Soliton solutions for coupled Schrödinger systems
with sign-changing potential

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Abstract In this paper, a class of coupled systems of nonlinear Schrödinger equations
with sign-changing potential, including the linearly coupled case, is considered. The existence
of non-trivial bound state solutions via linking methods for cones in Banach spaces
is proved.

Key words coupled Schrödinger system, sign-changing potential, cohomological index

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1 Introduction and main results

Recently, many mathematicians focused their attention to coupled nonlinear Schrödinger
systems. From the viewpoint of physics, coupled Schrödinger systems arise from the
models of a lot of natural phenomena. A typical example is the study of the dynamics of
coupled Bose-Einstein condensates and the following equation is derived

\begin{align}
\left\{
\begin{array}{l}
i \frac{\partial \psi_1}{\partial t} = (-\frac{\partial^2}{\partial x^2} + V_1 + U_{11}|\psi_1|^2 + U_{12}|\psi_2|^2)\psi_1 + \lambda \psi_2,
\end{array}
\right.
\end{align}

\begin{align}
\left\{
\begin{array}{l}
i \frac{\partial \psi_2}{\partial t} = (-\frac{\partial^2}{\partial x^2} + V_2 + U_{22}|\psi_2|^2 + U_{21}|\psi_1|^2)\psi_2 + \lambda \psi_1.
\end{array}
\right.
\end{align}

Such systems of equations also appear in nonlinear optical models and many other physical
contexts, see [7] for detail discussions. For such coupled systems, the solutions of the form
\(\psi_j = u_j \exp(i\omega_j t)\) (standing waves) are interesting, where \(u_j\) solve the following system

\begin{align}
\left\{
\begin{array}{l}
-\frac{\partial^2 u_1}{\partial x^2} + (V_1 + \omega_1)u_1 = -(U_{11}|u_1|^2 + U_{12}|u_2|^2)u_1 - \lambda u_2, \\
-\frac{\partial^2 u_2}{\partial x^2} + (V_2 + \omega_2)u_2 = -(U_{22}|u_2|^2 + U_{21}|u_1|^2)u_2 - \lambda u_1.
\end{array}
\right.
\end{align}

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In this paper, we will consider the following coupled system of nonlinear Schrödinger equations

\[
\begin{aligned}
-\Delta u_1 + (b_1(x) - \lambda V_1(x))u_1 &= W_t(x, u_1, u_2) + \lambda \gamma(x) u_2, \\
-\Delta u_2 + (b_2(x) - \lambda V_2(x))u_2 &= W_t(x, u_1, u_2) + \lambda \gamma(x) u_1,
\end{aligned}
\]  

(1.3)

\( u_1, u_2 \in H^1(\mathbb{R}^N) \),

here and in the sequel, \( V_i \in L^\infty(\mathbb{R}^N), \gamma \in L^\infty(\mathbb{R}^N) \), \( i = 1, 2 \), \( \nabla_x W = (W_t, W_s) \) is the gradient of \( W(x, t, s) \) with respect to \( z = (t, s) \in \mathbb{R}^2 \) and we will write \( W(x, z) = W(x, t, s) \) for convenience. We divide our discussions into two cases.

The non-radially symmetric case. We assume \( b_i(x) \) satisfying the following conditions

(B) for \( i = 1, 2 \), \( b_i \in C(\mathbb{R}^N) \), there exists a constant \( b_i^0 > 0 \) such that \( \inf_{x \in \mathbb{R}^N} b_i(x) \geq b_i^0 \), and the \( n \) dimensional Lebesgue measure \( \text{meas}\{x \in \mathbb{R}^N \mid b_i(x) \leq M\} < \infty \) for any \( M > 0 \).

We assume \( W \) satisfying the following conditions.

\( W_1 \) \( W \in C^1(\mathbb{R}^N \times \mathbb{R}^2) \), there exists \( p \in (2, 2^*) \) such that \( 0 \leq W(x, z) \leq C(1 + |z|^p) \), \n for \( \forall (x, z) \in \mathbb{R}^N \times \mathbb{R}^2 \), here, \( 2^* = \frac{2N}{N-2} \) if \( N > 2 \) and \( 2^* = +\infty \) if \( N = 1, 2 \),

\( W_2 \) \( \lim_{|z| \to \infty} \frac{W(x, z)}{|z|^2} = +\infty \) uniformly for \( x \in \mathbb{R}^N \),

\( W_3 \) \( W_t(x, 0, s) = 0, W_s(x, t, 0) = 0 \) for any \( x \in \mathbb{R}^N \), \( s \in \mathbb{R} \), \( t \in \mathbb{R} \), and \( \lim_{|z| \to 0} \frac{W(x, z)}{|z|^2} = 0 \) uniformly for \( x \in \mathbb{R}^N \),

\( W_4 \) \( W(x, z) = \nabla_z W(x, z) \cdot z - 2W(x, z) \), then there exists \( \theta \geq 1 \) such that \( \theta W(x, z) \geq W(x, \eta z), \forall (x, z) \in \mathbb{R}^N \times \mathbb{R}^2 \) and \( \eta \in [0, 1] \).

Remark. (1) From (W_4) and \( W(x, 0) = 0 \), we see that \( W(x, z) \geq 0 \) for any \( (x, z) \in \mathbb{R}^N \times \mathbb{R}^2 \) by taking \( \eta = 0 \). So we have \( \nabla_x W(x, z) \cdot z \geq 2W(x, z) \).

(2) From condition (W_3), when \( \lambda \gamma(x) \neq 0, \forall x \in \mathbb{R}^N \), for a non-trivial solution \( u = (u_1, u_2) \) of the problem \( \text{(1.3)} \), it is easy to see that \( u_1 \neq 0 \) and \( u_2 \neq 0 \), so \( u \) does not have an immediate counterpart for a single equation. We also remind that under the above conditions the potential \( b_i(x) - \lambda V_i(x) \) may change sign since \( \lambda \in \mathbb{R} \), see Theorem 1.1 below.

In this case, we have the following main result.

**Theorem 1.1** If (B) and (W_1)–(W_4) hold, the problem \( \text{(1.3)} \) possesses a non-trivial solution for every \( \lambda \in \mathbb{R} \).
The radially symmetric case. We assume that $b_i(x)$ satisfy the following condition

$$(B)_r \text{ for } i = 1, 2, b_i \in C(\mathbb{R}^N), \text{ there exists a constant } b_i^0 > 0 \text{ such that } \inf_{x \in \mathbb{R}^N} b_i(x) \geq b_i^0,$$

and $b_i$ are radially symmetric, i.e., $b_i(x) = b_i(|x|), \forall x \in \mathbb{R}^N,$

and $V_i(x), \gamma(x), W(x, z)$ further satisfy

$$(V)_r \text{ for } i = 1, 2, V_i(x) = V_i(|x|), \gamma(x) = \gamma(|x|), \forall x \in \mathbb{R}^N.$$

$$(W_3) \text{ } W(x, z) = W(|x|, z), \forall (x, z) \in \mathbb{R}^N \times \mathbb{R}^2.$$

For this case we have the following result.

**Theorem 1.2** If $(B)_r$, $(V)_r$ and $(W_1)$–$(W_5)$ hold, the problem $(1.3)$ possesses a non-trivial radially symmetric solution for every $\lambda \in \mathbb{R}$.

Next, we consider some special cases of $(1.3)$. Firstly, we consider some linearly coupled systems. Precisely, we assume that $W_i(x, t, s)$ does not depend on $s$ and $W_{a_i}(x, t, s)$ does not depend on $t$, that is to say one can write $(1.3)$ as

$$
\begin{cases}
-\Delta u_1 + (b_1(x) - \lambda V_1(x))u_1 = f(x, u_1) + \lambda \gamma(x)u_2,
-\Delta u_2 + (b_2(x) - \lambda V_2(x))u_2 = g(x, u_2) + \lambda \gamma(x)u_1,
\end{cases}
$$

$$(1.4) \quad u_1, u_2 \in H^1(\mathbb{R}^N).$$

In this case, we assume that $f, g \in C(\mathbb{R}^N \times \mathbb{R})$ satisfy

$$(f_1) \quad \exists p_1 \in (2, 2^*) \text{ such that } |f(x, t)| \leq C(1 + |t|^{p_1-1}), f(x, t)t \geq 0, \forall (x, t) \in \mathbb{R}^N \times \mathbb{R},$$

$$(f_2) \quad \text{set } F(x, t) = \int_0^t f(x, t)dt, \lim_{|t| \to \infty} \frac{F(x, t)}{|t|^2} = +\infty \text{ uniformly in } x \in \mathbb{R}^N,$$

$$(f_3) \quad \lim_{t \to 0} \frac{f(x, t)}{t} = 0 \text{ uniformly in } x \in \mathbb{R}^N,$$

$$(f_4) \quad F(x, t) = f(x, t)t - 2F(x, t), \text{ then there exists } \theta_1 \geq 1 \text{ such that } \theta_1 F(x, t) \geq F(x, \eta t),$$

$$(g_1) \quad \exists p_2 \in (2, 2^*) \text{ such that } |g(x, s)| \leq C(1 + |s|^{p_2-1}), g(x, s)s \geq 0, \forall (x, s) \in \mathbb{R}^N \times \mathbb{R},$$

$$(g_2) \quad \text{set } G(x, s) = \int_0^s g(x, s)ds, \lim_{|s| \to \infty} \frac{G(x, s)}{|s|^2} = +\infty \text{ uniformly in } x \in \mathbb{R}^N,$$

$$(g_3) \quad \lim_{s \to 0} \frac{g(x, s)}{s} = 0 \text{ uniformly in } x \in \mathbb{R}^N,$$

$$(g_4) \quad G(x, s) = g(x, s)s - 2G(x, s), \text{ then there exists } \theta_2 \geq 1 \text{ such that } \theta_2 G(x, s) \geq G(x, \eta s),$$

$$(g_5) \quad \forall (x, s) \in \mathbb{R}^N \times \mathbb{R} \text{ and } \eta \in [0, 1].$$
**Theorem 1.3** If (B), $(f_1)–(f_4)$ and $(g_1)–(g_4)$ hold, the problem (1.4) possesses a non-trivial solution for every $\lambda \in \mathbb{R}$.

**Proof.** Set $W(x,t,s) = F(x,t) + G(x,s)$, it is easy to see that $(W_1)$ and $(W_4)$ hold.

As for $(W_2)$, from $(f_2)$ and $(g_2)$, $\forall M > 0$, there exists $R > 0$ such that $\frac{F(x,t)}{|t|^2} > 2M$ when $|t| \geq R$ and $\frac{G(x,s)}{|s|^2} > 2M$ when $|s| \geq R$. Then

$$
\frac{F(x,t) + G(x,s)}{t^2 + s^2} \geq \frac{F(x,t) + G(x,s)}{2 \max(|t|^2, |s|^2)} > M
$$

when $\max(|t|, |s|) \geq R$. So $\lim_{|z| \to \infty} \frac{W(x,z)}{|z|^2} = +\infty$ uniformly for $x \in \mathbb{R}^N$.

From $(f_3)$, $(g_3)$ and the continuity of $f$ and $g$, we can see $f(x,0) = 0 = g(x,0)$, so $W_t(x,0,s) = 0, W_s(x,t,0) = 0$ for any $x \in \mathbb{R}^N$, $s \in \mathbb{R}, t \in \mathbb{R}$. Also from $(f_3)$ and $(g_3)$, we have $\lim_{|t| \to 0} \frac{F(x,t)}{|t|^2} = 0$ and $\lim_{|s| \to 0} \frac{G(x,s)}{|s|^2} = 0$, so

$$
0 \leq \frac{F(x,t) + G(x,s)}{|t|^2 + |s|^2} \leq \frac{F(x,t)}{|t|^2} + \frac{G(x,s)}{|s|^2} \to 0.
$$

So $(W_3)$ holds. From Theorem 1.1 we get the assertion.

As in Theorem 1.2, assuming that $f(x,t)$ and $g(x,s)$ further satisfy

$(f_5)$ $f(x,t) = f(|x|, t)$, for any $(x,t) \in \mathbb{R}^N \times \mathbb{R},$

$(g_5)$ $g(x,s) = g(|x|, s)$, for any $(x,s) \in \mathbb{R}^N \times \mathbb{R},$

also setting $W(x,t,s) = F(x,t) + G(x,s)$ and by the same reason as in the proof of Theorem 1.3 we have the following consequence.

**Theorem 1.4** If $(B)_r$, $(V)_r$, $(f_1)–(f_5)$ and $(g_1)–(g_5)$ hold, the problem (1.4) possesses a non-trivial radially symmetric solution for every $\lambda \in \mathbb{R}$.

By taking $f(x,t) = c_1(x)|t|^{p_1-2}t$ and $g(x,s) = c_2(x)|s|^{p_2-2}s$ with $c_i \in L^\infty(\mathbb{R}^N)$ and $\inf_{x \in \mathbb{R}^N} c_i(x) > 0$, $i = 1, 2$, we get the following system

$$
\begin{align*}
-\Delta u_1 + (b_1(x) - \lambda V_1(x))u_1 &= c_1(x)|u_1|^{p_1-2}u_1 + \lambda \gamma(x)u_2, \\
-\Delta u_2 + (b_2(x) - \lambda V_2(x))u_2 &= c_2(x)|u_2|^{p_2-2}u_2 + \lambda \gamma(x)u_1,
\end{align*}
$$

(1.5)

$u, v \in H^1(\mathbb{R}^N),$

then for $p_1, p_2 \in (2, 2^*)$, we have the following consequences.

**Corollary 1.5** If (B) holds, the problem (1.5) possesses a non-trivial solution for every $\lambda \in \mathbb{R}$.
Corollary 1.6 If \((B)_r\), \((V)_r\) hold, and \(c_i(x) = c_i(|x|)\) for any \(x \in \mathbb{R}^N\), \(i = 1, 2\), the problem \((1.5)\) possesses a non-trivial radially symmetric solution for every \(\lambda \in \mathbb{R}\).

Secondly, by taking \(W(x, t, s) = \frac{1}{4}t^4 + \frac{1}{2}t^2s^2 + \frac{1}{2}s^4\), we get the following systems

\[
\begin{align*}
-\Delta u_1 + (b_1(x) - \lambda V_1(x))u_1 &= u_1^3 + u_2^3 + \lambda \gamma(x)u_2, \\
-\Delta u_2 + (b_2(x) - \lambda V_2(x))u_2 &= u_2^3 + u_1^3 + \lambda \gamma(x)u_1, \\
u_1, u_2 &\in H^1(\mathbb{R}^N).
\end{align*}
\]

as consequences of Theorem 1.1 and 1.2, we have

Corollary 1.7 If \((B)\) holds, the problem \((1.6)\) possesses a non-trivial solution for every \(\lambda \in \mathbb{R}\).

Corollary 1.8 If \((B)_r\) and \((V)_r\) hold, the problem \((1.6)\) possesses a non-trivial radially symmetric solution for every \(\lambda \in \mathbb{R}\).

The study of linearly coupled Schrödinger systems from the mathematical point of view began very recently, see \([1, 3, 4, 7]\). In \([3]\), the authors proved the existence of positive ground state solution of the following system of nonlinear Schrödinger equations for \(0 < \lambda < 1\),

\[
\begin{align*}
-\Delta u + u &= (1 + a(x))|u|^{p-2}u + \lambda v, \\
-\Delta v + v &= (1 + b(x))|v|^{p-2}v + \lambda u, \\
u, v &\in H^1(\mathbb{R}^N),
\end{align*}
\]

with \(a, b \in L^\infty(\mathbb{R}^N)\), \(\lim_{|x| \to \infty} a(x) = \lim_{|x| \to \infty} b(x) = 0\), \(\inf_{\mathbb{R}^N} \{1 + a(x)\} > 0\), \(\inf_{\mathbb{R}^N} \{1 + b(x)\} > 0\) and \(a(x) + b(x) \geq 0\). In \([3]\), the authors devoted to the study the multi-bump solitons of the following system

\[
\begin{align*}
-\Delta u + u - u^3 &= \epsilon v, \\
-\Delta v + v - v^3 &= \epsilon u, \\
u, v &\in H^1(\mathbb{R}^N),
\end{align*}
\]

in \(\mathbb{R}^N\) with dimension \(N = 1, 2, 3\). In \([1]\), A. Ambrosetti studied the following two systems

\[
\begin{align*}
-u_1'' + u_1 &= (1 + \varepsilon a_1(x))u_1^3 + \gamma u_2, \\
-u_2'' + u_2 &= (1 + \varepsilon a_2(x))u_2^3 + \gamma u_1, \\
u_1, u_2 &\in H^1(\mathbb{R}),
\end{align*}
\]

\[
\begin{align*}
-\varepsilon^2 u_1'' + u_1 + U_1(x)u_1 &= u_1^3 + \gamma u_2, \\
-\varepsilon^2 u_2'' + u_2 + U_2(x)u_2 &= u_2^3 + \gamma u_1, \\
u_1, u_2 &\in H^1(\mathbb{R}),
\end{align*}
\]
and proved the existence of non-trivial solution for (1.9) under the conditions \( a_i \in L^\infty(\mathbb{R}) \),
\[
\lim_{{|x|\to \infty}} a_i(x) = 0, \; i = 1, 2, \; 0 < \gamma < 1, \; \gamma \neq 3/5, \; \text{and} \; (1.10)
\] possesses a solution concentrating at nondegenerate stationary points of the sum \( U_1 + U_2 \) when \( \varepsilon \to 0 \) under the conditions \( U_i \in L^\infty \) and \( \inf_{x \in \mathbb{R}} U_i(x) > -1, \; i = 1, 2 \). The main tools in [1, 3, 4] are the perturbation techniques, we refer [5] for detailed discussions about these methods. In [7], the following system was considered
\[
\begin{aligned}
-u'' + a(x)u_1 - b(x)u_2 &= c(x)H_1(u_1, u_2)u_1, \\
-u'' + d(x)u_2 - e(x)u_1 &= f(x)H_2(u_1, u_2)u_2, \\
u_1, u_2 &\in H^1(\mathbb{R}),
\end{aligned}
\tag{1.11}
\]
the authors got a non-trivial solution via Krasnosel’skii fixed point theory. We note that the potentials in systems (1.7)-(1.11) are positive.

To prove the main theorem, we deal with the existence problem of non-trivial solutions by variational methods. We first study an eigenvalue problem, whose eigenfunctions are solutions of (1.3) but without the nonlinear term, then the non-zero critical point of the functional related to the nonlinear perturbation of this eigenvalue problem is a weak solution of (1.3). To find the critical point, we use a critical point theorem developed by Degiovanni and Lancelotti in [10].

The rest of the paper is organized as follows. The variational setting is contained in section 2. In section 3, we study the eigenvalue problem. We prove that there exists a divergent sequence of eigenvalues which are defined by the cohomological index. We prove Theorem 1.1 and 1.2 in section 4.

2 Variational setting

Let \( H_1 := \{ u_1 \in H^1(\mathbb{R}^N) | \int_{\mathbb{R}^N} b_1(x)u_1^2dx < \infty \} \), then \( H_1 \) is a Hilbert Space with inner product \( \langle u_1, v_1 \rangle_1 = \int_{\mathbb{R}^N} (\nabla u_1 \cdot \nabla v_1 + b_1(x)u_1v_1)dx \) and norm \( \| u_1 \|^2_1 = \langle u_1, u_1 \rangle_1 \). Similarly, let \( H_2 := \{ u_2 \in H^1(\mathbb{R}^N) | \int_{\mathbb{R}^N} b_2(x)u_2^2dx < \infty \} \), then \( H_2 \) is a Hilbert Space with inner product \( \langle u_2, v_2 \rangle_2 = \int_{\mathbb{R}^N} (\nabla u_2 \cdot \nabla v_2 + b_2(x)u_2v_2)dx \) and norm \( \| u_2 \|^2_2 = \langle u_2, u_2 \rangle_2 \).

For the non-radially symmetric case, by the condition (B), \( H_1 \) and \( H_2 \) can be compactly embedded into \( L^p(\mathbb{R}^N) \), \( 2 \leq p < 2^* \) (see for example, [6, 17]). Set \( \mathcal{H} := H_1 \times H_2 \), then \( \mathcal{H} \) is a Hilbert space with inner product \( \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_1 + \langle \cdot, \cdot \rangle_2 \) and with norm \( \| u \|^2 = \| u_1 \|^2_1 + \| u_2 \|^2_2 \) for \( u = (u_1, u_2) \).

For the radially symmetric case, let \( H_{1,r} := \{ u_1 \in H_1 | u_1 \text{ is radially symmetric} \} \),


\( H_{2,r} := \{ u_2 \in H_2 | u_2 \text{ is radially symmetric} \} \), then \( H_{i,r} \) is a Hilbert Space with inner product \( \langle \cdot, \cdot \rangle_i \) and norm \( \| \cdot \|_i \) for \( i = 1, 2 \). By condition (B)\(_r\), \( H_{i,r} \) can be compactly embedded into \( L^p(\mathbb{R}^N) \), \( 2 \leq p < 2^* \) for \( i = 1, 2 \) (see \[6\] [17]). In this case, we set \( H_r := H_{1,r} \times H_{2,r} \), then \( H_r \) is a Hilbert space with inner product \( \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_1 + \langle \cdot, \cdot \rangle_2 \) and with norm \( \| u \|^2 = \| u_1 \|^2_1 + \| u_2 \|^2_2 \) for \( u = (u_1, u_2) \).

In order to prove \( \text{Theorem 1.1} \) we define a functional \( \Psi : H \rightarrow \mathbb{R} \) by

\[
\Psi(u) = E(u) - \lambda J(u) - P(u), \ u = (u_1, u_2) \in H,
\]

where

\[
E(u) = \frac{1}{2} \| u \|^2,
\]

\[
J(u) = \int_{\mathbb{R}^N} \left( \frac{1}{2} V_1(x)u_1^2 + \gamma(x)u_1u_2 + \frac{1}{2} V_2(x)u_2^2 \right) dx,
\]

and

\[
P(u) = \int_{\mathbb{R}^N} W(x, u)dx = \int_{\mathbb{R}^N} W(x, u_1, u_2)dx,
\]

then these four functionals are \( C^1 \), and for \( u = (u_1, u_2), \ v = (v_1, v_2) \in H \), there hold

\[
\langle E'(u), v \rangle = \int_{\mathbb{R}^N} (\nabla u_1 \cdot \nabla v_1 + b_1(x)u_1 v_1) dx + \int_{\mathbb{R}^N} (\nabla u_2 \cdot \nabla v_2 + b_2(x)u_2 v_2) dx,
\]

\[
\langle J'(u), v \rangle = \int_{\mathbb{R}^N} (V_1(x)u_1 v_1 + \gamma(x)u_2 v_1 + \gamma(x)u_1 v_2 + V_2(x)u_2 v_2) dx,
\]

\[
\langle P'(u), v \rangle = \int_{\mathbb{R}^N} (W_1(x, u_1, u_2)v_1 + W_2(x, u_1, u_2)v_2) dx,
\]

\[
\langle \Psi'(u), v \rangle = \langle E'(u), v \rangle - \lambda \langle J'(u), v \rangle - \langle P'(u), v \rangle.
\]

It is clear that critical points of \( \Psi \) are weak solutions of \( (1.3) \).

For the radially symmetric case, we can also define these four functionals and \( (2.16)-(2.19) \) hold, the only difference is the domain \( H \) of the functional \( \Psi \) is replaced by \( H_r \).

And the critical points of the functional \( \Psi \) are radially symmetric weak solutions of \( (1.3) \).

In order to find a critical point of \( \Psi \), we need the following critical point theorem. It was proved in \[10\], where the functional was supposed to satisfy the \( (PS) \) condition. Recently, in \[9\], the author extended it to more general case (the functional space is completely regular topological space or metric space). As observed in \[15\], if the functional space is a real Banach space, according to the proof of Theorem 6.10 in \[9\], the Cerami condition is sufficient for the compactness of the set of critical points at a fixed level and the first deformation lemma to hold (see \[10\]). So this critical point theorem still hold under the Cerami condition.
Theorem 2.1 (10) Let $\mathcal{H}$ be a real Banach space and let $C_-, C_+$ be two symmetric cones in $\mathcal{H}$ such that $C_+$ is closed in $\mathcal{H}$, $C_- \cap C_+ = \{0\}$ and
\[
i(C_+ \setminus \{0\}) = i(\mathcal{H} \setminus C_+) = m < \infty.
\]

Define the following four sets by
\[
D_- = \{u \in C_- | \|u\| \leq r_-, \}
\]
\[
S_+ = \{u \in C_+ | \|u\| = r_+ \},
\]
\[
Q = \{u + te | u \in C_-, t \geq 0, \|u + te\| \leq r_-, e \in H \setminus C_- \},
\]
\[
H = \{u + te | u \in C_-, t \geq 0, \|u + te\| = r_- \}.
\]

Then $(Q, D_- \cup H)$ links $S_+$ cohomologically in dimension $m + 1$ over $\mathbb{Z}_2$. Moreover, suppose $\Psi \in C^1(\mathcal{H}, \mathbb{R})$ satisfying the Cerami condition, and $\sup_{x \in D_- \cup H} \Psi(x) < \inf_{x \in S_+} \Psi(x)$, $\sup_{x \in Q} \Psi(x) < \infty$. Then $\Psi$ has a critical value $d \geq \inf_{x \in S_+} \Psi(x)$.

For convenience, let us recall the definition and some properties of the cohomological index of Fadell-Rabinowitz for a $\mathbb{Z}_2$-set, see [11, 12, 16] for details. For simplicity, we only consider the usual $\mathbb{Z}_2$-action on a linear space, i.e., $\mathbb{Z}_2 = \{1, -1\}$ and the action is the usual multiplication. In this case, the $\mathbb{Z}_2$-set $A$ is a symmetric set with $-A = A$.

Let $E$ be a normed linear space. We denote by $\mathcal{S}(E)$ the set of all symmetric subsets of $E$ which do not contain the origin of $E$. For $A \in \mathcal{S}(E)$, denote $\tilde{A} = A/\mathbb{Z}_2$. Let $\rho : \tilde{A} \to \mathbb{R}P^\infty$ be the classifying map and $\rho^* : H^*(\mathbb{R}P^\infty) = \mathbb{Z}_2[\omega] \to H^*(\tilde{A})$ the induced homomorphism of the cohomology rings. The cohomological index of $A$, denoted by $i(A)$, is defined by $\sup\{k \geq 1 : \rho^*(\omega^{k-1}) \neq 0\}$. We list some properties of the cohomological index here for further use in this paper. Let $A, B \in \mathcal{S}(E)$, there hold

(i1) (monotonicity) if $h : A \to B$ is an odd map, then $i(A) \leq i(B)$,

(i2) (continuity) if $C$ is a closed symmetric subset of $A$, then there exists a closed symmetric neighborhood $N$ of $C$ in $A$, such that $i(N) = i(C)$, hence the interior of $N$ in $A$ is also a neighborhood of $C$ in $A$ and $i(\text{int}N) = i(C)$,

(i3) (neighborhood of zero) if $V$ is bounded closed symmetric neighborhood of the origin in $E$, then $i(\partial V) = \dim E$. 

8
3 The eigenvalue problem

First we solve the eigenvalue problem

\[ E'(u) = \mu J'(u), \; u \in \mathcal{H}. \quad (3.20) \]

**Lemma 3.1** For any \( u = (u_1, u_2), v = (v_1, v_2) \in \mathcal{H} \), it holds that

\[ \langle E'(u) - E'(v), u - v \rangle \geq (\|u_1\|_1 - \|v_1\|_1)^2 + (\|u_2\|_2 - \|v_2\|_2)^2. \quad (3.21) \]

**Proof.** By direct computations, we have

\[
\langle E'(u) - E'(v), u - v \rangle \\
= \int_{\mathbb{R}^N} (|\nabla u_1|^2 + |\nabla v_1|^2 - 2\nabla u_1 \cdot \nabla v_1) \, dx + \int_{\mathbb{R}^N} b_1(x) (|u_1|^2 + |v_1|^2 - 2u_1 v_1) \, dx \\
+ \int_{\mathbb{R}^N} (|\nabla u_2|^2 + |\nabla v_2|^2 - 2\nabla u_2 \cdot \nabla v_2) \, dx + \int_{\mathbb{R}^N} b_2(x) (|u_2|^2 + |v_2|^2 - 2u_2 v_2) \, dx.
\]

From the definition of the norm in \( \mathcal{H} \), we can get

\[
\int_{\mathbb{R}^N} (|\nabla u_1|^2 + |\nabla v_1|^2 - 2\nabla u_1 \cdot \nabla v_1) \, dx + \int_{\mathbb{R}^N} b_1(x) (|u_1|^2 + |v_1|^2 - 2u_1 v_1) \, dx \\
= \|u_1\|_1^2 + \|v_1\|_1^2 - 2\langle u_1, v_1 \rangle_1 \geq \|u_1\|_1^2 + \|v_1\|_1^2 - 2\|\|u_1\|_1\|_1 (\|u_1\|_1 - \|v_1\|_1)^2, \quad (3.22)
\]

\[
\int_{\mathbb{R}^N} (|\nabla u_2|^2 + |\nabla v_2|^2 - 2\nabla u_2 \cdot \nabla v_2) \, dx + \int_{\mathbb{R}^N} b_2(x) (|u_2|^2 + |v_2|^2 - 2u_2 v_2) \, dx \\
= \|u_2\|_2^2 + \|v_2\|_2^2 - 2\langle u_2, v_2 \rangle_2 \geq \|u_2\|_2^2 + \|v_2\|_2^2 - 2\|\|u_2\|_2\|_2 (\|u_2\|_2 - \|v_2\|_2)^2. \quad (3.23)
\]

Now (3.22) and (3.23) imply (3.21).

**Lemma 3.2** If \( u_n \to u \) and \( \langle E'(u_n), u_n - u \rangle \to 0 \), then \( u_n \to u \) in \( \mathcal{H} \).

**Proof.** Since \( \mathcal{H} \) is a Hilbert space and \( u_n = (u_n, v_n) \to u = (u, v) \), we only need to show that \( \|u_n\| \to \|u\| \). Note that

\[
\lim_{n \to \infty} \langle E'(u_n) - E'(u), u_n - u \rangle = \lim_{n \to \infty} (\langle E'(u_n), u_n - u \rangle - \langle E'(u), u_n - u \rangle) = 0.
\]

By inequality (3.21) we have

\[
\langle E'(u_n) - E'(u), u_n - u \rangle \geq (\|u_n\|_1 - \|u\|_1)^2 + (\|v_n\|_2 - \|v\|_2)^2.
\]

So \( \|u_n\|_1 \to \|u\|_1, \|v_n\|_2 \to \|v\|_2 \) and hence \( \|u_n\| \to \|u\| \) as \( n \to \infty \) and the assertion follows.

**Lemma 3.3** \( J' \) is weak-to-strong continuous, i.e. \( u_n \to u \) in \( \mathcal{H} \) implies \( J'(u_n) \to J'(u) \).
Proof. Since \( u_n \to u \in \mathcal{H} \), \( u_n \to u \) in \( H_1 \). So \( u_n \to u \) in \( L^2(\mathbb{R}^N) \) because \( H_1 \) compactly embedded into \( L^2(\mathbb{R}^N) \). Similarly, we have \( v_n \to v \) in \( L^2(\mathbb{R}^N) \).

For any \( v = (\tilde{u}, \tilde{v}) \in \mathcal{H} \),

\[
\int_{\mathbb{R}^N} \tilde{u}^2 \, dx \leq \frac{1}{b_1} \int_{\mathbb{R}^N} b_1(x) \tilde{u}^2 \, dx \leq \frac{1}{b_1} \|\tilde{u}\|^2_1 \leq \frac{1}{b_1} \|v\|^2
\]

so \( \left( \int_{\mathbb{R}^N} \tilde{u}^2 \, dx \right)^\frac{1}{2} \leq C \|v\| \). Similarly, we have \( \left( \int_{\mathbb{R}^N} \tilde{v}^2 \, dx \right)^\frac{1}{2} \leq C \|v\| \). Then,

\[
|\langle J'(u_n) - J'(u), v \rangle| = \left| \int_{\mathbb{R}^N} (V_1(x)(u_n - u)\tilde{u} + \gamma(x)(v_n - v)\tilde{u} + \gamma(x)(u_n - u)\tilde{v} + V_2(x)(v_n - v)\tilde{v}) \, dx \right|
\]

\[
\leq \|V_1\|_\infty \left( \int_{\mathbb{R}^N} (u_n - u)^2 \, dx \right)^\frac{1}{2} \left( \int_{\mathbb{R}^N} \tilde{u}^2 \, dx \right)^\frac{1}{2} + \|\gamma\|_\infty \left( \int_{\mathbb{R}^N} (v_n - v)^2 \, dx \right)^\frac{1}{2} \left( \int_{\mathbb{R}^N} \tilde{u}^2 \, dx \right)^\frac{1}{2}
\]

\[
+ \|\gamma\|_\infty \left( \int_{\mathbb{R}^N} (u_n - u)^2 \, dx \right)^\frac{1}{2} \left( \int_{\mathbb{R}^N} \tilde{v}^2 \, dx \right)^\frac{1}{2} + \|V_2\|_\infty \left( \int_{\mathbb{R}^N} (v_n - v)^2 \, dx \right)^\frac{1}{2} \left( \int_{\mathbb{R}^N} \tilde{v}^2 \, dx \right)^\frac{1}{2}
\]

\[
\leq C \left( \int_{\mathbb{R}^N} (u_n - u)^2 \, dx \right)^\frac{1}{2} \|v\| + C \left( \int_{\mathbb{R}^N} (v_n - v)^2 \, dx \right)^\frac{1}{2} \|v\| \to 0,
\]

hence \( J'(u_n) \to J'(u) \).

Lemma 3.4 If \( u_n \to u \) in \( \mathcal{H} \), then \( J(u_n) \to J(u) \).

Proof.

\[
2|J(u_n) - J(u)| = |\langle J'(u_n), u_n \rangle - \langle J'(u), u \rangle|
\]

\[
= |\langle J'(u_n), u_n \rangle - \langle J'(u), u_n \rangle + \langle J'(u), u_n - u \rangle|
\]

\[
\leq \|J'(u_n) - J'(u)\| \|u_n\| + o(1).
\]

Because \( u_n \to u \), \( u_n \) is bounded. From Lemma 3.3 we have \( J(u_n) \to J(u) \).

In this section, we assume that \( V_1 \) and \( V_2 \) satisfy the following condition

\[
(**) \quad \text{meas}\{x \in \mathbb{R}^N | V_1(x) > 0\} > 0 \quad \text{or} \quad \text{meas}\{x \in \mathbb{R}^N | V_2(x) > 0\} > 0.
\]

Set \( \mathcal{M} = \{u \in \mathcal{H} | J(u) = 1\} \), by (**), we can see that \( \mathcal{M} \) is not empty, see also Lemma 3.4 below. Clearly, \( J(u) = \frac{1}{2} \langle J'(u), u \rangle \), so 1 is a regular value of the functional \( J \). Hence by the implicit theorem, \( \mathcal{M} \) is a \( C^1 \)-Finsler manifold. It is complete, symmetric, since \( J \) is continuous and even. Moreover, 0 is not contained in \( \mathcal{M} \), so the trivial \( \mathbb{Z}_2 \)-action on \( \mathcal{M} \) is free. Set \( \bar{E} = E|_{\mathcal{M}} \).

Lemma 3.5 If \( u \in \mathcal{M} \) satisfies \( \bar{E}(u) = \mu \) and \( \bar{E}'(u) = 0 \), then \( (\mu, u) \) is a solution of the functional equation \( J(u) = \mu \).
Proof. By Proposition 3.54 in [16], the norm of $\tilde{E}'(u) \in T_u^*M$ is given by
$$||\tilde{E}'(u)||_u^* = \min_{\nu \in \mathbb{R}} ||E'(u) - \nu J'(u)||^*$$
(3.20) and $\mu = \tilde{E}(u) = \frac{1}{2}(E'(u),u) = \frac{1}{2} \nu J'(u),u) = \frac{1}{2} \nu J(u) = \nu$. 

Lemma 3.6 $\tilde{E}$ satisfies the (PS) condition, i.e. if $(u_k)$ is a sequence on $M$ such that
$\tilde{E}(u_k) \to c$, and $\tilde{E}'(u_k) \to 0$, then up to a subsequence $u_k \to u \in M$ in $\mathcal{H}$.

Proof. First, from the definition of $E$, we can deduce that $(u_k)$ is bounded. Then, up
to a subsequence, $u_k$ converges weakly to some $u$, by Lemma 3.1, we have $J(u) = 1$, so
$u \in M$.

From $\tilde{E}'(u_k) \to 0$, we have $E'(u_k) - \nu_k J'(u_k) \to 0$ in $\mathcal{H}$ for a sequence of real numbers
$(\nu_k)$. So $\langle E'(u_k) - \nu_k J'(u_k),u_k \rangle \to 0$, thus we get $\nu_k \to c$. By Lemma 3.3 we have
$E'(u_k) \to c J'(u)$. Hence
$$\langle E'(u_k),u_k - u \rangle = \langle E'(u_k) - c J'(u),u_k - u \rangle + \langle c J'(u),u_k - u \rangle \to 0.$$ 

By Lemma 3.2 we obtain $u_k \to u$. \hfill \Box

Let $\mathcal{F}$ denote the class of symmetric subsets of $M$, $\mathcal{F}_n = \{ M \in \mathcal{F} | i(M) \geq n \}$ and
$$\mu_n = \inf_{M \in \mathcal{F}_n} \sup_{u \in M} E(u). \quad (3.24)$$

Since $\mathcal{F}_n \supset \mathcal{F}_{n+1}$, $\mu_n \leq \mu_{n+1}$.

Lemma 3.7 If (**) holds, then for every $\mathcal{F}_n$, there is a compact symmetric set $M \in \mathcal{F}_n$.

Proof. We follow the idea of the proof of Theorem 3.2 in [13]. Suppose
meas$\{ x \in \mathbb{R}^N | V_1(x) > 0 \} > 0$, it implies that $\forall n \in \mathbb{N}$, there exist $n$ open balls $(B_i)_{1 \leq i \leq n}$ in $\mathbb{R}^N$
such that $B_i \cap B_j = \emptyset$ for $i \neq j$ and meas$\{ x \in \mathbb{R}^N | V_1(x) > 0 \} \cap B_i > 0$. Approximating the
characteristic function $\chi_i$ of set $\{ x \in \mathbb{R}^N | V_1(x) > 0 \} \cap B_i$ by a $C^\infty$-function $u_i$ in $L^2(\mathbb{R}^N)$,
and require that the sequence $\{ u_i \}_{1 \leq i \leq n} \subseteq C^\infty(\mathbb{R}^N)$ satisfies $\int_{\mathbb{R}^N} V_1(x)|u_i|^2dx > 0$ for
all $i = 1, \cdots, n$ and supp $u_i \cap \text{supp } u_j = \emptyset$ when $i \neq j$. Set $u_i = (u_i,0) \in \mathcal{H}$, then
$J(u_i) = \frac{1}{2} \int_{\mathbb{R}^N} V_1(x)|u_i|^2dx > 0$. Normalizing $u_i$, we assume that $J(u_i) = 1$. Denote by
$U_n$ the space spanned by $(u_i)_{1 \leq i \leq n}$. $\forall u \in U_n$, we have $u = \sum_{i=1}^n \alpha_i u_i$ and $J(u) = \sum_{i=1}^n |\alpha_i|^2$. 

So $(J(u))^2$ defines a norm on $U_n$. Since $U_n$ is $n$ dimensional, this norm is equivalent to
$|| \cdot ||$. Thus $\{ u \in U_n | J(u) = 1 \} \subseteq M$ is compact with respect to the norm $|| \cdot ||$ and by
the property (i3) of cohomological index, \( i(\{ u \in U_n | J(u) = 1 \}) = n \). So \( \{ u \in U_n | J(u) = 1 \} \in F_n \). If \( \text{meas}\{ x \in \mathbb{R}^N | V_2(x) > 0 \} > 0 \), the proof is similar.

By Lemma 3.7, we have \( \mu_n < +\infty \), and by condition (B), there holds \( \mu_n \geq 0 \). Furthermore, by Lemma 3.6 and Proposition 3.52 in [10], we see that \( \mu_n \) is sequence of critical values of \( \tilde{E} \) and \( \mu_n \to +\infty \), as \( n \to \infty \). By Lemma 3.5, we get a divergent sequence of eigenvalues for problem (3.20). So we have the following result.

**Theorem 3.8** Under the condition (**) the problem (3.20) has an increasing sequence of eigenvalues for problem (3.20).

**Lemma 3.9** Under the condition (**) Set

\[
\rho_n = \inf_{K \subseteq F_n} \sup_{u \in K} E(u),
\]

where \( F_n = \{ K \subseteq F_n | K \text{ is compact} \} \). We have \( \mu_n = \rho_n \).

**Proof**. From Lemma 3.7, \( F_n \neq \emptyset \) and so \( \rho_n < +\infty \). It is obvious that \( \mu_n \leq \rho_n \). If \( \mu_n < \rho_n \), there is \( M \in \mathcal{F} \) such that \( \sup_{u \in M} E(u) < \rho_n \). The closure \( \overline{M} \) of \( M \) in \( \mathcal{M} \) is still in \( \mathcal{F} \). By the property (i2) of the cohomological index, we can find a small open neighborhood \( A \in \mathcal{F} \) of \( \overline{M} \) in \( \mathcal{M} \) such that \( \sup_{u \in A} E(u) < \rho_n \). As it was proved in the proof of Proposition 3.1 in [10], for every symmetric open subset \( A \) of \( \mathcal{M} \), there holds \( i(A) = \sup\{ i(K) | K \subseteq A \} \). So we can choose a symmetric compact subset \( K \subseteq A \) with \( i(K) \geq n \) and \( \sup_{u \in K} E(u) < \rho_n \). This contradicts to the definition of \( \rho_n \). Therefore, we have \( \mu_n = \rho_n \).

Set \( C_m = \{ u \in \mathcal{H} \setminus \{ 0 \} | E(u) \leq \mu_m J(u) \} \) and \( D_m = \{ u \in \mathcal{H} | E(u) < \mu_{m+1} J(u) \} \). It is clear that \( C_m, D_m \in \mathcal{S}(\mathcal{H}) \), i.e., \( C_m \) and \( D_m \) are symmetric subsets of \( \mathcal{H} \) and do not contain 0.

**Theorem 3.10** If \( \mu_m < \mu_{m+1} \) for some \( m \in \mathbb{N} \), then the cohomological indices satisfy

\[
i(C_m) = i(D_m) = m.
\]

**Proof**. Follow the idea of the proof of Theorem 3.2 in [10]. Suppose \( \mu_m < \mu_{m+1} \). If we set \( A_m = \{ u \in \mathcal{M} | E(u) \leq \mu_m \} \) and \( B_m = \{ u \in \mathcal{M} | E(u) < \mu_{m+1} \} \), by the definition (3.24), we have \( i(A_m) \leq m \). Assume that \( i(A_m) \leq m - 1 \). Then, by the property (i2) of the cohomological index, there exists a symmetric neighborhood \( N \) of \( A_m \) in \( \mathcal{M} \) satisfying \( i(N) = i(A_m) \). By the equivariant deformation theorem (see [8]), there exists \( \delta > 0 \) and an odd continuous map \( i : \{ u \in \mathcal{M} | E(u) \leq \mu_m + \delta \} \to \{ u \in \mathcal{M} | E(u) \leq \mu_m - \delta \} \cup N = N \).
Hence $i(u \in M \mid E(u) \leq \mu_m + \delta) \leq m - 1$. By (3.24), there exists $M \in F_m$ such that $\sup_{u \in M} E(u) < \mu_m + \delta$. So $M \subseteq \{u \in M \mid E(u) \leq \mu_m + \delta\}$ and thus $i(M) \leq m - 1$. This contradicts to the fact that $M \in F_m$. Thus we have $i(A_m) = m$. By 2-homogeneity of the functionals $E, J$, the map $h : C_m \rightarrow A_m$ with $h(u) = \frac{1}{\sqrt{J(u)}} u$ is odd, from the monotonicity (i1) of the cohomological index, we have $i(C_m) \leq m$. But it is clear that $A_m \subseteq C_m$, we have $i(C_m) \geq m$, so $i(C_m) = m$.

Since $A_m \subseteq B_m$ and $i(A_m) = m$, we have $i(B_m) \geq m + 1$. As in the proof of Lemma 3.9, there exists a symmetric, compact subset $K$ of $B_m$ with $i(K) \geq m + 1$. Since $\max_{u \in K} E(u) < \mu_{m+1}$, this contradicts to definition (3.24). So $i(B_m) = m$. Similar to the above arguments, we also have $i(D_m) = m$. \[\square\]

**Remark 3.11** If we consider the following eigenvalue problem,

$$E'(u) = \mu J'(u), \quad u \in H_r,$$

(3.27)

then all the results in this section still hold, we only need to replace the space $\mathcal{H}$ by $\mathcal{H}_r$.

### 4 Proof of the main theorems

Replacing $(\lambda, V_i, \gamma)$ with $(-\lambda, -V_i, -\gamma)$ if necessary, we can assume that $\lambda \geq 0$. First, we consider the case that condition $(\ast\ast)$ holds and there exists $m \geq 1$ such that $\mu_m \leq \lambda < \mu_{m+1}$. Set

$$C_+ = \{u \in \mathcal{H} \mid E(u) \geq \mu_m J(u)\},$$

(4.28)

$$C_- = \{u \in \mathcal{H} \mid E(u) \leq \mu_m J(u)\}.\quad (4.29)$$

It is easy to see that $C_-, C_+$ are two symmetric closed cones in $\mathcal{H}$ and $C_- \cap C_+ = \{0\}$. By (3.26) we have

$$i(C_- \setminus \{0\}) = i(C_m) = i(D_m) = i(\mathcal{H} \setminus C_+) = m.\quad (4.30)$$

**Lemma 4.1** There exist $r_+ > 0$ and $\alpha > 0$ such that $\Psi(u) > \alpha$ for $u \in C_+$ and $\|u\| = r_+$.

**Proof.** Let $\varepsilon > 0$ be small enough, from (W1) and (W3), we have $|W(x, z)| \leq \varepsilon|z|^2 + C_\varepsilon|z|^p$. 

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By the Sobolev embedding inequality, for $u = (u_1, u_2) \in C_+$, we can get

\[
\Psi(u) = E(u) - \lambda J(u) - P(u)
\]
\[
= E(u) - \frac{\lambda}{\mu_{m+1}} \cdot \mu_{m+1} J(u) - P(u)
\]
\[
\geq E(u) - \frac{\lambda}{\mu_{m+1}} E(u) - \varepsilon \int_{R^N} |u_1|^2 dx
\]
\[
- \varepsilon \int_{R^N} |u_2|^2 dx - C_\varepsilon \int_{R^N} |u_1|^p dx - C_\varepsilon \int_{R^N} |u_2|^p dx
\]
\[
\geq E(u) - \frac{\lambda}{\mu_{m+1}} E(u) - \varepsilon \int_{R^N} b_1(x) |u_1|^2 dx - \frac{\varepsilon}{2} \int_{R^N} b_2(x) |u_2|^2 dx
\]
\[
- C_\varepsilon \int_{R^N} |u_1|^p dx - C_\varepsilon \int_{R^N} |u_2|^p dx
\]
\[
\geq (1 - \frac{\lambda}{\mu_{m+1}} - 2 \max(\frac{\varepsilon}{m}, \frac{\varepsilon}{2})) E(u) - C_\varepsilon \int_{R^N} |u_1|^p dx - C_\varepsilon \int_{R^N} |u_2|^p dx
\]
\[
\geq \frac{1}{2}(1 - \frac{\lambda}{\mu_{m+1}} - 2 \max(\frac{\varepsilon}{m}, \frac{\varepsilon}{2})) \|u\|^2 - C \|u\|^p.
\] (4.31)

We remind that in the second inequality of (4.31), the condition (B) has been applied. Since $p > 2$, the assertion follows.

Since $\lambda \geq \mu_m$, by (W_1) it holds that

\[
\Psi(u) \leq 0, \ \forall u \in C_-
\] (4.32)

Set $R^+ = [0, +\infty)$. Following the idea of the proof of Theorem 4.1 in [10], we have

**Lemma 4.2** Let $e = (e_1, e_2) \in H \setminus C_-$, there exists $r_- > r_+$ such that $\Psi(u) \leq 0$ for $u \in C_- + R^+ e$ and $\|u\| \geq r_-.$

**Proof.** Define another norm on $H$ by $\|u\|_V^2 := \int_{R^N} (|V_1(x)| + |\gamma(x)| + 1)|u|^2 dx + \int_{R^N} (|V_2(x)| + |\gamma(x)| + 1)|v|^2 dx$ for $u = (u, v)$. Then the same reason as the proof of Theorem 4.1 in [10], there exists some constant $b > 0$ such that $\|u + te\| \leq b \|u + te\|_V$ for every $u \in C_-$, $t \geq 0$ and some $b > 0$. That is

\[
\int_{R^N} (|\nabla(u + te_1)|^2 + b_1(x)|u + te_1|^2) dx + \int_{R^N} (|\nabla(v + te_2)|^2 + b_2(x)|v + te_2|^2) dx
\]
\[
\leq b^2 \int_{R^N} (|V_1(x)| + |\gamma(x)| + 1)|u + te_1|^2 dx + b^2 \int_{R^N} (|V_2(x)| + |\gamma(x)| + 1)|v + te_2|^2 dx.
\] (4.33)

Let $\{u_k\}$ be a sequence such that $\|u_k\| \to +\infty$ and $u_k \in C_- + R^+ e$. Set $v_k = (u_k, v_k) := \frac{u_k}{\|u_k\|}$, then, up to a subsequence, $\{v_k\}$ converges to some $v = (u_0, v_0)$ weakly in $H$ and $u_k \to u_0$, $v_k \to v_0$ a.e. in $R^N$. Note that Lemma 4.4 is also true for functional $\int_{R^N} (|V_1(x)| + |\gamma(x)| + 1)|u|^2 dx + \int_{R^N} (|V_2(x)| + |\gamma(x)| + 1)|v|^2 dx$, $u = (u, v) \in H$, it follows from (4.33) that $\int_{R^N} (|V_1(x)| + |\gamma(x)| + 1)|u_0|^2 dx + \int_{R^N} (|V_2(x)| + |\gamma(x)| + 1)|v_0|^2 dx \geq \frac{1}{b^2}$.

So $|v| \neq 0$ on a positive measure set $\Omega_0$. By (W_2) we have

\[
\lim_{k \to \infty} \frac{W(x, u_k(x))}{\|u_k\|^2} = \lim_{k \to \infty} \frac{W(x, u_k, v_k(x))}{\|u_k\|^2 \|v_k(x)\|^2} |v_k(x)|^2 = +\infty, \ x \in \Omega_0.
\]
By \((W_1)\) and Fatou lemma we can get
\[
\frac{\int_{\mathbb{R}^N} W(x, u_k(x))dx}{\|u_k\|^2} \to +\infty, \text{ as } k \to \infty.
\]
By the arbitrariness of the sequence \(\{u_k\}\), we have
\[
\frac{\int_{\mathbb{R}^N} W(x, u(x))dx}{\|u\|^2} \to +\infty
\]
as \(\|u\| \to +\infty\) and \(u \in C_+ + \mathbb{R}^+e\). Noting that
\[
\Psi(u) = \frac{1}{2} - \frac{\lambda J(u)}{\|u\|^2} - \frac{\int_{\mathbb{R}^N} W(x, u(x))dx}{\|u\|^2}
\]
and by conditions (B) and (V), for \(u = (u, v) \in \mathcal{H}\)
\[
\frac{|J(u)|}{\|u\|^2} \leq C(\int_{\mathbb{R}^N} |u|^2dx + \int_{\mathbb{R}^N} |v|^2dx) \leq C(\int_{\mathbb{R}^N} b_1(x)|u|^2dx + \int_{\mathbb{R}^N} b_2(x)|v|^2dx) \leq C,
\]
the assertion follows from (4.34), (4.35) and (4.36).

**Lemma 4.3** \(\Psi\) satisfies the Cerami condition, i.e., for any sequence \(\{u_k\}\) in \(\mathcal{H}\) satisfying \((1 + \|u_k\|)\Psi'(u_k) \to 0\) and \(\Psi(u_k) \to c\) possesses a convergent subsequence.

**Proof.** Let \(\{u_k\}\) be a sequence in \(\mathcal{H}\) satisfying \((1 + \|u_k\|)\Psi'(u_k) \to 0\) and \(\Psi(u_k) \to c\). We claim that \(\{u_k\}\) is bounded in \(\mathcal{H}\). Otherwise, if \(\|u_k\| \to \infty\), we consider \(v_k := \frac{u_k}{\|u_k\|}\).

Then, up to subsequence, we get \(v_k \to v\) in \(\mathcal{H}\) and \(v_k \to v\) a.e. in \(\mathbb{R}^N\).

If \(v \neq 0\) in \(\mathcal{H}\), since \(\Psi'(u_k)u_k \to 0\), that is to say
\[
\|u_k\|^2 - \lambda J'(u_k) \cdot u_k - \int_{\mathbb{R}^N} \nabla_z W(x, u_k(x)) \cdot u_k dx = \|u_k\|^2 - 2\lambda J(u_k) - \int_{\mathbb{R}^N} \nabla_z W(x, u_k(x)) \cdot u_k dx \to 0,
\]
from (4.36), we have \(\frac{|J(u_k)|}{\|u_k\|^2} \leq C\), so by dividing the left hand side of (4.37) with \(\|u_k\|^2\) there holds
\[
\left| \int_{\mathbb{R}^N} \frac{\nabla_z W(x, u_k(x)) \cdot u_k(x)}{\|u_k\|^2} dx \right| \leq C'
\]
for some constant \(C' > 0\). On the other hand, Since \(v(x) \neq 0\) in some positive measure set \(\Omega \subset \mathbb{R}^N\), so \(v_k(x) \neq 0\) for large \(k\), and \(|u_k(x)| \to +\infty\) as \(k \to \infty\), for any fixed \(x \in \Omega\). So by \((W_2)\), we have
\[
\lim_{k \to \infty} \frac{|v_k(x)|^2 2W(x, u_k(x))}{\|u_k\|^2} = +\infty, \ \forall x \in \Omega.
\]
By Remark (1) before Theorem 1.1 we have
\[
\nabla_z W(x, u_k(x)) \cdot u_k(x) \geq 2W(x, u_k(x)).
\]
So as $k \to +\infty$, we have
\[
\int_{\mathbb{R}^N} \frac{\nabla_z W(x, \mathbf{u}_k(x)) \cdot \mathbf{u}_k(x)}{\| \mathbf{u}_k \|^2} \, dx = \int_{\{ \mathbf{v}_k \neq 0 \}} 2W(x, \mathbf{u}_k(x)) \frac{\nabla_z W(x, \mathbf{u}_k(x)) \cdot \mathbf{u}_k(x)}{\| \mathbf{u}_k \|^2} \, dx
\]
\[
\geq \int_{\mathbb{R}^N} \chi_{\{ \mathbf{v}_k \neq 0 \}}(x)|\mathbf{v}_k(x)|^2 \frac{2W(x, \mathbf{u}_k(x))}{\| \mathbf{u}_k \|^2} \, dx \geq \int_{\Omega} \chi_{\{ \mathbf{v}_k \neq 0 \}}(x)|\mathbf{v}_k(x)|^2 \frac{2W(x, \mathbf{u}_k(x))}{\| \mathbf{u}_k \|^2} \, dx \to \infty,
\]
this contradicts to (4.38). There is another explanation about the above estimate. We observe that there exists $\delta > 0$ such that $\text{meas}\{ x \in \Omega \,|\, |\mathbf{v}(x)| \geq \delta \} > 0$. Otherwise, $\forall n \in \mathbb{N}$,
$\text{meas}\{ x \in \Omega \,|\, |\mathbf{v}(x)| \geq \frac{1}{n} \} = 0$. Set $\Omega_n = \{ x \in \Omega \,|\, |\mathbf{v}(x)| \geq \frac{1}{n} \}$, then in $\Omega \setminus \bigcup_{n=1}^{+\infty} \Omega_n$, there holds $\mathbf{v}(x) = 0$. But $\Omega \setminus \bigcup_{n=1}^{+\infty} \Omega_n$ and $\Omega$ have the same measure, it is impossible. We may assume $\text{meas} \Omega < +\infty$, by Egorov’s theorem, there exists a positive measure subset $\Omega_0$ of $\{ x \in \Omega \,|\, |\mathbf{v}(x)| \geq \delta \}$ such that $\mathbf{v}_k$ uniformly convergent to $\mathbf{v}$, so for $k \geq K$ with $K$ large, there holds $|\mathbf{v}_k(x)| \geq \delta/2$ in $\Omega_0$. Thus (4.39) holds in $\Omega_0$. So there holds
\[
\int_{\{ \mathbf{v}_k(x) \neq 0 \}} |\mathbf{v}_k(x)|^2 \frac{\nabla_z W(x, \mathbf{u}_k(x)) \cdot \mathbf{u}_k(x)}{\| \mathbf{u}_k \|^2} \, dx \geq \int_{\Omega_0} |\mathbf{v}_k(x)|^2 \frac{2W(x, \mathbf{u}_k(x))}{\| \mathbf{u}_k \|^2} \, dx \to \infty.
\]
If $\mathbf{v} = 0$ in $\mathcal{H}$, inspired by [14], we choose $t_k \in [0, 1]$ such that $\Psi(t_k \mathbf{u}_k) := \max_{t \in [0, 1]} \Psi(t \mathbf{u}_k)$. For any $\beta > 0$ and $\tilde{\mathbf{v}}_k := (4\beta)^{1/2} \mathbf{v}_k \to 0$, by Lemma [3.3] and the compactness of $P'$ (see Lemma 1.22 in [18]) we have that $J(\tilde{\mathbf{v}}_k) \to 0$ and $\int_{\mathbb{R}^N} W(x, \tilde{\mathbf{v}}_k(x)) \, dx = P(\tilde{\mathbf{v}}_k) = P(\xi_k \tilde{\mathbf{v}}_k) = \langle P'(\xi_k \tilde{\mathbf{v}}_k), \tilde{\mathbf{v}}_k \rangle = \langle P'(\xi_k \tilde{\mathbf{v}}_k) - P'(0), \tilde{\mathbf{v}}_k \rangle + \langle P'(0), \tilde{\mathbf{v}}_k \rangle \to 0$ as $k \to \infty$, here $\xi_k \in (0, 1)$. So there holds
\[
\Psi(t_k \mathbf{u}_k) \geq \Psi(\tilde{\mathbf{v}}_k) = 2\beta - \lambda J(\tilde{\mathbf{v}}_k) - \int_{\mathbb{R}^N} W(x, \tilde{\mathbf{v}}_k(x)) \, dx \geq \beta,
\]
when $k$ is large enough. By the arbitrariness of $\beta$, it implies that
\[
\lim_{k \to \infty} \Psi(t_k \mathbf{u}_k) = \infty. \tag{4.40}
\]
Since $\Psi(0) = 0$, $\Psi(\mathbf{u}_k) \to c$, we have $t_k \in (0, 1)$. By the definition of $t_k$,
\[
\langle \Psi'(t_k \mathbf{u}_k), t_k \mathbf{u}_k \rangle = 0. \tag{4.41}
\]
From (4.40), (4.41), we have
\[
\Psi(t_k \mathbf{u}_k) - \frac{1}{\theta} \langle \Psi'(t_k \mathbf{u}_k), t_k \mathbf{u}_k \rangle = \int_{\mathbb{R}^N} \left( \frac{1}{2} \nabla_z W(x, t_k \mathbf{u}_k(x)) \cdot t_k \mathbf{u}_k(x) - W(x, t_k \mathbf{u}_k(x)) \right) \, dx \to \infty. \tag{4.42}
\]
By (W$_4$), there exists $\theta \geq 1$ such that
\[
\int_{\mathbb{R}^N} \left( \frac{1}{2} \nabla_z W(x, \mathbf{u}_k(x)) \cdot \mathbf{u}_k(x) - W(x, \mathbf{u}_k(x)) \right) \, dx \geq \frac{1}{\theta} \int_{\mathbb{R}^N} (\nabla_z W(x, t_k \mathbf{u}_k(x)) \cdot t_k \mathbf{u}_k(x) - W(x, t_k \mathbf{u}_k(x))) \, dx, \tag{4.43}
\]

Hence
\[
\int_{\mathbb{R}^N} \left( \frac{1}{2} \nabla_z W(x, u_k(x)) \cdot u_k(x) - W(x, u_k(x)) \right) \, dx \to \infty. \tag{4.44}
\]

On the other hand,
\[
\int_{\mathbb{R}^N} \left( \frac{1}{2} \nabla_z W(x, u_k(x)) \cdot u_k(x) - W(x, u_k(x)) \right) \, dx = \Psi(u_k) - \frac{1}{2} \langle \Psi'(u_k), u_k \rangle \to c. \tag{4.45}
\]

(4.44) and (4.45) are contradiction. Hence \(\{u_k\}\) is bounded in \(\mathcal{H}\). So up to a subsequence, we can assume that \(u_k \rightharpoonup u\) for some \(H\).

Since \(\Psi'(u_k) = E'(u_k) - \lambda J'(u_k) - P'(u_k) \to 0\) and \(J', P'\) are compact, we have that \(E'(u_k) \to \lambda J'(u) + P'(u)\) in \(\mathcal{H}\). So
\[
\langle E'(u_k), u_k - u \rangle = \langle E'(u_k) - (\lambda J'(u) + P'(u)), u_k - u \rangle + \langle \lambda J'(u) + P'(u), u_k - u \rangle \to 0.
\]

By Lemma 3.2, \(u_k \to u\) in \(\mathcal{H}\).

**Remark 4.4** If we replace the space \(\mathcal{H}\) by \(\mathcal{H}_r\), then Lemma 4.1, 4.2, 4.3 also hold.

**Proof of Theorem 1.1** Define \(D_-, S_+, Q, H\) as Theorem 2.1, then from Lemma 4.1 \(\Psi(u) \geq \alpha > 0\) for every \(u \in S_+\), from Lemma 4.2 \(\Psi(u) \leq 0\) for every \(u \in D_- \cup H\) and \(\Psi\) is bounded on \(Q\). Applying Lemma 4.3, it follows from Theorem 2.1 that \(\Psi\) has a critical value \(d \geq \alpha > 0\). Hence \(u\) is a non-trivial weak solution of (1.1).

For the cases \(0 \leq \lambda < \mu_1\) or \(V_1^+(x) \equiv 0 \equiv V_2^+(x)\), set \(C_- = \{0\}\) and \(C_+ = \mathcal{H}\), it is easy to see that the arguments above are valid. The proof of Theorem 1.1 is complete.

**Proof of Theorem 1.2** By Remarks 3.11 and 4.4, the proof is the same as that of Theorem 1.1 we only need to replace the space \(\mathcal{H}\) by \(\mathcal{H}_r\).

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