Probing inequivalent forms of Leggett-Garg inequality in subatomic systems

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We study various formulations of Leggett-Garg inequality (LGI), specifically, the Wigner and Clauser-Horne forms of LGI, in the context of subatomic systems, in particular, three flavor neutrino as well as meson systems. For the neutrinos, some of these inequalities can be written completely in terms of experimentally measurable probabilities. Hence, the Wigner and Clauser-Horne forms of LGI are found to be more suitable as compared to the standard LGI from the experimental point of view for the neutrino system. Further, these inequalities exhibit maximum quantum violation around the energies roughly corresponding to the maximum neutrino flux. The meson system being inherently a decaying system, allows one to see the effect of decoherence on the extent of violation of various inequalities. Decoherence is observed to reduce the degree of violation, and hence the nonclassical nature of the system.

I. Introduction

In 1985, Leggett and Garg proposed an interesting set of inequalities for testing the status of macrorealism in quantum theory. The concept of macrorealism consists of two main assumptions which seem reasonable in our everyday world: (a) Macrorealism per se: If a macroscopic system has two or more macroscopically distinguishable ontic states available to it, then the system remains in one of those states at all instants of time. (b) Noninvasive measurability: The definite ontic state of the macrosystem is determined without affecting the state itself or its possible subsequent dynamics. Let us consider the measurement of a dichotomic observable $M$ having outcomes $\pm 1$ performed at time $t_1$, $t_2$ and $t_3$, which in turn can be considered as the measurement of the observables $M_1$, $M_2$, and $M_3$ respectively. Then the measurement of the observables $M_1$, $M_2$, and $M_3$ should produce definite outcomes $+1$ or $-1$ at all instants of time from the assumptions of macrorealism per se. Noninvasive measurability condition says that the outcomes of the measurement of $M_2$ or $M_3$ remain unaffected due to the measurement of $M_1$ and so on. One can then formulate the standard Leggett-Garg inequalities (LGIs) as

$$K_3 = m_1m_2\langle M_1M_2 \rangle + m_2m_3\langle M_2M_3 \rangle - m_1m_3\langle M_1M_3 \rangle - 1 \leq 0,$$

(1)

where $m_1, m_2, m_3 = \pm 1$. It is well studied that in quantum theory ($K_3)Q > 0$ for a suitable choice of observables, even for a qubit system. The LGIs have been investigated in various studies both on the theoretic [2] as well as experimental [8-11] fronts. In recent times, various other formulations of LGIs, viz., Entropic LGI [12, 13], Wigner [14] and Clauser-Horne [15] form of LGIs has also been proposed. A new variant of LGIs has also been proposed providing the quantum violation up to the algebraic maximum [16]. Note that the assumptions of macrorealism per se and non-invasive measurability imply the existence of joint probability distribution in a macrorealist model. From the assumptions of joint probability and non-invasive measurability, we obtain the pairwise statistics of measurement of $M_2$ and $M_3$ having outcome $m_2$ and $m_3$ as $P(m_2, m_3) = \sum_{m_1=\pm} P(m_1, m_2, m_3)$ and similarly for others. We can write the expression, $P(-m_1, m_2) + P(m_1, m_3) - P(m_2, m_3) = P(-m_1, m_2, -m_3) + P(m_1, -m_2, m_3)$. By invoking the non-negativity of the probability, Wigner form of LGIs can be derived as

$$P(m_2, m_3) - P(-m_1, m_2) - P(m_1, m_3) \leq 0.$$  

(2)

One can obtain eight variants Wigner form of LGIs from [2]. Similarly, sixteen more inequalities can be derived from

$$P(m_1, m_3) - P(m_1, -m_2) - P(m_2, m_3) \leq 0,$$

(3)

$$P(m_1, m_2) - P(m_2, -m_3) - P(m_3, m_3) \leq 0.$$  

(4)

Thus one has twenty four variants of Wigner form of LGI characterized by different measurement settings. This richness turns out to be very useful especially in systems where experimental constraints put limitation on arbitrary preparation and detection process, viz., in subatomic systems like neutrinos and mesons. Some of us have recently shown that Wigner form of LGIs are stronger than the standard LGIs [17, 18].

The single marginal statistics of the measurement of the observable, for example, probability of getting outcome, when $M_2$ measurement is performed can be obtained as $P(m_2) = \sum_{m_1, m_3=\pm} P(m_1, m_2, m_3)$ and similarly for $P(m_1)$ and $P(m_3)$. By combining single and pair-wise statistics, we can get the expression, $P(m_1, m_3) + P(m_2) - P(m_1, m_2) - P(m_2, m_3) = P(m_1, -m_2, m_3) + P(-m_1, m_2, -m_3)$, which gives

$$P(m_1, m_2) + P(m_2, m_3) - P(m_1, m_3) - P(m_2) \leq 0.$$  

(5)
Inequality \([5]\) can lead to eight variants of Clauser-Horne form of LGIs \([15]\). Similarly, sixteen more inequalities can be derived in this manner. In compact notation, we can write,

\[
P(m_1, m_3) + P(m_1, m_2) - P(m_2, m_3) - P(m_1) \leq 0, \tag{6}
\]

\[
P(m_1, m_3) + P(m_2, m_3) - P(m_1, m_2) - P(m_3) \leq 0. \tag{7}
\]

Note that in the Wigner form of LGIs only pair-wise probabilities are involved but in Clauser-Horne form of LGIs single probabilities are also involved along with pair-wise ones. Wigner and Clauser-Horne forms of LGIs can be shown to be equivalent to standard LGIs in macrorealist model, but inequivalent in quantum theory \([15]\).

Recently, the study of LGIs and their variants has gained significant interest in the context of subatomic systems, particularly, flavor oscillations in neutrinos and mesons \([6, 7, 19]\). This sets the tone for the present work.

Here, we probe Wigner and Clauser-Horne forms of LGIs in the context of three flavour neutrino and meson systems. In both the formulation of LGIs, most of the inequalities contain non-measurable terms, as in the case of the standard LGIs \([7]\). However, in the context of neutrino oscillations, we find that some of these inequalities can be expressed solely in terms of the experimentally measurable quantities, i.e., neutrino survival and transition probabilities.

The plan of the paper is as follows. We briefly revisit the formalism of neutrino and meson systems, relevant to our work. This is followed by a study of the different forms of LGIs on these systems. We finally make our conclusions.

\section*{II. Dynamics of neutrino and meson systems}

In this section, we briefly review the dynamics of neutrino system in the context of three flavor neutrino oscillations. We also discuss the time evolution of the neutral meson \((K^0)\). The neutrino state time evolution is unitary, however the meson system being decaying in nature is a non-unitary system and is dealt with using the approach of open quantum systems \([20]\).

\subsection*{A. Three flavor neutrino system}

When dealing with the three flavor scenario of neutrino oscillation \([21]\), one represents a general neutrino state either in the flavor basis \(\{\nu_\alpha\} (\alpha = e, \mu, \tau)\) or in the mass basis \(\{\nu_k\} (k = 1, 2, 3)\)

\[
|\Psi\rangle = \sum_{\alpha = e, \mu, \tau} \nu_\alpha |\nu_\alpha\rangle = \sum_{k=1,2,3} \nu_k |\nu_k\rangle. \tag{8}
\]

The expansion coefficient in the two representations are connected by the so called Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix as follows

\[
\nu_\alpha = \sum_{k=1,2,3} U_{\alpha,k} \nu_k. \tag{9}
\]

Here, \(U_{\alpha,k}\) are the element of the PMNS matrix. Later can be parametrized in many ways, one that is often used in the literature \([22, 23]\) is given below

\[
U = \begin{pmatrix}
    c_{12}c_{13} & -s_{12}s_{13}e^{i\delta} & s_{12}c_{13} \\
    -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}e^{-i\delta} \\
    s_{13}s_{23} - c_{13}c_{23}s_{13}e^{i\delta} & c_{13}c_{23} - s_{13}s_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix}. \tag{10}
\]

Here \(c_{ij} = \cos \theta_{ij}, \ s_{ij} = \sin \theta_{ij}\), and the parameters \(\theta_{ij}\) and \(\delta\) are the mixing angles and the CP violating phase \([24]\), respectively. The Eq. \([9]\) can be written as \(\nu_\alpha = U \nu_k\) with \(\nu_\alpha\) and \(\nu_k\) being the column vectors of the expansion coefficients. The massive eigenstates evolve according to the Schrodinger equation, such that \(\nu_k(t) = E \nu_k(0)\). Here \(E = \text{diag} \{e^{iE_1t}, e^{iE_2t}, e^{iE_3t}\}\) is the diagonal matrix and \(E_1, E_2\) and \(E_3\) are the energies corresponding to the massive eigenstates \(|\nu_1\rangle, |\nu_2\rangle\) and \(|\nu_3\rangle\), respectively. One can now connect the flavor state at time \(t = 0\) and some later time \(t\) by the following relation

\[
\nu_\alpha(t) = E U E^{-1} \nu_\alpha(0) = U_f(t) \nu_\alpha(0). \tag{11}
\]

We will call \(U_f(t)\) the flavor evolution operator, which take a flavor state at time \(t = 0\) to the state at some later time \(t\). It is worth mentioning here that the above formalism is valid only for the neutrino propagation in vacuum. In order to carry out the analysis in the context of the neutrino experiments, one has take into account the matter effect as well. A detailed account on how to construct the time evolution operator in such a case, can be found in \([25, 27]\) and references therein.

\subsection*{B. Neutral meson \(K^0 - K^0\) system}

In this subsection, we revisit the formalism of the operator sum representation which is an important tool used
to describe the dynamics of the decaying neutral meson system. This will be followed by a discussion in the context of \( K^0 - \bar{K}^0 \) system.

**Operator sum representation:** The time evolution of a closed system can be described by a unitary operator. However, this is not true for an open system and one often resorts to what is called the *operator sum representation* (OSR) in terms of the Kraus operators \[^{25}\]. The OSR has proved to be a powerful tool for dealing with open quantum systems \[^{20,29,33}\]. The total Hilbert space is \( \mathcal{H}_S \otimes \mathcal{H}_E \) with the constraint that the system and environment start in the product state at time \( t = 0 \), that is, \( \rho(0) = \rho_S \otimes \rho_E \). The time evolution of the combined system is then governed by the unitary operator \( U_{SE}(t) \) as follows

\[
\rho(t) = U_{SE}(t)\rho(0)U_{SE}^\dagger(t). \tag{12}
\]

Usually one is interested in the dynamics of the system of interest and the environmental degrees of freedom are traced out

\[
\rho_S(t) = Tr_E\{U_{SE}(t)\rho(0)U_{SE}^\dagger(t)\}. \tag{13}
\]

One may write this reduced state in the following representation

\[
\rho_S(t) = \sum_i \mathcal{K}_i(t)\rho_S(0)\mathcal{K}_i^\dagger(t). \tag{14}
\]

The unitary nature of \( U_{SE}(t) \) ensures that \( \sum_i \mathcal{K}_i(t)\mathcal{K}_i^\dagger(t) = 1 \), implying that the evolution of \( \rho_S(t) \) has a Kraus representation and is completely positive.

**Time evolution of \( K^0 - \bar{K}^0 \) meson system:** Here, we spell out the open system dynamics of the \( K^0 - \bar{K}^0 \) system \[^{33}\]. The Hilbert space of the total system is given by the direct sum \( \mathcal{H}_{K^0} \oplus \mathcal{H}_0 \) \[^{34,39}\] spanned by the orthonormal vectors \( |K^0\rangle \), \( |\bar{K}^0\rangle \) and \( |0\rangle \) (denoting the vacuum state)

\[
|K^0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\bar{K}^0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad |0\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{15}
\]

The states \( \{|K^0\rangle, |\bar{K}^0\rangle\} \) are the eigenstates of the strangeness operator \( \hat{S} \); \( \hat{S}|K^0\rangle = |K^0\rangle \), \( \hat{S}|\bar{K}^0\rangle = -|\bar{K}^0\rangle \), \( \hat{S}|0\rangle = 0 \). These are related to charge-parity \((CP)\) eigenstates \( \{|K_1^0\rangle, |K_2^0\rangle\} \) as follows

\[
|K_1^0\rangle = \frac{|K^0\rangle + |\bar{K}^0\rangle}{\sqrt{2}}, \quad |K_2^0\rangle = \frac{|K^0\rangle - |\bar{K}^0\rangle}{\sqrt{2}}. \tag{16}
\]

The complete positivity demands the following OSR \[^{28}\]

\[
\rho(t) = \sum_{i=0}^5 \mathcal{K}_i(t)\rho(0)\mathcal{K}_i^\dagger(t), \tag{17}
\]

where the Kraus operators have the following form \[^{33}\]

\[
\mathcal{K}_0 = |0\rangle \langle 0|,
\mathcal{K}_1 = C_1 + \left[ |K^0\rangle \langle K^0| + |\bar{K}^0\rangle \langle \bar{K}^0| \right] + C_1\left[ \frac{1 + \epsilon}{1 - \epsilon} |K^0\rangle \langle \bar{K}^0| + \frac{1 - \epsilon}{1 + \epsilon} |\bar{K}^0\rangle \langle K^0| \right],
\mathcal{K}_2 = C_2 \frac{1}{1 + \epsilon} |0\rangle \langle K^0| + \frac{1}{1 - \epsilon} |0\rangle \langle \bar{K}^0|,
\mathcal{K}_3 = C_3 \frac{1}{1 + \epsilon} |0\rangle \langle 0| + C_3 \frac{1}{1 - \epsilon} \langle 0| 0\rangle,
\mathcal{K}_4 = C_4 \left( |K^0\rangle \langle K^0| + |\bar{K}^0\rangle \langle \bar{K}^0| \right) + \frac{1 + \epsilon}{1 - \epsilon} |K^0\rangle \langle \bar{K}^0| + \frac{1 - \epsilon}{1 + \epsilon} |\bar{K}^0\rangle \langle K^0|,
\mathcal{K}_5 = C_5 \left( |K^0\rangle \langle 0| + |\bar{K}^0\rangle \langle \bar{K}^0| \right) - \frac{1 + \epsilon}{1 - \epsilon} |K^0\rangle \langle 0| - \frac{1 - \epsilon}{1 + \epsilon} |\bar{K}^0\rangle \langle 0|.
\]

The coefficients appearing in the above equations are given by

\[
C_{1 \pm} = \frac{1}{2} \left[ e^{-(2imS + \Gamma^a + \lambda)t/2} \pm e^{-(2imL + \Gamma^a + \lambda)t/2} \right],
C_2 = \sqrt{\frac{1 + |\epsilon|^2}{2} \left( 1 - e^{-\Gamma_{ST}^a} - \delta^a \frac{1 - e^{-(\Gamma + \lambda - i\Delta m)t/2}}{1 - e^{-\Gamma_{LT}^a}} \right)},
C_{3 \pm} = \sqrt{\frac{1 + |\epsilon|^2}{2} \left( 1 - e^{-\Gamma_{LT}^a} \pm \left( 1 - e^{-(\Gamma + \lambda - i\Delta m)t/2} \right) \delta^a \right)},
C_4 = e^{-\Gamma_{ST}^a},
C_5 = e^{-\Gamma_{LT}^a}. \tag{18}
\]

Starting at time \( t = 0 \) with state \( \rho_{K^0}(0) = |K^0\rangle \langle K^0| \) or \( \rho_{\bar{K}^0}(0) = |\bar{K}^0\rangle \langle \bar{K}^0| \), the state at some later time \( t \), is given by

\[
\rho_{K^0}(t) = \frac{1}{2} e^{-\Gamma t} \begin{pmatrix}
2a_{ch}\epsilon^{-\lambda t} & \lambda \epsilon^{-\lambda t} \left( -a_{sh} + \epsilon^{-\lambda t} a_{ch} \right) & 0 \\
\left( \frac{1 + \epsilon}{1 + \epsilon} \right) \left( -a_{sh} + \epsilon^{-\lambda t} a_{ch} \right) & \left( \frac{1 + \epsilon}{1 + \epsilon} \right) \left( -a_{sh} + \epsilon^{-\lambda t} a_{ch} \right) & \lambda \epsilon^{-\lambda t} \left( -a_{sh} + \epsilon^{-\lambda t} a_{ch} \right) \\
0 & \lambda \epsilon^{-\lambda t} \left( -a_{sh} + \epsilon^{-\lambda t} a_{ch} \right) & \rho_{33}(t)
\end{pmatrix}. \tag{19}
\]
and

\[
\rho_{K^0}(t) = \frac{1}{2} e^{-\Gamma t} \begin{pmatrix}
|\frac{1}{\sqrt{2}}|^2 (a_{ch} - e^{-\lambda t} a_c) & \frac{1}{\sqrt{2}} (-a_{sh} + i e^{-\lambda t} a_c) & 0 \\
\frac{1}{\sqrt{2}} (-a_{sh} - i e^{-\lambda t} a_c) & a_{ch} + e^{-\lambda t} a_c & 0 \\
0 & 0 & \tilde{\rho}_{33}(t)
\end{pmatrix}.
\]

Here, \(a_{ch} = \cosh[\frac{\Delta\mu t}{2}]\), \(a_{sh} = \sinh[\frac{\Delta\mu t}{2}]\) and \(a_c = \cos[\Delta m t]\), \(a_s = \sin[\Delta m t]\) and \(\epsilon\) is the CP violating parameter. \(\Delta\Gamma = \Gamma_S - \Gamma_L\) is the difference of the decay width \(\Gamma_S\) (for \(K^0_S\)) and \(\Gamma_L\) (for \(K^0_L\)). \(\Gamma = \frac{1}{2}(\Gamma_L + \Gamma_H)\) is the average decay width. The mass difference between the long and short lived states is given by \(\Delta m = m_L - m_S\), where \(m_L\) and \(m_S\) are the masses of \(K^0_L\) and \(K^0_S\) states, respectively. The decoherence parameter \(\lambda\) is proportional to the strength of the interaction between the one particle system and its environment \([35]\). The above discussed formalism is used in the next sections to analyze Wigner and Clauser-Horne form of LGI for these systems.

### III. Quantum violation of Wigner and Clauser-Horne form of LGIs in neutrino system

We now study the relevant Wigner and Clauser-Horne forms of LGIs for the case of neutrino system, keeping in mind the experimental constraints. The inequalities should be casted in a form which is verifiable experimentally and at the same time leads to the maximum possible violation for the allowed parameter range. It turns out that for the case of neutrino system Wigner form of LGI given by inequality \([2]\) for the values of \(m_1 = -1\), \(m_2 = m_3 = +1\) is most suitable. With initial neutrino state \(|\nu_\mu\rangle\), we choose the dichotomic operator \(\hat{A} = 2|\nu_e\rangle\langle\nu_e| - I\), where \(I = \sum_{\alpha=e,\mu,\tau} |\nu_\alpha\rangle\langle\nu_\alpha|\).

The operator \(A\) amounts to asking whether the neutrino is found in flavor \(\nu_e\) (+1) or not (-1). With this setting, the standard LGI for three time measurement, turns out to be \(K_3 = 1 - 4\mathcal{P}_{\mu\mu}(t) + 4\mathcal{P}_{ee}(t)\mathcal{P}_{\mu\mu}(2t) + 4 \beta(t) [2]\). Here, \(\beta(t) = \text{Re}[U_f^{11}(t)U_f^{21}(t)U_f^{22}(t)U_f^{12}(t) + U_f^{31}(t)U_f^{32}(t)\tilde{U}_f^{12}(t)\tilde{U}_f^{12}(t)],\) a non-measurable term.

Here, \(U_f^{ij}(t)\) is \(ij\)th element of the matrix \(U_f\) defined in Eq. \([11]\). It is worth noting here that for subatomic systems less number of measurements are preferable due to experimental constraints. Therefore, three time LGI is most relevant for such systems. In contrast to the standard LGI, one of the variants of Wigner form of LGI (denoted here by \(W_Q\)) turns out to be independent of non-measurable terms which can be shown to be

\[
W_Q = \mathcal{P}_{ee}(t)\mathcal{P}_{\mu\mu}(2t) - \mathcal{P}_{\mu\mu}(2t) \leq 0.
\]

Here, \(\mathcal{P}_{\alpha\beta}(t)\) is the probability of transition from flavor state \(\nu_\beta\) to \(\nu_\alpha\) at time \(t\). This is a remarkable coincidence which has the potential to have positive impact on experimental investigations in the context of LGI violations in neutrino oscillations.

The suitable Clauser-Horne form of LGI, as found for the inequality \([5]\) for the values of \(m_1 = +1, m_2 = m_3 = -1\) and is denoted by \(CH_Q\)

\[
CH_Q = -\mathcal{P}_{\mu\mu}(t) + \mathcal{P}_{ee}(t)\mathcal{P}_{\mu\mu}(2t) \leq 0.
\]

Another useful Clauser-Horne form of LGI, \(CH'_Q\), can be obtained from the inequality \([7]\) for the values of \(m_1 = m_3 = -1, m_2 = +1\)

\[
CH'_Q = \mathcal{P}_{\mu\mu}(t) - \mathcal{P}_{ee}(2t)\mathcal{P}_{\mu\mu}(t) + \mathcal{P}_{\tau\mu}(t) + \mathcal{P}_{\mu\mu}(2t) + \mathcal{P}_{\tau\mu}(2t) - 1 \leq 0.
\]

The expressions for various probabilities appearing in the above equations can be seen from \([7, 13]\). Here, it is important to note that the quantum violation of the Clauser-Horne form of LGI given by inequality \([23]\) is larger than the violation shown by the inequality \([22]\) and the Wigner form of LGI (inequality \([21]\)) for the experimental set-up of DUNE. It is worth mentioning that \(\mathcal{P}_{\alpha\beta}(t)\) depend, apart from time, on parameters like mixing angles, mass square difference, energy of the neutrino and CP violating phase (for \(\alpha \neq \beta\)). In the ultra-relativistic limit, time can be approximated by the distance it travels, i.e., \(t \approx L\). Therefore, the Wigner parameter \(W_Q\) becomes a function of \(L\) and \(2L\). This implies that an experimental verification of this inequality would require two detectors to be placed at \(L\) and \(2L\), respectively. However, in the present day experimental setups, such a provision is not possible. This difficulty can be passed by replacing the \(2L\) dependence by \(L\) in such a way that \(\mathcal{P}_{\mu\mu}(2L, E) = \mathcal{P}_{\mu\mu}(L, E)\) for energy \(E\) within the experimentally allowed range. Such approach has been used to study Leggett Garg inequality in the context of experimental facilities like NOvA, T2K and DUNE \([19]\).

The standard LGI for the three flavor neutrino oscillation leads to some non-measurable terms as discussed above, forcing one to resort to the stationary assumption \([6, 7, 19]\). Wigner and Clauser-Horne forms of LGI as given in \([21, 22, 23]\) are completely in terms of measurable probabilities, without invoking the stationarity assumption. However, \([23]\) involves transition probabilities from flavor \(\nu_\mu\) to \(\nu_\tau\), which are beyond the scope of present experimental capabilities. The Wigner and Clauser-Horne forms of LGI may be advantageous over the standard LGI, since the maximum violation occurs at energies around the maximum neutrino flux.
FIG. 1. (color online) Standard LGI ($K_3$) (top panel) and Wigner form of LGI (bottom panel) (Eq. (21)) in neutrino system for different experimental set ups vz., $T2K$ (top), $NOνA$ (middle) and DUNE (bottom). Both the quantities are plotted with respect to the neutrino energy ($E_n$) in GeV. The baseline of 295 km, 810 km and 1300 km are used for $T2K$, $NOνA$ and DUNE experiments, respectively. The CP violating parameter $\delta = 0$ and the matter density parameter $\lambda \approx 1.01 \times 10^{-13}$ eV. The two inequalities show kind of complementary behavior in the sense that the range of energy for one does not show violation, the other does.

FIG. 2. (color online) Clauser-Horne form of Legget-Garg inequality, Eq.(22), in neutrino system for different experimental set ups vz., $T2K$ (left), $NOνA$ (middle) and DUNE (right). The quantity $CH'Q$ is plotted with respect to the neutrino energy $E_n$ and the CP violating phase $\delta$.

FIG. 3. (color online) Clauser-Horne form of Legget-Garg inequality in neutrino system for different experimental set ups vz., $T2K$ (left), $NOνA$ (middle) and DUNE (right). The quantity $CH'Q$ given by Eq.(23) is plotted with respect to the neutrino energy $E_n$.

IV. Quantum violation of Wigner and Clauser-Horne form of LGIs in K-meson system

Now, we discuss the relevant Wigner and Clauser-Horne forms of LGIs for the case of meson system which is inherently decaying in nature. The decoherence is controlled by the parameter $\lambda$ appearing in the Kraus operators. We assume that the initial state is $|K^o\rangle$ and the dichotomic operator is given by $\hat{O} = 2|K^o\rangle\langle K^o| - I$, with $|K^o\rangle\langle K^o| + |\bar{K}^o\rangle\langle \bar{K}^o| + |0\rangle\langle 0| = I$. The operator $\hat{O}$ is +1 or −1 depending on whether the measurement outcome is $|K^o\rangle$ or not. After analyzing all the possible forms of Wigner LGIs and Clauser-Horne form of LGIs for the meson system, we find the most appropriate is the one
Further, the most suitable form of Clauser-Horne form of LGI as compared to Wigner form of LGI. Further, the effect of decoherence is expectedly reducing the extent of violation of the two inequalities. The various parameter (defined in Sec. [11]) used in Fig. (4) are as follows: \( \tau = 1.889 \times 10^{-10} \) s, \( \Gamma = 5.59 \times 10^9 \) s\(^{-1} \), \( \Delta \Gamma = 1.174 \times 10^9 \) s\(^{-1} \), \( \lambda = 2.0 \times 10^8 \) s\(^{-1} \) and \( \Delta m = 5.320 \times 10^9 \) s\(^{-1} \). Also, \( Re[\epsilon] = 1.596 \times 10^{-3} \) and \( |\epsilon| = 2.228 \times 10^{-3} \) [39].

V. Conclusion

Various formulations of Leggett-Garg inequalities were studied in the neutrino and the K-meson systems. Neutrino dynamics is considered in three flavor scenario including matter and CP violation effects. The meson system is treated using the open system formalism. For neutrino system, it is found that some of the Wigner and Clauser-Horne forms of LGI are more suitable with respect to the standard LGIs from the experimental point of view, since these inequalities are in terms of experimentally measurable probabilities and the maximum violation is found to occur around the energies corresponding to the maximum neutrino flux. Specifically, we studied the violation of these inequalities in current running experiments like NO\( \nu \)A and T2K and also for the future coming experiment DUNE. Further in the context of mesons, the stationarity assisted LGIs would be more suitable from an experimental perspective. For meson system, enhanced violation is found in the case of Clauser-Horne form of LGI as compared to Wigner form of LGI. Since, the extent of violation of various forms of LGI corresponds to the degree of quantumness of the system, therefore, decoherence is expected to reduce the extent of violation of these inequalities. These features are nicely manifested in the meson system. The optimal forms of various LGIs for either neutrinos or mesons are seen to depend on measurement settings. The present work, therefore, provides scope for choosing various experimental setups for probing into the foundational issues in subatomic physics.

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