Global Well-posedness for the fourth order nonlinear Schrödinger equations with small rough data in high dimension

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Abstract: For $n \geq 2$, we establish the smooth effects for the solutions of the linear fourth order Shrödinger equation in anisotropic Lebesgue spaces with $\Box_k$-decomposition. Using these estimates, we study the Cauchy problem for the fourth order nonlinear Schrödinger equations with three order derivatives and obtain the global well posedness for this problem with small data in modulation space $M_{2,1}^{9/2}(\mathbb{R}^n)$.

Keywords: Global well-posedness, Fourth order nonlinear Schrödinger equations, small data.

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1 Introduction

In our earlier paper [30], we consider the Cauchy problem for the fourth order nonlinear Schrödinger equations with three order derivatives (4NLS)

$$iu_t + \Delta^2 u - \varepsilon \Delta u = F((\partial_x^6 u)_{|\alpha| \leq 3}, (\partial_x^6 \overline{u})_{|\alpha| \leq 3}), \quad u(0, x) = u_0(x), \quad (1.1)$$

where $\varepsilon \in \{0, 1\}$, $u$ is a complex valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$.

$$\Delta u = -\mathcal{F}^{-1} |\xi|^2 \mathcal{F} u, \quad \Delta^2 u = \mathcal{F}^{-1} |\xi|^4 \mathcal{F} u, \quad (1.2)$$

$F : \mathbb{C}^{\frac{1}{3n^3 + 2n^2 + \frac{11}{3}n + 2}} \rightarrow \mathbb{C}$ is a polynomial of the form

$$F(z) = P(z_1, \ldots, z_{\frac{1}{3}n^3 + 2n^2 + \frac{11}{3}n + 2}) = \sum_{m+1 \leq |\beta| \leq M+1} c_\beta z^\beta, \quad c_\beta \in \mathbb{C}. \quad (1.3)$$
2 \leq m \leq M, m, M \in \mathbb{N}. In this paper, we keep on studying this problem mainly with the method in [26].

The fourth order nonlinear Schrödinger equation, including its special forms, arise in deep water wave dynamics, plasma physics, optical communications (see [6]). A large amount of work has been devoted to the Cauchy problem of dispersive equations, such as [1, 3, 4, 5, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 21, 22] and references therein. In [21], by using the method of Fourier restriction norm, Segata studied a special fourth order nonlinear Schrödinger equation in one dimensional space. And the results have been improved in [12, 22].

In [26], Wang, Han and Huang discussed
\begin{equation}
\begin{aligned}
iu_t + \Delta x u = F(u, \bar{u}, \nabla u, \nabla \bar{u}), \\
u(0, x) = u_0(x)
\end{aligned}
\end{equation}
where $\Delta x u = \sum_{i=1}^n \varepsilon_i \partial_{x_i}^2 u$ and $\varepsilon_i \in \{-1, 1\}, i = 1, \ldots, n$. They proved (1.4) is global well-posed in modulation spaces $M_{2,1}^s(\mathbb{R}^n), s \geq 3/2$.

1.1 $M_{2,1}^s$ and $B_{2,1}^s$

In this paper, we are mainly interested in the cases that the initial data $u_0$ belongs to the modulation space $M_{2,1}^s$ for which the norm can be equivalently defined in the following way (cf. [7, 29, 28, 27]):
\begin{equation}
\| f \|_{M_{2,1}^s} = \sum_{k \in \mathbb{Z}^n} |\langle k \rangle|^{s} \| \mathcal{F} f \|_{L^2(Q_k)},
\end{equation}
where $\langle k \rangle = 1 + |k|, Q_k = \{ \xi : \arg \xi - k_i < 1/2, i = 1, \ldots, n \}$. For simplicity, we write $M_{2,1}^s = M_{0,2,1}^s$. Since only the modulation space $M_{2,1}^s$ will be used in this paper, we will not state the definition of the general modulation spaces $M_{p,q}^s, q'$. one can refer to Feichtinger [7]. Modulation spaces $M_{2,1}^s$ are related to the Besov spaces $B_{2,1}^s$ for which the norm is defined as follows:
\begin{equation}
\| f \|_{B_{2,1}^s} = \| \mathcal{F} f \|_{L^2(B(0,1))} + \sum_{j=1}^\infty 2^{sj} \| \mathcal{F} f \|_{L^2(B(0,2^{-j}) \setminus B(0,2^{-j-1}))},
\end{equation}
where $B(x_0, R) := \{ \xi \in \mathbb{R} : |\xi - x_0| \leq R \}$. It is known that there holds the following optimal inclusions between $B_{2,1}^{s+n/2}, M_{2,1}^s$ and $B_{2,1}^s$ (cf. [25, 23, 27]):
\begin{equation}
B_{2,1}^{s+n/2} \subset M_{2,1}^s \subset B_{2,1}^s.
\end{equation}
So, comparing $M_{2,1}^s$ with $B_{2,1}^{s+n/2}$, we see that $M_{2,1}^s$ contains a class of functions $u$ satisfying $\| u \|_{M_{2,1}^s} = \infty$ but $\| u \|_{B_{2,1}^{s+n/2}} \ll 1$. On the other hand, we can also find a class of rough functions $u$ satisfying $\| u \|_{B_{2,1}^s} = \infty$ but $\| u \|_{M_{2,1}^s} \ll 1$. We have $B_{2,1}^{s+n/2} \subset M_{2,1}^s \subset L^\infty \cap L^2$, this embedding is also optimal.
1.2 Main results

We now give our results, the notations used here can be found in the section 1.3.

**Theorem 1.1** Let \( n \geq 2, 2 \leq m \leq M < \infty, m > 8/n. \) Assume that \( u_0 \in M^{9/2}_{2,1} \) and \( \|u_0\|_{M^{9/2}_{2,1}} \leq \delta \) for some small \( \delta > 0. \) Then (1.1) has a unique global solution \( u \in C(\mathbb{R}, M^{9/2}_{2,1}) \cap X, \) where

\[
\|u\|_X = \sum_{\alpha=0,3} \sum_{i, \ell=1}^{n} \sum_{k \in \mathbb{Z}^n, |k_i| = k_{\max} > 4} (k_i)^3 \|\partial_{x_i} \square_k u\|_{L^\infty_{t} L^2_{x_i} L^2_{x_j} (x_j)_{j \neq i} L^2_2 (\mathbb{R}^{1+n})} \\
+ \sum_{\alpha=0,3} \sum_{i, \ell=1}^{n} \sum_{k \in \mathbb{Z}^n} (k_i)^{3/2-3/m} \|\partial_{x_i} \square_k u\|_{L^m_{t} \infty_{x_i} L^\infty_{x_j} L^\infty_2 (x_j)_{j \neq i} L^\infty_2 (\mathbb{R}^{1+n})} \\
+ \sum_{\alpha=0,3} \sum_{i=1, k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \|\partial_{x_i} \square_k u\|_{L^\infty_{t} L^2_{x_i} L^2_{x_j} (x_j)_{j \neq i} L^2_{x_j} (\mathbb{R}^{1+n})},
\]  

where \( k = (k_1, \ldots, k_n). \) Moreover, \( \|u\|_{X} \lesssim \delta. \)

In Theorem 1.1 if \( u_0 \in M^s_{2,1} \) with \( s > 9/2, \) then we have \( u \in C(\mathbb{R}, M^s_{2,1}). \) When the nonlinearity \( F \) has a simple form, say,

\[
iu_t + \Delta^2 u - \varepsilon \Delta u = \sum_{i=1}^{n} \lambda_i \partial_{x_i}^3 (u^{\kappa_i+1}), \quad u(0, x) = u_0(x),
\]

For this special forms, we have

**Theorem 1.2** Let \( n \geq 2, \kappa_i \geq 2 \vee \frac{8}{n}, \kappa_i \in \mathbb{N}, \lambda_i \in \mathbb{C}, \kappa = \min_{1 \leq i \leq n} \kappa_i. \) Assume that \( u_0 \in M^{3/2}_{2,1} \) and \( \|u_0\|_{M^{3/2}_{2,1}} \leq \delta \) for some small \( \delta > 0. \) Then (1.9) has a unique global solution \( u \in C(\mathbb{R}, M^{3/2}_{2,1}) \cap X, \) where

\[
\|u\|_{X_1} = \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}^n, |k_i| = k_{\max} > 4} (k_i)^3 \|\square_k u\|_{L^\infty_{t} L^2_{x_i} L^2_{x_j} (x_j)_{j \neq i} L^2_2 (\mathbb{R}^{1+n})} \\
+ \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}^n} (k_i)^{3/2-3/\kappa} \|\square_k u\|_{L^\infty_{t} L^\infty_{x_i} L^\infty_{x_j} L^\infty_2 (x_j)_{j \neq i} L^\infty_2 (\mathbb{R}^{1+n})} \\
+ \sum_{k \in \mathbb{Z}^n} (k_i)^{3/2} \|\square_k u\|_{L^\infty_{t} L^2_{x_i} L^2_{x_j} (x_j)_{j \neq i} L^2_{x_j} (\mathbb{R}^{1+n})},
\]

Moreover, \( \|u\|_{X_1} \lesssim \delta. \)

We remark that in Theorem 1.2 the same result holds if the nonlinear term \( \partial_{x_i}^3 (u^{\kappa_i+1}) \) is replaced by \( \partial_{x_i}^3 (|u|^{\kappa_i} u) \) \( (\kappa_i \in 2\mathbb{N}). \)
Corollary 1.3 Let \( n \geq 2, s > (n+3)/2 \). Let \( X \) and \( X_1 \) be as in Theorems 1.1 and 1.2, respectively. We have the following results.

(i) Let \( 2 \leq m \leq M < \infty, m > 8/n \). Assume that \( u_0 \in H^{s+3} \) and \( \| u_0 \|_{H^{s+1}} \leq \delta \) for some small \( \delta > 0 \). Then has a unique global solution \( u \in X \).

(ii) Let \( \kappa_i \geq 2 \vee \frac{2}{n}, \kappa_i \in \mathbb{N}, \lambda_i \in \mathbb{C} \). Assume that \( u_0 \in H^s \) and \( \| u_0 \|_{H^s} \leq \delta \) for some small \( \delta > 0 \). Then (1.9) has a unique global solution \( u \in X_1 \).

1.3 Notations

In this paper, we use the same notation as [26]. The following are some notations

\[ p \text{ denotes positive universal constants, which can be different at different places.} \]
\[ a \ll b \text{ stands for \( a \leq Cb \) for some constant } C > 1, \text{ a } \sim b \text{ means that } a \lesssim b \text{ and } b \lesssim a. \]
\[ \text{We write } a \wedge b = \min(a, b), a \vee b = \max(a, b), k_{max} = \max_{1 \leq i \leq n} |k_i|. \]
\[ \text{We denote by } p' \text{ the dual number of } p \in [1, \infty], \text{i.e., } 1/p + 1/p' = 1. \]

We will use Lebesgue spaces \( L^p(\mathbb{R}^n), \| \cdot \|_p := \| \cdot \|_{L^p} \), Sobolev spaces \( H^s = (I - \Delta)^{-s/2} L^2 \). Some properties of these function spaces can be found in [2] [24]. We will use the function spaces \( L^p_x L^q_t(\mathbb{R}^{n+1}) \) and \( L^p_x L^q_t(\mathbb{R}^{n+1}) \) for which the norms are defined by

\[ \| f \|_{L^p_x L^q_t(\mathbb{R}^{n+1})} = \left\| \left| f \right|_{L^p_x(\mathbb{R}^n)} \right\|_{L^q_t(\mathbb{R})}, \quad \| f \|_{L^p_x L^q_t(\mathbb{R}^{n+1})} = \left\| \left| f \right|_{L^p_x(\mathbb{R}^n)} \right\|_{L^q_t(\mathbb{R})}. \]

\[ L^p_{x,d}(\mathbb{R}^{n+1}) := L^p_x L^p_t(\mathbb{R}^{n+1}). \]

We denote by \( L^p_{x} L^p_{t} : L^p_{x} L^p_{t} := L^p_x L^p_{t}(\mathbb{R}^{1+n}) \) the anisotropic Lebesgue space for which the norm is defined by

\[ \| f \|_{L^p_x L^p_{t}(\mathbb{R}^{1+n})} := \left\| \left| f \right|_{L^p_x(\mathbb{R}^n)} \right\|_{L^p_x(\mathbb{R}^{1+n})}. \]

(1.11)

It is also convenient to use the notation \( L^p_{x_1} L^p_{x_2}, \ldots, L^p_{x_n} : L^p_{x_1} L^p_{x_2}, \ldots, L^p_{x_n} \).

For any \( 1 < k < n \), we denote by \( \mathcal{F}_{x_1, \ldots, x_k} \) the partial Fourier transform:

\[ (\mathcal{F}_{x_1, \ldots, x_k} f)(\xi_1, \ldots, \xi_k, x_{k+1}, \ldots, x_n) = \int_{\mathbb{R}^k} e^{-i(x_1\xi_1 + \ldots + x_k\xi_k)} f(x) dx_1 \ldots dx_k \]

(1.12)

and by \( \mathcal{F}_{-1}^{x_1, \ldots, x_k} \) the partial inverse Fourier transform, similarly for \( \mathcal{F}_{t,x} \) and \( \mathcal{F}_{-1}^{t, x} \).

\[ \mathcal{F} := \mathcal{F}_{x_1, \ldots, x_n}, \quad \mathcal{F}^{-1} := \mathcal{F}_{-1, x_1, \ldots, x_n}, \quad D^s_{x_i} = (-\partial^2_{x_i})^{s/2} = \mathcal{F}_{-1}^{-1} \xi_i \mathcal{F}_{xi} \]

expresses the partial Riesz potential in the \( x_i \) direction. \( \partial^s_{x_i} = \mathcal{F}_{-1}^{-1} (i\xi_i)^{-1} \mathcal{F}_{xi} \).

We will use the Bernstein multiplier estimate; cf. [2] [24]. For any \( r \in [1, \infty] \),

\[ \| \mathcal{F}^{-1} \varphi \mathcal{F} f \|_r \leq C \| \varphi \|_{H^s} \| f \|_r, \quad s > n/2. \]

(1.13)
We will use the frequency-uniform decomposition operators (cf. [29, 28, 27]). Let \( \{\sigma_k\}_{k \in \mathbb{Z}^n} \) be a function sequence satisfying
\[
\begin{align*}
\sigma_k(\xi) &\geq c, \quad \forall \xi \in Q_k, \\
\text{supp} \sigma_k &\subset \{\xi : |\xi - k| \leq \sqrt{n}\}, \\
\sum_{k \in \mathbb{Z}^n} \sigma_k(\xi) &\equiv 1, \quad \forall \xi \in \mathbb{R}^n, \\
|D^\alpha \sigma_k(\xi)| &\leq C_m, \quad \forall \xi \in \mathbb{R}^n, \quad |\alpha| \leq m \in \mathbb{N}.
\end{align*}
\]
(1.14)

Denote
\[
\Upsilon = \{\{\sigma_k\}_{k \in \mathbb{Z}^n} : \{\sigma_k\}_{k \in \mathbb{Z}^n} \text{ satisfies (1.14)}\}.
\]
(1.15)

Let \( \{\sigma_k\}_{k \in \mathbb{Z}^n} \in \Upsilon \) be a function sequence and
\[
\Box_k := \mathcal{F}^{-1} \sigma_k \mathcal{F}, \quad k \in \mathbb{Z}^n,
\]
(1.16)
which are said to be the frequency-uniform decomposition operators. One may ask the existence of the frequency-uniform decomposition operators. Indeed, let \( \rho \in \mathcal{S}(\mathbb{R}^n) \) and \( \rho : \mathbb{R}^n \to [0, 1] \) be a smooth radial bump function adapted to the ball \( B(0, \sqrt{n}) \), say \( \rho(\xi) = 1 \) as \( |\xi| \leq \sqrt{n}/2 \), and \( \rho(\xi) = 0 \) as \( |\xi| \geq \sqrt{n} \). Let \( \rho_k \) be a translation of \( \rho \): \( \rho_k(\xi) = \rho(\xi - k) \), \( k \in \mathbb{Z}^n \). We write
\[
\eta_k(\xi) = \rho_k(\xi) \left( \sum_{k \in \mathbb{Z}^n} \rho_k(\xi) \right)^{-1}, \quad k \in \mathbb{Z}^n.
\]
(1.17)

We have \( \{\eta_k\}_{k \in \mathbb{Z}^n} \in \Upsilon \). It is easy to see that for any \( \{\eta_k\}_{k \in \mathbb{Z}^n} \in \Upsilon \),
\[
\|f\|_{M_{\Box, s}^2} \sim \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{s} \|\Box_k f\|_{L^2(\mathbb{R}^n)}.
\]

We will use the function space \( \ell_{\Box}^s(L_t^p L_x^r(I \times \mathbb{R}^n)) \) which contains all of the functions \( f(t,x) \) so that the following norm is finite:
\[
\|f\|_{\ell_{\Box}^s(L_t^p L_x^r(I \times \mathbb{R}^n))} := \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{s} \|\Box_k f\|_{L_t^p L_x^r(I \times \mathbb{R}^n)}.
\]
(1.18)

For simplicity, we write \( \ell_{\Box}(L_t^p L_x^r(I \times \mathbb{R}^n)) = \ell_{\Box}^{1,0}(L_t^p L_x^r(I \times \mathbb{R}^n)) \).

This paper is organized as follows. In Section 2 we show the smooth effect estimates of the solutions of the fourth order linear Schrödinger equation in anisotropic Lebesgue spaces with \( \Box_k \)-decomposition. In Sections 3 and 4 we consider the frequency-uniform localized versions for the global maximal function estimates, the smooth effects with \( \Box_k \)-decomposition, together with their relations to the Strichartz estimates. In Sections 5 and 6 we prove our Theorems 1.2 and 1.1, respectively.
2 Smooth effects with $\Box_k$-decomposition

In this paper, we always denote

$$S(t) = e^{i[H(A^2 - \Delta) + \varepsilon \Delta]} = \mathcal{F}^{-1} e^{i[l(|\xi|^4 + |\xi|^2)]} \mathcal{F}, \quad \mathcal{A}f(t, x) = \int_0^t S(t - \tau) f(\tau, x) d\tau.$$ 

where $\varepsilon = 0, 1$.

**Proposition 2.1** For any $k = (k_1, ..., k_n) \in \mathbb{Z}^n$, $|k_i| = k_{\text{max}}$, $i = 1, ..., n$, we have

$$\| \Box_k \partial_{\xi_i}^3 \mathcal{A}f \|_{L^2_{\xi} L^2_{x} L^1_{\xi} L^2_{(\mathbb{R}^1 + n)}} \lesssim \| \Box_k f \|_{L^1_{x} L^2_{(\mathbb{R}^1 + n)}}$$

(2.1)

**Proof.** Firstly, we assume $\varepsilon = 0$. We only give the proof of the case $i = 1$, the other cases is identical due to the symmetry. Observing that

$$\partial_{\xi_1}^3 \mathcal{A}f = c \mathcal{F}_{\tau, \xi_1}^{-1} \frac{\xi_1^3}{|\xi|^4 - \tau} \mathcal{F}_{\tau} f$$

(2.2)

Using Plancherel’s identity, it is equivalent to prove

$$\left\| \mathcal{F}_{\xi_1}^{-1} \sigma_k(\xi) \xi_1^3 \mathcal{F}_{\xi_1} f \right\|_{L^2_{\xi_1} L^2_{(\mathbb{R}^1 + n)}} \lesssim \left\| \mathcal{F}_{\xi_1}^{-1} \sigma_k(\xi) f \right\|_{L^1_{\xi_1} L^2_{(\mathbb{R}^1 + n)}}$$

(2.3)

Using Young’s inequality, it is suffices to prove

$$\sup_{x_1, \tau, \xi_1(j \neq 1)} \left| \mathcal{F}_{\xi_1}^{-1} \frac{\sigma_k(\xi) \xi_1^3}{|\xi|^4 - \tau} \right| \lesssim C.$$  

(2.4)

We give the proof of (2.4) according to $\tau > 0$ or $\tau \leq 0$. Observing that in this case we have $|\xi_1| \sim \max |\xi_j|, j = 1, ..., n$, $|\xi| - k \leq \sqrt{n}$. Therefore, when $\tau \leq 0$ we have

$$\sup_{x_1, \tau, \xi_1(j \neq 1)} \left| \mathcal{F}_{\xi_1}^{-1} \frac{\sigma_k(\xi) \xi_1^3}{|\xi|^4 - \tau} \right| \lesssim \int_{|\xi_1| \sim k} \frac{1}{\xi_1} d\xi_1 \lesssim C$$

When $\tau > 0$, for simplicity we drop $\sigma_k(\xi)$. From $|\xi|^4 - \tau = (|\xi|^2 - \sqrt{\tau})(|\xi|^2 + \sqrt{\tau})$, and let $|\xi|^2 = \sum_{j=2}^n \xi_j^2$. We have

$$\int \frac{\xi_1^3}{|\xi|^4 - \tau} e^{i\xi_1 \xi_1} d\xi_1$$

$$= \int \frac{\xi_1^3}{(|\xi|^2 - \sqrt{\tau})(|\xi|^2 + \sqrt{\tau})} e^{i\xi_1 \xi_1} d\xi_1$$
\[
GWP \text{ for derivative } 4NLS
\]

\[
\int \frac{\xi_1^3}{(\xi_1^2 + |\xi|^2 - \sqrt{\tau})(\xi_1^2 + |\xi|^2 + \sqrt{\tau})} e^{ix_1 \xi_1} d\xi_1
\]

When \(|\xi|^2 - \sqrt{\tau} \geq 0\), we easily get the desired result.
When \(|\xi|^2 - \sqrt{\tau} < 0\), we let \(A^2 = -(|\xi|^2 - \sqrt{\tau})\) and \(B^2 = |\xi|^2 + \sqrt{\tau}\). We get

\[
\int \frac{\xi_1^3}{(\xi_1^2 + |\xi|^2 - \sqrt{\tau})(\xi_1^2 + |\xi|^2 + \sqrt{\tau})} e^{ix_1 \xi_1} d\xi_1 \tag{2.5}
\]

\[
= \int \frac{\xi_1}{\xi_1^2 - A^2} \cdot \frac{\xi_1^2}{\xi_1^2 + B^2} e^{ix_1 \xi_1} d\xi_1
\]

\[
= \frac{1}{2} \int \left( \frac{1}{\xi_1 + A} + \frac{1}{\xi_1 - A} \right) \frac{\xi_1^2}{\xi_1^2 + B^2} e^{ix_1 \xi_1} d\xi_1
\]

\[
= I + II
\]

For part I, we have

\[
I = \frac{1}{2} \int \frac{1}{\xi_1 + A} \frac{\xi_1^2 + B^2 - B^2}{\xi_1^2 + B^2} e^{ix_1 \xi_1} d\xi_1
\]

\[
= \frac{1}{2} \int \frac{1}{\xi_1 + A} e^{ix_1 \xi_1} d\xi_1 + \frac{1}{2} \int \frac{1}{\xi_1 + A} \frac{-B^2}{\xi_1^2 + B^2} e^{ix_1 \xi_1} d\xi_1
\]

\[
= I_1 + I_2
\]

The part \(I_1\) is bounded owing to Hilbert transform. For \(I_2\), by changes of variables, it suffices to show

\[
\sup_{x_1} \int \frac{1}{1 + \xi_1^2 + D^2 \xi_1^2} e^{ix_1 \xi_1} d\xi_1 \lesssim C \tag{2.6}
\]

where \(D = \frac{A}{B}\). Using the fact that \(\mathcal{F}(e^{-|x|})(\xi) = C \frac{1}{1 + |\xi|^2}\), we have

\[
\left\| \int \frac{1}{1 + \xi_1^2 + D^2 \xi_1^2} e^{ix_1 \xi_1} d\xi_1 \right\|_{L^\infty_{x_1}} = \left\| \mathcal{F}^{-1}_{\xi_1} \left( \frac{1}{1 + \xi_1} \right) \ast \mathcal{F}^{-1}_{\xi_1} \left( \frac{1}{1 + D^2 \xi_1^2} \right) \right\|_{L^\infty_{x_1}}
\leq \left\| \mathcal{F}^{-1}_{\xi_1} \left( \frac{1}{1 + \xi_1^2} \right) \right\|_{L^\infty_{x_1}} \left\| \mathcal{F}^{-1}_{\xi_1} \left( \frac{1}{1 + D^2 \xi_1^2} \right) \right\|_{L^\infty_{x_1}}
\leq \left\| \mathcal{F}^{-1}_{\xi_1} \left( \frac{1}{1 + \xi_1^2} \right) \right\|_{L^\infty_{x_1}} \left\| \frac{1}{D} e^{-\frac{ix_1 \xi_1^2}{D}} \right\|_{L^\infty_{x_1}}
\leq C
\]

The part \(I_2\) is similar to \(I\), so we get the result desired. Now we consider the case \(\epsilon = 1\). Comparing the proof of the case \(\epsilon = 0\), it suffices to show
\[ \sup_{x_1, \tau, \xi_j (j \neq 1)} \left| \mathcal{F}_{\xi_1}^{-1} \left( \sigma_k(\xi) \xi_1^3 \right) \right| \lesssim C. \]

When \( \tau \leq 0 \), the proof is identical to the case \( \varepsilon = 0 \). Observing that when \( \tau > 0 \), we can choose \( \tau_2 = -\frac{1}{2} + \sqrt{\frac{1}{4} + \tau} > 0 \) such that

\[ |\xi|^4 + |\xi|^2 - \tau = (|\xi|^2 - \tau)(|\xi|^2 + \tau_2 + 1) \quad (2.7) \]

which is turn to (2.5).

**Remark 2.2** We assume \( |k_i| = k_{\text{max}} \) in Prop 2.1. For the general case, see Section 3 for details.

**Proposition 2.3** For any \( k = (k_1, \ldots, k_n) \in \mathbb{Z}^n \), \( |k_i| = k_{\text{max}}, i = 1, \ldots, n \), we have

\[ \| \square_k D_x^{3/2} S(t) u_0 \|_{L^2_{\xi_1} L^2_{\xi_j} \neq 1 L^2_{\xi_1} (\mathbb{R}^{1+n})} \lesssim \| \square_k u_0 \|_{L^2} \quad (2.8) \]

**Proof.** As Prop 2.1, we only need to prove the case \( i = 1 \). By Plancherel’s identity, we have

\[ \| \square_k D_x^{3/2} S(t) u_0 \|_{L^2_{\xi_1} L^2_{\xi_j} \neq 1 L^2_{\xi_1} (\mathbb{R}^{1+n})} = \left\| \int \sigma_k(\xi) |\xi_1|^3 e^{it(|\xi|^4 + \varepsilon |\xi|^2)} \hat{u}_0(\xi) e^{ix_1 \xi_1} d\xi \right\|_{L^2_{\xi_1} L^2_{\xi_j} \neq 1 L^2_{\xi_1} (\mathbb{R}^{1+n})} \]

We can assume \( \xi_1 > 0 \), otherwise we let \( \xi_1' = -\xi_1 \). Making variables change \( \eta = |\xi|^4 + \varepsilon |\xi|^2 \) and using Plancherel’s identity, we have

\[ \left\| \int \sigma_k(\xi) \xi_1^3 e^{it|\xi|^4} \hat{u}_0(\xi) e^{ix_1 \xi_1} d\xi \right\|_{L^2_{\xi_1} L^2_{\xi_j} \neq 1 L^2_{\xi_1} (\mathbb{R}^{1+n})} \lesssim \left\| \int \sigma_k(\xi) \xi_1^3 |\xi|^2 e^{it\eta} \hat{u}_0(\xi(\eta)) e^{ix_1 \xi_1(\eta)} (|\xi|^2 + \varepsilon)^{-1} d\eta \right\|_{L^2_{\xi_1} L^2_{\xi_j} \neq 1 L^2_{\xi_1} (\mathbb{R}^{1+n})} \lesssim \left\| \int \sigma_k(\xi) \xi_1^3 \hat{u}_0(\xi(\eta)) (|\xi|^2 + \varepsilon)^{-1} |\xi|^2 + \varepsilon \right\|_{L^2_{\xi_1} L^2_{\xi_j} \neq 1 (\mathbb{R}^n)} \lesssim \left\| \sigma_k(\xi) \xi_1^3 \hat{u}_0(\xi)(|\xi|^2 + \varepsilon)^{-1/2} d\xi \right\|_{L^2_{\xi_1} L^2_{\xi_j} \neq 1 (\mathbb{R}^n)} = \| u_0 \|_{L^2} \]

\( \Box \)
By the duality of (2.8), we have the following

**Proposition 2.4** For any \( k = (k_1, ..., k_n) \in \mathbb{Z}^n \), \( |k| = k_{\text{max}}, i = 1, ..., n \), we have

\[
\left\| \Box_k \partial_x^3 f \right\|_{L^\infty_t L^2_x(\mathbb{R}^{1+n})} \lesssim \left\| \Box_k D^{3/2}_{x_i} f \right\|_{L^1_t L^2_{(x_1, x_2, ..., x_n)}(\mathbb{R}^{1+n})}. \quad (2.9)
\]

### 3 Linear estimates with \( \Box_k \)-decomposition

In this section we consider the smooth effect estimates, the maximal function estimates, the Strichartz estimates and their interaction estimates for the solutions of the fourth order Schrödinger equations by using the frequency-uniform decomposition operators. For convenience, we will use the following function sequence \( \{\sigma_k\}_{k \in \mathbb{Z}^n} \). Now we recall some results in [26].

**Lemma 3.1** Let \( \eta_k : \overline{\mathbb{R}} \rightarrow [0, 1] \) \((k \in \mathbb{Z})\) be a smooth-function sequence satisfying condition (1.14). Denote

\[
\sigma_k(x) := \eta_{k_1}(x_1) \cdots \eta_{k_n}(x_n), \quad k = (k_1, ..., k_n). \quad (3.1)
\]

Then we have \( \{\sigma_k\}_{k \in \mathbb{Z}^n} \in \Upsilon \).

**Lemma 3.2** For any \( \sigma \in \mathbb{R} \) and \( k = (k_1, ..., k_n) \in \mathbb{Z}^n \) with \( |k| \geq 4 \), we have

\[
\left\| \Box_k D^\sigma_{x_i} u \right\|_{L^{p_1}_{x_1} L^{p_2}_{x_2} \cdots L^{p_n}_{x_n} L^2_t(\mathbb{R}^{1+n})} \lesssim \langle k_i \rangle^\sigma \left\| \Box_k u \right\|_{L^{p_1}_{x_1} L^{p_2}_{x_2} \cdots L^{p_n}_{x_n} L^2_t(\mathbb{R}^{1+n})}. \quad (3.2)
\]

Replacing \( D^\sigma_{x_i} \) by \( \partial^\sigma_{x_i} \) \((\sigma \in \mathbb{N})\), the above inequality holds for all \( k \in \mathbb{Z}^n \).

**Remark 3.3** From the proof, we can get that the \( \lesssim \) in Lemma 3.2 can be strengthened as \( \sim \). In fact, taking \( v = D^\sigma_{x_i} u \) in (3.2) and noticing \( \sigma \in \mathbb{R} \), we can get the reversal inequality of (3.2).

The next lemma is essentially known, see [24, 29].

**Lemma 3.4** Let \( \Omega \subset \mathbb{R}^n \) be a compact set with \( \text{diam} \Omega < 2R \), \( 0 < p \leq q \leq \infty \). Then there exists a constant \( C > 0 \), which depends only on \( p, q \) such that

\[
\|f\|_q \leq C R^{n(1/p - 1/q)} \|f\|_p, \quad \forall f \in L^p_\Omega,
\]

where \( L^p_\Omega = \{ f \in S'(\mathbb{R}^n) : \text{supp} \hat{f} \subset \Omega, \|f\|_p < \infty \} \).
Here we emphasize that we can find a constant \( C > 0 \) uniformly holds for all \( k \in \mathbb{Z}^n \) in Lemma 3.4.

It is known that \( S(t) \) satisfy the following \( L^p - L^{p'} \) estimate:
\[
\| S(t)f \|_p \lesssim |t|^{-n/4(1-2/p)} \| f \|_{p'} , \quad |t| \geq 1; 2 \leq p \leq \infty, \tag{3.3}
\]
Using the same procedure as in \[29\], we have
\[
\| \Box_k S(t)f \|_p \lesssim \sum_{l \in \Lambda} \| \Box_{k+l}f \|_{p'} , \quad 2 \leq p \leq \infty \tag{3.4}
\]
where \( \Lambda = \{ l \in \mathbb{Z}^n : B(0, \sqrt{n}) \cap B(l, \sqrt{n}) \neq \emptyset \} \).

Combining (3.3) and (3.4), we have
\[
\| \Box_k S(t)f \|_p \lesssim (1 + |t|)^{-n/4(1-2/p)} \sum_{l \in \Lambda} \| \Box_{k+l}f \|_{p'} , \quad 2 \leq p \leq \infty \tag{3.5}
\]
Using (3.5) and following the procedure in \[28\], we get the following

**Lemma 3.5** Let \( 2 \leq p < \infty, \gamma \geq 2 \vee \gamma(p), \)
\[
\frac{4}{\gamma(p)} = n\left(\frac{1}{2} - \frac{1}{p}\right).
\]
Then we have
\[
\| S(t)f \|_{\ell^1_{\Box}(L^p(\mathbb{R}, L^p(\mathbb{R}^n)))} \lesssim \| f \|_{M_{2,1}(\mathbb{R}^n)},
\]
\[
\| \mathcal{A}f \|_{\ell^1_{\Box}(L^p(\mathbb{R}, L^p(\mathbb{R}^n))) \cap \ell^1_{\Box}(L^\infty(\mathbb{R}, L^2(\mathbb{R}^n)))} \lesssim \| f \|_{\ell^1_{\Box}(L^{p'}(\mathbb{R}, L^{p'}(\mathbb{R}^n)))).
\]
In particular, if \( 2 + 8/n \leq p < \infty \), then we have
\[
\| S(t)f \|_{\ell^1_{\Box}(L^p_{t,x}(\mathbb{R}^{1+n}))} \lesssim \| f \|_{M_{2,1}(\mathbb{R}^n)},
\]
\[
\| \mathcal{A}f \|_{\ell^1_{\Box}(L^p_{t,x}(\mathbb{R}^{1+n})) \cap \ell^1_{\Box}(L^\infty(\mathbb{R}, L^2(\mathbb{R}^{1+n})))} \lesssim \| f \|_{\ell^1_{\Box}(L^{p'}_{t,x}(\mathbb{R}^{1+n})).
\]

In \[13\], when \( n \geq 3 \) Ionescu and Kenig showed the following maximal function estimates:
\[
\| \Delta_k S(t)u_0 \|_{L^2_{t,x} L^\infty_{x,j \neq i} L^\infty_{x} (\mathbb{R}^{1+n})} \lesssim 2^{(n-1)k/2} \| \Delta_k u_0 \|_{L^2(\mathbb{R}^n)}. \tag{3.6}
\]
Combining their idea and the frequency-uniform decomposition operators as \[26\], we obtain the following

**Proposition 3.6** Let \( 8/n < q \leq \infty, \gamma \geq 2 \) and \( k = (k_1, \ldots, k_n) \in \mathbb{Z}^n \), we have
\[
\| \Box_k S(t)u_0 \|_{L^q_{t,x} L^\infty_{x,j \neq i} L^\infty_{x} (\mathbb{R}^{1+n})} \lesssim \langle k_{\max} \rangle^{3/q} \| \Box_k u_0 \|_{L^2(\mathbb{R}^n)}. \tag{3.7}
\]
Proof. It suffices to prove

\[ \left\| \int_{\mathbb{R}^n} \delta_k(\xi)e^{ix\xi}e^{it(|\xi|^4+\varepsilon|\xi|^2)}u_0 d\xi \right\|_{L_t^{\infty}L_x^\infty} \lesssim \langle k_{\text{max}} \rangle^{3/q} \| \Box u_0 \|_{L^2(\mathbb{R}^n)} \]  

(3.8)

By a standard \(TT^*\) method, it suffices to prove

\[ \left\| \int_{\mathbb{R}^{n-1} \times \mathbb{R}} \delta_k(\xi)e^{ix_1\xi_1}e^{ix_2\xi_2}e^{it(|\xi|^4+\varepsilon|\xi|^2)}u_0(\xi)d\xi_1d\xi_2 \right\|_{L_t^{\infty}L_x^{\infty}} \lesssim \langle k_{\text{max}} \rangle^{6/q} \]  

(3.9)

where \(\bar{k} = (k_2, ..., k_n), \bar{x} = (x_2, ..., x_n), \bar{\xi} = (\xi_2, ..., \xi_n)\). For convenience, we give the details of (3.9) when \(n = 2\). The general case can be treated similarly.

When \(n = 2\), we can write (3.9) as following

\[ \left\| \int_{\mathbb{R} \times \mathbb{R}} \delta_k(\xi)e^{ix_1\xi_1}e^{ix_2\xi_2}e^{it(|\xi|^4+\varepsilon|\xi|^2)}u_0(\xi)d\xi_1d\xi_2 \right\|_{L_t^{\infty}L_x^{\infty}} \lesssim \langle k_{\text{max}} \rangle^{6/q} \]  

(3.10)

Making variable changes \(\xi_j - k_j = \mu_j, j = 1, 2\), ie. \(\xi - k = \mu\), we have

\[ \left\| \int_{\mathbb{R} \times \mathbb{R}} \delta_0(\mu) \prod_{j=1}^{2} e^{ix_jk}e^{ix_j\mu_j}e^{it(|\xi+k|^4+\varepsilon|\xi+k|^2)}u_0(\mu+k)d\mu_1d\mu_2 \right\|_{L_t^{\infty}L_x^{\infty}} \lesssim \langle k_{\text{max}} \rangle^{6/q} \]  

(3.11)

We use the \(L\) denotes the left of (3.11) in later proof. Expanding the term \(|\xi+k|^4+\varepsilon|\xi+k|^2\), we obtain three sorts of terms.

Firstly, the term such as \(k_1^4, k_2^4, ..., 2(k_1k_2)^2\) have no relation with the integral variable \(\mu\). For these terms, we have \(|e^{itk_1^2}| = |e^{itk_2^2}| = ... = |e^{2it(k_1k_2)^2}| = 1\).

Secondly, noticing \(\mu \sim 0\), we can treat terms such as \(\mu_1^4, \mu_2^4, (\mu_1\mu_2)^2\) as the first case.

Finally, the main contribution to the \(L\) in the remainder terms such as \(\mu_1k_1^3, \mu_2k_2^3\). We treat this case according to \(|k_1| = k_{\text{max}}\) or \(|k_2| = k_{\text{max}}\). When \(|k_1| = k_{\text{max}}\), for (3.11), we need to prove

\[ L \lesssim \langle k_1 \rangle^{6/q} \]  

(3.12)

Using the above analysis about \(|\xi+k|^4+\varepsilon|\xi+k|^2\), we have

\[ L \approx \left\| \int_{\mathbb{R} \times \mathbb{R}} \delta_0(\mu) \prod_{j=1}^{2} e^{ix_jk}e^{ix_j\mu_j}e^{it(\mu_1k_1^4+\mu_2k_2^4)}u_0(\mu+k)d\mu_1d\mu_2 \right\|_{L_t^{\infty}L_x^{\infty}} \]  

(3.13)
In view of Lemma 3.1, we can write \( \Box_k = \mathcal{F}^{-1} \xi \eta_k(\xi) \), \( \mathcal{F} := \mathcal{F}^{-1} \xi \eta_k(\xi) \mathcal{F} \).

For one thing, in view of the decay of \( \Box_k S(t) \)

\[
\| \mathcal{F}^{-1} e^{it|\xi|^4 + |\xi|^2} \eta_k(\xi) \eta_k(\xi) \mathcal{F} \|_{L^p(R^{n-1})} \lesssim (1 + |t|)^{-(n-1)/4},
\]

\[
\| \mathcal{F}^{-1} e^{it|\xi|^4 + |\xi|^2} \eta_k(\xi) \|_{L^p(R)} \lesssim (1 + |t|)^{-(n-1)/4}.
\]

For other, integrating by part when \( x_1 \geq 4k^3_3 |t| + 1 \),

\[
\| \mathcal{F}^{-1} e^{i(x_1^3 + |\xi|^2)} \eta_k(\xi) \| \lesssim |x_1|^{-2}.
\]

Therefore, when \( x_1 \geq 4k^3_1 |t| + 1 \) we have

\[
L \lesssim \| |x_1|^{-2} \|_{L^{q/2}(R)} \lesssim C
\]  
(3.14)

When \( x_1 \leq 4k^3_3 |t| + 1 \), observing \((1 + |t|)^{-n/4} \lesssim (k^3_1)^{-n/4} (|k^3_1| + |x_1|)^{-n/4} \) we have

\[
L \lesssim \langle k^3_1 \rangle^{-1/2} \| (|k^3_1| + |x_1|)^{-n/4} \|_{L^{q/2}(R)}
\]

\[
\lesssim \langle k^3_1 \rangle^{6/q}
\]  
(3.15)

Observing the above argument also holds with change the place of \( k_1 \) and \( k_2 \) when \( |k_2| = k_{\text{max}} \). From this, we can see why the right of (3.7) is \( k_{\text{max}} \).

By duality of Proposition 3.6, we have the following:

**Proposition 3.7** For \( 2 \leq q \leq \infty, q > 8/n \) and \( k = (k_1, ..., k_n) \in \mathbb{Z}^n \), we have

\[
\left\| \Box_k \int_{\mathbb{R}} S(t - \tau)f(\tau) d\tau \right\|_{L^\infty L^2(R^{1+n})} \lesssim \langle k_{\text{max}} \rangle^{3/q} \| \Box_k f \|_{L^q_{t, \xi} L^{1}_x(R^{1+n})}.
\]  
(3.16)

In view of Propositions 2.1 and 2.4 we have

**Proposition 3.8** For any \( k = (k_1, ..., k_n) \in \mathbb{Z}^n \) and \( |k_i| = k_{\text{max}}, i = 1, ..., n \), we have

\[
\| \Box_k \mathcal{A} \partial^3_{x_i} f \|_{L^\infty_{t, \xi} L^2_{\xi}(R^{1+n})} \lesssim \| \Box_k f \|_{L^1_{t, \xi} L^2_{\xi}(R^{1+n})} \lesssim \| \Box_k f \|_{L^1_{t, \xi} L^2_{\xi}(R^{1+n})},
\]  
(3.17)

\[
\| \Box_k \mathcal{A} \partial^3_{x_i} f \|_{L^\infty_{t, \xi} L^2_{\xi}(R^{1+n})} \lesssim \langle k_{\text{max}} \rangle^{3/2} \| \Box_k f \|_{L^1_{t, \xi} L^2_{\xi}(R^{1+n})},
\]  
(3.18)

**Proof.** (3.17) holds by Proposition 2.1 directly. In the case \( |k_i| \geq 4 \), (3.18) holds by Proposition 2.3 and Lemma 3.2. In the case \( |k_i| \leq 3 \), in view of Proposition 2.4

\[
\| \Box_k \mathcal{A} \partial^3_{x_i} f \|_{L^\infty_{t, \xi} L^2_{\xi}(R^{1+n})} \lesssim \| D^{-3/2}_{x_i} \Box_k \mathcal{A} \partial^3_{x_i} f \|_{L^\infty_{t, \xi} L^2_{\xi}(R^{1+n})} \lesssim \| \Box_k f \|_{L^1_{t, \xi} L^2_{\xi}(R^{1+n})},
\]

which implies the result, as desired.

By the duality and Christ-Kiselev’s Lemma in anisotropic Lebesgue spaces 26, we have the following
Proposition 3.9 For \(2 \leq q \leq \infty, q > 8/n\) and \(k = (k_1, \ldots, k_n) \in \mathbb{Z}^n, |k_i| = k_{\text{max}}, i = 1, \ldots, n\), we have

\[
\| \Box_k \mathcal{A}^3 \psi \|_{L_{t}^{p}(\mathbb{R}^{1+n})} \lesssim \langle k_{\text{max}} \rangle^{3/2+3/q} \| \Box_k f \|_{L_{x}^{2}(\mathbb{R}^{1+n})}. \tag{3.19}
\]

Proposition 3.10 Let \(2 \leq r < \infty, 4/\gamma(r) = n(1/2 - 1/r)\) and \(\gamma > \gamma(r) \vee 2\). We have

\[
\| \Box_k \mathcal{A} f \|_{L_{t}^{p}(\mathbb{R}^{1+n})} \lesssim \| \Box_k u_0 \|_{L^2(\mathbb{R}^n)}, \tag{3.20}
\]

\[
\| \Box_k \mathcal{A}^2 \psi \|_{L_{t}^{p}(\mathbb{R}^{1+n})} \lesssim \| \Box_k f \|_{L_{t}^{2}(\mathbb{R}^{1+n})}. \tag{3.21}
\]

\[
\| \Box_k \mathcal{A}^3 \psi \|_{L_{t}^{p}(\mathbb{R}^{1+n})} \lesssim \langle k_{\text{max}} \rangle^{3/2} \| \Box_k f \|_{L_{t}^{2}(\mathbb{R}^{1+n})}, \tag{3.22}
\]

\[
\| \Box_k \mathcal{A}^3 \psi \|_{L_{t}^{p}(\mathbb{R}^{1+n})} \lesssim \langle k_{\text{max}} \rangle^{3/2} \| \Box_k f \|_{L_{t}^{2}(\mathbb{R}^{1+n})}. \tag{3.23}
\]

and for \(2 \leq q < \infty, q > 8/n, \alpha = 0, 3,\)

\[
\| \Box_k \mathcal{A}^\alpha \psi \|_{L_{t}^{p}(\mathbb{R}^{1+n})} \lesssim \langle k_{\text{max}} \rangle^{\alpha+3/q} \| \Box_k f \|_{L_{t}^{2}(\mathbb{R}^{1+n})}. \tag{3.24}
\]

Proof. (3.20) and (3.21) hold by 3.5 We now show (3.22). We use the same notations as in Proposition 3.9 By Lemmas 3.5 3.2 and Proposition 3.8

\[
\mathcal{L}_k(\mathcal{A}^3 f, \psi) \lesssim \langle k_{\text{max}} \rangle^{3/2} \| \mathcal{A} f \|_{L_{t}^{1/2}, \ldots, L_{t}^{2}(\mathbb{R}^{1+n})} \| \Box_k \psi \|_{L_{t}^{2}(\mathbb{R}^{1+n})} \lesssim \langle k_{\text{max}} \rangle^{3/2} \| \mathcal{A} f \|_{L_{t}^{1/2}, \ldots, L_{t}^{2}(\mathbb{R}^{1+n})} \| \mathcal{A} \psi \|_{L_{t}^{2}(\mathbb{R}^{1+n})}. \tag{3.25}
\]

By duality, (3.22) holds by (3.32) and Christ-Kiselev’s Lemma. Exchanging the roles of \(f\) and \(\psi\), we immediately have (3.23). (3.24) holds by Lemmas 3.5 3.2 and Proposition 3.7.

we summarize the main conclusion of this section as following:

Corollary 3.11 For \(8/n \leq p < \infty, 2 \leq q < \infty, q > 8/n\) and \(k = (k_1, \ldots, k_n) \in \mathbb{Z}^n, |k_i| = k_{\text{max}}, i = 1, \ldots, n\). We have

\[
\| D^{3/2} \Box_k \mathcal{A} f \|_{L_{t}^{p}(\mathbb{R}^{1+n})} \lesssim \| \Box_k u_0 \|_{L^2(\mathbb{R}^n)}, \tag{3.26}
\]

\[
\| \Box_k \mathcal{A} f \|_{L_{t}^{p}(\mathbb{R}^{1+n})} \lesssim \langle k_{\text{max}} \rangle^{3/2} \| \mathcal{A} f \|_{L_{t}^{2}(\mathbb{R}^{1+n})}. \tag{3.27}
\]

\[
\| \Box_k \mathcal{A}^2 f \|_{L_{t}^{p}(\mathbb{R}^{1+n})} \lesssim \| \mathcal{A} f \|_{L_{t}^{2}(\mathbb{R}^{1+n})}. \tag{3.28}
\]

\[
\| \Box_k \mathcal{A}^3 f \|_{L_{t}^{p}(\mathbb{R}^{1+n})} \lesssim \| \mathcal{A} f \|_{L_{t}^{2}(\mathbb{R}^{1+n})}. \tag{3.29}
\]
\[
\| k^\gamma \partial^3_t f \|_{L^3_t L^3_x} \leq \| L^3_t L^3_x (1 + \text{max} \| f \|_{L^3_t L^3_x}) \leq \| k^\gamma \partial^3_t f \|_{L^3_t L^3_x} (1 + \text{max} \| f \|_{L^3_t L^3_x})^q, \] (3.30)
\[
\| k^\gamma \partial^3_t f \|_{L^3_t L^3_x} \leq \| k^\gamma \partial^3_t f \|_{L^3_t L^3_x} (1 + \text{max} \| f \|_{L^3_t L^3_x})^q, \] (3.31)
\[
\| k^\gamma \partial^3_t f \|_{L^3_t L^3_x} \leq \| k^\gamma \partial^3_t f \|_{L^3_t L^3_x} (1 + \text{max} \| f \|_{L^3_t L^3_x})^q, \] (3.32)
\[
\| k^\gamma \partial^3_t f \|_{L^3_t L^3_x} \leq \| k^\gamma \partial^3_t f \|_{L^3_t L^3_x} (1 + \text{max} \| f \|_{L^3_t L^3_x})^q, \] (3.33)
\[
\| k^\gamma \partial^3_t f \|_{L^3_t L^3_x} \leq \| k^\gamma \partial^3_t f \|_{L^3_t L^3_x} (1 + \text{max} \| f \|_{L^3_t L^3_x})^q. \] (3.34)

4 Linear estimates with derivative interaction

Recall that in Prop. \[2.1\] we assume that \(|k_i| = k_{\text{max}}\) for any \(k \in \mathbb{Z}^n\). In view of (3.29) in Corollary \[3.11\], the operator \(\mathcal{A}\) in the space \(L^\infty_t L^2_x\) has succeeded in absorbing the partial derivative \(\partial^3_t\). However, it seem that \(\mathcal{A}\) can not deal with the partial derivative \(\partial^2_x\) in the space \(L^\infty_t L^2_x\). So, we need a new way to handle the interaction between \(L^\infty_t L^2_x\) and \(\partial^3_t\).

**Proposition 4.1** For \(i = 2, ..., n, 2 \leq q \leq \infty, q > 8/n\). Let \(4 \leq r < \infty, 2/\gamma(r) = n(1/2 - 1/r), \gamma > 2 \vee \gamma(r)4\).

For \(|k| = k_{\text{max}}\), we have

\[
\| k^\gamma \partial^3_t f \|_{L^3_t L^3_x} \leq \| k^\gamma \partial^3_t f \|_{L^3_t L^3_x} (1 + \text{max} \| f \|_{L^3_t L^3_x})^q. \] (4.1)
\[
\| k^\gamma \partial^3_t f \|_{L^3_t L^3_x} \leq \| k^\gamma \partial^3_t f \|_{L^3_t L^3_x} (1 + \text{max} \| f \|_{L^3_t L^3_x})^q. \] (4.2)

For \(|k| = k_{\text{max}}\), we have

\[
\| k^\gamma \partial^3_t f \|_{L^3_t L^3_x} \leq \| k^\gamma \partial^3_t f \|_{L^3_t L^3_x} (1 + \text{max} \| f \|_{L^3_t L^3_x})^q. \] (4.3)
\[
\| k^\gamma \partial^3_t f \|_{L^3_t L^3_x} \leq \| k^\gamma \partial^3_t f \|_{L^3_t L^3_x} (1 + \text{max} \| f \|_{L^3_t L^3_x})^q. \] (4.4)

**Proof.** From Proposition \[2.1\] we can get (4.1) directly. As before, we only give the proof when \(i = 2\). For (4.2), because of

\[
\mathcal{L}(\partial^3_t f, \psi) := \left\| \int_{\mathbb{R}} \left( \int_{\mathbb{R}} S(t - \tau) \partial^3_x \partial^3_t \psi(t) d\tau \right) dt \right\|
\leq \| \int_{\mathbb{R}} S(-\tau) \partial^3_x \partial^3_t \psi(t) d\tau \|_{L^2(\mathbb{R}^n)} \| \tilde{\partial}^3_t \psi(t) \|_{L^2(\mathbb{R}^n)}. \] (4.5)
By the Strichartz inequality and Proposition 2.4

\[ \mathcal{L}(\partial_{x_2}^3 f, \psi) \lesssim \| \partial_{x_2}^3 D_{-3/2} k f \|_{L^2_t L^\infty_x(\mathbb{R}^{1+n})} \| \psi \|_{L^1_t L^2_{x_1} L^2_{x_2} \ldots L^2_{x_n} L^2_t(\mathbb{R}^{1+n})}. \]  

(4.6)

By duality, (4.6) implies (4.2). The proof of (4.3) is similar. From Propositions 2.4, 3.7 and Lemma 3.2

\[ \mathcal{L}(\partial_{x_2}^3 \Box_k f, \psi) \leq \left\| \frac{1}{\mathbb{R}} \int S(-t) D_{-3/2} \Box_k f(t) \sigma dt \right\|_{L^2(\mathbb{R}^n)} \left\| \Box_k \frac{1}{\mathbb{R}} \int S(-t) \psi(t) \sigma dt \right\|_{L^2(\mathbb{R}^n)} \]

\[ \lesssim \left( k_{\text{max}} \right)^{3/2} \| \Box_k f \|_{L^2_t L^2_{x_1} L^2_{x_2} \ldots L^2_{x_n} L^2(\mathbb{R}^{1+n})} \left( k_{\text{max}} \right)^{3/4} \| \Box_k \psi \|_{L^2_t L^1_{x_1} L^1_{x_2} \ldots L^1_{x_n} L^1(\mathbb{R}^{1+n})} \]

\[ \lesssim \left( k_{\text{max}} \right)^{3/2} \left( k_{\text{max}} \right)^{3/4} \| \Box_k f \|_{L^2_t L^2_{x_1} L^2_{x_2} \ldots L^2_{x_n} L^2(\mathbb{R}^{1+n})} \| \Box_k \psi \|_{L^1_t L^1_{x_1} L^1_{x_2} \ldots L^1_{x_n} L^1(\mathbb{R}^{1+n})}. \]

(4.7)

(4.3) is the duality of (4.7). For (4.4), noticing that

\[ \mathcal{L}(\partial_{x_2}^3 \Box_k f, \psi) \leq \left\| \frac{1}{\mathbb{R}} \int S(-t) \partial_{x_2}^3 \Box_k f(t) \sigma dt \right\|_{L^2(\mathbb{R}^n)} \left\| \Box_k \frac{1}{\mathbb{R}} \int S(-t) \psi(t) \sigma dt \right\|_{L^2(\mathbb{R}^n)} \]

\[ \lesssim \left( k_{\text{max}} \right)^{3} \left( k_{\text{max}} \right)^{3} \| \Box_k f \|_{L^1_t L^1_{x_1} L^1_{x_2} \ldots L^1_{x_n} L^1(\mathbb{R}^{1+n})} \| \Box_k \psi \|_{L^2_t L^2_{x_1} L^2_{x_2} \ldots L^2_{x_n} L^2(\mathbb{R}^{1+n})}. \]

(4.8)

which implies (4.4), as desired.

\[ \square \]

**Lemma 4.2** Let \( \psi : [0, \infty) \to [0, 1] \) be a smooth bump function satisfying \( \psi(x) = 1 \) as \( |x| \leq 1 \) and \( \psi(x) = 0 \) if \( |x| \geq 2 \). Denote \( \psi_1(\xi) = \psi(\xi_2/2\xi_1), \psi_2(\xi) = 1 - \psi(\xi_2/2\xi_1) \), \( \xi \in \mathbb{R}^n \). Then we have for \( \sigma \geq 0 \),

\[ \sum_{k \in \mathbb{Z}^n, |k| \geq 4} \langle k_1 \rangle^\sigma \| \mathcal{F}^{-1}_{\xi_1, \xi_2} \psi_1 \mathcal{F}_{x_1, x_2} \Box_k \partial_{x_2}^3 \mathcal{A} f \|_{L^\infty_t L^2_{x_1} L^2_{x_2} \ldots L^2_{x_n} L^2(\mathbb{R}^{1+n})} \]

\[ \lesssim \sum_{k \in \mathbb{Z}^n, |k| \geq 4} \langle k_1 \rangle^\sigma \| \Box_k f \|_{L^1_t L^2_{x_1} L^2_{x_2} \ldots L^2_{x_n} L^2(\mathbb{R}^{1+n})}; \]  

(4.9)

and for \( \sigma \geq 3 \),

\[ \sum_{k \in \mathbb{Z}^n, |k| \geq 4} \langle k_1 \rangle^\sigma \| \mathcal{F}^{-1}_{\xi_1, \xi_2} \psi_2 \mathcal{F}_{x_1, x_2} \Box_k \partial_{x_2}^3 \mathcal{A} f \|_{L^\infty_t L^2_{x_1} L^2_{x_2} \ldots L^2_{x_n} L^2(\mathbb{R}^{1+n})} \]

\[ \lesssim \sum_{k \in \mathbb{Z}^n, |k| \geq 4} \langle k_1 \rangle^{\sigma-3} \langle k_2 \rangle^3 \| \Box_k f \|_{L^1_t L^2_{x_1} L^2_{x_2} \ldots L^2_{x_n} L^2(\mathbb{R}^{1+n})}. \]

(4.10)

**Proof.** For the terseness of proof, we let

\[ I = \| \mathcal{F}^{-1}_{\xi_1, \xi_2} \psi_1 \mathcal{F}_{x_1, x_2} \Box_k \partial_{x_2}^3 \mathcal{A} f \|_{L^\infty_t L^2_{x_1} L^2_{x_2} \ldots L^2_{x_n} L^2(\mathbb{R}^{1+n})}, \]
\[ II = \| \mathcal{F}^{-1}_{\xi_1, \xi_2} \psi_2 \mathcal{F}_{x_1, x_2} \Box_k \partial_{x_2}^3 \mathcal{A} f \|_{L^2_{\mathcal{X}_1} L^2_{\mathcal{X}_2}, \ldots, x_n} \| L^2_{\mathbb{R}^{1+n}}. \]

Firstly, we give the estimate of \( I \). Let \( \eta_k \) be as in Lemma 3.1. For \( k \in \mathbb{Z}^n \), \( |k_1| > 4 \), applying the almost orthogonality of \( \Box_k \), we have

\[ \begin{aligned}
I \lesssim & \sum_{|\ell_1|, |\ell_2| \leq 1} \left\| \mathcal{F}^{-1}_{\xi_1, \xi_2} \psi \left( \frac{\xi_2}{2\xi_1} \right) \left( \frac{\xi_2}{\xi_1} \right)^3 \prod_{i=1,2} \eta_{k_i + \ell_i} \mathcal{A} f \right\|_{L^2_{\mathcal{X}_1} L^2_{\mathcal{X}_2}, \ldots, x_n} \| L^2_{\mathbb{R}^{1+n}}. \\
\end{aligned} \]  

(4.11)

Denote

\[ (f \otimes g)(x) = \int_{\mathbb{R}^2} f(t, x_1 - y_1, x_2 - y_2, x_3, \ldots, x_n) g(t, y_1, y_2) dy_1 dy_2. \]  

(4.12)

We have for any Banach function space \( X \) defined on \( \mathbb{R}^{1+n} \),

\[ \| f \otimes g \|_X \leq \| g \|_L^1_{y_1, y_2}(\mathbb{R}^2) \left( \sup_{y_1, y_2} \| f(\cdot, \cdot, - y_1, - y_2, \cdot, \ldots, \cdot) \|_X. \]  

(4.13)

Hence, combining (4.11) with (4.12),

\[ \begin{aligned}
I \lesssim & \sum_{|\ell_1|, |\ell_2| \leq 1} \left\| \mathcal{F}^{-1}_{\xi_1, \xi_2} \psi \left( \frac{\xi_2}{2\xi_1} \right) \left( \frac{\xi_2}{\xi_1} \right)^3 \prod_{i=1,2} \eta_{k_i + \ell_i} \mathcal{A} f \right\|_{L^1(\mathbb{R}^2)} \left\| \Box_k \partial_{x_2}^3 \mathcal{A} f \right\|_{L^2_{\mathcal{X}_1} L^2_{\mathcal{X}_2}, \ldots, x_n} \| L^2_{\mathbb{R}^{1+n}}. \\
\end{aligned} \]  

(4.14)

Using Bernstein’s multiplier estimate, for \( |k_1| > 4 \), we have

\[ \left\| \mathcal{F}^{-1}_{\xi_1, \xi_2} \psi \left( \frac{\xi_2}{2\xi_1} \right) \left( \frac{\xi_2}{\xi_1} \right)^3 \prod_{i=1,2} \eta_{k_i + \ell_i} \mathcal{A} f \right\|_{L^1(\mathbb{R}^2)} \lesssim \sum_{|\alpha| \leq 2} \left\| D^\alpha \left[ \psi \left( \frac{\xi_2}{2\xi_1} \right) \left( \frac{\xi_2}{\xi_1} \right)^3 \prod_{i=1,2} \eta_{k_i + \ell_i} \mathcal{A} f \right] \right\|_{L^2(\mathbb{R}^2)} \lesssim 1. \]  

(4.15)

By Proposition 3.8 (4.14) and (4.15), we have

\[ I \lesssim \| \Box_k f \|_{L^1_{\mathcal{X}_1} L^2_{\mathcal{X}_2}, \ldots, x_n} \| L^2_{\mathbb{R}^{1+n}}. \]  

(4.16)

Next, we consider the estimate of \( II \). Using Proposition 4.1

\[ II \lesssim \| \mathcal{F}^{-1}_{\xi_1, \xi_2} \left( \frac{\xi_2}{\xi_1} \right) \psi_2 \mathcal{F}_{x_1, x_2} \Box_k \mathcal{A} f \|_{L^1_{\mathcal{X}_1} L^2_{\mathcal{X}_2}, \ldots, x_n} \| L^2_{\mathbb{R}^{1+n}}. \]  

(4.16)

\[ \begin{aligned}
II \lesssim & \sum_{|\ell_1|, |\ell_2| \leq 1} \left\| \mathcal{F}^{-1}_{\xi_1, \xi_2} \left( 1 - \psi \left( \frac{\xi_2}{2\xi_1} \right) \right) \left( \frac{\xi_2}{\xi_1} \right)^3 \prod_{i=1,2} \eta_{k_i + \ell_i} \mathcal{A} f \right\|_{L^1(\mathbb{R}^2)} \\
\end{aligned} \]
\[
\times \| \Box_k f \|_{L^1_{t_1} L^2_{x_2, ..., x_n} L^\infty_t (\mathbb{R}^{1+n})}. \tag{4.17}
\]

Notice that supp\(\psi_2 \subset \{ \xi : |\xi_2| \geq 2|\xi_1| \} \). If \(|k_1| > 4\), we have \(|k_2| > 6\) and \(|k_2| \geq |k_1|\) in the summation of the left-hand side of (4.11). So, \(\sum_{k \in \mathbb{Z}^n, |k_1| > 4} \langle k_2 \rangle^\sigma II \lesssim \sum_{k \in \mathbb{Z}^n, |k_1| > 4} \langle k_2 \rangle^{\sigma - 3} \langle k_1 \rangle^3 II.\)

Collecting (4.16) and (4.18), we get the result, as desired. \(\square\)

Lemma 4.3 Let \(k = (k_1, ..., k_n)\), \(2 \leq q \leq \infty\), \(q > 8/n\). Then we have for \(\sigma \geq 0\) and \(i, \alpha = 1, ..., n\),

\[
\sum_{k \in \mathbb{Z}^n, |k_1| = k_{\max} > 4} \langle k \rangle^\sigma \| \Box_k \partial_{x_i}^3 \mathcal{A} f \|_{L^q_{t_1} L^\infty_{x_2, ..., x_n} L^\infty_t (\mathbb{R}^{1+n})} \\
\lesssim \sum_{k \in \mathbb{Z}^n, |k_1| = k_{\max} > 4} \langle k_\alpha \rangle^{\sigma + 3/2 + 3/q} \| \Box_k f \|_{L^q_{t_1} L^2_{t_2, ..., x_n} L^2_t (\mathbb{R}^{1+n})}. \tag{4.19}
\]

**Proof.** First, we consider the case \(\alpha = 1\). In view of (3.30) and \(|k_1| = k_{\max} > 4\),

\[
\| \Box_k \partial_{x_i}^3 \mathcal{A} f \|_{L^q_{t_1} L^\infty_{x_2, ..., x_n} L^\infty_t (\mathbb{R}^{1+n})} \lesssim \sum_{|k_1|, |k_\ell| \leq 1} \left\| \mathcal{F}_{\xi_1, \xi_\ell}^{-1} \left( \frac{\xi_i}{\xi_1} \right)^3 \eta_{k_1 + \ell_1} (\xi_1) \eta_{k_\ell + \ell_\ell} (\xi_\ell) \right\|_{L^1_t (\mathbb{R}^2)} \times \| \Box_k \partial_{x_\ell}^3 \mathcal{A} f \|_{L^1_{t_1} L^\infty_{x_2, ..., x_n} L^\infty_t (\mathbb{R}^{1+n})}
\lesssim \langle k_1 \rangle^{3} \langle k_1 \rangle^{-3} \langle k_1 \rangle^{3/2 + 3/q} \| \Box_k f \|_{L^q_{t_1} L^2_{t_2, ..., x_n} L^2_t (\mathbb{R}^{1+n})}
\lesssim \langle k_1 \rangle^{3/2 + 3/q} \| \Box_k f \|_{L^q_{t_1} L^2_{t_2, ..., x_n} L^2_t (\mathbb{R}^{1+n})}. \tag{4.20}
\]

(4.20) implies the result, as desired. Next, we consider the case \(\alpha = 2\), the general case \(\alpha = 3, ..., n\) is similar. Notice that \(|k_2| = k_{\max} > 4\). By (4.13),

\[
\| \Box_k \partial_{x_1}^3 \mathcal{A} f \|_{L^q_{t_1} L^\infty_{x_2, ..., x_n} L^\infty_t (\mathbb{R}^{1+n})} \lesssim \sum_{|k_1|, |k_\ell| \leq 1} \left\| \mathcal{F}_{\xi_1, \xi_\ell}^{-1} \left( \frac{\xi_i}{\xi_2} \right)^3 \eta_{k_1 + \ell_1} (\xi_1) \eta_{k_\ell + \ell_\ell} (\xi_\ell) \right\|_{L^1_t (\mathbb{R}^2)} \times \| \Box_k \partial_{x_2}^3 \mathcal{A} f \|_{L^1_{t_1} L^\infty_{x_2, ..., x_n} L^\infty_t (\mathbb{R}^{1+n})}
\lesssim \langle k_1 \rangle^{3} \langle k_2 \rangle^{-3} \langle k_2 \rangle^{3/2 + 3/q} \| \Box_k f \|_{L^q_{t_1} L^2_{t_2, ..., x_n} L^2_t (\mathbb{R}^{1+n})}
\lesssim \langle k_2 \rangle^{3/2 + 3/q} \| \Box_k f \|_{L^q_{t_1} L^2_{t_2, ..., x_n} L^2_t (\mathbb{R}^{1+n})}. \tag{4.21}
\]

\(\square\)
Remark 4.4 From the proof of Lemma 4.3, we easily see that for \( \beta = 1, \ldots, n \)
\[
\sum_{k \in \mathbb{Z}^n, |k_\alpha| = k_{\max} > 4} \langle k \rangle^\sigma \| \Box_k \partial_x^3 \mathcal{A} f \|_{L_t^\infty L_x^\infty L_{x \neq i}^2 (\mathbb{R}^{1+n})} 
\lesssim \sum_{k \in \mathbb{Z}^n, |k_\alpha| = k_{\max} > 4} \langle k_\alpha \rangle^{\sigma + 3/2 + 3/q} \| \Box_k f \|_{L_t^1 L_x^2 \alpha \neq \alpha} L_t^2 L_x^2 (\mathbb{R}^{1+n}).
\] (4.22)

5 Proof of Theorem 1.2

Before we prove our main Theorem 1.1, we consider its special version—Theorem 1.2 As assumption, the nonlinear term takes the form
\[
F((\partial_x^\alpha u)_{|\alpha| \leq 3}, (\partial_x^\beta \bar{u})_{|\alpha| \leq 3}) = \sum_{i=1}^n \lambda_i \partial_x^{3} (u^{\kappa_i + 1}).
\]
Denote
\[
\rho_1 (u) = \sum_{i=1}^n \sum_{k \in \mathbb{Z}^n, |k_i| = k_{\max} > 4} \langle k_i \rangle^3 \| \Box_k u \|_{L_t^\infty L_x^2 \alpha \neq \alpha} L_t^2 (\mathbb{R}^{1+n}),
\]
\[
\rho_2 (u) = \sum_{i=1}^n \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2 - 3/\kappa} \| \Box_k u \|_{L_t^\infty L_x^\infty \alpha \neq i} L_t^\infty (\mathbb{R}^{1+n}),
\]
\[
\rho_3 (u) = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \| \Box_k u \|_{L_t^\infty L_x^2 \alpha \neq \alpha} L_t^2 (\mathbb{R}^{1+n}).
\]

Put
\[
X := \left\{ u \in \mathcal{S}'(\mathbb{R}^{1+n}) : \| u \|_X := \sum_{i=1}^3 \rho_i (u) \leq \delta_0 \right\}.
\]

Considering the following integral mapping:
\[
\mathcal{F} : u(t) \rightarrow S(t)u_0 - i\mathcal{A} \left( \sum_{i=1}^n \lambda_i \partial_x^{3} u^{\kappa_i + 1} \right).
\]

Firstly, we estimate \( \| S(t)u_0 \|_X \).

For simplicity, we denote
\[
\| u \|_{Y_i} = \sum_{k \in \mathbb{Z}^n, |k_i| = k_{\max} > 4} \langle k_i \rangle^3 \| \Box_k u \|_{L_t^\infty L_x^2 \alpha \neq \alpha} L_t^2 (\mathbb{R}^{1+n}).
\]

For the estimate of \( \rho_1 (u) \), it suffices to control \( \| \cdot \|_{Y_1} \). By (2.8) and Plancherel’s identity, we have
\[
\| S(t)u_0 \|_{Y_1} \lesssim \sum_{k \in \mathbb{Z}^n, |k_1| = k_{\max} > 4} \langle k_1 \rangle^3 \| \Box_k D_{x_1}^{-3/2} u_0 \|_{L^2 (\mathbb{R}^n)}
\]
\[ \lesssim \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \| \Box_k u_0 \|_{L^2(\mathbb{R}^n)}. \]

For \( \rho_2(u) \), using Prop 3.6 we obtain
\[ \| \Box_k S(t) u_0 \|_{L^t_x L^\infty_{\alpha(x)}} \lesssim \langle k \rangle^{3/4} \| \Box_k u_0 \|_{L^2(\mathbb{R}^n)}. \]

Therefore, we have
\[ \rho_2(S(t) u_0) \lesssim \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \| \Box_k u_0 \|_{L^2(\mathbb{R}^n)}. \]

For \( \rho_3(u) \), using Lemma 3.5, we have
\[ \rho_3(S(t) u_0) \lesssim \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \| \Box_k u_0 \|_{L^2(\mathbb{R}^n)}. \]

Secondly, we estimate
\[ \| \mathcal{A} (\sum_{i=1}^n \lambda_i \partial_{x_i}^3 u^{\kappa_1+1}) \|_X. \]

For the simplicity of proof, we denote
\[ S_{i,1}^{(i)} := \{ k^{(1)},...,k^{(\kappa_1+1)} \in \mathbb{Z}^n : |k_i^{(1)}| \wedge \ldots \wedge |k_i^{(\kappa_1+1)}| > 4 \}, \]
\[ S_{i,2}^{(i)} := \{ k^{(1)},...,k^{(\kappa_1+1)} \in \mathbb{Z}^n : |k_i^{(1)}| \wedge \ldots \wedge |k_i^{(\kappa_1+1)}| \leq 4 \}. \]

Using the frequency-uniform decomposition, we have
\[ u^{\kappa_1+1} = \sum_{k^{(1)},...,k^{(\kappa_1+1)} \in \mathbb{Z}^n} \Box_{k^{(1)}} u \ldots \Box_{k^{(\kappa_1+1)}} u \]
\[ = \sum_{S_{i,1}^{(i)}} \Box_{k^{(1)}} u \ldots \Box_{k^{(\kappa_1+1)}} u + \sum_{S_{i,2}^{(i)}} \Box_{k^{(1)}} u \ldots \Box_{k^{(\kappa_1+1)}} u. \tag{5.1} \]

Using (3.29) and (3.32), we obtain that
\[ \| \mathcal{A} \partial_{x_1}^3 u^{\kappa_1+1} \|_{Y_1} \lesssim \sum_{k \in \mathbb{Z}^n, |k| = k_{\max} > 4} \langle k \rangle \sum_{S_{i,1}^{(1)}} \| \Box_k (\Box_{k^{(1)}} u \ldots \Box_{k^{(\kappa_1+1)}} u) \|_{L^1_{x_1} L^2_{x_2} \ldots \mathbb{R}^n} + \sum_{k \in \mathbb{Z}^n, |k| = k_{\max} > 4} \langle k \rangle \langle k \rangle^{3/2} \sum_{S_{i,2}^{(1)}} \| \Box_k (\Box_{k^{(1)}} u \ldots \Box_{k^{(\kappa_1+1)}} u) \|_{L^{(2+\kappa)/(1+\kappa)}_{x}(\mathbb{R}^{1+n})} + I + II. \tag{5.2} \]

In view of the support property of \( \Box_k u \), we see that
\[ \Box_k (\Box_{k^{(1)}} u \ldots \Box_{k^{(\kappa_1+1)}} u) = 0, \text{ if } |k - k^{(1)} - \ldots - k^{(\kappa_1+1)}| \geq C. \tag{5.3} \]
Hence, by Lemma 3.2,

$$I \lesssim \sum_{k \in \mathbb{Z}^n, |k| = k_{\text{max}} > 4} (k_1)^3 \sum_{\mathcal{S}_{1,1}^{(1)}} \|\square_{k(1)} u \ldots \square_{k^{(\kappa_1+1)}} u\|_{L_{x_1}^1 L_{x_2}^2 \ldots L_{x_n}^2 (\mathbb{R}^{1+n})} \mathcal{X}_{|k^{(1)}| - \ldots - k^{(\kappa_1+1)}| \leq C}.$$  

(5.4)

By Hölder’s inequality and \(\|\Box k u\|_{L_x^\infty} \lesssim \|\Box k u\|_{L_x^2}\) uniformly holds for all \(k \in \mathbb{Z}^n\), we have

$$\|\Box_{k^{(1)}} u \ldots \Box_{k^{(\kappa_1+1)}} u\|_{L_{x_1}^1 L_{x_2}^2 \ldots L_{x_n}^2 (\mathbb{R}^{1+n})} \leq \|\Box_{k^{(1)}} u\|_{L_{x_1}^\infty L_{x_2}^2 \ldots L_{x_n}^2 (\mathbb{R}^{1+n})} \prod_{i=2}^{\kappa_1+1} \|\Box_{k^{(i)}} u\|_{L_{x_1}^\infty L_{x_2}^\infty \ldots L_{x_n}^\infty \cap L_{t}^\infty L_{x}^2 (\mathbb{R}^{1+n})}.$$  

Since \(|k - k^{(1)}| - \ldots - k^{(\kappa_1+1)}| \leq C\) implies that \(|k_1 - k^{(1)}_1| - \ldots - k^{(\kappa_1+1)}_1| \leq C\), we see that \(|k_1| \sim \max_{i=1, \ldots, \kappa_1+1} |k^{(i)}_1|\). Without loss of generality, we may assume that \(|k^{(1)}_{\kappa_1+1}| = \max_{i=1, \ldots, \kappa_1+1} |k^{(i)}_1|\) in the summation \(\mathcal{S}_{1,1}^{(1)}\) in (5.4) above. Therefore, we have

$$I \lesssim \sum_{k^{(1)} \in \mathbb{Z}^n, |k^{(1)}_1| \sim k_{\text{max}} > 4} \sum_{\mathcal{S}_{1,1}^{(1)}} \|\Box_{k^{(1)}} u\|_{L_{x_1}^\infty L_{x_2}^2 \ldots L_{x_n}^2 (\mathbb{R}^{1+n})} \times \sum_{k^{(2)}, \ldots, k^{(\kappa_1+1)} \in \mathbb{Z}^n} \prod_{i=2}^{\kappa_1+1} \|\Box_{k^{(i)}} u\|_{L_{x_1}^\infty L_{x_2}^\infty \ldots L_{x_n}^\infty \cap L_{t}^\infty L_{x}^2 (\mathbb{R}^{1+n})} \lesssim \rho_1(u)(\rho_2(u) + \rho_3(u))^{\kappa_1}.$$  

(5.5)

In view of (5.3) we easily see that \(|k_1| \leq C\) in II of (5.2). Hence,

$$II \lesssim \sum_{k \in \mathbb{Z}^n, |k| = k_{\text{max}} > 4} \sum_{\mathcal{S}_{1,1}^{(1)}} \|\Box_{k(1)} u \ldots \Box_{k^{(\kappa_1+1)}} u\|_{L_{x,t}^{2+\kappa} (\mathbb{R}^{1+n})} \mathcal{X}_{|k^{(1)}| - \ldots - k^{(\kappa_1+1)}| \leq C} \lesssim \mathcal{S}_{1,2}^{(1)} \sum_{i=1}^{\kappa_1+1} \sum_{\mathcal{S}_{1,2}^{(1)}} \|\Box_{k^{(i)}} u\|_{L_{x,t}^{2+\kappa}} \lesssim \mathcal{S}_{1,2}^{(1)} \|\mathcal{A} \partial_{x_1}^3 u^{\kappa_{1+1}}\|_{Y_1} \lesssim \rho_1(u)(\rho_2(u) + \rho_3(u))^{\kappa_1} + \rho_3(u)^{1+\kappa_1}.$$  

(5.6)

Hence, we have

$$\|\mathcal{A} \partial_{x_1}^3 u^{\kappa_{1+1}}\|_{Y_1} \lesssim \rho_1(u)(\rho_2(u) + \rho_3(u))^{\kappa_1} + \rho_3(u)^{1+\kappa_1}.$$  

(5.7)

Now, we turn to estimate \(\|\mathcal{A} \partial_{x_2}^3 u^{\kappa_{2+1}}\|_{Y_1}\). Let \(\psi_i\) be as in Lemma 4.2 For convenience, we write

$$P_i = \mathcal{F}_{\xi_1, \xi_2}^{-1} \psi_i \mathcal{F}_{x_1, x_2}, \quad i = 1, 2.$$  

(5.8)
We have
\[
\|a\partial_x^3 u_{t+1}^2\|_{Y_1} \lesssim \|P_1 \partial_x^3 a u_{t+1}^2\|_{Y_1} + \|P_2 \partial_x^3 a u_{t+1}^2\|_{Y_1} := III + IV.
\] (5.9)

Using the decomposition (5.1),
\[
III \lesssim \bigg\| P_1 \partial_x^3 a \sum_{\mathcal{S}_{k(1)}^{(1)}} (\Box_{k(1)} u \ldots \Box_{k(\kappa_2 + 1)} u) \bigg\|_{Y_1}
\]
\[
+ \bigg\| P_2 \partial_x^3 a \sum_{\mathcal{S}_{k(1), k(\kappa_2 + 1)}^{(2)}} (\Box_{k(1)} u \ldots \Box_{k(\kappa_2 + 1)} u) \bigg\|_{Y_1} := III_1 + III_2.
\] (5.10)

Using Lemma 4.2 and then taking the same way as in the estimate to (5.4), we get
\[
III_1 \lesssim \sum_{k \in \mathbb{Z}^n, |k| = k_{max} > 4} \langle k \rangle^3 \sum_{\mathcal{S}_{k(1)}^{(1)}} \|\Box_k (\Box_{k(1)} u \ldots \Box_{k(\kappa_2 + 1)} u)\|_{L_{1, \infty}^1 L_{x}^2 \ldots L_{n}^2(\mathbb{R}^{1+n})}
\]
\[
\lesssim \rho_1(u)(\rho_2(u) + \rho_3(u))^\kappa_2.
\] (5.11)

For the estimate of \(III_2\), observing the fact that \(\text{supp}\psi_1 \subset \{\xi : |\xi_2| \leq 4|\xi_1|\}\) and using the multiplier estimate, then applying (4.2), we have
\[
III_2 \lesssim \sum_{k \in \mathbb{Z}^n, |k| = k_{max} > 4, |k_2| \leq |k_1|} \langle k \rangle^3 \sum_{\mathcal{S}_{k(1)}^{(1)}} \|\Box_k (\Box_{k(1)} u \ldots \Box_{k(\kappa_2 + 1)} u)\|_{L_{1, \infty}^{2+\kappa_2}(\mathbb{R}^{1+n})}
\]
\[
\lesssim \rho_3(u)^{1+\kappa_2}.
\] (5.12)

We need to further control \(IV\). Using the decomposition (5.1),
\[
IV \leq \bigg\| P_2 \partial_x^3 a \sum_{\mathcal{S}_{k(1)}^{(2)}} (\Box_{k(1)} u \ldots \Box_{k(\kappa_2 + 1)} u) \bigg\|_{Y_1}
\]
\[
+ \bigg\| P_2 \partial_x^3 a \sum_{\mathcal{S}_{k(1), k(\kappa_2 + 1)}^{(2)}} (\Box_{k(1)} u \ldots \Box_{k(\kappa_2 + 1)} u) \bigg\|_{Y_1} := IV_1 + IV_2.
\] (5.13)

By Lemma 4.2
\[
IV_1 \lesssim \sum_{k \in \mathbb{Z}^n, |k| = k_{max} > 4} \langle k \rangle^3 \sum_{\mathcal{S}_{k(1)}^{(2)}} \|\Box_k (\Box_{k(1)} u \ldots \Box_{k(\kappa_2 + 1)} u)\|_{L_{1, \infty}^1 L_{x}^2 \ldots L_{n}^2(\mathbb{R}^{1+n})}.
\] (5.14)

By symmetry of \(k(1), \ldots, k(\kappa_2 + 1)\), we can assume that \(|k(1)| = \max_{1 \leq i \leq \kappa_2 + 1} |k_2^{(i)}|\) in \(\mathcal{S}_{2,1}^{(2)}\). Using the same way as in the estimate of \(I\), we have
\[
IV_1 \lesssim \sum_{\mathcal{S}_{2,1}^{(2)}, |k_2^{(1)}| > k_{max} > 4} \langle k_2^{(1)} \rangle^3 \|\Box_{k(1)} u \ldots \Box_{k(\kappa_2 + 1)} u\|_{L_{1, \infty}^1 L_{x}^2 \ldots L_{n}^2(\mathbb{R}^{1+n})}.
\] (5.15)
Using Hölder’s inequality, we have

\[
\|\Box_{k(1)} u \ldots \Box_{k(\kappa_2 + 1)} u\|_{L_{k\kappa_2 + 1}^1 L_{k\kappa_2 + 1}^1 L_{k\kappa_2 + 1}^1 (\mathbb{R}^{1+n})} \\
\lesssim \|\Box_{k(1)} u \Box_{k(2)} u \ldots \Box_{k(\kappa_2 + 1)} u\|^{1/2}_{L_{k\kappa_2 + 1}^2 (\mathbb{R}^{1+n})} \\
\times \|\Box_{k(1)} u \Box_{k(2)} u \ldots \Box_{k(\kappa_2 + 1)} u\|^{1/2}_{L_{k\kappa_2 + 1}^2 L_{k\kappa_2 + 1}^2 L_{k\kappa_2 + 1}^2 (\mathbb{R}^{1+n})} \\
\lesssim \|\Box_{k(1)} u\|_{L_{k\kappa_2 + 1}^\infty L_{k\kappa_2 + 1}^\infty L_{k\kappa_2 + 1}^\infty (\mathbb{R}^{1+n})} \\
\times \prod_{i=2}^{\kappa_2 + 1} \|\Box_{k(i)} u\|_{L_{k\kappa_2 + 1}^\infty L_{k\kappa_2 + 1}^\infty L_{k\kappa_2 + 1}^\infty (\mathbb{R}^{1+n})}.
\]  
(5.16)

Observing the inclusion \(L_{k\kappa_2 + 1}^\kappa L_{\xi-2}^\infty L_{\xi-2}^\infty \subset L_{k\kappa_2 + 1}^\kappa L_{\xi-2}^\infty L_{\xi-2}^\infty\), we immediately have

\[
IV_1 \lesssim \rho_1(u)(\rho_2(u) + \rho_3(u))^{1+\kappa_2}.
\]  
(5.17)

For the estimate of \(IV_2\), noticing the fact that \(\text{supp}\psi_2 \subset \{\xi : |\xi_2| \geq 2|\xi_1|\}\) and applying (4.2), we have

\[
IV_2 \lesssim \sum_{k \in \mathbb{Z}^n, |k_2| = k_{\text{max}} > 4} \langle k \rangle^{3/2} \sum_{S_2^{(2)}} \sum_{\Box_{k} (\Box_{k(1)} u \ldots \Box_{k(\kappa_2 + 1)} u)} \|\Box_{k} u\|_{L_{\xi-2}^{2+\kappa}/(1+\kappa)} (\mathbb{R}^{1+n}) \\
\lesssim \sum_{k \in \mathbb{Z}^n, |k_2| = k_{\text{max}} > 4} \sum_{S_2^{(2)}} \|\Box_{k} u\|_{L_{\xi-2}^{2+\kappa}/(1+\kappa)} (\mathbb{R}^{1+n}) \\
\lesssim \rho_3(u)^{1+\kappa_2}.
\]  
(5.18)

The treatment of the other terms in \(\rho_1(\cdot)\) is similar. Therefore, we have shown that

\[
\rho_1 \left( \mathcal{A} \left( \sum_{i=1}^{n} \lambda_i \partial_{x_i}^2 u^{\kappa_i + 1} \right) \right) \lesssim \sum_{i=1}^{n} (\rho_1(u)(\rho_2(u) + \rho_3(u))^{\kappa_i} + \rho_3(u)^{1+\kappa_i}).
\]  
(5.19)

For the estimate of \(\rho_2(\cdot)\). Denote

\[
\|u\|_{Z_i} = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2-3/\kappa} \|\Box_{k} u\|_{L_{\xi-2}^{2+\kappa}/(1+\kappa)} (\mathbb{R}^{1+n}) \cdot
\]  
(5.20)

We have

\[
\rho_2 \left( \mathcal{A} \left( \sum_{j=1}^{n} \lambda_j \partial_{x_j} u^{\kappa_j + 1} \right) \right) \lesssim \sum_{i=1}^{n} \mathcal{A} \left( \sum_{j=1}^{n} \lambda_j \partial_{x_j} u^{\kappa_j + 1} \right)_{Z_j}.
\]  
(5.21)
Observing the symmetry of $Z_1, \ldots, Z_n$, it suffices to consider the estimate of $\| \cdot \|_{z_i}$. Recall that $k_{\text{max}} := \max_{1 \leq i \leq n} |k_i|$. We have

$$
\|v\|_{Z_i} \leq \left( \sum_{k \in \mathbb{Z}^n, k_{\text{max}} > 4} + \sum_{k \in \mathbb{Z}^n, k_{\text{max}} \leq 4} \right) \langle k \rangle^{3/2-3/\kappa} \| \overset{\max}{\square}_{k} v \|_{L_{x_1}^{2} L_{x_2}^{\infty} \ldots L_{x_n}^{\infty} (\mathbb{R}^{1+n})} \\
:= \Gamma_1(v) + \Gamma_2(v).
$$

(5.22)

In view of (4.4) and Hölder’s inequality,

$$
\Gamma_2 \left( \mathcal{A} \left( \sum_{i=1}^{n} \lambda_i \partial_{x_i}^3 u^{\kappa_i+1} \right) \right) \lesssim \sum_{k \in \mathbb{Z}^n, |k_1| = k_{\text{max}} > 4} \| \overset{\max}{\square}_{k} \mathcal{A} \left( \sum_{i=1}^{n} \lambda_i \partial_{x_i}^3 u^{\kappa_i+1} \right) \|_{L_{x_1}^{2} L_{x_2}^{\infty} \ldots L_{x_n}^{\infty} (\mathbb{R}^{1+n})} \\
\lesssim \sum_{i=1}^{n} \sum_{k(1), \ldots, k(\kappa_i+1) \in \mathbb{Z}^n} \| \overset{\max}{\square}_{k(1)} u \ldots \overset{\max}{\square}_{k(\kappa_i+1)} u \|_{L_{t,x}^{2+\kappa_i} (\mathbb{R}^{1+n})} \cdots \| \overset{\max}{\square}_{k(\kappa_i+1)} u \|_{L_{t,x}^{2+\kappa_i} (\mathbb{R}^{1+n})} \\
\lesssim \sum_{i=1}^{n} \rho_3(u)^{\kappa_i+1}. 
$$

(5.23)

It is easy to see that

$$
\Gamma_1(v) \leq \left( \sum_{k \in \mathbb{Z}^n, |k_1| = k_{\text{max}} > 4} + \sum_{k \in \mathbb{Z}^n, |k_n| = k_{\text{max}} > 4} \right) \langle k \rangle^{3/2-3/\kappa} \| \overset{\max}{\square}_{k} v \|_{L_{x_1}^{2} L_{x_2}^{\infty} \ldots L_{x_n}^{\infty} (\mathbb{R}^{1+n})} \\
:= \Gamma_1^1(v) + \ldots + \Gamma_1^n(v).
$$

(5.24)

Collecting the decomposition (5.1), (4.4) and Lemma 4.3, we have

$$
\Gamma_1^i \left( \mathcal{A} \left( \sum_{i=1}^{n} \lambda_i \partial_{x_i}^3 u^{\kappa_i+1} \right) \right) \\
\lesssim \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}^n, |k_1| = k_{\text{max}} > 4} \| \overset{\max}{\square}_{k} \mathcal{A} \left( \sum_{i=1}^{n} \lambda_i \partial_{x_i}^3 u^{\kappa_i+1} \right) \|_{L_{x_1}^{2} L_{x_2}^{\infty} \ldots L_{x_n}^{\infty} (\mathbb{R}^{1+n})} \\
+ \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}^n, |k_1| = k_{\text{max}} > 4} \langle k \rangle^{3/2} \sum_{\mathcal{S}_{i,1}^{(1)}} \| \overset{\max}{\square}_{k(1)} u \ldots \overset{\max}{\square}_{k(\kappa_i+1)} u \|_{L_{t,x}^{2+\kappa_i/(1+\kappa_i)} (\mathbb{R}^{1+n})},
$$

(5.25)

Following the way as in (5.5) and (5.6), we can get

$$
\Gamma_1^i \left( \mathcal{A} \left( \sum_{i=1}^{n} \lambda_i \partial_{x_i}^3 u^{\kappa_i+1} \right) \right) \lesssim \sum_{i=1}^{n} \left( \rho_1(u) \rho_2(u) + \rho_3(u) \right)^{\kappa_i} + \rho_3(u)^{1+\kappa_i}.
$$

(5.26)
For the estimate of $\Gamma^2_1(\cdot)$. By Lemma 4.3 and (4.4),
\[
\Gamma^2_1\left(\mathcal{A}\left(\sum_{i=1}^{n} \lambda_i \partial^3_{x_i} u^{\kappa_i+1}\right)\right)
\lesssim \sum_{i=1}^{n} \mathbb{E}_{n, |f| = k_{\max} > 4} \langle k \rangle^{3/2-3/\kappa} \| \mathcal{A} \partial^3_{x_i} u^{\kappa_i+1} \|_{L^1_{t,x} L^\infty_{x_2,\ldots,x_n}(\mathbb{R}^{1+n})}
\lesssim \sum_{i=1}^{n} \mathbb{E}_{n, |f| = k_{\max} > 4} \langle k \rangle^{3/2} \| \mathcal{A} \partial^3_{x_i} u^{\kappa_i+1} \|_{L^1_{t,x} L^\infty_{x_2,\ldots,x_n}(\mathbb{R}^{1+n})}
\lesssim \sum_{i=1}^{n} \mathbb{E}_{n, |f| = k_{\max} > 4} \langle k \rangle^{3/2} \| \mathcal{A} \partial^3_{x_i} u^{\kappa_i+1} \|_{L^1_{t,x} L^\infty_{x_2,\ldots,x_n}(\mathbb{R}^{1+n})}
\]
(5.27)

This reduces the same estimate as $\Gamma^1_1(\cdot)$. The terms $\Gamma^i_1(\cdot)$ for $3 \leq i \leq n$ can be controlled in a similar way as $\Gamma^2_1(\cdot)$ due to symmetry. Therefore, we have shown that
\[
\left\| \mathcal{A}\left(\sum_{i=1}^{n} \lambda_i \partial^3_{x_i} u^{\kappa_i+1}\right) \right\|_{Z_1} \lesssim \sum_{i=1}^{n} \left( \rho_1(u)(\rho_2(u) + \rho_3(u))^{\kappa_i} + \rho_3(u)^{1+\kappa_i}\right).
\]
(5.28)

Now we turn to estimate $\rho_3(\mathcal{A} \partial^3_{x_i} u^{\kappa_i+1})$. Combining (3.21) with Lemma 3.2, we have
\[
\left\| \Box_k \mathcal{A} \partial^3_{x_i} f \right\|_{L^1_{t,x} L^2_{x_2,\ldots,x_n}(\mathbb{R}^{1+n})} \lesssim \left\| \Box_k \partial^3_{x_i} f \right\|_{L^1_{t,x} L^2_{x_2,\ldots,x_n}(\mathbb{R}^{1+n})}
\lesssim \langle k \rangle^{3} \| \Box_k f \|_{L^1_{t,x} L^2_{x_2,\ldots,x_n}(\mathbb{R}^{1+n})}.
\]
(5.29)

Collecting (5.1), (5.29) and (3.31), we have
\[
\rho_3(\mathcal{A} \partial^3_{x_i} u^{\kappa_i+1})
\lesssim \sum_{k \in \mathbb{Z}^n, |k| \leq 4} \langle k \rangle^{3} \langle k_{\max} \rangle^{3/2} \sum_{k(1),\ldots,k(n+1) \in \mathbb{Z}^n} \left\| \Box_k \left( \Box_k^{(1)} u \ldots \Box_k^{(n+1)} u \right) \right\|_{L^1_{t,x} L^2_{x_2,\ldots,x_n}(\mathbb{R}^{1+n})}
\lesssim \sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle^{3} \langle k_{\max} \rangle^{3/2} \sum_{k(1),\ldots,k(n+1) \in \mathbb{Z}^n} \left\| \Box_k \left( \Box_k^{(1)} u \ldots \Box_k^{(n+1)} u \right) \right\|_{L^1_{t,x} L^2_{x_2,\ldots,x_n}(\mathbb{R}^{1+n})}
\]
(5.30)

Whether $|k_1| = k_{\max} > 4$ or $|k_i| = k_{\max} > 4$, $i = 2, \ldots, n$, using Lemma 4.2 (5.5) and (5.6), we always have
\[
\rho_3(\mathcal{A} \partial^3_{x_i} u^{\kappa_i+1}) \lesssim \sum_{i=1}^{n} \left( \rho_1(u)(\rho_2(u) + \rho_3(u))^{\kappa_i} + \rho_3(u)^{1+\kappa_i}\right).
\]
(5.31)
Until now, we have shown that
\[ \| \mathcal{T} u \|_X \lesssim \| u_0 \|_{M_{3/2,1}} + \sum_{i=1}^n \| u \|_{X^{1+\kappa_i}}. \] (5.32)
Hence, Theorem 1.2 holds by a standard contraction mapping argument.

6 Proof of Theorem 1.1

When consider Theorem 1.1, we would like to follow some ideas as in the proof of Theorem 1.2. However, due to the nonlinearity contains the general terms \( \partial_x^\alpha u \) with \(|\alpha| \leq 3, m + 1 \leq |\beta| \leq M + 1\), the proof of Theorem 1.2 can not be directly applied. Inspired by Theorem 1.2, the space \( X' \) we need is likely to be as following:

\[ X' := \left\{ u \in \mathcal{S}'(\mathbb{R}^{1+n}) : \| u \|_X := \sum_{\ell=1}^3 \sum_{|\alpha| = 0}^3 \sum_{i=1}^n \varrho_\ell^{(i)} (\partial_x^\alpha u) \leq \delta \right\}. \]

where \( \alpha \) is a multi-index and
\[
\varrho_1^{(i)} (u) = \sum_{k \in \mathbb{Z}^n, |k_\ell| = k_{\max} > 4} \langle k_\ell \rangle^{3} \| \Box_k u \|_{L_2^\infty L_2^2 (x_j)_{j \neq i}} L_2^2 (\mathbb{R}^{1+n}), \\
\varrho_2^{(i)} (u) = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2 - 3/m} \| \Box_k u \|_{L_2^\infty L_2^\infty (x_j)_{j \neq i}} L_2^\infty (\mathbb{R}^{1+n}), \\
\varrho_3^{(i)} (u) = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \| \Box_k u \|_{L_2^{2+m} \cap L_2^\infty L_2^2 (\mathbb{R}^{1+n}).}
\]

However, when Comparing with the estimates we have established, we hope the space as following:

\[ X := \left\{ u \in \mathcal{S}'(\mathbb{R}^{1+n}) : \| u \|_X := \sum_{\ell=1}^3 \sum_{|\alpha| = 0, 3} \sum_{i=1}^n \varrho_\ell^{(i)} (\partial_x^\alpha u) \leq \delta \right\}. \]

Fortunately, we have the following:

**Lemma 6.1** \( X \) and \( X' \)-norm are equivalent.

**Proof.** Obviously, we have \( \| u \|_X \lesssim \| u \|_{X'}. \) For the reverse inequality, we only need show that when \(|\alpha| = 3, m + 1 \leq |\beta| \leq M + 1\), each term in the \( X' \) can be controlled by \( X \). Firstly, we consider \( \varrho_1^{(i)} (\partial_x^\alpha u), |\alpha| = 3 \). Without loss of generality, we can assume \( \partial_x^\alpha u = \partial_{x_l x_m x_o} u, l, m, o = 1, \ldots, n \). Then we have
\[
\varrho_1^{(i)} (\partial_{x_l x_m x_o} u) = \sum_{k \in \mathbb{Z}^n, |k_i| = k_{\max} > 4} \langle k_i \rangle^{3} \| \Box_k \partial_{x_l x_m x_o} u \|_{L_2^\infty L_2^2 (x_j)_{j \neq i}} L_2^2 (\mathbb{R}^{1+n})
\]
By symmetry, we can assume \( \langle k_l \rangle = \max\{ \langle k_l \rangle, \langle k_m \rangle, \langle k_o \rangle \} \) Using Lemma 3.2 and Remark 3.3 we have
\[
\begin{align*}
\| \Box_k \partial_{x_1 \cdots x_m} u \|_{L^\infty_t L^2_x (\mathbb{R}^{1+n})} & \lesssim \langle k_1 \rangle \langle k_m \rangle \langle k_o \rangle \| \Box_k u \|_{L^\infty_t L^2_x (\mathbb{R}^{1+n})} \\
& \lesssim \langle k_1 \rangle^3 \| \Box_k u \|_{L^\infty_t L^2_x (\mathbb{R}^{1+n})} \\
& \lesssim \| \Box_k \partial_{x_1}^3 u \|_{L^\infty_t L^2_x (\mathbb{R}^{1+n})}
\end{align*}
\]
Hence, we have
\[
\varrho_1^{(i)} (\partial_{x_1 \cdots x_m} u) \lesssim \varrho_1^{(i)} (\partial_{x_1}^3 u) \lesssim \| u \|_X.
\]
Secondly, the \( \varrho_2^{(i)} (\partial_{x_1}^3 u), |\alpha| = 3 \) can be treat in the same way as above.

Finally, we estimate \( \varrho_3^{(i)} (\partial_{x_1}^3 u), |\alpha| = 3 \). Noticing the denotation of \( \varrho_3^{(i)} (\partial_{x_1}^3 u) \), it suffices to show \( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \| \Box_k u \|_{L^\infty_t L^2_x (\mathbb{R}^{1+n})} \lesssim \| u \|_X \). As before, we also assume \( \partial_{x_1}^3 u = \partial_{x_1 x_2 x_m} u, l, m, o = 1, \ldots, n \) and \( \langle k_l \rangle = \max\{ \langle k_l \rangle, \langle k_m \rangle, \langle k_o \rangle \} \).

Using Sobolev imbedding Theorem, we have
\[
\begin{align*}
\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \| \Box_k \partial_{x_1}^3 u \|_{L^\infty_t L^2_x (\mathbb{R}^{1+n})} & \lesssim \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \| \Box_k \partial_{x_1 x_2 x_m} u \|_{L^\infty_t L^2_x (\mathbb{R}^{1+n})} \\
& \lesssim \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \| \Box_k \partial_{x_1}^3 u \|_{L^\infty_t L^2_x (\mathbb{R}^{1+n})} \\
& \lesssim \varrho_3^{(i)} (\partial_{x_1}^3 u) \\
& \lesssim \| u \|_X.
\end{align*}
\]

Considering the following mapping:
\[
\mathcal{F} : u(t) \rightarrow S(t) u_0 - i \mathcal{A} F((\partial_{x}^3 u)|_{\alpha| \leq 3}, (\partial_{x}^3 \bar{u})|_{\alpha| \leq 3})
\]
we will show that \( \mathcal{F} : X \rightarrow X \) is a contraction mapping.

Since \( \| u \|_X = \| \bar{u} \|_X \), we can assume that
\[
F((\partial_{x}^3 u)|_{\alpha| \leq 3}, (\partial_{x}^3 \bar{u})|_{\alpha| \leq 3}) = F(\partial_{x}^3 u)|_{\alpha| \leq 3}) = \sum_{m+1 \leq R \leq M+1} c_{R_0,R_1,R_2,R_3} u^{R_0} (\partial_{x}^{\alpha_1} u)^{R_1} (\partial_{x}^{\alpha_2} u)^{R_2} (\partial_{x}^{\alpha_3} u)^{R_3},
\]
where \( R = R_0 + |R_1| + |R_2| + |R_3| \), \( R_i \) and \( |\alpha_i| = i, (i = 1, 2, 3) \) are multi-index. For simplicity, we denote
\[
\begin{align*}
v_1 &= \ldots = v_{R_0} = u, \ldots, v_{R_0 + |R_1| + |R_2| + 1} = \ldots = v_{R} = \partial_{x}^{\alpha_3} u.
\end{align*}
\]
By (2.8), for $\alpha = 0, 3$,
\[ \partial^{(i)}_2 (\partial^\alpha x_j S(t) u_0) \lesssim \sum_{k \in \mathbb{Z}^n, \lvert k \rvert = k_{\text{max}} > 4} \langle k \rangle^{3/2} \langle k_j \rangle^3 \lVert \Box_k u_0 \rVert_{L^2(\mathbb{R}^n)} \leq \lVert u_0 \rVert_{M_{2,1}^{y/2}}. \]

By (3.27), (3.28), we have for $\alpha = 0, 3$,
\[ \partial^{(i)}_2 (\partial^\alpha x_j S(t) u_0) + \partial^{(i)}_3 (\partial^\alpha x_j S(t) u_0) \lesssim \lVert u_0 \rVert_{M_{2,1}^{y/2}}. \]

Hence,
\[ \lVert u \rVert_X \lesssim \lVert u_0 \rVert_{M_{2,1}^{y/2}}. \]

In order to estimate $\partial^{(i)}_1 (\mathcal{A} \partial^\alpha x_j (v_1 \ldots v_\tilde{R}))$, $i, j = 1, \ldots, n$, as before it suffices to estimate $\partial^{(i)}_1 (\mathcal{A} \partial^\alpha x_1 (v_1 \ldots v_\tilde{R}))$ and $\partial^{(i)}_1 (\mathcal{A} \partial^\alpha x_2 (v_1 \ldots v_\tilde{R}))$. Similarly as in (5.1), we will use the decomposition
\[ \Box_k (v_1 \ldots v_\tilde{R}) = \sum_{S_1^{(i)}} \Box_k \left( \Box_k (v_1 \ldots \Box_k u) v_\tilde{R} \right) + \sum_{S_2^{(i)}} \Box_k \left( \Box_k (v_1 \ldots \Box_k u) v_\tilde{R} \right), \tag{6.1} \]

where
\[ S_1^{(i)} := \{ k^{(1)}, \ldots, k^{(\tilde{R})} \in \mathbb{Z}^n : \lvert k^{(1)} \rvert \lor \ldots \lor \lvert k^{(\tilde{R})} \rvert > 4 \}, \]
\[ S_2^{(i)} := \{ k^{(1)}, \ldots, k^{(\tilde{R})} \in \mathbb{Z}^n : \lvert k^{(1)} \rvert \lor \ldots \lor \lvert k^{(\tilde{R})} \rvert \leq 4 \}. \]

In view of (3.17) and (3.23),
\[ \partial^{(1)}_1 (\mathcal{A} \partial^\alpha x_j (v_1 \ldots v_\tilde{R})) \lesssim \sum_{k \in \mathbb{Z}^n, \lvert k \rvert = k_{\text{max}} > 4} \langle k \rangle^{3} \sum_{S_1^{(i)}} \lVert \Box_k \left( \Box_k (v_1 \ldots \Box_k u) v_\tilde{R} \right) \rVert_{L^2_{x_1} L^2_{x_2}, \ldots, L^2_{x_n} (\mathbb{R}^{1+n})} \]
\[ + \sum_{k \in \mathbb{Z}^n, \lvert k \rvert = k_{\text{max}} > 4} \langle k \rangle^{3} \langle k_{\text{max}} \rangle^{3/2} \sum_{S_2^{(i)}} \lVert \Box_k \left( \Box_k (v_1 \ldots \Box_k u) v_\tilde{R} \right) \rVert_{L^2_{t, \tilde{x}} (\mathbb{R}^{1+n})} \]
\[ := I + II. \tag{6.2} \]

Similar to (5.5),
\[ I \lesssim \sum_{k^{(1)} \in \mathbb{Z}^n, \lvert k^{(1)} \rvert \sim k_{\text{max}} > 4} \langle k^{(1)} \rangle \lVert \Box_k (v_1) \rVert_{L^\infty_{t} L^2_{x_1} L^2_{x_2}, \ldots, L^2_{x_n} (\mathbb{R}^{1+n})} \]
\[
\sum_{k(2), \ldots, k(R) \in \mathbb{Z}^n} \mathbb{R} \prod_{i=2}^{\tilde{R}} \|\Box^{k(i)} v_i\|_{L_{x, t}^{\tilde{R}-1} L_2^{\infty} (\mathbb{R}^{1+n})}.
\]

By Hölder’s inequality and Lemma 3.4,

\[
\|\Box^{k(i)} v_i\|_{L_{x, t}^{R-1} L_2^{\infty} (\mathbb{R}^{1+n})} \leq \|\Box^{k(i)} v_i\|_{L_{x, t}^{R-1} L_2^{\infty} (\mathbb{R}^{1+n})} \|\Box^{k(i)} v_i\|_{L_{x, t}^{1} L_2^{\infty} (\mathbb{R}^{1+n})} \|\Box^{k(i)} v_i\|_{L_{x, t}^{\infty} L_2^{1} (\mathbb{R}^{1+n})}.\]

Owing to Lemma 3.2 and Remark 3.3, it suffices to consider four cases: \(v_i = u\), \(v_i = u_{x_j}\), \(v_i = u_{x_{jj}}\), and \(v_i = u_{x_{jjj}}\), \(j = 1, \ldots, n\). Collecting (6.3) and (6.4), we have

\[
I \lesssim \|u\|_{X}. \tag{6.5}
\]

Similar to (5.6), we see that \(|k_1| \leq C\) in the summation of \(II\). Again, in view of Hölder’s inequality and Lemma 3.4,

\[
\|\Box^{k(i)} v_1 \cdots \Box^{k(i)} v_{\tilde{R}}\|_{L_{x, t}^{R_i+1} (\mathbb{R}^{1+n})} \leq \prod_{i=1}^{\tilde{R}} \|\Box^{k(i)} v_i\|_{L_{x, t}^{R_i+1} (\mathbb{R}^{1+n})} \lesssim \prod_{i=1}^{\tilde{R}} \|\Box^{k(i)} v_i\|_{L_{x, t}^{R_i+1} \cap L_{x, t}^{\infty} L_2^1 (\mathbb{R}^{1+n})}. \tag{6.6}
\]

Hence, using a similar way as in (5.6),

\[
II \lesssim \|u\|_{X}. \tag{6.7}
\]

We now give the estimate of \(\varphi_1^{(1)} (\A \partial_x^3 (v_1 \cdots v_{\tilde{R}}))\). Since we have obtained the estimate in the case \(\alpha = 0\), it suffices to consider the case \(\alpha = 3\). Let \(\psi_i (i = 1, 2)\) be as in Lemma 4.2 and \(P_i = \mathcal{F}^{-1} \psi_i \mathcal{F}\). We have

\[
\varphi_1^{(1)} (\A \partial_x^3 (v_1 \cdots v_{\tilde{R}})) \leq \sum_{k \in \mathbb{Z}^n, |k_1| = k_{\max} > 4} \langle k_1 \rangle^3 |P_k \Box_k (\A \partial_x^3 (v_1 \cdots v_{\tilde{R}}))|_{L_{x_1}^{\infty} L_{x_2}^{2} \cdots L_{x_n}^{2}} + \sum_{k \in \mathbb{Z}^n, |k_1| = k_{\max} > 4} \langle k_1 \rangle^3 |P_k \Box_k (\A \partial_x^3 (v_1 \cdots v_{\tilde{R}}))|_{L_{x_1}^{\infty} L_{x_2}^{2} \cdots L_{x_n}^{2}} := III + IV. \tag{6.8}
\]
Using the decomposition (6.1),

\[ III \lesssim \sum_{k \in \mathbb{Z}^n, |k| = k_{\text{max}} > 4} \langle k_1 \rangle^3 \sum_{S_1^{(1)}} \left\| P_1 \Box_k (\mathcal{A} \partial_x^3 (\Box_k(\hat{u}_1 \ldots \Box_k(\hat{u}_R))) \right\|_{L^\infty_{X_1} L^2_{x_2} \ldots L^2_{x_n}} + \sum_{k \in \mathbb{Z}^n, |k| = k_{\text{max}} > 4} \langle k_1 \rangle^3 \sum_{S_1^{(1)}} \left\| P_1 \Box_k (\mathcal{A} \partial_x^3 (\Box_k(\hat{u}_1 \ldots \Box_k(\hat{u}_R))) \right\|_{L^\infty_{X_1} L^2_{x_2} \ldots L^2_{x_n}} := III_1 + III_2. \]  

(6.9)

By Lemma 4.2,

\[ III_1 \lesssim \sum_{S_1^{(1)}} \sum_{k \in \mathbb{Z}^n, |k| = k_{\text{max}} > 4} \langle k_1 \rangle^3 \left\| \Box_k (\Box_k(\hat{u}_1 \ldots \Box_k(\hat{u}_R))) \right\|_{L^1_{X_1} L^2_{x_2} \ldots L^2_{x_n}}. \]  

(6.10)

By symmetry, we may assume \( |k_1^{(1)}| = \max(|k_1^{(1)}|, \ldots, |k_1^{(\hat{R})}|) \) in \( S_1^{(1)} \). Hence,

\[ III_1 \lesssim \sum_{S_1^{(1)}, |k_1^{(1)}| \sim k_{\text{max}} > 4} \langle k_1^{(1)} \rangle^3 \langle k_{\text{max}} \rangle^{3/2} \sum \left\| \Box_k (\Box_k(\hat{v}_1 \ldots \Box_k(\hat{u}_R))) \right\|_{L^1_{X_1} L^2_{x_2} \ldots L^2_{x_n}} \lesssim \langle \theta_3^{(1)}(v_1) \rangle \prod_{i=2}^{\hat{R}} \langle \theta_2^{(1)}(v_i) \rangle \lesssim \| u \|_{L^\infty_X}. \]

(6.11)

Applying (4.12) and using a similar way as in (6.12),

\[ III_2 \lesssim \sum_{k \in \mathbb{Z}^n, |k| = k_{\text{max}} > 4, |k_2| \leq |k_1|} \langle k_1 \rangle^3 \langle k_{\text{max}} \rangle^{3/2} \sum \left\| \Box_k (\Box_k(\hat{v}_1 \ldots \Box_k(\hat{u}_R))) \right\|_{L^1_{X_1} L^2_{x_2} \ldots L^2_{x_n}} \lesssim \prod_{i=1}^{\hat{R}} \langle \theta_3^{(1)}(v_i) \rangle \leq \| u \|_{L^\infty_X}. \]

(6.12)

So, we have shown that

\[ III \lesssim \| u \|_{L^\infty_X}. \]

(6.13)

Now we estimate IV. Using the decomposition (6.1),

\[ IV \leq \sum_{k \in \mathbb{Z}^n, |k| = k_{\text{max}} > 4} \langle k_1 \rangle^3 \sum_{S_2^{(2)}} \left\| P_2 \Box_k (\mathcal{A} \partial_x^3 (\Box_k(\hat{u}_1 \ldots \Box_k(\hat{u}_R))) \right\|_{L^\infty_{X_1} L^2_{x_2} \ldots L^2_{x_n}} + \sum_{k \in \mathbb{Z}^n, |k| = k_{\text{max}} > 4} \langle k_1 \rangle^3 \sum_{S_2^{(2)}} \left\| P_2 \Box_k (\mathcal{A} \partial_x^3 (\Box_k(\hat{u}_1 \ldots \Box_k(\hat{u}_R))) \right\|_{L^\infty_{X_1} L^2_{x_2} \ldots L^2_{x_n}} := IV_1 + IV_2. \]  

(6.14)
By Lemma 4.2,
\[
IV_1 \lesssim \sum_{\mathcal{S}_2^{(2)}} \sum_{k \in \mathbb{Z}^n, |k_2| > k_{\text{max}} > 4} \langle k_2 \rangle^3 \| \Box_k (\Box_{k(1)} v_1 \cdots \Box_{k(\tilde{\rho}_1)} v_{\tilde{\rho}_1}) \|_{L_t^2 L_x^2 L^2_{1,2,\ldots, n}}.
\] (6.15)

In view of the symmetry, one can bound $IV_1$ by using the same way as that of $III_1$ and as in (5.14)–(5.17):
\[
IV_1 \lesssim \| u \|_{\tilde{R}^X}.
\] (6.16)

For the estimate of $IV_2$, we apply (4.2),
\[
IV_2 \lesssim \sum_{k \in \mathbb{Z}^n, |k_1| = k_{\text{max}} > 4} \langle k_1 \rangle^{3/2} \langle k_2 \rangle^3 \sum_{\mathcal{S}_2^{(2)}} \| P_2 \Box_k (\Box_{k(1)} v_1 \cdots \Box_{k(\tilde{\rho}_1)} v_{\tilde{\rho}_1}) \|_{L_t^{2+m/(1+m)} L_x^{2+m/(1+m)} (\mathbb{R}^{1+n})} \lesssim \| u \|_{\tilde{R}^X}.
\] (6.17)

Hence, in view of (6.16) and (6.17), we have
\[
IV \lesssim \| u \|_{\tilde{R}^X}.
\] (6.18)

Collecting (6.5), (6.7), (6.13) and (6.18), we have shown that
\[
\sum_{\alpha=0,3} \sum_{i,j=1}^n \varrho_1^{(i)} (\mathcal{A} \partial_{x_j} (u^{R_0} (\partial_{x_1} u)^{R_1} (\partial_{x_2} u)^{R_2} (\partial_{x_3} u)^{R_3})) \lesssim \| u \|_{\tilde{R}^X}.
\] (6.19)

For later estimate, we need a nonlinear mapping estimate

**Lemma 6.2** ([28], Lemma 7.1.) Let $s \geq 0$, $1 \leq p, p_i, \gamma, \gamma_i \leq \infty$ satisfy
\[
\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_N}, \quad \frac{1}{\gamma} = \frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_N}.
\] (6.20)

Then
\[
\sum_{k \in \mathbb{Z}^n} \langle k \rangle^s \| \Box_k (u_1 \cdots u_N) \|_{L_t^p L_x^p (\mathbb{R}^{1+n})} \lesssim \prod_{i=1}^N \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^s \| \Box_k u_i \|_{L_t^p L_x^p (\mathbb{R}^{1+n})} \right).
\] (6.21)

Firstly, we estimate $\varrho_2^{(1)} (\mathcal{A} (u^{R_0} (\partial_{x_1} u)^{R_1} (\partial_{x_2} u)^{R_2} (\partial_{x_3} u)^{R_3})))$ and $\varrho_3^{(1)} (\mathcal{A} (u^{R_0} (\partial_{x_1} u)^{R_1} (\partial_{x_2} u)^{R_2} (\partial_{x_3} u)^{R_3})))$. In view of (3.34) and (3.24),
\[
\sum_{j=2,3} \varrho_j^{(1)} (\mathcal{A} (u^{R_0} (\partial_{x_1} u)^{R_1} (\partial_{x_2} u)^{R_2} (\partial_{x_3} u)^{R_3})))
\]
\[ \lesssim \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \| \Box_k \left( u^{R_0} (\partial_{x_1} u) R_1 (\partial_{x_2} u) R_2 (\partial_{x_3} u) R_3 \right) \|_{L_{t,x}^{2+\frac{m}{2}} (\mathbb{R}^{1+n})}. \] (6.22)

We use Lemma 6.2 to control the right hand side of (6.22):

\[ \begin{aligned}
\lesssim & \prod_{i=1}^{m+1} \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \| \Box_k v_i \|_{L_{t,x}^{2+\frac{m}{2}} (\mathbb{R}^{1+n})} \right) \prod_{i=m+2}^{\tilde{R}} \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2} \| \Box_k v_i \|_{L_{t,x}^\infty (\mathbb{R}^{1+n})} \right) \\
\lesssim & \prod_{i=1}^{m+1} \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \| \Box_k v_i \|_{L_{t,x}^{2+\frac{m}{2}} (\mathbb{R}^{1+n})} \right) \prod_{i=m+2}^{\tilde{R}} \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \| \Box_k v_i \|_{L_{t}^\infty L_{x}^{4} (\mathbb{R}^{1+n})} \right) \\
\lesssim & \prod_{i=1}^{\tilde{R}} \mathcal{E}_3^{(1)} (v_i) \lesssim \| u \|_{\tilde{X}}. \end{aligned} \] (6.23)

Secondly, we estimate \( \mathcal{E}_2^{(1)} (\mathcal{A} \partial_{x_1}^3 (u^{R_0} (\partial_{x_1} u) R_1 (\partial_{x_2} u) R_2 (\partial_{x_3} u) R_3)). \)

\[ \begin{aligned}
\lesssim & \sum_{k \in \mathbb{Z}^n, \, k_{\text{max}} > 4} \langle k \rangle^{3/2-3/m} \| \Box_k \mathcal{A} \partial_{x_1}^3 (v_1 \ldots v_{\tilde{R}}) \|_{L_{t_1}^m L_{x_1}^\infty \ldots L_{x_n}^\infty (\mathbb{R}^{1+n})} \\
+ & \sum_{k \in \mathbb{Z}^n, \, k_{\text{max}} \leq 4} \langle k \rangle^{3/2-3/m} \| \Box_k \mathcal{A} \partial_{x_1}^3 (v_1 \ldots v_{\tilde{R}}) \|_{L_{t_1}^m L_{x_1}^\infty \ldots L_{x_n}^\infty (\mathbb{R}^{1+n})} \\
:= & V + VI. \end{aligned} \] (6.24)

By (3.24) and Lemma 6.2, we have

\[ \begin{aligned}
VI \lesssim & \sum_{k \in \mathbb{Z}^n} \| \Box_k (v_1 \ldots v_{\tilde{R}}) \|_{L_{t,x}^{2+\frac{m}{2}} (\mathbb{R}^{1+n})} \lesssim \| u \|_{\tilde{X}}. \end{aligned} \] (6.25)

It is easy to see that

\[ \begin{aligned}
V \lesssim & \left( \sum_{k \in \mathbb{Z}^n, \, |k_1| = k_{\text{max}} > 4} \ldots + \sum_{k \in \mathbb{Z}^n, \, |k_n| = k_{\text{max}} > 4} \right) \langle k \rangle^{3/2-3/m} \\
\times & \| \Box_k \mathcal{A} \partial_{x_1}^3 (v_1 \ldots v_{\tilde{R}}) \|_{L_{t_1}^m L_{x_1}^\infty \ldots L_{x_n}^\infty (\mathbb{R}^{1+n})} := \Upsilon_1 (u) + \ldots + \Upsilon_n (u). \end{aligned} \] (6.26)

Applying the decomposition (6.1), (4.4) and Lemma 4.3, we obtain that

\[ \begin{aligned}
\Upsilon_1 (u) \lesssim & \sum_{k \in \mathbb{Z}^n, \, |k_1| = k_{\text{max}} > 4} \langle k \rangle^{3} \sum_{\delta_1^{(1)}} \| \Box_k \left( \Box_{k(1)} v_1 \ldots \Box_{k(\delta_1)} v_{\tilde{R}} \right) \|_{L_{t_1}^{\delta_1} L_{x_1}^{2} \ldots L_{x_3}^{2} (\mathbb{R}^{1+n})} \\
+ & \sum_{k \in \mathbb{Z}^n, \, |k_1| = k_{\text{max}} > 4} \langle k \rangle^{9/2} \sum_{\delta_2^{(1)}} \| \Box_k \left( \Box_{k(1)} v_1 \ldots \Box_{k(\delta_2)} v_{\tilde{R}} \right) \|_{L_{t,x}^{\delta_2+1} (\mathbb{R}^{1+n})}, \end{aligned} \] (6.27)
which reduces to the case $\alpha = 3$ in (6.2). So,
\[ \Upsilon_1(u) \lesssim \|u\|_{X}^{\tilde{R}}. \] (6.28)

Again, in view of (4.4) and Lemma 4.3,
\[ \Upsilon_2(u) \lesssim \sum_{k \in \mathbb{Z}^n, |k| = k_{\text{max}} > 4} \langle k \rangle^{3/2} \sum_{\mathcal{S}(2)} \|\Box_k \left( \Box_k(v_1 \ldots \Box_k(v_{\tilde{R}})) \right) \|_{L^1_t L^2_{x_1 x_3 \ldots x_n} L^2_{(\mathbb{R}^{1+n})}} + \sum_{k \in \mathbb{Z}^n, |k| = k_{\text{max}} > 4} \langle k \rangle^{3/2} \sum_{\mathcal{S}(2)} \|\Box_k \left( \Box_k(v_1 \ldots \Box_k(v_{\tilde{R}})) \right) \|_{L^1_t L^2_{x_1} (\mathbb{R}^{1+n})}, \] (6.29)

which reduces to the same estimate as $\Upsilon_1(u)$. Using the same way as $\Upsilon_2(u)$, we can get the estimates of $\Upsilon_3(u), \ldots, \Upsilon_n(u)$. So,
\[ \rho_2^{(1)}(A \partial^3_{x_1} (v_1 \ldots v_{\tilde{R}})) \lesssim \|u\|_{X}^{\tilde{R}}. \] (6.30)

We need to further bound $\rho_2^{(1)}(A \partial^3_{x_i} (v_1 \ldots v_{\tilde{R}})), i = 2, \ldots, n$, which is essentially the same as $\rho_2^{(1)}(A \partial^3_{x_1} (v_1 \ldots v_{\tilde{R}})).$ Obviously, (6.24) holds if we substitute $\partial^3_{x_1}$ with $\partial^3_{x_i}$. Moreover, using (4.4), Lemmas 6.2 and 4.3, we easily get that
\[ \rho_2^{(1)}(A \partial^3_{x_i} (v_1 \ldots v_{\tilde{R}})) \lesssim \|u\|_{X}^{\tilde{R}}. \] (6.31)

By Lemma 3.2, (6.32), we see that
\[ \|\Box_k A \partial^3_{x_1} f\|_{L^\infty_t L^2_x \cap L^{2+m}_{t,x} (\mathbb{R}^{1+n})} \lesssim \langle k \rangle^3 \|\Box_k f\|_{L^{2+m}_{t,x} (\mathbb{R}^{1+n})}. \] (6.32)

Hence, in view of (3.22), repeating the procedure as in the estimates of $\rho_3(u)$ in Theorem 1.2 $\rho_3^{(1)}(A \partial^3_{x_1} (v_1 \ldots v_{\tilde{R}}))$ can be controlled by the right hand side of (6.27) and (6.25). By symmetry, the estimate of $\rho_3^{(1)}(A \partial^3_{x_i} (v_1 \ldots v_{\tilde{R}})), i = 2, \ldots, n$ is identical to $\rho_3^{(1)}(A \partial^3_{x_1} (v_1 \ldots v_{\tilde{R}}))$. Summarizing the estimates as in the above, we have shown that
\[ \|\mathcal{T} u\|_X \lesssim \|u_0\|_{M_{2,1}^{2/3}} + \sum_{m+1 \leq \tilde{R} \leq M+1} \|u\|_{X}^{\tilde{R}}. \] (6.33)

Therefore, the desired result holds by a standard contracting mapping argument.

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