On a problem of the elastic wedge-shaped body oscillations generated in its rib zone

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Abstract. In the paper there studies the wave process in the wedge-shaped body exited by the antiplane shear oscillation generators in the rib zone surface. Mathematical model construction is fulfilled by the mixed dynamic boundary value problem reduced to the boundary integral equation (BIE) about contact stresses in the domain of given oscillation generators. Previous author’s papers permit to construct BIE solution and reconstruct the wave field in the whole wedge-shaped medium. To estimate the correctness of application of the solution mentioned above the wedge-shaped medium with generators is transformed to the half-space. Then the solution of the mixed problem for the half-space is constructed by two independent methods: by the method based on BIE and by use the degenerated elliptical coordinates. Both solutions are in rigorous coincidence.

1. Introduction
The analysis of the exploitation reliability problem for the technological equipment under long dynamic loadings provides the wear and pre-fracture problem for the material of moving angular elements in the joining point zone of various mechanical constructions. The same problems arise in the nondestructive testing analyses of construction elements as well, in seismic prospects when analyzing the wave propagation in the wedge-shaped zones near the earth crust surface at al.
Thus the investigation of the stressed and deformable state in the joining point zone is of great interest not only theoretical but also practical one. Dynamic process problems of the elastic wedge-shaped medium are investigated by number of authors [1-8] at al. The mathematical modeling of such processes has been reduced to boundary integral equations (BIA), solvability problems having been studied in details [6-8] at al. In part, in connection with problems of contact interaction mentioned above ones are reduced to mixed dynamic contact problems for the elastic wedge-shaped bodies. Integral transform techniques in the dynamic elasticity permits in some cases permits to obtain analytical solutions of above problems [5-8] at al.
At the present paper it is succeed to construct the analytical solution of the BIA of the problem in question and to fulfill the inspection of such a solution. To valid the analytical solution of the BIE used in this paper, in part, the wedge of angle \( \alpha \) with oscillating sources on the rib zone surface is transformed to the elastic half-space \( \alpha \to \pi \) with the same sources. Validation of the exact solution for of the BIE relies on two independent methods:
1) constructing the solution of BIE based on the Bessel functions’ theory;
2) constructing the solution of BIE based on the Mathieu functions’ theory. Both methods adduce to the same results.

2. Steady antiplane oscillations of the homogeneous elastic wedge

Let us construct the exact solution of the problem of steady antiplane oscillations of the homogeneous elastic wedge of angle $\alpha$, generated by two sources distributed within the intervals $(0, b)$ and located symmetrically on its planes $\theta = 0, \theta = \alpha$ in the rib zone as it’s shown below in Fig.1. The rest of the boundary is assumed to be unloaded.

The problem is to find contact stresses in the distribution sources intervals $(0, b)$.

Let us search the displacements $U(x, y, t)$ in the described region under the harmonic oscillation $U(x, y, t) = u(x, y)e^{i\omega t}$, $\omega$ - frequency of oscillations. The problem in question may be formulated as the boundary value problem:

$$\Delta u + k^2u = 0, k^2 = D \omega^2 \mu^{-1}$$
$$\sigma_{\theta z} = \mu \frac{1}{r} \frac{\partial u}{\partial \theta} = 0, r > b, \theta = \pm \frac{\alpha}{2}$$
$$u = f(r), 0 < r < b, \theta = \pm \frac{\alpha}{2}, \infty$$

$$\frac{\partial u}{\partial n} - iku = o(r^{-1/2}), r = \sqrt{x^2 + y^2} \to \infty$$

Correlations (1) are Summerfield radiation conditions, $D$ is the density of the wedge, $\mu$ is shear modulus.

By means of Kontorovich-Lebedev integral transform technique [9] it may be found $\sigma_{\theta z}|_{\theta=0} = 0$ on the symmetry axis. The BIE of mixed boundary value problem may be reduced to that of antiplane oscillations of wedge angle $\frac{\alpha}{2}$ with free lower boundary:

$$K \tau = \int_0^b k(r, \rho) \tau (\rho) d\rho = f(r), 0 < r < b$$

$$k(r, \rho) = \frac{1}{i\kappa} \int_{-\infty + i\epsilon}^{\infty + i\epsilon} L^{-iz}(\kappa \rho)K^{-iz}(\kappa \rho) K(z)dz, \kappa = -ik$$

Figure 1. Antiplane oscillations in the rib zone of the wedge. Displacements are perpendicular to the figure plane.
where \( l_\nu(k\rho), K_\nu(k\rho) \) are modified Bessel functions \([9]\). Solvability problem of BIE (2) may be solved by method \([6-8]\) where it is considered the equation \((\varepsilon > 0)\):

\[
\int_0^b k(r, \rho) \tau(\rho) d\rho = f(r), a < r < b
\]

\[
k(r, \rho) = \frac{1}{i\varepsilon} \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} I_{-i\varepsilon}(k\rho)K_{-i\varepsilon}(k\rho) K(z)dz, \kappa = -ik
\]

(3)

Theorem Operator \( K \) of the left hand side (2) is uniquely inverted as operator acting in function spaces:

\[
K : W_{2}^{-1/2}(a, b) \rightarrow W_{2}^{1/2}(a, b)
\]

where \( W_{2}^\gamma(a, b), \gamma = \pm 1/2 \) are Sobolev-Slobodetsky spaces of fractional smoothness.

Such a result may be proved by method \([8]\) using the space \( H(a, b) \) of generalized solutions of the equation (3), one being obtained preliminary result:

\[
K : H(a, b) \rightarrow W_{2}^{-1/2}(a, b)
\]

\[
H(a, b) \subseteq W_{2}^{1/2}(a, b)
\]

The theorem result points out by proving the inverse embedding \( H(a, b) \supseteq W_{2}^{1/2}(a, b) \).

The structure of the solution (3) in the space \( W_{2}^{-1/2}(a, b) \) has the form as follows:

\[
\rho \tau(\rho) = \frac{1}{\pi i} \int_{\Gamma_1} K^{-1}(z)F(z) I_{-i\varepsilon}(k\rho)z dz + \\
+ \frac{1}{\pi i} \int_{\Gamma_2} \{ F_1(z)I_{-i\varepsilon}(k\rho)K_{-i\varepsilon}(\kappa a) + F_2(z)K_{-i\varepsilon}(\kappa a)\} \frac{z}{K_{-\varepsilon}(z)} dz
\]

(4)

Contour \( \Gamma_2 \) in (4) lays in the complex \( z \)-plane upper \( \Gamma_1(\Gamma_1 \Gamma_1 \Gamma_2 \Pi) \). \( \Pi \) is the regularity strip of the function \( K(z) \), containing real axis \( R^1 \). Function \( K_-(z) \) is the result of factorization \([10]\) of function \( K(z) \) in the form \( K(z) = K_+(z)K_-(z) \) with respect to \( R^1 \). Sought-for-functions \( X_{1,2}(z) \) are founded from the equations’ system:

\[
X_1(z)\varphi(z, b) = F_1(z)
\]

\[
X_2(z)\varphi(z, a) = F_2(z) + (VX_1)(z)
\]

\[
\varphi(z, b) = z K_{-i\varepsilon}(\kappa b)I_{-i\varepsilon}(\kappa b)
\]

\[
F_1(z) = \frac{1}{\pi^2 i} F_1(z)\frac{K_{-i\varepsilon}(\kappa b)I_{-i\varepsilon}(\kappa b) = F_1(z)\frac{K_{-i\varepsilon}(\kappa b)I_{-i\varepsilon}(\kappa b)}{K_{-\varepsilon}(z)} d\zeta \int_{\Gamma_1} \frac{F(\xi)\xi A_1(\xi)K(\xi)K_{-\varepsilon}(\kappa a)}{K_{-\varepsilon}(z)K_{-i\varepsilon}(\kappa a)} \xi d\zeta}
\]

\[
F_2(z) = \frac{1}{\pi^2 i} F_2(z)\frac{K_{-i\varepsilon}(\kappa b)I_{-i\varepsilon}(\kappa b)}{K_{-\varepsilon}(z)} d\zeta \int_{\Gamma_1} \frac{F(\xi)\xi A_2(\xi)K(\xi)K_{-i\varepsilon}(\kappa b)}{K_{-\varepsilon}(z)K_{-i\varepsilon}(\kappa b)} \xi d\zeta
\]

(5)

\[
(VX_1)(z) = \frac{1}{\pi^2 i} F_1(z)\frac{K_{-i\varepsilon}(\kappa b)I_{-i\varepsilon}(\kappa b)}{K_{-\varepsilon}(z)} d\zeta \int_{\Gamma_1} \frac{X_1(z)A_1(\xi)K(\xi)K_{-i\varepsilon}(\kappa b)}{K_{-\varepsilon}(z)K_{-i\varepsilon}(\kappa b)} \xi d\zeta
\]

Results of the theorem may be used to the boundary value problem under consideration and its BIE (3). Theorem conditions will be satisfied provided in (4) it is assumed \( f(r) \in W_{2}^{12}(0, b) \) and \( X_{1,2}(z) = o(|z|^\delta), \delta > 12 \) (\( |z| \to \infty \)).

To construct the solution for the BIA (3), to pass the limit in (4) \( a \to 0 \). Then \( X_2(z) \to 0 \) as \( a \to 0 \) because of the asymptotic behavior of the modified Bessel functions and the famous theorem due to Riemann-Lebegue on Fourier integral vanishing \([11]\). Expression of \( X_1(z) \) takes the form:

\[
X_1(z) = [z K_{-i\varepsilon}(\kappa b)I_{-i\varepsilon}(\kappa b)]^{-1} F_1(z)
\]

(6)
The consequent awkward transformations of contour integrals in the solution (5) on the base of its integrand asymptotic behavior results to the equality:

\[ \tau(\rho) = -2 \sum_{n=1}^{\infty} \left( \int_{0}^{\infty} f(r) H_{2n+1}(kr) \frac{dz}{r} \right) \]

To represent the integral (6) as the residues series, to transform one to the expression, where the integrand vanishes in the upper half-plane \( \text{Im} z > 0 \) (except for \( z_k \)-poles of \( K(z) \)) as follows:

\[ \sum_{n=1}^{\infty} \left( \int_{0}^{\infty} f(r) H_{2n+1}(kr) \frac{dz}{r} \right) \]

Let us develop the wedge to the half-space by the limit and use the circuit relations for Bessel functions described in [10]:

\[ I_{\nu}(\kappa p) = I_{\nu}(-ikp) = I_{\nu}(kp e^{-i\pi/2}) = e^{-i\pi/2} J_{\nu}(kp) \]

\[ K_{\nu}(\kappa p) = K_{\nu}(-ikp) = K_{\nu}(kp e^{-i\pi/2}) = \frac{\pi}{2} e^{i\pi/2} H_{\nu}^{(1)}(kp) \]

The relation (7) takes the form:

\[ \tau(\rho) = 2i \sum_{n=1}^{\infty} \left( \int_{0}^{\infty} f(r) H_{2n+1}(kr) \frac{dz}{r} \right) \]

In (9) \( J_{2n+1}(kp) \) is Bessel function of the first type, \( H_{2n+1}^{(1)}(kp) \) is Hankel function. To compare the exact solution (9) with one constructed by method [12], to make re-expanding the sum (9) by closed Mathieu functions’ system [13] connected with the degenerated elliptical coordinates on the contact segment \( r = |x| \leq b \) of the half-space boundary \( y = 0 \). To make such a re-expansion we use the auxiliary identity (S.9) proved in Supplementary data:

\[ \sum_{n=1}^{\infty} \left( \int_{0}^{\infty} f(r) H_{2n+1}(kr) \frac{dz}{r} \right) \]

According to uniform convergence series on the segment \( 0 \leq r \leq b \) in (9) we change the summation and integration order and transform the sums in the integrands by means of the auxiliary identity. There results the expression of the exact solution:

\[ \tau(\rho) = \frac{4}{b \sin \eta} \sum_{k=0}^{\infty} \left( \int_{0}^{\infty} f(r) \frac{s_{2k+1}(\eta', q)}{b \sin \eta} dr \right) \]

Transforming the expression in curly brackets and making the substitution \( r = b \cos \eta' \) we obtain the correlation:
\begin{equation}
\tau(\rho) = \frac{4}{b \sin \eta} \sum_{k=0}^{\infty} \gamma_{2m+1} \left( \frac{2}{\pi} N_{\eta}^{2m+1}(\xi q) \right) \text{se}_{2k+1}(\eta, q), \quad 0 \leq \eta \leq \frac{\pi}{2}, \quad q = \frac{k^2 b^2}{4}
\end{equation}

\[
\sin \eta = \sqrt{1 - (x/b)^2}, 0 \leq \eta \leq \frac{\pi}{2}, \quad q = \frac{k^2 b^2}{4}
\]

\[\gamma_{2m+1} = \frac{4}{\pi} \int_0^\pi f(b \cos \eta') \text{se}_{2k+1}(\eta', q) d\eta'
\]

**3. Comparison of solutions obtained by two methods**

To construct the solution of BIE (4) by method [12], to transform preliminary the BIE kernel in the case of half-space, using formulae for the modified Bessel functions [9] (\(\varepsilon > 0\)):

\[
k(r, \rho) = \frac{1}{i \pi} \int_{-\infty+ie}^{\infty+ie} t_{-\nu}(\kappa \rho) \text{K}_{-\nu}(\kappa \rho) \text{cth} \frac{\pi \xi}{2} d\xi = \frac{2}{\pi} \int_{-\infty}^{\infty} K_{-\nu}(\kappa \rho) K_{-\nu}(\kappa \rho) (1 + \chi \nu) d\nu =
\]

\[
= \frac{1}{\pi} \left\{ K_0(\kappa |r - \rho|) + K_0(\kappa |r + \rho|) \right\}
\]

The BIE on vibrations of the wedge-shaped medium of angle \(\alpha = \pi/2\) with free lower boundary takes the form (\(\tau(\xi) - \text{dimensionless contact stress, } |x| = r\)):

\[
\frac{1}{\pi} \int_{-b}^{b} \left\{ K_0(\kappa |x - \xi|) + K_0(\kappa |x + \xi|) \right\} \tau(\xi) d\xi = f(x), \quad 0 \leq x \leq b
\]

Using the symmetry of the initial problem under consideration, we extend the sought-for function \(\tau(x)\) and \(f(x)\) in (11) by the even way on the segment \(-b \leq x \leq b\). Then we adduce the BIE considered by method [12] :

\[
\frac{1}{\pi} \int_{-b}^{b} K_0(\kappa |x - x'|) \tau(x') dx' = f(x), \quad -b \leq x \leq b
\]

According to [12], it is introduced the function

\[
\Phi(x, y) = \frac{1}{\pi} \int_{-b}^{b} K_0(\kappa |x - x'|) \tau(x') d x'
\]

Function (12) obeys the mixed boundary value problem for Helmholtz equation both in Cartesian coordinates :

\[
\Delta \Phi + k^2 \Phi = 0
\]

\[
\Phi(x, y)|_{x=0,|y|<b} = f(x)
\]

and in elliptical coordinates \(x + iy = \text{bc}\cos(\xi + i\eta)\):

\[
\frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \eta^2} + 2 q (\text{ch}\, 2\xi - \cos 2\eta) \Phi = 0
\]

\[
\Phi(0, \eta) = f(b \cos \eta)
\]

\[
\Phi(\infty, \eta) = 0, \quad q = \frac{k^2 b^2}{4}
\]

Let us believe that function \(f(x)\) is even, according to loading type on the wedge boundary. Then on the segment \(|x| \leq b\) (in degenerate elliptical coordinates \(x = b \cos \eta, 0 \leq \eta \leq \pi\)) that function may be expanded by the Chebyshev polynomials of the second type \(U_{2k+1}(x/b)\) described in [9] in the form:

\[
f_{2k+1} = \frac{2}{\pi} \int_{-b}^{b} f(x) U_{2k+1}(x/b) \frac{dx}{\sqrt{b^2 - x^2}} = \frac{4}{\pi} \int_{0}^{\pi} f(x) U_{2k+1}(x/b) \frac{dx}{\sqrt{b^2 - x^2}} = \int_{0}^{\pi} f(b \cos \eta) \sin(2k + 1) \eta \, d\eta
\]

In the theory of periodic Mathieu functions

\[
\text{se}_{2m+1}(\eta, q) = \sum_{r=0}^{\infty} B^{2m+1}_{2r+1}(q) \sin(2r + 1) \eta
\]
there is the expansion:
\[ \sin(2\tau + 1)\eta = \sum_{m=0}^{\infty} B_{2\tau + 1}^{2m+1}(q) \text{se}_{2m+1}(\eta, q) \]

Then in the degenerate elliptical coordinates the expansion (13) takes the form:
\[ f(b \cos \eta) = \sum_{k=0}^{\infty} f_{2k+1} B_{2\tau + 1}^{2m+1}(q) \text{se}_{2m+1}(\eta, q) = \sum_{m=0}^{\infty} \gamma_{2m+1} \text{se}_{2m+1}(\eta, q) \]

\[ \gamma_{2m+1} = \int_{0}^{\pi} f(b \cos \eta) (\text{se}_{2m+1}(\eta, q) \sin(2k + 1)\eta) \, d\eta' = \frac{4}{\pi} \int_{0}^{\pi} f(b \cos \eta) \text{se}_{2m+1}(\eta, q) \, d\eta' \]  

(15)

Let us present \( \Phi(x, y) \) as the solution of the boundary value problem (13) in the form:
\[ \Phi(\xi, \eta) = \sum_{m=0}^{\infty} \frac{\gamma_{2m+1}(\xi, q)}{\gamma_{2m+1}(0, q)} \text{se}_{2m+1}(\eta, q) \]  

(16)

where coefficients \( \gamma_{2m+1} \) are determined in 15. Function [15] are satisfied all conditions (18) since
\[ \lim_{\xi \to \infty} \gamma_{2m+1}(q) = 0 \]. Then as a result by [13]
\[ \tau(x) = \tau(b \cos \eta) = \frac{1}{b|\sin|} |\eta| 0 \leq \eta \leq \frac{\pi}{2} \]
we adduce the correlation:
\[ \tau(b \cos \eta) = \frac{1}{b|\sin|} \sum_{m=0}^{\infty} \gamma_{2m+1}(\xi, q) \text{se}_{2m+1}(\eta, q) \]  

(17)

\[ \gamma_{2m+1} = \int_{0}^{\pi} f(b \cos \eta) \text{se}_{2m+1}(\eta, q) \, d\eta', \quad \sin \eta = \sqrt{1 - (x/b)^2} \]

Comparison of solutions (17) and (10) establishes its rigorous coincidence.

To reconstruct the wave field in the whole wedge-shaped medium we can apply the Green function method on the base of results [8]. For example, we can use Green function constructed in that paper in the form:
\[ G(\tau, \varphi | \rho, \psi) = \int_{0}^{\infty} \widehat{G}(\tau, \varphi, \psi) K_{-i\tau}(kr) L_{i\tau}(kr) \, d\tau \]

Results obtained at present paper may be used when considering another problems reduced to the BIE provided its principal part of operator coincides the operator of the left hand side of BIE (2),(3).

4. Supplementary data (Auxiliary identity)
In this part we prove an identity containing Bessel and Mathieu functions:
\[ \sum_{m=0}^{\infty} \frac{(2m+1)!}{\rho^{2m+1}} H_{2\tau + 1}(kp) f_{2\tau + 1}(kp') = \]

\[ = -2i \frac{1}{\pi} b^2 \sin \sin |\eta| \sum_{m=0}^{\infty} \frac{\partial}{\partial q} \frac{N_{2m+1}(kq)}{N_{2m+1}(0,q)} \text{se}_{2m+1}(\eta, q) \text{se}_{2m+1}(\eta', q) \rho = b \cos \eta, \rho' = b \cos |\eta'| \]

Mathieu functions appears when separating variables in the Helmholtz equation in elliptical coordinates considered in [9]:
\[ \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \eta^2} + 2q(\text{ch}2\xi - \cos2\eta) \Phi = 0, q = \frac{k^2 b^2}{4} - 0 \]  

(18)
In the case considered the elliptical coordinates \((\xi, \eta)\) in (18) are connected with Cartesian ones \((x, y)\) by virtue of correlations:
\[
\begin{align*}
  x &= b \cosh \xi \cos \eta \\
  y &= b \sinh \xi \sin \eta
\end{align*}
\]

or
\[
\begin{align*}
  x + iy &= b (\cosh (\xi + i \eta)) \\
  \xi > 0, \quad 0 \leq \eta \leq \pi
\end{align*}
\]

Contact segment \(|x| \leq b\) on the half-space boundary \(y = 0\) in the elliptical coordinates \((\xi, \eta)\) is described by the correlations:
\[
x = b \cos \eta, \quad \xi = 0
\]

Then solutions of the Helmholtz equation (18) may be taken in the form [13]
\[
\begin{align*}
  N_{2m+1}^{(1)}(\xi, q)\sin(2m+1)(\eta, q) &= s_{2m+1} \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{2m+1}(q) H_{2r+1}^{(1)}(kr) \sin(2r + 1) \theta \\
  S_{2m+1}^{(1)}(\xi, q)\sin(2m+1)(\eta, q) &= s_{2m+1} \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{2m+1}(q) J_{2r+1}(kr) \sin(2r + 1) \theta
\end{align*}
\]

In correlations (19) \(s_{2m+1}(\eta, q)\) are periodic Mathieu functions, \(S_{2m+1}(\xi, q), N_{2m+1}^{(1)}(\xi, q)\) are modified Mathieu functions. Coefficients \(B_{2r+1}^{2m+1}\) in Mathieu functions obey correlations:
\[
\sum_{m=0}^{\infty} B_{2r+1}^{2m+1} B_{2s+1}^{2m+1} = \delta_{rs} = \begin{cases} 0, & r \neq s \\ 1, & r = s \end{cases}
\]

Factors \(s_{2m+1}\) are determined by McLachlan [13] and adduced in [10] as well. Because of symmetry we can consider but right half-segment \(|x| \leq b, y = 0\), then set of points \(\nu = 0, \theta = 0\) in cylinder coordinates accords with elliptical ones \(\xi = 0, 0 \leq \eta \leq \frac{\pi}{2}\). Let us derive both sides of each equalities (19) with respect to the normal \(\nu\) while \(|x| \leq b, y = 0\) and then put \(\nu = 0\), using correlations \((H - \text{Lamé parameter for the elliptical coordinates})\):
\[
\frac{\partial}{\partial \nu} |_{\nu=0} = \frac{1}{r} \frac{\partial}{\partial \theta} |_{\theta=0, \pi} = \left(\frac{\partial}{\partial H} \frac{\partial}{\partial \xi}\right) |_{\xi=0} = \frac{1}{b \sinh \eta} \frac{\partial}{\partial \xi} |_{\xi=0}
\]

that results to the equalities:
\[
\begin{align*}
  s_{2m+1}^{-1} \frac{\partial}{\partial \sinh \eta} N_{2m+1}^{(1)}(\xi, q) |_{\xi=0} &= s_{2m+1} \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{2m+1}(q) \frac{2r+1}{r} H_{2r+1}^{(1)}(kp) \\
  s_{2m+1}^{-1} \frac{\partial}{\partial \sinh \eta} S_{2m+1}^{(1)}(\xi, q) |_{\xi=0} &= s_{2m+1} \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{2m+1}(q) \frac{2r+1}{r} J_{2r+1}(kp)
\end{align*}
\]

Inverting correlations (21) by virtue of (20), we obtain:
\[
\begin{align*}
  (-1)^r \frac{2r+1}{r} H_{2r+1}^{(1)}(kp) &= \frac{1}{b \cosh \eta} \sum_{m=0}^{\infty} s_{2m+1}^{-1} B_{2r+1}^{2m+1}(q) \frac{\partial}{\partial \xi} N_{2m+1}^{(1)}(\xi, q) |_{\xi=0} \sin(2m+1)(\eta, q) \\
  (-1)^r \frac{2r+1}{r} J_{2r+1}(kp) &= \frac{1}{b \cosh \eta} \sum_{m=0}^{\infty} s_{2m+1}^{-1} B_{2r+1}^{2m+1}(q) \frac{\partial}{\partial \xi} S_{2m+1}^{(1)}(\xi, q) |_{\xi=0} \sin(2m+1)(\eta, q)
\end{align*}
\]

Multiplying each other left hand sides and right hand sides of ones respectively and summing multiplications with respect to \(r = 0, 1, 2, ...\) in accordance with (21) we obtained the identity:
\[
\begin{align*}
  \sum_{r=0}^{\infty} \frac{(2r+1)^2}{pp^r} H_{2r+1}^{(1)}(kp) J_{2r+1}(kp') &= \frac{1}{b \sin \eta} \sum_{m=0}^{\infty} s_{2m+1}^{-2} \frac{\partial}{\partial \xi} N_{2m+1}^{(1)}(\xi, q) |_{\xi=0} \frac{\partial}{\partial \xi} S_{2m+1}^{(1)}(\xi, q) |_{\xi=0} \sin(2m+1)(\eta, q) \sin(2m+1)(\eta', q)
\end{align*}
\]

Among other things there established the correlations
\[
\frac{1}{s_{2m+1}} = -2i \frac{1}{\pi} \frac{\partial}{\partial \xi} N_{2m+1}^{(1)}(0, q) S_{2m+1}^{(1)}(0, q)
\]

It points out the next correlations from the determination of the modified Mathieu functions:
In the equality
\[ \text{Gey}_{2m+1}(\xi, q) - \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^m \left[ J_r(\sqrt{q} \xi) Y_{r+1}(\sqrt{q} \xi) - J_{r+1}(\sqrt{q} \xi) Y_r(\sqrt{q} \xi) \right] \]
we put \( \xi = 0 \), use the equality for Bessel functions:
\[ \sum_{r=0}^{\infty} \frac{1}{B_{2r+1}^m} \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^m \]
and obtain:
\[ \text{Gey}_{2m+1}(0, q) = \left( \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^m \right) \]

According to correlations for periodic Mathieu functions
\[ \text{se}_{2m+1}(\xi, q) = \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^m \sin(2r+1) \frac{\xi}{2} = \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^m \]
it points out the equalities from the (23):
\[ \text{Ne}_{2m+1}^{(1)}(0, q) = -2i \left( \frac{\pi}{\sqrt{q} B_{2m+1}^1} \right) \]
and then
\[ S_{2m+1}^{-1} = \frac{\pi}{\sqrt{q} B_{2m+1}^1} \frac{\text{se}_{2m+1}(\xi, q)}{\text{Ne}_{2m+1}^{(1)}(0, q)} \]
\[ S_{2m+1} = \frac{\text{se}_{2m+1}(\xi, q)}{\sqrt{q} B_{2m+1}^1} \]
The latter permit to transform the identity (22) and takes the form:
\[ \sum_{r=0}^{\infty} \frac{(2r+1)^2}{\rho \rho'} H_{2r+1}^{(1)}(kp)I_{2r+1}(kp') = \]
\[ = -2i \frac{1}{b^2 \sin \sin \eta} \frac{\sum_{m=0}^{\infty} \frac{1}{\pi} \frac{\partial}{\partial \xi} \text{Ne}_{2m+1}^{(1)}(\xi, q)|_{\xi=0} \text{se}_{2m+1}(\eta, q) \text{se}_{2m+1}(\eta', q)}{\text{Ne}_{2m+1}^{(1)}(0, q)} \]
\[ \rho = b \cos \eta, \rho' = b \cos \eta' \]

The established identity is just to be inspected.

5. Conclusion
1. It is constructed the exact solution of boundary value problem arising in a number of problems of exploitation reliability of equipment connected with the analysis of the joining point of technological elements of constructions.
2. Solvability of the problem under consideration are investigated in spaces of fractional smoothness.
3. Method validation of construing the exact solution of the problem in question is described and the structure of the solution of boundary value problem is justified
4. It is proved the auxiliary identity connected Bessel and Mathieu functions permitting to test the validity of the method for constructing the solutions suggested at present paper.
5. The exact solution of boundary value problem and its BIE permit to reconstruct the wave field in the whole wedge-shaped body.
6. Suggested method may be applied to another problems reduced to the BIE where its principal part of operator coincides the operator of the left hand side BIE (2), (3) under consideration.

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