Schrödinger equation with potential function vanishing exponentially fast

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ABSTRACT

Explicitly oscillating solutions of differential equation

\[ y'' + \left( \lambda + 20 \text{sech}^2 x \right) y = 0 \]

and its eigenpairs are obtained by calculating complex residues. Eigenpairs, spectral function and eigenfunction expansions are also reported for this specific Pöschl–Teller differential equation.

1. Introduction

In this paper, we consider the following second-order differential equation:

\[ y''(x) + (\lambda - q(x)) y(x) = 0, \quad (1) \]

with specific potential function \( q(x) = -20 \text{sech}^2 x \). Theory related to (1) is elaborately given in [1]. One may say that the only case to solve (1) explicitly for all \( \lambda \) is the case \( q(x) = 0 \) where the solutions are oscillating. In reality, this is not correct. There are indeed some examples which are not reported yet. One such example is

\[ y''(x) + \left( \lambda + 20 \text{sech}^2 x \right) y(x) = 0, \quad (2) \]

which is a special case of

\[ y''(x) + \left( \lambda + n(n + 1) \text{sech}^2 x \right) y(x) = 0 \]

with \( n = 4 \). This example is slightly different from what Titchmarsh does in the following example where its solutions involving Legendre functions

\[ y''(x) + \left( \lambda + \left( \frac{1}{2} - \frac{1}{4} \right) \text{sech}^2 x \right) y(x) = 0, \]

\[ x \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right). \]

For similar examples and results, see [1–8].

It is worth mentioning here how our interest in this title arises. There is a relationship between the Schrödinger equation and Sturm–Liouville differential equation (1). The Schrödinger equation arises from several partial differential equation appearing in physics by separation of variables [9]. For instance, for a single quantum-mechanical particle of mass \( m \) moving in one space dimension in a potential \( V(x) \), the time-dependent Schrödinger equation is

\[ i\hbar \partial_t \Psi = -\frac{\hbar^2}{2m} \partial_{xx} \Psi + V(x) \Psi. \]

Looking for separable solutions

\[ \Psi = \psi(x) e^{-iEt/\hbar}, \]

where \( E \) is energy eigenstates. Then one finds that

\[ -\frac{\hbar^2}{2m} \psi'' + V(x) \psi = E \psi. \]

After normalization, one obtains Equation (1) in the subject line above where the potential function \( q = (2m/\hbar^2)V(x) \), the eigenvalue parameter \( \lambda = (2m/\hbar^2)E \) and \( y = \psi \).

Therefore, if one considers \( x \rightarrow \infty \), then

\[ \text{sech}^2 x = \frac{4}{e^{2x} + 2 + e^{-2x}} \sim 4 e^{-2x}. \]

Hence, one may see that why we called title in the subject line above. It is also important to note that Equation (2) describes the diatomic molecular vibration and such potential also arises in the solutions of the Korteweg–de Vries equation [10, 11].

In general, it is not easy to obtain the solutions of Equation (1) as we obtain in our case by employing residues. Conventionally, solutions to Equation (1) are written either in terms of hypergeometric functions or as series. The purpose of this article is to provide analytical solutions of Equation (2) by using local complex residues.
2. Preliminaries

We now give some preliminaries. To get the expansion of an arbitrary function \( f(x) \) in terms of eigenfunctions one actually needs to know the following definitions and lemmas taken from [1, 2]. If \( \theta(x, \lambda) \) and \( \phi(x, \lambda) \) are the solutions of (1), with \( \alpha \) is real, satisfying
\[
\phi(0, \lambda) = \sin \alpha, \quad \phi'(0, \lambda) = -\cos \alpha, \\
\theta(0, \lambda) = \cos \alpha, \quad \theta'(0, \lambda) = \sin \alpha.
\]
(3)

Then Wronskian \( W_k(\phi, \theta) = \sin^2 \alpha + \cos^2 \alpha = 1 \). It is well known that the general solution of (1) is
\[
\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\phi(x, \lambda) \in L^2(0, \infty).
\]
The definition of the non-decreasing spectrum function is given by
\[
k(\lambda) = -\lim_{\delta \to 0} \int_0^{\lambda} \text{Im}(m(u + i\delta)) \, du.
\]
Lemma 2.1 (For proof, see [1]): For suitable class of functions on \((0, \infty)\) one has
\[
f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \phi(x, \lambda) \, d\lambda \int_{-\infty}^{\infty} \phi(t, \lambda) f(t) \, dt \\
+ \sum_{N=0}^{\infty} \phi_0(t, \lambda) \int_0^{\infty} \phi_N(t, \lambda) f(t) \, dt.
\]
If \( m(\lambda) \) does not have poles, then the summative terms in the expansion of \( f(x) \) disappear.

Lemma 2.2 (For proof, see [1]): Let \( q(x) \) be given. (If \( q(x) \) is an even function, then \( m_1(\lambda) = -m_2(\lambda) \)). Hence, for a suitable class of functions on \((-\infty, \infty)\) one has
\[
f(x) = \frac{1}{\pi} \left( \int_{-\infty}^{\infty} \theta(x, \lambda) \, d\lambda \int_{-\infty}^{\infty} \theta(y, \lambda) f(y) \, dy \\
+ \int_{-\infty}^{\infty} \phi(x, \lambda) \, d\lambda \int_{-\infty}^{\infty} \phi(y, \lambda) f(y) \, dy \right) \\
+ \sum_{N=0}^{\infty} \phi_0(t, \lambda) \int_0^{\infty} \phi_N(t, \lambda) f(t) \, dt,
\]
(4)

where
\[
\psi_1(x, \lambda) = \theta(x, \lambda) + m_1(\lambda)\phi(x, \lambda) \in L^2(-\infty, 0), \\
\psi_2(x, \lambda) = \theta(x, \lambda) + m_2(\lambda)\phi(x, \lambda) \in L^2(0, \infty), \\
\xi(\lambda) = -\lim_{\delta \to 0} \int_{-\infty}^{\lambda} \text{Im} \left( \frac{1}{m_1(u + i\delta) - m_2(u + i\delta)} \right) \, du, \\
\xi'(\lambda) = \text{q(x) is even} \lim \left( \frac{1}{2m_2(\lambda)} \right), \\
\zeta(\lambda) = -\lim_{\delta \to 0} \int_{-\infty}^{\lambda} \text{Im} \left( \frac{m_1(u + i\delta) m_2(u + i\delta)}{m_1(u + i\delta) - m_2(u + i\delta)} \right) \, du, \\
\zeta'(\lambda) = \text{q(x) is even} \lim \left( \frac{-1}{2m_2(\lambda)} \right).
\]
If \( m(\lambda) \) does not have poles, then the summative terms in (4) disappear.

3. Main results

We are now aiming to deal with the solution of (2) which is given by with \( s^2 = -\lambda \)
\[
y(x) = \cos^5 x \int_C \frac{\cosh(zs)}{(\sinh z - \sinh x)^3} \, dz,
\]
(5)
where \( C \) is just including the point \( z = x \) and excluding the other zeros of \( \sinh z - \sinh x \).

Theorem 3.1 (Case for any \( n \), see [2, 4]): The integral (5) solves Equation (2).

Proof:
\[
y'(x) = 5 \tanh(x) y(x) \\
+ 5 \cos^6 x \int_C \frac{\cosh(zs)}{(\sinh z - \sinh x)^6} \, dz, \\
y''(x) = 25 y(x) - 20 \sech^2 x y(x) \\
+ 55 \cos^5 x \sinh x \int_C \frac{\cosh(zs)}{(\sinh z - \sinh x)^6} \, dz \\
+ 30 \cos^7 x \int_C \frac{\cosh(zs)}{(\sinh z - \sinh x)^7} \, dz.
\]
Therefore,
\[
y''(x) + 20 \sech^2 x y(x) \\
= 5 \cos^5 x \int_C \frac{\cosh(zs)(5 \sinh^2 z + \sinh z \sinh x + 6)}{(\sinh z - \sinh x)^7} \, dz.
\]
(6)
Applying two times partial integration to (5) successively, one obtains
\[
y(x) = 5 \cos^5 x \int_C \frac{\sinh(zs) \cosh z}{(\sinh z - \sinh x)^6} \, dz \\
= \frac{5 \cos^5 x \int_C \cosh(zs) \left( 5 \sinh^2 z + \sinh z \sinh x + 6 \right)}{(\sinh z - \sinh x)^7} \, dz.
\]
(7)
By using \( s^2 = -\lambda \)
\[
\lambda y(x) = -5 \cos^5 x \int_C \frac{\cosh(zs)(5 \sinh^2 z + \sinh z \sinh x + 6)}{(\sinh z - \sinh x)^7} \, dz.
\]
Comparing (6) and (7), we see that
\[
y''(x) + (\lambda + 20 \sech^2 x) y(x) = 0.
\]
\[\blacksquare\]

Corollary 3.1: The factor \( \cosh(zs) \) in (5) plays little part in the argument. By replacing \( \cosh(zs) \) by \( \sinh(zs) \), the other linearly independent solution to Equation (2) is given by with \( s^2 = -\lambda \)
\[
y(x) = \cos^5 x \int_C \frac{\sinh(zs)}{(\sinh z - \sinh x)^5} \, dz.
\]
(8)

Theorem 3.2 (Case for any \( n \), see [2, 4]): The integral (8) solves Equation (2).

Proof: The proof is the same as Theorem 3.1. Hence, it is excluded.
4. Eigenpairs obtained by calculating the relevant residues

As a result of very laborious calculations, one obtains that the residue at \( z = x \) for (5) is
\[
\frac{\cosh(\lambda x)(9 + (15 + 45\pi^2 - 105) \tanh^2 x + s^4 - 10x^2 + 105 \tanh^3 x)}{4!(z - x)^3 \cosh^3 x} + \frac{s \sinh(\lambda x)((55 - 10\pi^2) \tanh x - 105 \tanh^3 x)}{4!(z - x)^3 \cosh^3 x}.
\]
Hence, by using \( s^2 = -\lambda \), one solution is
\[
y_1 = \cos(x\sqrt{\lambda})(9 + (-90 - 45\lambda) \tanh^2 x + \lambda^2 + 10\lambda + 105 \tanh^4 x) - \sqrt{\lambda} \sin(x\sqrt{\lambda})(55 + 10\lambda \tanh x - 105 \tanh^3 x).
\]
Similarly, by examining the residue of (8), one obtains second solution as
\[
y_2 = \sin(x\sqrt{\lambda})(9 + (-90 - 45\lambda) \tanh^2 x + \lambda^2 + 10\lambda + 105 \tanh^4 x) + \sqrt{\lambda} \cos(x\sqrt{\lambda})(55 + 10\lambda \tanh x - 105 \tanh^3 x).
\]
If \( Y(x, s) = y_1(x) - y_2(x) \), where \( y_1(x) \) and \( y_2(x) \) are given by (5) and (8) respectively, then
\[
Y(x, s) = \cosh^3 x \int_C \frac{e^{-sz}}{(\sinh z - \sin x)^3} dz.
\]

**Theorem 4.1 (Case for any \( n \), see [2, 3]):**

\[
Y_0(s) = \int_C \frac{e^{-sz}}{\sinh^2 z} dz = \frac{i\pi}{12}(3 + s)(1 + s)(s - 1)(s - 3),
\]
where the contour \( C \) is excluding all zeros of \( \cos z \) except \( z = \pi / 2 \). If \( Y(0, s) = 0 \), then \( \lambda = -1, -9 \).

**Proof:** If \( z = i(z - \pi / 2) \), then
\[
Y_0(s) = -e^{i\pi / 2} \int_C \frac{e^{-isz}}{\cos^3 z} dz = I_1 + I_2,
\]
where
\[
I_1 = -e^{i\pi / 2} \int_C \frac{\cos(sz)}{\cos^3 z} dz = -e^{i\pi / 2} f_1(s),
\]
\[
I_2 = -e^{i\pi / 2} \int_C \frac{\cos((s - 1)z + z)}{\cos^3 z} dz = -e^{i\pi / 2} \int_C \frac{\cos(s - 1)z \cos(z) - \sin(s - 1)z \sin(z)}{\cos^3 z} dz,
\]
integration by parts
\[
= -e^{i\pi / 2} \frac{1 + s - 1}{4} f_2(s - 1) = -e^{i\pi / 2} \frac{3 + s}{4} f_2(s - 1),
\]
by continuing this argument one gets
\[
l_1 = -e^{i\pi / 2} \frac{3 + s}{4} \frac{1}{3} \frac{1}{2} f_1(s - 4),
\]
where
\[
f_1(s - 4) = \int_C \frac{\cos(s - 4)z}{\cos z} dz = -2\pi i \cos \frac{(s - 4)p}{2}.
\]
After all,
\[
l_1 = \frac{2i e^{i\pi / 2} \pi}{4!} (3 + s)(1 + s)(s - 1)(s - 3) \cos \frac{(s - 4)p}{2}
\]
and
\[
l_2 = i e^{i\pi / 2} \int_C \frac{\sin(z)}{\cos^3 z} dz = i e^{i\pi / 2} f_2(s),
\]
\[
l_2 = i e^{i\pi / 2} \int_C \frac{\sin((s - 1)z + z)}{\cos^3 z} dz = i e^{i\pi / 2} \int_C \frac{\sin(s - 1)z \cos(z) + \sin(z) \cos(s - 1)z}{\cos^3 z} dz,
\]
partial integral implies that
\[
l_2 = i e^{i\pi / 2} \left(1 + \frac{s - 1}{4}\right) f_2(s - 1)
\]
keep continuing this argument,
\[
l_2 = i e^{i\pi / 2} \frac{3 + s}{4} \frac{1}{3} \frac{1}{2} \frac{1}{4} f_1(s - 4),
\]
where
\[
f_1(s - 4) = \int_C \frac{\sin(s - 4)p}{\cos z} dz = -2\pi i \sin \frac{(s - 4)p}{2}.
\]
That is,
\[
l_2 = \frac{2i e^{i\pi / 2} \pi}{4!} (3 + s)(1 + s)(s - 1)(s - 3) \sin \frac{(s - 4)p}{2}.
\]
By combining \( l_1 \) and \( l_2 \), one gets the expected result.
\[
l_1 + l_2 = \frac{2i e^{i\pi / 2} \pi}{4!} (3 + s)(1 + s)(s - 1)(s - 3) \cos \frac{(s - 4)p}{2} + \frac{2i e^{i\pi / 2} \pi}{4!} (3 + s)(1 + s)(s - 1)(s - 3) \sin \frac{(s - 4)p}{2}
\]
\[
= 2i \pi e^{i\pi / 2} \frac{e^{-i(s - 4)p / 2}}{4!} (3 + s)(1 + s)(s - 1)(s - 3)
\]
\[
= \pi e^{i\pi / 2} (3 + s)(1 + s)(s - 1)(s - 3)
\]
\[
= \pi \frac{1}{12} (3 + s)(1 + s)(s - 1)(s - 3).
\]
If \( Y(0) = 0 \), then \( (3 + s)(1 + s)(s - 1)(s - 3) = 0 \). Hence, \( s^2 = 1 \) or \( \lambda = -1 \) and \( s^2 = 9 \) or \( \lambda = -9 \). That completes the proof.
Theorem 4.2 (Case for any \( n \), see [2, 3]):
\[
\theta'(0, s) = 5 \int_C e^{-sz} \frac{dz}{\sinh^i z} = -\frac{i\pi}{12} (4 + s)(2 + s)s(s - 2)(s - 4)
\]
where the contour \( C \) is excluding all zeros of \( \cos z \) except \( z = \pi/2 \). If \( \theta'(0, s) = 0 \), then \( \lambda = -4, -16 \).

Proof: Proof is the same as Theorem 4.1. Hence, it is excluded. If \( \theta'(0, s) = 0 \), then \( s^2 = 4, 16 \) or \( \lambda = -4, -16 \). That completes the proof.

Theorem 4.3 (Case for any \( n \), see [2, 3]): \( Y(x, s) \to 0 \) as \( x \to \infty \).

Proof: Set \( z = i\omega \) and \( x = iy \). So
\[
Y(y, s) = \cos^5 y \int_{\sin w - \sin y} e^{-isw} \frac{dw}{(\sin w - \sin y)^2} = 2^{-5} \cos^5 y \int_{\sin w - \sin y} e^{-isw} \frac{dw}{(\sin w - \sin y)^2} = \int e^{-isw} (w - y)^5 dw.
\]
Let \( w - y = u \) and \( is = v \). Then one obtains
\[
Y(y, \lambda) = s^4 e^{-is} \int e^{-izv} v^5 \, dv = s^4 e^{-is} \int e^{-izv} v^5 \, dv.
\]
Therefore, this integral approaches zero as \( x \) goes to \( \infty \).

5. Eigenfunction expansions

If \( \theta(x, \lambda) \) and \( \phi(x, \lambda) \) are the solutions of (2) satisfying (3). Then one finds
\[
\phi(x, \lambda) = \frac{y_1(x, \lambda) \sin \alpha}{\sqrt{\lambda} (\lambda^2 + 10\lambda + 9)} - \frac{y_2(x, \lambda) \cos \alpha}{\sqrt{\lambda} (\lambda^2 + 20\lambda + 64)},
\]
\[
\theta(x, \lambda) = \frac{y_1(x, \lambda) \cos \alpha}{\sqrt{\lambda} (\lambda^2 + 10\lambda + 9)} + \frac{y_2(x, \lambda) \sin \alpha}{\sqrt{\lambda} (\lambda^2 + 20\lambda + 64)}.
\]

One needs to find spectral function \( k(\lambda) \). To get desired result, we have to find the asymptotics of (10) as \( x \to \infty \). If \( \text{Im}(\sqrt{\lambda}) > 0 \), then the asymptotics are
\[
\phi(x, \lambda) \sim e^{-i\lambda \sqrt{2}} M_1(\lambda),
\]
where
\[
M_1(\lambda) = \frac{\sqrt{\lambda} (\lambda^2 + 20\lambda + 64) (\lambda^2 - 35\lambda + 24) \sin \alpha}{\sqrt{\lambda} (\lambda^2 + 10\lambda + 9) (\lambda^2 + 20\lambda + 64)},
\]
and
\[
\theta(x, \lambda) \sim e^{-i\lambda \sqrt{2}} M(\lambda),
\]
where
\[
M(\lambda) = \frac{\sqrt{\lambda} (\lambda^2 + 20\lambda + 64) (\lambda^2 - 35\lambda + 24) \sin \alpha}{2 \sqrt{\lambda} (\lambda^2 + 10\lambda + 9) (\lambda^2 + 20\lambda + 64)},
\]
and
\[
\phi(x, -1) = \frac{\sinh x (45 \tanh x - \tanh^3 x)}{45},
\]
\[
\phi(x, -9) = \frac{\sinh (3x) (3 \tanh^2 x + \tanh^4 x)}{45},
\]
in particular, if \( \alpha = 0 \) and \( \lambda > 0 \) then one gets the continuous spectrum as
\[
k(\lambda) = \begin{cases} \frac{\sqrt{\lambda} (\lambda + 4)(\lambda + 16)}{(\lambda + 1)(\lambda + 9)}, & \lambda > 0 \\ 0, & \lambda < 0. \end{cases}
\]
Hence, with \( \alpha = 0 \), there exist two eigenvalues occurring at \( \lambda = -1 \) and \( \lambda = -9 \) and their associated eigenfunctions are
Hence, the eigenvalues are occurring at $\lambda = -4$ and $\lambda = -16$ and their associated eigenfunctions are

$$\phi(x, -4) = -\cosh(2x)[6 \tanh^2 x - 1 + 7 \tanh^4 x] - 2 \sinh(2x)[\tanh x - 7 \tanh^3 x],$$

$$\phi(x, -16) = \cosh(4x)[1 + 6 \tanh^2 x + \tanh^4 x] - 4 \sinh(4x)[\tanh x + \tanh^3 x].$$

So we once again conclude that if $\phi(x, \lambda)$ is (10) and $\alpha = \pi/2$, by using Lemma 2.1, then

$$f(x) = \frac{1}{\pi} \left\{ \int_{-\infty}^{\infty} \theta(x, \lambda) \, d\xi(\lambda) \int_{-\infty}^{\infty} \phi(y, \lambda) f(y) \, dy \right\} + c_3 \phi(x, -4) + c_4 \phi(x, -16).$$

Thus, one has formally proved the following theorem.

**Theorem 5.1:** If $\alpha = 0$, then there exist only two eigenvalues occurring at $\lambda = -1$ and $\lambda = -9$ and their associated eigenfunctions $\phi(x, -1)$ and $\phi(x, -9)$ are given in (13).

If $\alpha = \pi/2$, then there exist only two eigenvalues occurring at $\lambda = -4$ and $\lambda = -16$ and their associated eigenfunctions $\phi(x, -4)$ and $\phi(x, -16)$ are given in (14).

Consider the interval $(-\infty, \infty)$. Since $q(x)$ is an even function, we use definition of $\xi'(\lambda)$ and $\zeta'(\lambda)$ from Lemma 2.2, one gets

$$\xi'(\lambda) = \begin{cases} 2^{-1} \sqrt{\lambda(\lambda^2 + 20\lambda + 64)(\lambda^2 + 10\lambda + 9)} & \lambda > 0, \\ 0 & \lambda < 0 \end{cases},$$

$$\zeta'(\lambda) = \begin{cases} 2^{-1} \sqrt{\lambda(\lambda^2 + 20\lambda + 64)(\lambda^2 + 10\lambda + 9)} & \lambda > 0, \\ 0 & \lambda < 0. \end{cases}$$

If so, by using Lemma 2.2, one gets

$$f(x) = \frac{1}{\pi} \left\{ \int_{0}^{\infty} \theta(x, \lambda) \, d\xi(\lambda) \int_{-\infty}^{\infty} \phi(y, \lambda) f(y) \, dy \right\}$$

$$+ \int_{0}^{\infty} \phi(x, \lambda) \, d\zeta(\lambda) \int_{-\infty}^{\infty} \phi(y, \lambda) f(y) \, dy \right\} + c_3 \phi(x, -1) + c_4 \phi(x, -9)$$

$$+ c_3 \phi(x, -4) + c_4 \phi(x, -16),$$

where $c_1$, $c_2$, $c_3$ and $c_4$ are constants. $\phi(x, \lambda), \theta(x, \lambda), \xi'(\lambda)$ and $\zeta'(\lambda)$ are given by (10) and (15) respectively.

### 6. Conclusion

The explicit solutions of the Pöschl–Teller differential equation (2) and its eigenpairs are obtained by calculating complex residues. Eigenfunction expansions for arbitrary function $f(x)$ satisfying suitable conditions are also obtained. We concluded that differential equation (2) has oscillating solutions which are not reported in the literature.

### Disclosure statement

No potential conflict of interest was reported by the author.

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