On symmetry and asymmetry in a problem of shape optimization

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1 Introduction and main result

Let $\Omega$ be an open subset of finite measure in $\mathbb{R}^n$. For $1 < p, q, r < \infty$ we consider the following extremal problem:

$$\lambda(p, q, r; \Omega) = \inf \left\{ \frac{\|\nabla u\|_{L_p(\Omega)}}{\|u\|_{L_q(\Omega)}} : u \neq 0, u \in W^1_p(\Omega), \int_{\Omega} |u|^{r-2}u = 0 \right\}. \quad (1)$$

For $p = q = r = 2$ the problem (1) is called twisted Dirichlet eigenvalue problem, see [2]. We are interested in the so-called shape optimization problem: among $\Omega$ with fixed volume, to find the set which minimizes $\lambda(p, q, r; \Omega)$. In other words our problem is a type of isoperimetric inequality.

Even in one-dimensional case the problem (1) turned out to be rather complicated. In this case the problem is to define the sets of positivity and negativity of the minimizer to the problem

$$\min \left\{ \frac{\|u'\|_{L_p[-1,1]}}{\|u\|_{L_q[-1,1]}} : u(\pm 1) = 0, \int_{-1}^{1} |u|^{r-2}u = 0 \right\}. \quad (2)$$

It was mentioned in [8] that for $r = q$ the minimizer is always odd and thus the positivity and negativity sets are equal. On the other hand, authors of [8] discovered that in the Poincaré case $r = 2$ the situation is much more interesting: for $q \leq 2p$ the minimizer is also odd while for $q >> 1$ the symmetry breaking arises, and thus optimal partition of the interval $(-1, 1)$ is not equal.

After efforts of many mathematicians ([9], [3], [15], [11]) the final results in the problem (2) were established in [5], [17] for $r = 2$ and in [6] and recent paper [12] for general $r$. Namely, for $q \leq (2r - 1)p$ the minimizer of (2) is odd while $(2r - 1)p$ is the bifurcation point, and for $q > (2r - 1)p$ the minimizer is asymmetric.

Note that the problem (2) and equivalent statements in the Poincaré case arise also when estimating the eigenvalues in the Lagrange problem of the column stability

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In multidimensional case the problem (1) and similar problems are considerably less investigated. We mention the papers [13] and [18] where symmetry and asymmetry of the minimizers were proved, respectively, in a ball and in a square, for \( p = r = 2 \) and various values of \( q \).

The next important point is the paper [11], where the authors, dealing with the linear case \( p = q = r = 2 \), by a very tight analysis solved the shape optimization problem and proved that the only optimal shape is given by a pair of equal balls.

In a quite recent paper [7], authors tried to generalize this results to the case \( r = q \) which is the simplest one in one-dimensional problem. They claimed that for any \( p \in (1, \infty) \) and any admissible \( q \) the optimal shape is also given by a pair of equal balls. In the present paper we show that it is not the case, and that, in contrast with the problem (2), the symmetry breaking phenomenon arises here.

For any \( p \in [1, \infty) \) we define the critical Sobolev exponent \( p^* \) by relation

\[
\frac{1}{p^*} = \max \left\{ \frac{1}{p} - \frac{1}{n}; 0 \right\}.
\]

Our main result is as follows.

**Theorem 1.** Let \( 1 < p < \infty \) and \( 1 < q < p^*, \, 0 < r - 1 < p^* \). Then

1. The optimal shape in (1) is given only by a pair of disjoint balls.
2. For \( q = r - 1 \) the optimal shape is given only by a pair of equal balls.
3. For \( q > \tilde{q} \equiv \left( \frac{r-1}{n} + 1 \right)^2 p - \frac{(r-1)^2}{n} \) the optimal shape is given by a pair of non-equal balls.

**Corollary 1.** For any \( 1 < p < \infty \) and \( 1 < r - 1 < p^* \), for sufficiently large \( q < p^* \) the optimal shape in (1) is asymmetric.

**Corollary 2.** The relation \( q = r - 1 \) in Theorem 1, part 2, is the best possible in a sense. Namely, given \( \delta > 0 \), for any \( 1 < p < n \) the optimal shape in (1) is asymmetric provided \( q = r - 1 + \delta \) is sufficiently close to \( p^* \).

The last corollary evidently disproves [7, Theorem 1].

**Remark.** For \( n = 1 \) Theorem 1, part 3, gives \( \tilde{q} = r^2 p - (r - 1)^2 \) that is greater than the right exponent \( (2r - 1)p \), see [6] and [12]. To determine the true value of the bifurcation parameter \( \tilde{q} \) is an interesting open problem.

## 2 Proof of Theorem 1

Part 1 of Theorem 1 is quite standard. By compactness of embedding \( W^1_p(\Omega) \) into \( L_q(\Omega) \) and into \( L_{r-1}(\Omega) \), the infimum in (1) is attained. Let \( U \) be a minimizer. Define \( U_\pm = \max\{\pm U, 0\} \) and \( \Omega_\pm = \text{supp}(U_\pm) \). Applying the Schwarz symmetrization (see, e.g., [14]) to \( U_\pm \) separately we obtain

\[
\lambda(p, q, r; \Omega) \geq \lambda(p, q, r; B_+ \cup B_-),
\]

(3)
where \( B_\pm \) are disjoint balls equimeasurable with \( \Omega_\pm \), respectively. Moreover, by the Euler equations both symmetrized functions \( U_\pm^* \) have unique critical points. By [4], in this case the numerator in (1) strictly decreases under symmetrization, and thus the inequality in (3) is strict provided \( \Omega \neq B_+ \cup B_- \).

To prove part 2, we write the Euler equation for (1)
\[
- \text{div}(|\nabla U|^{p-2} \nabla U) = \lambda |U|^{q-2} U + \mu |U|^{r-2}
\]
(here \( \lambda \) and \( \mu \) are Lagrange’s multipliers) and note that for \( q = r - 1 \) we obtain the following equations for \( U_\pm \):
\[
- \text{div}(|\nabla U_\pm|^{p-2} \nabla U_\pm) = (\lambda \pm \mu) |U_\pm|^{q-2} U_\pm.
\]
We know from part 1 that \( \Omega = B_+ \cup B_- \) and functions \( U_\pm \) are radial. Therefore, \( U_\pm \) are in fact solutions of ODEs. By homogeneity, \( U_\pm \) can be obtained by dilation and multiplying by constant from a unique radial positive (generalized) solution of the boundary value problem
\[
- \text{div}(|\nabla v|^{p-2} \nabla v) = v^{q-1} \quad \text{in } B_1, \quad v|_{\partial B_1} = 0.
\]

Let \( R_\pm \) be the radii of \( B_\pm \). Then
\[
U_\pm(x) = c_\pm \cdot v(\frac{x}{R_\pm}),
\]
and (1) is reduced to the following minimization problem:
\[
\min \left\{ \frac{c_+^{q} R_+^{n-p} + c_-^{q} R_-^{n-p}}{(c_+^{q} R_+ + c_-^{q} R_-^p)^{\frac{q}{n}}} : c_+^{q-1} R_+^n = c_-^{q-1} R_-^n, \quad R_+ + R_- = C \equiv \frac{||\Omega||}{|B_1|}. \right\}
\]
Denoting \( x = c_+^{q-1} R_+^n \) and \( y = \frac{2}{c} R_+ - 1 \in (-1, 1) \), we rewrite (7) as minimization of
\[
F(y) \equiv \frac{(1+y)^{1-\frac{q}{n} - \frac{p}{n}} + (1-y)^{1-\frac{q}{n} - \frac{p}{n}}}{[(1+y)^{1-\frac{q}{n} - \frac{p}{n}} + (1-y)^{1-\frac{q}{n} - \frac{p}{n}}]^\frac{q}{n}}.
\]
Since \( q = r - 1 \), the denominator in (8) is constant. Since \( q < p^* \), the exponent in the numerator is negative. Thus, the function (8) is strictly convex, by symmetry its minimum is attained only at \( y = 0 \), and the statement follows.

To deal with part 3, we note that the radial positive solution of (5) solves the problem
\[
\inf \left\{ \frac{\|\nabla u\|_{L_\alpha(B_1)}}{\|u\|_{L_\alpha(B_1)}} : u \neq 0, \quad u \in \tilde{W}_p^1(B_1) \right\}.
\]
Therefore, if \( R_+ = R_- \) then the minimizer of (1) in \( B_+ \cup B_- \) satisfies (6).

Now we consider functions (3) for general \( R_\pm \) and show that even such simple variations can provide symmetry breaking.

We again arrive at (8) and write the Taylor series for the function \( F \) at zero:
\[
F(y) \approx 2^{1-\frac{q}{n}} \cdot \left( 1 + \frac{\gamma y^2}{2} \right), \quad \gamma = \left( \frac{p}{n} + \frac{p}{r-1} \right) \left( \frac{p}{n} + \frac{p}{r-1} - 1 \right) - \frac{p}{r-1} \left( \frac{q}{r-1} - 1 \right).
\]
For \( q > \hat{q} \) we have \( \gamma < 0 \), and the statement follows. \( \Box \)

**Proof of Corollaries 1 and 2.** For \( p \geq n \) the statement of Corollary 1 is trivial. Further, let \( p < n \). By elementary calculation we obtain that the exponent \( \hat{q} \) is increasing function of \( r-1 \in [0, p^*] \) and \( \hat{q} = p^* \) just for \( r-1 = p^* \). This gives both statements. \( \Box \)
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