SPECTRAL PROPERTIES OF THE RENORMALIZATION GROUP AT INFINITE TEMPERATURE

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Abstract. The renormalization group (RG) approach is largely responsible for the considerable success that has been achieved in developing a quantitative theory of phase transitions. Physical properties emerge from spectral properties of the linearization of the RG map at a fixed point. This article considers RG for classical Ising-type lattice systems. The linearization acts on an infinite-dimensional Banach space of interactions. At a trivial fixed point (zero interaction), the spectral properties of the RG linearization can be worked out explicitly, without any approximation. The results are for the RG maps corresponding to decimation and majority rule. They indicate spectrum of an unusual kind: dense point spectrum for which the adjoint operators have no point spectrum at all, only residual spectrum. This may serve as a lesson in what one might expect in more general situations.

1. Introduction

We consider renormalization group (RG) transformations for Ising-type lattice spin systems on $\mathbb{Z}^d$. Our original lattice is denoted by $L$ and our image lattice is denoted by $L'$. The image lattice $L'$ indexes a partition of $L$ into cubical blocks, all with the same cardinality $b^d$. Thus for each site $y$ in $L'$, there is a corresponding block $y^o$ that is a subset of $L$, given by

$$
y^o = \{ x : by_i - \frac{b-1}{2} \leq x_i \leq by_i + \frac{b-1}{2}, 1 \leq i \leq d \}
$$

for odd blocking factor $b$; and

$$
y^o = \{ x : by_i - \frac{b-2}{2} \leq x_i \leq by_i + \frac{b-2}{2}, 1 \leq i \leq d \}
$$

for even blocking factor $b$. More generally, for each subset $Y$ of $L'$, there is a corresponding union of blocks $Y^o$ that is a subset of $L$, given by

$$
y^o = \{ x : by_i - \frac{b-1}{2} \leq x_i \leq by_i + \frac{b-1}{2}, 1 \leq i \leq d \}
$$

for odd blocking factor $b$; and

$$
y^o = \{ x : by_i - \frac{b-2}{2} \leq x_i \leq by_i + \frac{b-2}{2}, 1 \leq i \leq d \}
$$

for even blocking factor $b$. More generally, for each subset $X$ of $L$, a spin variable $\sigma_X$ is assigned to each site $x$ in $X$, and a block spin variable $\sigma_Y^\prime$ is assigned to each site $y$ in $Y$. If $X$ is a finite subset of the original lattice, then $\sigma_X$ denotes the spin variable $\prod_{x \in X} \sigma_x$. Similarly, if $Z$ is a finite subset of the image lattice, then $\sigma_Y^\prime$ denotes the block spin variable $\prod_{z \in Z} \sigma_z$. The main physical properties of $L$ are encoded in the Hamiltonian $H(\sigma) = -\sum_X J(X) \sigma_X$, where $J$ is the original interaction defined on nonempty finite subsets of $L$. Likewise, the main physical properties of $L'$ are encoded in the Hamiltonian $H'(\sigma') = -\sum_Y J'(Y) \sigma_Y^\prime$, where $J'$ is the resulting interaction defined on nonempty finite subsets of $L'$. 

Here is the formal definition of the RG map:

$$
e^{\sum_Y J'(Y) \sigma_Y^\prime} \sum_{\sigma'} e^{\sum_Y J(Y) \sigma_Y} = \frac{\prod_{Y \in L'} T_y(\sigma, \sigma_Y') e^{\sum_X J(X) \sigma_X}}{\sum_\sigma e^{\sum_X J(X) \sigma_X}},
$$

where $\sum_{\sigma}$ and $\sum_{\sigma'}$ (normalized sums) denote the product probability measures on $\{+1, -1\}^L$ and $\{+1, -1\}^{L'}$, respectively, and $T_y(\sigma, \sigma_Y')$ denotes a specific RG probability kernel, which
There is a function $\phi_I \in $ the following, we restrict our attention to a special kind of deterministic probability kernel: $B$ and a normalization condition, $\sum\sigma$ and every $y$. Notice that because of (4) and (5),

$$\sum\sigma T_y(\sigma, +1) = \sum\sigma T_y(\sigma, -1) = 1.$$  

In the following, we restrict our attention to a special kind of deterministic probability kernel: There is a function $\phi_I(\sigma)$ that depends only on $\sigma$ through $y^o$, and $T_y(\sigma, \sigma'_y) = 2\delta(\phi_I(\sigma), \sigma'_y)$.

Our basic assumption is that the original interaction $J$ lies in a Banach space $B^r$, with norm

$$||J||_r = \sup_{x \in L} \sum_{X : x \in X} |J(X)| e^{r(l(x,X))},$$

where the constant $r \geq 0$, $d$ is a metric on $L$, and $l(x, X) = \sup\{d(x, y) : y \in X\}$, with the convention that $l(x, \emptyset) = 0$. Correspondingly, there is a paired Banach space $B^*_r$. As

$$|\sum_X J_1(X)J_2(X)| \leq \sum_X |J_1(X)| \sum_{X : x \in X} \frac{1}{|X|} |J_2(X)|$$

$$= \sum_{x \in L} \sum_{X : x \in X} \frac{1}{|X|} |J_1(X)||J_2(X)|$$

$$\leq \sum_{x \in L} \sup_{X : x \in X} \frac{1}{|X|} |J_2(X)| e^{-r(l(x,X))} \sum_{X : x \in X} |J_1(X)| e^{r(l(x,X))}$$

$$\leq \sup_{x \in L} \sum_{X : x \in X} |J_1(X)| e^{r(l(x,X))} \cdot \sum_{x \in L} \sup_{X : x \in X} \frac{1}{|X|} |J_2(X)| e^{-r(l(x,X))},$$

a suitable $B^*_r$ norm is defined by

$$||J||^*_r = \sum_{x \in L} \sup_{X : x \in X} \frac{1}{|X|} |J(X)| e^{-r(l(x,X))}.$$  

Notice that here $B^r$ is technically not the dual space of $B^*_r$, and $B^*_r$ is technically not the dual space of $B_r$. The spaces $B_r$ and $B^*_r$ are paired in the sense that each one is part of the dual space of the other, or in other words, each one consists of continuous linear functions defined on the other. We study the situation when $||J||_r = 0$ (indication of infinite temperature). We consider the spectrum of the linearization $L(J)$ of two commonly used RG transformations, decimation and deterministic majority rule with odd blocking factor. We show that this spectrum is of an unusual kind: dense point spectrum for which the adjoint operators $L^*(J)$ have no point spectrum at all, but only residual spectrum.

**Remark.** In this paper, spectrum is crudely divided into 3 types [1]: For a bounded linear operator $A$ acting on a Banach space $A : B \to B$,

1. $\lambda$ is in the point spectrum $\iff$ there exists $B \ni u \neq 0$, such that $(A - \lambda)u = 0$, i.e., $\text{Kernel}(A - \lambda I)$ is nontrivial.

2. $\lambda$ is in the residual spectrum $\iff$ $\lambda$ is not in the point spectrum, and $\overline{\text{Range}(A - \lambda I)} \neq B$. 


λ is in the continuous spectrum $\iff$ λ is not in the point spectrum or the residual spectrum, $\text{Range}(A - \lambda I) \neq B$, and $\text{Range}(A - \lambda I) = B$.

This definition is too simple to fully capture the notion of continuous spectrum, but it will be adequate for our purposes.

Israel [2] found the operator bound of $L(J)$ for decimation in a Banach algebra setting, but did not go into detail about the spectral type of this transformation. He also examined the operator bound of $L(J)$ for majority rule on the triangular lattice. These results are extended by the present investigation, which includes the spectral type of $L(J)$ and $L^*(J)$ for decimation (Theorems 3.3 and 3.10) and majority rule (Theorems 4.3 and 4.9). Even though this investigation is focused on the RG transformation acting on a system very close to a trivial interaction, it serves as a test case—after all, if it is reasonably difficult to compute the spectrum of the RG map, then one can get an idea of what to expect by computing in a simple case. If even this case has bizarre spectral properties, then it may serve as a lesson in what to expect in more general situations.

2. Some general results

Proposition 2.1. The renormalized coupling constants $J'$ are given by the expression

$$J'(Z) = \sum_{\sigma'} \sigma'_Z \log(W(\sigma')),$$

where $W(\sigma')$ is the frozen block spin partition function given by

$$W(\sigma') = \sum_{\sigma} \prod_{y \in L'} \frac{T_y(\sigma, \sigma_y') e^{\sum_X J(X)\sigma_X}}{\sum_{\sigma} T_y(\sigma, \sigma_y') e^{\sum_X J(X)\sigma_X}}.$$

Proof. In order to write down an explicit expression of $J'$, we use Fourier series on the group $\{+1, -1\}^{L'}$. If $H'(\sigma') = -\sum_Y J'(Y)\sigma_Y'$, then $J'(Z) = \sum_{\sigma'} -H'(\sigma')\sigma_Z'$. We see that

$$J'(Z) = \sum_{\sigma'} \sigma'_Z \log\left(\sum_{\sigma} \prod_{y \in L'} T_y(\sigma, \sigma_y') e^{\sum_X J(X)\sigma_X}\right)$$

$$+ \sum_{\sigma'} \sigma'_Z \log\left(\sum_{\sigma} e^{\sum_Y J'(Y)\sigma_Y'}\right) - \sum_{\sigma'} \sigma'_Z \log\left(\sum_{\sigma} e^{\sum_X J(X)\sigma_X}\right).$$

(11)

An important observation here is that $\log\left(\sum_{\sigma'} e^{\sum_Y J'(Y)\sigma_Y'}\right)$ and $\log\left(\sum_{\sigma} e^{\sum_X J(X)\sigma_X}\right)$ are constants with respect to $\sigma'_Z$; thus, when summing over all possible image configurations $\sigma'$, they both vanish. \Box

Proposition 2.2. Suppose the original interaction $J$ is at infinite temperature. Then for every subset $W$ of the original lattice and every subset $Z$ of the image lattice, the partial derivative $\frac{\partial J'(Z)}{\partial J(W)}$ of the RG transformation is given by the expression

$$\frac{\partial J'(Z)}{\partial J(W)} = \sum_{\sigma} \sum_{\sigma'} \prod_{y \in L'} T_y(\sigma, \sigma_y') \sigma_W \sigma'_Z.$$

(12)
Proof. We take the derivative of both sides of (12) with respect to \( J(W) \).

\[
\frac{\partial J'(Z)}{\partial J(W)} = \sum_\sigma \sigma'_Z \sum_\sigma \prod_{y \in L'} T_y(\sigma, \sigma'_y) e^{\sum_X J(X) \sigma_X} \sigma_W.
\]

(13)

When \( J \) is at infinite temperature, i.e., \( ||J||_r = 0 \), \( J(X) = 0 \) for every subset \( X \) of the original lattice. \( \square \)

**Definition 2.3.** For every subset \( Z \) of the image lattice, the linearization \( L(J) \) of the RG transformation for \( J \) at infinite temperature is given by a linear function of \( K \) (which indicates variation from infinite temperature),

\[
L(J)K(Z) = \sum_W \frac{\partial J'(Z)}{\partial J(W)} K(W),
\]

(14)

where \( W \) ranges over all finite subsets of the original lattice.

**Definition 2.4.** The adjoint of the linearization \( L^*(J) \) of the RG transformation for \( J \) at infinite temperature is characterized by the usual correspondence between adjoint operators,

\[
\sum_X K_1(X)L(J)K_2(X) = \sum_Y K_2(Y)L^*(J)K_1(Y),
\]

(15)

where \( X \) ranges over all finite subsets of the image lattice, and \( Y \) ranges over all finite subsets of the original lattice.

**Definition 2.5.** A constant pure magnetic field is one such that \( K(X) = 0 \) except for one-point sets \( \{x\} \), where \( K(\{x\}) = m \), a constant.

3. Spectrum of the linearization of decimation transformation and its adjoint at infinite temperature

**Proposition 3.1.** Consider decimation transformation with blocking factor \( b \) and a probability kernel defined by

\[
\phi_y(\sigma) = \sigma_{by},
\]

(16)

where \( by = b(y_1, ..., y_d) = (by_1, ..., by_d) \). Suppose the original interaction \( J \) is at infinite temperature. Then for every subset \( Z \) of the image lattice, the linearization \( L(J) \) of this transformation is given by the expression

\[
L(J)K(Z) = K(bZ),
\]

(17)

where \( bZ = \cup_{z \in Z} \{bz\} \).

Proof. We evaluate \( 12 \) explicitly:

\[
\frac{\partial J'(Z)}{\partial J(W)} = \sum_\sigma \delta(W, bZ) = \delta(W, bZ),
\]

(18)

where \( \delta \) is the Kronecker delta function. \( \square \)

**Proposition 3.2.** Consider the adjoint of decimation transformation with blocking factor \( b \) and a probability kernel defined by

\[
\phi_y(\sigma) = \sigma_{by}.
\]

(19)
Suppose the original interaction $J$ is at infinite temperature. Then for every subset $Z$ of the original lattice, the adjoint of the linearization $L^*(J)$ of this transformation is given by the expression

$$L^*(J)K(Z) = \begin{cases} K(Y) & \text{if } Z = bY; \\ 0 & \text{otherwise.} \end{cases}$$

(20)

**Proof.** We notice that in this case, \((15)\) becomes

$$\sum_X K_1(X)L(J)K_2(X) = \sum_X K_1(X)K_2(bX).$$

(21)

Without loss of generality, we assume $L^*(J)K(\{0\}) = 0$, which amounts to an index shift. ∎

**Theorem 3.3** (Israel). Suppose the original interaction $J$ is at infinite temperature. Then in the Banach Space $B_r$, the spectrum of the linearization of the decimation transformation $L(J)$ is all point spectrum, $|\lambda| \leq 1$.

**Proof.** The proof of this theorem follows from several propositions. ∎

**Proposition 3.4.** $|L(J)|| = 1$.

**Proof.** We check that for each fixed $x \in \mathcal{L}$, $\sum_{X: x \in X} |L(J)K(X)|e^{rl(x,X)} \leq ||K||r$, which would imply $||L(J)|| \leq 1$. By \((17)\),

$$\sum_{X: x \in X} |L(J)K(X)|e^{rl(x,X)} = \sum_{X: x \in X} |K(bX)|e^{rl(x,X)} \leq \sum_{X: bX \in X} |K(bX)|e^{rl(bx,bX)} \leq \sum_{X: bX \in X} |K(X)|e^{rl(bx,X)} \leq ||K||r.$$

(22)

The claim is verified when we realize that a constant pure magnetic field is an eigenvector with eigenvalue 1.

**Corollary 3.5.** Every eigenvalue $\lambda$ of $L(J)$ satisfies $|\lambda| \leq 1$.

**Proposition 3.6.** Every $|\lambda| \leq 1$ is an eigenvalue.

**Proof.** For a generic $\lambda$, we display one eigenvector here. In fact, with some further thought, it is not hard to show that there are infinitely many eigenvectors for each $\lambda$. The eigenvector $K$ is defined by

$$K(\{(b^n, 0, ..., 0)\}) = \lambda^n K(\{(1, 0, ..., 0)\}) = \lambda^n$$

(23)

for $n \geq 0$, and for all the other subsets $X$, $K(X)$ is set to zero. ∎

Moreover, we have stricter restrictions on the eigenvector $K$ that lies in a Banach space $B_r : r > 0$.

**Proposition 3.7.** For $\lambda \neq 0$ and for every finite subset $|X| > 1$, we must have $K(X) = 0$ for the eigenvector $K$.

**Proof.** This follows from the observation that we can always pick a site, say $x$, in $X$, such that $l(x,X) > 0$. As a result, $b^n x$ is a site in $b^n X$, and $l(b^n x, b^n X) = b^n l(x,X) > 0$. Since $L(J)K(X) = K(bX)$, we must have $K(b^n X) = \lambda^n K(X)$. Then due to the fact that $K$ is an eigenvector, we need to ensure that

$$|\lambda|^n |K(X)|e^{rb^n l(x,X)} < \infty.$$

(24)
The following statements concern translation-invariant Hamiltonians. In this case, it is believed that the RG map should be almost a contraction near the trivial fixed point. Almost means that except for a few degrees of freedom (maybe just one) it should be a contraction. If one restricts oneself to even interactions, then it should actually be a contraction—reflecting the fact that if we start with an even interaction in the very high temperature phase, then the RG map would drive it to the zero interaction.

**Proposition 3.8.** Restricted to the translation-invariant even subspace of $\mathcal{B}_r : r = 0$, the point spectrum of $L$ is $|\lambda| < 1$.

*Proof.* For $|\lambda| < 1$, the eigenvector $K$ may be defined by

$$K(\{x, y\}) = 1$$

for $x, y$ that are nearest-neighbors, and in general, $K(bX) = \lambda K(X)$. However, no such eigenvector would work for $|\lambda| = 1$. Suppose the nontrivial eigenvector $K(X) = m \neq 0$ for some finite subset $X : |X| > 1$. Due to translation-invariance, for arbitrary $n$, all sets $Y$ with the same shape as $b^nX$ will have $|K(Y)| = m$. In particular, there will be infinitely many subsets $Z$ containing 0 with $|K(Z)| = m$, which implies $||K||_r = \infty$. □

**Proposition 3.9.** Restricted to the translation-invariant even subspace of $\mathcal{B}_r : r > 0$, the point spectrum of $L$ is $\lambda = 0$.

*Proof.* This follows from Propositions 3.7 and 3.8. □

**Theorem 3.10.** Suppose the original interaction $J$ is at infinite temperature. Then in the Banach Space $\mathcal{B}_r^*$, the spectrum of the adjoint of the linearization of the decimation transformation $L^*(J)$ is all residual spectrum, $|\lambda| \leq 1$.

*Proof.* The proof of this theorem follows from several propositions. □

**Proposition 3.11.** $||L^*(J)|| \leq 1$.

*Proof.* By [20],

$$\sum_{x \in \mathcal{L}} \sup_{x \in X} \frac{1}{|X|} |L^*(J)K(X)| e^{-rl(x,X)} \leq \sum_{bX \in \mathcal{L}} \sup_{bX \in bX} \frac{1}{|X|} |K(X)| e^{-rl(bX,bX)}$$

$$\leq \sum_{x \in \mathcal{L}} \sup_{x \in X} \frac{1}{|X|} |K(X)| e^{-rl(x,x)}. \quad (26)$$

□

**Proposition 3.12.** For every $\lambda \neq 0$, there is no nontrivial eigenvector.

*Proof.* Fix an arbitrary finite subset $X$ of the infinite lattice, after a finite number of iterations of $L^*(J)$ (say $n$ times), $X$ will not be of the form $bY$ for some $Y$. Thus $\lambda^{n+1}K(X) = (L^*(J))^{n+1}K(X) = 0$, which implies $K(X) = 0$. □

**Proposition 3.13.** For $\lambda = 0$, there is no nontrivial eigenvector.

*Proof.* Suppose the nontrivial eigenvector $K(X) = m \neq 0$ for some finite subset $X$, then the crucial fact that we can always find $Y$, with $L^*(J)K(Y) = K(X)$ will do the job. As $L^*(J)K(Y) = \lambda K(Y) = 0$, we reach a contradiction. □

**Corollary 3.14.** In the Banach Space $\mathcal{B}_r^*$, the point spectrum of $L^*(J)$ is empty.
Proof of Theorem 3.10 continued. The only thing left to show now is that \( \text{Range}(\lambda I - L^*(J)) \neq B^*_r \) for \(|\lambda| \leq 1\). Define \( K((1, 0, \ldots, 0)) = 1 \), and \( K(X) = 0 \) for all other subsets \( X \). We will show that \( K \) cannot be approximated by any \( K' \) in \( \text{Range}(\lambda I - L^*(J)) \) within distance 1/2. To see this, note that for \( n \geq 0 \),

\[
K'((b^{n+1}, 0, \ldots, 0)) = \lambda S((b^{n+1}, 0, \ldots, 0)) - S((b^n, 0, \ldots, 0))
\]

(27)

for some \( S \) that lies in \( B^*_r \). Suppose

\[
\frac{1}{2} \geq ||K - K'||^*_r = \sum_{x \in \mathcal{L}} \sup_{x \in X} \left| K(X) - K'(X) \right| e^{-r_l(x,x)}
\]

\[
\geq \sum_{x=(b^n,0,\ldots,0)} \sup_{x \in X} \left| K(X) - K'(X) \right| e^{-r_l(x,x)}
\]

\[
\geq \sum_{n=0}^{\infty} |K((b^n, 0, \ldots, 0)) - K'((b^n, 0, \ldots, 0))|
\]

(28)

\[
= |\lambda S((1, 0, \ldots, 0))| - 1 + |\lambda S((b, 0, \ldots, 0))| - S((1, 0, \ldots, 0))| + \cdots
\]

(29)

Then, as \(|\lambda| \leq 1\), for any \( n \geq 0 \),

\[
\frac{1}{2} \geq |\lambda^{n+1} S((b^n, 0, \ldots, 0))| - |\lambda^n S((b^{n-1}, 0, \ldots, 0))| + \cdots
\]

\[
+ |\lambda^2 S((b, 0, \ldots, 0))| - |\lambda S((1, 0, \ldots, 0))| + |\lambda S((1, 0, \ldots, 0))| - 1|
\]

(30)

By the triangle inequality, this implies

\[
|\lambda^{n+1} S((b^n, 0, \ldots, 0))| - 1 \leq \frac{1}{2}
\]

(31)

which further implies

\[
|\lambda^{n+1} S((b^n, 0, \ldots, 0))| \geq \frac{1}{2}
\]

(32)

Using \(|\lambda| \leq 1\) again, we have

\[
|S((b^n, 0, \ldots, 0))| \geq \frac{1}{2}
\]

(33)

But then,

\[
||S||^*_r = \sum_{x \in \mathcal{L}} \sup_{x \in X} \left| S(X) \right| e^{-r_l(x,x)} \geq \sum_{n=0}^{\infty} |S((b^n, 0, \ldots, 0))| = \infty.
\]

(34)

Remark. Notice the similarity between the adjoint operators \( L(J)/L^*(J) \) in our Banach spaces and left/right translation in \( l^\infty/l^1 \). \( L(J) \) acts like left translation and \( L^*(J) \) acts like right translation on sequences \( (X, bX, \ldots) \) for all possible subsets \( X \). Moreover, ignoring multiplicity of the eigenvalues, the spectrum of \( L(J) \) is the same as that of left translation in \( l^\infty \), and the spectrum of \( L^*(J) \) is the same as that of right translation in \( l^1 \). This might be related to the fact that the norms in our Banach spaces are something like combinations of \( l^\infty \) and \( l^1 \) norms.
4. Spectrum of the linearization of majority rule transformation and its adjoint at infinite temperature

For notational convenience, in this section, we set $s = b^d$ and $\nu = (s^{-1})/2^{s-1}$.

**Proposition 4.1.** Consider majority rule transformation with odd blocking factor $b$ and a probability kernel defined by

$$\phi_y(\sigma) = \text{sign} \left( \sum_{x \in y^o} \sigma_x \right). \tag{35}$$

Suppose the original interaction $J$ is at infinite temperature. Then for every subset $Z$ of the image lattice, the linearization $L(J)$ of this transformation is given by the expression

$$L(J)K(Z) = \sum_{W: W \subset Z^o} \prod_{z \in Z} \chi(W \cap z^o)K(W), \tag{36}$$

where $\chi(W \cap z^o) = \sum_{\sigma} \sum_{\sigma'} T_z(\sigma, \sigma') \sigma_{W \cap z^o} \sigma'_{z^o}$.

**Proof.** We evaluate (12) explicitly:

$$\frac{\partial J'(Z)}{\partial J(W)} = \sum_{\sigma} \sigma_{W \setminus Z^o} \prod_{z \in Z} \sum_{\sigma'} T_z(\sigma, \sigma') \prod_{\sigma' \in \sigma_{W \cap z^o}} \chi(\sigma_{W \cap z^o} \sigma'_{z^o}). \tag{37}$$

Since $\sum_{\sigma} \sigma_{W \setminus Z^o} = 0$ for $W$ not completely contained inside $Z^o$, it follows that $W \subset Z^o$. \( \square \)

**Proposition 4.2.** Consider majority rule transformation with odd blocking factor $b$ and a probability kernel defined by

$$\phi_y(\sigma) = \text{sign} \left( \sum_{x \in y^o} \sigma_x \right). \tag{38}$$

Suppose the original interaction $J$ is at infinite temperature. Then for every subset $Z$ of the original lattice, the adjoint of the linearization $L^*(J)$ of this transformation is given by the expression

$$L^*(J)K(Z) = \prod_{W_n} \chi(W_n)K(\cup\{n\}), \tag{39}$$

where $Z = \cup W_n$ and $W_n \subset n^o$.

**Proof.** We notice that in this case, (15) becomes

$$\sum X K_1(X)L(J)K_2(X) = \sum X K_1(X) \sum Y: Y \subset X^o \prod x \in X \chi(Y \cap x^o)K_2(Y) = \sum Y: Y = \cup W_n \prod \chi(W_n)K_1(\cup\{n\}). \tag{40}$$

**Theorem 4.3.** Suppose the original interaction $J$ is at infinite temperature. Then in the Banach Space $B_r$, the spectrum of the linearization of the majority rule transformation $L(J)$ is all point spectrum, $|\lambda| \leq sv$.

**Proof.** The proof of this theorem follows from several propositions. \( \square \)
Proposition 4.4. Consider Ising-type spin system on an odd polygon $A$ with cardinality $|A|$. Fix a certain vertex $V$ and a certain subset $W$ of the vertices. If $\sigma'_a \in \{+1,-1\}$ satisfies $\sigma_A \sigma'_a > 0$, then

$$\left| \sum_{\sigma} \sigma_W \sigma'_a \right| \leq \sum_{\sigma} \sigma_V \sigma'_a = \binom{|A|-1}{\frac{|A|}{2}}/2^{|A|-1},$$

(41)

where $\binom{n}{k}$ is the binomial coefficient.

Proof. We first show that $\sum_{\sigma} \sigma_W \sigma'_a = 0$ for any $W$ with even cardinality. This is due to a symmetry argument. If there is a spin configuration with $\sigma_W \sigma'_a = 1$, then flipping the spins at every vertex, we will have a configuration with $\sigma_W \sigma'_a = (-1)^{|W|}(-1) = (-1)^{|W|+1} = -1$. Vice versa. Thus the total sum will be zero.

Next we investigate into the special case $\sum_{\sigma} \sigma_V \sigma'_a$ where $V$ is any fixed vertex. The explicit calculation is easy to carry out. Due to symmetry, we only consider $\sigma_V = 1$ in the following, and there are $|A| - 1$ vertices for which the spins are yet to be assigned.

1. $\sigma'_a = 1$, if there are more 1’s than −1’s in the overall spin configuration, i.e., as long as the number of −1’s does not exceed $\frac{|A| - 1}{2}$. It is not hard to see that there are $\binom{|A| - 1}{0} + \binom{|A| - 1}{1} + \cdots + \binom{|A| - 1}{\frac{|A|}{2}}$ of them.

2. $\sigma'_a = -1$, if there are more −1’s than 1’s in the overall spin configuration, i.e., as long as the number of −1’s exceeds $\frac{|A| - 1}{2}$. Again, it is not hard to see that there are $\binom{|A| - 1}{\frac{|A|}{2}} + \binom{|A| - 1}{\frac{|A| + 1}{2}} + \cdots + \binom{|A| - 1}{|A| - 1} = \binom{|A| - 1}{\frac{|A| + 1}{2}} + \binom{|A| - 1}{\frac{|A| - 1}{2}} + \cdots + \binom{|A| - 1}{0}$ of them.

In conclusion, when $\sigma_V = 1$, there are $\binom{|A| - 1}{|A| - 1}$ more spin configurations for $\sigma'_a$ to be 1 rather than to be −1. Similar result holds for $\sigma_V = -1$. Thus considering all possible spin configurations, there are $2\left(\binom{|A| - 1}{|A| - 1}\right)$ more spin configurations for $\sigma_V \sigma'_a$ to be 1 rather than to be −1. It follows that $\sum_{\sigma} \sigma_V \sigma'_a = \binom{|A| - 1}{\frac{|A|}{2}}/2^{|A|-1}$.

Finally we consider $\sum_{\sigma} \sigma_W \sigma'_a$ for any $W$ with odd cardinality. Without loss of generality, suppose $V \subset W$. For a fixed spin configuration, $\sigma_V \sigma'_a \neq \sigma_W \sigma'_a$ can only occur when there is an odd number of −1’s and an odd number of 1’s in the spin configuration for vertices in $W \setminus V$. For such a configuration, we notice the following important fact: Suppose it has the extra property that unequal numbers of −1’s and 1’s are assigned for the remaining $|A| - 1$ vertices of $AV$, then if we flip the spins at every vertex other than $V$, $\sigma_V \sigma'_a$ will change sign. Moreover, at the same time, the sign of $\sigma_W \sigma'_a$ also changes, so the total sum does not change. Therefore, we see that the difference in $\sum_{\sigma} \sigma_W \sigma'_a$ and $\sum_{\sigma} \sigma_V \sigma'_a$ can only be caused by the following scenario: Equal numbers of −1’s and 1’s are assigned for the remaining $|A| - 1$ vertices of $AV$, and there is an odd number of −1’s and an odd number of 1’s in the spin configuration for vertices in $W \setminus V$. It is not hard to see that there are at most $2\left(\binom{|A| - 1}{|A| - 1}\right)$ of them. Thus $\sum_{\sigma} \sigma_W \sigma'_a$ varies between $-\frac{\binom{|A| - 1}{|A| - 1}}{2^{|A|-1}}$ and $\frac{\binom{|A| - 1}{|A| - 1}}{2^{|A|-1}}$, and our claim follows. \qed

Proposition 4.5. $|L(J)| = sv$.

Proof. We check that for each fixed $x \in L$, $\sum_{X: x \in X} |L(J)K(X)|e^{r(x,X)} \leq sv||K||_r$, which would imply $|L(J)|| \leq sv$. As $x \in X$, $L(J)K(X)$ is a linear combination of $K(Y)$’s, each one with coefficient bounded above by $\nu$ by (14). Ignoring the coefficients of $K(Y)$’s, we can then collect terms according to which one of the sites in $x^o$ belongs to $Y$. (When $|Y \cap x^o| > 1$, $K(Y)$

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can be classified into either one of the $s$ groups.) Moreover, each $Y$ has size no smaller than $X$, the exponential factor changes to a larger quantity after the action of $L(J)$. We see that each collection is bounded above by $||K||_{\nu}$ by definition. The claim is verified when we realize that a constant pure magnetic field is an eigenvector with eigenvalue $sv$.

**Corollary 4.6.** Every eigenvalue $|\lambda| \leq sv$.

**Proposition 4.7.** Every $|\lambda| \leq sv$ is an eigenvalue.

**Proof.** For a generic $\lambda$, we display one eigenvector here. In fact, with some further thought, it is not hard to show that there are infinitely many eigenvectors for each $\lambda$. The eigenvector $K$ is defined by

$$K(\{(\frac{b-1}{2}, \ldots, \frac{b-1}{2})\}) = \lambda/\nu - (s - 1),$$

and

$$K(\{x\}) = 1$$

for $b \neq x \in 0^s$. In general, for $n \neq 0$, $K$ is defined by $sK(\{m\}) = \lambda/\nu K(\{n\})$ for $m \in n^s$. For all the other subsets $X$, $K(X)$ is set to zero. \hfill $\square$

**Corollary 4.8.** The spectrum of $L(J)$ diverges as $\sqrt{\frac{2s}{\pi}}$ as the blocking factor $b$ gets large.

**Proof.** This follows from an easy application of Stirling’s formula:

$$sv \sim \frac{s\sqrt{2\pi \cdot (s-1)(s-1)^{s-1}e^{-(s-1)}}}{2\pi \frac{s-1}{2} \frac{(s-1)^{s-1}e^{-(s-1)}}{2s-1}} \sim \sqrt{\frac{2s}{\pi}}.$$  \hfill (44)

**Theorem 4.9.** Suppose the original interaction $J$ is at infinite temperature. Then in the Banach Space $B_r^*$, the point spectrum of the adjoint of the linearization of the majority rule transformation $L^*(J)$ is empty. Moreover, every $|\lambda| \leq \nu$ is in the residual spectrum of $L^*(J)$.

**Proof.** The proof of this theorem follows from several propositions. \hfill $\square$

**Proposition 4.10.** For every $\nu \neq 0$ and $\nu \neq \nu$, there is no nontrivial eigenvector.

**Proof.** Fix an arbitrary finite subset $X$. For $\nu \neq 0$, $K(X)$ is either zero or a nonzero constant multiple of $K(\{0\})$ as a result of the action of $L^*(J)$. In particular, $\lambda K(\{0\}) = L^*(J) K(\{0\}) = \nu K(\{0\})$, which implies that $K(\{0\}) = 0$. \hfill $\square$

**Proposition 4.11.** For $\lambda = 0$, there is no nontrivial eigenvector.

**Proof.** Suppose the nontrivial eigenvector $K(X) = m \neq 0$ for some finite subset $X$, then the crucial fact that we can always find $Y$, with $L^*(J) K(Y)$ a nonzero constant multiple of $K(X)$ will do the job. As $L^*(J) K(Y) = \lambda K(Y) = 0$, we reach a contradiction. \hfill $\square$

**Proposition 4.12.** For $\nu = \nu$, every nontrivial eigenvector has norm infinity.

**Proof.** We must have $K(\{0\}) = m \neq 0$ in order for $K$ to be nontrivial. As

$$\nu K(\{x\}) = L^*(J) K(\{x\}) = \nu K(\{0\}),$$

for $x \in 0^s$, we see that $K(\{x\}) = m$ also. Following similar fashion, $K(\{n\}) = m$ for arbitrary $n$. But then, $||K||_r^* = \infty$. \hfill $\square$
Proof of Theorem 4.9 continued. The only thing left to show now is that $\text{Range}(\lambda I - L^*(J)) \neq \mathcal{B}_\nu^*$ for $|\lambda| \leq \nu$. Define $K(\{(0,0,\ldots,0)\}) = 1$, and $K(X) = 0$ for all other subsets $X$. We will show that $K$ can not be approximated by any $K'$ in $\text{Range}(\lambda I - L^*(J))$ within distance $1/4$. To see this, note that for $n \geq 0$,

$$K'(\{(b^{n+1},0,\ldots,0)\}) = \lambda S(\{(b^{n+1},0,\ldots,0)\}) - \nu S(\{(b^n,0,\ldots,0)\})$$  \hspace{1cm} (46)

for some $S$ that lies in $\mathcal{B}_\nu^*$. And in particular,

$$K'(\{0,\ldots,0\}) = (\lambda - \nu)S(\{0,\ldots,0\}).$$  \hspace{1cm} (47)

Suppose

$$\frac{1}{4} \geq ||K - K'||_p^* = \sum_{x \in \mathcal{L}} \sup_{X \in X} \frac{1}{|X|} |K(X) - K'(X)|e^{-rl(x,X)}$$

$$\geq |K(\{(0,0,\ldots,0)\}) - K'(\{(0,0,\ldots,0)\})| + \sum_{n=0}^{\infty} |K(\{(b^n,0,\ldots,0)\}) - K'(\{(b^n,0,\ldots,0)\})|$$

$$= |(\lambda - \nu)S(\{(0,0,\ldots,0)\}) - 1| + |\lambda S(\{(1,0,\ldots,0)\}) - \nu S(\{(0,0,\ldots,0)\})| + \cdots$$  \hspace{1cm} (48)

Then, as $|\lambda| \leq \nu \leq \frac{1}{2}$, for any $n \geq 0$,

$$\frac{1}{2} \geq |(\frac{\lambda}{\nu})^{n+1}(\lambda - \nu)S(\{(b^n,0,\ldots,0)\}) - (\frac{\lambda}{\nu})^{n}(\lambda - \nu)S(\{(b^{n-1},0,\ldots,0)\})| + \cdots$$

$$+ |\frac{\lambda}{\nu}(\lambda - \nu)S(\{(1,0,\ldots,0)\}) - (\lambda - \nu)S(\{(0,0,\ldots,0)\})| + |(\lambda - \nu)S(\{(0,0,\ldots,0)\}) - 1|.$$  \hspace{1cm} (49)

By the triangle inequality, this implies

$$|(\frac{\lambda}{\nu})^{n+1}(\lambda - \nu)S(\{(b^n,0,\ldots,0)\}) - 1| \leq \frac{1}{2},$$  \hspace{1cm} (50)

which further implies

$$|(\frac{\lambda}{\nu})^{n+1}(\lambda - \nu)S(\{(b^n,0,\ldots,0)\})| \geq \frac{1}{2}.$$  \hspace{1cm} (51)

Using $|\lambda| \leq \nu \leq \frac{1}{2}$ again, we have

$$|S(\{(b^n,0,\ldots,0)\})| \geq \frac{1}{2}.$$  \hspace{1cm} (52)

But then,

$$||S||_p^* = \sum_{x \in \mathcal{L}} \sup_{X \in X} \frac{1}{|X|} |S(X)|e^{-rl(x,X)} \geq \sum_{n=0}^{\infty} |S(\{(b^n,0,\ldots,0)\})| = \infty.$$  \hspace{1cm} (53)

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