Classical and Quantum Algorithms for Constructing Text from Dictionary Problem

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Abstract. We study algorithms for solving the problem of constructing a text (long string) from a dictionary (sequence of small strings). The problem has an application in bioinformatics and has a connection with the Sequence assembly method for reconstructing a long DNA sequence from small fragments. The problem is constructing a string \(t\) of length \(n\) from strings \(s_1, \ldots, s_m\) with possible intersections. We provide a classical algorithm with running time \(O(n + L + m (\log n)^2) = \tilde{O}(n + L)\) where \(L\) is the sum of lengths of \(s_1, \ldots, s_m\). We provide a quantum algorithm with running time \(O\left(n + \log n \cdot (\log m + \log \log n) \cdot \sqrt{m \cdot L}\right) = \tilde{O}\left(n + \sqrt{m \cdot L}\right)\). Additionally, we show that the lower bound for the classical algorithm is \(\Omega(n + L)\). Thus, our classical algorithm is optimal up to a log factor, and our quantum algorithm shows speed-up comparing to any classical algorithm in a case of non-constant length of strings in the dictionary.

Keywords: quantum computation, quantum models, quantum algorithm, query model, string constructing, sequence assembly, DNA constructing

1 Introduction

Quantum computing [18, 2, 1] is one of the hot topics in computer science of last decades. There are many problems where quantum algorithms outperform the best known classical algorithms [6, 8, 12, 11].

One of such problems are problems for strings. Researchers show the power of quantum algorithms for such problems in [16, 4, 19, 10].

In this paper, we consider the problem of constructing text from dictionary strings with possible intersections. We have a text \(t\) of length \(n\) and a dictionary \(s = s_1, \ldots, s_m\). The problem is constructing \(t\) only from strings of \(s\) with possible intersections. The problem is connected with the sequence assembly method for reconstructing a long DNA sequence from small fragments [17, 3].

We suggest a classical algorithm with running time \(O(n + L + m (\log n)^2) = \tilde{O}(n + L)\), where \(L = |s_1| + \cdots + |s_m|\), \(|s_i|\) is a length of \(s_i\) and \(\tilde{O}\) does not consider log factors. The algorithm uses segment tree [13] and suffix array [15] data structures, concepts of comparing string using rolling hash [9, 7] and idea of prefix sum [5].

The second algorithm is quantum. It uses similar ideas and quantum algorithm for comparing two strings with quadratic speed-up comparing to classical counterparts [10]. The running time for our quantum algorithm is

\[O\left(n + \log n \cdot (\log m + \log \log n) \cdot \sqrt{m \cdot L}\right) = \tilde{O}\left(n + \sqrt{m \cdot L}\right).\]

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Additionally, we show the lower bound in a classical case that is \( \Omega(n + L) \). Thus, we get the optimal classical algorithm in a case of \( m(\log n)^2 = O(L + n) \). It is true, for example, in a case of \( O(m) \) strings form \( s \) has length at least \( \Omega((\log n)^2) \) or in a case of \( m = o(\frac{n}{(\log n)^2}) \). In the general case, the algorithm is an optimal algorithm up to a log factor. The quantum algorithm is better than any classical counterparts in a case of \( \log n \cdot (\log m + \log \log n) \cdot \sqrt{m \cdot L} = o(L) \). It happens if \( O(m) \) strings from \( s \) has length at least \( \Omega((\log n \cdot (\log m + \log \log n)) \).

Our algorithm uses some quantum algorithms as a subroutine, and the rest part is classical. We investigate the problems in terms of query complexity. The query model is one of the most popular in the case of quantum algorithms. Such algorithms can do a query to a black box that has access to the sequence of strings. As a running time of an algorithm, we mean a number of queries to the black box.

The structure of the paper is the following. We present tools in Section 2. Then, we discuss the classical algorithm in Section 3.1 and quantum algorithm in Section 3.2. Section 4 contains lower bound.

2 Tools

Our algorithms uses several data structures and algorithmic ideas like segment tree\[13\], suffix array\[15\], rolling hash\[9\] and prefix sum \[5\]. Let us describe them in this section.

2.1 Preliminaries

Let us consider a string \( u = (u_1, \ldots, u_l) \) for some integer \( l \). Then, \( |u| = l \) is the length of the string. \( u[i, j] = (u_i, \ldots, u_j) \) is a substring of \( u \).

In the paper, we compare strings in lexicographical order. For two strings \( u \) and \( v \), the notation \( u < v \) means \( u \) precedes \( v \) in lexicographical order.

2.2 Rolling Hash for Strings Comparing

The rolling hash was presented in \[9\]. It is a hash function

\[
h_p(u) = \left( \sum_{i=1}^{[u]} index(u_i) \cdot K^{i-1} \right) \mod p,
\]

where \( p \) is some prime integer, \( K = |\Sigma| \) is a size of the alphabet and \( index(u_i) \in \{0, \ldots, K - 1\} \) is the index of a symbol \( u_i \) in the alphabet. For simplicity we consider binary alphabet. So, \( K = 2 \) and \( index(u_i) = u_i \).

We can use rolling hash and the fingerprinting method \[7\] for comparing two strings \( u, v \). Let us randomly choose \( p \) from the set of the first \( r \) primes, such that \( r \leq \frac{\max(|u|, |v|)}{\varepsilon} \) for some \( \varepsilon > 0 \). Due to Chinese Theorem and \[7\], the following statements are equivalent \( h_p(u) = h_p(v) \) and \( u = v \) with error probability at most \( \varepsilon \). If we have \( \delta \) invocations of comparing procedure, then we should choose \( \frac{\delta \cdot \max(|u|, |v|)}{\varepsilon} \) primes. Due to Chebishev’s theorem, the \( r \)-th prime number \( p_r \approx r \ln r \). So, if our data type for integers is enough for storing \( \frac{\delta \cdot \max(|u|, |v|)}{\varepsilon} \cdot (\ln(\delta) + \ln(\max(|u|, |v|)) - \ln(\varepsilon)) \), then it is enough for computing the rolling hash.
Additionally, for a string \( u \), we can compute prefix rolling hash, that is \( h_p(u[1, i]) \). It can be computed in \( O(|u|) \) running time using formula

\[
h_p(u[1, i]) = (h_p(u[1, i - 1]) + (2^{i-1} \mod p) \cdot u_i) \mod p \text{ and } h_p(u[1 : 0]) = 0.
\]

Assume, that we store \( K_i = 2^{i-1} \mod p \). We can compute all of them in \( O(|u|) \) running time using formula \( K_i = K_{i-1} \cdot 2 \mod p \).

Using precomputed prefix rolling hash for a string \( u \) we can compute rolling hash for any substring \( u[i, j] \) in \( O(1) \) running time by formula

\[
h_p(u[i, j]) = \left( \sum_{q=j}^{i} u_q \cdot 2^{q-1-(j-1)} \right) \mod p = \left( \sum_{q=j}^{i} u_q \cdot 2^{-q} \right) \cdot 2^{-j+1} \mod p =
\]

\[
= \left( \frac{1}{2} \right) (h_p(u[1, i]) - h_p(u[1, j - 1])) \cdot (2^{j-1}) \mod p.
\]

For computing the formula in \( O(1) \) we should precompute \( \mathcal{I}_i = 2^{-i} \mod p \). We can compute it in \( O(\log p + |u|) \) by the formula \( \mathcal{I}_i = \mathcal{I}_{i-1} \cdot 2^{-1} \mod p \) and \( \mathcal{I}_0 = 1 \). Due to Fermat’s little theorem \( 2^{-i} \mod p = 2^{p-2} \mod p \). We can compute it with \( O(\log p) \) running time using Exponentiation by squaring algorithm.

Let \( \text{ComputeKI}(\beta, p) \) be a procedure that computes \( K \) and \( \mathcal{I} \) up to the power \( \beta \) with \( O(\beta + \log p) \) running time. Let \( \text{ComputePrefixRollingHashes}(u, p) \) be a procedure that computes all prefix rolling hashes for a string \( u \) and store them.

Assume, that we have two strings \( u \) and \( v \) and already computed prefix rolling hashes. Then, we can compare these strings in lexicographical order in \( O(\log \min(|u|, |v|)) \) running time. The algorithm is following. We search the longest common prefix of \( u \) and \( v \), that is \( \text{lcp}(u, v) \). We can do it using binary search.

- If a \( \text{mid} \leq \text{lcp}(u, v) \), then \( h_p(u[1, \text{mid}]) = h_p(v[1, \text{mid}]) \).
- If a \( \text{mid} > \text{lcp}(u, v) \), then \( h_p(u[1, \text{mid}]) \neq h_p(v[1, \text{mid}]) \).

Using binary search we find the last index \( x \) such that \( h_p(u[1, x]) = h_p(v[1, x]) \) and \( h_p(u[1, x + 1]) \neq h_p(v[1, x + 1]) \). In that case \( \text{lcp}(u, v) = x \).

After that, we compare \( u_t \) and \( v_t \) for \( t = \text{lcp}(u, v) + 1 \). If \( u_t < v_t \), then \( u < v \); if \( u_t > v_t \), then \( u > v \); if \( |u| = |v| = \text{lcp}(u, v) \), then \( u = v \).

Binary search works with \( O(\log(\min(|u|, |v|))) \) running time because we have computed all prefix rolling hashes already.

Let \( \text{Compare}(u, v) \) be a procedure that compares \( u \) and \( v \) and returns \(-1\) if \( u < v \); \( 1 \) if \( u > v \); and \( 0 \) if \( u = v \).

### 2.3 Segment Tree with Range Updates

We consider a standard segment tree data structure [13] for an array \( a = (a_1, \ldots, a_l) \) for some integer \( l \). A segment tree for and array \( a \) can be constructed in \( O(l) \) running time. The data structure allows us to invoke the following requests in \( O(\log l) \) running time.
- **Update.** Parameters are three integers \(i, j, x\) \((1 \leq i \leq j \leq l)\). We assign \(a_q = \max(a_q, x)\) for \(i \leq q \leq j\).

- **Push.** We push all existing range updates.

- **Request.** For an integer \(i\) \((1 \leq i \leq l)\), we should return \(a_i\).

Let \(\text{constructSegmentTree}(a)\) be a function that constructs and returns a segment tree for an array \(a\) in \(O(l)\) running time.

Let \(\text{update}(st, i, j, x)\) be a procedure that updates a segment tree \(st\) in \(O(\log l)\) running time.

Let \(\text{push}(st)\) be a procedure that push all existing range updates for a segment tree \(st\) in \(O(l)\) running time.

Let \(\text{request}(st, i)\) be a function that returns \(a_i\) from a segment tree \(st\). The running time of the procedure is \(O(\log l)\). At the same time, if we invoke \(\text{push}\) procedure and after that do not invoke \(\text{update}\) procedure, then the running time of \(\text{request}\) is \(O(1)\).

### 2.4 Suffix Array

Suffix array [15] is an array \(\text{su}\) = \((\text{su}_1, \ldots, \text{su}_l)\) for a string \(u\) and \(l = |u|\). The suffix array is a lexicographical order for all suffixes of \(u\). Formally, \(u[\text{su}_i, l] < u[\text{su}_{i+1}, l]\) for any \(i \in \{1, \ldots, l-1\}\).

The suffix array can be computed in \(O(l)\) running time.

**Lemma 1 ([14]).** A suffix array for a string \(u\) can be constructed in \(O(|u|)\) running time.

Let \(\text{constructSuffixArray}(u)\) be a procedure that constructs a suffix array for a string \(u\).

### 3 Algorithms

Let us formally present the problem.

**Problem.** For some positive integers \(n\) and \(m\), we have a sequence of strings \(s = (s^1, \ldots, s^m)\). Each \(s^j = (s^j_1, \ldots, s^j_l) \in \Sigma^l\) where \(\Sigma\) is some finite size alphabet and \(l = |s^j|\). We call \(s\) dictionary. Additionally, we have a string \(t\) of length \(n\), where \(t = (t_1, \ldots, t_n) \in \Sigma^n\). We call \(t\) text. The problem is searching a subsequence \(s^{j_1}, \ldots, s^{j_z}\) and positions \(q_1, \ldots, q_z\) such that \(q_1 = 1, q_z = n - |s^{j_z}| + 1, q_j \leq q_{j-1} + |s^{j-1}|\) for \(j \in \{2, \ldots, z\}\). Additionally, \(t[q_j, q_j + |s^{j}| - 1] = s^{j}\) for \(j \in \{1, \ldots, z\}\).

For simplicity, we assume that \(\Sigma = \{0, 1\}\), but all results are right for any finite alphabet.

Informally, we want to construct \(t\) from \(s\) with possible intersections.

Firstly, let us present a classical algorithm.

#### 3.1 A Classical Algorithm

Let us present the algorithm. Let \(\text{long}_i\) be an index of a longest string from \(s\) that can start in position \(i\). Formally, \(\text{long}_i = j\) if \(s^j\) is a longest string from \(s\) such that \(t[i, i + |s^j| - 1] = s^j\). Let \(\text{long}_i = -1\) if there is no such string \(s^j\). If we construct such
Algorithm 1 \textsc{ConstructQI}(long). Constructing $Q$ and $I$ from $last$

\begin{verbatim}
t ← 1
\textcolor{gray}{i_1 ← long_1}
q_1 ← 1
\textcolor{gray}{left ← 2}
right ← |s^{i_1}| + 1
\textcolor{gray}{while q_t < n do}
    max_i ← left
    max_q ← -1
    if long_{left} > 0 then
        max_q ← left + |s^{long_{left}}| - 1
    end if
    for j ∈ \{left + 1, \ldots , right\} do
        if long_{j} > 0 and j + |s^{long_{j}}| - 1 > max_q then
            max_j ← j
            max_q ← j + |s^{long_{j}}| - 1
        end if
    end for
    if max_q = -1 or max_q < right then
        \textcolor{gray}{Break the While loop and return NULL ⊲ We cannot construct other part of the string t}
    end if
    t ← t + 1
    \textcolor{gray}{i_t ← long_{max_j}}
    q_t ← max_j
    left ← right + 1
    right ← max_q + 1
end while
\textcolor{gray}{return (Q, I)}
\end{verbatim}
array, then we can construct \( Q = (q_1, \ldots, q_z) \) and \( I = (i_1, \ldots, i_z) \) that is solution of the problem in \( O(n) \). A procedure \CONSTRUCTQI\( (\text{long}) \) from Algorithm 1 shows it. If there is no such decomposition of \( t \), then the procedure returns \( \text{NULL} \).

Let us discuss how to construct \( \text{long} \) array.

As a first step, we choose a prime \( p \) that is used for rolling hash. We choose \( p \) randomly from the first \( z = n \cdot m \cdot 4[\log_2 n]^2 \cdot \frac{1}{\varepsilon} \) primes. In that case, due to results from Section 2.2, the probability of error is at most \( \varepsilon \) in a case of at most \( m \cdot 4[\log_2 n] \) strings comparing invocations.

As a second step, we construct a suffix array \( \text{suf} \) for \( t \). Then, we consider an array \( a \) of pairs \((\text{len}, \text{ind})\). One element of \( a \) corresponds to one element of the suffix array \( \text{suf} \). After that, we construct a segment tree \( s_t \) for \( a \) and use \( \text{len} \) parameter of pair for maximum.

As a next step, we consider strings \( s^i \) for \( i \in \{1, \ldots, m\} \). For each string \( s^i \) we find the smallest index \( \text{low} \) and the biggest index \( \text{high} \) such that all suffixes \( t[\text{suf}_j, n] \) for \( \text{low} \leq j \leq \text{high} \) has \( s^i \) as a prefix. We can use binary search for this action. Because of sorted order of suffixes in suffix array, all suffixes with the prefix \( s^i \) are situated sequentially. As a comparator for strings, we use \COMPARISON\ procedure. Let us present this action as a procedure \SEARCHSEGMENT\( (u) \) in Algorithm 2. The algorithm returns \((\text{NULL}, \text{NULL})\) if no suffix of \( t \) contains \( u \) string as a prefix.

Then, we update values in the segment tree \( s_t \) by a pair \(|s^i|, i\).

After processing all strings from \((s^1, \ldots, s^m)\), the array \( a \) is constructed. We can construct \( \text{long} \) array using \( a \) and the suffix array \( \text{suf} \). We know that \( i \)-th element \((\text{len}, \text{ind})\) stores the longest possible string \( s^{\text{ind}} \) that starts from \( \text{suf}_i \). It is almost definition of \( \text{long} \) array. So we can put \( \text{long}_{\text{suf}_i} = \text{ind} \), if \( a_i = (\text{ind}, \text{len}) \).

Finally, we get the following Algorithm 3 for the text constructing from a dictionary problem.

Let us discuss properties of Algorithm 3.

**Theorem 2.** Algorithm 3 solves the text \( t \) constructing from a dictionary \( s = (s^1, \ldots, s^m) \) problem with \( O(n + L + m[\log n]^2 - \log \varepsilon) \) running time end error probability \( \varepsilon \) for some \( \varepsilon > 0 \), \( n = |t| \) and \( L = |s^1| + \cdots + |s^m| \). The running time is \( O(n + L + m[\log n]^2) \) in a case of \( \varepsilon = \text{const} \).

**Proof.** The correctness of the algorithm follows from construction.

Let us discuss running time of the algorithm. Note, that \( m \leq L = \sum_{j=1}^m |s^j| \) and \( 1 \leq |s^j| \leq n \) for \( j \in \{1, \ldots, m\} \).

Due to results from Section 2.2, \COMPUTEKI\ works with \( O(n + \log p) \) running time. Let us convert this statement.

\[
O(n + \log p) = O(n + \log (n \cdot \alpha/\varepsilon)) =
\]
\[
= O(n + \log n + 2\log \alpha - \log \varepsilon) = O(n + \log \alpha - \log \varepsilon) =
\]
\[
= O(n + \log (m \cdot 4[\log_2 n]^2) - \log \varepsilon) = O(n + \log m + \log \log n - \log \varepsilon) =
\]
\[
= O(n + \log m - \log \varepsilon)
\]

Due to results from Section 2.2, \COMPUTEPRWITHROLLINGHASHESES\ works in linear running time. Therefore, all invocations of \COMPUTEPRWITHROLLINGHASHESES\ procedure works in \( O(n + \sum_{j=1}^m |s^j|) = O(n + L) \) running time.
Algorithm 2 SEARCHSEGMENT(u). Searching a indexes segment of suffixes for \( t \) that have \( u \) as a prefix.

\[
\text{low} \leftarrow \text{NULL}, \text{high} \leftarrow \text{NULL} \\
l \leftarrow 1 \\
r \leftarrow n \\
\text{Found} \leftarrow \text{False} \\
\text{while } \text{Found} = \text{False} \text{ and } l \leq r \text{ do} \\
\quad \text{mid} \leftarrow \lfloor (l + r)/2 \rfloor \\
\quad \text{pref} \leftarrow t[\text{su}f_{\text{mid}}, \min(n, \text{su}f_{\text{mid}} + |u| - 1)] \\
\quad \text{pref1} \leftarrow t[\text{su}f_{\text{mid} - 1}, \min(n, \text{su}f_{\text{mid} - 1} + |u| - 1)] \\
\quad \text{compareRes} \leftarrow \text{COMPARE}([\text{pref}, u]) \text{, compareRes1} \leftarrow \text{COMPARE}([\text{pref1}, u]) \\
\quad \text{if } \text{compareRes} = 0 \text{ and } \text{compareRes1} = -1 \text{ then} \\
\quad \quad \text{Found} \leftarrow \text{true} \\
\quad \quad \text{low} \leftarrow \text{mid} \\
\quad \text{end if} \\
\quad \text{if } \text{compareRes} < 0 \text{ then} \\
\quad \quad \text{l} \leftarrow \text{mid} + 1 \\
\quad \text{end if} \\
\quad \text{if } \text{compareRes} \geq 0 \text{ then} \\
\quad \quad \text{r} \leftarrow \text{mid} - 1 \\
\quad \text{end if} \\
\text{end while} \\
\text{if } \text{Found} = \text{True} \text{ then} \\
\quad l \leftarrow 1 \\
\quad r \leftarrow n \\
\text{Found} \leftarrow \text{False} \\
\text{while } \text{Found} = \text{False} \text{ and } l \leq r \text{ do} \\
\quad \text{mid} \leftarrow \lfloor (l + r)/2 \rfloor \\
\quad \text{pref} \leftarrow t[\text{su}f_{\text{mid}}, \min(n, \text{su}f_{\text{mid}} + |u| - 1)] \\
\quad \text{pref1} \leftarrow t[\text{su}f_{\text{mid} + 1}, \min(n, \text{su}f_{\text{mid} + 1} + |u| - 1)] \\
\quad \text{compareRes} \leftarrow \text{COMPARE}([\text{pref}, u]) \text{, compareRes1} \leftarrow \text{COMPARE}([\text{pref1}, u]) \\
\quad \text{if } \text{compareRes} = 0 \text{ and } \text{compareRes1} = +1 \text{ then} \\
\quad \quad \text{Found} \leftarrow \text{true} \\
\quad \quad \text{high} \leftarrow \text{mid} \\
\quad \text{end if} \\
\quad \text{if } \text{compareRes} \leq 0 \text{ then} \\
\quad \quad \text{l} \leftarrow \text{mid} + 1 \\
\quad \text{end if} \\
\quad \text{if } \text{compareRes} > 0 \text{ then} \\
\quad \quad \text{r} \leftarrow \text{mid} - 1 \\
\quad \text{end if} \\
\text{end while} \\
\text{return } (\text{low}, \text{high})
Algorithm 3 The classical algorithm for the text $t$ constructing from a dictionary $s$ problem for an error probability $\varepsilon > 0$

\[
\begin{align*}
\alpha & \leftarrow m \cdot 4 \lceil \log_2 n \rceil^2 \\
r & \leftarrow n \cdot \alpha / \varepsilon \\
p & \in \mathbb{R} \{p_1, \ldots, p_r\} \\
\text{ComputeKI}(n, p) \\
\text{ComputePrefixRollingHashes}(t, p) \\
\text{for } j \in \{1, \ldots, m\} \text{ do} \\
\quad \text{ComputePrefixRollingHashes}(s^j, p) \\
\text{end for} \\
suf & \leftarrow \text{ConstructSuffixArray}(t) \\
a & \leftarrow [(0, -1), \ldots, (0, -1)] \\
st & \leftarrow \text{ConstructSegmentTree}(a) \\
\text{for } j \in \{1, \ldots, m\} \text{ do} \\
\quad (\text{low, high}) & \leftarrow \text{SearchSegment}(s^j) \\
\quad \text{if } (\text{low, high}) \neq (\text{NULL, NULL}) \text{ then} \\
\quad \quad \text{Update}(st, \text{low}, \text{high}, ([s^j], j)) \\
\text{end if} \\
\text{end for} \\
\text{Push}(st) \\
\text{for } i \in \{1, \ldots, n\} \text{ do} \\
\quad (\text{len, ind}) & \leftarrow \text{Request}(st, i) \\
\quad \text{long}_{suf_i} & \leftarrow \text{ind} \\
\text{end for} \\
(Q, I) & \leftarrow \text{ConstructQI}(\text{long}) \\
\text{return } (Q, I)
\end{align*}
\]

Due to Lemma 1, \text{ConstructSuffixArray} works in $O(n)$ running time. Initial-
\begin{align*}
\text{izing of } a \text{ does } O(n) \text{ steps. Due to results from Section 2.3, \text{ConstructSegmentTree} works in } O(n) \text{ running time.} \\
\text{SearchSegment procedure invokes \text{Compare procedure } O(\log n) \text{ times due to binary search complexity. \text{Compare procedure works in } O(\log n) \text{ running time. Therefore, \text{SearchSegment works in } O((\log n)^2) \text{ running time. Due to results from Section 2.3, \text{Update procedure works in } O(\log n) \text{ running time. Hence, the total complexity of processing all strings from the dictionary } s \text{ is } O(m \cdot ((\log n)^2 + \log n)) = O(m \cdot (\log n)^2).} \\
\text{The invocation of } \text{Push works in } O(n) \text{ running time due to results from Section 2.3. The invocation of \text{Request works in } O(1) \text{ running time because we do not invoke \text{Update after Push. Therefore, constructing of the array long takes } O(n) \text{ steps.} \\
\text{The running time of } \text{ConstructQI is } O(n) \text{ because we pass each element only once.} \\
\text{So, the total complexity of the algorithm is } \\
O \left( n + \log m \cdot \log \varepsilon + n + L + n + n + m(\log n)^2 + n + n + n \right) = \\
= O \left( n + L + m(\log n)^2 - \log \varepsilon \right).
\end{align*}

Let us discuss the error probability. We have $4 \cdot m \lceil \log_2 n \rceil$ invocations of \text{Compare procedure. Each invocation of \text{Compare procedure compares rolling hashes at most } \lceil \log_2 n \rceil \text{ times. Due to results from Section 2.2, if we compare strings of length at most } n \text{ using rolling hash } 4 \cdot m \lceil \log_2 n \rceil^2 \text{ times and choose } p \text{ from } r \text{ primes, then we}
get error probability at most $\varepsilon$. □

### 3.2 A Quantum Algorithm

Firstly, let us discuss a quantum subroutine. There is a quantum algorithm for comparing two strings in a lexicographical order with the following property:

**Lemma 3** ([10]). There is a quantum algorithm that compares two strings of length $k$ in lexicographical order with query complexity $O(\sqrt{k\log \gamma})$ and error probability $O\left(\frac{1}{\gamma^3}\right)$ for some positive integer $\gamma$.

Let $Q\text{COMPARE}\_\text{strings\_base}(u, v, l)$ be a quantum subroutine for comparing two strings $u$ and $v$ of length $l$ in lexicographical order. We choose $\gamma = m \log n$. In fact, the procedure compares prefixes of $u$ and $v$ of length $l$. $Q\text{COMPARE}\_\text{strings\_base}(u, v, l)$ returns 1 if $u > v$; it returns $-1$ if $u < v$; and it returns 0 if $u = v$.

Next, we use a $Q\text{COMPARE}(u, v)$ that compares $u$ and $v$ string in lexicographical order. Assume that $|u| < |v|$. Then, if $u$ is a prefix of $v$, then $u < v$. If $u$ is not a prefix of $v$, then the result is the same as for $Q\text{COMPARE}\_\text{strings\_base}(u, v, |u|)$. In the case of $|u| > |v|$, the algorithm is similar. The idea is presented in Algorithm 4.

#### Algorithm 4

The quantum algorithm for comparing two string in lexicographical order

```plaintext
if |u| = |v| then
   Result ← $Q\text{COMPARE}\_\text{strings\_base}(u, v, |u|)$
end if
if |u| < |v| then
   if Result = 0 then
      Result = -1
   end if
   Result ← $Q\text{COMPARE}\_\text{strings\_base}(u, v, |u|)$
end if
if |u| > |v| then
   Result ← $Q\text{COMPARE}\_\text{strings\_base}(u, v, |v|)$
   if Result = 0 then
      Result = 1
   end if
end if
return Result
```

Let us present a quantum algorithm for the text constructing form a dictionary problem. For the algorithm, we use the same idea as in the classical case, but we replace $\text{COMPARE}$ that uses the rolling hash function by $Q\text{COMPARE}$. In that case, we should not construct rolling hashes. Let $Q\text{SEARCH}\_\text{SEGMENT}$ be a quantum counterpart of $\text{SEARCH}\_\text{SEGMENT}$ that uses $Q\text{COMPARE}$.

The quantum algorithm is presented as Algorithm 5.

Let us discuss properties of Algorithm 5.

**Theorem 4.** Algorithm 5 solves the text $t$ constructing from a dictionary $s = (s^1, \ldots, s^m)$ problem in $O\left(n + \log n \cdot (\log m + \log \log n) \cdot \sqrt{m \cdot L}\right)$ running time end error probability $O\left(\frac{1}{m + \log n}\right)$. 

9
Algorithm 5 The quantum algorithm for the text $t$ constructing from a dictionary $s$ problem

```
suf ← CONSTRUCTSUFFIXARRAY($t$)
a ← [(0, −1), . . . , (0, −1)]  /* Initialization by 0-array*/
st ← CONSTRUCTSEGMENTTREE(a)
for $j \in \{1, \ldots , m\}$ do
    (low, high) ← QSEARCHSEGMENT($s_j$)
    UPDATE(st, low, high, (|$s_j$|, $j$))
end for
Push(st)
for $i \in \{1, \ldots , n\}$ do
    (len, ind) ← REQUEST(st, $i$)
    long suf$_i$ ← ind
end for
(Q, I) ← CONSTRUCTQI(long)
return (Q, I)
```

Proof. The algorithm does almost the same actions as the classical counterpart. That is why the correctness of the algorithm follows from Theorem 2.

Let us discuss the running time. Due to Theorem 2, the running time of the procedure CONSTRUCTSUFFIXARRAY is $O(n)$, the running time of the procedure CONSTRUCTSEGMENTTREE is $O(n)$, the running time of the procedure Push is $O(n)$, the running time of the array long construction is $O(n)$, the running time of CONSTRUCTQI is $O(n)$.

Due to Lemma 3, the running time of QCOMPARE for $s_j$ is $O(\sqrt{|s_j|}(\log m + \log \log n))$. The procedure QSEARCHSEGMENT invokes QCOMPARE procedure $O(\log n)$ times for each string $s^1, \ldots s^m$. So, the complexity of processing all strings from $s$ is

$$O \left( \log n \cdot (\log m + \log \log n) \cdot \sum_{j=1}^{m} \sqrt{|s_j|} \right)$$

Let us use the Cauchy-Bunyakovsky-Schwarz inequality and $L = \sum_{j=1}^{m} |s_j|$ equality for simplifying the statement.

$$\leq O \left( \log n \cdot (\log m + \log \log n) \cdot \sqrt{m \sum_{j=1}^{m} |s_j|} \right) = O \left( \log n \cdot (\log m + \log \log n) \cdot \sqrt{m \cdot L} \right).$$

The total running time is

$$O \left( n + n + n + \log n \cdot (\log m + \log \log n) \cdot \sqrt{m \cdot L} \right) =$$

$$= O \left( n + \log n \cdot (\log m + \log \log n) \cdot \sqrt{m \cdot L} \right)$$

Let us discuss the error probability. The algorithm invokes QCOMPARE procedure $2m[\log_2 n] \leq 2m(1 + \log_2 n)$ times. The success probability is

$$O \left( \left( 1 - \frac{1}{(m \log n)^3} \right)^{2m(1+\log_2 n)} \right) = O \left( \frac{1}{m \log n} \right).$$
4 Lower Bound

Let us discuss the lower bound for the running time of classical algorithms.

**Theorem 5.** Any randomized algorithm for the text $t$ constructing from a dictionary $s = (s^1, \ldots, s^m)$ problem works in $\Omega(n + L)$ running time, where $L = |s^1| + \cdots + |s^m|$.

**Proof.** Assume $L > n$. Let us consider $t = (t_1, \ldots, t_n)$ such that $t_{\lfloor n/2 \rfloor} = 1$ and $t_i = 0$ for all $i \in \{1, \ldots, n\}\backslash\{\lfloor n/2 \rfloor\}$.

Let $|s^i| \leq n/2$ for each $i \in \{1, \ldots, m\}$. Note, that in a general case $|s^i| \leq n$.

Therefore, we reduce the input data only at most twice. Assume that we have two options:

- all $s^i$ contains only 0s;
- there is $z$ such that we have two conditions:
  - all $s^i$ contains only 0s, for $i \in \{1, \ldots, m\}\backslash\{z\}$;
  - for $j_0 < |s^z|$, $s^z_{j_0} = 1$ and $s^z_j = 0$ for $j \in \{1, \ldots, |s^z|\}\backslash\{j_0\}$.

In a case of all 0s, we cannot construct the text $t$. In a case of existing 0 in the first half, we can construct $t$ by putting $s^z$ on the position $j_0 - \lfloor n/2 \rfloor + 1$ and we get 1 of the required position. Then, we complete $t$ by other 0-strings.

Therefore, the solution of the problem is equivalent to the search of 1 in unstructured data of size $L$. The randomized complexity of this problem is $\Omega(L)$ due to [4].

Assume $L < n$. Let $m = 1$, $|s^1| = 1$ and $s^1_1 = 1$. Assume that we have two options:

- $t$ contains only 1s;
- there is $g$ such that $t_g = 0$ and $t_j = 1$ for all $j \in \{1, \ldots, n\}\backslash\{g\}$.

In the first case, we can construct $t$ from $s$. In the second case, we cannot do it.

Here the problem is equivalent to search 0 among $n$ symbols of $t$. Therefore, the problem’s randomized complexity is $\Omega(n)$.

Finally, the total complexity is $\Omega(\max(n, L)) = \Omega(n + L)$. □

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