Compositional semantics for new paradigms: probabilistic, hybrid and beyond

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Abstract
Emerging computational paradigms, such as probabilistic and hybrid programming, introduce new primitive operations that often need to be combined with classic programming constructs. However, it still remains a challenge to provide a semantics to these features and their combination in a systematic manner.

For this reason, we introduce a generic, monadic framework that allows us to investigate not only which programming features a given paradigm supports, but also on how it can be extended with new constructs. By applying our method to the probabilistic and hybrid case, we list for example all binary program operations they possess, and show precisely when and if important axioms such as commutativity and idempotency hold. Using this framework, we also study the possibility of incorporating notions of failure and non-determinism, and obtain new results on this topic for hybrid and probabilistic programming.

Keywords  Probabilistic program, hybrid program, monad, semantics

1 Introduction
Probabilistic programming languages such as Church [9], Anglican [34] or Probabilistic C [21] have become increasingly popular in the last years, and although progress has been made in developing semantics for them many questions remain. In particular, how does one interpret combinations of probabilistic features with ‘classical’ features like error handling or non-determinism? Consider for example the program below, written in Probabilistic C-style.

```c
int main () {
    int a;
    int c_1=bernoulli(0.3);
    int c_2=bernoulli(0.6);
    printf("Please input an integer: ");
    scanf("%d", &a);
    if(a % 3 == 0){
        return c_1;
    } else if (a % 3 == 1){
        return c_2;
    } else {
        exit(EXIT_FAILURE);
    }
}
```

This program non-deterministically combines two probabilistic instructions – producing Bernoulli trials which return 1 with probability 0.3 (resp. 0.6) and 0 with probability 0.7 (resp. 0.4) – with an execution failure. By abstracting away from the C-style grammar and moving to an algebraic syntax, we want to understand how to interpret the expression

\[(1 +_3 0) + (1 +_6 0) + \text{abort}\]

where \(+_\lambda\) is the binary probabilistic choice operator with parameter \(\lambda \in [0, 1]\). It is easy enough to provide a semantics to the purely probabilistic instructions in terms of Markov kernels \(N \rightarrow D^N\), where \(D\) is the finitely supported distribution monad. But can we interpret \(+\) and \(\text{abort}\) in this semantics? And if not, how can we modify \(D\) to support these constructs? We aim to provide firm answers to this kind of question.

The challenge that we described above is not unique to probabilistic programming. Parallel to the latter, recent years have witnessed a flurry of research activity aiming to formalise the programmable features of hybrid systems [12, 23, 31], which require an orchestrated use of both classic program constructs and systems of differential equations. Consider, for example, the ‘C-style’ hybrid program below.

```c
int cool_or_heat () {
    int a;
    printf("Please input an integer: ");
    scanf("%d", &a);
    if(a == 0){
        // Heating up
        (dtemp = 1 & 3);
        return 0; // success
    } else if (a == 1){
        // Cooling down
        (dtemp = -1 & 3);
        return 0; // success
    } else {
        exit(EXIT_FAILURE);
    }
}
```

Depending on the input, the program increases or decreases a reactor’s temperature during three milliseconds or aborts:
there exists a global variable (temp) that registers the current temperature, the expression (dtemp = 1 & 3) dictates how the temperature is going to evolve for the next three milliseconds, and similarly for (dtemp = -1 & 3). Abstracting from the C-style grammar, we want to interpret the expression

\[(dtemp = 1 & 3) + (dtemp = -1 & 3) + \text{abort}\]

in a suitable manner, and this yields questions completely analogous to the probabilistic case. Purely hybrid programs can be naturally interpreted as functions \(\mathbb{R}^n \to H(\mathbb{R}^n)\) where \(H\) is the hybrid monad \([20]\). But it is not clear how to extend this semantics in a systematic manner so that it incorporates exceptions and non-determinism.

**Approach and contributions.** The paper is divided in two halves: in the first part we develop a framework – Kleisli representations – so that we can analyse program semantics in a systematic way. In the rest of the paper, we show how program semantics can be explicitly built in this framework, and use it to analyse the hybrid and probabilistic paradigms.

The framework is laid out in Section 3. It reinterprets Moggi’s idea \([19]\) of interpreting a program \(p\) as a Kleisli arrow \([p] : X \to TX\) and sequential composition as Kleisli composition by saying that the interpretation map is a monoid morphism from the monoid of programs to the monoid of endomorphisms of \(X\) in the Kleisli category of \(T\).

Representing an algebraic structure as a collection of endomorphisms is an idea very familiar to mathematicians and physicists alike, it sits at the heart of Representation Theory, a vast field of research which has extensively studied the representations of groups and Lie algebras as endomorphisms of vector spaces, with applications ranging from the classification of finite groups to quantum field theory \([11, 30]\). In this work we re-interpret denotations à la Moggi as representations in the usual mathematical understanding of the word. We will refer to a representation in the Kleisli category \(C_T\) of \(T\) as a Kleisli \(T\)-representation.

Representation theory provides a natural and useful way of thinking about program semantics: it captures both the algebraic aspects of the language and the coalgebraic aspects of its interpretation in a simple, well-known mathematical object, it separates the role of sequential composition from other operations on programs, it connects smoothly with the existing literature on algebraic effects \([24–27]\), and leads to new methods for building program semantic.

Whilst the Kleisli representation of sequential composition boils down to a natural transformation \(T \circ T \to T\), all other binary operations on programs will be interpreted as natural transformations \(T \times T \to T\), an idea first proposed by Plotkin and Power \([18, 19]\). Thus, in order to determine precisely which binary program operations are supported by a monad, we must completely enumerate all natural transformations \(T \times T \to T\). Section 4, introduces a set of techniques to do this for a large list of well-known monads. For example, we list all binary operations for the hybrid, probabilistic, non-deterministic, and partial paradigms.

Our next step is to study program axiomatics, i.e. the (im)possibility for a monad to support program operations satisfying some given axioms. This is done in Section 5 where we focus on commutativity, idempotence, units, and absorption. We give fine grained results on which axioms can be supported by Kleisli \(T\)-representations for different monads, which clarifies the types of operation specific computational paradigms support. We show for example that the hybrid paradigm does not admit a non-deterministic choice, that the probabilistic paradigm supports precisely one commutative idempotent operation and that it does not support failures.

Section 6 shows that combining monads with the Maybe monad \(M\) or the powerset monad \(P\) (i) provides a generic interpretation of tests, and (ii) yields richer monads with which to overcome representability issues highlighted in Section 5. We strengthen a well-known result from \([33]\) by showing that there cannot exist any monad structure on \(PD\) whatsoever. We also prove the existence of a distributive law of the non-empty powerset monad \(Q\) over the hybrid one. This allows to generate powerful, hybrid programming languages that mix non-deterministic assignments with differential predicates.

We assume basic knowledge of category theory and monads. All proofs can be found in the appendix.

## 2 Building semantics for hybrid and probabilistic programs

Let us illustrate the questions raised in the introduction and some of the solutions that we have developed for them by looking at two emerging programming paradigms.

**Hybrid programs** The distinguishing feature of hybrid programming is that it emphasises and makes explicit the interaction between digital devices and physical processes. This is becoming essential for the software engineer, because he/she needs more and more often to develop complex systems that are deeply intertwined with physical systems \([23, 32]\), e.g. cruise controllers, thermostats, etc.

Let us build a very simple hybrid programming language. Take a finite set of real-valued variables \(X = \{x_1, \ldots, x_n\}\) and denote by \(\text{At}(X)\) the set, given by the grammar

\[
\varrho = (x_1 := t, \ldots, x_n := t) \mid (\hat{x}_1 = t, \ldots, \hat{x}_n = t \& r)
\]

\[
t = r \cdot x \mid t + t
\]

where \(r\) is a real number and \(x \in X\). Then define \(\text{Hyb}(X)\) as the free monoid over \(\text{At}(X)\) given by the grammar

\[
p = a \in \text{At}(X) \mid \text{skip} \mid p;p
\]

and with the usual monoidal laws. One possible program is the composition \(a := 10; (p = v, \hat{v} = a \& 3)\), which, intuitively, sets the acceleration of a vehicle to \(10\text{m/s}^2\) and then makes it move during three milliseconds.
In Section 3, we will see that the semantics for this language comes naturally as a Kleisli representation
\[ [-] : \text{Hyb}(\mathbb{R}^3) \rightarrow \text{End}_{D}(\mathbb{R}^3) \]
where \( D \) is the hybrid monad [20] (whose definition is recalled in the following section). We now want to endow \( \text{Hyb}(\mathbb{R}^3) \) with other programming features, such as abort operations and non-deterministic choice. In the framework of Kleisli representations, this amounts to providing \( \text{End}_{D}(\mathbb{R}^3) \) a suitable algebraic structure that supports these constructs. Consider, for example, the language \( \text{Hyb}^0(\mathbb{R}^3) \) with syntax

\[ p = a \in \text{At}(\mathbb{R}^3) | \text{skip} | p; p | \emptyset \]

Our goal is to build a Kleisli representation
\[ [-] : \text{Hyb}^0(\mathbb{R}^3) \rightarrow \text{End}_{D}(\mathbb{R}^3) \]
for this language. Corollary 4.1 shows that this is impossible, because there does not exist a natural transformation \( \tau^0 : \text{End}_{D}(\mathbb{R}^3) \rightarrow H \) to interpret \( \emptyset \). In other words, pure hybrid computations do not support abort operations.

In order to surpass this obstacle, one may want to consider the partial hybrid monad \( H(-; 1) \). We show that this monad has precisely one natural transformation \( \tau^0 : \text{End}_{D}(\mathbb{R}^3) \rightarrow H(-; 1) \) to interpret \( \emptyset \) such that the axiom \( p; \emptyset = 0 \) holds. In contrast to classic paradigms, we prove that partial hybrid programs do not admit the axiom \( p; \emptyset = 0 \) and such is to be expected.

Suppose now that we wish to extend the language \( \text{Hyb}(\mathbb{R}^3) \) with a non-deterministic choice (+). As already mentioned, in our framework this amounts to finding a suitable natural transformation \( \tau^0 : \text{End}_{D}(\mathbb{R}^3) \rightarrow H \) to interpret \( \emptyset \). Using the set of techniques introduced in the paper, we list all such transformations and quickly discover that none of them is commutative. This means that in pure hybrid computations one cannot expect a non-deterministic choice operation.

To solve this issue, we show that there exists a distributive law of the non-empty powerset monad \( Q \) over \( H \). The monad \( QH \) inherits the natural transformation \( QH \times QH \rightarrow QH \) that takes unions and this allows to extend the language \( \text{Hyb}(\mathbb{R}^3) \) with a non-deterministic choice operation in the usual way.

**Probabilistic programs** Consider the simple probabilistic programming language \( \text{Prob} \) described by the syntax

\[ p = a \in \text{At} | \text{skip} | p; p | p + \lambda \ y \lambda \in [0, 1] \cap Q \]

where \( + \lambda \) is the probabilistic choice operation and \( \text{At} \) is a set of atomic programs. \( \text{Prob} \) satisfies the following axioms:

\[
\begin{align*}
(1) & \quad p; \text{skip} = \text{skip}; p = p \\
(2) & \quad p; (q + \lambda r) = (p; q) + \lambda (p; r) \\
(3) & \quad (p + \lambda q); r = (p; r) + \lambda (q; r) \\
(4) & \quad p + \lambda (q + \tau) = (p + \lambda q) + \lambda (p + \tau) \\
\end{align*}
\]

Let us call an algebraic structure for this signature and these equations a convex semiring. \( \text{Prob} \) is the free convex semiring over \( \text{At} \). The \( + \lambda \) fragment of such a structure is known as a convex algebra (modulo an extension to \( n \)-ary affine combinations, see e.g. [29]), and convex algebras are precisely the Eilenberg-Moore algebras for the distribution monad \( D : \text{Set} \rightarrow \text{Set} \). The set \( \text{End}_{D}(\mathbb{R}^3) \) can be equipped with a convex algebra structure inherited from \( D \) in the obvious way. In fact, \( \text{End}_{D}(\mathbb{R}^3) \) is also a convex semiring, i.e. all the axioms listed above also hold when \( \tau \) is interpreted as the Kleisli composition \( \circ \).

We now interpret the language above in terms of rational Markov kernels (i.e. maps \( X \rightarrow D_r X \) where \( D_r \) is the monad of rational probability distributions), by defining Kleisli \( D_r \)-representations as follows: for each \( a \in \text{At} \) choose an interpretation \( [a] : X \rightarrow D_r X \) for some state space \( X \). The semantics is then extended inductively:

\[
[a] \circ \eta_X \quad [p; q] = [p] \circ [q] \quad [p + \lambda q] = \lambda[p] + (1 - \lambda)[q]
\]

This Kleisli \( D_r \)-representation \([-] : \text{Prob} \rightarrow \text{End}_{D_r}(\mathbb{R}^3) \) is a convex semiring homomorphism. We now want to extend the language \( \text{Prob} \) with other operations on programs, such as non-deterministic choice (as in \( \text{ProbNet} \)) [6, 28], iteration, or parallel composition. Again, non-deterministic choice is enough to illustrate some goals and contributions of this paper. Consider the language \( \text{Prob}^+ \) whose syntax is

\[
p = a \in \text{At} | \text{skip} | \emptyset | p; p | p + \lambda \ y \lambda \in [0, 1] \cap Q
\]

and whose axioms are those of \( \text{Prob} \) together with those making \( (\cdot, 1, +, \emptyset) \) an idempotent semiring.

The most obvious strategy to provide a semantics for \( \text{Prob}^+ \) is to try to put a \( + \) operation satisfying the axioms above directly on \( \text{End}_{D_r}(\mathbb{R}^3) \) and then define Kleisli representations \( \text{Prob}^+ \rightarrow \text{End}_{D_r}(\mathbb{R}^3) \) in the same way as we did for \( \text{Prob} \).

Since \( + \) will in fine be interpreted by a natural transformation \( D_r \times D_r \rightarrow D_r \), our first task is to characterise these. We show in Theorem 4.10 that they are precisely the convex sum operations. It follows that by choosing the equally weighted convex sum \( _{1/2} \) we can equip \( \text{End}_{D_r}(\mathbb{R}^3) \) with an operation which satisfies all the axioms above apart from (10) and (11) – actually, we will see that this is the only possible choice. The axioms (10) and (11) fail because once again there are no natural transformations \( T^0 : \text{End}_{D_r}(\mathbb{R}^3) \rightarrow \text{End}_{D_r}(\mathbb{R}^3) \) to interpret \( \emptyset \) (Corollary 4.1). In particular there cannot exist any Kleisli \( D_r \)-representations of \( \text{Prob}^+ \). To overcome this obstacle we will, as in the hybrid case, consider a more complex monad in Section 6 – DM – which adds the missing notion of partial computation to \( D_r \) and restrict our attention to a particular class of non-deterministic instruction.

## 3 Kleisli representations

**Kleisli representation of monoids.** Given a monoid \( (M, \cdot, 1) \), a monad \( T : C \rightarrow C \) and a C-object \( X \), define a Kleisli \( T \)-representation of \( M \) in \( X \), or simply a Kleisli representation of \( M \) in \( X \) if there is no ambiguity, as a monoid homomorphism

\[
\rho : (M, \cdot, 1) \rightarrow (\text{End}_T)(X, \circ \tau, \eta_X)
\]
where $\circ_T$ is Kleisli composition and $\eta^T$ is the unit of $T$.

**Example 3.1** (Classical linear representations). Let $F : \text{Set} \to \text{Vect}$ be the functor building free vector spaces over some chosen field, and let $U : \text{Vect} \to \text{Set}$ be the corresponding forgetful functor. The composition $UF : \text{Set} \to \text{Set}$ is a monad, and for any group $G$, a Kleisli $UF$-representation of $G$ on a finite set $n$ is simply the usual notion of linear representation of $G$ on the $n$-dimensional vector space.

**Example 3.2** (Stochastic processes). The category $\text{Pol}$ is the category of Polish spaces, i.e., separable, completely metrisable topological spaces, and continuous maps. The Giry monad $G : \text{Pol} \to \text{Pol}$ associates to every Polish space $X$ the set of probability distributions on $X$ together with the topology of weak convergence, which is Polish. On morphisms, it associates to any continuous map $f : X \to Y$ the map $GF : GX \to GY, \mu \mapsto f_!(\mu)$ taking the pushforward of measures. A Kleisli $G$-representation of the monoid $([0, \infty), +, 0)$ of non-negative reals is a stochastic process.

**Example 3.3** (Hybrid systems). The functorial part of the hybrid monad $H : \text{Set} \to \text{Set}$ is defined by

$$H = \bigsqcup_{d \in \{0, \omega\}} \text{hom}([0, d], -)$$

Its unit is given by the equation $\eta_X(x) = (x, 0)$, with $x$ the constant function on $x$, and the multiplication is defined by $\mu_X(f, d) = (\theta_X \circ f, d) \oplus (f(d))$ with $\theta : H \to \Id$ the natural transformation that sends an evolution $(f, d)$ to $f(0)$ and $+: H \times H \to H$ the natural transformation that concatenates two evolutions. Intuitively, the multiplication will be used to concatenate the evolutions produced by two hybrid programs.

Recall from Section 2 the grammar $\text{At}(X)$ of atomic hybrid programs and denote the usual interpretation of a term $t$ over a valuation $\langle v_1, \ldots, v_n \rangle \in \mathbb{R}^n$ by $\llbracket t \rrbracket_{\langle v_1, \ldots, v_n \rangle}$ or simply $\llbracket t \rrbracket$ if the valuation is clear from the context. Since linear systems of ordinary differential equations always have unique solutions [22], there exists an interpretation map

$$\text{At}(X) \to \text{End}_H(\mathbb{R}^n)$$

that sends $(x_1 := t_1, \ldots, x_n := t_n)$ to the function $\mathbb{R}^n \to H(\mathbb{R}^n)$ defined by,

$$(v_1, \ldots, v_n) \mapsto \eta_{\mathbb{R}^n}(\llbracket t_1 \rrbracket, \ldots, \llbracket t_n \rrbracket)$$

and that sends $(x_1 = t_1, \ldots, x_n = t_n + \delta d)$ to the respective solution $\mathbb{R}^n \to (\mathbb{R}^n)[h, \omega]$ but restricted to $\mathbb{R}^n \to (\mathbb{R}^n)[h, \delta d]$. The free monoid extension of this interpretation map provides a Kleisli representation

$$\text{Hyb}(X) \to (\text{End}_H(\mathbb{R}^n), \circ, \eta_{\mathbb{R}^n})$$

which includes both assignments and differential equations. In the appendix we provide more details about this language, and give examples of other languages generated by the hybrid monad.

**Representing general varieties.** Consider a finitary variety $\mathcal{V}$ defined by a signature $\Sigma = \mathcal{F} \cup \{\text{skip}, \cdot\}$ with arity map ar : $\Sigma \to \mathbb{N}$, and a set of equations $E$ that contains the monoidal laws for $\{\text{skip}, \cdot\}$. Consider also a monad $T : \mathcal{C} \to \mathcal{C}$ on a category with products. As the reader may have guessed, a Kleisli $T$-representation of a $\mathcal{V}$-object $A$ will be a morphism $\rho : A \to \text{End}_{T}(X)$. But in which category? Since our starting point will always be a signature $\Sigma$, we define a Kleisli $T$-representation of a $\mathcal{V}$-object $A$ as a morphism $\rho : A \to \text{End}_{T}(X)$ in the category of $\Sigma$-algebras, that is to say a morphism $\rho$ which commutes with all the operations in the signature but whose codomain may not live in $\mathcal{V}$. The rationale for this choice of category is the following: a group representation is not a group homomorphism because the whole point of a representation is to map group elements to important mathematical objects which do not form a group, for example real-valued matrices. Similarly, requiring the representation map to be a $\mathcal{V}$-morphism would be too stringent and would drastically limit the choice of possible semantics. By defining a Kleisli representation as a $\Sigma$-algebra morphism we do not require that all equations in $E$ be valid in $\text{End}_{T}(X)$, but we do require that every operation in $\Sigma$ be interpretable in $\text{End}_{T}(X)$.

In order to endow $\text{End}_{T}(X)$ with a suitable algebraic structure from a signature $\Sigma$, we proceed in the footsteps of [24–26]: for every $\sigma \in \Phi$ consider the set of natural transformations $[C, C](\text{End}_{T^\sigma}(\sigma), T)$ (as usual we take $T^0 = 1$, the constant functor on the final object 1). For each $\sigma \in \Phi$ we choose an element $a^\sigma \in [C, C](\text{End}_{T^\sigma}(\sigma), T)$ and define the operation $[\sigma] : \text{End}_{T}(X)^{\text{at}(\sigma)} \to \text{End}_{T}(X)$ by

$$[\sigma](a_1, \ldots, a_{\text{at}(\sigma)}) = a^\sigma \circ (a_1, \ldots, a_{\text{at}(\sigma)})$$

(1)

We can now define a generic Kleisli representation as follows: a Kleisli $T$-representation of $A$ in $\text{C}_{T}$ is an assignment to every $\sigma \in \Phi$ of a natural transformation $a^\sigma : T^{\text{at}(\sigma)} \to T$ together with a $\Sigma$-algebra morphism $\rho : (A, \text{skip}, ;, \sigma \in \Phi) \to (\text{End}_{T}(X), \eta^T_X, \circ_T, ([\sigma])_{\sigma \in \Phi})$.

**Naturality and abstraction** The requirement that operations be interpreted by natural transformations could be seen as either too strong or too weak. Too strong because it is a very restrictive condition whose justification is not immediately obvious. Too weak because it is strictly weaker than the requirement of [24, 25] defining algebraic operations where compatibility conditions with the strength and multiplication of the monad are assumed. So why have we chosen to focus on naturality?

First, algebraic operations are in general too restrictive for our purpose. Examples of non-algebraic operations include exception-handling operators [26] (which we will address in Section 6), ‘true’ parallel composition axiomatised by the exchange law [8] and, more recently, examples from Game Logic [10]. Actually, our work is to a large degree a
systematic investigation of what can be said about the semantics of programs with operations that are not necessarily algebraic in the sense of Plotkin and Power – a research direction already mentioned in [25] – for a wide range of monads.

Second, it is important to be able to consider sub-representations and quotient representations, and, as we will show, naturality plays a key role in allowing these to be defined. Often we need to abstract away details of a representation in a large, fine-grained state space and build a representation in a coarser one, but in such a way that both representations ‘agree’ with each other. Formally, for a quotient map \( q : X \to Q \) and two Kleisli representations \( \rho : A \to \text{End}_T(X) \), \( \rho' : A \to \text{End}_T(Q) \) we need that the equation

\[
Tq \circ \rho(a) = \rho'(a) \circ q
\]

holds for all programs \( a \in A \). Abstracting and then interpreting should be the same as interpreting and then abstracting. The naturality of program operations allows a compositional construction of quotient representations because it allows to prove that the equation \( Tq \circ \rho(a) = \rho'(a) \circ q \) holds just by showing that it holds for atomic programs. For sequential composition, this follows from the naturality of \( \mu \):

\[
\begin{array}{c}
X \xrightarrow{\rho(a)} TX \xrightarrow{T \rho(b)} T^2X \xrightarrow{\mu_X} TX \\
\downarrow q \quad \downarrow Tq \quad \downarrow T^2q \\
Q \xrightarrow{\rho'(a)} TQ \xrightarrow{T \rho'(b)} T^2Q \xrightarrow{\mu_Q} TQ
\end{array}
\]

and for other operations the naturality requirement on \( \alpha : T \times T \to T \) makes the following diagram also commute.

\[
\begin{array}{c}
X \xrightarrow{(\rho(a), \rho(b))} TX \times TX \xrightarrow{\alpha_X} TX \\
\downarrow q \quad \downarrow Tq \quad \downarrow Tq \\
Q \xrightarrow{(\rho'(a), \rho'(b))} TQ \times TQ \xrightarrow{\alpha_Q} TQ
\end{array}
\]

So naturality allows to freely extend an abstraction from atomic programs to all programs in the language.

4 Interpreting constants and operations

4.1 Constants

The following result, despite its simplicity, gives a very general characterisation of natural transformations \( 1 \to T \).

**Theorem 4.1.** Let \( C \) be a category with an initial object \( \emptyset \) and an object \( 1 \) such that \( C(1, \_ ) \cong 1d \), then we have the sequence of bijections below:

\[
[C, C](1, F) \cong C(1, F\emptyset) \cong F\emptyset
\]

Note that in \( \text{Set} \) the result above is a trivial consequence of the Yoneda lemma, since \( 1 \) is representable as \( \text{hom}(\emptyset, \_ ) \).

**Example 4.2.**

1. The Maybe monad \( M \) has exactly one natural transformation \( \text{Set, Set}(1, M) = \emptyset = \{ \lambda x, \_ \} \).
2. \( \text{Set, Set}(1, D) = D\emptyset = \emptyset \), and since the category \( \text{Pol} \) of Polish spaces satisfies the assumptions of Theorem 4.1, it is also the case that \( \text{Pol, Pol}(1, G) = \emptyset \). Probabilistic programs do not support partial computations.
3. \( \text{Set, Set}(1, H) = H\emptyset = \emptyset \) and thus hybrid programs also do not support an interpretation of failure.

4.2 Operations

4.2.1 Coproducts of hom functors

We start with those functors that can be written as coproducts of hom functors, since they can be treated completely straightforwardly. Actually, using the notions of container and fibration, [2] already provides a powerful representation theorem for natural transformations \( T \times T \to T \) when \( T \) is one such functor. In order to keep this paper self-contained, however, we introduce a direct, equivalent result that does not need the notion of fibration nor the notion of container.

**Theorem 4.3.** If \( F : \text{Set} \to \text{Set} \) is a functor expressible as a coproduct of hom functors, i.e. if there exists a non-empty family \( (X_i)_{i \in I} \) of sets such that \( F = \coprod_{i \in I} \text{hom}(X_i, -) \), then

\[
\text{[Set, Set]}(F \times F, F) \cong \prod_{i,j \in I} F(X_i + X_j)
\]

**Example 4.4.** The maybe monad \( M \) can be written as the coproduct \( \text{hom}(\emptyset, -) + \text{hom}(1, -) \). It follows from Theorem 4.3 that the possible interpretations of a binary operation \( M^2 \to M \) are in bijective correspondence with the set \( M(2) \times M(1) \times M(1) \), in particular there are exactly 12 natural transformations \( M^2 \to M \). The ‘\( M(2)\)-coordinate’ specifies what a transformation does on pairs \( (x, y) \) with \( x, y \neq \_ \), viz. projecting to the left, to the right or mapping to \( \_ \), the first \( \text{M}(1)\)-coordinate specifies what happens to pairs \( (\_ , y) \), \( y \neq \_ \), viz. projecting to the left and the right, and similarly for the last coordinate and pairs \( (x, \_ ) \).

**Example 4.5.** Since \( H = \coprod_{d \in [0, \infty)} \text{hom}([0, d], -) \), the natural transformations \( H \times H \to H \) are in bijective correspondence with the set

\[
\prod_{i,j \in [0, \infty)} H([0, i] + [0, j])
\]

For an element \( s \) of this set, each \( (i, j)\)-coordinate \( s_{ij} \) dictates what the transformation \( \alpha^s \) does to pairs of evolutions with duration \( [0, i] \) and \( [0, j] \). In particular, it tells how the values in a given pair of evolutions \( (f, g) \) of duration \( [0, i] \) and \( [0, j] \) are distributed in the new evolution: one has the composition

\[
\begin{array}{c}
[0, k] \xrightarrow{s_{ij}} [0, i] + [0, j] \xrightarrow{[f, g]} X
\end{array}
\]

which makes clear that for every element \( a \in [0, k] \) the value \( \alpha^s_{ij}(f, g)(a) \) arises from one of the two starting functions \( f \) or \( g \) and an element in their respective domain (\( [0, i] \) or \( [0, j] \)).

To cover a broader spectrum of monads we need to introduce some more sophisticated mathematics.
4.2.2. The presentation of Set-valued functors. Let C be a small category and \( F : C \to \text{Set} \) be a functor. We define the category of elements of F, denoted \( \text{El}(F) \), as the category whose objects are pairs \((C, \alpha)\) where \( C \) is an object in \( C \) and \( \alpha \in FC \). There exists a morphism \( \hat{f} : (C, \alpha) \to (D, \beta) \) in \( \text{El}(F) \) whenever there exists a morphism \( f : C \to D \) such that \( Ff(\alpha) = \beta \). For every object \((C, \alpha)\) in \( \text{El}(F) \), we will define the orbit of \((C, \alpha)\) as all the objects \((D, \beta)\) which can be reached from \((C, \alpha)\) by a zigzag of morphisms in \( \text{El}(F) \). The decomposition of \( \text{El}(F) \) in orbits is key to understanding natural transformations involving \( F \), since naturality is only a constraint on objects in the same orbit. The category \( \text{El}(F) \) allows us to completely reconstruct \( F \). Moreover, this reconstruction process provides us with a presentation of \( F \) as a colimit of covariant hom functors.

**Theorem 4.6** ([17] L5). Let \( Y : C^{\text{op}} \to [\text{C}, \text{Set}] \) denote the Yoneda embedding, and \( U_F : \text{El}(F) \to C \) be the forgetful functor sending each pair \((A, \alpha)\) to the object \( A \), then

\[ F \cong \text{colim} \left( \text{El}(F)^{\text{op}} \xrightarrow{U^F} C^{\text{op}} \xrightarrow{Y} [\text{C}, \text{Set}] \right) \]

Theorem 4.6 gives us a way of presenting functors from a small category \( C \) to \( \text{Set} \) as a colimit of hom functors, but what we really need are presentations of functors \( \text{Set} \to \text{Set} \). To move from \( C \to \text{Set} \) to \( \text{Set} \to \text{Set} \) we need a few relatively well-known definitions. Recall that an object \( A \) in a category \( C \) is finitely presentable if the functor \( \text{hom}(A, -) \) preserves filtered colimits. In \( \text{Set} \) the finitely presentable objects are precisely the finite sets. A category \( C \) is called locally finitely presentable if it is cocomplete and contains a small subcategory \( C_{\omega} \) of finitely presentable objects such that every object \( A \) in \( C \) is the filtered colimit of the canonical diagram \( D_A : C_{\omega} \downarrow A \to C \) sending each arrow of \( C_{\omega} \downarrow A \) to its domain. The category \( \text{Set} \) is finitely presentable, since every set \( X \) can be written as a colim \( D_X \) with \( D_X : \omega \downarrow X \to \text{Set} \) and \( \omega \) the subcategory of \( \text{Set} \) consisting of elements \( n \in \omega \) (this simply says that every set is the union of its finite subsets). Finally, a functor is called finitary if it preserves filtered colimits. Finitary functors are entirely determined by their restriction to \( C_{\omega} \). In fact if \( F : C \to \text{Set} \) is a finite functor on a locally finitely presentable category and \( 1 : C_{\omega} \hookrightarrow C \) is the inclusion functor, then \( F \) can be written as the left Kan extension \( F = \text{Lan}_1(F_f) \), where \( F_f = F \circ 1 \). We refer the interested reader to the classic [3] for a full account of the theory of locally finitely presentable categories.

**Proposition 4.7.** Let \( F : \text{Set} \to \text{Set} \) be finitary and let \( 1 : \omega \to \text{Set} \) be the inclusion functor, then

\[ F \cong \text{colim} \left( \text{El}(F_f)^{\text{op}} \xrightarrow{U^F_{\text{op}}} \omega^{\text{op}} \xrightarrow{p_{\text{op}}} \text{Set}^{\text{op}} \xrightarrow{Y} [\text{Set}, \text{Set}] \right) \]

The following result is a simple application of Proposition 4.7 and the Yoneda lemma.

**Theorem 4.8.** Let \( F : \text{Set} \to \text{Set} \) be a finitary functor and let \( F_f \) denote its restriction to \( \omega \), then the set \([\text{Set}, \text{Set}](F \times F, F)\) is in one-to-one correspondence with the limit

\[ \lim \left( \text{El}(F_f \times F_f) \xrightarrow{F_f \circ 1 U} \text{Set} \right) \]

The hard work consists in computing the limit (2) above.

4.2.3. A classification result for some multiset functors.

Computing the limit (2) of Theorem 4.8 for general multiset-type monads (see their definition in the appendix) depends heavily on the choice of semiring and may prove extremely difficult. However, we do have an explicit characterisation in the following useful case. We say that a semiring \( S \) has the common integer divisor property if for any \( x, y \in S \) there exist an invertible element \( r \in S \) and \( m, n \in \mathbb{N} \) such that

\[ x = m \cdot r = r_1 + \ldots + r_m \quad y = n \cdot r \]

We will refer to \( r \) as a common integer divisor of \( x \) and \( y \). Clearly the semiring \( \mathbb{N} \) has this property since we can always pick \( r = 1 \) (which is trivially invertible) and \( m = x, n = y \). Similarly the semiring \( \mathbb{Q} \) has this property: given two rationals \( x = \frac{m_1}{n_1}, y = \frac{m_2}{n_2} \) we can choose \( r = \frac{n_1}{n_2} \) (which is invertible) and \( m = m_1 n_2, n = m_2 n_1 \). The semiring of real \( \mathbb{R} \) does not have this property: if \( \frac{1}{2} \) is irrational then there doesn’t exist an \( r \in \mathbb{R} \) with the desired property.

**Theorem 4.9.** Let \( S \) be a semiring with the common integer divisor property and let \( B_S \) be the multiset monad for \( S \), then \([\text{Set}, \text{Set}](B_S^o, B_S)\) is in one-to-one correspondence with the set of functions \( \phi : S^n \to S^n \).

4.2.4. Some results for the Giry monad

Due to its importance, we provide some detailed results for the Giry monad \( G \), which clarifies what can be expected of purely probabilistic program semantics. Before we turn to the full Giry monad, let us consider the rational distribution monad \( D_r \). The following can be shown using the same ideas as in the proof of Theorem 4.9.

**Theorem 4.10.** The only natural transformations \( D_r \times D_r \to D_r \) are the convex combinations \( x^\lambda, \lambda \in [0, 1] \cap \mathbb{Q} \).

Theorem 4.10 can be used to provide a full classification result for the Giry monad on \( \text{Pol} \). For this we use a set of criteria for functors \( F, G : \text{Pol} \to \text{Pol} \) developed in [4] and [5] under which it can be shown that

\[ [\text{Pol}, \text{Pol}](F, G) \cong [\text{Pol}_f, \text{Pol}_f](F_f, G_f) \]

where \( F_f \) is the restriction of \( F \) to the category \( \text{Pol}_r \), the category of finite Polish spaces (and similarly for \( G_f \)). These criteria restrict both the domain and the codomain functors. We refer the reader to [4] for more details; for our purpose it will be enough to say that the Giry monad \( G \) always satisfies the domain and codomain criteria (see [4, Prop. 5.1]) and that finite products of \( G \) satisfy the domain criteria (see [5,
We explore whether it is possible for a given monad \( T \) to satisfy \( \sigma(x, y) = \sigma(y, x) \) satisfiable in Kleisli \( T \)-representations. Finally, we will say that \( s = t \) is valid in Kleisli \( T \)-representations if it satisfies for any assignment \( a^\sigma : T^\omega(\sigma) \to T \).

### 5.1. Commutativity

We start with the case of coproducts of hom functors.

**Proposition 5.1.** If \( F = [\prod_{i \in I} \hom(X_i, -)] \), then a natural transformation \( \alpha : F^2 \to F \) given by an element \((s_{ij})_{i, j \in I} \in \prod_{i, j \in I} F(X_i + X_j) \) via Thm. 4.3 is commutative iff for all \( i, j \in I \), the equation below holds.

\[
\prod_{i, j \in I} [s_{ij} \circ s_{ji} : X_k \to X_j + X_i]
\]

In particular, for all \( i \in I \), \( s_{ii} \) must be the map with empty domain.

**Example 5.2.** 1. A natural transformation \( \alpha : M^2 \to M \) can only be commutative if its ‘M(2) coordinate’ (defined by Thm 4.3) lies in the second summand of \( M(2) \), i.e. if two elements different than failure are mapped to failure.

2. There is no commutative natural transformation for hybrid programs because there exists no real number \( d \in [0, \infty) \) such that \([0, d] = 0\).

For the other monads presented thus far we can look directly at the classification results provided in Section 4. We have for example:

1. The commutative natural transformations \( B_S^2 \to B_S \) are precisely given by the maps \( S^2 \to S \), i.e. the transformations choosing an equally weighted sum of multisets for each orbit.

2. The unique commutative natural transformation \( D_S^2 \to D_S \) is the average transformation \( + \frac{1}{2} \), and by the same argument as in Section 4.2 this is also the case for the Giry monad \( G \).

### 5.2. Idempotence

For coproducts of hom functors we have:

**Proposition 5.3.** If \( F = [\prod_{i \in I} \hom(X_i, -)] \), then a natural transformation \( \alpha : F^2 \to F \) given by an element \((s_{ij})_{i, j \in I} \in \prod_{i, j \in I} F(X_i + X_j) \) via Thm. 4.3 is idempotent iff for each \( i \in I \), \( s_{ii} \in F(X_i + X_i) \) is a map \( s_{ii} : X_i \to X_i + X_i \) such that \( \nabla \circ s_{ii} = \id \) where \( \nabla : X_i + X_i \to X_i \) is the codiagonal map.

**Example 5.4.** 1. A natural transformation \( \alpha : M^2 \to M \) is idempotent iff its ‘M(2) coordinate’ is in the first summand of \( M(2) \). In other words, if two elements different than failure are projected either to the left or to the right.
2. A natural transformation \( \alpha : H^2 \rightarrow H \) is idempotent if for every two evolutions \( (f, g) \) with the same duration \([0, d]\), \( \alpha^*(f, g) \) has domain \([0, d]\), and for every element \( a \in [0, d] \), \( \alpha^*(f, g)(a) = f(a) \) or \( \alpha^*(f, g)(a) = g(a) \).

It follows from Propositions 5.1 and 5.3 that the combination of commutativity and idempotence is never satisfiable for Kleisli \( T \)-representations when \( T \) is a non-constant coproduct of hom functors since there will then exist an \( X_i \neq \emptyset \) in the coproduct presentation of \( T \) for which Proposition 5.1 requires that \( s_{ij} \) be of type \( \theta \rightarrow X_i + X_j \) but for which Proposition 5.3 requires that \( s_{ij} \) be of type \( X_i \rightarrow X_i + X_j \). Thus no monad whose functor is a coproduct of hom functors can support a programming language with a commutative and idempotent binary operation on programs.

We summarise the characterisation of idempotent natural transformations for the other monads as follows:

1. The idempotent natural transformations \( B^2_{\emptyset} \rightarrow B_{\emptyset} \) are those which are defined as projections on pairs of multisets of equal weights. Note that as in the case of coproducts of hom functors, the combination of commutativity and idempotence is never satisfiable in Kleisli \( B_{\emptyset} \)-representations, since the commutative operations must take equally weighted sums, whereas the idempotent ones must put one weight to zero and the other to one on pairs of multisets of equal weights.

2. Theorem 4.10 guarantees that all natural transformation \( D^2_i \rightarrow D_i \) are idempotent, i.e. idempotence is valid over \( D_i \)-representations, and similarly for \( G \). Note that maps \( Q^2 \rightarrow Q^2 \) mapping each pair \( (q_1, q_2) \) to a pair \( (\lambda, 1 - \lambda) \) with \( \lambda \in \mathbb{Q} \cap [0, 1] \) also define the idempotent operations of \( B_{\emptyset} \).

3. Since there are no natural transformations \( 1 \rightarrow D \), there can be no interpretation of constants in Kleisli \( D \)-representations, and similarly for \( D_i \), \( G \) and \( H \).

4. The sub-distribution monad \( GM \) offers both probabilistic choice and partial computation but the unit axiom will be problematic for any binary operation: consider the sub-distributions on the singleton set \( 1 \), we need

\[
\alpha_1 \left( \frac{1}{n} \delta_i + \frac{n-1}{n} \delta_\ast \right) = \frac{1}{n} \delta_i + \frac{n-1}{n} \delta_\ast
\]

and

\[
\alpha_1 \left( \frac{1}{n} \delta_1 + \frac{n-1}{n} \delta_\ast \right) = \frac{1}{n} \delta_1 + \frac{n-1}{n} \delta_\ast
\]

This means that the function \( \phi \) defining \( \alpha \) (Thm. 4.12) must map \( \frac{0}{n} \rightarrow (0, 0) \) and \( \frac{1}{n} \rightarrow (1, 0) \) for each \( n \), which clearly cannot be continuous.

5.5. Absorption. The absorption law states that for every program \( p : X \rightarrow TX \) the equations below hold.

\[
p; \emptyset = \emptyset \text{ (left absorption)} \quad \emptyset; p = \emptyset \text{ (right absorption)}
\]

Even though they appear to be simple, to check their satisfiability in general Kleisli representations is a surprisingly complex issue. On the one hand, we have:

**Proposition 5.7.** The axiom \( p; \emptyset = \emptyset \) holds in every Kleisli representation.

On the other hand, the axiom \( p; \emptyset = \emptyset \) need not hold, and this is exemplified by the partial hybrid paradigm \( HM \) which is briefly introduced in the following section.

**Theorem 5.8.** Consider a monad \( T : \text{Set} \rightarrow \text{Set} \) with a natural transformation \( \theta : 1 \rightarrow T \). If the condition \( T\emptyset \equiv 1 \) holds then the axiom \( p; \emptyset = \emptyset \) holds as well.

6 Adding features by combining monads

6.1 Failure and Tests

There is a natural procedure for turning Kleisli representations which cannot interpret failure – such as those for the distribution or hybrid monads – into Kleisli representations which do support failure. More interestingly though, this procedure also allows to interpret tests, and in the case of probabilistic programs one recovers the interpretation of \( \text{if-then-else} \) commands first given by Kozen [14].

**Adding failure.** Recall from Section 4 that neither probabilistic nor hybrid systems can handle failure. As recently described in [1] for probabilistic systems, adding a distinguished state is an obvious strategy to overcome this shortcoming. We thus consider composing effect monads of interest with the Maybe monad \( M \). This can always be done as the following result [16] shows.

**Theorem 6.1.** For a monad \( T : \text{Set} \rightarrow \text{Set} \) there exists a distributive law \( \delta : MT \rightarrow TM \) defined by

\[
\delta_X = [T \eta_{X+1} \circ i_2]
\]
In particular $TM$ always forms a monad of which $M$ is a submonad. Combining monads in this way potentially addresses more than the problem of non-representability of failure: since every Set-monad $T$ is strong we can always find a natural transformation $\varphi^n : (-)^n \circ T \to T \circ (-)^n$ (we use the convention $(-)^0 = 1$ and $\varphi^0 = \eta^T_1$). Hence, if a Set-monad $S$ has a natural transformation $\alpha : S^n \to S$, then $TS$ also has a natural transformation of this type given by

$$(TS)^n \xrightarrow{\varphi^n S} T(S^n) \xrightarrow{T \alpha^n} TS$$

(4)

In other words, we can increase the list of available operations by composing monads. Note that the algebraic properties of $\alpha$ discussed in Section 5 are usually lost by the lifting described above. However if $T$ is commutative, then the lifting above will preserve associativity, commutativity and the existence of units. For $n = 0$, (4) shows that $TM$ always has a natural transformation

$$\eta^n_T : 1 \xrightarrow{\varphi^n_T} T1 \xrightarrow{T\alpha^n} TM$$

and we can thus always interpret failure in $TM$. In particular, the monads $DM$ and $HM$ support an abort operation $0$, and, as shown in Section 5, the axiom $0 : p = 0$ is valid in both paradigms. We also have the bijection $DM0 \cong D1 \cong 1$ and by an application of Theorem 5.8 this entails that the axiom $p : 0 = 0$ is valid in probabilistic programming. On the other hand, in the hybrid case we have the bijections $HM0 \cong H1 \cong [0, \infty]$ which say that one can always interpret an abort operation that produces failures for a given duration $[0, d]$. For each of these operations the axiom $p : 0 = 0$ cannot hold: on the left side, the duration of an evolution produced by $p$ is added to that of $0$. Hence, if the evolution of $p$ is greater than 0 the sum is greater than the duration of the evolution produced on the right side.

**Adding Tests.** The combination $TM$ also provides Kleisli $TM$-representations of useful fragments of Kleene algebras with tests (KAT) [15]. Here we will only consider the $+\text{-free}$ fragment of KATs, i.e. idempotent semirings with tests, or ISTs, which are defined as follows. An IST is a two-sorted structure $(S, B, +, ;, , 0, 1)$ such that $B \subseteq S$, $(S, +, ;, , 0, 1)$ is an idempotent semiring, and $(B, +, ;, , 0, 1)$ is a boolean algebra.

Let us first deal with tests: each test $b$ should be interpreted as a predicate $[b] : X \to 2$ on the state space, which is equivalent to a map

$$X \to X + 1, x \mapsto \begin{cases} x & \text{if } [b](x) = 1 \\ * & \text{else} \end{cases}$$

(5)

We will therefore interpret tests in $\End TM(X)$ as the maps $X \to TMX$ that can be built as $\eta^n_{MX} \circ f : X \to MX \to TMX$, where $f$ is of the shape (5). In particular, we automatically get a two-sorted structure on $\End TM(X)$. Note also that we can straightforwardly extend the single-sorted definition of Kleisli representation in Section 3 to a two-sorted definition so that KAT-like structures can be accommodated.

We now need to define a boolean algebra structure on tests in $\End TM(X)$. The constants are as expected: $[1]$ is the $T$-lifting of the unit of $M$, i.e. the unit of $TM$, and $[0]$ is the $T$-lifting of the unique natural transformation $T \to M$. The negation of a test $b$ is defined in the obvious way and it follows that $T = 0$. Conjunction is simply Kleisli composition. It is not hard to check by unravelling the definitions and using the properties of the distributive law $TM \times TM \to TM$ that for any two tests $a, b : [b] \circ TM [a] = [a ; b]$.

For disjunction, things are more complicated. We have seen in Theorem 4.3 and Section 5, that there is no idempotent commutative binary natural transformation $M \times M \to M$ with $1 \to M$ as unit. In particular we cannot expect the disjunction $\lor$ to be interpreted using a natural transformation $TM \times TM \to TM$ of the shape (4). In other words, simply composing $T$ with the Maybe monad does not give a non-deterministic choice in general.

However, there exists a natural and useful sub-class of programs in an IST for which composing $T$ with the Maybe monad does give non-deterministic choice. We call this subclass the $\text{if-then-else}$ fragment. It is given by:

$$p = b \in \text{Test } | \ a \in \text{At } | \ \text{skip} | \ 0 | p ; p | p + b p$$

where $p + b q$ reads ‘if $b$ then $p$ else $q$’. This restricted non-determinism has Kleisli $TM$-representations for any monad $T$. To see why, choose any of the 3 natural transformations $\alpha : M^2 \to M$ (Prop.5.5) for which $1 \to M$ is a unit, i.e. any of the transformations mapping pairs $(x, *)$ to $x$. With this choice made to resolve $M$-non-determinism, we define the Kleisli $TM$-representation of $p + b q$ in $\End TM(X)$ as:

$$[p + b q] := T\alpha_X \circ \varphi^2 \circ ([b; q], [b; q])$$

(6)

When $\varphi$ is derived from the strength of $T$ we can show that this Kleisli $TM$-representation of $\text{if-then-else}$ statements is sound in the sense that expected algebraic properties of the operators $-; -$ are preserved by the representation. It is routine by unravelling the definitions and using the explicit construction of strength in $\text{Set}$ to show:

**Theorem 6.2.** Let $a, b$ be tests and consider a Kleisli $TM$-representation in $\End TM(X)$ interpreting tests according to (5) and $\text{if-then-else}$ according to (6). Then,

$$[p + b p] = [p] \quad [p + b q] = [q + b p] \quad [p + 1 q] = [p] \quad [p + a(q + b r)] = [(p + ab)(q + \pi b r)]$$

In the case of probabilistic programs, the Kleisli $DM$-representation (6) of $\text{if-then-else}$ is precisely the one given by Kozen in [14].

---

1 Note the similarity with the axioms of convex algebras.
6.2 Non-determinism

In order to extend a given programming paradigm with non-deterministic features, one may also wish to combine the underlying monad $T$ with the powerset $P$ since the following two natural transformations always exist.

$$\alpha : PT \times PT \to PT, \quad \alpha_X(A, B) = A \cup B$$

$$0 : 1 \to PT, \quad 0_X(*) = \emptyset$$

These respect the unit, idempotence, commutativity, and associative laws for $\odot$. So, in principle, less well-behaved monads, like $H$ and $D$, together with $P$ could give rise to new monads that can be used to represent richer programming languages.

**A negative result for PD.** As already shown in [33], there is no distributive law $DP \to PD$. We now strengthen this result to ‘there does not exist any monad structure on PD whatsoever’ which answers the question recently raised in [13] about this problem. Following Theorem 4.1 we have:

**Lemma 6.3.** The only natural transformations $\eta : 1d \to PD$ are the constant transformation $\eta_X(x) = \emptyset$, and the natural transformation defined by $\eta_X(x) = \{\delta_x\}$.

By combining the result above, the monadic laws $\mu \odot \eta T = \mu \odot T \eta = 1d$, and adapting the proof of [33] we can show that:

**Theorem 6.4.** There is no monad structure on PD.

**A positive result for QH** On a more positive note, we can combine the non-empty powerset monad $Q$ with $H$ using the following theorem.

**Theorem 6.5.** There exists a distributive law $\delta : HQ \to QH$ defined by $\delta_X(f, d) = \{g, d \in H_X \mid g \in f\}$ where $g \in f$ is shorthand notation for the condition

$$\forall t \in [0, d]. g(t) \in f(t)$$

7 Conclusion

We have developed a general definition of program semantics as Kleisli representation, and studied whether monads describing various computational paradigms support Kleisli representations of classical programming constructs. We have shown that it is possible to explicitly and exhaustively answer this question by providing complete classification results of natural transformations of the type $(T^n) \to T$. In particular we have shown that monads for probabilistic and hybrid effects cannot possibly support the usual combination of failure and non-determinism, but the techniques developed are widely applicable. We have shown how combining effect monads with the Maybe monad allows the representation of a failure, tests and if-then-else statements, arguably representing a sweet spot between complexity and expressivity.
A Some programming languages generated by the hybrid monad

The programming language \( \text{Hyb}(X) \), analysed in Sections 2 and 3, is \textit{time-triggered}: a program \((\dot{x}_1 = t_1, \ldots, \dot{x}_n = t_n \land d)\) terminates precisely when the time of \(d\) is achieved. It is also possible to consider \textit{event-triggered} languages by forcing a program to terminate \textit{as soon as} a certain event occurs. For this, however, one needs to be very strict on the kinds of event allowed, otherwise there might not exist an earliest time at which an event happens.

Consider a finite set of real-valued variables \(X = \{x_1, \ldots, x_n\}\) and denote by \(\text{At}(X)\) the set given by the grammar

\[
\varphi = (x_1 := t, \ldots, x_n := t) \mid (\dot{x}_1 = t, \ldots, \dot{x}_n = t \land \psi),
\]

\[
t = r \mid t \times t + t,
\]

\[
\psi = t \leq t \mid t \leq t \mid \psi \land \psi \mid \psi \lor \psi
\]

where \(x \in X\). Then, for a predicate \(\psi\) the set \([\psi]\subseteq \mathbb{R}^n\) by,

\[
[t_1 \leq t_2] = \{(v_1, \ldots, v_n) \in \mathbb{R}^n \mid [t_1] \leq [t_2]\}
\]

\[
[t_1 \geq t_2] = \{(v_1, \ldots, v_n) \in \mathbb{R}^n \mid [t_1] \geq [t_2]\}
\]

\[
[\psi_1 \land \psi_2] = [\psi_1] \cap [\psi_2]
\]

\[
[\psi_1 \lor \psi_2] = [\psi_1] \cup [\psi_2]
\]

**Proposition A.1.** For every predicate \(\psi\) the set \([\psi]\) is closed in \(\mathbb{R}^n\).

**Proof.** Start with \([t_1 \leq t_2]\). We will to show that for every family of real numbers \((a_i, b_i)\) and \((c, d) \in \mathbb{R} \times \mathbb{R}\) the set

\[
A = \{(v_1, \ldots, v_n) \in \mathbb{R}^n \mid \sum_{i=1}^n a_i v_i + c \leq \sum_{i=1}^n b_i v_i + d\}
\]

is closed in \(\mathbb{R}^n\). Recall that the order relation \(\mathbb{R}_\leq \subseteq \mathbb{R} \times \mathbb{R}\) is closed in the Euclidean space \(\mathbb{R} \times \mathbb{R}\) and consider two maps

\[
f : \mathbb{R}^n \to \mathbb{R}\text{ defined by } f(v_1, \ldots, v_n) = \sum_{i=1}^n a_i v_i + c,
\]

\[
g : \mathbb{R}^n \to \mathbb{R}\text{ defined by } g(v_1, \ldots, v_n) = \sum_{i=1}^n b_i v_i + d
\]

Clearly both \(f\) and \(g\) are continuous since they can be written as compositions of multiplication and addition maps. Moreover, \(A = (f, g)^{-1}(\mathbb{R}_\leq)\) and therefore \(A\) is closed in \(\mathbb{R}^n\).

An analogous reasoning applies to \([t_1 \geq t_2]\) since the order relation \(\mathbb{R}_\geq \subseteq \mathbb{R} \times \mathbb{R}\) is also closed. Finally, for the cases that involve conjunction and disjunction apply the property of closed sets being closed under intersections and finite unions.

**Corollary A.2.** Let us consider a program \((\dot{x}_1 = t_1, \ldots, \dot{x}_n = t_n \land \psi)\), its solution \(\phi : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n\), and a valuation \((v_1, \ldots, v_n) \in \mathbb{R}^n\). If there exists a time instant \(t \in [0, \infty)\) such that \(\phi(v_1, \ldots, v_n, t) \in [\psi]\) then there exists a smallest time instant that also satisfies this condition.

**Proof.** Using Proposition A.1 we can easily show that the set \(\phi(v_1, \ldots, v_n, -)^{-1}([\psi]) \cap [0, t]\) is compact, and consequently that it has a minimum.

We can now introduce the following event-triggered programming language. Define the interpretation map

\[
\text{At}(X) \to \text{End}_{\eta}(\mathbb{R}^n)
\]

as the one that sends \((x_1 := t_1, \ldots, x_n := t_n)\) to the function \(\mathbb{R}^n \to \mathbb{H}(\mathbb{R}^n)\) defined by

\[
(v_1, \ldots, v_n) \mapsto \eta([t_1], \ldots, [t_n])
\]

and \((\dot{x}_1 = t_1, \ldots, \dot{x}_n = t_n \land \psi)\) to the function \(\mathbb{R}^n \to \mathbb{H}(\mathbb{R}^n)\) defined by

\[
(v_1, \ldots, v_n) \mapsto (\phi(v_1, \ldots, v_n, -), d)
\]

where \(d\) is the smallest time instant that intersects \([\psi]\) if \(\phi^{-1}(v_1, \ldots, v_n, -) \cap [\psi] \neq \emptyset\) (Corollary A.2) and 0 otherwise. The free monoid extension of this interpretation map induces a programming language

\[
p = a \in \text{At}(X) | \text{skip} | p;p
\]

whose semantics is a Kleisli representation.

**Example A.3** (Bouncing ball). Consider a bouncing ball dropped at a positive height \(p\) and with no initial velocity \(v\). Due to the gravitational acceleration \(g\), it falls into the ground and then bounces back up, losing part of its kinetic energy in the process. Consider the program,

\[
(p = v, \dot{v} = g \land p \leq 0 \land v \leq 0); (v := v \times -0.5)
\]

which is here denoted by \(b\). The composition

\[
(v := 0, p := 5) ; b ; b ; b
\]

encodes the action of dropping the ball at the height of five meters and letting it bounce exactly three times. The projection on \(p\) of this program yields the plot below.
Our next hybrid programming language is closely related to Höfner’s algebra of hybrid systems [12]. Recall the interpretation map $\text{At}(X) \to \text{End}_d(\mathbb{R}^n)$ for the event-triggered programming language and compose it with the function $\text{End}_d(\mathbb{R}^n) \to \text{End}_d(\mathbb{R}^n \times 2)$ that sends $f$ to the map $g$ defined by

\[ g(v, \bot) = \eta(v, \bot) \quad g(v, \top) = (h, d) \]

where $f(x) = (f(x, -), d)$ and $h(t) = (f(x, t), \bot)$ if $t \neq d$ and $h(t) = (f(x, t), \top)$ otherwise. The free monoid extension of the composition

\[ \text{At}(X) \to \text{End}_d(\mathbb{R}^n \times 2) \]

provides another Kleisli representation for the event-triggered programming language

\[ p = a \in \text{At}(X) \mid \text{skip} \mid p ; p \]

Its sequential composition now behaves essential like sequential composition in [12]: discrete assignments are applied precisely at the end of an evolution, and evolutions produced by two programs are concatenated.

**Example A.4** (Stopwatch). Let $t$ be a variable that denotes time and consider the program composition

\[
\begin{align*}
    t &:= 0; (i = 1 \& t = 5); t := 0; (i = 1 \& t = 10)
\end{align*}
\]

It yields the plot below.

---

## B Definition of multiset-type monads

Given a semiring $S$, the functorial part of the generalised multiset monad $B_S : \text{Set} \to \text{Set}$ is defined by

\[
\begin{align*}
    B_S X &= \{ \phi : X \to S \mid |\text{supp}(\phi)| < \omega \} \\
    B_S f : B_S X &\to B_S Y, \quad \phi \mapsto \lambda y, \sum_{x \in f^{-1}(y)} \phi(x)
\end{align*}
\]

The unit $\eta^B$ is defined at each $X$ by $\eta^B_X(x)(y) = \delta_{x}(y)$, i.e. 1 if $x = y$ and 0 otherwise. The multiplication $\mu^B$ is defined at each $X$ by $\mu^B_X(\Phi)(x) = \sum_{\phi \in \text{supp}(\Phi)} \Phi(\phi) \cdot \phi(x)$, where $\cdot$ is the semiring multiplication.

## C Proofs

**Proof of Theorem 4.1.**

For every $C$-object $X$ there exists a unique arrow $!_X : \emptyset \to X$, and thus any arrow $m : 1 \to F\emptyset$ can be extended to a morphism $F!_X \circ m : 1 \to FX$. The collection of all such morphisms forms a natural transformation $\alpha(m)$. Conversely, we can define $\phi : [C, C](1, F) \to C(1, F\emptyset), \beta \mapsto \beta_0$. It is clear that $\phi(\alpha(m)) = m$, to see that $\alpha(\phi(\beta)) = \alpha(\beta_0) = \beta$ use the naturality of $\beta$.

---

**Proof of Theorem 4.3.**

In $\text{Set}$ binary products distribute over arbitrary coproducts. It follows that

\[
F \times F = \bigcup_{i \in I} \text{hom}(X_i, -) \times \bigcup_{j \in J} \text{hom}(X_j, -)
\]

\[
= \bigcup_{i, j \in I} \text{hom}(X_i, -) \times \text{hom}(X_j, -) = \bigcup_{i, j \in I} \text{hom}(X_i + X_j, -)
\]

where the last step follows from the fact that the contravariant hom functor sends colimits to limits. It follows that

\[
[\text{Set}, \text{Set}](F \times F, F) = \bigcup_{i, j \in I} \text{hom}(X_i + X_j, -), F
\]

\[
\cong \bigcup_{i, j \in I} [\text{Set}, \text{Set}](\text{hom}(X_i + X_j, -), F)
\]

\[
= \bigcup_{i, j \in I} F(X_i + X_j)
\]

where the last step is an application of the Yoneda lemma. Given an element $s \in \prod_{i, j \in I} F(X_i + X_j)$ with components $s_{i, j} \in F(X_i + X_j)$, the associated natural transformation $\alpha^s$ is defined at each $Y$ by

\[
(a, b) \in \text{hom}(X_i, Y) \times \text{hom}(X_j, Y) \mapsto F[a, b](s_{ij})
\]

where $[a, b]$ is the coproduct map $X_i + X_j \to Y$.

---
Proof of Proposition 4.7.
We reason:
\[ F \cong \text{Lan}_n(F_f) \]
\[ (1) \text{Lan}_n \left( \colim \text{El}(F_f)^{\text{op}} \rightarrow \omega^{\text{op}} \rightarrow [\omega, \text{Set}] \right) \]
\[ (2) \text{colim} \left( \text{Lan}_n \text{El}(F_f)^{\text{op}} \rightarrow \omega^{\text{op}} \rightarrow [\omega, \text{Set}] \right) \]
\[ (3) \text{colim} \left( \text{El}(F_f)^{\text{op}} \rightarrow \omega^{\text{op}} \rightarrow [\omega, \text{Set}] \right) \]
where (1) is an application of Theorem 4.6, (2) takes advantage of \( \text{Lan}_n : [\omega, \text{Set}] \rightarrow [\text{Set}, \text{Set}] \) being a left adjoint and (3) uses the property \( \text{Lan}_n(\text{hom}(n, -) : \omega \rightarrow \text{Set}) \cong \text{hom}(n, -) : \text{Set} \rightarrow \text{Set} \).

Proof of Theorem 4.8.
We simply calculate:
\[ [\text{Set}, \text{Set}](F \times F, F) \]
\[ (1) \text{colim} \left( \text{El}(F_f \times F_f)^{\text{op}} \rightarrow \omega^{\text{op}} \rightarrow [\omega, \text{Set}] \right) \]
\[ (2) \text{lim} \left( \text{El}(F_f \times F_f)^{\text{op}} \rightarrow \omega^{\text{op}} \rightarrow [\omega, \text{Set}] \right) \]
\[ (3) \text{lim} \left( \text{El}(F_f \times F_f)^{\text{op}} \rightarrow \omega^{\text{op}} \rightarrow [\omega, \text{Set}] \right) \]
where (1) is an application of Proposition 4.7, (2) follows from the fact that the contravariant hom functor sends colimits to limits, and (3) is an instance of the fact that for every functor \( G : C \rightarrow \text{Set} \) there exists an isomorphism \[ [\text{Set}, \text{Set}](YG^{\text{op}}, F) \cong FG \] which follows from the Yoneda lemma.

Proof of Theorem 4.9.
For clarity we show the result for \( n = 2 \) but the same argument holds for any finite \( n \). Let \( \alpha : B_S \times B_S \rightarrow B_S \). Since we only consider weight functions with finite support the functor \( B_S \) is finitary and therefore Theorem 4.8 applies. Since we’re only considering multisets on the sets \( n \in \omega \), we will write a multiset on \( n \) simply as an \( n \)-tuple of elements of \( S \). Note that when \( B_S \) is applied to a multiset it creates a multiset of multisets which preserves the total mass of multisets. It follows that \( \text{El}(B_S \times B_S) \) has \( S^2 \) orbits since any pair \( (\ell_1, \ldots, \ell_n, s_1, \ldots, s_n) \) \( B_S(n) \times B_S(n) \) of multisets gets mapped to the pair \( (\ell_1, \ldots, \ell_n, s_1, \ldots, s_1) \) of their total mass under the map \( ! : n \rightarrow 1 \).

Let us now compute the limit (2). By definition, the orbits of \( \text{El}(B_S \times B_S) \) cannot communicate with one another via morphisms, thus to compute the limit (2) it is enough to characterize all the possible ‘threads’ in the image under \( B_S \circ U \) of each orbit. Each of these threads correspond to a possible definition of the natural transformation \( \alpha \) on the particular orbit in question. In particular, to specify \( \alpha \) completely one need to describe what it does on each orbit. To do this we choose a pair of arbitrary multisets over \( n \)
\[ ((r_1, \ldots, r_n), (s_1, \ldots, s_n)) \]
This choice fixes the orbit: we are now working in the orbit indexed by \( (\sum^n r_i, \sum^n s_i) \) (and no morphism can make us jump to another orbit). It follows from the common integer divisor property that we can find a common integer divisor \( q \in S \) for the finite sequence \( (r_1, \ldots, r_n, s_1, \ldots, s_n) \) and a sequence \( (l_1, \ldots, l_n, m_1, \ldots, m_n) \) with \( r_i = l_i q, s_i = m_i q, 1 \leq i \leq n \). Now let \( M = \sum^n l_i \) and \( N = \sum^n m_i \), assume w.l.o.g. that \( M \leq N \) (else reverse the roles of \( M \) and \( N \) in what follows). We can write the pair of multisets (8) as the image of the pair of multisets on \( M + N \)
\[ (\mu_1, \mu_2) = \left( \frac{M}{(q_1, \ldots, q_n, 0, \ldots, 0)}, \frac{N}{(0, \ldots, 0, q_1, \ldots, q_n)} \right) \]
under the map \( f : M + N \rightarrow n \) given by
\[ i \mapsto j \text{ if } \left\{ \begin{array}{l} 1 + \sum^n i \leq j \leq M + \sum^n j \text{ or,} \\ M + 1 + \sum^n k \leq i \leq i + M + \sum^n l \end{array} \right. \]
The pair of multisets (9) is invariant under all permutations on \( M + N \) of the shape \( (\pi)(\rho) \) with \( \pi \in \text{Perm}(M) \) and \( \rho \in \text{Perm}(N) \). It follows by naturality that the image of this pair of multisets under \( \sigma_{M+N} \) must also have this invariance property, and thus be of the shape
\[ \frac{M}{(s_1, \ldots, s_i, t, \ldots, t)} \]
for some \( s, t \in S \). Since \( q \) is invertible we can write \( s = qs' \) and \( t = qt' \) for some \( s', t' \in S \). By naturality we now have
\[ \alpha_n((r_1, \ldots, r_n), (s_1, \ldots, s_n)) = \alpha_n(f \times f(\mu_1, \mu_2)) = f(\alpha_{M+N}(\mu_1, \mu_2)) = f(qs', \ldots, qs', qt', \ldots, qt') = (l_1qs' + m_1qt', \ldots, l_nqs' + m_nqt') = (r_1s' + s_1t', \ldots, r_ns' + s_nt') \]
In other words, the image by \( \alpha_n \) is necessarily a weighted sum of the multisets. Thus \( (s', t') \) specifies an element in a thread of the limit (2) associated the orbit indexed by \( (\sum^n r_i, \sum^n s_i) \).

We now show that \( (s', t') \) in fact specifies an entire thread uniquely by showing that, once chosen, this weighting must constant across the entire orbit. This is easily done: take any other pair of multisets in the same orbit \( ((r'_1, \ldots, r'_n), (s'_1, \ldots, s'_n)) \), it follows from the common integer divisor property that we can find a common integer divisor \( q' \) for all the elements in \( ((r'_1, \ldots, r'_n), (s'_1, \ldots, s'_n)) \) and in \( (r_1, \ldots, r_n), (s_1, \ldots, s_n) \). By expressing all elements in terms of \( q' \) we can apply the same
trick as above and find that $\alpha_n$ and $\alpha_p$ will indeed take sums weighted by the same weight $(s', t')$.

We have thus shown that for a each choice $(\sum r_i, \sum s_i)$ of orbit, the corresponding elements of the limit (2) are given by a weighting scheme $(s, t) \in S^2$. It follows that the limit (2), or equivalently the possible specifications of $a$ across every orbit, is given by the set of maps $S^2 \rightarrow S^2$ mapping a pair of total weights to a weighting scheme.

In other words, a map $\phi : S^2 \rightarrow S^2$ defines the natural transformation which sends a tuple of multisets with total masses $(m_1, \ldots, m_n)$ to their sum weighted by the pair $\phi(m_1, \ldots, m_n)$. ■

**Proof of Theorem 4.11.**

It follows from (3) that we need only consider the set

$$[Pol_f, Pol_f] \rightarrow \left( \mathcal{G}_f \times \mathcal{G}_f, \mathcal{G}_f \right).$$

It is not hard to see that $\mathcal{G}_f n$ is homeomorphic to the $n-1$-dimensional simplex with the usual topology and that the set of rational probabilities $\mathcal{G}_f n$ on $n$ forms a dense subset of the $n$-dimensional simplex. It is trivial to adapt the proof of Theorem 4.10 to show that the natural transformations $\mathcal{G}_f \times \mathcal{G}_f \rightarrow \mathcal{G}_f$ are given by the convex combinations $\lambda^k$, where $\lambda$ can now range over any values in $[0, 1]$. It now follows from the fact that objects in $Pol$ are complete that these transformations extend by continuity (since the $(n-1)$-dimensional simplex is compact) to give us all the natural transformations $\mathcal{G}_f \times \mathcal{G}_f \rightarrow \mathcal{G}_f$, and thus all the natural transformation $\mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$. ■

**Proof of Theorem 4.12.**

It is not hard to check that the functor $\mathcal{G} \mathcal{M}$ satisfies both the domain and codomain conditions of [4]. It follows that

$$[Pol, Pol] \rightarrow \left( \mathcal{G} \mathcal{M} \times \mathcal{G} \mathcal{M}, \mathcal{G} \mathcal{M} \right)$$

is described by pairs of rationals $(r_1, r_2) \in [0, 1]^2$ (describing the weight assigned to the ‘+1’ component, or equivalently picking an $n-1$ hyperplane in $n$-dimensional simplex). For a fixed orbit labelled by $(r_1, r_2)$ we can use the proof of Theorem 4.9 to show that the only possible natural assignments $\alpha_n : \mathcal{G}_f n+1 \times \mathcal{G}_f n+1 \rightarrow \mathcal{G}_f n+1$ are weighted sums of the sub-distributions on $n$, i.e. weighted sums given by pairs $(q_1, q_2) \in [0, 1]^2$ such that $q_1 + q_2 \leq 1$. However, each $\alpha_n$ must be a continuous map, and thus vary continuously across the orbits, and the conclusion follows.

**Proof of Theorem 4.13.**

Consider any regular cardinal $\kappa > \lambda$, and the $\kappa$-accessible version $P_\kappa$ of $P$ (taking powersets of cardinality less that $\kappa$, see [3]). Theorem 4.8 generalises completely straightforward to $\kappa$-accessible functors, and $[Set, Set]((P_\kappa)^k, P_\kappa)$ is thus given by the limit (2) (with the inclusion functor $I$ suitably modified). To compute this limit, consider a set $X$ and a collection $(U_i)_{i \in \lambda}$ of subsets of $X$. It is easy to see, by considering what happens at the singleton $1$, that $\mathcal{E}(P_\kappa)^k$ has $2^k$-orbits; one for each element of $(P_\kappa 1)^k$. The object $(X, (U_i)_{i \in \lambda})$ in $\mathcal{E}(P_\kappa)^k$ belongs to the orbit indexed by $(i, (i_X[U_i])_{i \in \lambda})$, where $i_X : X \rightarrow 1$, i.e. to the orbit determined by a subset of indices $J \subseteq 2^\lambda$.

We will show that any thread in the limit (2) must pick an element $\bigcup_{k \in K} U_k$ in the copy of $P_\kappa(X)$ corresponding to $(X, (U_i)_{i \in \lambda})$, for some subset of indices $K \subseteq 2^\lambda$ such that $i \not\in K$ whenever $U_i = \emptyset$, and that this choice must be made consistently across the orbit. This will prove that a natural transformation $(P_\kappa)^k \rightarrow P_\kappa$ is entirely determined by non-increasing maps $2^k \rightarrow 2^k$ (mapping $J$ to $K$). To prove the claim consider the object

$$\left( \bigcup_{i \in \lambda} U_i \times \lambda \right) \in \mathcal{E}(P_\kappa)^k$$

where $\lambda_i = i$ if $U_i \neq \emptyset$ and $\emptyset$ else. It belongs to the same orbit as $(X, (U_i)_{i \in \lambda})$ as it is connected to it by the map $f : \bigcup_i U_i \times \lambda \rightarrow X$, defined by

$$\left( u, i \right) \mapsto \begin{cases} u & \text{if } u \in U_i \\ \text{any } u^i_i & \text{if } u \not\in U_i, U_i \neq \emptyset \\ \text{anything else} & \end{cases}$$

Note now that the object (10) is invariant under all endomorphisms which keep the $\lambda$-component constant. This means that for any thread in the limit (2), the component corresponding to the object (10) must contain $\bigcup_i U_i \times \lambda$, i.e. that it will be a union

$$\bigcup_{k \in K} \left( \bigcup_{i \in \lambda} U_i \times \lambda_k \right)$$

over some $K \subseteq \lambda$. Moreover, since $\lambda_k = \emptyset$ when $U_k = \emptyset$ we do indeed have that $J \supseteq K$ (for the obvious order on $2^\lambda$).

By pushing this component of the thread to the $(X, (U_i)_{i \in \lambda})$-component with $f$, we indeed get $\bigcup_{k \in K} U_k$ as claimed. It remains to check that the choice of $K \subseteq \lambda$ must be made consistently across the orbit. For this consider another object $(X, (V_i)_{i \in \lambda})$ in the same orbit (i.e. $V_i = \emptyset$ iff $U_i = \emptyset$). We can always build the object

$$(X^2, (U_i \times V_i)_{i \in \lambda})$$

which gets mapped to $(X, (U_i)_{i \in \lambda})$ and $(X, (V_i)_{i \in \lambda})$ by the projections maps $\pi_1, \pi_2 : X^2 \rightarrow X$. If the $(X, (U_i)_{i \in \lambda})$-component of a thread in the limit (2) is $\bigcup_{k \in K} U_k$, then the $(X^2, (U_i \times V_i)_{i \in \lambda})$-component of the thread must clearly be $\bigcup_{k \in K} U_i \times V_k$, and so the $(X, (V_i)_{i \in \lambda})$-component must be $\bigcup_{k \in K} V_k$. ■
Finally we need to show that our result for holds for the full powerset monad $P$. Suppose for the sake of contradiction that $\alpha : (P)^I \to P$ is not one of the transformations described above, then this must be witnessed at a set $X$, i.e. there must exist $(U_i)_{i \in I} \in (PX)^I$ such that $\alpha_X(U_i)_{i \in I}$ is not given by one of the transformations above. But since we can always find a regular cardinal $\kappa$ such that $P_\kappa X = PX$, this would define a natural transformation $(P_\kappa)^I \times P_\kappa \to P_\kappa$ which is not of the form described above, a contradiction.

Thus, intuitively, a non-increasing map $\phi : \mathbb{2}^n \to \mathbb{2}^n$ induces the natural transformation $P^n \to P$ that given an $n$-tuple of subsets $(X_1, \ldots, X_n)$ with total masses $(m_1, \ldots, m_n)$ it returns $\cup \{X_i \mid |X_i| \geq m_i, \ldots, m_n \} = 1}$.

**Proof of Proposition 5.1.**

If a natural transformation $\alpha^t$ is commutative then for every $a : X_i \to X$, $b : X_j \to X$ the equation,

$$\alpha^t(a, b) = [a, b] \circ s_{ji} = [b, a] \circ s_{ji}$$

must hold. In particular, the equation,

$$[i_2, i_1] \circ s_{ji} = [1, i_2] \circ s_{ji} = s_{ji} \quad (11)$$

holds for the injections $i_2 : X_i \to X_j + X_i, i_1 : X_j \to X_j + X_i$.

Let us now assume that Equation (11) holds. The goal is to show that for every elements $a : X_i \to X, b : X_j \to X$ the equation $[a, b] \circ s_{ji} = [b, a] \circ s_{ji}$ holds. Thus reason,

$$[b, a] \circ s_{ji} = [b, a] \circ [i_2, i_1] \circ s_{ji} = [a, b] \circ s_{ij}$$

**Proof of Proposition 5.3.**

If a transformation $\alpha^t$ is idempotent then for every $i \in I$, $a : X_i \to X$, the equation $\alpha^t(a, a) = [a, a] \circ s_{ii} = a : X_i \to X$ holds. In particular, we have

$$\nabla \circ s_{ii} = [\text{id}, \text{id}] \circ s_{ii} = \text{id} : X_i \to X_i \quad (12)$$

Now assume that Equation (12) holds and reason,

$$a = a \circ \text{id} = a \circ [\text{id}, \text{id}] \circ s_{ii} = [a, a] \circ s_{ii}$$

**Proposition C.1.** A natural transformation of the type $1 \to F = \bigsqcup_{i \in I} \text{hom}(X_i, -)$ factorizes through some inclusion $\text{hom}(X_k, -) \to F$

with $k \in I$ and $X_k = \emptyset$.

**Proof:** A natural transformation of the type $u : 1 \to F \cong \bigsqcup_{i \in I} \text{hom}(X_i, -)$ has in particular a map

$$u_\emptyset : 1 \to \bigsqcup_{i \in I} \text{hom}(X_i, \emptyset)$$

This entails the existence of an element $k \in I$ such that $X_k = \emptyset$ and $u_\emptyset(*) \in i_k(\text{hom}(X_k, \emptyset))$. Using naturality and the map $a_X : \emptyset \to X$, it follows that for every set $X$, $u_X(*)$ lives in the $k$-th summand of $\bigsqcup_{i \in I} \text{hom}(X_i, X)$.

**Proof of Proposition 5.5.**

If a natural transformation $\alpha^t : F \times F \to F$ has $u : 1 \to F$ as a unit then for every $a : X_i \to X$ the equation below holds

$$a = \alpha^t(a, x) = [a, x] \circ s_{ik} = \alpha^t(x, a) = [x, a] \circ s_{ki}$$

where $x = u_X(*) : X_k = \emptyset \to X$. In particular, one has

$$\text{id} = [\text{id}, \text{id}] \circ s_{ik} = [x, \text{id}] \circ s_{ki}$$

Since $m = [\text{id}, x] : X_i + \emptyset \to X_i$ and $n = [x, \text{id}] : \emptyset + X_i \to X_i$ we obtain,

$$\text{id} = m \circ s_{ik} = n \circ s_{ki} \quad (13)$$

Now assume that Equation (13) holds and reason,

$$a = a \circ \text{id} = a \circ m \circ s_{ik} = a \circ [\text{id}, x] \circ s_{ik} = [a, x] \circ s_{ik}$$

Using an analogous reasoning we obtain $a = [x, a] \circ s_{ki}$.  

**Proof of Proposition 5.7.**

It follows by the naturality of $0 : 1 \to T$ applied to the map $\eta_X : X \to TX$ that $T\eta_X \circ 0_X = 0_TX$, and thus

$$\mu_X \circ 0_TX = 0_X \circ \eta_X \circ 0_TX = 0_X \quad (14)$$

We then have for every Kleisli representation $[-]$ $[0; a] : = \mu_X \circ Ta \circ 0_X \circ i_X$

$$\mu_X \circ 0TX \circ i_X = \mu_X \circ 0_TX \circ i_X \quad \text{Naturality of 0}$$

$$= 0_X \circ i_X \quad \text{By Eq. (14)}$$

$$:= [0]$$

**Proof of Theorem 5.8.**

Assume the axiom $p \circ 0 = 0$ does not hold and recall the bijections below.

$$\begin{array}{c}
\text{Set, Set}(1, T) \cong \text{Set, Set}(\text{hom}(\emptyset, -), T) = T\emptyset
\end{array}$$

We will show that $T\emptyset \not\cong 1$. By assumption, there exists an element $x \in X$ such that

$$\mu_X \circ T(0_X \circ i_X) \circ p(x) \neq 0_X \circ i_X(x).$$

Hence, we have a natural transformation $\alpha : 1 \to T$ whose components are defined as

$$\begin{array}{c}
1 \xrightarrow{x} X \xrightarrow{p} TX \xrightarrow{T(0_X \circ i_X)} TTX \xrightarrow{\mu_X} TX
\end{array}$$

Since $\alpha_X(*) \neq 0_X(*)$ the conditions $T\emptyset \not\cong 1$ holds.

**Proof of Theorem 6.4.**

We proceed by contradiction. Assume that there exists a unit transformation $\eta : \text{id} \to PD$ and a multiplication transformation $\mu : (PD)^2 \to PD$ for which $PD$ is a monad. By definition of a monad we must have that at any $X$

$$\mu_X \circ \eta_{PDX} = \text{id}_{PDX} \quad (15)$$

and similarly,

$$\mu_X \circ PD\eta_X = \text{id}_{PDX}. \quad (16)$$

This set of equations gives us the action of $\mu_X$ on very specific inputs, and we will see that it is enough to generate a contradiction.
As shown in Lemma 6.3, \( \eta \) is either the constant natural transformation to \( \emptyset \) or is defined at \( x \in X \) by \( \eta_X(x) = \{ \delta_x \} \).

Assume first that \( \eta \) is the constant natural transformation to \( \emptyset \), and let \( X \) be any non-empty set. Since the cardinality of \( PDX \) is then at least 2, it is clear that there cannot exist a function \( \mu_X : PDX \rightarrow PDX \) such that Eq. 15 holds for any \( U \in PDX \):

\[
\mu_X \circ \eta_{PDX}(U) = \mu_X(\emptyset) = U.
\]

We therefore immediately get a contradiction if we assume that \( \eta \) is the constant natural transformation to \( \emptyset \).

Next we assume that \( \eta_X(x) = \{ \delta_x \} \) and follow an argument due to Plotkin. Consider the sets \( X = \{ a, b, c, d \} \) and \( Y = \{ a, b \} \), the map \( f : X \rightarrow Y \) defined by \( f(a) = f(c) = a, f(b) = f(d) = b \) and the element \( \left( \frac{1}{2} \delta_{\{a,b\}} + \frac{1}{2} \delta_{\{c,d\}} \right) \in PDYPX \). It is straightforward to compute that

\[
PDYPDf \left( \left( \frac{1}{2} \delta_{\{a,b\}} + \frac{1}{2} \delta_{\{c,d\}} \right) \right) = \left( \delta_{\{a,b\}} \right)
\]

It now follows by naturality and Eq. (15) that

\[
\left( \frac{1}{2} \delta_{\{a,b\}} + \frac{1}{2} \delta_{\{c,d\}} \right) = \mu_X(\left( \frac{1}{2} \delta_{\{a,b\}} + \frac{1}{2} \delta_{\{c,d\}} \right))
\]

and by naturality and Eq. (15), any distribution in \( \mu_X \left( \frac{1}{2} \delta_{\{a,b\}} + \frac{1}{2} \delta_{\{c,d\}} \right) \) must belong to the preimage of \( \{ \delta_a, \delta_b \} \) under \( PDf \), i.e. to

\[
\{ p\delta_a + (1 - p)\delta_c \mid p \in [0,1] \} \cup \{ p\delta_b + (1 - p)\delta_d \mid p \in [0,1] \}
\]

By considering the map \( g : X \rightarrow Y \) defined by \( g(a) = g(d) = a, g(b) = g(c) = b \), and following an argument similar to the one above, we also get

\[
PDYPDg \left( \left( \frac{1}{2} \delta_{\{a,b\}} + \frac{1}{2} \delta_{\{c,d\}} \right) \right) = \left( \delta_{\{a,b\}} \right)
\]

and thus by naturality and Eq. (15), any distribution in \( \mu_X \left( \frac{1}{2} \delta_{\{a,b\}} + \frac{1}{2} \delta_{\{c,d\}} \right) \) must also belong to the preimage of \( \{ \delta_a, \delta_b \} \) under \( PDg \), i.e. to

\[
\{ p\delta_a + (1 - p)\delta_d \mid p \in [0,1] \} \cup \{ p\delta_b + (1 - p)\delta_c \mid p \in [0,1] \}
\]

It follows that \( \mu_X \left( \frac{1}{2} \delta_{\{a,b\}} + \frac{1}{2} \delta_{\{c,d\}} \right) \) contains at most the elements

\[
\{ \delta_a, \delta_b, \delta_c, \delta_d \}
\]

Now consider the map \( h : X \rightarrow Z := \{ a, c \} \) defined by \( h(a) = h(b) = a, h(c) = h(d) = c \). By definition we have

\[
PDPHDh \left( \left( \frac{1}{2} \delta_{\{a,b\}} + \frac{1}{2} \delta_{\{c,d\}} \right) \right) = \left( \frac{1}{2} \delta_{\{a\}} + \frac{1}{2} \delta_{\{c\}} \right)
\]

It now follows by naturality and Eq. (16) that

\[
\{ \frac{1}{2} \delta_{\{a\}} + \frac{1}{2} \delta_{\{c\}} \} \rightarrow \mu_X(\left( \frac{1}{2} \delta_{\{a\}} + \frac{1}{2} \delta_{\{c\}} \right))
\]

and we immediately get a contradiction since

\[
Pdh(\{ \delta_a, \delta_b, \delta_c, \delta_d \}) = \{ \delta_a, \delta_c \}
\]

and \( \{ \frac{1}{2} \delta_a + \frac{1}{2} \delta_c \} \notin \{ \delta_a, \delta_c \} \).

\[\square\]

**Proof of Theorem 6.5.**

It is straightforward to prove that \( \delta : HQ \rightarrow QH \) is a natural transformation. So we will now show that the natural transformation \( \delta : HQ \rightarrow QH \) makes the following diagram commute.

\[
\begin{array}{c}
HQ \\
\delta \downarrow \\
QH
\end{array}
\]

Start with the upper triangle.

\[
\delta_X \circ HQ^X = \delta_X \circ ((\eta^Q \circ) \times \text{id}) = \eta^Q_{HX}
\]

We then proceed with the lower one.

\[
\delta_X \circ \eta^H_{QQX} = \{ \eta_X(a) \in HX \mid a \in - \} = Q \eta^H_{QX}
\]

Finally, we will show that the natural transformation \( \delta : HQ \rightarrow QH \) makes the following diagram commute.

\[
\begin{array}{c}
HQ \\
\delta \downarrow \\
QH \end{array}
\]

Start with the upper square. Consider an element \( (f,d) \in HQQX \). A straightforward calculation provides the following equations.

\[
\delta_X \circ H \mu_X (f,d) = \{ (g,d) \in HX \mid g \in \cup \circ f \}
\]

\[
\mu_{QX} \circ Q \delta_X \circ \delta_QX (f,d) = \bigcup \{ \delta_X(h,d) \mid (h,d) \in HQX \wedge h \in f \}
\]
We will show that both sets are indeed the same. For this, start with an element \((g, d) \in HX\), and reason in the following manner.
\[
g \in \cup \circ f
\]
\[
\iff \forall t \in [0, \infty). \ g(t) \in \cup \circ f
\]
\[
\iff \forall t \in [0, \infty). \ \exists \ Z_t \in f(t). \ g(t) \in Z_t
\]
\[
\iff \exists (h, d) \in HQX. \ g \in h \land h \in f
\]
\[
\iff \exists (h, d) \in HQX. \ (g, d) \in \delta_X(h, d) \land h \in f
\]
In order to keep the notation unburdened, and whenever no ambiguities arise, we will often use a pair \((f, d) \in HX\) as if it were simply the map \(f \in X^{[0, \infty)}\) which constantly outputs \(f(d)\) after \(d\) is achieved.

Let us now concentrate on the lower square. Consider an element \((f, d) \in HHQX\,\) and let \(e = \pi(d(f(d))).\) Then, a straightforward calculation shows that the following equations hold.
\[
\delta_X \circ \mu_{Q} (f, d)
\]
\[
= \{ (g, d + e) \in HX \mid g \in (\theta_{Q} \circ f, d) \oplus (f(d)) \}\]
\[
Q \circ \delta_{\mu_{H}} \circ \delta_{\mu_{X}} (f, d)
\]
\[
= \{ (\theta_{X} \circ g, d) \oplus (g(d)) \mid (g, d) \in HHX \land g \in \delta_{X} \circ f \}
\]
We will show that both sets are actually the same. Start with an element \((h, d + e) \in HX\), and reason as follows.
\[
h \in (\theta_{Q} \circ f, d) \oplus (f(d))
\]
\[
\iff \forall t \leq d . \ h(t) \in \theta_{Q} \circ f(t)
\]
\[
\land \forall t > d . \ h(t) \in (f(d))(t - d)
\]
\[
(\star)
\]
\[
\iff \forall t \leq d . \ h(t) \in Q \theta_{X} \circ \delta_{X} \circ f(t)
\]
\[
\land \exists (g, e) \in \delta_X(f(d)) . \forall t > d . \ h(t) = g(t - d)
\]
\[
\iff \exists (g, d) \in HHX . \ \forall t \leq d . \ g(t) \in \delta_X(f(t))
\]
\[
\land \theta_{X} \circ g(t) = h(t) \land \forall t > d . \ h(t) = (g(d))(t - d)
\]
\[
\iff \exists (g, d) \in HHX . \ (h, d + e) = (\theta_{X} \circ g, d) \oplus (g(d))
\]
\[
\land g \in \delta_{X} \circ f
\]
\[\blacksquare\]
Note that if instead of the functor \(Q\) one would consider the powerset, then the equivalence \((\star)\) would not hold. In particular, the equation
\[
\theta_{P}(X) \circ f = P \theta_{X} \circ \delta_{X} \circ f
\]
would not necessarily hold.