MULTIPLE REPRESENTATIONS OF REAL NUMBERS ON SELF-SIMILAR SETS WITH OVERLAPS

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Abstract. Let $K$ be the attractor of the following IFS
\[ \{ f_1(x) = \lambda x, f_2(x) = \lambda x + c - \lambda, f_3(x) = \lambda x + 1 - \lambda \}, \]
where $f_1(I) \cap f_2(I) \neq \emptyset, (f_1(I) \cup f_2(I)) \cap f_3(I) = \emptyset$, and $I = [0, 1]$ is the convex hull of $K$. Let $K \ast K = \{ x \ast y : x, y \in K \}$, where $\ast = +, -, \cdot$ or $\div$ (if $\ast = \div$, then $y \neq 0$). The main result of this paper is as follows: if $c \geq (1 - \lambda)^2$, then
\[ K \ast K = \{ x \ast y : x, y \in K, y \neq 0 \} = [0, \infty). \]

As a consequence, we also prove that the following conditions are equivalent:
(1) For any $u \in [0, 1]$, there are some $x, y \in K$ such that $u = x \cdot y$;
(2) For any $u \in [0, 1]$, there are some $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \in K$ such that
\[ u = x_1 + x_2 = x_3 - x_4 = x_5 \cdot x_6 = x_7 \div x_8; \]
(3) $c \geq (1 - \lambda)^2$.

1. Introduction

There are many methods which can represent real numbers. For instance, the $\beta$-expansions \cite{30, 2, 6, 7, 14, 13, 16}, the continued fractions \cite{12, 11}, the Liouville expansions \cite{5}, and so on. In this paper, we shall analyze a new representation, that is, the arithmetic representation of real numbers in terms of some self-similar sets with overlaps. Firstly, let us introduce some fundamental definitions and results. Let $A, B \subset \mathbb{R}$ be two non-empty sets. Define
\[ A \ast B = \{ x \ast y : x \in A, y \in B \}, \]
where $\ast$ is $+, -, \cdot$ or $\div$ (if $\ast = \div$, then we assume $y \neq 0$). We call $u = x \ast y$ an arithmetic representation in terms of $A$ and $B$. Steinhaus \cite{37} proved that
\[ C - C = \{ x - y : x, y \in C \} = [-1, 1], \]
where $C$ is the middle-third Cantor set. Recently, Athreya, Reznick and Tyson \cite{3} considered the multiplication on $C$, and proved that
\[ 17/21 \leq \mathcal{L}(C \cdot C) \leq 8/9, \]
where $\mathcal{L}$ denotes the Lebesgue measure. In \cite{28}, Jiang and Xi proved that $C \cdot C$ contains countably many intervals. Moreover, they also came up with a sufficient condition such that the image of $C \times C$ under some continuous functions contains a non-empty interior. The readers can find more results in \cite{48, 36, 21, 1, 40, 20, 24}. Athreya, Reznick and Tyson \cite{3} also investigated the division of $C$, namely the set $C \div C$, and proved that $C \div C$ is precisely the countably union of some closed
intervals. In [27], Jiang and Xi considered the representations of real numbers in $C = C = [-1, 1]$, i.e. let $x \in [-1, 1]$, define

$$S_x = \{(y_1, y_2) : y_1 - y_2 = x, \ (y_1, y_2) \in C \times C\}.$$ and

$$U_r = \{x : x(S_x) = r\}, r \in \mathbb{N}^+.$$ They proved that $\dim_H(U_r) = \frac{\log 2}{\log 3}$ if $r = 2^k$ for some $k \in \mathbb{N}$. Moreover,

$$0 < \mathcal{H}^s(U_1) < \infty, \mathcal{H}^s(U_{2^k}) = \infty, k \in \mathbb{N}^+,$$

where $s = \frac{\log 2}{\log 3}$. $U_{3 \cdot 2^k}$ is an infinitely countable set for any $k \geq 1$, where $\dim_H$ and $\mathcal{H}^s$ denote the Hausdorff dimension and Hausdorff measure, respectively. There are more general results in [27]. In this paper, we shall analyze the following self-similar set with overlaps [23], let $K$ be the attractor generated by the following IFS,

$$\{f_1(x) = \lambda x, f_2(x) = \lambda x + c - \lambda, f_3(x) = \lambda x + 1 - \lambda\},$$

where $f_1(I) \cap f_2(I) \neq \emptyset, (f_1(I) \cup f_2(I)) \cap f_3(I) = \emptyset,$ and $I = [0, 1]$ is the convex hull of $K$. The self-similar set $K$ is a typical example which allows serious overlaps. Many people analyzed this example from various aspects. Furstenberg conjectured that the following self-similar set with the equation

$$K_1 = \frac{K_1}{3} \cup \frac{K_1 + \alpha}{3} \cup \frac{K_1 + 2}{3}$$

has Hausdorff dimension for any irrational number $\alpha$. Hochman [22] made use of elegant methods from ergodic theory proving that this conjecture is correct. Keyon [29], Rao and Wen [39] proved that $\mathcal{H}^1(\Lambda) > 0$ if and only if $\lambda = p/q \in \mathbb{Q}$ with $p \equiv q \neq (0 \equiv 3)$. Feng and Lau [15] gave a multifractal analysis of $K$ when $K$ satisfies the weak separation property. Yao and Li [47] analyzed the points in $K$ with finite type condition, and an algorithm that can calculate the Hausdorff dimension of $K$ when $K$ is of finite type. In [9, 10], Dajani et al. analyzed the points in $K$ with multiple codings, and gave some examples such that the set of points with exactly 3 codings can be empty while the set of points with precisely 2 codings has the same Hausdorff dimension of the univoque set. In [18], Guo et al. in terms of some ideas from [35], considered the bi-Lipschitz equivalence of overlapping self-similar sets. In [25], Jiang, Wang and Xi gave a necessary condition that when $c = \lambda - \lambda^2$ the self-similar set $K$ is bi-Lipschitz equivalent to another self-similar set with the strong separation condition. In [38], Tian et al. proved that $K \cdot K = [0, 1]$ and if $c \geq (1 - \lambda)^2$. This result is sharp. Moreover, Jiang and Xi [28] also proved the following result. Suppose that $f$ is a continuous function defined on an open set $U \subset \mathbb{R}^2$. Denote the image

$$f_U(K, K) = \{f(x, y) : (x, y) \in (K \times K) \cap U\}.$$ If $\partial_x f, \partial_y f$ are continuous on $U$, and there is a point $(x_0, y_0) \in (K \times K) \cap U$ such that one of the following conditions is satisfied,

$$\max \left\{ \frac{1 - c - \lambda}{\lambda}, \frac{1 - \lambda}{1 - c} \right\} \leq \frac{\partial_y f(x_0, y_0)}{\partial_x f(x_0, y_0)} \leq \frac{1}{1 - c - \lambda}.$$
or
\[
\max \left\{ \frac{1 - c - \lambda}{\lambda}, \frac{1 - \lambda}{1 - c} \right\} < \left| \frac{\partial_x f_1(x_0,y_0)}{\partial_y f_1(x_0,y_0)} \right| < \frac{1}{1 - c - \lambda},
\]
then \( f_U(K,K) \) has a non-empty interior. We emphasize that it is difficult to obtain the above results if one utilizes the Newhouse thinkness theorem \([41]\). The main reason is that it is not easy to calculate the thinkness of \( K \) as there are very complicated overlaps in \( K \). For the Assouad dimension of \( K \) and the geodesic distance on \( K \times K \), we refer to \([32, 34, 46, 45, 33, 49, 42, 44, 43]\).

We have mentioned many results concerning with \( K \) from different perspective. In this paper we shall consider the multiple representations, i.e. addition, substraction, multiplication and division, on \( K \). This is the main motivation of this paper. In fact, similar analysis appears in the setting of \( \beta \)-expansions. For instance, the multiple \( \beta \)-expansions and simultaneous expansions are considered by Komornik, Pedicini, and Pethő \([31]\), Dajani, Jiang and Kempton \([8]\), Hare and Sidorov \([19, 20]\), Dajani et al. \([10, 9]\). The simultaneous expansions are related with the interior of the associated self-affine sets. For more applications of the simultaneous expansions, see \([17]\).

The following are the main results of this paper.

**Theorem 1.1.** Let \( K \) be the attractors of the following IFS
\[
\{f_1(x) = \lambda x, f_2(x) = \lambda x + c - \lambda, f_3(x) = \lambda x + 1 - \lambda\},
\]
where \( f_1(I) \cap f_2(I) \neq \emptyset, (f_1(I) \cup f_2(I)) \cap f_3(I) = \emptyset \), and \( I = [0,1] \) is the convex hull of \( K \). If \( c \geq (1 - \lambda)^2 \), then
\[
\frac{K}{\bar{K}} = \left\{ \frac{x}{y} : x, y \in K, y \neq 0 \right\} = [0,\infty).
\]

**Corollary 1.2.** Let \( K \) be the attractors defined in the above theorem. Then the following conditions are equivalent:

1. For any \( u \in [0,1] \), there are some \( x, y \in K \) such that \( u = x \cdot y \);
2. For any \( u \in [0,1] \), there are some \( x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \in K \) such that
\[
u = x_1 + x_2 = x_3 - x_4 = x_5 + x_6 = x_7 \div x_8;
\]
3. \( c \geq (1 - \lambda)^2 \).

This paper is arranged as follows. In section 2, we give the proofs of Theorem 1.1 and Corollary 1.2. In section 3, we give some remarks.

### 2. Proofs of Theorem 1.1 and Corollary 1.2

Let \( H = [0,1] \). For any \( (i_1, \cdots, i_n) \in \{1, 2, 3\}^n \), we call \( f_{i_1, \cdots, i_n}(H) = (f_{i_1} \circ \cdots \circ f_{i_n})(H) \) a basic interval of rank \( n \), which has length \( \lambda^n \). Denote by \( H_n \) the collection of all these basic intervals of rank \( n \). Let \( J \in H_n \), then \( \bar{J} = \bigcup_{i=1}^{3} I_{n+1,i} \), where \( I_{n+1,i} \subset H_{n+1} \) and \( I_{n+1,i} \subset J \) for \( i = 1, 2, 3 \). Let \( [A,B] \subset [0,1] \), where \( A \) and \( B \) are the left and right endpoints of some basic intervals in \( H_k \) for some \( k \geq 1 \), respectively. \( A \) and \( B \) may not be in the same basic interval. Let \( F_k \) be the collection of all the basic intervals in \([A,B]\) with length \( \lambda^k, k \geq k_0 \) for some \( k_0 \in \mathbb{N}^+ \), i.e. the union of all the elements of \( F_k \) is denoted by \( G_k = \bigcup_{i=1}^{t_k} I_{k,i} \), where \( t_k \in \mathbb{N}^+, I_{k,i} \subset H_k \) and \( I_{k,i} \subset [A,B] \). Clearly, by the definition of \( G_n \), it follows that \( G_{n+1} \subset G_n \) for any \( n \geq k_0 \). Similarly, suppose that \( M \) and \( N \) are the left and right endpoints of some basic intervals in \( H_k \). Denote by \( G'_k \) the union of all the
basic intervals with length $\lambda^k$ in the interval $[M, N]$, i.e. $G'_k = \bigcup_{j=1}^{t'_k} J_{k,j}$, where $t'_k \in \mathbb{N}^+$, $J_{k,j} \in H_k$ and $J_{k,j} \subset [M, N]$.

Very useful is the following lemma. It comes from [3] and [38]. For the convenience of readers, we give the detailed proof.

Lemma 2.1. Suppose $U \subset \mathbb{R}^2$ is a non-empty open set. Let $F : U \to \mathbb{R}$ be a continuous function. Suppose $A$ and $B$ ($M$ and $N$) are the left and right endpoints of some basic intervals in $H_{k_0}$ for some $k_0 \geq 1$ respectively such that $[A, B] \times [M, N] \subset U$. Then $K \cap [A, B] = \cap_{n=k_0} G_n$, and $K \cap [M, N] = \cap_{n=k_0} G'_n$. Moreover, if for any $n \geq k_0$ and any two basic intervals $I \subset G_n$, $J \subset G'_n$ such that

$$F(I, J) = F(\tilde{I}, \tilde{J}),$$

then $F(K \cap [A, B], K \cap [M, N]) = F(G_{k_0}, G'_{k_0})$.

Proof. By the construction of $G_n$ ($G'_n$), i.e. $G_{n+1} \subset G_n$ ($G'_{n+1} \subset G'_n$) for any $n \geq k_0$, it follows that

$$K \cap [A, B] = \cap_{n=k_0} G_n \text{ and } K \cap [M, N] = \cap_{n=k_0} G'_n.$$ 

The continuity of $F$ yields that

$$F(K \cap [A, B], K \cap [M, N]) = \cap_{n=k_0} F(G_n, G'_n).$$

In terms of the relation $G_{n+1} = \tilde{G}_n$, $G'_{n+1} = \tilde{G}'_n$ and the condition in the lemma, it follows that

$$F(G_n, G'_n) = \cup_{1 \leq i \leq t_n} \cup_{1 \leq j \leq t'_n} F(I_{n,i}, J_{n,j}) = \cup_{1 \leq i \leq t_n} \cup_{1 \leq j \leq t'_n} F(\tilde{I}_{n,i}, \tilde{J}_{n,j}) = F(\cup_{1 \leq i \leq t_n} \tilde{I}_{n,i}, \cup_{1 \leq j \leq t'_n} \tilde{J}_{n,j}) = F(G_{n+1}, G'_{n+1}).$$

Therefore, $F(K \cap [A, B], K \cap [M, N]) = F(G_{k_0}, G'_{k_0}).$ \hfill \qedsymbol

The following two lemmas are trivial. We shall use them frequently in the remaining paper.

Lemma 2.2. Let $K$ be the attractors of the following IFS

$$\{ f_1(x) = \lambda x, f_2(x) = \lambda x + c - \lambda, f_3(x) = \lambda x + 1 - \lambda \},$$

where $f_1(I) \cap f_2(I) \neq \emptyset$, $(f_1(I) \cup f_2(I)) \cap f_3(I) = \emptyset$, and $I = [0, 1]$ is the convex hull of $K$. Then $\lambda \leq c \leq 2\lambda, c + \lambda < 1$.

Lemma 2.3. The region satisfies the condition

$$\left\{ \begin{array}{c} (1 - \lambda)^2 \leq c < 1 - \lambda \\ \lambda \leq c \leq 2\lambda \end{array} \right.$$ 

is the orange region in Figure 1. Moreover, for any $(\lambda, c)$ in the orange region,

$$2c + \lambda - 1 \geq 0, 2\lambda + c - 1 \geq 0,$$

see the right picture in Figure 1.

Lemma 2.4. Let $I = [a, a + t], J = [b, b + t]$ be two basic intervals, where $b \geq a$. Suppose $a \geq 1 - c - \lambda$, if $c \geq (1 - \lambda)^2$, then $f(I, J) = f(\tilde{I}, \tilde{J})$, where $f(x, y) = \frac{x}{y}$. 


Proof. Note that \( \tilde{I} = [a, a+c] \cup [a+(1-\lambda)t, a+t], \tilde{J} = [b, b+c] \cup [b+(1-\lambda)t, b+t] \).

Clearly, \( f(I, J) = \left[ \frac{a}{b} + \frac{a + t}{b} \right] \),

\( f(\tilde{I}_1, \tilde{J}) = J_1 \cup J_2 \cup J_3 \cup J_4, \)

where

\[
J_1 = \left[ \frac{a}{b+c} \frac{a+c}{b+(1-\lambda)t} \right] = [e_1, h_1],
\]

\[
J_2 = \left[ \frac{a}{b+c} \frac{a+c}{b} \right] = [e_2, h_2],
\]

\[
J_3 = \left[ \frac{a+(1-\lambda)t}{b} \frac{a+t}{b+(1-\lambda)t} \right] = [e_3, h_3],
\]

\[
J_4 = \left[ \frac{a+(1-\lambda)t}{b+c} \frac{a+t}{b} \right] = [e_4, h_4].
\]

Since \( b \geq a \), it follows that

\[
e_3 - e_2 = \frac{t}{(b+c)(b+t)}(-a(1-c) + (b+c)(1-\lambda)) \geq 0,
\]

Note that \( f(I, J) = f(\tilde{I}, \tilde{J}) \) if and only if

\[
\begin{align*}
h_1 - e_2 & \geq 0, \\
h_2 - e_3 & \geq 0, \\
h_3 - e_4 & \geq 0.
\end{align*}
\]

Now we prove these inequalities.
Case 1. 

\[ h_1 - e_2 = \frac{t}{(b+ct)(b+t-\lambda t)}(a\lambda - a + c^2 t + ac + bc) \]
\[ \geq \frac{t}{(b+ct)(b+t-\lambda t)}(a\lambda - a + c^2 t + ac + ac) \]
\[ = \frac{t}{(b+ct)(b+t-\lambda t)}(a(\lambda + 2c - 1) + c^2 t). \]

Therefore, we need to assume \( \lambda + 2c - 1 \geq 0. \)

Case 2. 

\[ h_2 - e_3 = \frac{t}{b(b+ct)}(a+b(c+\lambda-1)+ct) \geq 0 \text{ as } b \geq a \geq 1 - c - \lambda. \]

Case 3. 

\[ h_3 - e_4 = \frac{a+t}{b+(1-\lambda)t} - \frac{a+(1-\lambda)t}{b+ct} \]
\[ = \frac{t}{(b+ct)(b+t-t\lambda)}(a\lambda - t - a + b\lambda + 2t\lambda - t\lambda^2 + ac + ct) \]
\[ \geq \frac{t}{(b+ct)(b+t-t\lambda)}(tc - (1-\lambda)^2) + a(2\lambda + c - 1)). \]

If \( c - (1-\lambda)^2 \geq 0 \) and \( 2\lambda + c - 1 \geq 0 \), then \( h_3 - e_4 \geq 0. \) By Lemmas 2.2 and 2.3 and the condition \( c - (1-\lambda)^2 \geq 0 \), it follows that if \( c - (1-\lambda)^2 \geq 0 \), then \( 2\lambda + c - 1 \geq 0, \lambda + 2c - 1 \geq 0. \)

Lemma 2.5. If \( \frac{3-\sqrt{5}}{2} \leq \lambda < 1 \), then \( \frac{K}{K} = [0, \infty). \)

Proof. Note that \( 1 - \lambda \geq 1 - c - \lambda \), by Lemma 2.4 we may take \( I = J = [1-\lambda, 1]. \)

Therefore, by Lemma 2.1

\[ \frac{K}{K} \supset f([1-\lambda, 1], [1-\lambda, 1]) = \left[ 1 - \lambda, \frac{1}{1-\lambda} \right]. \]

Since \( \frac{3-\sqrt{5}}{2} \leq \lambda \leq 1/2 \), it follows that \( \frac{\lambda}{1-\lambda} \geq 1 - \lambda. \) Therefore,

\[ [0, \infty) = \{0\} \cup \bigcup_{\lambda=-\infty}^{\infty} \lambda \left[ 1 - \lambda, \frac{1}{1-\lambda} \right] \subset \frac{K}{K} \subset [0, \infty). \]

Note that in the orange region (see Figure 1), we have

\[ c - \lambda^2 \geq 1 - c - \lambda. \]

In fact, \( y = \frac{1}{2}(x^2 - x + 1) \) takes the minimum value at \( \frac{3-\sqrt{5}}{2} \) on the interval \([0, 1]\), i.e. \( c_{\text{min}} \geq \frac{3-\sqrt{5}}{2} \) if \((\lambda, c)\) is in the orange region of Figure 1. Therefore, we can make use of Lemma 2.4. Let

\[ I = J = J_1 \cup J_2 \cup J_3. \]
where

\[ J_1 = [c - \lambda^2, c] \]
\[ J_2 = [1 - \lambda, 1 - \lambda + \lambda c] \]
\[ J_3 = [1 - \lambda^2, 1] \).

Then it is easy to check that \( f(I, J) = \bigcup_{i=1}^{9} L_i \), where

\[ L_1 = \left[ c - \lambda^2, \frac{c}{1 - \lambda^2} \right], L_2 = \left[ \frac{c - \lambda^2}{1 - \lambda + c\lambda}, \frac{c}{1 - \lambda} \right], L_3 = \left[ 1 - \lambda, \frac{1 - \lambda + \lambda c}{1 - \lambda^2} \right] \]
\[ L_4 = [*, *], L_5 = [*, *], L_6 = \left[ 1 - \lambda^2, \frac{1}{1 - \lambda^2} \right] \]
\[ L_7 = \left[ 1 - \frac{\lambda^2}{1 - \lambda + c\lambda}, \frac{1}{1 - \lambda} \right], L_8 = \left[ \frac{1 - \lambda}{c}, \frac{1 - \lambda + \lambda c}{c - \lambda^2} \right], L_9 = \left[ \frac{1 - \lambda^2}{c}, \frac{1}{c - \lambda^2} \right]. \]

We arrange \( L_i = [i_l, i_r], 1 \leq i \leq 9 \) from left to right, where “l”, “r” denote the words left and right, respectively. Here

\[ L_4 = \left[ \frac{c - \lambda^2}{c}, \frac{c}{1 - \lambda^2} \right], L_5 = \left[ \frac{1 - \lambda}{1 - \lambda + c\lambda}, \frac{1 - \lambda + \lambda c}{1 - \lambda} \right], \]

provided that \( \frac{1 - \lambda}{1 - \lambda + c\lambda} \geq \frac{c - \lambda^2}{c} \). If \( \frac{1 - \lambda}{1 - \lambda + c\lambda} < \frac{c - \lambda^2}{c} \), then

\[ L_4 = \left[ \frac{1 - \lambda}{1 - \lambda + c\lambda}, \frac{1 - \lambda + \lambda c}{1 - \lambda} \right], L_5 = \left[ \frac{c - \lambda^2}{c}, \frac{c}{1 - \lambda^2} \right]. \]

The reason why \( i_l < (i + 1)_l, i = 1, 2, 3, 5, 6, 7, 8 \) is due to following lemma.

**Lemma 2.6.** Let \( L_i = [i_l, i_r], 1 \leq i \leq 9 \) be the intervals defined as above. Then

1. \( 2_l < 3_t \) if and only if \( c \leq \frac{\lambda^2 + (1 - \lambda)^2}{1 - (1 - \lambda)^2} \).
2. \( \max \{4_t, 5_l\} < 6_t \) if and only if \( \max \left\{ \frac{c - \lambda^2}{c}, \frac{1 - \lambda}{1 - \lambda + c\lambda} \right\} < 1 - \lambda^2 \).
3. \( 7_l < 8_t \) if and only if \( \lambda + c < 1 \).
4. \( \frac{1}{1 - \lambda + c\lambda} > 1, c \geq 1 \).

**Proof.** It suffices to show that \( c \leq \frac{\lambda^2 + (1 - \lambda)^2}{1 - (1 - \lambda)^2} \). Note that the above inequality is equivalent to

\[ \lambda^2(c - 1) \leq (1 - \lambda)(1 - c - \lambda). \]

The left side is negative while the right is positive. \( \square \)

**Lemma 2.7.** Let \( K \) be the attractors of the following IFS

\[ \{ f_1(x) = \lambda x, f_2(x) = \lambda x + c - \lambda, f_3(x) = \lambda x + 1 - \lambda \}, \]

where \( f_1(I) \cap f_2(I) \neq \emptyset, (f_1(I) \cup f_2(I)) \cap f_3(I) = \emptyset, \) and \( I = [0, 1] \) is the convex hull of \( K \). If \( c \geq (1 - \lambda)^2 \) and \( 0 < \lambda \leq \frac{3 - \sqrt{5}}{2} \), then

\[ \frac{K}{K} \supset \left[ c - \lambda^2, \frac{1}{c - \lambda^2} \right]. \]
Lemma 2.8. Suppose \((\lambda, c)\) satisfies the conditions in Lemma 2.7. If

\[
\frac{1 - \lambda}{1 - \lambda + \lambda c} \geq \frac{c - \lambda^2}{c},
\]

then \(L_i \cap L_{i+1} \neq \emptyset, 1 \leq i \leq 8.\)

Proof. Note that

\[L_4 = \left[ \frac{c - \lambda^2}{c}, \frac{c}{c - \lambda^2} \right], L_5 = \left[ \frac{1 - \lambda}{1 - \lambda + \lambda c}, \frac{1 - \lambda + \lambda c}{1 - \lambda} \right],\]

if \(\frac{1 - \lambda}{1 - \lambda + \lambda c} \geq \frac{c - \lambda^2}{c}.

Case 1.

\[1_r - 2_t = \frac{c}{1 - \lambda^2} - \frac{c - \lambda^2}{1 - \lambda + \lambda c} = -\frac{\lambda}{(\lambda^2 - 1) (c\lambda - \lambda + 1)} (c^2 + c\lambda - c - \lambda^3 + \lambda) \geq 0.\]

Note that \(c < 1 - \lambda < 1 - \lambda^2\) and \(\lambda \geq (1 - c - \lambda)\) (Lemma 2.3), therefore,

\[
\lambda(1 - \lambda^2) \geq c(1 - c - \lambda) \iff c^2 + c\lambda - c - \lambda^3 + \lambda \geq 0.
\]

Case 2.

\[2_r - 3_t = \frac{c}{1 - \lambda} - (1 - \lambda) = \frac{1}{1 - \lambda} (-\lambda^2 + 2\lambda + c - 1) = \frac{1}{1 - \lambda} (c - (1 - \lambda)^2) \geq 0.
\]

Case 3.

\[3_r - 4_t = \frac{1 - \lambda + \lambda c}{1 - \lambda^2} - \frac{c - \lambda^2}{c} = -\frac{1}{c \lambda^2 - 1} \left( c^2 + c\lambda - c - \lambda^3 + \lambda \right) \geq 0.
\]

Case 4.

\[4_r - 5_t = \frac{c}{c - \lambda^2} - \frac{1 - \lambda}{1 - \lambda + \lambda c} = \frac{\lambda}{c^2 \lambda - c\lambda^3 - c\lambda + c + \lambda^3 - \lambda^2} \geq 0.
\]

Case 5.

\[5_r - 6_t = \frac{1 - \lambda + \lambda c}{1 - \lambda} - (1 - \lambda^2) = -\frac{\lambda}{\lambda - 1} (-\lambda^2 + \lambda + c) \geq 0.
\]

Case 6.

\[6_r - 7_t = \frac{1}{1 - \lambda^2} - \frac{1 - \lambda^2}{1 - \lambda + \lambda c} = -\frac{\lambda}{(\lambda^2 - 1) (c\lambda - \lambda + 1)} (-\lambda^3 + 2\lambda + c - 1) \geq 0.
\]

Here \(-\lambda^3 + 2\lambda + c - 1 \geq -\lambda^2 + 2\lambda + c - 1 = c - (1 - \lambda)^2.

Case 7.

\[7_r - 8_t = \frac{1}{1 - \lambda} - \frac{1 - \lambda}{c} = -\frac{1}{c (\lambda - 1)} (-\lambda^2 + 2\lambda + c - 1) \geq 0.
\]

Case 8.

\[8_r - 9_t = \frac{1 - \lambda + \lambda c}{c - \lambda^2} - \frac{1 - \lambda^2}{c} = \frac{\lambda}{c c - \lambda^2} (c^2 + c\lambda - c - \lambda^3 + \lambda) \geq 0.
\]
Lemma 2.9. Suppose $(\lambda, c)$ satisfies the conditions in Lemma 2.7. If
\[
\frac{1 - \lambda}{1 - \lambda + \lambda c} < \frac{c - \lambda^2}{c},
\]
then $L_i \cap L_{i+1} \neq \emptyset, 1 \leq i \leq 8$.

Proof. If \( \frac{1 - \lambda}{1 - \lambda + \lambda c} < \frac{c - \lambda^2}{c} \), then
\[
L_4 = \left[ \frac{1 - \lambda}{1 - \lambda + \lambda c}, \frac{1 - \lambda + \lambda c}{1 - \lambda} \right], \quad L_5 = \left[ \frac{c - \lambda^2}{c}, \frac{c}{c - \lambda^2} \right].
\]

With a similar discussion of Lemma 2.8 it suffices to prove the following three cases.

Case 1.
\[
3r - 4l = \frac{1 - \lambda + \lambda c}{1 - \lambda^2} - \frac{1 - \lambda}{1 - \lambda + \lambda c} = -\frac{\lambda}{(\lambda^2 - 1)(c\lambda - \lambda + 1)} (c^2\lambda - 2c\lambda + 2c - \lambda^2 + 2\lambda - 1) \geq 0.
\]

Here
\[
c^2\lambda - 2c\lambda + 2c - \lambda^2 + 2\lambda - 1 \geq 0
\]
is equivalent to $c(c\lambda - 2\lambda + 2) \geq (1 - \lambda)^2$. Since $c \geq (1 - \lambda)^2$, it suffices to prove that
\[
c\lambda - 2\lambda + 2 \geq 1.
\]
By the assumption $0 < \lambda \leq \frac{3 - \sqrt{5}}{2}$, the above inequality holds.

Case 2.
\[
4r - 5l = \frac{1 - \lambda + \lambda c}{1 - \lambda} - \frac{c - \lambda^2}{c} = -\frac{1}{c\lambda - 1} (c^2 - \lambda^2 + \lambda) \geq 0.
\]

Case 3.
\[
5r - 6l = \frac{c}{c - \lambda^2} - (1 - \lambda^2) = \frac{\lambda^2}{c - \lambda^2} (-\lambda^2 + c + 1) \geq 0.
\]

Proof of Lemma 2.7. Lemma 2.7 follows from Lemmas 2.8, 2.9, 2.4, and 2.1.

Theorem 2.10. If $c \geq (1 - \lambda)^2$, then $\frac{K}{K} = [0, \infty)$.

Proof. If $\frac{3 - \sqrt{5}}{2} \leq \lambda < 1$, then by Lemma 2.5 $\frac{K}{K} = [0, \infty)$. If $0 < \lambda < \frac{3 - \sqrt{5}}{2}$ and $c \geq (1 - \lambda)^2$, in terms of Lemma 2.7 we have $\frac{K}{K} \supset [c - \lambda^2, \frac{1}{c - \lambda^2}]$. We prove
\[
\frac{\lambda}{c - \lambda^2} - (c - \lambda^2) \geq 0.
\]
Recall $c \leq 2\lambda$, if we can show $2\lambda \leq \lambda^2 + \sqrt{\lambda}$, then we prove

$$\frac{\lambda}{c - \lambda^2} - (c - \lambda^2) \geq 0.$$  

However, $2\lambda \leq \lambda^2 + \sqrt{\lambda}$ is equivalent to

$$\lambda^3 - 4\lambda^2 + 4\lambda - 1 = (\lambda - 1)(\lambda^2 - 3\lambda + 1) \leq 0,$$

which is a consequence of $0 < \lambda < \frac{3 - \sqrt{5}}{2}$. Therefore, in terms of

$$\frac{\lambda}{c - \lambda^2} - (c - \lambda^2) \geq 0,$$

we conclude that

$$[0, \infty) = \{0\} \cup \cup_{K=\infty}^{\infty} \lambda^k [c - \lambda^2, \frac{1}{c - \lambda^2}] \subset \frac{K}{K} \subset [0, \infty).$$

\[\square\]

**Lemma 2.11.** Let $I = [a, a + t], J = [b, b + t]$ be two basic intervals, where $b \geq a$.
If $c \geq (1 - \lambda)^2$, then $f(I, J) = f(\tilde{I}, \tilde{J})$, where $f(x, y) = x + y$.

**Proof.** Note that $\tilde{I} = [a, a + ct] \cup [a + (1 - \lambda)t, a + t], \tilde{J} = [b, b + ct] \cup [b + (1 - \lambda)t, b + t]$.
Clearly,

$$f(\tilde{I}, \tilde{J}) = J_1 \cup J_2 \cup J_3,$$

where

$$J_1 = [a + b, a + b + 2ct] = [1_1, 1_r],$$

$$J_2 = [a + b + (1 - \lambda)t, a + ct + b + t] = [2_l, 2_r],$$

$$J_3 = [a + b + 2(1 - \lambda)t, a + b + 2t] = [3_l, 3_r].$$

By virtue of Lemma 2.3 it follows that

$$1_r - 2_l = a + b + 2ct - (a + b + (1 - \lambda)t) = t(2c + \lambda - 1) \geq 0,$$

$$2_r - 3_l = a + ct + b + t - (a + b + 2(1 - \lambda)t) = t(c + 2\lambda - 1) \geq 0.$$  

\[\square\]

**Lemma 2.12.** If $c \geq (1 - \lambda)^2$, then $K + K = [0, 2]$.

**Proof.** By Lemmas 2.1 and 2.11. Take $I = J = [c - \lambda, c] \cup [1 - \lambda, 1]$. Therefore, for $f(x, y) = x + y$, we have

$$f(I, J) = [2(c - \lambda), 2c] \cup [c + 1 - 2\lambda, 1 + c] \cup [2(1 - \lambda), 2].$$

By Lemma 2.3 we conclude that

$$2c - (c + 1 - 2\lambda) = c + 2\lambda - 1 \geq 0, 1 + c - (2(1 - \lambda)) = c + 2\lambda - 1 \geq 0.$$  

Since $c \leq 2\lambda$, it follows that

$$[0, 2] = \{0\} \cup \cup_{K=0}^{\infty} \lambda^k [2(c - \lambda), 2] \subset \frac{K}{K} \subset [0, 2].$$  

\[\square\]
Remark 2.13. We may give another proof of this result. Note that $K + K$ is a self-similar set, namely,

$$K + K = \left\{ \sum_{i=1}^{\infty} \frac{a_i}{q^i} : a_i \in \{0, d_1, d_2, 2d_1, d_1 + d_2, 2d_2\} \right\},$$

where $d_1 = \frac{c}{\lambda} - 1$, $d_2 = \frac{1}{\lambda} - 1$, $q = \frac{1}{\lambda}$. The IFS of $K + K$ is $\{g_i\}_{i=1}^6$, where

$$g_1(x) = \frac{x}{q}, g_2(x) = \frac{x + d_1}{q}, g_3(x) = \frac{x + d_2}{q},$$

$$g_4(x) = \frac{x + 2d_1}{q}, g_5(x) = \frac{x + d_1 + d_2}{q}, g_6(x) = \frac{x + 2d_2}{q}.$$ 

Let $E = [0, 2]$. It is easy to check that if $2 \lambda + c - 1 \geq 0$ and $2c + \lambda - 1 \geq 0$, then

$$\bigcup_{i=1}^{6} g_i(E) = [0, 2].$$

Therefore, if $c \geq (1 - \lambda)^2$ (Lemma 2.3), then $K + K = [0, 2]$.

We shall use this idea to prove the following result.

**Lemma 2.14.** If $c \geq (1 - \lambda)^2$, then $K - K = [-1, 1]$.

**Proof.** First,

$$K - K = \left\{ \sum_{i=1}^{\infty} \frac{a_i}{q^i} : a_i \in \{-d_2, -d_1, d_1 - d_2, 0, d_2, d_1, d_2\} \right\},$$

where $d_1 = \frac{c}{\lambda} - 1$, $d_2 = \frac{1}{\lambda} - 1$, $q = \frac{1}{\lambda}$. The IFS of $K + K$ is $\{h_i\}_{i=1}^7$, where

$$h_1(x) = \frac{x - d_2}{q}, h_2(x) = \frac{x - d_1}{q}, h_3(x) = \frac{x + d_1 - d_2}{q},$$

$$h_4(x) = \frac{x}{q}, h_5(x) = \frac{x + d_2 - d_1}{q}, h_6(x) = \frac{x + d_1}{q}, h_7(x) = \frac{x + d_2}{q}.$$ 

Let $M = [-1, 1]$. It is not difficult to check that if $2c + \lambda - 1 \geq 0$ and $2 \lambda + c - 1 \geq 0$, then

$$\bigcup_{i=1}^{7} h_i(M) = [-1, 1].$$

In other words, if $c \geq (1 - \lambda)^2$ (which implies $2c + \lambda - 1 \geq 0$ and $2 \lambda + c - 1 \geq 0$), then $K - K = [-1, 1]$.

**Proof of Corollary 1.2.** First, in [38], Tian et al. proved

$$K \cdot K = [0, 1]$$

if and only if $c \geq (1 - \lambda)^2$.

Therefore, (2) $\Rightarrow$ (1) $\Leftrightarrow$ (3). By Lemmas 2.12, 2.14 and Theorem 1.1, (3) $\Rightarrow$ (2), we are done.
3. Final remarks

We pose the following questions

**Question 3.1.** Let $K$ be the attractor of the following IFS
$$\{f_1(x) = \lambda x, f_2(x) = \lambda x + c - \lambda, f_3(x) = \lambda x + 1 - \lambda\},$$
where $f_1(I) \cap f_2(I) \neq \emptyset$, $(f_1(I) \cup f_2(I)) \cap f_3(I) = \emptyset$, and $I = [0, 1]$ is the convex hull of $K$. Then whether the following two conditions are equivalent

1. For any $u \in [0, 1]$, there are some $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \in K$ such that
   $$u = x_1 + x_2 = x_3 - x_4 = x_5 \cdot x_6 = x_7 \div x_8 = \sqrt{x_9} + \sqrt{x_{10}};$$
2. $c \geq (1 - \lambda)^2$.

**Question 3.2.** Give a necessary and sufficient condition such that $\frac{K}{K} = [0, \infty)$.

**Question 3.3.** Give a necessary and sufficient condition such that $\sqrt{K} + \sqrt{K} = [0, 2]$.

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