Hierarchy of coherent structures in turbulent channel flow

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Abstract. By analyzing a database of fully developed turbulent channel flow at the friction Reynolds number $Re_\tau = 4179$, we investigate the sustaining mechanism of a hierarchy of coherent structures in the wall-bounded turbulence. For this purpose, we decompose the turbulent fields into different scales by a band-pass filter. Using the filtered velocity and velocity gradients, we identify the hierarchy of coherent structures to observe that the largest-scale structures at each distance from the wall are composed of quasi-streamwise vortices and low-speed streaks. Since these are similar to well-known coherent structures in the buffer layer, they are likely to be maintained by the self-sustaining process. In contrast, structures smaller than the distance from the wall distribute isotropically. These observations are also confirmed by using a conditional sampling method. Moreover, quantifying the scale-dependent contributions to the vortex stretching and energy transfer, we show that the largest-scale coherent structures are strongly affected by the mean shear, whereas smaller-scale vortices are generated by the energy cascading events. Incidentally, in large-scale ejection regions (i.e. large-scale low-speed streaks), the cascading events are stronger than in large-scale sweep regions.

1. Introduction
Near-wall turbulence is sustained by the so-called self-sustaining process (SSP) [1, 2], namely, streamwise vortices induce low- and high-speed streaks by the advection of the streamwise momentum, while an instability induces the meandering of streaks, causing the regeneration of streamwise vortices. It is also known that these quasi-streamwise vortices are inclined in the wall-normal and spanwise directions, and they are located in a staggered array [3]. When the Reynolds number is low, there is no scale separation and the SSP explains the sustaining mechanism of the turbulence very well. However, as the Reynolds number increases, larger-scale vortices appear in addition to the near-wall coherent vortices. We emphasize that we cannot identify larger-scale vortices in terms of the velocity gradients. For example, the yellow objects in figure 1 are the positive isosurfaces of the second invariant of the velocity gradient tensor in turbulent channel flow at the friction Reynolds number $Re_\tau = 4179$ (see §2.1 for the details of the database). The visualized vortices are at the smallest scale. On the other hand, looking at the (blue) isosurfaces of the fluctuating streamwise velocity, we can see larger-scale motions. In other words, the quantities related to the velocity are appropriate for extracting largest-scale structures, whereas those related to its gradients are appropriate for extracting smallest-scale structures. The large-scale velocity structures have been extensively investigated in relation to large-scale vortices. For example, in a turbulent boundary layer, hairpin vortex packets form the
Figure 1. Vortices and low-speed structures visualized by the isosurfaces of the second invariant \((Q^+ = 3.0 \times 10^{-3})\) of the velocity gradient and by the streamwise fluctuating velocity \((u^+ = -3)\), respectively. The grid width on the wall indicates 1000 wall units. The flow is from lower left to upper right. A subdomain (half in the spanwise direction and full in the streamwise directions) of the computational domain is visualized.

source of bulge structures in the outer layer, which are one type of large-scale motions (LSM) [4, 5]. Lee et al. [6] also investigated the relation to the hairpin vortices and showed that the creation of very-large-scale motions (VLSM) is associated with the merging events of hairpin vortex packets. Incidentally, Kevin et al. [7] found that large-scale quasi-streamwise vortices exist along the side of large-scale low-speed structures.

In our previous study [8], we extracted the hierarchy of multiscale vortices in a turbulent boundary layer by coarse-graining the simulated velocity fields. Since the coarse-graining enables us to quantify the scale-dependent contributions, evaluating the scale-dependent contributions of the vortex stretching, we revealed the generation mechanism of multiscale vortices. Namely, vortices as large as the height are stretched by the mean shear, whereas smaller-scale vortices are stretched by the larger-scale vortices. The latter generation mechanism is consistent with a picture of the energy cascade in turbulence in a periodic cube [9, 10]. Incidentally, by using over-damped large-eddy simulations (LES), Hwang and his coworkers [11, 12, 13] extracted the attached eddies in the log and outer layers and showed that they were self-similar structures composed of quasi-streamwise vortices and streaks. The most important ingredient in both of our study [8] and the over-damped LES studies [11, 12, 13] is the analysis of coarse-grained turbulent fields.

In the present study, we investigate the hierarchical structures in turbulent channel flow at \(Re_\tau = 4179\) [14]. The Reynolds number of the flow is much higher than in the turbulent boundary layer examined in our previous study [8]. The purposes of the present study are (i) to show the similarity in the sustaining mechanism of the hierarchy of vortices in these two wall-bounded turbulent flows, and (ii) to show the relationship between vortex generation processes and the energy cascade. For these purposes, we extract the hierarchy not only of vortices but also of the velocity. We showed, for a turbulent boundary layer [8], that small-scale vortices in the log layer were stretched predominantly by the twice-larger-scale vortices, but we did not show how the energy was transferred. In the rest of the present paper, we show the mechanism of the generation of coherent vortices and of energy transfer in the log and buffer layers of fully developed wall-bounded turbulence.
2. Methods

2.1. Numerical databases
To reveal hierarchical structures near a wall, we investigate data of a direct numerical simulation of turbulent channel flow at the friction Reynolds number $Re_τ = 4179$ [14]. The simulation was conducted by integrating the Navier-Stokes equations for an incompressible fluid in terms of the wall-normal component of the vorticity and its Laplacian [15]. For the spatial discretization, the Fourier spectral method was used in the wall-parallel directions, whereas seven-point compact finite difference schemes [16] were used in the wall-normal direction. The sides of the computational domain are $L_x = 2\pi h$, $L_y = 2h$ and $L_z = \pi$, where $x$, $y$ and $z$ denote the streamwise, wall-normal and spanwise directions, respectively, and $h$ denotes the channel half-width. This computational domain is large enough to obtain one-point statistics [14] of fully developed turbulence, and the Taylor-length-based Reynolds number at $y/h \approx 0.4$ is approximately 200.

2.2. Scale-decomposition
To explicitly extract the hierarchy of flow structures in turbulence, we employ a filter corresponding to the Fourier band-pass filter for the velocity. First, we apply a Gaussian filter

$$u_i(\sigma)(x) = C \int_V \hat{u}_i(x') \exp \left( -\frac{2}{\sigma^2} (x - x')^2 \right) \, dx'$$

(1)

to the fluctuating velocity $\hat{u}_i$. Here, $\sigma$ denotes the filter scale and $C$ is the coefficient to ensure that the integration of the kernel is unity. For the wall-normal direction, we use the method proposed by Lozano-Durán et al. [17] that the filtering operation is extended by reflecting the filter at the walls and the sign of $u_2 (= v)$ is inverted. Since $u_i(\sigma)$ contains the information for all scales larger than $\sigma$, the filter corresponds to a low-pass filter of the Fourier modes of the velocity. We take the difference between the low-pass filtered fields at two different scales, i.e.

$$u_i(\sigma)(x) = u_i(\sigma)(x) - u_i(2\sigma)(x),$$

(2)

which was also used in our previous study [18] to examine the scale-dependent contributions of the enstrophy production rates in a turbulent boundary layer. This filter corresponds to a band-pass filter of the Fourier modes in the sense that $u_i(\sigma)$ has the contributions from only around the scale $\sigma$. We evaluate a scale-decomposed quantity $\cdot(\sigma)$ (for example, scale-decomposed strain-rate tensor $S_{ij}(\sigma)$) from $u_i(\sigma)$.

3. Results

3.1. Hierarchy of vortices and low-speed structures
As mentioned in the introduction, when we visualize the positive isosurfaces of the second invariant $Q$ of the velocity gradient tensor, only the smallest-scale vortices are captured (see figure 1a). This is because quantities related to the velocity gradient are predominantly determined by the smallest-scale flow structures. On the other hand, when we visualize, for example, the negative isosurfaces of the fluctuating streamwise velocity, structures as large as the distance from the wall are captured (see figure 1b). This is because quantities related to the velocity are predominantly determined by the largest-scale structures.

Figure 1 shows an obvious scale separation between the structures associated with vortices and velocity. Therefore, to extract arbitrary-scale structures, we employ the band-pass filter defined by (2). Figure 2 shows the isosurfaces of the second invariant $Q(\sigma)$ (yellow) of the velocity gradient tensor and the streamwise velocity $u(\sigma)$ (blue) which are evaluated from the fields filtered at different scales (a) $\sigma^+ = 960$, (b) 240 and (c) 60. The vortices and low-speed
Figure 2. The hierarchy of vortices and low-speed structures at the same instant and location as in figure 1. Isosurfaces of the second invariant $Q^{\sigma}$ (yellow) of the velocity gradient tensor and the streamwise velocity $u^{\sigma}$ (blue) filtered at (a) $\sigma^+ = 960$, (b) 240 and (c) 60. Thresholds of $Q^{\sigma^+}$ are (a) $6.0 \times 10^{-7}$, (b) $1.4 \times 10^{-5}$ and (c) $2.0 \times 10^{-4}$, and the threshold of $u^{\sigma^+}$ is $-0.5$. The grid width on the wall indicates 1000 wall units.
Figure 3. An example of the hierarchy composed of quasi-streamwise vortices and a low-speed streak for the scales (a) $\sigma^+=960$, (b) $\sigma^+=240$ and (c) $\sigma^+=60$. A small subdomain in figure 2 is shown. The grid width on the wall is the same as each filter scale $\sigma$. The flow is from lower left to upper right.

Figure 4. Averaged distribution of $Q[\sigma]$ and $u[\sigma]$ around intense vortical structures for the largest scale at each height, namely, (a) $\sigma^+=y_r^+=960$, (b) $\sigma^+=y_r^+=240$ and (c) $\sigma^+=y_r^+=60$. The thresholds of the yellow isosurfaces are (a) $Q[^{+\sigma}]=4.5 \times 10^{-7}$, (b) $2.1 \times 10^{-6}$ and (c) $4.0 \times 10^{-5}$, and those of the blue ones are (a) $u[^{+\sigma}]=-0.5$, (b) $-0.16$ and (c) $-0.11$. The black arrows indicate the center $(0, y_r, 0)$ of the reference frame in which the conditional average is taken. The grid width on the wall is $\sigma^+ (= y_r^+)$. The flow is from lower left to upper right. The blue arrow in (a) indicates the direction of positive $\omega_x$.

Structures are hierarchical. We emphasize that we cannot extract them from the simulated field (figure 1) without the filtering. Looking at the largest scale (figure 2a), we notice that vortices are quasi-streamwise but they are also inclined to the wall-normal direction, and that largest-scale streaks are located by these quasi-streamwise vortices. This combination of quasi-streamwise vortices and meandering streaks is reminiscent of the coherent structures near the wall that were found by Jeong et al. [3] for low-Reynolds-number turbulence. The observed largest streaks correspond to a VLSM and the quasi-streamwise vortices to LSM, which were observed by Hwang [11] in the over-damped LES. Moreover, not only for the largest scale but also for smaller scales (figure 2b,c), we observe similar structures. Evidence is shown in figure 3, where we crop them in rectangular boxes whose faces are parallel to the computational domain. We see that the relationship between the vortices and streaks is quite similar irrespective of the scale. In other words, the largest-scale structures at each height ($\sigma \sim y$), i.e. attached structures, are the coherent structures composed of quasi-streamwise vortices and low-speed streaks irrespective of the height $y$.

Although it is difficult to observe such hierarchical structures in the real space of simulated turbulent fields without scale-decomposition (or coarse-graining), we can easily find the self-similar hierarchy of structures (quasi-streamwise vortices and low-speed streaks) with it. To investigate whether the observed structures are dominant, we evaluate averaged distributions of
Figure 5. Different views of the conditionally averaged objects of a vortex with the scale \( \sigma^+ = 60 \) at heights (a,b,c) \( y_r^+ = 960 \), (d,e,f) 240 and (g,h,i) 60. The thresholds are constant values (a,b,c) \( Q_{\text{rms}}^{[\sigma]} = 4.0 \times 10^{-5} \), (d,e,f) \( 3.4 \times 10^{-5} \) and (g,h,i) \( 2.0 \times 10^{-5} \), which all correspond to \( Q_{\text{rms}}^{[\sigma]}(y_r) \).

\( Q^{[\sigma]} \) and \( u^{[\sigma]} \) with a given scale \( \sigma \) at a given height \( y_r \) around the intense vortical structures. For this purpose, under the conditions that \( Q^{[\sigma]} \) at a fixed height \( y_r \) is larger than the standard deviation \( Q_{\text{rms}}^{[\sigma]} \) at the same height \( y_r \) and the streamwise vorticity \( \omega_x^{[\sigma]} \) is positive, we take averages of \( Q^{[\sigma]} \) and \( u^{[\sigma]} \) around the points satisfying the condition. Here, the latter condition of \( \omega_x^{[\sigma]}(y_r) > 0 \) is imposed to break the spanwise symmetry. We emphasize that, since this conditional sampling corresponds to a moving average over the points satisfying the condition, the shape of the structures obtained by this conditional average does not correspond to the shape of typical coherent structures. In other words, they are the averaged distributions of \( Q^{[\sigma]} \) and \( u^{[\sigma]} \) around the intense structures (\( Q^{[\sigma]}(y_r) > Q_{\text{rms}}^{[\sigma]}(y_r) \)). We show in figure 4 the isosurfaces of the conditional averages of \( Q^{[\sigma]} \) (yellow) and \( u^{[\sigma]} \) (blue) for the filter scales (a) \( \sigma^+ = 960 \), (b) 240 and (c) 60. Note that, since the heights \( y_r^+ \) where the conditions are imposed are as large as the filter scales [namely, (a) \( y_r^+ = 960 \), (b) 240 and (c) 60], the obtained structures are the largest scale possible at each height. In other words, they are attached to the wall. As expected from the previous observation in figure 3, the averaged structures for the largest scale (i.e. attached structures) at each height are also similar irrespective of the heights. For example, the vortices are quasi-streamwise and inclined to the wall-normal and slightly to the spanwise directions. More quantitatively, the inclination angle (i.e. the angle with respect to the wall of the line connecting the two points where the plane perpendicular to the streamwise direction and the isosurface of \( Q^{[\sigma]} \) are tangent) are (a) 43, (b) 40 and (c) 31 degrees. The low-speed structures are also consistent with the observation in figure 3. Namely, a low-speed streak at each height tends to locate on the left side of a vortex at the scale with positive streamwise vorticity.

Next, to investigate the small-scale vortices away from the wall, we show in figure 5 the different views of the isosurfaces of \( Q^{[\sigma]} \) for a scale \( \sigma^+ = 60 \) at different heights (a,b,c) \( y_r^+ = 960 \), (d,e,f) 240 and (g,h,i) 60. Here, we choose the same thresholds (the standard deviation at \( y_r \)) of the isosurfaces. Since the vortex shown in figure 5(g,h,i), which is a large scale in the sense that
Figure 6. Average large-scale ($\sigma^+ = 960$) structures under the condition of the existence of active small-scale ($\sigma^+_{\text{cond}} = 60$) vortices ($Q^{|\sigma|_{\text{cond}}} > 10Q^{|\sigma|_{\text{rms}}}$) at the height $y^+_r = 960$. Thresholds are $\omega^{|\sigma|} = 1.0 \times 10^{-4}$ (yellow), $\omega^{|\sigma|} = -1.5 \times 10^{-4}$ (green), $u^{|\sigma|} = -0.15$ (blue) and $v^{|\sigma|} = 3.0 \times 10^{-2}$ (red). The black arrows indicate the center $(0, y_r, 0)$ of the reference frame in which the conditional average is taken. The grid width on the wall indicates $\sigma^+ (= y^+_r = 960)$. The flow is from lower left to upper right. The blue arrow indicates the direction of positive $\omega_x$.

$\sigma = y_r$, is identical to the one in figure 4(c), we reconfirm that the vortex is inclined to both the wall-normal (see figure 5h) and spanwise directions (see figure 5i). In contrast, detached structures are more spherical (see figure 5a,b,c). However, this observation does not imply that the vortical structures themselves are spherical but implies that they are distributed isotropically. The fact is consistent with the conclusion by Jiménez [19] that smaller-scale vortices away from the wall decouple from the mean shear and become isotropic. In our previous study [8], we also showed that smaller-scale vorticity became less aligned to the mean-flow stretching direction in a turbulent boundary layer.

Before closing this subsection, we investigate the spatial relationship between small-scale vortices and large-scale structures. For this purpose, we take a sampling of the large-scale ($\sigma^+ = 960$) fields under the condition that small-scale ($\sigma^+_{\text{cond}} = 60$) vortices away from the wall ($y^+_r = 960$) exist. Here, we set $Q^{|\sigma|_{\text{cond}}}(y_r) > 10Q^{|\sigma|_{\text{rms}}}(y_r)$ as the condition so that we can see the correlation between large-scale structures and intense small-scale vortices. Note that, since the imposed condition cannot break the spanwise symmetry, obtained structures are always symmetric in the spanwise direction. Figure 6 shows the isosurfaces of the conditionally averaged quantities of $\omega^{|\sigma|}$, $u^{|\sigma|}$ and $v^{|\sigma|}$. Looking at the yellow (positive $\omega^{|\sigma|}$) and green (negative $\omega^{|\sigma|}$) objects and the dot $(0, y_r, 0)$ indicated by the black arrow where the small-scale vortices exist, we can see that small-scale vortices away from the wall are more likely to exist in the strong upflow (red) induced by these large-scale streamwise vortices (yellow and green). Note again that this does not necessarily mean the existence of counter-rotating vortices. In addition, since a large-scale low-speed streak exists in the upflow region, large-scale ejection events occur in this region. Thus, intense small-scale vortices are likely to be in large-scale ejection regions. This is consistent with the observation by previous authors [20, 21, 22] that a cluster of small-scale vortices resides in a large-scale attached ejection.

There are two main results in this subsection. One is that the largest-scale structures at each height (i.e. attached structures) are composed of quasi-streamwise vortices and a low-speed streak (figures 3 and 4). The other is that the orientation of small-scale vortices away from the wall is isotropic (figure 5), and small-scale vortices are more likely to exist in the large-
scale upflow induced by quasi-streamwise vortices (figure 6). In other words, in terms of the momentum transfer, intense small-scale vortices are in the large-scale ejection. However, this does not necessarily imply that they are carried from the wall. In the next subsection, we investigate the sustaining mechanism of these small-scale vortices and its relation to the energy cascade.

3.2. Sustaining mechanism: vortex stretching and energy cascade

The transport equation for the enstrophy $\omega^2/2$ is expressed by

$$\frac{1}{2} \frac{D \omega_i^2}{Dt} = \omega_i S_{ij} \omega_j + \nu \omega_i \nabla^2 \omega_i,$$

where $\omega_i$ is the vorticity, $S_{ij}$ is the strain-rate tensor and $\nu$ is the kinematic viscosity. Only when the vortex stretching term is positive, is the enstrophy amplified, because the viscous term $\nu \omega_i \nabla^2 \omega_i$ weakens the enstrophy. Therefore, to investigate the generation mechanism of vortices, we focus on the vortex stretching term. Since strain rates at all scales simultaneously contribute to the stretching of vorticity at a given scale, we decompose $\omega_i$ and $S_{ij}$ into various scales and introduce a quantity

$$g_f(\sigma_S \rightarrow \sigma_\omega; y) = \frac{\omega_i^{[\sigma_\omega]} S_{ij}^{[\sigma_S]} \omega_j^{[\sigma_\omega]}}{\omega_i^{[\sigma_\omega]^2}}.$$ (4)

Here, $\omega_i^{[\sigma_\omega]}$ is the fluctuating vorticity filtered at the scale $\sigma_\omega$, and $S_{ij}^{[\sigma_S]}$ is the fluctuating strain rates filtered at $\sigma_S$. Hence, $g_f(\sigma_S \rightarrow \sigma_\omega)$ indicates the contribution of the production rates of the enstrophy at $\sigma_\omega$ from strain rates at $\sigma_S$. If $g_f(\sigma_S \rightarrow \sigma_\omega)$ is positive (or negative), it implies the stretching (or contraction) of the vorticity. Similar quantities were defined by Goto et al. [10] for periodic turbulence and by ourselves [8] for a turbulent boundary layer. We also define the contribution from the mean shear to the stretching of vortices at $\sigma_\omega$ by

$$g_m(\sigma_\omega; y) = \frac{\omega_i^{[\sigma_\omega]} \overline{S_{ij} \omega_j^{[\sigma_\omega]}}}{\omega_i^{[\sigma_\omega]^2}},$$ (5)

where $\overline{S_{ij}}$ is the mean rate of strain, in which $\overline{S_{12}}$ and $\overline{S_{21}}$ are non-zero for channel flow. Note that (4) and (5) differ only in the source of the stretching (i.e. $S_{ij}^{[\sigma_S]}$ and $\overline{S_{ij}}$). We have quantitatively evaluated these contributions for filtered fields of a turbulent boundary layer [8], and have concluded that vortices smaller than approximately one-fifth of the height are predominantly stretched by strain rates at a scale twice as large as themselves, whereas larger vortices are stretched directly by the mean shear. Here, we examine the ratio

$$\Gamma_\omega(\sigma_S \rightarrow \sigma_\omega; y) = \frac{g_f(\sigma_S \rightarrow \sigma_\omega; y)}{g_m(\sigma_\omega; y)},$$ (6)

between averages $g_f$ and $g_m$ to investigate their respective dominance in the turbulent channel flow. When $\Gamma_\omega > 1$ (and $g_m > 0$), the vortices with the scale $\sigma_\omega$ are predominantly stretched by strain rates with the scale $\sigma_S$, whereas, when $0 < \Gamma_\omega < 1$, the vortices with the scale $\sigma_\omega$ are stretched by the mean shear more significantly than strain rates at the scale $\sigma_S$. When $\Gamma_\omega < 0$, they are contracted by strain rates at $\sigma_S$ on average. Note that $\Gamma_\omega$ is a function of $y$ (as well as $\sigma_S$ and $\sigma_\omega$). Here, when taking the average at a fixed $y$, we impose two conditions. One is the condition that the stretched vorticity is in rotational regions ($Q^{[\sigma_\omega]}(y) > Q^{[\sigma_\omega]}_{\text{rms}}(y)$). The
Figure 7. The ratio $\Gamma_{\omega}(\sigma_S \rightarrow \sigma_\omega)$ defined by (6) between the contribution of strain rates with scale $\sigma_S$ to the stretching of vortices with scale $\sigma_\omega^+ = 30(\circ), 60(\Box), 120(\triangle), 240(\bullet)$ and $480(\blacksquare)$ and the contribution of the mean flow at heights (a) $y^+ = 960$, (b) 240 and (c) 60. The larger symbols correspond to the self-contribution ($\sigma_S = \sigma_\omega$). Black solid lines indicate the average conditioned by the large-scale upflow ($v[\sigma_{\text{cond}}] > v_{\text{rms}}[\sigma_{\text{cond}}]$ at $\sigma_{\text{cond}}^+ = 960$ at each height $y_r$), whereas blue dashed lines indicate the average conditioned by the large-scale downflow ($v[\sigma_{\text{cond}}] < -v_{\text{rms}}[\sigma_{\text{cond}}]$).
other is the condition whether the stretched vorticity is in the upflow (i.e. large-scale ejection; \( v^{[\sigma_{\text{cond}}]}(y) > v^{[\sigma_{\text{rms}}]}(y) \)) or downflow (i.e. large-scale sweep; \( v^{[\sigma_{\text{cond}}]}(y) < -v^{[\sigma_{\text{rms}}]}(y) \)) for a large scale \( \sigma_{\text{cond}} = 960 \). The latter condition is imposed to examine the origin of the observation that smaller-scale vortices away from the wall are more likely to exist in the largest-scale upflow regions (figure 6).

We show, in figure 7, \( \Gamma_\omega \) in the upflow case (black solid lines) and the downflow case (blue dashed lines) for (a) \( y^+ = 960 \), (b) 240 and (c) 60. We confirm that \( \Gamma_{\text{rms}} \) is always positive, and the sign of \( \Gamma_\omega \) is determined by the sign of \( \Gamma_{\text{rms}} \). Since the contributions in the upflow and downflow cases are quantitatively similar, first we describe observations common to both the cases. Looking at the open circles in figure 7(a) for a small-scale (\( \sigma_\omega^+ = 30 \)) at a location \( (y^+ = 960) \) near the upper boundary of the log layer, the contributions from fluctuating strain rates at the scales 1–8 times larger than \( \sigma_\omega^+ (= 30) \) are dominant (\( \Gamma_{\omega} > 1 \)). In particular, the contribution \( \Gamma_{\omega}(2\sigma_\omega \rightarrow \sigma_\omega) \) from the twice larger scale is the largest. This is also the case for the other small scales \( \sigma_\omega^+ = 60 \) (open squares) and 120 (open triangles). Moreover, the fact \( \Gamma_{\omega} < 0 \) for \( \sigma_S \leq \sigma_\omega/2 \) implies that the vortices are likely to be contracted by smaller-scale strain rates on average. On the other hand, for vortices \( (\sigma \sim y) \) whose size is of the order of the distance from the wall, the contributions from the mean shear are more significant than those from the fluctuating strain rates at any scales \( (\Gamma_{\omega} \leq 1) \). In addition, the results for a location \( (y^+ = 240) \) around the lower boundary of the log layer (figure 7b) show a similar tendency that small-scale vortices are stretched by the twice larger scale the most, whereas large-scale vortices are stretched directly by the mean shear.

Next, let us look at the contributions in the buffer layer \( (y^+ = 60; \text{figure 7c}) \) where the hierarchy of vortices does not exist. If we refer to \( \sigma \sim y \) as “large scale”, there are only large-scale vortices in the buffer layer. Although, among the contributions from the fluctuating strain rates, the twice larger scale contributes most significantly, the contributions from the mean shear is always more important (\( \Gamma_{\omega}(2\sigma_\omega \rightarrow \sigma_\omega) < 1 \)).

Next, we discuss the difference in the large-scale upflow and downflow. \( \Gamma_\omega(2\sigma_\omega \rightarrow \sigma_\omega) \) indicated by the black solid lines (in the upflow case) for small-scale vortices in the log layer (i.e. \( \sigma_\omega^+ = 30 \) in figure 7b and \( \sigma_\omega^+ = 30, 60 \) and 120 in figure 7c) are larger than the blue dashed lines (in the downflow case). The results show that the contributions from the fluctuations (rather than the mean shear) in upflow regions are larger than those in downflow regions. This is reasonable because, in upflow regions, the source of the vorticity is more likely to be carried from the wall. In other words, small-scale vortices in the large-scale low-speed streak (i.e. the large-scale ejection region) are not simply advected by the largest-scale vortices but they are stretched by 1–8 times larger vortices.

We have so far evaluated the scale-dependent enstrophy production rates and have shown that small-scale vortices away from the wall are stretched predominantly by the twice larger-scale vortices rather than the mean shear. This generation mechanism of the hierarchy of vortices seems consistent with the notion of the energy cascade, namely, the scale-by-scale energy transfer from larger to smaller scales. However, the creation of smaller-scale vortices does not necessarily correspond to the energy transfer to smaller scales. This issue was investigated in Ref. [9, 23] for periodic turbulence, but here we examine a different approach. The transport equation of the (mean) turbulent kinetic energy \( K = \tilde{u}_i \tilde{u}_i/2 \) is given by

\[
\frac{\partial K}{\partial t} = \frac{\partial}{\partial x_j} \left\{ -\overline{u}_j K - \frac{1}{2} \tilde{u}_j \tilde{u}_j^2 - \frac{1}{\rho} \tilde{u}_j \tilde{p} + \nu \left( \frac{\partial K}{\partial x_j} + \frac{\partial}{\partial x_i} \tilde{u}_j \tilde{u}_i \right) \right\} - \tilde{u}_j \gamma_i \frac{\partial \tilde{u}_i}{\partial x_j} - 2\nu \tilde{S}_{ij} \tilde{S}_{ij},
\]

where \( \tilde{S}_{ij} \) is the fluctuating strain-rate tensor \( \left( \partial \tilde{u}_j/\partial x_i + \partial \tilde{u}_i/\partial x_j \right)/2 \). The terms in \( \partial/\partial x_j \{ \cdots \} \) in (7) are the mean-flow advection, turbulent advection, velocity-pressure correlation, and viscous diffusion terms, respectively. The last term \(-2\nu \tilde{S}_{ij} \tilde{S}_{ij}\) is the viscous dissipation term.
These terms do not contribute to the production of the turbulent energy. On the other hand, $-\bar{u}_j\bar{u}_i\partial \overline{u_i}/\partial x_j$ represents the production of $K$ due to the mean flow. To investigate inter-scale energy transfer, we must decompose the velocity in (7) into scales. Recently, Kawata and Alfredsson [24] used a decomposition to analyze inter-scale transfer of the Reynolds-stress in plane Couette flow. They decompose the velocity into large-scale $u^{(L)}$ and small-scale $u^{(S)}$ parts by sharp Fourier filtering in the spanwise wave number. Since the Fourier filter is orthogonal, their cross-correlation $u^{(L)}u^{(S)}$ vanishes. Then we can derive the transport equation of the energy for the large- and small-scale parts. For example, the transport equation of the small-scale energy \((K^{(S)} = u^{(S)}_i u^{(S)}_j/2)\) is given by

$$\frac{\partial K^{(S)}_i}{\partial t} = \frac{\partial}{\partial x_j}\left\{-\bar{u}_j K^{(S)} - \frac{1}{\rho} u^{(S)}_j p^{(S)} + \nu \left( \frac{\partial K^{(S)}_i}{\partial x_j} + \frac{\partial}{\partial x_i} u^{(S)}_j u^{(S)}_i \right) \right\}$$

$$- \bar{u}_j^{(S)} u^{(S)}_i \frac{\partial \overline{u_i}}{\partial x_j} - 2 \nu S^{(S)}_{ij} + D^{(S)} + Tr^{(S)}, \quad (8)$$

where

$$D^{(S)} = -\frac{1}{2} \frac{\partial}{\partial x_j} \left\{ u^{(S)}_i u^{(S)}_j + u^{(S)}_i u^{(L)}_j + 2 u^{(S)}_i u^{(L)}_j \right\} \quad (9)$$

and

$$Tr^{(S)} = -u^{(S)}_i u^{(S)}_j \frac{\partial u^{(L)}_i}{\partial x_j} - \left( -u^{(L)}_i u^{(L)}_j \frac{\partial u^{(S)}_i}{\partial x_j} \right) = -u^{(S)}_i u^{(S)}_j S^{(L)}_{ij} - \left( -u^{(L)}_i u^{(L)}_j \delta^{(S)}_{ij} \right) \quad (10)$$

denote the inter-scale interaction related to the spatial redistribution of $K^{(S)}$ and the energy transfer between large and small scales. In particular, $-u^{(S)}_i u^{(S)}_j S^{(L)}_{ij}$ is the energy transfer from the large to small scales, while $-u^{(L)}_i u^{(L)}_j \delta^{(S)}_{ij}$ is that from small to large scales. Since we use the Gaussian filter in the present study, our decomposition is not orthogonal. Therefore, the transport equation for the turbulent energy $K^{(\sigma)}$ at a given scale can be complicated. However, taking into account that $Tr^{(S)}$ implies the energy transfer from the large to the small scale, we introduce a quantity

$$t_f(\sigma_S \rightarrow \sigma_u; y) = \frac{1}{u^{(\sigma_u)}_i} \left\{ -u^{[\sigma_u]}_i u^{[\sigma_S]}_j \delta^{(\sigma_S)}_{ij} - \left( -u^{[\sigma_S]}_i u^{[\sigma_u]}_j \delta^{(\sigma_S)}_{ij} \right) \right\} \quad (11)$$

and interpret this as the energy transfer rate from strain rates with the scale $\sigma_S$ to the $\sigma_u$-scale energy. If $t_f$ is positive, the $\sigma_S$-scale flow transfers the energy to the $\sigma_u$-scale flow; otherwise, the $\sigma_S$-scale flow reduces the energy with the scale $\sigma_u$. Similarly to the investigation of the scale-dependent enstrophy production rates, we also define the energy transfer from the mean flow to a given scale $\sigma_u$, i.e.

$$t_m(\sigma_u; y) = \frac{u^{[\sigma_u]}_i u^{[\sigma_u]}_j \delta^{(\sigma_S)}_{ij}}{u^{(\sigma_u)}_i^2}, \quad (12)$$

and the ratio

$$\Gamma_u(\sigma_S \rightarrow \sigma_u; y) = \frac{t_f(\sigma_S \rightarrow \sigma_u; y)}{t_m(\sigma_u; y)} \quad (13)$$
Figure 8. The ratio $\Gamma_u(\sigma_S \rightarrow \sigma_u)$ defined by (13) between the contribution of strain rates with scale $\sigma_S$ to the energy transfer at scale $\sigma_u^+ = 30(\circ), 60(\square), 120(\triangle), 240(\bullet)$ and $480(\blacksquare)$ and the contribution of the mean flow at given heights (a) $y^+ = 960$, (b) 240 and (c) 60. Black solid lines indicate the average conditioned by the large-scale upflow ($v_{\sigma_{\text{cond}}}^{+} > v_{\text{rms}}^{+}$ at $\sigma_{\text{cond}}^+ = 960$ at each height $y_r$), whereas blue dashed lines indicate the average conditioned by the large-scale downflow ($v_{\sigma_{\text{cond}}}^{+} < -v_{\text{rms}}^{+}$).
between the two averages. With a similar interpretation to (6), when \( \Gamma_u > 1 \) (and \( \Gamma_m > 0 \)), \( \sigma_S \)-scale flow transfers more energy to the \( \sigma_u \)-scale flow structures than the mean flow, whereas, when \( 0 < \Gamma_u < 1 \), the mean flow transfers more energy to \( \sigma_u \)-scale structures. When \( \Gamma_u < 0 \), the \( \sigma_u \)-scale energy is reduced by \( \sigma_S \)-scale flow on average. Therefore, by evaluating the scale-dependent energy transfer rates, we can discuss the inter-scale and mean-flow energy transfer. Similarly to the enstrophy production rate, we take the averages at a fixed \( y \) conditioned by the large-scale (\( \sigma_{\text{cond}}^+ = 960 \)) upflow (\( v|\sigma_{\text{cond}}| \langle y \rangle > v|\sigma_{\text{rms}}| \langle y \rangle \)) and downflow (\( v|\sigma_{\text{cond}}| \langle y \rangle < -v|\sigma_{\text{rms}}| \langle y \rangle \)) in the wall-normal direction.

We show, in figure 8, \( \Gamma_u \) evaluated for (a) \( y^+ = 960 \), (b) \( y^+ = 240 \) and (c) \( y^+ = 60 \). We have confirmed that \( \Gamma_m \) for the parameters shown in this figure is always positive. The results for \( y^+ = 960 \), which is in the log layer, show that \( \Gamma_u \) is similar to \( \Gamma_w \) shown in figure 7. For example, for the small-scale energy (\( \sigma_u^+ = 60, 120 \) and 240) in both upflow and downflow cases, the twice larger-scale strain rate contributes the most. Here, for \( \sigma_u^+ = 30 \), we cannot observe the trend that the contribution from twice larger scale is the largest because this scale is too small (\( \sigma_u/\eta < 10 \), where \( \eta \) is the Kolmogorov scale). Moreover, the contributions from the smaller scale (\( \sigma_S < \sigma_u \)) are negative. This implies that the smaller-scale (\( \sigma_S \)) reduces the energy at the larger scale (\( \sigma_u \)). In other words, the direction of the energy transfer is forward on average. This implies that the creation of small-scale vortices indeed corresponds to the energy cascading events. On the other hand, for “large scales” (\( \sigma_u^+ = 480 \)), although the contributions from the twice larger scale are the largest among the fluctuations, the energy transfer from the mean flow is more important (\( \Gamma_u < 1 \)). The results for the other heights \( y^+ = 240 \) (in the lower log layer) and \( y^+ = 60 \) (in the buffer layer) are similar to the results for \( y^+ = 960 \), if we refer to the scales comparable to the distance from the wall as “large scales”. Namely, at any height in turbulent channel flow, the large-scale energy is directly transferred from the mean flow, whereas smaller-scale energy is generated by an energy cascading process.

Incidentally, in the log layer (for example, open circles figure 8a), the comparison between black solid lines and blue dashed lines shows that the energy cascading process is more active in large-scale upflow regions than downflow ones. This is similar to the observation in figure 7, which shows the contributions of enstrophy production rates. These results also support the idea that the energy cascading process sustains small-scale structures in the region away from the wall.

It is also important to observe again that the balance of the scale-dependent contributions to the vortex stretching (figure 7) and to the energy transfer (figure 8) is similar to each other. This result gives a direct evidence that the vortex-stretching process plays a major role in the energy cascade [9, 10, 23, 25, 26]. Although we have interpreted \( \omega_i^{[\sigma_u]} S_{ij}^{[\sigma_s]} \omega_j^{[\sigma_u]} \) and \( u_i^{[\sigma_u]} u_j^{[\sigma_u]} S_{ij}^{[\sigma_s]} \) as the enstrophy production and the energy transfer between the two scales, this aspect of our work needs more careful consideration. In particular, it is an important future issue to examine spatial correlations between the enstrophy production at a given scale and energy transfer to that scale.

4. Conclusions

To reveal the hierarchy of coherent structures in wall-bounded turbulence and to understand its sustaining mechanism, we have analyzed the database of fully developed turbulent channel flow at \( Re_x = 4179 \) [14]. The key to our analyses is the scale decomposition using a band-pass filter, which is composed of the combination (2) of the Gaussian filters at two different scales.

Using the band-pass filtered velocity and its gradients, we visualized the hierarchy of vortices and low-speed structures (figure 2). An important observation is that the largest-scale structures at each height are composed of low-speed streaks and quasi-streamwise vortices located at the sides of the streak in a staggered manner (figure 3). The structures (figure 4) obtained by the
simple conditional average are consistent with this observation. These observations are also similar to those in the over-damped LES [11, 12, 13]. Moreover, since the observed structures (figure 3) are similar to the coherent structures in the buffer layer [3], the present result suggests that the hierarchy of quasi-streamwise vortices and low-speed streaks is maintained by SSP. Here, we reemphasize that these quasi-streamwise vortices are the largest vortices at each height in the sense that their size is comparable to the distance from the wall. By evaluating the scale-dependent contributions of the vortex stretching (6) and the energy transfer (13), we have shown that these largest-scale vortices are stretched predominantly by the mean shear (figure 7) and the largest-scale energy is transferred directly from the mean shear (figure 8). These contributions of the mean shear may be incorporated in the SSP. In contrast, smaller-scale vortices (i.e. smaller than the distance from the wall) are stretched by strain-rate fields at scales one to eight time larger (for example, see open symbols in figure 7a). In particular, the contribution from the twice larger scale is most significant. These trends are similar to the results of the energy transfer (for example, see open symbols in figure 8a). Incidentally, vortices at a given scale are contracted by smaller scales (for example, see open symbols for \( \sigma_S < \sigma_u \) in figure 7a), and at the same time, the energy of the given scale is reduced by the smaller scales (for example, see open symbols for \( \sigma_S < \sigma_u \) in figure 8a). These results in terms of the vortex stretching and the energy transfer are consistent with the picture of the energy cascade. In addition, we have shown that the cascading events are stronger in the large-scale upflow (ejection) regions. This is because the source of the vorticity, rather than the small-scale vortices themselves, are carried from the wall.

Thanks to the analysis by using the database of high-Reynolds-number turbulent channel flow [14], we have developed our understanding on the sustaining mechanism of wall-bounded turbulence. However, there remain several future issues. (i) Although the analyses of the vortex stretching are reasonable and consistent with the picture of the energy cascade, the arguments regarding the inter-scale energy transfer are preliminary. We need further examination, in particular, of the physical meaning of the quantity defined by (11). It is also unclear in the present analysis whether or not the energy transfer to smaller scales occurs at the same spatial locations where smaller-scale vortices are stretched and amplified. (ii) The largest-scale structures at each height are likely to be affected by each other. We need to investigate the interactions between them. (iii) We have shown the generation mechanism of the hierarchy of multi-scale vortices in the log and buffer layers. The shown results for the channel flow are similar to those for a turbulent boundary layer [8]. However, we have not yet shown the sustaining mechanism for vortices in the outer layer. This is an interesting problem because there would appear to be a difference between channel flow and turbulent boundary layers.

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