SHRINKING SIMPLICIAL SUBDIVISIONS, STRONG BARYCENTERS, AND LIMIT SETS OF CODIMENSION ONE QUASI-CONVEX SUBGROUPS

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Abstract. We show that quasi-convex subgroups of negatively curved manifold groups with codimension one have nicely embedded limit sets in the visual boundary if the complement of the limit sets admits what we call strong barycenters, a property related to the absence of large diameter sets with ‘positive curvature’. Furthermore, we show that the same result can be obtained if, in the complement of the limit set, simplicial complexes can be subdivided in a way that ‘shrinks’ them metrically. This provides us with two sufficient geometric conditions for the limit set of a quasi-convex subgroup to be nicely embedded.

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1. Introduction

Let $M$ be a closed, $n+1$-dimensional, negatively curved Riemannian manifold and $G$ its fundamental group. A question of interest in both geometric group theory and manifold theory is when $G$ can appear as the fundamental group of a negatively curved manifold of constant negative curvature. As was shown by Gromov and Thurston \cite{GT87}, there are infinitely many groups for which the manifold $M$ cannot be replaced by a manifold of constant negative curvature. One way to study this question is to study the action of $G$ on the universal covering of $M$, which we denote by $X$. Various people have
provided different characterisations on when the action of $G$ on $X$ is conjugate to an action on a constant curvature hyperbolic space by using entropy, topological dimension, and growth [BK02, BK04, Ham90, Kin15]. All these results require to understand the action of the group $G$ on $X$ as a whole. If $G$ has some interesting subgroups, one may hope that such subgroups can be used to break the process of determining one of these invariants into several steps. In this paper, we will concern ourselves with some topological questions that arise if $G$ admits a quasi-convex subgroup $H$, which has ‘codimension one’ in the sense that its limit set $\Lambda(H) \subset \partial_\infty X$ in the visual boundary of $X$ cuts the boundary into several connected components.

If the action of $G$ on $X$ were conjugate to an action on $H_{n+1}$, meaning that there exists a geometric action of $G$ on $H_{n+1}$ and a $G$-equivariant quasi-isometry from $X$ to $H_{n+1}$, then we could conjugate the action of $H$ to an action on $H_{n+1}$ as well. This puts some restrictions on what the limit set of $H$ can look like, both in terms of its topological type and the way it embeds into $\partial_\infty X$. The easiest example for this is the case when the limit set of $H$ is a sphere of codimension one, i.e. $\Lambda(H) \equiv S^{n-1} \subset \partial_\infty X \equiv S^n$.

Basic homology theory allows us to see that in this case, $\partial_\infty X \setminus \Lambda(H)$ has two connected components. However, if $n > 2$, such an embedding could be wild, meaning that the connected components of the complement may not be simply connected. (If $n = 2$, the Jordan curve theorem tells us that the complement consists of two discs. In higher dimensions, this is not true. Counter-examples include Alexanders horned sphere.) If the geometry of the embedding of $H$ in $G$ is to help us recognizing whether $G$ acts nicely on a constant curvature space, the topology of its boundary can be expected to be nice. This leads us to the central question we discuss in this paper, which is when the connected components of $\partial_\infty X \setminus \Lambda(H)$ are simply connected.

We provide two sufficient conditions for this, which are somewhat different in nature and allow for different interpretations.

Given $M, G, X, H, \text{ and } \Lambda(H)$ as above, we denote $C := C(\Lambda(H)) \subset X$ to be the closed convex hull of $\Lambda(H)$ in $X$. Since $X$ is negatively curved and $C$ is convex and closed, there exists a continuous closest-point projection map $p_C : X \to C$. This map admits a continuous extension to $\partial_\infty X$ that can be obtained as follows: Let $C_\epsilon$ denote the $\epsilon$-neighbourhood of $C$. By [Val76], the topological boundary $\partial C_\epsilon \subset X$ is a $C^{1,1}$-manifold. In particular, for every $\epsilon > 0$, there exists a unit normal vector field $N$ on $\Sigma_\epsilon$ that points outwards of $C_\epsilon$. These normal vector fields together form a continuous vector field on $X \setminus C$ that are orthogonal to $\Sigma_\epsilon$ for all $\epsilon > 0$. If we look at the geodesic flow $\Phi^N$ induced by $N$, we see that it preserves the fibres of $\pi_C$, i.e. $\pi_C(\Phi^N_t(p)) = \pi_C(p)$ for all $p \in X \setminus C$. We extend $\pi_C$ by setting $\pi_C(\xi) = \pi_C(p)$, where $\xi \in \partial_\infty X \setminus \Lambda(H)$ and $p \in X \setminus C$ such that the flow line $\Phi^N_t(p)$ represents $\xi$. This provides us with a continuous map $\pi_C : X \cup \partial_\infty X \setminus \Lambda(H) \to C$. 


Let $D \subset C$ be a fundamental domain of the action of $H$ on $C$. We define the adjacency of $D$ by

$$A(D) := \bigcup_{h \in H : hD \cap \overline{D} \neq \emptyset} hD.$$  

Fix a connected component $Z$ in $\partial_\infty X \setminus \Lambda(H)$ and let $\Sigma_R$ denote the connected component of the topological boundary $\partial C_R$ that satisfies $\lim_{t \to \infty} \Phi_t^N(\Sigma_R) = Z$ (see section 2 for details). For every $R > 0$, we define the $R$-preimage of $A(D)$ by

$$A(D)_R := \Sigma_R \cap \pi_C^{-1}(A(D)).$$

Analogously, we put

$$A(D)_\infty := Z \cap \pi_C^{-1}(A(D)).$$

Our main results state that, if $\Sigma_R$ or $Z$ satisfies one of two geometric properties in a sufficiently large area that contains $A(D)_R$ or $A(D)_\infty$ for some fundamental domain, then $Z$ is contractible. The first property is the existence of strong $\lambda$-barycenters.

**Definition.** Let $Z$ be a metric space, $\frac{1}{2} \leq \lambda < 1$, and $p_1, \ldots, p_n \in Z$ a finite collection of points. We call a point $b \in Z$ a $\lambda$-barycenter of $\{p_1, \ldots, p_n\}$ if

$$\forall 1 \leq i \leq n : d(b, p_i) \leq \lambda \cdot \text{diam}(\{p_1, \ldots, p_n\}).$$

If, additionally, $q_1, \ldots, q_m \in Z$, we call $b \in Z$ a $\lambda$-barycenter of $\{p_1, \ldots, p_n\}$ relative to $\{q_1, \ldots, q_m\}$ if it is a $\lambda$-barycenter of $\{p_1, \ldots, p_n\}$ and, additionally,

$$\forall 1 \leq j \leq m : d(b, q_j) \leq \text{diam}(q_1, p_1, \ldots, p_n).$$

Let $\Delta > 0$. We say that $Z$ has $\lambda$-barycenters up to diameter $\Delta$, if every finite set of points in $Z$ with diameter at most $\Delta$ has a $\lambda$-barycenter. We say that $Z$ has strong $\lambda$-barycenters up to diameter $\Delta$ if for any two finite sets $P, Q \subset Z$ such that $\text{diam}(P) \leq \Delta$ and $\text{diam}(P \cup Q) \leq 2\Delta$, there exists a $\lambda$-barycenter of $P$ relative to $Q$.

See section 3.2 for a discussion of this property. If $\Sigma_R$ or $Z$ has strong $\lambda$-barycenter up to a sufficiently large diameter, this allows us to construct a contraction of $Z$. This is our first main result.

**Theorem A.** Let $M$ be a closed, negatively curved manifold, $X$ its universal covering, $G := \pi_1(M)$ and $H < G$ a quasi-convex subgroup such that $\Lambda(H)$ cuts $\partial_\infty X$ into several connected components.

Let $Z$ be a connected component of $\partial_\infty X \setminus \Lambda(H)$ and let $H_0 < H$ be the subgroup that preserves $Z$. Let $C := C(\Lambda(H_0)) \subset X$ be the convex hull.

Suppose there exist $R > 0$, $\Delta > 0$, $\frac{1}{2} \leq \lambda < 1$ and a fundamental domain $D$ of the action of $H_0$ on $C$ with compact closure such that $\Sigma_R$ has strong $\lambda$-barycenters up to diameter $\Delta$ and $\text{diam}(A(D)_R) \leq \Delta$ with respect to the metric on $X$. Then $Z$ is contractible. In particular, $\Sigma_{R'}$ is contractible for all $R' > 0$ as well.
If there exist \( o \in X, \Delta > 0, \frac{1}{2} \leq \lambda < 1 \) and a fundamental domain \( D \) of the action of \( H_0 \) on \( C \) with compact closure such that \( Z \) has strong \( \lambda \)-barycenters up to diameter \( \Delta \) and \( \text{diam}(A(D)_\infty) \leq \Delta \) with respect to the visual metric \( \rho_o \), then \( \tilde{Z} \) is contractible.

The key idea of the proof is to construct a continuous retraction from \( C_\varepsilon \) to \( \Sigma_\varepsilon \). We can do so locally on every sufficiently small ball in \( C_\varepsilon \); the challenge lies in patching these local retractions together. We arrange the different local maps in a simplicial complex and show that, if we can mimic the process of taking the first barycentric subdivision with some good metric properties on \( \Sigma_R \), then the local maps can be patched together to yield a retraction \( C_\varepsilon \rightarrow \Sigma_\varepsilon \). Since \( C_\varepsilon \) is convex, it is contractible and contractibility of \( \Sigma_R \) for all \( R \in (0, \infty] \) follows.

The assumption that \( \Sigma_R \) has strong \( \lambda \)-barycenters up to a suitable diameter is needed to mimic the first barycentric subdivision. Instead of using the first barycentric subdivision, one could use any other finite subdivision, as long as certain metric properties remain satisfied. This yields our second sufficient condition to contractibility of \( Z \).

**Definition.** Let \((Z, d)\) be a metric space, \( S \) a finite simplicial complex, \( \iota : S^{(0)} \rightarrow Z \) a map, and \( \lambda < 1 \). Let \( S' \subset S \) be a subcomplex that contains all vertices of \( S \). Let \( S'_{\text{sub}} \) be a finite subdivision of \( S' \) and suppose \( \iota \) extends to the vertices of \( S'_{\text{sub}} \). We call the pair \((S'_{\text{sub}}, \iota)\) a \( \lambda \)-shrinking subdivision of \( S' \) in \((S, \iota)\) if the following two properties hold:

1. If \( \tilde{\sigma} \) is a simplex in \( S'_{\text{sub}} \) and \( \sigma \) the unique least-dimensional simplex in \( S' \) containing \( \tilde{\sigma} \), then \( \text{diam}(\iota(\tilde{\sigma}^{(0)})) \leq \lambda \text{diam}(\iota(\sigma^{(0)})) \).

2. For every simplex \( \sigma \) in \( S \), we have that \( \text{diam}(\iota(\sigma^{(0)}_{\text{sub}})) \leq \text{diam}(\iota(\sigma^{(0)})) \), where \( \sigma^{(0)}_{\text{sub}} = \sigma^{(0)} \cup (\sigma \cap S'_{\text{sub}}^{(0)}) \) denotes the set of all vertices of \( \sigma \) in \( S \) together with all vertices in \( \sigma \) that appear in the subdivision \( S'_{\text{sub}} \).

Note that we require this property for every simplex in \( S \), not just in \( S' \).

**Definition.** Let \((Z, d)\) be a metric space, \( \lambda < 1 \) and \( \Delta > 0 \). We say that \( Z \) admits \( \lambda \)-shrinking subdivisions up to diameter \( \Delta \), if for every finite simplicial complex \( S \), every map \( \iota : S^{(0)} \rightarrow Z \) such that for all simplices \( \sigma \) in \( S \), \( \text{diam}(\iota(\sigma^{(0)})) \leq \Delta \), for every subcomplex \( S' \) of \( S \) containing \( S^{(0)} \), for every (finite) \( \lambda \)-shrinking subdivision \((S'_{\text{sub}}, \iota)\), and every simplex \( \sigma \) in \( S \) not contained in \( S' \), there exists a finite subdivision \( \tilde{S}_{\text{sub}} \) of \( S' \cup \sigma \) and an extension of \( \iota \) to the vertices of \( \tilde{S}_{\text{sub}} \) such that \((\tilde{S}_{\text{sub}}, \iota)\) is an extension of \((S'_{\text{sub}}, \iota)\) and it is also \( \lambda \)-shrinking.

See section 3.3 for a discussion of this property. The same idea of proof that we will use to prove Theorem A can be used to prove

**Theorem B.** Let \( M \) be a closed, negatively curved manifold, \( X \) its universal covering, \( G := \pi_1(M) \) and \( H \subset G \) a quasi-convex subgroup such that \( \Lambda(H) \) cuts \( \partial_\infty X \) into several connected components.
Let \( Z \) be a connected component of \( \partial_\infty X \setminus \Lambda(H) \) and let \( H_0 < H \) be the subgroup that preserves \( Z \). Let \( C := C(\Lambda(H_0)) \subset X \) be the convex hull.

Suppose there exist \( R > 0, \Delta > 0, \frac{1}{2} \leq \lambda < 1 \) and a fundamental domain \( \Delta \) of the action of \( H_0 \) on \( C \) with compact closure such that \( \Sigma_R \) admits \( \lambda \)-shrinking subdivisions up to diameter \( \Delta \) and \( \text{diam}(A(D)_R) \leq \Delta \) with respect to the metric on \( X \). Then \( Z \) is contractible. In particular, \( \Sigma_{R'} \) is contractible for all \( R' > 0 \) as well.

If there exist \( o \in X, \Delta > 0, \frac{1}{2} \leq \lambda < 1 \) and a fundamental domain \( \Delta \) of the action of \( H_0 \) on \( C \) with compact closure such that \( Z \) admits \( \lambda \)-shrinking subdivisions up to diameter \( \Delta \) and \( \text{diam}(A(D)_{\infty}) \leq \Delta \) with respect to the visual metric \( \rho_o \), then \( Z \) is contractible.

Given a connected component \( Z \) of \( \partial_\infty X \setminus \Lambda(H) \) and \( H_0 < H \) the subgroup that preserves \( Z \), there are some immediate consequences if \( Z \) is contractible. Since \( H_0 \) acts freely and cocompactly on \( Z \) (see the beginning of section 3 for details), contractibility of \( Z \) implies that \( \mathbb{Z}/H_0 \) is a manifold \( K(H_0,1) \).

By Bestvina-Mess \([BM91]\), this implies that \( \Lambda(H_0) \) is a homology sphere. We thus obtain the following corollary.

**Corollary C.** Let \( M, X, G, H, Z, H_0 \) be as in Theorem \( \text{B} \). Suppose they satisfy the additional hypotheses of either Theorem \( \text{A} \) or \( \text{B} \). Then, \( \Lambda(H_0) \) is a homology sphere.

While the proofs of Theorem \( \text{A} \) and \( \text{B} \) are similar, the sufficient conditions are quite different in their use. Having strong \( \lambda \)-barycenters is a far less flexible property than admitting \( \lambda \)-shrinking subdivisions. However, strong \( \lambda \)-barycenters seem to be connected to control on curvature bounds that may be accessible when studying concrete classes of spaces. When looking at \( \lambda \)-shrinking subdivisions, it seems far more likely that the property has a homological interpretation that may allow us to conclude contractibility of the component \( Z \) via homological properties of \( G \) and \( H \).

The rest of the paper is structured as follows. In section 2, we introduce and recall some notation and basic facts about geometry of negatively curved spaces and simplicial complexes that we will need. In section 3, we show how the retraction \( C_{\epsilon} \to \Sigma_{\epsilon} \) is constructed up to one final gap. In sections 3.2 and 3.3, we show how strong \( \lambda \)-barycenters and \( \lambda \)-shrinking subdivisions allow us to close that gap respectively. During all this, we will work exclusively with \( \Sigma_R \). In section 4, we show how the arguments from the previous section generalise to the boundary, which finishes the proof of both Theorems.

**Acknowledgments:** The authors are grateful to Fanny Kassel and Alessandro Sisto for several helpful discussions and suggestions. The second author thanks Ivan Beschastnyi and Veronica Fantini for discussing a previous version of the retraction-construction. The first author was supported by NSF.
grant DMS-2052801. The second author has been funded by the SNSF grant 194996.

2. Preliminaries

2.1. Geometry in non-positive curvature. Part of the statements we make in this subsection hold for CAT(−1) or even CAT(0) spaces. Since we exclusively work with Riemannian manifolds, we will contain our discussion to that setting. Let $X$ be a Riemannian manifold. We say that $X$ is geodesically complete if every geodesic can be extended to a geodesic defined on all of $\mathbb{R}$ (in other words, the exponential map at a point $p \in X$ is defined for all of $T_pX$). Let $X$ be a Hadamard manifold, i.e. a simply connected, geodesically complete Riemannian manifold such that all sectional curvatures are non-positive. We say that $X$ has pinched negative curvature if there exist $0 < a < b$ such that all sectional curvatures on $X$ lie in the interval $[-b^2, -a^2]$. Assume from now on that the curvature of $X$ lies in an interval $[-b^2, -1]$. (We can always rescale a space with pinched negative curvature to have curvature at most $-1$.)

Given a Riemannian geodesic $\gamma : I \to X$, which is defined on an interval $I$, we call $\gamma$ a geodesic segment if $I$ is bounded, a geodesic ray if $I = (-\infty, c]$ or $I = [c, \infty)$ for some $c \in \mathbb{R}$ and a geodesic line if $I = \mathbb{R}$. We will frequently use the fact that, in non-positively curved spaces, for any geodesic $\gamma$ and any closed convex set $C$, the function $t \mapsto d(C, \gamma(t))$ is convex. Furthermore, if $C$ consists of a single point, this map is strictly convex.

For Hadamard manifolds, one can define a boundary at infinity as follows: Two geodesic rays $\gamma, \tilde{\gamma}$ are called asymptotically equivalent if they share an unbounded interval on which both are defined and $\lim_{|t| \to \infty} d(\gamma(t), \tilde{\gamma}(t)) < \infty$. The visual boundary of $X$ is defined by

$$\partial_\infty X := \{ \gamma \text{ geodesic ray in } X \}/\text{asymptotic equivalence}.$$ If a geodesic ray $\gamma : [c, \infty)$ represents a point $\xi \in \partial_\infty X$, we also write $\gamma(\infty) = \xi$ (and we analogously define $\gamma(-\infty)$ if $\gamma$ is defined on $(-, c]$.) A basic fact about Hadamard manifolds is that for every $p \in X$, $\xi \in \partial_\infty X$, there exists a unique geodesic ray $\gamma$ with $\gamma(0) = p$ and $\gamma(\infty) = \xi$. We call this the geodesic from $p$ to $\xi$. Given two points $p, q \in X \cup \partial_\infty X$, we denote the unique geodesic from $p$ to $q$ by $\gamma_{pq}$.

The space $\overline{X} := X \cup \partial_\infty X$ can be equipped with a topology, called cone topology such that $\overline{X}$ is compact, $X \subset \overline{X}$ is open, and its topology inherited as a subspace of $\overline{X}$ is the same as the manifold-topology. This topology is generated by the following open sets: We take all open sets in $X$ and for a fixed $o \in \overline{X}$ and all $\epsilon > 0, R > 0, \xi \in \partial_\infty X$, we add

$$U_{o, \epsilon, R}(\xi) := \{ p \in \overline{X} | d(\gamma_{op}(R), \gamma_{\xi}(R)) < \epsilon \} \setminus B_R(o),$$
where $B_R(o)$ denotes the closed ball of radius $R$, centered at $o$. Going over all $\xi, \epsilon, R$, these sets, together with all open sets in $X$, form a basis for a topology, which is called the cone topology. It is a basic fact that the cone
topology does not depend on the choice of the base point \( o \). The restriction of this topology to \( \partial_\infty X \) is also called cone topology, or sometimes visual topology.

For every point \( o \in X \), there exists a canonical map between the unit tangent sphere \( T^1_oX \) and the visual boundary. This map is given by sending every unit tangent vector at \( o \) to the unique geodesic ray obtained by moving forward along that unit tangent vector. One can check that for every \( o \in X \), this map is a homeomorphism. We will thus frequently identify points in the boundary with unit tangent vectors in \( X \) via these canonical maps.

For every point \( o \in X \), \( \xi, \eta \in \partial_\infty X \), we define the Gromov product:

\[
(\xi|\eta)_o := \lim_{t \to \infty} t - \frac{1}{2} d(\gamma_o\xi(t), \gamma_o\eta(t)).
\]

If \( X \) has curvature at most \(-1\), the expression

\[
\rho_o(\xi, \eta) := e^{-(\xi|\eta)_o},
\]

defines a metric on \( \partial_\infty X \), which depends on the base point \( o \). We call \( \rho_o \) a visual metric and the topology they induce on the visual boundary is the same as the one obtained from the cone topology.

Given a set \( S \subset X \), we define \( C(S) \) to be the convex hull of \( S \), i.e., the smallest convex set in \( X \) that contains \( S \). Similarly, if \( S \subset \overline{X} \), we define \( C(S) \) to be the smallest convex set in \( X \) that contains all (possibly bi-infinite) geodesics between points in \( S \). Given a convex set \( C \subset X \), we can consider all points in \( \partial_\infty X \) that can be represented by geodesic rays contained in \( C \) and refer to this as \( \partial_\infty C \). It is a result by Anderson (Theorem 3.3 in [And83]) that, if the curvature of \( X \) lies in \([−b^2, −1]\) for some \( b \geq 1 \), then for any closed set \( S \subset \partial_\infty X \), we have \( \partial_\infty C(S) = S \).

Let \( C \subset X \) be any subset and \( R > 0 \). We define the \( R \)-neighbourhood of \( C \) by

\[
N_R(C) := \{ p \in X | d(p, C) \leq R \}.
\]

A subset \( C \subset X \) is called quasi-convex if there exists a constant \( R \) such that for any two points \( p, q \in C \), the geodesic \( \gamma_{pq} \) is contained in \( N_R(C) \).

Let \( M \) be a closed Riemannian manifold with curvature in the interval \([-b^2, -1]\) and fundamental group \( G := \pi_1(M) \). It’s universal covering, which we denote by \( X \), is a Hadamard manifold with pinched negative curvature such that \( G \) acts properly, cocompactly by isometries on \( X \). Let \( H < G \) be a subgroup such that one (and hence all) orbits of \( H \) in \( X \) are quasi-convex. By Proposition 3.7 in [BH99], this implies that \( H \) is a hyperbolic group. We define the limit set of \( H \) by

\[
\Lambda(H) = \overline{Hx_0} \cap \partial_\infty X,
\]

where \( \overline{Hx_0} \) denotes the topological closure of the orbit of \( x_0 \) under the action of \( H \) in \( \overline{X} \) equipped with the cone topology. It is well-known that the limit set does not depend on the choice of \( x_0 \). For the rest of the paper, \( C \)
denotes the convex hull of $\Lambda(H)$. By Anderson, we know that $\partial_\infty C = \Lambda(H)$. Furthermore, the action of $H$ on $X$ preserves $C$ (otherwise, $hC \cap C$ would be a strictly smaller convex subset whose boundary contains $\Lambda(H)$) and $H$ acts cocompactly on $C$. Indeed, if it did not, one could construct a sequence in $C$ that moves farther and farther away from an orbit $Hx_0$. This sequence admits a subsequence that converges to a point that lies in $\Lambda(H)$ and in $\partial_\infty X \setminus \Lambda(H)$, a contradiction.

Let $\epsilon > 0$ and let $C_\epsilon$ be the $\epsilon$-neighbourhood of $C$. It turns out that $C_\epsilon$ offers some useful properties over $C$ that we will require. By $[\text{Wal76}]$ the topological boundary of $C_\epsilon$ is a $C^{1,1}$-manifold. Furthermore, we have the following result.

**Lemma 2.1.** For any $\epsilon > 0$, $C_\epsilon$ is strictly convex.

**Proof.** Suppose for contradiction that there exist $p, q \in C_\epsilon$ with geodesic

$$\gamma : [0, 1) \to X$$

from $p$ to $q$ such that $\text{Int}(\gamma) \not\subseteq \text{Int}(C_\epsilon)$. Then we can find $T$ such that $0 < T < 1$ and $\gamma(T)$ is in the frontier of $C_\epsilon$. Let $\rho(t) := d(\gamma(t), C)$, which is a convex function since $X$ is negatively curved. Furthermore, let $\pi_C : X \to C$ be the closest point projection to $C$, which is a continuous, well-defined map as $C$ is convex. We see that $\rho(T) = \epsilon$. The convexity of $\rho$ now implies that $\rho(t) = \epsilon$ for all $t \in [T, 1]$. However, geodesics in the frontier of $C_\epsilon$ are everywhere orthogonal to the fibers of $\pi_C$. This means we can find a flat strip in $X$ bounded by $\gamma([T, 1])$, its projection $(\pi_C \circ \gamma)([T, 1])$, and the geodesics from $\gamma(T)$ and $q = \gamma(1)$ to $C$. Since $X$ is negatively curved, this is a contradiction. $\square$

In addition to the above, $C_\epsilon$ has the following property: For any point $o \in C$, there exists a canonical homeomorphism between the topological boundary $\partial C_\epsilon \subset X$ and $\partial_\infty X \setminus \Lambda(H)$, which is obtained by sending every point $p \in \partial C_\epsilon$ to the geodesic ray obtained by extending the geodesic from $o$ to $p$ to a geodesic ray that starts at $o$. This map gives us an identification of the connected components of $\partial C_\epsilon$ and the connected components of $\partial_\infty X \setminus \Lambda(H)$. One easily checks that the identification of connected components does not depend on the choice of $o$ (provided that $o \in C$). In particular, every connected component $\Sigma_\epsilon$ of $\partial C_\epsilon$ can be canonically identified with a connected component $Z$ of $\partial_{\infty}X \setminus \Lambda(H)$, which we call the connected component corresponding to $\Sigma_\epsilon$. Furthermore, there exists a unique connected component $Y_\epsilon$ of $X \setminus C_\epsilon$ such that $\partial Y_\epsilon \cap \Sigma_\epsilon \neq \emptyset$. We call $Y_\epsilon$ the connected component of $X \setminus C_\epsilon$ corresponding to $\Sigma_\epsilon$. Suppose, $\epsilon, \epsilon' > 0$ and $\Sigma_\epsilon, \Sigma_{\epsilon'}$ be connected component of $\partial C_\epsilon$ and $\partial C_{\epsilon'}$ respectively. We say that $\Sigma_\epsilon$ and $\Sigma_{\epsilon'}$ correspond to each other if they correspond to the same component of $\partial_{\infty}X \setminus \Lambda(H)$.

### 2.2. Simplicial complexes

In this paper, we will only be concerned with combinatorial complexes, i.e. for any set of vertices, there can be at most one simplex that has exactly these vertices and no vertex appears twice as a corner of the same simplex. Given a $k$-simplex with vertices $(B_1, \ldots, B_{k+1})$, we...
also write $\sigma(B_1, \ldots, B_{k+1})$ for this $k$-simplex. We also say that $B_1, \ldots, B_{k+1}$ span this simplex. Given the standard $k$-simplex in $\mathbb{R}^{k+1}$, it consists of all convex combinations of the form $\sum_{i=1}^{k+1} \lambda_i e_i$, where the $e_i$ are the standard basis vectors. Analogously, we can write every point in the simplex $\sigma(B_1, \ldots, B_{k+1})$ as a convex combination of the vertices $B_i$, which we write as $\sum_{i=1}^{k+1} \lambda_i B_i$.

The simplicial complexes that we will encounter are obtained from coverings of spaces. Let $A \subset X$ be a subset in $X$ (where $X$ could be any topological space). Let $\mathcal{U}$ be a collection of open sets in $X$ that cover $A$. The multiplicity of $\mathcal{U}$ is defined as the least number $n$ such that every point in $A$ is contained in at most $n$ many elements of $\mathcal{U}$. We define the simplicial complex dual to $\mathcal{U}$ as follows: The set of vertices is the set $\mathcal{U}$ and a set of vertices $B_1, \ldots, B_{k+1} \in \mathcal{U}$ spans a $k$-simplex if and only if $\bigcap_{i=1}^{k+1} B_i \neq \emptyset$. If $\mathcal{U}$ has finite multiplicity, then its dual simplicial complex is locally finite and the multiplicity is equal to the dimension of the largest simplex in the dual complex.

We will have to work with subdivisions of simplicial complexes, in particular the barycentric subdivision. Given a simplicial complex $S$ with vertices $\{B_i | i \in I\}$, we can write every vertex in the first barycentric subdivision as $B_J$, where $J \subset I$ indexes a set of vertices in $S$ that span a simplex in $S$. Whenever the vertices indexed by $J$ generate a simplex in $S$, we denote that simplex by $\sigma_J$. Two vertices $B_J, B_{J'}$ are connected by an edge if and only if $J \subset J'$ or $J' \subset J$. A $k$-simplex in the first barycentric subdivision of $S$ is given by a sequence of sets $J_1 \subset \cdots \subset J_{k+1}$ such that for all $i$, the simplex $\sigma_{J_i}$ exists in $S$.

3. Retracting the convex hull to its boundary

Let $M$ be a closed Riemannian manifold with curvature in the interval $[-b^2, -1]$ and fundamental group $G$. Let $X$ denote the universal covering of $M$ and let $H < G$ be a quasi-convex subgroup. We denote the limit set of $H$ by $\Lambda(H)$ and $C := C(\Lambda(H))$ its convex hull. For all $\epsilon > 0$, let $C_{\epsilon}$ denote the $\epsilon$-neighbourhood of $C$. Fix a connected component $Z \subset \partial_\infty X \setminus \Lambda(H)$ and let $\Sigma_{\epsilon}, Y_{\epsilon}$ be the connected components of $\partial C_\epsilon, X \setminus C_\epsilon$ corresponding to $Z$ respectively.

Let $H_0 < H$ be the subgroup that preserves $Z$. The limit set of $H_0$ satisfies $\Lambda(H_0) = \partial Z$, where $\partial Z$ denotes the topological boundary of $Z$ in $\partial_\infty X$ with the visual topology. Since $H_0$ acts as a convergence group in its limit set, it is a quasi-convex subgroup of $H$, due to an argument of Kapovich-Kleiner [KK00] Theorem 8]. Note that $Z$ is a connected component in $\partial_\infty X \setminus \Lambda(H_0)$. In order to study contractibility of $Z$, we can work entirely with $H_0$. By abuse of notation, we thus replace $H$ by $H_0$ $C$ by the convex hull of $\Lambda(H_0)$ and adjust $C_{\epsilon}$ and $\Sigma_{\epsilon}$ accordingly. Since $H_0$ acts cocompactly on its convex hull and preserves $Z$, we can thus assume without loss of generality that $H$ not
only acts cocompactly on $C_\epsilon$, but also preserves $\Sigma_\epsilon$ and acts cocompactly on it.

3.1. Constructing the retraction. Our strategy to prove contractibility of $\Sigma_\epsilon$ relies on working with finite covers of a fundamental domain of the action of $H$ on $C_\epsilon$. We will frequently have to make sure that these covers interact well with their immediate vicinity, meaning translates of the cover by $H$ that are close to the original cover. We make a couple of definitions that will allow us to formulate this properly.

Let $K \subset X$ be a bounded subset and let $\mathcal{U}$ be a finite collection of bounded open sets in $X$ that covers $K$. We define the adjacency of $\mathcal{U}$ by

$$A_1(\mathcal{U}) := \{hB | B \in \mathcal{U}, h \in H, \exists B' \in \mathcal{U} : hB \cap B' \neq \emptyset\}.$$ 

We emphasise that the lack of restrictions on $h$ and $B'$ implies that the adjacency of $\mathcal{U}$ contains $\mathcal{U}$ itself. Since $H$ acts properly, and $\mathcal{U}$ is finite and consists of bounded open sets, its adjacency is also finite. Note that, if $B \in A_1(\mathcal{U})$ there may exist some elements $h \in H \setminus \{1\}$ such that $hB \in A_1(\mathcal{U})$ as well. We will sometimes call this the ‘partial action’ of $H$ on $A_1(\mathcal{U})$ and say that $h$ ‘acts’ on $B$ if $hB \in A_1(\mathcal{U})$. This ‘partial action’ extends to the dual complex of $A_1(\mathcal{U})$ where we will use similar notation, whenever it makes sense.

Given a subset $K \subset X$, we will denote its orbit under $H$ by $HK = \{hx | h \in H, x \in K\}$. Furthermore, given a collection $\mathcal{U}$ of subsets of $X$, we write $\bigcup \mathcal{U} := \bigcup_{B \in \mathcal{U}} B$.

We begin by proving the following, likely known in some form, lemma from topology.

**Lemma 3.1.** Let $X$ be a paracompact, Hausdorff topological space, $H$ a discrete group which acts properly by homeomorphisms on $X$, $K \subset X$ a closed subset and $\mathcal{U}$ a collection of open sets with compact closure in $X$ that covers $K$ and has finite multiplicity. Then there exists an open neighbourhood $U \subset \bigcup \mathcal{U}$ of $K$ and a continuous map $\Psi : \bigcup \mathcal{U} \cap HU \to S_{A_1(\mathcal{U})}$ such that for all $h \in H$, $q \in \bigcup \mathcal{U} \cap HU$ that satisfy $hq \in \bigcup \mathcal{U} \cap HU$, we have $\Psi(hq) = h\Psi(q)$.

**Proof.** Let $\overline{\mathcal{U}} := \{hB | h \in H, B \in \mathcal{U}\}$ and $S_{\overline{\mathcal{U}}}$ denote the simplicial complex dual to $\overline{\mathcal{U}}$. Note that, since all elements of $\mathcal{U}$ have compact closure, $\mathcal{U}$ has finite multiplicity, and $H$ acts properly by homeomorphisms, $\overline{\mathcal{U}}$ has finite multiplicity as well. Let $\mathcal{T}$ be an index set, such that we can write $\overline{\mathcal{U}} = \{B_i | i \in \mathcal{T}\}$. The action of $H$ on $\overline{\mathcal{U}}$ induces an action of $H$ on $\mathcal{T}$ and we obtain a subset $I \subset \mathcal{T}$ such that $\overline{\mathcal{U}} = \{B_i | i \in I\}$. We will prove the Lemma by constructing an $H$-equivariant, continuous map $\Psi : HU \to S_{\overline{\mathcal{U}}}$, which restricts to map from $\bigcup \mathcal{U} \cap HU$ to $S_{A_1(\mathcal{U})}$.

Since $K \subset X$ is closed and $\mathcal{U}$ is locally finite, the collection $\mathcal{U} \cup \{X \setminus K\}$ provides a locally finite cover of $X$ by open sets. Since $X$ is paracompact and Hausdorff, there exists a continuous partition of unity subordinate to
Let \( U \cup \{X \setminus K\} \), i.e. continuous maps \( x_i : X \to [0, 1] \) for \( i \in I \) and a map \( x_{X \setminus K} : X \to [0, 1] \) such that \( x_i|_{X \setminus B_i} \equiv 0 \) and

\[
\sum_{i \in I} x_i + x_{X \setminus K} \equiv 1.
\]

In particular, this implies that there exists an open neighbourhood \( U \subset \bigcup U \) of \( K \) such that for all \( q \in U \), there exists some \( i \in I \) with \( x_i(q) > 0 \).

We now use the family \( \{x_i|_{X \setminus K}\} \) to obtain an \( H \)-invariant partition of unity on \( HU \) subordinate to \( \overline{U} \). Let \( i \in I \) and consider the set \( hB_i = B_{hi} \in \overline{U} \). We define \( x_{hi} : X \to [0, 1] \) by \( x_{hi}(q) = x_i(h^{-1}q) \). Clearly, \( x_{hi} \) is continuous and satisfies \( x_{hi}|_{X \setminus hB_i} \equiv 0 \). Furthermore, since for every \( q \in U \) there exists an \( i \in I \) such that \( x_i(q) > 0 \), we find that for every \( q \in HU \), there exists \( i \in \overline{T} \) such that \( x_i(q) > 0 \). Since we also know from the beginning of the proof that \( \overline{U} \) is locally finite, we obtain that \( \sum_{i \in T} x_i > 0 \) is a locally finite sum. Renormalizing, we obtain functions

\[
\overline{x_i}(q) := \frac{x_i(q)}{\sum_{i \in T} x_i(q)},
\]

which are well-defined on \( HU \), form a continuous partition of unity subordinate to \( \overline{U} \), and satisfy \( \overline{x_{hi}}(hq) = \overline{x_i}(q) \).

We now define the map \( \Psi : HU \to S_{\overline{U}} \) by

\[
\Psi(q) := \frac{1}{\sum_{i \in T} \overline{x_i}(q)} \sum_{i \in T} \overline{x_i}(q) B_i,
\]

where \( B_i \) denotes the vertex in \( S_{\overline{U}} \) induced by the open set \( B_i \). This sum is locally finite by the construction before and thus, the map \( \Psi \) is well-defined and continuous. Furthermore, \( \Psi \) is \( H \)-equivariant, since

\[
\Psi(hq) = \frac{1}{\sum_{i \in T} \overline{x_i}(hq)} \sum_{i \in T} \overline{x_i}(hq) B_i
= \frac{1}{\sum_{i \in T} \overline{x_i}(q)} \sum_{i \in T} \overline{x_{hi}}(hq) B_{hi}
= \frac{1}{\sum_{i \in T} \overline{x_i}(q)} \sum_{i \in T} \overline{x_i}(q) (hB_i)
= h\Psi(x),
\]

where we recall that \( h \in H \) acts on \( S_{\overline{U}} \) by sending the vertex \( B_i \) to \( hB_i \) (a map which extends to the simplices since the cover \( \overline{U} \) is \( H \)-invariant).

We are left to show that for all \( x \in \bigcup U \cap HU \), we have \( \Psi(x) \in S_{A_1(\mathcal{U})} \), where we interpret the simplicial complex dual to \( A_1(\mathcal{U}) \) as a subcomplex of \( S_{\overline{U}} \) via its natural embedding. Indeed, if \( x \in \bigcup U \), then the only elements of \( \overline{U} \) that can contain \( x \) are elements of \( A_1(\mathcal{U}) \). Therefore, \( \Psi(x) \) is a convex
combination of vertices that lie in $S_{A_t(U)}$ and $\Psi$ restricts to a continuous map from $\bigcup \mathcal{U} \cap HU$ to $S_{A_t(U)}$. This proves the Lemma. \qed

Before we state the next Lemma, we introduce the following notation: Given two points $q \in X$, $\xi \in X \cup \partial X$, we denote the unit vector in $T_q X$ whose induced geodesic ray contains or represents $\xi$ by $\vec{q}\xi$.

**Lemma 3.2.** Let $\epsilon > 0$, $D$ a fundamental domain of the action of $H$ on $C_{\epsilon}$ with compact closure, $\Sigma_{\epsilon}$ a connected component of $\partial C_{\epsilon}$, and $Y_{\epsilon}$ the connected component of $X \setminus C_{\epsilon}$ that corresponds to $\Sigma_{\epsilon}$. Let $\mathcal{U}$ be a finite cover of an open neighbourhood of $\overline{D}$ and $S_1$ the dual simplicial complex of the adjacency $A_1(U)$. If there exists a continuous map $j : S_1 \rightarrow Y_{\epsilon}$ such that for all $h \in H$ and $x \in S_1$ that satisfy $hx \in S_1$, we have $j(hx) = hj(x)$, then there exists a continuous, $H$-invariant unit vector field $V$ on $C_{\epsilon}$ such that for all $q \in D$, the geodesic ray induced by $V(q)$ meets $j(q)$.

**Proof.** Applying Lemma 3.1 to $\mathcal{U}$, we obtain an open neighbourhood $U \subset \bigcup \mathcal{U}$ of $\overline{D}$ and a map $\Psi : \bigcup \mathcal{U} \cap HU \rightarrow S_1$, which satisfies $\Psi(hq) = h\Psi(q)$ for all $h \in H, q \in U$ such that $hq \in U$. In particular, if $hq \in U$, then $h\Psi(q) \in S_1$. Therefore, the composition $j \circ \Psi : \bigcup \mathcal{U} \cap HU \rightarrow Y_{\epsilon}$ satisfies $j(\Psi(hq)) = hj(\Psi(q))$ for all such $h$ and $q$. This, together with the facts that $H$ acts freely on $X$, $hY_{\epsilon} = Y_{\epsilon}$ for all $h \in H$, and $j \circ \Psi$ is defined on an open neighbourhood of a fundamental domain of $C_{\epsilon}$, allows us to extend $j \circ \Psi$ to a continuous, $H$-equivariant map $j \circ \Psi : C_{\epsilon} \rightarrow Y_{\epsilon}$, which is defined by $j \circ \Psi(hq) = hj \circ \Psi(q)$ for all $h \in H$ and $q \in U$.

We define the vector field $V$ on $C_{\epsilon}$ by putting $V(q) := \overrightarrow{q(j \circ \Psi)(q)}$, i.e. the unit vector at $q$ whose induced geodesic ray meets $j \circ \Psi(q)$. Due to the fact that $H$ acts on $X$ by isometries, $V$ inherits $H$-equivariance from $j \circ \Psi$. Since $j \circ \Psi$ is continuous and geodesic segments in negatively curved manifolds depend continuously on their endpoints, $V$ is continuous. \qed

As we will see later, if there exists a vector field $V$ as above that additionally satisfies $\angle(N(q), V(q)) < \frac{\pi}{2}$ for all $q \in \Sigma_{\epsilon}$, then $\Sigma_{\epsilon}$ is contractible (see Lemma 3.6). Our main task thus becomes the construction of a suitable map $j : S \rightarrow Y_{\epsilon}$. In order to provide a sufficient condition for its existence, we need some preparation. We start with notation: For all $\epsilon > 0$, let $N$ denote the unit normal vector field on $\Sigma_{\epsilon}$ that points outwards of $C_{\epsilon}$. This defines a continuous vector field on the open subset $\cup_{\epsilon > 0} Y_{\epsilon} \subset X$. Let $\Phi_t^N$ denote the geodesic flow along the vector field $N$. (Note that $\Phi_t^N$ sends $\Sigma_{\epsilon}$ to $\Sigma_{\epsilon+t}$.)

Let $D$ be a fundamental domain of the action of $H$ on $C_{\epsilon}$. We define the adjacency of $D$ by

$$A(D) := \bigcup_{h \in H : hD \cap \overline{\mathcal{T}} \neq \emptyset} h\overline{D}.$$ 

In order to construct $j$, we need to cover $\mathcal{T}$ with a suitable cover. The following Lemma (which contains only standard results that we summarize.
open sets of diameter at most $\delta$ are compact, and $D$ is a fundamental domain of the action of $H$ on $\mathcal{C}_e$ that has compact closure.

Then, there exist $\delta' > 0$ and $0 < \delta < \frac{\epsilon}{2}$ such that the following properties hold:

1. For all $q \in A(D) \cap \Sigma_e$, $p \in K$, and $p' \in B_{2\delta'}(p)$, we have
   \[ \angle(\overrightarrow{qp}, \overrightarrow{qp'}) \leq \frac{\alpha_0}{2}. \]

2. For all $q \in A(D) \cap \Sigma_e$, $q' \in B_{2\delta}(q)$, we have $d(\Phi^N_R(q), \Phi^N_R(q')) \leq \delta'$.

3. For all $q \in A(D) \cap \Sigma_e$, $q' \in B_{2\delta}(q)$ and $p \in K$, we have
   \[ |\angle(N(q), \overrightarrow{qp}) - \angle(N(q'), \overrightarrow{qp'})| \leq \frac{\alpha_0}{2}. \]

Furthermore, $\delta$ can be chosen so small that every finite cover $\mathcal{U}$ of $\overline{D}$ by open sets of diameter at most $\delta$ has a subcover (which we also denote by $\mathcal{U}$ that satisfies the following properties:

a) For all $B_i \in \mathcal{U}$, $B_i \cap \overline{D} \neq \emptyset$ and $B_i \cap C_e \subset A(D)$.

b) For all $h \in H \setminus \{1\}$ and $B_i, B_j \in \mathcal{U}$ with $B_i \cap B_j \neq \emptyset$, we have
   \((B_i \cup B_j) \cap h(B_i \cup B_j) = \emptyset\). (This includes the case $B_i = B_j$.)

c) For all $B_i, B_j \in \mathcal{U}$, $h \in H$ such that $hB_i \cap B_j \neq \emptyset$, we have $h\overline{D} \cap \overline{D} \neq \emptyset$. In particular, $hB_i \subset A(D)$.

Finally, any such cover has a refinement $\mathcal{U}'$ that still satisfies all the properties above and additionally, for all $B \in \mathcal{U}'$, we have that if $B \cap \Sigma_e \neq \emptyset$, then $B \cap \Sigma_e \cap \overline{D} \neq \emptyset$.

We emphasise that $\delta$ will not only depend on $R, \epsilon, \alpha_0, K$, and $D$, but also on $\delta'$.

Proof. Let $R, \epsilon, \alpha_0, K$, and $D$ be as in the Lemma. Since $\overline{D}$ is compact and $H$ acts properly, $A(D)$ is compact as well. Combining this with the facts that $K$ is compact, $N$ is continuous and radial vector fields are continuous, we find that for every $\alpha_0 \in (0, \frac{\epsilon}{2})$, there exists $\delta > 0$, such that for every $q \in A(D) \cap \Sigma_e$, $q' \in B_{2\delta}(q)$ and $p \in K$, we have
   \[ |\angle(N(q), \overrightarrow{qp}) - \angle(N(q'), \overrightarrow{qp'})| \leq \frac{\alpha_0}{2}. \]

In particular, we find $\delta$ such that inequality (3) is satisfied. Using the same continuity arguments and compactness, we find $\delta'$ such that for all $p \in K$, $p' \in B_{2\delta'}(p)$, and $q \in A(D) \cap \Sigma_e$, we have
   \[ \angle(\overrightarrow{qp}, \overrightarrow{qp'}) \leq \frac{\alpha_0}{2}. \]

Having obtained $\delta'$ that satisfies property (1), we can now choose $\delta$ sufficiently small so that property (2) is satisfied. Namely, since the geodesic
flow in $X$ is determined by an ODE, which depends continuously on its initial condition, and the vector field $N$ is continuous, we obtain that $\Phi^N_R(q)$ depends continuously on $q$, which allows for a suitable choice of $\delta$.

We are left to prove the statements about covers. Let $\mathcal{U}$ be a finite cover of $\overline{D}$ by open sets of diameter at most $\delta$. Replace $\mathcal{U}$ by the subcover consisting of all $B_i \in \mathcal{U}$ such that $B_i \cap \overline{D} \neq \emptyset$. Since $H$ acts properly on $X$, we can shrink $\delta$ if necessary so that for every set $B \subseteq C_\epsilon$ of diameter at most $\delta$ that satisfies $B \cap \overline{D} \neq \emptyset$, we have $B \subset A(D)$. This yields property a). Furthermore, properness of the action allows us to shrink $\delta$ even further so that for every set $B \subseteq C_\epsilon$ of diameter at most $2\delta$ and for every $h \in H \setminus \{1\}$, we have $hB \cap B = \emptyset$. If two elements $B_i, B_j \in \mathcal{U}$ have non-empty intersection, their union has diameter at most $2\delta$. Therefore, this property implies that $\mathcal{U}$ satisfies b). Finally, properness allows us to shrink $\delta$ such that for any $h \in H$ that satisfies $h\overline{D} \cap \overline{D} = \emptyset$, we have $d(h\overline{D}, \overline{D}) > \delta$. In particular, for all $B_i, B_j \in \mathcal{U}$ such that $B_i \cap B_j \neq \emptyset$, we have $d(h\overline{D}, \overline{D}) \leq \delta$ as $B_j \cap \overline{D} \neq \emptyset$ and thus $h\overline{D} \cap \overline{D} \neq \emptyset$. In particular, $hB_i \subset A(D)$. We obtain c).

We are left to show that $\mathcal{U}$ has a refinement $\mathcal{U}'$ that still has all the properties shown and additionally, if $B \in \mathcal{U}'$ such that $B \cap \Sigma_\epsilon \neq \emptyset$, then $B \cap \Sigma_\epsilon \cap \overline{D} \neq \emptyset$. Suppose $B_i \in \mathcal{U}$ such that $B_i \cap \Sigma_\epsilon \neq \emptyset$ but $B_i \cap \Sigma_\epsilon \cap \overline{D} = \emptyset$. Replace $B_i$ by $B'_i := B_i \setminus \Sigma_\epsilon$. By construction $B'_i$ is still open and the cover obtained by replacing $B_i$ with $B'_i$ still covers $\overline{D}$ and has all the properties discussed above. Additionally, every $B \in \mathcal{U}'$ with $B \cap \Sigma_\epsilon \neq \emptyset$ satisfies $B \cap \Sigma_\epsilon \cap \overline{D} \neq \emptyset$. This proves the Lemma.

Whenever we consider a covering $\mathcal{U}$ that satisfies the properties of Lemma 3.3, we mean that we have already chosen a refinement that satisfies all properties stated in the Lemma and we denote this refinement by $\mathcal{U}$.

In order to state the central Lemma for the construction of $j$, we need one more piece of notation.

**Definition 3.4.** Let $S$ be a simplicial complex and denote its set of vertices by $S^{(0)} = \{B_i | i \in I\}$. Suppose we have a map $\iota : S^{(0)} \to X$ that sends every vertex $B_i$ to a point $p_i \in X$. We define the convex hull of $\iota$ subordinate to $S$ by

$$C_S(\iota) := \bigcup_{B_1, \ldots, B_{k+1} \in S^{(0)}; \text{ } B_1, \ldots, B_{k+1} \text{ span a simplex in } S} C(\{\iota(B_1), \ldots, \iota(B_{k+1})\}),$$

i.e. $C_S(\iota)$ is the union of the convex hulls $C(\{\iota(B_1), \ldots, \iota(B_{k+1})\})$ where we go over all finite sets of vertices in $S$ that span a simplex in $S$.

**Lemma 3.5.** Let $\epsilon > 0$ and $D$ a fundamental domain of the action of $H$ on $C_\epsilon$ with compact closure. Let $R > \epsilon$, $a_0 \in (0, \frac{\epsilon}{2})$, and $K := \Phi^N_{R-a_0}(A(D) \cap \Sigma_\epsilon)$. Let $\delta' > 0, \delta > 0$ so that they satisfy the properties in Lemma 3.3 for this choice of $R, \epsilon, a_0, K, D$.

Let $\mathcal{U}$ be a finite collection of open sets with diameter at most $\delta$ that cover $\overline{D}$ and satisfy the properties in Lemma 3.3. Let $S_1$ be the simplicial complex

...
dual to $A_1(\mathcal{U})$ and let $\iota: S_1^{(0)} \to \Sigma_R$ be a map satisfying the following two properties: For all $B_i \in S_1^{(0)}$ and $h \in H$ satisfying $hB_i \in S_1^{(0)}$, we have $\iota(hB_i) = h\iota(B_i)$. If $B_i \in \mathcal{U}$ such that $B_i \cap \Sigma_\epsilon \neq \emptyset$, there exists $c_i \in B_i \cap \Sigma_\epsilon \cap \mathcal{D}$ such that $\iota(B_i) = \Phi^N_{R-\epsilon}(c_i)$.

Suppose there exist $R, \epsilon, D$ so that $\mathcal{U}$, and $\iota$ can be chosen with the properties above and satisfying

$$C_S(\iota) \cap C_\epsilon = \emptyset.$$ 

Then there exists a continuous, $H$-invariant vector field $V$ on $C_\epsilon$ such that for all $q \in \Sigma_\epsilon$, $\angle(V(q), N(q)) \leq \alpha_0 < \frac{\pi}{2}$.

Before we begin the proof, we want to remark some immediate consequences of the properties satisfied by $\iota$. The equation $\iota(hB_i) = h\iota(B_i)$ implies that $\iota(hB_i) = \Phi^N_{R-\epsilon}(hc_i)$. In particular, for $B_i \in A_1(\mathcal{U})$ such that $B_i \cap \Sigma_\epsilon \neq \emptyset$, we have a point $c_i \in B_i \cap \Sigma_\epsilon \cap A(D)$ such that $\iota(B_i) = \Phi^N_{R-\epsilon}(c_i)$. Furthermore, if $B_i = hB_j$ with $B_i, B_j \in A_1(\mathcal{U})$ with $B_j \cap \Sigma_\epsilon \neq \emptyset$, then $c_i = hc_j$.

Proof. Our strategy is to construct a map $j : S_1 \to C_{S_1}(\iota)$, which satisfies $j(hx) = hj(x)$ for all $h \in H$, $x \in S_1$ that satisfy $hx \in S_1$. By our assumption, $C_{S_1}(\iota) \subset Y_\epsilon$ and thus, once we have obtained $j$, Lemma 3.2 will give us a continuous, $H$-invariant vector field on $C_\epsilon$. To make sure that this vector field satisfies the stated estimate on angles, we construct our map $j$ appropriately.

In order to control what’s happening near $\Sigma_\epsilon$, we introduce the following notation. We define

$$\mathcal{U}_\Sigma := \{ B_i \in A_1(\mathcal{U}) | B_i \cap \Sigma_\epsilon \neq \emptyset \}$$

and we denote by $S_\Sigma$ the subcomplex of $S_1$ spanned by $\mathcal{U}_\Sigma$.

We first define $j$ on the vertices of $S_1$ by defining $j(B_i) = \iota(B_i)$ for all $B_i \in S_1^{(0)}$. By assumption, we see that $j(hx) = hj(x)$ whenever both sides of the equation make sense in $S_1$.

We extend $j$ to $S_1$ by extending $j$ to the $l$-skeleton of $S_1$ for all $l$ with induction over $l$. Furthermore, the extension of $j$ from the $l$-skeleton to the $(l+1)$-skeleton will be done by successively extending $j$ to one $(l+1)$-simplex after the other. The following will be our induction assumption.

**Induction Assumption.** Suppose $j$ has been defined on a subcomplex of $S_1$ that contains the $l$-skeleton of the complex and possibly some $(l+1)$-simplices such that $j(hx) = hj(x)$ for all $h \in H$, $x \in S_1$ that satisfy $hx \in S_1$ (in particular, $j$ is defined for both $x$ and $hx$). Furthermore, suppose that for all simplices $\sigma(B_1, \ldots, B_{k+1})$ on which $j$ is already defined, we have $j(\sigma(B_1, \ldots, B_{k+1})) \subset C(\iota(B_1), \ldots, \iota(B_{k+1}))$.

Clearly, the induction assumption is satisfied on the 0-skeleton. Now suppose the induction assumption is given for some $l$. Let $\tilde{\sigma} = \sigma(B_1, \ldots, B_{l+2})$ be a simplex in $S_1$ on whose interior $j$ is not yet defined. Since $j$ is already defined on the $l$-skeleton, we see that $j$ is already defined on $\partial \tilde{\sigma}$ and
\[ j(\partial \tilde{\sigma}) \subset C(\iota(B_1), \ldots, \iota(B_{+2})), \] which means the restriction of \( j \) to \( \partial \tilde{\sigma} \) defines an element of the \( l \)-th homotopy group \( \pi_l(C(\iota(B_1), \ldots, \iota(B_{+2}))) \). Since \( C(\iota(B_1), \ldots, \iota(B_{+2})) \) is convex, it is contractible. Therefore, there exists a continuous extension of \( j \) that maps \( \tilde{\sigma} \) into \( C(\iota(B_1), \ldots, \iota(B_{+2})) \). Furthermore, for every simplex in \( S_1 \) of the form \( h\tilde{\sigma} \) for some \( h \in H \), we extend \( j \) to \( h\tilde{\sigma} \) by \( j(hx) := hj(x) \). By the induction assumption, we know that for all \( x \in \partial \tilde{\sigma} \), the equation \( j(hx) = hj(x) \) is already satisfied and thus there is no conflict with any previous definition of \( j \). (If \( j \) was already defined on \( h\tilde{\sigma} \) in a previous induction step, then we already extended \( j \) to \( \tilde{\sigma} \) in that previous step, so no such conflict can occur in that way either. Furthermore, property b) of Lemma 3.3 guarantees that \( \tilde{\sigma} \) and \( h\tilde{\sigma} \) do not intersect, excluding the final source of contradictions in extending \( j \) this way.) Thus, \( j \) extends continuously to \( \tilde{\sigma} \) and all its translates.

Since \( S_1 \) is a finite simplicial complex, this procedure allows us to define a continuous map \( j : S_1 \to \mathcal{C}(\iota) \) such that \( j(hx) = hj(x) \), whenever \( x, hx \in S_1 \). Furthermore, by construction the map \( j \) satisfies for every simplex \( \sigma(B_1, \ldots, B_{+1}) \) that \( j(\sigma(B_1, \ldots, B_{+1})) \subset C(\iota(B_1), \ldots, \iota(B_{+1})) \).

We now prove that this last property makes the vector field induced by \( j \) satisfy the estimate of angles we require. Let \( q \in \Sigma_e \cap \mathcal{D} \) and let \( B_1, \ldots, B_{+1} \) be all elements of \( A_1(\mathcal{U}) \) that contain \( q \). (Note that for every element of the form \( hB_i \) with \( h \in H \), \( B_i \in \mathcal{U} \) that is not part of the list above, it cannot contain \( q \), even if \( hB_i \) is not in \( A_1(\mathcal{U}) \). If it did, then \( hB_i \) would intersect \( \mathcal{D} \), thus it would be an element of \( A_1(\mathcal{U}) \), thus it would be contained in the list \( B_1, \ldots, B_{+1} \) above.) The vector field \( V \) constructed in Lemma 3.2 is defined by \( V(q) = \tilde{\sigma}(\psi(q)) \). By definition of \( \psi \), \( \psi(q) \) is an element of the simplex \( \sigma(B_1, \ldots, B_{+1}) \in S_1 \). Therefore, \( j \circ \psi(q) \in C(\iota(B_1), \ldots, \iota(B_{+1})) \).

By assumption, since \( B_1, \ldots, B_{+1} \) all intersect \( \Sigma_e \), we have points \( c_i \in B_i \cap \Sigma_e \cap A(D) \) such that \( \iota(B_i) = \Phi^N_{R_i}(c_i) \). Since all \( B_i \) have diameter at most \( \delta \) and \( q \in \bigcap_{i=1}^{k+1} B_i \), we see that \( d(c_i, c_j) \leq 2\delta \) for all \( i, j \). By property (2) in Lemma 3.3 this implies that \( d(\iota(B_i), \iota(B_j)) \leq \delta' \), which implies that \( \text{diam}(\iota(B_1), \ldots, \iota(B_{+1})) \leq \delta' \). Since balls are convex in non-positively curved spaces, this implies that \( C(\iota(B_1), \ldots, \iota(B_{+1})) \subset \bigcap_{i=1}^{k+1} B_i \iota(\mathcal{U}) \).

Since \( \iota(B_i) = \Phi^N_{R_i}(c_i) \in K \) by definition of \( K \), properties (1) and (3) of Lemma 3.3 imply

\[
\angle(N(q), q\tilde{\sigma}(\psi(q))) \leq \angle(N(q), \overrightarrow{q\iota(B_1)}) + \frac{\alpha_0}{2} \\
\leq \angle(N(c_i), \overrightarrow{c_1\iota(B_1)}) + \frac{2\alpha_0}{2} = \alpha_0.
\]

By Lemma 3.2 we obtain a continuous, \( H \)-invariant vector field \( V \) on \( C_e \). By \( H \)-invariance of \( V \) and \( N \), the inequality \( \angle(N(q), V(q)) \leq \alpha_0 \) follows for all \( q \in \Sigma_e \). This proves the Lemma. \( \square \)
Lemma 3.6. Let $\epsilon > 0$ and $V$ a continuous vector field defined on $C_\epsilon$ such that for all $q \in C_\epsilon$, the geodesic ray induced by $V(q)$ crosses $\Sigma_\epsilon$ and for all $q \in \Sigma_\epsilon$, $\angle(V(q), N(q)) \leq \alpha$ for some $\alpha \in (0, \frac{\pi}{2})$. Then $\Sigma_\epsilon$ is contractible.

Proof. Given a point $q \in C_\epsilon$, we define $\gamma_q$ to be the geodesic ray $\exp_q(tV(q))$ induced by $V(q)$. We begin by defining a map $P : C_\epsilon \to \Sigma_\epsilon$ that sends $q \in C_\epsilon$ to the unique point where $\gamma_q$ intersects $\Sigma_\epsilon$. For $q \in C_\epsilon \setminus \Sigma_\epsilon$, this point is unique because $C_\epsilon$ is convex. If $q \in \Sigma_\epsilon$, then $V(q)$ points out of $C_\epsilon$. Therefore $\gamma_q$ leaves $C_\epsilon$ immediately and can never return by convexity. Thus, for $q \in \Sigma_\epsilon$, the unique intersection of $\gamma_q$ with $\Sigma_\epsilon$ is $q$ itself.

We claim that $P$ is continuous. Let $q_n \in C_\epsilon$ be a sequence converging to a point $q$. By continuity, this implies that $V(q_n) \to V(q)$ and thus $\gamma_{q_n} \to \gamma_q$ in compact-open topology.

Since $C_\epsilon$ is strictly convex, we have that for all $q \in C_\epsilon$, $\gamma_q$ intersects $\Sigma_\epsilon$ transversely. However, given a family of paths varying continuously in compact-open topology that intersects a submanifold of codimension one transversely, their intersection varies continuously. (One can see this by using local coordinates around the intersection point of $\gamma_q$.) Therefore, $P(q_n) \to P(q)$ and $P$ is continuous.

We now prove that all homotopy groups of $\Sigma_\epsilon$ vanish. The map $P$ is a retraction of $C_\epsilon \to \Sigma_\epsilon$, i.e. it is continuous and it fixes every point in $\Sigma_\epsilon$. This implies in particular that the inclusion map $\iota : \Sigma_\epsilon \hookrightarrow C_\epsilon$ induces injective maps $\iota_* : \pi_\ast$ on the homotopy groups, as $P_\ast$ is a left-inverse of $\iota_*$. Since $C_\epsilon$ is convex, all its homotopy-groups vanish and thus $\iota_*$ is an isomorphism. Whitehead’s theorem now implies that $\iota$ is a homotopy equivalence. The Lemma follows. 

\[ \square \]

3.2. Strong barycenters. As we have seen, the construction of the map $j$, and thus the proof of contractibility of $\Sigma_\epsilon$, relies on our ability to make sure that the image of $j$ is contained in $Y_\epsilon$. The proof of Lemma 3.5 shows that, if we assume the existence of a cover $U$ and a suitable map $\iota : A_1(U) \to Y$ such that $C_{S_\iota}(i) \cap C_\epsilon = \emptyset$, then we can construct such $j$. We will now provide a first sufficient condition as to when this assumption can be satisfied.

Before we provide the sufficient condition, we introduce some notation for barycentric subdivisions of simplicial complexes. Let $S$ be a simplicial complex. We denote its $n$-th barycentric subdivision by $(n)S$. Note that there is a canonical homeomorphism $i_n : S \to (n)S$. For a simplex $\sigma$ in $S$, its closed star $St(\sigma)$ is defined to be the closure of the union of all simplices in $S$ that intersect $\sigma$. Given a simplex $\sigma$, we write $St(\sigma)^{(0)}$ for the set of vertices in $St(\sigma)$ and $St(\sigma)^{(1)}$ for the set of vertices in the first barycentric subdivision of $St(\sigma)$.

Definition 3.7. Let $Z$ be a metric space, $\frac{1}{2} \leq \lambda < 1$, and $p_1, \ldots, p_n \in Z$ a finite collection of points. We call a point $b \in Z$ a $\lambda$-barycenter of $\{p_1, \ldots, p_n\}$
if

\[ \forall 1 \leq i \leq n : d(b, p_i) \leq \lambda \cdot \text{diam}(\{p_1, \ldots, p_n\}). \]

If, additionally, \( q_1, \ldots, q_m \in Z \), we call \( b \in Z \) a \( \lambda \)-barycenter of \( \{p_1, \ldots, p_n\} \) relative to \( \{q_1, \ldots, q_m\} \) if it is a \( \lambda \)-barycenter of \( \{p_1, \ldots, p_n\} \) and, additionally,

\[ \forall 1 \leq j \leq m : d(b, q_j) \leq \text{diam}(q_j, p_1, \ldots, p_n). \]

Let \( \Delta > 0 \). We say that \( Z \) has \( \lambda \)-barycenters up to diameter \( \Delta \) if every finite set of points in \( Z \) with diameter at most \( \Delta \) has a \( \lambda \)-barycenter. We say that \( Z \) has strong \( \lambda \)-barycenters up to diameter \( \Delta \) if for any two finite sets \( P, Q \subset Z \) such that \( \text{diam}(P) \leq \Delta \) and \( \text{diam}(P \cup Q) \leq 2\Delta \), there exists a \( \lambda \)-barycenter of \( P \) relative to \( Q \).

**Example 3.8.** Let \( X \) be a simply connected, geodesically complete Riemannian manifold such that all its sectional curvatures are at most 0 and let \( p_1, \ldots, p_n, q_1, \ldots, q_m \in X \). Without loss of generality, \( d(p_1, p_2) = \text{diam}(p_1, \ldots, p_n) \). Let \( \gamma \) be the geodesic from \( p_1 \) to \( p_2 \) and let \( b \) be the midpoint of \( \gamma \). Standard arguments with comparison triangles and Euclidean geometry together with the fact that the functions \( d(q_j, \gamma(t)) \) are convex show that \( b \) is a \( \sqrt{2} \)-barycenter of \( \{p_1, \ldots, p_n\} \) relative to \( \{q_1, \ldots, q_m\} \). We conclude that \( X \) has strong \( \frac{\sqrt{2}}{2} \)-barycenters up to any diameter. (Note that the same holds for any dense subset of \( X \).)

**Example 3.9.** Let \( Z \) be a circle of radius \( r \) embedded in the Euclidean plane, equipped with the metric it inherits from \( \mathbb{R}^2 \), and let \( x, y, z \) be three equidistant points on \( Z \). One easily checks that the only points \( b \in Z \) that satisfy \( \max(d(b, x), d(b, y), d(b, z)) \leq \text{diam}(x, y, z) \) are the points \( x, y, z \) themselves and there is no \( \lambda < 1 \) for which a \( \lambda \)-barycenter exists.

The diameter of any set of equidistant points on \( Z \) is equal to \( \sqrt{3}r \). As soon as we limit ourselves to sets with smaller diameter, we can find \( \lambda < 1 \) such that \( \lambda \)-barycenters exist. Therefore, \( Z \) has \( \lambda \)-barycenters up to diameter \( \Delta \) for every \( \Delta < \sqrt{3}r \). Note that \( Z \) does not have strong \( \lambda \)-barycenters up to \( \Delta \) close to \( \sqrt{3}r \), as for any two \( p_1, p_2 \in Z \) whose distance is slightly smaller than \( \sqrt{3}r \), we can choose \( q \) to be the antipodal point to the midpoint on the arc from \( p_1 \) to \( p_2 \). One easily sees that \( \{p_1, p_2\} \) cannot have a \( \lambda \)-barycenter relative to \( q \) for any \( \lambda < 1 \). However, if \( \Delta < \frac{\sqrt{3}}{2}r \), then \( P \cup Q \) is contained in a sufficiently small arc of the circle so that one can take the midpoint of the shortest arc that contains all elements of \( P \) as a \( \lambda \)-barycenter relative to \( Q \) for any \( \lambda \in [\frac{1}{2}, 1) \).

A similar example can be obtained by considering four equidistant points on a 2-sphere. This example also generalises to higher-dimensional spheres.

**Lemma 3.10.** Let \( \epsilon > 0 \), \( D \) a fundamental domain of the action of \( H \) on \( C_\epsilon \) with compact closure. For \( R > \epsilon \) denote by \( D_R := \Phi_{R-\epsilon}^N(D \cap \Sigma_\epsilon) \) the orthogonal ‘push-out’ of \( D \) to \( \Sigma_R \). Note that \( H \) acts on \( \Sigma_R \) and \( D_R \) is a fundamental domain of this action. Let \( \alpha_0 \in (0, \frac{\pi}{2}) \) and \( K = \Phi_{R-\epsilon}^N(A(D) \cap \Sigma_R) \)
Lemma 3.3 can be applied to show that for all \( \sigma \in \Sigma_c \) that are exactly the same when we replace \( h \) with respect to \( \sigma \) in Lemma 3.10, there exists a cover \( U \) of \( \mathbb{D} \) satisfying the properties in Lemma 3.3 with respect to \( \delta'' \) in the role of \( \delta' \).

Then there exist a cover \( U \) of \( \mathbb{D} \) satisfying the properties in Lemma 3.3 with respect to \( \delta'', \delta \), \( n \in \mathbb{N} \), and a map \( \iota : (n)S_1(0) \to K \) defined on the vertices of the \( n \)-th barycentric subdivision of \( S_1 \) such that \( \iota(hB_1) = h\iota(B_1) \) for all \( h \in H \), \( B_1 \in (n)S_1(0) \) that satisfy \( hB_1 \in (n)S_1(0) \) and \( C_n(S_1)(\iota) \cap C_\epsilon = \emptyset \).

Furthermore, the map \( \iota \) satisfies that for all vertices \( B_i \in U \) such that \( B_i \cap \Sigma_c \neq \emptyset \), there exists \( c_i \in B_i \cap \Sigma_c \) such that \( \iota(B_i) = \Phi^N_{R-c}(c_i) \) and for all simplices \( \sigma(B_1, \ldots, B_{k+1}) \) in \( S_{\Sigma} \) and for all vertices \( B_j \in (n)\sigma(B_1, \ldots, B_{k+1})(0) \), we have \( d(\iota(B_1), \iota(B_j)) \leq 2\delta' \).

**Corollary 3.11.** Suppose there exist \( R > \epsilon > 0 \), \( \Delta > 0 \), \( \frac{1}{2} \leq \lambda < 1 \), and a fundamental domain \( D \) of the action of \( H \) on \( C_\epsilon \) with compact closure such that \( \Sigma_R \) has strong \( \lambda \)-barycenters up to diameter \( \Delta \) and \( \text{diam}( \Phi^N_{R-c}(A(D) \cap \Sigma_c) ) \leq \Delta \). Then, there exists a continuous, \( H \)-invariant vector field \( V \) on \( C_\epsilon \) such that for all \( q \in \Sigma_c \), \( \angle(V(q), N(q)) \leq \alpha_0 \) for some \( \alpha_0 < \frac{\delta}{2} \).

**Proof of Corollary 3.11.** Let \( D \) be a fundamental domain of the action of \( H \) on \( C_\epsilon \) with compact closure, such that \( D \cap \Sigma_c = \Phi^N_{c-R}(D_R) \). By Lemma 3.10 there exists a cover \( U \) of \( \mathbb{D} \), \( n \in \mathbb{N} \), and \( \iota : (n)S_1 \to \Sigma_R \) such that \( C_n(S_1)(\iota) \cap C_\epsilon = \emptyset \). We now notice that the proof of Lemma 3.3 still works exactly the same when we replace \( S_1 \) with its \( n \)-th barycentric subdivision \((n)S_1 \). The only part that requires an extra argument is the estimate of angles that \( j \) needs to satisfy. For this, consider a point \( q \in \Sigma_c \) and let \( B_1, \ldots, B_{k+1} \) be the elements of \( A_1(U) \) that contain \( q \). Therefore, \( j_i \circ \Psi(q) \) is contained in a simplex of \( (n)S_1 \) that lies in the \( n \)-th barycentric subdivision of the simplex \( \sigma(B_1, \ldots, B_{k+1}) \). Let \( B_{l_1}, \ldots, B_{l_{k+1}} \) be the vertices in \( (n)S_1 \) spanning a simplex that contains \( i_n \circ \Psi(q) \). Due to the properties of \( \iota \) obtained in Lemma 3.10 we see that \( d(\iota(B_1), \iota(B_{l_j})) \leq 2\delta' \). Since \( j_i \circ (i_n(\Psi(q))) \in C(\iota(B_{l_1}), \ldots, \iota(B_{l_{k+1}})) \) by definition, we obtain that \( d(\iota(B_1), j_i(i_n(\Psi(q))) \leq 2\delta' \). Since \( \iota(B_1) = \Phi^N_{R-c}(c_1) \in K \) by Lemma 3.10 properties (1) and (3) of Lemma 3.3 can be applied to show

\[
\angle(N(q), q_j(i_n(\Psi(q)))) \leq \angle(N(q), q_j(B_1)) + \frac{\alpha_0}{2}
\]

\[
\leq \angle(N(c_1), c_1\iota(B_1)) + \alpha_0 = \alpha_0.
\]

Therefore, we obtain a continuous map \( j \circ i_n : S_1 \to C_n(S_1)(\iota) \), which is compatible with the action of \( H \) and satisfies the necessary estimate of angles. Since \( C_n(S_1)(\iota) \cap C_\epsilon = \emptyset \), we have that \( j \circ i_n(S_1) \subset Y_\epsilon \). Lemma 3.2
denote the simplicial complex dual to all simplices \( B \) most with the action on \( \delta \).

Proof of Lemma 3.10. Let \( R > \epsilon > 0 \), \( D \) a fundamental domain of the action of \( H \) on \( C_e \) with compact closure, \( D_R := \Phi_{R-\epsilon}(D \cap \Sigma_e), K := \Phi_{R-\epsilon}(A(D) \cap \Sigma_e), \alpha_0 \in (0, \frac{1}{2}) \) and \( \Delta > 0, \frac{1}{2} \leq \lambda < 1 \) such that \( \text{diam}(K) \leq \Delta \) and \( \Sigma_R \) has strong \( \lambda \)-barycenters up to diameter \( \Delta \).

Let \( \delta' \) be the constant obtained from Lemma 3.3 for our given \( R, \epsilon, \alpha_0, K, \) and \( D \), but choose it additionally sufficiently small such that \( \delta' \leq \frac{R-\epsilon}{2} \). Let \( \delta'' = (1-\lambda)\delta' \) and let \( \delta \) be a constant such that \( (\delta'', \delta) \) satisfy the properties of Lemma 3.3. Let \( \mathcal{U} \) be a finite collection of open sets in \( X \) of diameter at most \( \delta \) that cover \( \overline{D} \) and satisfy the properties in Lemma 3.3. Let \( S_1 \) be the simplicial complex dual to \( A_1(\mathcal{U}) \).

We first define a suitable map \( \iota : S_1^{(0)} \to K \). For every \( B_i \in \mathcal{U} \) such that \( B_i \cap \Sigma_e \neq \emptyset \), we find \( c_i \in B_i \cap \Sigma_e \cap \overline{D} \) by Lemma 3.3 and we define \( \iota(B_i) := \Phi_{R-\epsilon}(c_i) \). For all \( B_i \in \mathcal{U} \) such that \( B_i \cap \Sigma_e = \emptyset \), we define \( \iota(B_i) \) to be any choice of a point in \( D_R \). We extend \( \iota \) to \( A_1(\mathcal{U}) \) by \( \iota(hB_i) := h\iota(B_i) \) for all \( h \in H, B_i \in \mathcal{U} \) that satisfy \( hB_i \in A_1(\mathcal{U}) \). Since \( \mathcal{U} \) satisfies property c) of Lemma 3.3, we see that for every vertex \( hB_i \in A_1(\mathcal{U}) \), we have \( \iota(hB_i) \in K \). (If \( hB_i \in A_1(\mathcal{U}) \), then there exists an element \( B_j \in \mathcal{U} \) such that \( hB_i \cap B_j \neq \emptyset \). By property c), \( h\overline{D} \cap \overline{D} \neq \emptyset \) and therefore, \( h\overline{D}_R \subset K \).) This provides us with a map \( \iota : S_1^{(0)} \to K \). Because \( \text{diam}(K) \leq \Delta \) by assumption, this implies for all simplices \( \sigma \) in \( S_1 \) that \( \text{diam}(\iota(\sigma)) \leq \Delta \).

Let \( \overline{U} := \{ hB_i | h \in H, B_i \in \mathcal{U} \} \), which is a locally finite cover of \( C_e \) and let \( S \) denote the simplicial complex dual to \( \overline{U} \). There is a free simplicial action of \( H \) on \( S \), such that the canonical simplicial embedding \( S_1 \hookrightarrow S \) commutes with the action on \( S \) and the ‘partial action’ on \( S_1 \). Furthermore, \( S_1 \) contains a fundamental domain of the action. Using the free action of \( H \) on \( S \), we extend the map \( \iota : S_1^{(0)} \to K \) to an \( H \)-equivariant map \( \iota : S_1^{(0)} \to \Sigma_R \) that satisfies that for every simplex \( \sigma \) in \( S_1^{(0)} \), we have \( \iota(\sigma) \subset hK \) for some \( h \in H \) and \( \text{diam}(\iota(\sigma)) \leq \Delta \). We will extend \( \iota \) to the barycentric subdivisions of \( S \) and then restrict to the subdivisions of \( S_1 \).

Recall the notations for barycentric subdivisions we introduced in section 2.2 and at the beginning of this section. Let \( S \) be a locally finite simplicial complex with a free, cocompact simplicial action by \( H \). (Specifically, we want \( S \) to be \( (n)S \) for some \( n \in \mathbb{N} \), but what we are about to show holds more generally.) Denote the vertices of \( S \) by \( v_j \), indexed by some set \( I \). If \( J \subset I \) indexes a set of vertices that span a simplex in \( S \), we denote that

Remark 3.12. If one wishes to apply Lemma 3.5 more directly, one can show that the cover \( \mathcal{U} \) can be replaced with a cover \( (n)\mathcal{U} \) that still satisfies the properties in Lemma 3.3 and whose dual complex canonically embeds into \( (n)S_1 \). One can then apply Lemma 3.5 to \( (n)\mathcal{U} \) together with the map \( \iota \) provided by Lemma 3.10 to obtain the Corollary.
simplex by \( \sigma_J \) and the vertex in \((1)\tilde{S}\) corresponding to \(\sigma_J\) by \(v_J\). The proof of the Lemma is based on the following

**Claim.** Let \(\tilde{S}\) be a locally finite simplicial complex with a free, cocompact simplicial action by \(H\). Let \(\iota: \tilde{S}^{(0)} \to \Sigma_R\) be an \(H\)-equivariant map such that there exists \(\Delta_0 \in [0, \Delta]\) such that for every simplex \(\sigma\) in \(\tilde{S}\), we have \(\text{diam}(\iota(\sigma)) \leq \Delta_0 \leq \Delta\).

Then we can extend \(\iota\) to an \(H\)-equivariant map \(\tilde{\iota}: (1)\tilde{S}^{(0)} \to \Sigma_R\) such that for every simplex \(\sigma(v_{J_1}, \ldots, v_{J_{k+1}})\) in \((1)\tilde{S}\) with \(J_1 \subset \cdots \subset J_{k+1}\), we have

\[
\text{diam}(\iota(v_{J_1}), \ldots, \iota(v_{J_{k+1}})) \leq \lambda \text{diam} \left(\iota(\sigma^{(0)}_{J_{k+1}})\right) \leq \lambda \Delta_0.
\]

**Proof of Claim.** We will define \(\iota\) successively on vertices \(v_J\) by induction over the size of \(J\). To formulate our induction assumption precisely, we need some notation. Suppose \(\iota\) is defined on all vertices of \(\tilde{S}\) and on some additional vertices in \((1)\tilde{S}\). Let \(J = \{j_1, \ldots, j_{k+1}\}\) such that \(v_{j_1}, \ldots, v_{j_{k+1}}\) span a simplex \(\sigma_J\) in \(\tilde{S}\). At any step of the induction, denote by \(V_J \subset (1)\text{St}(\sigma_J)^{(0)}\) the set of vertices \(v_J\) in the first barycentric subdivision of \(\text{St}(\sigma_J)\) for which \(\iota\) is already defined and for which the simplex \(\sigma_{J \cup J'}\) exists in \(\tilde{S}\). Furthermore, we define \(W_J \subset (1)\sigma_J^{(0)}\) to be the set of vertices in the first barycentric subdivision of \(\sigma_J\) for which \(\iota\) is already defined. (For every index \(J\) of the first barycentric subdivision, the sets \(V_J, W_J\) are specific to the particular step of the induction under consideration. Whenever we speak of either of the two sets, we have to indicate which step of the induction we are taking them from.) Note that, if \(v_{J'} \in V_J\), then \(\sigma_J \in \text{St}(\sigma_{J'})\) due to the fact that \(\sigma_J, \sigma_{J'}\) are faces of the higher-dimensional simplex \(\sigma_{J \cup J'}\).

**Induction Assumption.** We define \(\iota\) inductively such that at every step it will satisfy the following for every index \(J\) for which \(\iota(v_J)\) is defined:

- For every \(h \in H\), \(\iota(hv_J)\) is defined and \(\iota(hv_J) = hu(v_J)\).
- For every index set \(J'\) that spans a simplex \(\sigma_{J'}\) in \(\tilde{S}\) (regardless of whether \(\iota(v_{J'})\) is defined),
  
  \[
  \text{diam} \left(\iota(W_{J'})\right) \leq \text{diam} \left(\iota(\sigma_{J'}^{(0)})\right).
  \]

- If \(J' \subsetneq J\), then \(\iota(v_{J'})\) is defined and
  
  \[
  d(\iota(v_{J'}), \iota(v_J)) \leq \lambda \text{diam}(\iota(\sigma_J^{(0)})).
  \]

Note that these properties are satisfied for \(\iota\) as we defined it on \(\tilde{S}^{(0)}\), where we are considering vertices \(v_J\) with \(|J| = 1\). Thus the start of the induction is given. Before we do the general induction-step, we show how the induction is done to extend \(\iota\) to vertices \(v_J\) with \(|J| = 2\). Suppose \(\iota\) is defined on \(\tilde{S}^{(0)}\) and on some vertices \(v_{J'}\) with \(|J'| = 2\) and suppose it satisfies the induction assumptions outlined above. Let \(J = \{i, j\}\) be the index of a vertex in \((1)\tilde{S}\) for which \(\iota\) is not yet defined. Denote the edge in \(\tilde{S}\) spanned by \(v_i, v_j\) by \(e\).
Figure 1. On the left, the map $\iota$ is defined for all vertices of the original simplicial complex and for all barycenters of edges. The points $p_1, \ldots, p_6, q_1, \ldots, q_4$ are images of vertices in the simplicial complex under $\iota$. As indicated, the vertices sent to the points $q_1, p_1, p_2, p_3$ span a simplex. On the right, we see the extension of $\iota$ to the barycenter $v_P$ of the simplex spanned by $p_1, p_2, p_3$. We have $V_{p_1,p_2,p_3} = \{p_1, \ldots, p_6, q_1, \ldots, q_4\}$ and $W_{p_1,p_2,p_3} = \{p_1, \ldots, p_6\}$. The extension is chosen to be a $\lambda$-barycenter of $\{p_1, \ldots, p_6\}$ relative to $\{q_1, \ldots, q_4\}$. Concretely, all distances $d(p_i, \iota(v_P))$ satisfy $d(p_i, \iota(v_P)) \leq \lambda \text{diam}(p_1, \ldots, p_6)$, while the distance to the points $q_j$ is controlled by $d(q_i, \iota(v_P)) \leq \text{diam}(q_i, p_1, \ldots, p_6)$.

Since $\text{diam}(K) \leq \Delta$ by assumption and we have proven that $\iota(\sigma_{J}(0)) \subset hK$ for some $h \in H$, we know that $\text{diam}(\iota(e(0))) \leq \Delta$. Furthermore, if $v_{J'} \in V_J$, then the index set $J' \cup \{i\}$ spans a simplex in $\tilde{S}$ by definition of $V_J$ and the set $W_{J' \cup \{i\}}$ taken at this moment of the induction satisfies

$$d(\iota(v_{J'}), \iota(v_i)) \leq \text{diam}(\iota(W_{J' \cup \{i\}})) \leq \text{diam}(\iota(\sigma_J(0) \cup \{i\})) \leq \Delta,$$

since $\iota$ sends all simplices in $\tilde{S}$ to sets of diameter at most $\Delta$ by induction assumption. We conclude that $\text{diam}(\iota(V_J)) \leq 2\Delta$.

Since $\Sigma_R$ has strong $\lambda$-barycenters up to diameter $\Delta$, there exists a point $p_J$ which is a $\lambda$-barycenter of $\iota(e(0))$ relative to $\iota(V_J \setminus e(0))$. We define $\iota(v_J) := p_J$ and immediately extend $\iota$ by $\iota(hv_J) := h\iota(v_J)$ to every edge in $\tilde{S}$ of the form $\sigma_{J'} = he$ for some $h \in H$. The way $p_J$ was chosen implies that
for every $v \in V_J$, we have
\[ d(\iota(v_J), \iota(v)) \leq \text{diam} \left( \iota(\sigma_{J'}(0)) \cup \{ \iota(v) \} \right) \]
and for both elements of $\{ v_i, v_j \} = \iota(0) = \sigma_{J'}(0)$,
\[ \max(d(\iota(v_J), \iota(v_i)), d(\iota(v_J), \iota(v_j))) \leq \lambda \text{diam} \left( \iota(\sigma_{J'}(0)) \right) . \]

We are left to show that the bound on $\text{diam}(\iota(W_{J'}))$ still holds for all index sets $J'$ that span a (possibly higher-dimensional) simplex $\sigma_{J'}$ in $\tilde{S}$. Let $W_{J'}^{\text{old}}$ denote the set $W_{J'}$ prior to extending $\iota$ to $v_J$ and let $W_{J'}^{\text{new}}$ denote the set $W_{J'}$ after extending $\iota$ to $v_J$ and its translates by $H$. Note that $W_{J'}^{\text{old}}$ and $W_{J'}^{\text{new}}$ only differ if $hJ \subseteq J'$ for some $h \in H$. Exploiting the fact that $H$ acts by isometries on $\Sigma_R$, we can assume without loss of generality that $J \subseteq J'$. Since we know that $\text{diam} \left( W_{J'}^{\text{old}} \right) \leq \text{diam} \left( \iota(\sigma_{J'}(0)) \right)$ (by the induction assumption), we only need to bound $d(\iota(v_J), \iota(v_j))$ for all $J \subseteq J'$ that satisfy $v_J \in W_{J'}^{\text{old}}$. Indeed, if $\iota(v_J)$ is defined and $J \subseteq J'$, then $v_J \in V_J$ prior to the extension to $v_J$ and the properties of $\lambda$-barycenters imply
\[ d(\iota(v_J), \iota(v_j)) \leq \text{diam} \left( \iota(e(0)) \cup \{ \iota(v) \} \right) \]
\[ \leq \text{diam} \left( \iota(W_{J'}^{\text{old}}) \right) \]
\[ \leq \text{diam} \left( \iota(\sigma_{J'}(0)) \right), \]
where we use the induction assumption to obtain the last inequality.

All inequalities obtained above are preserved under the action of $H$, as it acts by isometries and equivariantly w.r.t. $\iota$. Therefore, the extension of $\iota$ to $v_J$ and all its translations under $H$ satisfies the properties of the induction assumption. Since $\tilde{S}$ has a finite fundamental domain, we can use this extension procedure to extend $\iota$ to all vertices $v_J$ in $(1)\tilde{S}$ with $|J| \leq 2$.

We now show how to do the induction-step in general. Let $k \geq 3$ and suppose $\iota$ has been defined for all vertices $v_J$ with $|J| < k$ and for some vertices $v_{J'}$ with $|J| = k$ such that it satisfies the induction assumption. Let $J = \{ j_1, \ldots, j_k \}$ be the index of a vertex in $(1)\tilde{S}$ for which $\iota$ is not yet defined. First, we prove that $\text{diam}(\iota(1)\partial\sigma_J(0)) = \text{diam}(\iota(\sigma_J(0)))$. Indeed, since every vertex in $(1)\partial\sigma_J$ corresponds to a proper subset $J' \subset J$ and $\iota$ has been defined for $v_{J'}$ with $|J'| < k$, we have that $W_{J'} = (1)\partial\sigma_J(0)$ during this step of the induction. The induction assumption thus implies that $\text{diam}(\iota(1)\partial\sigma_J(0)) = \text{diam}(\iota(\sigma_J(0)))$. Furthermore, if $J'$ is an index set such that $\sigma_{J'}$ exists in $\tilde{S}$ and $v_{J'} \in V_J$, then for every $j \in J$, $J' \cup \{ j \}$ spans a simplex in $\tilde{S}$ and we have
\[ d(\iota(v_{J'}), \iota(v_j)) \leq \text{diam} \left( \iota(W_{J' \cup \{ j \}}) \right) \leq \text{diam} \left( \iota(\sigma_{J' \cup \{ j \}}(0)) \right) \leq \Delta. \]
We conclude that \( \text{diam}(\iota(V_J \cup V'_J)) \leq 2\Delta \).

Since \( \Sigma_R \) has strong \( \lambda \)-barycenters up to diameter \( \Delta \), there exists a point \( p_J \in \Sigma_R \) which is a \( \lambda \)-barycenter of \( \iota(\partial \sigma_J^{(0)}) \) relative to \( \iota(V_J \setminus \partial \sigma_J^{(0)}) \).

We define \( \iota(v_J) := p_J \) and immediately extend \( \iota \) by \( \iota(hv_J) := h\iota(v_J) \) to every vertex in \( \tilde{S} \) that can be written as \( hv_J \). The way \( p_J \) was chosen immediately implies that for every \( v \in V_J \),

\[
d(\iota(v_J), \iota(v)) \leq \text{diam}\left(\iota(\partial \sigma_J^{(0)}) \cup \{\iota(v)\}\right)
\]

and for every \( v_J' \in \partial \sigma_J^{(0)} \),

\[
d(\iota(v_J), \iota(v_J')) \leq \lambda \text{diam}\left(\iota(\partial \sigma_J^{(0)})\right).
\]

We are left to show that the bound on \( \text{diam}(\iota(W_J')) \) still holds for all index sets \( J' \) that span a simplex \( \sigma_{J'} \) in \( \tilde{S} \). For every such index set \( J' \), let \( W_{J'}^{\text{old}} \) denote the set \( W_{J'} \) prior to extending \( \iota \) to \( v_J \) and let \( W_{J'}^{\text{new}} \) denote the set \( W_{J'} \) after extending \( \iota \) to \( v_J \). As before, \( W_{J'}^{\text{old}} \) and \( W_{J'}^{\text{new}} \) only differ if \( hJ \subset J' \) for some \( h \in H \) and without loss of generality, \( J \subset J' \).

Since we know that \( \text{diam}(W_{J'}^{\text{old}}) \leq \text{diam}(\iota(\partial \sigma_J^{(0)})) \) (by the induction assumption), we only need to bound \( d(\iota(v_J), \iota(v_J')) \) for all \( J \subset J' \) that satisfy \( v_J \in W_{J'}^{\text{old}} \).

Indeed, if \( \iota(v_J) \) is defined and \( J \subset J' \), then \( v_J \in V_J \) and we continue from the inequality above

\[
d(\iota(v_J), \iota(v_J')) \leq \text{diam}\left(\iota(\partial \sigma_J^{(0)}) \cup \{\iota(v_J)\}\right)
\]

\[
\leq \text{diam}(\iota(W_{J'}^{\text{old}}))
\]

\[
\leq \text{diam}(\iota(\partial \sigma_J^{(0)})),
\]

where we use the induction assumption to obtain the last inequality. We thus see that the extension of \( \iota \) to \( v_J \) and its translates by \( H \) still satisfies the induction assumption.

We conclude that for every \( \ell \in \mathbb{N} \), we can extend \( \iota \) to all vertices \( v_J \) with \( \ell \) satisfying the properties stated in the induction assumption. In particular, \( \iota(hv_J) = h\iota(v_J) \) and for every simplex \( \sigma(v_{J_1}, \ldots, v_{J_{k+1}}) \) in \( \tilde{S} \) with index sets \( J_1 \subseteq \cdots \subseteq J_{k+1} \), we have

\[
\text{diam}(\iota(\sigma(v_{J_1}, \ldots, v_{J_{k+1}}))) \leq \lambda \text{diam}\left(\iota(\partial \sigma_{J_{k+1}}^{(0)})\right).
\]

This proves the Claim. \( \Box \)

Returning to the proof of the Lemma, we have a map \( \iota : S^{(0)} \to \Sigma_R \) such that \( \iota(hB_i) = h\iota(B_i) \) for all \( h \in H \), \( B_i \in \mathcal{U} \) and whenever \( B_i \in \mathcal{U} \) such that \( B_i \cap \Sigma \neq \emptyset \), then \( \iota(B_i) = \Phi_{R_i}^{-1}(c_i) \) for some \( c_i \in B_i \cap \Sigma \cap \overline{D} \). Similar, but not identical, to the notation of previous proofs, we define \( \mathcal{U}_\Sigma \) to be the set of all \( B \in \mathcal{U} \) such that \( B \cap \Sigma \neq \emptyset \). We define \( S_\Sigma \) to be the subcomplex of \( S \) spanned by \( \mathcal{U}_\Sigma \). (Previously, \( S_\Sigma \) denoted the ‘\( \Sigma \)-component’ of \( S_1 \). Because
we are working with the complex $S$ now, which consists of translates of $S_1$ by $H$. $S_\Sigma$ now denotes the ‘$\Sigma$-component’ of $S$. As we have shown earlier in the proof, property c) of Lemma 3.3 implies that for every simplex $\sigma$ in $S_1$, $\operatorname{diam}(\iota(\sigma(0))) \leq \Delta$. Furthermore, property (2) of Lemma 3.3 shows that, if $\sigma$ is a simplex in $S_\Sigma$, then $\operatorname{diam}(\iota(\sigma(0))) \leq \delta''$.

The claim proven above now implies that we can extend $\iota$ to a map $\iota : (1)S_1^0 \to \Sigma_R$ such that $\iota(hB_J) = h\iota(B_J)$ and for every simplex $\sigma(B_{J_1}, \ldots, B_{J_{k+1}})$ in $(1)S$, we have

$$\operatorname{diam}(\iota(B_{J_1}), \ldots, \iota(B_{J_{k+1}})) \leq \lambda \operatorname{diam}(\sigma_{J_{k+1}}) \leq \lambda \Delta.$$  

By induction, we see that repeated application of the Claim extends $\iota$ to an $H$-equivariant map $\iota : (n)S_1 \to \Sigma_R$ such that for every simplex $\sigma$ in $(n)S_1$, we have

$$\operatorname{diam}(\iota(\sigma(0))) \leq \lambda^n \Delta.$$  

Furthermore, if $\sigma$ is a simplex in $(n)S_\Sigma$, we have

$$\operatorname{diam}(\iota(\sigma(0))) \leq \lambda^n \min(\delta'', \Delta),$$  

because we have an additional bound on $\operatorname{diam}(\iota(\sigma(0)))$ if $\sigma$ is a simplex in $S_\Sigma$.

Choose $n$ sufficiently large such that $\lambda^n \Delta \leq \frac{R-\epsilon}{4}$ and recall that we made sure that $\delta'' < \delta' \leq \frac{R-\epsilon}{4}$ at the very beginning of the proof. We now show that for this $n$, $C_{(n)S_1}(\iota) \cap C_\epsilon = \emptyset$ (recall Definition 3.4). For every simplex $\sigma = (v_1, \ldots, v_{k+1})$ in $(n)S_1$, we know that $\operatorname{diam}(\iota(\sigma(0))) \leq \lambda^n \Delta \leq \frac{R-\epsilon}{4}$. Therefore, the convex hull $C(\iota(v_1), \ldots, \iota(v_{k+1}))$ is contained in a closed ball of radius $\frac{R-\epsilon}{4}$ centered at a point in $\Sigma_R \cap C(\iota(v_1), \ldots, \iota(v_{k+1}))$, say $\iota(v_1)$. Since $d(\Sigma_R, \Sigma_\epsilon) = R-\epsilon$, this closed ball cannot intersect $C_\epsilon$. Since $C_{(n)S_1}(\iota)$ is a finite union of convex hulls of this form, we see that it cannot intersect $C_\epsilon$.

We are left to prove the extra property of $\iota$ for simplices in $S_\Sigma$. Let $B_I$ be a vertex in $(n)S_\Sigma$. This vertex is a (possibly higher-order) barycenter
of a simplex $\sigma(B_1, \ldots, B_{m+1})$ in $S_{m}$. It is a standard fact on barycentric subdivisions that there exists $j \in \{1, \ldots, m+1\}$ and a sequence of vertices $B_j = v_0, v_1, \ldots, v_M = B_I$ such that $v_i$ is a vertex in $S_{m}$ and there exists an edge from $v_{i-1}$ to $v_i$ in $S_{m}$. (To find such a sequence, start at $v_M := B_I$ and find your way to some $B_j$.) By construction of $i$ and the choice of the sequence, $i(v_j)$ is a $\lambda$-barycenter of a set that contains the point $i(v_{i-1})$ and has diameter at most $\lambda^{i-1} \text{diam}(i(B_1), \ldots, i(B_{m+1})) \leq \lambda^{i-1} \delta''$. Thus, $d(i(v_{i-1}), i(v_i)) \leq \lambda^i \delta''$ and we estimate

$$d(i(B_1), i(B_j)) \leq \sum_{i=1}^{M} d(i(v_{i-1}), i(v_i))$$

$$\leq \sum_{i=1}^{M} \lambda^i \text{diam}(i(B_1), \ldots, i(B_{m+1}))$$

$$\leq \frac{1}{1-\lambda} \delta'' \leq \delta'.$$

Using the fact that $\text{diam}(i(B_1), \ldots, i(B_{m+1})) \leq \delta'' < \delta'$ again, we see that for every $i \in \{1, \ldots, m+1\}$, we have $d(i(B_1), i(B_i)) \leq 2\delta'$. This is the additional property of $i$ that we wanted to prove, which concludes the proof of the Lemma.

We are now able to prove a first version of our main result. Recall that $\Sigma_{\epsilon}$, which is a connected component of the topological boundary $\partial_{X} \subset X$ has a corresponding connected component in $\partial_{X} \setminus \Lambda(H)$, which is obtained via the homeomorphism discussed in section 2.1.

**Theorem 3.13.** Let $M$ be a closed, negatively curved manifold, $X$ its universal covering, $G := \pi_1(M)$ and $H < G$ a quasi-convex subgroup such that its limit set $\Lambda(H)$ cuts $\partial_{X} \setminus \Lambda(H)$ into several connected components.

Let $Z$ be a connected component of $\partial_{X} \setminus \Lambda(H)$ and let $H_0 < H$ be the subgroup that preserves $Z$. Let $C \subset X$ be the convex hull of $\Lambda(H_0)$, $\epsilon > 0$ and $C_{\epsilon}$ the $\epsilon$-neighbourhood of $C$. Let $\Sigma_{\epsilon}$ be the connected component of $\partial_{C_{\epsilon}}$ that corresponds to $Z$.

Suppose there exist $R > \epsilon$, $\Delta > 0$, $\frac{1}{2} \leq \lambda < 1$, and a fundamental domain $D$ of the action of $H_0$ on $C_{\epsilon}$ with compact closure such that $\text{diam}(\Phi_{R^{-1}}^N(A(D) \cap \Sigma_{\epsilon}) \leq \Delta$ and $\Sigma_{R}$ has strong $\lambda$-barycenters up to diameter $\Delta$. Then, $\Sigma_{\epsilon}$ is contractible. In particular, $Z$ is contractible as well.

**Proof.** Applying Corollary 3.11 to the fundamental domain $D$ and the constants $R, \epsilon, \Delta, \lambda$ from the Theorem, we find $0 < \alpha_0 < \frac{\epsilon}{2}$ and a continuous, $H$-invariant vector field $V$ on $C_{\epsilon}$ such that for all $q \in \Sigma_{\epsilon}$, $Z(V(q), N(q)) \leq \alpha_0$. Lemma 3.6 now implies that $\Sigma_{\epsilon}$ is contractible. Indeed, since $\Sigma_{R}$ is homeomorphic to $\Sigma_{\epsilon}$ for all $R' > 0$, we obtain that they are all contractible, including the connected component in $\partial_{X} \setminus \Lambda(H)$ corresponding to $\Sigma_{\epsilon}$. □
We find ourselves asking when $\Sigma_R$ admits strong $\lambda$-barycenters up to a sufficiently large diameter. Based on the examples presented earlier, we raise the following

**Question.** Suppose $\Sigma_R$ is $C^2$-differentiable and its second fundamental form is bounded by some constant $k$. Does there exist $\Delta$, depending only on $k$ and the dimension of $X$, such that $\Sigma_R$ has strong $\lambda$-barycenters up to diameter $\Delta$?

3.3. **Subdivisions with shrinking diameter.** Heuristically, having strong barycenters up to a sufficiently large diameter is a strong property. It turns out that we can formulate a weaker property that will still imply contractibility of $\Sigma$.

It is based on the following definitions.

Given a simplicial complex $S$ and a metric space $(Z, d)$, we will work with a map $\iota : S^{(0)} \to Z$. We will restrict this map to subsets and extend it to vertices of a subdivision of $S$. To avoid cluttered notation, we will note every restriction and extension of $\iota$ by $\iota$ as well.

**Definition 3.14.** Let $(Z, d)$ be a metric space, $S$ a finite simplicial complex, $\iota : S^{(0)} \to Z$ a map, and $\lambda < 1$. Let $S' \subset S$ be a subcomplex that contains all vertices of $S$. Let $S'_{sub}$ be a finite subdivision of $S'$ and suppose $\iota$ extends to the vertices of $S'_{sub}$. We call the pair $(S'_{sub}, \iota)$ a $\lambda$-shrinking subdivision of $S'$ in $(S, \iota)$ if the following two properties hold:

1. If $\tilde{\sigma}$ is a simplex in $S'_{sub}$ and $\sigma$ the unique least-dimensional simplex in $S'$ containing $\tilde{\sigma}$, then $\text{diam}(\iota(\tilde{\sigma})) \leq \lambda \text{diam}(\iota(\sigma))$.

2. For every simplex $\sigma$ in $S$, we have that $\text{diam}(\iota(\sigma)) \leq \text{diam}(\iota(\sigma_{sub}))$, where $\sigma_{sub} = \sigma \cup (\sigma \cap S^{(0)}_{sub})$ denotes the set of all vertices of $\sigma$ in $S$ together with all vertices in $\sigma$ that appear in the subdivision $S'_{sub}$.

Note that we require this property for every simplex in $S$, not just in $S'$.

**Definition 3.15.** Let $(Z, d)$ be a metric space, $\lambda < 1$ and $\Delta > 0$. We say that $Z$ admits $\lambda$-shrinking subdivisions up to diameter $\Delta$, if for every finite simplicial complex $S$, every map $\iota : S^{(0)} \to Z$ such that for all simplices $\sigma$ in $S$, $\text{diam}(\iota(\sigma)) \leq \Delta$, for every subcomplex $S'$ of $S$ containing $S^{(0)}$, for every (finite) $\lambda$-shrinking subdivision $(S'_{sub}, \iota)$, and every simplex $\sigma$ in $S$ not contained in $S'$, there exists a finite subdivision $\tilde{S}_{sub}$ of $S'$ and an extension of $\iota$ to the vertices of $\tilde{S}_{sub}$ such that $(\tilde{S}_{sub}, \iota)$ is an extension of $(S'_{sub}, \iota)$ and it is also $\lambda$-shrinking.

**Lemma 3.16.** Let $\epsilon > 0$, $D$ a fundamental domain of the action of $H$ on $C_\epsilon$ with compact closure. For $R > \epsilon$, denote by $D_R := \Phi^N_{R-\epsilon}(D \cap \Sigma_\epsilon)$ the orthogonal ‘push-out’ of $D$ to $\Sigma_R$. Let $\alpha_0 \in (0, \frac{\pi}{2})$ and $K = \Phi^N_{R-\epsilon}(A(D) \cap \Sigma_\epsilon)$, which is compact. Let $\delta' > 0$ and $0 < \delta < \frac{\pi}{2}$ be the constants obtained in Lemma 3.3 with respect to $R, \epsilon, \alpha_0, K,$ and $D$ above.

Suppose there exist $R > \epsilon$, $\Delta > 0$, and $\lambda < 1$ such that $\text{diam}(K) \leq \Delta$ and $\Sigma_R$ admits $\lambda$-shrinking subdivisions up to diameter $\Delta$ (with respect to
the metric on $X$). Let $\delta'' = (1 - \lambda)\delta'$ and shrink $\delta$ if necessary such that the pair $(\delta'', \delta)$ satisfies the properties of Lemma [3.22]

Then there exist a cover $U$ of $\mathcal{D}$ satisfying the properties of [3.3] a subdivision $S_{\text{sub}}$ of $S_1$ and a map $\iota : S_{\text{sub}} \to A(D_R)$ such that the subdivision is compatible with the action by $H$ (i.e. for every simplex $\sigma$ in $S_1$ and $h \in H$ such that $h\sigma$ is a simplex in $S_1$, the subdivision of $h\sigma$ in $S_{\text{sub}}$ is the $h$-translate of the subdivision of $\sigma$) and $\iota(hB) = h\iota(B)$ for all vertices in the subdivision of $S_{\text{sub}}$ such that $hB \in S_{\text{sub}}$. Furthermore,

$$C_{S_{\text{sub}}} (i) \cap C_\epsilon = \emptyset.$$ 

In addition, the map $\iota$ satisfies that for all vertices $B_i \in U$, such that $B_i \cap \Sigma_\epsilon \neq \emptyset$, there exists $c_i \in B_i \cap \Sigma_\epsilon \cap \mathcal{D}$ such that $\iota(B_i) = \Phi^N_{R-\epsilon}(c_i)$ and for all simplices $\sigma'$ in $S_{\text{sub}}$ that are contained in a simplex $\sigma$ in $S_\Sigma$, there exists a vertex $B_i \in S_\Sigma$ such that $\iota(\sigma(0)) \subset B_{2\delta'}(\iota(B_i))$.

**Proof.** The proof is the same as the proof of Lemma [3.10] in that we successively subdivide simplices of $S$ and extend $\iota$, both in an $H$-equivariant way, until we have obtained a subdivision of $S$ such for all simplices $\sigma'$ in the subdivision contained in a simplex $\sigma$ of $S$, we have $\text{diam}(\iota(\sigma(0))) \leq \lambda \text{diam}(\iota(\sigma(0)))$. Iterating this procedure finitely many times, we can make it so that for every simplex $\sigma'$ in the repeated subdivision, $\text{diam}(\iota(\sigma'')) \leq \frac{R-\epsilon}{4}$. At that point $C_{S_{\text{sub}}}(i) \cap C_\epsilon = \emptyset$.

The estimate $\text{diam}(\iota(\sigma')) \leq 2\delta'$ is obtained as follows. Given a vertex $B_i$ in $S_{\text{sub}}$ that is contained in a simplex of $S_\Sigma$, we find a vertex $B_j$ in $S_\Sigma$ and a sequence of vertices $B_j = v_0, \ldots, v_M = B_i$ such that $v_i$ is a vertex in the $i$-th subdivision of the construction above and $v_{i-1}$ is a vertex in the least-dimensional simplex of the $(i-1)$-th subdivision that contains the vertex $v_i$. (As before, this sequence is constructed from $v_M$ backwards.) Using property (2) of Definition [3.14] we estimate $d(\iota(v_{i-1}), \iota(v_i)) \leq \text{diam}(\iota(\sigma_{i-1}))$, where $\sigma_{i-1}$ is the least-dimensional simplex in the $(i-1)$-th subdivision that contains $v_i$. Using property (1) of Definition [3.14] repeatedly, we see that $\text{diam}(\iota(\sigma_{i-1})) \leq \lambda^{i-1}\delta''$. Adding up all these distances, we obtain $d(\iota(B_j), \iota(B_j)) \leq \frac{1}{1-\lambda^j}\delta'' \leq \delta'$ for some $j$. This completes the proof. \qed

Using the same proof as in the previous section, we conclude

**Corollary 3.17.** Suppose there exist $R > \epsilon > 0$, $\Delta > 0$, $\lambda < 1$ and a fundamental domain $D$ of the action of $H$ on $C_\epsilon$ with compact closure such that, denoting $K := \Phi^N_{R-\epsilon}(A(D) \cap \Sigma_\epsilon)$, we have $\text{diam}(K) \leq \Delta$ and $\Sigma_R$ admits $\lambda$-shrinking subdivisions up to diameter $\Delta$. Then, $\Sigma_\epsilon$ is contractible. In particular, the connected component in $\partial_\infty X \setminus \Lambda(H)$ corresponding to $\Sigma_\epsilon$ is contractible as well.

Having $\lambda$-shrinking subdivisions is a significantly weaker and more flexible property than the existence of $\lambda$-barycenters. The down side is that it is more of a deus-ex-machina-property for which it is unclear when it actually holds.
It is based on the observation that the construction of the map $j$ requires the ability to subdivide ‘metric’ simplicial complexes (i.e. pairs $(S, i)$) in an $H$-equivariant way while uniformly shrinking their diameter. This suggests that one could interpret the admittance of $\lambda$-shrinking subdivisions in a metric space as the property that all ‘metric’ simplicial complexes $(S, i)$ in the space, whose simplices do not exceed a certain diameter, can be expressed as a sum of simplices with arbitrarily small diameter. We therefore raise the following

**Question.** Can admittance of $\lambda$-shrinking subdivisions (up to some diameter) be expressed in terms of a suitable homology-theory for metric spaces?

4. **Obtaining contractibility from strong barycenters or shrinking subdivisions on the boundary**

Let $Z$ be a connected component of $\partial X \setminus \Lambda(H)$ and let $\Sigma_r$ be the connected component of $\partial C_r$ that corresponds to $Z$ (recall section 2.1). We have proven that the existence of strong $\lambda$-barycenters or $\lambda$-shrinking subdivisions on $\Sigma_R$ up to a suitably large diameter implies contractibility of $\Sigma_r$ and thus contractibility of $Z$. We now show how the same proofs with minor modification hold if we replace $\Sigma_R$ by $Z$. Indeed, interpreting $Z$ as $\Sigma_\infty$, we can follow the arguments presented in section 3 having to pay attention to a few points stated below.

One quickly notices that we have to work with distances on the visual boundary now. For this, we fix a base point $o \in X$ at the start of the proof and all distances in $\partial \infty X$ are with respect to the visual metric $\rho_o$. The choice of base point only matters with regards to the existence of strong $\lambda$-barycenters.

**Remark 4.1.** Lemma 3.3 still holds if we replace $\Sigma_R$ by $\Sigma_\infty := Z$. The key reason for this is the fact that the visual boundary $\partial \infty X$ with the visual topology, which is determined by any visual metric, is homeomorphic to the unit tangent sphere $T_q^1 X$ at any point in $X$ with its standard topology, which can be determined entirely in terms of angles. In order to obtain inequality $\star$, we then have to use the fact that the gradient vector fields of Busemann functions on Hadamard manifolds are continuous, instead of continuity of radial vector fields (see for example [HIH77]).

**Remark 4.2.** The definition of $C_S(\iota)$ in Definition 3.4 requires some minor modification. We have to replace the convex hull $C(\iota(B_1), \ldots, \iota(B_{k+1}))$ in $X$ by a convex hull in $X \cup \partial \infty X$. Namely, we have to consider the set $C(\iota(B_1), \ldots, \iota(B_{k+1})) \cup \partial \infty C(\iota(B_1), \ldots, \iota(B_{k+1}))$. By [And83], we know that this only adds back the points $\iota(B_i)$ that were in $Z$. Note that the convex hull, together with its boundary at infinity, is a subset of the compactification $X \cup \partial \infty X$ equipped with the visual topology.

Since $C_{S_1}(\iota)$ now contains elements in the visual boundary, we a-priori have to modify the assumption in Lemma 3.3 by

$$C_{S_1}(\iota) \cap (C_\epsilon \cup \Lambda(H)) = \emptyset.$$
Since \( t \) maps points into \( Z \), which does not intersect \( \Lambda(H) \), this modified assumption can be reduced back to the original assumption \( C_{S_1}(t) \cap C_\epsilon = \emptyset \).

**Remark 4.3.** In the proof of Lemma \( \text{3.5} \), the extension of \( j \) to \( S_1 \) requires a comment. When extending \( j \) to \( S_1 \), we used the fact that \( C(t(B_1), \ldots, t(B_{t+2})) \) is convex and thus contractible. Now, we have to work with \( C(t(B_1), \ldots, t(B_{t+2})) \cup \partial C(t(B_1), \ldots, t(B_{t+2})) = C(t(B_1), \ldots, t(B_{t+2})) \cup \{ t(B_1), \ldots, t(B_{t+2}) \} \). Adding these extra points, we are still left with a connected and contractible set in \( X \cup \partial X \). Therefore, we can still define \( j \) by using the same argument and Lemma \( \text{3.5} \) still holds if \( t : S_1^{(0)} \rightarrow U \).

Finally, we need to replace the penultimate step in the proof of Lemma \( \text{3.10} \) where we showed that after a sufficient amount of subdivisions, \( C_{\gamma}(t) \cap C_\epsilon = \emptyset \). In Lemma \( \text{3.10} \), we used the fact that, if the diameter of the vertices of every simplex is sufficiently small (smaller than \( \frac{\delta}{4} \)), we can make sure that \( C_{\gamma}(t) \cap C_\epsilon = \emptyset \). This argument no longer works in a straightforward manner if the target of \( t \) is \( Z \), as the relationship between the metric \( \rho_o \) on \( Z \) and the metric on \( X \) is not as straightforward. We need to show that, if a finite set of points has sufficiently small diameter with respect to \( \rho_o \), then its convex hull does not intersect \( C_\epsilon \). This follows from the following result, which is probably known, but hard to find a reference for.

**Lemma 4.4.** Let \( o \in X, C \subset X \) a closed and convex subset, \( K \subset \partial X \setminus \partial X \) compact. Then there exists \( \delta' > 0 \) such that for all sets \( P \subset K \) with diameter at most \( \delta' \), \( C(P) \cap C = \emptyset \).

**Proof.** The key tool of the proof is Lemma 2.6 in [Bow94], which states that the convex hull of a closed set \( P \subset X \cup \partial X \) is contained in the \( r \)-neighbourhood of the set of all geodesics between points in \( P \), where \( r \) depends only on the lower curvature bound of \( X \). Let \( r \) denote the constant \( \sigma \) in Lemma 2.6 and let \( \delta > 0 \) such that \( X \) is \( \delta \)-hyperbolic. Since \( K \) is compact, there exists a constant \( T > 0 \), such that for all geodesic rays \( \gamma \) that start at \( o \) and represent points in \( K \), we have that \( \gamma|_{[T, \infty)} \cap N_{r+3\delta}(C) = \emptyset \). Choose \( \delta' \) sufficiently small, such that whenever \( \rho_o(\xi, \tilde{\xi}) \leq \delta' \) and \( \gamma, \tilde{\gamma} \) are the unique geodesic rays starting at \( o \) representing \( \xi \) and \( \tilde{\xi} \) respectively, then \( d(\gamma(T), \tilde{\gamma}(T)) < \delta \).

Let \( P \subset K \) be a set of diameter at most \( \delta' \) with respect to \( \rho_o \). Replace \( P \) by its closure, which still satisfies the same bound on its diameter. Define

\[
\Gamma_{o,T} := \bigcup_{\gamma : \gamma|_{[T, \infty)} \cap C} \gamma|_{[T, \infty)},
\]

i.e. \( \Gamma_{o,T} \) considers all geodesic rays starting at \( o \) (at time zero) that represent points in \( P \) and restricts them to the interval \([T, \infty)\). Fix some \( \xi \in P \) and let \( \gamma \) be its geodesic representative starting at \( o \). We obtain

\[
\Gamma_{\gamma(T),0} := \bigcup_{\gamma' : \gamma'|_{[T, \infty)} \cap C} \gamma',
\]
which consists of all geodesic rays starting at \( \gamma(T) \) representing points in \( P \). Since \( X \) has non-positive curvature, \( \Gamma_\gamma(T),0 \) is contained in the \( \delta \)-neighbourhood of \( \Gamma_0,T \). By \( \delta \)-hyperbolicity, we see that the union of all bi-infinite geodesics between points in \( P \) is contained in the \( \delta \)-neighbourhood of \( \Gamma_0,T \). By Bowditch, the convex hull \( C(P) \) of \( P \) is contained in the \( r \)-neighbourhood of the union of these bi-infinite geodesics. We conclude that \( C(P) \) is contained in the \( r + 2\delta \)-neighbourhood of \( \Gamma_0,T \). Due to our choice of \( T \), this neighbourhood does not intersect \( C \). We conclude that \( C(P) \cap C = \emptyset \), which proves the Lemma.

Lemma 4.4 tells us that we can use the existence of strong \( \lambda \)-barycenters up to a sufficiently large diameter on the component \( U \) in the boundary to extend our map \( \iota \) to a sufficiently fine grid so that \( C_{(m)}(\iota) \cap C_\varepsilon = \emptyset \). The last step of the proof of Lemma 3.10 works the same way on \( Z \) as it did on \( \Sigma_R \). The arguments above also generalise Lemma 3.16.

To conclude our main result, we need one last piece of notation. Let \( Z \) be a connected component of \( \partial_\infty X \setminus \Lambda(H) \) and \( D \) a fundamental domain of \( C = C(\Lambda(H)) \). We define \( \pi_C : X \to C \) to be the closest point projection. Note that for every \( R > 0 \), if \( \Sigma_R \) corresponds to \( Z \), and \( p \in \Sigma_R \), the fibre of \( \pi_C \) that contains \( p \) also contains a geodesic ray that starts in \( C \), leaves \( C \) immediately and intersects every \( \Sigma_R \) orthogonally. In other words, the geodesic flow \( \Phi^N \) induced by the normal vector field \( N \) yields fibre-preserving homeomorphisms between \( \Sigma_R, \Sigma_R \) for all \( R \). We extend \( \pi_C \) to \( \partial_\infty X \setminus \Lambda(H) \) by defining \( \pi_C(\xi) = \pi_C(p) \), where \( p \in \Sigma_R \) for some \( R \) and the geodesic ray induced by \( \Phi^N(\xi) \) represents \( \xi \). Since \( \Phi^N \) preserves the fibres of \( \pi_C \), this does not depend on the choice of \( R \) and we obtain a continuous extension of \( \pi_C \) to \( \partial_\infty X \setminus \Lambda(H) \). We denote

\[
A(D)_\infty := \{ \xi \in Z | \pi_C(\xi) \in A(D) \}.
\]

**Theorem 4.5.** Let \( M \) be a closed, negatively curved manifold, \( X \) its universal covering, \( G := \pi_1(M) \), and \( H < G \) a quasi-convex subgroup such that its limit set \( \Lambda(H) \) cuts \( \partial_\infty X \) into several connected components.

Let \( Z \) be a connected component of \( \partial_\infty X \setminus \Lambda(H) \) and \( H_0 < H \) the subgroup that preserves \( Z \). If there exist \( o \in X \), \( \Delta > 0 \), \( \frac{1}{2} \leq \lambda < 1 \), and a fundamental domain \( D \) of the action of \( H_0 \) on \( C(\Lambda(H_0)) \) with compact closure such that \( Z \) has strong \( \lambda \)-barycenters up to diameter \( \Delta \) with respect to \( \rho_o \) and \( \text{diam}(A(D)_\infty) \leq \Delta \), then \( Z \) is contractible.

Analogously, we obtain a result if \( Z \) admits \( \lambda \)-shrinking subdivisions up to a sufficiently large diameter.

**Theorem 4.6.** Let \( M \) be a closed, negatively curved manifold, \( X \) its universal covering, \( G := \pi_1(M) \), and \( H < G \) a quasi-convex subgroup such that its limit set \( \Lambda(H) \) cuts \( \partial_\infty X \) into several connected components.

Let \( Z \) be a connected component of \( \partial_\infty X \setminus \Lambda(H) \) and \( H_0 < H \) the subgroup that preserves \( Z \). If there exist \( o \in X \), \( \Delta > 0 \), \( \frac{1}{2} \leq \lambda < 1 \), and a fundamental
domain $D$ of the action of $H_0$ on $C(\Lambda(H_0))$ with compact closure such that $Z$ admits $\lambda$-shrinking subdivisions up to diameter $\Delta$ with respect to $\rho_0$ and $\text{diam}(A(D)_{\infty}) \leq \Delta$, then $Z$ is contractible.

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