Anomalous bulk-boundary correspondence of topological magnon excitations in a spin chain

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Bulk-boundary correspondence, connecting bulk topology to robust edge states, is a generic feature of topological matter. Here, we discover unexpected sub-edge states and defect edge states of magnon excitations in a dimerized Heisenberg XXZ chain, which was supposed to be trivial within the topological band theory. We retrieve the topological origin of these sub-edge states by redefining a real-space representation of winding number, whose relation with sub-edge states is termed as anomalous bulk-boundary correspondence. Remarkably, even for single-magnon excitations, the longitudinal interaction still play a crucial role in the emergence of anomalous topological edge states. For multi-magnon excitations, even-magnon bound states are always topological trivial, but odd-magnon bound states may be topological nontrivial due to the interplay between the transverse dimerization and the longitudinal interaction. Our work opens an avenue for exploring topological magnon excitations in complement with the topological band theory.

Introduction.—Topological band theory underpins fertile topological states of matter [1, 2]. The topology of a bulk can be characterized by a topological invariant. Topological phase transition happens when the band gap is closed, otherwise the topological invariant remains unchanged under smooth deformations. According to the bulk-boundary correspondence (BBC) [3, 4], topological invariants of bulk bands are intimately related to robust edge states. Taking the Su-Schrieffer-Heeger (SSH) model [5, 6] as an example, the bulk topology characterized by a winding number \( \nu \) corresponds to \( \nu \) pairs of edge states under open boundary condition. Not limited to fermionic particles, the above wisdom has been widely applied to study topological bosonic excitations such as magnons [7–12], phonons [13] and photons [14].

Topological magnons, a kind of collective excitations over trivial ground states of magnetic materials, have provided new insights into topological states and potential applications such as topological magnon laser [15, 16], magnon spintronics [17, 18] and topological magnetic memory [19, 20]. Parallel to the electronic counterparts, magnon Hall effect was theoretically predicted and experimentally observed [21], and Dirac magnons [22–26], Weyl magnons [27–33], nodal line magnons [34, 35], topological magnon polarons [36–39], higher-order topological magnons [40, 41] were theoretically predicted. Because it is generally difficult to measure bulk topology of magnons, according to the BBC, the edge states have been utilized as an experimental signature. However, ubiquitous longitudinal interaction between spins will lead to magnon-magnon interaction, which may break the paradigm of topological band theory. It is still unclear whether the BBC is still applicable in a topological spin chain with longitudinal interaction.

In this Letter, we reveal that the conventional BBC is broken by strong longitudinal interaction in a dimerized Heisenberg XXZ chain. Remarkably, even for single-magnon excitations, anomalous sub-edge states and defect edge states may coexist in a system corresponding to a trivial SSH model under periodic boundary condition. This is because the strong longitudinal interaction gives rise to defects at boundaries, and make inter-cell and intra-cell couplings swap their positions. To explain the anomalous sub-edge states, we redefine a real-space representation of winding number by excluding the defects. We term the relation between such real-space representation of winding number and unexpected sub-edge states as the anomalous BBC. By calculating the real-space winding number, we give a topological phase diagram, whose phase boundaries can be analytically obtained via the transfer matrix method. Furthermore, topology of bound magnons sensitively depends on magnon-excitation numbers due to strong longitudinal interaction.

Model.—We consider a Heisenberg XXZ chain with transverse dimerization,

\[
\hat{H} = -\sum_{l=1}^{2L-1} \left\{ \left[ (J + (-1)^l \delta_0) \hat{S}^z_l \hat{S}^z_{l+1} + \text{H.c.} \right] + \Delta \hat{S}^x_l \hat{S}^x_{l+1} \right\}
\]

with the lattice index \( l \), the spin-\( \frac{1}{2} \) operators \( \hat{S}^{x,y,z}_l \), and the spin raising (lowing) operators \( \hat{S}^z_l = \hat{S}^z_l \pm i \hat{S}^y_l \). The system contains \( L \) cells in which each cell includes two adjacent spins. \( J \pm \delta_0 \) respectively denote the inter-cell and intra-cell spin-exchange strengths, and \( \Delta \) is the longitudinal interaction. Without loss of generality, we set \( J = 1 \) and \( \Delta > 0 \). For a large \( \Delta \), all spins down \( |↓↓\cdots↓⟩ \) is a ferromagnetic ground state, in which flipping a spin upward creates a magnon excitation. In contrast to the studies of topological ground states [42–48], topological magnon excitations are associated with ex-
cited states. Because the system (1) conserves the total $z$-direction magnetization, the subspaces with different magnon numbers are decoupled.

Viewing magnons as quasiparticles, the spin exchange and longitudinal interaction correspond to nearest-neighbor hopping and magnon-magnon interaction, respectively. At the first glance, since the magnon-magnon interaction is absent in single-magnon excitations, the topology of single-magnon excitations was supposed to behave as the celebrated SSH model [5, 6]. In such a SSH model, the bulk topology is characterized by winding number $\nu$, where $\nu = 1$ for $\delta_0 > 0$ and $\nu = 0$ for $\delta_0 \leq 0$. The conventional BBC indicates that a pair of edge states do (not) exist when the intra-cell spin exchange is weaker (stronger) than the inter-cell spin exchange. However, even for single-magnon excitations, we find that the conventional BBC becomes invalid in our system, in which the longitudinal interaction still play a crucial role.

**FIG. 1.** Single-magnon excitations. (a) and (b): single-magnon energy spectra respectively with $\Delta = 0.01$ and $\Delta = 100$. The eigenstate index is ordered for increasing values of the energy. The other parameters are chosen as $J = 1$, $\delta_0 = -0.5$ and $L = 50$. (c) and (d): spin magnetization distributions $S_l^z = \langle \Psi | S_l^z | \Psi \rangle$ for the isolated states far from the bottom band and lying in the energy gap in (b). The insets respectively correspond to the sketches of edge states.

**Topological single-magnon excitations.**—Below we discuss how longitudinal interaction affects topological edge states in single-magnon systems, which is ignored in the observation of topological magnon insulator states [49]. We show single-magnon energy spectra for weak and strong longitudinal interactions in Figs. 1(a) and 1(b), respectively. For $\Delta = 0.01$, there are only two separated energy bands, similar to the trivial SSH model. However, for $\Delta = 100$, four additional isolated edge states appear: the two below the bottom band (the first two states) correspond to a single magnon strongly confined at the leftmost or rightmost site, and the other two in the band gap (the $(L + 1)$-th and $(L + 2)$-th states) correspond to a single magnon strongly localized at the 2-th or $(2L - 1)$-th site, see the spin magnetization $S_l^z = \langle \Psi | S_l^z | \Psi \rangle$ in Figs. 1(c) and 1(d), respectively. The insets in Figs. 1(c) and (d) schematically display the two types of edge states at left, while their degenerate counterparts are symmetrically localized at right. Surprisingly, the edge states appear in the system of strong longitudinal interaction even when the bulk topology was supposed to be trivial. A question naturally arises: what is the emergence mechanism of these unexpected edge states?

To understand the origin of these edge states, we analyze their behaviors across the topological transition point $(J - \delta_0)/(J + \delta_0) = 1$ of the conventional SSH model. Fig. 2(a) shows the energy spectrum for $\Delta = 0.01$. Similar to a conventional SSH model, edge states appear in the band gap when $\delta_0 > 0$, which corresponds to nontrivial topology of the bulk band. However, for $\Delta = 100$, see Fig. 2(b), the edge states in the band gap appear when $\delta_0 < 0$, which corresponds to a topological trivial SSH model. These anomalous edge states in the band gap indicate that the conventional BBC is broken by the strong longitudinal interaction. Besides, the edge states below the bottom band always exist regardless of the dimerization strength $\delta_0$. This means that the edge states below the bottom band are not related to topology, but solely induced by the longitudinal interaction.

To further distinguish two types of edge states in Fig. 2(b), we calculate their inverse participation ratios (IPRs), $P_{\text{IPR}} = \sum_l |\psi_l|^4$. Fig. 2(c), we show the IPRs for the first and $(L + 1)$-th states. Here, $\psi_l$ is the amplitude of a magnon at the $l$-th site. The IPR quantifies the localization degree: $P_{\text{IPR}} \sim 1$ corresponds to the completely localized states, whereas $P_{\text{IPR}} \sim 0$ corresponds to the mostly extended states. The IPR of the first state always maintains 1 regardless of $(J - \delta_0)/(J + \delta_0)$, while the IPR of the $(L + 1)$-th state takes a tiny value before the critical point $(J - \delta_0)/(J + \delta_0) = 1$ as a feature of extended states, and then increases up to 1 after crossing the critical point, marking a transition to localized edge state.

The key to uncover these mysterious edge states is to revisit the mapping from spin-spin longitudinal interaction to magnon-magnon interaction. Taking the open three-spin chain as an example, single-magnon non-edge state $(|↓↑↓⟩)$ and edge states $(|↓↑↓⟩, |↓↓↓⟩)$ have onsite energies $\Delta/4 \pm \Delta/4$, respectively. The longitudinal interaction contributes an energy offset $\Delta/2$ between the magnon at the edge sites and the other bulk sites [50]. This means that the cooperation between the open boundary condition and the longitudinal interaction induces effectively on-site defects at the edge sites. A large $\Delta/2$ will trap a magnon at an end point to form the non-topological edge state, which can be dubbed as defect edge state. In the limit of $\Delta \to \infty$, the first and $(2L)$-th sites are decoupled from other bulk sites and can then be understood as vacuum. Thus the second and
edge states can be analytically given by the matrix method, the condition for the appearance of defect edge states. Importantly, by using transfer matrices, the real-space representation as

\[ \bar{\nu} = \frac{\nu_{\text{bottom}} + \nu_{\text{top}}}{2} \]

with \[ \nu_i = \frac{2}{L} \sum_{n \in \mathcal{C}} \langle \Psi_n | \hat{P} | \Psi_n \rangle \], (3)

where \( n \) is taken from a group of single-magnon bulk states in the \( \mathcal{C} \) (bottom or top) band. Similar to the scheme employed in non-Hermitian systems [52], the average of \( \nu_{\text{bottom}} \) and \( \nu_{\text{top}} \) is used to characterize topology. The winding number \( \bar{\nu} \) successfully witnesses the transition from topological trivial to nontrivial phases, see Fig. 2(d). To locate the transition point \( (J - \delta_0)/(J + \delta_0) = 1 \), we calculate \( \bar{\nu} \) for different lattice sizes: \( L = 50 \) (blue dotted line), \( L = 500 \) (green dashed-dotted line) and \( L = 5000 \) (red solid line). In the thermodynamic limit, \( \bar{\nu} \) will change suddenly at the critical point \( (J - \delta_0)/(J + \delta_0) = 1 \): \( \bar{\nu} = 1 \) for \( (J - \delta_0)/(J + \delta_0) > 1 \) and \( \bar{\nu} = 0 \) for \( (J - \delta_0)/(J + \delta_0) < 1 \). Thus the winding number \( \bar{\nu} \) can faithfully identify the appearance of anomalous topological edge states.

To revive the BBC, it is instructive to identify the anomalous topological edge states by renewing topological invariant. Depending on the strength of longitudinal interaction, the “topological interface” needs to be redefined when the first and last sites act as wall potentials. We first introduce the chiral displacement operator

\[ \hat{P} = \sum_{l' = 1}^{2L - 2l_0} \left[ \frac{l'}{2} \right] (-1)^{l' + 1} \hat{P}_{l'} \]

where \([x] \) means to round up the value of \( x \) and \( \hat{P}_{l'} = \hat{S}_{l'}^z + 1/2 \) is the magnon probability operator. The renormalized site index is set as \( l' = l - l_0 \) where \( l \) takes the value from 1 to \( 2L \). \( l_0 \) depends on whether defect edge states appear: \( l_0 = 1 \) (\( l_0 = 0 \)) in the presence (absence) of defect edge state. Importantly, by using transfer matrix method, the condition for the appearance of defect edge states can be analytically given by \( \Delta > \Delta_c \), with \( \Delta_c = 2(J + \delta_0) \) (see details in [51]). With the redefined “topological interface”, we renew the winding number in

![FIG. 2. Appearance of anomalous topological single-magnon excitations.](image)

Single-magnon excitation energy versus the transverse spin-exchange ratio \((J - \delta_0)/(J + \delta_0)\) for different longitudinal interaction strengths: (a) \( \Delta = 0.01 \) and (b) \( \Delta = 100 \) with \( L = 50 \). (c) The inverse participation ratio \( P_{\text{inv}} \) of the non-topological (blue line) and anomalous topological (red line) edge states in (b) as a function of \((J - \delta_0)/(J + \delta_0)\). (d) The winding number \( \bar{\nu} \) of the anomalous topological edge states in (b) as a function of \((J - \delta_0)/(J + \delta_0)\) for different \( L \).

(2L − 1)-th sites act as the new “boundaries” of a topological SSH lattice, in which the intra- and inter-cell spin exchanges are swapped. Consequently, the topology of the renormalized SSH model is opposite to the original trivial one, and topological edge states appear at the second and (2L − 1)-th sites. Hence, we term these sub-edge states in the band gap as anomalous topological edge states.

![FIG. 3. Topological phase diagram of single-magnon excitations.](image)

(a) The winding number \( \bar{\nu} \) as a function of \( \Delta \) and \((J - \delta_0)/(J + \delta_0)\) with \( L = 1000 \). The phase boundaries are marked by the white dashed lines, \( \Delta_c = 2(J + \delta_0) \) and \((J + \delta_0)/(J - \delta_0) = 1 \). (b) The band-edge gap \( \Delta E \) as a function of \( \Delta \). (c) The staggered magnetization of \((L + 1)\)-th state as a function of \( \Delta \). Our calculations are performed with \( \delta_0 = -0.5 \) (red solid line) and \( \delta_0 = 0.5 \) (blue dotted line) and \( L = 5000 \).

By calculating the winding number \( \bar{\nu} \), we give the topological phase diagram in the parameter space spanned by \( \Delta \) and \((J - \delta_0)/(J + \delta_0)\), see Fig. 3. Due to the longitudinal interactions, there appears a topological phase transition when the interaction strength increases. There are (no) topological edge states in the topological nontrivial (trivial) region, indicating that BBC is recovered. The boundary \( \Delta_c = 2(J + \delta_0) \) can also be exactly determined by the band-edge gap \( \Delta E \). In the thermodynamic limit, \( \Delta E \) naturally tends to zero in the topological trivial phase, and takes finite value due to the appearance of topological edge states in the topological nontrivial phase. We calculate the band-edge gap as a function...
of $\Delta$ by fixing $\delta_0 = -0.5$ and $\delta_0 = 0.5$, see Fig. 3(b). The band-edge gap can successfully identify the critical points, no matter for increasing from zero to finite value or vice versa.

Although the left-bottom and right-top regions of Fig. 3(a) share the same winding number $\bar{\nu} = 1$, they can be further distinguished by staggered magnetization,

$$m_{st}^z = \sum_{l=1}^{2L} (-1)^l \langle \psi | S_l^z | \psi \rangle.$$  \hspace{1cm} (4)

We choose the $(L+1)$-th state to analyze $m_{st}^z$ as a function of $\Delta$ for $(J - \delta_0)/(J + \delta_0) = 3$ with $\delta_0 = -0.5$ (red solid line) and $(J - \delta_0)/(J + \delta_0) = 1/3$ with $\delta_0 = 0.5$ (blue dotted line), see Fig. 3(c). Although $m_{st}^z$ vanishes in topological trivial phases, it takes finite negative (positive) value in the left-bottom (right-top) region (see details in [51]).

**Topological multi-magnon excitations.**—The longitudinal interaction provides not only boundary defects but also nearest-neighbor magnon-magnon interaction in multi-magnon interactions. Here, we only present results for multi-magnon bound states (see details in [51]) for unbound magnons. The bound states can be treated as a quasiparticle which is governed by an effective Hamiltonian (see details in [51]). Taking two magnons and three magnons as examples, the effective hopping strengths of a bound state in the bulk are given by

$$J_{\text{Eff}}^{(2)} = \frac{(J + \delta_0)(J - \delta_0)}{\Delta},$$  \hspace{1cm} (5)

$$J_{\text{Eff}}^{(3)} = \frac{(J - \delta_0)(J + \delta_0)(J + \delta_0(-1)^l)}{\Delta^2}.$$  \hspace{1cm} (6)

Because the effective hopping strengths of two-magnon and three-magnon bound states are respectively uniform and site-dependent, we can know that two-magnon bound states are topological trivial and three-magnon bound states may have topology inheriting from the SSH model. We generalize this result to an even-odd effect, that is, even-magnon bound states are trivial and odd-magnon bound states may have nontrivial topology.

We further analyze topological states in the three-magnon bound-state subspace. Fig. 4(a) shows the energy spectrum for three-magnon bound states. There are three types of three-magnon bound edge states: (i) bounded to the first site, (ii) bounded to the second site, and (iii) bounded to the third site with energy in the band gap, whose spin magnetizations and three-magnon correlations are respectively shown in Fig. 4(b, c, d). Here, the three-magnon correlations is defined as

$$C_{l,l+1,l+2} = \langle \psi | \hat{S}_{l+1}^+ \hat{S}_{l+2}^- \hat{S}_{l+2}^+ \hat{S}_{l+1}^- | \psi \rangle.$$  \hspace{1cm} (7)

The type-(i,ii) three-magnon bound states result from the emergent defects at the first and second sites, while the type-(iii) three-magnon bound states is due to nontrivial topology of the bulk band (see details in [51]).

**Conclusion.**—We uncover the anomalous BBC of topological magnon excitations in a dimerized Heisenberg XXZ chain. Unlike the conventional BBC, we explore the appearance of anomalous topological edge states at the sub edges, which can be explained by redefining the real-space winding number. Unlike the conventional topological band theory, the topological phase transition induced by the longitudinal interaction is accompanied with bulk-edge gap closing rather than bulk-band gap closing. The interplay among the longitudinal interaction, the transverse dimerization and the magnon-magnon correlations results to exotic topological magnon excitations which are able to be flexibly detected via spin dynamics (see details in [51]). We demonstrates that longitudinal interactions provide a versatile tool to engineer topological states, which have been overlooked for a long time. It deserves further study to generalize our method to a variety of topological magnon excitations, such as magnon Chern insulators and higher-order topological magnon excitations.

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[51] See Supplemental Material for more details on the derivation of the critical point \( \Delta_c \), the staggered magnetization as a function of \( \Delta \) and \( (J - \delta_\theta)/(J + \delta_\theta) \), two-magnon and three-magnon states, and the dynamical detection of magnon-excitation states.

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[53] There are two degenerated topological edge states in topological nontrivial phases where the \((L + 1)\)-th state is used to represent the left topological edge states.
Supplemental Material:

Anomalous bulk-boundary correspondence of topological magnon excitations in a spin chain

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DERIVATION OF THE CRITICAL POINT

Here we implement the transfer matrix approach [1–3] to derive the critical point $\Delta_c$. For an $2L$-lattice system, the $N_m$-magnon Hilbert space can be spanned by the basis $\mathcal{B}^{(N_m)} = \{ |l_1 l_2 \ldots l_{N_m} \rangle = \hat{S}_{l_1}^+ \hat{S}_{l_2}^+ \cdots \hat{S}_{l_{N_m}}^+ \prod_{m=1}^{2L} | \downarrow \rangle \}$ with $1 \leq l_1 < l_2 < \ldots < l_{N_m} \leq 2L$. An $N_m$-magnon wave function can be written as $|\Psi\rangle = \sum_{l_1 < l_2 < \ldots < l_{N_m}} \psi_{l_1,l_2,\ldots,l_{N_m}} |l_1 l_2 \ldots l_{N_m} \rangle$. For simplicity, we consider single-magnon excitations in a semi-infinite spin chain and an energy constant $\Delta N_m = \frac{\Delta}{2}(2L-1)$ with $N_m = 1$ is ignored. The symbols $v$ and $w$ are used as $v = -(J - \delta_0)$ and $w = -(J + \delta_0)$. According to the eigen-equation $(\hat{H} - E)|\Psi\rangle = 0$, the wave function relation between different sites is established in the form:

$$w\psi_0 + \left( -\frac{\Delta}{2} - E \right) \psi_1 + v\psi_2 = 0$$
$$v\psi_1 - E\psi_2 + w\psi_3 = 0$$
$$w\psi_2 - E\psi_3 + v\psi_4 = 0$$
$$\vdots$$
$$w\psi_{2L-3} - E\psi_{2L-2} + w\psi_{2L-1} = 0$$
$$w\psi_{2L-2} - E\psi_{2L-1} + v\psi_{2L} = 0.$$  (S1)

From the recurrence relation (S1), we directly read off the transfer matrix in the site basis. Multiplying the transfer matrices we find for the entire chain:

$$\begin{pmatrix}
\psi_{2L} \\
\psi_{2L-1}
\end{pmatrix} = M^{L-1} M_0 \begin{pmatrix}
\psi_1 \\
\psi_0
\end{pmatrix}$$  (S2)

with

$$M_0 = \begin{pmatrix}
\frac{\Delta/2 + E}{v} & -w \\
0 & 0
\end{pmatrix},
M = \begin{pmatrix}
\frac{E^2}{vw} & -\frac{w}{v} & -E \\
\frac{w}{v} & \frac{E}{w} & -\frac{E}{w}
\end{pmatrix}.$$  (S3)

One can obtain the critical point $\Delta_c$ in Figs.3(b) and (c) of red solid lines in the main paper by exploring the appearance of the left anomalous topological edge state. The initial conditions are set as $L \to \infty$, $\psi_0 = 0$, $\psi_1 = 1$. In the thermodynamic limit $L \to \infty$, the bulk-bulk energy keeps the same as the conventional SSH model

$$\hat{H}_k = (v + w \cos k)\sigma_x + w \sin k \sigma_y$$  (S4)

with the energy

$$E_k = \pm \sqrt{v^2 + w^2 + 2vw \cos k}.$$  (S5)

Since the minimum energy in the upper band is $w - v$, the anomalous topological edge state appears for an energy $E_a = w - v - \delta E$ with $\delta E \to 0^+$. We first deal with

$$\begin{pmatrix}
\frac{\Delta/2 + E}{v} & -w \\
1 & 0
\end{pmatrix} \begin{pmatrix}
\psi_1 \\
\psi_0
\end{pmatrix} = \begin{pmatrix}
\frac{\Delta/2 + E}{v} \\
1
\end{pmatrix} = \Phi.$$  (S6)
By solving $Mu_\pm = \epsilon_\pm u_\pm$, the corresponding eigenvalues $\epsilon_\pm$ and eigenstates $u_\pm$ are captured with $\epsilon_\pm = (\alpha \pm \sqrt{\alpha^2 - 4})/2$ and $\epsilon_+ \epsilon_- = 1$. Here $\alpha = E^2 - w^2 - \Delta^2$. After expanding the state $\Phi$ as $\Phi = Au_+ + Bu_-$, the wave functions at the two ends of the spin chain are related with

$$
\begin{pmatrix}
\psi_{2L} \\
\psi_{2L-1}
\end{pmatrix} = Ae^L u_+ + Be^{-L} u_-
$$

(S7)

Considering the energy $E_\alpha$ of our focused state, it exists $\epsilon_\pm \approx (-2 - r \pm \sqrt{r^2 + 4})/2$ for $r \approx 2(w - v)\delta E$. It has $\epsilon_- < 1$ and $\epsilon_+ > 1$ due to $r > 0$, which indicates $A = 0$ so that

$$
\begin{pmatrix}
\psi_{2L} \\
\psi_{2L-1}
\end{pmatrix} = \epsilon_-^{L-1} u_-
$$

(S8)

ensures the existence of the left anomalous topological edge state. Meanwhile, it has $\Phi = u_-$ which allows us to obtain

$$
\frac{\Delta/2 + E}{w} \approx \frac{\epsilon_- w + v}{E}.
$$

(S9)

Further, we obtain the relation

$$
\Delta = 2 \left[ \frac{(-2 - r - \sqrt{r^2 + 4r})w/2 + v^2}{w - v - \delta E} - (w - v - \delta E) \right].
$$

(S10)

When $\delta E \to 0^+$, the critical point yields $\Delta_c = -2w = 2(J + \delta_0)$.

For the same parameters as Figs.3(b) and (c) of red solid lines in the main paper, the critical point for the appearance of the non-topological edge state is analytically obtained in the same manner with an energy $E_\alpha = v + w - \delta E$ and $\delta E \to 0^+$. It is surprising to find that the critical points of the two kinds of edge states are the same with $\Delta_c = 2(J + \delta_0)$. Numerically, we can also witness the appearance of non-topological edge states via energy gap and staggered magnetization, as shown in Fig. S5. $\Delta E = E_3 - E_2$ represents the energy gap between the third and second eigenstates which keeps a vanishingly small value when $\Delta < \Delta_c$ with $\Delta_c \approx 1$. Distinct from the anomalous topological edge states finally lying on the middle of two bulk bands, the non-topological edge states linearly increases with $\Delta$ when $\Delta > \Delta_c$ under the influence of the longitudinal interaction $\Delta$ when the spin up locates on two outmost sites, as shown in Fig. S5(a). The stagger magnetization in Fig. S5(b) reflects, after crossing the critical point $\Delta_c \approx 1$, the left non-topological edge state $|\Psi_{ln}\rangle$ considerably distributes at odd sites as $\Delta$ increases. The numerical results are well consistent with analytical ones. By employing the same procedure, when $(J - \delta_0)/(J + \delta_0) < 1$ with $\delta_0 = 0.5$, the critical points for the appearance of non-topological edge states and the disappearance of topological edge state are also analytically given with $\Delta_c = 2(J + \delta_0)$. Under the guidance of the critical point for the appearance of non-topological edge states, the parameter dependence of the value of $l_0$ in Eq.(2) in the main paper yields the relation:

$$
l_0 = \begin{cases} 
1 & \text{if } \Delta > \Delta_c \\
0 & \text{if } \Delta \leq \Delta_c 
\end{cases}.
$$

(S11)

The critical line $\Delta_c = 2(J + \delta_0)$ is added into Fig.3(a) in the main paper with white dashed line.

**STAGGERED MAGNETIZATION**

To deepen the understanding of topological phase diagram of single-magnon excitations in Fig.3 in the main paper, Fig. S6 manifests the staggered magnetization of $(L + 1)$-th state as a function of the longitudinal interaction $\Delta$ and the transverse spin-exchange ratio $(J - \delta_0)/(J + \delta_0)$ with $L = 500$. $(L + 1)$-th state is used to denote the left topological edge state once the topological edge states exist. The analytical critical lines $\Delta_c = 2(J + \delta_0)$ and $(J - \delta_0)/(J + \delta_0) = 1$ are added into Fig. S6 with white dashed lines. The analytical results agree well with the numerical ones. In a weak longitudinal-interaction regime, the topological characterization behaves like the conventional SSH model: topological nontrivial phase for $(J - \delta_0)/(J + \delta_0) < 1$ while for $(J - \delta_0)/(J + \delta_0) > 1$ it turns to be topological trivial phase. $m_z(\Delta) \approx -1$ indicates the corresponding left topological edge state nearly distributes at odd sites where single-magnon excitation mainly locates at the first site and decays in other odd sites. Tuning the longitudinal interaction strong enough accompanied with the appearance of the non-topological edge states, topological edge states exist for $(J - \delta_0)/(J + \delta_0) > 1$ while disappear for $(J - \delta_0)/(J + \delta_0) < 1$ where the conventional bulk-boundary correspondence is broken. However, differently, such left topological edge states mainly distribute at even sites with $m_z(\Delta) \approx 1$. 
Provided two-magnon excitations ($N_m = 2$) where two spins point up and all the other spins point down, a richer two-magnon energy spectrum can be captured under the influence of strong longitudinal interaction, with eigenstates having the possibility of both magnon excitations locked on the same or different edges, one locked and the other free, and both free either as single magnons or as bound pairs. Considering the two-magnon basis $B^{(2)} = \{ |l_1 l_2 \rangle = \tilde{S}_{l_1}^{+} \tilde{S}_{l_2}^{+} \prod_{l=1}^{2L} |\downarrow\rangle \}$ with $1 \leq l_1 < l_2 \leq 2L$, the two-magnon states can be written as $|\Psi\rangle = \sum_{l_1 < l_2} \psi_{l_1 l_2} |l_1 l_2 \rangle$.

We identify the eigenstates via the two-magnon correlation function

$$C_{ij} = \langle \psi | \tilde{S}_i^{+} \tilde{S}_j^{+} \tilde{S}_i^{-} \tilde{S}_j^{-} | \psi \rangle.$$  \hfill (S12)

$i$ and $j$ denote the lattice sites and span from 1 to $2L$. The two-magnon correlation functions at two specific lines $i = j \pm d$ in the $(i,j)$ plane characterize the two-magnon bound states, where $d$ relies on the specific form of the longitudinal interaction. Various typical unbound states are exhibited via the two-magnon correlations in Fig. S7. $i$ and $j$ coordinates correspond to the positions of the first and second magnon excitations, and the color signifies the probability of two-magnon excitations to occupy $i$-th and $j$-th sites. Fig. S7(a) exhibits one magnon remains localized at the left end point whereas the other mainly at the opposite sub edge [(2L-1)-th site]. Fig. S7(b) represents an
FIG. S6. The staggered magnetization of \((L+1)\)-th state as a function of the longitudinal interaction \(\Delta\) and the transverse spin-exchange ratio \((J-\delta_0)/(J+\delta_0)\) for \(L = 500\). The white dashed lines correspond to the analytical critical lines \(\Delta_c = 2(J+\delta_0)\) and \((J-\delta_0)/(J+\delta_0) = 1\).

FIG. S7. Two-magnon unbound states for a strong longitudinal interaction \((\Delta = 100)\). (a)-(d) are respectively correlation distributions of various focused states. The other parameters are chosen as \(J = 1\), \(\delta_0 = -0.5\) and \(L = 24\).

Taking aforementioned two unbound states in Figs. S7(a) and (b) as examples, we understand their physical origin when extending the system from one-magnon to two-magnon excitations. For two indistinguishable magnons, the...
exact two-magnon state of our system
\[ H = -\sum_{i=1}^{2L-1} \left\{ \left( J + (-1)^i \delta_0 + H.c. \right) \hat{S}_i^z \hat{S}_{i+1}^z + \Delta \hat{S}_i^x \right\} , \tag{S13} \]
can be written as \(|\Psi\rangle = \sum_{l_1<\cdots<l_2} \psi_{l_1l_2} |l_1l_2\rangle \) where \(|\psi_{l_1l_2}\rangle^2\) describes the probability for one magnon occupying \(l_1\)-th site whereas the other at the \(l_2\)-th site. \(\{ |l_1l_2\rangle \} \) can viewed as two indistinguishable magnon basis with \(1 \leq l_1 < l_2 \leq 2L\). The two-magnon Hilbert space dimension is \(2L(2L-1)/2\). It allows to expand such exact two-magnon state into the two distinguishable magnon basis as \(|\tilde{\Psi}\rangle = \sum_{l_1l_2} \tilde{\psi}_{l_1l_2} |l_1l_2\rangle \) with \(\tilde{\psi}_{l_1l_2} = \psi_{l_1l_2} / \sqrt{2-\delta_{l_1l_2}}\), \(1 \leq l_1 \leq 2L\) and \(1 \leq l_2 \leq 2L\). There exists \(\tilde{\psi}_{l_1l_2} = \tilde{\psi}_{l_2l_1}\). Refer to the construction of higher-order topological states \([4]\), a two-magnon state \(|\tilde{\Psi}\rangle = \sum_{l_1l_2} \tilde{\psi}_{l_1l_2} |l_1l_2\rangle\) may be constructed in terms of the single-magnon state \(|\Psi^{\text{one}}\rangle = \sum_{l_1} \psi_{l_1}^1 |l_1\rangle\) in the form \(|\tilde{\Psi}\rangle = |\Psi^{\text{one}1}\rangle \otimes |\Psi^{\text{one}2}\rangle + |\Psi^{\text{one}2}\rangle \otimes |\Psi^{\text{one}1}\rangle\)

\(\tilde{\psi}_{l_1l_2} = \psi^{\text{one}1}_{l_1} \psi^{\text{one}2}_{l_2} + \psi^{\text{one}2}_{l_1} \psi^{\text{one}1}_{l_2}\). \tag{S14}

To distinguish two types of edge states, we respectively label left (right) non-topological edge states with \(|\Psi^{\text{in}}\rangle\) \((|\Psi^{\text{in}}\rangle)\) while for left (right) anomalous topological edge states with \(|\Psi^{\text{an}}\rangle\) \((|\Psi^{\text{an}}\rangle)\). According to the correlation properties of two-magnon state in Fig. S7(a), one can observe the first magnon is localized at the left end point and the other magnon mainly distributes at the right sub-edge site \(\{2L-1\}-\text{th site}\), or vice versa. Combining the left non-topological edge state \(|\Psi^{\text{in}1}\rangle\) with right anomalous topological edge state \(|\Psi^{\text{an}2}\rangle\), we can construct such two-magnon state with the following structure:

\(\tilde{\psi}_{l_1l_2} = \psi^{\text{in}1}_{l_1} \psi^{\text{an}2}_{l_2} + \psi^{\text{an}1}_{l_1} \psi^{\text{in}2}_{l_2}\). \tag{S15}

Our numerical calculation demonstrates the validity of the constructed two-magnon state and the exact two-magnon state yielding \(|\langle \tilde{\Psi} | \tilde{\Psi} \rangle| = 1\). This state is represented as a special type of two-magnon anomalous topological edge state where the longitudinal interaction creates the effective potential that trap one magnon at one end point and the other magnon forms a topological sub-edge states of the remaining sites. Owing to the longitudinal interaction, two magnons oppositely distribute at one outermost site and the other sub edge instead of bound together. Therefore, we provide a systematical method to construct two-magnon anomalous topological edge states. Similarly, the unbound state in Fig. S7(b) can also be explained effectively by the single-magnon states. Fig. S7(b) corresponds to a two-magnon state with two magnons respectively locking on two outermost sites \(|\Psi^{\text{an}1}\rangle\) and \(|\Psi^{\text{an}2}\rangle\), or vice versa. But for the states in Figs. S7(c) and (d), the constructed states are invalid due to the important role of \(\Delta\) in these two unbound states. Fig. S7(c) displays one magnon is edge-localized at the left end point, and the other one keeps spatially extended from the third site to \(\{2L-1\}-\text{th site}\) due to the presence of \(\Delta\), rather than fully delocalized along the chain from the second site to \(\{2L-1\}-\text{th site}\). The zero distribution on minor-diagonal lines \(j = i \pm 1\) in Fig. S7(d) implies that two magnons both free as single magnons but they cannot stay at the nearest-neighbor sites at the same time under the influence of \(\Delta\).

Two-magnon excitations tend to be bound together as a whole under the influence of strong longitudinal interaction, where two nearest-neighbor spins pointing in the same direction with in between an arbitrary number of spins with opposite orientation are energetically bound and form a new localized effective spin. The bound states of magnons constitute a central part of the energy spectrum. As a consequence, one expects the deep understanding of the interplay between the longitudinal interaction and the transverse dimerization for strongly interacting magnons.

For a strong longitudinal interaction, we obtain the bound-state subspace consisting of two-magnon bound states in Fig. S8(a). The eigenstate index on the horizontal axis is ordered for increasing values of the energy. The parameters are chosen as \(J = 1\), \(\delta_0 = -0.5\), \(\Delta = 100\) and \(L = 24\). The bound-state subspace contains four isolated bands which are ordered respectively for increasing values of the energy. We extract one state of the first band to compute the correlation function in Fig. S8(b), which indicates eigenstates with two magnons bound at one end point as a two-magnon bound edge state. The correlation properties of states in second band show the bound magnons mostly distribute at the one sub edge to form a two-magnon bound sub-edge state [Fig. S8(c)]. The two remaining continuum bands include eigenstates with two magnons both free as bound pairs [Fig. S8(d)]. The emergence of bound edge states is because the longitudinal interaction creates potentials trapping one magnon in a end point and binding the other together. But the presence of two-magnon bound sub-edge states is still unclear.

It is worth exploring the origination of such two-magnon bound sub-edge states. We employ the many-body perturbation theory to capture an effective model of bound magnons. For \(|\Delta| \gg |J|, |\delta_0|\), the system (S13) is divided into two parts \(\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}\) where the longitudinal interaction term \(\hat{H}_{\text{int}}\) plays a central role as a dominating term
whereas the transverse spin-exchange term \( \hat{H}_0 \) is treated as a perturbation to the dominating term with

\[
\hat{H}_\text{int} = -\Delta \sum_{l=1}^{2L-1} \hat{S}_l^z \hat{S}_{l+1}^z
\]

(S16)

and

\[
\hat{H}_0 = -\sum_{l=1}^{2L-1} \left\{ [J + (-1)^l \delta_0] \hat{S}_l^+ \hat{S}_{l+1}^- + \text{H.c.} \right\}.
\]

(S17)

The dominating term \( \hat{H}_\text{int} \) is separated into two subspaces. The first subspace includes two classes of states: (1) the states \( \{|l, l+1\rangle \} \) with \( l = 1, 2L-1 \) and \( E = -\frac{\Delta}{2} + \Delta \), (2) the states \( \{|l, l+1\rangle \} \) with \( 1 < l < 2L-1 \) and \( E = -\Delta + \Delta \). The complementary subspace consists of (3) the states \( \{|j, k\rangle \} \) with \( j < k , j \neq 1, k \neq 2L \) and \( E = C \), (4) the states \( \{|1, 2L\rangle \} \) with \( 2 < k < 2L \), \( \{|j, 2L\rangle \} \) with \( 1 < j < 2L-1 \) and \( E = -\Delta/2 + \Delta \), and (5) the states \( \{|1, 1\rangle \} \) and \( E = -\Delta + \Delta \). \( C = \Delta N_m - \frac{\Delta}{2}(2L-1) \) is a constant independent on the site with \( N_m = 2 \). We project the system (S13) onto the first subspace for an effective model \( \hat{H}_\text{eff}^{(2)} = \hat{h}_0 + \hat{h}_1 + \hat{h}_2 \) via a perturbation expansion up to the second order. In the lowest order, we have

\[
\hat{h}_0 = -\sum_{l=1}^{2L-1} A_l |l, l+1\rangle \langle l, l+1| + C
\]

(S18)

where \( A_l = \frac{4\Delta}{2} \) for \( l = 1, 2L-1 \) and \( A_l = \Delta \) for \( 1 < l < 2L-1 \). The first order \( \hat{h}_1 \) is equal to zero. The second-order effective Hamiltonian reads

\[
\hat{h}_2 = -\sum_{l} B_l |l, l+1\rangle \langle l, l+1|
-\sum_{l} D_l |l+1, l+2\rangle \langle l+1, l+2| + \text{H.c.}
\]

(S19)

where \( B_l = \frac{2(J+\delta_0)(-1)^{l+1}}{\Delta} \) for \( l \neq 1, 2L-2, 2L-1 \), \( B_l = \frac{(J+\delta_0)^2}{\Delta} \) for \( l = 1, 2L-1 \) and \( \frac{3(J+\delta_0)^2}{\Delta} \) for \( l = 2, 2L-2 \); \( D_l = \frac{3(J+\delta_0)(J-\delta_0)}{2\Delta} \) for \( l = 1, 2L-2 \) and \( D_l = \frac{(J+\delta_0)(J-\delta_0)}{\Delta} \) for \( 1 < l < 2L-2 \). After introducing the notation.
\[ |G_l| = |l, l + 1|, \] 
the effective Hamiltonian within a many-body perturbation theory up to second order reads as 

\[
\hat{H}_{\text{Eff}}^{(2)} = - \left\{ \sum_{l=1}^{2L-1} (A_l + B_l)|G_l|/|G_l| - C \right\} + \left\{ \sum_{l=1}^{2L-2} D_l |G_{l+1}|/|G_l| + \text{H.c.} \right\}. \tag{S20}
\]

The energy spectrum from the effective model \( \hat{H}_{\text{Eff}}^{(2)} \) is added into Fig. S8(a) with red circles. It can be observed that the numerical results (black dots) from the Hamiltonian \( \text{(S13)} \) perfectly agree with analytical ones (red circles) given by the effective model \( \hat{H}_{\text{Eff}}^{(2)} \) \( \text{(S20)} \). It is clearly shown that the effective potentials at the end points \([1\text{-th and } 2L\text{-th sites} \) and sub edges \([2\text{-th and } 2L-1\text{-th sites}] \) contribute to the two-magnon bound edge and sub-edge states. The effective hopping strength \( D_l \) no longer supports the SSH-type structure. The position-dependent potential forms two continuum bands. Therefore two-magnon bound sub-edge states belong to non-topological sub-edge states rather than topological edge states.

### THREE-MAGNON STATES

Within three-particle Hilbert space, the energy spectrum hosts scattering states, two-body bound states, and three-body bound states. The remarkable change of state distribution is caused by the interplay between the longitudinal interaction and the transverse dimerization. For the three-magnon basis \( \mathcal{B}^{(3)} \) 

\[
\mathcal{B}^{(3)} = \left\{ |l_1l_2l_3| = S_{l_1}^+ S_{l_2}^+ S_{l_3}^+ \prod_{i=1}^{2L} |\downarrow\rangle \right\}
\]

with \( 1 \leq l_1 < l_2 < l_3 \leq 2L \), the three-magnon eigenstates can be expanded as 

\[
|\Psi\rangle = \sum_{l_1 < l_2 < l_3} \psi_{l_1l_2l_3} |l_1l_2l_3\rangle.
\]

We identify the eigenstates via the three-magnon correlation functions 

\[
C_{ijk} = \langle \Psi | S_j^+ S_k^+ S_i^+ S_i^- S_k^- S_j^- | \Psi \rangle
\]

and spin magnetization distributions. \( i, j, \) and \( k \) coordinates respectively correspond to the positions of the first, second, and third magnons, and the color represents the probability of three magnons to occupy \( i\text{-th, } j\text{-th and } k\text{-th sites} \). The minor diagonal lines 

\[
(x = y \pm d = z \pm 2d, x = y \pm 2d = z \pm d, x = y \pm d = z \mp d) \text{ in the } (i, j, k) \text{ space serve as a signature of the three-magnon bound states.}
\]

Specifically, \( d \) depends on the longitudinal interaction such as \( d = 1 \) for the nearest-neighbor one in the system \( \text{(S13)} \).

Below we first concentrate on investigating the three-magnon bound states that synchronously flip the spins on the \( l\text{-th, } (l + 1)\text{-th and } (l + 2)\text{-th sites} \) from the ferromagnetic ground states \( \prod_{i=1}^{2L} |\downarrow\rangle \) under the strong longitudinal interaction. In the case of the three-magnon bound states, one can refer to a detailed construction of the effective model \( \hat{H}_{\text{Eff}}^{(2)} \) to derive an effective Hamiltonian \( \hat{H}_{\text{Eff}}^{(3)} \) for the three-magnon bound states. For a strong longitudinal interaction, one can treat the transverse spin-exchange term \( \hat{h}_0 \) as perturbation to the longitudinal one \( \hat{H}_{\text{Int}} \). The three-magnon bound-state subspace of the dominant term \( \hat{H}_{\text{Int}} \) is spanned by states \( |G_l\rangle = |l, l + 1, l + 2\rangle \) possessing 

\[
E = -5\Delta + \mathcal{C} \quad (l = 1, 2L - 2) \quad \text{and} \quad E = -2\Delta + \mathcal{C} \quad (2 \leq l \leq 2L - 3), \quad \mathcal{C} = \Delta N_m - \frac{J}{2}(2L - 1) \quad \text{is a constant independent on the lattice site with } N_m = 3.
\]

We now project the three-magnon system \( \text{(S13)} \) into such bound-state subspace by implementing the perturbation analysis. After a detailed calculation, the lowest order reads as 

\[
\hat{h}_0 = - \sum_{l=1}^{2L-2} A_l |G_l\rangle \langle G_l| + \mathcal{C} \tag{S21}
\]

where \( A_l = \frac{J}{2}\Delta \) for \( l = 1, 2L - 2 \) and \( A_l = 2\Delta \) for \( 1 < l < 2L - 2 \). The first order of the perturbation is equal to zero. The second order of the perturbation analysis is given by 

\[
\hat{h}_2 = - \sum_{l=1}^{2L-2} B_l |G_l\rangle \langle G_l|
\]

where \( B_l = \frac{(J-\delta_0)^2}{\Delta} \) for \( l = 1, 2L - 2 \) and \( B_l = \frac{(J+\delta_0)^2}{\Delta} \) for \( 1 < l < 2L - 2 \). Obviously, the second-order process contributes to the on-site potential especially for the distinguished effective on-site potential at sub edges. The third-order perturbation reads as 

\[
\hat{h}_3 = - \sum_{l=1}^{2L-3} D_l |G_{l+1}\rangle \langle G_l| + \text{H.c.} \tag{S23}
\]
where \( D_l = \frac{5(J-\delta_l)(J+\delta_l)(J-\delta_l)}{2\Delta^2} \) for \( l = 1, 2L - 3 \), \( D_l = \frac{(J-\delta_l)(J+\delta_l)(J-\delta_l)}{\Delta^2} \) for \( l = 2, 2L - 4 \) and \( D_l = \frac{(J-\delta_l)(J+\delta_l)(J-\delta_l)}{\Delta^2} \) for \( 2 < l < 2L - 4 \). The third-order term reorganizes the effective hopping strength but still supports the SSH-type structure. At last, the three-magnon bound states obey an effective Hamiltonian up to the third order

\[
\hat{H}_{\text{Eff}}^{(3)} = - \sum_{l=1}^{2L-2} (A_l + B_l) |G_l| \langle G_l| - C \right) + \sum_{l=1}^{2L-3} D_l |G_{l+1}| \langle G_l| + \text{H.c.}.
\]

The energy spectrum from the effective model \( \hat{H}_{\text{Eff}}^{(3)} \) is added into Fig.4(a) with red circles in the main paper. It can be observed that the numerical results (black dots) from the Hamiltonian (S13) agree well with analytical ones (red circles) given by the effective model \( \hat{H}_{\text{Eff}}^{(3)} \). We can conclude that the three-magnon bound sub-edge states in Fig.4(c) in the main paper are results of the effective on-site potential at sub edge (second site). The effective SSH-type hopping strength is responsible for the three-magnon bound next-sub edge (third site) state in Fig.4(d) in the main paper, that is, belongs to a type of anomalous topological edge states of bound magnons.

Apart from the three-magnon bound states discussed above, different types of three-magnon unbound states in three-magnon energy spectrum are exhibited. We utilize the correlation function to identify various typical three-magnon unbound states. Compared to the bound states, there lacks an effective model to clarify the three-magnon unbound states where three magnons cannot be treated as a whole. Three unbound magnon states suffer from the problem that two of three magnons form bound state and unbound with the third magnon. Therefore, the problem of how the three-magnon unbound states behave requires a further analysis.

The upper row in Fig. S9 displays the correlation of various typical states for three unbound magnons. Moreover, spin magnetization \( S_l^z = \langle \Psi | S_l^z | \Psi \rangle \) in the bottom row also provides an auxiliary perspective to better understand state distributions which significantly marks the magnon distribution in each site. Correlation function in Fig. S9(a) reveal magnons distribute at three corners, indicating the two magnons respectively at the two end points and the third one mainly at the third site. Since the existence of two-magnon bound states, the third one is limited from the third site to \((2L - 2)\)-th site. The topological nontrivial phase supports the topological edge states mainly distributing at the third site and \((2L - 2)\)-th site, respectively. Meanwhile, the third magnon can also become a bulk state extending along a chain from the third site to \((2L - 2)\)-th site, see Fig. S9(b). Fig. S9(c) corresponds to one magnon at the

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**FIG. S9.** Three-magnon unbound states for a strong longitudinal interaction (\( \Delta = 100 \)). (a)-(d) display respectively correlation distributions of various focused states with \( C_{ijk} \geq 0.0005 \). The other parameters are chosen as \( J = 1 \), \( \delta_0 = 0.5 \) and \( L = 24 \). The corresponding spin magnetization distributions are shown in the bottom row.
right end point and the remaining two freely distributing along a chain from the second site to \((2L - 2)\)-th site. In addition, we observe three magnons freely distribute at a chain from the second site to \((2L - 1)\)-th site in Fig. S9(d).

DETECTION OF MAGNON-EXCITATION STATES VIA SPIN DYNAMICS

FIG. S10. The time evolution of the spin magnetization distribution \(S_l^z(t) = \langle \Psi(t) | \hat{S}_l^z | \Psi(t) \rangle\). The top, middle, and bottom rows correspond to single-magnon excitations \(N_m = 1\) with \(\delta_0 = -0.5\), two-magnon excitations \(N_m = 2\) with \(\delta_0 = -0.5\), and three-magnon excitations \(N_m = 3\) with \(\delta_0 = 0.5\), respectively. For all three rows, the chosen initial states are shown from left to right, respectively. The evolved time is set as \(t = 2000\). The other parameters are chosen as \(J = 1\) and \(L = 24\).

Spin dynamics provides a method for dynamically detecting magnon-excitation states. Refer to the above analysis, fruitful magnon-excitation states appear as tuning intrinsic system parameters and magnon-excitation numbers. We calculate the time evolution of spin magnetization distribution \(S_l^z(t) = \langle \Psi(t) | \hat{S}_l^z | \Psi(t) \rangle\) by making use of the time-evolving block decimation (TEBD) algorithm \([5, 6]\). Initially, magnon excitations are located at different lattice sites as shown in Fig. S10 from \(N_m = 1\) to 3 magnons. The existence of non-topological edge states and topological edge state offers promising applications for manipulating the spin transports and designing the magnon spintronic devices.

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