The alternative to classical mass renormalization for tube-based self-force calculations

Andrew H Norton

Department of Mathematical Sciences, and CUDOS1, University of Technology Sydney, Australia

E-mail: andrew.norton@aei.mpg.de

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Abstract
To date, classical mass renormalization has been invoked in all tube-based self-force calculations, thus following the method introduced in Dirac’s 1938 calculation of the electromagnetic self-force for the classical radiating electron. In this paper, a new tube method is described that does not rely on a mass renormalization procedure. As a result, exact self-force calculations become possible for classical radiating systems of finite size. A new derivation of the Lorentz–Dirac equation is given and the relationship between the new tube method and the classical mass renormalization procedure is explained. It is expected that a similar tube method could be used to obtain rigorous results in the gravitational self-force problem.

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1. Introduction

The classical mass renormalization procedure was introduced 70 years ago by Dirac [1] and has since featured in all tube-based calculations of self-forces on radiating particles. This includes not only calculations involving the electromagnetic self-force on a radiating charge in Minkowski spacetime, as exemplified by [1–17], but also electromagnetic self-force calculations for a charged particle in a curved background spacetime [18, 19] and the tube-based approach to the gravitational self-force problem, as taken in [20] and reviewed in [21–24]. Nevertheless, the mathematical status of the classical mass renormalization procedure has remained unclear and justification for its use has principally been that it has given results that agree with other self-force methods. As recently emphasized in [25], this is particularly true of the gravitational self-force problem for which the concept of a point particle makes even less sense than in the electromagnetic case.

1 Centre for Ultrahigh bandwidth Devices for Optical Systems.
This paper describes a tube method for determining the self-force on a radiating particle or system of finite nonzero size. Dirac’s tube method is extended so that the calculations go through with a worldtube that has, at all stages of the calculation, a nonzero radius. The physical picture that emerges is very different to that which accompanies the classical mass renormalization procedure. In retrospect, one finds that the renormalization procedure is most easily understood as a short-cut that involves assuming, rather than deriving, a certain intermediate result (our equation (22)) that is required for a proper derivation of the self-force.

The new tube method is presented here for the problem of electromagnetic self-force in Minkowski spacetime and, as an example, a derivation of the Lorentz–Dirac equation is given that makes no mention of mass renormalization. Since the method is based on quite general considerations of energy–momentum conservation, the same approach ought to be applicable in any radiation dynamics problem for which a well-defined self-force exists. In particular, it seems likely that a similar tube method could be used to obtain rigorous results in the gravitational self-force problem.

1.1. Background

Dirac’s tube method for calculating the self-force on a radiating particle was used in [1] to derive what is now known as the Lorentz–Dirac equation of motion for a classical radiating particle of charge $q$ and mass $m$ in a background electromagnetic field $F$,

$$mc \ddot{z}^\alpha = q F^\alpha_\beta \dot{z}^\beta + \frac{q^2}{4\pi \epsilon_0 c^3} \left( \delta^\alpha_\beta + \dot{z}^\alpha \dot{z}^\beta \right) \ddot{z}^\beta, \tag{1}$$

where the worldline has equation $x = z(s)$ in Minkowski coordinates, overdots denote differentiation with respect to $s$, and the parametrization is such that $\dot{z}^\alpha \dot{z}_\alpha = -1$.

The calculation that led Dirac to (1) can be described as follows. One starts by deriving a balance relation that follows from energy–momentum conservation, as applied to the electromagnetic field within a worldtube containing the radiating particle. The balance relation expresses the fact that the rate of change of 4-momentum flowing within the worldtube must equal the rate at which electromagnetic energy–momentum is radiated through the wall of the tube. The derivation of Dirac’s balance relation is lengthy but straightforward. It is easily found (for a given worldtube) using computer algebra and takes the form of a power series in the tube radius $r$, which is taken to be small compared with the radius of curvature of the particle worldline (the inverse of the particle acceleration).

The worldtube exists only to facilitate the calculations, so the tube radius cannot enter into the sought-after equation of motion. Dirac’s approach for eliminating the tube radius was to let $r \to 0$, in which case only quantities determined by the particle worldline can appear in the final result. The difficulty with this idea is that because the particle must remain enclosed within the worldtube, it must then be modelled as being point-like. The electromagnetic field is then singular on the particle worldline, causing the required $r \to 0$ limit to be neither defined nor physically sensible. In fact, to obtain the desired result (1) using this approach, one must formally subtract out the divergent Coulomb field energy of the point charge by assigning the particle an infinite negative bare mass—the procedure that has come to be known as classical mass renormalization.

To avoid the absurdities of classical mass renormalization, one needs a way to eliminate the tube radius from Dirac’s balance relation without taking $r \to 0$. Moreover, simply taking $r$ to be small and neglecting $O(r)$ terms does not quite work, as this would result in a finite mass renormalization that is $r$-dependent. In effect, one would be saying that the field–energy contribution to the mass of the particle somehow depended on the radius of an imaginary tube that was introduced purely as a mathematical convenience.
In this paper, we shall see how to cancel all of the \( r \)-dependent terms from the Dirac balance relation for any classical radiating system. This is done by making better use of the information available in the problem, including the fact that the background electromagnetic field must satisfy the source-free Maxwell equations throughout the spacetime region occupied by the radiating particle or system. The worldtube used in the calculations will have a nonzero radius and there will be no need to assume any mass renormalization, finite or otherwise.

1.2. Organisation of the paper

The main ideas and results are presented in section 2, where the tube method is described with reference to the Lorentz–Dirac example, that is, for the problem of determining the self-force for a charged particle in Minkowski spacetime. As far as possible, all distracting calculations have been deferred to later sections. In particular, the geometry of the worldtube is dealt with in section 3 and the electromagnetic field calculations appear in section 4. A Mathematica notebook for all of the calculations is available as an electronic supplement [26].

In principle, the worldtube could contain any classical radiating system, so it should be kept in mind that the only parts of section 2 that are truly specific to the Lorentz–Dirac example are the values quoted for the coefficients in the two series expansions of \( \partial_s P_{\text{cap}}(s, r) \) (equations (14) and (23)).

The unit system used in this paper is the SI and the Minkowski metric is taken to be \( \eta = \text{diag}(-1, 1, 1, 1) \). Thus, any equation here can be converted to the geometrized Gaussian CGS units used by Hawking and Ellis [27], Misner, Thorne and Wheeler [28] and Wald [29], for example, by setting \( c = G = 1 \) and \( 4\pi \epsilon_0 = 1 \) (hence also \( \mu_0 = 4\pi \)).

2. The tube method

We start by supposing the particle has a physically sensible (but unknown) classical structure, for example, as a soliton in some nonlinear field theory that reduces to Maxwell electrodynamics in its weak-field limit, or perhaps as some classical electron model [16]. Here, physically sensible is taken to mean there exists a conserved stress–energy tensor \( T \) for this underlying theory, and that the particle has a finite size. Thus, we suppose

\[
\nabla \cdot T = 0, \tag{2}
\]

\[
T = \begin{cases} 
T_{\text{matter}} & \text{for } r \leq r_0, \\
T_{\text{e.m.}} & \text{otherwise},
\end{cases} \tag{3}
\]

where \( T_{\text{e.m.}} \) is a purely electromagnetic (vacuum) stress–energy tensor and the radius \( r_0 \) defines, for some predetermined level of accuracy, the extent of the matter fields or size of the particle structure. A precise value for this structure radius is not needed. It will suffice to assume that such an \( r_0 \) exists, so that the particle can be considered to lie within a worldtube of radius \( r_0 \). The worldline \( C \) on which this tube is centred will serve as a reference worldline for the particle, about which the asymptotic field of the particle shall be prescribed, usually in terms of a multipole expansion [10, 15, 30, 31].

The simplest such prescription is that the charged particle has an asymptotic field that exactly coincides with that of an electric monopole, as given by the retarded Liénard–Wiechert potential for a point-source with worldline \( C \). Later, we shall see that this case leads directly to the Lorentz–Dirac equation (1). More generally, the asymptotic field of the (nonzero size) charged particle would include higher order multipoles, and the corresponding self-force would differ from the Lorentz–Dirac result.
Energy-momentum balance is applied to the tube $\Sigma_{\text{tube}}(s, r)$ in the asymptotic region $r \gg r_0$, with the particle structure taken to be zero outside a tube of radius $r_0$, but is otherwise unknown.

The stress-energy tensor associated with the particle structure is taken to be zero outside a tube of radius $r_0$, and the self-force on the radiating particle thereby calculated using conservation of energy–momentum over the 4-volume $V(s, r)$.

The worldtube for which Dirac’s balance relation is to be derived will be chosen to have a tube radius $r$ that is much larger than the structure radius $r_0$, so that it lies in the asymptotic field of the particle, where the prescribed multipole expansion can be used. On the other hand, we still take $r$ to be small compared with the scale set by the radius of curvature of $C$, that is $|r\kappa_1| \ll 1$, where $\kappa_1$ (defined by (38)) is the first Frenet–Serret curvature of $C$. The situation described is shown in figure 1.

Let $C$ be a smooth timelike worldline with equation $x = z(s)$ where $x^\alpha, \alpha = 0, \ldots, 3$, are Minkowski coordinates with metric $\eta = \text{diag}(-1, 1, 1, 1)$ and the parameter $s$ is proper distance ($c \times$ proper time) along $C$, so that $\dot{z} \cdot \dot{z} = -1$. In section 3, spherical polar tube coordinates $\{s, r, \vartheta, \phi\}$ are constructed using the Frenet frame of $C$, such that the worldline parameter $s$ coincides on $C$ with the spacetime coordinate $s$, and $C$ has the equation $r = 0$. The various integration surfaces that are needed can then be specified in terms of coordinate surfaces. The worldtube $\Sigma_{\text{tube}}(s, r)$, the tube-cap $\Sigma_{\text{cap}}(s, r)$, their intersection 2-sphere $\Omega(s, r)$ and the 4-volume $V(s, r)$ are defined by

$$\Sigma_{\text{tube}}(\bar{s}, \bar{r}) = \{s \in [0, \bar{s}], r = \bar{r}\},$$  \hspace{1cm} (4)  

$$\Sigma_{\text{cap}}(\bar{s}, \bar{r}) = \{s = \bar{s}, r \in [0, \bar{r}]\},$$  \hspace{1cm} (5)  

$$\Omega(\bar{s}, \bar{r}) = \{s = \bar{s}, r = \bar{r}\},$$  \hspace{1cm} (6)  

$$V(\bar{s}, \bar{r}) = \{s \in [0, \bar{s}], r \in [0, \bar{r}]\}.$$(7)

Figure 1. The worldtube $\Sigma_{\text{tube}}(s, r)$ used in the self-force calculations. The tube is of length $s$ and has radius $r \gg r_0$ where $r_0$ is the structure radius of the particle. The momentum $P_{\text{cap}}(s, r)$ includes unknown contributions from the stress–energy tensor associated with the particle structure. Nevertheless, expressions for the derivatives $\partial_s P_{\text{tube}}(s, r)$ and $\partial_r P_{\text{cap}}(s, r)$ can still be evaluated in the asymptotic region $r \gg r_0$, and the self-force on the radiating particle thereby calculated using conservation of energy–momentum over the 4-volume $V(s, r)$.
where in each case the angular coordinates assume the range \((θ, ϕ) ∈ [0, π] × [0, 2π]\).

Figure 1 shows these various structures for the time-symmetric tube coordinate system defined in section 3.1.2. For this coordinate system, the worldtube is of Dirac type, meaning that the tube caps are orthogonal to \(C\), as used in [1]. The calculations can also, for example, be based on retarded (advanced) tube coordinates, in which case the tube caps are parts of forward (backward) null cones, as described in section 3.1.3. The results quoted in the present section are for the Dirac-type worldtube of figure 1.

Let \(P_{\text{cap}}(s, r)\) be the momentum flux through \(Σ_{\text{cap}}(s, r)\), and let \(P_{\text{tube}}(s, r)\) be the (outward) momentum flux across the worldtube \(Σ_{\text{tube}}(s, r)\). In section 3.2.1, it is shown that

\[
P_{\text{cap}}(s, r) = \frac{1}{c} \int_{r'=0}^r \int_{Ω(s', r')} T \cdot ν_{\text{cap}} dΩ',
\]

\[
P_{\text{tube}}(s, r) = \frac{1}{c} \int_{s'=0}^s \int_{Ω(s', r')} T \cdot ν_{\text{tube}} dΩ',
\]

where \(dΩ\) is the element of area for the 2-sphere \(Ω(r, s)\) and the vectors \(ν_{\text{cap}}\) and \(ν_{\text{tube}}\) (given by (62) and (65)) are normals for the integration 3-surfaces. Observe that expressions (8) and (9) depend on \(r\) and \(s\) respectively, only through the upper limits of the outer integrals. Differentiating (8) with respect to \(r\) and (9) with respect to \(s\) therefore gives

\[
\frac{∂P_{\text{cap}}(s, r)}{∂r} = \frac{1}{c} \int_{Ω(s, r')} T \cdot ν_{\text{cap}} dΩ',
\]

\[
\frac{∂P_{\text{tube}}(s, r)}{∂s} = \frac{1}{c} \int_{Ω(s, r')} T \cdot ν_{\text{tube}} dΩ'.
\]

Following Dirac, equation (11) becomes a relation involving \(P_{\text{cap}}(s, r)\) by considering energy–momentum conservation for the fields within the 4-volume \(V(s, r)\). Dirac assumes that these fields are purely electromagnetic, so that \(T = T_{\text{e.m.}}\) all the way up to the worldline \(C\), but in our case only the asymptotic field of the particle is known to us. The particle structure is unknown, as is the stress–energy tensor \(T_{\text{matter}}\) associated with this structure. Nevertheless, Gauss’ theorem still applies and in section 3.2 it is shown that by integrating (2) over the 4-volume \(V(s, r)\) one has

\[
P_{\text{cap}}(s, r) = P_{\text{cap}}(0, r) - P_{\text{tube}}(s, r).
\]

Differentiating (12) with respect to \(s\) and making use of (11) then gives

\[
\frac{∂P_{\text{cap}}(s, r)}{∂s} = -\frac{1}{c} \int_{Ω(s, r')} T \cdot ν_{\text{tube}} dΩ'.
\]

We now use the fact that the worldtube has radius \(r \gg r_0\). The 2-sphere \(Ω(s, r)\) over which the integral in (13) is evaluated therefore lies in the asymptotic field of the particle, where the stress–energy tensor is of known electromagnetic form. After replacing \(T\) in (13) by \(T_{\text{e.m.}}\), one then expands the integrand \(T_{\text{e.m.}} \cdot ν_{\text{tube}}\), as a power series in the tube radius \(r\), which although large with respect to the structure radius \(r_0\), is still assumed to be small with respect to the scale set by the worldline curvature \((|rκ_1| \ll 1)\). The angular integrations then involve trigonometric polynomials in \([θ, ϕ]\) and may be evaluated exactly. In section 4, these calculations are described in further detail for the Lorentz–Dirac example, in which case \(T_{\text{e.m.}}\) is taken to be the stress–energy tensor of an electromagnetic field that is the sum of a background field \(F\) and an electric monopole (retarded Liénard–Wiechert) field for the particle. The result is Dirac’s balance relation (equation (18) in [1]),

\[
\frac{∂P_{\text{cap}}(s, r)}{∂s} = -\frac{1}{c} \int_{Ω(s, r')} T_{\text{e.m.}} \cdot ν_{\text{tube}} dΩ = \sum_{k=-1}^{∞} a_k(s) r^k,
\]

5
where the first few coefficients in the series are found to be [26]

\[ a_{-1}(s) = -\frac{q^2}{8\pi \epsilon_0 c} \kappa_1 N_1, \]  
\[ a_0(s) = q F \cdot N_0 + \frac{q^2}{6\pi \epsilon_0 c} (\kappa_1 N_1 + \kappa_1 \kappa_2 N_2), \]  
\[ a_1(s) = \frac{q^2}{48\pi \epsilon_0 c} \left( 2\kappa_1^3 N_0 + (3\kappa_1^2 + 4\kappa_1 \kappa_2^2 - 4\kappa_1) N_1 \right. \]  
\[ - 4(\kappa_1 \kappa_2 + 2\kappa_2 \kappa_1) N_2 - 4\kappa_1 \kappa_2 \kappa_3 N_3 \left. \right). \]

Here \( N_i \) and \( \kappa_a \) are respectively the Frenet–Serret frame vectors and curvatures of \( C \) (see section 3.1.1 for definitions). For \( k \geq 2 \), the coefficients \( a_k(s) \) involve successively higher derivatives of \( F \) evaluated on \( C \) and become rapidly more complicated.

A similar calculation is used to evaluate (10). Again, on noting that \( \Omega(s, r) \) lies in the asymptotic field of the particle we replace \( T \) by \( T_{e.m.} \). Expand the integrand as a power series in \( r \) and then do the angular integrations. The result is

\[ \frac{\partial P_{\text{cap}}(s, r)}{\partial r} = \frac{1}{c} \int_{\Omega(s, r)} T_{e.m.} \cdot \nu_{\text{cap}} \, d\Omega = \sum_{k=-2}^{\infty} b_k(s) r^k, \]

where the first few coefficients in the series are

\[ b_{-2}(s) = \frac{q^2}{8\pi \epsilon_0 c} N_0, \]  
\[ b_{-1}(s) = 0, \]  
\[ b_0(s) = -\frac{q^2}{48\pi \epsilon_0 c} \left( 3\kappa_1^2 N_0 - 4\kappa_1 N_1 - 4\kappa_1 \kappa_2 N_2 \right). \]

Assuming convergence of the series, we now integrate (18) as a partial differential equation in independent variables \( s, r \) to get

\[ P_{\text{cap}}(s, r) = p(s) - b_{-2}(s) r^{-1} + \sum_{k=1}^{\infty} b_{k-1}(s) r^k / k, \]

where the quantity \( p(s) \) (independent of \( r \)) appears as the integration ‘constant’. It turns out that if the tube caps \( \Sigma_{\text{cap}}(s, r) \) are orthogonal to the worldline \( C \), as they are for the Dirac-type tube being used here, then this integration constant can be directly identified with the momentum of the particle (the situation for a worldtube with null-cone caps is covered in section 2.2). Differentiating (22) with respect to \( s \) finally gives

\[ \frac{\partial P_{\text{cap}}(s, r)}{\partial s} = \frac{\partial}{\partial s} \left( p(s) - b_{-2}(s) r^{-1} + \sum_{k=1}^{\infty} b_{k-1}(s) r^k / k \right). \]

We have thus found a second expression for the quantity \( \partial_s P_{\text{cap}}(s, r) \), in addition to Dirac’s balance relation (14). Subtracting (23) from (14) then gives

\[ 0 = (a_{-1} + b_{-2}) r^{-1} + (a_0 - \dot{p}) + \sum_{k=1}^{\infty} (a_k - b_{k-1} / k) r^k. \]

Since \( r \) is arbitrary, vanishing of this series entails that all coefficients in the series are zero. The \( r^0 \)-term gives an equation to be satisfied by \( p(s) \), whereas the terms involving \( r \) vanish.
identically\(^2\). Using the Mathematica code \([26]\), one can check this is so to at least \(O(r^5)\). We remark that to explicitly verify that the coefficient of \(r^k\) is identically zero requires, for \(k \geq 2\), that on \(C\) the background field \(F\) must satisfy the source-free Maxwell equations and also derivatives of these equations (up to a derivative order that increases with \(k\)). The vanishing of the \(r^0\)-term in (24) gives the momentum balance equation, \(p = a_0\). For the Lorentz–Dirac example, the coefficient \(a_0\) is given by (16) and this equation becomes

\[
p(s) = q F \cdot N_0 + \frac{q^2}{6 \pi \varepsilon_0 c} (\kappa_1 N_1 + \kappa_1 \kappa_2 N_2).
\]

(25)

This is an exact statement of momentum balance for the case in which the asymptotic field of the charged particle is equal to the retarded field of an electric monopole.

Note that \(p(s)\) is still unspecified. To obtain an equation of motion from (25), one must specify how the particle momentum is to be defined in terms of the geometry of the worldline and the internal degrees of freedom of the radiating system or particle. The Lorentz–Dirac equation is obtained by assuming (as does Dirac) that the particle has no internal degrees of freedom and that \(p(s)\) takes the simplest possible form,

\[
p(s) = cmN_0(s).
\]

(26)

Then (25) becomes the Lorentz–Dirac equation, the equivalence of (25) and (1) being easily established using the Frenet–Serret equations (38)–(41).

There are several logical gaps in the above derivation of the Lorentz–Dirac equation, but these gaps are not related to evaluating the self-force. If one knows the asymptotic electromagnetic field of the extended particle (even if this field is defined so as to depend on internal degrees of freedom of the particle) then the tube method can be used to derive an exact statement of momentum balance analogous to (25) that defines the self-force. The gaps only become evident when attempting to derive an equation of motion from this balance equation. For example, definitions of the momentum and centre-of-mass worldline of an extended particle have been given by Dixon in [32–34]. But our \(p(s)\) is defined as the \(r\)-independent term (the integration constant) in the series (22) for \(P_{\text{cap}}(s, r)\).

2.1. Classical mass renormalization

The classical mass renormalization procedure can be understood as a short-cut that amounts to assuming an \(O(r)\) approximation to our expression (22). To see this, we reproduce here the argument used in [1] to get from Dirac’s balance relation (14) to the result (25).

The traditional interpretation of the quantity \(P_{\text{cap}}(s, r)\) has been that in the limit \(r \rightarrow 0\) this represents a ‘bare momentum’ of the particle. Assume now that as \(r \rightarrow 0\) the bare momentum takes an almost familiar form, being the product of the 4-velocity of the particle with a bare mass function \(m_B(r)\). Thus, (compare equation (20) in [1]) let

\[
P_{\text{cap}}(s, r) = cm_B(r)N_0(s) + O(r).
\]

(27)

One now chooses the bare mass to be negative and diverging as \(r \rightarrow 0\) in exactly the manner required so as to cancel the problematic \(r^{-1}\) term appearing on the right-hand side of

\[\text{For } k \neq 0, \text{ the coefficients } a_k(s) \text{ are therefore perfect differentials. Having noticed this before obtaining (24), the author wrote a Mathematica package for symbolic integration of Frenet frame expressions such as (17). This package is available as part of [26].}\]
Moreover, the difference between these divergent terms must account for the physically observable mass of the particle. Thus, (compare equation (21)) in [1] one postulates that

$$ cm(r) = -\frac{q^2}{8\pi \epsilon_0 c} \frac{1}{r} + cm, \quad (28) $$

where $m$ is now referred to as the ‘renormalized’ mass of the particle. Substituting (28) into (27) gives

$$ P_{\text{cap}}(s, r) = cmN_0 - \frac{q^2}{8\pi \epsilon_0 c} \frac{N_0}{r} + O(r), \quad (29) $$

which is recognized as being the O($r$) approximation to (22), for a particle momentum given by (26). So an O($r$) approximation to the Lorentz–Dirac equation is obtained by following the steps previously given. The remaining $r$-dependent terms are then eliminated from the equation of motion by taking $r \to 0$, in which case (28) becomes an infinite mass renormalization.

The renormalization procedure evidently depends on being able to antidifferentiate with respect to $s$ any singular terms that appear on the right-hand side of (13) so that these can be absorbed into a redefined momentum on the left-hand side of (13). In 1944, Bhabha and Harish-Chandra [4] showed that for point particles possessing any multipole moments whatsoever, this can always be done. Using our present notation, what they found is that (compare equation (19)) in [4],

$$ \frac{\partial}{\partial r} \int_{\Omega(s, r)} T \cdot v_{\text{tube}} \, d\Omega \frac{\partial}{\partial s} \int_{\Omega(s, r)} T \cdot v_{\text{cap}} \, d\Omega = 0, \quad (30) $$

which is the relation $\left(\frac{\partial}{\partial r} - \frac{\partial}{\partial s}\right)P_{\text{cap}}(s, r) = 0$ expressed in terms of (10) and (13). They go on to observe that if $T$ can be expanded as a power series in $r$, then (30) shows that, with the exception of the $r^0$-term, all coefficients in the series for (13) (which they call ‘the inflow’) are perfect differentials. In a later paper [5], the authors refer to this result as their inflow theorem. In particular, in [4] they say ‘the singular terms being perfect differentials can always be compensated by the addition of suitable terms to $P_{\text{cap}}(s, r)$’. These two papers are both concerned with point particle theory (the phrase ‘point particles’ even appears in both titles) and there is no indication that the authors were aware of the full significance of their inflow theorem: if all of the perfect differentials, not just the singular terms, are ‘compensated’ then all dependence on $r$ vanishes, exactly as it did in equation (24). There is then no reason to take $r \to 0$, and the resulting balance equation (the counterpart of (25)) is exact for finite radius worldtubes. Reference [4] has received little attention (an exception is [10] page 361, but again, in the context of renormalization for point particles) and the fact that the inflow theorem underlies a finite radius tube method seems to have been overlooked.

2.2. Identification of the particle momentum

It was remarked earlier that being able to identify the integration constant in (22) as the particle momentum depended on having chosen to use a worldtube with caps that are orthogonal to the worldline. To see how a result that is independent of tube cap geometry can come about, one can do the calculations using a variety of worldtubes, with differing tube cap geometries, then demand that the result not depend on any particular choice of worldtube. Thus, in section 3 we define a 1-parameter family of worldtubes, with parameter $\epsilon \in [-1, 1]$. The tube caps vary with $\epsilon$, interpolating between forward null cones for $\epsilon = 1$ and backward null cones for $\epsilon = -1$. The Dirac-type worldtube, with caps orthogonal to $C$, corresponds to $\epsilon = 0$.

The original text refers to a quantity $A(\tau)$ that corresponds to our $P_{\text{cap}}(s, r)$ in (13).
In this more general setting, the integration constant in (22) depends on \( \epsilon \), and thus \( p(s) \) in (22) is to be replaced by \( p_\epsilon(s) \). In place of (25) one then finds that

\[
\dot{p}_\epsilon(s) = q F \cdot N_0 + \frac{q^2}{6\pi\varepsilon_0 c} \left( -\epsilon\kappa_1^2 N_0 + (1 - \epsilon) (\kappa_1 N_1 + \kappa_1\kappa_2 N_2) \right). \tag{31}
\]

An equation that is independent of \( \epsilon \) is obtained by integrating the \( \epsilon \)-dependent terms and absorbing them into the definition of the integration constant. Thus, by setting

\[
p_\epsilon(s) = p(s) - \frac{q^2}{6\pi\varepsilon_0 c} \epsilon\kappa_1 N_1, \tag{32}
\]

one recovers the relation (25).

2.3. A sanity check—the weak energy condition

In section 2.1, we saw that if the classical mass renormalization procedure is fully implemented using Bhabha and Harish-Chandra’s inflow theorem, then one recovers (in a somewhat awkward manner) the finite radius tube method of section 2. Taking \( r \to 0 \) is then quite pointless and only serves to make the theory mathematically ill-defined and physically nonsensical. In particular, the bizarre notion of negative bare mass (equations (28) and (29)) arises because the worldtube is too small to fit the particle within it. Using the weak energy condition, this can be turned around to determine a minimum possible size for the extended particle or system by finding the smallest worldtube that could enclose a system with the same prescribed multipole properties.

If at every point the stress–energy tensor obeys the inequality \( T_{\alpha\beta} u^\alpha u^\beta \geq 0 \) for any timelike vector \( u \), then \( T \) is said to satisfy the weak energy condition [27]. This is equivalent to the energy density being non-negative for any observer. Assuming the weak energy condition for \( T \), one has from (8) that

\[
-N_0 \cdot P_{\text{cap}}(s, r) = \frac{1}{c} \int_{r=0}^{r} \int_{\Omega(s, r')} T_{u^u u^u} N_0 \cdot N_0 \cdot d\Omega \cdot dr' \geq 0, \tag{33}
\]

where we have used a Dirac-type worldtube, for which (62) gives \( \nu_{\text{cap}} = -N_0 \). Substituting expression (22) for \( P_{\text{cap}} \), which we saw in section 2.2 is specifically for a Dirac-type worldtube, one has

\[
-N_0 \cdot p + N_0 \cdot b_{-2}(s) r^{-1} + O(r) \geq 0. \tag{34}
\]

If the asymptotic field of the particle is that of an electric monopole, then \( b_{-2}(s) \) is given by (19). In this case, after dropping \( O(r) \) terms and rearranging, (34) can be written as

\[
r \geq \frac{1}{8\pi\varepsilon_0 mc^2} \frac{q^2}{(-N_0 \cdot p)} . \tag{35}
\]

It is interesting to see what this result says about classical models for the electron. In the Lorentz–Dirac model, the momentum is assumed to be tangent to the worldline of the particle and thus, \( cm / (-N_0 \cdot p) = 1 \). Inequality (35) then becomes \( r \geq r_e/2 \), where \( r_e \) is the so-called classical electron radius [36],

\[
r_e \equiv \frac{1}{4\pi\varepsilon_0 m_e c^2} \approx 2.8 \times 10^{-15} \text{ m}, \tag{36}
\]

where \( q = -e \) is the electron charge and \( m = m_e \) is the electron mass. However, it is known experimentally [37] that the minimum tube radius for an electron is at least \( 10^5 \) times smaller than \( r_e/2 \). Not surprisingly, it follows that the Lorentz–Dirac classical electron model would be completely inadequate for describing the physics in such experiments. On the other hand,
our result (35) cannot be used to rule out all classical models of a *spinning* electron. In a classical spin model the worldline is typically a spacetime helix. The particle momentum is not tangent to the worldline, but instead, is directed along the helix axis. The factor $cm/(-N_0 \cdot p)$ appearing in (35) can therefore be arbitrarily small if the circular motion of the charge centre is close to the speed of light\(^4\). Such models arise naturally in second and higher order Lagrangian mechanics (for example, see [38, 39]).

3. Geometry of the worldtube

At least two obvious choices present themselves for the worldtube. One is a worldtube with caps orthogonal to \( C \), as used by Dirac [1] and the other is a worldtube with caps that are parts of forward null cones based on \( C \), sometimes called a Bhabha tube and used, for example, in [4, 12, 17]. In either case, \( \Sigma_{\text{tube}}(s, r) \) and \( \Sigma_{\text{cap}}(s, r) \) can be defined by equations (4) and (5) as coordinate surfaces in a system of spherical polar tube coordinates \( \{ s, r, \vartheta, \varphi \} \) centred on the worldline \( C \), with the angles \( \{ \vartheta, \varphi \} \) defined with respect to the Frenet frame of \( C \).

3.1. Spherical polar tube coordinates

3.1.1. The Frenet frame of \( C \). Recall that the worldline \( C \) has the equation \( x = z(s) \), where \( x^\alpha \) are Minkowski coordinates and \( s \) is proper distance along \( C \). Let the Frenet frame vectors of \( C \) be

\[
N_\beta(s) = N_\beta^\alpha(s) e_\alpha, \tag{37}
\]

where \( e_\alpha \) are Minkowski basis vectors, \( e_\alpha \cdot e_\beta = \eta_{\alpha \beta} \). The defining relations for the Frenet frame are that \( N_0 = \dot{z} = dz/ds \) is the future-directed unit tangent to \( C \); that the frame \( N_\beta \) is orthonormal and has the same orientation as the Minkowski frame \( e_\beta \); and that the frame \( N_\beta \) satisfies the Frenet–Serret equations

\[
N_0 = \kappa_1 N_1, \tag{38}
\]

\[
N_1 = \kappa_1 N_0 + \kappa_2 N_2, \tag{39}
\]

\[
N_2 = -\kappa_2 N_1 + \kappa_3 N_3, \tag{40}
\]

\[
N_3 = -\kappa_3 N_2. \tag{41}
\]

The scalar fields \( \kappa_\beta(s) \) are the Frenet–Serret curvatures of \( C \). Differentiation with respect to proper distance \( s \) is indicated by an overdot. The orthogonality conditions for the Frenet frame are \( N_\beta \cdot N_\gamma = \eta_{\beta \gamma} \), where \( \eta = \text{diag}(-1, 1, 1, 1) \), or in terms of components, \( N_\alpha^\beta N_\beta^\gamma \eta_{\sigma \tau} = \eta_{\alpha \gamma} \). The matrix inverse of \( N_\alpha^\beta \) is therefore given by its Minkowski transpose, \( N^{\alpha \beta} = \eta^{\beta \gamma} N^\gamma_\beta \eta_{\alpha \delta} \).

Finally, one has the following relation between the Minkowski basis covectors and the Frenet coframe,

\[
e^\alpha = N^\alpha_\beta e_\beta, \tag{42}
\]

where \( e^\alpha = \eta^{\alpha \beta} e_\beta \) and \( N^\alpha = \eta^{\beta \gamma} N^\gamma_\beta \).

\(^4\) Calculating in the momentum rest frame, one has \( p = (cm, 0) \) and \( N_0 = \gamma(1, v/c) \) where \( \gamma = (1 - (v/c)^2)^{-\frac{1}{2}} \) and \( v \) is the 3-velocity of the circular motion. Then, \( cm/(-N_0 \cdot p) = \gamma^{-1} \).
3.1.2. Time-symmetric tube coordinates. Time-symmetric tube coordinates \( \{s, r, \vartheta, \varphi\} \), centred on the worldline \( C \) are defined by the coordinate transformation

\[
x = z(s) + rn(s, \vartheta, \varphi).
\]

Here \( x = x^a e_a \) and \( z(s) = z^a(s)e_a \) are Minkowski position vectors, and \( n \) is a unit vector orthogonal to \( C \), parametrized in terms of polar angles as

\[
n = N_1(s) \sin \vartheta \cos \varphi + N_2(s) \sin \vartheta \sin \varphi + N_3(s) \cos \vartheta.
\]

The surfaces \( s = \text{const} \). are 3-planes orthogonal to \( C \) at \( x = z(s) \). Within these 3-planes the coordinates \( \{r, \vartheta, \varphi\} \) are standard spherical polar coordinates, with the polar angles measured with respect to the directions defined by the Frenet frame at \( x = z(s) \). We call these coordinates ‘time-symmetric’ rather than ‘instantaneous rest-frame’ or similar because the Frenet frame is Fermi-propagated (non-rotating) along \( C \) only if \( \kappa_2 = \kappa_3 = 0 \). Note also that implicit in (43) is the identification \( N_i(s, r, \vartheta, \varphi) \equiv N_i(s) \), that is, the frame at \( x = x(s, r, \vartheta, \varphi) \) is obtained by parallel transport of \( N_i(s) \) from the point \( x = z(s) \). Thus, the Frenet frame, originally defined only on \( C \), is henceforth considered to be a frame field on Minkowski spacetime.

3.1.3. Retarded (advanced) tube coordinates. Retarded (advanced) tube coordinates are obtained by taking the \( s = \text{const} \). surfaces to be the forward (backward) null cones based on \( C \). The coordinates \( \{s, r, \vartheta, \varphi\} \) are defined by the transformation

\[
x = z(s) + r(n(s, \vartheta, \varphi) + \varepsilon N_0(s)),
\]

where \( \varepsilon = 1 \) gives the retarded coordinate system and \( \varepsilon = -1 \) gives the advanced system, as shown in figure 2. Transformation (45) can also be interpreted as defining a continuous 1-parameter family of tube coordinate systems. Since \( \varepsilon = 0 \) corresponds to the time-symmetric tube coordinates (43), we may deal with all interesting cases at once by carrying \( \varepsilon \in [-1, 1] \) as a parameter through our calculations.

Figure 2. The tube caps are \( s = \text{const} \). surfaces in one of the tube coordinate systems defined by the transformation \( x = z + r(n + \varepsilon N_0) \). Shown here are the null caps for \( \varepsilon = \pm 1 \) and the Dirac-type cap for \( \varepsilon = 0 \) (the \( r = \text{const} \). worldtubes in this figure coincide for different \( \varepsilon \) only because the figure has been drawn for a source in linear motion).

This frame field is only needed in the vicinity of \( C \), where the tube coordinate metric \( g \) is non-singular. For the coordinate transformation (45), which includes (43) for \( \varepsilon = 0 \), one has \( \sqrt{-\det(g)} = r^2 \sin \vartheta (1 + \varepsilon \kappa_1 (1 - \varepsilon^2) \sin \vartheta \cos \varphi) \) near \( C \), so \( g \) is non-singular if \( |\kappa_1| < (1 - \varepsilon^2)^{-1} \).
3.2. Gauss’ formula for the worldtube

We require Gauss’s formula specialized to integrating over the tube volume (7). Writing the boundary of $V(s, r)$ as $\partial V(s, r) = \Sigma_{\text{cap}}(s, r) - \Sigma_{\text{cap}}(0, r) + \Sigma_{\text{tube}}(s, r)$ one has that for any $C^1$ vector field $X$

$$\int_{V(s, r)} \nabla \cdot X \, dV(s, r) = \int_{\Sigma_{\text{cap}}(s, r)} X \cdot d\Sigma_{\text{cap}}(s, r) - \int_{\Sigma_{\text{cap}}(0, r)} X \cdot d\Sigma_{\text{cap}}(0, r) + \int_{\Sigma_{\text{tube}}(s, r)} X \cdot d\Sigma_{\text{tube}}(s, r).$$

(46)

In sections 3.2.2–3.2.4, it is shown that these surface integrals can be evaluated as

$$\int_{\Sigma_{\text{cap}}(s, r)} X \cdot d\Sigma_{\text{cap}}(s, r) = \int_{r' = 0}^{r} \int_{\Omega(s', r')} X \cdot \nu_{\text{cap}} \, d\Omega \, dr',$$

(47)

$$\int_{\Sigma_{\text{tube}}(s, r)} X \cdot d\Sigma_{\text{tube}}(s, r) = \int_{s' = 0}^{s} \int_{\Omega(s, r')} X \cdot \nu_{\text{tube}} \, d\Omega \, ds'.$$

(48)

Here $d\Omega$ is the volume element of the 2-sphere $\Omega(s, r)$. It follows that the orientations of $\Sigma_{\text{tube}}(s, r)$ and the final $(s > 0)$ tube cap $\Sigma_{\text{cap}}(s, r)$ are given by the directions of the normals $\nu_{\text{tube}}$ and $\nu_{\text{cap}}$ respectively, and for (46) to hold these are to be chosen so that $X \cdot \nu$ is positive if $X$ is a vector that points out of $V(s, r)$ (see, for example, section 2.8 in [27] or appendix B.2 in [29]). The initial $(s = 0)$ tube cap is then reverse-oriented with respect to $\nu_{\text{cap}}$ since $X \cdot \nu_{\text{cap}}$ is negative for outward pointing vectors on $\Sigma_{\text{cap}}(0, r)$. This is in accordance with the sign of the corresponding term in (46). The normals are given by (62) and (65).

3.2.1. Energy–momentum conservation. Let $T = T^{\alpha\beta} e_\alpha \otimes e_\beta$ be the conserved stress–energy tensor (3). The four vector fields (for $\alpha = 0, \ldots, 3$) defined by $X^{(\alpha)} = e^\alpha \cdot T = T^{\alpha\beta} e_\beta$ are then divergence free, $\nabla \cdot X^{(\alpha)} = 0$. Applying Gauss’s formula (46) to each of these vector fields gives the four conservation equations,

$$0 = P^{\alpha}_{\text{cap}}(s, r) - P^{\alpha}_{\text{cap}}(0, r) + P^{\alpha}_{\text{tube}}(s, r),$$

(49)

where

$$P^{\alpha}_{\text{cap}}(s, r) = \frac{1}{c} \int_{\Sigma_{\text{cap}}(s, r)} X^{(\alpha)} \cdot d\Sigma_{\text{cap}}(s, r),$$

(50)

$$P^{\alpha}_{\text{tube}}(s, r) = \frac{1}{c} \int_{\Sigma_{\text{tube}}(s, r)} X^{(\alpha)} \cdot d\Sigma_{\text{tube}}(s, r).$$

(51)

The momentum flux 4-vectors are defined by $P^{\alpha}_{\text{cap}} = e_\mu P^{\alpha}_{\text{cap}}$ and $P^{\alpha}_{\text{tube}} = e_\mu P^{\alpha}_{\text{tube}}$. Equations (8) and (9) then follow from (47) and (48), and the energy–momentum conservation law (12) follows from (49).

3.2.2. The directed hypersurface element $d\Sigma$. Suppose we are given a hypersurface $\Sigma$ defined in terms of spherical polar tube coordinates. That is, $\Sigma$ is parametrized by coordinates $y^A, A = 1, 2, 3$ and has equations of the form $(s, r, \theta, \phi) = (f^A(y^A))$. We shall derive here the Frenet frame expression for the directed surface element of $\Sigma$.

In Minkowski coordinates the directed hypersurface element is $d\Sigma = e^\mu d\Sigma_{\mu}$, where the 3-forms $d\Sigma_{\mu}$ are given by

$$d\Sigma_{\mu} = \frac{1}{3!} e_{\mu\nu\delta\gamma} \, dx^\nu \wedge dx^\delta \wedge dx^\gamma.$$
Here $\varepsilon_{\mu\nu\rho}$ is the fully antisymmetric Levi-Civita tensor $(\varepsilon_{0123} = 1$ in Minkowski coordinates). For $\Sigma$ parametrized by coordinates $y^A$, the 1-forms $dx^a$ in (52) can be evaluated as

$$dx^a = \frac{\partial}{\partial y^A} (e^a \cdot x) dy^A = e^a \cdot \frac{\partial x}{\partial y^A} dy^A.$$  (53)

The vectors $\partial x/\partial y^A$ that appear here are to be calculated by differentiating (45), so are naturally obtained as linear combinations of the Frenet frame vectors. Let these linear combinations be

$$\frac{\partial x}{\partial y^A} = N_j X^j_A.$$  (54)

Using (42), equation (53) now becomes

$$dx^a = e^a \cdot N_j X^j_A dy^A = N_j^a X^j_A dy^A.$$  (55)

Again using (42), one has $d\Sigma = N^\mu N_i^\mu d\Sigma_\mu$. Substituting (55) into (52) then gives

$$d\Sigma = \frac{1}{3!} N^\mu \varepsilon_{\mu\nu\rho} N^\nu_a N^\rho_b N^\sigma_c X^a_A X^b_B X^c_C dy^A \wedge dy^B \wedge dy^C,$$

and because the Frenet frame is orthonormal and has the same orientation as the Minkowski frame,

$$\varepsilon_{\mu\nu\rho} N^\mu_a N^\nu_a N^\rho_a N^\sigma_a = \varepsilon_{iabc}.$$  (56)

The hypersurface element therefore simplifies to

$$d\Sigma = N^i \varepsilon_{iabc} X^a_1 X^b_2 X^c_3 dy^1 \wedge dy^2 \wedge dy^3,$$  (57)

which is the required Frenet frame expression.

3.2.3. Directed surface element for a tube cap. The tube cap $\Sigma_{\text{cap}}(s, r)$ is defined by (5) as part of the $s = \text{const}.$ coordinate surface. For $\varepsilon = 0$ the caps are orthogonal to $C$, whereas for $\varepsilon = \pm 1$ they are null cones based on $C$. As parameters for the tube cap, one may take $\{y^A\} = \{r, \vartheta, \varphi\}$. The derivatives $\partial x/\partial y^A$ are found using (45),

$$\frac{\partial x}{\partial r} = n + \varepsilon N_0,$$  (58)

$$\frac{\partial x}{\partial \vartheta} = r(N_1 \cos \vartheta \cos \varphi + N_2 \cos \vartheta \sin \varphi - N_3 \sin \vartheta),$$  (59)

$$\frac{\partial x}{\partial \varphi} = r(-N_1 \sin \vartheta \sin \varphi + N_2 \sin \vartheta \cos \varphi).$$  (60)

From these expressions one can read off the coefficients $X^j_A$, as defined by (54). The directed hypersurface element (57) is found to be

$$d\Sigma = v_{\text{cap}} \, \text{d}\Omega \wedge \text{d}r,$$  (61)

where $\text{d}\Omega = r^2 \sin \vartheta \, \text{d}\vartheta \wedge \text{d}\varphi$ and the normal is

$$v_{\text{cap}} = -(N_0 + \varepsilon n),$$  (62)

where the sign of $v_{\text{cap}}$ has been chosen as described in section (3.2).
3.2.4. Directed surface element for the worldtube. The worldtube $\Sigma_{\text{tube}}(s, r)$ is defined by (4) as part of the $r = \text{const.}$ coordinate surface. As parameters for the worldtube, one may take $\{y^A\} = \{s, \vartheta, \varphi\}$. The derivatives $\partial x/\partial y^A$ that are needed to evaluate the surface element are (59) and (60) together with

$$\frac{\partial x}{\partial s} = N_0 + r(\dot{n} + \varepsilon \kappa_1 N_1),$$

where

$$\dot{n} = (\kappa_1 N_0 + \kappa_2 N_2) \sin \vartheta \cos \varphi + (-\kappa_2 N_1 + \kappa_3 N_3) \sin \vartheta \sin \varphi - \kappa_3 N_2 \cos \vartheta.$$

For the coordinate ordering specified by $\{y^A\} = \{s, \vartheta, \varphi\}$, one finds that (57) gives

$$d\Sigma = -v_{\text{tube}} d\Omega \wedge ds,$$

where the normal

$$v_{\text{tube}} = n + r \kappa_1 \sin \vartheta \cos \varphi (n + \varepsilon N_0)$$

has its sign chosen as described in section (3.2).

4. The stress–energy tensor

For $r > r_0$, equation (3) defines $T$ to be the stress–energy tensor of an electromagnetic field. We take this field to be the sum of a background field $F$ and a prescribed retarded field $F_{\text{ret}}$ for the particle or radiating source that we are interested in,

$$F_{\text{tot}} = F + F_{\text{ret}}.$$

The background field $F$ will include any applied field, any fields generated by other moving charged particles and also any radiation from the source that has been reflected back to the source position by boundaries or material media. Thus, $F$ will appear in the calculations as an unspecified solution of the Maxwell equations, known to be source-free throughout the spacetime region occupied by the particle or system of interest.

The stress–energy tensor corresponding to $F_{\text{tot}}$ is $T_{\text{e.m.}} = T_{\text{e.m.}}^i N_i \otimes N_j$, where the frame components are

$$T_{\text{e.m.}}^{ij} = \frac{1}{\mu_0} \left( F_{\text{tot}}^{ab} F_{\text{tot}}^{ij} \eta_{ab} - \frac{1}{4} \eta^{ij} F_{\text{tot}}^{cd} F_{\text{tot}}^{cd} \eta_{ab} \eta_{cd} \right).$$

For the Lorentz–Dirac charged particle, the field of the radiating source is taken to be the retarded Liénard–Wiechert field.

4.1. The Liénard–Wiechert field

The retarded Liénard–Wiechert potential for the electromagnetic field of a point charge $q$ with worldline $C$ is

$$A(x) = \frac{q}{4\pi \varepsilon_0 c} \frac{\hat{z}_{\text{ret}}(x)}{R(x)},$$

where the vector and scalar fields $\hat{z}_{\text{ret}}(x)$ and $R(x)$ are defined as follows. Let the retarded proper distance $s_{\text{ret}}(x)$ be the scalar field that gives the worldline parameter value of the source-point on $C$ for a retarded field at $x$. Thus, $s_{\text{ret}}(x)$ is defined so that $x$ lies on the forward null cone based at $z(s_{\text{ret}}(x))$, and

$$\xi(x) = x - z(s_{\text{ret}}(x))$$

This last contribution to the background field accounts for changes in spontaneous emission rates and effects thereof, such as bandgaps in photonic crystals.
is therefore a future directed null vector field. Then \( \dot{z}_{\text{ret}}(x) = \dot{z}(s_{\text{ret}}(x)) \) is defined as the unit tangent to \( C \) parallel propagated from the retarded source point \( z_{\text{ret}}(x) = z(s_{\text{ret}}(x)) \). Similarly, one defines \( \ddot{z}_{\text{ret}}(x) = \ddot{z}(s_{\text{ret}}(x)) \). The scalar field \( R \) is the radial distance of \( x \) from the source, as measured in the instantaneous rest-frame of the source at the source-point \( z_{\text{ret}} \).

\[
R = -\dot{z}_{\text{ret}} \cdot \zeta. \tag{70}
\]

The scalar and 3-vector potentials are defined by \( A^\alpha = (\Phi/c, A) \). For a charge at rest, \( \dot{v} = (1, 0, 0, 0) \), and (68) defines \( \Phi \) to be the Coulomb potential.

The electromagnetic field tensor corresponding to (68) is

\[
F_{\text{ret}ij} = A_{\beta,\mu} - A_{\mu,\beta}. \tag{71}
\]

The coordinate derivatives of the potential can be calculated using the relation

\[
\dot{s}_{\text{ret}}(x) \equiv -\zeta R^{-1} = -k_\beta \tag{72}
\]

that follows from differentiating the equation \( \zeta^\alpha \zeta^\nu \eta_{\mu\nu} = 0 \) with respect to \( x^\beta \). Here we have introduced the null vector \( k = \zeta R^{-1} \), satisfying the normalization \( k \cdot \dot{z}_{\text{ret}} = -1 \). The Frenet frame components of the field tensor are then given by \( F_{\text{ret}}^{ij} = F_{\text{ret}ij}\gamma_{\mu}N^\mu_{ij} \). One finds that

\[
F_{\text{ret}}^{ij} = \frac{q}{4\pi \epsilon_0 c^2} \left( \left( \dot{z}_{\text{ret}} k^j - \dot{z}_{\text{ret}} k^i \right)(1 + aR)R^{-2} + \left( \zeta_{\text{ret}} k^j - \zeta_{\text{ret}} k^i \right)R^{-1} \right), \tag{73}
\]

where \( a \) is the acceleration dependent scalar field

\[
a = \ddot{z}_{\text{ret}} \cdot k. \tag{74}
\]

The frame components of the derivative fields \( \zeta_{\text{ret}} = \zeta_{\text{ret}} N_j \) and \( \ddot{z}_{\text{ret}} = \ddot{z}_{\text{ret}} N_j \) can be read from the series expressions (76) and (77) of the following section. Likewise, \( \zeta \) is given by the series (75) and the scalars \( R \) and \( a \) then follow from (70) and (74).

### 4.2. Tube coordinate expressions for retarded fields

Recall that the retarded proper distance \( s_{\text{ret}}(x) \) is such that \( x \) lies on the forward null cone based at \( z(s_{\text{ret}}(x)) \). The retarded tube coordinate system is constructed so that \( s_{\text{ret}}(x) = s \). In any other tube coordinate system, one must solve for \( s_{\text{ret}}(x) \) using the condition that \( \zeta(x) \), given by (69), is a null vector field.

For points sufficiently close to \( C \), the separation between \( s \) and \( s_{\text{ret}}(x) \) will be small. Thus, making use of (45) and writing \( s_{\text{ret}}(x) = s + \Delta \), one expands (69) as

\[
\zeta = r (n + \epsilon N_0) - (\dot{z} \Delta + \ddot{z} \Delta^2/2!) + \ddot{z} \Delta^3/(3!) + O(\Delta^4)
\]

\[
= r (n + \epsilon N_0) - N_0 \Delta - N_1 \kappa_1 \Delta^2/2! - (N_0 \kappa_1^2 + N_1 \kappa_1 + N_2 \kappa_1 \kappa_2) \Delta^3/3! + O(\Delta^4). \tag{75}
\]

Let \( \Delta = \sum_{k=1}^{\infty} c_k r^k \) for coefficients \( c_k = c_k(s, \vartheta, \phi) \) to be determined. Substituting the series for \( \Delta \) into (75) gives \( \zeta \) as a series in \( r \). Then, by forming the series for the null expression \( \zeta \cdot \dot{\zeta} \) and equating coefficients to zero, one obtains a sequence of equations that can be solved in turn for the unknown coefficients \( c_k \). The first coefficient is found to be \( c_1 = \epsilon \pm 1 \). Since the proper distance parameter has been taken to increase to the future, the two solutions for \( c_1 \) give \( s_{\text{ret}}(x) = s - r(1 - \epsilon) + O(r^2) \) and \( s_{\text{adv}}(x) = s + r(1 + \epsilon) + O(r^2) \). That is, the retarded solution corresponds to \( c_1 = \epsilon - 1 \). For brevity, let \( \mu = \epsilon - 1 \). The first few coefficients in the series for \( \Delta \) are then

\[
c_1 = \mu, \quad c_2 = \frac{1}{2} \mu^2 \kappa_1 \sin \vartheta \cos \phi, \quad c_3 = \frac{\mu^3}{24} (-4 + 3\mu) \kappa_1^2 + 3(4 + \mu) (\kappa_1 \cos \varphi \sin \vartheta)^2 + 4 \sin \vartheta (\kappa_1 \kappa_2 \sin \varphi + \kappa_1 \cos \varphi). \]


Observe that in retarded tube coordinates (for which \( \mu = \varepsilon - 1 = 0 \)), one has \( \kappa_0 = 0 \) and thus \( s_{\text{ret}}(x) = s \), as expected.

Using the above solution for \( s_{\text{ret}}(x) = s + \Delta \), the fields \( \dot{s}_{\text{ret}}(x) \) and \( \ddot{s}_{\text{ret}}(x) \), needed in expression (73) for the frame components of the retarded electromagnetic field, can be obtained by expansion in \( r \). One finds that

\[
\dot{s}_{\text{ret}}(x) = N_0 + \mu \kappa_1 r + \frac{\mu^2}{2} \left( N_0 \kappa_1^2 + N_1 \left( \kappa_1^2 \sin \vartheta \cos \varphi + \kappa_1 \right) + N_2 \kappa_1 \kappa_2 \right) r^2 + O(r^3),
\]

\[
\ddot{s}_{\text{ret}}(x) = N_1 \kappa_1 + \mu \left( N_0 \kappa_1^2 + N_1 \kappa_1 + N_2 \kappa_1 \kappa_2 \right) r
\]

\[
+ \frac{\mu^2}{2} \left( N_0 \kappa_1 \left( \kappa_1^2 \sin \vartheta \cos \varphi + 3 \kappa_1 \right) + N_1 \left( \kappa_1^3 - \kappa_1 \kappa_2^2 + \kappa_1 \kappa_1 \sin \vartheta \cos \varphi + \kappa_1 \right) 
\]

\[
+ N_2 \left( \kappa_1^2 \kappa_2 \sin \vartheta \cos \varphi + 2 \kappa_1 \kappa_2 + \kappa_1 \kappa_2 \right) + N_3 \kappa_1 \kappa_2 \kappa_3 \right) r^2 + O(r^3).
\]

The resulting expressions for \( F_{\text{ret}}^{ij} \) can be found in [26].

### 4.2.1. Electromagnetic field in retarded tube coordinates.

The field \( F_{\text{ret}} \) is particularly simple in retarded tube coordinates because in this case

\[
\dot{s}_{\text{ret}} = N_0, \quad \ddot{s}_{\text{ret}} = \kappa_1 N_1, \quad k = n + N_0, \quad R = r, \quad a = \kappa_1 \sin \vartheta \cos \varphi.
\]

The frame components of an electromagnetic field tensor are identified with the electric and magnetic fields in that frame by

\[
(F_{\text{ret}}^{ij}) = \begin{pmatrix}
0 & E_1/c & E_2/c & E_3/c \\
-E_1/c & 0 & B_3 & -B_2 \\
-E_2/c & -B_3 & 0 & B_1 \\
-E_3/c & B_2 & -B_1 & 0
\end{pmatrix},
\]

for example, \( F_{\text{ret}}^{0i} = E_1/c \). The electric and magnetic fields for \( F_{\text{ret}} \) with respect to the Frenet frame defined by the retarded tube coordinate system are easily found from (73) using (78),

\[
E = \frac{q}{4\pi \varepsilon_0} \left( \frac{n}{r^2} + (n \sin \vartheta \cos \varphi - N_1) \frac{\kappa_1}{r} \right),
\]

\[
B = \frac{q}{4\pi \varepsilon_0 c} \left( -N_2 \cos \vartheta + N_3 \sin \vartheta \sin \varphi \right) \frac{\kappa_1}{r}.
\]

The magnetic field in the frame associated with time-symmetric tube coordinates is non-singular. The singular field (81) can be traced to mixing of \( E \) and \( B \) by a boost, of approximate magnitude \( r \kappa_1 \), that relates these two different frame fields.

### 4.3. Expansion of the background field near \( \mathcal{C} \)

The values of the background field \( F = F^{\alpha\beta} e_\alpha \otimes e_\beta = F^{ij} N_i \otimes N_j \) on the worldtube cross-sections \( \Omega(r, s) \) are to be approximated in terms of frame components of the Minkowski derivatives of \( F(x) \) at the point \( z(s) \) on the worldline \( \mathcal{C} \). By Taylor expansion,

\[
F_{\text{ret}}^{ij}(s, r, \vartheta, \varphi) = [F^{ij}]_{r=0} + r \left[ \frac{\partial F^{ij}}{\partial r} \right]_{r=0} + \frac{r^2}{2!} \left[ \frac{\partial^2 F^{ij}}{\partial r^2} \right]_{r=0} + \cdots.
\]
The frame field $N_i(s)$ is independent of $r$, so the radial derivatives can be calculated as
\[
\frac{\partial F^{ij}}{\partial r} = \frac{\partial x^{\mu}}{\partial r} \frac{\partial}{\partial x^{\mu}} (F^{\alpha\beta}) N_i^\alpha N_j^\beta = \frac{\partial}{\partial r} (x^k N_k) (F^{ij}) N_i^\alpha N_j^\beta = \frac{\partial x^k}{\partial r} F^{ij}_k.
\]
where $x^k$ and $F^{ij}_k$ are respectively the frame components of the Minkowski position vector $x = x^\beta e_\beta = x^k N_k$ and the first Minkowski coordinate derivative of $F$. Similarly, for example,
\[
\frac{\partial^2 F^{ij}}{\partial r^2} = \frac{\partial x^k}{\partial r} \frac{\partial x^l}{\partial r} F^{ij}_{kl} + \frac{\partial^2 x^k}{\partial r^2} F^{ij}_k.
\]
From (45) one sees that $\frac{\partial x^k}{\partial r} = N_k \cdot (n + \epsilon N_0) = n^k + \epsilon \delta^k_0$, where the components of $n(s, \vartheta, \phi)$ are given by (44). The second and higher $r$-derivatives of $x^k$ are zero. The required Taylor expansion is therefore
\[
F^{ij}(s, r, \vartheta, \phi) = F^{ij}(s) + r (n^k + \epsilon \delta^k_0) F^{ij}_k(s) + \frac{r^2}{2!} (n^k + \epsilon \delta^k_0) (n^l + \epsilon \delta^l_0) F^{ij}_{kl}(s) + \cdots,
\]
where field values on $C$ are denoted, for example, as $F^{ij}(s) = [F^{ij}(s, r, \vartheta, \phi)]_{r=0} = N_i^\alpha(s) N_j^\beta(s) [F^{\alpha\beta}(x)]_{x=x_i(s)}$.

5. Conclusion

This paper describes a tube method for calculating the self-force on an extended radiating system. The new method is presented for the special case of electromagnetic self-force in Minkowski spacetime, but it seems likely that similar techniques could be used to derive a tube method applicable to a radiating system of finite size in a curved background spacetime. In particular, a finite radius tube method may offer a mathematically sound alternative to having to invoke a mass renormalization in the tube-based gravitational self-force calculations of [20]. In contrast to the classical mass renormalization procedure, all steps in our tube method are physically well motivated and could be justified to any required degree of mathematical rigour.

For finite-size electromagnetic systems in Minkowski spacetime, we find that if the field outside the radiating system is known exactly (for example, as a prescribed multipole expansion) then our tube method gives an exact expression for the self-force. This self-force coincides with that derived by the classical mass renormalization procedure, so in this sense, it has been proved that the renormalization procedure gives correct results in Minkowski spacetime. However, as discussed in section 2.1, the reason why classical mass renormalization works is that the point particle limit (tube radius $r \to 0$) becomes superfluous if renormalization is implemented in full using the Bhabha and Harish-Chandra inflow theorem. In fact, a fully implemented renormalization scheme would be exactly equivalent to our finite radius tube method. Thus, taking $r \to 0$ only serves to introduce mathematical inconsistencies and obscure the underlying physics. The classical mass renormalization procedure is therefore conceptually flawed, in so far as it is associated with a point particle limit.

The finite radius tube method follows from Dirac’s balance relation together with our expression (22) for the momentum flux through the cap of the worldtube. Assuming the weak energy condition for the total energy–momentum tensor $T$, expression (22) can also be used to determine the smallest possible size of an extended radiating system that has an asymptotic
field with a given multipole structure. In section 2.3, this was done for a particle with an asymptotic field equal to that of an electric monopole. We found that if such a particle is governed by the Lorentz–Dirac equation and if its mass and charge are set equal to that of the electron, then this particle is at least a thousand times larger than the length scale of any possible electron structure, else the weak energy condition is violated. On the other hand, the experimental limits on electron structure size are consistent with the weak energy condition as applied to helical worldline models of a classical spinning electron.

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