On the volume of the John-Löwner ellipsoid

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Abstract. We find an optimal upper bound on the volume of the John ellipsoid of a \( k \)-dimensional section of the \( n \)-dimensional cube, and an optimal lower bound on the volume of the Löwner ellipsoid of a projection of the \( n \)-dimensional cross-polytope onto a \( k \)-dimensional subspace. We use these results to give a new proof of Ball’s upper bound on the volume of a \( k \)-dimensional section of the hypercube, and of Barthe’s lower bound on the volume of a projection of the \( n \)-dimensional cross-polytope onto a \( k \)-dimensional subspace. We settle equality cases in these inequalities. Also, we describe all possible vectors in \( \mathbb{R}^n \), whose coordinates are the squared lengths of a projection of the standard basis in \( \mathbb{R}^n \) onto a \( k \)-dimensional subspace.

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1. Introduction

In [1], Fritz John proved that each convex body in \( \mathbb{R}^k \) contains a unique ellipsoid of maximal volume. John characterized all convex bodies \( K \) such that the ellipsoid of maximal volume in \( K \) is the Euclidean unit ball, \( \mathcal{E}_k \).

Theorem 1.1 (F. John). The Euclidean ball is the ellipsoid of maximal volume contained in a convex body \( K \in \mathbb{R}^k \) iff \( \mathcal{E}_k \subset K \) and, for some \( n \geq k \), there are Euclidean unit vectors \((u_i)^n_i\) on the boundary of \( K \), and positive numbers \((c_i)^n_i\) for which

\[
(1) \sum_{i=1}^n c_i u_i = 0;
\]

\[
(2) \sum_{i=1}^n c_i u_i \otimes u_i = I_k, \text{ the identity on } \mathbb{R}^k.
\]

Keith Ball in [5] added the converse part to this theorem.

Theorem 1.2 (K. Ball). Let \((u_i)^n_i\) be a sequence of unit vectors in \( \mathbb{R}^k \) and \((c_i)^n_i\) be a sequence of positive numbers satisfying (1) and (2).

Then the set \( K = \{ x \in \mathbb{R}^k | \langle x, u_i \rangle \leq 1, i \in \overline{1,n} \} \) contains a unique ellipsoid of maximal volume, which is the Euclidean unit ball.

The use of vectors \((u_i)^n_i\) and positive numbers \((c_i)^n_i\) satisfying (1) and (2) appears to be extremely powerful in a range of problems in convex analysis, including (see [6]): tight bounds on the volume ratio and on the outer volume ratio for centrally-symmetric convex bodies, and optimal upper bounds on the volume of a \( k \)-dimensional section of the \( n \)-cube.

In this short paper we study a simple alternative description of the vectors \((u_i)^n_i \subset \mathbb{R}^k\) and positive numbers \((c_i)^n_i\) satisfying (2).

Definition 1.3. We will say that some vectors \((v_i)^n_i \subset H\) give a unit decomposition in a \( k \)-dimensional vector space \( H \) if

\[
\sum_{i=1}^n v_i \otimes v_i = I_H,
\]
where $I_H$ is the identity operator in $H$.

Clearly, non-zero vectors $(v_i)_1^n \subset \mathbb{R}^k$ give a unit decomposition in $\mathbb{R}^k$ iff the vectors 
\[
\left( \frac{v_i}{|v_i|} \right)_1^n \subset \mathbb{R}^k
\]
and positive numbers $(|v_i|^2)_1^n$ satisfy (2).

In Lemma 2.5, the set of all possible vectors of positive reals $(c_1, \cdots, c_n)$, which we can get from condition (2). That this result may be interesting for finding optimal bounds in different geometric inequalities, including the Brascamb–Lieb inequality.

Using a geometric approach, we will give in Theorem 3.3 an optimal upper bound on the volume of the John ellipsoid of a projection of the n-dimensional cross-polytope onto a k-dimensional subspace.

In Section 4 we give a new proof for Ball’s upper bound on the volume of a convex body and for Barthe’s lower bound on the volume of a projection of the n-dimensional cross-polytope onto a k-dimensional subspace. Using a geometric approach, we will give in Theorem 3.3 an optimal upper bound on the volume of the Löwner ellipsoid of a projection of the n-dimensional cross-polytope onto a k-dimensional subspace.

In this section, using a simple linear algebra observation we introduce a description of sets of vectors, whose last $n-k$ coordinates are zero. For convenience, we will consider $(v_i)_1^n \subset \mathbb{R}^k \subset \mathbb{R}^n$ to be $k$-dimensional vectors.

**Lemma 2.1.** The following assertions are equivalent:

1. vectors $(v_i)_1^n \subset \mathbb{R}^k$ give a unit decomposition in $\mathbb{R}^k$;
2. there exists an orthonormal basis $\{f_1, \cdots, f_n\} \subset \mathbb{R}^n$ such that $v_i$ is the orthogonal projection of $f_i$ onto $\mathbb{R}^k$, for any $i \in 1, n$;
3. $\text{Lin}\{v_1, \cdots, v_n\} = \mathbb{R}^k$ and the Gram matrix $\Gamma$ of vectors $\{v_1, \cdots, v_n\} \subset \mathbb{R}^k$ is the matrix of a projection operator from $\mathbb{R}^n$ onto the linear hull of the rows of the matrix $A = [v_1, \cdots, v_n]$.
4. the $k \times n$ matrix $A = [v_1, \cdots, v_n]$ is a sub-matrix of an orthogonal matrix of order $n$.

**Proof.**

1) (4) $\Rightarrow$ (3).

Since $A$ is a sub-matrix of an orthogonal matrix, we have that $\text{rk} A = k$. Therefore, $\text{Lin}\{v_1, \cdots, v_n\} = \mathbb{R}^k$.

Let $\Gamma$ be the Gram matrix of vectors $\{v_1, \cdots, v_n\}$ and $P$ be the matrix of a projection operator from $\mathbb{R}^n$ onto the linear hull of the rows of the matrix $A = [v_1, \cdots, v_n]$. Since the rows of $A$ form an orthonormal basis of their linear hull $H_k$, we can identify $Pe_i$ and $v_i$ in this basis of $H_k$. Therefore,

\[
P_{ij} = \langle Pe_i, e_j \rangle = \langle P^2 e_i, e_j \rangle = \langle Pe_i, Pe_j \rangle = \langle v_i, v_j \rangle = \Gamma_{ij}.
\]

2) (3) $\Rightarrow$ (2).

Let the Gram matrix $\Gamma$ of the vectors $\{v_1, \cdots, v_n\}$ be the matrix of a projection operator onto $H_k$. By the last identity, we have that $\langle \Gamma e_i, \Gamma e_j \rangle = \langle v_i, v_j \rangle$. But if two sets $S_1$ and $S_2$ of vectors have the same Gram matrix, then there exists an orthogonal transformation of the space that maps vectors of $S_1$ to $S_2$. Indeed, each step in the Gram-Schmidt process for both
systems are identical, that means that any orthogonal transformation which maps the Gram-
Schmidt orthonormalization of $S_1$ to the Gram-Schmidt orthonormalization of $S_2$ maps $S_1$ to $S_2$. Therefore, with a proper choice of the orthonormal basis in $\mathbb{R}^n$, we can identify vectors $v_i$ and $Pe_i$, for $i \in \overline{1,n}$, ans subspaces $H_k$ and $\mathbb{R}^k = \text{Lin}\{v_1, \cdots, v_n\}$.  

3)(2) $\Rightarrow$ (1)  

Let $P$ be the projection from $\mathbb{R}^n$ onto $\mathbb{R}^k$. Let $v_i = Pf_i$. For an arbitrary vector $x \in \mathbb{R}^k$ we have $Px = x$ and therefore  

$$
\left(\sum_{i=1}^{n} v_i \otimes v_i\right) x = \sum_{i=1}^{n} \langle v_i, x \rangle v_i = \sum_{i=1}^{n} \langle Pf_i, x \rangle v_i = \sum_{i=1}^{n} \langle f_i, Px \rangle v_i = P\left(\sum_{i=1}^{n} \langle f_i, x \rangle f_i\right) = Px = x.
$$

4)(1) $\Rightarrow$ (4)  

For $i \in \overline{1,k}$ we have $e_j = \sum_{i=1}^{n} \langle v_i, e_j \rangle v_i$. Therefore,  

$$
1 = |e_j|^2 = \sum_{i=1}^{n} \langle v_i, e_j \rangle^2 = \sum_{i=1}^{n} v_i[j]^2,
$$

and  

$$
0 = \langle e_j, e_k \rangle = \sum_{i=1}^{n} \langle v_i, e_j \rangle \langle v_i, e_k \rangle = \sum_{i=1}^{n} v_i[j]v_i[k],
$$

where $v_i[j]$ is the $j$’th coordinate of the vector $v_i$ in the given basis.

That is, the rows of the $k \times n$ matrix $[v_1, \cdots, v_n]$ form an orthonormal system of $k$ vectors in $\mathbb{R}^n$. \hfill \Box

As a direct consequence of Lemma 2.1 we get

**Corollary 2.2.** Let $(u_i)^n_i$ be a sequence of unit vectors in $\mathbb{R}^k$ and $(c_i)^n_i$ be a sequence of positive numbers satisfying (2). Then the set $K = \{x \in \mathbb{R}^k ||\langle x, u_i \rangle| \leq 1, i \in \overline{1,n}\}$ is an affine image of a $k$-dimensional section of $\square^n$.

**Definition 2.3.** We will say that a vector $C = (c_1, \cdots, c_n)$ is realizable in $\mathbb{R}^k$ if there exist vectors $(v_i)^n_i$ which give a unit decomposition in $\mathbb{R}^k$ such that $c_i = |v_i|^2$, $i \in \overline{1,n}$.

Now we are going to describe all possible realizable vectors in $\mathbb{R}^k$. For this purpose, we need to use the following standard notation.

**Definition 2.4.** Let $a$ and $b$ be non-negative vectors in $\mathbb{R}^n$. The vector $a$ majorizes the vector $b$, which we denote by $a \succ b$, if the sum of the $k$ largest entries of $a$ is at least the sum of the $k$ largest entries of $b$, for every $k \in \overline{1,n}$, and the sums of all entries of $a$ and $b$ are equal.

**Lemma 2.5.** A vector $(c_1, \cdots, c_n)$ is realizable iff  

$$
(1, \cdots, 1, 0, \cdots, 0) \succ (c_1, \cdots, c_n).
$$

**Proof.**  

Let $(c_1, \cdots, c_n)$ be a realizable vector. By definition and by Lemma 2.1, there are vectors $(v_i)^n_i \subset \mathbb{R}^k$ that give a unit decomposition in $\mathbb{R}^k$ such that the diagonal entries of their Gram matrix $\Gamma$ are $(c_i)^n_i$, and $\Gamma$ is the matrix of a projection operator from $\mathbb{R}^n$ onto some $k$-dimensional subspace $H_k$.

So the vector $(c_1, \cdots, c_n)$ is realizable iff there exists a projection operator from $\mathbb{R}^n$ onto some $k$-dimensional subspace with $(c_1, \cdots, c_n)$ on the main diagonal. Applying Horn’s theorem [7], which asserts that a vector $(c_1, \cdots, c_n)$ can be the main diagonal of a Hermitian matrix
with a vector of eigenvalues \((\lambda_1, \cdots, \lambda_n)\) iff \(\lambda > c\), to the vector \((1, \cdots, 1, 0, \cdots, 0)\) we complete the proof. \(\square\)

3. Estimation for the Volume of the Löwner-John Ellipsoid

Before stating the next result, we recall that the John ellipsoid of a convex body \(K\) is the ellipsoid of maximal volume contained in \(K\), and the Löwner ellipsoid of a convex body \(K\) is the ellipsoid of minimal volume containing \(K\). We use \(\mathcal{E}_{H_k}\) and \(\mathcal{E}_{H_k}^n\) to denote the Löwner ellipsoid of \(\hat{\mathcal{O}}^n|H_k\) and the John ellipsoid of \(\mathcal{O}^n \cap H_k\), respectively.

**Lemma 3.1.** Suppose vectors \((v_i)_i \subset \mathbb{R}^k\) give a unit decomposition in \(\mathbb{R}^k\). Then for any ellipsoid \(\mathcal{E}\) with the center in the origin that covers all vectors \((v_i)_i\), we have

\[
\text{vol }\mathcal{E} \geq \left(\frac{k}{n}\right)^{\frac{k}{2}} \text{ vol }\mathcal{E}_k,
\]

where \(\mathcal{E}_k\) is the unit ball in \(\mathbb{R}^k\). The bound is tight. The inequality becomes an equality iff \(|v_i|^2 = \frac{k}{n}\), for all \(i \in \overline{1,n}\).

**Proof.**

For a positive-definite operator \(A\) on \(\mathbb{R}^k\), we know that the volume of the ellipsoid \(\{x \in \mathbb{R}^k | \langle Ax, x \rangle \leq 1\}\) is \(\frac{\text{vol}^2 \mathcal{E}_k}{\det A}\). To prove our lemma, it is enough to show that for any positive-definite operator \(A\) such that \(\langle Av_i, v_i \rangle \leq 1\) for all \(i \in \overline{1,n}\) we have \(\det A \leq \left(\frac{n}{k}\right)^k\).

Fix a positive-definite operator \(A\) on \(\mathbb{R}^k\) such that \(\langle Av_i, v_i \rangle \leq 1\) for all \(i \in \overline{1,n}\). We can choose an orthonormal basis in \(\mathbb{R}^k\) such that \(A = \text{diag}\{\lambda_1, \cdots, \lambda_k\}\) in this basis. Let \(v'_i\) be the coordinate vector of \(v_i\) in the new basis for each \(i \in \overline{1,n}\). We can rewrite the inequality \(\langle Av_i, v_i \rangle \leq 1\) in the following form:

\[
\sum_{j=1}^{k} \lambda_j v'_i[j]^2 \leq 1,
\]

where \(v'_i[j]\) is the \(j\)'th coordinate of the vector \(v'_i\) in the given basis.

Summing up the inequalities (3.2) for all \(i \in \overline{1,n}\) and using the observation that \(\sum_{j=1}^{n} v'_i[j]^2 = 1\) (see Lemma 2.1), we get

\[
\sum_{i=1}^{k} \lambda_i \leq n.
\]

Applying the AM-GM inequality, we get

\[
\det A = \prod_{i=1}^{k} \lambda_i \leq \left(\frac{\sum_{i=1}^{k} \lambda_i}{k}\right)^k \leq \left(\frac{n}{k}\right)^k.
\]

According to Lemma 2.5, the vector \(\left(\frac{k}{n}, \cdots, \frac{k}{n}\right)\) is realizable. It is clear that in this case inequality (3.1) becomes an equality. Moreover, by properties of the AM-GM inequality, we have an equality in (3.1) iff \(\lambda_i = \frac{n}{k}\), for all \(i \in \overline{1,n}\). By the inequality (3.2), there is an equality in (3.1) iff \(|v_i|^2 = \frac{k}{n}\), for all \(i \in \overline{1,n}\).

\(\square\)

Fix a \(k\)-dimensional subspace \(H_k\) in \(\mathbb{R}^n\). Let \(P : \mathbb{R}^n \to H_k\) be the projection onto \(H_k\). Since \(\hat{\mathcal{O}}^n|H_k\) is the absolute convex hull of the vectors \(v_i = Pe_i\) for \(i \in \overline{1,n}\) that give us a unit
decomposition in $H_k$, Lemma 3.1 implies that the volume of the L"owner ellipsoid for $\Diamond^n|H_k$ is at least $\left(\frac{k}{n}\right)^{\frac{k}{2}} \text{vol}\mathcal{E}_k$.

To settle the reverse case, we need to recall the following simple duality arguments.

For a given $k$-dimensional subspace $H_k$ in $\mathbb{R}^n$, we can consider the space $H_k \subset (\mathbb{R}^n)^* = \mathbb{R}^n$ itself to be the dual space for $H_k$. Indeed, $H_k$ is a $k$-dimensional space consisting of all linear functionals on $H_k$ with the proper linear structure, and the restriction of the Euclidean norm in $\mathbb{R}^n$ onto $H_k$ generates the operator norm on $H_k$.

For the sake of completeness we give a proof of the following well-known fact.

Lemma 3.2. Let $H_k$ be a $k$-dimensional subspace of $\mathbb{R}^n$. Assume the dual space $H_k^*$ for $H_k$ is $H_k$ itself. For a convex body $K \in \mathbb{R}^n$ containing the origin in the interior, we have

$$ (K \cap H_k)^o = K^o|H_k, $$

where we understand $K \cap H_k$ as a subset of $H_k$, and its polar set as a subset of $H_k^* = H_k$.

Proof. We use $H_k^\perp$ to denote the orthogonal complement of $H_k$ in $\mathbb{R}^n$.

1) Let us show that $(K \cap H_k)^o \supset K^o|H_k$. Fix a functional $p \in K^o|H_k$. Since $p$ belongs to the projection of $K^o$, there is a functional $p^\perp \subset H_k^\perp$ such that $p + p^\perp \in K^o$. By definition of the polar body, we have $\langle p + p^\perp, x \rangle \leq 1$ for any $x \in K$. In particular, for any $x \in K \cap H_k$, we have

$$ 1 \geq \langle p + p^\perp, x \rangle = \langle p, x \rangle + \langle p^\perp, x \rangle = \langle p, x \rangle. $$

This means that $p \in (K \cap H_k)^o$.

2) Let us show that $(K \cap H_k)^o \subset K^o|H_k$. Suppose for a contradiction that there is a functional $p \in (K \cap H_k)^o$ such that $p \notin K^o|H_k$. By the hyperplane separation theorem, there exists a vector $y \in H_k$ such that

$$ \langle p, y \rangle > 1 $$

and

$$ \langle q, y \rangle \leq 1 \quad \text{for all} \quad q \in K^o|H_k. $$

Clearly, $\langle y, q^\perp \rangle = 0$ for any $q^\perp \in H_k^\perp$. Combining this and the inequality (3.4), we get

$$ \langle q, y \rangle \leq 1 $$

for all $q \in K^o$. By the definition of the polar set, we obtain $y \in (K^o)^o = K$. So $y \in K$ and $y \in H_k$, therefore $y \in K \cap H_k$. This contradicts the inequality (3.3).

\square

For a given convex centrally-symmetric body $K$ with the center at the origin, by symmetry and duality arguments, we have that the polar ellipsoid of the John ellipsoid of $K$ is the L"owner ellipsoid of $K^o$.

Summarizing the arguments of section 3, we obtain.

Theorem 3.3. For any $1 \leq k \leq n$ we have

$$ \frac{\text{vol} \mathcal{E}_{H_k}}{\text{vol} \mathcal{E}_k} \geq \left(\frac{k}{n}\right)^{\frac{k}{2}} \quad \text{and} \quad \frac{\text{vol} \mathcal{E}_{H_k}}{\text{vol} \mathcal{E}_k} \leq \left(\frac{n}{k}\right)^{\frac{k}{2}}. $$

The bounds are sharp. That is, there exists a subspace $H_k$ such that the two inequalities are simultaneously hold as equalities.
4. Bounds on the volume of a section of $\Box^n$ and a projection of $\Diamond^n$

K. Ball, in his fundamental paper [8], proved the following inequality

\begin{equation}
\frac{\text{vol}(\Box^n \cap H_k)}{\text{vol } \Box^k} \leq \left(\frac{n}{k}\right)^\frac{k}{2}.
\end{equation}

F. Barthe in [3] proved the dual inequality

\begin{equation}
\frac{\text{vol}(\Diamond^n|H_k)}{\text{vol } \Diamond^k} \geq \left(\frac{k}{n}\right)^\frac{k}{2}.
\end{equation}

One can see that both inequalities become equalities when $k | n$ and $H_k$ is determined by the system of linear equations

\begin{equation}
x_{\frac{j}{k} j+i_1} = x_{\frac{j}{k} j+i_2}, \quad \text{where} \quad j \in \{0, k-1\} \quad \text{and} \quad 1 \leq i_1, i_2 \leq \frac{n}{k}.
\end{equation}

Using Theorem 3.3, we are going to give another proof of the inequalities (4.1) and (4.2), and settle the equality case.

**Theorem 4.1.** For any $k$-dimensional subspace of $\mathbb{R}^n$, we have

\begin{equation}
\frac{\text{vol}(\Box^n \cap H_k)}{\text{vol } \Box^k} \leq \left(\frac{n}{k}\right)^\frac{k}{2} \quad \text{and} \quad \frac{\text{vol}(\Diamond^n|H_k)}{\text{vol } \Diamond^k} \geq \left(\frac{k}{n}\right)^\frac{k}{2}.
\end{equation}

The bounds are optimal iff $k | n$.

**Proof.**

Using the Brascamb–Lieb inequality, K. Ball [8] proved that among all $k$-dimensional convex centrally-symmetric bodies, the $k$-cube has the greatest volume ratio (i.e., $(\frac{\text{vol } K}{\text{vol } E})^\frac{1}{n}$, where $E$ is the John ellipsoid of $K$). This means that

\begin{equation}
\frac{\text{vol}(\Box^n \cap H_k)}{\text{vol } \Box^k} \leq \frac{\text{vol } E_{H_k}}{\text{vol } E_k}.
\end{equation}

The dual case for the outer volume ratio (i.e. $(\frac{\text{vol } E}{\text{vol } K})^\frac{1}{n}$, where $E$ is the Löwner ellipsoid of $K$) was resolved using Barthe’s reverse Brascamb–Lieb inequality [6]. It was shown that $\Diamond^k$ has the biggest outer volume ratio among all $k$-dimensional convex centrally-symmetric bodies. Therefore

\begin{equation}
\frac{\text{vol}(\Diamond^n|H_k)}{\text{vol } \Diamond^k} \geq \frac{\text{vol } E_{H_k}}{\text{vol } E_k}.
\end{equation}

Combining (4.4) and (4.5) with the inequalities from the assertion of Theorem 3.3, we obtain (4.1) and (4.2).

We now prove that the bounds are optimal only if $k | n$.

In [3] Proposition 10, Barthe proved that whenever the volume ratio for a convex centrally-symmetric body $K \subset \mathbb{R}^k$ equals the volume ratio for $\Box^k$, then $K$ is an affine $k$-dimensional cube (or parallelotope). Also, he proved that if a centrally-symmetric convex body $K \subset \mathbb{R}^k$ has the extremal inner volume ratio, then $K$ is an affine cross-polytope. These arguments imply that $\Box^n \cap H_k$ is an affine cube and $\Diamond^n|H_k$ is an affine cross-polytope in the equality cases for the inequalities (4.1) and (4.2), respectively.

Using the fact that $\Box^k$ is the polar set of $\Diamond^k$ and employing Lemma 3.2, we obtain that for any given subspace $H_k$ equality holds in (4.1) if and only if equality holds in (4.2). Hence, it is enough to settle equality only for the inequality (4.2).

Suppose for a given $H_k$ we have equality in (4.2). Then $\Diamond^n|H_k$ is an affine $k$-dimensional cross-polytope. Let $P$ be the projection from $\mathbb{R}^n$ onto $H_k$, and $v_i = Pe_i$. It is easy to see that each vertex of $\Diamond^n|H_k$ is identical to at least one of the vectors $v_i$, where $i \in \overline{1,n}$. The proof of Lemma 3.1 yields that all lengths $|v_i|$, for $i \in \overline{1,n}$, are the same. From this and the triangle
inequality, we conclude that all vectors \( v_i, i \in \mathbb{1}, n \), are vertices of the affine cross-polytope \( \Diamond^n | H_k \). So, each vertex of \( \Diamond^n | H_k \) is identical to some \( v_i \), and conversely, each \( v_i \) is a vertex of \( \Diamond^n | H_k \).

Denote by \( \ell_i, i \in \mathbb{1}, k \), lines in \( H_k \) that pass through vertices of the affine cross-polytope \( \Diamond^n | H_k \). We showed that for any \( i \in \mathbb{1}, n \) there exists \( j \in \mathbb{1}, k \) such that \( v_i \in \ell_j \). Hence, there exist vectors \( d_j \in \ell_j, j \in \mathbb{1}, k \), such that \( I_{H_k} = \sum_{i=1}^{n} v_i \otimes v_i = \sum_{j=1}^{k} d_j \otimes d_j \). This means that the vectors \( d_j, j \in \mathbb{1}, k \), give us a unit decomposition in \( H_k \). By the assertion 4 of Lemma 2.1, we have that the vectors \( d_j, j \in \mathbb{1}, k \), form an orthonormal basis in \( H_k \). Therefore, all \( k \) sums \( \sum_{v_i \in \ell_j} |v_i|^2, j \in \mathbb{1}, k \), equals 1. As mentioned above, all lengths \( |v_i|, \) for \( i \in \mathbb{1}, n \), are the same. Consequently, the same number of vectors \( v_i, i \in \mathbb{1}, n \), lies on each line \( \ell_j, j \in \mathbb{1}, k \). That is, \( k|n| \).

\[ \square \]

**Remark 4.2.** Up to coordinate permutation and up to change of the sign of coordinates, Theorem 4.1 implies that equality in (4.1) and (4.2) is attained when \( H_k \) is determined by (4.3).

We should note that Ball’s and Barthe’s proofs of the inequalities (4.1) and (4.2) used the same arguments as the proofs of (4.4) and (4.5). However, we believe that it may be of interest how our result reveals the connection between Theorem 4.1 and the volume of the Löwner and the John ellipsoid.

We conjecture the following.

**Conjecture 4.3.**

\[ (4.6) \quad \frac{\text{vol}(\Diamond^n | H_k)}{\text{vol} \Diamond^k} \geq 2^{\frac{k-n}{2}}. \]

*The bound is optimal when \( 2k \geq n \).*

This is the dual statement for another Ball’s upper bound on the volume of a \( k \)-dimensional section of \( \square^n \):

\[ (4.7) \quad \frac{\text{vol}(\square^n \cap H_k)}{\text{vol} \square^k} \leq 2^{\frac{n-k}{2}}. \]

We note that inequalities (4.2) and (4.6) follow from the well-known Mahler conjecture and inequalities (4.1) and (4.7), respectively.

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