PLURISUBHARMONIC FUNCTIONS WITH WEAK SINGULARITIES

S.BENELKOURCHI, V.GUEDJ AND A.ZERIAHI

Dedicated to Professor C.O. Kiselman
on the occasion of his retirement

ABSTRACT. We study the complex Monge-Ampère operator in bounded hyperconvex domains of \(\mathbb{C}^n\). We introduce several classes of weakly singular plurisubharmonic functions: these are functions of finite weighted Monge-Ampère energy. They generalize the classes introduced by U.Cegrell, and give a stratification of the space of (almost) all unbounded plurisubharmonic functions. We give an interpretation of these classes in terms of the speed of decreasing of the Monge-Ampère capacity of sublevel sets and solve associated complex Monge-Ampère equations.

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1. INTRODUCTION

In two seminal papers [Ce 1,2], U.Cegrell was able to define and study the complex Monge-Ampère operator \((dd^c)^n\) on special classes of unbounded plurisubharmonic functions in a hyperconvex domain in \(\mathbb{C}^n\).

Since we are considering a new and important scale of classes of plurisubharmonic functions with finite weighted Monge-Ampère energy, we find it convenient to introduce new notations which reflect our intuition. Therefore we have to modify some of the classical ones to avoid confusions.

Let \(\Omega \subset \mathbb{C}^n\) be a bounded hyperconvex domain. The first important class considered by Cegrell (denoted by \(E_0(\Omega)\) in [Ce1]), is the class \(T(\Omega)\) of plurisubharmonic “test functions” on \(\Omega\), i.e. the convex cone of all bounded plurisubharmonic functions \(\varphi\) defined on \(\Omega\) such that \(\lim_{z \to \zeta} \varphi(z) = 0\), for every \(\zeta \in \partial \Omega\), and \(\int_\Omega (dd^c \varphi)^n < +\infty\). Besides this class, we will need the following classes introduced in [Ce1], [Ce2].

- The class \(DMA(\Omega)\) is the set of plurisubharmonic functions \(u\) such that for all \(z_0 \in \Omega\), there exists a neighborhood \(V_{z_0}\) of \(z_0\) and \(u_j \in T(\Omega)\) a decreasing sequence which converges towards \(u\) in \(V_{z_0}\) and satisfies \(\sup_j \int_\Omega (dd^c u_j)^n < +\infty\). U.Cegrell has shown [Ce 2] that the operator \((dd^c)^n\) is well defined on \(DMA(\Omega)\) and continuous under decreasing limits. The class \(DMA(\Omega)\) is stable under taking maximum and it is the largest class with these properties (Theorem 4.5 in [Ce 2]). Actually this class, introduced and denoted by \(E(\Omega)\) by U.Cegrell ([Ce 2]), turns out to coincide with the domain of definition of the complex Monge-Ampère operator on \(\Omega\) as was shown by Z.Blocki [Bl 1,2];
• the class $\mathcal{F}(\Omega)$ is the “global version” of $DMA(\Omega)$: a function $u$ belongs to $\mathcal{F}(\Omega)$ if there exists $u_j \in T(\Omega)$ a sequence decreasing towards $u$ in all of $\Omega$, which satisfies $\sup_j \int_\Omega (dd^c u_j)^n < +\infty$;

• the class $\mathcal{F}_a(\Omega)$ is the set of functions $u \in \mathcal{F}(\Omega)$ whose Monge-Ampère measure $(dd^c u)^n$ is absolutely continuous with respect to capacity i.e. it does not charge pluripolar sets;

• the class $\mathcal{E}^p(\Omega)$ (respectively $\mathcal{F}^p(\Omega)$) is the set of functions $u$ for which there exists a sequence of functions $u_j \in T(\Omega)$ decreasing towards $u$ in all of $\Omega$, and so that $\sup_j \int_\Omega (-u_j)^p (dd^c u_j)^n < +\infty$ (respectively $\sup_j \int_\Omega [1 + (-u_j)^p] (dd^c u_j)^n < +\infty$).

One purpose of this article is to use the formalism developed in [GZ] in a compact setting to give a unified treatment of all these classes. Given an increasing function $\chi : \mathbb{R}^- \to \mathbb{R}^-$, we consider the set $\mathcal{E}_\chi(\Omega)$ of plurisubharmonic functions of finite $\chi$-weighted Monge-Ampère energy. These are functions $u \in PSH(\Omega)$ such that there exists $u_j \in T(\Omega)$ decreasing to $u$, with

$$\sup_{j \in \mathbb{N}} \int_\Omega (-\chi) \circ u_j (dd^c u_j)^n < +\infty.$$  

It will be shown that $\mathcal{E}_\chi(\Omega) \subset DMA(\Omega)$.

Many important properties follow from the elementary observation that the Monge-Ampère measures $1_{\{u > -j\}} (dd^c u_j)^n$ strongly converge towards $(dd^c u)^n$ in the set $\Omega \setminus \{u = -\infty\}$, when $u_j := \max(u, -j)$ are the ”canonical approximants” of $u$.

**Theorem A.** If $u \in DMA(\Omega)$, then for all Borel sets $B \subset \Omega \setminus \{u = -\infty\}$,

$$\int_B (dd^c u)^n = \lim_{j \to \infty} \int_{B \cap \{u > -j\}} (dd^c u_j)^n,$$

where $u_j := \max(u, -j)$ are the canonical approximants.

We establish this result in section 2 and derive several consequences. This yields in particular simple proofs of quite general comparison principles.

The classes $\mathcal{E}_\chi(\Omega)$ have very different properties, depending on whether $\chi(0) = 0$ or $\chi(0) \neq 0$, $\chi(-\infty) = -\infty$ or $\chi(-\infty) \neq -\infty$, $\chi$ is convex or concave. We study these in section 3 and give a capacitary interpretation of them in section 4. Let us stress in particular Corollary 4.3 which gives an interesting characterization of the class $\mathcal{E}^p(\Omega)$ of U.Cegrell, in terms of the speed of decreasing of the capacity of sublevel sets:

**Proposition B.** For any real number $p > 0$,

$$\mathcal{E}^p(\Omega) = \left\{ \varphi \in PSH^- (\Omega); \int_0^{+\infty} (-\varphi)^{n+p-1} Cap_{\Omega}(\{\varphi < -t\}) dt < +\infty \right\}.$$  

Here $Cap_{\Omega}$ denotes the Monge-Ampère capacity introduced by E. Bedford and B.A. Taylor ([BT1]). Of course $\mathcal{E}^p(\Omega) = \mathcal{E}_\chi(\Omega)$, for $\chi(t) := -(-t)^p$.

Our formalism allows us to consider further natural subclasses of $PSH(\Omega)$, especially functions with finite “high-energy” (when $\chi$ increases faster than polynomials at infinity). We study in section 5 the range of the Monge-Ampère operator on these classes. Given a positive finite Borel measure $\mu$.
on \( \Omega \), we set 
\[
F_\mu(t) := \sup\{\mu(K); K \subset \Omega \text{ compact}, \ Cap_\Omega(K) \leq t\}, t \geq 0.
\]
Observe that \( F := F_\mu \) is an increasing function on \( \mathbb{R}^+ \) which satisfies 
\[
\mu(K) \leq F(Cap_\Omega(K)), \quad \text{for all Borel subsets } K \subset X.
\]
The measure \( \mu \) does not charge pluripolar sets iff \( F(0) = 0 \).

When \( F(x) \lesssim x^\alpha \) vanishes at order \( \alpha > 1 \), S. Kolodziej has proved [K2] that the equation \( \mu = (dd^c \varphi)^n \) admits a unique continuous solution with \( \varphi|_{\partial \Omega} = 0 \). If \( F(x) \lesssim x^\alpha \) with \( 0 < \alpha < 1 \), it follows from the work of U. Cegrell [Ce 1] that there is a unique solution in some class \( \mathcal{F}^p(\Omega) \).

Another objective of this article is to fill in the gap between Cegrell’s and Kolodziej’s results, by considering all intermediate dominating functions \( F \). Write \( F(x) = x[\varepsilon(-\ln x/n)]^n \) where \( \varepsilon : \mathbb{R}^+ \to [0, \infty[ \) is nonincreasing.

Our second main result is:

**Theorem C.** Assume for all compact subsets \( K \subset \Omega \), 
\[
\mu(K) \leq F_\varepsilon(Cap_\Omega(K)), \quad \text{where } F_\varepsilon(x) = x[\varepsilon(-\ln x/n)]^n.
\]
Then there exists a unique function \( \varphi \in \mathcal{F}(\Omega) \) such that \( \mu = (dd^c \varphi)^n \) and 
\[
\text{Cap}_\Omega(\{ \varphi < -s \}) \leq \exp(-nH^{-1}(s)), \quad \text{for all } s > 0,
\]
where \( H^{-1} \) is the reciprocal function of \( H(x) = e \int_0^x \varepsilon(t)dt + s_0(\mu) \).

Note in particular that when \( \mu \leq \text{Cap}_\Omega \) (i.e. \( \varepsilon \equiv 1 \)), then \( \mu = (dd^c \varphi)^n \) for a function \( \varphi \in \mathcal{F}(\Omega) \) such that \( \text{Cap}_\Omega(\{ \varphi < -s \}) \) decreases exponentially fast. Simple examples show that this bound is sharp (see [BGZ]).

For similar results in the case of compact Kähler manifolds, we refer the reader to [GZ], [EGZ], [BGZ].

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## 2. Canonical approximants

We let \( \text{PSH}(\Omega) \) denote the set of plurisubharmonic functions on \( \Omega \) (psh for short), and fix \( u \in \text{PSH}(\Omega) \). E.Bedford and B.A.Taylor have defined in [BT 2] the non pluripolar part of the Monge-Ampère measure of \( u \): the sequence \( \mu_u^{(j)} := 1_{\{u < -j\}}(dd^c \max\{u, -j\})^n \) is a nondecreasing sequence of positive measures. Its limit \( \mu_u \) is the “nonpluripolar part of \( (dd^c u)^n \)”, defined as,
\[
\mu_u(B) = \lim_{j \to \infty} \int_{B \cap \{u > -j\}} (dd^c \max\{u, -j\})^n,
\]
for any Borel set \( B \subset \Omega \).

In general \( \mu_u \) is not locally bounded near \( \{ u = -\infty \} \) (see e.g. [K]), but if \( u \in \text{DMA}(\Omega) \) then \( \mu_u \) is a regular Borel measure:

**Theorem 2.1.** If \( u \in \text{DMA}(\Omega) \), then for all Borel sets \( B \subset \Omega \setminus \{ u = -\infty \} \),
\[
\int_B (dd^c u)^n = \lim_{j \to \infty} \int_{B \cap \{u > -j\}} (dd^c u_j)^n,
\]
where \( u_j := \max(u, -j) \). In particular, \( \mu_u = 1_{\{u > -\infty\}} (dd^c u)^n \).

The measure \((dd^c u)^n\) puts no mass on pluripolar sets \( E \subset \{u > -\infty\} \).

**Proof.** Note that this convergence result is local in nature, hence we can assume, without loss of generality, that \( u \in \mathcal{F}(\Omega) \). For \( s > 0 \) consider the psh function \( h_s := \max(u/s + 1, 0) \). Observe that \( h_s \) increases to the Borel function \( 1_{\{u > -\infty\}} \) and \( \{h_s = 0\} = \{u \leq -s\} \). We claim that

\[
h_s ((dd^c \max(u, -s))^n) = h_s ((dd^c u)^n), \quad \text{for all } s > 0,
\]

in the sense of measures on \( \Omega \).

Indeed, recall that we can find a sequence of continuous test functions \( u_k \) in \( T(\Omega) \) decreasing towards \( u \) (see Theorem 2.1 in [Ce 2]). It follows from Proposition 5.1 in [Ce 2] that \( h_s ((dd^c \max(u, -s))^n) \) converges weakly to \( h_s ((dd^c \max(u, -s))^n) \) and \( h_s ((dd^c u)^n) \) converges weakly to \( h_s ((dd^c u)^n) \) as \( k \to \infty \).

Since \( \max(u_k, -s) = u_k \) on \( \{u_k > -s\} \), which is an open neighborhood of the set \( \{u > -s\} \), we infer

\[
h_s ((dd^c \max(u, -s))^n) = h_s ((dd^c u)^n),
\]
as claimed.

Observe that

\[
h_s ((dd^c \max(u, -s))^n) = h_s 1_{\{u > -s\}} ((dd^c u)^n) = h_s \mu_u^{(s)}
\]

increases as \( s \uparrow +\infty \) towards \( 1_{\{u > -\infty\}} \mu_u = \mu_u \), as follows from the monotone convergence and Radon-Nikodym theorems. Similarly \( h_s ((dd^c u)^n) \) converges to \( 1_{\{u > -\infty\}} ((dd^c u)^n) \). Thus \( \mu_u = 1_{\{u > -\infty\}} ((dd^c u)^n) \), this shows the desired convergence on any Borel set \( B \subset \Omega \setminus \{u = -\infty\} \).

Note that if \( u \in \mathcal{F}_a(\Omega) \) then \( \int_B ((dd^c u)^n) = \lim_{j \to \infty} \int_B ((dd^c u_j)^n) \), for all Borel subsets \( B \subset \Omega \) (see Theorem 3.4).

As an application, we give a simple proof of the following general version of the comparison principle (see also [NP]).

**Theorem 2.2.** Let \( u \in DMA(\Omega) \) and \( v \in PSH^-(\Omega) \). Then

\[
1_{\{u > v\}} ((dd^c u)^n) = 1_{\{u > v\}} ((dd^c u)_\max(u, v)^n)
\]

**Proof.** Set \( u_j := \max(u, -j) \) and \( v_j := \max(v, -j) \). Recall from [BT 2] that the desired equality is known for bounded psh functions,

\[
1_{\{u_j > v_j + 1\}} ((dd^c u_j)^n) = 1_{\{u_j > v_j + 1\}} ((dd^c \max(u_j, v_j + 1))^n)
\]

Observe that \( \{u > v\} \subset \{u_j > v_j + 1\} \), hence

\[
1_{\{u_j > v_j + 1\}} \cdot 1_{\{u_j > v_j + 1\}} ((dd^c u_j)^n) = 1_{\{u_j > v_j\}} \cdot 1_{\{u_j > v_j + 1\}} ((dd^c \max(u_j, v_j + 1))^n)
\]

\[= 1_{\{u > v\}} \cdot 1_{\{u > v\}} ((dd^c \max(u, v, -j))^n).
\]

It follows from Theorem 2.1 that \( 1_{\{u > -j\}} ((dd^c u_j)^n) \) converges in the strong sense of Borel measures towards \( \mu_u = 1_{\{u > -\infty\}} ((dd^c u)^n) \). Observe that \( 1_{\{u > v\}} 1_{\{u > -\infty\}} = 1_{\{u > v\}} \). We infer, by using Theorem 2.1 again with \( \max(u, v) \), that

\[
1_{\{u > v\}} ((dd^c u)^n) = 1_{\{u > v\}} ((dd^c \max(u, v))^n).
\]

\[\square\]
The following result has been proved by U.Cegrell \cite{Ce3}. We provide here a simple proof using Theorem 2.2, yet another consequence of the fact that the Monge-Ampère measures \(1_{\{u > j\}}(dd^{c}u)^{n}\) strongly converge towards \(1_{\{u > -\infty\}}(dd^{c}u)^{n}\) when \(u_{j} := \max(u, -j)\) are the “canonical approximants” (Theorem 2.1).

**Corollary 2.3.** Let \(\phi \in \mathcal{F}(\Omega)\) and \(u \in DMA(\Omega)\) such that \(u \leq 0\). Then
\[
\int_{\{\phi < u\}} (dd^{c}u)^{n} \leq \int_{\{\phi < u\} \cup \{\phi = -\infty\}} (dd^{c}\phi)^{n}.
\]

**Proof.** Since \(\psi := \max\{u, \phi\} \in \mathcal{F}(\Omega)\) and \(\phi \leq \psi\) on \(\Omega\), it follows that
\[
\int_{\Omega} (dd^{c}\psi)^{n} \leq \int_{\Omega} (dd^{c}\phi)^{n}.
\]
Indeed this is clear when \(\phi \in T(\Omega)\) by integration by parts and follows by approximation when \(\phi \in \mathcal{F}(\Omega)\) (see \cite{Ce2}).

We infer by using Theorem 2.2,
\[
\int_{\{\phi < u\}} (dd^{c}u)^{n} = \int_{\{\phi < u\}} (dd^{c}\max(u, \phi))^{n}
\]
\[
= \int_{\Omega} (dd^{c}\max(u, \phi))^{n} - \int_{\{\phi \geq u\}} (dd^{c}\max(u, \phi))^{n}
\]
\[
\leq \int_{\Omega} (dd^{c}\phi)^{n} - \int_{\{\phi > u\}} (dd^{c}\phi)^{n} - \int_{\{\phi = u\}} (dd^{c}\max(u, \phi))^{n}
\]
\[
\leq \int_{\{\phi \leq u\}} (dd^{c}\phi)^{n}.
\]
Now take \(0 < \varepsilon < 1\) and apply the previous result to get
\[
\int_{\{\varepsilon \phi < u\}} (dd^{c}u)^{n} \leq \int_{\{\varepsilon \phi \leq u\}} (dd^{c}\varepsilon \phi)^{n} = \varepsilon \int_{\{\varepsilon \phi \leq u\}} (dd^{c}\phi)^{n}.
\]

The desired inequality follows by letting \(\varepsilon \to 1\), since \(\{\varepsilon \phi < u\}\) increases to \(\{\phi < u\}\) and \(\{\varepsilon \phi \leq u\}\) increases to \(\{\phi < u\} \cup \{\phi = -\infty\}\). \(\square\)

Note that Corollary 2.3 is still valid when \(\phi, u \in DMA(\Omega)\) under the condition \(\{\phi < u\} \subseteq \Omega\).

The following comparison principle is due to U.Cegrell (see Theorem 5.15 in \cite{Ce2} and Theorem 3.7 in \cite{Ce3}).

**Corollary 2.4.** Let \(\phi \in \mathcal{F}_{a}(\Omega)\) and \(u \in DMA(\Omega)\), such that \((dd^{c}\phi)^{n} \leq (dd^{c}u)^{n}\). Then \(u \leq \phi\).

In particular if \((dd^{c}u)^{n} = (dd^{c}\phi)^{n}\) with \(u, \phi \in \mathcal{F}_{a}(\Omega)\), then \(u = \phi\).

**Proof.** The proof is a consequence of Corollary 2.3 and follows from standard arguments (see e.g. \cite{BT1} for bounded psh function). \(\square\)

Note that the result still holds when \(u \in DMA(\Omega)\) is such that \((dd^{c}u)^{n}\) vanishes on pluripolar sets and \(u \geq v\) near \(\partial \Omega\). However it fails in \(\mathcal{F}(\Omega)\) (see \cite{Ce2} and \cite{Z}).

Now, as another consequence of Theorem 2.2, we provide the following result which will be useful in the sequel:
Corollary 2.5. Fix $\varphi \in \mathcal{F}(\Omega)$. Then for all $s > 0$ and $t > 0$,

$$
(2.1) \quad t^n \text{Cap}_\Omega(\{\varphi < -s - t\}) \leq \int_{\{\varphi < -s\}} (dd^c \varphi)^n \leq s^n \text{Cap}_\Omega(\{\varphi < -s\}).
$$

In particular

$$
(2.2) \quad \int_\Omega (dd^c \varphi)^n = \lim_{s \downarrow 0} s^n \text{Cap}_\Omega(\varphi < -s) = \sup_{s > 0} s^n \text{Cap}_\Omega(\varphi < -s).
$$

Moreover a negative function $u \in \text{PSH}(\Omega)$ belongs to $\mathcal{F}(\Omega)$ if and only if

$$
\sup_{s > 0} s^n \text{Cap}_\Omega(u < -s) < +\infty.
$$

The inequalities (2.1) was proved for psh test functions in [K3] (see also [CKZ] and [EGZ]). For $\varphi \in \mathcal{F}(\Omega) \cap L^\infty(\Omega)$, it follows by approximation and quasi-continuity. In the general case, it can be deduced using Theorem 2.1. The last assertion follows easily from (2.1). It was first obtained in ([B]).

3. Weighted energy classes

Definition 3.1. Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be an increasing function. We let $\mathcal{E}_\chi(\Omega)$ denote the set of all functions $u \in \text{PSH}(\Omega)$ for which there exists a sequence $u_j \in \mathcal{T}(\Omega)$ decreasing to $u$ in $\Omega$ and satisfying

$$
\sup_{j \in \mathbb{N}} \int_\Omega (\chi \circ u_j)(dd^c u_j)^n < \infty.
$$

This definition clearly contains the classes of U.Cegrell:

- $\mathcal{E}_\chi(\Omega) = \mathcal{F}(\Omega)$ if $\chi$ is bounded and $\chi(0) \neq 0$;
- $\mathcal{E}_\chi(\Omega) = \mathcal{E}_p(\Omega)$ if $\chi(t) = -(-t)^p$;
- $\mathcal{E}_\chi(\Omega) = \mathcal{F}_p(\Omega)$ if $\chi(t) = -1 - (-t)^p$.

We will give hereafter interpretation of the classes $\mathcal{F}(\Omega) \cap L^\infty(\Omega)$ and $\mathcal{F}_a(\Omega)$ in terms of weighted-energy as well.

Let us stress that the classes $\mathcal{E}_\chi(\Omega)$ are very different whether $\chi(0) \neq 0$ (finite total Monge-Ampère mass) or $\chi(0) = 0$.

To simplify we consider in this section the case $\chi(0) \neq 0$, so that all functions under consideration have a well defined Monge-Ampère measure of finite total mass in $\Omega$. Note however that many results to follow still hold when $\chi(0) = 0$.

Proposition 3.2. Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be an increasing function such that $\chi(-\infty) = -\infty$ and $\chi(0) \neq 0$. Then

$$
\mathcal{E}_\chi(\Omega) \subset \mathcal{F}_a(\Omega).
$$

In particular the Monge-Ampère measure $(dd^c u)^n$ of a function $u \in \mathcal{E}_\chi(\Omega)$ is well defined and does not charge pluripolar sets. More precisely,

$$
\mathcal{E}_\chi(\Omega) = \{ u \in \mathcal{F}(\Omega) / \chi \circ u \in L^1((dd^c u)^n) \}.
$$

Proof. Fix $u \in \mathcal{E}_\chi(\Omega)$ and $u_j \in \mathcal{T}(\Omega)$ a defining sequence such that

$$
\sup_j \int_\Omega \chi(u_j)(dd^c u_j)^n < +\infty.
$$

The condition $\chi(0) \neq 0$ implies that $\mathcal{E}_\chi(\Omega) \subset \mathcal{F}(\Omega)$. In particular the Monge-Ampère measure $(dd^c u)^n$ is well defined. It follows from the upper semi-continuity of $u$ that $-\chi(u)(dd^c u)^n$ is bounded from above by any
cluster point of the bounded sequence $-\chi(u_j)(dd^c u_j)^n$. Therefore $\int_\Omega (-\chi) \circ u(dd^c u)^n < +\infty$, in particular $(dd^c u)^n$ does not charge the set $\{ \chi(u) = -\infty \}$, which coincides with $\{ u = -\infty \}$, since $\chi(-\infty) = -\infty$. It follows therefore from Theorem 2.1 that the measure $(dd^c u)^n$ does not charge pluripolar sets. To prove the last assertion, it remains to show the reverse inclusion

$$E_\chi(\Omega) \supset \{ u \in \mathcal{F}(\Omega) / \chi \circ u \in L^1((dd^c u)^n) \}.$$ 

So fix $u \in \mathcal{F}(\Omega)$ such that $\chi \circ u \in L^1((dd^c u)^n)$. It follows from [K 1] that there exists, for each $j \in \mathbb{N}$, a function $u_j \in \mathcal{T}(\Omega)$ such that $(dd^c u_j)^n = 1_{\{ u_j > \rho \}}(dd^c u)^n$, where $\rho \in \mathcal{T}(\Omega)$ any defining function for $\Omega = \{ \rho < 0 \}$. Observe that $(dd^c u)^n \geq (dd^c u_{j+1})^n \geq (dd^c u_j)^n$. We infer from Corollary 2.4 that $(u_j)$ is a decreasing sequence and $u \leq u_j$. The monotone convergence theorem thus yields

$$\int_\Omega (-\chi) \circ u_j (dd^c u_j)^n = \int_\Omega (-\chi) \circ u_j 1_{\{ u_j > \rho \}}(dd^c u)^n \to \int_\Omega (-\chi) \circ u (dd^c u)^n < +\infty,$$

so that $u \in E_\chi(\Omega)$.

There is a natural partial ordering of the classes $E_\chi(\Omega)$: if $\chi = O(\ddot{\chi})$ then $E_\ddot{\chi}(\Omega) \subset E_\chi(\Omega)$. Classes $E_\chi(\Omega)$ provide a full scale of subclasses of $PSH^\omega(\Omega)$ of unbounded functions, reaching, “at the limit”, bounded plurisubharmonic functions.

**Proposition 3.3.**

$$\mathcal{F}(\Omega) \cap L^\infty(\Omega) = \bigcap_{\chi \neq 0} E_\chi(\Omega),$$

where the intersection runs over all increasing functions $\chi : \mathbb{R}^- \to \mathbb{R}^-$. Note that it suffices to consider here those functions $\chi$ which are concave.

**Proof.** One inclusion is clear. Namely if $u \in \mathcal{F}(\Omega) \cap L^\infty(\Omega)$ and $u_j \in \mathcal{T}(\Omega)$ are decreasing to $u$, then for any $\chi$ as above,

$$\int_\Omega -\chi(u_j)(dd^c u_j)^n \leq \left[ \sup_\Omega |\chi(u)| \right] \int_\Omega (dd^c u)^n < +\infty.$$ 

Conversely, assume $u \in \mathcal{F}(\Omega)$ is unbounded. Then the sublevel sets $\{ u < t \}$ are non empty for all $t < 0$, hence we can consider the function $\chi$ such that

$$t \mapsto \chi'(t) = \frac{1}{(dd^c u)^n(\{ u < t \})},$$

for all $t < 0$.

The function $\chi$ is clearly increasing. Moreover $(dd^c u)^n$ has finite (positive) mass, hence $\chi'(t) \geq \frac{1}{(dd^c u)^n(\Omega)}$. This yields $\chi(-\infty) = -\infty$. Now

$$\int_\Omega (-\chi) \circ u (dd^c u)^n = \int_0^{+\infty} \chi'(-s)(dd^c u)^n(\{ u < -s \}) ds = +\infty.$$ 

This shows that if $u \in E_\chi(\Omega)$ for all $\chi$ as above, then $u$ has to be bounded. □

When $u \in E_\chi(\Omega) \subset \mathcal{F}_u(\Omega)$, the canonical approximants $u_j := \max(u, -j)$ yield strong convergence properties of weighted Monge-Ampère operators:
Theorem 3.4. Let $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function such that $\chi(-\infty) = -\infty$ and $\chi(0) \neq 0$. Fix $u \in \mathcal{E}_\chi(\Omega)$ as set $w^j = \max(u,-j)$. Then for each Borel subset $B \subset \Omega$,
\[
\lim_{j \to +\infty} \int_B \chi(w^j)(ddc^{j}w^j)^n = \int_B \chi(u)(ddc^u)^n.
\]

Moreover if $(u_j)_{j \in \mathbb{N}}$ is any decreasing sequence in $\mathcal{E}_\chi(\Omega)$ converging to $u$ such that $\sup_j \int_{\Omega} |\chi(u_j)|(ddc^{j}u_j)^n < +\infty$, then
\[
\lim_{j \to +\infty} \int_{\Omega} \chi(u_j)(ddc^j u_j)^n = \int_{\Omega} \chi(u)(ddc^u)^n.
\]

Let us stress that this convergence result is stronger than Theorem 5.6 in [Ce 1]: on one hand we produce here an explicit (and canonical) sequence of bounded approximants, on the other hand the convergence holds in the strong sense of Borel measures. Moreover the $\chi$–energy is continuous under decreasing sequences of plurisubharmonic functions with uniformly bounded $\chi$–energies.

Proof. We first show that $(ddc^w)^n$ converges towards $(ddc^u)^n$ “in the strong sense of Borel measures”, i.e. $(ddc^w)^n(B) \to (ddc^u)^n(B)$, for any Borel set $B \subset \Omega$. Observe that for $j \in \mathbb{N}^*$ fixed and $0 < s < j$, $\{u < -s\} = \{u_j < -s\}$. It follows from Corollary 2.5 that
\[
\int_{\Omega} (ddc^w)^n = \int_{\Omega} (ddc^u)^n.
\]

Therefore
\[
\int_{\{u \leq -j\}} (ddc^w)^n = \int_{\Omega} (ddc^w)^n - \int_{\{u > -j\}} (ddc^w)^n
\]
\[
= \int_{\Omega} (ddc^u)^n - \int_{\{u > -j\}} (ddc^u)^n = \int_{\{u \leq -j\}} (ddc^u)^n.
\]

Thus if $B \subset \Omega$ is a Borel subset,
\[
\left| \int_{B} (ddc^w)^n - \int_{B} (ddc^u)^n \right| \leq \int_{\{u \leq -j\}} (ddc^w)^n + \int_{\{u \leq -j\}} (ddc^u)^n
\]
\[
\leq 2 \int_{\{u \leq -j\}} (ddc^u)^n \to 0, \text{ as } j \to +\infty.
\]

The proof that $\chi \circ w^j(ddc^w)^n$ converges strongly towards $\chi \circ u(ddc^u)^n$ goes along similar lines, once we observe that
\[
\int_{\{u \leq -j\}} -\chi \circ w^j(ddc^w)^n = -\chi(-j) \int_{\{u \leq -j\}} (ddc^w)^n
\]
\[
= -\chi(-j) \int_{\{u \leq -j\}} (ddc^u)^n \leq \int_{\{u \leq -j\}} -\chi \circ u(ddc^u)^n.
\]

To prove the second statement we proceed as in [GZ]. Observe that the statement is true for uniformly bounded sequences of plurisubharmonic functions by Bedford and Taylor convergence theorems. For the general case, we first consider an increasing function $\tilde{\chi} : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\tilde{\chi} = o(\chi)$
The integer $k$ decreases towards $u$ of $u$ of $I$ follows easily from the following inequalities

$$\phi$$

Proposition 4.2. The classes $\chi$ where $\chi$ follows Definition 4.1.

and $A.T$ aylor in [BT 1]. Given $K$ sublevel sets. $\chi$ property for $\chi$ and prove the convergence of the $\tilde{\chi}$--energies. Indeed, for $k \in \mathbb{N}$ define the canonical approximants

$$u^k_j := \sup \{u_j, -k\}, \text{ and } u^k := \sup \{u, -k\}.$$  

The integer $k$ being fixed, the sequence $(u^k_j)_{j \in \mathbb{N}}$ is uniformly bounded and decreases towards $u^k$, hence the $\tilde{\chi}$--energies of $u^k_j$ converge to the $\tilde{\chi}$--energy of $u^k$ as $j \to +\infty$. Thus we will be done if we can show that the $\tilde{\chi}$--energies of $u^k_j$ converge to the $\tilde{\chi}$--energy of $u_j$ uniformly in $j$ as $k \to +\infty$. This follows easily from the following inequalities

$$I(j, k) := \left| \int_{\Omega} \tilde{\chi}(u^k_j)(dd^c u^k_j)^n - \int_{\Omega} \tilde{\chi}(u_j)(dd^c u_j)^n \right|$$

$$\leq \int_{\{u_j \leq -k\}} -\tilde{\chi}(u^k_j)(dd^c u^k_j)^n + \int_{\{u_j \leq -k\}} -\tilde{\chi}(u_j)(dd^c u_j)^n$$

$$\leq \frac{\tilde{\chi}(-k)}{\chi(-k)} \left( \int_{\{u_j \leq -k\}} -\chi(u^k_j)(dd^c u^k_j)^n + \int_{\{u_j \leq -k\}} -\chi(u_j)(dd^c u_j)^n \right)$$

$$\leq 2 \frac{\tilde{\chi}(-k)}{\chi(-k)} \int_{\Omega} -\chi(u_j)(dd^c u_j)^n \leq 2 M \frac{\tilde{\chi}(-k)}{\chi(-k)},$$

where $M := \sup_j \int_{\Omega} -\chi(u_j)(dd^c u_j)^n < +\infty$ and the last inequality follows from previous computations.

For the general case, observe that $0 \leq f := -\chi(u) \in L^1((dd^c u)^n)$ by Proposition 3.2. Then it follows easily by an elementary integration theory argument that there exists an increasing function $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t \to +\infty} h(t)/t = +\infty$ and $h(f) \in L^1((dd^c u)^n)$ (see [RR]). Thus $u \in \mathcal{E}_{\chi_1}(\Omega)$, where $\chi_1(t) := -h(-\chi(t))$ for $t < 0$ and $\chi = o(\chi_1)$ and the continuity property for $\chi$--energies follows from the previous case.

\[ \square \]

4. Capacity estimates

Of particular interest for us here are the classes $\mathcal{E}_{\chi}(\Omega)$, where the weight $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ has fast growth at infinity. It is useful in practice to understand these classes through the speed of decreasing of the capacity of sublevel sets.

The Monge-Ampère capacity has been introduced and studied by E.Bedford and A.Taylor in [BT 1]. Given $K \subset \Omega$ a Borel subset, it is defined as

$$\text{Cap}_\Omega(K) := \sup \left\{ \int_K (dd^c u)^n / u \in PSH(\Omega), -1 \leq u \leq 0 \right\}.$$  

Definition 4.1.

$$\hat{\mathcal{E}}_{\chi}(\Omega) := \left\{ \varphi \in PSH(\Omega) / \int_0^{+\infty} t^n \chi'(-t) \text{Cap}_\Omega(\{\varphi < -t\}) dt < +\infty \right\}.$$  

The classes $\mathcal{E}_{\chi}(\Omega)$ and $\hat{\mathcal{E}}_{\chi}(\Omega)$ are closely related:

Proposition 4.2. The classes $\hat{\mathcal{E}}_{\chi}(\Omega)$ are convex and stable under maximum: if $\varphi \in \hat{\mathcal{E}}_{\chi}(\Omega)$ and $\psi \in PSH^-(\Omega)$, then $\max(\varphi, \psi) \in \hat{\mathcal{E}}_{\chi}(\Omega)$.  

One always has \( \hat{\mathcal{E}}_{\chi}(\Omega) \subset \mathcal{E}_{\chi}(\Omega) \), while
\[
\mathcal{E}_{\chi}(\Omega) \subset \hat{\mathcal{E}}_{\chi}(\Omega), \quad \text{where} \quad \hat{\chi}(t) = \chi(2t).
\]

\textbf{Proof.} The convexity of \( \hat{\mathcal{E}}_{\chi}(\Omega) \) follows from the following simple observation: if \( \varphi, \psi \in \hat{\mathcal{E}}_{\chi}(\Omega) \) and \( 0 \leq a \leq 1 \), then
\[
\{a\varphi + (1 - a)\psi < -t\} \subset \{\varphi < -t\} \cup \{\psi < -t\}.
\]
The stability under maximum is obvious.

Assume \( \varphi \in \hat{\mathcal{E}}_{\chi}(\Omega) \). We can assume without loss of generality \( \varphi \leq 0 \) and \( \chi(0) = 0 \). Set \( \varphi_j := \max(\varphi, -j) \). It follows from Corollary 2.5 that
\[
\int_{\Omega} (-\chi) \circ \varphi_j (dd^c \varphi_j)^n = \int_0^{+\infty} \chi'(-t) (dd^c \varphi_j)^n (\varphi_j < -t) dt \\
\leq \int_0^{+\infty} \chi'(-t) t^n \text{Cap}_\Omega(\varphi < -t) dt < +\infty,
\]
This shows that \( \varphi \in \mathcal{E}_{\chi}(\Omega) \). The other inclusion goes similarly, using the second inequality in Corollary 2.5.

Observe that \( \mathcal{E}_{\hat{\chi}}(\Omega) \subset \hat{\mathcal{E}}_{\chi}(\Omega) \), with \( \hat{\chi}(t) = \chi(2t) \), as follows by applying inequalities of Corollary 2.5 with \( t = s \).

\[\square\]

Observe that \( \mathcal{E}_{\hat{\chi}}(\Omega) = \mathcal{E}_{\chi}(\Omega) \) when \( \chi(t) = -(t)^p \). We thus obtain a characterization of U.Cegrell’s classes \( \mathcal{E}^p(\Omega) \) in terms of the speed of decreasing of the capacity of sublevel sets. This is quite useful since this second definition does not use the Monge-Ampère measure of the function (nor of its approximants):

\textbf{Corollary 4.3.}
\[
\mathcal{E}^p(\Omega) = \left\{ \varphi \in \text{PSH}^-(\Omega) \mid \int_0^{+\infty} t^{n+p-1} \text{Cap}_\Omega(\{\varphi < -t\}) dt < +\infty \right\}.
\]

This also provide us with a characterization of the class \( \mathcal{F}_a(\Omega) \):

\textbf{Corollary 4.4.}
\[
\mathcal{F}_a(\Omega) = \bigcup_{\chi(0) \neq 0, \chi(-\infty) = -\infty} \mathcal{E}_{\chi}(\Omega).
\]

As we shall see in the proof, it is sufficient to consider here functions \( \chi \) that are convex.

\textbf{Proof.} The inclusion \( \supset \) follows from Proposition 3.2. To prove the reverse inclusion, it suffices to show that if \( u \in \mathcal{F}_a(\Omega) \) then there exists a function \( \chi \) such that \( u \in \mathcal{E}_{\chi}(\Omega) \): this is because \( \cup \mathcal{E}_{\chi} = \cup \hat{\mathcal{E}}_{\chi} \). Set
\[
h(t) := t^n \text{Cap}_\Omega(\{u < -t\}) \quad \text{and} \quad \tilde{h}(t) := \sup_{s > t} h(s), \quad t > 0
\]
The function \( \tilde{h} \) is bounded, decreasing and converges to zero at infinity. Consider \( \chi(t) := \frac{-1}{\sqrt{h(-t)}} \) for all \( t < 0 \). Thus \( \chi : \mathbb{R}^- \to \mathbb{R}^- \) is convex increasing,
with \( \chi(0) \neq 0 \) and \( \chi(-\infty) = -\infty \). Moreover \[
\int_0^{+\infty} t^n \chi'(t) \text{Cap}_\Omega(\{ \varphi < -t \}) dt \leq \frac{1}{2} \int_0^{+\infty} \frac{-\tilde{h}'(s)}{\tilde{h}^{1/2}(s)} ds = \tilde{h}^{1/2}(0) < +\infty,
\]
as follows from Corollary 2.5. 

Let us observe that a negative psh function \( u \) belongs to \( \mathcal{F}(\Omega) \) if and only if \( \tilde{h}(0) < +\infty \) (see Corollary 2.5).

We end up this section with the following useful observation. Let \( \chi : \mathbb{R}^- \to \mathbb{R}^- \) be a non-constant concave increasing function. Its inverse function \( \chi^{-1} : \mathbb{R}^- \to \mathbb{R}^- \) is convex, hence for all \( \varphi \in \text{PSH}(\Omega) \), the function \( \chi^{-1} \circ \varphi \) is plurisubharmonic,

\[
dd^c \chi^{-1} \circ \varphi = (\chi^{-1})' \circ \varphi \, dd^c \varphi + (\chi^{-1})'' \, d\varphi \wedge d^c \varphi \geq 0.
\]

Now \( \text{Cap}_\Omega(\{ \chi^{-1} \circ \varphi < -t \}) = \text{Cap}_\Omega(\{ \varphi < \chi(-t) \}) \)
decreases (very) fast if \( \chi \) has (very) fast growth at infinity. Thus \( \chi^{-1} \circ \varphi \) belongs to some class \( \mathcal{E}_\chi(\Omega) \), where \( \hat{\chi} \) is completely determined by \( \chi \) and has approximately the same growth order. This shows in particular that the class \( \mathcal{E}_\chi(\Omega) \) characterizes pluripolar sets, whatever the growth of \( \chi \).

**Theorem 4.5.** Let \( P \subset \Omega \) be a (locally) pluripolar set. Then for any concave increasing function \( \chi : \mathbb{R}^- \to \mathbb{R}^- \) with \( \chi(-\infty) = -\infty \), there exists \( \varphi \in \mathcal{E}_\chi(\Omega) \) such that

\[
P \subset \{ \varphi = -\infty \}.
\]

In particular we can choose \( \varphi \in \mathcal{E}_{\exp}(\Omega) \), where \[
\mathcal{E}_{\exp}(\Omega) := \left\{ \varphi \in \mathcal{F}(\Omega); \int_\Omega e^{-\tilde{\varphi}} (dd^c \varphi)^n < +\infty \right\}.
\]

5. **The range of the complex Monge-Ampère operator**

Throughout this section, \( \mu \) denotes a fixed positive Borel measure of finite total mass \( \mu(\Omega) < +\infty \) which is dominated by the Monge-Ampère capacity. We want to solve the following Monge-Ampère equation

\[
(dd^c \varphi)^n = \mu, \text{ with } \varphi \in \mathcal{F}(\Omega),
\]

and measure how far the (unique) solution \( \varphi \) is from being bounded, by assuming that \( \mu \) is suitable dominated by the Monge-Ampère capacity.

Measures dominated by the Monge-Ampère capacity have been extensively studied by S.Kolodziej in [K 1,2,3]. The main result of his study, achieved in [K 2], can be formulated as follows. Fix \( \varepsilon : \mathbb{R} \to [0, \infty[ \) a continuous decreasing function and set \( F_\varepsilon(x) := x[\varepsilon(-\ln x/n)]^n \). If for all compact subsets \( K \subset \Omega \),

\[
\mu(K) \leq F_\varepsilon(\text{Cap}_\Omega(K)), \text{ and } \int_0^{+\infty} \varepsilon(t) dt < +\infty,
\]

then \( \mu = (dd^c \varphi)^n \) for some continuous function \( \varphi \in \text{PSH}(\Omega) \) with \( \varphi|_{\partial \Omega} = 0 \).

The condition \( \int_0^{+\infty} \varepsilon(t) dt < +\infty \) means that \( \varepsilon \) decreases fast enough towards zero at infinity. This gives a quantitative estimate on how fast \( \varepsilon(-\ln \text{Cap}_\Omega(K)/n) \), hence \( \mu(K) \), decreases towards zero as \( \text{Cap}_\Omega(K) \to 0 \).
When \( \int_{+\infty}^{+\infty} \varepsilon(t)dt = +\infty \), it is still possible to show that \( \mu = (dd^c \varphi)^n \) for some function \( \varphi \in \mathcal{F}(\Omega) \), but \( \varphi \) will generally be unbounded. We now measure how far it is from being so:

**Theorem 5.1.** Assume for all compact subsets \( K \subset \Omega \),

\[
\mu(K) \leq F_\varepsilon \left( \text{Cap}_\Omega(K) \right).
\]

Then there exists a unique function \( \varphi \in \mathcal{F}(\Omega) \) such that \( \mu = (dd^c \varphi)^n \), and

\[
\text{Cap}_\Omega(\{ \varphi < -s \}) \leq \exp(-nH^{-1}(s)), \text{ for all } s > 0,
\]

Here \( H^{-1} \) is the reciprocal function of \( H(x) = e^{\int_0^x \varepsilon(t)dt + \varepsilon(0) + \mu(\Omega)^{1/n}} \).

In particular \( \varphi \in \mathcal{E}_\chi(\Omega) \) with \( -\chi(-t) = \exp(nH^{-1}(t/2)) \).

For examples showing that these estimates are essentially sharp, we refer the reader to section 4 in [BGZ].

**Proof.** The assumption on \( \mu \) implies in particular that it vanishes on pluripolar sets. It follows from [Ce 2] that there exists a unique \( \varphi \in \mathcal{F}_a(\Omega) \) such that \( (dd^c \varphi)^n = \mu \). Set

\[
f(s) := -\frac{1}{n} \log \text{Cap}_\Omega(\{ \varphi < -s \}), \forall s > 0.
\]

The function \( f \) is increasing and \( f(+\infty) = +\infty \), since \( \text{Cap}_\Omega \) vanishes on pluripolar sets.

It follows from Corollary 2.5 and (5.1) that for all \( s > 0 \) and \( t > 0 \),

\[
t^n \text{Cap}_\Omega(\varphi < -s - t) \leq \mu(\varphi < -s) \leq F_\varepsilon(\text{Cap}_\Omega(\{ \varphi < -s \})).
\]

Therefore

\[
(5.2) \quad \log t - \log \varepsilon \circ f(s) + f(s) \leq f(s + t).
\]

We define an increasing sequence \( (s_j)_{j \in \mathbb{N}} \) by induction. Setting

\[
s_{j+1} = s_j + e \varepsilon \circ f(s_j), \text{ for all } j \in \mathbb{N}.
\]

The choice of \( s_0 \). We choose \( s_0 \geq 0 \) large enough so that \( f(s_0) \geq 0 \). We must insure that \( s_0 = s_0(\mu) \) can chosen to be independent of \( \varphi \). It follows from Corollary 2.5 that

\[
\text{Cap}_\Omega(\{ \varphi < -s \}) \leq \frac{\mu(\Omega)}{s^n}, \forall s > 0
\]

hence \( f(s) \geq \log s - 1/n \log \mu(\Omega) \). Therefore \( f(s_0) \geq 0 \) if \( s_0 = \mu(\Omega)^{1/n} \).

The growth of \( s_j \). We can now apply (5.2) and get \( f(s_j) \geq j + f(s_0) \geq j \). Thus \( \lim_j f(s_j) = +\infty \). There are two cases to be considered.

If \( s_\infty = \lim_{s_j} s_j \in \mathbb{R}^+ \), then \( f(s) \equiv +\infty \) for \( s > s_\infty \), i.e. \( \text{Cap}_\Omega(\varphi < -s) = 0 \), \( \forall s > s_\infty \). Therefore \( \varphi \) is bounded from below by \( -s_\infty \), in particular \( \varphi \in \mathcal{E}_\chi(\Omega) \) for all \( \chi \).
Assume now (second case) that \( s_j \to +\infty \). For each \( s > 0 \), there exists \( N = N_s \in \mathbb{N} \) such that \( s_N \leq s < s_{N+1} \). We can estimate \( s \mapsto N_s \),

\[
s \leq s_{N+1} = \sum_{j=0}^{N} (s_{j+1} - s_j) + s_0 = \sum_{j=0}^{N} \varepsilon \circ f(s_j) + s_0
\]

\[
\leq e \sum_{j=0}^{N} \varepsilon(j) + s_0 \leq e \int_{0}^{\infty} \varepsilon(t) dt + \bar{s}_0 =: H(N),
\]

where \( \bar{s}_0 = s_0 + e \varepsilon(0) \). Therefore \( H^{-1}(s) \leq N \leq f(s_N) \leq f(s) \), hence

\[
\operatorname{Cap}_\Omega(\varphi < -s) \leq \exp(-nH^{-1}(s)).
\]

Set now \( g(t) = -\chi(-t) = \exp(nH^{-1}(t)/2) \). Then

\[
\int_{0}^{+\infty} t^n g'(t) \operatorname{Cap}_\Omega(\varphi < -t) dt
\]

\[
\leq \frac{n}{2} \int_{0}^{+\infty} t^n \frac{1}{\varepsilon(H^{-1}(t)) + s_0} \exp(-nH^{-1}(t)/2) dt
\]

\[
\leq C \int_{0}^{+\infty} (t + 1)^n \exp(n(\alpha - 1)t) dt < +\infty.
\]

This shows that \( \varphi \in \mathcal{E}_\chi(\Omega) \) where \( \chi(t) = -\exp(nH^{-1}(-t)/2) \). \( \square \)

Observe that the proof above gives easily an a priori uniform bound of the solution of \( (dd^c \varphi)^n = \mu \), when \( \mu \) is a finite Borel measure on \( \Omega \) satisfying (5.1) with \( \int_{0}^{+\infty} \varepsilon(t) dt < +\infty \) (see also [K2]). Indeed it follows from the above estimates that \( \varphi \geq -s_\infty \), where

\[
s_\infty \leq e \int_{0}^{+\infty} \varepsilon(t) dt + e \varepsilon(0) + \mu(\Omega)^{1/n}.
\]

We now generalize U.Cegrell’s main result [Ce 1].

**Theorem 5.2.** Let \( \chi : \mathbb{R}^- \to \mathbb{R}^- \) be an increasing function such that \( \chi(-\infty) = -\infty \). Suppose there exists a locally bounded function \( F : \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( \limsup_{t \to +\infty} F(t)/t < 1 \), and

\[
(5.3) \quad \int_{\Omega} (-\chi) \circ u \, d\mu \leq F(E_\chi(u)), \quad \forall u \in T(\Omega),
\]

where \( E_\chi(u) := \int_{\Omega} (-\chi) \circ u(dd^c u)^n \) denotes the \( \chi \)-energy of \( u \).

Then there exists a function \( \varphi \in \mathcal{E}_\chi(\Omega) \) such that \( \mu = (dd^c \varphi)^n \).

**Proof.** The assumption on \( \mu \) implies in particular that it vanishes on pluripolar sets. It follows from [Ce 2] that there exists a function \( u \in T(\Omega) \) and \( f \in L^1_{\text{loc}}((dd^c u)^n) \) such that \( \mu = f(dd^c u)^n \).

Consider \( \mu_j := \min(f, j)(dd^c u)^n \). This is a finite measure which is bounded from above by the Monge-Ampère measure of a bounded function. It follows therefore from [K 1] that there exist \( \varphi_j \in T(\Omega) \) such that

\[
(dd^c \varphi_j)^n = \min(f, j)(dd^c u)^n.
\]

The comparison principle shows that \( \varphi_j \) is a decreasing sequence. Set \( \varphi = \lim_{j \to -\infty} \varphi_j \). It follows from (5.3) that \( E_\chi(\varphi_j)(F(E_\chi(\varphi_j)))^{-1} \leq 1 \), hence \( \sup_{j \geq 1} E_\chi(\varphi_j) < \infty \). This yields \( \varphi \in \mathcal{E}_\chi(\Omega) \).
We conclude now by continuity of the Monge-Ampère operator along decreasing sequences that \((dd^c \varphi)^n = \mu\).

When \(\chi(t) = -(t)^p\) (class \(F^p(\Omega)\), \(p \geq 1\), the above result was established by U.Cegrell in [Ce 1]. Condition (5.3) is also necessary in this case, and the function \(F\) can be made quite explicit: there exists \(\varphi \in F^p(\Omega)\) such that \(\mu = (dd^c \varphi)^n\) if and only if \(\mu\) satisfies (5.3) with \(F(t) = Ct^{p/(p+n)}\), for some constant \(C > 0\).

Actually the measure \(\mu\) satisfies (5.3) for \(\chi(t) = -(t)^p\), and \(F(t) = C \cdot t^{p/(p+n)}\), \(p > 0\) if and only if \(F^p(\Omega) \subset L^p(\mu)\) (see [GZ]).

We finally remark that this condition can be interpreted in terms of domination by capacity.

**Proposition 5.3.** If \(F^p(\Omega) \subset L^p(\mu)\), then there exists \(C > 0\) such that
\[
\mu(K) \leq C \cdot \text{Cap}^\alpha_{\Omega}(K)^{\frac{p}{p+n}}, \quad \text{for all } K \subset \Omega.
\]

Conversely if \(\mu(\cdot) \lesssim \text{Cap}^\alpha_{\Omega}(\cdot)\) for some \(\alpha > p/(p+n)\), then \(F^p(\Omega) \subset L^p(\mu)\).

**Proof.** The estimate \(\text{(5.3)}\) applied to \(u = u^*_K\), the relative extremal function of the compact \(K\), yields
\[
\mu(K) = \int_{\Omega} 1_K \cdot d\mu \leq \int_{\Omega} (-u^*_K)^p d\mu \leq C \cdot \left( \int_{\Omega} (-u^*_K)^p (dd^c u^*_K)^n \right)^{\frac{p}{p+n}} = C \cdot \left( \text{Cap}^\alpha_{\Omega}(K) \right)^{\frac{p}{p+n}}.
\]

Conversely, assume that \(\mu(K) \leq C \cdot \text{Cap}^\alpha_{\Omega}(K)\) for all compact \(K \subset \Omega\), where \(\alpha > p/(n+p)\) then \(\text{(5.3)}\) is satisfied. Indeed, if \(u \in F^p(\Omega)\), then
\[
\int_{\Omega} (-u)^p d\mu = p \int_1^\infty t^{p-1} \mu(u < -t) dt + O(1)
\]
\[
\leq C \cdot p \int_1^\infty t^{p-1} \left( \text{Cap}^\alpha_{\Omega}(u < -t) \right)^{\alpha} dt + O(1)
\]
\[
\leq C \cdot \left( \int_1^\infty t^{n+p-1} \text{Cap}^\alpha_{\Omega}(u < -t) dt \right)^\alpha \cdot \left( \int_1^\infty t^{\beta p-1} \alpha(n+p-1) \beta \right)^\beta + O(1),
\]
where \(\alpha + \beta = 1\). The first integral converges by Corollary 4.3, the latter one is finite since \(p - 1 - \alpha(n+p-1) > \alpha - 1 = -\beta\). \(\square\)

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Slimane Benelkourchi, Vincent Guedj and Ahmed Zeriahi
Institut de Mathématiques de Toulouse,
Laboratoire Emile Picard,
Université Paul Sabatier
118 route de Narbonne
31062 TOULOUSE Cedex 09 (FRANCE)
benel@math.ups-tlse.fr
guedj@math.ups-tlse.fr
zeriahi@math.ups-tlse.fr