“BOTTOM OF THE WELL” SEMI-CLASSICAL TRACE INVARIANTS

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Abstract. Let $\hat{H}$ be an $h$-admissible pseudodifferential operator whose principal symbol, $H$, has a unique non-degenerate global minimum. We give a simple proof that the semi-classical asymptotics of the eigenvalues of $\hat{H}$ corresponding to the “bottom of the well” determine the Birkhoff normal form of $H$ at the minimum. We treat both the resonant and the non-resonant cases.

1. Introduction

Let $X$ be an $n$-dimensional manifold and $\hat{H}$ a self-adjoint zeroth order semiclassical $\Psi$DO acting on the space of half-densities, $|\Omega|^{1/2}(X)$. We will assume that the principal symbol, $H(x, \xi)$, of $\hat{H}$ has a unique non-degenerate global minimum, $H = C$, at some point $(x_0, \xi_0)$, and that outside a small neighborhood of $(x_0, \xi_0)$ $H$ is bounded from below by $C + \delta$, for some $\delta > 0$. We will also assume that at $(x_0, \xi_0)$ the subprincipal symbol of $\hat{H}$ vanishes. From these assumptions one can deduce that on an interval

$$C < E < C + \epsilon, \quad \epsilon < \delta,$$

the spectrum of $\hat{H}$ is discrete and consists of eigenvalues:

$$E_i(h), \quad 1 \leq i \leq N(h),$$

where

$$N(h) \sim (2\pi h)^{-n} \text{Vol} \{ (x, \xi) : H(x, \xi) \leq C + \epsilon \}.$$

In addition, we will make a non-degeneracy assumption on the Hessian of $H(x, \xi)$ at $(x_0, \xi_0)$. Choose a Darboux coordinate system centered at $(x_0, \xi_0)$ such that

$$H(x, \xi) = C + \sum_{i=1}^{n} \frac{u_i}{2}(x_i^2 + \xi_i^2) + \cdots.$$

In this paper we present a short proof of the following theorem:

Theorem 1.1. Assume that the $u_i$’s are linearly independent over the rationals and that the subprincipal symbol of $\hat{H}$ vanishes at $(x_0, \xi_0)$. Then the eigenvalues, $\{E_i(h)\}$, determine the Taylor series of $H$ at $(x_0, \xi_0)$ up to symplectomorphism or, in other words, determine the Birkhoff canonical form of $H$ at $(x_0, \xi_0)$.
Our results are closely related to some recent results of [4] on the Schrödinger operator, \( \hat{H} = \hbar^2 \Delta + V \), which show that the “bottom of the well” spectral asymptotics determines the Taylor series of \( V \) at \( x_0 \). They are also related to inverse spectral results of [2], [6], [3] and [5]. In these papers it is shown that if

\[
\exp tv_H, \quad v_H = \sum \frac{\partial H}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial}{\partial \xi_i}
\]

is the classical dynamical system on \( T^*M \) associated with \( H \) and \( T_\gamma \) is the period of a periodic trajectory \( \gamma \) of this system, the asymptotic behavior of the wave trace, \( \text{trace} \left( \exp -\frac{it}{\hbar} \hat{H} \right) \), at \( NT_\gamma, N \in \mathbb{Z} \), determines the Birkhoff canonical form of (1.4) in a formal neighborhood of \( \gamma \). (In the first two papers we have cited, results of this type are proved for standard \( \PsiDOs \), and [3] and [5] are versions of these results in the semiclassical setting.) It turns out that if the trajectory, \( \gamma \), is replaced by a fixed point of the system (1.4) and, in particular, if this fixed point is a non-degenerate minimum of \( H(x, \xi) \), the recovery of the Birkhoff canonical form from the spectral data, (1.1), can be greatly simplified. Our goal, in this short note, is to show why.

We also obtain nearly-optimal results in the resonant case, see §4.

2. The semi-classical Birkhoff canonical form

We quickly review here the construction of the semi-classical Birkhoff canonical form of \( \hat{H} \). We follow the exposition in [1] and refer to that paper for details.

Performing a preliminary microlocalization and conjugation by an \( \hbar \)-FIO, we can assume that \( \hat{H} \) is an operator on \( \mathbb{R}^n \), and that the global minimum \((x_0, \xi_0)\) is the origin \((0,0)\) \( T^*\mathbb{R}^n \). Let us denote by

\[
[H] = \sum_{\alpha, \beta, k} x^\alpha \xi^\beta \hbar^k
\]

the Taylor series of the full Weyl symbol of \( \hat{H} \), where the monomial \( x^\alpha \xi^\beta \hbar^j \) has degree \(|\alpha| + |\beta| + 2j\). In fact we can assume that

\[
[H] = \sum_i \frac{\hbar_i}{2} (x_i^2 + \xi_i^2) + \cdots
\]

where the dots indicate terms of degree three and higher. Notice that

\[
[H]|_{\hbar=0} = \text{the Taylor series of the principal symbol of } \hat{H}.
\]

Let \( \hat{H}_2 \) denotes the Weyl quantization of

\[
H_2 := \sum_i \frac{\hbar_i}{2} (x_i^2 + \xi_i^2),
\]

and set

\[
\hat{H} = \hat{H}_2 + \hat{L}.
\]

To construct the quantum Birkhoff canonical form of \( \hat{H} \), one conjugates \( \hat{H} \) by suitable Fourier integral operators in order to successively make higher-order terms
in $L$ commute with $\hat{H}_2$. The resulting series is the quantum Birkhoff canonical form, $H_{\text{can}}$ of $H$.

The non-resonance condition implies that we can write $H_{\text{can}}$ in the form:

$$\hat{H}_2 = \hat{H}_2 + F(P_1, \ldots, P_n, \hbar), \quad P_i = \hbar^2 D_i^2 + x_i^2$$

with $F$ an $\hbar$-admissible symbol whose Taylor series is of the form

$$F = \sum_{|r| \geq 1} c_r(\hbar)p^r,$$

where $p_i = \xi_i^2 + x_i^2$, $r = (r_1, \ldots, r_n)$,

$$c_r(\hbar) = \hbar^{|r|-1}(c_{r,0} + \cdots)$$

and $c_{r,0} = 0$ for $|r| = 1$ (so that all the monomials in $[F] - H_2$ have degree $\geq 3$).

Theorem 1.1 is a direct consequence of the following:

**Theorem 2.1.** Under the assumptions of Theorem 1.1, the eigenvalues, $(1.1)$, determine the semi-classical Birkhoff canonical form of $\hat{H}$.

### 3. The proof of Theorem 2.1

The first step in our argument is more or less identical with that of [4], [3] and [5]. Assume without loss of generality that $C = 0$, and let $\rho \in C_0^\infty(\mathbb{R})$ be equal to one on the interval $[-1/2, 1/2]$ and zero outside the interval $[-1, 1]$. Then for $\epsilon$ small the $\rho$-truncated wave trace

$$\text{Tr}(t, \hbar) := \text{trace} \rho \left( \hat{H} \frac{\epsilon^{it\hat{H}}}{\hbar} \right)$$

is equal modulo $O(\hbar^\infty)$ to the $\rho$-truncated wave trace for the Birkhoff canonical form,

$$\text{Tr}(t, \hbar) := \text{trace} \rho \left( \hat{H}_{\text{can}} \frac{\epsilon^{it\hat{H}_{\text{can}}}}{\hbar} \right).$$

The truncated wave trace admits an asymptotic expansion $\text{Tr}(t, \hbar) \sim a_0(t) + a_1(t)\hbar + \cdots$ as $\hbar \to 0$. This follows from the method of stationary phase and fact that for each $t$ the operator $\rho(\epsilon^{-1}\hat{H})e^{it\hbar^{-1}\hat{H}}$ is an $\hbar$-Fourier integral operator. Writing the truncated trace as an oscillatory integral, for each $t$ the phase has a unique critical point, corresponding to the absolute minimum $(x_0, \xi_0)$ which is a fixed point of the classical flow. Since the cutoff operator $\rho \left( \frac{H_{\text{can}}}{\epsilon} \right)$ is microlocally equal to the identity in a neighborhood of $(x_0, \xi_0)$, the asymptotic expansion of the cutoff trace is independent of $\rho$, provided $\rho \in C_0^\infty$ is identically equal to one near zero.

The truncated trace of the Birkhoff canonical form equals

$$\text{Tr}(t, \hbar) = \sum_{k \in (\mathbb{Z}_+)^n} \rho \left( \frac{H_{\text{can}}(\hbar(k+1/2), \hbar)}{\epsilon} \right) e^{it\hbar^{-1}H_{\text{can}}(\hbar(k+1/2), \hbar)}.$$
However, since \( \rho \) is identically equal to one in a neighborhood of zero, as a power series in \( \hbar \)

\[
\text{(3.4)} \quad \text{Tr}(t, \hbar) \sim \sum_{k \in (\mathbb{Z}_+)^n} e^{i t \hbar^{-1} H_{\text{can}}(\hbar(k+1/2), \hbar)}.
\]

We now rewrite (3.4) in a more amenable form using a variant of the “Zelditch trick” (see [6]).

**Proposition 3.1.** For any choice of \( \rho \) as above and for \( \epsilon \) small, as \( \hbar \to 0 \)

\[
\text{(3.5)} \quad \text{Tr}(t, \hbar) \sim \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \left( \sum_{|r| \geq 1} \hbar^{|r|-1} c_r(h) \left( \frac{1}{t} D_\theta \right)^r \right)^m \frac{e^{it \sum_j \theta_j}}{\Pi_j(1 - e^{it \theta_j})} \bigg|_{\theta = u},
\]

where

\[
D_\theta = -i \left( \frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_n} \right)
\]

and the right-hand side of (3.5) is understood as a power series in \( \hbar \).

**Proof.** Recalling that \([\hat{H}_2, \hat{F}] = 0\),

\[
\text{Tr}(t, \hbar) \sim \sum_{k \in (\mathbb{Z}_+)^n} e^{i t u \cdot (k+1/2)} \langle k | e^{i t \hbar^{-1} \hat{F}} | k \rangle,
\]

where \( \{|k\} \) is an orthonormal basis of eigenvectors of the canonical \( n \)-torus representation on \( L^2(\mathbb{R}^n) \), and \( u \cdot (k + 1/2) = \sum_{j=1}^n u_j (k_j + 1/2) \). For each \( k \), the Taylor expansion, \([\hat{F}], \) gives us an asymptotic expansion

\[
\text{(3.6)} \quad \langle k | e^{i t \hbar^{-1} \hat{F}} | k \rangle = \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \left( \sum_{|r| \geq 1} \hbar^{|r|-1} c_r(h) \right)^m \sum_{r \geq 1} \left( \frac{\hbar^{|r|-1} c_r(h)(k+1/2)^r}{1 + \epsilon} \right)^m.
\]

Let us introduce the variables \( \theta = (\theta_1, \ldots, \theta_n) \) and write:

\[
(k + 1/2)^r e^{i t u \cdot (k+1/2)} = \left( \frac{1}{t} D_\theta \right)^r e^{it \theta \cdot (k+1/2)} \bigg|_{\theta = u}.
\]

Then

\[
\text{Tr}(t, \hbar) \sim \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \sum_{k \in (\mathbb{Z}_+)^n} \rho \left( \frac{F(h(k+1/2), \hbar)}{\epsilon} \right) \left( \sum_{r \geq 1} \hbar^{|r|-1} c_r(h) \left( \frac{1}{t} D_\theta \right)^r \right)^m \left( \frac{1}{1 - e^{it \theta_j}} \right) \bigg|_{\theta = u},
\]

Finally, for each \( m \) (summing a geometric series)

\[
\sum_{k \in (\mathbb{Z}_+)^n} \left( \sum_{r \geq 1} \hbar^{|r|-1} c_r(h) \left( \frac{1}{t} D_\theta \right)^r \right)^m \left( \frac{e^{it \sum_j \theta_j}}{\Pi_j(1 - e^{it \theta_j})} \right) \bigg|_{\theta = u} =
\]

\[
= \left( \sum_{r \geq 1} \hbar^{|r|-1} c_r(h) \left( \frac{1}{t} D_\theta \right)^r \right)^m \left( \frac{e^{it \sum_j \theta_j}}{\Pi_j(1 - e^{it \theta_j})} \right) \bigg|_{\theta = u},
\]

and the result follows. \( \Box \)
We will show that the $m = 0$ term in the series on the right-hand side of (3.5) suffices to determine the $c_r(\hbar)$. More precisely:

**Theorem 3.2.** From the coefficients of $\hbar^s$, $s \leq \ell$, in the series in $\hbar$

$$V(t, \hbar) = \sum_{|r| \geq 1} \hbar^{|r|-1} c_r(\hbar) \left( \frac{1}{l} D_\theta \right)^r e^{it \frac{1}{l} \sum_j \theta_j} \bigg|_{\theta = u}$$

one can determine the coefficients of $\hbar^s$, $s \leq \ell$, in $c_r(\hbar)$ for all $r$.

**Proof.** Let $\rho$ be a cutoff function as before, and $\hat{\varphi} = \rho$. Integrating (3.7) against $\delta_n \varphi(\delta t)$ and essentially reversing the previous calculation, we find:

$$\tilde{V}(\hbar, \hbar) = \sum_k \left( \sum_{|r|} \hbar^{|r|-1} c_r(\hbar)(k + 1/2)^r \right) \rho \left( \epsilon^{-1} u \cdot (k + 1/2) \right)$$

$$= \sum_k \hbar^{-1} F(h(k + 1/2)) \rho \left( \epsilon^{-1} u \cdot (k + 1/2) \right).$$

Letting

$$c_r(\hbar) = \sum_{i=0}^{\infty} c_{r,i} \hbar^{|r|-1+i},$$

we can rearrange (3.8) in increasing powers of $\hbar$ (using the variable $\ell = 2|r| - 2 + i$ for the exponent of $\hbar$):

$$\tilde{V}(\epsilon, \hbar) = \sum_{\ell=0}^{\infty} \hbar^\ell \sum_{j=0}^{[\hbar]} \sum_{|r|=j+1} c_{r,\ell-2j} \left( \sum_k (k + 1/2)^r \rho \left( \epsilon^{-1} u \cdot (k + 1/2) \right) \right).$$

Now arrange the numbers

$$u_k = u \cdot (k + 1/2), \quad k \in (\mathbb{Z}_+)^n$$

in strictly increasing order (which is possible because there are no repetitions among them):

$$0 < \nu_1 = u_k|_{k=0} < \nu_2 < \cdots.$$

Let us write: $\nu_s = u_k|_{k=s}$. Now vary $\epsilon$ in (3.9), starting with a very small value. Gradually increasing $\epsilon$, we can arrange that the coefficient of $\hbar^\ell$ in (3.9) is

$$\sum_{j=0}^{[\hbar]} \sum_{|r|=j+1} c_{r,\ell-2j} \sum_{s=1}^m (k^{(s)} + 1/2)^r \rho \left( \epsilon^{-1} \nu_s \right)$$

for any given $m$. Therefore, by an inductive argument on $m$ we can recover the values of the polynomial

$$p_\ell(x) = \sum_{j=0}^{[\hbar]} \sum_{|r|=j+1} c_{r,\ell-2j} (x + 1/2)^r$$

at all $k \in (\mathbb{Z}_+)^n$. But these values determine the polynomial, and therefore its coefficients. □
Now we show that the asymptotic expansion of the trace, \( \text{Tr}(t, \hbar) \), determines \( V \):

**Theorem 3.3.** From the coefficients of \( \hbar^s, s \leq \ell \), in the expansion \( \text{Tr}(t, \hbar) \) one can determine the coefficients of \( \hbar^s, s \leq \ell \) in the series \( V(t, \hbar) \).

**Proof.** We proceed by induction on \( \ell \).

The coefficient of \( \hbar \) in \( V \) coincides with the coefficient of \( \hbar \) in (3.5), since all terms in the sum (3.5), except the first are of order \( O(\hbar^m), m > 1 \).

By theorem 3.2 the coefficient of \( \hbar \) in \( V \) enables us to determine the coefficient of \( \hbar \) in \( c_r(\hbar) \), and hence the coefficient of \( \hbar^2 \) in the second summand of (3.5). But the coefficient of \( \hbar^2 \) in the first summand coincides with the coefficient of \( \hbar^2 \) in \( V \), so the coefficients of \( \hbar \) and \( \hbar^2 \) in (3.5) determine the coefficient of \( \hbar^2 \) in \( V \). It is clear that this procedure can be continued indefinitely. \( \square \)

Theorem 3.3 is an immediate consequence of theorems 3.2 and 3.3.

4. The resonant case

We now consider the case when the frequencies \( u_1, \ldots, u_n \) are not linearly independent over \( \mathbb{Q} \). Following [1], let us introduce the number

\[
d = \min \{ |\alpha|, \alpha \in \mathbb{Z}^n \setminus \{0\} \mid \alpha \cdot u = 0 \}
\]

which is a measure of the rational relations among the frequencies (here \( |\alpha| = \sum_{j=1}^{n} |\alpha_j| \)). We will make use below of the following observation:

**Lemma 4.1.** Among the eigenvalues of \( \hat{H}_2 \) of the form:

\[
\lambda_k = k \cdot (u + 1/2) \quad \text{with} \quad |k| < d/2
\]

there are no repetitions (i.e., the mapping \( k \mapsto \lambda_k \) is 1-1 in the range \( |k| < d/2 \)).

**Proof.** If \( k \cdot (u + 1/2) = k' \cdot (u + 1/2) \), then \( (k - k') \cdot u = 0 \) and therefore \( |k - k'| \geq d \). The conclusion now follows from the triangle inequality. \( \square \)

Continuing to assume that the subprincipal symbol vanishes at the absolute minimum, the semi-classical Birkhoff canonical form in the resonant case has the following structure (see [1]):

\[
H_{can} = H_2 + F + K
\]

where

1. \( F = F(p_1, \ldots, p_n, \hbar) \) where \( F \) is a polynomial in all variables of degree at most \( \left\lfloor \frac{d-1}{2} \right\rfloor \).

2. \( [K] \) is a power series with monomials \( \hbar^j x^\alpha \xi^\beta \) where \( |\alpha| + |\beta| + 2j > d \) and \( [\hat{H}_2, K] = 0 \).

In this section we prove the following:
Theorem 4.2. If \( d \) is even the eigenvalues determine the entire semi-classical canonical form, \( F(x, \hbar) \). If \( d \) is odd, those eigenvalues determine the semi-classical canonical form except for the monomials of maximal degree, \( \left\lfloor \frac{d-1}{2} \right\rfloor \).

Except for a few additional complications, the method of proof is the same as in the non-resonant case. We begin by checking that the asymptotic expansion of the truncated trace can be treated by the same methods as before, up to a sufficiently high order in \( \hbar \):

**Proposition 4.3.** In the resonant case, the expansion \( (3.5) \) is valid modulo \( O(\hbar^{\left\lfloor \frac{d}{2} \right\rfloor}) \).

**Proof.** Once again we write the trace as a sum of diagonal matrix elements over a normalized basis \( \{ |k\rangle \} \) of eigenfunctions of the standard representation of the \( n \)-torus, splitting off the \( H_2 \) part (which is possible since \( \hat{F} + \hat{K} \) commutes with \( H_2 \)):

\[
\text{Tr}(t, \hbar) \sim \sum_{k \in \mathbb{Z}^n} e^{it\cdot \left( k + \frac{1}{2} \right)} \langle k | e^{i\hbar^{-1}(\hat{F} + \hat{K})} | k \rangle.
\]

We next expand the exponential in its Taylor series. We want to show that every term involving \( \hat{K} \) is \( O(\hbar^{\left\lfloor \frac{d}{2} \right\rfloor}) \).

A term involving \( \langle k | (\hat{F} + \hat{K})^m | k \rangle \)

is a sum of terms of the form

\[
\langle k | \hat{F}_1 \hat{K}_1 \cdots \hat{F}_s \hat{K}_s | k \rangle
\]

where the \( F_j \) are powers of \( F \) and the \( K_j \) are powers of \( K \). Therefore, the \( K_j \) are sums of monomials \( \hbar^j x^\alpha \xi^\beta \) where \( 2j + |\alpha| + |\beta| > d \), just as is \( K \). Let us express those monomials in terms of raising and lowering operators,

\[
A_{\alpha\beta} = z^\alpha \xi^\beta,
\]

\[
z = x + i\xi.
\]

Then \( (4.2) \) is a linear combination of terms of the form

\[
\hbar^{\sum_{i=1}^s j_i} \langle k | \hat{F}_1 A_{\alpha_1\beta_1} \cdots \hat{F}_s A_{\alpha_s\beta_s} | k \rangle
\]

where, for each \( i \),

\[
2j_i + |\alpha_i| + |\beta_i| > d.
\]

Now recall that (i) the \( \hat{F}_j \) are diagonal in the basis \( \{ |k\rangle \} \) and (ii) the \( \widehat{A_{\alpha\beta}} \) act on the basis vectors by:

\[
\widehat{A_{\alpha\beta}} |k\rangle = \hbar^{\beta} c_{\alpha\beta} |k + \alpha - \beta\rangle
\]

where \( c_{\alpha\beta} \) is a constant whose value we won’t need. Therefore, a diagonal matrix element of the sort \( (4.3) \) is zero unless

\[
\sum_{i=1}^s \alpha_i - \beta_i = 0,
\]
in which case \(O(\hbar^{j+\sum_i |\beta^i|})\) where \(j = \sum_i j_i\). However, \(|\alpha^i| + |\beta^i| > d - 2j_i\) for each \(i\) and
\[
\sum_{i=1}^s \alpha^i - \beta^i = 0 \Rightarrow \left| \sum_{i=1}^s \alpha^i \right| = \left| \sum_{i=1}^s \beta^i \right|.
\]
Therefore, \(\sum_i |\beta^i| \geq [sd/2] - j\) and so \(O(\hbar^{[sd/2]})\) is valid. It follows that all diagonal matrix elements to which \(\hat{K}\) contributes are at least \(O(\hbar^{d/4})\).

**Lemma 4.4.** \(F(h(x+1/2), h)\) is a polynomial in \(h\) of degree at most \([d-1]/2\), and if we write
\[
F(h(x+1/2), h) = \sum_{j=0}^{[d-1]/2} h^j F_j(x)
\]
the power series expansion of \(\text{Tr}(t, h)\) determines the values \(F_j(k)\) for all \(k \in (\mathbb{Z}_+)^n\) such that \(|k| < d/2\), for all \(j \leq [d-1]/2\) if \(d\) is even and for all \(j < [d-1]/2\) if \(d\) is odd.

**Proof.** The first statement follows from the general form of \(F\).

By theorems 3.3 and 3.2, for any \(\ell\) the first \(\ell\) terms of the expansion of \(\text{Tr}(t, h)\) determine the first \(\ell\) terms of \(F(x)\), provided we replace \(F\) by \(F + \hat{K}\). But, by the previous proposition, the expansion of \(\text{Tr}(t, h)\) mod \(O(\hbar^{[sd/2]})\) is insensitive to what \(K\) is. Therefore, \(F(x)\) remains valid mod \(O(\hbar^{d/4})\), where \(F\) now stands for the part of the canonical form we are determining from the spectrum.

If \(d\) is even
\[
[d-1]/2 < d/2,
\]
and so it follows that the expansion of \(\text{Tr}\) determines the sums
\[
\sum_k h^{-1} F(h(k + 1/2)) \rho(\epsilon^{-1} u \cdot (k + 1/2)).
\]
Now we proceed as before, letting \(\epsilon\) grow starting at a very small value. Since the eigenvalues are all different, we can determine the polynomial in \(h\), \(F(h(x + 1/2))\), evaluated at each \(k\) with \(|k| < d/2\). If \(d\) is odd we must discard the term \(F_j\) with \(j = [d-1]/2\).

Since \(F_j\) is a polynomial of degree at most \([d-1]/2\), the proof of theorem 4.2 is completed by the following result:

**Lemma 4.5.** Let \(f(x_1, \ldots, x_n)\) be a polynomial of degree \(N\). Then \(f\) is completely determined by its values at the points
\[
(k_1 + 1/2, \ldots, k_n + 1/2),
\]
for all \(k\) such that \(|k| \leq N\) and \(k_j \geq 0\).
Proof. The proof is by induction on the number of variables. The case $n = 1$ is trivial. Assume the result is true for polynomials of $n - 1$ variables, and let

$$f = f_N(x_2, \ldots , x_n) + f_{N-1}(x_2, \ldots , x_n)x_1 + \cdots + f_0 x_1^N.$$  

Note that degree $f_i = i$.

Evaluating $f$ at $(k+1/2, 1/2, \ldots , 1/2)$, $0 \leq k \leq N$ determines $f_i(1/2, 1/2, \ldots , 1/2)$, $i = 0, \ldots , N$, and in particular determines $f_0$.

Evaluating $f - f_0 x_1^N$ at $(k+1/2, k_2 + 1/2, \ldots , k_n + 1/2)$, $0 \leq k \leq N - 1$, $k_2 + \cdots + k_n \leq 1$ determines $f_i(k_2 + 1/2, \ldots , k_n + 1/2)$ at all $k_2 + \cdots + k_n \leq 1$ and in particular determines $f_1$.

Evaluating $f - f_1 x_1^{N-1} - f_0 x_1^N$ at $(k+1/2, k_2 + 1/2, \ldots , k_n + 1/2)$, $0 \leq k \leq N - 2$, $k_2 + \cdots + k_n \leq 2$ determines $f_i(k_2 + 1/2, \ldots , k_n + 1/2)$ at all $k_2 + \cdots + k_n \leq 2$ and in particular determines $f_2$. Etc.

\[ \Box \]

When $d$ is odd our methods recover the classical Birkhoff normal form of $H$ except for its monomials of top degree, $\frac{d-1}{2}$.

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