Numerical Approximation of Real Functions and One Minkowski’s Conjecture on Diophantine Approximations

Nikolaj M. Glazunov

Glushkov Institute of Cybernetics NAS 03187 Ukraine Kiev-187 Glushkov prospekt 40
Email: glanm@d105.icyb.kiev.ua

Abstract

In this communication I consider the applications of several kinds of approximations of real functions to the problem of verified computation (reliable computing) of the range of implicitly defined real function $x_{n+1} = G(x_1, \ldots, x_n)$, where dependency $F(x_1, \ldots, x_{n+1}) = 0$ is defined on some compact domain by a sufficiently smooth real function $F(x_1, \ldots, x_{n+1})$. Constructive version of Kolmogorov-Arnold and implicit function theorems, results about floating-point approximation, floating-point approximations which give lower-bound and upper-bound estimates of some real functions, and approximate algebraic computation are used for the purpose. The rigorous theory can be build on the base of analysis on manifolds over floating points domains. In the text we demonstrate our approach on examples.

Introduction

In [1] the notion of Approximate Algebraic Computation (AAC) is formulated and shortly discussed. The subject matter of this talk lies in the area between the geometry of numbers and the analysis of real functions and approximate algebraic computation of the functions. More specifically I want to discuss approximate algebraic computation aspects of relation between the geometry of parametric minima of convex and distance functions and the analysis of functions which determine critical determinants. Let

$$|\alpha x + \beta y|^p + |\gamma x + \delta y|^p \leq c |\det(\alpha \delta - \beta \gamma)|^{p/2},$$

be a diophantine inequality defined for a given real $p > 1$; hear $\alpha, \beta, \gamma, \delta$ are real numbers with $\alpha \delta - \beta \gamma \neq 0$. H. Minkowski in his monograph [2] raise the question about minimum constant $c$ such that the inequality has integer solution other than origin. This Minkowski’s problem can be reformulated as a conjecture concerning the critical determinant of the region $|x|^p + |y|^p \leq 1$, $p > 1$. Mentioned mathematical problems are closely connected with Diophantine Approximation.
Also I want to proposed algorithms for approximate algebraic computation. By the computation of the algorithms A. Malishev and I have investigated the Minkowski’s conjecture and proposed the strengthen Minkowski’s analytic (MAS) conjecture. For verified computation (reliable computing) we used interval analyses. Methods and algorithms for interval evaluation of explicitly and implicitly defined real functions are used. In the paper we briefly consider following topics:
critical determinant of a body and diophantine approximation;
the problem;
Minkowski’s analytic conjecture;
Kolmogorov-Arnold’s theorem;
computer experiments and strengthen Minkowski’s analytic conjecture;
interval-analytic methods;
algorithms for approximate algebraic computation and implementation.

2 Critical Determinant of a Body and Diophantine Approximation

Critical determinant is one of the main notion of the Geometry of Numbers. Recall the definitions [3]. Let \( D \) be a set and \( \Lambda \) be a lattice with base \( \{a_1, \ldots, a_n\} \) in \( \mathbb{R}^n \). A lattice \( \Lambda \) is admissible for body \( D \) (\( D - \text{admissible} \)) if \( D \cap \Lambda = \emptyset \) or 0. Let \( d(\Lambda) \) be the determinant of \( \Lambda \). The infimum \( \Delta(D) \) of determinants of all lattices admissible for \( D \) is called the critical determinant of \( D \); if there is no \( D - \text{admissible} \) lattices then puts \( \Delta(D) = \infty \). A lattice \( \Lambda \) is critical if \( d(\Lambda) = \Delta(D) \).

Critical determinant is closely connected with diophantine approximation, solving inequalities \( F(x_1, \ldots, x_n) < c \) in integer numbers \( x_1, \ldots, x_n \) (with some restrictions, for instance, \( x = (x_1, \ldots, x_n) \neq 0 \)). Usually in the geometry of numbers the function \( F(x) \) is a distance function. A real function \( F(x) \) defined on \( \mathbb{R}^n \) is distance function if

(i) \( F(x) \geq 0, x \in \mathbb{R}^n, F(0) = 0; \)
(ii) \( F(x) \) is continuous;
(iii) \( F(x) \) is homogenous: \( F(\lambda x) = \lambda F(x), \lambda \in \mathbb{R} \).

The problem of solving of diophantine inequality \( F(x) < c \), with a distance function \( F \) are investigated.

Let \( \overline{M} \) be the closure of a set \( M \) and \( \#P \) be the number of elements of a finite set \( P \). An open set \( S \subset \mathbb{R}^n \) is a star body if \( S \) includes the origin of \( \mathbb{R}^n \) and for any ray \( r \) beginning in the origin \( \#(r \cap (\overline{M} \setminus M)) \leq 1 \). If \( F(x) \) is a distance function then the set

\[ M_F = \{ x : F(x) < 1 \} \]

is a star body.

One of the main particular case of a distance function is the case of convex symmetrical function \( F(x) \) which with conditions (i) - (iii) satisfies the additional conditions
(iv) $F(x + y) \leq F(x) + F(y)$;
(v) $F(-x) = F(x)$.

\section*{3 The Problem}

In considering the question of the minimum value taken by the expression $|x|^p + |y|^p$, with $p \geq 1$, at points, other that the origin, of a lattice $\Lambda$ of determinant $d(\Lambda)$, Minkowski \cite{13} shows that the problem of determining the maximum value of the minimum for different lattices may be reduced to that of finding the minimum possible area of a parallelogram with one vertex at the origin and the three remaining vertices on the curve $|x|^p + |y|^p = 1$. The problem with $p = 1, 2$ and $\infty$ is trivial: in these cases the minimum areas are $1/2$, $\sqrt{3}/2$ and $1$ respectively. Let $D_p \subset \mathbb{R}^2 = (x, y), p > 1$ be the 2-dimension region:

$$|x|^p + |y|^p < 1.$$ 

Let $\Delta(D_p)$ be the critical determinant of the region. Using analytic parameterization Cohn \cite{3} gives analytic formulation of Minkowski’s conjecture. Let

$$\Delta(p, \sigma) = (\tau + \sigma)(1 + \tau^p)^{\frac{1}{p}} (1 + \sigma^p)^{-\frac{1}{p}}, \quad (1)$$

be the function defined in the domain

$$D_p : \infty > p > 1, \ 1 \leq \sigma \leq \sigma_p = (2^p - 1)^{\frac{1}{p}},$$

of the $\{p, \sigma\}$ plane, where $\sigma$ is some real parameter; here $\tau = \tau(p, \sigma)$ is the function uniquely determined by the conditions

$$A^p + B^p = 1, \ 0 \leq \tau \leq \tau_p,$$

where

$$A = A(p, \sigma) = (1 + \tau^p)^{-\frac{1}{p}} - (1 + \sigma^p)^{-\frac{1}{p}}$$

$$B = B(p, \sigma) = \sigma(1 + \sigma^p)^{-\frac{1}{p}} + \tau(1 + \tau^p)^{-\frac{1}{p}},$$

$\tau_p$ is defined by the equation

$$2(1 - \tau_p)^p = 1 + \tau_p^p, \ 0 \leq \tau_p \leq 1.$$ 

In this case needs to extend the notion of parameter variety to parameter manifold. The function $\Delta(p, \sigma)$ in region $D_p$ determines the parameter manifold.

Minkowski’s analytic conjecture:

For any real $p$ and $\tau$ with conditions $p > 1, p \neq 2, 0 < \tau < \tau_p$

$$\Delta(p, \sigma) > \min(\Delta(p, 1), \Delta(p, \sigma_p)).$$
For investigation of properties of function $\Delta(p, \sigma)$ which are need for proof of Minkowski’s conjecture \[13, 3\] we considered the value of $\Delta = \Delta(p, \sigma)$ and its derivatives $\Delta'_\sigma, \Delta''_{\sigma^2}, \Delta'_p, \Delta''_{\sigma p}, \Delta'''_{\sigma^2 p}$ on some subdomains of the domain $D_p \subseteq [0, 1, 2, 3]$. The analytical computation of the derivatives is a problem of computer algebra.

4 The Theorem of Kolmogorov-Arnold

Let $F(x_1, \cdots, x_{n+1}) = 0, n > 1$ be a sufficiently smooth real function of two variables. The rough form of the theorem of Kolmogorov-Arnold states that:

**Theorem** (Kolmogorov, Arnold)

*Any sufficiently smooth real function can be represented as a superposition of functions of two variables.*

Let us demonstrate the constructive version of the theorem on examples of function $\Delta = \Delta(p, \sigma)$ and it’s derivatives. At first expressing $\Delta'_\sigma, \Delta''_{\sigma^2}, \Delta'_p, \Delta''_{\sigma p}, \Delta'''_{\sigma^2 p}$ in terms of a sum of derivatives of “atoms” $s_i = \sigma^{p-i}, t_i = \tau^{p-i}, a_i = (1 + \sigma^p)^{-i-\frac{p}{p}}, b_i = (1 + \tau^p)^{-i-\frac{p}{p}}, A = b_0 - a_0, B = \tau b_0 + \sigma a_0, \alpha_i = A^{p-i}, \beta_i = B^{p-i} (i = 0, 1, 2, \ldots)$. Then by the implicit function theorem computing $\tau = \tau(p, \sigma)$ by means of the following iteration process:

$$\tau_{i+1} = (1 + \tau_i^p)^{\frac{1}{p}}(1 - ((1 + \tau_i^p)^{-\frac{1}{p}} - (1 + \sigma^p)^{-\frac{1}{p}})\tau)^{\frac{1}{p}} - \sigma(1 + \sigma^p)^{-\frac{1}{p}}),$$

For approximate computation of the expression for $\tau_p$ we apply the following iteration:

$$(\tau_p)_{i+1} = 1 - (2^{-\frac{1}{p}})(1 + (\tau_p)^p)^{\frac{1}{p}}, p > 1, (\tau_p)_0 \in [0, 0.36].$$

So we really have represented the function $\Delta$ as the function $\Delta(p, \sigma)$ of two variables. The same fact is true for it’s derivatives Now we can compute expressions for $\Delta, \Delta'_\sigma, \Delta''_{\sigma^2}, \Delta'_p, \Delta''_{\sigma p}, \Delta'''_{\sigma^2 p}$ by means of approximate algebraic computations.

5 Strengthen Minkowski’s analytic conjecture

Based on some theoretical evidences and results of mentioned computation A.V. Malishev and author proposed

*Strengthen Minkowski’s analytic (MAS) conjecture:*
For given \( p > 1 \) and increasing \( \sigma \) from 0 to \( \sigma_p \), the function \( \Delta(p, \sigma) \)
1) increase strictly monotonous if \( 1 < p < 2 \) and \( p \geq p^{(1)} \),
2) decrease strictly monotonous if \( 2 \leq p \leq p^{(2)} \),
3) has a unique maximum on the segment \((1, \sigma_p)\); until the maximum \( \Delta(p, \sigma) \) increase strictly monotonous and then decrease strictly monotonous if \( p^{(2)} < p < p^{(1)} \);
4) constant, if \( p = 2 \),
here \( p^{(1)} > 2 \) is a root of equation \( \Delta''(\sigma) = 0 \).

It seems that conjecture (MAS) did not proven for any parameter \( p \) except trivial \( p = 2 \).

6 Interval-analytic methods

Let \( X = (x_1, \ldots, x_n) = ([x_{11}, x_{12}], \ldots, [x_{n1}, x_{n2}] \) be the n-dimensional real interval vector with \( x_{ii} \leq x_i \leq x_{ii} \) ("rectangle" or "box"). The interval evaluation of a function \( G(x_1, \ldots, x_n) \) on an interval \( X \) is the interval \([G, \overline{G}]\) such that for any \( x \in X \), \( G(x) \in [G, \overline{G}] \).

The interval evaluation is called optimal if \( G = \min G \), and \( \overline{G} = \max G \) on the interval \( X \).

In the communication I consider the case \( n = 2 \). It is sufficient for Minkowski’s conjecture. For the purpose we used modified variant of the method \([7]\) which we called Malyshev’s Method:

Let \( D \) be a subdomain of \( D_p \). Under evaluation in \( D \) a mentioned function the domain is covered by rectangles of the form \([p, \overline{p}; \sigma, \overline{\sigma}]\).

In the case of formula (1) expressing \( \Delta''_\sigma, \Delta''_p, \Delta''_{\sigma_p}, \Delta''_{\sigma\sigma_p} \) in terms of a sum of derivatives of "atoms" \( s_i = \sigma^{p-1}, t_i = \tau^{p-1}, a_i = (1 + \sigma^{p-1})^{-i} - 1, b_i = (1 + \tau^{p-1})^{-i} - 1, A = b_0 - a_0, B = b_0 + \sigma a_0, \alpha_i = A^{p-1}, \beta_i = B^{p-1} (i = 0, 1, 2, \ldots) \), one applies the rational interval evaluation to construct formulas for lower bounds and upper bounds of the functions, which in the end can be expressed in terms of \( p, \overline{p}, \sigma, \overline{\sigma}, \overline{\sigma}, \overline{\sigma}, \); here the bounds \( \underline{\sigma}, \overline{\sigma}, \) are obtained with the help of the iteration process:

\[
\mathcal{L}_{i+1} = (1 + \overline{p}^{p-1}) \frac{\beta}{\alpha} ((1 - ((1 + \overline{p}^{p-1})^{-\frac{1}{p}} - (1 + \overline{\sigma}^{p-1})^{\frac{1}{p}})) \mathcal{L}_i - \sigma (1 + \overline{\sigma}^{p-1})^{-\frac{1}{p}}),
\]

\[
\mathcal{T}_{i+1} = (1 + \overline{p}^{p-1}) \frac{\beta}{\alpha} ((1 - ((1 + \overline{p}^{p-1})^{-\frac{1}{p}} - (1 + \overline{\sigma}^{p-1})^{\frac{1}{p}})) \mathcal{T}_i - \sigma (1 + \overline{\sigma}^{p-1})^{-\frac{1}{p}}).
\]

By the result of A. Kolmogorov and V. Arnold, any sufficiently smooth real function can be represented as a superposition of functions of two variables.
As interval computation is the enclosure method, we have to put:

\[ [\tau, \overline{\tau}] = [\tau_N, \overline{\tau_N}] \cap [\tau_0, \overline{\tau_0}] . \]

\( N \) is computed on the last step of the iteration.
For initial values we may take: \([\tau_0, \overline{\tau_0}] = [0, 0.36] \).

7 Algorithms for approximate algebraic computation

Below each of 6 first algorithms has 4 different forms:
(i) approximate algebraic computation of the given expression in a given point at the floating point representation;
(ii) approximate algebraic computation of a lower-bound estimate of the given expression in a given point at the floating point representation;
(iii) approximate algebraic computation of a upper-bound estimate of the given expression in a given point at the floating point representation;
(iv) approximate algebraic computation of an interval evaluation of the given expression over given intervals at the floating point representation;
Here we give names, input and output of algorithms for interval evaluation only. But all algorithms for Minkowski conjecture are implemented and tested.

Algorithm TPV
Input: An implicitly defined function \( \tau_p \) from Paragraphs 3 and 6. \([p, \overline{p}; \sigma, \overline{\sigma}] \).
Method: Shortly described in Paragraphs 3 and 6.
Output: The interval evaluation of \( \tau_p \).

Algorithm TAU
Input: Implicitly defined function \( \tau \) from Paragraphs 3 and 6. \([p, \overline{p}; \sigma, \overline{\sigma}] \).
Method: Shortly described in Paragraphs 3 and 6.
Output: The interval evaluation of \( \tau \).

Algorithm L0V
Input: Function \( l^0 = \Delta(p, \sigma) - \Delta_p^{(0)} \) from Paragraphs 3 and 6. \([p, \overline{p}; \sigma, \overline{\sigma}] \).
Method: Shortly described in Paragraphs 3 and 6.
Output: The interval evaluation of \( l^0 \).

Algorithm L1V
Input: Function \( l^1 = \Delta(p, \sigma) - \Delta_p^{(1)} \) from Paragraphs 3 and 6. \([p, \overline{p}; \sigma, \overline{\sigma}] \).
Method: Shortly described in Paragraphs 3 and 6.
Output: The interval evaluation of $l^1$.

Algorithm $GV$
Input: A function $g(p, \sigma)$ which has the same sign as function $\Delta'_\sigma$.
$[p, \overline{p}; \sigma, \overline{\sigma}]$.
Method: Shortly described in Paragraphs 3 and 6.
Output: The interval evaluation of $g(p, \sigma)$.

Algorithm $HV$
Input: A function $h(p, \sigma)$ which is the partial derivative by $\sigma$ the function $g(p, \sigma)$.
$[p, \overline{p}; \sigma, \overline{\sigma}]$.
Method: Shortly described in Paragraphs 3 and 6.
Output: The interval evaluation of $h(p, \sigma)$.

Next two algorithms are described in [8].

Algorithm $MonotoneFunction$
Input: A real function $F(x, y)$ monotonous by $x$ and by $y$.
Interval $[x, \overline{x}; y, \overline{y}]$.
Output: The interval evaluation of $F$.

Algorithm $RationalFunction$
Input: A rational function $R(x, y)$. Interval $[x, \overline{x}; y, \overline{y}]$.
Output: The interval evaluation of $R$.

Implementation

Algorithms are implemented on PL/1(O) and (partly) on C and C++. There is also an implementation of some of the algorithms on Reduce3.3-3.X in T. Sasaki’s package [4] of arbitrary precision rational arithmetic.

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