Abstract. A classical and useful way to study controllability problems is the moment method developed by Fattorini-Russell [12, 13], and based on the construction of suitable biorthogonal families. Several recent problems exhibit the same behavior: the eigenvalues of the problem satisfy a uniform but rather ‘bad’ gap condition, and a rather ‘good’ but only asymptotic one. The goal of this work is to obtain general and precise upper and lower bounds for biorthogonal families under these two gap conditions, and so to measure the influence of the ‘bad’ gap condition and the good influence of the ‘good’ asymptotic one. To achieve our goals, we extend some of the general results of Fattorini-Russell [12, 13] concerning biorthogonal families, using complex analysis techniques developed by Seidman [36], Güichal [20], Tenenbaum-Tucsnak [37] and Lissy [26, 27].

1. Introduction

1.1. Presentation of the subject.

Biorthogonal families are a classical tool in analysis. In particular, they play a crucial role in the so-called moment method, which was developed by Fattorini-Russell [12, 13] to study controllability for parabolic equations.

Given any sequence of nonnegative real numbers, \((\lambda_n)_{n \geq 1}\), we recall that a sequence \((\sigma_m)_{m \geq 1}\) is biorthogonal to the sequence \((e^{\lambda_n t})_{n \geq 1}\) in \(L^2(0, T)\) if

\[
\forall m, n \geq 1, \quad \int_0^T \sigma_m(t)e^{\lambda_n t} \, dt = \begin{cases} 
1 & \text{if } m = n \\
0 & \text{if } m \neq n .
\end{cases}
\]

The goal of this paper is to provide explicit and precise upper and lower bounds for the biorthogonal family \((\sigma_m)_{m \geq 1}\) under the following gap conditions:

- a ‘global gap condition’:
  \[
  \forall n \geq 1, \quad 0 < \gamma_{\text{min}} \leq \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \leq \gamma_{\text{max}},
  \]

- and an ‘asymptotic gap condition’:
  \[
  \forall n \geq N^*, \quad \gamma_{\text{min}}^* \leq \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \leq \gamma_{\text{max}}^*,
  \]

where \(\gamma_{\text{max}}^* - \gamma_{\text{min}}^* < \gamma_{\text{max}} - \gamma_{\text{min}}\).

Before explaining why we are interested in such a question, let us describe some of the main results of the literature on this subject.
1.2. The context.

Among the most important applications of biorthogonal families to control theory are those to the null controllability and sensitivity of control costs to parameters. Major contributions in such directions are the following:

- Fattorini-Russell [12, 13], Hansen [21], and Ammar Khodja-Benabdallah-González Burgos-de Teresa [1] studied the existence of biorthogonal sequences and their application to controllability for various equations;
- for nondegenerate parabolic equations and dispersive equations, Seidman [35], Guichal [20], Seidman-Avdonin-Ivanov [36], Miller [31], Tenenbaum-Tucsnak [37], and Lissy [26, 27] studied the dependence of the null controllability cost $C_T$ with respect to the time $T$ (as $T \to 0$, the so-called 'fast control problem') and with respect to the domain, obtaining extremely sharp estimates of the constants $\sigma(\Omega)$ and $C(\Omega)$ that appear in $e^{\sigma(\Omega)/T} \leq C_T \leq e^{C(\Omega)/T}$;
- Coron-Guerrero [8], Glass [17], Lissy [27] investigated the vanishing viscosity problem:
  \[
  \begin{cases}
  y_t + My_x - \varepsilon y_{xx} = 0, & x \in (0, L), \\
  y(0, t) = f(t),
  \end{cases}
  \]
  obtaining sharp estimates of the null controllability cost with respect to the time $T$, the transport coefficient $M$, the size of the domain $L$, and the diffusion coefficient $\varepsilon$;
- in [5, 6], we studied the dependence of the controllability cost with respect to the degeneracy parameter $\alpha$ for the degenerate parabolic equation
  \[
  u_t - (x^\alpha u_x)_x = 0, \quad x \in (0, \ell);
  \]
  and recently Gueye-Lissy [19] studied a 1-D parabolic-hyperbolic degenerate equation.

There is a common feature in these works: they depend on some parameter $\rho$, and this parameter forces the eigenvalues to satisfy (1. 1) (sometimes after normalization) with gap bounds $\gamma_{\min}(p)$ and $\gamma_{\max}(p)$ such that $\gamma_{\min}(p) \to 0$ and/or $\gamma_{\max}(p) \to \infty$.

This fact makes it necessary to have general and precise estimates with respect to the main parameters that appear in the problem.

In [6], we proved the following general result: given $T > 0$ and a family $(\lambda_n)_{n \geq 1}$ of nonnegative real numbers that satisfy the 'global gap condition' (1. 1), then:

- every family $(\sigma_m)_{m \geq 1}$, biorthogonal to $(e^{\lambda_n t})_{n \geq 1}$ in $L^2(0, T)$, satisfies the lower estimate
  \[
  ||\sigma_m||^2_{L^2(0, T)} \geq b_m e^{-2\lambda_n T} e^{2\gamma_{\max} T},
  \]
  with an explicit value of $b_m = b_m(T, \gamma_{\max}, m)$ (rational in $T$);
- there exists a family $(\sigma_m)_{m \geq 1}$, biorthogonal to $(e^{\lambda_n t})_{n \geq 1}$ in $L^2(0, T)$, that satisfies the upper estimate
  \[
  ||\sigma_m||^2_{L^2(0, T)} \leq B_m e^{-2\lambda_n T} e^{2\gamma_{\min} T} e^{C \sqrt{m}},
  \]
  with an explicit value of $B_m = B_m(T, \gamma_{\min}, m)$ (rational in $T$).

This generates several comments:

- concerning the results:
– the bounds (1.3) and (1.4) above describe quite precisely the behavior of the biorthogonal family, in particular in short time; (however, we were not able to make appear the exponential factor $e^{C \frac{\sqrt{\gamma_{\min}}}{\lambda_m}}$ in (1.3)), and it would be interesting to know if (1.3) could be improved;
– estimate (1.4) is in the spirit of [12, 13] but the dependence with respect to $T$ when $T \to 0^+$ is completely explicit;

• concerning the assumptions:
  – assumption (1.1) is very practical to understand the behavior of the biorthogonal family: $\gamma_{\min}$ allows us to estimate $(\sigma_m)_{m \geq 1}$ from above, while $\gamma_{\max}$ allows us to estimate $(\sigma_m)_{m \geq 1}$ from below;
  – assumption (1.1) is in the spirit of the asymptotic development of the eigenvalues used in Tenenbaum-Tucsnak [37] or Lissy [26, 27]:

\[
\lambda_n = rn^2 + O(n).
\]
(Note that (1.1) and (1.5) are close, but one can easily find sequences that satisfy one of them without satisfying the other:
* for example the family \( \{n^2, n^2 + \sqrt{n}\} \) satisfies (1.5) with \( r = \frac{1}{4} \), but does not satisfy (1.1) (the gaps $\sqrt{\lambda_{n+1}} - \sqrt{\lambda_n}$ are not uniformly bounded from below by a positive constant);
* and, on the contrary, (1.5) forces the sequence \( (\sqrt[\lambda_n])_n \) to converge (to $\sqrt{r}$), while one can construct sequences satisfying (1.1) for which \( (\sqrt[\lambda_n])_n \) does not converge, considering, e.g., the recurrence formula $\sqrt[\lambda_{n+1}] = \sqrt[\lambda_n] + \varepsilon_n$ with $\varepsilon_n \in \{1, 2\}$, and choosing $\varepsilon_n$ to be constant on sufficiently large intervals, in such a way that 1 and 2 are limits of subsequences of \( (\sqrt[\lambda_n])_n \).

1.3. Motivations and main results of this paper.
Even though the aforementioned results give a fairly good picture of the properties of the family $\{\sigma_m\}_m$, some delicate issues remain to be analyzed and will be addressed in this paper. For instance, one would like to understand the dependence of the family $\{\sigma_m\}_m$ with respect to relevant parameters that come into play. Typical examples of such problems are the following ones.

• For the 1D degenerate parabolic equation

\[
\begin{align*}
u_t - (x^\alpha u_x)_x &= 0, \quad x \in (0, \ell),
\end{align*}
\]
the eigenvalues $\lambda_{\alpha,n}$ of the associated elliptic operator (with suitable boundary conditions) can be expressed using the zeros of Bessel functions ([18]) and depend on the degeneracy parameter $\alpha \in (0, 2)$. One can then prove (see [5, 7]) that the global gap condition (1.1) is satisfied only with

\[
\gamma_{\max}(\alpha) \geq c(2 - \alpha)^{2/3},
\]
with $c > 0$, while the asymptotic gap condition (1.2) is satisfied with

\[
\gamma_{\max}^*(\alpha) \leq c^*(2 - \alpha),
\]
where $c^* > 0$, after the rank

\[
N^*(\alpha) = \frac{1}{2 - \alpha};
\]
in this case

\[
\frac{\gamma_{\max}(\alpha)}{\gamma_{\max}^*(\alpha)} \to +\infty \quad \text{as} \quad \alpha \to 2^-;
\]
hence it is natural to think that the better asymptotic gap (1. 2) could be used to improve the estimate (1. 3) of the associated biorthogonal sequences, but the fact that $N^*(\alpha) \to +\infty$ as $\alpha \to 2^-$

is certainly to be taken into account.

• In 2D problems such as the Grushin equation (see [2, 3]), where the solution is decomposed into Fourier modes, one has to give uniform bounds for a certain sequence of elliptic problems, the eigenvalues of which satisfy (1. 1) and (1. 2) with some $\gamma_{\min}(m)$, $\gamma_{\min}^*(m)$ and $N^*(m)$ such that

$$\frac{\gamma_{\min}(m)}{\gamma_{\min}^*(m)} \to 0 \text{ as } m \to \infty$$

and

$$N^*(m) \to +\infty \text{ as } m \to \infty;$$

once again, it is natural to think that the better asymptotic gap (1. 2) could be used to improve the estimate (1. 4) of the associated biorthogonal sequence, but the fact that $N^*(m) \to +\infty$ as $m \to \infty$ is certainly to be taken into account.

The above discussion motivates the general question whether estimates (1. 3) and (1. 4) can be improved when (1. 1) is combined with the asymptotic condition (1. 2). This is exactly what we prove in this paper: roughly speaking, (1. 3) and (1. 4) hold true replacing $\gamma_{\min}$ by $\gamma_{\min}^*$ and $\gamma_{\max}$ by $\gamma_{\max}^*$. Moreover, the fact that the 'good' gap condition (1. 2) holds true only after the $N^*$ first eigenvalues has a cost, and we obtain a precise estimate for that cost. Our main results (Theorem 2.1 and 2.2) are the following: under (1. 2), we prove that:

• every family $(\sigma_m)_{m \geq 1}$, biorthogonal to $(e^{\lambda_n t})_{n \geq 1}$ in $L^2(0, T)$, satisfies the lower estimate

\begin{equation}
\|\sigma_m\|_{L^2(0, T)}^2 \geq b^*_m e^{-2\lambda_m T} e^{\frac{2}{C_0 \sqrt{\gamma_{\min}}}},
\end{equation}

where the 'cost' $b^*_m = b^*_m(T, \gamma_{\max}, \gamma_{\max}^*, N^*, m)$ is a rational function of $T$ that we determine explicitly, and

• there exists a biorthogonal family that satisfies

\begin{equation}
\|\sigma_m\|_{L^2(0, T)}^2 \leq B^*_m e^{-2\lambda_m T} e^{\frac{C_0}{\sqrt{\gamma_{\min}}}} e^{\frac{C_0 \sqrt{\lambda_m}}{\gamma_{\min}^*}},
\end{equation}

where $C_0 > 0$ is a universal constant and $B^*_m(T, \gamma_{\min}, \gamma_{\min}^*, N^*, m)$ is a rational function of $T$ that we determine explicitly.

Let us observe that the presence of the exponential factors $e^{\frac{2}{C_0 \sqrt{\gamma_{\max}}}}$ and $e^{\frac{C_0 \sqrt{\lambda_m}}{\gamma_{\min}^*}}$ in (1. 6) and (1. 7) is quite natural and has already been pointed out by Seidman-Avdonin-Ivanov [36], Tenenbaum-Tucsnak [37], and Lissy [26, 27] (see also Haraux [22] and Komornik [24] for a closely related context). On the other hand, the precise estimate of the behavior of $b^*_m$ and $B^*_m$ with respect to parameters, that we develop in this paper, is completely new and will be crucial for the sensitivity analysis of control costs to be performed in [7].

Our proofs are based on complex analysis techniques and hilbertian methods developed by Seidman-Avdonin-Ivanov [36] and Güichal [20]. We have also used an idea from Tenenbaum-Tucsnak [37] and Lissy [26, 27], based on the introduction of an extra parameter depending on $T$ and the gap conditions.

Finally, as mentioned above, the gap conditions (1. 1)-(1. 2) are practical to obtain such estimates from above and below, but one could investigate more general
conditions, of the form
(1. 8) \[ \alpha_n n \leq \lambda_{n+1} - \lambda_n \leq \beta_n n, \]
with \( \liminf_n \alpha_n > 0 \) and \( \limsup \beta_n < \infty \). This would include the case of assumptions (1. 1) and (1. 5) and could allow to study more general problems.

1.4. Plan of the paper.

The paper is organized as follows:

- in section 2, we state our results;
- section 3 is devoted to the proof of Theorem 2.1 (construction of a biorthogonal family and derivation of upper bounds);
- section 4 is devoted to the proof of Theorem 2.2 (lower bounds for biorthogonal families).

2. Setting of the problem and main results

2.1. Existence of a suitable biorthogonal family and upper bounds.

We will prove the following result, that in some sense precises results of Fattorini and Russell [12, 13] (in short time) and are in the spirit of results of Tenenbaum-Tucsnak [37], Lissy [26, 27] (with a slightly weakened assumption on the eigenvalues).

**Theorem 2.1.** Assume that

\[ \forall n \geq 1, \quad \lambda_n \geq 0, \]

and that there is some \( 0 < \gamma_{\text{min}} < \gamma_{\text{min}}^* \) such that

(2. 1) \[ \forall n \geq 1, \quad \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \geq \gamma_{\text{min}}, \]

and

(2. 2) \[ \forall n \geq N^*, \quad \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \geq \gamma_{\text{min}}^*. \]

Denote

(2. 3) \[ M^* := (1 - \frac{\gamma_{\text{min}}}{\gamma_{\text{min}}^*})(N^* - 1). \]

Then there exists a family \( (\sigma_n^+)_{m \geq 1} \) which is biorthogonal to the family \( (e^{\lambda_n t})_{n \geq 1} \) in \( L^2(0, T) \):

(2. 4) \[ \forall m, n \geq 1, \quad \int_0^T \sigma_m^+(t)e^{\lambda_n t} dt = \delta_{mn}. \]

Moreover, it satisfies: there is some universal constant \( C \) independent of \( T, \gamma_{\text{min}}, \gamma_{\text{min}}^*, N^* \) and \( m \) such that, for all \( m \geq 1 \), we have

(2. 5) \[ \| \sigma_m^+ \|^2_{L^2(0,T)} \leq e^{-2\lambda_m T} e^{\gamma_{\text{min}}^*/\gamma_{\text{min}}} e^{C \frac{\lambda_{N^*}}{\gamma_{\text{min}}^*}} B^*(T, \gamma_{\text{min}}, \gamma_{\text{min}}^*, N^*, m), \]

where

(2. 6) \[ B^*(T, \gamma_{\text{min}}, \gamma_{\text{min}}^*, N^*, m) \]

\[
= \begin{cases} 
C_u \left( \frac{(8M^*)^m}{(\lambda_m(\gamma_{\text{min}})^2T)^m} + 1 \right) C_u M^* e^{C_u \frac{\lambda_{N^*}}{\gamma_{\text{min}}^*} \sqrt{\gamma_{\text{min}}}} \left( 1 + \frac{\lambda_{N^*}}{\gamma_{\text{min}}^*} T^2 \right) & \text{if } T \leq \frac{1}{\gamma_{\text{min}}^*} \\
C_u \left( \frac{(\gamma_{\text{min}}^*)^m}{\lambda_m^m} (8M^*)^m \right) + 1 \right) C_u M^* e^{C_u \frac{\lambda_{N^*}}{\gamma_{\text{min}}^*} \sqrt{\gamma_{\text{min}}}} \left( (\gamma_{\text{min}})^2 + (\gamma_{\text{min}}^*)^3 \right) & \text{if } T \geq \frac{1}{\gamma_{\text{min}}^*} 
\end{cases}. 
\]
Remark 2.1. Theorem 2.1 completes and improves several earlier results, in particular Theorem 1.5 of Fattorini-Russell [13] and [6] expliciting the dependence of the $L^2$ bound with respect to $\gamma_{\text{min}}$, $\gamma_{\text{min}}^*$ and in short time. It is useful in several problems, in which $\gamma_{\text{min}} \to 0$ with respect to some parameter, which occurs in several cases, see, e.g. [14], [2]. We will use the construction that was used by Seidman, Avdonin and Ivanov in [36], which has the advantage to be completely explicit (which is not the case for the construction of [12, 13, 14, 21, 1], since there is a contradiction argument), combined with some ideas coming from the construction of Tenenbaum-Tucsnak [37] and Lissy [26], adding some parameter, in order to obtain precise results.

2.2. General lower bounds.

We generalize a result of Güichal [20] to prove the following

Theorem 2.2. Assume that

$$\forall n \geq 1, \lambda_n \geq 0,$$

and that there are $0 < \gamma_{\text{min}} \leq \gamma_{\text{max}}^* \leq \gamma_{\text{max}}$ such that

$$\forall n \geq 1, \quad \gamma_{\text{min}} \leq \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \leq \gamma_{\text{max}},$$

and

$$\forall n \geq N_*, \quad \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \leq \gamma_{\text{max}}^*.$$

Then any family $(\sigma_m^*)_{m \geq 1}$ which is biorthogonal to the family $(e^{\lambda_n t})_{n \geq 1}$ in $L^2(0,T)$ (hence that satisfies (2. 4)) satisfies:

$$\forall m, n \geq 1, \quad \int_0^T \sigma_m^*(T-t)e^{\lambda_n(T-t)} dt = \delta_{mn},$$

and

$$\forall m, n \geq 1, \quad \int_0^T \left( \sigma_m^* - \infty \right) e^{-\lambda_n t} dt = \delta_{mn},$$

hence the family $(s_m)_{m \geq 0}$ defined by

$$s_m(t) := \sigma_m^* - \infty e^{\lambda_n T}$$

is biorthogonal to the family $(e^{-\lambda_n t})_{n \geq 1}$ in $L^2(0,T)$. Now extend $s_m$ by 0 outside $(0,T)$, and consider its Fourier transform

$$\forall z \in \mathbb{C}, \quad F(s_m)(z) := \int_0^\infty s_m(t) e^{-izt} dt.$$
For all \( m \geq 1 \), \( \mathcal{F}(s_m) \) is the Fourier transform of a compactly supported function, hence it is an entire function over \( \mathbb{C} \), and it satisfies
\[
\forall m, n \geq 1, \quad \mathcal{F}(s_m)(-i\lambda_n) = \delta_{mn},
\]
and it is of exponential type:
\[
|\mathcal{F}(s_m)(z)| \leq \left( \int_0^T |s_m(t)| \, dt \right) e^{T|z|};
\]
and also
\[
\mathcal{F}(s_m)(z) = \int_{-T/2}^{T/2} s_m(\tau + \frac{T}{2}) e^{-iz(\tau + \frac{T}{2})} \, d\tau,
\]
hence
\[
\mathcal{F}(s_m)(-z) e^{-iz\frac{T}{2}} = \int_{-T/2}^{T/2} s_m(\tau + \frac{T}{2}) e^{iz\tau} \, d\tau,
\]
and
\[
|\mathcal{F}(s_m)(-z) e^{-iz\frac{T}{2}}| \leq \left( \int_{-T/2}^{T/2} |s_m(\tau + \frac{T}{2})| \, d\tau \right) e^{\frac{T}{2}|z|}.
\]
Now we recall the Paley-Wiener theorem ([39]): if \( f : \mathbb{C} \rightarrow \mathbb{C} \) is an entire function of exponential type, such that there exist nonnegative constants \( C, A \) such that
\[
\forall z \in \mathbb{C}, \quad |f(z)| \leq Ce^{A|z|},
\]
and if \( f \in L^2(\mathbb{R}) \), then there exists \( \phi \in L^2(-A, A) \) such that
\[
f(z) = \int_{-A}^A \phi(t) e^{izt} \, dt.
\]
One of the objects of [36] is to prove the existence of a sequence \( (f_m)_m \) of entire functions satisfying
\[
(3.1) \quad \begin{cases} 
    \forall m, n \geq 1, & f_m(-i\lambda_n) = \delta_{mn}, \\
    \forall z \in \mathbb{C}, & |f_m(-z) e^{-iz\frac{T}{2}}| \leq C_m e^{\frac{T}{2}|z|}, \\
    \forall m \geq 1, & f_m \in L^2(\mathbb{R})
\end{cases}
\]
(see Theorem 2 and Lemma 3 in [36]) under some general assumptions on the sequence \( (\lambda_n)_n \). If we can apply such a result in our context (hence with our sequence \( (\lambda_n)_n \)), then the two last properties together with the Paley-Wiener theorem will imply that there exists some \( \phi_m \in L^2(-\frac{T}{2}, \frac{T}{2}) \) such that
\[
f_m(-z) e^{-iz\frac{T}{2}} = \int_{-T/2}^{T/2} \phi_m(\tau) e^{iz\tau} \, d\tau,
\]
hence
\[
f_m(z) = \int_0^T \phi_m(t - \frac{T}{2}) e^{-izt} \, dt,
\]
and then
\[
\int_0^T \phi_m(t - \frac{T}{2}) e^{-\lambda_n t} \, dt = f_m(-i\lambda_n) = \delta_{mn},
\]
hence \( \phi_m(t - \frac{T}{2}) \) will be biorthogonal to the family \( (e^{-\lambda_n t})_n \), and \( (\sigma_m^+(t))_m \) defined by
\[
\sigma_m^+(t) = \phi_m(t - \frac{T}{2}) e^{-\lambda_n T}
\]
will be biorthogonal to the family \( (e^{\lambda_n T})_n \) in \( L^2(0, T) \), as desired. Moreover
\[
\|\sigma_m^+\|_{L^2(0, T)}^2 = e^{-2\lambda_n T} \int_{-T/2}^{T/2} \phi_m(t)^2 \, dt \leq C e^{-2\lambda_n T} \|f_m\|_{L^2(\mathbb{R})}^2
\]
using the Parseval theorem.

Now, it remains to construct such entire functions $f_m$. The idea is to consider the natural infinite product that satisfies the first condition of (3.1), $f_m(-i\lambda_n) = \delta_{mn}$, and to multiply it by a so-called ‘mollifier’, in such a way that the other two conditions of (3.1) will be also satisfied. Hence one has to estimate the growth of the natural infinite product, and then to choose a choose a suitable mollifier. This is what is performed in [36]. For our problem, our task will be to add the dependency into the parameters $\gamma_{\text{min}}$, $\gamma_{\text{min}}^*$ and $T$, and to understand specifically the behaviour of the natural infinite product, the mollifier and at the end of $\|\sigma^+_m\|_{L^2(0,T)}$ with respect to $\gamma_{\text{min}}$ and $T$. We will modify a little the construction of [36], in order to obtain optimal results in our context, see Lemma 3.4, and specifically the definition (3.19) of the mollifier, where the additionnal parameter $N'$ will be chosen of the size $\frac{1}{T(\gamma_{\text{min}})^2}$, see (3.30).

3.2. The counting function.

Consider

$$\forall \rho > 0, \quad N_n(\rho) := \text{card} \{ k, 0 < |\lambda_n - \lambda_k| \leq \rho \}.$$ 

We prove the following:

**Lemma 3.1.** a) Assume that the gap assumption (2.1) is satisfied; then

$$(3.2) \quad \forall n \geq 0, \forall \rho > 0, \quad N_n(\rho) \leq 2 \frac{\sqrt{\rho}}{\gamma_{\text{min}}}.$$ 

b) Assume that the gap assumptions (2.1)-(2.2) are satisfied; then

- when $n = N^*$:

$$(3.3) \quad \forall \rho > 0, \quad N_{N^*}(\rho) \leq \begin{cases} \frac{2\sqrt{\rho}}{\gamma_{\text{min}}} + \frac{\sqrt{\rho}}{\gamma_{\text{min}}} & \text{if } \rho \leq \lambda_{N^*}, \\ N^* - 1 + \frac{\sqrt{\rho}}{\gamma_{\text{min}}} & \text{if } \rho \geq \lambda_{N^*} \end{cases}.$$ 

- when $n > N^*$:

$$(3.4) \quad \forall n > N^*, \forall \rho > 0, \quad N_n(\rho) \leq \begin{cases} \frac{2\sqrt{\rho}}{\gamma_{\text{min}}} & \text{if } \rho \leq \lambda_n - \lambda_{N^*}, \\ N^* - n + \frac{\sqrt{\rho}}{\gamma_{\text{min}}} & \text{if } \lambda_n - \lambda_{N^*} \leq \rho \leq \lambda_n, \\ \rho - 1 + \frac{\sqrt{\rho}}{\gamma_{\text{min}}} & \text{if } \rho \geq \lambda_n \end{cases}.$$ 

- when $n < N^*$

- when $\lambda_n \leq \lambda_{N^*} - \lambda_n$, then

$$(3.5) \quad \forall \rho > 0, \quad N_n(\rho) \leq \begin{cases} \frac{2\sqrt{\rho}}{\gamma_{\text{min}}} & \text{if } \rho \leq \lambda_n, \\ N^* - n + \frac{\sqrt{\rho}}{\gamma_{\text{min}}} & \text{if } \lambda_n \leq \rho \leq \lambda_{N^*} - \lambda_n, \\ N^* - 1 + \frac{\sqrt{\rho}}{\gamma_{\text{min}}} & \text{if } \rho \geq \lambda_{N^*} - \lambda_n \end{cases}.$$ 

- when $\lambda_n \geq \lambda_{N^*} - \lambda_n$, then

$$(3.6) \quad \forall \rho > 0, \quad N_n(\rho) \leq \begin{cases} \frac{2\sqrt{\rho}}{\gamma_{\text{min}}} & \text{if } \rho \leq \lambda_{N^*} - \lambda_n, \\ N^* - n + \frac{\sqrt{\rho}}{\gamma_{\text{min}}} & \text{if } \lambda_{N^*} - \lambda_n \leq \rho \leq \lambda_n, \\ N^* - 1 + \frac{\sqrt{\rho}}{\gamma_{\text{min}}} & \text{if } \rho \geq \lambda_n \end{cases}.$$ 

**Remark 3.1.** The main point in (3.3)-(3.6) is to observe that $N_n$ behaves as $\frac{\sqrt{\rho}}{\gamma_{\text{min}}}$ as $\rho \to +\infty$, and precising the needed additionnal constants.

**Proof of Lemma 3.1.** Take $k > n$. Then

$$\lambda_k - \lambda_n = \sqrt{\lambda_k} - \sqrt{\lambda_n} = (\sqrt{\lambda_k} - \sqrt{\lambda_n})(\sqrt{\lambda_k} + \sqrt{\lambda_n}),$$
and the gap assumption (2.1) insures that
\[ \sqrt{\lambda_k} - \sqrt{\lambda_n} \geq (k-n)\gamma_{\min}, \quad \sqrt{\lambda_k} + \sqrt{\lambda_n} \geq (k-n)\gamma_{\min} + 2\sqrt{\lambda_n}, \]
hence
\[ \lambda_k - \lambda_n \geq (k-n)^2\gamma_{\min}^2, \]
and
\[ k > n \text{ and } \lambda_k - \lambda_n \leq \rho \implies k - n \leq \frac{\sqrt{\rho}}{\gamma_{\min}}. \]
Similarly,
\[ k < n \text{ and } \lambda_n - \lambda_k \leq \rho \implies n - k \leq \frac{\sqrt{\rho}}{\gamma_{\min}}. \]
Hence
\[ N_n(\rho) \leq 2\sqrt{\frac{\rho}{\gamma_{\min}}}. \]
This proves (3.2).

Now we prove (3.3)-(3.6): let us introduce
\[ N_n^+ (\rho) := \text{card} \{ k > n, \lambda_k - \lambda_n \leq \rho \}, \quad N_n^- (\rho) := \text{card} \{ k < n, \lambda_n - \lambda_k \leq \rho \}. \]
We distinguish the three cases.

- **When** \( n = N^* \): from the previous study, we see that
  \[ N_n^+ (\rho) \leq \frac{\sqrt{\rho}}{\gamma_{\min}}, \quad \text{and} \quad N_n^- (\rho) \leq \begin{cases} \sqrt{\frac{\rho}{\gamma_{\min}}} & \text{if } \rho \leq \lambda_{N^*} - \lambda_n \\ N^* - 1 & \text{if } \rho \geq \lambda_{N^*} \end{cases}, \]
  this gives that
  \[ N_{N^*} (\rho) \leq \begin{cases} \sum_{k=n+1}^{\rho} N_{n+1}^- (\rho) + \sqrt{\frac{\rho}{\gamma_{\min}}} \quad & \text{if } \rho \leq \lambda_{N^*} \\ N^* - 1 + \sqrt{\frac{\rho}{\gamma_{\min}}} & \text{if } \rho \geq \lambda_{N^*} \end{cases}, \]
  which gives (3.3).

- **When** \( n > N^* \): now we have
  \[ N_n^+ (\rho) \leq \frac{\sqrt{\rho}}{\gamma_{\min}}, \]
  and
  \[ N_n^- (\rho) \leq \begin{cases} \sqrt{\frac{\rho}{\gamma_{\min}}} & \text{if } \rho \leq \lambda_n - \lambda_{N^*} \\ \frac{\sqrt{\rho}}{\gamma_{\min}} & \text{if } \lambda_n - \lambda_{N^*} \leq \rho \leq \lambda_n \\ n - 1 & \text{if } \rho \geq \lambda_n \end{cases}, \]
  which gives (3.4).

- **When** \( n < N^* \): now we have
  \[ N_n^- (\rho) \leq \begin{cases} \sqrt{\frac{\rho}{\gamma_{\min}}} & \text{if } \rho \leq \lambda_n \\ n - 1 & \text{if } \rho \geq \lambda_n \end{cases}, \]
  and
  \[ N_n^+ (\rho) \leq \begin{cases} \sqrt{\frac{\rho}{\gamma_{\min}}} & \text{if } \rho \leq \lambda_{N^*} - \lambda_n \\ N^* - n + \sqrt{\frac{\rho}{\gamma_{\min}}} & \text{if } \rho \geq \lambda_{N^*} - \lambda_n \end{cases}, \]
  hence when \( \lambda_n \leq \lambda_{N^*} - \lambda_n \) we have
  \[ N_n (\rho) \leq \begin{cases} \frac{2\sqrt{\rho}}{\gamma_{\min}} & \text{if } \rho \leq \lambda_n \\ n - 1 + \sqrt{\frac{\rho}{\gamma_{\min}}} & \text{if } \lambda_n \leq \rho \leq \lambda_{N^*} - \lambda_n \\ N^* - 1 + \sqrt{\frac{\rho}{\gamma_{\min}}} & \text{if } \rho \geq \lambda_{N^*} - \lambda_n \end{cases}, \]
  which gives (3.5), and similar estimates when \( \lambda_n \geq \lambda_{N^*} - \lambda_n \), which give (3.6). \(\square\)
Before going further, let us give another estimate of the counting function, which reveals to be more practical and more natural, since it gives a better understanding of the role of the different parameters:

**Lemma 3.2.** Assume that the gap assumptions (2. 1)-(2. 2) are satisfied; then

- when $n = N^*$:

  $$\forall \rho > 0, \quad N_{N^*}(\rho) \leq \begin{cases} \frac{\sqrt{\gamma}}{\gamma_{\min}} \frac{\sqrt{\gamma}}{\gamma_{\min}} + (1 - \frac{\gamma_{\min}}{\gamma_{\max}})(N^* - 1) + 2 \frac{\sqrt{\gamma}}{\gamma_{\min}} & \text{if } \rho \leq \lambda_{N^*} \\ \frac{\sqrt{\gamma}}{\gamma_{\min}} \frac{\sqrt{\gamma}}{\gamma_{\min}} + (1 - \frac{\gamma_{\min}}{\gamma_{\max}})(N^* - 1) + 2 \frac{\sqrt{\gamma}}{\gamma_{\min}} & \text{if } \rho \geq \lambda_{N^*} \end{cases}$$

- when $n > N^*$:

  $$\forall \rho > 0, \quad N_n(\rho) \leq \begin{cases} \frac{2\sqrt{\gamma}}{\gamma_{\min}} \frac{\sqrt{\gamma}}{\gamma_{\min}} + \frac{\sqrt{\gamma}}{\gamma_{\min}} & \text{if } \rho \leq \lambda_n - \lambda_{N^*} \\ (1 - \frac{\gamma_{\min}}{\gamma_{\max}})(N^* - 1) + 2 \frac{\sqrt{\gamma}}{\gamma_{\min}} & \text{if } \rho \geq \lambda_n \end{cases}$$

- when $n < N^*$:

  $$\forall \rho > 0, \quad N_n(\rho) \leq \begin{cases} \frac{2\sqrt{\gamma}}{\gamma_{\min}} \frac{\sqrt{\gamma}}{\gamma_{\min}} + \frac{\sqrt{\gamma}}{\gamma_{\min}} & \text{if } \rho \leq \max\{\lambda_n, \lambda_{N^*} - \lambda_n\} \\ (1 - \frac{\gamma_{\min}}{\gamma_{\max}})(N^* - 1) + (1 + \sqrt{2}) \frac{\sqrt{\gamma}}{\gamma_{\min}} & \text{if } \rho \geq \max\{\lambda_n, \lambda_{N^*} - \lambda_n\} \end{cases}$$

and also

$$\forall \rho > 0, \quad N_n(\rho) \leq \begin{cases} \frac{2\sqrt{\gamma}}{\gamma_{\min}} \frac{\sqrt{\gamma}}{\gamma_{\min}} + \frac{\sqrt{\gamma}}{\gamma_{\min}} & \text{if } \rho \leq \lambda_{N^*} \\ (1 - \frac{\gamma_{\min}}{\gamma_{\max}})(N^* - 1) + 2 \frac{\sqrt{\gamma}}{\gamma_{\min}} & \text{if } \rho \geq \lambda_{N^*} \end{cases}$$

**Remark 3.2.** Lemma 3.2 enlightens the role of the quantity $(1 - \frac{\gamma_{\min}}{\gamma_{\max}})(N^* - 1)$ (denoted $M^*$ in (2. 3)); when $\gamma_{\min} = \gamma_{\max}$ or if $N^* = 1$, this quantity is equal to zero, and we logically find estimates similar to the ones of Lemma 3.1 (i.e. the '1 gap condition'); in the more interesting case where $\gamma_{\min} < \gamma_{\max}$ and $N^* > 1$, this quantity measures the increase of the counting function with respect to the '1 gap condition'.

Let us note also that we expect that (3. 9) holds true with 2 instead of $1 + \sqrt{2}$, however we could not prove it in full generality.

**Proof of Lemma 3.2.**

- When $n = N^*$, it is sufficient to note that

  $$\sqrt{\lambda_{N^*}} \geq \gamma_{\min}(N^* - 1),$$

  hence, when $\rho \geq \lambda_{N^*}$, we have

  $$N^* - 1 = (1 - \frac{\gamma_{\min}}{\gamma_{\max}})(N^* - 1) + \frac{\gamma_{\min}}{\gamma_{\max}}(N^* - 1) \leq (1 - \frac{\gamma_{\min}}{\gamma_{\max}})(N^* - 1) + \frac{\sqrt{\lambda_{N^*}}}{\gamma_{\min}} \leq (1 - \frac{\gamma_{\min}}{\gamma_{\max}})(N^* - 1) + \frac{\sqrt{\gamma}}{\gamma_{\min}};$$

  this and (3. 3) imply (3. 7).

- When $n > N^*$: when $\rho \geq \lambda_n$, we have

  $$\sqrt{\lambda_n} \geq \sqrt{\lambda_{N^*}} + (n - N^*)\gamma_{\min} \geq (N^* - 1)\gamma_{\min} + (n - N^*)\gamma_{\min},$$

  hence

  $$n - 1 \leq \sqrt{\frac{\lambda_n}{\gamma_{\min}}} + (1 - \frac{\gamma_{\min}}{\gamma_{\max}})(N^* - 1) \leq \sqrt{\frac{\rho}{\gamma_{\min}}} + (1 - \frac{\gamma_{\min}}{\gamma_{\max}})(N^* - 1);$$

  this estimate and (3. 4) imply (3. 8).
• When \( n < N^* \), we obtain (3.10) proceeding in the same way: when \( \rho \leq \lambda N^* \), then clearly \( N_n(\rho) \) is less than the number of terms that would be at both sides, for which the gap of their square root would be \( \gamma_{min} \), hence

\[
N_n(\rho) \leq 2 \sqrt{\frac{\rho}{\gamma_{min}}};
\]

when \( \rho \geq \lambda N^* \), then clearly one has all the \( N^* - 1 \) first terms, and the others, for which the gap of their square root is \( \gamma_{min} \), hence

\[
N_n(\rho) \leq N^* - 1 + \frac{\sqrt{\rho}}{\gamma_{min}};
\]

but then

\[
N^* - 1 = (1 - \frac{\gamma_{min}}{\gamma_{min}})(N^* - 1) + \frac{\gamma_{min}}{\gamma_{min}}(N^* - 1) 
\leq (1 - \frac{\gamma_{min}}{\gamma_{min}})(N^* - 1) + \frac{\sqrt{\lambda N^*}}{\gamma_{min}}(N^* - 1) + \frac{\sqrt{\rho}}{\gamma_{min}},
\]

which gives (3.10);

• finally we prove (3.9): in the same way, if \( \rho \leq \max\{\lambda_n, \lambda N^* - \lambda_n\} \) one has immediately

\[
N_n(\rho) \leq \frac{2 \sqrt{\rho}}{\gamma_{min}};
\]

when \( \rho \geq \max\{\lambda_n, \lambda N^* - \lambda_n\} \), then we already know from (3.5) and (3.6) that

\[
N_n(\rho) \leq N^* - 1 + \frac{\sqrt{\rho}}{\gamma_{min}},
\]

hence

\[
N_n(\rho) \leq (1 - \frac{\gamma_{min}}{\gamma_{min}})(N^* - 1) + \frac{\gamma_{min}}{\gamma_{min}}(N^* - 1) + \frac{\sqrt{\rho}}{\gamma_{min}};
\]

since

\[
\sqrt{\lambda N^*} \geq \gamma_{min}(N^* - 1)
\]

and

\[
\rho \geq \max\{\lambda_n, \lambda N^* - \lambda_n\} \implies \rho \geq \frac{1}{2}(\lambda_n + (\lambda N^* - \lambda_n)) = \frac{1}{2}\lambda N^*;
\]

we deduce that

\[
\gamma_{min}(N^* - 1) \leq \sqrt{\lambda N^*} \leq \sqrt{2\rho};
\]

hence

\[
N_n(\rho) \leq (1 - \frac{\gamma_{min}}{\gamma_{min}})(N^* - 1) + (1 + \sqrt{2})\frac{\sqrt{\rho}}{\gamma_{min}},
\]

which is (3.9).

This concludes the proof of Lemma 3.2. \( \square \)

### 3.3. A Weierstrass product.

Motivated by [36], we consider

\[
(3.11) \quad F_m(z) := \prod_{k=1,k \neq m}^{\infty} \left(1 - \left(\frac{i z - \lambda_m}{\lambda_k - \lambda_m}\right)^2\right).
\]

Then the growth in \( k \) of \( \lambda_k \) ensures that this infinite product converges uniformly over all the compact sets, hence \( F_m \) is well-defined and entire over \( C \). Moreover

\[
F_m(-i\lambda_n) = \prod_{k=1,k \neq m}^{\infty} \left(1 - \left(\frac{\lambda_n - \lambda_m}{\lambda_k - \lambda_m}\right)^2\right) = \begin{cases} 
0 & \text{if } m \neq n, \\
1 & \text{if } m = n,
\end{cases}
\]
hence

\[(3.12)\quad \forall m, n \geq 1, \quad F_m(-i\lambda_n) = \delta_{mn}.
\]

We are going to estimate the growth of $F_m$. We prove the following

**Lemma 3.3.** a) Assume that the gap assumption (2.1) is satisfied. Then the function $F_m$ satisfies the following growth estimate: there is some uniform constant $C_u$ (independent of $m$, $\gamma_{\text{min}}$, and $z$) such that

\[(3.13)\quad \forall z \in \mathbb{C}, \quad |F_m(z)| \leq e^{C_u \sqrt{\gamma_{\text{min}}}} e^{C_u \sqrt{|z|}}.
\]

b) Assume that the gap assumptions (2.1)-(2.2) are satisfied. Then the function $F_m$ satisfies the following growth estimate: there is some uniform constant $C_u$ (independent of $m$, $\gamma_{\text{min}}$, $\gamma_{\text{min}}^*$ and $z$), such that

\[(3.14)\quad \forall m \geq 1, \forall z \in \mathbb{C}, \quad |F_m(z)| \leq B_m q_m(|z|) e^{C_u \sqrt{|z|}},
\]

with

\[(3.15)\quad B_m \leq e^{C_u \sqrt{\gamma_{\text{min}}}} e^{C_u \sqrt{\lambda_{\text{max}}}}
\]

and

\[(3.16)\quad q_m(|z|) \leq \left(3 + 2 \frac{|z|^2}{\lambda_{\text{max}}} \right)^{M^*},
\]

where $M^*$ has been defined in (2.3).

**Remark 3.3.** The main point in Lemma 3.3 is to obtain estimates of the growth of $F_m$ in $e^{C_u \sqrt{\gamma_{\text{min}}}}$ under (2.1)-(2.2), with explicit constants (given in (3.15) and (3.16)), that will help us in the following. Comparing with (3.13), this gives a better idea of the improvement brought by 'large' gap $\gamma_{\text{min}}^*$ and of the price to pay due to the 'small' gap $\gamma_{\text{min}}$ for the $N^*$ first eigenvalues. In fact we will first prove the following better estimates: (3.14) holds true with

\[(3.17)\quad B_m = \begin{cases} e^{\left(\frac{\delta}{\gamma_{\text{min}}} + \frac{C_u}{\gamma_{\text{min}}}\right) \sqrt{\gamma_{\text{min}}}} \sqrt{\lambda_{\text{max}} - \lambda_{\text{min}}} & \text{if } m < N^* \\ e^{\left(\frac{\delta}{\gamma_{\text{min}}} + \frac{C_u}{\gamma_{\text{min}}}\right) \sqrt{\lambda_{\text{max}}}} & \text{if } m = N^* \\ e^{\left(\frac{\delta}{\gamma_{\text{min}}} + \frac{C_u}{\gamma_{\text{min}}}\right) \sqrt{\lambda_{\text{min}} - \lambda_{\text{max}}}} & \text{if } m > N^* \end{cases}
\]

and

\[(3.18)\quad q_m(|z|) = \begin{cases} \left(3 + 2 \frac{|z|^2}{\lambda_{\text{max}}} \right)^{M^*} & \text{if } m < N^* \\ \left(3 + 2 \left(\frac{|z|^2}{\lambda_{\text{max}}} + \frac{|z|^2}{\lambda_{\text{min}}}\right) \right)^{M^*} & \text{if } m = N^* \\ \left(3 + 2 \frac{|z|^2}{\lambda_{\text{max}}} \right)^{M^*} & \text{if } m > N^* \end{cases}
\]

and this easily implies (3.15) and (3.16).

**Proof of Lemma 3.3.** Note that

\[F_m(z - i\lambda_m) = \prod_{k=1, k \neq m} \left(1 + \frac{z^2}{(\lambda_k - \lambda_m)^2}\right),
\]

hence (following [36])
Then we distinguish several cases:

- Under only (2. 1) we deduce from (3. 2) that

\[
2 \int_0^\infty \frac{N_m(\rho)}{\rho} \frac{|z|^2}{|z|^2 + \rho^2} d\rho \leq \frac{4}{\gamma_{\min}} \int_0^\infty \frac{1}{\sqrt{\rho}} \frac{|z|^2}{|z|^2 + \rho^2} d\rho = \left( \frac{4}{\gamma_{\min}} \int_0^\infty \frac{1}{\sqrt{s}} \frac{1}{1 + s^2} ds \right) \sqrt{|z|}.
\]

Then changing \( z - i\lambda_m \) into \( z \),

\[
\ln |F_m(z)| \leq \left( \frac{4}{\gamma_{\min}} \int_0^\infty \frac{1}{\sqrt{s}} \frac{1}{1 + s^2} ds \right) (\sqrt{|z|} + \sqrt{\lambda_m}),
\]

which gives (3. 13).

- Under (2. 1)-(2. 2) and when \( m = N^* \), we derive from (3. 7) that

\[
2 \int_0^\infty \frac{N_N^*(\rho)}{\rho} \frac{|z|^2}{|z|^2 + \rho^2} d\rho \leq 2 \int_0^{\lambda_{N^*}} \frac{N_N^*(\rho)}{\rho} \frac{|z|^2}{|z|^2 + \rho^2} d\rho + 2 \int_{\lambda_{N^*}}^\infty \frac{N_N^*(\rho)}{\rho} \frac{|z|^2}{|z|^2 + \rho^2} d\rho
\]

\[
\leq 2 \int_0^{\lambda_{N^*}} \frac{1}{\gamma_{\min}} \frac{|z|^2}{\rho(1 + s^2)} d\rho + 2 \int_{\lambda_{N^*}}^\infty \frac{1}{\gamma_{\min}} \frac{|z|^2}{\rho(1 + s^2)} d\rho
\]

\[
\leq 2 \int_0^{\lambda_{N^*}} \frac{1}{\gamma_{\min}} \frac{1}{(1 + s^2)} ds + 2 \int_{\lambda_{N^*}}^\infty \frac{1}{\gamma_{\min}} \frac{1}{(1 + s^2)} ds
\]

\[
\leq 2 \frac{1}{\gamma_{\min}} \int_0^{\lambda_{N^*}} \frac{|z|^2}{\rho(1 + s^2)} d\rho + 2 \lambda_{N^*} \frac{\ln(1 + |z|^2 + \lambda_{N^*}^2)}{\lambda_{N^*}} + 4 \sqrt{|z|} \int_0^\infty \frac{1}{\sqrt{s}} \frac{1}{1 + s^2} ds
\]

\[
\leq 4 \lambda_{N^*} \frac{1}{\gamma_{\min}} + M^* \ln(1 + |z|^2 + \lambda_{N^*}^2) + c_u \sqrt{|z|} + \sqrt{\lambda_{N^*}},
\]

Then changing \( z - i\lambda_{N^*} \) into \( z \),

\[
\ln |F_N^*(z)| \leq 4 \sqrt{\lambda_{N^*}} \left( \frac{1}{\gamma_{\min}} + \frac{1}{\gamma_{\min}^*} \right) + M^* \ln(1 + 2|z|^2 + \lambda_{N^*}^2) + c_u \sqrt{|z|} + \sqrt{\lambda_{N^*}},
\]

which gives (3. 14) with the \( B_m \) and \( q_m \) given in (3. 17) and (3. 18).

- Under (2. 1)-(2. 2) and when \( m > N^* \), applying the same method, we derive from (3. 8) that

\[
2 \int_0^\infty \frac{N_m(\rho)}{\rho} \frac{|z|^2}{|z|^2 + \rho^2} d\rho \leq 4 \left( \frac{1}{\gamma_{\min}} + \frac{2}{\gamma_{\min}^*} \right) \sqrt{\lambda_m} - \frac{4}{\gamma_{\min}} \sqrt{\lambda_m - \lambda_{N^*}}
\]

\[
+ M^* \ln(1 + |z|^2 + \lambda_{N^*}^2) + c_u \sqrt{|z|} + \sqrt{\lambda_{N^*}}.
\]

Then changing \( z - i\lambda_m \) into \( z \), we obtain (3. 14) with the related \( B_m \) and \( q_m \) given in (3. 17) and (3. 18).
The Proof of Lemma 3.4 follows by elementary analysis techniques. In the following we will make the following choices:

A sequence of holomorphic functions satisfying

\[ 3.5. \]

Then changing \( z - i\lambda_m \) into \( z \), we obtain (3. 14) with the related \( B_m \) and \( q_m \) given in (3. 17) and (3. 18). \( \square \)

3.4. A suitable mollifier.

Motivated by [36], we made in [6] the following construction: consider \( T' > 0, \ N' > 1, \ a_k := \frac{C_{N', T'}}{k} \) with

\[ C_{N', T'} := \frac{T'}{2 \sum_{k=N'}^{\infty} \frac{1}{k^2}}, \]

in order that

\[ \sum_{k=N'}^{\infty} a_k = \frac{T'}{2}, \]

and finally

\[ (3. 19) \quad P_{N', T'}(z) := e^{izT'} \prod_{k=N'}^{\infty} \cos(a_k z). \]

Then we have the following

**Lemma 3.4. ([6])**

1. The regularity and the growth of \( P_{N', T'} \) over \( \mathbb{C} \): The function \( P_{N', T'} \) is entire over \( \mathbb{C} \) and satisfies

\[ (3. 20) \quad \begin{cases} P_{N', T'}(0) = 1, \\ \forall z \in \mathbb{C} \text{ such that } \exists z \geq 0, \ |P_{N', T'}(z)| \leq 1, \\ \forall z \in \mathbb{C}, \ |e^{-izT'} P_{N', T'}(z)| \leq e^{iz T'}. \end{cases} \]

2. The behaviour of \( P_{N', T'} \) over \( \mathbb{R} \): there exist \( \theta_0 > 0, \theta_1 > 0 \), both independent of \( N' \) and \( T' \), such that \( P_{N', T'} \) satisfies

\[ (3. 21) \quad \begin{cases} \left( \frac{C_{N', T'}|x|}{\theta_0} \right)^{1/2} + 1 \geq N' \implies \ln |P_{N', T'}(x)| \leq -\theta_1 \left( \frac{C_{N', T'}|x|}{\theta_0} \right)^{1/2}, \\ \left( \frac{C_{N', T'}|x|}{\theta_0} \right)^{1/2} + 1 \leq N' \implies \ln |P_{N', T'}(x)| \leq -\theta_2 \left( \frac{C_{N', T'}|x|}{\theta_0} \right)^{2}. \end{cases} \]

3. The behaviour of \( P_{N', T'} \) over \( i\mathbb{R} \): there is some constant \( \theta_2 > 0 \), independent of \( N' \) and \( T' \), such that \( P_{N', T'} \) satisfies

\[ (3. 22) \quad \forall x \in \mathbb{R}_+, \quad P_{N', T'}(ix) \geq e^{-\theta_2 \sqrt{C_{N', T'} x}}. \]

The Proof of Lemma 3.4 follows by elementary analysis techniques. In the following we are going to use the mollifier \( P_{N', T'} \) to construct the biorthogonal family.

3.5. A sequence of holomorphic functions satisfying (3. 1).

Consider

\[ (3. 23) \quad \forall m \geq 0, \forall z \in \mathbb{C}, \quad f_{m, N', T'}(z) := F_m(z) \frac{P_{N', T'}(-z)}{P_{N', T'}(i\lambda_m)}. \]

We will make the following choices:
• for $T'$:
\[
T' := \min\{T, \frac{1}{(\gamma^*_{\min})^2}\};
\]

• for $N'$; we choose it such that
\[
N' \geq 2 + \frac{\theta_3}{(\gamma^*_{\min})^2 T'}
\]

with a suitable $\theta_3$ (independent of $T > 0$ and of $m \geq 0$, and given in (3.29)).

Then we will prove the following

Lemma 3.5. When $T'$ and $N'$ satisfy (3.24) and (3.25), the functions $f_{m,N',T'}$ are entire and satisfy the following properties:

• for all $m, n \geq 1$, we have
\[
f_{m,N',T'}(-i\lambda_n) = \delta_{mn};
\]

• for all $m \geq 1$, for all $\varepsilon > 0$, there exists $C_{m,N',T',\varepsilon} > 0$ such that
\[
\forall z \in \mathbb{C}, \quad |f_{m,N',T'}(-z)e^{-izT'}| \leq C_{m,N',T',\varepsilon} e^{(\frac{2}{T'} + \varepsilon)|z|};
\]

• for all $m \geq 1$, $f_{m,N',T'} \in L^2(\mathbb{R})$.

Then we will be in position to apply the Paley-Wiener theorem and to construct the desired biorthogonal sequence.

Proof of Lemma 3.5. First, the function $f_{m,N',T'}$ is well-defined since $P_{N',T'} > 0$ on $i\mathbb{R}_+$, and is entire since $F_m$ and $P_{N',T'}$ are entire. Next, using (3.12), we have (3.26). Next, concerning the exponential type: using (3.14) and (3.20), we have
\[
|f_{m,N',T'}(-z)e^{-izT'}| = |F_m(-z)||P_{N',T'}(z)e^{-izT'}|e^{-iz\frac{1}{P_{N',T'}(i\lambda_m)}} \leq \frac{1}{P_{N',T'}(i\lambda_m)} B_m q_m(|z|) e^{\frac{C_u \sqrt{|z|}}{\gamma^*_{\min}} e^{(\frac{2}{T'} + \varepsilon)|z|}};
\]

but for all $\varepsilon > 0$ we have
\[
C_u \frac{\sqrt{|z|}}{\gamma^*_{\min}} = C_u \frac{\sqrt{\varepsilon|z|}}{\gamma^*_{\min} \sqrt{\varepsilon}} \leq \frac{C_u}{2(\gamma^*_{\min})^2 \varepsilon} + \frac{\varepsilon}{2}|z|,
\]
and
\[
q_m(|z|) \leq C_m' e^{\frac{\varepsilon|z|}{2}}
\]
which imply (3.27). Finally, concerning the behaviour over $\mathbb{R}$, we deduce from (3.13), (3.21) and (3.22) that, if $|x|$ is large enough, then
\[
|f_{m,N',T'}(x)| \leq \frac{1}{P_{N',T'}(i\lambda_m)} B_m q_m(|x|) e^{\frac{C_u \sqrt{|x|}}{\gamma^*_{\min}} e^{-\frac{q_m}{C_{N',T'}|x|}}} \leq \frac{C_u}{\gamma^*_{\min}} e^{-\frac{q_m}{C_{N',T'}|x|}} \left(\frac{C_{N',T'}|x|}{q_m}\right)^{1/2},
\]

hence $f_{m,N',T'} \in L^2(\mathbb{R})$ if
\[
\frac{C_u}{\gamma^*_{\min}} - \frac{\theta_1}{8} \left(\frac{C_{N',T'}}{\theta_0}\right)^{1/2} < 0,
\]
which is true choosing $T'$ and $N'$ satisfying (3.24) and (3.25): indeed,
\[
C_{N',T'} = \frac{T'}{2} \sum_{k=N'}^{1} \frac{1}{k^2},
\]
and
\[
\frac{1}{N'} = \int_{N'}^{\infty} \frac{1}{y^2} \, dy \leq \sum_{k=N'}^{\infty} \frac{1}{k^2} \leq \int_{N'-1}^{\infty} \frac{1}{y^2} \, dy = \frac{1}{N' - 1};
\]
hence
\[(3.28)\]
\[
\frac{(N' - 1)T'}{2} \leq C_{N',T'} \leq \frac{N'T'}{2}.
\]
Hence, if
\[(3.29)\]
\[
(N' - 1)T' > \frac{\theta_3}{(\gamma_{\min}^*)^2} \quad \text{with} \quad \theta_3 := \frac{2\theta_0 C_u^2}{\theta_1^2},
\]
we obtain that \(f_{m,N',T'} \in L^2(\mathbb{R})\). And one easily verifies that \(T', N'\) satisfying (3.24) and (3.25) satisfy also (3.29). This completes the proof of Lemma 3.5. \(\square\)

3.6. The resulting biorthogonal sequence.

With our choices, the function \(x \mapsto f_{m,N',T'}(-x)e^{-ixT'/2}\) is in \(L^2(\mathbb{R})\), and we can consider its Fourier transform \(\phi_{m,N',T'}\):
\[
\phi_{m,N',T'}(\xi) := \frac{1}{2\pi} \int \! f_{m,N',T'}(-x)e^{-ix\xi}e^{-ix\xi} \, dx.
\]
It is well-defined since \(f_{m,N',T'} \in L^2(\mathbb{R})\), and the Paley-Wiener theorem ([39] p. 100) shows that \(\phi_{m,N',T'}\) is compactly supported in \([-\frac{T'}{2} - \varepsilon, \frac{T'}{2} + \varepsilon]\) (thanks to (3.27)). Since this is true for all \(\varepsilon > 0\), \(\phi_{m,N',T'}\) is compactly supported in \([-\frac{T'}{2}, \frac{T'}{2}]\).

To obtain good results, we will choose \(N'\) satisfying the stronger property:
\[(3.30)\]
\[
2 + \frac{\theta_3}{(\gamma_{\min}^*)^2T'} \leq N' \leq 4 + \frac{\theta_3}{(\gamma_{\min}^*)^2T'}.
\]
Then we have the following

Lemma 3.6. Take \(T'\) and \(N'\) satisfying (3.24) and (3.30), and consider
\[(3.31)\]
\[
\sigma_{m,N',T'}^+(t) := \phi_{m,N',T'}\left(\frac{T'}{2} - t\right)e^{-\lambda_mT}.
\]
Then the family \((\sigma_{m,N',T'}^+)^{m \geq 1}\) is biorthogonal to the family \((e^{\lambda_m t})^{n \geq 1}\) in \(L^2(0,T)\):
\[(3.32)\]
\[
\forall m, n \geq 1, \quad \int_0^T \sigma_{m,N',T'}^+(t)e^{\lambda_n t} \, dt = \delta_{mn}.
\]
Moreover, it satisfies: there is some universal constant \(C_u\) independent of \(T, \gamma_{\min}, \gamma_{\min}^*, N^*\) and \(m\) such that, for all \(m \geq 1\), we have
\[(3.33)\]
\[
\|\sigma_{m,N',T'}^+\|_{L^2(0,T)} \leq C_u e^{-2\lambda_m T} e^{C_u \gamma_{\min}^*} B(T, \gamma_{\min}, \gamma_{\min}^*, N^*, m),
\]
where \(B(T, \gamma_{\min}, \gamma_{\min}^*, N^*, m)\) is given by (2.6).

Proof of Lemma 3.6. The Fourier inversion theorem gives that
\[
f_{m,N',T'}(-x)e^{-ixT'/2} = \int_\mathbb{R} \phi_{m,N',T'}(\xi)e^{ix\xi} \, d\xi = \int_{-T'/2}^{T'/2} \phi_{m,N',T'}(\xi)e^{ix\xi} \, d\xi.
\]
Then
\[
\int_0^T \sigma_{m,N',T'}^+(t)e^{\lambda_m t} \, dt = \int_0^T \phi_{m,N',T'}\left(\frac{T'}{2} - t\right)e^{-\lambda_m T} e^{\lambda_m t} \, dt
\]
\[
= e^{-\lambda_m T} \int_{-T'/2}^{T'/2} \phi_{m,N',T'}(\xi)e^{i\lambda_m(\xi - \frac{T'}{2})} \, d\xi = e^{-\lambda_m T} e^{i\lambda_m \frac{T'}{2}} \int_{-T'/2}^{T'/2} \phi_{m,N',T'}(\xi)e^{-\lambda_m \xi} \, d\xi
\]
\[
= e^{-\lambda_m T} e^{i\lambda_m \frac{T'}{2}} f_{m,N',T'}(-i\lambda_m)e^{i\lambda_m \frac{T'}{2}} = f_{m,N',T'}(-i\lambda_m)e^{\lambda_m T} = \delta_{mn}.
\]
This gives (3.32). Concerning (3.33), we note that the Parseval equality gives
(3. 34) \[ \int_{\mathbb{R}} |f_{m, N', T'}(x)|^2 \, dx = \int_{\mathbb{R}} |f_{m, N', T'}(-x)|^2 \, dx \]
\[ = 2\pi \int_{\mathbb{R}} |\phi_{m, N', T'}(\xi)|^2 \, d\xi = 2\pi \int_{-T/2}^{T/2} |\phi_{m, N', T'}(\xi)|^2 \, d\xi. \]
Hence
\[ \|\sigma_{m, N', T'}\|^2_{L^2(0, T)} = e^{-2\lambda_m T} \int_{-T/2}^{T/2} |\phi_{m, N', T'}(\xi)|^2 \, d\xi = \frac{1}{2\pi} e^{-2\lambda_m T} \int_{\mathbb{R}} |f_{m, N', T'}(x)|^2 \, dx. \]

We need to estimate precisely the last integral. Denote
\[ X_{N', T'} := \frac{\theta_0(N' - 1)^2}{C_{N', T'}}. \]

Using (3. 13), (3. 21) and (3. 22), we have
\[ \int_{\mathbb{R}} |f_{m, N', T'}(x)|^2 \, dx = \int_{|x| \leq X_{N', T'}} |f_{m, N', T'}(x)|^2 \, dx + \int_{|x| > X_{N', T'}} |f_{m, N', T'}(x)|^2 \, dx \]
\[ \leq 2 e^{2\beta_1 \sqrt{C_{N', T'}}} \| \phi_{m, N', T'} \|^2_{L^2(0, T)} e^{-2\beta_1 \left( \frac{C_{N', T'}^2}{\tilde{\eta}^2} \right)^2} \]
\[ + \int_{X_{N', T'}}^\infty q_m(x) e^{2\beta_1 \sqrt{X_{N', T'}}} e^{-2\beta_1 \left( \frac{C_{N', T'}^2}{\tilde{\eta}^2} \right)^2} \, dx =: I_m^{(1)} + I_m^{(2)}. \]

First we estimate \( I_m^{(1)} \); we denote \( \theta_i \) various constants independent of all the other parameters; we have
\[ \int_{0}^{X_{N', T'}} q_m(x) e^{2\beta_1 \sqrt{X_{N', T'}}} e^{-2\beta_1 \left( \frac{C_{N', T'}^2}{\tilde{\eta}^2} \right)^2} \, dx \]
\[ \leq q_m(X_{N', T'}) e^{2\beta_1 \sqrt{X_{N', T'}}} \int_{0}^{X_{N', T'}} e^{-2\beta_1 \left( \frac{C_{N', T'}^2}{\tilde{\eta}^2} \right)^2} \, dx \]
\[ \leq C_{\eta} q_m(X_{N', T'}) e^{2\beta_1 \sqrt{X_{N', T'}}} \left( \frac{N'}{T'} \right)^{3/2} \frac{1}{C_{N', T'}} \]
\[ \leq C_{\eta} q_m(X_{N', T'}) e^{2\beta_1 \sqrt{X_{N', T'}}} \left( \frac{1}{T'} + \frac{1}{(T')^{3/2} \tilde{\eta}_{\min}^2} \right). \]

Using (3. 24), (3. 28) and (3. 30), we have
\[ X_{N', T'} \leq \theta_5 \left( \frac{1}{T'} + \frac{1}{(\gamma_{\min})^2 (T')^2} \right), \]
\[ \frac{\sqrt{X_{N', T'}}}{\gamma_{\min}^*} \leq \theta_5 \left( 1 + \frac{1}{(\gamma_{\min})^2 (T')^2} \right), \quad \text{and} \quad \sqrt{C_{N', T'}} \leq \frac{\theta_5}{\gamma_{\min}^*}; \]

hence
\[ I_m^{(1)} \leq C_{\eta} e^{\theta_5 \frac{\sqrt{X_{N', T'}}}{\gamma_{\min}^*}} B_m q_m(\theta_4 \left( \frac{1}{T'} + \frac{1}{(\gamma_{\min})^2 (T')^2} \right)^2) e^{\frac{\theta_8}{\tilde{\eta}_{\min}^2 (T')^2} \left( \frac{1}{T'} + \frac{1}{(T')^{3/2} \tilde{\eta}_{\min}^2} \right)}. \]

To conclude, we will use the following basic remark:
\[ y \in [0, 1] \implies (1 + y)^n \leq 2^n, \quad \text{and} \quad y \geq 1 \implies (1 + y)^n = y^n (1 + \frac{1}{y})^n \leq 2^n y^n, \]
hence
\[ y \geq 0 \implies (1 + y)^n \leq 2^n (1 + y^n), \quad \text{and} \quad a, b \geq 0 \implies (a + b)^n \leq 2^n (a^n + b^n). \]
Since (from (3. 24))

\[ \frac{1}{T^4} \leq \frac{1}{\left(\gamma_{\min}^*\right)^2(T')^2}, \]

we obtain that:

\[ q_m(\theta_4(\frac{1}{T^4} + \frac{1}{\left(\gamma_{\min}^*\right)^2(T')^2})^2) \leq \left(3 + \frac{2}{\lambda_m} \left(\frac{2\theta_4}{\left(\gamma_{\min}^*\right)^2(T')^2}\right)^2\right)^{2M^*} \]

\[ \leq 2^{2M^*} \left(3^{2M^*} + \left(\frac{2}{\lambda_m} \left(\frac{2\theta_4}{\left(\gamma_{\min}^*\right)^2(T')^2}\right)^2\right)^{2M^*}\right) \]

\[ \leq C_u^{2M^*} \left(1 + \frac{1}{\left(\lambda_m(\gamma_{\min}^*)^2(T')^2\right)^{4M^*}}\right), \]

then

\[ \forall m \geq N^*, \quad I_m^{(\downarrow)} \leq \frac{C_u}{(T')^{\gamma_{\min}^*}} \cdot \frac{\theta_4 \sqrt{N} \gamma_{\min}^*}{\gamma_{\min}^*} \cdot \frac{\theta_7}{\gamma_{\min}^*} \cdot \frac{\lambda_m}{\gamma_{\min}^*} \cdot \frac{B_m^2}{L^{\gamma_{\min}^*}} \cdot \frac{\theta_8}{\gamma_{\min}^*} \cdot \frac{\lambda_m}{\gamma_{\min}^*} \cdot \frac{C_u}{\gamma_{\min}^*} \cdot \frac{1}{\gamma_{\min}^*}. \]

Next we estimate \( I_m^{(\uparrow)} \). Denote

\[ L := \frac{2\theta_1}{2^{3/2}} \left(\frac{C_n \cdot T'}{\theta_0}\right)^{1/2} \cdot \frac{2C_u}{\gamma_{\min}^*}. \]

One can easily check that

\[ \frac{1}{L} \leq \frac{C_u}{T^\gamma_{\min}^*}. \]

Then

\[ I_m^{(\uparrow)} = 2e^{2\theta_2} \sqrt{C_n \cdot T'} \lambda_m \cdot B_m^2 \int_{X_{N', T'}} q_m(x)^2 \cdot e^{\frac{2C_u}{\gamma_{\min}^*}} \cdot e^{-\frac{2\theta_4}{2^{3/2}} \left(\frac{C_n \cdot T'}{\theta_0}\right)^{1/2}} \cdot dx \]

\[ = 2e^{2\theta_2} \sqrt{C_n \cdot T'} \lambda_m \cdot B_m^2 \int_{X_{N', T'}} q_m(x)^2 \cdot e^{-L \sqrt{T'}} \cdot dx \]

\[ \leq 2e^{\theta_7} \frac{\sqrt{N}}{\gamma_{\min}^*} \gamma_{\min}^* \lambda_m \int_0^\infty q_m(x)^2 \cdot e^{-L \sqrt{T'}} \cdot dx \]

\[ = 2e^{\theta_7} \frac{\sqrt{N}}{\gamma_{\min}^*} \gamma_{\min}^* \lambda_m \int_0^\infty q_m\left(\frac{t^2}{L^2}\right)^2 \cdot e^{-t} \cdot dt. \]

Recalling that

\[ \int_0^\infty t^k e^{-t} \cdot dt = k!, \]

we obtain

\[ I_m^{(\uparrow)} \leq 2^{2M^*} \left(3^{2M^*} + \left(\frac{2}{(\lambda_m)^2L^2}\right)^{2M^*} \cdot e^{-t} \cdot dt \right) \]

\[ = 2^{2M^*} \left(3^{2M^*} + \left(\frac{2^{2M^*} (8M^*)!}{L^{8M^*} (\lambda_m)^{4M^*}}\right) \leq C_u^{2M^*} \left(1 + \frac{1}{\left(\lambda_m(\gamma_{\min}^*)^2(T')^2\right)^{4M^*} (8M^*)!}\right), \]

\[ \leq \frac{\theta_7}{\gamma_{\min}^*} \frac{\lambda_m}{\gamma_{\min}^*} \frac{B_m^2}{L^{\gamma_{\min}^*}} \frac{\theta_8}{\gamma_{\min}^*} \frac{\lambda_m}{\gamma_{\min}^*} \frac{C_u}{\gamma_{\min}^*} \frac{1}{\gamma_{\min}^*}. \]
Finally, we see that there exists some $C_u$ independent of $m$, $\gamma_{\min}$, $\gamma_{\min}^*$, $N^*$ and $T$ such that
\[
\|\sigma_{m,N,T}^+\|^2_{L^2(0,T)} \leq C_u e^{-2\lambda_m T} \int_{\min}^{\lambda_{\min}} e^{\frac{\lambda_{\min} - \lambda}{\gamma_{\min}} e^{\frac{C_u}{\gamma_{\min}^{2/3}} \left( \frac{1}{(T')^{3/2}} + \frac{1}{(\gamma_{\min}^*)^2 (T')^2} \right)}} \left( 1 + \frac{1}{\lambda_m (\gamma_{\min}^*)^2 (T')^2} \right)^{M^*} (8 M^*)! \, d^{M^*}
\]
which gives (3. 33) and completes the proof of Lemma 3.6 and of Theorem 2.1. □

4. Proof of Theorem 2.2

4.1. A lower bound for any biorthogonal family.

Denote $E(\Lambda, T)$ the smallest closed subspace of $L^2(0, T)$ containing the functions $\varepsilon_{\lambda_n} : s \in (0, T) \mapsto e^{-\lambda_n s}$, $n \geq 1$.

It follows from (2. 7) that
\[
\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty,
\]
and then it is well-known ([33, 32]) that $E(\Lambda, T)$ is a proper subspace of $L^2(0, T)$. Moreover, given $m \geq 1$, denote $\Lambda_m := (\lambda_k)_{k \neq m}$, and $E(\Lambda_m, T)$ the smallest closed subspace of $L^2(0, T)$ containing the functions $\varepsilon_{\lambda_k}$, with $k \geq 1$ and $k \neq m$ (it does not include $\varepsilon_{\lambda_m}$). Then consider $p_m$ the orthogonal projection of $\varepsilon_{\lambda_m}$ on $E(\Lambda_m, T)$, and $d_{T,m}$ the distance between $\varepsilon_{\lambda_m}$ and $E(\Lambda_m, T)$: we have
\[
d_{T,m}^2 = \inf_{p \in E(\Lambda_m, T)} \|\varepsilon_{\lambda_m} - p\|^2_{L^2(0,T)} = \int_0^T (e^{-\lambda_m s} - p_m(s))^2 \, ds.
\]

Then $\varepsilon_{\lambda_m} - p_m$ is orthogonal to $E(\Lambda_m, T)$, which implies that
\[
\forall n \neq m, \quad \int_0^T (e^{-\lambda_m s} - p_m(s)) e^{-\lambda_n s} \, ds = 0,
\]
and
\[
\int_0^T (e^{-\lambda_m s} - p_m(s)) e^{-\lambda_m s} \, ds = \int_0^T (e^{-\lambda_m s} - p_m(s))(e^{-\lambda_m s} - p_m(s)) \, ds = d_{T,m}^2.
\]

Hence consider
\[
\sigma_{m}^-(s) := \frac{e^{-\lambda_m s} - p_m(s)}{d_{T,m}^2}.
\]

the sequence of functions $(\sigma_{m}^-)_{m \geq 1}$ is a biorthogonal family for the set $(\varepsilon_{\lambda_n})_{n \geq 1} = (e^{-\lambda_n t})_{n \geq 1} \in L^2(0, T)$.

Moreover it is optimal in the following sense: if $(\tilde{\sigma}_{m}^-)_{m \geq 1}$ is another biorthogonal family for the set $(\varepsilon_{\lambda_n})_{n \geq 1}$ in $L^2(0, T)$, then for all $m \geq 1$, $\tilde{\sigma}_{m}^- - \sigma_{m}^-$ is orthogonal to all $\varepsilon_{\lambda_n}$, hence to $E(\Lambda, T)$, hence to $\sigma_{m}$ since $\sigma_{m} \in E(\Lambda, T)$. Hence
\[
\|\tilde{\sigma}_{m}^-\|_{L^2(0,T)}^2 = \|\sigma_{m}^-\|_{L^2(0,T)}^2 + \|\tilde{\sigma}_{m}^- - \sigma_{m}^-\|_{L^2(0,T)}^2 \geq \|\sigma_{m}^-\|_{L^2(0,T)}^2.
\]

Therefore
\[
\|\tilde{\sigma}_{m}^-\|_{L^2(0,T)} \geq \|\sigma_{m}^-\|_{L^2(0,T)} = \frac{1}{d_{T,m}}.
\]

Hence $\frac{1}{d_{T,m}}$ is a lower bound of every biorthogonal sequence $(\tilde{\sigma}_{m}^-)_{m \geq 1}$; and a bound from above for $d_{T,m}$ gives a bound from below for every biorthogonal sequence.
At last, we note that if the sequence of functions \((\tilde{\sigma}_m^+)_{m \geq 1}\) is a biorthogonal family for the set \((e^{\lambda_n t})_{n \geq 1}\) in \(L^2(0, T)\), then
\[
\int_0^T \tilde{\sigma}_m^+(T - s)e^{\lambda_m T}e^{-\lambda_m s} \, ds = \delta_{mn},
\]
hence \((\tilde{\sigma}_m^+(T - s)e^{\lambda_m T})_m\) is biorthogonal for the set \((e^{-\lambda_n t})_{n \geq 1}\) in \(L^2(0, T)\). This implies that
\[
(4.4) \quad \| \tilde{\sigma}_m^+ \|_{L^2(0, T)} \geq e^{-\lambda_m T} d_{T,m}.
\]

Hence \(e^{-\lambda_m T} d_{T,m}\) is a lower bound of every biorthogonal sequence \((\tilde{\sigma}_m^+)_m \geq 1\). In the following (Lemma 4.4), we provide a bound from above for \(d_{T,m}\), that will give a bound from below for every biorthogonal sequence \((\tilde{\sigma}_m^+)_m \geq 1\).

4.2. A general result for sums of exponentials.

Clearly,
\[
d_{T,m} \leq \| e^{-\lambda_m s} - p(s) \|_{L^2(0, T)}
\]
for all \(p \in E(\Lambda_m, T)\). The idea used in Güichal [20] is to choose a particular element \(p \in E(\Lambda_m, T)\) in order to provide an upper bound of \(d_{T,m}\). The first thing to note is the following: consider \(M \geq m\) and
\[
q(s) := \sum_{i=1}^{M+1} A_i e^{-\lambda_i s}
\]
with coefficients \(A_1, \ldots, A_{M+1}\). Then \(q \in E(\Lambda_m, T)\) if and only if \(A_m = 0\), and when \(A_m \neq 0\), then
\[
\frac{1}{A_m} q(s) = e^{-\lambda_m s} + \sum_{i=1}^{m-1} \frac{A_i}{A_m} e^{-\lambda_i s} + \sum_{i=m+1}^{M+1} \frac{A_i}{A_m} e^{-\lambda_i s},
\]
hence
\[
(4.5) \quad \| \frac{1}{A_m} q(s) \|_{L^2(0, T)} \geq d_{T,m}.
\]

We will choose the coefficients \(A_1, \ldots, A_{M+1}\) so that
\[
q(0) = q'(0) = q''(0) = \cdots = q^{(M-1)}(0) = 0, \quad q^{(M)}(0) = 1.
\]
The following lemma is essentially extracted from Güichal [20]:

**Lemma 4.1.** Consider \(M \geq 0\), and \(0 < \lambda_1 < \cdots < \lambda_{M+1}\).

a) There exist coefficients \(A_1, \ldots, A_{M+1}\) so that the function \(q\) defined by
\[
q(s) := \sum_{i=1}^{M+1} A_i e^{-\lambda_i s}
\]
satisfies
\[
\begin{align*}
q(0) &= 0 \\
q'(0) &= 0 \\
q''(0) &= 0 \\
& \vdots \\
q^{(M-1)}(0) &= 0 \\
q^{(M)}(0) &= 1.
\end{align*}
\]
The coefficients are given by the following formulas:

\[(4. 6) \quad \forall k \in \{1, \cdots, M + 1\}, \quad A_k = \frac{1}{\prod_{i=1, i \neq k}^{M+1} (\lambda_i - \lambda_k)}.\]

b) With this choice of coefficients, we have

\[(4. 7) \quad \forall s > 0, \quad 0 < q(s) \leq \frac{s^M}{M!} e^{-\lambda_1 s}.\]

The only difference with G"uichal [20] is the estimate (4. 7) which is more precise than the one obtained in [20], Lemma 4:

\[\forall s > 0, \quad 0 < q(s) < \frac{s^M}{M!}.\]

In the following, we prove (4. 7), and in a sake of completeness, we give the main arguments for part a) of Lemma 4.1.

**Proof of Lemma 4.1.**

a) We write the linear system

\[
\begin{align*}
0 &= q(0) = \sum_{i=1}^{M+1} A_i \\
0 &= q'(0) = \sum_{i=1}^{M+1} -\lambda_i A_i \\
0 &= q''(0) = \sum_{i=1}^{M+1} (-\lambda_i)^2 A_i \\
\vdots & \quad \vdots \\
0 &= q^{(M-1)}(0) = \sum_{i=1}^{M+1} (-\lambda_i)^{(M-1)} A_i \\
1 &= q^{(M)}(0) = \sum_{i=1}^{M+1} (-\lambda_i)^M A_i.
\end{align*}
\]

This can be written

\[(4. 8) \quad \begin{pmatrix}
1 & 1 & \cdots & \cdots & 1 \\
-\lambda_1 & -\lambda_2 & \cdots & \cdots & -\lambda_{M+1} \\
(-\lambda_1)^2 & (-\lambda_2)^2 & \cdots & \cdots & (-\lambda_{M+1})^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(-\lambda_1)^M & (-\lambda_2)^M & \cdots & \cdots & (-\lambda_{M+1})^M
\end{pmatrix}
\begin{pmatrix}
A_1 \\
A_2 \\
A_3 \\
\vdots \\
A_{M+1}
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
\vdots \\
1
\end{pmatrix}.
\]

The \((M + 1) \times (M + 1)\) matrix \(A\) that appears in the left hand side of (4. 8) is invertible: indeed, its determinant is of Vandermonde type, and

\[\det A = \prod_{k < l} \left( -\lambda_l - (-\lambda_k) \right) = \prod_{k < l} (\lambda_k - \lambda_l) \neq 0.\]

Hence the system (4. 8) is invertible, and the Cramer’s formula gives

\[A_k = \frac{\det B}{\det A},\]

where \(B\) is the \((M + 1) \times (M + 1)\) matrix obtained from \(A\) putting the right-hand side member of (4. 8) at the place of the \(k\)th-column of \(A\). But then, we can develop \(\det B\) with respect to the \(k\)th-column and we find again a Vandermonde determinant. Then using the formula of Vandermonde determinant, one gets (4. 6).

b) We prove (4. 7) by induction. When \(M = 0\), (4. 7) is true. Assume that it is true for some \(M\), and let us prove that it is true for \(M + 1\): take

\[q(s) := \sum_{i=1}^{M+2} A_i e^{-\lambda_i s},\]

where the coefficients \(A_1, \cdots, A_{M+2}\), are chosen so that

\[q(0) = q'(0) = q''(0) = \cdots = q^{(M)}(0) = 0, \quad q^{(M+1)}(0) = 1.\]
Then the Taylor developments of $q$ and $q'$ say that
\[ q(s) = \frac{s^{M+1}}{(M+1)!} + O(s^{M+2}) \quad \text{and} \quad q'(s) = \frac{s^M}{M!} + O(s^{M+1}) \quad \text{as} \quad s \to 0. \]

Consider
\[ \tilde{q}(s) = e^{-2\lambda s} \frac{d}{ds} (q(s)e^{\lambda s}). \]
Then
\[ \tilde{q}(s) = e^{-2\lambda s} \frac{d}{ds} \left( \sum_{i=1}^{M+2} A_i e^{(\lambda M + 2 - \lambda_i)s} \right) \]
\[ = e^{-2\lambda s} \left( \sum_{i=1}^{M+2} A_i (\lambda M + 2 - \lambda_i) e^{(\lambda M + 2 - \lambda_i)s} \right) \]
\[ = \sum_{i=1}^{M+2} A_i (\lambda M + 2 - \lambda_i) e^{-(\lambda M + \lambda_i)s}. \]

But the last term in the series is clearly equal to 0, hence $\tilde{q}$ is a sum of $M + 1$ exponentials. Moreover,
\[ \tilde{q}(s) = q'(s) e^{-\lambda M + 2s} + \lambda M + 2q(s) e^{-\lambda M + 2s} \]
\[ = \left( \frac{s^M}{M!} + O(s^{M+1}) \right) e^{-\lambda M + 2s} + \lambda M + 2 \left( \frac{s^{M+1}}{(M+1)!} + O(s^{M+2}) \right) e^{-\lambda M + 2s} \]
\[ = \frac{s^M}{M!} + O(s^{M+1}) \quad \text{as} \quad s \to 0. \]

Hence
\[ \tilde{q}(0) = \tilde{q}'(0) = \tilde{q}''(0) = \cdots = \tilde{q}^{(M-1)}(0) = 0, \quad \tilde{q}^{(M)}(0) = 1, \]
and we can apply the induction assumption to $\tilde{q}$: then
\[ 0 < \tilde{q}(s) \leq \frac{s^M}{M!} e^{-(\lambda M + 1)s}. \]

We deduce first that $s \mapsto q(s)e^{\lambda M + 2s}$ is increasing. Since its value in 0 is 0, then $q$ is positive on $(0, +\infty)$. Next, we obtain that
\[ \frac{d}{ds} (q(s)e^{\lambda M + 2s}) \leq \frac{s^M}{M!} e^{(2\lambda M + 2 - \lambda M + 2 - \lambda_1)s} \]
\[ = \frac{s^M}{M!} e^{(\lambda M + 2 - \lambda_1)s} \leq \frac{d}{ds} \left( \frac{s^{M+1}}{(M+1)!} e^{(\lambda M + 2 - \lambda_1)s} \right), \]

hence by integration,
\[ q(s)e^{\lambda M + 2s} \leq \frac{s^{M+1}}{(M+1)!} e^{(\lambda M + 2 - \lambda_1)s}, \]

hence
\[ q(s) \leq \frac{s^{M+1}}{(M+1)!} e^{-\lambda_1 s}, \]

which completes the induction argument and the proof of Lemma 4.1. \qed
4.3. A precise estimate of the remaining part of the exponential function.

It turns out that we will need an estimate for the remaining part of the exponential function

\[ \sum_{n=N}^{\infty} \frac{x^n}{n!} \]

in function of \( x \) and \( N \). We prove the following general and precise result:

**Lemma 4.2.** We have the following estimates:

\[(4.9) \quad \forall N \geq 1, \forall x \geq 0, \quad \frac{1}{N!} \left( \frac{x}{1+x} \right)^N e^x \leq \sum_{n=N}^{\infty} \frac{x^n}{n!} \leq C_1 N \left( \frac{x}{1+x} \right)^N e^x, \]

where

\[ C_1 = \max_{x \in \mathbb{R}_+} \frac{(1-e^{-x})(1+x)}{x}. \]

**Proof of Lemma 4.2.** Denote

\[ f_N(x) := \sum_{n=N}^{\infty} \frac{x^n}{n!}. \]

Let us prove by induction that

\[ \forall N \geq 0, \forall x \geq 0, \quad f_N(x) \geq \frac{1}{N!} \left( \frac{x}{1+x} \right)^0 e^x. \]

First, of course \( f_0(x) = e^x \), and then

\[ f_0(x) \geq \frac{1}{0!} \left( \frac{x}{1+x} \right)^0 e^x. \]

Next, assume that

\[ \forall x \geq 0, \quad f_N(x) \geq \frac{1}{N!} \left( \frac{x}{1+x} \right)^N e^x. \]

We note that

\[ f_{N+1}(x) = f_N(x), \]

and

\[ \frac{d}{dx} \left( \frac{1}{(N+1)!} \left( \frac{x}{1+x} \right)^{N+1} e^x \right) = \frac{1}{(N+1)!} \left( \frac{x}{1+x} \right)^N e^x \left( \frac{N+1}{(1+x)^2} + \frac{x}{1+x} \right). \]

The study of the variations of the function \( x \mapsto \frac{N+1}{(1+x)^2} + \frac{x}{1+x} \) gives

\[ \forall x \geq 0, \quad 1 - \frac{1}{4(N+1)} \leq \frac{N+1}{(1+x)^2} + \frac{x}{1+x} \leq N+1, \]

hence

\[ \frac{d}{dx} \left( \frac{1}{(N+1)!} \left( \frac{x}{1+x} \right)^{N+1} e^x \right) \leq \frac{N+1}{(N+1)!} \left( \frac{x}{1+x} \right)^N e^x = \frac{1}{N!} \left( \frac{x}{1+x} \right)^N e^x. \]

Then

\[ f'_{N+1}(x) = f_N(x) \geq \frac{d}{dx} \left( \frac{1}{(N+1)!} \left( \frac{x}{1+x} \right)^{N+1} e^x \right), \]

and since the values at 0 are 0, we obtain that

\[ \forall x \geq 0, \quad f_{N+1}(x) \geq \frac{1}{(N+1)!} \left( \frac{x}{1+x} \right)^{N+1} e^x. \]

This proves the first part of (4.9).

For the second part (which is not necessary for us here), we note that

\[ \forall x \geq 0, \quad f_1(x) \leq C_1 \left( \frac{x}{1+x} \right)^1 e^x. \]
Assume that
\[ \forall x \geq 0, \quad f_N(x) \leq C_1 N \left( \frac{x}{1+x} \right)^N e^x. \]
Then
\[ \frac{d}{dx} \left( \left( \frac{x}{1+x} \right)^{N+1} e^x \right) = \left( \frac{x}{1+x} \right)^N e^x \left( \frac{N+1}{(1+x)^2} + \frac{x}{1+x} \right) \geq \left( 1 - \frac{1}{4(N+1)} \right) \left( \frac{x}{1+x} \right)^N e^x. \]
Hence
\[ f'_{N+1}(x) = f_N(x) \leq C_1 N \left( \frac{x}{1+x} \right)^N e^x \leq \frac{C_1 N}{1 - \frac{1}{4(N+1)}} \frac{d}{dx} \left( \left( \frac{x}{1+x} \right)^{N+1} e^x \right). \]
To conclude, note that
\[ \forall N \geq 1, \quad \frac{N}{1 - \frac{1}{4(N+1)}} \leq N + 1 : \]
indeed,
\[ (N+1)(1 - \frac{1}{4(N+1)}) = N + 1 - \frac{1}{4} = N + \frac{3}{4} \geq N. \]
Hence, we obtain that
\[ \forall x \geq 0, \quad f_{N+1}(x) \leq C_1 (N+1) \left( \frac{x}{1+x} \right)^{N+1} e^x, \]
which concludes the induction, and the proof of (4.9).

4.4. Consequence: a bound from above for the distance \( d_{T,m} \).

As a consequence of the upper estimate (4.5) for the distance and of Lemma 4.1, we obtain the following inequality: for all \( m \geq 1 \), for all \( M \geq m \), we have

\[ d_{T,m} \leq \left( \prod_{i=1, i \neq m}^{M+1} |\lambda_i - \lambda_m| \right) \left( \int_0^T \frac{s^{2M}}{M!^2} e^{-2\lambda_1 s} ds \right)^{1/2}. \]
It remains to estimate the terms that appear in the right hand side. This is the object of the next sections, and it is based on the gap conditions (2.7) and (2.8).

4.4.a. Estimate under the uniform gap condition (2.7).

We prove the following:

**Lemma 4.3.** Assume that \((\lambda_n)_n\) satisfies (2.7). Denote
\[ k_* := \left\lfloor \frac{2\sqrt{\lambda_1}}{\gamma_{\text{max}}} \right\rfloor + m + 2 \]
and
\[ C(T, \gamma_{\text{max}}, \lambda_1, m) = \frac{6\sqrt{1+2T\lambda_1}}{\pi^2 \sqrt{2T}} \frac{(k_* - 1)!}{(m+k_*+3)!} \frac{(T\gamma_{\text{max}}^{k_*+2})}{(m-1)!} \frac{(1+T\gamma_{\text{max}}^{k_*+3})}{(m+k_*+3)}. \]
Then
\[ \forall m \geq 1, \quad \frac{1}{d_{T,m}} \geq C(T, \gamma_{\text{max}}, \lambda_1, m) \frac{e^{\frac{1}{\gamma_{\text{max}}}}}{\sqrt{\pi}}. \]

**Proof of Lemma 4.3.** Of course
\[ \int_0^T \frac{s^{2M}}{M!^2} e^{-2\lambda_1 s} ds \leq \frac{T^{2M+1}}{M!^2 (2M+1)}, \]
an, on the other hand,
\[ \int_0^T \frac{s^{2M}}{M!^2} e^{-2\lambda_1 s} ds \leq \frac{T^{2M}}{M!^2} \int_0^T e^{-2\lambda_1 s} ds \leq \frac{T^{2M}}{M!^2} \frac{1-e^{-2\lambda_1 T}}{2\lambda_1}. \]
Hence
\[ \int_0^T \frac{s^{2M}}{M!^2} e^{-2\lambda_1 s} \, ds \leq \frac{T^{2M}}{M!^2} \inf \left\{ \frac{T}{2M + 1}, \frac{1 - e^{-2\lambda_1 T}}{2\lambda_1} \right\}. \]

But it is easy to check that
\[ \forall a, b > 0, \quad \inf \{a, \frac{1}{b}\} \leq \frac{2a}{1 + ab}. \]

Indeed, \( \inf \{a, \frac{1}{b}\} = a \) if \( ab \leq 1 \), and in this case \( 1 + ab \leq 2 \), hence \( a(1 + ab) \leq 2a \).

On the other hand, when \( ab \geq 1 \), \( \inf \{a, \frac{1}{b}\} = \frac{1}{b} \), and \( 1 + ab \leq 2ab \). We deduce that
\[ (4.12) \quad \left( \int_0^T \frac{s^{2M}}{M!^2} e^{-2\lambda_1 s} \, ds \right)^{1/2} \leq \frac{T^M}{M! \sqrt{2M + 1 + 2\lambda_1 T}}. \]

Now it remains to estimate the product
\[ \prod_{i=1, i \neq m}^{M+1} |\lambda_i - \lambda_m| = \left( \prod_{i=1, i \neq m}^{M+1} |\sqrt{\lambda}_i - \sqrt{\lambda}_m| \right) \left( \prod_{i=1, i \neq m}^{M+1} (\sqrt{\lambda}_i + \sqrt{\lambda}_m) \right). \]

We derive from (2.7) first that
\[ |\sqrt{\lambda}_i - \sqrt{\lambda}_m| \leq \gamma_{\text{max}} |i - m|, \]
and next that
\[ \sqrt{\lambda}_i + \sqrt{\lambda}_m \leq 2\sqrt{\lambda}_1 + \gamma_{\text{max}} (i + m). \]

Hence
- first
\[ \prod_{i=1, i \neq m}^{M+1} (\sqrt{\lambda}_i + \sqrt{\lambda}_m) \leq \prod_{i=1, i \neq m}^{M+1} (2\sqrt{\lambda}_1 + \gamma_{\text{max}} (i + m)) \leq \gamma_{\text{max}}^M (M + 1 + \left[ \frac{2\sqrt{\lambda}_1}{\gamma_{\text{max}}}, m + 1 \right]!) \left( \frac{2\sqrt{\lambda}_1}{\gamma_{\text{max}}}, m + 1 \right)! = c^{(+)} \gamma_{\text{max}}^M (M + k_*)! \]
  with
  \[ c^{(+)} = \frac{1}{\left( \frac{2\sqrt{\lambda}_1}{\gamma_{\text{max}}}, m + 1 \right)!} \quad \text{and} \quad k_* := \left[ \frac{2\sqrt{\lambda}_1}{\gamma_{\text{max}}}, m + 2 \right]; \]
- next
\[ \prod_{i=1, i \neq m}^{M+1} |\sqrt{\lambda}_i - \sqrt{\lambda}_m| \leq \prod_{i=1, i \neq m}^{M+1} \gamma_{\text{max}} |i - m| \]
  \[ = \gamma_{\text{max}}^M (m - 1)! (M - (m - 1))! = c^{(-)} \gamma_{\text{max}}^M (M - (m - 1))! \]
  with
  \[ c^{(-)} = (m - 1)!. \]

Combining this with (4.12), we derive from (4.10)
\[ d_{T,m} \leq c^{(+)} c^{(-)} \frac{\sqrt{2T}}{\sqrt{2M + 1 + 2\lambda_1 T}} (T \gamma_{\text{max}}^2)^M (M + k_*)!(M - m + 1)! \frac{M!}{M!}. \]

Denote
\[ c_* := c^{(+)} c^{(-)} \frac{\sqrt{2T}}{\sqrt{1 + 2\lambda_1 T}}. \]
Then, to conclude, we note that
\[ \frac{1}{d_{T,m}} = \frac{6}{\pi^2} \sum_{M=m+1}^{\infty} \frac{1}{(M-m)^2} \frac{1}{d_{T,m}} \]
\[ \geq \frac{6}{\pi^2 c_*} \sum_{M=m+1}^{\infty} \frac{1}{(M-m)^2} \frac{M!}{(M+k_*)!(M-m+1)!} \left( \frac{1}{T_{\gamma_{\text{max}}}^2} \right)^M \]
\[ \geq \frac{6}{\pi^2 c_*} \sum_{M=m+1}^{\infty} \frac{1}{(M+k_*)+2)!} \left( \frac{1}{T_{\gamma_{\text{max}}}^2} \right)^M \]
\[ = \frac{6}{\pi^2 c_*} (T_{\gamma_{\text{max}}}^2)^{k_*+2} \sum_{n=m+k_*+3}^{\infty} \frac{1}{n!} \left( \frac{1}{T_{\gamma_{\text{max}}}^2} \right)^n. \]

And using Lemma 4.2, we obtain that (4.11). This gives the expected exponential behaviour in \(1/(T_{\gamma_{\text{max}}}^2)\). In the following we take care of the asymptotic gap \(\gamma_{\text{max}}^*\).

4.4.b. Estimate under the uniform gap condition (2.7) and the asymptotic gap condition (2.8).

Now, taking into account the 'asymptotic gap' given by (2.8), we will be able to improve the previous estimate, roughly speaking replacing \(\gamma_{\text{max}}^2\) by \((\gamma_{\text{max}}^*)^2\) in the exponential factor.

**Lemma 4.4.** Assume that \((\lambda_n)\) satisfies (2.7)-(2.8). Then
\[ (4.13) \quad \frac{1}{d_{T,m}} \geq b^*(T,\gamma_{\text{max}},\gamma_{\text{max}}^*,N_*,\lambda_1,m) e^{\frac{T}{2}\gamma_{\text{max}}^*} \]
where \(b^*\) is given by
- when \(m \leq N_*\), we have
  \[ (4.14) \quad b^*(T,\gamma_{\text{max}},\gamma_{\text{max}}^*,N_*,\lambda_1,m) = C^* \frac{1}{\sqrt{T}} \frac{T (\gamma_{\text{max}}^*)^{2(K_*+K_*^*+2)}}{(1 + (T_{\gamma_{\text{max}}^*}))^{N_*+K_*+K_*^*+3}}, \]
  where
  \[ C^* = \frac{c_u (\gamma_{\text{max}}^*)^{2(N_*-1)}}{C^*(+) C^*(-)} \frac{1}{(N_*+K_*+K_*^*+3)!}, \]
and \(C^*(+), C^*(-)\), \(K_*\) and \(K_*^*\) are given respectively in (4.17), (4.20), (4.18) and (4.21);
- when \(m > N_*\), we have
  \[ (4.15) \quad b^*(T,\gamma_{\text{max}},\gamma_{\text{max}}^*,N_*,\lambda_1,m) = \tilde{C}^* \frac{1}{\sqrt{T}} \frac{T (\gamma_{\text{max}}^*)^{2(K_*+K_*^*+2)}}{(1 + (T_{\gamma_{\text{max}}^*}))^{m+K_*+K_*^*+3}}, \]
  where
  \[ \tilde{C}^* = \frac{c_u}{\tilde{C}^*(+) \tilde{C}^*(-)} \frac{1}{(m+K_*+3)!}, \]
  where \(\tilde{C}^*(+), \tilde{C}^*(-)\) and \(K_*\) are given respectively in (4.23), (4.25) and (4.18).

**Proof of Lemma 4.4.** The starting point is of course (4.10) and (4.12). Concerning the estimate of the product, we proceed in the same way as previously, distinguishing several cases.

a) First we investigate what can be said when \(m \leq N_* < M + 1\): in this case,
We deduce from (4.10), (4.12), (4.16) and (4.24) that
\[ \leq \gamma_{\max}^* (i - N_*) + 2 \sqrt{\lambda_1} + (m + N_*) \gamma_{\max}; \]

hence
\[
\begin{align*}
\prod_{i=1, i \neq m}^{M+1} (\sqrt{\lambda_i} + \sqrt{\lambda_m}) &= \left( \prod_{i=1, i \neq m}^{\hat{N}_s} (\sqrt{\lambda_i} + \sqrt{\lambda_m}) \right) \left( \prod_{i=\hat{N}_s + 1}^{M+1} (\sqrt{\lambda_i} + \sqrt{\lambda_m}) \right) \\
&\leq \left( \prod_{i=1, i \neq m}^{\hat{N}_s} (2\sqrt{\lambda_i} + \gamma_{\max}(i + m)) \right) \left( \prod_{i=\hat{N}_s + 1}^{M+1} (2\sqrt{\lambda_i} + (m + N_*) \gamma_{\max} + \gamma_{\max}^*(i - N_*)) \right) \\
&\leq C^+(\gamma_{\max}^*)^M (M + 1 - N_* + [\frac{2\sqrt{\lambda_1} + (N_* + m) \gamma_{\max}}{\gamma_{\max}}] + 1)! \\
&= C^+(\gamma_{\max}^*)^M (M + K_*),
\end{align*}
\]

with
\[
C^+ = \frac{\gamma_{\max}^N N - 1}{(N_* + m + [\frac{2\sqrt{\lambda_1}}{\gamma_{\max}}] + 1)!} \\
\left( \frac{2\sqrt{\lambda_1} + (N_* + m) \gamma_{\max}}{\gamma_{\max}} \right) + 1)!(2 + [\frac{2\sqrt{\lambda_1}}{\gamma_{\max}}] + 1)
\]

and
\[
K_* := [\frac{2\sqrt{\lambda_1} + (N_* + m) \gamma_{\max}}{\gamma_{\max}}] - N_* + 2;
\]

next, similarly we have
\[
\begin{align*}
\prod_{i=1, i \neq m}^{M+1} |\sqrt{\lambda_i} - \sqrt{\lambda_m}| &= \left( \prod_{i=1, i \neq m}^{\hat{N}_s} |\sqrt{\lambda_i} - \sqrt{\lambda_m}| \right) \left( \prod_{i=\hat{N}_s + 1}^{M+1} |\sqrt{\lambda_i} - \sqrt{\lambda_m}| \right) \\
&\leq \left( \prod_{i=1, i \neq m}^{\hat{N}_s} \gamma_{\max} |i - m| \right) \left( \prod_{i=\hat{N}_s + 1}^{M+1} \gamma_{\max}^* (i - N_*) + \gamma_{\max} (N_* - m) \right) \\
&\leq C^-(\gamma_{\max}^*)^M (M - N_* + 2 + [\frac{\gamma_{\max}}{\gamma_{\max}^*} (N_* - m)])! \\
&= C^-(\gamma_{\max}^*)^M (M + K'_*),
\end{align*}
\]

with
\[
C^- = \frac{\gamma_{\max}^N (m - 1)! (N_* - m)!}{(1 + [\frac{\gamma_{\max}}{\gamma_{\max}^*} (N_* - m)])!}
\]

and
\[
K'_* := [\frac{\gamma_{\max}}{\gamma_{\max}^*} (N_* - m)] - N_* + 2;
\]

We deduce from (4.10), (4.12), (4.16) and (4.24) that
\[
d_{T,m} \leq C^+ C^- \frac{\sqrt{2T}}{\sqrt{1 + 2T \lambda_1}} \frac{(M + K_*)! (M + K'_*)!}{M!} (T (\gamma_{\max}^*)^2)^M.
\]

Denote
\[
C_* := C^+ C^- \frac{\sqrt{2T}}{\sqrt{1 + 2T \lambda_1}}.
\]

Hence
\[
d_{T,m} \leq C_* \frac{(M + K_*)! (M + K'_*)!}{M!} (T (\gamma_{\max}^*)^2)^M.
\]
Then, as we did before, we have

\[
\frac{1}{d_{T,m}} = \frac{6}{\pi^2} \sum_{M=N_*+1}^{\infty} \frac{1}{(M-N_*)^2} \sum_{n=M}^{\infty} \frac{1}{n!} \frac{M!}{(M-N_*)^2 (M+K_*)! (M+K'_*)!} \left( \frac{1}{T(\gamma_{max})^2} \right)^M.
\]

Note that

\[
\frac{1}{(M-N_*)^2 (M+K_*)! (M+K'_*)!} \geq \frac{1}{(M+1) \cdots (M+K'_*)}. 
\]

Hence

\[
\frac{1}{d_{T,m}} \geq \frac{6}{\pi^2 C_*} \sum_{M=N_*+1}^{\infty} \frac{1}{(M+K_*)^2} \sum_{n=M}^{\infty} \frac{1}{n!} \frac{M!}{(M+K_*)^2} \left( \frac{1}{T(\gamma_{max})^2} \right)^M 
= \frac{6}{\pi^2 C_*} \left( T(\gamma_{max})^2 \right)^{K_*+K'_*+2} \sum_{n=N_*+K_*+K'_*+3}^{\infty} \frac{1}{n!} \left( \frac{T(\gamma_{max})^2}{n!} \right)^n.
\]

Applying Lemma 4.2 to (4. 22), we obtain

\[
\frac{1}{d_{T,m}} \geq \frac{6}{\pi^2 C_*} \left( N_*+K_*+K'_*+3 \right) \left( 1+X \right)^{N_*+K_*+K'_*+3} \left( \frac{1}{X} \right). 
\]

with

\[
X = T(\gamma_{max})^2. 
\]

This concludes the proof of Lemma 4.4 when \( m \leq N_* \).

b) In the same way, if \( m > N_* \), we have

- first

\[
\prod_{i=1, i \neq m}^{M+1} \left( \sqrt{\lambda_i} + \sqrt{\lambda_m} \right) = \left( \prod_{i=1}^{N_*} \left( \sqrt{\lambda_i} + \sqrt{\lambda_m} \right) \right) \left( \prod_{i=N_*+1, i \neq m}^{M+1} \left( \sqrt{\lambda_i} + \sqrt{\lambda_m} \right) \right) 
\leq \left( \prod_{i=1}^{N_*} \left( 2\sqrt{\lambda_1} + \gamma_{max}(i+m) \right) \left( \prod_{i=N_*+1, i \neq m}^{M+1} \left( 2\sqrt{\lambda_1} + (m+N_*)\gamma_{max} + \gamma_{max}(i-N_*) \right) \right) 
\leq \left( \gamma_{max} \right)^{N_*} \left( N_* + \left( \frac{2\sqrt{\lambda_1}}{\gamma_{max}} \right) + 1 \right) \left( \frac{M+1-N_* + \left( \frac{2\sqrt{\lambda_1}}{\gamma_{max}} (N_*+m) \gamma_{max} \right) + 1}{\left( \frac{2\sqrt{\lambda_1} (N_*+m) \gamma_{max}}{\gamma_{max}} + 1 \right) (m - N_* + \left( \frac{2\sqrt{\lambda_1} (N_*+m) \gamma_{max}}{\gamma_{max}} + 1 \right) + 1} \right) 
\leq \left( \gamma_{max} \right)^{N_*} \left( N_* + \left( \frac{2\sqrt{\lambda_1}}{\gamma_{max}} \right) + 1 \right) \left( \frac{M+1-N_* + \left( \frac{2\sqrt{\lambda_1}}{\gamma_{max}} (N_*+m) \gamma_{max} \right) + 1}{\left( \frac{2\sqrt{\lambda_1} (N_*+m) \gamma_{max}}{\gamma_{max}} + 1 \right) (m - N_* + \left( \frac{2\sqrt{\lambda_1} (N_*+m) \gamma_{max}}{\gamma_{max}} + 1 \right) + 1} \right) 
\]

with

\[
\tilde{C}(+) = \left( \frac{\gamma_{max}}{\gamma_{max}} \right)^{N_*} \left( N_* + \left( \frac{2\sqrt{\lambda_1}}{\gamma_{max}} \right) + 1 \right) \left( \frac{1}{\left( \frac{2\sqrt{\lambda_1} (N_*+m) \gamma_{max}}{\gamma_{max}} + 1 \right) (m - N_* + \left( \frac{2\sqrt{\lambda_1} (N_*+m) \gamma_{max}}{\gamma_{max}} \right) + 1} \right). 
\]
\begin{itemize}
  \item next, \[
  \prod_{i=1, i \neq m}^{M+1} \sqrt{\lambda_i - \sqrt{\lambda_m}} = \left( \prod_{i=1}^{N_m} \sqrt{\lambda_i - \sqrt{\lambda_m}} \right) \left( \prod_{i=N_m+1, i \neq m}^{M+1} \sqrt{\lambda_i - \sqrt{\lambda_m}} \right)
  \leq \left( \prod_{i=1}^{N_m} \gamma_{\max} |i - m| \right) \left( \prod_{i=N_m+1, i \neq m}^{M+1} \gamma_{\max}^* |i - m| \right)
  = (\gamma_{\max})^N, (\gamma_{\max}^*)^{M-N} (m - 1)! (M + 1 - m)!
  = \tilde{C}(-) (\gamma_{\max}^*)^M (M + 1 - m)!
  \]
  with \[
  \tilde{C}(-) = \frac{(\gamma_{\max})^N, (m - 1)!}{\gamma_{\max}^*}.
  \]
  \item then we can conclude:
  \[
  d_{T,m} \leq \tilde{C}_* \frac{(M + K_*)! (M + 1 - m)!}{M!} T^M (\gamma_{\max}^*)^{2M}
  \]
  with \[
  \tilde{C}_* = \frac{\sqrt{2T}}{\sqrt{1 + 2T \lambda_1}} C^{(+)} C^{(-)};
  \]
  \end{itemize}

then, in the same way,
\[
\frac{1}{d_{T,m}} = \frac{6}{\pi^2} \sum_{M=m+1}^{\infty} \frac{1}{(M-m)^2} \frac{1}{d_{T,m}} \geq \frac{6}{\pi^2 C_*} \sum_{M=m+1}^{\infty} \frac{1}{(M-m)^2} \frac{M!}{(M+K_*)!} \frac{1}{(\gamma_{\max}^*)^M} \frac{1}{T(\gamma_{\max}^*)^2}^M \geq \frac{6}{\pi^2 C_*} \sum_{M=m+1}^{\infty} \frac{1}{(M+K_*+2)!} \frac{1}{(\gamma_{\max}^*)^M} \frac{1}{T(\gamma_{\max}^*)^2}^M \geq \frac{6}{\pi^2 C_*} \left( T(\gamma_{\max}^*)^2 \right)^{K_*+2} \sum_{n=m+K_*+3}^{\infty} \frac{1}{n!} \left( \frac{1}{1 + T(\gamma_{\max}^*)^2} \right)^n e^{1/(T(\gamma_{\max}^*)^2)}.\]

This concludes the proof of Lemma 4.4 when $m > N_*$. \hfill $\Box$

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