Unified Theory of Ideals

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Unified field theories try to merge the gauge groups of the Standard Model into a single group. Here we lay out something different. We give evidence that the Standard Model can be reformulated simply in terms of numbers in the algebra \( \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O} \), as with the earlier work of Dixon \[1\].

Gauge bosons and the fermions they act on are unified together in the same algebra, as are the Lorentz transformations and the objects they act on. The theory aims to unify \textit{everything} into the algebra \( \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O} \). To set the foundation, we show this to be the case for a single generation of left-handed particles. In writing the theory down, we are not building a vector space structure, and then placing \( \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O} \) numbers in as the components. On the contrary, it is the vector spaces which come out of \( \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O} \).

**Introduction.** The Standard Model “has emerged as the best distillation of decades of research” \[2\]. Despite its experimental feats, though, it continues to be a complicated theory. The Standard Model is based on a curious list of groups (Lorentz, \( U(1) \), \( SU(2) \), \( SU(3) \)) acting on an even greater multitude of vector spaces. Surely a theory which works so reliably deserves a concise description.

We aim to show that this more concise description is provided by the division algebras. The division algebras are by no means new to physics; most theory, both classical and quantum, is described already in terms of the real, \( \mathbb{R} \), and complex numbers, \( \mathbb{C} \). Furthermore, the group \( SU(2) \) is everywhere in physics, and its connection to the quaternions, \( \mathbb{H} \), is well known. It is nearly irresistible to ask if the octonions, \( \mathbb{O} \), the last of the set of four normed division algebras over \( \mathbb{R} \), have a calling in nature. Certainly several have thought so: \[1\], \[3\]–\[10\], but for the most part, the octonions have remained hidden from mainstream physics.

In \[1\], Dixon gives the first well-known proposal for the connection between the Standard Model and the full algebra \( \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O} \). The work presented here may be viewed as an independent development of the same basic idea as \[1\], which differs in the algebra’s implementation. In \[3\], an octonionic representation of a Clifford algebra in four dimensions is demonstrated, while \[4\], \[5\], \[6\] and \[7\] show structure of the Standard Model provided by \( \mathbb{O} \). Baez and Huerta, \[8\], explain how supersymmetry arises from the existence of the division algebras. References \[9\] and \[10\] give both an introduction to \( \mathbb{O} \): while \[9\] notes applications to quantum logic, special relativity and supersymmetry, \[10\] discusses the algebra of observables and non-associative gauge theory.

More often than not, though, the octonions are passed by in haste because they are non-associative, and hence at times temperamental. As we will show, this property is in fact a gift, which will allow us to begin to streamline the Standard Model’s complex structure.

This paper uses what we call an \textit{ideal representation}: a group representation expressed in terms of the action of an algebra on one of its (generalized) ideals. We show that Lorentz group- and \( SU(3) \otimes SU(2) \otimes U(1) \)-representations of the Standard Model can be redrawn in terms of ideal representations of \( \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O} \).

**Prerequisites.** Little algebraic background is needed to understand the following pages, so we provide it here.

For a more generous introduction to the octonions see \[9\].

Any three imaginary units on a directed line segment in Figure 1 act as if they were a quaternionic triple.

\[
i^2 = j^2 = k^2 = ijk = -1,
\]

from which we get \( ij = -ji = k \), \( jk = -kj = i \), \( ki = -ik = j \).

The generic element of \( \mathbb{C} \otimes \mathbb{H} \) is written \( a + bi + cj + dk \) where \( a, b, c, d \in \mathbb{C} \). \( i, j, k \) follow the non-commutative quaternionic multiplication rules

\[
j^2 = k^2 = i^2 = -ij = -ji = k \quad jk = -kj = i \quad ki = -ik = j
\]

The generic element of \( \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O} \) is \( \sum_{n=0}^{7} A_n e_n \), with the \( A_n \in \mathbb{C} \). The \( e_n \) are octonionic imaginary units \( e_n^2 = -1 \), apart from \( e_0 = 1 \), which multiply according to Figure 11.

Any three imaginary units on a directed line segment in Figure 11 act as if they were a quaternionic triple. For example, \( e_6 e_1 = -e_1 e_6 = e_5 \), \( e_1 e_3 = -e_3 e_1 = e_5 \), \( e_5 e_6 = -e_6 e_5 = e_1 \), \( e_4 e_1 = -e_1 e_4 = e_2 \), etc. Octonionic multiplication harbours various symmetries, such as index doubling symmetry: \( e_i e_j = e_k \Rightarrow e_2 i e_2 j = e_k \), which can be seen by rotating Figure 11 by 120 degrees \( \Phi \).

For a more generous introduction to the octonions see \[9\] and \[10\].

The generic element of \( \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O} \) is \( \sum_{n=0}^{7} B_n e_n \), where the \( B_n \in \mathbb{C} \otimes \mathbb{H} \). Imaginary units of the different division algebras always commute with each other; explicitly, the

\[
B_n e_n = B_n e_0 e_n = B_n e_n = e_n B_n e_0 = e_n B_n
\]
complex \(i\) commutes with the quaternionic \(i, j, k\), all four of which commute with the octonionic \(e_n\).

We define a subalgebra \(I\) of an algebra \(A\) to be an ideal if \(m(a, \psi) \in I, \forall \psi \in I\) and for any \(a \in A\), where \(m\) is multiplication. Directly from the definition it can be seen that an ideal is the algebra’s resilient subspace, whose elements persist no matter what \(a\) is multiplied upon them \([12]\).

In what is to follow, we will find the invariant subspaces of \(\mathbb{C} \otimes \mathbb{H}\) under three separate actions of the algebra on itself: \(m(a, \psi) = a\psi\), \(m(a, \psi) = a \bar{\psi} \dagger\) and \(m(a, \psi) = a \bar{\psi} a\). Only in the first case, \(m(a, \psi) = a\psi\), does this give an ideal according to its standard definition. For \(m(a, \psi) = a \bar{\psi}\) and \(m(a, \psi) = a \bar{\psi}\), it would also be possible to call these invariant subspaces under an action of an algebra on itself, instead of ideals. However, this is cumbersome, and obscures the fact that conceptually they are simply generalized ideals, provided that we are permitted to consider multiplication to be a map which takes two elements in the algebra and returns a third.

**Ideals in \(\mathbb{C} \otimes \mathbb{H}\).** In writing the well known Lorentz representations of \(\mathbb{C} \otimes \mathbb{H}\) \([12, 13]\), we can see the surprising extent of this unification. The groups are unified with the vectors they transform, and further, all of the different particles are born from the same meagre algebra.

We start with ideals under left multiplication, from which Weyl spinors will arise:

\[
\psi' = a \psi,
\]

Any number in \(\mathbb{C} \otimes \mathbb{H}\) can be written \(c_1 + i k c_2 + c_3 i k c_4 + c_5 i k c_6 + c_7 i k c_8\), where the \(c_n \in \mathbb{C}\). It is amazing but true that the forms of \(\psi_1 = c_1 + i k c_2 + c_3 i k c_4\) and \(\psi_2 = c_5 + i k c_6 + c_7 i k c_8\) are stable against multiplication by any \(a\) in \(\mathbb{C} \otimes \mathbb{H}\):

\[
\begin{align*}
&\psi_1 = a (c_1 + i k c_2 + c_3 i k c_4) = c'_1 + i k c'_2 + c'_3 i k c'_4 \equiv \psi'_1, \\
&\psi_2 = a (c_5 + i k c_6 + c_7 i k c_8) = c'_5 + i k c'_6 + c'_7 i k c'_8 \equiv \psi'_2,
\end{align*}
\]

where \(\psi'_1\) and \(\psi'_2\) are two ideals under left multiplication. Clearly this should have consequences for mass, as it causes transitions between left and right handed spinors.

Perhaps the most compelling aspect of Unified Theory of Ideals is that it takes care of the patchwork which is usually involved when carrying out calculations. As an example, we now show an especially clean way to conjugate Weyl spinors.

When Weyl spinors are introduced in many textbooks, it is noted that the “complex conjugate” of the left column vector \(\psi_L\) transforms as a right-handed spinor, and vice-versa. However, the “complex conjugate” is not really the complex conjugate. The ad-hoc matrix \(\epsilon = -i s_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) is deployed to enforce that things work out, and as a result, the “complex conjugate” is defined as \(\epsilon \psi_L\). Clumsier still, in order to return back to \(\psi_L\), one must import an extra factor of \(-1\) so that \(\psi_L = -\epsilon (\epsilon \psi_L)^*\), instead of just taking the “complex conjugate” twice.

In the ideal representation, however, the true complex conjugate here knows exactly what to do: \(\psi_L^* = \psi_L^*[\uparrow L]^* + \psi_L^*[\downarrow L]^* = \psi_L^*[\uparrow R] - \psi_L^*[\uparrow R]\), with no white lies necessary.

A new multiplication rule leads to the 4-vector ideal representation:
Any number in $\mathbb{C} \otimes \mathbb{H}$ can be written as a sum of hermitian $\bar{h} = h_0 + h_1i + h_2j + h_3k$ and anti-hermitian $\tilde{h} = ih_4 + h_5i + h_6j + h_7k$ parts, where the $h_n \in \mathbb{R}$, and the hermitian conjugate $a^\dagger \equiv \bar{a}^\ast$. The element $\bar{a}$ is the parity conjugate of $a$, obtained by sending $i, j, k \leftrightarrow -i, -j, -k$ and reversing the order of multiplication. As $a \bar{h} a^\dagger$ is hermitian and $a \bar{h} a^\dagger$ is antihermitean for any $a \in \mathbb{C} \otimes \mathbb{H}$, it is clear that $\mathbb{C} \otimes \mathbb{H}$ splits again into two ideals, this time under the multiplication $m(a, \bar{a}) = a \bar{h} a^\dagger$.

It is shown in [13] that there exists another representation of the Lorentz group which uses this double-sided multiplication. Including again the complex conjugate representation, we have

$$e^{is\bar{h}} e^{-is^j} = h', \quad e^{-is} h^* e^{is} = h^{*'},$$

where $s$ is the same as before. The antihermitean case follows analogously. Matching components, one finds that $h$ transforms as a contravariant four-vector, and $h^*$ as a covariant four-vector. For example, momentum $p = p_0 + p_1i + p_2j + p_3k$ under a rotation about $k$ by an angle $\theta$ is given by

$$p' = e^{-\frac{\theta}{2}p} \bar{e} \bar{p} e^{\frac{\theta}{2}p} = (\cos \frac{\theta}{2} - k \sin \frac{\theta}{2}) p (\cos \frac{\theta}{2} + k \sin \frac{\theta}{2}) = p_0 + (p_1 \cos \theta + p_2 \sin \theta) i i + (p_2 \cos \theta - p_1 \sin \theta) j j + p_3 k k,$$

as expected. As shown in [13], the scalar $\frac{1}{2} (pq + \bar{p}q)$ can now form between a covariant vector $p$ and contravariant vector $q$, which is simply the real part of $pq$. Indeed, when $q = p^\ast$, this gives $p_0^2 - p_1^2 - p_2^2 - p_3^2$.

Scalars and field strength tensors are now shown to come from the multiplication rule

$$\phi' = a \bar{\phi} \bar{a^\dagger}.$$

$\mathbb{C} \otimes \mathbb{H}$ can be split yet again into ideals of the form $\phi = (F^{01} + iF^{03}) i + (F^{13} + iF^{03}) j + (F^{21} + iF^{03}) k$, $F^{mn} \in \mathbb{R}$, which weather the multiplication $a \bar{h} a^\dagger$ from any element of the algebra. As $\phi' = e^{is} \phi e^{is} = \phi$, we see that $\phi$ transforms as a Lorentz scalar. In [12] and [13] it is shown that indeed, massless spin-one bosons are represented by $\mathbb{F}$. Under the Lorentz transformations,

$$e^{is} F e^{-is} = F', \quad e^{-is}\bar{F} e^{is} = \bar{F}'.$$

$F' = (B^1 + iE^1) i + (B^2 + iE^2) j + (B^3 + iE^3) k$ gives the familiar field strength $F_{\mu\nu}$, $\bar{F}$ gives $\bar{F}^{\mu\nu}$, and the Hodge dual $\star F^{\mu\nu}$ is simply $-iF$.

As we have now the spin 0, 1/2 and 1 representations of the Lorentz group (summarized in Figure 2), we have accounted for all of the local spacetime degrees of freedom of the Standard Model.

Bilinears and other scalars can now arise from combining the various ideal representations, whose Lorentz transformations fit together like puzzle pieces. Take for example the real part of the Lorentz group, which has four solutions as opposed to the usual spin up and down pair. For $c = \frac{1}{2}$, we have $\frac{1}{2} = [\uparrow L]$ and $\frac{1}{2} = [\uparrow R]$. For $c = -\frac{1}{2}$, we have $\frac{1}{2} = [\downarrow L]$ and $\frac{1}{2} = [\downarrow R]$. This doubling of solutions has often caused work with $\mathbb{C} \otimes \mathbb{H}$ to be abandoned.

Replacing the quaternionic imaginary $i$ by $e_7$ gives another operator $O = \frac{e_7}{2}$, this time in $\mathbb{C} \otimes \mathbb{O}$. Recall that the octonions can be thought of as a net of quaternionic triples, where each
line in Figure 11 behaves like \{i, j, k\}. As $e_7$ is a member of three such lines, we might expect more than four solutions to the eigenvalue equation $\frac{d}{d\nu} e_7 = c e_7$. In fact there are eight solutions, suggestively named here in Figure 14. They span all of $\mathbb{C} \otimes O$, trivially making them an ideal.

Complex linear combinations of the sequences are equivalent to the set of 8 by 8 complex matrices, as also stated in 11 and 12. In this space, we find the unique set

$$
\begin{align*}
\lambda_1 &= \frac{1}{2} (\mathbf{157} + \mathbf{347}) \\
\lambda_2 &= -\frac{1}{2} (\mathbf{13} + \mathbf{35}) \\
\lambda_3 &= \frac{1}{2} (\mathbf{13} - \mathbf{457}) \\
\lambda_4 &= -\frac{1}{2} (\mathbf{257} + \mathbf{467}) \\
\lambda_5 &= \frac{1}{2} (\mathbf{24} + \mathbf{56}) \\
\lambda_6 &= -\frac{1}{2} (\mathbf{167} + \mathbf{237}) \\
\lambda_7 &= \frac{1}{2} (\mathbf{12} + \mathbf{36}) \\
\lambda_8 &= \frac{1}{2\sqrt{3}} (\mathbf{137} - \mathbf{457} + \mathbf{267}),
\end{align*}
$$

which annihilate leptons and map quarks according to the Gell-Mann matrices. These generators obey the analogue of the commutation relations

$$
\left[ \frac{\lambda_i}{2}, \frac{\lambda_j}{2} \right] = i c_{\ell mn} \frac{\lambda_m}{2} f \quad \forall f,
$$

where $c_{\ell mn}$ are the structure constants of $su(3)$. The difference from the usual commutation relations is that we are accustomed to working out commutators independently of the space they are acting on. Here, the operators are applied to each and every fermion.

**Commuting with this set are the $SU(2)$ generators**

$$
\tau_1 = -\mathbf{124}, \quad \tau_2 = -\mathbf{356}, \quad \tau_3 = i e_7.
$$

In the remaining space which commutes with both of these groups, we have the $U(1)$ generator of weak hypercharge, and lepton and baryon numbers

$$
\begin{align*}
Y_1 &= \frac{1}{6} \left( \mathbf{137} + \mathbf{267} + \mathbf{457} \right) \\
L &= \frac{1}{8} \left( 1 + i e_7 + i e_{13} - i e_{26} - i e_{45} - \mathbf{137} - 3 \mathbf{267} - 3 \mathbf{457} \right) \\
B &= \frac{1}{4} + \frac{1}{12} \mathbf{137} + \frac{1}{12} \mathbf{267} + \frac{1}{12} \mathbf{457}.
\end{align*}
$$

**Speculations.** This work shows that the $SU(3) \otimes SU(2) \otimes U(1)$ gauge bosons fit into the algebra, but as with the Standard Model, it leaves some mysteries hanging: Why is $SU(3) \otimes SU(2) \otimes U(1)$ nature’s favourite, over any other set of Lie groups? Or is it? Why does $SU(2)$ transform the left and not the right?

The Standard Model makes no attempt to answer these questions, but as we have seen already in the discrete colour symmetry example of Figure 10, Unified Theory of Ideals just might. Fortunately, the octonionic symmetries have been well studied; the first place to look for gauge groups is in $G_2$, the exceptional Lie group which forms the automorphism group of the octonions [3], as was proposed in [2], and subsequently in [1].
Acting on the fermion basis vectors, it is possible to find ladder operators which obey the correct anticommutation relations for both the $\mathbb{C} \otimes \mathbb{H}$ and $\mathbb{C} \otimes \mathbb{O}$ parts of the theory. This suggests that the theory might already be quantized, free of theoretical intervention. These anticommutation relations do not need Dirac delta functions. Furthermore, the operators raise and lower states relative to each other, not with respect to the vacuum. If we can use these ladder systems to replace the current framework of QFT, we might finally be able to slip out of the conceptual contortions caused by its zero particle state and Dirac delta functions.

Finally we mention that the operators of this theory appear to live in the Clifford algebra $\mathbb{O}$ generated by the imaginaries of the algebra acting successively on the fermions. This can be seen already in the bosonic operator solutions of this paper. A twist to the story is that a degeneracy in the Clifford algebra acts to cut it in half, so that the sequences of $e_s$ which make up the operators can only come in three levels. It would be interesting to see if these three levels could ultimately explain the three generations of the Standard Model.

Of course all of the above will require further scrutiny in the coming months.

Apart from this current work, Seth Lloyd is leading the development of the theory of Division Networks, a model for quantum gravity in the form of a lattice gauge theory, which is written in the Unified Theory of Ideals formalism.

**Conclusion.** Unified Theory of Ideals puts forward the idea that all of the particles of the Standard Model are simply numbers in the same algebra. This more powerful form of unification aims to describe all of the gauge and spacetime degrees of freedom, using only the 32 complex dimensions of $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$.

**Acknowledgement.** Thank you to: S. Lloyd at MIT for suggesting the name of ideal representations during my visit there, S. Lloyd and T. Konopka for steering me away from some major stylistic disasters, P.L. Mana, B. Hartmann and F. Caravelli for their comments, G. Dixon and J. Koeplinger for telling me of reference [1]. A special thanks to R. Sorkin for his careful insight during this paper’s final stages. Thank you to NSERC, the University of Waterloo and the Perimeter Institute. Research at Perimeter Institute for Theoretical Physics is supported in part by the Government of Canada through NSERC and by the Province of Ontario through MRI.

[1] G. Dixon, *Division Algebras: Octonions, Quaternions, Complex Numbers and the Algebraic Design of Physics*, Kluwer Academic Publishers (1994)

[2] C. Burgess, G. Moore, *The Standard Model: A Primer*, Cambridge University Press (2007)

[3] S. De Leo, K. Abdel-Khalek, *Octonionic Representations of GL (8, R) and GL (4, C)*, J.Math.Phys. 38 (1997) 582–598 [arXiv:hep-th/9607140]

[4] C. Manogue, T. Dray, *Dimensional Reduction*, Mod. Phys. Lett. A14, 9397 (1999) [arXiv:hep-th/9807044]

[5] C. Manogue, T. Dray, *Quaternionic Spin*, Clifford Algebras and their Applications in Mathematical Physics, Birkhäuser, (2000) [arXiv:hep-th/9910010]

[6] F.D.T. Smith, *Hermann Jordan Triple Systems, the Standard Model plus Gravity, and Alpha = 1/137.03608*, Phys.Rev. D49 (1994) 3779-3782 [arXiv:hep-th/9302030]

[7] M. Gümaydın, F. Görsey, *Quark Structure and the Octonions*, J. Math. Phys., Vol. 14, No. 11 (1973)

[8] J. Baez, J. Huerta, *Division Algebras and Supersymmetry* [arXiv:0909.0551]

[9] J. Baez, *The Octonions*, Bull. Amer. Math. Soc. 39 (2002), 145-205 [arXiv:math/0105155]

[10] O. Okubo, *Introduction to Octonion and Other Non-Associative Algebras in Physics*, CUP (1995)

[11] C. Furey, In preparation (2010)

[12] D. Hestenes, *Spaceetime Algebra*, Gordon and Breach Science Publishers (1966)

Here it is stated that Marcel Riesz knew that a spinor is an element of a minimal ideal in a Clifford algebra: M. Riesz, *Dixième Congrès des Mathématiciens Scandinaves*, Copenhagen, (1946) p.123

[13] M. Greiter, D. Schuricht, *Imaginary in all Directions*, Eur.J.Phys. 24 (2003) 397 [arXiv:math-ph/0309061]

[14] Further work relating octonions and Clifford Algebras: A. Bilge, T. Dereci, S. Koçak, *The Geometry of Self-Dual Two-Forms*, J.Math.Phys. 38 (1997) 4804-4814 [arXiv:hep-th/9605060]

S. Okubo, Representations of Clifford Algebras and its Applications, Math.Jap. 41 (1995) 59-79 [arXiv:hep-th/9408165]

J. Schray, C. Manogue, *Octonionic Representations of Clifford Algebras and Triality* [arXiv:hep-th/9407179]

[15] As the ideal will pull any element into itself, it can be thought of as an algebra’s version of a black hole. This is an idea further developed in [11], which details the quantum gravity project of which the present paper is a byproduct. The idea is to identify matter with a web of algebraic expressions. Given that algebraic expressions supply their own causal structure, they can exist self-sufficiently, without the need for an underlying space-time.

[16] In this paper, we show all of the spacetime degrees of freedom coming from the quaternionic part, and all of the gauge degrees of freedom coming from the octonionic part. Reference [1] uses the octonionic part to find different $SU(3)$ and $U(1)$ representations, in accordance with [2], and uses right multiplication of the quaternionic part to get $SU(2)$. (2)