ON SOME NONADMISSIBLE SMOOTH IRREDUCIBLE REPRESENTATIONS FOR $\text{GL}_2$

DANIEL LE

Abstract. Let $p > 2$ be a prime. We give examples of smooth absolutely irreducible representations of $\text{GL}_2(\mathbb{Q}_p^3)$ over $\mathbb{F}_{p^3}$ which are not admissible.

1. Introduction

Smooth representations of $p$-adic reductive groups arise naturally in the theory of automorphic forms. Smooth here means that every vector is invariant under an open subgroup. Classical finite-dimensionality results for automorphic forms imply admissibility: the invariants of the representation under any open subgroup is finite-dimensional. Both of these notions make sense for a base field of any characteristic. Representation theory over base fields of positive characteristic has attracted considerable attention in recent years because of its connection to congruences of automorphic forms and the modularity of Galois representations.

In the recent groundbreaking work [AHHV17], smooth, irreducible, admissible mod $p$ representations of connected reductive $p$-adic groups are classified in terms of supercuspidal representations, closely mirroring the earlier theory in characteristic not equal to $p$. For a base field of characteristic different from $p$, it is known from [Vig96, II.2.8] moreover that every smooth irreducible representation of a connected reductive $p$-adic group is admissible. [AHHV] Question 1 asks whether a similar statement is true for mod $p$ representations. We provide a negative answer, at least when $p > 2$.

Theorem 1.1. Let $p > 2$. There exists a smooth absolutely irreducible $\text{GL}_2(\mathbb{Q}_p^3)$-representation over $\mathbb{F}_{p^3}$ which is not admissible.

It will be clear from the construction that there are infinitely many such representations. Moreover, similar constructions exist for unramified extensions of larger degree, but we content ourselves with describing the simplest example.

Admissibility is a desirable property, in part because it implies that the irreducible representation has a central character, admits Hecke eigenvalues for weights, and has an endomorphism ring of finite dimension over the base field. [AHHV] Question 2, Question 8 ask whether irreducible mod $p$ representations must have central characters and Hecke eigenvalues. The representations that we construct have central characters and Hecke eigenvalues (matching certain supersingular representations), and so we do not answer these questions. However, by restricting scalars for a representation we construct, we also prove the following.

Theorem 1.2. There exists a smooth irreducible $\text{GL}_2(\mathbb{Q}_p^3)$-representation over $\mathbb{F}_{p^3}$ whose endomorphisms contain $\mathbb{F}_p$. 
Of course, such a representation cannot be absolutely irreducible as the endomorphism ring over $F_p$ would contain $F_p \otimes_{F_q} F_p$.

We now make brief remarks on the construction. Irreducible mod $p$ representations are typically rather difficult to construct, much less nonadmissible ones. Global constructions coming from the theory of automorphic forms always give admissible representations and parabolic induction preserves admissibility. However, the Bruhat–Tits tree and the diagrams of Paškūnas give a powerful method of constructing mod $p$ representations of $p$-adic $GL_2$ with fixed $K$-socle where $K$ is the maximal compact subgroup. [BP12] uses this close control of the $K$-socle to prove both irreducibility and admissibility for many representations that they construct. The main idea of this paper is that the control of the $K$-socle can also be used to prove irreducibility and nonadmissibility. We construct an infinite-dimensional diagram that gives rise to a nonadmissible $GL_2(Q_p^2)$-representation, and prove irreducibility using the methods of ibid.

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1.2. Notation. Let $p > 2$ and let $q$ be $p^f$ for a positive integer $f$. Fix an algebraic closure $F_p$ of $F_q$. If $V$ is an $F_q$-vector space, let $V_{F_p}$ denote $V \otimes_{F_q} F_p$.

Let $G$ be $GL_2(Q_q)$, $Z$ the center of $G$, $K$ be $GL_2(Z_q)$, and $I$ (resp. $I_1$) the preimage in $K$ of the upper triangular matrices (resp. unipotent upper triangular matrices) in $GL_2(F_q)$ under the natural reduction map. Let $\Pi \in G$ be the matrix $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$. Then $\Pi$ normalizes $I$ and the normalizer $N(I)$ of $I$ is $IZ \sqcup IZ\Pi$. Moreover, we have an isomorphism

$$N(I)/\langle \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \rangle \simeq I \rtimes \mathbb{Z}/2$$

(1.1)

$$\Pi \mapsto (\text{id}, 1).$$

(1.2)

For a character $\chi$ of $IZ$, let $\chi^\kappa$ be the character of $IZ$ given by precomposing $\chi$ by $\Pi$-conjugation. If $V$ is an $IZ$-representation and $\chi$ a character of $IZ$, we let $V^\chi$ be the $\chi$-isotypic part of $V$.

2. Diagrams

2.1. Diamond diagrams. A diagram is a triple $(D_0, D_1, r)$ where $D_0$ is a smooth $KZ$-representation, $D_1$ is a smooth $N(I)$-representation, and $r$ is an $IZ$-equivariant map $D_1 \to D_0$. A diagram is a basic 0-diagram if the induced map $r : D_1 \to D_0^\kappa$ is an isomorphism.

Let $\rho : G_{Q_p} \to GL_2(F_p)$ be a generic continuous irreducible representation in the sense of [BP12, Definition 11.7] (such representations exist with the assumption
that \( p > 2 \). Let \( \mathcal{D}(\rho) \) be the set of Serre weights defined in \[BP12\] §11. To \( \rho \), \[BP12\] Theorem 13.8 attaches a family of basic 0-diagrams. We fix for the rest of the paper a basic 0-diagram \( (D_0(\rho), D_1(\rho), r) \) in this family which is defined over \( \mathbb{F}_q \). That is \( D_0(\rho) \) and \( D_1(\rho) \) are finite dimensional \( K \) and \( N(I) \)-representations over \( \mathbb{F}_q \), respectively, and \( (D_0(\rho)_F\), \( D_1(\rho)_F\), \( r) \) is a member of the family constructed in \textit{loc. cit.} We note that the Jordan–Hölder factors of \( D_0(\rho) \) are multiplicity free by \[BP12\] Theorem 13.8], so that in particular, \( D_1(\rho) \) is a multiplicity free semisimple \( I\mathbb{Z} \)-representation. Then \( r \) identifies \( D_1(\rho) \) with \( D_0(\rho)^{\delta_1} \) as \( I\mathbb{Z} \)-representations, which we will identify implicitly. Recall that there is a direct sum decomposition \( D_0(\rho) = \bigoplus_{\sigma \in \mathcal{D}(\rho)} D_{0, \sigma}(\rho) \).

Recall from \[BP12\] Lemma 11.4 and the paragraph thereafter that there is a bijection

\[
2^{\mathbb{Z}/f} \to \mathcal{D}(\rho) \\
J \mapsto \sigma_J
\]

Define an automorphism \( \delta : 2^{\mathbb{Z}/f} \to 2^{\mathbb{Z}/f} \) by \( j \in \delta(J) \) if and only if \( j + 1 \in J \) (resp. \( j + 1 \notin J \)) for \( j \neq 0 \) (resp. \( j = 0 \)). This “shift then flip at \( j = 0 \)” is denoted \( \delta_1 \) in \[BP12\] §15.

We introduce one final piece of notation. For \( 0 \leq s \leq q - 1 \), let

\[
S_s := \sum_{\lambda \in \mathbb{F}_q} \lambda^s \begin{pmatrix} |\lambda| & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{F}_q[K].
\]

**Proposition 2.1.** Let \( v \) be a nonzero element in \( D_1(\rho)^{\chi} \). Then there is a unique \( 0 \leq s(\chi) \leq q - 1 \) such that \( S_s(\chi)(v) \) is a nonzero element of \( (\text{soc}_K D_0(\rho))^{\delta_1} \).

**Proof.** Since \( D_1(\rho)^{\chi} \) is one-dimensional, the \( K \)-representation generated by \( v \) has irreducible socle (cf. \[Bre11\] Proposition 5.1). The result now follows from \[BP12\] Lemma 2.7. \( \square \)

Define a linear map

\[
S : D_1(\rho) \to (\text{soc}_K D_0(\rho))^{\delta_1}
\]

which maps a nonzero \((I\mathbb{Z}, \chi)\)-eigenvector \( v \) to \( S_s(\chi)v \).

We recall the following result.

**Proposition 2.2.** Let \( \chi_J \) be the \( I \)-character of \( \sigma_J^{\delta_1} \). Then \( S \circ \Pi \) gives an isomorphism \( D_1(\rho)^{\chi_J} \) to \( D_1(\rho)^{\chi_{\delta_1}(J)} \) for all \( J \in 2^{\mathbb{Z}/f} \).

**Proof.** This follows from \[BP12\] Lemma 15.2 (see also the proof of \[Bre11\] Proposition 5.1)). \( \square \)

2.2. An infinite diagram. In this section, we let \( f \) be 3. Let \( D_0 \) be the \( K \)-representation \( \oplus_{i \in \mathbb{Z}} D_{0,i} \), where there is a fixed isomorphism \( D_{0,i} \cong D_0(\rho)^{\delta_1} \). Let \( \nu_i \) be the inclusion \( D_0(\rho) \subset D_{0,i} \subset D_0 \). For \( v \in D_0(\rho) \), we denote \( \nu_i(v) \) by \( v_i \).

Let \( D_1 \) be \( D_0^{\delta_1} \). Let \( \lambda = (\lambda_i)_{i \in \mathbb{Z}} \) be in \( \prod_{i \in \mathbb{Z}} \mathbb{F}_p^\times \). For such a \( \lambda \), we now define an action of \( N(I) \) on \( D_1 \) such that \( \Pi^2 \) acts trivially. By \[BP12\], it suffices to define an involution on \( D_1 \) taking \( D_1^\chi \) to \( D_1^{\chi^*} \) for every character \( \chi \) of \( I\mathbb{Z} \). We will denote this involution by \( \Pi \).

Let \( \chi^+ \) be the \( I\mathbb{Z} \)-character of the space \( \sigma_1^{\delta_1} \), and \( \chi^- \) be the \( I\mathbb{Z} \)-character of the space \( \sigma_{\{0,1\}}^{\delta_1} \) (as usual \( \Pi^2 \) acts trivially).
Proposition 2.3. There is an IZ-character $\chi_1$ (resp. $\chi_2$) such that both of the spaces $D_{0,\sigma_2}(\rho)^{\chi_1}$ and $D_{0,\sigma_1}(\rho)^{\chi_2}$ (resp. $D_{0,\sigma_{(0,1)}}(\rho)^{\chi_2}$ and $D_{0,\sigma_{(0)}}(\rho)^{\chi_2}$) are nonzero.

Proof. This follows from an explicit check using [BP12 Corollary 14.10 and Lemma 15.2]. In the notation of [BP12 §11], we have that $\sigma_{(2)}$ corresponds to

$$ (\lambda_0(r_0), \lambda_1(r_1), \lambda_2(r_2)) = (r_0, p - 2 - r_1, r_2 + 1) $$

and $\sigma_{(0,1)}$ corresponds to

$$ (\lambda_0(r_0), \lambda_1(r_1), \lambda_2(r_2)) = (p - 1 - r_0, r_1 + 1, p - 2 - r_2). $$

Then $\chi_1$ corresponds to

$$ (\mu_0(\lambda_0(r_0)), \mu_1(\lambda_1(r_1)), \mu_2(\lambda_2(r_2))) = (p - 2 - r_0, p - 1 - r_1, r_2 + 1) $$

and $\chi_2$ corresponds to

$$ (\mu_0(\lambda_0(r_0)), \mu_1(\lambda_1(r_1)), \mu_2(\lambda_2(r_2))) = (p - r_0, r_1 + 1, r_2). $$

$\square$

In fact, the characters $\chi_1$ and $\chi_2$ are uniquely described by the properties in Proposition 2.3 but we will not use this. As we will see, the only property that we will need is that $\chi_1$ (resp. $\chi_2$) is a character in $D_{0,\sigma_{(2)}}(\rho)^{\chi_1}$ (resp. $D_{0,\sigma_{(0,1)}}(\rho)^{\chi_2}$), which is not in $(\sigma_{(2)})^{I_1}$ (resp. $(\sigma_{(0,1)})^{I_1}$). The exact choices and formulas of Proposition 2.3 will not be important, and we include them only for the sake of concreteness.

If $v \in D_1(\rho)^{\chi}$ with $\chi \notin \{\chi_+, \chi_0, \chi_-, \chi_0^\pm, \chi_1, \chi_1^\pm\}$, we define

$$ \overline{\Pi}(v_i) = (\Pi v)_i. $$

If $v \in D_1(\rho)^{\chi^+}$, then we define

$$ \overline{\Pi}(v_i) = (\Pi v)_{i+1}. $$

If $v \in D_1(\rho)^{\chi^-}$, then we define

$$ \overline{\Pi}(v_i) = (\Pi v)_{i-1}. $$

If $v \in D_1(\rho)^{\chi_0}$, then we define

$$ \overline{\Pi}(v_i) = \lambda_i(\Pi v)_i. $$

This now uniquely defines an $\mathbb{F}_p$-linear involution $\overline{\Pi}$ of $D_1$, and it takes $D_1^{\chi^+}$ to $D_1^{\chi^-}$ for every character $\chi$ of IZ as desired.

Let $D(\lambda)$ be the basic 0-diagram $(D_0, D_1, \text{can})$ with the above actions, where can denotes the canonical inclusion $D_1 \subset D_0$. We define an $\mathbb{F}_p$-linear map $\overline{S} : D_1 \to \text{soc}_K D_0^{I_1}$ by the formula $\overline{S} t_i = t_i S$, where $S$ is as defined in [2.1]
3. The construction

For the purposes of notation, we review the proof of the following result, which is a special case of [BP12, Theorem 9.8], although we work over \( F_q \) rather than \( \mathbb{F}_p \).

**Theorem 3.1.** There exists a smooth \( G \)-representation \( \tau \) over \( F_q \) such that

- there is an injection of diagrams \( (D_0(\rho), D_1(\rho), r) \subset (\tau|_{KZ}, \tau|_{IZ}, \text{id}) \);
- \( \tau \) is generated as a \( G \)-representation by the image of \( D_0(\rho) \); and
- the induced injection \( \text{soc}_K D_0(\rho) \hookrightarrow \text{soc}_K \tau \) is an isomorphism.

**Proof.** Let \( \Omega \) be the \( K \)-injective envelope of \( D_0(\rho)|_K \). We give \( \Omega \) a \( KZ \)-action by demanding that \( \Pi^2 \) acts trivially. There is an idempotent \( e \in \text{End}_I(\Omega) \) such that \( e(\Omega)|_I \) is an \( I \)-injective envelope of \( D_1(\rho) \). There is a decomposition of \( e(\Omega)|_I \) as a direct sum

\[ \bigoplus \chi \Omega_{\chi}, \]

where \( \chi \) runs over the \( I \)-characters in \( D_1(\rho) \) and \( \Omega_{\chi} \) is an \( I \)-injective envelope of the \( \chi \)-isotypic part of \( D_1(\rho) \). By [BP12, Lemma 9.5], there is an \( F_q \)-linear map \( e(\Omega) \to e(\Omega) \) which intertwines the action and \( \Pi \)-conjugate action of \( IZ \), extends the action of \( \Pi \) on \( D_1(\rho) \), and whose restriction to \( \Omega_{\chi} \) for each \( \chi \) above gives a map

\[ \Omega_{\chi} \to \Omega_{\chi'} \cdot \]

This gives an action of \( N(I) \) on \( e(\Omega) \). There is also an action of \( N(I) \) on \( (1 - e)(\Omega) \) by [BP12, Lemma 9.6]. This gives an action of \( N(I) \) on \( \Omega \) whose restriction to \( I \) is compatible with the action coming from \( KZ \) on \( \Omega \). By [Paš04, Corollary 5.18], this gives an action of \( G \) on \( \Omega \). We then take \( \tau \) to be the \( G \)-representation generated by \( D_0(\rho) \).

**Theorem 3.2.** There exists a smooth \( G \)-representation \( \pi \) such that

- if \( \lambda \in \prod_{i \in \mathbb{Z}} \mathbb{F}_q^\times \), then \( \pi \) is defined over \( \mathbb{F}_q \);
- there is an injection of diagrams \( D(\lambda) \subset (\pi|_{KZ}, \pi|_{IZ}, \text{id}) \);
- \( \pi \) is generated as a \( G \)-representation by the image of \( D_0(\rho) \); and
- the induced injection \( \text{soc}_K D_0(\rho) \hookrightarrow \text{soc}_K \pi \) is an isomorphism.

**Proof.** Let \( \Omega \) be the \( K \)-injective envelope of \( D_0(\rho)|_K \) as in the proof of Theorem [3.1]. We give \( \Omega \) a \( KZ \)-action by demanding that \( \Pi^2 \) acts trivially. Recall the definitions of \( e \in \text{End}_I(\Omega) \) and \( \Omega_{\chi} \) from the proof of Theorem [3.1]. Now let \( \Omega_{\infty} \) be the \( KZ \)-representation \( \bigoplus_{i \in \Omega_{\chi}} \Omega_i \) where there is a fixed isomorphism \( \Omega_i \cong \Omega_{\chi}|_{KZ} \).

Let \( \iota_i \) be the \( KZ \)-injection \( \Omega \subset \Omega_i \subset \Omega_{\infty} \). To define an action of \( N(I) \) on \( \Omega_{\infty} \), it suffices to define an involution, which we call \( \overline{\iota} \), on \( \Omega_{\infty} \) which intertwines the action and \( \Pi \)-conjugate action of \( IZ \). For each \( i \in \mathbb{Z} \), define \( \overline{\iota_i(1 - e)}(\Omega) \) to be \( \iota_i \circ \Pi_i(\Omega) \). For \( \chi \notin \{ \chi_+, \chi_-, \chi_0, \chi_1 \} \), we define \( \overline{\iota_i|_{\Omega_{\chi}}} \) to be \( \iota_i \circ \Pi_i|_{\Omega_{\chi}} \). We define \( \overline{\iota_i(1 - e)(\Omega)} \rightarrow \Omega_{\infty} \cdot \overline{\iota_i(1 - e)(\Omega)} \) to be \( \iota_i \circ \chi \Pi_i(\Omega_{\chi}) \). This completely determines the \( \mathbb{F}_p \)-linear involution \( \overline{\iota} \). It is easy to see that the defined action of \( N(I) \) on \( \Omega_{\infty} \) extends the action of \( N(I) \) on \( D_1 \). By [Paš04, Corollary 5.18], this gives an action of \( G \) on \( \Omega_{\infty} \). If \( \lambda \in \prod_{i \in \mathbb{Z}} \mathbb{F}_q^\times \), then this action is defined over \( \mathbb{F}_q \). Then if we let \( \pi \) be the \( G \)-subrepresentation of \( \Omega_{\infty} \) generated by \( D_0 \), \( \pi \) satisfies the required hypotheses. Indeed, we have that \( \text{soc}_K \Omega_{\infty} \rightarrow \text{soc}_K \pi \rightarrow \text{soc}_K D_0 \), and \( \pi \) is defined over \( \mathbb{F}_q \) if \( \Omega_{\infty} \) is. \( \square \)
Let $D_{0,l}(\rho)$ and $D_{0,ll}(\rho)$ be $D_{0,\sigma_{\{2\}}}(\rho) \oplus D_{0,\sigma_{\{0,1\}}}(\rho)$ and $\oplus J D_{0,\sigma_{J}}(\rho)$, respectively, where the sum is over
$$J \in \{\emptyset, \{0\}, \{0,2\}, \{0,1,2\}, \{1\}, \{2\}\}.$$  
(This partition $2^{\mathbb{Z}/3} = J \cup J^c$ corresponds to $\delta$-orbits.) We now recall the following special case of [BP12, Theorem 19.10(i)], since the arguments play a crucial role in the proof of Theorem 3.4.

**Theorem 3.3.** Any $G$-representation $\tau$ satisfying the hypotheses in Theorem 3.1 is absolutely irreducible.

**Proof.** Let $\tau' \subset \tau_p$ be a nonzero $G$-subrepresentation. Since $\text{soc}_K \tau_p \cong \text{soc}_KD_{0}(\rho)\tau_p$, there is a $J$ such that $\text{Hom}_K(\sigma_J, \tau')$ is nonzero. Then by [BP12, Lemma 19.7], we have the inclusion $D_{0,\sigma_{\{J\}}}(\rho)\tau_p \subset \tau'$. Repeating this, we obtain an inclusion of one of $D_{0,l}(\rho)\tau_p$ and $D_{0,ll}(\rho)\tau_p$ in $\tau'$. Then either $(\tau')^I\sim \lambda_1$ or $(\tau')^I\sim \lambda_2$ is nonzero. Applying II, we see that they both must be nonzero so that $D_{0,\sigma_{\{J\}}}(\rho)\tau_p$ and $D_{0,\sigma_{\{2\}}}(\rho)\tau_p$ are both in $\tau'$. Repeating the earlier argument, we have that $D_{0}(\rho)\tau_p \subset \tau'$. Since $\tau_p$ is generated by $D_{0}(\rho)\tau_p$, we have that $\tau' = \tau_p$. □

The following is the main result of this section.

**Theorem 3.4.** If $\lambda_0 \in F_q$ and $\lambda_i \neq \lambda_0$ for all $i \neq 0$, then any $G$-representation $\pi$ satisfying the hypotheses in Theorem 3.1 is irreducible. If moreover the $F_q$-span of $(\lambda_i)_i$ in $\mathbb{F}_p$ is $\mathbb{F}_q$, then $\pi$ is irreducible as a $G$-representation over $F_q$.

**Proof.** Let $\pi'$ be a nonzero $G$-subrepresentation of $\pi$ seen as a representation over $F_q$ by restriction of scalars. Since $\text{soc}_K \pi' \subset D_{0}$, there exists $\sigma \in \mathcal{D}(\rho)$ such that $\text{Hom}_K(\sigma, \pi')$ is nonzero. Then there exists a $(c_i)_i$ in $\oplus_i \mathbb{F}_p$ such that
$$\left(\sum_i c_i t_i\right)(D_{0,\sigma}(\rho)) \cap \pi' \neq 0.$$  

**Lemma 3.5.** Suppose that $\sigma \in \mathcal{D}(\rho)$ and $(d_i)_i \in \oplus_i \mathbb{F}_p$ are elements such that
$$\left(\sum_i d_i t_i\right)(D_{0,\sigma}(\rho)) \cap \pi' \neq 0.$$  

Then for any $j \in \mathbb{Z}$,
$$\left(\sum_i d_i t_{i+j}\right)(D_{0}(\rho)) \subset \pi'.$$

**Proof.** We assume that $\sigma$ is $\sigma_0$, as the other cases are similar. Then as in the proof of Theorem 3.3, we see from repeatedly applying $\Pi$ that
$$\left(\sum_i d_i t_{i+j}\right)(D_{0,ll}(\rho)) \subset \pi'$$
for $j > 0$. Since for each $j > 0$, we have that
$$\left(\sum_i d_i t_{i+j}\right)(D_{0,ll}(\rho)^{2_j}) \subset \pi',$$
we have that
$$\left(\sum_i d_i t_{i+j}\right)(D_{0,l}(\rho)^{2_j}) \subset \pi'.$$
for \( j > 0 \). Again repeatedly applying \( \widetilde{S}\Pi \), we see that
\[
\left( \sum_i d_i \pi_i \right)(D_{0,1}(\rho)) \subset \pi'
\]
for all \( j \in \mathbb{Z} \). Then since
\[
\left( \sum_i d_i \pi_i \right)(D_{0,1}(\rho)^{\chi_2}) \subset \pi'
\]
for all \( j \in \mathbb{Z} \), we have that
\[
\left( \sum_i d_i \pi_i \right)(D_{0,11}(\rho)^{\chi_2}) \subset \pi'
\]
for all \( j \in \mathbb{Z} \). We conclude that
\[
\left( \sum_i d_i \pi_i \right)(D_{0,11}(\rho)) \subset \pi'
\]
for all \( j \in \mathbb{Z} \) by again repeatedly applying \( \widetilde{S}\Pi \).

In the proof of the next lemma, we will use the following notation. For \((d_i)_i \in \oplus_{i \in \mathbb{Z}} \mathbb{F}_p\), let \( \#(d_i)_i \) be the cardinality of \( \{ i \in \mathbb{Z} | d_i \neq 0 \} \).

**Lemma 3.6.** There is a nonzero constant \( c \in \mathbb{F}_p \) such that \( c \sigma_0(D_{0,\sigma(z)}(\rho)) \subset \pi' \).

**Proof.** Fix nonzero elements \( v^1 \in D_1(\rho)^{\chi_1} \) and \( v^2 \in D_1(\rho)^{\chi_2} \). One checks that \((S\Pi)^2 v^1 \) and \( S\Pi v^2 \) are nonzero elements in \( \sigma_1^{(1)} \subset D_1(\rho) \) using the definition of \( \chi_1 \) and \( \chi_2 \) and Proposition 2.2. Thus, there exists a scalar \( \mu \in \mathbb{F}_q^\times \) such that
\[
(S\Pi)^2 v^1 = \mu S\Pi v^2.
\]
Then by the definition of the action of \( \Pi \) on \( D_1 \), we have that
\[
(S\Pi)^2 v^1 = \lambda \mu S\Pi v^2
\]
for all \( i \in \mathbb{Z} \).

By Lemma 3.3, there exists a nonzero \((c_i)_i \) in \( \oplus_{i \in \mathbb{Z}} \mathbb{F}_p \) such that
\[
\left( \sum_i c_i \pi_i \right)D_0(\rho) \subset \pi'.
\]
Assume that \((c_i)_i \) is minimal among such elements of \( \oplus_{i \in \mathbb{Z}} \mathbb{F}_p \). It suffices to show that \((c_i)_i = 1 \) by Lemma 3.3. By Lemma 3.3, we can also assume that \( c_0 \) is nonzero.

Since \( \sum_i c_i v^1_i \) and \( \sum_i c_i v^2_i \) are in \( \pi' \), then by the first paragraph, we have that
\[
\sum_i c_i ((S\Pi)^2 v^1_i - \lambda_0 \mu S\Pi v^2_i) = \sum_i (\lambda_i - \lambda_0) c_i S\Pi v^2_i
\]
is in \( \pi' \). We see from Lemma 3.3 that
\[
\left( \sum_i c_i' \pi_i \right)D_{0,\sigma(z)}(\rho) \cap \pi' \neq 0
\]
for \( c_i' = (\lambda_i - \lambda_0) c_i \). Since the \( \lambda_i \neq \lambda_0 \) for \( i \neq 0 \) and \( c_0 \neq 0 \), \((c_i')_i = #(c_i) - 1 \). Since we assumed that \((c_i)_i \) is minimal, we must have that \( #(c_i)_i = 1 \). \( \square \)
We now complete the proof of Theorem 3.4. By Lemma 3.5, it suffices to show that \( c \) in Lemma 3.6 can be taken to be any element of \( \overline{\mathbb{F}}_p \). If \( \pi' \) is a subrepresentation of \( \pi \) over \( \mathbb{F}_p \), this is clear. Now assume that the \( \mathbb{F}_q \)-span of \( (\lambda_i)_i \) is \( \overline{\mathbb{F}}_p \). By Lemma 3.5 \( c_i j(D_0(\rho)^{\chi_1}) \subset \pi' \) for all \( j \in \mathbb{Z} \). By applying \( \Pi \) to \( c_i j(D_0(\rho)^{\chi_1}) \), we see that \( c \) can be taken to be \( c \lambda_j \) for all \( j \in \mathbb{Z} \). Since \( (c \lambda_j)_i \) spans \( \overline{\mathbb{F}}_p \) over \( \mathbb{F}_q \), we are done.

Note that since \( D_0 \) is not admissible, any \( \pi \) as in Theorem 3.2 is not admissible. Taking \( \lambda \in \prod_{i \in \mathbb{Z}} \mathbb{F}_q^\times \), Theorem 3.4 implies Theorem 1.1. Since the endomorphisms of any such \( \pi \) must contain \( \overline{\mathbb{F}}_p \), taking \( (\lambda_i)_i \) to span \( \overline{\mathbb{F}}_p \) over \( \mathbb{F}_q \) and restricting scalars of \( \pi \) to \( \mathbb{F}_q \), Theorem 3.4 implies Theorem 1.2.

References

[AHHV] N. Abe, G. Henniart, F. Herzig, and M.-F. Vignéras, Questions on mod p representations of reductive p-adic groups.

[AHHV17], A classification of irreducible admissible mod p representations of p-adic reductive groups, J. Amer. Math. Soc. 30 (2017), no. 2, 495–559. MR 3600042

[BP12] Christophe Breuil and Vytautas Paškūnas, Towards a modulo p Langlands correspondence for GL_2, Mem. Amer. Math. Soc. 216 (2012), no. 1016, vi+114. MR 2931521

[Bre11] Christophe Breuil, Diagrammes de Diamond et \((\phi, \Gamma)\)-modules, Israel J. Math. 182 (2011), 349–382. MR 2783977

[Paš04] Vytautas Paškūnas, Coefficient systems and supersingular representations of GL_2(F), Mém. Soc. Math. Fr. (N.S.) (2004), no. 99, vi+84. MR 2128381 (2005m:22017)

[Vig96] Marie-France Vignéras, Représentations l-modulaires d’un groupe réductif p-adique avec \( l \neq p \), Progress in Mathematics, vol. 137, Birkhäuser Boston, Inc., Boston, MA, 1996. MR 1395151