An algorithm of computing special values of Dwork’s $p$-adic hypergeometric functions in polynomial time

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Abstract

Dwork’s $p$-adic hypergeometric function is defined to be a ratio $sF_{s-1}(t)/sF_{s-1}(tp)$ of hypergeometric power series. Dwork showed that it is a uniform limit of rational functions, and hence one can define special values on $|t|_p = 1$. However to compute the value modulo $p^n$ in the naive method, the bit complexity increases by exponential when $n \to \infty$. In this paper we present a certain algorithm whose complexity increases at most $O(n^4(\log n)^3)$. The idea is based on the theory of rigid cohomology.

1 Introduction

Let $s \geq 2$ be an integer. Let $a = (a_1, \ldots, a_s) \in \mathbb{Z}_p^s$. Let

$$F_a(t) := sF_{s-1}\left(\frac{a_1}{1}, \ldots, \frac{a_s}{1}; t\right) = \sum_{n=0}^\infty \frac{(a_1)_n}{n!} \cdots \frac{(a_s)_n}{n!} t^n \in \mathbb{Z}_p[[t]]$$

be the hypergeometric power series where $(\alpha)_n$ denotes the Pochhammer symbol,

$$(\alpha)_n := \alpha(\alpha+1) \cdots (\alpha+n-1), \quad (\alpha)_0 := 1.$$ 

Let $p$ be a prime number. For $\alpha \in \mathbb{Z}_p$, let $\alpha'$ be the Dwork prime which is defined to be $(\alpha + k)/p$ with $k \in \{0, 1, \ldots, p-1\}$ such that $\alpha + k \equiv 0 \mod p$. The ratio

$$\mathcal{F}_a^{Dw}(t) := \frac{F_a(t)}{F_a(tp)}, \quad a := (a'_1, \ldots, a'_s)$$

is called Dwork’s $p$-adic hypergeometric function. In his seminal paper [Dw], Dwork discovered a sequence of rational functions which converges $\mathcal{F}_a^{Dw}(t)$. More precisely let $F_a(t) = \sum_i A_i t^i$ and $F_a(tp) = \sum_i A'_i t^i$. For a power series $f(t) = \sum_{i \geq 0} a_i t^i$, we denote $f(t)_{<m} := \sum_{i<m} a_i t^i$ the truncated polynomial. Then the Dwork congruence (Theorem 2.1)
asserts,
\[ F_{D_w}(t) \equiv \sum_{i=0}^{p-1} A_i t^i \mod p\mathbb{Z}_p[[t]], \]
\[ F_{D_w}(t) \equiv \frac{\sum_{i=0}^{p^2-1} A_i t^i}{\sum_{i=0}^{p-1} A'_i t^i} \mod p^2\mathbb{Z}_p[[t]], \]
\[ F_{D_w}(t) \equiv \frac{\sum_{i=0}^{p^3-1} A_i t^i}{\sum_{i=0}^{p^2-1} A'_i t^i} \mod p^3\mathbb{Z}_p[[t]], \]
\[ \vdots \]
Thus \( F_{D_w}(t) \) is a \( p \)-adic holomorphic function in the sense of Krasner, and one can define the special value at \( t = \alpha \in \mathbb{C}_p := \hat{\mathbb{Q}}_p \) by
\[ F_{D_w}(t)|_{t=\alpha} = \lim_{n \to \infty} \left( \frac{\sum_{i=0}^{p^n-1} A_i \alpha^i}{\sum_{i=0}^{p^n-1} A'_i \alpha^i} \right) \tag{1.1} \]
under the condition
\[ \left| \sum_{i=0}^{p^n-1} A_i \alpha^i \right|_p = 1, \quad \forall \ n \geq 1 \tag{1.2} \]
where \( | \cdot |_p \) denotes the \( p \)-adic valuation on \( \mathbb{C}_p \). One finds that the degrees of polynomials in the limit increase by exponential order, and then the coefficients \( A_i, A'_i \) get larger very quickly,
\[ (a)_p^n \sim (p^n)! \sim e^{p^n(n \log p - 1)} \quad \text{ (Stirling)}. \]
We note that the bit complexity for computing \( (a)_p^n \) is \( O(n^2 p^{2n}) \) (by the naive multiplication algorithm). This causes a serious difficulty for the purpose of explicit computations.

The aim of this paper is to present a certain algorithm for computing the special values in case \( s = 2 \), away from (1.1).

**Main result.** Let \( N, M \geq 2 \) be integers. Let \( a \in \frac{1}{N}\mathbb{Z}, \ b \in \frac{1}{M}\mathbb{Z} \) with \( 0 < a, b < 1 \). Suppose that \( p > \max(N, M) \) (hence \( p \neq 2 \)). Let \( W = W(F_p) \) be the Witt ring of \( F_p \). Let \( \alpha \in W^\times \setminus (1 + pW) \) be an arbitrary element satisfying (1.2). Then there is an algorithm of computing the special value
\[ F_{a,b}(t)|_{t=\alpha} \mod p^n W \]
such that the bit complexity (for fixed \( a, b, p, \alpha \)) is at most \( O(n^4 (\log n)^3) \) as \( n \to \infty \).

The algorithm is displayed in §5.2.

Let us see the examples in case that \( a = b = 1/2 \) and \( p^n = 5^{20} \sim 9.5367 \times 10^{13} \). It is almost impossible to compute
\[ \left( \frac{1}{2} \right)_{5^{20}-1}, \left( \frac{1}{2} \right)_{5^{20}-2}, \ldots \]
modulo $5^{20}$ by an ordinary PC in a direct way, because they are too large. On the other hand, our algorithm allows to compute in a few seconds, e.g.

$$F_{Dw}^{12}(t)_{t=2} \equiv 7213582472073 \mod 5^{20}$$
$$F_{Dw}^{12}(t)_{t=3} \equiv 22359491081212 \mod 5^{20}$$
$$F_{Dw}^{12}(t)_{t=4} \equiv 65856465245823 \mod 5^{20}$$

The algorithm is elementary (so that computers can read and work), while the idea and proofs are entirely arithmetic geometry. The central role in this paper is played by the hypergeometric fibration

$$f : X \longrightarrow \text{Spec} A = \text{Spec} W[t, (t - t^2)^{-1}], \quad f^{-1}(t) = \{(1 - x^N)(1 - y^M) = t\}$$

and its 1st rigid cohomology

$$H^1_{\text{rig}}(X_{\mathbb{F}_p}/A_{\mathbb{F}_p}).$$

Dwork’s $F_{Dw}^{a,b}(t)$ appears in the representation matrix of the Frobenius $\Phi$ on the rigid cohomology. The geometry of the hypergeometric fibration imposes the conditions such as “$p > \max(N, M)$”. The key point is the fact that the entries of $\Phi$ have nice $p$-adic expansions according to [KT, Theorem 2.1], e.g.

An entry of $\Phi \equiv \frac{\text{polynomial of degree} \leq p e_n + p}{(1 - c t^p)(1 - t)^{p e_n}} \mod p W[[t]]$

$$e_n := \max\{k \in \mathbb{Z}_{\geq 1} \mid \text{ord}_p(p^k/k!) < n\} \sim \frac{p - 1}{p - 2} n.$$ 

See Theorem [5.1] for the detail. Thus one can compute the special values from the above, away from (1.1). We hope to obtain a generalization of the algorithm for $F_{Dw}^{a,b}(t)$ with $s \geq 3$, by discussing the rigid cohomology of a higher dimensional fibration

$$(1 - x_1^{N_1}) \cdots (1 - x_s^{N_s}) = t,$$

though I have not worked out.

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2 Dwork’s $p$-adic Hypergeometric functions

Let $p$ be a prime number. Let $\mathbb{Z}_p$ be the ring of $p$-adic integers, and $\mathbb{Q}_p$ the fractional field. Let $\overline{\mathbb{C}}_p$ be the completion of $\mathbb{Q}_p$. Write $O_{\overline{\mathbb{C}}_p} = \{|x|_p \leq 1\}$ the valuation ring.
2.1 Definition

For an integer \( n \geq 0 \), we denote by \((\alpha)_n\) the Pochhammer symbol, which is defined by

\[
(\alpha)_n := \alpha(\alpha + 1) \cdots (\alpha + n - 1), \quad (\alpha)_0 := 1.
\]

Let \( s \geq 2 \) be an integer. For \((a_1, \ldots, a_s) \in \mathbb{Q}_p^s\) and \((b_1, \ldots, b_{s-1}) \in (\mathbb{Q}_p \setminus \mathbb{Z}_{\leq 0})^{s-1}\), the hypergeometric power series is defined to be

\[
s {F}_{s-1} \left( \frac{a_1, \ldots, a_s}{b_1, \ldots, b_{s-1}} ; t \right) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_s)_n}{(b_1)_n \cdots (b_{s-1})_n} \frac{t^n}{n!} \in \mathbb{Q}_p[[t]].
\]

In this paper, we only consider the series

\[
F_\mathbf{a}(t) := s {F}_{s-1} \left( \frac{a_1, \ldots, a_s}{1, \ldots, 1} ; t \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_s)_n}{n!} t^n \in \mathbb{Z}_p[[t]]
\]

for \( \mathbf{a} = (a_1, \ldots, a_s) \in \mathbb{Z}_p^s\).

For \( \alpha \in \mathbb{Z}_p \), let \( \alpha' \) denote the Dwork prime, which is defined to be \( (\alpha + k)/p \) where \( k \in \{0, 1, \ldots, p-1\} \) such that \( \alpha + k \equiv 0 \mod p \). Define the \( i \)-th Dwork prime by \( \alpha^{(i)} = (\alpha^{(i-1)})' \) and \( \alpha^{(0)} := \alpha \). Write \( \mathbf{a}' = (a_1', \ldots, a_s') \) and \( \mathbf{a}^{(i)} = (a_1^{(i)}, \ldots, a_s^{(i)}) \). Dwork’s \( p \)-adic hypergeometric function is defined to be a power series

\[
\mathcal{F}_\mathbf{a}^{\text{Dw}}(t) := \frac{F_\mathbf{a}(t)}{F_\mathbf{a}'(t^p)} \in \mathbb{Z}_p[[t]].
\]

A slight modification is

\[
\mathcal{F}_\mathbf{a}^{\text{Dw}, \sigma}(t) := \frac{F_\mathbf{a}(t)}{F_\mathbf{a}'(t^\sigma)} \in W[[t]]
\]

for a \( p \)-th Frobenius \( \sigma \) on \( W[[t]] \) given by \( \sigma(t) = c t^p \), \( c \in pW \), where \( W = W(\overline{\mathbb{F}}_p) \) is the Witt ring of \( \mathbb{F}_p \).

2.2 Dwork’s congruence relations

In general, neither of the power series \( F_\mathbf{a}(t) \mod p\mathbb{Z}_p[[t]] \) or \( \mathcal{F}_\mathbf{a}^{\text{Dw}}(t) \mod p\mathbb{Z}_p[[t]] \) terminate (e.g. [A] (4.28)). Therefore one cannot substitute \( t = 0 \in W \) in \( F_\mathbf{a}(t) \) or \( \mathcal{F}_\mathbf{a}^{\text{Dw}}(t) \) directly unless \( \alpha \in pW \). In his seminal paper [Dw], Dwork showed that there is a sequence of rational functions which converges to \( \mathcal{F}_\mathbf{a}^{\text{Dw}}(t) \), namely it is a \( p \)-adic analytic function in the sense of Krasner.

**Theorem 2.1 (Dwork’s congruence relations)** For a power series \( f(t) = \sum_{i \geq 0} a_i t^i \), we denote \( f(t)_{<k} = [f(t)]_{<k} = \sum_{0 \leq i < k} a_i t^i \) the truncated polynomial. Let \( \sigma(t) = c t^p \) with \( c \in 1 + pW \). Then

\[
\mathcal{F}_\mathbf{a}^{\text{Dw}, \sigma}(t) \equiv \frac{F_\mathbf{a}(t)_{<p^m}}{[F_\mathbf{a}'(t^\sigma)]_{<p^m}} \mod p^n W[[t]] \quad (2.1)
\]
for any \( n \geq 1 \). Hence for \( \alpha \in \mathcal{O}_{\mathbb{C}_p} \) satisfying

\[
F_\mathcal{D}^{\sigma}(t)\big|_{t=\alpha} \not\equiv 0 \mod m_{\mathbb{C}_p}, \quad \forall n \geq 1
\]

(2.2)

where \( m_{\mathbb{C}_p} := \{ |x|_p < 1 \} \) is the maximal ideal, one can define a special value of \( F_\mathcal{D}^{\sigma}(t) \) at \( t = \alpha \) by

\[
F_\mathcal{D}^{\sigma}(t)\big|_{t=\alpha} = F_\mathcal{D}^{\sigma}(\alpha) = \lim_{n \to \infty} \left( \frac{F_\mathcal{D}^{\sigma}(t)}{\left( F_\mathcal{D}^{\sigma}(ct) \right)^{\left( ct \right)^n}} \right)_{t=\alpha}.
\]

Remark 2.2 One cannot substitute \( t = \alpha \) in \( F_\mathcal{D}(t) \) since it is not a \( p \)-adic analytic function. For example, suppose \( \mathfrak{a}' = \mathfrak{a} \) and \( p \neq 2 \), the following is wrong!

\[
F_\mathcal{D}^{\sigma}(-1) = \frac{F_\mathcal{D}(-1)}{F_\mathcal{D}(-1)^p} = \frac{F_\mathcal{D}(-1)}{F_\mathcal{D}(-1)} = 1.
\]

Proof. When \( c = 1 \), this is proven in [Dw, p.37, Thm. 2, p.45]. The general case can be reduced to the case \( c = 1 \) in the following way. Since \( F_\mathcal{D}^{\sigma}(t) = F_\mathcal{D}^{\sigma}(t) \cdot F_\mathcal{D}^{\sigma}(t^p) / F_\mathcal{D}^{\sigma}(ct^p) \), it is enough to show that

\[
\frac{F_\mathcal{D}(t)}{F_\mathcal{D}^{\sigma}(ct)} = \frac{F_\mathcal{D}(t)}{F_\mathcal{D}^{\sigma}(ct)^{\left( ct \right)^n}} \mod p^{n+1}W[\lceil t \rceil]
\]
in general. Let \( F_\mathcal{D}(t) = \sum_i A_i t^i \). Then the above is equivalent to that

\[
\sum_{i+j=m, i,j \geq 0} A_{i+p^n}(c^j A_j) - A_i(c^{j+p^n} A_{j+p^n}) \equiv 0 \mod p^{n+1}
\]

for any \( m \geq 0 \). However this is obvious as \( c^p \equiv 1 \mod p^{n+1} \). \( \square \)

Corollary 2.3

\[
F_\mathcal{D}(t)_{\mid_{t=\alpha}}^{\sigma} \equiv F_\mathcal{D}(t)_{\mid_{t=\alpha}}^{\sigma}(F_\mathcal{D}(t)_{\mid_{t=\alpha}}^{\sigma})^{p} \cdots (F_\mathcal{D}(t^{(n-1)})_{\mid_{t=\alpha}}^{\sigma})^{p^{n-1}} \mod p\mathbb{Z}_p[\lceil t \rceil]. \quad (2.3)
\]

The condition (2.2) holds if and only if

\[
F_\mathcal{D}(t)_{\mid_{t=\alpha}}^{\sigma} \not\equiv 0 \mod m_{\mathbb{C}_p}, \quad \forall i \geq 0.
\]

Moreover we have

\[
F_\mathcal{D}(t)_{\mid_{t=\alpha}}^{\sigma} \in W[t, h(t)^{-1}]^{\wedge} := \lim_{n \to \infty} \left( W/p^n W[t, h(t)^{-1}] \right), \quad h(t) := \prod_{i=0}^{N} F_\mathcal{D}(t)_{\mid_{t=\alpha}}^{\sigma}(t)^{p} \quad (2.5)
\]

with some \( N \gg 0 \). In particular this is a \( p \)-adic analytic function in the sense of Krasner.

Proof. It follows from (2.1) that one has

\[
\frac{F_\mathcal{D}(t)_{\mid_{t=\alpha}}^{\sigma}}{\left( F_\mathcal{D}(t^{(p)})_{\mid_{t=\alpha}}^{\sigma} \right)^{\left( t^{(p)} \right)^n}} = \frac{F_\mathcal{D}(t)_{\mid_{t=\alpha}}^{\sigma}}{\left( F_\mathcal{D}(t)_{\mid_{t=\alpha}}^{\sigma} \right)^{\left( t^{(p)} \right)^{p^{n-1}}}} = F_\mathcal{D}(t)_{\mid_{t=\alpha}}^{\sigma} \mod p\mathbb{Z}_p[\lceil t \rceil].
\]

Then one can show (2.3) by induction on \( n \). Notice that a set \( \{ F_\mathcal{D}(t)_{\mid_{t=\alpha}}^{\sigma} \mod p \}_{i \geq 0} \) of polynomials with \( \mathbb{F}_p \)-coefficients has a finite cardinal. Therefore (2.4) is a condition for finitely many \( i \)'s. (2.3) is now immediate. \( \square \)
Theorem 2.4 Let $f^{(j)}(t) = \frac{d^j}{dt^j} f(t)$ denote the $j$-th derivative. Then

$$\frac{F_{a}^{(j)}(t)}{F_{a}(t)} = \frac{F_{a}^{(j)}(t) \mod p^n}{F_{a}(t) \mod p^n} \mod p^n \mathbb{Z}_p[[t]]$$

for all $n \geq 1$. Hence

$$\frac{F_{a}^{(j)}(t)}{F_{a}(t)} \in W[t, h(t)^{-1}]$$

is a $p$-adic analytic function in the sense of Krasner, and one can define the special values by (2.6).

Proof. [Dw] p.45, Lem.3.4. □

3 Hypergeometric Fibrations

For a smooth scheme $X$ over a commutative ring $A$, we denote by $H_{\text{dR}}^*(X/A) := H^*_{\text{zar}}(X, \Omega^\bullet_{X/A})$ the algebraic de Rham cohomology groups.

3.1 Setting

Let $N, M \geq 2$ be an integer. Let $W$ be a commutative ring such that $NM$ is invertible. Suppose that $W$ contains a primitive $\text{lcm}(N, M)$-th root of unity. Later we shall take $W$ to be the Witt ring of a perfect field of characteristic $p$. Let $P := P_1 \times P_1 \times P_1$ be the product of the projective lines over $W$ with homogeneous coordinates $(X_0, X_1) \times (Y_0, Y_1) \times (T_0, T_1)$. We use inhomogeneous coordinates $x := X_1/X_0$, $y := Y_1/Y_0$, $t := T_1/T_0$ and $z := x^{-1}$, $w := y^{-1}$, $s := t^{-1}$. Let $Y_s \subset P$ be the closed subscheme defined by a homogeneous equation

$$T_0(X_0^N - X_1^N)(Y_0^M - Y_1^M) = T_1 X_0^N Y_0^M$$

over $W$. Let

$$f_s : Y_s \longrightarrow \mathbb{P}^1 = \text{Proj} W[T_0, T_1]$$

be the projection onto the 3rd line. Put $A := W[t, (t-t^2)^{-1}]$, $U := \text{Spec} A \subset \mathbb{P}^1$ and

$$X := f_s^{-1}(U) = \text{Spec} A[x, y]/((1 - x^N)(1 - y^M) - t).$$

Then $X \rightarrow U$ is smooth and projective, and a geometric fiber is a connected smooth projective curve of genus $(N - 1)(M - 1)$ (e.g. the Hurwitz formula). An open set $Y_s \setminus f_s^{-1}(s = 0)$ is smooth over $W$ where “$s = 0$” denotes the closed subscheme $\text{Spec} W[s]/(s) \subset \mathbb{P}^1$. There are singular loci $\{s = 1 - z^N = w = 0\}$ and $\{s = z = 1 - w^M = 0\}$ in the affine open set

$$\text{Spec} W[s, z, w]/(s(1 - z^N)(1 - w^M) - z^N w^M) \subset Y_s.$$
All the singularities are of type \( xy = z^k \) where \( k = N \) or \( k = M \). One can resolve them according to Propositions 7.1 in Appendix B. The fiber \( f_s^{-1}(0) \) at \( t = 0 \) is a relative simple normal crossing divisor (abbreviated NCD) over \( W \) (see Appendix B for the definition), and all components are \( \mathbb{P}^1 \). The fiber \( f_s^{-1}(1) \) at \( t = 1 \) is an integral divisor which is smooth outside the point \( (x, y, t) = (0, 0, 1) \). The normalization of \( f_s^{-1}(1) \) is the Fermat curve \( z^N + w^M = 1 \). In a neighborhood of the point \( (x, y, t) = (0, 0, 1) \), the fiber \( f_s^{-1}(1) \subset Y_s \) is defined by \( x^N + y^M - x^N y^M = 0 \iff (x')^N + y^M = 0, x' := x(1 - y^M)^{\frac{1}{N}} \). One can further resolve it according to Propositions 7.2 in Appendix B.

Summing up the above, we have a smooth projective \( W \)-scheme \( Y \) with a fibration

\[
f : Y \longrightarrow \mathbb{P}^1 = \text{Proj} W[T_0, T_1]
\]

which satisfies the following conditions. Let \( D_0 := f^{-1}(0), D_1 := f^{-1}(1) = \sum n_i D_{1,i} \) and \( D_\infty := f^{-1}(\infty) = \sum_j m_j D_{\infty,j} \) denote the fibers at \( \text{Spec} W[t]/(t) \), \( \text{Spec} W[t]/(t-1) \) and \( \text{Spec} W[s]/(s) \) respectively.

(i) \( f \) is smooth over \( U = \text{Spec} W[t, (t-t^2)^{-1}] \subset \mathbb{P}^1 \), and \( X = f^{-1}(U) = f^{-1}(U) \).

(ii) \( D_0 \) and \( \sum_i D_{1,i} \) and \( \sum_j D_{\infty,j} \) are simple relative NCD’s over \( W \).

(iii) The multiplicities \( n_i \) of \( D_1 \) are either of \( 1, iN, jM \) with \( i \in \{1, \ldots, M\}, j \in \{1, \ldots, N\} \).

(iv) The multiplicities of \( m_j \) of \( D_\infty \) are integers \( \leq \max(N, M) \).

(v) Any components of \( D_0 \) or \( D_\infty \) are \( \mathbb{P}^1 \). There is a unique component of \( D_1 \) which is not \( \mathbb{P}^1 \). It is the Fermat curve \( z^N + w^M = 1 \).

Let \( \mu_n := \{ \zeta \in W^\times | \zeta^n = 1 \} \) denote the group of \( n \)-th roots of 1. For \( (\zeta_1, \zeta_2) \in \mu_N \times \mu_M \), the morphism \( (x, y, t) \rightarrow (\zeta_1 x, \zeta_2 y, t) \) extends to an automorphism on \( Y \) or \( X \), which we write by \( [\zeta_1, \zeta_2] \).

### 3.2 \( H^1_{\text{dr}}(X/A) \)

Let \( U_0 \) and \( U_1 \) be the affine open sets of \( X \) defined by \( X_0 Y_0 \neq 0 \) and \( X_1 Y_1 \neq 0 \) respectively,

\[
U_0 = \text{Spec} A[x, y]/((1 - x^N)(1 - y^M) - t),
\]

\[
U_1 = \text{Spec} A[z, w]/((1 - z^N)(1 - w^M) - tz^N w^M).
\]

Then \( X = U_0 \cup U_1 \). For \( i \in \{1, \ldots, N - 1\} \) and \( j \in \{1, \ldots, M - 1\} \) let

\[
\omega_{ij} := N \frac{x^{i-1} y^{j-M}}{1 - x^N} dx = -M \frac{x^{i-N} y^{j-1}}{1 - y^M} dy
\]

be rational relative 1-forms on \( X/A \).

**Lemma 3.1** \( \omega_{ij} \in \Gamma(X, \Omega^1_{X/A}) \).
Proof. Multiplying $x^i y^j$ on

$$t(1-t) \frac{y^{-M}}{1-x^N} \frac{dx}{x} = t \frac{dx}{x} - MN^{-1}(1-x^N) \frac{dy}{y}$$

one sees $\omega_{ij} \in \Gamma(U_0, \Omega^1_{X/A})$. Similarly, using an equality

$$\frac{1}{1-z^N} \frac{dz}{z} = (1-(1-t^{-1})(1-w^M)) \frac{dz}{z} - MN^{-1} \frac{dw}{w}$$

one sees $\omega_{ij} \in \Gamma(U_1, \Omega^1_{X/A})$. □

Lemma 3.2 Let $H^1(X, \mathcal{O}_X)$ be the Zariski cohomology which is isomorphic to the cokernel of the Cech complex

$$\delta : \Gamma(U_0, \mathcal{O}_X) \oplus \Gamma(U_1, \mathcal{O}_X) \to \Gamma(U_0 \cap U_1, \mathcal{O}_X), \quad (u_0, u_1) \mapsto u_1 - u_0.$$ Write $[f] := f \mod \text{Im} \delta \in H^1(\mathcal{O}_X)$. Then $H^1(X, \mathcal{O}_X)$ is generated as $A$-module by elements

$$[x^i y^{j-M}], \quad i \in \{1, \ldots, N-1\}, \quad j \in \{1, \ldots, M-1\}.$$ Moreover for any integers $k, l$, there is an element $\alpha \in A$ such that $[x^{i+kN} y^{j+lM}] = \alpha [x^i y^{j-M}]$ in $H^1(X, \mathcal{O}_X)$.

Proof. We first note that if $k, l \leq 0$ or $k, l \geq 0$ then $[x^k y^j] = 0$ by definition. Let $i, j$ be integers such that $1 \leq i \leq N-1$ and $1 \leq j \leq M-1$. Since $1-t = x^N + y^M - x^N y^M$, one has

$$(1-t)^k [x^i y^{j-M}] = [x^{kN} x^i y^{j-M}] = [x^{i+kN} y^{j-M}], \quad \forall k \geq 0. \quad (3.2)$$

Let $l \geq 1$. Then $(1-t)x^{i-N} y^{j-lM} = x^i y^{j-M} + x^{i-N} y^{j-(l-1)M} - x^i y^{j-(l-1)M}$, and this implies

$$[x^i y^{j-lM}] = [x^i y^{j-(l-1)M}], \quad \forall l \geq 2, \quad (3.3)$$

and for $l = 1$

$$[x^i y^{j-M}] + [x^{i-N} y^j] = 0. \quad (3.4)$$

We claim

$$[x^{i+kN} y^{j-lM}] \in A[x^i y^{j-M}], \quad \forall k \geq 0, \quad l \geq 1. \quad (3.5)$$

If $l = 1$, this is nothing other than (3.2). If $k = 0$, this follows from (3.3). Suppose $k \geq 1$ and $l \geq 2$. Then

$$(1-t)[x^{i+(k-1)N} y^{j-lM}] = [x^{i+kN} y^{j-lM}] + [x^{i+(k-1)N} y^{j-(l-1)M}] - [x^{i+kN} y^{j-(l-1)M}]$$

Hence (3.5) follows by induction on $k + l$. In the same way, one can show $[x^{i-kN} y^{j+lM}] \in A[x^{i-N} y^j]$ for all $k \geq 1$ and $l \geq 0$. Therefore $[x^{i-kN} y^{j+lM}] \in A[x^i y^{j-M}]$ by (3.4). This completes the proof. □
Proposition 3.3  

1. $\Gamma(X, \Omega^1_{X/A})$ is a free $A$-module with basis

$$ \omega_{ij}, \quad i \in \{1, \ldots, N-1\}, \ j \in \{1, \ldots, M-1\}. $$

2. $H^1(X, \mathcal{O}_X)$ is a free $A$-module with basis

$$ [x^iy^j-M], \quad i \in \{1, \ldots, N-1\}, \ j \in \{1, \ldots, M-1\}. $$

Proof. For a point $s \in U = \text{Spec} A$, we denote the residue field by $k(s)$, and write $X_s := X \times_A \text{Spec} k(s)$. Let $q = 0, 1$. Since $\dim_{k(s)} H^q(\Omega^{1-q}_{X_s/k(s)}) = (N-1)(M-1)$ is constant with respect to $s$, one can apply [Ha, III,12.9], so that $H^q(X, \Omega^1_{X/A})$ is a locally free $A$-module and the isomorphism $H^q(\Omega^{1-q}_{X_s/A}) \otimes k(s) \cong H^q(\Omega^{1-q}_{X_s/k(s)})$ follows. Obviously $\omega_{ij}|_{X_s} \neq 0$ and they are linearly independent over $k(s)$ since each $\omega_{ij}$ belongs to the distinct simultaneous eigenspace with respect to $\mu_N \times \mu_M$. Noticing that $\dim H^0(\Omega^1_{X_s/k(s)}) = (N-1)(M-1)$, one sees that $\{\omega_{ij}|_{X_s}\}_{i,j}$ forms a $(s)$-basis of $H^0(\Omega^1_{X_s/k(s)})$, and hence that $\{\omega_{ij}\}_{i,j}$ forms a $A$-basis of $H^0(A, \Omega^1_{X/A})$ by Nakayama’s lemma. This completes the proof of (1). In a similar way, the assertion (2) follows by using Lemma 3.2.

The algebraic de Rham cohomology $H^1_{\text{dR}}(X/A)$ is described in terms of the Cech complexes. Let

$$ \Gamma(U_0, \mathcal{O}) \oplus \Gamma(U_1, \mathcal{O}_X) \xrightarrow{d} \Gamma(U_0, \Omega^1_{X/A}) \oplus \Gamma(U_1, \Omega^1_{X/A}) $$

be a commutative diagram where $d$ is the differential map and $\delta$ is given by $(u_0, u_1) \mapsto u_1 - u_0$. Then the de Rham cohomology $H^1_{\text{dR}}(X/A)$ is isomorphic to the cohomology of the total complex. In particular, an element of $H^1_{\text{dR}}(X/A)$ is given as the representative of a cocycle

$$ (f) \times (\omega_0, \omega_1) \in \Gamma(U_0 \cap U_1, \mathcal{O}_X) \times \Gamma(U_0, \Omega^1_{X/A}) \oplus \Gamma(U_1, \Omega^1_{X/A}) $$

which satisfies $df = \omega_1 - \omega_0$. Let $\omega_{ij} \in \Gamma(X, \Omega^1_{X/A})$ be as in Proposition 3.3. We denote by the same notation $\omega_{ij}$ the element of $H^1_{\text{dR}}(X/A)$ via the natural map $\Gamma(X, \Omega^1_{X/A}) \to H^1_{\text{dR}}(X/A)$, which is the representative of a cocycle

$$ (0) \times (\omega_{ij}|_{U_0}, \omega_{ij}|_{U_1}). $$

We construct a lifting

$$ \eta_{ij} := (x^iy^j-M) \times (\eta_{ij}^0, \eta_{ij}^1) \in H^1_{\text{dR}}(X/A) $$

of $[x^iy^j-M] \in H^1(\mathcal{O}_X)$ in the following way. A direct computation yields

$$ (j-M)(1-t)x^iy^j-M-1dy - d(x^iy^j-M) = - \left( \frac{(j-M)t}{M} + \frac{i}{N}(1-x^N) \right) \omega_{ij}. $$
Note \( x^{-N}y^{-M}dy = -z^{N-i}w^{M-j-1}dw \in \Gamma(U_1, \Omega_{X/A}^1) \), and the right hand side lies in \( \Gamma(U_0, \Omega_{X/A}^1) \) by Lemma 3.1. Therefore we put
\[
\eta_{ij}^0 := -\left( \frac{(j - M)t}{M} + \frac{i}{N}(1 - x^N) \right) \omega_{ij}, \quad \eta_{ij}^1 := -(j - M)(1 - t)z^{N-i}w^{M-j-1}dw,
\]
then we get the desired cocycle (3.6). By Proposition 3.3 (2) together with liftings (3.6), the natural map \( H^1_{\text{dR}}(X/A) \to H^1(\mathcal{O}_X) \) is surjective, and hence one has an exact sequence
\[
0 \to \Gamma(X, \Omega_{X/A}^1) \to H^1_{\text{dR}}(X/A) \to H^1(X, \mathcal{O}_X) \to 0.
\]
Thus we get the following theorem.

**Theorem 3.4** \( H^1_{\text{dR}}(X/A) \) is a free \( A \)-module with basis
\[
\omega_{ij}, \eta_{ij} \quad i \in \{1, \ldots, N - 1\}, \; j \in \{1, \ldots, M - 1\}.
\]

### 3.3 \( H^1(\mathcal{Y}, \Omega_{\mathcal{Y}/W[[\lambda]]}(\log D)) \)

Let \( \lambda \) be an indeterminate. Let \( \text{Spec}W[[\lambda]] \to \mathbb{P}^1 \) be the morphism induced by \( \lambda = t, 1 - t \) or \( t^{-1} \). Let
\[
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
\text{Spec}W[[\lambda]] & \xrightarrow{f} & \mathbb{P}^1
\end{array}
\]
be the base change. Let \( D \subset \mathcal{Y} \) denote the central fiber, namely \( D = D_0, D_1 \) or \( D_{\infty} \) by the notation in §3.1. The reduced part \( D_{\text{red}} \) is a relative simple NCD over \( W \). Put \( \mathcal{Y}' := \mathcal{Y} \setminus D \). Define a \( \mathcal{O}_{\mathcal{Y}} \)-module
\[
\Omega_{\mathcal{Y}/W[[\lambda]]}(\log D) := \text{Coker} \left[ \mathcal{O}_{\mathcal{Y}} \frac{d\lambda}{\lambda} \to \Omega_{\mathcal{Y}/W}(\log D) \right]
\]
and consider the cohomology group
\[
H^1(\mathcal{Y}, \Omega_{\mathcal{Y}/W[[\lambda]]}(\log D)) := H^1_{\text{zar}}(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}} \to \Omega_{\mathcal{Y}/W[[\lambda]]}(\log D)).
\]

**Proposition 3.5** If \( N!M! \) is invertible in \( W \), then \( \Omega_{\mathcal{Y}/W[[\lambda]]}(\log D) \) is a locally free \( \mathcal{O}_{\mathcal{Y}} \)-module.

**Proof.** If \( N!M! \) is invertible in \( W \), then each multiplicity of \( D \) is invertible in \( W \) (see §3.1). The assertion can be checked locally on noticing that \( f \) is given by \( (x_1, x_2) \mapsto \lambda = x_1^{r_1}x_2^{r_2} \) with \( r_1, r_2 \) integers which are invertible in \( W \). \( \square \)
**Theorem 3.6** Suppose that $W$ is an integral domain of characteristic zero, and that $N!M!$ is invertible in $W$. Put

$$H_\lambda := \text{Im}[H^1(\mathcal{O}, \Omega^\bullet_{\mathcal{O}/W[[\lambda]]}(\log D)) \to H^1_{\text{dR}}(\mathcal{O}/W((\lambda)))],$$

$$\text{Fil}^1 H_\lambda := \text{Im}[\Gamma(\mathcal{O}, \Omega^1_{\mathcal{O}/W[[\lambda]]}(\log D)) \to H^1_{\text{dR}}(\mathcal{O}/W((\lambda)))].$$

Then $H_\lambda$ and $\text{Fil}^1 H_\lambda$ are free $W[[\lambda]]$-modules of rank 2 and 1 respectively. More precisely, the following holds.

1. If $\lambda = t$, then $\text{Fil}^1 H_\lambda$ has a $W[[\lambda]]$-basis $\{\omega_{ij}\}$ and $H_\lambda$ has a $W[[\lambda]]$-basis $\{\omega_{ij}, \eta_{ij}\}$ where $(i, j)$ runs over the pairs of integers such that $1 \leq i \leq N-1$ and $1 \leq j \leq M-1$.

2. If $\lambda = s = t^{-1}$, then $\text{Fil}^1 H_\lambda$ has a $W[[\lambda]]$-basis $\{\omega_{ij}\}$ and $H_\lambda$ has a $W[[\lambda]]$-basis $\{\omega_{ij}, s\eta_{ij}\}$.

3. If $\lambda = 1 - t$, set

$$\omega^*_{ij} := \begin{cases} \omega_{ij} & i/N + j/M \geq 1 \\ (1-t)\omega_{ij} & i/N + j/M < 1, \end{cases}$$

$$\eta^*_{ij} := \begin{cases} \eta_{ij} & i/N + j/M \geq 1 \\ (1-i/N - j/M)t\omega_{ij} - \eta_{ij} & i/N + j/M < 1. \end{cases}$$

Then $\text{Fil}^1 H_\lambda$ has a $W[[\lambda]]$-basis $\{\omega^*_{ij}\}$ and $H_\lambda$ has a $W[[\lambda]]$-basis $\{\omega^*_{ij}, \eta^*_{ij}\}$.

The proof of Theorem 3.6 shall be given in later sections.

### 3.4 Preliminary on Proof of Theorem 3.6

Let $U_{kl} = \mathcal{O} \cap \{X_kY_l \neq 0\}$, $k, l \in \{0, 1\}$ be an affine open set. Then $\mathcal{O} = \bigcup_{k=0,1} \bigcup_{l=0,1} U_{kl}$. The cohomology group $H^1(\mathcal{O}, \Omega^\bullet_{\mathcal{O}/W[[\lambda]]}(\log D))$ is isomorphic to the cohomology of the total complex of the double complex

$$\bigoplus \Gamma(U_{ab}, \mathcal{O}/\mathcal{O}) \xrightarrow{d} \bigoplus \Gamma(U_{ab}, \Omega^1_{\mathcal{O}/W[[\lambda]]}(\log D))$$

An element of $H^1(\mathcal{O}, \Omega^\bullet_{\mathcal{O}/W[[\lambda]]}(\log D))$ is represented by a cocycle

$$(f_{ab,cd}) \times (\alpha_{ab}) \in \bigoplus \Gamma(U_{ab} \cap U_{cd}, \mathcal{O}/\mathcal{O}) \times \bigoplus \Gamma(U_{ab}, \Omega^1_{\mathcal{O}/W[[\lambda]]}(\log D))$$

which satisfies $f_{ab,ef} = f_{ab,cd} + f_{cd,ef}$ and

$$\alpha_{cd}|_{U_{ab} \cap U_{cd}} - \alpha_{ab}|_{U_{ab} \cap U_{cd}} = d(f_{ab,cd}).$$
If we replace $\mathcal{O}_Y$ with $\mathcal{O}_X$ and $\Omega^1_{\mathcal{Y}/W[[\lambda]]}(\log D)$ with $\Omega^1_{\mathcal{X}/W((\lambda))}$ in the above, we obtain the algebraic de Rham cohomology group $H^1_{\text{dR}}(\mathcal{X}/W((\lambda)))$. Let $\omega_{ij}$, $\eta_{ij} \in H^1_{\text{dR}}(X/A)$ be as in (3.1) and (3.6). Then $\omega_{ij} |_{\mathcal{X}} \in H^1_{\text{dR}}(\mathcal{X}/W((\lambda)))$ is represented by

$$
(0) \times (\omega_{ij}|_{U_{ab}}) \in \bigoplus \Gamma(U_{ab} \cap U_{cd}, \mathcal{O}_X) \times \bigoplus \Gamma(U_{ab}, \Omega^1_{\mathcal{X}/W((\lambda))}).
$$

A cocycle which represents $\eta_{ij} |_{\mathcal{X}}$ is given as follows. We note that

$$x^{i-N}y^{j-M-1}dy = -z^{N-i}w^{M-j-1}dw \in \Gamma(U_{11}, \Omega^1_{X/A})$$

and

$$x^{i-N}y^{j-M-1}dy = -x^{i-N}w^{M-j-1}dw$$

$$= -(x^N + (1-t)w^M - x^Nw^M)x^{i-N}w^{M-j-1}dw$$

$$= -(1 - w^M)x^{i}w^{M-j-1}dw - (1-t)t^{-1}N\frac{1}{M}(1-w^M)^2x^{i-1}w^{M-j}dx$$

$$\in \Gamma(U_{01}, \Omega^1_{X/A})$$

where the 2nd equality follows from $1 = x^N + (1-t)w^M - x^Nw^M$. On the other hand $x^{i-N}y^{j-M-1}dy \notin \Gamma(U_{00}, \Omega^1_{X/A})$ while we have (3.7). Moreover $x^{i-N}y^{j-M-1}dy \notin \Gamma(U_{10}, \Omega^1_{X/A})$ while we have

$$(j-M)z^{N-i}y^{j-M-1}dy - (1-t)d(z^{N-i}y^{j-M}) = z^N\left(\frac{(j-M)t}{M} + \frac{(i-2N)}{N}(1-t)(1-z^N)\right)\omega_{ij}.$$ 

Therefore we put

$$\eta_{ij}^{11} := -(j-M)(1-t)z^{N-i}w^{M-j-1}dw,$$

$$\eta_{ij}^{01} := -(j-M)(1-t)x^{i-N}w^{M-j-1}dw,$$

$$\eta_{ij}^{00} := -\left(\frac{(j-M)t}{M} + \frac{i}{N}(1-x^N)\right)\omega_{ij},$$

$$\eta_{ij}^{10} := (1-t)z^N\left(\frac{(j-M)t}{M} + \frac{(i-2N)}{N}(1-t)(1-z^N)\right)\omega_{ij}$$

$$= (1-y^M + z^Ny^M)\left(\frac{(j-M)t}{M} + \frac{(i-2N)}{N}(1-t)(1-z^N)\right)\omega_{ij}$$

and

$$f_{00,11} = f_{00,01} := x^{i}y^{j-M}, \quad f_{10,11} = f_{10,01} := (1-t)^2z^{2N-i}y^{j-M} = (1-t)(1-y^M + z^Ny^M)z^{N-i}y^{j-M},$$

$$f_{01,11} := 0, \quad f_{00,10} := x^{i}y^{j-M} - (1-t)^2z^{2N-i}y^{j-M} = (1-x^N)(x^Ny^M - 2x^Ny^M)z^{2N-i}y^j$$

and $f_{11,00} := -f_{00,11}$ etc. Then we get a cocycle

$$(f_{ab,cd}) \times (\eta_{ij}^{ab}) \in \bigoplus \Gamma(U_{ab} \cap U_{cd}, \mathcal{O}_X) \times \bigoplus \Gamma(U_{ab}, \Omega^1_{\mathcal{X}/W((\lambda))})$$

which represents $\eta_{ij} |_{\mathcal{X}} \in H^1_{\text{dR}}(\mathcal{X}/W((\lambda)))$. 

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3.5 Deligne’s canonical extension

Let \( j : \text{Spec} \mathbb{C}((\lambda)) \to \text{Spec} \mathbb{C}[[\lambda]] \). Let \( (\mathcal{H}, \nabla) \) be an integrable connection on \( \text{Spec} \mathbb{C}((\lambda)) \). There is a unique subsheaf \( \mathcal{H}_e \subset \mathcal{H} \) which satisfies the following conditions (cf. \[Z\] (17)).

(D1) \( \mathcal{H}_e \) is a free \( \mathbb{C}[[\lambda]] \)-module such that \( j^{-1} \mathcal{H}_e = \mathcal{H} \),

(D2) the connection extends to have log pole, \( \nabla : \mathcal{H}_e \to \frac{d\lambda}{\lambda} \otimes \mathcal{H}_e \),

(D3) each eigenvalue \( \alpha \) of \( \text{Res}(\nabla) \) satisfies \( 0 \leq \text{Re}(\alpha) < 1 \), where \( \text{Res}(\nabla) \) is the map defined by a commutative diagram

\[
\begin{array}{ccc}
\mathcal{H}_e & \xrightarrow{\nabla} & \mathcal{H}_e \\
\downarrow & & \downarrow \\
\mathcal{H}_e \otimes \mathcal{H}_e & \xrightarrow{\text{Res}(\nabla)} & \mathcal{H}_e \otimes \mathcal{H}_e.
\end{array}
\]

The extended bundle \( (\mathcal{H}_e, \nabla) \) is called Deligne’s canonical extension.

Let \( g : V \to \text{Spec} \mathbb{C}[[\lambda]] \) be a projective flat morphism which is smooth over \( \text{Spec} \mathbb{C}((\lambda)) \). Let \( D \) be the central fiber. Suppose that \( D_{\text{red}} \) is a NCD. We define a locally free \( \mathcal{O}_V \)-module

\[
\Omega^1_{V/\mathbb{C}[[\lambda]]}(\log D) := \text{Coker} \left( \mathcal{O}_V \frac{d\lambda}{\lambda} \to \Omega^1_{V/\mathbb{C}}(\log D) \right)
\]

and \( \Omega^k_{V/\mathbb{C}[[\lambda]]}(\log D) := \wedge^k \Omega^1_{V/\mathbb{C}[[\lambda]]}(\log D) \).

Let \( U : V \setminus D \) and let \( (\mathcal{H}, \nabla) = (H^1_{\text{dR}}(U/\mathbb{C}((\lambda))), \nabla) \) be the Gauss-Manin connection on \( \text{Spec} \mathbb{C}((\lambda)) \). Then Deligne’s canonical extension of \( \mathcal{H} \) is given as follows (\[S\], (2.18)–(2.20)),

\[
\mathcal{H}_e \cong H^1(V, \Omega^1_{V/\mathbb{C}[[\lambda]]}(\log D)).
\]

Moreover \( \exp(-2\pi i \text{Res}_P(\nabla)) \) agrees with the monodromy operator on \( H_\mathcal{C} = \text{Ker}(\nabla^an) \) around \( \lambda = 0 \) (cf. \[S\], (2.21)).

We turn to our family \( \mathcal{Y} \to \text{Spec} \mathbb{W}[[\lambda]] \). Let \( K := \text{Frac}(\mathbb{W}) \) be the fractional field. The characteristic of \( K \) is zero by the assumption in Theorem \[3.6\]. Put \( \mathcal{Y}_K := \mathcal{Y} \times_{\mathbb{W}[[\lambda]]} K[[\lambda]], \mathcal{X}_K := \mathcal{X} \times_{\mathbb{W}[[\lambda]]} K[[\lambda]] \) and \( D_K := D \times_{\mathbb{W}} K \). Let \( (H^1_{\text{dR}}(\mathcal{X}_K/K((\lambda))), \nabla) \) be the Gauss-Manin connection on \( \text{Spec} K((\lambda)) \).

**Proposition 3.7** Let \( \nabla : H^1_{\text{dR}}(X/A) \to \text{Adt} \otimes H^1_{\text{dR}}(X/A) \) be the Gauss-Manin connection. Then

\[
(\nabla(\omega_{ij}), \nabla(\eta_{ij})) = dt \otimes \begin{pmatrix} \omega_{ij} & \eta_{ij} \end{pmatrix} \begin{pmatrix} 0 & (1-i/N)(1-j/M) \\ (t-t^2)^{-1} & (1-i/N-j/M)(1-t)^{-1} \end{pmatrix}.
\]

**Proof.** \([A]\) Proposition 4.15. \(\square\)
**Proposition 3.8** Put Deligne’s canonical extension

\[ H_{\lambda,K} := H^1(\mathcal{O}_K, \Omega^*_{\mathcal{O}_K/K[\lambda]}(\log D_K)) \subset H^1_{\text{dR}}(\mathcal{X}_K/K((\lambda))). \]  \hspace{1cm} (3.9)

Then the \(K[\lambda]\)-basis is given as follows.

1. If \(\lambda = t\), then
   \[ H_{\lambda,K} = \bigoplus_{i,j} K[[\lambda]]\omega_{ij} \oplus K[[\lambda]]\eta_{ij} \]
   where \((i, j)\) runs over the pairs of integers such that \(1 \leq i \leq N - 1\) and \(1 \leq j \leq M - 1\).

2. If \(\lambda = s = t^{-1}\), then
   \[ H_{\lambda,K} = \bigoplus_{i,j} K[[\lambda]]\omega_{ij} \oplus K[[\lambda]]s\eta_{ij} \]

3. If \(\lambda = 1 - t\), then
   \[ H_{\lambda,K} = \bigoplus_{i,j} K[[\lambda]]\omega^*_{ij} \oplus K[[\lambda]]\eta^*_{ij} \]
   where \(\omega^*_{ij}\) and \(\eta^*_{ij}\) are as in Theorem 3.6 (3).

**Proof.** The condition (D1) is obvious by Theorem 3.4. It is straightforward from Proposition 3.7 that (D2) and (D3) are satisfied in each case. \(\square\)

### 3.6 Proof of Theorem 3.6 (1), (2)

We prove Theorem 3.6 in case \(\lambda = t\) and in case \(\lambda = s = t^{-1}\). Write \(\eta'_{ij} = \eta_{ij}\) in case \(\lambda = t\) and \(\eta'_{ij} = \lambda\eta_{ij}\) in case \(\lambda = s\). Recall from Theorem 3.4 the fact that

\[ H^1_{\text{dR}}(\mathcal{X}/W((\lambda))) \cong W((\lambda)) \otimes_A H^1_{\text{dR}}(X/A) \]

is a free \(W((\lambda))\)-module with basis \(\{\omega_{ij}, \eta_{ij}; 1 \leq i \leq N - 1, 1 \leq j \leq M - 1\}\). It follows from Proposition 3.8 that

\[ H_{\lambda} \subset \bigoplus_{i,j} K[[\lambda]]\omega_{ij} + K[[\lambda]]\eta^0_{ij} \subset H^1_{\text{dR}}(\mathcal{X}_K/K((\lambda))). \]

Therefore

\[ H_{\lambda} \subset \bigoplus_{i,j} W[[\lambda]]\omega_{ij} + W[[\lambda]]\eta^0_{ij} \]  \hspace{1cm} (3.10)

as \(K[[\lambda]] \cap W((\lambda)) = W[[\lambda]]\). We show the opposite inclusion, namely

\[ \omega_{ij}, \eta^0_{ij} \in H_{\lambda}. \]  \hspace{1cm} (3.11)
We first show $\omega_{ij} \in H_\lambda$. There is an integer $m \geq 0$ such that $\lambda^m \omega_{ij} \in \Gamma(\mathcal{Y}, \Omega_{\mathcal{Y}/W[[\lambda]]}^1(\log D))$. On the other hand $\omega_{ij} \in \Gamma(\mathcal{Y}_K, \Omega_{\mathcal{Y}/K[[\lambda]]}^1(\log D))$. Note that $\Omega_{\mathcal{Y}/W[[\lambda]]}^1(\log D)$ is a locally free $\mathcal{O}_{\mathcal{Y}}$-module (Proposition 3.5). Moreover one can check that the map $a$ in the following diagram is injective.

$$
\begin{array}{c}
0 \to \Omega_{\mathcal{Y}/W[[\lambda]]}^1(\log D) \xrightarrow{\lambda^m} \Omega_{\mathcal{Y}/W[[\lambda]]}^1(\log D) \to 0
\end{array}
$$

Therefore we have $\omega_{ij} \in \Gamma(\mathcal{Y}, \Omega_{\mathcal{Y}/W[[\lambda]]}^1(\log D))$ by diagram chase.

Next we show $\eta^i_{ij} \in H_\lambda$. Recall from (3.8) the cocycle which represents $\eta_{ij}$,

$$(f_{ab,cd}) \times (\eta^i_{ij}) \in \bigoplus \Gamma(U_{ab} \cap U_{cd}, \mathcal{O}_\mathcal{X}) \times \bigoplus \Gamma(U_{ab}, \Omega_{\mathcal{Y}/W[[\lambda]]}^1(\log D)).$$

Therefore it is enough to show

$$f_{ab,cd} \in \Gamma(U_{ab} \cap U_{cd}, \mathcal{O}_\mathcal{Y}), \quad \eta^i_{ij} \in \Gamma(U_{ab}, \Omega_{\mathcal{Y}/W[[\lambda]]}^1(\log D)).$$

in case $\lambda = t$, and

$$sf_{ab,cd} \in \Gamma(U_{ab} \cap U_{cd}, \mathcal{O}_\mathcal{Y}), \quad s\eta^i_{ij} \in \Gamma(U_{ab}, \Omega_{\mathcal{Y}/W[[\lambda]]}^1(\log D)).$$

in case $\lambda = s$. However we have shown that $\omega_{ij} \in \Gamma(\mathcal{Y}, \Omega_{\mathcal{Y}/W[[\lambda]]}^1(\log D))$. Thus this is immediate from the explicit descriptions in \textit{3.4}. This completes the proof of \textit{3.11} and hence Theorem 3.6 (1), (2).

\subsection*{3.7 Proof of Theorem 3.6 (3)}

Let $\lambda = 1 - t$. By the same discussion as in \textit{3.6}, one can show

$$H_\lambda \subset \bigoplus_{i,j} W[[\lambda]] \omega^*_{ij} + W[[\lambda]] \eta^*_{ij},$$

and hence it is enough to show

$$\omega^*_{ij}, \eta^*_{ij} \in H_\lambda.$$  \hfill (13.13)

If $i/N + j/M \geq 1$, then the same discussion as the proof of \textit{3.11} works. Suppose that $i/N + j/M < 1$. The same discussion still works for showing $\omega^*_{ij} = \lambda \omega_{ij} \in H_\lambda$. The rest is to show that

$$\eta^*_{ij} = (1 - i/N - j/M) t \omega_{ij} - \eta_{ij} \in H_\lambda.$$  \hfill (13.14)

Recall from (3.8) the cocycle which represents $\eta_{ij}$,

$$(f_{ab,cd}) \times (\eta^i_{ij}) \in \bigoplus \Gamma(U_{ab} \cap U_{cd}, \mathcal{O}_\mathcal{X}) \times \bigoplus \Gamma(U_{ab}, \Omega_{\mathcal{X}/W[[\lambda]]}^1(\log D)).$$

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Hence
\[(f_{ab,cd}) \times ((\eta^*_i)^{ab}) := (f_{ab,cd}) \times ((1 - i/N - j/M)t_{ij} - \eta^*_{ij})\]
represents \(\eta^*_i \in H_{1,\mathcal{H}}(\mathcal{X}/W(\lambda))\). Each \(f_{ab,cd}\) obviously belongs to \(\Gamma(U_{ab,cd}; \Theta_\mathcal{Y})\). Therefore it is enough to show that each \((\eta^*_i)^{ab} \in \Gamma(U_{ab}, \Omega^1_{\mathcal{Y}/W[|\lambda|]}(\log D))\).

\[
\begin{align*}
(\eta^*_i)^{11} &= \left(1 - \frac{i}{N} - \frac{j}{M}\right)t_{ij} + (j - M)(1 - t)z^{N-i}w^{M-j-1}dw, \\
(\eta^*_i)^{01} &= \left(1 - \frac{i}{N} - \frac{j}{M}\right)t_{ij} + (j - M)(1 - t)x^{i-N}w^{M-j-1}dw, \\
(\eta^*_i)^{10} &= \left(1 - \frac{i}{N} - \frac{j}{M}\right)t_{ij} - (1 - t)z^{N} \left(\frac{(j - M)t}{M} + \frac{(i - 2N)}{N}(1 - t)(1 - z^{N})\right)\omega_{ij}, \\
(\eta^*_i)^{00} &= \left(\frac{(j - M)t}{M} + \frac{i}{N}(1 - x^{N})\right)\omega_{ij} + \left(1 - \frac{i}{N} - \frac{j}{M}\right)t_{ij} - \frac{i}{N}x^{N}\omega_{ij} \\
&= \frac{i}{N}(1 - t - 2x^{N})\omega_{ij}.
\end{align*}
\]

Multiplying \(Nz^{N-i}y^j\) on an equality
\[
t\frac{y^{-M}}{1 - z^{N}} = \frac{M}{N}(1 - t)(1 - z^{N})\frac{dy}{y} + t\frac{dz}{z}
\]
one has
\[
\omega_{ij} = M(1 - t)(1 - z^{N})z^{N-i}y^{j-1}dy + Ntz^{N-i-1}y^{j}dz.
\]
The shows \(\omega_{ij} \in \Gamma(U_{10}, \Omega^1_{\mathcal{Y}/W[|\lambda|]}(\log D))\). Similarly, using equalities
\[
t\frac{x^{-N}}{w^{M} - 1} = -\frac{N}{M}(1 - t)(1 - w^{N})\frac{dx}{x} - t\frac{dw}{w}
\]
\[
\frac{1}{1 - z^{N}} = (1 - (1 - t^{-1})(1 - w^{M}))\frac{dz}{z} - \frac{M}{N}\frac{dw}{w},
\]
one has
\[
\omega_{ij} \in \Gamma(U_{10} \cup U_{01} \cup U_{11}, \Omega^1_{\mathcal{Y}/W[|\lambda|]}(\log D)).
\]
We thus have \((\eta^*_i)^{ab} \in \Gamma(U_{ab}, \Omega^1_{\mathcal{Y}/W[|\lambda|]}(\log D))\) for \((a, b) = (0, 1), (1, 0)\) and \((1, 1)\). The rest is the case \((a, b) = (0, 0)\), namely we show
\[
\omega_{ij}^* = (1 - t)\omega_{ij}, x^{N}\omega_{ij} \in \Gamma(U_{00}, \Omega^1_{\mathcal{Y}/W[|\lambda|]}(\log D)),
\]
(note that \(\omega_{ij}\) no longer belongs to \(\Gamma(U_{00}, \Omega^1_{\mathcal{Y}/W[|\lambda|]}(\log D))\)). However the former is already shown in (3.13), and the latter follows from an equality
\[
x^{N}\omega_{ij} = -Mx^{i}y^{j-1}\frac{dy}{1 - y^{M}} = -Mt^{-1}(1 - x^{N})x^{i}y^{j-1}dy.
\]
This completes the proof of (3.14) and hence Theorem 3.6 (3).
4 Rigid cohomology and Dwork’s $p$-adic Hypergeometric functions

Let $W = W(k)$ be the Witt ring of a perfect field $k$ of characteristic $p > 0$. Let $A$ be a faithfully flat $W$-algebra. We mean by a $p^n$-th Frobenius on $A$ an endomorphism $\sigma$ such that $\sigma(x) \equiv x^{p^n} \mod pA$ for all $x \in A$ and that $\sigma$ is compatible with the $p^n$-th Frobenius on $W$. We also write $x^\sigma$ instead of $\sigma(x)$.

For a $W$-algebra $A$ of finite type, we denote by $A^\dagger$ the weak completion. Namely if $A = W[T_1, \ldots, T_n]$, then $A^\dagger = W[T_1, \ldots, T_n]^\dagger$ is the ring of power series $\sum a_\alpha T^\alpha$ such that for some $r > 1$, $|a_\alpha| p^{|\alpha|} \to 0$ as $|\alpha| \to \infty$, and if $A = W[T_1, \ldots, T_n]/I$, then $A^\dagger = W[T_1, \ldots, T_n]^\dagger/IW[T_1, \ldots, T_n]^\dagger$.

We denote by $\log(x)$ the logarithmic function defined by the power series

$$\log(x) = -\sum_{n=1}^{\infty} \frac{(1 - x)^n}{n}.$$ 

4.1 Rigid cohomology

Let $W = W(k)$ be the Witt ring of a perfect field $k$ of characteristic $p > 0$. Put $K := \text{Frac}(W)$ the fractional field. For a flat $W$-scheme $V$, we denote $V_K := V \times_W K$ and $V_k := V \times_W k$. For a flat $W$-ring $A$, we denote $A_K := A \otimes_W K$ and $A_k := A \otimes_W k$ as well.

Let $A$ be a smooth $W$-algebra, and $X$ a smooth $A$-scheme. Thanks to the theory due to Berthelot et al, the rigid cohomology groups

$$H^*_\text{rig}(X_k/A_k)$$

are defined. We refer the book [LS] for the general theory of rigid cohomology. Here we list the required properties. Let $A^\dagger$ be the weak completion of $A$, and $A^\dagger_K := A^\dagger \otimes_W K$. We fix a $p$-th Frobenius $\sigma$ on $A^\dagger$.

- $H^*_\text{rig}(X_k/A_k)$ is a finitely generated $A^\dagger_K$-module.
- (Frobenius) The $p$-th Frobenius $\Phi$ on $H^*_\text{rig}(X_k/A_k)$ (depending on $\sigma$) is defined in a natural way. This is a $\sigma$-linear endomorphism:
  $$\Phi(f(t)x) = \sigma(f(t))\Phi(x), \quad \text{for } x \in H^*_\text{rig}(X_k/A_k), f(t) \in A^\dagger_K.$$ 
- (Comparison with de Rham cohomology) There is the comparison isomorphism with the algebraic de Rham cohomology,
  $$H^*_\text{rig}(X_k/A_k) \cong H^*_{\text{dR}}(X_K/A_K) \otimes_{A_K} A^\dagger_K.$$ 
- (Comparison with crystalline cohomology) Let $\alpha$ be a $W$-rational point of $\text{Spec}A$ (i.e. a $W$-homomorphism $\alpha : A \to W$). Let $X_\alpha := X \times_{A,\alpha} W$ denote the fiber at $\alpha$. There is the comparison isomorphism with the crystalline cohomology,
  $$H^*_\text{rig}(X_k/A_k) \otimes_{A^\dagger_K,\alpha} K \cong H^*_{\text{crys}}(X_\alpha/k/W) \otimes \mathbb{Q}.$$
If $\alpha$ satisfies $\sigma^{-1}(m_\alpha A^\dagger_K) = m_\alpha A^\dagger_K$ where $m_\alpha \subset A$ denotes the ideal defining $\alpha$, then $\Phi_\alpha := \Phi \mod m_\alpha A^\dagger_K$ agrees with the $p$-th Frobenius on the crystalline cohomology.

Let $\mathcal{Y}$ be a proper flat scheme over $W[[t]]$ which is smooth over $W((t))$. Let the central fiber $D$ at $t = 0$. Put $\mathcal{X} := \mathcal{Y} \setminus D$. Suppose that $D_{\text{red}}$ is a relative NCD over $W$ and the multiplicities of components of $D$ are prime to $p$. Then there is the comparison isomorphism with the log crystalline cohomology with log pole $D$ ([Ka Theorem (6.4)]),

$$H^*_{\text{log-crys}}((\mathcal{Y}_{\mathbb{F}_p}, D_{\mathbb{F}_p})/(W[[t]], (t))) \cong H^*_{\text{zar}}(\mathcal{Y}, \Omega^*_{\mathcal{Y}/W[[t]]}(\log D)). \tag{4.1}$$

Fix a $p$-th Frobenius $\hat{\sigma}$ on $W[[t]]$ given by $\hat{\sigma}(t) = ct^p$ with some $c \in 1 + pW$. Then the $\hat{\sigma}$-linear $p$-th Frobenius $\Phi_{\text{crys}}$ on the crystalline cohomology group is defined in a natural way. Let $A \rightarrow W((t))$ be a $W$-homomorphism, and $A^\dagger \rightarrow W((t))^\wedge$ the induced homomorphism where $W((t))^\wedge$ denotes the $p$-adic completion. Suppose that there is an isomorphism $\mathcal{X} \cong X \times_A W((t))$ and that $\sigma$ and $\hat{\sigma}$ are compatible under the map $A^\dagger \rightarrow W((t))^\wedge$. Then the Frobenius $\Phi$ agrees with $\Phi_{\text{crys}}$ under the natural map

$$H^*_{\text{log-crys}}((\mathcal{Y}_{\mathbb{F}_p}, D_{\mathbb{F}_p})/(W[[t]], (t))) \rightarrow H^*_{\text{dR}}(X_K/A_K) \otimes A W((t))^\wedge \cong H^*_{\text{rig}}(X_K/A_K) \otimes A W((t))^\wedge. \tag{4.2}$$

### 4.2 Explicit description of $\Phi$ by overconvergent functions

Let

$$\begin{align*}
X & \rightarrow Y \\
| & \downarrow f \\
U = \text{Sp}A & \rightarrow \mathbb{P}^1
\end{align*}$$

be the fibration in §3.1. In what follows we work over the Witt ring $W = W(\mathbb{F}_p)$ with $p > \max(N, M)$. Put $K := \text{Frac}W$ the fractional field.

Let $c \in 1 + pW$ be fixed, and let $\sigma : A^\dagger \rightarrow A^\dagger$ be the $p$-th Frobenius given by $t^\sigma = ct^p$. Let

$$H^1_{\text{rig}}(X_{\mathbb{F}_p}/A_{\mathbb{F}_p})$$

be the rigid cohomology group, and $\Phi$ the $\sigma$-linear $p$-th Frobenius. We shall give an explicit description of $\Phi$.

**Lemma 4.1** Let $\text{Spec}W[[t]] \rightarrow \mathbb{P}^1$, and put $\mathcal{Y} := Y \times_{\mathbb{P}^1} \text{Spec}W[[t]]$ and $D := f^{-1}(0) \subset \mathcal{Y}$ the central fiber. Put $\mathcal{X} := \mathcal{Y} \setminus D$. Then the natural map

$$H^1(\mathcal{Y}, \Omega^*_{\mathcal{Y}/W[[t]]}(\log D)) \rightarrow H^1(\mathcal{Y}, \Omega^*_{\mathcal{Y}/W[[t]]}(\log D)) \otimes W((t)) \cong H^1_{\text{dR}}(\mathcal{X}/W((t)))$$

is injective.
\textbf{Proof.} It is enough to show that \( H^1(\mathcal{O}_\mathcal{Y}, \Omega^*_\mathcal{Y}/\mathcal{W}[t][\log D]) \) is \( t \)-torsion free. There is an exact sequence
\[
0 \rightarrow \Gamma(\Omega^1_{\mathcal{Y}/\mathcal{W}[t]}(\log D)) \rightarrow H^1(\mathcal{O}_\mathcal{Y}, \Omega^*_\mathcal{Y}/\mathcal{W}[t][\log D]) \rightarrow H^1(\mathcal{O}_\mathcal{Y}).
\]
The 1st term is \( t \)-torsion free by Proposition 3.5. We show that \( H^1(\mathcal{O}_\mathcal{Y}) \) is a free \( \mathcal{W}[t] \)-module. By [Ha, III,12.9], it is enough to show that \( \dim_{\kappa(s)} H^1(Y_s, \mathcal{O}_{Y_s}) = (N-1)(M-1) \) for any point \( s \in \text{Spec} \mathcal{W}[t] \) where \( \kappa(s) \) is the residue field, and \( Y_s := \mathcal{Y} \times_{\mathcal{W}[t]} \kappa(s) \). If \( t \) is invertible in \( \kappa(s) \), then \( Y_s \) is a smooth fiber, and then one has \( \dim_{\kappa(s)} H^1(Y_s, \mathcal{O}_{Y_s}) = (N-1)(M-1) \) as \( g(Y_s) = (N-1)(M-1) \). If \( t = 0 \) in \( \kappa(s) \), then \( Y_s = D_s := D \times_{\mathcal{W}} \kappa(s) \) is a simple NCD, and then one can directly show that \( \dim_{\kappa(s)} H^1(D_s, \mathcal{O}_{D_s}) = (N-1)(M-1) \).

The Frobenius \( \sigma \) extends on the Frobenius on \( K((t)) \) as \( \sigma(t) = ct^p \). Let \( \Phi_{\text{crys}} \) be the crystalline Frobenius on
\[
H^1_{\text{log-crys}}((\mathcal{O}_\mathcal{Y}, D_{\mathcal{Y}})/(\mathcal{W}[t]), (t)) \cong H^1(\mathcal{O}_\mathcal{Y}, \Omega^*_\mathcal{Y}/\mathcal{W}[t][\log D]).
\]
Let \( 1 \leq i \leq N - 1 \) and \( 1 \leq j \leq M - 1 \) be integers. Put \( a_i := 1 - i/N \) and \( b_j := 1 - j/M \). Let
\[
F_{ij}(t) = F_{a_i b_j}(t) = F_1\left(\frac{a_i b_j}{1}; t\right)
\]
be the hypergeometric power series. It follows from Theorem 3.6 that the elements
\[
\tilde{\omega}_{i,j} := \frac{1}{F_{ij}(t)}\omega_{i,j}, \quad \tilde{\eta}_{h,j} := -t(1-t)^{a_i+b_j} F_{ij}(t) \omega_{i,j} + (1-t)^{a_i+b_j-1} F_{ij}(t) \eta_{h,j} \quad (4.3)
\]
forms a \( \mathcal{W}[t] \)-basis of
\[
H^1(\mathcal{Y}, \Omega^*_\mathcal{Y}/\mathcal{W}[t][\log D]) \cong \text{Im}[H^1(\Omega^*_\mathcal{Y}/\mathcal{W}[t][\log D]) \to H^1_{\text{dR}}(\mathcal{Y}/W((t)))]
\]
where the isomorphism follows from Lemma 4.1.

\textbf{Theorem 4.2} Let \( \tau_{ij}(t) \in \mathbb{Q}[t] \) be defined by
\[
\frac{d}{dt}\tau_{ij}(t) = \frac{1}{t} \left( 1 - \frac{1}{(1-t)^{a_i+b_j} F_{ij}(t)^2} \right), \quad \tau_{ij}(0) = 0. \quad (4.4)
\]
Let \( \psi_p(z) \) be the \( p \)-adic digamma function introduced in [A, §2] (see also Appendix A). Put
\[
\tau_{ij}^{(\sigma)}(t) = -2\gamma_p - \psi_p(a_i) - \psi_p(b_j) + p^{-1} \log(c) + \tau_{ij}(t) - p^{-1} \tau^{(\sigma)}_{i'j'}(t') \in K[[t]] \quad (4.5)
\]
where \( i' \in \{1, \ldots, N-1\} \) and \( j' \in \{1, \ldots, M-1\} \) are integers such that \( i'p \equiv i \mod N \) and \( j'p \equiv j \mod M \). Then
\[
\Phi_{\text{crys}}(\tilde{\omega}_{i'j'}) = p \tilde{\omega}_{ij} + p \tau^{(\sigma)}_{ij}(t) \tilde{\eta}_{h,j} \quad (4.6)
\]
\[
\Phi_{\text{crys}}(\tilde{\eta}_{i'j'}) = \tilde{\eta}_{h,j}. \quad (4.7)
\]
Since $\Phi_{\text{crys}}$ agrees with $\Phi$ under the natural map \((4.2)\), Theorem \[4.2\] implies the following.

**Theorem 4.3** Write $f'(t) = \frac{d}{dt}f(t)$ for a power series $f(t)$. We define

\[
A_{ij}(t) := \frac{F_{ij}(t^\sigma)}{F_{ij}(t)} - t(1-t)^{a_i+b_j}F_{ij}'(t)F_{ij}'(t^\sigma)\tau_{ij}(t)
\]

\[
C_{ij}(t) := (1-t)^{a_i+b_j}F_{ij}(t)F_{ij}'(t^\sigma)\tau_{ij}(t)
\]

\[
B_{ij}(t) := pt^\sigma(1-t^\sigma)\frac{F_{ij}'(t^\sigma)}{F_{ij}'(t^\sigma)}A_{ij}(t) - t(1-t)^{a_i+b_j}F_{ij}''(t)\tau_{ij}(t)
\]

\[
D_{ij}(t) := pt^\sigma(1-t^\sigma)\frac{F_{ij}'(t^\sigma)}{F_{ij}'(t^\sigma)}C_{ij}(t) + (1-t)^{a_i+b_j}F_{ij}''(t)\tau_{ij}(t)
\]

Under the comparison isomorphism

\[
H^1_{\text{rig}}(X_{\varphi}/A_{\varphi}) \cong H^*_{\text{dR}}(X/A) \otimes_A A^!_{\varphi},
\]

the $p$-th Frobenius $\Phi$ is described as follows,

\[
(\Phi(\omega_{ij}'), \Phi(\eta_{ij}')) = (\omega_{ij}, \eta_{ij}) \begin{pmatrix} pA_{ij} & B_{ij} \\ pC_{ij} & D_{ij} \end{pmatrix}.
\]

**Corollary 4.4** All the power series $\tau_{ij}(t)$, $A_{ij}(t)$, $B_{ij}(t)$, $C_{ij}(t)$ and $D_{ij}(t)$ lie in the ring $W[[t]]$. In particular, $A_{ij}(t)$, $B_{ij}(t)$, $C_{ij}(t)$ and $D_{ij}(t)$ lie in the ring $A^!_K \cap W[[t]] = A^! \cap W[[t]]$.

**Proof.** Noticing that \((4.3)\) forms a $W[[t]]$-basis, the fact that $\tau_{ij}(t) \in W[[t]]$ is immediate from Theorem \[4.2\] \((4.6)\) together with the fact that $\Phi_{\text{crys}}(\Gamma(\Omega^1_{\varphi}/W[[t]])(\log D)) \subset pH^1(\Omega^1_{\varphi}/W[[t]])(\log D))$.

The others follows from this and the definition. \[\square\]

**Remark 4.5** I don’t know a direct proof of Corollary \[4.4\] (without $p$-adic cohomology).

**Remark 4.6** Note that $a_{ij} = (a_{ij})'$ and $b_{ij} = (b_{ij})'$ (Dwork prime). In particular $n_i := a_i - pa_i$ and $m_j := b_j - pb_j$ are integers $\leq 0$. We have

\[
\det \begin{pmatrix} pA_{ij} & B_{ij} \\ pC_{ij} & D_{ij} \end{pmatrix} = p^\frac{(1-t)^{a_i+b_j-1}}{(1-t^\sigma)^{a_i+b_j-1}} = p^\frac{(1-t)^{n_i+m_j-1}}{(1-t^\sigma)} \left( \frac{1-t^p}{1-t^\sigma} \right)^{a_i+b_j} (4.8)
\]

with

\[
\left( \frac{1-t^p}{1-t^\sigma} \right)^{a_i+b_j} = \sum_{n=0}^{\infty} p^n \binom{a_i+b_j}{n} u(t)^n \in (W[t], (1-t^{-1})^+) \times \quad (4.9)
\]

where we put $\frac{(1-t)^p}{(1-t^\sigma)} = 1 + pu(t)$. In particular

\[
\det \begin{pmatrix} pA_{ij} & B_{ij} \\ pC_{ij} & D_{ij} \end{pmatrix} \bigg|_{t=\alpha} = p \times (\text{unit})
\]

for $\alpha \in W^\times \setminus (1+pW)$. 

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4.3 Proof of Theorem 4.2 (4.7)

For integers \( k, l \) with \( N \nmid k \) and \( M \nmid l \) which do not necessarily satisfy that \( 1 \leq k \leq N - 1 \) and \( 1 \leq l \leq M - 1 \), \( \omega_{kl} \) denotes \( \omega_{k_0 l_0} \) where \( k_0 \in \{1, \ldots, N - 1\} \) and \( l_0 \in \{1, \ldots, M - 1\} \) such that \( k \equiv k_0 \) mod \( N \) and \( l \equiv l_0 \) mod \( M \). We apply the same convention to symbols \( \eta_{kl} \), \( \tau_{kl}(t) \), \( a_k, b_l \) etc.

Let
\[
\nabla : H^1(\mathcal{Y}, \Omega^*_\mathcal{Y}/W[[t]](\log D)) \longrightarrow \frac{dt}{t} \otimes H^1(\mathcal{Y}, \Omega^*_\mathcal{Y}/W[[t]](\log D))
\]
be the Gauss-Manin connection. By Proposition 3.7 (or [A] Prop 4.15),
\[
(\nabla(\tilde{\omega}_{i,j}) \quad \nabla(\tilde{\eta}_{h,j})) = dt \otimes (\tilde{\omega}_{i,j} \quad \tilde{\eta}_{h,j}) \begin{pmatrix} 0 & 0 \\ t^{-1}(1 - t)^{-a_l - b_j} F_{ij}(t)^{-2} & 0 \end{pmatrix}.
\]

Using this, one can show
\[
\text{Ker}(\nabla) = \bigoplus_{i,j} W \tilde{\eta}_{ij}.
\]

Since \( \nabla \Phi_{\text{crys}} = \Phi_{\text{crys}} \nabla \), one has
\[
\Phi_{\text{crys}}(\tilde{\eta}_{ij}) = \sum_{k,l} \alpha_{kl} \tilde{\eta}_{kl}
\]
with some constants \( \alpha_{kl} \in W \). Let \( i : D \to \mathcal{Y} \) be the embedding. Let \( h \) be the composition as follows
\[
H^1(\Omega^*_\mathcal{Y}/W[[t]](\log D)) \xrightarrow{\text{h}} H^1(\mathcal{Y}, \mathcal{O}_\mathcal{Y}) \xrightarrow{\tau} H^1(D, \mathcal{O}_D).
\]

Recall from [3.1] that \( D = f^{-1}(0) \) is a simple relative NCD, and the irreducible components are \( \{D_x = \zeta_1, D_y = \zeta_2 \mid \zeta_1 \in \mu_N, \zeta_2 \in \mu_M\} \) where \( D_x = \zeta_1 := \{x = \zeta_1\} \) and \( D_y = \zeta_2 := \{y = \zeta_2\} \) and \( \mu_n := \{\zeta \in W \mid \zeta^n = 1\} \). Put \( P(\zeta_1, \zeta_2) := D_x = \zeta_1 \cap D_y = \zeta_2 \) a single point, and \( P := \{P(\zeta_1, \zeta_2)\}_{\zeta_1, \zeta_2} \subset D \). There is an exact sequence
\[
\bigoplus_{\zeta_1} H^0(\mathcal{O}_{D_x = \zeta_1}) \oplus \bigoplus_{\zeta_2} H^0(\mathcal{O}_{D_y = \zeta_2}) \oplus \bigoplus_{\zeta_1, \zeta_2} H^0(\mathcal{O}_{P(\zeta_1, \zeta_2)}) \xrightarrow{\delta} H^1(\mathcal{O}_D) \to 0
\]
arising from an exact sequence
\[
0 \to \mathcal{O}_D \xrightarrow{j} \bigoplus_{\zeta_1} \mathcal{O}_{D_x = \zeta_1} \oplus \bigoplus_{\zeta_2} \mathcal{O}_{D_y = \zeta_2} \xrightarrow{u} \bigoplus_{\zeta_1, \zeta_2} \mathcal{O}_{P(\zeta_1, \zeta_2)} \to 0
\]
where \( j \) is the pull-back and \( u \) is the map which sends \( (f_{\zeta_1})_{\zeta_1} \times (g_{\zeta_2})_{\zeta_2} \) to \( ((g_{\zeta_2} - f_{\zeta_1})|_{P(\zeta_1, \zeta_2)})_{\zeta_1, \zeta_2} \).
Lemma 4.7 Let 
\[ e_{ij} := (\zeta_1^i \zeta_2^j)_{\zeta_1, \zeta_2} \in \bigoplus_{\zeta_1, \zeta_2} H^0(\mathcal{O}_{P(\zeta_1, \zeta_2)}) \]
be an element for \( i, j \in \mathbb{Z} \). Then \( \delta(e_{ij}) = h(\tilde{\eta}_{ij}) \) for \( i \in \{1, \ldots, N-1\} \) and \( j \in \{1, \ldots, M-1\} \). In particular \( h \otimes \mathbb{Q} \) is bijective.

Proof. Recall from (3.8) the cocycle \((f_{ab,cd}) \times (\eta_{ij}^{ab})\) where
\[
\begin{align*}
    f_{00,11} &= f_{00,01} := x^iy^{j-M}, \\
    f_{10,11} &= f_{10,01} := (1-t)^2z^{2N-i}y^{j-M} = (1-t)(1-y^M+z^N y^M)z^{N-i}y^{j-M}, \\
    f_{01,11} &= 0, \\
    f_{00,10} &= x^iy^{j-M}-(1-t)^2z^{2N-i}y^{j-M} = (1-x^N)(x^N y^M-2x^N y^M)z^{2N-i}y^j.
\end{align*}
\]
Note that \( D \subset U_{00} \cup U_{11} \). We have \( h(\tilde{\eta}_{ij}) = [(x^iy^{j-M}|_D)] \) under the isomorphism
\[
H^1(\mathcal{O}_D) \cong \text{Coker} \left[ \bigoplus_{(a,b)=(0,0),(1,1)} \Gamma(U_{ab}, \mathcal{O}_D) \.st \frac{d}{d} \Gamma(U_{00,11}, \mathcal{O}_D) \right]
\]
where \( U_{00,11} := U_{00} \cap U_{11} \) and \( d(f_{00,11}) := f_{11} - f_{00} \). A diagram chase
\[
\begin{array}{ccc}
    \bigoplus_{\zeta_1} \Gamma(U_{ab}, \mathcal{O}_{D_{x=\zeta_1}}) \times \bigoplus_{\zeta_2} \Gamma(U_{ab}, \mathcal{O}_{D_{y=\zeta_2}}) & \xrightarrow{u} & \bigoplus_{\zeta_1, \zeta_2} \Gamma(\mathcal{O}_{P(\zeta_1, \zeta_2)}) \\
    \downarrow{d} & & \downarrow{d} \\
    \Gamma(U_{00,11}, \mathcal{O}_D) & \xrightarrow{u} & \Gamma(U_{00,11}, \mathcal{O}_{D_{x=\zeta_1}}) \times \bigoplus_{\zeta_2} \Gamma(U_{00,11}, \mathcal{O}_{D_{y=\zeta_2}}) \\
    \downarrow{d} & & \downarrow{d} \\
    h(\tilde{\eta}_{ij}) = (x^iy^{j-M}|_D) & \xrightarrow{u} & (\zeta_1^i \zeta_2^j x^i, 0) \xrightarrow{u} \zeta_1^i \zeta_2^j \\
\end{array}
\]
yields \( \delta(e_{ij}) = h(\tilde{\eta}_{ij}) \). The last statement is an exercise of linear algebra. \( \square \)

We turn to the proof of (4.7). Apply \( \Phi_{\text{crys}} \) on the equality \( h(\tilde{\eta}_{ij}) = \delta(e_{ij}) \) in Lemma 4.7. Since \( h \) and \( \delta \) are compatible with respect to the action of \( \Phi_{\text{crys}} \), one has
\[
h\Phi(\tilde{\eta}_{ij}) = \sum_{k,l} \alpha_{kl} h(\tilde{\eta}_{kl}) = \delta\Phi_{\text{crys}}(e_{ij})
\]
by (4.11). On the other hand
\[
\Phi_{\text{crys}}(e_{ij}) = (\epsilon_{ip} \epsilon_{jp})_{\zeta_1, \zeta_2} = e_{ip,jp} \in \bigoplus_{\zeta_1, \zeta_2} H^0(\mathcal{O}_{P(\zeta_1, \zeta_2)})
\]
by definition of \( \Phi_{\text{crys}} \). Therefore one has
\[
\sum_{k,l} \alpha_{kl} h(\tilde{\eta}_{kl}) = \delta(e_{ip,jp}) = h(\tilde{\eta}_{ip,jp}),
\]
and hence \( \alpha_{kl} = 1 \) if \( (k, l) = (ip, jp) \) in \( \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z} \) and \( = 0 \) otherwise. This completes the proof of (4.7).
4.4 Proof of Theorem 4.2 (4.6)

For \((\zeta_1, \zeta_2) \in \mu_N \times \mu_M\), we denote by \([\zeta_1, \zeta_2]\) the automorphism of \(\mathcal{W}\) given by \((x, y, t) \mapsto (\zeta_1 x, \zeta_2 y, t)\). Since \([\zeta_1, \zeta_2]\Phi_{\text{crys}} = \Phi_{\text{crys}}[\zeta_1, \zeta_2]\), one has

\[
\Phi_{\text{crys}}(\tilde{\omega}_{ij}) \in W[[t]]\tilde{\omega}_{ip,jp} + W[[t]]\tilde{\eta}_{ip,jp}.
\]

One can further show that there is \(g_{ij}(t) \in W[[t]]\) such that

\[
\Phi_{\text{crys}}(\tilde{\omega}_{ij}) = p\omega_{ip,jp} + g_{ij}(t)\tilde{\eta}_{ip,jp}
\]

(4.12)

(this can be proved in the same way as the proof of \([A, \text{Lemma 4.5}])\). Thus our goal is to show \(g_{ij}(t) = p\tau_{ip,jp}(t)\). Apply \(\nabla\) on (4.12). It follows from (4.10) that we have

\[
\text{LHS} = \nabla \Phi_{\text{crys}}(\tilde{\omega}_{ij})
= \Phi_{\text{crys}} \nabla (\tilde{\omega}_{ij})
= \Phi_{\text{crys}} \left( (1 - t)^{-a_i - b_j} F_{ij}(t)^{-2} \frac{dt}{t} \otimes \tilde{\eta}_{ij} \right)
= p(1 - t^\sigma)^{-a_i - b_j} F_{ij}(t^\sigma)^{-2} \frac{dt}{t} \otimes \tilde{\eta}_{ip,jp} \quad \text{(by Theorem 4.2 (4.7))}
\]

and

\[
\text{RHS} = p(1 - t)^{-a_ip - b_j} F_{ip,jp}(t)^{-2} \frac{dt}{t} \otimes \tilde{\eta}_{ip,jp} + g_{ij}(t) dt \otimes \tilde{\eta}_{ip,jp}.
\]

Hence

\[
g'_{ij}(t) = \frac{p}{t} \left[ \frac{1}{(1 - t^\sigma)^{a_i + b_j} F_{ij}(t^\sigma)^2} - \frac{1}{(1 - t)^{a_ip + b_j} F_{ip,jp}(t)^2} \right]
\]

or equivalently

\[
g_{ij}(t) = p(C_{ij} + \tau_{ip,jp}(t) - p^{-1}\tau_{ij}(t^\sigma))
\]

(4.13)

with \(C_{ij}\) a constant. The rest is to show

\[
C_{ij} = -2\gamma_p - \psi_p(a_{ip}) - \psi_p(b_{jp}) + p^{-1}\log(c).
\]

(4.14)

To do this, we recall from \([A, 4.6]\) the regulator formula.

For \((\nu_1, \nu_2) \in \mu_N(K) \times \mu_M(K)\), let

\[
\xi = \xi(\nu_1, \nu_2) = \left\{ \frac{x - 1}{x - \nu_1}, \frac{y - 1}{y - \nu_2} \right\} \in K_2(X)
\]

(4.15)

be a \(K_2\)-symbol. The symbol \(\xi\) defines the 1-extension

\[
0 \rightarrow H^1(X/A)(2) \rightarrow M_\xi(X/A) \rightarrow A \rightarrow 0
\]

in the category of Fil-\(F\)-MIC(\(A\)) (see \([A, 4.5]\) or \([AM, 2.5]\) for the notation). Let \(e_\xi \in \text{Fil}^0 M_\xi(X/A)_d\mathbb{R}\) be the unique lifting of \(1 \in A\). Let \(E^{(ij)}_k(t) \in W[[t]]\) be defined by

\[
e_\xi - \Phi_{\text{crys}}(e_\xi) = -N^{-1}M^{-1} \sum_{i=1}^{N-1} \sum_{j=1}^{M-1} (1 - \nu_1^{-i})(1 - \nu_2^{-j})[E^{(ij)}_1(t)\tilde{\omega}_{ij} + E^{(ij)}_2(t)\tilde{\eta}_{ij}].
\]

(4.16)
Then one of the main results in [A] is
\[ E_1^{(ij)}(t) = -\mathcal{F}_{a_i,b_j}(t) \]  
([A] Theorem 4.18) where \( \mathcal{F}_{\sigma}^{(\sigma)}(t) \) is the \( p \)-adic hypergeometric function of log type introduced in [A] \( \S 3 \).

We turn to the proof of (4.14). Apply \( \nabla \) on (4.16). Noticing that \( \Phi_{\text{crys}} \nabla = \nabla \Phi_{\text{crys}} \) and
\[ \nabla(e^t) = d\log(\xi) = N^{-1}M^{-1} \sum_{i=1}^{N-1} \sum_{j=1}^{M-1} (1 - \nu_1^{-i})(1 - \nu_2^{-j}) \frac{dt}{t}\omega_{ij}, \]
one has a differential equation
\[ t \frac{d}{dt} E_2^{(ij)}(t) + (1 - t)^{-a_i-b_j} F_{ij}(t)^{-2} E_1^{(ij)}(t) = p^{-1} F_{i'j'}(t') g_{i'j'}(t') \]
by (4.10) and (4.12) where \( i', j' \) are integers such that \( i' \in \{1, \ldots, N - 1\} \) with \( p i' \equiv i \mod N \) and \( j' \in \{1, \ldots, M - 1\} \) with \( p j' \equiv j \mod M \). Substitute \( t = 0 \) in the above. We have
\[ E_1^{(ij)}(0) = p^{-1} g_{i'j'}(0) = C_{i'j'}. \]

By (4.17),
\[ E_1^{(ij)}(0) = -\mathcal{F}_{a_i',b_j'}^{(\sigma)}(0) = -2\gamma_p - \psi_p(a_i') - \psi_p(b_j') + p^{-1} \log(c), \]
and hence (4.14) as required. This completes the proof of Theorem 4.2 (4.6).

5 Computing Dwork’s \( p \)-adic Hypergeometric functions

In this section, we shall give an algorithm for computing special values of Dwork’s \( p \)-adic hypergeometric functions whose bit complexity increases at most \( O(n^4 \log^3 n) \) as \( n \to \infty \).

5.1 \( p \)-adic expansions of \( A_{ij}(t), B_{ij}(t), C_{ij}(t), D_{ij}(t) \)

We keep the setting in \( \S 4.2 \) Recall Theorem 4.3
\[ \begin{pmatrix} \Phi(\omega_{ij}) & \Phi(\eta_{ij}) \end{pmatrix} = \begin{pmatrix} \omega_{ij} & \eta_{ij} \end{pmatrix} \begin{pmatrix} pA_{ij}(t) & B_{ij}(t) \\ pC_{ij}(t) & D_{ij}(t) \end{pmatrix} \]
with \( A_{ij}(t), B_{ij}(t), C_{ij}(t), D_{ij}(t) \in A^1 \cap W[[t]] \) (Corollary 4.3). According to [KT] Theorem 2.1, the overconvergent functions \( A_{ij}(t), \ldots, D_{ij}(t) \) have nice \( p \)-adic expansions, and this is the key fact in our algorithm. For the sake of the completeness, we here give the necessary statement with the self-contained proof.
**Theorem 5.1** For an integer $n \geq 1$, define

$$e_n := \max \{ k \in \mathbb{Z}_{\geq 1} \mid \text{ord}_p(p^k/k!) < n \}.$$

Then

$$pA_{ij}(t) \equiv \frac{(\text{polynomial of degree } \leq pe_n + p)}{(1 - t^p)(1 - t)^{pe_n}} \mod p^nW[[t]],$$

$$B_{ij}(t) \equiv \frac{(\text{polynomial of degree } \leq pe_n + 2p)}{(1 - t^p)(1 - t)^{pe_n}} \mod p^nW[[t]],$$

$$pC_{ij}(t) \equiv \frac{(\text{polynomial of degree } \leq pe_n + p - 1)}{(1 - t^p)(1 - t)^{pe_n}} \mod p^nW[[t]],$$

$$D_{ij}(t) \equiv \frac{(\text{polynomial of degree } \leq pe_n + 2p - 1)}{(1 - t^p)(1 - t)^{pe_n}} \mod p^nW[[t]].$$

**Remark 5.2** Since $p \neq 2$ by the assumption, $e_n < \infty$ for any $n \geq 1$. More precisely

$$e_n \sim \frac{p - 1}{p - 2}n \quad \text{as } n \to \infty.$$

**Remark 5.3** The degrees $pe_n + p$ etc. are not optimal.

**Proof.** (cf. [KT, p.11–13]). Let $\lambda = t, 1 - t$ or $t - 1$. Let $\sigma_\lambda$ be the $p$-th Frobenius on $W[[\lambda]]$ given by $\sigma_\lambda(\lambda) = \lambda^p$. Note that $\sigma_\lambda$ induces the $p$-th Frobenius on $A^\dagger = W[t, (t - t^2)^{-1}]^\dagger$. Let $\Phi_\lambda$ denote the $\sigma_\lambda$-linear Frobenius on

$$H^1_{\text{rig}}(X_{\mathbb{F}_p}/A_{\mathbb{F}_p}) \cong H^1_{\text{dR}}(X/A) \otimes A_A^\dagger K.$$

Let $\sigma$ be the Frobenius given by $\sigma(t) = ct^p$ and $\Phi$ the $\sigma$-linear Frobenius as in §4.2. Then the relation with $\Phi_\lambda$ is given as follows ([EK, 6.1], [Ke, 17.3.1]).

$$\Phi(x) - \Phi_\lambda(x) = \sum_{k=1}^{\infty} \frac{(\lambda^\sigma - \lambda^p)^k}{k!} \Phi_\lambda \partial_\lambda^k x, \quad x \in H^1_{\text{dR}}(X/A) \otimes A_A^\dagger K \quad (5.1)$$

where $\partial_\lambda := \nabla_{d/\partial \lambda}$. Let $\lambda = 1 - t$. Since $\lambda^\sigma - \lambda^p = pw(\lambda) \in pW[\lambda]$, (5.1) yields

$$\Phi(x) - \Phi_\lambda(x) = \sum_{k=1}^{\infty} \frac{p^k}{k!}w(t)^k \Phi_\lambda \partial_\lambda^k x.$$

Note that $\Phi_\lambda(H_\lambda) \subset H_\lambda$ while $\Phi(H_\lambda) \nsubseteq H_\lambda$. Since $\partial_\lambda^k(H_\lambda) \subset \lambda^{-k}H_\lambda$, one has $\Phi_\lambda \partial_\lambda^k(H_\lambda) \subset \lambda^{-kp}H_\lambda$ for all $k \geq 0$, and hence

$$\Phi(x) \in \sum_{k=0}^{\infty} \frac{p^k}{k!}\lambda^{-kp} H_\lambda.$$
We thus have
\[
\Phi(x) \in \lambda^{-p\sigma}H_\lambda + p^n\hat{H}_\lambda, \quad \forall n \geq 1, \forall x \in (H^{\dagger}_{\text{dr}}(X/A) \otimes_A A_{K}^{\dagger}) \cap H_\lambda \tag{5.2}
\]
if \( \lambda = 1 - t \) where \( \hat{H}_\lambda \) is the \( p \)-adic completion of \( H_\lambda \otimes_{W[[\lambda]]} W((\lambda)) \). Let \( \lambda = t^{-1} \). In this case, since \( \sigma_\lambda(t) = t^p \), the Frobenius \( \Phi_\lambda \) acts on the \( W[[\lambda]] \)-lattice \( H_\lambda \). Hence
\[
\Phi_\lambda(H_\lambda) \subset H_\lambda, \quad \lambda = t^{-1}. \tag{5.3}
\]

Let us prove Theorem 5.1. Since \( A_{ij}, B_{ij}, C_{ij}, D_{ij} \in A^1 \cap W[[t]] \), one can write
\[
pA_{ij}(t) \mod p^nW[[t]] = \frac{pF^A_{ij}(t)}{(1 - t^\sigma)(1 - t)^{d^A_{ij}}},
\]
\[
B_{ij}(t) \mod p^nW[[t]] = \frac{F^B_{ij}(t)}{(1 - t^\sigma)(1 - t)^{d^B_{ij}}},
\]
\[
pC_{ij}(t) \mod p^nW[[t]] = \frac{pF^C_{ij}(t)}{(1 - t^\sigma)(1 - t)^{d^C_{ij}}},
\]
\[
D_{ij}(t) \mod p^nW[[t]] = \frac{F^D_{ij}(t)}{(1 - t^\sigma)(1 - t)^{d^D_{ij}}},
\]
in \( W/p^nW[[t]] \) with \( F^A_{ij}(t), F^B_{ij}(t), \ldots \in W/p^nW[t] \) polynomials and \( d^A_{ij}, d^B_{ij}, \ldots \in \mathbb{Z}_{\geq 0} \).

Let \( \lambda = t^{-1} \). Then \( H_\lambda \) is a free \( W[[t]] \)-module with basis \( \{\omega_{ij}, \lambda \eta_{ij}\} \) (Theorem 3.6 (2)). Therefore it follows from (5.3) that the entries of the \( 2 \times 2 \)-matrix in below lie in \( W[[\lambda]] \),
\[
(\Phi(\omega_{i'j'}) \quad \Phi(\lambda \eta_{i'j'})) = (\omega_{ij} \quad \lambda \eta_{ij}) \begin{pmatrix} pA_{ij} & \lambda \sigma B_{ij} \\ p\lambda^{-1}C_{ij} & \lambda \sigma \lambda^{-1} D_{ij} \end{pmatrix}.
\]
This implies
\[
\begin{align*}
\deg(pF^A_{ij}) & \leq d^A_{ij} + p \\
\deg(F^B_{ij}) & \leq d^B_{ij} + 2p \\
\deg(pF^C_{ij}) & \leq d^C_{ij} + p - 1 \\
\deg(F^D_{ij}) & \leq d^D_{ij} + 2p - 1.
\end{align*} \tag{5.4}
\]
Next we give upper bounds of \( d^A_{ij}, d^B_{ij}, d^C_{ij} \) and \( d^D_{ij} \). Let \( \lambda = 1 - t \) and let \( \omega_{ij}^*, \eta_{ij}^* \) be the basis of \( H_\lambda \) in Theorem 3.6 (3). Let
\[
(\Phi(\omega_{i'j'}^*) \quad \Phi(\eta_{i'j'}^*)) = (\omega_{ij}^* \quad \eta_{ij}^*) \begin{pmatrix} pA^*_{ij} & B^*_{ij} \\ pC^*_{ij} & D^*_{ij} \end{pmatrix}.
\]
It follows from (5.2) that we have
\[
(1 - t)^{p\sigma}pA^*_{ij}, (1 - t)^{p\sigma}B^*_{ij}, (1 - t)^{p\sigma}pC^*_{ij}, (1 - t)^{p\sigma}D^*_{ij} \in W[[\lambda]] + p^nW((\lambda))^\wedge. \tag{5.5}
\]
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If $i/N + j/M \geq 1$, then $(\omega_{ij}^*, \eta_{ij}^*) = (\omega_{ij}, \eta_{ij})$, and if $i/N + j/M > 1$, then

$$(\omega_{ij}^*, \eta_{ij}^*) = (\omega_{ij}, \eta_{ij}) \begin{pmatrix} 1 - t & lt \\ 0 & -1 \end{pmatrix},$$

where $l := 1 - i/N - j/M$. Therefore if $i/N + j/M \geq 1$ and $i'/N + j'/M \geq 1$, then

$$(pA_{ij}^* B_{ij}^* \quad B_{ij}^* D_{ij}^*) = (pA_{ij}^* B_{ij}^* \quad B_{ij}^* D_{ij}^*).$$

By (5.5), we have $d_{ij}^A, d_{ij}^B, d_{ij}^C, d_{ij}^D \leq pe_n$. If $i/N + j/M < 1$ and $i'/N + j'/M \geq 1$, then

$$(pA_{ij} B_{ij} \quad B_{ij} D_{ij}) = (1 - t \quad lt) \begin{pmatrix} 1 - t^\sigma & lt^\sigma \\ 0 & -1 \end{pmatrix}^{-1}$$

By (5.5), we have $d_{ij}^A \leq pe_n - 1$ and $d_{ij}^B, d_{ij}^C, d_{ij}^D \leq pe_n$. If $i/N + j/M \geq 1$ and $i'/N + j'/M < 1$, then

$$(pA_{ij} B_{ij} \quad B_{ij} D_{ij}) = (1 - t \quad lt) \begin{pmatrix} 1 - t^\sigma & lt^\sigma \\ 0 & -1 \end{pmatrix}^{-1}$$

By (5.5), we have $d_{ij}^A, d_{ij}^B, d_{ij}^C, d_{ij}^D \leq pe_n$. In any case one has

$$d_{ij}^A, d_{ij}^B, d_{ij}^C, d_{ij}^D \leq pe_n. \quad (5.6)$$

Theorem 5.1 follows from (5.4) and (5.6). \qed

### 5.2 Algorithm for computing Dwork’s $p$-adic hypergeometric functions

For $a, b \in \mathbb{Q}$, let

$$F_{ab}(t) = {}_2F_1 \left( \begin{array}{c} a, b \\ 1 \end{array} ; t \right)$$

be the hypergeometric power series. We give an algorithm for computing the special values

$$F_{ab}^{\text{Dw}, \sigma}(t) := \frac{F_{ab}(t)}{F_{a'b'}(t^\sigma)}, \quad F_{ab}^{\prime}(t) = \frac{F_{ab}(t)}{F_{ab}(t)},$$

(5.7)
at $\alpha \in W^\times \setminus (1 + pW)$ modulo $p^n$.

**Notation.** Let $N, M \geq 2$ be integers, and $p > \max(N, M)$ a prime. Let $W = W(\overline{F}_p)$. Let $a, b \in \mathbb{Q}$ satisfy that $a \in \frac{1}{N}\mathbb{Z}$ and $b \in \frac{1}{M}\mathbb{Z}$ and $0 < a, b < 1$. Let $a'$ denote the Dwork prime (see [2.1]). Let $c \in 1 + pW$, and let $\sigma : \overline{W}[t] \to \overline{W}[t]$ be the $p$-th Frobenius given by $\sigma(t) = ct^p$. Following the notation in (4.5) and Theorem 4.3, we define

\[
\frac{d}{dt} \tau_{ab}(t) = \frac{1}{t} \left( 1 - \frac{1}{(1-t)^{a+b}F_{ab}(t)^2} \right), \quad \tau_{ab}(0) = 0,
\]

\[
\tau_{ab}^{(\sigma)}(t) = -2\gamma_p - \psi_p(a) - \psi_p(b) + p^{-1}\log(c) + \tau_{ab}(t) - \tau_{ab}^{(\sigma)}(t^\sigma) \in \overline{W}[t],
\]

and

\[
A_{ab}(t) := \frac{F_{a'b'}(t^\sigma)}{F_{ab}(t)} - t(1-t)^{a+b}F_{ab}(t)F_{a'b'}(t^\sigma)\tau_{ab}^{(\sigma)}(t)
\]

\[
C_{ab}(t) := (1-t)^{a+b-1}F_{ab}(t)F_{a'b'}(t^\sigma)\tau_{ab}^{(\sigma)}(t)
\]

\[
B_{ab}(t) := pt^\sigma (1-t^\sigma)F_{a'b'}(t^\sigma)A_{ab}(t) - t^\sigma(1-t)^{a+b}F_{ab}(t)
\]

\[
D_{ab}(t) := pt^\sigma (1-t^\sigma)F_{a'b'}(t^\sigma)C_{ab}(t) + \frac{(1-t)^{a+b-1}F_{ab}(t)}{1-t^\sigma F_{a'b'}(t^\sigma)}.
\]

Let $a^{(k)}$ denote the $k$-th Dwork prime. Put

\[
F^{(k)}(t) := F_{a^{(k)}b^{(k)}}(t), \quad A^{(k)}_{\sigma}(t) := A_{a^{(k)}b^{(k)}}(t), \ldots, D^{(k)}_{\sigma}(t) := D_{a^{(k)}b^{(k)}}(t),
\]

\[
\mathcal{D}F^{(k)}(t) := \left( \frac{F^{(k)}(t)}{F^{(k)}(t)} \right)'.
\]

for $k \geq 0$. Note that $F^{(k)}(t)$ and $\mathcal{D}F^{(k)}(t)$ do not depend on $\sigma$. We put

\[
E^{(k)}_{\sigma}(t) := \frac{(1-t)^{a^{(k)}b^{(k)}}}{(1-t^\sigma)^{a^{(k+1)}b^{(k+1)}}} = (1-t)^{m_k} \left( \frac{(1-t)^p}{1-t^\sigma} \right)^{a^{(k+1)}b^{(k+1)}}
\]

where $m_k := a^{(k)} - pa^{(k+1)} + b^{(k)} - pb^{(k+1)} \in \mathbb{Z}_{\leq 0}$. Note $E^{(k)}_{\sigma}(t) \in (A^\times)^{\times}$. We have

\[
D^{(k)}_{\sigma}(t) = pt^\sigma (1-t^\sigma)C^{(k)}_{\sigma}(t)\mathcal{D}F^{(k+1)}(t^\sigma) + \frac{1-t^\sigma}{1-t} E^{(k)}_{\sigma}(t)\mathcal{D}W_{\sigma}(k)(t)
\]

and

\[
H^{(k)}_{\sigma}(t) = \left( \frac{pA^{(k)}_{\sigma}(t)}{pC^{(k)}_{\sigma}(t)} \right) \left( \frac{B^{(k)}_{\sigma}(t)}{D^{(k)}_{\sigma}(t)} \right) \left( t^\sigma (1-t^\sigma)\mathcal{D}F^{(k+1)}(t^\sigma) \right) - 1
\]

\[
= \frac{1-t^\sigma}{1-t} E^{(k)}_{\sigma}(t)\mathcal{D}W_{\sigma}(k)(t) \left( t(1-t)\mathcal{D}F^{(k)}(t) \right) - 1
\]

(5.9)

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Algorithm

Let $m \geq 1$ be the smallest integer such that $(a^{(m)}, b^{(m)}) = (a, b)$. Let $\alpha \in W^\times \setminus (1 + pW)$ be an arbitrary element satisfying

$$F(k)(t)_{|t=\alpha} \not\equiv 0 \mod pW, \quad 0 \leq k \leq m - 1.$$ 

Let $\sigma(t) = ct^p$ with $c \in 1 + pW$ arbitrary. The algorithm for computing (5.7) is the following.

**Step 1.** Let $\beta \in W^\times \setminus (1 + pW)$ satisfy

$$F(k)(t)_{|t=\beta} \not\equiv 0 \mod pW, \quad 0 \leq k \leq m - 1.$$ 

In **Step 3**, we shall take $\beta = t^\sigma|_{t=\alpha} = c\alpha^p$. Let $\sigma_\beta(t) = \beta^{1-p}t^p$ so that we have $t^{\sigma_\beta}|_{t=\beta} = \beta$.

Then we compute the special values

$$pA^{(k)}_{\sigma_\beta}(\beta), \ pC^{(k)}_{\sigma_\beta}(\beta), \ B^{(k)}_{\sigma_\beta}(\beta), \ D^{(k)}_{\sigma_\beta}(\beta) \mod p^nW$$

for each $k = 0, 1, \ldots, m - 1$. One can do it in the following way. Compute the power series

$$(1 - t^{\sigma_\beta})(1 - t)^{pe_n}A^{(k)}_{\sigma_\beta}(t)$$

until the degree $pe_n + p$, say $F^A(t)$. Then it follows from Theorem 5.1 that

$$pA^{(k)}_{\sigma_\beta}(\beta) \equiv \frac{pF^A(t)}{(1 - t^{\sigma_\beta})(1 - t)^{pe_n}} \mod p^nW[[t]]$$

and hence

$$pA^{(k)}_{\sigma_\beta}(\beta) \equiv \frac{pF^A(\beta)}{(1 - \beta)^{pe_n+1}} \mod p^nW.$$ 

The other values are obtained in the same way.

**Step 2.** We mean $H^{(l)}_{\sigma_\beta} = H^{(l_0)}_{\sigma_\beta}$ for arbitrary $l \in \mathbb{Z}$ where $l_0 \in \{0, 1, \ldots, m - 1\}$ such that $l \equiv l_0 \mod m$.

Compute an eigenvector $u_\beta$ of a $2 \times 2$-matrix

$$H^{(k-m)}_{\sigma_\beta}(\beta) \cdots H^{(k-2)}_{\sigma_\beta}(\beta)H^{(k-1)}_{\sigma_\beta}(\beta)$$

whose eigenvalue is a unit. This is unique up to scalar. Indeed, it follows from (5.9) that the vector

$$\begin{pmatrix} \beta(1 - \beta) & F^{(k)}(\beta) \\ -1 \end{pmatrix}$$

is an eigenvector of $H^{(1)}_{\sigma_\beta}(\beta) \cdots H^{(m)}_{\sigma_\beta}(\beta)$ whose eigenvalue is

$$\prod_{k=0}^{m-1} E^{(k)}_{\sigma_\beta}(\beta)D_{\sigma_\beta}^{(k)}(\beta) \in W^\times.$$ 

(5.11)
The other eigenvalue is not a unit as $\det(H^{(0)}_{\sigma}(\beta) \cdots H^{(m-1)}_{\beta}(\beta)) = p^m \times \text{(unit)}$ by Remark 4.8 (actually the determinant is equal to $p^m$). Therefore (5.10) is characterized as the eigenvector with the unique eigenvalue which is a unit. We thus have the special value

$$D F^{(k)}(\beta) = \frac{F'_{a(b|b)}(t)}{F'_{a(b|b)}(t)} |_{t=\beta} \mod p^n W$$

for each $k$.

**Step 3.** Let $\sigma(t) = c t^p$ be as in the beginning. Take $\beta = t^\sigma |_{t=\alpha} = c \alpha^p$ in Step 2. We have

$$D F^{(1)}(\beta) = D F^{(1)}(t^\sigma) |_{t=\alpha} \mod p^n W.$$ 

Compute the special values

$$p C^{(0)}_{\sigma}(\alpha), D^{(0)}_{\sigma}(\alpha), E^{(0)}_{\sigma}(\alpha) \mod p^n W$$

according to Step 1, and

$$E^{(0)}_{\sigma}(\alpha) \mod p^n W$$

utilizing the expansion

$$\left( \frac{1 - t^p}{1 - t^\sigma} \right)^{a+b} = \sum_{n=0}^{\infty} p^n \binom{a+b}{n} u(t)^n, \quad \frac{(1 - t^p}{1 - t^\sigma} = 1 + p u(t).$$

Substitute $t = \alpha$ in (5.8). Then we have the special value

$$\mathcal{D} F_D W, \sigma^{(0)}(\alpha) = \frac{F_{ab}(t)}{F'_{u|b}(t^\sigma)} |_{t=\alpha} \mod p^n W$$

as $E^{(0)}_{\sigma}(\alpha) \in W^\times$.

### 5.3 Bit Complexity

We give an upper estimate of the bit complexity of the algorithm displayed in §5.2.

We review the notion of the bit complexity. A general reference is the text book [BZ]. The bit of a natural number $N$ is defined to be the number of digits of $N$ in binary notation, so it is at most $\log_2(N + 1)$. The bit of $N!$ is at most $\log_2(N! + 1) \sim (\log 2)^{-1} N \log N$ (Stirling). The bit complexity of an algorithm is defined to be the number of single operations to complete the algorithm. The bit complexity of $(1\text{-digit}) \pm (1\text{-digit})$ or $(1\text{-digit}) \times (1\text{-digit})$ is 1 by definition. We denote by $M(n, m)$ the bit complexity of multiplication $(n\text{-digits}) \times (m\text{-digits})$. We write $M(n) = M(n, n)$. By the naive multiplication algorithm, $M(n, m)$ is $O(nm)$, which means that there is a constant $C$ such that $M(n, m) \leq Cnm$ when $n, m \to \infty$. We sum up the basic results.

- For integers $i, j \geq 0$, the bit complexity of $i \pm j$ is $O(\max(\log i, \log j))$. 

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• The bit complexity of $i \cdot j$ is $M(\log i, \log j)$ (which is at most $O(\log i \log j)$).

• The bit complexity for computing the remainder $(i \mod j)$ is $M(\log i, \log j)$.

Let $a$ be a fixed rational number. Then the bit complexity of $(a)_i$ is at most

$$
\sum_{n=1}^{i} M(n \log n, \log n) \leq O(i^2(\log i)^2) \quad (5.12)
$$

by computing it in the following way

$$(a)_i = (a + i - 1) \cdot (a)_{i-1}, \quad (a)_{i-1} = (a + i - 2) \cdot (a)_{i-2}, \ldots$$

Let $a_i, b_j$ be rational numbers whose denominators and numerators are less than $k$. Let $f(t) = \sum_{i=0}^{n} a_i t^i$ and $g(t) = \sum_{j=0}^{n} b_j t^j$. Then the bit complexity of computing $f(t) \pm g(t)$ is $O(n \log k)$ The bit complexity of computing $f(t)g(t)$

$$n^2 M(\log k) + O(n^2 \log(kn)) < O(n^2(\log n + (\log k)^2)) \quad (5.13)$$

on noticing that the coefficients of $f(t)g(t)$ are ratios of integers at most $nk$.

Let us see the bit complexity of our algorithm in 5.2. Fix $p, a, b, c$ and $\alpha$. We need to compute the power series

$$
\tau_{ab}^{(\sigma)}(t), \quad A_{\sigma}^{(k)}(t), \ldots, D_{\sigma}^{(k)}(t), \quad E_{\sigma}^{(k)}(t) \quad (5.14)
$$

until the degree $pe_n + 2p \sim p(p-1)/(p-2)n$. First of all, the bit complexities of computing the constants

$$
\gamma_p + \psi_p(a^{(k)}), \quad \gamma_p + \psi_p(b^{(k)}), \quad \log c
$$

modulo $p^n$ are small (cf. Appendix A), so that we can ignore them. Moreover the power series $E_{\sigma}^{(k)}(t)$ is simple, so we can also ignore the bit complexity of computing it.

We observe the bit complexity of computing $\tau_{ab}^{(\sigma)}(t)$. We work in a ring

$$K[t]/(t^{pe_n+2p+1}), \quad K := \text{Frac} W.$$

We begin with the truncated polynomials

$$F_{ab}(t) \in K[t]/(t^{pe_n+2p+1}).$$

By (5.12), the bit complexities of computing all the coefficients are at most

$$\sum_{i=0}^{pe_n+2p} O(i^2(\log i)^2) < O(n^3(\log n)^2).$$

Next we need compute

$$\frac{1}{F_{ab}(t)} = (1 + f)(1 + f^2) \cdots (1 + f^{2^d}) \in K[t]/(t^{pe_n+2p+1}), \quad f := 1 - F_{ab}(t) \quad (5.15)$$
where \( d := \lfloor \log_2(p \varepsilon_n + 2p) \rfloor + 1 \sim \log_2 n. \) The denominators and numerators of the coefficients of \( f^k \) for \( k \leq p \varepsilon_n + 2p \) are at most

\[
\sum_{i_1 + \cdots + i_k = l, i_r \geq 1} (i_1! \cdots i_k!)^2 < (l!)^2 \left( \frac{l - 1}{k - 1} \right) < (l!)^2 l^{p \varepsilon_n + 2p} < (n!)^2 n^{cn} \tag{5.16}
\]

with \( c > 0 \) a constant. Hence the bit complexities of computing \( f^2, \ldots, f^{2d} \) are at most

\[
O(n^2 (\log(n^2) n)^2) = O(n^4 (\log n)^2)
\]

by (5.13), and hence the bit complexity of computing (5.15) is

\[
O(dn^4 (\log n)^2)) = O(n^4 (\log n)^3).
\]

Summing up the above, the bit complexity of computing \( \tau^{(\sigma)}_{ab}(t) \) is \( O(n^4 (\log n)^3). \)

The power series of \( A^{(k)}(t), \ldots, D^{(k)}(t) \) are obtained by applying standard arithmetic operations (addition, subtraction and multiplication) on polynomials whose coefficients are ratios of integers at most (5.16). Therefore the bit complexities do not exceed \( O(n^4 (\log n)^3). \) All the algorithms in \textbf{Step 1}, \ldots, \textbf{Step 3} are standard arithmetic operations on the coefficients in the polynomials (5.14). One concludes that the total bit complexity of the algorithm in \textsection 5.2 is \( O(n^4 (\log n)^3). \)

6 Appendix A : \( p \)-adic polygamma functions

We give a brief review of \( p \)-adic polygamma functions introduced in \cite{A} \textsection 2.

Let \( r \in \mathbb{Z} \) be an integer. For \( z \in \mathbb{Z}_p \), define

\[
\tilde{\psi}^{(r)}_p(z) := \lim_{n \to z} \sum_{1 \leq k \leq n, \nu \nmid k} \frac{1}{k^{r+1}} \tag{6.1}
\]

where “\( n \to z \)” means the limit with respect to the \( p \)-adic metric. The existence of the limit follows from the fact that

\[
\sum_{1 \leq k < p^n, \nu \nmid k} k^m = \begin{cases} 
-p^{n-1} & p \geq 3 \text{ and } (p - 1) | m \\
2^{n-1} & p = 2 \text{ and } 2 | m \\
1 & p = 2 \text{ and } n = 1 \\
0 & \text{otherwise} 
\end{cases} \tag{6.2}
\]

modulo \( p^n \). Thus \( \tilde{\psi}^{(r)}_p(z) \) is a \( p \)-adic continuous function on \( \mathbb{Z}_p \). Define the \( p \)-adic Euler constant by

\[
\gamma_p := - \lim_{n \to \infty} \frac{1}{p^n} \sum_{0 \leq j < p^n, \nu \nmid j} \log(j), \quad (\log = \text{Iwasawa log}).
\]
We define the $r$-th $p$-adic polygamma function to be

$$
\psi_p^{(r)}(z) := \begin{cases} 
-\gamma_p + \tilde{\psi}_p^{(0)}(z) & r = 0 \\
-\zeta_p(r+1) + \tilde{\psi}_p^{(r)}(z) & r \neq 0 
\end{cases}
$$

(6.3)

where $\zeta_p(r+1)$ is the special value of the $p$-adic zeta function (see [A, Lem 2.3]). If $r = 0$, we also write $\psi_p(z) = \psi_p^{(0)}(z)$ and call it the $p$-adic digamma function.

Concerning Dwork’s $p$-adic hypergeometric functions, we need to compute the special values of $\tilde{\psi}_p(z) = \psi_p(z) + \gamma_p$ modulo $p^n$ (cf. (4.5)). To do this, the sum (6.1) is not useful because the number of terms increases with exponential order by (6.2). However we can avoid this difficulty by using the following theorem.

**Theorem 6.1 ([A, Thm.2.5])** Let $0 \leq i < N$ be integers and suppose $p \nmid N$. Then

$$
\tilde{\psi}_p^{(r)} \left( \frac{i}{N} \right) = N^r \sum_{\varepsilon \in \mu_N \setminus \{1\}} (1 - \varepsilon^{-i}) \ln_1^{(p)}(\varepsilon) 
$$

(6.4)

where $\ln_k^{(p)}(z)$ are the $p$-adic polylogarithmic functions (cf. [A, §2.1]).

Let $r = 0$. Then

$$
\ln_1^{(p)}(z) = -p^{-1} \log \frac{(1 - z)^p}{1 - z^p} = \sum_{n=1}^{\infty} \frac{p^{n-1}}{n} w(z)^n, \quad w(z) := p^{-1} \left( 1 - \frac{(1 - z)^p}{1 - z^p} \right).
$$

Using this expansion, one can compute $\tilde{\psi}_p(i/N) \mod p^n$ without (6.1).

7 Appendix B : Resolution of Singularities

Let $W$ be a commutative ring. Let $X$ be a smooth $W$-scheme of relative dimension $d \geq 2$ or its completion along a closed subscheme. A divisor $D$ is called a relative normal crossing divisor (abbreviated relative NCD) over $W$ if it is locally defined by $x_1 \cdots x_s = 0$ where $(x_1, \ldots, x_d)$ is a local coordinates over $W$. Further $D$ is called simple if each component is smooth over $W$.

**Proposition 7.1** Let $n > 0$ be an integer which is invertible in $W$. Let

$$
X := \text{Spec} W[[x, y, s]]/(sx - y^n) \supset C := \text{Spec} W[[x, y, s]]/(s, y^n).
$$

(7.1)

Then there is a proper morphism $\rho : X' \to X$ satisfying the following. Put $D := \rho^{-1}(C)$.

- $X'$ is smooth over $W$, and $X' \setminus D \xrightarrow{\cong} X \setminus C$, 
- $D = E_1 + 2E_2 + \cdots + (n - 1)E_{n-1} + nC'$ where $E_i$ are exceptional curves and $C'$ is the proper transform of $C$.

- $E_1 + E_2 + \cdots + E_{n-1} + C'$ is a simple relative NCD over $W$. The figure is as follows.

\[ \begin{array}{c}
E_{n-1} \\
\vdots \\
E_2 \\
E_1 \\
\end{array} \]

\[ C' \]

\[ nC' \]

\[ \begin{array}{c}
(O) \\
E_1 \\
(n - 1)E_2 \\
\end{array} \]

\[ 4C' \]

\[ 3E_2 \\
2E_3 \\
E_1 \\
\end{array} \]

\[ \text{Figure. } n \geq 3 \]

\[ \text{Figure. } n = 4 \]

\textbf{Proof.} Let $\rho_1 : X_1 \rightarrow X$ be the blow-up with center $(x, y, s) = (0, 0, 0)$. Then $X_1$ is covered by affine open sets

\[ U_1 = \text{Spec} W[[x, y, s]][y_1, s_1]/(s_1 - x^{n-2}y_1^n, xy_1 - y, xs_1 - s) \]

\[ \cong \text{Spec} W[[x, y, s]][y_1]/(xy_1 - y, x^{n-1}y_1^n - s), \]

\[ U_2 = \text{Spec} W[[x, y, s]][x_2, y_2]/(x_2 - s^{n-2}y_2^n, sy_2 - y, sx_2 - x) \]

\[ \cong \text{Spec} W[[x, y, s]][y_2]/(sy_2 - y, s^{n-1}y_2^n - x), \]

\[ U_3 = \text{Spec} W[[x, y, s]][x_3, s_3]/(s_3x_3 - y^{n-2}, yx_3 - x, ys_3 - s). \]

$U_1$ and $U_2$ are smooth over $W$. If $n = 2$, there is a unique exceptional curve $E$ such that $E \cap U_1 = \{x = 0\}$, and $\rho_1^{-1}(C) = E + 2C'$ where $C'$ is the proper transform of $C$. $X_1$ is smooth over $W$ and $E + C'$ is a simple relative NCD, so we are done. If $n \geq 3$, then the divisor $D_1 := \rho_1^{-1}(C) = E_1 + (n - 1)E_2 + nC'$ is as follows.
If $n = 4$, then $X_2$ is smooth over $W$ and $\rho_2^{-1}(D_1) = E_1 + 3E_2 + 2E_3 + 4C'$ is as in the figure where $E_3$ is the unique exceptional curve. So we are done. If $n \geq 5$, then $D = \rho_2^{-1}(D_1) = (n-1)E_2 + 2E_3 + (n-2)E_4 + nC'$ is as follows.

\[
\begin{array}{c|c}
(n-1)E_2 & nC' \\
(n-2)E_4 & E_1 \\
\end{array}
\]

Figure. $n \geq 5$

If $n = 5$ then $X_2$ is smooth over $W$, and so we are done. If $n \geq 6$, there is a singular point $O$. In a neighborhood of $O$, $X_2$ is defined by an equation $s_4x_4 = y^{n-4}$, and $E_3 = \{y = x_4 = 0\}$, $E_4 = \{y = s_4 = 0\}$ and $E_2 = \{s_4y^2 = 0\}$. Then we take the blowing-up at $O$. Continuing this, we finally obtain $\rho : X' \rightarrow X$ with $X'$ a smooth $W$-scheme such that $\rho^{-1}(C) = E_1 + 2E_2 + \cdots + (n-1)E_{n-1} + nC'$ and $E_1 + \cdots + E_{n-1} + C'$ is a simple relative NCD over $W$. □

**Proposition 7.2** Let $N, M > 0$ be integers which are invertible in $W$. Let $d = \gcd(N,M)$. Suppose that $W$ contains a primitive $d$-th root of unity. Let

\[
X := \text{Spec} W[[x,y]] \supset C := \text{Spec} W[[x,y]]/(x^N + y^M). \quad (7.2)
\]

Then there is a proper morphism $\rho : X' \rightarrow X$ satisfying the following. Put $D := \rho^{-1}(C)$.

- $X'$ is smooth over $W$, and $X' \setminus D \xrightarrow{\sim} X \setminus C$,
- $D = \sum n_iD_i$ with $D_i$ smooth over $W$. Moreover $D = \sum D_i$ is a simple relative NCD over $W$, and the multiplicities $n_i$ are either of

  \[ 1, \ iN, \ jM, \ i \in \{1, \ldots, M\}, \ j \in \{1, \ldots, N\}. \]

The figure of $\sum_i D_i$ is as follows, where $C'$ is the proper transform of $C$ which has $d$-components.

\[
\begin{array}{c|c|c|c}
& & & C' \\
\cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

**Proof.** For integers $a, b, c, d \geq 0$, we denote by $I(a, b; c, d)$ the divisor $\text{Spec} W[[x,y]]/(x^ay^b(x^c + y^d))$ in $\text{Spec} W[[x,y]]$. Our goal is to compute the embedded resolution of $I(0,0; N, M)$.

Let $D = \text{Spec} W[[x,y]]/(x^ay^b(x^c + y^d)) = aD_x + bD_y + D_s \subset X$ where $D_x := \{x = 0\}$, $D_y := \{y = 0\}$ and $D_s := \{x^c + y^d = 0\}$. Let $\rho : X' \rightarrow X$ be the blow-up with center $(x,y) = (0,0)$. Then $D' := \rho^{-1}(D) = (a + b + c)E + aD'_x + bD'_y + D'_s$ where $D'_s$...
denotes the proper transform of $D_s$ and $E$ the exceptional curve. In case $c < d$, there
is a unique point $O$ which is not normal crossing, and it is locally given by an equation
$x^{a + b + c} y^{d - c} = 0$, namely $I(a, a + b + c; c, d - c)$. The multiplicities of $D'$ are
$1, a, b, a + b + c$. In case $c > d$, there is also a unique point $O$ such that the divisor $D'$ around $O$
is $I(a + b + c)E + aD_x' + bD_y' + D_s'$ satisfies that $E + D_x' + D_y' + D_s'$ is a
simple relative NCD over $W$, and $D_s'$ has $c$-components (see the figure). In this case we stop
the resolution.

Case $c < d$

\[
\begin{array}{c}
O \\
\downarrow \\
bD_y' \\
aD_x' \\
\downarrow \\
D_s'
\end{array}
\begin{array}{c}
(a + b + c)E \\
\end{array}
\]

Case $c = d$

\[
\begin{array}{c}
\cdots \\
\downarrow \\
bD_y' \\
aD_x' \\
\downarrow \\
D_s'
\end{array}
\begin{array}{c}
(a + b + c)E \\
\end{array}
\]

Define

\[
(I(a, b; c, d))' := \begin{cases} 
I(a, a + b + c; c, d - c) & c \leq d \\
I(a + b + d; b; c - d, d) & c > d \\
I(a, b; c, d) & cd = 0 
\end{cases}
\]

and $I^{(0)} = I$, $I^{(i)} = (I^{(i-1)})'$. We begin with $I(0, 0; N, M)$ and consider a sequence
$I(a_i, b_i; c_i, d_i) := (I(0, 0; N, M))^{(i)}$

\[I(0, 0; N, M), I(a_1, b_1; c_1, d_1), \ldots, I(a_n, b_n; c_n, d_n)\]

until $c_n d_n = 0$. This corresponds to the sequence of blowing ups at $O$’s as above

\[X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_0 = X\]

such that the inverse image of $C$ in $X_n$ is supported in a relative simple NCD. Moreover let $D_i \subset X_i$ be the inverse image of $C$. Then the multiplicities of $D_i$ are either of
$1, a_1, \ldots, a_i, b_1, \ldots, b_i$. Therefore if we show Lemma 7.3 below (which is a simple lemma
in elementary number theory), then it ends the proof of Proposition 7.2. □
Lemma 7.3  Let $N, M \geq 1$ be integers and let $I(a_i, b_i; c_i, d_i) := (I(0, 0; N, M))^{(i)}$ be defined by (7.3). Let $n$ be the minimal integer such that $c_n d_n = 0$.

(1) There are integers $A_i, B_i, C_i, D_i \geq 0$ such that $a_i = A_i M, b_i = B_i N, c_i = C_i N - A_i M, d_i = D_i M - B_i N$.

(2) $A_i, B_i, C_i, D_i$ are non-decreasing sequences, and $A_n, D_n \leq N$ and $B_n, C_n \leq M$.

Proof. (1) The assertion is clear for $i = 0$ by putting $(A_0, B_0, C_0, D_0) := (0, 0, 1, 1)$. Suppose that the assertion holds for $i$. By definition

\[
(a_{i+1}, b_{i+1}, c_{i+1}, d_{i+1}) = \begin{cases} 
(a_i, a_i + b_i + c_i, c_i - d_i) & c_i \leq d_i \\
(a_i + b_i + d_i, b_i, c_i - d_i, d_i) & c_i > d_i
\end{cases}
\]

Hence the assertion holds by putting

\[
(A_{i+1}, B_{i+1}, C_{i+1}, D_{i+1}) := \begin{cases} 
(A_i, B_i + C_i, C_i, A_i + D_i) & c_i \leq d_i \\
(A_i + D_i, B_i, B_i + C_i, D_i) & c_i > d_i
\end{cases}
\]

(7.4)

(2) The former assertion is obvious from (7.3). We show $A_n, D_n \leq N$ and $B_n, C_n \leq M$. The algorithm $(c_0, d_0) \rightarrow (c_1, d_1) \rightarrow \cdots \rightarrow (c_n, d_n)$ is the Euclidean algorithm. Therefore $(c_n, d_n) = (0, \gcd(N, M))$ or $(\gcd(N, M), 0)$. In case $(c_n, d_n) = (0, \gcd(N, M))$, $A_n, B_n, C_n, D_n$ are characterized as the minimal positive integers satisfying $C_n N = A_n M$ and $D_n M - B_n N = \gcd(N, M)$. Hence it turns out that $A_n, D_n \leq N$ and $B_n, C_n \leq M$. The conclusion is the same also in case $(c_n, d_n) = (\gcd(N, M), 0)$. \qed

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