INFINITELY MANY SOLUTIONS FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH SLOW DECAYING OF POTENTIAL

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Abstract. In the paper we prove the multiplicity existence of both nonlinear Schrödinger equation and Schrödinger system with slow decaying rate of electric potential at infinity. Namely, for any $m, n > 0$, the potentials $P, Q$ have the asymptotic behavior

\[
\begin{align*}
P(r) &= 1 + \frac{a}{r^m} + O\left(\frac{1}{r^{m+\theta}}\right), \quad \theta > 0 \\
Q(r) &= 1 + \frac{b}{r^n} + O\left(\frac{1}{r^{n+\tilde{\theta}}}\right), \quad \tilde{\theta} > 0
\end{align*}
\]

then Schrödinger equation and Schrödinger system have infinitely many solutions with arbitrarily large energy, which extends the results of [37] for single Schrödinger equation and [30] for Schrödinger system.

1. Introduction. In this paper, we study nonlinear Schrödinger equation and Schrödinger system with symmetric and slow decaying electric potentials.

1.1. Nonlinear Schrödinger equation. The nonlinear Schrödinger equation (NLSE) plays an important role in theoretical physics. It is a classical field equation whose principal applications are to the propagation of light in nonlinear optical fibers and planar waveguides and to Bose-Einstein condensates. The equation also appears in the studies of small-amplitude gravity waves, the Langmuir waves in hot plasmas and many others.

For the NLSE of the form

\[ i\hbar \psi_t = -\Delta \psi + Q(x)\psi - |\psi|^{p-1}\psi \]  

where $i$ is the imaginary unit, $\hbar$ is the reduced Planck constant and $\psi(x, t)$ is the wave function, we consider its standing waves, namely solutions of the form $\psi(x, t) = e^{iEt/\hbar}u(x)$. So $\psi(x, t)$ satisfies [1] if and only if $u(x)$ solves the nonlinear elliptic problem

\[ \Delta u - V(x)u + u^p = 0 \]

where $V(x) = Q(x) + E$. This kind of problems has been widely investigated in the past.

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For instance, we consider the following singularly perturbed problem
\[ \varepsilon^2 \Delta u - V(x)u + u^p = 0. \] 
(2)

It is known that concentrated solutions, as \( \varepsilon \) goes to 0, are greatly affected by the potential \( V \), for example the number of the critical points of \( V \) (see \[1, 11, 12, 17, 33\] and references therein), the type of the critical points of \( V \) (see \[9, 20, 28\] and references therein), or near higher dimensional stationary sets of other auxiliary potentials, see also \[2, 13, 24, 34\] and references therein.

On the other hand, if we study the no perturbation problem (2), then the existence of the least energy solution depends on the condition
\[ \inf_{x \in \mathbb{R}^N} V(x) < \lim_{|x| \to \infty} V(x), \]
see \[14, 22, 23, 31\] and references therein. Otherwise, one has to find solutions in a higher energy level. In 2010, Wei-Yan \[37\] considered the following problem
\[ \Delta u - V(|x|)u + u^p = 0, \quad u > 0 \quad \text{in} \quad \mathbb{R}^N, \] 
(3)
where \( p \) is subcritical. And they proved that this problem admits infinitely many solution provided
\[ V(|x|) = V_0 + \frac{a}{r^m} + O \left( \frac{1}{r^{m+\theta}} \right), \quad \text{for some} \quad a > 0, \quad \theta > 0, \quad V_0 > 0 \quad \text{and} \quad m > 1. \]

Note that the problem (3) has no any parameter itself, hence the result of Wei-Yan is surprised and very interesting. After Wei-Yan’s work, there are two problems. One is that can we remove the symmetry on potential \( V \), which is almost done by del Pino-Wei-Yao \[16\] using the so called intermediate reduction method. The other is whether the condition \( m > 1 \) is the best. In other words, can we extend to any positive number \( m \)? In the paper we will give an affirmative answer.

From now on, we assume that the potential \( V(x) = V(|x|) = V(r) \) satisfying the condition:

There are constants \( a > 0, \ m > 0, \ \theta > 0, \ V_0 > 0 \) such that
\[ V(r) = V_0 + \frac{a}{r^m} + O \left( \frac{1}{r^{m+\theta}} \right) \] 
(V)
as \( r \to +\infty \). Without loss of generality, we may assume that \( V_0 = 1 \).

The corresponding energy is
\[ I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(|x|)u^2) - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}. \] 
(4)

Our main result in this paper can be stated as the following

**Theorem 1.1.** If the potential \( V(r) \) satisfies (V), then the problem (3) has infinitely non-radial positive solutions.

**Remark 1.2.** The main improvement comparing to Theorem 1.1 in \[37\] is that we can allow \( m \) to be any positive number. The key reason is that in \[37\], they choose \( L^2 \)-norm to estimate error and the perturbation term. Hence to ensure the energy produced by perturbation term small enough, \( m > 1 \) is necessary and can not be extend to any positive number \( m \) in their frame. In our case, we will choose weighted \( L^\infty \)-norm to get the solvability of the nonlinear problem. After careful analysis, we find \( m > 0 \) is suitable to get the solutions.
1.2. **Nonlinear Schrödinger system.** We consider the following nonlinear Schrödinger system

\[
\begin{aligned}
-\Delta u + P(|x|)u &= \mu_1 u^3 + \beta v^2 u \quad x \in \mathbb{R}^3, \\
-\Delta v + Q(|x|)v &= \mu_2 v^3 + \beta u^2 v \quad x \in \mathbb{R}^3,
\end{aligned}
\]  

(5)

These type of systems arises when one considers standing wave solutions of time-dependent $N$-coupled Schrödinger systems, for $j = 1, \cdots, M$,

\[
-\partial_t \Phi_j = \Delta \Phi_j - V_j(x)\Phi_j + \mu_j |\Phi_j|^2 \Phi_j + \Phi_j \sum_{k=1,k\neq j}^M \beta_{jk} |\Phi_k|^2 \quad \text{in } \mathbb{R}^3.
\]

These systems of equations, also known as coupled Gross-Pitaevskii equations, have applications in many physical problems such as in nonlinear optics and in Bose-Einstein condensates theory [29] with $M$ different hyperfine states denoted by $|j\rangle$.

Physically, $\Phi_j$ are the wave functions of the corresponding condensates; the coefficients $\mu_j$ and $\beta_{jk} = \beta_{kj}$ are the intraspecies and interspecies scattering lengths, respectively. The sign of $\beta_{jk}$ determines whether the interactions of the states $|j\rangle$ and $|k\rangle$ are repulsive or attractive. In the attractive case the components of a vector solution tend to go along with each other, leading to synchronization. In the repulsive case, the components tend to segregation from each other, leading to phase separations. Besides, the internal actions of the single state $|j\rangle$ are attractive if $\mu_j > 0$.

System of nonlinear Schrödinger equations (5), especially the case $P = Q = 1$, has been extensively studied in recent years. Lin-Wei [21] prove the existence of the least energy solution when $\beta_{12}$ is less than some positive $\beta_0$, and get its asymptotic behavior for the singular perturbation case. Phase separation was also considered in several cases, see details in [5, 6, 15, 27, 32, 36] and references therein, as the coupling constant $\beta_{12}$ tends to negative infinity in the repulsive case. On the other hand, for the attractive case, that is $\beta_{12} > 0$, the least energy solution in [21] exhibits synchronization, i.e. the peaks of $u_1$ and $u_2$ go together. We refer other synchronized solutions to [4] under suitable conditions.

In 2013, using the basic idea in [37], Peng-Wang [30] considers the following Schrödinger system

\[
\begin{aligned}
-\Delta u + P(|x|)u &= \mu_1 u^3 + \beta v^2 u \quad x \in \mathbb{R}^3, \\
-\Delta v + Q(|x|)v &= \mu_2 v^3 + \beta u^2 v \quad x \in \mathbb{R}^3,
\end{aligned}
\]  

(6)

where $\mu_1, \mu_2 > 0$, $\beta \in \mathbb{R}$. When the potentials $P, Q$ satisfy the assumptions

(P): There are constants $a > 0, m > 1, \theta > 0$ such that, as $r \to \infty$,

\[
P(r) = 1 + \frac{a}{r^m} + O\left(\frac{1}{r^{m+\theta}}\right),
\]

(7)

(Q): There are constants $b > 0, n > 1, \tilde{\theta} > 0$ such that, as $r \to \infty$,

\[
Q(r) = 1 + \frac{b}{r^n} + O\left(\frac{1}{r^{n+\tilde{\theta}}}\right),
\]

(8)

then they proved that the above system admits infinitely many synchronized and segregated solutions. Here we should point out that since the authors used the same frame as that in [37], $m, n > 1$ is necessary. On the other hand, the symmetry conditions for $P$ and $Q$ have also been removed by Ao-Wang-Yao [3].
In this paper, we want to remove the restriction $m, n > 1$ and build the following result.

**Theorem 1.3.** Suppose that $P(r), Q(r)$ satisfy (7), (8) respectively, but with $m, n > 0$. Then there exists a decreasing sequence $\{\beta_k\} \subset (-\sqrt{\mu_1\mu_2}, 0)$, with $\beta_k \to -\sqrt{\mu_1\mu_2}$, such that for any $\beta \in (-\sqrt{\mu_1\mu_2}, 0) \cup (0, \min\{\mu_1, \mu_2\}) \cup (\max\{\mu_1, \mu_2\}, \infty)$, $\beta \neq \beta_k$ and for any $K \in \mathbb{N}$, Problem (6) possesses infinitely many non-radial positive synchronized solutions $(u_K, v_K)$ whose energy can be arbitrarily large, provided one of the following two conditions holds:

(i) $m < n$, $a > 0$ and $b \in \mathbb{R}$; or $m > n$, $a \in \mathbb{R}$ and $b > 0$;
(ii) $m = n$, $aB + bC > 0$ where $B, C$ are positive numbers in the energy expansion (64).

**Theorem 1.4.** Suppose that $P(r), Q(r)$ satisfy (7), (8) correspondingly with $m = n > 0$, $a > 0, b > 0$. Then there exists $\beta_0 > 0$ such that for $\beta < \beta_0$, Problem (6) has infinitely many non-radial positive segregated solutions $(u_K, v_K)$ whose energy can be arbitrarily large.

**Remark 1.5.** In Theorem 1.3 and Theorem 1.4 we just need $m, n$ to be positive, which improves Theorem 1.1 and Theorem 1.2 in [30].

2. Preliminaries. Throughout the paper, we need the following facts. The first is well-known which comes from [17, 18, 19].

**Lemma 2.1.** If $1 < p < \infty$ for $N = 2$ and $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$, then every positive solution of the problem

$$
\Delta w - w + w^p = 0, \quad w > 0 \text{ in } \mathbb{R}^N, \quad w \in H^1(\mathbb{R}^N)
$$

has the form $\tilde{w}(\cdot - \xi)$ for some $\xi \in \mathbb{R}^N$, where $\tilde{w}(x) = w(|x|) = w(r) \in C^\infty(\mathbb{R}^N)$ is the unique positive radial solution which satisfies

$$
\lim_{r \to \infty} r^{\frac{N-1}{p-1}} e^r w(r) = c_{N,p}, \quad \lim_{r \to \infty} \frac{w'(r)}{w(r)} = -1. \tag{10}
$$

Here $c_{N,p}$ is a positive constant depending only on $N$ and $p$. Furthermore, the Morse index of $w$ is one and $w$ is non-degenerate in the sense that

$$
\ker (\Delta - 1 + pw^{p-1}) \cap L^\infty(\mathbb{R}^N) = \text{Span} \{\partial_{x_1} w, \ldots, \partial_{x_N} w\}.
$$

As before, we set

$$
x_j = \left(r \cos \frac{2(j-1)\pi}{K}, \ r \sin \frac{2(j-1)\pi}{K}, 0\right), \quad j = 1, \ldots, K,
$$

where $0$ is the zero vector in $\mathbb{R}^{N-2}$ and

$$
r \in \Lambda_K := \left[\left(\frac{m}{2\pi} - \sigma\right) K \ln K, \left(\frac{m}{2\pi} + \sigma\right) K \ln K\right], \tag{11}
$$

for $0 < \sigma < \frac{m}{2\pi}$ but fixed.

It is easy to see that

$$
\rho := \min_{j \neq k} |x_j - x_k| = 2r \sin \frac{\pi}{K} \to \infty, \quad \text{for any } r \in \Lambda_K, \text{ as } K \to \infty.
$$

Actually, the order of $\rho$ is between $(m - 2\pi\sigma) \ln K$ and $(m + 2\pi\sigma) \ln K$ when $K$ tends to infinity.
Now we can define our approximation

\[ W_r(x) = \sum_{j=1}^{K} U_j(x) \quad \text{where } U_j(x) = w(x - x_j), \quad (12) \]

and divide the whole space \( \mathbb{R}^N \) into \( K + 1 \) parts:

\[ \Omega^\ell_j = \left\{ x \in \mathbb{R}^N \bigg| |x - x_j| = \min_{1 \leq k \leq K} |x - x_k| \leq \frac{\ell \rho}{2} \right\}, \quad \forall \ j = 1, \ldots, K, \]

and

\[ \Omega^\ell_{K+1} = \mathbb{R}^N \setminus \bigcup_{j=1}^{K} \Omega^\ell_j. \]

Then the interior of \( \Omega^\ell_j \cap \Omega^\ell_k = \emptyset \) for any \( j \neq k \).

Next lemma gives out the important estimates which will be used later again and again.

**Lemma 2.2.** There are positive constants \( K_0 \) and \( C \) (independent of \( K \)) such that for all \( K \geq K_0, \ell \in \mathbb{Z}_+, \) we have, for any \( x \in \Omega^\ell_{j_0}, \)

\[ \sum_{j=1}^{K} w(x - x_j) \leq C \ell w(x - x_{j_0}), \quad \sum_{j=1}^{K} e^{-\gamma |x - x_j|} \leq C \ell e^{-\gamma |x - x_{j_0}|}, \quad (13) \]

and

\[ \sum_{j \neq j_0} w(x_j - x_{j_0}) \leq C w(\rho), \quad \sum_{j \neq j_0} e^{-\gamma |x_j - x_{j_0}|} \leq C e^{-\gamma \rho} \quad (14) \]

for any positive number \( \gamma. \)

**Proof.** We just notice that in our case \( \rho = 2r \sin \frac{\pi}{K} \) and the result comes from Lemma 3.5 and Corollary 3.6 in [10].

**Remark 2.3.** \( W_r \) is bounded in \( \mathbb{R}^N. \) In fact, if \( x \in \Omega^\ell_{j_0}, j_0 = 1, \ldots, K, \) then by [13] \( W_r \leq C \ell w(x - x_{j_0}) \leq C \ell. \) Otherwise, \( x \in \Omega^\ell_{K+1}, \) then

\[ W_r \leq C \sum_{j=1}^{K} e^{-|x - x_j|} \leq CK e^{-\frac{\ell \rho}{2}} \leq CK e^{-\frac{m \ell}{4} \ell^N K} \leq C \]

if we choose integer \( \ell \) such that \( \ell > \frac{4}{m}. \)

Now let us turn to the asymptotic expansion of energy.

**Proposition 2.4.** There exists a small positive number \( \gamma > 0 \) such that

\[ I(W_r) = K \left( A + \frac{B_1}{r^{m}} - B_2 \left( \frac{r}{K} \right)^{-\frac{N-1}{2}} e^{-\gamma} + O \left( \frac{1}{K^m + \gamma} \right) \right), \quad (15) \]

where \( A, B_1 \) and \( B_2 \) are all positive constants.

**Proof.** The proof is almost from Proposition A.3 in [37] except one thing. On page 438 in [37], it is got that

\[ \sum_{j=2}^{K} \int_{\mathbb{R}^N} U_j^p U_j = \tilde{B}_2 \sum_{j=2}^{K} e^{-\gamma |x_1 - x_j|} + O \left( \sum_{j=2}^{K} e^{-(1+\gamma)|x_1 - x_j|} \right). \]

In fact, according to the decaying rate of \( w, \) we will obtain that

\[ \sum_{j=2}^{K} \int_{\mathbb{R}^N} U_j^p U_j = \tilde{B}_2 \sum_{j=2}^{K} |x_1 - x_j|^{-\gamma} e^{-\gamma |x_1 - x_j|} + O \left( \sum_{j=2}^{K} e^{-(1+\gamma)|x_1 - x_j|} \right), \]
which gives the third term in (15) not the simple form $B_2 e^{-\frac{2\pi r}{K}}$ in [37].

Similar computations can also be found in Lemma 3.9 of [16].

3. Proof of Theorem 1.1. The aim of this section is to obtain Theorem 1.1 by the Lyapunov-Schmidt reduction method. First, we introduce

$$W(x) = \sum_{j=1}^{K} e^{-\eta|x-x_j|},$$

for some $\eta \in (0, p - 1)$ chosen later, and define the weighted norm

$$\|h\|_* = \sup_{x \in \mathbb{R}^N} W^{-1}(x)|h(x)|.$$  (17)

By the definition, one can get that

$$\|h\|_{L^\infty} \leq C\|h\|_*,$$

and

$$\|h\|_{L^q} \leq C K^{\frac{1}{q}} \|h\|_*,$$

for any $q > 1$.

In fact, with the help of (13), it holds that

$$\|h\|_{L^\infty} \leq \sup_{x \in \mathbb{R}^N} \sum_{j=1}^{K} e^{-\eta|x-x_j|} \|h\|_*$$

$$\leq \begin{cases} 
C \ell e^{-\eta|x_{j_0}|} \|h\|_* & \text{for } x \in \Omega_{j_0}, j_0 = 1, \ldots, K, \\
CK e^{-\frac{\eta}{\ell} x} \|h\|_* & \text{for } x \in \Omega_{K+1},
\end{cases}$$

if we choose $\ell > \frac{1}{\eta m}$ independent of $K$. On the other hand, it is obvious

$$\|h\|_{L^q} \leq \|h\|_* \|W\|_{L^q}.$$  (18)

According to (13) and Cauchy inequality,

$$\|W\|_{L^q}^q = \int_{\mathbb{R}^n} \left( \sum_{j=1}^{K} e^{-\eta|x-x_j|} \right)^q dx$$

$$= \sum_{i=1}^{\ell} \int_{\Omega_i} \left( \sum_{j=1}^{K} e^{-\eta|x-x_j|} \right)^q dx + \int_{\Omega_{K+1}} \left( \sum_{j=1}^{K} e^{-\eta|x-x_j|} \right)^q dx$$

$$\leq C \ell^q \sum_{i=1}^{K} e^{-\eta q|x-x_i|} dx + K^{q-1} \int_{\Omega_{K+1}} \sum_{j=1}^{K} e^{-\eta q|x-x_j|} dx$$

$$\leq C \ell^q K + CK^{q-1} e^{-\frac{\eta}{\ell^q} q} \leq CK,$$

with $\ell > \frac{16}{\eta m}$ independent of $K$.

Remark 3.1. From (18), it can be concluded that

$$\|W\|_{L^q} \leq C K^{\frac{1}{q}},$$

for any $q \in [1, \infty)$ and any $\eta > 0$,  (19)

which will play important role later.

Let

$$Z_j = \frac{\partial U_j}{\partial r},$$

$j = 1, \ldots, K,$
denote $x = (x', x'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$ and define

\[ H_s = \left\{ u : u \in \mathbb{R}^N, u \text{ is even in } x_k, k = 2, \ldots, N, \forall j = 1, \ldots, K - 1, \right. \]
\[ \left. u(r \cos \theta, r \sin \theta, x'') = u \left( r \cos \left( \theta + \frac{2\pi j}{K} \right), r \sin \left( \theta + \frac{2\pi j}{K} \right), x'' \right) \right\}. \]

(20)

Recall the approximation (12) and we want to find the solutions with the form $u = W_r + \phi \in H_s$, which leads to

\[ L(\phi) + E + N(\phi) = 0, \]

(21)

where

\[ L(\phi) = \Delta \phi - V \phi + pW_r^{p-1} \phi, \]

\[ E = -(V - 1)W_r + W_r^K - \sum_{j=1}^{K} U^p_j, \]

\[ N(\phi) = (W_r + \phi)^p - W_r^p - pW_r^{p-1} \phi. \]

(22)

The following lemma is standard, we may refer to Lemma 4.3 and Remark 6 in [16]. Similar discussion can be found in Section 3 and Section 4, pages 1936-1946 of [25], see also Proposition 2.1 in [4].

Lemma 3.2. Let $\eta \in (0, 1)$ be fixed. There exists $K_0$ such that for all $K \geq K_0$, $h \in H_s$ with $\|h\|_s < \infty$, there exists a unique function $\phi \in H^2(\mathbb{R}^N)$ and a unique multiplier $c_0$ satisfying

\[ \begin{cases} 
L(\phi) = h + c_0 \sum_{j=1}^{K} Z_j, & \phi \in H_s, \\
\int_{\mathbb{R}^N} U^{p-1}_j Z_j \phi = 0, & \text{for any } j = 1, \ldots, K.
\end{cases} \]

(23)

Furthermore,

\[ \|\phi\|_s + c_0 \leq C \|h\|_s, \quad \|\phi\|_{H^1(\mathbb{R}^N)} \leq C \|h\|_{L^2} \leq CK^\frac{1}{2} \|h\|_s. \]

(24)

where $C$ is a positive constant independent of $K$.

Based on Lemma 3.2 we do not solve the equation (21) directly. Indeed, we consider the following problem

\[ \begin{cases} 
L(\phi) = -E - N(\phi) + c_0 \sum_{j=1}^{K} Z_j, & \phi \in H_s, \\
\int_{\mathbb{R}^N} U^{p-1}_j Z_j \phi = 0, & \text{for any } j = 1, \ldots, K.
\end{cases} \]

(25)

Proposition 3.3. Assume that $V(r)$ satisfies (V). There exist $K_0$ and positive constant $C$ independent of $K$ such that for all $K \geq K_0$, the problem (25) has a unique solution $\phi_r$. Moreover, the mapping $r \to \phi_r$ is $C^1$ and

\[ \|\phi_r\|_s \leq C \left( \frac{1}{r^m} + \frac{1}{K_{\min}^{1, \frac{m}{2}} m - \sigma_0} \right) \]

(26)

where $\sigma_0 > 0$ small enough.
Proof. The proof can be divided into several steps.

**Step 1.** We need to estimate \( \|E\|_\ast \).

Recall that

\[
E = -(V - 1)W_r + \left( W_p^r - \sum_{j=1}^{K} U_j^r \right) := E_1 + E_2,
\]

and let us do the estimates term by term.

If \( x \in \Omega_{j_0}^\ell \) for some \( j_0 = 1, \ldots, K \), \( \ell \) is to be determined later but always does not depend on \( K \), then \( |x - x_{j_0}| \leq \frac{\ell}{2} \) and

\[
|x| \geq r \left( 1 - \ell \sin \frac{\pi}{K} \right) \geq \frac{r}{2} \rightarrow \infty \quad \text{as} \quad K \rightarrow \infty.
\]

Hence by (13) and (V)

\[
|E_1| \leq C |V(x) - 1| \ell w(x - x_{j_0}) \leq \frac{C\ell}{m} e^{-|x-x_{j_0}|} \leq \frac{C\ell}{m} \sum_{j=1}^{K} e^{-\eta|x-x_j|}. \tag{27}
\]

For \( x \in \Omega_{K+1}^\ell \), then for all \( j = 1, \ldots, K \), \( |x - x_j| \geq \frac{\ell}{2} \) and

\[
|E_1| \leq CW_r = C \sum_{j=1}^{K} U_j \leq C \sum_{j=1}^{K} e^{-|x-x_j|} \leq C \sum_{j=1}^{K} e^{-\eta|x-x_j|} \leq CW e^{-(1-\eta)\frac{\ell}{2}} \tag{28}
\]

\[
\leq CW e^{-(1-\eta)\frac{\ell}{2} \frac{2r}{\pi} \frac{\pi}{K}} \leq CW e^{-(1-\eta)\frac{\ell}{2m} \ln K} \leq \frac{C}{K^m} W
\]

if we choose \( \ell \) large enough, for example, \( \ell = \left\lceil \frac{2\pi}{1-\eta} \right\rceil + 1 \). Here the notation \( \lceil \alpha \rceil \) means to take the largest integer not more than \( \alpha \). Thus (27) and (28) result to

\[
\|E_1\|_\ast \leq C \left( r^{-m} + K^{-m} \right). \tag{29}
\]

Now we turn to \( E_2 \). Also we first consider \( x \in \Omega_{j_0}^\ell \) for some \( j_0 = 1, \ldots, K \), obviously for any \( j \neq j_0 \), \( |x - x_j| \geq \frac{\ell}{2} \rho \). Then we have

\[
|E_2| \leq C \left( U_{j_0}^{p-1} \sum_{j \neq j_0} U_j + \sum_{j \neq j_0} U_j^p \right) \leq C \left( e^{-(p-1)|x-x_{j_0}|} \sum_{j \neq j_0} e^{-|x-x_j|} + \sum_{j \neq j_0} e^{-p|x-x_j|} \right). \tag{30}
\]

Now we discuss two cases very carefully since \( \eta \in (0, p - 1) \), which plays the key role to remove the restriction \( m > 1 \).

First, if \( p - \eta \geq 2 \), then

\[
e^{-(p-1)|x-x_{j_0}|} \sum_{j \neq j_0} e^{-|x-x_j|} \leq e^{-(1+\eta)|x-x_{j_0}|} \sum_{j \neq j_0} e^{-|x-x_j|} \leq e^{-\eta|x-x_{j_0}|} \sum_{j \neq j_0} e^{-|x-x_j|}
\]

where we have used $|x - x_{j_0}| + |x - x_j| \geq |x_j - x_{j_0}|$ in the last inequality. Thus by 
\[ e^{-(p-1)|x-x_{j_0}|} \sum_{j \neq j_0} e^{-|x-x_j|} \leq CW e^{-\rho} \leq CW K^{-m+\sigma_0} \]  
(31)

where $\sigma_0$ is small and fixed if $\sigma$ is small enough. If $1 < p - \eta < 2$, note that $|x - x_j| \geq \max\{\frac{\ell}{2}, |x - x_{j_0}|\}$ for any $j \neq j_0$. Thus it holds that 
\[ e^{-(p-1)|x-x_{j_0}|} \sum_{j \neq j_0} e^{-|x-x_j|} \leq e^{-(p-1-\eta)|x-x_{j_0}|} e^{-\eta|x-x_j|} \sum_{j \neq j_0} e^{-(p-2-\eta)|x-x_j|} \]  
\[ \leq CW \sum_{j \neq j_0} e^{-(p-1-\eta)|x-x_{j_0}|} e^{(p-2-\eta)\rho} \]  
\[ \leq CW e^{-(p-1-\eta)\rho+(p-2-\eta)\rho} \leq CW K^{-\frac{p-\eta}{2}m+\sigma_0}. \]  
(33)

On the other hand, in either case we always have 
\[ \sum_{j \neq j_0} e^{-p|x-x_j|} \leq e^{-(p-\eta)\rho} \sum_{j \neq j_0} e^{-\eta|x-x_j|} \leq CW K^{-\frac{p-\eta}{2}m+\sigma_0}. \]  
(34)

Now we come to consider the case $x \in \Omega^2_{K+1}$. Since for all $j = 1, \ldots, K$, $|x - x_{j_0}| \geq \frac{\ell}{2}$, it can be got quickly that 
\[ |E_2| \leq C \left( \sum_{j=1}^{K} e^{-(1-\eta)p|x-x_j|} \right)^{p} + C \sum_{j=1}^{K} e^{-(p-\eta)|x-x_j|} \]  
\[ \leq CK^{p-1} W e^{-(p-\eta)\rho} + CW e^{-(p-\eta)\frac{\rho}{2}} \leq CW K^{-m}, \]  
(35)

if we choose for instance $\ell = \frac{2p\rho(m+p)}{m} + 1$.

From (31)-(35) it follows that 
\[ \|E_2\|_* \leq C \left( \frac{1}{r^m} + \frac{1}{K^{\min\{1,\frac{p-2}{2}\}m-\sigma_0}} \right). \]

**Step 2.** We need to estimate $N(\phi)$.

Let us denote 
\[ \mathcal{B} = \left\{ \phi \in L^\infty(\mathbb{R}^N) \big| \|\phi\|_* \leq C \left( \frac{1}{r^m} + \frac{1}{K^{\min\{1,\frac{p-2}{2}\}m-\sigma_0}} \right) \right\}. \]

Without loss of generality, we may assume that $\phi \in \mathcal{B}$, then by mean value theorem, 
\[ |N(\phi)| \leq C\|\phi\|_{\min\{2,p\}}, \]
which leads to 
\[ \|N(\phi)\|_* \leq C\|\phi\|^{\min\{2,p\}}_* \]
Similarly, for any $\phi_1, \phi_2 \in \mathcal{B}$, 
\[ \|N(\phi_1) - N(\phi_2)\|_* \leq \frac{1}{2}\|\phi_1 - \phi_2\|_* \]
provided $K$ large.

**Step 3.** Once we get the sharp estimates of $\|E\|_*$ and $\|N(\phi)\|_*$, we can use the contraction mapping to obtain the proper $\phi_r$. The process is very standard and is well known.

Let $\phi_r$ be the mapping obtained in Proposition 3.3 define

$$F(r) = I(W_r + \phi_r).$$

Then it is well understood that if $r$ is a critical point of $F(r)$, then $c_0 = 0$ in [25], which means that $u = W_r + \phi_r$ is really a solution of [3]. For this we need to get the asymptotic behavior of $F(r)$. Note that the estimate of $F(r)$ is similar as that in [37], but the method is totally different because here we use the weighted $L^\infty$-norm instead of $H^1$-norm.

**Proposition 3.4.** When $K$ large enough, there exists $\bar{\sigma} > 0$ small enough, such that

$$F(r) = K \left( A + \frac{B_1}{r^m} - B_2 \left( \frac{r}{K} \right)^{-\frac{N-1}{2}} e^{-\frac{r^2}{N-2}} + O \left( \frac{1}{K^{m+\bar{\sigma}}} \right) \right).$$

**Proof.**

$I(W_r + \phi_r) = I(W_r) + \int_{\mathbb{R}^N} (\nabla W_r \cdot \nabla \phi_r + V(r)W_r \phi_r) + \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla \phi_r|^2 + V(r)\phi_r^2)$

$$- \frac{1}{p+1} \int_{\mathbb{R}^N} [(W_r + \phi_r)^{p+1} - W_r^{p+1}]$$

$$= I(W_r) + \int_{\mathbb{R}^N} (-\Delta W_r + V(r)W_r - W_r^p) \phi_r + O (\|\phi_r\|_{H^1}^2)$$

$$- \frac{1}{p+1} \int_{\mathbb{R}^N} [(W_r + \phi_r)^{p+1} - W_r^{p+1} - (p+1)W_r^p \phi_r]$$

$$:= I(W_r) + I_1 + I_2 + I_3,$$

where $I_j$, $j = 1, 2, 3$ is defined by the last equality.

By [18] and Proposition 3.3

$$|I_1| \leq \int_{\mathbb{R}^N} |E \phi_r| \leq \|E\|_* \|\phi_r\|_* \int_{\mathbb{R}^N} W^2$$

$$\leq CK \left( \frac{1}{r^{2m}} + \frac{1}{K^{\min(2,p-n)m-2m_0}} \right).$$

(37)

According to the useful estimate [24],

$$|I_2| \leq C\|\phi_r\|_{H^1}^2 \leq C\|E + N(\phi_r)\|_{L^2}^2 \leq CK\|E + N(\phi_r)\|_*^2$$

$$\leq CK \left( \frac{1}{r^{2m}} + \frac{1}{K^{\min(2,p-n)m-2m_0}} \right).$$

(38)

and since $W_r$ is bounded, $p+1 \geq 2$

$$|I_3| \leq C \int_{\mathbb{R}^N} (W_r^{p-1} \phi_r^2 + \phi_r^{p+1})$$

$$\leq C\|\phi_r\|_*^2 \int_{\mathbb{R}^N} W_r^{p-1}W^2 + C\|\phi_r\|_*^{p+1} \int_{\mathbb{R}^N} W^{p+1}$$

$$\leq C\|\phi_r\|_*^2 \int_{\mathbb{R}^N} W^2 + CK\|\phi_r\|_*^{p+1}$$
Combining all the above, we deduce that

\[ I(W_r + \phi_r) = I(W_r) + O \left( K \left( \frac{1}{r^{2m}} + \frac{1}{K^{\min(2,p-\eta)m-2\sigma_0}} \right) \right). \]

Hence

\[ F(r) = I(W_r + \phi_r) = K \left( A + \frac{B_1}{r^m} - B_2 \left( \frac{r}{K} \right)^{-\frac{N-1}{2}} e^{-\frac{2\pi r}{K}} + O \left( \frac{1}{K^{m+\gamma}} \right) \right), \]

Hence

\[ F(r) = I(W_r + \phi_r) = K \left( A + \frac{B_1}{r^m} - B_2 \left( \frac{r}{K} \right)^{-\frac{N-1}{2}} e^{-\frac{2\pi r}{K}} + O \left( \frac{1}{K^{m+\gamma}} \right) \right) \]

\[(40)\]

since we fix \( \eta \in (0, p-1) \) and \( \sigma \) is small enough.

Recall that \( \Lambda_K = \left[ \left( \frac{m}{2\pi} - \sigma \right) K \ln K, \left( \frac{m}{2\pi} + \sigma \right) K \ln K \right] \) where \( \sigma \) is very small but fixed. Consider

\[ \max \{ F(r) : r \in \Lambda_K \}. \]

Let

\[ S_r = \frac{B_1}{r^m} - B_2 \left( \frac{r}{K} \right)^{-\frac{N-1}{2}} e^{-\frac{2\pi r}{K}}, \]

then

\[ S_r' \left( \frac{m}{2\pi} - \sigma \right) K \ln K \geq 0, \quad S_r' \left( \frac{m}{2\pi} + \sigma \right) K \ln K \leq 0. \]

So \( S_r \) has a maximum point

\[ s_k = \left( \frac{m}{2\pi} + o(1) \right) K \ln K \]

which is an interior point of \( \Lambda_K \). Thus there is an \( r_K \) such that

\[ F(r_K) = \max_{r \in \Lambda_K} F(r) \]

which is also an interior point of \( \Lambda_K \). Hence \( W_{r_K} + \phi_{r_K} \) is a solution of Problem \([3] \), which finishes the proof of Theorem \([1,3] \) \( \Box \)

4. Proof of Theorem \([1,3] \) The aim of this section is to find the non-radial positive solutions of

\[ \begin{cases} 
\Delta u - P(|x|) u + \mu_1 u^\alpha + \beta v^2 u = 0 & x \in \mathbb{R}^3, \\
\Delta v - Q(|x|) v + \mu_2 v^\alpha + \beta u^2 v = 0 & x \in \mathbb{R}^3.
\end{cases} \]

(41)

It is easy to check that \((u, v)\) is the solution of the problem (41) if and only if it is a critical point of the energy

\[ I(u, v) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + P(|x|)u^2 + |\nabla v|^2 + Q(|x|)v^2) \]

\[ - \frac{1}{4} \int_{\mathbb{R}^3} (\mu_1 u^4 + \mu_2 v^4) - \frac{\beta}{2} \int_{\mathbb{R}^3} u^2 v^2. \]

Recall the unique solution \( w \) of

\[ \Delta w - w + w^p = 0, \quad w > 0 \quad \text{in} \quad \mathbb{R}^N, \quad w \in H^1(\mathbb{R}^N), \]

define

\[ (U, V) = (\alpha w, \gamma w) \quad \text{with} \quad \alpha = \sqrt{\frac{\mu_2 - \beta}{\mu_1 \mu_2 - \beta^2}} \quad \text{and} \quad \gamma = \sqrt{\frac{\mu_1 - \beta}{\mu_1 \mu_2 - \beta^2}}. \]
then \((U, V)\) is the solution of
\[
\begin{align*}
\Delta u - u + \mu_1 u^3 + \beta v^2 u &= 0 \quad \text{in } \mathbb{R}^3, \\
\Delta v - v + \mu_2 v^3 + \beta u^2 v &= 0 \quad \text{in } \mathbb{R}^3.
\end{align*}
\]
Let
\[
U_j(x) = U(|x - x_j|) = \alpha w(|x - x_j|), \quad V_j(x) = V(|x - x_j|) = \gamma w(|x - x_j|),
\]
and
\[
U_r = \sum_{j=1}^{K} U_j, \quad V_r = \sum_{j=1}^{K} V_j,
\]
where \(x_j, j = 1, 2, \ldots, K\) is given by
\[
x_j = \left( r \cos \frac{2(j - 1)\pi}{K}, r \sin \frac{2(j - 1)\pi}{K}, 0 \right).
\]
Consider \(r\) in the set
\[
\Lambda_K := \left[ \left( \frac{\min\{m, n\}}{2\pi} - \sigma \right) K \ln K, \left( \frac{\min\{m, n\}}{2\pi} + \sigma \right) K \ln K \right]
\]
for \(0 < \sigma < \frac{\min\{m, n\}}{4\pi}\) but fixed. Here we still denote \(\rho = \min_{j \neq k} \{|x_j - x_k|\}\).

We hope to find the solution of the form \((u, v) = (U_r + \phi_r, V_r + \psi_r) \in H_s \times H_s\) defined in \((40)\) which reduces that
\[
L \left( \begin{array}{c}
\phi \\
\psi
\end{array} \right) + E + N \left( \begin{array}{c}
\phi \\
\psi
\end{array} \right) = 0
\]
where
\[
L \left( \begin{array}{c}
\phi \\
\psi
\end{array} \right) = \left( \begin{array}{c}
L_1 \\
L_2
\end{array} \right) = \left( \begin{array}{c}
\Delta \phi - P(|x|) \phi + 3\mu_1 U_r^2 \phi + \beta V_r^2 \phi + 2\beta U_r V_r \phi \\
\Delta \psi - Q(|x|) \psi + 3\mu_2 V_r^2 \psi + \beta U_r^2 \psi + 2\beta U_r V_r \psi
\end{array} \right),
\]
\[
E = S \left( \begin{array}{c}
U_r \\
V_r
\end{array} \right) = \left( \begin{array}{c}
S_1 \\
S_2
\end{array} \right) = \left( \begin{array}{c}
\Delta U_r - P(|x|) U_r + \mu_1 U_r^3 + \beta V_r^2 U_r \\
\Delta V_r - Q(|x|) V_r + \mu_2 V_r^3 + \beta U_r^2 V_r
\end{array} \right),
\]
and
\[
N \left( \begin{array}{c}
\phi \\
\psi
\end{array} \right) = \left( \begin{array}{c}
N_1 \\
N_2
\end{array} \right) = \left( \begin{array}{c}
3\mu_1 U_r \phi^2 + \mu_1 \phi^3 + \beta(2V_r \phi \psi + \psi^2 U_r + \psi^2 \phi) \\
3\mu_2 V_r \psi^2 + \mu_2 \psi^3 + \beta(2U_r \phi \psi + \phi^2 V_r + \phi^2 \psi)
\end{array} \right).
\]
We will not solve the problem \((43)\) directly. In fact, we consider the following project problem
\[
\begin{align*}
L \left( \begin{array}{c}
\phi \\
\psi
\end{array} \right) + E + N \left( \begin{array}{c}
\phi \\
\psi
\end{array} \right) &= c_0 \left( \sum_{j=1}^{K} Y_j w_j^2 \right), \\
\int_{\mathbb{R}^3} \sum_{j=1}^{K} w_j^2 (Y_j \phi + Z_j \psi) &= 0,
\end{align*}
\]
where
\[
Y_j = \frac{\partial U_j}{\partial r}, \quad Z_j = \frac{\partial V_j}{\partial r}, \quad j = 1, 2, \ldots, K.
\]
In order to get the solvability of (47), first we consider the linear problem

\[
\begin{aligned}
L(\phi, \psi) &= \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + c_0 \begin{pmatrix} \sum_{j=1}^{K} Y_j w_j^2 \\ \sum_{j=1}^{K} Z_j w_j^2 \end{pmatrix}, \\
\int_{\mathbb{R}^3} \sum_{j=1}^{K} w_j^2 (Y_j \phi + Z_j \psi) &= 0.
\end{aligned}
\] (48)

Once we know that the problem (48) is uniquely solvable, then by contraction mapping theorem one can obtain the solution of (47). Finally to ensure that \( c_0 = 0 \) we need to find the critical points of energy \( G(r) = I(U_r + \phi_r, V_r + \psi_r) \). The process is very standard and we do not repeat step by step. Here we just give out several key estimates.

With the weighted function \( W \) of (16) in mind, for \( h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \) define

\[
\| h \|_* = \sup_{i=1,2} \sup_{x \in \mathbb{R}^3} W^{-1}(x)|h_i(x)|.
\]

Lemma 4.1. Let \( \eta \in (0,1) \) be fixed. There exists \( K_0 \) such that for all integer \( K \geq K_0 \), \( \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in H^1 \times H^1 \) with \( \left\| \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\|_* < \infty \), any solution \( \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in (H^2(\mathbb{R}^3))^2 \) and \( c_0 \) of (48) satisfies

\[
\left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|_* + c_0 \leq C \left\| \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\|_*,
\]

where \( C \) is a positive constant independent of \( K \).

Proof. Multiply the first equation of (48) by \( Y_k \) for some \( k = 1,2,\ldots, K \) and integrate, we obtain

\[
\int_{\mathbb{R}^3} (\Delta Y_k - P(|x|)Y_k + 3\mu_1 U_r^2 Y_k + \beta V_r^2 Y_k) \phi + \int_{\mathbb{R}^3} 2\beta U_r V_r \psi Y_k
= \int_{\mathbb{R}^3} h_1 Y_k + c_0 \sum_{j=1}^{K} w_j^2 Y_j Y_k. \quad (49)
\]

Now let us give the estimates step by step. First direct computations lead to

\[
\left| \int_{\mathbb{R}^3} h_1 Y_k \right| \leq C \| h_1 \|_* \int_{\mathbb{R}^3} Y_k \leq C \| h_1 \|_*
\]

and

\[
c_0 \int_{\mathbb{R}^3} \sum_{j=1}^{K} w_j^2 Y_j Y_k = c_0 \int_{\mathbb{R}^3} w_k^2 Y_k^2 + c_0 \int_{\mathbb{R}^3} \sum_{j \neq k} w_j^2 Y_j Y_k = c_0 \alpha^2 \gamma_0 + o(c_0)
\]

where \( \gamma_0 = \int_{\mathbb{R}^3} w^2(u')^2 \) is a universal positive constant and we have used the following property

\[
\left| \int_{\mathbb{R}^3} \sum_{j \neq k} w_j^2 Y_j Y_k \right| \leq C \int_{\mathbb{R}^3} \sum_{j \neq k} e^{-3|x-x_j|} e^{-|x-x_k|}
\leq C \int_{\mathbb{R}^3} \sum_{j \neq k} e^{-\frac{1}{2}|x_j-x_k|} e^{-\frac{1}{2}|x-x_k|}
\]
from which by similar computations one may get that

\[ L.H.S. = \int_{\mathbb{R}^3} \left[ (1 - P(|x|)) Y_k + 3 \mu_2 (U_r^2 - U_k^2) Y_k + \beta (V_r^2 - V_k^2) Y_k \right] \phi \]

\[ + \int_{\mathbb{R}^3} 2 \beta (U_r V_r - U_k V_k) Z_k \phi - \int_{\mathbb{R}^3} 2 \beta U_r V_r Z_k \phi + \int_{\mathbb{R}^3} 2 \beta U_r V_r Y_k \psi \]  

\[ := J_1 + J_2 - \int_{\mathbb{R}^3} 2 \beta U_r V_r Z_k \phi + \int_{\mathbb{R}^3} 2 \beta U_r V_r Y_k \psi \]

where \( J_1, J_2 \) are defined by the first two integrations. On the other hand, multiply the second equation by \( Z_k \) and integrate to deduce that

\[ \int_{\mathbb{R}^3} (\Delta Z_k - Q Z_k + 3 \mu_2 V_r^2 Z_k + \beta U_r^2 Z_k) \psi + \int_{\mathbb{R}^3} 2 \beta U_r V_r \phi Z_k \]

\[ = \int_{\mathbb{R}^3} h_2 Z_k + c_0 \sum_{j=1}^{K} w_j^2 Z_j Z_k, \]

from which by similar computations one may get that

\[ \int_{\mathbb{R}^3} 2 \beta U_r V_r \phi Z_k = - \int_{\mathbb{R}^3} (\Delta Z_k - Q(|x|) Z_k + 3 \mu_2 V_r^2 Z_k + \beta U_r^2 Z_k) \psi \]

\[ + O (\|h_2\|_\ast) - c_0 \gamma^2 \gamma_0 + o(c_0) \]

\[ = - \int_{\mathbb{R}^3} [(1 - Q(|x|)) Z_k + 3 \mu_2 (V_r^2 - V_k^2) Z_k + \beta (U_r^2 - U_k^2) Z_k] \psi \]

\[ + \int_{\mathbb{R}^3} 2 \beta U_k V_k Y_k \psi + O (\|h_2\|_\ast) - c_0 \gamma^2 \gamma_0 + o(c_0) \]

Putting (51) into (50), we have

\[ L.H.S. = J_1 + J_2 + \int_{\mathbb{R}^3} [(1 - Q(|x|)) Z_k + 3 \mu_2 (V_r^2 - V_k^2) Z_k + \beta (U_r^2 - U_k^2) Z_k] \psi \]

\[ + \int_{\mathbb{R}^3} 2 \beta (U_r V_r - U_k V_k) Y_k \psi + O (\|h_2\|_\ast) - c_0 \gamma^2 \gamma_0 + o(c_0) \]

\[ := J_1 + J_2 + J_3 + J_4 + O (\|h_2\|_\ast) - c_0 \gamma^2 \gamma_0 + o(c_0) \]

where \( J_3, J_4 \) are given by the two integrations. Now we gather them to obtain

\[ (\alpha^2 + \gamma^2) \gamma_0 c_0 + o(c_0) = O (\|h\|_\ast) \]

(52)

Since the forms of \( J_1 \) and \( J_3 \) are almost the same, so are \( J_2 \) and \( J_4 \). Hence here we just give the details of \( J_1 \) and \( J_2 \). Let us consider \( J_1 \) term by term. It holds that

\[ \left| \int_{\mathbb{R}^3} (1 - P) Y_k \phi \right| \leq \left( \int_{\{|x| \leq \frac{5}{2}\}} + \int_{\{|x| > \frac{5}{2}\}} \right) \left| (1 - P) Y_k \phi \right| \]

\[ \leq C \int_{\{|x| \leq \frac{5}{2}\}} e^{-\frac{1}{2} |x - x_k|} |\phi| + \frac{C}{r^m} \|\phi\|_\ast \int_{\mathbb{R}^3} |Y_k| \]

\[ \leq C \|\phi\|_\ast \int_{\{|x| \leq \frac{5}{2}\}} e^{-\frac{1}{2} |x - x_k|} e^{-\frac{1}{2} \frac{5}{2}} + \frac{C}{r^m} \|\phi\|_\ast \]

\[ \leq C \|\phi\|_\ast \left( e^{-\frac{1}{2}} + \frac{1}{r^m} \right) \leq \frac{C}{r^m} \|\phi\|_\ast \]
since for $|x| \leq \frac{r}{2}$, $|x - x_k| \geq |x_k| - |x| \geq \frac{r}{2}$ and $K$ large.

$$\left| \int_{\mathbb{R}^3} (U_r^2 - U_k^2) Y_k \phi \right| \leq \int_{\mathbb{R}^3} \sum_{j \neq k} U_j^2 Y_k \phi + \int_{\mathbb{R}^3} \sum_{i \neq j} U_i U_j Y_k \phi$$

$$\leq C \| \phi \| \left( \int_{\mathbb{R}^3} \sum_{j \neq k} e^{-2|x - x_j|/|x - x_k|} dx + 2 \int_{\mathbb{R}^3} \sum_{j \neq k} U_j Y_k \phi + \int_{\mathbb{R}^3} \sum_{i \neq j} U_i U_j Y_k \phi \right)$$

$$\leq C \| \phi \| \left( e^{-\frac{1}{2} \rho} + \int_{\mathbb{R}^3} \sum_{j \neq k} e^{-|x - x_j|/|x - x_k|} \sum_{i \neq k} e^{-\frac{1}{2} |x_i - x_k|} \right)^2$$

$$\leq C \| \phi \| e^{\rho}.$$

Similarly,

$$\left| \int_{\mathbb{R}^3} (V_r^2 - V_k^2) Y_k \phi \right| \leq C \| \phi \| e^{-\frac{1}{2} \rho}.$$

Thus we have

$$|J_1| \leq C \| \phi \| \left( \frac{1}{r^m} + e^{-\frac{1}{2} \rho} \right).$$

and, by the same procedure,

$$|J_3| \leq C \| \psi \| \left( \frac{1}{r^m} + e^{-\frac{1}{2} \rho} \right).$$

As for $J_2$, we have

$$|J_2| \leq C \int_{\mathbb{R}^3} \left( U_r \sum_{j \neq k} V_j + V_k \sum_{j \neq k} U_j \right) |Z_k \phi|$$

$$\leq C \| \phi \| \int_{\mathbb{R}^3} \left( U_k \sum_{j \neq k} V_j + V_k \sum_{j \neq k} U_j + \sum_{i \neq k} U_i \sum_{j \neq k} V_j \right) e^{-|x - x_k|}$$

$$\leq C \| \phi \| \int_{\mathbb{R}^3} \left[ \sum_{j \neq k} e^{-2|x - x_k|} e^{-|x - x_j|} + \left( \sum_{i \neq k} e^{-|x - x_i|} e^{-\frac{1}{2} |x_i - x_k|} \right)^2 \right]$$

$$\leq C \| \phi \| \int_{\mathbb{R}^3} \left[ \sum_{j \neq k} e^{-|x - x_k|} e^{-|x_k - x_j|} + \left( \sum_{i \neq k} e^{-\frac{1}{2} |x_k - x_i|} e^{-\frac{1}{2} |x - x_k|} \right)^2 \right]$$

$$\leq C \| \phi \| e^{-\frac{1}{2}}.$$ 

The same estimate holds for $J_4$.

Finally putting them into (52), we get the desired result that

$$|c_0| \leq C \left( \left( \frac{h_1}{h_2} \right) \right) + C \left( \frac{1}{r^{\min \{m, n\}}} + e^{-\frac{1}{2} \rho} \right) \left( \| \phi \| \right).$$

(55)
Next, we argue by contradiction. Suppose that there exist \( \left( \frac{\phi_K}{\psi_K} \right) \) and \( h_K \) satisfying (48) and
\[
\| h_K \|_* \to 0, \quad \left\| \left( \frac{\phi_K}{\psi_K} \right) \right\|_* = 1
\]
as \( K \to \infty \). For simplicity, we drop the subscript \( K \) in the following.

We want to get that for any \( x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^{K} B(x_j, \tau) \), the following key estimates hold
\[
|\phi(x)| \leq C \left( \left\| \left( \frac{L_1}{L_2} \right) \right\|_* + \sup_{j=1, \cdots, K} \| \phi \|_{L^\infty(B(x_j, \tau))} + \sup_{j=1, \cdots, K} \| \psi \|_{L^\infty(B(x_j, \tau))} \right) W,
\]
\[
|\psi(x)| \leq C \left( \left\| \left( \frac{L_1}{L_2} \right) \right\|_* + \sup_{j=1, \cdots, K} \| \phi \|_{L^\infty(B(x_j, \tau))} + \sup_{j=1, \cdots, K} \| \psi \|_{L^\infty(B(x_j, \tau))} \right) W,
\]
where \( B(x_j, \tau) \) means the ball with the center \( x_j \), the radius \( \tau \) is to be determined later independent of \( K \) and \( L_1, L_2 \) are defined in (44).

To prove the above estimates, the first observation is, for any \( x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^{K} B(x_j, \tau) \)
\[
|U_r(x)| \leq D_0 w(\tau), \quad |V_r(x)| \leq D_0 w(\tau)
\]
which comes from (13), see also [16] and \( D_0 \) is independent of \( K \) and \( \tau \).

Obviously for \( \tau \) fixed, W.O.L.G. \( P(|x|), Q(|x|) \geq \frac{1}{4} \) in \( \mathbb{R}^3 \setminus \bigcup_{j=1}^{K} B(x_j, \tau) \). Thus
\[
L_P(W) := \Delta W - P(|x|)W \leq (\eta^2 - \frac{3}{4})W \leq -\frac{1}{2} W
\]
if we choose \( 0 < \eta < \frac{1}{2} \). Here \( W \) is the weighted function, that is, \( W = \sum_{j=1}^{K} e^{-\eta|x-x_j|} \). Similarly
\[
L_Q(W) := \Delta W - Q(|x|)W \leq -\frac{1}{2} W.
\]

Define
\[
F(x) = \left[ 2\|L_P(\phi)\|_* + e^{\eta r} \sup_{j=1, \cdots, K} \| \phi \|_{L^\infty(B(x_j, \tau))} \right] W(x),
\]
then
\[
L_P(\pm \phi - F) = \pm L_P(\phi) - L_P(F)
\]
\[
\geq \pm L_P(\phi) + \|L_P(\phi)\|_* W + \frac{1}{2} e^{\eta r} \sup_{j=1, \cdots, K} \| \phi \|_{L^\infty(B(x_j, \tau))} W \geq 0.
\]

On the boundary \( \partial \left( \bigcup_{j=1}^{K} B(x_j, \tau) \right) \), \( \pm \phi \leq F \). By maximum principle we can get that for any \( x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^{K} B(x_j, \tau) \),
\[
|\phi(x)| \leq \left[ 2\|L_P(\phi)\|_* + e^{\eta r} \sup_{j=1}^{K} \| \phi \|_{L^\infty(B(x_j, \tau))} \right] W(x).
\]
The same analysis tell us that
\[
|\psi(x)| \leq \left[ 2\|L_Q(\psi)\|_* + e^{\eta r} \sup_{j=1}^{K} \| \psi \|_{L^\infty(B(x_j, \tau))} \right] W(x).
\]

Using the definition of \( L_P \) and \( L_1 \), one may reduce that
\[
|\phi(x)| \leq 2 \left( \|L_1\|_* + 3\mu_1 \|U_r^2 \phi\|_* + \beta \|V_r^2 \phi\|_* + 2\beta \|U_r V_r \psi\|_* \right) W(x)
\]
\[
+ e^{\eta r} \sup_{j=1, \cdots, K} \| \phi \|_{L^\infty(B(x_j, \tau))} W(x).
\]
If \( x \in B(x_j, \tau) \) for some \( j = 1, 2, \ldots, K \), \[13\] leads to
\[
U_\tau^2|\phi|W^{-1} \leq C_0 \sup_j \|\phi\|_{L^\infty(B(x_j, \tau))} U_\tau^2 W^{-1} \leq C_0 \sup_j \|\phi\|_{L^\infty(B(x_j, \tau))}.
\]

Here and in the following denoting \( C_0 \) as constant which is independent of \( K, \tau \) and may be different from line to line.

Otherwise, \( x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^K B(x_j, \tau) \)
\[
U_\tau^2(x)|\phi(x)|W^{-1}(x) \leq \|\phi W^{-1}\|_{L^\infty(\mathbb{R}^3 \setminus \bigcup_{j=1}^K B(x_j, \tau))} D_0^2 w^2(\tau) \leq C_0 e^{-2\tau} \|\phi W^{-1}\|_{L^\infty(\mathbb{R}^3 \setminus \bigcup_{j=1}^K B(x_j, \tau))}.
\]

Hence
\[
3\mu_1 \|U_\tau^2 \phi\|_\star \leq C_0 \sup_{j=1}^K \|\phi\|_{L^\infty(B(x_j, \tau))} + C_0 e^{-2\tau} \|\phi W^{-1}\|_{L^\infty(\mathbb{R}^3 \setminus \bigcup_{j=1}^K B(x_j, \tau))}.
\]

With almost the same computations,
\[
\beta \|V_\tau^2 \phi\|_\star \leq C_0 \sup_{j=1}^K \|\phi\|_{L^\infty(B(x_j, \tau))} + C_0 e^{-2\tau} \|\phi W^{-1}\|_{L^\infty(\mathbb{R}^3 \setminus \bigcup_{j=1}^K B(x_j, \tau))}
\]
and
\[
2\beta \|U_\tau V_\tau \psi\|_\star \leq C_0 \sup_{j=1}^K \|\psi\|_{L^\infty(B(x_j, \tau))} + C_0 e^{-2\tau} \|\psi W^{-1}\|_{L^\infty(\mathbb{R}^3 \setminus \bigcup_{j=1}^K B(x_j, \tau))}.
\]

Till now we have proved that, for any \( x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^K B(x, \tau) \),
\[
|\phi(x)W^{-1}(x)| \leq C \left[ \|L_1\|_\star + e^{\eta \tau} \sup_j \|\phi\|_{L^\infty(B(x_j, \tau))} + \sup_j \|\psi\|_{L^\infty(B(x_j, \tau))} \right] + C_0 e^{-2\tau} \left( \|\phi W^{-1}\|_{L^\infty(\mathbb{R}^3 \setminus \bigcup_{j=1}^K B(x_j, \tau))} + \|\psi W^{-1}\|_{L^\infty(\mathbb{R}^3 \setminus \bigcup_{j=1}^K B(x_j, \tau))} \right),
\]
so does \( \psi W^{-1} \). Now by taking \( \tau \) large enough such that \( C_0 e^{-\eta \tau} = \frac{1}{4} \), it is easily to get that
\[
|\phi(x)| + |\psi(x)| \leq C \left( \left\| \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \right\|_\star + \sup_j \|\phi\|_{L^\infty(B(x_j, \tau))} + \sup_j \|\psi\|_{L^\infty(B(x_j, \tau))} \right) W(x)
\]
for any \( x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^K B(x_j, \tau) \). So we get \[57\]
With \[59\], \( c_0 \to 0 \) and thus
\[
\left\| L \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\|_\star \to 0.
\]

For any \( j = 1, 2, \ldots, K \),
\[
\sup_{B(x_j, \tau)} |\phi(x)W^{-1}(x)| \leq \|\phi\|_{L^\infty(B(x_j, \tau))} \sup_{B(x_j, \tau)} e^{n|x-x_j|} \leq e^{\eta \tau} \|\phi\|_{L^\infty(B(x_j, \tau))}
\]
Combining \[57\], we obtain
\[
\left\| \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\|_\star \leq C \left\| L \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\|_\star + C \sup_{j=1}^K \left\| \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\|_{L^\infty(B(x_j, \tau))}.
\]

By the assumptions, there exists a subsequence of \( x_j \) such that
\[
\left\| \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\|_{L^\infty(B(x_j, \tau))} \geq C.
\]
Because of
\[ \left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|_{L^\infty(B(x_j, \tau))} \leq C \left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|_* = C, \]
using the standard elliptic estimates together with Ascoli-Arzela theorem, we may have
\[ \begin{pmatrix} \phi(\cdot + x_j) \\ \psi(\cdot + x_j) \end{pmatrix} \to \begin{pmatrix} \phi_\infty \\ \psi_\infty \end{pmatrix} \]
in any compact subset of \( \mathbb{R}^3 \) solving
\[ \begin{cases} 
\Delta \phi - \phi + 3\mu_1 U^2 \phi + \beta V^2 \phi + 2\beta UV \psi = 0, \\
\Delta \psi - \psi + 3\mu_2 V^2 \psi + \beta U^2 \psi + 2\beta UV \phi = 0, \\
\int_{\mathbb{R}^3} w^2 \left( \frac{\partial U}{\partial x_1} \phi_\infty + \frac{\partial V}{\partial x_1} \psi_\infty \right) = 0. 
\end{cases} \]
By Proposition 2.3 in [30] and symmetry, the only possibility is
\[ \begin{pmatrix} \phi_\infty \\ \psi_\infty \end{pmatrix} = \tilde{C} \begin{pmatrix} \theta(\beta) w'(r) \\ \beta \alpha \gamma^2 \end{pmatrix} \]
and then \( \tilde{C} \) satisfies
\[ \tilde{C}(\alpha \theta(\beta) + \gamma) \int_{\mathbb{R}^3} w^2(|x|)(w'(|x|))^2 = 0 \]
where
\[ \theta(\beta) = \frac{\mu_2 - \beta}{\mu_1 - \beta} > 0. \]
Now it is easy to see that \( \tilde{C} = 0 \), a contradiction to (61).

\[ \square \]

**Lemma 4.2.** With \( P, Q \) satisfying [4], [5] correspondingly. Then for \( K \) large, we have
\[ \left\| S \begin{pmatrix} U_r \\ V_r \end{pmatrix} \right\|_* \leq C \left( \frac{1}{\min(m,n)} + \frac{1}{K^{\min(m,n)-\sigma_0}} \right), \] (62)
where \( C \) is independent of \( K \) and \( \sigma_0 \) is arbitrarily small.

**Proof.** According to the equation satisfied by \( U_r \), it is easy to see that
\[ S_1 = -(P(|x|) - 1)\alpha \sum_{j=1}^K w(|x - x_j|) + \mu_1 \alpha^3 \left( \sum_{j=1}^K w(x - x_j) \right)^3 - \sum_{j=1}^K w^3(|x - x_j|) \]
\[ + \beta \alpha \gamma^2 \left[ \sum_{j=1}^K w(|x - x_j|) \left( \sum_{l=1}^K w(|x - x_l|) \right)^2 - \sum_{j=1}^K w^3(|x - x_j|) \right]. \] (63)

Up to a constant, it is easily observed that the first two terms in \( S_1 \) have the same form as \( E \) in step 1 of Proposition 3.3. Hence
\[ \left\| -(P(|x|) - 1) \sum_{j=1}^K w(|x - x_j|) \right\|_* \leq \frac{C}{r^m} + \frac{C}{K^m} \]
and in this case \( p = 3 \) leads to
\[ \left\| \left( \sum_{j=1}^K w(x - x_j) \right)^3 - \sum_{j=1}^K w^3(|x - x_j|) \right\|_* \leq \frac{C}{r^m} + \frac{C}{K^{m-\sigma_0}}. \]
Next let us consider the interaction term. Obviously
\[
\sum_{j=1}^{K} w(|x - x_j|) \left( \sum_{l=1}^{K} w(|x - x_l|) \right)^2 - \sum_{j=1}^{K} w^2(|x - x_j|)
\]
\[
= \sum_{j=1}^{K} w(|x - x_j|) \left( \sum_{l \neq j} w^2(|x - x_l|) + \sum_{i \neq k} w(|x - x_i|)w(|x - x_k|) \right).
\]
Assume that \( x \in \Omega_{j_0}^f \) for some \( j_0 = 1, 2, \ldots, K \), then
\[
\sum_{j=1}^{K} w(|x - x_j|) \sum_{l \neq j} w^2(|x - x_l|)
\]
\[
= w(|x - x_{j_0}|) \sum_{l \neq j_0} e^{-2|x - x_l|} + \sum_{j \neq j_0} w(|x - x_j|) \sum_{l \neq j} w(|x - x_l|)w(|x - x_{j_0}|)
\]
\[
\leq Ce^{-|x - x_{j_0}|} \sum_{l \neq j_0} e^{-2|x - x_l|} + \sum_{j \neq j_0} w(|x - x_j|) \sum_{l \neq j} w(|x - x_l|)w(|x - x_{j_0}|)
\]
\[
\leq C \sum_{j \neq j_0} e^{-|x - x_j|} e^{-|x - x_{j_0}|} + C \sum_{j \neq j_0} e^{-|x - x_j|} e^{-|x - x_{j_0}|} \sum_{l \neq j} e^{-|x - x_l|}
\]
\[
\leq C \sum_{j \neq j_0} e^{-|x - x_j|} e^{-|x - x_{j_0}|} + C \sum_{j \neq j_0} e^{-|x - x_j|} \sum_{l \neq j} e^{-\eta |x - x_l|}
\]
\[
\leq CW \sum_{j \neq j_0} e^{-|x - x_j|} + CW \sum_{j \neq j_0} e^{-|x - x_{j_0}|}
\]
\[
\leq CW e^{-\rho} \leq \frac{C}{K^{m-\sigma_0}}.
\]
At the same time, for any \( i \neq j_0, |x - x_i| \geq \frac{1}{2} |x_i - x_{j_0}|. \)
\[
\sum_{j=1}^{K} w(|x - x_j|) \sum_{i \neq k} w(|x - x_i|)w(|x - x_k|)
\]
\[
= \sum_{j=1}^{K} w(|x - x_j|) \left( 2 \sum_{i \neq k, i \neq j_0} w(|x - x_i|)w(|x - x_{j_0}|) + \sum_{i \neq k, i \neq j_0} w(|x - x_i|)w(|x - x_k|) \right)
\]
\[
\leq C \sum_{j=1}^{K} w(|x - x_j|) \sum_{i \neq j_0} e^{-|x_i - x_{j_0}|} + C \sum_{j=1}^{K} w(|x - x_j|) \sum_{k \neq j_0} w(|x - x_k|) \sum_{i \neq j_0} w(|x - x_i|)
\]
\[
\leq C \sum_{j=1}^{K} w(|x - x_j|) \sum_{i \neq j_0} e^{-|x_i - x_{j_0}|} + C \sum_{j=1}^{K} w(|x - x_j|) \left( \sum_{i \neq j_0} e^{-\frac{1}{2} |x_i - x_{j_0}|} \right)^2
\]
\[
\leq C \sum_{j=1}^{K} w(|x - x_j|) e^{-\rho} \leq \frac{CW}{K^{m-\sigma_0}}.
\]
So the remaining domain is \( \Omega_{K+1}^f \). In this domain, \( |x - x_j| \geq \frac{\ell \rho}{2} \) for any \( j = 1, 2, \ldots, K \). Then
\[
\sum_{j=1}^{K} w(|x - x_j|) \left( \sum_{l \neq j} w^2(|x - x_l|) + \sum_{i \neq k} w(|x - x_i|)w(|x - x_k|) \right)
\]
by taking $\ell = 1 + \frac{3}{\min(m,n)}$ but independent of $K$.

In conclusion,

$$\|S_1\|_* \leq C\left(\frac{1}{r_m} + \frac{1}{K_{\min(m,n)} - \sigma_0}\right).$$

Similarly, we also have that

$$\|S_2\|_* \leq C\left(\frac{1}{r_m} + \frac{1}{K_{\min(m,n)} - \sigma_0}\right).$$

The proof is completed.  

With these two lemmas in hand, we can use the contraction mapping theorem to get that problem (47) has a unique solution $(\phi_r, \psi_r)$ with

$$\|\begin{pmatrix} \phi_r \\ \psi_r \end{pmatrix}\|_* \leq C\left(\frac{1}{r_{\min(m,n)}} + \frac{1}{K_{\min(m,n)} - \sigma_0}\right).$$

On the other hand, according to Proposition A.2 in [30],

$$I(U_r, V_r) = K\left[A + \frac{aB}{r^n} + \frac{bC}{r^n} - (D + \beta H) e^{-\frac{2\pi}{r}} \frac{K}{r} + O\left(\frac{1}{r^m K^\sigma} + \frac{1}{r^n K^\sigma} + e^{-\frac{2\pi}{r}}\right)\right]$$

where

$$A = \frac{\mu_1 + \mu_2 - 2\beta}{4(\mu_1 \mu_2 - \beta^2)} \int_{\mathbb{R}^3} w^4, \quad B = \frac{\alpha^2}{2} \int_{\mathbb{R}^3} w^2, \quad C = \frac{\gamma^2}{2} \int_{\mathbb{R}^3} w^2,$$

$D$ and $H$ are all positive numbers independent of $K$.

Hence one may reduce the energy expansion.

**Proposition 4.3.**

$$F(r) = I(U_r, \phi_r, V_r, \psi_r)$$

$$= K\left[A + \frac{aB}{r^n} + \frac{bC}{r^n} - (D + \beta H) e^{-\frac{2\pi}{r}} \frac{K}{r} + O\left(\frac{1}{r^m K^\sigma} + \frac{1}{r^n K^\sigma} + e^{-\frac{2\pi}{r}}\right)\right].$$  

(64)

**Proof.** Since in our case the interaction and the power are so good and we can expand the energy directly by explicit formula,

$$F(r) = I(U_r, V_r) + \int_{\mathbb{R}^3} (\nabla U_r \cdot \nabla \phi_r + P(|x|) U_r \phi_r - \mu_1 U_r^3 \phi_r - \beta U_r V_r^2 \phi_r)$$

$$+ \int_{\mathbb{R}^3} (\nabla V_r \cdot \nabla \psi_r + Q(|x|) V_r \psi_r - \mu_2 V_r^3 \psi_r - \beta V_r U_r^2 \psi_r)$$

$$+ \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla \phi_r|^2 + P(|x|) \phi_r^2 - 3\mu_1 U_r^2 \phi_r^2 - \beta V_r^2 \phi_r^2 - 2\beta U_r V_r \psi_r \phi_r)$$

$$+ \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla \psi_r|^2 + Q(|x|) \psi_r^2 - 3\mu_2 V_r^2 \psi_r^2 - \beta U_r^2 \psi_r^2 - 2\beta U_r V_r \phi_r \psi_r)$$

$$- \frac{1}{4} \int_{\mathbb{R}^3} \mu_1 \left[(U_r + \phi_r)^4 - U_r^4 - 4U_r^3 \phi_r - 6U_r^2 \phi_r^2\right].$$
where 5.

Proof of Theorem 1.4. In this section, we just give out the difference between the synchronized case and segregated case.

Denote $U_\mu$ be the unique solution of

$$\begin{cases}
  \Delta u - u + \mu u^3 = 0, & u > 0 \text{ in } \mathbb{R}^3, \\
  u(0) = \max_{x \in \mathbb{R}^3} u(x), & u(x) \in H^1(\mathbb{R}^3).
\end{cases}$$

Actually $U_\mu = \frac{1}{\sqrt{\mu}} w$.

Recall

$$x_j = \left( r \cos \frac{2(j-1)\pi}{K}, r \sin \frac{2(j-1)\pi}{K}, 0 \right), \quad j = 1, \ldots, K$$

and define $y_j$ as

$$y_j = \left( R \cos \frac{2(j-1)\pi}{K}, R \sin \frac{2(j-1)\pi}{K}, 0 \right), \quad j = 1, \ldots, K,$$

where

$$\langle r, R \rangle \in \left[ \left( \frac{m}{2\pi} - \sigma \right) K \ln K, MK \ln K \right] \times \left[ \left( \frac{m}{2\pi} - \sigma \right) K \ln K, MK \ln K \right]$$

provided $\sigma$ small and $M > 0$ large. Then one may easily observe that

$$\min_{i \neq j} \{|x_i - x_j|\} = |x_1 - x_2| = 2r \sin \frac{\pi}{K}, \quad \min_{i \neq j} \{|y_i - y_j|\} = |y_1 - y_2| = 2R \sin \frac{\pi}{K}$$

and

$$\min_{i,j} \{|x_i - y_j|\} = |x_1 - y_1| = \sqrt{(R - r \cos \frac{\pi}{K})^2 + r^2 \sin^2 \frac{\pi}{K}} \geq 2\sqrt{R} r \sin \frac{\pi}{2K}.$$ 

Let

$$\bar{U}_j(x) = U_{\mu_1}(x - x_j), \quad \bar{V}_j(x) = U_{\mu_2}(x - y_j), \quad j = 1, 2, \ldots, K$$

and define our approximation

$$\bar{U}_r = \sum_{j=1}^K \bar{U}_j, \quad \bar{V}_R = \sum_{j=1}^K \bar{V}_j.$$
Indeed, we want to find the solutions of the form \((\bar{U}_r + \phi, \bar{V}_R + \psi) \in H_x \times H_s\). Thus we get the same form of \(L(\phi, \psi), E(\bar{U}_r, \bar{V}_R), N(\phi, \psi)\) in (44), (45) and (46) with \((\bar{U}_r, \bar{V}_R)\) instead of \((U_r, V_r)\).

In the segregated case, we need the following weighted function
\[
\bar{W}(x) = \sum_{j=1}^{K} \left( e^{-\eta|x-x_j|} + e^{-\eta|x-y_j|} \right)
\]
and for \(h = (h_1, h_2)\) define
\[
\|h\|_{**} = \sup_{i=1,2} \sup_{x \in \mathbb{R}^3} |h_i(x)|.
\]

Since the progress is almost the same as that in the synchronized case, here we just prove the following result

**Lemma 5.1.** With \(P, Q\) satisfying (7), (8) correspondingly and \(m = n\). Then for \(K\) large, we have
\[
\|\bar{E}\|_{**} = \left\| S \left( \bar{U}_r, \bar{V}_R \right) \right\|_{**} \leq C \left( \frac{1}{r^m} + \frac{1}{R^m} + \frac{1}{K^{m-\sigma_0}} + \frac{e^{-|x_1-y_1|}}{|x_1-y_1|} \right),
\]
where \(C\) is independent of \(K\) and \(\sigma_0\) is arbitrarily small if \(\sigma\) is small enough.

**Proof.** Note that
\[
\bar{E} = S \left( \bar{U}_r, \bar{V}_R \right) = \begin{pmatrix}
\Delta \bar{U}_r - P(|x|) \bar{U}_r + \mu_1 \bar{U}_r^3 + \beta \bar{V}_R^2 \bar{U}_r \\
\Delta \bar{V}_R - Q(|x|) \bar{V}_R + \mu_2 \bar{V}_R^3 + \beta \bar{U}_r^2 \bar{V}_R
\end{pmatrix}
\]
\[
= \begin{pmatrix}
(1 - P(|x|)) \bar{U}_r + \mu_1 \bar{U}_r^3 - \mu_1 \sum_{j=1}^{K} \bar{U}_j^3 + \beta \bar{V}_R^2 \bar{U}_r \\
(1 - Q(|x|)) \bar{V}_R + \mu_2 \bar{V}_R^3 - \mu_2 \sum_{j=1}^{K} \bar{V}_j^3 + \beta \bar{U}_r^2 \bar{V}_R
\end{pmatrix}. \tag{66}
\]

With the same computations in Step 1 of Proposition 3.3
\[
\left\| (1 - P(|x|)) \bar{U}_r + \mu_1 \bar{U}_r^3 - \mu_1 \sum_{j=1}^{K} \bar{U}_j^3 \right\|_{**} \leq C \left( \frac{1}{r^m} + \frac{1}{K^{m-\sigma_0}} \right)
\]
and
\[
\left\| (1 - Q(|x|)) \bar{V}_R + \mu_2 \bar{V}_R^3 - \mu_2 \sum_{j=1}^{K} \bar{V}_j^3 \right\|_{**} \leq C \left( \frac{1}{R^m} + \frac{1}{K^{m-\sigma_0}} \right),
\]
since with the definition of \(W\),
\[
\|h\|_{**} \leq \|h\|_*,
\]
which is also true with \(x_j\) in \(W\) replaced by \(y_j\) correspondingly.

So we just give the estimates to the term \(\bar{V}_R^2 \bar{U}_r\) in details. Then the term \(\bar{U}_r^2 \bar{V}_R\) follows. Let
\[
\tilde{\rho} := \min\{|x_1-y_1|, |x_1-x_2|, |y_1-y_2|\} \geq \frac{m \ln K}{3}
\]
if \( \delta \) is sufficiently small and denote \( x_{K+j} = y_j, j = 1, 2, \ldots, K \), then we can divide the whole space \( \mathbb{R}^3 \) into the following

\[
\overline{\Omega}_j^t = \left\{ x : |x - x_j| = \min_{1 \leq i \leq 4K} \{|x - x_i| \} \leq \frac{\ell_0}{2} \right\}, \quad j = 1, \ldots, 2K
\]

and \( \overline{\Omega}_{2K+1} = \mathbb{R}^3 \setminus \bigcup_{j=1}^{2K} \overline{\Omega}_j^t \).

If \( x \in \overline{\Omega}_{j0}^t \) for some \( j_0 = 1, 2, \ldots, K \), then

\[
|x - y_j| \geq \begin{cases} 
|x - x_{j0}|, & \text{if } |x - x_{j0}| \geq \frac{1}{2}|x_1 - y_1| \\
|x_{j0} - y_j| - |x - x_{j0}|, & \text{if } |x - x_{j0}| \leq \frac{1}{2}|x_1 - y_1| 
\end{cases} \geq \frac{1}{2}|x_1 - y_1|.
\]

Thus

\[
\sum_{j=1}^{K} \bar{U}_j \left( \sum_{j=1}^{K} \bar{V}_j \right) \leq C \ell \left( \sum_{j=1}^{K} e^{-\eta |x - x_{j0}|} \left( \sum_{j=1}^{K} e^{-\frac{(1+n)}{2} |x - x_{j0}|} |V_j|^{1+n} V_j^{1+n} \right) \right)^2
\]

\[
\leq C \ell \left( \sum_{j=1}^{K} e^{-\eta |x - x_{j0}|} \left( \sum_{j=1}^{K} e^{-\frac{(1+n)}{2} |y_j - x_{j0}|} |e^{-\frac{(1+n)}{2} |x_1 - y_1|} \cdot \frac{1}{|x - y_j|^{1+n}} \right) \right)^2
\]

\[
\leq C \ell \left( \sum_{j=1}^{K} e^{-\frac{(1+n)}{2} |y_j - x_{j0}|} |e^{-\frac{(1+n)}{2} |x_1 - y_1|} \cdot \frac{1}{|x - y_j|^{1+n}} \right)^2
\]

\[
\leq C \ell \left( \sum_{j=1}^{K} e^{-\frac{(1+n)}{2} |y_j - x_{j0}|} |e^{-\frac{(1+n)}{2} |x_1 - y_1|} \cdot \frac{1}{|x - y_j|^{1+n}} \right)^2
\]

\[
\leq C \ell \left( \sum_{j=1}^{K} e^{-\frac{(1+n)}{2} |x_1 - y_1|} \cdot \frac{1}{|x - y_j|^{1+n}} \right)^2
\]

\[
\leq C \ell \left( \sum_{j=1}^{K} e^{-\frac{(1+n)}{2} |x_1 - y_1|} \cdot \frac{1}{|x - y_j|^{1+n}} \right)^2
\]

\[
= C \ell \mathcal{W} \left( \sum_{j=1}^{K} e^{-\frac{(1+n)}{2} |x_1 - y_1|} \cdot \frac{1}{|x - y_j|^{1+n}} \right)^2
\]

For \( x \in \overline{\Omega}_{j0+K}^t \), then similarly \( |x - x_j| \geq \frac{1}{2}|x_1 - y_1| \) and

\[
\bar{V}_{k}^2 \bar{U}_r \leq C \sum_{j=1}^{K} w(|x - x_j|) \left( \sum_{j=1}^{K} w(|x - y_j|) \right)^2
\]

\[
\leq C \ell^2 \sum_{j=1}^{K} w(|x - x_j|) w^2(|x - y_{j0}|)
\]

\[
\leq C \ell^2 \sum_{j=1}^{K} \frac{e^{-|x_1 - y_1|}}{|x - x_j|} e^{-|x_1 - y_1|} |e^{-|x_1 - y_1|} |
\]

\[
\leq C \ell^2 \sum_{j=1}^{K} \frac{e^{-|x_1 - y_1|}}{|x_1 - y_1|} \mathcal{W} \leq C \ell^2 \mathcal{W} e^{-|x_1 - y_1|} \frac{|x_1 - y_1|}{|x_1 - y_1|}.
\]
Finally, for $x \in \Omega_{2K+1}$, then for all $j = 1, 2, \ldots, K, \ |x - x_j| \geq \frac{\ell \rho}{2}, |x - y_j| \geq \frac{\ell \rho}{2}$ which leads to
\[
\tilde{V}_R^2 \tilde{U}_r \leq C \sum_{j=1}^K w(|x - x_j|) \left( \sum_{j=1}^K w(|x - y_j|) \right)^2 \leq C W K^2 e^{-\ell \rho} \leq C W K^{-m}
\]
by taking $\ell = 3 + \frac{6}{m}$. Combining the above, we finish the estimate of $\tilde{V}_R^2 \tilde{U}_r$. From the above, the lemma is concluded. \hfill $\Box$

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