Gravity and Yang-Mills theory: two faces of the same theory?

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Abstract

We introduce a gauge and diffeomorphism invariant theory on the Yang-Mills phase space. The theory is well defined for an arbitrary gauge group with an invariant bilinear form, it contains only first class constraints, and the spacetime metric has a simple form in terms of the phase space variables. With gauge group $SO(3, C)$, the theory equals the Ashtekar formulation of gravity with a cosmological constant. For Lorentzian signature, the theory is complex, and we have not found any good reality conditions. In the Euclidean signature case, everything is real. In a weak field expansion around de Sitter spacetime, the theory is shown to give the conventional Yang-Mills theory to the lowest order in the fields. We show that the coupling to a Higgs scalar is straightforward, while the naive spinor coupling does not work. We have not found any way of including spinors that gives a closed constraint algebra. For gauge group $U(2)$, we find a static and spherically symmetric solution.

PACS: 04.20.Fy, 04.20.+h

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1 Introduction

The Ashtekar formulation of Einstein gravity is a canonical description on the \( so(3) \) Yang-Mills phase space \([1]\). This formulation has received a lot of attention mainly due to the fact that the constraints in the theory have a very simple form, making it possible to restart the old attempts at a canonical quantization of gravity. There is, however, another aspect of the Ashtekar’s variables that make them very interesting already at the classical level. Merely the fact that gravity can be so successfully formulated in terms of Yang-Mills type variables, could be an indication that there exist a unified description of gravity and Yang-Mills theory. Furthermore, there are a few puzzling features of the Ashtekar’s variables that could be the clues that lead us to the correct theory. For instance: 1. Why does the theory has to be complex for Lorentzian signature? 2. Why do we only need half the Lorentz group in the construction? 3. How come the coupling to Yang-Mills theory seems more unnatural than the other matter couplings?

It could happen that the answer to the two first questions are the same: perhaps the other half of the Lorentz group is tied up serving as an internal symmetry group for some other gauge interaction \((su(2))\), and in order to find the pure gravity theory, we need to split \( so(1,3) \cong su(2,C) \oplus su(2,C) \) which necessarily means introducing complex fields. (See \([3]\) for a treatment of an \( so(1,3) \) Ashtekar formulation.)

In this paper, we will present a generalization of the Ashtekar formulation. It is a generalization in the sense that we now can use an arbitrary gauge group as internal symmetry group. The model contains only first class constraints and the spacetime metric has a simple form in terms of the basic fields. There are, however, several reasons that make us believe that this is not the unified theory we are looking for. For instance, we still need complex fields in order to get Lorentzian signature on the metric. Moreover, since the theory seems to work perfectly alright for an arbitrary gauge group, the theory does not tell us anything about what gauge symmetries we should have. With this perspective, the theories described in \([3]\) appear to be much more interesting ones, since they seem to give Lorentzian signature with real fields, and the construction singles out specific gauge groups. So although the examples chosen in \([3]\) were shown to describe nothing more than complex Einstein gravity, there might still exist other gauge groups that, by using the construction in \([3]\), give physically interesting unified theories. There is, however, another reason that, so far, makes the construction presented here superior to the one used in \([3]\): the theory presented in this paper gives the standard Yang-Mills theory when we expand around de Sitter spacetime for weak fields. Weak fields here means; weak compared to the energy scale set by the dimension-full constant in the theory: the cosmological constant. It is actually a necessary requirement, in this type of theories, that one has to have a dimension-full constant included. The reason is that in these theories the metric and the Yang-Mills ”electric-field” are roughly speaking comparable objects, and since these two fields have different dimensions, we need a dimension-full constant for a rescaling of one of the fields\(^2\). Moreover, when doing a weak field expansion of the theory (to compare it to conventional Yang-Mills theory on approximately Minkowskian spacetime) this constant

\(^2\)Without a dimensionfull constant one would have to use the Planck constant – already at the classical level – in order to be able to unify a metric-type variable and a YM-type electric field. Also, without a cosmological constant term in the Hamiltonian, it is hard to see how the conventional Einstein-YM Hamiltonian could be produced in the weak field limit of the theory.
will be the natural energy scale to define what is meant by ”weak fields”. In the theory we study in this paper, this constant is the cosmological constant, which from cosmological considerations normally is considered to correspond to a very low energy-density. In practice this means that we will only get the conventional Yang-Mills theory for fields that are weak w.r.t the cosmological constant energy-scale. This seems to be a serious drawback for any unified description where the energy scale is set by the cosmological constant. Thus, if we believe in this kind of unification – with the cosmological constant – we would expect to get corrections to e.g. Maxwell’s theory already for very low field strengths (that is if we isolate the experiment from all background energy densities, since a slowly varying background energy density will manifest itself as a cosmological constant, and thereby increasing the relevant energy scale). There is, however, another possibility: by instead introducing a dimension-full constant that correspond to a very high energy density – or very short length – we would not get into conflict with cosmology, and Yang-Mills theory would be correct up to much higher energies. This is actually what one does in string theory when one introduces a classical string-tension that corresponds to a very short length scale. To do this in practice, in a unified theory of the type discussed in this paper, one could modify the Hamiltonian constraint by adding terms that are of higher order in the magnetic field.

The outline of the paper is: in section 2 we present the problem of generalizing the Ashtekar formulation, and in section 3 we introduce the explicit model we study in this paper, do a constraint analysis and give a short discussion of the problem of finding good reality conditions. In section 4, we do a weak field expansion of the Hamiltonian around de Sitter spacetime, and show that the conventional YM Hamiltonian is contained in this theory. Section 5 shows that it is straightforward to include a Higgs scalar field in the general theory, while spinor fields seem to be problematic to incorporate into the unified formulation. We derive an explicit static and spherically symmetric solution to the $U(2)$ theory in section 6. In appendix A, we derive the Hamiltonian for the conventional Yang-Mills theory, and finally in appendix B, we introduce dimensions and units in the theories.

Throughout this paper, we try to give all conventions needed in the relevant section. As a general rule, our index conventions are: $\alpha, \beta, \gamma$, .... spacetime indices, $a, b, c$, ... spatial indices, $i, j, k$, ... $so(3)$ indices in the vector representation, $I, J, K$, ... gauge indices (other than $so(3)$) in the vector representation. $A, B, C$, .... are used both as Yang-Mills gauge indices in the vector representation as well as $su(2)$ spinor indices.

Also, compared to for instance [2], we have rescaled the Ashtekar variables according to: $A_{ai} \rightarrow -iA_{ai}, E^{ai} \rightarrow iE^{ai}$. These conventions are the ones that minimizes the number of complex ”i’s” in the theory. (Actually, for pure gravity without a cosmological constant, no explicit ”i” appears in the formulation.) These conventions are also the most natural from the unification point of view.

## 2 Generalizing Ashtekar’s variables

In an attempt to find a unified theory of gravity and Yang-Mills theory (YM), on Yang-Mills phase space, a natural starting point is the generalization of Ashtekar’s variables to other gauge groups. The Ashtekar formulation is a canonical description of (3+1)-
dimensional gravity on the $SO(3, C)$ Yang-Mills phase space \cite{1}, \cite{2}. If the unified theory is found from a generalization of the Ashtekar Hamiltonian, we are guaranteed that the pure gravity theory will appear when we remove the YM part of the theory. That is, when we choose the gauge group to be $SO(3, C)$ again. But perhaps this is a too naive expectation, perhaps we instead should look for a more sophisticated construction were the pure gravity part is more non-trivially embedded. This would for instance be the case if we, in the formulation of the model, used special objects, features etc that only exist for the unified gauge group”. See e.g. \cite{3} for examples of that. For examples of that. In this paper, we examine the naive construction.

The Ashtekar Hamiltonian for pure gravity with a cosmological constant is:

$$H_{tot} = NH + NaH_a + \Lambda \mathcal{G}^i$$

$$H := -\frac{1}{4} \epsilon_{abc} \epsilon_{ijk} E^{ai} E^{bj} (B^{ck} + \frac{2i\lambda}{3} E^{ck}) \approx 0$$

$$H_a := \frac{1}{2} \epsilon_{abc} E^{bi} B^c_i \approx 0$$

$$\mathcal{G}_i := D_a E^a_i = \partial_a E^a_i + f_{ijk} A^j_a E^k_i \approx 0$$

The index-conventions are: $a, b, c, ...$ are spatial indices on the three dimensional hyper surface, and $i, j, k, .....$ are $SO(3, C)$ gauge-indices in the vector representation. Gauge-indices are raised and lowered with an invariant bilinear form of the Lie-algebra (the ”group-metric”), here chosen to be the Cartan-Killing form, $\delta_{ij}$. The basic conjugate fields are $A_{ai}$ and $E^{bj}$, which satisfy the fundamental Poisson bracket: \{ $A_{ai}(x), E^{bj}(y)$ \} = $\delta^b_i \delta_i^j \delta^a_j (x - y)$. $A_{ai}$ is a gauge connection, and $E^{ai}$ is often referred to as the ”electric field”, and it equals the densitized triad, in a solution. The other fields in the theory, $N, N^a, \Lambda_i$ are Lagrange multiplier fields whose variations impose the constraints $H, H_a$ and $\mathcal{G}_i$. $B^{ai}$ is the ”magnetic field”:

$$B^{ai} := \epsilon^{abc} F^{ai}_{bc} = \epsilon^{abc} (2\partial_b A^c_i + f_{ijk} A^j_a A^k_i),$$

and $\lambda$ is the cosmological constant. Finally, $\epsilon_{ijk}$ and $\epsilon_{abc}$ are the totally antisymmetric epsilon-symbols: $\epsilon_{123} := 1$ for both of them.

The reason why it is a non-trivial task to generalize this Hamiltonian to other gauge groups can be found in the $\epsilon_{ijk}$ in the Hamiltonian constraint, $H$. If one sees this object as an epsilon-symbol, it is only well-defined for three dimensional Lie-algebras. If one instead sees this $\epsilon_{ijk}$ as the structure constant of $SO(3)$ (which it is, with proper normalization), the introduction of other gauge groups will not give a closed constraint algebra for the constraints given in \((2.1)-(2.4)\). The problem is the Poisson bracket $\{ H, H \}$ which will not weakly vanish for arbitrary gauge groups. So, the theory would need additional constraints, possibly of second class, and that would further complicate the attempts at a canonical quantization of it.

To evade these difficulties, we see three different strategies: I. eliminate the epsilon completely from $H, H_a$ and $\mathcal{G}_i$. II. generalize the epsilon-symbol to be well-defined for higher dimensional gauge groups as well, or III. replace the $\epsilon_{ijk}$ with $f_{ijk}$, the structure constant of the Lie-algebra, and use only gauge groups that gives a closed constraint algebra.

I. This method has been described in \cite{4}. One multiplies the Hamiltonian constraint, $H$ with a scalar function containing an $\epsilon_{ijk}$, and then uses the $\epsilon - \delta$ identity to remove all $\epsilon$'s from the formulation. The resulting constraints will in general give a closed algebra.
for arbitrary gauge groups.

**II.** The generalization of the $\epsilon_{ijk}$ to higher dimensional gauge groups can be done by using the three-dimensionality of the underlying hyper surface. Since there exist an totally antisymmetric tensor density $\epsilon_{abc}$ on the hyper surface, one can easily construct a Lie-algebra valued $\epsilon_{ijk}$ as follows:

$$\epsilon_{ijk}(V^d) := \frac{\epsilon_{abc}V^a_iV^b_jV^c_k}{\sqrt{\det(V^d_dV^e_e)}}$$  \hspace{1cm} (2.5)

where $V^{ai}$ is a trio of Lie-algebra valued vector fields. Note that this method actually is contained in I.

**III.** To use this method, one needs a Lie-algebra whose structure constants satisfy a certain type of identity. The only Lie-algebras we know of that can be used here, are $so(4), so(1,3), su(2), so(3), so(1,2)$ and the Kac-Moody affine extension of these. See [3] for an example of a theory based on the Kac-Moody extension of $so(1,3)$.

In this paper, we will study a model based on a generalization of type II (or I).

### 3 The arbitrary gauge group theory

In this section, we will study one Hamiltonian that can be found from the Ashtekar Hamiltonian (2.1)-(2.4) by using method II (or I) in the previous section. First, we describe the Hamiltonian, valid for arbitrary gauge groups, then we calculate the constraint algebra and show that the theory contains only first class constraints, and finally, we give the geometrical interpretation of the fields. There is also a short discussion of the (absence of) reality conditions.

#### 3.1 The Hamiltonian

We start by generalizing the Ashtekar Hamiltonian using method II, and choosing the vector fields $V^{ai}$ to be $E^{ai}$. There is nothing unique about this choice, and the reason for it is that we know that the standard gravity-Yang-Mills coupling is of sixth order in the electric fields, meaning that we need $V^{ai}$ to be linear in $E^{ai}$ if we should recover the standard coupling in this theory. Then, $V^{ai} = E^{ai}$ is the simplest choice. Thus, the only change from (2.1)-(2.4) is that $\mathcal{H}$ now becomes:

$$\mathcal{H} := -\frac{1}{4} \epsilon_{abc} \epsilon_{IJK}(E^{dL}) E^{ai} E^{bj} (B^{cK} + \frac{2i\lambda}{3} E^{cK}) \approx 0$$  \hspace{1cm} (3.1)

where

$$\epsilon_{IJK}(E^{dL}) := \frac{\epsilon_{abc}E^a_I E^b_J E^c_K}{\sqrt{\det(E^{dLE^e_e})}}$$  \hspace{1cm} (3.2)

There is also the invisible change that the gauge indices $I, J, K, ...$ now take values $1, 2, ... N$, where $N$ is the dimension of the Lie-algebra. Also, we suppose that there exist an invariant bilinear form of the Lie-algebra that can be used to contract the gauge indices.
(For the semi-simple classical groups, this form can always be chosen to be the Cartan-Killing form.) Note that the definition of $\epsilon_{IJK}(E^{\mu})$ requires a non-zero $\text{det}(E^aI E^bJ)$, which later will be shown to correspond to a non-degenerate metric. Since we will have to allow the fields to be complex, we really get a sign-ambiguity from the square-root in the $\epsilon_{IJK}(E^{\mu})$. This sign-ambiguity just corresponds to a sign change of the total space-time metric. If one wants to avoid this inconvenient square-root, one could instead use $\epsilon_{abc} E^aI E^bJ E^cK$ as denominator. This would give an inequivalent $\epsilon_{IJK}(E^{\mu})$, which, however, coincides with the conventional $\epsilon_{IJK}$ for $SO(3)$. We will choose (3.2) as our definition.

Before we go on and study the Hamiltonian, a few words should be said about $\epsilon_{IJK}(E^{\mu})$ itself. This object has the following important properties: it is totally antisymmetric, it reduces to the conventional, constant epsilon-symbol for three dimensional algebras, and it satisfies an "$\epsilon - \delta$" identity on the three dimensional subspace (in the Lie-algebra) spanned by $E^aI$. The antisymmetricity follows from antisymmetric property of $\epsilon_{abc}$, and by using the three dimensional identities

$$\epsilon_{abc} E^aI E^bJ E^cK = \text{det}(E^{\mu}) \epsilon^{IJK}$$

$$\text{det}(E^aI E^bJ) = (\text{det}(E^{\mu}))^2$$

it follows that $\epsilon_{IJK}(E^{\mu}) = \pm \epsilon_{IJK}$ for three-dimensional algebras. To give the "$\epsilon - \delta$" identity, we first need to define a few objects:

$$q^{ab} := -E^aI E^bJ$$

$$q_{ab} := \frac{1}{2} \left| q^{ef} \right| \epsilon_{abc} \epsilon_{def} q^{ef} q^{df} \Rightarrow q_{ab} q^{bc} = \delta_a^c$$

$$E^I_a := -q_{ab} E^bJ \Rightarrow E^I_a E^bJ = \delta_a^b$$

$$\tilde{\delta}^I_J := E^I_a E^aJ$$

The $\tilde{\delta}^I_J$ is a projection operator that projects down to the three dimensional space spanned by the $E^aI$. With these definitions, the "$\epsilon - \delta$" identity becomes

$$\epsilon_{IJK}(E^{\mu}) \epsilon^{LMN}(E^{\mu}) = \tilde{\delta}_L^I \tilde{\delta}_M^J \tilde{\delta}_N^K$$

where the square brackets denotes antisymmetrization of all indices inside, without factors.

### 3.2 Constraint analysis

Returning to the Hamiltonian, one needs to check that the theory is consistent in the sense that an initial field configuration that satisfy all constraint will continue to do so. That is the same as requiring the time evolution of the constraints to be weakly vanishing (zero, modulo all constraints). And for Hamiltonians which only consist of constraints, this corresponds to checking the constraint algebra. If all constraints are first class, i.e all Poisson brackets between constraints result in linear combinations of constraints, the time-evolution of the constraints will weakly vanish.

In calculating the constraint algebra in a gauge and diffeomorphism invariant theory, it is often convenient to first identify the generators of gauge transformations and diffeomorphisms on the hyper surface. Since these generators have a simple action on all gauge
and diffeomorphism covariant objects, it is a simple task to calculate all Poisson brackets containing these generators. For theories formulated on Yang-Mills phase space it is well known that the generator of gauge transformations is the Gauss law constraint, \( \mathcal{G}_I \), and that the generator of spatial diffeomorphisms is given by \( \mathcal{H}_a := \mathcal{H}_a - \Lambda^I_a \mathcal{G}_I \), where \( \mathcal{G}_I \) and \( \mathcal{H}_a \) are given in (2.3) and (2.4). (This is true in arbitrary spacetime dimensions \( > 2 \), and for arbitrary gauge groups. See e.g [3].) To prove this, consider the transformations generated on the basic fields \( A^I_a \) and \( E^a_I \). (Square brackets here denote smearing over the hyper surface: \( \mathcal{G}_I[\Lambda] := \int_\Sigma d^3x \Lambda^I \mathcal{G}_I \).)

\[
\delta^G I^I \equiv \{ E^a_I, \mathcal{G}^J[I_a] \} = f^I_{JK} \Lambda^J E^a_K
\]

(3.10)

\[
\delta^G I^I_a = \{ A^I_a, \mathcal{G}^J[I_a] \} = -\mathcal{D}_a \Lambda^I
\]

(3.11)

which shows that \( \mathcal{G}_I \) is indeed the generator of gauge transformations, and

\[
\delta^\mathcal{H}_a E^a_I := \{ E^a_I, \mathcal{H}_b[N^b] \} = N^b \partial_b E^a_I - E^b_I \partial_b N^a + E^a_I \partial_b N^b = \mathcal{L}_{N^b} E^a_I
\]

(3.12)

\[
\delta^\mathcal{H}_a A^I_a := \{ A^I_a, \mathcal{H}_b[N^b] \} = N^b \partial_b A^I_a + A^I_a \partial_b N^b = \mathcal{L}_{N^b} A^I_a
\]

(3.13)

prove that \( \mathcal{H}_a \) generates spatial diffeomorphisms. Then it is clear how any gauge and diffeomorphism covariant function of \( A^I_a \) and \( E^a_I \) transforms:

\[
\delta^\mathcal{G}_I[A^I] \Phi_{ab}(E^a_I) = \Lambda^J \Phi^K_a(E^a_I, A^I_a) f_{JKI}
\]

(3.14)

\[
\delta^\mathcal{H}_a[N^a] \Phi_{ab}(E^a_I) = \mathcal{L}_{N^a} \Phi_{ab}(E^a_I)
\]

(3.15)

Thus, knowing that \( \mathcal{G}_I \) and \( \mathcal{H} \) are gauge-covariant, and that \( \mathcal{G}_I, \mathcal{H}_a \) and \( \mathcal{H} \) all are tensor densities, the Poisson brackets containing \( \mathcal{G}_I \) and \( \mathcal{H}_a \) are easily calculated:

\[
\{ \mathcal{G}_I[I_a], \mathcal{G}_J[I_a] \} = \mathcal{G}_K[f_{KIJ} A^J]
\]

(3.16)

\[
\{ \mathcal{G}_I[I_a], \mathcal{H}_a[N^a] \} = \mathcal{G}_I[-\mathcal{L}_{N^a} A^I]
\]

(3.17)

\[
\{ \mathcal{G}_I[I_a], \mathcal{H}[N] \} = 0
\]

(3.18)

\[
\{ \mathcal{H}_a[N^a], \mathcal{H}_b[M^b] \} = \mathcal{H}_a[-\mathcal{L}_{M^b} N^a]
\]

(3.19)

\[
\{ \mathcal{H}_a[N^a], \mathcal{H}[N] \} = \mathcal{H}[-\mathcal{L}_{N} N^a]
\]

(3.20)

This leaves only one Poisson bracket left to calculate: \( \{ \mathcal{H}[N], \mathcal{H}[M] \} \). This is a rather complicated calculation but it simplifies to note that the result must be antisymmetric in \( N \) and \( M \), meaning that it is only terms containing derivatives of these test functions that survive. Furthermore, by first checking that \( \{ \det(E^a_I E^b_I)[N], \mathcal{H}[M] \} = (N \leftrightarrow M) = 0 \), one sees that we only have two contributions to the end result:

\[
\{ \mathcal{H}[N], \mathcal{H}[M] \} = \{ B^b_I [\frac{-N}{4 \sqrt{-\det(q^{ab})}} e^{abcdacq^{ce}q^{df}E^a_I}], E^q_J [-\frac{M}{4 \sqrt{-\det(q^{ab})}} e^{abcdacq^{ce}q^{df}E^a_I}] \} +
\]

\[
\{ B^b_I [\frac{-N}{4 \sqrt{-\det(q^{ab})}} e^{abcdacq^{ce}q^{df}E^a_I}], E^q_J [-\frac{M}{4 \sqrt{-\det(q^{ab})}} e^{abcdacq^{ce}q^{df}E^a_I}] \} = 0
\]

(3.21)
where we in the last step used $\epsilon_{abc}q^{bd}q^{ce} = \det(q^{ab})q_{af}\epsilon^{fde}$.

Thus the constraint algebra closes and the theory is complete and consistent, in this sense. Furthermore, the constraints satisfy the general constraint algebra that is required for any diffeomorphism invariant theory, with a metric. In ref. [3] Hojman et al showed that in any canonical formulation of a diffeomorphism invariant theory, with a metric, where one can find a set of constraints $C_a$ and $C$ generating spatial diffeomorphisms and time-like diffeomorphisms, these constraints will always satisfy the algebra above, with $C_a = \mathcal{H}_a$ and $C = \mathcal{H}$. It was also shown, as a consistency requirement of path-independence of deformations, that the spatial metric has to appear as a structure function in the Poisson bracket corresponding to (3.21) above. In our case, this means that the densitized spatial metric on the hyper surface is given by

$$q^{ab} := -E^a_l E^b_l.$$ (3.22)

However, note that we, at this point, cannot say anything about the physical relevance of this metric. Is this the metric that defines the causal structure of the theory? So far, we do not have any answer to that question, it should, however, be clear that this is the only consistent choice for a metric. If it later is shown that this theory allows non-causal propagation with respect to the light cones of the metric, the theory is unphysical and must be abandoned.

Now, the constraint algebra gave us the spatial metric. What about the complete spacetime metric? Knowing the spacetime metric is the same as knowing how the spatial metric evolves from one hyper surface to another, which corresponds to the time-evolution of the spatial metric. The following is true

$$\{q^{ab}, \mathcal{H}[N]\} = -2Ng^{\frac{1}{2}}(K^{ab} - g^{ab}K^{cc})$$ (3.23)

where $g^{ab} = \frac{1}{\sqrt{|q^{ab}|}}q^{ab}$ and $K^{ab}$ is the extrinsic curvature of the hyper surface embedded in spacetime. This, together with the conventional expression for the extrinsic curvature in terms of the spacetime metric, makes it straightforward to find the form of the spacetime metric in terms of the fields in the theory:

$$\tilde{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta} = \left( \frac{-N}{N^a} - NE^a_l E^b_l - \frac{N^a N^b}{N} \right).$$ (3.24)

With such a simple form of the metric, it is an easy task to put the Lorentz-signature condition on the basic fields. (In [4] another generalization of the Ashtekar Hamiltonian was given, but for that model, the metric had a very complicated form, making it almost impossible to put the signature condition directly on the basic fields.) Here, Lorentz-signature corresponds to $q^{ab}$ positive definite.

It is at this stage the need for complex fields appears (for Lorentzian signature). If $q^{ab}$ is positive definite, $\epsilon_{IJK}(E_L^d)$ becomes complex, unless $E^a_l$ is imaginary. In both cases, the formulation becomes complex. If one tries to rescale the Hamiltonian by multiplying with $\sqrt{|E^a_l E^b_l|}$, the result is that the right-hand side of (3.21) changes so that the spatial metric becomes $-|E^{cK}E^d_K| E^a_l E^b_l$, a metric that by construction cannot be positive definite for real fields. For Euclidean signature, $q^{ab}$ is required to be negative definite, which can be accomplished without ever having to introduce complex fields. Thus, the Lorentzian case still needs complex fields regardless of the signature of the "group-metric", 8
and the Euclidean case works perfectly all right with real fields. This does not, however, mean that the hope of finding a real (non-complex) unified theory is completely dead. It may be that there exist another type of modification of the Ashtekar variables that gives a real theory, perhaps a modification that only works for a very special gauge group with some particular feature. After all, that is precisely what we want; a problem in the theory that can only be solved by choosing the correct gauge group.

### 3.3 Reality conditions

If we want to have a consistent and physically sensible theory in the Lorentzian case, we must allow the fields to be complex, and, furthermore, we need good reality conditions that will give us real physical quantities. The question is then, what objects are physical, and should therefore always be real in a solution? Normally, one says that the metric is physical while for instance a Yang-Mills connection is not considered physical. The reason for this is that the connection is not gauge-invariant and therefore cannot be a physical measurable object, while the metric normally is considered to be measurable, in some sense. Also, if one consider the underlying Lorentzian spacetime to be real (which one normally does), the invariant proper time $d\tau^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ will be complex if the metric is not real. However, seen from the point of view of canonical formulations of theories with local symmetries, the diffeomorphism symmetry is on the same footing as the gauge-symmetry. And, according to Dirac, no objects that transform under the transformations generated by the first class constraint should be considered physical. This means that only gauge and diffeomorphism invariant objects are to be considered measurable. And, if one really scrutinizes what is done when the metric is ”measured”, one realizes that the metric always is coupled to some matter fields or test-particles, meaning that it is really the diffeomorphism invariant combination of metric and matter that is measured.

This discussion is included to show that it is not so obvious how to impose the reality conditions. Of course, the safe way is to require all ”electric” and ”magnetic” fields to be real, although that may be a too severe restriction on the theory. (Actually, in the Ashtekar formulation for pure gravity, one cannot require the ”magnetic” field to be real since it equals the self dual part of the Riemann tensor, in a solution.)

A more sensible compromise is to say that the metric and all gauge-invariant combinations of the Yang-Mills part should be real. But already here we run into problems; how can one impose these conditions on the Yang-Mills part without breaking the full symmetry. That is, in the full theory, where the fields take values in the unified Lie-algebra, there is no splitting between gravity and the Yang-Mills fields, so how can one impose reality conditions that only restricts one of the sectors? And, a unified theory where the reality conditions can only imposed after the symmetry has been broken, seems rather artificial.

To us, all these complications and the uglification of the theory due to the need for complex fields and reality conditions, indicate that this theory is not the unified description of gravity and Yang-Mills theory we are looking for. Instead, we believe that there probably exists another (real?) unified theory, possibly constructed with the use of a, more or less, unique gauge group, and that this formulation could be very close to the model presented here. Finally, since we do not have any good reality conditions, we cannot really say that we have a theory for Lorentzian signature; we have a theory for Euclidean signature or complex gravity.
4 Weak-field expansion

A minimum requirement on a unified theory of gravity and Yang-Mills theory is that the Einstein theory of gravitation and the standard Yang-Mills theory appear in some limit of the theory. We already know that the unified model, presented in the previous section, reduces to the Ashtekar formulation for Einstein gravity with a cosmological constant if the gauge group is chosen to be $SO(3, C)$. Thus, what is left to check is if the conventional Yang-Mills theory appears in some limit of the theory, and the natural limit to consider is weak fields around approximate flat spacetime. That is, we know that the $U(1)$ Yang-Mills theory on Minkowski spacetime describes electromagnetism to a very high accuracy. So, in order not to get in conflict with that observation, we need to find the $U(1)$ Yang-Mills theory when we expand the unified theory around approximate Minkowski spacetime (approximate, since the non-zero cosmological constant forces us to expand around de Sitter spacetime instead).

To do this expansion, we assume that the unified symmetry is broken down to a direct-product symmetry: $G^{tot} = SO(3, C) \times G^{YM}$ where $SO(3, C)$ is supposed to be the gravitational symmetry group, and $G^{YM}$ is the internal symmetry group for the Yang-Mills part. The gauge indices are; $i, j, k, ...$ denote $SO(3, C)$ indices, while $A, B, C, ...$ are the $G^{YM}$ indices. For instance: $E^a_i E^b_i = E^a_i E^b_i + E^a_A E^b_B$.

We do not need to use the explicit de Sitter solution to find the expansion, it suffices to know that

$$B^{ai} = -\frac{2i\lambda}{3} E^{ai}$$

for the de Sitter solution, in terms of Ashtekar variables. See section 6.2. The exact solution will be denoted by a bar on top of the symbols, and the perturbations around it with lower case letters.

$$E^{ai} = \bar{E}^{ai} + e^{ai}$$
$$E^{aA} = \bar{E}^{aA} + e^{aA}$$
$$B^{ai} = \bar{B}^{ai} + b^{ai} = -\frac{2i\lambda}{3} \bar{E}^{ai} + b^{ai}$$
$$B^{aA} = \bar{B}^{aA} + b^{aA}$$
$$N = \bar{N} + n$$
$$N^a = \bar{N}^a + n^a$$
$$\Lambda_I = \bar{\Lambda}_I + \lambda_I$$

Here, we have used $\bar{E}^{ai}$ and the fact that the Yang-Mills fields vanishes in the de Sitter solution. We have also picked a coordinate-system where the shift vector $\bar{N}^a$ vanishes. Now, weak-field expansion here means: $e^{ai} \ll \bar{E}^{ai}, b^{ai} \ll \lambda \bar{E}^{ai}$, $e^{aA} e^{bA} \ll \bar{q}^{ab} := -\bar{E}^{ai} \bar{E}^{bi}$ and $e^{aA} b^{aA} \ll \lambda \bar{E}^{ai} \bar{E}^{bi}$. Since we are mainly interested in the Yang-Mills part, we do not write out the pure gravity perturbation (although, it is a straightforward task to use the above expansion to get the gravitational part as well):

$$N^a H = \bar{N}^a H^{(2)}_{YM} + N^a H^{GR} + O(e^4, b^4, ...)$$
$$N^a H_a = \bar{N}^a H_a^{GR} + O(e^4, b^4, ...)$$
$$\Lambda^I G_I = \lambda^I G^{(2)YM}_I + \lambda^I G^{GR}_I + O(e^4, b^4, ...)$$
where
\[ \tilde{\mathcal{H}}_{YM}^{(2)} = \frac{N}{4} \frac{1}{\sqrt{-|\bar{q}^{ab}|}} (\epsilon_{abc} \epsilon_{def} \bar{q}^{ad} \bar{q}^{be} e^{-A} (b_A^f + \frac{2i\lambda}{3} e_A^f)) \] (4.4)
\[ \lambda^A G_{A}^{(2)YM} = \lambda^A D_a e_A^a = \lambda^A (\partial_a e_A^a + f_{ABC} a_B^a e^aC) \]

and the \( O(e^A, b_A, ...) \) term includes all the higher order terms. \( |\bar{q}^{ab}| \) is the determinant of the spatial metric \( \bar{q}^{ab} := -\bar{E}^{ai} \bar{E}^{bi} \). These expressions are to be compared to the conventional Yang-Mills total Hamiltonian on any fixed background [6, 7]:
\[ H_{\text{tot}}^{\text{conv}} = \frac{N}{4\sqrt{|\bar{q}^{ab}|}} \epsilon_{abc} \epsilon_{def} \bar{q}^{ad} \bar{q}^{be} (e^{-A} e_A^f + \frac{1}{4} b^c A b^f_A) + N_a \frac{1}{2} \epsilon_{abc} e^A b^c_A + \lambda^A D_a e^A_a \] (4.5)

To get exact agreement, we perform a canonical transformation in the unified theory:
\[ \tilde{e}^A := e^A - \frac{3i}{4\lambda} b^A, \text{ and } a_A \text{ unchanged. With this, (4.4) becomes:} \]
\[ \tilde{\mathcal{H}}_{YM}^{(2)} = \frac{-\bar{N}}{2\sqrt{|\bar{q}^{ab}|}} \epsilon_{abc} \epsilon_{def} \bar{q}^{ad} \bar{q}^{be} (\frac{\lambda}{3} e^{-A} e_A^f + \frac{3}{16\lambda} b^c A b^f_A) \] (4.6)

where we see that the physical Yang-Mills fields are:
\[ e_{A}^{a,\text{phys}} = \sqrt{\frac{2\lambda}{3}} e^{-A} \] (4.7)
\[ b_{A}^{a,\text{phys}} = \sqrt{\frac{3}{2\lambda}} b^{-A} \]

and that the unified theory (2.1) reproduces the conventional Yang-Mills theory to lowest order. This means that in a weak field expansion around de Sitter spacetime, the rescaled Yang-Mills fields (4.7) will be governed by the Yang-Mills equations of motion. For a \( U(1) \) Yang-Mills field, we know that these equations of motion are Maxwell’s equations which are very well experimentally confirmed. So the question is, for what energy scales does this unified theory predict significant corrections to the Maxwell’s equations? For a very small \( \lambda \) (since \( \lambda \) has dimension inverse length square, we really mean; on length scales where \( \lambda r^2 \ll 1 \) we know that the de Sitter metric is approximately the Minkowski metric. So, \( \bar{E}^{ai} \bar{E}^{bi} \approx \delta^{ab} \) in cartesian coordinates. The weak field expansion is then good for
\[ e_{A}^{a,\text{phys}} e_{A,\text{phys}}^b \ll \frac{2\lambda}{3} \delta^{ab} \] (4.8)

With an experimental upper bound on \( \lambda \) of \( 10^{-62} \ m^{-2} \) [8] this restricts the electric field to be much weaker than \( 10^{-4} V/m! \) This seems to be a severe problem for this theory: it predicts large corrections to Maxwell’s equations already for rather modest field strengths. Note however that the cosmologically constant here really just is a representative for any slowly varying background energy density. This means that in an experiment in a lab here on earth we must include in \( \lambda \) all the contributions coming from e.g thermal energy. (In room temperate air, the heat energy-density is about \( 10^{-40} \ m^{-2} \) in natural units, which means that the restriction on the electric field increases to \( 10^{8} V/m. \)
5 Matter couplings

Although it is an interesting achievement by itself to find a consistent unification of gravity and Yang-Mills type interactions, the theory is not of much use if it doesn't allow the introduction of matter couplings (at least spinors). In this section we show that the introduction of a Higgs type scalar field (for arbitrary gauge group) is straightforward, while spinors cannot be included by a simple generalization of the conventional spinor coupling. The scalar field coupling is found by a trivial generalization of the standard scalar field coupling in the Ashtekar formulation [6]. We show that the constraint algebra continues to be closed without the introduction of further constraints. Moreover, the simple form of the spacetime metric \((3.24)\) still holds. We have not looked for/found any reality conditions here either.

5.1 Scalar field

As mentioned above, the coupling to the scalar field is found by trivially generalizing the scalar field coupling in the conventional Ashtekar formulation [6]. Here, we consider the scalar field to take values in the vector representation of the unified Lie-algebra. If one wants to treat scalar fields which transforms as scalars under the internal symmetry group as well, this is easily accomplished by simply dropping the internal index on the scalar field. Here is the Hamiltonian.

\[
H_{\text{tot}} = \int_{\Sigma} d^3x \left( N(\mathcal{H}^G + \mathcal{H}^\Phi) + N^a(\mathcal{H}_a^G + \mathcal{H}_a^\Phi) + \Lambda^I(\mathcal{G}_I^G + \mathcal{G}_I^\Phi) \right) 
\]

The fundamental Poisson bracket for the scalar field \(\Phi_I\) and its momenta \(\Pi_I\) is:

\[
\{\Phi_I(x), \Pi_J(y)\} = \delta^I_J\delta^3(x - y).
\]

To calculate the constraint algebra, we again use the shortcut described in section 3.2; first identify the generators of gauge transformations and spatial diffeomorphisms, then calculate all Poisson brackets containing these generators using the transformation-properties of the constraints.

First, consider the transformations generated by the total Gauss law constraint, \(\mathcal{G}_I^{\text{tot}} := \mathcal{G}_I^G + \mathcal{G}^\Phi_I\). (We also define \(\mathcal{H}_a^{\text{tot}}\) and \(\mathcal{H}^{\text{tot}}\) in an analogous way) Since the matter part \(\mathcal{G}^\Phi_I\) does not depend on \(A_{al}\) or \(E^a_l\), the transformations of them are unaltered, and given in (3.10) and (3.11). For \(\Phi_I\) and \(\Pi_I\), we get

\[
\delta^{\mathcal{G}_I^{\text{tot}}} \Phi_I := \{\Phi_I, \mathcal{G}_I^{\text{tot}}[\Lambda^J]\} = f^{IJK} \Lambda_J \Phi_K
\]

\[
\delta^{\mathcal{G}_I^{\text{tot}}} \Pi_I := \{\Pi_I, \mathcal{G}_I^{\text{tot}}[\Lambda^J]\} = f_{IJK} \Lambda^J \Pi^K
\]
which shows that \( \mathcal{G}^{\text{tot}}_I \) generates gauge transformations on all the basic fields. Then, we define \( \mathcal{H}_a := \mathcal{H}_a^{\text{tot}} - A_{at} \mathcal{G}^{\text{tot}}_I \). The matter part of this constraint is manifestly independent of \( E^aI \), and the terms containing both \( A_{at} \) as well as matter fields cancel, implying that the transformations on \( A_{at} \) and \( E^aI \) are still given by (3.12) and (3.13). The action on the matter fields are

\[
\delta \mathcal{H}_a^{\text{tot}} \Phi^I := \{ \Phi^I, \mathcal{H}_a^{\text{tot}}[N^a] \} = N^a \partial_a \Phi^I = \mathcal{L}_{N^a} \Phi^I \tag{5.10}
\]

\[
\delta \mathcal{H}_a^{\text{tot}} \Pi_I := \{ \Pi_I, \mathcal{H}_a^{\text{tot}}[N^a] \} = \partial_a (N^a \Pi_I) = \mathcal{L}_{N^a} \Pi_I \tag{5.11}
\]

which again shows that \( \mathcal{H}_a^{\text{tot}} \) is the true generator of spatial diffeomorphisms. With these results, we do not even have to write out the Poisson brackets containing \( \mathcal{G}^{\text{tot}}_I \) and \( \mathcal{H}_a^{\text{tot}} \). The result is already given in (3.16)-(3.20). The last Poisson bracket is \{\mathcal{H}^{\text{tot}}[N], \mathcal{H}^{\text{tot}}[M]\}, and to do this calculation one should again notice that only terms containing derivatives of \( N \) and \( M \) will survive. Thus, an inspection of the Hamiltonian constraint gives that the only terms that could give a non-zero result are: \{\mathcal{H}^{G}[N], \mathcal{H}^{G}[M]\}, \{\mathcal{H}^{G}[N], \frac{1}{2}q^{ab}D_a \Phi^I D_b \Phi_I[M]\} \sim (N \leftrightarrow M) \text{ and } \{\frac{1}{2} \Pi^I \Pi_I[N], \frac{1}{2}q^{ab}D_a \Phi^I D_b \Phi_I[M]\} \sim (N \leftrightarrow M). The final result is

\[
\{\mathcal{H}^{\text{tot}}[N], \mathcal{H}^{\text{tot}}[M]\} = \mathcal{H}_a^{\text{tot}}[q^{ab}(N \partial_b M - M \partial_b N)],
\tag{5.12}
\]

showing that the constraint algebra is closed. Thus, the theory described by (5.1) is complete and consistent. From (5.12) it is also clear that the spatial metric is still given by \( q^{ab} = -E^aI E^bI \). Furthermore, since the total structure of the algebra is completely similar to the matter-free case, it follows that the entire spacetime metric is again given by (3.24).

### 5.2 Spinor field

We have not managed to find a spinor coupling that gives a closed constraint algebra for the unified theory. Here, we just want to show what the problem is of using the obvious generalization of the conventional spinor coupling.

Since we know the form of the conventional gravity-spinor coupling in Ashtekar’s variables [3], the obvious first attempt to include spinors into the unified theory, is to generalize that coupling. Thus, we replace the gravitational part of the Hamiltonian constraint with (3.1), and replace the Pauli-matrices – the spin-\( \frac{1}{2} \) representation of \( \text{so}(3) \) – with the representation \( T^A_I \) for the unified gauge Lie-algebra.

The Hamiltonian is

\[
H^{\text{tot}} = \int d^3x \left( N(\mathcal{H}^G + \mathcal{H}^{Sp}) + N^a(\mathcal{H}^G_a + \mathcal{H}^{Sp}_a) + \Lambda^I (\mathcal{G}^I_I + \mathcal{G}^{Sp}_I) \right) \tag{5.13}
\]

\[
\mathcal{H}^G := -\frac{1}{4} \epsilon_{abc} \epsilon_{IJK} (E^aI)E^bJ(B^cK + \frac{2i\lambda}{3}E^cK) \tag{5.14}
\]

\[
\mathcal{H}^{Sp} := -\sqrt{2} E^aI T^A_I B^a \Pi^A \mathcal{D}_a \lambda_B \tag{5.15}
\]

\[
\mathcal{H}_a := \frac{1}{2} \epsilon_{abc} E^bI B^cI \tag{5.16}
\]

\[
\mathcal{H}^{Sp}_a := \Pi^A \mathcal{D}_a \lambda_A \tag{5.17}
\]

\[
\mathcal{G}_I := \mathcal{D}_a E^aI = \partial_a E^aI + f_{IJK} A^a_a E^aK \tag{5.18}
\]

\[
\mathcal{G}^{Sp}_I := -\frac{1}{2} T^A_I \Pi^A \lambda_B \tag{5.19}
\]
The fundamental spinor field, $\lambda_A$ and its conjugate momenta, $\Pi^B$ satisfy the basic Poisson brackets:

$$\{\Pi^B(x), \lambda_A(y)\} = \delta^R_A \delta^3(x-y) \quad \{\lambda_A(x), \Pi^B(y)\} = \delta^R_A \delta^3(x-y)$$  \hspace{1cm} (5.20)

Here, $A, B, C, ...$ are spinor indices in the representation space of the unified Lie-algebra, $I, J, K, ...$ denote Lie-algebra indices in the vector representation, and $T^I_{IA}B$ is a representation of the unified Lie-algebra. The spinors are taken to be grassman odd: e.g $\lambda_A \Psi_B = - \Psi_B \lambda_A$. The action of the covariant derivative on the spinors is:

$$D_a \lambda_A = \partial_a \lambda_A - \frac{1}{\sqrt{2}} A_{al} T^B_{IA} \lambda_B$$  \hspace{1cm} (5.21)

$$D_a \Pi^A = \partial_a \Pi^A + \frac{1}{\sqrt{2}} A_{al} T^B_{IA} \Pi^B$$  \hspace{1cm} (5.22)

Now, it is again straightforward to check that $G^{\text{tot}}_I$ and $\tilde{H}^{\text{tot}}_a := H^{\text{tot}}_a - A^I_a G^{\text{tot}}_I$ generate gauge transformations and spatial diffeomorphisms, respectively. We also knows that all constraints are diffeomorphism-covariant, and that $G^{\text{tot}}_I$ and $H^{\text{tot}}_a$ are gauge-covariant. This means again that the crucial calculation is $\{H^{\text{tot}}[N], H^{\text{tot}}[M]\}$. Here, however, we run into trouble; this last Poisson bracket fails to close the algebra. We get

$$\{H^{\text{tot}}[N], H^{\text{tot}}[M]\} = H^G_{ab} q^{ab} (N \partial_b M - M \partial_b N) + \int_{\Sigma} d^3x \left( \sqrt{2} (M \partial_a N - M \partial_a M) E^{aI} E^{bJ} \Pi^A D_b \lambda_B \times \right)$$

$$\left( \epsilon_{IK} (E^D) T^C_{A} + \sqrt{2} T_{IA} B T_{JB}^D \right).$$  \hspace{1cm} (5.23)

The reason why the constraint algebra closes when the Lie-algebra is $so(3)$ and the $T^I_{IA}B$'s are the Pauli-matrices, is that for that case $\epsilon_{IJK} (E^D) = f_{IJK}$ and we have the identity $T_{IA} B T_{IB}^C = - \frac{1}{2} \delta^A C - \frac{1}{\sqrt{2}} f_{IK} T^K_{IA} C$, meaning that we get a cancelation of "bad" terms in (5.23). Thus, although it is possible to find other Lie-algebras whose representations satisfy identities of the above type, it does not seem possible to cancel the term containing $\epsilon_{IJK} (E^D)$, for higher dimensional Lie-algebras. Therefore, it seems that the naive generalization of the conventional gravity-spinor coupling fails to give a closed constraint algebra for Lie-algebras of dimension $> 3$.

The solution to this problem might be that there exist another spinor coupling that gives a correct constraint algebra and reduces to the conventional one when the Lie-algebra is chosen to be $so(3)$. We have, however, not managed to find such a coupling.

### 6 Static, spherically symmetric solution to the $U(2)$ theory.

In some cases, it is much more rewarding to study an explicit solution to a theory than to study the full theory. It may, for instance, happen that some unphysical features of the theory are completely obvious in the explicit solution, while it is very hard to notice them in the full theory.
In this section, we will therefore derive the static and spherically symmetric solution to the unified model, for gauge group $U(2)$, and compare it to the normal Reissner-Nordström solution (with a cosmological constant). The Reissner-Nordström solution is a static and spherically symmetric solution to the conventional Einstein-Maxwell theory. In subsection (6.1) we will give the ansatz for the static and spherically symmetric fields, and also derive the quantities that are common for the conventional and the unified theory. In subsections (6.2) and (6.3) we derive the solutions for these two different theories.

6.1 Static and spherically symmetric ansatz

To find the spherically symmetric ansatz for the fields, we will use the results in [9]. In this reference, one requires that the Lie-derivative along the three rotational vector fields, when acting on the fields, will correspond to a constant gauge transformation. That is

$$\mathcal{L}_{L^a} V = \{V, G_i[A^i]\}$$

(6.1)

where $L^a$ denote the three rotational vector fields, $V$ represent any field in the theory, and $A^i$ is a constant gauge-parameter. By solving these equations, an ansatz was found. We will use the same ansatz here, but first we perform a gauge rotation to minimize the number of trigonometric functions in the fields. By using the $SO(3)$ group element

$$G = \begin{pmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{pmatrix}$$

(6.2)

we transform the ansatz in [9], and find the following expressions

$$E_i^r = E_1 \sin \theta V_i$$
$$E_i^\theta = E_2 \sin \theta Z_i + E_3 \sin \theta W_i$$
$$E_i^\phi = -E_2 W_i + E_3 Z_i$$
$$A_i^r = A_1 V^i$$
$$A_i^\theta = \frac{1}{2} A_3 W^i + \frac{1}{2} A_2 Z^i$$
$$A_i^\phi = \cos \theta V^i - \frac{1}{2} A_2 \sin \theta W^i + \frac{1}{2} A_3 \sin \theta Z^i$$

(6.3-6.8)

for the $so(3,C)$ valued fields $E_{ai}$ and $A_{ai}$. The index $r, \theta$ and $\phi$ denote the normal spherical coordinates. The internal vectors $V^i, W^i$ and $Z^i$ span a constant right-handed orthonormal basis for the internal space. For instance: $V^i = (1,0,0)$, $W^i = (0,1,0)$ and $Z^i = (0,0,1)$ is a good choice. $E_1, E_2, E_3, A_1, A_2$ and $A_3$ are all functions of $r$. When performing the gauge transformation on the connection $A_{ai}$, one must first choose a three dimensional representation for the Lie-algebra, use the transformation rule:

$$A_a^G = G A_a G^T - (\partial_a G) G^T$$

(6.9)

and then go back to the vector representation.

Now, since the magnetic field has the same properties as the electric field under Lie-derivative and gauge-transformations, it follows that $B_{ai}$ gets the same ansatz as $E_{ai}$.
above. By using the ansatz for $A_{ai}$, the following relations can be derived:

\[
B_1 = \frac{1}{2}(A_2^2 + A_3^2 - 4) \quad (6.10)
\]

\[
B_2 = A_1A_2 - A_3' \quad (6.11)
\]

\[
B_3 = A_2' + A_1A_3 \quad (6.12)
\]

The prime denotes derivative w.r.t the $r$-coordinate. The other fields in the theory are restricted by the spherical symmetry to have the following form:

\[
N = \frac{\bar{N}(r)}{\sin \theta} \quad (6.13)
\]

\[
N^a = (N'(r), 0, 0) \quad (6.14)
\]

\[
A_{0i} = A(r)V_i \quad (6.15)
\]

\[
A_0 = a(r) \quad (6.16)
\]

\[
E^a = \sin \theta(E'(r), 0, 0) \quad (6.17)
\]

\[
A_a = (A_r(r), 0, 0) \Rightarrow B^a = (0, 0, 0) \quad (6.18)
\]

The two theories that will be compared here, are the conventional minimally coupled Einstein-Maxwell theory and the unified theory for gauge group $U(2) \cong SO(3) \times U(1)$. For the conventional theory, the gravity Hamiltonian is given by (2.1) and the Maxwell part by (A.8) with gauge group $U(1)$. The Hamiltonian for the unified theory is given by (3.1). We notice that the only difference lies in the Hamiltonian constraint, meaning that we can treat the other constraints without specifying to which theory they belong. We list the result for the $SO(3)$ and the $U(1)$ Gauss law, as well as the vector constraint:

\[
\mathcal{G}_i = \partial_a E^a_i + f_{ijk} A^j_a E^{ak} = \sin \theta V_i(E_1' + E_2A_3 - A_2E_3) \quad (6.19)
\]

\[
\mathcal{G} = \partial_a E^a = E'^r \quad (6.20)
\]

\[
\mathcal{H}_a = \frac{1}{2} \epsilon_{abc}(E^b E^c_i + E^b E^c) = \delta_a^r \sin \theta(E_2B_3 - E_3B_2) \quad (6.21)
\]

### 6.2 Conventional theory

In the conventional theory, the Hamiltonian constraint is

\[
\mathcal{H} = -\frac{1}{4} \epsilon_{abc}\epsilon_{ijk} E^{ai} E^{bj}(B^{ck} + \frac{2i\lambda}{3} E^{ck}) + \frac{1}{2} \sqrt{|q^{ab}|} q_{ab}(E^a E^b + \frac{1}{4} B^a B^b) \quad (6.22)
\]

where $q^{ab} := -E^{ai} E^b_i$ and $q_{ab}$ is its inverse. We also know that the densitized spacetime metric is given by

\[
\tilde{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta} = \begin{pmatrix}
\frac{1}{N} & \frac{N^a}{N} \\
\frac{N^a}{N} & -NE^a E^b_i - \frac{N^a N^b}{N}
\end{pmatrix} \quad (6.23)
\]

and its determinant is

\[
\sqrt{-g} = N\sqrt{-|E^a E^b_i|} \quad (6.24)
\]
Note that we will assume the spatial metric $q^{ab}$ to be positive definite and real. Using the ansatz given in the previous subsection, we get

$$\mathcal{H} = \sin^2 \theta \left( \frac{i}{2} \frac{E_2^2 + E_3^2}{E_1} (E^r)^2 - \frac{1}{2} B_1 (E_2^2 + E_3^2) - E_1 (E_3B_3 + E_2B_2) - i\lambda E_1 (E_2^2 + E_3^2) \right)$$

(6.25)

Now it is straightforward to derive all the equations of motion, but before we write out the result, we will fix the remaining gauges. It is four gauges that need to be fixed: one component of the $SO(3)$ rotations, the radial part of the spatial diffeomorphisms, the $U(1)$ symmetry, and the time-like diffeomorphisms. The gauges we choose are

$$A_1 = 0$$
$$A_3 = 0$$
$$A_r = 0$$
$$E_1 = -ir^2$$

(6.26) (6.27) (6.28) (6.29)

We will not go into details about these choices. It is, however, straightforward to check that $A_1 = 0$ can be reached by an $SO(3)$ transformation from a generic field configuration. Similarly, $A_3 = 0$ and $A_r = 0$ can always be reached by transformations generated by $\mathcal{H}$ and $\mathcal{G}$, respectively. The reason for the choice $E_1 = -ir^2$ is that we want to recover the standard Reissner-Nordström form of the solution (in Schwarzschild coordinates). This choice is, however, not completely generic. There is another gauge choice that gives a physically inequivalent solution: $E_1 = constant$. But since we are only interested in comparing the standard Reissner-Nordström type solution, here, we will only study the former choice.

With these choices, the remaining equations are

$$E^r = q = constant$$
$$E_2 = 0$$
$$N^r = 0$$
$$A_2 = -\frac{2ir}{E_3}$$
$$A'_2 = -\frac{i}{2} \left( \frac{1}{2} A_2^2 - 2 \right) \frac{E_3}{r^2} - i\lambda E_3 - \frac{i}{2} \frac{E_3q^2}{r^4}$$
$$A' = \tilde{N}(E_3 A'_2 + i\lambda E_3^2 - \frac{i}{2} \frac{E_3q^2}{r^4})$$
$$A_2 A = \tilde{N}(E_3 \left( \frac{1}{2} A_2^2 - 2 \right) - ir^2 A'_2 + 2\lambda r^2 E_3 + \frac{E_3q^2}{r^2})$$
$$A = \frac{\tilde{N}}{2} A_2 E_3 + \frac{i}{E_3} \partial_r (\tilde{N}r^2 E_3)$$
$$a' = \frac{\tilde{N} E_3^2}{r^2} q$$

(6.30) (6.31) (6.32) (6.33) (6.34) (6.35) (6.36) (6.37) (6.38)

Using (6.33) and (6.34) it is possible to solve for $E_3$ and $A_2$:

$$E_3 = -\frac{ir}{\xi(r)}$$

(6.39)
\[ A_2 = 2\xi(r) \]  
\[ \xi(r) := \sqrt{1 - \frac{C}{r} - \frac{\lambda r^2}{3} + \frac{q^2}{2r^2}} \]  

where \( C \) is a constant of integration. With this solution at hand, (6.35) and (6.36) give \( \tilde{N} \) and \( A \):

\[ \tilde{N} = D \frac{\xi^2(r)}{r^2} \]  
\[ A = -\frac{D}{2} \partial_r \xi^2(r) \]

where \( D \) is another constant of integration. Now, using the expression for the metric (6.23), the line-element becomes:

\[ ds^2 = -D\xi^2(r) dt^2 + \xi^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \]

which is the standard Reissner-Nordström solution in Schwarzschild coordinates. The conventional choices for \( D \) and \( C \) are: \( D = 1 \) and \( C = 2M \). The first choice just corresponds to a normalization of the time-coordinate, and the second one is related to the fact that \( M \) is the ADM-mass if \( \lambda = 0 \).

### 6.3 The unified U(2) theory

Since the gauge group \( U(2) \) locally is isomorphic to \( SO(3) \times U(1) \), we choose to split the theory into these parts directly from the beginning. Here, we have the problem of using a non semi-simple group, meaning that the Cartan-Killing form is degenerate. Therefore, we will instead choose \( g_{IJ} = \delta_{IJ} \) as our bilinear form. Note, however, that there is nothing that stops us from changing sign of the \( U(1) \) component of this "group-metric". In the end, these choices are simply related by an imaginary rescaling of the \( U(1) \) fields.

The Hamiltonian for the unified theory is

\[ \mathcal{H} = -\frac{1}{4} \epsilon_{abc} \epsilon_{IJK} (E^{aI}) (E^{bJ}) (E^{cK}) + \frac{2i\lambda}{3} E^{cK} = \frac{i}{2} \sqrt{q^{ab}} \left( q_{ab} E^{aI} B^{ib} - 2i\lambda \right) \]

where \( q^{ab} = -E^{aI} E^{bI} = -E^{ai} E^{bi} - E^a E^b \) and \( q_{ab} q^{bc} = \delta^c_a \). The \( U(2) \) indices are denoted \( I, J, K, \ldots \), and for \( SO(3) \) we use \( i, j, k, \ldots \).

The densitized spacetime metric is now given by

\[ \tilde{g}^{\alpha\beta} = \sqrt{-g} g^{\alpha\beta} = \left( \begin{array}{c} -\frac{1}{N} \frac{N^a}{N} - N E^{aI} E^{bI} \frac{N^a}{N} \end{array} \right) \]

and its determinant is

\[ \sqrt{-g} = N \sqrt{-|E^{aI} E^{bI}|} \]

To not get into trouble about the double-valueness of the square root for complex fields, we will always arrange the argument of the square root to be real and positive definite. Then, we choose the positive branch. The other branch just corresponds to \( N \rightarrow -N \).
With the static, spherically symmetric ansatz given in section 6.1, the Hamiltonian constraint becomes:

\[
H = \frac{i}{2} \sqrt{- (E_1^2 + (E_r)^2)(E_2^2 + E_3^2)} \sin^2 \theta \left( \frac{E_1B_1}{E_1^2 + (E_r)^2} + 2 \frac{E_2B_2 + E_3B_3}{E_2^2 + E_3^2} + 2i \lambda \right) \tag{6.48}
\]

Now, it is again straightforward to derive the equations of motion from (6.19)-(6.21), (6.48), but before writing them out, we fix the remaining gauges:

\[
A_1 = 0 \quad A_3 = 0 \quad E_1 = -i \sqrt{r^4 + (E_r)^2} \quad A_r = 0 \tag{6.49}
\]

The reason for these choices are the same as for the conventional theory: the two first can always be reached from a generic field-configuration with transformations generated by the Gauss law and the Hamiltonian constraint, respectively. The third choice is due to the wish to recover the solution in Schwarzschild-like coordinates. Note, however, that this choice is not completely generic. There exist another choice \(E_1 = \text{const} \) that gives a non-equivalent solution.

With these gauge choices, the constraints and the equations of motion become:

\[
E_2 = 0 \quad N^r = 0 \quad E^r = \tilde{q} = \text{constant} \tag{6.50}
\]

\[
A_2 = -\frac{2ir^3}{E_3\sqrt{r^4 + q^2}} \tag{6.51}
\]

\[
A'_2 = \frac{iE_3 \sqrt{r^4 + q^2}}{2r^4} \left( \frac{1}{2} A_2^2 - 2 \right) - iE_3 \lambda \tag{6.52}
\]

\[
A' = -\frac{i\tilde{N}}{2} \frac{2q^2 + r^4}{r^6} E_3 \left( \frac{1}{2} A_2^2 - 2 \right) \tag{6.53}
\]

\[
A_2A = i\tilde{N} r^2 A'_2 \tag{6.54}
\]

\[
A = \frac{\tilde{N}}{2} \frac{\sqrt{r^4 + q^2}}{r^2} E_3 A_2 + i \frac{1}{E_3} \partial_r (\tilde{N} r^2 E_3) \tag{6.55}
\]

Using (6.58) and (6.54), \(E_3\) and \(A_2\) are easily determined:

\[
E_3 = -\frac{i r^3}{\sqrt{r^4 + q^2}} \tilde{\xi}(r) \tag{6.56}
\]

\[
A_2 = 2 \tilde{\xi}(r) \tag{6.57}
\]

where we have defined

\[
\tilde{\xi}(r) := \sqrt{1 - \frac{\tilde{C}}{r} - \frac{1}{r} I(r)} \tag{6.58}
\]

\[
I(r) := \int_r^\infty ds \sqrt{\lambda s^4 + q^2} \tag{6.59}
\]

and \(\tilde{C}\) is a constant of integration. Then, (6.56) and (6.57) give us \(\tilde{N}\) and \(A\):

\[
\tilde{N} = \frac{i D \sqrt{r^4 + q^2} \tilde{\xi}^2(r)}{r^4} \tag{6.60}
\]

\[
A = \frac{i D \sqrt{r^4 + q^2}}{2r^2} \partial_r \tilde{\xi}^2(r) \tag{6.61}
\]
where $\tilde{D}$ is another constant of integration. Finally, one must check (6.55), which is satisfied with the solution given above.

Altogether, we now get the line-element

$$ds^2 = -\tilde{D}\xi^2(r)dt^2 + \frac{r^4}{r^4 + \tilde{q}^2}\xi^{-2}(r)dr^2 + r^2d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (6.64)$$

Far away from the charge, where $r^2 \gg \tilde{q}$, we can expand $g_{tt}$ and $g_{rr}$:

$$g_{tt} = -\tilde{D}\left(1 - \frac{\tilde{C}}{r} - \frac{\lambda r^2}{3} - \frac{\lambda \tilde{q}^2}{2r^2} + \mathcal{O}\left(\frac{\tilde{q}^4}{r^8}\right)\right) \quad (6.65)$$

$$g_{rr} = \left(1 - \frac{\tilde{C}}{r} - \frac{\lambda r^2}{3} - \frac{5\lambda \tilde{q}^2}{6r^2} + \mathcal{O}\left(\frac{\tilde{q}^4}{r^8}\right)\right)^{-1} \quad (6.66)$$

Similarly, we may expand the metric close to the charge $r^2 \ll \tilde{q}$

$$g_{tt} = -\tilde{D}\left(1 - \frac{\tilde{C}}{r} - \frac{\lambda r^4}{5\tilde{q}} + \mathcal{O}\left(\frac{r^6}{\tilde{q}^3}\right)\right) \quad (6.67)$$

$$g_{rr} = \frac{r^4}{\tilde{q}^2}(1 - \frac{r^4}{\tilde{q}^2})\left(1 - \frac{\tilde{C}}{r} - \frac{\lambda r^4}{5\tilde{q}} + \mathcal{O}\left(\frac{r^6}{\tilde{q}^3}\right)\right)^{-1} \quad (6.68)$$

Thus, we see that the solution coincides with the conventional Schwarzschild-de Sitter solution when $\tilde{q} = 0$, and it resembles the Reissner-Nordström solution for $r^2 \gg \tilde{q}$. Note, however, that the numerical factors in the electric charge-terms do not agree. Remember also that the physical charge is $q_{\text{phys}} \sim \sqrt{\tilde{q}}$. We also see that the wormhole like feature of the solution that was found in the (2+1)-dimensional case [10] does not appear here. That is because the sign inside the square root $\sqrt{r^4 + \tilde{q}^2}$ is a plus-sign, whereas the corresponding square root in (2+1)-dimensions had a minus-sign. This sign is, however, closely related to the choice of bilinear invariant form of the Lie-algebra, and by choosing this bilinear form to be $g_{IJ} = \text{diag}(1, 1, 1, -1)$ we would have recovered the same solution with $q^2 \to -q^2$. In the true theory, this sign will depend on how the $U(1)$ and $SO(3)$ algebras are embedded in the full algebra. We have not tried to extend this solution beyond the horizons.

### 7 Conclusions

We conclude this paper by noting that the unified model presented here is interesting mainly because it gives the conventional Yang-Mills theory in the weak field limit. However, this theory still has a few unwanted features, as e.g. the need for complex fields for Lorentzian spacetime, and the incapability of selecting a unique gauge group. Therefore, we do not believe that this model is to be taken serious as a candidate theory for the unified description of gravity and Yang-Mills theory. Instead, we hope for a modification of this theory that will solve the above mentioned problems.

**Acknowledgements**

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We thank Ingemar Bengtsson and Fernando Barbero for discussions and suggestions. S.C is grateful to U.G.C (India) and Council for International Exchange of Scholars (CIES), for the opportunity to visit the Center for Gravitational Physics and Geometry, Penn State University. S.C’s work was supported in part by the grant from CIES, grant no. 17263. P.P’s work was supported by the NFR (Sweden) contract no. F-PD 10070-300, and Per Erik Lindahls Fond, Kungliga Vetenskapsakademien.
A Legendre transforms

Here, we will derive the Hamiltonian formulation of the Yang-Mills Lagrangian.

The Lagrangian density for the Yang-Mills theory is

\[ \mathcal{L}^{YM} = \frac{k}{4} \sqrt{-g} g^\alpha{}_{\beta} g^\delta{}_{\epsilon} F^A{}_{\alpha\delta} F^A{}_{\beta\epsilon} \]  

where \( g^\alpha{}_{\beta} \) is the inverse metric, \( g \) the determinant of the metric, and \( F^A{}_{\alpha\beta} \) is the YM field strength in the vector representation. We also have an arbitrary multiplicative constant \( k \) included. First, we do a \((3+1)\)-decomposition:

\[ \mathcal{L}^{YM} = \frac{k}{2} \sqrt{-g} \left( g^{00} g^{ab} F^A{}_{0bA} - g^{0a} g^{0b} F^A{}_{0bA} + 2 g^{0a} g^{bc} F^A{}_{acA} + \frac{1}{2} g^{ab} g^{cd} F^A{}_{acA} F^A{}_{bdA} \right) \]  

and then define the momenta canonically conjugate to the YM connection \( A_{aA} \):

\[ E_a^A := \frac{\delta \mathcal{L}^{YM}}{\delta \dot{A}_a^A} = k \sqrt{-g} \left( g^{00} g^{ab} F^A_{b0A} - g^{0b} g^{0a} F^A_{0bA} + g^{0b} g^{ac} F^A_{bca} \right) \]  

Now we introduce the ADM-decomposition of the metric:

\[
\begin{align*}
  g^{00} & = - \frac{1}{M^2} \\
  g^{0a} & = \frac{N^a}{M^2} \\
  g^{ab} & = h^{ab} - \frac{N^a N^b}{M^2} \Rightarrow \sqrt{-g} = \frac{M}{\sqrt{|h^{ab}|}}
\end{align*}
\]  

Then we invert the relation (A.3):

\[ F^A{}_{ab} = - \frac{M}{k} \sqrt{|h^{ab}|} h_{bc} E^c^A + \frac{1}{2} N^e \epsilon_{ebd} B^d^A \]  

where \( h^{ab} \) is the inverse to \( h_{ab} \), and we have defined the YM magnetic field: \( B^a^A := \epsilon^{abc} F^A{}_{bc} \). Thus, the total YM Hamiltonian density becomes

\[
\mathcal{H}^{tot} = \mathcal{H}_a^A \dot{A}_a^A - \mathcal{L}^{YM} = \mathcal{H}_a^A \left( F^A_{0a} + D_a A_0^A \right) - \mathcal{L}^{YM} = - \frac{M}{2k} \sqrt{|h^{ab}|} h_{ab} E^a^A E^b^A \\
- \frac{kM}{8} \sqrt{|h^{ab}|} h_{ab} B^a^A B^b_A + \frac{1}{2} N^a \epsilon_{abc} E^b^A B^c_A - A_{0A} D_a E^a^A
\]  

Finally, to make contact with the Ashtekar variables, we define

\[
\begin{align*}
  q^{ab} & := - E^{ai} E^{b}_i := \frac{h^{ab}}{|h^{cd}|} \\
  N & := M \sqrt{|h^{ab}|}
\end{align*}
\]  

which gives the final Hamiltonian

\[
\begin{align*}
  H^{YM} & = \int \Sigma d^3x \left( N \mathcal{H}^{YM} + N^a \mathcal{H}_a^Y M - A_{0A} \mathcal{G}_A^Y \right) \\
  \mathcal{H}_a^Y & = - \frac{1}{2} \sqrt{|q^{ab}|} q_{ab} \left( \frac{1}{k} E^a^A E^b_A + \frac{k}{4} B^a^A B^b_A \right) \\
  \mathcal{H}_a^Y & = \frac{1}{2} \epsilon_{abc} E^b^A B^c_A \\
  \mathcal{G}_A^Y & = D_a E^a^A = \partial_a E^a_A + f_{ABC} A^B_a E^{AC}
\end{align*}
\]
If one picks an invariant bilinear form for the YM Lie-algebra that is positive definite, one normally considers a negative $k$ in order to get a positive definite Hamiltonian.

## B Dimensions and units

Here, we want to introduce dimensions and a unit-system in the theory discussed in this paper. The normal convention to put some or all of the fundamental constants of nature equal to one, is very useful in doing calculations. However, as soon as one wants to start comparing with reality, it is often more convenient to express the results in a system of units that one is used to. We have chosen to use the SI unit-system here. In this system, the fundamental units for length, mass, time and charge are meter (m), kilogram (kg), second (s) and Coulomb (C). (Charge is equal to current times time, and often it is convenient to use Ampere (A) instead of Coulomb as a fundamental unit. The following is true: $1\text{C}=1\text{As}$.) To introduce units in the theory, we need a set of dimension-full constants: the gravitational constant, $G = 6.67 \times 10^{-11}\text{m}^3\text{s}^{-2}\text{kg}^{-1}$, the speed of light, $c = 3 \times 10^8\text{m/s}$, the Planck constant, $\hbar = 1.05 \times 10^{-34}\text{kgm}^2\text{s}^{-1}$. When using the SI-units, one also needs the dielectric constant for vacuum, $\varepsilon_0 = 8.85 \times 10^{-12}\text{As}^2\text{m}^{-3}$.

Now, we know that a good action should have dimension energy$\times$time, which has units: $[S] = \text{kg m}^2\text{s}^{-2}$. Here, the square brackets denote the units of the enclosed object. $S$ stands for the action. If we then assume that all our spacetime coordinates have dimension length (we choose a coordinate system where this is true), the metric becomes dimension-less, and the Riemann tensor gets dimension $\text{length}^{-2}$. Consequently, the Einstein-Hilbert action for gravity gets dimension $\text{length}^2$: $[g_{\alpha\beta}] = 1\quad [R_{\alpha\beta\delta}] = \frac{1}{m^2}\quad [d^4x] = m^4 \Rightarrow [S_{EH}] = [d^4x]\sqrt{-g}[R] = m^2 \quad (B.1)$

To get the correct dimension on the action, we need to multiply it with a constant of dimension $\text{mass}^{-3}\text{time}^2$. The combination $\frac{c^3}{G}$ will do the job, meaning that the dimensionally correct action is (throughout this paper, we have neglected the conventional factors of $\pi$)

$$S = \frac{c^3}{G} \int_M d^4x \sqrt{-g}R \quad (B.2)$$

To find out the correct dimension for the Maxwell field, $A_{\alpha}$, we start from the expression of the Lorentz-force on a charged particle in an electromagnetic field:

$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B}) \quad (B.3)$$

Here, $Q$ is the charge, $\vec{F}$ the force, $\vec{E}$ the electric field, $\vec{v}$ the velocity, and $\vec{B}$ the magnetic field. Since the unit for force is $1\text{N} = 1\text{kgm}\text{s}^{-2}$, it follows that the units for the electromagnetic fields are $[\vec{E}] = \frac{\text{kgm}}{\text{As}}$ and $[\vec{B}] = \frac{\text{kgm}}{\text{As}}$. With $\vec{B} = \nabla \times \vec{A}$, the unit for the vector potential becomes: $[A_{\alpha}] = [\vec{A}] = \frac{\text{kgm}}{\text{As}}$. (To make contact with the conventional units for measuring electromagnetic fields, one can use the relations: $1\text{V} = \frac{\text{kgm}^2}{\text{As}^2}$ and $1\text{T} = \frac{\text{kg}}{\text{As}}$.)

Now, it is an easy task to calculate the units of the Maxwell action:

$$[S_{Max}] = [d^4x]\sqrt{-g}[g^{\alpha\beta}\partial_{\gamma}F_{\alpha\beta}][F_{\alpha\beta}F_{\beta\gamma}] = \frac{kg^2m^4}{A^2s^4} \quad (B.4)$$
Thus, we need to multiply this action with a constant with units $A^2 s^3$, and the natural choice for this constant is $\epsilon_0 c$. so, the total action for the coupled Einstein-Maxwell theory becomes

$$S^{\text{tot}} = \frac{c^3}{G} \int_M d^4x \sqrt{-g} (R - 2\lambda) - \epsilon_0 c \int_M d^4x \sqrt{-g} g^{\alpha\beta} g^{\delta\epsilon} F_{\alpha\delta} F_{\beta\epsilon} \quad (B.5)$$

Our conventions for the gravitational fields are that we keep all metric variables $(E_{ai}, N, N^a)$ dimension-less, and let the connection have dimension length$^{-1}$. By doing that, we do not need to introduce $G$ into the pure gravity theory. Effectively this means that we have pulled out the factor $\frac{G}{c^3}$ from the total action, and we instead get a factor $\epsilon_0 G c$ multiplying the Maxwell action. And, since the Hamiltonian formulation of the Maxwell (Yang-Mills) action is found without this factor, we must introduce this factor in the Hamiltonian constraint according to (A.8):

$$\mathcal{H}^{\text{Max}} = \sqrt{|q_{ab}|} q_{ab} \left( \frac{c^2}{\epsilon_0 G} E^a E^b + \frac{1}{c^2} \frac{\epsilon_0 G}{4} B^a B^b \right) \quad (B.6)$$

(To further complicate things, note that the physical electric field really is: $E^a = \frac{\epsilon_0 G}{c^3} E^a_{\text{phys}}$ due to the rescaling of the momenta in the Legendre transform (A.3)) Now, it is straightforward to compare this Hamiltonian with the result of the weak-field expansion of the unified model (4.6). Thus, we see that

$$\sqrt{\lambda} e^a \sim \sqrt{\frac{c^2}{\epsilon_0 G}} E^a = \sqrt{\frac{c^2}{\epsilon_0 G}} \frac{\epsilon_0 G}{c^3} E^a_{\text{phys}} \quad (B.7)$$

$$\frac{1}{\sqrt{\lambda}} b^a \sim \sqrt{\frac{\epsilon_0 G}{c^2}} B^a_{\text{phys}} \quad (B.8)$$

meaning that

$$\delta^{ab} \gg e^a e^b \sim \frac{\epsilon_0 G}{c^3} E^a_{\text{phys}} E^b_{\text{phys}} \quad (B.9)$$

$$\lambda^2 \delta^{ab} \gg b^a b^b \sim \frac{\epsilon_0 G \lambda}{c^2} B^a_{\text{phys}} B^b_{\text{phys}} \quad (B.10)$$

Numerically this means: $E^a_{\text{phys}} \ll 10^{-14} V/m$ and $B^a_{\text{phys}} \ll 10^{-12} T$! Where we have used the value $10^{-62} m^{-2}$ for the cosmological constant. Note that the restriction on the electromagnetic fields scales as $\sqrt{\lambda}$.

Finally, we want to say a few word about the coupling to spinor fields. Normally when one couples spinors to electromagnetism, one uses the minimal coupling

$$\mathcal{D}_\alpha \Psi = \partial_\alpha \Psi + i e k A_\alpha \Psi \quad (B.11)$$

where $e$ is the charge of the electron an $k$ is some dimension-full constant used to get consistent dimensions for both terms. Now, by using the units for $A_\alpha$ given above $[A_\alpha] = \frac{kg}{M^2}$, we see that $k$ must have units: $[k] = \frac{e}{kg m^2}$, the units of $\text{action}^{-1}$. The only reasonable choice for such a constant is $\frac{1}{h}$. At first sight it seems strange that one has to introduce Planck's constant already in the classical theory, however, one may also have the viewpoint that the concept of spinors naturally belongs to the quantum theory anyway. That is, it does not make much sense to talk about a classical spinor. Furthermore, when one
couples a charged point-particle to electromagnetism, the term corresponding to (B.11) is really: $p_\alpha - eA_\alpha$, which in the quantization introduces the $\hbar$ by the definition $\hat{p}_\alpha = i\hbar \partial_\alpha$.

However, when we introduce the $U(1)$ covariant derivative in the unified model, we do not have any elementary charge or Planck constant available. Instead, combinations of the other constants will appear in the coupling. From (5.21) we see that

$$D_a \Psi = \partial_a \Psi + i A_a \Psi$$

(B.12)

for the $U(1)$ connection in the unified model, and we also know the relation between this connection and the physical Maxwell connection:

$$A_a \sim \sqrt{\frac{\lambda \epsilon_0 G}{c^2}} A_a^{phys}$$

(B.13)

Hence, the coupling constant $\frac{e}{\hbar}$ has been replaced by $\sqrt{\frac{\lambda \epsilon_0 G}{c^2}}$. These two constants do not numerically agree (unless the cosmological constant is extremely large), but that is really irrelevant since the value of this constant is renormalized in the quantum theory.

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