Morita–Rieffel Equivalence and Spectral Theory for Integrable Automorphism Groups of C*-Algebras

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ABSTRACT. Given a C*-dynamical system \((A, G, \alpha)\), we discuss conditions under which subalgebras of the multiplier algebra \(M(A)\) consisting of fixed points for \(\alpha\) are Morita–Rieffel equivalent to ideals in the crossed product of \(A\) by \(G\). In case \(G\) is abelian we also develop a spectral theory, giving a necessary and sufficient condition for \(\alpha\) to be equivalent to the dual action on the cross-sectional C*-algebra of a Fell bundle. In our main application we show that a proper action of an abelian group on a locally compact space is equivalent to a dual action.

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1. Introduction.

The object under study in the present work is a C*-dynamical system \((A, G, \alpha)\), that is, a strongly continuous action \(\alpha\) of a locally compact topological group \(G\) on a C*-algebra \(A\). The questions we discuss fall broadly in two categories, the first one being Rieffel’s project, initiated in [R4] and recently continued in [R5], of defining a generalized fixed point algebra, under suitable hypothesis, and proving it to be Morita–Rieffel\(^1\) equivalent to an ideal in the reduced crossed product algebra. The second issue we discuss is spectral theory for \(G\) abelian.

To describe what we mean by spectral theory suppose that \(G\) is not only abelian but also compact. In this very simple case it is easy to show (see e.g. [E1: 2.5]) that \(A\) may be decomposed as the closure of the direct sum \(\bigoplus_{x \in \hat{G}} B_x\), where for each \(x\) in the Pontryagin dual \(\hat{G}\) of \(G\), the spectral subspace \(B_x\) is the closed subspace of \(A\) formed by the elements \(a \in A\) such that \(\alpha_t(a) = \langle x, t \rangle a\), for all \(t \in G\). As usual we denote by \(\langle x, t \rangle\) the duality between \(\hat{G}\) and \(G\).

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1 Given the enormous contribution made by Rieffel on the subject of Morita equivalence in the context of C*-algebras, I believe that the term “strong Morita equivalence” should be re-coined “Morita–Rieffel equivalence”. In fact I was told that similar terms were already employed in the 1998 Great Plains Operator Theory Symposium (GPOTS) held at Kansas State University.
By spectral theory we mean a theory encompassing generalizations of the above decomposition including situations where $G$ is not compact. Returning for a moment to the compact case, one may prove that the spectral subspaces $B_x$ defined above satisfy
\[ B_x B_y \subseteq B_{xy}, \quad \text{and} \quad B_x^* = B_{x^{-1}}, \]
for all $x$ and $y$ in $\hat{G}$. It therefore follows that the collection $B = \{B_x\}_{x \in \hat{G}}$ forms a Fell bundle over the discrete group $\hat{G}$ (see [FD] for a comprehensive treatment of the theory of Fell bundles, also referred to as $C^*$-algebraic bundles). One may then prove [E4: 4.7] that $A$ is isomorphic to $C^*(B)$, the cross-sectional $C^*$-algebra of $B$ [FD: VIII.17.2]. Moreover the natural isomorphism between $A$ and $C^*(B)$ turns out to be covariant for the dual action [E2: Section 5] of $G$ on $C^*(B)$. In short, every compact abelian action is equivalent to a dual action.

If $G$ is abelian but no longer compact one can still consider the dual action of $G$ on the cross-sectional $C^*$-algebra of any given Fell bundle over the (no longer discrete) dual group $\hat{G}$. We therefore take this to be the model action for spectral theory and hence when a given action is shown to be equivalent to a dual action we shall consider that the goal of spectral theory has been achieved.

One of the reasons one might benefit from proving an action to be a dual action is that Fell bundles can be classified [E3: Theorem 7.3] up to stable equivalence by a twisted partial action of the base group on the unit fiber algebra. The successful completion of this program is therefore likely to provide a deep insight on the behavior of the given dynamical system.

In one of our main results, namely Corollary 11.15, we give a necessary and sufficient condition for an action to be equivalent to a dual action, thus solving a problem considered in [E2]. According to Theorem 5.5 in [E2], in the case of a dual action $A$ contains a dense set of $\alpha$-integrable elements: an element $a \in A$ is said to be $\alpha$-integrable [E2] (see also [R5]) if, for every $b \in A$, the functions $t \mapsto \alpha_t(a)b$ and, $t \mapsto b\alpha_t(a)$ are unconditionally integrable in the sense that the Bochner integrals over compact subsets of $G$ form a converging net (Definition 2.1 below). Accordingly, the necessary and sufficient conditions of 11.15 are expressed in terms of the existence of certain sets of $\alpha$-integrable elements.

We are then able to verify that these conditions hold for a proper action on a locally compact topological space and hence we obtain (Corollary 12.5) that proper actions are equivalent to dual actions.

At first glance it may seem that the questions related to the generalized fixed point algebra and Morita–Rieffel equivalence bear no relation to spectral theory but these turn out to be intimately related subjects. Among the indications that this is so is Theorem 10.6 below, according to which Rieffel’s project may be satisfactorily carried out when the action in question is a dual action. In particular, the role of the generalized fixed point algebra is played by the unit fiber algebra which appears as a subalgebra of the multiplier algebra and is fixed by the dual action.

At the root of the relationship between Rieffel’s project mentioned above and spectral theory is the notion of relative continuity which we would now like to describe in some detail. Following [E2] and [R5] (although we sometimes use different notation and terminology) we let $\mathcal{P}_\alpha$ be the set of positive $\alpha$-integrable elements and $\mathcal{N}_\alpha = \{a \in A : a^* a \in \mathcal{P}_\alpha\}$. It turns out that $\mathcal{N}_\alpha$ is a hereditary cone [E2: 6.6] which implies that $\mathcal{N}_\alpha$ is a left ideal in $A$. Moreover, it can be shown without much difficulty that $\mathcal{N}_\alpha$ is a right Hilbert $M_\alpha(A)$–module, where $M_\alpha(A)$ refers to the subalgebra of the multiplier algebra $M(A)$ consisting of the fixed points for $\alpha$. The $M_\alpha(A)$–valued inner-product $\langle a, b \rangle_R$ is given for any $a, b \in \mathcal{N}_\alpha$ by the strict-unconditional integral of $\alpha_t(a^* b)$, with respect to $t$.

It is implicit in Rieffel’s early work on this subject [R4] that $\mathcal{N}_\alpha$, with its Hilbert module structure, hides the clue to the Morita–Rieffel equivalence between the generalized fixed point algebra, which has yet to be defined, and ideals in the crossed product algebra.

Our starting point is a certain representation of the Hilbert module $\mathcal{N}_\alpha$ as bounded operators from $\mathcal{H}$ to $L_2(G, \mathcal{H})$, where $\mathcal{H}$ is any fixed representation space for $A$ (and hence also for $M(A)$). Precisely speaking, we define a linear map
\[ \zeta : \mathcal{N}_\alpha \to \mathcal{B}(\mathcal{H}, L_2(G, \mathcal{H})) \]
which satisfies $\zeta(a)m = \zeta(am)$, and $\langle a, b \rangle_R = \zeta(a)^* \zeta(b)$ for all $a, b \in \mathcal{N}_\alpha$ and $m \in M_\alpha(A)$. Once $\mathcal{N}_\alpha$ is concretely represented, our ability to answer questions related to it greatly increase. In particular it
is immediate that the algebra of generalized compact operators \([R1]\) is the closed linear span of the set \(\{\zeta(a)\zeta(b)^*: a, b \in \mathcal{N}_\alpha\}\), which is a subset of \(\mathcal{B}(L_2(G, \mathcal{H}))\).

Since the reduced crossed product \(A\rtimes_{\alpha, r} G\) is also an algebra of operators on \(L_2(G, \mathcal{H})\), it makes sense to ask whether or not

\[
\zeta(a)\zeta(b)^* \in A\rtimes_{\alpha, r} G
\]

for a given pair of elements \(a, b \in \mathcal{N}_\alpha\). We show in 13.5 that in an important example this fails more often than it holds. When \(G\) is abelian we give a necessary and sufficient condition for 1.1 to hold in terms of the Fourier coefficients \(E_x(p)\) of the element \(p := a^*a\), defined by

\[
E_x(p) = \int_G <x, t> \alpha_t(p) \, dt, \quad x \in \hat{G},
\]

and that of \(q := b^*b\). Precisely, we show in Theorem 7.5 that 1.1 holds if and only if

\[
\lim_{z \to a} \|E_{xz}(p)E_y(q) - E_x(p)E_{zy}(q)\| = 0,
\]

uniformly in \(x\) and \(y\) in \(\hat{G}\). A first consequence is that 1.1 depends only on the absolute value of \(a\) and \(b\). Secondly it brings to light a relation between \(\alpha\)-integrable elements: given any pair of \(\alpha\)-integrable elements \(p\) and \(q\) we say that \(p\) and \(q\) are relativley continuous, and denote it as \(p \sim Rq\), if 1.2 holds.

The relation \(\sim R\) is not reflexive, transitive, or symmetric (see 13.5) but it enjoys some surprising properties (see 8.3, 8.4, and 8.5) such as

\[
0 \leq a \leq b \sim Rc \Rightarrow a \sim Rc,
\]

whenever \(a, b,\) and \(c\) are \(\alpha\)-integrable.

If one is to find a submodule \(\mathcal{N}\) of \(\mathcal{N}_\alpha\) which serves as the imprimitivity bimodule for a Morita–Rieffel equivalence involving a subalgebra of \(A\rtimes_{\alpha, r} G\), then it would be convenient if \(\mathcal{N}\) satisfied \(\zeta(\mathcal{N})\zeta(\mathcal{N})^* \subseteq A\rtimes_{\alpha, r} G\). Therefore, for all \(a, b \in \mathcal{N}\) it must be that \(a^*a \sim Rc b^*b\) and hence, by the polarization formula, that \(p \sim Rc\) for any \(p, q \in \mathcal{N}^*\mathcal{N}\).

We then see that \(\mathcal{N}^*\mathcal{N}\) must be a relatively continuous set in the sense that its elements are mutually relatively continuous. The role of relative continuity in relation to Morita–Rieffel equivalence thus becomes clear. In particular one of the main hypothesis in Theorem 9.2, our main result related to Morita–Rieffel equivalence, is that a certain set is relatively continuous. With respect to Theorem 11.14, one of our main results in spectral theory, relatively continuous sets also play a crucial role. In particular the continuity of the norm and multiplication on the Fell bundle we construct there are derived from the relatively continuous set \(W\) in the hypothesis of 11.14.

Relatively continuous sets are therefore crucial for our arguments to be carried out in both fronts. However, it could be argued that requiring relative continuity beforehand diminishes the scope of application of our main results. It would therefore be highly desirable to produce large relatively continuous sets as a consequence of the existence of large (say dense) sets of \(\alpha\)-integrable elements (see questions 9.4, 9.5, and 11.16). However, except for Section 12, which deals with commutative algebras, we have nothing to offer in this respect.

As the main application of our results we discuss in Section 12 the case of a classical dynamical system \((C_0(X), G, \alpha)\) based on an abelian group, therefore assuming as much commutativity as possible.

By a result of Rieffel \([R5: 4.7]\) the linear span of the positive \(\alpha\)-integrable elements is dense in \(C_0(X)\) if and only if \(\alpha\) is a proper action in the usual sense. Therefore it seems reasonable to restrict our attention to proper actions. In this case it turns out that every \(f\) in \(C_c(X)\) is relatively continuous with respect to any other \(\alpha\)-integrable element (Proposition 12.3) and hence it deserves to be called absolutely continuous. This implies that \(C_c(X)\) is a relatively continuous set and hence we are able to apply the general theory
developed in the previous sections to show the already mentioned fact that proper actions are equivalent to dual actions.

Given the essential role played by absolutely continuous elements in our treatment of classical dynamical systems one could suspect them to play a similar role in other situations. However, in the case of the action of \( \mathbb{Z} \) on the algebra \( \mathcal{K} \) of compact operators given by conjugation by the powers of the bilateral shift, we prove in 13.6 that the only positive absolutely continuous element is the zero operator. So it is not clear, in general, how to produce a canonical relatively continuous set.

This explains, in retrospect, why it is so important to postulate the existence of the dense subalgebra \( A_0 \) in Definition 1.2 of [R4]. Since relative continuity is a relative relation, there seems not to be a universal construction of \( A_0 \) that works. Exploring the example of the action of \( \mathbb{Z} \) on \( \mathcal{K} \), mentioned above, we are able to give a precise reason why this is so by showing that a maximal relatively continuous cone is not unique.

It is noteworthy that several tools from the Harmonic Analysis of abelian groups are used in this work. While this is quite natural with respect to spectral theory, it is not so much so in relation to the aspects of Morita–Rieffel equivalence that we discuss. In fact, the very statement of our main spectral theory result (Theorem 11.14) makes no sense if \( G \) is not commutative, given the use of the Pontryagin dual. However, Rieffel’s project of generalizing the concept of fixed point algebra is meaningful for non-commutative groups and in fact many such examples have already been treated [R4]. This indicates either that our option to use tools from Harmonic Analysis is a poor one, or that the stakes are higher than previously thought!

2. Preliminaries.

Throughout this article we will fix a \( C^* \)-algebra \( A \) and a strongly continuous action
\[
\alpha : G \to \text{Aut}(A)
\]
of a locally compact group \( G \) on \( A \). We will also suppose, without loss of generality, that \( A \) is a non-degenerate \( C^* \)-algebra of operators on a Hilbert space \( \mathcal{H} \).

Recall from [E2: 2.3] that a function \( f : G \to A \) is said to be unconditionally integrable if it is Bochner integrable [Y: V.5] over every relatively compact subset \( K \subseteq G \), and the net
\[
\left\{ \int_K f(t) \, dt \right\}_{K \in \mathcal{K}}
\]
converges in the norm topology of \( A \), where \( \mathcal{K} \) is the directed set of all relatively compact subsets of \( G \) ordered by inclusion.

Although this is not relevant for us here we should remark that it is possible to prove that the notion of unconditional integrability coincides with that of Pettis integrability at least for bounded continuous functions.

If \( f \) is unconditionally integrable we let
\[
\int_G^u f(t) \, dt := \lim_K \int_K f(t) \, dt.
\]
The superscript “\( u \)” in the integral above is meant to remind us that we are speaking of an unconditional integral. According to [E2: 6.1] we say that an element \( a \) in \( A \) is \( \alpha \)-integrable if for every \( b \) in \( A \) one has that the functions
\[
t \in G \mapsto \alpha_t(a) b \in A, \quad \text{and} \quad t \in G \mapsto b \alpha_t(a) \in A
\]
are unconditionally integrable (see also [R5]). In this case we denote by
\[
\int_G^{su} \alpha_t(a) \, dt
\]
the multiplier \((L, R)\) of \(A\) given by

\[
L(b) = \int_G^u \alpha_t(a) b \, dt, \quad \text{and} \quad R(b) = \int_G^u b \alpha_t(a) \, dt,
\]

for every \(b \in A\). The superscript “\(su\)” standing for \textit{strict unconditional} integral.

For simplicity, in order to deal with left and right multiplication at the same time, we will often adopt the following alternative definition of \(\alpha\)-integrability:

\section{Definition.} An element \(a \in A\) is said to be \(\alpha\)-integrable if, given any pair \(b, c\) of elements of the multiplier algebra \(M(A)\) of \(A\) such that either \(b \in A\) and \(c = 1\), or \(b = 1\) and \(c \in A\), we have that the map \(t \in G \mapsto b \alpha_t(a) c \in A\) is unconditionally integrable (compare \([R5: 2.2]\)).

\section{Remark.} We should remark that our notion of \(\alpha\)-proper elements, taken from \([E2: 6.1]\), is related to Rieffel’s notion of \(\alpha\)-\textit{proper} elements \([R5: 4.1]\) as opposed to what is called \textit{order-integrable} in \([R5: 1.4]\). See also \([R5: 4.4]\).

Throughout this work we shall adopt the following:

\section{Notation.}

(i) \(\mathcal{P}_\alpha = \{a \in A_+ : a \text{ is } \alpha\text{-integrable}\}\),

(ii) \(\mathcal{N}_\alpha = \{a \in A : a^* a \in \mathcal{P}_\alpha\}\),

(iii) \(\mathcal{M}_\alpha = \text{span}(\mathcal{P}_\alpha)\).

We will denote also by \(\alpha\) the usual (not necessarily strongly continuous) extension of the action \(\alpha\) to \(M(A)\). The subset of \(M(A)\) formed by the fixed points under \(\alpha\) will be denoted \(M_\alpha(A)\). It is easy to see that for any \(\alpha\)-integrable element \(a\) one has that \(\int_G^u \alpha_t(a) \, dt \in M_\alpha(A)\).

\section{Proposition.}

(i) \(\mathcal{P}_\alpha\) is a hereditary cone,

(ii) \(\mathcal{M}_\alpha\) consists of \(\alpha\)-integrable elements,

(iii) \(\mathcal{P}_\alpha, \mathcal{N}_\alpha,\) and \(\mathcal{M}_\alpha\) are invariant under \(\alpha\),

(iv) \(\mathcal{N}_\alpha M_\alpha(A) \subseteq \mathcal{N}_\alpha\).

\textbf{Proof.} That \(\mathcal{P}_\alpha\) is hereditary is precisely the content of \([E2: 6.6]\). See also \([R5: 2.7]\). It is obvious that a linear combination of \(\alpha\)-integrable elements is again \(\alpha\)-integrable and hence (ii) holds.

Let \(a\) be any \(\alpha\)-integrable element and let \(s \in G\). Then for every \(K \in \mathcal{K}\) and \(b, c\) as in 2.1 we have that

\[
\int_K b \alpha_t(a_s)(a) c \, dt = \int_K b \alpha_{ts}(a) c \, dt = \Delta(s^{-1}) \int_K b \alpha_t(a) c \, dt,
\]

where \(\Delta\) is the modular function\(^2\) of \(G\). We then see that \(\alpha_s(a)\) is \(\alpha\)-integrable. This clearly implies that \(\mathcal{P}_\alpha\) is \(\alpha\)-invariant and, since \(\mathcal{N}_\alpha\) and \(\mathcal{M}_\alpha\) are built in terms of \(\mathcal{P}_\alpha\), it follows that they are also \(\alpha\)-invariant. We leave the proof of (iv) to the reader. \(\square\)

\section{Remark.} Regarding 2.4.(ii) we should remark that Rieffel \([R5: 8.9]\) has found an example of a self-adjoint \(\alpha\)-integrable element \(a\) such that \(|a|\) is not \(\alpha\)-integrable. This means that \(a_+\) and \(a_-\) cannot both be \(\alpha\)-integrable. A short argument involving the fact that \(\mathcal{P}_\alpha\) is hereditary shows that \(a\) is not a linear combination of positive \(\alpha\)-integrable elements. Therefore, in Rieffel’s example, \(\mathcal{M}_\alpha\) is strictly smaller than the set of all \(\alpha\)-integrable elements.

The following result, which will be used for our \(\mathcal{P}_\alpha, \mathcal{N}_\alpha,\) and \(\mathcal{M}_\alpha\), holds in a much greater generality. For the sake of completeness we shall give a detailed proof of it although it has appeared implicitly in the literature. See for example \([P: 5.1.2]\), \([KR: 7.5.2]\), and \([R5: \text{Section 1}]\).

\(^2\) With the convention that \(\int_G f(ts) \, dt = \Delta(s^{-1}) \int_G f(t) \, dt,\) and \(\int_G f(t^{-1}) \, dt = \int_G f(t) \Delta(t^{-1}) \, dt.\)
2.6. Proposition. Let \( \mathcal{P} \) be any hereditary cone in any \( C^* \)-algebra \( A \). Define
\[
\mathcal{N} = \{ a \in A : a^* a \in \mathcal{P} \}, \quad \text{and} \quad \mathcal{M} = \text{span}(\mathcal{P}).
\]

Then
- (i) \( \mathcal{N} \) is a left ideal in \( A \),
- (ii) \( \mathcal{M} = \mathcal{N}^* \mathcal{N} \) (linear span of products, no closure),
- (iii) \( \mathcal{M} \cap A_+ = \mathcal{P} \),
- (iv) \( \mathcal{M} \) is a hereditary *-subalgebra of \( A \).

Proof. Given \( a, b \in \mathcal{N} \) note that
\[
(a + b)^*(a + b) \leq (a + b)^*(a + b) + (a - b)^*(a - b) = 2a^* a + 2b^* b \in \mathcal{P}.
\]
Since \( \mathcal{P} \) is hereditary we conclude that \( (a + b)^*(a + b) \in \mathcal{P} \) and hence that \( a + b \in \mathcal{N} \). That \( \mathcal{N} \) is closed under complex multiplication is evident, so we see that \( \mathcal{N} \) is a linear subspace of \( A \).

If \( c \in A \) and \( a \in \mathcal{N} \) we have that \( (ca)^*ca = a^*c^*ca \leq \|c\|^2 a^* a \in \mathcal{P} \). Since \( \mathcal{P} \) is hereditary we see that \( ca \in \mathcal{N} \), thus proving that \( \mathcal{N} \) is a left ideal.

Given \( a, b \in \mathcal{N} \) we have by the polarization identity \( a^* b = \frac{1}{4} \sum_{k=0}^{3} i^{-k} (a + i^k b)^* (a + i^k b) \) that \( a^* b \in \text{span}(\mathcal{P}) = \mathcal{M} \), and hence that \( \mathcal{N}^* \mathcal{N} \subseteq \mathcal{M} \). Conversely, let \( a \in \mathcal{P} \). Then we have by definition that \( b : = a^{1/2} \in \mathcal{N} \), and hence \( a = b^* b \in \mathcal{N}^* \mathcal{N} \). This shows that \( \mathcal{P} \subseteq \mathcal{N}^* \mathcal{N} \) and hence that \( \mathcal{M} = \text{span}(\mathcal{P}) \subseteq \mathcal{N}^* \mathcal{N} \). This proves (ii).

Speaking of (iii) note that \( \mathcal{M} \cap A_+ \supseteq \mathcal{P} \) by definition. Conversely, let \( a \in \mathcal{M} \cap A_+ \), and write \( a \) as \( a = \sum_{i=1}^{n} \lambda_i p_i \) with \( p_i \in \mathcal{P} \) and \( \lambda_i \in \mathbb{C} \). Since \( a = (a + a^*)/2 \) we may assume that the \( \lambda_i \in \mathbb{R} \). Therefore we have
\[
0 \leq a = \sum_{i=1}^{n} \lambda_i p_i \leq \sum_{i=1}^{n} |\lambda_i| p_i \in \mathcal{P},
\]
from which it follows that \( a \in \mathcal{P} \). This proves (iii).

As for (iv) we have by (ii) that \( \mathcal{M} \mathcal{M} = \mathcal{N}^* \mathcal{N} \mathcal{N}^* \mathcal{N} \subseteq \mathcal{N}^* \mathcal{N} \mathcal{N} \mathcal{N} = \mathcal{M} \), where the penultimate step follows by (i). This shows that \( \mathcal{M} \) is an algebra and the remaining assertions in (iv) are now evident. \( \square \)

Boosting up the conclusion of 2.4.(iv) we have:

2.7. Proposition. \( \mathcal{N}_\alpha \) is a right pre–Hilbert \( \mathcal{M}_\alpha(A) \)–module for the usual multiplication, and inner-product given by
\[
\langle a, b \rangle_R := \int_{\mathbb{G}} \alpha_t(a^* b) \, dt,
\]
for \( a, b \in \mathcal{N}_\alpha \).

Proof. It now suffices to check that
\[
\langle a, bm \rangle_R = \langle a, b \rangle_R m,
\]
for all \( a, b \in \mathcal{N}_\alpha \) and \( m \in \mathcal{M}_\alpha(A) \), but this follows easily by inspection. \( \square \)

In dealing with Hilbert modules it is always convenient to have a Hilbert space representation. The following result is intended to provide us with one.

2.8. Theorem. For each \( a \in \mathcal{N}_\alpha \) there exists a bounded linear transformation \( \zeta(a) : \mathcal{H} \rightarrow L_2(\mathbb{G}, \mathcal{H}) \) such that for every \( v \in \mathcal{H} \) and \( t \in G \),
\[
\zeta(a)v|_t = \Delta(t)^{-1/2} a|_t^{-1} v.
\]
Moreover the map
\[
\zeta : a \in \mathcal{N}_\alpha \mapsto \zeta(a) \in \mathcal{B}(\mathcal{H}, L_2(\mathbb{G}, \mathcal{H}))
\]
is an isometric representation of the module \( \mathcal{N}_\alpha \) in the sense that for any \( a, b \in \mathcal{N}_\alpha \), and \( m \in \mathcal{M}_\alpha(A) \),
- (i) \( \|\zeta(a)\| = \|a\|_{\mathcal{N}_\alpha} \), where as usual \( \|a\|_{\mathcal{N}_\alpha} = \|\langle a, a \rangle_R\|^{1/2} \),
- (ii) \( \zeta(a)m = \zeta(am) \),
- (iii) \( \langle a, b \rangle_R = \langle \zeta(a)^* \zeta(b) \rangle \).
2.10. Corollary. Proof. Let \( a \in \mathcal{N}_a \) and \( v \in \mathcal{H} \). Recall that \( A \) is a non-degenerate algebra of operators on \( \mathcal{H} \). Therefore by the Cohen-Hewitt factorization theorem [HR: 32.22], for every \( \varepsilon > 0 \) there exists \( b \in A \) and \( w \in \mathcal{H} \) such that \( v =bw, \|b\| \leq 1, \) and \( \|w-v\| < \varepsilon \). So

\[
\|\zeta(a)v\|^2 = \int_G \|\Delta(t)^{-1/2} \alpha_t^{-1}(a)v\|^2 \, dt = \int_G \|\alpha_t(a)v\|^2 \, dt = \int_G \langle \alpha_t(a)bw, v \rangle \, dt = \\
= \left\langle \left( \int_G \alpha_t(a^*a)b \, dt \right) w, v \right\rangle \leq \left\| \int_G \alpha_t(a^*a)b \, dt \right\| \left\| w \right\| \left\| v \right\| \leq \left\| \int_G \alpha_t(a^*a) \, dt \right\| \left\| b \right\| \left\| w \right\| \left\| v \right\| \leq \|a\|_{\mathcal{N}_a}^2 (\|v\| + \varepsilon) \left\| v \right\|.
\]

Therefore, since \( \varepsilon \) is arbitrary, we have that \( \|\zeta(a)v\| \leq \|a\|_{\mathcal{N}_a} \|v\| < \infty \). It follows that \( \zeta(a)v \) is indeed in \( L_2(G, \mathcal{H}) \), that \( \zeta(a) \) is a bounded map, and that \( \|\zeta(a)\| \leq \|a\|_{\mathcal{N}_a} \).

In order to prove (ii) let \( m \in M_r(A) \) and note that for all \( t \) in \( G \) we have

\[
\zeta(a)mv|_t = \Delta(t)^{-1/2} \alpha_t^{-1}(a)mv = \Delta(t)^{-1/2} \alpha_t^{-1}(am)v = \zeta(am)v|_t.
\]

As for (iii), let \( a, b \in \mathcal{N}_a \). Fixing \( c \in A \), and \( v \) and \( w \) in \( \mathcal{H} \), we have

\[
\langle \zeta(a)^* \zeta(b)cv, w \rangle = \langle \zeta(b)cv, \zeta(a)w \rangle = \int_G \Delta(t)^{-1} \langle \alpha_t^{-1}(b)cv, \alpha_t^{-1}(a)w \rangle \, dt = \\
= \int_G \langle \alpha_t(a^*b)cv, w \rangle \, dt = \left\langle \left( \int_G \alpha_t(a^*b) \, dt \right) cv, w \right\rangle = \langle \langle a, b \rangle_R cv, w \rangle.
\]

Since \( A \) is non-degenerate this shows that \( \zeta(a)^* \zeta(b) = \langle a, b \rangle_R \). It now remains to show the equality in (i) but this follows easily since

\[
\|\zeta(a)\|^2 = \|\zeta(a)^* \zeta(a)\| = \|\langle a, a \rangle_R\| = \|a\|_{\mathcal{N}_a}^2.
\]

Many issues become greatly simplified once we have an isometric representation of a Hilbert module, such as the one constructed in our previous result. For instance:

2.9. Corollary.

(i) The completion of \( \mathcal{N}_a \), which we henceforth denote by \( \mathcal{X}_a \), can be identified with the closure of \( \zeta(\mathcal{N}_a) \) within \( B(\mathcal{H}, L_2(G, \mathcal{H})) \).

(ii) If \( T \) is a bounded linear operator on \( L_2(G, \mathcal{H}) \) such that both \( T \mathcal{X}_a \subseteq \mathcal{X}_a \) and \( T^* \mathcal{X}_a \subseteq \mathcal{X}_a \) then the map \( S \in \mathcal{X}_a \mapsto TS \in \mathcal{X}_a \) is an adjointable operator [JT: 1.1.7] on \( \mathcal{X}_a \).

(iii) The algebra \( \mathcal{K}(\mathcal{X}_a) \) of generalized compact operators [R1] on \( \mathcal{X}_a \) can be identified with the closed linear span of \( \mathcal{X}_a \mathcal{X}_a^* \) within \( B(L_2(G, \mathcal{H})) \).

(iv) \( \mathcal{X}_a \) is a ternary ring of operators [Z] in the sense that \( \mathcal{X}_a \mathcal{X}_a^* \mathcal{X}_a \subseteq \mathcal{X}_a \).

Proof. Left to the reader. \( \square \)

We immediately obtain the following:

2.10. Corollary. Let \( D \) be the subalgebra of \( M_r(A) \) given by \( D = \mathcal{X}_a^* \mathcal{X}_a \) (closed linear span) and let \( E \) be the algebra of operators on \( L_2(G, \mathcal{H}) \) given by \( E = \mathcal{X}_a \mathcal{X}_a^* \). Then \( \mathcal{X}_a \) is an imprimitivity bimodule between \( E \) and \( D \), which are therefore Morita–Rieffel equivalent [R1].
We would now like to briefly describe the left regular representation of the crossed product, mainly to fix our notation. See [P] for details. Consider the representation

\[ \pi : A \to \mathcal{B}(L_2(G, \mathcal{H})) \]

given by

\[ \pi(a)\xi(t) = \alpha^{-1}_t(a)\xi(t) \]

for all \( a \in A \), \( \xi \in L_2(G, \mathcal{H}) \) and \( t \in G \). Also, let \( \Lambda \) be the representation of \( G \) on \( \mathcal{B}(L_2(G, \mathcal{H})) \) given by \( \Lambda = \lambda \otimes \text{id} \), where \( \lambda \) is the left regular representation of \( G \) on \( L_2(G) \) and we have identified \( L_2(G, \mathcal{H}) \cong L_2(G) \otimes \mathcal{H} \). It is well known [P: 7.7.1] that the pair \((\pi, \Lambda)\) is a covariant representation of the \( C^*\)-dynamical system \((A, G, \alpha)\) and that \( \pi \times \Lambda : A \rtimes_{\alpha, r} G \to \mathcal{B}(L_2(G, \mathcal{H})) \) is a faithful representation, provided that \( G \) is an amenable group [P: 7.7.5 and 7.7.7]. In any case \( \pi \times \Lambda \) is called the regular representation of \( A \rtimes_{\alpha, r} G \) and its range is the so called reduced crossed product \( A \rtimes_{\alpha, r} G \).

We stress that \( A \rtimes_{\alpha, r} G \) is thus a concrete algebra of operators on \( L_2(G, \mathcal{H}) \).

Observe that the algebra \( E := \alpha_{\alpha}A\alpha_{\alpha}^* \), mentioned above, is also an algebra of operators on \( L_2(G, \mathcal{H}) \). This raises the question as to whether there is any relationship between \( A \rtimes_{\alpha, r} G \) and \( E \). This turns out to be the most dramatic question in the present subject. We shall have more to say about it in what follows.

For simplicity we let

\[ \rho := \pi \times \Lambda. \]

For \( f \) in \( C_c(G, A) \) (seen as a subalgebra of \( A \rtimes_{\alpha} G \)), \( \xi \) in \( C_c(G, \mathcal{H}) \) (seen as a subspace of \( L_2(G, \mathcal{H}) \)), and \( t \) in \( G \), we have [P: 7.7.1],

\[ \rho(f)\xi(t) = \int_G \alpha^{-1}_t(f(s))\xi(s^{-1}t) \, ds = \int_G \alpha^{-1}_t(f(ts))\xi(s) \, ds = \int_G \Delta(s)^{-1}\alpha^{-1}_t(f(ts^{-1}))\xi(s) \, ds. \]

It follows that \( \rho(f) \) is an “integral operator” with “kernel”

\[ k(t, s) = \Delta(s)^{-1}\alpha^{-1}_t(f(ts^{-1})), \]

meaning that

\[ \rho(f)\xi(t) = \int_G k(t, s)\xi(s) \, ds. \]

One easily checks that \( k \) satisfies

\[ k(tr, sr) = \Delta(r)^{-1}\alpha^{-1}_t(k(t, s)), \]

for all \( t, s, r \in G \). Later we will investigate integral operators again.
3. Integrable elements.

This section is intended to discuss a few technical results about unconditional integrability to be used below. Our main reference for what follows is [E2]. We begin by quoting a few consequences of [E2], in a form suitable for our purposes.

3.1. Theorem. Let \( f: G \to A \) be an unconditionally integrable map.

(i) For every \( \phi \in L_\infty(G) \) one has that the pointwise product \( \phi f \) is also unconditionally integrable.

(ii) There exists \( M \geq 0 \) such that

\[
\left\| \int_G \phi(t) f(t) \, dt \right\| \leq M \|\phi\|, \quad \phi \in L_\infty(G).
\]

(iii) For every \( \varepsilon > 0 \) there exists a compact set \( K \subseteq G \) such that for every compact subset \( L \) of \( G \) with \( K \cap L = \emptyset \),

\[
\left\| \int_L \phi(t) f(t) \, dt \right\| \leq \varepsilon \|\phi\|, \quad \phi \in L_\infty(G).
\]

Proof. The first statement is [E2:2.8]. The second follows easily from (i) and [E2:2.7]. Finally, (iii) is precisely [E2:2.9]. \( \square \)

3.2. Proposition. Let \( a \in A \) be \( \alpha \)-integrable. Then for every \( \phi \in L_\infty(G) \) the map \( t \mapsto \phi(t)\alpha_t(a) \) is strictly-unconditionally integrable. In addition there exists a constant \( M \geq 0 \) such that

\[
\left\| \int_G \phi(t)\alpha_t(a) \, dt \right\| \leq M \|\phi\|, \quad \phi \in L_\infty(G).
\]

Proof. For each \( \phi \in L_\infty(G) \) consider the maps

\[
L_\phi(b) = \int_G \phi(t)\alpha_t(a) b \, dt, \quad \text{and} \quad R_\phi(b) = \int_G \phi(t)b\alpha_t(a) \, dt, \quad b \in A.
\]

It is clear that the pair \((L_\phi, R_\phi)\) defines a multiplier of \( A \). Since multipliers are automatically bounded we have, in particular, that \( L_\phi \in \mathcal{B}(A, A) \). We claim that the set \( \{L_\phi : \|\phi\| \leq 1\} \) is pointwise bounded. In fact, fixing \( b \in A \), the map \( t \mapsto \alpha_t(a) b \) is unconditionally integrable and hence by 3.1.(ii) there exists a constant \( M \) such that

\[
\|L_\phi(b)\| = \left\| \int_G \phi(t)\alpha_t(a) b \, dt \right\| \leq M \|\phi\| \leq M,
\]

provided that \( \|\phi\| \leq 1 \). By the uniform boundedness principle there exists a constant \( N \) such that \( \|L_\phi\| \leq N \|\phi\| \) for all \( \phi \in L_\infty(G) \) and hence,

\[
\left\| \int_G \phi(t)\alpha_t(a) b \, dt \right\| = \|L_\phi(b)\| \leq \|L_\phi\| \|b\| \leq N \|\phi\| \|b\|,
\]

for all \( b \in A \) and \( \phi \in L_\infty(G) \). This concludes the proof. \( \square \)

3.3. Definition. Given an \( \alpha \)-integrable element \( a \in A \) we shall denote by \( \|a\|_1 \) the smallest constant \( M \) for which the inequality in 3.2 holds.

If \( a \) is \( \alpha \)-integrable one could attempt to introduce a different “\( L_1 \)-norm” of \( a \) by setting \( \|a\|'_1 := \left\| \int_G \alpha_t(|a|) \, dt \right\| \). However this does not work because \( |a| \) may not be \( \alpha \)-integrable as remarked in 2.5. See also Rieffel’s observation following [R5:1.1]
3.4. Proposition. Let $f : G \to A$ be an unconditionally integrable map and let $\{\phi_i\}_i \subseteq L_\infty(G)$ be a bounded net converging to $\phi \in L_\infty(G)$ uniformly over compact subsets of $G$. Then

$$\lim_i \int_G^u \phi_i(t) f(t) \, dt = \int_G^u \phi(t) f(t) \, dt$$

in the norm topology of $A$.

Proof. Given $\varepsilon > 0$ let $M$ be as in 3.1.(ii) and $K$ as in 3.1.(iii). Choose an index $i_0$ such that for all $i \geq i_0$ one has that $\sup_{t \in K} \|\phi_i(t) - \phi(t)\| \leq \varepsilon$. Therefore for $i \geq i_0$

$$\left\| \int_G^u \phi_i(t) f(t) \, dt - \int_G^u \phi(t) f(t) \, dt \right\| \leq \left\| \int_K (\phi_i(t) - \phi(t)) f(t) \, dt \right\| + \left\| \int_{G \setminus K} (\phi_i(t) - \phi(t)) f(t) \, dt \right\| \leq M \|\chi_{K}(\phi_i - \phi)\| + \varepsilon \|\phi_i - \phi\| \leq M\varepsilon + 2\varepsilon \sup_i \|\phi_i\|.$$  

This concludes the proof. \qed

3.5. Proposition. Let $f \in C_c(G, A)$. Then

$$\left( \int_G f(t) \, dt \right)^* \left( \int_G f(t) \, dt \right) \leq |\text{supp}(f)| \int_G f(t)^* f(t) \, dt,$$

where $|\text{supp}(f)|$ refers to the Haar measure of the support of $f$.

Proof. Let $S$ be the support of $f$ and recall that $A$ is an operator algebra on the Hilbert space $H$. For $v \in H$ we have

$$\left\langle \left( \int_S f(t) \, dt \right)^* \left( \int_S f(t) \, dt \right), v, v \right\rangle = \left\| \int_S f(t) v \, dt \right\|^2 \leq \left\langle \int_S \|f(t) v\| \, dt \right\|^2 =$$

$$= \left( \int_S \langle f(t) v, f(t) v \rangle^{1/2} \, dt \right)^2 = \left( \int_S \langle f(t)^* f(t) v, v \rangle^{1/2} \, dt \right)^2 \leq$$

$$\leq \left( \int_S 1 \, dt \right) \left\langle \left( \int_S f(t)^* f(t) v, v \right), v \right\rangle = |\text{supp}(f)| \left\langle \left( \int_S f(t)^* f(t) \, dt \right) v, v \right\rangle,$$

where the penultimate step is Hölder’s inequality. Since $v$ is arbitrary, the proof is concluded. \qed

3.6. Proposition. Let $a$ be an $\alpha$-integrable element of $A$ and let $g \in C_c(G)$. Then the element $a' \in A$ defined by $a' = \int_G g(t)\alpha_t(a) \, dt$ is $\alpha$-integrable as well. In addition, if $\phi \in L_\infty(G)$ then

$$\int_G^{su} \phi(t)\alpha_t(a') \, dt = \int_G^{su} (\phi * g)(t)\alpha_t(a) \, dt.$$  

Proof. For the first assertion we have to show that the net

$$\left\{ \int_K b\alpha_s(a') c \, ds \right\}_{K \in K}$$

converges in the norm of $A$, whenever $b \in A$ and $c = 1$, or $b = 1$ and $c \in A$. We have

$$\int_K b\alpha_s(a') c \, ds = \int_K b\alpha_s \left( \int_G g(t)\alpha_t(a) \, dt \right) c \, ds = \int_K \int_G g(t)b\alpha_{st}(a) c \, dt \, ds =$$
Let and converges uniformly over compacts to the constant function taking the value but since We claim that if We remain under the assumption that on the Hilbert space the space is a locally compact group. We would now like to relate the space $\pi$ (introduced in 2.9.(i)) to the reduced crossed product algebra $A\rtimes_{\pi, r} G$.

Thus, to prove the statement it suffices to show that

$$a' := \int_G |g(t)|^2 \alpha_t(a^* a) \, dt$$

is $\alpha$-integrable, since the set of positive $\alpha$-integrable elements forms a hereditary cone by 2.4.(i). But the integrability of $a'$ follows from 3.6. □

4. The left structure of $X_\alpha$.

We remain under the assumption that $(A, G, \alpha)$ is a $C^*$-dynamical system, where $A$ is a non-degenerate $C^*$-algebra of operators on a Hilbert space $H$, and $G$ is a locally compact group. We would now like to relate the space $X_\alpha$ (introduced in 2.9.(i)) to the reduced crossed product algebra $A\rtimes_{\alpha, r} G$.

Recall that $(\pi, \Lambda)$, defined near the end of section 2, is a covariant representation of the $C^*$-dynamical system $(A, G, \alpha)$ on the Hilbert space $L_2(G, H)$, and that $\rho$ denotes the representation of $A\rtimes_{\alpha} G$ given by $\rho = \pi \times \Lambda$.

4.1. Lemma. If $a \in N_\alpha$ then

(i) $\pi(b) \zeta(a) = \zeta(ba)$ for all $b \in A$.
(ii) $\Lambda_t \zeta(a) = \Delta(t)^{1/2} \zeta(\alpha_t(a))$ for all $t \in G$.
(iii) Let $g \in C_c(G)$ and define $a' = \int_G g(t) \Delta(t)^{1/2} \alpha_t(a) \, dt$. Then $\zeta(a') = (\int_G g(t) \Lambda_t \, dt) \zeta(a)$ (observe that $a' \in N_\alpha$ by 3.7).
(iv) Let $g$ and $a'$ be as above and take $c \in A$. Consider the function $f \in L_1(G, A)$ defined by $f(t) = g(t)c$. Then $\rho(f) \zeta(a) = \zeta(ca')$ (observe that $ca' \in N_\alpha$ by 2.6.(i)).
Proof. Let \( v \in \mathcal{H} \) and \( s \in \mathcal{G} \). Then

\[
\pi(b)\zeta(a)v|_s = \alpha_s^{-1}(b) \left( \zeta(a)v|_s \right) = \Delta(s)^{-1/2} \alpha_s^{-1}(b) \alpha_s^{-1}(a) v = \\
= \Delta(s)^{-1/2} \alpha_s^{-1}(ba)v = \zeta(ba)v|_s,
\]
proving (i). As for (ii) we have

\[
\Lambda_t \zeta(a)v|_s = \zeta(a)v|_{t^{-1}s} = \Delta(t^{-1}s)^{-1/2} \alpha^{-1}_{t^{-1}s}(a) v = \\
= \Delta(t)^{1/2} \Delta(s)^{-1/2} \alpha_s^{-1}(\alpha_t(a))v = \Delta(t)^{1/2} \zeta(\alpha_t(a))v|_s.
\]

With respect to (iii) it is certainly tempting to apply \( \zeta \) to both sides in the definition of \( \alpha' \) and use (ii) but this would require some sort of continuity property for \( \zeta \) which we do not seem to have. Instead, let \( v \in \mathcal{H} \) and \( \xi \in L_2(\mathcal{G}, \mathcal{H}) \). Then

\[
\left< \left( \int_G g(t)\Lambda_t dt \right) \zeta(a)v, \xi \right> = \int_G \left< \left( \int_G g(t)\Lambda_t dt \right) \zeta(a)v|_s, \xi(s) \right> ds = \\
= \int_G \left< \int_G g(t) \left( \zeta(a)v|_{t^{-1}s} \right) dt, \xi(s) \right> ds = \\
= \int_G \left< \int_G g(t) \Delta(t^{-1}s)^{-1/2} \alpha^{-1}_{t^{-1}s}(a) v dt, \xi(s) \right> ds = \\
= \int_G \int_G g(t) \Delta(t)^{1/2} \Delta(s)^{-1/2} \langle \alpha_{s-t}(a)v, \xi(s) \rangle dt ds.
\]

On the other hand

\[
\langle \zeta(a')v, \xi \rangle = \int_G \left< \zeta(a')v|_s, \xi(s) \right> ds = \int_G \left< \Delta(s)^{-1/2} \alpha_s^{-1}(a')v, \xi(s) \right> ds = \\
= \int_G \left< \Delta(s)^{-1/2} \alpha_s^{-1} \left( \int_G g(t) \Delta(t)^{1/2} \alpha_t(a) dt \right) v, \xi(s) \right> ds = \\
= \int_G \int_G \Delta(s)^{-1/2} g(t) \Delta(t)^{1/2} \langle \alpha_{s-t}(a)v, \xi(s) \rangle dt ds.
\]

This proves (iii). As for (iv) we have

\[
\rho(f)\zeta(a) = \left( \int_G \pi(f(t))\Lambda_t dt \right) \zeta(a) = \pi(c) \left( \int_G g(t)\Lambda_t dt \right) \zeta(a) = \pi(c)\zeta(a') = \zeta(ca'),
\]
proving (iv).

\( \square \)

### 4.2. Corollary

Let \( \mathcal{N} \) be a subset of \( \mathcal{N}_\alpha \) such that

(i) for every \( g \in C_c(\mathcal{G}) \) and \( a \in \mathcal{N} \) one has that \( \int_G g(t)\alpha_t(a) dt \in \mathcal{N} \), and

(ii) there exists a dense subset \( \mathcal{D} \) of \( A \) such that \( \mathcal{D}\mathcal{N} \subseteq \mathcal{N} \).

Then \( (A \rtimes_{\alpha, r} G) \mathcal{X} \subseteq \mathcal{X} \), where \( \mathcal{X} \) is the closure of \( \zeta(\mathcal{N}) \) in \( \mathcal{X}_\alpha \).

**Proof.** Let \( f(t) = g(t)c \), where \( g \in C_c(\mathcal{G}) \) and \( c \in \mathcal{D} \). Then, by 4.1.(iv) we have that \( \rho(f)\zeta(\mathcal{N}) \subseteq \zeta(\mathcal{N}) \), and hence that \( \rho(f)\mathcal{X} \subseteq \mathcal{X} \). Since the linear span of the set of all \( \rho(f)'s \) of the above form is dense in \( A \rtimes_{\alpha, r} G \), the conclusion follows.

\( \square \)
4.3. Corollary. One has that \((A \times_{\alpha,r} G) \mathcal{X}_\alpha \subseteq \mathcal{X}_\alpha\)

Proof. Follows at once from 4.2 once we note that \(\mathcal{N}_\alpha\) satisfies the required hypothesis by 3.7 and 2.6.(i) \(\square\)

We therefore see that \(\mathcal{X}_\alpha\) is a left \((A \times_{\alpha,r} G)\)-module under the composition of operators. Moreover, in view of 2.9.(ii), all operators in \(A \times_{\alpha,r} G\) act as adjointable operators on \(\mathcal{X}_\alpha\).

If we let \(E = \mathcal{X}_\alpha \mathcal{X}_\alpha^*\), as before, we have that \((A \times_{\alpha,r} G) E \subseteq E\) and hence that \((A \times_{\alpha,r} G) \cap E\) is an ideal in \(A \times_{\alpha,r} G\). It would seem natural to conjecture that this is the ideal one would like to show is Morita–Rieffel equivalent to the (still not yet defined) generalized fixed point algebra, as in [R5: Section 6]. A particularly intriguing question is:

4.4. Question. Is there a linear subspace \(\mathcal{X}\) of \(\mathcal{X}_\alpha\) such that \(\mathcal{X} \mathcal{X}^* = (A \times_{\alpha,r} G) \cap E\)?

A related question is whether or not \(\mathcal{X}_\alpha\) is also a Hilbert module over \(A \times_{\alpha,r} G\). Clearly this would be the case if we knew that \(A \times_{\alpha,r} G\) contains the range of the left inner-product on \(\mathcal{X}_\alpha\), given by

\[\langle T, S \rangle_L = TS^*, \quad T, S \in \mathcal{X}_\alpha.\]

In particular, one could ask:

4.5. Question. Given a pair of elements \(a, b \in \mathcal{N}_\alpha\), how could one determine if \(\zeta(a) \zeta(b)^*\) belongs to \(A \times_{\alpha,r} G\)?

This question, to which we will give a satisfactory answer in the abelian group case, resides in the heart of the matter and will dominate our attention in the remaining sections of this work. In order to explore it further we need a deeper understanding of integral operators.

5. Integral and Laurent operators.

Inspired by 2.12 we would now like to establish a precise notion of integral operators in our context. For this purpose let \(k\) be any continuous \(\mathcal{B}(\mathcal{H})\)-valued function on \(G \times G\) and suppose that for all \(\xi \in C_c(G, \mathcal{H})\) the function \(\eta: G \rightarrow \mathcal{H}\), given by

\[\eta(t) = \int_G k(t, s)\xi(s) \, ds, \quad t \in G,\]

is in \(L_2(G, \mathcal{H})\). Suppose further that there exists a constant \(M \geq 0\) such that \(\|\eta\|_2 \leq M \|\xi\|_2\) for every \(\xi\) as above. This implies the existence of a bounded operator \(T\) on \(L_2(G, \mathcal{H})\) satisfying

\[T\xi = \int_G k(t, s)\xi(s) \, ds, \quad \xi \in C_c(G, \mathcal{H}), \quad t \in G.\]

5.2. Definition. By an integral operator on \(L_2(G, \mathcal{H})\) we shall mean any bounded operator \(T\) that satisfies 5.1 for some continuous function \(k: G \times G \rightarrow \mathcal{B}(\mathcal{H})\). In this case \(k\) will be called the kernel of \(T\).

If a general theory of integral operators is desired one should probably relax the requirement that the integral kernel \(k\) be continuous. For the applications we have in mind, however, the definition given is enough, apart from the fact that it greatly simplifies the study of our operators.

Whereas it is highly unlikely that a necessary and sufficient condition on \(k\) will ever be found for \(T\) to be bounded, we will be able to use the concept of integral operators quite profitably. Of course, in the applications, the boundedness of \(T\) must be derived from other sources. This is akin to studying an operator on a Hilbert space via its matrix with respect to an orthonormal basis although no one really knows how to characterize boundedness in terms of matrices.

Observe that 2.12 shows that \(\rho(f)\) is an integral operator for any \(f \in C_c(G, A)\).

Inspired by 2.13, and in analogy with the case of the trivial action of \(\mathbb{Z}\) on \(\mathbb{C}\), we make the following:

5.3. Definition. A Laurent operator is an integral operator \(T\) on \(L_2(G, \mathcal{H})\) whose kernel \(k\) takes values in \(A \subseteq \mathcal{B}(\mathcal{H})\) and satisfies 2.13, namely that \(k(tr, sr) = \Delta(r)^{-1} \alpha_r^{-1} (k(t, s))\) for all \(t, s, r \in G\).
Given a Laurent operator with kernel $k$, define

$$f(r) := \alpha_r(k(r, e)), \quad r \in G.$$  

Taking into account the right hand side of 2.11, let us compute

$$\Delta(s)^{-1} \alpha_r^{-1}(f(ts^{-1})) = \Delta(s)^{-1} \alpha_r^{-1}(\alpha_{ts^{-1}}(k(ts^{-1}, e))) = \Delta(s)^{-1} \alpha_r^{-1}(k(ts^{-1}, e)) = k(t, s).$$

Therefore $k$ satisfies 2.11 with respect to $f$. The reader should however be warned that, in general, the function $f$ defined above for a Laurent operator need not be in $C_c(G, A)$, or even in $L_1(G, A)$. This is related to the remark at the end of section 6 in [R5].

5.4. Definition. Given a Laurent operator $T$ with kernel $k$ we say that the (continuous) function $f: G \to A$ given by $f(r) = \alpha_r(k(r, e))$ is the symbol of $T$.

It is then obvious that for any $f \in C_c(G, A)$ the symbol of $\rho(f)$ is $f$. We also note that:

5.5. Proposition. Let $T$ be a Laurent operator with symbol $f$. Suppose that $f$ is in $C_c(G, A)$. Then $T$ belongs to $A \rtimes_{\alpha_r} G$.

Proof. By 5.1 it is clear that two Laurent operators with the same symbol, and hence also the same kernel, coincide. Since both $T$ and $\rho(f)$ are Laurent operators with symbol $f$, we must have $T = \rho(f)$. □

One of the reasons we are interested in Laurent operators is as follows:

5.6. Proposition. Suppose $a, b \in N_\alpha$ one has that $\varsigma(a)\varsigma(b)^\ast$ is a Laurent operator with symbol

$$f(r) = \Delta(r)^{-1/2}a\alpha_r(b^\ast).$$

Proof. Let $\xi \in C_c(G, \mathcal{H})$ and $v \in \mathcal{H}$. Then

$$\langle \varsigma(b)^\ast \xi, v \rangle = \langle \varsigma, \varsigma(b)v \rangle = \int_G \langle \varsigma(s), \Delta(s)^{-1/2} \alpha_{bs}^{-1}(b)v \rangle ds =$$

$$= \int_G \langle \Delta(s)^{-1/2} \alpha_{bs}^{-1}(b^\ast)\xi(s), v \rangle ds.$$

So we see that $\varsigma(b)^\ast \xi$ is given by

$$\varsigma(b)^\ast \xi = \int_G \Delta(s)^{-1/2} \alpha_{bs}^{-1}(b^\ast)\xi(s) ds,$$

Given $t \in G$ we then have

$$\varsigma(a)\varsigma(b)^\ast \big|_t = \Delta(t)^{-1/2} \alpha_t^{-1}(a)\varsigma(b)^\ast = \Delta(t)^{-1/2} \alpha_t^{-1}(a) \int_G \Delta(s)^{-1/2} \alpha_{bs}^{-1}(b^\ast)\xi(s) ds =$$

$$= \int_G \Delta(ts)^{-1/2} \alpha_t^{-1}(a)\alpha_s^{-1}(b^\ast)\xi(s) ds,$$

from where we conclude that $\varsigma(a)\varsigma(b)^\ast$ is the integral operator with kernel

$$k(t, s) = \Delta(ts)^{-1/2} \alpha_t^{-1}(a)\alpha_s^{-1}(b^\ast).$$

It is easy to see that $k$ satisfies 2.13 and hence that $\varsigma(a)\varsigma(b)^\ast$ is a Laurent operator. Its symbol is given by

$$f(r) = \alpha_r(k(r, e)) = \Delta(r)^{-1/2}a\alpha_r(b^\ast).$$

Referring to the question posed at the end of section 4, namely whether $\varsigma(a)\varsigma(b)^\ast \in A \rtimes_{\alpha_r} G$ for $a, b \in N_\alpha$, we may give an affirmative answer in a very simple case:

5.7. Proposition. Suppose $a, b \in N_\alpha$ are such that the map $r \mapsto a\alpha_r(b^\ast)$ is compactly supported. Then $\varsigma(a)\varsigma(b)^\ast \in A \rtimes_{\alpha_r} G$.

Proof. This is an immediate consequence of 5.6 and 5.5. □
6. Abelian groups and the Fourier transform.

In the general case there is not much more we can say about the question of whether \( \zeta(a)\zeta(b)^* \) belongs to \( A\times_G A \) for \( a,b \in \mathcal{N}_\alpha \), as mentioned at the end of Section 4. This is the main obstacle to defining the generalized fixed point algebra and proving it to be Morita–Rieffel equivalent to an ideal in \( A\times_G G \).

We shall therefore restrict our attention to the special case of abelian groups and hence we assume from now on that \( G \) is abelian. In particular \( G \) is amenable and hence \( \rho \) establishes an isomorphism between \( A\times_G G \) and \( A\times_G A \) \([P; 7.7.7]\). We will therefore identify these algebras without further notice, remarking, however, that we will be much more interested in the concrete algebra \( A\times_G G \) of operators on \( L_2(G, \mathcal{H}) \) rather than in the abstract \( C^* \)-algebra \( A\times_\alpha G \).

Let \( \hat{G} \) be the Pontrjagin dual of \( G \). Given \( x \in \hat{G} \) and \( t \in G \) we will denote the value of the character \( x \) on \( t \) by \( <x,t> \).

6.1. Definition. Let \( a \in A \) be \( \alpha \)-integrable and let \( x \in \hat{G} \). The Fourier coefficient\(^3\) \( E_x(a) \) of \( a \) is the element of \( M(A) \) given by

\[
E_x(a) = \int_G <x,t> \alpha_t(a) \, dt.
\]

It is easy to prove \([E2; 6.4]\) that \( E_x(a) \) belongs to the \( x \)-spectral subspace of \( M(A) \) defined by

\[
M_x(A) = \{ m \in M(A) : \alpha_t(m) = <x,t>m, \ t \in G \}.
\]

In particular \( M_x(A) \) is consistent with our previous notation for the set of fixed points for \( \alpha \) within \( M(A) \).

As an immediate consequence of 3.2 we have:

6.2. Proposition. Let \( a \) be \( \alpha \)-integrable. Then for every \( x \in \hat{G} \) one has that \( \| E_x(a) \| \leq \| a \|_1 \).

As in the classical case we have:

6.3. Proposition. (See also \([E2; 6.3]\)). For each \( \alpha \)-integrable element \( a \in A \) the Fourier transform

\[
x \in \hat{G} \mapsto E_x(a) \in M(A)
\]

is uniformly continuous in the strict topology of \( M(A) \). That is, given \( b,c \in M(A) \) such that either \( b \in A \) and \( c = 1 \), or \( b = 1 \) and \( c \in A \), one has

\[
\lim_{z \to c} \sup_{x \in \hat{G}} \| bE_{zx}(a)c - bE_x(a)c \| = 0.
\]

\(^3\) When working with abelian groups one often has to make choices, as is the case of the complex conjugation in the definition of the Fourier transform: one could very well develop all of classical Harmonic Analysis defining the Fourier transform of a complex function \( f \) by \( \hat{f}(x) = \int_G <x,t> f(t) \, dt \), as opposed to the more usual \( \hat{f}(x) = \int_G <x,t> \overline{f(t)} \, dt \). Likewise, given an action of \( G \) on a space \( X \) one usually induces an action on the algebra of functions on \( X \) by the formula \( \alpha_t(f)(x) = f(t^{-1}x) \), but when \( G \) is commutative one has the option of dropping the inversion of \( t \). This extra freedom has a price: although there are no right or wrong choices, some of them often cause an excessive number of inverses and complex conjugations, which one feels should not be there. Often an attempt to back up and change conventions reveals only too late that the undesirable inverses and conjugations pop up in greater numbers further on.

The convention adopted here for the Fourier transform (without the complex conjugation) takes into account that the action of a compact abelian group \( G \) on itself by left multiplication, once induced to \( C(G) \) via the formula \( \alpha_t(f)(x) = f(t^{-1}x) \), ought to satisfy \( E_x(f) = \delta_{x,y}f \), when \( f(\cdot) = <y,\cdot> \), hence agreeing with the prevailing convention for the Fourier transform (with the complex conjugation). This in turn seems to indicate the need to include the complex conjugation in the definition of \( M_x(A) \) above. These choices seem rather reasonable but they have the somewhat unpleasant consequence of forcing us to part with tradition with respect to the dual action (see e.g \([P; 7.8.3]\)) to be defined in section 10.

In fact there are stronger reasons for adopting our conventions: among these we would like to mention that Proposition 11.12 would have to suffer some rather unnatural modifications in order to survive under seemingly natural alternative conventions. The same goes for Proposition 6.5.
Proof. Suppose by contradiction that this is not so. Then there exists \( \varepsilon > 0 \) and nets \( \{ z_i \}_i \) and \( \{ x_i \}_i \) in \( \hat{G} \) such that \( z_i \to 0 \) and \( \| bE_{z_ix_i}(a)c - bE_{x_i}(a)c \| \geq \varepsilon \). Observe that

\[
\begin{align*}
bE_{z_ix_i}(a)c - bE_{x_i}(a)c &= \int_G ( <z_i,x_i,t> - <x_i,t> ) b\alpha_t(a)c \, dt \\
&= \int_G ( <z_i,t> - 1 ) <x_i,t> b\alpha_t(a)c \, dt = \int_G \varphi(t) b\alpha_t(a)c \, dt,
\end{align*}
\]

where \( \varphi(t) = ( <z_i,t> - 1 ) <x_i,t> \). By definition of the topology on \( \hat{G} \) we have that \( <z_i,t> \to 1 \) uniformly over compact sets and hence \( \varphi(t) \to 0 \), also uniformly over compacts. Using 3.4 we thus obtain

\[
\int_G \varphi(t) b\alpha_t(a)c \, dt \to 0,
\]

therefore arriving at a contradiction. \( \square \)

For future use we now collect some important properties of the Fourier transform.

6.4. Proposition. Let \( a, b \in A \) be \( \alpha \)-integrable, let \( x, y \in \hat{G} \), and let \( m \in M_y(A) \). Then

(i) \( E_x(a^*) = E_{x^{-1}}(a^*) \),

(ii) \( ma \) is \( \alpha \)-integrable and \( mE_x(a) = E_{yx}(ma) \),

(iii) \( am \) is \( \alpha \)-integrable and \( E_{x}(a)m = E_{xy}(am) \),

(iv) \( E_x(a)E_y(b) = E_{xy}(aE_y(b)) = E_{xy}(E_x(a)b) \).

Proof. Left to the reader. \( \square \)

Our next result is related to the classical result according to which the Fourier transform of a convolution is the pointwise product of the corresponding transforms. As usual we denote by \( \hat{g} \) the Fourier transform of a function \( g \in L_1(G) \), i.e.,

\[
\hat{g}(x) = \int_G \langle x, t \rangle g(t) \, dt, \quad x \in \hat{G}.
\]

6.5. Proposition. Given an \( \alpha \)-integrable element \( a \in A \) and \( g \in C_c(G) \) let

\[
a' = \int_G g(t)\alpha_t(a) \, dt.
\]

Then for every \( x \) in \( \hat{G} \) one has that \( E_x(a') = \hat{g}(x)E_x(a) \).

Proof. Recall initially that \( a' \) is \( \alpha \)-integrable by 3.6. Also from 3.6 we have

\[
E_x(a') = \int_G \langle x, t \rangle \alpha_t(a') \, dt = \int_G (\phi*g)(t)\alpha_t(a) \, dt,
\]

where \( \phi(t) = \langle x, t \rangle \). On the other hand

\[
(\phi*g)(t) = \int_G \phi(ts^{-1})g(s) \, ds = \int_G \langle x, t \rangle \frac{1}{s} g(s) \, ds = \langle x, t \rangle \hat{g}(x).
\]

Therefore

\[
E_x(a') = \int_G \langle x, t \rangle \hat{g}(x)\alpha_t(a) \, dt = \hat{g}(x)E_x(a).
\]

The following is the Fourier inversion Theorem for our context.
6.6. Proposition. Let $a$ be an $\alpha$-integrable element of $A$ whose Fourier transform is absolutely integrable, that is, such that $\int_G \|E_x(a)\| \, dx < \infty$. Then for every $t$ in $G$ one has that

$$\int_G \langle x, t \rangle E_x(a) \, dx = a_t(a).$$

Proof. Initially observe that the map $x \mapsto E_x(a)$ is continuous in the strict topology of $M(A)$ by 6.3, and hence the integral in the statement is well defined as a strict integral. Therefore, in order to prove the statement, it is enough to show that for every $b \in A$ and vectors $\xi, \eta \in \mathcal{H}$ (recall that $A$ is represented as a non-degenerate algebra of operators on $\mathcal{H}$) one has that

$$\int_G \langle x, t \rangle \langle E_x(a)b\xi, \eta \rangle \, dx = \langle a_t(a)b\xi, \eta \rangle.$$ 

Consider the function

$$\psi : t \in G \mapsto \langle a_t(a)b\xi, \eta \rangle \in \mathbb{C}.$$ 

Since $a$ is $\alpha$-integrable we have that $\psi$ is in $L_1(G)$. The inverse Fourier transform of $\psi$ is clearly given by

$$\hat{\psi}(x) = \langle E_x(a)b\xi, \eta \rangle.$$ 

By hypothesis $\psi$ is integrable and hence the result follows from the classical Fourier inversion Theorem [HR: 31.44.(c)].

7. The dual unitary group.

For a given $x$ in $\hat{G}$ consider the unitary operator $V_x$ on $L_2(G, \mathcal{H})$ defined by

$$V_x\xi |_t = \langle x, t \rangle \xi(t), \quad \xi \in L_2(G, \mathcal{H}), \quad t \in G.$$ 

It is well known that the correspondence $x \mapsto V_x$ is a strongly continuous unitary representation of $\hat{G}$ on $L_2(G, \mathcal{H})$ which we shall call the dual unitary group.

Let $T$ be an integral operator with kernel $k$. Then for every $\xi \in C_c(G, \mathcal{H})$ and $t \in G$ we have

$$V_xTV_x^{-1}\xi |_t = \langle x, t \rangle \int_G k(t, s) \langle x^{-1}, s \rangle \xi(s) \, ds = \int_G \langle x, ts^{-1} \rangle k(t, s) \xi(s) \, ds,$$

and hence we see that $V_xTV_x^{-1}$ is an integral operator whose kernel is given by $k_x(t, s) = \langle x, ts^{-1} \rangle k(t, s)$. It is also evident that if $T$ is a Laurent operator then so is $V_xTV_x^{-1}$. In this case let $f$ be the symbol of $T$. Then the symbol $f_x$ of $V_xTV_x^{-1}$ is given by

$$f_x(r) = \alpha_r(k_x(r, e)) = \langle x, r \rangle \alpha_r(k(r, e)) = \langle x, r \rangle f(r), \quad r \in G.$$ 

In particular, given $f \in C_c(G, A)$ we saw that $\rho(f)$ is the Laurent operator with symbol $f$ and hence $V_x\rho(f)V_x^{-1}$ is the Laurent operator with symbol

$$\hat{\alpha}_x(f)(r) := \langle x, r \rangle f(r), \quad r \in G.$$ 

In other words

$$V_x\rho(f)V_x^{-1} = \rho(\hat{\alpha}_x(f)).$$ 

Observe that $\hat{\alpha}$ is, up to a sign convention, the dual action of $\hat{G}$ on $A \rtimes_\alpha G$, as defined in [P: 7.8.3], and hence we see that the pair $(\rho, V)$ is a covariant representation of the dual $C^*$-dynamical system $(A \rtimes_\alpha G, \hat{G}, \hat{\alpha})$.

Again using [P: 7.8.3] we have that the dual action is strongly continuous and hence for each operator $T$ in $A \rtimes_\alpha G$ the map

$$x \in \hat{G} \mapsto V_xTV_x^{-1} \in \mathcal{B}(L_2(G, \mathcal{H}))$$ 

is norm continuous.
7.1. **Definition.** We say that a bounded operator $T \in \mathcal{B}(L_2(G, \mathcal{H}))$ is **continuous with respect to $V$**, or just **$V$-continuous**, if the map given by $x \in \hat{G} \mapsto V_x TV^{-1}_x \in \mathcal{B}(L_2(G, \mathcal{H}))$ is continuous in norm.

So, for any $f$ in $C_c(G, A)$, the operator $\rho(f)$ is a $V$-continuous Laurent operator. We would now like to prove a converse of this statement, in which we will use the following:

7.2. **Lemma.** There exists a net $\{g_i\}_{i \in \Lambda} \subseteq L_1(G)$ such that $\hat{g}_i$ has compact support and $\|g_i\|_1 = 1$ for all $i \in \Lambda$, and also such that $g_i * f$ converges uniformly to $f$ for any bounded uniformly continuous function $f$ from $G$ to any Banach space $X$.

**Proof.** Let $\mathcal{I}(G)$ be the ideal in $L_1(G)$ (under convolution) formed by all $g \in L_1(G)$ such that $\hat{g}$ has compact support. Since $L_1(G)$ satisfies Ditkin’s condition [HR: 39.29] we have that $\mathcal{I}(G)$ is dense in $L_1(G)$.

The usual argument of taking functions supported in smaller and smaller neighborhoods of the unit of $G$ provides a net as required, except that $\hat{g}_i$ may not have compact support. For each $(i, n) \in \Lambda \times \mathbb{N}$ choose $g_{i,n} \in \mathcal{I}(G)$ such that $\|g_i - g_{i,n}\| \leq 1/n$. One may now easily prove that the net $\{g_{i,n}\}_{(i,n) \in \Lambda \times \mathbb{N}}$ satisfies the required properties.

This brings us to the main result of this section:

7.3. **Theorem.** Let $T$ be a Laurent operator. Then $T$ belongs to $A \ltimes_o G$ if and only if $T$ is continuous with respect to the dual group $V$.

**Proof.** The “only if” part has already been verified in the discussion before 7.1 so let us deal with the converse and hence we suppose that $T$ is a $V$-continuous Laurent operator. Therefore the expression

$$\tau(x) := V_x TV^{-1}_x$$

defines a norm continuous $\mathcal{B}(L_2(G, \mathcal{H}))$-valued function on $\hat{G}$. For $x, y \in \hat{G}$ note that

$$\|\tau(x) - \tau(y)\| = \|V_x TV^{-1}_x - V_y TV^{-1}_y\| = \|T - V^{-1}_x V_y TV^{-1}_x\| = \|\tau(e) - \tau(x^{-1}y)\|.$$ 

Therefore the continuity of $\tau$ at the identity group element $e$ implies that $\tau$ is in fact uniformly continuous on $\hat{G}$. Reversing the roles of $G$ and $\hat{G}$ in 7.2, let $\{g_i\}$ be a net in $L_1(G)$ such that $\hat{g}_i$ has compact support in $G$ and such that $\tau_i := g_i * \tau$ converges uniformly to $\tau$. Note that

$$\tau_i(x) = \int_G g_i(y) \tau(y^{-1}x) dy = \int_G g_i(y)V_y^{-1}V_x TV^{-1}_x V_y dy = V_x \left( \int_G g_i(y)V_y^{-1}TV_y dy \right) TV^{-1}_x.$$

Let us further investigate the operator

$$T_i := \tau_i(e) = \int_G g_i(y)V_y^{-1}TV_y dy$$

appearing above. For this purpose let $\xi, \eta \in C_c(G, \mathcal{H})$ and note that

$$\langle T_i \xi, \eta \rangle = \int_G g_i(y) \langle TV_y \xi, V_y \eta \rangle dy = \int_G g_i(y) \int_G \int_G \langle k(t, s) \left( V_y \xi \right)_t, V_y \eta \rangle_k ds dt dy = 
\int_G \int_G \int_G g_i(y) \langle ty, t^{-1}s \rangle \langle k(t, s) \xi(s), \eta(t) \rangle ds dt dy = 
\int_G \int_G \hat{g}_i(t^{-1}s) \langle k(t, s) \xi(s), \eta(t) \rangle ds dt.$$

It follows that $T_i$ is an integral operator with kernel $k_i(t, s) := \hat{g}_i(t^{-1}s)k(t, s)$. It is evident that $k_i$ satisfies 2.13 so that $T_i$ is a Laurent operator. The symbol $f_i$ of $T_i$ may be computed in terms of the symbol $f$ of $T$ as follows:

$$f_i(r) = \alpha_r(k_i(r, e)) = \hat{g}_i(r^{-1})\alpha_r(k(r, e)) = \hat{g}_i(r^{-1})f(r).$$

Since $f$ is continuous and $\hat{g}_i \in C_c(G)$ we have that $f_i$ is in $C_c(G, A)$. So by 5.5, $T_i$ belongs to $A \ltimes_o G$. Finally, given that $\tau_i$ converges uniformly to $\tau$ we have that

$$T = \lim_i \tau_i(e) = \lim_i T_i,$$

and hence $T \in A \ltimes_o G$ as well. \qed
Given \( a, b \in \mathcal{N}_a \) we have seen in 5.6 that \( \zeta(a)\zeta(b)^* \) is a Laurent operator. We may therefore apply the above result to determine whether or not this operator belongs to \( A \rtimes_a G \). Before that we need the following:

**7.4. Lemma.** Let \( a, b \in \mathcal{N}_a \). Then for all \( x \in \hat{G} \) we have

\[
\zeta(a)^*V_x\zeta(b) = E_x(a^*b).
\]

**Proof.** For \( v, w \in \mathcal{H} \) and \( c \in A \) we have

\[
\langle \zeta(a)^*V_x\zeta(b)cv, w \rangle = \langle V_x\zeta(b)cv, \zeta(a)w \rangle = \int_G \langle <x, t> \alpha_{t^{-1}}(b)cv, \alpha_{t^{-1}}(a)w \rangle \, dt =
\]

\[
= \int_G \langle <x, t> \alpha_t(a^*b)cv, w \rangle \, dt = \langle E_x(a^*b)cv, w \rangle.
\]

Observe that, since \( G \) is abelian, its modular function \( \Delta \) is identically 1 and hence was omitted in the calculation above. Since \( A \) is non-degenerate the proof is complete. \( \square \)

The following result is intended to answer the question posed at the end of section 4 for the abelian group case, giving a necessary and sufficient condition for \( \zeta(a)\zeta(b)^* \) to belong to \( A \rtimes_a G \), for a given pair \( a, b \in \mathcal{N}_a \).

**7.5. Theorem.** Let \( a, b \in \mathcal{N}_a \) and denote \( p := a^*a \) and \( q := b^*b \). Then the following are equivalent:

1. \( \zeta(a)\zeta(b)^* \) belongs to \( A \rtimes_a G \),
2. \( \|E_{xz}(p)E_y(q) - E_x(p)E_{zy}(q)\| \) converges to zero, uniformly in \( x \) and \( y \), as \( z \to e \),
3. One has that

\[
\lim_{z \to e} \|E_z(p)E_e(q) - E_x(p)E_e(q)\| = 0
\]

and

\[
\lim_{z \to e} \|E_z(p)E_{z^{-1}}(q) - E_e(p)E_e(q)\| = 0.
\]

**Proof.** Observe that, by 7.4, for all \( x, y, \) and \( z \) in \( \hat{G} \) we have

\[
\|E_{xz}(p)E_y(q) - E_x(p)E_{zy}(q)\| \leq \|\zeta(a)^*V_{xz}\zeta(a)\zeta(b)^*V_y\zeta(b) - \zeta(a)^*V_x\zeta(b)^*V_{zy}\zeta(b)\| \\
\leq \|\zeta(a)^*V_x\| \|V_{xz}\zeta(a)\zeta(b)^* - \zeta(a)\zeta(b)^*V_x\| \|V_y\zeta(b)\| = \\
= \|\zeta(a)\| \|V_x\zeta(a)\zeta(b)^*V_x^{-1} - \zeta(a)\zeta(b)^*\| \|\zeta(b)\|.
\]

Suppose that \( \zeta(a)\zeta(b)^* \in A \rtimes_a G \). Then by 7.3 we have that \( \zeta(a)\zeta(b)^* \) is a \( V \)-continuous operator and hence (ii) holds.

That (ii)\( \Rightarrow \) (iii) follows by taking \( x = y = e \) on the one hand, and \( x = e \) and \( y = z^{-1} \) on the other. In order to verify (iii)\( \Rightarrow \) (i) we claim that

\[
\lim_{x \to e} \|V_x\zeta(a)\zeta(b)^*V_x^* - \zeta(a)\zeta(b)^*\| = 0.
\]

Observe that the \( C^* \)-identity \( \|a^*a\| = \|a\|^2 \) implies that \( \|aa^*a\| = \|a\|^3 \) which is the form we choose to evaluate the norm in the limit above. We have

\[
\left( V_x\zeta(a)\zeta(b)^*V_x^* - \zeta(a)\zeta(b)^* \right) \left( V_x\zeta(a)\zeta(b)^*V_x^* - \zeta(a)\zeta(b)^* \right) \left( V_x\zeta(a)\zeta(b)^*V_x^* - \zeta(a)\zeta(b)^* \right) = \\
= V_x\zeta(a) \left( \zeta(b)^*\zeta(b)\zeta(a)^*\zeta(a) - \zeta(b)^*V_x^*\zeta(b)\zeta(a)^*V_x\zeta(a) \right) \zeta(b)^*V_x^* +
\]

\[
= V_x\zeta(a) \left( \zeta(b)^*\zeta(b)\zeta(a)^*\zeta(a) - \zeta(b)^*V_x^*\zeta(b)\zeta(a)^*V_x\zeta(a) \right) \zeta(b)^*V_x^* +
\]

\[
= V_x\zeta(a) \left( \zeta(b)^*\zeta(b)\zeta(a)^*\zeta(a) - \zeta(b)^*V_x^*\zeta(b)\zeta(a)^*V_x\zeta(a) \right) \zeta(b)^*V_x^* +
\]
8.1. Definition. If \( \zeta(a) \zeta(b) \zeta(a)^* V_x \zeta(a) - \zeta(b)^* V_x \zeta(b) \zeta(a)^* \zeta(a) \) \( \zeta(b)^* V_x^* \) +
\(+ \zeta(a) \zeta(b)^* \zeta(b) \zeta(a)^* \zeta(a) - \zeta(b)^* \zeta(b) \zeta(a)^* V_x \zeta(a) \) \( \zeta(b)^* \) +
\(+ \zeta(a) \zeta(b)^* \zeta(b) \zeta(a)^* \zeta(a) - \zeta(b)^* V_x \zeta(b) \zeta(a)^* V_x \zeta(a) \) \( \zeta(b)^* \) =
\( V_x \zeta(a) \left( E_c(q)E_c(p) - E_{x^{-1}}(q)E_{x}(p) \right) \zeta(b)^* V_x^* + \zeta(a) \left( E_c(q)E_c(p) - E_x(q)E_c(p) \right) \zeta(b)^* V_x^* +
+ V_x \zeta(a) \left( E_{x^{-1}}(q)E_c(p) - E_x(q)E_{x^{-1}}(p) \right) \zeta(b)^* + \zeta(a) \left( E_x(q)E_x(p) - E_{x^{-1}}(q)E_{x^{-1}}(p) \right) \zeta(b)^* \),
Taking adjoints and using 6.4.(i) we conclude from (iii) that the above tends to zero as \( z \to e \). This proves our claim and hence that the function
\( \tau : x \in \hat{G} \mapsto V_x \zeta(a) \zeta(b)^* V_x^* \in \mathcal{B}(L_2(G, \mathcal{H})) \)
is continuous at \( x = e \). As in the proof of 7.3 this implies that \( \tau \) is uniformly continuous on \( \hat{G} \) and hence that \( \zeta(a) \zeta(b)^* \) is \( V \)-continuous. That \( \zeta(a) \zeta(b)^* \in A_{\alpha, G} \) then follows from 5.6 and 7.3. \( \square \)

8. Relative continuity.
As already mentioned, one of the crucial aspects of this subject is the question of whether or not \( \zeta(a) \zeta(b)^* \in A_{\alpha, G} \), for given \( a \) and \( b \) in \( \mathcal{N}_\alpha \). Theorem 7.5, which gives necessary and sufficient conditions for this to happen, will therefore acquire a special relevance to us. We will see that a deep understanding of this issue is the basis for the study of a certain Morita–Rieffel equivalence as well as the spectral theory which we plan to develop.

8.1. Definition. Let \( a, b \in A \) be \( \alpha \)-integrable elements. We say that the pair \((a, b)\) is relatively continuous, and denote it by \( a \overset{rc}{\sim} b \), if \( \| E_{xz}(a)E_y(b) - E_x(a)E_{zy}(b) \| \to 0 \) uniformly in \( x \) and \( y \), as \( z \to e \).

Exercising the new terminology we have the following immediate consequence of 7.5:

8.2. Corollary. If \( a, b \in \mathcal{N}_\alpha \) then \( a^*a \overset{rc}{\sim} b^*b \) if and only if \( \zeta(a)\zeta(b)^* \in A_{\alpha, G} \).

We do not claim that the relation of being relatively continuous is reflexive, symmetric, or transitive. Instead we have:

8.3. Proposition. Let \( a, b, \) and \( c \) be \( \alpha \)-integrable elements.
(i) If \( a \overset{rc}{\sim} c \) and \( b \overset{rc}{\sim} c \) then \( a + b \overset{rc}{\sim} c \).
(ii) If \( a \overset{rc}{\sim} b \) then \( b^* \overset{rc}{\sim} a^* \).
(iii) If \( a \overset{rc}{\sim} b \) then for every \( w \in \hat{G} \) and every \( m \in M_w(A) \) one has that both \( ma \overset{rc}{\sim} b \) and \( a \overset{rc}{\sim} bm \).
(iv) If \( a \overset{rc}{\sim} c \) and \( c \overset{rc}{\sim} b \) then for every \( w \in \hat{G} \) we have that \( aE_w(c) \overset{rc}{\sim} b \) and \( a \overset{rc}{\sim} E_w(c)b \).

Proof. The first assertion is trivial. In order to prove (ii) note that by 6.4.(i) we have for all \( x, y, z \in \hat{G} \) that
\[ (E_{xz}(b^*)E_y(a^*) - E_x(b^*)E_{zy}(a^*))^* = -(E_{y^{-1}x^{-1}}(a)E_{x^{-1}}(b) - E_{y^{-1}}(a)E_{x^{-1}x^{-1}}(b)), \]
from where (ii) follows easily. As for (iii), using 6.4.(ii), we have for all \( x, y, z \in \hat{G} \) that
\[ \| E_{xz}(ma)E_y(b) - E_{xz}(ma)E_{zy}(b) \| = \| mE_{w^{-1}xz}(a)E_y(b) - mE_{w^{-1}x}(a)E_{zy}(b) \| \leq \| m \| \| E_{w^{-1}xz}(a)E_y(b) - E_{w^{-1}x}(a)E_{zy}(b) \|. \]
This proves that $ma \sim b$. The proof that $a \sim bm$ goes along similar lines. With respect to (iv) observe that by 6.4(iv) we have
\[
\|E_{xw}(aE_w(c))E_y(b) - E_x(aE_w(c))E_{zy}(b)\| = \\
= \|E_{xw^{-1}}(a)E_w(c)E_y(b) - E_{xw^{-1}}(a)E_w(c)E_{zy}(b)\| \leq \\
\leq \|E_{xw^{-1}}(a)E_w(c)E_y(b) - E_{xw^{-1}}(a)E_{zw}(c)E_y(b)\| + \\
+ \|E_{xw^{-1}}(a)E_{zw}(c)E_y(b) - E_{xw^{-1}}(a)E_w(c)E_{zy}(b)\| \leq \\
\leq \|E_{xw^{-1}}(a)E_w(c)E_y(b) - E_{xw^{-1}}(a)E_{zw}(c)E_y(b)\| \leq \\
\|E_{xw^{-1}}(a)E_w(c)E_y(b) - E_{xw^{-1}}(a)E_{zw}(c)E_y(b)\| + \|a\|_1 \|E_{zw}(c)E_y(b) - E_w(c)E_{zy}(b)\|,
\]
where we have used 6.2 in the last step. This proves that $aE_w(c) \sim b$. Again the proof that $a \sim E_w(c)b$ follows similarly. \qedhere

Relative continuity enjoys a curious hereditary property which we discuss next.

8.4. Proposition. Let $a, b, c \in A$ be positive $\alpha$-integrable elements such that $a \leq b \sim c$. Then $a \sim c$.

Proof. Write $a = a^*_1a_1$, $b = b^*_1b_1$, and $c = c^*_1c_1$, for $a_1, b_1, c_1 \in A$. Since $a \leq b$ we have for all $v$ in $\mathcal{H}$ that $\|a_1(v)\| = \langle a_1^*a_1(v), v \rangle^{1/2} \leq \langle b_1^*b_1(v), v \rangle^{1/2} = \|b_1(v)\|$. It therefore follows that there exists a bounded operator $T$ on $\mathcal{H}$ with $\|T\| \leq 1$ such that $a_1 = Tb_1$.

Even though $T$ may not be in $A$ we would like to make sense of the expression $\pi(T)$ and also to show that $\pi(T)\zeta(b_1) = \zeta(Tb_1)$, mimicking 4.1(i). In order to accomplish this let us suppose, without loss of generality, that there is a strongly continuous unitary representation $U$ of $G$ on $\mathcal{H}$ such that $\alpha_t(a) = U_tau_t^{-1}$, for all $a$ in $A$. We may then define
\[
\pi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(L^2(G, \mathcal{H})),
\]
by
\[
\pi(S)\xi = U_t^{-1}SU_t\xi(t), \quad S \in \mathcal{B}(\mathcal{H}), \quad \xi \in L^2(G, \mathcal{H}), \quad t \in G,
\]
extending the representation $\pi$ of section 2. Observe that $\pi(S)\xi$ is a measurable function on $G$, and thus represents an element of $L^2(G, \mathcal{H})$, because $U$ is strongly continuous. We now claim that $\pi(T)\zeta(b_1) = \zeta(a_1)$. In fact, given $v \in \mathcal{H}$ and $t \in G$ we have
\[
\zeta(a_1)v|_t = a_t^{-1}(Tb_1)v = U_t^{-1}Tb_1U_tv = U_t^{-1}TU_t\alpha_t^{-1}(b_1)v = \\
= U_t^{-1}TU_t\left(\zeta(b_1)v|_t\right) = \left(\pi(T)\zeta(b_1)v\right)|_t,
\]
thus proving our claim. In order to show that $a \sim c$ let $x, y, z \in \hat{G}$ and observe that by 7.4
\[
\|E_{xz}(a)E_y(c) - E_x(a)E_{zy}(c)\| = \\
= \|\zeta(a_1)^*V_z\zeta(a_1)\zeta(c_1)^*V_y\zeta(c_1) - \zeta(a_1)^*V_z\zeta(a_1)\zeta(c_1)^*V_{zy}\zeta(c_1)\| \leq \\
\leq \|\zeta(a_1)^*V_z\|\|V_{\pi(T)}\zeta(b_1)\zeta(c_1)^* - \pi(T)\zeta(b_1)\zeta(c_1)^*V_z\|\|V_{\pi(T)}\zeta(c_1)\| = \ldots
\]
It is easy to show that $\pi(T)$ commutes with $V_z$. Therefore the above equals
\[
\ldots = \|\zeta(a_1)\|\|\pi(T)\zeta(b_1)\zeta(c_1)^* - \pi(T)\zeta(b_1)^{1/2}\zeta(c_1)^*V_z\|\|\zeta(c_1)\| \leq \\
\leq \|\zeta(a_1)\|\|\pi(T)\|\|\zeta(b_1)\zeta(c_1)^* - \zeta(b_1)^{1/2}\zeta(c_1)^*V_z\|\|\zeta(c_1)\|.
\]
Since, by hypothesis we have that $b^*b_1 \sim c_1^*c_1$ we conclude by 8.2 that $\zeta(b_1)\zeta(c_1)^* \in A \rtimes_s G$ and hence that this is a $V$-continuous operator. It follows that the last expression displayed above tends to zero as $z \rightarrow e$ and hence that $a \sim c$. \qedhere
8.5. Proposition. Let $a$ and $b$ be $\alpha$-integrable elements such that $a \preceq b$. Then, for each $g \in C_*(G)$ one has that $a' \preceq b$, where $a' = \int_G g(t)\alpha(t)\,dt$.

Proof. Given $x, y, z \in \hat{G}$ we have, using 6.5, that
\[
\|E_{xz}(a')E_y(b) - E_x(a')E_{zy}(b)\| = \|\hat{g}(xz)E_{xz}(a)E_y(b) - \hat{g}(x)E_x(a)E_{zy}(b)\| \leq \\
\leq \|\hat{g}(xz)E_{xz}(a)E_y(b) - \hat{g}(x)E_x(a)E_{zy}(b)\| + \|\hat{g}(x)E_{xz}(a)E_y(b) - \hat{g}(x)E_x(a)E_{zy}(b)\| \leq \\
\leq \|\hat{g}(xz) - \hat{g}(x)\| \|a\| \|b\| + \|\hat{g}(x)\| \|E_{xz}(a)E_y(b) - E_x(a)E_{zy}(b)\|
\]
which tends to zero, uniformly in $x$ and $y$, as $z \to e$, because $\hat{g}$ is uniformly continuous and bounded. \hfill \Box

We will often be concerned with sets of mutually relatively continuous elements. For this reason we make the following:

8.6. Definition. Let $\mathcal{W}$ be a set of $\alpha$-integrable elements.

(i) We say that $\mathcal{W}$ is a relatively continuous set if for every $a, b \in \mathcal{W}$ one has that $a \preceq b$.

(ii) We say that $\mathcal{W}$ is spectrally invariant if, given $a \in \mathcal{W}$ and $x \in \hat{G}$, one has that $E_x(a)\mathcal{W} \subseteq \mathcal{W}$ and $WE_x(a) \subseteq \mathcal{W}$.

(iii) The spectrally invariant hull of $\mathcal{W}$, denoted $\mathcal{W}$, is the intersection of all spectrally invariant sets of $\alpha$-integrable elements containing $\mathcal{W}$.

Observe that the intersection of any number of spectrally invariant sets is again spectrally invariant and so is $\mathcal{W}$.

We should also note that the elements of a relatively continuous set satisfy $a \preceq a$, which is by no means automatic.

8.7. Proposition. Let $\mathcal{W}$ be a relatively continuous set of $\alpha$-integrable elements. Then the spectrally invariant hull $\mathcal{W}$ of $\mathcal{W}$ is also relatively continuous.

Proof. In order to avoid repetition, during the course of this proof $a, b,$ and $c,$ with or without subscripts, will always refer to elements of $\mathcal{W}$, and $x, y,$ and $z$ will denote elements of $\hat{G}$.

Given $n$-vectors $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{x} = (x_1, \ldots, x_n)$ let
\[
E_{\vec{x}}(\vec{a}) = E_{x_1}(a_1) \cdots E_{x_n}(a_n).
\]
If $\vec{b} = (b_1, \ldots, b_m)$ and $\vec{y} = (y_1, \ldots, y_m)$ is another such pair of vectors consider the set $E_{\vec{y}}(\vec{a})WE_{\vec{y}}(\vec{b})$. Its elements are thus of the form $u = E_{\vec{x}}(\vec{a})cE_{\vec{y}}(\vec{b})$. Observe that, by 6.4, $u$ is $\alpha$-integrable and for $z \in \hat{G}$ we have
\[
E_z(u) = E_{\vec{x}}(\vec{a})E_{\vec{x}}(\vec{c})E_{\vec{y}}(\vec{b}),
\]
where $z' = (x_1 \ldots x_n)^{-1}z(y_1 \ldots y_m)^{-1}$. It is thus clear that the union of all sets of the form $E_{\vec{x}}(\vec{a})WE_{\vec{y}}(\vec{b})$, as above, is spectrally invariant and hence, being the smallest among those which contain $\mathcal{W}$, coincides with $\mathcal{W}$.

It therefore suffices to show that any two elements
\[
u = E_{\vec{x}}(\vec{a})cE_{\vec{y}}(\vec{b}), \text{ and } u' = E_{\vec{x}'}(\vec{a}')c'E_{\vec{y}'}(\vec{b}')
\]
of the above form satisfy $\nu \preceq u'$. In view of 8.3.(iii) it suffices to consider the case where
\[
u = cE_{\vec{y}}(\vec{b}), \text{ and } u' = E_{\vec{x}'}(\vec{a}')c'.
\]
In order to prove this we use induction on $m = |\vec{b}| + |\vec{a}'|$, where $|\cdot|$ denotes the number of coordinates of a given vector. If $m = 0$ this follows from the hypothesis. Now, suppose that $m > 0$ and hence either $|\vec{b}| > 0$ or $|\vec{a}'| > 0$. Suppose first that $n := |\vec{b}| > 0$ and let $\vec{b}' = (b_1, \ldots, b_{n-1})$ and $\vec{y}' = (y_1, \ldots, y_{n-1})$ so that $E_{\vec{y}}(\vec{b}) = E_{\vec{y}'}(\vec{b}')E_{y_n}(b_n)$. By the induction hypothesis we have that
\[
cE_{\vec{y}'}(\vec{b}') \preceq b_n \preceq u'.
\]
By 8.3.(iv) we then conclude that $\nu \preceq u'$. If $|\vec{b}| = 0$ then $|\vec{a}'| > 0$ and a similar argument may be used to complete the proof. \hfill \Box
8.8. Definition. For any subset $X \subseteq A$ we denote by $\text{Alg}(X, \alpha)$ the smallest $\alpha$-invariant sub-$C^*$-algebra of $A$ containing $X$.

8.9. Proposition. If $\mathcal{W} \subseteq A$ is a set of $\alpha$-integrable elements then $\text{Alg}(\mathcal{W}, \alpha) = \text{Alg}(\hat{\mathcal{W}}, \alpha)$, where $\hat{\mathcal{W}}$ is the spectrally invariant hull of $\mathcal{W}$.

Proof. Let $B$ be an $\alpha$-invariant sub-$C^*$-algebra of $A$. We claim that the set $B_1$ of $\alpha$-integrable elements in $B$ is spectrally invariant. In fact, given $a, b \in B_1$, $x \in \hat{G}$, and $t \in G$ we have that $\langle x, t \rangle \alpha_t(a)b \in B$ and hence the unconditional integral of this expression with respect to $t$, namely $E_x(a)b$, also belongs to $B$. Since $E_x(a)b$ is $\alpha$-integrable by 6.4.(ii) we have that $E_x(a)b \in B_1$. Similarly one shows that $bE_x(a) \in B_1$, hence proving our claim.

Taking $B = \text{Alg}(\mathcal{W}, \alpha)$ in the argument above we thus conclude that $\hat{\mathcal{W}} \subseteq \text{Alg}(\mathcal{W}, \alpha)$, from which it follows that $\text{Alg}(\hat{\mathcal{W}}, \alpha) \subseteq \text{Alg}(\mathcal{W}, \alpha)$. The converse inclusion is trivial. \hfill \Box

By definition a cone is a subset of $A$ which is closed under addition and multiplication by positive (i.e. $\geq 0$) scalars. We would now like to study relative continuity for cones. Recall that $\mathcal{P}_\alpha$ denotes the hereditary cone of all positive $\alpha$-integrable elements of $A$.

8.10. Proposition. Let $\mathcal{P} \subseteq \mathcal{P}_\alpha$ be a relatively continuous cone which is maximal among the relatively continuous cones contained in $\mathcal{P}_\alpha$. Then

(i) $\mathcal{P}$ is hereditary.

(ii) If $g \in C_c(G)$ is a positive function and $a \in \mathcal{P}$ then $a' \in \mathcal{P}$, where $a' = \int_G g(t)\alpha_t(a)\, dt$.

Proof. It is useful to keep in mind that relative continuity is a symmetric relation among self-adjoint elements by 8.3.(ii).

Suppose that $0 \leq a \leq b$, where $a \in A$ and $b \in \mathcal{P}$. By 2.4.(i) we have that $a$ is $\alpha$-integrable. Define $\mathcal{P} = \{\lambda a + p : \lambda \geq 0, p \in \mathcal{P}\}$, which then turns out to be a cone consisting of $\alpha$-integrable positive elements. We claim that $\mathcal{P}$ is relatively continuous. In order to prove this it clearly suffices to show that $a \sim_\mathcal{P} a$ and that $a \sim_\mathcal{P} p$ for all $p \in \mathcal{P}$. Since $\mathcal{P}$ is relatively continuous we have that $a \leq b \sim p$ for every $p \in \mathcal{P}$. So $a \sim p$ by 8.4. It now follows that $a \leq b \sim a$ and hence that $a \sim a$. This proves that $\mathcal{P}$ is relatively continuous. Since $\mathcal{P}$ is maximal by hypothesis, we conclude that $\mathcal{P} = \mathcal{P}$ and hence that $a \in \mathcal{P}$.

To prove the second assertion let $\mathcal{P} = \{\lambda a + p : \lambda \geq 0, p \in \mathcal{P}\}$, which is a subcone of $\mathcal{P}_\alpha$ by 3.6. Again we claim that $\mathcal{P}$ is relatively continuous and to prove it we must once more show that $a' \sim_\mathcal{P} a'$ and that $a' \sim_\mathcal{P} p$ for all $p \in \mathcal{P}$. Since $a \in \mathcal{P}$ we have that $a \sim_\mathcal{P} p$ and hence 8.5 implies that $a' \sim_\mathcal{P} p$. In particular, taking $p = a$ we have that $a \sim_\mathcal{P} a'$ and 8.5 finally gives $a' \sim_\mathcal{P} a'$. This proves that $\mathcal{P}$ is relatively continuous and the conclusion follows as above. \hfill \Box

8.11. Corollary. Let $\mathcal{W}$ be a relatively continuous set of positive $\alpha$-integrable elements. Then $\mathcal{W}$ is contained in a relatively continuous cone $\mathcal{P}$ satisfying 8.10.(i-ii).

Proof. It is easy to see that the set $\mathcal{P}_0$ formed by all linear combinations with positive coefficients of elements of $\mathcal{W}$ is a relatively continuous cone. By Zorn’s Lemma let $\mathcal{P}$ be a maximal relatively continuous cone containing $\mathcal{P}_0$. Then $\mathcal{P}$ satisfies the required properties by 8.10. \hfill \Box
9. Morita–Rieffel equivalence.

In this section we will apply the knowledge gained so far to the problem of establishing a Morita–Rieffel equivalence between subalgebras of \( M_c(A) \) and ideals of \( A \rtimes \alpha G \).

For the time being we shall fix a relatively continuous cone \( P \subseteq P_\alpha \) satisfying 8.10.(i–ii). Let

\[
\mathcal{N} = \{ a \in A : a^*a \in P \}, \quad \text{and} \quad \mathcal{M} = \text{span}(P).
\]

Since \( P \) is supposed hereditary we have by 2.6 that \( \mathcal{N} \) is a left ideal of \( A \), clearly contained in \( \mathcal{N}_\alpha \), and \( \mathcal{M} \) is a subalgebra of \( A \) contained in \( \mathcal{M}_\alpha \).

We shall denote by \( \mathcal{X} \) the closure of \( \zeta(\mathcal{N}) \) in \( \mathcal{X}_\alpha \).

9.1. Proposition. One has that \( (A \rtimes \alpha G) \mathcal{X} \subseteq \mathcal{X} \).

Proof. We will obviously use 4.2, observing that the second hypothesis required there holds for \( D = A \) since \( \mathcal{N} \) is a left ideal. As for 4.2.(i) let \( a' = \int_G g(t) \alpha_t(a) \, dt \) where \( g \in C_c(G) \) and \( a \in \mathcal{N} \). Observe that, by 3.5, we have

\[
a^*a' \leq |\text{supp}(g)| \int_G |g(t)|^2 \alpha_t(a^*a) \, dt.
\]

Since \( |g(t)|^2 \) is a positive compactly supported function on \( G \) and \( a^*a \in P \) we have that the right hand side above gives an element in \( P \) (by the fact that \( P \) satisfies 8.10.(i–ii)). Also, since \( P \) is hereditary we have that \( a^*a' \in P \) and hence that \( a' \in \mathcal{N} \). This shows that \( \mathcal{N} \) satisfies the requirements in 4.2 and hence the proof is complete.

9.2. Theorem. Let \( P \) be a relatively continuous cone consisting of positive \( \alpha \)-integrable elements of \( A \). Suppose that \( P \) satisfies 8.10.(i–ii) (e.g. if \( P \) is maximal). Set \( \mathcal{N} = \{ a \in A : a^*a \in P \} \) and \( \mathcal{M} = \text{span}(P) \), and let \( \mathcal{X} \) be the closure of \( \zeta(\mathcal{N}) \) within \( \mathcal{X}_\alpha \). Then \( \mathcal{X} \) is a left Hilbert \( (A \rtimes \alpha G) \)–module and hence it establishes a Morita–Rieffel equivalence between \( \mathcal{X}^*\mathcal{X} \), which is a subalgebra of \( M_c(A) \) coinciding with \( E_c(\mathcal{M}) \), and the ideal \( \mathcal{X}^*\mathcal{X}^\ast \) in \( A \rtimes \alpha G \).

Proof. By the last Proposition we see that \( \mathcal{X} \) is a left \( (A \rtimes \alpha G) \)–module under multiplication of operators. Given \( T, S \in \mathcal{X} \) we claim that \( T S^* \in A \rtimes \alpha G \). In fact, by definition of \( \mathcal{X} \) it is enough to show that \( \zeta(a)\zeta(b)^* \in A \rtimes \alpha G \) for all \( a, b \in \mathcal{N} \). Since \( a^*a \) and \( b^*b \) belong to \( P \), and \( P \) is relatively continuous, we have that \( a^*a \sim b^*b \) and hence the claim follows from 8.2. Therefore \( \mathcal{X} \) is a left Hilbert \( A \rtimes \alpha G \)–module for the inner-product \( (T|S)_L := TS^* \), and hence \( \mathcal{X}^*\mathcal{X}^\ast \) is an ideal in \( A \rtimes \alpha G \).

By 2.8.(iii) we have that \( \zeta(a)^*\zeta(b) = E_c(a^*b) \) and hence, using 2.6.(ii), we have

\[
\zeta(\mathcal{N})^*\zeta(\mathcal{N}) = E_c(\mathcal{N}^*\mathcal{N}) = E_c(\mathcal{M}) \subseteq M_c(A).
\]

It follows that \( \mathcal{X}^*\mathcal{X} = E_c(\mathcal{M}) \).

Using our last result in combination with 8.11 we obtain:

9.3. Corollary. Let \( \mathcal{W} \) be a relatively continuous set of positive \( \alpha \)-integrable elements. Then there exists a Morita–Rieffel equivalence between a subalgebra of \( M_c(A) \) containing \( E_c(\mathcal{W}) \) and an ideal in \( A \rtimes \alpha G \).

This brings about the following:

9.4. Question. Suppose that \( P_\alpha \) is dense in \( A_+ \). Does if follow that there exists a relatively continuous cone \( P \) which is also dense in \( A_+ \)?

9.5. Question. Is the subalgebra \( \mathcal{X}^*\mathcal{X} \) of \( M_c(A) \) always the same for different maximal relatively continuous cones?

If the above question is answered affirmatively then one should probably call \( \mathcal{X}^*\mathcal{X} \) the “generalized fixed point algebra” of \( \alpha \). We shall have more to say about this question in what follows.
10. Dual action on Fell bundles.

We would now like to discuss an important example of $C^\ast$-dynamical system, namely that of the dual action on the cross-sectional $C^\ast$-algebra of a Fell bundle.

This action is particularly well behaved from the point of view we wish to discuss and, in particular, question 9.4 will be answered affirmatively.

Another specially important aspect of this example is that the concept of spectral subspace is built in, as the bundle fibers. For this reason we will take this example as the model for our spectral theory to be developed in the next section.

Let $G$ be an abelian group with Pontryagin dual $\hat{G}$ and let $B = \{B_x\}_{x \in \hat{G}}$ be a Fell bundle over $\hat{G}$ (see [FD] for a comprehensive treatment of the theory of Fell bundles, also referred to as $C^\ast$-algebraic bundles). Let $C^\ast(B)$ be the cross sectional $C^\ast$-algebra of $B$ [FD: VIII.17.2] defined to be the enveloping $C^\ast$-algebra of the Banach *-algebra $L_1(B)$ formed by the integrable sections [FD: VIII.5.2].

As before we will fix a faithful representation of $C^\ast(B)$ on a Hilbert space $H$ and hence will think of $C^\ast(B)$ as an algebra of operators on $H$.

Denote by $C_c(B)$ the dense sub-algebra of $L_1(B)$ formed by the continuous, compactly supported sections [FD: II.14.2]. We remark that our notation differs from [FD] with respect to $C_c(B)$.

As described in [FD: VIII.5.8] (see also [E2: Section 5]) each $b_x$ in each fiber $B_x$ of $B$ defines a multiplier of $C^\ast(B)$ such that

\[(b_x f)(y) = b_x f(x^{-1} y), \quad \text{and} \quad (fb_x)(y) = f(yx^{-1})b_x,\]

for all $f \in L_1(B)$ and $y \in \hat{G}$.

The dual action of $G$ on $C^\ast(B)$, which we denote simply by $\alpha$, is determined by the expression

\[\alpha_t(f)\big|_x = \langle x, t \rangle f(x),\]

for all $f \in L_1(B)$, $t \in G$, and $x \in \hat{G}$. See [E2: Section 5] for details but beware of the change in conventions which we attempted to explain in footnote 3. It is easy to see that the natural extension of $\alpha$ to the multiplier algebra $M(C^\ast(B))$ satisfies

\[\alpha_t(b_x) = \langle x, t \rangle b_x,\]

for all $b_x$ in any $B_x$. Therefore $B_x \subseteq M_x(C^\ast(B))$ for all $x \in \hat{G}$. Since this inclusion may be proper, one may argue over which set deserves to be called the "$x$-spectral subspace" for the dynamical system $(C^\ast(B), G, \alpha)$.

We believe that $B_x$ is the "correct" choice and we hope to convince the reader of this in what follows. In particular we propose defining the "generalized fixed point algebra" to be $B_x$ among the many subalgebras of $M_x(C^\ast(B))$ which occur in examples.

An argument in favor of this point of view is related to the Takai duality Theorem: given an action of $\hat{G}$ on a $C^\ast$-algebra $A$ consider the corresponding semi-direct product bundle $B$ [FD: VIII.4.2]. Its cross sectional algebra is $A \rtimes \hat{G}$ and the dual action, as defined above, coincides with the classical notion of dual action [P: 7.8.3], up to a sign convention. Moreover the crossed product by the dual action is isomorphic to $A \otimes K(L_2(\hat{G}))$ [P: 7.9.3]. Thus, if we want the "generalized fixed point algebra" to be Morita–Rieffel equivalent to the crossed product (see [R4] and [R5: Section 6]) a natural choice is to take it to be $A$, which is precisely the unit fiber of the semi-direct product bundle!

According to Theorem 5.5 of [E2] every element of $C^\ast(B)$ of the form $p = f^*f$, with $f \in C_c(B)$, is $\alpha$-integrable and in addition\(^4\),

\[(10.1) \quad E_x(p) = p(x),\]

where the term in the right hand side is to be interpreted as a multiplier of $C^\ast(B)$ as explained above. It then follows that $C_c(B)$ is contained in $N_\alpha$.

---

\(^4\) Observe that [E2] uses a different sign convention both for the dual action and for the Fourier coefficients. Nevertheless these differences compensate each other in such a way that the statement of Theorem 5.5 in [E2] remains valid as it stands.
10.2. Proposition. The subset \( C_c(\mathcal{B})^2 \subseteq C^*(\mathcal{B}) \) of all linear combinations of products of elements in \( C_c(\mathcal{B}) \) is a relatively continuous set of \( \alpha \)-integrable elements.

Proof. As seen above \( a^*a \) is \( \alpha \)-integrable for all \( a \in C_c(\mathcal{B}) \). By the polarization formula it is then easy to see that \( a^*b \) is \( \alpha \)-integrable for all \( a, b \in C_c(\mathcal{B}) \). Since \( C_c(\mathcal{B}) \) is self-adjoint we then conclude that any element of \( C_c(\mathcal{B})^2 \) is \( \alpha \)-integrable.

We now have to prove that \( a^*b \overset{rc}{\sim} c^*d \) for all \( a, b, c, d \in C_c(\mathcal{B}) \). Again by the polarization formula it is enough to verify the case \( a^*a \overset{rc}{\sim} b^*b \). In order to do this we shall use the implication (iii)\( \Rightarrow \) (ii) of 7.5. So let \( p = a^*a \) and \( q = b^*b \) and observe that by 10.1 we have

\[
\lim_{x \to c} \|E_x(p)E_c(q) - E_c(p)E_x(q)\| = \lim_{x \to c} \|p(x)q(e) - p(e)q(x)\|,
\]

and

\[
\lim_{x \to e} \|E_x(p)E_{x^{-1}}(q) - E_c(p)E_x(q)\| = \lim_{x \to e} \|p(x)q(x^{-1}) - p(e)q(e)\|.
\]

Since the norm and multiplication are continuous on \( \mathcal{B} \), and both \( p \) and \( q \) are continuous sections, we see that the limits above vanish. This shows that 7.5.(iii) holds and hence also 7.5.(ii), concluding the proof. \( \square \)

As a consequence we have:

10.3. Corollary. Let \( a, b \in C_c(\mathcal{B}) \). Then \( \zeta(a)\zeta(b)^* \) belongs to \( C^*(\mathcal{B}) \times_{\alpha} G \).

Proof. By the previous result we have that \( a^*a \overset{rc}{\sim} b^*b \). So the conclusion follows from 8.2. \( \square \)

Observe that \( C_c(\mathcal{B})^2 \cap C^*(\mathcal{B})_+ \) is dense in \( C^*(\mathcal{B})_+ \), and hence \( C^*(\mathcal{B})_+ \) contains a dense cone which is relatively continuous. This answers question 9.4 affirmatively.

There is a very natural Hilbert module associated to any given Fell bundle. That is the \( B_e \)-module, denoted \( L_2(\mathcal{B}) \), defined to be the completion of \( C_c(\mathcal{B}) \) under the \( B_e \)-valued inner-product defined by the integral \( \int_G f(x)^*g(x) \, dx \), for \( f \) and \( g \) in \( C_c(\mathcal{B}) \). Our next result is intended to relate it to the present situation. Recall that \( \langle f, g \rangle_R = \int_G \alpha_t(f^*g) \, dt \), as defined in 2.7.

10.4. Proposition. Let \( f, g \in C_c(\mathcal{B}) \). Then \( \langle f, g \rangle_R = \int_G f(x)^*g(x) \, dx \).

Proof. By the polarization identity it suffices to consider \( f = g \). In this case we have

\[
\langle f, f \rangle_R = \int_G \alpha_t(f^*f) \, dt = E_c(f^*f)^{\circ} \text{(10.1)} \quad (f^*f)(e) = \int_G f(x)^*f(x) \, dx.
\]

We therefore conclude that the inclusion \( C_c(\mathcal{B}) \subseteq \mathcal{N}_\alpha \) respects the corresponding \( B_e \) valued inner-products and hence that \( L_2(\mathcal{B}) \) is isomorphic to the closure of \( \zeta(C_c(\mathcal{B})) \) within \( \mathcal{N}_\alpha \), as Hilbert \( B_e \)-modules.

10.5. Proposition. Let \( \mathcal{X} \) be the subset of \( \mathcal{X}_\alpha \) obtained by closing \( \zeta(C_c(\mathcal{B})) \). Then

\[
(C^*(\mathcal{B}) \times_{\alpha} G) \mathcal{X} \subseteq \mathcal{X}.
\]

Proof. We will derive this from 4.2, observing that 4.2.(ii) is satisfied for \( \mathcal{D} = C_c(\mathcal{B}) \). In order to prove 4.2.(i), let \( a \in C_c(\mathcal{B}) \) and denote the closed support of \( a \) by \( K \). Denote by \( C_K(\mathcal{B}) \) the subset of \( C_c(\mathcal{B}) \) formed by the continuous sections of \( \mathcal{B} \) vanishing on \( G \setminus K \). Equipped with the supremum norm \( C_K(\mathcal{B}) \) is a normed space and in fact a Banach space by [FD: II.13.13]. Consider the map

\[
\psi : t \in G \mapsto \alpha_t(a) \in C_K(\mathcal{B}),
\]

(that \( \alpha_t(a) \) is in \( C_K(\mathcal{B}) \) follows from [FD: II.13.14]). We claim that \( \psi \) is continuous. In fact, if a net \( \{t_i\} \) converges to \( t \) in \( G \), then

\[
\lim_{i} \sup_{x \in K} \| <x, t_i> - <x, t> \| = 0
\]
by the Pontryagin–van Kampen duality Theorem [HR: 24.8], from which the claim follows easily. Given $g$ in $C_c(G)$ we then have that the Bochner integral

$$\int_G g(t)\alpha_t(a)\,dt$$

converges and hence defines an element $a' \in C_K(\mathcal{B})$. Since the inclusions

$$C_K(\mathcal{B}) \to L_1(\mathcal{B}) \to C^*(\mathcal{B})$$

are obviously continuous we conclude that the above integral, if seen as the integral of a $C^*(\mathcal{B})$–valued function, also converges to $a'$. It therefore follows that $C_c(\mathcal{B})$ satisfies the hypothesis of 4.2, concluding the proof.

The following result is a generalization of the Takai duality for the kind of actions we are dealing with. It is perhaps also an argument in favor of defining the generalized fixed point algebra to be the unit fiber algebra, in the case of a dual action.

**10.6. Theorem.** Let $\mathcal{X}$ be the closure of $\zeta(C_c(\mathcal{B}))$ as above. Then $\mathcal{X}$ is a Hilbert bimodule over the unit fiber algebra $B_e$ on the right hand side and $C^*(\mathcal{B}) \rtimes_\alpha G$ on the left. Moreover $\overline{\mathcal{X}\mathcal{X}^*}$ is an ideal in $C^*(\mathcal{B}) \rtimes_\alpha G$ which is Morita–Rieffel equivalent to $B_e$. The imprimitivity bimodule implementing this equivalence may be taken to be $\mathcal{X}$.

**Proof.** As seen above $C_c(\mathcal{B}) \subseteq N_\alpha$ and $B_e \subseteq M_e(C^*(\mathcal{B}))$. Using 2.8.(ii) we then have that $\zeta(C_c(\mathcal{B}))B_e \subseteq \zeta(C_c(\mathcal{B}))$ and hence that $\mathcal{X}B_e \subseteq \mathcal{X}$. In other words $\mathcal{X}$ is a right $B_e$–module. From 10.5 we have that $(C^*(\mathcal{B}) \rtimes_\alpha G)\mathcal{X} \subseteq \mathcal{X}$ and hence $\mathcal{X}$ is a left $(C^*(\mathcal{B}) \rtimes_\alpha G)$–module.

For $T, S \in \mathcal{X}$ consider the inner-products

$$(T|S)_R = T^*S, \quad (T|S)_L = TS^*.$$

By 2.8.(iii) and 10.4 we have that the range of $(\cdot|\cdot)_R$ is contained in $B_e$, and by 10.3, that the range of $(\cdot|\cdot)_L$ is contained in $C^*(\mathcal{B}) \rtimes_\alpha G$. Thus $\mathcal{X}$ is a Hilbert $(C^*(\mathcal{B}) \rtimes_\alpha G) – B_e$–bimodule.

We claim that $\overline{\mathcal{X}\mathcal{X}^*}$ coincides with $B_e$. In fact, given a positive element $b \in B_e$ choose by [FD: Appendix C] a continuous section $f$ of $\mathcal{B}$ such that $f(0) = b^{1/2}$. Given $\varepsilon > 0$ let $\Omega$ be a neighborhood of $0$ in $\hat{G}$ such that $x \in \Omega$ implies that $\|f^*(x)f(x) - b\| < \varepsilon$. Also take $g \in C_c(\hat{G})$ such that $\text{supp}(g) \subseteq \Omega$ and $\int_{\hat{G}} |g(x)|^2\,dx = 1$. It follows that the section $f'$, given by the pointwise product $f' = gf$, is in $C_c(\mathcal{B})$ and that

$$\left\| \int_{\hat{G}} f'(x)^*f'(x)\,dx - b \right\| < \varepsilon.$$

On the other hand we have that

$$\int_{\hat{G}} f'(x)^*f'(x)\,dx \overset{(10.4)}{=} \langle f', f' \rangle_R \overset{(2.8)}{=} \zeta(f')^*\zeta(f') \in \mathcal{X}^*\mathcal{X},$$

proving that $b$ is in $\overline{\mathcal{X}\mathcal{X}^*}$. This proves our claim. The remaining statements are routine. $\square$
11. Spectral theory.

As seen above, dual actions in the context of Fell bundles gives rise to well behaved $C^*$-dynamical systems from the point of view of the questions we propose to treat. It is therefore desirable to characterize the abelian group actions which arise as such. Another important reason why one would like to describe an action as a dual action is the classification Theorem for Fell bundles \([E_3]:7.3\) according to which every Fell bundle is stably isomorphic to the semi-direct product bundle for a twisted partial action of the base group on the unit fiber algebra.

Precisely speaking, given a strongly continuous action $\alpha$ of an abelian group $G$ on a $C^*$-algebra $A$, satisfying suitable hypothesis, we wish to find a Fell bundle $\mathcal{B}$ over the dual group $\hat{G}$ such that $A$ is isomorphic to $C^*(\mathcal{B})$ under an isomorphism which puts $\alpha$ in correspondence with the dual action of $G$ on $C^*(\mathcal{B})$. Recall from 10.2 that dual actions contain a dense relatively continuous set of integrable elements. It is therefore natural to require the existence of relatively continuous sets, which we do next. For convenience we have chosen to require certain conditions, namely (iv) and (v) below, which will eventually be removed when we state the last and main Theorem of this section.

11.1. Hypothesis. We shall assume, until further notice, that we are given a subset $\mathcal{W} \subseteq A$ such that

(i) $\mathcal{W}^* = \mathcal{W}$,
(ii) $\mathcal{W}$ consists of $\alpha$-integrable elements,
(iii) $\mathcal{W}$ is relatively continuous,
(iv) $\mathcal{W}$ is spectrally invariant, and
(v) $\mathcal{W}$ is a linear subspace of $A$.

Our main goal will be to construct a Fell bundle $\mathcal{B}$ over $\hat{G}$ whose cross-sectional algebra $C^*(\mathcal{B})$ is isomorphic to $Alg(\mathcal{W}, \alpha)$ (Definition 8.8) under an isomorphism which is covariant for the dual action of $G$ on $C^*(\mathcal{B})$ and the action $\alpha$ on $A$.

11.2. Definition. For each $x$ in $\hat{G}$ let $B_x$ be defined to be the closure of the set

$$\{E_x(a) : a \in \mathcal{W}\}$$

within $M_x(A)$.

Since $\mathcal{W}$ is a linear space, it is clear that $B_x$ is a Banach space.

11.3. Proposition. For any $x, y \in \hat{G}$ one has that $B_xB_y \subseteq B_{xy}$ and $B_x^* = B_{x^{-1}}$.

Proof. Given $a, b \in \mathcal{W}$ we have by 6.4.(iv) that

$$E_x(a)E_y(b) = E_{xy}(aE_y(b)).$$

Since $\mathcal{W}$ is spectrally invariant we have that $aE_y(b)$ is in $\mathcal{W}$ and hence that $E_x(a)E_y(b)$ belongs to $B_{xy}$. This proves the first statement. The second statement follows immediately from by 6.4.(i) and the assumption that $\mathcal{W}$ is self-adjoint. 

So we see that the collection $\mathcal{B} = \{B_x\}_{x \in \hat{G}}$ forms a Fell bundle over the group obtained by replacing the topology of $\hat{G}$ with the discrete topology. In order to make $\mathcal{B}$ into a Fell bundle over $\hat{G}$ (with its own topology) we will use \([FD]:13.18\). That is, we must provide a linear space $\Gamma$ of sections of $\mathcal{B}$ such that

(a) for each $f \in \Gamma$ the numerical function $x \mapsto ||f(x)||$ is continuous on $\hat{G}$, and
(b) for each $x$ in $\hat{G}$ the set $\{f(x) : f \in \Gamma\}$ is dense in $B_x$.

11.4. Proposition. Let $\Gamma$ be the linear space of sections $f$ of the form $f(x) = E_x(a)$ for $a \in \mathcal{W}$. Then $\Gamma$ satisfies (a) and (b) above. Therefore there exists a unique topology on the disjoint union $\mathcal{B}$ of all $B_x$’s making it into a Banach bundle and such that $x \mapsto E_x(a)$ is a continuous section for all $a \in \mathcal{W}$. 

Proof. Property (b) is immediate from the definition of \( B_x \). With respect to (a) let \( f \) be defined by \( f(x) = E_x(a) \), where \( a \in W \). Then
\[
\|f(x)\|^2 = \|f(x)^* f(x)\| = \|E_{x^{-1}}(a^*) E_x(a)\|.
\]
We claim that the map
\[
x \mapsto E_{x^{-1}}(a^*) E_x(a)
\]
is continuous as an \( M(A) \)-valued function on \( \hat{G} \). This will imply that \( \|f(x)\| \) depends continuously on \( x \).

In order to prove our claim observe that \( a^* \cong a \) because \( W \) is self-adjoint and relatively continuous. So, for every \( x_0 \) in \( \hat{G} \) we have, with \( \delta = x_0 x^{-1} \), that
\[
\lim_{x \to x_0} \left\| E_{x^{-1}}(a^*) E_x(a) - E_{x_0^{-1}}(a^*) E_{x_0}(a) \right\| = \lim_{z \to 0} \left\| E_{z^{-1}}(a^*) E_z(a) - E_{z_0^{-1}}(a^*) E_{z_0}(a) \right\| = 0.
\]
The last sentence in the statement follows readily from the already mentioned Theorem 13.18 of [FD]. □

11.5. Proposition. With the multiplication operation \( B_x \times B_y \to B_{xy} \) induced by the corresponding operation on \( M(A) \), \( B \) is a Banach algebraic bundle (as defined in [FD: VIII.2.2]).

Proof. The only non-trivial axiom left to be verified is the continuity of the multiplication with respect to the bundle topology. In order to verify it we use [FD: VIII.2.4]. That is, it suffices to show that, given sections
\[
\beta(x) = E_x(a), \quad \gamma(x) = E_x(b), \quad x \in \hat{G},
\]
with \( a, b \in W \), one has that the map
\[
(x, y) \in \hat{G} \times \hat{G} \mapsto \beta(x) \gamma(y) \in B
\]
is continuous. In order to prove continuity at a given \( (x_0, y_0) \in \hat{G} \times \hat{G} \) we will use [FD: II.13.12] and hence all we must do is show that there exists a continuous section \( \delta \) such that \( \delta(x_0 y_0) = \beta(x_0) \gamma(y_0) \) and
\[
\lim_{(x, y) \to (x_0, y_0)} \| \beta(x) \gamma(y) - \delta(x y) \| = 0.
\]
We take \( \delta \) to be the section
\[
\delta(x) = E_x(a E_{y_0}(b)), \quad x \in \hat{G}.
\]
Observing that \( a E_{y_0}(b) \) belongs to \( W \) by spectral invariance we have that \( \delta \) is continuous. We have, using 6.4.(iv), that
\[
\| \beta(x) \gamma(y) - \delta(x y) \| = \| E_x(a) E_y(b) - E_{xy}(a E_{y_0}(b)) \| = \| E_x(a) E_{y_0^{-1} y_0}(b) - E_{xy y_0^{-1}}(a) E_{y_0}(b) \|,
\]
which tends to zero as \( z := y y_0^{-1} \to e \) (and hence also as \( (x, y) \to (x_0, y_0) \)) because \( a \cong b \). This concludes the proof. □

It is clear that the adjoint operation is continuous for the bundle topology by [FD: VIII.3.2.(vi')], and hence \( B \) is in fact a Banach *-algebraic bundle. Given that the \( C^* \)-identity \( (\|b^* b\| = \|b\|^2) \) obviously holds on \( B \) we conclude that:

11.6. Proposition. \( B \) is a Fell bundle (also referred to as a \( C^* \)-algebraic bundle in [FD: VIII.16.2]).

This concludes one of the major steps in the development of our spectral theory which is the description of the spectral subspaces and of the global topology represented by the Fell bundle \( B \). The next step is to relate the cross-sectional \( C^* \)-algebra of \( B \) back to \( A \).
11.7. Proposition. The inclusion map $\kappa : B \mapsto B(H)$ is a $^*$-representation of $B$ (in the sense defined in \[FD\: VIII.9.1\]).

Proof. The continuity in \[FD\: VIII.8.2.(iv)\] is the only point which we still need to verify. In order to do so let $\{u_i\}$ be a net in $B$ converging to some $u_0 \in B$. Also let $\xi \in \mathcal{H}$ be a generic vector in $\mathcal{H}$. In order to prove the continuity of our representation we must therefore prove that $u_i \xi \rightarrow u_0 \xi$ in the norm topology of $\mathcal{H}$.

Let $x_i$ be such that $u_i \in B_{x_i}$. Given $\varepsilon > 0$, choose a continuous section of $\mathcal{B}$ of the form $f(x) = E_x(a)$, with $a \in \mathcal{W}$, such that $\|f(x_0) - u_0\| < \varepsilon$. We have

$$
\|u_i \xi - u_0 \xi\| \leq \|u_i \xi - f(x_i) \xi\| + \|f(x_i) \xi - f(x_0) \xi\| + \|f(x_0) \xi - u_0 \xi\| \leq \\
\|u_i - f(x_i)\| \|\xi\| + \|f(x_i) - f(x_0)\| \xi\| + \varepsilon \|\xi\|.
$$

Since $f$ is continuous, and so is the norm on $\mathcal{B}$, we have that $\|u_i - f(x_i)\| \rightarrow \|u_0 - f(x_0)\| < \varepsilon$. The proof will then be concluded once we show that $\|f(x_i) \xi - f(x_0) \xi\| \rightarrow 0$. Write $\xi = b \xi' \in \mathcal{B}$ for $b \in A$ and $\xi' \in \mathcal{H}$ by the Cohen-Hewitt factorization Theorem [\HR\: 32.22] and observe that

$$
\|f(x_i) \xi - f(x_0) \xi\| = \|E_{x_i}(a)b \xi' - E_{x_0}(a)b \xi'\| \leq \|E_{x_i}(a)b - E_{x_0}(a)b\| \|\xi'\|,
$$

which converges to zero by 6.3. \hfill \Box

We now wish to show that a certain set of sections is dense in $L_1(B)$. In order to do so we need a result from Classical Harmonic Analysis for which we have found no specific reference and hence we prove it below:

11.8. Lemma. Let $J(G) = \{g \in C_c(G) : \hat{g} \in L_1(\hat{G})\}$. Then $J(G)$ is dense in $C_0(\hat{G})$.

Proof. Observe that $J(G)$ is a $^*$-subalgebra of $L_1(G)$ (under convolution). Therefore $\hat{J}(G)$ is a $^*$-subalgebra of $C_0(\hat{G})$ (under pointwise multiplication). We claim that $\hat{J}(G)$ satisfies the hypothesis of the Stone-Weierstrass theorem. In fact, suppose that $x_1, x_2 \in \hat{G}$ are such that $x_1 \neq x_2$. Let $t_0 \in G$ be such that $<x_1, t_0> \neq <x_2, t_0>$ and take closed convex cones $C_1$ and $C_2$ in $\mathcal{C}$ such that $<x_i, t_0>$ is in the interior of $C_i$ for $i = 1, 2$, and also such that $C_1 \cap C_2 = \{0\}$. Let $\Omega$ be a relatively compact open neighborhood of $t_0$ in $G$ such that

$$
t \in \Omega \Rightarrow <x_i, t> \in C_i,
$$

for $i = 1, 2$. Using [\HR\: 39.16] choose $\psi \in L_1(\hat{G})$ such that $\hat{\psi}(t_0) = 1$ and $\hat{\psi}(G \setminus \Omega) = \{0\}$. Replacing $\psi$ by $\psi^* \psi$ we may suppose that $\hat{\psi}(t) \geq 0$ for all $t \in G$. Let $g = \hat{\psi}$. Then it is clear that $g \in J(G)$ and moreover we have

$$
\hat{g}(x_i) = \int_G <x_i, t> g(t) \, dt = \int_\Omega <x_i, t> g(t) \, dt \in C_i.
$$

Since the integrand is continuous and lies in the interior of $C_i$ for $t = t_0$, it also follows that $\hat{g}(x_1) \neq 0$. Therefore $\hat{g}(x_1) \neq \hat{g}(x_2)$, concluding the proof. \hfill \Box

11.9. Proposition. Let $J(G)$ be as in 11.8 and let $\mathcal{S}$ be the linear span of the set of sections $\eta$ of $\mathcal{B}$ of the form

$$
\eta(x) = \hat{g}(x) E_x(a), \quad x \in \hat{G},
$$

where $a \in \mathcal{W}$ and $g \in J(G)$. Then $\mathcal{S}$ is a dense subset of $L_1(B)$ and hence also of $C^*(B)$.

Proof. By 11.4 and [\FD\: II.13.14] we have that each $\eta$ of the above form is a continuous section of $\mathcal{B}$. Since $\|E_x(a)\| \leq \|a\|_1$, and $\hat{g} \in L_1(\hat{G})$ it follows that $\eta \in L_1(B)$. Thus $\mathcal{S} \subseteq L_1(B)$.

Given $h \in J(G)$ and $\eta$ as above we have

$$
\hat{h}(x) \eta(x) = \hat{h}(x) \hat{g}(x) E_x(a) = (\hat{h} \hat{g})(x) E_x(a), \quad x \in \hat{G}.
$$

Since $J(G)$ is an algebra under convolution we conclude that $\hat{h} \eta$ belongs to $\mathcal{S}$. This says that $\mathcal{S}$ is invariant under pointwise multiplication by $\hat{h}$ for every $h \in J(G)$. In other words $J(G)\mathcal{S} \subseteq \mathcal{S}$.

Since $J(G)$ is dense in $C_0(\hat{G})$ by 11.8, we get $C_0(\hat{G})\mathcal{S} \subseteq \mathcal{S}$, where the closure is taken in $L_1(B)$. One may now easily prove that $\overline{\mathcal{S}} \cap C_c(B)$ satisfies the hypotheses of [\FD\: II.15.10], thus concluding the proof. \hfill \Box
The integrated form of $\kappa$ \cite[VIII.11.6]{FD} is the representation of the Banach $\ast$-algebra $L_1(B)$ on $\mathcal{H}$, which we also denote by $\kappa$, given by

$$\kappa(f)v = \int_{\hat{G}} f(x)v\,dx, \quad f \in L_1(B), \quad v \in \mathcal{H}.$$ 

Since $C^\ast(B)$ is defined \cite[VIII.17.2]{FD} to be the enveloping $C^\ast$-algebra of $L_1(B)$ we have that $\kappa$ extends to $C^\ast(B)$.

**11.10. Proposition.** $\kappa(C^\ast(B))$ coincides with $\text{Alg}(\mathcal{W}, \alpha)$ (Definition 8.8). Moreover, viewing $\kappa$ as a $\ast$-homomorphism from $C^\ast(B)$ to $A$, we have that $\kappa$ is covariant with respect to the dual action of $G$ on $C^\ast(B)$, henceforth denoted by $\beta$, and the given action $\alpha$ on $A$.

**Proof.** Let $a \in \mathcal{W}$ and let $g \in \mathcal{J}(G)$. We then have that the element

$$a' := \int_{\hat{G}} g(t)\alpha_t(a)\,dt$$

clearly belongs to $\text{Alg}(\mathcal{W}, \alpha)$ and is $\alpha$-integrable by 3.6. Observe that the section

$$\eta : x \in \hat{G} \mapsto E_x(a') = \int_{\hat{G}} \hat{g}(x)E_x(a)\,dx \in B_x$$

is in the space $S$ introduced in 11.9. In fact $S$ is spanned by the section of this form. Since $\eta \in L_1(B)$ we have by 6.6 that

$$a' = \int_{\hat{G}} E_x(a')\,dx = \int_{\hat{G}} \eta(x)\,dx = \kappa(\eta).$$

This shows that $a' \in \kappa(C^\ast(B))$ as well as that $\kappa(S) \subseteq \text{Alg}(\mathcal{W}, \alpha)$. Using 11.9 we then conclude that $\kappa(C^\ast(B)) \subseteq \text{Alg}(\mathcal{W}, \alpha)$.

Applying \cite[39.16]{HR} (to the dual group $\hat{G}$) we may find $g \in \mathcal{J}(G)$ such that $\|a' - a\|$ is arbitrarily small. This shows that $a$ lies in the range of $\kappa$ and hence that $\mathcal{W} \subseteq \kappa(C^\ast(B))$. Summarizing our findings so far we have

$$\mathcal{W} \subseteq \kappa(C^\ast(B)) \subseteq \text{Alg}(\mathcal{W}, \alpha).$$

Let $a \in \mathcal{W}$, $g \in \mathcal{J}(G)$, and consider $a'$ and $\eta$ defined in terms of $a$ and $g$ as above. We claim that $\alpha_t(\kappa(\eta)) = \kappa(\beta_t(\eta))$ for all $t \in G$, where $\beta$ refers to the dual action. In order to prove it observe that

$$\kappa(\beta_t(\eta)) = \int_{\hat{G}} \beta_t(\eta)\,dx = \int_{\hat{G}} <x,t>\eta(x)\,dx =$$

$$= \int_{\hat{G}} <x,t> E_x(a')\,dx \overset{(6.6)}{=} \alpha_t(a') = \alpha_t(\kappa(\eta)).$$

This proves our claim and, since $S$ is dense in $C^\ast(B)$, it also proves the last assertion in the statement. In addition we conclude that the range of $\kappa$ is invariant under $\alpha$ and hence, in view of 11.11, we have that $\kappa(C^\ast(B)) = \text{Alg}(\mathcal{W}, \alpha)$.

Consider the representation $\kappa \otimes \lambda$ of $B$ on $L_2(\hat{G}, \mathcal{H})$. This is given by

$$(\kappa \otimes \lambda)(b_x) = \kappa(b_x) \otimes \lambda_x, \quad x \in \hat{G}, \quad b_x \in B_x,$$

where $\lambda$ refers to the left regular representation of $\hat{G}$. As before we also denote by $\kappa \otimes \lambda$ the corresponding integrated representation of $C^\ast(B)$.

Recall from section 2 that $\pi$ is the representation of $A$ on $L_2(G, \mathcal{H})$ given by $\pi(a)\xi|_t = \alpha_t^{-1}(a)\xi(t)$, for $a \in A$, $\xi \in L_2(G, \mathcal{H})$, and $t \in G$. 
11.12. **Proposition.** Let $a' := \int_G g(t) \alpha_t(a) \, dt$, where $a \in W$ and $g \in \mathcal{J}(G)$, and let $\eta$ be the element of $C^*(B)$ represented by the section $\eta(x) = \tilde{g}(x)E_x(a) = E_x(a')$ for $x \in \hat{G}$. Then the following diagram commutes

$$
\begin{array}{ccc}
L_2(G, \mathcal{H}) & \xrightarrow{\pi(a')} & L_2(G, \mathcal{H}) \\
\mathcal{F} & & \mathcal{F} \\
\downarrow & & \downarrow \\
L_2(\hat{G}, \mathcal{H}) & \xrightarrow{(\kappa \otimes \lambda)\eta} & L_2(\hat{G}, \mathcal{H})
\end{array}
$$

where $\mathcal{F}$ stands for the Fourier Transform.

**Proof.** Let $\xi \in C_c(G, \mathcal{H})$ and $x \in \hat{G}$. Then

$$
\mathcal{F}\pi(a')\xi|_x = \int_G \overline{<x,t>}\pi(a')\xi|_t \, dt = \int_G \overline{<x,t>}\alpha_t^{-1}(a')\xi(t) \, dt = \int_G \overline{<y^{-1}x,t>}\xi(t) \, dt \, dy = \int_G E_y(a')\xi|_y \, dy = \int_G (E_y(a') \otimes \lambda_y)\xi|_y \, dy = (\kappa \otimes \lambda)\eta\xi|_x.
$$

Since $x$ is arbitrary we have that $\mathcal{F}\pi(a')\xi = ((\kappa \otimes \lambda)\eta)\mathcal{F}\xi$. Since $C_c(G, \mathcal{H})$ is dense in $L_2(G, \mathcal{H})$ the proof is concluded. \hfill \Box

11.13. **Proposition.** The representation $\kappa$ of $C^*(B)$ defined above is faithful and hence $C^*(B)$ is covariantly isomorphic to $\text{Alg}(W, \alpha)$.

**Proof.** Let $a' := \int_G g(t) \alpha_t(a) \, dt$, where $a \in W$ and $g \in \mathcal{J}(G)$. Set $\eta(x) = \tilde{g}(x)E_x(a) = E_x(a')$ for $x \in \hat{G}$, so that $\eta \in \mathcal{S}$ and $\kappa(\eta) = a'$, as before. Consider the composition of maps

$$
C^*(B) \xrightarrow{\kappa} A \xrightarrow{\pi} B(L_2(G, \mathcal{H})) \xrightarrow{Ad_{\mathcal{F}}} B(L_2(\hat{G}, \mathcal{H})),
$$

where $Ad_{\mathcal{F}}$ is the conjugation by the Fourier transform. We then have that

$$
Ad_{\mathcal{F}}(\pi(\kappa(\eta))) = \mathcal{F}\pi(a')\mathcal{F}^* = (\kappa \otimes \lambda)(\eta),
$$

where the last step follows from 11.12. Since the set of $\eta$'s considered span $\mathcal{S}$, and since $\mathcal{S}$ is dense in $C^*(B)$, we conclude that the above composition of maps coincides with $\kappa \otimes \lambda$, which is a faithful representation of $C^*(B)$ by [E5: 3.6]. This shows that $\kappa$ is one to one and hence the proof is complete. \hfill \Box

The following Theorem subsumes our findings in this section and is one of our main results. We observe that whereas we have worked above under the assumption that $W$ is a spectrally invariant linear space, these hypotheses are not needed below.

11.14. **Theorem.** Let $\alpha$ be a strongly continuous action of a locally compact abelian group $G$ on a $C^*$-algebra $A$ and let $W$ be a subset of $A$ such that

(i) $W^* = W$,

(ii) $W$ consists of $\alpha$-integrable elements (Definition 2.1) and

(iii) $W$ is relatively continuous (Definition 8.6).

Then there exists a Fell bundle $B$ over $\hat{G}$ such that $C^*(B)$ is isomorphic to the smallest $\alpha$-invariant sub-$C^*$-algebra of $A$ containing $W$ under an isomorphism which is covariant with respect to $\alpha$ and the dual action of $G$ on $C^*(B)$.
Proof. Let \( \tilde{W} \) be the spectrally invariant hull of \( W \). Observing the description of \( \tilde{W} \) given in the proof of 8.7 it is clear that \( \tilde{W} \) is self-adjoint. Also by 8.7 we have that \( \tilde{W} \) is relatively continuous, and hence \( \tilde{W} \) satisfies 11.1.(i–iv). It follows that \( \text{span}(\tilde{W}) \) satisfies all of the conditions in 11.1.

Note that, by 8.9 we have that \( \text{Alg}(W, \alpha) = \text{Alg}(\tilde{W}, \alpha) = \text{Alg}(\text{span}(\tilde{W}), \alpha) \). The conclusion then follows from the previous results applied to \( \text{span}(\tilde{W}) \).

Combining the previous result with 10.2 we obtain the following:

11.15. Corollary. Let \((A, G, \alpha)\) be a \( C^* \)-dynamical system where \( G \) is abelian. A necessary and sufficient condition for \( \alpha \) to be equivalent to a dual action is that \( A \) contains a dense, self-adjoint, relatively continuous set of \( \alpha \)-integrable elements.

We have not been able to determine the extent to which condition (iii) in 11.14 is really necessary and hence we leave open the following:

11.16. Question. Suppose that the set of \( \alpha \)-integrable elements is dense in \( A \), or even that the set of linear combinations of positive \( \alpha \)-integrable elements is dense\(^5\). Does it follow that there exists a dense subset \( W \) of \( \alpha \)-integrable elements which is relatively continuous?

12. Classical dynamical systems.

In this section we shall apply the results obtained so far to the case of a classical dynamical system, that is, a group action on a locally compact topological space. Given that our methods were developed for abelian groups we shall let, throughout this section, \( G \) be a locally compact abelian group and

\[
\alpha : (t, p) \in G \times X \mapsto tp \in X
\]

be a continuous action of \( G \) on a locally compact space \( X \). One therefore gets a strongly continuous action of \( G \) on the \( C^* \)-algebra \( A = C_0(X) \), which we will also denote by \( \alpha \), given by

\[
\alpha_t(f)\big|_p = f(t^{-1}p), \quad t \in G, \quad f \in C_0(X), \quad p \in X.
\]

Let \( f \in C_0(X) \) be an \( \alpha \)-integrable element. Then, seeing \( M(C_0(X)) \) as the set \( C_b(X) \) of bounded continuous functions on \( X \), we have that \( E_{\alpha}(f) \) is represented by the function

\[
E_{\varepsilon}(f)(p) = \int_G f(t^{-1}p) \, dt, \quad p \in X.
\]

By a simple change of variables we see that \( E_{\varepsilon}(f) \) is constant on the orbits of \( X \) under \( G \) and hence it defines, by passage to the quotient, a continuous function on the orbit space \( X/G \), even if this not a locally compact or Hausdorff space! In any case \( X/G \) carries the quotient topology and the the quotient map

\[
Q : X \to X/G
\]

is continuous.

12.1. Definition. We shall denote by \( C_0(X/G) \) the space of continuous complex valued functions \( f \) on \( X/G \) such that for every \( \varepsilon > 0 \), there exists a compact subset \( K \subseteq X \) such that \( |f(q)| < \varepsilon \), for all \( q \in (X/G) \setminus Q(K) \).

That is, \( C_0(X/G) \) is defined in the usual way, except that when it comes to considering compact subsets of \( X/G \) we take only the images of compact subsets of \( X \) under the quotient map. Equivalently, \( C_0(X/G) \) consists of the continuous functions \( f \) on \( X \) which are constant along every orbit and such that for every \( \varepsilon > 0 \), there exists a compact subset \( K \subseteq X \) such that \( |f(p)| < \varepsilon \), for every \( p \in X \) outside the orbit \( O(K) \) of \( K \).

\(^5\) This is Rieffel’s tentative definition of proper actions [R5: 4.5].
12.2. Lemma. Let $f$ be an $\alpha$-integrable element of $C_0(X)$. Suppose that $f$ is positive and that $E_e(f)$ belongs to $C_0(X/G)$. Then $f$ is absolutely continuous in the sense that $f \overset{ac}{\sim} g$ for every $\alpha$-integrable $g$.

Proof. Fix an $\alpha$-integrable element $g \in C_0(X)$. Let $\varepsilon > 0$ be given and take a compact subset $K \subseteq X$ such that $|E_e(f)(p)| < \varepsilon/(2\|g\|_1)$, for every $p \in X \setminus O(K)$. Observe that for all $p \in X$ and $w \in \hat{G}$ we have

$$|E_w(f)(p)| \leq \int_G |<w, t>| f(t^{-1}p) dt = \int_G f(t^{-1}p) dt = E_x(f)(p).$$

Thus $|E_w(f)(p)| < \varepsilon/(2\|g\|_1)$, for all $p \in X \setminus O(K)$. For any such $p$, and any $x, y, z \in \hat{G}$, we therefore have that

$$\left|\left(E_{xyz}(f)E_y(g) - E_x(f)E_{zy}(g)\right)\right|_p \leq |E_{xyz}(f)(p)|\|g\|_1 + |E_x(f)(p)|\|g\|_1 \leq \varepsilon.$$

Recall that for every $w \in \hat{G}$, one has that $E_w(f) \in M_w(C_0(X))$ and hence for any $p \in X$ and $s \in G$, $E_w(f)(s^{-1}p) = <w, s>E_w(f)(p)$. It follows that

$$(E_{xyz}(f)E_y(g) - E_x(f)E_{zy}(g))\big|_{s^{-1}p} = \left\langle xz, \alpha \right\rangle \left(E_{xyz}(f)E_y(g) - E_x(f)E_{zy}(g)\right)\big|_p,$$

for every $x, y, z \in \hat{G}$. For brevity we will use the abbreviation

$$\Delta_{x,y}^z = E_{xyz}(f)E_y(g) - E_x(f)E_{zy}(g).$$

We have therefore proven that, for every $x, y, z \in \hat{G}$

(i) if $p \notin O(K)$ then $|\Delta_{x,y}^z(p)| \leq \varepsilon$, and
(ii) if $p \in X$ and $s \in G$ then $\Delta_{x,y}^z(s^{-1}p) = \overline{<xyz, s>}\Delta_{x,y}^z(p)$.

Let $h \in C_0(X)$ be such that $h(p) = 1$ for every $p \in K$, so that, using (ii),

$$\sup_{p \in O(K)} |\Delta_{x,y}^z(p)| = \sup_{p \in K} |\Delta_{x,y}^z(p)| \leq \|\Delta_{x,y}^z h\| = \|E_{xyz}(f)E_y(g)h - E_x(f)E_{zy}(g)h\| \leq$$

$$\leq \|E_{xyz}(f)E_y(g)h - E_x(f)E_y(g)h\| + \|E_x(f)E_y(g)h - E_x(f)E_{zy}(g)h\| \leq$$

$$\leq \|g\|_1 \|E_{xyz}(f)h - E_x(f)h\| + \|f\|_1 \|E_y(g)h - E_{zy}(g)h\|.$$

By 6.3 there exists a neighborhood $\Omega$ of $e$ in $\hat{G}$ such that the above is less than $\varepsilon$ for all $z \in \Omega$ and all $x, y \in G$. We conclude that, for $z \in \Omega$,

$$\sup_{x,y \in \hat{G}} \|E_{xyz}(f)E_y(g) - E_x(f)E_{zy}(g)\| = \sup_{p \in X} \sup_{x, y \in \hat{G}} |\Delta_{x,y}^z(p)| \leq \varepsilon,$$

and hence that $f \overset{ac}{\sim} g$. □

Recall from [R5: 4.7] that $\alpha$ is a proper action in the usual sense\(^6\) if and only if the linear span of the set of positive $\alpha$-integrable elements is dense in $C_0(X)$ (see remark 2.2 regarding a difference between our terminology and that adopted in [R5]). In fact, for a proper action any $f$ in $C_c(X)$ is $\alpha$-integrable (see [R4 : Section 1]).

12.3. Proposition. If $\alpha$ is proper then any $f$ in $C_c(X)$ is absolutely continuous (as defined in 12.2).

---

\(^6\) A group action is said to be proper when the map $(t, x) \in G \times X \mapsto (tx, x) \in X \times X$ is proper, i.e the inverse image of compact sets is compact.
Proof. Let $f \in C_c(X)$. It is then clear that $E_c(f)$ is compactly supported in $X/G$ and, in particular, that $E_c(f) \in C_0(X/G)$. Therefore if $f$ is also positive the conclusion follows from 12.2. In the general case observe that $f$ is a linear combination of four positive elements in $C_c(X)$ each of which is absolutely continuous. □

Let $\mathcal{P} = C_0(X)_+ \cap C_c(X)$. Then by the above result $\mathcal{P}$ is a relatively continuous set. It is also clear that $\mathcal{P}$ is a hereditary cone and satisfies 8.10.(ii). We may then use 9.2 to derive the following well known result (see [G], [R3], [R5]):

12.5. Corollary. Let $\alpha$ be a proper action of the abelian group $G$ on the locally compact topological space $X$. Then the closure of $\zeta(C_c(X))$ is the imprimitivity bimodule for a Morita–Rieffel equivalence between $C_0(X/G)$ and an ideal in $C_0(X) \times \alpha G$.

Proof. Let $\mathcal{P} = C_0(X)_+ \cap C_c(X)$ as above. Then it is clear that the sets $\mathcal{N}$ and $\mathcal{M}$ constructed in terms of $\mathcal{P}$ as in 9.2 both coincide with $C_c(X)$. By 9.2 all we need to verify is that the closure of $E_n(C_c(X))$ coincides with $C_0(X/G)$. As observed in the proof of 12.3 we have that $E_n(C_c(X)) \subseteq C_c(X/G)$. The conclusion then follows easily from the Stone–Weierstrass Theorem. □

Let us now consider applying Theorem 11.14 to the present situation. For this, observe that taking $\mathcal{W} = C_c(X)$, all of the hypotheses of 11.14 hold. Moreover, since $\mathcal{W}$ is dense in $C_0(X)$, we have that $Alg(\mathcal{W}, \alpha) = C_0(X)$. Theorem 11.14 then immediately implies:

12.4. Corollary. Let $\alpha$ be a proper action of a locally compact abelian group $G$ on a locally compact topological space $X$. Then $\alpha$ is equivalent to a dual action. That is, there exists a Fell bundle $\mathcal{B}$ over $G$ such that $C_0(X)$ is covariantly isomorphic to $C^*(\mathcal{B})$ with respect to the dual action of $G$ on $C^*(\mathcal{B})$.

13. An example.

Throughout this article we have been working under the tacit assumption that dual actions are the “good guys” among $C^*$-dynamical systems, and we hope to have given enough evidence to convince the reader of this. Nevertheless we would now like to explore a specific example of dual action which will bring to light some of the complexities that may be found as well as some of the inherent difficulties in answering the questions that we have posed.

Our example will in fact be that of a classical dual action as described in [P: 7.8.3]. Consider the action of the unit circle $S^1$ on $C(S^1)$ by translation. The crossed product algebra is well known to be isomorphic to the algebra $\mathcal{K}$ of compact operators on $\ell_2(\mathbb{Z})$. It is also well known that the dual action of $\mathbb{Z}$ on $\mathcal{K}$ is given by conjugation by the powers of the forward bilateral shift (this is in fact a consequence of Takai duality since the first action described above is already a dual action with respect to the trivial action of $\mathbb{Z}$ on a point).

The action we wish to discuss is thus the action of $\mathbb{Z}$ on $\mathcal{K}$ given by

$$\alpha_n(T) = U^nTU^{-n}, \quad n \in \mathbb{Z}, \quad T \in \mathcal{K},$$

where $U$ is the forward bilateral shift on $\ell_2(\mathbb{Z})$.

In order to avoid an excessive use of the Fourier transform we will identify the Hilbert spaces $\ell_2(\mathbb{Z})$ and $L_2(S^1)$, henceforth denoted simply by $\mathcal{H}$, by considering the usual orthonormal basis $\{e_n\}_{n \in \mathbb{Z}}$ of the latter given by $e_n(z) = z^n$ for all $z$ in $S^1$. Under this identification $U$ becomes the operator of pointwise multiplication by $z$:

$$U\xi|_z = z\xi(z), \quad \xi \in L_2(S^1), \quad z \in S^1.$$

13.1. Proposition. Let $P$ be a rank-one projection on $\mathcal{H}$. Then $P$ is $\alpha$-integrable if and only $P$ has the form

$$P(\xi) = \langle \xi, \phi \rangle \phi, \quad \xi \in \mathcal{H},$$

where $\|\phi\|_2 = 1$ and $\phi$ belongs to $L_\infty(S^1)$, viewed as a subspace of $L_2(S^1)$. In that case the series $\sum_{n \in \mathbb{Z}} \alpha_n(P)$ converges strictly-unconditionally to the operator of pointwise multiplication by $|\phi|^2$. 


Proof. Suppose that $P$ has the form mentioned in the statement for some $\phi \in L_\infty(S^1)$ and denote by $M_\phi$ the operator on $\mathcal{H}$ given by pointwise multiplication in $\phi$. Also, given any finite subset $J \subseteq \mathbb{Z}$, let $Q_J$ denote the orthogonal projection onto span$\{e_n : n \in J\}$. We then have for all $\xi \in \mathcal{H}$ that

$$M_\phi Q_J M_\phi^*(\xi) = M_\phi \left( \sum_{n \in J} \langle M_\phi^* \xi, e_n \rangle e_n \right) = \sum_{n \in J} \langle \xi, P e_n \rangle P e_n = \sum_{n \in J} \langle \xi, U^n(\phi) \rangle U^n(\phi) = \sum_{n \in J} U^n P U^{-n}(\xi) = \sum_{n \in J} \alpha_n(P)(\xi).$$

It follows that $M_\phi Q_J M_\phi^* = \sum_{n \in J} \alpha_n(P)$.

If $T$ is any compact operator on $\mathcal{H}$ we have that $\lim_J Q_J T = \lim_J T Q_J = T$, where the limit is with respect to the directed set formed by all compact, i.e., finite, subsets $J \subseteq \mathbb{Z}$. Therefore, still assuming that $T$ is compact,

$$\lim_J \sum_{n \in J} \alpha_n(P)T = \lim_J M_\phi Q_J M_\phi^* T = M_\phi M_\phi^* T = M|_{\phi^2};$$

and similarly if $T$ is on the left hand side. It follows that the series $\sum_{n \in \mathbb{Z}} \alpha_n(P)$ converges strictly-unconditionally to $M|_{\phi^2}$ proving that $P$ is $\alpha$-integrable, and also proving the last sentence in the statement.

Conversely suppose that the rank-one projection $P$ is $\alpha$-integrable and let $\phi$ be a vector in the range of $P$ of norm one. Thus $P(\xi) = \langle \xi, \phi \rangle \phi$, for $\xi \in \mathcal{H}$. Given $i, j$ in $\mathbb{Z}$ we have

$$\langle Pe_j, e_i \rangle = \langle e_j, \phi \rangle \phi, e_i \rangle = \widehat{\phi}(i) \overline{\phi}(j) = \overline{\phi}(i) \overline{\phi}(-j),$$

where $\overline{\phi}$ is the pointwise complex conjugate of $\phi$. Let $T = \sum_{k \in \mathbb{Z}} \alpha_k(P)$, where the series converges strictly-unconditionally and hence also in the weak-operator topology. We then have

$$\langle Te_j, e_i \rangle = \sum_{k \in \mathbb{Z}} \langle \alpha_k(P)e_j, e_i \rangle = \sum_{k \in \mathbb{Z}} \langle U^k P U^{-k} e_j, e_i \rangle = \sum_{k \in \mathbb{Z}} \langle Pe_{j-k}, e_{i-k} \rangle = \sum_{k \in \mathbb{Z}} \overline{\phi}(i-k) \overline{\phi}(k-j) = \left( \overline{\phi} * \overline{\phi} \right)(i-j) = \overline{\phi}(i-j).$$

It follows that the matrix of $T$ with respect to the canonical basis $\{e_n \}_{n \in \mathbb{Z}}$ is a Laurent matrix (in the classical sense $[H:241]$) with symbol $|\phi|^2$. Therefore, since $T$ is bounded, we must have that $\phi \in L_\infty(S^1)$.

13.2. Remark. The first important consequence to be drawn from this result is that Definition 6.4 in [R5] leads to a generalized fixed point algebra bigger than desired. In fact, adopting that definition, by 13.1 we would have that the generalized fixed point algebra will be formed by all multiplication operators by measurable bounded functions on $S^1$ and hence will be isomorphic to $L_\infty(S^1)$. However, since we are in a situation where the classical Takai duality Theorem holds, there seems to be little doubt that the "correct" definition of generalized fixed point algebra should yield $C(S^1)$ instead. In addition our example illustrates that taking the integrals of higher powers of integrable elements is not the correct approach either since the trouble here is caused by idempotent operators! We should note that this was also indicated in [E2: Section 7].

As a simple consequence of 13.1 we can give a precise characterization of positive integrable elements:

13.3. Proposition. Let $T$ be a positive compact operator on $\mathcal{H}$. Then $T$ is $\alpha$-integrable if and only if there exists a sequence $\{\lambda_n\}_n$ of positive real numbers converging to zero and a pairwise orthogonal sequence $\{\phi_n\}_n \subseteq L_\infty(S^1)$ such that $\sum_{n=1}^{\infty} \lambda_n |\phi_n|^2$ is pseudo-summable in $L_\infty(S^1)$ (meaning that the finite sums are uniformly bounded $[E2:2.5]$), and

$$T = \sum_n \lambda_n P_n,$$

where $P_n$ is the projection onto the subspace of $\mathcal{H}$ spanned by $\phi_n$. 

\[\square\]
Prove. Suppose that $T$ is $\alpha$-integrable. Using the spectral Theorem for compact self-adjoint operators write $T = \sum_n \lambda_n P_n$, where $\lambda_n > 0$ for all $n$, $\lim_n \lambda_n = 0$, and the $P_n$ are pairwise orthogonal rank-one projections. Since the set of $\alpha$-integrable elements is a hereditary cone by 2.4, we have that each $P_n$ is $\alpha$-integrable and hence, by the above Lemma, there are bounded measurable functions $\phi_n$, necessarily pairwise orthogonal, such that $P_N^\phi_n$ is the projection onto the one dimensional space spanned by $\phi_n$. For each integer $N$ we have that $0 \leq \sum_{n=1}^N \lambda_n P_n \leq T$, and hence

$$0 \leq E_c(\sum_{n=1}^N \lambda_n P_n) = \sum_{n=1}^N \lambda_n |\phi_n|^2 \leq E_c(T),$$

which implies that $\sum_{n=1}^N \lambda_n |\phi_n|^2$ is uniformly bounded with $N$.

Conversely, suppose that $T$ is of the above form and let us prove it to be $\alpha$-integrable. For a given finite subset $J \subseteq \mathbb{Z}$ we have

$$\sum_{k \in J} \alpha_k(T) = \sum_{k \in J} \sum_{n \in \mathbb{Z}} \lambda_n \alpha_k(P_n) = \sum_{n \in \mathbb{Z}} \lambda_n \sum_{k \in J} \alpha_k(P_n) = \sum_{n \in \mathbb{Z}} \lambda_n M_{\phi_n} Q_J M_{\phi_n}^* \leq \sum_{n \in \mathbb{Z}} \lambda_n |\phi_n|^2,$$

where $Q_J$ is as in the proof of 13.1. It follows that the net $\{\sum_{k \in J} \alpha_k(T)\}_J$ is bounded and since it is also increasing it must converge strongly. The strong topology can be easily shown to coincide with the strict topology on bounded subsets of $K$ and hence we have shown that our net converges strictly. This completes the proof.

Let us now consider the problem mentioned near the end of Section 4 as to whether $\zeta(a)\zeta(b)^*$ belongs to the crossed product algebra for a given pair of elements $a, b \in \mathcal{N}_\sigma$. We will restrict our attention to rank-one projections since this will suffice to illustrate a point to be made below and also because the behavior of general elements of $K$ is mirrored in the behavior of rank-one projections, as seen in 13.3. We must first compute the Fourier transform:

13.4. Proposition. Let $\phi \in L_\infty(S^1)$ with $\|\phi\|_2 = 1$, and let $P$ be the projection onto the subspace of $\mathcal{H}$ spanned by $\phi$. Then for each $z$ in the Pontryagin dual $S^1$ of $\mathbb{Z}$ one has

$$E_z(P) = M_\phi W_z M_\phi^*,$$

where $W_z$ is the diagonal operator given on the canonical basis by

$$W_z(e_n) = z^n e_n, \quad n \in \mathbb{Z}.$$

Proof. In order to prove the statement it suffices to show that the operators above have identical matrices with respect to the canonical basis $\{e_n\}_{n \in \mathbb{Z}}$. Recalling that $E_z(P) = \sum_{n \in \mathbb{Z}} z^n \alpha_n(P)$, where the sum converges strictly-unconditionally, we have

$$\langle E_z(P) e_j, e_i \rangle = \sum_{n \in \mathbb{Z}} z^n \langle U^n P U^{-n} e_j, e_i \rangle = \sum_{n \in \mathbb{Z}} z^n \langle P e_{j-n}, e_{i-n} \rangle = \sum_{n \in \mathbb{Z}} z^n \overline{\phi(i-n)} \phi(j-n).$$

On the other hand

$$\langle M_\phi W_z M_\phi^* e_j, e_i \rangle = \langle W_z M_\phi^* e_j, M_\phi^* e_i \rangle = \sum_{n \in \mathbb{Z}} z^n \langle M_\phi^* e_j, e_n \rangle \overline{\langle M_\phi^* e_i, e_n \rangle} = \sum_{n \in \mathbb{Z}} z^n \overline{\phi(i-n)} \phi(j-n),$$

concluding the proof. \qed
13.5. Proposition. Let $\phi$ and $\psi$ be in $L_\infty(S^1)$ with $\|\phi\|_2 = \|\psi\|_2 = 1$ and denote by $P$ and $Q$ the projections onto the one-dimensional subspaces of $\mathcal{H}$ spanned by $\phi$ and $\psi$ respectively. Then the following are equivalent

(i) $\zeta(P)\zeta(Q)^*$ belongs to $\mathcal{K}_{\alpha\alpha}\mathbb{R}$,

(ii) $P \lesssim Q$,

(iii) $\overline{\phi}\psi \in C(S^1)$.

Proof. Given that $P = P^*P$ and similarly for $Q$, the equivalence between (i) and (ii) follows immediately from 8.2. Therefore we need only prove that (ii) and (iii) are equivalent. During the course of this proof we will denote the multiplication operator $M_f$ simply by $f$ for each $f$ in $L_\infty$. Let $\tau$ be the (discontinuous) action of $S^1$ on $L_\infty(S^1)$ given by

$$\tau_z f|_w = f(zw), \quad z, w \in S^1, \quad f \in L_\infty(S^1),$$

and observe that for all $f$ in $L_\infty(S^1)$ one has $W_z f W_z^* = \tau(f)$. Given $x, y, z \in S^1$ we have

$$\|E_{z\phi}(P)E_y(Q) - E_z(P)E_{z\phi}(Q)\| \leq \|\phi W_z \phi^* W_y \psi^* - \phi W_x \phi^* W_{z\phi} \psi^*\| \leq \|\phi W_z\| \|W_z \phi^* \psi - \phi^* W_z \phi^* \psi\| + \|W_y\| \|W_y \psi^*\| \leq \|\phi\|_\infty \|\psi\|_\infty \|\tau_z(\overline{\phi}\psi) - \overline{\phi}\psi\|.$$

Thus, assuming that $\overline{\phi}\psi \in C(S^1)$, we have that $P \lesssim Q$. Conversely, suppose that $P \lesssim Q$. Then for all $z$ in $S^1$ we have

$$\|\tau_z(\overline{\phi}\psi) - \overline{\phi}\psi\|^2 = \|(W_z \phi^* \psi W_z^* - \phi^* \psi)(W_z \phi^* \psi W_z^* - \phi^* \psi)\| \leq \||W_z \phi^* \psi W_z^* - W_z \phi^* \psi W_z^* - W_z \phi^* \psi W_z^*\| \leq 2\|\phi\|_\infty \|\psi\|_\infty \|	au_z(\overline{\phi}\psi) - \overline{\phi}\psi\|,$$

which tends to zero as $z \to 1$ by (ii). This shows that $\overline{\phi}\psi$ is continuous. \qed

Given that absolutely continuous elements played an important role in the case of classical dynamical systems, studied above, it is interesting to characterize them in the present situation. However we have:

13.6. Proposition. Let $T$ be a positive $\alpha$-integrable element of $\mathcal{K}$. Suppose that $T$ is absolutely continuous in the sense that $T \lesssim S$ for all $\alpha$-integrable $S \in K$. Then $T = 0$.

Proof. Suppose by contradiction that $T \neq 0$ and pick a nonzero vector $\phi$ in $\mathcal{H}$ such that $T(\phi) = \|T\| \phi$. Also let $P$ be the orthogonal projection onto the one-dimensional space spanned by $\phi$, so that $\|T\| P \leq T$. So $P$ is $\alpha$-integrable by 2.4.(i) and hence $\phi \in L_\infty(S^1)$ by 13.1.

Given $\psi \in L_\infty(S^1)$ let $Q$ be the orthogonal projection onto the one-dimensional subspace of $\mathcal{H}$ spanned by $\psi$. Then

$$\|T\| P \leq T \lesssim Q$$

which implies by 8.4 that $P \lesssim Q$. Therefore, using 13.5 we have that $\overline{\phi}\psi \in C(S^1)$. But this can only happen for all $\psi$ in $L_\infty(S^1)$ if $\phi = 0$. \qed
As already mentioned, $K$ is the crossed product algebra $C(S^1) \rtimes \tau$, where $\tau$ is given by translation. Therefore, $K$ is also the cross-sectional $C^*$-algebra for the semi-direct product bundle [FD: VIII.4] $B$ whose total space is $C(S^1) \times S^1$, carrying the bundle operations

$$(f, z) \cdot (g, w) = (f \tau_z(g), zw), \quad \text{and} \quad (f, z)^* = (\tau_z^{-1}(f), z^{-1}),$$

for $f, g \in C(S^1)$ and $z, w \in S^1$. Consider the representation of $B$ on $\mathcal{H} = L_2(S^1)$ given by

$$\rho : (f, z) \in B \mapsto M_f W_z \in \mathcal{B}(\mathcal{H}),$$

where $W_z$ is as in 13.4. The integrated form of $\rho$ will also be denoted by $\rho$. Since we know that $C^*(B)$ is isomorphic to $K$, a simple algebra, any representation of it will be faithful and hence so is $\rho$. We shall then identify $C^*(B)$ with its image under $\rho$.

Let $\eta$ be in $C_c(B)$, so that $\eta(z) = (f_z, z)$, for $z \in S^1$, where $z \mapsto f_z$ is a continuous $C(S^1)$-valued function on $S^1$. We write $f_z(w)$ as $f(z, w)$. In the latter form $f$ represents a continuous function on $S^1 \times S^1$. Observe that

$$\rho(\eta)\xi|_z = \int_{S^1} f(w, z)\xi(wz) \, dw = \int_{S^1} f(zw^{-1}, z)\xi(w) \, dw,$$

for $\xi \in C_c(S^1) \subseteq \mathcal{H}$. Therefore $\rho(\eta)$ is the (classical) integral operator with kernel $k(z, w) = f(wz^{-1}, z)$. Since the transformation $(z, w) \mapsto (wz^{-1}, z)$ is a homeomorphism of $S^1 \times S^1$, we have that $C_c(B)$ (or rather its image under $\rho$) consists exactly of the integral operators with continuous symbols.

For example, if $P$ is the rank one projection onto the space spanned by $\phi \in \mathcal{H}$ with $\|\phi\|_2 = 1$, we have that

$$P\xi|_z = (\xi, \phi)_z = \int_{S^1} \phi(z)\overline{\phi(w)}\xi(w) \, dw,$$

so that $P$ is also an integral operator. Its kernel, given by $k(z, w) = \phi(z)\overline{\phi(w)}$, is clearly continuous if and only if $\phi$ is a continuous function. Therefore $P$ belongs to $C_c(B)$ if and only if $\phi \in C(S^1)$.

Fix a measurable function $\delta \in L_\infty(S^1)$ such that $|\delta(z)| = 1$ for all $z$. The multiplication operator $M_\delta$ is then a unitary operator on $\mathcal{H}$ which commutes with the bilateral shift and hence the map

$$\Delta : T \in \mathcal{K} \mapsto M_\delta TM_\delta^{-1} \in \mathcal{K}$$

is an automorphism of $\mathcal{K}$ which commutes with $\alpha$. It follows that all aspects of the dynamical system $(\mathcal{K}, \mathbb{Z}, \alpha)$ are left invariant under $\Delta$. For example, if $T$ is $\alpha$-integrable then so is $\Delta(T)$ and

$$E_z(\Delta(T)) = \Delta(E_z(T)), \quad z \in S^1,$$

and so on.

If $T$ is an integral operator with kernel $k$, observe that

$$\Delta(T)\xi|_z = M_\delta TM_\delta^{-1}\xi|_z = \int_{S^1} \delta(z)k(z, w)\overline{\delta(w)}\xi(w) \, dw, \quad \xi \in C_c(S^1),$$

and hence that $\Delta(T)$ is the integral operator with kernel

$$k'(z, w) = \delta(z)k(z, w)\overline{\delta(w)}.$$

So we see that $\Delta(C_c(B))$ consists exactly of the integral operators whose kernels have the above form for a continuous function $k$.

If $\delta$ is sufficiently discontinuous we may then have $\Delta(C_c(B)) \neq C_c(B)$. Nevertheless, given that $\Delta$ commutes with $\alpha$, one can use $\Delta(C_c(B))$ in place of $C_c(B)$ in 10.6 in order to obtain a Morita–Rieffel equivalence between $B_c$ and an ideal of $K \rtimes_\alpha \mathbb{Z}$. Let

$$P_1 = C_c(B)^2 \cap C^*(B)_+, \quad \text{and} \quad P_2 = \Delta(C_c(B))^2 \cap C^*(B)_+.$$

...
We claim that $\mathcal{P}_1 \cup \mathcal{P}_2$ is not necessarily relatively continuous. In fact, let $P$ be the rank-one projection onto the space spanned by a unit vector $\phi$ (for the 2-norm) belonging to $C(S^1)$. Then it is easy to see that $\Delta(P)$ is the projection onto the span of $\delta \phi$. Let $Q$ be another rank-one projection whose range is spanned by a unit vector $\psi \in C(S^1)$, then as seen above

- $P \not\subseteq Q$ because $\overline{\delta \phi \psi} \in C(S_1)$,
- $\Delta(P) \not\subseteq \Delta(Q)$ because $\overline{\delta \phi \psi} = \overline{\phi \psi} \in C(S_1)$.

However, it may happen that $P \not\subseteq \Delta(Q)$ because nothing guarantees that $\overline{\phi \psi} \in C(S_1)$.

One of the main lessons to be learned from this example is that a maximal relatively continuous cone $\mathcal{P}$, as discussed in 8.10 and 9.2, is not unique. In fact, by Zorn’s Lemma we may take for each $i = 1, 2$, a maximal such cone containing $\mathcal{P}_i$, say $\mathcal{P}_i$, and we must then have $\mathcal{P}_1 \neq \mathcal{P}_2$, or else $\mathcal{P}_1 \cup \mathcal{P}_2$ is relatively continuous.

This shows that a choice has to be made somewhere if question 9.4 is to be answered affirmatively. Clearly the same goes for Question 11.16.

Let $X_i = \overline{\zeta(N_i)}$, for $i = 1, 2$, where $N_i$ is constructed from $\mathcal{P}_i$ as in 9.2. It is easy to see that $X_1^* X_1 = X_2^* X_2$ so that we fall short of giving a counter-example for Question 9.5 even if we knew (which we don’t) that the $\mathcal{P}_i$ were maximal.

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