Partial non-renormalisation of the stress-tensor four-point function in $N = 4$ SYM and AdS/CFT

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Abstract

We show that, although the correlator of four stress-tensor multiplets in $N = 4$ SYM is known to have radiative corrections, certain linear combinations of its components are protected from perturbative renormalisation and remain at their free-field values. This result is valid for weak as well as for strong coupling and for any gauge group. Our argument uses Intriligator’s insertion formula, and includes a proof that the possible contact term contributions cannot change the form of the amplitudes.

Combining this new non-renormalisation theorem with Maldacena’s conjecture allows us to make a prediction for the structure of the corresponding correlator in AdS supergravity. This is verified by first considerably simplifying the strong coupling expression obtained by recent supergravity calculations, and then showing that it does indeed exhibit the expected structure.
1 Introduction

Much effort has been devoted over the years to the study of the dynamical aspects of quantum field theory. Weak coupling expansions have been pushed to high orders providing useful insight, however, the strong coupling regime of most theories has remained elusive. Nevertheless, important information can be and has been obtained from the study of various quantities whose weak and strong coupling behaviour is accessible. Such quantities are axial \[1\] and conformal anomalies \[2, 3\], and also coupling constants that are not renormalised as one goes from weak to strong coupling \[4, 5, 6\]. The emergence of the latter quantities has been particularly noted in the context of the AdS/CFT correspondence conjectured by Maldacena \[7, 8, 9\]. One of the most exciting features of this conjecture is that strong coupling information for certain quantum field theories can be obtained from tree-level supergravity calculations. Therefore, with the explicit strong coupling results for various correlation functions in hand, one can identify “non-renormalised” quantities which remain the same both in the weak and the strong coupling regimes.

So far the strongest evidence in favour of this conjecture has been gathered in the context of \(N = 4\) SYM theory with gauge group \(SU(N_c)\) \[10, 11, 4, 12, 5, 13, 14\]. This is not surprising since the study of this theory has a long history and its dynamics is relatively well understood. Moreover, the success of this programme has prompted more thorough investigations of \(N = 4\) SYM within field theory, and in particular a search for non-renormalisation theorems for the correlators involving only “short” operators, i.e. certain class of gauge invariant composite operators which do not depend on all the Grassmann variables in superspace. The most typical example, first introduced in \[15\], are the series of operators obtained by tensoring the \(N = 4\) SYM field strength considered as a Grassmann analytic harmonic superfield. They were identified with short multiplets of \(SU(2,2/4)\) and their correspondence with the K-K spectrum of IIB supergravity was established in \[16\]. Short multiplets are important in the AdS/CFT correspondence because they have protected conformal dimensions and therefore allow a reliable comparison between quantities computed in the bulk versus quantities derived in the CFT \[17\].

Recently it has been found that, for correlators of short multiplets, the absence of radiative corrections is a rather common phenomenon; by now non-renormalisation theorems have been established not only for two- and three-point functions \[5, 18, 13, 19, 20, 21, 22, 23\], but also for so-called “extremal” \[24, 25, 26\] and “next-to-extremal” \[26, 27, 28\] correlators; these are \(n\)-point functions obeying certain conditions on the conformal dimensions of the operators involved.

At the same time, some correlators of short operators are known to acquire quantum corrections beyond tree level. The simplest example is the four-point function of \(N = 4\) SYM supercurrents, as has been shown by explicit computations at two loops in \[29, 30, 31\], and at three loops in \[32, 33\]. Nevertheless, even in this case one can still formulate a “partial non-renormalisation” theorem which is the main subject of this paper.

The work presented here is part of an ongoing investigation of this four-point function, which grew out of a line of work initiated by P.S. Howe and P.C. West. Their original aim was to study the implications of superconformal covariance for correlators satisfying Grassmann and harmonic analyticity constraints. In references \[15, 34\] a systematic investigation of the superconformal Ward identities and their consequences for Greens functions of \(N = 2\) and \(N = 4\) short operators
was initiated \(^4\). The \(N = 2\) operators are gauge-invariant products of the hypermultiplet and the \(N = 4\) operators are gauge-invariant products of the \(N = 4\) field strength. A number of interesting results were derived \([15, 34]\), in particular: The \(N = 4\) SYM field strength is a covariantly analytic scalar superfield from which the aforementioned set of analytic gauge-invariant operators can be built; the two- and three-point Greens functions of these operators were determined up to constants \([15, 18]\) (see also \([5, 15, 21]\)); the set of all non-nilpotent analytic superconformal invariants was given \([34]\). Finally, in \([40]\) it was shown how to derive certain differential constraints on the correlator of four stress-tensor multiplets using only the superconformal algebra and some general properties of the \(N = 2\) \([11]\) or \(N = 4\) \([12]\) harmonic superspace formulations of \(N = 4\) SYM. (These constraints are reproduced and their general solution is obtained in the Appendix to the present paper.)

However, we will see that considerably stronger restrictions can be derived for this amplitude if not only pure symmetry-based arguments are employed, but also some direct input from field theory making use of the explicit form of the \(N = 4\) SYM Lagrangian written down in terms of \(N = 2\) superfields. The essential tool in our analysis is Intriligator’s “insertion” (or “reduction”) formula, which relates the above four-point function to a five-point function involving the original operators and the gauge-invariant \(N = 2\) SYM Lagrangian. This procedure, introduced in the present context in \([13]\), has turned out both more efficient and more powerful than the direct approach to four-point functions: The lowest term of the five-point function obtained in this way in the context of \(N = 2\) harmonic superspace is a nilpotent superconformal covariant of mixed chiral-analytic type. Such objects are more strongly constrained than the original non-nilpotent amplitudes. The insertion approach has been constructively used in the harmonic superspace derivation or rederivation of some of the non-renormalisation theorems mentioned before \([21, 24, 28]\).

In the past the applicability of Intriligator’s insertion formula has been questioned on the grounds that possible contact term contributions might spoil any prediction based on it \([8, 13]\). While contact term contributions to correlators can usually be consistently ignored, it had been pointed out in \([8]\) that this is not obvious if the insertion procedure is used: It involves an integration which may promote a contact term to a regular term. Here we resolve this issue by finding necessary conditions for the existence of a five-point contact term with the required properties and by giving its most general allowed form. It then becomes clear that the regular term which it produces upon integration is compatible with our non-renormalisation statement.

The insertion formula has also proved very useful in explicit quantum calculations at two and three loops because it allows to apply superconformal covariance arguments to significantly reduce the complexity of Feynman diagram calculations of the correlator discussed here in \(N = 2\) harmonic superspace \([13, 22]\). The results obtained show a remarkable pattern: certain linear combinations of the amplitudes are protected from perturbative renormalisation, and thus remain at their free-field values. As we show in the present paper, this property can be generalised to every loop order and thus to a non-perturbative result, irrespectively of the choice of the gauge group. Combining this new non-renormalisation theorem with Maldacena’s conjecture we can make a prediction for the structure of the strong coupling limit obtained from AdS supergravity.

AdS supergravity calculations have been initiated in \([11, 1, 12, 13, 15, 14, 24]\) and most of the relevant methodology was developed in these articles. Yet, there the focus was on correlators whose CFT counterparts involve the top components \(F^2, \bar{F} \bar{F}\) of the \(N = 4\) stress-tensor multiplet, since these are more readily accessible from the supergravity side. On the contrary, on

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\(^4\)Similar works analysing constrained \(N = 1\) superfields are \([15, 26, 37, 38, 39]\).
the CFT side it is easiest to investigate the lowest components (the physical scalars) of the same multiplet. So, until recently the results on both sides were difficult to compare. With the completion of the computation of the quartic terms in the supergravity effective action, a strong coupling limit for the lowest component of the SYM stress-energy tensor four-point function became available [46]. However, it is presented in a form which still contains parameter integrals. To verify our non-renormalisation prediction, we first bring this result into a completely explicit form, involving only logarithms and dilogarithms of the conformal cross ratios. The “non-renormalised part” is then indeed found to agree with the free-field SYM amplitude.

The organisation of the paper is as follows: In Section 2, exploiting $SO(6)$ and conformal covariance and point-permutation symmetry we show how to reduce the lowest component of the correlator of four stress tensors in $N = 4$ SYM to that of a single $N = 2$ hypermultiplet correlator. Section 3 provides a minimum of information about the $N = 2$ harmonic superspace formulation [41] of $N = 4$ SYM. Section 4 is central, and contains the proof of our “partial non-renormalisation theorem” based on the insertion formula. This section also includes the necessary study of the possible contact terms. In Section 5 we verify that the AdS/CFT correspondence holds for the non-renormalised part of the correlator. In the Appendix we state the differential constraints on this four-point function found by the more abstract analysis of [40]. We give their general solution, from which it is evident that these constraints are weaker than the ones obtained in Section 4.

2 The $N = 4$ SYM four-point stress-tensor correlator

Here we show how one can compute the leading scalar term of the $N = 4$ four-point function of four SYM supercurrents (stress-tensor multiplets) from the leading scalar term of an $N = 2$ hypermultiplet four-point function. This section is based on [30] but we also derive some additional restrictions on the amplitude following from point-permutation symmetry.

We recall that in $N = 4$ superspace the $N = 4$ field strength superfield $W^A$, $A = 1, \ldots, 6$ transforms under the vector representation of the R symmetry group $SO(6) \sim SU(4)$. This superfield satisfies an on-shell constraint reducing it to six real scalars, four Majorana spinors and a vector.

The $N = 4$ supercurrent is given by

$$T^{AB} = W^A W^B - \frac{1}{6} \delta^{AB} W^C W^C$$

(2.1)

(the trace over the Yang-Mills indices is implied). It is in the symmetric traceless $20$ of $SO(6)$ and is conserved as a consequence of the on-shell constraints on $W^A$.

The four-point function we are going to consider is

$$G^{(N=4)} = \langle T^{A_1 B_1} T^{A_2 B_2} T^{A_3 B_3} T^{A_4 B_4} \rangle$$

(2.2)

where the numerical subscripts indicate the point concerned. This function can be expressed in terms of $SO(6)$ invariant tensors multiplied by scalar factors which are functions of the coordinates. Given the symmetry of $G^{(N=4)}$, the only $SO(6)$ invariant tensor that can arise is the Kronecker $\delta$, and there are two modes of hooking the indices up, each of which can occur in
three combinations making six independent amplitudes in all. Thus, for the leading component in the $\theta$ expansion we have

$$G^{(N=4)}|_{\theta=0} = a_1(s,t)\frac{(\delta_{12})^2(\delta_{34})^2}{x_{12}^4 x_{34}^4} + a_2(s,t)\frac{(\delta_{13})^2(\delta_{24})^2}{x_{13}^4 x_{24}^4} + a_3(s,t)\frac{(\delta_{14})^2(\delta_{23})^2}{x_{14}^4 x_{23}^4}$$

$$\quad + b_1(s,t)\frac{\delta_{13}\delta_{14}\delta_{23}\delta_{24}}{x_{12}^2 x_{14}^2 x_{23}^2 x_{24}^2} + b_2(s,t)\frac{\delta_{12}\delta_{14}\delta_{32}\delta_{34}}{x_{12}^2 x_{14}^2 x_{32}^2 x_{34}^2} + b_3(s,t)\frac{\delta_{12}\delta_{13}\delta_{12}\delta_{43}}{x_{12}^2 x_{13}^2 x_{12}^2 x_{43}^2}$$

where $x_{pq}^2 = (x_p - x_q)^2$ and, for example,

$$(\delta_{12})^2(\delta_{34})^2 = \delta_{\{A_2B_2\}}^\{A_1B_1\}\delta_{\{A_3B_3\}}^\{A_4B_4\}, \quad \delta_{13}\delta_{14}\delta_{23}\delta_{24} = \delta_{\{A_2B_4\}}^\{A_1B_2\}\delta_{\{A_3B_3\}}^\{B_1B_1\},$$

and where the braces denote tracefree symmetrisation at each point.

When writing down (2.3) we have taken into account the conformal covariance of the correlator. In each term we have introduced the corresponding scalar propagator structure which has the required conformal weight of the correlator. This implies that the coefficient functions $a_{1,2,3}, \ b_{1,2,3}$ are conformal invariants, so they depend on the two conformal cross-ratios

$$s = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad t = \frac{x_{12}^2 x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \quad (2.5)$$

Next, we notice the point-permutation symmetry of the correlator (2.2). To implement it it is sufficient to require invariance under two permutations, for instance,

$$1 \to 3 : s \to t \quad \text{and} \quad 1 \to 2 : s \to \frac{s}{t}, \ t \to \frac{1}{t}. \quad (2.6)$$

This leads to the following constraints:

$$a_1(s,t) = a_3(t,s) = a_1(s/t, 1/t)$$
$$a_2(s,t) = a_2(t,s) = a_3(s/t, 1/t)$$
$$b_1(s,t) = b_3(t,s) = b_1(s/t, 1/t)$$
$$b_2(s,t) = b_2(t,s) = b_3(s/t, 1/t) \quad (2.7)$$

So, the six coefficients in the $N = 4$ amplitude are in fact reduced to only two independent ones, one of the $a_i$ and one of the $b_i$. Now we shall show that one can determine these two functions by studying a certain $N = 2$ component of the $N = 4$ correlator. Let us see what happens when one reduces $N = 4$ supersymmetry to $N = 2$. The first step is to decompose the $SO(6)$ vector $W^A$ in a complex basis as $3 + \overline{3}$ under $SU(3)$ and further as $2 + 1 + \overline{2} + \overline{1}$ under $SU(2)$:

$$W^A \rightarrow (W^i \equiv \phi^i, W^3 \equiv W, W^i \equiv \phi^i, W_3 \equiv \overline{W}). \quad (2.8)$$

Here $W$ is the $N = 2$ SYM field-strength and $\phi^i$ ($i = 1, 2$) is the $N = 2$ matter hypermultiplet.

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5 An alternative explanation is provided by a double OPE. Indeed, each pair of operators is decomposed into six irreps of $SO(6)$, $20 \times 20 = 1 + 15 + 20 + 84 + 105 + 175$. Then the double OPE, being diagonal, gives rise to six structures.

6 In section we shall show that superconformal covariance allows us to restore the complete $\theta$ dependence of (2.2), given its leading component.
From the on-shell constraints on $W^A$ it is easy to derive that these superfields (evaluated at $\theta_3 = \theta_4 = 0$) obey the corresponding constraints,

$$\bar{D}_i W = 0$$  \hspace{1cm} (2.9)

and

$$D^{(i}_\alpha \phi^j) = 0 , \quad \bar{D}_i \alpha^j - \frac{1}{2} \delta^j_i \bar{D}_k \phi^k = 0 .$$  \hspace{1cm} (2.11)

Eq. (2.9) means that $W$ is a chiral superfield (a kinematic constraint) while eq. (2.10) is the Yang-Mills equation of motion, and eqs. (2.11) are the field equations of the hypermultiplet.

In $N=2$ harmonic superspace (see section 3 for a review) all $SU(2)$ indices are projected out by harmonic variables thus obtaining objects which carry $U(1)$ charge but are singlets under $SU(2)$. For instance, the harmonic superfield $q^+$ and its conjugate $\bar{q}^+$ are related to the ordinary hypermultiplet $N=2$ superfield $\phi_i$ by

$$q^+ = u^+_i \phi^i , \quad \bar{q}^+ = u^{+i} \bar{\phi}_i$$  \hspace{1cm} (2.12)

on shell.

For our purpose of identifying the coefficients in the $N=4$ amplitude it will be sufficient to restrict $T^{AB}$ to the following $N=2$ projections involving only hypermultiplets:

$$T_{ij} = \bar{\phi}_i \phi_j \ ( \text{at points 1 and 3}) , \quad T^{ij} = \phi^j \bar{\phi}^i \ ( \text{at points 2 and 4}) .$$  \hspace{1cm} (2.13)

Then we multiply the resulting hypermultiplet correlators by $u^+_i u^+_j$ at each point and obtain

$$G^{(N=2)} = \langle \bar{q}^+ q^+ | \bar{q}^+ q^+ | \bar{q}^+ q^+ | \bar{q}^+ q^+ \rangle .$$  \hspace{1cm} (2.14)

Its leading component is

$$G^{(N=2)}|_{\theta=0} = a_1(s,t) \frac{(12)^2(34)^2}{x_{12}^4 x_{34}^2} + a_3(s,t) \frac{(14)^2(23)^2}{x_{14}^4 x_{23}^2} + b_2(s,t) \frac{(12)(23)(34)(41)}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2}$$  \hspace{1cm} (2.15)

where, for example,

$$(12) = u^{+i} u^{+j} \epsilon_{ij} .$$

It is now clear that if we know the coefficient functions in the $N=2$ correlator (2.15), using the symmetry properties (2.7) we can obtain all the six coefficients in the $N=4$ amplitude (2.3). Recalling that the symmetry of the correlator under the exchange of points 1 $\leftrightarrow$ 3 (or 2 $\leftrightarrow$ 4) relates the coefficients $a_1$ and $a_3$ to each other, we conclude that the correlator (2.15) is, in principle, determined by two a priori independent functions of the conformal cross-ratios.

This result is, of course, completely general, and must hold perturbatively as well as non-perturbatively. In the following we will study the radiative corrections to this correlator, and will show that they are given in terms of a single independent coefficient function; the ratios of the “quantum” parts of the functions $a_1, a_3, b_2$ are completely fixed:

$$a_1^q \text{ corr.} = s\mathcal{F}(s,t) , \quad a_3^q \text{ corr.} = t\mathcal{F}(s,t) , \quad b_2^q \text{ corr.} = (1-s-t)\mathcal{F}(s,t)$$

(see eq. (4.18) below). To this end we have to supplement the pure symmetry arguments given above by some knowledge about the dependence of the hypermultiplet correlator on the Grassmann variables (G-analyticity), superconformal covariance and a new, dynamical property, the so-called harmonic (H-)analyticity.

\footnote{Strictly speaking, eqs. (2.9)-(2.11) hold only for an Abelian gauge theory and in the non-Abelian case a gauge connection needs to be included. However, the constraints satisfied by the gauge-invariant bilinears that we shall consider are in fact the same in the Abelian and non-Abelian cases.}
3 The hypermultiplet in harmonic superspace

In the preceding section we showed that all the information about the $N = 4$ correlator (2.2) is contained in the $N = 2$ one (2.14). There is another reason why we prefer to work with $N = 2$ rather than $N = 4$ superfields. The non-renormalisation theorem that we are going to prove in section 4 is based on the insertion formula originating in the standard off-shell Lagrangian formulation of field theory. The absence of an off-shell formulation of $N = 4$ SYM theory makes it difficult to justify this procedure in $N = 4$ superspace. However, there exists an \textit{off-shell} reformulation of the $N = 4$ theory in terms of $N = 2$ harmonic superfields [11].

We start by a brief review of the formulation of the $N = 2$ hypermultiplet in harmonic superspace (the reader may wish to consult [17] for the details). It can be described as a superfield in the Grassmann (G-)analytic superspace [41] with coordinates $x^\alpha_A, \theta^{+\alpha}, \bar{\theta}^{+\dot{\alpha}}, u^{\pm}_i$. Here $u^{\pm}_i$ are the harmonic variables which form a matrix of formulation of field theory. The absence of an off-shell formulation in section 4 is based on the insertion formula originating in the standard off-shell Lagrangian Grassmann (G-)analytic superspace [41] with coordinates $x^\alpha_A, \theta^{+\alpha}, \bar{\theta}^{+\dot{\alpha}}, u^{\pm}_i$. Here $u^{\pm}_i$ are the harmonic variables which form a matrix of $SU(2)$ and parametrise the sphere $S^2 \sim SU(2)/U(1)$. A harmonic function $f^{(p)}(u^{\pm}_i)$ of $U(1)$ charge $p$ is a function of $u^{\pm}_i$ which is invariant under the action of the group $SU(2)$ (acting on the index $i$ of $u^{\pm}_i$) and homogeneous of degree $p$ under the action of the group $U(1)$ (acting on the index $\pm$ of $u^{\pm}_i$). Such functions have infinite harmonic expansions on $S^2$ whose coefficients are $SU(2)$ tensors (multispinors). The superspace is called G-analytic since it only involves half of the Grassmann variables, the $SU(2)$-covariant harmonic projections $\theta^{+\alpha} = u^{+i}\bar{\theta}_i^{\alpha}, \bar{\theta}^{+\dot{\alpha}} = u^{+i}\bar{\theta}_i^{\dot{\alpha}}$. In a way, this is the generalisation of the familiar concept of a left- (or right-)handed chiral superfield depending on a different half of the Grassmann variables, either the left-handed ($\theta_\alpha$) or right-handed ($\bar{\theta}^{\dot{\alpha}}$) one.

In this framework the hypermultiplet is described by a G-analytic superfield of charge $+1$, $q^+(x_A, \theta^+, \bar{\theta}^+, u)$ (and its conjugate $\bar{q}^+(x_A, \theta^+, \bar{\theta}^+, u)$ where $\bar{\cdot}$ is a special conjugation on $S^2$ preserving G-analyticity). G-analyticity can also be formulated as differential constraints on the superfield:

$$D^+_\alpha q^+ = \bar{D}^+_\dot{\alpha} q^+ = 0$$

(3.1)

where $D^+_\alpha = u^{+i}_i D^i_\alpha$, $\bar{D}^+_\dot{\alpha} = u^{+i}_i D^i_{\dot{\alpha}}$. Note the similarity between (3.1) and the chirality condition (2.3). In fact, both of them are examples of what is called a “short multiplet” in the AdS/CFT language [16].

It is well-known that this $N = 2$ supermultiplet cannot exist off shell with a finite set of auxiliary fields [48]. This only becomes possible if an infinite number of auxiliary fields (coming from the harmonic expansion on $S^2$) are present. On shell these auxiliary fields are eliminated by the harmonic (H-)analyticity condition (equation of motion)

$$D^{++} q^+ = 0 .$$

(3.2)

Here $D^{++}$ is the harmonic derivative on $S^2$ (the raising operator of the group $SU(2)$ realised on the $U(1)$ charges, $D^{++} u^+ = 0$, $D^{++} u^- = u^+$).

The reader can better understand the meaning of eq. (3.2) by examining the general solution to the H-analyticity condition on a (non-singular) harmonic function of charge $p$:

$$D^{++} f^{(p)}(u^{\pm}_i) = 0 \Rightarrow \begin{cases} f^{(p)} = 0 & \text{if } p < 0 ; \\ f^{(p)} = u^{+}_{i_1} \ldots u^{+}_{i_p} f^{i_1 \ldots i_p} & \text{if } p \geq 0 . \end{cases}$$

(3.3)
In other words, the solution only exists if the charge is non-negative and it is a polynomial of degree \( p \) in the harmonics \( u^+ \). The coefficient \( f^{i_1 \cdots i_p} \) forms an irrep of \( SU(2) \) of isospin \( p/2 \). Thus, \( H \)-analyticity is just an \( SU(2) \) irreducibility condition having the form of a differential constraint on the harmonic functions.

Now it becomes clear why the combination of the \( G \)-analyticity constraints (3.1) with the \( H \)-analyticity one (3.2) is equivalent to the original on-shell hypermultiplet constraints (2.11). Indeed, from (3.3) one derives (2.12) and then, by removing the arbitrary harmonic commuting variables from both \( q^+ \) and \( D^+, \bar{D}^+ \), one arrives at (2.11).

The crucial advantage of the harmonic superspace formulation is that the equation of motion (3.2) can be derived from an off-shell action given by an integral over the \( G \)-analytic superspace:

\[
S_{HM} = - \int dud^4x A d^2\theta d^2\bar{\theta} \tilde{q}^+ D^{++} q^+ .
\]

(3.4)

This is the starting point for quantisation of the theory in a straightforward way [49]. In particular, one can introduce the following propagator (two-point function):

\[
\langle \tilde{q}^+_a (1) q^+_b (2) \rangle = \frac{(12)}{4\pi^2 x_{12}^2} \delta_{ab} .
\]

(3.5)

Here

\[ x_{12} = x_{A1} - x_{A2} + \frac{4i}{(12)} [(1-2) \theta_1^+ \bar{\theta}_1^+ + (2-1) \theta_2^+ \bar{\theta}_2^+ + \theta_1^+ \bar{\theta}_2^+ + \theta_2^+ \bar{\theta}_1^+] \]

is a supersymmetric coordinate difference and, e.g., \((1-2) = u_1^{-i} u_2^{+j} \epsilon_{ij}\). This propagator satisfies the Green’s function equation (suppressing the color indices),

\[
D^{++}_{1} \langle \tilde{q}^+(1) q^+(2) \rangle = \delta^4(x_{12}) \delta^2(\theta_1^+ - \theta_2^+) \delta^2(\bar{\theta}_1^+ - \bar{\theta}_2^+) \delta(u_1, u_2) .
\]

(3.6)

In deriving (3.6) one makes use of the following property of the harmonic derivative of a singular harmonic function:

\[
D^{++}_{1} = \frac{4i}{(12)} \delta(u_1, u_2) .
\]

Let us now assume that the space-time points 1 and 2 are kept apart, \( x_1 \neq x_2 \). Then the right-hand side of eq. (3.6) vanishes and the two-point function (3.5) becomes \( H \)-analytic:

\[
D^{++}_{1} \langle \tilde{q}^+(1) q^+(2) \rangle = 0 \quad \text{if} \quad x_1 \neq x_2 .
\]

(3.7)

This property can be extended to any correlation function involving gauge-invariant composite operators made out of hypermultiplets, \( O = \text{Tr}[ (\tilde{q}^+)^p (q^+)^q] \):

\[
D^{++}_{1} \langle O(1) \ldots \rangle = 0 \quad \text{if} \quad x_1 \neq x_2, x_3, \ldots .
\]

(3.8)

In reality eq. (3.8) is a Schwinger-Dyson equation based on the free field equation (3.2). Hence, its right-hand side contains contact terms like in (3.6), which vanish if the space-time points are kept apart. So, \( H \)-analyticity is a dynamical property of such \( N = 2 \) correlators holding away from the coincident points. It will play an important rôle in the next section.

Finally, just a word about the other ingredient of the \( N = 4 \) theory, the \( N = 2 \) SYM multiplet. Here we do not need to go through the details of how it is formulated in harmonic superspace and subsequently quantised [50]. We just recall that the corresponding field strength is a (left-handed) chiral superfield which is harmonic independent, \( D^{++} W = 0 \). The \( N = 2 \) SYM action is given by the chiral superspace integral

\[
S_{N=2 \text{ SYM}} = \frac{1}{4\tau^2} \int d^4x_L d^4\theta \text{ Tr } W^2
\]

(3.9)
where $\tau$ is the (complex) gauge coupling constant.

4 Superconformal covariance and analyticity as the origin of non-renormalisation

In this section we are going to derive the constraints on the $G$-analytic correlator $G^{(N=2)}$ following from superconformal covariance and H-analyticity. Applied to the lowest component in its $\theta$ expansion (see (2.13)), H-analyticity simply means irreducibility under $SU(2)$, from which one easily derives the three independent $SU(2)$ tensor structures. Further, conformal covariance implies that their coefficients are arbitrary functions of the conformal cross-ratios (in the particular case (2.13) point-permutation symmetry reduces the number of independent functions to two).

The combined requirements of superconformal covariance in $G$-analytic superspace and H-analyticity have further, far less obvious consequences which arise at the next level of the $\theta$ expansion of the correlator. They have been derived in [40] where it was found that one of the three coefficient functions remains unconstrained while two linear combinations of them have to satisfy certain first-order linear differential constraints. The solution of the latter allows for some functional freedom which cannot be fixed on general grounds. A summary of these constraints as well as their explicit solution is given in the Appendix.

Alternatively [13], one can apply a more efficient procedure which, as it turns out, also completely fixes the above freedom. It is based on the insertion formula [13] (for an explanation in the context of $N = 2$ harmonic superspace see [43, 26]). This formula relates the derivative of a 4-point correlator of the type (2.14) with respect to the (complex) coupling constant $\tau$ to a 4 + 1-point correlator obtained by inserting the $N = 2$ SYM action (3.9):

$$\frac{\partial}{\partial \tau} G^{(N=2)} \sim \int d^4 x_{L_0} d^4 \theta_0 \langle Tr W^2(0) | q^+ q^+ | q^+ q^+ | q^+ q^+ | q^+ q^+ \rangle .$$

(4.1)

Recall that unlike the matter superfields $q^+$ which are $G$-analytic and harmonic-dependent off shell, $W$ is chiral and harmonic-independent. The integral in the insertion formula (4.1) goes over the chiral insertion point 0. As we shall see later on, the combination of chirality with $G$-analyticity, in addition to conformal supersymmetry and H-analyticity, impose strong constraints on this five-point function.

Let us try to find out what could be the general form of a $4 + 1$-point correlator $\Gamma^{(0/2,2,2,2)}$ which is chiral at point 0 and $G$-analytic at points 1, $\ldots$, 4 (with $U(1)$ charges +2), has the corresponding superconformal properties and is also H-analytic,

$$D^+ r + \Gamma^{(0/2,2,2,2)} = 0 , \quad \text{if } x_r \neq x_s , \quad r,s = 1, \ldots, 4 .$$

(4.2)

In particular, it should carry a certain R weight. Indeed, the expansion of the matter superfield $q^+ = \phi^j(x) u^+_j + \ldots$ starts with the physical doublet of scalars of the $N = 2$ hypermultiplet which have no R weight. At the same time, the $N = 2$ SYM field strength $W = \ldots + \theta \sigma^{\mu \nu} \theta F_{\mu \nu}(x)$ contains the YM field strength $F_{\mu \nu}$ (R weight 0) in a term with two left-handed $\theta$'s, so the R

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8 In fact, there exists an alternative form given by the right-handed chiral integral $\int d^4 x d^4 \theta \ Tr \ W^2$. In a topologically trivial background the coupling constant becomes real and the two forms are equivalent (up to a total derivative).
weights of $W$ and of the Lagrangian equal 2 and 4, respectively. From (1.1) it is clear that this weight is compensated by that of the chiral superspace measure $d^4x_Ld^4\theta$, so the correlator on the left-hand side of eq. (1.1) has the expected weight 0.

The task now is to explicitly construct superconformal covariants of R weight 4 out of the coordinates of chiral superspace $x_{L0}$, $\theta_0^\alpha$ at the insertion point 0 and of G-analytic harmonic superspace $x_{Ar}$, $\theta_r^{+\alpha}$, $\bar{\theta}_r^{+\dot{\alpha}}$, $u_{ri}^\pm$, $r = 1, \ldots, 4$ at the matter points. To this end we need to know the transformation properties of these coordinates under $Q$ and $S$ supersymmetry (parameters $\epsilon$ and $\eta$, correspondingly). Since we are only interested in the leading term in the $\theta$ expansion, it is sufficient to examine the linearised transformations of the Grassmann variables $[51]$:

\[
\delta \theta^{i\alpha} = \epsilon^{i\alpha} + x^\alpha L_\beta \bar{\eta}_\beta^i + O(\theta^2), \quad \delta \theta^{+\alpha} = u^+_i \epsilon^{i\alpha} + x^\alpha A_\beta \bar{\eta}_\beta^i u^+_i + O(\theta^2), \quad \delta \bar{\theta}^{+\dot{\alpha}} = u^+_i \epsilon^{i\dot{\alpha}} - u^+_i \eta^i_{\dot{\alpha}} x^\beta A_{\beta} + O(\theta^2). \tag{4.3}
\]

We remark that $Q$ supersymmetry acts as a simple shift, whereas the $S$ supersymmetry terms shown in (4.3) are shift-like. Let us assume for the moment that we stay away from any singularities in $x$-space (we shall come back to this important point in a moment). Then we can use the four left-handed parameters $\epsilon^{i\alpha}$ and $x^\alpha \bar{\eta}_\beta^i$ to shift away four of the six left-handed spinors $\theta_0^{i\alpha}$ and $\theta_r^{+\alpha}$. This means that our correlator effectively depends, to lowest order, on two left-handed spinor coordinates. We can make this counting argument more explicit by forming combinations of the $\theta$'s which are invariant under $Q$ supersymmetry and under the shift-like part of $S$ supersymmetry. $Q$ supersymmetry suggests to use the differences

\[
\theta_0^{i\alpha} = \theta_0^{i\alpha} u^+_r - \theta_r^{+\alpha}, \quad \delta_Q \theta_0^{i\alpha} = 0, \quad r = 1, \ldots, 4. \tag{4.4}
\]

Then we can form the following two cyclic combinations of three $\theta_0^{i\alpha}$:

\[
(\xi_{12r})_\dot{\alpha} = (12)(x_{0r}^{-1}\theta_{0r})_\dot{\alpha} + (2r)(x_{01}^{-1}\theta_{01})_\dot{\alpha} + (r1)(x_{02}^{-1}\theta_{02})_\dot{\alpha}, \quad r = 3, 4 \tag{4.5}
\]

where $x_{0r} \equiv x_{L0} - x_{Ar}$ are translation-invariant and $(rs) \equiv u^+_ru^+_su^+_t\bar{\epsilon}_{ij}$ are $SU(2)$-invariant combinations of the space-time and harmonic coordinates, correspondingly. It is now easy to check that $\xi_{12r}$ are Q and S invariant to lowest order (i.e., shift-invariant):

\[
\delta_Q S \xi_{12r} = O(\theta^2). \tag{4.6}
\]

Here one makes use of the harmonic cyclic identity

\[
(rs)t_i + (st)r_i + (tr)s_i = 0. \tag{4.7}
\]

As a side remark we point out that the above counting of $Q$ and $S$ shift-invariant Grassmann variables can also be applied to the four-point function (2.14). It depends on four G-analytic $\theta^+$'s and their conjugates $\bar{\theta}^+$. This number equals that of the $Q$ and $S$ supersymmetry parameters, therefore we conclude that there exists no invariant combination (under the assumption that we keep away from the coincident space-time points). In other words, there are no nilpotent superconformal G-analytic covariants having the properties of the correlator (2.14). This means that given the lowest component (2.13) in the $\theta$ expansion of (2.14) and using superconformal transformations we can uniquely reconstruct the entire correlator (2.14). A similar argument applies to the $N = 4$ correlator (2.2).
Let us now inspect the structure of the correlator $\Gamma^{(0)2,2,2}$. As we noted earlier, it has R weight 4. In superspace the only objects carrying R weight are the odd coordinates, so the $\theta$ expansion of our correlator must start with the product of four left-handed $\theta$'s, i.e., the correlator should be nilpotent [21, 43]. Further, superconformal covariance requires that the shift-like transformations (4.3) do not produce structures with less than four $\theta$'s, so we must use all the four available shift-invariant combinations (4.5) (notice that they have R weight 1, even though they are right-handed spinors). Thus, we can write down the leading term in the correlator in the following form:

$$\Gamma^{(0)2,2,2} = (12)^{-2} \xi_{1235124}^2 F(x, u) + O(\theta^5) . \tag{4.8}$$

The coefficient function $F$ depends on the space-time and harmonic variables and carries vanishing $U(1)$ charges, due to the explicit harmonic prefactor $(12)^{-2}$.

The nilpotent prefactor in (4.8) can be expanded in terms of $\theta_0$ and $\theta^+$. In fact, what contributes in the insertion formula (4.1) is just the purely chiral term in it:

$$\xi_{1235124}^2 = (\theta_0)^4 R' \frac{\partial^4 G^{(N=2)}}{\partial T} + \text{terms containing } \theta^+ \tag{4.9}$$

where

$$R' = (12)^2(34)^2 x_{14}^2 x_{23}^2 + (14)^2(23)^2 x_{12}^2 x_{34}^2 + (12)(23)(34)(41) (x_{12}^2 x_{34}^2 - x_{13}^2 x_{24}^2 - x_{14}^2 x_{23}^2) \tag{4.10}$$

Here the polynomial $R'$ has been written in two different ways using the harmonic cyclic identity (4.7). Note that the superconformal covariant (4.9) is completely permutation symmetric in the points 1, ..., 4, although this is not obvious from the form of the left hand side.

Next, we substitute (4.9), (4.10) in (4.1) and carry out the integration over the insertion point. The $\theta_0$ integral is trivial due to the Grassmann delta function $(\theta_0)^4$ in (4.9). The result is

$$\frac{\partial}{\partial T} G^{(N=2)}(s, t) \sim \mathcal{F}(s, t) \left[ s \frac{(12)^2(34)^2}{x_{12}^4 x_{34}^4} + t \frac{(14)^2(23)^2}{x_{14}^4 x_{23}^4} + (1 - s - t) \frac{(12)(23)(34)(41)}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}} \right] \tag{4.11}$$

where

$$\mathcal{F}(s, t) = \frac{1}{s} \int d^4 x_0 \frac{x_{12}^4 x_{34}^4 x_{14}^4 x_{23}^4}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} F(x, u) \tag{4.12}$$

is an arbitrary conformally invariant four-point function (the factor $1/s$ is introduced for convenience).

The reason why we have not indicated any harmonic dependence on the left-hand side of (4.12) has to do with our last requirement, namely, the H-analyticity condition (4.1) on the four-point amplitude (4.11):

$$D_\tau^{++} [(12)^2(34)^2 \mathcal{F}] = D_\tau^{++} [(14)^2(23)^2 \mathcal{F}] = D_\tau^{++} [(12)(23)(34)(41) \mathcal{F}] = 0 . \tag{4.13}$$
Since the function $\mathcal{F}$ has vanishing $U(1)$ charges, from (3.3) one easily derives that it must be harmonic independent, i.e., an $SU(2)$ singlet:

$$\mathcal{F} = \mathcal{F}(s,t) .$$  \hspace{1cm} (4.14)

In the analysis so far we have not taken into account possible contact terms in the 4+1 correlator $\Gamma^{(0/2,2,2,2)}$. As pointed out in [8], they may become important in the context of the insertion formula (4.11). Concerning the G-analytic points 1, . . . , 4, we have decided to keep away from the coincident points, and this assumption allowed us to impose H-analyticity. Thus, we never see contact terms of the type $\delta^4(x_{rs})$, $r, s = 1, . . . , 4$. However, there can also be contact terms involving the insertion point 0 and only one of the matter points, i.e., singularities of the type $(\partial_0)^n \delta^4(x_{0r})$. Since we are supposed to integrate over $x_0$ in the insertion formula (4.11), such terms may result in a non-contact contribution to the left-hand side. Therefore we must investigate them in detail.

In order to do this we have to adapt our construction of nilpotent superconformal covariants. As before, we can profit from the available $S$—supersymmetry freedom to shift away two of the four $Q$—invariant combinations $\theta_{br}$, but this time we should be careful not to use singular (in space-time) coordinates transformations. Let us suppose that we are dealing with a contact term containing, e.g., $(\partial_0)^n \delta^4(x_{04})$. In the vicinity of the singular point $x_0 \sim x_4$ the matrix $x_{04}$ is not invertible anymore, so we cannot shift away $\theta_{04}$. However, since the four matter points are kept apart, the matrices $x_{01}$, $x_{02}$ and $x_{03}$ still are invertible. This allows us to shift away, e.g., $\theta_{01}$ and $\theta_{02}$. Then, just as before, we will be left with only two spinors, $\theta_{03}$ and $\theta_{04}$, all of which must be used to construct the nilpotent covariant of R weight 4. Thus, the latter is unique. This counting can also be done in a manifestly $Q$— and $S$—covariant way. We can still use the shift-invariant combination $\xi^{123}$ from (4.13) but not $\xi^{124}$ because it is now singular. Then we replace the latter by

$$\hat{\theta}_{04} = \frac{x_{04}}{(12)} \xi^{124} = \theta_{04} + \frac{(24)}{(12)} x_{04} x_{01}^{-1} \theta_{01} + \frac{(41)}{(12)} x_{04} x_{02}^{-1} \theta_{02} .$$  \hspace{1cm} (4.15)

which is regular at $x_{04} = 0$. Thus, the contact analog of (4.8) is

$$\Gamma^{(0/2,2,2,2)}_{\text{contact}} = (\partial_0)^n \delta^4(x_{04}) \xi^{123}_{123} \hat{\theta}_{04}^2 F_{\text{contact}}(x,u) + O(\theta^5 \bar{\theta}) .$$  \hspace{1cm} (4.16)

(note that if there are no derivatives on the delta function, we will only see the first term in (4.13), but we need the other two in the general case). Next, using the identity $\hat{\theta}_{04}^2 = -x_{04}^2 \xi^{124}_{124}/(12)^2$ and (4.13) in the insertion formula we obtain a contribution to $G^{(N=2)}$ having exactly the same form as (4.11) but where now

$$\mathcal{F}_{\text{contact}}(s,t) = -\frac{1}{s} \int d^4 x_0 \ (\partial_0)^n \delta^4(x_{04}) \frac{x_{12}^4 x_{14}^2 x_{23}^2}{x_{01}^2 x_{02}^2 x_{03}^2} F_{\text{contact}}(x,u) .$$  \hspace{1cm} (4.17)

As before, H-analyticity implies that this function must be harmonic independent.

In the above construction of contact contributions we have singled out the matter point 4 but any other matter point can be obtained by cyclic permutations (it is obvious that two such terms cannot help each other to achieve superconformal covariance). Remarking that the harmonic

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9Contact terms of the ultra-local type and their effect on the non-renormalisation of two- and three-point functions have been studied in [43].
polynomial \( R' \) in (4.11) is invariant under such permutations, we conclude that the possible contact terms do not alter the form of the four-point function (4.11).

We remark that this argument also removes a possible loophole which so far had remained in the proofs of the non-renormalisation theorems for extremal and next-to-extremal four-point functions as presented in [26, 28].

Thus the combination of superconformal covariance, chirality, G- and H-analyticity within the context of the insertion formula (4.1) restricts the freedom in any four-point correlator of the type \( G^{(N=2)} \) to a single function of the conformal cross-ratios. We recall that (4.1) only gives the derivative of \( G^{(N=2)} \) with respect to the coupling. This means that from (4.11) and (2.15) we can read off the quantum corrections to the three coefficients:

\[
\begin{align*}
    a_1^{q \text{ corr.}} &= s \mathcal{F}(s, t), \\
    a_2^{q \text{ corr.}} &= t \mathcal{F}(s, t), \\
    b_2^{q \text{ corr.}} &= (1 - s - t) \mathcal{F}(s, t).
\end{align*}
\]  

Using the symmetry constraints (2.7) we find that \( \mathcal{F} \) should satisfy

\[
\mathcal{F}(s, t) = \mathcal{F}(t, s) = \frac{1}{t} \mathcal{F}(\frac{s}{t}, \frac{1}{t}),
\]

and we determine the remaining coefficients \( a_2, b_1 \) and \( b_3 \). Finally, we express all the six coefficients in the \( N = 4 \) correlator (2.3) in terms of a single unconstrained function of the cross-ratios (with the symmetry properties (4.19)):

\[
\begin{align*}
    a_1 &= A_0 + s \mathcal{F}(s, t), \\
    a_2 &= A_0 + t \mathcal{F}(s, t), \\
    a_3 &= A_0 + t \mathcal{F}(s, t), \\
    b_1 &= B_0 + (s - t - 1) \mathcal{F}(s, t), \\
    b_2 &= B_0 + (1 - s - t) \mathcal{F}(s, t), \\
    b_3 &= B_0 + (t - s - 1) \mathcal{F}(s, t),
\end{align*}
\]

The constants correspond to the free-field part which cannot be determined by the insertion formula. The direct tree-level computation including disconnected diagrams yields respectively:

\[
A_0 = \frac{4(N_c^2 - 1)^2}{(2\pi)^8}, \quad B_0 = \frac{16(N_c^2 - 1)}{(2\pi)^8}.
\]

The result for the connected correlator is obtained from this by setting \( A_0 = 0 \).

Another way to state our non-renormalisation theorem is to say that in (4.20) one finds linear combination of the coefficient functions which are non-zero at the free-field level, but vanish for all radiative corrections.

5 The supergravity result and the AdS/CFT correspondence

According to the AdS/CFT correspondence, the connected part of the correlator of four stress-tensor multiplets in \( N = 4 \) SYM in the limit \( N_c \to \infty \) and \( \lambda = g^2 N_c \) large but fixed should match\footnote{It is a separate issue whether contact terms of this type can actually arise in a supergraph calculation. So far experience at two and three loops [13, 32] has shown that this is not the case. Nevertheless, the possibility remains open.}
a certain tree-level correlator in $N = 8$ AdS supergravity in five dimensions. More precisely, the leading component of the CFT correlator (2.3) that we are discussing should correspond to the correlator of four sets of supergravity scalars lying in the 20 of SO(6). The latter has recently been computed by Arutyunov and Frolov [46]. The purpose of this section is to first rewrite the result of [46] in a much simpler form and then compare it to eqs. (4.20) from Section 4. We find a perfect match between the CFT prediction and the explicit AdS result which, in our opinion, is new evidence for the validity of the correspondence conjecture.

According to [46], the two independent coefficients of the (connected) amplitude are, e.g.,

$$\frac{a_1}{x_{12}^2 x_{34}^2} = \frac{32 N_c^2}{2^8 \pi^{10}} \left[ -\frac{1}{2} \frac{1}{x_{34}^2} D_{2211} + \left( \frac{t}{s} + \frac{1}{s} \right) x_{12}^2 D_{3322} + \frac{3}{2} x_{12}^2 D_{2222} \right] \quad (5.1)$$

and

$$\frac{b_2}{x_{12}^2 x_{34}^2 x_{14}^2 x_{23}^2} = \frac{32 N_c^2}{2^8 \pi^{10}} \left[ \left( \frac{t}{s} + \frac{s}{t} - \frac{1}{t} - 6 \right) D_{2222} + 4 x_{12}^2 D_{3322} + 4 x_{14}^2 D_{3322} \right. \\
+ x_{12}^2 D_{2211} \left( \frac{1}{x_{12}^2 x_{34}^2} - \frac{1}{x_{14}^2 x_{23}^2} \right) + x_{14}^2 D_{2112} \left( \frac{1}{x_{14}^2 x_{23}^2} - \frac{1}{x_{12}^2 x_{34}^2} \right) \\
+ x_{13}^2 D_{2121} \left( \frac{1}{x_{13}^2 x_{12}^2} + \frac{1}{x_{14}^2 x_{23}^2} \right) \right] \quad (5.2)$$

and the four others can be obtained from these by symmetry according to equations (2.7). The scaling weights in the subscript of the $D$-functions refer to points 1 through 4 in the obvious order. These functions were first introduced in [14]; they denote the basis integrals appearing in tree-level four-point calculations in AdS supergravity. They can be written in various forms, of which the most useful one for our purpose is the following [52]:

$$D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} = K \int_0^\infty dt_1 \cdots dt_4 \Delta_1^{\Delta_1-1} \cdots \Delta_4^{\Delta_4-1} S_t^{\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 / 2} \exp \left[ -\frac{1}{S_t} \sum_{i<j} t_i t_j x_{ij}^2 \right] \quad (5.3)$$

Here $S_t = t_1 + t_2 + t_3 + t_4$, and the prefactor is (specialising to the 4 + 1 - dimensional case)

$$K = \frac{\pi^2 \Gamma(\Delta_1 + \cdots + \Delta_4) - 2)}{2 \Gamma(\Delta_1) \cdots \Gamma(\Delta_4)}. \quad (5.4)$$

Using this representation for $D_{1111}$ and integrating out the global scaling variable $S_t$, one immediately obtains the Feynman parameter representation of the standard one-loop box integral with external “momenta” $x_{12}, x_{23}, x_{34}, x_{41}$. As is well-known, this integral can be expressed in terms of logarithms and dilogarithms [53]; following [54], we write the result as

$$\frac{1}{K_{1111}} D_{1111} = \frac{1}{x_{12}^2 x_{24}^2} \Phi^{(1)}(s, t), \quad (5.5)$$

where

$$\Phi^{(1)}(s, t) = \frac{1}{\lambda} \left\{ 2 \left( \text{Li}_2(-\rho s) + \text{Li}_2(-\rho t) \right) + \ln \frac{t}{s} \ln \frac{1 + \rho t}{1 + \rho s} + \ln(\rho s) \ln(\rho t) + \frac{\pi^2}{3} \right\} \quad (5.6)$$
\[ \lambda(s,t) = \sqrt{(1-s-t)^2 - 4st}, \quad \rho(s,t) = 2(1-s-t+\lambda)^{-1}. \] (5.7)

\( \text{Li}_2 \) denotes the dilogarithm function.

Moreover, from the representation (5.3) it is also obvious that all the D-functions occurring in the formulas above for \( a_1 \) and \( b_2 \) can be obtained from \( D_{1111}^{K_{1111}} \) by appropriate differentiations with respect to the \( x_{ij}^2 \) (this fact was already noted in [14]). Applying this algorithm we rewrite \( a_1, b_2 \) in term of (third-order) differential operators acting on the basic function \( \Phi^{(1)}(s,t) \). Next, note that

\[ \partial_s \Phi^{(1)}(s,t) = \frac{1}{\lambda^2} \left( \Phi^{(1)}(s,t)(1-s+t) + 2 \ln(s) - \ln(t) \right) \frac{(s+t-1)}{s} \] (5.8)

and similarly with \( s \leftrightarrow t \). These identities are sufficient to express inductively arbitrary derivatives of \( \Phi^{(1)} \) with respect to \( s, t \) by \( \Phi^{(1)} \) itself and logarithmic terms. The final result of this procedure is

\[ a_1 = s \mathcal{F}(s,t), \quad b_2 = \frac{16N_c^2}{(2\pi)^8} + (1-s-t) \mathcal{F}(s,t) \] (5.10)

where

\[ \mathcal{F}(s,t) = -\frac{16N_c^2}{(2\pi)^8} \lambda^6 \left\{ \Phi^{(1)}(s,t) 12 s t [(1+s+t)\lambda^2 + 10 st] \right. \\
\quad + [\ln(s) 2 s [(1+t^2-s-st+10t)\lambda^2 + 30 st(1+t-s)] \\
\quad + [\ln(t) 2 t [(1+s^2-t-st+10s)\lambda^2 + 30 st(1+s-t)] \\
\quad \left. + [(1+s+t)\lambda^4 + 20 st \lambda^2] \right\}. \] (5.11)

The supergravity result (5.9), (5.10) exactly matches the form of our general prediction (4.20). The first term on the right-hand side of (5.10) agrees, to leading order in \( N_c \), with the connected free-field part (4.21); the function \( \mathcal{F} \), which is the sum of all quantum corrections in the appropriate limit, has the required permutation symmetry properties and occurs with the expected simple prefactors.

6 Conclusions

The correlator of four stress-tensor multiplets in \( N = 4 \) SYM originally contains six amplitudes which can be reduced to only two independent functions quite trivially, just by exploiting the obvious symmetries. On grounds of superconformal invariance, Grassmann and harmonic analyticity we show that the quantum part of the amplitudes is in fact universal to all of them. It is given by one a priori arbitrary function of the conformal cross-ratios which occurs in all six amplitudes with simple prefactors.

This property had been observed in weak coupling perturbation theory at two and three loops and we verify it for strong coupling by simplifying the supergravity result of [46]. We regard

11Here we assume \( \lambda^2 > 0 \); the case \( \lambda^2 < 0 \) requires an appropriate analytic continuation.
12As it is not clear whether the gauge group in the AdS/CFT correspondence is of unitary or special unitary type, the supergravity calculation [46] assumed \( U(N_c) \) for simplicity.
this as new evidence in favour of the AdS/CFT conjecture. It provides also a highly non-trivial check on the results of [46].

Alternatively, one can turn the argument around: The supergravity result does obey constraints originating from harmonic superspace. This makes us believe that there exists an appropriate harmonic superspace formulation of AdS supergravity in which all these properties become manifest.

We remark that $N = 4$ supersymmetry has not played any particular rôle here. What we actually used was $N = 2$ superconformal invariance, a property which a large class of $N = 2$ finite theories posses [56]. We would need $N = 4$ conformal supersymmetry if we wish to reconstruct the correlator of four stress tensors (the top component in the $N = 4$ SYM multiplet) from that of the scalar composites considered here. If we restrict ourselves to finite $N = 2$ theories, we can still claim a non-renormalisation property of the correlator of four hypermultiplet bilinears.

We have carried out the analysis of contact terms in the context of the reduction formula to the extent necessary for our present purposes. By an explicit construction of the relevant type of contact terms we have shown that, at least at the four-point level, their appearance in the Intriligator formula would not alter the form of the result, which remains proportional to the universal polynomial $R'$. This also removes the only possible loophole remaining in the proof of the non-renormalisation theorems for extremal and next-to-extremal four-point functions as presented in [26, 28]. Those contact terms could, however, in principle have invalidated the two-and three-loop computations presented in [43, 32]. Since their results have been confirmed by independent calculations [24, 8] we conclude that, up to the three-loop level, contact terms of the “malignant” type are absent.

Finally, it would be of obvious interest to study whether the universality property found here has non-trivial implications for the operator product expansion.

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7 Appendix: Differential constraints on the four-point function

The main result of this paper is that the four-point correlator (2.2) is determined by a single function of the space-time variables, apart from simple terms reflecting the free-field part of the amplitudes. The argument we gave makes use of the insertion formula (4.1) and therefore explicitly distinguishes free-field and quantum parts.

Alternatively [40], one can directly study the consequences of superconformal covariance and $H$-analyticity on the higher-level terms in the $\theta$ expansion of $N = 2$ correlators of the type (2.14). The constraints found in this way in [40] are reproduced here and solved explicitly. As before, we find a function $F(s,t)$ contributing to all six amplitudes in the same way as shown in (1.21), but there is additionally an a priori arbitrary function of one variable, named $h$ in the following. We express the free-field part of the amplitudes $a_1, b_2$ in terms of these functions.
The expression for \( h \) found in this way is not zero, whereby it is clear that it cannot be omitted on general grounds. This, in our opinion, explains why the direct approach to the four-point function inevitably yields weaker constraints.

We start by introducing the following two linear combinations of the coefficients \( a_1, a_3, b_2 \) of the \( N = 2 \) correlator (2.15)\(^{13}\):

\[
\alpha = \frac{1}{s} a_1 + \frac{s-1}{t^2} a_3 + \frac{1}{t} b_2, \quad \beta = \frac{1-s-t}{t^2} a_3 - \frac{1}{t} b_2 .
\] (7.1)

As shown in [40], the full implementation of H-analyticity combined with superconformal covariance leads to the following constraints:

\[
\beta_s = -s\alpha_s - t\alpha_t - \alpha , \quad (7.2)
\]

\[
\beta_t = s\alpha_s + (t - 1)\alpha_t + \alpha . \quad (7.3)
\]

These first-order coupled differential equations have an integrability condition in the form of a second-order equation for each function:

\[
\Delta \alpha = \Delta \beta = 0 \quad (7.4)
\]

where

\[
\Delta = s\partial_{ss} + t\partial_{tt} + (s + t - 1)\partial_{st} + 2\partial_s + 2\partial_t . \quad (7.5)
\]

We substitute

\[
\alpha(s, t) = \frac{A(s, t)}{\lambda} , \quad \beta(s, t) = \frac{B(s, t)}{\lambda} \quad (7.6)
\]

where \( \lambda \) has been defined in (5.7). Then we perform a transformation to the variables

\[
\xi = \frac{2s}{1-s-t+\lambda} ,
\]

\[
\eta = \frac{2t}{1-s-t+\lambda} . \quad (7.7)
\]

whose inverse is \(^{14}\)

\[
s = \frac{\xi}{(1 + \xi) (1 + \eta)} ,
\]

\[
t = \frac{\eta}{(1 + \xi) (1 + \eta)} ,
\]

\[
\lambda = \frac{1 - \xi\eta}{(1 + \xi) (1 + \eta)} > 0 . \quad (7.8)
\]

This brings the first equation in (7.4) to its “normal form”:

\[
A\xi\eta = 0 . \quad (7.9)
\]

Its general solution obviously is

\[
A(\xi, \eta) = f(\xi) + g(\eta) \quad (7.10)
\]

\(^{13}\)The cross-ratios used in [40] are related to (2.5) as follows: \( s' = t/s, t' = 1/s. \)

\(^{14}\)When inverting (7.7) we have to remember that \( \lambda \) (5.7) is defined as a positive square root. Correspondingly, the choice of sign in the last of eqs. (7.8), as well as the form of the first two, determines the allowed domain of the variables \( \xi, \eta \). The use of such variables has been studied in [55].
where \( f, g \) are arbitrary functions. So, we have found

\[
\alpha(\xi, \eta) = \frac{1}{\lambda} [f(\xi) + g(\eta)].
\]  

(7.11)

Given \( \alpha \), it is not difficult to solve for \( \beta \) from the first-order system (7.2), (7.3) (the integration constant can be absorbed into a redefinition of \( f, g \)). The result is

\[
\beta(\xi, \eta) = -\frac{1}{\lambda} \left[ \frac{\xi f(\xi)}{1+\xi} + \frac{g(\eta)}{1+\eta} \right].
\]  

(7.12)

At this point we should recall that \( a_1, a_3, b_2 \) appearing in (7.1) in fact originate from the \( N=4 \) correlator (2.3) and are therefore subject to the symmetry requirements (2.7). Under the permutations (2.6) we have

\[
1 \to 3 : \xi \to \eta \quad \text{and} \quad 1 \to 2 : \xi \to -1/(1+\eta), \eta \to -(1+\xi)/\xi.
\]  

(7.13)

Taking this into account leads to identifying the two functions in (7.11), (7.12):

\[
f(\xi) = -\frac{\xi g(\xi)}{3(1+\xi)} \equiv h(\xi).
\]  

(7.14)

Further, the two independent coefficients, e.g., \( a_1 \) and \( b_2 \) become

\[
a_1 = \frac{\xi}{1-\xi\eta} \left[ F(\xi, \eta) + 2h(\xi) - h(\eta) + h(-1/(1+\xi)) + h(-1/(1+\eta)) \right],
\]

\[
b_2 = \frac{1+\xi\eta}{1-\xi\eta} \left[ F(\xi, \eta) + h(-1/(1+\xi)) + h(-1/(1+\eta)) \right] - \frac{1-2\xi\eta}{1-\xi\eta} [h(\xi) + h(\eta)].
\]  

(7.15)

Here the functions \( F(\xi, \eta) \) and \( h(\xi) \) satisfy the following constraints:

\[
F(\xi, \eta) = F(\eta, \xi) = F(-1/(1+\eta), -(1+\xi)/\xi),
\]

\[
h(\xi) + h(-1+\xi)/\xi + h(-1/(1+\xi)) = C_q - (A_0 + B_0),
\]  

(7.16)

where the constants on the r.h.s. of the second equation are the free-field values from equation (4.20), possibly plus a quantum correction \( C_q \). The functions \( F, h \) may be split into a free-field part

\[
h_0(\xi) = \frac{1}{3} (A_0(\xi + \frac{1}{\xi}) - B_0)
\]

\[
F_0(\xi, \eta) = \frac{A_0}{3} \frac{-\xi^3 + 3\xi + 1}{\xi(\xi + 1)} + \frac{A_0}{3} \frac{-\eta^3 + 3\eta + 1}{\eta(\eta + 1)} + B_0.
\]  

(7.17)

and a quantum part \( h_q, F_q \).

In conclusion, the direct approach to the four-point correlator (2.14) leaves more freedom than the one from Section 4. Indeed, comparing (7.15) with (4.19), (4.20) we see that in the latter case \( F = F_q/\lambda \) and the quantum part of \( h \) vanishes, which is our non-renormalisation result. The fact that the argument given in this Appendix applies to both the free-field and quantum part explains why it cannot be as strong as the one based on the insertion formula.
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