Study of COVID-19 epidemiological evolution in India with a multi-wave SIR model

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The global pandemic due to the outbreak of COVID-19 ravages the whole world for more than two years in which all the countries are suffering a lot since December 2019. In order to control this ongoing waves of epidemiological infections, attempts have been made to understand the dynamics of this pandemic in deterministic approach with the help of several mathematical models. In this article characteristics of a multi-wave SIR model have been studied which successfully explains the features of this pandemic waves in India. Stability of this model has been studied by identifying the equilibrium points as well as by finding the eigen values of the corresponding Jacobian matrices. Complex eigen values are found which ultimately give rise to the oscillatory solutions for the three categories of populations, say, susceptible, infected and removed. In this model, a finite probability of the recovered people for becoming susceptible again is introduced which eventually lead to the oscillatory solution in other words. The set of differential equations has been solved numerically in order to obtain the variation for numbers of susceptible, infected and removed people with time. In this phenomenological study, finally an additional modification is made in order to explain the aperiodic oscillation which is found necessary to capture the feature of epidemiological waves particularly in India.

I. INTRODUCTION

The outbreak of COVID-19 in December 2019, due to the spreading of infectious corona virus named SARS-CoV-2 finally triggered to a global pandemic in which all the countries are suffering a lot till to date. This highly contagious disease has been transmitted to millions of people globally where a fraction of infected people is succumbed to it eventually. Most of the countries witness multiple peaks of epidemiological infections in its evolution which is counted as number of waves. Among the most affected countries, for example, countries in European union (CEU), USA, Russia and Canada experience five successive epidemiological waves while the third wave is going on in India, Indonesia and Brazil. Which means that five peaks of the epidemiological infections are found in CEU, USA, Russia and Canada, while three distinct peaks are noted in India, Indonesia and Brazil. However, peaks of epidemiological infections in India are sharper than those found in Brazil. So, it is obvious that characteristics of these epidemiological waves found in different countries are not similar. Widths and heights of the peaks are different as well as the separation between them are not equal. It is observed that the same individual gets infected multiple time during this pandemic in all countries.

In order to study the dynamics of the pandemic in a deterministic approach, several models have been proposed based on the classic SIR model introduced by Kermack and Mckendrick in 1927. [1] Most of the models are mainly formulated to study the characteristics of the first wave of the pandemic. Those are known as SEIR, SIQR and their hybrids which are all derived from the single-wave SIR (SWSIR) model. [2–6] It is termed as SWSIR model because of the fact that this primitive version in general yields a solitary peak in its evolution. In those attempts, effects of quarantine, isolation, latent time of infection and other factors are taken into account where several predictions and forecasts are available. [7, 8] Nonetheless, periodic outbreak of disease in terms of SIR, SIQR, and SEIQR models are studied analytically in great details. [4, 12] Evolution of the COVID-19 pandemic for different countries have been studied with the help of those models where origin of differences in its features are explained. Case studies on few countries are available. [14,20]

Hence, a multi-wave SIR (MWSIR) model is formulated by modifying the SWSIR model in order to explain the dynamics of epidemiological infections including the origin of multiple waves found in the infection pattern. This is a fact that the same individual has been infected repeatedly during this pandemic period. So in this model, probability of the same individual for getting infected multiple times after curing is taken into account. [21] Time delay between successive infections of the same individual is introduced in the MWSIR model. In this work, periodic outbreak of COVID-19 in India will be studied, and so, the daily new cases in terms of seven-day rolling average in India during January 30, 2020 to February 3, 2022 (736 days) are shown in Figure 1. It reveals that the duration of those waves are different, so, it is not perfectly periodic. The approximate duration of first and second waves are $T_1 = 370$ and $T_2 = 300$ days, respectively, while the third wave crosses 66 days ($T_3 > 66$) by this time. Hence, separation between adju-
cent peaks are different. In addition to that width and height of the peaks are unequal.

In the section II a MWSIR model is considered which gives rise to almost periodic epidemiological oscillation. The stability of this model is studied in section III analytically while its periodic feature is studied numerically in section IV. In order to produce aperiodic oscillation, this model has been further modified. The epidemiological oscillation produced by this model is compared with the daily new COVID-19 cases in India. A discussion based on those results is available in section V.

II. MULTI-WAVE SIR MODEL

In order to explain the epidemiological waves of infection due to the spreading of COVID-19 in India, a multi-wave SIR model is formulated where the total population is divided into three dynamic sub-populations, those are known as susceptible, infected and removed. Susceptible population is constituted by the individuals those are otherwise healthy but have a probability for having infected at any time in future. The intermediate stage is comprised of infected population where the individual is instantaneously contagious. In the terminal stage individuals are either recovered or succumbed to the disease, however, jointly referred to as removed population. As the ongoing COVID-19 pandemic exhibits periodic outbreak of the disease globally, a new model is necessary to study the dynamics of this pandemic.

Therefore, in order to understand the nature of periodic outbreak, a MWSIR model has been formulated by modifying the primitive version of SIR model as described below. In this model the recovered population has a nonzero probability to get infected again. As a result, the susceptible population gets enriched with time which is measured in terms of infected population with a time delay, \( \tau \). Thus, the multi-wave SIR model is defined by a set of coupled first-order nonlinear differential equations (Eq 1):

\[
\begin{align*}
\frac{dS}{dt} &= \mu N - \alpha S(t)I(t) + \gamma I(t-\tau) - \mu S(t), \\
\frac{dI}{dt} &= \alpha S(t)I(t) - \beta I(t) - \mu I(t), \\
\frac{dR}{dt} &= \beta I(t) - \gamma I(t-\tau) - \mu R(t),
\end{align*}
\]

where \( S(t), I(t), \) and \( R(t) \) are the number of susceptible, infected and removed people at time \( t \). The positive constants \( \mu, \alpha, \beta \) and \( \gamma \) are the rates of birth per individual, infection, removed and susceptibility, respectively. The model incorporates the fact that the recovered people become susceptible again after the mean period \( \tau \). However, \( I(t-\tau) = 0 \), when \( t < \tau \). The SWSIR model will be restored when \( \mu = 0 \), and \( \gamma = 0 \). The system in this formulation is closed in a sense that the total population, \( N = S(t) + I(t) + R(t) \), does not change with time, which means the rate at which individual suffers natural mortality is also given by the parameter \( \mu \). In other words, mean lifespan of individual turns out to be \( 1/\mu \). In the same way, \( 1/\beta \) can be regarded as the mean recovery time for the infected individual. The flowchart of this model is shown in Figure 2.

The basic reproduction number, \( R_0 \) plays an important role in epidemiology, which is defined as average number of new infections per infected individual. So, in this three-compartment system, it is equal to the transmission rate multiplied by the infectious period, \( R_0 = \alpha/(\beta + \mu) \). The value of \( R_0 \) is crucial to have an idea about how does the disease flow in the whole population and at the same time it provides clue to control its spreading.

III. STABILITY ANALYSIS OF MULTI-WAVE SIR MODEL

In order to determine the stable equilibrium points, the rate of change of \( S(t), I(t) \) and \( R(t) \) with time are made zero. The trivial solution leads to \( I(t) = 0 \) at any time, which corresponds to disease-free equilibrium (DFE). Mathematically, this point is expressed as \( (S^*, I^*, R^*)_0 = (N, 0, 0) \), where the entire population becomes susceptible but no infected individual. This is true for both \( t < \tau \) and \( t \geq \tau \). Anyway, this DFE is insignificant as it does not correspond to dynamics of epidemic by any means.

![Flowchart for the MWSIR model.](Image)
The nontrivial solution leads to $I(t) \neq 0$, at finite time, but, $I(t \to \infty) \to 0$, which corresponds to endemic equilibrium and that is observed in SWSIR model. However, in this MWSIR model, nontrivial solutions mean $I(t) > 0$, at any time. So these dynamic equilibrium points appear periodically with time without showing any signature of ending if there is no damping. These equilibrium points are marked by $(S^*, I^*, R^*) = \left( \frac{\beta N}{\alpha}, \frac{\beta N}{\alpha} - \frac{\beta + \mu}{\alpha}, \frac{\beta N}{\alpha} - \frac{\beta + \mu}{\alpha} \right)$, when $t < \tau$, however, for $t \geq \tau$, $(S^*, I^*, R^*)(t)_{\text{atr}} = \left( \frac{\beta + \mu}{\alpha}, \frac{\alpha N}{\alpha} - \frac{\beta + \mu}{\alpha}, \frac{\alpha N}{\alpha} - \frac{\beta + \mu}{\alpha} \right)$.

Hence, equilibrium points are static when $t < \tau$, but, dynamic when $t \geq \tau$.

In order to understand the dynamics close to the equilibria, Jacobian matrix $(J)$ is constructed which can be expressed at the respective equilibrium points. Jacobian matrix has the form
\[
\begin{bmatrix}
-(\alpha I^* + \mu) & -\alpha S^* & 0 \\
\alpha I^* & \alpha S^* - \beta - \mu & 0 \\
0 & \beta & -\mu
\end{bmatrix}.
\]

The eigenvalues of Jacobian matrix $(\lambda_0, \lambda_{\pm})$ can be given as
\[
\begin{align*}
\lambda_0 &= -\mu, \\
\lambda_{\pm} &= -\alpha(I^* - S^*) + \beta + 2\mu \pm \sqrt{\alpha^2(I^* - S^*)^2 + \beta^2 - 2\alpha\beta(S^* + I^*)}. 
\end{align*}
\]

For the DFE, $(S^*, I^*, R^*)_{\text{tr}} = (N, 0, 0)$, the eigenvalues are
\[
\begin{align*}
\lambda_0 &= -\mu, \\
\lambda_{+} &= \alpha N - \beta - \mu, \\
\lambda_{-} &= -\mu.
\end{align*}
\]

As a result, it will behave like a stable point when $\alpha < (\beta + \mu)/N$, which is similar to the SWSIR model. It is obvious that the eigenvalues will be complex for the general case when
\[
\alpha \beta > \frac{\alpha^2(I^* - S^*)^2 + \beta^2}{2(S^* + I^*)}.
\]

It is expected that variations of $S(t)$, $I(t)$, and $R(t)$ will be oscillatory around these points.

**IV. NUMERICAL RESULTS**

As the values of $S(t)$, $I(t)$, and $R(t)$ cannot be determined analytically by solving the set of equations, Eq 3 they have been solved numerically. The oscillations of $S(t)$, $I(t)$, and $R(t)$ are noted for long time and it is observed that all are oscillating with almost constant amplitude. Four successive waves are shown in Fig 3, where it is observed that the mean time period of oscillations, $T$, is always greater than $\tau$. As no damping factor is introduced here, amplitudes of the respective waves for $S(t)$, $I(t)$, and $R(t)$ do not change with time.

The time period of oscillations for $S(t)$, $I(t)$, and $R(t)$ are obviously the same, although it decreases with the number of cycles. However the difference, $T - \tau$ vanishes with the increase of $\tau$. This feature is shown in Fig 4 where variations of time period $T$ along with relative difference between $T$ and $\tau$ or $(T - \tau)/\tau$ with $\tau$ are plotted.

**FIG. 3:** Four successive waves are shown here those are generated numerically by solving MWSIR model (Eq 3). Values of the parameters: $S(0) = 10$, $I(0) = 1$, $R(0) = 0$, $N = 11$, $\alpha = 0.5$, $\beta = 1$, $\gamma = 1$, $\tau = 10$, $\mu = 0$. Time period of oscillation is marked by $T$.

**FIG. 4:** Variation of $T$ and $(T - \tau)/\tau$ with $\tau$, in the MWSIR model for $S(0) = 10$, $I(0) = 1$, $R(0) = 0$, $N = 11$, $\alpha = 0.5$, $\beta = 1$, $\gamma = 1$, $\mu = 0$.

The variation of $T$ with $\alpha$ and $\beta$ for fixed value of $\tau$ is shown in Figure 4. It clearly indicates that $T > \tau$, for any values of $\alpha$ and $\beta$. Moreover, the difference, $T - \tau$
In order to capture this specific aperiodic feature, the value of \( \gamma \) has been split up for three different waves in the MWSIR model as

\[
\gamma = \begin{cases} 
0, & t < \tau, \\
\gamma_1, & \tau \leq t < \nu \tau, \\
\gamma_2, & t \geq \nu \tau, 
\end{cases}
\]

where \( \gamma_1 = 0.166, \gamma_2 = 0.056, \nu = 2.78, \) and \( \tau = 175 \) days. The relation, \( \gamma_2 < \gamma_1 \), corresponds to the fact that height of third peak is lower than the second one. The value, \( \beta = 1/14 \), is kept fixed as it corresponds to the mean recovery period of COVID-19, \( 1/\beta = 14 \) days. The numerical data obtained by solving the MWSIR model is plotted in red dashed line and compared with the daily new COVID-19 cases (seven-day rolling average) in India shown in blue solid line. A very good agreement is found when the rate of infection is assumed very small (\( \alpha = 1.78 \times 10^{-7} \)).

This phenomenological study predicts that 16.6% of individuals infected during January 30, 2020 to December 5, 2020 (first wave) become susceptible again for further infection. Similarly, 5.6% of individuals infected during December 5, 2020 to May, 27, 2021 (second wave) become susceptible again for the next infection. This reduction may account the effects of quarantine, isolation, vaccination and other preventive measures. However, as the solutions of the non-linear equations are highly sensitive to the initial conditions, like \( S(0), I(0), \) and \( R(0) \), estimated values of the parameters, \( \alpha, \gamma, \nu \) and \( \tau \) are likely to change dramatically with the change of initial values. Which on the other hand will change the predictions for obvious reason.

V. DISCUSSION

In order to study the behaviour of ongoing pandemic of COVID-19 in India mathematically, a modified version of SWSIR in India is proposed here which is found successful to reproduce the features of the multiple epidemiological waves observed in this country. The model is deigned in such a fashion that it is able to yield periodic as well as aperiodic epidemiological waves with varying widths and heights by tuning its parameters. As the SARS-CoV-2 virus responsible for the COVID-19 is highly infectious, the spreading of this disease cannot be controlled easily by means of simple preventive measures like quarantine, isolation, lock down and even vaccination. As a result, in due course, a sizable fraction of recovered people gets infected multiple times leading to multiple waves of the epidemiological infection. In order to explain this particular feature, the SIR model has been modified in such a manner that the recovered people has a finite chance for becoming susceptible again after their recovery in this modified version. Estimated values of the parameters obtained in this study will be useful for imposing restrictions to control the pandemic.
It will become helpful even for making accurate forecast and prediction. However, in this simple model no time delay for the incubation period of the pathogens within the human body is considered, and the effect of quarantine, isolation, vaccination and other macroscopic measures are not take into account. So, it is expected that more accurate prediction can be made with the help of this model by accommodating the effect of quarantine, isolation, vaccination, lock-down and other factors in its modified version. Which means that the discrepancy between theoretical estimation and daily new cases can be removed further by formulating multi-wave SEIR, SIQR, SEIQR and other hybrid models. Obviously, this model can be employed for examining the characteristics of epidemiological waves found in other countries as well.

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