Propagation property and its application to inverse scattering for fractional powers of the negative Laplacian

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Abstract

Enss (1983) proved a propagation estimate for the usual free Schrödinger operator that turned out later to be very useful for inverse scattering in the work of Enss–Weder (1995). Since then, this method has been called the Enss–Weder time-dependent method. We study the same type of propagation estimate for the fractional powers of the negative Laplacian and, as with the Enss–Weder method, we apply our estimate to inverse scattering. We find that the high-velocity limit of the scattering operator uniquely determines the short-range interactions.

Keywords: scattering theory, inverse problem, fractional Laplacian
MSC2010: 81Q10, 81U05, 81U40

1 Introduction

For $1/2 \leq \rho \leq 1$, the fractional powers of the negative Laplacian as self-adjoint operator acting on $L^2(\mathbb{R}^n)$ is defined by the Fourier multiplier with symbol

$$\omega_{\rho}(\xi) = |\xi|^{2\rho}/(2\rho).$$

(1.1)

We denote this operator by

$$H_{0,\rho} = \omega_{\rho}(D_x),$$

(1.2)
where \( D_x = -i \nabla_x = -i(\partial_{x_1}, \ldots, \partial_{x_n}) \). More specifically, we can represent \( H_{0, \rho} \) by the Fourier integral operator

\[
H_{0, \rho} \phi(x) = (\mathcal{F}^* \omega_\rho(\xi) \mathcal{F} \phi)(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} \omega_\rho(\xi)(\mathcal{F} \phi)(\xi) d\xi / (2\pi)^n / 2 = \int_{\mathbb{R}^2n} e^{i(x-y) \cdot \xi} \omega_\rho(\xi) \phi(y) dyd\xi / (2\pi)^n
\]

for \( \phi \in \mathcal{D}(H_{0, \rho}) = H^{2\rho}(\mathbb{R}^n) \), which is the Sobolev space of order \( 2\rho \). In particular, if \( \rho = 1 \), then \( H_{0, 1} \) is the free Schrödinger operator \( \omega_1(D_x) = -\Delta_x / 2 = -\sum_{j=1}^n \partial_{x_j}^2 / 2 \). If \( \rho = 1/2 \), then \( H_{0, 1/2} \) is the massless relativistic Schrödinger operator \( \omega_{1/2}(D_x) = \sqrt{-\Delta_x} \).

In Section 2, we prove the following Enss-type propagation estimate for \( e^{-itH_{0, \rho}} \).

Throughout this paper, \( F(\cdots) \) is the usual characteristic function of the set \( \{ \cdots \} \). We denote the smooth characteristic function \( \chi \in C^\infty(\mathbb{R}^n) \) by

\[
\chi(x) = \begin{cases} 1 & |x| \geq 2, \\ 0 & |x| \leq 1. \end{cases}
\]

**Theorem 1.1.** Let \( f \in C^\infty_0(\mathbb{R}^n) \) with \( \text{supp} \ f \subset \{ \xi \in \mathbb{R}^n \mid |\xi| \leq \eta \} \) for some given \( \eta > 0 \). Choose \( v \in \mathbb{R}^n \) such that \( |v| > \eta \) and

\[
\begin{cases} 
16n(1 - \rho)(|v| - \eta)^{2\rho - 2} \eta \leq |v|^{2\rho - 1} & 1/2 \leq \rho < 1, \\
8\eta \leq |v| & \rho = 1.
\end{cases}
\]

For \( t \in \mathbb{R} \) and \( N \in \mathbb{N} \), the following estimate holds.

\[
\left\| \chi \left( \frac{x - (\nabla_\xi \omega_\rho)(v)t}{|v|^{2\rho - 1}|t|/4} \right) e^{-itH_{0, \rho}} f(D_x - v) F \left( |x| \leq \frac{|v|^{2\rho - 1}|t|}{16} \right) \right\| \leq C_N (1 + |v|^{2\rho - 1}|t|)^{-N},
\]

where \( \| \cdot \| \) stands for the operator norm on \( L^2(\mathbb{R}^n) \), and the constant \( C_N > 0 \) also depends on the dimension \( n \) and the shape of \( f \).

Enss [5] proved the following estimate for the free Schrödinger operator

\[
\left\| F \left( |x - vt| \geq \frac{|v||t|}{4} \right) e^{-itD_x^2/2} f(D_x - v) F \left( |x| \leq \frac{|v||t|}{16} \right) \right\| \leq C_N (1 + |v||t|)^{-N}.
\]

This estimate was proved not only for the spheres but more generally for the measurable subsets of \( \mathbb{R}^n \) (see Proposition 2.10 in Enss [5]). Before considering
Theorem 1.1 further, we discuss the meaning of the estimate (1.7). From the perspective of classical mechanics, \( D_x \) represents the momentum or equivalently the velocity of the particle of unit mass. On the left-hand side of (1.7), \( D_x \) is localized to the neighborhood of \( v \) by the cut-off function \( f \). Therefore, along the time evolution of the propagator \( e^{-itD_x^2/2} \), the position of the particle behaves according to

\[
x \sim D_x t \sim vt.
\]

Because the behavior of the points on the sphere is the same, the center of the sphere moves toward \( vt \) from the origin

\[
\left\{ x \in \mathbb{R}^n \mid |x| \leq \frac{|v||t|}{16} \right\} \sim \left\{ x \in \mathbb{R}^n \mid |x - vt| \leq \frac{|v||t|}{16} \right\}.
\]

We extract an interpretation of the estimate (1.7) from these observations. The behavior of the sphere (1.9) makes the characteristic functions on both sides of (1.7) disjoint. Thus, this gives rise to the decay associated with time and velocity. Theorem 1.1 is the fractional Laplacian version of (1.7). From \((\nabla_\xi \omega_\rho)(v) = |v|^{2\rho-2}v\), the case where \( \rho = 1 \) in (1.6) is essentially equivalent to (1.7). Conversely, if \( \rho = 1/2 \) in (1.6), the decay on the right-hand side does not involve \( |v| \). However, this does not conflict with the physical meaning. In the case where \( \rho = 1/2 \), the system is relativistic. In this system, the particle does not have a mass, and its velocity is the speed of light, which is normalized to 1. Therefore, the decay cannot include the velocity \( v \).

Spectral analysis for the relativistic Schrödinger operator was initiated by Weder [21], following which Umeda [16, 17] studied the resolvent estimate and mapping properties associated with the Sobolev spaces. Wei [24] studied the generalized eigenfunctions. Weder [22] analyzed the spectral properties of the fractional Laplacian for the massive case, and Watanabe [19] studied the Kato-smoothness. Giere [7] investigated the scattering theory and proved the asymptotic completeness of the wave operators for short-range perturbations. Recently, Kitada [11, 12] constructed the long-range theory.

In Section 3, we assume that the dimension of the space satisfies \( n \geq 2 \). As an application of Theorem 1.1 we consider a multidimensional inverse scattering. The high-velocity limit of the scattering operator uniquely determines the interaction potentials that satisfy the short-range condition below by using the Enss–Weder time-dependent method (Enss–Weder [6]).

**Assumption 1.2.** \( V \in C^1(\mathbb{R}^n) \) is real-valued and for \( \gamma > 1 \), satisfies

\[
|\partial_{\beta}^\beta V(x)| \leq C_{\beta}(x)^{-\gamma-|\beta|}, \quad |\beta| \leq 1,
\]

where the bracket of \( x \) has the usual definition \( \langle x \rangle = \sqrt{1 + |x|^2} \).
For the full Hamiltonian \( H_\rho = H_{0,\rho} + V \), where \( V \) belongs to the class stated above, the existence of the wave operators

\[
W_\rho^\pm = \lim_{t \to \pm \infty} e^{itH_\rho} e^{-itH_{0,\rho}}
\]  

and their asymptotic completeness have already been proved in Kitada [11, 12]. Thus, we can define the scattering operator \( S_\rho = S_\rho(V) \) by

\[
S_\rho = (W_\rho^+)^* W_\rho^-.
\]

(1.12)

Under these situations, the following uniqueness theorem can be proved.

**Theorem 1.3.** Let \( V_1 \) and \( V_2 \) be interaction potentials which satisfy Assumption 1.2. If \( S_\rho(V_1) = S_\rho(V_2) \), then \( V_1 = V_2 \) holds for \( \frac{1}{2} < \rho \leq 1 \).

We note that \( \rho = \frac{1}{2} \) is excluded in this theorem. As mentioned before, if \( \rho = \frac{1}{2} \), the system is relativistic and the speed of light is always equal to 1, that is, \( |v| \equiv 1 \). The Enss–Weder time-dependent method is also called the high-velocity method. As the name suggests, deriving the uniqueness of the interaction potentials requires the limit of \( |v| \). Therefore, this method does not combine well with relativistic phenomena (see also Jung [10]).

In Enss–Weder [6], the estimate (1.7) was demonstrated to be very useful for inverse scattering and the Enss–Weder time-dependent method was developed. Since then, the uniqueness of the interaction potentials for various quantum systems has been studied by many authors (Weder [23], Jung [10], Nicoleau [13, 14, 15], Adachi–Maehara [4], Adachi–Kamada–Kazuno–Toratani [2], Valencia–Weder [18], Adachi–Fujiwara–Ishida [3], and Ishida [9]). This paper is motivated by their results. In particular, Enss–Weder [6] first proved the uniqueness of the potentials for \( \rho = 1 \) by applying (1.7). Jung [10] treated \( \rho = \frac{1}{2} \) using a different approach. Naturally, we cannot consider the limit of the velocity in this case. However, Jung [10] obtained the uniqueness without using an estimate of the type (1.6). Thus, Theorem 1.3 represents an interpolation between the results of Enss–Weder [6] and Jung [10].

**2 Propagation Property**

In this section, we prove Theorem 1.1. Regarding estimate (1.7), the idea of Enss [5] is very simple and understandable. The Galilean transformation in the direction of \( v \) enables a reduction to a static system, and iterations of the integration by parts, by taking the points of stationary phase into account, leads to (1.7). However, in our case, these ingredients do not work well because of the fractional
powers. Instead, our main strategy is the asymptotic expansion of the symbolic calculus of pseudo-differential theory.

Here, we recall several basics of the calculus of pseudo-differential operators. They are recounted from standard textbooks. For \( m \in \mathbb{R} \), let \( S_{1,0}^m \) be the Hörmander symbol class, that is, we say \( p \in S_{1,0}^m \) if and only if \( p \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_\xi^n) \) and, for any multi-indices \( \beta \) and \( \beta' \),

\[
|\partial_x^{\beta'} \partial_\xi^\beta p(x, \xi)| \leq C_{\beta\beta'} \langle \xi \rangle^{-m - |\beta|}
\]

are satisfied. Then, the pseudo-differential operator \( p(x, D_x) \) with symbol \( p \in S_{1,0}^m \) is defined by

\[
p(x, D_x) \phi(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} p(x, \xi) (\mathcal{F} \phi)(\xi) d\xi / (2\pi)^n / 2
\]

for \( \phi \in \mathcal{S}(\mathbb{R}^n) \) which is the Schwartz functional space. When \( p \in S_{1,0}^m \), we denote the semi-norm \( |p|_{m,k} \) by

\[
|p|_{m,k} = \sup_{x, \xi \in \mathbb{R}^n} \sum_{|\beta| + |\beta'| \leq k} \langle \xi \rangle^{-m - |\beta|} |\partial_x^{\beta'} \partial_\xi^\beta p(x, \xi)|.
\]

If \( p_1 \in S_{1,0}^{m_1} \) and \( p_2 \in S_{1,0}^{m_2} \), then the symbol of the product \( p_1p_2 = q \in S_{1,0}^{m_1 + m_2} \) has the following asymptotic expansion

\[
q(x, \xi) = \sum_{|\beta| \leq N - 1} \partial_\xi^\beta p_1(x, \xi) \times (-i \partial_x)^\beta p_2(x, \xi) / \beta! + r_N(x, \xi),
\]

where the remainder \( r_N \) satisfies \( r_N \in S_{1,0}^{m_1 + m_2 - N} \) and

\[
|\partial_x^{\beta'} \partial_\xi^\beta r_N(x, \xi)| \\
\leq C_{\beta\beta'} \sum_{|\alpha| = N} |\partial_\xi^\alpha p_1|_{m_1 - N, M + |\beta| + |\beta'|} |\partial_x^{\alpha} p_2|_{m_2, M + |\beta| + |\beta'|} \langle \xi \rangle^{m_1 + m_2 - N - |\beta|}.
\]

for some \( M \in \mathbb{N} \) (Chapter 8 in Wong [20]). Moreover, by the \( L^2 \)-boundedness theorem, if \( m_1 + m_2 - N \leq 0 \), then there exists \( K \in \mathbb{N} \) such that the operator-norm of \( r_N \) is estimated by

\[
\|r_N(x, D_x)\| \leq C_N |r_N|_{m_1 + m_2 - N, K} \\
\leq C_N \sup_{x, \xi \in \mathbb{R}^n} \sum_{|\beta| + |\beta'| \leq K} \langle \xi \rangle^{-m_1 - m_2 + N + |\beta|} |\partial_x^{\beta'} \partial_\xi^\beta r_N(x, \xi)| \\
\leq C_N \sup_{x, \xi \in \mathbb{R}^n} \sum_{|\beta| + |\beta'| \leq K} |\partial_\xi^\beta p_1|_{m_1 - N, M + |\beta| + |\beta'|} |\partial_x^{\alpha} p_2|_{m_2, M + |\beta| + |\beta'|} (2.6)
\]

(Theorem 3.36, Lemma 3.37–3.39 and Remark 3.40 in Abels in [1]).
Proof of Theorem 1.1. The left-hand side of (1.6) is bounded uniformly in \( t \) and \( v \). Therefore, it is sufficient to prove
\[
\left\| \chi \left( \frac{x - (\nabla \xi \omega_{\rho})(v) t}{|v|^{2\rho - 1}|t|/4} \right) e^{-itH_0,\rho} f(D_x - v) F \left( |x| \leq \frac{|v|^{2\rho - 1}|t|}{16} \right) \right\| \leq C_N(|v|^{2\rho - 1}|t|)^{-N} \tag{2.7}
\]
for \( |v|^{2\rho - 1}|t| \geq 1 \). By using the unitary translations, we have the following relations
\[
e^{iv \cdot x} D_x e^{-iv \cdot x} = D_x - v, \tag{2.8}
e^{it \omega_{\rho}(D_x + v)} x e^{-it \omega_{\rho}(D_x + v)} = x + (\nabla \xi \omega_{\rho})(D_x + v)t. \tag{2.9}
\]
We thus compute that
\[
\chi \left( \frac{x - (\nabla \xi \omega_{\rho})(v) t}{|v|^{2\rho - 1}|t|/4} \right) e^{-itH_0,\rho} f(D_x - v)
= e^{iv \cdot x} \chi \left( \frac{x - (\nabla \xi \omega_{\rho})(v) t}{|v|^{2\rho - 1}|t|/4} \right) e^{-it \omega_{\rho}(D_x + v)} f(D_x) e^{-iv \cdot x}
= e^{iv \cdot x} e^{-it \omega_{\rho}(D_x + v)} \chi \left( \frac{x + (\nabla \xi \omega_{\rho})(D_x + v)t - (\nabla \xi \omega_{\rho})(v)t}{|v|^{2\rho - 1}|t|/4} \right) f(D_x) e^{-iv \cdot x}. \tag{2.10}
\]
The strategy of our proof is as follows. The momentum operator \( D_x \) can move inside the compact region only because \( f \) is compactly supported. Therefore, \((\nabla \xi \omega_{\rho})(D_x + v)\) and \((\nabla \xi \omega_{\rho})(v)\) almost cancel when \( |v| \) is sufficiently large, and the function \( \chi \) in (2.10) behaves as though
\[
\chi \left( \frac{x + (\nabla \xi \omega_{\rho})(D_x + v)t - (\nabla \xi \omega_{\rho})(v)t}{|v|^{2\rho - 1}|t|/4} \right) \sim \chi \left( \frac{x}{|v|^{2\rho - 1}|t|/4} \right). \tag{2.11}
\]
We now justify this strategy. Because \( |\xi| \leq \eta \) on the support of \( f \), we have
\[
|\xi + v| \geq |v| - |\xi| \geq |v| - \eta > 0. \tag{2.12}
\]
This inequality implies
\[
\chi \left( \frac{x + (\nabla \xi \omega_{\rho})(\xi + v)t - (\nabla \xi \omega_{\rho})(v)t}{|v|^{2\rho - 1}|t|/4} \right) f(\xi) \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_\xi^n). \tag{2.13}
\]
Moreover, when \( 1/2 \leq \rho < 1 \),
\[
| (\nabla \xi \omega_{\rho})(\xi + v) - (\nabla \xi \omega_{\rho})(v) | \leq \int_0^1 |(\nabla_{\xi}^2 \omega_{\rho})(v + \theta \xi)| d\theta |\xi| \tag{2.14}
\]
\[(\nabla_x^2 \omega_\rho)(v + \theta \xi) = \max_{1 \leq j \leq n} \sum_{k=1}^{n} |(\partial_{\xi_j} \partial_{\xi_k} \omega_\rho)(v + \theta \xi)| \leq 2n(1 - \rho)(|v| - \eta)^{2\rho - 2}. \quad (2.15)\]

hold for $|\xi| \leq \eta$, where $\nabla_x^2 \omega_\rho$ denotes the Hessian matrix of $\omega_\rho$. In the case where $\rho = 1$, it is clear that

\[|(\nabla_x \omega_1)(\xi + v) - (\nabla_x \omega_1)(v)| = |\xi|. \quad (2.16)\]

We thus obtain, for $1/2 \leq \rho \leq 1$ and $|v|$ which satisfies \((1.5)\),

\[|(\nabla_x \omega_\rho)(\xi + v) - (\nabla_x \omega_\rho)(v)| \leq |v|^{2\rho - 1}/8. \quad (2.17)\]

It follows from \((2.17)\) that

\[
|x| \geq |x + (\nabla_x \omega_\rho)(\xi + v)t - (\nabla_x \omega_\rho)(v)t| - |(\nabla_x \omega_\rho)(\xi + v) - (\nabla_x \omega_\rho)(v)||t| \\
\geq |v|^{2\rho - 1}|t|/4 - |v|^{2\rho - 1}|t|/8 = |v|^{2\rho - 1}|t|/8, \quad (2.18)
\]

on the supports of $f$ and $\chi$. This means that

\[
\chi \left( \frac{x + (\nabla_x \omega_\rho)(\xi + v)t - (\nabla_x \omega_\rho)(v)t}{|v|^{2\rho - 1}|t|/4} \right) f(\xi) \\
= \chi \left( \frac{x + (\nabla_x \omega_\rho)(\xi + v)t - (\nabla_x \omega_\rho)(v)t}{|v|^{2\rho - 1}|t|/4} \right) f(\xi) \chi \left( \frac{x}{|v|^{2\rho - 1}|t|/16} \right) \quad (2.19)
\]

because $\chi(x/(|v|^{2\rho - 1}|t|/16)) = 1$ by \((2.18)\). However, in the pseudo-differential calculus, the product of the symbols is not equal to the symbol of the product. The additional asymptotic error terms arise. By the product formula \((2.4)\), the symbol of \((2.19)\) becomes

\[
\sum_{|\beta| \leq N - 1} \frac{1}{\beta!} \partial_x^\beta \left\{ \chi \left( \frac{x + (\nabla_x \omega_\rho)(\xi + v)t - (\nabla_x \omega_\rho)(v)t}{|v|^{2\rho - 1}|t|/4} \right) f(\xi) \right\} \times (-i\partial_x)^\beta \chi \left( \frac{x}{|v|^{2\rho - 1}|t|/16} \right) + R_N(t, x, \xi) \quad (2.20)
\]

for any $N \in \mathbb{N}$. All terms with $|\beta| \leq N - 1$ vanish due to another characteristic function

\[
\left\{ \partial_x^\beta \chi \left( \frac{x}{|v|^{2\rho - 1}|t|/16} \right) \right\} F \left( |x| \leq \frac{|v|^{2\rho - 1}|t|}{16} \right) = 0. \quad (2.21)
\]

Next, we consider the remainder term $R_N$. Because $f$ is compactly supported,

\[
\chi \left( \frac{x + (\nabla_x \omega_\rho)(\xi + v)t - (\nabla_x \omega_\rho)(v)t}{|v|^{2\rho - 1}|t|/4} \right) f(\xi) \in S_{-\infty}^{-\infty} = \bigcap_{-\infty < m < \infty} S_{1,0}^m \quad (2.22)
\]
To apply the Enss–Weder time-dependent method, we have to assume that
3 Uniqueness of Interactions
holds. Clearly, χ(x/(|v|^{2ρ−1}|t|/16)) ∈ S_{1,0}^0 also holds. In particular, we see that
\[ \left| \partial_x^\beta \chi \left( \frac{x + (\nabla_x \omega_p)(\xi + v)t - (\nabla_x \omega_p)(v)t}{|v|^{2\rho−1}|t|/4} \right) \right| \leq C_\beta |v|^{−(2\rho−1)} \leq C_\beta \] (2.23)
for all β with |β| ≥ 1. Here, C_β > 0 is independent of t and v. Therefore, it is sufficient to focus only on the derivative at x. By the estimate of the remainder (2.6), there exists N′ ∈ N such that
\[ \|R_N(t, x, D_x)\| \leq C_N \sum_{0 \leq j \leq N'} (|v|^{2\rho−1}|t|)^{-j} \sum_{N \leq j \leq N + N'} (|v|^{2\rho−1}|t|)^{-j} \leq C_N(|v|^{2\rho−1}|t|)^{-N} \] (2.24)
because |v|^{2\rho−1}|t| ≥ 1. This completes the proof. \(\square\)

3 Uniqueness of Interactions

To apply the Enss–Weder time-dependent method, we have to assume that n ≥ 2 and that ρ > 1/2 from here on. The following Radon transformation-type reconstruction formula enables Theorem 1.3 to be proved. We devote ourselves to proving Theorem 3.1 in this section. Contrary to Enss–Weder [6], the key calculation in our proof is the pseudo-differential asymptotic expansion as in Theorem 1.1.

Theorem 3.1. Let v ∈ \(\mathbb{R}^n\) be given and let \(\hat{v} = v/|v|\). Suppose that η > 0, and that \(\Phi_0, \Psi_0 \in L^2(\mathbb{R}^n)\) such that \(\mathcal{F}\Phi_0, \mathcal{F}\Psi_0 \in C_0^\infty(\mathbb{R}^n)\) with supp \(\mathcal{F}\Phi_0, \mathcal{F}\Psi_0 \subset \{\xi \in \mathbb{R}^n \mid |\xi| \leq \eta\}\). Let \(\Phi_v = e^{iv \cdot x}\Phi_0, \Psi_v = e^{iv \cdot x}\Psi_0\). Then
\[ |v|^{2\rho−1}(i(S_\rho - 1)\Phi_v, \Psi_v) = \int_{-\infty}^{\infty} (V(x + \hat{v}t)\Phi_0, \Psi_0)dt + o(1) \] (3.1)
holds as |v| → ∞ for any V which satisfies Assumption 1.2, where \((\cdot, \cdot)\) is the scalar product of \(L^2(\mathbb{R}^n)\).

We first prepare the propagation estimate of the following integral form. In the proof of this proposition, we can see that Theorem 1.1 plays an important role. While \|\cdot\| also indicates the norm in \(L^2(\mathbb{R}^n)\), for simplicity, we do not distinguish between the notations for the usual \(L^2\)-norm and its operator norm in this paper.

Proposition 3.2. Let v and \(\Phi_v\) be as in Theorem 3.1. Then
\[ \int_{-\infty}^{\infty} \|V(x)e^{-itH_0,v}\Phi_v\|dt = O(|v|^{1−2\rho}) \] (3.2)
holds as |v| → ∞ for any V which satisfies Assumption 1.2.
Proof. The original idea of this proof is given in Lemma 2.2 of Enss–Weder [6]. We extend it to the case of the fractional powers of the negative Laplacian. Choose $f \in C_0^\infty(\mathbb{R}^n)$ such that $\mathcal{F} \Phi_0 = f \mathcal{F} \Phi_0$ and $\text{supp } f \subset \{ \xi \in \mathbb{R}^n \mid |\xi| \leq \eta \}$. Then the relation

$$\Phi_v = e^{iw \cdot x} \mathcal{F}^* f(\xi) \mathcal{F} \Phi_0 = e^{iw \cdot x} f(D_x) \Phi_0 = f(D_x - v) \Phi_v$$

(3.3) follows. We compute

$$\|V(x)e^{-itH_{0,\rho}} \Phi_v\| = \|V(x)e^{-itH_{0,\rho}} f(D_x - v) \Phi_v\| \leq I_1 + I_2,$$

(3.4) where $I_1$ and $I_2$ are given by

$$I_1 = \left\| V(x) \left\{ 1 - \chi \left( \frac{x - (\nabla \xi \omega_\rho)(v)t}{|v|^{2\rho-1}|t|/4} \right) \right\} e^{-itH_{0,\rho}} f(D_x - v) \Phi_v \right\|,$$

(3.5)

$$I_2 = \left\| V(x) \chi \left( \frac{x - (\nabla \xi \omega_\rho)(v)t}{|v|^{2\rho-1}|t|/4} \right) e^{-itH_{0,\rho}} f(D_x - v) \Phi_v \right\|.$$  

(3.6)

When $|x - (\nabla \xi \omega_\rho)(v)t| \leq |v|^{2\rho-1}|t|/2$ holds, we have

$$|x| \geq |(\nabla \xi \omega_\rho)(v)t| - |x - (\nabla \xi \omega_\rho)(v)t| \geq |v|^{2\rho-1}|t|/2.$$  

(3.7)

By virtue of the decay condition on $V$ in (1.10) and inequality (3.7), $I_1$ can be estimated as follows

$$\int_{-\infty}^{\infty} I_1 dt \leq C \int_{0}^{\infty} \langle |v|^{2\rho-1}|t| \rangle^{-\gamma} dt = C|v|^{1-2\rho} \int_{0}^{\infty} \langle \tau \rangle^{-\gamma} d\tau = O(|v|^{1-2\rho})$$

(3.8)

because $\gamma > 1$, where we changed the integral variable by $\tau = |v|^{2\rho-1}t$. We next estimate $I_2$. By inserting

$$F \left( |x| \leq \frac{|v|^{2\rho-1}|t|}{16} \right) + F \left( |x| > \frac{|v|^{2\rho-1}|t|}{16} \right) = 1$$

(3.9)

between $f(D_x - v)$ and $\Phi_v$, $I_2$ is estimated so that $I_2 \leq I_{2,1} + I_{2,2}$ where $I_{2,1}$ and $I_{2,2}$ are given by

$$I_{2,1} = C \left\| \chi \left( \frac{x - (\nabla \xi \omega_\rho)(v)t}{|v|^{2\rho-1}|t|/4} \right) e^{-itH_{0,\rho}} f(D_x - v) F \left( |x| \leq \frac{|v|^{2\rho-1}|t|}{16} \right) \right\|,$$

(3.10)

$$I_{2,2} = C \left\| F \left( |x| > \frac{|v|^{2\rho-1}|t|}{16} \right) \Phi_0 \right\|.$$  

(3.11)

By applying Theorem 1.1 to $I_{2,1}$ with $N = 2$

$$\int_{-\infty}^{\infty} I_{2,1} dt \leq C \int_{0}^{\infty} \langle |v|^{2\rho-1}|t| \rangle^{-2} dt = C|v|^{1-2\rho} \int_{0}^{\infty} \langle \tau \rangle^{-2} d\tau = O(|v|^{1-2\rho})$$

(3.12)
is obtained. $I_{2,2}$ also provides the same estimate of \((3.12)\). Indeed, $I_{2,2}$ satisfies

$$I_{2,2} \leq C \left\| F \left( \left| x \right| > \frac{|v|^{2\rho-1}}{16} |t| \right) \right\| \langle x \rangle^{-2} \left\| \langle x \rangle^2 \Phi_0 \right\| \leq C \langle |v|^{2\rho-1} |t| \rangle^{-2} \tag{3.13}$$

because $\Phi_0 \in \mathcal{S}(\mathbb{R}^n)$ by the assumption. Therefore, we obtain

$$\int_{-\infty}^{\infty} I_{2,2} \, dt = O(|v|^{1-2\rho}). \tag{3.14}$$

From \((3.8)\), \((3.12)\), and \((3.14)\), it follows that

$$\int_{-\infty}^{\infty} \| V(x) e^{-itH_{0,\rho} \Phi_v} \| \, dt \leq \int_{-\infty}^{\infty} (I_1 + I_{2,1} + I_{2,2}) \, dt = O(|v|^{1-2\rho}). \tag{3.15}$$

\(\square\)

**Corollary 3.3.** Let $v$ and $\Phi_v$ be as in Theorem 3.1. Then

$$\|(W_{\rho}^\pm - 1) e^{-itH_{0,\rho} \Phi_v} \| = O(|v|^{1-2\rho}) \tag{3.16}$$

holds as $|v| \to \infty$ uniformly for $t \in \mathbb{R}$.

**Proof.** The proof is similar to that of Corollary 2.3 in Enss–Weder [6], and therefore is sketched as follows. The difference between $W_{\rho}^\pm$ and 1 can be represented by the following integral form

$$(W_{\rho}^\pm - 1) e^{-itH_{0,\rho} \Phi_v} = \int_0^{\pm\infty} \partial_\tau e^{i\tau H_{\rho} \Phi_v} e^{-i\tau H_{0,\rho} \Phi_v} d\tau e^{-itH_{0,\rho}}$$

$$= i \int_0^{\pm\infty} e^{i\tau H_{\rho}} V(x) e^{-i(\tau + t)H_{0,\rho}} d\tau = i \int_t^{\pm\infty} e^{i(\tau' - t)H_{\rho}} V(x) e^{-i\tau' H_{0,\rho}} d\tau'. \tag{3.17}$$

In the last equation, we changed the integral variable $\tau' = \tau + t$. By using Proposition 3.2, we have

$$\|(W_{\rho}^\pm - 1) e^{-itH_{0,\rho} \Phi_v} \| \leq \int_{-\infty}^{\infty} \| V(x) e^{-i\tau' H_{0,\rho} \Phi_v} \| d\tau' = O(|v|^{1-2\rho}). \tag{3.18}$$

\(\square\)

We are ready to prove the reconstruction theorem.

**Proof of Theorem 3.1.** As in the proof of Corollary 3.3, we represent the difference between $W^+$ and $W^-$ by the integral

$$W^+_{\rho} - W^-_{\rho} = \int_{-\infty}^{\infty} \partial_\tau e^{i\tau H_{\rho} \Phi_v} e^{-itH_{0,\rho} \Phi_v} \, dt = i \int_{-\infty}^{\infty} e^{i\tau H_{\rho}} V(x) e^{-itH_{0,\rho} \Phi_v} \, dt. \tag{3.19}$$
Recall the intertwining property $e^{-itH_\rho}W_\rho^\pm = W_\rho^\pm e^{-itH_0,\rho}$. We can then compute

$$i(S_\rho - 1)\Phi_v = i(W_\rho^+ - W_\rho^-)^*W_\rho^\pm\Phi_v$$

$$= \int_{-\infty}^\infty e^{itH_0,\rho}V(x)e^{-itH_0,\rho}W_\rho^-\Phi_vdt = \int_{-\infty}^\infty e^{itH_0,\rho}V(x)W_\rho^-e^{-itH_0,\rho}\Phi_vdt \quad (3.20)$$

and

$$|v|^{2\rho-1}(i(S_\rho - 1)\Phi_v, \Psi_v) = |v|^{2\rho-1}\int_{-\infty}^\infty (V(x)W_\rho^-e^{-itH_0,\rho}\Phi_v, e^{-itH_0,\rho}\Psi_v)dt$$

$$= |v|^{2\rho-1}\int_{-\infty}^\infty I_v(t)dt + R_v, \quad (3.21)$$

where we defined $I_v(t)$ and $R_v$ in (3.21) by

$$I_v(t) = (V(x)e^{-itH_0,\rho}\Phi_v, e^{-itH_0,\rho}\Psi_v), \quad (3.22)$$

$$R_v = |v|^{2\rho-1}\int_{-\infty}^\infty ((W_\rho^- - 1)e^{-itH_0,\rho}\Phi_v, V(x)e^{-itH_0,\rho}\Psi_v)dt. \quad (3.23)$$

Proposition 3.2 and Corollary 3.3 immediately give

$$R_v = O(|v|^{1-2\rho}). \quad (3.24)$$

Thus far, the proof has been roughly parallel to that in Enss–Weder [6]. However, the principal part of (3.21) demands further rigorous scrutiny. We first divide the integral as follows

$$|v|^{2\rho-1}\int_{-\infty}^\infty I_v(t)dt = |v|^{2\rho-1}\left(\int_{|t|<|v|^{-\sigma}} + \int_{|t|>|v|^{-\sigma}}\right)I_v(t)dt, \quad (3.25)$$

where $\sigma > 2\rho - 1$ is independent of $t$ and $v$. We will later determine an upper bound on $\sigma$. Because $I_v(t)$ is uniformly bounded in $t$ and $v$, the integral on $|t| < |v|^{-\sigma}$ is

$$|v|^{2\rho-1}\int_{|t|<|v|^{-\sigma}} |I_v(t)|dt \leq C|v|^{2\rho-1-\sigma}. \quad (3.26)$$

We next consider the integral on $|t| \geq |v|^{-\sigma}$, which is represented by

$$|v|^{2\rho-1}\int_{|t|>|v|^{-\sigma}} I_v(t)dt = |v|^{2\rho-1}\int_{|t|>|v|^{-\sigma}} (V(x + (\nabla_\xi\omega_\rho)(v)t)\Phi_0, \Psi_0)dt$$

$$+ |v|^{2\rho-1}\int_{|t|>|v|^{-\sigma}} \{I_v(t) - (V(x + (\nabla_\xi\omega_\rho)(v)t)\Phi_0, \Psi_0)\} dt. \quad (3.27)$$
We note that $|\nabla_x \omega_\rho(v)| = |v|^{2\rho-2}$. After the change of the integral variable $\tau = |v|^{2\rho-1}t$, the first term of the right-hand side of (3.27) converges

$$
|v|^{2\rho-1} \int_{|t| \geq |v|^{-\sigma}} (V(x + (\nabla_x \omega_\rho)(v)t)\Phi_0, \Psi_0) dt
$$

$$
= \int_{|\tau| \geq |v|^{2\rho-1-\sigma}} (V(x + \hat{v}\tau)\Phi_0, \Psi_0) d\tau \to \int_{-\infty}^{\infty} (V(x + \hat{v}\tau)\Phi_0, \Psi_0) d\tau \quad (3.28)
$$
as $|v| \to \infty$ because we assumed that $2\rho - 1 - \sigma < 0$. This also indicates that

$$
|v|^{2\rho-1} \int_{|t| \geq |v|^{-\sigma}} (V(x + (\nabla_x \omega_\rho)(v)t)\Phi_0, \Psi_0) dt
$$

$$
= \int_{-\infty}^{\infty} (V(x + \hat{v}t)\Phi_0, \Psi_0) dt + O(|v|^{2\rho-1-\sigma}) \quad (3.29)
$$

by the uniformly boundedness of $(V(x + \hat{v}t)\Phi_0, \Psi_0)$. Recall the relations (2.8) and (2.9). We then have

$$
I_v(t) = (V(x + (\nabla_x \omega_\rho)(D_x + v)t)\Phi_0, \Psi_0). \quad (3.30)
$$

Therefore, as in the proof of Theorem 1.1 we try to derive the order of decay in the second term on the right-hand side of (3.27) from the nearly cancellation of $(\nabla_x \omega_\rho)(\xi + v)$ and $(\nabla_x \omega_\rho)(v)$ on the support of $\mathcal{F}\Phi_0$. In our assumptions, $V$ belongs to $C^1(\mathbb{R}^n)$, however we can compute

$$
V\left(x + (\nabla_x \omega_\rho)(\xi + v)t\right) - V\left(x + (\nabla_x \omega_\rho)(v)t\right)
$$

$$
= \int_0^1 (\nabla_x V)\left(x + (\nabla_x \omega_\rho)(v)t + \theta\left((\nabla_x \omega_\rho)(\xi + v) - (\nabla_x \omega_\rho)(v)\right)\right)\cdot((\nabla_x \omega_\rho)(\xi + v) - (\nabla_x \omega_\rho)(v)) t d\theta \quad (3.31)
$$
as the pseudo-differential symbolic calculus in the Fourier integral. We particularly note that the second- and higher-order derivatives of $V$ do not appear on the right-hand side of (3.31) because $(\nabla_x \omega_\rho)(\xi + v) - (\nabla_x \omega_\rho)(v)$ does not include $x$. Let $f_1, f_2 \in C^\infty_0(\mathbb{R}^n)$ satisfy $\mathcal{F}\Phi_0 = f_1\mathcal{F}\Phi_0$ and $f_1 = f_2f_1$. Then $\Phi_0 = f_2(D_x) f_1(D_x) \Phi_0$ holds. We define $g_{j,v}$ by

$$
g_{j,v}(\xi) = \{(\partial_\xi \omega_\rho)(\xi + v) - (\partial_\xi \omega_\rho)(v)\} f_1(\xi) \quad (3.32)
$$

for $1 \leq j \leq n$ and, as in (2.14), (2.15) and (2.16)

$$
|\partial_\xi^\beta g_{j,v}(\xi)| \leq C_\beta |v|^{2\rho-2} \quad (3.33)
$$
follows for any $\beta$. We also define the vector-valued function $\psi_v$ by

$$\psi_v(t, x, \xi) = x + (\nabla_\xi \omega_\rho)(v)t + \theta\{(\nabla_\xi \omega_\rho)(\xi + v) - (\nabla_\xi \omega_\rho)(v)\}t$$

(3.34)

to avoid complicated notation. Now, to estimate the second term in (3.27), we only have to consider the following norm, which includes the integrand of the $j$-th term on the right-hand side of (3.31),

$$|t||(|\partial_{x_j} V)(\psi_v(t, x, D_x))f_2(D_x)g_{j,v}(D_x)\Phi_0| \leq J_1 + J_2,$$

(3.35)

where $J_1$ and $J_2$ are given by

$$J_1 = |t||(|\partial_{x_j} V)(\psi_v(t, x, D_x))f_2(D_x)\chi\left(\frac{x}{|v|^{2\rho-1}|t|/4}\right) g_{j,v}(D_x)\Phi_0|,$$

$$J_2 = |t||(|\partial_{x_j} V)(\psi_v(t, x, D_x))f_2(D_x)\left\{1 - \chi\left(\frac{x}{|v|^{2\rho-1}|t|/4}\right)\right\} g_{j,v}(D_x)\Phi_0|.$$

(3.36)

For $J_1$, we insert

$$F\left(|x| > \frac{|v|^{2\rho-1}|t|}{4}\right) + F\left(|x| \leq \frac{|v|^{2\rho-1}|t|}{4}\right) = 1$$

(3.37)

between $g_{j,v}(D_x)$ and $\Phi_0$. Then $J_1$ is estimated so that $J_1 \leq J_{1,1} + J_{1,2}$, where $J_{1,1}$ and $J_{1,2}$ are given by

$$J_{1,1} = C|t||F\left(|x| > \frac{|v|^{2\rho-1}|t|}{4}\right)\Phi_0|,$$

$$J_{1,2} = C|t|\left|\chi\left(\frac{x}{|v|^{2\rho-1}|t|/4}\right) g_{j,v}(D_x)F\left(|x| \leq \frac{|v|^{2\rho-1}|t|}{4}\right)\Phi_0\right|.$$

(3.38)

(3.39)

The estimate of $J_{1,1}$ is almost the same as (3.13). However, in this estimate, we choose $\nu \in \mathbb{R}$ such that

$$J_{1,1} \leq C|t||F\left(|x| > \frac{|v|^{2\rho-1}|t|}{4}\right)\|\langle x \rangle^{-\nu}\| \|\langle x \rangle^\nu\Phi_0\| \leq C|t|\langle|v|^{2\rho-1}|t|\rangle^{-\nu}.$$  

(3.40)

Therefore, for $\nu > 2$, we obtain

$$|v|^{2\rho-1}\int_{|t|>|v|^{-\sigma}} J_{1,1}dt \leq C|v|^{2\rho-1}\int_{|t|>|v|^{-\sigma}} |t|\langle|v|^{2\rho-1}|t|\rangle^{-\nu} dt$$

$$\leq C|v|^{-(2\rho-1)(\nu-1)}\int_{|v|^{-\sigma}} t^{-\nu+1} dt = O(|v|^{-(2\rho-1)(\nu-1)+\sigma(\nu-2)}).$$

(3.41)

Although this estimate holds for any $\nu > 2$, the exponent is better when $\nu$ is closer to 2 because

$$-(2\rho-1)(\nu-1) + \sigma(\nu-2) = (\nu-2)\{\sigma - (2\rho-1)\} + 1 - 2\rho$$

(3.42)
and \( \sigma > 2\rho - 1 \). In the estimate of \( J_{1,2} \), we compute the following commutator by using the pseudo-differential product formula (2.4)

\[
\chi \left( \frac{x}{|v|^{2\rho-1}|t|/4} \right), g_{j,x}(\xi)
\]

\[
= - \sum_{1 < |\beta| < N-1} \frac{1}{\beta!} \partial_\xi^\beta g_{j,x}(\xi) \cdot (-i\partial_x)^\beta \chi \left( \frac{x}{|v|^{2\rho-1}|t|/4} \right) + R_N(t, x, \xi) \quad (3.43)
\]

for any \( N \in \mathbb{N} \). As in the proof of Theorem 1.1, the disjointness of two characteristic functions means that, for \( 0 \leq |\beta| \leq N - 1 \),

\[
\left\{ \partial_\xi^\beta \chi \left( \frac{x}{|v|^{2\rho-1}|t|/4} \right) \right\} F \left( |x| \leq \frac{|v|^{2\rho-1}|t|}{4} \right) = 0. \quad (3.44)
\]

Therefore, \( J_{1,2} \) only has the remainder term \( R_N \). To estimate \( R_N \), we divide the integral again

\[
|v|^{2\rho-1} \left| \int_{|t| \geq |v|^{-\sigma}} J_{1,2} dt \right| = |v|^{2\rho-1} \left( \int_{|v|^{-\sigma} \leq |t| < |v|^{1-2\rho}} + \int_{|t| \geq |v|^{1-2\rho}} \right) J_{1,2} dt. \quad (3.45)
\]

By using the \( L^2 \)-boundedness (2.6), when \( |t| \geq |v|^{1-2\rho} \), \( R_N(t, x, D_x) \) is estimated as

\[
\| R_N(t, x, D_x) \| \leq C_N |v|^{2\rho-2} \left( |v|^{2\rho-1} |t| \right)^{-N} \quad (3.46)
\]

because \( |v|^{2\rho-1} |t| > 1 \) holds in this case, where \( |v|^{2\rho-2} \) comes from (3.33). We therefore compute, for \( N \geq 3 \)

\[
|v|^{2\rho-1} \left| \int_{|t| \geq |v|^{1-2\rho}} J_{1,2} dt \right| = C |v|^{2\rho-1} \int_{|t| \geq |v|^{1-2\rho}} |t| \| R_N(t, x, D_x) \| dt 
\leq C_N |v|^{2(2\rho-1)-1-(2\rho-1)N} \int_{|v|^{1-2\rho}}^\infty t^{-N+1} dt = O(|v|^{-1}). \quad (3.47)
\]

In contrast, when \( |v|^{-\sigma} \leq |t| < |v|^{1-2\rho} \), there exists \( \tilde{N} \geq N \) such that

\[
\| R_N(t, x, D_x) \| \leq C_N |v|^{2\rho-2} \left( |v|^{2\rho-1} |t| \right)^{-\tilde{N}} \quad (3.48)
\]

because \( |v|^{2\rho-1} |t| \leq 1 \) holds, and

\[
|v|^{2\rho-1} \left| \int_{|v|^{-\sigma} \leq |t| < |v|^{1-2\rho}} J_{1,2} dt \right| = C |v|^{2\rho-1} \int_{|v|^{-\sigma} \leq |t| < |v|^{1-2\rho}} |t| \| R_N(t, x, D_x) \| dt 
\leq C_N |v|^{2(2\rho-1)-1-(2\rho-1)\tilde{N}} \int_{|v|^{-\sigma}}^{|v|^{1-2\rho}} t^{-\tilde{N}+1} dt 
= O(|v|^{-1}) + O(|v|^{2(2\rho-1)-1-(2\rho-1)\tilde{N}+\sigma(\tilde{N}-2)}) \quad (3.49)
\]
is obtained. This estimate holds for any $\tilde{N} \geq N \geq 3$. However, the best exponent is the smallest $\tilde{N}$ because

$$2(2\rho - 1) - 1 - (2\rho - 1)\tilde{N} + \sigma(\tilde{N} - 2) = (\tilde{N} - 2)\{\sigma - (2\rho - 1)\} - 1.$$  \hfill (3.50)

From (3.41), (3.42), (3.47), (3.49), and (3.50), we have

$$|v|^{2\rho - 1} \int_{|t| \geq |v|^\sigma} J_1 dt = O(|v|^{v - 2(\sigma - (2\rho - 1)) + 1 - 2\rho}) + O(|v|^{(\tilde{N} - 2)\{\sigma - (2\rho - 1)\} - 1}).$$  \hfill (3.51)

We next consider $J_2$. On the supports of $f_2$ and $1 - \chi$,

$$|\psi_v(t, x, \xi)| \geq |v|^{2\rho - 1}|t| - |x| - |(\nabla \omega_\rho)(\xi + v) - (\nabla \omega_\rho)(v)||t| \geq |v|^{2\rho - 1}|t|/2 - |v|^{2\rho - 1}|t|/8 = 3|v|^{2\rho - 1}|t|/8 \geq |v|^{2\rho - 1}|t|/4$$  \hfill (3.52)

holds for large $|v|$, here we used (2.17). This says that

$$f_2(\xi) \left\{ 1 - \chi \left( \frac{x}{|v|^{2\rho - 1}|t|/4} \right) \right\} = \chi \left( \frac{\psi_v(t, x, \xi)}{|v|^{2\rho - 1}|t|/8} \right) f_2(\xi) \left\{ 1 - \chi \left( \frac{x}{|v|^{2\rho - 1}|t|/4} \right) \right\},$$  \hfill (3.53)

because $\chi(\psi_v(t, x, \xi)/(|v|^{2\rho - 1}|t|/8)) = 1$ by (3.52). However, symbolically, (3.53) is

$$\sum_{|\beta| \leq M - 1} \frac{1}{\beta!} \left\{ \partial_\xi^\beta \chi \left( \frac{\psi_v(t, x, \xi)}{|v|^{2\rho - 1}|t|/8} \right) \right\} \times f_2(\xi)(-i\partial_x)^\beta \left\{ 1 - \chi \left( \frac{x}{|v|^{2\rho - 1}|t|/4} \right) \right\} + R_M(t, x, \xi)$$  \hfill (3.54)

for any $M \in \mathbb{N}$ by using the asymptotic product formula (2.24) again. We note that

$$\left| \partial_\xi^\beta \chi \left( \frac{\psi_v(t, x, \xi)}{|v|^{2\rho - 1}|t|/8} \right) \right| |f_2(\xi)| \leq C_\beta |v|^{-2\rho - 1} \leq C_\beta$$  \hfill (3.55)

for any $\beta$ with $|\beta| \geq 1$ and $C_\beta$ is independent of $t$. Therefore, for $0 \leq |\beta| \leq M - 1$, it is sufficient to consider

$$\chi \left( \frac{\psi_v(t, x, \xi)}{|v|^{2\rho - 1}|t|/8} \right) f_2(\xi) \times \partial_x^\beta \left\{ 1 - \chi \left( \frac{x}{|v|^{2\rho - 1}|t|/4} \right) \right\}$$  \hfill (3.56)

only. The term which includes (3.56) is estimated to be

$$|t| \left\| (\partial_x V)(\psi_v(t, x, D_x)) \chi \left( \frac{\psi_v(t, x, D_x)}{|v|^{2\rho - 1}|t|/8} \right) \right\| \times \left\| \partial_x^\beta \left\{ 1 - \chi \left( \frac{x}{|v|^{2\rho - 1}|t|/4} \right) \right\} \right\| \|g_{j,v}(D_x)\| \leq C|t|(|v|^{2\rho - 1}|t|)^{-1 - \gamma}(|v|^{2\rho - 1}|t|)^{-|\beta|/2(2\rho - 2)},$$  \hfill (3.57)
here we used the decay condition on $V$ in (1.10) and the estimate of $g_{j,v}$ in (3.33). We then compute the following integral

$$|v|^{2\rho-1} \int_{|t| > |v|^{-\sigma}} |t|(|v|^{2\rho-1}|t|)^{-1-\gamma}(|v|^{2\rho-1}|t|)^{-\beta} |v|^{2\rho-2} dt$$

$$\leq C|v|^{2(2\rho-1)-(2\rho-1)(1+\gamma)-(2\rho-1)|\beta|} \int_{|v|^{-\sigma}}^{\infty} t^{-\gamma-|\beta|} dt$$

$$= O(|v|^{2(2\rho-1)-(2\rho-1)(1+\gamma)-(2\rho-1)|\beta|+\sigma(\gamma+|\beta|-1)}) \quad (3.58)$$

because $\gamma > 1$. The decay exponent in (3.58) is represented by

$$2(2\rho-1) - 1 - (2\rho-1)(1+\gamma) - (2\rho-1)|\beta| + \sigma(\gamma + |\beta| - 1)$$

$$=(\gamma + |\beta| - 1)\{\sigma - (2\rho - 1)\} - 1. \quad (3.59)$$

Because $\sigma > 2\rho - 1$, the top term between $0 \leq |\beta| \leq M - 1$ is $|\beta| = M - 1$. To estimate the term involving $R_M$, we have to divide the integral into $|v|^{-\sigma} < |t| < |v|^{1-2\rho}$ and $|t| \geq |v|^{1-2\rho}$ once more. By the same argument in (2.24), there exists $M' \in \mathbb{N}$ such that

$$\|R_M(t,x,D_x)\| \leq C_M \sum_{0 \leq j \leq M'} (|v|^{2\rho-1}|t|)^{-j} \times \sum_{M \leq j \leq M+M'} (|v|^{2\rho-1}|t|)^{-j}, \quad (3.60)$$

where, in the summation of $0 \leq j \leq M'$, we used the following boundedness again

$$\left| \partial_\xi^\beta \left( \frac{\psi_v(t,x,\xi)}{|v|^{2\rho-1}|t|/8} \right) \right| |f_2(\xi)| \leq C_\beta \quad (3.61)$$

for any $\beta$. Therefore, when $|t| \geq |v|^{1-2\rho}$, we have

$$\|R_M(t,x,D_x)\| \leq C_M (|v|^{2\rho-1}|t|)^{-M} \quad (3.62)$$

because $|v|^{2\rho-1}|t| \geq 1$. On the other hand, in the case where $|v|^{-\sigma} \leq |t| < |v|^{1-2\rho}$,

$$\|R_M(t,x,D_x)\| \leq C_M (|v|^{2\rho-1}|t|)^{-\bar{M}} \quad (3.63)$$

is obtained because $|v|^{2\rho-1}|t| < 1$, here we put $\bar{M} = M + 2M'$. From (3.62), (3.63), and (3.33), it follows that

$$|v|^{2\rho-1} \left( \int_{|v|^{-\sigma} \leq |t| < |v|^{1-2\rho}} + \int_{|t| \geq |v|^{1-2\rho}} \right) |t||\|R_M(t,x,D_x)\|\|g_{j,v}(D_x)\| dt$$

$$= O(|v|^{(\bar{M}-2)(\sigma-(2\rho-1))-1}) \quad (3.64)$$
for $M \geq 3$. The computation in (3.64) is quite similar to (3.47) and (3.49) (see also (3.50)). By (3.58), (3.59), and (3.64), $J_2$ is estimated to be
\[ |v|^{2\rho - 1} \int_{|t| \geq |v|^{-\sigma}} J_2 dt = O(|v|^\gamma + 1) \left| \sigma - (2\rho - 1) \right|^{-1} + O(|v|^\gamma \left| \sigma - (2\rho - 1) \right|^{-1}), \] (3.65)
here we fixed $|\beta| = 2$ in (3.58) and (3.59), because we can choose $M = 3$. By combining (3.24), (3.26), (3.29), (3.51), and (3.65), we obtain
\[ |v|^{2\rho - 1} (i(S_\rho - 1)\Phi_v, \Psi_v) = \int_{-\infty}^{\infty} (V(x + \hat{v}t)\Phi_0, \Psi_0) dt \]
\[ + O(|v|^{1-2\rho}) + O(|v|^{2\rho - 1 - \sigma}) \]
\[ + O(|v|^{\nu - 2} (\sigma - (2\rho - 1))^{1 - 2\rho}) + O(|v|^{\tilde{N} - 2} (\sigma - (2\rho - 1))^{-1}) \]
\[ + O(|v|^{\gamma + 1} (\sigma - (2\rho - 1))^{-1}) + O(|v|^{\tilde{M} - 2} (\sigma - (2\rho - 1))^{-1}) \] (3.66)
as $|v| \to \infty$. We evaluate these error exponents. It is clear that $2\rho - 1 - \sigma < 0$ and, that $1 - 2\rho < (\nu - 2) (\sigma - (2\rho - 1)) + 1 - 2\rho < 0$ because we can choose $\nu - 2 > 0$ to be sufficiently small, independent of the size of $\sigma$. Therefore, to complete this proof, we need to ensure $\sigma$ satisfies $(\tilde{N} - 2) (\sigma - (2\rho - 1)) - 1 < 0$, $(\gamma + 1) (\sigma - (2\rho - 1)) - 1 < 0$ and $(\tilde{M} - 2) (\sigma - (2\rho - 1)) - 1 < 0$ on condition that $\sigma > 2\rho - 1$. To do that, it suffices to choose $\sigma$ such that
\[ 2\rho - 1 < \sigma < 2\rho - 1 + \min\{1/(1 + \gamma), 1/(\tilde{N} - 2), 1/\tilde{M} - 2\} \] (3.67)
for $\gamma > 1$, $\tilde{N} \geq 3$ and $\tilde{M} \geq 3$. This completes the proof. \hfill $\square$

From the Plancherel formula associated with the Radon transformation (see Helgason [8]), the proof of Theorem 1.3 can be performed in the same way as in Theorem 1.1 of Enss–Weder [6]. We thus omit the proof here.

Acknowledgments. This work was partially supported by the Grant-in-Aid for Young Scientists (B) #16K17633 from JSPS. Moreover, the author would like to thank the late Professor Hitoshi Kitada for many valuable discussions and comments.

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