Semi-Exclusive Heavy Quark Production
in Charged-Current Deep Inelastic Scattering

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To Cristina and Augusto, my parents

Theory is when you know everything but nothing works.
Practice is when everything works but no one knows why.
If practice and theory are combined, nothing works and no one knows why.

Albert Einstein

An expert is a person who has made
all the mistakes that can be made
in a very narrow field.

Niels Bohr
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Abstract

The main aim of this research consists in the description of Deep Inelastic Scattering (DIS) processes mediated by Charged Currents (CC), producing a massive quark in the final state.

First of all, we have carried out the semi-exclusive hadronic cross section parametrised by multi-differential structure functions, including in the treatise also the mass of the charged leptons.

Structure functions are reconstructed by means of the parton model, introducing Partonic Distribution Functions (PDF) and Fragmentation Functions (FF); in the context of Quantum Chromodynamics (QCD), we have obtained the functional form of the semi-exclusive coefficient functions for heavy quark production up to Next-to-Leading Order (NLO).

The obtained results have been therefore numerically implemented in order to carry out a quantitative analysis, in view of a more complete phenomenological investigation.

In this way we have pointed out that the semi-exclusive result obtained with techniques at fixed order (NLO) is quantitatively inadequate in some kinematic regions, highlighting the need of contributions coming from higher orders and therefore showing the necessity of an all-order resummed calculation.

In order to make easier the inclusion of resummation, we have considered the possibility to perform convolutions of parton densities, coefficient functions and fragmentation functions directly inside the Mellin space. For this purpose we have written a C/C++ code allowing to calculate evolved Mellin transforms of some parametrisation of modern PDF, as CTEQ6 and MRST2001. The only missing element is the extension of the analytical results for semi-exclusive coefficient functions to the whole complex plane: in this case the numerical approach appears to be difficult and at the moment an implementation has not yet carried out.

The first application of our results is the description of Strange quark inside nucleons: the CC DIS process with production of a hadron containing Charm is very useful for this purpose because quark and antiquark distributions can be independently studied. The means offered by our semi-exclusive results allow to accurately reproduce experimentally accessible observables and therefore it is possible a precise measurement for both Strange partonic distribution and $s - \bar{s}$ asymmetry.
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A.1 Generic electroweak leptonic vertex.

A.2 Leptonic Tensor.

A.3 Hadronic Tensor: incoming and outgoing momenta.

A.4 Boson Propagator.
Introduction

In processes where the transferred momentum is high, the structure of nucleons is characterized by means of Parton Distribution Functions which describe neutrons and protons in terms of quarks and gluons, according to the parton model at first, and -more in general- to the QCD Factorization Theorem [1]. These distributions are a fundamental requirement to formulate predictions, based on a perturbative approach, for observable quantities of high energy physics and particles experiments.

In Deep Inelastic Scattering (DIS) processes the interaction of leptons with hadrons is under investigation: it is then possible to probe the inner structure of the latter. In particular when Charged Currents are exchanged ($W^\pm$ bosons), interactions involve (at Leading Order of the perturbative theory) only quarks with positive electrical charge ($u, c, t, \bar{d}, \bar{s}, \bar{b}$) or negative ($d, s, b, \bar{u}, \bar{c}, \bar{t}$); it is then possible to select interaction channels sensitive to specific partonic distributions, simply imposing further constraints to final state configurations and considering the suppression originated by CKM matrix elements. For example, for Charm quark production in DIS processes with Charged Currents and with neutrinos in initial state

$$\nu + N \rightarrow \ell + H_c + X$$

the only partonic distributions contributing - at Leading Order - are the ones of Strange and Down (CKM suppressed) quarks. Consequently this class of events is suitable for the study of Strange quark distribution inside nucleons; the same holds about the Strange antiquark distribution taking into account the analogous case where in the initial state there is an antineutrino, producing Charm antiquark in the final state.

Through DIS processes mediated by $W^\pm$ it is then possible to carry out information on quark and antiquark independently; this is favourable to investigate the asymmetry between $s$ and $\bar{s}$, suggested by recent experimental measurements [2], [3]. Among light quarks, the Strange distribution inside nucleons is the less accurate: following the previously illustrated approach, it is possible to study it in a dedicated and profitable manner.
In this thesis we have examined in depth the formalism of DIS processes mediated by Charged Currents (CC), including moreover the mass contributions of charged lepton participating in the interaction (first chapter). In the second chapter we have carried out the semi-exclusive partonic coefficient functions for heavy quark production up to Next-to-Leading Order, in the context of perturbative QCD; it has been the first check of the only result present in literature and we have confirmed it. The property of exclusiveness is important in order to carry out predictions on realistic observable and not to restrict only to inclusive quantities, so that the four-momentum of Charm in the final state can be investigated. In the third chapter we have shown the outcomes of a numerical implementation of the obtained results, analysing also its theoretical uncertainties. Finally in the fourth chapter we have described the Mellin Transform approach, with a formalism that could facilitate the numerical treatment of convolutions found and allowing a more natural inclusion of resummation effects.
Chapter 1

Charged-Current Deep Inelastic Scattering

The main aim of this chapter is to introduce to cross section calculation for Deep Inelastic Scattering (DIS) process mediated by Charged Currents. Only fundamental aspects have been reported here, whereas more complete and accurate passages are contained in appendix A, in this way the explanation should be more fluent and essential. Such an appendix is not simply a mere list of formulae but is integral part of the treatise.

In paragraph 1.1 the generic semi-exclusive cross section of DIS processes mediated by Charged Currents is carried out

\[
\frac{d^3\sigma}{dx dy dz} = \frac{G_F^2 ME}{\pi} \frac{M_W^2}{(Q^2 + M_W^2)^2} \left[ \sum_{k=1}^{5} c_k(x, y) \frac{dF_k}{dz}(x, y, z) \right]
\]

and in paragraph 1.2 is shown that, in the parton model context, functions \( dF_k/dz \) have structure

\[
\frac{dF_k}{dz}(x, y, z) = \sum_{a,h} \int_x^1 \frac{d\xi}{\xi} \int_z^1 \frac{d\zeta}{\zeta} \frac{dF_k^a}{d\zeta}(\xi, y, \zeta) f_a^a \left( \frac{x}{\xi} \right) D_{h} \left( \frac{z}{\zeta} \right)
\]

Finally, in paragraph 1.2.4 the formalism has been extended to hadrons containing heavy quarks and corrections originated from the mass of the initial state hadron have been taken into account.
1.1 Standard Approach

The formalism introduced at the beginning of DIS investigations is now standard and well consolidated; it originated when QCD did not exist yet and it has been formulated in the context of Quantum Relativistic Field Theories. DIS is a process involving scattering between leptons and hadrons (usually nucleons) and producing both hadronic \((X)\) and leptonic final states:

\[
\ell_{\text{in}}(k_{\text{in}}) + N(P) \rightarrow \ell_{\text{out}}(k_{\text{out}}) + X(p_X)
\]

Leptons \(\ell_{\text{in}}\) and \(\ell_{\text{out}}\) can be of the same kind (Neutral Currents exchange (NC), where mediating particles are \(\gamma\) and/or \(Z^0\)) or different (Charged Currents exchange (CC), mediated by \(W^\pm\)). A graphical sketch of a DIS process is shown in Fig. 1.1.

![Figure 1.1: Deep Inelastic Scattering (DIS).](image)

The general expression for a cross section is

\[
d\sigma = \frac{1}{4F} |T_{fi}|^2 (2\pi)^4 \delta^{(4)}(P + k_{\text{in}} - k_{\text{out}} - p_X) d\Phi
\]

(1.1)

where \(F = \sqrt{(P k_{\text{in}})^2 - m_N^2 m_{\ell_{\text{in}}}^2}\) is the flux factor, \(|T_{fi}|^2\) is the squared modulus of the amplitude summed over all final and initial states, averaged on the latter, \(d\Phi\) is the phase-space of the final state (for hadrons and leptons). Four-momentum conservation is assured by the \(\delta\)-Dirac.

From now on, we consider CC DIS for the particular process

\[
\nu_{\ell}(k_{\text{in}}) + N(P) \rightarrow \ell^-(k_{\text{out}}) + X(p_X)
\]

with massless neutrino (initial state, \(m_\nu = 0\)) and massive charged lepton (final state, \(m_\ell\)); the following formalism holds also for NC case with few
modifications. We are able to write the general structure of $T_{fi}$ in a simple way, representing the process as interaction of the leptonic current $j^\nu_L$ (Fig.A.1) with the hadronic one $J^\tau_H (r, s$ are the spins of the leptons, the ones of the hadrons have been understood):

$$T_{fi} = \left\langle k_{\text{out}}, s | j^\nu_L | k_{\text{in}}, r \rangle \right. \left. \times B_{\nu\tau}(0) \frac{g_W}{\sqrt{2}} \langle p_X | J^\tau_H | P \rangle \right. \left. \times \frac{\sqrt{g^2}}{\sqrt{2}} \right\rangle,$$

where $B_{\nu\tau}(0)$ is the $W$ propagator in unitary gauge ($\eta = 0$, appendix A.3), so

$$\sum_{\text{pol,X}} |T_{fi}|^2 = \frac{g^4_W}{4} \sum_{r,s} \left\langle k_{\text{in}}, r | j^\nu_L | k_{\text{out}}, s \rangle \left\langle k_{\text{out}}, s | j^\nu_L | k_{\text{in}}, r \rangle \right. \left. \times \frac{\sqrt{g^2}}{\sqrt{2}} \right\rangle \left\langle P | J^\nu_H | p_X \rangle \langle p_X | J^\tau_H | P \rangle \right. \left. \times \frac{\sqrt{g^2}}{\sqrt{2}} \right\rangle$$

$$= \frac{8G_F^2 M_W^4}{(q^2 - M_W^2)^2} L^{\mu\nu} T_{\mu\nu\tau \rho} W^{\rho\tau} \right\rangle$$

using the relation $\frac{g^2_W}{8M_W^2} = \frac{G_F}{\sqrt{2}}$ and introducing the leptonic ($L^{\mu\nu}$) and hadronic ($W^{\rho\tau}$) tensors. Next step consists in averaging on initial states: with regard to leptonic state, there is a neutrino (only one helicity state allowed), whereas for the hadronic one there is a nucleon (spin 1/2), then resulting in an overall factor 1/2 (tensors $L^{\mu\nu}$ and $W^{\rho\tau}$ are not averaged, as explicitly declared in appendices A.1 and A.2). We insert a further normalization factor $4\pi$ according to the usual definition of hadronic tensor ($W^{\rho\tau} = W^{\rho\tau}/4\pi$). Therefore the cross section 1.1 can be written as

$$d\sigma = \frac{G_F^2}{ME} \frac{M_W^4}{(Q^2 + M_W^2)^2} L^{\mu\nu} T_{\mu\nu\tau \rho} \left[ 4\pi W^{\rho\tau} (2\pi)^4 \delta^{(4)}(P + q - p_X) \right] d\Phi$$

choosing the reference system where the nucleon is at rest (that is $P = (M, 0, 0, 0)$), defining $Q^2 = -q^2 = -(k_{\text{in}} - k_{\text{out}})^2$ and $E$ is the energy of the incoming neutrino.
For CC case, the leptonic tensor is (A.1)

\[ L_{\mu \nu} = L_{\text{CC}}^{\mu \nu} = 2 [k_{\text{out}}^\mu k_{\text{in}}^\nu + k_{\text{in}}^\mu k_{\text{out}}^\nu - (k_{\text{in}} k_{\text{out}}) g^{\mu \nu}] + 2i \epsilon^{\mu \nu \alpha \beta} k_{\alpha}^\text{in} k_{\beta}^\text{out} \quad (1.4) \]

Within this approach, the nature of the interaction between nucleon and electroweak vector boson is supposed to be unknown, so that \( \hat{W}^{\rho \tau} \) is constructed as the most generic second rank Lorentz-invariant tensor. In appendix A.2 we derive the expression for this tensor and we report some interesting remarks.

\[
\hat{W}^{\rho \tau}(2\pi)^4 \delta^{(4)}(P + q - p_X) \equiv H^{\rho \tau} = -(2Pq)H_1 g^{\rho \tau} + 4H_2 P^\rho P^\tau \\
- 2iH_3 \epsilon_{\alpha \beta}^{\rho \tau} P^\alpha q^\beta + 2H_4 q^\rho q^\tau + 2H_5 (P^\rho q^\tau + q^\rho P^\tau)
\]

Figure 1.2: DIS kinematics.

At this point it is useful to describe DIS kinematics; through notation shown in Fig1.2 for momenta of the particles, we can finally carry out some relations. Remember that we have chosen to work in the reference system where the nucleon is at rest.

\[
Q^2 = -q^2 \\
k_{\text{in}}q = -\frac{Q^2 + m_\ell^2}{2} \\
k_{\text{out}}q = \frac{Q^2 - m_\ell^2}{2} \\
Pq = M(E - E') \equiv \nu \\
k_{\text{in}}k_{\text{out}} = \frac{Q^2 + m_\ell^2}{2} \\
Pk_{\text{in}} = ME \\
x = \frac{Q^2}{2\nu} \\
Pk_{\text{out}} = ME' \\
y = \frac{qP}{k_{\text{in}}P} = 1 - \frac{E'}{E}
\]

where \( q = k_{\text{in}} - k_{\text{out}} \) is the momentum carried by \( W \) boson, \( M \) and \( m_\ell \) are respectively the mass of the nucleon and the charged lepton, whereas \( E, E' \)
are respectively the energies of incoming ($\nu_\ell$) and outgoing ($\ell^-$) leptons; all the external particles have been considered on-mass-shell. We can separate the phase-space of 1.3 in hadronic ($d\Phi_X$) and leptonic part

$$d\Phi = d\Phi_X \left[ \frac{d^4k_{out}}{(2\pi)^3}\delta\left(k_{out}^2 - m_\ell^2\right) \right] = d\Phi_X \frac{yME}{2(2\pi)^3} d\phi dx dy$$

rewriting the leptonic phase-space in terms of $x$ and $y$ variables as explained in appendix A.4 (eq.A.12). Moreover phase-space $d\Phi_X$ has been factorised in order to highlight the quantities $\alpha_i$: it is potentially possible to remain differential on them, if the explicit dependence is needed.

Making use of 1.5, in appendix A.3 we have calculated the contraction $L_{\mu\nu}^{CC} T_{\rho\tau} H^{\rho\tau}$ for a generic gauge parameter in $W$ propagator, finally choosing the unitary one; substituting this result (A.11) and 1.6 into 1.3, then the cross section can be written in a differential way as

$$\frac{d^{2+j}\sigma}{dxdy \prod_i d\alpha_i} = \frac{G_F^2 ME}{\pi} \frac{M_W^4}{(Q^2 + M_W^2)^2} \left\{ \left[ xy^2 + \frac{m_\ell^2 y}{2ME} \right] (1 + \chi_1(x,y)) \frac{d^j F_1}{\prod_i d\alpha_i} \\
+ \left[ \left( 1 - \frac{m_\ell^2}{4E^2} \right) - \left( 1 + \frac{M}{2E} \right) x \right] (1 + \chi_2(x,y)) \frac{d^j F_2}{\prod_i d\alpha_i} \\
+ \left[ xy \left( 1 - \frac{y}{2} \right) - \frac{m_\ell^2 y}{2ME} \right] (1 + \chi_3(x,y)) \frac{d^j F_3}{\prod_i d\alpha_i} \\
+ \left[ \frac{m_\ell^2(m_\ell^2 + Q^2)}{4M^2E^2x} \right] (1 + \chi_4(x,y)) \frac{d^j F_4}{\prod_i d\alpha_i} \\
- \left[ \frac{m_\ell^2}{ME} \right] (1 + \chi_5(x,y)) \frac{d^j F_5}{\prod_i d\alpha_i} \right\}$$

(1.7)

having integrated over flat angular variable $\phi \in [0, 2\pi)$, because no terms depend on such a quantity. Factor $d\hat{\Phi}_X$ has been included in the definition of functions

$$\frac{d^j F_k}{\prod_i d\alpha_i}(x, y, \alpha_i) \equiv A_k \int d\hat{\Phi}_X H_k$$

(1.8)

with $A_{1,5} = 4v$, $A_{2,3} = 8v$ and $A_4 = 4xv$, because of the chosen parametrisation, in accordance with appendix A.4. Result 1.7 agrees with literature for the inclusive case [3]; with abuse of notation (for sake of convenience and for historical reasons), $Q^2$ variable has been let expressed in the second side
of relation 1.7 and for analogous quantities, nevertheless it would be more correct to employ the equality \( Q^2 = 2MExy \), showing explicitly the full dependence on \( x \) and \( y \). Functions \( \chi_k(x,y) \) are reported in appendix A.3.

Inclusive Cross Section

In 1.7 is shown a cross section semi-exclusive on hadronic states, but it is possible to choose to obtain a fully inclusive result:

\[
\frac{d^2\sigma}{dxdy} = \frac{G_F^2 ME}{\pi} \frac{M_W^4}{(Q^2 + M_W^2)^2} \left[ \sum_{k=1}^{5} c_k(x,y)F_k \right]
\]

obviously identifying

\[
F_k(x,y) = \int \prod_i d\alpha_i \left( \frac{d^j F_k}{\prod_i d\alpha_i} \right) = A_k \int d\Phi X H_k
\]

and \( c_k \) coefficients are the ones in 1.7.
This does not change considerably the structure of previous results.

Massless Case

For the massless charged lepton case (obtained setting \( m_\ell = 0 \) in 1.7) the expression for coefficients \( c_k \) becomes simpler

\[
\frac{d^{2+2j}\sigma}{dx dy \prod_i d\alpha_i} = \frac{G_F^2 ME}{\pi} \frac{M_W^4}{(Q^2 + M_W^2)^2} \left\{ xy^2 \frac{d^j F_1}{\prod_i d\alpha_i} + \left( 1 - y + \frac{M}{2E} xy \right) \frac{d^j F_2}{\prod_i d\alpha_i} + xy \left( 1 - \frac{y}{2} \right) \frac{d^j F_3}{\prod_i d\alpha_i} \right\}
\]

Notice that terms containing \( H_4 \) and \( H_5 \) (i.e. \( F_4, F_5 \)) disappear: this because coefficients \( c_4 \) and \( c_5 \) cancel out; even a potential term \( H_6 \) (eq A.6) is not present as explained in appendix A.2. furthermore \( H_4 \) and \( H_5 \) exist only for the case of massive lepton, whereas \( H_6 \) does not contribute in any case.
Range of variability for x and y

The range for x and y variables is carried out from kinematics according to [1.5]:

\[
\frac{m_\ell^2}{2M(E - m_\ell)} \leq x \leq 1 \quad y_1 - y_2 \leq y \leq y_1 + y_2
\]

\[
y_1 = \frac{1 - m_\ell^2 \left(\frac{2ME}{2E_x} + \frac{1}{2E_x}\right)}{2 \left(1 + \frac{M_x}{2E}\right)}
\]

\[
y_2 = \sqrt{\left(1 - \frac{m_\ell^2}{2ME}\right)^2 - \frac{m_\ell^2}{E^2}}
\]

Charged Currents in general

Result [1.7] (and similarly [1.9, 1.10]) can be extended to the case of generic DIS process with charged current. As highlighted at the end of appendix A.1, swapping particles for antiparticles, the only variation for the whole study is a change of sign for \(c_3\) term. A charged lepton in the initial states needs a factor 1/2 because of the average on spin (instead of only one helicity state allowed for a neutrino). Then

\[
\frac{d^{2+j} \sigma^{CC}}{dxdy \prod_i d\alpha_i} = \frac{G_F^2 ME}{\pi} \frac{M_W^4}{(Q^2 + M_W^2)^2} \left[ \left(\frac{E^2}{E^2 - m_\ell^2}\right) \left(\frac{1 - (-1)^b P_\ell}{2}\right)\right]^a \left[ \sum_{k=1,2,4,5} c_k(x,y) \frac{d^j F_k}{\prod_i d\alpha_i} + (-1)^b c_3(x,y) \frac{d^j F_3}{\prod_i d\alpha_i} \right]
\]

(1.11)

where \(c_k\) coefficients are obviously those contained in [1.7] and \(a\) and \(b\) exponents have to be evaluated following the table:

| CC Process | a | b |
|------------|---|---|
| \(\nu_\ell + N \rightarrow \ell^- + X\) | 0 | 0 |
| \(\ell^- + N \rightarrow \nu_\ell^- + X\) | 1 | 1 |
| \(\bar{\nu}_\ell + N \rightarrow \ell^+ + X\) | 0 | 1 |
| \(\ell^+ + N \rightarrow \bar{\nu}_\ell + X\) | 1 | 0 |

Term \(E^2/(E^2 - m_\ell^2)\) comes from flux factor and phase-space (A.12), considering a massive lepton \((m_\ell)\) in initial state. We have also introduced the term \(P_\ell\) to potentially take into account the polarisation degree of the (charged)
lepton beams: \( P_\ell = (N_L - N_R)/(N_L + N_R) \) where \( N_L \) and \( N_R \) are respectively the number of particles with left-handed and right-handed polarisation. Unpolarized case corresponds to \( P_\ell = 0 \). Setting \( a = b = 0 \), result in 1.7 is recovered.

We have initially parametrised the unidentified physics of the hadronic vertex with a set \( \{ H_k \} \) of unknown functions and consequently the result can not be predictive but it depends on them; comparing the experimentally measured cross section to the parametrisation 1.7 (or a simplified expression: in fact the one for massless leptons, as shown in paragraph 1.2.3) functions \( d^3 F_k/\prod_i d\alpha_i \) can be carried out.

No hypothesis has been yet proposed on the nature of interactions behind the hadronic tensor, so some physical model has to be employed to describe it. Eventually we remark that there are many other ways to parametrise the cross section: all are comparable and they differ in expressions relating factors \( H_k \) to \( d^3 F_k/\prod_i d\alpha_i \), or in choosing \((x, y, \alpha_i)\) observables in the phase-space (appendix A.4).

1.2 Parton Model

1.2.1 Partonic Cross Section

Cross section 1.7 holds in general in the context of Quantum Relativistic Field Theories, with the support of Weinberg-Salam-Glashow Electroweak theory. Applying the parton model and QCD, we are trying to predict and describe the functional form of terms \( d^3 F_k/\prod_i d\alpha_i \).

The naive parton model is independent of QCD, having been hypothesised before than the latter had been formulated ([6],[7]). In fact it originated as quasi-classical model for DIS, based on the idea that hadrons could be described as a collection of independent point-like particles (so-called partons), having a small transverse momentum (and therefore negligible) and interacting with the incident leptons by exchanging a vector boson. Identifying quarks and gluons with partons, the quark-parton-model is made, so quantum numbers are assigned to partons in order to reproduce the ones for known hadrons. The interactions vertex between \( W \) boson and the nucleon can then be interpreted as in representation of Fig 1.3.

We can consequently work at partonic level exactly as in previous section, simply substituting for the momentum \( P \) of the nucleon, the momentum \( \hat{p} \) of the parton contained in it and interacting with the \( W \) boson.
We introduce the *partonic tensor*, whose properties and structure are identical to the hadronic one:

\[ h^{a}_{\mu\nu} = -(2\hat{p}q)h^{a}_{1}g_{\mu\nu} + 4h^{2}_{2}\hat{p}_{\mu}\hat{p}_{\nu} \]
\[ -2ih^{3}_{3}\epsilon^{\rho\sigma}_{\mu\nu}\hat{p}_{\rho}q_{\sigma} + 2h^{4}_{4}q_{\mu}q_{\nu} + 2h^{5}_{5}(\hat{p}_{\mu}q_{\nu} + q_{\mu}\hat{p}_{\nu}) \]  

(1.12)

where the \( a \) index indicates which kind of parton is taken into account.

We define the partonic quantities

\[ \hat{x} = \frac{Q^{2}}{2pq} \quad \hat{y} = \frac{\hat{p}q}{\hat{p}k} \quad \hat{v} = \hat{p}q \]  

(1.13)

analogously to (1.15) in order to contract (1.12) with the leptonic tensor (A.1) also the phase-space of final state contains a parton, therefore it has to be expressed by means of partonic quantities \((d\Phi_{X} \rightarrow d\varphi_{X})\):

\[ d\varphi_{X} = d\hat{\varphi}_{X} \prod_{i} d\hat{\chi}_{i} \]

Temporarily neglecting modifications to flux factor \( F \) (introduced in (1.1)), for the sake of simplicity, and going back over all steps of previous section, we obtain
\[
\frac{d^2 \hat{\sigma}}{d\hat{x}d\hat{y} \prod_i \hat{d}\alpha_i} = \frac{G_F^2 ME}{\pi} \frac{M_W^4}{(Q^2 + M_W^2)^2} \sum_a \left\{ \left[ \hat{x}\hat{y}^2 + \frac{m_i^2 \hat{y}}{2ME} \right] (1 + \chi_1(\hat{x}, \hat{y})) \frac{d\hat{F}_1^a}{\prod_i \hat{d}\alpha_i} 
\right.
\]

\[
+ \left[ \left( 1 - \frac{m_i^2}{4E^2} \right) - \hat{y} \right] (1 + \chi_2(\hat{x}, \hat{y})) \frac{d\hat{F}_2^a}{\prod_i \hat{d}\alpha_i} 
\]

\[
+ \hat{x}\hat{y} \left( 1 - \frac{\hat{y}}{2} \right) - \frac{m_i^2 \hat{y}}{4ME} \right] (1 + \chi_3(\hat{x}, \hat{y})) \frac{d\hat{F}_3^a}{\prod_i \hat{d}\alpha_i} 
\]

\[
+ \frac{\left[ \frac{m_i^2 (m_i^2 + Q^2)}{4M^2 E^2 \hat{x}} \right]}{(1 + \chi_4(\hat{x}, \hat{y})) \frac{d\hat{F}_4^a}{\prod_i \hat{d}\alpha_i}} 
\]

\[
- \left[ \frac{m_i^2}{ME} \right] (1 + \chi_5(\hat{x}, \hat{y})) \frac{d\hat{F}_5^a}{\prod_i \hat{d}\alpha_i} \right\}
\]

(1.14)

where, analogously to 1.8, it is has been defined

\[
\frac{d\hat{F}_k^a}{\prod_i \hat{d}\alpha_i}(\hat{x}, \hat{y}, \hat{\alpha}_i) \equiv \hat{A}_k \int d\hat{\phi}_X h_k^a(\hat{x}, \hat{y}, \hat{\phi}_X) 
\]

(1.15)

being \(\hat{A}_{1,5} = 4\hat{v}, \hat{A}_{2,3} = 8\hat{v}, \hat{A}_4 = 4x\hat{v}\).

### 1.2.2 Semi-Exclusive Processes

We leave now the general formalism to consider a specific case. We want to study a CC DIS with heavy quark production in the final state. According to perturbative QCD at LO (Fig. 2.1) the phase-space contains only one particle (the heavy quark) whereas at NLO (Fig. 2.6 and 2.7) there are two (an additional light quark or gluon). As shown in appendix A.6, for the NLO case there is only one degree of freedom, whilst obviously none at LO. Then in the following we will consider cross sections with at the most \(j = 1\) in 1.7 and 1.14 i.e. neglecting the trivial dependence (through \(\delta\)-Dirac) on non-free variables. We define \(z (\equiv \alpha_1\) in 1.7) and \(\zeta (\equiv \hat{\alpha}_1\) in 1.14) respectively the variables of interest for the hadronic and partonic level. In particular we remark that, under these assumptions, starting from 1.15 it is possible to obtain (with a little abuse of notation)

\[
\frac{d\hat{F}_k}{d\zeta}(\hat{x}, \hat{y}, \zeta) = A_k \int d\hat{\phi}_X h_k(\hat{x}, \hat{y}, \hat{\phi}_X, \zeta) \equiv A_k h_k(\hat{x}, \hat{y}, \zeta) \int d\hat{\phi}_X 
\]

(1.16)

implying that integration is performed on flat variables (that is \(h_k\) is independent of them) and on all those ones being not degrees of freedom; obviously in
\( h_k(\hat{x}, \hat{y}, \hat{\phi}_X, \zeta) \) the dependent variables \((\hat{\phi}_X)\) are replaced by relations carried out from \(\int d\hat{\phi}_X\). We call this approach \textit{semi-exclusive}, in order to highlight that the considered quantities are as differential (exclusive) as possible, compatibly with the degrees of freedom.

### 1.2.3 Parton-Hadron Correspondence

In nature only hadrons are experimentally accessible physical states, whereas partons have not been observed. Assuming partons on-mass-shell and collinear to the momentum of nucleon containing them, the hadronic cross section has the factorised form

\[
d\sigma(P) \sim \sum_a \int_0^1 d\xi f^a(\xi) d\hat{\sigma}_a(\hat{p}_a = \xi P)
\]

Parameter \(\xi \in (0,1)\) is the fraction of nucleon momentum carried by the parton of kind \(a\), whilst \(f^a\) describes how \(\xi\) is distributed and in practice it could be interpreted as probability density of interaction (through boson exchange) with a parton of kind \(a\) and momentum \(\hat{p}_a = \xi P\). Quantity \(f^a(\xi)\) is called Partonic Distribution Function (PDF) and it is non-perturbatively calculable.

In the final state we have imposed the presence of partons, then, similarly to PDF, also Fragmentation Functions (FF) have been introduced to phenomenologically describe the transitions from quarks to hadrons. Using the correspondence parton-hadron in a more rigorous way, flux factor \(F\) becomes \(F\xi\) and it holds

\[
\hat{x} = \frac{x}{\xi} \quad \hat{y} = y \quad \hat{\nu} = \xi \nu \quad \chi_k(\hat{x}, \hat{y}) = \chi_k(x, y)
\]

so we are able to write the hadronic cross section as convolution of the partonic one \((1.14)\) with PDF and FF:

\[
\frac{d^3\sigma}{dx dy dz} = \frac{G_E^2 M E}{\pi} \left\{ \sum_{k=1,3,4,5} c_k(x, y) \left[ \sum_a \sum_h \int_x^1 \frac{d\xi}{\xi} \int_z^1 \frac{d\zeta}{\zeta} dF^a_k(\xi, y, \zeta) f^a \left( \frac{x}{\xi} \right) D_h \left( \frac{z}{\zeta} \right) \right] \right\} (1.18)
\]
where \( c_k \) and \( \tilde{c}_2 \) are respectively the coefficients of terms \( d^j F_k \) and \( d^j F_2 \) in (1.14) whilst \( \mathcal{D}_h \) is a FF describing the manner of forming a hadron of kind \( h \), starting from a final state parton. In this investigation we assume that the hadron containing the heavy quark is originated directly from such a quark and not from other partons, being not predominant channels. Comparing cross section (1.18) with the one purely hadronic previously carried out (eq.1.7)

\[
\frac{d^3\sigma}{dxdydz} = \frac{G_F^2 M E}{\pi} \frac{M_W^4}{(Q^2 + M_W^2)^2} \left\{ \sum_{k=1}^{5} c_k(x, y) \frac{dF_k}{dz}(x, y, z) \right\}
\]

(1.19)

it is clear an explicit and immediate correspondence between hadronic and partonic level

\[
\frac{dF_k}{dz}(x, y, z) = \sum_{a,h} \int_x^1 \frac{d\xi}{\xi} \int_z^1 \frac{d\zeta}{\zeta} \frac{d\hat{F}_k^a}{d\zeta}(\xi, y, \zeta)f_a\left(\frac{x}{\xi}\right) \mathcal{D}_h\left(\frac{z}{\zeta}\right) \quad k = 1, 3, 4, 5
\]

\[
\frac{dF_2}{dz}(x, y, z) = \frac{\tilde{c}_2(x, y)}{c_2(x, y)} \sum_{a,b,h} \int_x^1 \frac{d\xi}{\xi} \int_z^1 \frac{d\zeta}{\zeta} \frac{d\hat{F}_2^a}{d\zeta}(\xi, y, \zeta)f_a\left(\frac{x}{\xi}\right) \mathcal{D}_h\left(\frac{z}{\zeta}\right)
\]

(1.20)

Notice that several \( \hat{F}_k^a \) correspond to each \( F_k \) because there are many kind of partons realising the sub-process shown in Fig.1.3 and to each one there is associated an opportune distribution \( f^a(\xi) \). Furthermore it is very interesting that no terms \( \frac{M^2}{Q^2} \hat{x} \hat{y} \) appear into \( \tilde{c}_2 \) coefficient for \( \hat{F}_2 \) (1.14): this because for an on-mass-shell nucleon it holds \( p^2 = M^2 \), whereas for a parton \( \hat{p}^2 = 0 \). However in QCD this fact is predicted by the factorization theorem asserting that only the dominant contribution can be reproduced by the perturbative theory: are therefore excluded terms proportional to \( \frac{M^2}{Q^2} \) \( x^2 y \) (term \( \frac{M^2}{Q^2} \) \( x^2 y \) is in fact equivalent to \( M^2/Q^2 \frac{x^2 y^2}{Q^2} \)). From the operative point of view there are no problems, because in practice such a term is negligible, being typically \( M \sim 1 GeV, 30 \leq E[GeV] \lesssim 600 \) (CCFR, NuTeV experiments [9], [10]) and also because of a suppression, being \( x < 1 \). Then we can tacitly assume \( \tilde{c}_2(x, y) \sim c_2(x, y) \); for the case of inclusive cross sections the difference is less than 2% ([11], page 3091). In view of this remark, one should be coherent with the chosen accuracy: in fact, with regard to experimentally accessible parameters, coefficients \( c_4 \) and \( c_5 \) are much smaller than that last term we have neglected, like other contributions inside \( c_1, c_2 \) and \( c_3 \). So expression in (1.10) has to be used as parametrisation for cross sections to carry out quantitative results; this also because functions \( F_4 \) and \( F_5 \) calculated in QCD (2.4.1) are quantitatively comparable to \( F_1, F_2, F_3 \). We remark that this fact does not come (only) from an issue of experimental precision, but it originates
from the limit of the parton model well highlighted by the factorization theorem of Perturbative QCD. Therefore, even with an exact treatment (1.7) for DIS considering massive charged lepton (the one coming directly by leptonic current in DIS), we intrinsically can not obtain an accuracy higher than the massless case, at the state of the theory.

For the case of heavy quark production in the final state, the convolution with $\mathcal{D}_h$ does not prevent from recovering directly the inclusive result; in fact thanks to the usual definition of FF normalization (holding for heavy quarks)

$$\sum_h \int_0^1 dt \mathcal{D}_h(t) = 1$$

it is possible to verify that

$$F_k(x, y) = \int_0^1 dz \left[ \frac{dF_k}{dz}(x, y, z) \right] = \int_0^1 dz \left[ \sum_{a, h} \int_x^1 \frac{d\xi}{\xi} \int_z^1 \frac{d\zeta}{\zeta} \frac{d\tilde{F}_k^a}{d\zeta}(\xi, y, \zeta) f^a \left( \frac{x}{\xi} \right) \mathcal{D}_h \left( \frac{z}{\zeta} \right) \right]$$

$$= \sum_a \int_x^1 \frac{d\xi}{\xi} f^a \left( \frac{x}{\xi} \right) \left[ \int d\zeta \tilde{F}_k^a(\xi, y, \zeta) \right]$$

$$= \sum_a \int_x^1 \frac{d\xi}{\xi} f^a \left( \frac{x}{\xi} \right) \tilde{F}_k^a(\xi, y)$$

We remark that the whole hadronic final state consists in the remainder of nucleon not directly participating to the process and in the one generated by the interaction with $W$ boson. The former is made of spectator-partons, obviously hadronizing through interactions with particles carrying colour charge in the process; nevertheless this contribution generally is not taken into account, because usually it remains in the beam pipe for experiments not-a-rest or it does not reach the detectors. In any case it is not experimentally selected: only the latter produces the “observed” final state(Fig.1A).
Figure 1.4: Typical configuration of a final state for a DIS process.
1.2.4 Slow Rescaling and Target Mass Correction

The main result we have carried out till now consists in the correspondence between hadronic and partonic structure functions as reported in 1.20. In order to enable a smoother comparison with literature and to use a more compact formalism, we adopt the convention

\[
\frac{dF_k}{dz}(x, y, z) = F_k(x, y, z)\quad k = 1, 4, 5
\]

\[
\frac{dF_2}{dz}(x, y, z) = 2x F_2(x, y, z)
\]

\[
\frac{dF_3}{dz}(x, y, z) = 2 F_3(x, y, z)
\]

\[
\mathcal{F}_k(x, y, z) = \sum_{a,h} \int_1^1 \frac{d\xi}{\xi} \int_1^1 \frac{d\zeta}{\zeta} \hat{\mathcal{F}}_a^k(\xi, y, \zeta) f_a^a \left( \frac{\chi}{\xi} \right) D_h \left( \frac{z}{\zeta} \right) \tag{1.21}
\]

where we have profitably redefined \( \chi = x \), \( \hat{\mathcal{F}}_a^k = \frac{d\hat{\mathcal{F}}_a^k}{d\zeta} \) for \( k = 1, 4, 5 \), \( \hat{\mathcal{F}}_2 = \frac{\xi}{2x} \frac{dF_2}{d\xi} \) and \( \hat{\mathcal{F}}_3 = \frac{1}{2} \frac{dF_3}{d\xi} \). Relations (1.21) are not the arrival point of our investigation and they need a more exhaustive examination.

**Heavy Quarks**

The goal of our analysis is to study hadron production in CC DIS processes; in particular we consider hadrons containing a heavy quark, then, to obtain a realistic and correct result, we assume that the parton produced in the final state is the heavy quark with mass \( m \) becoming a hadron later on. As shown in detail in the following (see remarks after 2.22), it is natural to introduce an adimensional parameter \( \lambda \), defined as

\[
\lambda = \frac{Q^2}{Q^2 + m^2} \tag{1.22}
\]

where as usually \( Q^2 \) is the quantity in [1.5]. Relations (1.21) expressed by means of \( x \), properly hold only for the case of massless quark production; investigating more in depth, we have found that the right link between partonic and hadronic level is carried out substituting \( x \to x/\lambda \), that is by means of a slow-rescaling “prescription” to take into account also the mass of the produced quark. The massless quark case can be recovered for \( \lambda = 1 \).
Target Mass Corrections

Relation \( \hat{p}_a = \xi P \) is usually introduced in an “infinite” reference system where masses of particles are negligible compared to momenta; it is a connection between hadronic and partonic view, correct if the mass of target-nucleon is disregarded. In fact, using light-cone formalism (appendix A.7), it is possible to demonstrate that \( \hat{p}^+ = \xi P^+ \), but the partonic momentum \( \hat{p}^- \) is not simply a rescaling of the “small” momentum \( P^- \) of the nucleon, i.e. \( \hat{p}^- \neq \xi P^- \) [4],[8].

Consequently Target Mass Corrections (TMC) are needed; they have non-perturbative nature. We introduce the Natchmann \( \eta \) [12] variable defined by

\[
\frac{1}{\eta} = \frac{1}{2x} (1 + \rho) \quad \rho = \sqrt{1 + \left(\frac{2Mx}{Q}\right)^2} \tag{1.23}
\]

(it holds for massless partons inside nucleons) and we describe the hadronic functions \( dF_k/dz \) imposing the substitution \( \chi \to \eta \) in relations 1.21 (and not into the coefficient of \( F_2 \), written on purpose as \( 2x \)).

In the collinear limit \( (\hat{p}_\perp = 0) \) new relations have found [11]

\[
\begin{align*}
\frac{dF_1}{dz}(x,y,z) &= \mathcal{F}_1(x,y,z) \\
\frac{dF_2}{dz}(x,y,z) &= 2x \frac{\lambda}{\rho^2} \mathcal{F}_2(x,y,z) \\
\frac{dF_3}{dz}(x,y,z) &= 2 \frac{\rho}{\lambda} \mathcal{F}_3(x,y,z) \\
\frac{dF_4}{dz}(x,y,z) &= \lambda \frac{1}{2} \left(1 - \rho^2\right) \mathcal{F}_2(x,y,z) + \mathcal{F}_4(x,y,z) + \frac{1}{\rho} \mathcal{F}_5(x,y,z) \\
\frac{dF_5}{dz}(x,y,z) &= \lambda \frac{1}{\rho^2} \mathcal{F}_2(x,y,z) + \frac{1}{\rho} \mathcal{F}_5(x,y,z)
\end{align*} \tag{1.24}
\]

where a mixing involving terms \( k = 4,5 \) appears. In this case \( \mathcal{F}_k(x,y,z) \) is defined as

\[
\mathcal{F}_k(x,y,z) = \sum_{a,h} \int_{1/\eta/\lambda}^{1} \frac{d\xi}{\xi} \int_{1/\zeta}^{1} \frac{d\zeta}{\zeta} \mathcal{F}_k^a(\xi,y,\zeta) f_a \left(\frac{\eta/\lambda}{\xi}\right) D_h \left(\frac{z}{\zeta}\right)
\]

Furthermore \( \lambda \) parameter (1.22) has been introduced in order to include also the treatment of massive quarks produced in final state. Relations 1.24 hold up to NLO for initial state partons collinear to nucleon \( (\hat{p}_\perp = 0) \) or up to LO,
if $\hat{p}_\perp \neq 0$; for the latter case it is possible an extension up to NLO completing with further contributions [13], [14] beyond the purpose of this investigation. It can be trivially verified that, if $M \to 0$, situation [1.21] is recovered from [1.24] with null TMC

$$\hat{p}_a \to \xi P \quad \eta \to x \quad \rho \to 1$$

$$\frac{dF_{1,4,5}}{dz} \to \mathcal{F}_{1,4,5} \quad \frac{dF_2}{dz} \to 2\frac{x}{\lambda} \mathcal{F}_2 \quad \frac{dF_3}{dz} \to 2\mathcal{F}_3$$

(1.25)

In short, in accordance with the hypothesis on the mass of the target-nucleon ($M_{\text{Nuc}}$) and the produced quark ($m$), $\chi$ has to be chosen following the table

| Quark Mass | Nucleon Mass | $M_{\text{Nuc}} = 0$ | $M_{\text{Nuc}} \neq 0$ |
|------------|--------------|---------------------|---------------------|
| $m = 0$    | $\chi = x$  | $\chi = x$          | $\chi = \eta$      |
| $m \neq 0$ | $\chi = x/\lambda$ | $\chi = x/\lambda$ | $\chi = \eta/\lambda$ |

or, without loss of generality, setting $\chi = \eta/\lambda$: opportune quantities are then included by default (according to the assumptions done).

As reported in [11], being the Charm mass not so different from the one of a nucleon, if corrections for massive Charm quark are taken into account, then also TMC should be included. Considering only the former would be not consistent; in fact, in some configurations of kinematics, the discrepancy can be even more than 25%. In Fig.1.5 it is illustrated the behaviour of $\rho$ and $\eta/x$ in the regions $0 \leq x \leq 0.5$ and $1 \leq Q[GeV] \leq 5$. 

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Figure 1.5: Target Mass Corrections - Dependence on $x$ and $Q$. 
Chapter 2

Coefficient Functions

In the previous chapter we have shown that the cross section is proportional to structure functions $\mathcal{F}_k$ \(^{(1.21)}\), having the form

$$\mathcal{F}_k = \sum_{a,h} \int_\chi^1 \frac{d\xi}{\xi} \int_z^1 \frac{d\zeta}{\zeta} \hat{\mathcal{F}}_k^a(\xi, y, \zeta) f^a\left(\frac{\chi}{\xi}\right) D_h\left(\frac{z}{\zeta}\right)$$

Now, in the context of perturbative QCD, we calculate the expression of the “Coefficient Functions” up to Next-to-Leading Order, using the decomposition

$$\hat{\mathcal{F}}_k^a = \hat{\mathcal{F}}_k^{a,\text{LO}} + \hat{\mathcal{F}}_k^{a,\text{NLO}}$$

For the treatment of the generic partonic tensor, unknown functions $h_k$ have been introduced in order to parametrise the cross section; assuming that QCD is the right theory to describe the structure of hadrons (and then in particular of the nucleons), we are carrying out the analytic form for this kind of functions. Still referring to CC DIS process $\nu_e + N \rightarrow \ell^- + X$ with heavy quarks production, we obtain the tensor $h_{\mu\nu}^{a,QCD}$ from amplitudes of Feynman diagrams with a parton of kind $a$ in the initial state. We introduce a set of projectors $P^{\mu\nu}_k$ (appendix A.5) so that

$$P^{\mu\nu}_k h_{\mu\nu}^a = p_k h_k^a$$

where $h_{\mu\nu}^a$ is the partonic tensor in \(^{(1.12)}\) and $p_k$ are opportune coefficients \(^{(A.15)}\). Therefore

$$h_k^a = \frac{P^{\mu\nu}_k h_{\mu\nu}^{a,QCD}}{p_k} = \frac{1}{4\pi} \frac{P^{\mu\nu}_k h_{\mu\nu}^{a,QCD}}{p_k}$$

(2.1)
identifying the QCD tensor with the partonic one and inserting the proper normalization factor $4\pi$, in accordance with the convention introduced in 1.3. Consequently it is possible to carry out the analytical form of every term $h^a_k$ and then to obtain the expression of $\hat{F}^a_k$ through relation 1.10

$$\hat{F}^a_k(\xi, y, \zeta) = \hat{A}_k h^a_k \int d\hat{\phi}_X = \frac{\hat{A}_k}{4\pi} \left( \frac{P_{\mu\nu} h^a_{QCD}}{p_k} \right) \int d\hat{\phi}_X$$ (2.2)

with $\hat{A}_{1,2,3,5} = 4\hat{\nu}$, $\hat{A}_4 = 4\xi\hat{\nu}$. Deriving from perturbative QCD, tensor $h^a_{\mu\nu}^{QCD}$ is calculable at every perturbative order; for each parton $a$, from now on definitely identified with quarks ($q$) or gluons ($g$), we can then expand

$$h^a_{\mu\nu}^{QCD} = h^a_{\mu\nu}^{LO} + h^a_{\mu\nu}^{NLO} + \ldots$$ (2.3)

At Next-to-Leading Order we can coherently write

$$\hat{F}^a_k(\xi, y, \zeta) = \hat{F}^a_{k,LO}(\xi, y, \zeta) + \hat{F}^a_{k,NLO}(\xi, y, \zeta)$$

with obvious notation. All calculations have been performed by hand and subsequently checked with the aid of FORM [15] or MATHEMATICA [16].

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1 As a consequence of 1.21 $\hat{A}_{2,3} = \hat{A}_{2,3}/2 = \hat{A}_{1,5}$

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2.1 Tools for Calculations

2.1.1 Dimensional Regularization

We have chosen the dimensional regularization approach to deal with ultra-violet and infrared divergences, met respectively in virtual and real contributions at NLO: we have also chosen to work in a Minkowski space-time with $D = 4 + 2\varepsilon$ dimensions. In order to obtain convergence of the expressions, $\varepsilon < 0$ is formally assumed for the ultraviolet case, whereas $\varepsilon > 0$ for the infrared one; in particular there are poles of kind $1/\varepsilon^n$ that, when $\varepsilon \to 0$, indicate the presence of a divergence in four dimensions. Unlike other regularization techniques, the dimensional one allows to maintain gauge and Lorentz invariance. Some prescriptions are needed in order to work in a $D$-dimensional space-time:

- rescaling the coupling constants by means of a parameter with dimension of a mass, to maintain the right dimensionality of the Lagrangian: $g \to g\mu^{\frac{4-D}{2}}$.
- using relations holding for an Algebra in $D$ dimensions (appendix B.1) and extending to $D$ dimensions also the phase-space (appendix A.6).
- paying attention particularly to $\gamma_5$ matrix inside the electroweak vertex $\gamma^\mu (1 - \gamma_5)^2$.

This last point is very delicate: in fact $\gamma_5 = i\epsilon_{\mu\nu\rho\tau}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\tau/4!$, however the Levi-Civita tensor is not well defined beyond four dimensions. In this case it is necessary to adopt a particular scheme of regularization to avoid lacks of foundation [9], [17], [18], [19]. We have chosen the HVBM scheme [20], [21], [22], where the prescription is

$$\{\gamma^\mu, \gamma_5\} = 0 \quad \mu = 0, 1, 2, 3 \quad [\gamma^\mu, \gamma_5] = 0 \quad \text{otherwise} \quad (2.4)$$

Such a trick splits the $D$-dimensional space-time in two sub-spaces of 4 and $D - 4$ dimensions respectively: the relation on the left of (2.4) is applied for the former, whereas the one on the right for the latter. In accordance with remarks in [20], for the case of of unpolarised DIS some simplifications appear, so $\gamma_5$ contributions are only those arising from the 4-dimensional sub-space.
2.1.2 Expansions in the Sense of Distributions

It must be paid particular attention to the manner to isolate singularities for the encountered quantities; in fact, to highlight the pole structure through $\varepsilon$ parameter, it is necessary to perform expansions in the sense of mathematical distributions (appendix B.2). In order to maintain the analysis as general as possible and subsequently to recover the massless case, we have to deal in an accurate way with the contributions not containing poles for massive case, but containing for the massless one. For example, the term

$$\frac{(1 - \xi)^{1+2\varepsilon}}{(1 - \lambda \xi)^{2+\varepsilon}}$$

is not singular for the massive case ($\lambda \neq 1$, $\lambda < 1$), so a Taylor expansion apparently seems to be opportune

$$\frac{(1 - \xi)^{1+2\varepsilon}}{(1 - \lambda \xi)^{2+\varepsilon}} = \frac{(1 - \xi)}{(1 - \lambda \xi)^2} \left[ 1 + \varepsilon \left( 2 \log(1 - \xi) - \log(1 - \lambda \xi) \right) + O(\varepsilon^2) \right]$$

When $\varepsilon \to 0$ we would obtain $\frac{(1-\xi)}{(1-\lambda^2)^2}$, however if $\lambda = 1$, that is recovering the massless limit, it becomes $(1 - \xi)^{-1}$, clearly diverging when integrated up to $\xi = 1$. Strategy of expansion therefore consists in considering the quantity $C = A + B$, where $A$ and $B$ are the exponents of the generic expression $(1 - \xi)^A(1 - \lambda \xi)^B$; if $C(\varepsilon = 0) < 0$ we need an expansion in the sense of mathematical distributions, else such a term would be definitely finite also for the massless case. With regard to the previous example, $C(\varepsilon = 0) = -1$ so, expanding in distributional sense, (about the definition of $K_A$, see paragraph 2.3)

$$(1 - \xi)^{1+2\varepsilon}(1 - \lambda \xi)^{-2-\varepsilon} = \delta(1 - \xi) \left[ \frac{1}{\varepsilon} - \frac{(1 - \lambda)^\varepsilon}{\varepsilon} - \frac{1 + \lambda}{\lambda} K_A - \frac{(1 - \lambda)^\varepsilon}{\lambda} \right]$$

$$+ \left[ \frac{1 - \xi}{(1 - \lambda \xi)^2} \right]_+ + O(\varepsilon)$$

from which immediately both massless

$$(1 - \xi)^{-1+\varepsilon} = \frac{1}{\varepsilon} \delta(1 - \xi) + \frac{1}{(1 - \xi)_+} + O(\varepsilon)$$

and massive case

$$\frac{(1 - \xi)}{(1 - \lambda \xi)^2} = -\delta(1 - \xi) \left[ \log(1 - \lambda) + \frac{1 + \lambda}{\lambda} K_A + \frac{1}{\lambda} \right] + \left[ \frac{1 - \xi}{(1 - \lambda \xi)^2} \right]_+$$

can be recovered.
2.1.3 Fragmentation Variable

In paragraph 1.2.3 we have shown that the hadronic cross section is given by a convolution of the partonic one with Parton Distribution Functions and Fragmentation Functions. For both PDF and FF the link between partonic and hadronic points of view occurs imposing a prescription connecting a hadronic observable quantity to an analogous partonic variable (not observable). For FF the choice of carrying out such a correspondence is quite free. As an example, a way consists in considering the energy of the final state hadron and imposing that is a fraction of the one carried by the originating parton. The same arguments hold for momentum or any components. Working with high momenta, all choices are almost equivalent, whereas when masses and energies become comparable, the schemes remarkably differ. In any case it has to be defined a fragmentation variable, that is the quantity of interest allowing a link between partonic and hadronic level. For our investigation we introduce the observable \[ z = \frac{P_H \cdot P_N}{q \cdot P_N} \] (2.5)

where \( P_H \) is the four-momentum of the produced hadron, \( P_N \) is the one of the initial state nucleon and \( q \) the one of \( W \) boson (1.5). In the reference system where the nucleon is at rest, previous expression reduces to \[ z = \frac{E_H}{E_W} \] where \( E_H \) and \( E_W \) are respectively the energies of the produced hadron and \( W \) boson. We can introduce an analogous quantity at parton level \[ \zeta = \frac{\hat{p}_b \cdot \hat{p}_a}{q \cdot \hat{p}_a} \] (2.6)

where \( \hat{p}_b \) is the momentum of the final state parton generating the hadron and \( \hat{p}_a \) the one of the initial state; imposing the prescription defined through \( z \), a fraction \( t \in (0,1) \) of \( \zeta \), the typical convolution as in 1.2.1 is recovered

\[
\int_{\zeta_{\text{min}}}^{1} d\zeta \hat{F}_{k}^{a}(\xi, y, \zeta) \int_{0}^{1} dt D_{h}(t) \delta(z - t\zeta) = \int_{\zeta_{\text{max}}(z, \zeta_{\text{min}})}^{1} \hat{F}_{k}^{a}(\xi, y, \zeta) D_{h}\left(\frac{z}{\zeta}\right) \frac{d\zeta}{\zeta}
\]

Notice that \( z = t\zeta \) correctly holds if \( \hat{p} = \xi P_N \) (that is neglecting Target Mass Corrections (1.2.4)) and that momentum \( P_H \) of the final state hadron is a fraction \( t \) of the originating parton (\( P_H = t\hat{p}_b \)); this last hypothesis entails that masses are negligible with respect to the considered momenta, being

\[
\hat{p}_b = \left(\sqrt{m_b^2 + |\hat{p}_b|^2}, \hat{p}_b\right) \quad P_H = \left(\sqrt{m_H^2 + |P_H|^2}, P_H\right)
\]
After the choice of the hadronic fragmentation variable, the Coefficient Functions have to be rewritten by means of the corresponding partonic quantity. Dealing with the partonic phase-space (appendix A.6) we have introduced the variable $\hat{w} = (1 + \cos \theta)/2$ where $\theta$ is the scattering angle between the produced heavy quark (mass $m$) and the straight line formed by the momentum ($q$) of $W$ and of the initial state parton ($p$), in the reference system of the centre of mass ($p + q = 0$). In the following paragraphs we calculate the functional form for Coefficient Functions in terms of $\hat{w}$ variable, obtaining then $F_{\alpha k}(\xi, y, \hat{w})$; in order to convolve with FF, it is necessary to perform a change of variables to get $\zeta$ as defined in 2.6:

$$\hat{w} = \left[ \zeta - \frac{(1 - \lambda)\xi}{(1 - \lambda\xi)} \right] \frac{(1 - \lambda\xi)}{(1 - \xi)} \quad \text{d}\hat{w} = \frac{(1 - \lambda\xi)}{(1 - \xi)} \text{d}\zeta$$

$$\hat{w}_{\text{min}} = 0 \rightarrow \zeta_{\text{min}} = \frac{(1 - \lambda)\xi}{(1 - \lambda\xi)}$$

$$\hat{w}_{\text{max}} = 1 \rightarrow \zeta_{\text{max}} = 1$$

It is very important to pay attention to such an operation, because we are not dealing with ordinary functions, but with distributions; in fact the following rules hold

$$\delta(1 - \hat{w})\text{d}\hat{w} \rightarrow \delta(1 - \zeta)\text{d}\zeta$$

$$\frac{1}{(1 - \hat{w})_+}\text{d}\hat{w} \rightarrow \frac{1}{(1 - \zeta)_+}\text{d}\zeta$$

$$\left[ \frac{\log(1 - \hat{w})}{1 - \hat{w}} \right]_+\text{d}\hat{w} \rightarrow \left[ \frac{\log(1 - \zeta)/(1 - \zeta_{\text{min}})}{1 - \zeta} \right]_+\text{d}\zeta$$

Introducing distribution $\oplus$ defined by

$$\int_{\zeta_{\text{min}}}^{1} \text{d}\zeta f(\zeta)[g(\zeta)]_\oplus \equiv \int_{\zeta_{\text{min}}}^{1} \text{d}\zeta [f(\zeta) - f(1)]g(\zeta)$$

Furthermore, being $\zeta_{\text{min}} \rightarrow 1$ when $\xi \rightarrow 1$,

$$\delta(1 - \xi)f(\zeta, \xi) = \delta(1 - \xi)\delta(1 - \zeta) \int_{\zeta_{\text{min}}}^{1} f(t, \xi) dt \bigg|_{\xi = 1}$$
2.2 Leading Order

At the first perturbative order only the Feynman diagram in Fig. 2.1 contributes to the scattering amplitude.

\[ h_{q,\text{LO}}^{a\mu\nu} = \frac{1}{2} \left[ \left( p_{\mu} p_{\nu} - (p' p') g_{\mu\nu} \right) - 2 i \epsilon_{\alpha\beta \mu\nu} p_{\alpha} p'_{\beta} \right] |V_{ij}|^2 \]  

(2.11)

and from four-momentum conservation \((p' = p + q)\), it follows \(h_{q,\text{LO}}^{a\mu\nu} = 0\) and \(h_{k=1,2,3,5}^{q,\text{LO}} = |V_{ij}|^2 / 4\pi\), in accordance with contractions in 2.1. Such a result is clear also at first sight comparing 2.11 to 1.12. The phase-space for the final state of a massive quark is given by

\[ d\phi_X^{\text{LO}} = \frac{d^4 p'}{(2\pi)^3} \delta^+(p'^2 - m^2) (2\pi)^4 \delta^{(4)} (p' - q - p) \equiv d\hat{w} \int d\hat{\phi}_X^{\text{LO}} \]

\[ = 2\pi \frac{\lambda}{Q^2} \frac{1}{\delta(1 - \xi)} \delta(1 - \hat{w}) d\hat{w} \]

as referred in A.23.

Finally we obtain the expression of the Coefficient Functions, according with 2.2

\[ \hat{F}_{k}^{a,\text{LO}}(\xi, y, \hat{w}) = \frac{\delta_{k}^{a}}{4\pi} h_{k}^{a,\text{LO}} \int d\hat{\phi}_X^{\text{LO}} \]

\[ = (1 - \delta_{a4}) \delta_{aq} |V_{ij}|^2 \delta(1 - \xi) \delta(1 - \hat{w}) \]  

(2.12)

At LO it does not exist any contribution coming from gluons, then \(h_{k}^{g,\text{LO}} = 0\) \(\forall k\): factor \(\delta_{aq}\) takes it into account.
2.3 Next-to-Leading Order

At Next-to-Leading Order Feynman diagrams are of two kinds: in addition to the quark channel already present at LO, a new channel appears where an initial state gluon generates a quark/antiquark pair by splitting (Fig. 2.7); one of these partons interacts then with W boson. We have therefore to distinguish between quark-channel and gluon-channel. The contribution directly coming from quarks is illustrated in Fig 2.6 and consists in a radiative correction of LO, that is in gluon emission from a quark.

To obtain the QCD partonic tensor at NLO, we have to sum, for each production channel, the amplitudes of two Feynman diagrams and calculate the squared modulus. As expected in Quantum Field Theory, these contributions diverge if integrated on the whole phase-space, because of soft and collinear singularities; introducing the dimensional regularization formalism (paragraph 2.1.1) divergences are carried out as $\varepsilon$ poles and can be analytically treated. Obviously at NLO also virtual diagrams in Fig 2.2 and Fig 2.3 have to be included; the latter cancel soft divergences of quark channel leaving only the collinear ones, according to KLN theorem ([24], [25]).

In accordance with appendix A.6, the phase-space for real corrections at NLO is

$$d\varphi_X^{NLO} = \frac{1}{8\pi} \left( \frac{Q^2 + m^2}{4\pi} \right)^\varepsilon \frac{1}{\Gamma(1 + \varepsilon)} \hat{w}^\varepsilon (1 - \hat{w})^{-\varepsilon} (1 - \xi)^{1+2\varepsilon} (1 - \lambda \xi)^{-1-\varepsilon} d\hat{w}$$

expressed through $\xi$ and $\hat{w}$ variables defined in appendix A.6

$$\xi = \frac{x}{\lambda} \quad \lambda = \frac{Q^2}{Q^2 + m^2} \quad \hat{w} = \frac{1 + \cos \theta}{2}$$

where $\theta$ is the scattering angle between the heavy quark (mass $m$) and the straight line of momentum $(q)$ of W boson and initial state parton $(p)$, in the centre of mass of the system of reference $(p + q = 0)$; $x$ and $\lambda$ have been respectively introduced in 1.13 and 1.22.

Using the same variables for the phase-space, some invariants can be rewrit-
ten as

\[
2pp' = \frac{Q^2}{\lambda} \left[ \hat{w} \left( 1 - \lambda \right) \left( 1 - \hat{w} \right) \right]
\]

\[
2pl = \frac{Q^2 (1 - \xi)(1 - \hat{w})}{\xi (1 - \lambda \xi)}
\]

\[
2p'l = \frac{Q^2 (1 - \xi)}{\xi}
\]

in agreement to kinematics convention shown in figures 2.6 and 2.7. For sake of completeness, we introduce other definitions and expressions used in the following

\[
\hat{s} = (p + q)^2 = \frac{Q^2}{\lambda} \left( 1 - \lambda \xi \right)
\]

\[
m^2 = Q^2 \left( 1 - \frac{\lambda}{\lambda} \right)
\]

\[
K_A = \frac{(1 - \lambda)}{\lambda} \log(1 - \lambda)
\]

\[
\text{Li}_2(\lambda) = - \int_0^1 \frac{\log(1 - \lambda t)}{t} \, dt
\]

Having carried out calculations using \( \hat{w} \) variable, it is possible to move to the fragmentation variable \( \xi \) in accordance with paragraph 2.1.3.

Figure 2.2: CC DIS: Self-energy (Virtual NLO contribution).

Figure 2.3: CC DIS: Vertex correction (Virtual NLO contribution).
2.3.1 Virtual Contribution

In this section we carry out the contribution of virtual diagrams appearing at NLO. We can distinguish between self-energy (Fig. 2.2) or vertex (Fig. 2.3) contributions. For diagram in Fig. 2.4 we can write the amplitude as

$$M_V^\nu = -i \frac{g_w}{\sqrt{2}} V_{ij} \bar{\pi}_{\alpha i}(p') \Lambda_\nu \frac{1 - \gamma_5}{2} \pi_{\alpha j}(p)$$

identifying

$$\Lambda_\nu \equiv \left(\frac{g_s}{\mu^\varepsilon}\right)^2 C_F \int \frac{d^D k}{(2\pi)^D k^2} \frac{1}{k^2} \gamma^\rho \frac{(p' + k + m)}{(p + k)^2 - m^2} \gamma_\nu \frac{(\not{k} + \not{p})}{(p + k)^2} \gamma^\rho$$

and using the dimensional regularization as usual ($D = 4 + 2\varepsilon$, $g_s \to g_s\mu^{-\varepsilon}$, $d^4k \to d^Dk$); we have chosen the Feynman gauge for the propagator of the virtual gluon $-ig_{\rho\sigma}/k^2$, because the result is unchanged if all external lines are on-mass-shell \[26\], even using a more general gauge of the form \[A.7\]. The structure of numerator inside the integral comprises scalar, vector and:

![Figure 2.4: Kinematics of Vertex-correction.](image-url)
tensorial (of second rank) terms; this situation is considerably more complicated than the case for massless particles, not simply because of the appearance of a mass for the outgoing particle, but also because of the mass-asymmetry between incoming and outgoing fermionic lines. Calculating the integral, the result is

$$\Lambda_\nu = \frac{\alpha_s}{4\pi} C_F \frac{1}{\Gamma(1+\varepsilon)} \left( \frac{Q^2+m^2}{4\pi \mu^2} \right)^\varepsilon \left[ C_0\gamma_\nu + \frac{C_p}{m} p_\nu + \frac{C_q}{m} q_\nu \right]$$

being

$$C_0 = -\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} - 8 - K_A + \frac{(1-\lambda)^\varepsilon}{\varepsilon^2} \left[ 1 - 2\varepsilon + \left( 4 + 2\text{Li}_2(\lambda) - \frac{\pi^2}{6} \right) \varepsilon^2 \right]$$

$$C_p = 2K_A$$

$$C_q = -2 \left( \frac{1-\lambda+K_A}{\lambda} \right)$$

(2.15)

This is the result for the vertex correction in Fig. 2.4. Now self-energy contribution in Fig. 2.2 has to be calculated; a way to include it into the previous treatment consists in renormalization of fermionic wave function. Renormalization constant is defined on-mass-shell:

$$z_i^{-1} \equiv 1 - \frac{d\Sigma}{dp_i} \bigg|_{p_i=m_i}$$

being $\Sigma$ the Green function of self-energy, whereas $p_i, m_i$ are respectively momenta and masses of the considered particles.

For the outgoing fermionic (massive) line

$$z_{p'} = 1 + \frac{\alpha_s}{4\pi} C_F \left( \frac{m^2}{4\pi \mu^2} \right)^\varepsilon \frac{1}{\Gamma(1+\varepsilon)} \left[ \frac{3}{\varepsilon} - 4 \right]$$

whereas for the (massless) one incoming, trivially $z_p = 1$.

Consequently the renormalized vertex is carried out by

$$\frac{\sqrt{z_p z_{p'}} (\gamma_\nu + \Lambda_\nu) (1 - \gamma_5)}{2} \equiv (\gamma_\nu + \Lambda^R_\nu) \frac{(1 - \gamma_5)}{2}$$

$$\Lambda^R_\nu = \frac{\alpha_s}{4\pi} C_F \frac{1}{\Gamma(1+\varepsilon)} \left( \frac{Q^2+m^2}{4\pi \mu^2} \right)^\varepsilon \left[ C_R\gamma_\nu + \frac{C_p}{m} p_\nu + \frac{C_q}{m} q_\nu \right]$$

paying attention to the fact that also the Born level diagram has to be taken into account in renormalization

$$C_R = -\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} - 8 - K_A + \frac{(1-\lambda)^\varepsilon}{\varepsilon^2} \left[ 1 - \frac{\varepsilon}{2} + \left( 2 + 2\text{Li}_2(\lambda) - \frac{\pi^2}{6} \right) \varepsilon^2 \right]$$

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Figure 2.5: Cut diagrams for the virtual contribution, with renormalized vertex.

In order to obtain the whole virtual contribution at NLO, we can describe the QCD tensors corresponding to cut diagrams in Fig. 2.5; being one the complex conjugate of the other, their sum is twice the real contribution of each one. Then QCD tensor for the virtual contribution is

$$h_{\mu\nu}^V = 2|V_{ij}|^2 \text{Re} \left\{ \text{Tr} \left[ \not{p} \gamma_\mu \frac{(1 - \gamma_5)}{2} (\not{p} + m) \Lambda_{\nu} \frac{(1 - \gamma_5)}{2} \right] \right\} \quad (2.16)$$

neglecting the electroweak coupling constant as in other QCD tensors, because such a factor has been already taken into account in eq.1.2.

All contractions with projectors in A.15 can be calculated and the virtual contribution of Coefficient Functions is carried out according to 2.2; the phase-space is the same as for the Born case A.22.

$$\hat{F}_{kn,V}(\xi, y, \hat{w}) = \delta_{aq} C_F \frac{\alpha_s}{2\pi} \frac{1}{\Gamma(1 + \varepsilon)} \left( \frac{Q^2 + m^2}{4\pi\mu^2} \right) ^\varepsilon |V_{ij}|^2 \epsilon_k \delta(1 - \xi) \delta(1 - \hat{w})$$

(2.17)

with

| $k$ | 1 | 2 | 3 | 4 | 5 |
|-----|---|---|---|---|---|
| $\epsilon_k$ | $C_R$ | $C_R + C_p/2$ | $C_R$ | 0 | $C_R + C_q/2$ |

Our result essentially agrees with the one referred in [26]$^2$.

$^2$In this article there are two mistakes about virtual contributions for $k = 2$ and $k = 5$. In eq. A.12 and A.15 of [26] the wrong coefficients 1/4 must be substituted for 1/2; this correction also holds for the first column of table I inside the same paper.
2.3.2 Quark Channel

The scattering amplitudes sum for the quark-channel case is

\[ M_Q = -ig_s T_{k_k'} V_{ij} \bar{u}^\langle s \rangle (p') u^\langle r \rangle (p) \]

\[ \left[ \gamma_\mu \frac{1 - \gamma_5}{2} \frac{i (\hat{p} - \hat{l})}{2(p - l)^2} \gamma^\rho + \gamma^\rho \frac{i (\hat{p}' + m + \hat{l})}{(p' + l)^2 - m^2} \gamma_\mu \frac{1 - \gamma_5}{2} \right] u^\langle r \rangle (p) \epsilon^{(t)}_\rho (l) \]

where \( s \) and \( r \) are spin indices of the quark and \( t \) of the gluon, \( T_{k_k'} \) is the colour matrix associated with gluon emission from a quark and \( g_s \) the strong coupling constant, whereas \( V_{ij} \) is the CKM matrix element for the transition from a quark of flavour \( j \) to \( i \). Notice that the electroweak coupling constant has not been enclosed because already taken into account in expression 1.2.

Taking the squared modulus of \( M_Q \), summing over polarisations \((r,s,t)\) and colours of quarks \((k, k')\) and of emitted gluon \((a)\), then

\[ h_{\mu\nu} \equiv \sum_{t,s,r,a,k,k'} |M_Q|^2 = |V_{ij}|^2 \alpha_s 4\pi C_F \]

\[ \text{Tr} \left\{ (p' + m) \left[ \gamma_\mu \frac{1 - \gamma_5}{2} \frac{i (\hat{p} - \hat{l})}{(2p - l)^2} \gamma^\rho + \gamma^\rho \frac{i (\hat{p}' + m + \hat{l})}{(2p' + l)^2 - m^2} \gamma_\mu \frac{1 - \gamma_5}{2} \right] \sum_t \epsilon^{(t)}_\rho (l) \bar{\epsilon}^{(t)}_\tau (l) \right\} \]

Choosing an axial gauge for the gluon field, i.e. of the form

\[ \sum_t \epsilon^{(t)}_\rho (l) \bar{\epsilon}^{(t)}_\tau (l) = -g_{\rho\tau} + \frac{n_{\rho\tau} + l_{\rho\tau}}{nl} \]
where \( n \) is a generic vector with \( n^2 = 0 \), \( np = 0 \), and \( l \) is the momentum of the gluon, it is easy to verify that the terms in 2.18 give null contribution when contracted with elementary tensors contained in projectors [A.15]. We can therefore take into account only the contribution coming from \(-g_{\mu\nu}\):

\[
h_{\mu\nu}^Q = -|V_{ij}|^2 \alpha_s A_4 \pi C_F \text{Tr} \left\{ \gamma_\mu \left( \gamma_5 \dot{p} + m \right) \frac{(1 - \gamma_5)}{2} \frac{(\dot{p} - l)}{(-2pl)} \gamma_\rho + \gamma_\rho \frac{(\dot{p} + m + l)}{2 p'(l)} \gamma_\mu \frac{(1 - \gamma_5)}{2} \right\}
\]

(2.19)

At this point all contractions between \( h_{\mu\nu}^Q \) and elementary tensors \( g_{\mu\nu} \), \( p' p' \), \( p' q' \), \( q' q' \), \( \epsilon_{\alpha\beta} p' q' \), \( q' q' \), \( \epsilon_{\alpha\beta} p' q' \) can be evaluated and each term \( P_{\mu\nu}^k h_{\mu\nu}^Q \) can be reconstructed as linear combination of them.

**Example of Contraction**

We illustrate the contraction \(-g_{\mu\nu} h_{\mu\nu}^Q\) as an example, because this is the most complicated contribution and is very rich of particularities.

\[
-g_{\mu\nu} h_{\mu\nu}^Q = |V_{ij}|^2 \alpha_s A_4 \pi C_F 4 (1 + \epsilon) \left\{ \frac{2pl}{2l'} + \frac{2p'l}{2pl} \right\} - 2\epsilon + 2 \frac{2pp' + 2pl - 2p'l}{2pl 2p'l} \left( 2pp' - m^2 \right) \left( \frac{2p'l}{2pl} \right)
\]

(2.13)

Trivially the previous expression can be connected to that in [9, 27] for the massless case \((m \to 0)\) or equivalently \(\lambda \to 1\). Adding the phase-space \(\int \varphi_X^{NLO} \) terms of the form \((1 - \xi)^a (1 - \lambda \xi)^b (1 - \hat{\omega})^c\) are generated: they have to be managed as mathematical distributions according to previous remarks (paragraph 2.1.2) in order to isolate singularities. Expanding up to \(\epsilon\) order (excluded)
\[-g^\mu\nu h^{Q}_{\mu\nu} \int \hat{\phi}^NLO_X = \frac{\alpha_s}{2\pi} C_F |V_{ij}|^2 \frac{4\pi(1+\varepsilon)}{\Gamma(1+\varepsilon)} \left( \frac{Q^2 + m^2}{4\pi\mu^2} \right)^\varepsilon \left\{ \right. \]

\[
(1-\hat{w}) \left[ \frac{1-\xi}{(1-\lambda\xi)^2} \right] + \frac{(1-\xi)}{(1-\hat{w})} + 2\frac{(1-\xi)}{(1-\hat{w})+(1-\xi)} + \delta(1-\xi)\delta(1-\hat{w}) \left[ \frac{2}{\epsilon^2} - \frac{3}{\epsilon} - \frac{(1-\lambda)}{\epsilon^2} \left( 1 - \frac{\epsilon}{2} - 2\epsilon^2 + \frac{\pi}{6}\epsilon^2 + 2\text{Li}_2(\lambda)\epsilon^2 \right) \right] \]

\[
+ \delta(1-\hat{w}) \left\{ (1+\xi^2) \left[ \frac{2\log(1-\xi) - \log(1-\lambda\xi)}{1-\xi} \right] + (1+\xi^2) \frac{\log \xi}{1-\xi} \right\} \]

\[
+ (1-\xi) + \frac{1}{\epsilon} \left[ \frac{1+\xi^2}{1-\xi} + \right] \}

\[
+ [1-(1-\lambda)\delta(1-\xi) \left\{ (1+\hat{w}^2) \left[ \frac{\log(1-\hat{w})}{1-\hat{w}} \right] + (1+\hat{w}^2) \frac{\log \hat{w}}{(1-\hat{w})} \right\} \]

\[
+ (1-\hat{w}) + \frac{1}{\epsilon} \left[ \frac{1+\hat{w}^2}{1-\hat{w}} + \right] \}

\[
+ (1-\lambda)^\varepsilon \delta(1-\xi) \left\{ \left[ \hat{w} \frac{\log(1-\hat{w})}{1-\hat{w}} \right] + \left[ \hat{w} \log \hat{w} \right] + \frac{\hat{w}}{1-\hat{w}} \left[ \frac{\hat{w}}{1-\hat{w}} \right] \right\} \]

\[
- (1-\hat{w}) \delta(1-\xi) \left[ \frac{1+\lambda}{\lambda} K_A + \frac{(1-\lambda)\varepsilon}{\lambda} \right] \}
\]

Soft singularities cancellation is worked out adding virtual contributions; contraction of tensor \(-g^\mu\nu\) with the virtual one (2.16) gives

\[-g^\mu\nu h^V_{\mu\nu} \int \hat{\phi}^LO_X = C_F \frac{\alpha_s}{2\pi} \frac{4\pi(1+\varepsilon)}{\Gamma(1+\varepsilon)} \left( \frac{Q^2 + m^2}{4\pi\mu^2} \right)^\varepsilon |V_{ij}|^2 \delta(1-\xi)\delta(1-\hat{w}) \]

\[
\left\{ \right. \]

\[
- \frac{2}{\epsilon^2} + \frac{3}{\epsilon} - 8 - K_A + \frac{1-\lambda + K_A}{\lambda} \]

\[
- \frac{(1-\lambda)^\varepsilon}{\epsilon^2} \left( -1 + \frac{\epsilon}{2} - 2\epsilon^2 + \frac{\pi^2}{6}\epsilon^2 - 2\text{Li}_2(\lambda)\epsilon^2 \right) \}
\]

then

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\[-g^{\mu\nu} \left( h_{\nu}^{Q} \int \hat{\phi}_{X}^{NLO} + h_{\nu}^{V} \int \hat{\phi}_{X}^{LO} \right) \right\} = C_{F} \frac{\alpha_{s} 4\pi (1 + \varepsilon)}{2\pi \Gamma(1 + \varepsilon)} \left( \frac{Q^{2} + m_{\lambda}^{2}}{4\pi \mu^{2}} \right) \varepsilon |V_{ij}|^{2} \}

\begin{align*}
(1 - \hat{w}) \left[ \frac{1 - \xi}{(1 - \lambda \xi)^{2}} \right]_{+} + \frac{1 - \xi}{(1 - \hat{w})_{+}} + 2 \frac{w}{(1 - \hat{w})_{+}} \frac{\xi}{(1 - \xi)_{+}} - \delta(1 - \xi) \delta(1 - \hat{w}) \left\{ \left( 8 + K_{A} - \frac{1 - \lambda + K_{A}}{\lambda} + (1 - \lambda)^{\varepsilon} \right) \left( -4 + \frac{\pi^{2}}{3} \right) \right\} \\
+ \delta(1 - \hat{w}) \left\{ (1 + \xi^{2}) \left[ \frac{2 \log(1 - \xi) - \log(1 - \lambda \xi)}{1 - \xi} \right]_{+} - (1 + \xi^{2}) \frac{\log \xi}{1 - \xi} \\
+ (1 - \xi) + \frac{1}{\varepsilon} \left[ \frac{1 + \xi^{2}}{1 - \xi} \right]_{+} \right\} \\
+ [1 - (1 - \lambda)^{\varepsilon}] \delta(1 - \xi) \left\{ (1 + \hat{w}^{2}) \left[ \frac{\log(1 - \hat{w})}{1 - \hat{w}} \right]_{+} + (1 + \hat{w}^{2}) \frac{\log \hat{w}}{1 - \hat{w}} \\
+ (1 - \hat{w}) + \frac{1}{\varepsilon} \left[ \frac{1 + \hat{w}^{2}}{1 - \hat{w}} \right]_{+} \right\} \\
+ (1 - \hat{w}) \delta(1 - \xi) \left\{ \left[ \frac{\hat{w} \log(1 - \hat{w})}{1 - \hat{w}} \right]_{+} + \left[ \frac{\hat{w} \log \hat{w}}{1 - \hat{w}} \right]_{+} + \frac{1}{\varepsilon} \left[ \frac{\hat{w}}{1 - \hat{w}} \right]_{+} \right\} \\
- (1 - \hat{w}) \delta(1 - \xi) \left[ \frac{1 + \lambda}{\lambda} K_{A} + \frac{(1 - \lambda)^{\varepsilon}}{\lambda} \right] \right\} \}

\textbf{Massless Result}

For \( m = 0 \) (\( \Rightarrow \lambda = 1, \xi = \hat{x} \)) the previous expression becomes

\[-g^{\mu\nu} \left( h_{\nu}^{Q} \int \hat{\phi}_{X}^{NLO} + h_{\nu}^{V} \int \hat{\phi}_{X}^{LO} \right) \right\} = C_{F} \frac{\alpha_{s} 4\pi (1 + \varepsilon)}{2\pi \Gamma(1 + \varepsilon)} \left( \frac{Q^{2} + m_{\lambda}^{2}}{4\pi \mu^{2}} \right) \varepsilon |V_{ij}|^{2} \}

\begin{align*}
\frac{1 - \hat{w}}{(1 - \hat{x})^{2}}_{+} + \frac{1 - \hat{x}}{(1 - \hat{w})_{+}} + 2 \frac{\hat{w}}{(1 - \hat{w})_{+}} \frac{\hat{x}}{(1 - \hat{x})_{+}} - 8\delta(1 - \hat{x}) \delta(1 - \hat{w}) \\
+ \delta(1 - \hat{w}) \left\{ (1 + \hat{x}^{2}) \left[ \frac{\log(1 - \hat{x})}{1 - \hat{x}} \right]_{+} - (1 + \hat{x}^{2}) \frac{\log \hat{x}}{1 - \hat{x}} \\
+ (1 - \hat{x}) + \frac{1}{\varepsilon} \left[ \frac{1 + \hat{x}^{2}}{1 - \hat{x}} \right]_{+} \right\} \\
+ \delta(1 - \hat{x}) \left\{ (1 + \hat{w}^{2}) \left[ \frac{\log(1 - \hat{w})}{1 - \hat{w}} \right]_{+} + (1 + \hat{w}^{2}) \frac{\log \hat{w}}{1 - \hat{w}} \\
+ (1 - \hat{w}) + \frac{1}{\varepsilon} \left[ \frac{1 + \hat{w}^{2}}{1 - \hat{w}} \right]_{+} \right\} \right\}

(2.20)
We note that some poles remain; their coefficient is proportional to the Altarelli-Parisi splitting functions

\[ P_{qq}^{(0)}(t) = C_F \left[ \frac{1 + t^2}{1 - t} \right]_+ \]

It is easy to verify that these singularities are collinear and therefore can be assimilated into PDF and FF (respectively for poles of \( P_{qq}^{(0)}(\hat{x}) \) and \( P_{qq}^{(0)}(\hat{w}) \)) thanks to collinear factorization theorem \[28\]. We can rewrite

\[
\frac{1}{\Gamma(1 + \varepsilon)} \left( \frac{Q^2}{4\pi\mu^2} \right)^\varepsilon \frac{P_{qq}^{(0)}(t)}{\varepsilon} = \frac{P_{qq}^{(0)}(t)}{\varepsilon} \left[ \frac{1}{\varepsilon} + \gamma_e - \log 4\pi + \log \left( \frac{Q^2}{\mu^2} \right) \right] + O(\varepsilon)
\]

and we can choose the \( \overline{MS} \) scheme prescribing for removing the term

\[
P_{qq}^{(0)}(t) \left[ \frac{1}{\varepsilon} + \gamma_e - \log 4\pi \right]
\]

leaving consequently the dependence on two arbitrary factorization scales \( \mu \) (\( \mu_F^{PDF} \) and \( \mu_F^{FF} \)) inside the term \(-g^{\mu
u} (h_{\mu\nu}^{Q} \int \tilde{\varphi}_{X}^{NLO} + h_{\mu\nu}^{V} \int \tilde{\varphi}_{X}^{LO})\), finite from now on. For sake of consistency, the convolutions obtained in \[1.21\] must involve PDF and FF defined using such a scheme.

**Massive Result:** \( m \neq 0 \) (\( \lambda \neq 1 \))

\[
- g^{\mu\nu} \left( h_{\mu\nu}^{Q} \int \hat{\varphi}_{X}^{NLO} + h_{\mu\nu}^{V} \int \hat{\varphi}_{X}^{LO} \right) = C_F \frac{\alpha_s}{2\pi} \frac{4\pi(1 + \varepsilon)}{\Gamma(1 + \varepsilon)} \left( \frac{Q^2 + m^2}{4\pi\mu^2} \right)^\varepsilon |V_{ij}|^2 \left\{ \right.
\]

\[
(1 - \hat{w}) \left[ \frac{1 - \xi}{1 - \lambda \xi^2} \right]_+ + \frac{(1 - \xi)}{(1 - \hat{w})_+} + 2 \frac{\hat{w}}{(1 - \hat{w})_+} \left[ \frac{\xi}{1 - \xi} \right]_+ - \delta(1 - \xi) \delta(1 - \hat{w}) \left\{ 4 + \frac{\pi^2}{3} + K_A - \frac{1 - \lambda + K_A}{\lambda} \right\}
\]

\[
+ \delta(1 - \hat{w}) \left\{ (1 + \xi^2) \left[ \frac{2 \log(1 - \xi) - \log(1 - \lambda \xi)}{1 - \xi} \right] + (1 + \xi^2) \frac{\log \xi}{1 - \xi} \right. \]

\[
+ \left. (1 - \xi) + \frac{1}{\varepsilon} \left[ \frac{1 + \xi^2}{1 - \xi} \right]_+ \right\}
\]

\[
+ \delta(1 - \xi) \left\{ - \log(1 - \lambda) \left( \left[ \frac{1 + \hat{w}^2}{1 - \hat{w}} \right]_+ + \left[ \frac{\hat{w}}{1 - \hat{w}} \right]_+ \right) + \hat{w} \log(1 - \hat{w}) \right. \]

\[
+ \left. \left[ \frac{\hat{w} \log \hat{w}}{1 - \hat{w}} \right]_+ + \frac{1}{\varepsilon} \left[ \frac{\hat{w}}{1 - \hat{w}} \right]_+ - (1 - \hat{w}) \left[ \frac{1 + \lambda}{\lambda} K_A + \frac{1}{\lambda} \right] \right}\}
\]

\[3\] Apparently in \[2.20\] there would be a multiplicative factor \((1 + \varepsilon)\) not yet taken into account: when expanded, it seems to be able to generate other finite contributions mixing with the singular terms. Actually this fact does not happen because coefficients in \[A.16\] exactly cancel this factor.
As for the massless case, it remains a collinear singularity for the term \( \frac{1 + \xi^2}{1 - \xi} \): it is included into PDF as shown before. In this case, however, it does not exist an analogous term to be assimilated into FF and then an arbitrary scale \( \mu_F^E \) does not appear; at phenomenological level this implies that it is possible to use FF not depending on an energetic scale. Nevertheless it remains apparently uncancelled a pole of the form

\[
\frac{1}{\epsilon} \delta(1 - \xi) \left( \frac{\hat{w}}{1 - \hat{w}} \right)_+ \tag{2.21}
\]

The explanation of this fact comes from the structure of the phase-space at Born level: as referred in appendix A.6 “angular” \( \hat{w} \) variable is not well defined when the final state quark is produced at rest \( p' = 0 \Rightarrow \xi = 1 \). It is therefore necessary to give meaning to this term either passing to a better defined \( \zeta \) variable (A.19) or by means of prescription A.21 so

\[
\delta(1 - \xi)g(\xi, \hat{w}) = \delta(1 - \xi)\delta(1 - \hat{w}) \left[ \int_0^1 d\alpha g(\xi, \alpha) \right]_{\xi=1}
\]

and since

\[
\int_0^1 d\alpha g_+ (\alpha) = 0
\]

the anomalous pole disappears: this is therefore a phase-space effect. After \( \overline{\text{MS}} \) subtraction, it is then possible to rewrite\(^4\)

\[
- g^{\mu\nu} \left( h^Q_{\mu\nu} \int \hat{\Phi}_{\chi}^{NLO} + h^V_{\mu\nu} \int \hat{\Phi}_{\chi}^{LO} \right) \bigg|_{\overline{\text{MS}}} = C_F \frac{\alpha_s}{2\pi} 4\pi(1 + \epsilon)|V_{ij}|^2 \left\{ \right.
\]

\[
(1 - \hat{w}) \left[ \frac{1 - \xi}{(1 - \lambda \xi)^2} \right]_+ + \frac{(1 - \xi)}{(1 - \hat{w})_+} + 2 \frac{\hat{w}}{(1 - \hat{w})_+} \frac{\xi}{(1 - \xi)_+} - \delta(1 - \xi)\delta(1 - \hat{w}) \left\{ 4 + \frac{\pi^2}{3} + \frac{1}{2\lambda} + \frac{(1 + 3\lambda)}{\lambda} K_A - \frac{1 - \lambda + K_A}{\lambda} \right\} \left(1 - \xi\right) + \delta(1 - \hat{w}) \left\{ (1 + \xi^2) \left[ \frac{2\log(1 - \xi) - \log(1 - \lambda \xi)}{1 - \xi} \right]_+ - (1 + \xi^2) \log \xi \frac{1}{1 - \xi} \right. 
\]

\[
+ \left(1 - \xi\right) + \log \left( \frac{Q^2 + m^2}{\mu^2} \right) \left[ \frac{1 + \xi^2}{1 - \xi} \right]_+ \right\} \right\} \right. \tag{2.22}
\]

We want to remark that it is not possible to recover the massless result \((2.20)\) simply taking the limit of the massive one \((2.22)\); this because of the

\(^4\)Also for this case, it holds what explained in note 3.
non-commutativity of limits

\[
\lim_{\lambda \to 1^-} \lim_{\epsilon \to 0^+} (1 - \lambda)^\epsilon = 1 \quad \lim_{\epsilon \to 0^+} \lim_{\lambda \to 1^-} (1 - \lambda)^\epsilon = 0
\]

Notice that \(\hat{x}\) is the variable of distributions that we identify with Altarelli-Parisi splitting functions for the massless case, whereas \(\xi\) for the massive case: this suggests that the relevant quantity at parton level for the massive case is a *slow rescaling* of \(\hat{x}\), that is \(\xi = \hat{x}/\lambda\) \cite{11,26}. 
Quark Coefficient Functions

NLO quark-channel contribution to Coefficient Functions can be expressed in the form

\[ \hat{F}^{q, \text{NLO}}_k(\xi, y, \zeta) = \frac{\alpha_s(\mu^2)}{2\pi} \left[ (1 - \delta_k) \mathcal{H}_0^q(\xi, y, \zeta, \mu^2_F, \lambda) + \Delta^q_k(\xi, \zeta, \lambda) \right] |V_{ij}|^2 \]

(2.23)

\[ \mathcal{H}_0^q(\xi, y, \zeta, \mu^2_F, \lambda) = \delta(1 - \zeta) \left\{ P_{qq}^{(0)}(\xi) \log \left( \frac{Q^2 + m^2}{\mu^2_F} \right) + C_F \left[ 1 - \xi + (1 + \xi^2) \left\{ \frac{2 \log(1 - \xi) - \log(1 - \lambda \xi)}{(1 - \xi)} \right\} - \frac{(1 + \xi^2)}{(1 - \xi)} \log \xi \right] \right\} \]

\[ - C_F \delta(1 - \xi) \delta(1 - \zeta) \left[ 4 + \frac{\pi^2}{3} + \frac{1}{2\lambda} + \frac{(1 + 3\lambda) K_A}{2\lambda} \right] \]

\[ + C_F \left\{ \frac{1 - \xi}{(1 - \zeta)_+} + (1 - \zeta) \left[ \frac{1 - \lambda \xi}{1 - \xi} \right]^2 \left[ \frac{1 - \xi}{(1 - \lambda \xi)^2} \right] + \frac{2\xi}{(1 - \xi)_+ (1 - \zeta)_+} \left[ (1 - \zeta) \frac{1 - \lambda \xi}{1 - \xi} + 2\xi \left( \frac{1 - \xi}{1 - \xi} \right) \right] \right\} \]

\[ \Delta^q_1(\xi, \zeta, \lambda) = 0 \]

\[ \Delta^q_2(\xi, \zeta, \lambda) = 2C_F \left\{ \delta(1 - \xi) \delta(1 - \zeta) \frac{K_A}{2} \right\} \]

\[ - \left[ \xi(1 - 3\lambda) \left( 1 - (1 - \zeta) \frac{1 - \lambda \xi}{1 - \xi} \right) + (1 - \lambda) \right] \right\} \]

\[ \Delta^q_3(\xi, \zeta, \lambda) = 2C_F \left\{ (1 - \xi) \left[ 1 - (1 - \zeta) \frac{1 - \lambda \xi}{1 - \xi} \right] - (1 - \lambda \xi) \right\} \]

\[ \Delta^q_4(\xi, \zeta, \lambda) = 2C_F \xi \]

\[ \Delta^q_5(\xi, \zeta, \lambda) = 2C_F \left\{ \xi \left[ 1 - (1 - \xi) \frac{1 - \lambda \xi}{1 - \xi} \right] + \lambda \xi \zeta \right\} \]
2.3.3 Gluon Channel

\[ M^G = -i g_s T^a_{kk'} V_{ij} \bar{u}^{(s)}_{out}(p') \]

\[ \gamma_\mu \left( \frac{1 - \gamma_5}{2} i (p - l) \right) \gamma^\rho + \gamma_\rho \left( \frac{1 - \gamma_5}{2} \right) \right] v_{out}^{(r)} (l) \epsilon^{(t)} (p) \]

Figure 2.7: CC DIS: gluon-channel (NLO Real contribution).

The sum of scattering amplitudes for this channel is

\[ \mathcal{M}^G = -i g_s T^a_{kk'} V_{ij} \bar{u}^{(s)}_{out}(p') \]

\[ \left[ \gamma_\mu \left( \frac{1 - \gamma_5}{2} i (p - l) \right) \gamma^\rho + \gamma_\rho \left( \frac{1 - \gamma_5}{2} \right) \right] v_{out}^{(r)} (l) \epsilon^{(t)} (p) \]

therefore the QCD tensor can be carried out as the previous case, paying however attention to colour factors (\( T_F = 1/2 \)); remarks on contributions coming from gluon (axial) gauge still hold.

\[ h^G_{\mu\nu} = -|V_{ij}|^2 \alpha_s 4\pi T_F \text{Tr} \left\{ \right. \\
\left. \gamma_\mu \left( \frac{1 - \gamma_5}{2} i (p - l) \right) \gamma^\rho + \gamma_\rho \left( \frac{1 - \gamma_5}{2} \right) \right\} \]

\[ (p' + m) \left[ \gamma_\nu \left( \frac{1 - \gamma_5}{2} i (p - l) \right) \gamma^\rho + \gamma_\rho \left( \frac{1 - \gamma_5}{2} \right) \right] \]

\[ \gamma_\nu \left( \frac{1 - \gamma_5}{2} i (p - l) \right) \gamma^\rho + \gamma_\rho \left( \frac{1 - \gamma_5}{2} \right) \]

\[ \left[ \gamma_\rho \left( \frac{1 - \gamma_5}{2} i (p - l) \right) \gamma^\rho + \gamma_\rho \left( \frac{1 - \gamma_5}{2} \right) \right] \]

\[ \left\{ \right. \\
\left. \gamma_\nu \left( \frac{1 - \gamma_5}{2} i (p - l) \right) \gamma^\rho + \gamma_\rho \left( \frac{1 - \gamma_5}{2} \right) \right\} \]

This result can be directly obtained also from \[2.19\], making use of symmetry of Feynman diagrams under crossing among particles of initial and final state:

\[ h^G_{\mu\nu}(p, l, p') = -h^Q_{\mu\nu}(-l, -p, p') \]

At this point all contractions between the gluonic tensor \( h^G_{\mu\nu} \), and the elementary ones contained in projectors \( A_{15} \) can be calculated; subsequently
relations 2.14 and phase-space 2.13 are employed (in this case the massless particle corresponds to a light quark). Expansions (in the sense of distributions) encountered in this channel are easier than in quark-channel, in fact there are not infrared singularities excepting the collinear one coming from the initial state parton. It is possible to check that the only pole has the form

\[
\frac{1}{\Gamma(1+\varepsilon)} \left( \frac{Q^2 + m^2}{4\pi \mu^2} \right)^\varepsilon \frac{P_{qq}^{(0)}(t)}{\varepsilon} = P_{qq}^{(0)}(t) \left[ \frac{1}{\varepsilon} + \gamma_e - \log 4\pi + \log \left( \frac{Q^2 + m^2}{\mu^2} \right) \right]
\]

setting

\[
P_{qq}^{(0)}(t) = T_F \left[ t^2 + (1-t)^2 \right]
\]

Adopting the \(\overline{MS}\) scheme, such a singularity is removed, including it into PDF as shown in the previous sections. Finally functions \(h_k^G\) have been carried out according to relations 2.2 and to the structure of projectors A.15.

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Gluon Coefficient Functions

In the following we report the expression of NLO contribution to Coefficient Functions for the gluon-channel, using the same formalism of the quark case.

\[
\hat{\mathcal{H}}_{k,NLO}^{g}(\xi, y, \zeta) = \frac{\alpha_s(\mu_R^2)}{2\pi} \left[ (1 - \delta_{k4})\mathcal{H}^{g}_0(\xi, y, \zeta, \mu_F^2, \lambda) + \Delta^{g}_{k}(\xi, \zeta, \lambda) \right] |V_{ij}|^2
\]

\[
\mathcal{H}^{g}_0(\xi, y, \zeta, \mu_F^2, \lambda) = \delta(1 - \zeta) \left\{ p_{qq}^{(0)}(\xi) \left[ \log \left( \frac{Q^2 + m^2}{\mu_F^2} \right) + \log \left( \frac{1 - \xi^2}{\zeta(1 - \lambda \xi)} \right) \right] + \xi(1 - \xi) \right\} + \left[ \frac{1}{(1 - \zeta)^\oplus} + \frac{1}{\zeta} \right] p_{qq}^{(0)}(\xi)
\]

\[
\Delta^{g}_{k}(\xi, \zeta, \lambda) = \frac{-\xi^2}{\zeta^2}(1 - \lambda)(1 - 2\lambda) + \frac{2\xi}{\zeta}(1 - \lambda)(1 - 2\lambda \xi) + 2\xi \lambda(1 - \lambda \xi) - 1
\]

\[
\Delta^{g}_{2}(\xi, \zeta, \lambda) = \frac{\xi^2}{\zeta^2}(1 - \lambda)(1 - 6\lambda + 6\lambda^2) + \frac{6\xi \lambda}{\zeta}(1 - \lambda)(1 - 2\lambda \xi)
\]

\[
+ \lambda \left[ 6\lambda \xi(1 - \lambda \xi) - 1 \right]
\]

\[
\Delta^{g}_{3}(\xi, \zeta, \lambda) = \frac{\xi^2}{\zeta^2}(1 - \lambda)(1 - 2\lambda) - \frac{2}{\zeta} \left[ p_{qq}^{(0)}(\xi) + \xi(1 - \xi - \lambda \xi)(1 - \lambda) \right]
\]

\[
\Delta^{g}_{4}(\xi, \zeta, \lambda) = 2 \left[ (1 - \xi) - (1 - \lambda) \xi \frac{1 - \zeta}{\zeta} \right]
\]

\[
\Delta^{g}_{5}(\xi, \zeta, \lambda) = \Delta^{g}_{1}(\xi, \zeta, \lambda) + 2\xi \left\{ - \frac{(1 - \lambda)^2 \xi}{\zeta^2} + \frac{(1 - \lambda)}{\zeta}(1 - 2\lambda \xi)
\right.
\]

\[
+ \lambda(1 - \lambda \xi) + \lambda \left[ (1 - \xi) - (1 - \lambda) \xi \frac{1 - \zeta}{\zeta} \right] \}
\]
2.4 NLO Coefficient Functions

2.4.1 Semi-Exclusive Coefficient Functions

We own of all the contributions to enunciate the complete result for semi-exclusive Coefficient Functions up to Next-to-Leading Order:

\[ \hat{F}^a_k(\xi, y, \zeta) = \hat{F}^a_{k,LO}(\xi, y, \zeta) + \hat{F}^a_{k,NLO}(\xi, y, \zeta) \]

where \( \hat{F}^a_{k,LO}(\xi, y, \zeta) \) has been carried out in 2.12

\[ \hat{F}^a_{k,LO}(\xi, y, \zeta) = (1 - \delta_{k4})\delta_{aq}\delta(1 - \xi)\delta(1 - \zeta)|V_{ij}|^2 \]

so \( \hat{F}^a_{k,NLO}(\xi, y, \zeta) \) can be expressed as

\[ \hat{F}^a_{k,NLO}(\xi, y, \zeta) = \frac{\alpha_s(\mu^2)}{2\pi} \hat{H}^a_k(\xi, y, \zeta, \mu^2, \lambda)|V_{ij}|^2 \]

in accordance with the notation introduced in 2.28 and 2.29

With respect to terms where \( a = q, g \) and \( k = 1, 2, 3 \), these expressions fully agree with literature [29, 20]. Inclusive Coefficient Functions can be recovered integrating over \( \zeta \) (we obtained it also from a dedicated study). In this way it has been possible to verify the correctness of the semi-exclusive results for \( a = q, g \) with \( k = 4, 5 \). [20, 26]

2.4.2 Inclusive Coefficient Functions

Inclusive Coefficient Functions \( \hat{\mathcal{I}}^a_k \) are carried out integrating directly the semi-exclusive ones:

\[ \hat{\mathcal{I}}^a_k(\xi, y) = \int_{\frac{1}{(1 - \xi)}}^1 d\zeta \hat{F}^a_k(\xi, y, \zeta) = \hat{F}^a_{k,LO}(\xi, y) + \hat{F}^a_{k,NLO}(\xi, y) \]

\[ = \hat{F}^a_{k,LO}(\xi, y) + \frac{\alpha_s(\mu^2)}{2\pi} \hat{I}^a_{k,NLO}(\xi, y)|V_{ij}|^2 \]

According with previous paragraph then

\[ \hat{F}^a_{k,LO} = (1 - \delta_{k4})\delta_{aq}\delta(1 - \xi)|V_{ij}|^2 \]

whereas at Next-to-Leading Order the following expressions hold

\[ \hat{F}^a_{k,NLO}(\xi, y, \zeta) \]

\[ \hat{F}^a_{k,NLO}(\xi, y, \zeta) = (1 - \delta_{k4})\delta_{aq}\delta(1 - \xi)|V_{ij}|^2 \]

\[ \hat{F}^a_{k,NLO}(\xi, y, \zeta) = (1 - \delta_{k4})\delta_{aq}\delta(1 - \xi)|V_{ij}|^2 \]

\[ \hat{F}^a_{k,NLO}(\xi, y, \zeta) = (1 - \delta_{k4})\delta_{aq}\delta(1 - \xi)|V_{ij}|^2 \]

\[ \hat{F}^a_{k,NLO}(\xi, y, \zeta) = (1 - \delta_{k4})\delta_{aq}\delta(1 - \xi)|V_{ij}|^2 \]

\[ \hat{F}^a_{k,NLO}(\xi, y, \zeta) = (1 - \delta_{k4})\delta_{aq}\delta(1 - \xi)|V_{ij}|^2 \]

\[ \hat{F}^a_{k,NLO}(\xi, y, \zeta) = (1 - \delta_{k4})\delta_{aq}\delta(1 - \xi)|V_{ij}|^2 \]

In that paper the modern approach of dimensional regularization, prescribing that gluon helicity has be set to \( D - 2 \), has not be taken into account.
Inclusive Quark Coefficient Functions

\[ \hat{I}_{q,NLO}^{4} = C_F \left[ 1 - \frac{\xi^2(1 - \lambda)^2}{(1 - \lambda \xi)^2} \right] \]

\[ \hat{I}_{q,NLO}^{k} = C_F \left\{ a_k \delta(1 - \xi) + b_{1,k} \left[ \frac{1}{1 - \xi} \right] + b_{2,k} \left[ \frac{1}{1 - \lambda \xi} \right] + b_{3,k} \left[ \frac{1 - \xi}{(1 - \lambda \xi)^2} \right] + c^q \right\} \]

\[ c^q = \left[ \frac{1 + \xi^2}{1 - \xi} \right] \log \left( \frac{Q^2 + m^2}{\mu_F^2} \right) - (4 + \frac{\pi^2}{12} + \frac{3K_A}{2\lambda}) \delta(1 - \xi) \]

\[ - (1 + \xi^2) \log \left( \frac{\xi}{1 - \xi} \right) + (1 + \xi^2) \left[ \frac{2 \log(1 - \xi) - \log(1 - \lambda \xi)}{1 - \xi} \right] \]

| k   | a_k   | b_{1,k} | b_{2,k} | b_{3,k} |
|-----|-------|---------|---------|---------|
| 1   | 0     | 1 - 4\xi + \xi^2 | \xi(1 - \xi) | 1/2     |
| 2   | K_A   | 2 - 2\xi^2 - 2/\xi | 2/\xi - 1 - \xi | 1/2     |
| 3   | 0     | -1 - \xi^2 | 1 - \xi | 1/2     |
| 5   | -(1 - \lambda + K_A)/\lambda | -1 - \xi^2 | 3 - 2\xi - \xi^2 | \xi - 1/2 |

Inclusive Gluon Coefficient Functions

\[ \hat{I}_{g,NLO}^{4} = 2(1 - \xi) - 2(1 - \lambda)\xi \log \left( \frac{1 - \lambda \xi}{(1 - \lambda \lambda)} \right) \]

\[ \hat{I}_{g,NLO}^{k} = \left\{ \xi(1 - \xi)C_{1,k} + C_{2,k} + (1 - \lambda)\xi \log \left( \frac{1 - \lambda \xi}{(1 - \lambda \xi)} \right) (C_{3,k} + \lambda \xi C_{4,k}) + C^g_k \right\} \]

\[ C^g_k = \left[ \frac{\xi^2 + (1 - \xi^2)}{2} \right] \log \left( \frac{Q^2 + m^2}{\mu_F^2} \right) + (-1)^{d_{k3}} \log \left( \frac{(1 - \xi)^2}{(1 - \lambda \xi)^2} \right) \]

| k   | C_{1,k} | C_{2,k} | C_{3,k} | C_{4,k} |
|-----|---------|---------|---------|---------|
| 1   | 4 - 4(1 - \lambda) | \frac{(1 - \lambda \xi)}{(1 - \lambda \lambda)} - 1 | 2        | -4      |
| 2   | 8 - 18(1 - \lambda) + 12(1 - \lambda)^2 | \frac{(1 - \lambda \xi)}{(1 - \lambda \lambda)} - 1 | 6\lambda | -12\lambda |
| 3   | 2(1 - \lambda) | 0 | -2(1 - \xi) | 2       |
| 5   | 8 - 10(1 - \lambda) | \frac{(1 - \lambda \xi)}{(1 - \lambda \lambda)} - 1 | 4        | -10     |
Chapter 3

Numerical Results and Analysis

In the following we resume the obtained results during the extensive study carried out in the previous chapters; subsequently we will leave the general CC DIS case to address exclusively the events of kind

\[ \nu_{\mu} + N \rightarrow \mu^- + H_c + X \]

where \( H_c \) is a hadron containing Charm quark. Such a process is suitable to investigate Strange quark distribution inside nucleons; in particular we will introduce the observable

\[ \frac{x}{\lambda} \lambda^{s_{\text{eff}}} (x, y, z) = \frac{1}{2} G_{F}^{2} \frac{\pi}{M_{W}^{4}} \left( Q^{2} + M_{W}^{2} \right)^{2} \left| V_{cs} \right|^{-2} \frac{d^{3} \sigma}{dxdydz} \]

and we will study the dependence on some parameters.

It will be shown that \( s_{\text{eff}} \) is related to Strange quark distribution in nucleons and that an analogous quantity for antiquark distribution can be introduced; in fact, because of properties of Charged Currents, it is possible independently to probe the quark and antiquark distributions.
3.1 General Result

Semi-Exclusive cross section for the generic CC DIS process

\[ \nu_e + N \to \ell^- + H + X \]

where the charged lepton is assumed to be massive and the hadronic state \( H \) comes from heavy quark production, can be expressed as

\[
\frac{d^3\sigma}{dxdydz} = \frac{G_F^2 M}{\pi} \frac{M_W^4}{(Q^2 + M_W^2)^2} \left[ \sum_{k=1}^{5} c_k(x, y) \frac{dF_k}{dz}(x, y, z) \right]
\]

being \( x, y, M, E (Q^2 = 2MEXy) \) as defined in relations \(1.5\), \( z \) an observable quantity for the hadron containing the massive quark \(2.5\) and coefficients \( c_k(x, y) \) the same of functions \( F_k \) inside expressions in \(1.7\).

Extension for the analogous CC DIS processes

\[\ell^- + N \to \nu_e + H + X\]
\[\bar{\nu}_e + N \to \ell^+ + H + X\]
\[\ell^+ + N \to \bar{\nu}_e + H + X\]

is trivially performed changing the sign of \( c_3 \) and/or averaging on initial spin states as illustrated in paragraph \(1.1\). According to remarks of section \(1.2\) the charged lepton \( \ell^\pm \) should in practice be assumed massless for electron and muon, being \( m_e \sim 5 \cdot 10^{-4} \text{ GeV} \), \( m_\mu \sim 0.1 \text{ GeV} \), whereas for the Tau lepton \( (m_\tau \sim 1.8 \text{ GeV}) \) some contributions coming from the mass are not necessarily negligible. Then, for electrons and muons, coefficients \( c_k \) simplify and terms \( dF_k/dz \) disappear for \( k = 4, 5 \).

Functions \( dF_k/dz \) can be parametrised as

\[
\frac{dF_1}{dz}(x, y, z) = \mathcal{F}_1 \\
\frac{dF_2}{dz}(x, y, z) = 2x \frac{x}{\lambda} \frac{\mathcal{F}_2}{\rho^2} \\
\frac{dF_3}{dz}(x, y, z) = 2 \frac{\mathcal{F}_3}{\rho} \\
\frac{dF_4}{dz}(x, y, z) = \frac{1}{\lambda} \left( 1 - \frac{1}{\rho} \right)^2 \mathcal{F}_2 + \mathcal{F}_4 + \frac{1 - \rho}{\rho} \mathcal{F}_5 \\
\frac{dF_5}{dz}(x, y, z) = \frac{\mathcal{F}_5}{\rho} + \frac{1 - \rho}{\lambda \rho^2} \mathcal{F}_2
\]

\[
\mathcal{F}_k(x, y, z) = \sum_a \int_0^1 \frac{d\xi}{\xi} \int_0^1 \frac{d\zeta}{\zeta} \left[ \hat{F}_k^a(\xi, y, \zeta) \hat{f} \left( \frac{\chi}{\zeta}, \mu_F^2 \right) \mathcal{D}_h \left( \frac{z}{\zeta} \right) \right] \frac{d\zeta}{\zeta}
\]

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being $\zeta_{min} = (1 - \lambda)\xi/(1 - \lambda\xi)$ whereas $\chi$ and $\rho$ have been defined in section 1.2.4 depending on hypothesis about TMC.

It is possible to calculate parton structure functions using perturbative QCD; at Leading + Next-to-Leading Order we obtain

$$\hat{F}_a^k(\xi, y, \zeta) = \left[ \delta_{aq}(1 - \delta_{k4}) \delta(1 - \xi) \delta(1 - \zeta) + \frac{\alpha_s(\mu_R^2)}{2\pi} \mathcal{H}_k^a(\xi, \zeta, \mu_F^2, \lambda) \right] |V_{ij}|^2$$

where parton $a$ is a quark or a gluon, and expressions $\mathcal{H}_k^a$ are given in 2.23 and 2.25. The inclusive formalism holds neglecting the dependence on $z$ in the previous expressions and with

$$\hat{F}_k^a(x, y) = \sum_a \int_x^1 \frac{d\xi}{\xi} \hat{F}_k^a(\xi, y) f_a^a \left( \frac{\chi}{\xi}, \mu_F^2 \right)$$  \hspace{1cm} (3.3)$$

where $\hat{F}_k^a(\xi, y) = \int_{\zeta_{min}}^1 d\zeta \hat{F}_k^a(\xi, y, \zeta)$, so we can write

$$\hat{F}_k^a(\xi, y) = \left[ \delta_{aq}(1 - \delta_{k4}) \delta(1 - \xi) + \frac{\alpha_s(\mu_R^2)}{2\pi} \hat{I}_k^{a,NLO}(\xi, y) \right] |V_{ij}|^2$$

as for the Inclusive Coefficient Functions $\hat{I}_k^{a,NLO}$ in paragraph 2.4.2.
3.2 Charm Production and Strange Distribution in Nucleons

Exploiting the general and complete analytical result we can now dedicate ourselves to the study of a particular process. We consider the production of $H_c$ hadrons, containing Charm quark(s), through CC DIS events with muonic neutrinos in initial state, supposing that Charm has mass $m_c$:

$$\nu_\mu + N \rightarrow \mu^- + H_c + X$$

Neglecting the muon mass ($m_\mu = 0$; see 1.10), the semi-exclusive cross-section is

$$\frac{d^3\sigma(\nu)}{dxdydz} = \frac{G_F^2 ME}{\pi} \frac{M_W^4}{(Q^2 + M_W^2)^2} \left\{ xy^2 F_1 + (1-y) \frac{2}{\rho^2} x F_2 + xy \left(1 - \frac{y}{2}\right) \frac{2}{\rho} F_3 \right\}$$

(3.4)

being $F_k$ defined as reported in 3.2. We define the observable $^6$

$$x \Lambda^\text{eff} \left(\frac{x}{\Lambda}, y, z\right) = \frac{1}{2} \frac{\pi}{G_F^2 ME} \frac{(Q^2 + M_W^2)^2}{M_W^4} |V_{cs}|^{-2} \frac{d^3\sigma^\nu}{dxdydz}$$

(3.5)

experimentally measured [10], [30].

First of all we analyse such a quantity taking into account only QCD contributions up to LO. We have chosen to study events having a final state whose the leptonic current has negative charge ($\mu^-$), so we can use the convention that $W^+$ is emitted from the leptonic vertex and is absorbed into the partonic vertex. Because of electric charge conservation, $W^+$ boson can interact only with quarks having charge $-1/3$ (or antiquarks of charge $-2/3$) turning it into a quark of charge $+2/3$ (or correspondingly into an antiquark of charge $+1/3$); the transition probability from a flavour $j$ to $i$ is governed by the CKM matrix elements $V_{ij}$. Imposing that a Charm is produced in the final state, then contributions coming from antiquarks with charge $-2/3$ are avoided; in the initial state we can neglect Beauty quark, since its presence inside nucleons is strongly suppressed because of the high mass ($m_b \sim 4.5 \text{ GeV} \gg M_N \sim 1 \text{ GeV}$). Then only Down and Strange quarks are probed by $W$ boson (Fig.3.1).

---

$^6$Notice that 3.5 is not a simple choice of normalization for the cross section in 3.4 because $Q^2$ depends on both $x$ and $y$, so the multiplicative factor in 3.5 is not a constant; nevertheless in the kinematics region where $Q \ll M_W(\sim 80 \text{ GeV})$ this dependence is negligible.
Taking into account the fact that, in general, a target is not simply made by only one kind of nucleon, but rather it is a mixing of isotopes from many chemical elements (therefore having different mass and atomic numbers), in practice it can be thought as a collection of protons and neutrons having average mass number $A$ and atomic number $Z$. Supposing true and exact the strong isospin symmetry between proton and neutron (that is we assume that distributions $d_N$, $u_N$, $s_N$ for the neutron are respectively equal to $u_P$, $d_P$, $s_P$ for the proton), we can finally rewrite observable 3.5 as

$$x \lambda s_{\text{LO}}^{\text{eff}} \left( \frac{x}{\chi}, y, z \right) = \frac{1}{\rho^2} \frac{x}{\lambda} \left[ |V_{cd}|^2 \left( \frac{Z d_P(\chi) + (A - Z) u_P(\chi)}{A} \right) + s_P(\chi) \right] \cdot \left[ 1 - y \left( 1 - \frac{\lambda \rho}{2(3 - \rho)} \right) \right] \mathcal{P}(z)$$

(3.6)

because at LO $\mathcal{F}_{1,2,3} = \sum_a f^a(\chi) \mathcal{P}(z)$. In CHARM experiment marble ($\text{CaCO}_3$) has been used: it corresponds to an isoscalar target, that is $Z = (A - Z)$; neglecting Target Mass Corrections ($\chi = x/\lambda$, $\rho = 1$), then

$$x \lambda s_{\text{LO}}^{\text{eff}} \left( \frac{x}{\chi}, y, z \right) = \frac{x}{\lambda} \left[ |V_{cd}|^2 \left( \frac{d_P + u_P}{2} \right) + s_P \right] \left[ 1 - \frac{m^2_\lambda}{2 M_{Ex}} \right] \mathcal{P}(z)$$

(3.7)

related to the quantity investigated in [10]; dependence of $d_P$, $u_P$, and $s_P$ on $x/\lambda$ and $\mu_F^2$ scale is understood. Setting $|V_{cd}| \simeq 0.220$ and $|V_{cs}| \simeq 0.974$ [31],[32] it follows $|V_{cd}|^2/|V_{cs}|^2 \sim 0.05$; $s_{\text{LO}}^{\text{eff}}$ can be therefore considered to all intents and purposes a Strange quark distribution inside nucleons. Nevertheless we have to pay attention to special configurations (typically high $x$) where contribution of $u$ and $d$ could be not negligible in spite of CKM suppression. To this aim in Fig. 3.2 and 3.3 some comparisons between Strange and non-Strange component contained in 3.7 have been shown.
Figure 3.2: For CTEQ 6 PDF set [33], with an energy scale $Q^2 = 4 \, \text{GeV}^2$, distributions $x_s(x, Q^2)$ (continuous line) and $x \left| \frac{V_{cd}}{V_{cs}} \right|^2 \left( d(x, Q^2) + u(x, Q^2) \right)$ (dashed line) are shown for the case of an isoscalar target.

Figure 3.3: Using CTEQ 6 PDF set [33] the ratio $R_{\text{eff}} = \frac{s(x, Q^2)}{\left| \frac{V_{cd}}{V_{cs}} \right|^2 \left( d(x, Q^2) + u(x, Q^2) \right)}$ is carried out varying the energetic scale $Q^2$ ($4 \, \text{GeV}^2$, $100 \, \text{GeV}^2$, $900 \, \text{GeV}^2$) in the region $x < 0.1$. 
3.2.1 Experimental Context

Neutrino beams are usually generated by collisions of proton beams on a fixed target, typically Beryllium; pions and kaons are produced in the event and a large fraction of them decays into muons and muonic neutrinos, through semi-leptonic channels. Neutrino beam for CCFR experiment \cite{2,34} is made of about 86.4% $\nu_\mu$, 11.3% $\bar{\nu}_\mu$, 2.3% $\nu_e$ and $\bar{\nu}_e$. Energies of these beams cover a wide range: for CCFR from 30 to 300 GeV. The target material, on which neutrinos are sent to investigate DIS, depends on the experiment: Iron for CCFR (NuTeV now) at Fermilab, and for CDHSW at CERN; marble ($\text{CaCO}_3$) for CHARM, glass for CHARM II, Neon or Deuterium for BEBC, all at CERN. Unlike marble and Deuterium, Iron is not an isoscalar target. Nuclear effects should be included because the events typically consist in scattering on heavy targets: Fermi motion inside nucleons, EMC effect and recombination of gluons among nucleons of close nuclei (Nikolaev and Zakharov, 1975; Mueller and Qiu, 1986), . . .

For DIS processes using neutrino beams, the typical range for $x$ and $Q^2$ are respectively $0.01 < x < 0.8$ and $0.1 \text{GeV}^2 < Q^2 < 100 \text{GeV}^2$. A review of measurements in experiments dedicated to DIS with neutrino beams is given in \cite{32}.

Charm production through CC DIS is experimentally studied observing dimuonic events; in fact Charm quark, straight after creation, hadronizes (typically) into mesons ($D^0, D^+, D^{*0}, \ldots$) that can decay (also indirectly, $D^{*0} \rightarrow D^0 \ldots \rightarrow \nu_\mu + \mu^+ + \ldots$) into neutrino and anti-muon:

$$\nu_\mu + N \xrightarrow{W^+} \mu^- + H_\epsilon + X_1 \xrightarrow{\epsilon} \nu_\mu + \mu^+ + X_2$$

The whole final state, made of a muon pair of opposite charge, is then a very strong signal of Charm production through CC DIS, or correspondingly that Strange quark distribution inside nucleons has been probed.
3.2.2 Test

As initial test to check the correctness of our numerical implementation, we have tried to reproduce results in [35]; we remind in the following all the employed assumptions:

- Charm mass set to \( m_c = 1.5 \text{ GeV} \)
- Incident neutrino energy set to \( E = 192 \text{ GeV} \)
- Renormalization scale (\( \mu_R \)) in the strong coupling constant \( \alpha_s \) chosen to be equal to the factorization one (\( \mu_F \)) for PDF and Coefficient Functions; both set to \( \mu_F^2 = \mu_R^2 = Q^2 + m_c^2 \)
- GRV-94 PDF [36], with \( \overline{MS} \) scheme
- Peterson FF [37] (see paragraph 3.3.2)
- for LO result, PDF defined at LO have been coherently employed, whereas the value \( \epsilon_{P}^{LO} = 0.20 \) set for FF;
  for NLO result, PDF defined at NLO and \( \epsilon_{P}^{NLO} = 0.06 \) for FF
- we have considered an isoscalar target; contributions coming from \( Up \) and \( Down \) quarks have been neglected
- CKM matrix elements have been set to \( |V_{cd}| = 0.220 \) and \( |V_{cs}| = 0.974 \);
  for coherence with the previous point, \( |V_{cd}| = 0 \) for the contribution coming from the gluon (i.e. gluon splitting in \( d, \bar{d} \) pair suppressed)
- Target Mass Corrections disregarded

The obtained results have been illustrated in Fig.3.4, 3.5: they agree with literature [35]. Introducing also TMC (1.2.4), it can be verified that such a kind of corrections are absolutely negligible in the range of \( x \) and \( Q^2 \) here explored.
Figure 3.4: $\tilde{g}_{\text{eff}}(\tilde{x}, y, z)$ for $x = 0.015$ and $Q^2 = 2.4 \text{ GeV}^2$. $\text{Born}(\text{LO})$ means that expression 3.7 has been used with the corresponding value $\epsilon^\text{LO}$ and PDF defined at LO. With notation $\text{Born}(\text{NLO})$, it means that the analysis has been carried out at NLO using the definition in 3.5 PDF at NLO and $\epsilon^\text{NLO}$.

Figure 3.5: $\tilde{g}_{\text{eff}}(\tilde{x}, y, z)$ for $x = 0.125$ and $Q^2 = 17.9 \text{ GeV}^2$. $\text{Born}(\text{LO})$ means that expression 3.7 has been used with the corresponding value $\epsilon^\text{LO}$ and PDF defined at LO. With notation $\text{Born}(\text{NLO})$, it means that the analysis has been carried out at NLO using the definition in 3.5 PDF at NLO and $\epsilon^\text{NLO}$. 

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This simple test is satisfactory to point out the role of the appropriate value for \( \alpha_s \) in order to compare the numerical results. In [29, 35] the routine used to calculate \( \alpha_s \) at the scale of \( Z^0 \) mass gives a value of 0.109, whereas our routine, using a modern value for \( \Lambda_{QCD} \) (and being able to change thresholds, \( \Lambda_{QCD} \) scales and many other parameters) gives \( \alpha_s(M_{Z^0}^2) \sim 0.118 \).

At lower scales the discrepancy between results of the two routines increases: for graph in Fig.3.4 (\( Q^2 = 2.4 GeV^2 \), \( m_c = 1.5 GeV \)) we obtain \( \alpha_s(Q^2 + m_c^2) \sim 0.240 \), whereas using our routine \( \alpha_s(Q^2 + m_c^2) \sim 0.289 \). In this case therefore we are under-evaluating NLO contributions of about 20%; in Fig.3.6 the comparison between these last two results is shown.

![Figure 3.6](image_url)

Figure 3.6: Same plot as Fig.3.4 for \( x_{\text{eff}}(x, y, z) \) at LO+NLO, evaluated setting \( \alpha_s(Q^2 + m_c^2) \sim 0.240 \) (continuous line) and \( \alpha_s(Q^2 + m_c^2) \sim 0.289 \) (dashed line).

---

7We would like to thank very much Dr. Stephan Kretzer to have provided us his numerical code to carry out a comparison.
3.3 Analysis

Interpretation of $s_{\text{eff}}$ observable as quantity proportional to Strange quark distribution inside nucleons does not hold beyond the Leading Order; in fact Parton Distribution Functions are convoluted with Coefficient Functions (not trivial at NLO) so besides contributions of light quarks $u$, $d$, $s$, even the one of gluon appears.

In order to study the distribution of Strange quark inside nucleons by means of $s_{\text{eff}}$ observable, it must be subtracted from $s_{\text{exp}}^{\text{eff}}$, defined as in (3.5) and experimentally measured, the contribution coming from the gluon (and even the one from $u$ and $d$ quarks, if not negligible) predicted by the theory, using the Semi-Exclusive Coefficient Functions at NLO in (2.25) (2.23).

As pointed out at the beginning of this work, DIS mediated by Charged Currents allows to independently probe quark and antiquark distributions; all what we have written therefore holds also for the process

$$\bar{\nu}_\mu + N \rightarrow \mu^+ + H_\pi + X$$

where $H_\pi$ is a hadron containing Charm.

In this case it is possible to define the observable $\bar{s}_{\text{eff}}$ analogously to (3.5)

$$\frac{x}{\lambda} s_{\text{eff}} \left( \frac{x}{\lambda}, y, z \right) = 1 \frac{\pi}{2 G_F^2 M E} \left( \frac{Q^2 + M_W^2}{M_W^4} \right)^2 |V_{cs}|^{-2} \frac{d^3 \sigma(\pi)}{dx dy dz}$$

where the cross section is derived from the observations on CC DIS in general (paragraph 1.1)

$$\frac{d^3 \sigma(\tau)}{dx dy dz} = \frac{G_F^2 M E}{\pi} \frac{M_W^4}{(Q^2 + M_W^2)^2} \left\{ xy^2 F_1 + (1-y) \frac{2 x}{\rho^2 \lambda} F_2 - xy \left( 1-y \right) \frac{2}{\rho} F_3 \right\}$$

By means of CC DIS is then possible to study independently distributions of Strange and anti-Strange quarks; this is useful to point out an eventual asymmetry between the two distributions, as hypothesised by recent experimental results [38], [32], [2].

From Fig. 3.4 and 3.5 it has been found that NLO contribution coming from gluon-channel ($\text{Gluon(NLO)}$) is comparable to the one at LO coming directly from Strange quark ($\text{Born(NLO)}$); this because the gluon distribution inside nucleons is quantitatively high and it balances the suppression originated from the powers of $\alpha_s$. Finally we remark that $s_{\text{eff}}$ (and then also the cross section) becomes negative when $z \gtrsim 0.85$: quantitatively such a behaviour is obviously non-physical, but it indicates the need of resumming terms of higher order.
In the following we show the behaviour for observable $(x/\lambda)s_{eff}(x/\lambda, Q^2, z)$ with respect to parameters and models; as setting, we assume

- $m_c = 1.5 \text{ GeV}$, $E = 192 \text{ GeV}$, $|V_{cd}| = 0.220$, $|V_{cs}| = 0.974$
- instead of GRV-94 PDF we use a more modern set: CTEQ-6 \[33\] with \(\overline{\text{MS}}\) scheme; we do not have a definition at LO, so the result carried out is only the one at NLO
- we use Peterson FF \[3.9\], with $\epsilon^{NLO}_P = 0.06$
- $\mu^2_P = \mu^2_R = Q^2 + m_c^2$
- isoscalar target and contributions from $u$, $d$
- $\alpha_s$ generated by the same LHA-PDF package \[39\] used for PDF; for CTEQ-6: $\alpha_s(M^2_{Z^0}) = 0.118$, $\alpha_s(Q^2_0 + m_c^2) = 0.271$
  with $Q^2_0 = 2.4 \text{ GeV}^2$ and $m_c = 1.5 \text{ GeV}$
- Target Mass Corrections disregarded

Furthermore we consider only the case $x = 0.015$ and $Q^2 = 2.4 \text{ GeV}^2$.

Numerical error-bars have not been shown in graphs because not visible, being negligible ($< 1\%$).
3.3.1 PDF Dependence

We want to verify the hypothesis that Strange quark distribution is dominant for this process with respect to one for other quarks, at least in the range of \(x\) and \(Q^2\) here taken into account: distributions of Up and Down (\(|V_{cd}| = 0.220\)) have been added to results of Fig. 3.4.

![Graphs showing PDF contributions for GRV-94 PDF set](image)

Figure 3.7: Contributions to \(\xi_s(x, Q^2, z)\) for GRV-94 PDF set: comparison between the case without Up and Down quarks (\(|V_{cd}| = 0\), continuous line) and the case including them (\(|V_{cd}| = 0.220\), dashed line).

As illustrated in Fig. 3.7, contributions coming from Up and Down for GRV-94 PDF are not at all negligible: an accurate analysis must take them into account. In Fig. 3.2 and 3.3 we have shown that for CTEQ-6 PDF (for the chosen values \(x\) and \(Q^2\)) Up and Down are suppressed; this agrees with results in Fig. 3.8.

Finally we want to highlight the strong dependence on the PDF set: in Fig. 3.9 the behaviour of \((x/\lambda)s_{s,eff}(x/\lambda, Q^2, z)\) is shown for PDF sets contained in the LHA-PDF package [39].
Figure 3.8: Graphs of $\xi s_{\text{eff}}(x, Q^2, z)$ with and without contributions coming from distributions for $Up$ and $Down$ quarks, using GRV-94 and CTEQ-6 PDF sets.

Figure 3.9: Comparison of the behaviour for $\xi s_{\text{eff}}(x, Q^2, z)$ obtained for different PDF sets: CTEQ-6, MRST-2001, Fermi-2002, Alekhin, Botje.
3.3.2 Dependence on Fragmentation Functions

In paragraph 1.2.3 we have introduced in a generic way the Fragmentation Functions in order to link the partonic to the hadronic level, analogously to PDF; with regard to massive quarks production in paragraph 2.3.2 we have seen that FF are independent of a factorization scale $\mu_F^2$. Now we show some models.

Fragmentation Functions employed to reproduce numerical results exposed till now is the Peterson one\[37\]:

$$D(z, \varepsilon_P) = N \frac{1}{z [1 - z^{-1} - \varepsilon_P (1 - z)^{-1}]^2}$$

(3.9)

Normalization factor $N$ is found imposing $\int_0^1 dz D(z, \varepsilon_P) = 1$ and obviously it depends on the choice of $\varepsilon_P$. To perform an analysis at LO, the relevant value of $\varepsilon_P$ corresponds to $\varepsilon_P = 0.20 \pm 0.04$ (taken by a comparison with experimental data\[10\]); for an analysis at NLO the suggested value corresponds to $\varepsilon_P = 0.06 \pm 0.03$ (from non-DIS data\[40\],\[41\]). Finally most recent investigations propose values of about $\varepsilon_P^{NLO} \approx 0.02 \pm 0.035$\[42\],\[43\].

In Fig. 3.10 $(x/\lambda) s_{\text{eff}}(x/\lambda, Q^2, z)$ is displayed for $\varepsilon_P$ equal to 0.04, 0.06, 0.08.

![Figure 3.10: Dependence of $s_{\text{eff}}(x/\lambda, Q^2, z)$ on $\varepsilon_P$ parameter.](image-url)
A very widely employed FF in literature is the one of Collins-Spiller [44]:

$$D(z, \varepsilon_{CS}) = N \left[ \frac{1 - z}{z} + \frac{(2 - z)\varepsilon_{CS}}{1 - z} \right] \frac{1 + z^2}{[1 - z^{-1} - \varepsilon_{CS}(1 - z)^{-1}]^2}$$  \hspace{1cm} (3.10)

For a NLO analysis, in [45] is reported an average value $\varepsilon_{CS} = 0.13 \pm 0.08$.

Figure 3.11: Behaviour of $\xi_{\text{eff}} (x, Q^2, z)$ obtained for a Collins-Spiller FF.

At last we take into account a Kartvelishvili FF [46], $D(z, \alpha) = N z^\alpha (1 - z)$.

Figure 3.12: Behaviour of $\xi_{\text{eff}} (x, Q^2, z)$ obtained for a Kartvelishvili FF.
So far we have tacitly assumed that it is possible to describe hadronization of the produced heavy quark simply introducing a Fragmentation Function and convolve with Semi-Exclusive Coefficient Functions. It is important to notice that actually this approach is not rigorously correct \textit{a priori}. First of all, as pointed out in paragraph 2.1.3, being massive the quark, it is arbitrary to choose a way to rescale partonic quantities (as energy, four-momentum, momentum, etc.) to describe the corresponding hadronic quantities: only when the transverse momentum is high, differences among the many approaches are negligible. Secondarily when the transverse momenta are low, it is not correct to assume that fragmentation is really independent of the remainder of the event; in general it is not, because the single quark fragmenting must exchange colour charge with the rest of particles of the process and only when the transverse momentum is high, these effects are suppressed.

### 3.3.3 Dependence on Charm Quark Mass

![Graph](image)

Figure 3.13: Dependence of $\xi s_{eff}(\xi, Q^2, z)$ from the mass of Charm quark.
3.3.4 Scale Dependence

Until now factorization scale $\mu_F^2$ has been set $Q^2 + m_c^2$ in PDF and Coefficient Functions; the same has been done with regard to the renormalization scale $\mu_R^2$, appearing as argument of the coupling constant $\alpha_s$. In order to investigate the dependence of $\frac{x}{\lambda}s_{\text{eff}}(x/\lambda, Q^2, z)$ on these two scales, we introduce a compact notation rewriting $\mu_{F,R}^2 = (Q^2 + m_c^2)r_{F,R}$. In graph 3.14 there are shown the results for $r_F = 1 = r_R$ and the envelopment given by maxima and minima for $r_{F,R} \in \{0.5, 1, 2\}$ independently varying (excluding the cases $\{r_F, r_R\} = \{0.5, 2\}$ and $\{r_F, r_R\} = \{2, 0.5\}$).

![Graph 3.14: Dependence of $\frac{x}{\lambda}s_{\text{eff}}(x/\lambda, Q^2, z)$ on the factorization ($\mu_F^2$) and renormalization ($\mu_R^2$) scales. The continuous line is the result for $\mu_F^2 = \mu_R^2 = (Q^2 + m_c^2)$ whereas the upper (lower) line is the envelopment given by maxima (minima) obtained varying the parameters $r_{F,R} = \mu_{F,R}^2/(Q^2 + m_c^2)$ in $\{0.5, 1.0, 2.0\}$, excluding combinations $\{r_F, r_R\} = \{0.5, 2\}$ and $\{r_F, r_R\} = \{2, 0.5\}$.

Figure 3.14: Dependence of $\frac{x}{\lambda}s_{\text{eff}}(x/\lambda, Q^2, z)$ on the factorization ($\mu_F^2$) and renormalization ($\mu_R^2$) scales. The continuous line is the result for $\mu_F^2 = \mu_R^2 = (Q^2 + m_c^2)$ whereas the upper (lower) line is the envelopment given by maxima (minima) obtained varying the parameters $r_{F,R} = \mu_{F,R}^2/(Q^2 + m_c^2)$ in $\{0.5, 1.0, 2.0\}$, excluding combinations $\{r_F, r_R\} = \{0.5, 2\}$ and $\{r_F, r_R\} = \{2, 0.5\}$. 
Chapter 4

Mellin Space

In this chapter we show that working in Mellin space allows to elementarily deal with convolutions of Coefficient Functions, PDF, FF and with evolution of Parton Distribution Functions; in particular the formalism is very advantageous in order to include resummation.

The main difficulties to implement Mellin transforms for Coefficient Functions carried out in chapter 2 have been pointed out.

Main result of this work essentially consists in the analytical calculation of functions $F_k(\chi,y,z)$ that we found to have the form (eq.3.2)

$$F_k(\chi,y,z) = \sum_a \int_1^{\chi} \int_{\max\{z,\zeta\min\}}^1 \hat{F}_a(k)(\xi,\mu^2) f_a(\chi,\mu^2) D(z) d\xi d\zeta$$  \hspace{1cm} (4.1)

We define the Mellin Transform (MT) $h(N)$, being $N \in \mathbb{C}$, for a function $h(t)$ with support in $t \in [0,1]$:

$$h(N) \equiv \int_0^1 dt t^{N-1} h(t)$$

Taking MT of (4.1) with respect to $\chi$ and $z$ variables, neglecting for sake of simplicity the dependence of Coefficient Functions from $y$ and $\mu^2$, we obtain

$$F_k(N,M) = \sum_a \int_1^1 d\chi \chi^{N-1} \int_0^1 dzz^{M-1} \hat{F}_k(\chi,z)$$

$$= \sum_a \int_1^1 f_a(N,\mu^2) D(M) \int_0^1 d\xi \xi^{N-1} \int_{\zeta\min}^1 d\zeta \zeta^{M-1} \hat{F}_a(\xi,\zeta)$$

with $\zeta\min = (1-\lambda)\xi/(1-\lambda\xi)$; setting

$$G_a(N,M) \equiv \int_0^1 d\xi \xi^{N-1} \int_{\zeta\min}^1 d\zeta \zeta^{M-1} \hat{F}_a(\xi,\zeta)$$  \hspace{1cm} (4.2)
we can rewrite the equality in a factorised form

\[ F_k(N, M) = \sum_a f^a(N, \mu^2) \mathcal{D}(M) g^a_k(N, M) \]  

(4.3)

A very important reason to work in Mellin space concerns resummation. In chapter 3 we have highlighted that \( \xi s_{\text{eff}}(\xi, Q^2, z) \) (deduced just from perturbative NLO calculation) has no quantitative meaning in the region \( z \gtrsim 0.85 \): in fact it is fundamental to consider also resummation in order to work out accurate and physically meaningful results. Using MT formalism an observable \( \sigma \) resummed up to Next-to-Leading Log (NLL) + Next-to-Leading Order (NLO) is given by

\[ \sigma^{(\text{Res})}(N) = \left[ \sigma^{(\text{NLL})}(N, \alpha_s) - \sigma^{(\text{NLL})}(N, \alpha_s) \right] + \sigma^{(\text{NLO})}(N, \alpha_s) \]  

(4.4)

where \( \sigma^{(\text{NLL})}(N, \alpha_s) \mid_{\alpha_s} \) represents the expansion of \( \sigma^{(\text{NLL})}(N, \alpha_s) \) truncated to \( \alpha_s \) order. Unlike NLO, theoretical calculation up to NLL is analytically feasible only in Mellin space: consequently, inside such a space, the matching between NLO and NLL contributions is carried out in a natural and elementary way [47].

### 4.1 Analytical Treatment and Numerical Implementation

In Mellin space convolutions have therefore a simplified structure allowing to represent functions \( F_k \) as simple product of factors. Nevertheless the experimental measurement for \( F_k \) is done through variables \( \chi, z \) and not through conjugate variables \( N \) and \( M \), then the Inverse-Mellin-Transform (IMT) of 4.3 has to be taken in order to compare with \( F_k(\chi, z) \).

IMT \( h(t) \) for a function \( h(N) \) is defined as

\[ h(t) \equiv \frac{1}{2\pi i} \int_C dN t^{-N} h(N) \]

where \( C \) is a path passing on the right of all singularities of the analytical continuation of \( h(N) \) to the complex plane. So

\[ F_k(\chi, z) = \frac{1}{(2\pi i)^2} \int_{C_1} dN \chi^{-N} \int_{C_2} dM z^{-M} F_k^a(N, M) \]  

(4.5)

At mathematical level, the most elementary path to evaluate the inverse-transform is the one illustrated on the left in Fig.4.1 however it is not usable
for numerical implementations (for usually encountered functions), because there is not suppression of contributions coming from the integrand when the imaginary part tends to infinity: therefore it is not possible to achieve a (settled) convergence for the result \[48\]. In practice a contour like the one on the right of Fig.4.1 is employed, or its deformations to obtain results having a faster convergence.

![Figure 4.1: Some integration paths to evaluate an Inverse-Mellin-Transform.](image)

Notice that evaluating a function \( g(N) \) along the first path, the values Re\((N)\) are definitely on the right of whatever real number initially fixed; for the second path this is no more true and originates great issues about treatment of the double integral defined in 4.2. In order to illustrate the problem in an incisive manner, an example is shown without loss of generality.

Calculating the Mellin transform for \( g(t) = t^\alpha \):

\[
g(N) = \int_0^1 dt t^{N-1} t^\alpha = \frac{t^{N+\alpha}}{N + \alpha} \bigg|_0^1
\]

In the region where Re\((N) > -\text{Re}(\alpha)\) the result is well defined and equal to \(1/(N + \alpha)\); however, when Re\((N) < -\text{Re}(\alpha)\), a divergence is found evaluating integral in 0. In order to avoid this problem, MT is evaluated for Re\((N) > -\text{Re}(\alpha)\) and the result is extended by analytical continuation to the whole complex plane, at the most excepting isolated poles on the real axis. This explains why it is not possible numerically to carry out Mellin transforms for contours not fully contained in the region on the right of a fixed real number. Calculation in 4.2 has to be analytically performed and then the result extended to the complex plane, because of requirements on the integration path.
We consider the double integral \[ g(\xi, \zeta) = \frac{1}{(1-\xi)+1} \] for the term contained in \( G^a_k(N,M) \) of \[ 4.2 \] coming from Semi-Exclusive Coefficient Functions in \[ 2.23 \]

\[
\begin{align*}
g(N,M) &= \int_0^1 \frac{d\xi}{(1-\xi)^{N-1}} \int_0^1 \frac{d\zeta}{(1-\xi)^{M-1}} g(\xi, \zeta) \\
&= \int_0^1 \frac{\xi^{N-1}}{(1-\xi)^{N-1}} \int_0^1 \frac{d\zeta}{(1-\xi)^{M-1}} \tag{4.7}
\end{align*}
\]

Making use of the expression

\[
(1-t^k) = (1-t) \sum_{j=1}^{k} t^{j-1}
\]

a solution in compact form is found

\[
g(N,M) = - \sum_{k=1}^{M-1} \frac{1}{k} \sum_{j=1}^{k} 2 \frac{\Gamma(M+k+j)}{\Gamma(M+k+1)} \Psi(M+N+k+j) (1-\lambda)^{M-1} \tag{4.8}
\]

but it is difficult to analytically extend to the complex plane for \( M \) index: therefore such a result is not usable at the moment.

An alternative solution is given by

\[
g(N,M) = S_1(M-1) \Psi(N) + \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left[ \frac{\Gamma(M+k+j)}{\Gamma(M+k+1)} \Psi(M+N+k+j) (1-\lambda)^{M-1} \right] \frac{\lambda^j}{j!} (1-\lambda)^{k+1} \tag{4.9}
\]

where \( \Psi(t) \) function is the logarithmic derivative of \( \Gamma(t) \) function

\[
\Psi(t) = \frac{d}{dt} \log \Gamma(t) = \frac{\Gamma'(t)}{\Gamma(t)} \tag{4.10}
\]

whereas \( S_1(t) \) is the analytical continuation of the harmonic series to the whole complex plane

\[
S_1(t) = \sum_{k=1}^{t} \frac{1}{k} = \Psi(t+1) + \gamma_e \tag{4.11}
\]
where \( \gamma_e \approx 0.57721 \ldots \) is the Euler constant.

Being \( \int_1^{\infty} d\zeta h_{\mp}(\zeta) = 0 \) because of plus-distribution properties, it is verified from (4.7) that \( g(N,1) = 0 \); in fact expressions (4.8) and (4.9) satisfy this requirement. To check the correctness, these expressions have been evaluated for values of \( N \) and \( M \) integer and positive, varying \( \lambda \), and they have been compared (Fig. 4.2) to the analogous result numerically derived from (4.7).

![Figure 4.2: Expression of \( g(N, M) \) in (4.7) evaluated analytically (4.9) and by numerical integration; \( \lambda = 0.516 \) \( (Q^2 = 2.4\text{ GeV}^2, m_c = 1.5\text{ GeV}) \).](image)

Extension of result (4.9) to the complex plane is immediate because there are only isolated poles on the real axis: such an expression is then usable with a generic contour to calculate an inverse-transform. Numerical implementation of this contribution involves great complications. First of all the integration path must be good enough to guarantee stability and fast convergence for inverse-transform of \( S_1 \) and \( \Psi \) functions. Furthermore the right evaluation of \( g(N, M) \) for \( N \) and \( M \) with real and/or imaginary parts “large” in absolute value is hard; in fact the terms of the sum are made of contribution tending to balance (first line in (4.9) with the second one) and the nearly cancellation of large numbers implicates a not trivial control of accuracy. A method for
fast converging of the Inverse-Mellin-Transform inside a restricted region of $N$ and $M$ is therefore needed or it is mandatory to use multiple [49, 50] and adjustable precision in order to deal with sums containing a high number of terms; for the second case in particular, there is also the problem that some coefficients can become very big and go out the maximum range of a computer.

It can be shown that through $g(N, M)$ and few other contributions it is possible to write, in principle, the whole Coefficient Functions. Terms proportional to $\delta(1 - \zeta)$ contained in 2.23 and 2.25 give results that can be deduced from relations in [51]. For term

$$\left(1 - \zeta\right) \left(1 - \frac{\lambda \zeta}{1 - \zeta}\right)^2 \left[\frac{1 - \xi}{(1 - \lambda \xi)^2}\right]_+$$

in 2.23 we have to explicitly calculate the double integral 4.2. By means of the identity (as distribution)

$$1 = \frac{(1 - \xi)(1 - \zeta)}{(1 - \xi)_+(1 - \zeta)_+}$$

any terms of remaining contributions in Coefficient Functions can be rewritten in the form

$$r(\xi, \zeta) = \sum_{a,b} C_{ab} \frac{\xi^a \zeta^b}{(1 - \xi)_+(1 - \zeta)_+}$$

where $C_{ab}$ is an opportune numerical coefficient, containing at the most powers of $\lambda$. Using the properties of Mellin transforms, if $h(N)$ is the MT of $h(t)$, then the one for $t^\alpha h(t)$ is given by $h(N + \alpha)$, so

$$r(N, M) = \sum_{a,b} C_{ab} g(N + a, M + b)$$

Theoretically it is possible to obtain the MT of the whole Coefficient Functions in this way. In practice however such an approach is opportune only if a fast and accurate routine is found to calculate $g(N, M)$; otherwise it is preferable to evaluate the double integral 4.2 for each single term contained in $\mathcal{G}_k^a(\xi, y, z)$, running into the same problems encountered evaluating $g(N, M)$, optimizing algorithms according to the necessities.

Complications in evaluating integrals 4.2 originates from the fact that we are dealing with Coefficient Functions both differential and for massive quarks.
For the massless case \((\lambda = 1)\), expression 4.2 in fact reduces to

\[
G^a_k(N, M) = \int_0^1 d\xi \xi^{N-1} \int_0^1 d\zeta \zeta^{M-1} \hat{F}^a_k(\xi, \zeta)
\]

\[
= \sum_j \left[ \int_0^1 d\xi \xi^{N-1} \Phi^a_{j,k}(\xi) \right] \left[ \int_0^1 d\zeta \zeta^{M-1} \Phi^a_{j,k}(\zeta) \right]
\]

that is \(G^a_k(N, M)\) becomes a sum of terms factorised as product of two independent Mellin transforms.

The inclusive case involves, by definition, only one variable and it is simply recovered setting \(M = 1\) in 4.2: \(G^a_k(N, 1)\) is then turned out to be the Mellin transform of expressions in paragraph 2.4.2. For both cases, thanks to MT of functions and most recurring distributions [51], there are not particular issues.

For the situation differential and massive at the same time, the inner integral in 4.2 depends on \(\xi\) variable: this is the root cause for the encountered complexity.

### 4.2 PDF Treatment

In the previous section we have pointed out the difficulties of a numerical implementation to calculate \(F_k(\chi, z)\) structure functions using the Mellin space formalism. The advantage of such an approach justifies the investment on investigation about the described problems, even if effective tools to numerically calculate relation 4.1 already exist.

Besides being the natural environment to handle resummation 4.4 working in Mellin space is convenient also to deal with PDF.

#### 4.2.1 Evolution Equation

Dependence of PDF on the scale \(\mu^2\) is regulated by the matrix evolution equation DGLAP ([9], [52]), coupling \(2n_f+1\) distribution functions of quarks, antiquarks and gluons (\(n_f\) is the flavours number)

\[
\mu^2 \frac{\partial}{\partial \mu^2} \left[ \begin{array}{c} q_i(x, \mu^2) \\ g(x, \mu^2) \end{array} \right] = \alpha_s(\mu^2) \sum_{q, g} \int_x^1 \frac{d\xi}{\xi} \left[ \begin{array}{ccc} P_{q_i q_j}(\xi) & P_{q_i g}(\xi) & P_{g g}(\xi) \\ \end{array} \right] \left[ \begin{array}{c} q_j(x/\xi, \mu^2) \\ P_{q_j q_i}(\xi) \\ P_{q_j g}(\xi) \end{array} \right]
\]

\[(4.12)\]
where \( q_i \) and \( q_j \) can vary along all the freedom degrees of quarks. We introduce combinations \( q_{NS} \) of non-singlet:

\[
\begin{align*}
V_4 &= q_i^- \\
T_3 &= u^+ - d^+ \\
T_8 &= u^+ + d^+ - 2s^+ \\
T_{15} &= u^+ + d^+ + s^+ - 3c^+ \\
T_{24} &= u^+ + d^+ + c^+ - 4b^+ \\
T_{35} &= u^+ + d^+ + s^+ + c^+ + b^+ - 5t^+
\end{align*}
\]

\[(4.13)\]

with \( q_i^\pm = q_i \pm \bar{q}_i \), where \( q_i = u, d, c, s, t, b \) obviously are quark flavours; for these combinations the matrix equation \( 4.12 \) decouples in \( 2n_f - 1 \) independent scalar equations

\[
\begin{align*}
\mu^2 \frac{\partial}{\partial \mu^2} q_{NS}^V(x, \mu^2) &= \frac{\alpha_s(\mu^2)}{2\pi} \int_x^1 \frac{d\xi}{\xi} P_{qq}^-(\xi) q_{NS}^V(x/\xi, \mu^2) \\
\mu^2 \frac{\partial}{\partial \mu^2} q_{NS}^T(x, \mu^2) &= \frac{\alpha_s(\mu^2)}{2\pi} \int_x^1 \frac{d\xi}{\xi} P_{qq}^+(\xi) q_{NS}^T(x/\xi, \mu^2)
\end{align*}
\]

having defined \( P^\pm = P_{qq}^{NS} \pm P_{q\bar{q}}^{NS} \), where \( P^{NS} \) have been introduced by decomposing

\[
P_{q\bar{q}k} = \delta_{ik} P_{qq}^{NS} + P_{q\bar{q}}^{S} \quad P_{q\bar{q}k} = \delta_{ik} P_{q\bar{q}}^{NS} + P_{q\bar{q}}^{S}
\]

We construct the only singlet distribution as

\[
\Sigma = \sum_i q_i^+ = \sum_i (q_i + \bar{q}_i)
\]

(4.15)

Its evolution is coupled to the distribution of the gluon

\[
\mu^2 \frac{\partial}{\partial \mu^2} \left[ \frac{\Sigma(x, \mu^2)}{g(x, \mu^2)} \right] = \frac{\alpha_s(\mu^2)}{2\pi} \int_x^1 \frac{d\xi}{\xi} \left[ \begin{array}{cc}
P_{qq}^+(\xi) & 2n_f P_{qg}^+(\xi) \\
P_{qg}^+(\xi) & P_{gg}^+(\xi)
\end{array} \right] \left[ \begin{array}{c}
\Sigma(x/\xi, \mu^2) \\
g(x/\xi, \mu^2)
\end{array} \right]
\]

\[(4.16)\]

so now

\[
\begin{align*}
P_{qq} &= P^+ + n_f \left( P_{qq}^S + P_{q\bar{q}}^S \right) \\
P_{qg} &= P_{qg} = P_{\bar{q}g} \\
P_{gg} &= P_{gg} = P_{g\bar{g}}
\end{align*}
\]

What has been worked out so far allows to simplify the matrix structure of evolution equations, but the base structure of such a kind of integral-differential equations is left unchanged.
Applying MT with respect to $x$ variable for evolution equations (4.14) and (4.16) it follows

\[
\mu^2 \frac{\partial}{\partial \mu^2} q_{NS}^V(N, \mu^2) = \frac{\alpha_s(\mu^2)}{2\pi} \gamma_{qq}(N) q_{NS}^V(N, \mu^2) \\
\mu^2 \frac{\partial}{\partial \mu^2} q_{NS}^T(N, \mu^2) = \frac{\alpha_s(\mu^2)}{2\pi} \gamma_{qq}^+(N) q_{NS}^T(N, \mu^2)
\]

(4.17)

\[
\mu^2 \frac{\partial}{\partial \mu^2} \left[ \frac{\Sigma(N, \mu^2)}{g(N, \mu^2)} \right] = \frac{\alpha_s(\mu^2)}{2\pi} \left[ \begin{array}{cc} \gamma_{qq}(N) & 2n_f \gamma_{qq}(N) \\ \gamma_{qq}(N) & \gamma_{gg}(N) \end{array} \right] \left[ \begin{array}{c} \Sigma(N, \mu^2) \\ g(N, \mu^2) \end{array} \right]
\]

(4.18)

where anomalous dimensions $\gamma_{ab}^{(\pm)} (a, b \in \{q, g\})$ are given by

\[
\gamma_{ab}^{(\pm)}(N, \alpha_s) = \int_0^1 dx x^{N-1} P_{ab}^{(\pm)}(x, \alpha_s)
\]

(4.19)

We have highlighted the dependence of $P_{ab}$ (and therefore of $\gamma_{ab}$) also on $\alpha_s$ and consequently on the scale $\mu^2$ therein contained. Under Mellin transform convolution integrals become multiplications, so an analytical solution is easily carried out; for the case of non-singlet it follows (with opportune $\gamma(N, t)$)

\[
q_{NS}(N, \mu^2) = q_{NS}(N, \mu_0^2) \exp \left\{ \int_{\mu_0^2}^{\mu^2} \frac{dt}{t} \frac{\alpha_s(t)}{2\pi} \gamma(N, t) \right\}
\]

(4.20)

For the case of singlet $\Sigma$ and of $g$ it is possible to proceed analogously even if it is a little bit more complicated because of the reciprocal dependence. Inverting relations (4.13) and (4.15) distributions of $n_f$ quarks, $n_f$ antiquarks and gluon $g$ are obtained. We want to point out that, because of (4.20) in Mellin space PDF have the form

\[
f^a(N, \mu^2) = f^a(N, \mu_0^2) \mathcal{E} \left( \mu^2, \mu_0^2, N \right)
\]

\[
\mathcal{E} \left( \mu^2, \mu_0^2, N \right) = \exp \left\{ \int_{\mu_0^2}^{\mu^2} \frac{dt}{t} \frac{\alpha_s(t)}{2\pi} \gamma(N, t) \right\}
\]

(4.21)

where $f^a(N, \mu_0^2)$ is the distribution function of a parton inside a nucleon at the fixed scale $\mu_0^2$, whereas $\mathcal{E}$ is the operator of evolution from a scale $\mu_0^2$ to $\mu^2$ [48], [53], [54].
4.2.2 Modellization

As continuation of the results of the previous paragraph, we have implemented a numerical code enabling to obtain values of PDF for an arbitrary scale \( \mu^2 > \mu_0^2 \), freely choosing a parametrisation for \( f_a(N, \mu_0^2) \) (or equivalently for \( f_a(\chi, \mu_0^2) \), taking then the MT); to achieve this goal we partially made use of a code written by Dr. S. Weinzierl [55].

As preliminary test to verify the good working, we have reproduced the MRST-2001 PDF set calculating Mellin transforms for parton distributions parametrised at the scale \( \mu_0^2 = 1 \text{ GeV}^2 \), illustrated in [56];

\[
\begin{align*}
    x u_V(x) &= 0.158 x^{-0.29} (1 - x)^{3.33} (1 + 5.61x^{0.5} + 55.49x) \\
    x d_V(x) &= 0.040 x^{0.27} (1 - x)^{3.88} (1 + 52.73x^{0.5} + 30.65x) \\
    x S(x) &= 0.222 x^{-0.26} (1 - x)^{7.10} (1 + 3.42x^{0.5} + 10.30x) \\
    x g(x) &= 1.900 x^{0.09} (1 - x)^{3.70} (1 + 1.26x^{0.5} - 1.43x) - 0.21x^{-0.33} (1 - x)^{10}
\end{align*}
\]

with a structure for light flavours of the sea given by

\[
\begin{align*}
    2 \pi &= 0.4S - \Delta \\
    2 \bar{d} &= 0.4S + \Delta \\
    2 \pi &= 0.2S \Delta \\
    x \Delta &= x(\bar{d} - \bar{u}) = 1.195 x^{1.24} (1 - x)^{9.10} (1 + 14.05x - 45.52x^2)
\end{align*}
\]

We have compared our results at many scales \( \mu^2 \) to the ones provided by LHA-PDF [39] library: in Fig.4.3 we show a comparison for distributions of some partons at \( \mu^2 = 100 \text{ GeV}^2 \) as scale.

After that we have implemented also parametrisations for CTEQ6 [33] and the ones given by ZEUS collaboration.

An interesting possibility consists in being able to define also asymmetric distributions with regard to quark/antiquark pairs coming from gluon splitting of the sea; in particular this holds for \( s \) and \( \bar{s} \), up until now assumed equal by prescription in standard PDF sets.

Rewriting (4.3) in the form

\[
\mathcal{F}_k(N, M) = \sum_a f^a(N, \mu_0^2) \mathcal{C}(\mu_0^2, \mu^2) \mathcal{D}(M) \mathcal{G}^a(N, M) \quad (4.22)
\]

we can also carry out \( \mathcal{F}_k(\chi, z) \) numerically calculating the Inverse-Mellin Transform with respect to \( N \) and \( M \); through a double integral therefore is obtained the results for the two convolutions of Coefficient Functions with FF and PDF, including their evolutions, without solving by means of numerical methods integral-differential equations (4.14) and (4.16) furthermore depending on the chosen functional form for \( f(\chi, \mu_0^2) \).
Figure 4.3: Behaviour of $xf(x,\mu^2)$ provided by LHA-PDF (points) and by our code (continuous line), for a scale $\mu^2 = 100 \text{ GeV}^2$ and for each parton (up, down, strange, charm, gluon). At the top inside the figure, parametrisation MRST-2001 is taken into account, whereas at the bottom, the CTEQ-6 one.
Conclusions and Outlook

In this research we have handled in a general way the Deep Inelastic Scattering processes mediated by Charged Currents. Following the classical approach to DIS, we have parametrised an exclusive hadronic cross section with multi-differential structure functions \( (d^j F_j / \prod_i d\alpha_i) \), including in the analysis the mass of charged leptons.

In the parton model context, we have reconstructed the structure functions from the Partonic Distribution Functions (PDF) and Fragmentation Functions (FF), and in the framework of Quantum Chromodynamics (QCD) we have carried out the semi-exclusive Coefficient Functions up to Next-to-Leading Order for heavy quark production.

This result was already partially known \[29\],\[20\]; we can therefore confirm it independently and now we have the full analytic control of the model.

The obtained results have been numerically implemented using C/C++ language, in order to carry out quantitative analysis for a more complete phenomenological study. In the meantime, the theoretical uncertainties coming from physical inputs (heavy quark mass, Fragmentation Functions, parton densities, ...) or from non-physical parameters (as factorization and renormalization scales) have been investigated.

In this way we have then verified that the semi-exclusive result obtained through calculations at fixed order (NLO) is quantitatively inadequate in some kinematic regions, pointing out the need of considering also contributions originated by higher orders and then the necessity of all-orders-resummed calculation.

In order to simplify the inclusion of resummation in the analysis, we have considered the possibility to perform the convolution among Partonic Densities, Coefficient Functions and Fragmentation Functions directly in Mellin space. For this purpose we have written a C/C++ code returning evolved Mellin moments of certain parametrisations for modern PDF sets, as CTEQ6 and MRST2001.

The only open item is then the numerical treatment of analytical expressions extended to the whole complex plane for the moments of the semi-exclusive
Coefficient Functions. Results for this ingredient seem to be hard to obtain numerically and their implementation has not yet been completed.

As first application of our results, we have considered the investigation of Strange quark distribution inside nucleons. The CC DIS process with production of a hadron containing Charm is actually very useful to this purpose, because it allows to study independently the distributions of quarks and antiquarks. Therefore our semi-exclusive results permit to precisely reproduce experimentally accessible observables and to accurately measure partonic distribution of Strange quark and $s - \bar{s}$ asymmetry.

Although the usual $x$-space convolution is satisfactory with regard to the described approach, in the future we will try to carry out an effective numerical implementation of Mellin transforms for the semi-exclusive Coefficient Functions. This will allow to include in the (NLO) analysis the resummation up to Next-to-Leading Logarithm (NLL) in a natural way, so eventually we will have a theoretical result reliable and accurate in the whole kinematic region of interest.

A further development of the numerical implementation should be the inclusion of decay of charmed hadrons in muons, using a Montecarlo approach. This would enable to simulate in detail di-muonic events experimentally measured (3.2.1) and therefore to describe the phenomenology in a more accurate and complete way.
Appendix A

DIS - Details and Formalism

A.1 Electroweak Leptonic Tensor

The most generic electroweak leptonic vertex is (neglecting overall coupling constants)

\[ \langle \ell_{\text{out}}, s | j_L^\mu | \ell_{\text{in}}, r \rangle = \overline{\psi}_{\text{out}}(k_{\text{out}}) \Gamma^\mu \psi_{\text{in}}^r(k_{\text{in}}) \]

\[ = \overline{\psi}_{\text{out}}(k_{\text{out}}) \gamma^\mu \frac{(V - A \gamma_5)}{2} \psi_{\text{in}}^r(k_{\text{in}}) \]

where \( V \) and \( A \) are the appropriate couplings of the leptonic current to the electroweak bosons.

![Generic electroweak leptonic vertex.](image)

Keeping the squared modulus and summing over the polarization states \((s, r)\) of the leptons, it is straightforward to obtain

\[ L^{\mu \nu} = (V^2 + A^2) \left[ k_{\text{out}}^\mu k_{\text{in}}^\nu + k_{\text{in}}^\mu k_{\text{out}}^\nu - (k_{\text{in}} k_{\text{out}})^{\mu \nu} g_{\mu \nu} \right] 
+ 2iVA \epsilon^{\mu \alpha \nu \beta} k_{\alpha}^\mu k_{\beta}^\nu + (V^2 - A^2) m_{\text{in}} m_{\text{out}} g^{\mu \nu} \]

(A.1)

the most generic leptonic tensor.
• Electromagnetic case is carried out for $V = 2$, $A = 0$ and obviously $m_{k_{\text{in}}} = m_{k_{\text{out}}} \equiv m$:

$$L_{\text{EM}}^{\mu\nu} = 4 \left[ k_{\text{out}}^{\mu} k_{\text{in}}^{\nu} + k_{\text{in}}^{\mu} k_{\text{out}}^{\nu} - (k_{\text{in}} k_{\text{out}}) g^{\mu\nu} \right] + 4m^2 g^{\mu\nu} \quad (A.2)$$

• Charged Electroweak case is carried out for $V = 1$, $A = 1$ and the masses are allowed to be different (however their contributions disappear):

$$L_{\text{CC}}^{\mu\nu} = 2 \left[ k_{\text{out}}^{\mu} k_{\text{in}}^{\nu} + k_{\text{in}}^{\mu} k_{\text{out}}^{\nu} - (k_{\text{in}} k_{\text{out}}) g^{\mu\nu} \right] + 2i\epsilon^{\mu\alpha\beta\gamma} k_{\alpha}^{\text{in}} k_{\beta}^{\text{out}} \quad (A.3)$$

• Neutral Electroweak case is carried out for $V = g_V$, $A = g_A$ and equal masses ($m_{k_{\text{in}}} = m_{k_{\text{out}}} \equiv m$):

$$L_{\text{NC}}^{\mu\nu} = \left( g_V^2 + g_A^2 \right) \left[ k_{\text{out}}^{\mu} k_{\text{in}}^{\nu} + k_{\text{in}}^{\mu} k_{\text{out}}^{\nu} - (k_{\text{in}} k_{\text{out}}) g^{\mu\nu} \right] + 2i g_V g_A \epsilon^{\mu\alpha\beta\gamma} k_{\alpha}^{\text{in}} k_{\beta}^{\text{out}} + \left( g_V^2 - g_A^2 \right) m^2 g^{\mu\nu} \quad (A.4)$$

where $g_V$ and $g_A$ are the (leptonic) fermion couplings to $Z^0$.

If the current is made of antiparticles rather than particles, then, instead of $j_L^\mu$, we must use

$$\Gamma^\mu \rightarrow C^{-1} j_L^\mu C = -\gamma^\mu \frac{(V + A\gamma_5)}{2}$$

In this way the result comes from $(A.1)$ simply performing the change $A \rightarrow -A$ ($C$ is the charge-conjugation operator; see [54] pages 298 and 303). Notice that no averaging over the initial states (of spin) has been done.

![Figure A.2: Leptonic Tensor.](80)
A.2 Hadronic Tensor

In (1.3) we found the term
\[ \hat{W}^\alpha_\beta (2\pi)^4 \delta^{(4)}(P+q-p_X) \equiv \frac{W^\alpha_\beta}{4\pi} (2\pi)^4 \delta^{(4)}(P + q - p_X) \]

\[ = \frac{1}{4\pi} \sum_{\text{pol},X} \langle P|J_H^{\dagger}(0)|X\rangle \langle X|J_H^\alpha(0)|P\rangle (2\pi)^4 \delta^{(4)}(P + q - p_X) \]

where polarisation states are understood and the tensor has not been averaged over initial spin states (we have done expressly in the main calculation); rewriting \( \delta\text{-Dirac} \) as inverse Fourier transform

\[ \delta^{(4)}(P + q - p_X) = \frac{1}{(2\pi)^4} \int d^4t e^{i(P+q-p_X)t} \]

then

\[ H^\alpha_\beta \equiv \hat{W}^\alpha_\beta (2\pi)^4 \delta^{(4)}(P + q - p_X) \]

\[ = \frac{1}{4\pi} \int d^4t e^{iqt} \sum_{\text{pol},X} \langle P|e^{iPt}J_H^{\dagger}(0)e^{-ipXt}|X\rangle \langle X|J_H^\alpha(0)|P\rangle \]

\[ = \frac{1}{4\pi} \int d^4t e^{iqt} \langle P|J_H^{\dagger}(t)J_H^\alpha(0)|P\rangle \]

\[ = \frac{1}{4\pi} \int d^4t e^{iqt} \langle P| \left[ J_H^{\dagger}(t),J_H^\alpha(0) \right]|P\rangle \]

that is \( H^\alpha_\beta \) can be expressed by means of the four-momenta of the particles of the process. Physics of vertex involving the nucleon is supposed unknown and then it is described in the most general way with a second rank Lorentz-invariant tensor.

It is possible to construct such a tensor using only the momenta of incoming particles in the vertex (Fig.A.3); in fact the final state is fixed by four-momentum conservation \( (p' = p + q) \).

\[ H^\alpha_\beta = -W_1 g^{\alpha_\beta} + W_2 p^\alpha p^\beta \]

\[ - iW_3 \epsilon^{\alpha_\beta}_{\rho\tau} p^\rho q^\tau + W_4 q^\alpha q^\beta \]

\[ + W_5 (p^\alpha q^\beta + q^\alpha p^\beta) + iW_6 (p^\alpha q^\beta - q^\alpha p^\beta) \]

(A.5)

(cf. [58], page 532) where \( W_k \) depend on scalar quantities made of \( p \) and \( q \) (i.e. \( W_k = W_k(p^2, pq, q^2) \)). It is interesting to notice that there are not \( \gamma \)-matrices: in fact we have already summed over spin states ([57], page 180).
Figure A.3: Hadronic Tensor: incoming and outgoing momenta.

If we suppose that interaction is purely electromagnetic, then antisymmetric
terms disappear (because of parity conservation) and the current is preserved
\( q_{\alpha} H^{\alpha \beta} = q_{\beta} H^{\alpha \beta} = 0 \); in this case the tensor structure is simpler:

\[
H^{\alpha \beta}_{em} = W_1 \left( -g^{\alpha \beta} + \frac{q^{\alpha} q^{\beta}}{q^2} \right) + W_2 \left( p^{\alpha} - \frac{pq}{q^2} q^{\alpha} \right) \left( p^{\beta} - \frac{pq}{q^2} q^{\beta} \right)
\]

For parity violating processes (i.e. weak) with current conservation (i.e. for
massless particles) then \[27\]

\[
H^{\alpha \beta}_{ew} = W_1 \left( -g^{\alpha \beta} + \frac{q^{\alpha} q^{\beta}}{q^2} \right) + W_2 \left( p^{\alpha} - \frac{pq}{q^2} q^{\alpha} \right) \left( p^{\beta} - \frac{pq}{q^2} q^{\beta} \right) - iW_3 \epsilon_{\rho \tau}^{\alpha \beta} p^\rho q^\tau
\]

For the case of DIS with Charged Currents (parity violation), with massive
final states \( m_{p'} \neq 0 \Rightarrow \) not conserved current), then we must use the most
generic expression \[A.5\] It is straightforward to demonstrate that term \( W_6 \)
ever contributes when contracted even with the most generic electroweak
leptonic tensor \[A.1\], because of symmetries and anti-symmetries of involved
tensors, being \( q^{\mu} = k^{\mu}_{in} - k^{\mu}_{out} \); then the term \( i(p^{\alpha} q^{\beta} - q^{\alpha} p^{\beta})W_6 \) can be defini-
tively neglected.

Eventually we can adopt as hadronic tensor

\[
H^{\alpha \beta} = - (2pq)H_1 g^{\alpha \beta} + 4H_2 p^{\alpha} p^{\beta} - 2iH_3 \epsilon_{\rho \tau}^{\alpha \beta} p^\rho q^\tau
+ 2H_4 q^{\alpha} q^{\beta} + 2H_5 (p^{\alpha} q^{\beta} + q^{\alpha} p^{\beta})
\]  

(A.6)

that is, we have chosen a particular normalization for the coefficients.
A.3 Vector Bosons Propagator and Tensors Contraction

The most general expression for a massive \((M)\) 1-spin boson propagator is

\[
B_{\rho\sigma}(\eta) = -i \left( \frac{g_{\rho\sigma} - q_{\rho} q_{\sigma} / M^2}{q^2 - M^2 + i\epsilon} + \frac{q_{\rho} q_{\sigma} / M^2}{q^2 - M^2 / \eta + i\epsilon} \right)
\]

\[
= -i \left[ \frac{g_{\rho\sigma}}{q^2 - M^2 + i\epsilon} - \frac{(1 - \eta^{-1})q_{\rho} q_{\sigma}}{(q^2 - M^2 + i\epsilon)(q^2 - M^2 / \eta + i\epsilon)} \right]
\]

(A.7)

where \(\eta \in [0, 1]\) is the Stueckelberg gauge parameter (58, page 698). Massless case is well defined \([A.7]\) and easily recovered in the limit \(M \to 0\).

\[
\begin{align*}
\frac{\rho}{\sigma} B_{\rho\sigma}(\eta) \\
\end{align*}
\]

Figure A.4: Boson Propagator.

We calculate contraction \(L^{\mu\nu}T_{\mu\nu\rho\tau}H^{\rho\tau}\) for CC DIS, as defined respectively in

\[1.2, A.3, A.6\]

\[
L^{\mu\nu}B_{\mu\rho}(\eta)B_{\nu\tau}(\eta)H^{\rho\sigma} = L^{\mu\nu}H_{\mu\nu} + (1 - \eta^{-1})^2 \frac{(q_{\mu} L_{\rho}^{\nu})(q_{\rho} H^{\tau\tau})}{(q^2 - M_W^2 / \eta)^2}
\]

\[
- (1 - \eta^{-1}) \left[ \frac{(q_{\rho} L_{\mu}^{\nu})(q_{\nu} H^{\tau\tau}) + (q_{\mu} L_{\tau}^{\nu})(q_{\nu} H^{\rho\tau})}{(q^2 - M_W^2 / \eta)} \right]
\]

(A.8)

Notice that for the massless lepton case \((m_\ell = 0)\), only the term \(L^{\mu\nu}H_{\mu\nu}\) contributes because of \(q_{\rho} L^{\mu\nu} = 0 = q_{\mu} L^{\mu\nu}\); also for the Feynman gauge case \((\eta = 1)\) only the first term contributes, nevertheless it still depends on the lepton mass.

According to conventions of chapter 11 (see 1.5)

\[
L^{\mu\nu}H_{\mu\nu} = 8 \frac{ME}{y} \left\{ \left( xy^2 + \frac{m_\ell^2 y}{2ME} \right) H_1 v + \left[ \left( 1 - \frac{m_\ell^2}{4E^2} \right) - \left( 1 + \frac{M}{2E} x \right) y \right] 2H_2 v \right\} 2H_3 v + \left[ xy \left( 1 - \frac{y}{2} \right) - \frac{m_\ell^2 y}{4ME} \right] 2H_4 v + \left[ \frac{m_\ell^2 (m_\ell^2 + Q^2)}{4M^2 E^2} \right] H_5 v - \left[ \frac{m_\ell^2}{ME} \right] H_6 v
\]

(A.9)
We have computed the contribution of the remaining two terms in \( A.8 \) and it is easy to show that the full result (A.10) is obtained from A.9 simply multiplying each coefficient of \( H_k \) by \( [1 + \chi_k(\eta, x, y)] \), where

\[
\chi_1 = -\frac{m_2^2(1-\eta^{-1})}{(Q^2 + M_W^2/\eta)^2} \frac{2\eta}{(\eta - 1)^2} \frac{4m_1^2 E^2 y}{(\eta Q^2 + M_W^2)^2} [4(y-1) E^2 + m_1^2 + Q^2] \left[ 1 - \frac{(\eta - 1)^2}{(\eta Q^2 + M_W^2)^2} y(m_1^2 + Q^2) \right]
\]

\[
\chi_2 = \frac{m_1^2}{(\eta Q^2 + M_W^2)^2} \frac{2\eta}{(\eta - 1)^2} \frac{4m_1^2 E^2 y}{(\eta Q^2 + M_W^2)^2} [4(y-1) E^2 + m_1^2 + Q^2] \left[ 1 - \frac{(\eta - 1)^2}{(\eta Q^2 + M_W^2)^2} y(m_1^2 + Q^2) \right]
\]

\[
\chi_3 = 0
\]

\[
\chi_4 = \frac{(1-\eta)Q^2}{(\eta Q^2 + M_W^2)^2} (Q^2 + 2M_W^2 + \eta Q^2)
\]

\[
\chi_5 = -\frac{(\eta - 1)^2}{(\eta Q^2 + M_W^2)^2} \left[ Q^2 + (Q^2 + m_1^2) \frac{y}{2} \right] + \frac{(\eta - 1)^2}{(\eta Q^2 + M_W^2)^2} \left[ Q^2 (Q^2 + m_1^2) \frac{y}{2} \right]
\]

(notice that \( Q^2 \) depends on \( y \) through \( Q^2 = 2M_E x y \)). Factors \( \chi_j \) are originated from terms \( q_\rho q_\tau / M_W^2 \) of \( W \) boson propagator whilst terms containing \( m_\mu \) in A.9 come directly from kinematics.

To carry out a calculation, it is necessary to choose a particular gauge: for example \( \eta = 1 \) corresponds to Feynman gauge, \( \eta = \infty \) to Lorentz gauge, \( \eta = 0 \) to unitary gauge and so on. To avoid to introduce contributions of ghosts, we prefer to choose a physical gauge as the unitary one is ([59], chapter VI, section B), so finally we obtain

\[
\chi_1(\eta = 0) = \frac{m_1^2(Q^2 + 2M_W^2)}{2M_W^4}
\]

\[
\chi_2(\eta = 0) = -\frac{m_1^2 E^2 y [4M_W^2 + y(Q^2 + m_1^2)]}{M_W^4 [4(y-1) E^2 + m_1^2 + Q^2]}
\]

\[
\chi_3(\eta = 0) = 0
\]

\[
\chi_4(\eta = 0) = \frac{Q^2(Q^2 + 2M_W^2)}{M_W^4}
\]

\[
\chi_5(\eta = 0) = \frac{Q^2}{M_W^2} + \frac{(Q^2 + M_W^2)(Q^2 + m_1^2)}{2M_W^4} y
\]

This result agrees with the one reported in [4].
A.4 Structure Functions

In this appendix, we briefly illustrate the result of the contraction between hadronic and leptonic tensors; furthermore we analyse the phase-space for the charged lepton in the final state and we give some considerations about the general structure of functions \( d^j F_k / \prod_i d\alpha_i \).

Calculating contraction A.8 in unitary gauge \( (\chi_k = \chi_k(\eta = 0), A.10) \), the full result is

\[
L^{\mu\nu}B_{\mu\rho}(\eta = 0)B_{\nu\tau}(\eta = 0)H^{\rho\tau} = 2ME\left\{ xy^2 + \frac{m_\tau^2 y}{2ME} \right\} (1 + \chi_1)H_44v + \left[ \left( 1 - \frac{m_\tau^2}{4E^2} \right) - \left( 1 + \frac{M}{2E} \right) \right] y \\
\cdot (1 + \chi_2)H_28v + \left[ xy \left( 1 - \frac{y}{2} \right) - \frac{m_\tau^2 y}{4ME} \right] (1 + \chi_3)H_38v + \left[ \frac{m_\tau^2(m_\tau^2 + Q^2)}{4M^2E^2} \right] (1 + \chi_4)H_44v - \left[ \frac{m_\tau^2}{ME} \right] (1 + \chi_5)H_54v \right\} \tag{A.11}
\]

We introduce here the parametrisation \( f_k = A_kH_k \), with \( A_{1,5} = 4v, A_{2,3} = 8v, A_4 = 4xv \). Obviously many others exist; we have chosen it in order to obtain 1.7.

Charged Massive Lepton Phase Space

\[
d\Phi = \frac{d^4k_{\text{out}}}{(2\pi)^3} \delta_+ \left( k_{\text{out}}^2 - m_\tau^2 \right) = \frac{d^3k_{\text{out}}}{(2\pi)^3 2\sqrt{|k_{\text{out}}|^2 + m_\tau^2}} = \frac{\kappa^2 d\kappa d(1 - \cos \theta) d\phi}{2(2\pi)^3 \sqrt{\kappa^2 + m_\tau^2}}
\]

using polar coordinates and with \( \kappa \equiv |k_{\text{out}}| \).

We identify \( \theta \) as angle between the incoming lepton and the outgoing one, that is between \( k_{\text{in}} \) and \( k_{\text{out}} \). At this point we can choose a variables transformation

\[
\begin{align*}
x &= \frac{Q^2}{2Pq} = \frac{2E\left( \sqrt{\kappa^2 + m_\tau^2} - \kappa \cos \theta \right) - m_\tau^2}{2M\left( E - \sqrt{\kappa^2 + m_\tau^2} \right)} \\
y &= \frac{Pq}{P_{k_{\text{in}}}} = 1 - \frac{\sqrt{\kappa^2 + m_\tau^2}}{E}
\end{align*}
\]

according to kinematics in 1.5. It is then straightforward to calculate the Jacobian and to express muon phase-space in the form

\[
d\Phi = \frac{1}{(2\pi)^3} \frac{yME}{2} d\phi dx dy \tag{A.12}
\]
It is interesting to notice that both massive and massless muon cases give the same result. Phase-space $d\hat{\Phi}$ is then combined with $[A.11]$ and re-inserted in the cross section to obtain $[1.7]$ (and $[1.14]$ with the appropriate partonic quantities). Obviously other choices are allowed so, the most general cross section can be written

$$d^2\sigma = \frac{G_F^2 ME}{\pi} \frac{M_W^4}{(Q^2(\Upsilon, \Sigma) + M_W^2)^2} \left\{ \sum_{k=1}^{5} c_k(\Upsilon, \Sigma)f_k(\Upsilon, \Sigma)d\hat{\Phi}_X \right\} J_{\Upsilon\Sigma}d\Upsilon d\Sigma$$

with opportune $\Upsilon$, $\Sigma$, $c_k$, $f_k$, $J_{\Upsilon\Sigma}$ (Jacobian of transformation), whilst $d\Phi_X$ is the phase-space for hadronic final state $X$. If we choose to be completely inclusive over such a state, we can then identify functions $F_k$ with \( \int d\Phi_X f_k \); else rewriting $X$ phase-space as

$$d\Phi_X = d\hat{\Phi}_X \prod_i^j d\alpha_i \delta(\alpha_i - g_i(\Phi_X)) = \hat{\Phi}_X \prod_i^j d\alpha_i \quad (A.13)$$

where $g_i$ functions define $\alpha_i$ variables (observables), it is then natural to identify

$$\frac{d^j F_k}{\prod_i^j d\alpha_i} = \int d\hat{\Phi}_X f_k \quad (A.14)$$

obtaining a more exclusive result.
\section{Projectors $P^\mu_\nu$}

In order to work in the context of dimensional regularization, the expressions of projectors $P^\mu_\nu$ in a $D$-dimensional space-time are given by

\begin{align}
P^\mu_\nu_1 &= -(2pq)^2g^\mu_\nu + 4Q^2p^\mu p^\nu + 2(2pq)(p^\mu q^\nu + q^\mu p^\nu) \\
P^\mu_\nu_2 &= -Q^2(2pq)^2g^\mu_\nu + 4(D - 1)Q^4p^\mu p^\nu \\
&+ 2Q^2(D - 1)(2pq)(p^\mu q^\nu + q^\mu p^\nu) + (D - 2)(2pq)^2q^\mu q^\nu \\
P^\mu_\nu_3 &= -i\epsilon^\mu_\nu_\sigma_\tau p^\sigma q^\tau \\
P^\mu_\nu_4 &= p^\mu p^\nu \\
P^\mu_\nu_5 &= -(2pq)^2g^\mu_\nu + 4(D - 1)Q^2p^\mu p^\nu + D(2pq)(p^\mu q^\nu + q^\mu p^\nu) \\
\end{align}

When $D = 4 + 2\varepsilon$ their contractions with the partonic tensor $h^a_\mu_\nu$ \textit{(1.12)} provide

\[ P^\mu_\nu_5 h^a_\mu_\nu = p_k(\varepsilon, 2pq)h^a_k \]

where $p_k$ coefficients are

\begin{align}
p_1(\varepsilon, 2pq) &= 2(1 + \varepsilon)(2pq)^3 \\
p_2(\varepsilon, 2pq) &= 2(1 + \varepsilon)(2pq)^4 \\
p_3(\varepsilon, 2pq) &= (2pq)^2 \\
p_4(\varepsilon, 2pq) &= \frac{(2pq)^2}{2} \\
p_5(\varepsilon, 2pq) &= 2(1 + \varepsilon)(2pq)^3 \end{align} \textit{(A.16)}

It is important to remember that $q$ is the four-momentum of $W$ boson exchanged, whilst $p$ is the one of the initial state parton.

A similar formalism is used in \cite{26}.
A.6 Partonic Phase-Space

The Phase-Space (PS) for two bodies, suitable to describe a final state containing a massive particle (heavy quark, $p'$) and one massless (quark or gluon, $k'$), has the form

$$(PS)_{[2]} = \frac{d^D p'}{(2\pi)^{D-1}} \delta_+ (p'^2 - m^2) \frac{d^D k'}{(2\pi)^{D-1}} \delta_+ (k'^2) (2\pi)^D \delta^{(D)} (k' + p' - q - p)$$

(A.17)

expressed in a $D$-dimensional space-time. Degrees of freedom enumeration: 2 x 4 coordinates for momenta, minus 2 conditions for mass-shell, minus 4 because of four-momentum conservation, minus 1 rotational symmetry axis around the centre of mass [23]; at last only one independent not-trivial variable remains.

We are going to carry out an expression depending on the only one degree of freedom.

Integrating $\delta^{(D)}$ through $d^D k'$, we obtain $k' = p + q - p'$ and PS becomes

$$\int d^D k'(PS)_{[2]} = \frac{d^D p'}{(2\pi)^{D-2}} \delta_+ (p'^2 - m^2) \delta_+ ((p + q - p')^2)$$

Rewriting $d^D p' = dp'_0 d^{D-1} p'$ and integrating first $\delta$-Dirac, we recover on-mass-shell relation $p'_0 = \sqrt{p'^2 + m^2}$. Performing a variables change introducing (hyper-)spherical coordinates, then the previous expression becomes

$$\int d^D k' dp'_0 (PS)_{[2]} = \frac{|p'|^{D-2}}{2\sqrt{|p'|^2 + m^2}} d|p'| d\theta | \sin \theta |^{D-3} \frac{d\Omega_{D-3}}{(2\pi)^{D-2}} \delta_+ ((p + q - p')^2)$$

$\theta$ angle has not been defined because we have not yet chosen a framework; in literature $\theta$ is often chosen as angle between the massive quark momentum and the straight line given by $p + q = 0$ (i.e. working in the system reference of the centre of mass).

Introducing $\hat{w} \equiv \frac{(1 + \cos \theta)}{2} \in [0, 1]$, $\hat{s} \equiv (p + q)^2$ and integrating $\delta$-Dirac with $d|p'|$ (so $|p'| = \frac{\hat{s} - m^2}{2\sqrt{\hat{s}}}$), a compact result is obtained (remember $D = 4 + 2\varepsilon$)

$$d\varphi^NLO_X \equiv \int d^D k' dp'_0 d|p'| d\Omega_{D-3} (PS)_{[2]} = dw \left[ \int d\varphi^NLO_X \right]$$

$$= \frac{1}{8\pi} \frac{1}{\Gamma(1 + \varepsilon)} \left( \frac{\hat{s} - m^2}{\hat{s}} \right) \left[ \frac{(\hat{s} - m^2)^2}{4\pi\hat{s}} \right]^{-\varepsilon} \hat{w}^\varepsilon (1 - \hat{w})^\varepsilon dw$$

(A.18)
where we have already integrated the factor \(d\Omega_{D-3}\) according with G.11 in [27]. Through A.18 we emphasize the dependence only on the not-trivial variable \(\hat{w}\); such a result is in agreement with eq.4 in [23].

At this point one can integrate modulus-squared amplitudes over the whole PS to carry out an inclusive result either to remain differential in order to convolve with Fragmentation Functions. In the latter case, it is fundamental to notice that \(\hat{w}\) is not well defined: in fact it is not Lorentz invariant and furthermore it is not a fragmentation variable, that it is not possible to join in a natural way the final state parton momentum and/or energy to the corresponding observed hadron (par. 2.1.3). At Born level it is clear that \(\hat{w}\) is not well defined: in the reference system where \(p' = p + q = 0\), the massive quark produced is at rest and then it makes not sense to define a scattering angle with respect to whatever direction. Also at NLO there are difficulties in defining \(\hat{w}\) (2.21). So, in order to avoid these drawbacks, a further variable change has to be done

\[
\zeta \equiv \frac{p' \cdot p}{q \cdot p}
\]  

(A.19)
as in [29], where \(p\) is the momentum of initial state parton and \(q\) the one of \(W\) boson. The link between \(\zeta\) and \(\hat{w}\) is given by

\[
\zeta = \left[ \frac{\hat{w}(1 - \xi) + (1 - \lambda)\xi}{(1 - \lambda\xi)} \right]
\]  

(A.20)

that becomes an identity in the massless limit (\(m = 0 \rightarrow \lambda = 1\)). Working with \(\zeta\) is mathematically correct, however it is also possible to use \(\hat{w}\) in a distributional sense, prescribing

\[
f(\hat{w}, p' = 0) \equiv \delta(1 - \hat{w}) \left[ \int_0^1 f(t, p') dt \right]_{p' = 0}
\]  

(A.21)

where the artificial value \(\hat{w} = 1\) has been set in order that such a contribution is always included in convolutions; nevertheless A.21 is not a mere unjustified prescription, but it follows relating the results obtained using \(\zeta\) to those ones with \(\hat{w}\) and using eq.18 in [20]. This last remark is very important and it solves the issue about the appearance of an anomalous pole in [21].
Finally we illustrate PS for a single massive particle in the final state

\[
(PS)_{[1]} = \frac{d^4 p'}{(2\pi)^3} \delta_+ \left( p'^2 - m^2 \right) (2\pi)^4 \delta^{(4)} \left( p' - q - p \right)
\]  

(A.22)

We introduce the definition of \( \zeta \) with the prescription \( \delta \left( \zeta - \frac{p' \cdot p}{q \cdot p} \right) \) and we decompose the space-phase in this way

\[
(PS)_{[1]} = \left[ \frac{d^4 p'}{(2\pi)^3} \delta_+ \left( p'^2 - m^2 \right) (2\pi)^4 \delta^{(4)} \left( p' - q - p \right) \delta \left( \zeta - \frac{p' \cdot p}{q \cdot p} \right) \right] d\zeta
\]

\[
\equiv d\hat{\phi}^{LO}_X d\zeta
\]

(A.23)

Then easily

\[
\hat{\phi}_X^{LO} \equiv \left[ \int d\hat{\phi}^{LO}_X \right] d\zeta = 2\pi \frac{\lambda}{Q^2} \delta \left( 1 - \xi \right) \delta \left( 1 - \hat{w} \right) d\zeta
\]

(A.24)

being

\[
\xi = \frac{\hat{x}}{\lambda} \quad \lambda = \frac{Q^2}{Q^2 + m^2}
\]

where \( \hat{x} \) has been previously defined in 1.13 and \( \lambda \) introduced in 1.22.

Eventually the space-phase can be expressed through \( \hat{w} \) variable as

\[
d\hat{\phi}_X^{LO} = \left[ \int d\hat{\phi}_X^{LO} (\hat{w}) \right] d\hat{w}
\]

with

\[
\int d\hat{\phi}_X^{LO} (\hat{w}) = 2\pi \frac{\lambda}{Q^2} \delta \left( 1 - \xi \right) \delta \left( 1 - \hat{w} \right)
\]
A.7 Light-Cone Variables

Light-Cone coordinates are defined starting from a change of variables from the usual \((0, x, y, z)\) coordinates. Given a vector \(V^\mu\), its light-cone components are defined as

\[
V^\pm = \frac{V_0 \pm V_z}{\sqrt{2}}, \quad V^\perp = (V_x, V_y)
\]

then it is possible to write \(V^\mu = (V^+, V^-, V^\perp)\).

It can be easily verified that Lorentz invariant scalar products have the form

\[
V \cdot W = V^+ W^- + V^- W^+ - V^\perp \cdot W^\perp
\]

\[
V \cdot V = 2V^+ V^- - |V^\perp|^2
\]

This set of coordinates has the useful property to transform very simply under boosts along the \(z\)-axis; when a vector is boosted along this direction, light-cone coordinates show what are the large and small component of the vector (momentum).

In fact, boosting the coordinates along the \(z\) axis, a new vector \(V'\mu\) is obtained: expressing it through ordinary \((0, x, y, z)\) components

\[
V'_0 = \frac{V_0 + vV_z}{\sqrt{1 - v^2}}, \quad V'_z = \frac{vV_0 + V_z}{\sqrt{1 - v^2}}, \quad V'_x = V_x, \quad V'_y = V_y
\]

whereas using light-cone coordinates

\[
V'^\pm = V^\pm e^{\pm \psi}, \quad V'^\perp = V^\perp
\]

where \(v\) is the velocity defining the boost and the hyperbolic angle \(\psi\) is set to \(\frac{1}{2} \log \left(\frac{1 + v}{1 - v}\right)\), so \(v = \tanh \psi\); notice that if two boosts of parameters \(\psi_1\) and \(\psi_2\) are applied, the result is an overall boost \(\psi_1 + \psi_2\).

Now we apply the previous results to DIS: we fix \(P^2 = M^2\) and \(q^2 = -Q^2\) as vector-basis. Using light-cone coordinates then

\[
P^\mu = \left(\frac{M^2}{2P^+}, 0, 0\right), \quad q^\mu = \left(-\frac{Q^2}{2\eta P^+}, \frac{Q^2}{2\eta P^+}, 0\right)
\]

where \(\eta\) is defined through the implicit equation

\[
2qP = \frac{Q^2}{\eta} - \eta M^2
\]
giving \[1.23\] choosing \(\hat{p}^\perp = 0\), we define a class of reference systems so-called *collinear* \[\text{[III]}\]. In general a vector \(\hat{p}\) can be decomposed in accordance with

\[
\hat{p}^\mu = C_P P^\mu + C_q q^\mu
\]

Setting the constrains \(\hat{p}^2 = 0\) and \(\hat{p}^+ = \xi P^+\), then it is straightforward to carry out

\[
C_P = \frac{Q^2}{Q^2 + \eta^2 M^2 \xi}, \quad C_q = -\frac{M^2 \eta}{Q^2 + \eta^2 M^2 \xi}
\]

and finally

\[
\hat{p}^\mu = (\xi P^+, 0, 0)
\]

It is manifest that \(\hat{p}^- \neq \xi P^-\).

Our last remark is that rewriting \(P = (\sqrt{p^2 + m^2}, px, py, p)\) by means of light-cone coordinates

\[
P^\pm = p \pm \sqrt{p^2 + m^2}
\]

then \(P^+ > P^-\) is always satisfied (from this the definition of \(P^+\) as “large” component and of \(P^-\) as “small”).
Appendix B
Mathematical Tools

B.1 Clifford Algebra

Rules for the Dirac algebra in a $D$-dimensional space-time are

\[
\begin{align*}
\gamma_\mu \phi \gamma^\mu &= -(D - 2) \phi \\
\gamma_\mu \phi \gamma^\mu &= +4ab + (D - 4) \phi \phi \\
\gamma_\mu \phi \gamma^\mu &= -2\phi \phi - (D - 4) \phi \phi
\end{align*}
\] (B.1)

B.2 Distributions and Expansions

Some of the following expansions can be found in [26] and [20].

\[
\begin{align*}
\hat{w}^\varepsilon(1 - \hat{w})^\varepsilon &= 1 + \varepsilon \log \hat{w} + \varepsilon \log(1 - \hat{w}) + O(\varepsilon) \\
\hat{w}^\varepsilon(1 - \hat{w})^{1+\varepsilon} &= (1 - \hat{w}) (1 + \varepsilon \log \hat{w} + \varepsilon \log(1 - \hat{w})) + O(\varepsilon) \\
\hat{w}^\varepsilon(1 - \hat{w})^{-1+\varepsilon} &= \frac{\delta(1 - \hat{w})}{\varepsilon} + \frac{1}{(1 - \hat{w})_+} + \varepsilon \frac{\log \hat{w}}{1 - \hat{w}} + \varepsilon \left[ \frac{\log(1 - \hat{w})}{1 - \hat{w}} \right]_+ + O(\varepsilon^2) \\
\hat{w}^{1+\varepsilon}(1 - \hat{w})^{-1+\varepsilon} &= \frac{\delta(1 - \hat{w})}{\varepsilon} + \frac{\hat{w}}{(1 - \hat{w})_+} + \varepsilon \frac{\hat{w} \log \hat{w}}{1 - \hat{w}} + \varepsilon \hat{w} \left[ \frac{\log(1 - \hat{w})}{1 - \hat{w}} \right]_+ + O(\varepsilon^2) \\
\xi^{-\varepsilon}(1 - \xi)^{-1+2\varepsilon}(1 - \lambda \xi)^{-\varepsilon} &= \delta(1 - \xi) \left[ \frac{1}{\varepsilon} - \frac{(1 - \lambda)^\varepsilon}{2\varepsilon} - (1 - \lambda)^\varepsilon \text{Li}_2(\lambda) \right] \\
&\quad + \left[ \frac{1}{1 - \xi} \right]_+ - \varepsilon \frac{\log \xi}{1 - \xi} + \varepsilon \left[ \frac{2 \log(1 - \xi) - \log(1 - \lambda \xi)}{1 - \xi} \right]_+ + O(\varepsilon^2)
\end{align*}
\]
\[(1 - \xi)^2(1 - \lambda \xi)^{-1 - \varepsilon} = \delta(1 - \xi) \left[ \frac{1}{\varepsilon} - \frac{(1 - \lambda)^{\varepsilon}}{\varepsilon} - K_A \right] + \left[ \frac{1}{1 - \lambda \xi} \right]_+ + O(\varepsilon)\]

\[(1 - \xi)^{1 + 2\varepsilon}(1 - \lambda \xi)^{-2 - \varepsilon} = \delta(1 - \xi) \left[ \frac{1}{\varepsilon} - \frac{(1 - \lambda)^{\varepsilon}}{\varepsilon} - \frac{1 + \lambda}{\lambda} K_A - \frac{(1 - \lambda)^{\varepsilon}}{\lambda} \right] + \left[ \frac{1 - \xi}{(1 - \lambda \xi)^2} \right]_+ + O(\varepsilon)\]

\[\xi^{-\varepsilon}(1 - \xi)^{1 + 2\varepsilon}(1 - \lambda \xi)^{-\varepsilon} = (1 - \xi) \left[ 1 + \varepsilon \log \left( \frac{(1 - \xi)^2}{(1 - \lambda \xi) \xi} \right) \right] + O(\varepsilon^2)\]

Plus distributions are defined as

\[
\int_0^1 d\xi f(\xi) [g(\xi)]_+ \equiv \int_0^1 d\xi [f(\xi) - f(1)] g(\xi)
\]
\[
\int_{\hat{\zeta}_{\min}}^1 d\hat{\zeta} f(\hat{\zeta}) [g(\hat{\zeta})]_+ \equiv \int_{\hat{\zeta}_{\min}}^1 d\hat{\zeta} [f(\hat{\zeta}) - f(1)] g(\hat{\zeta})
\]
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