Universal thermal conductivity in the vortex state of cuprate superconductors

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(March 22, 2022)

We formulate an effective low energy theory for the fermionic excitations in \textit{d}-wave superconductors in the presence of periodic vortex lattices. These can be modeled by an effective free Dirac Hamiltonian with renormalized velocities and possibly a small mass term. In the presence of random nonmagnetic impurities this will result in universal (i.e. field and disorder strength independent) thermal and spin conductivities with values different from those occurring in the Meissner state.

At low energies physics of the cuprate superconductors is dominated by fermionic excitations in the vicinity of the four nodes of the \textit{d}_{x^2-y^2} superconducting order parameter. Formally, these low energy excitations can be described as four species of relativistic massless Dirac fermions. Perhaps the most spectacular manifestation of these Dirac-like excitations is the appearance of universal conductivities at low temperatures. This phenomenon was predicted first by Lee \cite{1} for electrical conductivity and later extended to spin and thermal conductivities \cite{2}. “Universal” in this context means that as a result of the linear dispersion of Dirac quasiparticles conductivities become independent of the scattering rate below certain temperature scale (which itself is non-universal and set by the scattering rate). Measurement of these universal conductivities thus provides information about the intrinsic properties of the underlying clean system. Thermal conductivity has a special significance since its universal value is unaffected by vertex and Fermi liquid corrections \cite{3}. Experimental measurements of thermal conductivity at sub-Kelvin temperatures confirmed the existence of the universal regime in samples of YBa\textsubscript{2}Cu\textsubscript{3}O\textsubscript{6.9} (YBCO) \cite{4} and Bi\textsubscript{2}Sr\textsubscript{2}CaCu\textsubscript{2}O\textsubscript{8} \cite{5} and provided values of the Dirac anisotropy ratio \(\alpha_D = v_F/\Delta\) in agreement with other probes, such as the angle-resolved photoemission. To date, no observation of universal conductivity in charge or spin channel has been reported.

In the present paper we argue that, under certain conditions, universal conductivities may also appear in the vortex state by an effective free Dirac Hamiltonian with renormalized parameters. Since the \(T \to 0\) behavior is entirely determined by this low energy Hamiltonian, computation of conductivities follows essentially the same path as in the Meissner state \cite{6,7} and yields the aforementioned universal behavior.

The above sketch captures the essential physics of our result and would be exact within the linearized model. The reality is somewhat more complicated. Calculations beyond the linearized model \cite{8-10,11-13} indicate that internal nodal scattering and nonlinear terms in the full Hamiltonian can modify the band structures discussed above by displacing or creating additional Dirac nodes or producing small gaps at the Dirac point. Such “massive” Dirac fermions have been argued to give rise to quantized thermal Hall conductivity \(\kappa_{xy}\) \cite{12-15}. We demonstrate below that, surprisingly, even such massive Dirac fermions give rise to universal \textit{longitudinal} conductivities, provided that the scattering is in the unitary limit.

Thermal conductivity in the vortex state of a \textit{d}-wave superconductor has been addressed previously by number of authors \cite{14-17}. In these works the effect of vortices has been treated within semiclassical “Volovik” approximation \cite{18} which tends to capture the essential physics in many situations but whose domain of validity remains under debate \cite{19}. Here, for the first time, we present a fully quantum treatment of the longitudinal thermal and spin conductivities in the vortex lattice. Such quantum treatment is essential when addressing the behavior of very clean systems in the \(T \to 0\) limit.

Our starting point is the Bogoliubov-deGennes (BdG) Hamiltonian linearized near a single node of the \textit{d}-wave order parameter,

\[
\mathcal{H} = v_F(\hat{p}_x + a_x)\sigma_3 + v_\Delta(\hat{p}_y + a_y)\sigma_1 + v_Fv_{sz} + U(r)\sigma_3. \tag{1}
\]

Here \(\sigma_i\) are the Pauli matrices, \(\hat{p} = -\nabla\), and we have already performed the singular gauge transformation \cite{9} to “unwind” the phase of the superconducting order parameter \(\Delta = \Delta_0 e^{-i\varphi}\). Dirac fermions are coupled to the “Berry” gauge field \(a = \frac{1}{2}(\nabla \varphi_A - \nabla \varphi_B)\), and to the “Doppler” gauge field \(v_\Delta = \frac{1}{2}(\nabla \varphi_A + \nabla \varphi_B - 2eA/c)\) which formally enter Hamiltonian (1) as vector and scalar potentials respectively. In a static vortex lattice these
low energy density of states, with \( \phi \) of the vortex lattice, and satisfy results. For the energy we obtain points is established [7,8,12], it is straightforward to effect of the vortex lattice is to renormalize the Dirac spectrum remains Dirac-like at the lowest energies; the main standard band structure techniques [7,8,11]. As illustrated turrems Hamiltonian (1) can be diagonalized using stan-

\[ \chi^0_k \] is Hamiltonian (4) can be diagonalized using standard band structure techniques [2,4]. As illustrated in Fig. 1, for inversion symmetric vortex lattice, spectrum remains Dirac-like at the lowest energies; the main effect of the vortex lattice is to renormalize the Dirac velocities \( v_F \) and \( v_\Delta \). Once the presence of the nodal points is established [2,4], it is straightforward to understand the linear dispersion. The wavefunctions can be written in the Bloch form, \( \Psi_k(r) = e^{ik\cdot r}\chi_k(r) \), where \( \chi_k(r) \) is a two component spinor periodic on the unit cell. \( \chi_k(r) \) is an eigenstate of the Bloch Hamiltonian \( \hat{H}(k) = e^{-ik\cdot r}\hat{H}_0 e^{ik\cdot r} \), where \( \hat{H}_0 \) is Hamiltonian (4) with the disorder term set to zero. At zero energy there is a doublet of degenerate states, \( |\chi_0^\pm\rangle \), satisfying \( \hat{H}(0)|\chi_0^\pm\rangle = 0 \). These zero energy states provide a convenient basis for a perturbative expansion in the vicinity of the node: we may write

\[ \hat{H}(k) = \hat{H}(0) + (v_F k_x \sigma_3 + v_\Delta k_y \sigma_1), \]

and treat the second term as a small perturbation to \( \hat{H}(0) \) [2]. We find that, near the nodal point, a simple first order degenerate perturbation theory yields very accurate results. For the energy we obtain

\[ E_k^{(1)} = \pm \sqrt{(v_F k_x)^2 + (v_\Delta k_y)^2}, \]

where \( \tilde{v}_F \) and \( \tilde{v}_\Delta \) are renormalized Dirac velocities to be specified shortly. Higher order terms produce \( O(k^2) \) corrections to the energy and are unimportant at small \( k \), in agreement with Fig. 1.

The above considerations imply that very accurate description of the low energy excitations of the system in the absence of disorder can be achieved by projecting its Hamiltonian (4) onto the subspace spanned by its two zero energy eigenstates. We now include the effects of disorder by projecting the full Hamiltonian (4) onto this subspace. Formally this is accomplished by introducing a projector \( \mathcal{P} = \sum_{k,\nu=\pm} |k\nu\rangle \langle k\nu| \) with \( |k,\pm\rangle = e^{ik\cdot r}|\chi_0^\pm\rangle \). The effective low energy Hamiltonian \( \hat{H}_{\text{eff}} \) reads

\[ \hat{H}_{\text{eff}}^\mu = \delta_{kk'} [v_F k_x (n_1 \cdot \tau) + v_\Delta k_y (n_2 \cdot \tau)] + U_{kk'} \cdot \tau, \]

with \( U_{\mu\nu}(G) = \langle \chi_0^\mu | \sigma_{\mu} e^{-iG\cdot \hat{r}} |\chi_0^\nu\rangle \). Summation over the reciprocal lattice vectors \( G \) arises because of the periodicity of \( \chi_0^\mu(k) \) and will in general complicate disorder averaging. However, numerical evaluation of \( U_{\mu\nu}(G) \) indicates that the sum in Eq. (4) is dominated by the uniform \( (G = 0) \) term. In the following we therefore drop all but this uniform term, i.e. \( U_{\mu\nu}(G) \rightarrow \delta_{G=0} n_1 \cdot \tau \). This approximation allows for simple disorder averaging.

For vortex lattices with inversion symmetry it is easy to show [3] that \( n_1 \cdot n_2 = 0 \). In such a case, in the absence of disorder, \( \hat{H}_{\text{eff}} \) has spectrum (4) with \( \tilde{v}_F = v_F |n_1| \) and \( \tilde{v}_\Delta = v_\Delta |n_2| \). It is then also possible to choose as a basis such linear combination of the degenerate states \( |\chi_0^\pm\rangle \) that \( v_F (n_1 \cdot \tau) = \tilde{v}_F \tau_3 \) and \( v_\Delta (n_2 \cdot \tau) = \tilde{v}_\Delta \tau_1 \) and write the effective low energy Hamiltonian as

\[ \hat{H}_{\text{eff}}^\mu = \delta_{kk'} [\tilde{v}_F k_x \tau_3 + \tilde{v}_\Delta k_y \tau_1 + m_D \tau_2] + U_{kk'} \tau_3. \]

Here \( U_{kk'} = u_0 |n_1| \sum_{\alpha} e^{-\alpha |k-k'|} \) and we added by hand the \( m_D \) “mass” term to model the small gap which according to Refs. [10,12] can open up at the Dirac point as a result of internodal scattering or nonlinear terms neglected in (4). In the absence of disorder the spectrum of Hamiltonian (5) is \( E_k = \pm \sqrt{(\tilde{v}_F k_x)^2 + (\tilde{v}_\Delta k_y)^2 + m_D^2} \).

The effective Hamiltonian (5) is valid at energies \( E \ll \omega_H = \pi v_F v_\Delta (H/2\Phi_0) \) where \( \Phi_0 = hc/2e \) is the flux quantum. Taking the YBCO values \( v_F = 2.5 \times 10^5 \text{m/s} \) and \( \alpha_D = 14 \) we find \( \omega_H = 25K \sqrt{H/1T} \). Hamiltonian (5) should be valid at sub-Kelvin temperatures relevant to the low-T heat conduction experiments [2,4].

Around the single Dirac node the bare Matsubara Green’s function can be written as
\[
\tilde{G}_0(\mathbf{k}, i\omega_n) = \frac{i\omega_n + \tilde{v}_Fk_x\tau_3 + \tilde{v}_Dk_y\tau_1 + m_D\tau_2}{(i\omega_n)^2 - (\epsilon_k^2 + m_D^2)} \quad (6)
\]
where \(\omega_n = (2n+1)\pi T\) and \(\epsilon_k^2 = (\tilde{v}_Fk_x)^2 + (\tilde{v}_Dk_y)^2\). The impurities alter the bare Green’s function by introducing a Matsubara self-energy \(\Sigma(i\omega_n)\). In the spirit of Ref. [4] we assume that all but the scalar component of \(\Sigma(i\omega_n)\) can be neglected or absorbed into dispersion or pairing [22]. Hence the dressed Green’s function becomes
\[
\tilde{G}(\mathbf{k}, i\omega_n) = \tilde{G}_0(\mathbf{k}, i\omega_n - \Sigma(i\omega_n)). \quad (7)
\]
Retarded Green’s functions are obtained by analytically continuing \(\tilde{G}_{\text{ret}}(\mathbf{k}, \omega) = \tilde{G}(\mathbf{k}, i\omega_n \rightarrow \omega + i\delta)\) and the impurity scattering rate is defined as \(\gamma(\omega) = -\text{Im} \Sigma_{\text{ret}}(\omega)\).

Within the self-consistent t-matrix approximation the self energy is given by [22,23]
\[
\Sigma(i\omega_n) = \Gamma g_0(i\omega_n)/[\epsilon^2 - \Gamma_0(i\omega_n)], \quad (8)
\]
with \(\Gamma = n_i/\pi\rho_0, n_i\) the impurity density, \(\rho_0\) the normal state DOS; \(\epsilon = \cot \delta_0\) with \(\delta_0\) the scattering phase shift, and \(g_0(i\omega_n) = (2\pi\rho_0)^{-1}N_k \sum_k \text{Tr} \tilde{G}(\mathbf{k}, i\omega_n)\), \(N\) the number of Dirac nodes. Eq. (6) self-consistently determines the frequency dependent scattering rate \(\gamma(\omega)\). At low temperatures we are interested in \(\gamma_0 \equiv \gamma(\omega \rightarrow 0)\). In the Born limit \((c \gg 1)\) we find
\[
\gamma_0^2 \approx -m_D^2 + \Lambda^2\epsilon^{-4m^2\tilde{v}_F\tilde{v}_\Delta}/N^2 \quad (9)
\]
where \(\Lambda\) is the upper cutoff of the order of maximum superconducting gap. In the massless case this equation always has a real solution for \(\gamma_0\) implying finite DOS as \(\omega \rightarrow 0\) and universal conductivities, albeit below exponentially small temperatures [1]. When \(m_D > 0\) there is no real solution below the critical impurity concentration \(n_i^c = 2\pi^2\epsilon^2\tilde{v}_F\tilde{v}_\Delta\rho_0/[N\ln(\Lambda/m_D)]\); in the massive case weak disorder cannot fill in the gap and produce universal conductivities.

In the unitary limit \((c \rightarrow 0)\), we find
\[
\gamma_0^2 \approx \pi^2\tilde{v}_F\tilde{v}_\Delta \Gamma \left[\frac{\Lambda^2}{\gamma_0^2 + m_D^2}\right]^{-1} \quad (10)
\]
This equation has real solution for arbitrarily small impurity concentrations. In the presence of unitary scattering the system will exhibit finite DOS at \(\omega > 0\) and, as we show below, universal conductivities even when \(m_D > 0\). Physically this is a consequence of impurity bound states forming inside the gap [23]. Overlap between these states leads to formation of impurity band capable of carrying the quasiparticle current.

We proceed by computing the spin conductivity \(\sigma^s\) which is simply related to the heat conductivity by Wiedemann-Franz law [13,24,29]: \(\kappa/T = (\pi^2k_B^2/3\gamma^2)\sigma^s\). From now on we shall assume that \(\gamma_0 > 0\), which according to the discussion above is guaranteed in the unitary scattering limit. Neglecting vertex corrections [4] the static spin conductivity reads
\[
\sigma^s = \frac{N\gamma_0^2\tilde{v}_F^2 + \tilde{v}_\Delta^2}{4\pi^2\tilde{v}_F\tilde{v}_\Delta} \quad (11)
\]
where \(s = 1/2\) is the coupling constant for spin current and
\[
K^s(\omega) = \int \frac{d^2k}{(2\pi)^2} (\tilde{v}_F^2\text{Tr}[\tilde{G}_{\text{ret}}(\mathbf{k}, \omega)\tau_3\tilde{G}_{\text{ret}}(\mathbf{k}, \omega)\tau_3]
+ \tilde{v}_\Delta^2\text{Tr}[\tilde{G}_{\text{ret}}(\mathbf{k}, \omega)\tau_1\tilde{G}_{\text{ret}}(\mathbf{k}, \omega)\tau_1], \quad (12)
\]
with \(\tilde{G}_{\text{ret}}(\mathbf{k}, \omega) = \text{Im}\tilde{G}_{\text{ret}}(\mathbf{k}, \omega)\). In the limit \(T \rightarrow 0\) we can take \(-\partial n_F(\omega)/\partial \omega \rightarrow \delta(\omega)\). Substituting
\[
\tilde{G}_{\text{ret}}(\mathbf{k}, 0) = \frac{\gamma_0 - m_Di\tau_2}{\epsilon_k^2 + m_D^2 + \gamma_0^2} \quad (13)
\]
into Eq. (12) and performing the traces we find
\[
K^s(0) = \frac{\gamma_0 - m_Di\tau_2}{\epsilon_k^2 + m_D^2 + \gamma_0^2} \quad (14)
\]
The remarkable feature of this result is that the Dirac mass and the scattering rate enter only in the combination \(m_D^2 + \gamma_0^2\). This is somewhat counterintuitive since in the single particle spectrum the two tend to have opposite effects: mass depletes the low energy DOS while \(\gamma_0\) enhances it.

By power counting the integral in Eq. (14) is seen to be independent of \(m_D^2 + \gamma_0^2\), implying universal conductivity just as in the massless case. We obtain
\[
\sigma^s = \frac{N\gamma_0^2\tilde{v}_F^2 + \tilde{v}_\Delta^2}{4\pi^2\tilde{v}_F\tilde{v}_\Delta} \quad (15)
\]
and by Wiedemann-Franz law
\[
\kappa/T = \frac{N\gamma_0^2\tilde{v}_F^2 + \tilde{v}_\Delta^2}{4\pi^2\tilde{v}_F\tilde{v}_\Delta} \times \frac{\gamma_0}{\tilde{v}_\Delta} \approx \frac{N\gamma_0^2\tilde{v}_F^2 + \tilde{v}_\Delta^2}{4\pi^2\tilde{v}_F\tilde{v}_\Delta} \times \frac{\gamma_0}{\tilde{v}_\Delta} \quad (16)
\]
where \(\tilde{\alpha}_D = \tilde{v}_F/\tilde{v}_\Delta \gg 1\) is the renormalized Dirac anisotropy. Direct computation of \(\kappa\) yields identical result.

At fields \(H_{c1} \ll H \ll H_{c2}\) the only scale in the problem is the intervortex separation \(l\), implying that to leading order \(\tilde{\alpha}_D\) is field independent [24] yet distinct from \(H = 0\) value. Thermal conductivity [24] is universal: it is independent of disorder and magnetic field \(H\). Numerical calculations within linearized model [23,29] show that \(\tilde{\alpha}_D > \alpha_D\) and thus we expect the quasiparticle contribution to the thermal conductivity to be enhanced in the presence of vortices [24].

Fig. 4 summarizes our result for the field dependence of the longitudinal thermal conductivity in the \(T \rightarrow 0\) limit in the presence of periodic inversion symmetric vortex lattice. In the Meissner state \((H < H_{c1} )\) magnetic field is excluded from the sample and thermal conductivity assumes its zero field universal value [2] \(\kappa_0/T = (k_B^2/3)(e_F/e_\Delta)\). Above \(H_{c1}\) formation of the vortex lattice causes renormalization of the Dirac velocities and possiblegapping or creation of additional
The associated universal conductivity is $\kappa/T = (Nk_B^2/12)(\nu_F/\nu_D)$. It is worth noting that this type of behavior has been recently observed in the ultra-pure single crystals of YBCO.

We conclude by discussing the range of validity of our considerations. The crucial assumption is that of perfectly periodic Bravais vortex lattice with inversion symmetry. Within the linearized model this guarantees the existence of Dirac quasiparticles at low energies. This assumption should be well satisfied in the ultra-pure single crystals of YBCO. Inclusion of internodal scattering and nonlinear effects neglected in the linearized model, in general can produce small gaps at the Dirac points or introduce additional Dirac nodes in the spectrum. With the mass term present our effective Hamiltonian is sufficiently general to include any such effects. A remarkable result is that, in the presence of unitary scatterers (or sufficiently high concentration of Born impurities) even massive Dirac fermions give rise to the universal conductivities. We note in passing that this implies universal conductivities for situations in which the Dirac nodes have been gapped by other mechanisms, such as in $d_{x^2-y^2} + id_{xy}$ state. This conclusion holds within the self-consistent t-matrix approximation which corresponds to the saddle point analysis in the replicated field theory treatment of Ref. [3]. Ultimately, at the lowest energies, fluctuations around this saddle point will drive the DOS to zero. It appears, however, that this regime has not yet been accessed experimentally, except perhaps in samples of underdoped YBCO.

Linearized models indicate that for bare Dirac anisotropies $\alpha_D \gtrsim 10$ the band structure in the vortex lattice becomes essentially one dimensional, i.e. $\tilde{\alpha}_D \to \infty$. In such a case our effective Hamiltonian would have vanishing domain of validity. Calculations using the full BdG Hamiltonian however show that the renormalization of the anisotropy is much less severe.

In less pure samples we expect disorder to destroy periodic vortex lattice. It has been argued that universal field independent conductivity also emerges in the random vortex arrays at high fields $H \gg H_{c2}(T/T_c)^2$. This type of behavior was indeed observed at $T \lesssim 8K$ [20]. At sub-Kelvin temperatures, on the other hand, it was found in less pure samples that $\kappa/T \sim \sqrt{H}$ [21], consistent with semiclassical treatments which neglect quasiparticle scattering from vortices. From this perspective our result for clean samples sketched in Fig. 2 represents a nontrivial universal limit with a simple prediction that is testable by experiment. Although the size of the jump at $H_{c1}$ depends on the details its presence is a robust qualitative prediction of the present theory.

The authors are indebted to R. Hill, C. Lupien and L. Taillefer for discussing their data on ultra-pure YBCO crystals prior to publication, and are grateful to M.-R. Li, A. Melikyan and Z. Tešanović for discussions. This work was supported in part by NSF grant DMR-9415549 (O.V.) and by NSERC (M.F.)