A remark on the Restricted Isometry Property in Orthogonal Matching Pursuit

Qun Mo and Yi Shen

Abstract—This paper demonstrates that if the restricted isometry constant \( \delta_{K+1} \) of the measurement matrix \( A \) satisfies

\[
\delta_{K+1} < \frac{1}{\sqrt{K+1}},
\]

then a greedy algorithm called Orthogonal Matching Pursuit (OMP) can recover every \( K \)-sparse signal \( x \) in \( K \) iterations from \( Ax \). By contrast, a matrix is also constructed with the restricted isometry constant

\[
\delta_{K+1} = \frac{1}{\sqrt{K}}
\]

such that OMP can not recover some \( K \)-sparse signal \( x \) in \( K \) iterations. This result positively verifies the conjecture given by Dai and Milenkovic in 2009.

Index Terms—compressed sensing, restricted isometry property, orthogonal matching pursuit, sparse signal reconstruction.

I. INTRODUCTION

C OMPRESSIVE sensing is a new type of sampling theory. It shows that it is highly possible to reconstruct sparse signals and images from what was previously believed to be incomplete information [2]. Let \( x \in \mathbb{R}^n \) be a signal, we want to recover it from a linear measurement

\[
Ax = y,
\]

where \( A \) is a given \( m \times n \) measurement matrix. In general, if \( m < n \), the solution of (1) is not unique. To recover \( x \) uniquely, some additional assumptions on \( x \) and \( A \) are needed. We are interested in the case when \( x \) is sparse. Let \( |x|_0 \) denote the number of nonzero entries of \( x \). We say that a vector \( x \) is \( K \)-sparse when \( |x|_0 \leq K \). To recover such a signal \( x \), a natural choice is to seek a solution of the \( l_0 \) minimization problem

\[
\min_x ||x||_0 \quad \text{subject to} \quad Ax = y
\]

where \( A \) and \( y \) are known. To ensure the \( K \)-sparse solution is unique, we would like to use the restricted isometry property introduced by Candès and Tao in [3]. A matrix \( A \) satisfies the restricted isometry property of order \( K \) with the restricted isometry constant \( \delta_K \) if \( \delta_K \) is the smallest constant such that

\[
(1 - \delta_K)||x||_2^2 \leq ||Ax||_2^2 \leq (1 + \delta_K)||x||_2^2
\]

holds for all \( K \)-sparse signal \( x \). If \( \delta_{2K} < 1 \), the \( l_0 \) minimization problem has a unique \( K \)-sparse solution [3]. The \( l_0 \) minimization problem is equal to the \( l_1 \) minimization problem when \( \delta_{2K} < \sqrt{2} - 1 [1] \). Recently, Mo and Li have improved the sufficient condition to \( \delta_{2K} < 0.4931 [8] \).

OMP is an effective greedy algorithm for seeking the solution of the \( l_0 \) minimization problem. Basic references for this method are [6], [10] and [11]. For a given \( m \times n \) matrix \( A \), we denote the matrix with indices of its columns in \( \Omega \) by \( A_\Omega \). We shall use the same way to deal with the restriction \( x_\Omega \) of the vector \( x \). Let \( e_i \) be the \( i \)th coordinate unit vector in \( \mathbb{R}^n \). The iterative algorithm below shows the framework of OMP.

\[
\text{Input:} \quad A, \ y
\]

\[
\text{Set:} \quad \Omega_0 = \emptyset, \ r_0 = y, \ j = 1
\]

while not converge

\[
- \quad \Omega_j = \Omega_{j-1} \cup \arg \max_{i \in [Ae_i]} |\langle Ae_i, r_{j-1} \rangle|
- \quad x_j = \arg \min_{z \in \mathbb{R}^n} ||A_{\Omega_j}z - y||_2
- \quad r_j = y - A_{\Omega_j}x_j
- \quad j = j + 1
\]

end while

\[
\hat{x}_{\Omega_j} = x_j, \ \hat{x}_{\Omega_j^C} = 0
\]

Return \( \hat{x} \)

Davenport and Wakin have proved that \( \delta_{K+1} < \frac{1}{\sqrt{K+1}} \) is sufficient for OMP to recover any \( K \)-sparse signal in \( K \) iterations [3]. Later, Liu and Temlyakov have improved the condition to \( \delta_{K+1} < \frac{1}{\sqrt{K+1} + 1} [7] \). By contrast, Dai and Milenkovic have conjectured that there exist a matrix with \( \delta_{K+1} \leq \frac{1}{\sqrt{K}} \) and a \( K \)-sparse vector for which OMP fails in \( K \) iterations. This conjecture has been confirmed via numerical experiments in [5] for the case \( K = 2 \). The main results of this paper are consist of two parts.

- We prove that

\[
\delta_{K+1} \leq \frac{1}{\sqrt{K+1} + 1}
\]

is sufficient for OMP to exactly recover every \( K \)-sparse \( x \) in \( K \) iterations.

- For any given \( K \geq 2 \), we construct a matrix with

\[
\delta_{K+1} = \frac{1}{\sqrt{K}}
\]

where OMP fails for at least one \( K \)-sparse signal in \( K \) iterations.

II. PRELIMINARIES

Before going further, we introduce some notations. Suppose \( x \) is a \( K \)-sparse signal in \( \mathbb{R}^n \). In the rest of this paper, we
assume that
\[ \mathbf{x} = (x_1, \ldots, x_k, 0, \ldots, 0) \]
where \( x_i \neq 0, \, i = 1, 2, \ldots, k, \, k \leq K \). For a given \( K \)-sparse signal \( \mathbf{x} \) and a given matrix \( \mathbf{A} \), we define
\[ S_i := \langle \mathbf{A} e_i, \mathbf{A} \mathbf{x} \rangle, \quad i = 1, \ldots, n. \]
Denote \( S_0 := \max_{i \in \{1, \ldots, K\}} |S_i| \). The following lemma is useful in our analysis.

**Lemma 2.1:** Suppose that the restricted isometry constant \( \delta_{K+1} \) of a matrix \( \mathbf{A} \) satisfies
\[ \delta_{K+1} < \frac{1}{\sqrt{K+1}}, \]
then \( S_0 > |S_i| \) for \( i > K \).

**Proof:** By Lemma 2.1 in [1], we have
\[ |S_i| = |\langle \mathbf{A} e_i, \mathbf{A} \mathbf{x} \rangle| \leq \delta_{K+1} \| \mathbf{x} \|_2 \quad \text{for all} \quad i > K. \tag{2} \]
For the given \( K \)-sparse \( \mathbf{x} \), we obtain
\[ \langle \mathbf{A} \mathbf{x}, \mathbf{A} \mathbf{x} \rangle = \left\langle \mathbf{A} \sum_{i=1}^{K} x_i \mathbf{e}_i, \mathbf{A} \mathbf{x} \right\rangle \]
\[ = \sum_{i=1}^{K} x_i \langle \mathbf{A} e_i, \mathbf{A} \mathbf{x} \rangle \]
\[ = \sum_{i=1}^{K} x_i S_i. \]
It follows
\[ (1 - \delta_{K+1}) \| \mathbf{x} \|_2^2 \leq \langle \mathbf{A} \mathbf{x}, \mathbf{A} \mathbf{x} \rangle \]
\[ = \sum_{i=1}^{K} x_i S_i \]
\[ \leq S_0 \| \mathbf{x} \|_1 \]
\[ \leq S_0 \sqrt{K} \| \mathbf{x} \|_2. \]
This implies
\[ \frac{(1 - \delta_{K+1}) \| \mathbf{x} \|_2}{\sqrt{K}} \leq S_0. \tag{3} \]
It follows from (2) and (3) that the lemma holds.

III. MAIN RESULTS

This section establishes the main results of this paper.

**Theorem 3.1:** Suppose that \( \mathbf{A} \) satisfies the restricted isometry property of order \( K + 1 \) with the restricted isometry constant
\[ \delta_{K+1} < \frac{1}{\sqrt{K+1}}, \]
then for any \( K \)-sparse signal \( \mathbf{x} \), OMP will recover \( \mathbf{x} \) from \( \mathbf{y} = \mathbf{A} \mathbf{x} \) in \( K \) iterations.

**Proof:** Consider the first iteration, the sufficient condition for OMP choosing an index from \( \{1, \ldots, K\} \) is
\[ S_0 > |S_i| \quad \text{for all} \quad i > K. \]
By Lemma 2.1 \( \delta_{K+1} < \frac{1}{\sqrt{K+1}} \) guarantees the success of the first iteration. OMP makes an orthogonal projection in each iteration. By induction, it can be proved that OMP selects a different index from \( \{1, 2, \ldots, K\} \) in each iteration. \( \blacksquare \)

**Theorem 3.2:** For any given positive integer \( K \geq 2 \), there exist a \( K \)-sparse signal \( \mathbf{x} \) and a matrix \( \mathbf{A} \) with the restricted isometry constant
\[ \delta_{K+1} = \frac{1}{\sqrt{K}}, \]
for which OMP fails in \( K \) iterations.

**Proof:** For any given positive integer \( K \geq 2 \), let
\[ \mathbf{A} = \begin{pmatrix} I_K & \frac{1}{\sqrt{K}} & \cdots & \frac{1}{\sqrt{K}} \\ \frac{1}{\sqrt{K}} & \frac{\sqrt{K}}{K} & \cdots & \frac{1}{\sqrt{K}} \\ & \cdots & \cdots & \cdots \\ \frac{1}{\sqrt{K}} & \frac{1}{\sqrt{K}} & \cdots & \frac{\sqrt{K}}{K} \end{pmatrix}_{(K+1) \times (K+1)}. \]
By simple calculation, we get
\[ \mathbf{A}^T \mathbf{A} = \begin{pmatrix} I_K & \frac{1}{\sqrt{K}} & \cdots & \frac{1}{\sqrt{K}} \\ \frac{1}{\sqrt{K}} & \frac{\sqrt{K}}{K} & \cdots & \frac{1}{\sqrt{K}} \\ & \cdots & \cdots & \cdots \\ \frac{1}{\sqrt{K}} & \frac{1}{\sqrt{K}} & \cdots & \frac{\sqrt{K}}{K} \end{pmatrix}_{(K+1) \times (K+1)} \]
where \( \mathbf{A}^T \) denotes the transpose of \( \mathbf{A} \). It is obvious that the eigenvalues \( \{\lambda_i\}_{i=1}^{K+1} \) of \( \mathbf{A}^T \mathbf{A} \) are
\[ \lambda_1 = \cdots = \lambda_{K-1} = 1, \quad \lambda_K = 1 - \frac{1}{\sqrt{K}} \quad \text{and} \quad \lambda_{K+1} = 1 + \frac{1}{\sqrt{K}}. \]
Therefore, the restricted isometry constant \( \delta_{K+1} \) of \( \mathbf{A} \) is \( \frac{1}{\sqrt{K}} \). Let
\[ \mathbf{x} = (1, 1, \ldots, 1, 0)^T \in \mathbb{R}^{K+1}. \]
We have
\[ S_i = \langle \mathbf{A} e_i, \mathbf{A} \mathbf{x} \rangle = 1 \quad \text{for all} \quad i \in \{1, \ldots, K+1\}. \]
This implies OMP fails in the first iteration. Since OMP chooses one index in each iteration, we conclude that OMP fails in \( K \) iterations for the given matrix \( \mathbf{A} \) and the vector \( \mathbf{x} \). \( \blacksquare \)

**Remark 3.3:** It is challenging to design a measurement matrix having a very small restricted isometry constant \( \delta_{K+1} \); and Theorem 3.2 shows that this kind of requirement is necessary. However, if we select multiple indices per iteration, we can recover the \( K \)-sparse signal given in Theorem 3.2 in \( K \) iterations. Actually, this technique has been widely used in many related greedy pursuit algorithms, such as CoSaMP [9] and Subspace Pursuit algorithm [4].

REFERENCES

[1] E. J. Candès, “The restricted isometry property and its implications for compressed sensing,” C. R. Math. Acad. Sci., Ser. I, vol. 346, pp. 589-592, 2008.
[2] E. J. Candès, J. Romberg, and T. Tao, “Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information,” IEEE Trans. Inform. Theory, vol. 52, no.2, pp. 489-509, 2006.
[3] E. J. Candès and T. Tao, “Decoding by linear programming,” IEEE Trans. Inform. Theory, vol. 51, no.12, pp. 4203-4215, 2005.
[4] W. Dai and O. Milenkovic, “Subspace pursuit for compressive sensing signal reconstruction,” IEEE Trans. Inform. Theory, vol. 55, no. 5, pp. 2230-2249, 2009.
[5] M. A. Davenport and M. B. Wakin, “Analysis of orthogonal matching pursuit using the restricted isometry property,” IEEE Trans. Inform. Theory, vol. 56, no. 9, pp. 4395-4401, 2010.
[6] G. Davis, S. Mallat, and M. Avellaneda, “Adaptive greedy approximation,” *J. Constr. Approx.*, vol. 13, pp. 57-98, 1997.

[7] E. Liu and V. N. Temlyakov, “Orthogonal super greedy algorithm and applications in compressed sensing,” preprint, 2010.

[8] Q. Mo and S. Li, “New bounds on the restricted isometry constant $\delta_{2k}$,” *Appl. Comput. Harmon. Anal.*, vol. 31, no. 3, pp. 460-468, 2011.

[9] D. Needell and J. A. Tropp, “CoSaMP: Iterative signal recovery from incomplete and inaccurate samples,” *Appl. Comput. Harmon. Anal.*, vol. 26, no. 3, pp. 301-321, 2009.

[10] Y. C. Pati, R. Rezaiifar, and P. S. Krishnaprasad, “Orthogonal Matching Pursuit: Recursive function approximation with applications to wavelet decomposition,” in *Proc. 27th Ann. Asilomar Conf. on Signals, Systems and Computers*, Nov. 1993.

[11] J. A. Tropp, “Greed is good: Algorithmic results for sparse approximation,” *IEEE Trans. Inform. Theory*, vol. 50, no. 10, pp. 2231-2242, 2004.

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