Non-perturbative Expansion, Renormalons, and $\tau$ Decay

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Abstract

We analyze inclusive $\tau$ decay using a modified version of the $a$ expansion, a non–perturbative technique in which the effective coupling is analytic in the infrared region. The modification involves renormalization group improvement of the integrand in a spectral representation for the $D$–function prior to implementing the $a$ expansion. The advantage of this approach is that it enables us to monitor the structure of the induced power corrections and to ensure that these are consistent with the operator product expansion. Numerically the method agrees well with experiment: the comparison is made with the physical quantity $R_Z$ using $R_\tau$ as input.

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I. INTRODUCTION

Inclusive $\tau$–decay provides an ideal arena for extracting information about the low energy domain of the strong interactions, in particular the QCD coupling constant $\alpha_s(M_\tau^2)$. A detailed theoretical analysis of this process has been presented in [1] (see also [2–4]). There has been recent interest in trying to estimate the intrinsic uncertainty in such results due to the finite order truncation of the perturbative series. Attempts at going beyond this limitation have included the use of analytic continuation to resum so-called $\pi^2$–corrections [10–17], and infrared renormalon resummations [18–20] (see also [21–29]).

The starting point for the theoretical analysis is the expression [1]

$$ R_\tau = \frac{\Gamma(\tau^{-} \rightarrow \nu_{\tau}\text{hadrons}(\gamma))}{\Gamma(\tau^{-} \rightarrow \nu_{\tau}\nu_{e}e^{-}(\gamma))} = 2 \int_{0}^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + \frac{2s}{M_\tau^2}\right) \tilde{R}(s), \quad (1) $$

where $\tilde{R}(s) = \text{Im} \Pi_{A,V}(s + i\epsilon)/\pi$ (see e.g. [1]). The integral is not amenable to standard perturbation theory as it runs over small values of momentum. However, using Cauchy’s theorem, one may rewrite this result as a contour integral in the $s$–plane with a contour running clockwise round the circle $|s| = M_\tau^2$,

$$ R_\tau = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} (1 - z)^3(1 + z) \tilde{D}(M_\tau^2 z), \quad (2) $$

which naively appears to avoid the low momentum region. However, this trick requires the Adler $D$–function, $D(Q^2) = -Q^2 d\Pi(Q^2)/dQ^2$, to have specific analytic properties, namely to be an analytic function in the $Q^2$–plane except for a cut along the negative real axis. This property is broken in perturbation theory due to the presence of the Landau pole at $Q^2 = \Lambda^2_{QCD}$. Consequently all perturbative treatments are sensitive to the prescription for dealing with the Landau pole.

For this reason, in a previous publication [30], an analysis of $\tau$–decay was presented making use of an effective running coupling constant which is infrared (IR) finite and respects the above mentioned analytic properties [31,32]. While numerically quite successful, a conceptual problem with this approach was that one had little knowledge of the structure of induced non-perturbative contributions, in particular power corrections. In this letter we present a revised formalism which, by first analysing the resummation ambiguity of the perturbative series, allows us much greater control over the induced power corrections. In Section 2 we explain this modification, and in Section 3 apply it to the analysis of $\tau$–decay, extracting a value of $R_Z$ by running the effective coupling up to the $Z$–scale. Conclusions and future applications of this technique are discussed in Section 4.

II. THE $A$–EXPANSION FORMALISM

The nonperturbative expansion method proposed in [31,32] (see also [33,34]) allows one to systematically study the low energy regime of QCD and evaluate the integral (1) either

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1 All definitions are as in [30], and we use the standard convention $Q^2 = -q^2$ where $q$ is the current momentum.
directly or via use of Cauchy’s theorem as in [2, 30, 33]. The method is based on a new expansion parameter \( a \) connected with \( \alpha_s \) by the equation

\[
\lambda \equiv \frac{\alpha_s}{4\pi} = \frac{g^2}{(4\pi)^2} = \frac{1}{C(1-a)^3}, \tag{3}
\]

where \( C \) is a positive variational parameter fixed via data from meson spectroscopy [32]. An arbitrary Green function may be expanded as a power series in \( a \), a result achieved in practice by a resummation of a truncated perturbative series, replacing \( \alpha_s \) by \( a \) via the appropriate expansion of (3). It is clear from (3) that for all positive values of \( \lambda \) the parameter \( a \) lies in the region \( 0 \leq a < 1 \).

The \( Q^2 \) evolution of \( a \) is defined by the renormalization group (RG) equation,

\[
f(a) = f(a_0) + \frac{2\beta_0}{C} \ln \frac{Q^2}{Q^2_0}, \tag{4}
\]

where \( a_0 = a(Q^2_0) \). We shall work at \( O(a^5) \) which, from the expansion of (3), allows the use of perturbative results at \( O(\lambda^2) \). Calculation of the non-perturbative \( \beta \)–function at this order [32] leads to

\[
f(a) = \frac{1}{5(5+3B)} \sum_{i=1}^{3} x_i J(a, b_i), \tag{5}
\]

where

\[
J(a, b) = \frac{-2}{a b} - \frac{4}{a b^2} - \frac{12}{a b} - \frac{9}{(1-a)(1-b)} + \frac{4 + 12b + 21b^2}{b^3} \ln a
\]

\[
+ \frac{30 - 21b}{(1-b)^2} \ln(1-a) - \frac{(2+b)^2}{b^3(1-b)^2} \ln(a-b). \tag{6}
\]

In this expression

\[
x_i = \frac{1}{(b_i - b_j)(b_i - b_k)}, \tag{7}
\]

where \( ijk = 123 \) and cyclic permutations, the values of \( b_i \) are the solutions of the equation

\[
1 + 9a/2 + 2(6+a)a^2 + 5(5+3B)a^3 = 0,
\]

where \( B = \beta_1/(2C\beta_0) \), and \( \beta_0 \) and \( \beta_1 \) are the perturbative 1– and 2–loop coefficients of the \( \beta \)–function.

For subsequent analysis of \( \tau \)–decay it is useful to separate the higher order terms of the \( D \)–function via the definition

\[
D(Q^2) = 3 \sum f Q^2_f (1 + 4\lambda_{\text{eff}}(Q^2)), \tag{8}
\]

where one may consider the contribution to \( \lambda_{\text{eff}} \) either from perturbation theory or from the \( a \)–expansion as appropriate. A direct application of the latter to \( \tau \)–decay, as in [30,33], then corresponds to defining \( \lambda_{\text{eff}} \) via the appropriate expansion of (3). However, the conceptual difficulty with this procedure is that one has little knowledge of the induced non-perturbative
corrections contained in the variational series. In order to have control over this aspect we must first analyse the structure of the RG improved perturbative series.

To do this in a manner which respects the analytic properties of the $D$–function we use the standard spectral representation,

$$D(Q^2, \lambda) = Q^2 \int_0^\infty ds \frac{1}{(s + Q^2)^2} R(s, \lambda),$$

(9)

At next–to–leading–log (NLL) order $R(s)$ is given by

$$R(s) = 3 \sum_f Q^2_f \left[ 1 + 4\lambda + \left( a_1 - a_2 \ln \frac{s}{\mu^2} \right)^2 \lambda^2 \right],$$

(10)

where

$$a_1 = \frac{2}{3} [3365 - 22f - 8\zeta(3)(33 - 2f)], \quad a_2 = 4\beta_0,$$

(11)

It is now convenient to perform an integration by parts in (9), which results in the expression

$$\lambda_{\text{eff}}(t, \lambda) = \int_0^\infty d\tau \omega(\tau) \left[ \lambda + \frac{1}{4}(a_1 + a_2 - a_2 \ln(t\tau)) \lambda^2 \right],$$

(12)

where $t = Q^2/\mu^2$ and we have used (8). The weight function $\omega(\tau)$, given by

$$\omega(\tau) = \frac{2\tau}{(1 + \tau)^3},$$

(13)

describes the distribution of virtuality. One may now use RG improvement under the integral, first discussed in [37], with the knowledge that $R(s)$ obeys the same homogeneous RG equation as $D$. Using the two–loop $\beta$–function $\beta(\lambda) = -\beta_0 \lambda^2 - \beta_1 \lambda^3$, at NLL order we obtain

$$\lambda_{\text{eff}}(t, \lambda) = \int_0^\infty d\tau \omega(\tau) \left[ \frac{\lambda}{1 + \beta_0 \lambda \ln \tau} - \frac{\beta_1}{\beta_0} \lambda^2 \ln(1 + \beta_0 \lambda \ln \tau) \right] + \frac{\lambda^2}{(1 + \beta_0 \lambda \ln \tau)^2},$$

(14)

where $d = (1/4)(a_1 + a_2(1 - \ln k))$ and $\lambda \equiv \lambda(t)$. This expression is now in a form which on inspection will exhibit the divergences associated with large orders in perturbation theory. We observe that the virtuality distribution function (13) exactly coincides with the function used in [25] and is numerically very close to that obtained in [27] (see Figure 1), which in contrast to the present construction was obtained via an all–orders resummation of renormalon contributions in the large $\beta_0$ limit.

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Footnote:

2 In Eq. (14), we have restored the factor $k$ representing the renormalization scheme dependence. In the so–called V–scheme $k_V=1$, and in the $\overline{\text{MS}}$–scheme which we shall use throughout $k_{\overline{\text{MS}}} = \exp(-5/3)$ (see, e.g. [13]).
This connection with renormalons is illustrated more clearly by performing a formal
Borel transformation on (14), whereby we obtain

$$\lambda_{\text{eff}}(t, \lambda) = \int_0^\infty db \exp \left(-\frac{b}{\lambda(t, \lambda)}\right) B(b),$$  \hspace{1cm} (15)

with

$$B(b) = \left[1 + \left(\frac{\beta_1}{2\beta_0} + 2d\right)b\right] \Gamma(1 + b\beta_0)\Gamma(2 - b\beta_0) + \frac{\beta_1}{4\beta_0}b.$$  \hspace{1cm} (16)

This Borel function exhibits the correct infrared and ultraviolet renormalon poles for the
$D$–function, although not the full branch structure [38,39]. Nevertheless, the fact that the
poles are correctly positioned implies that that the resummation ambiguity associated with
the first IR renormalon is $O(1/Q^4)$, which is consistent with the lowest dimension vacuum
condensate operator in the operator product expansion (OPE) for the $D$–function. [4]

This result is important, as we may now conclude that introduction of the $a$–expansion at
this stage, given convergence of the variational series, as has been proven in simpler systems
[40–42] may only induce those non-perturbative power corrections required to cancel this

$^3$Note that the partial integration performed to obtain (12) which generates the appropriate
virtuality distribution is also responsible for the removal of the first IR singularity at $b = 1/\beta_0$
required for consistency with the OPE.

$^4$By convergence we mean strictly absolute convergence of the sequence of approximants for the
$D$–function $\{D_1(O(a), C_1), D_2(O(a^2), C_2), \ldots, D_N(O(a^N), C_N)\}$. 

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FIG. 1. The virtuality distribution functions $\tau \omega(\tau)$ taken from Ref. [27] (solid line) and the
function (13) multiplied by a factor of $\tau$ (dashed line) versus $\ln \tau$. 

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perturbative ambiguity. Thus, from the discussion above these corrections will only start at $O(1/Q^4)$ and will therefore be consistent with an OPE treatment.

Introducing the $a$–expansion for the $D$–function in this modified spectral representation leads to the replacement of (14) by

$$\lambda_{eff} = \int_{0}^{\infty} d\tau \omega(\tau) \tilde{\lambda},$$

(17)

where to $O(a^5)$,

$$\tilde{\lambda} = \frac{a^2}{C} + \frac{3a^3}{C} + \frac{a^4}{C} \left[ 6 + \frac{d}{C} \right] + \frac{a^5}{C} \left[ 10 + 6 \frac{d}{C} \right].$$

(18)

The running expansion parameter $a = a(Q^2)$ is determined via (4), and the variational parameter has been determined as $C = 21.5$ at $O(a^5)$ (and also as $C = 4.1$ at $O(a^3)$) \cite{32}. The $D$–function constructed in this way may now be utilised in an analysis of $\tau$–decay.

III. $\tau$–DECAY ANALYSIS

Making use of the contour integral representation (2) we isolate the QCD contribution by defining $R_\tau = R_\tau^{(0)}(1 + \Delta R_\tau)$, with

$$R_\tau^{(0)} = 3(|V_{ud}|^2 + |V_{us}|^2) S_{EW}.$$  

(19)

The electroweak factor and the CKM matrix elements are $S_{EW} = 1.0194$, $|V_{ud}| = 0.9753$, and $|V_{us}| = 0.221$ respectively, taken from \cite{1}. Using the relations (8) and (17), and Cauchy’s theorem, we obtain

$$\Delta R_\tau = 48 \int_{0}^{M_\tau^2} \frac{ds}{M_\tau^2} \left( \frac{s}{M_\tau^2} \right)^2 \left( 1 - \frac{s}{M_\tau^2} \right) \tilde{\lambda}(ks),$$  

(20)

which has a modified kinematic factor as compared to the standard relation (2) with the maximum shifted to $s = (2/3)M_\tau^2$.

Extracting the experimental value of $R_\tau$ from \cite{43}, as described in \cite{4}, we obtain $R_\tau^{exp} = 3.64 \pm 0.02$. Using this as input we obtain the effective coupling $\alpha_s(M_\tau^2) = 0.31 \pm 0.01$ at $O(a^5)$. This result is not directly comparable with perturbative extractions of $\alpha_s$, as it also includes various non–perturbative corrections, and a correction due to the removal of the Landau pole. Nevertheless we note that this quantity is lower than most extractions in fixed order perturbation theory, and that this shift is consistent with expectations from renormalon resummations \cite{20,44}.

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5Note that a $1/Q^2$ correction may also be induced in the running coupling by removal of the Landau pole \cite{20}. However this has a purely perturbative origin.

6In this case we take the u,d, and s quarks to be massless and thus ignore threshold effects.
It is more consistent to consider only physical quantities, and therefore we use the experimental result for $R_\tau$ to run the coupling up to the $Z$-scale and extract $R_Z$. In order to evaluate this quantity we apply the matching procedure in the physical timelike channel where, at least to leading order, the change in the number of active quarks is easily associated with the threshold for pair production. The effective coupling and its derivative are required to be continuous at the threshold points, which is achieved by solving the continuity equations for $C_f$ and $a^f_0$. Using the standard heavy quark masses, $m_c = 1.6$ GeV and $m_b = 4.5$ GeV, and applying this matching procedure we find

$$R_Z = 20.96 \pm 0.01 \quad \text{at } O(a^3),$$  

$$= 20.83 \pm 0.01 \quad \text{at } O(a^5),$$

which, at $O(a^5)$, is within one standard deviation of the experimental result $R_Z = 20.77 \pm 0.07$ \cite{43}. Note that this is a considerably better fit to the data than using the technique of \cite{30} when one accounts for the change in the data over the intervening period.

### IV. CONCLUDING REMARKS

In the present letter we have described a development of the technique introduced in \cite{31,32} which allows significantly greater control over the structure of induced non-perturbative corrections. This has allowed an application to $\tau$–decay and the resulting normalisation of the effective coupling results in very good agreement with experimental data at the $Z$-scale.

The intriguing possibility raised by this technique is that, as discussed in section 2, provided one assumes convergence of the variational series, the induced power corrections will be consistent with the existence of vacuum condensate operators in the OPE. Thus one may wonder whether this formalism may allow an investigation of the meson spectrum in the framework of QCD sum rules which does not explicitly include arbitrary condensate parameters (see e.g. \cite{45,46}), the assumption being that these corrections, at least in some averaged sense, are automatically induced by convergence of the series. The results of an investigation of this kind will be presented in a forthcoming publication \cite{47}.

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