leq \varepsilon_0$ for all $t \in \mathbb{R}$, see (3.2), and $h$ is a $C^1$ function on the compact ball of radius $\delta$ in complex plane to $H^2$. Hence, there exists a constant $C > 0$ such that $\|\psi_E(t)\|_{H^2} \leq C\varepsilon_0$ for all $t \in \mathbb{R}$, and, by regularity arguments, see for example II Theorem 8.1.1] we get the same estimate for the $H^3$ norm with a possible larger constant $C$.

Consider now the nonlinear operator in (3.8):

$$N(u)(t) = \int_0^t \Omega(t, s)[P_c i F_2(\psi_E(s), u(s)) - Dh|_{a(s)} \varepsilon \langle \psi_0, i F_2(\psi_E(s), u(s)) \rangle] ds$$

In order to apply a contraction mapping argument for (3.8) we use the following Banach spaces. Let $p_1 = 2 + \alpha_1$, $p_2 = 2 + \alpha_2$, then 

$$Y_t = \left\{ u \in C(\mathbb{R}, L^2(\mathbb{R}^N)) \cap L^{p_1}(\mathbb{R}^N) \cap L^{p_2}(\mathbb{R}^N) : \sup_{t \in \mathbb{R}} (1 + |t|) N(\frac{1}{2} - \frac{1}{p_1}) \|u(t)\|_{L^{p_1}} < \infty, \sup_{t \in \mathbb{R}} \frac{(1 + |t|)^{m_1}}{\log(2 + |t|)} \|u(t)\|_{L^{p_2}} < \infty, \sup_{t \in \mathbb{R}} \|u(t)\|_{L^2} < \infty \right\} \quad (3.10)$$

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endowed with the norm
\[ ||u||_{Y_i} = \max\{\sup_{t \in \mathbb{R}}(1 + |t|)^{N\left(\frac{1}{2} - \frac{1}{1+\alpha_2}\right)}||u||_{L^p}, \sup_{t \in \mathbb{R}}(1 + |t|)^{n_3}\left(\sup_{t \in \mathbb{R}}(1 + |t|)^{\frac{n_3}{m_1}}||u||_{L^m}, \sup_{t \in \mathbb{R}}||u||_{L^2}\right) \]
for \( i = 1, 2, 3 \), where \( n_{1,2} = N\left(\frac{1}{2} - \frac{1}{p}\right) \), \( n_3 = \frac{N_3}{m_1} - 1 \), \( m_1 = m_3 = 0 \) and \( m_2 = 1 \).

**Lemma 3.1** Consider the cases:

1. \( N \left(\frac{\alpha_1 - 1}{2} + \frac{1}{2 + \alpha_2}\right) > 1 \);  
2. \( N \left(\frac{\alpha_1 - 1}{2} + \frac{1}{2 + \alpha_2}\right) = 1 \);  
3. \( N \left(\frac{\alpha_1 - 1}{2} + \frac{1}{2 + \alpha_2}\right) < 1 \);

and assume that (3.9) holds for some \( \sigma > N/2 \). Then, for each case number \( i \), \( N : Y_i \to Y_i \) is well defined, and locally Lipschitz, i.e., there exists \( C_i > 0 \), such that

\[ ||Nu_1 - Nu_2||_{Y_i} \leq C_i(||u_1||_{Y_1} + ||u_2||_{Y_1} + ||u_1||^{\alpha_1}_{Y_1} + ||u_2||^{\alpha_1}_{Y_1} + ||u_1||^{\alpha_2}_{Y_1} + ||u_2||^{\alpha_2}_{Y_1})||u_1 - u_2||_{Y_1}. \]

Note that the Lemma finishes the proof of Theorem 3.1. Indeed, \( \alpha_1 \) and \( \alpha_2 \) determine in which case \( i = 1, 2, 3 \) we are. But in all of them, if we denote:

\[ v = \Omega(t,0)r(0), \]

then:

\[ ||v||_{Y_i} \leq C_0||r(0)||_{L^{p_2} \cap L^2} \leq C_0(1 + ||\psi_0||_{L^{p_2}}) \left(\sup_{t \leq 0}||u_0||_{L^{p_2} \cap L^2}\right), \]

where \( C_0 = \max\{C_2, C_2\} \), see Theorem 3.2. We choose \( \varepsilon_0 \) in the hypotheses of theorem 3.1 such that \( R = 2||v||_{Y_1} \) satisfies

\[ Lip = 2\bar{C_i}(R + R^{\alpha_1} + R^{\alpha_2}) < 1, \]

where \( \bar{C_i} \) is given by the above Lemma. In this case the integral operator given by the right hand side of the (3.9):

\[ K(r) = v - N(r) \]

leaves the closed ball \( B(0,R) = \{ z \in Y : ||z||_{Y_1} \leq R \} \) invariant and it is a contraction on it with Lipschitz constant \( Lip \). Consequently the equation (3.9) has a unique solution in \( B(0,R) \). In particular, \( r(t) \) satisfies the \( L^p \) estimates claimed in the Theorem 3.1. We now have two solutions of (3.9), one in \( C(\mathbb{R}, H^1) \) from classical well posedness theory and one in \( C(\mathbb{R}, L^2 \cap L^{p_1} \cap L^{p_2}) \), \( p_1 = 2 + \alpha_1 \), \( p_2 = 2 + \alpha_2 \) from the above argument. Using uniqueness and the continuous embedding of \( H^1 \) in \( L^2 \cap L^{p_1} \cap L^{p_2} \), we infer that the solutions must coincide. Therefore, the time decaying estimates in the space \( Y_1 \) hold also for the \( H^1 \) solution.

**Proof of Lemma 3.1** Let us first consider the difference

\[ F_2(\psi_E, u_1) - F_2(\psi_E, u_2) = g(\psi_E + u_1) - g(\psi_E + u_2) - Dg|_{\psi_E}[u_1] + Dg|_{\psi_E}[u_2] \]

\[ = \int_0^1 (Dg(\psi_E + u_2 + \tau(u_1 - u_2))|u_1 - u_2| - Dg(\psi_E)|u_1 - u_2|)d\tau \]

\[ = \int_0^1 \int_0^1 D^2g(\psi_E + s(u_2 + \tau(u_1 - u_2)) + \tau(u_1 - u_2))|u_1 - u_2|dsd\tau \quad (3.10) \]

For a fixed \( (t,x) \in \mathbb{R} \times \mathbb{R}^N \), the inside integral is a line integral connecting \( z_1 = \psi_E(x) \) and \( z_2 = \psi_E(x) + u_2(x) + \tau(u_1(t,x) - u_2(t,x)) \). Note that, from (1.3) and (1.4), \( D^2g(\cdot) \) is integrable along any segment in the complex plane. Using the equivariance under rotations (1.3) we will reduce the line integral to a horizontal segment. First let us observe the behavior of \( Dg \) and \( D^2g \) under rotation:

\[ Dg(z)[e^{i\theta}\beta] = \lim_{\varepsilon \to 0} g(z + \varepsilon e^{i\theta}\beta) - g(z) = e^{i\theta} \lim_{\varepsilon \to 0} \frac{g(e^{-i\theta}z + \varepsilon\beta) - g(e^{-i\theta}z)}{\varepsilon} = e^{i\theta} Dg(e^{-i\theta}z)[\beta] \]
Hence, depending on the powers $\alpha$ and $\beta$, where the $\alpha$ term appears only when $\alpha > 1$, the $A_3$ term appears only when $\alpha < 1$. Now let us consider the difference $N_{u_1} - N_{u_2}$

\[
(N_{u_1} - N_{u_2})(t) = -i \int_0^t \Omega(t, s) P_{c} [F_2(\psi_E, u_1) - F_2(\psi_E, u_2)] ds + \int_0^t \Omega(t, s) D h|_{a(s)} f(\psi_0, F_2(\psi_E, u_1) - F_2(\psi_E, u_2)) ds
\]

**$L^p$ Estimate :**

\[
\|N_{u_1} - N_{u_2}\|_{L^p} \leq \int_0^{|t|} \|\Omega(t, s)\|_{L^p_{s} \rightarrow L^p_s} (\|A_1\|_{L^p} + \|A_2\|_{L^p} + \|A_3, 4\|_{L^p_s}) ds
\]

\[
+ \int_0^{|t|} \|\Omega(t, s)\|_{L^p_s \rightarrow L^p_s} (\|D h|_{a(s)} f|_{L^p_s} (\|\psi_0, A_1\|_{B_1} + \|\psi_0, A_2\|_{B_2} + \|\psi_0, A_3, 4\|_{B_3, 4})) ds
\]

From (1.3) we have $|D^2 g(z)| \leq C(|z|^{\alpha - 1} + |z|^{\alpha_2 - 1})$. For $z_2 \neq z_1 \in \mathbb{C}$ let $z_2 - z_1 = |z_2 - z_1| e^{i \theta}$.

Then:

\[
|\langle g(z_2) - g(z_1) \rangle[\beta]| \leq \left| \int_0^1 D^2 g (z_1 + s(z_2 - z_1))[z_2 - z_1][\beta] ds \right|
\]

\[
\leq \left| \int_0^1 e^{i \theta} D^2 g(e^{-i \theta} z_1 + s e^{-i \theta} (z_2 - z_1))[e^{-i \theta} (z_2 - z_1)][e^{-i \theta} \beta] ds \right|
\]

\[
\leq \int_0^1 C (|e^{-i \theta} z_1 + s z_2 - z_1|^{\alpha - 1} + |e^{-i \theta} z_1 + s z_2 - z_1|^{\alpha_2 - 1})|z_2 - z_1||\beta| ds
\]

Now for $0 < \alpha < 1$ we have

\[
\int_0^1 |e^{-i \theta} z_1 + s z_2 - z_1|^{\alpha - 1} ds \leq \int_0^1 |\Re z_1 + s z_2 - z_1|^{\alpha - 1} ds \leq |z_2 - z_1|^\alpha
\]

while for $\alpha > 1$,

\[
\int_0^1 |e^{-i \theta} z_1 + s z_2 - z_1|^{\alpha - 1} ds \leq \max_{t \in [0, 1]} |e^{-i \theta} z_1 + t z_2 - z_1|^{\alpha - 1} \int_0^1 1 ds \leq \max\{|z_1|^{\alpha - 1}, |z_2|^{\alpha - 1}\}
\]

Hence, depending on the powers $\alpha_1$ and $\alpha_2$ we have

\[
|F_2(\psi_E, u_1) - F_2(\psi_E, u_2)| \leq C \left[ (|u_1|^{\alpha_1} + |u_2|^{\alpha_1}) |u_1 - u_2| + (|u_1|^{\alpha_2} + |u_2|^{\alpha_2}) |u_1 - u_2| + |\psi_E|^{\alpha_2 - 1} |u_1 + u_2| |u_1 - u_2| + |\psi_E|^{\alpha_1 - 1} |u_1 + u_2| |u_1 - u_2| \right]
\]

(3.11)
The term $A_1$ satisfies via H"{o}lder inequality:
\[
\|(u_1|^{\alpha_1} + |u_2|^{\alpha_2})|u_1 - u_2\|_{L^p_2} \leq \left(\|u_1\|_{L^p_1}^{\alpha_1} \|u_1\|_{L^2}^{(1-\theta)\alpha_1} + \|u_2\|_{L^p_1}^{\alpha_2} \|u_2\|_{L^2}^{(1-\theta)\alpha_2}\right) \\
\times \|u_1 - u_2\|_p \|u_1 - u_2\|_{L^2}^{1-\theta},
\]
where $\frac{1}{p_2} = (1 + \alpha_1)(\frac{1-\theta}{2} + \frac{\theta}{p_1})$, $0 \leq \theta \leq 1$. Using Theorem 1.2 we get
\[
\int_0^{[t]} \|\Omega(t, s)\|_{L^p_2 \rightarrow L^p_2} \|A_1\|_{L^p_2} ds \\
\leq \int_0^{[t]} \frac{C(p_2)}{|t-s|^{N(\frac{\alpha_1}{2} - \frac{\alpha_2}{p_2})}} \frac{\|u_1\|_{Y_1}^{\alpha_1} + \|u_2\|_{Y_1}^{\alpha_2}}{(1 + |s|)^{N(\frac{\alpha_1}{2} + \frac{\alpha_2}{p_2})}} ds \\
\leq \frac{C(p_2)C_3[\log(2 + |t|)]^{m_2}}{(1 + |t|)^{m_1}} \left(\|u_1\|_{Y_1}^{\alpha_1} + \|u_2\|_{Y_1}^{\alpha_2}\right) \|u_1 - u_2\|_{Y_1},
\]
where the different decay rates $n_i$ depend on the case number in the hypotheses of this Lemma:
1. corresponds to $N(\frac{\alpha_2}{2} + \frac{1}{p_2}) > 1$, and in this case
\[
C_3 = \sup_{t \in \mathbb{R}} (1 + |t|)^{N(\frac{\alpha_2}{2} + \frac{1}{p_2})} \int_0^{[t]} \frac{1}{|t-s|^{N(\frac{\alpha_2}{2} - \frac{1}{p_2})}} \frac{1}{(1 + |s|)^{N(\frac{\alpha_2}{2} + \frac{1}{p_2})}} ds < \infty;
\]
2. corresponds to $N(\frac{\alpha_2}{2} + \frac{1}{p_2}) = 1$, and in this case
\[
C_3 = \sup_{t \in \mathbb{R}} \frac{(1 + |t|)^{\frac{N(\frac{\alpha_2}{2} - \frac{1}{p_2})}{\log(2 + |t|)}}}{\log(2 + |t|)} \int_0^{[t]} \frac{1}{|t-s|^{N(\frac{\alpha_2}{2} - \frac{1}{p_2})}} \frac{1}{(1 + |s|)^{N(\frac{\alpha_2}{2} + \frac{1}{p_2})}} ds < \infty;
\]
3. corresponds to $N(\frac{\alpha_2}{2} + \frac{1}{p_2}) < 1$, and in this case
\[
C_3 = \sup_{t \in \mathbb{R}} (1 + |t|)^{N(\frac{\alpha_2}{2} + \frac{1}{p_2}) - 1} \int_0^{[t]} \frac{1}{|t-s|^{N(\frac{\alpha_2}{2} - \frac{1}{p_2})}} \frac{1}{(1 + |s|)^{N(\frac{\alpha_2}{2} + \frac{1}{p_2})}} ds < \infty.
\]
To estimate the term containing $A_2$, observe that via H"{o}lder inequality
\[
\|(u_1|^{\alpha_2} + |u_2|^{\alpha_2})|u_1 - u_2\|_{L^p_2} \leq \left(\|u_1\|_{L^p_2}^{\alpha_2} + \|u_2\|_{L^p_2}^{\alpha_2}\right) \|u_1 - u_2\|_{L^p_2}
\]
since $\frac{1}{p_2} = \frac{1 + \alpha_2}{p_2}$. Again, using Theorem 1.2 we have
\[
\int_0^{[t]} \|\Omega(t, s)\|_{L^p_2 \rightarrow L^p_2} \|A_2\|_{L^p_2} ds \quad (3.13)
\]
\[
\leq \int_0^{[t]} \frac{C(p_2)}{|t-s|^{N(\frac{\alpha_2}{2} - \frac{1}{p_2})}} \frac{[\log(2 + |s|)]^{(1+\alpha_2)m_2}}{(1 + |s|)^{(1+\alpha_2)m_1}} \|(u_1|^{\alpha_2} + |u_2|^{\alpha_2})|u_1 - u_2\|_{Y_1} ds \\
\leq \frac{C(p_2)C_4C_5[\log(2 + |t|)]^{m_2}}{(1 + |t|)^{m_1}} \|(u_1|^{\alpha_2} + |u_2|^{\alpha_2})|u_1 - u_2\|_{Y_1},
\]
where $C_5 = \sup_{m_1, m_2} \frac{(1 + \alpha_2)^{m_2}}{[\log(2 + |t|)]^{(1+\alpha_2)m_2}} \int_0^{[t]} \frac{[\log(2 + |s|)]^{(1+\alpha_2)m_2}}{|t-s|^{N(\frac{\alpha_2}{2} - \frac{1}{p_2})}} (1 + |s|)^{(1+\alpha_2)m_1} ds < \infty$ since $(1 + \alpha_2)n_1 > 1$.

As for the $A_{3,4}$ terms note that they only appear when $\alpha_2 > 1$ respectively $\alpha_1 > 1$. To estimate them observe that
\[
\|(\psi_E|^{\alpha_1-1} + \psi_E|^{\alpha_2-1})(|u_1| + |u_2|)|u_1 - u_2\|_{L^p_2} \leq \|(\psi_E|^{\alpha_1-1} + \psi_E|^{\alpha_2-1})\|_{L^\infty} \|(u_1|_{L^p_2} + |u_2|_{L^p_2})\|_{L^p_2} \|u_1 - u_2\|_{L^p_2}
\]
where \( \frac{1}{p} + \frac{2}{p_2} = \frac{1}{p_1} \). By Theorem 4.4 we have for each case number \( i \) and \( u_1, u_2 \in Y_i \):

\[
\int_0^t \| \Omega(t, s) \|_{L_{p_1'}^2 \rightarrow L_{p_2}^2} \| A_{3,4} \|_{L_{p_1'}^2} \, ds
\]

\[
\leq \int_0^t \frac{C(p_2)}{|t-s|^{N\left(\frac{1}{2} - \frac{1}{p_1} \right)}} \| \psi_E \|_{L_{\alpha}} \| \psi_E \|_{L_{\alpha_2}} \frac{(\log(2 + |s|))^{2m_1}}{(1 + |s|)^{2m_1}} \| u_1 \|_{Y_i} \| u_2 \|_{Y_i} \| u_1 - u_2 \|_{Y_i} \, ds
\]

\[
\leq \frac{C(p_2)C_1C_2}{(1 + |t|)^{N\left(\frac{1}{2} - \frac{1}{p_1} \right)}} \| u_1 \|_{Y_i} \| u_2 \|_{Y_i} \| u_1 - u_2 \|_{Y_i}
\]

where \( C_1 = \sup_i \| \psi_E \|_{L_{\alpha}} \| \psi_E \|_{L_{\alpha_2}} \) and \( C_2 = \sup_i \int_0^t \frac{(\log(2 + |s|))^{2m_1}}{(1 + |s|)^{2m_1}} \, ds < \infty \) since \( 2n_i > 1 \) (for \( p_2 = \alpha_2 + 2 > 3 \), and \( \alpha_1 \) satisfying (1.4)). The uniform bounds in \( t \in \mathbb{R} \) for \( \| \psi_E \|_{L_{(\alpha_1)^{-1}}} \), \( j = 1, 2 \) follow from (3.9).

For \( B \) terms we have:

\[
|B_1| \leq \| \psi_0 \|_{L_{p_2}} \| A_1 \|_{L_{p_2}'}, \quad |B_2| \leq \| \psi_0 \|_{L_{p_2}} \| A_2 \|_{L_{p_2}'}, \quad \text{and} \quad |B_{3,4}| \leq \| \psi_0 \|_{L_{p_2}} \| A_{3,4} \|_{L_{p_2}'}.
\]

Note that

\[
\| |u_1|_{\alpha_1} + |u_2|_{\alpha_1} \|_{L_{p_1}'} \leq \left( \| u_1 \|_{L_{p_2}} + \| u_2 \|_{L_{p_2}} \right) \| u_1 - u_2 \|_{L_{p_1}}
\]

since \( \frac{1}{p_1} = \frac{1 + \alpha_1}{p_1} \). Using Theorem 4.1 we have

\[
\int_0^t \| \Omega(t, s) \|_{L_{p_2}^2} \| Dh|_{\alpha(s)} \|_{L_{p_2}^2} \| \psi_0 \|_{L_{p_1}} \| A_1 \|_{L_{p_1}'} \, ds
\]

\[
\leq \int_0^t \frac{C(p_2)C_2\| \psi_0 \|_{L_{p_1}}\left( \| u_1 \|_{Y_i}^\alpha + \| u_2 \|_{Y_i}^\alpha \right) \| u_1 - u_2 \|_{Y_i}}{|t-s|^{N\left(\frac{1}{2} - \frac{1}{p_1} \right)}(1 + |s|)^{N\left(\frac{1}{2} - \frac{1}{p_1} \right)}(1 + |s|)^{\alpha_1}} \, ds
\]

\[
\leq \frac{C(p_2)C_1C_2}{(1 + |t|)^{N\left(\frac{1}{2} - \frac{1}{p_1} \right)}} \left( \| u_1 \|_{Y_i}^\alpha + \| u_2 \|_{Y_i}^\alpha \right) \| u_1 - u_2 \|_{Y_i},
\]

where we used

\[
N \left( \frac{1}{2} - \frac{1}{p_1} \right) (\alpha_1 + 1) > 1
\]

because \( p_1 = \alpha_1 + 2 \) and \( \alpha_1 \) satisfies (1.4), and the uniform estimates

\[
\| Dh|_{\alpha(s)} \|_{L_{p_2}^2} \leq C_2, \quad \text{for all} \ s \in \mathbb{R},
\]

which follow from \( h \) being \( C^1 \) on \( a \in \mathbb{C}, \ |a| \leq \delta \), with values in \( L_{p_2}^2 \), see Proposition 2.1 and \( |a(s)| \leq \varepsilon_0 \leq \delta \) for all \( s \in \mathbb{R} \), see (3.2).

Now

\[
\int_0^t \| \Omega(t, s) \|_{L_{p_2}^2} \| Dh\|_{L_{p_2}^2} \| \psi_0 \|_{L_{p_2}^2} \| A_2 \|_{L_{p_2}'} \, ds
\]

is estimated as (3.13), and

\[
\int_0^t \| \Omega(t, s) \|_{L_{p_2}^2} \| Dh\|_{L_{p_2}^2} \| \psi_0 \|_{L_{p_2}^2} \| A_{3,4} \|_{L_{p_2}'} \, ds
\]

is estimated as (3.14).

\begin{itemize}
  \item \( L_{p_1}^n \) Estimate :
\end{itemize}
\[ \| Nu_1 - Nu_2 \|_{L^p_1} (t) \leq \| \int_0^t \Omega(t,s)[F_2(\psi_E(s), u_1(s)) - F_2(\psi_E(s), u_2(s))]ds \|_{L^p_1} \]

\[ + \quad C \int_0^{|t|} \| \Omega(t,s) \|_{L^2_1 \rightarrow L^p_1} \| Dh|_{\alpha(s)} \|_{L^2_{B_1}} \left( \| (\psi_0, A_1) \|_{B_1} + \| (\psi_0, A_2) \|_{B_2} + \| (\psi_0, A_{3,4}) \|_{B_{3,4}} \right) ds \]

The second integral is estimated as in the previous \( L^p \) estimates on \( B_1, B_2 \) and \( B_{3,4} \) to obtain the required bound. For the first integral moving the norm inside the integration and applying \( L^p_1 \rightarrow L^p_1 \) estimates for \( \Omega(t,s) \) and \( (3.11) \) for the nonlinear term would require the control of \( A_2 \) and \( A_{3,4} \) in \( L^p_1 \). The latter, unfortunately, can no longer be interpolated between \( L^2 \) and \( L^p_2 \). To avoid this difficulty we separate and treat differently the part of the nonlinearity having an \( A_2 \) and \( A_{3,4} \) like behavior by decomposing \( \mathbb{R}^N \) in two disjoint measurable sets related to the inequality \( (3.11) \):

\[ V_1(s) = \{ x \in \mathbb{R}^N \mid |F_2(\psi_E(s), u_2(s,x)) - F_2(\psi_E(s), u_1(s,x))| \leq C(A_2(s,x) + A_{3,4}(s,x)) \}, \]

\[ V_2(s) = \mathbb{R}^N \setminus V_1(s) \]

On \( V_2(s) \), using polar representation of complex numbers, we further split the nonlinear term into:

\[ F_2(\psi_E(s), u_1(s,x)) - F_2(\psi_E(s), u_2(s,x)) = e^{i\theta(s,x)}C(A_2(s,x) + A_{3,4}(s,x)) \]

\[ + e^{i\theta(s,x)} \left[ |F_2(\psi_E(s), u_1(s,x)) - F_2(\psi_E(s), u_2(s,x))| - C(A_2(s,x) + A_{3,4}(s,x)) \right] \]

where, due to inequality \( (3.11) \), \( |G(s,x)| \leq CA_1(s,x) \) on \( V_2(s) \). Then we have:

\[ \int_0^t \Omega(t,s)[F_2(\psi_E(s), u_1(s)) - F_2(\psi_E(s), u_2(s))]ds = \int_0^t \Omega(t,s)(1 - \chi(s))G(s)ds \]

\[ + \int_0^t \Omega(t,s)|\chi(s)(F_2(\psi_E(s), u_1(s)) - F_2(\psi_E(s), u_2(s))) + (1 - \chi(s))e^{i\theta(s)}C(A_2(s) + A_{3,4}(s))|ds, \]

where \( \chi(s) \) is the characteristic function of \( V_1(s) \). Now

\[ \| \int_0^t \Omega(t,s)(1 - \chi(s))G(s)ds \|_{L^p_1} \leq \int_0^{|t|} \| \Omega(t,s) \|_{L^p_{i_1} \rightarrow L^p_{i_1}} C \| A_1(s) \|_{L^p_{i_1}} ds \]

and estimates as in the previous step for \( A_1 \) give the required decay, see \( (3.15) \) and the inequalities following it. For \( I(t) \) we use interpolation:

\[ \| I(t) \|_{L^p_1} \leq \| I(t) \|_{L^2_1}^{\frac{1}{p_1} - \frac{\theta}{p_2}} \| I(t) \|_{L^{p_2}_{i_2} \rightarrow L^{p_2}_{i_2}}^{\frac{1}{p_2} - \theta} \left( \int_0^{|t|} \| \Omega(t,s) \|_{L^p_{i_2} \rightarrow L^p_{i_2}} \| A_2 + A_{3,4} \|_{L^p_{i_2}} ds \right)^{\theta} \]

where \( \frac{1}{p_1} = \frac{1}{\theta} + \frac{\theta}{p_2} \). We know from previous step that the above integral decays as \( (1 + |t|)^{-N \left( \frac{1}{2} - \frac{1}{p_2} \right)} \) and below we will show its \( L^2 \) norm will be bounded. Therefore, since \( \theta N \left( \frac{1}{2} - \frac{1}{p_2} \right) = N \left( \frac{1}{2} - \frac{1}{p_1} \right) \) we have:

\[ \sup_t (1 + |t|)^{N \left( \frac{1}{2} - \frac{1}{p_1} \right)} \| I(t) \|_{L^p_1} < \infty \]

and the \( L^p_1 \) estimates are complete.

- \( L^2 \) Estimate:
We will use $L^2 \to L^2$ bounds for $\Omega(t,s)$, see Theorem 1.2 to control the $B_{1-4}$ terms. For the $A_{1-4}$ terms we avoid $L^2 \to L^2$ bounds because that would require us to control the $L^{2(\alpha_2+1)}$ norm of functions in $Y_t$ which is impossible since it can no longer be interpolated between the norms in $L^2$ and $L^{p_2}$, $p_2 = \alpha_2 + 2$. Instead we use the decomposition:

$$\Omega(t,s) = e^{-iH(t-s)}P_c + \tilde{T}(t,s) + (T(t,s) - \tilde{T}(t,s))$$

where for $t \geq s$

$$\tilde{T}(t,s) = \int_s^{\min\{t,s+1\}} e^{-iH(t-\tau)}P_c g_u(\tau)e^{-iH(\tau-s)}P_c d\tau$$

$$= e^{-iH(t-s)}P_c \int_s^{\min\{t,s+1\}} e^{-iH(t-s)}P_c e^{iH(\tau-s)}P_c g_u(\tau)e^{-iH(\tau-s)}P_c d\tau$$

while for $t < s$

$$\tilde{T}(t,s) = \int_s^{\max\{t,s-1\}} e^{-iH(t-\tau)}P_c g_u(\tau)e^{-iH(\tau-s)}P_c d\tau$$

$$= e^{-iH(t-s)}P_c \int_s^{\max\{t,s-1\}} e^{iH(\tau-s)}P_c g_u(\tau)e^{-iH(\tau-s)}P_c d\tau$$

For $e^{-iH(t-s)}P_c$ and $\tilde{T}(t,s)$ we will use Stricharz estimates, while for $T(t,s) - \tilde{T}(t,s)$ we will use $L^{p'} \to L^2$ estimates, see Theorem 1.2. All in all we have:

$$\|N u_1 - N u_2\|_{L^2} \leq \int_0^{|t|} \|\Omega(t,s)\|_{L^2} \|Dh\|_{L^2}(|B_1| + |B_2| + |B_{3,4}|) ds$$

$$+ \int_0^{|t|} \|T(t,s) - \tilde{T}(t,s)\|_{L^{p_1}} \|A_1\|_{L^{p'_1}} ds$$

$$+ \int_0^{|t|} \|T(t,s) - \tilde{T}(t,s)\|_{L^{p_2}} \|A_2\|_{L^{p'_2}} ds$$

$$+ \int_0^{|t|} \|e^{-iH(t-s)}P_c A_1(1) ds\|_{L^2} + \int_0^{|t|} \|e^{-iH(t-s)}P_c (A_2(s) + A_{3,4}(s)) ds\|_{L^2}$$

$$+ \int_0^{|t|} \|\tilde{T}(t,s)A_1 ds\|_{L^2} + \int_0^{|t|} \|\tilde{T}(t,s)(A_2 + A_{3,4}) ds\|_{L^2}$$

First three integrals are estimated similar to the previous cases. We deduce that this integrals are uniformly bounded by:

$$\tilde{C}_i(\|u_1\|_{Y_t}^{\alpha_i} + \|u_2\|_{Y_t}^{\alpha_i} + \|u_1\|_{Y_t}^{\alpha_i} + \|u_2\|_{Y_t}^{\alpha_i})\|u_1 - u_2\|_{Y_t}$$

For the fourth integral we use Stricharz estimate:

$$\sup_{t \in \mathbb{R}} \int_0^t e^{-iH(t-s)}P_c A_1 ds\|_{L^2} \leq C_S \left( \int_{\mathbb{R}} \|A_1(s)\|_{L^{p'_1}} ds \right)^{\frac{1}{\gamma_1}}$$

where $\frac{1}{\gamma_1} + \frac{1}{\gamma_1} = 1$, and $\frac{2}{\gamma_1} = N \left( \frac{1}{2} - \frac{1}{p_1} \right)$. Furthermore we have

$$\|A_1\|_{L^{p'_1}} \leq C_1 \int_{\mathbb{R}} \left[ \frac{ds}{(1 + |s|)^{N(1+\alpha_i)\gamma_1(4 - \frac{1}{p_1})}} \right]^{\frac{1}{\gamma_1}} (\|u_1\|_{Y_t}^{\alpha_i} + \|u_2\|_{Y_t}^{\alpha_i}) \|u_1 - u_2\|_{Y_t}$$

$$\leq C_1 C_{10} (\|u_1\|_{Y_t}^{\alpha_i} + \|u_2\|_{Y_t}^{\alpha_i}) \|u_1 - u_2\|_{Y_t}$$

(3.17)
where \( C_{10} = \int_{\mathbb{R}} \frac{ds}{(1 + |s|)^{N(1 + \alpha_1) \gamma_1^{\alpha_1}}} < \infty \) since \( N(1 + \alpha_1) \gamma_1^{\alpha_1} \left( \frac{1}{2} - \frac{1}{p_1} \right) > 1 \).

Similarly, for the fifth integral:

\[
\sup_{t \in \mathbb{R}} \left\| \int_{0}^{t} e^{-iH(t-s)} P_c (A_2 + A_{3,4}) ds \right\|_{L^2} \leq C_s \left[ \left( \int_{\mathbb{R}} \| A_2(s) \|_{L^{p_2}}^2 ds \right)^{\frac{1}{2}} + \left( \int_{\mathbb{R}} \| A_{3,4}(s) \|_{L^{p_2}}^2 ds \right)^{\frac{1}{2}} \right]
\]

where \( \frac{1}{p_2} + \frac{1}{q_2} = 1 \), and \( \frac{2}{q_2} = N \left( \frac{1}{2} - \frac{1}{p_2} \right) \). Using again the estimates we obtained before for \( A_2 \) and \( A_{3,4} \) we get:

\[
\| A_{3,4} \|_{L^{p_2}} \leq C_{11} \left[ \int_{\mathbb{R}} \frac{(\log(2 + |s|))(1 + \alpha_2 m_1 \gamma_2^2) ds}{(1 + |s|)^{2n_1 \gamma_2^2}} \right]^{\frac{1}{2}} \left( \| u_1 \|_{Y_i} + \| u_2 \|_{Y_i} \right)\| u_1 - u_2 \|_{Y_i}
\]

\[
\leq C_{11} C_{8} (\| u_1 \|_{Y_i} + \| u_2 \|_{Y_i}) \| u_1 - u_2 \|_{Y_i}
\]

(3.18)

where \( C_{8} = \int_{\mathbb{R}} \frac{(\log(2 + |s|))(1 + \alpha_2 m_1 \gamma_2^2) ds}{(1 + |s|)^{2n_1 \gamma_2^2}} < \infty \) since \( 2n_1 \gamma_2^2 > 1 \) and:

\[
\| A_2 \|_{L^{p_2}} \leq C_{12} \left[ \int_{\mathbb{R}} \frac{(\log(2 + |s|))(1 + \alpha_2 m_1 \gamma_2^2) ds}{(1 + |s|)^{1 + \alpha_2 m_1 \gamma_2^2}} \right]^{\frac{1}{2}} \left( \| u_1 \|_{Y_i}^2 + \| u_2 \|_{Y_i}^2 \right)\| u_1 - u_2 \|_{Y_i}
\]

\[
\leq C_{9} (\| u_1 \|_{Y_i}^2 + \| u_2 \|_{Y_i}^2) \| u_1 - u_2 \|_{Y_i}
\]

(3.19)

where \( C_{9} = \int_{\mathbb{R}} \frac{(\log(2 + |s|))(1 + \alpha_2 m_1 \gamma_2^2) ds}{(1 + |s|)^{1 + \alpha_2 m_1 \gamma_2^2}} < \infty \) since \( (1 + \alpha_2) m_1 \rho_1^2 > 1 \).

Now for the last two integrals consider

\[
\tilde{A}_i(t) = \| \int_{0}^{t} \tilde{T}(t, s) A_i(s) ds \|_{L^2}, \quad i = 1, 4
\]

Fix \( t \geq 0 \) and \( i \in \{1, 4\} \). The case \( t < 0 \) is treated analogously. We have

\[
\tilde{A}_i(t) \leq \sup_{\| \tilde{v} \|_{L^2} = 1} \left| \langle \tilde{u}, \int_{0}^{t} \tilde{T}(t, s) A_i(s) ds \rangle \right|
\]

\[
\leq \sup_{\| \tilde{v} \|_{L^2} = 1} \int_{0}^{t} \left| \langle e^{iH(t-s)} P_c \tilde{v}, \int_{s}^{\min\{t, s+1\}} e^{-iH(\tau-s)} P_c g_\alpha(\tau) e^{-iH(\tau-s)} P_c A_i(s) d\tau \rangle \right| ds
\]

\[
\leq \sup_{\| \tilde{v} \|_{L^2} = 1} \int_{0}^{t} \| e^{iH(t-s)} P_c \tilde{v} \|_{L^p} \| e^{-iH(\tau-s)} P_c g_\alpha(\tau) e^{-iH(\tau-s)} P_c A_i \|_{L^{p'}} ds
\]

\[
\leq \sup_{\| \tilde{v} \|_{L^2} = 1} \int_{0}^{t} \| e^{iH(t-s)} P_c \tilde{v} \|_{L^p} C \sup_{\tau \in [s, s+1]} \| g_\alpha(\tau) \|_{L^1} \| A_i \|_{L^{p'}} ds
\]

where we used the Fourier multiplier type estimate \( \| e^{iH(t-s)} F e^{-iH(\tau-s)} P_c \|_{L^p \rightarrow L^{p'}} \leq C \| \tilde{F} \|_{L^1} \) for all \( 1 \leq p \leq \infty \) and \( |\tau - s| \leq 1 \); see Appendix in [7, Theorem 5.2]. Note that by Strichartz estimates there exist a fixed constant \( C > 0 \) such that for all \( \tilde{v} \in L^2 \):

\[
\| e^{iH(t-s)} \tilde{v} \|_{L^{2} \tilde{L}_{p}^{2}} \leq C \| \tilde{v} \|_{L^2}
\]

and using (3.14), (3.18) and (3.19) for \( A_i \|_{L^{p'}}, L^p \) we get that \( \tilde{A}_i(t) \) are bounded uniformly for \( t \in \mathbb{R} \).

The \( L^2 \) estimates are now complete and the proofs of Lemma 3.1 and Theorem 3.1 are finished. □

To obtain Corollary 3.3 in the case (i), i.e.

\[
\alpha_1 \geq \frac{4}{N}, \quad \text{or} \quad p_2 = \alpha_2 + 2 < \frac{2N}{2 + N - N\alpha_1}
\]
we use Riesz-Thorin interpolation for $2 < p < p_2$ while for $p > p_2$ we return to the integral form of the equation for $r(t)$:

$$r(t) = \Omega(t, 0)r(0) - \int_0^t \Omega(t, s) P_s [iF_2(\psi_E, r) - Dh_{\alpha(t)}(\psi_0, iF_2(\psi_E, r))] ds = \Omega(t, 0)r(0) - N(r),$$

see (3.8), and use the same arguments as in $L^{p_2}$ estimates for Lemma 3.1 with $u_1 = r$, $u_2 = 0$ and $p_2$ replaced by an arbitrary $p$, $p_2 < p < 2N/(N - 2)$. Same works for $p_2 < p < 2N/(N - 2)$ in the cases (ii) and (iii), i.e.

$$\alpha_1 < \frac{4}{N} \quad \text{and} \quad p_2 = \alpha_2 + 2 = \frac{2N}{2 + N - N\alpha_1},$$

respectively

$$\alpha_1 < \frac{4}{N} \quad \text{and} \quad p_2 = \alpha_2 + 2 > \frac{2N}{2 + N - N\alpha_1}.$$ 

In these last two cases, for $\alpha_1 + 2 = p_1 < p < p_2$ we use the arguments for $L^{p_1}$ estimates in Lemma 3.1 with $p_1$ replaced by $p$. Riesz-Thorin interpolation gives the required estimates for $2 < p < p_1$ and finishes the proof of Corollary 3.1.

For the proof of Corollary 3.2 we return to the dynamics on the center manifold given by equation (3.4):

$$i \frac{da}{dt} = E(|a(t)|)a(t) + \langle \psi_0, g(\psi_E(t) + r(t)) - g(\psi_E(t)) \rangle.$$

Changing to $\tilde{a}(t) = e^{i \int_0^t E(|a(s)|) ds} a(t)$ which eliminates the fast phase oscillations of the complex valued function $a(t)$, and using the symmetries of $g(z)$ with respect to rotations of the complex plane (1.5), we get:

$$i \frac{d\tilde{a}}{dt} = \langle \psi_0, \tilde{g}(\tilde{\psi}_E(t) + \tilde{r}(t)) - g(\tilde{\psi}_E(t)) \rangle = \langle \psi_0, Dg_{\tilde{\psi}_E(t)}[\tilde{r}(t)] + F_2(\tilde{\psi}_E(t), \tilde{r}(t)) \rangle,$$

where $\tilde{\psi}_E(t) = e^{i \int_0^t E(|a(s)|) ds} \psi_E(t) = \tilde{a}(t)\psi_0 + h(\tilde{a}(t))$ and $\tilde{r}(t) = e^{i \int_0^t E(|a(s)|) ds} r(t)$. Now the right hand side of (3.20) is integrable in time on $t \in [-\infty, \infty]$. Indeed, for the nonlinear term we use (3.11) with $u_1 = \tilde{r}$ and $u_2 \equiv 0$ to get:

$$\|\langle \psi_0, F_2(\tilde{\psi}_E(t), \tilde{r}(t)) \rangle\| \leq C \left[ \|\psi_0\|_{L^2} \|\tilde{r}\|_{L^2}^{\alpha_1 + 1} + \|\psi_0\|_{L^2} \left( \|\tilde{r}\|_{L^2}^{\alpha_2 + 1} + \|\tilde{\psi}_E\|^{\alpha_1 - 1} + \|\tilde{\psi}_E\|^{\alpha_2 - 1} \|\tilde{r}\|_{L^2}^2 \right) \right] \leq \frac{C}{(1 + |t|)^\beta}$$

where

$$\beta = \min \left\{ (\alpha_1 + 1)N \left( \frac{1}{2} - \frac{1}{p_1} \right), (\alpha_2 + 1)N \left( \frac{1}{2} - \frac{1}{p_2} \right), 2N \left( \frac{1}{2} - \frac{1}{p_2} \right) \right\} > 1,$$

see the estimates for $\|r(t)\| = |\tilde{r}(t)|$ in Theorem 3.1.

For the linear term we are forced to use $L^2_{x,\sigma}$ estimates. By (2.5), (2.4), and (1.3) combined with Cauchy-Schwarz and H"older inequalities:

$$\|\langle \psi_0, Dg_{\tilde{\psi}_E(t)}[\tilde{r}(t)] \rangle\| \leq \|\psi_0\|_{L^2} \left( \|\tilde{r}\|_{L^2} \left( \|x\| < x^\sigma \|\tilde{\psi}_E(t)\|_{L^\infty} + \|x\| > x^\sigma \|\tilde{\psi}_E(t)\|_{L^\infty} \right) \right) \|x\| > x^{-\sigma} \|\tilde{r}(t)\|_{L^2} \leq C \|r(t)\|_{L^2_{x,\sigma}},$$

where the uniform in time bounds for the norms involving $|\tilde{\psi}_E(t)| = |\psi_E(t)|$ follow from (3.4). For the
$L^2_{-\alpha}$ estimate of $r(t)$ we turn to (3.8) which combined with (3.11) where $u_1 = r$ and $u_2 \equiv 0$ gives:

$$
\|r(t)\|_{L^2_{-\alpha}} \leq \|\Omega(t,0)\|_{L^\infty_{-\alpha} \to L^2_{-\alpha}} \|r(0)\|_{L^\infty} + \|\Omega(t,s)\|_{L^\infty_{-\alpha} \to L^2_{-\alpha}} \|r(s)\|_{L^2_{-\alpha}}^{\alpha_1+1} \, ds + \|\Omega(t,s)\|_{L^\infty_{-\alpha} \to L^2_{-\alpha}} \|r(s)\|_{L^2_{-\alpha}}^{\alpha_2+1} \, ds + \|\Omega(t,s)\|_{L^\infty_{-\alpha} \to L^2_{-\alpha}} \|r(0)\|_{L^2_{-\alpha}} \, ds
$$

(3.22)

Now, from the weighted estimates for $\Omega(t,s)$ respectively (3.16), and from (3.21) we get that the last two integrals decay like $(1 + |t|)^{-\tilde{\beta}}$, where

$$
\tilde{\beta} = \min \left\{ N \left( \frac{1}{2} - \frac{1}{p_3} \right), (\alpha_1 + 1) N \left( \frac{1}{2} - \frac{1}{(\alpha_1 + 1)p_3} \right) \right\} > 1.
$$

(3.23)

such that $p_3 \leq p$ with $p$ given in the hypotheses of Corollary 3.2. Then we use $q_0 = p_3'$ and split the first two integrals on the right hand side of (3.22) into integrals from zero to $|t| - 1$ where we choose $q_1 = q_2 = p_3'$ and integrals from $|t| - 1$ to $|t|$ where we choose $q_1 = p_1'$, $q_2 = p_2'$. This way the first three terms on the right hand side of (3.22) decay like $(1 + |t|)^{-\tilde{\beta}}$ where

$$
\tilde{\beta} = \min \left\{ N \left( \frac{1}{2} - \frac{1}{p_3} \right), (\alpha_1 + 1) N \left( \frac{1}{2} - \frac{1}{(\alpha_1 + 1)p_3} \right) \right\} > 1.
$$

Remark 3.2 The restriction $\alpha_1 > 4/N$ is necessary for the existence of $p_3$ with the properties (3.23) which in turn insures the integrability in time of the first two integrals in (3.22). Such a restriction was not needed for the integrability of (3.21), the nonlinear terms on the right hand side of (3.22). Consequently a decomposition that removes the linear term on the right hand side of (3.20) will also remove this restriction to asymptotic stability.

All in all we now have

$$
\frac{d\tilde{a}}{dt} = -i\langle \psi_0, Dg|_{\bar{\psi}_E(t)} \hat{f}(t) \rangle + F_2(\bar{\psi}_E(t), \hat{r}(t)) - i\langle \psi_0, Dg|_{\bar{\psi}_E(t)} \hat{f}(t) \rangle + F_2(\bar{\psi}_E(t), \hat{r}(t)) \rangle \leq \frac{C}{(1 + |t|)^{\beta}} , \beta > 1.
$$

Consequently there exist $\tilde{a}_{\pm \infty} \in \mathbb{C}$ such that

$$
\lim_{t \to \pm \infty} \tilde{a}(t) = \tilde{a}_{\pm \infty}, \quad |\tilde{a}(t) - \tilde{a}_{\pm \infty}| \leq \frac{C_1}{(1 + |t|)^{\beta - 1}}.
$$

Moreover, because $E(|a|), |a| \leq \varepsilon_0$ is a $C^1$ function hence Lipschitz, we get $E(t) = E(|a(t)|) = E(|\tilde{a}(t)|) \to E(|a_{\pm \infty}|)$ as $t \to \pm \infty$ and the function

$$
\theta(t) = \left\{ \begin{array}{ll}
\frac{1}{2} \int_0^t E(|a(s)|) - E(|a_+|) \, ds & \text{if } t \geq 1 \\
\frac{1}{2} \int_0^{-t} E(|a(s)|) - E(|a_-|) \, ds & \text{if } t \leq -1
\end{array} \right.
$$

converges to zero as $t \to \pm \infty$. Finally, from $\psi_E(t) = a(t)\psi_0 + h(a(t))$, by the continuity of $h(a)$ and its equivariance with respect to rotations see Proposition 2.1, we get:

$$
\lim_{t \to \pm \infty} \|\psi_E(t) - e^{-it(E(|a_{\pm \infty}|) + \theta(t))}\psi_E(|a_{\pm \infty}|)\|_{H^2 \cap L^2_{-\alpha}} = 0.
$$

and the proof of Corollary 3.2 is finished.

In the next section we obtain the estimates for the propagator $\Omega(t,s), t, s \in \mathbb{R}$ of (3.7). Note that they were essential in proving Theorem 3.1 and Corollaries 3.1 and 3.2.

17
4 Linear Estimates

Consider the linear Schrödinger equation with a potential in four and five space dimensions:

\[ i \frac{\partial u}{\partial t} = (-\Delta + V(x)) u \]

\[ u(0) = u_0. \]

If \( V \) satisfies hypothesis (H1) (i) 1. and (ii) it is known, see [13] Example 7.8], that for \( N = 4, 5 \), and \( \sigma > N/2 \), there exists a constant \( C_N > 0 \) such that

\[ \| e^{-iHt} P_c u_0 \|_{L^2_{\sigma}} \leq C_N \| u_0 \|_{L_2} \]  \tag{4.1}

where \( P_c \) is the projection onto the continuous spectrum of \( H = -\Delta + V \).

In addition, if \( V \) satisfies (H1) (i) 1., 3. and (ii) then for each \( 2 \leq p \leq \infty \), \( 1/p' + 1/p = 1 \) there exists a constant \( C_p > 0 \) such that:

\[ \| e^{-iHt} P_c u_0 \|_{L^p} \leq \frac{C_p}{|t|^{N(\frac{1}{p'} - \frac{1}{p})}} \| u_0 \|_{L^{p'}}, \quad N = 4, 5 \]  \tag{4.2}

see for example [4].

We would like to extend these estimates to the linearized dynamics around the time dependent motion on center manifold. We consider the linear equation with initial data at time \( s \) in the range of \( P_c \):

\[ i \frac{dz}{dt} = Hz(t) + P_c Dg|_{\psi_E(t)}[z(t)] + i Dh|_{\psi_E(t)}(\psi_0, Dg|_{\psi_E(t)}[z(t)]) \]

\[ z(s) = v \in \text{range} P_c \]

where \( Dg|_{\psi_E}[z] = \frac{d}{dz} g(\psi_E + \varepsilon z)|_{\varepsilon = 0} = \frac{\partial}{\partial \varepsilon} g(u)|_{u = \psi_E z} + \frac{\partial}{\partial \varepsilon} g(u)|_{u = \psi_E \tau}. \)

By Duhamel’s principle we have:

\[ z(t) = e^{-iH(t-s)} P_c v - \int_s^t e^{-iH(t-\tau)} P_c \left( i Dg|_{\psi_E(\tau)}[z(\tau)] - Dh|_{\psi_E(\tau)}(\psi_0, i Dg|_{\psi_E(\tau)}[z(\tau)]) \right) d\tau \]  \tag{4.3}

In the next theorems we will extend estimates of type (4.1)-(4.2) to the operators

\[ \Omega(t,s)v = z(t), \quad \text{and} \quad T(t,s) = \Omega(t,s) - e^{-iH(t-s)} P_c, \]

relying on the fact that \( \psi_{E(t)} \) is small and localized in space, see [3,9]. The arguments can be extended for large \( \psi_{E(t)} \) provided for a certain fixed solution \( \psi_E \) of (2.2) we have \( \inf_{\theta \in \mathbb{R}} \| \psi_{E(t)} - e^{i\theta} \psi_E \|_{H^1} \) is small uniformly in \( t \in \mathbb{R} \), see [6]. We start with weighted estimates. While the approach is similar to the one in [10], see also [7], we include the proofs for completeness.

**Theorem 4.1** Fix \( \sigma > N/2 \), and \( \frac{2N}{N-2} < q_2 < \frac{2N}{N-4} \). There exists \( \varepsilon_1(q_2) > 0 \) such that if \( \| \psi \| < \varepsilon \) for all \( t \in \mathbb{R} \), then there are constants \( C_\sigma, C_p, C \) and \( C(q_2) > 0 \) with the property that
for any \( t, s \in \mathbb{R} \) the following estimates hold:

\[
\begin{align*}
(i) \quad & \| \Omega(t, s) \|_{L^2_{t,s} \to L^2_\sigma} \leq \frac{C_\sigma}{(1 + |t - s|)^2}; \\
(ii) \quad & \| \Omega(t, s) \|_{L^2_{t,s} \to L^2} \leq \frac{C_\sigma}{|t - s|^{(N - 1)/p}}; \quad \text{for all } 2 \leq p < \frac{2N}{N - 2}; \\
(iii) \quad & T(t, s) \in L^2(\mathbb{R}, L^2 \to L^2_{-\sigma}) \cap L^\infty(\mathbb{R}, L^2 \to L^2_{-\sigma}); \\
(iv) \quad & \| \Omega(t, s) \|_{L^p_{t,s} \to L^2_\sigma} \leq \frac{C_\sigma}{|t - s|^{N(\frac{2}{p} - 1)}} \quad \text{for all } 2 \leq p \leq q_2; \\
\| T(t, s) \|_{L^p_{t,s} \to L^2_\sigma} \leq \begin{cases} 
C & \text{for } |t - s| \leq 1 \text{ and } 2 \leq p \leq \frac{2N}{N - 2}, \\
\frac{C_\sigma}{|t - s|^{N(\frac{2}{p} - 1)}} & \text{for } |t - s| \leq 1 \text{ and } \frac{2N}{N - 2} < p \leq q_2, \\
\frac{C_\sigma}{(1 + |t - s|)^{N(\frac{2}{p} - 1)}} & \text{for } |t - s| > 1 \text{ and } 2 \leq p \leq q_2.
\end{cases}
\end{align*}
\]

Proof of Theorem 4.1

Fix \( s \in \mathbb{R} \) and let \( q_1 = \frac{2N}{N - 2} \).

(i) By definition, we have \( \Omega(t, s)v = z(t) \) where \( z(t) \) satisfies equation (4.3). We are going to prove the estimate by showing that the nonlinear equation (4.3) can be solved via contraction principle argument in an appropriate functional space. To this extent let us consider the functional space

\[ X_1 := \{ u \in C(\mathbb{R}, L^2_{-\sigma}(\mathbb{R}^N)) : \sup_{t \in \mathbb{R}}(1 + |t - s|)^{\frac{N}{2}} \| u(t) \|_{L^2_\sigma} < \infty \} \]

endowed with the norm

\[ \| u \|_{X_1} := \sup_{t \in \mathbb{R}}(1 + |t - s|)^{\frac{N}{2}} \| u(t) \|_{L^2_\sigma} < \infty \]

Note that the inhomogeneous term in (4.3) \( z_0 = e^{-iH(t-s)}P_v v \) satisfies \( z_0 \in X_1 \) and

\[ \| z_0 \|_{X_1} \leq C_N \| v \|_{L^2_\sigma} \]

because of (4.3). We collect the \( z \) dependent part of the right hand side of (4.3) in a linear operator

\[ L(s) : X_1 \to X_1 \]

\[ [L(s)z](t) = - \int_s^t e^{-iH(t-\tau)} P_v (iDg|_{\psi_E(\tau)} [z(\tau)] - Dh|_{\alpha(\tau)} i\langle \psi_0, Dg|_{\psi_E(\tau)} [z(\tau)] \rangle) d\tau \]

We will show that \( L \) is a well defined bounded operator from \( X_1 \) to \( X_1 \) whose operator norm can be made less or equal to \( 1/2 \) by choosing \( \epsilon_1 \) sufficiently small. Consequently \( Id - L \) is invertible and the solution of the equation (4.3) can be written as \( z = (Id - L)^{-1}z_0 \). In particular

\[ \| z \|_{X_1} \leq (1 - \| L \|)^{-1} \| z_0 \|_{X_1} \leq 2 \| z_0 \|_{X_1} \]

which in combination with the definition of \( \Omega \), the definition of the norm \( X_1 \) and the estimate (4.4), finishes the proof of (i).

By computing the \( L^2_\sigma \) norm of both the left hand side and right hand side of (4.5), for \( t > s \) we have:

\[
\begin{align*}
\| [L(s)z](t) \|_{L^2_\sigma} & \leq \int_s^t \| e^{-iH(t-\tau)} P_v \|_{L^2_{t-\tau} \to L^2_\sigma} \left[ \| Dg|_{\psi_E} [z] \|_{L^2_\sigma} + \| Dh|_{\alpha(\tau)} \|_{C \to L^2} \| \psi_0, Dg|_{\psi_E} [z] \| \right] d\tau \\
& \leq \int_s^t \| e^{-iH(t-\tau)} P_v \|_{L^2_{t-\tau} \to L^2_\sigma} \left[ \| Dg|_{\psi_E} [z] \|_{L^2_\sigma} + \| Dh|_{\alpha(\tau)} \|_{C \to L^2} \| \psi_0 \|_{L^2} \| Dg|_{\psi_E} [z] \|_{L^2} \right] d\tau
\end{align*}
\]
On the other hand, from (2.4), (2.5), and (1.3) we obtain:

\[ \|Dg|_{\psi E}[z]\|_{L^2_{\sigma}} \leq \|\langle x \rangle^{\alpha_1} (|\psi E|^{\alpha_1} + |\psi E|^{\alpha_2})\|_{L^\infty} \|z\|_{L^2_{\sigma}} \leq \varepsilon_1 \|z\|_{L^2_{\sigma}} \] \quad (4.6)

\[ \|Dg|_{\psi E}[z]\|_{L^2} \leq \|\langle x \rangle^{\alpha} (|\psi E|^{\alpha_1} + |\psi E|^{\alpha_2})\|_{L^\infty} \|z\|_{L^2_{\sigma}} \leq \varepsilon_1 \|z\|_{L^2_{\sigma}} \] \quad (4.7)

Also

\[ \|Dh|_{a(\tau)}\|_{C_{-L^2_{\sigma}}} \leq C_2, \quad \text{for all } \tau \in \mathbb{R} \]

which follow from \( h \) being \( C^1 \) on \( a \in C, |a| \leq \delta \), with values in \( L^2_{\sigma} \), see Proposition (2.4) and \( |a(\tau)| \leq \varepsilon_0 \leq \delta \) for all \( \tau \in \mathbb{R} \), see (3.2).

Using the last three relations, as well as the estimate (4.1) and the fact that \( t > s \) for all \( i \), we already know that (4.3) has a unique solution in \( L^2_{\sigma} \). Indeed, using (4.2) and (4.9) we obtain that

\[ \sup_{t > s} \|L(s)z(t)\|_{L^2_{\sigma}} \leq \varepsilon_1 \sup_{t > s} \|1 + |t - s|\|z\|_{L^2_{\sigma}} \]

\[ \leq \varepsilon_1 \sup_{t > s} \|1 + |t - s|\|z\|_{X_1} \]

\[ \leq C \varepsilon_1 \|z\|_{X_1} \]

Similar arguments lead to \( \sup_{t < s} \|1 + |t - s|\|z\|_{L^2_{\sigma}} \leq C \varepsilon_1 \|z\|_{X_1} \), hence \( \|L(s)\|_{X_1 \rightarrow X_1} \leq C \varepsilon_1 \). Now choosing \( \varepsilon_1 \) small enough we get that \( L(s) \) is a contraction operator on the Banach space \( X_1 \), therefore:

\[ \|\Omega(t, s)\|_{L^2_{\sigma} \rightarrow L^2_{\sigma}} \leq \frac{C}{(1 + |t - s|)^{\varepsilon_1}} \]

\[(ii) \text{ From part } (i) \text{ we already know that (4.3) has a unique solution in } L^2_{\sigma}. \]

We are going to show that the right had side of (4.3) in \( L^p \). Indeed, using (4.2) and \( L^2_{\sigma} \hookrightarrow L^p \) we have:

\[ \|e^{-iH(t-s)}P_c v\|_{L^p} \leq \frac{C_p}{|t - s|^{N(\frac{1}{2} - \frac{1}{p})}} \|v\|_{L^2_{\sigma}}. \] \quad (4.8)

The remaining term satisfies for \( t > s \):

\[ \|L(s)z(t)\|_{L^p} \leq \int_s^t \|e^{-iH(t-\tau)}P_c\|_{L^p - L^p} \left[ \|Dg|_{\psi E}[z]\|_{L^p} + \|Dh\|_{C_{-L^p}} \|\psi_0, Dg|_{\psi E}[z]\| \right] d\tau \]

\[ \leq \int_s^t \frac{C}{|t - \tau|^{N(\frac{1}{2} - \frac{1}{p})}} \left[ \|\langle x \rangle^{\alpha} \psi E\|_{L^p} + \|Dh\|_{L^p} \|\langle x \rangle^{\alpha} \psi E\|_{L^\infty} \right] \|z(\tau)\|_{L^2_{\sigma}} d\tau \]

\[ \leq \int_s^t \frac{C}{|t - \tau|^{N(\frac{1}{2} - \frac{1}{p})}} \|v\|_{L^2_{\sigma}} d\tau \]

\[ \leq \frac{C}{|t - s|^{N(\frac{1}{2} - \frac{1}{p})}} \|v\|_{L^2_{\sigma}}, \quad \text{for all } 2 \leq p < \frac{2N}{N - 2} \]

and same estimate can be obtained for \( t < s \). Plugging (4.8) and (4.9) into (4.3) we get:

\[ \|z(t)\|_{L^p} \leq \frac{C}{|t - s|^{N(\frac{1}{2} - \frac{1}{p})}} \|v\|_{L^2_{\sigma}}, \quad \text{for all } t \in \mathbb{R} \]

which by the definition \( \Omega(t, s)v = z(t) \) finishes the proof of part (ii).

\[(iii) \text{ Denote: } T(t, s)v = W(t) \] \quad (4.10)
then, by plugging in (4.3), $W(t)$ satisfies the following equation:

$$W(t) = -i \int_s^t e^{-iH(t-\tau)} P\bar{c} Dg\langle e^{-iH(\tau-s)} P\bar{c} v \rangle d\tau$$

$$+ \int_s^t e^{-iH(t-\tau)} P\bar{c} Dh|_{\alpha(t)} i\langle \psi_0, Dg|_{\psi_E} e^{-iH(\tau-s)} P\bar{c} v \rangle d\tau$$

$$+ [L(s)W](t)$$

Consequently

$$\langle x \rangle^{-\sigma} W(t) = - \int_s^t \langle x \rangle^{-\sigma} e^{-iH(t-\tau)} P\bar{c} \left( iDg|_{\psi_E} e^{-iH(\tau-s)} P\bar{c} v \right) - Dh|_{\alpha(t)} i\langle \psi_0, Dg|_{\psi_E} e^{-iH(\tau-s)} P\bar{c} v \rangle) d\tau$$

$$- \int_s^t \langle x \rangle^{-\sigma} e^{-iH(t-\tau)} P\bar{c} \left( iDg|_{\psi_E} W(\tau) - Dh|_{\alpha(t)} i\langle \psi_0, Dg|_{\psi_E} W(\tau) \rangle \right) d\tau.$$ 

Then, by (4.1):

$$\| \langle x \rangle^{-\sigma} W(t) \|_{L^2_0 L^2_0} \leq \left\| \int_s^t \frac{C}{(1 + |t - \tau|)^{\frac{N}{2}}} (1 + \| Dh \|_{L^2_0} \| \psi_0 \|_{L^2_0}) \| \langle x \rangle^{\sigma} Dg|_{\psi_E} e^{-iH(\tau-s)} P\bar{c} v \|_{L^2_0} d\tau \right\|_{L^2_0}$$

$$+ \left\| \int_s^t \frac{C}{(1 + |t - \tau|)^{\frac{N}{2}}} (1 + \| Dh \|_{L^2_0} \| \psi_0 \|_{L^2_0}) (\| \langle x \rangle^{2\sigma} g_u \|_{L^\infty} + \| \langle x \rangle^{2\sigma} g_u \|_{L^\infty}) \| \langle x \rangle^{-\sigma} W(\tau) \|_{L^2_0} d\tau \right\|_{L^2_0}$$

$$\leq \varepsilon_1^{\alpha_1} C \| K \|_{\mathcal{L}^1} \| v \|_{L^2_0} + \varepsilon_1^{\alpha_1} C \| K \|_{\mathcal{L}^1} \| \langle x \rangle^{-\sigma} W \|_{L^2_0 L^2_0}$$

where we used Young inequality $\| K * f \|_{L^2} \leq \| K \|_{\mathcal{L}^1} \| f \|_{L^2}$, with $K(t) = (1 + |t|)^{-\frac{N}{2}} \in L^1_{\mathbb{R}}$, while for the term

$$\langle x \rangle^{\sigma} Dg|_{\psi_E} e^{-iH(\tau-s)} P\bar{c} v = \langle x \rangle^{\sigma} g_u \langle x \rangle^{-\sigma} e^{-iH(\tau-s)} P\bar{c} v + \langle x \rangle^{\sigma} g_u \langle x \rangle^{\sigma} e^{-iH(\tau-s)} P\bar{c} v$$

we used $\| \langle x \rangle^{2\sigma} g_u \|_{L^\infty}$ and $\| \langle x \rangle^{2\sigma} g_u \|_{L^\infty}$ is uniformly bounded in $t$ by $\varepsilon_1^{\alpha_1}$ since $|g_u| = |g_0| \leq C(\| \psi_E \|_{\mathcal{A}} + \| \psi_E \|_{\mathcal{A}})$ and the Kato smoothing estimate $\| \langle x \rangle^{-\sigma} e^{-iH(\tau-s)} P\bar{c} v \|_{L^2(\mathbb{R}, L^2_0)} \leq C \| v \|_{L^2_0}$. Choosing $\varepsilon_1 < 1/(C \| K \|_{\mathcal{L}^1})$ we get $\| \langle x \rangle^{-\sigma} W \|_{L^2_0 L^2_0} < \infty$. In other words $T(t, s) \in L^2(\mathbb{R}, L^2 \to L^2_0).$

Similarly, using now (4.2) with $p = 2$ and $u_0 = v$, we obtain for $t > s$:

$$\| \langle x \rangle^{\sigma} W(t) \|_{L^2_0} \leq \int_s^t \frac{C}{(1 + |t - \tau|)^{\frac{N}{2}}} (1 + \| Dh \|_{L^2_0} \| \psi_0 \|_{L^2_0}) \| \langle x \rangle^{\sigma} Dg|_{\psi_E} e^{-iH(\tau-s)} P\bar{c} v \|_{L^2_0} d\tau$$

$$+ \int_s^t \frac{C}{(1 + |t - \tau|)^{\frac{N}{2}}} (1 + \| Dh \|_{L^2_0} \| \psi_0 \|_{L^2_0}) (\| \langle x \rangle^{2\sigma} g_u \|_{L^\infty} + \| \langle x \rangle^{2\sigma} g_u \|_{L^\infty}) \| \langle x \rangle^{-\sigma} W(\tau) \|_{L^2_0} d\tau$$

$$\leq \varepsilon_1^{\alpha_1} C \| v \|_{L^2} + \varepsilon_1^{\alpha_1} C \sup_{\tau \in \mathbb{R}} \| \langle x \rangle^{-\sigma} W(\tau) \|_{L^2_0}$$

Same argument works for $t < s$. Then passing to supremum over $t \in \mathbb{R}$ on the left hand side we get the required estimate provided $\varepsilon_1$ is small enough.

(iv) By definition of $T(t, s)$ (4.10) and the similarity between $t > s$ and $t < s$ estimates it is sufficient to prove that the solution of (4.11) satisfies

$$\| W(t) \|_{L^2_0} \leq \begin{cases} C \| v \|_{L^\infty_0} \frac{C}{[t-s]^{\frac{N}{2} - \frac{N}{2} - 1}} & \text{for } s \leq t \leq s + 1 \\ C \| v \|_{L^\infty_0} \frac{C}{(1 + |t-s|)^{\frac{N}{2}}} & \text{for } t > s + 1 \\ \end{cases}$$
The estimates for $2 \leq p \leq q_2$ are then obtained by Riesz-Thorin interpolation. Let us also observe that it suffices to obtain estimates only for the forcing terms in (4.11):

$$f(t) = -i \int_s^t e^{-iH(t-\tau)} P_c Dg|\psi_E[e^{-iH(\tau-s)} P_c v]d\tau$$  \hspace{1cm} (4.12)

$$\hat{f}(t) = \int_s^t e^{-iH(t-\tau)} P_c Dh|_{a(\tau)}i\langle \psi_0, Dg|e^{-iH(\tau-s)} P_c v]\rangle d\tau$$  \hspace{1cm} (4.13)

because then we will be able to do the contraction principle in the functional space in which $f(t), \hat{f}(t)$ are, and thus obtain the same decay for $W$ as for $f(t)$ and $\hat{f}(t)$. This time we will consider the functional spaces (recall that $s \in \mathbb{R}$ is a fixed number)

$$X_1 = \left\{ u \in C([s-1, s+1], L^2_{-\sigma}(\mathbb{R}^N)) : \sup_{|t-s|\leq 1} \|u(t)\|_{L^2_{-\sigma}} < \infty \right\}$$

$$X_2 = \left\{ u \in C(\mathbb{R}, L^2_{-\sigma}(\mathbb{R}^N)) : \sup_{|t-s|>1} (1 + |t-s|)^N(\frac{1}{2} - \frac{3}{2q})\|u(t)\|_{L^2_{-\sigma}} < \infty \right\}$$

endowed with the norms

$$\|u\|_{X_1} = \sup_{|t-s|\leq 1} \|u(t)\|_{L^2_{-\sigma}}$$

$$\|u\|_{X_2} = \max \left\{ \sup_{|t-s|\leq 1} |t-s|^{N(\frac{1}{2} - \frac{3}{2q})-1}\|u(t)\|_{L^2_{-\sigma}}, \sup_{|t-s|>1} (1 + |t-s|)^N(\frac{1}{2} - \frac{3}{2q})\|u(t)\|_{L^2_{-\sigma}} \right\}$$

First we will investigate the short time behavior of the forcing terms. If $s < t \leq s + 1$:

$$\|f(t)\|_{L^2_{-\sigma}} \leq \|\langle x \rangle^{-\sigma} \int_s^t e^{-iH(t-\tau)} P_c Dg|\psi_E[e^{-iH(\tau-s)} P_c v]d\tau\|_{L^2_{-\sigma}}$$

$$\leq \|\langle x \rangle^{-\sigma}\|_{L^0} \int_s^t \|e^{-iH(t-\tau)} P_c g_u e^{-iH(\tau-s)} P_c v\|_{L^2_{-\sigma}} d\tau$$

$$+ \|\langle x \rangle^{-\sigma}\|_{L^0} \int_s^{s + \frac{t-s}{2}} \|e^{-iH(t+s-2\tau)} e^{-iH(\tau-s)} P_c g_u e^{iH(\tau-s)} P_c v\|_{L^2_{-\sigma}} d\tau$$

$$+ \int_s^{s + \frac{t-s}{2}} \|e^{-iH(t-\tau)} P_c g_u L^2_{-\sigma} \langle x \rangle^{-\sigma} g_u e^{-iH(\tau-s)} P_c v\|_{L^2_{-\sigma}} d\tau$$

$$\leq \int_s^t \frac{C}{|t-s|^{N(\frac{1}{2} - \frac{3}{2q})}} \sup_{\tau \in [s,t]} \|\hat{g}_u(\tau)\|_{L^1_{-\sigma}} \|v\|_{L^{q_2} \sigma} d\tau$$

$$+ \int_s^{s + \frac{t-s}{2}} \frac{C}{|t+s-2\tau|^{N(\frac{1}{2} - \frac{3}{2q})}} \sup_{\tau \in [s,t]} \|\hat{g}_u(\tau)\|_{L^1_{-\sigma}} \|v\|_{L^{q_2} \sigma} d\tau$$

$$+ \int_s^{t + \frac{t-s}{2}} (1 + |t-\tau|)^N(\frac{1}{2} - \frac{3}{2q}) \|\langle x \rangle^{-\sigma} g_u \|_{L^0} \|e^{-iH(\tau-s)} P_c v\|_{L^q_{-\sigma}} d\tau$$

$$\leq \frac{C \sup_{\tau \in [s,t]} (\|\hat{g}_u(\tau)\|_{L^1_{-\sigma}} + \|\hat{g}_u(\tau)\|_{L^0} + \|\langle x \rangle^{-\sigma} g_u(\tau)\|_{L^q_{-\sigma}})}{|t-s|^{N(\frac{1}{2} - \frac{3}{2q})-1}}$$

where we used the Fourier multiplier type estimates:

$$\|e^{-iH(\tau-s)} F(x) e^{iH(\tau-s)}\|_{L^p(\mathbb{R}^N)} \leq C \|\hat{F}\|_{L^1(\mathbb{R}^N)}, \text{ for all } |\tau-s| \leq 1, \text{ and } 1 \leq p \leq \infty$$  \hspace{1cm} (4.14)
where $\hat{F}$ is the Fourier transform of $F$ with respect to the variable $x \in \mathbb{R}^N$ and $C > 0$ is a fixed constant see [7, Theorem 5.2]. Similarly we obtain $\|\hat{f}\|_{L_{t,s}^2 \sigma} \leq C\|v\|_{L_{t,s}^q}$ for $q_1 = \frac{2N}{N-2}$ and $|t-s| \leq 1$. For the second term we have:

$$
\|\hat{f}(t)\|_{L_{t,s}^2 \sigma} \leq \int_s^t \|e^{-iH(t-\tau)}P_c v\|_{L_{t,s}^2 \sigma} \|DH|_{\sigma}(\tau)\|L_{t,s}^2\|\langle \psi_0, Dg|_{\psi E} [e^{-iH(t-\tau)}P_c v]\rangle| \\
\leq \int_s^t \frac{C}{(1+|t-\tau|)^\frac{N}{2}} \|DH|_{\sigma}(\tau)\|L_{t,s}^2\|\langle \psi_0, g_u e^{-iH(t-\tau)}P_c v\rangle| + |\langle \psi_0, g_u e^{-iH(t-\tau)}P_c u\rangle| \\
\leq \int_s^t \frac{C\|DH\|}{(1+|t-\tau|)^\frac{N}{2}} \sup_{\tau \in [s,t]} (\|g_u(\tau)\|_{L_{t,s}^1} + \|\hat{g_u}(\tau)\|_{L_{t,s}^1}) \|\psi_0\|_{L_{t,s}^q} \|v\|_{L_{t,s}^q} \leq \frac{C\|v\|_{L_{t,s}^q}}{(1+|t-s|)^\frac{2}{N-2}}.
$$

where we also used $\psi_0 \in H^2 \hookrightarrow L^{q_2}$. Similarly $\|\hat{f}\|_{L_{t,s}^2 \sigma} \leq \frac{C\|v\|_{L_{t,s}^q}}{(1+|t-s|)^\frac{2}{N-2}} \leq C\|v\|_{L_{t,s}^q}$ for $q_1 = \frac{2N}{N-2}$.

For $t > s + 1$ we will split these two integral in two parts to be estimated differently:

$$
f(t) = \int_s^{s+\frac{1}{2}} \underbrace{\int_s^t}_{I_1} + \int_{s+\frac{1}{2}}^t \underbrace{\int_{s+\frac{1}{2}}^t}_{I_2} -ie^{-iH(t-\tau)}P_c v d\tau \|DH|_{\sigma}(\tau)\|L_{t,s}^2 \langle \psi_0, Dg|_{\psi E} [e^{-iH(t-\tau)}P_c v]\rangle d\tau
$$

and

$$
\hat{f}(t) = \int_s^{s+\frac{1}{2}} \underbrace{\int_s^t}_{I_1} + \int_{s+\frac{1}{2}}^t \underbrace{\int_{s+\frac{1}{2}}^t}_{I_2} e^{-iH(t-\tau)}P_c DH|_{\sigma}(\tau)\|\psi_0, Dg|_{\psi E} [e^{-iH(t-\tau)}P_c v]\rangle d\tau.
$$
Then we have:

\[
\|I_1\|_{L^2_\sigma} \leq \|\langle x \rangle^{-\sigma} \int_s^{s+\frac{1}{2}} -ie^{-iH(t-s)} P_c DG|\psi_E[e^{-iH(t-s)} P_c v]|d\tau\|_{L^2}
\]

\[
\leq \|\langle x \rangle^{-\sigma}\|_{L^\beta} \int_s^{s+\frac{1}{2}} \|e^{-iH(t-s)} e^{iH(t-s)} P_c g_0 e^{-iH(t-s)} P_c v\|_{L^{q_2}} d\tau
\]

\[
+ \|\langle x \rangle^{-\sigma}\|_{L^\beta} \int_s^{s+\frac{1}{2}} \|e^{-iH(t+s+2\tau)} e^{-iH(t-s)} P_c g_0 e^{iH(t-s)} P_c v\|_{L^{q_2}} d\tau
\]

\[
\leq \|\langle x \rangle^{-\sigma}\|_{L^\beta} \int_s^{s+\frac{1}{2}} \left|\langle \overline{\psi}_s \rangle \right|^{\frac{1}{2}} \|e^{iH(t-s)} P_c g_0 e^{-iH(t-s)} P_c v\|_{L^{q_2}} d\tau
\]

\[
+ \|\langle x \rangle^{-\sigma}\|_{L^\beta} \int_s^{s+\frac{1}{2}} \left|\langle \overline{\psi}_s \rangle \right|^{\frac{1}{2}} \|e^{-iH(t-s)} P_c g_0 e^{iH(t-s)} P_c v\|_{L^{q_2}} d\tau
\]

\[
\leq C(q_2) \|\langle x \rangle^{-\sigma}\|_{L^\beta} \left( \int_s^{s+\frac{1}{2}} \left( \|\overline{\psi}_s \|_{L^1} + \|\overline{\psi}_s \|_{L^1} \right) \|\overline{\psi}_s \|_{L^{q_2}} d\tau \right)
\]

\[
\leq C(q_2) \|\langle x \rangle^{-\sigma}\|_{L^\beta} \left( \int_s^{s+\frac{1}{2}} \left( \|\overline{\psi}_s \|_{L^1} + \|\overline{\psi}_s \|_{L^1} \right) \|\overline{\psi}_s \|_{L^{q_2}} d\tau \right)
\]

\[
\leq C(q_2) \|\langle x \rangle^{-\sigma}\|_{L^\beta} \left( \int_s^{s+\frac{1}{2}} \left( \|\overline{\psi}_s \|_{L^1} + \|\overline{\psi}_s \|_{L^1} \right) \|\overline{\psi}_s \|_{L^{q_2}} d\tau \right)
\]

For the second integral we have

\[
\|I_2\|_{L^2_\sigma} \leq \|\langle x \rangle^{-\sigma} \int_s^{s+\frac{1}{2}} e^{-iH(t-\tau)} P_c Dg|\psi_E[e^{-iH(t-s)} P_c v]|d\tau\|_{L^2_\sigma}
\]

\[
\leq \int_s^{s+\frac{1}{2}} \left|\langle \overline{\psi}_s \rangle \right|^{\frac{1}{2}} \|e^{-iH(t-\tau)} P_c g_0 e^{-iH(t-s)} P_c v\|_{L^{q_2}} d\tau
\]

\[
\leq C(q_2) \|\langle x \rangle^{-\sigma}\|_{L^\beta} \int_s^{s+\frac{1}{2}} \left( \|\overline{\psi}_s \|_{L^1} + \|\overline{\psi}_s \|_{L^1} \right) \|\overline{\psi}_s \|_{L^{q_2}} d\tau
\]

For the second forcing term \( \tilde{f}(t) \), we use again \( \psi_0 \in H^2 \hookrightarrow L^{q_2} \):

\[
\|II_1\|_{L^2_\sigma} \leq \int_s^{s+\frac{1}{2}} \left( \|e^{\pm iH(t-s)} P_c v\|_{L^{q_2}} \right) d\tau
\]

\[
\leq \left( \|e^{\pm iH(t-s)} P_c v\|_{L^{q_2}} \right) \int_s^{s+\frac{1}{2}} \left( \|\overline{\psi}_s \|_{L^1} + \|\overline{\psi}_s \|_{L^1} \right) \|\overline{\psi}_s \|_{L^{q_2}} d\tau
\]

\[
\leq C \|\langle x \rangle^{-\sigma}\|_{L^\beta} \left( \int_s^{s+\frac{1}{2}} \left( \|\overline{\psi}_s \|_{L^1} + \|\overline{\psi}_s \|_{L^1} \right) \|\overline{\psi}_s \|_{L^{q_2}} d\tau \right)
\]

\[
II_2 \text{ is estimated exactly the same way as } I_2. \text{ Let us observe that the above estimates are for the case } t > s + 1. \text{ Because of that we can replace the } C/|t-s| \text{ term by } C/(1+|t-s|) \text{ in the } I_1, I_2 \text{ and } II_2 \text{ integrals. The estimates for } s - 1 \leq t \leq s \text{ respectively } t < s - 1 \text{ are obtained in the same way as the ones for } s \leq t \leq s + 1 \text{ respectively } t > s + 1.\]
Theorem 4.1 is now completely proven. □

The next step is to obtain estimates for \( \Omega(t,s) \) and \( T(t,s) \) in unweighted \( L^p \) spaces.

**Theorem 4.2** Fix \( \sigma > N/2 \) and \( \frac{2N}{N-2} < q_2 < \frac{2N}{N-4} \). Assume that \( \| x > 4 \psi_{E(t)} \|_{L^\infty} < \varepsilon_1(q_2) \), for all \( t \in \mathbb{R} \) (where \( \varepsilon_1(q_2) \) is the one used in Theorem 4.1). Then there exist constants \( C_2, C'_2 \) and \( C_p \) such that for all \( t, s \in \mathbb{R} \) the following estimates hold:

\[
\begin{align*}
(i) \quad & \| \Omega(t,s) \|_{L^2 \to L^2} \leq C_2, \quad \| T(t,s) \|_{L^2 \to L^2} \leq C_2 \\
(ii) \quad & \| T(t,s) \|_{L^p \to L^p} \leq \begin{cases} 
C_p \left( \frac{\varepsilon}{|t-s|^{N(p-1)/2}} \right) & \text{for } |t-s| \leq 1 \\
C_p \left( \frac{\varepsilon}{|t-s|^{N(p-1)/2}} \right) & \text{for } |t-s| > 1 \\
\end{cases}, \quad \text{for } 2 \leq p \leq q_2 < \frac{2N}{N-4} \\
(iii) \quad & \| \Omega(t,s) \|_{L^p \to L^p} \leq \frac{C_p}{|t-s|^{N(p-2)/2}}, \quad \text{for } 2 \leq p \leq \frac{2N}{N-2} \\

\end{align*}
\]

where

\[
\tilde{T}(t,s) = \begin{cases} 
- \int_s^{\min(t,s+1)} e^{-iH(t-\tau)} P_c g_a e^{iH(\tau-s)} P_c d\tau, & \text{if } t \geq s, \\
- \int_s^{\max(t,s-1)} e^{-iH(t-\tau)} P_c g_a e^{iH(\tau-s)} P_c d\tau, & \text{if } t < s.
\end{cases}
\]

**Proof of Theorem 4.2** Fix \( s \in \mathbb{R} \). Because of the estimate (4.2) and relation \( \Omega(t,s) = T(t,s) + e^{-iH(t-s)} P_c \), it suffices to prove the theorem for \( T(t,s) \). Throughout this proof we will repeatedly use the equations defining \( T(t,s) \) (4.10) - (4.11) where the linear operator \( L(s) \) is given by (4.5). Note that we have already denoted the remaining forcing terms in (4.11) by \( f \) respectively \( \tilde{f} \) see (4.12) and (4.13).

(i) To estimate the \( L^2 \) norm we will use the following duality argument:

\[
\| f(t) \|_{L^2}^2 = \langle f(t), f(t) \rangle = \int_s^t \int_s^t \langle e^{-iH(t-\tau)} P_c g_a e^{iH(t-s)} P_c v, e^{-iH(t-\tau')} P_c Dg|\psi_E[e^{-iH(\tau')} P_c v] \rangle d\tau' d\tau \\
= \int_s^t \int_s^t \langle Dg|\psi_E[e^{-iH(t-\tau')} P_c v], e^{-iH(\tau')} P_c Dg|\psi_E[e^{-iH(t-s)} P_c v] \rangle d\tau' d\tau \\
= \int_s^t \int_s^t \langle \langle x \rangle^\sigma Dg|\psi_E[e^{-iH(t-\tau')} P_c v], \langle x \rangle^\sigma e^{-iH(\tau')} P_c Dg|\psi_E[e^{-iH(t-s)} P_c v] \rangle d\tau' d\tau \\
\leq \int_s^t \int_s^t \| Dg|\psi_E[e^{-iH(t-\tau')} P_c v] \|_{L^2} \| e^{-iH(\tau')} P_c Dg|\psi_E[e^{-iH(t-s)} P_c v] \|_{L^2} d\tau' d\tau \\
\leq \int_s^t \int_s^t \| Dg|\psi_E[e^{-iH(t-s)} P_c v] \|_{L^2} \| e^{-iH(t-s)} P_c Dg|\psi_E[e^{-iH(t-s)} P_c v] \|_{L^2} d\tau' d\tau \\
\leq C \| \langle x \rangle^\sigma Dg|\psi_E[e^{-iH(t-s)} P_c v] \|_{L^2} \| e^{-iH(t-s)} P_c Dg|\psi_E[e^{-iH(t-s)} P_c v] \|_{L^2} d\tau' d\tau \\
\leq \| K \|_{L^1} \| \langle x \rangle^\sigma Dg|\psi_E[e^{-iH(t-s)} P_c v] \|_{L^2} \| e^{-iH(t-s)} P_c Dg|\psi_E[e^{-iH(t-s)} P_c v] \|_{L^2} \leq C \| v \|_{L^2} \leq \| \Omega(t,s) \|_{L^2} \leq C \| v \|_{L^2} < \infty
\]

At the last line we used Young inequality \( \| K f \|_{L^2} \leq \langle K, f \rangle_{L^1} \| f \|_{L^2} \) with \( K(t) = (1+|t|)^{-\frac{\sigma}{2}} \in L^1 \) and for the term \( \langle x \rangle^\sigma Dg|\psi_E[e^{-iH(t-s)} P_c v] = \langle x \rangle^\sigma (g_a e^{-iH t} P_c v + g_a e^{iH t} P_c v) \) we used the Kato smoothing estimate \( \| (x)^{-\sigma} e^{-iH t} P_c v \|_{L^2(\mathbb{R}, L^2)} \leq C \| v \|_{L^2} \), together with the uniform bounds in time \( \| (x)^{2\sigma} g_a(\tau') \|_{L^\infty}, \| (x)^{2\sigma} g_a(\tau') \|_{L^\infty} \leq \)
\[ C_{\xi}^{\alpha_1} \text{ since } |g_{\xi}(\tau')|, \ |g_{\xi}(\tau')| \leq C(|\psi_{E(\tau')}|^{\alpha_1} + |\psi_{E(\tau')}|^{\alpha_2}). \text{ Similarly we have,} \]

\[ \| \hat{f} \|_{L^2}^2 = \int_s^t \int_s^t \langle e^{-iH(t-\tau')} P_c \Delta \| \psi_0, Dg|_{\psi_E}|e^{-iH(s-\tau')} P_c v) \rangle dt' d\tau \]

\[ \leq \int_s^t \int_s^t \| Dh\|_{L^2} \| \psi_0 \|_{L^2} \| Dg|_{\psi_E}|e^{-iH(s-\tau')} P_c v) \|_{L^2} \]

\[ \times \frac{C}{(1 + |s - \tau'|)^{N/2}} \| Dh\|_{L^2} \| \psi_0 \|_{L^2} \| Dg|_{\psi_E}|e^{-iH(s-\tau')} P_c v) \|_{L^2} \]

\[ \leq C\| v \|_{L^2}^2 < \infty \]

We will estimate \( L^2 \) norm of \( L(s) \) see \( \text{[15]} \) in the same way as for \( f \):

\[ \| L(s)W \|_{L^2}^2 = \langle L(s)W, L(s)W \rangle \]

\[ = \int_s^t \int_s^t \left( \langle Dg|_{\psi_E}|W(\tau) \rangle - Dh\| |\psi_0, Dg|_{\psi_E}|W(\tau) \rangle, e^{-iH(s-\tau')} P_c (Dg|_{\psi_E}|W(\tau')) - Dh\| |\psi_0, Dg|_{\psi_E}|W(\tau')) \right) dt' d\tau \]

\[ \leq \int_s^t \int_s^t (\| x \|^2 g_u \|_{L^2} + \| x \|^2 g_u \|_{L^2}) (1 + \| Dh\|_{L^2} \| \psi_0 \|_{L^2}) \| x \|^{-2} W \|_{L^2} \]

\[ \times \int_s^t CK(\tau' - \tau^{'}) (\| x \|^2 g_u \|_{L^2} + \| x \|^2 g_u \|_{L^2}) (1 + \| Dh\|_{L^2} \| \psi_0 \|_{L^2}) \| x \|^{-2} W \|_{L^2} \]

\[ \leq C\| K \|_{L^1} \| x \|^{-2} W \|_{L^2}^2 < \infty \]

By Theorem \( \text{[13]} (iii) \), \( \| x \|^{-2} W \|_{L^2} L^2 < \infty \).

Therefore we conclude \( \| T(t, s) \|_{L^2 \rightarrow L^2} \leq C \) and \( \| \Omega(t, s) \|_{L^2 \rightarrow L^2} \leq C \)

(iii) It suffices to prove the estimates for \( p = 2 \) and \( q_2 \). The estimates for \( 2 < p < q_2 \) will follow from Riesz-Thorin interpolation. We will also assume \( t > s \) since the case \( t < s \) can be treated similarly.

We start with short time estimates, \( s < t \leq s + 1 \) where the difficult part is to remove the non-integrable singularities of \( e^{-iHt} P_c \) at \( t = 0 \), see \( \text{[12]} \) for \( p > 2N/(N - 2) \), which appears in the convolution integrals in \( \text{[13]} \). For this purpose we will use the Fourier multiplier estimates \( \text{[14]} \). Let us first investigate the short time behavior of the terms \( f(t) \) and \( \hat{f}(t) \). In what follows \( p = 2 \) and
\[ p = q_2 > 2N/(N - 2) : \]

\[ \| f(t) \|_{L^p} = \| \int_s^t e^{-iH(t-\tau)} P_c Dg[\psi_e[e^{-iH(\tau-s)} P_c v]] d\tau \|_{L^p} \]

\[ \leq \int_s^t \| e^{-iH(t-s)} \|_{L^{p'} \to L^p} \| e^{iH(\tau-s)} P_c g_e e^{-iH(\tau-s)} P_c v \|_{L^{p'}} d\tau \]

\[ + \int_{s+}^{t-} \| e^{-iH(t+\tau-s-2\tau)} \|_{L^{p'} \to L^p} \| e^{iH(\tau-s)} P_c g_e e^{iH(\tau-s)} P_c \bar{v} \|_{L^{p'}} d\tau \]

\[ + \int_{t+}^{t-} \| e^{-iH(t+\tau-s-2\tau)} \|_{L^{p'} \to L^p} \| e^{-iH(\tau-s)} P_c g_e e^{iH(\tau-s)} P_c \bar{v} \|_{L^{p'}} d\tau \]

\[ \leq \int_s^t \frac{C_p}{|t-s|^{N(\frac{1}{2} - \frac{1}{p})}} C \| \hat{g}_u(\tau) \|_{L^1} \| v \|_{L^{p'}} d\tau + \int_{s+}^{t-} \frac{C_p}{|t-s-2\tau|^{N(\frac{1}{2} - \frac{1}{p})}} C \| \hat{g}_u(\tau) \|_{L^1} \| v \|_{L^{p'}} d\tau \]

\[ + \int_{t+}^{t-} \frac{C_p}{|t-\tau|^{N(\frac{1}{2} - \frac{1}{p})}} C \| \hat{g}_u(\tau) \|_{L^1} \| v \|_{L^{p'}} d\tau \]

\[ \leq \tilde{c} \max \{ \sup_{\tau \in \mathbb{R}} \| \hat{g}_u(\tau) \|_{L^{\infty}}, \sup_{\tau \in \mathbb{R}} \| \hat{g}_u(\tau) \|_{L^1}, \sup_{\tau \in \mathbb{R}} \| \hat{g}_u(\tau) \|_{L^1} \} \| v \|_{L^{p'}} \]

For \( \tilde{f} \) we use the Sobolev imbeddings \( \| F \|_{L^p} \leq C \| F \|_{H^2} \) for \( 2 \leq p \leq \frac{2N}{N-1} \) together with \( \| e^{-iH} F \|_{H^2} \leq C \| F \|_{H^2} \):

\[ \| \tilde{f}(t) \|_{L^p} \leq \int_s^t C \| Dh_{a(\tau)} \|_{C \to H^2} \| \psi_0, Dg[\psi_e[e^{-iH(\tau-s)} P_c v]] \| d\tau \]

\[ \leq \int_s^t \| Dh_{a(\tau)} \|_{C \to H^2} \| \langle e^{-iH(\tau-s)} \psi_0, e^{iH(\tau-s)} g_e e^{-iH(\tau-s)} P_c v \rangle \| d\tau \]

\[ + \| \langle e^{-iH(\tau-s)} \psi_0, e^{iH(\tau-s)} g_e e^{iH(\tau-s)} P_c \bar{v} \rangle \| d\tau \]

\[ \leq \int_s^t \| Dh_{a(\tau)} \|_{C \to H^2} \| e^{iH(\tau-s)} \psi_0 \|_{L^p} \| e^{iH(\tau-s)} g_e e^{-iH(\tau-s)} P_c v \|_{L^{p'}} d\tau \]

\[ + \int_s^t \| Dh_{a(\tau)} \|_{C \to H^2} \| e^{-iH(\tau-s)} \psi_0 \|_{L^p} \| e^{-iH(\tau-s)} g_e e^{iH(\tau-s)} P_c \bar{v} \|_{L^{p'}} d\tau \]

\[ \leq C \sup_{\tau \in \mathbb{R}} \| Dh_{a(\tau)} \|_{C \to H^2} \| \hat{g}_u(\tau) \|_{L^1} \| \hat{g}_u(\tau) \|_{L^1} \| \psi_0 \|_{H^2} \| t-s \| \| v \|_{L^{p'}} \leq \frac{\tilde{c} \| v \|_{L^{p'}}}{|t-s|^{N(\frac{1}{2} - \frac{1}{p})}} \]

We now move to the short time estimate: \( s < t \leq s + 1 \) of \( \| L(s)W \|_{L^p} \), \( p = 2 \), \( q_2 \), see (4.3) for the definition of the integral operator \( L(s) \). The main difference compared to the \( f \) and \( \tilde{f} \) terms is the fact that the singularity at \( \tau = s \) is integrable due to Theorem 4.1 part (iv):

\[ [L(s)W](t) = -i \int_s^t e^{-iH(t-\tau)} P_c Dg[\psi_\tau[W(\tau)]] d\tau + \int_s^t e^{-iH(t-\tau)} P_c Dh_{a(\tau)} i\langle \psi_0, Dg[\psi_e[W(\tau)]] \rangle d\tau \]
where
\[
\| \int_s^t e^{-iH(t-\tau)} P_e D\psi \| \| D(P_e W(\tau)) d\tau \|_{L^p} \leq \int_s^t C\| e^{-iH(t-\tau)} P_e \|_{H^2_{\tau}} \| D\psi \|_{C_{\tau}} \| D(P_e W(\tau)) \| d\tau.
\]
\[
\leq \int_s^t C\| D\psi \|_{C_{\tau}} \| e^{-iH(t-\tau)} P_e \|_{L^2_{\tau}} d\tau.
\]
\[
\leq \|\psi\|_{L^2} \sup_{\tau \in \mathbb{R}} \{\| D\psi \|_{C_{\tau}} \| e^{-iH(t-\tau)} P_e \|_{L^2_{\tau}} \} \int_s^t \frac{C \| \psi \|_{L^p}}{|\tau - s|^{N(1/2 - 1/2)}} d\tau \leq \frac{C \| \psi \|_{L^p}}{|t - s|^{N(1/2 - 1/2)}}
\]

However to remove the non-integrable singularity at \( \tau = t \) in the remaining integral we need to plug in (4.11) in it:
\[
\int_s^t e^{-iH(t-\tau)} P_e D\psi \| W(\tau) \| d\tau
\]
\[
= \int_s^t e^{-iH(t-\tau)} P_e \psi \left( \int_s^t e^{-iH(\tau')} P_e (-iD\psi [e^{-iH(\tau') - P_e}] + D\psi \| e^{-iH(\tau') - P_e}] + D\psi \| e^{-iH(\tau') - P_e}] \right) d\tau'
\]
\[
+ \int_s^t e^{-iH(\tau - \tau')} P_e (-iD\psi [e^{-iH(\tau') - P_e}] + D\psi \| e^{-iH(\tau') - P_e}] \right) d\tau
\]
\[
+ \int_s^t e^{-iH(\tau - \tau')} P_e (-iD\psi [e^{-iH(\tau') - P_e}] + D\psi \| e^{-iH(\tau') - P_e}] \right) d\tau'
\]
\[
+ \int_s^t e^{-iH(\tau - \tau')} P_e (-iD\psi [e^{-iH(\tau') - P_e}] + D\psi \| e^{-iH(\tau') - P_e}] \right) d\tau'
\]
All the terms will be either of the following forms
\[
\begin{align*}
L_1 &= \int_s^t e^{-iH(t-\tau)} P_e \psi \int_s^\tau e^{-iH(\tau')} P_e X(\tau') d\tau' d\tau \\
L_2 &= \int_s^t e^{-iH(t-\tau)} P_e \psi \int_s^\tau e^{-iH(\tau')} P_e \overline{X(\tau')} d\tau' d\tau \\
\tilde{L}_1 &= \int_s^t e^{-iH(t-\tau)} P_e \psi \int_s^\tau e^{-iH(\tau')} P_e D\psi \| e^{-iH(\tau') \psi X(\tau')} d\tau' d\tau \\
\tilde{L}_2 &= \int_s^t e^{-iH(t-\tau)} P_e \psi \int_s^\tau e^{-iH(\tau')} P_e D\psi \| e^{-iH(\tau') \psi X(\tau')} d\tau' d\tau
\end{align*}
\]
where \( X(\tau) \) can be either of \(-i\overline{\psi} e^{-iH(\tau') \psi X(\tau')} \psi \), \(-i\overline{\psi} e^{-iH(\tau') \psi X(\tau')} \psi \overline{\psi} \), \(-i\overline{\psi} W(\tau') \), or \(-i\overline{\psi} W(\tau') \). We can now remove the singularity of \( e^{-iH(t-\tau)} P_e \|_{L^p \rightarrow L^p} \) at \( \tau = t \) via (4.11):
• For $X(\tau') = -ig_\alpha e^{-iH(\tau'-s)}P_cv$ we have:

$$
\|L_1\|_{L^p} \leq \int_s^t \left\| e^{-iH(t-\tau')} P_c g_\alpha e^{iH(\tau')} \right\|_{L^p} \int_s^\tau \left\| e^{-iH(t-s)} P_c g_\alpha e^{iH(\tau'-s)} P_c v \right\|_{L^p} d\tau' d\tau
\leq \int_s^t \left\| \tilde{g}_u \right\|_{L^1} \int_s^\tau \frac{C}{|t-s|^{N(\frac{7}{4}-\frac{1}{p})}} \left\| \tilde{g}_u \right\|_{L^1} \left\| v \right\|_{L^p} d\tau' d\tau
\leq \int_s^t \frac{C}{|t-s|^{N(\frac{7}{4}-\frac{1}{p})}} \left\| \tilde{g}_u \right\|_{L^1} \left\| v \right\|_{L^p} d\tau' d\tau
\leq \int_s^t \frac{C}{|t-s|^{2N(\frac{7}{4}-\frac{1}{p})-2}} \left\| \tilde{g}_u \right\|_{L^1} \left\| v \right\|_{L^p} d\tau' d\tau
$$

$$
\|L_2\|_{L^p} \leq \int_s^t \left\| e^{-iH(t-\tau')} P_c \right\|_{L^p} \int_s^\tau \left\| e^{-iH(t-s)} e^{-iH(\tau'-s)} P_c g_\alpha e^{iH(\tau'-s)} P_c \tilde{v} \right\|_{L^p} d\tau' d\tau
\leq \int_s^t \frac{C}{|t-s|^{N(\frac{7}{4}-\frac{1}{p})}} \left\| \tilde{g}_u \right\|_{L^p} \int_s^\tau \frac{C}{|t-s|^{N(\frac{7}{4}-\frac{1}{p})}} \left\| \tilde{g}_u \right\|_{L^1} \left\| v \right\|_{L^p} d\tau' d\tau
\leq \int_s^t \frac{C}{|t-s|^{2N(\frac{7}{4}-\frac{1}{p})-2}} \left\| \tilde{g}_u \right\|_{L^1} \left\| v \right\|_{L^p} d\tau' d\tau
$$

$$
\|\tilde{L}_{1,2}\|_{L^p} \leq \int_s^t \left\| e^{-iH(t-\tau')} P_c g_\alpha e^{iH(\tau')} \right\|_{L^p} \int_s^\tau C \left\| D\bar{b} \right\|_{C-H^2} \left\| \psi_0, g_\alpha e^{-iH(\tau'-s)} P_c v \right\| d\tau' d\tau
\leq \int_s^t \left\| \tilde{g}_u(\tau) \right\|_{L^1} \int_s^\tau C \left\| e^{iH(\tau'-s)} \psi_0, e^{iH(\tau'-s)} g_\alpha e^{-iH(\tau'-s)} P_c v \right\| d\tau' d\tau
\leq \int_s^t \left\| \tilde{g}_u(\tau) \right\|_{L^1} \int_s^\tau C \left\| e^{iH(\tau'-s)} \psi_0 \right\|_{L^p} \left\| e^{iH(\tau'-s)} g_\alpha e^{-iH(\tau'-s)} P_c v \right\|_{L^p} d\tau' d\tau
\leq C |t-s|^2 \left\| v \right\|_{L^p}
$$

• For $X(\tau') = -ig_\alpha e^{iH(\tau'-s)}P_cv$ in $L_1$ we first change the order of integration then split and use (4.14):

$$
\|L_1\|_{L^p} \leq \int_s^{s+\frac{t-s}{2}} \int_s^{s+\frac{t-s}{2}} \left\| e^{-iH(t-\tau')} P_c g_\alpha e^{iH(\tau')} e^{-iH(t+s-2\tau')} P_c e^{-iH(\tau'-s)} P_c g_\alpha e^{iH(\tau'-s)} P_c v \right\|_{L^p} d\tau d\tau'
\leq \int_s^{s+\frac{t-s}{2}} \int_s^{s+\frac{t-s}{2}} \left\| \tilde{g}_u \right\|_{L^1} \frac{C}{|t+s-2\tau'|^{N(\frac{7}{4}-\frac{1}{p})}} \left\| \tilde{g}_u \right\|_{L^1} d\tau d\tau'
\leq \int_s^{s+\frac{t-s}{2}} \int_s^{s+\frac{t-s}{2}} \frac{C}{|t+s-2\tau'|^{N(\frac{7}{4}-\frac{1}{p})}} \left\| \tilde{g}_u \right\|_{L^1} \frac{C}{|t-s|^{N(\frac{7}{4}-\frac{1}{p})}} d\tau d\tau'
\leq \frac{C \left\| v \right\|_{L^p}}{|t-s|^{2N(\frac{7}{4}-\frac{1}{p})-2}}
$$

For $L_2$ we do not change the order of integration but we have to split both integrals to avoid
singularities:

\[
\|L_2\|_{L^p} \leq \int_{t}^{t-\frac{1}{T}} \|e^{-iH(t-\tau)}P_{c}||_{L^p}d\tau'
\]

\[
\times \left[ \|g_u\|_{L^p} \int_{t}^{t+\frac{1}{T}} \|e^{-iH(2\tau-\tau-s)}P_{c}g_u||_{L^p}d\tau' \right]
\]

\[
+ \int_{t}^{t-\frac{1}{T}} \|e^{-iH(t-\tau)}P_{c}g_u||_{L^p}d\tau'
\]

\[
\times \left[ \int_{t}^{t+\frac{1}{T}} \|e^{-iH(2\tau+2\tau'-s)}e^{iH(\tau'-s)}P_{c}g_u||_{L^p}d\tau' \right]
\]

\[
+ \int_{t}^{t-\frac{1}{T}} \|e^{-iH(t-2\tau+2\tau'-s)}e^{iH(\tau'-s)}P_{c}g_u||_{L^p}d\tau'
\]

\[
\leq \int_{t}^{t} \left[ \left( \frac{C}{|t-2\tau+2\tau'-s|^{N\left(\frac{4}{2}-\frac{1}{p}\right)}} \right) \|\hat{g}_u\|_{L^1} \|v\|_{L^p}d\tau'
\]

\[
+ \int_{t}^{t} \left( \frac{C}{|t-2\tau+\tau'|^{N\left(\frac{4}{2}-\frac{1}{p}\right)}} \|\hat{g}_u\|_{L^1} \|v\|_{L^p} \right)\|v\|_{L^p}d\tau'
\]

\[
+ \int_{t}^{t} \left( \frac{C}{|t-2\tau+2\tau'-s|^{N\left(\frac{4}{2}-\frac{1}{p}\right)}} \|\hat{g}_u\|_{L^1} \|v\|_{L^p} \right)\|v\|_{L^p}d\tau'
\]

\[
\leq \frac{C\|v\|_{L^p}}{|t-s|^{2N\left(\frac{4}{2}-\frac{1}{p}\right)-3}}
\]

\[\hat{L}_1 \text{ and } \hat{L}_2 \text{ are estimated as in the previous case.}\]

• For \(X(\tau') = -ig_uW(\tau')\) and \(-ig_uW(\tau')\) we will change the order of the integration and use Theorem 4.1 part (iv):

\[
\|L_1\|_{L^p} \leq \int_{s}^{t} \int_{\tau'}^{t} \|e^{-iH(t-\tau)}P_{c}g_u||_{L^p}d\tau\|e^{-iH(t-\tau')}P_{c}||_{L^p}d\tau'
\]

\[
\leq \int_{s}^{t} \left[ \int_{\tau'}^{t} \|\hat{g}_u(\tau)||_{L^1} \|\langle x \rangle^\sigma g_u(\tau')||_{L^p}d\tau' \right]\|W(\tau')||_{L^2}d\tau'
\]

\[
\leq \sup_{\tau' \in [s,t]} \|\hat{g}_u(\tau)||_{L^1} \sup_{\tau' \in [s,t]} \|\langle x \rangle^\sigma g_u(\tau')||_{L^p} \int_{s}^{t} \left( \frac{C}{|t-\tau'|^{N\left(\frac{4}{2}-\frac{1}{p}\right)-1}} \frac{C\|v\|_{L^p}}{|t-s|^{2N\left(\frac{4}{2}-\frac{1}{p}\right)-3}} \right) d\tau'
\]

\[
\leq \frac{C\|v\|_{L^p}}{|t-s|^{2N\left(\frac{4}{2}-\frac{1}{p}\right)-3}}
\]

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\[ \|L_2\|_{L^p} \leq \int_s^t \int_{t'}^{t''} ||e^{-iH(t-\tau)} P_{c'} ||_{L^{p'} \rightarrow L^p} \|g_{u} e^{-iH(t-\tau)} P_{c} g_{u} W(\tau') d\tau' \|_{L^{p'} d\tau d\tau'} \\
\quad + \int_s^t \int_{t'}^{t''} ||e^{-iH(t-\tau)} P_{c} g_{u} e^{iH(t-\tau')} ||_{L^{p'} \rightarrow L^p} ||e^{-iH(t+\tau'-2\tau')} P_{c} g_{u} W(\tau') ||_{L^{p'} d\tau d\tau'} \\
\quad \leq \int_s^t \int_{t'}^{t''} \frac{C}{|t-\tau|^{N\left(\frac{1}{q}-\frac{1}{p}\right)}} \|g_{u} ||_{L^{2\sigma}} ||e^{-iH(\tau-\tau')} ||_{L^{2} \rightarrow L^{2}} \|g_{u} W(\tau') ||_{L^2 d\tau d\tau'} \\
\quad + \int_s^t \int_{t'}^{t''} \frac{C}{|t-\tau|^{N\left(\frac{1}{q}-\frac{1}{p}\right)-1}} \|\langle x \rangle^\sigma g_{u}(\tau') \|_{L^\infty} \|W(\tau') ||_{L^2,\sigma} d\tau' \\
\quad \leq \frac{C}{|t-s|^{2N\left(\frac{1}{q}-\frac{1}{p}\right)-3}} \\
\]
For the middle integrals, $I_2$ and $I_{2'}$, we simply use \eqref{e:1.2} combined with \eqref{e:1.19}. For the remaining integrals we remove the singularities at $\tau = s$ respectively at $\tau = t$ as in the above short time estimates. More precisely, for $p > 2N/(N - 2)$, we have:

$$
\|I_2\|_{L^p} \leq \int_{s+1/4}^{t-1/4} \left\|e^{-iH(t-\tau)} P_c \|\|_{L^{p'} \to L^p} (\|g_u e^{-iH(s)} P_c v\|_{L^{p'}} + \|g_u e^{-iH(t-s)} P_c\|_{L^{p'}}) \right\|
$$

$$
\leq \int_{s+1/4}^{t-1/4} \frac{C_p}{|t - \tau|^{N(\frac{1}{2} - \frac{1}{p})}} (\|g_u\|_{L^{p'}} + \|g_u\|_{L^{p'}}) \|e^{-iH(t-s)} P_c v\|_{L^p} d\tau
$$

$$
\leq \sup_{\tau \in \mathbb{R}} \{\|g_u(\tau)\|_{L^{p'}} + \|g_u(\tau)\|_{L^{p'}}\} \int_{s+1/4}^{t-1/4} \frac{C_p}{|t - \tau|^{N(\frac{1}{2} - \frac{1}{p})}} \frac{C_p}{|t - s|^{N(\frac{1}{2} - \frac{1}{p})}} d\tau
$$

$$
\leq C \frac{\|v\|_{L^{p'}}}{|t - s|^{N(\frac{1}{2} - \frac{1}{p})}}
$$

$$
\|I_{2'}\|_{L^p} \leq \int_{s+1/2}^{t-1/2} \|e^{-iH(t-\tau)} P_{c'} \|\|_{L^{p'} \to L^p} \|Dh|_{\alpha(\tau)}\|_{L^{p'}} \|\psi_0\|_{L^2} \left(\|g_u\|_{L^{p'} \to L^p} + \|g_u\|_{L^{p'} \to L^p}\right) \|e^{-iH(t-s)} P_c v\|_{L^p} d\tau
$$

$$
\leq \|\psi_0\|_{L^2} \sup_{\tau \in \mathbb{R}} \{\|Dh|_{\alpha(\tau)}\|_{L^{p'}} \left(\|g_u(\tau)\|_{L^{p'}} + \|g_u(\tau)\|_{L^{p'}}\right)\} \int_{s+1/2}^{t-1/2} \frac{C_p}{|t - \tau|^{N(\frac{1}{2} - \frac{1}{p})}} \frac{C_p}{|t - s|^{N(\frac{1}{2} - \frac{1}{p})}} d\tau
$$

$$
\leq C \frac{\|v\|_{L^{p'}}}{|t - s|^{N(\frac{1}{2} - \frac{1}{p})}}
$$

$$
\|I_1\|_{L^p} \leq \int_{s}^{s+1/4} \|e^{-iH(t-s)} P_c \|\|_{L^{p'} \to L^p} \|e^{iH(t-s)} P_c g_u e^{-iH(s)} P_c v\|_{L^{p'}} d\tau
$$

$$
+ \int_{s}^{s+1/4} \|e^{-iH(t+s-2\tau)} \|\|_{L^{p'} \to L^p} \|e^{-iH(s)} P_c g_u e^{iH(s)} P_c v\|_{L^{p'}} d\tau
$$

$$
\leq \int_{s}^{s+1/4} \frac{C_p}{|t - s|^{N(\frac{1}{2} - \frac{1}{p})}} C \|\hat{g}_u\|_{L^1} \|v\|_{L^{p'}} d\tau + \int_{s}^{s+1/4} \frac{C_p}{|t + s - 2\tau|^{N(\frac{1}{2} - \frac{1}{p})}} C \|\hat{g}_u\|_{L^1} \|v\|_{L^{p'}} d\tau
$$

$$
\leq C \sup_{\tau \in \mathbb{R}} \{\|\hat{g}_u(\tau)\|_{L^1} + \|\hat{g}_u(\tau)\|_{L^1}\} \int_{s}^{s+1/4} \frac{C_p}{|t - s|^{N(\frac{1}{2} - \frac{1}{p})}} + \frac{C_p}{|t - s - 1/2|^{N(\frac{1}{2} - \frac{1}{p})}} d\tau
$$

$$
\leq C \frac{\|v\|_{L^{p'}}}{|t - s|^{N(\frac{1}{2} - \frac{1}{p})}}
$$

$$
\|I_3\|_{L^p} \leq \int_{t-1/4}^{t} \|e^{-iH(t-\tau)} P_c g_u e^{iH(t-\tau)} \|\|_{L^{p'} \to L^p} \|e^{-iH(s)} P_c v\|_{L^p} d\tau
$$

$$
+ \int_{t-1/4}^{t} \|e^{-iH(t-\tau)} P_c g_u e^{iH(t-\tau)} \|\|_{L^{p'} \to L^p} \|e^{-iH(t+s-2\tau)} P_c v\|_{L^p} d\tau
$$

$$
\leq \int_{t-1/4}^{t} C \|\hat{g}_u\|_{L^1} \frac{C_p}{|t - s|^{N(\frac{1}{2} - \frac{1}{p})}} d\tau + \int_{t-1/4}^{t} C \|\hat{g}_u\|_{L^1} \frac{C_p}{|t + s - 2\tau|^{N(\frac{1}{2} - \frac{1}{p})}} d\tau
$$

$$
\leq C \sup_{\tau \in \mathbb{R}} \{\|\hat{g}_u(\tau)\|_{L^1} + \|\hat{g}_u(\tau)\|_{L^1}\} \int_{t-1/4}^{t} \frac{C_p}{|t - s|^{N(\frac{1}{2} - \frac{1}{p})}} + \frac{C_p}{|t - s - 1/2|^{N(\frac{1}{2} - \frac{1}{p})}} d\tau
$$

$$
\leq C \frac{\|v\|_{L^{p'}}}{|t - s|^{N(\frac{1}{2} - \frac{1}{p})}}
$$
\[ \|II_1\|_{L^p} \leq \int_s^{s+1/2} \left| e^{-iH(t-t')} P_c \right|_{L^{p'} \rightarrow L^p} \left| \left| \left( \langle \psi_0, Dg \rangle_{\mathcal{E}} e^{-iH(t-t')} P_c v \right) \right|_v \right|_{L^p} dt' \]
\[ \leq \int_s^{s+1/2} \frac{C_p}{|t-t'|^{N\left(\frac{1}{2} - \frac{1}{p} \right)}} \left| \left| \left( \langle \psi_0, Dg \rangle_{\mathcal{E}} e^{-iH(t-t')} P_c v \right) \right|_v \right|_{L^p} \frac{1}{\left| \left| \left( \langle \psi_0, Dg \rangle_{\mathcal{E}} e^{-iH(t-t')} P_c v \right) \right|_v \right|_{L^p}} d\tau \]
\[ \leq \frac{C_p}{|t-t'|^{N\left(\frac{1}{2} - \frac{1}{p} \right)}} \int_s^{s+1/2} \left| \left| \left( \langle \psi_0, Dg \rangle_{\mathcal{E}} e^{-iH(t-t')} P_c v \right) \right|_v \right|_{L^p} \frac{1}{\left| \left| \left( \langle \psi_0, Dg \rangle_{\mathcal{E}} e^{-iH(t-t')} P_c v \right) \right|_v \right|_{L^p}} d\tau \]
\[ \leq \frac{C}{|t-t'|^{N\left(\frac{1}{2} - \frac{1}{p} \right)}} \int_s^{s+1/2} \left| \left| \left( \langle \psi_0, Dg \rangle_{\mathcal{E}} e^{-iH(t-t')} P_c v \right) \right|_v \right|_{L^p} \frac{1}{\left| \left| \left( \langle \psi_0, Dg \rangle_{\mathcal{E}} e^{-iH(t-t')} P_c v \right) \right|_v \right|_{L^p}} d\tau \]
\[ \leq \frac{C}{|t-t'|^{N\left(\frac{1}{2} - \frac{1}{p} \right)}} \cdot \frac{1}{|t-t'|^{N\left(\frac{1}{2} - \frac{1}{p} \right)}} \int_s^{s+1/2} \left| \left| \left( \langle \psi_0, Dg \rangle_{\mathcal{E}} e^{-iH(t-t')} P_c v \right) \right|_v \right|_{L^p} \frac{1}{\left| \left| \left( \langle \psi_0, Dg \rangle_{\mathcal{E}} e^{-iH(t-t')} P_c v \right) \right|_v \right|_{L^p}} d\tau \]

Similarly we will investigate the long time behavior, \( t > s + 1 \), of \( L(s)W \). We split it into three integrals with \( s_1 = \min\{s + 1, t - 1/16\} \):

\[ [L(s)W](t) = \int_s^{s_1} + \int_{s_1}^{t - 1/16} + \int_{t - 1/16}^{t} e^{-iH(t-t')} P_c \left( -iDg \langle \psi_{E(t)} \rangle_\mathcal{E} \left[ W(t) \right] \right) \left. d\tau \right|_{L_3} \]

Due to Theorem 4.14 part (iv) and \( W(t) = T(t, s, v) \), the integral \( L_3 \) has an integrable singularity at \( \tau = s \) while \( L_4 \) has no singularities. A combination of 4.12, estimates in Theorem 4.11 part (iv), and 4.19 gives the required result for \( L_3 \) and \( L_4 \). In \( L_5 \) we will first remove the singularity at \( \tau = t \) in a similar manner we did it for short time estimates. More precisely, for \( p = q_2, 2N/(N - 2) < q_2 < 2N/(N - 4) \), we have:

\[ [L_5](t) = \int_s^{s_1} \int_{s_1}^{t - 1/16} + \int_{t - 1/16}^{t} \]

\[ \leq \int_s^{s_1} \left| e^{-iH(t-t')} P_c \left|_{L^{p'} \rightarrow L^p} \right| \left( \left| \langle \psi_0, Dg \rangle_{\mathcal{E}} e^{-iH(t-t')} P_c v \right| \right|_v \right|_{L^p} \frac{1}{|t-t'|^{N\left(\frac{1}{2} - \frac{1}{p} \right)}} d\tau \]

\[ \leq \sup_{\tau \in \mathbb{R}} \left\{ \left( 1 + \left| \left| \left( \langle \psi_0, Dg \rangle_{\mathcal{E}} e^{-iH(t-t')} P_c v \right) \right| \right|_v \right|_{L^p} \right\} \left[ \left| \left| \left( \langle \psi_0, Dg \rangle_{\mathcal{E}} e^{-iH(t-t')} P_c v \right) \right| \right|_v \right|_{L^p} \frac{1}{\left| \left| \left( \langle \psi_0, Dg \rangle_{\mathcal{E}} e^{-iH(t-t')} P_c v \right) \right| \right|_v \right|_{L^p}} d\tau \]

\[ \leq \frac{C}{\left| t-t' \right|^{N\left(\frac{1}{2} - \frac{1}{p} \right)}} \int_s^{s_1} \frac{C_p}{\left| t-t' \right|^{N\left(\frac{1}{2} - \frac{1}{p} \right)}} d\tau \]
We estimate $L_4$ exactly as $L_3$ but now $\|W(\tau)\|_{L^2_x} \leq C_p(1 + |\tau - s|)^{-N(\frac{2}{d} - \frac{2}{p})}$, see Theorem 4.14 part (iv), and the convolution estimate (4.13) is now employed to yield exactly the same result. For $L_5$, one of the integrands has no singularities:

$$\| \int_{t-\frac{1}{\tau}}^{t} e^{-iH(t-\tau)} P_c Dh_{a(\tau)} i(\psi_0, Dg_{\psi E}[W(\tau)]) d\tau \|_{L^p} \leq \int_{t-\frac{1}{\tau}}^{t} C \| e^{-iH(t-\tau)} P_c H^2 \| H^2 \| Dh_{a(\tau)} \|_{H^2} \| \langle \psi_0, Dg_{\psi E}[W(\tau)] \rangle \| d\tau \leq \int_{t-\frac{1}{\tau}}^{t} C \| Dh_{a(\tau)} \|_{H^2} \| \psi_0 \|_{L^2}(\| \langle x \rangle g_{u, L^\infty} \| + \| \langle x \rangle g_{u, L^\infty} \|) \| W(\tau) \|_{L^2_x} d\tau \leq \| \psi_0 \|_{L^2} \sup_{\tau \in \mathbb{R}} \| Dh_{a(\tau)} \|_{H^2} (\| \langle x \rangle g_{u, L^\infty} \| + \| \langle x \rangle g_{u, L^\infty} \|) \int_{t-\frac{1}{\tau}}^{t} \frac{C \| v \|_{L^{p'}}}{|t - s|^{N(\frac{2}{d} - \frac{2}{p})}} d\tau \leq \frac{C \| v \|_{L^{p'}}}{|t - s|^{N(\frac{2}{d} - \frac{2}{p})}}.$$

However to remove the non-integrable singularity at $\tau = t$ in the remaining integral we need to plug in (4.11) in it:

$$\int_{t-\frac{1}{\tau}}^{t} e^{-iH(t-\tau)} P_c Dg_{\psi E}[W(\tau)] d\tau = \int_{t-\frac{1}{\tau}}^{t} e^{-iH(t-\tau)} P_c g_{u} \left( \int_{t}^{t} e^{-iH(t-\tau')} P_c (-iDg_{\psi E}[e^{-iH(\tau'-s)} P_c v] + Dh_i(\psi_0, Dg_{\psi E}[e^{-iH(\tau'-s)} P_c v])) d\tau' + \int_{t}^{t} e^{-iH(t-\tau')} P_c (-iDg_{\psi E}[W(\tau')] + Dh_i(\psi_0, Dg_{\psi E}[W(\tau')])) d\tau' \right) d\tau + \int_{t-\frac{1}{\tau}}^{t} e^{-iH(t-\tau')} P_c g_{u} \left( \int_{t}^{t} e^{iH(t-\tau')} P_c (-iDg_{\psi E}[e^{-iH(\tau'-s)} P_c v] + Dh_i(\psi_0, Dg_{\psi E}[e^{-iH(\tau'-s)} P_c v])) d\tau' + \int_{t}^{t} e^{iH(t-\tau')} P_c (-iDg_{\psi E}[W(\tau')] + Dh_i(\psi_0, Dg_{\psi E}[W(\tau')])) d\tau' \right) d\tau$$

We will add $e^{iH(t-\tau)}$ and $e^{-iH(t-\tau)}$ terms after $g_{u}$ and $g_{u}$'. Then all the terms will be similar to $L_1, L_2, L_1^\prime,$ and $L_2^\prime$, see (4.13) – (4.18). After separating the the inside integrals into pieces, we will estimate short time step integrals exactly the same way we did for short time behavior by using estimate (4.14), and the other integrals will be estimated using the usual norms. For completeness we show below how each term is treated:

- For $X(\tau') = -ig_{u} e^{-iH(\tau'-s)} P_c v$ we have

  $$\| L_1 \|_{L^p} \leq \int_{t-\frac{1}{\tau}}^{t} \| e^{-iH(t-\tau)} P_c g_{u} e^{iH(t-\tau)} \|_{L^p \rightarrow L^p} \left[ \int_{s}^{s+\frac{1}{t}} \| e^{-iH(t-s)} P_c \|_{L^p \rightarrow L^p} \| e^{iH(t'-s)} P_c g_{u} e^{-iH(t'-s)} P_c v \|_{L^{p'}} d\tau' + \int_{s}^{s+\frac{1}{t}} \| e^{-iH(t'-s)} P_c \|_{L^p \rightarrow L^p} \| g_{u} e^{-iH(t'-s)} P_c v \|_{L^p} d\tau' \right] d\tau + \int_{t-\frac{1}{\tau}}^{t} \| e^{-iH(t-\tau)} P_c g_{u} e^{iH(t-\tau)} \|_{L^p \rightarrow L^p} \| e^{-iH(t-\tau)} P_c v \|_{L^p} d\tau' \leq \frac{C \| v \|_{L^{p'}}}{|t - s|^{N(\frac{2}{d} - \frac{2}{p})}}.$$
\[ \| L_2 \|_{L^p} \leq \int_{t-rac{1}{16}}^{t} \left[ \left( \int_s^{s+\frac{1}{4}} \| e^{-iH(t-s)^2/2} P_c g u e^{iH(t-s)} \|_{L^p} \| e^{-iH(t-s)} P_c g u e^{iH(t-s)} \|_{L^p'} \| e^{-iH(t-s)} P_c \bar{v} \|_{L^p'} \| e^{-iH(t-s)} P_c \bar{v} \|_{L^p'} \right) dt' \right] \]
\[ \times \left[ \left( \int_s^{s+\frac{1}{4}} \| e^{-iH(t-s)} P_c \|_{L^p'} \| e^{-iH(t-s)} P_c \|_{L^p} \| e^{-iH(t-s)} P_c \bar{v} \|_{L^p'} \| e^{-iH(t-s)} P_c \bar{v} \|_{L^p'} \right) dt' \right] \]
\[ + \int_{s+\frac{1}{4}}^{t} \left( \left( \int_s^{s+\frac{1}{4}} \| e^{-iH(t-s)^2/2} P_c g u e^{iH(t-s)} \|_{L^p} \| e^{-iH(t-s)} P_c g u e^{iH(t-s)} \|_{L^p'} \| e^{-iH(t-s)} P_c \bar{v} \|_{L^p'} \| e^{-iH(t-s)} P_c \bar{v} \|_{L^p'} \right) dt' \right] \]
\[ + \int_{s+\frac{1}{4}}^{t} \left( \left( \int_s^{s+\frac{1}{4}} \| e^{-iH(t-s)^2/2} P_c g u e^{iH(t-s)} \|_{L^p} \| e^{-iH(t-s)} P_c g u e^{iH(t-s)} \|_{L^p'} \| e^{-iH(t-s)} P_c \bar{v} \|_{L^p'} \| e^{-iH(t-s)} P_c \bar{v} \|_{L^p'} \right) dt' \right] \]
\[ \leq \frac{C \| v \|_{L^p'}}{|t-s|^{\frac{N}{2}+\frac{1}{p}}} \]

\[ \| \bar{L}_1 \|_{L^p} \leq \int_{t-rac{1}{16}}^{t} \left[ \left( \int_s^{s+\frac{1}{4}} \| e^{-iH(t-s)^2/2} P_c g u e^{iH(t-s)} \|_{L^p} \| e^{-iH(t-s)} P_c g u e^{iH(t-s)} \|_{L^p'} \| e^{-iH(t-s)} P_c \bar{v} \|_{L^p'} \| e^{-iH(t-s)} P_c \bar{v} \|_{L^p'} \right) dt' \right] \]
\[ + \int_{s+\frac{1}{4}}^{t} \left( \left( \int_s^{s+\frac{1}{4}} \| e^{-iH(t-s)^2/2} P_c g u e^{iH(t-s)} \|_{L^p} \| e^{-iH(t-s)} P_c g u e^{iH(t-s)} \|_{L^p'} \| e^{-iH(t-s)} P_c \bar{v} \|_{L^p'} \| e^{-iH(t-s)} P_c \bar{v} \|_{L^p'} \right) dt' \right] \]
\[ \leq \frac{C \| v \|_{L^p'}}{|t-s|^{\frac{N}{2}+\frac{1}{p}}} \]

\[ \bar{L}_2 \] is treated exactly the same as \( \bar{L}_1 \) except that in the decomposition of the inside integral 4\tau - 3t is used instead of \( t - 1/4 \).

- For \( X(t') = -i g u e^{iH(t'-s)} P_c \bar{v} \) we have

\[ \| L_1 \|_{L^p} \leq \int_{t-\frac{1}{16}}^{t} \left[ \left( \int_s^{s+\frac{1}{4}} \| e^{-iH(t-s)^2/2} P_c g u e^{iH(t-s)} \|_{L^p} \| e^{-iH(t-s)} P_c g u e^{iH(t-s)} \|_{L^p'} \| e^{-iH(t-s)} P_c \bar{v} \|_{L^p'} \| e^{-iH(t-s)} P_c \bar{v} \|_{L^p'} \right) dt' \right] \]
\[ + \int_{s+\frac{1}{4}}^{t} \left( \left( \int_s^{s+\frac{1}{4}} \| e^{-iH(t-s)^2/2} P_c g u e^{iH(t-s)} \|_{L^p} \| e^{-iH(t-s)} P_c g u e^{iH(t-s)} \|_{L^p'} \| e^{-iH(t-s)} P_c \bar{v} \|_{L^p'} \| e^{-iH(t-s)} P_c \bar{v} \|_{L^p'} \right) dt' \right] \]
\[ \leq \frac{C \| v \|_{L^p'}}{|t-s|^{\frac{N}{2}+\frac{1}{p}}} \]
\[ \|L_2\|_{L^p} \leq \int_{t-\frac{1}{4}}^{t} \left\| e^{-iH(t-\tau)} P_c g_u e^{iH(t-\tau)} \right\|_{L^p} \int_{s}^{s+\frac{1}{4}} \left\| e^{-iH(t-2\tau+2\tau'-s)} e^{iH(\tau'-s)} P_c g_u e^{-iH(\tau'-s)} P_c v \right\|_{L^p} d\tau' \\
+ \int_{s+\frac{1}{4}}^{t} \left\| e^{-iH(t-2\tau+\tau')} P_c \right\|_{L^p} \int_{s}^{t} \left\| e^{iH(t-\tau')} P_c v \right\|_{L^p} d\tau' \\
+ \int_{t-\frac{1}{4}}^{t} \left\| e^{-iH(t+\tau'-2\tau)} P_c g_u e^{iH(t+\tau'-2\tau)} e^{-iH(t-2\tau+2\tau'-s)} P_c v \right\|_{L^p} d\tau' \right] d\tau \\
\leq \frac{C\|v\|_{L^p'}}{|t-s|^{N(\frac{1}{2}-\frac{\beta}{p})}} \\
\]

\( \hat{L}_1 \) and \( \hat{L}_2 \) are treated as in the previous case.

- For \( X(\tau') = -ig_u W(\tau') \) and \( X(\tau') = -ig_u W(\tau') \) we will separate the \( L_1 \) term into three integrals. For the first integral we will use short time \( L^2 \) estimate for \( W \). Also note that one can obtain the same estimates for \( |t-s| \leq \frac{1}{4} \) and \( |t-s| > \frac{1}{4} \) in the Theorem 4.1 part (iv). For the last integral we will change the order of the integration:

\[ \|L_1\|_{L^p} \leq \int_{t-\frac{1}{4}}^{t} \| e^{-iH(t-\tau)} P_c g_u(\tau) e^{iH(\tau)} \right\|_{L^p} \int_{s}^{\tau} \left\| e^{-iH(t-\tau')} P_c \right\|_{L^p} \int_{s}^{\tau} \left\| g_u W(\tau') \right\|_{L^p'} d\tau' d\tau \\
\leq \sup_{\tau \in \mathbb{R}} \|\hat{g}_u(\tau)\|_{L^1} \int_{t-\frac{1}{4}}^{t} \int_{s}^{\tau} \frac{C_p}{|t-\tau|^{N(\frac{1}{2}-\frac{\beta}{p})}} \left\| e^{\gamma_x} g_u \right\|_{L^2} \left\| W(\tau') \right\|_{L^\infty} d\tau' d\tau \\
\leq \int_{t-\frac{1}{4}}^{t} \left[ \int_{s}^{s+\frac{1}{4}} \frac{C_p}{|t-\tau|^{N(\frac{1}{2}-\frac{\beta}{p})}} \frac{\|v\|_{L^p'}}{|t-\tau|^{N(\frac{1}{2}-\frac{\beta}{p})}-1} d\tau' + \int_{s+\frac{1}{4}}^{t} \frac{C_p}{|t-\tau|^{N(\frac{1}{2}-\frac{\beta}{p})}} \frac{\|v\|_{L^p'}}{(1+|\tau'-s|)^{N(\frac{1}{2}-\frac{\beta}{p})}} d\tau' \right] d\tau \\
+ \int_{t-\frac{1}{4}}^{t} \int_{s}^{t} \frac{C_p}{|t-\tau|^{N(\frac{1}{2}-\frac{\beta}{p})}} \frac{\|v\|_{L^p'}}{(1+|\tau'-s|)^{N(\frac{1}{2}-\frac{\beta}{p})}} d\tau' d\tau' \\
\leq \frac{C\|v\|_{L^p'}}{|t-s|^{N(\frac{1}{2}-\frac{\beta}{p})}} \\
\]

Similar to \( L_1 \) we will split \( L_2 \) in three integrals. In the first and last we use estimate (4.14) and
we also change the order of integration in the last integral:

\[
\|L_2\|_{L^p} \leq \int_{t-rac{2\tau_1}{3}}^t \|e^{-iH(t-\tau)} P_{e} g_{u} e^{iH(t-\tau)}\|_{L^p} \left( \int_{s}^{t-rac{2\tau_1}{3}} \|e^{-iH(t-\tau'-2\tau)} P_{e}\|_{L^p} \|\|g_{u} W(\tau')\|_{L^p} d\tau' \right) d\tau
\]

\[
+ \int_{t-rac{2\tau_1}{3}}^t \int_{s}^{t-rac{2\tau_1}{3}} \|e^{-iH(t-\tau)} P_{e} g_{u} e^{iH(t-\tau)}\|_{L^p} \|\|g_{u} W(\tau')\|_{L^p} d\tau' d\tau
\]

\[
\leq \frac{C}{|t-s|^{N\left(\frac{4}{3} - \frac{2}{p}\right)}} \|v\|_{L^p}
\]

The \(L_1\) and \(L_2\) terms are estimated as in the previous cases, more precisely:

\[
\|L_2\|_{L^p} \leq \int_{t-rac{2\tau_1}{3}}^t \|e^{-iH(t-\tau)} P_{e} g_{u} e^{iH(t-\tau)}\|_{L^p} \left( \int_{s}^{t-rac{2\tau_1}{3}} \|e^{-iH(t-\tau'-2\tau)} P_{e}\|_{L^p} \|\|g_{u} W(\tau')\|_{L^p} d\tau' \right) d\tau
\]

where we used

\[
\|\psi_0, g_{u} W(\tau')\| \leq \|\psi_0\|_{L^2} \sup_{\tau' \in \mathbb{R}} \{\|\mathcal{A}^2 g_{u}(\tau')\|_{L^\infty}\} \|W(\tau')\|_{L^2_{\sigma}} \leq \begin{cases} \frac{C}{|t-s|^{N\left(\frac{4}{3} - \frac{2}{p}\right)}} & \text{if } |\tau' - s| \leq 1 \\ \frac{C}{|t-s|^{N\left(\frac{4}{3} - \frac{2}{p}\right)}} & \text{if } |\tau' - s| > 1 \\ \end{cases}
\]

This finishes the proof of (ii).

(iii) The case \(p = 2\) has already been proven in part (i). It remains to show the estimate for \(p = \frac{2N}{N-2}\) since the ones for \(2 < p < \frac{2N}{N-2}\) follow from Riesz-Thorin interpolation. We will again use the definition (4.10) and expansion (4.11) together with notations (4.12)-(4.13), see (4.5) for the definition of \(L(s)\). We will treat the \(t \geq s\) case as the \(t < s\) one can be treated similarly.

For the \(f\) term let us first consider \(s \leq t \leq 1\). Recall that

\[
\bar{T}(t, s) = -i \int_s^{\text{min}(t, s+1)} \frac{e^{-iH(t-\tau)} P_{e} g_{u}(\tau)}{e^{-iH(t-s)} P_{e}} d\tau
\]
Then for this time interval the forcing term corresponding to $f$ of the operator $T(t, s) - \tilde{T}(t, s)$ becomes

$$I_1 = -i \int_s^t e^{-iH(t-\tau)} P_c g_\alpha e^{iH(\tau-s)} P_c \tilde{v} d\tau$$

(4.20)

For fixed $t$ and $s$ we have

$$\|I_1\|_{L^2} = \int_s^t e^{-iH(t-\tau)} P_c g_\alpha e^{iH(\tau-s)} P_c \tilde{v} d\tau \leq \|e^{-iH(t-s)} P_c e^{iH(\tau-s)} P_c \tilde{v} d\tau\|_{L^2}$$

\[\leq \|e^{-iH(t-s)} P_c \|_{L^2 \rightarrow L^2} \int_s^t e^{2iH \tau} P_c q(\tau) d\tau \leq C \|q(\tau)\|_{L^2([s, s+1], L^{p'})} \leq C \|v\|_{L^{p'}}\]

at the last step we used end point Strichartz estimates [5 Theorem 1.2] and the fact that $\|q(\tau)\|_{L^{p'}} \leq C \|\tilde{g}_\alpha(\tau)\|_{L^1} \|v\|_{L^{p'}}$, see (1.14).

For the long time we split $f$ as follows

$$f = \int_{I_2}^s + \int_{I_2}^t -ie^{-iH(t-\tau)} P_c Dg \psi_E [e^{-iH(\tau-s)} P_c \tilde{v}] d\tau$$

Then $I_1$ is estimated exactly as (4.20) above and for $I_2$ we have via Strichartz estimates for the admissible pair $(\gamma, \rho)$, $\gamma \geq 2$ fixed but $\gamma \neq 1/2$:

$$\|I_2\|_{L^2} \leq C \left( \int_{s+1}^{\infty} \|g_\alpha e^{-iH(\tau-s)} P_c \tilde{v}\|_{L^{2'}} d\tau \right)^{\frac{1}{2}} + C \left( \int_{s+1}^{\infty} \|g_\alpha e^{iH(\tau-s)} P_c \|_{L^{2'}} d\tau \right)^{\frac{1}{2}}$$

$$\leq C \left( \int_{s+1}^{\infty} \|g_\alpha\|_{L^{2'}} \|e^{-iH(\tau-s)} P_c \tilde{v}\|_{L^{2'}} d\tau \right)^{\frac{1}{2}} + C \left( \int_{s+1}^{\infty} \|g_\alpha\|_{L^{2'}} \|e^{iH(\tau-s)} P_c \tilde{v}\|_{L^{2'}} d\tau \right)^{\frac{1}{2}}$$

$$\leq C \left( \int_{s+1}^{\infty} \|\tilde{v}\|_{L^{2'}} d\tau \right)^{\frac{1}{2}} < \infty$$

where $1/\beta + 1/p = 1/p'$ and we used $N\left(\frac{1}{2} - \frac{1}{p}\right) \gamma' > 1$ for $p = \frac{2N}{N-2}$ and $\gamma \neq 2$.

Similarly for the other forcing term we have:

$$\hat{f}(t) = \int_{I_1}^s + \int_{I_2}^t e^{-iH(t-\tau)} P_c Dh |_{\alpha(\tau)} \langle \psi_0, Dg \psi_E [e^{-iH(\tau-s)} P_c \tilde{v}] \rangle d\tau.$$
For the $II_2$ term we have again via Strichartz estimates for the admissible pair $(\gamma, \rho)$, $\gamma \geq 2$ fixed but $\gamma \neq \infty$:
\[
\|II_2\|_{L^2} \leq C \left( \int_{s+1}^{\infty} \| Dh(\psi_0, g_u e^{-iH(\tau-s)} P_c v) \|_{L^2}^{\gamma'} d\tau \right)^{\frac{1}{\gamma'}} + C \left( \int_{s+1}^{\infty} \| Dh(\psi_0, g_u e^{iH(\tau-s)} P_c v) \|_{L^2}^{\gamma'} d\tau \right)^{\frac{1}{\gamma'}} \\
\leq C \left( \int_{s+1}^{\infty} \| Dh(\psi_0, g_u, e^{-iH(\tau-s)} P_c v) \|_{L^{2\rho}}^{\gamma'} \|\|_{L^{2\rho}} d\tau \right)^{\frac{1}{\gamma'}} \\
+ C \left( \int_{s+1}^{\infty} \| Dh(\psi_0, g_u, e^{iH(\tau-s)} P_c v) \|_{L^{2\rho}}^{\gamma'} \|\|_{L^{2\rho}} d\tau \right)^{\frac{1}{\gamma'}} \\
\leq C \left( \int_{s+1}^{\infty} \frac{d\tau}{(1 + |\tau-s|^{\frac{2N}{N-2}})^{\gamma'}} \right)^{\frac{1}{\gamma'}} < \infty
\]
where $1/\beta + 1/p = 1/\rho'$.

Similarly we can estimate $L(s)W$:
\[
\|L(s)W(t)\|_{L^2} \leq C \left( \int_{s}^{\infty} \| Dg(\psi_0, W(\tau)) \|_{L^2}^{\gamma'} d\tau \right)^{\frac{1}{\gamma'}} + C \left( \int_{s}^{\infty} \| Dg(\psi_0, W(\tau)) \|_{L^2}^{\gamma'} d\tau \right)^{\frac{1}{\gamma'}} \\
\leq C \left( \int_{s}^{\infty} \| Dg(\psi_0, W(\tau)) \|_{L^2}^{\gamma'} d\tau \right)^{\frac{1}{\gamma'}} \\
+ C \left( \int_{s}^{\infty} \| Dg(\psi_0, W(\tau)) \|_{L^2}^{\gamma'} d\tau \right)^{\frac{1}{\gamma'}} \\
\leq C \left( \int_{s}^{\infty} \frac{d\tau}{(1 + |\tau-s|^{\frac{2N}{N-2}})^{\gamma'}} \right)^{\frac{1}{\gamma'}} < \infty
\]
Hence $T(t, s) - \tilde{T}(t, s) : L^{\rho'} \to L^2$ is bounded for $p = \frac{2N}{N-2}$ and by part (i) for $p = 2$. By Riesz-Thorin interpolation it is bounded for any $2 \leq p \leq \frac{2N}{N-2}$. This finishes the proof of part (iii) and the theorem.

\[
\square
\]

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