Non-perturbative renormalisation with domain wall fermions

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We present results from a study of the renormalisation of both quark bilinear and four-quark operators for the domain wall fermion action, using the non-perturbative renormalisation technique of the Rome-Southampton group. These results are from a quenched simulation, on a $16^3 \times 32$ lattice, with $\beta = 6.0$ and $L_s = 16$.

1. Introduction

Domain wall fermions are increasingly being used for many studies of physically interesting quantities. It is therefore important to have reliable values for the renormalisation coefficients of the operators needed in these calculations. Here we will discuss the application of the nonperturbative renormalisation technique of the Rome-Southampton group, the Regularisation Independent (RI) scheme, to quark bilinears and four-quark operators using the domain wall fermion action. Rather than concentrating on the numerical values for the renormalisation coefficients, the emphasis will be on highlighting tests of the chiral properties of domain wall fermions and on interesting aspects of the non-perturbative renormalisation technique.

All the results presented are from a study using a lattice with dimensions $16^3 \times 32$, an extent in the fifth dimension of 16 points, a domain wall height ($M_5$) of 1.8 and a Wilson gauge action with $\beta = 6.0$.

2. The RI scheme

A full description of the RI scheme may be found in [1], but the relevant details will be quickly summarised here.

The matrix element of the operator of interest, between external quark states, at high momenta, in a fixed gauge (in this case Landau gauge was used) is calculated. The external legs of this matrix element are then amputated. A renormalisation condition may be defined by requiring that, for the renormalised operator at given scale, a chosen spin and colour projection of this quantity is equal to its free case value.

3. Quark Bilinears

For the case of flavour non-singlet fermion bilinear operators, we write

$$[\bar{\mathcal{U}} \Gamma_i \mathcal{D}]_{\text{ren}} = Z_{\Gamma_i} [\bar{\mathcal{U}} \Gamma_i \mathcal{D}]$$  \hspace{1cm} (1)

with $\Gamma_i = \{1, \gamma_\mu, \gamma_5, \gamma_\mu \gamma_5, \sigma_{\mu\nu}\}$, where $i$ represents whatever indices the gamma matrices have. Taking projections of the amputated matrix elements with the same gamma matrices as in the definition of the operator, normalised such that this trace is unity in the free theory, gives the renormalisation condition

$$1 = \frac{Z_{\Gamma_i}}{Z_q} \Lambda_{\Gamma_i} \bigg|_{m_{\text{ren}}=0}$$  \hspace{1cm} (2)

with

$$\Lambda_{\Gamma_i} = \frac{1}{\text{Tr}[\Gamma_i \Gamma_i]} \text{Tr} [\Gamma_i (\bar{\mathcal{U}} \Gamma_i \mathcal{D} u(p) d(p))_{\text{AMP}}].$$  \hspace{1cm} (3)

Invariance of the action under axial flavour transformations requires that $Z_A = Z_V$ and $Z_S = Z_P$. The extent to which these relations hold for domain wall fermions is therefore an excellent test of their chiral properties. Figure 1 shows data for $\Lambda_A - \Lambda_V$ versus $(ap)^2$ for several values of the
input mass, $m_f$. While there is a small difference between $\Lambda_A$ and $\Lambda_V$ at low momenta, this difference becomes smaller as either the momentum becomes larger or the mass becomes smaller, suggesting that the bulk of this difference is due to the effects of chiral symmetry being spontaneously broken by the vacuum and softly broken by the mass respectively. Within the statistics there is therefore no signal for a difference between $Z_A$ and $Z_V$. Figure 2 shows $\Lambda_S$, $\Lambda_P$ and their ratio, after the mass poles have been subtracted, showing that the any splitting between $Z_S$ and $Z_P$ is smaller than the resolution of this study.

Figure 1. A plot of $\Lambda_A - \Lambda_V$ versus $(ap)^2$, showing that there is no significant difference between $Z_A$ and $Z_V$.

and their ratio after the mass poles have been subtracted by fitting the mass dependence. Again, within the accuracy of the method there is no signal for a splitting of $Z_S$ and $Z_P$.

4. Four Quark Operators

The renormalisation of four-quark operators is much more complicated than that of the quark bilinears considered above. Mixing both among the four-quark operators and between these operators and lower dimensional operators must be considered. However, these operators are of much interest phenomenologically, in particular because of their relevance for the calculation of $B_K$ and matrix elements of the $\Delta S = 1$ Hamiltonian.

4.1. $\Delta S = 2$ Hamiltonian

The $\Delta S = 2$ Hamiltonian, which is relevant for calculating $B_K$, is one of the simplest four-quark renormalisation problems, as mixing with lower dimensional operators is not possible. It consists of the operators $O_{AB} = \bar{s} \Gamma_A d \bar{s} \Gamma_B d$, where $\Gamma_A$ and $\Gamma_B$ are arbitrary gamma matrices except for the condition that there be no free indices.

As with the bilinears, renormalisation conditions may be placed on these operators by considering their amputated matrix elements between external quark states. As mixing between four-quark operators is possible, a single renormalisation condition no longer suffices; however, by varying the projectors used, all the renormalisation coefficients can be fixed.

To completely specify the renormalisation condition, the momentum configuration for the external quark states must also be chosen. For such calculations to be useful, however, we are required to use the same momenta configuration used in existing perturbative matching calculations, which take all the momenta to be equal.

The renormalisation structure of these operators is strongly constrained by chiral symmetry. Previously presented results have confirmed that the parity conserving part of the operator $\bar{s} \gamma_\mu (1-\gamma_5) d \bar{s} \gamma_\mu (1-\gamma_5) d$ does not significantly mix with any of the four wrong chirality operators it could possibly mix with. Chiral symmetry also predicts that the renormalisation factor for the parity conserving and parity breaking components of the operator are equal. The ra-
ratio of these factors is shown in Figure 3, with no evidence for chiral symmetry breaking visible for moderately high momenta.

Figure 3. The ratio of the parity positive and parity negative renormalisation constants for the $\Delta S = 2$ Hamiltonian. No evidence of explicit chiral symmetry breaking can be seen.

Another prediction of chiral symmetry is that the operators $\hat{O}^\pm = \bar{s}d\gamma^\mu d \pm \bar{u}\gamma^\mu u\pi$ do not mix with one another. The element of the inverse mixing matrix resulting from a naive application of the RI scheme renormalisation conditions to this case is shown in Figure 4 and, as can be seen, this is significantly non-zero. The mass behaviour is also very distinctive, with the signal being further away from zero the smaller $m_f$ is. This is not, however, a signal for a large mixing between these two operators, but an interesting systematic effect in the matrix elements we are calculating.

A simple understanding of this effect can be gained by noting that while we are interested in the matrix elements of operators between quark states, we are still evaluating this matrix element in a theory with propagating pseudo-goldstone bosons. We may approximately represent the effects of the pseudo-goldstone bosons by an effective interaction of the form

$$C \left\{ \bar{\pi}d\gamma_5 u + \bar{u}\gamma_5 d\pi \right\},$$

with each such interaction between the quark states and the pseudo-goldstone bosons giving rise to a propagator of the form

$$\frac{1}{(p - q)^2 + M_\pi^2},$$

where $p$ and $q$ are the momenta on the incoming and outgoing quarks.

In the case we are considering, the problem is that all the external quark states have equal momenta and any interaction between them through the vertex in Equation 4 will give a contribution to the matrix element $\propto 1/M_\pi^2 \equiv 1/m_{\text{quark}}$.

Figure 4 shows the same quantity as Figure 3, for $m_f = 0.02$, except that the momenta of the incoming external states, $p$, and outgoing states, $q$, can be different. The data on the graph is such that for each point $p^2 = q^2$ with four distinct blocks of values of $p^2$ being shown, while $(p - q)^2$ varies between the different points. For all of the points except the four that are distinctly lower, $p \neq q$, while for those four points $(p - q)^2 = 0$. The points for which $(p - q)^2 \neq 0$ are clearly suppressed as is expected from Equation 5.

4.2. $\Delta S = 1$ Hamiltonian

The mixing problem for the $\Delta S = 1$ Hamiltonian is much more complicated that for the $\Delta S = 2$ Hamiltonian, both in the number of four-quark operators that may mix and because of the presence of mixing with lower dimensional operators. The latter must be subtracted before the standard RI scheme conditions are applied, and
Figure 5. The same quantity as shown in Figure 4 except that the momenta for the external states is such that the pseudo-goldstone boson pole is suppressed for all but four points on this graph.

it is the feasibility of this subtraction for domain wall fermions that we will touch on here.

The fact that the RI conditions are applied in a fixed gauge and for off-shell quark states, makes the problem of mixing with lower dimensional operators very difficult [7], with many operators that must be subtracted. However, this number is greatly reduced if chiral symmetry is respected. Also, when calculating the renormalisation coefficients in the context of a practical study, perturbation theory must be relied upon for the running of the effective Hamiltonian and scheme matching. Operators that do not appear in these calculations (because they are higher order in perturbation theory) need not be considered unless they are power divergent in the lattice spacing.

As it is chiral symmetry that makes performing this subtraction feasible, here we will give one example of a subtraction coefficient and its chiral properties. The $\Delta S = 1$ operator $O_6 = (s_\alpha d_\beta)_L \sum_q (q_\beta q_\alpha)_R$ may mix with the lower dimensional operator $\bar{s}d$. If chiral symmetry is not respected, then the needed subtraction coefficient is cubically power divergent in the lattice spacing with only subleading mass dependence. Chiral symmetry, however, constrains the leading behaviour of this coefficient to be $\propto (1/a^2)(m_s + m_d)$. Figure 5 shows the behaviour of this subtraction coefficient versus $m_f$, for a single momentum value. While this does not extrapolate exactly to zero at the same point as the renormalised mass ($m = -m_{\text{res}}$), using a linear extrapolation, the constant piece is small and is of the order of potential subleading effects.

Figure 6. The coefficient of $\bar{s}d$ for its subtraction from $O_6$. The strong mass dependence is a sign that, to a good approximation, chiral symmetry is respected.

5. Conclusions

In conclusion, for the lattice parameters used in this study, domain wall fermions have excellent chiral properties. Also, the non-perturbative renormalisation technique of the Rome-Southampton group seems to work well as long as one is careful when choosing the momentum configuration used.

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