Automatic Structures and Boundaries for Graphs of Groups

Walter D. Neumann and Michael Shapiro*

Abstract. We study the synchronous and asynchronous automatic structures on the fundamental group of a graph of groups in which each edge group is finite. Up to a natural equivalence relation, the set of biautomatic structures on such a graph product bijects to the product of the sets of biautomatic structures on the vertex groups. The set of automatic structures is much richer. Indeed, it is dense in the infinite product of the sets of automatic structures of all conjugates of the vertex groups. We classify these structures by a class of labelled graphs which “mimic” the underlying graph of the graph of groups. Analogous statements hold for asynchronous automatic structures. We also discuss the boundaries of these structures.

1. Introduction

Given a group \( G \), there is a natural equivalence relation on the set of synchronous or asynchronous automatic structures on \( G \). Namely, two such structures, \( L \) and \( L' \) are equivalent (written \( L \sim L' \)) if there is a constant \( K \) so that whenever a word of \( L \) and a word of \( L' \) represent the same element of \( G \), these two words asynchronously \( K \)-fellow travel each other. (For definitions, see below.) This leads [NS1] to introduce \( S\mathfrak{A}(G) \), \( BS\mathfrak{A}(G) \), \( \mathfrak{A}(G) \) and \( B\mathfrak{A}(G) \), the sets of (respectively) automatic, biautomatic, asynchronously automatic, and asynchronously biautomatic structures on \( G \) up to equivalence.

Currently, information is fairly scarce about these sets. \( \mathfrak{A}(G) \) has been computed for \( G \) virtually abelian, virtually free ([NS1]), or virtually a surface group ([B]). In the latter two cases it is a single point. \( S\mathfrak{A}(G) = BS\mathfrak{A}(G) \) and is a single point if \( G \) is word hyperbolic. \( BS\mathfrak{A}(G) \) has been computed, and \( S\mathfrak{A}(G) \) is fairly well understood when \( G \) is a geometrically finite hyperbolic group ([NS2]). By contrast, \( \mathfrak{A}(G) \) is very large and poorly understood when \( G \) is the fundamental group of a closed hyperbolic 3-manifold group which fibers over the circle. Notice that here \( G \) is an HNN-extension of a hyperbolic surface group. However, its unique automatic structure does not arise from the automatic structure on the surface group. Indeed, the surface group is not rational in this automatic structure.

In this paper, we will study these sets when \( G = \pi_1(\mathcal{Y}) \) is the fundamental group of a finite graph of groups \( \mathcal{Y} \) in which each edge group is finite. The assumption of finite edge groups turns out to ensure that each conjugate of a vertex group is rational with respect to any (synchronous or asynchronous) automatic structure on \( G \). Consequently, there is a

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natural map
$$\mathfrak{A}(G) \to \prod_{H \in \mathcal{H}} \mathfrak{A}(H),$$
where $\mathcal{H}$ denotes the set of conjugates of vertex groups. One of our main results is that this map is injective with dense image (Theorem 3.9). Moreover, $\mathfrak{SA}(G)$ is just the inverse image of $\prod_{H \in \mathcal{H}} \mathfrak{SA}(H)$ under this map. In [NS2] there is an analogous injection for synchronous automatic structures on a geometrically finite hyperbolic group with maximal parabolics playing the role that vertex groups play here.

Now if $H \in \mathcal{H}$, we have $H = gG_V g^{-1}$ where $G_V$ is a vertex group. Given such a $g$, it is natural to look for “minimal” $h$ so that $H = hG_V h^{-1}$. We shall see that if $H \neq G_V$ there is an $F_E$ orbit of such values, where $F_E$ is the edge group associated to an edge $E$ incident at $V$. Thus, to specify an asynchronous automatic structure on $G$ it is only necessary to specify a choice of structure $[L_h] \in \mathfrak{A}(G_V)$ for each such $h$, for this, in turn, specifies a structure on $H = hG_V h^{-1}$. In a sense which we shall make clear, the choice of $L_h$ must be equivariant with respect to the action of $F_E$ on the set of such $h$. This allows us to classify $\mathfrak{A}(G)$ in terms of maps which we call regular deployments (Theorems 3.3 and 3.8).

There is a more concrete way to classify $\mathfrak{A}(G)$, and that is in terms of objects which we call minimal special $\mathcal{Y}$-graphs. Roughly, a $\mathcal{Y}$-graph $\mathcal{X}$ is a finite labelled graph which maps onto the underlying graph of $\mathcal{Y}$. Each vertex of $\mathcal{X}$ is labelled by an equivalence class of structures on the corresponding vertex group of $\mathcal{Y}$. Each edge from this vertex is labelled by a rational subset of this vertex group. The labelling must be equivariant in terms of actions of the edge groups. In the case of a biautomatic or asynchronously biautomatic structure, a most efficient $\mathcal{Y}$-graph is essentially the underlying graph of $\mathcal{Y}$ with biautomatic or asynchronously biautomatic structures at each vertex. In particular, this gives bijections
$$\text{B}\mathfrak{A}(G) \to \prod_{V \in \text{vert}(\mathcal{Y})} \text{B}\mathfrak{A}(G_V)$$
and
$$\text{BS}\mathfrak{A}(G) \to \prod_{V \in \text{vert}(\mathcal{Y})} \text{BS}\mathfrak{A}(G_V),$$
where $G_V$ denotes the vertex group at $V$.

As classifying objects these $\mathcal{Y}$-graphs have several advantages over regular deployments. They are finite objects and they are easy to construct. Unlike regular deployments, $\mathcal{Y}$-graphs admit “local” modifications. Finally, a $\mathcal{Y}$-graph is easily turned into a generalized finite state automaton for the structure which it determines.

In the final section of this paper we describe the boundary of an asynchronous or synchronous automatic structure on $G = \pi_1(\mathcal{Y})$. It is a “tree completion” of the disjoint union of the boundaries for the automatic structures on the conjugates of the vertex groups. The tree in question is the tree on which $G$ acts with quotient $\mathcal{Y}$ (see for example [Se]).

The assumption of finite edge groups in this paper may seem restrictive. However, as the hyperbolic 3-manifold example mentioned above shows, it is necessary to ensure that the vertex groups are rational. In fact, even if one restricts to abelian edge groups,
the Heisenberg group is an example of the fundamental group of such a graph of groups where the vertex group has plentiful automatic structures but the group itself is not even asynchronously automatic. Another example is $F_2 \times \mathbb{Z}$, which can be seen either as $F_2 \ast F_2$ or $\mathbb{Z}^2 \ast \mathbb{Z}^2$. In work in preparation we show that $\mathbb{S}(F_2 \times \mathbb{Z})$ is quite large and unlikely to yield to classification by techniques like the current ones.

2. Background and definitions.

We start with a finitely generated group $G$ and a map from a finite set $A = \{a_i\}$ into $G$ denoted by $a_i \mapsto \overline{a}_i$. The set of all finite strings $w = a_{i_1} \ldots a_{i_n}$ on elements of $A$ (including the empty string) forms a monoid under the operation of concatenation. We denote this monoid by $A^*$. We define the length of $w = a_{i_1} \ldots a_{i_n} \in A^*$ to be $n$ and denote this by $\ell(w)$. (The length of the empty word is 0.) The map $a_i \mapsto \overline{a}_i$ extends to a unique monoid homomorphism from $A^*$ to $G$ and we denote this extension by $w \mapsto \overline{w}$. We will assume that this map is onto. We will also assume that $A$ is supplied with an involution denoted by $a_i \mapsto a_i^{-1}$ and that the evaluation map respects this, that is, $\overline{a_i^{-1}} = (\overline{a_i})^{-1}$. This allows us to form the Cayley graph of $G$ with respect to $A$, $\Gamma = \Gamma_A$. The vertices of $\Gamma_A$ are the elements of $G$. There is a directed edge from $g$ to $g'$ labelled by $a \in A$ exactly when $g' = ga$. Thus there is exactly an $A$'s worth of edges emanating from each vertex of $G$. Since $A$ is finite, $\Gamma$ is locally finite. Since $A$ generates $G$, $\Gamma$ is path connected. By making each edge of $\Gamma$ isometric with the unit interval, we endow $G$ with a metric $d_A = d$ called the word metric. That is, the distance between two points of $\Gamma$ is defined to be the length of the shortest path connecting them. $G$ acts on $\Gamma$ by left translation, and this action preserves distance. We take the length of an element of $G$ to be its distance from the identity, that is $\ell(g) = d(1, g)$. A word $w \in A^*$ determines a path in $\Gamma$, which we also denote by $w$, as follows. The path $w$ maps the interval $[0, \ell(w)]$ into $\Gamma$ by following at unit speed along the edge path in $\Gamma$ based at 1 and labelled by $w$. We extend this to a map of $[0, \infty)$ by setting $w(t) = \overline{w}$ for $t \geq \ell(w)$.

We call a subset of $A^*$ a language. A language $L$ is a normal form if $\overline{L} = G$. Note that we do not require $L \to G$ to be an injection. We will say that a normal $L$ has the asynchronous fellow traveller property if there is a constant $K$ so that given $w, w' \in L$ with $d(\overline{w}, \overline{w'}) \leq 1$, there are monotone maps $\phi, \psi$ of $[0, \infty)$ onto itself so that for all $t$, $d(\phi(t), \psi(t)) \leq K$. We say $L$ has the synchronous fellow traveller property if $\phi$ and $\psi$ can be chosen to be the identity.

Given two normal forms $L, L'$, each with the asynchronous fellow traveller property, we will say that they are equivalent and write $L \sim L'$ if $L \cup L'$ has the asynchronous fellow traveller property. We denote the equivalence class of $L$ by $[L]$.

Recall that a finite state automaton $A$ with alphabet $A$ is a finite directed graph on a vertex set $S$ (called the set of states) with each edge labelled by an element of $A$ and such that different edges leaving a vertex always have different labels. Moreover, a start state $s_0 \in S$ and a subset of accepted states $T \subset S$ are given. A word $w \in A^*$ is in the language $L$ accepted by $A$ if and only if it defines a path starting from $s_0$ and ending in an accept state in this graph. We may assume there is no “dead state” in $S$ (a state not accessible from $s_0$ or from which no accepted state is accessible). Eliminating such states does not change the language $L$ accepted by $A$. 

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A language is *regular* if it is accepted by some finite state automaton.

We will also need the concept of a *non-deterministic finite state automaton*. The difference is that a non-deterministic finite state automaton is allowed to have several start states instead of just one, different edges from a vertex may have the same label, and edges are allowed to have empty label (such edges are called *ε-transitions*). A word is accepted by such an automaton if it labels a path from a start state to an accept state. This path is allowed to traverse ε-transitions. It is a standard result that the language of words accepted by a non-deterministic finite state automaton is a regular language.

We shall also have occasion to use generalized finite state automata. A *generalized finite state automaton* is defined just like a non-deterministic finite state automaton except that the edges are labelled by regular sublanguages of $A^*$ rather than by elements of $A$. This machine accepts a word $w$ if $w$ can be written as $w_1 \ldots w_k$ such that there is a corresponding directed edge path $e_1 \ldots e_k$ from a start state to an accept state such that $w_i$ is in the language labelling $e_i$ for each $i$. It is a standard fact that this language of accepted words is regular.

A *(synchronous) automatic structure* for $G$ is a regular normal form with the synchronous fellow traveller property. It is a result of [ECHLPT] that every automatic structure has a sublanguage which bijects to $G$. Notice that if $L \subset L'$ and $L$ is an automatic structure, then $L \sim L'$. We will take the following as our definition of asynchronous automatic structure. An asynchronous automatic structure for $G$ is a rational normal form with the asynchronous fellow traveller property. This is not exactly equivalent to the use of the term in [ECHLPT]. Rather, these are the non-deterministic asynchronous automatic structures of [S1]. Since every non-deterministic asynchronous automatic structure contains an equivalent asynchronous automatic structure which bijects to $G$, we will make no further distinction between the two. We will call an asynchronous automatic structure $L$ asynchronously biautomatic if there is a constant $K$ so that if $w, w' \in L$ with $\overline{w} = aw'$ where $a \in A \cup \{1\}$, then there are reparameterizations $\phi$ and $\psi$ so that for all $t$, $d(\phi(t), aw'(\psi(t))) \leq K$. (Here $aw(\cdot)$ is the translate of $w(\cdot)$ by $a$.) We will call an automatic structure biautomatic if there exists $K$ so that $\phi$, $\psi$ can be taken to be the identity. We take $S\mathfrak{A}(G)$, $B\mathfrak{A}(G)$, $B\mathfrak{A}(G)$ and $B\mathfrak{A}(G)$ to be respectively the sets of automatic, biautomatic, asynchronously automatic, and asynchronously biautomatic structures on $G$ up to equivalence.

Given an asynchronous or synchronous automatic structure $L$, we say that $S \subset G$ is $L$-rational if $\{w \in L : \overline{w} \in S\}$ is regular. It is a result of [NS1] that $L$-rationality depends only on the equivalence class of $L$. Using the techniques of [GS], one sees that if $H$ is an $L$-rational subgroup of $G$, then $L$ induces an equivalence class of asynchronous respectively synchronous automatic structures on $H$.

**Convention.** We have pointed out that any asynchronous automatic structure contains an equivalent one that bijects to $G$, so there is certainly no loss of generality in assuming that all our structures are finite-to-one. Since this simplifies some proofs, we will assume it from now on.

3. Graphs of groups with finite edge groups
Let $\mathcal{Y}$ be a graph of groups. We start by fixing notation. The underlying graph $Y$ of $\mathcal{Y}$ is a connected graph made of a finite collection of vertices and a finite collection of unoriented edges. We consider each unoriented edge as a pair of oriented edges and denote the initial and terminal vertices of an oriented edge $E$ by $\partial_0 E$ and $\partial_1 E$. The reverse of an edge $E$ is denoted $E^{-1}$. To each vertex $V$ is associated a group $G_V$ and to each edge $E$ is associated a group $F_E$ with $F_E = F_{E^{-1}}$. Further, to each edge $E$ is associated a pair of injections $\partial_0 : F_E \to G_{\partial_0 E}$ and $\partial_1 : F_E \to G_{\partial_1 E}$, which are exchanged when $E$ is replaced by $E^{-1}$.

Such a graph may be seen as instructions for building a group by repeated free products with amalgamation and HNN-extensions. To do this one takes a maximal tree $T \subset Y$. Inductively one forms free product with amalgamation for each edge of $T$. One then performs an HNN-extension for each edge not on $T$. The resulting group is determined up to isomorphism by the graph of groups. We refer to it as the fundamental group of $\mathcal{Y}$, denoted $\pi_1(\mathcal{Y})$. For details see [Se], for example.

We describe a normal form for the elements of $G = \pi_1(\mathcal{Y})$. We take the maximal tree $T$ to be fixed throughout. We also choose a fixed base vertex $V_0 \in \mathcal{Y}$.

**Definition.** For each edge $E$ of $Y$ we have an element $t_E \in G = \pi_1(\mathcal{Y})$ as follows: $t_E$ is the stable letter associated to the edge $E$ if $E$ is not in $T$ and $t_E = 1$ if $E$ is in $T$. In particular, $t_{E^{-1}} = t_E^{-1}$. Then each element of $G$ can be written in the normal form

$$h = g_0 t_{E_1} g_1 \cdots t_{E_m} g_m$$

where:

1. $E_1 \ldots E_m$ is a path in $Y$ starting at the base vertex $V_0$;
2. $g_0 \in G_{V_0}$ and $g_i \in G_{\partial_i E_i}$ for $i = 1, \ldots, m$;
3. if $E_{i+1} = E_i^{-1}$ then $g_i \notin \partial_1(F_{E_i})$. This expression is unique up to the following two operations:
   - one can add or delete terminal words consisting of trivial $t_{E_i}$’s (subject to condition (3); such words are bounded in length by the diameter of the maximal subtree $T$);
   - one can replace $g_i t_{E_i} g_{i+1}$ by $(g_i \partial_0(f)) t_{E_i} (\partial_1(f^{-1}) g_{i+1})$ for $f \in F_{E_i}$.

We shall assume from now on that all edge groups $F_E$ are finite. Note that if $\mathcal{Y}$ includes an edge $E$ with $\partial_1 F_E = G_{\partial_1 E}$ then this edge can be collapsed without changing $\pi_1(\mathcal{Y})$ unless the edge is a loop, say $\partial_0 E = \partial_1 E = V$. In this case we may also eliminate $E$ by replacing $G_V$ by $G_V \rtimes \mathbb{Z}$. This $G_V \rtimes \mathbb{Z}$ is virtually cyclic so $\mathfrak{A}(G_V \rtimes \mathbb{Z})$ consists of a single point (cf. [NS1]). Thus, from the point of view of computing asynchronous automatic structures on $G$ in terms of asynchronous structures on the vertex groups, this simplification of $\mathcal{Y}$ is harmless. If $\mathcal{Y}$ has no edge with $\partial_1 F_E = G_{\partial_1 E}$ we will say $\mathcal{Y}$ is reduced.

To simplify later notation we define an extended graph $\hat{\mathcal{Y}}$ by adding a new base edge $E_0$ to $\mathcal{Y}$, going from a new vertex (which we will never need to refer to) to the base vertex $V_0$. We put $F_{E_0} = \{1\}$. Using the above normal form we define for each edge $E$ of $\hat{\mathcal{Y}}$: 

$$\mathcal{G}_E = \{ (\ = ) \cap \mathcal{E}_\infty \ldots \cap \mathcal{E}_3 \} \in \mathcal{G} \text{ as in } (*) : \mathcal{D} \geq t, \mathcal{E}_0 = \mathcal{E}, \mathcal{D} \in \partial_\infty(\mathcal{F}_1) \}.$$ 

In particular, $\mathcal{G}_{E_0} = \{ \infty \}$. We stress that the only role of the base edge $E_0$ is to support the notation $\mathcal{G}_{E_0}$. We do not include the reverse edge $E_0^{-1}$. 

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The significance of these sets is that, as we will discuss in detail in section 5, the disjoint union $\bigsqcup G_\mathcal{E}/\mathcal{F}_\mathcal{E}$ over $E \in \text{edge} \hat{\mathcal{Y}}$ is in one-one correspondence with the vertices of the tree on which $G$ acts with quotient $\mathcal{Y}$ and vertex and edge stabilizers given by the data of $\mathcal{Y}$. If $\mathcal{Y}$ is reduced then each conjugate of a vertex group stabilizes precisely one vertex of this tree, so the disjoint union $\bigsqcup G_\mathcal{E}/\mathcal{F}_\mathcal{E}$ is a set of representatives for the conjugates of vertex groups. For our present purposes we formulate this as the following lemma.

**Lemma 3.1.** Suppose $\mathcal{Y}$ is reduced. Let $H = gG_V g^{-1} \in \mathcal{H}$. Then $H$ determines an edge $E$ of $\hat{\mathcal{Y}}$ with $\partial_1 E = V$ and $h \in G_\mathcal{E}$ so that $H = hG_V h^{-1}$. If $E'$ with $\partial_1 E' = V$ and $h' \in G_{\mathcal{E}'}$ also satisfy $h' G_V h'^{-1} = H$ then $E = E'$ and $h' \in h\partial_1 F_E$.

**Proof.** The edge $E$ and $h \in G_\mathcal{E}$ can be found from $g$ in the following manner. If $V = V_0$ and $g \in G_{V_0}$ we take $h = 1 \in G_\mathcal{E}$. Otherwise, write $g$ in normal form $(*)$, and delete any final portion of $g$ lying in $G_V$. Call the resulting expression $h$. If the last letter of $h$ lies in some $G_{V'}$, we take $E$ to be the last edge in the path in $T'$ from $V'$ to $V$. If the last letter of $h$ is the stable letter of an edge $E_1$ not in $T$, we take $E = E_1$ if $\partial_1 E_1 = V$ and otherwise we take $E$ to be the last edge in the path in $T$ from $\partial_1 E_1$ to $V$.

Clearly $H = hG_V h^{-1}$. The uniqueness statement about $E$ and $h$ follows by noting that $h$ (up to the action of $\partial_1 F_E$) is visible as the first half of the normal form of any $h x h^{-1} \in hG_V h^{-1}$ with $x \notin \partial_1 F_E$. \hfill \blacksquare

**Definition.** Whether $\mathcal{Y}$ is reduced or not, we define a *deployment* to be a map

$$
\psi : \bigsqcup_{E \in \text{edge} \hat{\mathcal{Y}}} G_\mathcal{E} \rightarrow \bigsqcup_{V \in \text{vert} \mathcal{Y}} \mathfrak{A}(G_\mathcal{V})
$$

with finite image taking $G_\mathcal{E}$ to $\mathfrak{A}(G_{\partial_1 E})$ for each $E$ and with the following equivariance property: the restriction of $\psi$ to $G_\mathcal{E}$ is $F_E$-equivariant in the sense that $\psi(h) = f \psi(h f)$ for $f \in \partial_1 F_E$.

There is some ambiguity in the notation $\psi(h)$, for a group element $h$ can be in more than one $G_\mathcal{E}$. Thus we are implicitly thinking of $h$ as an element of the disjoint union $\bigsqcup G_\mathcal{E}$. The particular $G_\mathcal{E}$ intended should be clear from context.

We wish to see that an arbitrary deployment determines an equivalence class of (possibly non-regular) languages with the fellow traveller property which map onto $G$. We start by choosing a convenient alphabet.

**Definition.** The above normal form for elements $G$ gives embeddings of the vertex groups $G_V \subset G$. We can take a generating set $A$ for $G$ which is a union of generating sets for the vertex groups together with a generator $t_E$ for each edge $E \notin T$. We denote by $A_V$ the subset of all elements of $A$ evaluating into $G_V$. We may choose our generators such that for each edge $E$ of $\mathcal{Y}$ we have a subset of $A$ which bijects to $\partial_1 F_E$. By identifying any duplicates, we can assume that each element of a group $\partial_1 F_E \subset G$ is represented by exactly one letter of $A$. We refer to the subset of $A$ evaluating into $\partial_1 F_E$ as $A_E$. In particular, $A_E = A_{E^{-1}}$ for $E \in T$. Also, there is a unique element $e \in A$ which evaluates to $1 \in G$ and is in every $A_E$. We call $A$ a *convenient alphabet for $G*.
Let \( \psi \) be a deployment. For each \( h \in \mathcal{G}_E \), \( \psi(h) \) is a class in \( \mathfrak{A}(G_{\partial_1 E}) \). Choose a language \( L_\psi(h) \in \psi(h) \) for each \( \psi(h) \). We thus have \( L_\psi(h) = L_\psi(h') \) whenever \( \psi(h) = \psi(h') \). We assume our alphabet \( A \) is convenient and each \( G_V \)-language is over the alphabet \( A_V \). We take \( t_E \) to be the empty word for \( E \in T \). Recall that \( E_0 \) denotes the "base edge" that we added to \( Y \) with \( \partial_1 E_0 = V_0 \). We denote the unique element of \( \mathcal{G}_E \), by 1 and define

\[
L_\psi = \{ u_0 t_{E_1} \ldots t_{E_m} u_m : m \geq 0, \ E_1 \ldots E_m \text{ forms a path based at } V_0, \ u_0 \in L_\psi(1), u_i \in L_\psi(\bar{u}_0 t_{E_1} \ldots t_{E_i}) \text{ for } i \geq 1, \ 
\]

if \( E_{i+1} = E_i^{-1} \) then \( \bar{u}_i \notin \partial_1 F_E \}\).

In particular, \( \bar{u}_0 \in G_{V_0}, \bar{u}_i \in G_{\partial_1 E_i} \) for \( i \geq 1 \).

**Lemma 3.2.** \( L_\psi \) has the asynchronous fellow traveller property and is determined up to equivalence by \( \psi \).

**Proof.** We first show that \( L_\psi \) has the asynchronous fellow traveller property. The fellow traveller constant will be \( 1 + \max\{ \delta_h \} \) where \( \delta_h \) is a fellow traveller constant for \( \hat{L}_\psi(h) := \bigcup_{f \in A_E} fL_\psi(h \bar{f}) \). This union is an automatic structure by the equivariance of \( \psi \).

So suppose \( w, w' \in L_\psi \) with \( \bar{w} = \bar{w}' a, a \in A \). These words determine based edge paths \( p \) and \( p' \) in \( Y \) up to terminal segments lying in \( T \).

First suppose we can take \( p = p' \) so that \( w = u_0 t_{E_1} \ldots t_{E_m} u_m \) and \( w' = u'_0 t_{E_1} \ldots t_{E_m} u'_m \). If \( m = 0 \) then \( w = u_0 \) and \( w' = u'_0 \) both lie in \( L_\psi(1) \) so they fellow-travel. Otherwise, \( \bar{u}_0 \) and \( \bar{u}'_0 \) differ at most by an element of \( \partial_0 F_{E_1} \). Thus \( \bar{u}_0 = \bar{u}'_0 f_0 \) with \( f_0 \in A_{E_1} \). Hence, again, \( u_0 \) and \( u'_0 \) asynchronously fellow travel with the fellow traveller constant of \( L_\psi(1) \).

More generally, \( u_0 t_{E_1} \ldots u_{i-1} t_{E_i} = u'_0 t_{E_1} \ldots u'_{i-1} t_{E_i} g_i \) with \( g_i \in A_{E_i} \). We assume inductively that the two word have asynchronously fellow-travelled to this point with fellow-traveller constant \( \delta \) as above. If we put \( h = u'_0 t_{E_1} \ldots u'_{i-1} t_{E_i} \) then \( u'_i \in L_\psi(h \bar{f}_i) \subset L_\psi(h) \) and \( g_i u_i \in g_i L_\psi(h \bar{f}_i) \subset L_\psi(h) \). Also, \( \bar{g}_i \bar{u}_i = u'_i \bar{f}_i \) where \( f_i \in A_{E_i} \) is such that \( f_i t_{E_{i+1}} = t_{E_{i+1}} g_{i+1} \). Thus \( g_i u_i \) and \( u'_i \) asynchronously fellow-travel with the fellow-traveller constant of \( L_\psi(h) \). Hence \( u_0 t_{E_1} \ldots t_{E_i} u_i \) and \( u'_0 t_{E_1} \ldots t_{E_i} u'_i \) asynchronously \( \delta \)-fellow-travel. Thus, by induction, \( w \) and \( w' \) asynchronously \( \delta \)-fellow-travel.

We must now examine the case where we cannot take \( p = p' \). In this case, we can choose the paths \( p \) and \( p' \) so that (say) \( p' \) is an initial segment of \( p \). Then the previous case will apply to \( w \) and \( w'a \).

Notice that the above argument also shows that \( [L_\psi] \) did not depend on the choices \( \{ L_\psi(h) \} \). For if we are given choices \( \{ L'_\psi(h) \} \) giving \( L'_\psi \), we repeat the argument using \( \{ L''_\psi(h) \} \) where \( L''_\psi(h) = L_\psi(h) \cup L'_\psi(h) \) and observe that \( L_\psi \cup L'_\psi \subset L''_\psi \).

**Definition.** We say that \( \psi \) is a regular deployment if \( L_\psi \) is regular, and hence an asynchronous automatic structure, for some choice of languages in the classes in \( \text{Im}(\psi) \). We will see in the proof of the following theorem that the word “some” in this definition can be replaced by “any”.

**Theorem 3.3.** The map \( \psi \mapsto [L_\psi] \) gives a bijection \{regular deployments\} \( \rightarrow \mathfrak{A}(G) \).
Proof. We start by constructing the inverse map. That is, we construct a deployment \( \psi_L \) from \([L] \in A(G)\).

Lemma 3.4 [BGSS]. Let \( L \subset A^* \) be a finite to one rational structure for a group \( G \). Then for each \( g \in G \) there are only finitely many \( y \in A^* \) so that for some \( x, z \in A^* \), \( xyz \in L \) and \( \overline{y} = g \).

Proof. We suppose not, and let \( A \) be a finite state automaton for the language \( L \). We then have \( x_1y_1z_1, x_2y_2z_2, \ldots \in L \) with \( \overline{y_1} = \overline{y_2} = \ldots \). Among these we can find infinitely many \( x_1' y_1' z_1', x_2' y_2' z_2', \ldots \) so that each of \( x_1', \ldots \) labels a path from the start state of \( A \) to a common state of \( A \). Among these we can find infinitely many \( x_1'' y_1'' z_1'', x_2'' y_2'' z_2'', \ldots \) so that each of \( x_1'' y_1'', \ldots \) labels a path from the start state of \( A \) to a common state of \( A \). But then \( x_1'' y_1'' z_1'', \ldots \in L \) with \( \overline{x_1'' y_1'' z_1''} = \overline{x_1' y_1' z_1'} = \ldots \), contradicting the assumption that \( L \) is finite to one.

Lemma 3.5. Let \( A \) be a convenient alphabet for \( G = \pi_1(Y) \). Given an asynchronous automatic structure \( L' \) on \( G \), we can choose an asynchronous automatic structure \( L \) so that \( L \sim L' \) and \( L \subset A^* \). Moreover \( L \) can be chosen so that if \( y \in A^* \), \( xyz \in L \) and \( \overline{y} \in \partial_1 F_E \) for some edge \( E \) of \( Y \), then \( y \) has the form \( e^m f e^n \) with \( e, f \in A \) and \( \overline{y} = 1 \).

Proof. It is an observation of [NS1], based on a result of [ECHLPT], that given arbitrary monoid generating sets \( A \) and \( B \) for a group \( G \) and an synchronous or asynchronous automatic structure \( L' \subset B^* \) for \( G \), there is a synchronous respectively asynchronous automatic structure \( L'' \subset A^* \) with \( L'' \sim L' \). Choose \( A \) as in the lemma and \( L'' \) as just described. By the previous lemma, there are only finitely many \( y \in A^* \) so that \( xyz \in L'' \) and \( \overline{y} \) is in some \( \partial_1 F_E \). We consider such words that are not already in \( A\{e\}^* \). For each such word \( y \) there is a unique word \( w_y \in A\{e\}^* \) with the same length and value.

Let \( A'' \) be a finite state automaton for the language \( L'' \). Considering it as a finite graph and replacing it by a finite cover if necessary, we may assume that every path in \( A'' \) labelled by one of these words \( y \) is embedded. We now construct a nondeterministic machine as follows: wherever we see a path labelled by one of the words \( y \) we add a new path from the beginning point of this path to its end point labelled by \( w_y \). Call the language of this machine \( N \). The language we seek is obtained from the language of \( N \) by removing the regular language of all words containing one of the words \( y \) as a subword and is hence regular. Since two such subwords may be adjacent, we can only ensure that a word with value \( \overline{f} \) has the form \( e^m f e^n \). If we wish a language in which the \( e \)'s do not occur, we simply replace each edge of a machine labelled \( e \) by an \( e \)-transition.

We choose \( L \) as in Lemma 3.5. Then each element of \( L \) has a decomposition

\[
\begin{align*}
 w = u_0 t_{E_1} \ldots t_{E_m} u_m \quad (**) 
\end{align*}
\]

where
- \( t_{E_i} \) is the empty word if \( E_i \in T \);
- \( E_1 \ldots E_m \) is a path in \( Y \) starting at \( V_0 \);
Let $A$ states, $R$ which have a path to an accept state of $A$ path to an accept state of $A$.

There are finitely many distinct languages $L$ on $h$.

For each $h \in G_E$, we define

$$N_h = \{ v \in A^* : \exists w \in L \text{ edge path decomposed as in (***) above such that}$$

for some $i$, $E_i = E, v = u_i, u_0 t_{E_1} \ldots u_{i-1} t_{E_i} = h \},$$

and set

$$L_h = \bigcup_{f \in A_E} f N_{hF}.$$ (As usual we think of $h$ as lying in $\prod G_E$. This saves wear and tear on subscripts. In the next section it will be helpful to write $L_{E,h}$ instead.)

**Lemma 3.6.** For each $h$ in $G_E$, $L_h$ is an asynchronous automatic structure for $G_{A_E}$. There are finitely many distinct languages $L_h$. The equivalence class of $L_h$ depends only on $h$ and the equivalence class of $L$. If $L$ is a synchronous automatic structure, then so is $L_h$. The assignment $h \mapsto [L_h]$ is $F_E$-equivariant in the following sense: for $f \in F_E$ we have $[L_h] = f[L_{hf}]$. In particular this assignment is a deployment $\psi_L$ which depends only on $[L]$.

**Proof.** Assume that $h \in G_E$. We first check that $N_h$ is regular. We will express $N_h$ as the union of two sublanguages, determined by whether the $E_{i+1}$ in the definition of $N_h$ equals $E^{-1}$ or not, and build a nondeterministic finite state automaton for each of these languages (the nondeterminism will consist only in possibly having several start states).

The edge path decomposition of $L$-words induces similar decompositions for subwords of $L$-words, which we will use in the following.

Let $A$ be a finite state automaton for $L$. There are only finitely many words in the prefix-closure of $L$ which evaluate to $h$. We take those which have an edge path decomposition ending with $t_E$. We let $S_h$ be the collection of states of $A$ reached by these words. Let $R'_E$ be the collection of states of $A$ which are accept states or have a path to an accept state of $A$ labelled by a word that is not in $(A_{\partial_1 E})^*$ and has edge path decomposition (as just described for $L$-subwords) $t_{E'} \ldots$ with $\partial_0 E' = \partial_1 E$ and $E' \neq E^{-1}$. Let $A'_h$ be the nondeterministic machine obtained from $A$ by making $S_h$ the set of start states, $R'_E$ the set of accept states, and deleting all arrows labelled by letters not in $A_{\partial_1 E}$. Let $N'_h$ be the language accepted by the machine $A'_h$. Let $R''_E$ be the collection of states of $A$ which have a path to an accept state of $A$ labelled by a word that is not in $(A_{\partial_1 E})^*$ and has edge path decomposition $t_{E^{-1}} \ldots$. Let $A''_h$ be defined like $A'_h$ but using $R''_E$ instead of $R'_E$ and let $N''_h$ be the corresponding language. Then $N_h = N'_h \cup (N''_h - \{ e \}^* A_E \{ e \}^*)$. It is thus a regular language. Moreover, it is determined by the $E$ and the subset $S_h$ of the states of $A$, so there are only finitely many different languages $N_h$.

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The language $L_h$ is now a finite union of regular languages, hence regular. Moreover, it is determined by $E$ and the family of subsets $S_{hT}, f \in A_E$, of the states of $\mathcal{A}$, so there are a finite number of these languages.

We show that $L_h$ surjects onto $G_{\partial E}$. For $g \in G_{\partial E}$ let $w \in L$ be a word with value $hg$. If $v$ is the largest terminal segment of $w$ which lies in $(A_{\partial E})^*$, then $w$ decomposes as $ut_Ev$ with $ut_E = h\overline{f}$ and $\overline{f} = f^{-1}g$ with $f \in A_E$. Then $fv \in fN_{hT} \subset L_h$ and $\overline{f}v = g$.

We next show $L_h$ has the asynchronous fellow traveller property. If not, we could find $f_i v_i \in f_i N_{hT} \subset L_h, f'_i v'_i \in f'_i N_{hT} \subset L_h$, for $i = 1, 2, \ldots$, so that $d(f_i v_i, f'_i v'_i)$ is bounded, but for any $K$, there is some $i$ so that $f_i v_i$ and $f'_i v'_i$ do not asynchronously $K$-fellow travel. We would then have $u_1 v_1 w_1, u'_1 v'_1 w'_1, u_2 v_2 w_2, u'_2 v'_2 w'_2, \ldots \in L$ with $h f_i = u_i, h f'_i = u'_i$ for each $i$. We can replace each $w_i$ and $w'_i$ with $x_i$ and $x'_i$ of bounded length. Then for each $i$, $u_i v_i x_i, u'_i v'_i x'_i \in L, d(u_i v_i x_i, u'_i v'_i x'_i) = d(h f_i v_i x_i, h f'_i v'_i x'_i) = d(f_i v_i x_i, f'_i v'_i x'_i)$ is bounded, and yet there is no $K$ so that each of the pairs $u_i v_i x_i$ and $u'_i v'_i x'_i$ asynchronously $K$-fellow travel. This contradicts the assumption that $L$ is an asynchronous automatic structure.

The same argument shows $L_h$ is synchronous if $L$ is. If $L \sim L'$ we can apply the argument to $L \cup L'$ to see that, for a fixed $h$, $[L_h]$ depends only on $[L]$. Finally, the equivariance property is immediate from the definition of $L_h$.

**Proof of Theorem 3.3 continued.** It is clear that $\psi \mapsto [L_\psi]$ maps $\{\text{regular deployments}\} \to \mathfrak{A}(G)$. We shall show below that $[L] \mapsto \psi_{[L]}$ maps $\mathfrak{A}(G) \to \{\text{regular deployments}\}$. This is easy to see that these maps are mutual inverses.

For suppose we start with $[L] \in \mathfrak{A}(G)$. For $h \in G_E$ we take $L_h = N_h \cup L_{h \sim L_h} \cap L_{h'}$. Since this is a finite union of equivalent asynchronous automatic structures on $G_{\partial E}$, it is itself an asynchronous automatic structure. We use the languages $L_h \in \psi_{[L]}(h)$ to define $L_{[L_h]}$. Then $L_{[L_h]}$ contains the language $L$ so certainly $[L] = [L_{[L_h]}]$.

Similarly, if $\psi$ is a regular deployment then $\psi_{[L_\psi]}(h) = [(L_\psi)_h]$ and $(L_\psi)_h$ contains the language $eL_{[L_\psi]}(h)$. Thus $\psi_{[L_\psi]}(h) = [L_{[L_\psi]}(h)$ for each $h$. That is, $\psi_{[L_\psi]}(h) = \psi(h)$ for all $h$, so $\psi_{[L_\psi]} = \psi$.

So, to complete the proof of 3.3, we need only prove that if $L$ is an asynchronous automatic structure on $G$ then the deployment $\psi_{[L]}$ is regular. Let $\psi = \psi_{[L]}$. We shall describe a nondeterministic finite state automaton $\mathcal{T}$ for $L_\psi$, thus showing $L_\psi$ is a regular language, so $\psi$ is a regular deployment.

We assume $L$ is as in Lemma 3.5. The following is our key lemma.

**Lemma 3.7.** Given $K > 0$, there exists a finite state automaton $S = S_K$ with the following properties:

1. It accepts any word $w \in A^*$;
2. Suppose $w$ is a word with value $h$ which asynchronously $K$-fellow travels a word of $L$ with the same value. Then the final state reached by $w$ in the machine $S$ tells one for each edge $E$ of $\hat{Y}$ whether $h \in G_E$, and if so, what the corresponding language $L_h$ is. In particular, if $L'$ is the prefix closure of $L$, then the language $\{w \in L' : L_\pi = L_h\}$ is regular for any $h \in \bigsqcup G_E$.

**Proof.** Recall that the language $L_h$ is determined by $E$ and the map $f \mapsto S_hT$ of $A_E$ to the power set of the set of states of $\mathcal{A}$. Moreover, at any point along a path $w$, $S_h$ is the set of $\mathcal{A}$ states reached by paths in $\mathcal{A}$ having the same value $h$ as our path’s current value.
and decomposing as $u t_E$. Thus it behooves us to modify our machine $A$ to make “visible”, the invisible $t_E$’s when $E \in T$. So suppose we take the alphabet $A \cup \{r_E : E \in T\}$, and let $s_E = t_E$ if $E \notin T$, $s_E = r_E$ if $E \in T$. Then the reader can check that the language

$$\{u_0 s_{E_1} \ldots s_{E_m} u_m : w = u_0 t_{E_1} \ldots t_{E_m} u_m \text{ is an edge path decomposition of } w \in L\}$$

is regular. (Here is a sketch proof. Add a loop with label $r_E$ to every vertex of $A$ for every $E \in T$. This machine accepts the language obtained from $L$ by adding arbitrary subwords in these $r_E$’s. The desired language is obtained from this by deleting any word that has one of a certain finite collection of prohibited subwords; it is hence regular.)

We take a deterministic machine for this language and replace each $r_E$ edge with a $t_E$ edge which we take as an $\epsilon$ transition. This is a machine for $L$ and we assume $A$ is of this form.

Now suppose $w$ $K$-fellow travels some path of $L$ with the same value. Then any word of $L$ with the same value $(K + K_L)$-fellow travels $w$, where $K_L$ is the fellow traveller constant for $L$. It thus suffices to keep track at each step along $w$ of what $A$ states have been reached by paths which $(K + K_L)$-fellow travelled ours and have final value in a $K$-neighborhood of our current value, and, when one of these has the same value as $w$, whether its final edge in $A$ is a $t_E$ edge. That is, the information we must keep track of is an element of $\text{Maps}(B, \mathcal{P}(S \times \mathcal{E}(A)))$, where $B$ is a ball of radius $K + K_L$ in the Cayley graph, $S$ and $E(A)$ are the sets of states and edges of $A$ respectively, and $\mathcal{P}(\cdot)$ denotes the power set.

More precisely, to keep track of the desired information we use a finite state automaton $S$ with $\text{Maps}(B, \mathcal{P}(S \times \mathcal{E}(A)))$ as set of states. For $\alpha, \beta \in \text{Maps}(B, S \times \mathcal{P}(\mathcal{E}(A)))$ and $a$ in our alphabet $A$, $S$ has an edge labelled $a$ from $\alpha$ to $\beta$ if and only if each $\beta(g)$ consists of the set of pairs $(s, e)$ such that there exists a path in $B \cup aB$ from a point $g_1 \in B$ to $ag$ labelling a path in $A$ from a state $s'$ with $(s', e')$ in $\alpha(g_1)$ to $s$ with final edge $e$. As start state we take the element $\sigma$ with $\sigma(g)$ equal to the set of pairs $(s, e)$ so that $s$ reachable in $A$ from the start state by paths with value $g$ and final edge $e$. Any word $w \in A^*$ then defines a path in $S$ from the start state. If $w$ asynchronously $K$-fellow travels an element of $L$ with the same value then this path ends in a state $\alpha$ with $p(\alpha(f)) = S_{w, f}$ for each $f \in A_E$ where $p$ denotes projection onto the first factor. This state $\alpha$ thus gives the desired information.

We now return to the proof that $L_\psi$ is regular for $\psi = \psi_L$. We choose regular languages $L_{\psi(h)}$ for each $\psi(h)$ to define the language $L_\psi$. By proving that this $L_\psi$ is regular, we will also have proved the remark preceding Theorem 3.3.

We have already shown that $L$ asynchronously fellow travels one choice of $L_\psi$ (namely, the one with $L_{\psi(h)} = \hat{L}_h$). Hence, by Lemma 3.2 it asynchronously fellow travels any choice of $L_\psi$. Let $K$ be the fellow traveller constant for our particular choice. Let $S$ be the machine of the above lemma.

Let $A'$ be the disjoint union of machines for the languages $L_{\psi(h)}$. We shall construct a nondeterministic machine for the language $L_\psi$ by adding some arrows to the product machine of $A'$ and $S$. Namely, for each state $s$ of this product machine and each edge $E$ of $\mathcal{Y}$ we will add an arrow labelled $t_E$ from $s$ to the following state $t$, if it exists. The
\( S \) component of \( t \) is the one determined by the \( t_E \)-transition from the \( S \)-component of \( s \).

The \( A' \) component is the start state of the machine for \( L_{ht_E} \), where \( \psi(h t_E) = [L_{ht_E}] \) is determined by \( E \) and the \( S \) component of \( s \) as in the above lemma. It is easy to see that this machine performs as advertised.

There is an entirely analogous version of Theorem 3.3 for synchronous automatic structures.

**Theorem 3.8.** The bijection of Theorem 3.3 restricts to a bijection between \( \mathbb{SA}(G) \) and the set of regular deployments whose images lie in \( \coprod_{V \in \text{vert} \mathcal{Y}} \mathbb{SA}(G_V) \).

**Proof.** Given a synchronously automatic structure \( L \) on \( G \), Lemma 3.6 ensures that the deployment \( \psi_L \) takes its image in \( \coprod_{V \in \text{vert} \mathcal{Y}} \mathbb{SA}(G_V) \). It remains to check that if \( \psi \) is a regular deployment whose image lies in \( \coprod_{V \in \text{vert} \mathcal{Y}} \mathbb{SA}(G_V) \) then we can find a synchronously automatic structure \( L \sim L_\psi \).

So suppose \( \psi \) is such a deployment. We choose synchronous automatic structures with uniqueness \( L_{\psi(h)}(h) \) for each class in \( \text{Im} \psi \). For each such language and each edge \( E' \), if the language occurs as \( L_{\psi(h)} \) with \( h \in G_E \) and \( \partial_1 E = \partial_0 E' \), we choose a set of \( F_{E'} \)-coset representatives in the language \( L_{\psi(h)} \). To do this we order our alphabet \( A \). This induces a total order \( \prec \) on \( A^* \) by ordering first on length and then by lexicographic order for words of a given length. Let

\[
L_{\psi(h),E'} = \{ w \in L_{\psi(h)} : w \text{ is } \prec \text{-minimal among } w \text{ with } \overline{w} \in \overline{w} F_{E'} \}.
\]

By [BGSS], each of these languages is regular. We take

\[
L = \{ u_0 t_{E_1} \ldots t_{E_m} u_m \mid \overline{u_0} \in G_{V_0}, \overline{u_i} \in G_{\partial_1 E_i} \text{ for } i > 0, \overline{u_i} \in L'_{\psi(u_0 t_{E_1} \ldots t_{E_i})}, E_{i+1} \text{ for } i < m, u_m \in L_{\psi(u_1 \ldots t_{E_m})} \}.
\]

Notice that by the normal form for graphs of groups, \( L \) bijects to \( G \). Further, \( L \subset L_\psi \), hence \( L \sim L_\psi \). To see that \( L \) is regular, one builds a product machine based on \( S \) and \( A' \), where here \( A' \) is the disjoint union of machines for the languages \( L'_{\psi(h),E'} \), and modifies this to accept \( L_{\psi(h)} \) in the final factor. Finally we wish to see that \( L \) has the synchronous fellow traveller property. We repeat the argument that \( L_\psi \) has the asynchronous fellow traveller property, but with the following observation. Suppose \( w = u_0 t_{E_1} \ldots t_{E_m} u_m \) and \( w' = u'_0 t_{E'_1} \ldots t_{E'_{m'}} u'_{m'} \), with \( w, w' \in L \) and \( \overline{w} = \overline{w'} \). Let \( p = E_1 \ldots E_m, p' = E'_{1} \ldots E'_{m'} \). If \( p = p' \), then by our choice of coset representatives, \( u_0 t_{E_0} \ldots t_{E_m} = u'_0 t_{E'_0} \ldots t_{E'_{m'}} \), and since \( u_m \) and \( u_{m'} \) synchronously fellow travel, so, too, do \( w \) and \( w' \). On the other hand, if we cannot choose the decompositions so that \( p = p' \), then (say) \( p' \) is an initial segment of \( p \) and we in fact have \( w' = wa \).

**Theorem 3.9.** Let \( L \) be an asynchronous automatic structure on \( G = \pi_1(\mathcal{Y}) \). Then each subgroup \( H \subset G \) conjugate to a vertex group is \( L \)-rational, and hence has an induced automatic structure \( L_H \), determined up to equivalence by \( [L] \). Let \( \mathcal{H} \) be the set of all conjugates of vertex groups. Then the map \([L] \mapsto ([L_H])_{H \in \mathcal{H}} \) defines maps

\[
\mathbb{A}(G) \rightarrow \coprod_{H \in \mathcal{H}} \mathbb{A}(H), \quad \mathbb{SA}(G) \rightarrow \coprod_{H \in \mathcal{H}} \mathbb{SA}(H),
\]
which are injective and have dense image in the product topology.

Proof. We replace \( \mathcal{G} \) by a reduced graph of groups (this is defined just before Lemma 3.1). This does not affect any vertex with non-trivial \( \mathfrak{A}(G_V) \), and hence does not change the validity of the theorem. Let \( L \subset A^* \) be an asynchronous automatic structure on \( G \) chosen according to Lemma 3.5. Let \( H = hG_Vh^{-1} \) with \( h \in \mathcal{G}_E \) be as in Lemma 3.1 and denote \( V = \partial_1 E \). By considering the edge path decomposition one sees that any \( w \in L \) with \( \pi \in H - h(\partial_1 F_E)h^{-1} \) has the form \( w = xyz \) with \( \pi, (\pi)^{-1} \in h\partial_1 F_E \) and \( y \in (A_V)^* \). Since there are just finitely many possibilities for \( x \) and \( z \), the set of \( L \)-words of this form is a regular sub-language of \( L \). Thus \( H - h(\partial_1 F_E)h^{-1} \) is rational, whence \( H \) is, since \( h(F_E)h^{-1} \) is finite.

We also see that \( \{ w \in L : \pi \in H \} \sim hLh^{-1} \). Thus \( [L_H] = h\psi_L(h)h^{-1} \). In particular, the map \( H \mapsto [L_H] \) determines \( \psi_L \) and hence \( [L] \), so the map \( \mathfrak{A}(G) \to \prod_{H \in \mathcal{H}} \mathfrak{A}(H) \) is injective. The statement that this map has dense image is the statement that, if we specify a structure \( L_H \) for finitely many \( H \), there is a structure \( L \) on \( G \) which realizes these \( L_H \)'s. We defer the proof of this to the next section.

The statement of the theorem in the synchronous case follows from the above together with Theorem 3.8.

Remark. We can turn \( L_h \) itself into an automatic structure on \( H \) if we use the evaluation map \( A_V \rightarrow H \) given by \( a \mapsto h\pi h^{-1} \). With this interpretation, \( [L_h] = [L_H] \).

4. \( \mathcal{G} \)-graphs

Regular deployments are unsatisfactory as classifying objects. This is for several reasons. First is the fact that it is hard to specify an arbitrary deployment and there is no convenient way to tell \( a \) priori whether a deployment is in fact regular. This fact is reflected strongly in the second, namely, that regularity is non-local in following sense. If one changes the value of a regular deployment on one element \( h \in \prod \mathcal{G}_E \) one is likely to obtain a non-regular deployment.

In this section we introduce a classifying object which avoids these deficiencies.

Definition. Given a finite graph of groups \( \mathcal{G} \) with finite edge groups, a \( \mathcal{G} \)-graph \( \mathcal{X} \) is a finite directed labelled graph \( X \) with the following additional structure:

- A map \( \pi : X \rightarrow Y \) of underlying graphs is given. A vertex \( v \) of \( \mathcal{X} \) with \( \pi(v) = V \) is called a \( V \)-vertex and an edge \( e \) of \( \mathcal{X} \) with \( \pi(e) = E \) is called an \( E \)-edge. This is called the \( \mathcal{G} \)-type of \( v \) or \( e \).
- A vertex \( v_0 \) of \( \mathcal{X} \) is chosen as start vertex and every vertex of \( \mathcal{X} \) can be reached by a directed path from this start vertex. (We may assume that \( v_0 \) is a \( V_0 \)-vertex, where \( V_0 \) is the base vertex for \( \mathcal{G} \) chosen in Section 3.)
- Each \( V \)-vertex \( v \) is labelled by an element \([L_v] \in \mathfrak{A}(G_V)\).
- Each edge \( e \) out of \( v \) is labelled by an \([L_e] \)-rational subset \( S_e \) of \( G_V \). For each edge \( E \) of \( \mathcal{G} \) out of \( V \), the labels on the \( E \)-edges out of \( v \) are disjoint. Their union is \( G_V \) if \( v = v_0 \) or if \( v \) has an incoming edge of \( \mathcal{G} \)-type other than \( E^{-1} \). If \( v \neq v_0 \) and all incoming edges at \( v \) are \( E^{-1} \)-edges, their union is \( G_V - \partial_0 F_E \).
- For each \( V \)-vertex \( v \) and each edge \( E \) out of \( V \), there is a \( F_E \)-action on \( \mathcal{X} \) which fixes all vertices except those reached by one \( E \)-edge from \( v \). This action respects
labels in the following sense. For a vertex \( v' \) reached by an \( E \)-edge from \( v \) we have 
\[ L_f(v') = f(L_v) \]. For an edge \( e \) departing \( v \) with label \( S_e \subset G_V \) the edge \( fe \) has label 
\[ S_{fe} = S_e f^{-1} \]. For an edge \( e \) departing a vertex reached by an edge from \( v \) we have 
\[ S_{fe} = f S_e \].

Given a \( \mathcal{Y} \)-graph, \( X' \), the following choices determine a language \( L_X \) for \( G = \pi_1(\mathcal{Y}) \).
Choose a convenient generating set \( A \) for \( G \). For each vertex \( v \) of \( X' \), choose an asynchronous automatic structure \( L_V \subset (A_{\pi(v)})^* \) in the class associated to \( v \). For an edge \( e \) departing \( v \) let \( L_e \) be the sublanguage of words of \( L_V \) that represent elements of the set \( S_e \). Let \( T \) be a maximal spanning tree in \( \mathcal{Y} \). For each edge \( E \) of \( \mathcal{Y} \) let \( t_E \) be as in Section 3, that is, it is the corresponding stable letter if \( E \notin T \) and the empty word if \( E \in T \). Then \( L_X \) is

\[
L_X = \{ u_0 t_{\pi(e_1)} \ldots t_{\pi(e_m)} u_m : e_1 \ldots e_m \text{ is a path in } X \text{ from the start vertex,} \\
u_k \in L_{e_{k+1}} \text{ for } k = 0, \ldots, k - 1, \\
\quad u_m \in L_{\partial_1 e_m}, \\
\quad u_{k+1} \notin \partial_0(F_{\pi(e_{k+1})}) \text{ if } \pi(e_{k+1}) = \pi(e_k)^{-1} \}
\]

**Theorem 4.1.** The above language \( L_X \) is an asynchronous automatic structure on \( \pi_1(\mathcal{Y}) \) and depends, up to equivalence, only on the \( \mathcal{Y} \)-graph \( X' \).

Every asynchronous automatic structure on \( \pi_1(\mathcal{Y}) \) is equivalent to one constructed as above.

**Proof.** The proof that \( L_X \) has the asynchronous fellow traveller property and is determined up to equivalence by \( X' \) is just like the proof of the analogous statement for a language determined by a deployment (Lemma 3.2), and is left to the reader. (In fact, it is not hard to see that \( L_X \) is contained in a language determined by the following deployment \( \psi \). Given \( h \in G_E \), we find a path \( p \) in \( X \) from the start vertex of \( X \) whose final edge \( e \) has \( \pi(e) = E \), and the language determined by \( p \) contains a word with value \( h \). We take \( \psi(h) = [L_{\partial_1 e}] \).

We must check that \( L_X \) is regular. To do this it is helpful to modify \( L_X \) by redefining \( t_E \) for each \( E \in T \) temporarily to be a new letter which evaluates to \( 1 \in G \), rather than the empty word. We first turn \( X \) into a generalized finite state automaton \( A_X \). We do this by subdividing each edge \( e \) of \( X \) into two edges. We label the first of these by \( L_e \) and the second by \( t_{\pi(e)} \). The start state of \( A_X \) is the start vertex of \( X \). We take all vertices of \( A_X \) to be accept states. The language of this machine contains \( L_X \). In fact, \( L_X \) is exactly the sublanguage of words containing no substring of the form \( t_E u t_E^{-1} \) with \( u \in L_V \), \( \pi(v) = \partial_1 E \), \( \pi \in \partial_1 F_E \). Since there are finitely many such strings, \( L_X \) is regular as required. If we now replace each letter \( t_E \) with \( E \in T \) by the empty word we get our original \( L_X \) back, and it is still regular.

We must check that every asynchronous automatic structure arises as above. Suppose \( L \) is an asynchronous automatic structure on \( G \). We assume that our language \( L \) and alphabet \( A \) are as in Lemma 3.5. We shall construct a \( \mathcal{Y} \)-graph \( X \) for \( L \) of a rather special type. The start vertex will have no incoming edges and each vertex other than the start vertex will have incoming edges all of one \( \mathcal{Y} \)-type.
Let \( \hat{\mathcal{Y}} \) be \( \mathcal{Y} \) extended by a base edge as in Section 3, and for \( E \) an edge of \( \hat{\mathcal{Y}} \), let \( G_E \) be as defined in Section 3. We refer to the edge path decomposition of elements of \( L \) described before Lemma 3.6. For \( h \in G_E \) we define

\[
N_{E,h} = \{ v \in A^* : \exists w \in L \text{ with edge path decomposition } w = u_0 t_{E_1} \ldots t_{E_m} u_m \\
such that for some } i, E_i = E, u_0 t_{E_1} \ldots u_{i-1} t_{E_i} = h \\
v = u_i t_{E_{i+1}} \ldots t_{E_m} u_m \},
\]

and

\[
L_{E,h} = \bigcup_{\tilde{f} \in \partial_1 f_E} f N_{E,h\tilde{f}}.
\]

We claim that for each \( h \in G_E \) the language \( N_{E,h} \) is regular. Let \( \mathcal{A} \) be a machine for \( L \). As in the proof of Lemma 3.6, we let \( S_h \) be the set of states of \( \mathcal{A} \) reached by words in the prefix closure of \( L \) which evaluate to \( h \) and have an edge path decomposition ending in \( t_E \). Then \( N_{E,h} \) is the language accepted by the machine obtained from \( \mathcal{A} \) by making \( S_h \) the set of start states. Thus \( N_{E,h} \) is regular. It is also determined by the finite set of states \( S_h \), so there are finitely many different languages \( N_{E,h} \). Thus there are also finitely many languages \( L_{E,h} \), and they are regular. The language \( L_{E,h} \) is determined by the map \( f \mapsto S_{h\tilde{f}} \) of \( \mathcal{A} \) to the power set of the set of states of \( \mathcal{A} \).

The subset of \( G \) onto which the language \( L_{E,h} \) evaluates depends only on \( E \): it is the set of elements of \( G \) whose normal form decomposition with base vertex \( \partial_1 E \) cannot start with \( t_{E-1} \). Note that, except for a finite number of elements of \( G \), if an element is distance 1 from this set then it is also in this set. It therefore makes sense to talk about the asynchronous fellow traveller property for \( L_{E,h} \), even though this language does not surject to \( G \). We claim that \( L_{E,h} \) has this property. For suppose \( f v \in f N_{E,h\tilde{f}} \) and \( f' v' \in f' N_{E,h\tilde{f}'} \) with \( \tilde{f}, \tilde{f}' \in \partial_1 F_E \) and \( d(\tilde{f} v, \tilde{f}' v') \leq 1 \). Then there exist \( u, u' \) with \( \overline{u} = h\tilde{f} \) and \( \overline{u'} = h\tilde{f}' \) so that \( u w \in L \) and \( u' v' \in L \). Since \( w, u' v' \in L \) and \( d(\overline{u w}, \overline{u' v'}) = d(\tilde{f} v, \tilde{f}' v') \leq 1 \), \( w \) and \( u' v' \) asynchronously fellow-travel. It follows that after reparameterization, \( f v \) and \( f' v' \) also asynchronously fellow-travel.

For fixed \( E \) and \( h, h' \in G_E \) it therefore also makes sense to ask if \( L_{E,h} \sim L_{E,h'} \). We define

\[
\hat{L}_{E,h} = \bigcup_{L_{E,h}, L_{E,h'} \sim L_{E,h}} L_{E,h'}.
\]

Since this is a finite union of regular equivalent languages, it is also regular with the asynchronous fellow-traveller property.

Each \( \hat{L}_{E,h} \) induces an asynchronous automatic structure \( \hat{L}_{E,h} \) on \( G_{\partial_1 E} \). Namely, we define

\[
\hat{L}_{E,h} = \{ u \in A^* : \exists v \in A^*, u v \in \hat{L}_{E,h}, \overline{u v} \in G_{\partial_1 E} \}.
\]

It is easy to see that this language is an asynchronous automatic structure on \( G_{\partial_1 E} \). In fact, it is just \( \bigcup_{L_{E,h}, L_{E,h'} \sim L_{E,h}} L_{E,h'} \), where \( L_{E,h'} \) is the \( L_{h'} \) of Lemma 3.6 (we are now making the edge \( E \) explicit in our notation).
We are now prepared to describe the $\mathcal{Y}$-graph, $\mathcal{X}$ of the Theorem. For each $E \in \text{edge } \hat{\mathcal{Y}}$ it has a vertex for each $\hat{\mathcal{E}}^{E,h}$. The vertex corresponding to $\hat{\mathcal{E}}^{E,h}$ projects to $\partial_1 E$ under $\pi$, and is labelled by $[\hat{\mathcal{L}}_{E,h}]$. We take the vertex corresponding to $\hat{\mathcal{L}}^{E_0,1}$ to be the start vertex. Suppose that $E$ and $E'$ are edges of $\mathcal{Y}$ with $\partial_1 E = \partial_0 E'$. There is an edge $E'$ from the vertex for $\hat{\mathcal{L}}^{E,h}$ to the vertex for $\hat{\mathcal{L}}^{E',h'}$ if the set

$$S_{E'} = \{ g \in G_{\partial_1 E} : \hat{\mathcal{L}}^{E',h'} = \hat{\mathcal{L}}^{E,h}g \}$$

is not empty. In this case $E'$ is labelled by $S_{E'}$ and projects to $E'$ under $\pi$.

We must check that this defines a $\mathcal{Y}$-graph. That is, we must see that the set $S_{E'}$ is well defined, that it is $\hat{\mathcal{E}}^{E,h}$-rational, that the labels on the $E'$ edges out of the vertex for $\hat{\mathcal{L}}^{E,h}$ partition $G_{\partial_1 E}$ if $E' \neq E^{-1}$ and partition $G_{\partial_1 E} - \partial_1 F_E$ if $E' = E^{-1}$, and that $\mathcal{X}$ has the appropriate equivariance properties.

So suppose $g \in G_{\partial_1 E}$. We check that $\hat{\mathcal{L}}^{E',h}$ depends only on $\hat{\mathcal{L}}^{E,h}$ and $g$, and does not depend on $h$. We first show that $\hat{\mathcal{L}}^{E,h}$ depends only on $\hat{\mathcal{L}}^{E,h}$ and $g$. Now $w \in L^{E',hg} = \bigcup_{f \in A_{E'}} fNE',hg\bar{f}$ if and only if $w = fv$ with $f \in A_{E'}$ and there is $u$ with $\pi = hgf\bar{f}$ and $uv \in L$, and $u$ ending with $i_{E'}$ in some edge path decomposition of $uv$. If there is such a $u$, it has the form $u = xy$ with $\pi = hgf\bar{f}$, $y = \bar{f'}g\bar{f}$, where $\bar{f'}$ in $\partial_1 F_E$. Thus $w = fv \in L^{E',hg}$ if and only if we find $f'yv \in L^{E,h}$ with $f'y = g\bar{f}$. (The reader might want to draw a picture.) Thus $\hat{\mathcal{L}}^{E',h}$ depends only on $L^{E,h}$ and $g$, and not on $h$. A similar argument shows that $L^{E',hg} \sim L^{E',h'g}$ if $L^{E,h} \sim L^{E,h'}$. So by taking the appropriate unions we see that $\hat{\mathcal{L}}^{E',h}$ depends only on $\hat{\mathcal{L}}^{E,h}$ and $g$, and thus $S_{E'}$ is well defined.

The fact that $S_{E'}$ is $\hat{\mathcal{E}}^{E,h}$-rational will follow from the existence of the machine $S$ of Lemma 3.7. Recall that that machine does the following: if $w$ is a word with value $h \in G_E$ which fellow travels some $L$-word with the same value then the state of $S$ reached by $w$ determines the language $L_E,h$ (called $L_h$ in Lemma 3.7). The way it does this is by determining the map $f \mapsto S_{h\bar{f}}$ of $A_E$ to the power set of the set of states of $A$. But, as we saw above, this map also determines $L^{E,h}$, and hence $\hat{\mathcal{L}}^{E,h}$. Thus $L^{E,h}$ can be replaced by $\hat{\mathcal{L}}^{E,h}$ in Lemma 3.7.

Now let $u$ be an $L$-word for $h$ and $v$ be a $\hat{\mathcal{L}}^{E,h}$-word for $g$. Then, by construction of $\hat{\mathcal{L}}^{E,h}$, the word $w = uv$ fellow travels an $L$-word for $hg$. We can thus test $w$ with the machine $S$ to see if $\hat{\mathcal{L}}^{E',hg} = \hat{\mathcal{L}}^{E',h'}$. That is, $g \in S_{E'}$ if and only if the word $v$ for $g$ labels a path in $S$ from the state reached by $u$ to a state that determines the language $\hat{\mathcal{L}}^{E',h'}$. This is a regular condition, so $S_{E'}$ is $\hat{\mathcal{E}}^{E,h}$-rational, as required.

The labels on the $E'$-edges out of the vertex for $\hat{\mathcal{L}}^{E,h}$ are disjoint, since $\hat{\mathcal{L}}^{E,h}$ and $g$ determine $\hat{\mathcal{L}}^{E,h}g$, and their union is clearly $G_{\partial_1 E} - \partial_1 F_E$ if $E' = E^{-1}$ and $G_{\partial_1 E}$ otherwise.

We must construct the $F_{E'}$ action and show that it respects labels. To this end, let $E''$ be an edge of $Y$ with $\partial_1 E' = \partial_0 E''$. We suppose that there is an edge $E''$ from the vertex for $\hat{\mathcal{L}}^{E',h'}$ to the vertex for $\hat{\mathcal{L}}^{E'',h''}$. If $\bar{f} \in F_{E'}$, we let $f$ carry the vertex for $\hat{\mathcal{L}}^{E',h'} = \hat{\mathcal{L}}^{E,h}g\bar{f}$ to the vertex $\hat{\mathcal{L}}^{E',hg\bar{f}}$. As we have seen, this is well defined. The label at this vertex is $[L_{E',h\bar{f}}]$, which is $[fL_{E',h}]$ as required. It now follows that the action of $\bar{f}$ on vertices induces an action on edges out of the vertex for $\hat{\mathcal{L}}^{E,h}$, and this action respects edge labels,
for if \( g \in S_{E'} \) then \( g\overline{f^{-1}} \) labels an edge from the vertex for \( \widehat{L}^{E,h} \) to the \( \overline{f} \) image of the vertex for \( \widehat{L}^{E',h'} \). In particular, we have \( S_{\overline{f}E'} = S_{E'\overline{f^{-1}}} \) as required. In the same way, the \( \overline{f} \) action on vertices induces an action on edges whose initial vertex is moved by \( \overline{f} \) carrying (say) \( E''_i \) to (say) \( \overline{f}E''_i \) so that \( S_{\overline{f}E''_i} = \overline{f}S_{E''_i} \). This completes the proof that \( X \) is indeed a \( Y \)-graph.

Finally, we must check that \( L \) is equivalent to \( L_X \). It is an easy induction on free product length that \( L \) and \( L_X \) determine the same deployment. Alternatively, one may note that \( L_X \) is equivalent to \( L \) since it contains \( L \) as a sublanguage. □

Remark. Call a \( Y \)-graph \( X \) special if it satisfies:

- the start vertex \( v_0 \) has no incoming edges;
- each vertex \( v \neq v_0 \) has incoming edges of just one \( Y \)-type.

Then the \( Y \)-graph \( X \) for \( L \) constructed in the above proof is special, and it is not hard to verify that it is minimal with this property, in the sense that any other special \( Y \)-graph \( X' \) defining a language equivalent to \( L \) can be mapped to \( X \) by a graph mapping that respects the \( Y \)-type of vertices and edges, respects vertex labels, and also respects edge labels in the sense that the rational set associated to an edge of \( X' \) is contained in the rational set associated to corresponding edge of \( X \). Thus minimal special \( Y \)-graphs actually classify asynchronous automatic structures on \( G = \pi_1(Y) \). They are, however, not always efficient classifying objects, in that one can often find a much smaller non-special \( Y \)-graph to describe the same structure, as we will now describe.

Let \( A \) be a convenient alphabet for \( G = \pi_1(Y) \). Suppose \( X \) is a \( Y \)-graph and let \( L_X \subset A^* \) be, as in Theorem 4.1, the language of words labelling paths in \( X \) from the start vertex. For any vertex \( v \) of \( X \) we can define similarly the language \( L_v \) of words in \( A^* \) that label paths starting at \( v \). (If \( X \) is as constructed in the above proof and \( v \) is the vertex corresponding to \( \widehat{L}^{E,h} \) then \( L_v \sim \widehat{L}^{E,h} \).) Now suppose that for some vertex \( V \) of \( Y \) we have \( V \)-vertices \( v \) and \( v' \) of \( X \) such that \( L_v \cup L_{v'} \) has the asynchronous fellow-traveller property. We can then attempt to create a smaller \( Y \)-graph by identifying the vertices \( v \) and \( v' \) of \( X \). If \( E \) is an incoming edge at \( V \) then we have a \( F_E \)-action on \( X \) which permutes the \( V \)-vertices, so we must do this identification equivariantly. There is no guarantee that we can do this, for if \( L_v \cup L_{v'} \) and \( L_{v'} \cup L_{v''} \) have the asynchronous fellow-traveller property, we cannot deduce that \( L_v \cup L_{v''} \) does. However, if we can do this identification equivariantly, we obtain a smaller \( Y \)-graph for [\( L \)].

Even if one can collapse \( X \) as above, there may be several inequivalent ways of doing so. Indeed, it is not hard to find an example of a graph of groups \( Y \) with finite edge groups for which the \( Y \)-graph \( X \) constructed in the proof of Theorem 4.1 can be collapsed to several inequivalent “minimal” \( Y \)-graphs.

There are some situations in which the above collapse is clearly possible. For example, if one has a \( Y \)-graph \( X \) with no “special” vertices — that is, every vertex \( v \) has more than one \( Y \)-type of incoming edge — then every \( L_v \) surjects to \( G \) so the condition that \( L_v \cup L_{v'} \) have the asynchronous fellow-traveller property defines an equivalence relation on the vertices of \( X \). It is then easy to see that there is a \( Y \)-graph that can be obtained by collapsing \( X \) as above.

Another case is when the language \( L \) is asynchronously biautomatic. Each \( L_v \) is
equivalent to a sublanguage of a translate of \( L \). But, by definition of biautomaticity, any translate of \( L \) is equivalent to \( L \). We can thus collapse all \( V \)-vertices to a single vertex for each \( V \). This collapse is clearly equivariant and extends trivially to edges, so we see that there is a \( \mathcal{Y} \)-graph \( \mathcal{X} \) whose underlying graph \( X \) is isomorphic to the underlying graph \( Y \) of \( \mathcal{Y} \). This proves part of:

**Theorem 4.2.** \( L \) is an asynchronously biautomatic structure on \( \pi_1(\mathcal{Y}) \) if and only if \( L \) has a \( \mathcal{Y} \)-graph \( \mathcal{X} \) for which \( \pi \) is an isomorphism of the underlying graphs of \( \mathcal{X} \) and \( \mathcal{Y} \) and the structure at each vertex \( v \) of \( \mathcal{X} \) is an asynchronously biautomatic structure for \( G_{\pi(v)} \).

The corresponding statement holds also with “asynchronously biautomatic” replaced by “biautomatic”.

**Proof.** We first point out that a asynchronous or synchronous biautomatic structure on \( G \) induces asynchronous or synchronous biautomatic structures on the vertex groups, since they are rational subgroups. Thus the labels on the vertices of the above \( \mathcal{Y} \)-graph \( \mathcal{X} \) are as claimed.

Now suppose we have a \( \mathcal{Y} \)-graph \( \mathcal{X} \) as in the theorem. We check that the ensuing structure \( L \) is asynchronously biautomatic. So suppose \( A \) is a convenient alphabet we have chosen languages at each vertex, and suppose also that \( w = u_0 t_{E_1} \ldots t_{E_m} u_m \) is an edge path decomposition of a word in the resulting language. We must show that if \( a \in A \), the word for \( a\overline{w} \) asynchronously fellow travels \( w \). There are several cases. We use the same symbol for a vertex of \( \mathcal{Y} \) and the corresponding vertex of \( \mathcal{X} \).

We suppose first that \( u_0 \in (A_{V_0})^* \) is not empty and \( \overline{a} \in G_{V_0} \). We take \( u_0' \) to be the word in \( L_{V_0} \) for \( \overline{au} \). Then \( w' = u_0' t_{E_1} \ldots t_{E_m} u_m \) is an accepted word for \( \overline{aw} \). Since \( L_{V_0} \) is asynchronously biautomatic, \( w' \) and \( \overline{aw} \) asynchronously fellow travel as required. If \( L_{V_0} \) is biautomatic, they synchronously fellow travel.

We now suppose that \( u_0 \) is non-empty and that \( \overline{a} \notin G_{V_0} \). Then we have \( w' = au_0 t_{E_1} \ldots t_{E_m} u_m \) an accepted word and again \( \overline{aw} \) and \( w' \) appropriately fellow travel, unless it happens that \( a = t_{E_i}, \overline{u_0} \in \partial_1 F_{E_i} \), and the first letter after \( u_0 \) in \( w \) is \( t_{E_i} = t_{E_{i-1}} \). In this case \( w' = f'u_{i+1} \ldots u_m \), for suitable \( f' \in A_{E_{i-1}} \), is the word we seek, and again \( \overline{aw} \) and \( w' \) synchronously or asynchronously fellow travel as required.

The remaining cases are similar and are left to the reader.

**Proof of Theorem 3.9 (completed).** We need to show the map of Theorem 3.9 has dense image. Suppose that every vertex group \( G_V \) has an asynchronous automatic structure. We will first show that \( G = \pi_1(\mathcal{Y}) \) has at least one asynchronous automatic structure (this follows from the methods of [S2], but we give a proof here for completeness). We shall need the following lemma.

**Lemma 4.3.** Suppose that \( E \) is an edge of \( \mathcal{Y} \), with \( \partial_0 E = V \), and suppose we are given \([L_V] \in \mathfrak{A}(G_V)\). Then there is a partition of \( G_V \) into distinct \([L_V]\)-rational sets \( S_F, F \in F_E \) so that \( F_E \) acts on the right to permute these sets \( \{S_F\} \).

**Proof.** We assume \( L_V \in [L_V] \) is a structure with uniqueness. We take \( L' \) to be those words \( w \in L_V \) that are least in dictionary order among the words that evaluate into \( \overline{a} \). Since \( F_E \) is finite, and it is easy to check dictionary order by means of a finite state automaton,
the language
\[
L'' = \{(u, v) \in L_V \times L_V : \overline{u} \in \overline{F}_E \text{ and } u \text{ precedes } v\}
\]
is the language of an asynchronous two tape automaton. It follows that \(L' = L_V - p_2(L'')\) is regular. (Here \(p_2\) denotes projection onto the second factor.) We take \(S_1 = L\), and for each \(\overline{t} \in F_E\), we take \(S_{\overline{t}} = S_1 \overline{t}\). It is easy to check that each of these is \(L_V\)-rational. ■

We now construct a special \(\mathcal{Y}\)-graph \(\mathcal{X}'\). This graph will have a vertex for each edge \(E\) of \(\widehat{\mathcal{Y}}\) and each element of \(\partial_1 F_E\). For fixed \(E\) these vertices will constitute a \(\partial_1 F_E\)-orbit, and they will be labelled by the orbit of structures \([\partial_1 f L_V] \in \mathfrak{A}(G_V), f \in F_E\). The above lemma allows us to put in edges in an equivariant fashion to complete the \(\mathcal{Y}\)-graph \(\mathcal{X}'\). Let \([L']\) be the structure determined by \(\mathcal{X}'\).

Recall that \(\mathcal{H}\) is the set of conjugates of vertex groups in \(G = \pi_1(\mathcal{Y})\) and we are trying to show that the map
\[
L \mapsto (L_H)_{H \in \mathcal{H}} : \mathfrak{A}(G) \to \prod_{H \in \mathcal{H}} \mathfrak{A}(H)
\]
has dense image. We must show that if \(H_1, \ldots, H_n\) are distinct groups in \(\mathcal{H}\) and we are given \([L_{H_1}], \ldots, [L_{H_n}]\) in \(\mathfrak{A}(H_1), \ldots, \mathfrak{A}(H_n)\), we can find \([L] \in \mathfrak{A}(G)\) so that \([L]\) induces \([L_{H_1}], \ldots, [L_{H_n}]\) on \(H_1, \ldots, H_n\). We shall modify the structure \(L'\) described above to do what is required.

As discussed at the beginning of the proof of Theorem 3.9, we may assume that \(\mathcal{Y}\) is reduced. For each \(i = 1, \ldots, n\) choose \(E_i\) and \(h_i \in \mathcal{G}_E\) as in Lemma 3.1 with \(H_i = h_i G_{\partial_1 F_E} h_i^{-1}\). For each \(i\) the normal form for \(h_i\) determines a path in \(\mathcal{Y}\) starting at the base vertex \(V_0\). Inclusion of these paths in each other induces a partial order on the \(h_i\) and hence on the \(H_i\). We may assume the ordering \(H_1, \ldots, H_n\) respects this partial order. We will describe a modification of \(\mathcal{X}'\) to make the structure on \(H_i\) equal to the desired one without changing the structure on any \(H_j\) which is earlier in the partial order. Repeating this iteratively for \(i = 1, \ldots, n\) then proves the theorem.

Thus suppose \(i\) is chosen and write \(h = h_i, H = H_i\). By taking a cover of \(\mathcal{Y}\) if necessary, we may assume that the path \(\sigma\) in \(\mathcal{Y}\) determined by \(h\) is embedded. There is an induced covering of \(\mathcal{X}'\) and we replace \(\mathcal{X}'\) by this covering. We choose a lift \(\sigma'\) of the path \(\sigma\) to \(\mathcal{X}'\).

Suppose \(\sigma'\) has length at least 2 (we leave the case that it is shorter to the reader). Let the final two edges of \(\sigma'\) be \(e'\) and \(e\), so \(\partial_1 e' = \partial_0 e = w\) say. Let \(\pi(e') = E', \pi(e) = E\), \(\pi(w) = W\). There is a word \(u_0 \ldots t_{E'} u t_E\) labelling the path \(\sigma'\) and evaluating to \(h\). Then \(\overline{w} \in S_{\overline{e}}\). We delete \(\overline{w}\) from \(S_{\overline{e}}\) and establish a new edge out of \(v\) to a new vertex. We label the new edge with \(\{\overline{u}\}\) and the new vertex with \([h^{-1} L_H h]\). We let the edges out of this new vertex duplicate the edges out of \(\partial_1 e\). Likewise, for each \(f \in \partial_1 F_E\) we delete \(\overline{w}f\) from \(S_{\overline{f}(e)}\) and establish a new edge with label \(\{\overline{w}f\}\) to a new vertex labelled \([hf^{-1} L_H hf]\). For each of the vertices \(f(w)\) with \(f \in \partial_1 F_E\) we perform the same operation, constructing new edges to the vertices we have just added. This produces a new \(\mathcal{Y}\)-graph for a structure which induces the desired structure on \(H = hG_V h^{-1}\) and has not changed the induced structure on any earlier \(H_j\). ■
Remark. In the proof of Theorem 4.1, we were required to show that a $\mathcal{Y}$-graph $\mathcal{X}$ determines a regular language $L_{\mathcal{X}}$, and to do this, we turned $\mathcal{X}$ into a generalized finite state automaton $A_{\mathcal{X}}$ which almost accepted the language in question. To obtain the desired language, we only needed to delete those words containing subwords of the form $t_E f t_{E^{-1}}$ where $f \in \partial_1 F_E$. In fact, there is a straightforward procedure for turning a $\mathcal{Y}$-graph $\mathcal{X}$ into a generalized finite state automaton which accepts $L_{\mathcal{X}}$ itself. The method here is to build a generalized finite state automaton $B_{\mathcal{X}}$ whose underlying graph projects to that of $A_{\mathcal{X}}$. For each vertex of $v$ of $\mathcal{X}$, and each edge $E$ into $v$, there are two vertices in $B_{\mathcal{X}}$. One of these is reached only by elements of $\partial_1 F_E$, and there are no $\pi(E^{-1})$ edges out of this vertex. The other is reached by all elements not in $\partial_1 F_E$. This latter has a full armamentarium of edges out of it. The interested reader may wish to fill in the details along the lines of the proofs of Lemmas 1.1 and 3.1 in [S2].

5. The boundary

We recall the boundary of an asynchronous automatic structure, as defined in [NS1]. Let $L \subset A^*$ be an asynchronous automatic structure on a group $G$. As usual, we assume $L$ is finite to one. An $L$-ray is an infinite word $w \in A^\infty$, all of whose initial segments are initial segments of $L$-words. Two rays are equivalent if they asynchronously fellow travel (at a distance that may depend on the rays). The boundary of $L$ is the set $\partial L$ of equivalence classes of rays with the following topology. For an $L$-rational subset $R$ of $G$, define $\partial R$ to be the set of rays which fellow travel $R$ (that is, travel in a bounded neighborhood of $R$; the bound may depend on the ray). These sets form a basis of closed sets for a topology on $\partial L$. This boundary can be attached to $G$: the sets $\text{cl } R := R \cup \partial R$ are a basis of closed sets for a topology on $\text{cl } G := G \cup \partial L$ which has $\partial L$ as a closed subspace and $G$ as an open discrete subspace. (This topology is the “rational topology” of [NS1]. Other topologies on $\partial L$ are also discussed there.)

In [NS1] the “rehabilitated boundary” $\hat{\partial L}$ is also discussed, which appears to be an appropriate notion for groups with large abelian subgroups. A subset $\sigma$ of $\partial L$ is called an abstract simplex if, for any choice of a neighborhood in $\text{cl } G$ for each point of $\sigma$, the intersection of these neighborhoods is non-empty. This makes $\partial L$ into a topological abstract simplicial complex, the geometric realization of which is the rehabilitated boundary $\hat{\partial L}$.

Let $B$ be a tree and $B' = B \cup C$ be its end compactification. Given a continuous map $p$ of a space $X$ to $B$, we define the tree completion of $X$ with respect to $p$ as the disjoint union

$$X' = X \cup C,$$

with the smallest topology for which $X$ is a subspace and the induced map $p': X' \to B'$ is continuous. It is an easy exercise to see that $X'$ is compact if and only if $p$ is a proper map and is Hausdorff if and only if $X$ is Hausdorff.

Now let $\mathcal{Y}$ be a graph of groups with finite edge stabilizers and $G = \pi_1(\mathcal{Y})$. By [Se], there is a $G$-tree $B$ with $B/G$ equal to the underlying graph $Y$ of $\mathcal{Y}$ and with edge and vertex stabilizers given by the data of $\mathcal{Y}$. Let $\pi: B \to Y$ be the projection. The stabilizer of a vertex $v$ of $B$ is therefore a conjugate $H_v$ of the vertex group $G_{\pi(v)}$. 

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As in section 3, for \([L] \in \mathfrak{A}(G)\) and \(H\) a conjugate of a vertex group, \([L_H]\) denotes the induced structure on \(H\).

**Theorem 5.1.** Let \(X\) be the disjoint union of boundaries \(\partial L_{H,v}\), indexed by the vertices of \(B\), and \(p: X \rightarrow \text{vert}\ B \subset B\) the obvious map. Then the boundary \(\partial L\) is the tree completion \(X^*\). The analogous statement holds also for rehabilitated boundaries.

**Proof.** We first describe \(B\), following Serre [Se]. It has vertices \(\coprod_{V \in \mathcal{V}} G/G_V\). We denote the vertex determined by \(V\) and \([g] \in G/G_V\) by \(g\tilde{V}\). For each \(E \in \text{edge}\ Y\) and each \([g] \in G/\partial_0 F_E\) there is an edge, denoted \(g\tilde{E}\), from \(g\partial_0 E\) to \(g\tilde{E}\tilde{E}^\dagger E\). (Thus the reverse of the edge \(g\tilde{E}\) is the edge determined by \(gt_EE^{-1}\); this is slightly different notation from [Se].) Serre shows that \(B\) is a tree and the quotient by the obvious action of \(G\) is \(Y\).

We sketch the proof that \(B\) is a tree. We can choose a base vertex for \(B\) as the vertex \(v_0 = 1\tilde{V}_0\), where \(V_0\) is the base vertex for \(Y\). A normal form representation \(h = g_1t_{E_1}g_1\ldots t_{E_m}g_m\) with \(E_m = E\) for an element of \(G\), as defined early in section 3, determines a path in \(B\) from the base vertex \(v_0\) of \(B\) to the vertex \(v = h\tilde{E}\). This path consists of the sequence of edges \(g_0\tilde{E}_1\ldots, g_{m-1}\tilde{E}_m\). Thus \(B\) is a connected graph. Moreover, since we can right-multiply \(h\) by an element of \(G_{\partial_1 E}\) without changing \(v\), we can assume \(g_m \in \partial_1 F_E\). Then \(h \in \mathcal{G}_E\) and \(h\) is determined up to the right-action of \(F_E\). Thus we see that \(v\) is actually determined by \(E\) and an element of \(\mathcal{G}_E/F_E\). Now it not hard to see that any path without back-tracking from \(v_0\) to \(v\) gives a normal form representation for \(h\), and uniqueness of normal forms up to the operations mentioned in section 3 leads to uniqueness of such paths, showing that \(B\) is a tree.

This also shows that the vertices of \(B\) are in one-one correspondence with \(\coprod_{E \in \mathcal{Y}} \mathcal{G}_E/F_E\). From this point of view the stabilizer of the vertex \(v = h\tilde{E}\) corresponding to \(h \in \mathcal{G}_E/F_E\) is \(H_v = hG_{\partial_1 E}h^{-1}\), and the induced language \(L_{H_v}\) is equivalent to the language \(L_h\) on \(G_{\partial_1 E}\) (see Remark at end of section 3).

Now suppose \(L\) is an asynchronous automatic structure on \(G\). Since the boundary \(\partial L\) only depends on the equivalence class of \(L\), we may assume that \(L\) is chosen as in Lemma 3.5. We may also assume it is prefix-closed. Note that if \(R\) is an \(L\)-rational subset of \(G\) and \(R'\) is its “prefix-closure” (i.e., \(R' = N'\), where \(N'\) is the prefix closure of the set \(N\) of \(L\)-words evaluating into \(R\)), then \(R'\) lies in a bounded neighborhood of \(R\), so \(\partial R' = \partial R\). As bound one may take the diameter of a finite state automaton for \(N\). Thus, in discussing the topology on \(\partial L\) we need only consider “prefix-closed” rational subsets of \(G\).

For any word \(u \in L\), the shortest edge path decomposition (see definition preceding Lemma 3.6) determines a shortest normal form representative for \(\overline{u}\), and hence, as above, a simple path \(\gamma_{\overline{u}}\) from the base vertex \(v_0\) in \(B\). If \(u_1\) is a subword of \(u\) then \(\gamma_{\overline{u_1}}\) is a subpath of \(\gamma_{\overline{u}}\). It follows that an \(L\)-ray \(w\) determines a simple path \(\gamma_{\overline{w}}\) in \(B\), which is a finite or infinite path according as longer and longer initial segments of \(w\) eventually all evaluate into a fixed \(hG_V\) or not. We shall need the following Lemma.

**Lemma 5.2.** 1. For any \(k > 0\) there exists \(K > 0\) such that if \(u, u' \in L\) satisfy \(d(\overline{u}, \overline{u'}) \leq k\) then, by deleting at most the last \(K\) letters from \(u\) and \(u'\) one may obtain words \(u_0\) and \(u'_0\) with \(\overline{u_0} = \overline{u'_0}\).
2. If the $L$-ray $w$ fellow travels a subset $S \subset G$ then every initial segment of $\gamma_w$ appears as an initial segment of some $\gamma_g$, $g \in S$. The converse holds if $\gamma_w$ is infinite and $S = \overline{R}$ with $R \subset L$ prefix-closed.

Proof. 1. Since $L$ is finite-to-one, there exists a function $\phi : \mathbb{N} \to \mathbb{N}$ such that any terminal segment $u_1$ of an $L$-word with $\text{len}(u_1) > \phi(k)$ satisfies $d(\overline{w}, 1) > k$. Let $k_1 = \max\{d(f, 1) : f \in F_E \text{ for some edge } E \text{ of } \mathcal{Y}\}$ and define $K = \phi(k + k_1)$. Now suppose that $u, u' \in L$ satisfy $d(\overline{w}, u') \leq k$ but do not satisfy the conclusion of the lemma. Let $\gamma$ be the longest common segment of $\overline{\gamma}\nu$ and $\overline{\gamma}\mu$, and let $u_0$ and $u'_0$ be the longest initial segments of $u$ and $u'$ with $\gamma_0 = \gamma_0' = \gamma$. Write $u = u_0u_1$, $u' = u'_0u'_1$. At least one of $u_1$ and $u'_1$, say $u_1$, has length greater than $k$. By choice of $K$, the distance of $\overline{w_0}$ to $\overline{w} = \overline{w_0w_1}$ exceeds $k + k_1$, so the distance from $\overline{w}$ to $\overline{w_0}f$ exceeds $k$ for any $f$ in an edge group. But, by considering the normal form of $u^{-1}u'$ one sees that the shortest path in the Cayley graph from $u'$ to $\overline{w}$ must pass through $\overline{w_0}f$ for some $f \in \partial F_E$, where $E$ is the first edge of $u_1$ in the edge path decomposition of $u = u_0u_1$. This path hence has length exceeding $k$, contradicting $d(\overline{w}, u') \leq k$.

2. Part 1 of the lemma shows that if $w$ fellow travels a subset $S$ of $G$ then there exist arbitrarily long initial segments $u$ of $w$ with $\gamma_\nu$ equal to an initial segment of a path $\gamma_g$ with $g \in S$. But every initial segment of $\gamma_w$ is an initial segment of some such $\gamma_\nu$, so the first sentence of Lemma 5.2.2 is proved.

Conversely, suppose $\gamma_w$ is infinite and $S = \overline{R}$ with $R \subset L$ prefix-closed and suppose every initial segment of $\gamma_w$ is an initial segment of some $\gamma_g$ with $g \in S$. For a given initial segment $\gamma$ of $\gamma_w$, choose such a $g = \nu$ with $u \in R$ and let $u_0$ and $w_0$ be the initial segments of $u$ and $w$ corresponding to $\gamma$. Then $\overline{w_0}$ and $\overline{w_0}$ differ by an element of the edge group for the final edge of $\gamma$, so $u_0$ and $w_0$ fellow travel. Since $u_0$ travels in $S$ and $w_0$ is an arbitrarily long initial segment of $w$, the result follows. ■

We return to the proof of Theorem 5.1 for the boundary $\partial L$. If two rays $w$ and $w'$ fellow travel then their paths $\gamma_w$ and $\gamma_{w'}$ in $B$ are equal by Lemma 5.2.1. Moreover, if $\gamma_w$ and $\gamma_{w'}$ are equal and infinite then $w$ and $w'$ do fellow travel by Lemma 5.2.2. Thus rays $w$ with $\gamma_w$ infinite determine a subset of $\partial L$ that bijects to the set $C$ of infinite rays in $B$. This is the same as the set of ends of $B$.

Suppose now $\gamma_w$ is finite, say it ends at the vertex of $B$ determined by $[h] \in G_e/F_E$. Then by cutting $w$ at the point where it has determined the whole path $\gamma_w$, we write $w$ in the form $w_0u$ with $\overline{w_0} = h^f$ for some $f \in A_E$, and $fu$ a ray in $L_h$. If $w'$ is another ray with the same path $\gamma_w$ we decompose it likewise as $w_0'u'$ with $\overline{w_0'} = h^{f'}$ so that $f'u'$ is a ray in $L_h$. Then $w$ and $w'$ fellow travel if and only if the rays $fu$ and $f'u'$ fellow travel. We thus get a copy of $\partial L_h$ in $\partial L$. We have thus shown that, as a set, $\partial L$ is as claimed in the theorem.

It remains to verify that the topology is correct. Consider $v \in \text{vert } B$. Let the simple path in $B$ from $v_0$ to $v$ be $\gamma$. The set $R_v := \{u \in L : \gamma_u = \gamma\}$ is regular. Hence, $\overline{R_v} \subset G$ and its complement are both rational. The rays which fellow-travel $\overline{R_v}$ define the image of $\partial L_{H_v}$ in $\partial L = X \cup C$ and the rays that fellow travel the complement of $\overline{R_v}$ define the complement of $\partial L_{H_v}$. It follows that $\partial L_{H_v}$ is an open and closed subset of $\partial L$. Moreover, rational subsets of $\overline{R_v}$ correspond to rational subsets of $H_v$, so $\partial L_{H_v}$ carries the appropriate
Now suppose $w$ is a ray for which $\gamma_w$ is infinite, so $w$ represents an element of $C \subset \partial L$. Suppose $S = \overline{R} \subset G$ is a rational subset with $R \subset L$ prefix-closed, and suppose $[w]$ is in the set $U$ of equivalence classes of rays that fail to fellow-travel $S$. Then Lemma 5.2.2 implies that there is some initial segment $\gamma$ of $\gamma_w$ which does not appear as an initial segment of any $\gamma_g$, $g \in S$. Let $S_\gamma$ be the set of $g \in G$ such that $\gamma_g$ has $\gamma$ as an initial segment and let $U_\gamma \subset \partial L$ be the set of equivalence classes of rays which fail to fellow-travel $G - S_\gamma$. Then, $[w] \in U_\gamma \subset U$, so these sets $U_\gamma$ form a neighborhood basis for $[w] \in \partial L$. But $U_\gamma$ is the set of equivalence classes of rays $w$ such that $\gamma_w$ has $\gamma$ as an initial segment. This defines the topology on $\partial L$ claimed in the theorem.

To see the analogous statement for the rehabilitated boundary we must show that every non-trivial abstract simplex in $\partial L$ is an abstract simplex of some $\partial L_{H_v} \subset \partial L$ and vice versa. It is easy to see that an abstract simplex of $\partial L_{H_v}$ is one for $\partial L$. Thus, we must show that if $x, y$ are points of $\partial L$ which do not lie in some common $\partial L_{H_v}$, then they do not form an abstract simplex, that is, they have disjoint neighborhoods in $\operatorname{cl} G$. By what was said above, a set of the form $\operatorname{cl} \overline{R}_v$ is an open and closed subset of $\operatorname{cl} G$ whose intersection with $\partial L$ is $\partial L_{H_v}$. These sets are disjoint for different $v$’s, so they provide disjoint neighborhoods for $x$ and $y$ lying in distinct sets $\partial L_{H_v}$. Suppose just one of $x$ and $y$ lies in a $\partial L_{H_v}$, say $x \in \partial L_{H_v}$ and $y \in C$. Then $y = [w]$ with $\gamma_w$ infinite, so we can choose an initial segment $\gamma$ of $\gamma_w$ which is not an initial segment of the path in $B$ from $v_0$ to $v$. The set $\operatorname{cl} S_\gamma$ is an open and closed neighborhood of $y$ which is disjoint from the neighborhood $\operatorname{cl} \overline{R}_v$ of $x$. Finally, if $x = [w]$ and $y = [w']$ are distinct points of $C$ and $\gamma$ and $\gamma'$ are initial segments of $\gamma_w$ and $\gamma_w'$ which are longer than the longest common initial segment of $\gamma_w$ and $\gamma_w'$ then $\operatorname{cl} S_\gamma$ and $\operatorname{cl} S_{\gamma'}$ are disjoint open and closed neighborhoods of $x$ and $y$.

Remark. $\operatorname{cl} G$ can also be seen as a tree completion. For $g \in G$ the path $\gamma_g$ ends in a vertex $v_g$ of $B$, so we get a map $p_G : G \to \operatorname{vert} B \subset B$. If $v$ is the vertex determined by $[h] \in G_{E} / F_{E}$, then $p^{-1}(v) = h(G_{h,E} - \overline{F_{E}})$ (except that $p^{-1}(v_0) = G_{v_0}$ rather than $G_{v_0} - \{1\}$). That is, up to a finite set $p^{-1}(v)$ is just a translate of the group $G_{\pi(v)}$ on which the language $L_h$ (equivalent to $L_{H_v}$) is defined, so we can attach to $p^{-1}(v)$ the boundary $\partial L_h \simeq \partial L_{H_v}$. This gives us a topology on $G \cup \bigsqcup_{v \in \operatorname{vert} B} \partial L_{H_v}$ and a map of this space to $\operatorname{vert} B \subset B$. Then $\operatorname{cl} G$ is its tree completion. We leave the details to the reader.

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The Ohio State University
Department of Mathematics
Columbus, OH 43210

City College
Department of Mathematics
New York, NY 10031