UNIQUE CONTINUATION PROPERTIES FOR SOLUTIONS TO THE CAMASSA-HOLM EQUATION AND OTHER NON-LOCAL EQUATIONS.

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Abstract. It is shown that if $u(x, t)$ is a solution of the initial value problem for the Camassa-Holm equation which vanishes in an open set $\Omega \subset \mathbb{R} \times [0, T]$, then $u(x, t) = 0$, $(x, t) \in \mathbb{R} \times [0, T]$. This result also applies to solutions of the initial periodic boundary value problems associated to the Camassa-Holm equation. The argument of proof can be placed in a general setting to extend the above results to a class of non-linear non-local 1-dimensional models which includes the Degasperis-Procesi equation.

1. Introduction

In this work we shall study the Camassa-Holm (CH) equation

$$\partial_t u + 3u \partial_x u - \partial_t \partial_x^2 u = 2\partial_x u \partial_x^2 u + u \partial_x^3 u, \quad t, x \in \mathbb{R}. \quad (1.1)$$

The CH equation (1.1) was first noted by Fuchssteiner and Fokas [17] in their work on hereditary symmetries. Later, it was written explicitly and derived physically as a model for shallow water waves by Camassa and Holm [5], who also examined its solutions. It also appears as a model in nonlinear dispersive waves in hyperelastic rods [10].

The CH equation (1.1) has received extensive attention due to its remarkable properties, among them the fact that it is a bi-Hamiltonian completely integrable model (see [1], [5], [8], [28], [29], [30] and references therein).

The CH equation possesses “peakon” solutions [5]. In the case of a single peakon this solitary wave solution can be written as

$$u_c(x, t) = ce^{-|x-ct|}, \quad c > 0. \quad (1.2)$$

The multi-peakon solutions exhibit the “elastic” collision property that reflect their soliton character (see [2]).

1991 Mathematics Subject Classification. Primary: 35Q51 Secondary: 37K10.
Key words and phrases. Camassa-Holm equation, unique continuation.

The first author was partially supported by CNPq and FAPERJ/Brazil.
It is convenient to write the CH equation (1.1) in the following (formally) equivalent form

$$\partial_t u + u \partial_x u + \partial_x (1 - \partial_x^2)^{-1} (u^2 + \frac{1}{2}(\partial_x u)^2) = 0, \ t, \ x \in \mathbb{R}. \quad (1.3)$$

The initial value problem (IVP) as well as the initial periodic boundary value problem (IPBVP) associated to the equation (1.3) has been extensively examined. In particular, in [25] and [31] strong local well-posedness (LWP) of the IVP was established in the Sobolev space

$$H^s(\mathbb{R}) = (1 - \partial_x^2)^{-s/2}L^2(\mathbb{R}), \ s > 3/2.$$ 

The peakon solutions do not belong to these spaces. In fact, 

$$\phi(x) = e^{-|x|} \notin W^{p,1+1/p}(\mathbb{R}) \quad \text{for any} \quad p \in [1, \infty), \quad (1.4)$$

where $$W^{s,p}(\mathbb{R}) = (1 - \partial_x^2)^{-s/2}L^p(\mathbb{R})$$ with $$W^{s,2}(\mathbb{R}) = H^s(\mathbb{R}).$$

However,

$$\phi(x) = e^{-|x|} \in W^{1,\infty}(\mathbb{R}),$$

where $$W^{1,\infty}(\mathbb{R})$$ denotes the space of Lipschitz functions.

In [6] it was proved that if $$u_0 \in H^1(\mathbb{R})$$ with $$u_0 - \partial_x^2 u_0 \in M^+(\mathbb{R}),$$ where $$M^+(\mathbb{R})$$ denotes the set of positive Radon measures with bounded total variation, then the IVP for the CH equation (1.3) has a global weak solution $$u \in L^\infty((0, \infty) : H^1(\mathbb{R})).$$

An improvement of the previous result was obtained in [9] by showing that if $$u_0 \in H^1(\mathbb{R})$$ with $$u_0 - \partial_x^2 u_0 \in M^+(\mathbb{R}),$$ then the IVP for the CH equation (1.3) has a unique solution

$$u \in C([0, \infty) : H^1(\mathbb{R})) \cap C^1((0, \infty) : L^2(\mathbb{R}))$$

satisfying that $$y(t) \equiv u(\cdot, t) - \partial_x^2 u(\cdot, t) \in M^+(\mathbb{R})$$ is uniformly bounded in $$[0, \infty).$$

In [33] the existence of a $$H^1$$-global weak solution for the IVP for the CH equation (1.3) for data $$u_0 \in H^1(\mathbb{R})$$ was established.

In [6] and [7] (see also [25]) there were deduced conditions on the data $$u_0 \in H^3(\mathbb{R})$$ such that the corresponding local solution $$u \in C([0, T) : H^3(\mathbb{R}))$$ of the IVP associated to the CH (1.3) blows up in finite time by showing that

$$\lim_{t \uparrow T} \|\partial_x u(\cdot, t)\|_\infty = \infty,$$

corresponding to the breaking of waves. Observe that $$H^1$$-solutions of the CH equation (1.1) satisfy the conservation law

$$E(u)(t) = \int_{-\infty}^{\infty} (u^2 + (\partial_x u)^2)(x, t)dx = E(u_0),$$
so that the $H^1$-norm of the solutions remains invariant within the existence interval.

More recently, in [3] and [4] the existence and uniqueness, respectively, of a $H^1$ global solution for the CH equation (1.3) was settled.

For other well-posedness results see also [18], [19] and references therein.

We shall describe the class of solutions which we will be working with. First, we consider the IVP and recall a result found in [27] motivated by an early work [16] for the IPBVP:

**Theorem 1.1.** Given $u_0 \in X \equiv H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, there exist a non-increasing function $T = T(\|u_0\|_X) > 0$ and a unique solution $u = u(x,t)$ of the IVP associated to the CH equation (1.3) such that

\[ u \in Z_T \equiv C([-T,T] : H^1(\mathbb{R})) \cap L^\infty([-T,T] : W^{1,\infty}(\mathbb{R})) \cap C^1((-T,T) : L^2(\mathbb{R})), \tag{1.5} \]

with

\[ \sup_{[-T,T]} \|u(\cdot,t)\|_X = \sup_{[-T,T]} (\|u(\cdot,t)\|_{1,2} + \|u(\cdot,t)\|_{1,\infty}) \leq c\|u_0\|_X, \]

for some universal constant $c > 0$. Moreover, given $R > 0$, the map $u_0 \mapsto u$, taking the data to the solution, is continuous from the ball \{ $u_0 \in X : \|u_0\|_X \leq R$ \} into $Z_{T(R)}$.

**Remark 1.2.** The strong notion of LWP introduced in [23] does not hold in this case. This notion includes existence, uniqueness, persistence property, namely that if $u_0 \in Y$, then $u \in C([0,T] : Y)$, and that the map data $\mapsto$ solution is locally continuous from $Y$ to $C([0,T] : Y)$. In particular, this strong version of LWP guarantees that the solution flow defines a dynamical system in $Y$.

As it was mentioned before it holds in $H^s(\mathbb{R})$ with $s > 3/2$, where the peakon solutions are not included. In Theorem 1.1 by assuming that $u_0 \in X = H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, it is proved that the solution flow defines a dynamical system in $H^1(\mathbb{R})$, with the peakons belonging to this class.

In this work we are interested in unique continuation properties of solutions of the CH equation. Thus, we recall two theorems deduced in [20] concerning unique continuation and decay persistence properties of solutions of the IVP for the CH equation:

**Theorem 1.3 (20).** Assume that for some $T > 0$ and $s > 3/2$, $u \in C([0,T] : H^s(\mathbb{R})) \cap C^1((0,T) : H^{s-1}(\mathbb{R}))$.
is a strong solution of the IVP associated to the CH equation (1.3). If for some \( \alpha \in (1/2, 1) \), \( u_0(x) = u(x, 0) \) satisfies
\[
|u_0(x)| = o(e^{-x}) \quad \text{and} \quad |\partial_x u_0(x)| = O(e^{-\alpha x}), \quad \text{as} \quad x \uparrow \infty, \tag{1.6}
\]
and there exists \( t_1 \in (0, T] \) such that
\[
|u(x, t_1)| = o(e^{-x}), \quad \text{as} \quad x \uparrow \infty,
\]
then \( u \equiv 0 \).

Roughly, Theorem 1.3 is optimal:

**Theorem 1.4** ([20]). Assume that for some \( T > 0 \) and \( s > 3/2 \),
\[
u \in C([0, T] : H^s(\mathbb{R})) \cap C^1((0, T) : H^{s-1}(\mathbb{R}))
\]
is a strong solution of the IVP associated to the CH equation (1.3). If for some \( \theta \in (0, 1) \), \( u_0(x) = u(x, 0) \) satisfies
\[
|u_0(x)|, \quad |\partial_x u_0(x)| = O(e^{-\theta x}), \quad \text{as} \quad x \uparrow \infty,
\]
then
\[
|u(x, t)|, \quad |\partial_x u(x, t)| = O(e^{-\theta x}), \quad \text{as} \quad x \uparrow \infty,
\]
uniformly in the time interval \([0, T]\).

**Remark 1.5.** In [27] Theorem 1.3 and Theorem 1.4 were extended to the class considered in Theorem 1.1.

Our first result in this work is:

**Theorem 1.6.** Let \( u = u(x, t) \) be a solution of the IVP associated to the CH equation (1.3) in the class described in Theorem 1.1. If there exists an open set \( \Omega \subset \mathbb{R} \times [0, T] \) such that
\[
u(x, t) = 0, \quad (x, t) \in \Omega, \tag{1.7}
\]
then \( u \equiv 0 \).

**Remark 1.7.** (i) A stronger version of this result has been established for the Korteweg-de Vries (KdV) equation, 1-dimensional non-linear Schrödinger (NLS) equation and the Benjamin-Ono (BO) equation in [32], [21] and [24] respectively. More precisely, it was proven there that if \( u_1, u_2 \) are two solutions of these equations which agree in an open set \( \Omega \subset \mathbb{R} \times [0, T] \), then they are identical.

Our approach is simpler than the ones in [32], [21] and [24] but it does not allow to obtain the above mention result. It applies only to a single solution of the CH equation since it depends on the whole structure of the equation.

Roughly speaking, this is related to the unique continuation property known for these equations under assumptions of decay at infinity at two
different times. For the KdV and 1-dimensional NLS equations results are known for the difference of two solutions $u_1, u_2$, see [12], [13] and references therein. However, the corresponding results for the BO and the CH equations require that $u_2(x, t) \equiv 0$, see [15], [26] and references therein for the BO equation and [20] for the CH equation.

In the periodic case, analogous LWP results as those in Theorem 1.1 were previously obtained in [16]. More precisely:

**Theorem 1.8.** Given $u_0 \in X \equiv H^1(S) \cap W^{1,\infty}(S)$, there exist a non-increasing function $T = T(\|u_0\|_X) > 0$ and a unique solution $u = u(x, t)$ of the IPBVP associated to the CH equation (1.3) such that

$$u \in Y_T \equiv C([-T, T] : H^1(S)) \cap L^\infty([-T, T] : W^{1,\infty}(S)) \cap C^1((-T, T) : L^2(S)),$$

with

$$\sup_{[-T, T]} \|u(\cdot, t)\|_X = \sup_{[-T, T]} (\|u(\cdot, t)\|_{1,2} + \|u(\cdot, t)\|_{1,\infty}) \leq c \|u_0\|_X,$$

for some universal constant $c > 0$. Moreover, given $R > 0$, the map $u_0 \mapsto u$, taking the data to the solution, is continuous from the ball \{ $u_0 \in X : \|u_0\|_X \leq R$ \} into $Y_T(R)$.

Our next theorem extends the results in Theorem 1.6 for the IVP to the IPBVP for the CH equation (1.3):

**Theorem 1.9.** Let $u = u(x, t)$ be a solution of the IPBVP associated to the CH equation (1.3) in the class described in Theorem 1.8. If there exists an open set $\Omega \subset S \times [0, T]$ such that

$$u(x, t) = 0, \quad (x, t) \in \Omega,$$

then $u \equiv 0$.

**Remark 1.10.** In the proof of Theorem 1.6 and Theorem 1.9 the only condition on the structure of integrand term in (1.3)

$$(u^2 + \frac{1}{2} (\partial_x u)^2)(x, t)$$

needed will be that it is non-negative. Hence, the same proof provides a similar result for any equation of the form

$$\partial_t u + u^k \partial_x u + g(u, \partial_x u) + \partial_x (1 - \partial_x^2)^{-1} h(u, \partial_x u) = 0, \quad k \in \mathbb{Z}^+, \quad (1.10)$$

with

$$g(\cdot, \cdot), \quad h(\cdot, \cdot) \quad \text{smooth} \quad g(0, 0) = h(0, 0) = 0, \quad (1.11)$$

and

$$h(x, y) > 0, \quad \forall (x, y) \neq (0, 0). \quad (1.12)$$
If \( h(\cdot, \cdot) = h(\cdot) \) one requires that \( h(x) > 0 \) whenever \( x \neq 0 \).

More precisely,

**Theorem 1.11.** (i) The result in Theorem 1.6 applies to any solution \( u(\cdot, \cdot) \) in the class (1.5) of the IVP associated to the equation (1.10) satisfying the hypotheses (1.11) and (1.12).

(ii) The result in Theorem 1.9 applies to any solution \( u(\cdot, \cdot) \) in the class (1.8) of the IPBVP associated to the equation (1.10) satisfying the hypotheses (1.11) and (1.12).

**Remark 1.12.** It is easy to see that the LWP in Theorem 1.1 and Theorem 1.8 for the IVP and the IPBVP resp. for the CH (1.3) extend to the equation in (1.10).

**Remark 1.13.** The class of equations described in (1.10)-(1.12) includes the Degasperis-Procesi (DP) equation (11)

\[
\partial_t u + 4u\partial_x u - \partial_t \partial_x^2 u = 3\partial_x u \partial_x^2 u + u \partial_x^3 u, \quad t, x \in \mathbb{R}, \tag{1.13}
\]

which can be written as

\[
\partial_t u + u \partial_x u + \frac{3}{2} \partial_x (1 - \partial_x^2)^{-1}(u^2) = 0,
\]

and also includes the so called \( b \)-equations (14)

\[
\partial_t u + (b + 1)u\partial_x u - \partial_t \partial_x^2 u = b\partial_x u \partial_x^2 u + u \partial_x^3 u
\]

which can be written as

\[
\partial_t u + u \partial_x u + \partial_x (1 - \partial_x^2)^{-1}\left( \frac{b}{2} u^2 + \frac{3 - b}{2} (\partial_x u)^2 \right) = 0, \quad b \in [0, 3]. \tag{1.14}
\]

Notice that for \( b = 2 \) in (1.14) one gets the CH equation meanwhile for \( b = 3 \) in (1.14) one obtains the DP equation, the only bi-hamiltonian and integrable models in this family, see [22].

The rest of this work is organized as follows: Section 2 contains the proofs of Theorem 1.6 and Theorem 1.9. It is also shown how the argument in the proofs of Theorem 1.6 and Theorem 1.9 can be extended to prove Theorem 1.11.

### 2. Proof of Theorem 1.6 and Theorem 1.9

First we shall prove Theorem 1.6.

**Proof of Theorem 1.6.** We recall that

\[
(1 - \partial_x^2)^{-1} h(x) = \frac{1}{2} \left( e^{-|\cdot|} * h \right)(x), \quad h \in L^2(\mathbb{R}). \tag{2.1}
\]
From the hypothesis it follows that
\[
\left(u^2 + \frac{\partial_x u^2}{2}\right)|_\Omega \equiv 0, \tag{2.2}
\]
and from the equation (1.3) one gets
\[
\partial_x \left((1 - \partial_x^2)^{-1} \left(u^2 + \frac{\partial_x u^2}{2}\right)\right)|_\Omega \equiv 0. \tag{2.3}
\]
Thus, \(\exists t^* \in (0, T)\) and \(I = [a, b], \ a < b, \ [a, b] \times \{t^*\} \subset \Omega\) such that defining
\[
F(x) := \partial_x \left((1 - \partial_x^2)^{-1} \left(u^2 + \frac{\partial_x u^2}{2}\right)\right)(x, t^*) \tag{2.4}
\]
and
\[
f(x) := (u^2 + \frac{(\partial_x u)^2}{2})(x, t^*) \tag{2.5}
\]
one has that
\[
F \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C(\mathbb{R}), \quad f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \tag{2.6}
\]
with
\[
F(x) = f(x) = 0, \quad x \in [a, b]. \tag{2.7}
\]
We observe that for any \(y \notin [a, b]\)
\[
- \sgn(b - y) e^{-|b-y|} > - \sgn(a - y) e^{-|a-y|}. \tag{2.8}
\]
Hence,
\[
F(b) = -\frac{1}{2} \int_{-\infty}^{\infty} \sgn(b - y) e^{-|b-y|} f(y) dy \geq -\frac{1}{2} \int_{-\infty}^{\infty} \sgn(a - y) e^{-|a-y|} f(y) dy = F(a), \tag{2.9}
\]
with \(f \geq 0\) and
\[
F(b) = F(a) \quad \text{if and only if} \quad f \equiv 0. \tag{2.10}
\]
Since, \(F(b) = F(a) = 0\), we obtain the desired result.

**Remark 2.1.** One can give a different proof by showing that \(F(\cdot)\) defined in (2.4) is differentiable in \((a, b)\), (see [27]), with
\[
F'(x) = (e^{-|\cdot|} * f)(x) - f(x), \quad x \in (a, b),
\]
(since \(\partial_x^2(1 - \partial_x^2)^{-1} = (1 - \partial_x^2)^{-1} - 1\)). Therefore, since \(f(x) = 0\) for any \(x \in [a, b]\) and \(f \geq 0\) on \(\mathbb{R}\) one has that
\[
F'(x) \geq 0, \quad x \in (a, b),
\]
with
\[ F'(x) = 0 \text{ if and only if } f \equiv 0. \]

Recalling (2.7), i.e. \( F(a) = F(b) = 0 \), one gets the result.

**Proof of Theorem 1.9.** The proof is similar to that given for the IVP in Theorem 1.6. The only difference is to show that the equivalent inequality in (2.8) is satisfied in \( S \simeq \mathbb{R}/\mathbb{Z} \simeq [0,1) \).

We recall that if \( h \in L^2(S) \), then
\[ \partial_x(1 - \partial_x^2)^{-1}h(x) = (\partial_x G \ast h)(x) \]
where
\[ G(x) = \frac{\cosh(x - \lfloor x \rfloor - 1/2)}{2 \sinh(1/2)}, \quad x \in \mathbb{R}, \quad (2.11) \]
and \( \lfloor \cdot \rfloor \) denotes the greatest integer function. Observe that \( G \) is differentiable in \( \mathbb{R} - \mathbb{Z} \).

Thus, here \( G(x) \) plays the role of the (Green) function \( e^{-|x|}/2 \) on the line (for \( (1 - \partial_x^2) \)).

Hence, to obtain the equivalent expression to (2.8) one has to show: if \( 0 < a < b < 1 \), then
\[ \partial_x G(b - y) > \partial_x G(a - y), \quad y \in [0,1] - [a,b]. \quad (2.12) \]

Since
\[ \partial_x G(x) = \frac{\sinh(x - \lfloor x \rfloor - 1/2)}{2 \sinh(1/2)}, \]
it suffices to see that if \( y \in [0,1] - [a,b] \), then
\[ \sinh \left( b - y - \lfloor b - y \rfloor - \frac{1}{2} \right) > \sinh \left( a - y - \lfloor a - y \rfloor - \frac{1}{2} \right). \]

By combining that:
\[ \begin{align*}
\text{if } & y \in [0,a], \text{ then } [b - y] = [a - y] = 0, \\
\text{if } & y \in [b,1], \text{ then } [b - y] = [a - y] = -1,
\end{align*} \quad (2.13) \]
and the fact that \( \sinh(\cdot) \) is strictly increasing the proof is concluded. \( \square \)

The proof of Theorem 1.11 will be omitted since the argument follows same lines as the proofs of Theorem 1.6 and Theorem 1.9 given in detail above.
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