Isometric Path Partition Problem on Tree Derived Architectures

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Abstract. An isometric path partition of a graph $G$ is defined as a set of isometric paths that partition the vertex set of $G$. The problem of isometric path partition is to obtain a minimum isometric path partition of $G$. The isometric path partition number of a graph $G$, denoted by $ip_p(G)$ is the minimum cardinality of isometric path partition in $G$. In this paper, we compute the isometric path partition number of certain tree derived architectures like hypertrees, 1-rooted sibling trees, $k$-rooted sibling trees, $l$-complete binary trees, and $l$-sibling trees.

1. Introduction

Let $G = (V, E)$ be a simple, finite and undirected connected graph with the vertex set $V$ and edge set $E$. For graph theoretical terminologies we refer to Bondy & Murty [2]. A shortest path between any pair of vertices is referred to, as an isometric path. It is also referred as geodesic. The problem of isometric path has wide application in the field of communication networks more particularly in broadcasting problems. A set of isometric paths which cover the vertex set $V(G)$ is said to be an isometric path cover. A minimum cardinality isometric path cover is referred to an isometric path cover number that is denoted by $ip_c(G)$. A path partition of a graph $G$ is a collection of vertex disjoint paths whose union is $V(G)$. An isometric path partition is defined as a set of vertex disjoint isometric paths whose union is the vertex set of $G$. We denote a set of isometric paths that induces an isometric path partition of $G$ by $S_n$. The isometric path partition number of $G$ is the minimum cardinality among all such isometric path partitions and it is denoted by $ip_p(G)$. A diametral isometric path is an isometric path whose length is equal to the diameter of the graph.

In the recent few years, the problem of the isometric path problems has received much attention. The concept of the isometric path cover was first used by Aggarwal et al. [1] while designing the algorithm for VLSI layouts. Later, the isometric path cover number has been computed for trees, cycles, the grid, complete bipartite graphs, the block graphs, the Cartesian product of paths, complete $r$-partite graphs, hypercube $Q_r$, and Cartesian product of 2 or 3 complete graphs. A lower bound for the isometric path cover number was given by Fisher and Fitzpatrick [3]. The isometric path partition problem has been proved to be $NP$-complete by Paul Manuel in [6]. Also he has determined the isometric path partition number of the $(r \times s)$-dimensional grid, the $(r \times r)$-dimensional torus and the $r$-dimensional benes network. In this paper, we have computed the isometric path partition number of hypertrees, 1-rooted sibling trees, $k$-rooted sibling trees, $l$-complete binary trees and $l$-sibling trees.
2. Hypertree

A basic skeleton of a hypertree is a complete binary tree \( T_n \) of height \( n \). The labeling of nodes are as follows: Assign the label 1 to the root vertex (represents level 0). The left and right children of a parent vertex \( r \) are labeled as \( 2r \) and \( 2r + 1 \) respectively (Refer figure1). In a hypertree, new links are drawn horizontal (referred as hyperedge) and two vertices are joined in the same level \( i \) of the tree if their label difference is \( 2^i \). We denote an \( n \)-level hypertree as \( HT(n) \). It has \( 2^{n+1} - 1 \) vertices and \( 3(2^{n} - 1) \) edges. Indra Rajasingh et al [9] have given another representation for \( HT(n) \) (Refer figure 2(a)). The root-fault hypertree is obtained from hypertree by removing the root vertex of \( HT(n) \). It is denoted by \( HT^*(n) \), \( n \geq 2 \) [5] (Refer figure 2(b)).

![Diagram](image1.png)

**Figure 1. Two equivalent representation of \( HT(3) \).**

2.1. Isometric path partition number of Hypertree \( HT(n) \)

**Input:** Hypertree \( HT(n) \), \( n \geq 2 \).

**Algorithm:** Label the vertices of \( HT(n) \) from the root vertex and proceed to the next higher levels from left to right as 1, 2, 3, ... , \( 2^{n+1} - 1 \). Let \( P_{n,m} \) and \( P'_{n,m} \) denote the isometric paths \( P_n \) contained in \( S_m \) of \( HT(n) \) and \( S'_m \) denotes isometric path partition of \( HT(n) \setminus \{1\} \).

(i) Select \( HT(2) \). Then \( S_2 = \{ P_{2,4}, P_{2,3} \} \) where \( P_{2,4} : 4 \rightarrow 6 \rightarrow 7 \) and \( P_{2,3} : 1 \rightarrow 2 \rightarrow 5 \).

(ii) Select \( HT(3) \). Then \( S_3 = \{ P_{3,6}, P'_{3,6}, P_{3,2}, P_{3,1} \} \) where \( P_{3,6} : 8 \rightarrow 12 \rightarrow 6 \rightarrow 3 \rightarrow 7 \rightarrow 15 \), \( P'_{3,6} : 13 \rightarrow 9 \rightarrow 4 \rightarrow 2 \rightarrow 5 \rightarrow 11 \), \( P_{3,2} : 10 \rightarrow 14 \) and \( P_{3,1} : 1 \).

(iii) Select \( HT(2) \setminus \{1\} \). Then \( S'_2 = \{ P_{2,4}, P_{2,2} \} \).

(iv) Select \( HT(3) \setminus \{1\} \). Then \( S'_3 = \{ P_{3,6}, P_{3,6}, P_{3,2} \} \).

(v) Select \( HT(4) \). Then \( S_4 = \{ P_{4,8}, P'_{4,8}, P_{4,2}, P_{4,1} \} \cup \{ 2 \text{ copies of } (S'_2) \} \).

(vi) Inductively \( S_n = \{ P_{n,2n}, P'_{n,2n}, P_{n,2}, P_{n,1} \} \cup \{ 2 \text{ copies of } (S'_2 \cup S'_3 \cup \ldots \cup S'_{n-2}) \} \).

**Output:** \( ip_{n}(HT(n)) = n^2 - 3n + 4 \), \( n \geq 2 \).

**Proof of correctness:** In \( HT(n) \), identify the two disjoint diametral isometric paths of length \( 2n - 1 \) covering all the boundary vertices of \( HT(n) \) except the root vertex (Refer figure 2(a)). Then the removal of the two diametral isometric paths results in the disconnected graph with the components (i) an isolated vertex (root vertex) (ii) a \( P_i \) path and (iii) 2 copies of \( HT(i) \setminus \{1\} \) for \( 2 \leq i \leq n - 2 \) (also called as Root-fault hypertree \( HT^*(i) \)). Also we observe that \( ip_{n}(HT(2) \setminus \{1\}) = ip_{n}(HT(2)) \) and \( ip_{n}(HT(i) \setminus \{1\}) = ip_{n}(HT(i)) - 1 \) for \( 3 \leq i \leq n - 2 \) (Refer figure 2(a)). Hence by induction, the isometric path partition of \( HT(n) \) is obtained by the union of the isometric path partitions of all the components along with the two diametral isometric paths of \( HT(n) \).

**Theorem 2.1.1:** Let \( G \) be the hypertree \( HT(n) \), \( n \geq 2 \). Then \( ip_{n}(G) = n^2 - 3n + 4 \), \( n \geq 2 \).

**Corollary 2.1.2:** Let \( G \) be the root-fault hypertree \( HT^*(n) \), \( n \geq 2 \). Then \( ip_{n}(G) = n^2 - 3n + 3 \), \( n \geq 2 \).
3. 1-Rooted sibling tree

Sibling tree is obtained from complete binary tree by adding new edges between the left and right children of the same parent vertex [5] (Refer figure 3(a)). It is denoted by $ST_n$. The 1-rooted sibling tree $ST_n^1$ is attained by adding a pendant vertex to the root vertex of sibling tree (Refer figure 3(b)). The edges between the left and right children of the same parent vertex are referred as sibling edges. It has $2^n$ vertices and $3 \cdot 2^{n-1} - 2$ edges [7].

3.1. Isometric path partition number of 1-Rooted sibling tree $ST_n^1$

Input: 1-rooted sibling tree $ST_n^1$, $n \geq 2$.

Algorithm: Label the vertices of $ST_n^1$ from the root vertex as 1, 2, 3, ... , $2^n$, where the children of the vertex $r$ are labeled as $2r$ and $2r + 1$ respectively. Let $P_{m,n}$ denote the isometric paths $P_n$ contained in $S_m$ of $ST_n^1$ and $S_n'$ denotes isometric path partition of $ST_n^1 \setminus \{1\}$. Denote the three isolated vertices of $ST_3^1 \setminus \{1\}$ by $P_{3,1}$, $P_{3,1}'$ and $P_{3,1}''$.

(i) Select $ST_2^1$. Then $S_2 = \{P_{2,3}, P_{2,1}\}$ where $P_{2,3} : 1 \rightarrow 2 \rightarrow 3$ and $P_{2,1} : 4$.

(ii) Select $ST_3^1$. Then $S_3 = \{P_{3,4}, P_{3,2}, P_{3,1}, P_{3,1}'\}$ where $P_{3,4} : 5 \rightarrow 3 \rightarrow 4 \rightarrow 8$, $P_{3,2} : 1 \rightarrow 2$, $P_{3,1} : 6$ and $P_{3,1}' : 7$.

(iii) Select $ST_2^1 \setminus \{1\}$. Then $S_2' = \{P_{2,2}, P_{2,1}\}$.

(iv) Select $ST_3^1 \setminus \{1\}$. Then $S_3' = \{P_{3,4}, P_{3,1}, P_{3,1}', P_{3,1}''\}$.

(v) Select $ST_4^1$. Then $S_4 = \{P_{4,6}, P_{4,2}, P_{4,1}, P_{4,1}'\} \cup \{2 \text{ copies of } (S_2')\}$.

(vi) Inductively, $S_n = \{P_{n,2(n-1)}, P_{n,2}, P_{n,1}, P_{n,1}'\} \cup \{2 \text{ copies of } (S_2' \cup S_3' \cup ... \cup S_{n-2}')\}$. 

Output: $ip_n(ST_n^1) = 2^{n-1}, n \geq 2$.

Proof of correctness: In $ST_n^1$, consider the diametral isometric path passing through the vertices ($2^{n-1} + 1$, $2^{n-2} + 1$, ..., 5, 3, 4, 8, ..., $2^{n-1}$, 2$^n$). On removing the diametral isometric path, the graph $ST_n^1$ gets disconnected with the components (i) two isolated vertices (each represents a $P_1$ path) (ii) a $P_2$
path and (iii) two copies of \( ST^i \setminus \{1\} \) for \( 2 \leq i \leq n - 2 \) (also called as Sibling tree \( ST \)). It is observed that \( ip_{\rho}(ST^i \setminus \{1\}) = ip_{\rho}(ST^i) \) for \( 2 \leq i \leq n - 2 \) (Refer figure 4). Hence by induction, the isometric path partition of \( ST^i \) is the union of the isometric path partitions of all the above components together with the diametral isometric path.

**Theorem 3.1.1:** Let \( G \) be a 1-rooted sibling tree \( ST^1_n, n \geq 2 \). Then \( ip_{\rho}(G) = 2^{n-1}, n \geq 2 \).

**Corollary 3.1.2:** Let \( G \) be a sibling tree \( ST_n, n \geq 2 \). Then \( ip_{\rho}(G) = 2^{n-1}, n \geq 2 \).

**4. k-Rooted sibling tree**

The \( k \)-rooted sibling tree \( ST^k_n \) is attained from \( k \) vertex disjoint 1-rooted sibling tree \( ST^1_n \) with the roots say \( r_1, r_2, \ldots, r_k \) and adding edges \((r_i, r_{i+1}), 1 \leq i \leq k - 1\) (Refer figure 5). The diameter of \( ST^k_n \) is \( 2n + k - 1 \) for \( k \geq 2 \) [7].

**4.1. Isometric path partition number of \( k \)-rooted sibling tree \( ST^k_n \)**

**Input:** \( k \)-rooted sibling tree \( ST^k_n \), \( k, n \geq 2 \).

**Algorithm:** Identify the diametral isometric path of \( ST^k_n \). Then removal of the vertices covered by the diametral isometric path, results in a disconnected subgraph. The collection of isometric path partitions of the above disconnected graph components together with the diametral isometric path gives the isometric path partition of \( ST^k_n \).

**Output:** \( ip_{\rho}(ST^k_n) = k 2^{n-1} - 1, \) for \( k, n \geq 2 \).

**Proof of correctness:** Consider \( ST^k_n \), in which there exist a diametral isometric path of length \( 2n + k - 1 \). Next on removing the diametral isometric path, the graph \( ST^k_n \) gets disconnected with the following components (i) 2 isolated vertices (each represents a \( P_1 \) path) (ii) 2 copies of \( ST^i_i \) for \( 2 \leq i \leq n - 1 \) and (iii) \( k - 2 \) copies of \( ST^1_n \) (Refer figure 5). Hence the collection of the isometric path partitions of all the above components together with the diametral isometric path provides the output.

**Theorem 4.1.1:** Let \( G \) be a \( k \)-rooted sibling tree \( ST^1_n \), \( k, n \geq 2 \). Then \( ip_{\rho}(G) = k 2^{n-1} - 1, \) for \( k, n \geq 2 \).
5. **l-Complete binary tree**

A graph obtained from the two copies of complete binary tree \( T_n \) say \( T_1, T_2 \) by joining each vertex of \((n - 1)\)th level of \( T_1 \) to the corresponding \((n - 1)\)th level vertex of \( T_2 \) is referred as \( l \)-complete binary tree and is denoted by \( l-T_n \) \cite{5} (Refer figure 6).

5.1. **Isometric path partition number of \( l \)-complete binary tree \( l-T_n \)**

**Input:** \( l \)-complete binary tree \( l-T_n \), \( n \geq 2 \).

**Algorithm:** First identify the two disjoint diametral isometric paths covering all the boundary vertices of \( l-T_n \). Then removal of the vertices covered by the above diametral isometric paths, results in a disconnected subgraph. The collection of isometric path partitions of the above disconnected graph components together with the diametral isometric paths gives the isometric path partition of \( l-T_n \).

**Output:**

\[
\text{ip}_p(l-T_n) = \begin{cases} 
\frac{1}{3}(2^n - 2) & \text{if } n \text{ is odd} \\
\frac{1}{3}(2^n + 2) & \text{if } n \text{ is even}
\end{cases}
\]

**Proof of correctness:** Consider \( l-T_n \), in which there exist the two disjoint diametral isometric paths of length \( 2(n - 1) \) covering all the boundary vertices of \( l-T_n \). Next on removing such diametral isometric paths, the graph \( l-T_n \) gets disconnected into the components namely two copies of \( l-T_i \), \( 2 \leq i \leq n - 2 \) (Refer figure 6). The collection of the isometric path partitions of all the above components together with the diametral isometric paths gives the output.

**Theorem 5.1.1:** Let \( G \) be a \( l \)-complete binary tree \( l-T_n \), \( n \geq 2 \). Then \( \text{ip}_p(G) = \begin{cases} 
\frac{1}{3}(2^n - 2) & \text{if } n \text{ is odd} \\
\frac{1}{3}(2^n + 2) & \text{if } n \text{ is even}
\end{cases} \)
6. l-Sibling tree

A graph obtained from the two copies of rooted sibling tree $ST_n^k$, say $ST_1^k$, $ST_2^k$ by drawing new edges between each vertex of $(n - 1)^{th}$ level of $ST_1^k$ and the corresponding $(n - 1)^{th}$ level vertex of $ST_2^k$ is referred as l-sibling tree and is denoted by $l-ST_n^k$ [5] (Refer figure 7).

6.1. Isometric path partition number of l-sibling tree $l-ST_n^k$

Input: l-sibling tree $l-ST_n^k$, $k, n \geq 2$.

Algorithm: Label the vertices of $l-ST_n^k$ as $1, 2, \ldots, 2^{n+1}, 2^{n+1} + 1, \ldots, 2(2^{n+1}), 2(2^{n+1}) + 1, \ldots, 3(2^{n+1}), \ldots, (k-1)2^{n+1}, ((k-1)2^{n+1}) + 1, \ldots, k2^{n+1}$ (Refer figure 7). First identify the two disjoint diametral isometric paths covering all the boundary vertices of $l-ST_n^k$. Then removal of the vertices covered by the above diametral isometric paths, results in a disconnected subgraph. The collection of the isometric path partitions of all the above components together with the diametral isometric paths gives the output.

Output: $ip_p(l-ST_n^k) = k2^{n-1} - 1$, for $k, n \geq 2$.

Proof of correctness: Consider $l-ST_n^k$, in which there exist the two disjoint diametral isometric paths of length $2n + k$ covering all the boundary vertices of $l-ST_n^k$ (Refer figure 8). Next on removing such diametral isometric paths, the graph $l-ST_n^k$ gets disconnected with the components (i) a $P_2$ path (ii) two copies of $(l-ST_n^k) \setminus \{1, 2^{n+1}\}$ for $2 \leq i \leq n - 1$ and (iii) $k - 2$ copies of $(l-ST_n^k) \setminus \{1, 2^{n+1}\}$. It is observed that, for $2 \leq i \leq n$, $ip_p(l-ST_n^k) \setminus \{1, 2^{n+1}\}) = ip_p((l-ST_n^k))$ (Refer Figure 8). Hence the collection of the isometric path partitions of all the above components together with the diametral isometric paths gives the result.

Theorem 6.1.1: Let $G$ be a l-sibling tree $l-ST_n^k$, $k, n \geq 2$. Then $ip_p(G) = k2^{n-1} - 1$, for $k, n \geq 2$. 

Figure 7. $l - ST_2^3$. 

Figure 8. Isometric path partition of $l - ST_5^3$. 

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7. Conclusion
The problem of isometric path partition is to find the minimum cardinality isometric path partition of a given graph $G$. Finding a generalized algorithm to compute the isometric path partition number of interconnection networks is quite challenging. In this paper, we have determined the isometric path partition number of certain tree derived architectures like hypertrees, 1-rooted sibling trees, $k$-rooted sibling trees, $l$-complete binary trees, and $l$-sibling trees.

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