Polarization Properties of the ”Photon Pistol”

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Abstract

The deterministic single-photon emission by means of STIRAP through the atoms with degenerate levels is studied. The expression for the polarization matrix of the emitted photon is obtained and its dependence on the polarization of the driving laser field and on the initial atomic state is examined.

1 Introduction

The application of quantum optical devices in quantum information processing, specifically the efforts in constructing quantum memories [1, 2, 3, 4] and quantum networks [5, 6, 7], is an active research area. The necessary part of such devices, employing the photons as flying q-bits, is the deterministic single-photon emitter - the "photon pistol". The most promising scheme for the controlled generation of a single photon is based on the technique of vacuum stimulated Raman scattering involving adiabatic passage (STIRAP), proposed in [8] and then realized experimentally in [9, 10, 11, 12]. In these experiments the three-level Λ - type atom was adiabatically passing through a high-finesse cavity, one branch of the atomic Λ - type transitions was in resonance with the quantized cavity field, while the other one was driven by the coherent laser field. In course of interaction with the fields inside the cavity the atom emitted a single photon. The two polarization degrees of freedom of the emitted photon provide the most natural way to encode the q-bit, however, in the early experiments [9, 10] the polarization of the emitted photons was not controlled, nor detected, and the degeneracy of resonant levels was not taken into account. In the subsequent experiments [11, 12] the magnetic field directed along the cavity axis was applied to lift the degeneracy of atomic levels and the frequencies of the driving field and of the cavity modes were adjusted to realize the non-degenerate Λ-scheme with three Zeeman states, which allowed to produce the circularly polarized photons. The objective of this paper is to calculate the polarization state of the emitted photon and to study its dependence on the polarization of the driving laser field and on the atomic initial state for the arbitrary values of the level angular momenta and to look for the schemes enabling to produce photons with tailored polarizations.

The STIRAP with the three-level atom with non-degenerate levels is well described in the reviews and textbooks (see, e.g., [13, 14]). In case of non-degenerate levels there exists the only dark state – the superposition of the
ground state $a$ and some metastable state $b$ – uncoupled to the excited level $c$. In course of Raman scattering the atom is adiabatically transferred from the initial state $a$ to the target state $b$. The STIRAP with degenerate atomic levels and with classical coherent resonant fields was studied in [15], were it was shown that unlike the non-degenerate case the population from the initial ground state $a$ is not always totally transferred to the target level $b$, but only if the order of degeneracy of this target state is not less, than the order of degeneracy of the initial state. In our previous papers [16, 17] we have studied the single-photon emission via non-adiabatic vacuum stimulated Raman scattering of short pulses on the atoms with degenerate levels and with totally or partially resolved hyperfine structure, the polarization of the cavity mode was assumed to be well-determined and the dependence of the photon emission probability on the mutual orientation of polarizations of the driving pulse and that of the cavity mode was calculated. In the present paper the adiabatic Raman scattering (STIRAP) is considered, and it is assumed that the cavity equally sustains both polarization modes.

In section 2 the interaction model and the evolution operator for this model in the adiabatic approximation are described, while in sections 3 and 4 the instantaneous eigenvectors of the interaction operator, which determine the evolution operator, are constructed and classified. In case of degenerate levels not only the number of these eigenvectors increases, but there appear the new types of these eigenvectors, non-existing in case of non-degenerate levels, like the dark states, which atomic part belongs to only one of the lower levels $a$ or $b$, and the bright states, which couple the excited level $c$ with only one of the lower levels $a$ or $b$. In section 5 the formula for the polarization matrix of the emitted photon is obtained and in section 6 this formula is used for calculation of the photon polarization matrix for the transitions with the angular momenta $J_a = J_c = 3, J_b = 2$ and $J_b = J_c = 1, J_a = 2$, corresponding to the transitions between the hyperfine structure components of the electronic levels $5S_{1/2}$ and $5P_{3/2}$ of the $^{85}\text{Rb}$ and $^{87}\text{Rb}$ atoms, which were employed in the experiments [9, 11, 12, 13].

2 Evolution operator

Let us consider the coherent laser field with the carrier frequency $\omega_c$, which is in resonance with the frequency $\omega_{\alpha 0}$ of an optically allowed transition $J_a \rightarrow J_c$ between the ground state $J_a$ and the excited state $J_c$, while the quantized cavity field with the carrier frequency $\omega$ is in resonance with the frequency $\omega_0$ of an optically allowed transition $J_b \rightarrow J_c$ between the long-lived state $J_b$ and the same excited state $J_c$ (Fig.1). Here $J_a, J_b$ and $J_c$ are the values of the angular momenta of the levels. The coherent laser field is characterized by the electric field strength

$$E_c = e_c(t)\mathbf{l}_c e^{-i\omega_c t} + c.c.,$$

(1)
while the quantized field of the cavity in the interaction representation is described by the operator:

\[
\hat{E} = e(t)(\hat{a}_1 l_1 + \hat{a}_2 l_2)e^{-i\omega t} + h.c.,
\]

where \(e_c(t)\) and \(l_c\) are the slowly varying amplitude and the unit polarization vector of the laser field, \(e(t)\) is the slowly varying amplitude of the cavity field, \(l_1\) and \(l_2\) are the two unit orthogonal vectors of the two polarization modes of this field, \(\hat{a}_1\) and \(\hat{a}_2\) are the photon annihilation operators for this modes. The temporal dependence \(e(t)\) of the cavity field amplitude appears due to the motion of the atom through the cavity or to some tailored alteration of cavity parameters. The equation for the slowly-varying density matrix \(\hat{\rho}\) of the system, which consists of a single three-level atom and two-mode cavity field, in the rotating-wave approximation and in case of Raman resonance \(\omega_{c0} - \omega_c = \omega_0 - \omega = \Delta\) is as follows:

\[
\frac{d}{dt}\hat{\rho} = \frac{i}{2} \left[ \hat{V}(t), \hat{\rho} \right],
\]

\[
\hat{V}(t) = -2\Delta \hat{P}_c + \hat{G}(t) + \hat{G}^\dagger(t),
\]

\[
\hat{G}(t) = \Omega_a(t)\hat{g}_a + \Omega_b(t)\hat{g}_b,
\]

\[
\hat{g}_b = \hat{g}_{b1} \hat{a}_1^\dagger + \hat{g}_{b2} \hat{a}_2^\dagger.
\]

Here \(\hat{P}_c\) is the projector on the subspace of the atomic excited level \(J_c\), \(\Omega_a(t) = 2|d_a|e_c(t)/\hbar\) and \(\Omega_b(t) = 2|d_b|e(t)/\hbar\) are the reduced Rabi frequencies for the coherent laser field and for the cavity field, \(d_a = d(J_a,J_c)\) and \(d_b = d(J_b,J_c)\) are the reduced matrix elements of the electric dipole moment operator for the transitions \(J_a \rightarrow J_c\) and \(J_b \rightarrow J_c\), while

\[
\hat{g}_a = \hat{g}_a l_c^\dagger, \quad \hat{g}_b = \hat{g}_b l_i^\dagger, \quad i = 1, 2,
\]

Figure 1: The level diagram.
\( \hat{g}_a \) and \( \hat{g}_b \) are the dimensionless electric dipole moment operators for the transitions \( J_a \to J_c \) and \( J_b \to J_c \). These operators are expressed through Wigner 3J-symbols and partial atomic operators

\[
\hat{P}_{m_a m_b}^{J_a J_\beta} = |J_\alpha m_\alpha > < J_\beta m_\beta|, \quad \alpha, \beta = a, b, c,
\]

in a following way \[19]\):

\[
\hat{g}_a = \sum_{m_a, m_c, q} (-1)^{J_a - m_a} t_{cq}^* \begin{pmatrix} J_a & 1 & J_\alpha \\ -m_a & q & m_c \end{pmatrix} \hat{P}_{m_a m_c}^{J_a J_\alpha}, \tag{8}
\]

\[
\hat{g}_{bi} = \sum_{m_b, m_c, q} (-1)^{J_b - m_b} t_{iq}^* \begin{pmatrix} J_b & 1 & J_\beta \\ -m_b & q & m_c \end{pmatrix} \hat{P}_{m_b m_c}^{J_b J_\beta}, \tag{9}
\]

where \( t_{cq} \) and \( t_{iq} \) are the circular components of polarization vectors \( \mathbf{l}_c \) and \( \mathbf{l}_1 \).

The solution of the equation \[3\] is expressed through the evolution operator \( \hat{S}(t) \):

\[
\hat{\rho}(t) = \hat{S}(t)\hat{\rho}(0)\hat{S}^+(t). \tag{10}
\]

In the adiabatic approximation (see, e.g., \[13\]) the evolution operator \( \hat{S}(t) \) is defined by the instantaneous eigenvectors \( |v_k(t) > \) and eigenvalues \( \lambda_k(t) \) of the interaction operator \( \hat{V}(t) \):

\[
\hat{S}(t) = \sum_k \exp\{i\phi_k(t)\}|v_k(t) > < v_k(0)|, \tag{11}
\]

\[
\phi_k(t) = \frac{1}{2} \int_0^t \lambda_k(t')dt'. \tag{12}
\]

### 3 Bright states

Since only the processes of a single photon emission are discussed in the present paper it is sufficient to limit the system space to the subspace with the basis vectors \( |J_a m_a > |0, 0 >, |J_b m_b > |1, 0 >, |J_c m_c > |0, 0 >, \) where \( |J_a m_a >, |J_b m_b > \) and \( |J_c m_c > \) denote the atomic Zeeman states, while \( |n_1, n_2 > (n_{1,2} = 0, 1) \) are the field number states with \( n_1 \) photons in the first polarization mode and \( n_2 \) – in the second. This subspace with the dimension \( N = 2(J_a + 2J_b + J_c + 2) \) constitutes the invariant subspace of the interaction operator \( \hat{V}(t) \), so that its matrix represents itself a square hermitian \( N \times N \) matrix. The states \( |a > |0, 0 > \) and \( |c > |0, 0 > \), which atomic part belongs to the level \( a \) or \( c \), may be represented as columns with \( 2J_a + 1 \) or \( 2J_c + 1 \) elements correspondingly, while the states \( |b_1 > |1, 0 > + |b_2 > |0, 1 >, \) which atomic part belongs to the level \( b \), may be represented as columns with \( 2(2J_b + 1) \) elements:

\[
|b_1 > |1, 0 > + |b_2 > |0, 1 > = \begin{pmatrix} |b_1 > \\ |b_2 > \end{pmatrix}.
\]
Then the operator $\hat{g}_a$ will be represented by the $(2J_a + 1) \times (2J_c + 1)$ matrix, while the operator $\hat{g}_b$ will be represented by the $2(2J_b + 1) \times (2J_c + 1)$ matrix

$$\hat{g}_b = \begin{pmatrix} g_{b1} \\ g_{b2} \end{pmatrix},$$

were each block $\hat{g}_bi$ $(i = 1, 2)$ represents itself a $(2J_b + 1) \times (2J_c + 1)$ matrix.

In order to find out the instantaneous eigenvectors $|v_k(t)\rangle$ and eigenvalues $\lambda_k(t)$ of the interaction operator $\hat{V}(t)$ let us start with the eigenvectors of the operator $\hat{G}^i(t)\hat{G}(t)$, which acts at the subspace of the upper atomic level $c$. Let us denote as $|D^c_n\rangle$ the states, if there are any, which are simultaneously the eigenvectors of both operators $\hat{g}_a^{\dagger}\hat{g}_a$ and $\hat{g}_b^{\dagger}\hat{g}_b$ with zero eigenvalues:

$$\hat{g}_a^{\dagger}\hat{g}_a|D^c_n\rangle = \hat{g}_b^{\dagger}\hat{g}_b|D^c_n\rangle = 0,$$  \hspace{1cm} (13)

the number of such states being $N^d_c$. These states remain uncoupled to the lower atomic levels $a$ and $b$. Next, let us consider the states $|C^a_n\rangle$, which are the eigenvectors of operator $\hat{g}_a^{\dagger}\hat{g}_a$, with non-zero eigenvalues and at the same time the eigenvectors of the operator $\hat{g}_b^{\dagger}\hat{g}_b$ with zero eigenvalues:

$$\hat{g}_a^{\dagger}\hat{g}_a|C^a_n\rangle = c^2_{an}|C^a_n\rangle, \hspace{0.3cm} c_{an} > 0, \hspace{0.3cm} \hat{g}_b^{\dagger}\hat{g}_b|C^a_n\rangle = 0,$$  \hspace{1cm} (14)

the number of such states being $N^a_c$. Similarly, the states $|C^b_n\rangle$, coupled to the lower atomic level $b$ only, are the eigenvectors of the operator $\hat{g}_b^{\dagger}\hat{g}_b$ with non-zero eigenvalues, which are at the same time the eigenvectors of the operator $\hat{g}_a^{\dagger}\hat{g}_a$ with zero eigenvalues:

$$\hat{g}_b^{\dagger}\hat{g}_b|C^b_n\rangle = c^2_{bn}|C^b_n\rangle, \hspace{0.3cm} c_{bn} > 0, \hspace{0.3cm} \hat{g}_a^{\dagger}\hat{g}_a|C^b_n\rangle = 0,$$  \hspace{1cm} (15)

the number of such states being $N^b_c$. Finally, let us consider the states $|C_n(t)\rangle$, which satisfy the inequations:

$$\hat{g}_a|C_n(t)\rangle \neq 0, \hspace{0.5cm} \hat{g}_b|C_n(t)\rangle \neq 0,$$

at any time. These states, coupled to both lower atomic levels $a$ and $b$, may be obtained as the eigenvectors of the operator $\hat{G}^i(t)\hat{G}(t)$ with non-zero eigenvalues at any time:

$$\hat{G}^i(t)\hat{G}(t)|C_n(t)\rangle = c^2_n(t)|C_n(t)\rangle, \hspace{0.3cm} c_n(t) > 0,$$  \hspace{1cm} (16)

the number of such states being $N_c$. All the temporally independent $N^c_d$ states $|D^c_n\rangle$, $N^a_{c,b}$ states $|C^a_n\rangle$, and temporally dependent $N_c$ states $|C_n(t)\rangle$ constitute the complete $(N^c_d + N^a_c + N^b_c + N_c = 2J_c + 1)$ orthonormal set of states with the atomic part belonging to the atomic level $c$ at any instant of time.

The bright states

$$|F^{a,b}_n\rangle = \frac{1}{c_{a,bn}}\hat{g}_{a,b}|C^{a,b}_{n}\rangle, \hspace{0.3cm} n = 1, \ldots, N^{a,b}_c,$$  \hspace{1cm} (17)
eigenvectors of the operator \( \hat{b} \) which satisfy the equation

\[
|n\rangle \rightarrow \alpha_n |n\rangle, \quad n = 1, ..., N_c, \quad (18)
\]

coupled to the states \(|C_{n,a}^{a,b}\rangle\) and \(|C_n(t)\rangle\) by electric dipole transitions, also form an orthonormal set of states, as it follows from (17)-(18). The atomic part of the temporally independent states \(|F_n^{a,b}\rangle\) belongs to the subspace of the only one lower level \(a\) or \(b\), while the atomic part of the temporally dependent states \(|F_n(t)\rangle\) belongs to the subspace of both lower levels \(a\) and \(b\).

With the introduction of the states \(|D_n^{c}\rangle\), \(|C_{n,a}^{a,b}\rangle\), \(|C_n(t)\rangle\) and \(|F_n^{a,b}\rangle\), \(|F_n(t)\rangle\), the interaction operator \(\hat{V}(t)\) may be easily diagonalized. It has \(N^d_c\) eigenvectors \(|D_n^{c}\rangle\), then \(2(N^a_c + N^b_c)\) eigenvectors \(|V^{(\pm)}_{a,b,n}(t)\rangle\), which are linear superpositions of states \(|F_n^{a,b}\rangle\) and \(|C_{n,a}^{a,b}\rangle\), and \(2N_c\) eigenvectors \(|V^{(\pm)}_{n}(t)\rangle\) which are linear superpositions of states \(|F_n(t)\rangle\) and \(|C_n(t)\rangle\). All these states \(|D_n^{c}\rangle\), \(|V^{(\pm)}_{a,b,n}(t)\rangle\) and \(|V^{(\pm)}_{n}(t)\rangle\) constitute the orthonormal set of \(N^d = 2(2J_c + 1) - N^d_c\) eigenvectors of the operator \(\hat{V}(t)\) with non-zero eigenvalues.

4 Dark states

The other \(N^d = N - N^f\) eigenvectors \(|D_k(t)\rangle\) of the interaction operator \(\hat{V}(t)\) obtain zero eigenvalues:

\[
\hat{V}(t)|D_k(t)\rangle = 0, \quad k = 1, ..., N_d.
\]

These states with the atomic part belonging to the subspace of the lower atomic levels \(a\) and \(b\) remain uncoupled to the upper level \(c\). These states – dark states – satisfy the equation:

\[
\hat{G}^\dagger(t)|D_k(t)\rangle = 0, \quad k = 1, ..., N_d. \quad (19)
\]

As it follows from (19), all the dark states \(|D_k(t)\rangle\) are orthogonal to all the states \(|D_n^{c}\rangle\), \(|V^{(\pm)}_{a,b,n}(t)\rangle\) and \(|V^{(\pm)}_{n}(t)\rangle\).

Among all the dark states let us distinguish first the dark states \(|D_n^{a}\rangle\), which atomic part belongs to the lower level \(a\) only. These states are time independent and may be obtained as the eigenvectors of the operator \(\hat{g}_a\hat{g}_a^\dagger\) with zero eigenvalues:

\[
\hat{g}_a\hat{g}_a^\dagger|D_n^{a}\rangle = 0, \quad k = 1, ..., N^d_a. \quad (20)
\]

All the other eigenvectors \(|A_n\rangle\) of the operator \(\hat{g}_a\hat{g}_a^\dagger\) with non-zero eigenvalues, which satisfy the equation

\[
\hat{g}_a\hat{g}_a^\dagger|A_n\rangle = a_n^2|A_n\rangle, \quad a_n > 0, \quad n = 1, ..., N_a, \quad (21)
\]

constitute the subspace orthogonal to the subspace of the dark states \(|D_n^{a}\rangle\), its dimension being \(N_a = 2J_a + 1 - N^d_a\). Similarly the dark states \(|D_n^{b}\rangle\), which atomic part belongs to the lower level \(b\) only, may be obtained as the eigenvectors of the operator \(\hat{g}_b\hat{g}_b^\dagger\) with zero eigenvalues:

\[
\hat{g}_b\hat{g}_b^\dagger|D_n^{b}\rangle = 0, \quad k = 1, ..., N^d_b. \quad (22)
\]
The \( N_b = 2(2J_b+1) - N^d_b \) eigenvectors \( |B_n> \) of the operator \( \hat{g}_b \hat{g}_b^\dagger \) with non-zero eigenvalues:

\[
\hat{g}_b \hat{g}_b^\dagger |B_n> = b_n^2 |B_n>, \quad b_n > 0, \quad n = 1, \ldots, N_b,
\]

(23)

constitute the subspace orthogonal to the subspace of the dark states \( |D^b_k> \).

Let us now consider the dark states \( |D^{ab}_k(t)> \), which atomic part belongs to both atomic lower levels \( a \) and \( b \). These states satisfy the equation

\[
\hat{G}^\dagger(t) |D^{ab}_k(t)> = 0, \quad k = 1, \ldots, N^d_{ab},
\]

(24)

while

\[
\hat{g}_a^\dagger |D^{ab}_k(t)> \neq 0, \quad \hat{g}_b^\dagger |D^{ab}_k(t)> \neq 0.
\]

The temporal dependence of these states may be immediately obtained from the equation (24):

\[
|D^{ab}_k(t)> = Z_k(t) [\Omega_a(t) |A^d_k> - \Omega_b(t) |B^d_k>],
\]

where \( Z_k(t) \) is the normalization factor, while \( |A^d_k> \) and \( |B^d_k> \) are temporally independent states, which atomic parts belong to the levels \( a \) and \( b \) correspondingly and which satisfy the equation

\[
\hat{g}_a^\dagger |A^d_k> = \hat{g}_b^\dagger |B^d_k> \neq 0.
\]

(25)

Introducing the matrix

\[
\hat{D}_b = \sum_{n=1}^{N_b} \frac{1}{b_n^2} |B_n><B_n|,
\]

(26)

containing only the eigenvectors \( |B_n> \) of matrix \( \hat{g}_b \hat{g}_b^\dagger \) with non-zero eigenvalues, we may write the equation (25) as follows:

\[
|B^d_k> = \hat{D}_{ba} |A^d_k>, \quad \hat{D}_{ba} = \hat{D}_b \hat{g}_b \hat{g}_a^\dagger.
\]

(27)

Now we may define the orthonormal set of states \( |A^d_k> \) as the eigenvectors of the hermitian matrix \( \hat{D}^\dagger_{ba} \hat{D}_{ba} \) with non-zero eigenvalues:

\[
\hat{D}^\dagger_{ba} \hat{D}_{ba} |A^d_k> = a^2_{dk} |A^d_k>, \quad a_{dk} > 0.
\]

(28)

Then, as it follows from (27) and (28), the states \( a_{dk}^\dagger |B^d_k> \) also constitute the orthonormal set, so that the orthonormal set of dark states \( |D^{ab}_k(t)> \) may be expressed through the eigenvectors \( |A^d_k> \) of the equation (28) by the formula:

\[
|D^{ab}_k(t)> = \frac{\Omega_a(t) \hat{D}_{ba} |A^d_k> - \Omega_b(t) |A^d_k>}{\sqrt{\Omega_a^2(t) + a_{dk}^2 \Omega_a^2(t)}}.
\]

(29)
5 Single photon emission

All the eigenvectors of the interaction operator $\hat{V}(t)$, comprising the set of states $|D_{cn}^c\rangle$, $|V_{a_{cm}}^{(+)}(t)\rangle$, $|V_{n}^{(+)}(t)\rangle$ and the set of dark states $|D_{a}^{b} \rangle$, $|D_{a}^{b}(t)\rangle$, constitute the complete orthonormal set of states, which determines the evolution operator $\hat{U}$ in the adiabatic approximation. Initially the atom is at its lower level $a$, while the cavity field is in its vacuum state, the initial atomic density matrix of the atom+field system being $\hat{\rho}_0$. The classical coherent laser field is adiabatically switched on, while the interaction with the quantum field is adiabatically switched off in the time interval $T$, so that:

$$\Omega_a(0) = 0, \quad \Omega_a(T) = \Omega_{a0},$$
$$\Omega_b(0) = \Omega_{b0}, \quad \Omega_b(T) = 0.$$

The atomic part of the initial states $|V_{a}^{(\pm)}(0)\rangle$ belongs to the atomic levels $b$ and $c$ only, so that only the states $|V_{a}^{(\pm)}(t)\rangle$ and the dark states $|D_{a}^{b} \rangle$ and $|D_{a}^{b}(t)\rangle$ will contribute to the evolution operator $\hat{U}$. After the STIRAP process is finished at $t = T$ the atomic part of the states $|V_{a}^{(\pm)}(T)\rangle$ belongs to the levels $a$ and $c$, the atomic part of the states $|D_{a}^{b} \rangle$ belongs to the level $a$ and the atomic part of the states $|D_{a}^{b}(T)\rangle$ belongs to the level $b$, so that only the presence of the dark states $|D_{a}^{b}(t)\rangle$ in the evolution operator contributes to the probability of the photon emission, while the presence of the states $|V_{a}^{(\pm)}(t)\rangle$ and $|D_{a}^{b} \rangle$ in the evolution operator reduces this probability. The total probability $w$ of the single photon emission is

$$w = \langle 0, 1|\hat{\rho}_F|1, 0 \rangle + \langle 0, 1|\hat{\rho}_F|0, 1 \rangle,$$

while the polarization of the photon is described by the $2 \times 2$ polarization matrix

$$\sigma = \frac{1}{w} \begin{pmatrix} \langle 0, 1|\hat{\rho}_F|1, 0 \rangle & \langle 0, 1|\hat{\rho}_F|0, 1 \rangle \\ \langle 0, 1|\hat{\rho}_F|1, 0 \rangle & \langle 0, 1|\hat{\rho}_F|0, 1 \rangle \end{pmatrix}.$$

Here

$$\hat{\rho}_F = tr_A \left\{ \hat{U} \hat{\rho}_0 \hat{U}^\dagger \right\}, \quad \hat{U} = \hat{D}_{ba} \hat{P}_a^d,$$

the trace is carried out over the atomic variables, the operator $\hat{D}_{ba}$ is defined by (22) and

$$\hat{P}_a^d = \sum_{k=1}^{N_{ab}^d} \frac{1}{a_{dk}} |A_k^d\rangle \langle A_k^d|,$$

where $|A_k^d\rangle$ and $a_{dk}$ are defined by (25).

6 Discussion

Let the quantization axis $Z$ be directed along the cavity axis and let us choose the two unit polarization vectors of the quantum field as the two unit vectors
along the Cartesian axes $X$ and $Y$:

$$l_1 = e_x, \quad l_2 = e_y,$$

while the driving field propagates along the axis $Y$ so that its polarization vector $l_c$ belongs to the plane $XZ$. In case of linearly polarized driving field

$$l_c = e_z \cos \psi + e_x \sin \psi.$$

The photon polarization matrix (31) may be expressed through the Stokes parameters $\xi_n$ ($n = 1, 2, 3$) [20]:

$$\sigma = \frac{1}{2} \left( \begin{array}{cc} 1 + \xi_3 & \xi_1 - i \xi_2 \\ \xi_1 + i \xi_2 & 1 - \xi_3 \end{array} \right),$$

which determine the photon total degree of polarization

$$P = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}.$$ 

In the experiments [9, 10] the resonant levels with angular momenta $J_a = J_c = 3$, $J_b = 2$ were the hyperfine structure components of the electronic levels $5S_{1/2}$ and $5P_{3/2}$ of the $^{85}\text{Rb}$ atom, however the polarization of the driving field and the initial atomic state were not specified. So let the atom be initially in the equilibrium state

$$\hat{\rho}_a^0 = \frac{\hat{P}_a}{2J_a + 1} |0, 0 <\!< 0, 0|,$$

where $\hat{P}_a$ is the projector on the subspace of the atomic ground state $a$. Then we obtain from the numeric calculations based on the formulae (30)-(33), that the photon emission probability $w = 0.857$ is less than unity and does not depend on the angle $\psi$ between the cavity axis $Z$ and the polarization vector $l_c$ of the driving field, while the emitted photon is almost unpolarized, its degree of polarization is zero $P = 0$ with the $\pi$-polarized driving field ($\psi = 0$) and $P = 0.07$ with the $\sigma$-polarized field ($\psi = \pi/2$). Now let us assume that the atom is initially prepared in a pure Zeeman state $|J_a = 3, m_a = 0 >$ with zero value of the angular momentum projection $m_a$ on the quantization axis $Z$, like in the experiment [18]. Then we obtain from (30)-(33) the dependence of the photon emission probability $w$ on the angle $\psi$ presented at Figure 2. In this case with the $\pi$-polarized driving field ($\psi = 0$) the photon is not emitted at all $w = 0$, while it is emitted with the unit probability $w = 1$ with the $\sigma$-polarized field ($\psi = \pi/2$) and at an angle $\psi = 0.685 \text{ rad } (39^\circ)$, at $\psi = \pi/2$ its degree of polarization being $P = 0.43$ and at $\psi = 0.685$ it is $P = 0.80$.

In the experiments [11, 12] the magnetic field directed along the cavity axis $Z$ was applied to lift the degeneracy of atomic levels and the cavity field was tuned in resonance with the transition between the Zeeman state $|F' = 1, m_{F'} = 0 >$ of the hyperfine structure component $F' = 1$ of the excited $5P_{3/2}$ electronic level of $^{87}\text{Rb}$ atom and one of the Zeeman states $|F = 1, m_F = \pm 1 >$ of the hyperfine
structure component $F = 1$ of the ground level $5S_{1/2}$, while the linearly $\sigma$-polarized driving field was in resonance with the transition between the same Zeeman state $|F' = 1, m_{F'} = 0 >$ of the excited level and the other Zeeman state $|F = 1, m_F = \mp 1 >$ of the ground level, so that the emitted photons were circularly polarized, as it is evident from selection rules.

In the experiment [18] the resonant levels with angular momenta $J_b = J_c = 1$, $J_b = 2$ were the hyperfine structure components of the electronic levels $5S_{1/2}$ and $5P_{3/2}$ of the $^{87}Rb$ atom and the atom was initially prepared in the pure Zeeman state $|J_a = 2, m_a = 0 >$, while the driving field was linearly $\pi$-polarized. The results of numeric calculations of the photon emission probability $w$ and the degree of polarization $P$ versus the angle $\psi$ between the cavity axis $Z$ and the polarization vector $\mathbf{l}_c$ of the driving field for these transitions are presented at Figure 3. In case of $\pi$-polarized ($\psi = 0$) driving field, as it was in the experiment, the photon is emitted with unit probability $w = 1$, but it is fully unpolarized $P = 0$. The photon becomes unpolarized after the averaging over the states of the atom, however in the experiment [18] the same atom was driven twice in the time interval less than decoherence time to produce the sequence of two photons with strongly correlated polarizations. As it may be seen from Figure 3, with the same experimental setup but with the driving field linearly $\sigma$-polarized ($\psi = \pi/2$) the emitted photon will be fully polarized $P = 1$ (linearly polarized along the axis $Y$ as follows from calculations), though the emission probability in this case is reduced to $w = 0.25$. To obtain the linearly polarized photons with unit probability the scheme with lower values of angular momenta $J_a = 0$, $J_b = J_c = 1$, will be suitable. In this case the $\sigma$-polarized driving field will produce the photons linearly polarized along the axis $Y$ with unit probability.

Figure 2: The photon emission probability $w$ on the transitions with the angular momenta $J_a = J_c = 3$, $J_b = 2$ versus the angle $\psi$ between the cavity axis $Z$ and the polarization vector $\mathbf{l}_c$ of the driving field in case of initial pure atomic state $m_a = 0$ with zero angular momentum projection on the quantization axis $Z$.
Figure 3: The photon emission probability $w$ (solid line) and the degree of polarization $P$ (dashed line) on the transitions with the angular momenta $J_b = J_c = 1$, $J_a = 2$ versus the angle $\psi$ between the cavity axis $Z$ and the polarization vector $\mathbf{l}_c$ of the driving field in case of initial pure atomic state $m_a = 0$ with zero angular momentum projection on the quantization axis $Z$.

7 Conclusions

In the present paper the general formulae for calculation of the polarization matrix of a single photon emitted in the microcavity by a single three-level $\Lambda$-type atom with degenerate levels, driven adiabatically by the classical laser field from the ground state $a$ to the long-lived state $b$ through the excited state $c$, are obtained for the arbitrary polarization of the driving field and arbitrary values of the angular momenta $J_a$, $J_b$ and $J_c$ of the resonant atomic levels.

For the transitions with the angular momenta $J_a = J_c = 3$, $J_b = 2$ and $J_b = J_c = 1$, $J_a = 2$, corresponding to the transitions between the hyperfine structure components of the electronic levels $5S_{1/2}$ and $5P_{3/2}$ of the $^{85}\text{Rb}$ and $^{87}\text{Rb}$ atoms, which were employed in the experiments [9, 10, 18], and for the atom initially prepared at pure Zeeman state with zero projection $m_a = 0$ on the quantization axis, the dependencies of the photon emission probability $w$ and degree of polarization $P$ on the angle $\psi$ between the cavity axis and the polarization vector $\mathbf{l}_c$ of the driving field were calculated numerically. In case of transitions with $J_a = J_c = 3$, $J_b = 2$, the unit emission probability may be obtained at the angles $\psi = 39^\circ$ and $\psi = 90^\circ$ with the degrees of polarization $P = 0.80$ and $P = 0.43$ correspondingly. In case of transitions with $J_b = J_c = 1$, $J_a = 2$, at $\psi = 0$ the fully unpolarized photon is emitted with unit probability, while at $\psi = 90^\circ$ fully linearly polarized photon is emitted with the probability $w = 0.25$. 

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The fully linearly polarized photon emitted with unit probability may be obtained by means of $\sigma$-polarized driving field ($\psi = 90^\circ$) with lower values of the level angular momenta $J_b = J_c = 1$, $J_a = 0$.

Acknowledgements
Author is indebted for financial support of this work to Russian Ministry of Science and Education (grant 2407).

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