The Two-Point Connection Problem for a Sub-Class of the Heun Equation

R. Williams
Mathematics Section,
International Center for Theoretical Physics, Trieste, Italy

D. Batic
Department of Mathematics,
University of the West Indies, Kingston 6, Jamaica

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The present article discusses the two point connection problem for Heun’s differential equation. We employ a contour integral method to derive connection matrices for a sub-class of the Heun equation containing 3 free parameters. Explicit expressions for the connection coefficients are found.

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I. INTRODUCTION

Schäfke and Schmidt [1], [2], and [3] studied two point connection problems between pairs of solutions around neighboring singularities using a contour integral approach based on the Cauchy Integral Formula. In [3], they studied in particular the connection problem between pairs of solutions around regular singularities. They obtained expressions for the connection coefficients as a limit of a sequence involving the coefficients in the Frobenius expansion of the solution around 0. The shortcoming of this method is that it assumes these coefficients are known. For the Hypergeometric equation (see [4]), this is not a problem as the coefficients satisfy a two-term recurrence relation which is easy to solve. However this is not true for the Heun equation. The required coefficients are solutions of a three-term recurrence relation for which there is no known explicit solution in the general case. In this paper, we will modify the methods used by [3] to fully solve the connection problem for a subclass of the Heun Equation for which this recurrence relation can be solved explicitly. We give explicit expressions for the connection coefficients.

The Heun equation is an increasingly important equation which appears more and more frequently in the literature. Much of the work surrounding the Heun function involves finding integral representations. Several integral representations for the Hypergeometric function are known. These provide a successful strategy for solving the two-point connection problem for the Hypergeometric equation (see [4] for details). In this paper we solve the two-point connection for a subclass of Heun equation without using any integral representations, thus illustrating the power of the strategy employed by Schäfke and Schmidt.

We consider the two-point connection problem for the Heun equation [5] given by

\[
\frac{d^2 y}{dz^2} + \left( \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a} \right) \frac{dy}{dz} + \frac{\alpha \beta \gamma \delta - q}{z(z-1)(z-a)} y = 0,
\]

(1)

with \(a, q \in \mathbb{C}\) and where \(\epsilon = \alpha + \beta + 1 - \gamma - \delta\) and \(a \neq 0, 1\). It is well known that equation (1) has regular singularities at 0, 1, \(a\), and \(\infty\). Furthermore, equation (1) has a Frobenius solution which is regular for \(|z| < \min\{1, |a|\}\) and is denoted \(Hl(a, q; \alpha, \beta, \gamma, \delta; z)\), the local Heun function. Note that \(Hl\) is normalized so that \(Hl(a, q; \alpha, \beta, \gamma, \delta; 0) = 1\). The coefficients in the expansion

\[
Hl(a, q; \alpha, \beta, \gamma, \delta; z) = \sum_{k=0}^{\infty} A_k z^k, \quad |z| < \min\{1, |a|\}
\]

satisfy the well known [5] recurrence relation

\[
\begin{align*}
0 &= a A_1 \gamma - q A_0, \\
0 &= a P_k A_{k+1} - [Q_k + q] A_k + R_k A_{k-1}, \quad k \geq 1,
\end{align*}
\]

(2)
where \( P_k = (k + 1)(k + \gamma) \), \( Q_k = k(k - 1 + \gamma)(1 + a) + k(a\delta + \epsilon) \), \( R_k = (k - 1 + \alpha)(k - 1 + \beta) \), and \( A_0 = 1 \). Maier \( [6] \) described the fundamental pairs of local Frobenius solutions to equation \((1)\) and gave various relations satisfied by the local Heun function. We denote these pairs of solutions by \( \{y_0, y_1\} \), \( \{y_11, y_12\} \), \( \{y_{\infty1}, y_{\infty2}\} \), \( \{y_{a1}, y_{a2}\} \), and \( \{y_{\infty1}, y_{\infty2}\} \) where

\[
y_0(z) = Hl(a, q; \alpha, \beta, \gamma, \delta; z),
\]

\[
y_1(z) = Hl(1 - a, \alpha\beta - q; \alpha, \beta, \gamma, \delta; 1 - z),
\]

\[
y_2(z) = (z - a)^{1 - \epsilon} Hl \left( \frac{a}{a - 1}, -q + a(\gamma + \delta - \alpha)(\gamma + \delta - \beta) + \gamma(\epsilon - 1); \gamma + \delta - \beta, \gamma + \delta - \alpha, 2 - \epsilon, \delta; \frac{z - a}{1 - a} \right),
\]

\[
y_{\infty1}(z) = z^{-\alpha} Hl \left( \frac{1}{a}, q + a(\alpha + 1 - \gamma - \delta) + \delta - \beta); \alpha, \alpha + 1 - \gamma, 1 + \alpha - \beta, \delta; z^{-1} \right),
\]

\[
y_{\infty2}(z) = z^{-\beta} Hl \left( \frac{1}{a}, q + \beta(\beta + 1 - \gamma - \delta) + \delta - \alpha); \beta, \beta + 1 - \gamma, 1 + \beta - \alpha, \delta; z^{-1} \right).
\]

In order for these pairs to be linearly independent and well-defined, we require that \( \gamma, \delta, \epsilon, \alpha - \beta \notin \mathbb{Z} \).

\[
y_{11}(z) = Hl(1, a, \alpha\beta - q; \alpha, \beta, \gamma, \delta; 1 - z),
y_{12}(z) = (1 - z)^{1 - \delta} Hl(1 - a, \alpha\beta - q + (1 - \delta)(\epsilon + (1 - a)\gamma); 1 - a - \delta; 1 + \beta - \delta, 2 - \delta, \gamma; 1 - z)
\]

\[
y_{a1}(z) = Hl \left( \frac{a}{a - 1}, \frac{a\alpha\beta - q}{a - 1}; \alpha, \beta, \alpha + 1 - \gamma - \delta, \delta; \frac{z - a}{1 - a} \right),
y_{a2}(z) = (z - a)^{1 - \epsilon} Hl \left( \frac{a}{a - 1}, -q + a(\gamma + \delta - \alpha)(\gamma + \delta - \beta) + \gamma(\epsilon - 1); \gamma + \delta - \beta, \gamma + \delta - \alpha, 2 - \epsilon, \delta; \frac{z - a}{1 - a} \right),
y_{\infty1}(z) = z^{-\alpha} Hl \left( \frac{1}{a}, q + a(\alpha + 1 - \gamma - \delta + \delta - \beta); \alpha, \alpha + 1 - \gamma, 1 + \alpha - \beta, \delta; z^{-1} \right),
y_{\infty2}(z) = z^{-\beta} Hl \left( \frac{1}{a}, q + \beta(\beta + 1 - \gamma - \delta + \delta - \alpha); \beta, \beta + 1 - \gamma, 1 + \beta - \alpha, \delta; z^{-1} \right).
\]

II. PRELIMINARIES

In this paper we consider the subclass of equation \((1)\) where \( \delta = (\alpha + \beta + 1 - \gamma)/2, q = 0, \) and \( a = -1 \). That is, we consider the Fuchsian equation

\[
\frac{d^2y}{dz^2} + \left( \frac{\gamma}{z} + \frac{\delta}{z - 1} + \frac{\delta}{z + 1} \right) \frac{dy}{dz} + \frac{a\beta}{(z - 1)(z + 1)} y = 0.
\]

\[
(3)
\]

Remark II.1 Note that for this subclass of the Heun Equation, the methods employed in \([3]\) cannot be immediately applied since they assumed that the differential equation has no other singularity in the closed disk \( \{ z \in \mathbb{C} : |z| \leq 1 \} \) besides 0 and 1, whereas \([3]\) has singularities at -1, 0, and 1.

It is not difficult to see that \([2]\) becomes

\[
A_{k+1} = \frac{(k - 1 + \alpha)(k - 1 + \beta)}{(k + 1)(k + \gamma)} A_{k-1}, \quad k \geq 1.
\]

Whence we obtain

\[
A_{2n} = \frac{(\frac{\alpha}{\gamma})_n (\frac{\beta}{\gamma})_n}{n! (\frac{\gamma + 1}{2})_n}, \quad n \geq 0,
\]

\[
(4)
\]

and \( A_{2n+1} = 0, \quad \forall n \geq 0 \). Hence,

\[
Hl(-1, 0; \alpha, \beta, \gamma, \delta; z) = \sum_{n=0}^{\infty} \frac{(\frac{\alpha}{\gamma})_n (\frac{\beta}{\gamma})_n}{n! (\frac{\gamma + 1}{2})_n} z^{2n}.
\]

\[
(5)
\]
Equation (3) has fundamental pairs of solutions given by

\[ y_{01}(z) = H_l(-1; 0; \alpha, \beta, \gamma, \delta; z) \]
\[ y_{02}(z) = z^{1+\gamma} H_l(-1, 0; 1+\alpha-\gamma, 1+\beta-\gamma, 2-\gamma, \delta; z), \]
\[ y_{11}(z) = H_l(2, 0; \alpha, \beta, \delta, \gamma, 1-z), \]
\[ y_{+2}(z) = (1-z)^{-1-\delta} H_l(2, \alpha+1-\delta; 1+\alpha-\delta; 1+\beta-\delta, 2-\delta, \gamma; 1-z), \]
\[ y_{-1}(z) = H_l(1/2, \alpha+1/2, \gamma+\delta; 1+\alpha-\gamma, 1+\beta-\gamma, 2-\gamma, 1); \]
\[ y_{-2}(z) = (z+1)^{1-\delta} H_l(1/2, (\gamma+\delta-\alpha); 1+\alpha-\gamma, 1+\beta-\gamma, 2-\gamma, 1); \]
\[ y_{\infty 1}(z) = z^{-\alpha} H_l(-1, 0; 1+\alpha-\gamma, 1+\beta-\gamma, 2-\gamma, \delta; z^{-1}), \]
\[ y_{\infty 2}(z) = z^{-\beta} H_l(-1, 0; 1+\alpha-\gamma, 1+\beta-\gamma, 2-\gamma, \delta; z^{-1}). \]

(6)

**Remark II.2** Note that any solution of (1) may be analytically continued along any path in \( \mathbb{C} - \{0, 1, a\} \) and the analytic continuation is a solution (see for example Theorem 3.7.2 in [7] or 10.1 in [5]). If two paths are homotopic then the continuation is unique by the Monodromy Theorem (see [7] or [5]). Thus, if the domain \( D \subset \mathbb{C} - \{0, 1, a\} \) is simply connected and has non-empty intersection with the open disc \( \{z \in \mathbb{C} : |z| < 1\} \) then in particular it is not difficult to see that \( y_{01} \) has a unique analytic extension to \( D \).

We will also find the following results helpful in proving our main result.

**Lemma II.3** If \( \Re \alpha, \Re \beta > 0 \), then

\[ \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \, dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \]

**Proof.** This is a standard result about the Euler-Beta Function. For a proof, see, for example, Section 1.5 in [4].

**Lemma II.4 (Asymptotics of ratio of two gamma functions)**

In the intersection of the sectors \( |\arg(z+a)| < \pi \) and \( |\arg(z+b)| < \pi \), we have

\[ \frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b}, \quad z \to \infty \]

**Proof.** This is a standard result which may be readily derived from Stirling’s Series. For an alternative proof, see pg. 118 in [11].

**Lemma II.5** Let \( 1 < \rho < 2 \) and \( \alpha \in \mathbb{C} \). If \( k > \Re \alpha \) and \( \Re \alpha > 0 \), then

\[ \int_1^\rho (z-1)^\alpha z^{-k-1} \, dz = \frac{\Gamma(k-\alpha)}{\Gamma(k+1)} + \mathcal{O}(\rho^{-k}), \quad k \to \infty. \]

Let \( F : B_1(1) \to \mathbb{C} \) be holomorphic. Then,

\[ \int_1^\rho (z-1)^\alpha z^{-k-1} F(z) \, dz = \mathcal{O}(k^{-\Re \alpha - 1}), \quad k \to \infty. \]

where the powers in the above integral take their principal values.

**Proof.** Observe that

\[ \int_1^\rho (z-1)^\alpha z^{-k-1} \, dz = \int_1^\infty (z-1)^\alpha z^{-k-1} \, dz - \int_\rho^\infty (z-1)^\alpha z^{-k-1} \, dz. \]

Since \( k > \Re \alpha \) and \( \Re \alpha > 0 \), using the transformation \( z = 1/t \) we obtain

\[ \left| \int_\rho^\infty (z-1)^\alpha z^{-k-1} \, dz \right| = \left| \int_0^{\frac{1}{\rho}} t^{k-1-\alpha} (1-t)^\alpha \, dt \right| \leq \int_0^{\frac{1}{\rho}} x^{k-1-\Re \alpha} \, dx = \frac{\rho^{-k+\Re \alpha}}{k-\Re \alpha}. \]
Hence,
\[ \int_{\rho}^{\infty} (z - 1)^{\alpha} z^{-k-1} dz = \mathcal{O}(\rho^{-k}), \quad k \to \infty. \]

Also,
\[ \int_{1}^{\infty} (z - 1)^{\alpha} z^{-k-1} dz \overset{z=1/t}{=} \int_{0}^{1} (1 - t)^{\alpha} t^{k-\alpha-1} dt \overset{II.3}{=} \frac{\Gamma(k-\alpha)\Gamma(\alpha+1)}{\Gamma(k+1)}. \]

We prove now the second part of the lemma. Notice that since \( F \) is holomorphic we may find an \( M \in \mathbb{R}^+ \) such that
\[ \left| \int_{1}^{\rho} (z - 1)^{\alpha} z^{-k-1} F(z) dz \right| \leq M \int_{1}^{\rho} x^{-k-1} (x-1)^{R_\alpha} dx, \]
\[ \overset{z=1/t}{=} M \int_{0}^{1} t^{k-1-R_\alpha} (1-t)^{R_\alpha} dt \overset{II.3}{=} \frac{M \Gamma(k-R_\alpha)\Gamma(R_\alpha+1)}{\Gamma(k+1)}. \]

Using Lemma II.3 we get
\[ \int_{1}^{\rho} (z - 1)^{\alpha} z^{-k-1} F(z) dz = \mathcal{O}(k^{-R_\alpha-1}), \quad k \to \infty. \]

This concludes the proof. \( \Box \)

III. SOLUTION OF THE TWO-POINT CONNECTION PROBLEM

We consider simultaneously the two-point connection problem between 0 and 1 and between 0 and -1. That is we seek coefficients \( c_{11}, c_{12}, c_{11}, c_{12} \) such that
\[ y_{01} = c_{11} y_{1+1} + c_{12} y_{1+2} \quad (7) \]
\[ y_{01} = c_{11} y_{1-1} + c_{12} y_{1-2} \quad (8) \]

Note however, that \( c_{11}^+ \) and \( c_{11}^- \) may be easily found. We recall the well-known result [4]
\[ \lim_{z \to 1^+} F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}. \]

Using this relation and taking the limit of (7) as \( z \to 1 \) and assuming \( \Re(1-\delta) > 0 \) we obtain
\[ c_{11}^+ = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{n! (\frac{1}{2}+\frac{1}{2})_n} = \lim_{z \to 1^+} F \left( \frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma+1}{2}; z \right) = \frac{\Gamma(\frac{\alpha+1}{2} - \frac{\alpha}{2} - \frac{\beta}{2})\Gamma(\frac{\gamma+1}{2})}{\Gamma(\frac{\alpha+1}{2} - \frac{\alpha}{2})\Gamma(\frac{\frac{\gamma+1}{2}}{2} - \frac{\beta}{2})}, \]
and similarly taking the limit as \( z \to -1 \) of (8) we obtain
\[ c_{11}^- = \frac{\Gamma(\frac{\alpha+1}{2} - \frac{\alpha}{2} - \frac{\beta}{2})\Gamma(\frac{\gamma+1}{2})}{\Gamma(\frac{\alpha+1}{2} - \frac{\alpha}{2})\Gamma(\frac{\frac{\gamma+1}{2}}{2} - \frac{\beta}{2})} = c_{11}^+. \]

We compute \( c_{12}^+ \) and \( c_{12}^- \) with the following theorem.

**Theorem III.1** Let \( y_{01}, y_{1+1}, y_{1+2}, y_{-1}, y_{-2} \) be as in (7) and \( c_{11}^+, c_{12}^+, c_{11}^-, c_{12}^- \) be as in (7) and (8). Furthermore let \( \Re(1-\delta) > 0 \), then we have
\[ c_{12}^+ = 2^{1-\delta} \frac{\Gamma(\delta-1)\Gamma(\frac{\gamma+1}{2})}{\Gamma(\frac{\delta}{2})\Gamma(\frac{\gamma+1}{2})} = c_{12}^-; \]
FIG. 1: Integration contour showing components of $C_{\sigma,\phi}$

**Proof.** For simplification, first observe that $y_{01}$ is of the form

$$y_{01}(z) = \sum_{k=0}^{\infty} d_k z^k.$$

Let $C_{\sigma,\phi} = C_{\sigma,\phi}^0 + L_{1,\sigma,\phi}^1 - C_{\sigma,\phi}^2 + L_{2,\sigma,\phi}^3 - C_{\sigma,\phi}^{-1} + L_{4,\sigma,\phi}^2$ be the contour shown in FIG. 1 where $C_{\sigma,\phi}^0 = H_{\sigma,\phi}^0 + H_{\sigma,\phi}^1$ and $1 < \rho < 2$. By the Cauchy Integral Formula we have for any $k \in \mathbb{N}_0$, and $\sigma, \phi > 0$ sufficiently small

$$d_k = \frac{1}{2\pi i} \int_{C_{\sigma,\phi}} z^{-k-1} \hat{y}_{01}(z) \, dz$$

where $\hat{y}_{01}$ is the unique analytic extension of $y_{01}$ to the simply connected set $\mathbb{C} - ((-\infty, -1] \cup [1, +\infty))$ guaranteed to exist by Remark II.2. In particular,

$$\hat{y}_{01}(z) = c_{11}^+ y_{+1}(z) + c_{12}^+ y_{+2}(z), \quad z \in L_{1,\sigma,\phi}^1 \cup C_{\sigma,\phi}^+ \cup L_{2,\sigma,\phi}^2,$$

$$\hat{y}_{01}(z) = c_{11}^- y_{-1}(z) + c_{12}^- y_{-2}(z), \quad z \in L_{3,\sigma,\phi}^3 \cup C_{\sigma,\phi}^- \cup L_{4,\sigma,\phi}^4,$$

where we take the principle value of the powers occurring in $y_{+2}$ and $y_{-2}$. Notice that the left hand side of (9) above does not depend on $\sigma$ or $\phi$. Hence we consider the limit of the expression on the right hand side as $\sigma, \phi \to 0$. This
limit if it exists should be equal to $d_k$. So

$$2\pi i d_k = \lim_{\sigma \to 0} \left( \lim_{\phi \to 0} \left( \int_{C_{k,\phi}} z^{-k-1} y_01(z) dz \right) \right) = I^0_k + I^1_k + I^2_k$$

where

$$I^0_k = I^0_{k,1} - I^0_{k,2} - I^0_{k,3}$$

$$I^1_k = \lim_{\sigma \to 0} \left( \lim_{\phi \to 0} \left( \int_{L_{k,\phi}^+} z^{-k-1} (c_{11} y_1 + c_{12} y_2)(z) dz - \int_{L_{k,\phi}^-} z^{-k-1} (c_{11} y_1 + c_{12} y_2)(z) dz \right) \right)$$

$$I^2_k = \lim_{\sigma \to 0} \left( \lim_{\phi \to 0} \left( \int_{L_{k,\phi}^+} z^{-k-1} (c_{11} y_1 - c_{12} y_2)(z) dz - \int_{L_{k,\phi}^-} z^{-k-1} (c_{11} y_1 - c_{12} y_2)(z) dz \right) \right)$$

and

$$I^0_{k,1} = \lim_{\sigma \to 0} \left( \lim_{\phi \to 0} \left( \int_{C_{k,\phi}} z^{-k-1} y_01(z) dz \right) \right)$$

$$I^0_{k,2} = \lim_{\sigma \to 0} \left( \lim_{\phi \to 0} \left( \int_{C_{k,\phi}} z^{-k-1} (c_{11} y_1 + c_{12} y_2)(z) dz \right) \right)$$

$$I^0_{k,3} = \lim_{\sigma \to 0} \left( \lim_{\phi \to 0} \left( \int_{C_{k,\phi}} z^{-k-1} (c_{11} y_1 - c_{12} y_2)(z) dz \right) \right)$$

First we deal with $I^0_k$. In particular, we will show that $I^0_k = O(\rho^{-k})$. Note that we have the following parametrizations

$$C_{k,\phi,\sigma}^+ : \{z(\theta) = 1 + \sigma e^{i\theta}, \quad \phi \leq \theta \leq 2\pi - \phi \}$$

$$C_{k,\phi,\sigma}^- : \{z(\theta) = -1 + \sigma e^{i\theta}, \quad \phi - \pi \leq \theta \leq \pi - \phi \}$$

Since $y_{+1}$ is holomorphic in $\{z : |z - 1| < 1\}$ and $y_{-1}$ is holomorphic in $\{z : |z + 1| < 1\}$ we obtain

$$I^0_{k,2} = c_{12} \lim_{\sigma \to 0} \left( \lim_{\phi \to 0} \left( \int_{C_{k,\phi,\sigma}^+} z^{-k-1} y_{+1}(z) dz \right) \right)$$

$$I^0_{k,3} = c_{12} \lim_{\sigma \to 0} \left( \lim_{\phi \to 0} \left( \int_{C_{k,\phi,\sigma}^-} z^{-k-1} y_{-1}(z) dz \right) \right)$$

We express $y_{+2}$ and $y_{-2}$ as $y_{+2}(z) = (1 - z)^{1-\delta} f_1(z)$ and $y_{-2}(z) = (z + 1)^{1-\delta} f_2(z)$ where $f_1, f_2$ are holomorphic functions in the discs of radius 1 centered at 1, -1 respectively. Thus, we obtain

$$I^0_{k,2} = c_{12} \lim_{\sigma \to 0} \left( \lim_{\phi \to 0} \left( \int_{C_{k,\phi,\sigma}^+} z^{-k-1} (1 - z)^{1-\delta} f_1(z) dz \right) \right)$$

Let $M$ be such that $|f_1(z)|, |f_2(z)| < M$, $\forall z \in \{z \in \mathbb{C} : |z - 1| < 1\}$ (M is guaranteed to exist since $F_1$ is holomorphic in an open disc centered at 1), then

$$\left| \int_{C_{k,\phi,\sigma}^+} z^{-k-1} (1 - z)^{1-\delta} f_1(z) dz \right| = \int_{\phi}^{2\pi - \phi} \left| (1 + \sigma e^{i\theta})^{-k-1} (-\sigma e^{i\theta})^{1-\delta} f_1(1 + \sigma e^{i\theta}) \sigma e^{i\theta} d\theta \right|$$

$$\leq M 2^{k+1} \int_{\phi}^{2\pi - \phi} \sigma^{2-\Re \delta} d\theta = M 2^{k+1} 2^{2-\Re \delta} (\pi - \phi) \sigma^{2-\Re \delta}$$

The second inequality follows from the fact that for $\sigma$ sufficiently small $|1 + \sigma e^{i\theta}| > 1/2$. Since $\Re(1 - \delta) > 0$, this implies that $I^0_{k,2} = 0$. It may similarly be shown that $I^0_{k,3} = 0$. Note that $y_{01}$ may be extended analytically to simply connected domains $D_1$ and $D_2$ containing $H^0_0$ and $H^1_0$ respectively. By the continuity of these extensions, their absolute values have a common upper bound $M \in \mathbb{R}^+$ on $H^0_0$ and $H^1_0$. Since these extensions also extend $y_{01}$
and since $H^2_\sigma \subset H^0_{\sigma}$ and $H^1_\sigma \subset H^1_{\sigma}$, this bound also holds for $z \in H^0_{\sigma} \cup H^1_{\sigma} = C^0_{\sigma, \phi}$. Thus by the M-L Formula (see for example (9) page 83 in [9]) we have

$$\left| \int_{C^0_{\sigma, \phi}} z^{-k-1} y_0(z)dz \right| \leq 2\pi M \rho^{-k}$$

Hence $I^0_k = I^0_{k,1} = O(\rho^{-k})$. We now give parametrizations of the contours $L^1_{\sigma, \phi}$, $L^2_{\sigma, \phi}$, $L^3_{\sigma, \phi}$, and $L^4_{\sigma, \phi}$.

$$L^1_{\sigma, \phi} : z(r) = 1 + re^{i\phi}, \quad L^2_{\sigma, \phi} : z(r) = 1 + re^{-i\phi}, \quad L^3_{\sigma, \phi} : z(r) = -1 - re^{-i\phi}, \quad L^4_{\sigma, \phi} : z(r) = -1 - re^{i\phi}$$

where the parameter $r$ runs $\sigma \leq r \leq \sqrt{\cos^2 \phi + \rho^2 - 1 - \cos \phi}$. Using the above parametrizations, and taking the limits we see that

$$I^1_k = c_{12} \int_0^{\rho-1} (1+r)^{-k-1}(e^{-2\pi i (1-\delta)} - 1)y_2(1+r) dr \equiv c_{12} \frac{1}{1 - e^{-2\pi i (1-\delta)}} J^1_k$$

and similarly

$$I^2_k = c_{12} \int_0^{\rho-1} (-1-r)^{-k-1}(1 - e^{-2\pi i (1-\delta)}) y_2(-1-r) dr \equiv c_{12} \frac{1}{1 - e^{-2\pi i (1-\delta)}} J^2_k$$

where

$$J^1_k = \int_{\rho}^{1} x^{-k-1} y_2(x) dx, \quad J^2_k = \int_{-\rho}^{-1} x^{-k-1} y_2(x) dx$$

Now, we rewrite $y_2$ and $y_2$ as follows

$$y_2(z) = \left( \sum_{j=0}^{m} G_j (1-z)^{1-\delta+j} + (1-z)^{2+m-\delta} F_1(z) \right), \quad y_2(z) = \left( \sum_{j=0}^{m} H_j (1+z)^{1-\delta+j} + (1+z)^{2+m-\delta} F_2(z) \right)$$

where $F_1$, $F_2$ are analytic functions in the discs of radii 1 centered at $1$, $-1$ respectively and $G_0 = 1 = H_0$. Using the representation for $y_2$ and $y_2$ found in (10), we obtain

$$J^1_k = \left( \sum_{j=0}^{m} G_j \int_{\rho}^{1} x^{-k-1} (1-x)^{1-\delta+j} dx + \int_{\rho}^{1} x^{-k-1} (1-x)^{2+m-\delta} F_1(x) dx \right)$$

$$J^2_k = \left( \sum_{j=0}^{m} H_j \int_{-\rho}^{-1} x^{-k-1} (1+x)^{1-\delta+j} dx + \int_{-\rho}^{-1} x^{-k-1} (1+x)^{2+m-\delta} F_2(x) dx \right)$$

Hence if $\Re(1-\delta) > 0$, we may apply Lemma II.5 to obtain

$$J^1_k = -\sum_{j=0}^{m} G_j \exp(\pi i (1-\delta+j)) \frac{\Gamma(k-j-1 + \delta) \Gamma(2-\delta+j)}{\Gamma(k+1)} + O(\rho^{-k}) + O(k^{-\Re(3+m-\delta)})$$

and similarly

$$J^2_k \equiv \sum_{j=0}^{m} H_j (-1)^{-k-1} \int_{1}^{\rho} v^{-k-1} (1-v)^{1-\delta+j} dv + (-1)^{-k-1} \int_{1}^{\rho} v^{-k-1} (1-v)^{2+m-\delta} F_2(-v) dv$$

$$= \sum_{j=0}^{m} H_j (-1)^{-k-1} \exp(\pi i (1-\delta+j)) \frac{\Gamma(k-j-1 + \delta) \Gamma(2-\delta+j)}{\Gamma(k+1)} + O(\rho^{-k}) + O(k^{-\Re(3+m-\delta)})$$
Hence, when we multiply by the ratio $\Gamma(k+1)/\Gamma(k-1+\delta)$

$$\frac{\Gamma(k+1)}{\Gamma(k-1+\delta)} d_k = \mathcal{O}(\rho^{-k}) + \frac{\Gamma(k+1)}{\Gamma(k-1+\delta) 2\pi i} \left[I^1_k + I^2_k\right]$$
$$= \mathcal{O}(\rho^{-k}) + \frac{\Gamma(k+1)}{\Gamma(k-1+\delta) 2\pi i} \left[c_{i2}^+ L_k^1 + c_{i2}^- L_k^2\right]$$

Also, using (4) we obtain

$$\mathcal{O}(\rho^{-k}) + \mathcal{O}(k^{-m-1}) + \frac{\Gamma(k+1)}{\Gamma(k-1+\delta) 2\pi i} \left[1 - \exp(-2\pi i(1-\delta))\right] \frac{1}{2\pi i} \left[c_{i2}^+ L_k^1 + c_{i2}^- L_k^2\right]$$

Furthermore using the fact that $(\alpha)_{n} = \Gamma(\alpha+n)/\Gamma(\alpha)$ we obtain

$$\lim_{n \to \infty} \frac{\Gamma(2n+1)}{\Gamma(2n-1+\delta)} d_{2n} = \frac{(c_{i2}^+ + c_{i2}^-)}{\Gamma(\delta-1)}$$

Also, taking $k = 2n + 1$ (i.e. $k$ odd) we obtain

$$\lim_{n \to \infty} \frac{\Gamma(2n+2)}{\Gamma(2n+\delta)} d_{2n+1} = \frac{(c_{i2}^+ - c_{i2}^-)}{\Gamma(\delta-1)}$$

Using (4) we obtain

$$c_{i2}^+ + c_{i2}^- = \lim_{n \to \infty} \frac{\Gamma(2n+1)\Gamma(\delta-1)(\frac{\delta}{2})_n}{\Gamma(2n-1+\delta) n!(\frac{\delta}{2}+1)_n}, \quad c_{i2}^+ - c_{i2}^- = 0 \implies c_{i2}^+ = c_{i2}^-$$

Furthermore using the fact that $(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)$ we obtain

$$c_{i2}^+ + c_{i2}^- = \frac{\Gamma(\alpha/2)\Gamma(\beta/2)}{\Gamma(\alpha/2)\Gamma(\beta/2)} \lim_{n \to \infty} \frac{\Gamma(2n+1)\Gamma(\alpha/2+n)\Gamma(\beta/2+n)}{\Gamma(2n+1+\delta)\Gamma(n+1)\Gamma(\gamma+1/2+\delta)}$$

This completes the proof. □

IV. THE CONNECTION MATRICES

Maier [4] showed the following relations hold

$$y_{11}(z) = z^{1-\gamma} H_1(1-a,-q + \alpha \beta + (\gamma - 1)(1-a)\delta; 1 + \alpha - \gamma, 1 + \beta - \gamma, \delta, 2 - \gamma; 1 - z),$$

$$y_{a1}(z) = \left(\frac{z}{a}\right)^{1-\gamma} H_1\left(\frac{a}{a-1}, Q; 1 + \alpha - \gamma, 1 + \beta - \gamma, \alpha + \beta + 1 - \gamma - \delta; \frac{z-a}{1-a}\right),$$

where

$$Q = \frac{a(1 + \alpha - \gamma)(1 + \beta - \gamma) - q - (1 - \gamma)(\alpha + \beta + 1 - \gamma + (a - 1)\delta)}{a - 1},$$

and

$$y_{11}(z) = z^{-\alpha} H_1\left(1 - \frac{1}{a}, -q + \alpha[(a-1)\delta + \beta]; \alpha, \alpha + 1 - \gamma, \delta, 1 + \alpha - \beta; 1 - z^{-1}\right),$$
Using the above relations it follows, after some computations, that the connection matrices we seek are
\[
y_{a1}(z) = (-z)^{-a} Hl \left( \frac{1}{1-a}; \frac{-q + a[(\alpha + 1 - \gamma - \delta)(1-a) + \beta]}{1-a}, \alpha, \alpha + 1 - \gamma, \epsilon, \delta; \frac{z^{-1} - a}{1-a} \right).
\]

Indeed the transformation \( z = 1/t, w(z) = f(t) \), and \( f(t) = t^a \phi(t) \) is a symmetry of equation \( \square \). Applying the transformation, \( \square \) is transformed into a Heun equation with singularities at \( \{0, 1, 1/a, \infty \} \) and that the r.h.s. above is a solution to \( \square \) in a neighborhood of \( 1/a \). In particular if \( a = -1 \) then we have another solution in a neighborhood of \( -1 \). Hence the r.h.s. above must be a linear combination of \( y_{a1} \) and \( y_{a2} \) in the case \( a = -1 \). Comparing the behaviours of the functions, it becomes clear that the assertion above is true. Let
\[
q_1(\alpha, \beta, \gamma) = \frac{\Gamma\left(\frac{\alpha + 1}{2} - \frac{\beta}{2} - \frac{\gamma}{2}\right)\Gamma\left(\frac{\gamma + 1}{2}\right)}{\Gamma\left(\frac{\alpha + 1}{2} - \frac{\beta}{2}\right)\Gamma\left(\frac{\gamma + 1}{2} - \frac{\beta}{2}\right)},
q_2(\alpha, \beta, \gamma) = 2^{1-a} \frac{\Gamma(\delta - 1)\Gamma\left(\frac{\gamma + 1}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{\gamma}{2}\right)}.
\]

Then
\[
Hl(-1, 0; \alpha, \beta, \gamma, \delta; z) = q_1(\alpha, \beta, \gamma) Hl(2, \alpha \beta; \alpha, \beta, \delta, \gamma; 1 - z) + q_2(\alpha, \beta, \gamma)(1 - z)^{1-\delta} Hl(2, \alpha \beta + (1 - \delta)(\delta + 2\gamma); 1 + \alpha - \delta; 1 + \beta - \delta; 2 - \delta; \gamma; 1 - z),
\]
and
\[
Hl(-1, 0; \alpha, \beta, \gamma, \delta; z) = q_1(\alpha, \beta, \gamma) Hl \left( \frac{1}{2}; \frac{\alpha \beta}{2}; \alpha, \beta, \delta, \gamma; \frac{z + 1}{2} \right) + q_2(\alpha, \beta, \gamma) \left( \frac{z + 1}{2} \right)^{1-\delta} Hl \left( \frac{1}{2}; \frac{\alpha \beta}{2} + (1 - \delta) \left( \gamma + \frac{\delta}{2} \right); 1 + \alpha - \delta, 1 + \beta - \delta, 2 - \delta, \gamma; \frac{z + 1}{2} \right).
\]

Using the above relations it follows, after some computations, that the connection matrices we seek are
\[
C_{0+} = \begin{pmatrix} q_1(\alpha, \beta, \gamma) & q_2(\alpha, \beta, \gamma) \\ q_1(1 + \alpha - \gamma, 1 + \beta - \gamma, 2 - \gamma) & q_2(1 + \alpha - \gamma, 1 + \beta - \gamma, 2 - \gamma) \end{pmatrix},
\]
\[
C_{0-} = \begin{pmatrix} (-)^{1-\gamma} q_1(\alpha, \beta, \gamma) & (-)^{1-\gamma} q_2(\alpha, \beta, \gamma) \\ (-)^{1-\gamma} q_1(1 + \alpha - \gamma, 1 + \beta - \gamma, 2 - \gamma) & (-)^{1-\gamma} q_2(1 + \alpha - \gamma, 1 + \beta - \gamma, 2 - \gamma) \end{pmatrix},
\]
\[
C_{\infty+} = \begin{pmatrix} q_1(\alpha, \alpha + 1 - \gamma, 1 + \alpha - \beta) & (-)^{\delta-1} q_2(\alpha, \alpha + 1 - \gamma, 1 + \alpha - \beta) \\ q_1(\beta, \beta + 1 - \gamma, 1 + \beta - \alpha) & (-)^{\delta-1} q_2(\beta, \beta + 1 - \gamma, 1 + \beta - \alpha) \end{pmatrix},
\]
\[
C_{\infty-} = \begin{pmatrix} (-)^{-\gamma} q_1(\alpha, \alpha + 1 - \gamma, 1 + \alpha - \beta) & (-)^{-\gamma} q_2(\alpha, \alpha + 1 - \gamma, 1 + \alpha - \beta) \\ (-)^{-\gamma} q_1(\beta, \beta + 1 - \gamma, 1 + \beta - \alpha) & (-)^{-\gamma} q_2(\beta, \beta + 1 - \gamma, 1 + \beta - \alpha) \end{pmatrix}.
\]

**Remark IV.1** These matrices remarkably provide a way for us to express \( y_{+1} \) and \( y_{-1} \) linearly in terms of \( y_{01} \) and \( y_{02} \). This is interesting because the coefficients appearing in \( y_{+1} \) and \( y_{-1} \) satisfy a much more general three-term recurrence relation (and perhaps much more difficult to solve) than that of the coefficients of \( y_{01} \) and \( y_{02} \). Hence it would be difficult to express \( y_{+1} \) and \( y_{-1} \) in closed form if one only had \( \square \) to rely on. In actuality, our matrices allow \( y_{+1} \) and \( y_{-1} \) to be expressed in closed form in terms of \( y_{01} \) and \( y_{02} \), which have closed form expressions.

**V. CONCLUSION**

In summary, we have modified the methods used by [8] to explicitly solve the two-point connection problem for a subclass of the Heun equation. To the best of our knowledge, this is the first time explicit expressions for the connection coefficients for even a subclass of the Heun equation has been given. We emphasize that the subclass of the Heun equation we have considered has 3 free parameters (just as many as the Hypergeometric equation). Our results will likely have many applications in a diverse range of fields, notably in mathematical physics where the solutions of important equations may be given in terms of Heun functions. The connection matrices we have found here will enable the construction of global analytic solutions of such equations. Take, for example, the work of [12] where the most general class of potential was given such that the solution of the one-dimensional Schrödinger equation may be expressed in terms of Heun functions. Our results should enable the computation of bound states and energy eigenvalues, and the study of scattering and tunneling phenomena for a subclass of the potentials derived by [12].
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