Abstract

Baxter operators are constructed for quantum spin chains with deformed \( \mathfrak{sl}_2 \) symmetry. The parallel treatment of Yang-Baxter operators for the cases of undeformed, trigonometrically and elliptically deformed symmetries presented earlier and relying on the factorization regarding parameter permutations is extended to the global chain operators following the scheme worked out recently in the undeformed case.
1 Introduction

We consider periodic quantum spin chains with integrable dynamics where the single-site quantum states form an infinite-dimensional irreducible representation of the trigonometrically or elliptically deformed $\mathfrak{sl}_2$ symmetry algebra characterized by the representation parameter or spin $\ell$. The set of commuting quantum observables can be represented by the transfer matrix $t(u)$ in a generating function form. The task of finding the spectrum and eigenstates of these conserved charges can be treated by the algebraic Bethe ansatz method. The Bethe equation is related to the Baxter equation, being a difference equation involving the transfer matrix and the Baxter operator $Q(u)$. The concept of Baxter operators has been introduced by Baxter [1] in analyzing the eight-vertex model. Baxter operators have been constructed for various models, e.g. in [2, 12, 14]. General algebraic schemes of construction have been formulated in [6, 13]. The case of non-compact representations has been addressed in particular in [3, 7, 14, 16]. This concept provides an alternative way of solution, in particular a Baxter operator allows to construct the separated variable representation [17, 18].

In a number of papers a systematization and reformulation of known results on integrable quantum systems and essential progress has been achieved in view of the case of non-compact representations appearing in application to gauge field theory. In [24] the approach of constructing Baxter operators based on general Yang-Baxter operators has been presented in detail for the undeformed $\mathfrak{sl}_2$ symmetry. The non-compact representation case is understood as the generic one and the case of finite-dimensional representation is obtained in the limit where $2\ell$ approaches non-negative integer values. The relation of this to another approach worked out in [25] has been investigated in detail [26].
The general Yang-Baxter operators serve as local building blocks for global spin chain operators. Their construction has been formulated in [27] in a uniform way for the three cases of the symmetry algebra \( sl_2 \) undeformed, trigonometrically and elliptically deformed. The factorization regarding the permutation of representation parameters is the essential feature of this construction. The features of factorization have been noticed earlier in studies of the chiral Potts model [19–22]. By the results of [30] the formulation of the elliptic case has been essentially completed. The particular permutation operator intertwining representation \( \ell \) and \(-\ell - 1\) of Sklyanin algebra was taken in [27] in the form of a series relying on [40] (see also [41]) which is well defined rather in the finite-dimensional case but not in the infinite-dimensional one. The form based on the elliptic beta integral [31, 32, 37] avoids this problem and is easier to handle.

In the present paper we extend the parallel treatment of the three cases from the operators related to the chain sites to the global chain operators. For the results in [24] referring to the generic infinite-dimensional representation case we present here the corresponding extensions to these two cases of deformation. The constructions in [24] result in explicit expressions for the relevant operators and their action. Analogous explicit results are given here for the deformed cases.

In Section 2 a summary of the relevant relations of [24, 27] is given formulating simultaneously the relations which hold uniformly in all three cases. We present the general scheme which is suited for the three cases of symmetry algebra and allows to construct Baxter Q-operators, general transfer matrices and to establish their commutativity and factorization properties. As a new result we derive a formula relating both Baxter Q-operators. In the Sections 3 and 4 we specify the general formulae and work out the details for the cases of \( q \)-deformation and elliptic deformation, respectively. Some more details, less important for comparison of the three cases, but useful and potentially important in further investigations, have been put into the Appendix.

## 2 Factorized Yang-Baxter operators and Baxter equation

Building blocks in the construction of the quantum systems of integrable spin chains are the R-operators depending on spectral parameter \( u \), intertwining the tensor product of two representations

\[
\mathcal{R}_{12}(u) : \mathcal{V}_1 \otimes \mathcal{V}_2 \to \mathcal{V}_1 \otimes \mathcal{V}_2 \quad (2.1)
\]

and obeying the Yang-Baxter relation,

\[
\mathcal{R}_{12}(u - v) \mathcal{R}_{13}(u) \mathcal{R}_{23}(v) = \mathcal{R}_{23}(v) \mathcal{R}_{13}(u) \mathcal{R}_{12}(u - v). \quad (2.2)
\]

All operators act in the tensor product of three spaces \( \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \), where e.g. \( \mathcal{R}_{12} \) acts non-trivially in the tensor product of the first and the second spaces \((2.1)\) and as identity operator on the remaining space \( \mathcal{V}_3 \).

We consider irreducible representations of \( sl_2 \) and its trigonometric and elliptic deformations which are parameterized by spin \( \ell \). The representations are infinite dimensional for generic complex number \( \ell \), i.e. for \( \ell \neq n/2 \), \( n = 0, 1, 2 \cdots \), but finite \( n + 1 \) dimensional for \( \ell = n/2 \). Initially \((2.2)\) is written without restrictions on the representations involved. In the following we shall preserve the notation \( \mathcal{R}_{12}(u) \) for the general R-operator which by definition acts on the tensor product of two infinite dimensional representations

\[
\mathcal{R}_{12}(u|\ell_1, \ell_2) : \mathcal{V}_{\ell_1} \otimes \mathcal{V}_{\ell_2} \to \mathcal{V}_{\ell_1} \otimes \mathcal{V}_{\ell_2}
\]

and respects the Yang-Baxter relation \((2.2)\) in the space \( \mathcal{V}_{\ell_1} \otimes \mathcal{V}_{\ell_2} \otimes \mathcal{V}_{\ell_3} \)

\[
\mathcal{R}_{12}(u - v|\ell_1, \ell_2) \mathcal{R}_{13}(u|\ell_1, \ell_3) \mathcal{R}_{23}(v|\ell_2, \ell_3) = \mathcal{R}_{23}(v|\ell_2, \ell_3) \mathcal{R}_{13}(u|\ell_1, \ell_3) \mathcal{R}_{12}(u - v|\ell_1, \ell_2). \quad (2.3)
\]
Along with the previous case the cases of R-operators with one or both tensor factors finite dimensional are to be considered. If one factor is the fundamental spin $\frac{1}{2}$ representation $\mathbb{C}^2$ the R-operator is called L-operator and if both are the fundamental spin $\frac{1}{2}$ representation then one has the fundamental $\mathcal{R}$-matrix, $\mathcal{R}(u) : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$. The latter can be considered as the source of the relevant algebra and co-algebra relations. Indeed, the particular case of the relation (2.2) where the representations labeled by 1, 2 are the fundamental ones $\mathbb{V}_1 = \mathbb{V}_2 = \mathbb{C}^2$,

$$\mathcal{R}_{ij,km}(u-v) L_{ns}(u) L_{mp}(v) = L_{is}(v) L_{jp}(u) \mathcal{R}_{sp,nm}(u-v)$$  \hspace{1cm} (2.4)

the given $\mathcal{R}$ fixes the commutation relations of the matrix elements of the involved L-operator. Here summation over indices $i, j, \cdots = 1, 2$ is assumed. We have in mind the situation where these matrix elements generate the generic irreducible representation related to one site of the spin chain.

$$L(u|\ell) : \mathbb{V}_\ell \otimes \mathbb{C}^2 \to \mathbb{V}_\ell \otimes \mathbb{C}^2$$

It is distinguished from other solutions by a rather simple dependence on spectral parameter $u$ and linearity in the generators of the symmetry algebra.

The co-algebra structure is also defined because the relation (2.4) still holds if a solution $L(u)$ is substituted by the matrix product $L_1(u) L_2(u) \cdots L_N(u)$ acting on the tensor product representation related to the sites 1, 2, $\cdots$, $N$. Besides of the case related to irreducible representations of $s\ell_2$ or its trigonometric and elliptic deformations other solutions of (2.4) with the same $s\ell_2$ are known, which emerge as degeneracy limits of the former. They have been considered recently in [25] and also in [26] concerning the undeformed $s\ell_2$; the present study will not touch this case.

Further, the case of (2.2) if representations labeled by 1, 2 are generic spin $\ell_1$ and spin $\ell_2$ representations correspondingly $\mathbb{V}_1 = \mathbb{V}_{\ell_1}$, $\mathbb{V}_2 = \mathbb{V}_{\ell_2}$ but the representation 3 is fundamental $\mathbb{V}_3 = \mathbb{C}^2$

$$\mathbb{R}_{12}(u-v|\ell_1,\ell_2) L_1(u|\ell_1) L_2(v|\ell_2) = L_2(v|\ell_2) L_1(u|\ell_1) \mathbb{R}_{12}(u-v|\ell_1,\ell_2)$$  \hspace{1cm} (2.5)

can be read as the defining relation for the general R-operator with the given L-operator.

At this point it is convenient to adopt notations showing that representations of the symmetry algebra with parameters $\ell$ and $-\ell - 1$ are equivalent, since the corresponding values of Casimir operators are equal. For this reason we join the spectral parameter $u$ and spin parameter $\ell$ into two independent linear combinations $u_1, u_2$ such that

$$u_1 \leftrightarrow u_2 \sim \ell \leftrightarrow -\ell - 1$$  \hspace{1cm} (2.6)

and denote the L-operator also by $L(u_1, u_2)$. For example in the case of undeformed $s\ell_2$ considered in [21] we have $u_1 = u - \ell - 1$, $u_2 = u + \ell$. In the cases of deformed symmetries the corresponding relations will be specified below. Further we refer to both $u_1, u_2$ as spectral parameters. Correspondingly the general R-operator in (2.5) appears as depending on the parameters $u_1, u_2, v_1, v_2$ where

$$u_1 \leftrightarrow u_2 \sim \ell_1 \leftrightarrow -\ell_1 - 1 \hspace{1cm}; \hspace{1cm} v_1 \leftrightarrow v_2 \sim \ell_2 \leftrightarrow -\ell_2 - 1$$.

Rewriting now (2.7) in terms of $R_{12} = P_{12} R_{12}$, where $P_{12}$ is the operator of permutation of the tensor factors,

$$R_{12}(u_1, u_2|v_1, v_2) L_1(u_1, u_2) L_2(v_1, v_2) = L_1(v_1, v_2) L_2(u_1, u_2) R_{12}(u_1, u_2|v_1, v_2)$$  \hspace{1cm} (2.7)

we see the action of R-operator on the product of L-operators appearing as the permutation a pair of parameters $(u_1, u_2)$ in the first space L-operator with a pair $(v_1, v_2)$ in the second space L-operator and represents a permutation $s$ in the set of four parameters

$$s : u = (v_1, v_2, u_1, u_2) \mapsto (u_1, u_2, v_1, v_2)$$. 

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It turns out to be useful to study operators corresponding to other permutation operations on this set of parameters. An arbitrary permutation of four parameters can be constructed out of three elementary transpositions $s^i$ ($i=1,2,3$) interchanging a pair of adjacent parameters only. For them we find an operator representation $s^i \mapsto S^i(u)$ with the composition rule $s^i s^j \mapsto S^i(s^j u) S^j(u)$ such that

$$S^i(u) L_1(u_1, u_2) L_2(v_1, v_2) = L_1(u'_1, u'_2) L_2(v'_1, v'_2) S^i(u) ; \quad s^i u = (u'_1, u'_2, v'_1, v'_2)$$

Furthermore, operator relations corresponding to the symmetric group defining relations $s^i s^j = 1$, $s^1 s^2 s^1 = s^2 s^1 s^2$, $s^3 s^2 s^3 = s^2 s^3 s^2$ have to be satisfied. The operators $S^1, S^2, S^3$ have been constructed for the three cases of symmetry algebra in [27,30]. Each operator effectively depends on one parameter. Their defining relations have the form

$$S^1(v_2 - v_1) L_2(v_1, v_2) = L_2(v_2, v_1) S^1(v_2 - v_1), \quad (2.8)$$
$$S^2(u_1 - v_2) L_1(u_1, u_2) L_2(v_1, v_2) = L_1(v_2, u_2) L_2(v_1, v_1) S^2(u_1 - v_2), \quad (2.9)$$
$$S^3(u_2 - u_1) L_1(u_1, u_2) = L_1(u_2, v_1) S^3(u_2 - u_1). \quad (2.10)$$

The binary relation $S^i(a) S^i(-a) = 1$ and the triple Coxeter relations

$$S^1(a) S^2(a + b) S^1(b) = S^2(b) S^1(a + b) S^2(a), \quad (2.11)$$
$$S^3(a) S^2(a + b) S^3(b) = S^2(b) S^3(a + b) S^2(a) \quad (2.12)$$
do hold implying that we have indeed an operator representation of the symmetric group. Particularly $S^0(0) = 1$. Since the defining relations (2.8) and (2.10) are essentially identical and in view of (2.7) the operators $S^1(a)$ and $S^3(a)$ are two copies of the intertwining operator $W(a)$ of the symmetry algebra,

$$W(u_2 - u_1) S_a(\ell) = S_a(-\ell - 1) W(u_2 - u_1), \quad (2.13)$$

where $S_a(\ell)$ denote spin $\ell$ representation of the generators of the symmetry algebra, acting in the second $S^1(a) = W_2(a)$ and the first $S^3(a) = W_1(a)$ quantum spaces. It is worth mentioning that at (half)-integer $\ell$ ($u_2 - u_1 = 2\ell + 1 \in \mathbb{N}$) the intertwining operator $W$ has a non-trivial kernel coinciding with the $2\ell + 1$ dimensional invariant subspace. This will be shown below by rewriting this operator in a particular form valid in this case. For consistency of (2.13) the vectors annihilated by $W$ have to span an invariant subspace.

Out of elementary transposition operators more involved operators can be constructed. We need the operators $R^1$ and $R^2$ which satisfy the defining relations

$$R^1(u_1|v_1, v_2) L_1(u_1, u_2) L_2(v_1, v_2) = L_1(v_1, u_2) L_2(u_1, v_2) R^1(u_1|v_1, v_2), \quad (2.14)$$
$$R^2(u_1, u_2|v_2) L_1(u_1, u_2) L_2(v_1, v_2) = L_1(u_1, v_2) L_2(u_1, u_2) R^2(u_1, u_2|v_2) \quad (2.15)$$

and can be factorized as follows

$$R^1(u_1|v_1, v_2) = S^2(v_2 - v_1) S^1(u_1 - v_1) S^2(u_1 - v_2) = S^1(u_1 - v_2) S^2(u_1 - v_1) S^1(v_2 - v_1), \quad (2.16)$$
$$R^2(u_1, u_2|v_2) = S^2(u_2 - u_1) S^3(v_2 - v_2) S^2(u_1 - v_2) = S^3(u_1 - v_2) S^2(u_2 - v_2) S^3(u_2 - u_1). \quad (2.17)$$

Finally using $R^1$ and $R^2$ we factorize the general R-operator

$$R(u_1, u_2|v_1, v_2) = R^1(u_1|v_1, u_2) R^2(u_1, u_2|v_2) = R^2(v_1, u_2|v_2) R^1(u_1|v_1, v_2). \quad (2.18)$$

Notice that we have two factorized representations for the R-operator and that their consistency follows from Coxeter relations (2.11) and (2.12).
Now we proceed to global operators building them out of local ones considered above. It is well known that the physics of a homogeneous periodic spin chains can be obtained from the transfer matrix defined from the L-operators by
\[ t(u) = \text{tr} L_1(u) L_2(u) \cdots L_N(u), \tag{2.19} \]
where the lower index \( k \) refers to the local quantum space at the \( k \)-th site and the trace is taken over an auxiliary space \( \mathbb{C}^2 \), because it is the generating function of the set of commuting operators, \( [t(u), t(v)] = 0 \), as the consequence of (2.4). In the homogeneous case we restrict to \( \ell_1 = \ell_2 = \ldots = \ell_N = \ell \).

In a similar manner we build the general transfer matrix substituting locally the L-operators by general R-operators
\[ T_s(u|\ell) = \text{tr}_0 R_{10}(u|\ell, s) R_{20}(u|\ell, s) \cdots R_{N0}(u|\ell, s). \tag{2.20} \]
Whereas the matrix trace in (2.19) concerns the fundamental representation now the trace is to be taken in the generic infinite dimensional representation \( \mathbb{V}_s \) of spin \( s \) labeled by index 0. We assume \( \ell \) to be fixed and often omit it using the notation \( T_s(u) \). Similarly out of the local operators \( R^1 \) and \( R^2 \) we build the following traces of monodromies
\[ Q_1(u - v_1|\ell) = \text{tr}_0 R^1_{10}(u_1|v_1, u_2) \cdots R^1_{N0}(u_1|v_1, u_2), \tag{2.21} \]
\[ Q_2(u - v_2|\ell) = \text{tr}_0 R^2_{10}(u_1, u_2|v_2) \cdots R^2_{N0}(u_1, u_2|v_2). \tag{2.22} \]

The form of the spectral parameter dependence in the two previous formulae follows from corresponding property of local building blocks and can be seen from (2.16) and (2.17).

The introduced operators happen to be commutative
\[ [t(u), Q_i(v)] = 0 \; ; \; [T_s(u), Q_k(v)] = 0 \; ; \; [Q_i(u), Q_k(v)] = 0 \; ; \; [P, Q_k(u)] = 0 \; \; ; \; i, k = 1, 2 \tag{2.23} \]
where \( P = P_{12}P_{13} \cdots P_{1N} \) is the cyclic permutation along the closed chain, and they respect factorization relations
\[ P \cdot T_s(u - v) = Q_1(u - v_1) Q_2(u - v_2) = Q_2(u - v_2) Q_1(u - v_1) \tag{2.24} \]
where \( v_1, v_2 \) are linear combinations of \( v \) and \( s \) analogous to the case of \( u_1, u_2, u, \ell \) (2.6). Corresponding proofs rely on Yang-Baxter like relations and can be found in [24].

In [27] we have obtained uniformly the general Yang-Baxter operators and their factors related to parameter permutations in the cases of \( s\ell_2 \) symmetry undeformed and with quantum and elliptic deformation. Starting from the well-known fundamental \( R \)-matrices for each case, we have formulated the L-operators with the matrix elements embedded in the algebra generated by Heisenberg conjugated pairs \( z, \partial \). We have noticed the factorized form in all considered cases
\[ L(u_1, u_2) = [u] V^{-1}(z, u_2) D(z, \partial) V(z, u_1), \tag{2.25} \]
where \([u]\) is a function of spectral parameter, \( V \) and \( D \) are two by two matrices with operator entries. The formulation of the spin chain takes such pairs \( z_i, \partial_i, i = 1, \ldots, N \) for each site and an additional one for the auxiliary space labeled as \( i = 0 \). Our operator constructions rely first on the algebraic relations, based on the algebra generated by the mentioned canonical pairs. The results appear as expressions in these generators or as related integral operators with integration over \( z_i \). We did not investigate all aspects of detailed definition of these constructions as operators in functional spaces with bilinear forms, i.e. using the word operator in view of the related physics we are not claiming a construction completed in the latter sense.
As suggested by the notation we consider representations by functions of $z$. In the undeformed and $q$-deformed cases the representation modules of spin $\ell$ can be considered as spanned by monomials of non-negative powers with 1 as lowest weight vector. Whereas the above algebraic construction is done in the frame of Heisenberg pairs in the elliptic case as well we do not provide a detailed description of the representations in this case as embedded in Heisenberg algebra representations. The known facts about these representation are mentioned in Section 4. Actually, the detailed form of the representation is not needed in the construction. The above trace should be defined by the embedding in the Heisenberg algebra representation.

In the next step we have to prove that the introduced traces of monodromies $Q_1$ and $Q_2$ (2.21), (2.22) are indeed Baxter Q-operators. For this goal it remains to check that they do respect Baxter equation. Above we have derived factorization and commutativity properties of operators $T_s$, $Q_1$, $Q_2$ from appropriate local relations for their building blocks. Similarly we derive the Baxter equation from a local relation. Starting from the defining relation for $R^2$ (2.15) we shall obtain the following local relation in the space $V_\ell \otimes V_s \otimes \mathbb{C}^2$

$$Z_0^{-1} \cdot [R_{k0}^2(u) L_k(u_1, u_2)] \cdot Z_0 = \left( \kappa^{-1} \cdot [R_{k0}^2(u + \delta)] \cdots 0 \kappa \Delta(u_1, u_2) \cdot [R_{k0}^2(u - \delta)] \right).$$

Here we use the shorthand notation $R^2(u) = R^2(u_1, u_2[0])$. The index $k$ refers to the corresponding local quantum space $V_\ell$ in the spin chain site, and the index 0 refers to the infinite dimensional auxiliary space $V_s$. $Z_0$ denotes a certain auxiliary matrix acting in the space $V_s \otimes \mathbb{C}^2$. $\kappa$ and $\delta$ are some constants and $\Delta(u_1, u_2)$ is a symmetric function of the spectral parameters: $\Delta(u_1, u_2) = \Delta(u_2, u_1)$. In the undeformed case we have $\Delta(u_1, u_2) = (u_1u_2)^N$, where $N$ is the number of sites, and in the deformed cases the corresponding deformed modifications of this expression appears. The matrix element above the diagonal denoted by ellipsis in (2.26) is not indicated explicitly since we do not need it for our purposes. Detailed calculations leading to (2.26) will be done in Subsections 3.3 and 4.4 for the cases of $q$-deformation and elliptic deformation, respectively. Considerations in both cases follow the general strategy and are very similar as in [24]. They do not use the explicit expression for operator $R^2$ or for its building blocks $S^2$, $S^3$, but only their properties. The calculations in both cases use only:

1. The defining relation for $R^2$ (2.15).
2. The second factorization of $R^2$ (2.17) in the product of elementary permutation operators.
3. The factorization formula for L-operator (2.25).
4. The general property of the operator $R^2$: $[R^2_{i12}, z_2] = 0$ that follows from (2.17).
5. Several recurrence relations for $S^2$ and $S^3$, connecting $S^i(a \pm \delta)$ with $S^i(a)$ at $i = 2, 3$.

As a by-product we obtain a set of peculiar recurrence relations for intertwining operators of the symmetry algebra. These relations are similar to the recurrence relations used for the evaluation of the q-beta-integral [35] and elliptic beta-integral [31][32].

In the elliptic case the recurrence relations give rise to a factorized form for the intertwining operator suitable for finite-dimensional representations of the Sklyanin algebra. It is an alternative form to the one proposed by A. Zabrodin in [39].

Having the local relation (2.26) it is rather straightforward to produce the corresponding global relation. We form the monodromy $R_{10}^2(u) \cdots R_{N0}^2(u) L_1(u) \cdots L_N(u)$, apply $N$ times the local relation (2.26) obtaining the product of $N$ triangular matrices with operator entries and calculate the traces over the auxiliary two dimensional space $\mathbb{C}^2$ and the auxiliary infinite dimensional space $V_s$ and obtain the Baxter equation for $Q_2(u)$

$$t(u) Q_2(u) = \kappa^{-N} Q_2(u + \delta) + \kappa^N \Delta^N(u_1, u_2) Q_2(u - \delta).$$

(2.27)
Similarly starting from the local relation for $R^1$ (2.14) it is possible to obtain the Baxter equation for $Q_1(u)$. However, going beyond the review of [24], here we establish a useful relation between the two Baxter operators $Q_1$ and $Q_2$. To make the presentation more transparent we use the following notations for the elementary transposition operators

$$S^1(a) = W_2(a) ; \quad S^2(a) = S_{12}(a) ; \quad S^3(a) = W_1(a),$$

where we take into account (2.13). Lower indices on the right hand side refer to spaces where the corresponding operators act nontrivially. Taking into account (2.17) we rewrite (2.22) as follows

$$Q_2(u|\ell) = \text{tr}_0 \, P_{10} W_1(u_1) S_{10}(u_2) W_1(u_2 - u_1) \cdots P_{N0} W_N(u_1) S_{N0}(u_2) W_N(u_2 - u_1) =$$

$$= \text{tr}_0 \, P_{10} W_1(u_1) S_{10}(u_2) \cdots P_{N0} W_N(u_1) S_{N0}(u_2) \cdot T(u_2 - u_1),$$

where we have introduced the operator

$$T(a) = W_1(a) \cdots W_N(a)$$

which is a product of intertwining operators referring to the $N$ sites of the spin chain. Consequently the property $T(a)T(-a) = 1$ holds. Similarly due to (2.16) the first Q-operator (2.21) takes the form

$$Q_1(u|-\ell-1) = \text{tr}_0 \, P_{10} W_0(u_2 - u_1) S_{10}(u_2) W_0(u_1) \cdots P_{N0} W_0(u_2 - u_1) S_{N0}(u_2) W_0(u_1) =$$

$$= T(u_2 - u_1) \cdot \text{tr}_0 \, P_{10} S_{10}(u_2) W_0(u_1) \cdots P_{N0} S_{N0}(u_2) W_0(u_1).$$

Notice that in $Q_1$ we choose the representation parameter in quantum space to be $-\ell - 1$ but not $\ell$ that corresponds to the transposition of spectral parameters $u_1 \leftrightarrow u_2$ (2.6). Then taking into account the cyclicity of the trace it is easy to see that the traces of monodromies in (2.29) and (2.30) coincide. Thus we conclude that the two Baxter operators are related by the similarity transformation

$$T(u_2 - u_1) \cdot Q_2(u|\ell) = Q_1(u|-\ell-1) \cdot T(u_2 - u_1)$$

and consequently the Baxter equation for $Q_1(u)$ has exactly the same form as (2.27).

Finally in [24] we have shown that the trace of any monodromy of the form

$$A = P_{10} A(z_1, \partial_1|z_0) \cdot P_{20} A(z_2, \partial_2|z_0) \cdots P_{N0} A(z_N, \partial_N|z_0)$$

can be easily calculated

$$\text{tr}_0 \, A = P \cdot A(z_1, \partial_1|z_2) \cdot A(z_2, \partial_2|z_3) \cdots A(z_N, \partial_N|z_0)|_{z_0 \rightarrow z_1}.$$ 

The same formula is valid if $A$ is an integral operator. The Baxter operator $Q_2$ constructed out of $R^2$ fits into this formula since, as we have already mentioned, $R^2_{12}$ commutes with the variable $z_2$.

In analogy to the undeformed case, using (2.33) in $q$-case and elliptic case we shall obtain an explicit formula for the action of $Q_2(u)$ on a particular state appearing as a generating function by its dependence on auxiliary parameters.

### 3 Trigonometric deformation case

#### 3.1 Trigonometric L-operator and parameter permutation operators

Now we are going to apply the above strategy to the $q$-deformed symmetry $U_q(\mathfrak{sl}_2)$. We are interested in infinite dimensional representations of the algebra on Verma modules, i.e. the representation space
\( \forall \ell \) coincides with the space of polynomials \( \mathbb{C}[z] \) and the generators of the algebra are realized as finite-difference operators. Details can be found in \([27]\). The L-operator respecting (2.4) has the factorized form

\[
L(u_1, u_2) = \begin{pmatrix}
1 & 1 \\
zq^{-u_2} & zq^{u_2}
\end{pmatrix}
\begin{pmatrix}
q^{z\partial_z+1} & 0 \\
0 & q^{-z\partial_z-1}
\end{pmatrix}
\begin{pmatrix}
u_1 & -z^{-1} \\
-q^{-u_1} & z^{-1}
\end{pmatrix}
\] (3.1)

where the two sets of parameters are related according to (2.6) as

\[
u_1 = u - \ell - 1 ; \quad u_2 = u + \ell.
\] (3.2)

Further we quote the operators of elementary permutations \( S^1, S^2, S^3 \) being the building blocks of the general R-operator and constructed in \([27]\).

- \( S^1(a) \) and \( S^3(a) \) \([2.8], \ [10]\) are two copies of the operator \( W(a) \) acting nontrivially in the second and the first quantum spaces, respectively \([2.28]\). It has the explicit form

\[
W(a) = \frac{q^2}{z^a} \cdot \frac{(q^{2z\partial_z+2-2a}; q^2)}{(q^{2z\partial_z+2}; q^2)} \cdot q^{-az\partial_z}.
\] (3.3)

In the latter formula \((x; q^2)\) denotes the infinite \( q \)-product \( (A.1) \). \( W(2\ell + 1) \) intertwines representations parameterized by \( \ell \) and \(-\ell - 1 \) \([2.13]\). Let us mention that in addition to the indispensable symmetric group relations we have \( W(a)W(-a) = 1 \) and furthermore the exponential property: \( W(a)W(b) = W(a+b) \).

- The operator \( S^2 \) defined by \([2.9]\) acts nontrivially in the tensor products of two quantum spaces. We choose it in the following form

\[
S^2(a) = z^2 \cdot \frac{(\frac{2}{z^a}q^{1-a}; q^2)}{(\frac{2}{z^1}q^{1+a}; q^2)}.
\] (3.4)

Let us stress that the defining relations \([2.13]\) and \([2.9]\) do not fix uniquely the operators \( W(a) \) and \( S^2(a) \). Indeed, let us consider the equation for the operator \( S(u) \)

\[
S(u) \cdot L_1(u_1, u_2) L_2(v_1, v_2) = L_1(u'_1, u'_2) L_2(v'_1, v'_2) \cdot S(u)
\] (3.5)

where \((v'_1, v'_2, u'_1, u'_2)\) is a permutation of the set \( u = (v_1, v_2, u_1, u_2) \). It is clear that if we multiply any solution \( S \) of the latter equation by an arbitrary operator \( \varphi \) such that

\[
[z_1, \varphi] = [z_2, \varphi] = 0 ; \quad [q^{2z_1\partial_z}, \varphi] = [q^{2z_2\partial_z}, \varphi] = 0
\] (3.6)

we obtain another solution of \([3.5]\). In particular we are free to multiply \([3.4]\) by an arbitrary multiplicatively-periodic function \( \varphi(z_1, z_2) \) with multiplicative period equal to \( q^2 \): \( \varphi(q^2z_1, z_2) = \varphi(z_1, q^2z_2) = \varphi(z_1, z_2) \). In this way we find another solution of \([2.9]\) which happens to be useful for us,

\[
S^2(a) = z^2 \cdot \frac{(\frac{2}{z^a}q^{1-a}; q^2)}{(\frac{2}{z^1}q^{1+a}; q^2)}.
\] (3.7)

Having on disposal the operators of elementary permutations we find the operators \( R^1, R^2 \). In \([42]\) they have been constructed in another way solving directly the system of operator relations \([2.14], \ [2.15]\). In Appendix \([13]\) we show that the two constructions do agree.

In \([27]\) Coxeter relation \([2.11], \ [2.12]\) have been proved using the series expansion for \([3.3]\) and the \( q \)-summation formula. In Appendix \([14]\) we show that Coxeter relations follow from the pentagon formula \((A.5)\) only.
The general R-operator concerns the generic symmetry algebra representations \(\ell_1, \ell_2\). The analytic dependence on these representation parameters contains in the limits to integer or half-integer values simpler objects like L-operator. In [24] we have shown how to extract the L-operator from \(\mathbb{R}(u|\ell, s)\) at \(s = \frac{1}{2}\) restricting it to invariant subspace \(\mathcal{V}_\ell \otimes \mathbb{C}^2\) in the case of undeformed \(s\ell_2\) symmetry. In Appendix D we present the analogous calculation in the case of \(q\)-deformation.

3.2 Trigonometric recurrence relations

Now we are going to establish several recurrence relations, which relate the elementary permutation operators with shifted arguments, i.e. we show how to connect \(S^i(a)\) and \(S^i(a \pm 1)\) \((i = 1, 2, 3)\). We shall need such relations for proving Baxter equation.

First we consider the recurrence relations for \(S^1, S^3\) which are in fact two copies of the operator \(W(a)\). We have

\[
-q^{-\frac{1}{2}} W(a + 1) = W(a) \frac{1}{z} \left( q^z \partial_z - q^{-z} \partial_z \right) = \frac{1}{z} \left( q^z \partial_z - q^{-z} \partial_z \right) W(a) .
\] (3.8)

It can be easily checked using the explicit expression (3.3).

In passing we notice that starting from \(W(0) = 1\) we obtain a factorized representation for \(W(n)\) if \(n\) is nonnegative integer

\[
W(n) = q^{\frac{n}{2}} \left( \frac{1}{z} \left( q^z \partial_z - q^{-z} \partial_z \right) \right)^n .
\] (3.9)

By this it is easy to see that the action of \(W(n)\) annihilates \(z^k, k = 0, 1, ..., n - 1\) spanning the invariant subspace in the case \(2\ell + 1 = n \in \mathbb{N}\).

Further we quote two matrix relations,

\[
-q^\frac{1}{2} (q^a - q^{-a}) W(a - 1) \begin{pmatrix} 1 & 1 \end{pmatrix} = (-z, 1) W(a) \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} q^z \partial_z + 1 & 0 \\ 0 & q^{-z} \partial_z - 1 \end{pmatrix} ,
\] (3.10)

\[
q^\frac{1}{2} (q^a - q^{-a}) z^{-1} W(a - 1) \begin{pmatrix} 1 & -1 \end{pmatrix} = \begin{pmatrix} q^z \partial_z + 1 & 0 \\ 0 & q^{-z} \partial_z - 1 \end{pmatrix} \begin{pmatrix} q^a - z^{-1} & 0 \\ 0 & z^{-1} \end{pmatrix} W(a) \begin{pmatrix} 1 & -1 \end{pmatrix} .
\] (3.11)

In order to see that the right hand side of (3.10) is proportional to the row \(\begin{pmatrix} 1 & 1 \end{pmatrix}\) we multiply the intertwining relation (2.13) \(W(a) L(0, a) = L(a, 0) W(a)\) by the row \((-z, 1)\) on the left obtaining \((-z, 1) W(a) L(0, a) = (0, 0)\) which is equivalent (3.1) to

\[
(-z, 1) W(a) \begin{pmatrix} 1 & 1 \\ zq^{-a} & zq^a \end{pmatrix} \begin{pmatrix} q^z \partial_z + 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 ,
\]

and this implies the linear dependence of the two equations in the system (3.10). Consequently verifying the system of two relations in (3.10) one needs to check only the first one which can be done easily taking into account the explicit expression (3.3) for \(W(a)\). Similarly the second system (3.11) of operator relations can be proven. In fact verifying Baxter equation we shall need (3.11) only.

The needed recurrence relation for \(S^2\) (3.1) are

\[
q^{z \partial_z} S^2(a) q^{-z \partial_z} = z_1^{-1} \left( 1 - \frac{z}{2} q^{-a} \right) S^2(a + 1) ,
\] (3.12)

\[
q^{z \partial_z} S^2(a) q^{-z \partial_z} = (z_1 - z q^a) S^2(a - 1) ,
\] (3.13)

\[
q^{-z \partial_z} S^2(a) q^{z \partial_z} = (z_1 - z q^{-a}) S^2(a - 1) .
\] (3.14)

The latter simple relations could be considered not worth to be displayed here. However they have direct analogs in the elliptic case which are much more involved as we shall see in Subsection 4.2.
### 3.3 Trigonometric Baxter equation

Now we are ready to proceed to the Baxter equation for $Q_2$ \eqref{Eq2.22}. It is constructed from several copies of local operator $R^2$ which respects the relation \eqref{Eq2.15} interchanging parameters $u_2 \leftrightarrow v_2$ in the product of two L-operators. Further it will be convenient for us to exploit its second factorized representation in \eqref{Eq2.17}.

We derive the Baxter equation from the appropriate local relation for building blocks of the transfer matrix $t(u)$ \eqref{Eq2.19} and of $Q_2$ \eqref{Eq2.22}. More exactly we find expressions for the diagonal elements of the matrix (cf. \eqref{Eq2.26})

\[
\begin{pmatrix}
1 & 0 \\
-z_1 & 1
\end{pmatrix}
R^2_{12}(u_1, u_2|v_2) L_1(u_1, u_2) \begin{pmatrix}
1 & 0 \\
-z_2 & 1
\end{pmatrix}
\tag{3.15}
\]

in the limit $v_2 = 0$. As we shall see shortly in the examined limit the matrix element below diagonal turns to zero. Starting from \eqref{Eq2.15}, substituting the factorized form of the L-operator \eqref{Eq3.1} and taking into account the commutativity of $R^2$ with $z_2$ one obtains

\[
R^2(u_1, u_2|v_2) L_1(u_1, u_2) \begin{pmatrix}
1 & 0 \\
-z_2 & 1
\end{pmatrix}
q^{z_2 \partial_2} R^2(u_1, u_2|v_2) q^{-z_2 \partial_2}
\begin{pmatrix}
1 & 0 \\
-z_2 & 1
\end{pmatrix}
q^{z_2 \partial_2} R^2(u_1, u_2|v_2) q^{-z_2 \partial_2}
\tag{3.16}
\]

or in the other form

\[
R^2(u_1, u_2|v_2) L_1(u_1, u_2) = L_1(u_1, v_2) \begin{pmatrix}
1 & 0 \\
-z_2 & 1
\end{pmatrix}
q^{z_2 \partial_2} R^2(u_1, u_2|v_2) q^{-z_2 \partial_2}
\begin{pmatrix}
1 & 0 \\
-z_2 & 1
\end{pmatrix}
q^{z_2 \partial_2} R^2(u_1, u_2|v_2) q^{-z_2 \partial_2}
\tag{3.17}
\]

These are the two main relations in the present calculation.

We start with the matrix element below diagonal in \eqref{Eq3.15}. Taking into account \eqref{Eq3.1} and

\[
(-z_1, 1) L_1(u_1, 0) \sim (-z_1, 1) \begin{pmatrix}
1 & 1 \\
-z_1 & z_1
\end{pmatrix} = (0, 0)
\]

we conclude that it is equal to zero due to \eqref{Eq3.16}.

Then consider the first diagonal matrix element in \eqref{Eq3.15} $(1, 0) R^2(u_1, u_2|0) L_1(u_1, u_2) \begin{pmatrix} 1 \\ z_2 \end{pmatrix}$. We are going to show that it is proportional to the operator $R^2$ with shifted arguments. From \eqref{Eq3.16} one gets

\[
R^2(u) L_1(u_1, u_2) \begin{pmatrix} 1 \\ z_2 \end{pmatrix} = L_1(u_1, 0) \begin{pmatrix} 1 & 0 \\ z_2 & z_2 q^{-u_2} \end{pmatrix}
q^{z_2 \partial_2} R^2(u) q^{-z_2 \partial_2}
\begin{pmatrix} 1 & 0 \\ z_2 & z_2 q^{-u_2} \end{pmatrix},
\]

where we use the shorthand notation $R^2(u) = R^2(u_1, u_2|0)$. Further we note that due to the previous formula, the second factorization formula \eqref{Eq2.17} for $R^2(u)$

\[
q^{z_2 \partial_2} R^2(u) q^{-z_2 \partial_2} = S^3(u_1) q^{z_2 \partial_2} S^2(u_2) q^{-z_2 \partial_2} S^3(u_2 - u_1),
\tag{3.18}
\]

and the intertwining relation \eqref{Eq2.10} the wanted matrix element takes the form

\[
S^3(u_1) \cdot (1, 0) L_1(0, u_1) \begin{pmatrix} 1 \\ z_2 q^{-u_2} \end{pmatrix} \cdot q^{z_2 \partial_2} S^2(u_2) q^{-z_2 \partial_2} S^3(u_2 - u_1).
\]
Taking into account (3.11) we notice that the underlined matrix composition in the latter formula takes the form
\[
(1, 1) \begin{pmatrix} \frac{z_1}{z_1} q^{z_1 \partial_{z_1} + 1} & 0 \\ 0 & q^{-z_1 \partial_{z_1} - 1} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot (1 - \frac{z_1}{z_1} q^{-u_2})
\]
and due to the recurrence formula (3.12) for the argument shift in \(S^2\) we obtain that the matrix element is equal to
\[
S^3(u_1) \frac{1}{z_1} \left( q^{z_1 \partial_{z_1} - q^{-z_1 \partial_{z_1}}} \right) S^2(u_2 + 1) S^3(u_2 - u_1).
\]
In the final step we use the recurrence formula for the operator \(S^3\) (3.8) to shift its argument and obtain the first diagonal matrix element
\[
-q^{-1/2} S^3(u_1 + 1) S^2(u_2 + 1) S^3(u_2 - u_1).
\]

Consider the second diagonal matrix element in (3.15): \((-z_1, 1) R^2(u_1, u_2) L_1(u_1, u_2) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). Due to (3.17) it is equal to
\[
\frac{1}{z_2(q^{u_2} - q^{-u_2})} (-z_1, 1) L_1(u_1, v_2) \left( \frac{1}{z_2 q^{-u_2}} \frac{1}{z_2 q^{u_2}} \right) \left( \frac{-q z_2 \partial_{z_2} R^2(u_1, u_2 | v_2) q^{-z_2 \partial_{z_2}}}{q^{-z_2 \partial_{z_2}} R^2(u_1, u_2 | v_2) q^{z_2 \partial_{z_2}}} \right),
\]
where we need to take \(v_2 \to 0\) carefully. In this limit we have
\[
\frac{1}{z_2(q^{u_2} - q^{-u_2})} (-z_1, 1) \left( \frac{1}{z_1 q^{-v_2}} \frac{1}{z_1 q^{v_2}} \right) \to \frac{z_1}{2 z_2} (-1, 1)
\]
and consequently taking into account the factorization of the L-operator (3.11), performing the argument shift in \(S^2\) by means of (3.13), (3.14)
\[
\left( \frac{1}{z_2 q^{-u_2}} \frac{1}{z_2 q^{u_2}} \right) \left( \frac{-q z_2 \partial_{z_2} S^2(u_2) q^{-z_2 \partial_{z_2}}}{q^{-z_2 \partial_{z_2}} S^2(u_2) q^{z_2 \partial_{z_2}}} \right) = z_2(q^{u_2} - q^{-u_2}) \left( \frac{1}{z_1} \right) S^2(u_2 - 1),
\]
we see that in the limit \(v_2 \to 0\) (3.19) takes the form
\[
\frac{z_1}{2 z_2} (-1, 1) \left( \frac{z_1 \partial_{z_1} + 1}{z_1 \partial_{z_1} - 1} \right) \left( \frac{q^{u_1}}{-q^{-u_1}} \frac{z_1^{-1}}{-z_1^{-1}} \right) S^3(u_1) \left( \frac{1}{z_1} \right) z_2(q^{u_2} - q^{-u_2}) S^2(u_2 - 1) S^3(u_2 - u_1).
\]
In the previous formula the underlined expression can be transformed using the matrix recurrent relation (3.11) and finally we find that the wanted matrix element is equal to
\[
-q^{-1/2} (q^{u_1} - q^{-u_1})(q^{u_2} - q^{-u_2}) S^3(u_1 - 1) S^2(u_2 - 1) S^3(u_2 - u_1).
\]
Now the calculation of the second diagonal matrix element can be completed in a similar way using the other recurrent formula (3.10) instead of (3.11).

The result of the computation is the following matrix (3.15) (cf. (2.26))
\[
\begin{pmatrix} 1 & 0 \\ -z_1 & 1 \end{pmatrix} R^2(u) L_1(u_1, u_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{z_2} = \begin{pmatrix} -q^{-1/2} R^2(u + 1) \\ 0 \end{pmatrix} \ldots \begin{pmatrix} 1 \\ -q^{1/2} (q^{u_1} - q^{-u_1})(q^{u_2} - q^{-u_2}) R^2(u - 1) \end{pmatrix}.
\]
As it has been explained in Section 2 we readily obtain the Baxter equation (cf. (2.27))
\[
t(u) Q_i(u) = (-)^N q^{-N} Q_i(u + 1) + (-)^N q^N \Delta^N(u_1, u_2) Q_i(u - 1), \quad i = 1, 2
\]
where \(\Delta(u_1, u_2) = (q^{u_1} - q^{-u_1})(q^{u_2} - q^{-u_2})\) is symmetric. The Baxter equation for \(Q_1\) follows from the fact that both \(Q\)-operators are connected by similarity transformation (2.31).
3.4 Explicit action of the trigonometric Q-operator

Here we are going to establish an explicit formula for the operator $Q_2$ (2.22) by computing its action on the generating function of the symmetry algebra representation in the quantum space of the spin chain states. It turns out that due to (2.33) it is sufficient to know the action by $R^2$ on the generating function of the representation in one site of the spin chain. The latter observation simplifies the task considerably reducing the global problem to a local one. Moreover the local problem can be solved up to a constant without any reference to an explicit expression for the operator $R^2$ (like (2.17)) but using only the its defining relation (2.15). We need the explicit expression for $R^2$ only in the last step for fixing a constant multiplier. Let us remind that in (2.15) the involved $L$-operators act in two different quantum spaces. Taking into account (2.4) we see that $L_1 \cdot L_2$ defines a co-product like structure on the quantum algebra $U_q(sl_2)$. Further in (2.15) we shift the four parameters $u_i \rightarrow u_i + \lambda$, $v_i \rightarrow v_i + \lambda$ ($i = 1, 2$) which does not change the operator $R^2$, extract the matrix element below diagonal, consider the asymptotics $\lambda \rightarrow +\infty$ and obtain the following relation

$$R^2(u_1, u_2 | v_2) \left[ S_i^+(u_1 - u_2 + 1) K_2(-v_1 - 1) + K_1^{-1}(u_2) S_2^+(v_1 - v_2 + 1) \right] =$$

$$\left[ S_i^+(u_1 - v_2 + 1) K_2(-v_1 - 1) + K_1^{-1}(v_2) S_2^+(v_1 - u_2 + 1) \right] R^2(u_1, u_2 | v_2)$$

(3.21)

where we use the shorthand notations $S^+(\Delta) = z[z\partial_z + a]$ ; $K(a) = q^{\alpha \partial_z - a}$. In (3.21) the following co-product like expression appears depending on some spectral parameters

$$(\Delta S^+)(a, b, c, d) = S_i^+(a) K_2(-b) + K_1^{-1}(c) S_2^+(d).$$

Then from (3.21) it follows that

$$R^2(u_1, u_2 | v_2) F(\lambda(\Delta S^+)(u_1 - u_2 + 1, v_1 + 1, u_2, v_1, v_1 - u_2 + 1)) \cdot 1 =$$

$$= F(\lambda(\Delta S^+)(u_1 - v_2 + 1, v_1 + 1, v_2, v_1, v_1 - u_2 + 1)) R^2(u_1, u_2 | v_2) \cdot 1$$

(3.22)

where $F$ is any function. Further we choose it to be the $q$-series

$$F(x) = \sum_{k \geq 0} \frac{q^{\frac{n(n+1)}{2}}}{(q^2; q^2)_n} (q^{-1} - q)^n x^n.$$

(3.23)

In order to calculate $F(\lambda(\Delta S^+)) \cdot 1$ we need the following formula

$$(\Delta S^+)(a, b, c, d) = \sum_{k=0}^{n} \frac{(q^{2a}; q^2)_k (q^{2d}; q^2)_n (q^{-1} - q)^n (q^2; q^2)_k (q^2; q^2)_n}{(q^{-1} - q)^n} q^{(b-a)k} z_1^k z_2^{n-k}$$

which can be established easily by induction. Then by means of the $q$-binomial formula (A.2) one obtains straightforwardly\footnote{Here and below we adopt the notation $(a, b, c, \cdots | q^2) = (a; q^2)(b; q^2)(c; q^2) \cdots$.}

$$F(\lambda(\Delta S^+)(a, b, c, d)) = (q^{1+a+b \lambda z_1}, q^{1+c+d \lambda z_2}; q^2).$$

Thus we see that $F(\lambda(\Delta S^+)) \cdot 1$ is a product of two factors, the first being a function of $z_1$ and the second being a function of $z_2$. This happens due to the special choice of the function $F$ (3.23). This factorization is crucial for our purpose, because by the commutativity of $R^2$ with $z_2$ the result (3.22) can be rewritten as

$$R^2_{12}(u) \left( \frac{q^{-2\ell \lambda z_1; q^2}}{q^{2\ell \lambda z_1; q^2}} \right) = e \cdot \left( \frac{q^{1+u-\ell \lambda z_1, q^{-u+\ell \lambda z_2}; q^2}}{q^{1+u-\ell \lambda z_1, q^{1+c+d \lambda z_2}; q^2}} \right)$$

(3.24)
where we choose \( v_1 = -2, v_2 = 0 \) and calculate the constant \( c = R^2(u) \cdot 1 \) using one of the possible forms of \( R^2 \):

\[
c = R^2(u) \cdot 1 = q^{-u_1 u_2 + \frac{u_2^2}{2}} \frac{(q^{2+2u_1-2u_2}; q^2)}{(q^{2+2u_1}; q^2)}.
\]

With this form of the action of \( R^2 \) we proceed to the action of \( Q_2 \). At this point it is convenient to renormalize the operator \( Q_2 \): \( Q_2(u) \rightarrow c^{-N} \cdot Q(u) \), such that the Baxter equation takes the form

\[
t(u) Q(u) = \Delta_+(u) Q(u + 1) + \Delta_-(u) Q(u - 1),
\]

where \( \Delta_{\pm}(u) = (q^{u\pm\ell} - q^{-u\pm\ell})^N \). Now we see that due to (2.33) the Baxter operator \( Q(u) \) acts on the generating function of spin chain states as follows

\[
Q(u) \cdot \prod_{i=1}^{N} \frac{(q^{-2\ell} \lambda_i z_i; q^2)}{(q^{2+2\ell} \lambda_i z_i; q^2)} = \prod_{i=1}^{N} \frac{(q^{-u-\ell} \lambda_i q^{-u+\ell} \lambda_{i+1} z_i; q^2)}{(q^{2+u+\ell} \lambda_i q^{-u+\ell} \lambda_{i+1} z_i; q^2)},
\]

where as usual we assume cyclicity, \( N + 1 \equiv 1 \).

In Appendix E we propose an alternative proof of (3.24) which uses only the Coxeter relation (2.12) and is very similar to the derivation of the analogous formula in the elliptic case in Subsection 4.3.

Thus for a spin chain with \( q \)-deformed symmetry algebra we have constructed a pair of Baxter Q-operators, proved the corresponding Baxter equation and found an explicit formula for one of them. Such operators respect commutativity (2.23) and factorization (2.24) property as it has been explained in Section 2 on the basis of general arguments.

4 Elliptic deformation case

4.1 Elliptic L-operator and parameter permutation operators

Now we proceed to the most intricate example of deformation applying the general arguments of Section 2. The elliptically deformed \( s\ell_2 \) symmetry algebra has been introduced in [28, 29] and is called Sklyanin algebra. Its defining relations are equivalent to (2.4) where the numerical \( R \)-matrix is due to Baxter and appeared first in his solution of the eight-vertex model [1].

Further we are interested in infinite-dimensional representations of the algebra in the space of meromorphic even functions of one complex variable [29]. It is realized by second order finite difference operators which are constructed out of Jacobi theta functions (see Appendix A), depend on two deformation parameters \( \eta, \tau \) and on spin \( \ell \) being an arbitrary complex number. Choosing this operator representation of the algebra generators one can obtain the factorized form for the L-operator [27, 40, 41]

\[
L(u_1, u_2) = \frac{1}{\theta_1(2z)} M(z \mp u_2) \begin{pmatrix} e^{\eta \theta_2} & 0 \\ 0 & e^{-\eta \theta_2} \end{pmatrix} N(z \mp u_1)
\]

(4.1)

where we denote matrices involving theta functions as follows

\[
M(a \mp b) = \begin{pmatrix} (a - b)_3 & -(a + b)_3 \\ -(a - b)_4 & (a + b)_4 \end{pmatrix}, \quad N(a \mp b) = \begin{pmatrix} (a + b)_4 & (a - b)_3 \\ (a - b)_4 & (a + b)_3 \end{pmatrix},
\]

(4.2)

and specify the relation between two sets of spectral parameters \( (u_1, u_2) \) and \( (u, \ell) \) according to (2.0) as

\[
u_1 = \frac{u}{2} - \eta \ell - \eta, \quad u_2 = \frac{u}{2} + \eta \ell.
\]

(4.3)
All notations and properties of theta functions relevant for us are collected in Appendix A. Let us note that by \( (A.6) \) the matrices \( M \) and \( N \) \( (4.2) \) are inverse to each other
\[
M(a + b)N(a + b) = 2\theta_1(2a)\theta_1(2b) \cdot 1.
\] (4.4)

This L-operator \( (4.1) \) has been used in \( [27] \) to construct the elementary permutation operators \( S^i(a) \) \((i = 1, 2, 3)\) and corresponding general R-operator respecting \( (2.7) \). However it has been demonstrated in \( [30] \) that it is more convenient to work with a slightly modified version of \( (4.1) \). Due to invariance of Baxter’s \( R \)-matrix: \( \sigma_3 \otimes \sigma_3 R(u) = R(u) \sigma_3 \otimes \sigma_3 \), we conclude that the L-operator multiplied by Pauli matrix \( \sigma_3 \) on the left \( \sigma_3 L(u) \) solves \( (2.4) \) as well and, consequently, can be substituted as L-operator. This transformation, \( L(u) \rightarrow \sigma_3 L(u) \), corresponds to a certain automorphism of the Sklyanin algebra. Further we solve the RLL-relation \( (2.7) \) in the form
\[
R_{12}(u_1, u_2|v_1, v_2)\sigma_3 L_1(u_1, u_2)\sigma_3 L_2(v_1, v_2) = \sigma_3 L_1(v_1, v_2)\sigma_3 L_2(u_1, u_2) R_{12}(u_1, u_2|v_1, v_2)
\]
where the L-operator is given in \( (4.1) \). The same substitution \( L(u) \rightarrow \sigma_3 L(u) \) should be done also in the other formulae in Section 2: \( (2.8), (2.9), (2.10), (2.14), (2.15) \).

Now we are ready to present the operators of elementary permutations \( S^1, S^2 \) and \( S^3 \). Their construction and consequently the integrability structure of the spin chain is based on the elliptic gamma function \( \Gamma(z|\tau, 2\eta) \). Further we use for it the shorthand notation \( \Gamma(z) \). Its definition and properties relevant for us are collected in Appendix A.

- The operator \( S^2 \) acts nontrivially in the both quantum spaces and is defined by the operator relation \( (2.9) \) where \( L(u) \rightarrow \sigma_3 L(u) \) \( (4.1) \). One of the possible solutions of the this relation which is suitable for our purposes has the form \( [2] \) \( [30] \)
\[
S^2(a) = \Gamma(\mp z_1 \mp z_2 + a + \eta + \frac{\pi}{2})
\] (4.5)

A similar expression appeared in \( [27] \) with some additional exponentials and without the shift by \( \frac{\pi}{2} \). Exactly the same expression was used in \( [38] \) for the formulation of the star-triangle relation.

- The infinite-dimensional representations of the Sklyanin algebra parameterized by \( \ell \) and \( \ell - 1 \) are equivalent since the Casimir operators take coinciding numerical values for both representations. The corresponding intertwining operator \( W(\eta(2\ell + 1)) \) \( (2.13) \) can be realized as an integral operator \( [30] \) on the space of even functions
\[
W(a)\Phi(z) = \int_0^1 dx \mu(x)e^{-\frac{\pi i}{\eta}(\mp z + x + a)}\frac{\Gamma(\mp z \mp x - a)}{\Gamma(-2a)}\Phi(x) ; \mu(x) = \frac{e^{2\pi i x^2}}{\Gamma(\mp 2x)}.
\] (4.6)
where \( \mu(x) \) is the integration measure and the constant is \( C = \frac{1}{2}(e^{4\pi i \eta}, e^{4\pi i \eta})(e^{2\pi i \tau}, e^{2\pi i \tau}). \)

It is remarkable that a similar operator was used in \( [33] \) for the construction of the integral Bailey transformation. The formula which is equivalent to the binary relation of permutation group \( W(a)W(-a) = 1 \) was proved in \( [34] \) in the context of integral Bailey transformation. Unlike the \( q \)-deformed case the exponential property is missing. The operators \( S^1(a) \) and \( S^3(a) \) \( (2.8), (2.10) \) are two copies of the intertwining operator \( W(a) \) which act nontrivially in the second and the first quantum spaces, respectively \( (2.28) \).

Finally, the indicated operators \( S^1, S^2, S^3 \) respect Coxeter relations \( (2.11), (2.12) \) \( [30] \) that is the direct consequence of the elliptic beta integral evaluation formula by V. Spiridonov \( [31] \).
For comparison to [27] we remark that the elementary permutation operators has been constructed there using the L-operator (4.1). The resulting operator $S^2$ is not symmetric under $z_1 \leftrightarrow z_2$ contrary to (4.3). Further, the intertwining operator $W$ of Sklyanin algebra representations has been taken in the form of the operator series constructed by A. Zabrodin. Despite of the fact that the existence of this series in the infinite-dimensional representation case is elusive the Coxeter relations has been proven by means of formal manipulations on the operator series using the Frenkel-Turaev summation formula [49].

4.2 Elliptic recurrence relations

Now we proceed to recurrence relations which connect $S^i(a)$ and $S^i(a \pm \eta)$ ($i = 1, 2, 3$). As we shall see their form is very similar to the undeformed and the trigonometric case (3.2).

We start with recurrence relations for the intertwining operator $W(a)$ (4.6)

$$R(\tau) e^{-\pi \eta}(z)_i W(a + \eta) = W(a) \frac{1}{\theta_1(2z)} \left( (z - a)_i e^{\eta \partial_z} - (z + a)_i e^{-\eta \partial_z} \right), \quad (4.7)$$

$$R(\tau) e^{-\pi \eta} W(a + \eta)(z)_i = \frac{1}{\theta_1(2z)} \left[ (z + a + \eta)_i e^{\eta \partial_z} - (z - a - \eta)_i e^{-\eta \partial_z} \right] W(a), \quad (4.8)$$

where $i = 3, 4$ and $R(\tau)$ is a constant (see Appendix A). The second formula is a consequence of the first one due to $W(a)W(-a) = 1$.

(4.7), (4.8) can be rewritten in matrix form

$$R(\tau) e^{-\pi \eta}(z)_3 W(a + \eta) = W(a) \frac{1}{\theta_1(2z)} M(z \mp a) \left( e^{\eta \partial_z}, e^{-\eta \partial_z} \right), \quad (4.9)$$

$$R(\tau) e^{-\pi \eta} W(a + \eta)(z)_3 = \frac{1}{\theta_1(2z)} \left( e^{\eta \partial_z}, -e^{-\eta \partial_z} \right) N(z \mp a) W(a). \quad (4.10)$$

Given the recurrence relations (4.9) and (4.10) it is easy to deduce that $W(a)$ is an intertwining operator, i.e. that it respects: $W(a) L(0, a) = L(a, 0) W(a)$. Indeed due to (4.9) and (4.6) the left hand side takes the form

$$W(a)L(0, a) = W(a) \frac{1}{\theta_1(2z)} M(z \pm a) \left( e^{\eta \partial_z}, e^{-\eta \partial_z} \right) \otimes ((z)_4, (z)_3) = c \left( (z)_3, -(z)_4 \right) W(a + \eta) \otimes ((z)_4, (z)_3),$$

where $c = R(\tau) e^{-\pi \eta}$, and due to (4.10) and (4.6) the right hand side takes the same form

$$L(a, 0)W(a) = \left( \frac{(z)_3}{(z)_4} \right) \otimes \frac{1}{\theta_1(2z)} \left( e^{\eta \partial_z}, -e^{-\eta \partial_z} \right) N(z \pm a) W(a) = c \left( (z)_3, -(z)_4 \right) \otimes W(a + \eta) ((z)_4, (z)_3).$$

The other two identities for $W(a)$ (4.6) of interest in the following are:

$$-2 R(\tau)^{-1} e^{\pi \eta} \theta_1(2a) W(a - \eta) \left( \begin{array} {c} 1 \\ 1 \end{array} \right) = \left( \begin{array} {cc} e^{\eta \partial_z} & 0 \\ 0 & e^{-\eta \partial_z} \end{array} \right) N(z \mp a) W(a) \left( (z)_3, -(z)_4 \right), \quad (4.11)$$

$$-2 R(\tau)^{-1} e^{\pi \eta} \theta_1(2a) W(a - \eta) \left( \begin{array} {c} 1 \\ -1 \end{array} \right)^T = ((z)_4, (z)_3) W(a) \left( \frac{1}{\theta_1(2z)} M(z \mp a) \left( e^{\eta \partial_z}, 0 \right) \begin{array} {c} 1 \\ 0 \end{array} \right) \left( e^{-\eta \partial_z}, 0 \right). \quad (4.12)$$

To see that the right hand side of (4.12) is proportional to the row $(1, -1)$ it is sufficient to multiply $W(a) L(0, a) = L(a, 0) W(a)$ by the row $(\frac{(z)_3}{(z)_4}, (z)_3)$ from the left to obtain $(\frac{(z)_3}{(z)_4}, (z)_3) W(a) L(0, a) = (0, 0)$ and to substitute (4.11) with the result

$$((z)_4, (z)_3) W(a) \frac{1}{\theta_1(2z)} M(z \mp a) \left( e^{\eta \partial_z}, 0 \right) \left( \begin{array} {c} 1 \\ 1 \end{array} \right) = 0.$$
Consequently verifying the system of two relations in (3.10) (or in (3.11)) one needs to check only the first one. In the following Subsection proving Baxter equation we will use only (4.11).

We also need a series of relations for $S^2$ (4.5)

\[
e^{\eta\theta_{z_2}} S^2(a) e^{-\eta\theta_{z_2}} = -R^{-2}(\tau) e^{-\frac{8\pi i}{\tau} \theta_4^{-1}(z_1 - z_2 + a)} S^2(a + \eta), \tag{4.13}
\]

\[
e^{\eta\theta_{z_2}} S^2(a) e^{-\eta\theta_{z_2}} = -R^2(\tau) e^{\frac{8\pi i}{\tau} \theta_4(\mp z_1 + z_2 + a)} S^2(a - \eta), \tag{4.14}
\]

\[
e^{-\eta\theta_{z_2}} S^2(a) e^{\eta\theta_{z_2}} = -R^2(\tau) e^{\frac{8\pi i}{\tau} \theta_4(\mp z_1 - z_2 + a)} S^2(a - \eta). \tag{4.15}
\]

All these recurrence relations can be easily proven taking into account the explicit expression for intertwining operator $W (\tau)$. For example we will prove here relation (4.8). Let us consider

\[
\left[ (z + b)_i e^{\eta\theta_{z_2}} - (z - b)_i e^{-\eta\theta_{z_2}} \right] W(b - \eta) \Phi(z), \tag{4.16}
\]

where we apply the finite difference operator to the kernel of the integral operator $W(b - \eta)$ (4.6) and perform argument shifts in the elliptic gamma functions by means of (A.9) obtaining

\[
\frac{R(\tau)e^{-\pi i \eta}}{\theta_1(-2b)} \int_0^1 dx \mu(x) \frac{e^{-\frac{8\pi i}{\tau}(x^2 + x^2)}}{\Gamma(-2b)} [(z + b)_i \theta_1(z \mp x - b) - (z - b)_i \theta_1(-z \mp x - b)] \Gamma(z \mp x - b) \Phi(x).
\]

Further simplifying the combination of theta functions in the latter formula using (A.7)

\[(z + b)_i \theta_1(z \mp x - b) - (z - b)_i \theta_1(-z \mp x - b) = (x)_i \theta_1(2z) \theta_1(-2b)\]

we see that (4.16) is equal to $R(\tau) e^{-\pi i \eta} \theta_1(2z) W(b) (z) \Phi(z)$ in accordance with (4.8).

In a similar way using the formula for the argument shift in the elliptic gamma function (A.9) and the formulae (4.3) (or (4.7)) relating theta functions with quasi periods $\tau$ and $\tau'$ it is not difficult to check the other recurrence relations. Let us emphasize that we do not need Riemann identities for theta functions for verifying the recurrence relations. Therefore we do not need them in proving that $W (\tau)$ is indeed the intertwining operator.

### 4.3 Factorization of the intertwining operator

Now we digress from the main line of our construction aiming at Baxter Q-operators for generic values of $2\ell$ and show an interesting application of the above recurrence relations if $2\ell + 1 = n \in \mathbb{N}$. Since $W(0) = 1$ using (4.7), (4.8) we factorize the intertwining operator $W(\eta)$ (2.13) in the case of representations with parameters $\ell = \frac{n-1}{2}$ and $-\ell - 1 = -\frac{n-1}{2}$ at $n = 1, 2, \cdots$ into a product of $n$ simpler finite-difference operators,

\[
W(\eta) = \prod_{k=0}^{n-1} \frac{1}{\theta_1(2z)^{\frac{1}{2}} \theta_1(2z)} [(z - \eta k)_i e^{\eta \theta_{z_2}} - (z + \eta k)_i e^{-\eta \theta_{z_2}}] = \]

\[
e^{2 \cdot \prod_{k=0}^{n-1} \frac{1}{\theta_1(2z)^{\frac{1}{2}} \theta_1(2z)} [(z + \eta n - \eta k)_i e^{\eta \theta_{z_2}} - (z - \eta n + \eta k)_i e^{-\eta \theta_{z_2}}] \cdot \frac{1}{(z)_i^4}; \quad i = 3, 4 \tag{4.17}
\]

where $c = R^{-1}(\tau) e^{\pi i \eta}$. For illustration we write this explicitly for $n = 1, 2$ corresponding to spin $\ell = 0$ and $\ell = \frac{1}{2}$ respectively.

\[
W(\eta) = c \cdot \frac{1}{\theta_1(2z)} \left[ e^{\eta \theta_{z_2}} - e^{-\eta \theta_{z_2}} \right],
\]

\[
W(2\eta) = c^2 \cdot \frac{1}{\theta_1(2z)} \left[ e^{\eta \theta_{z_2}} - e^{-\eta \theta_{z_2}} \right] \cdot \frac{1}{\theta_1(2z)} \left[ (z - \eta)_i e^{\eta \theta_{z_2}} - (z + \eta)_i e^{-\eta \theta_{z_2}} \right].
\]
Expanding the latter formula we obtain a sum of four finite difference operators which can be simplified further by means of (A.7)

$$W(2\eta) = \frac{c^2}{\theta_1(2z - 2\eta)\theta_1(2z)\theta_1(2z + 2\eta)} \left[ \theta_1(2z - 2\eta)e^{2\eta\partial_z} - \frac{\theta_1(4\eta)}{\theta_1(2\eta)}\theta_1(2z) + \theta_1(2z + 2\eta)e^{-2\eta\partial_z} \right].$$

Similarly using (A.7) it is rather straightforward to rewrite (4.17) in the form of a sum

$$W(\eta n) = c^n \sum_{k=0}^{n} (-k)^k \binom{n}{k} \frac{\theta_1(2z + 2\eta n - 4\eta k)}{\prod_{j=0}^{n} \theta_1(2z - 2\eta k + 2\eta j)} e^{(n-2k)\eta\partial_z}.$$  

The latter expression for the intertwining operator at (half)-integer spin appeared first in [39] and has been derived directly from the integral operator representation in [30].

Let us note that this factorized representation (4.17) for $W(\eta n)$ is rather special form since all factors contain the third or the fourth Jacobi theta function simultaneously, and that a set of similar ones follows from the recurrence relations (4.7), (4.8). Their linear combinations can be arranged in compact formulae taking into account (A.6)

$$\prod_{k=0}^{n-1} \theta_1(z \mp a_k) \cdot W(\eta n) = c^n \prod_{k=0}^{n-1} \frac{1}{\theta_1(2z)} \left[ \theta_1(z - \eta k \mp a_k) e^{\eta\partial_z} - \theta_1(z + \eta k \mp a_k) e^{-\eta\partial_z} \right],$$

$$W(\eta n) \cdot \prod_{k=0}^{n-1} \theta_1(z \mp a_k) = c^n \prod_{k=0}^{n-1} \frac{1}{\theta_1(2z)} \left[ \theta_1(z + \eta n - \eta k \mp a_k) e^{\eta\partial_z} - \theta_1(z - \eta n + \eta k \mp a_k) e^{-\eta\partial_z} \right]$$

where $a_0, \ldots, a_{n-1}$ denote arbitrary parameters.

The irreducible representation of the Sklyanin algebra at (half)-integer spin $\ell = \frac{n-1}{2}$ is $n$-dimensional and it can be realised in the space $\Theta^+_{2n-2}$ of even theta functions of order $2n-2$. By the factorized representation of $W(\eta n)$ we see that its action annihilates this irreducible representation space. Let us demonstrate this fact by means of the recurrence relations. The set of $n$ functions

$$(z^k \frac{1}{3} (z^4)^{n-k}) ; \ k = 0, 1, \ldots, n - 1$$

form a basis in the space $\Theta^+_{2n-2}$. Then applying $n - 1$ times (4.8) and taking into account that

$$W(\eta n) \cdot 1 = \frac{c}{\theta_1(2z)} \left[ e^{\eta\partial_z} - e^{-\eta\partial_z} \right] \cdot 1 = 0$$

we obtain that $W(\eta n) \cdot (z^k \frac{1}{3} (z^4)^{n-k}) = 0$.

### 4.4 Elliptic Baxter equation

Now we have at our disposal all necessary identities to prove the Baxter equation. The calculation in this Subsection repeats step by step the one of the Subsection 3.3 devoted to the $q$-deformed case. Our aim is to obtain the corresponding local relation underlying Baxter equation. We compute the matrix elements of (cf. (2.26))

$$\text{R}^2(u) \sigma_3 L(u_1, u_2) \left( \frac{1}{(z_1)_4} - (z_1)_3 \right)$$

starting from the defining relation for $\text{R}^2$ (2.15) where as before $\text{R}^2(u) = \text{R}^2(u_1, u_2 | 0)$. Let us remind that in all formulae we multiply the $L$-operator (4.11) by the Pauli matrix $\sigma_3$ on the left:
$\Lambda(u) \rightarrow \sigma_3 \Lambda(u)$. We start from (2.15) for elliptic deformation case, use the factorized form of the L-operator (4.1) and take into account the commutativity of $R^2$ with $z_2$ to obtain

$$R^2(u_1, u_2|v_2) \sigma_3 \Lambda_1(u_1, u_2) = \sigma_3 \Lambda_1(u_1, v_2) \sigma_3 \Lambda(z_2 \mp u_2) \begin{pmatrix} e^{\eta \partial_{z_2}} R^2 e^{-\eta \partial_{z_2}} & 0 \\ 0 & e^{-\eta \partial_{z_2}} R^2 e^{\eta \partial_{z_2}} \end{pmatrix}$$

or in the other form due to (4.4)

$$R^2(u_1, u_2|v_2) \sigma_3 \Lambda_1(u_1, u_2) = \sigma_3 \Lambda_1(u_1, v_2) = \sigma_3 \Lambda(z_2 \mp u_2) \times$$

$$\begin{pmatrix} e^{\eta \partial_{z_2}} R^2(u_1, u_2|v_2) e^{-\eta \partial_{z_2}} \\ 0 \\ e^{-\eta \partial_{z_2}} R^2(u_1, u_2|v_2) e^{\eta \partial_{z_2}} \end{pmatrix} N(z_2 \mp v_2) \frac{1}{2 \theta_1(2z_2) \theta_1(2v_2)}.$$ (4.20)

These are the two main relations in the following calculation.

We start with the matrix element below diagonal of (4.18). Taking into account (2.1),

$$(z_1)_4, -(z_1)_3) \Lambda(u_1, 0) \sim ((z_1)_4, -(z_1)_3) \Lambda(z_1) = (0, 0),$$

and (4.19) at $v_2 = 0$ we conclude that it is equal to zero in agreement with general statement (2.26).

Further let us consider the first diagonal element in (4.18): $(1, 0) R^2(u) \sigma_3 \Lambda(u_1, u_2) \frac{(z_2)_3}{(z_2)_4}$.

From (4.19) at $v_2 = 0$ we see that it is equal to

$$(1, 0) \Lambda_1(u_1, 0) \sigma_3 \Lambda(z_2 \mp u_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\eta \partial_{z_2}} R^2(u) e^{-\eta \partial_{z_2}}.$$ 

Then we take into account that (2.17) $e^{\eta \partial_{z_2}} R^2(u) e^{-\eta \partial_{z_2}} = S^3(u_1) e^{\eta \partial_{z_2}} S^2(u_2) e^{-\eta \partial_{z_2}} S^3(u_2 - u_1)$ and move $S^3(u_1)$ to the left by means of intertwining relation (2.10) $S^3(u_1) \Lambda(0, u_1) = \Lambda(u_1, 0) S^3(u_1)$ and obtain

$$S^3(u_1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{\theta_1(2z_1)} M(z_1 \mp u_1) \begin{pmatrix} e^{\eta \partial_{z_1}} & 0 \\ 0 & e^{-\eta \partial_{z_1}} \end{pmatrix} N(z_1) \sigma_3 \frac{(z_2 - u_2)_3}{(z_2 - u_2)_4} e^{\eta \partial_{z_2}} S^2(u_2) e^{-\eta \partial_{z_2}} S^3(u_2 - u_1).$$

The underlined matrix in the previous formula is equal to $2 \theta_1(z_1 + z_2 - u_2) \theta_1(z_1 - z_2 + u_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in view of (2.12) and (A.6). Thus using the recurrence relation (4.13) we find that the wanted matrix element is equal to

$$-2 R^{-2}(\tau) e^{-\frac{\pi i \tau}{2}} S^3(u_1) \frac{1}{\theta_1(2z_1)} \left[(z_1 - u_1)_3 e^{\eta \partial_{z_1}} - (z_1 + u_1)_3 e^{-\eta \partial_{z_1}}\right] S^2(u_2 + \eta) S^3(u_2 - u_1).$$

Finally using the recurrence relation (1.11) for the intertwining operator $S^3$ in the underlined factor we have (2.14)

$$-2 R^{-2}(\tau) e^{-\pi i \tau - \frac{\pi i}{2}} (z_1)_3 R^2_{12}(u + 2\eta).$$

Consider the second diagonal matrix element in (4.18): $((z_1)_4, -(z_1)_3) R^2(u) \sigma_3 \Lambda(u_1, u_2) \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

Due to (4.20) it is equal to

$$\left((z_1)_4, -(z_1)_3\right) \Lambda_1(u_1, u_2) \sigma_3 \Lambda(z_2 \mp u_2) \times$$

$$\begin{pmatrix} e^{\eta \partial_{z_2}} R^2(u) e^{-\eta \partial_{z_2}} & 0 \\ 0 & e^{-\eta \partial_{z_2}} R^2(u) e^{\eta \partial_{z_2}} \end{pmatrix} \frac{(z_2 + v_2)_3}{(z_2 - v_2)_3} \frac{1}{2 \theta_1(2z_2) \theta_1(2v_2)}.$$
where we have to take carefully the limit \( v_2 = 0 \). To do this we notice that (4.2), (A.6)

\[
\frac{1}{\theta_1(2v_2)} ((z_1)_4, (z_1)_3) M(z_1 \mp v_2) = \frac{2}{\theta_1(2v_2)} \left( \frac{\theta_1(2z_1 - v_2) \theta_1(v_2)}{\theta_1(2z_1 + v_2) \theta_1(v_2)} \right) \to \theta_1(2z_1) \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \text{ at } v_2 \to 0.
\]

Thus the wanted matrix element takes the form (4.1)

\[
\frac{1}{2} \left( \begin{array}{cc} 1 & 0 \\ 1 & e^{-\eta \theta_1} \end{array} \right) N(z_1 \mp u_1) S^3(u_1) \sigma_3 M(z_2 \mp u_2) \left( \begin{array}{cc} e^{\eta \theta_1} S^2(u_2) e^{-\eta \theta_1} & (z_2)_3 \\ e^{\eta \theta_1} S^2(u_2) e^{-\eta \theta_1} & \theta_1(2z_2) \end{array} \right) S^3(u_2 - u_1).
\]

Further we take into account the recurrence relations (4.14), (4.15) and (4.2), (A.6) and find that the underlined matrix can be written as follows

\[
-R^2(\tau) e^{\eta \theta_1} S^2(u_2 - \eta) \frac{1}{2} N(z_2 \mp u_2) \left( \begin{array}{c} (z_1)_3 \\ (z_1)_4 \end{array} \right).
\]

Since the matrices \( M \) and \( N \) are inverse to each other (4.14): \( M(z_2 \mp u_2) N(z_2 \mp u_2) = 2 \theta_1(2u_2) \theta_1(2z_2) \mathbb{1} \), the wanted matrix element takes the form

\[
-\frac{1}{2} R^2(\tau) e^{\eta \theta_1} \theta_1(2u_2) \theta_1(2z_2) \left[ \begin{array}{cc} 1 & 0 \\ 1 & e^{-\eta \theta_1} \end{array} \right] N(z_1 \mp u_1) S^3(u_1) \left( \begin{array}{c} (z_1)_3 \\ -(z_1)_4 \end{array} \right) S^2(u_2 - \eta) S^3(u_2 - u_1).
\]

Using the recurrence relation (4.11) we finally obtain the wanted matrix element

\[
-2R^{-1}(\tau) e^{\eta \theta_1} \theta_1(2u_1) \theta_1(2u_2) \left( \begin{array}{c} (z_1)_4 \\ (z_1)_3 \end{array} \right) R^2(u - 2\eta),
\]

that completes the calculation of (4.18).

Inserting the permutation operator \( P_{12} \) and implementing the similarity transformation in (4.18) we obtain that (cf. (2.20))

\[
Z_2 \mathbb{R}^2_{12}(u) \sigma_3 L_1(u_1, u_2) Z_2^{-1} = \begin{pmatrix} 2 \kappa^{-1} R^2_1(u + 2\eta) & \cdots \\ 0 & 2 \kappa \theta_1(2u_1) \theta_1(2u_2) \mathbb{R}^2_{12}(u - 2\eta) \end{pmatrix}
\]

(4.21)

where as usual \( \mathbb{R}^2_{12} = P_{12} \mathbb{R}^2_{12} \), the constant is \( \kappa = -R(\tau)e^{\eta \theta_1} \frac{\eta \theta_1}{\tau} \) and \( Z_2 \) stands for \( Z_2 = \begin{pmatrix} (z_2)_3 & 0 \\ (z_2)_4 & -(z_2)_3 \end{pmatrix} \). According to the final step explained in Section 2 we obtain immediately the Baxter equation (cf. (2.24))

\[
t(u) Q_2(u) = 2^N \kappa^{-N} Q_2(u + 2\eta) + 2^N \kappa^N \Delta^N(u_1, u_2) Q_2(u - 2\eta)
\]

(4.22)

where we used the notation \( \Delta(u_1, u_2) = \theta_1(2u_1) \theta_1(2u_2) \) for a symmetric function. Here the transfer matrix \( t(u) \) (2.19) is constructed out of \( \sigma_3 L(u) \) according to our adopted convention.

### 4.5 Explicit action of the elliptic \( Q \)-operator

In the \( q \)-deformation case we have found in Subsection 3.4 that the operator \( \mathbb{R}^2_{12} \) acts in a simple way on a certain function of the variable \( z_1 \) and of the auxiliary parameter \( \lambda \) (3.22). In Appendix D we also show that this formula can be obtained at least formally using the Coxeter relation (2.12) only. Now we are going to deduce the analogous result in the elliptic case. Using the formulation with the intertwining operator \( W \) being an integral operator (1.6) we have a solid base to deduce an elliptic analog of the formula (3.24) from Coxeter relation because the latter is equivalent to the elliptic beta integral evaluation formula.
Thus we start from the Coxeter relation (2.12) \(S^2(u_2-u_1)S^3(u_2)S^2(u_1)S^3(u_1-u_2) = S^3(u_1)S^2(u_2)\) and apply both sides to \(\delta(z_1-z_3)\). Using the integral operator for \(S^1, S^3\) (4.6) we assume the position variables taking real values.

Since (4.3), (4.6)

\[S^3(a) \cdot \delta(z_1-z_3) = \frac{C e^{-\frac{\pi i}{3}z_1^2 + \frac{\pi i}{3}z_2^2}}{\Gamma(-2a)\Gamma(\mp 2z_3)} \Gamma(\mp z_1 \mp z_3 - a) ; \quad S^2(a) \delta(z_1-z_3) = S^2(a)|_{z_1 \rightarrow z_3} \delta(z_1-z_3)\]

and \(R^2(u) = S^2(u_2-u_1)S^3(u_2)S^2(u_1)\) (2.17) we obtain immediately the wanted local formula

\[R^2_{12}(u) \cdot e^{-\frac{\pi i}{3}z_1^2 + \frac{\pi i}{3}z_2^2} = e \cdot e^{-\frac{\pi i}{3}z_1^2} \Gamma(\mp z_1 \mp z_3 - \frac{u}{2} + \eta \ell + \eta) \Gamma(\mp z_2 \mp z_3 + \frac{u}{2} + \eta \ell + \eta + \frac{\tau}{2})\]

where we take into account definition of spectral parameters (4.3) and denote \(e = \frac{\Gamma(4\eta \ell + 2\eta)}{\Gamma(u + 2\eta \ell + 2\eta)}\).

We proceed to the action of \(Q_2\). For convenience we renormalize \(Q_2\): \(Q_2(u) \rightarrow e^{-N} \cdot Q(u)\), such that the Baxter equation takes the form

\[t(u) Q(u) = \Delta_+(u) Q(u + 2\eta) + \Delta_-(u) Q(u - 2\eta)\]

where \(\Delta_{\pm}(u) = 2^N e^{-\pi i N \theta_1 N (\pi i u - 2\pi i n \ell + \pi i u / 2)}\). Now we see that due to (2.33) the Baxter operator \(Q(u)\) acts on the generating function depending on arbitrary \(\lambda_1, \cdots, \lambda_N\) as follows

\[Q(u) \cdot \prod_{i=1}^{N} e^{-\frac{\pi i}{3}z_i^2} \Gamma(\mp z_i + \lambda_i + 2\eta \ell + \eta) = \prod_{i=1}^{N} e^{-\frac{\pi i}{3}z_i^2} \Gamma(\mp z_i \mp \lambda_i + \frac{u}{2} + \eta \ell + \eta + \frac{\tau}{2}) \Gamma(\mp z_i + \lambda_i + 1 - \frac{u}{2} + \eta \ell + \eta).\]

A function similar to \(\Gamma(\mp z \mp \lambda + 2\eta \ell + \eta)\) was used in [36] as a generating function for the infinite-dimensional module of the Sklyanin algebra.

5 Summary

In our approach the case of spin chains where the one-site states form irreducible infinite-dimensional representations is the basic one for the construction. This generic case with \(\ell \neq \frac{n}{2}, n = 0, 1, 2, \cdots\) is addressed here where the \(s\ell_2\) symmetry is deformed in trigonometric or elliptical way. In the preceding paper concerning the undeformed symmetry also the case of finite dimensional representations at the chain sites has been treated by investigating the limits where \(2\ell\) approaches nonnegative integer. The presented here discussion of the deformed cases does not cover this finite-dimensional representation case. The expressions for \(R^1, R^2\) in terms of \(q\)-Gamma functions obtained in the trigonometric case (Appendix B) allow to study the integer limit in analogy to the undeformed case. Missing the analogous form for \(R^1, R^2\) in the elliptic case we face an obstacle to proceed here by analogy. Nevertheless in Subsection 4.3 we have given a detailed discussion of the intertwining operator of the Sklyanin algebra representations at integer and half-integer spin. In the trigonometric case we have realized the representations on the space of polynomials and the corresponding Yang-Baxter operators have been represented as functions of Weyl pairs. In the elliptic we have formulated operators as integral ones.
The idea of looking for such building blocks and relations in the undeformed case which have immediate counterparts in the deformed cases provides the guideline through the increasing complexity. The factors of the general Yang-Baxter operator related to the elementary transpositions of representation parameters turn out to be the appropriate building blocks for this purpose.

In this way we have extended the parallel treatment of quantum spin chains with symmetry bases on the algebra \( \mathfrak{sl}_2 \) without deformation, with trigonometric and elliptic deformations beyond the local (one-site) operators considered earlier to the global chain operators. Besides of the well-known transfer matrix further generating functions of conserved charges have been considered, in particular the general transfer matrix \( T_s(u) \) and two Baxter operators, \( Q_1, Q_2 \). The scheme of their construction and the proof of the Baxter equation, which was applied in the preceding paper to the undeformed case, has been shown here to work in the deformed cases as well. The construction results in explicit expressions, in particular the explicit form of the action of \( Q_2 \) on a generating function of spin chain states has been provided for all cases.

Note that in the elliptic case the \( Q \)-operator was constructed by A. Zabrodin in [40] using a different method. It will be very interesting to relate both approaches.

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Appendices

A Special functions for trigonometric and elliptic deformations

In this Appendix we collect some useful formulae concerning the special functions which we need in our calculations. The standard \( q \)-products involving a complex number \( q (|q| < 1) \) are defined as

\[
(x; q^2) = \prod_{i=0}^{\infty} (1 - x q^{2i}), \quad (x; q^2)_k = \prod_{i=0}^{k-1} (1 - x q^{2i}).
\]  

(A.1)

The \( q \)-binomial formula at \( |z| < 1 \)

\[
\sum_{n \geq 0} \frac{(a; q^2)_n}{(q^2; q^2)_n} z^n = \frac{(az; q^2)}{(z; q^2)}
\]  

(A.2)

produces the expansions

\[
(x; q^2) = \sum_{k \geq 0} \frac{(-)^k q^{k(k-1)}}{(q^2; q^2)_k} x^k, \quad (x; q^2)^{-1} = \sum_{k \geq 0} \frac{x^k}{(q^2; q^2)_k}.
\]  

(A.3)

If operators \( u \) and \( v \) form a Weyl pair: \( uv = q^2 vu \) then \([43,45]\)

\[
(u; q^2)(v; q^2) = (u + v; q^2),
\]  

(A.4)

\[
(v; q^2)(u; q^2) = (u; q^2)(-v u; q^2)(v; q^2).
\]  

(A.5)
We use standard definitions of Jacobi theta functions \( \theta_n(z|\tau) \) \( (n=1,\cdots,4) \) (see e.g. \[35\][36]) and the shorthand notations \( \theta_1(z) = \theta_1(z|\tau) \), \( \theta_4(z) = \theta_4(z|\tau) \) for theta functions with quasi period \( \tau \) and the shorthand notations \( (z)_3 = \theta_3(z|\frac{\tau}{2}) \), \( (z)_4 = \theta_4(z|\frac{\tau}{2}) \) for theta functions with quasi period \( \frac{\tau}{2} \). Recall that the first theta function is odd \( \theta_1(z) = -\theta_1(z) \) and the other three Jacobi theta functions are even. All Jacobi theta functions are connected by argument shifts. For example

\[
\theta_1(z + \frac{\tau}{2}) = i e^{-\pi i z} e^{-\frac{\pi i \tau}{2}} \theta_4(z) .
\]

Theta functions with quasi periods \( \tau \) and \( \frac{\tau}{2} \) are related by the bilinear relations

\[
2 \theta_1(x \mp y) = (x)_4(y)_3 - (y)_4(x)_3 , \quad 2 \theta_4(x \mp y) = (x)_4(y)_3 + (y)_4(x)_3 ,
\]

where we adopt the notation \( \theta_\alpha(x \mp y) = \theta_\alpha(x + y) \theta_\alpha(x - y) \). As an immediate consequence of \[A.6\] we obtain the formulae

\[
(y)_i \theta_1(x \mp z) - (x)_i \theta_1(y \mp z) = (z)_i \theta_1(x \mp y) \quad \text{where} \quad i = 3,4.
\]

The elliptic gamma function is defined by the double-infinite product \[32\][47][48]

\[
\Gamma(z|\tau,2\eta) = \prod_{n,m=0}^{\infty} \frac{1 - e^{2\pi i (\tau(n+1)+2\eta(m+1)-z)}}{1 - e^{2\pi i (\tau n+2\eta m+z)}} \tag{A.8}
\]

for \( \text{Im} \tau > 0, \text{Im} \eta > 0 \). In our study we only need its transformation property under the shift of the argument

\[
\Gamma(z + 2\eta) = R(\tau) e^{\pi i z} \theta_1(z) \Gamma(z) , \tag{A.9}
\]

where \( R(\tau) = -i e^{-\frac{\pi i \tau}{2}} (e^{2\pi i \tau};e^{2\pi i \tau})^{-1} \).

\[ \text{B} \quad \text{The operators} \ R^1 \text{ and} \ R^2 \text{ in the trigonometric case} \]

Here we shall establish relations between several explicit expressions for operators \( R^1 \) and \( R^2 \) which respect relations \[2.14\] and \[2.15\], respectively. In \[2.16\] and \[2.17\] we have cited several expressions for them. Let us consider \( R^2 \) and take into account \[3.3\], \[3.4\]

\[
S^2(a)S^3(a + b)S^2(b) = c \cdot \frac{(u_3;q^2)(v_1;q^2)(v_2;q^2)}{(u_1;q^2)(v_1^2;q^2)} q^{-(a+b)z_1 \theta_1} (v_2;q^2)^{-1} \frac{(u_2;q^2)}{(u_4;q^2)} \tag{B.1}
\]

where \( c \) is the constant \( c = q^2 \frac{a^2-b^2}{\tau} \) and \( u_i \) and \( v_j \) form Weyl pairs: \( u_i, v_j = q^2 v_j u_i \),

\[
u_1 = \frac{z_2}{z_1} q^{1+a} ; \quad u_2 = \frac{z_2}{z_1} q^{1-b} ; \quad u_3 = \frac{z_2}{z_1} q^{1-a} ; \quad u_4 = \frac{z_2}{z_1} q^{1+b} ; \quad v_1 = q^{2z_1 \theta_1 + 2 - 2a} ; \quad v_2 = q^{2z_1 \theta_1 + 2 + 2b}.
\]

We are going to rewrite \[B.1\] in several equivalent forms. By means of the pentagon relation \[A.5\]

\[
(u;q^2)^{-1}(v;q^2) = (v - u) (u;q^2)(v;q^2)^{-1}
\]

we have

\[
B.1 = c \cdot (u_3;q^2)(v_1 - v_1 u_1;q^2) \frac{(u_1;q^2)^{-1} q^{-(a+b)z_1 \theta_1}}{(u_2;q^2)} (u_2 - v_2 u_2;q^2)^{-1} (u_4;q^2)^{-1} .
\]

Further we note that underlined expression in the previous formula is equal to \( q^{-(a+b)z_1 \theta_1} \) and use Schützenberger formula \[A.4\] to rewrite it as follows

\[
B.1 = c \cdot (u_3 + v_1 - v_1 u_1;q^2) q^{-(a+b)z_1 \theta_1} (u_4 + v_2 - v_2 u_2;q^2)^{-1} . \tag{B.2}
\]
Thus we have transformed the original expression $[B.1]$ containing $6$ $q$-exponents to the expression with $2$ $q$-exponents.

Further we shall obtain another expression with $4$ $q$-exponents. We use the pentagon formula $[A.5]$ $(\mathbf{v} - \mathbf{v} \mathbf{u}; q^2) = (\mathbf{u}; q^2)^{-1} (\mathbf{v}; q^2) (\mathbf{u}; q^2)$ to transform the previous formula as follows

$$c \cdot (\mathbf{u}_1 - \mathbf{v}_1^{-1} \mathbf{u}_3; q^2)^{-1} (\mathbf{v}_1; q^2) (\mathbf{u}_1 - \mathbf{v}_1^{-1} \mathbf{u}_3; q^2) q^{-(a+b)z_1 \partial_1} (\mathbf{u}_2 - \mathbf{v}_2^{-1} \mathbf{u}_4; q^2) (\mathbf{v}_2; q^2)^{-1} (\mathbf{u}_2 - \mathbf{v}_2^{-1} \mathbf{u}_4; q^2)$$

and take into account that the underlined expression is equal to $q^{-(a+b)z_1 \partial_1}$ to obtain

$$c \cdot (\mathbf{u}_1 - \mathbf{v}_1^{-1} \mathbf{u}_3; q^2)^{-1} (\mathbf{v}_1; q^2) q^{-(a+b)z_1 \partial_1} (\mathbf{v}_2; q^2)^{-1} (\mathbf{u}_2 - \mathbf{v}_2^{-1} \mathbf{u}_4; q^2). \quad [B.3]$$

As the result we have found that $[B.1]$ can be rewritten in two equivalent forms $[B.2]$ and $[B.3]$ which appeared in [42]

- $S^2(a) S^3(a + b) S^2(b) = c \cdot (U(a); q^2) q^{-(a+b)z_1 \partial_1} (U(-b); q^2)^{-1}, \quad [B.4]$ where

  \[ U(a) = \frac{z_2}{z_1} q^{1-a} + q^2 z_1 \partial_1 + 2 - 2a - \frac{z_2}{z_1} q^2 z_1 \partial_1 + 1 - a. \]

- $S^2(a) S^3(a + b) S^2(b) = c \cdot (q^{1+a} \mathbf{u}; q^2)^{-1} q^{-z_2 z_1 \partial_1} (q^2 - 2a \mathbf{u} \mathbf{v}; q^2)^{-1} (q^2 - 2b \mathbf{u} \mathbf{v}; q^2) \left( q^{-b} \mathbf{u}; q^2 \right), \quad [B.5]$ where

  \[ \mathbf{u} = \frac{z_2}{z_1} (1 - q^{-2z_1 \partial_1}) \quad ; \quad \mathbf{v} = q^{2z_1 \partial_1}. \]

Now we turn to the operator $R^1 [2.16]$. At first let us mention that we cannot rewrite it in the form like $[B.1]$, because the multipliers $z_2^a$ and $z_1^b$ from $S^2(a)$ and $S^2(b)$ $[3.4]$ do not compensate the multiplier $z_2^{-a-b}$ from $S^1(a + b)$ $[3.3]$. To overcome this difficulty we remind that the defining relation (2.9) does not fix uniquely the operator $S^2$: we can multiply $S^2$ by an arbitrary function $\varphi$ which respects $[3.6]$. Now we choose $S^2$ $[3.7]$ instead of $S^2$ $[3.4]$. Since $S^2 \to S^2$ and $S^3 \to S^1$ amount to the change $z_1 \leftrightarrow z_2$ the above calculation for $R^2$ is suitable as well for $R^1$ after the indicated change.

Let us remark that in the original form of operators $S^1, S^2, S^3$ they respect Coxeter relations $[2.11], [2.12]$ and are sufficient to build the general $R$-operator. However $[B.5]$ is crucial when we restrict the general $R$-operator to the invariant subspace in order to reproduce the $L$-operator. Thus if we dealt with the original set of elementary operators $S^1, S^2, S^3$ only we would not be able to reproduce the standard $L$-operator $[3.1].$

C From pentagon to Coxeter relations

Here we shall prove the Coxeter relations $[2.11], [2.12]$ for the elementary intertwining operators $S^1, S^2, S^3$ in the case of $q$-deformation using only the pentagon relation $[A.5]$ for a Weyl pair

$$\mathbf{u} = \frac{z_2}{z_1} q \quad ; \quad \mathbf{v} = q^{2z_1 \partial_1 + 2} \quad ; \quad \mathbf{u} \mathbf{v} = q^{2} \mathbf{u} \mathbf{v}.$$ 

We start with the right hand side of $[2.12]$, take into account $[3.4], [3.3]$ and $c = q^{\frac{a^2}{2} - \frac{b^2}{2}},$

$$S^2(b) S^3(a + b) S^2(a) = c \cdot \frac{(uq^{-b}; q^2)}{(uq^b; q^2)} (\mathbf{v} q^{-2a}; q^2)^{-1} \cdot q^{-(a+b)z_1 \partial_1} \cdot (\mathbf{v} q^{-2b}; q^2) \frac{(uq^{-a}; q^2)}{(uq^a; q^2)}. \quad [C.1]$$
Then we apply twice the pentagon relation (A.5) in the left hand side of the previous formula

\[(uq^{-b}; q^2)^2 (uq^b; q^2)^{-1} (vq^{2a}; q^2)^{-1} = (uq^{-b}; q^2) (vq^{2a}; q^2)^{-1} (-v \ u \ q^{2a+b}; q^2)^{-1} (uq^b; q^2)^{-1} =
\[(vq^{2a}; q^2)^{-1} (uq^{-b}; q^2) (-v \ u \ q^{2a+b}; q^2)^{-1} (uq^b; q^2)^{-1}\]

(C.2)

and also in the right hand side

\[(vq^{-2b}; q^2) (uq^{-a}; q^2) (uq^a; q^2)^{-1} = (uq^{-a}; q^2) (-v \ u \ q^{-a-2b}; q^2) (vq^{-2b}; q^2) (uq^a; q^2)^{-1} =
\[(uq^{-a}; q^2) (-v \ u \ q^{-a-2b}; q^2) (-v \ u \ q^{-2b}; q^2)^{-1} (uq^a; q^2)^{-1} (vq^{-2b}; q^2).\]

(C.3)

Then returning to (C.1) we note that the double underlined factors in (C.2) and (C.3) cancel each other. Thus we rewrite (C.1) as

\[c \cdot (vq^{2a}; q^2)^{-1} (uq^{-b}; q^2) (-v \ u \ q^{-b}; q^2) \cdot q^{-(a+b)z_1 \partial_1} \cdot (-v \ u \ q^a; q^2)^{-1} (uq^a; q^2)^{-1} (vq^{-2b}; q^2) =
\]

and apply the pentagon relation (A.5) to obtain that the left hand side of (2.12) is

\[= c \cdot (vq^{2a}; q^2)^{-1} (v; q^2) (uq^{-b}; q^2) \cdot q^{-2(a+b)z_1 \partial_1} \cdot (uq^a; q^2)^{-1} (v; q^2)^{-1} (vq^{-2b}; q^2) = S^3(a) S^2(a+b) S^3(b).\]

Similarly we prove the second Coxeter relation (2.11) using a Weyl pair

\[\overline{u} = q^{2z_2 \partial_2 + 2} ; \ \overline{v} = \frac{z_2}{z_1} q ; \ \overline{u} \ \overline{v} = q^2 \ \overline{v} \ \overline{u}.\]

In view of (3.3), (3.4) the left hand side of (2.11) has the explicit form \((c = q^{(a+b)^2})\)

\[S^1(a) S^2(a+b) S^1(b) = c z_1^{a+b} \cdot \frac{(\overline{u}; q^2)}{(uq^{2a}; q^2)} (vq^b; q^2)^{-1} \cdot \frac{1}{z_2^{a+b}} q^{-(a+b)z_2 \partial_2} \cdot (\overline{v} q^{-a}; q^2)^{-1} (uq^{-2b}; q^2)^{-1} (u; q^2).\]

(C.4)

As before applying the pentagon relation twice in the left hand side and in the right hand side of the latter formula we have

\[(\overline{u}; q^2) (\overline{u} q^{2a}; q^2)^{-1} (vq^b; q^2)^{-1} = (vq^b; q^2)^{-1} (\overline{u}; q^2) (-v \ u \ q^{2a+b}; q^2)^{-1} (\overline{u} q^{2a}; q^2)^{-1}\]

\[(\overline{v} q^{-a}; q^2) (^{2b}; q^2) (\overline{u} q^{-2b}; q^2) (\overline{u}; q^2)^{-1} = (\overline{u} q^{-2b}; q^2) (-v \ u \ q^{-a-2b}; q^2) (-v \ u \ q^{-a}; q^2)^{-1} (\overline{u} q^{-2a}; q^2)^{-1} (\overline{u} q^{-a}; q^2)^{-1},\]

where double underlined factors cancel each other when we substitute both above formulæ in (C.4). Thus (C.4) acquires the form

\[c z_1^{a+b} \cdot (vq^b; q^2)^{-1} (\overline{u}; q^2) (-v \ u \ q^{-b}; q^2) \cdot \frac{1}{z_2^{a+b}} q^{-(a+b)z_2 \partial_2} \cdot (\overline{v} \ u \ q^a; q^2)^{-1} (\overline{u}; q^2)^{-1} (vq^{-a}; q^2),\]

and using the pentagon relation (A.5) two more times we have finally

\[c z_1^{a+b} (vq^b; q^2)^{-1} (\overline{v} q^{-b}; q^2) (\overline{u}; q^2) \frac{1}{z_2^{a+b}} q^{-(a+b)z_2 \partial_2} (\overline{u}; q^2)^{-1} (vq^a; q^2)^{-1} (vq^{-a}; q^2) = S^2(b) S^1(a+b) S^1(a).\]

D  L-operator recovered from the general R-operator in the trigonometric case

The operator \(R_{12}(u \ell, s)\) acts in the tensor product \(V_\ell \otimes V_s \cong \mathbb{C}[z_1] \otimes \mathbb{C}[z_2]\) of two infinite-dimensional spaces. At (half)-integer \(s\) the space \(V_s\) contains an invariant finite-dimensional subspace \(\mathbb{C}^{2s+1}\).
Now we choose \( s = \frac{1}{2} \) and restrict the general R-operator to the subspace \( \mathcal{V}_\ell \otimes \mathbb{C}^2 \) of functions of the form
\[
\Psi(z_1, z_2) = \phi(z_1) + \psi(z_1) z_2 ,
\]
where \( \phi \) and \( \psi \) are polynomials. For technical reasons it will be convenient to start with \( 2s = 1 - \varepsilon \) and to consider the limit \( \varepsilon \to 0 \) in the second factorized form (2.18) of the R-operator
\[
R(u_1, u_2 | v_1, v_2) = R^2(v_1, u_2 | v_2) R^1(u_1 | v_1, v_2) ,
\]
where
\[
u_1 = u - \ell - 1 \; ; \; v_2 = u + \ell \; ; \; v_1 = -1 - \frac{1}{2} + \frac{\varepsilon}{2} \; ; \; v_2 = \frac{1}{2} - \varepsilon .
\]
The operators \( R^1 \) and \( R^2 \) are taken in the form (3.5) with four \( q \)-exponential factors. We start with
\[
R^1(u_1 | v_1, v_2) = (q^{3-\varepsilon} u; q^2)^{-1} q^{(v_1-u_1)z_2} \partial_{z_2} (q^{-2+2\varepsilon} v; q^2) (q^{v_2-u_1+1} u; q^2) ,
\]
where
\[
u = \frac{z_1}{z_2} (1 - q^{-2z_2} \partial_{z_2}) \; ; \; \nu = q^{2z_2} \partial_{z_2} ,
\]
and act on the function (D.1). It is clear that due to the special dependence of the operator \( R^1 \) on the variable \( z_1 \) for our purposes it will be sufficient to apply the operator to the monomials \( 1 \) and \( z_2 \). Further we will need the formulae (A.3)
\[
(x; q^2) = 1 - \frac{x}{1 - q^2} + O(x^2) \; ; \; (x; q^2)^{-1} = 1 + \frac{x}{1 - q^2} + O(x^2) .
\]
Thus due to \( u \cdot 1 = 0 \) we have \( R^1(u_1 | v_1, v_2) \cdot 1 = \frac{(q^{-2+2\varepsilon} u; q^2)}{(q^{2u_1-2v_2+2} q^2)} \). Similarly due to \( u^2 \cdot z_2 = 0 \) we have
\[
(q^{v_2-u_1+1} u; q^2) \cdot z_2 = z_2 + q^{v_2-u_1-1} z_1 \; ; \; (q^{3-\varepsilon} u; q^2)^{-1} \cdot z_2 = z_2 - q^{1-\varepsilon} z_1
\]
and consequently after some trivial algebra we obtain
\[
R^1(u_1 | v_1, v_2) \cdot z_2 = \frac{(q^{2\varepsilon} u; q^2)}{(q^{2u_1-2v_2+2}; q^2)} [q^{u_1-v_2+1} (1 - q^{2v_2-2u_1-4}) \cdot z_1 + q^{u_1-v_2} (1 - q^{2u_1-2v_2+2}) \cdot z_2] + O(\varepsilon^2) .
\]
Thus we see that \( R^1 \cdot 1 = O(\varepsilon) \) and \( R^1 \cdot z_2 = O(\varepsilon) \). It is due to the factor \( (q^{-2+2\varepsilon} v; q^2) \) in (D.4). Consequently, to obtain \( R \cdot \Psi(z_1, z_2) \) at \( \varepsilon = 0 \) we only need to extract the simple poles contributions from \( R^2 \).

Then we consider the second factor in (D.2):
\[
R^2(v_1, u_2 | v_2) = (q^{v_2-v_1+1} u; q^2)^{-1} q^{(v_2-u_2)z_1 \partial_{z_1}} \frac{(q^{v_2-2u_2+2} v; q^2)}{(q^{-2+2\varepsilon} v; q^2)} (q^{3-\varepsilon} u; q^2) ,
\]
where
\[
u = \frac{z_1}{z_2} (1 - q^{-2z_1} \partial_{z_1}) \; ; \; \nu = q^{2z_1} \partial_{z_1} .
\]
In the previous formula only the factor \( (q^{-2+2\varepsilon} v; q^2)^{-1} \) can produce poles. As far as we are interested only in singular contributions from \( R^2 \) we can choose \( \varepsilon = 0 \) in the other factors of \( R^2 \). The operator \( R^2 \) acts trivially on the variable \( z_2 \) but it acts in a rather nontrivial way on the functions \( \phi(z_1) \). Further we apply it to monomials \( z_1^m \). Due to
\[
\nu \cdot z_1^m = (-)^k q^{k(k-1)} - 2mk \sum_{m-k} (q^2; q^2)_m z_1^{m-k} z_2^k \]

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To prove this we start with the Coxeter relation (2.12) and substitute the explicit expression for $S_2$ of the Baxter $Q$-operator (3.25). The present derivation uses Coxeter relation (2.12) only. At first

Here we present an alternative proof of the local relation (3.24) which produces an explicit formula for the rightmost factor in (D.7) at $\varepsilon = 0$.

Taking into account (D.6) and $\mathfrak{u}^2 \cdot z_1 = 0$ we obtain

$$(q^{u_2-v_1+1} \mathfrak{u}; q^2)^{-1} \cdot z_1 = z_1 - q^{u_2-v_1-1} z_2,$$

and finally after some trivial algebra we find the simple pole contribution to $R^2(v_1, u_2|v_2) \cdot z_1^m$ is equal

$$\frac{(q^{2v_1-2u_2+2}; q^2)^{-m}}{(q^{2v_1}; q^2) (1 - q^2)}\left[ q^{v_2-v_1-1}(1 - q^{2m}) \cdot z_1 z_2^{m-1} - (1 - q^{2v_1-2u_2+2m}) \cdot z_2^m \right] + O(1),$$

where the first term drops out at $m = 0$. Thus we are ready to calculate the restriction of the R-operator to the subspace $R_{12}$

$$z_1^m \rightarrow \frac{(q^{2(-u-L+\frac{1}{2}); q^2})^{-m}}{(q^{2(u-L-\frac{1}{2}); q^2})} q^{u-L-\frac{1}{2}} \left[ (q^{u+L-m+\frac{1}{2}} - q^{-u-L+m-\frac{1}{2}}) \cdot z_2^m - (q^m - q^{-m}) \cdot z_1 z_2^{m-1} \right],$$

$$z_1^m z_2 \rightarrow \frac{(q^{2(-u+L+\frac{1}{2}); q^2})^{-m}}{(q^{2(u-L-\frac{1}{2}); q^2})} q^{u-L-\frac{1}{2}} \left[ -(q^{-m-2L} - q^{m+2L}) \cdot z_2^m + (q^{u+m-L+\frac{1}{2}} - q^{-u-m+L-\frac{1}{2}}) \cdot z_1 z_2^{m+1} \right].$$

Taking into account permutation $\mathbb{R}_{12} = P_{12} R_{12}$ and choosing the basis in the space $\mathbb{C}^2$ as follows

$$e_1 = -z_2; \quad e_2 = 1$$

we finally conclude that the reduction of the R-operator coincides with L-operator (3.1)

$$\mathbb{R}_{12} (u | \ell, s = \frac{1}{2}) \big|_{\mathfrak{V}_{\ell} \otimes \mathbb{C}^2} = \frac{(q^{2(-u-L+\frac{1}{2}); q^2})^{-m}}{(q^{2(u-L-\frac{1}{2}); q^2})} q^{u-L-\frac{1}{2}} \cdot L (u + \frac{1}{2} | \ell).$$

### E From the Coxeter relation to the explicit $Q$-operator action

Here we present an alternative proof of the local relation (3.24) which produces an explicit formula of the Baxter $Q$-operator (3.25). The present derivation uses Coxeter relation (2.12) only. At first taking into account the factorization of $R^2$ (2.17) we rewrite (3.24) as follows

$$(q^{2z_1 \partial_{z_1} + 2 + 2b}; q^2) \cdot (q^{1-b \zeta_1}; q^2) \cdot (q^{-a \zeta_1}; q^2) = \frac{(q^{2+2b}; q^2) \cdot (q^{1-b \zeta_1}; q^2) \cdot (q^{1-a \zeta_1}; q^2) \cdot (q^{-2a \zeta_1}; q^2)}{(q^{2+2a \zeta_1}; q^2) \cdot (q^{1+b \zeta_1}; q^2) \cdot (q^{1+a \zeta_1}; q^2) \cdot (q^{-a \zeta_1}; q^2)}. \quad (E.1)$$

To prove this we start with the Coxeter relation (2.12) and substitute the explicit expression for $S^2$, $S^3$ (3.3), (3.4)

$$\frac{(v q^{2b}; q^2)}{(v q^{2a+b}; q^2)} \frac{(u q^{-b}; q^2)}{(u q^{2a+b}; q^2)} q^{-(a+b) z_1 \partial_{z_1}} \frac{(v; q^2)}{(v q^{2b}; q^2)} = \frac{(u q^{-b}; q^2)}{(u q^{2a+b}; q^2)} \frac{(v; q^2)}{(v q^{2a+2b}; q^2)} q^{-(a+b) z_1 \partial_{z_1}} \frac{(u q^{a}; q^2)}{(u q^{a}; q^2)}. \quad (E.2)$$
where $u$ and $v$ form a Weyl pair: $u = \frac{z_2}{z_3} q$; $v = q^{2z_3} \delta_{z_1} + 2$; $uv = q^2 vu$. Then we apply both sides of the identity (E.2) to the delta function in series expansion $\sum_{k=-\infty}^{+\infty} \left( \frac{z_1}{z_3} \right)^k = z_3 \delta(z_1 - z_3)$. The series represents actually the $\delta$ distribution on the unit circle, i.e. with the restriction to the variables $|z_i| = 1$. Here we apply the series expansion formally ignoring this restriction.

Thus in the left hand side we have

$$q^{-(a+b)z_1 \partial_{z_1}} \left( \frac{v; q^2}{(v q^{2b}; q^2)} \right) \cdot z_3 \delta(z_1 - z_3) = \sum_{k \geq 0} \frac{(q^{2+2k}; q^2)}{(q^{2+2b+2k}; q^2)} q^{-(a+b)k} \left( \frac{z_1}{z_3} \right)^k = \frac{(q^2; q^2)}{(q^{2+2b}; q^2)} \left( \frac{z_3 q^{2-a+b}; q^2}{z_3 q^{2-a-b}; q^2} \right)$$

where $q$-binomial formula (A.2) is used on the last step. Let us note that summation over all integer $k$ reduces to summation over non-negative $k$ due to the factor $(q^{2+2k}; q^2)$. Similarly in the right hand side we have

$$= \frac{(u q^{-a}; q^2)}{(u q^a; q^2)} \bigg|_{z_1 \to z_3} \sum_{k \geq 0} \frac{(q^{2+2k}; q^2)}{(q^{2+2a+2b+2k}; q^2)} q^{-(a+b)k} \left( \frac{z_1}{z_3} \right)^k = \frac{(q^2; q^2)}{(q^{2+2a}; q^2)} \left( \frac{z_3 q^{2+a-b}; q^2}{z_3 q^{2-a-b}; q^2} \right)$$

Comparing both sides of the equation we finally arrive at (E.1).

References.

[1] R. J. Baxter, Exactly solved models in statistical mechanics, Academic press, London, 1982; Partition function of the eight vertex lattice model, Annals Phys. 70 (1972) 193 [Annals Phys. 281 (2000) 187].

[2] V. V. Bazhanov and Yu. G. Stroganov, Chiral Potts model as a descendant of the six vertex model, J. Statist. Phys. 59 (1990) 799.

[3] M. Gaudin and V. Pasquier, The periodic Toda chain and a matrix generalization of the Bessel function’s recursion relations, J. Phys. A 25 (1992) 5243.

[4] A. Y. Volkov, Quantum lattice KdV equation, Lett. Math. Phys. 39 (1997) 313. arXiv:hep-th/9509024.

[5] L. D. Faddeev, R. M. Kashaev and A. Y. Volkov, Strongly coupled quantum discrete Liouville theory. I. Algebraic approach and duality, Commun. Math. Phys. 219 (2001) 199 [hep-th/0006156].

[6] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, Integrable structure of conformal field theory. I-III, Commun. Math. Phys. 177 (1996) 381 [hep-th/9412229], 190 (1997) 247 [hep-th/9604044], 200 (1999) 297 [hep-th/9805008].

[7] S. E. Derkachov, Baxter’s $Q$-operator for the homogeneous XXX spin chain, J. Phys. A 32 (1999) 5299. arXiv:solv-int/9902015.

[8] V. B. Kuznetsov, M. Salerno and E. K. Sklyanin, Quantum Backlund transformation for the integrable DST model, J. Phys. A 33 (2000) 171 [solv-int/9908002].

27
[9] G. P. Pronko, *On the Baxter’s Q operator for the XXX spin chain*, Commun. Math. Phys. 212 (2000) 687. [arXiv:hep-th/9908179].

[10] E. K. Sklyanin, *Backlund transformations and Baxter’s Q-operator*, [arXiv:nlin/0009009].

[11] M. Rossi and R. Weston, *A Generalized Q operator for U(q)(affine sl(2)) vertex models*, J. Phys. A 35 (2002) 10015. [arXiv:math-ph/0207004].

[12] V. B. Kuznetsov, V. V. Mangazeev and E. K. Sklyanin, *Q-operator and factorised separation chain for Jack’s symmetric polynomials*, Indag. Math. 14 (2003) 451. [arXiv:math/0306242].

[13] A. Antonov and B. Feigin, *Quantum group representations and Baxter equation*, Phys. Lett. B 392 (1997) 115 [hep-th/9603105].

[14] M. Kirch and A. N. Manashov, *Noncompact SL(2,R) spin chain*, JHEP 0406 (2004) 035 [hep-th/0405030].

[15] C. Korff, *Representation Theory and Baxter’s TQ equation for the six-vertex model. A pedagogical overview*, [arXiv:cond-mat/0411758]; *A Q-operator identity for the correlation functions of the infinite XXZ spin-chain*, J. Phys. A 38 (2005) 6641 [hep-th/0503130].

[16] A. G. Bytsko and J. Teschner, *Quantization of models with non-compact quantum group symmetry: Modular XXZ magnet and lattice sinh-Gordon model*, J. Phys. A 39 (2006) 12927 [hep-th/0602093].

[17] E. K. Sklyanin, *Quantum Inverse Scattering Method*, in Quantum Groups and Quantum Integrable Systems, (Nankai lectures), ed. Mo-Lin Ge, pp. 63-97, World Scientific Publ., Singapore 1992, [hep-th/9211111]. *The quantum Toda chain*, Lecture Notes in Physics, vol. 226, Springer, 1985, pp.196–233; *Functional Bethe ansatz*, in “Integrable and superintegrable systems”, ed. B.A. Kupershmidt, World Scientific, 1990, pp.8–33;

[18] V. V. Bazhanov, G. P. Korchemsky and A. N. Manashov, *Noncompact Heisenberg spin magnets from high-energy QCD. I: Baxter Q-operator and separation of variables*, Nucl. Phys. B 617 (2001) 375, JHEP 0307 (2003) 047. [arXiv:hep-th/0210216].

[19] R. M. Kashaev, V. V. Mangazeev and T. Nakanishi, *Yang-Baxter equation for the sl(n) chiral Potts model*, Nucl. Phys. B 362 (1991) 563.

[20] V. V. Bazhanov, R. M. Kashaev, V. V. Mangazeev and Y. G. Stroganov, *\(Z(N)x)^{**(n-1)}\) generalization of the chiral Potts model*, Commun. Math. Phys. 138 (1991) 393.

[21] E. Date, M. Jimbo, K. Miki and T. Miwa, *Generalized chiral Potts models and minimal cyclic representations of \(U-q(gl(n,C))\)*, Commun. Math. Phys. 137 (1991) 133.

[22] Tarasov V., *Cyclic monodromy matrices for sl(n) trigonometric R matrices*, Commun. Math. Phys. 158 (1993) 459–481, [arXiv:hep-th/9211105].

[23] S. E. Derkachov and A. N. Manashov, *Factorization of the transfer matrices for the quantum sl(2) spin chains and Baxter equation*, J. Phys. A 39 (2006) 4147 [nlin/0512047 [nlin-si]].

[24] D. Chicherin, S. Derkachov, D. Karakhanyan and R. Kirschner, *Baxter operators for arbitrary spin*, Nucl. Phys. B 854 (2012) 393 [arXiv:1106.4991 [hep-th]].

[25] V. V. Bazhanov, T. Lukowski, C. Meneghelli and M. Staudacher, *A Shortcut to the Q-Operator*, J. Stat. Mech. 1011 (2010) P11002 [arXiv:1005.3261 [hep-th]].
[26] D. Chicherin, S. Derkachov, D. Karakhanyan and R. Kirschner, Baxter operators for arbitrary spin II, Nucl. Phys. B 854 (2012) 433 [arXiv:1107.0643 [hep-th]].

[27] S. Derkachov, D. Karakhanyan and R. Kirschner, Yang-Baxter R operators and parameter permutations, Nucl. Phys. B 785 (2007) 263 [arXiv:0703076 [hep-th]].

[28] E. K. Sklyanin, On some algebraic structures related to Yang-Baxter equation, Funkz. Analiz i ego Pril. 16 (1982), 27-34.

[29] E. K. Sklyanin, On some algebraic structures related to Yang-Baxter equation: representations of the quantum algebra, Funkz. Analiz i ego Pril. 17 (1983), 34-48.

[30] S. E. Derkachov and V. P. Spiridonov, Yang-Baxter equation, parameter permutations, and the elliptic beta integral, arXiv:1205.3520 [math-ph].

[31] V. P. Spiridonov, On the elliptic beta function, Uspekhi Mat. Nauk 56 (1) (2001), 181-182 (Russian Math. Surveys 56 (1) (2001), 183-186).

[32] V. P. Spiridonov, Essays on the theory of elliptic hypergeometric functions, Uspekhi Mat. Nauk 63 (3) (2008), 3–72 (Russian Math. Surveys 63 (3) (2008), 405–472). arXiv:0805.3135 [math.CA].

[33] V. P. Spiridonov, A Bailey tree for integrals, Teor. Mat. Fiz. 139 (2004), 104–111 (Theor. Math. Phys. 139 (2004), 536–541), math.CA/0312502.

[34] V. P. Spiridonov and S. O. Warnaar, Inversions of integral operators and elliptic beta integrals on root systems, Adv. Math. 207 (2006), 91–132.

[35] G. E. Andrews, R. Askey, and R. Roy, Special Functions, Encyclopedia of Math. Appl. 71, Cambridge Univ. Press, Cambridge, 1999.

[36] V. P. Spiridonov, Continuous biorthogonality of the elliptic hypergeometric function, Algebra i Analiz 20 (5) (2008), 155–185 (St. Petersburg Math. J. 20 (5) (2009) 791–812), arXiv:0801.4137 [math.CA].

[37] V. P. Spiridonov, Elliptic beta integrals and solvable models of statistical mechanics, Contemp. Math. 563 (2012) 181 arXiv:1011.3798 [hep-th].

[38] V. V. Bazhanov and S. M. Sergeev, A Master solution of the quantum Yang-Baxter equation and classical discrete integrable equations, [arXiv:1006.0651 [math-ph]].

[39] A. Zabrodin, On the spectral curve of the difference Lamé operator, Int. Math. Research Notices, no. 11 (1999), 589-614, arXiv:math/9812161 [math.QA]

[40] A. Zabrodin, Commuting difference operators with elliptic coefficients from Baxter’s vacuum vectors, J. Phys. A 33 (2000) 3825 math/9912218 [math-qa].

[41] A. Zabrodin, Intertwining operators for Sklyanin algebra and elliptic hypergeometric series, J. Geom. Phys. 61 (2011) 1733 arXiv:1012.1228 [math-ph].

[42] S. E. Derkachov, D. Karakhanyan and R. Kirschner, Baxter Q-operators of the XXZ chain and R-matrix factorization, Nucl. Phys. B 738 (2006) 368 arXiv:hep-th/0511024

[43] L. D. Faddeev and R. M. Kashaev, Quantum Dilogarithm, Mod. Phys. Lett. A 9 (1994) 427 hep-th/9310070.
[44] A. Y. Volkov, *Noncommutative hypergeometry*, Commun. Math. Phys. 258 (2005) 257 [math/0312084 [math.QA]].

[45] A. N. Kirillov, *Dilogarithm identities*, Prog. Theor. Phys. Suppl. 118 (1995) 61 [hep-th/9408113].

[46] A. Erdelyi et al., *Higher transcendental functions*, vol. 2, New York 1955

[47] S. N. M. Ruijsenaars, *First order analytic difference equations and integrable quantum systems*, J. Math. Phys. 38 (1997) 1069-1146

[48] G. Felder and A. Varchenko, *The elliptic gamma function and SL(3,Z)*, Adv. Math. 156 (2000) 44 [math/9907061 [math.QA]].

[49] I. Frenkel and V. Turaev, *Elliptic solutions of the Yang-Baxter equation and modular hypergeometric functions*, The Arnold-Gelfand Mathematical Seminars (Cambridge, MA:Birkhauser Boston)(1997), pp. 171–204