BOUNDS ON THE JOINT AND GENERALIZED SPECTRAL RADIUS OF HADAMARD GEOMETRIC MEAN OF BOUNDED SETS OF POSITIVE KERNEL OPERATORS

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Abstract. Let $\Psi_1, \ldots, \Psi_m$ be bounded sets of positive kernel operators on a Banach function space $L$. We prove that for the generalized spectral radius $\rho$ and the joint spectral radius $\tilde{\rho}$ the inequalities

\[
\rho \left( \Psi_1^{\frac{1}{m}} \circ \cdots \circ \Psi_m^{\frac{1}{m}} \right) \leq \rho(\Psi_1 \Psi_2 \cdots \Psi_m),
\]

\[
\tilde{\rho} \left( \Psi_1^{\frac{1}{m}} \circ \cdots \circ \Psi_m^{\frac{1}{m}} \right) \leq \rho(\Psi_1 \Psi_2 \cdots \Psi_m)^{\frac{1}{m}}
\]

hold, where $\Psi_1^{\frac{1}{m}} \circ \cdots \circ \Psi_m^{\frac{1}{m}}$ denotes the Hadamard (Schur) geometric mean of the sets $\Psi_1, \ldots, \Psi_m$.

1. Introduction

In [34], X. Zhan conjectured that, for non-negative $n \times n$ matrices $A$ and $B$, the spectral radius $\rho(A \circ B)$ of the Hadamard product satisfies

\[
\rho(A \circ B) \leq \rho(AB),
\]

where $AB$ denotes the usual matrix product of $A$ and $B$. This conjecture was confirmed by K.M.R. Audenaert in [3] via a trace description of the spectral radius. Soon after, this inequality was reproved, generalized and refined in different ways by several authors ([18], [19], [27], [28], [26], [7], [13]). Applying a fact that the Hadamard product is a principal submatrix of the Kronecker product (i.e., by applying the technique used by R.A. Horn and F. Zhang of [18]), Z. Huang proved that

\[
\rho(A_1 \circ A_2 \circ \cdots \circ A_m) \leq \rho(A_1 A_2 \cdots A_m)
\]

for $n \times n$ non-negative matrices $A_1, A_2, \ldots, A_m$ (see [19]). The author of the current paper extended the inequality (1.1) to non-negative matrices that define bounded operators on Banach sequence spaces in [26]. Moreover, in [26, Theorem 3.16] he generalized this inequality to the setting of the generalized and the joint spectral radius of bounded sets of such non-negative matrices. In the proofs certain results on the Hadamard product from [11] and [25] were used.

Earlier, A.R. Schep was the first one to observe that the results [11] and [25] are applicable in this context (see [27] and [28]). In particular, in [27, Theorem

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he proved that the inequality
\[ \rho \left( A^\left(\frac{1}{2}\right) \circ B^\left(\frac{1}{2}\right) \right) \leq \rho(AB)^{\frac{1}{2}} \] (1.2)
holds for positive kernel operators on $L^p$ spaces. Here $A^\left(\frac{1}{2}\right) \circ B^\left(\frac{1}{2}\right)$ denotes the Hadamard geometric mean of operators $A$ and $B$. In \[13,\] Theorem 3.1, R. Drnovšek and the author, generalized this inequality and proved that the inequality
\[ \rho \left( A_1^{\left(\frac{1}{m}\right)} \circ A_2^{\left(\frac{1}{m}\right)} \circ \cdots \circ A_m^{\left(\frac{1}{m}\right)} \right) \leq \rho(A_1 A_2 \cdots A_m)^{\frac{1}{m}} \] (1.3)
holds for positive kernel operators $A_1, \ldots, A_m$ on an arbitrary Banach function space.

The article is partly expository, since it also includes some new proofs of known results. It is organized as follows. In the second section we introduce some definitions and facts, and we recall some results from \[11\] and \[25\], which we will need in our proofs. In the third section we give a new proof of a key result from \[11\] and \[25\] (Theorem 3.1) and recall what this result actually means in the setting of the generalized and the joint spectral radius (Theorem 3.3). In our main result (Theorem 3.4) we generalize the inequality (1.3) to the setting of the generalized and the joint spectral radius of bounded sets of positive kernel operators on an arbitrary Banach function space.

2. Preliminaries

Let $\mu$ be a $\sigma$-finite positive measure on a $\sigma$-algebra $M$ of subsets of a non-void set $X$. Let $M(X, \mu)$ be the vector space of all equivalence classes of (almost everywhere equal) complex measurable functions on $X$. A Banach space $L \subseteq M(X, \mu)$ is called a Banach function space if $f \in L$, $g \in M(X, \mu)$, and $|g| \leq |f|$ imply that $g \in L$ and $\|g\| \leq \|f\|$. Throughout the article, it is assumed that $X$ is the carrier of $L$, that is, there is no subset $Y$ of $X$ of strictly positive measure with the property that $f = 0$ a.e. on $Y$ for all $f \in L$ (see \[33\]).

Standard examples of Banach function spaces are Euclidean spaces, the space $c_0$ of all null convergent sequences (equipped with the usual norms and the counting measure), the well-known spaces $L^p(X, \mu)$ ($1 \leq p \leq \infty$) and other less known examples such as Orlicz, Lorentz, Marcinkiewicz and more general rearrangement-invariant spaces (see e.g. \[5\], \[8\] and the references cited there), which are important e.g. in interpolation theory. Recall that the cartesian product $L = E \times F$ of Banach function spaces is again a Banach function space, equipped with the norm $\|(f, g)\|_L = \max\{\|f\|_E, \|g\|_F\}$.

By an operator on a Banach function space $L$ we always mean a linear operator on $L$. An operator $A$ on $L$ is said to be positive if it maps nonnegative functions to nonnegative ones, i.e., $AL_+ \subseteq L_+$, where $L_+ = \{f \in L : f \geq 0 \text{ a.e.}\}$. Given operators $A$ and $B$ on $L$, we write $A \geq B$ if the operator $A - B$ is positive. Recall that a positive operator $A$ is always bounded, i.e., its operator norm
\[ \|A\| = \sup\{\|Ax\|_L : x \in L, \|x\|_L \leq 1\} = \sup\{\|Ax\|_L : x \in L_+, \|x\|_L \leq 1\} \] (2.1)
is finite. Also, its spectral radius \( \rho(A) \) is always contained in the spectrum.

An operator \( A \) on a Banach function space \( L \) is called a kernel operator if there exists a \( \mu \times \mu \)-measurable function \( a(x, y) \) on \( X \times X \) such that, for all \( f \in L \) and for almost all \( x \in X \),

\[
\int_X |a(x, y)f(y)| \, d\mu(y) < \infty \quad \text{and} \quad (Af)(x) = \int_X a(x, y)f(y) \, d\mu(y).
\]

One can check that a kernel operator \( A \) is positive iff its kernel \( a \) is non-negative almost everywhere. Observe that (finite or infinite) non-negative matrices that define operators on Banach sequence spaces are a special case of positive kernel operators (see e.g. [26], [13], [12] and the references cited there). It is well-known that kernel operators play a very important, often even central, role in a variety of applications from differential and integro-differential equations, problems from physics (in particular from thermodynamics), engineering, statistical and economic models, etc (see e.g. [20], [4], [21], [10] and the references cited there). For the theory of Banach function spaces and more general Banach lattices we refer the reader to the books [33], [5], [1], [2].

Let \( A \) and \( B \) be positive kernel operators on \( L \) with kernels \( a \) and \( b \) respectively, and \( \alpha \geq 0 \). The Hadamard (or Schur) product \( A \circ B \) of \( A \) and \( B \) is the kernel operator with kernel equal to \( a(x, y)b(x, y) \) at point \( (x, y) \in X \times X \) which can be defined (in general) only on some order ideal of \( L \). Similarly, the Hadamard (or Schur) power \( A^{(\alpha)} \) of \( A \) is the kernel operator with kernel equal to \( (a(x, y))^{\alpha} \) at point \( (x, y) \in X \times X \) which can be defined only on some order ideal of \( L \).

Let \( A_1, \ldots, A_n \) be positive kernel operators on a Banach function space \( L \), and \( \alpha_1, \ldots, \alpha_n \) positive numbers such that \( \sum_{j=1}^n \alpha_j = 1 \). Then the Hadamard weighted geometric mean \( A = A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_n^{(\alpha_n)} \) of the operators \( A_1, \ldots, A_n \) is a positive kernel operator defined on the whole space \( L \), since \( A \leq \alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n \) by the inequality between the weighted arithmetic and geometric means. Let us recall the following result which was proved in [11, Theorem 2.2] and [25, Theorem 5.1].

**Theorem 2.1.** Let \( \{A_{ij}\}_{i=1, j=1}^{k,m} \) be positive kernel operators on a Banach function space \( L \). If \( \alpha_1, \alpha_2, \ldots, \alpha_m \) are positive numbers such that \( \sum_{j=1}^m \alpha_j = 1 \), then the positive kernel operator

\[
A := (A_{11}^{(\alpha_1)} \circ \cdots \circ A_{1m}^{(\alpha_m)}) \cdots (A_{k1}^{(\alpha_1)} \circ \cdots \circ A_{km}^{(\alpha_m)})
\]

satisfies the following inequalities

\[
A \leq (A_{11} \cdots A_{k1})^{(\alpha_1)} \circ \cdots \circ (A_{1m} \cdots A_{km})^{(\alpha_m)},
\]

\[
\|A\| \leq \|A_{11} \cdots A_{k1}\|^{\alpha_1} \cdot \cdots \|A_{1m} \cdots A_{km}\|^{\alpha_m},
\]

\[
\rho(A) \leq \rho(A_{11} \cdots A_{k1})^{\alpha_1} \cdots \rho(A_{1m} \cdots A_{km})^{\alpha_m}.
\]

The following result is a special case of Theorem 2.1.

**Theorem 2.2.** Let \( A_1, \ldots, A_m \) be positive kernel operators on a Banach function space \( L \), and \( \alpha_1, \ldots, \alpha_m \) positive numbers such that \( \sum_{j=1}^m \alpha_j = 1 \). Then we have

\[
\|A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_m^{(\alpha_m)}\| \leq \|A_1\|^{\alpha_1} \cdot \|A_2\|^{\alpha_2} \cdots \|A_m\|^{\alpha_m}
\]

(2.5)
\[
\rho(A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_m^{(\alpha_m)}) \leq \rho(A_1)^{\alpha_1} \rho(A_2)^{\alpha_2} \cdots \rho(A_m)^{\alpha_m}. \tag{2.6}
\]

Recall also that the above results on the spectral radius and operator norm remain valid under less restrictive assumption \(\sum_{j=1}^{m} \alpha_j \geq 1\) in the case of (finite or infinite) non-negative matrices that define operators on sequence spaces ([16], [11], [25], [26], [13]).

Let \(\Sigma\) be a bounded set of bounded operators on \(L\). For \(m \geq 1\), let
\[
\Sigma^m = \{A_1A_2 \cdots A_m : A_i \in \Sigma\}.
\]

The generalized spectral radius of \(\Sigma\) is defined by
\[
\rho(\Sigma) = \lim_{m \to \infty} \sup_{\alpha \in \Sigma^m} \|A\|^{1/m} \tag{2.7}
\]
and is equal to
\[
\rho(\Sigma) = \sup_{m \in \mathbb{N}} \|A\|^{1/m}. \tag{2.8}
\]

The joint spectral radius of \(\Sigma\) is defined by
\[
\hat{\rho}(\Sigma) = \lim_{m \to \infty} \sup_{A \in \Sigma^m} \|A\|^{1/m}.
\]

It is well known that \(\rho(\Sigma) = \hat{\rho}(\Sigma)\) for a precompact set \(\Sigma\) of compact operators on \(L\) (see e.g. [30], [31], [22]), in particular for a bounded set of complex \(n \times n\) matrices (see e.g. [6], [14], [29], [9], [23]). This equality is called the Berger-Wang formula or also the generalized spectral radius theorem (for an elegant proof in the finite dimensional case see [9]). However, in general \(\rho(\Sigma)\) and \(\hat{\rho}(\Sigma)\) may differ even in the case of a bounded set \(\Sigma\) of compact positive operators on \(L\) as the following example from [29] shows. Let \(\Sigma = \{A_1, A_2, \ldots\}\) be a bounded set of compact operators on \(L = l^2\) defined by \(A_k e_k = e_{k+1}, (k \in \mathbb{N})\) and \(A_k e_j = 0\) for \(j \neq k\). Then \((A_1 A_2 \cdots A_{k})^2 = 0\) for arbitrary \(k \in \mathbb{N}\) and any subset \(\{i_1, i_2, \ldots, i_k\} \subset \mathbb{N}\). Thus \(\rho(\Sigma) = 0\). Since
\[
A_m A_{m-1} \cdots A_1 e_1 = e_{m+1}, \quad m \in \mathbb{N},
\]
\[
A_m A_{m-1} \cdots A_1 e_j = 0, \quad j \neq 1,
\]
we have \(\hat{\rho}(\Sigma) \geq \limsup_{m \to \infty} \|A_m \cdots A_1\|^{1/m} = 1\) and so \(\rho(\Sigma) \neq \hat{\rho}(\Sigma)\).

In [17], the reader can find an example of two positive non-compact weighted shifts \(A\) and \(B\) on \(L = l^2\) such that \(\rho(\{A, B\}) = 0 < \hat{\rho}(\{A, B\})\).

The theory of the generalized and the joint spectral radius has many important applications for instance to discrete and differential inclusions, wavelets, invariant subspace theory (see e.g. [6], [9], [32], [30], [31] and the references cited there). In particular, \(\hat{\rho}(\Sigma)\) plays a central role in determining stability in convergence properties of discrete and differential inclusions. In this theory the quantity \(\log \rho(\Sigma)\) is known as the maximal Lyapunov exponent (see e.g. [32]).

We will use the following well known facts that
\[
\rho(\Sigma^m) = \rho(\Sigma)^m, \quad \hat{\rho}(\Sigma^m) = \hat{\rho}(\Sigma)^m, \quad \rho(\Psi \Sigma) = \rho(\Sigma \Psi) \quad \text{and} \quad \hat{\rho}(\Psi \Sigma) = \hat{\rho}(\Sigma \Psi),
\]
where \(\Psi \Sigma = \{AB : A \in \Psi, B \in \Sigma\}\) and \(m \in \mathbb{N}\).
3. Results

First we provide a new proof of the inequality (2.4) (based on its special case (2.6)) by applying the method of proof of the inequality (1.3) from [13, Theorem 3.1].

**Theorem 3.1.** Let \( \{A_{ij}\}_{i,j=1}^{k,m} \) be positive kernel operators on a Banach function space \( L \) and let \( \alpha_1, \alpha_2, \ldots, \alpha_m \) be positive numbers such that \( \sum_{j=1}^{m} \alpha_j = 1 \). Then the inequality (2.4) holds.

**Proof.** If \( A_1, \ldots, A_k \) are positive kernel operators on \( L \), then the block matrix

\[
T = T(A_1, A_2, \ldots, A_k) := \begin{bmatrix}
0 & A_1 & 0 & \cdots & 0 & 0 \\
0 & 0 & A_2 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & A_{k-1} \\
A_k & 0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}
\]

defines a positive kernel operator on the cartesian product of \( k \) copies of \( L \). Since \( T^k \) has a block diagonal form

\[
T^k = \text{diag} \left( A_1A_2 \cdots A_k, A_2A_3 \cdots A_kA_1, A_3A_4 \cdots A_kA_1A_2, \ldots, A_kA_1A_2 \cdots A_{k-1} \right),
\]

we have \( \rho(T)^k = \rho(T^k) = \rho(A_1A_2 \cdots A_k) \).

Now define \( T_i := T(A_{1i}, A_{2i}, \ldots, A_{ki}) \) for \( i = 1, 2, \ldots, m \). Then \( \rho(T_i) = \rho(A_{1i}A_{2i} \cdots A_{ki})^{1/k} \) for each \( i = 1, 2, \ldots, m \). Using the inequality (2.6) we obtain that

\[
\rho \left( T_{1}^{(\alpha_1)} \circ T_{2}^{(\alpha_2)} \circ \cdots \circ T_{m}^{(\alpha_m)} \right) \leq \rho(T_1)^{\alpha_1} \rho(T_2)^{\alpha_2} \cdots \rho(T_m)^{\alpha_m},
\]

which proves the inequality (2.4). \( \square \)

**Remark 3.2.** In the case of non-negative matrices that define operators on a Banach sequence space (see e.g. [26], [13], [11], [25] for exact definitions), the same proof works for positive numbers \( \alpha_1, \alpha_2, \ldots, \alpha_m \) such that \( \sum_{j=1}^{m} \alpha_j \geq 1 \).
Let $\Psi_1, \ldots, \Psi_m$ be bounded sets of positive kernel operators on a Banach function space $L$ and let $\alpha_1, \ldots, \alpha_m$ be positive numbers such that $\sum_{i=1}^m \alpha_i = 1$. Then the bounded set of positive kernel operators on $L$, defined by

$$\Psi_{\alpha_1} \circ \cdots \circ \Psi_{\alpha_m} = \{ A_{\alpha_1}^{\alpha_1} \circ \cdots \circ A_{\alpha_m}^{\alpha_m} : A_1 \in \Psi_1, \ldots, A_m \in \Psi_m \},$$

is called the weighted Hadamard (Schur) geometric mean of sets $\Psi_1, \ldots, \Psi_m$. The set $\Psi_{\alpha_1}^{1/\alpha_1} \circ \cdots \circ \Psi_{\alpha_m}^{1/\alpha_m}$ is called the Hadamard (Schur) geometric mean of sets $\Psi_1, \ldots, \Psi_m$.

A version of the following result on the generalized and the joint spectral radius was stated in [26, Theorem 3.4] and [25, Corollary 5.3] only in the case of bounded sets of non-negative matrices that define operators on Banach sequence spaces, however the same proof works in our more general setting by applying the inequalities (2.4) and (2.3). The proof is included for the convenience of the reader.

**Theorem 3.3.** Let $\Psi_1, \ldots, \Psi_m$ be bounded sets of positive kernel operators on a Banach function space $L$ and let $\alpha_1, \ldots, \alpha_m$ be positive numbers such that $\sum_{i=1}^m \alpha_i = 1$. Then we have

$$\rho(\Psi_{\alpha_1}^{(\alpha_1)} \circ \cdots \circ \Psi_{\alpha_m}^{(\alpha_m)}) \leq \rho(\Psi_1^{\alpha_1} \cdots \rho(\Psi_m^{\alpha_m}), \tag{3.1}$$

and

$$\hat{\rho}(\Psi_{\alpha_1}^{(\alpha_1)} \circ \cdots \circ \Psi_{\alpha_m}^{(\alpha_m)}) \leq \hat{\rho}(\Psi_1^{\alpha_1} \cdots \hat{\rho}(\Psi_m^{\alpha_m}). \tag{3.2}$$

**Proof.** Let $A \in (\Psi_{\alpha_1}^{(\alpha_1)} \circ \cdots \circ \Psi_{\alpha_m}^{(\alpha_m)})^l, l \in \mathbb{N}$. Then there are $A_{ik} \in \Psi_k, i = 1, \ldots, l, k = 1, \ldots, m$ such that

$$A = (A_{i1}^{\alpha_1} \circ \cdots \circ A_{im}^{\alpha_m}) \cdots (A_{l1}^{\alpha_1} \circ \cdots \circ A_{lm}^{\alpha_m}).$$

By Theorem 2.1 we have

$$\rho(A) \leq \rho(A_{1l1}^{\alpha_1} \cdots A_{lm}^{\alpha_m}). \tag{3.3}$$

Since $A_{ik} \in \Psi_k$ for all $k = 1, \ldots, m$, (3.3) implies (3.1).

By replacing $\rho(\cdot)$ with $\| \cdot \|$ in the proof above, we obtain the inequality (3.2), which completes the proof. \hfill \Box

Now we prove our main result, which is a generalization of the inequality (1.3). It can also be considered as a kernel version of [26, Theorem 3.16], which holds for bounded sets of non-negative matrices that define operators on Banach sequence spaces.

**Theorem 3.4.** Let $\Psi_1, \ldots, \Psi_m$ be bounded sets of positive kernel operators on a Banach function space $L$. Then we have

$$\rho\left(\Psi_{\alpha_1}^{1/\alpha_1} \circ \cdots \circ \Psi_{\alpha_m}^{1/\alpha_m}\right) \leq \rho(\Psi_1 \cdots \Psi_m)^{1/m}. \tag{3.4}$$

and

$$\hat{\rho}\left(\Psi_{\alpha_1}^{1/\alpha_1} \circ \cdots \circ \Psi_{\alpha_m}^{1/\alpha_m}\right) \leq \hat{\rho}(\Psi_1 \cdots \Psi_m)^{1/m}. \tag{3.5}$$
Proof. To prove (3.4) we will show that
\[ \rho \left( \Psi_1^{1/m} \circ \cdots \circ \Psi_m^{1/m} \right)^m \leq \rho (\Psi_1 \Psi_2 \cdots \Psi_m). \]

Take \( A \in \left( \Psi_1^{1/m} \circ \cdots \circ \Psi_m^{1/m} \right)^{mk} \). Then \( A = A_1 A_2 \cdots A_k \), where
\[ A_i = \left( A_{i11}^{1/m} \circ A_{i12}^{1/m} \circ \cdots \circ A_{i1m}^{1/m} \right) \left( A_{i21}^{1/m} \circ A_{i22}^{1/m} \circ \cdots \circ A_{i2m}^{1/m} \right) \cdots \]
\[ \cdots \left( A_{im1}^{1/m} \circ A_{im2}^{1/m} \circ \cdots \circ A_{imm}^{1/m} \right) \]
for some \( A_{ij1} \in \Psi_1, \ldots, A_{ijm} \in \Psi_m \) and all \( j = 1, \ldots, m, i = 1, \ldots, k \). Then
\[ A_i = \left( A_{i11}^{1/m} \circ A_{i12}^{1/m} \circ \cdots \circ A_{i1m}^{1/m} \right) \left( A_{i22}^{1/m} \circ \cdots \circ A_{i2m} \circ A_{i21} \right) \cdots \]
\[ \cdots \left( A_{i1m} \circ A_{i1} \circ \cdots \circ A_{i1m_1} \right). \]

By (2.2) we have
\[ A = A_1 A_2 \cdots A_k \leq B_1^{1/m} \circ B_2^{1/m} \circ \cdots \circ B_m^{1/m}, \]
where
\[ B_1 = \prod_{i=1}^k A_{i11} A_{i12} \cdots A_{i1m} \in (\Psi_1 \Psi_2 \cdots \Psi_m)^k, \]
\[ B_2 = \prod_{i=1}^k A_{i12} A_{i23} \cdots A_{i2m} \in (\Psi_2 \Psi_3 \cdots \Psi_1)^k, \]
\[ \cdots \]
\[ B_m = \prod_{i=1}^k A_{i1m} A_{i21} \cdots A_{imm} \in (\Psi_m \Psi_1 \cdots \Psi_{m-1})^k. \]

By Theorem 2.2 we have
\[ \rho (A) \leq \rho (B_1)^{1/m} \rho (B_2)^{1/m} \cdots \rho (B_m)^{1/m}, \]
which implies
\[ \rho \left( \Psi_1^{1/m} \circ \cdots \circ \Psi_m^{1/m} \right)^m \leq (\rho (\Psi_1 \Psi_2 \cdots \Psi_m) \rho (\Psi_2 \Psi_3 \cdots \Psi_1) \cdots \rho (\Psi_m \Psi_1 \cdots \Psi_{m-1}))^{1/m} \]
\[ = \rho (\Psi_1 \Psi_2 \cdots \Psi_m). \]
This proves (3.4) and the inequality (3.5) is proved similarly. \( \square \)

The following special case of Theorem 3.4 generalizes (1.2).
Corollary 3.5. Let $\Psi$ and $\Sigma$ be bounded sets of positive kernel operators on a Banach function space $L$. Then we have
\[
\rho \left( \Psi^{\left(\frac{1}{2}\right)} \circ \Sigma^{\left(\frac{1}{2}\right)} \right) \leq \rho(\Psi \Sigma)^{\frac{1}{2}}
\] (3.6)
and
\[
\hat{\rho} \left( \Psi^{\left(\frac{1}{2}\right)} \circ \Sigma^{\left(\frac{1}{2}\right)} \right) \leq \hat{\rho}(\Psi \Sigma)^{\frac{1}{2}}
\] (3.7)

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