Formulation of Supersymmetry on a Lattice as a Representation of a Deformed Superalgebra

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Abstract

The lattice superalgebra of the link approach is shown to satisfy a Hopf algebraic supersymmetry where the difference operator is introduced as a momentum operator. The breakdown of the Leibniz rule for the lattice difference operator is accommodated as a coproduct operation of (quasi)triangular Hopf algebra and the associated field theory is consistently defined as a braided quantum field theory. Algebraic formulation of path integral is perturbatively defined and Ward-Takahashi identity can be derived on the lattice. The claimed inconsistency of the link approach leading to the ordering ambiguity for a product of fields is solved by introducing an almost trivial braiding structure corresponding to the triangular structure of the Hopf algebraic superalgebra. This could be seen as a generalization of spin and statistics relation on the lattice. From the consistency of this braiding structure of fields a grading nature for the momentum operator is required.

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1 Introduction

We consider that a constructive definition of regularized supersymmetric field theory is getting increasingly important. There are several reasons: Phenomenologically there is an expectation that superparticles might be discovered by LHC experiment in the near future. If the supersymmetry becomes reality, obviously we need to formulate supersymmetric field theory constructively. The formulation should provide a basis for the numerical study of nonperturbative supersymmetry phenomenology. It is natural to expect that a lattice formulation of supersymmetry may play a crucial rôle just like the lattice QCD is playing an important rôle as the only numerical mean for the strong interaction phenomenology. We expect that a fermionic counterpart of QCD region may exist in the new energy scale.

Secondly, it is not obvious that the lattice fermion problems \[1, 2\] are well understood from the lattice regularization point of view. It is, however, a general consensus that the chiral fermion problem is solved for lattice QCD \[3, 4, 5\]. One may say that species doublers of chiral fermion on a lattice are lattice artifacts so that it would have been better if they were not there. It was, however, claimed that these extra species doubler degrees of freedom is exactly the needed one and corresponds to the extended twisted supersymmetry degrees of freedom \[6, 7, 8, 9\]. It was shown that the twisted supersymmetry can be derived by the Dirac-Kähler twisting procedure \[10\] in any dimensions: \( \mathcal{N} = 2 \) in two dimensions, \( \mathcal{N} = 4 \) in three dimensions, and \( \mathcal{N} = 4 \) in four dimensions which coincides with the twisting derived by Marcus \[11\]. In these formulations the fermionic internal degrees of freedom can be defined semilocally on a lattice to be compatible with differential form nature of Dirac-Kähler fermion \[6, 7, 8, 9, 12, 13\]. This type of the correspondence has been anticipated by the old works \[14\]. It turns out that the lattice Dirac-Kähler fermion formulation \[15\] was, however, proved to be equivalent to the staggered fermion formulation \[16\] with an introduction of mild noncommutativity between differential forms and fields to accommodate the modified Leibniz rule for lattice difference operator. These results suggest that the regularization of fermions on a lattice naturally leads to a necessity of supersymmetry in a fundamental way.

In the path integral formulation of field theory, fermionic fields are treated as Grassmann odd variables and thus have an anti-commuting nature in compatible with spin and statistics theorem which requires Lorentz invariance exactly \[17\]. Since the Lorentz invariance is broken on the lattice it is not obvious that this anti-commuting nature of fermions at the lattice constant level is the mandatory requirement. In this paper we explore a possibility that commuting and anti-commuting nature of fields are modified with an introduction of mild noncommutativity in compatible with the lattice Leibniz rule of difference operator. This may be identified as a generalization of spin and statistics on a lattice.

Trials for the formulation of supersymmetry on a lattice have a long history. Since lattice does not have infinitesimal translatational invariance, there are various difficulties to formulate supersymmetry algebra which includes an infinitesimal translation generator. To overcome the difficulties, various approaches and formulations have been proposed so far. If we focus on the treatment of the algebraic aspects of lattice supersymmetry, there are essentially three possible approaches:

1. keeps the continuum superalgebra approximately with lattice corrections;

2. keeps exactly only a subalgebra of the continuum superalgebra which doesn’t contain the momentum operators;

3. deforms the continuum superalgebra into a lattice version of superalgebra, and keeps exactly the full sector of this lattice superalgebra.

There is a long list of many trials of the first approach summarized in \[18, 19, 20\] and the previous
references are therein, where superalgebra is kept up to the lattice corrections. There are some
later developments [21, 22, 23, 24]. In this approach one has to see how the superalgebra is
restored in the continuum limit. In order to see the recovery of supersymmetry for the whole
range of coupling constant, it is inevitable to find reliable methods for numerical analyses.
Influenced by the developments of renormalization group analyses and Ginsparg-Wilson relation
of chiral symmetry analyses for lattice QCD, there have been recent systematic applications of
the methods to supersymmetric models [25, 26, 27].

In the second approach [28, 29, 30, 31, 32, 33, 34, 35, 36], only one or two nilpotent su-
percharges are preserved exactly. It is again particularly important to examine how the full
continuum supersymmetry is recovered in the continuum limit. In this approach the importance
of accidental symmetry is stressed so that the recovery of supersymmetry in the infrared region
is expected due to the suppression of relevant operators by the partial exactness of the total
supersymmetry algebra [37, 38]. In fact in some specific cases it can be shown that only the
part of supersymmetry which is realized on the lattice is enough to suppress non-manageable
fine-tuning in the continuum limit [28, 30, 31, 32, 39, 38]. One can then use such models as
constructive definition of corresponding continuum supersymmetric models. We should, how-
ever, note that in extracting the sector of superalgebra which can be preserved on the lattice,
extended supersymmetry and its twisting procedure [39, 10] play an important rôle.

In the third approach [6, 7, 8, 12, 13, 34, 40], one defines lattice version of superalgebra
where the momentum operators in the continuum superalgebra are replaced by finite difference
operators on the lattice. This seems the most natural and naïve deformation of continuum
algebra. Nevertheless, it is not straightforward due to the following obvious reason. Since a
finite difference operator, not being an infinitesimal operator, is not rigorously an element of
algebra, that deformed “superalgebra” is not strictly an algebra in the usual sense. This is
actually not simply a terminological issue, but is crucial in the formulation. Namely, finite
difference operator does not obey the Leibniz rule, which is nothing but equivalent to say that
the operator is not an element of algebra. On the other hand, because of the nilpotency of
Grassmann parameters, “normal” supercharge would always obey exactly the Leibniz rule. This
mismatch of natures of finite difference operators and supercharges makes the naïve realization
of lattice deformed superalgebra in the above sense difficult. What we are going to follow in this
paper is in fact the formulation struggling to give an answer to this difficult situation, which
was originally proposed in [6, 7, 8]. We call this formulation as link approach as a whole or
DKKN formalism when we have more stress on the algebraic aspect of the formulation. In this
approach, we introduce the notion of modified Leibniz rule to overcome this difficulty.

Despite of the invention of the modified Leibniz rule, the link approach still faces its incom-
pleteness. Firstly, it is rather unclear whether this additionally introduced modified Leibniz rule
is totally consistent and acceptable to express the “symmetry” of a quantum field theory, because
it is in any case different from the standard Lie algebraic symmetry. An answer to this issue
we will present that this modification is indeed consistently introduced, utilizing the fact that
the deformed algebra forms a Hopf algebra which generalizes the Lie algebraic symmetry, and
that a quantum field theory which has a Hopf algebraic symmetry can be constructed at least
perturbatively thanks to the previously formulated framework known as braided quantum field
theory (BQFT) [11]. For this argument, it is important to identify the DKKN superalgebra as a
rigorous Hopf algebra which obeys a set of axioms to prescribe the Hopf algebra. It is also crucial
to correctly determine the braiding structure on the representation space of the Hopf algebra.
With the use of the BQFT formulation, we can derive a series of Ward–Takahashi identities
Corresponding to the Hopf algebraic symmetry, which would give a clear physical interpretation
of the deformed symmetry.

The second and most crucial aspect to be clarified in the link approach is on the “inconsis-
tency” raised in [12], which claims that the modified Leibniz rule inevitably leads to a problem
due to the ordering ambiguity caused essentially by the asymmetric nature of the deformation. In order to clarify the ordering problem, a matrix formulation for one dimensional model was explicitly analysed. It was shown that there is no ambiguity at the superfield level but the problem remains at the component level \[40\]. Fortunately, our Hopf algebraic description also resolves this problem: given the Hopf algebraic symmetry together with the appropriate treatment of the braiding structure of BQFT, we will show that this difficulty no longer exists. The braiding structure, which could be interpreted as a kind of generalized statistics or a “mild noncommutativity”, is again the key ingredient for this argument. The treatment for the gauge theory is outside of the scope of this paper.

In the recent investigations it is stressed that the second approach formulated by orbifold construction and the third approach of the link construction are equivalent \[34, 35\]. It was already noticed that a particular choice of a constant vector parameter makes the scalar supercharge shiftless and \( N = D = 2 \) super Yang-Mills action of the link construction coincides with that of orbifold construction \[7\]. It was, however, stressed that supercharges carrying shifts are not supersymmetry invariant \[34, 35\]. Therefore in the symmetric choice of the parameters in the link construction, all the supercharges carry shifts and thus no supersymmetry exists, although the corresponding action has larger discrete chiral and spacetime symmetries than that of orbifold construction \[35\]. This criticism is due to the non-standard definition of the shifted (anti-)commutators. In this paper we intend to stress exactly on this point that all supercharges of the link construction preserves deformed lattice supersymmetry exactly, where shifting nature plays a crucial rôle.

Recently it was shown as a no-go theorem that a proper definition of product of lattice fields naturally leads to a breakdown of Leibniz rule of lattice differential operator under the conditions of translational invariance and locality on the lattice \[43\]. It was also shown that an extension of blocked symmetry transformation realizing Ginsparg-Wilson relation to supersymmetric case leads to only consistent solution of SLAC-type derivative \[44\] which is in fact consistent with the above no-go theorem \[27\]. It is known, however, that SLAC-derivative is a highly non-local differential operator. It should be noted that the modified nature of the lattice Leibniz rule or equivalently the deformation of supersymmetry algebra is compatible with the results of those analyses.

This paper is organized as follows. In Section 2 we will make a brief review of the fundamental structures of the link formulation in somewhat generalized form and its above mentioned difficulties. In Section 3 we will give a description how to treat the superalgebra on the lattice as a deformed/modified algebra in the scheme of the Hopf algebra theory. We will list all the necessary and sufficient formulae which form the whole structures of a Hopf algebra. We also derive the explicit form of the braiding which is necessary for the total consistency of the representation. Twisting of our Hopf algebra will be discussed, too, which naturally explains our lattice theory should have the braiding or equivalent noncommutativity. We will illustrate how a quantum field theory with this Hopf algebraic symmetry can be perturbatively defined entirely based on the general formulation of BQFT. As a concrete example we show a two dimensional \( N = 2 \) Wess-Zumino model. We then give the conclusion of the paper and discuss some remaining issues in the last section. In the appendix we give a brief summary of Hopf algebra to fix the notation and terminology appeared in the text.
2 General Framework of the Link Formulation of the Dirac–Kähler Twisted Supersymmetry on a Lattice

2.1 Generality of the Formulation

The principle of the link approach \([6, 7, 8]\) to a realization of a supersymmetric theory on a lattice is based on a simple assumption that the superalgebra in the continuum

\[
\{Q_A, Q_B\} = 2\tau_{AB}P_\mu, \\
\{Q_A, P_\mu\} = \{P_\mu, P_\nu\} = 0 
\] (2.1)

has some natural counterpart on the lattice

\[
\{Q^\text{lat}_A, Q^\text{lat}_B\} = 2\tau_{AB}P^\text{lat}_\mu, \\
\{Q^\text{lat}_A, P^\text{lat}_\mu\} = \{P^\text{lat}_\mu, P^\text{lat}_\nu\} = 0. 
\] (2.2)

Here \(\tau^\mu_{AB}\) is just a constant coefficient, and \(Q^\text{lat}_A\) and \(P^\text{lat}_\mu\) are understood both as deformed operators on the lattice which come back to \(Q_A\) and \(P_\mu\), respectively, in the naïve continuum limit;

\[
\lim_{a \to 0} Q^\text{lat}_A = Q_A, \quad \lim_{a \to 0} P^\text{lat}_\mu = P_\mu. 
\] (2.3)

We require that

\[
\sum_x P^\text{lat}_\mu \phi(x) = 0 
\] (2.4)

for the “momentum” operator \(P^\text{lat}_\mu\) and any field \(\phi(x)\) on the lattice. This is because, in the continuum, the general superinvariance of Lagrangian is up to total divergence which vanishes under the integral, and the same structure should be true for the “exact” supersymmetry on the lattice, for which the property above is necessary. We would also require the translational invariance and (semi-)locality for the operator \(P^\text{lat}_\mu\) so that the whole theory would have these properties. Another possible requirement might be the Hermiticity (or the reflection (Osterwalder–Schrader) positivity \([45]\) of transfer matrices on the lattice \([47]\)), but we don’t force it here because it is related to the subtlety of the doubling phenomenon \([11, 2]\) for which we defer the discussion to later sections.

The simplest candidates for the “momentum” operator \(P^\text{lat}_\mu\) would be the finite difference operators on the lattice,

\[
P^\text{lat}_\mu = i\partial_{\mp \mu}, \quad i\partial_{s \mu}, \quad \text{etc.}, 
\] (2.5)

where

\[
\partial_{\mp \mu} \phi(x) := \frac{1}{a}\left(\phi(x + a\hat{\mu}) - \phi(x)\right) \quad \text{(forward difference operator)}, 
\] (2.6)

\[
\partial_{-\mu} \phi(x) := \frac{1}{a}\left(\phi(x) - \phi(x - a\hat{\mu})\right) \quad \text{(backward difference operator)}, 
\] (2.7)

\[
\partial_{s \mu} \phi(x) := \frac{1}{2}\left(\partial_{+ \mu} + \partial_{- \mu}\right) \phi(x) \quad \text{(symmetric difference operator)}, 
\] (2.8)

\[
= \frac{1}{2a}\left(\phi(x + a\hat{\mu}) - \phi(x - a\hat{\mu})\right) 
\]

where \(\hat{\mu}\) is the unit vector to the direction of \(x^\mu\) and \(a\) is the lattice constant\(^2\). The symmetric difference is self anti-Hermitian: \((\partial_{s \mu})^\dagger = -\partial_{s \mu}\), while the others are anti-Hermitian conjugate to

\(^1\)The notation here is schematic; indices \(A\) and \(B\) could contain both spinor and internal d.o.f., and their conjugate as well.

\(^2\)We always keep the lattice constant \(a\) explicitly in this paper unless otherwise specified.
and each other; \((\partial_{\pm \mu})^\dagger = -\partial_{\mp \mu}\). An immediate consequence of using these finite difference operators is that they break the Leibniz rule, or put it milder, obey the modified Leibniz rule as in
\[
\partial_{\pm \mu}(\varphi_1 \varphi_2)(x) = \partial_{\pm \mu} \varphi_1(x) \varphi_2(x) + \varphi_1(x \pm a \hat \mu) \partial_{\pm \mu} \varphi_2(x) \\
= \partial_{\pm \mu} \varphi_1(x) \varphi_2(x \pm a \hat \mu) + \varphi_1(x) \partial_{\pm \mu} \varphi_2(x) \\
= \partial_{\pm \mu} \varphi_1(x) \varphi_2(x) + \varphi_1(x) \partial_{\pm \mu} \varphi_2(x) \pm a \partial_{\pm \mu} \varphi_1(x) \partial_{\pm \mu} \varphi_2(x),
\]
and
\[
\partial_{\pm \mu}^\prime(\varphi_1 \varphi_2)(x) = \partial_{\pm \mu}^\prime \varphi_1(x) \varphi_2(x - a \hat \mu) + \varphi_1(x + a \hat \mu) \partial_{\pm \mu}^\prime \varphi_2(x) \\
= \partial_{\pm \mu}^\prime \varphi_1(x) \varphi_2(x + a \hat \mu) + \varphi_1(x - a \hat \mu) \partial_{\pm \mu}^\prime \varphi_2(x) \\
= \partial_{\pm \mu}^\prime \varphi_1(x) \varphi_2(x) + \varphi_1(x) \partial_{\pm \mu}^\prime \varphi_2(x) + \frac{a}{2} \left( \partial_{+ \mu} \varphi_1(x) \partial_{+ \mu} \varphi_2(x) - \partial_{- \mu} \varphi_1(x) \partial_{- \mu} \varphi_2(x) \right).
\]

Although superficially the breaking term in each case of the Leibniz rule is proportional to the lattice constant \(a\), it is not in general of higher order in the continuum limit, due to the contributions from the cut off scale region of the momentum \(\partial_{\pm \mu} \varphi(x) \sim O(1/a)\) \cite{24}. Note also that the last term of \((2.10)\) is proportional to a total difference \(\partial_{- \mu}(\partial_{+ \mu} \varphi_1(x) \partial_{+ \mu} \varphi_2(x))\), so that one may consider that this breaking of the Leibniz rule is irrelevant under a summation over the whole lattice sites. But this is true only for the product of two fields, so that might be a good property only in a free theory, not in an interacting case. (Even in the free case there is an associated doubler problem for the anti-Hermitian symmetric difference. We will see this later in more detail.) One might also try to impose a constraint on the fields to make the breaking terms vanish, but this would only result in a nonlocal formulation \cite{18}. Thus, as long as we use the simple difference operators \((2.5)\), we can’t naively neglect the breaking of the Leibniz rule. In fact, it is more generally shown \cite{27, 43} that we have to admit the breaking of the Leibniz rule of any “momentum” operators on a lattice, unless we allow nonlocal operators like so-called SLAC derivative \cite{44} or, say, many multiflavors. These facts are already enough for the lattice counterpart of the superalgebra \((2.2)\) to lose the nature of strict Lie superalgebra, which is the most evident and crucial obstacle to formulate supersymmetry entirely based on the superalgebra on a lattice.

One possibility to overcome the situation is to interpret the superalgebra on the lattice \((2.2)\) as a “deformed” Lie superalgebra with the deformation parameter that vanishes in the continuum limit. This is in fact the basic strategy in the link approach as we can see in what follows.

Since the r.h.s. of \((2.2)\) obeys the modified Leibniz rule, it is natural to deform the algebra so that the generators in the l.h.s. also obeys a modified Leibniz rule. In the link approach, the central ansatz is that the supercharge \(Q^\text{lat}_A\) obeys the Leibniz rule of the form\footnote{Here \(|\varphi|\) is 0 or 1, depending on whether \(\varphi\) is bosonic or fermionic, respectively.}
\[
Q^\text{lat}_A(\varphi_1 \varphi_2)(x) = Q^\text{lat}_A \varphi_1(x) \varphi_2(x) + (-1)^{|\varphi_1|} \varphi_1(x + a_A) Q^\text{lat}_A \varphi_2(x),
\]
where \(x + a_A\) is to be interpreted as denoting an extended lattice site which goes to \(x\) in the naive continuum limit. Introducing a translation or shift operator \(T_{a_A}\) in a “fundamental” representation such that
\[
T_{a_A} \varphi(x) = \varphi(x + a_A),
\]
the Leibniz rule can be written as
\[
Q^\text{lat}_A(\varphi_1 \varphi_2)(x) = Q^\text{lat}_A \varphi_1(x) \varphi_2(x) + (-1)^{|\varphi_1|} T_{a_A} \left( \varphi_1 T_{a_A}^{-1} Q^\text{lat}_A \varphi_2 \right)(x),
\]
i.e.
\[
T_{a_A}^{-1} Q^\text{lat}_A(\varphi_1 \varphi_2)(x) = \left( T_{a_A}^{-1} Q^\text{lat}_A \varphi_1 \right)(x) \varphi_2(x - a_A) + (-1)^{|\varphi_1|} \varphi_1(x) \left( T_{a_A}^{-1} Q^\text{lat}_A \varphi_2 \right)(x),
\]
(2.13)
showing that the operator \( T_{aA}^{-1}Q_A^{\text{lat}} \) obeys a slightly different modified Leibniz rule. We may also write it in a symmetric form as
\[
T_{aA}^{-1/2}Q_A^{\text{lat}}(\varphi_1 \varphi_2)(x) = (T_{aA}^{-1/2}Q_A^{\text{lat}} \varphi_1)(x)\varphi_2(x - a_A/2) + (-1)^{|\varphi_1|}\varphi_1(x + a_A/2)(T_{aA}^{-1/2}Q_A^{\text{lat}} \varphi_2)(x),
\]
which is still a modified version of Leibniz rule. We could have begun with a little more generalized modification such as
\[
Q_A^{\text{lat}}(\varphi_1 \varphi_2)(x) = Q_A^{\text{lat}} \varphi_1(x)\varphi_2(x + a'_A) + (-1)^{|\varphi_1|}\varphi_1(x + a_1^A)Q_A^{\text{lat}} \varphi_2(x),
\]
but with a redefinition of \( Q_A^{\text{lat}} \) to \( T_{a_A}^{-1}Q_A^{\text{lat}} \) it is always equivalent to the original one (2.11), which can be seen in a similar fashion as in the above. Such a redefinition only causes a total difference in the algebra (2.2), hence the original form (2.11) suffices in general.

The field in the fundamental representation of the translation/shift operator (2.12) could be interpreted as a normal function on the lattice. If, by contrast, we introduce the “adjoint” representation of the translation/shift operator as in
\[
T_{aA} \varphi(x)T_{aA}^{-1} = \varphi(x + a_A),
\]
the Leibniz rule (2.11) can be written as
\[
Q_A^{\text{lat}}(\varphi_1 \varphi_2)(x) = Q_A^{\text{lat}} \varphi_1(x)\varphi_2(x) + (-1)^{|\varphi_1|}T_{aA} \varphi_1(x)T_{aA}^{-1}Q_A^{\text{lat}} \varphi_2(x),
\]
i.e.
\[
T_{aA}^{-1}Q_A^{\text{lat}}(\varphi_1 \varphi_2)(x) = T_{aA}^{-1}Q_A^{\text{lat}} \varphi_1(x)\varphi_2(x) + (-1)^{|\varphi_1|}\varphi_1(x)T_{aA}^{-1}Q_A^{\text{lat}} \varphi_2(x).
\]
Now we can see that the operator \( T_{a_A}^{-1}Q_A^{\text{lat}} \) obeys the usual exact Leibniz rule. We could write this further as
\[
T_{aA}^{-1}Q_A^{\text{lat}}(\varphi_1 \varphi_2)(x)T_{a'_A} = T_{aA}^{-1}Q_A^{\text{lat}} \varphi_1(x)T_{a'_A} \varphi_2(x - a'_A) + (-1)^{|\varphi_1|}\varphi_1(x)T_{aA}^{-1}Q_A^{\text{lat}} \varphi_2(x)T_{a'_A},
\]
which shows the operator \( T_{a_A}^{-1}Q_A^{\text{lat}} \) obeys a different Leibniz rule. We would thus again find that the original modified Leibniz rule (2.11) itself is equivalent to the more general form (2.15), and even to the usual Leibniz rule (2.17) with a suitable redefinition of the operator \( Q_A^{\text{lat}} \). Notice that in this adjoint representation such a redefinition would change the algebra (2.2) in a nontrivial way, except for the case \( a'_A = a_A \), for which the algebra, or more precisely the “momentum” operator, would remain unchanged up to a total difference so that the “momentum” operator still obeys the modified Leibniz rule. In other words, unless \( a'_A = a_A \), we have a possibility to redefine both the operators \( Q_A^{\text{lat}} \) and \( P_{\mu}^{\text{lat}} \) so as to follow the usual Leibniz rule for which the usual representation would exist. This fact may play an important rôle for an explicit representation of the lattice superalgebra. Another point to observe is that, in the adjoint representation, field itself should be identified as an operator or a matrix which formally belongs, together with the shift operator, to an algebra (which would be a universal enveloping algebra of a Lie superalgebra, just as in a canonical quantization scheme.).

The operator \( T_{aA} \), when multiplying from the left/right on a field, changes the property under a commutation of the field. For instance, suppose \( \varphi_1(x) \) and \( \varphi_2(x) \) commute with each other; \( \varphi_1(x)\varphi_2(x) = \varphi_2(x)\varphi_1(x) \). Then \( T_{aA} \varphi_1(x) \) and \( \varphi_2(x) \) no longer commute strictly, but commute with a shift in the sense that \( T_{aA} \varphi_1(x)\varphi_2(x) = T_{aA} \varphi_2(x)T_{aA}^{-1}T_{aA} \varphi_1(x) = \varphi_2(x + a_A)T_{aA} \varphi_1(x) \). This type of mild noncommutative nature doesn’t simply occur in the fundamental case since \( (T_{aA} \varphi_1(x)\varphi_2(x) = \varphi_2(x)(T_{aA} \varphi_1(x)) \). We will see in the next chapter how the modified Leibniz rules in the fundamental representation (2.11), (2.13) and (2.14) can be more systematically treated in the framework of Hopf algebraic symmetry. Here in what follows we continue the general description mainly with the adjoint representation case.
Suppose now $Q_A^\text{lat}$ also belongs to this same algebra that $T_{AA}$ and $\varphi(x)$ form. The fact that the combination $T_{AA}^{-1}Q_A^\text{lat}$ in (2.17) follows the usual Leibniz rule thus motivates us to write formally

$$T_{AA}^{-1}Q_A^\text{lat} =: \hat{Q}_A^\text{lat} \equiv i \text{ad}(\hat{Q}_A^\text{lat}),$$

which acts on a field as

$$T_{AA}^{-1}Q_A^\text{lat}(\varphi(x)) = i \text{ad}(\hat{Q}_A^\text{lat})\varphi(x) := i\{\hat{Q}_A^\text{lat}, \varphi(x)\}_{(-1)|\varphi|+1},$$

or

$$Q_A^\text{lat}(\varphi(x)) = iT_{AA}\{\hat{Q}_A^\text{lat}, \varphi(x)\}_{(-1)|\varphi|+1} =: i \text{ad}^{\text{lat}}(Q_A^\text{lat})\varphi(x),$$

(2.20)

It can also be written as

$$Q_A^\text{lat}\varphi(x) = iT_{AA}\hat{Q}_A^\text{lat}\varphi(x) - (1)^{|\varphi|}iT_{AA}\varphi(x)T_{AA}^{-1}T_{AA}\hat{Q}_A^\text{lat}$$

$$= iQ_A^\text{lat}\varphi(x) - (1)^{|\varphi|}\varphi(x + a_A)Q_A^\text{lat} =: i\{Q_A^\text{lat}, \varphi(x)\}_{(-1)|\varphi|+1} =: i \text{ad}^{\text{lat}}(Q_A^\text{lat})\varphi(x),$$

(2.21)

where $Q_A^\text{lat} := T_{AA}\hat{Q}_A^\text{lat}$. In this last equation we have defined a kind of deformed adjoint operation $\text{ad}^{\text{lat}}$ which was referred to as the shifted (anti-)commutator in the DKKN formalism. It illustrates the general fact that an operator which obeys a modified Leibniz rule could be expressed with a shifted (anti-)commutator. We have, however, introduced objects like $\hat{Q}_A^\text{lat}$, $Q_A^\text{lat}$ and respectively their (anti-)commutator and shifted (anti-)com mutator $\text{ad}(\hat{Q}_A^\text{lat})$ and $\text{ad}^{\text{lat}}(Q_A^\text{lat})$ only in a formal way, neither specified the explicit forms nor even justified the existence of them. So far we have only found that $T_{AA}^{-1}Q_A^\text{lat} = \hat{Q}_A^\text{lat}$ obeys the usual Leibniz rule, which would be regarded as a normal operator, and that $Q_A^\text{lat}$ would be expressed as $Q_A^\text{lat} = T_{AA}\hat{Q}_A^\text{lat}$. The point here is the following: As we mentioned above, we assume that the shift parameter $a_A$ reduces to zero in the naive continuum limit. Correspondingly the translation/shift operator $T_{AA}$ would go to unity in the limit: $T_{AA} \rightarrow \text{1}$, and thus the formal expression $Q_A^\text{lat} = iT_{AA}\text{ad}(\hat{Q}_A^\text{lat}) = i \text{ad}^{\text{lat}}(Q_A^\text{lat})$ reduces to the normal (anti-)commutator $Q_A = i \text{ad}(Q_A^\text{lat})$. This implies that normal (anti-)commutators in the continuum, if used in any algebraic expressions, should be simply replaced with the shifted (anti-)commutators on the lattice to accommodate the modified Leibniz rule (2.11). This reminds us of the correspondence principle between the Poisson bracket in the classical theory and the commutator in the quantum theory. We are motivated by this analogy to think the lattice version of the superalgebra of a “quantization” of the continuum superalgebra. This viewpoint of the formulation is discussed in the next chapter.

Let us move on to the algebra (2.2). Here, for generality, we consider the modified Leibniz rule (2.15). The l.h.s. of the algebra applies on a product $\varphi_1\varphi_2$ as in

$$\{Q_A^\text{lat}, Q_B^\text{lat}\}(\varphi_1\varphi_2)(x) = \{Q_A^\text{lat}, Q_B^\text{lat}\}\varphi_1(x)\varphi_2(x + a_A^r + a_B^r) + \varphi_1(x + a_A^l + a_B^l)\{Q_A^\text{lat}, Q_B^\text{lat}\}\varphi_2(x)$$

$$= \sum_\mu 2\tau_{AB}^\mu \left( P_{\mu}^\text{lat}\varphi_1(x)\varphi_2(x + a_A^r + a_B^r) + \varphi_1(x + a_A^l + a_B^l)P_{\mu}^\text{lat}\varphi_2(x) \right),$$

while the r.h.s. as in

$$2\tau_{AB}^\mu P_{\mu}^\text{lat}(\varphi_1\varphi_2)(x) = \sum_\mu 2\tau_{AB}^\mu \left( P_{\mu}^\text{lat}\varphi_1(x)\varphi_2(x + a_A^r + a_B^r) + \varphi_1(x + a_A^l + a_B^l)P_{\mu}^\text{lat}\varphi_2(x) \right),$$

(2.23)

where $a_A^l$ and $a_A^r$ are, depending on the choice of $P_{\mu}^\text{lat}$,

$$(a_A^l, a_A^r) = \begin{cases} (\pm a, 0) & \text{or} & (0, \pm a) & \text{for } P_{\mu}^\text{lat} = i\partial_{\pm\mu}, \\ (+a, -a) & \text{or} & (-a, +a) & \text{for } P_{\mu}^\text{lat} = i\partial_{\mu}. \end{cases}$$

(2.24)

$^4$Here in the equation below $[A, B]_\pm := AB \pm BA$.  

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The algebra (2.2) requires that these two equations to be equal. As we can easily find, the first necessary condition is that the coefficient \( \tau^\mu_{AB} \) should have the form

\[
\tau^\mu_{AB} = \tau_{AB} \delta^\mu \mu(A,B)
\]  

(2.25)

for a certain vector index \( \mu(A,B) \) uniquely determined by the combination of spinor indices \( A \) and \( B \). Namely, only one, at most, of \( D \) “momenta” \( P^1, \ldots, P^D \) could appear in the r.h.s. of the algebra for each combination of \( A \) and \( B \). Then the corresponding algebra in the continuum would be such that \( \{Q_A,Q_B\} = 2\tau_{AB}P_{\mu(A,B)} \), which violates the Lorentz covariance of the algebra except that \( A \) or \( B \) also has a “vector”, or more precisely not just a spinor, index. We know such a basis of indices in which a supercharge has a “vector” index, namely as the basis of twisted supersymmetry \[39\] \[10\]. In fact in the link formalism and also in the other approaches the twisted basis for the spinor indices is adopted, and it is the twisted version of extended supersymmetry, and is put on the lattice in such a way that the theory becomes free from the mismatch of d.o.f. between fermions and bosons so that comes to meet the “flavor” of the extended supersymmetry, and is put on the lattice in such a way that the theory becomes free from the mismatch of d.o.f. between fermions and bosons so that comes to meet the nonperturbative criterion for supersymmetry which reads that the partition function becomes unity. We will see this point again later.

At any rate suppose the coefficient \( \tau^\mu_{AB} \) satisfies the condition (2.25). The condition that (2.22) coincides with (2.23) leads in this case to

\[
a^1_A + a^1_B = a^1 \mu(A,B), \quad a^r_A + a^r_B = a^r \hat{\mu}(A,B).
\]  

(2.26)

That these have consistent solutions for \( a^1_A \) is the second necessary condition for the link formalism to work. Recalling that for an operator \( Q^A_{\mu} \) which satisfies the modified Leibniz rule (2.15) the combination \( T^{-1}_A Q^A_{\mu} \) follows the usual Leibniz rule in the adjoint representation, we may consider the corresponding algebra

\[
\{T^{-1}_A Q^A_{\mu} T a^r_A, T^{-1}_B Q^B_{\mu} T a^r_B\} = 2\tau_{AB} T^{-1}_A T^{-1}_B P^A_{\mu(\hat{A},B)} T a^r_A T a^r_B,
\]

(2.27)

where we have assumed that the condition (2.26) is met. The relation (2.26) assures that the operator in the r.h.s. also follows the usual Leibniz rule. In fact, we find

\[
T a^r_A T a^1_B = T a^r \bar{\mu}(A,B), \quad T a^r_A T a^r_B = T a^r \hat{\mu}(A,B),
\]

(2.28)

and we may write \( P^A_{\hat{A},B} \) as, up to the lattice constant and other constant factors,

\[
P^A_{\hat{A},B} = \text{Ad}(T a^r \bar{\mu}(A,B)) - \text{Ad}(T a^r \hat{\mu}(A,B)),
\]

i.e.

\[
P^A_{\hat{A},B} \varphi(x) = T a^r \bar{\mu}(A,B) \varphi(x) T^{-1}_A a^r \bar{\mu}(A,B) - T a^r \hat{\mu}(A,B) \varphi(x) T^{-1}_A a^r \hat{\mu}(A,B),
\]

(2.29)

where we define \( \text{Ad}(T a^r \bar{\mu}(A,B)) \varphi(x) = T a^r \bar{\mu}(A,B) \varphi(x) T^{-1}_A a^r \bar{\mu}(A,B) \) which could be compared with the definition of \( \text{ad}(\hat{Q}^A_{\mu}) \) in (2.20). Then

\[
T a^r_A T^{-1}_A T^{-1}_B P^A_{\hat{A},B} T a^r_A T a^r_B = - \text{ad}(T a^r \hat{\mu}(A,B) T a^r \bar{\mu}(A,B)),
\]

i.e.

\[
T a^r_A T^{-1}_A P^A_{\hat{A},B} \varphi(x) T a^r_A T a^r_B = - \left( T a^r \hat{\mu}(A,B) T a^r \bar{\mu}(A,B) \varphi(x) - \varphi(x) T a^r \hat{\mu}(A,B) T a^r \bar{\mu}(A,B) \right),
\]

(2.30)

We denote the spacetime dimension as \( D \).
which is a normal commutator. Notice that, as mentioned before, this redefinition of the “momentum” operator is possible only for $a^r_A \neq a^r_A$, which is assured here by the requirement that (2.26) holds. We have seen that all “generators” in the algebra (2.27) follow the usual Leibniz rule, so that would give a basis for the construction of supersymmetry on the lattice in a manner quite parallel to that of the continuum.

2.2 Twisted Basis and the Doubling of Chiral Fermion

When one regularizes chiral fermions on the lattice species doublers of chiral fermions inevitably appear [1, 2]. It was shown that the naïve fermion formulation where the continuum differential operators in the Dirac action is naïvely replaced by the lattice difference operator can be spin diagonalized and leads to the staggered fermion formulation [48] which is shown to be essentially equivalent [49, 50] to Kogut-Susskind fermion formulation [51]. The equivalence of the staggered fermion formulation and the Dirac–Kähler fermion has been proved exactly with an introduction of mild noncommutativity between differential forms and fields [16]. This means that all these lattice fermion formulations are equivalent where the mild noncommutativity seems to play an important rôle. Among these fermion formulations the Dirac–Kähler fermion formulation has clear geometrical correspondence with respect to the fields since the differential form and simplex of lattice have one to one correspondence.

The claim of the Dirac–Kähler twisting procedure is that these species doublers are not just lattice artifacts but fundamental d.o.f. for the regularization of fermions [10]. It is exactly these d.o.f. which constitute the twisted extended supersymmetry: $\mathcal{N} = 2$ in two dimensions, $\mathcal{N} = 4$ in three dimensions, and $\mathcal{N} = 4$ in four dimensions. The four dimensional Dirac–Kähler twisting procedure coincides with the twisting derived by Marcus [11]. These arguments apply in higher dimensions, too, requiring that in $D$ dimensions, which has $2^{D/2}$ (on-shell) doubler’s degeneracy, should be treated with $\mathcal{N} = 2^{D/2}$ extended supersymmetry. (In two dimensions $\mathcal{N} = 2$, for example, does not correspond to the number of total charges and thus it is sometimes denoted as $\mathcal{N} = (2, 2)$ instead.)

In the Dirac–Kähler twisting procedure spinor suffix and flavor suffix constitute the scalar, vector, tensor.. nature of the super charges. In other words the flavor d.o.f. which are originally the species doublers d.o.f. is now identified as the extended supersymmetry d.o.f.. The corresponding suffix can be rotated by the internal R-symmetry generator of extended supersymmetry. In this way the internal d.o.f. plays the rôle of changing spin of the fields. The mechanism how the spin and the internal rotation are related should be understood from lattice point of view. This issue is fundamentally related to the spin and statistics problem on the lattice. Since the Lorentz invariance is broken on the lattice it is natural to expect that the (anti-)commuting nature of fields will be modified.

Let us begin with the two dimensional case. Here we only consider the simplest cases. Superalgebra in the Dirac–Kähler twisted basis on the lattice is given as

$$\{Q^\text{lat}_\mu, Q^\text{lat}_\nu\} = P^\text{lat}_\mu, \quad \{\tilde{Q}^\text{lat}_\mu, Q^\text{lat}_\nu\} = -\epsilon_{\mu\nu}P^\text{lat}_\nu’, \quad \{\text{others}\} = 0,$$

(2.31)

which is the twisted version of $\mathcal{N} = (2, 2)$ superalgebra in two dimensions. We have put a prime on the second “momentum” operator to distinguish from the first one, since there is an ambiguity for the lattice “momentum” operator as explained above. Note in each commutator the r.h.s. contains only one “momentum” operator for each given combination of indices, which is necessary for the algebraic consistency as claimed in the preceding section. The reason we specified $\mathcal{N} = (2, 2)$ is that then the corresponding supermultiplet contains four fermions, which has the same (on-shell) d.o.f. as that of the Dirac–Kähler/staggered fermions which originate the doubler’s d.o.f. on the lattice in two dimensions.
The shift variable condition \( (2.26) \) reads in this case
\[
\dot{a}^{1r} + a_{\mu}^{1r} = a^{1r} \hat{\mu}, \quad \ddot{a}^{1r} + a_{\mu}^{1r} = \epsilon_{\mu\nu} a^{1r} \hat{\nu}.
\] (2.32)

With the same argument given in the original link formalism these lead to \( a^{1r} + a_{1}^{1r} + a_{2}^{1r} + \dot{a}^{1r} = (a^{1r} + a^{1r}) \hat{1} = (a^{1r} + a^{1r}) \hat{2} \), which is only possible if \( a^{1r} = -a_{1}^{1r} \). In our simple choices of the “momentum” operators, it implies that, due to \( (2.24) \),
\[
\begin{align*}
P_{\mu}^{\text{lat}} &= i \partial_{\pm \mu}, \\
P^{\text{lat}}_{\hat{\mu}} &= i \partial_{\mp \mu},
\end{align*}
\] or
\[
P_{\mu}^{\text{lat}} = P^{\text{lat}}_{\hat{\mu}} = i \partial_{\mp \mu}.
\] (2.33)

The former possibility was considered in the original link formulation, whereas the latter one, although a solution for the consistency, might not be so good from the viewpoint of the doubling issue: it would create, if naively used, the doubling degeneracy again. In any of these cases, the shift conditions become
\[
a^{1r} + a_{\mu}^{1r} = a^{1r} \hat{\mu}, \quad \ddot{a}^{1r} + a_{\mu}^{1r} = -\epsilon_{\mu\nu} a^{1r} \hat{\nu},
\] (2.34)

which are four conditions with one constraint, so three remaining conditions in total, for four shift variables. It thus seems that one shift variable could be free. In view of the lattice structure, however, this free parameter should not be irrational, otherwise it would lead to uncountable number of “dual” lattice points, which spoils the lattice regularization! Though still any rational numbers are allowed, it is easy to see we will then have unnecessary d.o.f. again or unnatural lattice structure, except for the case when this free parameter is fixed to zero or half the lattice constant. These choices of the free parameter were referred to as the asymmetric and symmetric choices, respectively, in the link formalism.

Similarly in four dimensions, we take the superalgebra
\[
\{Q^{\text{lat}}_{\mu}, Q^{\text{lat}}_{\nu}\} = P^{\text{lat}}_{\tau \mu}, \quad \{Q^{\text{lat}}_{\mu \nu}, Q^{\text{lat}}_{\rho \sigma}\} = \delta_{\mu \nu, \rho \sigma} P^{\text{lat}}_{\tau \rho} - \delta_{\mu \tau, \rho \nu} P^{\text{lat}}_{\nu \rho},
\] (2.35)

and the other commutators all vanish. This is the Dirac–Kähler twisted superalgebra of \( N = 4 \) which is required, as explained above, from the general argument on the fermionic d.o.f. We can show that these combinations of \( P^{\text{lat}}_{\pm \mu} \) indeed lead to the Leibniz rule conditions for the shift variables which have nontrivial set of solutions \([7]\).

### 2.3 The Claimed Inconsistency

What is intriguing in the link formalism is the algebraic structure based on the modified Leibniz rule for the symmetry operators. If a suitable representation of this algebra is unambiguously obtained, it seems at first sight that it gives a formulation of supersymmetry on the lattice. It turns out, however, such a representation would conflict with the conventional component field path integral formulation on the lattice. This problem can be seen as the fact that, although supertransformations of single component fields are well-defined, supertransformations of products of fields becomes sensitive to the order of the fields in the products. If such an order is uniquely determined, it is nothing harmful. However, we have no criteria to introduce such an order on the conventional lattice, thus we have a serious difficulty that supertransformations are not totally defined in a unique and consistent manner as transformations of path integral variables. In fact, this difficulty is claimed as an inconsistency of the link formalism in \([32]\). The criticism is two-folded: one is for the non-gauge theories \([6]\), the other is, also investigated in a similar attitude in \([34]\), for the case of gauge theories \([7,8]\), and both are summarized as that the
supercharges in the link formalism add nontrivial link structure on component fields changing the original link nature of the fields in an ordering sensitive way.

Let us see these arguments more explicitly. In the link formalism, scalar fields \( \phi(x) \) defined on sites of the lattice are naturally assumed to be commutative

\[
\phi_1(x)\phi_2(x) = \phi_2(x)\phi_1(x).
\]

(2.36)

Applying the supertransformation on the both sides of this equation, we have, from the left hand side, that

\[
Q_A^{\text{lat}}(\phi_1(x)\phi_2(x)) = \psi_1A(x)\phi_2(x + a_A^1) + \phi_1(x + a_1^A)\psi_2A(x),
\]

(2.37)

where \( \psi_{1,A}(x) := Q_A^{\text{lat}}\phi_{1,2}(x) \), and from the right,

\[
Q_A^{\text{lat}}(\phi_2(x)\phi_1(x)) = \psi_2A(x)\phi_1(x + a_1^A) + \phi_2(x + a_1^A)\psi_1A(x).
\]

(2.38)

These two equations must be the same as they are the transformations of one and the same quantity; otherwise the supertransformations on products of fields aren’t uniquely defined. But actually these two conflicts with each other if fermions \( \psi_{1,2,A} \) are also assumed to be simple (anti-)commuting objects: the term containing \( \psi_{1,A}(x) \) in the first equation has the factor \( \phi_2(x + a_1^A) \), whereas in the second has \( \phi_2(x + a_1^A) \), and they are different unless \( a_1^A = a_1^A \), which, however, wouldn’t lead to the consistent solution for the shift variable conditions as already explained in the previous sections. The discrepancy between these two equations cannot be expressed as a total difference, so that it gives an essential obstacle for the invariance of any possible action. It causes similar difficulties also in the gauge theory actions.

In the following chapters, we will propose a possible solution to the above mentioned first criticism for the non-gauge theories by introducing the following mild noncommutativity [5]:

\[
\varphi_A(x)\varphi_B(y) = (-1)^{|\varphi_A||\varphi_B|}\varphi_B(y + a_A)\varphi_A(x - a_B),
\]

(2.39)

where \( \varphi_A(x) \) and \( \varphi_B(y) \) carry a shift \( a_A \) and \( a_B \), respectively. In fact we can easily confirm that the expressions of (2.37) and (2.38) coincide if we identify that \( \psi_{1,2,A} \) carry a shift \( x_1^A = x_2^A \) while \( \phi_{1,2} \) carry no shift and they satisfy the noncommutative relation (2.39). The key point is to treat each field as a noncommutative object, or an object with nontrivial statistics, to uniquely define the ordering which is necessary to avoid the conflict.

If we introduce the noncommutative nature for the fields as in (2.39), the formulation of field theory should be modified from the conventional definition in such a way that any algebraic manipulation of fields and operators should be compatible with the new deformed supersymmetry. In the following we show that it is possible to define a new lattice field theory which has the exact deformed supersymmetry with Hopf algebraic nature. Addressing the similar questions to the gauge theories is out of the scope of this paper.

3 Hopf Algebraic Structure of the Lattice Superalgebra

In this section, we investigate the “lattice superalgebra” from a yet different algebraic viewpoint, namely in terms of Hopf algebra. As has been developed in recent years, extending the notion of the symmetry in a field theory to the Hopf algebraic one brings us still useful frameworks in some specific cases especially in noncommutative theories [11, 52, 53, 54, 55]. A slightly different applications are found in [16]. In the current interest, the superalgebra on the lattice is understood as a deformed algebra on the lattice and forming a Hopf algebra. This identification assures us of the mathematical consistency of the deformed algebra. Using the general scheme called braided quantum field theory [11, 54], we will show that the field theory whose symmetry
is prescribed by the deformed algebra can be constructed at least perturbatively. The deformed symmetry leads to the corresponding Ward–Takahashi identities on the lattice, which may serve as a good physical interpretation of the deformed symmetry itself.

Appendix \[A\] is devoted to a brief mathematical basis on Hopf algebra and summarizing our notation and terminology.

### 3.1 Lattice Superalgebra as a Hopf Algebra

We begin with the lattice superalgebra \( \text{2.2} \), or those in the twisted basis \( \text{2.31} \), \( \text{2.35} \). Here we treat these as abstract Lie superalgebra and denote as \( \mathcal{A} \), so that \( P_{\mu}^{\text{lat}}, Q_{A}^{\text{lat}} \in \mathcal{A} \). We then introduce the space of fields on the lattice as \( X = X_{e} \oplus X_{o} \), where \( X_{e} \) consists of all bosonic fields and \( X_{o} \) of all fermionic fields. We need a multiplication/product of fields to construct a field theory, which is in general noncommutative. We assume here this multiplication is associative for our current application. Thus the space \( X \) is supposed to be an associative graded algebra. However, as a quantum field theory, products of fields, i.e. composite fields, could be clearly distinguished from the single fields, i.e. elementary fields, because the elementary fields are the variables of path integral (if any defined), or the ones obeying the canonical (anti-)commutation relations, whose behaviour is clearly different from that of the composite fields. We thus denote by \( X \) the elementary fields and extend it to the formal space of all tensor products of the elementary fields to include any composite fields:

\[
\hat{X} := \bigoplus_{n=0}^{\infty} X^{n}, \quad X^{0} := X_{e}^{0} \oplus X_{o}^{0}, \quad X^{n} := X \otimes \cdots \otimes X, \quad (3.1)
\]

where \( X_{e}^{0} \) and \( X_{o}^{0} \) are the space of bosonic and fermionic constant functions, respectively. Multiplications/products of fields are naturally defined in \( \hat{X} \) as \( m(\varphi \otimes \varphi') = \varphi \cdot \varphi' \in \hat{X} \) (\( \varphi, \varphi' \in \hat{X} \)).

We now consider general action (see Appendix A) of \( \mathcal{A} \) on the space of fields \( \hat{X} \). We denote the action of an operator \( a \in \mathcal{A} \) as \( a \cdot \). With the successive actions, we are naturally led to the notion of an (associative) universal enveloping algebra \( \mathcal{U}(\mathcal{A}) \) of \( \mathcal{A} \), as in \( (a \cdot b) \cdot := a \cdot (b \cdot) := a \cdot (b \cdot) \), with \( a, b \in \mathcal{A} \) and \( a \cdot b \in \mathcal{U}(\mathcal{A}) \). We also introduce the identity operator \( 1 \) as a unit element of the universal enveloping algebra. We may define the unit map by \( \eta(c) := c 1 \), \( c \in \mathbb{C} \).

Even on the lattice, actions or representations of the operators \( Q_{A}^{\text{lat}} \) and \( P_{\mu}^{\text{lat}} \) on elementary fields would be well-defined with no difficulty. We denote these formally as

\[
Q_{A}^{\text{lat}} \cdot \varphi(x) = (Q_{A}^{\text{lat}} \varphi)(x), \quad P_{\mu}^{\text{lat}} \cdot \varphi(x) = (P_{\mu}^{\text{lat}} \varphi)(x), \quad \varphi \in \hat{X}. \quad (3.2)
\]

Explicit form of \( Q_{A}^{\text{lat}} \cdot \varphi \) depends on the model we take. An example is listed in the appendix\[B\]. As for the expression \( P_{\mu}^{\text{lat}} \varphi \), we could essentially take some of the difference operators as in \( \text{2.3} \), but it turns out that lattice momentum operator \( P_{\mu}^{\text{lat}} \) should carry a nontrivial grading structure, which is required from the Hopf algebraic consistency. We will see this point in the following subsection.

Actions on the trivial/constant fields are also easily defined as in

\[
Q_{A}^{\text{lat}} \cdot f = 0, \quad P_{\mu}^{\text{lat}} \cdot f = 0, \quad f \in X_{e}^{0}. \quad (3.3)
\]

As a matter of convention, we write these equations in terms of a map \( \epsilon \) called counit as in

\[
Q_{A}^{\text{lat}} \cdot f = \epsilon(Q_{A}^{\text{lat}})f = 0, \quad \text{i.e.} \quad \epsilon(Q_{A}^{\text{lat}}) = 0, \quad P_{\mu}^{\text{lat}} \cdot f = \epsilon(P_{\mu}^{\text{lat}})f = 0, \quad \text{i.e.} \quad \epsilon(P_{\mu}^{\text{lat}}) = 0. \quad (3.4)
\]

The essential nontriviality comes in the actions of the operators on composite fields, i.e. product of the elementary fields, due to the failure of the usual Leibniz rule. The link formalism manages this difficulty with the introduction of appropriate deformation or modification of
Leibniz rules when the operators act on the composite fields. Mathematically, this is understood as equipping the universal enveloping algebra $U(A)$ with an additional structure, the coproduct/comultiplication, denoted by $\Delta$. To be specific, consider the actions of $Q^\text{lat}_A$ and $P^\mu_\mu$ on a product of two elementary fields $\varphi_1(x)$, $\varphi_2(x) \in X$. Introducing the modified Leibniz rule (2.15) and (2.16) is equivalent to defining these actions to be

$$Q^\text{lat}_A \triangleright (\varphi_1(x) \cdot \varphi_2(x)) := m \left( \Delta(Q^\text{lat}_A) \triangleright (\varphi_1(x) \otimes \varphi_2(x)) \right),$$

$$P^\mu_\mu \triangleright (\varphi_1(x) \cdot \varphi_2(x)) := m \left( \Delta(P^\mu_\mu) \triangleright (\varphi_1(x) \otimes \varphi_2(x)) \right),$$

(3.5)

together with the coproducts

$$\Delta(Q^\text{lat}_A) = Q^\text{lat}_A \otimes T_{a_A} + (-1)^F \cdot T_{a^*_A} \otimes Q^\text{lat}_A,$$

$$\Delta(P^\mu_\mu) = P^\mu_\mu \otimes T_{a^*_\mu} + T_{a^*_\mu} \otimes P^\mu_\mu,$$

(3.6)

where $F$ is the fermion number operator with which $(-1)^F$ takes care of the statistics factors, and the shift operator $T_b$, which is also assumed to belong to $U(A)$, acts as

$$T_b \triangleright \varphi(x) := \varphi(x + b).$$

(3.7)

For these operators we set

$$\epsilon(T_b) = 1, \quad \Delta(T_b) = T_b \otimes T_b,$$

(3.8)

and

$$\epsilon((-1)^F) = 1, \quad \Delta((-1)^F) = (-1)^F \otimes (-1)^F.$$

(3.9)

Note that these definitions are natural, since the counit essentially prescribes the action on a constant, whereas the coproduct defines the action on a product. We also list, though obvious, the action of the identity operator $1$ on $X$. It must be, by definition, such that $1 \triangleright \varphi = \varphi$, $\varphi \in X$. On a constant field, $f = 1 \triangleright f = \epsilon(1)f$, $f \in X^0$, so that

$$\epsilon(1) = 1.$$

(3.10)

On a product of elementary fields, $\varphi_1 \cdot \varphi_2 = 1 \triangleright (\varphi_1 \cdot \varphi_2) = m \left( \Delta(1) \triangleright (\varphi_1 \otimes \varphi_2) \right)$, so that

$$\Delta(1) = 1 \otimes 1.$$

(3.11)

Counit $\epsilon$ and coproduct $\Delta$ has to satisfy some consistency conditions. First, we note that any single elementary field $\varphi$ might be expressed as a product of unity and itself; $\varphi = m(1 \otimes \varphi) = m(\varphi \otimes 1)$. The action should be uniquely determined regardless of this reinterpretation of the degrees of product. More specifically, this requires that

$$(Q^\text{lat}_A \varphi)(x) = Q^\text{lat}_A \triangleright \varphi(x) = Q^\text{lat}_A \triangleright m(1 \otimes \varphi(x)) = m \left( \Delta(Q^\text{lat}_A) \triangleright (1 \otimes \varphi(x)) \right)$$

$$= m \left( (Q^\text{lat}_A \triangleright 1) \otimes (T_{a^*_A} \triangleright \varphi(x)) + (T_{a^*_A} \triangleright 1) \otimes (Q^\text{lat}_A \triangleright \varphi(x)) \right)$$

$$= m \left( (\epsilon(Q^\text{lat}_A)1) \otimes (T_{a^*_A} \triangleright \varphi(x)) + (r(T_{a^*_A})1) \otimes (Q^\text{lat}_A \triangleright \varphi(x)) \right)$$

$$= m(1 \otimes (Q^\text{lat}_A \varphi)(x)) = (Q^\text{lat}_A \varphi)(x),$$

(3.12)

which is consistently realized. The other consistency also holds:

$$(Q^\text{lat}_A \varphi)(x) = Q^\text{lat}_A \triangleright \varphi(x) = Q^\text{lat}_A \triangleright m(\varphi(x) \otimes 1) = m \left( \Delta(Q^\text{lat}_A) \triangleright (\varphi(x) \otimes 1) \right)$$

$$= m \left( (Q^\text{lat}_A \triangleright \varphi(x)) \otimes (T_{a^*_A} \triangleright 1) + ((-1)^F \cdot T_{a^*_A} \triangleright \varphi(x)) \otimes (Q^\text{lat}_A \triangleright 1) \right)$$

$$= m \left( (Q^\text{lat}_A \triangleright \varphi(x)) \otimes (\epsilon(T_{a^*_A})1) + ((-1)^F T_{a^*_A} \triangleright \varphi(x)) \otimes (\epsilon(Q^\text{lat}_A)1) \right)$$

$$= m((Q^\text{lat}_A \varphi)(x) \otimes 1) = (Q^\text{lat}_A \varphi)(x).$$

(3.13)
which shows that (3.17) holds for $Q$. As for $T_b$,

$$\varphi(x + b) = T_b \triangleright \varphi(x) = T_b \triangleright m(1 \otimes \varphi(x)) = m\left(\Delta(T_b) \triangleright (1 \otimes \varphi(x))\right)$$

$$= m\left((T_b \triangleright 1) \otimes (T_b \triangleright \varphi(x))\right)$$

$$= m\left(\epsilon(T_b)1 \otimes \varphi(x + b)\right) = m(1 \otimes \varphi(x + b)) = \varphi(x + b),$$

which is again consistent. These results show that the definitions of counit and coproduct in (3.4), (3.6), (3.8) are compatible to the trivial unital structure of the algebra $\hat{X}$. Second consistency condition is so-called the \textit{coassociativity}. Since the multiplication on $\hat{X}$ is associative, actions on products of three elementary fields should respect this associativity. This requires the coassociativity for the coproduct. It also means the action on products of three elementary fields is defined in a natural way as in

$$m \circ (m \otimes \text{id}) \circ (\Delta \otimes \text{id}) \circ \Delta(Q_A^{\text{lat}}) \triangleright \left(\left(\varphi_1(x) \otimes \varphi_2(x)\right) \otimes \varphi_3(x)\right)$$

$$= Q_A^{\text{lat}} \triangleright \left(\left(\varphi_1(x) \cdot \varphi_2(x)\right) \cdot \varphi_3(x)\right)$$

$$= Q_A^{\text{lat}} \triangleright \left(\varphi_1(x) \cdot \varphi_2(x) \cdot \varphi_3(x)\right)$$

$$= Q_A^{\text{lat}} \triangleright \left(\varphi_1(x) \cdot (\varphi_2(x) \cdot \varphi_3(x))\right)$$

$$= m \circ (\text{id} \otimes m) \circ (\text{id} \otimes \Delta) \circ \Delta(Q_A^{\text{lat}}) \triangleright \left(\varphi_1(x) \otimes \left(\varphi_2(x) \otimes \varphi_3(x)\right)\right).$$

Since the product $m$ is associative,

$$m \circ (m \otimes \text{id}) = m \circ (\text{id} \otimes m),$$

it requires that

$$(\Delta \otimes \text{id}) \circ \Delta(Q_A^{\text{lat}}) = (\text{id} \otimes \Delta) \circ \Delta(Q_A^{\text{lat}}).$$

The same condition should follow for $P_{\mu}^{\text{lat}}$ and $T_b$. These conditions are indeed satisfied for the coproducts in the present case. Using (3.6), we computed

$$(\Delta \otimes \text{id}) \circ \Delta(Q_A^{\text{lat}}) = (\text{id} \otimes \Delta)(Q_A^{\text{lat}} \otimes T_{a_A} + (-1)^F \cdot T_{a_A} \otimes Q_A^{\text{lat}})$$

$$= (Q_A^{\text{lat}} \otimes T_{a_A} + (-1)^F \cdot T_{a_A} \otimes Q_A^{\text{lat}}) \otimes T_{a_A} + ((-1)^F \cdot T_{a_A} \otimes (-1)^F \cdot T_{a_A}) \otimes Q_A^{\text{lat}}$$

$$= Q_A^{\text{lat}} \otimes T_{a_A} \otimes T_{a_A} + (-1)^F \cdot T_{a_A} \otimes Q_A^{\text{lat}} \otimes T_{a_A} + (-1)^F \cdot T_{a_A} \otimes (-1)^F \cdot T_{a_A} \otimes Q_A^{\text{lat}},$$

and

$$(\text{id} \otimes \Delta) \circ \Delta(Q_A^{\text{lat}}) = (\text{id} \otimes \Delta)(Q_A^{\text{lat}} \otimes T_{a_A} + (-1)^F \cdot T_{a_A} \otimes Q_A^{\text{lat}})$$

$$= Q_A^{\text{lat}} \otimes (T_{a_A} \otimes T_{a_A}) + (-1)^F \cdot T_{a_A} \otimes (Q_A^{\text{lat}} \otimes T_{a_A} + (-1)^F \cdot T_{a_A} \otimes Q_A^{\text{lat}})$$

$$= Q_A^{\text{lat}} \otimes T_{a_A} \otimes T_{a_A} + (-1)^F \cdot T_{a_A} \otimes Q_A^{\text{lat}} \otimes T_{a_A} + (-1)^F \cdot T_{a_A} \otimes (-1)^F \cdot T_{a_A} \otimes Q_A^{\text{lat}},$$

which shows that (3.17) holds for $Q_A^{\text{lat}}$. We have thus found unambiguously that

$$Q_A^{\text{lat}} \triangleright \left(\varphi_1(x) \cdot \varphi_2(x) \cdot \varphi_3(x)\right)$$

$$= (Q_A^{\text{lat}} \varphi_1(x) \cdot \varphi_2(x + a_A^i) \cdot \varphi_3(x + a_A^j) + (-1)^{|\varphi_1|} \varphi_1(x + a_A^i) \cdot (Q_A^{\text{lat}} \varphi_2(x) \cdot \varphi_3(x + a_A^j) + (-1)^{|\varphi_2|} \varphi_2(x + a_A^j) \cdot (Q_A^{\text{lat}} \varphi_3(x).$$

\footnote{Here we use the relation $\Delta((-1)^F \cdot T_b) = \Delta((-1)^F) \cdot \Delta(T_b)$, which will be explained shortly.}
As an example, we compute

\[
P^\text{lat}_\mu \triangleright (\varphi_1(x) \cdot \varphi_2(x) \cdot \varphi_3(x))
\]
\[
= (P^\text{lat}_\mu \varphi_1)(x) \cdot \varphi_2(x + a^\dagger \hat{\mu}) \cdot \varphi_3(x + a^\dagger \hat{\mu}) + \varphi_1(x + a^\dagger \hat{\mu}) \cdot (P^\text{lat}_\mu \varphi_2)(x) \cdot \varphi_3(x + a^\dagger \hat{\mu}) + \varphi_1(x + a^\dagger \hat{\mu}) \cdot \varphi_2(x + a^\dagger \hat{\mu}) \cdot (P^\text{lat}_\mu \varphi_3)(x).
\]

(3.21)

Similarly, \(T_b\) satisfies the coassociativity, for

\[
(\Delta \otimes \text{id}) \circ \Delta(T_b) = (\Delta \otimes \text{id})(T_b \otimes T_b) = (T_b \otimes T_b) \otimes T_b
\]
\[
= T_b \otimes (T_b \otimes T_b) = (\text{id} \otimes \Delta)(T_b \otimes T_b) = (\text{id} \otimes \Delta)\Delta(T_b),
\]

so that

\[
T_b \triangleright (\varphi_1(x) \cdot \varphi_2(x) \cdot \varphi_3(x)) = \varphi_1(x + b) \cdot \varphi_2(x + b) \cdot \varphi_3(x + b).
\]

(3.23)

The result for \((-1)^F\) would be obvious.

Now that we have shown that our operators of the prime interest, \(Q_A^\text{lat}, P^\text{lat}_\mu, T_b\) are well defined concerning to the actions on the elementary fields, constants, and products of two or three elementary fields, we find that any other actions are also consistently defined (needless to say actions as well as the maps introduced above are all linear). In particular the actions on any numbers of elementary fields can be computed inductively using the coassociativity. We need further the actions of products of operators. As we started above, the product of operators is defined concerning to the actions on the elementary fields, constants, and products of two or more elementary fields can be computed inductively using the coassociativity. We need further the actions of products of operators. As we started above, the product of operators is defined as an operator of the successive applications of each operator in the product. On the elementary fields, it is easily understood, because it is nothing but the definition. On the trivial (i.e. constant) fields, this implies a consistency on the counit map, as

\[
\epsilon(a \cdot b)f = (a \cdot b) \triangleright f = a \triangleright ob \triangleright f = \epsilon(a)\epsilon(b)f,
\]

(3.24)
i.e.

\[
\epsilon(a \cdot b) = \epsilon(a)\epsilon(b).
\]

(3.25)

Similarly, the product of operators should act on a product of elementary fields with the successive operations with

\[
m\left(\Delta(a \cdot b) \triangleright (\varphi_1 \otimes \varphi_2)\right) = (a \cdot b) \triangleright (\varphi_1 \cdot \varphi_2) = a \triangleright ob \triangleright (\varphi_1 \cdot \varphi_2) = a \triangleright m\left(\Delta(b) \triangleright (\varphi_1 \otimes \varphi_2)\right)
\]
\[
= m\left(\Delta(a) \triangleright \Delta(b) \triangleright (\varphi_1 \otimes \varphi_2)\right) = m\left(\Delta(a) \cdot \Delta(b) \triangleright (\varphi_1 \otimes \varphi_2)\right),
\]

so implies

\[
\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b).
\]

(3.27)

As an example, we compute

\[
\Delta(Q_A^\text{lat} \cdot Q_B^\text{lat}) = \Delta(Q_A^\text{lat}) \cdot \Delta(Q_B^\text{lat})
\]
\[
= (Q_A^\text{lat} \otimes T_{a_A^\dagger}) + (-1)^F \cdot T_{a_A^\dagger} \otimes Q_A^\text{lat} \cdot (Q_B^\text{lat} \otimes T_{a_B^\dagger}) + (-1)^F \cdot T_{a_B^\dagger} \otimes Q_B^\text{lat}
\]
\[
= Q_A^\text{lat} \cdot Q_B^\text{lat} \otimes T_{a_A^\dagger} \cdot T_{a_B^\dagger} + Q_A^\text{lat} \cdot (-1)^F \cdot T_{a_B^\dagger} \otimes T_{a_A^\dagger} \cdot Q_B^\text{lat}
\]
\[
+ (-1)^F \cdot T_{a_A^\dagger} \cdot Q_B^\text{lat} \otimes Q_A^\text{lat} \cdot T_{a_B^\dagger} + (-1)^F \cdot T_{a_B^\dagger} \cdot (-1)^F \cdot T_{a_A^\dagger} \otimes Q_A^\text{lat} \cdot Q_B^\text{lat}.
\]

(3.28)
Other simple examples are
\[
\Delta(Q_A^\text{lat} \cdot P^\text{lat}_\mu) = \Delta(Q_A^\text{lat}) \cdot \Delta(P^\text{lat}_\mu) \\
= (Q_A^\text{lat} \otimes T_{a_A} + (-1)^F \cdot T_{a_A}^* \otimes Q_A^\text{lat}) \cdot (P^\text{lat}_\mu \otimes T_{a_\mu} + T_{a_\mu} \otimes P^\text{lat}_\mu) \\
= Q_A^\text{lat} \cdot P^\text{lat}_\mu \otimes T_{a_A}^* \cdot T_{a_\mu} + Q_A^\text{lat} \cdot T_{a_A} \otimes T_{a_\mu}^* \cdot P^\text{lat}_\mu \\
+ (-1)^F \cdot T_{a_A} \cdot P^\text{lat}_\mu \otimes Q_A^\text{lat} \cdot T_{a_\mu} + (-1)^F \cdot T_{a_A} \cdot T_{a_\mu} \otimes Q_A^\text{lat} \cdot P^\text{lat}_\mu,
\]
and
\[
\Delta(T_b \cdot Q_A^\text{lat}) = \Delta(T_b) \cdot (Q_A^\text{lat}) = (T_b \otimes T_b) \cdot (Q_A^\text{lat} \otimes T_{a_A} + (-1)^F \cdot T_{a_A}^* \otimes Q_A^\text{lat}) \\
= T_b \cdot Q_A^\text{lat} \otimes T_b \cdot T_{a_A} + T_b \cdot (-1)^F \cdot T_{a_A} \otimes T_b \cdot Q_A^\text{lat},
\]
\[
\Delta(T_b \cdot T_c) = \Delta(T_b) \cdot \Delta(T_c) = (T_b \otimes T_b) \cdot (T_c \otimes T_c) = T_b \cdot T_c \otimes T_b \cdot T_c.
\]

Let us recall now the superalgebra (2.2), and introduce a natural algebra with respect to \(T_b\) as in
\[
\{Q_A^\text{lat}, Q_B^\text{lat}\} = 2\tau^{\mu} P^\text{lat}_\mu, \\
\{Q_A^\text{lat}, P^\text{lat}_\mu\} = \{P^\text{lat}_\mu, P^\text{lat}_\nu\} = 0, \\
\{Q_A^\text{lat}, T_b\} = \{P^\text{lat}_\mu, T_b\} = T_b, T_c = 0.
\]
The last relations are in a way obvious, and states that
\[
Q_A^\text{lat} \varphi(x + b) = T_b(Q_A^\text{lat} \varphi(x)), \quad T_b \varphi(x + c) = T_c \varphi(x + b) = \varphi(x + b + c) = T_{b+c} \varphi(x),
\]
and similar for \(P^\text{lat}_\mu\). We also list here the obvious algebra for \((-1)^F\):
\[
\{Q_A^\text{lat}, (-1)^F\} = [P^\text{lat}_\mu, (-1)^F] = [T_b, (-1)^F] = 0, \quad (-1)^F \cdot (-1)^F = 1.
\]
From these relations and using (3.28), we find that
\[
\Delta(\{Q_A^\text{lat}, Q_B^\text{lat}\}) = \{Q_A^\text{lat}, Q_B^\text{lat}\} \otimes T_{a_A^*} \cdot T_{a_B^*} + T_{a_A^*} \cdot T_{a_B^*} \otimes \{Q_A^\text{lat}, Q_B^\text{lat}\},
\]
reproducing the general result we found in (2.22). Just as an additional explicit check of the consistency, we compute the action of the product \(Q_A^\text{lat} \cdot Q_B^\text{lat}\) on the product of three fields \(\varphi_1 \cdot \varphi_2 \cdot \varphi_3\), which is given by the object
\[
(\Delta \otimes \text{id}) \circ \Delta(Q_A^\text{lat} \cdot Q_B^\text{lat}) = (\Delta \otimes \text{id}) \circ (\{\Delta(Q_A^\text{lat}), \Delta(Q_B^\text{lat})\}) \\
= \Delta(Q_A^\text{lat} \cdot Q_B^\text{lat}) \otimes T_{a_A^*} \cdot T_{a_B^*} + \Delta(Q_A^\text{lat} \cdot (-1)^F \cdot T_{a_B}^* \otimes T_{a_A} \cdot Q_B^\text{lat} \\
- \Delta(Q_B^\text{lat} \cdot (-1)^F \cdot T_{a_B^*} \otimes Q_A^\text{lat} \cdot T_{a_A^*}^* \otimes Q_A^\text{lat} \cdot Q_B^\text{lat}
\]
and then this can be computed using (3.28), (3.30) and (3.31). This of course leads to
\[
(\Delta \otimes \text{id}) \circ \Delta(\{Q_A^\text{lat}, Q_B^\text{lat}\}) = (\Delta \otimes \text{id}) \circ (\{\Delta(Q_A^\text{lat}), \Delta(Q_B^\text{lat})\}) \\
= \{Q_A^\text{lat}, Q_B^\text{lat}\} \otimes T_{a_A^*} \cdot T_{a_B^*} + \Delta(T_{a_A^*} \cdot T_{a_B}^*) \otimes \{Q_A^\text{lat}, Q_B^\text{lat}\} \\
+ T_{a_A^*} \cdot T_{a_B^*} \otimes Q_A^\text{lat} \cdot Q_B^\text{lat} \\
+ T_{a_A^*} \cdot T_{a_B}^* \otimes \{Q_A^\text{lat}, Q_B^\text{lat}\} \\
+ T_{a_B}^* \cdot T_{a_A} \otimes \{Q_A^\text{lat}, Q_B^\text{lat}\}.
\]

Equations (3.10), (3.25) and (3.11), (3.27) naturally require that the counit and coproduct, respectively, are both consistent to the structure of the algebra \(\mathcal{U}(\mathcal{A})\), i.e. both algebra maps
We can compute the actions of any operators on any fields in a consistent manner. Mathematically, all these features assure that our lattice superalgebra actually forms a bialgebra.

Notice that our bialgebra is a mixture of both algebra-like elements, like $Q_A^{\text{lat}}$ or $P_\mu^{\text{lat}}$, and group-like elements, like $T_b$. The latter have their inverse, $T_b^{-1}$. The former would also have a sort of inverse, $-Q_A^{\text{lat}}$ and $-P_\mu^{\text{lat}}$, implying the naïve connection between group and algebra. In fact, we need one more ingredient, namely an antipode, to claim that the DKKN lattice superalgebra is a Hopf algebra, and it is essentially a map to give the “inverse” element for each operator. It would be introduced as a linear map such that satisfies the identity

$$\cdot \circ (S \otimes \text{id}) \circ \Delta = \cdot \circ (\text{id} \otimes S) \circ \Delta = \eta \circ \epsilon,$$

(3.38)

where we have used the notation $\cdot (a \otimes b) = a \cdot b$ for the product of operators. We define it explicitly, on the single operators, as

$$S(Q_A^{\text{lat}}) = -T_a^{-1}_A \cdot (-1)_{F} \cdot Q_A^{\text{lat}} \cdot T_{a'}^{-1}_A,$$

$$S(P_\mu^{\text{lat}}) = -T_{a'}^{-1}_{\mu} \cdot P_\mu^{\text{lat}} \cdot T_a^{-1}_{\mu},$$

$$S(T_b) = T_b^{-1},$$

$$S((-1)_{F}) = (-1)^{-F} = (-1)^{F},$$

(3.39)

and extend it so that it becomes linear and anti-algebraic in the sense $S(a \cdot b) = S(b) \cdot S(a)$, $S(1) = 1$, $(a, b \in U(A))$. In fact it is shown that the anti-algebraic nature automatically follows if the identity (3.35) holds for the antipode. Here we just see what this identity implies in our superalgebra, without digging into the detail. Applying the first two terms of (3.38) on $Q_A^{\text{lat}}$, we find

$$\cdot \circ (S \otimes \text{id}) \circ \Delta(Q_A^{\text{lat}}) = \cdot \circ (S \otimes \text{id}) \big( Q_A^{\text{lat}} \otimes T_{a'}^{-1}_A + (-1)^{F} \cdot T_{a'}^{-1}_A \otimes Q_A^{\text{lat}} \big)$$

$$= (-T_{a'}^{-1}_A \cdot (-1)^{F} \cdot Q_A^{\text{lat}} \cdot T_{a'}^{-1}_A \otimes Q_A^{\text{lat}} + T_a^{-1}_A \cdot (-1)^{F} \otimes Q_A^{\text{lat}})$$

$$= -T_a^{-1}_A \cdot (-1)^{F} \cdot Q_A^{\text{lat}} + T_a^{-1}_A \cdot (-1)^{F} \cdot Q_A^{\text{lat}} = 0,$$

(3.40)

and

$$\cdot \circ (\text{id} \otimes S) \circ \Delta(Q_A^{\text{lat}}) = \cdot \circ (\text{id} \otimes S) \big( Q_A^{\text{lat}} \otimes T_{a'}^{-1}_A + (-1)^{F} \cdot Q_A^{\text{lat}} \otimes T_{a'}^{-1}_A \big)$$

$$= (Q_A^{\text{lat}} \otimes T_{a'}^{-1}_A - (-1)^{F} \cdot T_{a'}^{-1}_A \otimes Q_A^{\text{lat}} \cdot T_a^{-1}_A \cdot (-1)^{F} \cdot Q_A^{\text{lat}} \cdot T_a^{-1}_A)$$

$$= Q_A^{\text{lat}} \cdot T_a^{-1}_A = Q_A^{\text{lat}} \cdot T_a^{-1}_A = 0,$$

(3.41)

while the last terms gives

$$\eta \circ \epsilon(Q_A^{\text{lat}}) = 0.$$

(3.42)

Thus the identity (3.38) holds for the operator $Q_A^{\text{lat}}$ with the definition (3.39). Similar calculations show that $P_\mu^{\text{lat}}$ also obeys the identity. As for $T_b$, we compute

$$\cdot \circ (S \otimes \text{id}) \circ \Delta(T_b) = \cdot \circ (S \otimes \text{id})(T_b \otimes T_b) = (T_b^{-1} \otimes T_b) = 1,$$

(3.43)

and

$$\cdot \circ (\text{id} \otimes S) \circ \Delta(T_b) = \cdot \circ (\text{id} \otimes S)(T_b \otimes T_b) = (T_b \otimes T_b^{-1}) = 1,$$

(3.44)

whereas

$$\eta \circ \epsilon(T_b) = 1,$$

(3.45)

These conditions are the same as imposing the product $m$ and unit $\eta$ should be coalgebra maps.

We use here $S((-1)^{F} \cdot T_b) = S(T_b) \cdot S((-1)^{F})$ as explicitly shown as (3.40).
again showing the consistency. Let us calculate the antipodes of products of operators with the use of the identity (3.38). Applying the l.h.s. of the identity on $T_b \cdot T_c$,

$$\cdot \circ (S \otimes \text{id}) \circ \Delta(T_b \cdot T_c) = \cdot \circ (S \otimes \text{id})(T_b \cdot T_c \otimes T_b \cdot T_c) = \cdot \circ \left( S(T_b \cdot T_c) \otimes T_b \cdot T_c \right)$$

$$= S(T_b \cdot T_c) \cdot (T_b \cdot T_c),$$

and the r.h.s.

$$\eta \circ \epsilon(T_b \cdot T_c) = \eta(\epsilon(T_b)\epsilon(T_c)) = \eta(1) = 1,$$

so that the identity reads

$$S(T_b \cdot T_c) = (T_b \cdot T_c)^{-1} = T_c^{-1} \cdot T_b^{-1} = S(T_c) \cdot S(T_b),$$

showing the anti-algebraic nature of the antipode. Just in a similar manner can we show generally

$$S(g_1 \cdots g_n) = S(g_n) \cdots S(g_1), \quad g_i = T_b \quad \text{or} \quad (1)^{\mathbb{F}}.$$

Applying next on $T_b \cdot Q_A^\text{lat}$, the l.h.s. is

$$\cdot \circ (S \otimes \text{id}) \circ \Delta(T_b \cdot Q_A^\text{lat}) = \cdot \circ (S \otimes \text{id})(T_b \cdot Q_A^\text{lat} \otimes T_b \cdot T_a^\text{lat} + T_b \cdot (-1)^{\mathbb{F}} \cdot T_a^\text{lat} \otimes T_b \cdot Q_A^\text{lat})$$

$$= S(T_b \cdot Q_A^\text{lat}) \cdot T_b \cdot T_a^\text{lat} + S(T_b \cdot (-1)^{\mathbb{F}} \cdot T_a^\text{lat}) \cdot T_b \cdot Q_A^\text{lat},$$

while the r.h.s. is

$$\eta \circ \epsilon(T_b \cdot Q_A^\text{lat}) = \eta(\epsilon(T_b)\epsilon(Q_A^\text{lat})) = \eta(0) = 0,$$

thus the identity gives

$$S(T_b \cdot Q_A^\text{lat}) = -S(T_b \cdot (-1)^{\mathbb{F}} \cdot T_a^\text{lat}) \cdot T_b \cdot Q_A^\text{lat} \cdot T_a^{-1} \cdot T_b^{-1} = -T_a^{-1} \cdot (-1)^{\mathbb{F}} \cdot Q_A^\text{lat} \cdot T_a^{-1} \cdot T_b^{-1} = S(Q_A^\text{lat}) \cdot S(T_b).$$

Now we proceed to the calculation for $Q_A^\text{lat} \cdot Q_B^\text{lat}$: the l.h.s. reads

$$\cdot \circ (S \otimes \text{id}) \circ \Delta(Q_A^\text{lat} \otimes Q_B^\text{lat})$$

$$= \cdot \circ (S \otimes \text{id})(Q_A^\text{lat} \cdot Q_B^\text{lat} \otimes T_a^\text{lat} \cdot T_b^\text{lat} + Q_A^\text{lat} \cdot (-1)^{\mathbb{F}} \cdot T_a^\text{lat} \otimes T_b^\text{lat})$$

$$+ (-1)^{\mathbb{F}} \cdot T_a^\text{lat} \cdot Q_B^\text{lat} \otimes Q_A^\text{lat} \cdot T_a^\text{lat} + (-1)^{\mathbb{F}} \cdot T_a^\text{lat} \cdot (-1)^{\mathbb{F}} \cdot T_a^\text{lat} \otimes Q_A^\text{lat} \otimes Q_B^\text{lat}$$

$$= S(Q_A^\text{lat} \cdot Q_B^\text{lat}) \cdot T_a^\text{lat} \cdot T_b^\text{lat} + S(Q_A^\text{lat} \cdot (-1)^{\mathbb{F}} \cdot T_a^\text{lat}) \cdot T_a^\text{lat} \cdot Q_B^\text{lat}$$

$$+ S((-1)^{\mathbb{F}} \cdot T_a^\text{lat} \cdot Q_B^\text{lat}) \cdot Q_A^\text{lat} \cdot T_a^\text{lat} + S((-1)^{\mathbb{F}} \cdot T_a^\text{lat} \cdot (-1)^{\mathbb{F}} \cdot T_b^\text{lat}) \cdot Q_A^\text{lat} \cdot Q_B^\text{lat},$$

and the r.h.s.

$$\eta \circ \epsilon(Q_A^\text{lat} \cdot Q_B^\text{lat}) = \eta \circ (\epsilon(Q_A^\text{lat})\epsilon(Q_B^\text{lat})) = \eta(0) = 0,$$

so that the identity requires that

$$S(Q_A^\text{lat} \cdot Q_B^\text{lat}) \cdot T_a^\text{lat} \cdot T_b^\text{lat} = T_a^{-1} \cdot (-1)^{\mathbb{F}} \cdot T_a^{-1} \cdot (-1)^{\mathbb{F}} \cdot Q_A^\text{lat} \cdot T_a^{-1} \cdot T_a^\text{lat} \cdot Q_B^\text{lat}$$

$$+ T_a^{-1} \cdot (-1)^{\mathbb{F}} \cdot Q_B^\text{lat} \cdot T_a^{-1} \cdot T_a^\text{lat} \cdot (-1)^{\mathbb{F}} \cdot Q_A^\text{lat} \cdot T_a^\text{lat}$$

$$- T_a^{-1} \cdot (-1)^{\mathbb{F}} \cdot T_a^{-1} \cdot (-1)^{\mathbb{F}} \cdot Q_A^\text{lat} \cdot Q_B^\text{lat},$$

i.e.

$$S(Q_A^\text{lat} \cdot Q_B^\text{lat}) = T_a^{-1} \cdot (-1)^{\mathbb{F}} \cdot Q_B^\text{lat} \cdot T_a^{-1} \cdot T_a^\text{lat} \cdot (-1)^{\mathbb{F}} \cdot Q_A^\text{lat} \cdot T_a^\text{lat} = S(Q_B^\text{lat}) \cdot S(Q_A^\text{lat}),$$

thus we find that the anti-algebraic nature of the antipode map holds regardless of the fermionic nature of the supercharges. The other identity $\cdot \circ (\text{id} \otimes S) = \eta \circ \epsilon$ also gives the same consequences.
3.2 Shift Structure as a Braiding

Here we explain why we need the braiding or shift structure in the space of fields, beginning with a simple illustration. Suppose we are considering a normal supersymmetry with a bosonic field $\varphi$ and a fermionic field $\psi$. Needless to say, bosonic fields commute with any other fields, while fermionic fields anticommute only with other fermions. Now take a supertransformation $Q\varphi = \chi$ with a normal supercharge $Q$ which is supposed to obey the Leibniz rule $Q(\varphi_1\varphi_2) = Q\varphi_1\varphi_2 + (-1)^{|\varphi_1|}\varphi_1Q\varphi_2$. In the Hopf algebraic description we may say that it has the coproduct $\Delta(Q) = Q \otimes 1 + (-1)^F 1 \otimes Q$ as before. We “know” that the field $\chi$ is fermionic, as a field of supertransformation of a boson $\varphi$. The point is that this fact is indeed inevitable; we are forced to chose $\chi$ to be fermionic for the algebraic consistency. In fact, note that the quantity $Q(\phi\psi) = \chi\psi + \phi(Q\psi)$ is equal to $Q(\psi\phi) = (Q\psi)\phi - \psi\chi$, because $\phi$ is defined as a boson, i.e. $\phi\psi = \psi\phi$. Comparing these two relations, we find $\chi\psi + \psi\chi = (Q\psi)\phi - \phi(Q\psi)$, which is zero again due to that $\phi$ is bosonic. This results in that $\chi\psi - \psi\chi$, “proving” that $\chi$ is a fermion. The essence for this proof is twofold: the one is the coproduct structure of the transformation operator $Q$, especially the factor $(-1)^F$, and the other is the covariance of exchanging fields under the transformation $Q$, namely, when we exchange the order of a product of fields and then apply the transformation with $Q$, the result is the same as the quantity obtained by first applying the transformation with $Q$ on the product and then exchanging the order of the transformed object. The property of fields under exchanging the order in a product is nothing but the statistics of the fields. Here we have just seen a natural and obvious fact that the statistics of fields should be consistent to the algebraic structure and covariance of transformations which apply on the fields. It might be still worth stressing it, however, because it is the reason we need the braiding for our present application of the Hopf algebraic symmetry on fields. It is also the reason of that we think of the braiding as giving a generalized statistics. We are going to investigate these issues in detail in the following.

Let us introduce the general notion of exchanging the order of fields. We denote the exchanged object of $\varphi_1 \otimes \varphi_2$ as

$$\Psi_{X_1, X_2} (\varphi_1 \otimes \varphi_2), \quad \varphi_1 \in X_1, \ \varphi_2 \in X_2. \quad (3.56)$$

The map $\Psi$ is called a braiding when it satisfies some natural consistency conditions (see appendix A). The trivial braiding is given as the normal transposition, and, in the application to the link formalism, we assume that the scalar fields on sites of the lattice would have the trivial braiding nature;

$$\Psi_{X_s, X_s} (\phi_1 \otimes \phi_2) = \phi_2 \otimes \phi_1, \quad \phi_1, \ \phi_2 \in X_s : \text{scalar fields on sites.} \quad (3.57)$$

Repeating the argument above, we may apply $Q_{\text{lat}}^A$ on the product of scalar fields, or equivalently, take the action of coproduct of $Q_{\text{lat}}^A$ as

$$\Delta(Q_{\text{lat}}^A) \triangleright (\phi_1 \otimes \phi_2) = (Q_{\text{lat}}^A \phi_1) \otimes \phi_2 (x + a_1^r) + \phi_1 (x + a_1^l) \otimes (Q_{\text{lat}}^A \phi_2). \quad (3.58)$$
Similarly on the exchanged product,

\[
\Delta(Q^\text{lat}_A) \triangleright (\phi_2 \otimes \phi_1) = \Delta(Q^\text{lat}_A) \triangleright \Psi_{X_A,X_1}(\phi_1 \otimes \phi_2)
\]

\[
= (Q^\text{lat}_A \phi_2) \otimes \phi_1(x + a^r_A) + \phi_2(x + a^l_A) \otimes (Q^\text{lat}_A \phi_1).
\]

(3.59)

We now assume the covariance of the braiding under symmetry transformations, or, in other words, we assume that the braiding to be an intertwiner of the transformations. In the present case, this requires that

\[
\Delta(Q^\text{lat}_A) \triangleright \Psi_{X_A,X_1}(\phi_1 \otimes \phi_2) = \Psi'(\Delta(Q^\text{lat}_A) \triangleright (\phi_1 \otimes \phi_2)).
\]

(3.60)

The l.h.s. is given by (3.59), while the r.h.s. is

\[
\Psi'(\Delta(Q^\text{lat}_A) \triangleright (\phi_1 \otimes \phi_2)) = \Psi_{X_{1A},X_1}(Q^\text{lat}_A \phi_1 \otimes \phi_2(x + a^r_A))
\]

\[
+ \Psi_{X_A,X_{1A}}(\phi_1(x + a^l_A) \otimes (Q^\text{lat}_A \phi_2)),
\]

(3.61)

where we have denoted the space of fermionic fields of the index \( A \) as \( X_{1A} \) to which the transformed fields \( Q^\text{lat}_A \phi_{1,2} \) are to belong. Comparing these two equations, and noting that the fields \( \phi_1 \) and \( \phi_2 \) could be completely independent, we find the consequence, with a simple identification, should be

\[
\Psi_{X_{1A},X_1}(Q^\text{lat}_A \phi_1 \otimes \phi_2(x + a^r_A)) = \phi_2(x + a^l_A) \otimes (Q^\text{lat}_A \phi_1),
\]

\[
\Psi_{X_A,X_{1A}}(\phi_1(x + a^l_A) \otimes (Q^\text{lat}_A \phi_2)) = (Q^\text{lat}_A \phi_2) \otimes \phi_1(x + a^r_A).
\]

(3.62)

These are not the trivial braiding as in (3.57). Instead, these braiding mean that when we exchange the order of the fermion \( Q^\text{lat}_A \phi \) with the other field, it changes the argument of the other field by the amount \( a^l_A - a^r_A \) under the exchange from the left to the right, and by the opposite amount under the exchange from the right to the left. Recalling that the scalar fields obey the trivial braiding, we might interpret this fact as that the transformed fields, fermions, inherited the nontrivial braiding nature from the supercharge, which, in a way, shows the nontrivial braiding already in the structure of the coproduct. In fact, this kind of nontrivial braiding is referred to as the shifted commutation structure in the link formalism.

We have to emphasize here that the “claimed inconsistency” [42] explained in section 2 no longer appears with incorporating this nontrivial braiding in the non-gauged link formalism. Our approach which is purely based on the Hopf algebraic description clarifies the necessity of the braiding and shows how that problem criticized can be resolved.

To confirm how the things work, let us compute another example:

\[
\Delta(Q^\text{lat}_B) \triangleright \psi_{1A}(x) \otimes \phi_2(x) = (Q^\text{lat}_B \psi_{1A}(x) \otimes \phi_2(x + a^r_B)) - \psi_{1A}(x + a^l_B) \otimes \psi_{2B}(x),
\]

(3.63)

where \( \psi_{2B} := Q^\text{lat}_B \phi_2 \), and thus

\[
\Psi'(\Delta(Q^\text{lat}_B) \triangleright \psi_{1A}(x) \otimes \phi_2(x)) = \Psi_{X_{1B},X_1}(Q^\text{lat}_B \psi_{1A}(x) \otimes \phi_2(x + a^r_B))
\]

\[
- \Psi_{X_{1A},X_{1B}}(\psi_{1A}(x + a^l_B) \otimes \psi_{2B}(x)),
\]

(3.64)

whereas

\[
\Delta(Q^\text{lat}_B) \triangleright (\phi_2(x + a^l_A - a^r_A) \otimes \psi_{1A}(x)) = \Delta(Q^\text{lat}_B) \triangleright \Psi_{X_{1A},X_1}(\psi_{1A}(x) \otimes \phi_2(x))
\]

\[
= \psi_{2B}(x + a^l_A - a^r_A) \otimes \psi_{1A}(x + a^l_B)
\]

\[
+ \phi_2(x + a^l_A - a^r_A + a^l_B) \otimes (Q^\text{lat}_B \psi_{1A}(x)).
\]

(3.65)
Here $X_{AB}$ is such that $Q_B^\text{lat}\psi_A \in X_{AB}$. Assuming again the covariance
\[ \Psi'(\Delta(Q_B^\text{lat}) \triangleright (\psi_{1A}(x) \otimes \phi_2(x))) = \Delta(Q_B^\text{lat}) \triangleright \Psi_{X_{1A}X_s}(\psi_{1A}(x) \otimes \phi_2(x)), \]  
we obtain the following braiding relations:
\[
\begin{align*}
\Psi_{X_{AB},X_1}(Q_B^\text{lat}\psi_1A(x) \otimes \phi_2(x + a_B^r) & = \phi_2(x + a_A^1 - a_A^r + a_B^1) \otimes (Q_B^\text{lat}\psi_1A(x)), \\
\Psi_{X_{1A},X_B}(\psi_{1A}(x + a_B^1) & \otimes \psi_2B(x)) = -\psi_2B(x + a_A^1 - a_A^r) \otimes \psi_{1A}(x + a_B^1).
\end{align*}
\]  
Notice, in passing, from the first equation of \[3.67\], we have
\[
\begin{align*}
\Psi_{X_{AB},X_1}((Q_B^\text{lat}\psi_1A(x) \otimes \phi_2(x)) & = \phi_2(x + a_A^1 - a_A^r + a_B^1 - a_B^r) \otimes (Q_B^\text{lat}\psi_1A(x)), \\
\Psi_{X_{AB},X_1}((Q_A^\text{lat}\psi_1B(x) \otimes \phi_2(x)) & = \phi_2(x + a_B^1 - a_B^r + a_A^1 - a_A^r) \otimes (Q_A^\text{lat}\psi_1B(x)), \\
\end{align*}
\]  
so that, summing up these two,
\[
\Psi_{X_{AB},X_1}((Q_B^\text{lat}\psi_1A(x) \otimes \phi_2(x)) = 2\tau_{AB}^\mu \Psi_{X_{AB},X_1}(P_\mu^\text{lat}\phi_1(x) \otimes \phi_2(x))
\]  
where we have used the abbreviation $\varphi_Ao\ldots A_p : = Q_A^\text{lat} \cdots Q_A^\text{lat}\varphi_A$, which could just vanish, where $\varphi_A := \phi$. If we had introduced a scalar field which itself has nontrivial braiding/shift structure, this relation would have even been generalized.

The exchanging of a product of more than three fields should be naturally introduced. In the case of the trivial braiding,
\[
\Psi_{X_1\otimes X_2,X_3}((\phi_1 \otimes \phi_2) \otimes \phi_3) = \phi_3 \otimes (\phi_1 \otimes \phi_2) = \phi_3 \otimes \phi_1 \otimes \phi_2 = \Psi_{X_{1},X_3}((\phi_1 \otimes \phi_3) \otimes \phi_2)
\]  
so that in the general case we extend it to
\[
\Psi_{X_1\otimes X_2,X_3} = \Psi_{X_1,X_3} \circ \Psi_{X_2,X_3}, \quad \Psi_{X_1,X_2\otimes X_3} = \Psi_{X_{1},X_3} \circ \Psi_{X_{1},X_2}.
\]  
For example,
\[
\Psi_{X_1,X_2\otimes X_3}(\phi_1(x) \otimes \phi_2(x) \otimes \psi_{3A}(x)) = \phi_2(x) \otimes \psi_{3A}(x) \otimes \phi_1(x - a_A^r + a_A^l).
\]  
For the exchanging with the trivial or constant fields, we should impose
\[
\Psi_{X_0^1,X} = \Psi_{X,X_0} = \text{id},
\]  
where $X_0^1$ denotes the space of trivial bosonic fields. Using these rules, let us calculate one more example:
\[
(id \otimes \Delta) \circ \Delta(Q_A^\text{lat}) \triangleright (\phi_1 \otimes \phi_2 \otimes \psi_{3B})(x + a_A^l) = \psi_{1A}(x) \otimes \phi_2(x + a_A^l) \psi_{3B}(x + a_A^l)
\]
\[
+ \phi_1(x + a_A^l) \otimes \psi_{2A}(x) \otimes \psi_{3B}(x + a_A^l)
\]
\[
+ \phi_1(x + a_A^l) \otimes \phi_2(x + a_A^l) \otimes (Q_A^\text{lat}\psi_{3B})(x),
\]
These examples show that the braiding, i.e. the amount of shifts of the arguments of fields induced under exchanging, is additive; for a field \( \phi \).

\[ \text{We may thus introduce, in addition to the normal graded structure of fields, i.e. bosonic and fermionic statistics, the graded structure which we call the shift structure so that the space of elementary fields} X \text{ is decomposed in general as} \]

\[ X = \bigoplus_{\text{grading}} X_e \oplus X_o. \]  

(3.78)

The space of whole fields, \( \hat{X} \), is also decomposed with respect to the shift/grading structure the same way;

\[ \hat{X} = \bigoplus_{n=0}^{\infty} \bigoplus_{\text{grading}} X^n. \]

(3.79)

The field contents and their shift structure are determined in each model, mainly with the use of the Leibniz rule consistency conditions. We have to emphasize that this grading structure is especially crucial to define the explicit form of the “momentum” operator \( P_{\mu}^{\text{lat}} \). As mentioned at the beginning of the previous subsection, we might have started with taking a difference operator as its representation: \( (P_{\mu}^{\text{lat}} \phi)(x) = a^{-1}(\phi(x + a^{\dagger}_\mu) - \phi(x + a^\mu)) \). This, however, doesn’t satisfy the relation (3.69), since we have assumed that \( \phi \) obeys a trivial braiding and thus \( a^{-1}(\phi(x + a^{\dagger}_\mu) - \phi(x + a^\mu)) \) has the same trivial braiding. We thus need an expression like \( (P_{\mu}^{\text{lat}} \phi')(x) = a^{-1}(\phi'(x + a^{\dagger}_\mu) - \phi'(x + a^\mu)) \) for which \( \phi' \) has an additional grading to satisfy the relation (3.69). To give a consistent representation for them is important for the formulation and will be treated elsewhere. Here our claim is that the algebraic description presented here can still formalize a field theory with the Hopf algebraic symmetry even if we don’t have the
explicit representation for these graded fields and their “momentum” operators, as is seen in what follows.

Let us note also that our braiding satisfies that
\[ \Psi_{X_1,X_2} \circ \Psi_{X_2,X_1} = \text{id}, \]
(3.80)
or equivalently,
\[ \Psi_{X_2,X_1} = \Psi_{X_1,X_2}^{-1}. \]
(3.81)
In a standard mathematical terminology this kind of exchanging map \( \Psi \) isn’t referred to as a braiding, or one may distinguish it from the strictly braided case. Here we use the term braiding in a broader sense, allowing a type of simple nature (3.80). We emphasize that it is still nontrivial in the sense that \( \Psi \neq \tau \), where \( \tau \) is the simple transposition: \( \tau(\varphi_1 \otimes \varphi_2) = \varphi_2 \otimes \varphi_1 \). In fact, our braiding is a transposition plus some shifts of the arguments of fields up to the statistics factors. This should be compared with the statistics of usual bosons and fermions; for that case the braid is nothing but the simple exchanging up to the statistics. We could therefore describe these facts as that the fields which represent our Hopf algebraic \( \text{lattice superalgebra} \) naturally obtain a braiding structure which expresses slightly more generalized statistics than the usual one.

According to the general discussion (see appendix A), it seems that the simple braiding structure (3.80) might be given as an explicit formula (A.20) when the corresponding Hopf algebra is triangular. We find that this is indeed the case at least formally; our symmetry algebra could be identified as a triangular Hopf algebra with an additional grading structure, and the braiding (3.70) be given with the corresponding (quasi-)triangular structure \( R \). To see this, let us first introduce a formal expression for the shift operator \( T_b \)
\[ T_b = \exp(b^\mu \partial_\mu). \]
(3.82)
We write this as if the continuum derivative operator \( \partial_\mu \) were introduced on the lattice; however it must be understood as a formal operator and only well-defined when exponentiated to give the lattice proper operator \( T_b \). We may impose
\[ \Delta(\partial_\mu) = \partial_\mu \otimes 1 + 1 \otimes \partial_\mu, \quad \epsilon(\partial_\mu) = 0, \quad S(\partial_\mu) = -\partial_\mu, \]
(3.83)
which should be interpreted as formal equivalents of the relations (3.82) and (3.83) for \( T_b \). We then recall that the generator \( Q_A^{\text{lat}} \) has a kind of grading as an amount of the shift \( a_A := a_A^1 - a_A^r \) induced under exchanging \( Q_A^{\text{lat}} \varphi \) with other fields. We may express this fact with introducing another operator \( L^\mu \) such that
\[ a[L^\mu, Q_A^{\text{lat}}] = (a_A)^\mu Q_A^{\text{lat}}, \quad \text{i.e.} \quad [L^\mu, Q_A^{\text{lat}}] = l_A^\mu Q_A^{\text{lat}}, \]
(3.84)
where \( l_A^\mu = a^{-1}(a_A)^\mu \). Since \( P_\mu^{\text{lat}} \) is given as \( P_\mu^{\text{lat}} \sim \{Q_A^{\text{lat}}, Q_B^{\text{lat}}\} \), it also has the grading as in
\[ a[L^\mu, P_\nu^{\text{lat}}] = a_P(\hat{\nu})^\mu P_\nu^{\text{lat}} = a_P \delta_\nu^\mu P_\nu^{\text{lat}}, \quad \text{i.e.} \quad [L^\mu, P_\nu^{\text{lat}}] = l_P \delta_\nu^\mu P_\nu^{\text{lat}}, \]
(3.85)
where \( a_P := a^1 - a^r \) and \( l_P := a^{-1}a_P \). We list the other relations
\[ [L^\mu, T_b] = [L^\mu, (-1)^F] = [L^\mu, L^\nu] = 0, \]
(3.86)
where the first two are due to the fact that neither \( T_b \) nor \( (-1)^F \) induces shift and the latter one is automatic because of the “Abelian” nature of (3.81), (3.85) and the others. For completeness, we set
\[ \Delta(L^\mu) = L^\mu \otimes 1 + 1 \otimes L^\mu, \quad \epsilon(L^\mu) = 0, \quad S(L^\mu) = -L^\mu. \]
(3.87)
\(^9\)A well-known example of generalized statistics is that of anyons, for which the exchanging map is strictly braided in general. Our statistics is thus more like the usual statistics than the anyonic one.
Now let
\[ R := \exp(aL^\mu \otimes \partial_\mu - a\partial_\mu \otimes L^\mu + i\pi F \otimes F). \] (3.88)

We can show that this formal operator \( R \in \mathcal{U}(\mathcal{A}) \otimes \mathcal{U}(\mathcal{A}) \) is invertible and satisfies the relations
\[
\tau \circ \Delta(h) = R \cdot \Delta(h) \cdot R^{-1}, \\
(\Delta \otimes \text{id}) R = R_{13}R_{23}, \quad (\text{id} \otimes \Delta) R = R_{13}R_{12}.
\] (3.89)

(See appendix A for the notation.) Notice first that \( R^{-1} \) is given as
\[
R^{-1} = \exp(-aL^\mu \otimes \partial_\mu + a\partial_\mu \otimes L^\mu + i\pi F \otimes F)
\] (recall that \( F \) only gives integer numbers), and so that
\[
R_{21} = \exp(a\partial_\mu \otimes L^\mu - aL^\mu \otimes \partial_\mu + i\pi F \otimes F) = R^{-1}.
\] (3.91)

As for the first relation in (3.89), compute
\[
R \cdot \Delta(h) \cdot R^{-1} = \sum_{n=0}^{\infty} \frac{1}{n!}(\text{ad}(L))^n \Delta(h),
\] (3.92)

where we have written \( L := aL^\mu \otimes \partial_\mu - a\partial_\mu \otimes L^\mu + i\pi F \otimes F \) just for simplicity, and used \( \text{ad} \) to denote the Lie derivative. For \( h = Q_{\text{lat}}^A \),
\[
\text{ad}(L) \Delta(Q_{\text{lat}}^A) = [aL^\mu \otimes \partial_\mu, -a\partial_\mu \otimes L^\mu + i\pi F \otimes F, Q_{\text{lat}}^A \otimes T_{a'_A} + (-1)^F \cdot T_{a^+_A} \otimes Q_{\text{lat}}^A] \\
= [aL^\mu \otimes \partial_\mu + i\pi F \otimes F, Q_{\text{lat}}^A \otimes T_{a'_A} + a\partial_\mu \otimes L^\mu + i\pi F \otimes F, (-1)^F \cdot T_{a^+_A} \otimes Q_{\text{lat}}^A] \\
= a[L^\mu, Q_{\text{lat}}^A] \otimes \partial_\mu \cdot T_{a'_A} + i\pi F, Q_{\text{lat}}^A \otimes [F, Q_{\text{lat}}^A] + (-a\partial_\mu \otimes L^\mu + i\pi F \otimes F, (-1)^F \cdot T_{a^+_A} \otimes Q_{\text{lat}}^A] \\
= Q_{\text{lat}}^A \otimes (a\partial_\mu \otimes i\pi F) \cdot T_{a'_A} + (-a\partial_\mu \otimes i\pi F) \cdot (-1)^F T_{a^+_A} \otimes Q_{\text{lat}}^A,
\] (3.93)

so that
\[
(\text{ad}(L))^n \Delta(Q_{\text{lat}}^A) = Q_{\text{lat}}^A \otimes ((a\partial_\mu \otimes i\pi F)^n \cdot T_{a'_A} + (-a\partial_\mu \otimes i\pi F)^n \cdot (-1)^F \cdot T_{a^+_A} \otimes Q_{\text{lat}}^A).
\] (3.94)

We therefore obtain
\[
R \cdot \Delta(Q_{\text{lat}}^A) \cdot R^{-1} \\
= Q_{\text{lat}}^A \otimes \exp((a\partial_\mu \otimes i\pi F) \cdot T_{a'_A} + \exp(-a\partial_\mu \otimes i\pi F) \cdot (-1)^F \cdot T_{a^+_A} \otimes Q_{\text{lat}}^A) \\
= Q_{\text{lat}}^A \otimes (-1)^F \cdot T_{a^+_A} \otimes T_{a'_A} \otimes Q_{\text{lat}}^A = \tau \circ \Delta(Q_{\text{lat}}^A),
\] (3.95)

since
\[
\exp(\pm (a\partial_\mu \otimes i\pi F) ) = \exp(\pm (a^1_{\text{lat}} \otimes \partial_\mu) \cdot \exp(\mp (a^1_{\text{lat}} \otimes \partial_\mu) \cdot \exp(i\pi F) = T_{a^+_A} \cdot T_{a'_A} \cdot (-1)^F.
\] (3.96)

A simpler calculation leads to similar result for \( h = P_{\mu}^\text{lat} \) too. For \( h = T_{b}, (-1)^F, L^\mu \), it is rather clear that
\[
R \cdot \Delta(h) \cdot R^{-1} = \Delta(h) = \tau \circ \Delta(h).
\] (3.97)
Thus the first equation in \((3.89)\) indeed holds for the choice \((3.88)\) of \(\mathcal{R}\). The second relation follows as
\[
(\Delta \otimes \text{id})\mathcal{R} = \exp(a\Delta(L^\mu) \otimes \partial_\mu - a\Delta(\partial_\mu) \otimes L^\mu + i\pi\Delta(\mathcal{F}) \otimes \mathcal{F})
\]
\[
= \exp(aL^\mu \otimes 1 \otimes \partial_\mu - a\partial_\mu \otimes 1 \otimes L^\mu + i\pi\mathcal{F} \otimes 1 \otimes \mathcal{F})
\]
\[
\quad + a1 \otimes L^\mu \otimes \partial_\mu - a1 \otimes \partial_\mu \otimes L^\mu + i\pi1 \otimes \mathcal{F} \otimes \mathcal{F})
\]
\[
= \exp(aL^\mu \otimes 1 \otimes \partial_\mu - a\partial_\mu \otimes 1 \otimes L^\mu + i\pi\mathcal{F} \otimes 1 \otimes \mathcal{F})
\]
\[
\quad \cdot \exp(a1 \otimes L^\mu \otimes \partial_\mu - a1 \otimes \partial_\mu \otimes L^\mu + i\pi1 \otimes \mathcal{F} \otimes \mathcal{F}) = \mathcal{R}_{13} \cdot \mathcal{R}_{23}.
\]
The third one is almost the same.

We have thus shown that the formal operator \(\mathcal{R}\) given as \((3.88)\) is a quasitriangular structure and, due to \((3.91)\), our lattice superalgebra is identified as a triangular Hopf algebra. The whole spaces of fields, as representation spaces of a triangular Hopf algebra, would be braided by \(\mathcal{R}\) as in
\[
\Psi = \tau \circ \mathcal{R} \triangleright,
\]
which agrees with our formula \((3.70)\) as now seen. We need the representation of \(L^\mu\) on the elementary fields. First for normal scalar fields \(\{\phi, \ldots\}\) let
\[
L^\mu \triangleright \phi = 0 \cdot \phi = 0.
\]
(3.100)
For the other fields in the irreducible supermultiplet to which the above bosonic fields belong, the actions of \(L^\mu\) are automatically determined by the algebra \((3.84)\). For instance, on \(\psi_{A} := Q_{A}^{\text{lat}} \phi\), we find
\[
L^\mu \triangleright \psi_{A} = L^\mu \triangleright (Q_{A}^{\text{lat}} \phi) = ([L^\mu, Q_{A}^{\text{lat}}] + Q_{A}^{\text{lat}} \cdot L^\mu) \triangleright \phi = l_{A}^{\mu} Q_{A}^{\text{lat}} \phi = l_{A}^{\mu} \psi_{A}.
\]
(3.101)
Then inductively, we find for \(\varphi_{A_{1} \cdots A_{n}} = Q_{A_{1}}^{\text{lat}} \cdots Q_{A_{n}}^{\text{lat}} \phi\) that
\[
L^\mu \triangleright \varphi_{A_{1} \cdots A_{n}} = \big(l_{A_{1}} + \cdots + l_{A_{n}}\big)^{\mu} \varphi_{A_{1} \cdots A_{n}}.
\]
(3.102)
These relations express explicitly the grading structure of fields explained above. We thus compute
\[
\mathcal{R} \triangleright (\varphi_{A_{1} \cdots A_{p}}(x) \otimes \varphi_{B_{1} \cdots B_{q}}(y))
\]
\[
= \exp\left(1 \otimes ((a_{A_{1}} + \cdots + a_{A_{p}})^{\mu} \partial_\mu - (a_{B_{1}} + \cdots + a_{B_{q}})^{\mu} \partial_\mu) \otimes 1 + i\pi pq 1 \otimes 1 \right)
\]
\[
\triangleright (\varphi_{A_{1} \cdots A_{p}}(x) \otimes \varphi_{B_{1} \cdots B_{q}}(y))
\]
\[
= (-1)^{pq} (1 \otimes T_{a_{A_{1}} + \cdots + a_{A_{p}}}) (T_{-1}^{a_{B_{1}} + \cdots + a_{B_{q}}} \otimes 1) \triangleright (\varphi_{A_{1} \cdots A_{p}}(x) \otimes \varphi_{B_{1} \cdots B_{q}}(y))
\]
\[
= \varphi_{A_{1} \cdots A_{p}} \left(x - \sum_{i=1}^{p} a_{B_{i}}\right) \otimes \varphi_{B_{1} \cdots B_{q}} \left(y + \sum_{i=1}^{q} a_{A_{i}}\right).
\]
(3.103)
Since here \(a_{A_{i}} = a_{A_{i}}^{1} - a_{A_{i}}^{-1}\), etc., we have shown that the equation \((3.99)\) does reproduce the general braiding rule \((3.70)\).

It is worth pointing out that our quasitriangular structure \(\mathcal{R}\) can be written as
\[
\mathcal{R} = \chi_{21} \cdot \mathcal{R}_{0} \cdot \chi^{-1}, \quad \mathcal{R}_{0} := \exp(i\pi \mathcal{F} \otimes \mathcal{F}),
\]
(3.104)
with some invertible operator \(\chi \in \mathcal{U}(A) \otimes \mathcal{U}(A)\) which satisfies so-called the 2-cocycle condition
\[
(\chi \otimes 1) \cdot (\Delta \otimes \text{id}) \chi = (1 \otimes \chi) \cdot (\text{id} \otimes \Delta) \chi,
\]
(3.105)
and the counital condition

\[(\epsilon \otimes \text{id})\chi = (\text{id} \otimes \epsilon)\chi = 1.\] (3.106)

Such an operator is not necessarily unique. We take one specific example to illustrate it:

\[\chi := \exp(\partial_\mu \otimes L_1^\mu + a L^\mu \otimes \partial_\mu),\]

\[\chi_{21} = \exp(a L_1^\mu \otimes \partial_\mu + a \partial_\mu \otimes L^\mu),\] \[\chi^{-1} = \exp(-a \partial_\mu \otimes L_1^\mu - a L^\mu \otimes \partial_\mu),\] (3.107)

where we have introduced two more operators \(L_1^\mu\) and \(L^\mu\) such that \(L^\mu = L_1^\mu - L^\mu\), namely,

\[a[L_1^\mu, Q_1^{\text{lat}}] = (a_1^\mu)^\mu Q_1^{\text{lat}},\] etc., \[\] (3.108)

with coproduct, counit and antipode formulae similar to those of \(L^\mu\). It is easy to see that \(3.103\) actually holds for this operator \(\chi\). The cocycle condition is fulfilled as

\[(\text{l.h.s.}) = \exp(a \partial_\mu \otimes L^\mu \otimes 1 + a L^\mu \otimes \partial_\mu \otimes 1) \cdot \exp(a \Delta(\partial_\mu) \otimes L^\mu + a \Delta(L^\mu) \otimes \partial_\mu)
\]

\[= \exp(a \partial_\mu \otimes L^\mu \otimes 1 + a L^\mu \otimes \partial_\mu \otimes 1
\]

\[+ a \partial_\mu \otimes 1 \otimes L^\mu + a 1 \otimes \partial_\mu \otimes L^\mu + a L^\mu \otimes 1 \otimes \partial_\mu + a 1 \otimes L^\mu \otimes \partial_\mu)
\]

\[= \exp(1 \otimes a \partial_\mu \otimes L^\mu + 1 \otimes a L^\mu \otimes \partial_\mu)
\]

\[= \exp(1 \otimes a \partial_\mu \otimes L^\mu + 1 \otimes a L^\mu \otimes \partial_\mu)\cdot \exp(a \partial_\mu \otimes \Delta(L^\mu) + a L^\mu \otimes \Delta(\partial_\mu))
\]

\[= (\text{r.h.s.}),\] (3.109)

while the counitality is clear because \(\epsilon(\partial_\mu) = \epsilon(L_1^\mu) = 0\). We thus conclude from these results that our lattice superalgebra \(U(A)\) with the quasitriangular structure \(\mathcal{R}\) could be understood as so-called the twist by the cocycle element \(\chi\) of some other Hopf algebra \(U(A)_0\) with the simple quasitriangular structure \(\mathcal{R}_0\). The "untwisted" Hopf algebra \(U(A)_0\) has the same algebra and counit as those of \(U(A)\) but its coproduct and antipode are such that

\[\Delta(h) = \chi \cdot \Delta_0(h) \cdot \chi^{-1},\]

\[S(h) = U \cdot S_0(h) \cdot U^{-1},\] \[U := (\text{id} \otimes S)\chi,\] \[U^{-1} = (S \otimes \text{id})\chi^{-1}.\] (3.110)

Thus for \(h = T_b, (-1)^\mathcal{F}, L^\mu\), we find \(\Delta_0(h) = \Delta(h)\), whereas for \(h = Q_\text{lat}^A, P_\mu^\text{lat}\), we can show that

\[\Delta_0(Q_\text{lat}^A) = Q_\text{lat}^A \otimes 1 + (-1)^\mathcal{F} \otimes Q_\text{lat}^A,\]

\[\Delta_0(P_\mu^\text{lat}) = P_\mu^\text{lat} \otimes 1 + 1 \otimes P_\mu^\text{lat}.\] (3.111)

Since in the present case

\[U = \exp(-a L_+^\mu \cdot \partial_\mu),\]

\[U^{-1} = \exp(a L_+^\mu \cdot \partial_\mu),\]

\[L_+^\mu := L_1^\mu + L^\mu,\] (3.112)

antipodes as well remain unchanged for \(h = T_b, (-1)^\mathcal{F}, L_1^\mu\), but changed again for \(h = Q_\text{lat}^A, P_\mu^\text{lat}\):

\[S_0(Q_\text{lat}^A) = -(-1)^\mathcal{F} \cdot Q_\text{lat}^A,\]

\[S_0(P_\mu^\text{lat}) = -P_\mu^\text{lat},\] (3.113)

as seen with the use of

\[U \cdot Q_\text{lat}^A \cdot U^{-1} = \exp((a_1^\mu + a_\mu^\tau) \mu \partial_\mu) \cdot Q_\text{lat}^A = T_{\partial_\mu}^A \cdot T_{a_\mu^\tau}^A \cdot Q_\text{lat}^A\] (3.114)

and of similar for \(P_\mu^\text{lat}\).
We have found that the (un)twisted Hopf algebra \((\mathcal{U}(A)_\theta, \mathcal{R}_0)\) becomes much simpler and has the form of a normal universal enveloping Lie superalgebra of normal supersymmetry. This result might seem confusing because under the twisting the algebraic structure of the original Hopf algebra remains the same and operators themselves don’t take any transformations; if such simpler Hopf algebra exits, could we just begin with it without taking the deformed one \((\mathcal{U}(A), \mathcal{R})\)? Actually we can equally formulate the whole story with the simpler Hopf algebra \((\mathcal{U}(A)_\theta, \mathcal{R}_0)\), but notice that this twisting transformation is only possible with the nontrivial “charge” or “grading” operators \(L^{1,x} \mu\) at our disposal, and that the twisted Hopf algebra keeps them as well. On our original Hopf algebra \((\mathcal{U}(A), \mathcal{R})\), these have a natural interpretation as those assigning how fields are geometrically put on the lattice and how operators affect on such a geometrical structure. On the twisted Hopf algebra \((\mathcal{U}(A)_\theta, \mathcal{R}_0)\), this kind of interpretation is less clear since \(\Delta_0, S_0, \text{etc.}\), just have normal structure and, nevertheless, these operators \(L^{1,x} \mu\) must be included for the whole algebra to be represented exactly. This last observation would be quite crucial, particularly when compared with the no-go theorem presented in \([43]\), since in the twisted algebra the “momentum” operator obeys the exact, not modified, Leibniz rule for \(\hat{X}\) (see appendix \([A]\)), \(h \in \mathcal{U}(A)_\theta\) can act covariantly on products of fields for \(\hat{X}_\theta\) only with the product

\[
m_0 := m \circ \chi \triangleright \tag{3.115}
\]

(note that the twisting from \((\mathcal{U}(A), \mathcal{R})\) to \((\mathcal{U}(A)_\theta, \mathcal{R}_0)\) is given with \(\chi^{-1}\)), as in

\[
h \triangleright m_0(\varphi \otimes \varphi') = m_0(\Delta_0(h) \triangleright (\varphi \otimes \varphi')). \tag{3.116}
\]

Suppose that this product \(m_0\) is “commutative” in the sense that

\[
m_0 \circ \Psi_0 = m_0, \quad \text{i.e.} \quad m_0 \circ \tau \circ \mathcal{R}_0 \triangleright = m_0, \tag{3.117}
\]

which means commutative up to the statistics factor induced by \(\mathcal{R}_0\). This assumption would be natural because the twisted algebra \((\mathcal{U}(A)_\theta, \mathcal{R}_0)\) has the simple Hopf algebraic structure which is symmetric under exchanging orders of any objects. It turns out that then the multiplication \(m\) is again commutative up to the nontrivial statistics \(\Psi\) (thus noncommutative in the standard sense):

\[
m \circ \Psi = m_0 \circ \chi^{-1} \triangleright \circ \tau \circ \mathcal{R} \triangleright = m_0 \circ \tau \circ (\chi_2^{-1} \cdot \mathcal{R} \cdot \chi) \triangleright \circ \chi^{-1} \triangleright = m_0 \circ \Psi_0 \circ \chi^{-1} \triangleright = m_0 \circ \chi^{-1} \triangleright = m. \tag{3.118}
\]

This consequence in a way shows that multiplication rule should incorporate the statistics in the obvious manner so that it becomes commutative up to the statistics. When the statistics is itself nontrivial, this notion of the commutativity up to the statistics may be expressed as just a noncommutativity in the standard sense. In our case, we have

\[
\varphi_{A_1 \ldots A_p}(x) \cdot \varphi_{B_1 \ldots B_q}(y) = (-1)^{pq} \varphi_{B_1 \ldots B_q}(y + \sum_{i=1}^{p} a_{A_i}) \cdot \varphi_{A_1 \ldots A_p} \left( x - \sum_{i=1}^{q} a_{B_i} \right). \tag{3.119}
\]

We regard it as the consequence of either the lattice-deformed statistics, or the mild noncommutativity, and may use the notation \(\varphi * \varphi'\) to emphasize its noncommutative nature.
We finally recall that the space of fields on the lattice $\hat{X}$, defined in (3.1), forms an algebra. It actually forms a Hopf algebra in a natural way \[41, 54]\:

$$
m(\varphi_1 \otimes \varphi_2) = \varphi_1 \cdot \varphi_2 \quad \text{(product)},
\eta(1) = 1 \quad \text{(unit)},
\Delta(\varphi) = \varphi \otimes 1 + 1 \otimes \varphi, \quad \Delta(1) = 1 \otimes 1 \quad \text{(coproduct)},
\epsilon(\varphi) = 0, \quad \epsilon(1) = 1 \quad \text{(counit)},
S(\varphi) = -\varphi, \quad S(1) = 1 \quad \text{(antipode)},
$$

(3.120)

where $\varphi \in X$. This Hopf algebraic structure shouldn’t be confused with that of the symmetry operators $U(A)$ acting on $\hat{X}$. In addition to these Hopf algebraic structure, the space $\hat{X}$ has the braiding/shift structure $\Psi$ which obeys the consistency conditions (3.72) and (3.74). With the use of the braiding, the Hopf algebraic structure is extended to the whole field space $\hat{X}$; coproduct, counit, and antipode of a product of two elementary fields $\varphi_1, \varphi_2 \in X$ are defined by

$$
\Delta(\varphi_1 \cdot \varphi_2) := (m \otimes m) \circ (\text{id} \otimes \Psi \otimes \text{id})(\Delta(\varphi_1) \otimes \Delta(\varphi_2)),
\epsilon(\varphi_1 \cdot \varphi_2) := \epsilon(\varphi_1) \epsilon(\varphi_2),
S(\varphi_1 \cdot \varphi_2) := m \circ \Psi(S(\varphi_1) \otimes S(\varphi_2)),
$$

(3.121)

and generalized inductively to any products in $\hat{X}$. One of the most crucial nature for this braiding structure is that it must be covariant under the symmetry operations. In fact we recall that the braiding structure is inevitable only for the covariant consistency under the Hopf algebraic symmetry: $a \triangleright \circ \Psi = \Psi \circ a \triangleright$, $a \in U(A)$. With all these properties, the space $\hat{X}$ is called a braided Hopf algebra, or, more precisely, Hopf algebra in a braided category. We thus claim that the link formalism naturally treats the space of fields as a braided Hopf algebra with a Hopf algebraic symmetry, for which the general BQFT formalism can apply. We now see this application in the next subsection.

### 3.3 Perturbative Definition of Supersymmetry on the Lattice as a Braided Quantum Field Theory

Following the general theory of BQFT given in \[41\], we are now constructing a lattice theory which has the Hopf algebraic symmetry introduced in the previous subsections. Before going to concrete examples, let us here briefly review the general framework. The crucial ingredient to define a quantum field theory is the path integral. For defining a perturbation theory it is enough to introduce it as a formal Gaussian integral, such that the total functional derivative under it is supposed to be zero. We therefore need the functional derivative, which is defined as below.

We now introduce the functional derivative with respect to $\varphi \in X$ as in

$$
\frac{\delta}{\delta \varphi(x)} \varphi(y) = \delta^D(x - y). \quad (3.122)
$$

Following more abstract definition in \[41\], we write this as

$$
\text{ev} \left( \frac{\delta}{\delta \varphi(x)} \otimes \varphi(y) \right) := \frac{\delta}{\delta \varphi(x)} \varphi(y), \quad (3.123)
$$

introducing the evaluation map $\text{ev}$. It is a kind of natural contraction of $X$ and $X^*$, where $X^*$ is the dual space to $X$ composed of $\delta/\delta \varphi$. Similarly we might introduce the opposite one, a kind of completeness relation, as

$$
\text{coev}(\lambda) := \lambda \sum_x \varphi(x) \otimes \frac{\delta}{\delta \varphi(x)}. \quad (3.124)
$$

29
These maps are characterized with the identities:

\[(ev \otimes id)(id \otimes coev) = id_{X^*}, \quad (id \otimes ev)(coev \otimes id) = id_X.\]  

(3.125)

The functional derivative can be naturally extended to the one which acts on the whole space of fields \(\hat{X}\) as in the following way. On a product of two elementary fields \(\varphi_1, \varphi_2 \in X\), the functional derivative acts with the use of a braided Leibniz rule as in

\[
\frac{\delta}{\delta \varphi(x)} (\varphi_1(x_1) \cdot \varphi_2(x_2)) = \frac{\delta}{\delta \varphi(x)} \varphi_1(x_1) \cdot \varphi_2(x_2) + \left[ \Psi^{-1} \left( \frac{\delta}{\delta \varphi(x)} \otimes \varphi_1(x_1) \right) \right] \left( 1 \otimes \varphi_2(x_2) \right).
\]

(3.126)

On products of more than three fields are extended inductively. Needless to say, the derivative trivially commutes with a constant field (see (3.74)), and gives zero when it acts on a constant. More rigorous definition of the functional derivative is given in \([41, 54]\).

Now we can introduce a Gaussian integration with the following property:

\[
\int \frac{\delta}{\delta \varphi} \left( O[\varphi] e^{-S_0} \right) = 0, \quad O[\varphi] \in \hat{X}, \quad \frac{\delta}{\delta \varphi} \in X^*,
\]

(3.127)

where \(\exp(-S_0) \in \hat{X}\) is the corresponding Gaussian factor. In the application to the field theory, \(S_0\) is interpreted as the free part of the action. Notice that this integration is formally understood as the one which satisfies the property (3.127) without referring to its real values. This way of abstract definition is already enough to define a perturbation theory and to compute correlation functions with arbitrary order, since for such computations only the ratio of the integral to another integral, partition function, is needed (this is nothing different from the path integral in a usual field theory), and that ratio can be computed only with these algebraic properties.

We now introduce a kind of propagator. Letting

\[
\frac{\delta}{\delta \varphi(x)} e^{-S_0} = -\gamma \left( \frac{\delta}{\delta \varphi(x)} \right) e^{-S_0},
\]

(3.128)

we define an object \(\gamma: X^* \to X\). More specifically, it is given as

\[
\gamma \left( \frac{\delta}{\delta \varphi(x)} \right) = \frac{\delta}{\delta \varphi(x)} S_0,
\]

(3.129)

which roughly corresponds to the inverse propagator, so that the propagator is in a way given as \(\gamma^{-1}\). This naïve argument can be justified shortly.

The free \(n\)-point correlation function is now defined by

\[
Z^{(0)}_n(\alpha_n) := \frac{\int \alpha_n e^{-S_0}}{\int e^{-S_0}}, \quad \alpha_n \in X^n.
\]

(3.130)

The superscript \((0)\) stands for the free theory. In this definition, the denominator, denoted here tentatively as \(Z^{(0)}\), might be interpreted as the free partition function, but in the general case we don’t have any definition to directly compute it as mentioned above. Still this definition is enough to calculate the correlation functions of any order. To see this argument, notice first that

\[
\alpha_n \varphi e^{-S_0} = \alpha_n \gamma^{-1}(\varphi) e^{-S_0} = -\alpha_n \gamma^{-1}(\varphi) (e^{-S_0}),
\]

(3.131)

where we have used the definition (3.128) and the fact that \(\gamma^{-1}(\varphi) \in X^*\) and so is a functional derivative. We then find, using the braided Leibniz rule, that

\[
-\alpha_n \gamma^{-1}(\varphi) (e^{-S_0}) = -\left( \gamma^{-1}(\varphi) \alpha_n^{\varphi} e^{-S_0} \right) = -\gamma^{-1}(\varphi) \alpha_n^{\varphi} e^{-S_0} - \gamma^{-1}(\varphi) \alpha_n^{\varphi} e^{-S_0}
\]

(3.132)
where we have denoted the “shifted” field as \( \varphi^{\alpha_n} \) and \( \alpha_n^e \), with the superscripts implying the amount of shifts\(^{10}\). We thus find
\[
\int \alpha_n \varphi e^{-S_0} = - \int \left( \gamma^{-1}(\varphi^{\alpha_n})(\alpha_n^e e^{-S_0}) - \gamma^{-1}(\varphi^{\alpha_n})(\alpha_n^e) e^{-S_0} \right) = \int \gamma^{-1}(\varphi^{\alpha_n})(\alpha_n^e) e^{-S_0}.
\]
(3.133)

In the second equality, the first term vanishes because its a total derivative under the path integral. We therefore obtain a basic formula
\[
Z^{(0)}(\alpha_n \varphi) = Z^{(0)}(\gamma^{-1}(\varphi^{\alpha_n})(\alpha_n^e)).
\]
(3.134)

For example, putting \( \alpha = 1 \) \((n = 0)\) in the formula above (3.134), it is clear that
\[
Z_1^{(0)}(\varphi) = 0.
\]
(3.135)

The simplest nontrivial example is given for \( n = 1 \) by taking \( \alpha_1 = \varphi_1(x_1) \in X \) and \( \varphi = \varphi_2(x_2) \in X \) in (3.134), so that
\[
Z_2^{(0)}(\varphi_1(x_1)\varphi_2(x_2)) = Z_2^{(0)}(\gamma^{-1}(\varphi_2(x_2 + a_{\varphi_1}))(\varphi_1(x + a_{\varphi_2}))) = \gamma^{-1}(\varphi_2(x_2 + a_{\varphi_1}))(\varphi_1(x + a_{\varphi_2})).
\]
(3.136)

The other formulae can be computed inductively using (3.134). The general results are summarized as follows:
\[
Z_2^{(0)} = ev \circ (\gamma^{-1} \otimes id) \circ \Psi,
\]
(3.137)
\[
Z_{2n}^{(0)} = (Z_2^{(0)})^n \circ [2n - 1]_{\Psi} !!,
\]
(3.138)
\[
Z_{2n+1}^{(0)} = 0,
\]
(3.139)

where
\[
[2n - 1]_{\Psi} !! := ([1]_{\Psi} \otimes id^{2n-1}) \circ ([3]_{\Psi} \otimes id^{2n-3}) \circ \cdots \circ ([2n - 1]_{\Psi} \otimes id),
\]
(3.140)
\[
[n]_{\Psi} := id^n + id^{n-2} \otimes \Psi^{-1} + \cdots + \Psi_{1,n-1}^{-1}.
\]

These are the Wick’s theorem in the BQFT formalism.

When an interaction is turned on, we can treat the theory perturbatively. Let the action be \( S = S_0 + \lambda S_{int} \). The \( n \)-point correlation function now reads
\[
Z_n(\alpha_n) := \int \frac{\alpha_n e^{-S_0}}{e^{-S}} = \frac{\int \alpha_n e^{-S_0} (1 - \lambda S_{int} + \cdots) e^{-S_0}}{(1 - \lambda S_{int} + \cdots) e^{-S_0}}, \quad \alpha_n \in X^n.
\]
(3.141)

Dividing both the numerator and denominator by the “partition function” \( Z^{(0)} \), we find
\[
Z_n = \frac{Z_n^{(0)} - \lambda Z_{n+k}^{(0)} \circ (id^n \otimes S_{int}) + \frac{1}{2} \lambda^2 Z_{n+2k}^{(0)} \circ (id^n \otimes S_{int} \otimes S_{int}) + \cdots}{1 - \lambda Z_k^{(0)} \circ S_{int} + \frac{1}{2} \lambda^2 Z_{2k}^{(0)} \circ (S_{int} \otimes S_{int}) + \cdots},
\]
(3.142)

\(^{10}\)This notational simplicity can only apply to our present case for the specific braiding/shift structure. The general expression with general braiding \( \Psi \) is given in [11].
where \( k \) is the order of the interaction \( S_{\text{int}} \), i.e. \( S_{\text{int}} \in X^k \), and we have put a map \( S_{\text{int}} : \mathbb{C} \rightarrow X^k \) with the abuse of notation.

Let us give an example to see the formalism above more explicitly. We here consider \( \mathcal{N} = (2,2) \) Wess–Zumino model in two dimensions in the Dirac-Kähler twisted basis. Superalgebra is given as before

\[
\{ Q_{\text{lat}}^\mu, Q_{\text{lat}}^\nu \} = i \partial_{+\mu}, \quad \{ \tilde{Q}_{\text{lat}}^\mu, Q_{\text{lat}}^\nu \} = -i \epsilon_{\mu\nu} \partial_{-\nu}.
\]

(3.143)

Bosonic fields include scalars \( \phi, \sigma \) and auxiliary fields \( \tilde{\phi}, \tilde{\sigma} \), whereas fermionic fields are \( \psi, \tilde{\psi}, \psi_{\mu} \).

Supertransformations are given in Appendix B. The action is given as

\[
S = \sum_x \left[ (\partial_{+\mu}^\text{lat} \sigma)(x - a\hat{\mu})(\partial_{-\mu}^\text{lat} \phi)(x) - \tilde{\sigma}(x + a_1 + a_2) \cdot \tilde{\phi}(x) - i\psi(x - a) \cdot \partial_{-\mu}^\text{lat} \psi_{\mu}(x) - i\epsilon_{\mu\nu} \tilde{\psi}(x - \tilde{a}) \cdot \partial_{+\mu}^\text{lat} \psi_{\nu}(x) - \partial_{\sigma} W(x + a_1 + a_2) \cdot \tilde{\phi}(x) - \partial_{\sigma} V(x + a + \tilde{a}) \cdot \tilde{\sigma}(x) + \partial_{\sigma}^2 V(x + a + \tilde{a}) \cdot \tilde{\psi}(x + \tilde{a}) \cdot \tilde{\psi}(x) - i\epsilon_{\mu\nu} \partial_{\sigma}^2 W(x + a_1 + a_2) \cdot \psi_{\mu}(x + a_\nu) \cdot \psi_{\nu}(x) \right],
\]

(3.144)

where \( W \) and \( V \) are potentials in the twisted basis. The invariance of the action can be unambiguously seen using the modified Leibniz rule taking care of the specific “staggered” configurations of arguments of the fields as well as of the mildly generalized statistics (3.119).

### 3.4 Ward–Takahashi Identities

Here we follow [54]. The invariance of the correlation functions can be written as

\[
Z_n(a \triangleright \chi) = \epsilon(a) Z_n(\chi), \quad a \in \mathcal{U}(A), \quad \chi \in X^n,
\]

(3.145)

which is the Ward–Takahashi identity corresponding to the Hopf algebraic symmetry \( \mathcal{U}(A) \). Just as in the usual field theory, the invariance of the correlation functions follows from the invariance of the action. One obvious difference from the usual case is that, with the nontrivial braiding, the symmetry operators must act on the fields in a manner consistent to the braiding structure. In fact it is shown that the identity (3.145) follows when the following four conditions are satisfied [54]:

1. Invariance of the free action:

\[
a \triangleright \gamma^{-1}(\varphi) = \gamma^{-1}(a \triangleright \varphi).
\]

(3.146)

2. Invariance of the interaction:

\[
a \triangleright S_{\text{int}} = \epsilon(a) S_{\text{int}}.
\]

(3.147)

3. Covariance of the braiding:

\[
\Psi(a \triangleright (X_1 \otimes X_2)) = a \triangleright \Psi(X_1 \otimes X_2).
\]

(3.148)

4. Invariance of the delta function:

\[
ev(a \triangleright (X^* \otimes X)) = \epsilon(a) ev(X^* \otimes X).
\]

(3.149)

In our current application, the general formula of Ward–Takahashi identity (3.145) naturally gives the correct identities on the lattice. It is important that the general formula (3.145) can be proved unambiguously only using the algebraic relations.
3.5 Nonperturbative Definition?

In this section, we first extracted the essential requirements for the symmetry operators in the link formalism, concluding that this symmetry is Hopf algebraic. Then we utilized the general framework of BQFT formulated in [41], showing that supersymmetric theory on a lattice in the link formalism can be treated with a formal definition of path integral. This path integral approach, however, only gives a perturbative formulation in general, due to the lack of explicit definition of the path integral. As a field theory on a lattice, this situation wouldn’t be satisfactory at all, especially for the application to numerical simulations. It is known in some cases one can define a “braided integral” explicitly [56]. We might be able to apply such an approach to the current problem to define a rigorous path integral on the lattice, which, if possible, should give the nonperturbative definition in this formulation based on the Hopf algebraic symmetry. As we have shown in subsection 3.2, it is also crucial to accommodate the explicit representation of grading nature for the lattice momentum operator.

4 Conclusion and Discussion

We have shown how the link formalism is treated as a field theory on a lattice with deformed or modified algebraic symmetry. The deformation of the algebra is indeed identified as the one naturally treated in the framework of the Hopf algebra. We showed this argument explicitly, defining the corresponding Hopf algebraic structures of the supersymmetry algebra for the link formalism. The modified Leibniz rule, which is the crucial notion in the original link formalism, was incorporated as the coproduct structure of the Hopf algebra, whose consistency is assured with the other relevant structures of the algebra. The Hopf algebra introduced this way in fact turned out to be a (quasi)triangular Hopf algebra, which has a nontrivial universal $R$-matrix. When represented on the space of fields, this quasitriangular structure inevitably induce a nontrivial statistics, or a noncommutativity, which is the key ingredient for the consistent representation. With these algebraic descriptions, we could identify the link formalism as a representation theory of a quasitriangular Hopf algebra. On the other hand, it is known that there is a general scheme to construct a quantum field theory which has a Hopf algebraic symmetry, called braided quantum field theory. We applied this general formulation to the link formalism. The construction is purely algebraic. In particular, it defines a path integral using only algebraic properties. One can show that it still gives a well-defined perturbative description of the theory, providing full methods for calculating correlation functions in any order. It also gives a concise formulation to derive the possible Ward–Takahashi identities corresponding to the Hopf algebraic symmetry. We therefore realized the link formalism as a quantum field theory which has the quasitriangular Hopf algebraic symmetry at least in the perturbative sense.

From the consistency of the Hopf algebraic structure, it is required that the lattice momentum operator which is proportional to the difference operator should carry a grading compatible with the shifting nature of the difference operator. In this paper we have not given a concrete representation of this grading structure which may be needed to give an explicit nonperturbative definition of this formulation. We leave this issue for the future investigation.

The algebraic inconsistency pointed out in [12], which is connected with the ordering ambiguity of component fields when applying supersymmetry transformation, is solved by the introduction of braiding structure according to the notion of coproduct for the lattice super charges and the momentum operator in Hopf algebra.

It is then important to ask the question how the continuum limit of this formulation is realized. If one can formulate the braided quantum field theory which respects the Hopf algebraic structure as a concrete representation for modified path integral, the twisted lattice supersymmetry will be kept in the continuum limit since the lattice twisted supersymmetry is exactly
kept. As we have shown the lattice supersymmetry is kept in the perturbative level of braided quantum field theory. It is still nontrivial question how the symmetry is recovered even in the nonperturbative level. In any case we expect that fine tuning is not needed to keep the supersymmetry in the continuum limit if the formulation of deformed supersymmetry algebra is concretely constructed.

In the formulation of orbifold construction of lattice field theories only a subset of lattice super charges in particular the nilpotent scalar super charge which corresponds to the shiftless charge in the link construction is exactly preserved on the lattice [28, 29, 30, 31, 32, 33, 34, 35, 36]. The lattice super algebra in this case is identified as the same as the continuum twisted supersymmetry algebra. It was stressed that the super charges carrying a shift break the lattice super symmetry in the sense of the continuum twisted superalgebra [34, 35]. Our claim in this paper is that these supercharges carrying the shift may break the continuum twisted supersymmetry but preserve exactly the Hopf algebraic supersymmetry. Thus in the link approach all the lattice super charges are claimed to preserve exactly in the framework of Hopf algebraic supersymmetry. The supersymmetry algebra is deformed from the continuum twisted supersymmetry to Hopf algebraic supersymmetry.

We have not considered the gauge extension of deformed supersymmetry in this paper. It was pointed out that there is similar ordering ambiguity for the lattice super Yang-Mills formulation of link approach [42]. We consider that this problem can be solved similar as non-gauge case by identifying the lattice supersymmetry with gauge symmetry of link approach as Hopf algebraic symmetry. There is, however, yet another problem in the gauge extension; the loss of the gauge invariance due to the link nature of the lattice super charges. A possible solution was proposed by introducing covariantly constant super parameters \( \eta_A \): 

\[
\{ \nabla_B, \eta_A \} = 0
\]

where \( \nabla_B \) is super covariant derivative. This is highly nontrivial relation in the sense that the fermionic parameter \( \eta_A \) carrying a shift should carry an internal space-time dependence caused by the super covariant derivative \( \nabla_B \) to keep the covariant constancy. Here we may consider that fermionic link variables are defined on the links of internal space-time. In other words the space-time distortion of internal space-time may compensate the required dependence of the fermionic parameter. There is a possibility that gravity may play a role in these questions.

It has been pointed out that the breakdown of the Leibniz rule for the lattice difference operator is inevitable under reasonable assumptions for algebraic property on the lattice [43]. Recent renormalization group analyses confirms this statement from different point of view [27]. In order to realize supersymmetry algebra which includes the momentum operator on the lattice it is most natural to introduce the difference operator in the lattice supersymmetry algebra. While the exact supersymmetry of continuum supersymmetry algebra was realized only for the nilpotent super charge which is the scalar part of the twisted supersymmetry algebra but does not include the crucial momentum dependence. We claim that the deformation of the Lie algebraic continuum supersymmetry to Hopf algebraic supersymmetry on the lattice is inevitable to accommodate the difference operator in the algebra.

It is obviously very important to find concrete representation of the Hopf algebraic supersymmetry algebra on the lattice to obtain "modified" path integral definition of the QFT with this particular braiding structure. As we have already shown this type of mild noncommutativity with shifting nature may be well accommodated by a matrix formulation of lattice noncommutativity [40, 57]. This part of concrete proposal with the necessary formulation of graded momentum lattice operator will be given elsewhere. It would be also interesting to compare the formulation of the link approach with other noncommutative approach [58] and non-lattice formulations [59, 60].
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A Brief Summary of Hopf Algebra

Here we briefly list the axioms of Hopf algebra and some related notions which are used in this article. For rigorous and complete descriptions, see, for example, [62, 63, 61].

A.1 Hopf Algebra

A Hopf algebra over a field $k (= \mathbb{C} \text{ or } \mathbb{R})$ is a vector space $H$ over $k$ which has the following properties 1, 2, 3 and 4.

1. $H$ is a unital associative algebra, so that

- it has a $k$-linear multiplication (or product) map\[11\]

$$
\cdot : H \otimes H \to H, \quad (h_1 \otimes h_2) = h_1 \cdot h_2, \quad (A.1)
$$

which is associative

$$
\cdot \circ (\cdot \otimes \text{id}) = \cdot \circ (\text{id} \otimes \cdot), \quad \text{i.e.} \quad (h_1 \cdot h_2) \cdot h_3 = h_1 \cdot (h_2 \cdot h_3); \quad (A.2)
$$

- it has unit element $1$ which satisfies $1 \cdot h = h \cdot 1 = h$, whose existence can be formally expressed as the existence of a $k$-linear map

$$
\eta : k \to H, \quad \eta(\lambda) = \lambda 1, \quad \lambda \in k. \quad (A.3)
$$

2. $H$ is a coalgebra. Namely,

- it has a $k$-linear map called coproduct:

$$
\Delta : H \to H \otimes H, \quad \Delta(h) = \sum_i h_i^{(1)} \otimes h_i^{(2)}, \quad h_i^{(1)}, h_i^{(2)} \in H, \quad (A.4)
$$

which satisfies the coassociativity\[12\]

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta, \quad \text{i.e.} \quad h_{(1)}^{(1)} \otimes h_{(1)}^{(2)} \otimes h_{(2)}^{(1)} = h_{(1)}^{(1)} \otimes h_{(2)}^{(1)} \otimes h_{(2)}^{(2)}; \quad (A.5)
$$

- it has also another $k$-linear map called counit

$$
\epsilon : H \to k, \quad (A.6)
$$

which obeys the relation

$$
(\epsilon \otimes \text{id}) \circ \Delta = (\text{id} \otimes \epsilon) \circ \Delta = \text{id}, \quad \text{i.e.} \quad \epsilon(h_{(1)}) h_{(2)} = \epsilon(h_{(2)}) h_{(1)} = h. \quad (A.7)
$$

3. These structures of algebra and coalgebra are compatible with each other. Namely,

\[11\text{In what follows we take, unless otherwise specified, } h, h_1, h_2, \ldots , \text{ to be arbitrary elements of } H.
\[12\text{We use below much simpler abbreviation } \Delta(h) = h_{(1)} \otimes h_{(2)} \text{ known as the Sweedler’s notation.}
• the coproduct and the counit are both algebra maps:

\[ \Delta(h_1 \cdot h_2) = \Delta(h_1) \cdot \Delta(h_2), \quad \epsilon(h_1 \cdot h_2) = \epsilon(h_1)\epsilon(h_2). \quad (A.8) \]

4. \( H \) has one more map called \textit{antipode}:

• it has a \( k \)-linear map

\[ S : H \to H, \quad (A.9) \]

which obeys the identity

\[ \cdot(S \otimes \text{id}) \circ \Delta = \cdot(\text{id} \otimes S) \circ \Delta = \eta \circ \epsilon, \quad \text{i.e.} \quad S(h_{(1)}) \cdot h_{(2)} = h_{(1)} \cdot S(h_{(2)}) = \epsilon(h) \mathbb{1}. \quad (A.10) \]

If the \( k \)-linear space \( H \) satisfies these properties but not 1, it is called a \textit{bialgebra}.

### A.2 Quasitriangular Structure

A Hopf algebra \( H \) is said to be \textit{quasitriangular} if there exists an invertible element \( \mathcal{R} \in H \otimes H \) which satisfies

\[ \tau \circ \Delta h = \mathcal{R} \cdot (\Delta h) \cdot \mathcal{R}^{-1}, \]

\[ (\Delta \otimes \text{id}) \mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{23}, \quad (\text{id} \otimes \Delta) \mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{12}, \quad (A.11) \]

where

\[ \mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}, \]

\[ \mathcal{R}_{12} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \otimes \mathbb{1}, \quad \mathcal{R}_{13} = \sum \mathcal{R}^{(1)} \otimes \mathbb{1} \otimes \mathcal{R}^{(2)}, \quad \mathcal{R}_{23} = \sum \mathbb{1} \otimes \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}, \quad (A.12) \]

and \( \tau : H \otimes H \to H \otimes H \) is the transposition map

\[ \tau(h_1 \otimes h_2) = h_2 \otimes h_1, \quad h_1, h_2 \in H. \quad (A.13) \]

The element \( \mathcal{R} \), if exists, is called the \textit{quasitriangular structure} or \textit{universal R-matrix}.

If a quasitriangular structure \( \mathcal{R} \) of a quasitriangular Hopf algebra \( H \) obeys further the following condition, the Hopf algebra is said to be \textit{triangular}:

\[ \mathcal{R}_{21} \mathcal{R} = \mathbb{1} \otimes \mathbb{1}, \quad \text{i.e.} \quad \mathcal{R}_{21} = \mathcal{R}^{-1}, \quad \text{where} \quad \mathcal{R}_{21} = \sum \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)}. \quad (A.14) \]

### A.3 Action on Algebras

A (left) \textit{action} of a Hopf algebra \( H \) on an associative algebra \( X \) is a representation \( \rho : H \to \text{Lin}(X) \), where \( \text{Lin}(X) \) is the algebra of linear maps on \( X \), which satisfies the covariance in the following sense

\[ \left\{ \begin{array}{ll}
 h \triangleright (\varphi \cdot \varphi') &= m(\Delta(h) \triangleright (\varphi \otimes \varphi')), \\
 h \triangleright \mathbb{1} &= \epsilon(h) \mathbb{1},
\end{array} \right. \quad \varphi, \varphi' \in X. \quad (A.15) \]

We have here introduced the notation \( h \triangleright \varphi := \rho(h)(\varphi) \), the product \( m \) of \( X \) with the abbreviation \( \varphi \cdot \varphi' := m(\varphi \otimes \varphi') \), and the unit \( \mathbb{1} \in X \).
A.4 Braiding

Let us consider a formal collection of representation spaces \((1, X, Y, Z, \cdots)\) of a Hopf algebra \(H\) together with the collection of tensor products of the representation spaces \((1 \otimes X \cong X \otimes 1 \cong X, X \otimes Y, (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z), \cdots)\) for which \(H\) acts with the coproduct structure \(((\Delta h) \triangleright (\varphi \otimes \chi), h \in H, \varphi \in X, \chi \in Y)\). If there exists an invertible intertwiner (isomorphism)

\[
\Psi_{X,Y} : X \otimes Y \to Y \otimes X, \quad \Psi_{X,Y}(\Delta(h) \triangleright (\varphi \otimes \chi)) = \Delta(h) \triangleright \Psi_{X,Y}(\varphi \otimes \chi)
\]

(A.16)

with the properties

\[
\Psi_{X \otimes Y, Z} = \Psi_{X,Z} \circ \Psi_{Y,Z}, \quad \Psi_{X,Y \otimes Z} = \Psi_{X,Z} \circ \Psi_{X,Y},
\]

(A.17)

it unambiguously relates the two representations on \(X \otimes Y\) and \(Y \otimes X\). It should be compatible with any maps which intertwine the representation spaces as in

\[
\Psi_{Z,W} \circ (g_{XZ} \otimes g_{YW}) = (g_{YW} \otimes g_{XZ}) \circ \Psi_{X,Y}, \quad g_{XZ} : X \to Z, \quad g_{YW} : Y \to W.
\]

(A.18)

We call this isomorphism \(\Psi\) a braid. Strictly speaking, a braid should be such that

\[
\Psi \circ \Psi \neq \text{id}, \quad \text{or} \quad \Psi_{X,Y} \neq \Psi_{Y,X}^{-1},
\]

(A.19)

which means there are two distinct ways in relating \(X \otimes Y\) to \(Y \otimes X\). It gives a nontrivial rule of exchanging factors of a tensor product, and generalizes the statistics of the representation spaces. If, on the other hand, it satisfies \(\Psi \circ \Psi = \text{id}\), the isomorphism is more like a simple transposition and said to be symmetric.

When the Hopf algebra \(H\) is quasitriangular, we can express the braiding more explicitly using the quasitriangular structure \(R\) of \(H\) and the transposition map \(\tau\) as in

\[
\Psi_{X,X'}(\varphi \otimes \varphi') = \tau \circ R \triangleright (\varphi \otimes \varphi'), \quad \varphi \in X, \varphi' \in X'.
\]

(A.20)

This indeed becomes an invertible intertwiner (A.16) and satisfies the conditions (A.17) and (A.18). We find that

\[
\Psi_{X,X'} \circ \Psi_{X',X} = \tau \circ R \triangleright (\tau \circ R \triangleright) = \tau((R \cdot R_{21}) \triangleright \circ \tau).
\]

Thus the condition (A.19), that is for \(\Psi\) to be strictly braided, is equivalent to

\[
R \cdot R_{21} \neq 1 \otimes 1,
\]

(A.21)

namely that the universal \(R\)-matrix is really quasitriangular. Equivalently, a symmetric isomorphism \(\Psi\) corresponds to the triangular structure \(R \cdot R_{21} = 1 \otimes 1\).

A.5 Twist

Let \(H\) be a Hopf algebra. An invertible element \(\chi \in H \otimes H\) is called a \((2-)\)cocycle when it satisfies that

\[
(\chi \otimes 1)(\Delta \otimes \text{id})\chi = (1 \otimes \chi)(\text{id} \otimes \Delta)\chi \quad \text{((2-)cocycle condition)}.
\]

(A.22)

A cocycle \(\chi\) is said to be counital if

\[
(\epsilon \otimes \text{id})\chi = 1 \quad \text{and} \quad (\text{id} \otimes \epsilon)\chi = 1 \quad \text{(counital condition)}.
\]

(A.23)

For a quasitriangular Hopf algebra \((H, R)\) and a counital 2-cocycle \(\chi\), there exists a new Hopf algebra \((H_\chi, R_\chi)\) which has

---

\(^{13}\)The notion of braiding would be most suitably defined in terms of category theory. Here instead we just give a simple and intuitive description.

\(^{14}\)Here actually only one of the two conditions is suffice.
• the same algebra and counit as those for \((H,\mathcal{R})\),
• coproduct: \(\Delta_h = \chi(\Delta h)\chi^{-1}\),
• antipode: \(S_h = U(S_h)U^{-1}\), where \(U = \cdot(id \otimes S)\chi, U^{-1} = \cdot(S \otimes id)\chi^{-1}\),
• quasitriangular structure: \(\mathcal{R}_h = \chi_{21}\mathcal{R}\chi^{-1}\), where \(\chi_{21} = \tau(\chi)\) with \(\tau\) given in (A.13).

The process obtaining the Hopf algebra \((H_\chi,\mathcal{R}_\chi)\) from the original one \((H,\mathcal{R})\) is called a twist with the element \(\chi\) called a twist element. If \((H,\mathcal{R})\) is triangular, so is \((H_\chi,\mathcal{R}_\chi)\).

When a Hopf algebra \(H\) acts on an associative algebra \(X\) covariantly as in (A.15), the twisted Hopf algebra \(H_\chi\) with a twist element \(\chi\) acts covariantly on a new algebra \(X_\chi\) with a new product
\[
\varphi \ast \varphi' := m \circ \chi^{-1} \triangleright (\varphi \otimes \varphi')
\]
and with the same unit. The new product \(\ast\) is associative and in general noncommutative even if the original product \(\cdot\) is commutative.

### B \(\mathcal{N} = (2, 2)\) Wess–Zumino Model in Two Dimensions

We list here the explicit supertransformation formulae for \(\mathcal{N} = (2, 2)\) Wess–Zumino model in two dimensions. The superalgebra is
\[
\{Q^{\text{lat}}_1, Q^{\text{lat}}_\mu\} = P^{\text{lat}}_{+\mu}, \quad \{\tilde{Q}^{\text{lat}}_1, Q^{\text{lat}}_\mu\} = -\epsilon_{\mu\nu} P^{\text{lat}}_{-\nu},
\]
with the other commutators just vanishing. The field contents are \(\{\phi, \sigma, \psi, \psi_\mu, \tilde{\psi}, \tilde{\phi}, \tilde{\sigma}\}\), for which the supertransformations are as follows:

- \(Q^{\text{lat}}_1 \phi = 0\), \(Q^{\text{lat}}_\mu \phi = \psi_\mu\), \(\tilde{Q}^{\text{lat}}_1 \phi = 0\),
- \(Q^{\text{lat}}_1 \psi_\mu = i\partial_{+\mu} \phi\), \(Q^{\text{lat}}_\mu \psi_\mu = -\epsilon_{\mu\nu} \tilde{\phi}\), \(\tilde{Q}^{\text{lat}}_1 \psi_\mu = -i\epsilon_{\nu\mu} \partial_{-\mu} \phi\),
- \(Q^{\text{lat}}_1 \tilde{\phi} = -i\epsilon_{\mu\nu} \partial_{+\mu} \psi_\nu\), \(Q^{\text{lat}}_\mu \tilde{\phi} = 0\), \(\tilde{Q}^{\text{lat}}_1 \tilde{\phi} = i\partial_{-\mu} \psi_\mu\),
- \(Q^{\text{lat}}_1 \tilde{\sigma} = -\psi\), \(Q^{\text{lat}}_\mu \tilde{\sigma} = 0\), \(\tilde{Q}^{\text{lat}}_1 \tilde{\sigma} = -\tilde{\psi}\),
- \(Q^{\text{lat}}_1 \tilde{\psi} = 0\), \(Q^{\text{lat}}_\mu \tilde{\psi} = -i\partial_{+\mu} \sigma\), \(\tilde{Q}^{\text{lat}}_1 \tilde{\psi} = -i\partial_{-\mu} \psi_\mu\),
- \(Q^{\text{lat}}_1 \tilde{\sigma} = i\epsilon_{\mu\nu} \partial_{-\nu} \psi\), \(Q^{\text{lat}}_\mu \tilde{\sigma} = i\epsilon_{\mu\nu} \partial_{-\nu} \psi\) + \(i\partial_{+\mu} \tilde{\psi}\).

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