The generalized Kupershmidt deformation for constructing new integrable systems from integrable bi-Hamiltonian systems

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Abstract Based on the Kupershmidt deformation for any integrable bi-Hamiltonian systems presented in [4], we propose the generalized Kupershmidt deformation to construct new systems from integrable bi-Hamiltonian systems, which provides a nonholonomic perturbation of the bi-Hamiltonian systems. The generalized Kupershmidt deformation is conjectured to preserve integrability. The conjecture is verified in a few representative cases: KdV equation, Boussinesq equation, Jaulent-Miodek equation and Camassa-Holm equation. For these specific cases, we present a general procedure to convert the generalized Kupershmidt deformation into the integrable Rosochatius deformation of soliton equation with self-consistent sources, then to transform it into a $t$-type bi-Hamiltonian system. By using this generalized Kupershmidt deformation some new integrable systems are derived. In fact, this generalized Kupershmidt deformation also provides a new method to construct the integrable Rosochatius deformation of soliton equation with self-consistent sources.
1 Introduction

It is known that one can construct a new integrable system starting from a bi-Hamiltonian system. Fuchssteiner and Fokas showed \(^{[1]}\) that compatible symplectic structures lead in natural way to hereditary symmetries, which provides a method to construct a hierarchy of exactly solvable evolution equations. Olver and Rosenau \(^{[2]}\) demonstrated that most integrable bi-Hamiltonian systems are governed by a compatible trio of Hamiltonian structures, and their recombination leads to integrable hierarchies of nonlinear equations.

Recently for the following KdV6 equation or nonholonomic deformation of KdV equation derived in \(^{[3]}\)

\[
\begin{align*}
  u_t &= u_{xxx} + 6uu_x - \omega_x, \quad (1a) \\
  \omega_{xxx} + 4u\omega_x + 2u_x\omega &= 0, \quad (1b)
\end{align*}
\]

Kupershmidt found \(^{[4]}\) that (1) can be converted into

\[
\begin{align*}
  u_t &= B_1(\frac{\delta H_3}{\delta u}) - B_1(\omega), \quad (2a) \\
  B_2(\omega) &= 0, \quad (2b)
\end{align*}
\]

where

\[
B_1 = \partial = \partial_x, \quad B_2 = \partial^3 + 2(u\partial + \partial u) \quad (3)
\]

are the two standard Hamiltonian operators of the KdV hierarchy and \(H_3 = u^3 - \frac{u_x^2}{2}\). In general, for a bi-Hamiltonian system

\[
\begin{align*}
  u_{tn} &= B_1(\frac{\delta H_{n+1}}{\delta u}) = B_2(\frac{\delta H_n}{\delta u}), \quad (4)
\end{align*}
\]

the ansatz \(^{[4]}\) provides a nonholonomic deformation of bi-Hamiltonian systems\(^{[4]}\):

\[
\begin{align*}
  u_{tn} &= B_1(\frac{\delta H_{n+1}}{\delta u}) - B_1(\omega), \\
  B_2(\omega) &= 0 \quad (5)
\end{align*}
\]
which is called as Kupershmidt deformation of bi-Hamiltonian systems. This deformation is conjectured to preserve integrability and the conjecture is verified in a few representative cases in [4].

In [5], we showed that the Kupershmidt deformation (2) of KdV equation is equivalent to the integrable Rosochatius deformation of KdV equation with self-consistent sources, and constructed the bi-Hamiltonian structure for the Kupershmidt deformation of KdV equation (2). The conjecture is then proved in [6] that the Kupershmidt deformation of a bi-Hamiltonian system is itself bi-Hamiltonian.

Rosochatius found that it would still keep the integrability to add a potential of the sum of inverse squares of the coordinates to that of the Neumann system [7]. The deformed system is called as Neumann-Rosochatius system. Then the Rosochatius deformation of Garnier system, Jacobi system and many constrained flows of soliton equations were constructed in [8, 9, 10]. This Rosochatius-type integrable systems have important physical applications [11, 12, 13]. However, these Rosochatius deformations are limited to few finite-dimensional integrable Hamiltonian systems (FDIHS). Recently, in [14] we proposed a systematic method for generalizing the integrable Rosochatius deformation for FDIHS to integrable Rosochatius deformation for infinite-dimensional integrable equations. Many integrable Rosochatius deformations of soliton equations with self-consistent sources and their Lax representations were presented in [14, 15].

In present paper, based on the Kupershmidt deformation (5), we propose the generalized Kupershmidt deformation (GKD) to construct new systems from integrable bi-Hamiltonian systems which provides a nonholonomic perturbation of the bi-Hamiltonian systems. The generalized Kupershmidt deformation is conjectured to preserve integrability. Although it is difficult to prove the integrability in general, the conjecture can be verified in many specific cases. Using KdV equation, Boussinesq equation, Jaulent-Miodek equation and Camassa-
Holm equation as examples, we present a general procedure to show how to convert these generalized Kupershmidt deformations into the Rosochatius deformations of soliton equations with self-consistent sources. These Rosochatius deformations of soliton equations with self-consistent sources possess the Lax representations, which are easy constructed by using the systematic method in [14, 15], and their stationary equations can be shown to be finite-dimensional integrable Hamiltonian systems in the Liouville’s sense [14, 15, 16]. Furthermore, for the specific Rosochatius deformations of soliton equations with self-consistent sources there is a general procedure to transform it into a $t$-type bi-Hamiltonian system by introducing the Jacobi-Ostrogradiski coordinates and taking the spatial variable $x$ as the evolution parameter according to [17, 18, 19, 20, 5]. These facts imply the integrability of the Rosochatius deformations of soliton equations with self-consistent sources. Indeed the generalized Kupershmidt deformation also provides a new method for obtaining the Rosochatius deformation of soliton equation with self-consistent sources, which is quite different from the method presented in [14, 15].

In section 2, we propose the the generalized Kupershmidt deformation (GKD) of bi-Hamiltonian system, by using GKD of KdV hierarchy to illustrate the formulae. Section 3 is devoted to a new integrable system obtained from the GKD of Camassa-Holm equation and shows how to transform the GKD of Camassa-Holm equation into the integrable Rosochatius deformation of Camassa-Holm equation with self-consistent sources. Section 4 treats the GKD of Boussinesq equation. The last section presents the GKD of Jaulent-Miodek hierarchy, and demonstrate how to convert the GKD of a integrable system into a bi-Hamiltonian system with $t$—type Hamiltonian operator by taking $x$ as evolution parameter and $t$ as 'spatial' variable.
2 The generalized Kupershmidt deformation of bi-Hamiltonian systems

Consider a hierarchy of soliton equations with bi-Hamiltonian structure

\[ u_t = B_1(\frac{\delta H_{n+1}}{\delta u}) - B_2(\frac{\delta H_n}{\delta u}), \]  

(6)

where \( B_1 \) and \( B_2 \) are two standard Hamiltonian operators. The associated spectral problem reads

\[ L(u)\phi = \lambda\phi. \]  

(7)

For the eigenvalue \( \lambda \), it is easy to find that the variational derivative of \( \lambda \)

\[ \frac{\delta \lambda}{\delta u} = f(\phi). \]

Assume that \( \lambda_j, j = 1, \ldots, N \) are \( N \) distinct real eigenvalues of (7), we have

\[ L\phi_j = \lambda_j\phi_j, \quad j = 1, 2, \ldots, N, \]

and we denote

\[ \frac{\delta \lambda_j}{\delta u} = \frac{\delta \lambda}{\delta u}|_{\lambda = \lambda_j} = f(\phi_j). \]

Based on the Kupershmidt deformation (5), we first generalize Kupershmidt deformation as follows

\[ u_t = B_1(\delta H_{n+1}) - B_1(\sum_{j=1}^{N} \omega_j), \]  

(8a)

\[ (B_2 - \alpha_jB_1)(\omega_j) = 0, \quad j = 1, 2, \ldots, N, \]  

(8b)

where \( \alpha_j \) are arbitrary constants, which also gives rise to a nonholonomic deformation of bi-Hamiltonian systems similar to the integrable KdV6’s type nonholonomic deformation of soliton equations. So provides a way to construct new systems from the bi-Hamiltonian systems. Furthermore, observe that \( \omega_j \) in (8a) are at the same position as \( \delta H_{n+1} \), and the eigenvalues \( \lambda_j \) are also the conserved quantities for \( H_n \), it is reasonable to take \( \omega_j = \frac{\delta \lambda_j}{\delta u} \).
and this setting is compatible with (8b). So we finally propose the generalized Kupershmidt deformation for a bi-Hamiltonian systems as follows

\[ u_{1n} = B_1 \left( \frac{\delta H_{n+1}}{\delta u} - \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u} \right), \quad (9a) \]

\[ (B_2 - \alpha_j B_1) \left( \frac{\delta \lambda_j}{\delta u} \right) = 0, \quad j = 1, 2, \ldots, N. \quad (9b) \]

As the conjecture made in [4], it is reasonable to conjecture the integrability of the new system (9). It seems that it is difficult to prove the integrability in general. However it can be verified in many specific cases. Using KdV equation, Boussinesq equation, Jaulent-Miodek equation and Camassa-Holm equation as examples, we will proceed to the general procedure to convert these generalized Kupershmidt deformations into the Rosochatius deformations of soliton equations with self-consistent sources. By using the systematic method in [14, 15], it is easy to construct the Lax representation for these Rosochatius deformations of soliton equations with self-consistent sources, and to show their stationary equations to be finite-dimensional integrable Hamiltonian systems in the Liouville’s sense [14, 15]. Furthermore, following the method proposed in [17, 18, 19, 20], we have presented a general procedure to transform the Rosochatius deformations of soliton equations with self-consistent sources into a bi-Hamiltonian system by introducing the Jacobi-Ostrogradski coordinates and taking the spacial variable \( x \) as the evolution parameter in [5]. We will use the GKD of Jaulent-Miodek equation to illustrate the general constructure. These facts imply the integrability of the generalized Kupershmidt deformation of soliton equation. So the deformation [5] and [9] provides a way to construct new integrable systems from integrable bi-Hamiltonian systems and to establish the integrable Rosochatius deformation of soliton equation with self-consistent sources in different way from that in [14, 15].

We now use the KdV hierarchy to illustrate the procedure. Consider the
Schrödinger eigenvalue problem

\[ \phi_{xx} + (u - \lambda)\phi = 0, \tag{10} \]

the associated KdV hierarchy read

\[ u_n = B_1 \left( \frac{\delta H_{n+1}}{\delta u} \right) = B_2 \left( \frac{\delta H_n}{\delta u} \right), \quad n = 1, 2, \ldots \tag{11} \]

where

\[ B_1 = \partial = \partial_x, \quad B_2 = \partial^3 + 2(u\partial + \partial u) \]

\[ H_{n+1} = \int b_{n+1} dx, \quad b_{n+1} = -\frac{2}{2n+1} R^n u, \quad R = -\frac{1}{4} \partial^2 - u + \frac{1}{2} \partial^{-1} u_x. \]

It is easy to find that

\[ \frac{\delta \lambda}{\delta u} = \varphi^2. \]

For \( N \) distinct eigenvalues \( \lambda_j \), consider the spectral problem

\[ \varphi_{jxx} + (u - \lambda_j)\varphi_j = 0, \quad j = 1, 2, \ldots, N. \]

We have

\[ \frac{\delta \lambda_j}{\delta u} = \varphi_j^2. \]

For \( n = 2, \alpha_j = \lambda_j \), \ref{8} gives rise to the following new integrable generalized KdV6 equation

\[ u_t = u_{xxx} + 6uu_x - \sum_{j=1}^{N} \omega_{jx}, \tag{12a} \]

\[ \omega_{jxxx} + 4u\omega_{jx} + 2u_x\omega_j - \lambda_j \omega_{jx} = 0, \quad j = 1, 2, \ldots, N. \tag{12b} \]

\ref{9b} yields

\[ 2\varphi_j [\varphi_{jxx} + (u - \lambda_j)\varphi_j]_x + 6\varphi_{jx} [\varphi_{jxx} + (u - \lambda_j)\varphi_j] = 0, \]

which immediately gives rise to

\[ \varphi_{jxx} + (u - \lambda_j)\varphi_j = \frac{\mu_j}{\varphi_j^3}, \]
where $\mu_j, j = 1, 2, \cdots, N$ are integral constants. When $n = 2$, $\alpha_j = \lambda_j$, (9) gives rise to the following generalized Kupershmidt deformation of KdV equation

$$u_t = \frac{1}{4}(u_{xxx} + 6uu_x) - \sum_{j=1}^{N} (\varphi_j^2)_x,$$

(13a)

$$\varphi_{jxx} + (u - \lambda_j)\varphi_j = \frac{\mu_j}{\varphi_j^2}, j = 1, 2, \cdots, N$$

(13b)

which is just the integrable Rosochatius deformation of KdV equation with self-consistent sources (RD-KdVHSCS) presented in [14]. When $\mu_j = 0$, $j = 1, \cdots, N$, (13) reduces to the KdV equation with self-consistent sources [21]. The Lax pair for (13) was constructed by a systematic method in [14]

$$\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}_x = U \begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}, \quad U = \begin{pmatrix}
0 & 1 \\
\lambda - u & 0
\end{pmatrix},$$

(14a)

$$\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}_t = V \begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix},$$

$$V = \begin{pmatrix}
-\frac{\mu}{4} & -\lambda + \frac{\lambda^2}{2} \\
-\lambda^2 - \frac{\mu}{4} \lambda - \frac{\mu^2}{16} - \frac{\mu^2}{8} + \frac{1}{2} \sum_{j=1}^{N} \varphi_j^2 & \frac{\mu}{4} - \frac{\mu^2}{8}
\end{pmatrix}$$

$$-\frac{1}{2} \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix}
\varphi_j \varphi_{jx} & -\varphi_j^2 \\
\varphi_j^2 \varphi_{jx} + \frac{\mu_j}{\varphi_j^2} & -\varphi_j \varphi_{jx}
\end{pmatrix}.$$ 

(14b)

The stationary equation of (13) reduces to the generalized integrable Hénon-Heiles system [14]. The bi-Hamiltonian structure for (13) was presented in [5]

### 3 The new integrable system obtained from the Camassa-Holm equation

The Camassa-Holm (CH) equation [1, 22, 23] read

$$m_t = B_1 \frac{\delta H_1}{\delta u} = B_2 \frac{\delta H_0}{\delta u} = -2u_xm - um_x, \quad m = u - u_{xx} + \omega$$

(15)
where
\[ B_1 = -\partial + \partial^3, \quad B_2 = m\partial + \partial m \]
are the two standard Hamiltonian operators of the CH equation and
\[ H_0 = \frac{1}{2} \int (u^2 + u_x^2) dx, \quad H_1 = \frac{1}{2} \int (u^3 + uu_x^2) dx. \]
The Lax pair for CH equation is
\[ \phi_{xx} = \left( \frac{1}{4} - \frac{1}{2} m\lambda \right) \phi, \quad (16a) \]
\[ \phi_t = \frac{1}{2} u_x \phi - \left( \frac{1}{\lambda} + u \right) \phi_x. \quad (16b) \]
For \( N \) distinct real eigenvalues \( \lambda_j \), consider the following spectral problem
\[ \varphi_{jxx} = \frac{1}{4} \varphi_j - \frac{1}{2} m\lambda_j \varphi_j, \quad j = 1, 2, \cdots, N. \quad (17) \]
It is easy to find that
\[ \frac{\delta \lambda}{\delta m} = \lambda \varphi^2, \quad \frac{\delta \lambda_j}{\delta m} = \lambda_j \varphi_j^2. \]
Then we obtain the following new nonholonomic deformation of the Camassa-Holm equation from (8) with \( \alpha_j = \frac{1}{\lambda_j} \)
\[ m_t = -2u_x m - um_x + \sum_{j=1}^{N} [\omega_{jx} - \omega_{jxxx}], \quad (18a) \]
\[ 2m\varphi_{jx} + m_x \omega_j + \lambda_j[\omega_{jx} - \omega_{jxxx}] = 0, \quad j = 1, 2, \cdots, N. \quad (18b) \]
Take \( \alpha_j = \frac{1}{\lambda_j} \), Eq. (9) leads to the following generalized Kupershmidt deformation of CH
\[ m_t = B_1(\frac{\delta H_1}{\delta m} - \sum_{j=1}^{N} \frac{1}{\lambda_j} \frac{\delta \lambda_j}{\delta m}) = -2u_x m - um_x + \sum_{j=1}^{N} [(\varphi_j^2)_x - (\varphi_j^2)_{xxx}], \quad (19a) \]
\[ (B_2 - \frac{1}{\lambda_j} B_1)(\frac{1}{\lambda_j} \frac{\delta \lambda_j}{\delta m}) = 0, \quad j = 1, 2, \cdots, N. \quad (19b) \]
Then (19b) yields
\[ 2\varphi_j(\varphi_{jxx} + \frac{1}{2} \lambda_j m\varphi_j - \frac{1}{4} \varphi_j)_x + 6\varphi_jx(\varphi_{jxx} + \frac{1}{2} \lambda_j m\varphi_j - \frac{1}{4} \varphi_j) = 0 \]
which immediately gives rise to

\[
\varphi_{jxx} = \frac{1}{4}\varphi_j - \frac{1}{2}m\lambda_j\varphi_j + \frac{\mu_j}{\varphi_j^3}.
\]

So Eq. (19) gives a new integrable system

\[
m_t = -2u_xm - um_x + \sum_{j=1}^{N}[(\varphi_j^2)_x - (\varphi_{j,x})_{xxx}], \quad (20a)
\]

\[
\varphi_{jxx} = \frac{1}{4}\varphi_j - \frac{1}{2}m\lambda_j\varphi_j + \frac{\mu_j}{\varphi_j^3}, \quad j = 1, 2, \cdots, N \quad (20b)
\]

which lax pair can be found by using the method in \cite{24}

\[
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
\]

\[
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
\]

\[
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
\]

\[
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
\]

\[
U = \left(\begin{array}{cc}
0 & 1 \\
\frac{1}{4} - \frac{1}{2}\lambda m & 0
\end{array}\right) \quad (21a)
\]

\[
V = \left(\begin{array}{cc}
\frac{u_x}{2} & -\frac{1}{\lambda} - u \\
\frac{u_x}{2} - \frac{m\mu}{2} & \frac{u}{2}
\end{array}\right) - \sum_{j=1}^{N} \frac{\lambda\lambda_j}{\lambda - \lambda_j} \left(\begin{array}{cc}
\varphi_j\varphi_{j,x} & -\varphi_j^2 \\
\varphi_j^2 + \frac{\mu_j}{\varphi_j} & -\varphi_j\varphi_{j,x}
\end{array}\right) \quad (21b)
\]

In fact Eq. (20) can also be regarded as the RD-CHESCS.

4 The generalized Kupershmidt deformation of Boussinesq equation

For the following third-order eigenvalue problem \cite{25}

\[
L\phi = \phi_{xxx} + v\phi_x + \left(\frac{1}{2}\nu_x + w\right)\phi = \lambda\phi, \quad (22)
\]

the associated Boussinesq equation is

\[
\begin{pmatrix}
v \\
w
\end{pmatrix}_t = B_1 \left(\begin{array}{c}
\frac{\delta H}{\delta v} \\
\frac{\delta H}{\delta w}
\end{array}\right) = B_2 \left(\begin{array}{c}
\frac{\delta H}{\delta v} \\
\frac{\delta H}{\delta w}
\end{array}\right) = \left(\begin{array}{c}
2w_x \\
-\frac{2}{3}\nu_x - \frac{1}{6}w_{xxx}
\end{array}\right), \quad (23)
\]
where

\[ B_1 = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \]

\[ B_2 = \frac{1}{3} \begin{pmatrix} 2\partial^3 + 2v\partial + v_x & 3w\partial + 2w_x \\ 3w\partial + w_x & -\frac{1}{6}(\partial^5 + 5v\partial^3 + \frac{15}{2}v_x\partial^2 + \frac{9}{2}v_{xx}\partial + 4v^2\partial + v_{xxx} + 4vv_{x}) \end{pmatrix}, \]

are the two standard Hamiltonian operators of the Boussinesq equation and

\[ H_1 = \int wdx, \quad H_2 = \int \left( \frac{1}{12}v_x^2 - \frac{1}{9}v^3 + w^2 \right) dx. \]

For \( N \) distinct real eigenvalues \( \lambda_j \), consider the following spectral problem and its adjoint spectral problem

\[ \varphi_{jxxx} + v\varphi_{jx} + \left( \frac{1}{2}v_x + w \right)\varphi_j = \lambda\varphi_j, \quad (24a) \]
\[ \varphi_{jxxx}^* + v\varphi_{jx}^* + \left( \frac{1}{2}v_x - w \right)\varphi_j^* = -\lambda\varphi_j^*, \quad j = 1, 2, \cdots, N. \quad (24b) \]

We have

\[ \frac{\delta\lambda_j}{\delta v} = \frac{3}{2}(\varphi_{jx}\varphi_j^* - \varphi_j\varphi_{jx}^*), \quad \frac{\delta\lambda_j}{\delta w} = 3\varphi_j\varphi_j^*. \]

The generalized Kupershmidt deformed Boussinesq equation is given by (9) with \( \alpha_j = \lambda_j \) as follows

\[ \begin{pmatrix} v \\ w \end{pmatrix}_t = B_1 \begin{pmatrix} \frac{\delta H_2}{\delta v} \\ \frac{\delta H_2}{\delta w} \end{pmatrix} - \sum_{j=1}^{N} \begin{pmatrix} \frac{\delta\lambda_j}{\delta v} \\ \frac{\delta\lambda_j}{\delta w} \end{pmatrix}, \quad (25a) \]
\[ (B_2 - \lambda_j B_1) \begin{pmatrix} \frac{\delta\lambda_j}{\delta v} \\ \frac{\delta\lambda_j}{\delta w} \end{pmatrix} = 0, \quad j = 1, 2, \cdots, N. \quad (25b) \]

Set

\[ f_j = \varphi_{jxxx} + v\varphi_{jx} + \left( \frac{1}{2}v_x + w \right)\varphi_j - \lambda_j\varphi_j, \]
\[ (B_2 - \lambda_j B_1) \begin{pmatrix} \delta\lambda_j \\ \delta\lambda_j \end{pmatrix} = \begin{pmatrix} \delta\varphi_j \\ \delta\varphi_j \end{pmatrix}, \quad j = 1, 2, \cdots, N. \]
Direct calculation gives

$$\mathcal{A}_j = \varphi_j^* f_{jx} + 2 \varphi_j^* f_j - (2 \varphi_j^* f_{jx} + \varphi_j^* f_j)^*, \quad (26a)$$

which leads to

$$f_j = \frac{\mu_j}{\varphi_j^2}, \quad f_j^* = -\frac{\mu_j}{\varphi_j^2}. \quad (26b)$$

Using (26b), we get

$$B_j = \frac{1}{3} (2v\varphi_j + 5\varphi_{jxx})(f_j^* + \mu_j^{\varphi_j^2}) - \frac{1}{3} (2v\varphi_j^* + 5\varphi_{jxx})(f_j - \mu_j^{\varphi_j^2}) - \frac{5}{6} \varphi_j^2 (f_j - \frac{\mu_j}{\varphi_j^2}),$$

$$= \frac{5}{6} \varphi_{jx} (f_j^* + \frac{\mu_j}{\varphi_j^2}) - \frac{1}{6} \varphi_j^2 (f_j - \frac{\mu_j}{\varphi_j^2})_{xx} - \frac{1}{6} \varphi_j^2 (f_j^* + \frac{\mu_j}{\varphi_j^2})_{xx} + \frac{2\mu_j}{3\varphi_j^2 \varphi_j^3}$$

$$\mathcal{B}_j = \frac{v\varphi_j^2 \varphi_j^3 (\varphi_j^* - \varphi_j) - \varphi_j^3 \varphi_j^3 + \varphi_j^3 (\varphi_{jxx}^2 - 2 \varphi_j^* \varphi_{jxx}) + 2 \varphi_j^3 \varphi_j^* \varphi_{jxx}}{3\varphi_j^3 \varphi_j^3}$$

which yields to $\mu_j = 0$. Thus, the generalized Kupershmidt deformed Boussinesq equation (26) gives the following integrable system

$$v_t = 2w_x - 3 \sum_{j=1}^{N} (\varphi_j \varphi_j^*)_x, \quad (28a)$$

$$w_t = \frac{1}{6} (4v \varphi_j + \varphi_{jxx}) - \frac{3}{2} (\varphi_j \varphi_j^* - \varphi_j \varphi_j^*_{jxx}), \quad (28b)$$

$$\varphi_{jxxx} + v \varphi_{jx} + \frac{1}{2} v_x + w) \varphi_j = \lambda_j \varphi_j, \quad (28c)$$

$$\varphi_j \varphi_{jxxx} + v \varphi_{jxx} + \frac{1}{2} v_x - w) \varphi_j^* = -\lambda_j \varphi_j^*, \quad j = 1, 2, \cdots, N \quad (28d)$$

which just is the Boussinesq equation with self-consistent sources and has the following Lax representation [26]

$$L_t = [\partial^2 + \frac{2}{3} v + \sum_{j=1}^{N} \varphi_j \partial^{-1} \varphi_j^*, L] \quad (29a)$$

$$L \psi = (\partial^3 + v \partial + \frac{1}{2} v_x + w) \psi = \lambda \psi, \quad (29b)$$

$$\psi_t = (\partial^2 + \frac{2}{3} v + \sum_{j=1}^{N} \varphi_j \partial^{-1} \varphi_j^*) \psi. \quad (29c)$$
5 The generalized Kupershmidt deformation of Jaulent-Miodek equation and its bi-Hamiltonian structure

5.1 The generalized Kupershmidt deformation of Jaulent-Miodek equation

The JM eigenvalue problem reads \[27]\[
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}_x = U \begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix},
U = \begin{pmatrix}
0 & 1 \\
-\lambda^2 + \lambda q + r & 0
\end{pmatrix},
\tag{30}
\]
the associated JM hierarchy is
\[
\begin{pmatrix}
q \\
r
\end{pmatrix}_t = B_1 \begin{pmatrix}
b_{n+2} \\
b_{n+1}
\end{pmatrix} = B_1 \left( \frac{\delta H_{n+1}}{\delta q}, \frac{\delta H_{n+1}}{\delta r} \right) = B_2 \left( \frac{\delta H_n}{\delta q}, \frac{\delta H_n}{\delta r} \right),
\]
where
\[
B_1 = \begin{pmatrix}
0 & 2\partial \\
2\partial & -q_x - 2q\partial
\end{pmatrix},
B_2 = \begin{pmatrix}
2\partial & 0 \\
0 & r_x + 2r\partial - \frac{1}{2}\partial^3
\end{pmatrix},
\]
\[
\begin{pmatrix}
b_{n+2} \\
b_{n+1}
\end{pmatrix} = L \begin{pmatrix}
b_{n+1} \\
b_n
\end{pmatrix}, \quad n = 1, 2, \ldots
\]
\[
b_0 = b_1 = 0, \quad b_2 = -1, \quad H_n = \frac{1}{n-1} (2b_{n+2} - qb_{n+1}).
\]
For \(N\) distinct real eigenvalues \(\lambda_j\), from the spectral problem
\[
\varphi_{1jx} = \varphi_{2j}, \quad \varphi_{2jx} = (-\lambda_j^2 + \lambda_j q + r)\varphi_{1j}
\]
we have
\[
\frac{\delta \lambda_j}{\delta q} = \frac{1}{2} \lambda_j \varphi_{1j}^2, \quad \frac{\delta \lambda_j}{\delta r} = \frac{1}{2} \varphi_{1j}^2.
\]
Similarly, the generalized Kupershmidt deformation (9) with $\alpha_j = \lambda_j$ for JM hierarchy gives rise to

$$
\begin{pmatrix}
q \\
r
\end{pmatrix}
_t = B_1 \left( \frac{\delta H_{x+1}}{\delta q} \right) + \sum_{j=1}^{N} \left( \frac{\delta \lambda_j}{\delta q} \right),
$$

(31a)

$$(B_2 - \lambda_j B_1) \left( \frac{\delta \lambda_j}{\delta r} \right) = 0, \quad j = 1, 2, \cdots, N.
$$

(31b)

The first equation in (31b) is an identity and second one in (31b) yields

$$
\varphi_{1j}(\varphi_{1j} - \lambda_j \varphi_{1j} + \lambda_j^2 \varphi_{1j}) + 3 \varphi_{1j}(\varphi_{1j} - \lambda_j \varphi_{1j} + \lambda_j^2 \varphi_{1j}) = 0
$$

which, by setting $\varphi_{2j} = \varphi_{1j}$, leads to

$$
\varphi_{2j} = (-\lambda_j^2 + \lambda_j q + r) \varphi_{1j} + \frac{\mu_j}{\varphi_{1j}}, \quad j = 1, 2, \cdots, N.
$$

Then the generalized Kupershmidt deformation with of JM equation (31) with $n = 3$ gives rise to the following integrable system

$$
q_t = -r_x - \frac{3}{2} q_{xx} + 2 \sum_{j=1}^{N} \varphi_{1j} \varphi_{2j},
$$

(32a)

$$
r_t = \frac{1}{4} q_{xxx} - q_x r - \frac{1}{2} q r_x + \sum_{j=1}^{N} \left[ 2(\lambda_j - q) \varphi_{1j} \varphi_{2j} - \frac{1}{2} q_x \varphi_{1j}^2 \right],
$$

(32b)

$$
\varphi_{1j} = \varphi_{2j}, \quad \varphi_{2j} = (-\lambda_j^2 + \lambda_j q + r) \varphi_{1j} + \frac{\mu_j}{\varphi_{1j}} j = 1, 2, \cdots, N.
$$

(32c)

which just is the integrable RD-JMESCS and has the Lax representation (14a) with $U$

$$
U = \begin{pmatrix}
0 & 1 \\
-\lambda^2 + \lambda q + r & 0
\end{pmatrix},
$$

$$
V = \begin{pmatrix}
\frac{1}{2} q_x & \lambda - \frac{1}{2} q \\
\lambda^3 - \frac{1}{2} q \lambda^2 - (\frac{1}{2} q^2 + r) \lambda + \frac{1}{4} q_{xx} - \frac{1}{4} q r & -\lambda - \frac{1}{2} q
\end{pmatrix}
$$

$$
+ \frac{1}{2} \begin{pmatrix}
0 & 0 \lambda \langle \Phi_1, \Phi_1 \rangle - (A \Phi_1, \Phi_1) - q \langle \Phi_1, \Phi_1 \rangle & 0
\end{pmatrix} + \frac{1}{2} \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix}
\phi_{1j} \phi_{2j} & -\phi_{1j}^2 \\
\phi_{2j}^2 + \frac{\mu_j}{\varphi_{1j}} & -\phi_{1j} \phi_{2j}
\end{pmatrix}
$$

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5.2 The bi-Hamiltonian structure of GKDJME

In the following, we will show how to construct the bi-Hamiltonian structure for the generalized Kupershmidt deformation of a bi-Hamiltonian system. We follow the method in [17]-[20] to construct the bi-Hamiltonian formalism with $t-$type Hamiltonian operator for GKDJME by taking $x$ as the evolution parameter and $t$ as the 'spatial' variable. We denote the inner product in $\mathbb{R}^N$ by $\langle ... \rangle$ and $\Phi_i = (\varphi_{i1}, \varphi_{i2}, \cdots, \varphi_{iN})^T$, $i=1,2$, $\mu = (\mu_1, \cdots, \mu_N)^T$, $\Lambda = diag(\lambda_1, \cdots, \lambda_N)$.

Eq. (32) can be written as

$$
\begin{pmatrix}
q \\
r
\end{pmatrix}
= B_1 \begin{pmatrix}
\frac{1}{8}q_{xx} - \frac{3}{4} qr - \frac{5}{16} q^3 + \frac{1}{2} \langle \Lambda \Phi_1, \Phi_1 \rangle \\
- \frac{1}{2} r - \frac{3}{8} q^2 + \frac{1}{2} \langle \Phi_1, \Phi_1 \rangle
\end{pmatrix}
$$

(33a)

$$
\varphi_{1jx} = \varphi_{2j}, \; \varphi_{2jx} = -\lambda_j^2 \varphi_{1j} + q \lambda_j \varphi_{1j} + r \varphi_{1j} + \mu_j \varphi_{1j}.
$$

(33b)

Notices that Kernel of $B_1$ is $(c_1 + \frac{1}{4} q c_2, c_2)^T$, we may rewrite (34) as

$$
\frac{1}{8} q_{xx} - \frac{3}{4} qr - \frac{5}{16} q^3 + \frac{1}{2} \langle \Lambda \Phi_1, \Phi_1 \rangle = c_1 + \frac{1}{4} q c_2, \; - \frac{1}{2} r - \frac{3}{8} q^2 + \frac{1}{2} \langle \Phi_1, \Phi_1 \rangle = c_2
$$

(34a)

$$
c_{1x} = \frac{1}{2} \partial_t (r + \frac{1}{4} q^2), \; c_{2x} = \frac{1}{2} \partial_t q.
$$

(34b)

By introducing

$$
q_1 = q, \; p_1 = -\frac{1}{8} q_x
$$

(35)
Eqs. (33b) and (34b) give rise to the t-type Hamiltonian form

\[ R_x = G_1 \frac{\delta F_1}{\delta R}, \]  

(36a)

where

\[ R = (\Phi_1^T, q_1, \Phi_2^T, p_1, c_1, c_2)^T, \]

\[ F_1 = -4p_1^2 - \frac{1}{16}q_1^4 - \frac{1}{2}q_1^2c_2 + q_1c_1 - c_2^2 + \frac{3}{8}q_1^2\langle \Phi_1, \Phi_1 \rangle - \frac{1}{2}q_1\langle \Lambda\Phi_1, \Phi_1 \rangle \]

\[ + \frac{1}{2}\langle \Phi_2, \Phi_2 \rangle + \frac{1}{2}\langle \Lambda^2\Phi_1, \Phi_1 \rangle + c_2\langle \Phi_1, \Phi_1 \rangle - \frac{1}{4}\sum_{j=1}^N \varphi_{1j}^4 + \frac{1}{2}\sum_{j=1}^N \frac{\mu_j}{\varphi_{1j}^2}, \]  

(36b)

and the t-type Hamiltonian operator \( G_1 \) is given by

\[
G_1 = \begin{pmatrix}
0 & I_{(N+1)\times(N+1)} & 0 & 0 \\
-I_{(N+1)\times(N+1)} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2}\partial_t \\
0 & 0 & \frac{1}{2}\partial_t & 0 
\end{pmatrix}.
\]  

(36c)

The modified Jaulent-Miodek (MJM) eigenvalue problem reads

\[
\begin{pmatrix}
\tilde{\psi}_1 \\
\tilde{\psi}_2
\end{pmatrix} = \tilde{U}(\tilde{u}, \lambda) \begin{pmatrix}
\tilde{\psi}_1 \\
\tilde{\psi}_2
\end{pmatrix}, \quad \tilde{U} = \begin{pmatrix}
-\tilde{r} & \lambda \\
-\lambda + \tilde{q} & \tilde{r}
\end{pmatrix}, \quad \tilde{u} = \begin{pmatrix}
\tilde{r} \\
\tilde{q}
\end{pmatrix}
\]

(37)

the associated MJM equation is of the form

\[
\tilde{u}_t = \begin{pmatrix}
\tilde{r} \\
\tilde{q}
\end{pmatrix}_t = \begin{pmatrix}
-\frac{1}{2}\tilde{q}_{xx} - \frac{1}{2}(\tilde{q}\tilde{r})_x \\
-2\tilde{r}\tilde{r}_x - \frac{3}{2}\tilde{q}\tilde{q}_x + \tilde{r}_{xx}
\end{pmatrix} = \tilde{B}_1 \frac{\delta \tilde{H}_2}{\delta \tilde{u}}
\]

where \( \tilde{B}_1 = \begin{pmatrix}
\frac{1}{2}\partial & 0 \\
0 & 2\partial
\end{pmatrix}, \quad \tilde{H}_2 = -\frac{1}{2}\tilde{q}_x\tilde{r} - \frac{1}{2}\tilde{q}\tilde{r}^2 - \frac{1}{8}\tilde{q}^3. \)

We have

\[
\frac{\delta \lambda}{\delta \tilde{u}} = \begin{pmatrix}
\tilde{\varphi}_1 \tilde{\varphi}_2 \\
\frac{1}{2}\tilde{\varphi}_1^2
\end{pmatrix}
\]

Then the Rosochatius deformation of MJM equation with self-consistent sources
(RD-MJMCS) is defined as

\[
\begin{pmatrix}
\ddot{r} \\
\ddot{q}
\end{pmatrix}_t = \hat{B}_1 \left( \frac{\delta \tilde{H}_2}{\delta \dot{u}} + \frac{\delta \lambda}{\delta \dot{u}} \right) = \hat{B}_1 \left( \begin{array}{c}
-\frac{1}{2} \dddot{q} - \dddot{q} \dddot{r} + \langle \tilde{\Phi}_1, \tilde{\Phi}_2 \rangle \\
-\frac{1}{2} \dddot{r}^2 - \frac{3}{8} \dddot{q}^2 + \frac{1}{2} \dddot{r} x + \frac{1}{2} \langle \tilde{\Phi}_1, \tilde{\Phi}_1 \rangle
\end{array} \right) \tag{38a}
\]

\[
\ddot{\varphi}_{1jx} = -\dddot{r} \dddot{\varphi}_{1j} + \lambda_j \dddot{\varphi}_{2j}, \quad \ddot{\varphi}_{2jx} = -\lambda_j \dddot{\varphi}_{1j} + \dddot{q} \dddot{\varphi}_{1j} + \dddot{r} \dddot{\varphi}_{2j} + \frac{\mu_j}{\lambda_j} \dddot{\varphi}_{3j}. \tag{38b}
\]

Since the Kernel of $\hat{B}_1$ is $(\tilde{c}_1, \tilde{c}_2)^T$, let

\[
-\frac{1}{2} \dddot{q} + \langle \tilde{\Phi}_1, \tilde{\Phi}_2 \rangle = \tilde{c}_1, \quad -\frac{1}{2} \dddot{r}^2 - \frac{3}{8} \dddot{q}^2 + \frac{1}{2} \dddot{r} x + \frac{1}{2} \langle \tilde{\Phi}_1, \tilde{\Phi}_1 \rangle = \tilde{c}_2,
\]

then RD-MJMCS can be written as a t-type Hamiltonian system

\[
\tilde{R}_x = \tilde{G}_1 \frac{\delta \tilde{F}_1}{\delta \tilde{R}} \tag{39a}
\]

where

\[
\tilde{F}_1 = -2 \tilde{p}_1 \tilde{c}_1 + \tilde{q}_1 \tilde{c}_2 + 2 \tilde{p}_1^2 \tilde{q}_1 + \frac{1}{8} \tilde{q}_1^3 + 2 \tilde{p}_1 \langle \tilde{\Phi}_1, \tilde{\Phi}_2 \rangle - \frac{1}{2} \tilde{q}_1 \langle \tilde{\Phi}_1, \tilde{\Phi}_1 \rangle + \frac{1}{2} \langle \tilde{\Phi}_2, \tilde{\Phi}_2 \rangle + \frac{1}{2} \langle \tilde{\Phi}_1, \tilde{\Phi}_1 \rangle + \sum_{j=1}^N \frac{\mu_j}{2 \lambda_j} \tilde{\Phi}_{1j}^2, \tag{39b}
\]

\[
\tilde{G}_1 = \begin{pmatrix}
0 & I_{(N+1) \times (N+1)} & 0 & 0 \\
-I_{(N+1) \times (N+1)} & 0 & 0 & 0 \\
0 & 0 & 2 \partial_t & 0 \\
0 & 0 & 0 & \frac{1}{2} \partial_t
\end{pmatrix}. \tag{39c}
\]

The Miura map relating systems (36) and (39), i.e. $R = M(\tilde{R})$, is given by

\[
\Phi_1 = \tilde{\Phi}_1, \quad \Phi_2 = \tilde{\Phi}_2 + 2 \tilde{p}_1 \tilde{\Phi}_1, \tag{40a}
\]

\[
q_1 = \tilde{q}_1, \quad p_1 = -\frac{1}{2} \tilde{q}_1 \tilde{p}_1 - \frac{1}{4} \langle \tilde{\Phi}_1, \tilde{\Phi}_2 \rangle + \frac{1}{4} \tilde{c}_1, \tag{40b}
\]

\[
c_1 = \frac{1}{2} \tilde{F}_1 + \partial_t \tilde{p}_1, \quad c_2 = \tilde{c}_2. \tag{40c}
\]

Denote

\[
M' \equiv \frac{D R}{D R^T}
\]

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where $\frac{DB}{DR^*}$ is the Jacobi matrix consisting of Frechet derivative of $M$, $M'^*$ denotes adjoint of $M'$. According to the standard procedure[29], applying the map $M$ to the first Hamiltonian structure of Eq.(39), we can generates the second Hamiltonian structure of Eq.(36)

$$G_2 = M \tilde{G}_1 M^* = 
\begin{pmatrix}
0 & 0 & \Lambda & -\frac{1}{4} \Phi_1 & \frac{1}{2} \Phi_2 & 0 \\
0 & 0 & 2 \Phi_1^T & -\frac{1}{2} q_1 & -4 p_1 - \partial_t & 0 \\
-\Lambda & 2 \Phi_1 & 0 & \frac{1}{4} \Phi_2 & g_{35} & 0 \\
\frac{1}{2} \Phi_1^T & \frac{1}{2} q_1 & -\frac{1}{4} \Phi_2^T & \frac{1}{8} \partial_t & g_{45} & 0 \\
-\frac{1}{2} \Phi_2^T & 4 p_1 - \partial_t & -g_{35} & g_{45} & 0 & \partial_t q_1 \\
0 & 0 & 0 & 0 & q_1 \partial_t & 2 \partial_t 
\end{pmatrix}
$$

(41)

where

$$g_{35} = \frac{1}{2} q_1 \Lambda \Phi_1 - \frac{1}{2} \Lambda^2 \Phi_1 - \frac{3}{8} q_1^2 \Phi_1 - c_2 \Phi_1 + \frac{1}{4} \Phi_1(\Phi_1, \Phi_1) + \left( \frac{\mu_1}{\varphi_{11}}, \cdots, \frac{\mu_N}{\varphi_{11N}} \right)^T$$

$$g_{45} = -\frac{1}{2} c_1 + \frac{1}{4} (\Lambda \Phi_1, \Phi_1) - \frac{3}{8} q_1 (\Phi_1, \Phi_1) + \frac{1}{2} q_1 c_2 + \frac{1}{8} q_1^3$$

$$g_{55} = \partial_t (\frac{1}{4} (\Phi_1, \Phi_1) - \frac{1}{2} c_2) + (\frac{1}{4} (\Phi_1, \Phi_1) - \frac{1}{2} c_2) \partial_t - \frac{1}{4} q_1 \partial_t q_1.$$

Thus we get the bi-Hamiltonian structure for Eq.(36a)-(36c)

$$R_x = G_1 \frac{\delta F_1}{\delta R} = G_2 \frac{\delta F_0}{\delta R}, \quad F_0 = 2 c_1.
$$

(42)

### 6 Conclusion

It is known that there were some methods to construct a new integrable system starting from a bi-Hamiltonian system. The main purpose of this paper is to propose the generalized Kupershmidt deformation (GKD) of bi-Hamiltonian systems to construct new systems from integrable bi-Hamiltonian systems which is conjectured to be integrable. We have not be able to prove the integrability of the generalized Kupershmidt deformation of bi-Hamiltonian systems in general. However, for many specific cases, such as for KdV equation, Boussinesq
equation, Jaulent-Miodek equation and Camassa-Holm equation, by using this generalized Kupershmidt deformation some new integrable systems are derived from integrable bi-Hamiltonian systems. We present a general procedure to convert this generalized Kupershmidt deformation of the bi-Hamiltonian systems into an integrable Rosochatius deformation of soliton equation with self-consistent sources, as well as to transform it into a t-type bi-Hamiltonian system. These imply that the generalized Kupershmidt deformation of bi-Hamiltonian systems provides a way to construct new integrable system from an integrable bi-Hamiltonian systems. On other hand the generalized Kupershmidt deformation of bi-Hamiltonian systems also offers a new method to obtain the integrable Rosochatius deformation of soliton equation with self-consistent sources, which is quite different from the method used before. In the further we will continue to study the integrability of the generalized Kupershmidt deformation of bi-Hamiltonian systems in general. We believe that the method in [6] is helpful for proving the bi-Hamiltonian structure of the generalized Kupershmidt deformation.

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