Nekrasov functions from exact Bohr–Sommerfeld periods: the case of $SU(N)$

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Abstract

We suggested in 2009 that the Nekrasov function with one non-vanishing deformation parameter $\epsilon$ is obtained by the standard Seiberg–Witten (SW) contour-integral construction. The only difference is that the SW differential $pdx$ is substituted by its quantized version for the corresponding integrable system, and contour integrals become exact monodromies of the wavefunction. This provides an explicit formulation of the earlier guess by Nekrasov and Shatashvili in 2009. In this paper, we successfully check this suggestion in the first order in $\epsilon^2$ and the first order in instanton expansion for the $SU(N)$ model, where the consistency of the so-deformed SW equations is already non-trivial.

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1. Introduction

Integrability plays a very important role in modern theoretical physics, because effective actions of quantum theories always exhibit integrability properties [3]. The basic reason for this is the freedom to change integration variables in functional integral. If this freedom is preserved on some ‘mini-superspace’ (moduli space) of coupling constants, the universality classes of effective actions are labeled by some simple and well-known integrable system in low spacetime dimensions. Today, there are a number of interesting examples where this phenomenon manifests itself. One of them is the Seiberg–Witten (SW) theory, describing the low-energy effective actions of 4D $\mathcal{N} = 2$ supersymmetric gauge theories [4]: universality classes in this case are labeled by 1D integrable systems [5] like Toda [5, 6], Calogero [7], Ruijsenaars [8] models and spin chains [9]. An alternative description of the SW theory is in terms of the Nekrasov functions [10], which originally appeared from an attempt to perform a regularized integration [11] over instanton moduli spaces with the help of the Duistermaat–Heckman (localization) technique [12]. Nowadays, the Nekrasov functions have
become an important class of special functions in string theory [13], generalizing the ordinary hypergeometric series in a nontrivial way [14], and the AGT conjecture [15] implies that they provide a good starting point to describe at least the entire set of 2D conformal blocks. All this makes the description of the Nekrasov functions in terms of integrability theory an important and urgent problem. Of course, from the general perspective, the Nekrasov functions are fragments of KP–Toda $\tau$-functions, closely related to discrete matrix models [16] and combinatorics of symmetric groups [17]. However, their relevance for the SW theory implies that there should be a relation to a much simpler class of 1D integrable systems. A first guess in this direction was made in a recent paper [2], where it was suggested that introducing the $\epsilon$ parameters corresponds in some way to a direct quantization of the integrability/SW relation of [5]. In [1], we provided an explicit description of this quantization procedure.

The SW theory [4] defines a prepotential $F_{\text{SW}}(\vec{a})$ from the system of equations:

$$a_i = \oint_{A_i} dS^{(0)} = \Pi_{A_i}^{(0)},$$

$$\frac{\partial F_{\text{SW}}(\vec{a})}{\partial a_i} = \oint_{B_i} dS^{(0)} = \Pi_{B_i}^{(0)},$$

where contour integrals are the Bohr–Sommerfeld (BS) periods of an associated 1D integrable system [5]. The claim of [1] is that Nekrasov’s prepotential $F(\vec{a}|\epsilon_1)$ with one $\epsilon$-parameter switched on (in principle, there can be arbitrary many such $\epsilon$-parameters, though Nekrasov [10] discusses just two) is defined by the same system (1), only the BS presymplectic differential $dS^{(0)} \approx \vec{p}d\vec{q}$ is substituted by its exact quantum counterpart: the one which defines the phase of exact wavefunction of the integrable system. To emphasize that the relevant moduli $\vec{a}$ are now different (deformed), we rewrite this system in the slightly different notation:

$$\alpha_i = \oint_{A_i} dS = \Pi_{A_i},$$

$$\frac{\partial F(\vec{a}|\epsilon)}{\partial \alpha_i} = \oint_{B_i} dS = \Pi_{B_i}.$$

The deformed BS periods are nothing but (Abelian) monodromies of the wavefunction.

In [1], we explicitly checked this suggestion (in the lowest orders of various expansions) only in the simplest $SU(2)$ case, when the relevant integrable system is the ordinary sine-Gordon. Though generalizations to $SU(N)$ Toda systems are well known to be straightforward, this is an important check to be done, because for $N > 2$ system (2) could be non-resolvable: no set of periods can be represented as a gradient of something. Consistency of the system cannot be proved with the help of ordinary Riemann’s theorem $T_{ij} = T_{ji}$ as in the case of the original SW theory, because, after the deformation, $dS$ is no longer a SW differential with the property $\delta(dS) = \text{holomorphic}$. Still, a memory of the spectral Riemann surface survives (it actually becomes modified only in the vicinity of ramification points), and we gave a technical argument at the end of [1] in favor of the consistency of (2), and now we check that this system is indeed consistent and, moreover, has $F(\vec{a}|\epsilon_1)$ as its solution. Like [1], we will make this check only in the first orders of expansions in $\epsilon_1^2$ and $\Lambda^{2N}$, and even this calculation is rather cumbersome. A better proof should, of course, be sought.

3 A similar relation of 2D supersymmetric theories and quantum integrable systems can be found in [18].

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To simplify the calculations, we exploit the existing knowledge about the SW theory and the Nekrasov functions as much as possible. Actually, we proceed in the following three steps.

**Step 1. SW periods** $\Pi^{(0)}$ and Nekrasov functions. The $SU(N)$ universality class of SW theory is labeled by a polynomial

$$K(p) = \sum_{k=0}^{N} u_k p^k = u_N \prod_{i=1}^{N} (p - \lambda_i).$$

The SW/Toda spectral curve is given then by

$$K(p) + \gamma \cos \phi = 0, \quad \gamma = \Lambda^N,$$

and the SW differential is

$$dS = p \, d\phi.$$  

The periods $\Pi^{(0)}$ can be calculated in various ways, either directly or with the help of the Picard–Fuchs equations. We, however, take the most economic and transparent way: we calculate $a_i(\vec{\lambda})$ directly from the definition, but take the difficult dual periods from the Nekrasov function

$$F(\vec{a}) = \lim_{\epsilon_1, \epsilon_2 \to 0} \epsilon_1 \epsilon_2 \log Z_{LNS}(\vec{a} | \epsilon_1, \epsilon_2).$$

**Step 2. WKB theory and deformed differential** $dS$. The deformed differential $dS$ is an exact solution to the deformed (quantized) equation (4); see the very last formula in [1]:

$$\left\{ K \left( -i\hbar \frac{\partial}{\partial \phi} \right) + \gamma \cos \phi \right\} \exp \left( \frac{i}{\hbar} \int d\phi \right) = 0,$$

and actually $\hbar = \epsilon_1$. WKB theory [19] provides an expansion of $dS = \sum_{k=0}^{\infty} \hbar^k P_k \, d\phi$, where $P_k = p(\phi)$ is the ‘classical’ momentum, that is, the root of (4), which is single valued on the spectral Riemann surface. A technically reasonable way to calculate the periods of $P_k \, d\phi$ with $k > 0$ is to represent $dS = \hat{O} \, dS^{(0)}$ as an action of some differential operator $\hat{O}$ (acting on parameters $u_i$ and $\gamma$): then $\Pi_C = \hat{O} \, \Pi_C^{(0)}$.

**Step 3. The check of the ‘exact BS’ suggestion of [1].** Finally one

- evaluates the deformed A-periods $\tilde{\alpha}(\vec{\lambda}) = \hat{O}[\tilde{\alpha}(\vec{\lambda})],$
- substitutes these deformed A-periods into the $\alpha$-derivatives of the known Nekrasov function $\mathcal{F}(\tilde{\alpha} | \epsilon_1) = \lim_{\epsilon_2 \to 0} \epsilon_1 \epsilon_2 \log Z_{LNS}(\tilde{\alpha} | \epsilon_1, \epsilon_2),$
- compares the result with deformed B-periods, obtained at step (1) from the $a$-derivatives of the SW–Nekrasov function $F_{SW}(\vec{a}) = \mathcal{F}(\vec{a} | \epsilon_1 = 0)$, i.e., with $\hat{O}[\partial_\alpha F_{SW}(\vec{a})]$.

In other words, we will prove the relation

$$\Pi_B(\hat{O}[\Pi_A^{(0)}(\lambda)]) = \hat{O}[\Pi_B^{(0)}(\Pi_A^{(0)}(\lambda))].$$

extracting $\Pi_B^{(0)}(\alpha)$ and $\Pi_B(\alpha)$ from the Nekrasov functions with vanishing and non-vanishing $\epsilon_1$, respectively, explicitly evaluating $\Pi_A^{(0)}(\lambda)$ and deriving operator $\hat{O}$ from WKB theory.

All these steps are actually easily computerized and higher-order corrections can also be analyzed after that. In this paper, however, we present as many formulae as possible explicitly, without recourse to computer calculations. In fact, there is a close similarity between emerging formulae and those familiar from various matrix-model calculations, especially from [20] and the theory of CIV–DV potentials [21].

We actually begin in section 2 from step 2, then proceed to step 1 in sections 3 and 4 and end with step 3 in section 5.
2. WKB theory and deformed differential dS

2.1. Conjugation of the differential operator

\[ e^{-i \int P \, dt} e^{i \int P \, dt} = p^n - i\hbar \frac{n(n-1)}{2} \dot{p} - \hbar^2 \left( \frac{n(n-1)(n-2)}{6} p^{n-3} \dot{p} + \frac{n(n-1)(n-2)(n-3)}{8} p^{n-4} \ddot{p}^2 \right) + O(\hbar^3), \tag{9} \]

where \( \dot{P} \equiv \partial P \), while prime is reserved for \( P \)-derivatives of \( P \)-dependent functions; see below.

2.2. Schrödinger equation (7) for the differential dS

For

\[ K(z) = \sum_{k=0}^{N} u_k z^k \tag{10} \]

one needs to solve

\[ (K(-i\hbar \partial) + \gamma \cos x) e^{i \int P \, dt} = 0. \tag{11} \]

Making use of (9), this can be rewritten as

\[ K(P) - \frac{i\hbar}{2} K''(P) \dot{P} - \hbar^2 \left( \frac{1}{6} K'''(P) \dot{P} + \frac{1}{8} K''''(P) \ddot{P}^2 \right) = -V(x) = -\gamma \cos x. \tag{12} \]

Substituting

\[ P = p + \hbar P_1 + \hbar^2 P_2 + O(\hbar^3), \tag{13} \]

one obtains

\[ K(p) = -V(x), \]
\[ P_1 = -i \frac{K''(p) \dot{p}}{2K'(p)} = -i \frac{1}{2} \partial (\log K'(p)), \]
\[ P_2 = \left( \frac{3K''}{8K^3} - \frac{K'''K''}{2K^2} + \frac{K'''}{8K^2} \right) \dot{p}^2 + \left( -\frac{K''}{4K^2} + \frac{K'''}{6K} \right) \ddot{p}, \tag{14} \]

\[ \ddots. \]

Here and below \( K \) with omitted argument denotes \( K(p) \), similarly \( K' = K'(p) \) and so on.

From the first equation it follows that

\[ \dot{p} = -\frac{V'}{K'}, \]
\[ \ddots. \]

and

\[ P_2 = \left( \frac{K''}{4K^3} - \frac{K'''}{6K^2} \right) V'' + \left( \frac{5K''}{8K^3} + \frac{2K'''}{3K^4} \right) V'^2. \tag{16} \]
2.3. Simplified expression for contour integrals

For contour integrals integration by parts is allowed, and this allows one to considerably simplify the integral of (16):

$$\Pi_{c}^{(2)} \equiv \hbar^{2} \oint_{c} P_{2} \, dx = \frac{\hbar^{2}}{24} \oint_{c} \left( \frac{K''}{K^{3}} - \frac{K'''}{K^{2}} \right) V'' \, dx. \quad (17)$$

For $K(p) = \frac{1}{2} p^{2} - E$, these formulae reproduce the standard WKB expressions used in [1].

2.4. Exact periods from BS periods and the operator $\hat{O}$

For $V(x) = \gamma \cos x$, one has $V'' = -V$. Further, from $K(p) = -V = -\gamma \cos x$ and (10) it follows that

$$\gamma \frac{\partial p}{\partial \gamma} = -\frac{V}{K'}, \quad \frac{\partial p}{\partial u_{j}} = -p^{j} \frac{1}{K'}, \quad \gamma \frac{\partial^{2} p}{\partial \gamma \partial u_{j}} = -\left( \frac{K''}{K^{3}} p^{j} - \frac{1}{K^{2}} \right) V. \quad (18)$$

and

$$\frac{\hbar^{2} \gamma}{24} \frac{\partial}{\partial \gamma} \left( \sum_{j} j(j-1)u_{j} \frac{\partial}{\partial u_{j-2}} \right) p = -\frac{\hbar^{2}}{24} \left( \frac{K''}{K^{3}} - \frac{K'''}{K^{2}} \right) V. \quad (19)$$

This means that for any closed contour $C$

$$\Pi_{C}^{(0)} + \Pi_{C}^{(2)} = \hat{O} \Pi_{C}^{(0)} = \left( 1 + \frac{\hbar^{2} \gamma}{24} \frac{\partial}{\partial \gamma} \sum_{j} j(j-1)u_{j} \frac{\partial}{\partial u_{j-2}} \right) \Pi_{C}^{(0)}. \quad (20)$$

3. Nekrasov functions

The Nekrasov functions are now reviewed in numerous papers [22]. They are obtained from the LNS contour multi-integrals [11], which in the simplest $SU(N)$ case look like

$$Z_{LNS}(\vec{a} | \epsilon) \equiv \sum_{\epsilon} \frac{1}{k!} \left( \epsilon \right)_{k} \oint \frac{d\psi_{I}}{2\pi i} \prod_{I=1}^{k} Q(\psi_{I}) \prod_{j=1}^{N} \prod_{a < b} \frac{\psi_{I}^{2} \prod_{a < b} (\psi_{I} - (\epsilon_{a} + \epsilon_{b})^{2})}{\prod_{a} (\psi_{I}^{2} - \epsilon_{a}^{2})}, \quad (21)$$

where the polynomial $Q$ depends on the matter content of the model, for pure gauge theory $Q(\varphi) = \Lambda^{2N}$. The crucial step was done in [10]: the integral was rewritten as an explicit sum over a collection of Young diagrams, which provided a practically useful expansion basis for various purposes.

The Nekrasov function for $SU(N)$ is given by

$$\mathcal{F}(a | \epsilon) = \mathcal{F}^{pert}(a | \epsilon) + \mathcal{F}^{inst}(a | \epsilon), \quad (22$$
As explained in the introduction, we evaluate the perturbative contribution for $\epsilon \neq 0$ looks nice only when the $a$-derivative is taken: 

$$-\frac{\partial F_{\text{even}}}{\partial a_i} = 2\epsilon_1 \sum_{j \neq i} \log \frac{\Gamma(1 + a_{ij}/\epsilon_1)}{\Gamma(1 - a_{ij}/\epsilon_1)}$$

$$= \sum_{j \neq i} 4a_{ij} \left\{ \left( \log \frac{a_{ij}}{\Lambda} - 1 \right) + \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)} \left( \frac{\epsilon_1}{a_{ij}} \right)^{2m} \right\}$$

$$= 4 \sum_{j \neq i} \left\{ a_{ij} \left( \log \frac{a_{ij}}{\Lambda} - 1 \right) + \frac{\epsilon_1^2}{12a_{ij}} + O\left(\epsilon_1^3\right) \right\}, \quad (23)$$

while the instanton part is a series in powers of $\gamma^2 = \Lambda^{2N}$, of which we will need only the first term (associated with the single-box Young diagrams)

$$F_{\text{inst}} = \frac{\Lambda^{2N}}{2a_N^2} \sum_{i=1}^{N} \prod_{j \neq i} \frac{1}{a_{ij}}(a_{ij} + \epsilon) + O(\Lambda^{4N})$$

$$= \frac{1}{2a_N^2} \sum_{i=1}^{N} \prod_{j \neq i} \frac{\Lambda^{2N}}{a_{ij}^2} \left\{ 1 + \epsilon^2 \left( \sum_{j \neq i} \frac{1}{a_{ij}'} + \sum_{j \neq k} \frac{1}{a_{jk}} \right) + O(\epsilon^3) \right\} + O(\Lambda^{4N}). \quad (24)$$

The SW prepotential $F_{\text{SW}}(\bar{a})$ is defined by the same formulae, only all terms with $\epsilon^2$ are omitted; see section 4.2.

4. SW/BS periods $\Pi^{(0)}$

As explained in the introduction, we evaluate the $A$-periods $a_i = \Pi^{(0)}(A^i)$ as functions of $\lambda_i$ and $\gamma$ directly, while the $B$-periods $\Pi^{(0)}(B_j)$ will be obtained from (4) by differentiating $F_{\text{SW}}(a_i)$ from the previous section and then substituting there $a_i(\lambda)$.

4.1. SW/BS $A$-periods $\bar{a}$ through the roots $\bar{\lambda}$

Shifting $\phi \rightarrow \phi - iN \log \Lambda$ in (4), one obtains

$$e^{i\phi} = - (2K(p) + \Lambda^{2N} e^{-i\phi}) = -2K(p) \left( 1 - \frac{\Lambda^{2N}}{4K(p)^2} \right). \quad (25)$$

Therefore,

$$\Pi^{(0)} = i \oint p d\phi = \oint \frac{p dK}{K} + \oint \frac{p dK}{2} = \sum_k \oint \frac{p dp}{p - \lambda_k} + \frac{\Lambda^{2N}}{4a_N^2} \oint \frac{dp}{\prod_k (p - \lambda_k)^2} \quad (26)$$

and

$$a_i = \Pi^{(0)}_k = \lambda_i - \frac{\Lambda^{2N}}{2a_N^2} \prod_{k \neq i} \frac{1}{\lambda_i - \lambda_k}. \quad (27)$$

4.2. SW/BS $B$-periods from the Nekrasov function

Putting $\epsilon = 0$ in formulae of section 3, one obtains

$$\Pi^{(0)}_{B_i} = -\frac{1}{4} \frac{\partial F_{\text{SW}}}{\partial a_i} = \sum_{j \neq i} a_{ij} \left( \log \frac{a_{ij}}{\Lambda} - 1 \right) - \frac{\Lambda^{2N}}{8a_N^2} \frac{\partial}{\partial a_j} \sum_{j=1}^{N} \frac{1}{\prod_{k \neq j} a_{jk}^2} + O(\Lambda^{4N})$$

$$= \sum_{j \neq i} a_{ij} \left( \log \frac{a_{ij}}{\Lambda} - 1 \right) + \frac{\Lambda^{2N}}{4a_N^2} \left( \frac{1}{\prod_{k \neq i} a_{ik}^2} \sum_{k \neq i} \frac{1}{\alpha_k} + \sum_{j \neq i} \frac{1}{a_{ij}} \prod_{k \neq i,j} a_{jk}^2 \right). \quad (28)$$
In order to apply operator $\hat{O}$, one needs the periods expressed through the roots $\vec{\lambda}$ or coefficients $\vec{a}$ rather than through the moduli $\vec{u}$. Thus, one needs to substitute $\vec{a}(\vec{\lambda})$ from (27) into (28)

$$\Pi_B^{(0)} = \sum_{j \neq i} a_{ij}(\lambda) \left( \log \frac{\lambda_{ij}}{\Lambda} - 1 \right) + \frac{\Lambda^{2N}}{4a_N^2} \left( \frac{1}{\prod_i \gamma_{ik}^2} \sum_i \frac{1}{\gamma_{ik}} + \sum_i \frac{1}{\lambda_{ij} \prod_{j \neq i} \gamma_{jk}^2} \right). \quad (29)$$

In the one-instanton approximation, the only difference between (29) and (28), except for a simple substitution $a_i \rightarrow \lambda_i$, is that the coefficient in front of logarithm is now $a_{ij}$, not $\lambda_{ij}$. The change of logarithm’s argument does not contribute.

5. Quantized SW prepotential and the Nekrasov function

We are now ready to act with operator (20),

$$\hat{O} = \left( 1 + \frac{\hbar^2\gamma}{24} \frac{\partial}{\partial \gamma} \sum_j j(j - 1) a_j \frac{\partial}{\partial u_{j-2}} + O(\hbar^4) \right) = 1 + \frac{\epsilon^2}{24} \hat{O}^{(2)} + O(\epsilon^4), \quad (30)$$

on (27) and (29), substitute the former one into the full Nekrasov functions (22)–(24) and compare its derivative with the latter one. The results coincide, thus validating the suggestion of [1] in the first order in $\Lambda^{2N}$ and $\epsilon^2$.

5.1. Specifics of the second-order approximation

Operator $\hat{O}^{(2)}$ acts only on the $\Lambda$-dependent ($\gamma = \Lambda^N$) quantities, and the $u$-differential operator can be conveniently expressed through the $\lambda$-derivatives:

$$\sum_{j=0}^{N} j(j - 1) a_j \frac{\partial}{\partial u_{j-2}} = -\sum_{m=1}^{N} K''(\lambda_m) \frac{\partial}{\partial \lambda_m}. \quad (31)$$

It can easily be tested by acting on $p(u_1)$ and using $K'(p) \frac{\partial p}{\partial u_j} = -p_j$.

Identity (8), which we want to prove, in the leading approximation can be rewritten as follows. Its left-hand side is

$$\Pi_B^{(0)} \left( \vec{a} + \frac{\epsilon^2}{24} \hat{O}^{(2)}[\vec{a}] \right) = \Pi_B^{(0)}(\vec{a}) + \frac{\epsilon^2}{24} \sum_{j=1}^{N} \hat{O}^{(2)}[a_j(\vec{\lambda})] \frac{\partial}{\partial a_j} \Pi_B^{(0)}(\vec{a})$$

$$+ \frac{\epsilon^2}{24} \left( 2 \sum_{j \neq i} \frac{1}{a_{ij}} + \frac{12\Lambda^{2N}}{a_N^2} \prod_{j \neq i} a_{ij} \left( \sum_{j \neq i} \frac{1}{a_{ij}} + \sum_{j < k} \frac{1}{a_{ij} a_{ik}} \right) \right), \quad (32)$$

while its right-hand side is

$$\Pi_B^{(0)}(\vec{a}) + \frac{\epsilon^2}{24} \hat{O}^{(2)}[\Pi_B^{(0)}(\vec{\lambda})]. \quad (33)$$

Thus what we prove in this paper is

$$\hat{O}^{(2)}[\Pi_B^{(0)}(\vec{\lambda})] - \sum_{j=1}^{N} \hat{O}^{(2)}[a_j(\vec{\lambda})] \frac{\partial}{\partial a_j} \Pi_B^{(0)}(\vec{a})$$

$$= 2 \sum_{j \neq i} \frac{1}{a_{ij}} + \frac{12\Lambda^{2N}}{a_N^2} \prod_{j \neq i} a_{ij} \left( \sum_{j \neq i} \frac{1}{a_{ij}} + \sum_{j < k} \frac{1}{a_{ij} a_{ik}} \right). \quad (34)$$
In the next subsection, we explicitly describe the check for $\Lambda$-independent terms in this formula. The single-instanton contributions, i.e., the terms with $\Lambda^{2N}$, also match at both sides, but formulae are somewhat lengthy and we do not present them in this paper.

5.2. Perturbative level

For the perturbative part of the Nekrasov function, the difference between $\vec{a}$ and $\vec{\lambda}$ is inessential. The $\hbar$-corrections ($\hbar = \epsilon_1$) to the $\Lambda$-independent piece in $\mathcal{F}(\vec{a}|\epsilon_1)$ arise from the action of deformation operator $\hat{O}$ on the logarithm in perturbative part of the SW prepotential:

$$-\hat{O} \frac{\partial \mathcal{F}}{\partial a_i} = \left(1 + \frac{\hbar^2}{24} \frac{\partial}{\partial \gamma} \sum_k k(k-1)u_k \frac{\partial}{\partial u_k} + \cdots \right) \sum_{j \neq i} 4a_{ij} \log \frac{a_{ij}}{\Lambda}$$

(35)

$$= 4 \sum_{j \neq i} \left\{ \lambda_{ij} \log \frac{\lambda_{ij}}{\Lambda} + \frac{\hbar^2}{24N} \left( \frac{K''(\lambda_i)}{K'(\lambda_i)} - \frac{K''(\lambda_j)}{K'(\lambda_j)} \right) + O(\hbar^4, \Lambda^2) \right\}.$$  

(36)

In the last line and in the remaining part of the calculation, we neglect all the dependences on $\gamma = \Lambda^N$; in this approximation, $a_i$ are just the roots $\lambda_i$ of the polynomial $K(p) = u_N \prod_{j=1}^N (p - \lambda_j)$ and

$$K'(\lambda_i) = u_N \prod_{j \neq i} \lambda_{ij},$$

$$K''(\lambda_i) = 2u_N \sum_{j \neq i} \left( \prod_{k \neq i,j} \lambda_{ik} \right)$$

(37)

and

$$\frac{K''(\lambda_i)}{K'(\lambda_i)} = 2 \sum_{k \neq i} \frac{1}{\lambda_{ik}}.$$ 

(38)

Using these formulae, one can check that (36) coincides with (23), provided $\hbar = \epsilon_1$:

$$\sum_{j \neq i} \left( \frac{K''(\lambda_i)}{K'(\lambda_i)} \right) - \frac{K''(\lambda_j)}{K'(\lambda_j)} \right) = 2N \sum_{j \neq i} \frac{1}{\lambda_{ij}}.$$ 

(39)

Indeed,

$$N = 2 : \frac{2}{\lambda_{12}} - \frac{2}{\lambda_{21}} = \frac{4}{\lambda_{12}},$$

$$N = 3 : \frac{2(\lambda_{12} + \lambda_{13})}{\lambda_{12}\lambda_{13}} - \frac{2(\lambda_{21} + \lambda_{23})}{\lambda_{21}\lambda_{23}} - \frac{2(\lambda_{31} + \lambda_{32})}{\lambda_{31}\lambda_{32}} = 6 \left( \frac{1}{\lambda_{12}} + \frac{1}{\lambda_{13}} \right),$$

(40)

$$\ldots.$$ 

6. Conclusion

In this paper, we reported the first check of the claim that the (degenerated) Nekrasov functions are neatly described by the deformation of the SW construction from quasiclassical to quantum integrable systems in the simplest non-Abelian case of the $SU(N)$ gauge theory or the $SL(N)$ affine Toda system. Switching from the quasiclassical BS periods to the exact quantum monodromies preserves consistency of the SW system of equations; thus, they can be used to define the deformed prepotential which coincides with Nekrasov’s $\mathcal{F}(\vec{a}|\epsilon_1)$ with $\epsilon_2 = 0.$
This seems to be in accordance with the original guess in [2]. We performed the check only in the first order, both in instanton corrections (in $\gamma^2 = \Lambda^{2N}$) and in the quantum deformation parameter $\bar{\gamma}^2 = \epsilon_1^2$; this case is already non-trivial. Of course, higher-order corrections deserve to be found as well. Generalizations to other models with other gauge groups and additional matter multiplets, especially to quiver theories, should also be examined. Of interest is also the similar study of the second deformation to $\epsilon_1, \epsilon_2 \neq 0$ and its relation to another important hypothesis: the AGT conjecture [15].

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References

[1] Mironov A and Morozov A 2009 arXiv:0910.5670
[2] Nekrasov N and Shatashvili S 2009 arXiv:0908.4052
[3] Morozov A 1994 Phys.—Usp. 37 1 (arXiv:hep-th/9303139)
Morozov A 1995 arXiv:hep-th/9502091
Morozov A 2005 arXiv:hep-th/0502010
Mironov A 1994 Int. J. Mod. Phys. A 9 4355 (arXiv:hep-th/9312212)
Mironov A 2002 Phys. Part. Nucl. 33 537 (arXiv:hep-th/9409190)
Mironov A 1998 Theor. Math. Phys. 114 127 (arXiv:q-alg/9711006)
[4] Seiberg N and Witten E 1994 Nucl. Phys. B 426 19–52 (arXiv:hep-th/9408099)
Seiberg N and Witten E 1994 Nucl. Phys. B 451 484–550 (arXiv:hep-th/9407087)
Argyres P and Shapere A 1996 Nucl. Phys. B 461 437–59 (arXiv:hep-th/9509175)
Sonnenschein J, Theisen S and Yankielowicz S 1996 Phys. Lett. B 367 145–50 (arXiv:hep-th/9510129)
[5] Gorsky A, Krichever I, Marshakov A, Mironov A and Morozov A 1995 Phys. Lett. B 355 135–40 (arXiv:hep-th/9505035)
[6] Martinec E and Warner N 1996 Nucl. Phys. 459 97 (arXiv:hep-th/9509161)
[7] Donagi R and Witten E 1996 Nucl. Phys. B 460 299–334 (arXiv:hep-th/9510101)
Martinelli E 1996 Phys. Lett. B 367 91–6 (arXiv:hep-th/9510204)
Gorsky A and Marshakov A 1996 Phys. Lett. B 374 218–24 (arXiv:hep-th/9510224)
Itoyama H and Morozov A 1996 Nucl. Phys. B 477 855–77 (arXiv:hep-th/9511126)
Itoyama H and Morozov A 1997 Nucl. Phys. B 491 529–73 (arXiv:hep-th/9512161)
Itoyama H and Morozov A 1996 arXiv:hep-th/9601168
[8] Nekrasov N 1998 Nucl. Phys. B 531 323–44 (arXiv:hep-th/9609219)
Braden H W, Marshakov A, Mironov A and Morozov A 1999 Phys. Lett. B 448 195 (arXiv:hep-th/9812078)
Braden H W, Marshakov A, Mironov A and Morozov A 1999 Nucl. Phys. B 558 371 (arXiv:hep-th/9902205)
[9] Gorsky A, Marshakov A, Mironov A and Morozov A 1996 Phys. Lett. B 380 75–80 (arXiv:hep-th/9603140)
Gorsky A, Marshakov A, Mironov A and Morozov A 1996 arXiv:hep-th/9604078
Gorsky A, Gukov S and Mironov A 1998 Nucl. Phys. B 517 409–61 (arXiv:hep-th/9707120)
Gorsky A and Mironov A 2000 arXiv:hep-th/0011197
[10] Nekrasov N 2004 Adv. Theor. Math. Phys. 7 831–64 (arXiv:hep-th/0206161)
[11] Moore G, Nekrasov N and Shatashvili S 1998 Nucl. Phys. B 534 549–611 (arXiv:hep-th/9711108)
Moore G, Nekrasov N and Shatashvili S 1998 arXiv:hep-th/9801061
LOSEV A, Nekrasov N and Shatashvili S 2000 Commun. Math. Phys. 209 97–121 (arXiv:hep-th/9712241)
LOSEV A, Nekrasov N and Shatashvili S 2000 Commun. Math. Phys. 209 77–95 (arXiv:hep-th/9803265)
[12] Semenov-Tian-Shansky M 1976 Izv. RAN, Ser. Phys. 40 562
Duistermaat J J and Heckman G J 1983 Ind. Math. 72 153
Atiyah M and Bott R 1984 Topology 23 1
Atiyah M F 1985 Asterisque 131 43
Witten E 1988 Commun. Math. Phys. 117 353
Witten E 1991 Int. J. Mod. Phys. A 6 2775–92
Aleksandrov A, Faddeev L and Shatashvili S 1989 J. Geom. Phys. 1 3
Blau M, Keskila-Vakkuri E and Niemi A 1990 Phys. Lett. B 246 92
Hietamaki A, Morozov A, Niemi A and Palo K 1991 Phys. Lett. B 263 417–24
Hietamaki A, Morozov A, Niemi A and Palo K 1991 Phys. Lett. B 271 365–71
Hietamaki A, Morozov A, Niemi A and Palo K 1992 Nucl. Phys. B 377 295–338
Hietamaki A, Morozov A, Niemi A and Palo K 1992 Int. J. Mod. Phys. B 6 2149–58
[13] Gerasimov A, Khoroshkin S, Lebedev D, Mironov A and Morozov A 1995
Int. J. Mod. Phys. A 10 2589–614 (arXiv:hep-th/9405011)
Alexandrov A, Mironov A and Morozov A 2004 Int. J. Mod. Phys. A 19 4127
Alexandrov A, Mironov A and Morozov A 2005 Theor. Math. Phys. 142 349 (arXiv:hep-th/0310113)
Alexandrov A, Mironov A, Morozov A and Putrov P 2008 arXiv:0811.2825
[14] Mironov A and Morozov A 2009 Phys. Lett. B 680 188–94 (arXiv:0908.2190)
[15] Alexandrov A, Mironov A and Morozov A 2009 arXiv:0908.2064
Mironov A and Morozov A 2009 Nucl. Phys. B 825 1–37 (arXiv:0908.2569)
Mironov A and Morozov A 2009 Phys. Lett. B 682 118–24 (arXiv:0909.3531)
Igari S and Nunez C 2009 arXiv:0908.3460
Nanoopoulos D and Xie D 2009 arXiv:0908.4409
Nanoopoulos D and Xie D 2009 arXiv:0911.1990
Alday L, Gaiotto D, Gukov S, Tachikawa Y and Verlinde H 2009 arXiv:0909.0495
Drukker N, Gomis J, Okuda T and Teschner J 2009 arXiv:0911.1105
Dijkgraaf R and Vafa C 2009 arXiv:0909.2453
Poghosian R 2009 arXiv:0909.3412
Gadde A, Pomoni E, Rastelli L and Razamat S 2009 arXiv:0910.2225
Bonelli G and Tanzini A 2009 arXiv:0909.4031
Alday L, Benni F and Tachikawa Y 2009 arXiv:0909.4776
Awata H and Yamada Y 2009 arXiv:0910.4431
Alba V and Morozov A 2009 arXiv:0911.0363
Wu J-F and Zhou Y 2009 arXiv:0911.1922
[16] Kharchev S, Marshakov A, Mironov A and Morozov A 1995 Int. J. Mod. Phys. A 10 2015
(arXiv:hep-th/9312210)
Eynard B 2008 J. Stat. Mech. 0807 P07023 (arXiv:0804.0381)
Klemm A and Gukov S 2009 Nucl. Phys. B 819 400–30 (arXiv:0810.4944)
[17] Nekrasov N and Okounkov A 2003 arXiv:hep-th/0306238
Mironov A, Morozov A and Natanzon S 2009 arXiv:0904.4227
[18] Nekrasov N and Shatashvili S 2009 Nucl. Phys., Proc. Suppl. B 192–3 91–112 (arXiv:0901.4744)
Nekrasov N and Shatashvili S 2009 arXiv:0901.4748
[19] Wentzel G 1926 Z. Phys. 38 518
Brillouin L 1926 Comptes Rendus 183 24
Kramers H A 1926 Z. Phys. 39 828
Zwaan A 1929 Arch. Neerl. Sci. 12 33
Dunham J L 1932 Phys. Rev. 41 713–20
[20] Alexandrov A, Mironov A and Morozov A 2005 Fortschr. Phys. 53 512–21 (arXiv:hep-th/0412205)
Mironov A 2006 Theor. Math. Phys. 146 63–72 (arXiv:hep-th/0506158)
[21] Dijkgraaf R and Vafa C 2002 Nucl. Phys. B 644 3 (arXiv:hep-th/0206255)
Dijkgraaf R and Vafa C 2002 Nucl. Phys. B 644 21 (arXiv:hep-th/0207106)
Dijkgraaf R and Vafa C 2002 arXiv:hep-th/0208048
Chekhov L and Mironov A 2003 Phys. Lett. B 552 293 (arXiv:hep-th/0209085)
Itoyama H and Morozov A 2003 Prog. Theor. Phys. 109 433–63 (arXiv:hep-th/0212032)
Itoyama H and Morozov A 2003 Int. J. Mod. Phys. A 18 3889–906 (arXiv:hep-th/0301136)
Chekhov L, Marshakov A, Mironov A and Vasiliev D 2003 arXiv:hep-th/0301071
Chekhov L, Mironov A, Mironov A and Vasiliev D 2005 Proc. Steklov Inst. Math. 251 254 (arXiv:hep-th/0506075)

[22] Flume R and Pogossian R 2003 Int. J. Mod. Phys. A 18 2541
Nakajima H and Yoshioka K 2003 arXiv:math/0306198
Nakajima H and Yoshioka K 2003 arXiv:math/0311058
Shadchin S 2006 SIGMA 2 008 (arXiv:hep-th/0601167)
Shadchin S 2005 arXiv:hep-th/0502180
Bellisai D, Fucito F, Tanzini A and Travaglini G 2000 Phys. Lett. B 480 365 (arXiv:hep-th/0002110)
Bruzzo U, Fucito F, Tanzini A and Travaglini G 2001 Nucl. Phys. B 611 205–26 (arXiv:hep-th/0008225)
Bruzzo U, Fucito F, Morales J and Tanzini A 2003 J. High Energy Phys. JHEP05(2003)054 (arXiv:hep-th/0211108)
Bruzzo U and Fucito F 2004 Nucl. Phys. B 678 638–55 (arXiv:math-ph/0310036)
Fucito F, Morales J and Pogossian R 2004 J. High Energy Phys. JHEP10(2004)037 (arXiv:hep-th/040890)