THE CLASSIFICATION OF HYPERELLiptIC THREEFOLDS

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Abstract. We complete the classification of hyperelliptic threefolds, describing in an elementary way the hyperelliptic threefolds with group $D_4$. These are algebraic and form an irreducible 2-dimensional family.

Introduction

A Generalized Hyperelliptic Manifold $X$ is defined to be a quotient $X = T/G$ of a complex torus $T$ by the free action of a finite group $G$ which contains no translations. We say that $X$ is a Generalized Hyperelliptic Variety if moreover the torus $T$ is projective, i.e., it is an Abelian variety $A$.

The main purpose of the present paper is to complete the classification of the Generalized Hyperelliptic Manifolds of complex dimension three. The cases where the group $G$ is Abelian were classified by H. Lange in [La01], using work of Fujiki [Fu88] and the classification of the possible groups $G$ given by Uchida and Yoshihara in [UY76]: the latter authors showed that the only possible non Abelian group is the dihedral group $D_4$ of order 8. This case was first excluded but it was later found that it does indeed occur (see [CD18] for an account of the story and of the role of the paper [DHS08]). Our paper is fully self-contained and show that the family described in [CD18] gives all the possible hyperelliptic threefolds with group $D_4$.

Our main theorem is the following

Theorem 0.1. Let $T$ be a complex torus of dimension 3 admitting a fixed point free action of the dihedral group

$$G := D_4 := \langle r, s \mid r^4 = 1, s^2 = 1, (rs)^2 = 1 \rangle,$$

such that $G = D_4$ contains no translations.

Then $T$ is algebraic. More precisely, there are two elliptic curves $E, E'$ such that:

1. $T$ is a quotient $T := T'/H$, $H \cong \mathbb{Z}/2$, where

$$T' := E \times E \times E' =: E_1 \times E_2 \times E_3,$$

$$H := \langle \omega \rangle, \quad \omega := (h + k, h + k, 0) \in T'[2],$$

and $h, k$ are 2-torsion element $h, k \in E[2]$, such that $h, k \neq 0, h + k \neq 0$;

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(II) there is an element \( h' \in E' \) of order precisely 4, such that, for \( z = (z_1, z_2, z_3) \in T' \):
\[
 r(z) = (z_2, -z_1, z_3 + h') = R(z_1, z_2, z_3) + (0, 0, h'),
 s(z) = (z_1 + h, -z_2 + k, -z_3) = S(z_1, z_2, z_3) + (h, k, 0).
\]
Conversely, the above formulae give a fixed point free action of the dihedral group \( G = D_4 \) which contains no translations.
In particular, we have the following normal form:
\[
 E = \mathbb{C}/(\mathbb{Z} + Z \tau), \quad E' = \mathbb{C}/(\mathbb{Z} + Z \tau'), \quad \tau, \tau' \in H := \{ z \in \mathbb{C} | \text{Im}(z) > 0 \},
 h = 1/2, k = \tau/2, h' = 1/4
 r(z_1, z_2, z_3) := (z_2, -z_1, z_3 + 1/4)
 s(z_1, z_2, z_3) := (z_1 + 1/2, -z_2 + \tau/2, -z_3).
\]
In particular, the Teichmüller space of hyperelliptic threefolds with group \( D_4 \) is isomorphic to the product \( \mathbb{H}^2 \) of two upper halfplanes.

1. Proof of the main theorem

We use the following notation: \( T = V/\Lambda \) is a complex torus of dimension 3, which admits a free action of the group
\[
 G = \langle r, s \rangle^4 = s^2 = (rs)^2 = 1 \cong D_4,
\]
such that the complex representation \( \rho: G \to \text{GL}(3, \mathbb{C}) \) is faithful.
A first observation is that the complex representation \( \rho \) of \( G \) must contain the 2-dimensional irreducible representation \( V_1 \) of \( G \) (else, \( \rho \) would be a direct sum of 1-dimensional representations: this, by the assumption on the faithfulness of \( \rho \), would imply that \( G \) is Abelian, a contradiction).
Hence we have a splitting \( V = V_1 \oplus V_2 \),
where \( V_2 \) is 1-dimensional, and we can choose an appropriate basis so that, setting \( \hat{R} := \rho(r), S := \rho(s) \), we are left with the two cases

**Case 1:** \[
 R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 \end{pmatrix},
\]

**Case 2:** \[
 R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

which are distinguished by the multiplicity of the eigenvalue 1 of \( S \).
Indeed \( R \) is necessarily of the form above, since the freeness of the \( G \)-action implies that \( \rho(g) \) must have eigenvalue 1 for every \( g \in G \).

**Lemma 1.1.** In both Cases 1 and 2, the complex torus \( T = V/\Lambda \) is isogenous to a product of three elliptic curves, \( T \sim \text{iso}. E_1 \times E_2 \times E_3 \), where \( E_i \subset T \), for \( i = 1, 2, 3 \) and \( E_1 \) and \( E_2 \) are isomorphic elliptic curves. In other words, writing \( E_j = W_j/\Lambda_j \), the complex torus \( T \) is isomorphic to
\[
 (E_1 \times E_1 \times E_3)/H, \quad H = \Lambda/(\Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3).
\]
Proof. Let $I$ be the identity of $T$.
In Case 1, we set $E := \ker(S - I)^0 = \text{im}(S + I)$, $E_3 := \ker(R - I)^0$ and 
$E_2 := R(E_1)$ (here, the superscript zero denotes the connected component of 
the identity). Then it is clear that $E_1 \cong E_2$, and that $T$ is isogenous to 
$E_1 \times E_2 \times E_3$.
In Case 2, we define similarly $E_2 := \ker(S + I)^0 = \text{im}(S - I)$, $E_3 := \ker(R - 
I)_0$ and $E_1 := R(E_2)$. We obtain again $E_1 \cong E_2$, and that $T$ is isogenous to 
$E_1 \times E_2 \times E_3$.

\[ \square \]

Lemma 1.2. Writing $E_j = W_j/\Lambda_j$, the following statements hold.

1. In Case 1, the lattice $\Lambda_2$ is equal to $W_2 \cap \Lambda$.
2. In Case 2, the lattice $\Lambda_1$ is equal to $W_1 \cap \Lambda$.

Proof. (1) Obviously, $E_2 = R(E_1) = W_2/R(\Lambda_1)$, i.e., $\Lambda_2 = R(\Lambda_1) \subset W_2 \cap \Lambda$.
On the other hand, $R(W_2 \cap \Lambda) \subset W_1 \cap \Lambda = \Lambda_1$, and applying the automorphism $R$ of $\Lambda$ gives $W_2 \cap \Lambda \subset R(\Lambda_1) = \Lambda_2$.

(2) Here, $E_1 = R(E_2) = W_1/R(\Lambda_2)$, i.e., $\Lambda_1 = R(\Lambda_2) \subset W_1 \cap \Lambda$. For the converse inclusion, observe $R(W_1 \cap \Lambda) \subset W_2 \cap \Lambda = \Lambda_2$, and applying $R$ yields again the result.

We can now choose coordinates on $V$ such that $r$ is induced by a transformation of the form
\[ r(z_1, z_2, z_3) = (z_2, -z_1, z_3 + c_3), \]
by choosing as the origin in $V_1$ a fixed point of the restriction of $r$ to $V_1$.
We can now view $r, s$ as affine self maps of $T$ induced by affine self maps of 
$E_1 \times E_2 \times E_3$ of the form
\[ r(z_1, z_2, z_3) = (z_2, -z_1, z_3 + c_3), \]
\[ s(z_1, z_2, z_3) := (z_1 + a_1, -z_2 + a_2, \pm z_3 + a_3), \]
and sending the subgroup $H$ to itself.

Lemma 1.3. The freeness of the action of the powers of $r$ is equivalent to: $H$ contains no element with last coordinate equal to $c_3$, or $2c_3$.
Moreover, $(0, 0, 4c_3) \in H$.

Proof. $r(z) = z$ is equivalent to $(z_1 - z_2, z_1 + z_2, -c_3) \in H$. However, the endomorphism
\[ (z_1, z_2) \mapsto (z_1 - z_2, z_1 + z_2) \]
of $E_1 \times E_2$ is surjective, hence $H$ cannot contain any element with last coordinate equal to $c_3$.
Since $r^2(z) = (-z_1, -z_2, z_3 + 2c_3)$, $r^2(z) = z$ is equivalent to $(-2z_1, -2z_2, 2c_3) \in 
H$, and we reach the similar conclusion that $H$ cannot contain any element with last coordinate equal to $2c_3$.
Finally, the condition that $r^4$ is the identity is equivalent to $(0, 0, 4c_3) \in H$.

\[ \square \]

Proposition 1.1. Case 2 does not occur.
Proof. Since we assume that
\[ s(z_1, z_2, z_3) := (z_1 + a_1, -z_2 + a_2, z_3 + a_3), \]
and that \( s^2 \) is the identity, it must be
\[ (2a_1, 0, 2a_3) \in H. \]
Consider now \( rs \):
\[ rs(z) = (-z_2 + a_2, -z_1 - a_1, z_3 + a_3 + c_3). \]
The condition that \((rs)^2\) is the identity is equivalent to:
\[ (a_1 + a_2, -(a_1 + a_2), 2(a_3 + c_3)) \in H. \]
This condition, plus the previous one, imply that
\[ (a_2 - a_1, -(a_1 + a_2), 2c_3) \in H, \]
contradicting Lemma 1.3. □

Henceforth we shall assume that we are in Case 1, and we can choose the
origin in \( E_3 \) so that
\[ s(z_1, z_2, z_3) := (z_1 + a_1, -z_2 + a_2, -z_3). \]

**Lemma 1.4.** If
\[ s(z_1, z_2, z_3) := (z_1 + a_1, -z_2 + a_2, -z_3), \]
then
\[ (2a_1, 0, 0) \in H \]
and \( H \) contains no element of the form
\[ (a_1, w_2, w_3). \]

**Proof.** The first condition is equivalent to \( s^2 \) being the identity, while the
second is equivalent to the condition that \( s \) acts freely, since \( s(z) = z \) is
equivalent to \((a_1, -2z_2 + a_2, -2z_3) \in H. \)

\[ \square \]

**Proposition 1.2.** For each \( \lambda \in \Lambda \) there exist \( \lambda' \in \Lambda, \lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2, \lambda_3 \in \Lambda_3 \), such that
\[ 2\lambda = \lambda_1 + \lambda', \quad 2\lambda' = \lambda_2 + \lambda_3 \]
More precisely, we even have:
\[ \Lambda \subset (1/2)\Lambda_1 + (1/2)\Lambda_2 + (1/4)\Lambda_3. \]

**Proof.** Let \( \lambda \in \Lambda \): we can write
\[ 2\lambda = (I + S)\lambda + (I - S)\lambda. \]
Furthermore, since \( \lambda' \in \text{im}(I - S) \), we obtain
\[ 2\lambda' = (I + R^2)\lambda' + (I - R^2)\lambda' \]
Hence, \( \lambda = \frac{\lambda_1}{4} + \frac{\lambda_2}{4} + \frac{\lambda_3}{4} \) for unique \( \lambda_j \in \Lambda_j. \)
Applying the automorphism $R$ of $\Lambda$ and the unicity of the $\lambda_j$ yields the result, since $R$ exchanges $\Lambda_1$ and $\Lambda_2$.

\[ \square \]

**Proposition 1.3.** We have $\Lambda \subset \frac{1}{2}\Lambda_1 + \frac{1}{2}\Lambda_2 + \frac{1}{2}\Lambda_3$.

**Proof.** For $\lambda \in \Lambda$ we can write $\lambda = \frac{\lambda_1}{2} + \frac{\lambda_2}{2} + \frac{\lambda_3}{2}$ for unique $\lambda_j \in \Lambda_j$. We now use the property $E_i \hookrightarrow T \Rightarrow \forall (0,0,d) \in H, d = 0$. Indeed, $2\lambda = \lambda_1 + \lambda_2 + \frac{\lambda_3}{2}$, hence $(0,0,[\frac{\lambda_3}{2}]) \in H$ and $\frac{\lambda_3}{2} = 0$ in $E_3$. Equivalently, there is an element $\lambda'_3 \in \Lambda_3$ with $\lambda_3 = \frac{\lambda'_3}{2}$.

\[ \square \]

**Lemma 1.5.** Consider the transformation $rs$:

$rs(z) = (-z_2 + a_2, -z_1 - a_1, -z_3 + c_3)$.

The condition that its square is the identity amounts to

$$(a_1 + a_2, -(a_1 + a_2), 0) \in H,$$

while the freeness of its action is equivalent to the fact that $H$ contains no element of the form

$$(w_1 - a_2, w_1 + a_1, w_3) \iff \forall (d_1, d_2, d_3) \in H: \ d_1 + a_2 \neq d_2 - a_1.$$

**Proof.** The first condition is straighforward, while the freeness of the action is equivalent to the non existence of $(z_1, z_2, z_3)$ such that

$$(z_1 + z_2 - a_2, z_2 + z_1 + a_1, 2z_3 - c_3) \in H.$$ 

As usual, we observe that for each $w_1, w_3$ there exist $z_1, z_2, z_3$ with $z_1 + z_2 = w_1, 2z_3 - c_3 = w_3$.

\[ \square \]

We put together the conclusions of Lemmas 1.3, 1.4, 1.5

- (i) $(0, 0, 4c_3) \in H$
- (ii) $(2a_1, 0, 0) \in H$
- (iii) $(a_1 + a_2, -a_1 - a_2, 0) \in H$, hence also $(a_1 - a_2, a_1 + a_2, 0) \in H$.

(1) $H$ contains no element of the form $(w_1, w_2, c_3)$,
(2) nor of the form $(w_1, w_2, 2c_3)$
(3) nor of the form $(a_1, w_2, w_3)$
(4) nor of the form $(w_1, w_2, w_3)$ with $w_1 + a_2 = w_2 - a_1$.

It follows from (iii) and (3) that $a_2 \neq 0$. While the condition that each element of $H$ which has two coordinates equal to zero is indeed zero (since $E_i$ embeds in $T$) imply

$2a_1 = 0, 4c_3 = 0$.

By conditions (1), (2), (3) the elements $a_1, c_3$ have respective orders exactly 2, 4. Moreover:
• (4) and (i) imply that $a_1 + a_2 \neq 0$
• (ii), (iii) and the fact that $H$ has exponent 2 implies $2a_2 = 2a_1 = 0$, $2a_1 + 2a_2 = 0$. Hence $a_1 \neq a_2$ are nontrivial 2-torsion elements.

We have thus obtained the desired elements
\[ h := a_1, k := a_2, h' := c_3. \]

It suffices to show that $H$ is generated by $\omega := (h + k, h + k, 0) = (a_1 + a_2, a_1 + a_2, 0)$.

Observe first that $\omega \in H$, by condition (iii).

Condition (4) implies that the first coordinate of an element of $H$ must be a multiple of $(a_1 + a_2)$: since it cannot equal $a_1$, by condition (3), and if it equals $a_2$, we can add $\omega$ and obtain an element of $H$ with first coordinate $a_1$. Using $R$, we infer that both coordinates must be a multiple of $(a_1 + a_2)$.

Possibly adding $\omega$, we may assume that $w_1 = 0$: then by (4) we conclude that also $w_2 = 0$. Finally, the condition that each element of $H$ which has two coordinates equal to zero is indeed zero, show that $H$ is then generated by $\omega$, as we wanted to show.

The last assertions of the main theorem follow now in a straightforward way (see [CC17] concerning general properties of Teichmüller spaces of hyperelliptic manifolds).

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