Quark Mass in the Sakai-Sugimoto Model of Chiral Symmetry Breaking

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Abstract

We re-analyze $D8$ brane embeddings in the geometry of a $D4$ brane wrapped on a circle that describe chiral symmetry breaking in a strongly coupled non-supersymmetric gauge theory. We argue that if the holographic fields are correctly interpreted, the original embeddings describe a complex quark mass and condensate in the theory. We show that in this interpretation when a quark mass is present there is a massive pseudo Goldstone boson (pion). A previously identified massless fluctuation is, we argue, not a physical state in the field theory. We also determine the behaviour of the quark condensate as a function of the quark mass.

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1 Introduction

A holographic model of chiral symmetry breaking in a strongly coupled gauge theory has been recently proposed by Sakai and Sugimoto [1, 2] (alternative string realizations of QCD-like chiral symmetry breaking can be found in [3]-[12]). Their model consists of probe $D8/\bar{D}8$ branes in the geometry of a stack of $N D4$ branes wrapped on a circle.

The basic set up, for the simplest configuration, is shown on the left side of Figure 1 - perturbatively the $D8$ and $\bar{D}8$ lie in the non-compact dimensions of the space but are point like and maximally separated on the wrapped circle. They intercept the $D4$ brane wrapped on the circle at the origin. Strings stretching between the $D8s$ and $D4$ provide massless chiral quark fields in the field theory and the $SU(N_f)$ gauge symmetry of the $D8$ and $\bar{D}8$ branes’ world volumes correspond to the chiral symmetry in the gauge theory.

When the full $D4$ brane geometry, with an induced horizon a distance $u_{KK}$ away from the $D4$, is included though, the $D8$ and $\bar{D}8$ brane prefer to combine as shown by the curve reaching into the horizon on the right hand side of Figure 1. There is a non-zero separation between the $D4$ and $D8$ branes suggesting the dynamical generation of a quark mass ($\Sigma_q = Tu_{KK}$). The separate symmetries of the $D8$ and $\bar{D}8$ are broken to a single vector symmetry on the combined world volume. The model therefore encodes chiral symmetry breaking in the pattern of QCD.

For this configuration which brings the $D4$ and $D8$ closest (presumably corresponding to a massless initial quark), the spectrum of fluctuations of the model, which correspond to bound states in the gauge theory, has been well explored in the literature [1, 2, 15]. A massless Goldstone field (essentially the pion) exists in the spectrum of the gauge field on the $D8$ brane.

However, we believe there is some confusion in the literature (eg in [1, 9, 10, 12-14]) about how to interpret the remainder of the embedding solutions shown on the right in Figure 1. For these solutions the minimum $D4$-$D8$ separation increases which suggests that a bare quark mass is being included. A naive analysis [1] seems to show though that the massless pion remains for these configurations. There is a growing mythology that one can not introduce a quark mass simply in this model and that the quark condensate is not fixed by the dynamics.

In this letter we wish to re-analyze these ‘massive’ embeddings and argue that they do in fact correspond to the presence of a bare quark mass and that there is no massless pion after all. We point out that in the weak coupling string construction there is generically an instability to the $D8$ and $\bar{D}8$ combining and breaking chiral symmetry. The symmetry breaking can not be solely dynamically driven therefore (except for the specific case of massless quarks when the $D8$ and $\bar{D}8$ are at anti-podal points on the circle). In the strong coupling holographic description we
will argue that the quark condensate and mass are described by a single, complex holographic field which we will identify. It is crucial to correctly identify the quark mass and condensate within this holographic field. We will show for example that the putative massless pion in these embeddings is in fact a spurious field corresponding to allowing the phase of the quark mass to vary along with the condensate’s phase. This is not a physical mode in the gauge theory.

2 The Sakai-Sugimoto Model - Perturbatively

Let us begin by considering the possible $D8$ and $\bar{D}8$ configurations perturbatively in the Sakai-Sugimoto model (ie in the left hand image of Figure 1). We will argue that for generic separations in $\tau$ the picture in Figure 1 is not the perturbative ground state and is therefore misleading. The straight D8 configurations do minimize the D8s’ world volumes but they do not take into account their mutual interactions (note formally one might think these were down in a large $N$ expansion but the presence of a quark mass is independent of the gauge dynamics and so one must include the interactions).

The crucial observation is that the system, which is non-supersymmetric, is unstable to the D8 and $\bar{D}8$ coming together and annihilating. The set up is equivalent to placing an electron and a positron on a circle where their attraction will bring them together to annihilate. The joining of the D8 and $\bar{D}8$ corresponds to chiral symmetry breaking.

To be definite about the field theory one should consider at least meta-stable configurations. There are two simple set ups. Firstly if the branes lie at antipodal points on the $S^1$, as discussed in the introduction, one would expect the configuration to be static (but still unstable) - here we will generate massless, chiral quarks. On the other hand if we lie the $D8$ and $\bar{D}8$ on top of each other they will annihilate and there will be no quarks - this is only achieved in the field theory at weak coupling by including an infinite quark mass.

For configurations between these two extremes we can try to place the $D8$ and $\bar{D}8$ at separate
values of $\tau$ and we might think this would interpolate between the massless and massive limit. The desire of the $D8$ and $\bar{D}8$ to annihilate suggests an inherent chiral symmetry breaking parameter is present. One would expect though that they would come together all along their length and completely annihilate. The mass term appears to want to grow and remove the quarks from the theory. This presumably reflects the fact that if one makes the mass term in a gauge theory a dynamical scalar then it can minimize the vacuum energy by removing the quarks from the theory.

When we look at the gravity dual of the field theory at strong coupling we will find that there are configurations where asymptotically the $D8$ and $\bar{D}8$ do lie at non-antipodal points (the right hand side of Figure 1). These configurations interpolate between the theory with massless quarks and the theory with no quarks - we think it entirely natural to associate these with the presence of a quark mass and will argue further for this interpretation below. Possibly at strong coupling these configurations are stabilized because the vacuum energy can be lowered by the formation of the quark condensate meaning the theory “likes” to have quarks present.

3 The Sakai-Sugimoto Model - Strong Coupling

The metric for the space around a stack of $D4$ branes in standard coordinates is

$$ds^2 = \left(\frac{u}{R}\right)^3 \left(dx_4^2 + f(u)d\tau^2\right) + \left(\frac{R}{u}\right)^3 \left(\frac{du^2}{f(u)} + u^2 d\Omega_4^2\right)$$

(1)

With $f(u) \equiv 1 - \left(\frac{u_{KK}}{u}\right)^3$. There is a nonzero four-form flux (not important for this analysis) and a dilaton $e^{-\phi} = g_s \left(\frac{u}{R}\right)^{-\frac{3}{4}}$.

Note the coordinate $\tau$ is periodic with the period given by $\delta \tau = \frac{4\pi}{3} \frac{R^3}{u_{KK}^2}$ forming a $S^1$ which is wrapped by the $D4$-branes. This compactification is necessary in order to make the spacetime smooth and complete. There is a horizon at $u = u_{KK}$ (where the radius of the $S^1 \rightarrow 0$) which means the co-ordinate $u$ is restricted to the range $[u_{KK}, \infty]$.

We will change variables to the radial coordinate $z$ where $1 + z^2 = \left(\frac{u}{u_{KK}}\right)^3$ so the geometry becomes

$$ds^2 = \left(\frac{u_{KK}}{R}\right)^3 \left(\sqrt{1 + z^2} dx_4^2 + \frac{z^2}{\sqrt{1 + z^2}} d\tau^2\right) + \left(\frac{R}{u_{KK}}\right)^3 u_{KK}^2 \left(\frac{4}{9}(1 + z^2)^{-\frac{2}{3}} dz^2 + (1 + z^2)^{\frac{1}{3}} d\Omega_4^2\right)$$

(2)
Figure 2: Some regular D8-brane embeddings in the z, τ-plane. We have set $R = 1$ and $u_{KK} = 1$ for the numerical plot.

We can find the embeddings of a probe D8-brane in the above background. These form a family of curves in the $(z, \tau)$-plane which we parameterize as $\tau(z)$. The Dirac-Born-Infeld (DBI) action for the embedding is

$$S_{DBI} = \int_{D8} d^8 \zeta \, e^{-\phi} \sqrt{-\text{Det} \left[ g_{ab} \right]}$$

(3)

This gives

$$S_{DBI} = \text{Vol}(S^4) \int d^4 x \int dz \, \frac{2}{3} g_s u_{KK}^5 \left( \frac{R}{u_{KK}} \right)^{\frac{5}{2}} (1 + z^2)^{\frac{3}{2}} \times \sqrt{1 + \frac{9}{4u_{KK}^2} \left( \frac{u_{KK}}{R} \right)^{\frac{3}{2}} z^2 (1 + z^2)^{\frac{4}{2}} \tau'(z)^2}$$

(4)

One finds the extremal configurations $\tau(z)$ for the D8 obey

$$\tau'(z) = \frac{2}{3} \left( \frac{R}{u_{KK}} \right)^{\frac{5}{2}} \frac{J}{\sqrt{u_{KK}^6 g_s^2 z^2 (1 + z^2)^2 - J^2 u_{KK}^2 z^2 (1 + z^2)^{\frac{1}{2}}}}$$

(5)

Here $J = g_s u_{KK}^4 z_0 (1 + z_0^2)^{\frac{5}{2}}$ is chosen effectively to make the gradient infinite at $z = z_0$. This point is the point of closest approach of the D8 to the horizon at $u = u_{KK}$.

This gives us a one-parameter family of embeddings where choosing a particular value of $z_0$ specifies one particular curve. Some examples are shown in Figure 2 for $z_0$ increasing in factors of $\sqrt{10}$. Note the curve for $z_0 = 0$ consists of two horizontal pieces at $\tau = \pm \frac{\pi}{6} \frac{R^2}{u_{KK}^2}$ plus a vertical piece at $z = 0$ connecting the two. The vertical piece lies on the horizon.

The large $z$ (UV) asymptotic behaviour of the solutions takes the form
\[ \tau = c - \frac{m}{z^3} \]  

We can determine the parameter \( m \) in terms of the value of \( z \) where the \( D8 \) and \( \bar{D}8 \) join \((z = z_0)\) as follows. At large \( z \) the expression for \( \tau'(z) \) becomes

\[ \tau'(z) = + \frac{2}{9} \frac{R^2}{uKKg_s} J z^4 \tag{7} \]

This can be integrated to give

\[ \tau(z) = c - \frac{2}{9} \frac{R^2}{uKKg_s} J \frac{z^3}{z_0} \tag{8} \]

c is the constant and one can see \( m = \frac{2}{9} \frac{R^2}{uKKg_s} J \equiv \frac{2}{9} \frac{R^2}{uKK} z_0 (1 + z_0^2) \frac{z_0}{6} \).

For any given value of \( c \) there is a unique regular embedding which fixes the parameter \( m \) (or equally for each value of \( m \) there is a unique value of \( c \)).

We have argued in Section 2, from the perturbative brane construction, that a mass is present for the non-anti-podal embeddings. Simply from the solutions it appears that the parameter \( m \) measures the quark mass. This parameter is zero for the embedding with the \( D8 \) and \( \bar{D}8 \) at antipodal points on the \( S^1 \). It is infinite as \( z_0 \to \infty \) and the quarks are removed from the theory by the \( D8 \) and \( \bar{D}8 \) lying on top of each other.

Equally the parameter \( c \) appears to measure the quark condensate (it is zero as the mass goes to infinity and largest at \( m = 0 \)).

This identification is not so straightforward though. Fluctuations of the D8 branes contain information about operators of the form \( \bar{q}_Lq_R \) from strings stretched between the D8 and \( \bar{D}8 \) but also operators in the adjoint of the left (or right) handed groups from strings with both ends on one brane. The asymptotic fluctuations therefore are most naturally associated with a coupling and vev for the operators \( \bar{q}_L D^{\mu} \gamma_\mu q_L \) and \( \bar{q}_R D^{\mu} \gamma_\mu q_R \) [9]. However, the linking of the D8 and \( \bar{D}8 \) brane show that these vevs are linked to the condensate and mass term for the quarks. Indeed in a gauge theory one would expect all dynamically generated couplings and masses to be determined by the bare Lagrangian quark’s mass. The identification of \( m \) and \( c \) with the quark condensate and mass is thus indirect but they do nevertheless provide a measure of those
quantities. We propose that this is why the mass shows up here as the normalizable mode and the condensate as the non-normalizable mode.

That these configurations describe the gauge theory at non-zero mass is what we would expect from the usual holographic arguments that if the embedding contains any information about the condensate (which it does because it provides the effective quark mass in the theory via the D8-D4 separation) it must also describe a quark mass. The reason is that if we write the mass terms in the field theory action as

\[ S = \int d^4x \left( m \bar{q}_L q_R + m^* \bar{q}_R q_L \right) \]  

then, requiring the action to be symmetry invariant, we see that \( m^* \) and \( \bar{q}_L q_R \) have the same global symmetry charges. The holographic field associated with the condensate operator shares these charges (there should be a unique operator-field map if the correspondence makes sense) and so necessarily one can’t distinguish \( m^* \) and \( \bar{q}_L q_R \). The presence of two constants in the solutions of the second order equation of motion also naturally match the double role. It would therefore be very surprising if a mass were not present in the configurations - if it weren’t one would have to search for a larger set of D8 embeddings, but there are no such extra embeddings.

We plot \( c \) which we take as a measure of the quark condensate as a function of the quark mass parameter \( m \) in Figure 3 - it takes a numerical value of \( \frac{\pi R^2}{3 u_{KK}} \) for zero quark mass and decreases monotonically as we increase the quark mass.

Figure 3: ‘Chiral condensate’ as a function of quark mass - the parameter \( c \) plotted against \( m \) (with \( R = u_{KK} = 1 \)).
3.1 Vacuum Manifold and Pions in the Massless Limit

Let us first of all review the origin of the pion in the construction with the embedding that brings the D8 branes to \( z = 0 \). The branes lie flat in this limit at antipodal points on the \( \tau \) circle down to \( z = 0 \).

If chiral symmetry is broken there should be a vacuum manifold with different points corresponding to the different possible phases on the quark condensate. The embedding function \( \tau(z) \) is a real number so how do we place a phase on the parameter \( c \) in its asymptotic solution? In [1] the phase was identified with the value of the gauge field \( A_z \) living on the D8 world volume. There is therefore a complex holographic field that describes the condensate

\[
\Phi = \tau(z)e^{iA_z} = |\langle \bar{q}_L q_R \rangle|e^{i\pi}
\]  

(10)

Note here that by \( \langle \bar{q}_L q_R \rangle \) we mean a holographic function of \( z \) that describes the condensate.

To identify the vacuum manifold we should find background solutions (that is, independent of the \( x_4 \) co-ordinates) for \( A_z(z, x_4) \) which correspond to different global choices of the phase \( \pi \). \( A_z \) is described by the DBI action including a U(1) gauge field, which at low energy has the Lagrangian density on the D8 world-volume

\[
L = e^{-\phi}\sqrt{-\det[P(g_{ab})]} \left( -1 - \frac{1}{4}F^{ab}F_{ab} \right)
\]  

(11)

For the massless D8-brane embedding we can take \( \tau(z) = \pm \frac{\delta \tau}{4} \) which evaluates to \( \pm \frac{\pi}{3} \frac{R^2}{u_{KK}} \). Physically, the vertical part of the D8-brane in this case can be neglected because it lies along the horizon where points separated in \( \tau \) are degenerate. Working on the upper branch of the D8-brane (\( \tau(z) = +\frac{\pi}{3} \frac{R^2}{u_{KK}} \)) the action then takes the simple form (neglecting the volume factor coming from the four-sphere angular coordinates - we are working with states of zero spin on the \( S^4 \) here)

\[
S = \frac{1}{2} \int_0^\infty dz \int d^4x \left( e^{-\phi}\sqrt{-g}g^{zz}g^{11} \right) \left( -(\partial_0 A_z)^2 + (\partial_1 A_z)^2 + (\partial_2 A_z)^2 + (\partial_3 A_z)^2 \right)
\]  

(12)

More explicitly this is

\[
S = \frac{3}{4}g_\ast u_{KK}^3 \left( \frac{R}{u_{KK}} \right)^{\frac{3}{2}} \int_0^\infty dz \int d^4x \left( 1 + z^2 \right) \left( -(\partial_0 A_z)^2 + (\partial_1 A_z)^2 + (\partial_2 A_z)^2 + (\partial_3 A_z)^2 \right)
\]  

(13)
It is apparent that $F^{a b}$ and hence the action vanishes if $A_z$ is the only non-zero field and if it is only a function of $z$. Any function of $z$ is allowed. This is an artifact of gauge freedom in the model and one should pick a gauge. For example one could gauge fix by including a term

$$\delta \mathcal{L} = \frac{1}{\xi} e^{-\phi} \sqrt{-\det[P(g_{a b})]} (\nabla_a A^a - \kappa(z))^2$$

(14)

where $\kappa(z)$ is any arbitrary function. Writing $A_z(z, x_4) \equiv g(z) \pi(x_4)$, there is sufficient freedom to pick any functional form of $g(z)$. We will follow the choice of Sakai and Sugimoto and pick

$$g(z) = \frac{C}{1 + z^2}$$

(15)

The solution contains the arbitrary multiplicative factor $C$ since the action is only quadratic in $A_z$. The freedom to pick the constant $C$ in this solution is the freedom to move on the vacuum manifold.

We can now identify the pion field. It should correspond to space-time $(x^\mu)$ dependent fluctuations around the vacuum manifold. In other words we look at solutions of the form

$$A_z(z, x) = \pi(x_4) \times \frac{2}{\sqrt{3\pi}} \frac{1}{1 + z^2}$$

(16)

Substituting this into the action (13) we find a canonically normalized kinetic term for a massless field.

$$S = \int d^4 x \frac{1}{2} (\partial^\mu \pi)^2$$

(17)

This is the pion - the Goldstone mode of the chiral symmetry breaking (although we call it a pion in QCD it is the $\eta'$ which is not a Goldstone due to anomaly arguments - at large $N$ this field is closer in spirit to the pions of QCD).

4 Massive Embeddings

The analysis of $A_z$ in the case of the embeddings that we claim also describe a quark mass appears initially to be identical to the massless case above. We can define, using the gauge freedom, a background configuration where
\[ A_z = \text{constant} \times \frac{1}{\sqrt{e^{-\phi \sqrt{-g_{zz} g_{11}}} \sqrt{1 + z^2}}} \]  
(18)

where the constant is chosen to make the coefficient of the kinetic term equal to \( \frac{1}{2} \) when we integrate over \( z \) to give a 4D action (as is apparent from (12))

We can again find a space-time dependent field

\[ A_z = \pi(x_4) \times \frac{1}{\sqrt{e^{-\phi \sqrt{-g_{zz} g_{11}}} \sqrt{1 + z^2}}} \]  
(19)

which on substitution back into the action appears to give a massless state. Is there therefore a Goldstone even for these embeddings hence invalidating our interpretation of there being a quark mass present?

In fact we have been cavalier. We must be careful to correctly identify what this massless field is. In particular we need an equation equivalent to (10) above telling us what \( A_z \) is holographically describing. Led by (10) the natural choice is

\[ \Phi = \tau(z)e^{iA_z} = \langle \bar{q}_L q_R \rangle + m^* \]  
(20)

We again mean on the right here holographic functions of \( z \) that describe the condensate and mass.

It is now immediately clear that changing \( A_z \) corresponds to changing the phase of both the quark condensate and the quark mass together. This is a spurious transformation that is indeed a flat direction in the potential in the space of such theories and the presence of the vacuum degeneracy and the ‘Goldstone’ is now seen to be natural. To be able to make this transformation though we must be able to change the mass parameter of the theory moving us to a different theory. It is not an allowed transformation with in any one theory. This is not a physical state.

It is worth noting that a holographic field of precisely the form in (20) is found in the chiral symmetry breaking models using D3/D7 [3] and D4/D6 systems [6]. As we argue is the case

\footnote{Note that naively \( A_z \) is singular in these ansaetze but the singularity at the point of closest approach \( z_0 \) in each case is a result of the singularity in the embedding function \( \tau(z) \). The D8 embedding is not really singular at these points though. The singularity results from using the coordinate \( z \) as the coordinate on the D8 world volume - this is clearly inappropriate at \( z_0 \). This in fact is also the case in the massless limit although there the embedding only deviates from a flat embedding at precisely the point \( z_0 \). One could work with \( \tau \) as the coordinate and express the embedding through a function \( z(\tau) \) and then \( z_0 \) would be singularity free but there would be a singularity at large \( z \). We will live with the coordinate induced singularity in our equations. The singularity also makes issues of whether \( \pi(z) \) is an even or odd function tricky - in fact though it is even since the embedding function is (this is necessary for the pion to be a pseudo-scalar - see [11]). The even property is again most easily seen by interchanging the roles of \( \tau \) and \( z \).}
here, those models have a continuous set of degenerate probe configurations even when the quark mass is non-zero - it is precisely the spurious symmetry identified above that moves between these configurations (in those models it is an explicit rotational symmetry of a spacetime plane). The presence of the spurious symmetry that transforms both the mass and the condensate is to be expected because one has the ability to determine the mass through the holographic field. This symmetry ought to be present in the Sakai-Sugimoto model and our interpretation correctly explains it.

To identify the true pion in the Sakai-Sugimoto model we must change the phase on the quark condensate whilst leaving the phase on the mass unchanged. It is clear from (20) that switching on the pion in this way will force a fluctuation in the embedding function $\tau(z)$ - since there is no flat direction for this embedding we expect the pion to acquire a mass. We write

$$\Phi = \langle \bar{q}_L q_R \rangle e^{i\pi(z,x_4)} + m^*$$

and rewrite this in modulus-argument form and expand to quadratic order in the pion field. This tells us via (22) how the embedding and the $A_z$ field are perturbed by the pion. The result is

$$\tau(z, x_4) = \langle \bar{q}_L q_R \rangle + m^* - \frac{\langle \bar{q}_L q_R \rangle m^* \pi(z,x_4)^2}{2(\langle \bar{q}_L q_R \rangle + m^*)}, \quad A_z(z, x_4) = \frac{\langle \bar{q}_L q_R \rangle \pi(z,x_4)}{\langle \bar{q}_L q_R \rangle + m^*}$$

Note that $A_z = \pi(z, x_4)$ in the massless case.

At this point one would like to identify which part of the background embedding function, which we will hence forth call $\tau_0(z)$, represents the $z$ dependence (renormalization group flow) of the condensate and which represents the $z$ dependence of the mass. In general there is no obvious way to make this split (a similar problem exists in other holographic models of chiral symmetry breaking as discussed in [4]). Let us simply write $\tau_0 = c(z) + m(z)$ where $c(z)$ and $m(z)$ are functions that asymptote to the two terms in (6).

To quadratic order in the pion field we have

$$\tau(z, x_4) = \tau_0 - \frac{c(z)m(z)\pi(z,x_4)^2}{2\tau_0}, \quad A_z(z, x_4) = \frac{c(z)\pi(z,x_4)}{\tau_0}$$

These describe perturbations to the embedding and gauge field such that substituting the above into the DBI action will give a term of order zero in $\pi$ which will recover the background embedding functions $\tau_0(z)$ and a term quadratic in $\pi$ giving linear dynamics to our pion field.
As before our strategy will be to write down a 5D action and use the gauge freedom to reduce the kinetic term to canonical form when integrated over \( z \) to give a 4D pion action. This time, for a massive embedding we will have an extra term in the 4D action (from the curvature of the embedding) which will generate a mass squared term.

The kinetic term in the 4D Lagrangian from the DBI action for the gauge field is now (separating variables so \( \pi(z, x_4) \equiv g(z)\pi(x_4) \))

\[
\mathcal{L}_{\text{kinetic}} = \int dz \frac{3}{4} g_s u_{KK}^3 \left( \frac{R}{u_{KK}} \right)^{\frac{3}{2}} \left( \frac{c(z)}{\tau_0(z)} \right)^2 \frac{(1 + z^2)^2 g(z)^2 \partial_\mu \pi(x_4) \partial^\mu \pi(x_4)}{\sqrt{1 + \frac{9}{4u_{KK}^2} \left( \frac{u_{KK}}{R} \right)^3 z^2 (1 + z^2)^{\frac{1}{2}} \tau'_0(z)^2}}
\]

To normalize our 4D kinetic term canonically we should choose

\[
g(z)^2 = \frac{4}{3} \frac{1}{g_s u_{KK}^3} \left( \frac{R}{u_{KK}} \right)^{\frac{3}{2}} \left( \frac{\tau_0(z)}{c(z)} \right)^2 \frac{1}{1 + \frac{1}{2} \int_0^\infty \frac{dx}{1 + x^2}}
\]

For the massive embedding there is though an additional term in the Lagrangian coming from the change in the embedding in (22) above. We find, expanding \( e^{-\phi} \sqrt{-g} \) which is (4) with the perturbation switched on (so we replace \( \tau(z) \) in (4) with \( \tau_0(z) - \frac{c(z)m(z)\pi(z,x_4)^2}{2\tau_0} \) and expand the radical as \( \sqrt{a + b} \sim \sqrt{a} + \frac{b}{2\sqrt{a}} \))

\[
\mathcal{L}_{\text{mass}} = \int dz \frac{3}{4} g_s u_{KK}^3 \left( \frac{u_{KK}}{R} \right)^{\frac{3}{2}} \frac{z^2 (1 + z^2) \tau'_0(z) \left( \frac{m(z)c(z)g(z)^2}{\tau_0(z)} \right)^{\frac{1}{2}}}{\sqrt{1 + \frac{9}{4u_{KK}^2} \left( \frac{u_{KK}}{R} \right)^3 z^2 (1 + z^2)^{\frac{1}{2}} \tau'_0(z)^2}} \pi(x_4)^2
\]

One can see that for the case of massless quarks, where \( \tau'_0(z) = 0 \), the expression correctly gives a vanishing pion mass. Away from that limit it will be non-zero and the pion becomes a pseudo-Goldstone boson. Unfortunately any explicit computation requires a deeper understanding of the split of \( \tau_0 \) into the parts \( m(z) \) and \( c(z) \). Nevertheless this computation provides support for our interpretation of the embeddings given above.

### 5 Summary

We have re-analyzed the family of embeddings of the Sakai-Sugimoto holographic model of chiral symmetry breaking. We have argued that the natural interpretation of the asymptotic
parameters is as the quark mass and condensate - this matches the perturbative string picture and the behaviour of the effective quark mass in the model. We have shown that in this identification a previous argument that all the embeddings have a massless pion is incorrect - the massless state is a spurious field corresponding to allowing the quark mass to change its phase. The true pion state develops a non-zero mass on these embeddings consistent with the interpretation that an explicit quark mass is present.

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