About the Equivalence of Cut-off and Renormalised Effective Field Theories

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Abstract

The equivalence of cut-off and renormalized effective field theories is demonstrated for the example of very low energy effective field theory for the nucleon-nucleon interaction.

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I. INTRODUCTION

It is widely believed that QCD is the correct theory of strong interactions. On the other hand nuclear forces are studied within different potential models. It is not clear whether or not these phenomenological approaches can be justified from fundamental theory. Effective field theory is thought as a bridge between QCD and potential models. Chiral perturbation theory serves as a low-energy effective field theory inspired by QCD.

There has been much recent interest in the EFT approach to the nucleon-nucleon scattering problem (see [4–10,13–16] and citations in these papers). The chiral perturbation theory approach for processes involving an arbitrary number of nucleons was formulated in [2,3]. Unlike purely pionic processes [1], for the $n$-nucleon problem power counting should be used for the “effective potentials” and not the full amplitudes. The effective potential is defined as a sum of time-ordered perturbation theory diagrams for the $T$-matrix excluding those with purely nucleonic intermediate states.

To find the full $S$-matrix one should solve a Lippmann-Schwinger equation (or Schröedinger equation) with this effective potential in place of the interaction Hamiltonian, and with only $n$-nucleon intermediate states [2].

The Lagrangian of effective field theory is highly non-renormalizable in the traditional sense but it contains all possible terms which are not suppressed by the symmetries of the theory and the ultraviolet divergences are absorbed into the parameters of the Lagrangian. Renormalization points are chosen of the order of external momenta $p$ or less. After renormalization, the effective cut-off is of order $p$ [3].

Renormalised effective field theory encounters quite severe (technical) problems: if one takes the potential up to some order and tries to iterate via the Lippmann-Schwinger equation one will encounter divergences. One could try to include counter-terms in the potential, but due to the non-renormalizability of the theory the inclusion of an infinite number of terms with more and more derivatives will be needed. One could even think that Weinberg’s power counting breaks down because higher order terms are strongly involved. But it should be remembered that power counting (for both amplitudes and the potentials) is valid after renormalization when the contributions of counter terms are already taken into account [2,3,4]. So, one has either to exactly solve (formally) the equation and after subtract divergences explicitly, or otherwise one should draw all relevant diagrams, subtract them and then sum these renormalised diagrams up. In recent papers [3,5] Kaplan et. al suggested a systematic method of summation of an infinite number of diagrams using dimensional regularization and the Power Divergent Subtraction scheme. But as was mentioned in the above cited papers for the external momenta exceeding 100 MeV it is difficult to justify suggested approximations, so for higher energies the problem of summation of renormalized diagrams remains open. Fortunately these problems can be overcome using cut-off theory. One can calculate up to any desired order, but there is a very crucial question: what is the relation between cut-off and renormalised theories? This question is addressed in a number of papers [3,13,17,18], but as yet the complete answer has not been determined. Moreover some authors question the validity and systematic character of cut-off theory calculations (see for example [3,13,20]). This work is an attempt to investigate some details about the equivalence of renormalized and cut-off theories.

Below the simple example of contact interaction of nucleons in $^1S_0$ wave is considered. The amplitude is renormalized by subtracting divergent integrals at some normalisation point and
its relation to the amplitude obtained from cut-off theory is studied. The numerical values of phase shifts obtained from renormalised and cut-off theories (without removing cut-off) are compared.

II. $P^2$ ORDER CALCULATIONS

For the very low energy nucleon-nucleon scattering processes the pions can be integrated out and the effective non-relativistic Lagrangian takes the following form [13]:

$$\mathcal{L} = N^\dagger i\partial_t N + N^\dagger \frac{\nabla^2}{2M} N - \frac{1}{2} C_S \left( N^\dagger N \right)^2 - \frac{1}{2} C_T \left( N^\dagger \sigma N \right)^2 - \frac{1}{2} C_2 \left( N^\dagger \nabla^2 N \right) \left( N^\dagger N \right) + h.c. + \ldots$$

(1)

where the nucleonic field $N$ is a two-spinor in spin space and a two-spinor in isotopic spin space and $\sigma$ are the Pauli matrices acting on spin indices. $M$ is the mass of nucleon and the ellipses refer to additional 4-nucleon operators involving two or more derivatives, as well as relativistic corrections to the propagator. $C_T$ and $C_S$ are couplings introduced by Weinberg [2,3], they are of dimension $(mass)^{-2}$ and $C_2$ is of the order $(mass)^{-4}$.

The leading order contribution to the 2-nucleon potential is

$$V_0(p, p') = C_S + C_T (\sigma_1, \sigma_2),$$

(2)

in the $^1S_0$ wave it gives:

$$V_0(p, p') = C$$

(3)

where $C = C_S - 3C_T$. The next to leading order contribution to the 2-nucleon potential in the $^1S_0$ channel takes the form:

$$V_2(p, p') = C_2 \left( p^2 + p'^2 \right)$$

(4)

The formal iteration of the potential $V_0 + V_2$ using the Lippmann-Schwinger equation gives for on-shell ($E = p^2/M$) $s$-wave $T$-matrix [4]:

$$\frac{1}{T(p)} = \frac{(C_2 I_3 - 1)^2}{C + C_2^2 I_5 + p^2 C_2 (2 - C_2 I_3)} - I(p)$$

(5)

$$I_n = -M \int \frac{d^3k}{(2\pi)^3} k^{n-3}, \quad I(p) = M \int \frac{d^3k}{(2\pi)^3} \frac{1}{p^2 - k^2 + i\eta} = I_1 - \frac{iMp}{4\pi},$$

(6)

where $p$ is the on-shell momentum and $I_1, I_3$ and $I_5$ are divergent integrals.
A. Renormalization by subtracting divergences

To renormalize (5) it is necessary to include contributions of an infinite number of counter-terms with higher and higher (up to infinity) derivatives [16]. While it is impossible to write down all these contributions explicitly it is quite straightforward to renormalize (5) by just subtracting divergent integrals. Before implementing this scheme it would be useful to write down the leading and $p^2$ order counter-terms.

One can write down the chiral expansion for $T$-matrix [13] (it is equivalent to an expansion of $T$ obtained from (3) in powers of $C_2$):

$$ T = \frac{C}{1 - CI(p)} + \frac{2p^2C_2 + 2CC_3}{(1 - CI(p))^2} + ... $$  (7)

and

$$ \frac{1}{T(p)} = -I(p) + \frac{1 - 2C_2I_3}{C} - \frac{2C_2p^2}{C^2} + ... $$  (8)

The final goal is to absorb divergences in (7) into $C$ and $C_2$ which are to be given by

$$ C = C^{(1)}(C_R) + C^{(2)}_2(C_R) + ... \quad C_2 = C^{(1)}_2(C_R) + ... $$  (9)

where $C_R$ and $C^{(2)}_2$ are renormalized coupling constants and $C^{(1)}(C_R)$, $C^{(2)}(C_R)$, $C^{(1)}_2(C_R)$... are functions of $C_R$.

To determine $C$ and $C_2$ in terms of $C_R$ and $C^{(2)}_2$ it is simpler to work with (8) and require:

$$ \frac{1 - 2C_2I_3}{C} - \frac{2C_2p^2}{C^2} - I(p) = \frac{1 - 2C^R_2(I_3 - \Delta_3)}{C_R} - \frac{2C^R_2p^2}{C^2_R} - (I(p) - \Delta) $$  (10)

Where $\Delta$ and $\Delta_3$ are divergent parts of $I(p)$ and $I_3$ integrals (with arbitrary finite contributions). Equating coefficients of different powers of $p$ one gets from (10):

$$ \frac{1 - 2C_2I_3}{C} = \frac{1 - 2C^R_2(I_3 - \Delta_3)}{C_R} + \Delta; \quad \frac{C_2}{C^2} = \frac{C^{(2)}_2}{C^2_R} $$  (11)

and from (11):

$$ C = \frac{-C^2_R \Delta - C_R \{1 - 2C^R_2(I_3 - \Delta_3)\} \pm C_R \left(8C^2_RI_3 + \left[1 - 2C^R_2(I_3 - \Delta_3) + C_R \Delta\right]^2\right)^{\frac{1}{2}}}{4C^2_RI_3} $$  (12)

In ordinary perturbation theory (expansion in terms of coupling constants) one has $C = C_R + ...$.

The non-perturbative expression (12) respects this perturbative expansion if the "+" sign is taken. Expanding the chosen solution in powers of $C^R$ and keeping only terms of first order one obtains:

$$ C = \frac{C_R}{1 + C_R \Delta} + \frac{2C_RC^{(2)}_2(I_3 - \Delta_3)}{(1 + C_R \Delta)^2} - \frac{2C_RC^{(2)}_2I_3}{(1 + C_R \Delta)^3} $$  (13)
and

\[ C_2 = \frac{C_R^2}{(1 + C_R \Delta)^2} \]  

(14)

Substituting (13) and (14) into (7) one gets a finite renormalized expression:

\[ T = \frac{C_R}{1 - C_R [I(p) - \Delta]} + \frac{2p^2 C_R^2 + 2C_R C_2^R (I_3 - \Delta_3)}{(1 - C_R [I(p) - \Delta])^2} \]  

(15)

Switching back to (3) one can apply the subtraction scheme analogous to the one originally used by Weinberg \[3\] and subtract divergent integrals at \( p^2 = -\mu^2 \). Integrals are divided into two parts:

\[ I_n = I_n \left( p^2 = -\mu^2 \right) + \left[ I_n - I_n \left( p^2 = -\mu^2 \right) \right] = I_n^d + I_n^R, \quad i = 3, 5 \]  

(16)

\[ I(p) = I \left( p^2 = -\mu^2 \right) + \left[ I(p) - I \left( p^2 = -\mu^2 \right) \right] = I^d + I^R(p) \]  

(17)

where \( I_n^d \) and \( I^d \) are divergent parts and are to be cancelled by contributions of counter-terms. To absorb all contributions of \( I_n^d \) and \( I^d \) in (3) one needs to include contributions of an infinite number of counter-terms with higher and higher order (up to infinity) derivatives \[10\]. While it is impossible to write down these counter terms explicitly, one can take their contributions into account by just neglecting \( I_n^d \) and \( I^d \) terms and replacing \( C \) and \( C_2 \) by renormalized couplings. Finally the amplitude is left with finite parts of integrals \( I_n^R \) and \( I^R(p) \) (note that \( I_n^R = 0 \)):

\[ \frac{1}{T} = \frac{1}{C_R + 2C_2^R p^2} - I^R(p) \]  

(18)

where

\[ I^R(p) = -\frac{M}{4\pi} \mu - \frac{M}{4\pi} ip \]  

(19)

Note that although the expression (18) was obtained in [14] using Power Divergent Subtractions, that scheme is completely different from the one applied in this work. The difference is clearly seen when pions are included explicitly. (It may be worth mentioning that the subtraction scheme used here is just one among an infinite number of possibilities).

Matching (18) to the effective range expansion

\[ \frac{1}{T} = -\frac{M}{4\pi} \left( -\frac{1}{a} + \frac{1}{2} r_e p^2 + 0 \left( p^4 \right) - ip \right) \]  

(20)

gives

\[ \frac{1}{C_R} + \frac{M}{4\pi} a = \frac{M}{4\pi a} \]  

(21)

\[ \frac{2C_2^R}{C_R} = \frac{Mr_e}{8\pi} \]  

(22)
and from (18)
\[
\frac{1}{T} = -\frac{M}{4\pi} \left( \frac{-\frac{1}{a}(1 - a\mu) - \frac{1}{2}r_ea\mu p^2 - ip}{1 - a\mu + \frac{1}{2}r_eap^2} \right) \tag{23}
\]

The result given in (23) does not depend on the regularization scheme.

Below a cut-off version of the above effective theory is considered and it is demonstrated that these two approaches are equivalent up to (including) \( p^2 \) order. However, before proceeding it should be specified what is understood by the equivalence of two approaches.

In terms of \( \nu \) expansion [13] the NN scattering amplitude is given by
\[
T = T_0 + T_1 + T_2 + T_3 + \ldots \tag{24}
\]
where \( T_0 \) is of leading order, \( T_1 \) is of sub-leading order etc. The expansion parameter is \( \sim Q/\Lambda \) where \( Q \) is of the order of external momenta and \( \Lambda \) is expected to be of the order of the mass of lightest particle which was integrated out. If \( Q << \Lambda \) then the first few terms of (24) should approximate the whole amplitude well. In accordance with power counting [2,3] higher order terms are expected to be small for renormalized theory. As for cut-off theory, it would be most satisfactory if by adjusting the values of the cut-off parameter and coupling constants (including additional couplings introduced into the effective Lagrangian to compensate the effects of the finite cut-off) it could reproduce order by order the series (24) given by renormalized theory. At any finite order the criteria of equivalence would be for cut-off theory to reproduce the renormalized amplitude exactly up to (and including) the order one is working with, the difference being of the order of terms neglected by the approximation. The above condition would guarantee that the amplitudes of renormalized and cut-off theory are in good agreement in the range of validity of approximation made in renormalized theory. It is not evident that it is possible to satisfy these criteria.

**B. Cut-off theory**

Effective potential with sharp cut-off has the following form:
\[
V^{(2)}(p', p) = \left\{ \bar{C} + \bar{C}_2 \left( p^2 + p'^2 \right) \right\} \theta(l - p)\theta(l - p') \tag{25}
\]
Here \( l \) is the cut-off parameter and \( \bar{C} \) and \( \bar{C}_2 \) depend on this parameter. \( l \) should be of the order of the mass of lightest particle which was integrated out [17]. It is not difficult to write down the solution of the Lippmann-Schwinger equation explicitly (see [10]):
\[
\frac{1}{T(p)} = \frac{(\bar{C}_2I_3^l - 1)^2}{\bar{C} + \bar{C}_2^2I_5^l + p^2C_2(2 - C_2I_3^l)} - I_l(p), \tag{26}
\]
where
\[
I_n^l \equiv -M \int \frac{d^3q}{(2\pi)^3} q^{n-3}\theta(l - q) = -\frac{M}{2\pi^2} \frac{l^n}{n} \tag{27}
\]
FIG. 1. Phase shifts calculated in $p^2$ order. Double line corresponds to the effective range expansion, solid line corresponds to the cut-off theory and long-dashed, dash-dotted and short-dashed lines correspond to $\mu = 0$, $\mu = 40$ and $\mu = 130$ MeV respectively.

$I_l(p) = M \int \frac{d^3q}{(2\pi)^3} \frac{1}{p^2 - q^2 + i\epsilon} \theta(l - q) = -\frac{M}{2\pi^2} \left[ l + \frac{p}{2} \ln \frac{1 - \frac{p}{l}}{1 + \frac{p}{l}} \right] - \frac{iM}{4\pi} \rho(l) \quad (28)$

Matching (28) to the effective range expansion (20) leads to:

\[
\left( 1 - \bar{C}_2 I_3^2 \right)^2 \bar{C} + \bar{C}_2^2 I_5^5 = \frac{M}{4\pi a} - \frac{Ml}{2\pi^2} \equiv x \quad (29)
\]

\[
\frac{C_2 (2 - C_2 I_3^2)}{C + C_2^2 I_5^5} = \frac{1}{x} \left( \frac{Mr_e}{8\pi} - \frac{M}{2\pi^2 l} \right) \equiv y \quad (30)
\]

and the solution of these equations for $\bar{C}$ and $\bar{C}_2$ gives:

\[
\bar{C}_2 = \frac{1}{I_3} \left[ 1 - \left( \frac{x}{x + y I_3} \right)^{\frac{1}{2}} \right] \quad (31)
\]

\[
\bar{C} = \frac{1}{x + y I_3} - \bar{C}_2^2 I_5^5 \quad (32)
\]

$\bar{C}_2$ was obtained by solving quadratic equation. Analogously to (12) the sign in solution was fixed respecting the structure of ordinary perturbation theory (expansion in coupling constants).

Substituting (31) and (32) into (26) one gets:

\[
\frac{1}{T} = -\frac{M}{4\pi} \left( \frac{-\pi + 2al}{\pi a + p^2 \rho^2 (\pi a l - 3)} + \frac{4\pi}{M} I_l(p) \right) \quad (33)
\]

Note that higher-order corrections to the cut-off expression are suppressed by powers of $p/l$ and hence are small for momenta well below the cut-off.
FIG. 2. Phase shifts calculated in $p^2$ order cut-off theory. Solid line corresponds to the effective range expansion. The lowest dashed line corresponds to cut-off parameter $l = 110$ MeV and subsequent lines correspond to the values $l = 120, 130, 140, 150, 160, 170$ MeV.

The solution of $a$ and $r_e$ from (21) and (22) (for some value of $\mu$) and substitution into (31) and (32) leads to a lengthy but simple relation between $\bar{C}$, $\bar{C}_2$ and $C_R$, $C_R^2$ the fulfilment of which guarantees that the cut-off and renormalised inverse $T$-matrices are equal up to (including) $p^2$ order. This equality is manifested by (23) and (33). Consequently in terms of $\nu$ expansion of the Feynman amplitude given in [13] the two amplitudes are equal up to (including) $\nu = 2$ order. Higher order corrections to the cut-off expression are suppressed by powers of cut-off parameter $l$ which should be taken of the order of lightest integrated particle, so they are of the order of terms which are neglected by the approximation taken in renormalized theory.

Substituting actual values for scattering length and effective range $a = -1/(8.4 \text{ MeV})$ and $r_e = 0.0137 \text{ MeV}^{-1}$ into (23) and (33) one can calculate the phase shifts. The results for $l = 130$ MeV and $\mu = 0, 40, 130$ MeV (Note that $\mu = 130$ MeV does not violate the power counting at least for the present problem) are plotted in FIG.1. As is seen from this graph the cut-off phase shifts are in good agreement with the ones of renormalised theory for all energies for which the second approach describes the results as effective range expansion well. Note that the expansion parameter for renormalized theory when pion is integrated out is expected to be of the order $p/m_\pi$ so one should not expect good agreement with data far beyond 40 MeV. The $\mu = 0$ graph shows the failure of $\overline{MS}$ renormalised theory encountered in [13]. One can also calculate the numerical values of cut-off coupling constants $\bar{C} \approx -1/(78.2 \text{ MeV})^2$ and $\bar{C}_2 \approx 1/(155.5 \text{ MeV})^4$.

To study the dependence of phase shifts on cut-off parameter the phase shifts for different values of this parameter are plotted in FIG.2. It is seen that phase shifts do not depend on cut-off up to momenta $\sim 50$ MeV.

Note that figures quite analogous to FIG.1 and FIG.2 but in different context and with different subsequent conclusions are given in [8].
III. $p^4$ ORDER CALCULATIONS

To estimate the corrections from the next orders let us consider the $p^4$ order potential:

$$V(p, p') = C + C_2 (p^2 + p'^2) + B (p^4 + p'^4) + B_1 p^2 p'^2$$

This potential can be written as separable one:

$$V(p, p') = \sum_{i,j=0}^{2} p'^{2i} \lambda_{ij} p^{2j}$$

where

$$\{\lambda_{ij}\}_{i,j=0}^{1} = \begin{pmatrix} C & C_2 & B \\ C_2 & B_1 & 0 \\ B & 0 & 0 \end{pmatrix}.$$  

The relativistic corrections are suppressed by the mass of the nucleon while $B$ and $B_1$ are expected to be of the order $\sim m_\pi^{-6}$ and consequently the relativistic corrections are not included. A straightforward generalisation of calculations with $p^2$-order potential given in [9] leads to the following expression:

$$\frac{1}{T} = \frac{N}{D} - I(p)$$

where

$$N = 1 - I_3 \left( 2 C_2 + 2 p^2 B + p^2 B_1 \right) - I_5 \left( 2 B + B_1 \right) + I_5^2 \left( 2 B B_1 + B^2 \right) + I_3 \left( 2 p^4 B^2 + 2 C_2 p^2 B + C_2^2 - C B_1 \right) + I_3 I_5 \left( 2 p^2 B B_1 + 2 p^2 B^2 + 2 C_2 B \right) - 2 B B_1 I_3 I_7 - B^2 B_1 I_3^2 I_9 + p^2 B^2 B_1 I_3^2 I_7 - B^2 B_1 I_3^3 - B^2 B_1 p^2 I_3 I_5^2 + 2 B^2 B_1 I_3 I_5 I_7$$

and

$$D = C + 2 C_2 p^2 + 2 p^4 B + p^4 B_1 - I_5 \left( 2 p^4 B B_1 + p^4 B^2 + C B_1 - C^2 \right) + I_3 \left[ \left( C B_1 - C_2^2 \right) p^2 - 2 C_2 p^4 B - p^6 B^2 \right] + I_7 \left( 2 p^2 B B_1 + 2 p^2 B^2 + 2 C_2 B \right) + B^2 I_9 + p^2 B^2 B_1 I_3 I_9 - p^4 B^2 B_1 I_3 I_7 - p^2 B^2 B_1 I_5 I_7 - B^2 B_1 I_5 I_9 + p^4 B^2 B_1 I_5^2 + B^2 B_1 I_7^2$$

A. Renormalization by subtracting divergences

Analogously to the $p^2$ order case one can renormalize [37] by subtracting divergent integrals at $p^2 = -\mu^2$ and get:

$$\frac{1}{T} = \frac{1}{C^R + 2 C_2^R p^2 + 2 p^4 B^R + p^4 B_1^R} - I^R(p)$$
where $C_R$, $C_R^2$, $B_R$ and $B_R^1$ are renormalised coupling constants and

$$I^R(p) = -\frac{M}{4\pi} \mu - \frac{M}{4\pi} ip$$ (41)

Comparing (40) to the effective range expansion

$$\frac{1}{T} = -\frac{M}{4\pi} \left( -\frac{1}{a} + \frac{1}{2} r_e p^2 + dp^4 + 0 \left( p^6 \right) - ip \right)$$ (42)

gives

$$\frac{1}{C_R} + \frac{M}{4\pi} \mu = \frac{M}{4\pi a}$$ (43)
$$\frac{2C_R^2}{C_R^2} = \frac{Mr_e}{8\pi}$$ (44)
$$\frac{4(C_R^2)^2}{C_R^2} = \frac{2B_R + B_R^1}{C_R^2} = -\frac{Md}{4\pi}$$ (45)

**B. Cut-off theory**

Introducing a sharp cut-off (factor of $\theta(l - p)\theta(l - p')$) into the potential (34) and solving Lippmann-Schwinger equation for the $T$-matrix one gets the expressions (37), (38), (39) with $I_n$ replaced by $I_n^l$, $I(p)$ replaced by $I_l(p)$ and cut-off dependent couplings $\bar{C}$, $\bar{C}_2$, $\bar{B}$ and $\bar{B}_1$.

For the purposes of this work one can take $\bar{B} = 0$. While simplifying calculations significantly this value is quite satisfactory as far as adjusting remaining parameters one can satisfy the equivalence criteria. Substituting this value one gets:

$$\frac{1}{T} = \frac{(1 - C_2 I_3)^2 - I_5 B_1 - I_3^2 C B_1 - I_3 B_1 p^2}{C - I_5 \left( C B_1 - C_2^2 \right) + I_3 \left( C B_1 - C_2^2 \right) p^2 + 2C_2 p^2 + p^4 B_1} - I_l(p)$$ (46)

Comparing (46) to the effective range expansion (42), after a lengthy but straightforward calculation one obtains:

$$\bar{B}_1 = \frac{y}{x + I_5 z}$$ (47)
$$\bar{C}_2 = \frac{1}{I_3} \left\{ \left( 1 - \bar{B}_1 I_5 \right) \pm \left[ \left( 1 - \bar{B}_1 I_5 \right)^2 - \frac{I_3 z}{x + I_3 z} + \bar{B}_1 I_3^2 + z I_3 I_5 \right]^{\frac{1}{2}} \right\}$$ (48)
$$\bar{C} = -\frac{2 I_5 \bar{C}_2}{I_3} + \frac{I_3 + z I_5}{I_3 x + I_3^2 z}$$ (49)

where
FIG. 3. Phase shifts calculated in $p^4$ order. Double line corresponds to the effective range expansion, solid line corresponds to the cut-off theory and dash-dotted, long-dashed and short-dashed lines correspond to $\mu = 0$, $\mu = 40$ and $\mu = 130$ MeV respectively.

\[ x = a_1 + I_5 \frac{a^2_2 + a_1 a_3}{a^2_1 + a_2 I_3} \]  \hfill (50)

\[ y = \frac{a^2_2 + a_1 a_3}{a^2_1 + a_2 I_3} \]  \hfill (51)

\[ z = \frac{a_1 a_2 - a_3 I_3}{a^2_1 + a_2 I_3} \]  \hfill (52)

and

\[ a_1 = \frac{M}{4\pi a} - \frac{M l}{2\pi^2} \]  \hfill (53)

\[ a_2 = \frac{M r_e}{8\pi} - \frac{M}{2\pi^2 l} \]  \hfill (54)

\[ a_3 = \frac{M d}{4\pi} - \frac{M}{6\pi^2 l^3} \]  \hfill (55)

Solving for $a$, $r_e$ and $d$ from (53)-(55) and substituting into (11)-(13) one obtains lengthy algebraic relations between $\bar{C}$, $\bar{C}_2$, $\bar{B}_1$ and $C_R$, $C_2^R$, $B_R$ and $B_1^R$, the fulfilment of which along the condition $\bar{B} = 0$ guarantees the equality of cut-off and renormalized inverse amplitudes up to (including) $p^4$ order, and consequently the $T$ matrices of two approaches are equal up to (including) $\nu = 4$ order. The higher order corrections to the cut-off expression are again suppressed by powers of $p/l$ and hence are small for momenta well below the cut-off.

Substituting the numerical values for $a = -1/(8.4$ MeV), $r_e = 0.0137$ MeV$^{-1}$, $d = 0$ (the first two terms in effective range expansion describe experimental data quite well so there is no need to determine $d$ from data at least for the purposes of this paper) and $l = 130$ MeV one can calculate coupling constants $\bar{C} \approx -1/(76.8$ MeV)$^2$, $\bar{C}_2 \approx 1/(135.2$ MeV)$^4$ (the sign “−” in (48) is again chosen respecting the structure of ordinary perturbation theory), $\bar{B}_1 \approx -1/(124.6$ MeV)$^6$. Using these values one calculates phase shifts from (46). These phase shifts
FIG. 4. Phase shifts calculated in $p^4$ order cut-off theory. Solid line corresponds to the effective range expansion. The lowest dashed line corresponds to cut-off parameter $l = 110$ MeV and subsequent lines correspond to the values $l = 120, 130, 140, 150, 160, 170$ MeV.

are compared with results of effective range expansion and also of \((\text{II})\) in FIG. 3. The phase shifts of cut-off theory coincide with ones obtained from renormalised theory for all momenta for which the second approach describes the data well.

In FIG.4 the phase shifts for different values of the cut-off parameter are plotted. It can be seen that phase shifts are cut-off independent up to $\sim 60$ MeV.

IV. CONCLUSIONS

In the simple example of low energy effective field theory for nucleon-nucleon scattering it was demonstrated that the cut-off theory can reproduce the results of the renormalised theory up to the order of accuracy of the considered approximation, the difference being of the order of neglected terms. This simple example serves as a demonstration of some more general considerations about cut-off field theories formulated below.

Using chiral power counting originally developed by Weinberg one can find the potential up to any desired order. Then to remove divergences one can impose cut-off regularization. The cut-off regularization destroys chiral and gauge symmetries and to restore them it is necessary to include additional terms into the Lagrangian (and consequently into the potential). Cut-off dependence of the physical quantities can be removed systematically by including additional terms in the Lagrangian [17].

The power-law divergences, which caused higher order operators to be involved in the renormalization of the diagrams obtained by iterating the low order potential, now emerge as powers of the cut-off parameter. As far as cut-off should be taken of the order of masses of particles which were integrated out, it should be clear that cut-off regularization does not respect power counting and it seems that imposing this regularization will destroy the whole machinery. (The problem cannot be solved by imposing a small cut-off as cut-off regularized integrals contain inverse powers of cut-off parameters as well). However the large factors which seem to threaten
power counting can be absorbed by redefining the couplings already included into the potential [18].

Fitting the parameters of the cut-off theory one can reproduce the results of the renormalised theory up to the order of accuracy determined by approximation made in the potential. The results of cut-off theory are as reliable as the ones of renormalised theory, the error being of the order of terms neglected in the potential. As far as the cut-off is of the order of the mass of lightest particle which was integrated out the higher order (in momenta) cut-off dependent corrections are suppressed by powers of this parameter. By increasing the cut-off parameter one could make the mentioned corrections smaller but for large cut-off it would be problematic to include the positive powers of the cut-off parameter in to a redefinition of the coupling constants. So, the equivalence between cut-off and renormalized theory can be achieved only for the cut-off of the order of the mass of lightest particle which was integrated out.

One should conclude that the doubts about consistency and systematic character of cut-off theories [8,19,20] are ungrounded. So the reasonable success of the cut-off chiral perturbation theory originally started with work [4] should not be a surprise. Although the cut-off theory is technically a little complicated it has a great advantage in that one can determine amplitudes from equations. Note that there are no self-contained equations for renormalised amplitudes in these non-renormalizable (in the traditional sense) effective field theories and one instead has to sum up renormalised diagrams.

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REFERENCES

[1] S. Weinberg, *Physica* **96 A** (1979) 327;
[2] S. Weinberg, Phys. Lett. **B251**, 288 (1990).
[3] S. Weinberg, Nucl. Phys. **B363**, 3 (1991).
[4] C. Ordonez, L. Ray and U. van Kolck, Phys. Rev. C **53**, 2086 (1996).
[5] T. D. Cohen, Phys. Rev. C **55**, 67 (1997).
[6] D. R. Phillips and T. D. Cohen, Phys. Lett. **B390**, 7 (1997).
[7] K. A. Scaldeferri, D. R. Phillips, C.-W. Kao, and T. D. Cohen, Phys. Rev. C **56**, 679-688, 1997.
[8] D. R. Phillips, [nucl-th/9804040](http://arxiv.org/abs/nucl-th/9804040);
[9] D. R. Phillips, S. R. Beane, and T. D. Cohen, [hep-th/9706070](http://arxiv.org/abs/hep-th/9706070);
[10] S. R. Beane, T. D. Cohen and D. R. Phillips, [nucl-th/9709062](http://arxiv.org/abs/nucl-th/9709062);
[11] U. van Kolck, [hep-ph/9711222](http://arxiv.org/abs/hep-ph/9711222);
[12] S. K. Adhikari, and A. Ghosh, J. Phys. A **30**, 6553 (1997); C. F. de Araujo, Jr., L. Tomio, and S. K. Adhikari, ibid., 4687; S. K. Adhikari, T. Frederico, and I. D. Goldman, Phys. Rev. Lett. **74**, 487 (1995).
[13] D. B. Kaplan, M. J. Savage, and M. B. Wise, Nucl. Phys. **B478**, 629 (1996);
[14] D. B. Kaplan, M. J. Savage, and M. B. Wise, [nucl-th/9801034](http://arxiv.org/abs/nucl-th/9801034);
[15] D. B. Kaplan, M. J. Savage, and M. B. Wise, [nucl-th/9802075](http://arxiv.org/abs/nucl-th/9802075);
[16] J. Gegelia, Phys. Lett. **B429**, 227 (1998), [nucl-th/9802038](http://arxiv.org/abs/nucl-th/9802038);
[17] G. P. Lepage, [nucl-th/9706029](http://arxiv.org/abs/nucl-th/9706029);
[18] G. P. Lepage, *What is Renormalization?*, in *From actions to Answers*, edited by T. DeGrand and D. Toussaint, World Scientific Press (Singapore, 1990).
[19] E. Epelbaoum, W. Glöckle, Ulf-G. Meissner [nucl-th/9801063](http://arxiv.org/abs/nucl-th/9801063);
[20] E. Epelbaoum, W. Glöckle, Ulf-G. Meissner [nucl-th/9804003](http://arxiv.org/abs/nucl-th/9804003).