A REMARK ON THE ALEXANDROV-FENCHEL INEQUALITY

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ABSTRACT. In this article, we give a complex-geometric proof of the Alexandrov-Fenchel inequality without using toric compactifications. The idea is to use the Legendre transform and develop the Brascamp-Lieb proof of the Prékopa theorem. New ingredients in our proof include an integration of Timorin’s mixed Hodge-Riemann bilinear relation and a mixed norm version of Hörmander’s $L^2$-estimate, which also implies a non-compact version of the Khovanskiǐ-Teissier inequality.

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1. INTRODUCTION

The classical Brunn-Minkowski inequality is an inequality on the volumes of convex bodies in $\mathbb{R}^n$. It plays an important role in many branches of mathematics, to quote from Gardner’s survey article [20]: "In a sea of mathematics, the Brunn-Minkowski inequality appears like an octopus, tentacles reaching far and wide...". A far reaching generalization of it is the Alexandrov-Fenchel inequality, which has many different proofs (see section 20.3 in [12]). In 1936, Alexandrov found a combinatorial proof and an analytic proof. The latter is a generalization of Hilbert’s 1910 proof ("Minkowskis Theorie von Volumen und Oberfläche") of the Brunn-Minkowski inequality. A simple algebraic proof (see [26] and [27]) based on the Bernstein-Kushnirenko theorem and the intersection theory on quasi-projective variety was given by Kaveh and Khovanskiǐ around 2008. For other interesting proofs and related results, see [22], [30], [18] and [13], to cite only a few.

The Brunn-Minkowski inequality also has a functional version, i.e. the Prékopa theorem [31] for convex functions, which was found by Prékopa in 1973. In 1976 [11], Brascamp and Lieb gave another proof of the Prékopa theorem, the main idea is to use the Brascamp-Lieb lemma (see Lemma 4.2) to reduce the Prékopa theorem to a weighted $L^2$-estimate of Hörmander type [23] (so called the Brascamp-Lieb inequality) for the minimal solution $u$ of

$$du = v.$$ 

In 1998, by a magic way of using Hörmander’s $\overline{\partial}-L^2$ estimate [23], Berndtsson [3] proved a complex version of the Prékopa theorem for plurisubharmonic functions. In 2005, inspired by [1], Cordero-Erausquin [15] discovered the relation between Berndtsson’s work and the Brascamp-Lieb proof. Shortly after that, a very general and useful theory (so called the complex Brunn-Minkowski theory) [6,5] behind the Brascamp-Lieb proof and Maitani-Yamaguchi’s result [29]
was established by Berndtsson. The main result in that theory is a deep and beautiful curvature formula for a certain direct image bundle, which has found many highly non-trivial applications in Kähler geometry and algebraic geometry, see \[6, 9, 8, 7, 4\] and references therein. Inspired by \[34\] and Berndtsson’s theory, in this paper we obtain a new complex-geometric proof of the Alexandrov-Fenchel inequality. The main idea is that the Brascamp-Lieb lemma (see Lemma 4.2) reduces the Alexandrov-Fenchel inequality to an $L^2$-estimate $||u|| \leq ||\theta||$ on $\mathbb{R}^n \times (\mathbb{R}^n/\mathbb{Z}^n)$ for the minimal solution of

$$
du = (d^c)^* \theta, \ d^c := i\partial - i\bar{\partial},$$

with respect to Timorin’s mixed norm (see \[53\] and \[51\]). The main advantage of this approach is that we can prove the $L^2$-estimate $||u|| \leq ||\theta||$ directly, without using the compactification theory. In fact, by Hörmander’s $L^2$-theory \[24, 17\], it is enough to construct a special complete Kähler metric on $\mathbb{R}^n \times (\mathbb{R}^n/\mathbb{Z}^n)$ (Lemma 7.1). Another advantage is that the $L^2$-estimate $||u|| \leq ||\theta||$ is true on a large class of non-compact manifolds, not only on $\mathbb{R}^n \times (\mathbb{R}^n/\mathbb{Z}^n)$. In \[21\] (p 21), Gromov suggested to study non-compact generalizations of the Khovanskiĭ-Teissier inequality.

Our approach generalizes the Khovanskiĭ-Teissier inequality to the following:

**Theorem 1.1.** Let $(X, \hat{\omega})$ be an $n$-dimensional complete Kähler manifold with finite volume. Let $\alpha_1, \cdots, \alpha_n$ be smooth $d$-closed semi-positive $(1, 1)$-forms such that $\alpha_j \leq \hat{\omega}$ on $X$ for every $1 \leq j \leq n$. Assume that $n \geq 2$. Put

$$T := \alpha_3 \wedge \cdots \wedge \alpha_n, \ T := 1, \text{ if } n = 2.$$  

Then

$$\left( \int_X \alpha_1 \wedge \alpha_2 \wedge T \right)^2 \geq \left( \int_X \alpha_1^2 \wedge T \right) \left( \int_X \alpha_2^2 \wedge T \right).$$

**Remark:** The above theorem can be seen as a special case of our main result (Theorem 3.1). Recall that a Hermitian manifold $(X, \hat{\omega})$ is said to be complete if there exists a smooth function, say

$$\rho : X \rightarrow [0, \infty),$$

such that $\rho^{-1}([0, c])$ is compact for every $c > 0$ and

$$|d\rho|_\omega(x) \leq 1, \quad \forall \ x \in X.$$  

In order to deduce the classical Alexandrov-Fenchel inequality from Theorem 1.1, we construct a special complete Kähler metric on $\mathbb{R}^n \times (\mathbb{R}^n/\mathbb{Z}^n)$ in Lemma 7.1 The whole paper is organized as follows.

**CONTENTS**

1. Introduction 1
2. Preliminaries 3
   2.1. Basic notions in convex geometry 3
   2.2. Alexandrov-Fenchel inequality 6
   2.3. Khovanskiĭ-Teissier inequality 7
3. Main theorem 9
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2. Preliminaries

2.1. Basic notions in convex geometry.

(1) A set $\Omega$ in $\mathbb{R}^n$ is said to be convex if the line segment between any two points in $\Omega$ lies in $\Omega$.

(2) We call a compact convex set, say $A$, with non-empty interior, say $A^\circ$, in $\mathbb{R}^n$ a convex body.

Let $A_0$, $A_1$ be two convex bodies in $\mathbb{R}^n$. We call

$$A_0 + A_1 := \{a_0 + a_1 \in \mathbb{R}^n : a_0 \in A_0, a_1 \in A_1\},$$

the Minkowski sum of $A_0$ and $A_1$. The Brunn-Minkowski theorem (see [20] for a nice survey) reads as follows:

**Theorem 2.1** (Brunn-Minkowski inequality). $|A_0 + A_1|^{1/n} \geq |A_0|^{1/n} + |A_1|^{1/n}$, where the absolute value of a convex body means its volume (Lebesgue measure).

**Remark:** The Brunn-Minkowski inequality is also true for compact non-convex sets with non-empty interior, see [23].

We will also need the following notion in convex geometry.
**Definition 2.1** (Legendre transform). Let $A$ be a convex body. Let $\psi$ be a smooth real-valued function on $A^o$. $\psi$ is said to be strictly convex if the Hessian matrix $(\psi_{jk})$ is positive definite at every point in $A^o$. We call

$$\psi^*(y) := \sup_{x \in A^o} x \cdot y - \psi(x), \quad x \cdot y := \sum_{j=1}^{n} x^j y^j,$$

the Legendre transform of $\psi$ (with respect to $A^o$).

**Proposition 2.2.** Let $\psi$ be a smooth strictly convex function that tends to infinity at the boundary of a convex body $A$. Then its Legendre transform $\psi^*$ is also smooth, strictly convex, moreover the gradient map of $\psi^*$

$$\nabla \psi^*(y) := (\partial \psi^*/\partial x^1, \ldots, \partial \psi^*/\partial x^n),$$

defines a diffeomorphism from $\mathbb{R}^n$ onto $A^o$.

**Proof.** It is enough to prove that the gradient map of $\psi$ defines a diffeomorphism from $A^o$ to $\mathbb{R}^n$, $\psi^*$ is smooth and $\nabla \psi^*$ is the inverse of $\nabla \psi$.

1. **\nabla \psi is a diffeomorphism from $A^o$ to $\mathbb{R}^n$.** Since $\psi$ is smooth and strictly convex, we know that $\nabla \psi$ is a local diffeomorphism.

   1. **\nabla \psi is injective:** assume that $\nabla \psi(x_1) = \nabla \psi(x_2) = y_0$, consider

   $$\psi^{y_0}(x) := \psi(x) - y_0 \cdot x,$$

   we know that $\psi^{y_0}$ is smooth, strictly convex and

   $$\nabla \psi^{y_0}(x_1) = \nabla \psi^{y_0}(x_2) = 0.$$

   Consider the restriction, say $g$, of $\psi^{y_0}$ to the line determined by $x_1$ and $x_2$, then $g$ is convex with critical points $x_1$ and $x_2$. Thus $g$ is a constant on the line segment from $x_1$ to $x_2$, moreover, strict convexity of $g$ implies $x_1 = x_2$. Thus $\nabla \psi$ is injective.

   2. **$\nabla \psi(A^o) = \mathbb{R}^n$: fix $y \in \mathbb{R}^n$, since $\psi^y$ tends to infinity at the boundary of $A$, strict convexity of $\psi$ implies that $\psi^y$ has a unique minimum point, say $x \in A^o$. Thus

   $$0 = \nabla \psi^y(x) = \nabla \psi(x) - y.$$

2. **$\psi^*$ is smooth.** Notice that

$$\psi^*(\nabla \psi(x)) = \nabla \psi(x) \cdot x - \psi(x).$$

Thus $\psi^* \circ \nabla \psi$ is a smooth, which implies that $\psi^*$ is smooth on $\mathbb{R}^n$.

3. **$\nabla \psi^*$ is the inverse of $\nabla \psi$.** Apply the differential to (2.4), we get that

$$(\nabla \psi^* \circ \nabla \psi(x)) \cdot (\psi_{jk}) = x \cdot (\psi_{jk}), \quad \forall x \in A^o.$$ 

Since $(\psi_{jk})$ is an invertible matrix function, the above formula gives $\nabla \psi^* \circ \nabla \psi = Id$. □
Remark: Put $\phi = \psi^*$. We know from the above proposition that $\nabla \phi$ is a diffeomorphism from $\mathbb{R}^n$ onto the interior of $A$, thus

\begin{equation}
|A| = \int_A dy = \int_{\mathbb{R}^n} MA(\phi) \, dx, \quad dx := dx^1 \wedge \cdots \wedge dx^n, \quad dy := dy^1 \wedge \cdots \wedge dy^n.
\end{equation}

where $MA(\phi) := \det(\phi_{jk})$ denotes the determinant of the Hessian of $\phi$. In case $A$ is the convex hull of a finite set, say $\{p_j\}_{1 \leq j \leq N} \subset \mathbb{R}^n$, one may choose $\phi(x) = \log \left( \sum_{j=1}^N e^{p_j \cdot x} \right)$.

For more results on convex function of the above type, see [36] and [21], see also [2] and [16] for the canonical choice of such $\phi$.

The following proposition is a generalization of (2.6).

**Proposition 2.3.** Let $\phi_1, \cdots, \phi_N$ be smooth strictly convex functions such that each $\nabla \phi_j$ is a diffeomorphism from $\mathbb{R}^n$ onto the interior of a convex body $A_j$. Then we have

\begin{equation}
|t_1 A_1 + \cdots + t_N A_N| = \int_{\mathbb{R}^n} MA(t_1 \phi_1 + \cdots + t_N \phi_N) \, dx, \quad t_j > 0, \quad \forall \ 1 \leq j \leq N.
\end{equation}

**Proof.** By induction on $N$, it suffices to show that

\begin{equation}
\nabla (\phi_1 + \phi_2)(\mathbb{R}^n) = A_1^\circ + A_2^\circ,
\end{equation}

where $A^\circ$ denotes the interior of $A$. Obviously we have $\nabla (\phi_1 + \phi_2)(\mathbb{R}^n) \subset A_1^\circ + A_2^\circ$. Thus it is enough to show that for every $y_1 \in A_1^\circ$ and every $y_2 \in A_2^\circ$, there exists $x_0 \in \mathbb{R}^n$ such that $\nabla (\phi_1 + \phi_2)(x_0) = y_1 + y_2$. Consider $\phi_j^b$ instead of $\phi_j$, one may assume that $y_1 = y_2 = 0$. Choose $x_1$ and $x_2$ such that

\begin{equation}
\nabla \phi_1(x_1) = \nabla \phi_2(x_2) = 0.
\end{equation}

Since $\phi_j$ is convex, we know that each $x_j$ is the minimum point of $\phi_j$. Thus strict convexity of $\phi_j$ implies that

\begin{equation}
\phi_j(x) \to \infty, \quad \text{as} \quad |x| \to \infty,
\end{equation}

i.e. each $\phi_j$ is proper. Thus $\phi_1 + \phi_2$ is also proper. Hence there exists a unique minimum point, say $x_0$, of $\phi_1 + \phi_2$. Thus $\nabla (\phi_1 + \phi_2)(x_0) = 0$. The proof is complete. $\square$

**Remark:** The above proposition implies that

\begin{equation}
p(t) := |t_1 A_1 + \cdots + t_n A_n|,
\end{equation}

is a polynomial of degree $n$. We call the coefficient of $t_1 \cdots t_n$ in the polynomial $p(t)$, i.e.

\begin{equation}
V(A_1, \cdots, A_n) := \frac{\partial^n |t_1 A_1 + \cdots + t_n A_n|}{\partial t_1 \cdots \partial t_n},
\end{equation}

the *mixed volume* of $A_1, \cdots, A_n$. 
2.2. Alexandrov-Fenchel inequality.

**Theorem 2.4 (Alexandrov-Fenchel inequality).** Let $A_1, \cdots, A_n$ be convex bodies in $\mathbb{R}^n$. Assume that $n \geq 2$. Then
\[
V(A_1, \cdots, A_n)^2 \geq V(A_1, A_1, A_3, \cdots, A_n)V(A_2, A_2, A_3, \cdots, A_n).
\]

The following lemma can be used to find equivalent forms of the Alexandrov-Fenchel inequality.

**Lemma 2.5.** Let $f$ be a positive smooth function on an open convex cone, say $K$, in $\mathbb{R}^N$. Assume that $f$ is $1$-homogeneous, i.e.
\[
f(tx) \equiv tf(x), \ \forall \ t > 0, \ x \in K.
\]

Then the following statements are equivalent:

A1: $f(x + y) \geq f(x) + f(y)$, $\forall \ x, y \in K$;

A2: $-f$ is convex;

A3: $-\log f$ is convex;

A4: For every $x', y' \in K$, $t \mapsto -\log f(tx' + (1 - t)y')$ is convex on $(0, 1)$.

**Proof.** Since $f$ is $1$-homogeneous, A1 implies
\[
f(tx + (1 - t)y) \geq tf(x) + (1 - t)f(y).
\]

Thus $A1 \Rightarrow A2$. Since
\[
(- \log f)\xi \xi = -\frac{f\xi}{f} + (f\xi)^2 f^2, \ f\xi = \sum \xi^j f_{x^j},
\]
we know $A2 \Rightarrow A3$. Since $A3 \Rightarrow A4$ is trivial, it is enough to show $A4 \Rightarrow A1$: notice that $A4$ implies
\[
f(tx' + (1 - t)y') \geq f(x')^t f(y')^{1-t}.
\]

Take
\[
x' = \frac{x}{f(x)}, \ y' = \frac{y}{f(y)}, \ t = \frac{f(x)}{f(x) + f(y)}.
\]

we get A1. The proof is complete. \qed

Apply the above lemma to the following function
\[
f(x) = V(A_x, A_x, A_3, \cdots, A_n)^{1/2}, \ A_x := x_1 A_1 + x_2 A_2,
\]
on $K := \mathbb{R}^2_+$. Notice that the square of
\[
f(x + y) \geq f(x) + f(y),
\]
is equivalent to
\[
V(A_x, A_y, A_3, \cdots, A_n)^2 \geq V(A_x, A_x, A_3, \cdots, A_n)V(A_y, A_y, A_3, \cdots, A_n).
\]

By the above lemma, we have
Proposition 2.6. The Alexandrov-Fenchel inequality is equivalent to the convexity of
\[ t \mapsto -\log V(A_t, A_3, \cdots, A_n), \quad A_t := tA_1 + (1-t)A_2, \]
on \((0, 1)\).

A generalized form of the Alexandrov-Fenchel inequality is also true.

Theorem 2.7. Let \(A_1, A_2, A_{m+1}, \cdots, A_n\), \(2 \leq m \leq n\), be convex bodies in \(\mathbb{R}^n\). Then the following function is convex on \((0, 1)\)
\[ t \mapsto -\log V(A_t, A_{m+1}, \cdots, A_n), \quad A_t := tA_1 + (1-t)A_2. \]

The above theorem is in fact equivalent to the Alexandrov-Fenchel inequality (see Theorem 7.4.5 in [32]).

2.3. Khovanski˘i-Jeissier inequality. We will use the following complex geometry interpretation of the volume function in Proposition 2.3.

Lemma 2.8. Let \(\phi_1, \cdots, \phi_N\) be smooth strictly convex functions such that each \(\nabla \phi_j\) is a diffeomorphism from \(\mathbb{R}^n\) onto the interior of a convex body \(A_j\). Let us look at \( \phi := \sum_{j=1}^{N} t_j \phi_j \), as a function on \(\mathbb{R}^n \times \mathbb{T}^n = \mathbb{C}^n / i\mathbb{Z}^n\), \( \mathbb{T} := \mathbb{R} / \mathbb{Z}, \ i := \sqrt{-1}, \)
i.e. \(\phi(x + iy) := \sum_{j=1}^{N} t_j \phi_j(x)\). Then we have
\[ \int_{\mathbb{R}^n} MA(\phi) \ dx = \int_{\mathbb{R}^n \times \mathbb{T}^n} \left( \frac{(dd^c \phi)^n}{n!} \right), \quad d^c := i\partial - i\bar{\partial}. \]

Proof. Since
\[ dd^c \phi = 2i\partial \bar{\partial} \phi = \frac{i}{2} \sum_{j,k=1}^{n} \phi_{jk} \ dz^j \wedge d\bar{z}^k, \quad z^j := x^j + iy^j, \]
where \(\phi_{jk} := \partial^2 \phi / \partial x^j \partial x^k\), we have
\[ \frac{(dd^c \phi)^n}{n!} = \det(\phi_{jk}) (dx^1 \wedge dy^1) \wedge \cdots \wedge (dx^n \wedge dy^n), \]
thus the lemma follows from the Fubini theorem and \( \int_{\mathbb{T}^n} dy = 1. \)

The above lemma implies

Lemma 2.9. Let \(\phi_1, \cdots, \phi_n\) be smooth strictly convex functions such that each \(\nabla \phi_j\) is a diffeomorphism from \(\mathbb{R}^n\) onto the interior of a convex body \(A_j\). Then we have the following mixed volume formula
\[ V(A_1, \cdots, A_n) = \int_{\mathbb{R}^n \times \mathbb{T}^n} dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n. \]
Proof. The previous lemma gives
\[ |\sum_{j=1}^{n} t_j A_j| = \int_{\mathbb{R}^n \times T^n} (dd^c \phi)^n \frac{n!}{n!}, \quad t_j > 0, \quad \forall \ 1 \leq j \leq n. \]

Notice that
\[ \frac{(dd^c \phi)^n}{n!} = \sum_{\alpha_1 + \cdots + \alpha_n = n} \frac{t_1^{\alpha_1} \cdots t_n^{\alpha_n}}{\alpha_1! \cdots \alpha_n!} (dd^c \phi_1)^{\alpha_1} \cdots (dd^c \phi_n)^{\alpha_n}, \]
and each term \((dd^c \phi_1)^{\alpha_1} \cdots (dd^c \phi_n)^{\alpha_n}\) is a positive \((n,n)\)-form, thus
\[ \int_{\mathbb{R}^n \times T^n} \frac{(dd^c \phi_1)^{\alpha_1} \cdots (dd^c \phi_n)^{\alpha_n}}{\alpha_1! \cdots \alpha_n!} < \infty. \]

Now we have
\[ |\sum_{j=1}^{n} t_j A_j| = \sum_{\alpha_1 + \cdots + \alpha_n = n} \frac{t_1^{\alpha_1} \cdots t_n^{\alpha_n}}{\alpha_1! \cdots \alpha_n!} \int_{\mathbb{R}^n \times T^n} (dd^c \phi_1)^{\alpha_1} \cdots (dd^c \phi_n)^{\alpha_n}, \]
and the lemma follows. \(\square\)

By the above lemma, we know that Theorem 2.7 is equivalent to the following:

**Theorem 2.10.** Let \(\phi_1, \phi_2, \phi_{m+1}, \cdots, \phi_n, 2 \leq m \leq n\), be smooth strictly convex functions such that each \(\nabla \phi_j\) is a diffeomorphism from \(\mathbb{R}^n\) onto the interior of a convex body \(A_j\). Then the following function is convex on \((0,1)\)
\[ t \mapsto -\log \int_{\mathbb{R}^n \times T^n} \omega^m \wedge T, \]
where
\[ \omega := tdd^c \phi_1 + (1-t)dd^c \phi_2, \quad T := dd^c \phi_{m+1} \wedge \cdots \wedge dd^c \phi_n. \]

Let us recall the following Khovanskiï-Teissier theorem.

**Theorem 2.11** (Khovanskiï-Teissier inequality). Let \(\omega_1, \cdots, \omega_n\) be Kähler forms on a compact Kähler manifold \(X\). Assume that \(n \geq 2\). Put
\[ T := \omega_3 \wedge \cdots \wedge \omega_n, \quad T := 1, \text{ if } n = 2. \]
Then
\[ \left( \int_X \omega_1 \wedge \omega_2 \wedge T \right)^2 \geq \left( \int_X \omega_1^2 \wedge T \right) \left( \int_X \omega_2^2 \wedge T \right). \]

By Lemma 2.5, we know that the Khovanskiï-Teissier inequality is equivalent to the \((m = 2)\) case convexity of
\[ t \mapsto -\log \int_X \omega_1 \wedge \omega_2 \wedge T, \quad \omega := t \omega_1 + (1-t)\omega_2, \quad T := \omega_{m+1} \wedge \cdots \wedge \omega_n. \]
Thus Theorem 2.10 can be seen as a Khovanskiï-Teissier inequality for \(\mathbb{R}^n \times T^n\).
Remark: The above equivalent description of the Khovanskiĭ-Teissier inequality was first used by Graham in his proof of the convexity of the interpolating function, see [19]. There are also other descriptions of the Khovanskiĭ-Teissier inequality. A very nice intersection theory description of its algebraic version can be found in [25] and [26]. In the Hodge theory description, the Khovanskiĭ-Teissier inequality is a direct application of the mixed generalization of the classical Hodge-Riemann bilinear relation (MHRR) for $(1,1)$-forms. MHRR for general $(p,q)$-forms on a compact Kähler manifold was first proved by Dinh-Nguyễn in [18] based on Timorin’s result [33] for the torus case, see also [13] for another approach that applies to general polarized Hodge-Lefschetz modules.

3. Main Theorem

Theorem 3.1. Let $(X, \hat{\omega})$ be an $n$-dimensional complete Kähler manifold with finite volume. Let $\alpha_1, \alpha_2, \alpha_m, \cdots, \alpha_n$, $2 \leq m \leq n$, be smooth $d$-closed semi-positive $(1,1)$-forms such that each $\alpha_j \leq \hat{\omega}$ on $X$. Then the following function is convex on $(0,1)$

$$ t \mapsto - \log \int_X \frac{\omega^m}{m!} \wedge T, \quad \omega := t\alpha_1 + (1-t)\alpha_2, $$

where $T := \alpha_{m+1} \wedge \cdots \wedge \alpha_n$, $T := 1$, if $n = m$.

By Lemma [2,5] in case $m = 2$, our main theorem is equivalent to Theorem [1,1] which is a non-compact generalization of the Khovanskiĭ-Teissier inequality.

About the proof of the main theorem. Put

$$ f(t) = - \log \int_X \frac{\omega^m}{m!} \wedge T. $$

Consider $\alpha_j + \epsilon \hat{\omega}$ instead of $\alpha_j$ and denote by $f^\epsilon$ the associated function. Then we have

$$ f = \lim_{\epsilon \to 0} f^\epsilon. $$

Thus it suffices to show that each $f^\epsilon$ is convex on $(0,1)$, i.e. one may assume that

$$ (3.1) \quad \frac{\hat{\omega}}{C} \leq \alpha_j \leq C\hat{\omega}, $$

for every $j$ in Theorem [3.1], where $C$ is a fixed positive constant. Then Theorem 3.1 follows from the following three lemmas.

Lemma 3.2. Assume that (3.1) is true. Define $G$ on $X$ such that

$$ \frac{d}{dt} \left( \frac{\omega^m}{m!} \wedge T \right) = -G \frac{\omega^m}{m!} \wedge T. $$

Then

$$ f_{tt} := \frac{d^2 f}{dt^2} = \int_X \left( G_t - (G - E_\mu(G))^2 \right) \, d\mu, $$

where

$$ d\mu := \frac{\omega^m}{m!} \wedge T, \quad E_\mu(G) := \int_X G \, d\mu. $$
Lemma 3.3. Assume that (3.1) is true. Then
\[
\int_X G_t d\mu = e^f ||\theta||^2_{T,\omega}, \quad \theta := \frac{d}{dt}\omega = \alpha_1 - \alpha_2,
\]
and
\[
\int_X (G - E_\mu(G))^2 d\mu = e^f ||G - E_\mu(G)||^2_{T,\omega},
\]
where \( || \cdot ||_{T,\omega} \) denotes the \( T \)-Hodge theory norm (see Definition 5.6). Moreover,
\[
T \wedge G = -\Lambda(T \wedge \theta),
\]
where \( \Lambda \) denotes the adjoint of \( \omega \wedge \cdot \) in \( T \)-Hodge theory.

Lemma 3.4. Assume that (3.1) is true. Then \( T \wedge (E_\mu(G) - G) \) is the \( L^2 \)-minimal solution of
\[
d(\cdot) = (d^c)^*(T \wedge \theta),
\]
with respect to the \( T \)-Hodge theory norm and
\[
||G - E_\mu(G)||_{T,\omega} \leq ||\theta||_{T,\omega}.
\]

4. Brascamp-Lieb Lemma

We shall use the Brascamp-Lieb lemma to prove Lemma 3.2.

4.1. Brascamp-Lieb proof of the Prékopa theorem. The following Prékopa theorem was found by Prékopa around 1973.

Theorem 4.1 (Prékopa’s theorem [31]). Let \( \phi \) be a smooth, strictly convex function of \((t, x)\) in \( \mathbb{R}^{n+1} \). Then
\[
t \mapsto -\log \int_A e^{-\phi(t,x)} d\lambda(x),
\]
is strictly convex on \( \mathbb{R} \), where \( A \) is a fixed convex body in \( \mathbb{R}^n \) and \( d\lambda(x) \) denotes the Lebesgue measure.

The Brascamp-Lieb proof in [11] contains three steps.

Step 1: The second order derivative of function (4.1) can be written as
\[
\int_A \phi_{tt} - (\phi_t - E_\nu(\phi_t))^2 d\nu,
\]
where
\[
d\nu := \frac{e^{-\phi(t,x)} d\lambda(x)}{\int_A e^{-\phi(t,x)} d\lambda(x)}, \quad E_\nu(\phi_t) := \int_A \phi_t d\nu.
\]

Step 2: Prove the following Brascamp-Lieb inequality:
\[
\int_{\mathbb{R}^n} (\phi_t - E_\nu(\phi_t))^2 d\nu \leq \int_{\mathbb{R}^n} \sum_{j,k=1}^n \phi_{tj} \phi_{tk} d\nu,
\]
where \((\hat{\phi}^{jk})\) denotes the inverse matrix of \((\phi_{jk})\).

**Step 3**: Use strict convexity of \(\phi\) to prove \(\phi_{tt} > \sum_{j,k=1}^{n} \phi_{tj} \hat{\phi}^{jk} \phi_{tk}\).

**Remark**: The first step follows from the following lemma (take \(dV = e^{-\phi} d\lambda\)). Since

\[
\phi_t - E_{\mu}(\phi_t)
\]

is the (weighted) \(L^2\)-minimal solution of \(d(\cdot) = d(\phi_t)\), an Hörmander type \(L^2\)-estimate gives step 2, see also [11] for a direct proof. For step 3, let \(D_{t,x}^x\) be the determinant of the full hessian matrix of \(\phi\), let \(D_x^x\) be the determinant of the hessian matrix of \(\phi\) as a function of \(x\), then

\[
\frac{D_{t,x}^x}{D_x^x} = \phi_{tt} - \sum_{j,k=1}^{n} \phi_{tj} \hat{\phi}^{jk} \phi_{tk}.
\]

Strict convexity of \(\phi\) implies \(D_{t,x}^x > 0\) and \(D_x^x > 0\). Thus Step 3 follows.

**Lemma 4.2** (Brascamp-Lieb lemma). Let \(A\) be a relatively compact open set in a smooth manifold \(X\). Let \(\{dV(t)\}_{t \in \mathbb{R}}\) be a smooth family of smooth volume forms on \(X\). Let us define \(G\) such that

\[
\frac{d}{dt} dV(t) = -G(t, x) dV(t), \quad (t, x) \in \mathbb{R} \times X.
\]

Then

\[
\frac{d^2}{dt^2} \left( -\log \int_A dV(t) \right) = \int_A \left( G_t - (G - E_{\mu}(G))^2 \right) d\mu,
\]

where

\[
d\mu := \frac{dV}{\int_A dV}, \quad E_{\mu}(G) := \int_A G d\mu.
\]

**Proof.** Since \(A\) is relatively compact, we have

\[
\frac{d}{dt} \left( -\log \int_A dV(t) \right) = \int_A G d\mu.
\]

Apply the differential again, we get

\[
\frac{d^2}{dt^2} \left( -\log \int_A dV(t) \right) = \int_A G_t d\mu + G \frac{d}{dt} d\mu.
\]

A direct computation gives

\[
\frac{d}{dt} d\mu = -G d\mu + E_{\mu}(G) d\mu,
\]

which implies \(\int_A G \frac{d}{dt} d\mu = -\int_A (G - E_{\mu}(G))^2 d\mu\). Thus the lemma follows. \(\square\)

**Remark**: In [6], Berndtsson proved that the Brascamp-Lieb lemma is essentially a subbundle curvature formula associated to a certain direct image bundle. Our main theorem can also be proved along this line, see [35, 34]. Other interesting formulas for the second order derivative of \(-\log \int dV\) can be found in [11].
4.2. **Proof of Lemma 3.2.** Notice that the Brascamp-Lieb lemma gives Lemma 3.2 if $X$ is compact. In case $X$ is non-compact we can not directly apply the Brascamp-Lieb lemma. In our case the main point is that

$$e^{-f} = \int_X \frac{\omega^m}{m!} \wedge T,$$

is a polynomial of degree $m$. The reason is that we can write

$$\frac{\omega^m}{m!} \wedge T = \sum_{j=1}^{m} i^j \Omega_j.$$

Then (3.1) implies that each $\int_X \Omega_j$ is finite and

$$e^{-f} = \sum_{j=1}^{m} \left( \int_X \Omega_j \right) i^j.$$

Thus in our case, $\int_X$ commutes with $\frac{d}{dt}$ and the Brascamp-Lieb lemma applies.

5. **Timorin’s $T$-Hodge Theory**

We shall use Timorin’s $T$-Hodge theory to prove Lemma 3.3. The motivation comes from the Brunn-Minkowski case, i.e. $T = 1$ and $X = \mathbb{R}^n \times \mathbb{T}^n$ (recall $\mathbb{T} := \mathbb{R}/\mathbb{Z}$).

5.1. **Brunn-Minkowski inequality.** By Lemma 2.5 we know that the Brunn-Minkowski inequality is equivalent to the convexity of

$$f : t \mapsto -\log |A_t|, \ A_t := tA_1 + (1 - t)A_2,$$

on $(0, 1)$. Let $\phi_1$ and $\phi_2$ be smooth strictly convex functions that tend to infinity at the boundary of $A_1$ and $A_2$ respectively. Put

$$\psi_1 := \phi_1^*, \ \psi_2 := \phi_2^*.$$

Proposition 2.2 gives

$$\nabla \psi_1(\mathbb{R}^n) = A_1^c, \ \nabla \psi_2(\mathbb{R}^n) = A_2^c.$$

Thus by Proposition 2.3 we have

$$|A_t| = \int_{\mathbb{R}^n} \det(\phi_{jk}) \, dx, \ \phi := t\psi_1 + (1 - t)\psi_2.$$

Apply the Brascamp-Lieb lemma to

$$dV = \det(\phi_{jk}) \, dx,$$

we get

$$(5.1) \quad f_{tt} = \int_{\mathbb{R}^n} G_t - (G - E_\mu(G))^2 \, d\mu,$$

where

$$d\mu := \frac{\det(\phi_{jk}) \, d\lambda(x)}{\int_{\mathbb{R}^n} \det(\phi_{jk}) \, d\lambda(x)}, \ E_\mu(G) := \int_{\mathbb{R}^n} G \, d\mu.$$

**Lemma 5.1.** $G = -\sum_{j,k=1}^{n} \phi_{tjk} \phi_{jk}$.
Proof. We use the fact that if $M(t)$ is a smooth family of positive definite matrices then
\[(\log \det M)_t = \text{Trace}(M^{-1}M_t).\]
Consider $M = (\phi_{jk})$ then $G = -\text{Trace}(M^{-1}M_t)$ and the lemma follows. \qed

**Lemma 5.2.** $G_t = \sum_{j,k,l,m=1}^n \phi_{tlm} \phi_{i} j^{l} \phi_{km}$.

**Proof.** If $M(t)$ is a smooth family of positive definite matrices then
\[(M^{-1})_t = -M^{-1}M_tM^{-1}.\]
Apply the above fact, we get
\[(\phi^{jk})_t = -\sum_{l,m=1}^n \phi_{tlm} \phi_{i} j^{l} \phi_{km}.\]
Moreover, Lemma [5.1] implies $G_t = -\sum_{j,k=1}^n \phi_{tjk}(\phi^{jk})_t$, thus the lemma follows. \qed

By Lemma [2.8] we have
\[f = -\log \int_{\mathbb{R}^n \times \mathbb{T}^n} \frac{(ddc \phi)^n}{n!}.\]
Consider $\omega = ddc \phi$. The above two lemmas give
\[G = -\Lambda \theta, \ G_t = |\theta|_{\omega}^2,\]
thus Lemma [3.3] is true in case $T = 1$ and $X = \mathbb{R}^n \times \mathbb{T}^n$.

### 5.2. T-Hodge Theory

In this subsection, we will introduce the $T$-Hodge theory behind the proof of Lemma [3.3]. The $T$-Hodge theory is an integration of Timorin’s work in [33], see the author’s notes [35] for a systematic study of the $T$-Hodge theory.

Denote by $V^{p,q}$ the space of smooth $(p,q)$-forms on an $n$-dimensional complex manifold $X$. Put
\[V := \bigoplus_{0 \leq p,q \leq n} V^{p,q}, \ V^k := \bigoplus_{p+q=k} V^{p,q}.\]

**Definition 5.1.** Let
\[T = \alpha_{m+1} \wedge \cdots \wedge \alpha_n,\]
be a finite wedge product of smooth positive $(1,1)$-forms on $X$. We call the Hodge theory on $V_T := \{T \wedge u : u \in V\}$ the $T$-Hodge theory.

For bidegree reason, we have
\[V_T = \bigoplus_{0 \leq p,q \leq m} V^{p,q}_T,\]
where $V^{p,q}_T$ denotes the space of forms that can be written as $T \wedge u$, where $u$ is a smooth $(p,q)$-form on $X$. Fix a smooth positive $(1,1)$-form $\omega$ on $X$. The $L$ operator
\[L : T \wedge u \mapsto \omega \wedge T \wedge u,\]
is well defined and maps $V^{p,q}_T$ to $V^{p+1,q+1}_T$. 


**Theorem 5.3** (Timorin's mixed hard-Lefschetz theorem). Put $V^k_T = \oplus_{p+q=k} V_T^{p,q}$ then

$$L^{m-k} : T \land u \mapsto T \land u \land \omega^{m-k}, \quad 0 \leq k \leq m,$$

defines an isomorphism from $V^k_T$ to $V^{2m-k}_T$.

**Proof.** By Theorem 4.2 in [35], we know that

$$A : u \mapsto T \land u \land \omega^{m-k},$$

defines an isomorphism from $V^k$ to $V^{2n-k}$. Hence $V^{2n-k} = V^{2m-k}_T$ and the following map

$$f_T : u \mapsto T \land u, \quad u \in V^k,$$

is injective. Thus $f_T$ defines an isomorphism from $V^k$ to $V^k_T$. Hence $L^{m-k} = A \circ f_T^{-1}$ is an isomorphism from $V^k_T$ to $V^{2m-k}_T$. □

**Definition 5.2.** We call $T \land u \in V^k_T$ a primitive $k$-form if $k \leq m$ and $L^{m-k+1}(T \land u) = 0$.

Theorem 5.3 implies:

**Theorem 5.4.** Every $T \land u \in V^k_T$ has an Lefschetz decomposition as follows:

$$(5.2) \quad T \land u = \sum_{r=0}^{j} L^r(T \land u_r), \quad \text{for some } 0 \leq j \leq m,$$

where each $T \land u_r$ is zero or primitive in $V^{k-2r}_T$. If $T \land u = 0$ then $T \land u_r = 0$ for every $r$.

**Proof.** By the isomorphism in Theorem 5.3 one may assume that $0 \leq k \leq m$. Notice that all forms in $V^0_T$ and $V^1_T$ are primitive. Assume that $2 \leq k \leq m$, Theorem 5.3 gives $\hat{u} \in V^{k-2}$ such that

$$L^{m-k+2}(T \land \hat{u}) = L^{m-k+1}(T \land u).$$

Put $u_0 = u - L\hat{u}$, then $T \land u_0$ is primitive and

$$T \land u = T \land u_0 + L(T \land \hat{u}).$$

Consider $\hat{u}$ instead $u$, the Lefschetz decomposition of $T \land u$ follows by repeating the above argument. If $T \land u = \sum_{r=0}^{j} L^r(T \land u_r) = 0$ then primitivity of $T \land u_r$ for $0 \leq r < j$ implies

$$0 = L^{m-k+j}(\sum_{r=0}^{j} L^r(T \land u_r)) = L^{m-k+2j}(T \land u_j),$$

which gives $T \land u_j = 0$ by Theorem 5.3 By induction on $j$, we get $T \land u_r = 0$ for every $r$. □

**Definition 5.3.** If $T \land u \in V^k_T$ is primitive then we define

$$*_s(L_r(T \land u)) := (-1)^{[k]} L_{m-r-k}(T \land u),$$

where

$$L_p := \frac{L_p}{p!}, \quad [k] := 1 + \cdots + k = \frac{k(k+1)}{2}.$$

$*_s$ extends to a $\mathbb{C}$-linear map $*_s : V_T \to V_T$, we call it the Lefschetz star operator on $V_T$. 

The Lefschetz star operator above is a generalization of the symplectic star operator, see [35] for the background.

**Definition 5.4.** Put $\Lambda = \ast_*^{-1}L\ast_*$, $B := [L, \Lambda]$. We call $(L, \Lambda, B)$ the $sl_2$-triple on $V_T$.

**Definition 5.5.** We call $\ast := \ast_s \circ J$ the Hodge star operator on $V_T$, where $J$ is the Weil-operator defined by $Ju = i^{p-q}u$ if $u \in V_T^{p,q}$.

Timorin’s mixed Hodge-Riemann bilinear relation [33] gives:

**Theorem 5.5.** For every non-zero $u \in V^k$, $0 \leq k \leq m$,

$$\int_X u \wedge \ast (T \wedge u) > 0,$$

where $\ast$ denotes the Hodge star operator on $V_T$.

**Proof.** Let $T \wedge u = \sum_{r=0}^{j} L_r(T \wedge u_r)$ be the Lefschetz decomposition of $T \wedge u$. By our assumption, the degree of $u$ is no bigger than $m$, thus Theorem 4.2 in [35] implies

$$u = \sum_{r=0}^{j} L_r u_r.$$

Now primitivity of $T \wedge u_r$ gives

$$u \wedge \ast (T \wedge u) = \sum_{r=0}^{j} (-1)^{[k-2r]} L_r L_{m+r-k} (T \wedge u_r) \wedge J(u_r).$$

By Theorem 4.1 in [35], if $u_r$ is not zero then

$$(-1)^{[k-2r]} L_r L_{m+r-k} (T \wedge u_r) \wedge J(u_r) > 0,$$

as a positive $(n, n)$-form. Thus the theorem follows. \qed

Let us define

$$||T \wedge u||^2 := ||u||^2_{T, \omega} := \int_X u \wedge \ast (T \wedge u), \ u \in V^k, \ 0 \leq k \leq m.$$

**Definition 5.6.** We call $||T \wedge u|| = ||u||_{T, \omega}$ the $T$-Hodge theory norm on $V_k^T$.

5.3. **Proof of Lemma 3.3** (3.3) follows directly from the definition of the $T$-Hodge theory norm. For (3.2), notice that

$$\frac{d}{dt} \left( \frac{\omega^m}{m!} \wedge T \right) = \theta \wedge \frac{\omega^{m-1}}{(m-1)!} \wedge T,$$

gives

(5.3) $$\left( \theta + G \frac{\omega}{m} \right) \wedge \frac{\omega^{m-1}}{(m-1)!} \wedge T = 0.$$

**Definition 5.7.** $\theta_0 := \theta + G \frac{\omega}{m}$, $\theta_1 := -\frac{G}{m}$, $\theta' := -\theta_0 \wedge \frac{\omega^{m-2}}{(m-2)!} + \theta_1 \wedge \frac{\omega^{m-1}}{(m-1)!}$. 

We have $\theta = \theta_0 + \theta_1 \omega$. (5.3) implies that $T \wedge \theta_0$ is primitive. Thus we have

$$T \wedge \theta' = *(T \wedge \theta) = *(T \wedge \theta).$$

Apply the derivative of (5.3) with respect to $t$, we get

$$(G_t \frac{\omega}{m} + G \frac{\theta}{m}) \wedge \frac{\omega^{m-1}}{(m-1)!} \wedge T + \theta_0 \wedge \theta \wedge \frac{\omega^{m-2}}{(m-2)!} \wedge T = 0,$$

thus

$$G_t \frac{\omega}{m} \wedge T = \theta_1 \theta \wedge \frac{\omega^{m-1}}{(m-1)!} \wedge T - \theta_0 \wedge \theta \wedge \frac{\omega^{m-2}}{(m-2)!} \wedge T$$

$$= \theta \wedge \theta' \wedge T = \theta \wedge *(T \wedge \theta),$$

which gives (3.2). Now it suffices to prove (3.4). Notice that Definition 5.4 gives

$$\Lambda(T \wedge \theta) = \Lambda(T \wedge \theta) = \Lambda(T \wedge \theta).$$

Thus (3.4) is true.

6. Hörmander $L^2$-estimate in $T$-Hodge theory

**Notation:** In this paper, $d^c$ and $(d^c)^*$ denote the adjoint of $d$ and $d^c$ with respect to the $T$-Hodge theory norm.

**Theorem 6.1.** Let $(X, \omega)$ be an $n$-dimensional complete Kähler manifold. Let

$$T := \alpha_{m+1} \wedge \cdots \wedge \alpha_n, \quad 2 \leq m \leq n,$$

be a finite wedge product of Kähler forms on $X$ such that (3.1) is true. Let $\theta$ be a smooth $d$-closed 2-form on $X$. Assume that the $T$-Hodge theory norm $||T \wedge \theta||$ is finite. Then there exists a smooth solution of

$$d(T \wedge u) = (d^c)^*(T \wedge \theta)$$

such that $||T \wedge u|| \leq ||T \wedge \theta||$.

**Proof.** The proof contains two steps.

**Step 1:** "A prior estimate"

$$||(T \wedge \alpha, (d^c)^*(T \wedge \theta))|^2 \leq ||T \wedge \theta||^2 Q(\alpha, \alpha),$$

for every smooth 1-form $\alpha$ with compact support in $X$, where

$$Q(\alpha, \alpha) := ||d(T \wedge \alpha)||^2 + ||(d^*T \wedge \alpha)||^2.$$

**Proof of Step 1:** Since

$$(T \wedge \alpha, (d^c)^*(T \wedge \theta)) = (d^c(T \wedge \alpha), T \wedge \theta),$$

it suffices to show the following $T$-geometry version of the Bochner-Kodaira-Nakano identity

$$||d(T \wedge \alpha)||^2 + ||d^*(T \wedge \alpha)||^2 = ||d^c(T \wedge \alpha)||^2 + ||(d^c)^*(T \wedge \alpha)||^2,$$

which is a special case of Theorem 4.8 in [33].
Step 2: By Step 1, we know that
\[ F : \alpha \mapsto (T \wedge \alpha, (d^c)^* (T \wedge \theta)), \]
is \(Q\)-bounded by \(\|T \wedge \theta\|\). Thus \(F\) extends to a bounded linear functional on the \(Q\)-completion, say \(H\), of the space of smooth 1-forms with compact support in \(X\). The Riesz representation theorem gives \(\beta \in H\) with
\[ Q(\beta, \beta) \leq \|T \wedge \theta\|^2, \tag{6.2} \]
such that
\[ Q(\alpha, \beta) = F(\alpha) = (T \wedge \alpha, (d^c)^* (T \wedge \theta)), \tag{6.3} \]
for every smooth 1-form \(\alpha\) with compact support in \(X\), where
\[ Q(\alpha, \beta) = (d(T \wedge \alpha), d(T \wedge \beta)) + (d^* (T \wedge \alpha), d^* (T \wedge \beta)). \tag{6.4} \]
Since \(H\) is a subspace of the space of currents, we have
\[ Q(\alpha, \beta) = (T \wedge \alpha, (dd^* + d^* d)(T \wedge \beta)). \tag{6.5} \]
Thus (6.3) and (6.5) together give
\[ (dd^* + d^* d)(T \wedge \beta) = (d^c)^* (T \wedge \theta), \]
in the sense of current. Let us define \(u\) such that \(T \wedge u = d^* (T \wedge \beta)\). Since \(dd^* + d^* d\) is elliptic, we know that \(\beta\) is smooth. Thus \(u\) is smooth. Notice that (6.2) gives
\[ \|T \wedge u\| \leq \|T \wedge \theta\|, \]
Thus it suffices to prove the following identity. \(\square\)

Lemma 6.2. \(d^* d(T \wedge \beta) \equiv 0\).

Proof. The \(T\)-Kähler identity \((d^c)^* = [d, \Lambda]\) (see section 4 in [35]) implies that
\[ d(d^c)^* + (d^c)^* d = 0. \]
Thus
\[ d(d^c)^* (T \wedge \theta) = -(d^c)^* d(T \wedge \theta) = 0. \]
Now we have
\[ dd^* d(T \wedge \beta) \equiv 0. \]
Since \(\hat{\omega}\) is complete, there exists a smooth exhaustion function, say \(\rho\), on \(X\) such that
\[ |d\rho|_{\hat{\omega}} \leq 1. \tag{6.6} \]
Let \(0 \leq \chi \leq 1\) be a smooth function on \(\mathbb{R}\) such that \(\chi \equiv 1\) on \((-\infty, 1)\) and \(\chi \equiv 0\) on \((2, \infty)\). Then for each \(\varepsilon > 0\), \(\chi(\varepsilon \rho)\) is a smooth function with compact support. Since
\[ (\chi^2(\varepsilon b)dd^* d(T \wedge \beta), d(T \wedge \beta)) = 0, \tag{6.7} \]
and
\[ \chi^2(\varepsilon b)dd^* d(T \wedge \beta) = d(\chi^2(\varepsilon b)d^* d(T \wedge \beta)) - 2d(\chi(\varepsilon b)) \wedge \chi(\varepsilon b)d^* d(T \wedge \beta), \]
we have
\[ ||\chi(\varepsilon b)d^* d(T \wedge \beta)||^2 = 2(d(\chi(\varepsilon b)) \wedge \chi(\varepsilon b)d^* d(T \wedge \beta), d(T \wedge \beta)). \tag{6.8} \]
Thus Lemma 6.2 follows from the following estimate
\[
\lim_{\varepsilon \to 0} ||d(\chi(\varepsilon b)) \wedge \chi(\varepsilon b) d^* (T \wedge \beta)|| = 0.
\]
The above estimate is easily seen to be true in case \( T = 1 \), see \([14]\). The general case will be proved in the appendix. \( \square \)

6.1. Proof of Lemma 3.4. By Lemma 3.3, we have
\[
d(T \wedge (E_\mu(G) - G)) = d\Lambda(T \wedge \theta) = [d, \Lambda](T \wedge \theta),
\]
By the Kähler identity in \( T \)-Hodge theory (section 4 in \([35]\)), we have \([d, \Lambda] = (d^c)^*\), thus \( T \wedge (E_\mu(G) - G) \) is a solution of
\[
d(\cdot) = (d^c)^*(T \wedge \theta).
\]
Notice that \( T \wedge (E_\mu(G) - G) \) is perpendicular to \( \ker d \), thus it is also the \( L^2 \)-minimal solution. By (3.1), for every fixed \( 0 < t < 1 \), \( \omega = t\alpha_1 + (1 - t)\alpha_2 \) is complete. Apply Theorem 6.1 to the case \( \omega = \omega \), Lemma 3.4 follows.

7. Proof of the Alexandrov-Fenchel inequality

Lemma 7.1. Put
\[
\psi(x) = \sum_{j=1}^n \log \frac{1}{1 + (x^j)^2} + C \log(1 + e^{x^j}), \quad C := 4(1 + e^{\sqrt{3}})^2 e^{\sqrt{3}}.
\]
Then \( \psi \) is strictly convex on \( \mathbb{R}^n \) and \( \nabla \psi(\mathbb{R}^n) \subset (-1, C + 1)^n \). Moreover, if we look at \( \psi \) as a function on \( \mathbb{R}^n \times \mathbb{T}^n \) then \( dd^c \psi \) is complete Kähler on \( \mathbb{R}^n \times \mathbb{T}^n \).

Proof. A direct computation gives
\[
\left( \log \frac{1}{1 + (x^j)^2} \right)_{x^j} = \frac{-2x^j}{1 + (x^j)^2},
\]
and
\[
\left( \log \frac{1}{1 + (x^j)^2} \right)_{x^j x^j} = \frac{2(x^j)^2 - 2}{(1 + (x^j)^2)^2} \geq \frac{1}{1 + (x^j)^2}, \quad \text{if } (x^j)^2 \geq 3.
\]
Since \( \log(1 + e^x) \) is convex, the above inequality gives
\[
\psi_{x^j x^j} \geq \frac{1}{1 + (x^j)^2} \quad \text{if } (x^j)^2 \geq 3.
\]
We also have
\[
\left( \log(1 + e^{x^j}) \right)_{x^j x^j} = \frac{e^{x^j}}{(1 + e^{x^j})^2} \geq \frac{e^{-\sqrt{3}}}{(1 + e^{\sqrt{3}})^2}, \quad \text{if } (x^j)^2 \leq 3.
\]
Thus
\[
C \left( \log(1 + e^{x^j}) \right)_{x^j x^j} \geq 4 \geq \frac{4}{1 + (x^j)^2}, \quad \text{if } (x^j)^2 \leq 3,
\]
which gives
\[ \psi_{x^j} \geq \frac{4}{1 + (x^j)^2} + \frac{2(x^j)^2 - 2}{(1 + (x^j)^2)^2} \geq \frac{2}{1 + (x^j)^2} \quad \text{if} \quad (x^j)^2 \leq 3. \]

Notice that \( \psi_{x^j x^k} = 0 \) if \( j \neq k \). Thus \( \psi \) is strictly convex and
\[ dd^c \psi \geq \sum_{j=1}^{n} \frac{1}{1 + (x^j)^2} dx^j \wedge dy^j, \]
on \( \mathbb{R}^n \times T^n \). Denote by \( g \) the associated Riemannian metric of \( dd^c \psi \), then we have
\[ g \geq g_0 := \sum_{j=1}^{n} \frac{1}{1 + (x^j)^2} (dx^j \otimes dx^j + dy^j \otimes dy^j). \]

Thus
\[ |dx^j|_g \leq |dx^j|_{g_0} = \sqrt{1 + (x^j)^2}. \]

Since \( d \log(1 + |x|^2) = \sum_{j=1}^{n} \frac{2x^j dx^j}{1 + |x|^2} \), we have
\[ |d \log(1 + |x|^2)|_g \leq \sum_{j=1}^{n} \frac{2|x^j|}{1 + |x|^2} |dx^j|_g \leq \sum_{j=1}^{n} \frac{2|x^j|}{1 + |x|^2} \sqrt{1 + (x^j)^2} \leq n. \]

Notice that \( \log(1 + |x|^2) \) is an exhaustion function on \( \mathbb{R}^n \times T^n \), the above inequality implies that \( dd^c \psi \) is complete Kähler. \( \nabla \psi(\mathbb{R}^n) \subset (-1, C + 1)^n \) follows from
\[ \psi_{x^j} = \frac{-2x^j}{1 + (x^j)^2} + C \frac{e^{x^j}}{1 + e^{x^j}}, \quad 2|x^j| \leq 1 + (x^j)^2, \quad 0 < \frac{e^{x^j}}{1 + e^{x^j}} < 1. \]
The proof is complete. \( \square \)

We shall use our main theorem and the above lemma to prove Theorem 2.10 which implies the Alexandrov-Fenchel inequality.

### 7.1. Proof of Theorem 2.10

Put
\[ \tilde{\phi} = \psi + \phi_1 + \phi_2 + \phi_{m+1} + \cdots + \phi_n. \]

The above lemma implies that \( \tilde{\omega} := dd^c \tilde{\phi} \) is complete on \( \mathbb{R}^n \times T^n \) and \( dd^c \phi_j \leq \tilde{\omega} \) for each \( j \). Moreover, by the above lemma, \( \nabla \psi(\mathbb{R}^n) \) is bounded, thus \( \nabla \tilde{\phi}(\mathbb{R}^n) \) is bounded and \( (X, \tilde{\omega}) \) has finite volume. We know that Theorem 2.10 follows from Theorem 3.1.

### 8. Appendix

#### 8.1. Compare the T-Hodge theory norm with the usual norm

For every smooth \( k \)-form \( u \), \( 0 \leq k \leq m \), on \( X \), let us define \( |u|^2_{T, \omega} \) such that
\[ u \wedge * (\frac{T}{\omega} \wedge u) = |u|^2_{T, \omega} \frac{\omega^m}{m!} \wedge T. \]

where \( * \) denotes the Hodge star operator on \( V_T \), see Definition 5.5.

**Definition 8.1.** We call \( |u|_{T, \omega} \) the pointwise T-norm of \( u \).
Lemma 8.1. Let $|T \wedge u_\omega|$ be the usual pointwise norm of $T \wedge u$ with respect to $\omega$. If $T = \omega^{n-m}$ then
\[
\frac{n!(n-m)!}{m!} |u|_T^2 \leq |T \wedge u|_\omega^2 \leq \frac{(n!)^2}{(m!)^2} |u|_T^2.
\]

Proof. By Definition 5.2, if $T = \omega^{n-m}$ then a form $T \wedge v \in V^k_T$ is primitive in $T$-Hodge theory if and only if $v$ is primitive with respect to $\omega$ in the usual sense. Let

$$T \wedge u := \sum_{r=0}^j L_r(T \wedge u_r) = \sum_{r=0}^j L_{n-m+r} u'_r, \quad u'_r := \frac{(n-m+r)!}{r!} u_r,$$

be the Lefschetz decomposition of $T \wedge u$. Then Definition 5.5 gives

$$\ast(T \wedge u) = \sum_{r=0}^j (-1)^{|k-2r|} L_{m-k+r}(T \wedge Ju_r).$$

Moreover,

$$\ast(T \wedge u) = \sum_{r=0}^j (-1)^{|k-2r|} L_{m-k+r}(Ju'_r),$$

where $\ast$ denotes the usual Hodge star operator. Recall that

$$T \wedge u \wedge \ast(T \wedge u) = |T \wedge u|_\omega^2 \frac{\omega^n}{n!}.$$

Thus the lemma follows. \qed

For general $T = \alpha_{m+1} \wedge \cdots \wedge \alpha_n$, we have:

**Lemma 8.2.** Assume that (3.1) is true. Then there exists a constant $C_1$ that only depends on $C, n, m$ such that

$$C_1^{-1} |u|_{T, \omega} \leq |T \wedge u|_\omega \leq C_1 |u|_{T, \omega}.$$

Proof. By Lemma 8.1 it suffices to compare $|u|_{T_0, \omega}$ with $|u|_{T_0, \hat{\omega}}$, where $T_0 := \hat{\omega}^{n-m}$. Fix an arbitrary point, say $z_0$, in $X$, let us choose local coordinates, say $\{z^j\}$, near $z_0$ such that

$$\hat{\omega}(z_0) = i \sum_{j=1}^n dz^j \wedge d\bar{z}^j.$$

With respect to the local coordinates $\{z^j\}$, we can identify the space of positive $(1,1)$-forms at $z_0$ with the space of positive definite $n$ by $n$ Hermitian matrices. We know that every positive definite $n$ by $n$ Hermitian matrix can be written as

$$A = OBO^*, \quad OO^* = I_n,$$

where $O^*$ denotes the conjugate transpose of $O$, $I_n$ is the identity matrix and $B$ is a diagonal matrix with positive eigenvalues. Moreover,

$$\frac{\omega(z_0)}{C} \leq \hat{\omega}(z_0) \leq C \omega(z_0)$$
if and only if each eigenvalue of the associated matrix of \( \omega(z_0) \) lies in \([1/C, C]\). Consider
\[
V := U(n) \times [1/C, C]^n,
\]
where \( U(n) := \{ O : OO^* = I_n \} \) is the unitary group. Every element, say \( v = (O, \lambda_1, \cdots, \lambda_n) \), in \( V \) represents a positive (1, 1)-form, say \( \omega^v \), at \( z_0 \) whose associated matrix is
\[
ODiag\{\lambda_1, \cdots, \lambda_n\}O^*.
\]
Consider the following map, say \( F \), from
\[
V^{n-m} := \underbrace{V \times \cdots \times V}_{n-m}
\]
to the space of Hermitian norms on \( ^k(C \otimes T_{z_0}^* X) \), \( 0 \leq k \leq m \), defined by
\[
(v^{m+1}, \cdots, v^n) \mapsto | | T_{z_0} \omega | | = \omega^{v^{m+1}} \wedge \cdots \wedge \omega^v.
\]
The lemma follows since \( V^{n-m} \) is compact and connected.

8.2. **Proof of estimate** (6.9). Let us write \( d^* d(\gamma \wedge \beta) \) as \( T \wedge \sigma \), where \( \sigma \) is a one-form. Then
\[
||d(\chi(\varepsilon \rho)) \wedge \chi(\varepsilon \rho)d^* d(T \wedge \beta)||^2 = \int_X | | d(\chi(\varepsilon \rho)) \wedge \chi(\varepsilon \rho)\sigma | |^2 T_{z_0} \omega^m \wedge T.
\]
By Lemma 8.2, we have
\[
| | d(\chi(\varepsilon \rho)) \wedge \chi(\varepsilon \rho)\sigma | | T_{z_0} \omega \leq C_1 | | d(\chi(\varepsilon \rho)) \wedge \chi(\varepsilon \rho)\sigma | | T_{z_0} \omega\|\omega^m \wedge T.
\]
Since \( | | d\rho | | \omega \leq 1 \), we have
\[
| | d(\chi(\varepsilon \rho)) \wedge \chi(\varepsilon \rho)d^* d(\beta \wedge T) | | \omega \leq (\sup X(\varepsilon \rho)) | | d(\chi(\varepsilon \rho)) \wedge \chi(\varepsilon \rho)d^* d(T \wedge \beta) | | \omega.
\]
Use Lemma 8.2 again, we get
\[
| | d(\chi(\varepsilon \rho)) \wedge \chi(\varepsilon \rho)\sigma | | T_{z_0} \omega \leq (\sup X(\varepsilon \rho)) | | d(\chi(\varepsilon \rho)) \wedge \chi(\varepsilon \rho)d^* d(T \wedge \beta) | | \omega.
\]
which gives
\[
| | d(\chi(\varepsilon \rho)) \wedge \chi(\varepsilon \rho)d^* d(T \wedge \beta) | | \leq (\sup X(\varepsilon \rho)) | | \chi(\varepsilon \rho)d^* d(T \wedge \beta) | |.
\]
By (6.8), then we have
\[
| | \chi(\varepsilon \rho)d^* d(T \wedge \beta) | | \leq 2 (\sup X(\varepsilon \rho)) | | \chi(\varepsilon \rho)d^* d(T \wedge \beta) | | \cdot | | T \wedge \theta | |,
\]
hence
\[
| | \chi(\varepsilon \rho)d^* d(T \wedge \beta) | | \leq (2\sup X(\varepsilon \rho)) | | T \wedge \theta | |,
\]
which gives
\[
| | d(\chi(\varepsilon \rho)) \wedge \chi(\varepsilon \rho)d^* d(T \wedge \beta) | | \leq 2(\sup X(\varepsilon \rho)^2 | | T \wedge \theta | |,
\]
thus (6.9) follows.
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