Quantum Dynamics of the Slow Rollover Transition in the Linear Delta Expansion

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Abstract

We apply the linear delta expansion to the quantum mechanical version of the slow rollover transition which is an important feature of inflationary models of the early universe. The method, which goes beyond the Gaussian approximation, gives results which stay close to the exact solution for longer than previous methods. It provides a promising basis for extension to a full field theoretic treatment.

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1 Introduction

Inflationary models of the early Universe rely on the slow evolution of an inflaton field $\varphi$ from the initial unstable vacuum state in which $\langle \varphi \rangle = 0$ to the final stable vacuum in which $\langle \varphi \rangle = \pm a$, say. The effective potential $V_{\text{eff}}(\varphi_c)$ giving rise to this transition has the generic form of a gentle hill centred at $\varphi_c = 0$ with minima at $\varphi_c = \pm a$.

The transition can be discussed at various levels of sophistication. At the most naïve level one can think classically in terms of a ball rolling slowly down the slope of the potential. The corresponding quantum-mechanical problem, which is the subject of the present paper, is the time-development of a state whose wave function is initially concentrated around the position of the maximum of the potential. The full treatment of the problem must, of course, be formulated within the framework of quantum field theory.

The first treatment of the quantum mechanical problem was given by Guth and Pi [3], who solved exactly the equation of motion for an initial Gaussian wave-function in an upside-down harmonic oscillator potential $V = -\frac{1}{2}kx^2$. This was followed by a paper by Cooper et al. [4], who used a Gaussian ansatz in the Dirac time-dependent variational

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principle for the standard symmetry-breaking potential \( V = \lambda (x^2 - a^2)^2 / 24 \). The resulting “Hartree-Fock” solution tracks the exact solution for a short time, but departs from it before the time at which \( \langle x^2 \rangle \) reaches its first maximum. Several years later Cheetham and Copeland [4] went beyond the Gaussian approximation by using an ansatz which included a second-order Hermite polynomial. This represented an improvement on the Hartree-Fock approximation, but still did not reproduce the first maximum in \( \langle x^2 \rangle \) of the exact wave-function.

In the present paper we tackle this problem afresh using the linear delta expansion. This is a method akin to perturbation theory, but with the crucial difference that the form of the unperturbed Hamiltonian \( H_0 \) is not fixed once and for all, but varied at each order in the expansion by some well-defined criterion. The role of the formal parameter \( \delta \) is simply to keep track of the order of the expansion. The method has the great advantage that its generalization to field theory is straightforward. We are motivated to apply it to this dynamical problem by its success with the static properties of the anharmonic oscillator, where it can be proved rigorously that it converges to the exact result when applied to the finite-temperature partition function[1] and the energy levels[2].

The relevant Hamiltonian (\( \hbar = 1 \)) is
\[
H = -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \lambda (x^2 - a^2)^2 / 24 + \text{const.}
\]
\[
= -\frac{1}{2} \frac{\partial^2}{\partial x^2} - \frac{1}{2} m^2 x^2 + g x^4,
\]
(1)
with \( m^2 = \lambda a^2 / 6 \) and \( g = \lambda / 24 \), which we split according to
\[
H = -\frac{1}{2} \frac{\partial^2}{\partial x^2} \pm \frac{1}{2} \mu^2 x^2 + \delta g (x^4 - \rho x^2),
\]
(2)
where \( 2g\rho = m^2 \mp \mu^2 \). That is, we choose as bare mass term \( \pm \frac{1}{2} \mu^2 x^2 \). The sign of the term as well as the value of \( \mu \) will be determined as functions of \( t \) after the perturbative expansion has been carried out to a given order (at which stage \( \delta \) is set equal to 1) by the criterion of minimal sensitivity (PMS) [5], namely that
\[
\frac{\partial \langle x^2 \rangle}{\partial \mu} = 0.
\]
(3)

Note that for either sign of the new mass term, \( \mu \) has a limited range. In case (i), when the mass term is \( -\frac{1}{2} \mu^2 \), we have \( 2g\rho = m^2 - \mu^2 \). The essence of the delta expansion is that the extra term \( -g\rho x^2 \) in the interaction should compensate as far as possible the original term \( gx^4 \), which means that \( \rho \) should be positive. Hence we require that \( \mu^2 < m^2 \). In case (ii), when the mass term is \( +\frac{1}{2} \mu^2 \), the same restriction will arise from the form of the zeroth-order solution and the initial wave-function.
2 Delta Expansion

The two cases need to be treated separately. Which of them is relevant at a given value of $t$ is determined by the PMS criterion.

2.1 Case (i)

In this case the bare Hamiltonian is

$$H_0 = -\frac{1}{2} \frac{\partial^2}{\partial x^2} - \frac{1}{2} \mu^2 x^2$$

(4)

It is useful to scale $x$ according to $x = y/\sqrt{\mu}$, so that

$$H_0 = -\frac{\mu}{2} \left( \frac{\partial^2}{\partial y^2} + y^2 \right).$$

(5)

Given that the initial wave-function is a Gaussian of the form $\psi(t = 0) = A \exp(-By^2)$, the zeroth-order equation of motion $H_0 \psi_0 = i\partial \psi_0 / \partial t$ can be solved exactly by a wave-function of the same form with $A$ and $B$ becoming functions of $t$. The equations they have to satisfy are

$$i\dot{B}/\mu = 2B^2 + \frac{1}{2},$$

$$i\dot{A}/\mu = AB,$$

(6)

with solutions

$$B = \frac{1}{2} \tan(\eta_0 - i\mu t)$$

$$A = \mathcal{N} (\cos(\eta_0 - i\mu t))^{-\frac{1}{2}}$$

(7)

where $\eta_0$ is determined by $B(t = 0) = \frac{1}{2} \tan \eta_0$ and the normalization constant $\mathcal{N}$ by $A(t = 0) = \mathcal{N} (\cos \eta_0)^{-\frac{1}{2}}$. This is precisely the solution of Guth and Pi for the upside-down oscillator, with $m$ replaced by $\mu$.

To obtain a systematic perturbative expansion in powers of $\delta$ it is useful to write $\psi = \varphi \exp(-B(t)y^2)$. The equation for $\varphi$ is then

$$i\dot{\varphi} = \mu \left[ B\varphi + 2By\varphi' - \frac{1}{2} \varphi'' + \delta \tilde{g}(y^4 - \tilde{\rho}y^2) \right],$$

(8)

where we have scaled $g$ and $\rho$ according to $g = \mu^3 \tilde{g}$ and $\rho = \tilde{\rho}/\mu$.

A general feature of perturbative expansions for a polynomial potential of degree $p$ is that $\varphi$ is also a polynomial, of degree $Np$ in $N$th order of the expansion. Thus in the present case, expanding $\varphi$ as $\varphi = \sum \delta^n \varphi_n$, the first-order part $\varphi_1$ is an (even) polynomial of
degree 4, which we write as $\varphi_1 = a + by^2 + cy^4$. The equations of motion for the coefficients $a$, $b$ and $c$ are

$$\begin{align*}
i\dot{a}/\mu &= Ba - b \\
i\dot{b}/\mu &= 5Bb - 6c - \tilde{\rho}\tilde{g}A \\
i\dot{c}/\mu &= 9Bc + \tilde{g}A,
\end{align*}$$

(9)

which can be solved successively in reverse order, using the solutions for $A$ and $B$ previously determined. The initial conditions at $t = 0$ are $a = b = c = 0$. Thus $c$ is given by

$$c = \frac{-i\tilde{g}N/8}{(\cosh\tilde{\theta})^{9/2}} \left\{ 3\tilde{\theta} + 2\sinh 2\tilde{\theta} + \frac{1}{4}\sinh 4\tilde{\theta} - c_0 \right\},$$

(10)

where $\tilde{\theta} = \mu t + i\eta_0$ and $c_0 = 3i\eta_0 + 2i\sin 2\eta_0 + (i/4)\sin 4\eta_0$.

Using this solution in the equation for $b$ we obtain

$$b = \frac{-i\tilde{g}N}{(\cosh\tilde{\theta})^{5/2}} \left\{ -\frac{1}{2}\tilde{\rho}(\tilde{\theta} + \frac{1}{2}\sinh 2\tilde{\theta}) + b_0 + \frac{3i}{4} \left[ 3\tilde{\theta}\tanh\tilde{\theta} + \cosh^2\tilde{\theta} - c_0\tanh\tilde{\theta} \right] \right\},$$

(11)

where

$$b_0 = \frac{i}{2}\tilde{\rho}\eta_0 + \frac{i}{4}\tilde{\rho}\sin 2\eta_0 + \frac{9i}{4}\eta_0 \tan\eta_0 - \frac{3}{4}c_0\tan\eta_0 - \frac{3i}{4}\cos^2\eta_0.$$

Finally, using this solution in the equation for $a$ we obtain

$$a = \frac{\tilde{g}N}{(\cosh\tilde{\theta})^{1/2}} \left\{ -\frac{1}{2}\tilde{\rho}\tanh\tilde{\theta} + b_0\tanh\tilde{\theta} - a_0 \\
+ \frac{3i}{4} \left[ 3 \left( -\frac{1}{2}\tilde{\theta}\sech^2\tilde{\theta} + \frac{1}{2}\tanh\tilde{\theta} \right) + \tilde{\theta} + \frac{1}{2}c_0\sech^2\tilde{\theta} \right] \right\},$$

(12)

where

$$a_0 = \frac{1}{2}\tilde{\rho}\eta_0 \tan\eta_0 + ib_0 \tan\eta_0 + \frac{3i}{4} \left[ \frac{3}{2} \left( i\eta_0\sec^2\eta_0 + i\tan\eta_0 \right) + i\eta_0 + \frac{1}{2}c_0\sec^2\eta_0 \right].$$

2.2 Case (ii)

The zeroth-order equations in this case are

$$\begin{align*}
i\dot{B}/\mu &= 2B^2 - \frac{1}{2}, \\
i\dot{A}/\mu &= AB,
\end{align*}$$

(13)

with solutions

$$\begin{align*}
B &= \frac{1}{2}\coth(\eta_0 + i\mu t) \\
A &= N(\sinh(\eta_0 + i\mu t))^{-\frac{1}{2}}.
\end{align*}$$

(14)
As mentioned in the Introduction, the restriction on \( \mu \) in this case comes from the form of \( B(t=0) \) and the form of the initial wave-function, which, in all the papers quoted, is taken as a minimal wave-packet appropriate to a positive mass term \( +\frac{1}{2}m^2x^2 \). In the present formulation this means that \( B(t=0) = (1/2)(m/\mu) \). But since \( B(t=0) = (1/2)\coth \eta_0 < 1/2 \), we have the same restriction on \( \mu \), namely \( \mu < m \), as in Case (i).

The first-order equations for \( a, b \) and \( c \) are identical in form to Eq. (9), but the driving terms \( A \) and \( B \) are now different.

The solution for \( c \) is now

\[
c = -\frac{i\tilde{g}N}{8(i \sin \tilde{\theta})^{9/2}} \left\{ 3\tilde{\theta} - 2 \sin 2\tilde{\theta} + \frac{1}{4} \sin 4\tilde{\theta} - c_0 \right\} \tag{15}
\]

where \( c_0 = -3i\eta_0 + 2i \sin \eta_0 - (i/4) \sinh 2\eta_0 \) and in this case \( \tilde{\theta} = \mu t - i\eta_0 \).

Using this solution in the equation for \( b \) we obtain

\[
b = \frac{\tilde{g}N}{(i \sin \tilde{\theta})^{5/2}} \left\{ -\frac{i}{2} \tilde{\rho}(\tilde{\theta} - \frac{1}{2} \sin 2\tilde{\theta}) + b_0 - \frac{3}{4} \left[ -3\tilde{\theta}\cot \tilde{\theta} + \cos^2 \tilde{\theta} + c_0\cot \tilde{\theta} \right] \right\}, \tag{16}
\]

where

\[
b_0 = \frac{1}{2} \tilde{\rho}\eta_0 - \frac{1}{4} \tilde{\rho} \sin \eta_0 - \frac{9}{4} \eta_0 \coth \eta_0 + \frac{3i}{4} c_0 \coth \eta_0 + \frac{3}{4} \cosh^2 \eta_0.
\]

Finally, using this solution in the equation for \( a \) we obtain

\[
a = \frac{\tilde{g}N}{(i \sin \tilde{\theta})^{1/2}} \left\{ \frac{1}{2} \tilde{\rho} \cot \tilde{\theta} + ib_0 \cot \tilde{\theta} + a_0 - \frac{3i}{4} \left[ \frac{3}{2} \tilde{\theta}\cosech^2 \tilde{\theta} + \frac{1}{2} \cot \tilde{\theta} - \tilde{\theta} - \frac{1}{2} c_0 \cosech^2 \tilde{\theta} \right] \right\}, \tag{17}
\]

where

\[
a_0 = \frac{1}{2} \tilde{\rho}\eta_0 \coth \eta_0 - b_0 \coth \eta_0 - \frac{3i}{4} \left[ \frac{3}{2} \eta_0 \cosech^2 \eta_0 + \frac{1}{2} \coth \eta_0 + \eta_0 - \frac{i}{2} c_0 \cosech^2 \eta_0 \right].
\]

We have checked these solutions by numerical integration using the Runge-Kutta method. This reveals that in Case (ii) care needs to be taken to ensure that we are on the appropriate branch of the square roots. At values of \( t \) where \( \sin \tilde{\theta} = -1 \), a naïve numerical evaluation will stay on the first sheet, thus giving rise to a discontinuity, whereas the true solution is, of course, continuous.

In fact, as we shall see, the coefficient \( a \) is not needed in the calculation of \( \langle x^2 \rangle^{1/2} \) to first order in \( \delta \), though it would, of course, be needed in higher order.

### 3 Variational Aspect

The expressions we have obtained all depend on the parameter \( \mu \) introduced in Eq. (2). The other essential aspect of the delta expansion is that such a parameter is determined
by some non-perturbative criterion, most frequently the principle of minimal sensitivity, Eq. (3).

To that end we need an expression for \( \langle x^2 \rangle \), which, given that the wave-function is a (complex) Gaussian with polynomial corrections, can be written down in closed form in terms of the coefficients \( A, B, a, b, c \). Thus to order \( \delta \),

\[
|\psi|^2 = \left[ |A|^2 + 2\delta \text{Re} \left\{ A^* (a + by^2 + cy^4) \right\} \right] e^{-\alpha y^2},
\]

where \( \alpha = 2\text{Re}B \), so that

\[
\langle y^2 \rangle = \frac{1}{2\alpha} \left[ 1 + \frac{2\delta}{|A|^2} \text{Re} \left\{ A^* \left( \frac{b}{\alpha} + \frac{3c}{\alpha^2} \right) \right\} \right].
\]

(19)

It is an interesting feature of the structure of the perturbative equations that the wave-function is automatically normalized to the order we are working. That is, \( \int \phi_0^* \phi_1 = 0 \). Thus the second equation is identical to the first, and not merely an \( O(\delta) \) approximation to it. The expectation value we seek is obtained on scaling by \( \mu \), i.e. \( \langle x^2 \rangle = \langle y^2 \rangle / \mu \).

At this stage we set \( \delta = 1 \) and apply Eq. (3). This has to be done for each time \( t \), and the result is that the chosen value \( \tilde{\mu} \) of \( \mu \) now becomes a function of \( t \), even though \( \mu \) was treated as a constant in the equations of motion. In the present case, since we are unable to go to very high orders in the expansion this is a more important property than the \( N \)-dependence of \( \tilde{\mu} \). The \( O(\delta^0) \) calculation does not have such a stationary point.

In Fig. 1 we show graphs of \( \langle x^2 \rangle_{\tilde{\mu}} \) for various values of \( t \). The parameters chosen are those used in Refs. [4] and [5], namely \( a = 5 \) and \( \lambda = 0.01 \) (which corresponds to a “large” dimensionless coupling constant[5]). We include both cases by plotting \( \langle x^2 \rangle_{\tilde{\mu}} \) as a function of \( \sigma \mu \), where \( \sigma = -1 \) for case (i) and +1 for case (ii). There is a well-defined maximum which moves steadily to the right as \( t \) increases, crossing over from case (i) to case (ii) at about \( t = 11 \). From these and similar graphs we extract the value of \( \tilde{\mu}(t) \), which is plotted in Fig. 2.

Using these values of \( \tilde{\mu}(t) \) we can then calculate \( \langle x^2 \rangle_{\tilde{\mu}(t)} \) from Eq. (19) as a function of \( t \). This is plotted in Fig. 3 along with the results obtained using the “Hartree-Fock” method of Ref. [4], the improved variational method of Ref. [5], the exact value of \( \langle x^2 \rangle_{\tilde{\mu}} \), obtained by Fourier transform and numerical integration[7], and finally the result of first-order perturbation theory. The latter corresponds to \( O(\delta) \) of the delta expansion, but with \( \mu \) fixed at \( m \) in case (i), and exemplifies the importance of the \( t \)-dependence of \( \tilde{\mu} \).

As can be seen, the delta-expansion result tracks the exact result for longer than either of the other variational calculations, essentially up to the point where \( \langle x^2 \rangle_{\tilde{\mu}} \) reaches its maximum, but then overshoots. A similar degree of accuracy in quantum field theory would mean that, to this order of the expansion, the inflationary period would be very well described, but the reheating process less so. To extend the range of the approximation to longer times a higher-order calculation would presumably be needed.
4 Discussion

Figure 3 is our main result, but it is also of interest to enquire how closely the calculated wave-function agrees with the exact result, since a well-known feature of variational methods is that quite reasonable values for expectation values such as $\langle x^2 \rangle$ can be obtained with rather inaccurate wave-functions. In fact our wave-function agrees rather well with the true wave-function up to $t \approx 6$, but begins to diverge from it thereafter, even though still giving good values for $\langle x^2 \rangle$. In Fig. 4 we plot the two values of $|\psi|^2$ versus $x$ for $t = 6$ and 8.

Various extensions of the present treatment are possible. Given the simplicity of the first-order equations resulting from the method it seems that the extension to $O(\delta^2)$ should be relatively straightforward, certainly if the integrations are performed numerically, and it would be interesting to see the improvement thereby achieved. As mentioned earlier, the wave-function would involve an even polynomial of order 8 multiplying the lowest-order Gaussian. Another possibility is the use of the original delta expansion, whereby the $x^4$ term in the potential is written as $x^{2(1+\delta)}$ and expanded as $x^2(1 + \delta \ln x^2 + \ldots)$. This expansion is known to converge for the energy levels of the anharmonic oscillator, and the $O(\delta)$ calculation for the present problem should be tractable. However, the disadvantage of this method is that its extension to field theory beyond first order becomes extremely difficult.

The most important extension is clearly to attempt to apply the methodology of the linear delta expansion to the quantum field theory problem. The importance of going beyond the Gaussian approximation has been emphasized in refs. [5] and [9], and technically the linear delta expansion is essentially a modified perturbation theory, modulo the crucial variational aspect.

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Figure 1: Graphs of $\langle x^2 \rangle^{1/2}$ versus $\sigma \mu$ for $t=5$, 9 and 13, where $\sigma = -1$ for case (i) and $\sigma = +1$ for case (ii). We have excluded the region $\mu < 0.05$ since there are severe round-off problems near $\mu = 0$. 
Figure 2: $\bar{\mu}$ versus $t$. The change-over from case (i) to case (ii) occurs between $t=11$ and $t=12$. 
Figure 3: $\langle x^2 \rangle^{1/2}$ versus $t$. First-order linear delta expansion (LDE) compared with the exact result (Exact), the variational calculations of Ref. [3] (HF) and Ref. [4] (CC), and first-order perturbation theory (PT).
Figure 4: Graphs of $|\psi|^2$ versus $x$ for $t=6$ and $8$. The solid line is the first-order LDE calculation and the dotted line is the exact result.