A Moment of Perfect Clarity I: 
The Parallel Census Technique*

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Abstract

We discuss the history and uses of the parallel census technique—an elegant tool in the study of certain computational objects having polynomially bounded census functions. A sequel [GH] will discuss advances (including [CNS95] and Glaßer [Gla00]), some related to the parallel census technique and some due to other approaches, in the complexity-class collapses that follow if NP has sparse hard sets under reductions weaker than (full) truth-table reductions.

1 Introduction

Have you ever used a pair of binoculars? Then you know the process one goes through to initially set the distance between the two eyepieces—sometimes the view may black out, yet if one goes too far the other way one has two circles of view that don’t coincide. However, there is a point where things are just right: All is crisply aligned and one can enjoy the view of that pileated woodpecker, at least if it has been so polite as to wait while one was playing with the interocular adjustment.

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The parallel census technique is very much like this: too far one way and our view blacks out, too far the other way and we get chaos from overlapping views, but at a certain magic point everything comes into focus.

2 The Parallel Census Technique

So, what is the parallel census technique? Loosely put—and since we are speaking of a flavor of approach loosely put is appropriate here—the parallel census technique refers to an approach that can be used when faced with a deterministically or nondeterministically recognizable type of objects of which one knows that one has at most polynomially many but one does not know exactly how many one has. The parallel census approach is to, in parallel, for each possible guess of the cardinality to ask, also in parallel, a bunch of questions to a nondeterministic set (machine) that will guess and check a number of objects matching the cardinality you guessed and that will test some bit of information about one of the objects. Our set of questions will be such that the questions corresponding to the correct cardinality will—when all their answers are viewed together—reveal everything about the objects: the overall cardinality and even the name of each object.

Put more succinctly, the parallel census technique is a way of gaining information via parallelized queries to nondeterministic classes. For example the following known result can be crisply seen via the parallel census technique.

Definition 2.1 A polynomial-time truth-table reduction (see [LLS75] or [GH]) is said to be exponentially length-decreasing if there is a constant $k$ such that, on all inputs $x$, each of the (at most polynomial number of) parallel queries generated on input $x$ is of length at most $k \log |x|$.

Theorem 2.2 (can be seen via the tools of [Har83, HIS85], see the discussion in Section 3) Every sparse NP set reduces via a parallel (i.e., truth-table), exponentially length-decreasing reduction to an NE set (NE = $\bigcup_k \text{NTIME}[2^{kn}]$).

Definition 2.3 [VV86] Let $Q(\cdot)$ be a one-argument boolean predicate. Let SAT be, as usual, the set of satisfiable boolean formulas.

$$\text{USAT}_Q(x) = \begin{cases} 
\text{SAT}(x) & \text{if the number of satisfying assignments of } x \text{ is } 0 \text{ or } 1 \\
Q(x) & \text{otherwise}
\end{cases}$$

Definition 2.4
a) \[\text{Val79}\] A function \(f\) is in \(#P\) iff there is some NPTM, \(N\), such that for each \(x\) it holds that \(N(x)\) (i.e., the computation of \(N\) on input \(x\)) has exactly \(f(x)\) accepting paths.

b) \[\text{Val76}\] A set \(B\) is in \(\text{UP}\) iff there is a \(#P\) function \(f\) satisfying \((\forall x)[f(x) \leq 1]\) such that \(B = \{x \mid f(x) > 0\}\).

c) \[\text{AR88}\] A set \(B\) is in \(\text{FewP}\) iff there is a \(#P\) function \(f\) satisfying \((\exists \text{ polynomial } q)(\forall x)[f(x) \leq q(|x|)]\) such that \(B = \{x \mid f(x) > 0\}\).

d) \[\text{CH90}\] A set \(B\) is in \(\text{Few}\) iff there is a polynomial-time predicate \(R(\cdot, \cdot)\) and a \(#P\) function \(f\) satisfying \((\exists \text{ polynomial } q)(\forall x)[f(x) \leq q(|x|)]\) such that \(B = \{x \mid R(x, f(x))\}\).

**Fact 2.5** \(P \subseteq \text{UP} \subseteq \text{FewP} \subseteq \text{Few}\).

**Theorem 2.6** (see Section 3 for a discussion of attribution) If \(L \in \text{Few}\) then \((\forall Q)[L \leq^p_{\text{tt}} \text{USAT}_Q]\), where \(\leq^p_{\text{tt}}\) denotes truth-table (i.e., parallel) reductions.

We defer a discussion of the history and attribution of Theorems 2.2 and 2.6 until after the proof of Theorem 2.6. The proof of Theorem 2.6 will provide a quintessential example of the parallel census technique.

**Proof (Theorem 2.6):** Let \(L\) be an arbitrary set from \(\text{Few}\). Let \(R(\cdot, \cdot), f,\) and \(q(\cdot)\) be as in Definition 2.4(d). With respect to \(#P\) function \(f\), let \(N\) be as in Definition 2.4(a), and without loss of generality assume that for each \(x\) all computation paths of \(N(x)\) are of length exactly \(p(|x|)\), where \(p\) is a polynomial and \((\forall w)[p(w) \geq 1]\).

As we discuss at the end of the proof, the proof will actually establish even the claim that the complete list of accepting paths of \(N(x)\) is computable in \(\text{FP}^\text{USAT}_Q\).

Consider the set

\[
J = \{\langle x, c, k, b \rangle \mid c \leq q(|x|) \land 1 \leq k \leq p(|x|) \land 1 \leq j \leq c \land b \in \{0, 1\} \land (\exists p_1 < p_2 < \cdots < p_c)[(\text{the } k^{\text{th}} \text{ bit of } p_j \text{ is } b) \land (\forall i : 1 \leq i \leq c)[p_i \text{ is an accepting path of } N(x)]\}.
\]

Let \(N'\) be the obvious, natural, NP machine for \(J\). Note that this machine will have the property that on each input of the form \(\langle x, f(x), \cdot, \cdot, \cdot \rangle\), machine \(N'\) will have either zero or one accepting path, as there will be just one valid guess for
Let \( \sigma_{N'} \) be a polynomial-time, parsimonious many-one reduction (Cook’s reduction can be implemented parsimoniously \cite{Gal74}) from questions about membership in \( J \) to boolean formulas. That is, for each string \( v \), the formula \( \sigma_{N'}(v) \) will have exactly as many satisfying assignments as \( N'(v) \) had accepting computation paths.

Let \( Q \) be any predicate. The reduction to implement \( L \leq_{tt}^p \text{USAT}_Q \) is as follows. On input \( x \), ask \text{USAT}_Q, \text{in parallel, all questions of the form}

\[
\sigma_{N'}(x, c, j, k, b)
\]

for all \( c \leq q(|x|), 1 \leq j \leq c, 1 \leq k \leq p(|x|), b \in \{0, 1\} \).

Note that the answers here are a bit magical. In particular, consider all answers associated (on input \( x \)) with a single value of \( c \), say, \( c' \). If \( c' > f(x) \), that is, if \( c' \) is greater than the number of accepting paths of \( N(x) \), then all the questions \( "\sigma_{N'}((x, c', j, k, b)) \in \text{USAT}_Q?" \) will get the answer “no” (since in this case \( N'(x, c', j, k, b) \) will have zero accepting paths, \( \sigma_{N'}((x, c', j, k, b)) \) will be unsatisfiable, so \text{USAT}_Q cannot contain it). Of course, if \( c' < f(x) \), that is, if \( c' \) is less than the number of accepting paths of \( N(x) \), then the answers to the questions \( "\sigma_{N'}((x, c', j, k, b)) \in \text{USAT}_Q?" \) will yield a big muddle of information, since the too-low guessed cardinality \( c' \) will allow multiple valid guesses of \( (p_1, p_2, \ldots, p_c) \) and so bits will be overlayed in ways that may potentially hide information. However, and this is the beautiful core of the parallel census technique, when \( c' = f(x) \), that is, when \( c' \) equals the number of accepting paths of \( N(x) \), then if \( f(x) = 0 \) every question of the form \( "\sigma_{N'}((x, c', j, k, b)) \in \text{USAT}_Q?" \) will get the answer “no,” and if \( f(x) > 0 \) then at least one question of the form \( "\sigma_{N'}((x, c', j, k, b)) \in \text{USAT}_Q?" \) will get the answer “yes,” and furthermore the answer to all questions of the form \( "\sigma_{N'}((x, c', j, k, b)) \in \text{USAT}_Q?" \) will in effect specify the entire list of accepting paths! That is, when \( c' = f(x) \) everything comes into focus—a moment of perfect clarity.

So, after we get back the huge list of answers in parallel, if all the answers are “no” we know that \( N(x) \) has zero accepting paths, and we accept exactly if \( R(x, 0) \). If at least one answer is yes, we consider the largest \( \hat{c} \) for which some query \( "\sigma_{N'}((x, \hat{c}, j, k, b)) \in \text{USAT}_Q?" \) received the answer “yes.” So, \( \hat{c} = f(x) \). So, we accept exactly if \( R(x, \hat{c}) \). This already proves our stated theorem, but note that a bit more holds. From the whole collection of answers to the questions \( "\sigma_{N'}((x, c', j, k, b)) \in \text{USAT}_Q?" \) we can reconstruct all the accepting paths of \( N(x) \). So not only is it true that \( L \leq_{tt}^p \text{USAT}_Q \), but in fact it even holds that the complete list of accepting paths of \( N(x) \) is computable in \( \text{FP}_{tt}^{\text{USAT}_Q} \).
From the very general claim expressed as Theorem 2.6, one can immediately conclude many of the other ways that the benefits of the parallel census technique are used as they relate to Few. Most particularly, it is immediate from Theorem 2.6 and Fact 2.5 that the following corollary holds.

**Corollary 2.7** If $(\exists Q)[\text{USAT}_Q \in P]$ then $P = \text{Few}$ (and thus $P = \text{UP}$ and $P = \text{FewP}$).

## 3 Comments and Attributions

We come to the issue of attribution. This is a bit tricky, as where one sees this as originating depends in part on how flexible one is in defining “this.” However, in terms of pointing to where the parallel census technique came to be seen as a key approach to reaching conclusions about FewP, that is relatively clear: Selman ([Sel90] and its journal version [Sel94]; see especially Proposition 6 of the former, which is Theorem 2 of the latter) and Toda (the proof of Theorem 3.10 of [Tod91]).

The result $(\exists Q)[\text{USAT}_Q \in P] \Rightarrow P = \text{FewP}$ is explicitly stated by Buhrman, Fortnow, and Torenvliet [BFT97], and is there attributed as “this was essentially proven in Toda’s paper [Tod91].” Since it is now known that $P = \text{FewP} \iff P = \text{Few}$ [Sel94], in light of this equivalence one can conclude Corollary 2.7 from this.

Theorem 2.6 may never have been stated before in the strong form it appears here, but it just reflects an attempt to distill to its core what is going on in the parallel census technique of Selman and Toda.

We mention that the fundamental machinery needed to exploit the idea behind the parallel census technique appears in a 1983 paper of Hartmanis (Theorem 2.1 of [Har83], and it reappears in Hartmanis, Immerman, and Sewelson [HIS85] as Theorem 1; see also [Sew83]). We say that the “fundamental machinery” is there as in both [Har83] and [HIS85], the proofs build sequential algorithms. However, if one closely examines the proofs, it becomes clear that (since the number of census values is polynomial) one can redo the algorithm so that it works via one big round of parallel queries. If one were to do this, what one would arrive at would be what is stated earlier in this paper as Theorem 2.2.

The name “parallel census technique” was coined in Arvind et al. [AHH+93], which noted the relation of the technique to the work of Hartmanis, Immerman, and Sewelson. Arvind et al. [AHH+93] also somewhat improves on the number of queries from Selman [Sel90], and generalizes the technique beyond NP.

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1Upward separation (equivalently referred to sometimes as “downward collapse”), the theme of the work of Hartmanis [Har83] and Hartmanis, Immerman, and Sewelson [HIS85], has been further studied—qualified, extended, etc.—in such papers as [HY84, AW90, All91, RRR94, HJ95, BG98, HHH99, BF99, HHH98].
Finally, as it will be useful in Part II’s discussion of sparse sets hard for NP with respect to various reductions, we state the following famous result of Valiant and Vazirani \cite{VV86}. This result gives a quite different conclusion from the hypothesis \((\exists Q)[\text{USAT}_Q \in P]\) than does Corollary \ref{corollary:usat-p}.

**Theorem 3.1** \cite{VV86} If \((\exists Q)[\text{USAT}_Q \in P]\) then \(R = \text{NP}\).

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