Asymptotic approximation for the solution to a semilinear parabolic problem in a thin star-shaped junction

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A semilinear parabolic problem is considered in a thin 3-D star-shaped junction that consists of several thin curvilinear cylinders that are joined through a domain (node) of diameter $O(\varepsilon)$.

The purpose is to study the asymptotic behavior of the solution $u_\varepsilon$ as $\varepsilon \to 0$, ie, when the star-shaped junction is transformed in a graph. In addition, the passage to the limit is accompanied by special intensity factors $\{\varepsilon^\alpha_i\}$ and $\{\varepsilon^\beta_i\}$ in nonlinear perturbed Robin boundary conditions.

We establish qualitatively different cases in the asymptotic behavior of the solution depending on the value of the parameters $\{\alpha_i\}$ and $\{\beta_i\}$. Using the multiscale analysis, the asymptotic approximation for the solution is constructed and justified as the parameter $\varepsilon \to 0$. Namely, in each case, we derive the limit problem ($\varepsilon = 0$) on the graph with the corresponding Kirchhoff transmission conditions (untypical in some cases) at the vertex, define other terms of the asymptotic approximation and prove appropriate asymptotic estimates that justify these coupling conditions at the vertex, and show the impact of the local geometric heterogeneity of the node and physical processes in the node on some properties of the solution.

KEYWORDS
approximation, asymptotic estimate, graph, Kirchhoff transmission conditions, nonlinear perturbed boundary condition, semilinear parabolic problem, thin star-shaped junction

1 | INTRODUCTION

We are interested in the study of evolution phenomena in junctions composed of several thin curvilinear cylinders that are joined through a domain of diameter $O(\varepsilon)$ (see Figure 1). The corresponding mathematical models are described by semilinear parabolic equations that allow to model a variety of biological and physical phenomena (reaction and diffusion processes in biology and biochemistry, heat-mass transfer, etc) in channels, junctions, and networks.

As we can see from Figure 1, a thin junction is shrunk into a graph as the small parameter $\varepsilon$, characterizing thickness of the thin cylinders and domain connecting them, tends to zero. Thus, the aim is to find the corresponding limit problem in this graph and prove the estimate for the difference between the solutions of these 2 problems.

A large amount of physical and mathematical articles and books dedicated to different models on graphs has been published for the last 3 decades.\textsuperscript{1-13} The main question arising in problems on graphs is point interactions at nodes of networks, ie, the type of coupling conditions at vertices of the graph.
Also, there is increasing interest in the investigation of the influence of a local geometric heterogeneity in vessels on the blood flow. This is both an aneurysm (a pathological extension of an artery like a bulge) and a stenosis (a pathological restriction of an artery). In Evju et al., the authors classified 12 different aneurysms and proposed a numerical approach for this study. The aneurysm models have been meshed with 800,000 to 1,200,000 tetrahedral cells containing 3 boundary layers. However, as was noted by the authors, “the question how to model blood flow with sufficient accuracy is still open.”

Because of those point interactions and local geometric irregularities, the reaction-diffusion processes, heat-mass transfer, and flow motions in networks possess many distinguishing features. A natural approach to explain the meaning of point interactions at vertices is the use of the limiting procedure mentioned above.

There are several asymptotic approaches to study such problems. As far as we have known, the paper was the first paper, where convergence results for linear diffusion processes in a region with narrow tubes were obtained with the help of the martingale-problem method of proving weak convergence. As a result, the standard gluing conditions (or so-called “Kirchhoff” transmission conditions) at the vertices of the graph were derived. Then, this probabilistic approach was generalized in Albeverio and Kusuoka.

The method of the partial asymptotic domain decomposition was proposed in Panasenko, and then, it was applied to different problems under the following assumptions: the uniform boundary conditions on the lateral surfaces of thin rectilinear cylinders, the right-hand sides, depend only on the longitudinal variable in the direction of the corresponding cylinder, and they are constant in some neighborhoods of the nodes and vertices. It follows from these papers that the main difficulty is the identification of the behavior of solutions in neighborhoods of the nodes.

To overcome this difficulty and to construct the leading terms of the elastic field asymptotics for the solution of the equations of anisotropic elasticity on junctions of thin three-dimensional beams, the following assumptions were made in Nazarov and Slutskii:

1. The first terms of the volume force \( f \) and surface load \( g \) on the rods satisfy special orthogonality conditions (see (3.5) and (3.6)), and the second term of the volume force \( f \) has an identified form and depends only on the longitudinal variable; similar orthogonality conditions for the right-hand sides on the nodes are satisfied (see 3.41) and the second term is a piecewise constant vector function (see 3.42). By these assumptions, the displacement field at each node can be approximated by a rigid displacement. As a result, the approximation does not contain boundary layer terms, ie, the asymptotic expansion is not complete a priori.

2. Similar approach was used for thin two-dimensional junctions in Nazarov and Slutskii.

There is a special interest in spectral problems on thin graph-like structures, since such problems have many applications. A fairly complete review on this topic has been presented in Kuchment. The main task is to study the possibility of approximating the spectra of different operators by the spectra of appropriate operators on the corresponding graph. The convergence of spectra for the Laplacians with different boundary conditions (Neumann, Dirichlet, and Robin) at various levels of generality was proved in the previous studies. In Kuchment and Zeng and Post, the authors took into account large protrusions at the vertices; as a result, different Kirchhoff conditions appeared depending on the value of the protrusion. It was demonstrated in Grieser that the type of the transmission conditions depends crucially on the boundary layer phenomenon in the vicinity of the nodes; in addition, the complete asymptotic expansions for the \( k \)th eigenvalue and the eigenfunctions were obtained there, uniformly for \( k \), in terms of scattering data on a noncompact limit space. Interesting multifarious transmission conditions are obtained in the limit passage for spectral problems on thin periodic honeycomb lattice. Numerical approach to deduce the vertex coupling conditions for the nonlinear Schrödinger equation on two-dimensional thin networks was proposed in Ueckera et al.
1.1 Novelty and method of the study

In the present paper, we continue to develop the asymptotic method proposed in our papers\(^{38,39}\) for linear elliptic problems, which does not need the above-mentioned assumptions. In addition, our approach gives the better estimate for the difference between the solution of the starting problem and the solution of the corresponding limit problem (compare (1) and (2) in Klevtsovskiy and Mel'nyk\(^{38}\)). Here, we have adapted this method to semilinear parabolic problems with nonlinear perturbed Robin boundary conditions

\[
\partial_t u + \varepsilon^n \kappa_1 (u_x, x_i, t) = \varepsilon^h \varphi_i (x, t)
\]

both on the boundaries of the thin curvilinear cylinders \((i \in \{1, 2, 3\})\) and on the boundary of the node \((i = 0)\), which depend on special intensity factors \(\varepsilon^n\) and \(\varepsilon^h\). We study the influence of these factors on the asymptotic behavior of the solution as \(\varepsilon \to 0\).

It turned out that the asymptotic behavior of the solution depends on the parameters \(\{a_i\}\) and \(\{\beta_i\}\) and essentially on the parameter \(a_0\) that characterizes the intensity of processes at the boundary of the node. It is natural to expect that physical processes on the node boundary provoke crucial changes in the whole process in the thin star-shaped junction; in particular, they can reject the traditional Kirchhoff transmission conditions at the vertex in some cases. We discover 3 qualitatively different cases in the asymptotic behavior of the solution. If \(a_0 > 0, \beta_0 > 0, a_i, \beta_i \geq 1, i \in \{1, 2, 3\}\), then, we have classical Kirchhoff transmission conditions in the limit as \(\varepsilon \to 0\). In the case \(a_0 = 0, \beta_0 = 0, a_i, \beta_i \geq 1, i \in \{1, 2, 3\}\), new gluing conditions at the vertex \(x = 0\) of the graph look as follows:

\[
\omega_0^{(1)} (0, t) = \omega_0^{(2)} (0, t) = \omega_0^{(3)} (0, t),
\]

\[
\pi h_1^2 (0) \frac{\partial \omega_0^{(1)}}{\partial x_1} (0, t) + \pi h_2^2 (0) \frac{\partial \omega_0^{(2)}}{\partial x_2} (0, t) + \pi h_3^2 (0) \frac{\partial \omega_0^{(3)}}{\partial x_3} (0, t) - |\Gamma_0|_2 \kappa_0 (\omega_0^{(1)} (0, t)) = - \int_{\Gamma_0} \varphi_i (\xi, t) \, d\sigma_\xi,
\]

where \(|\Gamma_0|_2\) is the Lebesgue measure of the boundary \(\Gamma_0\) of the node. If \(a_0 < 0\), the limit problem splits in 3 independent problems with the Dirichlet conditions. Also, interesting untypical Kirchhoff transmission conditions appear for other terms of the asymptotics.

To construct the asymptotic approximation for the solution in each case, we use the method of matching asymptotic expansions (see Il’in\(^{40}\)) with special cut-off functions. The approximation consists of 2 parts, namely, the regular part of the asymptotics located inside of each thin cylinder and the inner part of the asymptotics discovered in a neighborhood of the node. The terms of the inner part of the asymptotics are special solutions of boundary-value problems in an unbounded domain with different outlets at infinity. It turns out they have polynomial growth at infinity. Matching these parts, we derive the limit problem \((\varepsilon = 0)\) in the graph and the corresponding coupling conditions at the vertex.

Also, we have proved energetic estimates for the difference between the solution and the approximating function in each case. It should be stressed that the error estimates and convergence rate are very important both for justification of adequacy of one- or two-dimensional models that aim at description of actual three-dimensional thin bodies and for the study of boundary effects and effects of local (internal) inhomogeneities in applied problems. In addition, these estimates in our problem justify transmission conditions of Kirchhoff type at the vertices of metric graphs, which include coefficients that take into account the impact of the local geometric heterogeneity of the node and physical processes in the node (see eg, Equation 1.2).

Thus, our approach makes it possible to take into account various factors (eg, variable thickness of thin curvilinear cylinders, inhomogeneous nonlinear boundary conditions, and geometric characteristics of nodes) in statements of boundary-value problems on graphs.

The rest of this paper is organized as follows. The statement of the problem and features of the investigation are presented in Section 2. In Section 3, the existence and uniqueness of the weak solution is proved for every fixed value \(\varepsilon\). Also, a priori estimates and auxiliary inequalities are deduced there. In Section 4, we formally construct the leading terms both of the regular part of the asymptotics and the inner one in the case \(a_0 \geq 0, a_i \geq 1, \ i \in \{1, 2, 3\}\). Then, using the constructed terms, we build the approximation and prove the corresponding asymptotic estimates in Section 5. Section 6 shows us what will happen in the case \(a_0 < 0, a_i \geq 1, i \in \{1, 2, 3\}\). The main novelty is that the limit problem splits into 3 independent problems with the uniform Dirichlet condition at the vertex. In addition, the view of asymptotic ansatzes are very sensitive to the parameter \(a_0\). Here, we construct the approximation and prove the corresponding estimates for more typical and realistic subcases \(a_0 \in (-1, 0)\) and \(a_0 = -1\). The general case \(a_0 < -1\) is only discussed in Section 7, where we also analyze obtained results and consider research perspectives.
2 | STATEMENT OF THE PROBLEM

The model thin star-shaped junction $\Omega_\varepsilon$ consists of 3 thin curvilinear cylinders

$$\Omega_\varepsilon^{(i)} = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \varepsilon \ell_i^0 < x_i < \ell_i, \quad \sum_{j=1}^{3} (1 - \delta_{ij})x_j^2 < \varepsilon^2 h_i^2(x_i) \right\}, \quad i = 1, 2, 3,$$

that are joined through a domain $\Omega_\varepsilon^{(0)}$ (referred in the sequel “node”). Here, $\varepsilon$ is a small parameter; $\ell_i^0 \in (0, \frac{1}{3})$, $\ell_i \geq 1$, $i = 1, 2, 3$; the positive function $h_i$ belongs to the space $\mathcal{C}^1([0, \ell_i])$, and it is equal to some constants in neighborhoods of the points $x = 0$ and $x_i = 1(i = 1, 2, 3)$; the symbol $\delta_{ij}$ is the Kronecker delta, i.e., $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$.

The node $\Omega_\varepsilon^{(0)}$ (see Figure 2) is formed by the homothetic transformation with coefficient $\varepsilon$ from a bounded domain $\Xi^{(0)} \subset \mathbb{R}^3$, i.e., $\Omega_\varepsilon^{(0)} = \varepsilon \Xi^{(0)}$. In addition, we assume that its boundary contains the disks

$$\Upsilon_\varepsilon^{(i)(\varepsilon \ell_i^0)} = \left\{ x \in \mathbb{R}^3 : x_i = \varepsilon \ell_i^0, \quad \sum_{j=1}^{3} (1 - \delta_{ij})x_j^2 < \varepsilon^2 h_i^2(\varepsilon \ell_i^0) \right\}, \quad i = 1, 2, 3,$$

and denote $\Gamma_\varepsilon^{(0)} := \partial \Omega_\varepsilon^{(0)} \setminus \left\{ \bigcup_{i=1}^{3} \Upsilon_\varepsilon^{(i)(\varepsilon \ell_i^0)} \right\}$.

Thus, the model thin star-shaped junction $\Omega_\varepsilon$ (see Figure 3) is the interior of the union $\bigcup_{i=0}^{3} \Omega_\varepsilon^{(i)}$, and we assume that it has the Lipschitz boundary.

Remark 2.1. We can consider more general thin star-shaped junctions with arbitrary orientation of thin cylinders (their number can be also arbitrary). But to avoid technical and huge calculations and to demonstrate the main steps of the proposed asymptotic approach, we consider the case when the cylinders are placed on the coordinate axes.

**FIGURE 2** The node $\Omega_\varepsilon^{(0)}$

**FIGURE 3** The model thin star-shaped junction $\Omega_\varepsilon$, [Colour figure can be viewed at wileyonlinelibrary.com]
In $\Omega$, we consider the following semilinear parabolic problem:

$$
\begin{align*}
\partial_t u_e(x, t) - \Delta u_e(x, t) + k(u_e(x, t)) &= f(x, t), & (x, t) &\in \Omega, T, \\
\partial_t u_e(x, t) + \epsilon \sigma_0( u_e(x, t)) &= \epsilon \phi_e^0(x, t), & (x, t) &\in \Gamma^0(x), T, \\
\partial_t u_e(x, t) + \epsilon \sigma_1( u_e(x, t), x_1, t) &= \epsilon \phi_e^1(x, t), & (x, t) &\in \Gamma^1(x), T, \quad i = 1, 2, 3, \\
\end{align*}
$$

(2.1)

where $\Gamma^0 = \partial \Omega \cap \{ x \in \mathbb{R}^3 : \epsilon < x_i < \epsilon \}$, $T > 0$, $\partial_t = \partial / \partial t$, $\partial_i$ is the outward normal derivative, and the parameters $\{ \alpha_i \}_{i=0}^3 \subset \mathbb{R}$, $\beta_0 \geq 0$, $\beta_i \geq 1$, $i = 1, 3$. For the given functions $f, k, \{ \phi_e^i(\epsilon, \kappa) \}_{\kappa=0}^3$, we assume the following conditions:

C1. The function $f$ belongs to the space $C \left( \Omega_{a_0} \times [0, T] \right)$, and its restriction on the curvilinear cylinder $\Omega_{a_0}^i (i = 1, 2, 3)$ belong to the space $C^1 \left( \Omega_{a_0} \times [0, T] \right)$ (the space of all continuous functions having continuous derivatives with respect to variables $\bar{x}_i$ in $\Omega_{a_0}^i \times [0, T]$), where $a_0$ is a fixed positive number such that $\Omega_e \subset \Omega_{a_0}$ for all values of the small parameter $\epsilon \in (0, \epsilon_0)$ and

$$
\bar{x}_i = \begin{cases}
(x_2, x_3), & i = 1, \\
(x_1, x_3), & i = 2, \\
(x_1, x_2), & i = 3.
\end{cases}
$$

C2. The functions $\phi_e^0(x, t) := \phi^0 \left( \frac{x}{\epsilon}, t \right)$ and $\phi_e^i(x, t) := \phi^i \left( \frac{x}{\epsilon}, x_i, t \right)$, $i = 1, 2, 3$ belong to the spaces $C \left( \Omega_{a_0} \times [0, T] \right)$, $i \in \{0, 1, 2, 3\}$, respectively.

C3. The functions $\{ \kappa_i(s, x_i, t) \}_{i=1}^3$, $\{ x_i, x_i \} \in \mathbb{R} \times [0, \epsilon_i] \times [0, T]$ are continuous in their domains of definition and have the partial derivatives with respect to $s, k \in C^1(\mathbb{R}), k_0 \in C^1(\mathbb{R})$, and there exists a positive constant $k_+$ such that

$$
0 \leq k'(s) \leq k_+, \quad 0 \leq \kappa_i'(s) \leq k_+, \quad 0 \leq \partial_i \kappa_i(s, x_i, t) \leq k_+ \quad \text{for} \quad s \in \mathbb{R}
$$

(2.2)

uniformly with respect to $x_i \in [0, \epsilon_i]$ and $t \in [0, T]$, respectively.

a. If $\alpha_0 < 0$, then in addition, the function $\kappa_0$ is a $C^2$ function with bounded derivatives, there exists a constant $k_-$ such that $0 < k_- \leq \kappa_i'(s)$ for all $s \in \mathbb{R}$ and $\kappa_0(0) = 0$ (so-called condition of zero absorption).

Denote by $\mathcal{H}^*_\epsilon$ the dual space to the Sobolev space $\mathcal{H}_\epsilon = \{ u \in H^1(\Omega_e) : |u|_{L^2(\Omega_e)} = 0 \}, \quad i = 1, 2, 3$. Recall that a function $u_e \in L^2(0, T; \mathcal{H}_e)$, with $\partial_t u_e \in L^2(0, T; \mathcal{H}^*_\epsilon)$, is called a weak solution to the problem (2.1) if it satisfies the integral identity

$$
\int_{\Omega_e} \partial_t u_e v \ dx + \int_{\Omega_e} \nabla u_e \cdot \nabla v \ dx + \int_{\Omega_e} k(u_e) v \ dx + \epsilon \int_{\Gamma^0} \kappa_0(u_e) v \ d\sigma_x \\
+ \sum_{i=1}^3 \epsilon \int_{\Gamma^1} \kappa_i(u_e, x_i, t) v \ d\sigma_x = \int_{\Omega_e} f v \ dx + \sum_{i=0}^3 \epsilon \phi_e^i(v) \ d\sigma_x
$$

(2.3)

for any function $v \in \mathcal{H}_\epsilon$ and $\epsilon t \in (0, T)$, and $u_e|_{t=0} = 0$. It is known (see, eg, Showalter) that $u_e \in C([0, T]; L^2(\Omega_e))$, and thus the equality $u_e|_{t=0} = 0$ makes sense.

The aim of the present paper is to

- construct the asymptotic approximation for the solution to the problem (2.1) as the parameter $\epsilon \to 0$;
- derive the corresponding limit problem ($\epsilon = 0$);
- prove the corresponding asymptotic estimates from which the influence of the local geometric heterogeneity of the node $\Omega_e$ and physical processes inside will be observed;
- study the influence of the parameters $\{ \alpha_i, \beta_i \}_{i=0}^3$ on the asymptotic behavior of the solution.
2.1 Comments to the statement

To our knowledge, the first works on the study of boundary-value problems for reaction-diffusion equations were papers by Kolmogorov et al.⁴² and Fisher.⁴³ Standard assumptions for reaction terms of semilinear equations are as follows:

- \( \exists C > 0 \ \forall s_1, s_2 \in \mathbb{R}: |k(s_1) - k(s_2)| \leq C|s_1 - s_2| \)
- \( \exists C_1 > 0 \ \exists C_2 \geq 0 \ \forall s \in \mathbb{R}: k(s)s \geq C_1 s^2 - C_2 \)

This is sufficient for the existence and uniqueness of the weak solution. However, many physical processes, especially in chemistry and medicine, have monotonous nature. Therefore, it is natural to impose special monotonous conditions for nonlinear terms. In our case, we propose simple conditions in (2.2) which are easy to verify. For instance, the functions

\[
k(s) = \lambda s + \cos s \quad (\lambda \geq 1), \quad k(s) = \frac{\lambda s}{1 + \nu s} \quad (\lambda, \nu > 0)
\]

satisfy this condition. The last one corresponds to the Michaelis-Menten hypothesis in biochemical reactions and to the Langmuir kinetics adsorption models (see Conca et al and Pao⁴⁴,⁴⁵); it satisfies the zero-absorption condition.

From conditions in (2.2), it follows the following inequalities:

\[
k(0)s \leq k(s)s \leq k_+ s^2 + k(0)s, \quad k_0(0)s \leq k_0(s)s \leq k_+ s^2 + k_0(0)s,
\]

\[
\kappa_i(0, x_i, t)s \leq \kappa_i(s, x_i, t)s \leq k_+ s^2 + \kappa_i(0, x_i, t)s, \quad \text{for } s \in \mathbb{R}
\]

uniformly with respect to \((x_i, t) \in [0, \varepsilon_i] \times [0, T]\), respectively, \(i = 1, 2, 3\). For the case \(C3(a)\), we have

\[
k_- s^2 \leq \kappa_0(s)s \leq k_+ s^2 \quad \forall s \in \mathbb{R}.
\]

Doubtlessly both the function \(k\) and \(\kappa_0\) may also depend on \(x\) and \(t\). However, we have omitted this dependence to avoid cumbersome formulas, leaving it only for the functions \(\{\kappa_i\}_{i=1}^3\).

As will be seen from further calculations in the case when some parameter \(a_1 > 1\), the condition in (2.2) for the corresponding function \(\kappa_i\) can be weakened. In this case, it is sufficient that \(\kappa_i\) is continuous and there exist constants \(c_1 > 0, c_2 \geq 0\) such that for any \(s_1, s_2, \ s \in \mathbb{R}, x_i \in [0, \varepsilon_i], \ t \in [0, T] \):

\[
\left(\kappa_i(s_1, x_i, t) - \kappa_i(s_2, x_i, t)\right)(s_1 - s_2) \geq 0, \quad \left|\kappa_i(s, x_i, t)\right| \leq c_1\left(1 + |s|\right), \quad \kappa_i(s, x_i, t)s \geq -c_2.
\]

It should be noted here that the asymptotic behavior of solutions to the reaction-diffusion equation in different kind of thin domains with the uniform Neumann conditions was studied in Arrieta et al.⁴⁶ and Prizzi and Rybakowski.⁴⁷ The convergence theorems were proved under the following assumptions for the reaction term \(k\): In Arrieta et al.,⁴⁶ it is a \(C^2\) function with bounded derivatives and

\[
\lim \inf_{|s| \to +\infty} \frac{k(s)}{s} > 0;
\]

in Prizzi and Rybakowski,⁴⁷ it is a \(C^1\) function, \(|k'(s)| \leq C(1 + |s|^q)|, \quad q \in (0, +\infty)|, \quad \text{and the dissipative condition (2.6)}\) is satisfied. It is easy to see that from (2.5) it follows (2.6).

In a typical interpretation, the solution to the problem (2.1) denotes the density of some quantity (temperature, chemical concentration, the potential of a vector-field, etc) within the thin star-shaped junction \(\Omega_e\). The nonlinear Robin boundary conditions considered in this problem mean that there is some interaction between the surrounding density and the density just inside \(\Omega_e\). It is evident from the results we have presented that these conditions have a substantial influence on the asymptotic behavior of the solution. To study this influence, we introduce special intensity factors \(\varepsilon^{a_i}, \quad i \in \{0, 1, 2, 3\}\) for the interaction and \(\varepsilon^{b_i}\), where \(\beta_0 \geq 0, \beta_i \geq 1, i \in \{0, 1, 2, 3\}\), for the surrounding density.

Since in this paper we are more interested in the study of the influence of the boundary interaction at the boundary of the node, we take the parameter \(a_0\) from \(\mathbb{R}\) and the other ones from \([1, +\infty)\). The case when \(a_i < 1 (i \in \{1, 2, 3\})\) is only discussed in Section 7. If \(a_0 > 0\) and \(b_0 > 0\) (the interaction and surrounding density are small), then, we have classical Kirchhoff transmission conditions that also correspond to the uniform Neumann condition in the starting 3D problem. In the case \(a_0 = 0\) and \(b_0 = 0\), the interaction and surrounding density are bigger, and as a result, we arrive to (1.2) in the limit passage as \(\varepsilon \to 0\). If \(a_0 < 0\), the the interaction on the boundary of the node is more bigger and due to the zero-absorption condition, the limit problem splits in 3 independent problems with the Dirichlet conditions at \(x = 0\).
3 | EXISTENCE AND UNIQUENESS OF THE WEAK SOLUTION

To obtain the operator statement for the problem (2.1), we introduce the new norm \( \| \cdot \|_\epsilon \) in \( H_\epsilon \), which is generated by the scalar product

\[
(u, v)_\epsilon = \int_{\Omega_\epsilon} \nabla u \cdot \nabla v \, dx, \quad u, v \in H_\epsilon.
\]

Due to the uniform Dirichlet condition on \( Y^{(i)}(\epsilon_i) \), \( i = 1, 2, 3 \), the norm \( \| \cdot \|_\epsilon \) and the ordinary norm \( \| \cdot \|_{H^1(\Omega_\epsilon)} \) are uniformly equivalent, i.e., there exist constants \( C_1 > 0 \) and \( \epsilon_0 > 0 \) such that for all \( \epsilon \in (0, \epsilon_0) \) and for all \( u \in H_\epsilon \), the following estimate holds:

\[
\| u \|_\epsilon \leq \| u \|_{H^1(\Omega_\epsilon)} \leq C_1 \| u \|_\epsilon.
\] (3.1)

**Remark 3.1.** Here and in what follows, all constants \( \{ C_i \} \) and \( \{ c_j \} \) in inequalities are independent of the parameter \( \epsilon \).

Further, we will often use the inequalities

\[
\epsilon \int_{\Omega_\epsilon^\prime} \nabla v \cdot \nabla v \, dx \leq C_2 \left( \epsilon^2 \int_{\Omega_\epsilon^0} |\nabla v|^2 \, dx + \int_{\Omega_\epsilon^0} \nabla v^2 \, dx \right),
\]

\[
\int_{\Omega_\epsilon^0} \nabla v^2 \, dx \leq C_3 \left( \epsilon^2 \int_{\Omega_\epsilon^0} |\nabla v|^2 \, dx + \epsilon \int_{\Omega_\epsilon^0} \nabla^2 v \, dx \right) \quad \forall v \in H^1(\Omega_\epsilon^0), \quad (i \in \{1, 2, 3\})
\] (3.3)

proved in Mel’nyk.\(^{48}\) Let us prove similar inequalities for the node \( (i = 0) \).

**Proposition 3.1.** Let \( Q \) be a bounded domain in \( \mathbb{R}^3 \) with the smooth boundary \( \partial Q \). Then, there exists a positive constant \( C_2 > 0 \) that is independent of \( \epsilon \) such that for any function \( v \) from the space \( H^1(Q_\epsilon) \), the following inequalities hold:

\[
\epsilon \int_{\partial Q_\epsilon} \nabla v \cdot \nabla v \, d\sigma \leq C_2 \left( \epsilon^2 \int_{Q_\epsilon} |\nabla v|^2 \, dx + \int_{Q_\epsilon} \nabla v^2 \, dx \right) \quad \text{and} \quad \int_{Q_\epsilon} \nabla v^2 \, dx \leq C_3 \left( \epsilon^2 \int_{Q_\epsilon} |\nabla v|^2 \, dx + \epsilon \int_{\partial Q_\epsilon} \nabla^2 v \, d\sigma \right).
\] (3.4)

where \( Q_\epsilon := \epsilon Q \) is the homothetic transformation with the coefficient \( \epsilon \) of \( Q \).

**Proof.** Let \( r(\xi) := (r_1(\xi), r_2(\xi), r_3(\xi)), \quad \xi \in \mathbb{S} \subset \mathbb{R}^2 \) be a smooth parametrization of \( \partial Q \). Then, \( r_\epsilon := \epsilon r = (\epsilon r_1, \epsilon r_2, \epsilon r_3) \) is the parametrization of \( \partial Q_\epsilon \). Denote by \( \rho(r) := \sqrt{E \Sigma - F^2} \), where \( E = \sum_{i=1}^3 \left( \frac{\partial r_i}{\partial \xi_1} \right)^2, \quad G = \sum_{i=1}^3 \left( \frac{\partial r_i}{\partial \xi_2} \right)^2, \quad F = \sum_{i=1}^3 \frac{\partial r_1}{\partial \xi_2} \frac{\partial r_2}{\partial \xi_1} \). Then, \( \rho(r_\epsilon) = \epsilon^2 \rho(r) \). Using definition of the surface integral, we get

\[
\int_{\partial Q_\epsilon} \nabla v^2 \, d\sigma = \int_{Q} \left( r_\epsilon(\xi) \right) \rho(r_\epsilon(\xi)) \, d\xi = \epsilon^2 \int_{Q} \nabla^2 (\epsilon r(\xi)) \rho(r(\xi)) \, d\xi = \epsilon^2 \int_{\partial Q} \nabla^2 v(\xi) \, d\sigma
\] (3.5)

for all \( v \in H^1(Q_\epsilon) \), where \( v_\epsilon(\xi) := v(\epsilon \xi), \xi = (\xi_1, \xi_2, \xi_3), \) and \( x = \epsilon \xi \).

Taking into account the boundedness of the trace operator, we get,

\[
\exists c_0 > 0 : \quad \| v_\epsilon \|_{L^2(\partial Q_\epsilon)} \leq c_0 \| v_\epsilon \|_{H^1(Q_\epsilon)},
\]

where constant \( c_0 \) does not depend on \( v_\epsilon \), and the equality

\[
\epsilon^3 \left( \int_{Q} |\nabla v_\epsilon|^2 \, d\xi + \int_{Q} v_\epsilon^2 \, d\xi \right) = \epsilon^2 \int_{Q_\epsilon} |\nabla v|^2 \, dx + \int_{Q_\epsilon} v^2 \, dx,
\]

we obtain the first inequality in (3.4). By the same arguments, we can prove the second one. \( \square \)

It is easy to prove the inequality

\[
\int_{\Omega_\epsilon^0} \nabla v^2 \, dx \leq C_4 \epsilon \left( \int_{\Omega_\epsilon} |\nabla v|^2 \, dx + \int_{Y^{(i)}(\epsilon)} v^2 \, d\xi \right),
\]
and then, with the help of the first inequality in (3.4),
\[
\int_{\Omega^0_t} \nabla^2 v \, d\sigma_x \leq C_5 \left( \int_{\Omega} |\nabla v|^2 \, dx + \int_{\Gamma^0} \nabla^2 v \, dx \right) \tag{3.6}
\]
for all \( v \in H^1(\Omega_t) \) and \( i \in \{1, 2, 3\} \).

Define a nonlinear operator \( A_\varepsilon(t) : H_\varepsilon \to H_\varepsilon^* \) through the relation
\[
\langle A_\varepsilon(t)u, v \rangle_\varepsilon = \int_{\Omega_t} \nabla u \cdot \nabla v \, dx + \int_{\Omega_t} k(u)v \, dx + \varepsilon \int_{\Gamma^0} \kappa_0(u) v \, d\sigma_x + \sum_{i=1}^{3} \varepsilon_i \int_{\Gamma^0} \kappa_i(u, x, t) v \, d\sigma_x \quad \forall u, v \in H_\varepsilon,
\]
and the linear functional \( F_\varepsilon(t) \in H_\varepsilon^* \) by the formula
\[
\langle F_\varepsilon(t), v \rangle_\varepsilon = \int_{\Omega_t} f v \, dx + \sum_{i=0}^{3} \varepsilon_i \int_{\Gamma^0} \varphi^{(i)}_\varepsilon v \, d\sigma_x \quad \forall v \in H_\varepsilon,
\]
for \( \varepsilon \in \{0, 1\} \), \( \varepsilon \) is the duality pairing of \( H_\varepsilon^* \) and \( H_\varepsilon \).

Then, the integral identity (2.3) can be rewritten as follows:
\[
\langle \partial_t u_\varepsilon, v \rangle_\varepsilon + \langle A_\varepsilon(t)u_\varepsilon, v \rangle_\varepsilon = \langle F_\varepsilon(t), v \rangle_\varepsilon \quad \forall v \in H_\varepsilon, \tag{3.7}
\]
for \( \varepsilon \in \{0, 1\} \) and \( u_\varepsilon \rvert_{t=0} = 0 \).

To prove the well-posedness result, we verify some properties of the operator \( A_\varepsilon \) for a fixed value of \( \varepsilon \).

1. With the help of (2.4) and Cauchy inequality with \( \delta > 0 \) \( (ab \leq \delta a^2 + \frac{b^2}{4\delta}) \), we obtain
\[
\langle A_\varepsilon(t)v, v \rangle_\varepsilon \geq \int_{\Omega_t} |\nabla v|^2 \, dx + \int_{\Omega_t} k(0) v \, dx + \varepsilon \int_{\Gamma^0} \kappa_0(0) v \, d\sigma_x + \sum_{i=1}^{3} \varepsilon_i \int_{\Gamma^0} \kappa_i(0, x, t) v \, d\sigma_x
\geq \|v\|^2_2 - \delta \left( \int_{\Omega_t} |v|^2 \, dx + \int_{\Gamma^0} |v|^2 \, d\sigma_x + \varepsilon \sum_{i=1}^{3} \int_{\Gamma^0} |v|^2 \, d\sigma_x \right)
- \frac{1}{4\delta} \left( |k(0)|^2 |\Omega_t|^3 + \varepsilon^2 \kappa_0(0)^2 |\Gamma^0_t|^2 + \sum_{i=1}^{3} \varepsilon_i \max_{0 \leq \varepsilon \leq 1, 0 \leq t \leq T} (\kappa_i(0, x, t))^2 |\Gamma^0_t|^2 \right). \tag{3.8}
\]

Here and in what follows, \(|S|_n\) is the \(n\)-dimensional Lebesgue measure of a set \( S \). Then, using (3.1), (3.2), and (3.6) and recalling the assumption C3(a), we can select appropriate \( \delta \) such that
\[
\langle A_\varepsilon(t)v, v \rangle_\varepsilon \geq C_6 \|v\|^2_2 - C_7 \varepsilon^2 \left( 1 + \varepsilon^{2\alpha_0} + \sum_{i=1}^{3} \varepsilon_i^{2\alpha_i-2} \right) \quad \forall v \in H_\varepsilon.
\]

This inequality means that the operator \( A_\varepsilon \) is coercive for \( \varepsilon \in (0, T) \).

2. Let us show that it is strongly monotone for \( \varepsilon \in (0, T) \). Taking into account (2.2), we get
\[
\langle A_\varepsilon(t)u_1 - A_\varepsilon(t)u_2, u_1 - u_2 \rangle_\varepsilon \geq \|u_1 - u_2\|^2_2 \quad \forall u_1, u_2 \in H_\varepsilon.
\]

3. The operator \( A_\varepsilon \) is hemicontinuous for \( \varepsilon \in (0, T) \). Indeed, the real valued function
\[
[0, 1] \ni \tau \to \langle A_\varepsilon[u_1 + \tau v], u_2 \rangle_\varepsilon
\]
is continuous on \([0, 1]\) for all fixed \( u_1, u_2, v \in H_\varepsilon \) due to the continuity of the functions \( k, \{\kappa_i\}_{i=0}^{3} \) and Lebesgue dominated convergence theorem.

4. Let us prove that operator \( A_\varepsilon \) is bounded. Using Cauchy-Bunyakovsky integral inequality, (3.1) and (2.4), we deduce the following inequality:
\[
\langle A_\varepsilon u, v \rangle_\varepsilon \leq \int_{\Omega_t} \nabla u \cdot \nabla v \, dx + \int_{\Omega_t} (k_+ |u| + |k(0)|) |v| \, dx
+ \varepsilon \int_{\Gamma^0} (k_+ |u| + |k(0)|) |v| \, d\sigma_x + \sum_{i=1}^{3} \varepsilon_i \int_{\Gamma^0} (k_+ |u| + |\kappa_0(0)|) |v| \, d\sigma_x \tag{3.9}
\]
for any \( (3.6) \), and \( (3.10) \), it follows that 

\[
\text{Now, with the help of (3.4), we get}
\]

Now, let us consider the case \( C_3(a) \)

\[
\text{Now, with the help of (3.2) and (3.6), we obtain}
\]

\[
\text{Selecting appropriate }
\]

\[
\text{for any } \tau \in (0, T). \text{ Selecting appropriate } \delta > 0 \text{ and taking the conditions } C_1 \text{ to } C_3(a) \text{ into account, we obtain the uniform estimate}
\]

\[
\text{max}_{\tau \in [0, T]} \| u(\cdot, t) \|_{L^2(\Omega_\alpha)} + \| u(\cdot, T, t) \|_{L^2(\Omega_\alpha)}
\]

\[
\leq C_0 \left( \sqrt{T} \left( |k(0)| + \varepsilon^{\alpha_0} |\kappa_0(0)| + \sum_{i=1}^{3} \varepsilon^{\alpha_{i-1}} \max_{\Omega_\alpha \times [0, T]} |\kappa_i(0, x_i, t)| \right) + \varepsilon^{\beta_0} \| \varphi^{(0)} \|_{L^2(\Omega_\alpha \times [0, T])} + \sum_{i=1}^{3} \varepsilon^{\beta_{i-1}} \| \varphi_i^{(0)} \|_{L^2(\Omega_\alpha \times [0, T])} \right) \leq C_1 \varepsilon \tag{3.10}
\]

for all values of the parameters \( \left\{ \alpha_i \right\}_{i=0}^{3} \) and \( \beta_0 \geq 0, \beta_i \geq 1, i \in \{1, 2, 3\} \).

Now, let us consider the case \( C_3(a) \) \( (a_0 < 0) \). From the integral identity \( (2.3) \) and inequalities \( (2.4), (2.5), (3.2), (3.1), (3.6), \) and \( (3.10) \), it follows that

\[
\varepsilon^{\alpha_0} \int_{\Gamma_0^\alpha(0, T)} u^2 \, d\sigma \, dt \leq C_1 \left( \sqrt{T} \left( |k(0)| + \sum_{i=1}^{3} \varepsilon^{\alpha_{i-1}} \max_{\Omega_\alpha \times [0, T]} |\kappa_i(0, x_i, t)| \right) + \varepsilon^{\beta_0} \| \varphi^{(0)} \|_{L^2(\Gamma_0^\alpha \times [0, T])} + \sum_{i=1}^{3} \varepsilon^{\beta_{i-1}} \| \varphi_i^{(0)} \|_{L^2(\Gamma_0^\alpha \times [0, T])} \right) \leq C_2 \varepsilon^2.
\]

Now, with the help of (3.4), we get

\[
\int_{\Omega_0^\alpha(0, T)} u^2 \, dx \, dt \leq C_3 \left( \varepsilon^2 \int_{\Omega_0^\alpha(0, T)} |\nabla u|^2 \, dx \, dt + \varepsilon^{1-\alpha_0} \varepsilon^{\alpha_0} \int_{\Gamma_0^\alpha(0, T)} u^2 \, d\sigma \, dt \right) \leq C_4 \varepsilon^\theta, \tag{3.11}
\]

where \( \theta := \min\{4, 3 - a_0\} \). This means that

\[
\frac{1}{\varepsilon^3} \int_{\Omega_0^\alpha(0, T)} u^2 \, dx \, dt \leq C_4 \varepsilon^{\min\{1, -a_0\}} \to 0 \text{ as } \varepsilon \to 0. \tag{3.12}
\]
4 | FORMAL ASYMPTOTIC APPROXIMATION. THE CASE $\alpha_0 \geq 0$, $\alpha_i \geq 1$, $i \in \{1, 2, 3\}$

In this section, we assume that the functions $f$, $k$, $\{\varphi_i^{(0)}$, $k_i\}_{i=0}^3$ are smooth enough. Following the approach of Klevtsovsky and Mel’nyk,38 we propose ansatzes of the asymptotic approximation for the solution to the problem (2.1) in the following form:

1. The regular parts of the approximation

$$\omega_0^{(i)}(x_i, t) + \varepsilon \omega_1^{(i)}(x_i, t) + \varepsilon^2 u_2^{(i)} \left( \frac{X}{\varepsilon}, x_i, t \right) + \varepsilon^3 u_3^{(i)} \left( \frac{X}{\varepsilon}, x_i, t \right)$$

is located inside of each thin cylinder $\Omega_\varepsilon^{(i)}$, and their terms depend both on the corresponding longitudinal variable $x_i$ and so-called “fast variables” $\overline{x}_i$ ($i = 1, 2, 3$);

2. and the inner part of the approximation

$$N_0 \left( \frac{X}{\varepsilon}, t \right) + \varepsilon N_1 \left( \frac{X}{\varepsilon}, t \right) + \varepsilon^2 N_2 \left( \frac{X}{\varepsilon}, t \right)$$

is located in a neighborhood of the node $\Omega_0^{(i)}$.

4.1 | Regular parts

Substituting the representation (4.1) for each fixed index $i \in \{1, 2, 3\}$ into the differential equation of the problem (2.1), using Taylor formula for the function $f$ at the point $\overline{x}_i = (0, 0)$ for the function $k$ at $\omega_0^{(i)}$ and collecting coefficients at $\varepsilon^0$, we obtain

$$-\Delta_{x_i} u_2^{(i)}(\overline{x}_i, x_i, t) = -\frac{\partial \omega_0^{(i)}}{\partial t}(x_i, t) + \frac{\partial^2 \omega_1^{(i)}}{\partial x_i^2}(x_i, t) - k \left( \omega_0^{(i)}(x_i, t) \right) + f_0^{(i)}(x_i, t),$$

(4.3)

where $\overline{x}_i = \frac{x_i}{\varepsilon}$ and $f_0^{(i)}(x_i, t) := f(x, t)|_{x_i=(0,0)}$.

It is easy to calculate the outer unit normal to $\Gamma_\varepsilon^{(i)}$ :

$$\nu(x_i, \overline{x}_i) = \frac{1}{\sqrt{1 + \varepsilon^2 |h_i'(x_i)|^2}}(\nu_1(\overline{x}_i), \nu_2(\overline{x}_i)) = \begin{cases} 
\frac{-\varepsilon h_i'(x_i), \nu_1(\overline{x}_i), \nu_2(\overline{x}_i)}{\sqrt{1 + \varepsilon^2 |h_i'(x_i)|^2}}, & i = 1, \\
\frac{\nu_1(\overline{x}_i), -\varepsilon h_i'(x_i), \nu_2(\overline{x}_i)}{\sqrt{1 + \varepsilon^2 |h_i'(x_i)|^2}}, & i = 2, \\
\frac{\nu_1(\overline{x}_i), \nu_2(\overline{x}_i), -\varepsilon h_i'(x_i)}{\sqrt{1 + \varepsilon^2 |h_i'(x_i)|^2}}, & i = 3,
\end{cases}$$

where $\nu_i(\overline{x}_i)$ is the outward normal for the disk $\nu_i^{(0)}(x_i) := \{\overline{x}_i \in \mathbb{R}^2 : |\overline{x}_i| < h_i(x_i)\}$.

Taking the view of the outer unit normal into account and putting the sum (4.1) into the third relation of the problem (2.1), we get with the help of Taylor formula for the function $k_i$ the following relation:

$$\varepsilon \frac{\partial \nu_1^{(i)}}{\partial x_i}(\overline{x}_i, x_i, t) = \frac{h_i'(x_i)}{\Delta_{x_i}} \frac{\partial \omega_0^{(i)}}{\partial x_i}(x_i, t) - \varepsilon h_i'(x_i) \left( \omega_0^{(i)}(x_i, t), x_i, t \right) + \varepsilon \varphi_i^{(i)}(\overline{x}_i, x_i, t).$$

(4.4)

Relations (4.3) and (4.4) form the linear inhomogeneous Neumann boundary-value problem

$$\begin{cases}
-\Delta_{x_i} u_2^{(i)}(\overline{x}_i, x_i, t) = -\frac{\partial \omega_0^{(i)}}{\partial t}(x_i, t) + \frac{\partial^2 \omega_1^{(i)}}{\partial x_i^2}(x_i, t) - k \left( \omega_0^{(i)}(x_i, t) \right) + f_0^{(i)}(x_i, t), & \overline{x}_i \in Y_i(x_i), \\
\frac{\partial \nu_1^{(i)}}{\partial x_i}(\overline{x}_i, x_i, t) = h_i'(x_i) \frac{\partial \omega_0^{(i)}}{\partial x_i}(x_i, t) - \delta_{\nu,1} k_i \left( \omega_0^{(i)}(x_i, t), x_i, t \right) + \frac{\partial \nu_1^{(i)}}{\partial x_i}(\overline{x}_i, x_i, t), & \overline{x}_i \in \partial Y_i(x_i), \\
\langle u_2^{(i)}(\cdot, x_i, t) \rangle_{Y_i(x_i)} = 0,
\end{cases}$$

(4.5)
to define $u_i^{(i)}$. Here, $\langle u(\cdot, x, t) \rangle_{Y(x_i)} := \int_{Y(x_i)} u(\xi, x_i, t) d\xi$, the variables $(x, t)$ are regarded as parameters from $I_t^0 \times (0, T)$, where $I_t^0 := \{ x : x_i \in (\varepsilon, c^0), \quad \xi = (0, 0) \}$. We add the third relation in (4.5) for the uniqueness of a solution.

Writing down the necessary and sufficient conditions for the solvability of the problem (4.5), we derive the differential equation

$$
\pi h_i^2(x_i) \frac{\partial u_i^{(i)}}{\partial t}(x_i, t) - \pi \frac{\partial}{\partial x_i} \left( h_i^2(x_i) \frac{\partial u_i^{(i)}}{\partial x_i}(x_i, t) \right) + \pi h_i^2(x_i) k \left( \omega_0^{(i)}(x_i, t) \right) + 2\pi \delta_{n,1} h_i(x_i) k \left( \omega_1^{(i)}(x_i, t), x_i, t \right) = \pi h_i^2(x_i) f_0^{(i)}(x_i, t) + \delta_{n,1} \int_{\partial Y_i} \varphi^{(i)}(\xi, x_i, t) d\xi, \quad (x_i, t) \in I_t^0 \times (0, T),
$$

(4.6)

to define $\omega_0^{(i)} (i \in \{ 1, 2, 3 \})$.

Let $\omega_0^{(i)}$ be a solution of the differential equation (4.6) (its existence will be proved in the Section 4.2.1). Thus, there exists a unique solution to the problem (4.5) for each $i \in \{ 1, 2, 3 \}$.

For determination of the coefficients $u_3^{(i)}$, $i = 1, 2, 3$, we similarly obtain the following problems:

$$
\left\{ \begin{array}{l}
-\Delta_{\xi_i} u_3^{(i)}(\xi_i, x_i, t) = -\frac{\partial u_3^{(i)}}{\partial t}(x_i, t) + \frac{\partial^2 u_3^{(i)}}{\partial x_i^2}(x_i, t) - k \left( \omega_0^{(i)}(x_i, t) \right) \omega_1^{(i)}(x_i, t) + f_1^{(i)}(\xi_i, x_i, t), \quad \xi = Y_i(x_i), \\
\partial_{x_i} u_3^{(i)}(\xi_i, x_i, t) = h_i^2(x_i) \frac{\partial}{\partial x_i} \left( \omega_0^{(i)}(x_i, t) \right) - \delta_{n,1} h_i(x_i) k \left( \omega_1^{(i)}(x_i, t), x_i, t \right) \omega_3^{(i)}(x_i, t) \quad \xi \in \partial Y_i(x_i), \\
\langle u_3^{(i)}(\cdot, x_i, t) \rangle_{\gamma(x_i)} = 0
\end{array} \right.
$$

(4.7)

for each $i \in \{ 1, 2, 3 \}$. Here,

$$
f_1^{(i)}(\xi_i, x_i, t) = \sum_{j=1}^{3} (1 - \delta_{ij}) \xi_j \frac{\partial}{\partial x_j} f(x_i, t) |_{\xi=(0,0)}.
$$

Repeating the previous reasoning, we find that the coefficients $\{ \omega_3^{(i)} \}_{i=1}^3$ have to be solutions to the respective linear ordinary differential equation

$$
\pi h_i^2(x_i) \frac{\partial u_3^{(i)}}{\partial t}(x_i, t) - \pi \frac{\partial}{\partial x_i} \left( h_i^2(x_i) \frac{\partial u_3^{(i)}}{\partial x_i}(x_i, t) \right) + \pi h_i^2(x_i) k \left( \omega_0^{(i)}(x_i, t) \right) \omega_1^{(i)}(x_i, t) + 2\pi \delta_{n,1} h_i(x_i) k \left( \omega_0^{(i)}(x_i, t), x_i, t \right) \omega_3^{(i)}(x_i, t)
$$

$$
= \int_{Y_i(x_i)} f_1^{(i)}(\xi_i, x_i, t) d\xi_i - 2\pi \delta_{n,2} h_i(x_i) k \left( \omega_0^{(i)}(x_i, t), x_i, t \right) + \delta_{n,2} \int_{\partial Y_i(x_i)} \varphi^{(i)}(\xi_i, x_i, t) d\xi_i,
$$

$$(x_i, t) \in I_t^0 \times (0, T) \quad (i \in \{ 1, 2, 3 \}).
$$

(4.8)

## 4.2 | Inner part

To obtain conditions for the functions $\{ \omega_n^{(i)} \}_{i=1}^3$, $n \in \{ 0, 1 \}$ at the point $(0, 0, 0)$, we introduce the inner part of the asymptotic approximation (4.2) in a neighborhood of the node $\Omega^{(0)}$. If we pass to the “fast variables” $\xi = \frac{x}{\varepsilon}$ and tend $\varepsilon$ to 0, the domain $\Omega_\varepsilon$ is transformed into the unbounded domain $\Xi$ that is the union of the domain $\Xi^{(0)}$ and 3 semibounded cylinders

$$
\Xi^{(0)} = \{ \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_0 < \xi_i < +\infty, \quad [\xi_i] = h_i(0), \quad i = 1, 2, 3,
$$

ie, $\Xi$ is the interior of $\overline{\mathbb{U}^{(3)}}$ (see Figure 4).

Let us introduce the following notation for parts of the boundary of the domain $\Xi$:

$$
\Gamma_i = \{ \xi \in \mathbb{R}^3 : \xi_0 < \xi_i < +\infty, \quad [\xi_i] = h_i(0), \quad i = 1, 2, 3, \quad \Gamma_0 = \partial \Xi \setminus \bigcup_{i=1}^{3} \Gamma_i \}.
$$
Substituting (4.2) into the problem (2.1) and equating coefficients at the same powers of \( \varepsilon \), we derive the following relations for \( N_n, (n \in \{0, 1, 2\}) \):

\[
\begin{align*}
-\Delta_\varepsilon N_n(\xi, t) &= F_n(\xi, t), & \xi \in \Xi, \\
\partial_\nu N_n(\xi, t) &= B_n^{(i)}(\xi, t), & \xi \in \Gamma_0, \\
\partial_{\nu_i} N_n(\xi, t) &= B_n^{(i)}(\xi, t), & \xi \in \Gamma_i, \quad i = 1, 2, 3, \\
N_n(\xi, t) &\sim \omega_n^{(i)}(0, t) + \Psi_n^{(i)}(\xi, t), & \xi_i \to +\infty, & \xi_i \in Y_i(0), \quad i = 1, 2, 3.
\end{align*}
\]  

(4.9)

Here,

\[
F_0 \equiv F_1 \equiv 0, \quad F_2(\xi, t) = -\partial_t N_0 - k(N_0) + f(0, t), \quad \xi \in \Xi,
\]

\[
B_0^{(i)} \equiv 0, \quad B_1^{(i)} = -\delta_{\nu_0,0} \kappa_0(N_0) + \delta_{\nu_i,0} \varphi^{(i)}(\xi, t),
\]

\[
B_2^{(i)}(\xi, t) = -\delta_{\nu_0,0} \kappa_0(N_0) N_1 - \delta_{\nu_1,1} \kappa_0(N_0) + \delta_{\nu_i,1} \varphi^{(i)}(\xi_i, 0, t), \quad \xi \in \Gamma_0,
\]

\[
B_0^{(i)} \equiv B_1^{(i)} \equiv 0, \quad B_2^{(i)}(\xi, t) = -\delta_{\nu_1,1} \kappa_i(N_0, 0, t) + \delta_{\nu_i,1} \varphi^{(i)}(\xi_i, 0, t), \quad \xi \in \Gamma_i, \quad i = 1, 2, 3.
\]

The variable \( t \) is regarded as parameter from \((0, T)\). The right-hand sides in the differential equation and boundary conditions on \( \{\Gamma_i\} \) of the problem (4.9) are obtained with the help of the Taylor formula for the functions \( f, k \) and \( \varphi^{(i)} \) at the points \( x_0 = s = N_0 \), and \( x_i = 0, i = 1, 2, 3 \), respectively.

The fourth condition in (4.9) appears by matching the regular and inner asymptotics in a neighborhood of the node, namely, the asymptotics of the terms \( \{N_n\} \) as \( \xi_1 \to +\infty \) have to coincide with the corresponding asymptotics of the terms \( \{\omega_n^{(i)}\} \) as \( x_1 = \varepsilon \xi_i \to +0, i = 1, 2, 3 \), respectively. Expanding formally each term of the regular asymptotics in the Taylor series at the points \( x_i = 0 \) and collecting the coefficients of the same powers of \( \varepsilon \), we get

\[
\Psi_0^{(i)} \equiv 0, \quad \Psi_1^{(i)}(\xi, t) = \xi_i \frac{\partial \omega_0^{(i)}}{\partial \xi_i}(0, t), \quad i = 1, 2, 3,
\]

\[
\Psi_2^{(i)}(\xi, t) = \frac{\varepsilon_i^2}{2} \frac{\partial^2 \omega_0^{(i)}}{\partial \xi_i^2}(0, t) + \xi_i \frac{\partial \omega_0^{(i)}}{\partial \xi_i}(0, t) + u_2^{(i)}(\xi_i, 0, t), \quad i = 1, 2, 3.
\]  

(4.10)

A solution of the problem (4.9) at \( n = 1, 2 \) is sought in the form

\[
N_n(\xi, t) = \sum_{i=1}^{3} \Psi_n^{(i)}(\xi, t) \chi_i(\xi_i) + \bar{N}_n(\xi, t),
\]  

(4.11)

where \( \chi_i \in C^\infty(\mathbb{R}^+) \), \( 0 \leq \chi_i \leq 1 \) and

\[
\chi_i(\xi_i) = \begin{cases} 
0, & \text{if } \xi_i \leq 1 + \varepsilon_0, \\
1, & \text{if } \xi_i \geq 2 + \varepsilon_0,
\end{cases} \quad i = 1, 2, 3.
\]
Then, \( \bar{N}_n \) has to be a solution of the problem

\[
\begin{aligned}
-\Delta \xi \bar{N}_n(\xi, t) &= \bar{F}_n(\xi, t), \quad \xi \in \Xi, \\
\partial \xi \bar{N}_n(\xi, t) &= B_{1n}^{(0)}(\xi, t), \quad \xi \in \Gamma_0, \\
\partial \xi \bar{N}_n(\xi, t) &= B_{in}^{(i)}(\xi, t), \quad \xi \in \Gamma_i, \quad i = 1, 2, 3,
\end{aligned}
\]  

(4.12)

where

\[
\bar{F}_1(\xi, t) = \sum_{i=1}^{3} \left( \xi_i \frac{\partial \omega_0^{(i)}(0, t)}{\partial \xi_i} \chi''(\xi_i) + 2 \frac{\partial \omega_0^{(i)}(0, t)}{\partial \xi_i} \chi'(\xi_i) \right)
\]

\[
F_2(\xi, t) = \sum_{i=1}^{3} \left[ \left( \xi_i^2 \frac{d^2 \omega_0^{(i)}(0, t)}{dx_i^2} + \xi_i \frac{\partial \omega_0^{(i)}(0, t)}{\partial \xi_i} + a_0^{(i)}(\xi_i, 0, t) \right) \chi''(\xi_i) + 2 \left( \xi_i \frac{d^2 \omega_0^{(i)}(0, t)}{dx_i^2} - 2a_0^{(i)}(0, t) + \frac{\partial \omega_0^{(i)}(0, t)}{\partial \xi_i} \right) \chi'(\xi_i) \right]
\]

\[- \partial_i \bar{N}_0 - k(N_0) + \sum_{i=1}^{3} \left( \partial_i \omega_0^{(i)}(0, t) + k(\omega_0^{(i)}(0, t)) \right) \chi_i(\xi_i) + \left( 1 - \sum_{i=1}^{3} \chi_i(\xi_i) \right) f(0, t)\]

and

\[
B_{1n}^{(0)} = -\delta_{n,0} k_0(N_0) + \delta_{n,0} \varphi_0(\xi, t), \quad B_{2}^{(0)}(\xi, t) = -\delta_{n,0} k_0(N_0) N_1 - \delta_{n,1} k_0(N_0) + \delta_{n,1} \varphi_0(\xi, t),
\]

\[
B_{1}^{(i)} = 0, \quad B_{2}^{(i)}(\xi, t) = -\delta_{n,1} \left( k_1(N_0, 0, t) - k_1(\omega_0^{(i)}(0, t), 0, t) \chi_i(\xi_i) \right) + \delta_{n,1} \varphi_0(\xi_i, 0, t)(1 - \chi_i(\xi_i)),
\]

for \( i \in \{1, 2, 3\} \). In addition, we demand that \( \bar{N}_n \) satisfies the following stabilization conditions:

\[
\bar{N}_n(\xi, t) \to \omega_0^{(i)}(0, t) \quad \text{as} \quad \xi_i \to +\infty, \quad \bar{\xi}_i \in \mathcal{Y}_i(0), \quad i = 1, 2, 3.
\]  

(4.13)

The existence of a solution to the problem (4.12) in the corresponding energetic space can be obtained from general results about the asymptotic behavior of solutions to elliptic problems in domains with different exits to infinity (see, eg, Kondratiev and Oleinik\(^{49}\) and Nazarov\(^{50}\)). We will use approach proposed in Nazarov and Mel’nyk.\(^{50,51}\)

Let \( C_0^\infty(\Xi) \) be a space of functions infinitely differentiable in \( \Xi \) and finite with respect to \( \xi \), ie,

\[
\forall \, \forall \, \in C_0^\infty(\Xi) \quad \exists \, R > 0 \quad \forall \, \xi \in \Xi \quad \xi_i \geq R, \quad i = 1, 2, 3 : \quad v(\xi) = 0.
\]

We now define a space \( H := \left( C_0^\infty(\Xi), \, \| \cdot \|_H \right) \), where

\[
\| v \|_H = \sqrt{\int_{\Xi} |\nabla v(\xi)|^2 \, d\xi + \int_{\Xi} |v(\xi)|^2 \, d\xi + \int_{\Xi} |\nabla v(\xi)|^2 \, d\xi}.
\]

and the weight function \( \rho \in C_0^\infty(\Xi) \), \( 0 \leq \rho \leq 1 \) and

\[
\rho(\xi) = \begin{cases} 1, & \text{if} \quad \xi \in \Xi^{(0)}, \\
|\xi_i|^{-1}, & \text{if} \quad \bar{\xi}_i \geq \xi_0 + 1, \quad \xi \in \Xi^{(0)}, \quad i = 1, 2, 3.
\end{cases}
\]

**Definition 4.1.** A function \( \bar{N}_n \) from the space \( H \) is called a weak solution of the problem (4.12) if the identity

\[
\int_{\Xi} \nabla \bar{N}_n \cdot \nabla v \, d\xi = \int_{\Xi} \bar{F}_n v \, d\xi + \sum_{i=0}^{3} \int_{\Gamma_i} \bar{B}_{in}^{(i)} v \, ds
\]

holds for all \( v \in H \).

Similarly as in Mel’nyk,\(^{51}\) we prove the following proposition.

**Proposition 4.1.** Let \( \rho \bar{F}_n(\cdot, t) \in L^2(\Xi), \quad \bar{B}_{1n}^{(0)}(\cdot, t) \in L^2(\Gamma_0), \quad \rho \bar{F}_n(\cdot, t) \in L^2(\Gamma_i), \quad i = 1, 2, 3, \) for all \( t \in (0, T) \). Then, there exist a weak solution of problem (4.12) if and only if

\[
\int_{\Xi} \bar{F}_n \, d\xi + \sum_{i=0}^{3} \int_{\Gamma_i} \bar{B}_{in}^{(i)} \, ds = 0.
\]  

(4.14)
This solution is defined up to an additive constant. The additive constant can be chosen to guarantee the existence and uniqueness of a weak solution of problem (4.12) with the following differentiable asymptotics:

\[
\hat{N}_n(\xi, t) = \begin{cases} 
\mathcal{O}(\exp(-\gamma_1 \xi_1)) & \text{as } \xi_1 \to +\infty, \\
\delta_n^{(2)}(t) + \mathcal{O}(\exp(-\gamma_2 \xi_2)) & \text{as } \xi_2 \to +\infty, \\
\delta_n^{(3)}(t) + \mathcal{O}(\exp(-\gamma_3 \xi_3)) & \text{as } \xi_3 \to +\infty,
\end{cases}
\]  

(4.15)

where \(\gamma_i, i = 1, 2, 3\) are positive constants.

The values \(\delta_n^{(2)}\) and \(\delta_n^{(3)}\) in (4.15) are defined as follows:

\[
\delta_n^{(i)}(t) = \int_{\Xi} \mathcal{R}_i(\xi) \tilde{F}_n(\xi, t) d\xi + \sum_{j=0}^{3} \int_{\Gamma_j} \mathcal{R}_i(\xi) \tilde{F}_n(\xi, t) d\sigma_j, \quad i = 2, 3, \quad n \in \{0, 1, 2\},
\]

(4.16)

where \(\mathcal{R}_2\) and \(\mathcal{R}_3\) are special solutions to the corresponding homogeneous problem

\[-\Delta \mathcal{R} = 0 \text{ in } \Xi, \quad \partial \mathcal{R} = 0 \text{ on } \partial \Xi,
\]

(4.17)

for the problem (4.12).

**Proposition 4.2.** The problem (4.17) has 2 linearly independent solutions \(\mathcal{R}_2\) and \(\mathcal{R}_3\) that do not belong to the space \(\mathcal{H}\), and they have the following differentiable asymptotics:

\[
\mathcal{R}_2(\xi) = \begin{cases} 
-\frac{\xi_1}{\pi h_1(0)} + \mathcal{O}(\exp(-\gamma_1 \xi_1)) & \text{as } \xi_1 \to +\infty, \\
\frac{\xi_2}{\pi h_2(0)} + C_2^{(2)} + \mathcal{O}(\exp(-\gamma_2 \xi_2)) & \text{as } \xi_2 \to +\infty, \\
C_2^{(3)} + \mathcal{O}(\exp(-\gamma_3 \xi_3)) & \text{as } \xi_3 \to +\infty,
\end{cases}
\]

(4.18)

\[
\mathcal{R}_3(\xi) = \begin{cases} 
-\frac{\xi_1}{\pi h_1(0)} + \mathcal{O}(\exp(-\gamma_1 \xi_1)) & \text{as } \xi_1 \to +\infty, \\
C_2^{(2)} + \mathcal{O}(\exp(-\gamma_2 \xi_2)) & \text{as } \xi_2 \to +\infty, \\
\frac{\xi_2}{\pi h_2(0)} + C_3^{(3)} + \mathcal{O}(\exp(-\gamma_3 \xi_3)) & \text{as } \xi_3 \to +\infty.
\end{cases}
\]

(4.19)

Any other solution to the homogeneous problem, which has polynomial growth at infinity, can be presented as a linear combination \(c_1 + c_2 \mathcal{R}_2 + c_3 \mathcal{R}_3\).

**Proof.** The solution \(\mathcal{R}_2\) is sought in the form of a sum

\[
\mathcal{R}_2(\xi) = -\frac{\xi_1}{\pi h_1(0)} \chi_1(\xi_1) + \frac{\xi_2}{\pi h_2(0)} \chi_2(\xi_2) + \hat{\mathcal{R}}_2(\xi),
\]

where \(\hat{\mathcal{R}}_2 \in \mathcal{H}\) and \(\tilde{\mathcal{R}}_2\) is the solution to the problem (4.12) with right-hand sides

\[
F_2^{\xi}(\xi) = \begin{cases} 
-\frac{1}{\pi h_1(0)} \left( \frac{\xi_1}{\xi_1} \chi_1(\xi_1) \right)' + \chi_1'(\xi_1), & \xi \in \Xi^{(1)}, \\
-\frac{1}{\pi h_2(0)} \left( \frac{\xi_2}{\xi_2} \chi_2(\xi_2) \right)' + \chi_2'(\xi_2), & \xi \in \Xi^{(2)}, \\
0, & \xi \in \Xi^{(0)} \cup \Xi^{(3)}.
\end{cases}
\]

It is easy to verify that the solvability condition (4.14) is satisfied. Thus, by virtue of Proposition 4.1, there exist a unique solution \(\tilde{\mathcal{R}}_2 \in \mathcal{H}\) that has the asymptotics

\[
\tilde{\mathcal{R}}_2(\xi) = (1 - \delta_2) c_2^{(j)} + \mathcal{O}(\exp(-\gamma_j \xi_2)) \quad \text{as } \xi_j \to +\infty, \quad j = 1, 2, 3.
\]

Similarly, we can prove the existence of the solution \(\mathcal{R}_3\) with the asymptotics (4.19).
Obviously, that \( \mathfrak{R}_2 \) and \( \mathfrak{R}_3 \) are linearly independent and any other solution to the homogeneous problem, which has polynomial growth at infinity, can be presented as \( \epsilon_1 + \epsilon_2 \mathfrak{R}_2 + \epsilon_3 \mathfrak{R}_3. \)

\[ \square \]

**Remark 4.1.** To obtain formulas in (4.16), it is necessary to substitute the functions \( \hat{N}_n, \mathfrak{R}_2 \) and \( \hat{N}_n, \mathfrak{R}_3 \) in the second Green-Ostrogradsky formula

\[
\int_{\Xi_k} \left( \hat{N} \Delta x - \mathfrak{R} \Delta \hat{x} \right) d\xi = \int_{\sigma_{\Xi_k}} \left( \hat{N} \partial_{x_i} \mathfrak{R} - \mathfrak{R} \partial_{x_i} \hat{N} \right) d\sigma_x,
\]

respectively, and then pass to the limit as \( R \to +\infty. \) Here, \( \Xi_R = \Xi \cap \{ \xi : |\xi| < R, i = 1, 2, 3 \}. \)

### 4.2.1 Limit problem

The problem (4.9) at \( n = 0 \) is as follows:

\[
\begin{cases}
-\Delta x \mathfrak{R}_0(\xi, t) = 0, & \xi \in \Xi, \\
\partial_{x_i} \mathfrak{R}_0(\xi, t) = 0, & \xi \in \Gamma, i = 1, 2, 3, \\
\partial_{x_i} \mathfrak{R}_0(\xi, t) = 0, & \xi \in \Gamma, i = 1, 2, 3, \\
\mathfrak{R}_0(\xi, t) \to \omega_0(i)(0, t), & \xi_i \to +\infty, \ \bar{\xi}_i \in \Upsilon_i(0), i = 1, 2, 3.
\end{cases}
\]

It is easy to verify that \( \delta_0^{(2)} = \delta_0^{(3)} = 0 \) and \( \hat{N}_0 \equiv 0. \) Thus, this problem has a solution in \( \mathcal{H} \) if and only if

\[
\omega_0^{(1)}(0, t) = \omega_0^{(2)}(0, t) = \omega_0^{(3)}(0, t); \tag{4.20}
\]

in this case, \( \mathfrak{R}_0 = \hat{N}_0 = \omega_0^{(1)}(0, t). \)

In the problem (4.12) at \( n = 1, \) the solvability condition (4.14) reads as follows:

\[
\pi h^2_1(0) \frac{\partial \omega_0^{(1)}}{\partial x_1}(0, t) + \pi h^2_2(0) \frac{\partial \omega_0^{(2)}}{\partial x_2}(0, t) + \pi h^2_3(0) \frac{\partial \omega_0^{(3)}}{\partial x_3}(0, t) - \delta_{\omega_0}|\Gamma_0|_2 \xi_0(\omega_0^{(1)}(0, t)) = -d_0^*(t), \tag{4.21}
\]

where

\[
d_0^*(t) = \delta_{\omega_0} \int_{\Gamma_0} \varphi^{(0)}(\xi, t) d\sigma_x.
\]

Substituting (4.1) into the forth condition in (2.1) and neglecting terms of order of \( O(\epsilon) \), we arrive to the following boundary conditions:

\[
\omega_0^{(i)}(\xi, t) = 0, \quad i = 1, 2, 3. \tag{4.22}
\]

Thus, taking into account (4.6), (4.20), (4.21), and (4.22), we obtain for \( \omega_0^{(i)} \) the following semilinear problem:

\[
\begin{cases}
\pi h^2_1(\xi_1) \frac{\partial \omega_0^{(i)}}{\partial x_1}(\xi_1, t) - \pi \frac{\partial}{\partial x_1} \left( h^2_1(\xi_1) \frac{\partial \omega_0^{(i)}}{\partial x_1}(\xi_1, t) \right) + \pi h^2_1(\xi_1) k \left( \omega_0^{(i)}(\xi_1, t) \right) \\
+ 2\pi \delta_{\omega_0} h_1(\xi_1) k \left( \omega_0^{(i)}(\xi_1, t), x_1, t \right) = \bar{\mathfrak{R}}_0^{(i)}(\xi, t), \quad (\xi_1, t) \in I_i \times (0, T), i = 1, 2, 3,
\end{cases}
\]

where

\[
d_0^*(t) = \delta_{\omega_0} \int_{\Gamma_0} \varphi^{(0)}(\xi, t) d\sigma_x,
\]

for \( \omega_0^{(i)} \) the following semilinear problem:

\[
\begin{cases}
\pi h^2_1(\xi_1) \frac{\partial \omega_0^{(i)}}{\partial x_1}(\xi_1, t) - \pi \frac{\partial}{\partial x_1} \left( h^2_1(\xi_1) \frac{\partial \omega_0^{(i)}}{\partial x_1}(\xi_1, t) \right) + \pi h^2_1(\xi_1) k \left( \omega_0^{(i)}(\xi_1, t) \right) \\
+ 2\pi \delta_{\omega_0} h_1(\xi_1) k \left( \omega_0^{(i)}(\xi_1, t), x_1, t \right) = \bar{\mathfrak{R}}_0^{(i)}(\xi, t), \quad (\xi_1, t) \in I_i \times (0, T), i = 1, 2, 3,
\end{cases}
\]

where \( I_i := \{ x : x_1 \in (0, \xi_i), \ \bar{x}_i = (0, 0) \} \) and
\[
\hat{F}_0^{(i)}(x_i, t) := \pi h_i^2(x_i) f(x, t) \mid_{x_i=0} + \delta_{\beta,1} \int_{\partial T_{(x_i)}} \phi^{(i)}(\xi, x_i, t) \, d\xi, \quad x \in I_i. \tag{4.24}
\]

The problem (4.23) is called the limit problem for problem (2.1).

For functions
\[
\bar{\phi}(x) = \begin{cases} 
\phi^{(1)}(x_i), & x_1 \in I_1, \\
\phi^{(2)}(x_i), & x_2 \in I_2, \\
\phi^{(3)}(x_i), & x_3 \in I_3,
\end{cases}
\]
defined on the graph \( I = \overline{T_1} \cup \overline{T_2} \cup \overline{T_3} \), we introduce the Sobolev space
\[
\mathcal{H}_0 := \{ \bar{\phi} : \phi^{(i)} \in H^1(I_i), \quad \phi^{(i)}(\ell_i) = 0, \quad i = 1, 2, 3, \quad \text{and} \quad \phi^{(1)}(0) = \phi^{(2)}(0) = \phi^{(3)}(0) \}
\]
with the scalar product
\[
(\bar{\phi}, \bar{\psi})_0 := \sum_{i=1}^{3} \pi \int_{0}^{\ell_i} h_i^2(x_i) \frac{d \phi^{(i)}}{dx_i} \frac{d \psi^{(i)}}{dx_i} \, dx_i, \quad \bar{\phi}, \bar{\psi} \in \mathcal{H}_0.
\]

**Definition 4.2.** A function \( \tilde{\omega} \in L^2(0, T; \mathcal{H}_0) \), with \( \tilde{\omega}^{'} \in L^2(0, T; \mathcal{H}_0^*) \), is called a weak solution to the problem (4.23) if it satisfies the integral identity
\[
\pi \sum_{i=1}^{3} \int_{0}^{\ell_i} h_i^2(x_i) \partial_t \omega^{(i)}(x_i, t) \psi^{(i)}(x_i) \, dx_i + (\tilde{\omega}, \bar{\psi})_0 + \delta_{\alpha,0} |\Gamma_0|_2 \kappa_0 (\omega^{(1)}_0(0, t)) \psi^{(1)}(0)
\]
\[
+ \sum_{i=1}^{3} \left( \pi \int_{0}^{\ell_i} h_i^2(x_i) k(\omega^{(i)}(x_i, t)) \psi^{(i)}(x_i) \, dx_i + 2 \pi \delta_{\alpha,1} \int_{0}^{\ell_i} h_i(x_i) k_1(\omega^{(1)}(x_i, t), x_i, t) \psi^{(i)}(x_i) \, dx_i \right)
\]
\[
= d_0^*(t) \psi^{(1)}(0) + \sum_{i=1}^{3} \int_{0}^{\ell_i} F_0^{(i)}(x_i, t) \psi^{(i)}(x_i) \, dx_i \tag{4.25}
\]
for any function \( \bar{\psi} \in \mathcal{H}_0 \) and ae \( t \in (0, T) \), and \( \tilde{\omega}|_{t=0} = 0 \).

Similarly as was done in Section 3, the integral identity (4.25) can be rewritten as follows:
\[
\langle \partial_t \tilde{\omega}, \bar{\psi} \rangle_0 + \langle A_0(t) \tilde{\omega}, \bar{\psi} \rangle_0 = \langle F_0(t), \bar{\psi} \rangle_0,
\]
for all \( \bar{\psi} \in \mathcal{H}_0 \) and ae \( t \in (0, T) \), and \( \tilde{\omega}|_{t=0} = 0 \). Here, the nonlinear operator \( A_0(t) : \mathcal{H}_0 \rightarrow \mathcal{H}_0^* \) is defined through the relation
\[
\langle A_0(t) \phi^{(i)}, \psi^{(i)} \rangle_0 = (\bar{\phi}, \bar{\psi})_0 + \delta_{\alpha,0} |\Gamma_0|_2 \kappa_0 (\omega^{(1)}_0(0, t)) \psi^{(1)}(0)
\]
\[
+ \sum_{i=1}^{3} \left( \pi \int_{0}^{\ell_i} h_i^2(x_i) k(\phi^{(i)})(x_i) \psi^{(i)}(x_i) \, dx_i + 2 \pi \delta_{\alpha,1} \int_{0}^{\ell_i} h_i(x_i) k_1(\phi^{(1)}(x_i, t), x_i, t) \psi^{(i)}(x_i) \, dx_i \right)
\]
for all \( \bar{\phi}, \bar{\psi} \in \mathcal{H}_0, \) and the linear functional \( F_0(t) \in \mathcal{H}_0^* \) is defined by
\[
\langle F_0(t), \bar{\psi} \rangle_0 = d_0^*(t) \psi^{(1)}(0) + \sum_{i=1}^{3} \int_{0}^{\ell_i} F_0^{(i)}(x_i, t) \psi^{(i)}(x_i) \, dx_i \quad \forall \, \bar{\psi} \in \mathcal{H}_0.
\]

where \( \langle \cdot, \cdot \rangle_0 \) is the duality pairing of the dual space \( \mathcal{H}_0^* \) and \( \mathcal{H}_0 \).

Using (2.2) and (2.4), we can prove that the operator \( A_0 \) is bounded, strongly monotone, hemicontinuous, and coercive. As a result, the existence and uniqueness of the weak solution to the problem (4.23) follow directly from corollary 4.1 (see Showalter\textsuperscript{41}(chapter 3)).
4.2.2 | Problem for \( \{\tilde{\omega}_1\} \)

Let us verify the solvability condition (4.14) for the problem (4.12) at \( n = 2 \). Knowing that \( N_0 \equiv \omega_0^{(1)}(0, t) \) and taking into account the third relation in problem (4.5), the equality (4.14) can be rewritten as follows:

\[
\sum_{i=1}^{3} \int_{\xi_0}^{\xi_0+1} \int_{\xi_i}^{\xi_{i+1}} \frac{\partial^2 \omega_0^{(i)}(0, t) + \frac{\partial \omega_0^{(i)}(0, t)}{\partial x_i}}{\partial x_i} \chi'_i(\xi_j) \, d\xi_i \\
- \int_{\xi_0}^{\xi_0+2} (1 - \chi_i(\xi_j)) \left( \partial_t \omega_0^{(i)}(0, t) + k(\omega_0^{(i)}(0, t)) - f(0, t) \right) \, d\xi_i \\
- \int_{\xi_0}^{\xi_0+2} (1 - \chi_i(\xi_j)) \left( \frac{\partial \omega_0^{(i)}(0, t)}{\partial x_i} \phi'_{\omega}(\xi_j, 0, t) \right) \, d\xi_i \\
- \delta_{\alpha, 1} \int_{\Gamma_0} k_0' \omega_0^{(i)}(0, t) N_1(\xi, t) \, d\sigma_\xi - \delta_{\alpha, 1} \int_{\Gamma_0} k_0 \omega_0^{(i)}(0, t) \, d\sigma_\xi + \delta_{\beta, 1} \int_{\Gamma_0} \phi(0, t) \, d\sigma_\xi \\
- \int_{\Xi^{(i)}} (\partial_t \omega_0^{(i)}(0, t) + k(\omega_0^{(i)}(0, t)) - f(0, t)) \, d\xi = 0.
\]

Whence, integrating by parts in the first 3 integrals regarding (4.6), we obtain the following relations for \( \{\omega_0^{(i)}\}_{i=1}^{3} \):

\[
\sum_{i=1}^{3} \frac{\partial \omega_0^{(i)}(0, t)}{\partial x_i} = d_i^{(i)}(t),
\]

where

\[
d_i^{(i)}(t) = -\epsilon_0 \sum_{i=1}^{3} \left( \frac{\partial^2 \omega_0^{(i)}(0, t) + \frac{\partial \omega_0^{(i)}(0, t)}{\partial x_i}}{\partial x_i} \right) - f(0, t)
\]
\[+ 2\pi \delta_{\alpha, 1} h_i(0) \left( \frac{\partial \omega_0^{(i)}(0, t) + k(\omega_0^{(i)}(0, t)) - f(0, t)}{\partial x_i} \right) \]
\[+ \delta_{\alpha, 1} \int_{\Gamma_0} k_0 \omega_0^{(i)}(0, t) \, d\sigma_\xi + \left| \Xi^{(i)} \right| \left( \partial_t \omega_0^{(i)}(0, t) + k(\omega_0^{(i)}(0, t)) - f(0, t) \right).
\]

Hence, if the functions \( \{\omega_0^{(i)}\}_{i=1}^{3} \) satisfy (4.26), then there exist a weak solution \( \tilde{N}_2 \) of the problem (4.12). According to Proposition 4.1, it can be chosen in a unique way to guarantee the asymptotics (4.15). It remains to satisfy the stabilization conditions (4.13) at \( n = 1 \). For this, we represent a weak solution of the problem (4.12) in the following form:

\[ N_1(\xi, t) = \omega_1^{(1)}(0, t) + \hat{N}_1(\xi, t), \quad \xi \in \Xi^{(i)}. \]

Taking into account the asymptotics (4.15), we have to put

\[ \omega_1^{(1)}(0, t) = \omega_1^{(2)}(0, t) - \delta_1^{(1)}(t) = \omega_1^{(3)}(0, t) - \delta_1^{(3)}(t). \]

As a result, we get the solution of the problem (4.9) with the following asymptotics:

\[ N_i(\xi, t) = \omega_i^{(i)}(0, t) + \Psi_i^{(i)}(\xi, t) + O(\exp(-\gamma_i \xi_i)) \quad \text{as} \quad \xi_i \to +\infty, \quad i = 1, 2, 3. \]

Let us denote by

\[ G_i(\xi, t) := \omega_i^{(i)}(0, t) + \Psi_i^{(i)}(\xi, t), \quad (\xi, t) \in \Xi^{(i)} \times (0, T), \quad i = 1, 2, 3. \]

Remark 4.2. Due to (4.29), the function \( N_1 - G_1 \) are exponentially decrease as \( \xi_i \to +\infty, i = 1, 2, 3 \).

Relations (4.28) and (4.26) are the first and second transmission conditions for \( \{\omega_0^{(i)}\}_{i=1}^{3} \) at \( x = 0 \), respectively. Thus, the second term of the regular asymptotics \( \tilde{\omega}_1 \) is determined from the linear problem.
\[
\begin{aligned}
\pi h_i^2(x_i) \frac{\partial^2 u}{\partial x_i^2}(x_i, t) - \pi \frac{\partial}{\partial x_i} \left( h_i^2(x_i) \frac{\partial u}{\partial x_i}(x_i, t) \right) + \pi h_i^2(x_i) k' \left( \int_0^1 \frac{\partial^2 u}{\partial x_i^2}(x_i, t) \right) \hat{u}_{0i}(x_i, t) + 2\pi \delta_{ij} h_i(x_i) \partial_i k_1 \left( \int_0^1 \frac{\partial^2 u}{\partial x_j^2}(x_j, t) \right) \hat{u}_{0i}(x_i, t),
\end{aligned}
\]

where

\[
\hat{P}_{1i}(x_i, t) = \int_{\Gamma_i(x_i)} f_{1i}(y_i, x_i, t) d\Gamma_i - 2\pi \delta_{ij} h_i(x_i) \partial_i k_1 \left( \int_0^1 \frac{\partial^2 u}{\partial x_j^2}(x_j, t) \right) \hat{u}_{0i}(x_i, t) + \delta_{ij} \int_{\partial Y_i(x_i)} \phi_{ij}(y_i, x_i, t) d\Gamma_i,
\]

\[
(\hat{P}_{1i}, t) \in (0, T), \quad i = 1, 2, 3.
\]

The values \(\delta_1^{(2)}\) and \(\delta_1^{(3)}\) are uniquely determined (see Remark 4.1) by formula

\[
\delta_1^{(i)}(t) = \int_{\Xi} \mathcal{Y}_i(\xi) \sum_{j=1}^{N} \left( \xi_j \frac{\partial^2 \omega^{(j)}}{\partial x_j^2}(0, t) \chi_j'(\xi) + 2 \frac{\partial \omega^{(j)}}{\partial x_j}(0, t) \chi_j' \right) d\xi,
\]

\[
+ \int_{\Gamma_0} \mathcal{Y}_i(\xi) \left( - \delta_{n.i.0} \kappa_0 \omega^{(i)}(0, t) + \delta_{n.i.0} \phi^{(i)}(0, t) \right) d\sigma_i, \quad i = 2, 3.
\]

With the help of the substitutions \(\phi_1^{(i)}(x_i, t) = \omega^{(i)}(x_i, t), \phi_2^{(i)}(x_2, t) = \omega^{(i)}(x_2, t) - \delta^{(i)}(t)(1 - x_2/\ell_2^i), \phi_3^{(i)}(x_3, t) = \omega^{(i)}(x_3, t) - \delta^{(i)}(t)(1 - x_3/\ell_3^i),\) we reduce the problem (4.30) to the respective linear parabolic problem in the space \(L^2(0, T; H_0)\). Thus, the existence and uniqueness of the solution to the problem (4.30) follow from the classical theory of linear parabolic problems.

\section{5 | JUSTIFICATION}

With the help of \(\bar{o}_0, \bar{o}_1, N_i,\) and smooth cut-off functions defined by formulas

\[
\chi_{r_0}^{(i)}(x_i) = \begin{cases} 1, & \text{if } x_i \geq 3 \epsilon_0, \\ 0, & \text{if } x_i \leq 2 \epsilon_0, \end{cases} \quad i = 1, 2, 3,
\]

we construct the following asymptotic approximation:

\[
U_{r}^{(1)}(x, t) = \sum_{i=1}^{3} \chi_{r_0}^{(i)} \left( \frac{X_i}{\ell_0} \right) \left( \omega^{(i)}(x_i, t) + \epsilon \omega^{(i)}(x_i, t) \right) + \left( 1 - \sum_{i=1}^{3} \chi_{r_0}^{(i)} \left( \frac{X_i}{\ell_0} \right) \right) \left( \omega^{(1)}_0(0, t) + \epsilon N_1 \left( \frac{X}{\ell}, t \right) \right),
\]

\[
x \in \Omega_r \times (0, T),
\]

where \(\ell_0\) is a fixed number from the interval \(\left( \frac{2}{3}, 1 \right).\)

**Theorem 5.1.** Let assumptions made in the statement of the problem (2.1) are satisfied. Then, the sum (5.2) is the asymptotic approximation for the solution \(u_t\) to the boundary-value problem (2.1) if, \(\exists C_0 > 0 \forall \epsilon_0 > 0 \forall \epsilon \in (0, \epsilon_0) :\)

\[
\max_{\Omega_0 \cap (0, T)} \left\| U_{r}^{(1)}(\cdot, t) - u_t(\cdot, t) \right\|_{L^2(\Omega_0)} + \left\| U_{r}^{(1)} - u_t \right\|_{L^2(0, T; L^2(\Omega_0))} \leq C_0 \mu(\epsilon),
\]

where \(\mu(\epsilon) = o(\epsilon) \) as \(\epsilon \to 0\) and

\[
\mu(\epsilon) = \left( \epsilon_{1 + \frac{2}{3}} + \sum_{i=1}^{3} \left( 1 - \delta_{n,0} \right) + \left( 1 - \delta_{n,1} \right) + \left( 1 - \delta_{n,2} \right) + \left( 1 - \delta_{n,0} \right) + \left( 1 - \delta_{n,1} \right) + \left( 1 - \delta_{n,2} \right) \right).
\]
Proof. Substituting $U^{(0)}_\varepsilon$ in the equations and the boundary conditions of problem (2.1), we find

$$
\begin{cases}
\partial_t U^{(1)}_\varepsilon - \Delta_x U^{(1)}_\varepsilon + k U^{(1)}_\varepsilon - f = \hat{R}_\varepsilon & \text{in } \Omega_x \times (0, T), \\
\partial_t U^{(1)}_\varepsilon + \varepsilon \kappa_0 U^{(1)}_\varepsilon - \varepsilon \beta e \phi^{(0)}_\varepsilon = \hat{R}^{(0)}_\varepsilon & \text{on } \Gamma^{(0)}_\varepsilon \times (0, T), \\
\partial_t U^{(1)}_\varepsilon + \varepsilon \kappa_1 U^{(1)}_\varepsilon, x_i, t) - \varepsilon \beta e \phi^{(1)}_\varepsilon = \hat{R}^{(1)}_\varepsilon & \text{on } \Gamma^{(1)}_\varepsilon \times (0, T), \quad i = 1, 2, 3, \\
U^{(1)}_\varepsilon = 0 & \text{on } \Omega^{(0)}_\varepsilon \times (0, T), \quad i = 1, 2, 3,
\end{cases}
$$

(5.5)

where

$$
\hat{R}_\varepsilon(x, t) = - \sum_{i=1}^3 \left( 2e^{-\alpha} \frac{d^2 \chi^{(0)}_\varepsilon}{dx_i^2} (x_i) \right) \left( \frac{d \omega^{(0)}_\varepsilon(x_i, t)}{dx_i} - \frac{d \omega^{(0)}_\varepsilon(0, t)}{dx_i} \right) + \varepsilon \frac{d \omega^{(1)}_\varepsilon(x_i, t)}{dt} + \varepsilon \frac{d \omega^{(1)}_\varepsilon(0, t)}{dt} + \varepsilon \frac{d \omega^{(1)}_\varepsilon(x_i, t)}{dx_i} + \varepsilon \frac{d \omega^{(1)}_\varepsilon(0, t)}{dx_i} - f(x, t)
$$

and

$$
\hat{R}^{(0)}_\varepsilon(x, t) = \varepsilon \kappa_0 (U^{(1)}_\varepsilon(x, t) - \delta_{i=0} \kappa_0 (\phi^{(1)}_\varepsilon(0, t)) - \varepsilon \beta e \phi^{(0)}_\varepsilon(x, t) + \delta \beta e \phi^{(1)}_\varepsilon(x, t),
$$

$$
\hat{R}^{(1)}_\varepsilon(x, t) = - \frac{\varepsilon \kappa_1 (x_i)}{\sqrt{1 + \varepsilon^2 |h'_(x_i)|^2}} \left( \frac{d \omega^{(0)}_\varepsilon(x_i, t)}{dx_i} \right) + \varepsilon \frac{d \omega^{(1)}_\varepsilon(x_i, t)}{dx_i} + \varepsilon \frac{d \omega^{(1)}_\varepsilon(x_i, t)}{dx_i} + \varepsilon \kappa_1 (U^{(1)}_\varepsilon(x, t), x_i, t) - \varepsilon \beta e \phi^{(1)}_\varepsilon(x, t),
$$

$i = 1, 2, 3$.

Since

$$
\omega^{(0)}_\varepsilon(x_i, 0) = \omega^{(1)}_\varepsilon(x_i, 0) = 0, \quad x_i \in I_i, \quad i = 1, 2, 3,
$$

it follows from (4.9) at $n = 1$ that $N_1|_{t=0} = 0$. As result, asymptotic approximation (5.2) leaves no residuals in the initial condition, i.e,

$$
U^{(1)}_\varepsilon|_{t=0} = 0 \quad \text{in } \Omega_x.
$$

From (5.5), we derive the following integral relation:

$$
\int_{\Omega_x} \partial_t U^{(1)}_\varepsilon v \, dx + \int_{\Omega_x} \Delta_x U^{(1)}_\varepsilon \cdot \nabla v \, dx + \int_{\Omega_x} k(U^{(1)}_\varepsilon) v \, dx + \int_{\Omega_x} \kappa_0(U^{(1)}_\varepsilon) v \, d\sigma_x + \sum_{i=1}^3 \int_{I^{(1)}_\varepsilon} \kappa_i(U^{(1)}_\varepsilon, x_i, t) v \, d\sigma_x
$$

$$
+ 3 \int_{I^{(0)}_\varepsilon} \kappa(U^{(1)}_\varepsilon, x_i, t) v \, d\sigma_x - \int_{\Omega_x} f v \, dx - \sum_{i=0}^3 \epsilon \beta e \int_{I^{(0)}_\varepsilon} \phi^{(1)}_\varepsilon v \, d\sigma_x = R_e(v),
$$

for all $v \in L^2(0, T; \mathcal{H}_e)$ and $a t \in (0, T)$. Here,

$$
R_e(v) = \int_{\Omega_x} \hat{R}_e v \, dx + \sum_{i=0}^3 \int_{I^{(0)}_\varepsilon} \hat{R}^{(0)}_e v \, d\sigma_x.
$$

From (4.5) and (4.7), we deduce that integral identities
Using (5.7) and (5.8), we rewrite
\[ R_\epsilon \left( \eta, \frac{\partial \omega_0^{(i)}}{\partial \xi_1}, \frac{\partial^2 \omega_0^{(i)}}{\partial \xi_2^2} \right) = \int_{\gamma_1} \nabla_{\eta_1} \cdot \nabla_{\eta_1} \eta \, d\tilde{\eta}_1 - \int_{\partial \gamma_1} h_{14} \frac{\partial \omega_0^{(i)}}{\partial \xi_1} \eta \, d\tilde{\eta}_1 + \int_{\gamma_1} k_1 \left( \omega_0^{(i)} \right) \eta \, d\tilde{\eta}_1 + \frac{\partial \omega_0^{(i)}}{\partial \xi_1} \eta \, d\tilde{\eta}_1 \]
and
\[ \int_{\gamma_1} \left( - \frac{\partial \omega_1^{(i)}}{\partial t} + \frac{\partial^2 \omega_1^{(i)}}{\partial \xi_2^2} \right) \eta \, d\tilde{\eta}_1 = \int_{\gamma_1} \nabla_{\eta_1} \cdot \nabla_{\eta_1} \eta \, d\tilde{\eta}_1 - \int_{\partial \gamma_1} h_{14} \frac{\partial \omega_1^{(i)}}{\partial \xi_1} \eta \, d\tilde{\eta}_1 + \int_{\gamma_1} k_1' \left( \omega_1^{(i)} \right) \eta \, d\tilde{\eta}_1 + \frac{\partial \omega_1^{(i)}}{\partial \xi_1} \eta \, d\tilde{\eta}_1 \]

(5.7)

(5.8)

hold for all \( \eta \in H^1(Y_1(x_1)) \) and for all \( (x_i, t) \in I_1 \times (0, T), \quad i = 1, 2, 3, \)

Using (5.7) and (5.8), we rewrite \( R_\epsilon \) in the form
\[ R_\epsilon (v) = \sum_{j=1}^{12} R_{\epsilon,j} (v), \]

where
\[ R_{\epsilon,1} (v) = \int_{\Omega_1} \left( k_0 (U^{(1)}_\epsilon (x, t)) - \frac{3}{\epsilon} \chi_{\Gamma_0}^{(i)} \left( \frac{x_i}{\epsilon} \right) \left( k_0 \left( \omega_0^{(i)} (x_i, t) \right) + \epsilon k'_1 \left( \omega_0^{(i)} (x_i, t) \right) \omega_1^{(i)} (x_i, t) \right) \right) v(x) \, dx, \]

\[ R_{\epsilon,2} (v) = - \int_{\Omega_1} \left( f(x, t) - \frac{3}{\epsilon} \chi_{\Gamma_0}^{(i)} \left( \frac{x_i}{\epsilon} \right) \left( f_0^{(i)} (x_i, t) + \epsilon f_1^{(i)} \left( \frac{x_i}{\epsilon} \right) \right) \right) v(x) \, dx, \]

\[ R_{\epsilon,3} (v) = \epsilon \int_{\Omega_1} \left( k_0 (U^{(1)}_\epsilon (x, t)) - \frac{3}{\epsilon} \chi_{\Gamma_0}^{(i)} \left( \frac{x_i}{\epsilon} \right) \left( k'_0 \left( \omega_0^{(i)} (x_i, t) \right) \omega_1^{(i)} (x_i, t) \right) \right) v(x) \, d\sigma_x - \int_{\Omega_1} \left( 1 - \delta_{\beta,0} \right) \phi_\epsilon^{(j)} (x, t) \, v(x) \, d\sigma_x, \]

\[ R_{\epsilon,4} (v) = \frac{3}{\epsilon^a} \sum_{i=1}^{3} \int_{\Gamma_0^{(i)}} \left( \chi_{\epsilon}^{(i)} \left( \left( 1 - \chi_{\epsilon}^{(i)} \left( \frac{x_i}{\epsilon} \right) \right) \left( \delta_{\beta,1} \chi_{\epsilon}^{(i)} (x_i, t) \right) \right) \right) v(x) \, d\sigma_x, \]

\[ R_{\epsilon,5} (v) = - \frac{3}{\epsilon^a} \sum_{i=1}^{3} \int_{\Gamma_0^{(i)}} \left( 1 - \chi_{\epsilon}^{(i)} \left( \frac{x_i}{\epsilon} \right) \left( \delta_{\beta,2} \chi_{\epsilon}^{(i)} (x_i, t) \right) \right) \phi_\epsilon^{(j)} (x, t) \, d\sigma_x, \]

\[ R_{\epsilon,6} (v) = \int_{\Omega_1} \left( 1 - \frac{3}{\epsilon} \chi_{\epsilon}^{(i)} \left( \frac{x_i}{\epsilon} \right) \left( \frac{\partial \omega_0^{(i)} (0, t)}{\partial t} + \epsilon \frac{\partial \omega_1^{(i)} (0, t)}{\partial t} \left( \frac{x_i}{\epsilon} \right) \right) \right) v(x) \, dx, \]

\[ R_{\epsilon,7} (v) = \frac{3}{\epsilon^a} \sum_{i=1}^{3} \int_{\Gamma_0^{(i)}} \left( \frac{\partial \omega_0^{(i)}}{\partial \xi_i} (x_i, t) + \epsilon \frac{\partial \omega_1^{(i)}}{\partial \xi_i} (x_i, t) \right) v(x) \, d\sigma_x, \]

\[ R_{\epsilon,8} (v) = -2 \frac{3}{\epsilon^a} \sum_{i=1}^{3} \int_{\Omega_1} \frac{d \chi_{\epsilon}^{(i)} (x_i)}{dx_i} \left( \frac{\partial \omega_0^{(i)} (x_i, t)}{\partial x_i} - \frac{\partial \omega_0^{(i)} (0, t)}{\partial t} + \epsilon \frac{\partial \omega_1^{(i)} (x_i, t)}{\partial x_i} \right) v(x) \, dx, \]

\[ R_{\epsilon,9} (v) = - \frac{3}{\epsilon^a} \sum_{i=1}^{3} \int_{\Omega_1} \frac{d^2 \chi_{\epsilon}^{(i)} (x_i)}{dx_i^2} \left( \omega_0^{(i)} (x_i, t) - \omega_0^{(i)} (0, t) - \frac{\partial \omega_0^{(i)} (0, t)}{\partial x_i} + \epsilon \omega_1^{(i)} (x_i, t) - \epsilon \omega_1^{(i)} (0, t) \right) v(x) \, dx, \]
\[ R_{\varepsilon,10}(v) = -\varepsilon^2 \sum_{i=1}^{3} \int_{\Omega} \int_{0}^{T} \chi^{(i)}_{\varepsilon,0} \left( \frac{X_{i}}{\varepsilon^a} \right) \nabla_{\varepsilon,0} u_{2}^{(i)}(\xi, t) \cdot \nabla_{\varepsilon,0} v(x, t) \ d\xi_{i} \ dx, \]

\[ R_{\varepsilon,11}(v) = -\varepsilon^3 \sum_{i=1}^{3} \int_{\Omega} \int_{0}^{T} \chi^{(i)}_{\varepsilon,0} \left( \frac{X_{i}}{\varepsilon^a} \right) \nabla_{\varepsilon,0} u_{3}^{(i)}(\xi, t) \cdot \nabla_{\varepsilon,0} v(x, t) \ d\xi_{i} \ dx, \]

\[ R_{\varepsilon,12}(v) = -\sum_{i=1}^{3} \int_{\Omega} \left( 2\varepsilon^{-a} \frac{d\chi^{(i)}_{\varepsilon,0}}{d\xi_{i}}(\xi, t) - \frac{\partial G_{1}}{\partial \xi_{i}}(\xi, t) + \varepsilon^{1-2a} \frac{d^2\chi^{(i)}_{\varepsilon,0}}{d\xi_{i}^2}(\xi, t) \right) \left( N_{1}(\xi, t) - G_{1}(\xi, t) \right) \right|_{\xi_{i} = \frac{x_i}{\varepsilon}, \xi_{i} = \varepsilon} v(x) \ dx. \]

Let us estimate the value \( R_{\varepsilon} \). Using (3.2), (3.6), and (2.4), we deduce the following estimates:

\[ |R_{\varepsilon,j}(v)| \leq \tilde{C} \left( \sum_{i=1}^{3} \varepsilon \left( \max_{x_{i} \in \xi_{i}} h_{i}^{2}(x_{i}) \right) \right)^{2} \left\| v \right\|_{L^{2}(\Omega_{j})}, \quad j = 1, 2, \tag{5.9} \]

\[ |R_{\varepsilon,3}(v)| \leq \tilde{C} \sqrt{\left\| \frac{\Gamma_{0}}{12 \varepsilon} \left( \varepsilon^6 \varepsilon_{0} + \varepsilon^{6}(1 - \delta_{\varepsilon,0}) \right) \right\| v \right\|_{H^{1}(\Omega_{3})}, \tag{5.10} \]

\[ |R_{\varepsilon,4}(v)| \leq \tilde{C} \sum_{i=1}^{3} \left( 2\varepsilon^{2} \max_{x_{i} \in \xi_{i}} h_{i}(x_{i}) \right)^{2} \varepsilon \left( \sum_{i=1}^{3} h_{i}^{2}(0) \right)^{\frac{1}{2}} \left\| v \right\|_{H^{1}(\Omega_{4})}, \tag{5.11} \]

\[ |R_{\varepsilon,5}(v)| \leq \tilde{C} \sum_{i=1}^{3} \left( 2\varepsilon^{2} \max_{x_{i} \in \xi_{i}} h_{i}(x_{i}) \right)^{2} \varepsilon \left( \sum_{i=1}^{3} h_{i}^{2}(0) \right)^{\frac{1}{2}} \left\| v \right\|_{H^{1}(\Omega_{5})}, \tag{5.12} \]

\[ |R_{\varepsilon,6}(v)| \leq \tilde{C} \sqrt{\left\| \frac{\Gamma_{0}}{12 \varepsilon} \left( \varepsilon^6 \varepsilon_{0} + \varepsilon^{6}(1 - \delta_{\varepsilon,0}) \right) \right\| v \right\|_{H^{1}(\Omega_{6})}, \tag{5.13} \]

\[ |R_{\varepsilon,7}(v)| \leq \tilde{C} \sum_{i=1}^{3} \left( 2\varepsilon^{2} \max_{x_{i} \in \xi_{i}} h_{i}(x_{i}) \right)^{2} \varepsilon \left( \sum_{i=1}^{3} h_{i}^{2}(0) \right)^{\frac{1}{2}} \left\| v \right\|_{H^{1}(\Omega_{7})}, \tag{5.14} \]

\[ |R_{\varepsilon,8}(v)| \leq \tilde{C} \sum_{i=1}^{3} \left( 2\varepsilon^{2} \max_{x_{i} \in \xi_{i}} h_{i}(x_{i}) \right)^{2} \varepsilon \left( \sum_{i=1}^{3} h_{i}^{2}(0) \right)^{\frac{1}{2}} \left\| v \right\|_{H^{1}(\Omega_{8})}, \tag{5.15} \]

\[ |R_{\varepsilon,9}(v)| \leq \tilde{C} \varepsilon^{2} \left\| v \right\|_{L^{2}(\Omega_{9})}, \quad |R_{\varepsilon,10}(v)| \leq \tilde{C} \varepsilon^{3} \left\| v \right\|_{L^{2}(\Omega_{10})}. \tag{5.16} \]

Due to the exponential decreasing of functions \( N_{1} - G_{1} \) (see Remark 4.2) and the fact that the support of the derivative of \( \chi^{(i)}_{\varepsilon,0} \) belongs to the set \( \{ x_{i} : 2\varepsilon_{0} \varepsilon^{a} \leq x_{i} \leq 3\varepsilon_{0} \varepsilon^{a} \} \), we arrive that

\[ |R_{\varepsilon,12}(v)| \leq \tilde{C} \varepsilon^{1-2} \exp \left( -\frac{2\varepsilon_{0}}{\varepsilon^{1-2}} \min_{i=1,2,3} y_{i} \right) \left\| v \right\|_{L^{2}(\Omega_{12})}. \tag{5.17} \]

Subtracting the integral identity (2.3) from (3.4) and integrating over \( t \in (0, \tau) \), where \( \tau \in (0, T) \), we obtain

\[ \int_{0}^{\tau} \left( \left\{ \partial_{t} U_{t}^{(1)} - \partial_{t} u_{t}, v \right\} + \left\{ A_{t}(t) U_{t}^{(1)} - A_{t}(t) u_{t}, v \right\} \right) dt = \int_{0}^{\tau} R_{t}(v) \ dt \quad \forall v \in L^{2}(0, T; H_{t}). \tag{5.18} \]

Now, set \( v = U_{t}^{(1)} - u_{t} \) in (5.18). Then, taking into account that \( A_{t} \) is strongly monotone and (5.9) to (5.17), we arrive to the inequality

\[ \left\| U_{t}^{(1)}(\tau, \cdot) - u_{t}(\tau, \cdot) \right\|_{L^{2}(\Omega_{t})}^{2} + \left\| U_{t}^{(1)} - u_{t} \right\|_{L^{2}(0, \tau; H_{t})}^{2} \leq C \mu(\varepsilon) \left\| U_{t}^{(1)} - u_{t} \right\|_{L^{2}(0, \tau; H^{1}(\Omega_{t}))}, \]

whence thanks to (3.1), it follows (5.3).

\[ \square \]

**Corollary 5.1.** The differences between the solution \( u_{t} \) of problem (2.1) and the sum

\[ U_{t}^{(0)}(x, t) = \sum_{i=1}^{3} \chi_{\varepsilon,0}^{(i)} \left( \frac{X_{i}}{\varepsilon^a} \right) \omega_{0}^{(i)}(x_{i}, t) + \left( 1 - \sum_{i=1}^{3} \chi_{\varepsilon,0}^{(i)} \left( \frac{X_{i}}{\varepsilon^a} \right) \right) \omega_{0}^{(1)}(0, t), \quad x \in \Omega_{t} \times (0, T) \]

admit the following asymptotic estimate:
where $\mu(\varepsilon)$ is defined in (5.4), and $a$ is a fixed number from the interval $\left(\frac{2}{3}, 1\right)$.

In each thin cylinder $\Omega^{(i)}_{\varepsilon,a} := \Omega^{(i)} \cap \{x \in \mathbb{R}^3 : x_i \in I^{(i)}_{\varepsilon,a} := (3\varepsilon_0 a^i, \varepsilon_1)\}$, $(i = 1, 2, 3)$, the following estimate holds:

$$
\max_{t \in [0,T]} \| u_\varepsilon(\cdot, t) - U^{(i)}_\varepsilon(\cdot, t) \|_{L^2(\Omega^{(i)}_{\varepsilon,a})} + \| u_\varepsilon - U^{(0)}_\varepsilon \|_{L^2(0,T;H^1(\Omega^{(i)}_{\varepsilon,a}))} \leq \tilde{C}_0 \mu(\varepsilon),
$$

(5.19)

where $\{\omega_0^{(i)}\}_{i=1}^3$ is the solution of the limit problem (4.23).

In the neighborhood $\Omega^{(i)}_{\varepsilon,a} := \Omega_\varepsilon \cap \{x : x_i < 2\varepsilon_0 a^i, \ i = 1, 2, 3\}$ of the node $\Omega^{(i)}_{\varepsilon,a}$, we get estimates

$$
\| \nabla x u_\varepsilon - \nabla x N_1 \|_{L^2(\Omega^{(i)}_{\varepsilon,a} \times (0,T))} \leq \tilde{C}_4 \mu(\varepsilon).
$$

(5.21)

Proof. Denote by $\chi^{(i)}_{\varepsilon,a}(\cdot) := \chi^{(i)}_{\varepsilon_0}(\cdot)$ (the function $\chi^{(i)}_{\varepsilon_0}$ is determined in (5.1) and

$$
\| \psi \|_{\Omega}^* := \max_{t \in [0,T]} \| \psi(\cdot, t) \|_{L^2(\Omega)} + \| \psi \|_{L^2(0,T;H^1(\Omega))}.
$$

Using the smoothness of the functions $\{\alpha^{(i)}_1\}_{i=1}^3$ and the exponential decay of the functions $\{N_i - G_i\}, i = 1, 2, 3$, at infinity, we deduce the inequality (5.19) from estimate (5.3), namely,

$$
\max_{t \in [0,T]} \| u_\varepsilon - U^{(i)}_\varepsilon \|_{\Omega^{(i)}_{\varepsilon,a}} \leq \| u_\varepsilon - U^{(i)}_\varepsilon \|_{\Omega^{(i)}_{\varepsilon,a}}^* + \varepsilon \left| \sum_{i=1}^3 \chi^{(i)}_{\varepsilon,a} \alpha^{(i)}_1 + \left(1 - \sum_{i=1}^3 \chi^{(i)}_{\varepsilon,a}\right) N_1 \right|_{\Omega^{(i)}_{\varepsilon,a}}^* + \varepsilon \| N_1 \|_{\Omega^{(i)}_{\varepsilon,a}}\n$$

$$
\leq C_1 \mu(\varepsilon) + \varepsilon \left| \sum_{i=1}^3 \chi^{(i)}_{\varepsilon,a} \alpha^{(i)}_1 + \left(1 - \sum_{i=1}^3 \chi^{(i)}_{\varepsilon,a}\right) N_1 \right|_{\Omega^{(i)}_{\varepsilon,a}}^* + \varepsilon \| N_1 \|_{\Omega^{(i)}_{\varepsilon,a}}\n$$

$$
\leq C_1 \mu(\varepsilon) + \varepsilon \left| \sum_{i=1}^3 \chi^{(i)}_{\varepsilon,a} \alpha^{(i)}_1 + \left(1 - \sum_{i=1}^3 \chi^{(i)}_{\varepsilon,a}\right) N_1 \right|_{\Omega^{(i)}_{\varepsilon,a}}^* + \varepsilon \| N_1 \|_{\Omega^{(i)}_{\varepsilon,a}}\n$$

$$
+ \varepsilon \left| \sum_{i=1}^3 \alpha^{(i)}_1 \right|_{\Omega^{(i)}_{\varepsilon,a}} + \varepsilon \left| \sum_{i=1}^3 \chi^{(i)}_{\varepsilon,a} \alpha^{(i)}_1 + \left(1 - \sum_{i=1}^3 \chi^{(i)}_{\varepsilon,a}\right) (N_1 - G_1) \right|_{\Omega^{(i)}_{\varepsilon,a}}^* + \varepsilon \| N_1 \|_{\Omega^{(i)}_{\varepsilon,a}}\n$$

$$
+ \varepsilon \max_{t \in [0,T]} \| N_1 (\cdot, t) \|_{L^2(\mathbb{R}^3)} + \varepsilon ^{\xi} \| N_1 \|_{L^2(0,T;H^1(\mathbb{R}^3))} \leq \tilde{C}_0 \mu(\varepsilon)\n$$

Also, with the help of estimate (5.3), we derive

$$
\| u_\varepsilon - \alpha^{(i)}_0 \|_{\Omega^{(i)}_{\varepsilon,a}}^* \leq \| u_\varepsilon - U^{(i)}_\varepsilon \|_{\Omega^{(i)}_{\varepsilon,a}}^* + \varepsilon \| \alpha^{(i)}_1 \|_{\Omega^{(i)}_{\varepsilon,a}}^* \leq \tilde{C}_2 \mu(\varepsilon)\n$$

whence, we get (5.20).

The energetic estimate (5.21) in a neighborhood of the node $\Omega^{(i)}_{\varepsilon,a}$ follows directly from (5.3).

Using the Cauchy-Buniakovskii-Schwarz inequality and the continuously embedding of the space $H^1(I^{(i)}_{\varepsilon,a})$ in $C([3\varepsilon_0 a^i, \varepsilon_1])$, it follows from (5.20) the following corollary.

**Corollary 5.2.** If $h_i(x_i) \equiv h_i \equiv const, i = 1, 2, 3$, then

$$
\max_{t \in [0,T]} \| (E^{(i)}_\varepsilon u_\varepsilon)(\cdot, t) - \alpha^{(i)}_0(\cdot, t) \|_{L^2(I^{(i)}_{\varepsilon,a})} + \| E^{(i)}_\varepsilon u_\varepsilon - \alpha^{(i)}_0 \|_{L^2(0,T;C([3\varepsilon_0 a^i, \varepsilon_1]))} \leq C_5 \frac{\mu(\varepsilon)}{\varepsilon},
$$

(5.22)

where $\mu(\varepsilon)$ is defined in (5.4) and

$$
(E^{(i)}_\varepsilon u_\varepsilon)(x_i, t) = \frac{1}{\pi \varepsilon^2 h_i^2} \int_{Y^{(i)}(0)} u_\varepsilon(x,t) \, d\overline{x}_i.
$$
ASYMPTOTIC APPROXIMATION IN THE CASE $\alpha_0 < 0$, $\alpha_i \geq 1$, $i \in \{1, 2, 3\}$

Taking the main terms of the approximation in the form $\omega^{(i)}_0(x_i, t) + \varepsilon^2 u^{(i)}_2 \left( \frac{x_i}{\varepsilon}, x_i, t \right)$ and substituting it into (4.23), we obtain the same differential equation on each side $I_i (i \in \{1, 2, 3\})$ of the graph as in (4.23). Similarly as in the subsection 4.2 we arrive to the relation (4.20). But now, from (3.12) it follows that $\omega^{(i)}_0 (0, t) = 0$, $i \in \{1, 2, 3\}$, and consequently also $N_0 \equiv 0$. Thus, the limit problem (4.23) splits into the following 3 independent problems:

$$
\begin{aligned}
&\left\{ \begin{align*}
\pi h_i^2(x_i) \frac{d\omega^{(i)}_0}{dt}(x_i, t) - \pi \frac{\partial}{\partial x_i} \left( h_i^2(x_i) \frac{d\omega^{(i)}_0}{dx_i}(x_i, t) \right) + \pi h^2_i(x_i) k \left( \omega^{(i)}_0(x_i, t) \right) \\
+ 2\pi \delta_{i,1} h_0(x_i) k_1 \left( \omega^{(i)}_0(x_i, t), x_i, t \right) &= \hat{F}_0^{(i)}(x_i, t), \quad (x_i, t) \in I_1 \times (0, T), \\
\omega^{(i)}_0(0, t) &= \omega^{(i)}_0(\ell_i^1, t) = 0, \quad t \in (0, T), \\
\omega^{(i)}_0(x_i, 0) &= 0, \quad x_i \in I_i,
\end{align*} \right.
\end{aligned}
$$

where $\{\hat{F}_0^{(i)}\}_{i=1}^3$ are defined in (4.24).

To avoid cumbersome formulas and calculations, we consider here more typical and realistic 2 cases: $\alpha_0 \in (-1, 0)$ and $\alpha_0 = -1$. The general case $\alpha_0 < -1$ is discussed in Section 7.

6.1 | The case $\alpha_0 \in (-1, 0)$

For the regular parts of the approximation in each thin cylinder $\Omega_{x_i}^{(i)} (i \in \{1, 2, 3\})$, we propose the following ansatz:

$$
\sum_{n \in \mathbb{N}} \left( \varepsilon^n a^{(i)}_{n}(x_i, t) + \varepsilon^{n+2} a^{(i)}_{n+2} \left( \frac{x_i}{\varepsilon}, x_i, t \right) \right),
$$

where the index set $\mathcal{I} = \{0, -\alpha_0, -2\alpha_0, 1 + \alpha_0, 1, 1 - \alpha_0\}$, and for the inner part of the approximation in a neighborhood of the node $\Omega_{x_i}^{(i)}$, the ansatz looks as follows:

$$
\varepsilon^{-\alpha_0} V_{-a_0}(t) + \varepsilon^{-2\alpha_0} V_{-2a_0}(t) + \sum_{n \in \mathcal{I}} \left( \varepsilon^n V_n(t) + \varepsilon^n N_n \left( \frac{x_i}{\varepsilon}, t \right) \right),
$$

where the index set $\mathcal{I} = \{1, 1 - \alpha_0, 2 + \alpha_0, 2\}$.

Similarly as was done in the Section 4.1, we obtain the linear inhomogeneous Neumann boundary-value problems to define coefficients $f^{(i)}_{n+2}$:

$$
\begin{aligned}
&\left\{ \begin{align*}
-\Delta_{x_i} u^{(i)}_{n+2} &= \frac{\partial \omega^{(i)}_0}{\partial t} + \frac{\partial^2 a^{(i)}_n}{\partial x_i^2} - \delta_{n,0} k(\omega^{(i)}_0) - (1 - \delta_{n,0}) \left( k'(\omega^{(i)}_0) \omega^{(i)}_n + k''(\omega^{(i)}_0) K_n \left( \{a^{(i)}_{j} \}_{j=0}^n \right) \right) \\
&\quad + \delta_{0,n} f_0^{(i)} + \delta_{1,n} f_1^{(i)} \quad \text{in} \quad Y_i(x_i), \\
\partial_{x_i} u^{(i)}_{n+2} &= h_i^2 \frac{d\omega^{(i)}_0}{dx_i} - \sum_{m \in \mathcal{I}} \delta_{m,n+1} \delta_{\alpha_0} k_1(\omega^{(i)}_0, x_i, t) \\
&\quad + (1 - \delta_{m,n}) \left( \frac{1}{2} \delta_{k} k(\omega^{(i)}_0, x_i, t) \omega^{(i)}_{n-m} + \frac{1}{2} \delta_{2} k_i(\omega^{(i)}_0, x_i, t) K_{n-m} \left( \{a^{(i)}_{j} \}_{j=0}^{n-m} \right) \right) \\
&\quad + \partial_{x_i} f^{(i)}_{n+1} \quad \text{on} \quad \partial Y_i(x_i),
\end{align*} \right.
\end{aligned}
$$

where $n, j \in \mathcal{I}, \quad i \in \{1, 2, 3\}; \partial_{\alpha_0} k_1 = \partial^2 k_1 / \partial s^2$; and functions $K_n := K_n \left( \{z_i \}_{i=1}^n \right), \quad n \in \mathcal{I}$, are defined by the formulas

$$
K_0 \equiv K_{-\alpha_0} \equiv K_{1+\alpha_0} \equiv 0, \quad K_{1} = z_{1+\alpha_0} z_{-\alpha_0}, \quad K_{-2\alpha_0} = \frac{1}{2} z_{-\alpha_0}^2, \quad K_{1-\alpha_0} = z_1 z_{-\alpha_0} + z_{1+\alpha_0} z_{-2\alpha_0}.
$$

If $n - m \notin \mathcal{I}$, then $\omega^{(i)}_{n-m} \equiv 0$, $K_{n-m} \equiv 0$. Also, if $\alpha_i \neq 2 + \alpha_0$, then $\omega^{(i)}_{1+\alpha_0} \equiv 0$.

In (6.4), the right-hand sides $f_0^{(i)}$, $f_1^{(i)}$ are defined in the Section 4.1 and the variables $(x_i, t)$ are regarded as parameters from $I^2 \times (0, T)$. Also, we should add conditions $\langle u^{(i)}_{n+2}(\cdot, x_i, t) \rangle_{Y_i(x_i)} = 0$ to these problems to guarantee the uniqueness of the solution.
By the same way, as in Section 4.2, the coefficients $N_n$, $n \in \mathfrak{S}$ of the inner part of the asymptotics (6.3) are determined from the following relations:

\[
\begin{align*}
&\begin{cases}
-\Delta \xi N_n(\xi, t) = F_n(\xi, t), & \xi \in \Xi, \\
\partial_{\xi} N_n(\xi, t) = B_n^{(0)}(\xi, t), & \xi \in \Gamma_0, \\
\partial_{\xi_i} N_n(\xi, t) = B_n^{(i)}(\xi, t), & \xi \in \Gamma_i, \quad i = 1, 2, 3,
\end{cases}

V_n(t) + N_n(\xi, t) \sim \omega_n^{(i)}(0, t) + \Psi_n^{(i)}(\xi, t), & \xi_i \to +\infty, \quad \tilde{\xi}_i \in \Gamma_i(0), & i = 1, 2, 3.
\end{align*}
\]

(6.5)

Whence, using the representation (4.11) (at $n = \in \mathfrak{S}$), we get the problem

\[
\begin{align*}
&\begin{cases}
-\Delta \xi \tilde{N}_n(\xi, t) = \tilde{F}_n(\xi, t), & \xi \in \Xi, \\
\partial_{\xi} \tilde{N}_n(\xi, t) = \tilde{B}_n^{(0)}(\xi, t), & \xi \in \Gamma_0, \\
\partial_{\xi_i} \tilde{N}_n(\xi, t) = \tilde{B}_n^{(i)}(\xi, t), & \xi \in \Gamma_i, \quad i = 1, 2, 3,
\end{cases}

V_n(t) + \tilde{N}_n(\xi, t) \sim \omega_n^{(i)}(0, t) & \xi_i \to +\infty, \quad \tilde{\xi}_i \in \Gamma_i(0), & i = 1, 2, 3.
\end{align*}
\]

(6.6)

to determine $\tilde{N}_n$. As before, we demand that $\tilde{N}_n$ satisfies the following stabilization conditions:

\[
V_n(t) + \tilde{N}_n(\xi, t) \sim \omega_n^{(i)}(0, t) \quad \text{as} \quad \xi_i \to +\infty, \quad \tilde{\xi}_i \in \Gamma_i(0), \quad i = 1, 2, 3.
\]

(6.7)

The variable $t$ in (6.5) and (6.6) is regarded as parameter from $(0, T)$. The right-hand sides in the differential equations and boundary conditions on $\{\Gamma_i\}$ of the problems (6.5), (6.6), and the fourth conditions in (6.5) are similarly obtained as in Section 4.2. As a result, we get

\[
\Psi_n^{(i)}(\xi, t) = \frac{\partial \omega_n^{(i-1)}}{\partial \xi_i}(0, t), \quad n \in \mathfrak{S} \setminus \{2\}, \quad i = 1, 2, 3;
\]

\[
\Psi_2^{(0)}(\xi, t) = \frac{\partial \omega_n^{(i-1)}}{\partial \xi_i}(0, t) + \xi_i \frac{\partial \omega_n^{(i-1)}}{\partial \xi_i}(0, t) + u_2^{(i)}(\tilde{\xi}_i, 0, t), \quad i = 1, 2, 3;
\]

\[
F_n \equiv 0, \quad n \in \mathfrak{S} \setminus \{2\}, \quad F_2(\xi, t) = -k(0) + f(0, t),
\]

\[
\tilde{F}_n(\xi, t) = \sum_{i=1}^{3} \left[ \left( \xi_i \frac{\partial \omega_n^{(i-1)}}{\partial \xi_i}(0, t) \chi_i^{\prime\prime}(\xi_i) + 2 \frac{\partial \omega_n^{(i-1)}}{\partial \xi_i}(0, t) \chi_i^{\prime}(\xi_i) \right) + \left( 1 - \sum_{i=1}^{3} \chi_i(\xi_i) \right) \left( f(0, t) - k(0) \right) \right],
\]

\[
B_1^{(i)}(\xi, t) = \tilde{B}_1^{(i)}(\xi, t) = -\kappa_1^\prime(0)V_{-\kappa_1}(t) + \delta_{\kappa_1, 0}\phi^{(0)}(\xi, t),
\]

\[
B_1^{(i)}_{-\kappa_1}(\xi, t) = \tilde{B}_1^{(i)}_{-\kappa_1}(\xi, t) = -\kappa_1^\prime(0)V_{-\kappa_1}(t) + \frac{1}{2} \kappa_1^{\prime\prime}(0)V_{-\kappa_1}(t) + \delta_{\kappa_1, -\kappa_1}\phi^{(0)}(\xi, t),
\]

\[
B_2^{(i)}_{-\kappa_1}(\xi, t) = \tilde{B}_2^{(i)}_{-\kappa_1}(\xi, t) = -\kappa_0^\prime(0)(V_{1}(t) + N_{1}(\xi, t)) + \delta_{\kappa_0, -\kappa_1}\phi^{(0)}(\xi, t),
\]

\[
B_2^{(i)}(\xi, t) = \tilde{B}_2^{(i)}(\xi, t) = -\kappa_0^\prime(0)(V_{-\kappa_0}(t) + N_{-\kappa_0}(\xi, t)) - \kappa_0^\prime(0)(V_{1}(t) + N_{1}(\xi, t))V_{-\kappa_0}(t) + \delta_{\kappa_0, -\kappa_0}\phi^{(0)}(\xi, t),
\]

\[
B_2^{(i)}_{-\kappa_0}(\xi, t) = \tilde{B}_2^{(i)}_{-\kappa_0}(\xi, t) = -\kappa_0^\prime(0)(V_{1}(t) + N_{1}(\xi, t)) - \kappa_0^\prime(0)(V_{1}(t) + N_{1}(\xi, t))V_{-\kappa_0}(t) + \delta_{\kappa_0, -\kappa_0}\phi^{(0)}(\xi, t),
\]

\[
B_2^{(i)}_{\omega_2}(\xi, t) = \tilde{B}_2^{(i)}_{\omega_2}(\xi, t) = -\kappa_0^\prime(0)(V_{1}(t) + N_{1}(\xi, t)) - \kappa_0^\prime(0)(V_{1}(t) + N_{1}(\xi, t))V_{-\kappa_0}(t) + \delta_{\kappa_0, -\kappa_0}\phi^{(0)}(\xi, t),
\]

\[
B_2^{(i)}_{\omega_2}(\xi, t) = \tilde{B}_2^{(i)}_{\omega_2}(\xi, t) = -\kappa_0^\prime(0)(V_{1}(t) + N_{1}(\xi, t)) - \kappa_0^\prime(0)(V_{1}(t) + N_{1}(\xi, t))V_{-\kappa_0}(t) + \delta_{\kappa_0, -\kappa_0}\phi^{(0)}(\xi, t),
\]

The existence of a solution of the problem (6.6) in $H$ follows from Proposition 4.1. To satisfy solvability conditions (4.14) of the problem (6.6), we choose the values $V_{n-1-\kappa_0}$, $n \in \mathfrak{S}$ as follows: $V_{1+\kappa_0} \equiv 0,$
\[ V_{-a_0}(t) = \frac{1}{\kappa_0'(0) |\Gamma_0|_2} \left( \sum_{i=1}^{3} \frac{\partial \omega^{(i)}_{-a_0}(0, t)}{\partial \xi_i} (0, t) + \delta_{\beta,0} \int_{\Gamma_0} \phi^{(0)}(\xi, t) \, d\sigma_{\xi} \right), \]

\[ V_{-2a_0}(t) = \frac{1}{\kappa_0'(0) |\Gamma_0|_2} \left( \sum_{i=1}^{3} \frac{\partial \omega^{(i)}_{-a_0}(0, t)}{\partial \xi_i} (0, t) - \frac{1}{2} \kappa_0''(0) |\Gamma_0|_2 V_{-a_0}(t) + \delta_{\beta,-a_0} \int_{\Gamma_0} \phi^{(0)}(\xi, t) \, d\sigma_{\xi} \right), \]

\[ V_1(t) = \frac{1}{\kappa_0'(0) |\Gamma_0|_2} \left( \sum_{i=1}^{3} \frac{\partial \omega^{(i)}_{1}(0, t)}{\partial \xi_i} (0, t) - \frac{1}{2} \kappa_0''(0) \int_{\Gamma_0} \rho_i(\xi, t) \, d\sigma_{\xi} + \delta_{\beta,1} \int_{\Gamma_0} \phi^{(0)}(\xi, t) \, d\sigma_{\xi} \right), \]

\[ V_{1-a_0}(t) = \frac{1}{\kappa_0'(0) |\Gamma_0|_2} \left( \sum_{i=1}^{3} \frac{\partial \omega^{(i)}_{1-a_0}(0, t)}{\partial \xi_i} (0, t) - \frac{1}{2} \kappa_0''(0) \int_{\Gamma_0} \rho_i(\xi, t) \, d\sigma_{\xi} - \gamma_{\beta}(0) V_{-a_0}(t) \int_{\Gamma_0} (V_1(t) + \rho_i(\xi, t)) \, d\sigma_{\xi} \right) \]  

\[ + \varepsilon_0 \sum_{i=1}^{3} \left( \pi \omega_i^2(0) \left( k(0) - f(0, t) \right) + 2\pi \sum_{i=1}^{3} h_i(0) \rho_i(0, 0, t) - \delta_{\beta,1} \int_{\gamma_i(0)} \phi^{(i)}(\xi, t) \, d\sigma_{\xi} \right) \]

\[ - \left| \Xi^{(0)} \right|_1 \int_{\xi(0)} (k(0) - f(0, t)) + \delta_{\beta,1} \int_{\Gamma_0} \phi^{(0)}(\xi, t) \, d\sigma_{\xi} \right). \]  

(6.8)

Again, according to Proposition 4.1, the solution can be chosen in a unique way to guarantee the asymptotics (4.15) with values \( \delta_n^{(2)} \) and \( \delta_n^{(3)} \) (at \( n = n \in \mathbb{N} \)).

It remains to satisfy the stabilization conditions (6.7) at \( n \in \{1, \ldots, 1 - a_0\} \). Taking into account the asymptotics (4.15), we have to put

\[ \omega_n^{(1)}(0, t) = V_n(t), \quad \omega_n^{(2)}(0, t) = V_n(t) + \delta_n^{(2)}(t), \quad \omega_n^{(3)}(0, t) = V_n(t) + \delta_n^{(3)}(t), \quad n \in \{1, \ldots, 1 - a_0\}. \]  

(6.9)

As a result, we get the solution of the problem (6.5) with the following asymptotics:

\[ N_n(\xi, t) = -V_n(t) + \omega_n^{(i)}(0, t) + \mathcal{O}(\exp(-\gamma_i(\xi))) \quad \text{as} \quad \xi_i \to +\infty, \quad i = 1, 2, 3. \]  

(6.10)

To complete matching the regular and inner asymptotics, we put

\[ \omega_{1+a_0}(0, t) = 0, \quad \omega_{-a_0}(0, t) = V_{-a_0}(t), \quad \omega_{-2a_0}(0, t) = V_{-2a_0}(t), \quad i = 1, 2, 3. \]  

(6.11)

With the help of the necessary and sufficient condition for the solvability of the problem (6.4) and conditions (6.9) and (6.11), we get the following problems for \( \omega_n^{(1)}, \omega_n^{(2)}, \) and \( \omega_n^{(3)}(n \in \mathbb{N} \setminus \{0\}) : \)

\[
\begin{cases}
\pi h_i^2(x_i) \frac{\partial \omega^{(i)}_n(0, t)}{\partial t} - \pi \frac{\partial}{\partial x_i} \left( h_i^2(x_i) \frac{\partial \omega^{(i)}_n(0, t)}{\partial x_i} \right) + \pi k_i^2(x_i) \omega_n^{(i)}(0, t) = F_n^{(i)}(x_i, t), \quad (x_i, t) \in I_i \times (0, T), \\
\omega_n^{(i)}(0, t) = V_n(t) + \delta_n^{(i)}(t), \quad x_i \in I_i \\
\omega_n^{(i)}(x_i, 0) = 0, \quad x_i \in I_i \\
\omega_n^{(i)}(x_i, 0) = 0, \quad x_i \in I_i 
\end{cases}
\]

(6.12)

for each \( i \in \{1, 2, 3\} \). Here, the values \( V_n \) are defined in (6.8),

\[
F_n^{(i)}(x_i, t) = -\pi h_i^2(x_i) k_i''(\omega_n^{(i)}(x_i, t)) K_n \left( \left\{ \omega_n^{(i)}(x_i, t) \right\}_{j \in \mathbb{N}} \right) - 2\pi h_i(x_i) \sum_{m \in \mathbb{N}} \delta_{m,n} \left( \delta_{m,n} k_i \left( \omega_n^{(i)}(x_i, t), x_i, t \right) \right) \]

\[
+ (1 - \delta_{m,n}) \left( \delta_{m,n} k_i \left( \omega_n^{(i)}(x_i, t), x_i, t \right) \right) \]

\[
+ \delta_{1,n} \int_{\gamma_i(\xi)} \left( \rho_i(\xi, x_i, t) \right) \, d\sigma_{\xi} + \delta_{\beta,n+1} \int_{\gamma_i(\xi)} \phi^{(i)}(\xi, x_i, t) \, d\sigma_{\xi}, \quad (x_i, t) \in I_i \times (0, T), \quad i = 1, 2, 3;
\]
the values \( \delta^{(i)}_{1+a} = \delta^{(i)}_{-\alpha_0} = \delta^{(i)}_{-2\alpha_0} = 0, \) \( i \in \{1, 2, 3\} \), \( \delta^{(1)}_{1-a_0} = 0 \), and \( \delta^{(2)}_{1}, \delta^{(3)}_{1} \) and \( \delta^{(2)}_{1-a_0}, \delta^{(3)}_{1-a_0} \) are uniquely determined (see Remark 4.1) by formulas

\[
\begin{align*}
\delta^{(i)}_{1}(t) &= \int_\Xi \mathbf{G}_i(\xi) \sum_{j=1}^{3} \left( \xi_j \frac{\partial \omega^{(j)}_{\alpha_0}(0,t) \chi''(\xi_j)}{\partial x_j}(\xi_j) + 2 \frac{\partial \omega^{(j)}_{\alpha_0}(0,t) \chi'(\xi_j)}{\partial x_j}(\xi_j) \right) d\xi \\
&- \kappa''_0(0)V_{-a_0}(t) \int_{r_0} \mathbf{G}_i(\xi) d\sigma_x + \delta_{\beta,0} \int_{r_0} \mathbf{G}_i(\xi) \varphi^{(0)}(\xi,t) d\sigma_x, \quad i = 2, 3;
\end{align*}
\]

(6.13)

\[
\begin{align*}
\delta^{(i)}_{1-a_0}(t) &= \int_\Xi \mathbf{G}_i(\xi) \sum_{j=1}^{3} \left( \xi_j \frac{\partial \omega_{-\alpha_0}(0,t) \chi''(\xi_j)}{\partial x_j}(\xi_j) + 2 \frac{\partial \omega_{-\alpha_0}(0,t) \chi'(\xi_j)}{\partial x_j}(\xi_j) \right) d\xi \\
&- \left( \kappa''_0(0)V_{-2\alpha_0}(t) + \frac{1}{2} \kappa''_0(0)V_{-\alpha_0}^2(t) \right) \int_{r_0} \mathbf{G}_i(\xi) d\sigma_x + \delta_{\beta,-\alpha_0} \int_{r_0} \mathbf{G}_i(\xi) \varphi^{(0)}(\xi,t) d\sigma_x, \quad i = 2, 3,
\end{align*}
\]

(6.14)

where \( \mathbf{G}_i \) and \( \mathbf{G}_j \) are defined in Proposition 4.2.

The determination of the terms of the asymptotics is conducted according to the following scheme:

\[
\begin{align*}
&N_2 \quad \rightarrow \quad \{\omega^{(i)}_{1+a_0}\}_{i=1}^3 \\
&N_1 \quad \rightarrow \quad V_1 \quad \rightarrow \quad \{\omega^{(i)}_{1-a_0}\}_{i=1}^3 \\
&V_{-\alpha_0} \quad \rightarrow \quad \{\omega^{(i)}_{-\alpha_0}\}_{i=1}^3 \\
&V_{-2\alpha_0} \quad \rightarrow \quad \{\omega^{(i)}_{-2\alpha_0}\}_{i=1}^3
\end{align*}
\]

Comments to the scheme. The arrows indicate the order for determining the terms of the asymptotics. We start with elements \( \{\omega^{(i)}_{0}\}_{i=1}^3 \) (see (6.1)) and move across the arrows. Here the terms \( \{\omega^{(i)}_{n}\}_{i=1}^3, n \in \mathfrak{A} \setminus \{0\} \) and \( N_n, n \in \mathfrak{A} \) are determined from the problems (6.12) and (6.5), respectively; the values \( V_{n-1-a_0}, n \in \mathfrak{A} \) are defined in (6.8). If \( \alpha_j \neq 2 + a_0 \) for some \( j \in \{1, 2, 3\} \), then \( \alpha^{(i)}_{1+a_0} \equiv 0 \) (see (6.4) and comments below) and term \( \omega^{(i)}_{1+a_0} \) does not depend on \( \omega^{(i)}_{2+a_0} \). If \( \alpha_j \neq 2 + a_0 \) for all \( j \in \{1, 2, 3\} \), then the dashed arrows disappear and we don’t need to find the elements \( \{\omega^{(i)}_{-2a_0}\}_{i=1}^3 \). The approximation does not contain the terms \( N_{2+a_0} \) and \( N_2 \), they are only needed to find the values \( V_1 \) and \( V_{1-a_0} \).

Thus, the asymptotic approximation in the case \( a_0 \in (-1, 0) \) has the following form:

\[
U_{1-\alpha_0}(x,t) = \sum_{i=1}^{3} \chi^{(i)}_{\epsilon_0} \left( \frac{x_i}{\epsilon} \right) \left( \omega^{(i)}_{0}(x_i, t) + \epsilon^{-\alpha_0} \omega^{(i)}_{0}(x_i, t) + \epsilon \omega^{(i)}_{0}(x_i, t) + \epsilon^{1-\alpha_0} \omega^{(i)}_{1-a_0}(x_i, t) \right) \\
+ \left( 1 - \sum_{i=1}^{3} \chi^{(i)}_{\epsilon_0} \left( \frac{x_i}{\epsilon} \right) \right) \left( \epsilon^{-\alpha_0} V_{-a_0}(t) + \epsilon \left( V_1(t) + N_{1-\alpha_0} \left( \frac{x_i}{\epsilon}, t \right) \right) + \epsilon^{1-\alpha_0} \left( V_{-1-a_0}(t) + N_{2-a_0} \left( \frac{x_i}{\epsilon}, t \right) \right) \right),
\]

(6.15)

\( (x,t) \in \Omega_\epsilon \times (0,T) \),

where \( \alpha \) is a fixed number from the interval \( \left(\frac{2}{3}, 1\right) \) and \( \{\chi^{(i)}_{\epsilon_0}\}_{i=1}^3 \) are defined in (5.1).

**Theorem 6.1.** Let assumptions made in the statement of the problem (2.1) and \( C_3(a) \) are satisfied. Then, the sum (6.15) is the asymptotic approximation for the solution \( u_{\epsilon} \) to the boundary-value problem (2.1) ie, \( \exists C_0 > 0 \exists \epsilon_0 > 0 \forall \epsilon \in (0, \epsilon_0) : \)
where \( \mu_0(\varepsilon) = o(\varepsilon) \) as \( \varepsilon \to 0 \) and
\[
\mu_0(\varepsilon) = \left( \varepsilon^{1 + \frac{2}{3}} + \sum_{i=1}^{3} \left( (1 - \delta_{\alpha,1} - \delta_{\alpha,1-a_0}) \varepsilon^{x_i} + (1 - \delta_{\beta,1} - \delta_{\beta,1-a_0}) \varepsilon^{y_i} \right) + \varepsilon^{2+a_0} + \varepsilon^{1-a_0} + (1 - \delta_{\beta,0} - \delta_{\beta,-a_0}) \varepsilon^{y_{\beta+1}} \right).  \tag{6.17}
\]

**Proof.** The proof of Theorem 6.1 repeats the proof of Theorem 5.1. To avoid huge amount of calculations, we note the main differences.

The residual \( \hat{R}_\varepsilon \) in the differential equation in the whole domain \( \Omega \) and the residuals \( \tilde{R}_\varepsilon^{(i)} \) in the boundary conditions on the surfaces \( \Gamma_i \) of the thin cylinders \( \Omega^{(i)}_\varepsilon \ (i \in \{1, 2, 3\}) \) can be similarly obtained and estimated.

Let us consider the residual that asymptotic approximation (6.15) leaves in the boundary condition on the node. We get
\[
\partial_r U^{(1-a_0)}_\varepsilon + \varepsilon \kappa_{0} U^{(1-a_0)}_\varepsilon = \tilde{R}_\varepsilon^{(0)} \quad \text{on} \quad \Gamma^{(0)}_\varepsilon \times (0, T),
\]
where
\[
\tilde{R}_\varepsilon^{(0)}(x, t) = \varepsilon \kappa_{0} \left( U^{(1-a_0)}_\varepsilon(x, t) - \kappa'_{0}(0)V_{-a_0}(t) - \varepsilon \kappa''_{0}(0)V_{-2a_0}(t) - \varepsilon \frac{1}{2} \kappa''_{0}(0) V_2^{a_0}(t) \right)
+ (\delta_{\beta,0} + \delta_{\beta,-a_0} - 1) \varepsilon \kappa^{(0)}(x, t), \quad (x, t) \in \Gamma^{(0)}_\varepsilon \times (0, T).
\]

Denote by
\[
N^{(1-a_0)}_\varepsilon(x, t) := \varepsilon \kappa_{0} \left( V_{-a_0}(t) + \varepsilon \left( V_1(t) + N_1 \left( \frac{x}{\varepsilon}, t \right) \right) + \varepsilon \kappa_{0} \left( V_{-a_0}(t) + N_1 \left( \frac{x}{\varepsilon}, t \right) \right) \right).
\]

Taking into account that \( U^{(1-a_0)}_\varepsilon = N^{(1-a_0)}_\varepsilon \) on \( \Gamma^{(0)}_\varepsilon \) and using Taylor formula
\[
\kappa_{0} \left( U^{(1-a_0)}_\varepsilon \right) = \kappa'_{0}(0) N^{(1-a_0)}_\varepsilon + \int_{0}^{s} \left( N^{(1-a_0)}_\varepsilon - s \right) \kappa''_{0}(s) \; ds,
\]
we rewrite \( \tilde{R}_\varepsilon^{(0)} \) in the following form:
\[
\tilde{R}_\varepsilon^{(0)}(x, t) = \varepsilon \kappa_{0} \left( V_{-a_0}(t) + \varepsilon \left( V_1(t) + N_1 \left( \frac{x}{\varepsilon}, t \right) \right) + \varepsilon \kappa_{0} \left( V_{-a_0}(t) + N_1 \left( \frac{x}{\varepsilon}, t \right) \right) \right)
+ \varepsilon \kappa_{0} \left( N^{(1-a_0)}_\varepsilon \right) - s \kappa''_{0}(s) \; ds - \varepsilon \kappa_{0} \left( V_{-2a_0}(t) - \varepsilon \frac{1}{2} \kappa''_{0}(0) V_2^{a_0}(t) \right)
+ (\delta_{\beta,0} + \delta_{\beta,-a_0} - 1) \varepsilon \kappa^{(0)}(x, t), \quad (x, t) \in \Gamma^{(0)}_\varepsilon \times (0, T).
\]

With the help of (3.6), we obtain
\[
|R_{\varepsilon,3}(v)| = \left| \int_{\Gamma^{(0)}_\varepsilon} \tilde{R}_\varepsilon^{(0)} v \; d\sigma_x \right| \leq C \sqrt{\Gamma_{0} / \varepsilon} \left( \varepsilon^{1+a_0} + \varepsilon^{a_0} + \varepsilon \kappa^{(0)}(1 - \delta_{\beta,0} - \delta_{\beta,-a_0}) \right) \varepsilon \| v \|_{L^2(\Omega^{(0)})},
\]
for all \( v \in L^2(0, T; H_\varepsilon) \) and \( x \in (0, T) \).

\[ \Box \]

### 6.2 The case \( a_0 = -1 \)

In this case, we take ansatzes (4.1) for the approximation in each thin cylinder \( \Omega^{(i)}_\varepsilon \ (i \in \{1, 2, 3\}) \) and entirely repeat all calculations from the Section 4.1. In a neighborhood of the node \( \Omega^{(0)}_\varepsilon \), we consider only 1 term
\[
\varepsilon N_1 \left( \frac{x}{\varepsilon}, t \right).
\]
Similarly as in Section 4.2, we derive the following relations for $N_1$:

\[
\begin{aligned}
-\Delta_\xi N_1(\xi, t) &= 0, \\
\partial_\xi N_1(\xi, t) + \kappa_0'(0) N_1(\xi, t) &= B_1^{(0)}(\xi, t), \\
\partial_\xi N_1(\xi, t) &= 0,
\end{aligned}
\]

(6.18)

With the help of the representation (4.11) (at $n = 1$), we obtain the problem

\[
\begin{aligned}
-\Delta_\xi \tilde{N}_1(\xi, t) &= \tilde{F}_1(\xi, t), \\
\partial_\xi \tilde{N}_1(\xi, t) + \kappa_0'(0) \tilde{N}_1(\xi, t) &= \tilde{B}_1^{(0)}(\xi, t), \\
\partial_\xi \tilde{N}_1(\xi, t) &= 0,
\end{aligned}
\]

(6.19)

\[
\tilde{N}_1(\xi, t) \sim \omega_1^{(0)}(0, t) + \Psi_1^{(0)}(\xi, t),
\]

\[
\xi_i \to +\infty, \quad \bar{\xi}_i \in \Gamma_i(0), \quad i = 1, 2, 3.
\]

to determine $\tilde{N}_1$. Here, $(\Psi_1^{(0)})_{i=1}^3$, $\tilde{F}_1$ are the same as in Section 4.2, and $\tilde{B}_1^{(0)} = \tilde{B}_1^{(0)} = \delta_{\phi, 0} \varphi^{(0)}$.

Similarly as in Section 4.2, we introduce the space $\mathcal{H}$ and prove the existence of a unique weak solution to the problem (6.19). But in contrast to the problem (6.12), we have the Robin condition on $\Gamma_0$.

**Definition 6.1.** A function $\tilde{N}_1$ from the space $\mathcal{H}$ is called a weak solution of the problem (6.19) if the identity

\[
\int_{\Xi} \nabla \tilde{N}_1 \cdot \nabla v \, d\xi + \kappa_0'(0) \int_{\Xi} \tilde{N}_1 v \, d\sigma_\xi = \int_{\Xi} \tilde{F}_1 v \, d\xi + \int_{\Gamma_0} \tilde{B}_1^{(0)} v \, d\sigma_\xi
\]

holds for all $v \in \mathcal{H}$.

**Proposition 6.1.** Let $\rho^{-1} \tilde{F}_1(\cdot, t) \in L^2(\Xi)$, $\tilde{B}_1^{(0)}(\cdot, t) \in L^2(\Gamma_0)$ for ae $t \in (0, T)$. Then, there exists a unique weak solution of problem (6.19) with the following differentiable asymptotics:

\[
\tilde{N}_1(\xi, t) = \begin{cases}
\delta_1^{(1)}(t) + O(\exp(-\gamma_1 \xi_1)) & \text{as} \quad \xi_1 \to +\infty, \\
\delta_1^{(2)}(t) + O(\exp(-\gamma_2 \xi_2)) & \text{as} \quad \xi_2 \to +\infty, \\
\delta_1^{(3)}(t) + O(\exp(-\gamma_3 \xi_3)) & \text{as} \quad \xi_3 \to +\infty,
\end{cases}
\]

(6.20)

where $\gamma_i, i = 1, 2, 3$ are positive constants.

The values $(\delta_1^{(i)})_{i=1}^3$ in (6.20) are defined as follows:

\[
\delta_1^{(i)}(t) = \int_{\Xi} \mathcal{G}_i(\xi) \tilde{F}_1(\xi, t) \, d\xi + \int_{\Gamma_0} \mathcal{G}_i(\xi) \tilde{B}_1^{(0)}(\xi, t) \, d\sigma_\xi, \quad i = 1, 2, 3,
\]

(6.21)

where $\{\mathcal{G}_i\}_{i=1}^3$ are special solutions to the corresponding homogeneous problem

\[
-\Delta_\xi \mathcal{G} = 0 \quad \text{in} \quad \Xi, \quad \partial_\xi \mathcal{G} + \kappa_0'(0) \mathcal{G} = 0 \quad \text{on} \quad \Gamma_0, \quad \partial_\xi \mathcal{G} = 0 \quad \text{on} \quad \partial\Xi \setminus \Gamma_0
\]

(6.22)

for the problem (6.19).

**Proposition 6.2.** The problem (6.22) has 3 linearly independent solutions $\{\mathcal{G}_i\}_{i=1}^3$ that do not belong to the space $\mathcal{H}$ and they have the following differentiable asymptotics:

\[
\mathcal{G}_i(\xi) = \begin{cases}
C_i^{(1)} + \delta_{i1} \frac{\xi_i}{\mathcal{N}_i(0)} + O(\exp(-\gamma_1 \xi_1)) & \text{as} \quad \xi_1 \to +\infty, \\
C_i^{(2)} + \delta_{i2} \frac{\xi_i}{\mathcal{N}_i(0)} + O(\exp(-\gamma_2 \xi_2)) & \text{as} \quad \xi_2 \to +\infty, \\
C_i^{(3)} + \delta_{i3} \frac{\xi_i}{\mathcal{N}_i(0)} + O(\exp(-\gamma_3 \xi_3)) & \text{as} \quad \xi_3 \to +\infty,
\end{cases}
\]

(6.23)

Any other solution to the homogeneous problem, which has polynomial growth at infinity, can be presented as a linear combination $c_1\mathcal{G}_1 + c_2\mathcal{G}_2 + c_3\mathcal{G}_3$.

To satisfy the forth condition in (6.19), we have to put
As a result, we get the solution of the problem (6.18) with the following asymptotics:

\[ \omega_1^{(i)}(0, t) = \delta_1^{(i)}(t), \quad i = 1, 2, 3. \]  

(6.24)

Taking into account (6.24), we derive for each \( i \in \{1, 2, 3\} \) the problem

\[
\begin{align*}
\rho h^2_t(x_i) \frac{\partial \omega_1^{(i)}}{\partial t}(x_i, t) - \pi \frac{\partial}{\partial x_i} \left( h^2_t(x_i) \frac{\partial \omega_1^{(i)}}{\partial x_i}(x_i, t) \right) + \pi h^2_t(x_i) k \left( \frac{\omega_1^{(i)}(x_i, t)}{\omega_1^{(i)}(x_i, t)} \right) \omega_1^{(i)}(x_i, t) \\
+ 2\pi \delta_{\alpha, 1} h_t(x_i) \partial_{x_i} \omega_1^{(i)}(x_i, t, x_i, t) \omega_1^{(i)}(x_i, t) = \hat{F}_1^{(i)}(x_i, t), \quad (x_i, t) \in I_t \times (0, T),
\end{align*}
\]

(6.26)

to determine uniquely \( \omega_1^{(i)} \). Here, \( \hat{F}_1^{(i)} \) are defined in (4.31), and

\[
\delta_1^{(i)}(t) = \int_{\mathbb{R}_i(\xi)} \sum_{i=1}^{3} \frac{\partial \omega_0^{(i)}}{\partial x_j}(0, t) \left( \frac{x_i}{\varepsilon^a} \right) \frac{\partial}{\partial x_j} \omega_1^{(i)}(x_i, t) + \left( 1 - \sum_{i=1}^{3} R_1^{(i)}(0, t) \left( \frac{x_i}{\varepsilon^a} \right) \right) \varepsilon N_1 \left( \frac{x_i}{\varepsilon}, t \right), \quad (x_i, t) \in \Omega_\varepsilon \times (0, T),
\]

(6.27)

where \( \{\mathcal{R}_i\}_{i=1}^{3} \) are defined in Proposition 6.2.

With the help of \( \{\omega_0, \omega_1\}_{i=1}^{3} \), \( N_1 \) (see (6.1), (6.26), and (6.18), respectively), we construct the following asymptotic approximation:

\[
U_1^{(i)}(x, t) = \frac{3}{\varepsilon^a} \frac{x_i}{\varepsilon^a} \left( \omega_1^{(i)}(x_i, t) + \varepsilon \omega_1^{(i)}(x_i, t) \right) + \left( 1 - \frac{3}{\varepsilon^a} \frac{x_i}{\varepsilon^a} \right) \varepsilon N_1 \left( \frac{x_i}{\varepsilon}, t \right), \quad (x, t) \in \Omega_\varepsilon \times (0, T),
\]

(6.28)

where \( a \) is a fixed number from the interval \( \left( \frac{2}{3}, 1 \right) \) and \( \{\xi_i^{(i)}\}_{i=1}^{3} \) are defined in (5.1).

**Theorem 6.2.** Let assumptions made in the statement of the problem (2.1) and C3(a) are satisfied. Then, the sum (6.28) is the asymptotic approximation for the solution \( u \), to the boundary-value problem (2.1) i.e., \( \exists C_0 > 0 \exists \varepsilon_0 > 0 \forall \varepsilon \in (0, \varepsilon_0) \) :

\[
\max_{t \in [0, T]} \left\| U_1^{(i)}(\cdot, t) - u_\varepsilon(\cdot, t) \right\|_{L^2(H^a)} + \left\| U_1^{(i)} - u_\varepsilon \right\|_{L^2(0, T; H^a)} \leq C_0 \mu_0(\varepsilon),
\]

(6.29)

where \( \mu_0(\varepsilon) = o(\varepsilon) \) as \( \varepsilon \to 0 \) and

\[
\mu_0(\varepsilon) = \left( \varepsilon^{1+\frac{2}{a}} + \sum_{i=1}^{3} \left( (1 - \delta_{\alpha, 1}) \varepsilon^{q_i} + (1 - \delta_{\beta, 1}) \varepsilon^{\delta} \right) + (1 - \delta_{\beta, 0}) \varepsilon^{\beta + 1} \right).
\]

(6.30)

**Proof.** The proof of Theorem 6.2 repeats the proof of Theorem 5.1. The only difference is the residual on the boundary of the node, namely,

\[
R_1^{(0)}(x, t) = \varepsilon^{-1} k_0 \left( U_1^{(i)}(x, t) - k_0(0) N_1 \left( \frac{x_i}{\varepsilon}, t \right) + (\delta_{\beta, 0} - 1) \varepsilon^{\beta} \phi_\varepsilon^{(0)}(x, t) \right), \quad (x, t) \in \Gamma_\varepsilon^{(0)} \times (0, T).
\]

We estimate the value

\[
R_{\varepsilon, 3}(\nu) = \int_{\Gamma_\varepsilon^{(0)}} R_1^{(0)} \nu \ d\sigma_x
\]

with the help of (3.6) and Taylor formula. As a result, we get

\[
|R_{\varepsilon, 3}(\nu)| \leq \tilde{C} \sqrt{\int_{\Gamma_\varepsilon^{(0)}} (\varepsilon + \varepsilon^{\beta}(1 - \delta_{\beta, 0})) |\nu|_H^2(\Omega_\varepsilon)},
\]

for all \( \nu \in L^2(0, T; H_\varepsilon) \) and \( \varepsilon \in (0, T) \).

**Remark 6.1.** As we have argued, inequalities (5.20), (5.21), and (5.22), we can prove similar inequalities in the case \( \alpha_0 \in [-1, 0) \) using (6.16) and (6.29).
7 | COMMENTS

1. At first glance, it may seem that there is no difference between the nonlinear Robin condition (1.1) in the problem (2.1) and the corresponding linear Neumann condition, since the term \( \kappa_i(u_i, x_i, t) \) is multiplied by \( \varepsilon^n(i \in \{1, 2, 3\}) \). However, this is true only if \( \alpha_i > 1 \). If \( \alpha_i = 1 \), then, the new blow-up term

\[
2\pi h_i(x_i) \kappa_i\left(\omega_0^{(i)}(x_i, t), x_i, t\right),
\]

which takes into account the curvilinearity of the thin cylinder \( \Omega_i^{(i)} \) through the function \( h_i \), appears in the differential equation of the corresponding limit problem (see (4.23) and (6.1))

What happens when \( \alpha_i < 1 \) for some \( i \in \{1, 2, 3\} \) is, to be specific, we put \( \alpha_1 < 1 \). As in the case C3(a), we additionally suppose that there is a constant \( k_\alpha \) such that \( 0 < k_\alpha \leq \partial_x \kappa_i(s, x_i, t) \) for all \( s \in \mathbb{R} \) uniformly with respect to \( x_i \in [0, \epsilon_i] \) and \( t \in [0, T] \), and \( k_i(0, x_i, t) = 0 \). Then, from the integral identity (2.3) and inequalities (2.4), (3.2), (3.1), (3.6), and (3.10), it follows

\[
\varepsilon^n \int_{\Gamma^{(i)} \times (0, T)} u_i^2 \, ds \, dt \leq C_1 \left| k(0) \varepsilon + \varepsilon^{a+1} k_0(0) \right| + \sum_{l=2}^3 \varepsilon^n \max_{[0, \epsilon_i] \times [0, T]} \left| \kappa_l(x_i, t) \right| \\
+ \left\| f \right\| L^2(\Omega_i \times (0, T)) + \varepsilon^6 \left\| \varphi_i^{(0)} \right\| L^2(\Gamma_i^{(i)} \times (0, T)) + \sum_{l=3}^3 \varepsilon^6 \frac{\varepsilon^{a+1}}{\varepsilon^n} \left\| \varphi_i^{(l)} \right\| L^2(\Gamma_i^{(i)} \times (0, T)) \right\| u_l \right\| L^2(0, T; H_i) \leq C_2 \varepsilon^2.
\]

Now, with the help of (3.3), we get

\[
\int_{\Omega_i^{(i)} \times (0, T)} u_i^2 \, dx \, dt \leq C_3 \left( \varepsilon^2 \int_{\Omega_i^{(i)} \times (0, T)} \left| \nabla_x u_i \right|^2 \, dx \, dt + \varepsilon^{1-a_i} \varepsilon^{a_i} \int_{\Gamma_i^{(i)} \times (0, T)} u_i^2 \, ds \, dt \right) \leq C_4 \varepsilon^\theta,
\]

where \( \theta := \min\{4, 3 - a_i\} \). This means that

\[
\frac{1}{\varepsilon^2} \int_{\Omega_i^{(i)} \times (0, T)} u_i^2 \, dx \, dt \leq C_4 \varepsilon^\min\{2-ai\} \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0.
\] (7.1)

Due to (7.1), we conclude that \( \omega_0^{(i)} \equiv 0 \). If \( \alpha_1 < 0 \), then, we can state that there are 2 independent problems in (6.1) \((i = 2 \text{ and } i = 3)\) to determine \( \omega_0^{(2)} \) and \( \omega_0^{(3)} \). The view of the limit problem is still unknown for \( \alpha_1 \in (0, 1) \); we guess that the limit problem will depend on the parameter \( \alpha_0 \) in addition.

2. We can construct the asymptotic approximation for the solution also in the case \( \alpha_0 < -1 \), but with extra assumptions. Namely, if \( \alpha_0 \in [-q, -q + 1) \), \( q \in \mathbb{N} \), \( q \geq 2 \), we assume the following more stronger condition of zero absorption:

\[
\kappa_0 \in C^{q+1}(\mathbb{R}), \quad \frac{d^{q+1} \kappa_0}{ds^{q+1}} \in L^\infty(\mathbb{R}), \quad \kappa_0(0) = \frac{d^{q+1} \kappa_0}{ds^{q+1}}(0) = \cdots = \frac{d^{q+1} \kappa_0}{ds^{q+1}}(0) = 0, \quad \exists k_\alpha > 0 \quad \forall s \in \mathbb{R} \quad : \quad \frac{d^{q+1} \kappa_0}{ds^{q+1}}(s) \geq k_\alpha; \quad (7.2)
\]

in addition, if \( \alpha_0 \neq -q \), then

\[
f \in C^{q-1}_x \left( \overline{\Omega_0} \times [0, T] \right) \cap C^q_x \left( \Omega_0^{(i)} \times [0, T] \right), \quad k \in C^{q+1}(\mathbb{R}), \quad \frac{d^{q+1} \kappa_0}{ds^{q+1}} \in L^\infty(\mathbb{R}), \quad q_1 \in \{1, \ldots, q + 1\}, \quad (7.3)
\]

uniformly with respect to \( x_i \in [0, \epsilon_i] \) and \( t \in [0, T] \) \((i \in \{1, 2, 3\})\). It should be noted here that \( \kappa_0(0) \) is positive in C3(a).

For instance, the function

\[
k(s) = \frac{\lambda s^q}{1 + vs^q}, \quad s \in \mathbb{R} \quad (\lambda, v > 0),
\]

which occurs in applications (see Pao\(^4\)), satisfies the conditions in (7.2).

**Proposition 7.1.** Under conditions (7.2)

\[
\frac{1}{\varepsilon^2} \int_{\Omega_i^{(i)} \times (0, T)} u_i^2 \, dx \, dt \leq C_4 \varepsilon^\min\{1, -\frac{2a_1}{\varepsilon} - 1\} \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0.
\] (7.4)
Proof. With the help of Taylor formula (with Lagrange form of the remainder) and (7.2), we obtain

$$|\kappa_0(s)| s \geq \frac{k}{q!} |s|^{q+1}, \quad s \in \mathbb{R}.$$  

Knowing that $\kappa_0(s)s \geq 0$ for all $s \in \mathbb{R}$ (see (2.4), we get

$$\kappa_0(s) s \geq \frac{k}{q!} |s|^{q+1}, \quad s \in \mathbb{R}. \quad (7.5)$$

Similarly as in Section 3.1, from the integral identity (2.3) and inequalities (2.4), (3.2), (3.1), (3.6), and (3.10), it follows

$$\varepsilon^2 \int_{T_2^\varepsilon(0,T)} \kappa_0(u_e) u_e \ d\sigma_x dt \leq C_1 \varepsilon^2.$$  

Thanks to (7.5) and Hölder inequality, we get

$$\int_{T_2^\varepsilon(0,T)} u_e^2 \ d\sigma_x dt \leq \left( e^2 |\Gamma_0|_2^2 T \right)^{\frac{q+1}{q+2}} \left( \int_{T_2^\varepsilon(0,T)} |u_e|^{q+1} \ d\sigma_x dt \right)^{\frac{1}{q+2}} \leq C_2 \varepsilon^{2-\frac{2q}{q+1}}.$$  

Now, with the help of (3.4), we get

$$\int_{T_2^\varepsilon(0,T)} u_e^2 \ dx \ dt \leq C_3 \left( e^2 \int_{T_2^\varepsilon(0,T)} |\nabla_x u_e|^2 \ dx \ dt + \varepsilon \int_{T_2^\varepsilon(0,T)} u_e^2 \ d\sigma_x dt \right) \leq C_4 \varepsilon^\theta,$$

where $\theta := \min \{4, 3 - \frac{2q}{q+1}\}. \quad \square$

Thus, in consequence of (7.4), we have the same $3$ independent problems (6.1) to determine $\omega_0^{(1)}, \omega_0^{(2)},$ and $\omega_0^{(3)}$. Further, we can repeat the procedure from Section 6. The conditions (7.2) are needed for the solvability of the corresponding problems to determine the terms of the inner asymptotics. To prove asymptotic estimates in the case if $a_0 \neq -q$, we use (7.3).

3. From obtained results it follows that the asymptotic behavior of the solution essentially depends on the parameter $a_0$ characterizing the intensity of processes at the boundary of the node. If $a_0 > 0$ and $\beta_0 > 0$, then the limit problem (4.23) does not feel both those processes and the node geometry. In this case, to take into account all these factors on the global level, we propose to consider a system consisting of the limit problems (4.23) and (4.30) on the graph. The coefficients $\mathbf{d}_1^*, \mathbf{\delta}_1^{(2)}$, and $\mathbf{\delta}_1^{(3)}$ in the Kirchhoff transmission conditions of the problem (4.30) pay respect to all parameters $\{a_i\}_{i=0}^3, \{\beta_i\}_{i=0}^3$, and many other features (see formulas (4.27) and (4.32). This proposition is justified with the estimate (5.3) in Theorem 5.1.

The same observation holds for the cases $a_0 \in (-1,0)$ and $a_0 = -1$ despite the limit problem is split into 3 problems (6.1) and the problems (6.12) and (6.26) are independent at first glance. In fact, the nonuniform Dirichlet conditions at the vertex $x = 0$ in the problems (6.12) indicate the dependence of these problems both on previous solutions $\omega_0^{(1)}, \omega_0^{(2)},$ and $\omega_0^{(3)}$ ($n \in \mathfrak{M}$) and on other factors through the values $V_n$ and $\mathbf{\delta}_n^{(3)}$ (see (6.8), (6.13), and (6.14) for $a_0 \in (-1,0)$; in the case $a_0 = -1$, see the problems in (6.26) and formulas in (6.27).

4. Thanks to estimates (5.20) and (5.21), we get the zero-order approximation of the gradient (flux) of the solution

$$\nabla u_e(x,t) \sim \frac{\partial a_0^{(i)}}{\partial x_i}(x_i, t) \quad \text{as} \quad \varepsilon \to 0,$$

in each curvilinear cylinders $\Omega_{v,\varepsilon}^{(i)}(i = 1, 2, 3)$ and

$$\nabla u_e(x,t) \sim \nabla_\varepsilon N_i(\xi_i, t) \quad \text{as} \quad \varepsilon \to 0,$$

in the neighborhood $\Omega_{v,\varepsilon}^{(i)}$ of the node.

The estimate (5.21) is very important if we investigate processes occurring in a neighborhood of the node. In this case, in terms of practical application, we propose to apply numerical methods not to original problems in thin star-shaped
junctions, as was done for instance in Evju et al.\textsuperscript{14} without enough accuracy (see the Section 1) but to the corresponding problem for $N_1$ (see (4.9), (6.5) at $n = 1$, and (6.18).

5. An important task of existing multiscale methods is their stability and accuracy. The proof of the error estimate between the constructed approximation and the exact solution is a general principle that has been applied to the analysis of the efficiency of a multiscale method. In our paper, we have constructed and justified the asymptotic approximation for the solution to problem (2.1) and proved the corresponding estimates for different values of the parameters $\{a_i\}$ and $\{\beta_i\}$. It should be noted here that we do not assume any orthogonality conditions for the right-hand sides in the equation and in the nonlinear Robin boundary conditions.

The results obtained in Theorems 5.1, 6.1, and 6.2 and Corollaries 5.1 and 5.2 argue that depending on $\{a_i\}$ and $\{\beta_i\}$, it is possible to replace the complex boundary-value problem (2.1) with the corresponding limit problems (4.23) and (6.1) on the graph $I$ with sufficient accuracy measured by the parameter $\varepsilon$ characterizing the thickness and the local geometrical irregularity of the thin star-shaped junction $\Omega_\varepsilon$.

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