REPRESENTABILITY OF AFFINE ALGEBRAS OVER AN ARBITRARY FIELD

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Abstract. In a series of papers culminating in [16], summarized in [17], we used full quivers as tools in describing PI-varieties of algebras and providing a complete proof of Belov’s solution of Specht’s problem for affine algebras over an arbitrary Noetherian ring. In this paper, utilizing ideas from that work, we give a full exposition of Belov’s theorem [9] that relatively free affine PI-algebras over an arbitrary field are representable.

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1. Introduction

In this paper, utilizing ideas from [10], we give a full exposition of Belov’s theorem [6] that relatively free affine PI-algebras over an arbitrary field are representable. As in [10], the main tool in utilizing the combinatorics of polynomials is “hiking,” which however is more complicated here since it involves non-homogeneous polynomials, and is described below in several stages.

We work with algebras over a field $F$, with special emphasis to the possibility that $F$ is finite. A (noncommutative) polynomial is an element of the free associative algebra $F\{x\}$ on countably many generators. A polynomial identity (PI) of an algebra $A$ over $F$ is a noncommutative polynomial which vanishes identically for any substitution in $A$. We use [10] as a general reference for PIs. A T-ideal of $F\{x\}$ is an ideal $I$ of $F\{x\}$ closed under all algebra endomorphisms $F\{x\} \to F\{x\}$. We write $\text{id}(A)$ for the T-ideal of PIs of an algebra $A$.

Conversely, for any T-ideal $I$ of $F\{x\}$, each element of $I$ is a PI of the algebra $F\{x\}/I$, and $F\{x\}/I$ is relatively free, in the sense that for any PI-algebra $A$ with $\text{id}(A) \supseteq I$, and any $a_1, a_2, \ldots \in A$, there is a natural homomorphism $F\{x\}/I \to A$ sending $x_i \mapsto a_i$ for $i = 1, 2, \ldots$.

1.1. Representability.

An $F$-algebra $A$ is called representable if it is embeddable as an $F$-subalgebra of $M_n(K)$ for a suitable field $K \supseteq F$. Obviously any representable algebra is PI, but an easy counting argument of Lewin [30] leads to the existence of non-representable affine PI-algebras.

Nevertheless, the representability question of relatively free affine algebras has considerable independent interest, and the purpose of this paper is to give a full, self-contained proof of the following result proved by Kemer over any infinite field, and to elaborate Belov [6] (over a finite field):

**Theorem 1.1.** Every relatively free affine PI-algebra over an arbitrary field is representable.

Kemer obtained Theorem 1.1 over infinite fields by means of the following amazing results:

**Theorem 1.2 ([10, Theorem 4.66], [27]).**

1. Every affine PI-algebra over an infinite field (of arbitrary characteristic) is PI-equivalent to a finite dimensional (f.d.) algebra.

2. Every PI-algebra of characteristic 0 is PI-equivalent to the Grassmann envelope of a finite dimensional (f.d.) algebra.

In [12]–[16] we have provided a complete proof for the affine case of Specht’s problem in arbitrary positive characteristic. (The non-affine case has counterexamples, cf. [4, 5].) Together with Kemer’s solution in characteristic 0, this leads to:

**Theorem 1.3.** Any affine PI-algebra over an arbitrary commutative Noetherian ring satisfies the ACC on T-ideals.

**Remark 1.4.** For graded associative algebras [1] and various nonassociative affine algebras of characteristic 0, the finite basis of T-ideals has been established in the case when the operator algebra is PI [Iltyakov [24, 25] for alternative and Lie algebras, and
Vais and Zelmanov [34] for Jordan algebras) but the representability question remains open for nonassociative algebras, so representability presumably is more difficult. The obstacle is getting started via some analog of Lewin’s theorem [30], which is not yet available. Below [9] proved representability of alternative or Jordan algebras satisfying all identities of some finite dimensional algebra.

1.1.1. Plan of the proof of Theorem 1.1.

An immediate consequence of Theorem 1.2(1) is that every relatively free affine PI-algebra over an infinite field is representable, since it can be constructed with generic elements obtained by adjoining commutative indeterminates to the f.d. algebra. Kemer deduced from this the finite basis of T-ideals (the solution of Specht’s problem) over an infinite field, in which he applied combinatorial techniques to representable algebras.

The approach here for positive characteristic, following [6], is the reverse, where one starts with the solution of Specht’s problem and applies Noetherian induction to prove representability of affine relatively free PI-algebras over arbitrary fields (including finite fields). (These methods also work in characteristic 0, but rely on Kemer’s solution of Specht’s problem in characteristic 0, which in turn relies on his representability theorem in characteristic 0.)

Remark 1.5. We fix the following notation: We start with the free affine algebra $F\{x\} = F\{x_1, \ldots, x_\ell\}$ in $\ell$ indeterminates, and a T-ideal $I$. This gives us the relatively free algebra $A = F\{x\}/I$.

We say that the T-ideal $I$ is representable if the affine algebra $A$ is representable.

The proof of Theorem 1.1 goes along the following version of Noetherian induction:

We aim to show that every T-ideal $I$ is representable. In view of Lewin’s theorem [30], $I$ contains a representable T-ideal $I_0$, so we assume that $A_0 := F\{x\}/I_0$ is representable. In view of Theorem 1.2, we have a maximal representable T-ideal $I_1 \supseteq I_0$ of $A$ contained in $I$, which we aim to show is $I$. Assuming on the contrary that $I_1 \subset I$, we replace $I_0$ by $I_1$ and $A_0$ by $A_0/I_1$. This reduces us to the case where $A_0$ is representable but every nonzero T-ideal of $A_0$ contained in $I$ is not representable. Our goal is to arrive at a contradiction by finding a representable T-ideal $J \subseteq I_1$ which strictly contains $I$. We will do this by taking some $f \in I \setminus I_1$, i.e., $f \notin \text{id}(A_0)$, and finding $J$ inside the T-ideal generated by $f$. (This process will terminate because of the solution to Specht’s problem given in [16]. For this reason, we do not need to introduce parameters of the induction.)

The rest of this paper consists of the proof of Theorem 1.1 by means of Remark 1.5. The proof relies on the ideas of the proof of Kemer’s representability theorem given in [10, 11]. Much of this paper is devoted to elaborating the theory of [13] and [14], as described in [16, 17], and there is a considerable overlap with [6].

The proof of Theorem 1.3 in [16] is somewhat different from Kemer’s proof. In [13] we considered the full quiver of a representation of an associative algebra over a field, and determined properties of full quivers by means of a close examination of the structure of Zariski closed algebras, studied in [12]. Then we modified $f$ by means of a “hiking procedure” in order to force $f$ to have certain combinatorial properties, and used this to carve out a T-ideal $J$ from inside a given T-ideal; modding out $J$ lowers the quiver in some sense, and then one obtains Specht’s conjecture by induction.
Our approach here is similar, but with some variation. Here we need not mod out by \( J \), but need \( J \) to be representable. We start the same way, but one of the key steps fails, and we need a way of getting around it. In both instances, our techniques rely on the theory of full quivers of a f.d. algebra over a field, to be recalled, followed by adjunction of characteristic coordinates.

We introduce “critical” polynomials (Definition 4.3) which enable us to calculate characteristic coefficients using the combinatoric properties of polynomials, leading to our main tool:

**Theorem 1.6** (Canonization Theorem for Polynomials). Suppose \( f(x_1, \ldots, x_t) \) is a nonidentity of \( A_0 \) whose nonzero evaluation passes through all the blocks of the quiver, via the dominant branch. Then the T-ideal of \( f \) contains a critical nonidentity.

The proof of the Canonization Theorem for Polynomials is based on applying hiking to obtain more and more complicated polynomials while preserving the two “Kemer invariants” of the polynomial described in [10, 11], which underly the computational study of T-ideals. The basic operations of hiking, namely multiplying by a Capelli polynomial or replacing a radical element by a commutator element of the same form preserves the hypotheses of [11, Lemma 6.7.3], so we can measure the dimension of the semisimple part and the nilpotence index of the radical in terms of the Kemer invariants.

The Canonization Theorem for Polynomials will enable us to replace multiplication by characteristic coefficients in the Shirshov extension, with multiplication by elements of \( A_0 \).

2. **Preliminaries**

Let us review some of the techniques we need for the proof. The reader can refer to [17] for further details of all of this material.

2.1. **Linearization and quasi-linearization.**

The well-known linearization process of a polynomial can be described in two stages: First, writing a polynomial \( f(x_1, \ldots, x_n) \) as

\[
f(0, x_2, \ldots, x_n) + (f(x_1, \ldots, x_n) - f(0, x_2, \ldots, x_n)),
\]

one sees by iteration that any T-ideal is additively spanned by T-ideals of polynomials for which each indeterminate appearing nontrivially appears in each of its monomials, cf. [31, Exercise 2.3.7]. Then we define the **linearization process** by introducing a new indeterminate \( x'_i \) and passing to

\[
f(x_1, \ldots, x_i + x'_i, \ldots, x_m) - f(x_1, \ldots, x_i, \ldots, x_m) - f(x_1, \ldots, x'_i, \ldots, x_m).
\]

This process, applied repeatedly, yields a multilinear polynomial in the same T-ideal. In characteristic 0 the linearization process can be reversed by taking \( x'_i = x_i \), implying that every T-ideal is generated by multilinear polynomials. But this fails in positive characteristic, as exemplified by the Boolean identity \( x^2 - x \), so we need an alternative. To handle characteristic \( p > 0 \), Kemer [28] took a closer look, which we review from [17].

**Definition 2.1.** A function \( f \) is **i-quasi-linear** on \( A \) if

\[
f(\ldots, a_i + a'_i, \ldots) = f(\ldots, a_i, \ldots) + f(\ldots, a'_i, \ldots)
\]
for all $a_i, a'_i \in A$; $f$ is $A$-**quasi-linear** if $f$ is $i$-quasi-linear on $A$ for all $i$. When $A$ is understood, we just say **quasi-linear**.

Suppose $f(x_1, x_2, \ldots) \in F\{x\}$ has degree $d_i$ in $x_i$. The $i$-**partial linearization** of $f$ is

$$\Delta_i f := f(x_1, x_2, \ldots, x_{i,1} + \cdots + x_{i,d_i}, \ldots) - \sum_{j=1}^{d_i} f(x_1, x_2, \ldots, x_{i,j}, \ldots)$$

(1)

where the substitutions were made in the $i$ component, and $x_{i,1}, \ldots, x_{i,d_i}$ are new variables.

When $\Delta_i f(A) = 0$, then $f$ is $i$-quasi-linear on $A$, so we apply (1) at most $\deg_i f$ times repeatedly, if necessary, to each $x_i$ in turn, to obtain a nonzero polynomial that is $A$-quasi-linear in the $T$-ideal of $f$.

**Proposition 2.2** ([15] Corollary 2.13]). Assume $\text{char } F = p > 0$. For any polynomial $f$ which is not an identity of $A_0$, the $T$-ideal generated by $f$ contains a quasi-linear non-identity for which the degree in each indeterminate is a $p$-power.

### 2.2. Full quivers.

In this subsection $A_0$ is a representable affine algebra over a field $F$, i.e., $A_0 \subset M_n(K)$ with $K$ finite or algebraically closed, and we fix this particular representation. The closure of $A_0$ in $M_n(K)$ with respect to the Zariski topology [17 § 3.1] is PI-equivalent to $A_0$, so we assume throughout that $A_0$ is Zariski closed. In particular, when $F$ is infinite then we may assume $F = K$, cf. [17 Remark 3.1]. By Wedderburn’s Principal Theorem [32 Theorem 2.5.37], $A_0 = S \oplus J$ as vector spaces, where $J$ is the radical of $A_0$ and $S \cong A_0/J$ is a semisimple subalgebra of $A_0$. Thus $S$ is a direct product of matrix algebras $R_1 \times \cdots \times R_k$, which we want to view along the diagonal of $M_n(K)$, although perhaps with identification of coordinates, which are to be described graphically. By the Braun-Kemer-Razmyslov theorem, cf. [20], $J$ is nilpotent, so we take $t = t_{A_0}$ maximal such that $J^t \neq 0$.

We need an explicit description, but which may distinguish Morita equivalent algebras since matrix algebras of different size are not PI-equivalent. The full quiver of $A_0$ is a directed graph $\Gamma$, having neither loops, double edges, nor cycles, with the following information attached to the vertices and edges:

The vertices of the full quiver of $A_0$ correspond to the diagonal matrix blocks arising in the semisimple part $S$, whereas the arrows come from the radical $J$. Every vertex likewise corresponds to a central idempotent in a corresponding matrix block of $M_n(K)$.

- The vertices are ordered, say from $1$ to $k$, and an edge always takes a vertex to a vertex of higher order. There are identifications of vertices, called **diagonal gluing**, and identification of edges, called **off-diagonal gluing**. Gluing of vertices in full quivers is identical or Frobenius, as in $\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha q \end{pmatrix} : \alpha \in K \right\}$ where $|F| = q$ and $K = \overline{F}$.
- Each vertex is labeled with a roman numeral ($I$, $II$ etc.); glued vertices are labeled with the same roman numeral. A vertex can be either **filled** or **empty**.

The first vertex listed in a glued matrix block is also given a pair of subscripts — the **matrix degree** $n_i$ and the **cardinality** of the corresponding field extension of $F$ (which, when finite, is denoted as a power $q^{n_i}$ of $q = |F|$).
When the base field $F$ is finite, superscripts indicate the Frobenius twist between glued vertices, induced by the Frobenius automorphism $a \mapsto a^{q}$; this could identify $a^{q_1}$ with $a^{q_2}$ for powers $q_1, q_2$ of $q$ (or equivalently $a$ with $a^{q_2/q_1}$ when $q_1 < q_2$); we call this $(q_1, q_2)$-Frobenius gluing.

Off-diagonal gluing (i.e., gluing among the edges) includes Frobenius gluing (which only exists in nonzero characteristic) and proportional gluing with an accompanying scaling factor $\nu$.

Examples are given in [14]. Now we take some non-identity of $A_0$, say $f(x_1, \ldots, x_m) = \sum g_j(x_1, \ldots, x_m) \in \mathbb{I}$ for monomials $g_j$. An easy technical condition: Since the full quiver $\Gamma$ of $A_0$ could be replaced by the subquiver corresponding to the algebra generated by evaluations of all polynomials in $\mathbb{I}$, and then $f$ could be replaced by a sum of polynomials in $\mathbb{I}$, we may assume that $\Gamma$ passes through all blocks.

The numbers $\dim_F A_0$ and $t$ are crucial to the description of quivers, so we want these numbers to be reflected in the polynomial $f$. This is achieved by means of Kemer’s First Lemma ([11, Proposition 6.5.2]) and Kemer’s Second Lemma ([11, Proposition 6.6.31]). On the other hand, we need $f$ to be full on the f.d. algebra $A_0$ in the sense that some nonzero evaluation of $f$ passes through all the blocks of the quiver, via the dominant branch $\mathcal{B}$. This is achieved by means of Lemma [11, Proposition 6.7.3], called the Phoenix property. Applying these results to $f$ after hiking (to be described below), we assume throughout that $f$ is full, and that the conclusion of Kemer’s First Lemma and Kemer’s Second Lemma hold.

In view of Proposition 2.2 we may assume that $f$ is quasi-linear. When specializing $x_i$ to $A_0$, we write the substitutions $\bar{x}_i$ as sums of radical and semisimple elements; since $f(x_1, \ldots, x_m)$ is quasi-linear, we reduce the substitutions in $S + J$ to their component parts in $S \cup J$; we call these substitutions pure. Thus $f$ has a nonzero specialization where all substitutions $\bar{x}_i$ are pure. We fix this specialization and the notation $\bar{x}_1, \ldots, \bar{x}_m$. Any other specialization is denoted $\bar{x}_i'$.

Any pure semisimple substitution $\bar{x}_i$ is in $S$ and thus in a block (or in glued blocks) of some degree $n_i$, which we also call the degree of $\bar{x}_i$. A radical substitution $\bar{x}_i$ is somewhat more subtle. It is viewed as an edge connecting two vertices in blocks, say of degrees $n_{i_1}$ and $n_{i_2}$. If these blocks are not glued, then we call this substitution a bridge of degrees $n_{i_1}$ and $n_{i_2}$. A bridge is proper if $n_{i_1} \neq n_{i_2}$. A proper bridge connecting vertices of degree $n_i \neq n_j$ is an $\tilde{n}$-bridge if $n_i$ or $n_j$ is $\tilde{n}$. But there also is the possibility that a radical substitution connects two glued blocks of the same degree $\tilde{n}$, in which case we call it $\tilde{n}$-internal.

2.2.1. Review of the three canonization theorems for quivers.

Since arbitrary gluing is difficult to describe, we need some “canonization” theorems to “improve” the gluing. The first theorem shows that we have already specified enough kinds of gluing.

Theorem 2.3 (First Canonization Theorem, cf. [13 Theorem 6.12]). The Zariski closure of any representable affine PI-algebra $A_0$ has a representation for whose full quiver all gluing is proportional Frobenius.

For the Second Canonization Theorem we grade paths according to the following rule:
Definition 2.4. When $|F| = q < \infty$, we write $M_\infty$ for the multiplicative monoid \{1, q, q^2, \ldots, \epsilon\}, where \epsilon a = \epsilon$ for every $a \in M_\infty$. (In other words, \epsilon is the zero element adjoined to the multiplicative monoid \langle q \rangle.) Let $\overline{M}$ be the semigroup $M_\infty/\sim$ where $\sim$ is the equivalence relation obtained by matching the degrees of glued variables: When two vertices have a $(q_1, q_2)$-Frobenius twist, we identify $1$ with $q^k = \frac{\pi}{q_2}$ in the respective matrix blocks, and use $\overline{M}$ to grade the paths.

Definition 2.5. A full quiver is basic if it has a unique initial vertex $r$ and unique terminal vertex $s$, and all of its gluing above the diagonal is proportional Frobenius. A basic full quiver $\Gamma$ is canonical if any two paths from the vertex $r$ to the vertex $s$ have the same grade.

(Our notion of basic quiver has nothing to do with the notion of basic algebra in representation theory.)

Theorem 2.6 (Second Canonization Theorem, cf. [14, Theorem 3.7]). Any relatively free algebra is a subdirect product of algebras whose full quivers are basic.

Any basic full quiver $\Gamma$ of a representable relatively free algebra can be modified (via a change of base) to a canonical full quiver of an isomorphic algebra (i.e., relatively free algebra of the same variety).

In view of this result, we may reduce to the case that the full quiver of our polynomial $f$ is basic.

The Third Canonization Theorem [14, Theorem 3.12] describes what happens when one mods out a “nice” $T$-ideal, so is not relevant, since all we need is to find a representable $T$-ideal, which we do later by another method.

3. The Canonization Theorem for Polynomials

We have two languages: quivers and their representations on one hand, versus the combinatorial language of identities on the other hand. First we consider the geometrical aspect. A branch is a path $B$ in the quiver. The length of $B$ is its number of arrows, excluding loops, which equals its number of vertices (say $k$) minus 1. Thus, a typical branch has vertices of various matrix degree $n_j$, $j = 1, 2, \ldots, k$. We call $(n_1, \ldots, n_k)$ the degree vector [16, Definition 2.32] of the branch $B$. The descending degree vector is obtained by ordering the entries of the degree vector to put them in descending order lexicographically (according to the largest $n_j$ which appears in the distinct glued matrix blocks, excluding repetitions, taking the multiplicity into account in the case of Frobenius gluing). We write the descending degree vector as $(\pi(n)_1, \ldots, \pi(n)_k)$. Thus, $\pi(n)_1 = \max\{n_1, \ldots, n_k\}$. If $B$ appears in a nonzero specialization of a monomial of $f$, we call $B$ a branch of $f$.

We denote the largest $n_j$ appearing in the quiver as $\tilde{n}$.

Definition 3.1. A branch $B$ is dominant if it has the maximal possible number of $\tilde{n}$-bridges, has maximal possible length $k$ with regard to this property, has the maximal possible number of vertices of $\tilde{n}$-bridges among these in the lexicographic order, and then we continue down the line to $\tilde{n} - 1$ etc. The depth of a dominant branch $B$ is the number of times $\tilde{n}$ appears in its degree vector.
Our goal is somehow to force every nonzero evaluation of \( f \) into a dominant branch by considering each degree from \( \tilde{n} \) down in turn. Throughout, \( c_m \) here denotes the Capelli polynomial in \( 2m^2 \) indeterminates (denoted \( c_{2m^2} \) in [10]), which is alternating in \( m \) indeterminates and an identity of \( M_{m-1}(F) \) for any field \( F \); \( h_{m,i}(y) \) denotes a multilinear central polynomial \( h_{m,i}(y_1, \ldots, y_m) \) for \( M_m(F) \), in specific indeterminates \( y_1, \ldots, y_m \) which are all distinct. Evaluating \( h_{m,i} \) on semisimple matrices of degree \( < m \) is 0. We put

\[
 h_m = h_{m,1}h_{m,2} \cdots h_{m,t+1},
\]

the product of \( t+1 \) copies of distinct Capelli polynomials of the same degree \( 2m^2 \), and call the \( h_{m,j} \) the respective components of \( h \). We focus first on semisimple substitutions having matrix degree \( \tilde{n} \), and put \( h = h_{\tilde{n}} \).

**Lemma 3.2.** Any nonzero specialization of \( h \) has a component of solely semisimple substitutions (all of the same degree).

**Proof.** Otherwise every component has a radical substitution, so we have a product of \( t+1 \) radical elements, which is 0 by definition of \( t \). \( \square \)

Viewing a substitution of \( x_i \) as corresponding to an edge in the quiver, we have two degrees, one for each vertex.

**Definition 3.3.** An \( m \)-right substitution of \( x_i \) is one having degree including \( m \). An \( m \)-wrong substitution \( \overline{x}_i \) of \( x_i \) is one both of whose degrees differ from \( m \), where \( m \) appears as a degree of the substitution \( \overline{x}_i \).

We write right (resp. wrong) for \( \tilde{n} \)-right (resp. \( \tilde{n} \)-wrong).

In view of Lemma 3.2, a wrong substitution would lead to \( h \) having a component of semisimple substitutions in a matrix block of the wrong degree.

One delicate point: An internal radical bridge say from one matrix block of degree \( m \) to a different matrix block of degree \( m \) is technically “right” according to this definition, but must be dealt with.

**Remark 3.4.** Suppose \( f(x_1, \ldots, x_t) \) is a full nonidentity of \( A_0 \) whose nonzero evaluation passes through all the blocks of the quiver, via the dominant branch \( B \) say of degrees \( m_1, \ldots, m_k \) having some number \( k \) of bridges, and \( k' \) internal radical substitutions. By Theorem 2.6, any wrong nonzero substitution may be assumed to have \( k \) bridges since otherwise we apply induction to the number of semisimple components in the full quiver. On the other hand, we can take the nonzero evaluation with \( k' \) maximal, so then any wrong substitution has at most \( k' \) internal radical substitutions.

We work with a dominant branch \( B \) of \( \Gamma \) in \( f \). Our objective is to modify \( f \) to a nonidentity containing a Capelli component which enables us to use combinatorial methods to calculate characteristic coefficients in a Shirshov extension, with multiplication by elements of \( A_0 \). Here is one of our main results, enabling us to correspond quivers with properties of polynomials, and which leads directly to the representability theorem.

**Theorem 3.5** (Canonization Theorem for Polynomials). Suppose \( f(x_1, \ldots, x_t) \) is a full nonidentity of \( A_0 \). Then the \( T \)-ideal of \( f \) contains a critical nonidentity.
4. The proof of the Canonization Theorem for Polynomials

The proof of the Canonization Theorem for Polynomials is done in several stages:

1. Eliminate unwanted semisimple substitutions.
2. Make sure that the substitutions are in the “correct” semisimple components.
3. Provide a molecule inside the polynomial where we can compute the action of characteristic coefficients.

4.1. Unmixed case.

First, following Kemer, we dispose of the following easy case. We say a substitution is unmixed if it does not involve bridges, i.e., all substitutions are in a single Peirce component. Here we need only multiply by a Capelli polynomial of the matrix degree, and then proceed directly to the method of §6.

This aspect is crucial to our proof, since substitutions alone are not sufficient to take care of examples such as the non-finitely generated T-space of \([33]\) (generated by \(\{[x_1, x_2]x_1^{p^k-1}x_2^{p^k-1}, k \in \mathbb{N}\}\) in the Grassmann algebra with two generators; also see [22, 23]).

4.2. The mixed case: the hiking procedure.

To complete the proof of the Canonization Theorem for Polynomials, we must turn to the mixed case. The main feature in the proof of the Canonization Theorem for Polynomials is hiking. The notion of hiking passes from branches of quiver combinatorics of nonidentities, showing how to modify a non-identity of a T-ideal \(I\) to another non-identity in \(I\) whose algebraic operations leave us in the same quiver.

In our combinatorics we need to cope with the danger that our substitutions are wrong, or the base field of the semisimple component is of the wrong size. To prevent this, we make substitutions of multilinear polynomials for indeterminates inside \(f\), called hiking, which force the evaluations to become 0 in such situations. In other words, hiking replaces \(f\) by a more complicated polynomial in its T-ideal, which yields a zero valuation when we start with a wrong substitution in the original indeterminates of \(f\).

We have three kinds of variables:

- Core variables, used for exclusive absorption inside the radical (such as variables which appear in commutators with central polynomial),
- variables used for hiking,
- variables inside Capelli polynomials used for computing the actions of characteristic coefficients.

Example 4.1. An easy example of the underlying principle: If \(k = 2\) with \(n_1 > n_2\), then the quiver \(\Gamma\) consists of two blocks and an arrow connecting them, so we replace a variable \(y\) of \(f\) with a radical substitution by \(h_{n_1,1}[h_{n_1,2}, z]yh_{n_2}\). The corresponding specialization remains in the radical. Then we are ready to utilize the techniques given below in §7 to compute characteristic coefficients, bypassing the complications of hiking.

Suppose we have the polynomial \(f\), with a radical evaluation. We replace it and have a hiked polynomial. If \(\hat{g}\) belongs to the T-ideal generated by \(g\), then one of the variables in \(h\) must have a radical evaluation. After making the substitution we get a
new polynomial $g'$ of the same form as $g$. This is like the Phoenix property described in characteristic 0. But in general we need a rather intricate analysis.

**Definition 4.2.** Given a polynomial $f(x_1, \ldots, x_t)$ and another polynomial $g$, we write $f_{x_i \mapsto g}$ to denote that $g$ is substituted for $x_i$. We say that $f$ is **hiked** to $\tilde{f} := f_{x_i \mapsto g}$ at $x_i$ if $g$ is linear in $x_i$.

We call the replacement $g$ of $x_i$ a **molecule** of the hiked polynomial. A **complex molecule** is the product of molecules.

**Definition 4.3.** A polynomial is **docked** (of length $d$) if it can be written in the form

$$
\sum u \, g_{u,1}\tilde{n}(y)g_{u,2}\tilde{n}(y) \cdots g_{u,d}\tilde{n}(y)g_{u,d+1}
$$

for suitable polynomials $g_{u,i}$ (perhaps constant) in which the $y$ indeterminates do not occur. (In other words the $y$ indeterminates occur only in the $\tilde{n}(y)$.) Unfortunately, if the $x_i$ repeat then the molecules repeat, and thus the variables $y$ repeat.

A docked polynomial $f(x_1, \ldots, x_t; y, y', y''; z, z')$ is **critical** if any nonzero substitution of the $y_i$ is right.

Thus the docks are attached to molecules. If $f$ is hiked to various polynomials $f_j$ we also say it can be hiked to $\sum f_j$.

(Likewise for other indeterminates that appear once the hiking is initiated.)

**Remark 4.4.** First suppose that the depth $u = k$, i.e., all $n_j = \tilde{n}$, and there are no nonzero external radical substitutions. In other words, the only nonzero substitutions involve specializing all the $x_i$ to semisimple elements in blocks of degree $\tilde{n}$. Then we simply replace $f$ by $hf$, which trivially is docked, and the theorem is proved. So in the continuation, we assume that $u < k$, which means there is some nonzero substitution $f(x_1, \ldots, x_k)$ in our dominant branch $B$, for which some $x_i$ is an $\tilde{n}$-bridge. We fix $x_1, \ldots, x_k$ in what follows, and call it our **fundamental substitution**, with bridge at this $i$.

We prove a more technical version of the Canonization Theorem, to handle the mixed case.

**Theorem 4.5** (Hiking Theorem for Polynomials). Suppose $f(x_1, \ldots, x_t)$ is a full non-identity of $A_0$, possibly with mixed or pure substitutions. Then $f$ can be hiked to a critical nonidentity in which all of the substitutions of the $x_i$ are right.

5. Details of hiking

The proof of Theorem 4.5 is through a succession of hiking steps in order both to eliminate “wrong” substitutions and then docking, i.e., insert $\tilde{n}$ into the polynomial. The latter is achieved by replacing $z_i$ by $\tilde{n}z_i$ and $z'_i$ by $\tilde{n}z'_i$; i.e., we pass to $f_{z_i \mapsto \tilde{n}z_i; z'_i \mapsto \tilde{n}z'_i}$.

The hiking procedure requires three different stages.
5.1. Preliminary hiking.

Our initial use of hiking is to resolve some technical issues. First, we want to eliminate the effect of \((q_1, q_2)\)-Frobenius gluing for \(q_1 \neq q_2\), since it can complicate docking. Toward this end, we substitute \(z_t c_{n,t}^j(y)^{q_1/q_2}\) for \(z_t\), for each instance of Frobenius gluing. It makes the Frobenius gluing identical on \(f\).

We also need the base fields of the components all to be the same. When \(B'\) is another branch with the same degree vector, and the corresponding base fields for the \(i\)-th vertex of \(B\) and \(B'\) are \(n_i\) and \(n'_i\) respectively, we take \(t_i = q^n_i\) and replace \(x_i\) by \((c_{n_i}^i - c_{n_i})x_i\). This cuts off the specializations to matrices over finite fields of the wrong order.

5.2. First stage of hiking.

We have a quasi-linear nonidentity \(f\) of a Zariski closed algebra \(A_0\) which has a fundamental substitution in some branch \(B\), where \(x_{i'}\) in \(A_0\) is an \(\tilde{n}\)-bridge, corresponding to an edge in the full quiver whose initial vertex is labeled by \((n_i, t_{i+1})\) and whose terminal vertex is labeled by \((n_{i+1}, t_{i+1})\) where \(\tilde{n} = \max\{n_i, n_{i+1}\}\). We replace \(x_i\) by \(c_{n_i} z_{i} [x_{i1}, h_{\tilde{n}-1}] z_{i1+1} c_{n_i+1}\), (where as always the \(c_{n_i}\) involve new indeterminates in \(v\)), and \(z_i, z_{i1+1}\) also are new indeterminates which we call “docking indeterminates”); this yields a quasi-linear polynomial in which any substitution of \(x_i\) into a diagonal block of degree \(< n_i\) or a bridge which is not an \(n\)-bridge is 0. For each semisimple substitution \(x_i\) in a block of degree \(n_i\), taking \([x_{i1}, h_{n_{i1}}]\) yields 0. This removes all semisimple component substitutions in \(h\) of such \(x_i\) whose degree is too “small,” i.e., less than \(n_i\). For the time being, we could still have radical substitutions, but first stage hiking does prepare for their elimination in the second stage.

The number of extra \(\tilde{n}\)-bridges in a specialization of \(c_{n_{i1}} (v) z_{i1} [x_{i1}, h_{n_{i1}} (v)] z_{i1+1} c_{n_{i1}+1}(v)\) is called its (first stage) bridge contribution. (In other words, one takes the total number of bridges, and subtracts 1 if \(x_i\) is an \(\tilde{n}\)-bridge.) The term \([x_{i1}, h_{n_{i1}} (v)]\) is called the core of the bridge contribution.

**Lemma 5.1.** Any nonzero specialization of \(h\) is either \(\tilde{n}\)-semisimple, or its bridge contribution is positive.

**Proof.** By definition, if the bridge contribution is 0 then every substitution has to be semisimple or a \((j, j)\)-bridge for some \(j\). If \(\tilde{n}\) does not appear then the graph would have \(t\) such bridges. □

**Lemma 5.2.** After the first stage of hiking, a wrong specialization of an \(\tilde{n}\)-semisimple element cannot be \(m\)-semisimple for \(m < \tilde{n}\) unless its bridge contribution is at least 2.

**Proof.** When evaluating \(h_{\tilde{n}}\) on semisimple elements of degree \(m\) we get 0 unless we pass away from the \(m\)-semisimple component, which requires two bridges. □

**Lemma 5.3.** After the first stage of hiking, a wrong specialization of an \(\tilde{n}\)-semisimple element is either \(\tilde{n}\)-semisimple or its bridge contribution is at least 1.

**Proof.** When evaluating \(h_{\tilde{n}}\) on semisimple elements of degree \(m\) we get 0 unless we pass away from the \(m\)-semisimple component, which requires two bridges. □

The first stage of hiking does not instantly zero out bridges \(x_i\), but does prepare for their elimination in the second stage.
Appending the Capelli polynomials also sets the stage for eliminating other unwanted substitutions in the second stage.

After repeated applications of first stage hiking, we wind up with a new polynomial \( f(x_1, \ldots, x_i; v; z) \) where we still have our original indeterminates \( x_i \) but have adjoined new indeterminates from \( v \) and \( z \).

5.3. Second stage of hiking.

Example 5.4. To introduce the underlying principle, here is a slightly more complicated example. Consider the quiver of three arrows, from degree 2 to degree 1, degree 1 to degree 1, and finally from degree 1 to degree 1.

First we multiply on the left by \( c_4[h_{1,1}, z_1]z_2 \). The second substitution could have an unwanted position inside the first matrix block of degree 2, since \( c_4[h_{1,1}, z_1]z_2 \) can be evaluated in the larger component. We take \( f_{x_1 \to c_2 y' j x_1} - f_{x_1 \to c_2 y' j y} \), i.e., we multiply by a central polynomial \( h_2 \) on the left and subtract it from a parallel evaluation of \( h_2 \) on the right. The unwanted substitution then cancels out with the other substitution and leaves 0.

In the second stage of hiking, in the blended case, we arrange for all nonzero substitutions to be pure radical.

Suppose \( f(x_1, \ldots, x_i; y; z') \) is already hiked after the first stage. Suppose in the branch \( B \) the indeterminate \( z_i \) occurs of degree \( d_i \) and the indeterminate \( z_{u+1} \) occurs of degree \( d' \), where \( 1 \leq j \leq u \).

Proposition 5.5. There are three cases to consider:

1. There is a string \( x_{i-1} x_i \cdots x_j x_{j+1} \) where \( x_i, \ldots, x_j \) are all semisimple of the same degree \( x_{n_1} \), whereas \( x_{i-1}, x_{i+1}, x_{j+1} \) are both \( \bar{n} \)-bridges. We take the polynomial

   \[
   f_{z_i \to h_{a}(y')t_i z_i} - f_{z_{u+1} \to h_{a}(y')t_i z_{u+1}},
   \]

   where the branch \( B \) has depth \( u \) and \( t_i \) designates the maximal degree of \( x_i \) in a monomial of \( B \), where \( y' \) is a fresh new set of indeterminates

2. There is a string \( x_{i-1} x_i \cdots x_j x_{j+1} \) where \( x_i, \ldots, x_j \) are all semisimple of the same degree \( x_{n_1} \) whereas \( x_j x_{j+1} \) is an \( \bar{n} \)-bridge. We take the polynomial

   \[
   f_{z_i \to h_{a}(y')t_i z_{j}}.
   \]

3. There is a string \( x_{i-1} x_i \cdots x_k x_k \) where \( x_i, \ldots, x_{k-1} \) are all semisimple of the same degree \( x_{n_1} \) whereas \( x_{k-1} x_{k} \) is an \( \bar{n} \)-bridge. We take the polynomial

   \[
   f_{z_{j-1} \to h_{a}(y')t_{j-1} z_{j-1}}.
   \]

This hiking zeroes out semisimple evaluations of highest degree (\( \bar{n} \)), but not a radical evaluation at the \( u \) block.

Proof. (Note that (1) is the usual case, but we also need (2) and (3) to handle terms lying at the ends of the polynomial.) The expression (2) yields zero on a semisimple substitution, but not on a radical substitution, since exactly one of the two summands of (2) would be 0.

Lemma 5.6. The second stage of hiking forces any nonzero specialization of an \( \bar{n} \)-bridge also to be a \( \bar{n} \)-bridge.
Proof. In order to provide a nonzero value, at least one of its vertices must be of degree $\tilde{n}$. But if both were $\tilde{n}$ the evaluation would be 0, by Lemmas 5.1–5.3 and Remark 5.5. Thus we get an $\tilde{n}$-bridge.

**Lemma 5.7.** After the first and second stages of hiking, the positions of semisimple substitutions of degree $\tilde{n}$ are fixed; in other words, semisimple substitutions of degree $\tilde{n}$ are $\tilde{n}$-right.

Proof. Lemma 5.6 “uses up” all the places for $\tilde{n}$-bridges, since more $\tilde{n}$-bridges would yield a substitution contradicting the maximality of the number of $\tilde{n}$-bridges in $\mathcal{B}$. If $\mathcal{B}$ has no semisimple substitutions of degree $\tilde{n}$ then there is no room for any semisimple substitutions of degree $\tilde{n}$, and we are done.

But if $\mathcal{B}$ has a semisimple substitution of degree $\tilde{n}$, that substitution must border an $\tilde{n}$-bridge, fixing the order of the pair of indices in the $\tilde{n}$-bridge, and thus fixing the positions of all the gaps of index $\tilde{n}$ between $\tilde{n}$-bridges, so we are done.

**Remark 5.8.** Although this is taken care of in the proof, we can remove finite components simply by substituting $x_i^m - x_i$ for $x_i$, for suitable $\ell, m$.

**Proof of Theorem 4.5.** Just iterate the hiking procedure down from $\tilde{n}$.

**Proof of Theorem 3.5.** One obtains the dock by replacing $f$ by

\[ f_{z_0 \rightarrow c_0(y)z_{u_1}, z_{u + 1} \rightarrow c_{u}(y)z_1} \]

\[ f_{z_0 \rightarrow c_0(y)z_{u_1}, z_{u + 1} \rightarrow c_{u}(y)z_1} \]

**Example 5.9.** Let us run through the hiking procedure, taking

\[ A_0 = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix} \]

We have the full quiver

\[ I_1 \rightarrow II_2, \]

and take the nonidentity $f = x_1[x_2, x_3][x_4 + x_4][x_2, x_3][x_1^2]$. We have nonzero specializations with $\overline{x}_1$ in the first matrix component, $\overline{x}_2$ an external radical specialization, and $\overline{x}_3$ in the second matrix component, which we denote as $\mathcal{C} = \text{M}_2(K)$, but also we have a nonzero specialization of all variables into $\mathcal{C}$. To avoid this situation, we replace $f$ by

\[ f(c_3(y)zx_1z', x_2, x_3, x_4) = c_3(y)zx_1z'[x_2, x_3][x_4 + x_4][x_2, x_3][x_1^2]c_3(y)zx_1z'c_3(y)zx_1z'. \]

Now any specialization into $\mathcal{C}$ becomes 0, so we have eliminated some “wrong” specializations. For stage 2 we take

\[ f(x; y; y'; z; z') := f(c_3(y)c_3(y')^2zx_1z', x_2, x_3, x_4) - f(c_3(y)zx_1z'c_3(y'), x_2, x_3, x_4) \]

\[ = (c_3(y)c_3(y')^2zx_1z'[x_2, x_3][x_4 + x_4][x_2, x_3][x_1^2]c_3(y)c_3(y')^2zx_1z'c_3(y)zx_1z') \]

\[ - (c_3(y)zx_1z'c_3(y')[x_2, x_3][x_4 + x_4][x_2, x_3][x_1^2]c_3(y)zx_1z'c_3(y')zx_1z'c_3(y')), \]

where we see the specialization of highest degree to the first matrix component has been eliminated. We can eliminate the nonzero specializations of $h(y''')$ of degree 1 by
taking \( \tilde{f}(x; y; y'; c_3(y'')z; z') - \tilde{f}(x; y; y'; z; z'c_3(y'')) \) which leaves us only with a radical specialization and a critical polynomial with a single dock \( c_3(y'') \).

Note how quickly the polynomial becomes complicated even though we have hiked only one indeterminate.

**Remark 5.10.** Other examples of hiking are given in [16]. The main difference between the hiking procedure of this paper and that of stage 3 hiking of [16] is in the treatment of the Frobenius. Stage 4 hiking is analogous.

6. Characteristic coefficient-absorbing polynomials inside T-ideals

We have already pinpointed the \( x_i \) that must have substitutions into semisimple blocks of degree \( \tilde{n} \), in order to utilize the well-understood properties of semisimple matrices (especially the coefficients of their characteristic polynomials, which we call characteristic coefficients). We follow the discussion of coefficient-absorbing polynomials from [16, Theorem 4.26] and [17, §6.3], although we can skip much of it because we already have a docked polynomial.

Any matrix \( a \in M_n(K) \) can be viewed either as a linear transformation on the \( n \)-dimensional space \( V = K^{(n)} \), and thus having Hamilton-Cayley polynomial \( f_a \) of degree \( n \), or (via left multiplication) as a linear transformation \( a \) on the \( n^2 \)-dimensional space \( \tilde{V} = M_n(K) \) with Hamilton-Cayley polynomial \( f_{\tilde{a}} \) of degree \( n^2 \). The matrix \( a \) can be identified with the matrix \( \tilde{a} \), so its eigenvalues have the form \( \beta \otimes 1 = \beta \) for each eigenvalue \( \beta \) of \( a \). From this, we conclude:

**Proposition 6.1 ([14, Proposition 2.4]).** Suppose \( a \in M_n(F) \). Then the characteristic coefficients of \( a \) are integral over the \( F \)-algebra \( \hat{C} \) generated by the characteristic coefficients of \( \tilde{a} \).

**Proof.** The integral closure of \( \hat{C} \) contains all the eigenvalues of \( \tilde{a} \), which are the eigenvalues of \( a \), so the characteristic coefficients of \( \tilde{a} \) also belong to the integral closure. \( \square \)

Next we use hiking also to force the characteristic coefficients of the matrices to commute with each other. Using Theorem 3.35, we work with quasi-linear polynomials and pinpoint semisimple substitutions of degree \( \tilde{n} \), in order to utilize the well-understood properties of semisimple matrices (especially the characteristic coefficients).

Having obtained semisimple substitutions of degree \( \tilde{n} \), we have two ways of obtaining intrinsically the coefficients of the characteristic polynomial

\[
g_a = \lambda^n + \sum_{k=1}^{n-1} (-1)^k \alpha_k(a) \lambda^{n-k}
\]

of a matrix \( a \). Fixing \( k \), we write \( \alpha_k \) for \( \alpha_k(a) \), which we call the \( k \)-characteristic coefficient of \( a \). We want to extract these characteristic coefficients, by means of polynomials.
Definition 6.2. In any matrix ring $\mathbb{M}_n(W)$, we define

$$\alpha_{\text{mat}}(a) :=\sum_{j=1}^{n} \sum e_{j,i_1} a e_{i_2,i_2} a \cdots a e_{i_k,i_k} a e_{i_1,j},$$

the inner sum taken over all index vectors of length $k$.

We can also define the characteristic coefficients via polynomials.

Definition 6.3. Given a quasi-linear polynomial $f(x; y)$ in indeterminates labeled $x_i, y_i$, we say $f$ is characteristic coefficient-absorbing with respect to a full quiver $\Gamma$ if $f(A_0(\Gamma))^+$ absorbs multiplication by any characteristic coefficient of any element in each docked (diagonal) matrix block of a molecule of $A_0(\Gamma)$.

Lemma 6.4 (as in [15, Lemma 3.6]). Write the polynomial $f$ of Theorem 3.5 as a sum of homogeneous components $\sum f_j$. Each $f_j$ is characteristic coefficient absorbing in the blocks of degree $\tilde{n}$.

Proof. The proof can be formulated in the language of [10, Theorem J, Equation 1.19, page 27] (with the same proof), as follows, writing $T_{a,j}$ for the transformation given by left multiplication by $a$:

$$\alpha_k f(a_1, \ldots, a_t, r_1, \ldots, r_m) = \sum f(T_{a}^{k_1} a_1, \ldots, T_{a}^{k_t} a_t, r_1, \ldots, r_m),$$

summed over all vectors $(k_1, \ldots, k_t)$ with each $k_i \in \{0, 1\}$ and $k_1 + \cdots + k_t = k$, where $\alpha_k$ is the $k$-th characteristic coefficient of a linear transformation $T_a : V \to V$. □

Since the purpose of Lemma 6.4 was to obtain the conclusion (6), we merely assume (6).

Lemma 6.5. For any homogeneous polynomial $f(x_1, x_2, \ldots)$ quasi-linear in $x_1$ with respect to a matrix algebra $\mathbb{M}_n(F)$, satisfying (6), there is a polynomial $\hat{f}$ in the T-ideal generated by $f$ which is characteristic coefficient absorbing.

Proof. Take the polynomial of Lemma 6.4 □

Remark 6.6. Notation as in (6), the Cayley-Hamilton identity for $n \times n$ matrices is

$$0 = \sum_{k=0}^{n} (-1)^k \alpha_k f(a_1, \ldots, a_t, r_1, \ldots, r_m) \lambda^{n-k},$$

which is thus an identity in the T-ideal generated by $f$.

Definition 6.7. We call the identity

$$\sum_{k=0}^{n} (-1)^k \sum_{k_1 + \cdots + k_t = k} f(T_{a}^{k_1} a_1, \ldots, T_{a}^{k_t} a_t, r_1, \ldots, r_m) \lambda^{n-k},$$

obtained in Remark 6.6 the Hamilton-Cayley identity induced by $f$.

Definition 6.8. Fixing $0 \leq k < n$, we denote this implicit definition in Lemma 6.5 of $\alpha_k$, the $k$-th characteristic coefficient of $a$, as $\alpha_{\text{pol}}(a)$.
If the vertex corresponding to $r$ has matrix degree $n_i$, taking an $n_i \times n_i$ matrix $w$, we define $\alpha_{\text{pol}}^{\bar{q}}_u(w)$ as in the action of Definition 6.8 and then the left action

$$a_{u,v} \mapsto \alpha_{\text{pol}}^{\bar{q}}_u(w)a_{u,v}. \quad (7)$$

Likewise, for an $n_j \times n_j$ matrix $w$ we define the right action

$$a_{u,v} \mapsto a_{u,v}\alpha_{\text{pol}}^{\bar{q}}_v(w). \quad (8)$$

(However, we only need the action when the vertex is non-empty; we forego the action for empty vertices.)

**Remark 6.9** (For $f$ homogeneous.) Take $\hat{f}$ of Lemma 6.5, and one more indeterminate $y''$. There is a Capelli polynomial $\tilde{c}_{n_i^2}(y'')$ and $p$-power $\bar{q}$ such that

$$\tilde{c}_{n_i^2}(\alpha_k y'')x_i c_{n_i^2}(y'') = \alpha_k^q(y_1)c_{n_i^2}(y'')x_i c_{n_i^2}(y'') \quad (9)$$
on any diagonal block. Since characteristic coefficients commute on any diagonal block, we see from this that

$$\tilde{c}_{n_i^2}(y'')x_i c_{n_i^2}(y'')\tilde{c}_{n_i^2}(z)x_i c_{n_i^2}(z) - \tilde{c}_{n_i^2}(z)x_i c_{n_i^2}(z)\tilde{c}_{n_i^2}(y'')x_i c_{n_i^2}(y'') \quad (10)$$

vanishes identically on any diagonal block, where $z = \alpha_k y''$. One concludes from this that substituting (10) for $x_i$ would hike $f$ one step further. But there are only finitely many ways of performing this hiking procedure. Thus, after a finite number of hikes, we arrive at a polynomial in which we have complete control of the substitutions, and the characteristic coefficients defined via polynomials commute.

### 7. Resolving ambiguities for nonhomogeneous polynomials

When $f$ is homogeneous, we can skip this section. The non-homogeneous case is more delicate. Since we may be in nonzero characteristic, in the main situation our quasi-linear hiked polynomials are not homogeneous. In §6 we obtained actions on each monomial component separately, but we need to provide a uniform action on each of these components.

#### 7.1. Removing ambiguity of matrix degree for nonhomogeneous polynomials

First we want to make sure that we are working in the same matrix degree for each monomial.

**Definition 7.1.** A hiked polynomial is **uniform** if there is some indeterminate $x_i$ for which, in each of its monomials, the molecule obtained from hiking $x_i$ is semisimple of the same matrix degree.

Our objective in this section is to hike to a uniform polynomial. First we use §4.1 to dispose of the easy case where each hiked monomial has a semisimple molecule (Definition 4.2).

**Definition 7.2.** A radical element of a complex molecule is **isolated** if multiplication by any radical element on the left or right is zero.

**Remark 7.3.** The product of two isolated elements is 0, by definition.

**Proposition 7.4.** Any polynomial can be hiked to a uniform polynomial.
Proof. Multiply \( x_i \) by a new indeterminate \( x'_i \) and hike that. We are done unless it yields a radical substitution. Since \( J_{t+1} = 0 \), we get an isolated element after at most \( t \) hikes. \( \square \)

7.2. Removing ambiguities for nonhomogeneous polynomials having molecules of the same matrix degree.

We have just reduced to the case where all monomials have molecules of some \( x_i \) of the same matrix degree \( \tilde{n} \), but we still must contend with the possibility that \( x_i \) has different degrees in different monomials, and then our characteristic coefficient arguments work differently for the different monomials.

Take the finitely many matrix components \( R_i, 1 \leq i \leq k \), each of which has degree \( \tilde{n} \) and multiplicity \( q_i \). Multiplication by characteristic coefficients \( \alpha_i \) is integral over the multiplication by \( \alpha_i^{q_i} \).

We introduce a commuting indeterminate \( \lambda_i \) for each of the finitely many characteristic coefficients \( \alpha_i, i \in I \), define \( C' \) to be \( \hat{C}[\lambda_i : i \in I] \). Defining the action of \( \lambda_i \) on \( R_i \) to be the same as that of \( \alpha_i \), we have \( (\lambda_i - \alpha_i)R_i = 0 \).

Definition 7.5. Given matrices \( a_1, \ldots, a_t \), the symmetrized \((k; j)\) characteristic coefficient is the \( j \)-elementary symmetric function applied to the \( k \)-characteristic coefficients of \( a_1, \ldots, a_t \).

For example, taking \( k = 1 \), the symmetrized \((1, j)\) characteristic coefficients are

\[
\sum_{j=1}^{t} \text{tr}(a_j), \quad \sum_{j_1 > j_2} \text{tr}(a_{j_1}) \text{tr}(a_{j_2}), \quad \ldots, \quad \prod_{j=1}^{t} \text{tr}(a_j).
\]

Lemma 7.6. Any characteristic coefficient \( \alpha_k \) is integral over the ring with all the symmetrized characteristic coefficients adjoined.

Proof. If \( \alpha_{k,t} \) denotes the \( (t; j) \)-characteristic coefficient, then \( \alpha_k \) satisfies the usual polynomial \( \lambda^n + (-1)^j \sum_{j=1}^{n} \alpha_{k,j} \lambda^{t-j} \). \( \square \)

Remark 7.7. The reason that we need to introduce the symmetrized characteristic coefficient is that we might have several glued components and their molecules, which we cannot distinguish, so we need to find coefficients common to all of them.

Proposition 7.8. Take \( \hat{f} \) of Lemma 6.5. There is a uniform polynomial \( \tilde{f} \) hiked from \( \hat{f} \) which is characteristic coefficient absorbing.

Proof. \( \hat{f} \) has no semisimple evaluations into a maximal component, and thus (inductively) all evaluations of \( \hat{f} \) involving a maximal component have a “radical bridge,” i.e., \( \hat{f} \) has a monomial \( h \) where the indeterminates \( x_i \) specialize to external radical substitutions. In particular \( n_i = \tilde{n} = n_1 \).

We adjoin the characteristic values of all the complex molecules. Then we apply Proposition 7.4 to Lemmas 6.4 and 6.5 \( \square \)

8. Application of Shirshov’s theorem

Here is the connection to full quivers.
Definition 8.1. For a Zariski closed algebra $A \subseteq M_n(K)$ faithful over an integral domain $C$, we denote by $\hat{C}$ the algebra obtained by adjoining to $C$ the symmetrized characteristic coefficients of products of the Peirce components of the generic generators of $A$ (of length up to the bound of Shirshov's Theorem [10, Chapter 2]).

Characteristic Value Adjunction Theorem [16, Theorem 3.22]. Let $A_0$ denote the algebra obtained by adjoining to $A_0$ the characteristic matrix coefficients of products of the sub-Peirce components of the generic generators of $A_0$ (of length up to the bound of Shirshov’s Theorem [10, Chapter 2]), and let $\hat{C}$ be the algebra obtained by adjoining to $F$ these symmetrized characteristic coefficients. The T-ideal $I$ generated by the polynomial $\tilde{f}$ contains a nonzero T-ideal which is also an ideal of the algebra $\hat{A}_0$. Recall in view of Shirshov’s theorem that we only need to adjoin a finite number of elements to obtain $\hat{C}$.

Lemma 8.2. The algebra $A_0$ is a finite module over $\hat{C}$, and in particular is Noetherian and representable.

Proof. Let $\hat{C}'$ be the commutative algebra generated over $C$, by all the characteristic coefficients of (finitely many) products of the Peirce components of the generic generators of $A_0$, as in Definition 8.1. Clearly $\hat{C} \subseteq \hat{C}'$.

Enlarge $A_0$ to $A_0' = \hat{C}'A_0$, which is a finite module over $\hat{C}'$ in view of Shirshov’s Theorem. But $\hat{C}'$ is finite over $\hat{C}$, in view of Lemma 7.6 implying $A_0$ is finite over $\hat{C}$. Thus $A_0$ is Noetherian, and is representable by Anan’in’s Theorem [3]. □

Also, for any characteristic coefficient-absorbing polynomial $f$ with respect to the quiver of $A_0$, the Hamilton-Cayley identity induced by $f$ is an identity of $A_0$, and thus of $A_0$.

9. Conclusion of the proof of Theorem 1.1

Proof. Let $I$ be the T-ideal generated by $\tilde{f}$, and $I_1$ be the T-ideal of $\hat{A}_0$ generated by symmetrized $\tilde{q}$-characteristic coefficient-absorbing polynomials of $I$ in $\hat{A} := \hat{C}'A_0$. The ideal $I_0$ is representable by Lemma 8.2, implying $A_0 \cap I_1$ is representable, as desired. □

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