Calabi–Yau threefolds with non-Gorenstein involutions

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Funding information
Hongik University, Grant/Award Number: 2021; Hongik University Research Fund; National Research Foundation of Korea, Grant/Award Number: 2020R1A2C1A01004503

Abstract
The concept of non-Gorenstein involutions on Calabi–Yau threefolds is a higher dimensional generalization of non-symplectic involutions on K3 surfaces. We present some elementary facts about Calabi–Yau threefolds with non-Gorenstein involutions. We give a classification of the Calabi–Yau threefolds of Picard rank one with non-Gorenstein involutions, whose fixed locus is not zero-dimensional.

KEYWORDS
Calabi–Yau threefold, involution, Q-Fano threefold

MSC (2020)
14J32, 14J50, 14J45

1 INTRODUCTION

A Calabi–Yau manifold is a compact Kähler manifold with trivial canonical class such that the intermediate cohomology groups of its structure sheaf are trivial \( h^i(O_X) = 0 \) for \( 0 < i < n = \dim(X) \). If \( \rho \) is an involution on \( X \), then \( \rho^* \) acts as a multiplication by \( +1 \) or \( -1 \) on \( H^0(X, \Omega^n_X) \) since \( \dim H^0(X, \Omega^n_X) = 1 \). In this note, an involution \( \rho \) of a Calabi–Yau manifold will be said to be Gorenstein if \( \rho^*(\omega) = \omega \) for each \( \omega \in H^0(X, \Omega^n_X) \) and non-Gorenstein otherwise. The non-symplectic involutions on K3 surfaces, which are Calabi–Yau twofolds, are non-Gorenstein in this definition. Note that Enriques surfaces, which form an important class of algebraic surfaces, are quotients of K3 surfaces by fixed-point-free non-Gorenstein involutions. Non-Gorenstein (so non-symplectic) involutions on K3 surfaces have been classified by Nikulin [17–20] and they have a remarkable property of the mirror symmetry of lattice polarized K3 surfaces, which enabled Voisin [24] and Borcea [5] to build their mirror pairs of Calabi–Yau threefolds. They have also been used in producing many new examples of G2-manifolds [9, 13]. Seeing that there are rich geometries and applications concerning non-Gorenstein involutions on K3 surfaces, it seems natural to consider non-Gorenstein involutions on Calabi–Yau threefolds. In this note, we present some elementary findings about non-Gorenstein involutions on Calabi–Yau threefolds with some examples. It turns out that a non-Gorenstein involution is not fixed-point-free and has 16 fixed points if the fixed locus is zero-dimensional (Theorem 2.5). So, there are no smooth three-dimensional generalizations of Enriques surfaces. The next simplest Calabi–Yau threefolds with non-Gorenstein involutions would be the ones that are of the Picard rank one. We consider the case that the fixed locus is not zero-dimensional and the Picard rank of Calabi–Yau threefolds are one. Almost nothing is known for the classification of general Calabi–Yau threefolds, even for the case of the Picard rank one. However, ones with such non-Gorenstein involutions turn out to be closely related with Q-Fano threefolds with some specific cyclic singularities. With this relation, we show that there are finitely many families of such Calabi–Yau threefolds (Theorem 2.9) and give a classification of Calabi–Yau threefolds of Picard number one with non-Gorenstein involutions with respect to \( H^3, H \cdot c_2, \ldots \).
TABLE 1  Invariant of $X$’s when $Y$’s are smooth Fano threefolds.

| $N$ | $s$ | $H^3$ | $H \cdot c_2$ | $e$ | References |
|-----|-----|-------|---------------|----|------------|
| 0   | 2   | 4     | 52            | $-256$ | [11]       |
| 0   | 2   | 8     | 56            | $-176$ | [11]       |
| 0   | 2   | 12    | 60            | $-144$ | [11]       |
| 0   | 2   | 16    | 64            | $-128$ | [11]       |
| 0   | 2   | 20    | 68            | $-120$ | [11]       |
| 0   | 2   | 24    | 72            | $-116$ | [11]       |
| 0   | 2   | 28    | 76            | $-116$ | [11]       |
| 0   | 2   | 32    | 80            | $-116$ | [11]       |
| 0   | 2   | 36    | 84            | $-120$ | [11]       |
| 0   | 2   | 44    | 92            | $-128$ | [11]       |
| 0   | 4   | 2     | 32            | $-156$ | [11]       |
| 0   | 4   | 4     | 40            | $-144$ | [11]       |
| 0   | 4   | 6     | 48            | $-156$ | [11]       |
| 0   | 4   | 8     | 56            | $-176$ | [11]       |
| 0   | 4   | 10    | 64            | $-200$ | [11]       |
| 0   | 6   | 4     | 52            | $-256$ | [11]       |
| 0   | 8   | 2     | 44            | $-296$ | [11]       |

TABLE 2  Invariant of $X$’s when $Y$’s are Fano–Enriques threefolds.

| $N$ | $s$ | $H^3$ | $H \cdot c_2$ | $e$ | References |
|-----|-----|-------|---------------|----|------------|
| 8   | 2   | 4     | 28            | $-88$ | [16, 21]    |
| 8   | 2   | 8     | 32            | $-64$ | [16, 21]    |
| 8   | 4   | 2     | 20            | $-72$ | [16, 21]    |

$N$, and $s$ (Tables 1–3) with sharp bounds, where $H$ is an ample generator of the Picard group of the Calabi–Yau threefolds, $c_2$ is the second Chern class, $N$ is the number of isolated fixed points of the involutions, and $s$ is the involution index.

2  NON-GORENSTEIN INVOLUTIONS ON CALABI–YAU THREEFOLDS

Let $X$ be a Calabi–Yau threefold and $\rho$ be an involution on it. Let $p$ be a fixed point of $\rho$. Borrowing arguments from [8], one can show that there exists a local holomorphic coordinate system $z_1, z_2, z_3$ around $p$ such that

1. $\rho(z_1, z_2, z_3) = (z_1, -z_2, -z_3)$ if $\rho$ is Gorenstein and
2. $\rho(z_1, z_2, z_3) = (-z_1, -z_2, -z_3)$ or $\rho(z_1, z_2, z_3) = (z_1, z_2, -z_3)$ if $\rho$ is non-Gorenstein.

Hence, the fixed locus of $\rho$ is a disjoint union of smooth curves if $\rho$ is Gorenstein. When $\rho$ is non-Gorenstein, its fixed locus is a union of a set $P$ of isolated points and a disjoint union $S_X$ of smooth surfaces:

$$X^\rho = P \cup S_X.$$  

From now on, we assume that $\rho$ is non-Gorenstein. Let us take a quotient $Y = X/\langle \rho \rangle$ of $X$ by $\rho$ and $\varphi : X \to Y$ be the quotient map. Then, $Y$ is smooth at $\varphi(p)$ for $p \in S_X$. Let $p \in P$, then $Y$ has a singularity of type $\frac{1}{2}(1,1,1)$ at $\varphi(p)$. Since all the singularities are quotient singularities, $Y$ is $\mathbb{Q}$-factorial. Let $P = \{p_1, p_2, ..., p_k\}$ and $q_i = \varphi(p_i)$. Let $f : \tilde{X} \to X$ be the blow-up at $P$ with exceptional divisors $E_1, E_2, ..., E_k$ over $p_1, p_2, ..., p_k$ and $g : \tilde{Y} \to Y$ be the blow-up at $\varphi(P)$ with the exceptional divisors $F_1, F_2, ..., F_k$ over $q_1, q_2, ..., q_k$. Note that $F_i \simeq \mathbb{P}^2$ and $F_i|_{F_i} = \mathcal{O}_{F_i}(-2)$. $\tilde{Y}$ is now smooth. It is easy to
TABLE 3 Invariant of X’s when Y’s are \( \mathbb{Q} \)-Fano threefolds of index less than one.

| \( N \) | \( s \) | \( H^3 \) | \( H \cdot c_2 \) | \( e \) | References |
|---|---|---|---|---|---|
| 1 | 2 | 5 | 50 | -200 | [23] |
| 1 | 2 | 9 | 54 | -144 | [23] |
| 1 | 2 | 13 | 58 | | [23] |
| 1 | 2 | 17 | 62 | -108 | [14, 23] |
| 1 | 2 | 21 | 66 | -100, -104 | [14, 23] |
| 1 | 2 | 25 | 70 | -100 | [14, 23] |
| 1 | 2 | 29 | 74 | -96 | [14, 23] |
| 2 | 2 | 6 | 48 | -156 | [23] |
| 2 | 2 | 10 | 52 | | [23] |
| 2 | 2 | 14 | 56 | -96, -100 | [14, 23] |
| 2 | 2 | 18 | 60 | -84, -92 | [14, 23] |
| 2 | 2 | 22 | 64 | -92 | [14, 23] |
| 2 | 2 | 30 | 72 | -95 | [14, 23] |
| 3 | 2 | 7 | 46 | | [23] |
| 3 | 2 | 11 | 50 | | [23] |
| 3 | 2 | 15 | 54 | -76, -84 | [14, 23] |
| 3 | 2 | 19 | 58 | -76 | [14, 23] |
| 3 | 2 | 23 | 62 | | [23] |
| 4 | 2 | 8 | 44 | -88 | [14, 15, 23] |
| 4 | 2 | 12 | 48 | -68 | [14, 23] |
| 4 | 2 | 16 | 52 | -72 | [14, 23] |
| 5 | 2 | 9 | 42 | | [23] |
| 5 | 2 | 13 | 46 | | [23] |
| 5 | 2 | 17 | 50 | | [23] |
| 6 | 2 | 10 | 40 | | [23] |
| 6 | 2 | 14 | 44 | | [23] |
| 6 | 2 | 18 | 48 | | [23] |
| 7 | 2 | 15 | 42 | | [23] |
| 7 | 2 | 19 | 46 | | [23] |

see that \( \varphi \) can be extended to a double covering \( \tilde{\varphi} : \tilde{X} \to \tilde{Y} \), branched along

\[
B = S_Y \cup F_1 \cup \cdots \cup F_k,
\]

where \( S_Y = \varphi(S_X) \) and \( S_Y = g^{-1}(S_Y) \). We have the commutative diagram:

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{\varphi}} & \tilde{Y} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\varphi} & Y
\end{array}
\]

**Lemma 2.1.** Let \( Y = X/\langle \rho \rangle \) be the quotient, where \( X \) is a Calabi–Yau threefold and \( \rho \) is a non-Gorenstein involution. Then,

\[ h^i(Y, \mathcal{O}_Y) = 0 \text{ for } i > 0 \text{ and rank}(\text{Pic}(Y)) \leq \text{rank}(\text{Pic}(X)). \]

**Proof.** Note

\[ K_X \sim f^*(K_X) + 2 \sum_i E_i = 2 \sum_i E_i. \]
By the Hurwitz formula, we have

\[ K_{\tilde{X}} \sim \phi^*(K_{\tilde{Y}}) + S_{\tilde{X}} + \sum_i E_i, \]

where \( S_{\tilde{X}} = \phi^{-1}(S_{\tilde{Y}}). \) Note

\[ \phi_*(K_{\tilde{X}}) \sim 2K_{\tilde{Y}} + S_{\tilde{Y}} + \sum_i F_i. \]

But

\[ \phi_*(K_{\tilde{X}}) \sim \phi_*(2 \sum_i E_i) = 2 \sum_i \phi_*(E_i) = 2 \sum_i F_i. \]

Hence

\[ 2K_{\tilde{Y}} + S_{\tilde{Y}} \sim \sum_i F_i. \quad (2.1) \]

Let \( D = K_{\tilde{Y}} + S_{\tilde{Y}}, \) then

\[ 2D = 2K_{\tilde{Y}} + 2S_{\tilde{Y}} = (2K_{\tilde{Y}} + S_{\tilde{Y}}) + S_{\tilde{Y}} \sim \sum_i F_i + S_{\tilde{Y}}, \]

which is the branch locus of the double covering \( \phi : \tilde{X} \to \tilde{Y}. \) So,

\[ \phi_*(\mathcal{O}_{\tilde{X}}) \simeq \mathcal{O}_{\tilde{Y}} \oplus \mathcal{O}_{\tilde{Y}}(-D). \]

Accordingly, we have

\[
H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) \simeq H^i(\tilde{Y}, \phi_*(\mathcal{O}_{\tilde{X}})) \\
\simeq H^i(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) \oplus H^i(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(-D)) \\
= H^i(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) \oplus H^i(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(-K_{\tilde{Y}} - S_{\tilde{Y}})) \\
\simeq H^i(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) \oplus H^3-i(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(2K_{\tilde{Y}} + S_{\tilde{Y}}))^* \quad (\because \text{Serre duality}) \\
\simeq H^i(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) \oplus H^3-i \left( \tilde{Y}, \mathcal{O}_{\tilde{Y}} \left( \sum_i F_i \right) \right)^*. \quad (\because \text{Equation (2.1)})
\]

Let \( F = \bigcup_i F_i \) and consider the following exact sequence,

\[ 0 \to \mathcal{O}_\tilde{Y} \to \mathcal{O}_\tilde{Y} \left( \sum_i F_i \right) \to \mathcal{O}_F \left( \sum_i F_i \right) \to 0 \]

to induce an exact sequence

\[
H^{i-1} \left( F, \mathcal{O}_F \left( \sum_i F_i \right) \right) \to H^i \left( \tilde{Y}, \mathcal{O}_{\tilde{Y}} \right) \to H^i \left( \tilde{Y}, \mathcal{O}_{\tilde{Y}} \left( \sum_i F_i \right) \right) \to H^i \left( F, \mathcal{O}_F \left( \sum_i F_i \right) \right).
\]

Note \( \mathcal{O}_F \left( \sum_i F_i \right) \simeq \bigoplus_i \mathcal{O}_{F_i}(-2) \) and

\[ H^i \left( F, \mathcal{O}_F \left( \sum_i F_i \right) \right) \simeq \bigoplus_i H^i \left( F_i, \mathcal{O}_{F_i}(-2) \right) = 0 \]
for any $i$, which implies

$$H^i(\check{Y}, \mathcal{O}_{\check{Y}}) \simeq H^i\left(\check{Y}, \mathcal{O}_{\check{Y}}\left(\sum_i F_i\right)\right).$$

Therefore, we have

$$H^i(\check{X}, \mathcal{O}_{\check{X}}) \simeq H^i(\check{Y}, \mathcal{O}_{\check{Y}}) \oplus H^{3-i}\left(\check{Y}, \mathcal{O}_{\check{Y}}\left(\sum_i F_i\right)\right)^* \simeq H^i(\check{Y}, \mathcal{O}_{\check{Y}}) \oplus H^{3-i}(\check{Y}, \mathcal{O}_{\check{Y}})^*.$$ 

Accordingly,

$$h^i(\check{X}, \mathcal{O}_{\check{X}}) = h^i(\check{Y}, \mathcal{O}_{\check{Y}}) + h^{3-i}(\check{Y}, \mathcal{O}_{\check{Y}}). \quad (2.2)$$

We note

$$h^p(\check{X}, \mathcal{O}_{\check{X}}) = h^p(X, \mathcal{O}_X) = 0$$

for $p = 1, 2$. By letting $i = 0$ in Equation (2.2), we have $h^3(\check{Y}, \mathcal{O}_{\check{Y}}) = 0$ and by letting $i = 1$, we have

$$0 = h^1(\check{X}, \mathcal{O}_{\check{X}}) = h^1(\check{Y}, \mathcal{O}_{\check{Y}}) + h^2(\check{Y}, \mathcal{O}_{\check{Y}}),$$

which implies $h^1(\check{Y}, \mathcal{O}_{\check{Y}}) = h^2(\check{Y}, \mathcal{O}_{\check{Y}}) = 0$.

Finally, we have

$$h^i(Y, \mathcal{O}_Y) = h^i(\check{Y}, \mathcal{O}_{\check{Y}}) = 0$$

for $i > 0$.

Let $\check{\rho}$ be the involution on $\check{X}$ that is induced by the involution $\rho$ on $X$.

Borrowing arguments from [10] (pp. 6–7), one can show

$$\check{\phi}_* \Omega^p_\check{X} \simeq \Omega^p_\check{Y} \oplus \Omega^p_\check{Y}(\log B) \otimes L^{-1},$$

where $L$ is the anti-invariant part of the direct image of $\mathcal{O}_X$ to $\check{Y}$ under the action by $\check{\rho}$. Hence, we have

$$H^q(\check{X}, \Omega^p_\check{X}) \simeq H^q(\check{Y}, \phi_* \Omega^p_\check{X}) \simeq H^q(\check{Y}, \Omega^p_\check{Y}) \oplus H^q(\check{Y}, \Omega^p_\check{Y}(\log B) \otimes L^{-1}).$$

and so

$$\dim H^2(\check{X}) = \dim H^1(\check{X}, \Omega^1_\check{X}) \geq \dim H^1(\check{Y}, \Omega^1_\check{Y}) = \dim H^2(\check{Y}). \quad (2.3)$$

Now, we have $H^i(\mathcal{O}_X) = 0$, $H^i(\mathcal{O}_Y) = 0$ for $i = 1, 2$. So by the exponential sequence, we have $\text{Pic}(X) \simeq H^2(X, \mathbb{Z})$, $\text{Pic}(\check{X}) \simeq H^2(\check{X}, \mathbb{Z})$, $\text{Pic}(\check{Y}) \simeq H^2(\check{Y}, \mathbb{Z})$ and

$$\text{rank}(\text{Pic}(Y)) = \text{rank}(\text{Pic}(\check{Y})) - k$$

$$= \dim H^2(\check{Y}) - k$$

$$\leq \dim H^2(\check{X}) - k \quad (\because \ 2.3)$$

$$= \dim H^2(X) + k - k$$

$$= \text{rank}(\text{Pic}(X)),$$

that is, $\text{rank}(\text{Pic}(Y)) \leq \text{rank}(\text{Pic}(X))$. \qed
Let us consider some examples.

**Example 2.2.** Let $X$ be the intersection of the Fermat quadric and the Fermat quartic in $\mathbb{P}^5$. Then, $X$ is a Calabi–Yau threefold. Define an involution $\rho$ by

$$(x_0, x_1, x_2, x_3, x_4, x_5) \mapsto (-x_0, -x_1, -x_2, x_3, x_4, x_5).$$

Its fixed locus is composed of 16 isolated points—so zero-dimensional. Hence, $\rho$ is non-Gorenstein (see Remark 2.6). The quotient $Y$ has 16 singularities of type $\frac{1}{2}(1, 1, 1)$.

**Example 2.3.** There is a $K3$ surface $S$ with two commuting involutions $\sigma_1, \sigma_2$ such that $\sigma_1$ is a fixed point free non-symplectic involution and $\sigma_2$ is a symplectic involution (e.g., vol. 23 in [1]). Note that $\sigma_2$ has eight fixed points. Let $E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ be an elliptic curve with period $\tau$ and define involutions $\theta_1, \theta_2$ of $E$ by

$$\theta_1(x) = -x, \theta_2(x) = \frac{\tau}{2} - x,$$

respectively. Then, the involution $\sigma_1 \times \theta_1$ on $S \times E$ is fixed-point-free and the quotient $X := (S \times E)/\langle \sigma_1 \times \theta_1 \rangle$ is a Calabi–Yau threefold with infinite fundamental group. The involution $\sigma_2 \times \theta_2$ on $S \times E$ induces an involution $\rho$ on $X$ with 16 fixed points and the quotient $Y$ has 16 singularities of type $\frac{1}{2}(1, 1, 1)$. Note that $\rho$ is non-Gorenstein.

Let us note a simple fact about topology:

**Lemma 2.4.** Let $Z$ be a topological space, $q_1, q_2, \ldots, q_k$ be points on it and $Z^* = Z - \{q_1, q_2, \ldots, q_k\}$. If each point $q_i$ has a simply-connected neighborhood $U_i$ such that $U_i - \{q_i\}$ is open and path-connected, then the group homomorphism between fundamental groups:

$$\pi_1(Z^*) \to \pi_1(Z),$$

which is induced by the inclusion $Z^* \subset Z$, is surjective.

**Proof.** Let $Z_0 = Z^*$ and $Z_i = Z^* \cup \{q_1, q_2, \ldots, q_i\}$, then $Z_k = Z$.

We apply the Seifert–Van Kampen theorem to $Z_{i+1} = Z_i \cup U_{i+1}$, noting that $Z_i \cap U_{i+1} = U_{i+1} - \{q_{i+1}\}$. Since $U_{i+1}$ is simply-connected, the homomorphism

$$\psi_{i+1} : \pi_1(Z_i) \to \pi_1(Z_{i+1}),$$

induced by the inclusion $Z_i \subset Z_{i+1}$, is surjective. Since the homomorphism $\pi_1(Z^*) \to \pi_1(Z)$ is the composition

$$\psi_k \circ \psi_{k-1} \circ \cdots \circ \psi_0,$$

it is surjective. □

The same number 16 (the number of fixed points) in Examples 2.2 and 2.3 is not accidental:

**Theorem 2.5.** Let $\rho$ be a non-Gorenstein involution on a Calabi–Yau threefold $X$. Then, it is not fixed-point-free. Assume that its fixed locus is zero-dimensional. Then, the number of fixed points is 16. Furthermore, if $X$ has finite fundamental group, then the quotient $Y = X/\langle \rho \rangle$ is simply-connected.

**Proof.** By Lemma 2.1, $\chi(Y, \mathcal{O}_Y) = 1 - 0 + 0 = 1$. Suppose that $\rho$ is fixed-point-free, then $2\chi(Y, \mathcal{O}_Y) = \chi(X, \mathcal{O}_X) = 0$, which is a contradiction. So, $\rho$ is not fixed-point-free. Now, assume that its fixed locus is zero-dimensional and let $k$ be the number
of the fixed points. Note that \( \chi(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = \chi(Y, \mathcal{O}_Y) = 1 \). By the Riemann–Roch theorem, we have

\[
\chi(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = \frac{1}{24} c_1(\tilde{Y}) \cdot c_2(\tilde{Y}).
\]

By adjunction,

\[
c(\tilde{Y})|_{F_i} = (1 + F_i|_{F_i}) \cdot c(F_i) \text{ in } \bigoplus_{k=0}^2 H^{2k}(F_i, \mathbb{Z})
\]

and we have

\[
c_2(\tilde{Y})|_{F_i} = F_i|_{F_i} \cdot c_1(F_i) + c_2(F_i) = -6 + 3 = -3.
\]

From Equation (2.1), \( c_1(\tilde{Y}) = -\frac{1}{2} \sum_i F_i \) and we have

\[
\chi(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = \frac{1}{24} \left( -\frac{1}{2} \sum F_i \right) \cdot c_2(\tilde{Y})
\]

\[
= \frac{1}{48} \left( \sum_i c_2(\tilde{Y})|_{F_i} \right)
\]

\[
= \frac{1}{48} (-3k)
\]

\[
= \frac{k}{16}.
\]

So, we have \( k = 16 \).

Now, suppose that \( X \) has a finite fundamental group \( \pi_1(X) \). Let

\[
X^* = X - \{ p_1, p_2, \ldots, p_{16} \} \text{ and } Y^* = Y - \{ q_1, q_2, \ldots, q_{16} \},
\]

where \( p_i \)'s are the fixed points of \( \rho \) and \( q_i \)'s are the image of \( p_i \)'s in \( Y \), respectively. Applying the Seifert–Van Kampen theorem repeatedly as in the proof of Lemma 2.4, one can show \( \pi_1(X^*) \cong \pi_1(X) \)—one can take \( U_i \) so that \( U_i - \{ p_i \} \) is simply-connected. Hence, \( \pi_1(X^*) \) is also finite. By Lemma 2.4, we know that the homomorphism \( \pi_1(Y^*) \to \pi_1(Y) \) is surjective. So, \( \pi_1(Y) \) is also finite and \( Y \) has a projective universal covering \( \tilde{Y} \). Note that \( 2K_Y = 0 \) and \( \tilde{Y} \) has 16d singularities, where \( d \) is the degree of the covering \( \tilde{Y} \to Y \). Let \( \tilde{Y}^* = \tilde{Y} - \text{Sing}(\tilde{Y}) \), where \( \text{Sing}(\tilde{Y}) \) is the set of the singularities of \( \tilde{Y} \). By Theorem 1.1 in [14] (its proof works for the case that \( -2K_Y = 0 \) and \( S = \emptyset \)), there is a smooth threefold \( \tilde{X} \) with trivial canonical class such that there is a double covering \( \tilde{\phi} : \tilde{X} \to \tilde{Y} \), branched at \( \text{Sing}(\tilde{Y}) \). Let \( \tilde{X}^* = \tilde{\phi}^{-1}(\tilde{Y}^*) \), then \( \tilde{X}^* \to \tilde{Y}^* \) is an unbranched covering. Thus, \( \pi_1(X^*) \) is finite and so is \( \pi_1(\tilde{X}) \) since \( \pi_1(X^*) \cong \pi_1(X) \). Hence, \( h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0 \). Using

\[
\beta^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) \approx H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(K_{\tilde{X}}))^* = H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})^*,
\]

we have \( h^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) = h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0 \) and we conclude that \( \tilde{X} \) is also a Calabi–Yau threefold. The covering transformation of \( \tilde{X} \to \tilde{Y} \) is a non-Gorenstein involution on \( \tilde{X} \) and so \( \tilde{Y} \) has 16 singularities. Therefore, we have \( 16d = 16 \), that is, \( d = 1 \), which implies that \( \tilde{Y} \to Y \) is an isomorphism and so that \( Y \) is simply-connected. \( \square \)

If \( X \) has an infinite-fundamental group, then \( Y \) may not be simply-connected. In Example 2.3, let \( \tilde{Y} = (S \times E)/\langle \sigma_2 \times \theta_2 \rangle \), then it is easy to see that there is an unbranched double covering \( \tilde{Y} \to Y \). So, \( Y \) is not simply-connected.

There are fixed-point-free Gorenstein involutions on Calabi–Yau threefolds (see, e.g., [4, 6]). Recall the fixed locus of a Gorenstein involution is a disjoint union of smooth curves, so by Theorem 2.5, we make the following remark.

**Remark 2.6.** An involution on a Calabi–Yau threefold is Gorenstein if and only if it is fixed-point-free or its fixed locus is one-dimensional.
Now, let us consider some examples where the involutions have two-dimensional fixed locus.

**Example 2.7.** Let $\zeta = e^{2\pi \sqrt{-1}/3}$ and $E_3$ the elliptic curve whose period is $\zeta$. Let $\overline{X} = E_3^3/\langle \zeta \rangle$ be the quotient of $E_3^3$ by the scalar multiplication by $\zeta$. Then, $\overline{X}$ has 27 singularities of type $1/3(1, 1, 1)$. Blow up $\overline{X}$ at those singularities to get a smooth threefold $X$. It is known that $X$ is a Calabi–Yau threefold [3]. Let $\hat{0} \in \overline{X}$ be the image of $(0, 0, 0) \in E_3^3$ and $G$ be the exceptional divisor over $\hat{0}$ in the blow-up $X \to \overline{X}$. Then, $G \cong P^2$. The scalar multiplication on $E_3^3$ by $-1$ induces a non-Gorenstein involution $\rho$ on $X$. Then, $X/\rho = G \cup P$, where $P$ is a set of 31 points.

**Example 2.8.** Consider smooth the threefold $X$ in a weighted projective space $\mathbb{P}(1, 1, 1, 2, 5)$ with the equation

$$x^{10} + y^{10} + z^{10} + w^5 = t^2,$$

where $x$, $y$, $z$, $w$, and $t$ are homogeneous coordinates of weights 1, 1, 1, 2, and 5, respectively. Then, $X$ is a Calabi–Yau threefold. Define an involution $\rho$ by $(x, y, z, w, t) \mapsto (x, y, z, w, -t)$. Then, it is a non-Gorenstein involution and its fixed locus is:

$$X^\rho = \{(0, 0, 0, 0, 1)\} \cup S_X,$$

where $S_X = \{(x, y, z, w, t) \in X | x^{10} + y^{10} + z^{10} + w^5 = 0 \}$. It is not hard to see that the quotient $Y$ is isomorphic to $\mathbb{P}(1, 1, 1, 2)$ and the quotient map is the projection:

$$\varphi : X \to \mathbb{P}(1, 1, 1, 2),$$

$$(x, y, z, w, t) \mapsto (x, y, z, w).$$

Note that $Y = \mathbb{P}(1, 1, 1, 2)$ has a singularity of type $1/2(1, 1, 1)$ at $(0, 0, 0, 1)$. Note that $\mathbb{P}(1, 1, 1, 2)$ is a $\mathbb{Q}$-Fano threefold.

We call a $\mathbb{Q}$-Factorial threefold $\mathbb{Q}$-Fano when it has at worst terminal singularities and its anticanonical class is ample. Classifying general Calabi–Yau threefolds with non-Gorenstein involutions seems to be out of reach for now. In the case of the Picard rank one, one can get a boundedness result thanks to that of $\mathbb{Q}$-Fano threefolds of the Picard rank one.

**Theorem 2.9.** There are finitely many families of Calabi–Yau threefolds of Picard rank one that have non-Gorenstein involutions with two-dimensional fixed locus.

**Proof.** Let $X$ be a Calabi–Yau threefold of Picard rank one that has a non-Gorenstein involution $\rho$ with two-dimensional fixed locus and $Y = X/\langle \rho \rangle$. By Lemma 2.1, rank($\text{Pic}(Y)$) $\leq 1$ and so rank($\text{Pic}(Y)$) = 1. Using the same notation as in Lemma 2.1, we note that $2K_Y = g^*(-2K_Y) + \sum F_i$. By Equation (2.1), $g^*(-2K_Y) = S_Y$. Let $H'$ be an ample divisor of $Y$. Since $S_Y \neq \emptyset$ and it is disjoint from the exceptional divisors $F_i$’s, we have

$$H'^2 \cdot (-2K_Y) = g^*(H'^2) \cdot g^*(-2K_Y) = g^*(H'^2) \cdot S_Y > 0.$$

Since $Y$ has Picard rank one, $-2K_Y$ is an ample divisor and so $Y$ is a $\mathbb{Q}$-Fano threefold. Conversely, a $\mathbb{Q}$-Fano threefold of the Picard rank one with singularities of type only $1/2(1, 1, 1)$ has a Calabi–Yau threefold of the Picard rank one as its double covering [14–16]. Since there are finitely many families of $\mathbb{Q}$-Fano threefolds of the Picard rank one [12], there are finitely many families of Calabi–Yau threefolds of the Picard rank one that have non-Gorenstein involutions with two-dimensional fixed locus. \(\square\)

We remark that, for some Calabi–Yau threefold $X$ of the Picard rank one, there may exist two non-Gorenstein involutions $\rho_1, \rho_2$ on $X$ with two-dimensional fixed locus such that the pairs $(X, \rho_1), (X, \rho_2)$ are not deformation equivalent. Such an example is given in [16] (see comments before Table 2 also).

As shown in the proof of Theorem 2.9, the classification of Calabi–Yau threefolds of the Picard rank one that have non-Gorenstein involutions with two-dimensional fixed locus can be deduced from that of $\mathbb{Q}$-Fano threefolds of the Picard rank
one with singularities of type $\frac{1}{2}(1, 1, 1)$. The $\mathbb{Q}$-Fano threefolds of the Picard rank one with singularities of type $\frac{1}{2}(1, 1, 1)$ have been classified in several papers [2, 7, 21–23]. Let $Y$ be a $\mathbb{Q}$-Fano threefold of Picard rank one with $N$ singularities of type $\frac{1}{2}(1, 1, 1)$.

With the known classifications of $\mathbb{Q}$-Fano threefolds, we give a classification of Calabi–Yau threefolds $X$’s of Picard number one with non-Gorenstein involutions $\rho$’s whose fixed locus is not zero-dimensional; the classification is given with respect to numerical invariants $H^3, H \cdot c_2, e, N$ and involution index $s$ in Tables 1–3, where

1. $H$ is an ample generator of the Picard group of the Calabi–Yau threefold $X$,
2. $c_2$ is the second Chern class of $X$,
3. $e$ is the topological Euler characteristic of $X$,
4. $N$ is the number of isolated fixed points of $\rho$, and
5. $s$ is determined as follows: let $S_X$ be the two-dimensional fixed locus of $\rho$, then $S_X$ is numerically equivalent to $sH$ for some positive integer $s$.

These invariants can be easily calculated from those of $\mathbb{Q}$-Fano threefolds, whose formulas are given in [14–16]. For example, $H^3 = 2|H'|^3$, where $H'$ is a generator of the class group of $Y$.

If $N = 0$, then $Y$ is a smooth Fano threefold of Picard number one, whose classification is classical. Their invariants are listed in Table 1.

In case of $N > 0$, let $-2K_Y = r_Y H_Y$, where $H_Y$ is a primitive Cartier divisor and $r_Y \in \mathbb{N}$. The number $\frac{r_Y}{2}$ is called the Fano index of $Y$ and denoted by $F_Y$.

If the Fano index is greater than one, there is one single example which is $\mathbb{P}(1, 1, 1, 2)$ [7, 22]. We note that it is a quotient of a degree 10 Calabi–Yau hypersurface in $\mathbb{P}(1, 1, 1, 2, 5)$ by a non-Gorenstein involution (see Example 2.8). In this case, $N = 1, s = 10, H^3 = 1, H \cdot c_2 = 34, and e = -288$.

If the Fano index is one, such $\mathbb{Q}$-Fano threefolds are called Fano–Enriques threefolds and classified in [2, 21]. There are four families of them. Their Calabi–Yau double coverings were studied in [16]. It is notable that those Calabi–Yau double coverings are not simply-connected. Their invariants are listed in Table 2. Two of those $\mathbb{Q}$-Fano threefolds have the same Calabi–Yau threefold as their double coverings and that is why Table 2 has only three entries (not four).

If the Fano index is less than one, then it is equal to $\frac{1}{2}$. In this case, a classification was obtained in [23], with respect to the invariants $N, s, H^3$, and $H \cdot c_2$. The Calabi–Yau double coverings of those $\mathbb{Q}$-Fano threefolds were studied in [14, 15]. We remark that the possible Euler characteristics are not fully determined. Their invariants are listed in Table 3.

Remark 2.10 (for Table 3).

1. If $e$ is given in Table 3, then there is a known example of a Calabi–Yau threefold with those invariants; it is possible that there may be other examples with different topological Euler characteristics (but the other invariants are the same). Hence, the Euler characteristics are not fully determined.
2. If $e$ is missing, no examples are known yet.

From the tables and the invariants of the case in Example 2.8 (given between Tables 1 and 2), we can deduce the following sharp bounds for all Calabi–Yau threefolds of Picard number one with non-Gorenstein involutions whose fixed locus is not zero-dimensional:

$$1 \leq H^3 \leq 44,$$

$$20 \leq H \cdot c_2 \leq 92,$$

$$0 \leq N \leq 8,$$

and

$$2 \leq s \leq 10.$$
ACKNOWLEDGMENTS
The author is very thankful to the referees for making several valuable suggestions for the initial draft of this note. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2020R1A2C1A01004503) and 2021 Hongik University Research Fund.

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How to cite this article: N.-H. Lee, *Calabi–Yau threefolds with non-Gorenstein involutions*, Math. Nachr. **296** (2023), 3449–3458. https://doi.org/10.1002/mana.202100027