FIBONACCI NUMBERS AND TRIVALENT GRAPHS

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ABSTRACT. We study the Fibonacci and Lucas numbers and demonstrate how identities can be constructed by investigating trivalent graphs (an infinite graph with all vertices of degree 3) and splitting fields.

1. INTRODUCTION

This article is a continuation of [7]. The main purpose of the present article is to demonstrate how identities can be constructed. The second section is devoted to the study of a trivalent graph associated to the Fibonacci numbers. It turns out that many identities (old and new) become visible by applying our technique. The third section studies the splitting field of the characteristic polynomial of certain recurrences and such study again explains the existence of certain identities. The fourth section studies the identities of the subsequences of the Fibonacci sequence. To the best of our knowledge, our way of finding the following third-order identity (see Proposition 2.5) is not in the literature yet.

Proposition 2.5.

\[ F_n^3 = \frac{(F_{n-1}F_{n+1}F_{n+2} + F_{n-2}F_{n-1}F_{n+1})}{6} + \frac{(F_{n-2}F_{n+1}F_{n+2} - F_{n-2}F_{n-1}F_{n+2})}{2}. \] (1.1)

It seems that the technique we presented in the article will enable us to find more identities (see section 2.5).

2. FIBONACCI NUMBERS AND TRIVALENT GRAPHS

Let \( e_1, e_2, e_3 \) be arbitrary vectors placed in the following trivalent graph and let \( x \) be the vector given by

\[ x = 2(e_3 + e_2) - e_1. \] (2.1)

Such a vector \( x \) is said to be \( F \)-generated by \( e_3, e_2, \) and \( e_1 \) (in this order). Applying (2.1), we may construct an infinite sequence as follows:

\[ e_1, e_2, e_3, e_4 = 2(e_3 + e_2) - e_1, \ldots, e_{n+1} = 2(e_n + e_{n-1}) - e_{n-1}, \ldots. \] (2.2)

Denoted by \( F(e_1, e_2, e_3) \) the above sequence. In this section, we shall assign special values for \( e_1, e_2 \) and \( e_3 \) and present a very visual method to display various identities concerning the Fibonacci numbers. Our work is inspired by Conway [2] that makes the values of binary quadratic forms visible. To put our idea in short, the trivalent graph we defined brings the Fibonacci numbers that reveal certain identities close to one another and this makes such identities visible (see Proposition 2.2 for example).
2.1. **Trivalent graph for** $(1, 0, 0), (2, 1, 0)$ and $(2, 2, 1)$. We shall first recall some results in [7]. A function $x(n) : Z \rightarrow C$ is called an $F$-function if $x(n)$ satisfies the following recurrence relation.

$$x(n + 3) = 2(x(n + 2) + x(n + 1)) - x(n).$$

One can show by induction that $x(n) = (-1)^n$, $x(n) = F_{n+r}^2$ and $x(n) = L_{n+r}$, where $F_n$ and $L_n$ are the $n$-th Fibonacci and Lucas numbers respectively. Further, if $A(n)$ and $B(n)$ are $F$-functions, then $A(n) \pm B(n)$ are $F$-functions. Note that composition and product of $F$-functions are not $F$-functions. The following is clear.

**Lemma 2.1.** Let $A(n)$ and $B(n)$ be $F$-function. Suppose that $A(k) = B(k)$ for $k = 0, 1$ and $2$. Then $A(n) = B(n)$.

**Proposition 2.2.** Let $F_n$ and $L_n$ be the $n$-th Fibonacci and Lucas number. Then $F_{n-1}^2 - F_{n-4}F_n - F_{n+1}F_{n-2} = F_{n-2}^2$.

**Proof.** Recall first that $L_m = (-1)^m L_m$ and $F_m = (-1)^{m+1} F_m$. We shall assume that $n \geq 0$. The case $n \leq 0$ can be proved similarly. Denoted by $A(n)$ and $B(n)$ the right and left hand side of our identity. Since both $A(n)$ and $B(n)$ are $F$-functions and $A(k) = B(k)$ for $k = 0, 1, 2$, we may apply Lemma 2.1 and conclude that $A(n) = B(n)$. □

**Discussion.** We shall now explain how to see Proposition 2.2 as follows. It is a trivalent graph that reveals the identity. Let $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$. The first nine terms of $F(e_1, 2e_1 + e_2, 2e_1 + 2e_2 + e_3)$ are given as follows.

```
   2   7   44  298
  1  6  40  273
 10 15 104 778
```

One sees the following interesting pattern from the first nine terms of our sequence.

(i) The second entry is a product of two Fibonacci numbers $F_n F_{n+1}$.

(ii) The difference of the first and second entries of every vector is the square of a Fibonacci number $F_{n-2}^2$.

(i) and (ii) of the above suggest us to express the first entry in terms of Fibonacci and/or Lucas numbers and we did. It is given as follows.

(iii) The first entry of every vector is of the form $L_{n-1}^2 - F_{n-4}F_n$, where $L_k$ is the $k$-th Lucas number.

(i), (ii) and (iii) of the above implies that for the first nine vectors of the above graph, one has

$$L_{n-1}^2 - F_{n-4}F_n - F_n F_{n+1} = F_{n-2}^2,$$

which leads us the Proposition 2.2. Note also that

(iv) The sum of the third entry (product of two Fibonacci numbers) of the first $2k + 1$ terms is a square of a Fibonacci number.

(v) the second and third entries of every vector are product of Fibonacci numbers and their sum is another Fibonacci number.
2.2. **Trivalent graph for** \((1, 2), (2, 1)\) **and** \((1, 2)\). We shall present in this section two identities involved sum of Fibonacci numbers. As the proof of such identities is elementary, we shall present how such identities can be seen rather than how such identities can be proved.

**Proposition 2.3.**

(i) \(3(F_4 + F_{12} + F_{20} + \cdots + F_{4+8n}) = F_{4n+4}^2\).

(ii) \(1 + 3(F_6 + F_{14} + F_{22} + \cdots + F_{6+8n}) = F_{4n+5}^2\).

**Discussion.** Let \(e_1 = (1, 0)\), \(e_2 = (0, 1)\) and let \(u_1 = e_1 + 2e_2, u_2 = 2e_1 + e_2, u_3 = e_1 + 2e_2\). The first ten vectors \(u_1, u_2, \ldots, u_{10}\) of \(F(u_1, u_2, u_3)\) are given as follows. Note that \(e_1 + 2e_2\) appears twice in the beginning.

\[
\begin{array}{ccccccc}
{e_1 + 2e_2} & {e_1 + 2e_2} & {10} & {11} & {73} & {74} & {505} & {506} & \ldots \\
{2e_1 + e_2} & {5} & {4} & {29} & {28} & {194} & {193} & {1325} & {1324}
\end{array}
\]

One sees that

(i) The differences \(d_i\)'s of the entries of the vectors \((u_5, u_3), (u_9, u_7), (u_{13}, u_{11}), \cdots\) of the top half of the graph are three times of some Fibonacci numbers. The first three differences are \(10 - 1 = 9 = 3F_4\), \(505 - 73 = 432 = 3F_{12}\) and \(23761 - 3466 = 20295 = 3F_{20}\).

(ii) The sum of consecutive \(d_i\)'s are squares. \(9 = 3^2, 9 + 432 = 21^2, 9 + 432 + 20295 = 144^2\).

One sees that (i) and (ii) give us

\[3F_4 + 3F_12 = F^2_8, \quad 3F_4 + 3F_{12} + 3F_{20} = F^2_{12}, \tag{2.5}\]

which leads us to (i) of Proposition 2.3.

We now observe the bottom half of the above graph. The entries of the first few vectors gives the following.

\[1 + 3F_6 + 3F_{14} = (-1 + 2) + (-5 + 29) + (-194 + 1325) = 34^2 = F^2_5, \tag{2.6}\]

which leads us to (ii) of Proposition 2.3.

2.3. **Trivalent graph of** \((1, 0, -1), (2, 1, 1)\) **and** \((2, 1, 4)\). Similar to sections 2.1 and 2.2, the following identity can be obtained by study the trivalent graph of \((1, 0, -1), (2, 1, 1)\) and \((2, 1, 4)\).

**Proposition 2.4.** \(L_{2n+1}^2 - F_{n+1}^2 - (L_n^2 - F_{n-3}F_{n+1} + F_{2n-2}) = (-1)^{n-1}.\)

2.4. **More identities.** We list in this section a few identities that can be seen by studying certain trivalent graphs. Let \(u_1, u_2, u_3\) be three vectors. The identity generated by \(F(u_1, u_2, u_3)\) is given as follows.
(c1) \((1, 1, 2) (0, 1, -1) (1, 0, 2)\) : \(F_{2n} = F_{n+1}^2 - F_n^2\)

(c2) \((1, 1, 2) (0, 1, -1) (1, 0, 2)\) : \(F_{2n+1} = F_{n+1}^2 + F_n^2\)

(c3) \((1, 1, 2) (0, 1, -1) (1, 0, 2)\) : \(F_{n+2}F_{n-1} = F_{n+1}^2 - F_n^2\)

(c4) \((1, 1, 2) (0, 1, -1) (1, 0, 2)\) : \(F_n^2 - F_n - 2F_{n+2} = (-1)^n\)

(c5) \((0, 1, 0) (1, 0, 1) (1, 2, 3)\) : \(\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}\)

(c6) \((0, 1, 0) (1, 0, 1) (1, 2, 3)\) : \(F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1}\)

(c7) \((1, -1, 0) (0, 1, 0) (0, 0, 1)\) : \(F_n F_{n+3} = F_{n+1} F_{n+2} + (-1)^n\)

(c8) \((1, 1, 1) (0, 1, 0) (0, 0, 1)\) : \(F_n^2 - F_{n-1}^2 = F_n F_{n-1} + (-1)^n\)

(c9) \((-1, 0, 0) (3, 1, 0) (0, 0, 1)\) : \(F_{2n+1} + (-1)^n = F_{n-1} F_{n+1} + F_{n+1}^2\)

**Proof.** Note first that the left and right hand side of (c1)-(c9) are all \(F\)-functions. One may now prove the identities by applying Lemma 2.1. \(\square\)

### 2.5 Discussion

We have presented in this section how identities about Fibonacci numbers (old and new) can be seen by drawing a simple trivalent graph. Note that the graph we use to generate various identities here is just a subgraph of an infinite trivalent graph (see Appendix A of [7]). Such infinite trivalent graph may generate more identities. We are currently investigating this infinite graph and the related identities. Note that the vectors \(e_i\)'s in this article are vectors in \(\mathbb{R}^2\) or \(\mathbb{R}^3\) and the nonidentity constant we use in the recurrence relation \(e_n = c(e_{n-1} + e_{n-2}) - e_{n-3}\) is \(c = 2\). All these invariance can be replaced by other choices. For instance,

\[
e_n = 5e_{n-1} + 15e_{n-2} - 15e_{n-3} - 5e_{n-4} + e_{n-5}.
\]

(2.7)

Note that \(F_n^2\) satisfies the recurrence (2.1) and \(F_n^4\) satisfies the recurrence (2.7). In (2.7), let \(e_1 = (0, 0, 0, 1), e_2 = (0, 0, 1, 0), e_3 = (0, 0, 0, 0), e_4 = (0, 1, 0, 0)\) and \(e_5 = (1, 0, 0, 0)\). The first nine \(e_i\)'s are

\[
\begin{array}{cccccccc}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \\
0 & 0 & 0 & 0 & 1 & 5 & 40 & 260 & 1820 \\
0 & 0 & 0 & 1 & 0 & 15 & 60 & 520 & 3276 \\
0 & 1 & 0 & 0 & 0 & -5 & -24 & -195 & -1260 \\
1 & 0 & 0 & 0 & 0 & 1 & 5 & 40 & 260
\end{array}
\]

One sees easily that for each \(e_i = (a, b, -c, d), 6a, 2b, 2c\) and \(6d\) are product of four Fibonacci numbers. The expression \(a + b - c + d\) is a fourth power of a Fibonacci number. Similar to Propositions 2.2-2.4, one has the following third-order identity.

**Proposition 2.5.**

\[
F_n^3 = (F_{n-1}F_{n+1} + F_{n-2}F_{n-1}F_{n+1})/6 + (F_{n-2}F_{n+1} - F_{n-2}F_{n+1}F_{n+2})/2.
\]

(2.8)
To the best of our knowledge, the way we construct (2.8) is not in the literature yet. Note that our approach towards the study of the identities is computer-free and our technique always makes the identities visible. However, it is perhaps possible to implement this idea in a computer to visualise more identities. The recurrence relation one should consider is (see [5])

\[ e_{n+1} = - \sum_{r=1}^{n+1} (-1)^{r(r+1)/2} \frac{F_{n+1}F_n \cdots F_{n+2-r}}{F_rF_{r-1} \cdots F_1} e_{n+1-r}. \]  

(2.9)

The interested readers may want to apply the technique of induction and/or Binet’s formula to obtain a general formula for \( F_n^m \).

3. Characteristic polynomials and Splitting fields

Let \( \{u_0, u_1, u_2, \cdots \} \) be a sequence defined by the following recurrence relation \( u_0 = 0, u_1 = 1, \)

\[ u_r = pu_{r-1} - qu_{r-2}. \]  

(3.1)

where \( p \) and \( q \) are rational numbers. The characteristic polynomial of \( \{u_n^p\} \) is defined to be (see for examples, [1], [3], [4], [6])

\[ \Phi_n(p, q, x) = \sum_{i=0}^{n} (-1)^i q^{i(i-1)/2} (n|i) u^i, \]  

(3.2)

where \( (n|k)_u \) is the generalised binomial coefficient (see Appendix A). Note that \( (n|0)_u = 1 \) and \( (n|k)_u = u_nu_{n-1} \cdots u_{n-k+1}/u_ku_{k-1} \cdots u_1 \) for \( 1 \leq k \leq n \) if \( u_1u_2 \cdots u_k \neq 0 \). Let \( \sigma \) and \( \tau \) be roots of \( x^2 - px + q = 0. \) It is well known that

\[ \Phi_n(p, q, x) = \prod_{j=0}^{n} (x - \sigma^j \tau^{n-j}). \]  

(3.3)

In this section, we shall associated to the above factorisation some identities, which we propose to call them the Galois Identities of \( \Phi_n(p, q, x) \). These identities (see Proposition 3.4 and Corollary 3.6) must be well known among the experts. However, our interpretation of the existence of these identities maybe of some interest (see Discussion 3.7).

3.1. Fibonacci and Lucas numbers and \( L_n^2 - 5F_n^2 = 4(-1)^n. \) In (3.1), \( p = 1 \) and \( q = -1 \) give the Fibonacci numbers. Applying (3.3), it is clear that the Galois group of \( \Phi_n(1, -1, x) \) over \( \mathbb{Q} \) is \( \mathbb{Z}_2 \) and the splitting field of \( \Phi_n(1, -1, x) \) is \( \mathbb{Q}(\sqrt{5}) \). Recall another well known fact about the factorisation of \( \Phi_n(1, -1, x). \)

\[ \Phi_n(1, -1, x) = (-1)^n(x^2 - L_n x + (-1)^n)\Phi_{n-2}(1, -1, x). \]  

(3.4)

Since \( \Phi_n(1, -1, x) \) splits in \( \mathbb{Q}(\sqrt{5}) \) and \( x^2 - L_n x + (-1)^n = 0 \) splits in \( \mathbb{Q}(\sqrt{L_n^2 - 4(-1)^n}) \), one must have

\[ L_n^2 - 4(-1)^n = 5A_n^2, \]  

(3.5)

for some \( A_n \in \mathbb{N} \). This tells us that the difference between \( 4(-1)^n \) and the square of the Lucas number \( L_n \) must be five times a square \( A_n^2 \). As for why \( A_n \) must be \( F_n \), we note that both \( F_n^2 \) and \( L_n^2 \) satisfy the recurrence relation (2.3) and that \( L_n^2 - 4(-1)^n \) and \( F_n^2 \) have the
same initial values (see Lemma 2.1). As a consequence, we have just recover the following well
known identity by investigating the splitting field of \( \Phi_n(1, -1, x) \).

\[
L_n^2 - 4(-1)^n = 5F_n^2. \tag{3.6}
\]

**Discussion.** The above investigation suggests that (3.6) is not just a numerical coincidence
and can be viewed as the consequence of the fact that the Galois group of \( \Phi_n(1, -1, x) \) is \( \mathbb{Z}_2 \)
and that the splitting field of \( \Phi_n(1, -1, x) \) is \( \mathbb{Q}(\sqrt{5}) = \mathbb{Q}(\sqrt{L_n^2 + 4(-1)^n}) \).

**Remark.** It follows easily from (3.6) that the following identities hold, the first identity is
proved by Freitag and the second by Zeitlin and Filipponi (independently). See [4] for more
detail.

\[
\frac{L_n^2 - (-1)^a L_{n+a}^2}{F_n - (-1)^a F_{n+a}^2} = 5, \quad \frac{L_n^2 + L_{n+2a}^2 + 8(-1)^n}{F_n + F_{n+2a}^2} = 5. \tag{3.7}
\]

### 3.2. The Galois Identities of \( w_r = pw_{r-1} - qw_{r-2} \)

By (3.3), the splitting field of \( \Phi_n(p, q, x) \)
is \( \mathbb{Q}(\sqrt{p^2 - 4q}) \) and the Galois group of \( \Phi_n(p, q, x) \) is \( \mathbb{Z}_2 \) if and only if \( p^2 - 4q \) is not a perfect square in \( \mathbb{Q} \).

**Definition 3.1.** Let \( \sigma \) and \( \tau \) be given as in (3.3). Define \( \{w_n\} \) to be the sequence \( w_n = \sigma^n + \tau^n \).

**Lemma 3.2.** \( w_0 = 2, w_1 = p, w_r = u_{r+1} - qu_{r-1} \) and \( w_r = pw_{r-1} - qw_{r-2} \). Suppose that \( p^2 - 4q \neq 0 \). Then \( u_r = (w_{r+1} - qw_{r-1})/(p^2 - 4q) \).

**Proof.** It is clear that \( w_0 = 2 \) and that \( w_1 = p \). Applying Binet’s formula, one has \( w_r = u_{r+1} - qu_{r-1} \) and that \( w_r \) satisfies the recurrence \( w_r = pw_{r-1} - qw_{r-2} \).

\[
w_r = u_{r+1} - qu_{r-1} = pu_r - 2qu_{r-1}, \quad w_r-1 = w_r - qu_{r-2} = 2u_r - pw_{r-1}, \tag{3.8}
\]

we conclude that \( u_r = (w_{r+1} - qw_{r-1})/(p^2 - 4q) \). This completes the proof of the lemma. \( \square \)

**Lemma 3.3.** \( q^n, w_n^2 \) and \( u_n^2 \) satisfy the following recurrence

\[
X(m+3) = (p^2 - q)X(m+2) + (q^2 - p^2q)X(m+1) + q^3X(m). \tag{3.9}
\]

**Proof.** Applying (3.1), one can show easily that \( u_n^2 \) satisfies the recurrence (3.9). The rest can be verified similarly. \( \square \)

Suppose that \( p^2 - 4q \) is not a square in \( \mathbb{Q} \). Applying Galois Theory, the set of conjugates of \( \sigma^n \) over \( \mathbb{Q} \) is \( \{\sigma^n, \tau^n\} \). Hence the following holds for every \( n \in \mathbb{N} \).

\[
f_n(x) = (x - \sigma^n)(x - \tau^n) = x^2 - w_nx + q^n \in \mathbb{Q}[x]. \tag{3.10}
\]

It is clear that \( f_n(x) \) splits in \( \mathbb{Q}(\sqrt{w_n^2 - 4q^n}) \). Applying (3.3), \( f_n(x) \) splits in \( \mathbb{Q}(\sqrt{p^2 - 4q}) \). Hence

\[
w_n^2 - 4q^n = z^2(p^2 - 4q), \tag{3.11}
\]

for some \( z \in \mathbb{Q} \). The following proposition shows that the solution of (3.11) is \( z = u_n \), the recurrence we defined in (3.1).

**Proposition 3.4.** Let \( w_n \) be given as in Definition 3.1. Then \( w_n^2 - 4q^n = u_n^2(p^2 - 4q) \).

**Proof.** Since both \( w_n^2 - 4q^n \) and \( u_n^2(p^2 - 4q) \) satisfy the recurrence (3.9) and admit the same initial values, we have \( w_n^2 - 4q^n = u_n^2(p^2 - 4q) \). \( \square \)
The equation $x^2 + y^2 - z^2 = 4$ is solvable in $\mathbb{Z}$. Further, one may choose $y$ and $z$ in such a way that $py = 2z$ for any $p \in \mathbb{Z}$.

Proof. Let $q = 1$ and let $p$ be any integer. Applying Proposition 3.4, one has $w_n^2 + (2u_n)^2 - (pu_n)^2 = 4$ for all $n \geq 1$.

Corollary 3.6. $w_{2n} - 2q^n = u_n^2(p^2 - 4q)$. In particular, $L_{2n} - 2(-1)^n = 5F_n^2$.

Proof. Applying Definition 3.1, one has $w_n^2 = w_{2n} + 2q^n$. □

Discussion 3.7. The identities in Proposition 3.4 and Corollary 3.6 must be well known and we propose to call them the Galois identities associated to $\Phi_n(p, q, x)$. We would like to emphasise that Proposition 3.4 is not just a numerical coincidence but can be treated as the consequence of the fact that the splitting field of $\Phi_n(p, q, x)$ is $\mathbb{Q}(\sqrt{p^2 - 4q}) = \mathbb{Q}(\sqrt{w_n^2 - 4q^2})$. The most famous identity among all, of course, is (3.6).

4. Subsequences of the Fibonacci sequence

One sees easily that

$$F_{n+r} = L_r F_n + (-1)^{r+1} F_{n-r}. \quad (4.1)$$

The above is known as the multiple angle recurrence (see [8]). In the case $r = 2$, (4.1) gives $F_{n+2} = 3F_n - F_{n-2}$. It follows that both $x_n = F_{2n}$ and $y_n = F_{2n+1}$ ($\{x_n\}$ and $\{y_n\}$ are subsequences of the Fibonacci sequence) satisfy the recurrence $w_n = 3w_{n-1} - w_{n-2}$. Applying Lemma 3.3,

$$w_n^2 = 8w_{n-1}^2 - 8w_{n-2}^2 + w_{n-3}^2. \quad (4.2)$$

As a consequence,

$$F_{2n+k}^2 = 8F_{2(n-1)+k}^2 - 8F_{2(n-2)+k}^2 + F_{2(n-3)+k}^2 \quad (4.3)$$

where $k = 0$ or 1. Note that $r$ can be any integer and the idea can be generalised to higher order products.

5. Appendix A

Let $\sigma$ and $\tau$ be roots of $x^2 - px + q = 0$ (known as the characteristic polynomial of (3.1)). It is well known that

$$u_r = \sum_{i=1}^{r-1} \sigma^{r-i} \tau^i = f_r(\sigma, \tau), \quad (A1)$$

where $f_r(x, y)$ is the polynomial $\sum x^{r-1-i} y^i = (x^r - y^r)/(x - y)$. Let $G(z)$ be the following function.

$$G(z) = (1 - z^{m+n})(1 - z^{m+n-1}) \cdots (1 - z^{m+1}/(1 - z^n)(1 - z^{n-1}) \cdots (1 - z). \quad (A2)$$

$G(z)$ is known as the Gaussian binomial coefficient and is a polynomial in $z$ (a more powerful result actually implies that (A2) can be written as product of cyclotomic polynomials). One may apply this fact to show that that $(f_r f_r f_r \cdots f_{r-k+1})/(f_k f_k \cdots f_1) \in \mathbb{Z}[x, y]$ is a polynomial in $x$ and $y$. Denoted by $F(r, k, x, y)$ this polynomial. Define

$$(r|k)_u = F(r, k, \sigma, \tau). \quad (A3)$$
We call \((A3)\) the generalised binomial coefficient. It is clear that if \(u_1u_2 \cdots u_k \neq 0\). Then \((n|k)_u\) takes the following form.

\[
(n|k)_u = u_nu_{n-1} \cdots u_{n-k+1}/u_ku_{k-1}\cdots u_1. 
\]  

(A4)

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