SHOOTING A CLUB WITHFINITE CONDITIONS

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Abstract. We study cohabitation of the poset shooting a club through a given stationary subset of $\omega_1$ with finite conditions with other forcings.

Definition 1.

1. If $I \subset \omega_1$ is a countable interval of ordinals and $S \subset \omega_1$ is countable, we define $P_{S,I} = \{p : p$ is a finite function from $I$ to $S$ such that there exists $f : I \to S \cap \alpha$ increasing continuous, $\text{rng}(f)$ unbounded in $S, p \subset f\}$.

2. Let $S \subset \omega_1$ be stationary, $I = \omega_1 \setminus \alpha$ for some $\alpha < \omega_1$. Then $P_{S,I} = \{p : p$ is a finite function from $I$ to $S$ such that $\exists \alpha < \beta < \omega_1 \exists f : \beta \setminus \alpha \to S$ continuous increasing and $p \subset f\}$. The order is by inverse inclusion. $P_S = P_{S,\omega_1}$.

We will be interested in $P_S$ for various $S \subset \omega_1$ stationary, “shooting a club through $S$ with finite conditions”.

Lemma 1. If $p = \{\langle \alpha, \alpha \rangle\} \in P_S$ then $P_S \upharpoonright p = P_{S\cap \alpha, \alpha} \times P_{S\upharpoonright \omega_1 \setminus \alpha}$.

Corollary 1. $P_S$ is $S$-proper. [S]

Proof. Let $p \in P_S$, $N \prec H_\theta$ countable with $S, p \in N$ and $\alpha = N \cap \omega_1 \in S$. Then $q = p \cup \{\langle \alpha, \alpha \rangle\} \in P$ is a master condition for $N$ as in [B].

Corollary 2. $P_S$ is homogeneous.

Proof. If $p, q \in P_S$ find $N \prec H_\theta$ countable such that $p, q, S \in N$ and $\alpha = N \cap \omega_1 \in S$. Then $p_1 = p \cup \{\langle \alpha, \alpha \rangle\}$, $q_1 = q \cup \{\langle \alpha, \alpha \rangle\}$ are both in $P_S$ and due to the Lemma can be viewed as elements of $P_{S\cap \alpha, \alpha} \times P_{S\setminus \alpha, \omega_1 \setminus \alpha}$ with support only the first coordinate. But $P_{S\cap \alpha, \alpha}$ is a countable notion of forcing and so is homogeneous. Now it is easy to devise an automorphism of $P_S$ sending $p_1$ under $q_1$, proving homogeneity.

Fix $G \subset P_S$ generic.

Corollary 3. $r \in \omega_\omega \cap V[G]$ iff $\exists \alpha < \omega_1 \ r \in V[G \upharpoonright \alpha]$.

Corollary 4. If $V \subset W$, $\omega_1^V = \omega_1^W$ and $G \in W$ then $G \subset P_S$ is $V$-generic iff

1. $G : \omega_1 \to S$ is increasing and continuous

2. $\forall \alpha < \omega_1 G(\alpha) = \alpha$ implies $G \upharpoonright \alpha$ is $P_{S\cap \alpha, \alpha}$-generic.

Proof. Let $G$ has the above properties and $A \subset P_S$ is a maximal antichain in $V$. Choose $N \prec H_\theta$ in $W$ countable containing $G, A$. Then $A \cap N$ is a maximal antichain in $N \cap P_S$ and so it is met by (2). (1) actually follows from (2).
Lemma 2. If $T$ is a tree of height $\omega_1$ and $P_S \ni p \Vdash "\dot{b} is a branch through T"$ then $p \Vdash \dot{b} \in V$.

Proof. Let $\theta$ be large enough regular cardinal and $M < H_\theta$ be a countable submodel containing $S, p, T, \dot{b}, \alpha = M \cap \omega_1 \in S$. Set $p_1 = p \cup \{\langle \alpha, \alpha \rangle \}$. Find $p_2 \leq p_1$ deciding $\dot{b} \cap$ the $\beta^{th}$ level of $T$. Let $q = p_2 \cap M$. For any $r_1, r_2 \in M$, if $r_1, r_2 \leq q$ then both of them are comparable with $p_2$ and therefore any elements of $T \cap M$ forced by $r_1$ or $r_2$ into $\dot{b}$ must be linearly ordered in $T$. By elementarity, $c = \{x \in T : \exists r \leq q \ r \Vdash x \in \dot{b}\}$ is a branch of $T$ and consequently $q \Vdash \dot{b} = c$. Since $q \leq p$ and $c \in V$ we are done.

Lemma 3. If $Q$ is c.c.c. then $P_S \Vdash "Q is c.c.c."$.

Proof. Let $p \Vdash "\langle \dot{q}_\alpha : \alpha < \omega_1 \rangle is an antichain in Q"$. Fix a bijection $g : P_S \rightarrow \omega_1$ and find $\langle M_\alpha : \alpha < \omega_1 \rangle$, a continuous increasing chain of submodels of $H_\theta$ with $S, g, p, \langle \dot{q}_\alpha : \alpha < \omega_1 \rangle \in M_0$. Set $\dot{\beta}_\alpha = M_\alpha \cap \omega_1$ and find $\langle \dot{p}_\alpha : \alpha < \omega_1 \rangle$ such that $p_\alpha \leq p \cup \{\langle \dot{\beta}_\alpha, \dot{\beta}_\alpha \rangle \}$, $p_\alpha \Vdash \dot{q}_\alpha$ if $\dot{\beta}_\alpha \in S$. Define $f : S \cap \{\dot{\beta}_\alpha : \alpha < \omega_1 \} \rightarrow \omega_1$ by $f(\dot{\beta}_\alpha) = g(p_\alpha \cap M_\alpha)$. $f$ is defined on a stationary set and can be easily seen to be regressive, therefore we can find a stationary set $T \subset \omega_1$ and $q$ such that $\alpha \in T$ implies $p_\alpha \cap M_\alpha = q$. Now similarly to the proof of the $\Delta$-system lemma one can find $U \subset T$ of cardinality $\aleph_1$ such that $\alpha_1, \alpha_2 \in U$ implies $p_{\alpha_1}$ is comparable with $p_{\alpha_2}$ and thus $\langle \dot{q}_\alpha/p_\alpha : \alpha \in U \rangle \subset Q$ is an antichain giving contradiction with assumed c.c.c. of $Q$.

Lemma 4. Let $S \subset \omega_1$ be stationary and $J = NS_{\omega_1} + (\omega_1 \setminus S)$.

1. If $J$ is precipitous then $P_S \Vdash "NS_{\omega_1} is precipitous"$.
2. If $J$ is presaturated then $P_S \Vdash "NS_{\omega_1} is presaturated"$ iff $P(\omega_1)/I \Vdash "jS \subset \omega_1^\omega = \omega_1^{\omega[G]} is stationary", where j : V \rightarrow M is the canonical generic ultrapower$.

Remark. The situation in (2) is parallel to that of [BT]. Notice that $P_S \Vdash "NS_{\omega_1}$ is not $\omega_2$-saturated". (Choose $\langle g_\gamma : \gamma < \omega_2 \rangle$, a family of almost disjoint functions, $g_\gamma : \omega_1 \rightarrow S, g_\gamma(\alpha) > \alpha$, all $\alpha < \omega_1$, $\gamma < \omega_2$. If $G : \omega_1 \rightarrow S$ is the $P_S$-generic function then $\langle S_\gamma : \gamma < \omega_2 \rangle$, $S_\gamma = \{\alpha < \omega_1 : G(\alpha) = g_\gamma(\alpha)\}$ is a long antichain of stationary sets in $V[G]$.) As far as the second condition in (2) is concerned, let us say that a pair $Q, j$ is stationarily correct if $Q \Vdash "j : V \rightarrow M is elementary, $\kappa = crit(j)$ and $M \Vdash "T \subset j\kappa is stationary"$ iff $T$ is stationary”. We have proved that the nonstationary tower ultrapower as described in [W] is stationarily correct as well as the $NS_{\omega_1}$-generic ultrapower under $MA^+(\omega_1\text{-closed})$. It is consistent w.r.t. suitable hypotheses that $NS_{\omega_1}$ is c.c.c. destructibly $\omega_2$-saturated and stationarily correct or that there is $J \subset P(\omega_1)$, a normal $\omega_2$-saturated ideal which is not stationarily correct. It seems however that it is an open problem whether $NS_{\omega_1} + $ a single set can be presaturated and not stationarily correct. Thus the second condition in (2) is possibly empty.

Proof. (1) follows from the following two claims:

Fact 1. $P$ preserves maximal antichains of stationary subsets of $S$.

Fact 2. Let $p \in P$, $p \Vdash "\dot{f} : \dot{T}_0 \rightarrow Ord, \dot{T}_0 \subset \omega_1 stationary"$. Then there are $q < p, T_1, g : \omega_1 \rightarrow Ord$ such that $q \Vdash "\dot{T}_1 \subset \dot{T}_0$ is stationary and $\forall \alpha \in \dot{T}_1 \dot{f}(\alpha) = \dot{g}(\alpha)"$.

Let us fix an enumeration $E : P_S \rightarrow \omega_1$ and go on to prove the above facts. In the case of Fact 1, let $(T_i : i \in I)$ be a maximal antichains of stationary subsets of $S$. Then $j : V \rightarrow M$ is elementary, $\kappa = crit(j)$ and $M \Vdash "T \subset j\kappa is stationary"$ iff $T$ is stationary”. We have proved that the nonstationary tower ultrapower as described in [W] is stationarily correct as well as the $NS_{\omega_1}$-generic ultrapower under $MA^+(\omega_1\text{-closed})$. It is consistent w.r.t. suitable hypotheses that $NS_{\omega_1}$ is c.c.c. destructibly $\omega_2$-saturated and stationarily correct or that there is $J \subset P(\omega_1)$, a normal $\omega_2$-saturated ideal which is not stationarily correct. It seems however that it is an open problem whether $NS_{\omega_1} + $ a single set can be presaturated and not stationarily correct. Thus the second condition in (2) is possibly empty.
of \( S, p \in P_S \), \( p \Vdash \text{ "} \hat{U} \subset \omega_1 \text{ is stationary} \). Set \( \hat{U} = \{ \alpha \in S : \exists q \leq p \cup \{ \langle \alpha, \alpha \rangle \} q \Vdash \alpha \in \hat{U} \} \) and choose \( q_\alpha \) witnessing \( \alpha \in \hat{U} \) for \( \alpha \) in \( \hat{U} \). \( \hat{U} \) is stationary. Define \( F : \hat{U} \to \omega_1 \) by \( F(\alpha) = E(q_\alpha \upharpoonright \alpha). \) \( F \) is regressive on a stationary set and so we can find \( T \subset \hat{U} \) stationary such that \( F''T = \{ \beta \} \), some \( \beta < \omega_1 \). Let \( q = F^{-1}(\beta) \) and choose \( i \in I \) such that \( T \cap T_i \) is stationary. Then \( p > q \Vdash \text{ "} \hat{U} \cap \hat{T}_i \text{ is stationary} \" \). Choose \( r < q, r \Vdash \text{ "} \hat{C} \subset \omega_1 \text{ is a club} \" \). Choose \( M < H_0 \) countable for some large regular \( \theta \) such that \( r, \hat{C} \subset M \) and \( \alpha = M \cap \omega_1 \in T \). Then \( q_\alpha \) and \( r \) are compatible and their common lower bound forces \( \alpha \) into \( \hat{U} \cap \hat{T}_i \cap \hat{C} \) (notice \( q_\alpha \) is a master condition for \( M \)).

The proof of Fact 2 follows a similar pattern. Let \( p, T_0 \) be as in the Fact 2. Set \( \tilde{T}_0 = \{ \alpha \in S : \exists q \leq p \cup \{ \langle \alpha, \alpha \rangle \} q \Vdash \alpha \in \tilde{T}_0 \} \). For \( \alpha \in \tilde{T}_0 \) choose \( q_\alpha \leq p \cup \{ \langle \alpha, \alpha \rangle \} \) \( q_\alpha \Vdash \alpha \in \tilde{T}_0 \), \( q_\alpha \) decides \( \hat{f}(\alpha) \). Let \( F(\alpha) = E(q_\alpha \upharpoonright \alpha) \). \( F \) is regressive on a stationary set and we can find \( \beta, \tilde{T}_1 \subset S \) stationary, \( F''\tilde{T}_1 = \{ \beta \} \). Define \( \tilde{T}_1 = \{ \alpha \in \tilde{T}_1 : q_\alpha \in G \} \), where \( G \) is the generic filter and \( g : \tilde{T}_1 \to Ord, g(\alpha) = \) the unique \( \xi \) such that \( q_\alpha \Vdash \hat{f}(\alpha) = \xi \). Then as above \( p > F^{-1}(\beta) \Vdash \text{ "} \tilde{T}_1 \text{ is stationary} \" \) and \( \hat{f} \upharpoonright \tilde{T}_1 = \hat{g} \upharpoonright \tilde{T}_1 \). It is left to the reader to show now that (1) holds. To prove (2) we first observe

\textbf{Fact 3.} \( P(\omega_1)/J \ast jP_S \mid \langle \omega_1, \omega_1 \rangle \) is isomorphic to \( P_S \ast P(\omega_1)/\mathcal{N}S_{\omega_1} \).

To see this, let \( G \subset P(\omega_1)/I \) be \( V \)-generic and \( H \subset jP_S \mid \langle \omega_1, \omega_1 \rangle \) be \( V[G] \)-generic. Again we confuse \( H \) with \( \bigcup H : \omega^V_2 \to jS \). Set \( H' = H \upharpoonright \omega_1 \). Thus \( H' \) can be regarded as \( V \)-generic object for \( P_S \). The standard techniques give an extension of \( j \) : \( V \to M, j \in V[G], \) to \( \hat{j} : V[H'] \to M[H] \) in \( V[G][H] \). We set \( G' = \{ T \in P(\omega_1) \cap V[H'] : \omega_j \in \hat{j}T \} \) and claim that \( G' \subset P(\omega_1)/\mathcal{N}S_{\omega_1} \) is \( V[H'] \)-generic and moreover \( V[G][H] = V[H'][G'] \). To this end, fix \( f, T, p, (\tilde{T}_i : i \in I) \) such that \( T \subset S \) is stationary, \( T \Vdash \text{ "} \langle \omega_1, \omega_1 \rangle \in \tilde{p} \in jP_S, \tilde{p} = \{ \hat{f} \} \", f : T \to P_S, p \upharpoonright \omega_1 \Vdash \text{ "} \langle \tilde{T}_i : i \in I \rangle \text{ is a maximal antichain in } P(\omega_1)/\mathcal{N}S_{\omega_1} \. \) For \( i \in I \) set \( \tilde{T}_i = \{ \alpha \in S : \exists q \in P_S q \leq f(\alpha), q \Vdash \alpha \in \tilde{T}_i \} \). Since the \( \tilde{T}_i \)'s are forced to form a maximal antichain, there is an \( i \in I \) such that \( \tilde{T}_i \cap T \) is stationary. For each \( \alpha \in \tilde{T}_i \cap T \), choose \( q_\alpha \leq f(\alpha), q_\alpha \Vdash \alpha \in \tilde{T}_i \). Define \( F(\alpha) = E(q_\alpha \upharpoonright \alpha) \). \( F \) is regressive on a stationary set and we can find \( U \subset \tilde{T}_i \cap T, \beta < \omega_1 \text{ such that } U \text{ is stationary and } F''U = \{ \beta \} \). Let \( g : U \to P_S \) be defined by \( g(\alpha) = q_\alpha \). Then in \( P(\omega_1)/I \ast jP_S \mid \langle \omega_1, \omega_1 \rangle \) \( U, [g] \leq T, \text{ } p \text{ and } U, [g] \Vdash \omega_1 \in \hat{j}(\tilde{T}_i/H') \) and therefore \( G' \) meets the antichain given by \( \langle \tilde{T}_i : i \in I \rangle \). This proves the genericity. To reconstruct \( G, H \) from \( G', H' \), notice that \( G = G' \cap V \). If \( \hat{j} \) is the generic ultrapower of \( V[H'] \) by \( G' \), it is immediate that \( H = \hat{j}H' \).

(2) now follows: if \( jS \) is forced to be stationary, then \( jP_S \) will be a forcing in \( V[G] \) which does not collapse \( \omega_1^{V[G]} = \omega_2^{V[G]} \). If on the other hand \( jS \) can be nonstationary, let us say \( T \Vdash \text{ "} jS \text{ is nonstationary} \" \) then it is easy to find two disjoint closed unbounded subsets of \( \omega_2^{V} \) in \( V[G][H] = V[H'][G'] \) if \( T \subset G \subset G' \) and so \( \omega_2 \) was collapsed.

From now on, let \( Q_{\omega_1} \) denote the forcing adding \( \omega_1 \) Cohen reals. From our previous work, any real added by \( P_S \) is in some Cohen extension of the ground model. It is also not very hard to see that \( Q_{\omega_1} \) regularly embeds into \( P_S \). (See Corollary 5 for a rather complicated example how to do this.) It is natural to ask whether such embedding can reap all the real numbers of \( V[P_S] \), i.e. if we can have \( Q_{\omega_1} \subset P_S \) as a regular subalgebra so that \( P_S \Vdash \text{ "} \omega^\omega \cap V[G] = \omega^\omega \cap V[G \cap Q_{\omega_1}] \" \).

\textbf{Lemma 5.} Let \( H \subset Q_{\omega_1} \) be generic. In \( V[H] \) (actually in \( V[\hat{j}\omega_1^{V[H]}] \)) there is an
ω-distributive $S$-proper forcing $T$ such that $T \Vdash \text{“there is } G \subset P_S \text{ } V$-generic such that } ω^n \cap V[G] = ω^n \cap V[H]”

Proof. Work in $V[H]$ and define $T = \{g : \exists \alpha < ω_1, \{⟨α, α⟩\} \in P_S, g \subset P_{S, α} \text{ is generic over } V\}$ ordered by reverse inclusion. Certainly all $g ∈ T$ are hereditarily countable, thus coded by reals and $T ∈ V[ω^n]$. Choose $g ∈ T$, $⟨D_i : i < ω⟩$ a sequence of open dense subsets of $T$, and $M \prec H_θ, S, g, ⟨D_i : i < ω⟩, H ∈ M$, $M ∩ ω_1 ∈ S$. $\{T ∩ M\} \cup \{D ∩ T ∩ M : D ∈ M \text{ open dense }\}$ is a countable collection of hereditarily countable objects and as such belongs to some $V[H \upharpoonright α], α < ω_1$. In $V[H \upharpoonright α], T ∩ M \upharpoonright g$ is isomorphic to adding one Cohen real. Let us regard $H(α)$ as a subset of it. Then it is easy to show that $h = \bigcup H(α)$ is a $V$-generic subset of $P_{S, M∩ω_1}$ and a strongly $M$-generic condition under $g$, in particular $h ∈ \bigcup i<ω D_i$.

Due to the local genericity condition in Corollary 4, if $K \subset T$ is generic, $G = \bigcup K$ is a $V$-generic subset of $P_S$. The last thing to check is that $ω^n \cap V[G] = ω^n \cap V[H]$. To this aim, for $α < ω_1$ define $D_α = \{g ∈ T : H \upharpoonright α ∈ V[g]\}$. The following Subclaim will complete the proof.

Subclaim. Each $D_α$ is a dense subset of $T$.

Thus the reals coming from $P_S$ look exactly the same as the reals coming from $Q_{ω_1}$.

Lemma 6. Cons(ZFC+$κ$ Mahlo) implies Cons(ZFC+$∃S \subset ω_1$ stationary costationary and there is an embedding $Q_{ω_1} \prec P_S$ reaping all the reals of $V^{P_S}$.

Proof. Fix a Mahlo cardinal $κ$ and set $S = \{α < κ : α \text{ inaccessible }\}$. $Coll(ω, < κ)$ is homogeneous and so for every finite function $p$ from $κ$ to $κ$, either $Coll(ω, < κ) \Vdash \exists α < κ ∃ f : α \rightarrow κ$ increasing continuous with $f”α ⊂ S, p ∈ f”$ or $Coll(ω, < κ) \Vdash \neg ∃ α < κ ∃ f : α \rightarrow κ$ increasing continuous with $f”α ⊂ S, p ∈ f”$. (Notice that due to the $κ$-c.c. $Coll(ω, < κ)$ preserves stationarity of $S$.) Therefore we can define $P = \{p : p$ is a finite function and $Coll(ω, < κ) \Vdash \exists α < κ ∃ f : α \rightarrow κ$ increasing continuous with $f” α ⊂ S, p ∈ f”$ ordered by inclusion and be sure to get $Coll(ω, < κ) \Vdash \bigf = \bigf_S$.

Claim 1. $P \Vdash \text{“} κ = 8_1, G \text{ (confused with } \bigcap G) : κ \rightarrow S \text{ increasing continuous.”}$

Claim 2. $P \Vdash \text{“} ∃ H \subset Coll(ω, < κ)$ generic over $V, ω^n \cap V[H] = ω^n \cap V[G]\text{”}$

Proof. Fix $G \subset P$ generic and work in $V[G]$. Notice that as in the case of Lemma 1, (due to the easy factorization of $P$) $r ∈ ω^n ∩ V[G] ≜ r ∈ ω^n ∩ V[G \upharpoonright α]$, some $α < κ$. Consider the following poset $Y = \{h : h ∈ Coll(ω, < α) \text{ generic over } V \text{ for some } α ∈ rng(G)\}$ ordered by reverse inclusion. $Y$ is ω-closed by the closure of $rng(G)$: assume $h_0 > h_1 > \cdots > h_i > \cdots, i < ω$, is a decreasing sequence of elements in $Y$, $h_i ∈ Coll(ω, < α_i)$, some $α_i ∈ rng(G)$. Then $α = sup_i<ω α_i ∈ rng(G)$, $α$ is $V$-inaccessible and $h = \bigcup_i<ω h_i ∈ Coll(ω, < α)$ is generic over $V$, since if $A ⊂ Coll(ω, < α)$ is a maximal antichain in $V$, we have $|A| < α$ (in $V$) and thus for some $i < ω, A ⊂ Coll(ω, < α_i)$ and $A$ is met by a condition in $h_i$. For $α < κ$ define $D_α = \{h ∈ Y : G \upharpoonright α ∈ V[h]\}$. The following subclaim will finish the proof of the Claim 2 since $κ = 8_1, Y$ is ω-closed and any real in $V[G]$ is coded by an initial segment of $G$. 
Subclaim. Each $D_\alpha$ is dense in $Y$.

Now we can finish the proof of the Lemma. Fix $\dot{H}$, a $P$-name for a generic subset of $Coll(\omega, < \kappa)$ as in Claim 2. Fix $K \subseteq Coll(\omega, < \kappa)$ generic over $V$. We claim that $V[K]$ is a model of the wanted theory with our $S$. To prove it, choose $G \subseteq P_S = P$ generic over $V[K]$. By a mutual genericity argument, $H = \dot{H}/G \subseteq Coll(\omega, < \kappa)$ is generic over $V[K]$. In $V[K]$, however, $\kappa = \aleph_1$ and so $Coll(\omega, < \kappa)$ is isomorphic to $Q_{\omega_1}$. We view $H$ as a subset of $Q_{\omega_1}$ (transferred by some isomorphism of $Coll(\omega, < \kappa)$ and $Q_{\omega_1}$ in $V[K]$). The only thing left to check is that $\omega^\omega \cap V[K][G] = \omega^\omega \cap V[K][H]$. For that we use Corollary 3 and the most significant property of $H$, that $\{G \upharpoonright \alpha : \alpha < \kappa\} \subset V[H]$.

Lemma 7. If $C \subset \omega_1$ is a club then $P_{S \cap C} \subset P_S$. In fact, $P_S \Vdash \text{"if } \dot{D} \subset \omega_1 \text{ is the generic club then } \dot{C} \cap \dot{D} \text{ is } P_{S \cap C} \text{-generic club".}$

Proof. By the local genericity criterion in Corollary 4 it is enough to prove the following claim:

Claim 3. If $\alpha < \omega_1$ is indecomposable, $T \subset S \subset \alpha$, $T$ clunbounded in $S$ and $P_{T,\alpha}, P_{S,\alpha}$ are nonempty posets then $P_{S,\gamma} \Vdash \text{"if } \dot{D} \subset \alpha \text{ is the generic club then } \dot{D} \cap T \text{ is a } P_{T,\alpha}\text{-generic club".}$

Proof of the Claim. We first give two subclaims, then prove the Claim from them and complete the proof of the Lemma by proving the two subclaims.

Subclaim. If $\gamma$ is indecomposable, $T \subset S$, where $T$ is clunbounded in $S$, which is countable and $P_{T,\gamma}, P_{S,\gamma} \neq 0$ then $P_{S,\gamma} \Vdash \text{"o.t. } \dot{D} \cap T = \gamma\text{"}$, where $\dot{D}$ is the generic club through $S$.

Subclaim. If $I, J \subset \omega_1$ are countable intervals of ordinals, o.t.$J \leq \text{o.t. } I$ are both indecomposable, $T \subset S$, where $T$ is clunbounded in $S$, which is countable and $P_{T,\gamma}, P_{S,\gamma} \neq 0$ then for any $t \in P_{T,J}$ there is $s \in P_{S,I}$ such that $s \Vdash \text{"} t \text{ is a subset of the increasing enumeration of } \dot{T} \cap \dot{D} \text{ starting with } \text{min}(I)\text{"}$, $\dot{D}$ the generic club.

Now we can proceed to prove the Claim. For technical reasons we pretend that $\alpha \in S \cap T$ and any $p \in P_{S,\alpha}$ contains $\langle \alpha, \alpha \rangle$ (accordingly $\alpha \in \dot{D}$ then). Choose $p_0 \in P_{S,\alpha}$ arbitrary. We find $p \prec p_0$ and $q \in P_{T,\alpha}$ such that for any $q' \leq q$ there is $p' \prec p$ such that $p' \Vdash P_{S,\alpha} \text{"} q' \subset \text{the enumeration of } \dot{T} \cap T\text{"}$, proving the Claim. We build $\alpha = \alpha_0 > \alpha_1 > \ldots, p_0 = p_0 > p_1 > \ldots, \alpha = \gamma_0 > \gamma_1 > \ldots$ so that

1. $\alpha_i \in \text{dom}(p_i)$,
2. $p_i \Vdash_{P_{S,\alpha}} \text{"} p_i(\alpha_i) \in T \text{ is the } \gamma_i^{th} \text{ element of } \dot{T} \cap \dot{D} \text{"}$ where $\dot{D}$ is the generic club $\subset \alpha$,
3. $\text{dom}(p_{i+1} \upharpoonright p_i) \subset \alpha_i$,
4. o.t.$\alpha_i \setminus \alpha_{i+1}$ is indecomposable,
5. o.t.$\gamma_i \setminus \gamma_{i+1}$ is indecomposable,
6. $\gamma_i$ limit implies $\text{dom}(p_{i+1}) \cap (\alpha_{i+1}, \alpha_i) = 0$,
7. $\gamma_i$ successor implies $\gamma_{i+1}$ is the predecessor of $\gamma_i$.

This is easily done and must end at some $n < \omega$ since the $\alpha_i$'s form a descending sequence of ordinals. Set $q = \{\langle \gamma_i, p_i(\alpha_i) \rangle : i < n\}$ and $p = p_n$. We claim that $p, q$ are what we are looking for. First, $p \in P_{S,\alpha}$ countable containing

...
everything relevant and \( p \in g \subset P_{S,\alpha} \) generic over \( M \). Then by elementary absoluteness considerations \( T \cap rng(\bigcup g) \) is a club subset of \( T \) of ordertype \( \alpha \) such that \( q \) is a subset of its enumeration. Second, choose \( q' < q \) in \( P_{T,\alpha} \). Let us assume for simplicity that \( dom(q' \setminus q) \subset (\gamma_{i+1}, \gamma_i) \) for some \( i \). Then \( \gamma_i \) is limit by (7). Now we use the second subclaim with \( I = \alpha_i \setminus \alpha_{i+1}, J = \gamma_i \setminus \gamma_{i+1}, S \cap (p_{i+1}(\alpha_i), p_i(\alpha_i)) \) in the place of \( S \) and \( T \cap (p_{i+1}(\alpha_i), p_i(\alpha_i)) \) in the place of \( T \) on \( t = q' \setminus q \). The resulting \( s \) is easily seen to be such that \( P_{S,\alpha} \ni p \cup s = p' \parallel_{P_{S,\alpha}} \) “\( q' \) is a subset of the increasing enumeration of \( \hat{D} \cap \hat{T} \)."

To prove the first subclaim, let \( \gamma < \omega_1 \) be the least indecomposable such that there are \( T \subset S \) violating the statement. We distinguish two cases:

(1) \( \gamma \) is a limit of indecomposables. Let \( P_{S,\gamma} \ni p \parallel \) “\( o.t.\hat{D} \cap \hat{T} \) \( < \beta < \gamma \)” for some indecomposable \( \beta > \max(dom(p)) \). Let \( \xi \in T, \xi > \max(rng(p)) \) be such that \( P_{T \cap \xi, \beta} \neq 0 \). By indecomposability of \( \beta \) and \( \gamma \) \( p \cup \{\langle \beta, \xi \rangle \} \in P_{S,\gamma} \). By minimality of \( \gamma \), \( P_{S \cap \xi, \beta} \parallel \) “\( o.t.\hat{T} \cap \hat{D} = \beta \)” , therefore \( p \cup \{\langle \beta, \xi \rangle \} \parallel \) “\( o.t.\hat{T} \cap \hat{D} \geq \beta \)”, a contradiction.

(2) \( \gamma = \omega \beta \) for some \( \beta < \gamma \) indecomposable. Fix \( p \in P_{S,\gamma} \). We get \( q = p \cup \{\langle \delta, \xi \rangle \} \subset q \) such that \( q \parallel \) “\( o.t.\hat{T} \cap \hat{D} \cap (\xi \setminus \max(rng(p))) = \beta \)” . The contradiction then follows by a simple genericity argument. To get our \( q \), we set \( \delta = \max(dom(p)) + \beta \) and choose \( \xi \in T, \xi > \max(rng(p)) \) such that \( P_{T \cap \xi, \beta} \neq 0 \). By minimality of \( \gamma \) and indecomposability of \( \gamma, \beta \), it follows that \( q \) works.

The second subclaim is in fact a corollary to the first one. It is certainly enough to prove it for \( J = \gamma \leq \alpha = 1 \). For simplicity we assume that \( t = \langle \beta, \xi \rangle, \xi \in T \). Find \( \beta = \beta_0 > \beta_1 > \cdots > \beta_m-1 > 0 = \beta_m \) so that \( \forall i < m \text{ o.t.}(\beta_i \setminus \beta_{i+1}) \) is indecomposable and choose \( s \in P_{T,\gamma}, dom(s) = \{\beta_0 \ldots \beta_{m-1}\}, s(\beta) = \xi \). Then \( s \) is a member of \( P_{S,\alpha} \) as well and by the first claim it has the required property.

Let us evaluate the factor forcing \( P_S/P_{S \cap C} \). For definiteness, assume that \( C \subset \omega_1 \) is such that \( |S \setminus C| = \aleph_1 \). Let us fix \( H \subset P_{S \cap C} \) \( V \)-generic and in \( V[H] \), choose a continuous increasing sequence \( \langle M_\alpha : \alpha < \omega_1 \rangle \) of countable submodels of some \( H_\theta \) with \( C, S, H, H_\theta \subset M_\theta \). Define \( \beta_0 = 0 \) and for \( 0 < \alpha < \omega_1 \) let \( \beta_\alpha = M_\alpha \cap \omega_1 \). Define a forcing \( Q = \) the finite support product of \( P_{S \cap I_\alpha}/P_{S \cap C \cap I_\alpha} \) for \( \alpha < \omega_1 \) where \( I_\alpha = [\beta_\alpha, \beta_{\alpha+1}), \text{ the } V \)-generic subset of \( P_{S \cap C \cap I_\alpha} \) is just \( H \cap P_{S \cap C \cap I_\alpha} \) and the embedding \( P_{S \cap C \cap I_\alpha} \to P_{S \cap I_\alpha} \) is the one described in Claim 3 (modulo an ordinal shift) . Then it is not difficult to see that \( Q \) is isomorphic to \( Q_{\omega_1} \) in \( V[H] \), since the forcings standing in the finite support product are nontrivial and \( \aleph_0 \)-dense. One can easily prove that a \( V[H] \)-generic \( K \subset Q \) together with \( H \) gives a \( V \)-generic \( G \subset P_S \) such that \( V[G] = V[H][K] \).

**Corollary 5.** \( P_S = P_S \times Q_{\omega_1} \).

Since \( C \subset \omega_1 \) is as above, \( P_S = P_{S \cap C} * Q_{\omega_1} = P_{S \cap C} \times Q_{\omega_1} = P_{S \cap C} \times Q_{\omega_1} \times Q_{\omega_1} = P_S \times Q_{\omega_1} \).

**Corollary 6.** If \( S = T \) modulo \( NS_{\omega_1} \) then \( P_S = P_T \) (again, as Boolean algebras).

To see this, fix \( S, T \) as in the Corollary and choose \( C \subset \omega_1 \) club such that \( |S \setminus C| = |T \setminus C| = \aleph_1 \) and \( S \cap C = T \cap C \). Then \( P_S = P_{S \cap C} \times Q_{\omega_1} = P_{T \cap C} \times Q_{\omega_1} = P_T \) from the remarks preceding Corollary 5.
Corollary 7. Cons(\(\kappa\) Mahlo) implies Cons(\(Q_{\omega_1}\) embeds into \(P_{\omega_1}\) reaping all the reals).

The proof of the corollary is left to the reader. The model is \(V[K][L]\), where \(K \subset \text{Coll}(\omega, < \kappa)\) is generic as in Lemma 6 and \(K\) is a generic club through \(S = \{\alpha < \kappa : \alpha\) inaccessible in \(V\}\) using countable conditions. The key to the proof is to notice that \(V[K][L] \models P_{\omega_1} = P_S\) as Boolean algebras; the rest carries over from Lemma 6.

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