Hermitian-Yang-Mills equations and pseudo-holomorphic bundles on nearly Kähler and nearly Calabi-Yau twistor 6-manifolds

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Abstract

We consider the Hermitian-Yang-Mills (HYM) equations for gauge potentials on a complex vector bundle $\mathcal{E}$ over an almost complex manifold $X^6$ which is the twistor space of an oriented Riemannian manifold $M^4$. Each solution of the HYM equations on such $X^6$ defines a pseudo-holomorphic structure on the bundle $\mathcal{E}$. It is shown that the pull-back to $X^6$ of any anti-self-dual gauge field on $M^4$ is a solution of the HYM equations on $X^6$. This correspondence allows us to introduce new twistor actions for bosonic and supersymmetric Yang-Mills theories. As examples of $X^6$ we consider homogeneous nearly Kähler and nearly Calabi-Yau manifolds which are twistor spaces of $S^4$, $\mathbb{C}P^2$ and $B_4$, $\mathbb{C}B_2$ (real 4-ball and complex 2-ball), respectively. Various explicit examples of solutions to the HYM equations on these spaces are provided. Applications in flux compactifications of heterotic strings are briefly discussed.
1 Introduction

In the recent paper [1], the flux compactifications of type IIA string theory of the form $\text{AdS}_4 \times X^6$ with nearly Kähler internal spaces $X^6 = \text{Sp}(2)/\text{Sp}(1) \times \text{U}(1)$ or $\text{SU}(3)/\text{U}(1) \times \text{U}(1)$ were considered (see also [2]) and it was found that the Kaluza-Klein decoupling for the original $\text{AdS}_4$ vacua requires that the above-mentioned internal spaces are substituted by the nearly Calabi-Yau spaces $\text{Sp}(1,1)/\text{Sp}(1) \times \text{U}(1)$, $\text{SU}(1,2)/\text{U}(1) \times \text{U}(1)$ or by their compact analogues obtained by quotienting out the internal manifolds by a discrete group. In our paper, we describe various solutions of the Hermitian-Yang-Mills equations on all these four coset spaces $X^6$ that can further be used in heterotic string compactifications with non-trivial background fluxes [3]-[5].

Since their discovery, more than ten years ago, tractable flux compactifications in string theory have become a very active area of research (see e.g. [6, 7, 8] for reviews and references). This has been particularly explored in type IIB theory (see e.g. [9]-[12]), and some efforts have been devoted to moduli-fixing problem in the case of type IIA compactifications (see e.g. [13, 14, 15]) where metric fluxes can arise partially from the T-duality of NS fluxes [16, 17]. Compactifications in the presence of fluxes can be described in the language of $G$-structures on $d$-dimensional manifolds: $\text{SU}(3)$ structure for dimension $d=6$, $G_2$ for $d=7$ and $\text{Spin}(7)$ for $d=8$ (see e.g. [18]-[22] and references therein). Note that 6-manifolds with $\text{SU}(3)$ structure (i.e. $\text{SU}(3)$ holonomy of the spin connection with torsion proportional to the $H$-field) can be described in terms of conditions on torsion classes of these manifolds [19]. Due to the inclusion of the $H$-field in the geometry of the internal manifold as torsion, we have to deal with non-Kähler and in some cases non-complex manifolds.

Most research in flux compactifications was done in type II string theories (see e.g. [6]-[8] and [23]-[25] for more recent results). However, fluxes in heterotic string theory, which play a prominent role in stringy model building, have been considered as well (see e.g. [3, 4, 5], [26]-[34] and references therein). Historically, heterotic flux compactifications have been known for quite some time, starting with the works [35] in the mid-1980’s. In heterotic string compactifications one has the freedom to choose a gauge bundle since the simple embedding of the spin connection into the gauge connection is ruled out for compactifications with $dH \neq 0$. For the torsionful background, the allowed gauge bundle is restricted by the Hermitian-Yang-Mills equations [36, 37] and by the Bianchi identity for the $H$-field (anomaly cancellation). The existence of such vector bundles on some non-Kähler complex 3-folds, their stability and the procedure of solving the Hermitian-Yang-Mills equations were discussed e.g. in [28, 31, 32, 33]. Here we consider the procedure of solving the Hermitian-Yang-Mills equations on the homogeneous nearly Kähler spaces $\text{Sp}(2)/\text{Sp}(1) \times \text{U}(1)$, $\text{SU}(3)/\text{U}(1) \times \text{U}(1)$ and nearly Calabi-Yau spaces $\text{Sp}(1,1)/\text{Sp}(1) \times \text{U}(1)$, $\text{SU}(1,2)/\text{U}(1) \times \text{U}(1)$ which may serve as a local model for compact case obtained by quotienting out by a discrete group.

The above-mentioned four manifolds are twistor spaces of the four-dimensional sphere $S^4$, projective plane $\mathbb{C}P^2$ and balls $B_4 = \text{Sp}(1,1)/\text{Sp}(1) \times \text{Sp}(1)$, $\mathbb{C}B_2 = \text{SU}(1,2)/\text{SU}(1) \times \text{U}(2)$ endowed with nonintegrable almost complex structure. Hence, complex vector bundles over these twistor spaces can carry a pseudo-holomorphic structure but not the holomorphic ones. That is why we begin our discussion with the notion of the pseudo-holomorphic bundles [38] and their relations with the Hermitian-Yang-Mills equations. Then we consider the twistor space $T(M^4)$ of an oriented Riemannian 4-manifold $M^4$ along with the canonical projection $\pi : T(M^4) \to M^4$ and a complex vector bundle $E$ with an anti-self-dual connection $A$. We show that any anti-self-dual gauge field $F = dA + A \wedge A$ on $M^4$ uplifted to the gauge field $\hat{F} := \pi^* F$ on $T(M^4)$ provides a solution to the Hermitian-Yang-Mills equations on $T(M^4)$. This correspondence allows us to introduce new
twistor actions for bosonic and $\mathcal{N}=4$ supersymmetric Yang-Mills theories.

Specializing to the cases $M^4 = S^4$, $\mathbb{C}P^2$, $B_4$ and $\mathbb{C}B_2$, we describe Kähler, nearly Kähler and nearly Calabi-Yau structures on the twistor spaces $T(M^4)$ for these four cases. Various explicit solutions of the Hermitian-Yang-Mills equations on $T(M^4)$ will be written down and their applications in flux compactifications of heterotic strings will be briefly discussed.

2 Pseudo-holomorphic bundles and Hermitian-Yang-Mills equations

Notation. Let $X^{2n}$ be an oriented Riemannian $2n$-dimensional manifold and $\{e^a\}$ a local orthonormal basis of $T^*X^{2n}$, $a = 1, \ldots, 2n$. For $p$-forms on $X^{2n}$ we use the notation

$$ F_p = \frac{1}{p!} F_{a_1\ldots a_p} e^{a_1\ldots a_p} \quad \text{with} \quad e^{a_1\ldots a_p} := e^{a_1} \wedge \ldots \wedge e^{a_p}, $$

(2.1)

$$ (\ast F_p)_{a_1\ldots a_{2n-p}} = \frac{1}{p!} e_{a_1\ldots a_{2n-p} b_1\ldots b_p} F^{b_1\ldots b_p} \iff F_p \wedge \ast F_p = \frac{1}{p!} F_{a_1\ldots a_p} F^{a_1\ldots a_p} \operatorname{vol}_{2n}, $$

(2.2)

where $\ast$ is the Hodge star operator and $\operatorname{vol}_{2n} = e_1 \wedge \ldots \wedge e_{2n}$. We also use notation [19]

$$ (F_p \downarrow S_p)_{b_1\ldots b_q} = \frac{1}{p!} (F_p)_{a_1\ldots a_p} (S_p)_{a_1\ldots a_p b_1\ldots b_q} $$

(2.3)

that exploits the underlying Riemannian metric $ds^2 = \delta_{ab} e^a e^b$ with the convention that $e^{12} e^{123} = e^3$ etc.

Pseudo-holomorphic bundles. Consider an oriented $2n$-dimensional manifold $X^{2n}$ with an almost complex structure $J$ and a complex vector bundle $E$ over $X^{2n}$ endowed with a connection $A$. According to R.Bryant [38], a connection $A$ on $E$ is said to define a pseudo-holomorphic structure if it has curvature $F = dA + A \wedge A$ of type (1,1) with respect to $(\text{w.r.t.) } J$, i.e.

$$ \mathcal{F}^{0,2} = 0 = \mathcal{F}^{2,0}. $$

(2.4)

One can endow the bundle $E$ with a Hermitian metric and choose $A$ to be compatible with the Hermitian structure on $E$. If, in addition, $\omega$ is an almost Hermitian structure on $(X^{2n}, J)$ and

$$ \omega \downarrow \mathcal{F} = i \lambda \operatorname{Id}_E $$

(2.5)

with $\lambda \in \mathbb{R}$, the connection $A$ is said to be (pseudo-)Hermitian-Yang-Mills [38]. We shall consider (2.5) with $\lambda = 0$, i.e. assume $c_1(E) = 0$ since for a bundle with field strength $\mathcal{F}$ of non-zero degree one can obtain a zero-degree bundle by considering $\mathcal{F} = \mathcal{F} - \frac{1}{k} (\operatorname{tr} \mathcal{F}) \cdot 1_k$, where $k = \operatorname{rank} E$.

Hermitian-Yang-Mills equations. The Hermitian-Yang-Mills (HYM) equations\footnote{We omit the prefix ‘pseudo’ for conformity with the literature on string compactifications.} read

$$ \mathcal{F}^{0,2} = 0 \quad \text{and} \quad \omega \downarrow \mathcal{F} = 0. $$

(2.6)

In the special case of an almost complex 4-manifold $X^4$ with a metric $g$ they coincide with the anti-self-dual Yang-Mills (ASDYM) equations

$$ \ast \mathcal{F} = -\mathcal{F}, $$

(2.7)
where $*$ is the Hodge operator. Note that (2.7) is valid on manifolds $(M^4, g)$ which are not necessarily almost complex manifolds.

Recall that there are various generalizations of the first order ASDYM equations (2.7) to higher dimensions [39]-[43] with some explicit solutions (see e.g. [44]). In particular, in $d=2n=6$ one can consider the equations [43]

$$\ast F = - \omega \wedge F, \quad (2.8)$$

where $\omega$ is a two-form. Differentiating (2.8), we obtain

$$d(\ast F) + A \wedge \ast F - \ast F \wedge A + \ast H \wedge F = 0, \quad (2.9)$$

where the 3-form $H$ is defined by the formula

$$H := \ast d\omega. \quad (2.10)$$

Equations (2.9) differ from the standard Yang-Mills equations by the last term with a 3-form $H$ which can be identified with a totally antisymmetric torsion. These equations naturally appear in string theory.

For manifolds $X^6$ with an almost complex structure $J$ the equations (2.8) can be rewritten in the form (2.6) with an almost Hermitian structure $\omega$. To each solution $A$ of the HYM equation (2.6) there corresponds a pseudo-holomorphic structure on the vector bundle $E$ over $X^6$. In the case of integrable almost complex structure $J$ the first equation in (2.6) defines a holomorphic structure on $E$ and the second equation in (2.6) is the requirement of (semi)stability of the bundle $E$ [36, 37]. Thus, for complex manifolds the HYM connections $A$ (solutions to (2.6)) describe (semi)stable holomorphic bundles $E$. It would be interesting to generalize the notion of stability to pseudo-holomorphic bundles $E$ and to learn whether the second equation in (2.6) is also equivalent to an expected stability of $E$.

3 Twistor correspondence and pseudo-holomorphic bundles

**Twistor space of $M^4$.** Let us consider an oriented real four-manifold\(^2\) with a Riemannian metric $g$ and the principal bundle $P(M^4, SO(4))$ of orthonormal frames over $M^4$. The twistor space $T(M^4)$ of $M^4$ can be defined as an associated bundle [45]

$$T(M^4) = P \times_{SO(4)} SO(4)/U(2). \quad (3.1)$$

with the canonical projection

$$\pi : T(M^4) \to M^4. \quad (3.2)$$

The fibres of this bundle are two-spheres $S^2_\tau \cong SO(4)/U(2)$ which parametrize almost complex structures on the tangent spaces $T_x M^4$. As a real manifold, $T(M^4)$ has dimension six.

Another (equivalent) definition of $T(M^4)$ is obtained by considering the vector bundle $\Lambda^2 T^* M^4$ of two-forms on $M^4$. Using the Hodge operator (2.2), one can split $\Lambda^2 T^* M^4$ into the direct sum

\(^2\)It is not necessary that this manifold is almost complex. For instance, there is not any almost complex structure on the four-sphere $S^4$. 

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\( \Lambda^2 T^* M^4 = \Lambda^2_+ T^* M^4 \oplus \Lambda^2_\mp T^* M^4 \) of the subbundles of self-dual and anti-self-dual two-forms on \( M^4 \). Then the twistor space \( T(M^4) \) of \( M^4 \) can be defined as the unit sphere bundle

\[
T(M^4) = S_1(\Lambda^2_+ T^* M^4)
\]

in the vector bundle \( \Lambda^2_+ T^* M^4 \).

Note that while a manifold \( M^4 \) admits in general no almost complex structure, its twistor space \( T(M^4) \) can always be equipped with two natural almost complex structures. The first, \( J = J_+ \), introduced in [45], is integrable if and only if the Weyl tensor of \( g \) on \( M^4 \) is anti-self-dual, while the second \( J = J_- \), introduced in [46], is never integrable. In fact, \( J_+ \) and \( J_- \) differ only on \( S^2 \cong \mathbb{C}P^1 \leftrightarrow T(M^4) \) \( (J_+|_{\mathbb{C}P^1} = -J_-|_{\mathbb{C}P^1}) \) and coincide on \( T^2 \). Twistor spaces \( T(M^4) \) with an almost complex structure \( J \) can be considered as a particular case of almost complex manifolds \( X \) discussed in section 2 in the context of the pseudo-holomorphic bundles and the HYM equations.

**Pull-back of complex vector bundles from \( M^4 \) to \( T(M^4) \).** Let \( E \) be a rank \( k \) complex vector bundle over \( M^4 \) and \( A \) a connection one-form (gauge potential) on \( E \) with the curvature \( F = dA + A \wedge A \) (gauge field). Suppose that the gauge field \( F \) satisfies the ASDYM equations (2.7). Bundles \( E \) with such a connection \( A \) are called anti-self-dual [45]. Using the projection (3.2), we pull \( E \) back to a bundle \( \hat{E} := \pi^* E \) over \( T(M^4) \). In accordance with the definition of a pull-back, the connection \( \hat{A} := \pi^* A \) on \( \hat{E} \) is flat along the fibres \( \mathbb{C}P^1_\pm \) of the bundle (3.2) and we can set the components of \( \hat{A} \) along the fibres equal to zero. Thus, restrictions of the smooth vector bundle \( \hat{E} \) to fibres \( \mathbb{C}P^1_\pm \) of projection \( \pi \) are holomorphically trivial for each \( x \in M^4 \).

It was shown in [45] that if the Weyl tensor of \( (M^4, g) \) is anti-self-dual then the almost complex structure \( J = J_+ \) on the twistor space \( T(M^4) \) is integrable and \( T(M^4) \) inherits the structure of a complex analytic 3-manifold. Furthermore, it was proven that an anti-self-dual bundle \( E \) over \( M^4 \) lifts to a holomorphic bundle \( \hat{E} \) over complex \( T(M^4) \) defined by the equation \( \hat{F}^{0,2} = 0 \), where \( \hat{F} := \pi^* F \) is the pull-back to \( \hat{E} \) of an anti-self-dual (ASD) gauge field \( F \) on \( E \). In [45] it was also mentioned in a remark that one can introduce a Hermitian metric on \( T(M^4) \) such that \( \hat{F} \) will be orthogonal to the Hermitian form. However, the HYM equations were introduced later [36, 37] in a different context.

**Generalized twistor correspondence.** The essence of the canonical twistor approach is to establish a correspondence between four-dimensional space-time \( M^4 \) and complex twistor space \( T(M^4) \) of \( M^4 \). Using this correspondence, one transfers data given on \( M^4 \) to data on \( T(M^4) \) and vice versa. In twistor theory one considers holomorphic objects \( h \) on \( T(M^4) \) (Čech cohomology classes, holomorphic vector bundles etc.) and transforms them to objects \( f \) on \( M^4 \) which are constrained by some differential equations [47, 48, 45, 49]. Thus, the main idea of twistor theory is to encode solutions of some differential equations on \( M^4 \) in holomorphic data on the complex twistor space \( T(M^4) \) of \( M^4 \). In particular, solutions of the ASDYM equations on manifolds \( M^4 \) with the ASD Weyl tensor correspond to holomorphic vector bundles \( \hat{E} \) over \( T(M^4) \). However, in Donaldson theory [50] one considers the ASDYM equations on manifolds \( M^4 \) whose Weyl tensor is not restricted and it is desirable to have a twistor description of this generic case.\(^3\)

In [53] it has been shown that the vortex equations on a compact Riemann surface \( \Sigma \) are equivalent to the ASDYM equations on \( \Sigma \times \mathbb{C}P^1 \) and to the Hermitian-Yang-Mills equations on the

\(^3\)This desire is supported by ideas of the theory of harmonic maps where never integrable almost complex structure \( J_- \) on twistor spaces play a key role [46, 51]. For a recent review of this subject see e.g. [52].
twistor space $T(M^4)$ of $M^4 = \Sigma \times \mathbb{C}P^1$. In the general case, the manifold $M^4$ is not anti-self-dual and an almost complex structure on $T(M^4)$ is not integrable.\(^4\) Here, we show that this generalized twistor correspondence holds for the case of an arbitrary oriented 4-manifold $M^4$. Namely, we describe a correspondence between Hermitian vector bundles $E$ with ASD connections $A$ on an oriented 4-manifold $M^4$ and pseudo-holomorphic vector bundles $\hat{E}$ over an almost complex twistor space $T(M^4)$.

**Almost complex structure on $T(M^4)$.** We fix an open subset\(^5\) $\mathcal{U}$ of $M^4$ with coordinates $\{x^\mu\}$, $\mu = 1, \ldots, 4$, and an open subset $U = \mathbb{C}P^1 \setminus \{\infty\}$ of $\mathbb{C}P^1$ with a local complex coordinate $\zeta$. Then $U \times U$ is an open subset of $T(M^4)$. Note that over $U$ there exists a section $J = (J^\mu_\nu)$ of the bundle (3.2) (a local almost complex structure) and this allows to introduce forms of type $(p, q)$ w.r.t. $J$. Globally such an almost complex structure $J$ on $M^4$ may not exist but this is not necessary for all twistor constructions.

Let $\{\vartheta^\mu\}$ represents some orthonormal coframe on $\mathcal{U} \subset M^4$, i.e. $ds^2 = \delta_{\mu\nu}\vartheta^\mu\vartheta^\nu$. Using the canonical form of a local almost complex structure $J$, we introduce forms

$$\theta^1 := \vartheta^1 + i\vartheta^2, \quad \theta^2 := \vartheta^3 + i\vartheta^4, \quad \theta^i := \vartheta^{1} - i\vartheta^2 \quad \text{and} \quad \theta^\bar{i} := \vartheta^3 - i\vartheta^4,$$  

which provide a local basis of orthonormal $(1,0)$-forms w.r.t. $J$. Then one can introduce forms

$$\omega^1 := \frac{1}{(1 + \zeta\bar{\zeta})^2}(\theta^1 - \zeta\theta^2), \quad \omega^2 := \frac{1}{(1 + \zeta\bar{\zeta})^2}(\theta^2 + \zeta\theta^1) \quad \text{and} \quad \omega^3 := \frac{1}{1 + \zeta\bar{\zeta}}(d\zeta - \Gamma^i_+ L^i_\zeta),$$  

which may serve as the definition of an almost complex structure $J$ on $T(M^4)$ such that

$$J \omega^i = i \omega^i \quad \text{for} \quad i = 1, 2, 3.$$  

Here $\Gamma_+ = (\Gamma^i_+)$ is the self-dual part of the Levi-Civita connection on $M^4$ and $L^i_\zeta$ are holomorphic components of vector fields $L_i = L^i_\zeta\partial_\zeta + L^\bar{i}_\zeta\bar{\partial}_\zeta$ on fibres $\mathbb{C}P^1 \hookrightarrow T(M^4)$ which give a realization of the generators of the group $SU(2)$ acting on $\mathbb{C}P^1 = SU(2)/U(1)$. One can take e.g. $L_1 - iL_2 = -2i$, $L_1 + iL_2 = 2i\zeta^2$ and $L_3 = -2i\zeta$. Note that the forms (3.5) extend to a basis of globally defined forms on $T(M^4)$ of type $(1,0)$ w.r.t. $J$. That is why our discussion does not depend on the choice of local coordinates, forms etc.

**From ASDYM on $M^4$ to HYM on $T(M^4)$.** For the curvature $F = dA + A \wedge A$ of the vector bundle $E \to M^4$ we have

$$F = \frac{1}{2}(F + *F) + \frac{1}{2}(F - *F) =: F^+ + F^-,$$  

where in the basis (3.4) of local forms

$$F^+ = \frac{1}{2}(F_{11} + F_{22})(\theta^{11} + \theta^{22}) + F_{12}\theta^{12} + F_{1\bar{2}}\theta^{1\bar{2}},$$  

$$F^- = \frac{1}{2}(F_{11} - F_{22})(\theta^{11} - \theta^{22}) + F_{12}\theta^{12} + F_{2\bar{1}}\theta^{2\bar{1}}.$$  

\(^4\)In a special case, when the twistor geometry becomes integrable (holomorphic), the vortex equations on $\Sigma$ appear as a commutator of two auxiliary linear differential operators with a ‘spectral’ parameter, i.e. become integrable.

\(^5\)This subset may coincide with a point $x \in M^4$.  

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with $\theta^{11} := \theta^1 \wedge \theta^1$, $\theta^{12} := \theta^1 \wedge \theta^2$ etc. Here $F^+$ and $F^-$ are self-dual and anti-self-dual parts of the curvature $F$, respectively.

For the pull-back $\hat{F}^\pm := \pi^* F^\pm$ of the two-forms (3.8) and (3.9) on $M^4$ to $T(M^4)$ we obtain
\begin{align}
\hat{F}^+ &= \frac{1}{2} (\hat{F}_{11} + \hat{F}_{22}) (\omega^{11} + \omega^{22}) + \hat{F}_{12} \omega^{12} + \hat{F}_{1\bar{2}} \omega^{1\bar{2}}, \\
\hat{F}^- &= \frac{1}{2} (\hat{F}_{11} - \hat{F}_{22}) (\omega^{11} - \omega^{22}) + \hat{F}_{12} \omega^{1\bar{2}} + \hat{F}_{1\bar{2}} \omega^{12},
\end{align}
where $\omega^{11} := \omega^1 \wedge \omega^1$, $\omega^{12} := \omega^1 \wedge \omega^2$ etc. Note that
\begin{align}
\omega^{12} &= \frac{1}{1 + \zeta} [\theta^{12} + \bar{\zeta} (\theta^{11} + \theta^{22}) + \zeta^2 \theta^{1\bar{2}}], \quad \omega^{1\bar{2}} = \theta^{1\bar{2}}, \\
\omega^{21} &= \frac{1}{1 + \zeta} [\theta^{21} - \bar{\zeta} (\theta^{11} + \theta^{22}) + \zeta^2 \theta^{2\bar{2}}], \quad \omega^{2\bar{1}} = \theta^{2\bar{1}},
\end{align}
as one can easily derive from (3.5). Also we have
\begin{align}
\hat{F}_{12} &= \frac{1}{1 + \zeta} [F_{12} + \bar{\zeta} (F_{11} + F_{22}) + \bar{\zeta}^2 F_{1\bar{2}}], \quad \hat{F}_{12} = F_{12}, \\
\hat{F}_{1\bar{2}} &= \frac{1}{1 + \zeta} [F_{1\bar{2}} - \zeta (F_{11} + F_{22}) + \zeta^2 F_{2\bar{1}}], \quad \hat{F}_{1\bar{2}} = F_{2\bar{1}}, \\
\hat{F}_{11} + \hat{F}_{22} &= \frac{1}{1 + \zeta} [(1 - \zeta) (F_{11} + F_{22}) - 2 \zeta F_{1\bar{2}} + 2 \bar{\zeta} F_{2\bar{1}}], \quad \hat{F}_{11} - \hat{F}_{22} = F_{11} - F_{22},
\end{align}
and by construction
\begin{align}
\hat{F}_{i3} = \hat{F}_{i3} = 0 \quad \text{and} \quad \text{h.c.}
\end{align}
for $i = 1, 2, 3$.

Using (3.5), we can introduce on $T(M^4)$ an almost Hermitian form
\begin{align}
\omega = \frac{i}{2} \left( \omega^1 \wedge \omega^1 + \omega^2 \wedge \omega^2 + \varepsilon \omega^3 \wedge \omega^3 \right),
\end{align}
where $\varepsilon = \pm 1$.\(^6\) Then for $\hat{F} = \hat{F}^-$ from (3.11) it follows that
\begin{align}
\hat{F}^{0,2} = 0
\end{align}
and
\begin{align}
\omega \cdot \hat{F} = 0.
\end{align}
Thus, anti-self-dual gauge fields $F = F^-$ on the vector bundle $E \to M^4$ are pulled back to the gauge fields $\hat{F}$ on the vector bundle $\hat{E}$ over the twistor space $T(M^4)$ which satisfy the Hermitian-Yang-Mills equations (3.16), (3.17) on $T(M^4)$ without demanding integrability of an almost complex structure (3.6). In its turn, such gauge fields $\hat{F}$ define a pseudo-holomorphic structure on the vector bundle $\hat{E}$ which is holomorphically trivial on $\mathbb{CP}^2_x \hookrightarrow T(M^4)$ for each $x \in M^4$. Conversely, any such pseudo-holomorphic bundle $\hat{E} \to T(M^4)$ corresponds to a solution $A$ of the ASDYM equations on $M^4$.

\(^6\)Note that the metric for $\varepsilon = -1$ will have Hermitian signature $(2,1)$. Later we shall see that $\varepsilon = -1$ can be a proper choice for manifolds $M^4$ of negative scalar curvature.
4 Kähler geometry on twistor spaces of $S^4$ and $B_4$

Here, as $M^4$ we consider the four-sphere $S^4$ with a metric $g$ of constant positive curvature and the open four-ball $B_4$ with a metric $g$ of constant negative curvature. In the next section 5 we shall consider the projective plane $\mathbb{C}P^2$ with the Fubini-Study metric and the complex two-ball $\mathbb{C}B_2$ with the metric of constant negative holomorphic sectional curvature. All these spaces $M^4$ are homogeneous manifolds (coset spaces) as well as their twistor spaces $T(M^4)$. Although the geometry of these spaces is well-known, we describe it here by using local coordinates for fixing our notation. We also need this for self-consistency and further applications.

**Manifolds $S^4$ and $B_4$ as coset spaces.** Let us consider the group $\text{Sp}(2)$ as a subgroup of $\text{SU}(4)$ and the group $\text{Sp}(1,1)$ as a subgroup of $\text{SU}(2,2)$ defined as 4×4 matrices $Q$ such that

$$Q^\dagger \eta Q = Q \eta Q^\dagger = \eta$$

where $\varepsilon = 1$ for $\text{Sp}(2) \subset \text{SU}(4)$ and $\varepsilon = -1$ for $\text{Sp}(1,1) \subset \text{SU}(2,2)$. We consider $S^4$ and $B_4$ as coset spaces

$$S^4 = \text{Sp}(2)/\text{Sp}(1) \times \text{Sp}(1) \quad \text{and} \quad B_4 = \text{Sp}(1,1)/\text{Sp}(1) \times \text{Sp}(1)$$

of positive and negative scalar curvature, respectively. Then one can consider $\text{Sp}(2)$ fibred over $S^4$ and $\text{Sp}(1,1)$ fibred over $B_4$ as principal bundles

$$\text{Sp}(2) \to S^4 \quad \text{and} \quad \text{Sp}(1,1) \to B_4,$$

both with the structure group $\text{Sp}(1) \times \text{Sp}(1)$.

Let us consider local sections of the fibrations (4.3) which are given by 4×4 matrices

$$Q := f^{-\frac{i}{2}} \begin{pmatrix} 1_2 & -\varepsilon x \\ x^\dagger & 1_2 \end{pmatrix} \quad \text{and} \quad Q^{-1} = \eta Q^\dagger \eta = f^{-\frac{i}{2}} \begin{pmatrix} 1_2 & \varepsilon x \\ -x^\dagger & 1_2 \end{pmatrix},$$

where

$$x = x^\mu \tau_\mu, \quad x^\dagger = x^\mu \tau^\dagger_\mu, \quad f := 1 + \varepsilon x^\dagger x = 1 + \varepsilon r^2 = 1 + \varepsilon \delta_{\mu\nu} x^\mu x^\nu,$$

and matrices

$$(\tau_\mu) = (-i \sigma_i, 1_2) \quad \text{and} \quad (\tau^\dagger_\mu) = (i \sigma_i, 1_2)$$

obey

$$\tau^\dagger_\mu \tau_\nu = \delta_{\mu\nu} \cdot 1_2 + \eta^i_{\mu\nu} i \sigma_i (= \delta_{\mu\nu} \cdot 1_2 + \eta_{\mu\nu}), \quad \{\eta^i_{\mu\nu}\} = \{\varepsilon^i_{jk}, \mu = j, \nu = k; \delta^i_j, \mu = j, \nu = 4\},$$

$$\tau_\mu \tau^\dagger_\nu = \delta_{\mu\nu} \cdot 1_2 + \bar{\eta}^i_{\mu\nu} i \sigma_i (= \delta_{\mu\nu} \cdot 1_2 + \bar{\eta}_{\mu\nu}), \quad \{\bar{\eta}^i_{\mu\nu}\} = \{\varepsilon^i_{jk}, \mu = j, \nu = k; -\delta^i_j, \mu = j, \nu = 4\}. \quad (4.7)$$

Here $\{x^\mu\}$ are local coordinates on $U \subset S^4$ or $B_4$. Note that we will consistently combine formulae for both these spaces with the help of $\varepsilon = \pm 1$. Matrices (4.4) are representative elements for cosets (4.2) encoding information about their differential geometry.

**(Anti-)self-dual gauge fields.** For $M^4 = S^4$ or $B_4$, let us consider a flat connection $A$ on the trivial vector bundle $M^4 \times \mathbb{C}^4 \to M^4$ given by the one-form

$$A = Q^{-1} dQ = \begin{pmatrix} A^- & -\varepsilon \phi \\ \phi^i & A^+ \end{pmatrix},$$

(4.8)
where from (4.4) we obtain

\[ A^- = \frac{\xi}{T} \eta_{\mu} x^\mu dx^\nu =: \begin{pmatrix} \alpha_- & -\beta_- \\ \beta_- & -\alpha_- \end{pmatrix} \in su(2), \]

(4.9)

\[ A^+ = \frac{\xi}{T} \eta_{\mu} x^\mu dx^\nu =: \begin{pmatrix} \alpha_+ & -\beta_+ \\ \beta_+ & -\alpha_+ \end{pmatrix} \in su(2), \]

(4.10)

\[ \phi = \frac{1}{f} dx = -i \frac{1}{f} \left( \frac{dx_2 + i dx_4}{dx_1 + i dx_2} \right) dx_1 - i \frac{1}{f} \left( \frac{dz}{dy} - i \frac{d\bar{z}}{d\bar{y}} \right) = -i \frac{1}{2\Lambda} \left( \theta^2 + \theta^1 \right), \]  

(4.11)

with

\[ \alpha_- = \frac{\xi}{T} \bar{y} dy + \bar{z} dz - y d\bar{y} - z d\bar{z}, \quad \beta_- = \frac{\xi}{T} (y dz - z dy), \]

\[ \alpha_+ = \frac{\xi}{T} \bar{y} dy + z d\bar{z} - y d\bar{y} - \bar{z} dz, \quad \beta_+ = \frac{\xi}{T} (y d\bar{z} - \bar{z} dy), \]

\[ \theta^1 := \frac{2\Lambda dy}{1 + \epsilon r^2}, \quad \theta^2 := \frac{2\Lambda dz}{1 + \epsilon r^2}, \quad \theta^1 := \frac{2\Lambda d\bar{z}}{1 + \epsilon r^2}. \]  

(4.12)

Here, the bar denotes complex conjugation. Note that the real parameter \( \Lambda \) can be identified with the “radius” of \( M^4 = S^4 \) or \( B_4 \).

The connection (4.8) is flat, i.e.

\[ F = dA + A \wedge A = \begin{pmatrix} F^- \varepsilon \phi \wedge \phi^\dagger & -\varepsilon (d\phi + \phi \wedge A^+ + A^- \wedge \phi) \\ d\phi^\dagger + A^+ \wedge \phi^\dagger + \phi^\dagger \wedge A^- & F^+ \varepsilon \phi^\dagger \wedge \phi \end{pmatrix} = 0, \]  

(4.15)

where \( F^\pm = dA^\pm + A^\pm \wedge A^\pm \). From (4.15) we get

\[ F^- = \varepsilon \phi \wedge \phi^\dagger = -\frac{\varepsilon}{4\Lambda} \begin{pmatrix} \theta^1 \wedge \theta^1 - \theta^2 \wedge \theta^2 & 2\theta^1 \wedge \theta^2 \\ -2\theta^1 \wedge \theta^2 & -\theta^1 \wedge \theta^1 + \theta^2 \wedge \theta^2 \end{pmatrix}, \]

(4.16)

\[ F^+ = \varepsilon \phi^\dagger \wedge \phi = -\frac{\varepsilon}{4\Lambda^2} \begin{pmatrix} \theta^1 \wedge \theta^1 + \theta^2 \wedge \theta^2 & 2\theta^1 \wedge \theta^2 \\ -2\theta^1 \wedge \theta^2 & -\theta^1 \wedge \theta^1 - \theta^2 \wedge \theta^2 \end{pmatrix}. \]  

(4.17)

One can easily see that \( *F^\pm = \pm F^\pm \), i.e. \( F^+ \) and \( F^- \) are self-dual (SD) and anti-self-dual (ASD) gauge fields on a rank-2 complex vector bundle \( E \to M^4 \), respectively. They can be identified with SD and ASD parts of the Riemann tensor of the metric \( ds^2 = \theta^1 \theta^1 + \theta^2 \theta^2 \).

**Twistor manifolds of \( S^4 \) and \( B_4 \) as coset spaces.** Let us consider the Hopf bundle

\[ S^3 \to S^2 \]  

(4.18)

over the Riemann sphere \( S^2 \cong CP^1 \) and the one-monopole connection \( a \) on the bundle (4.18) having in the local coordinate \( \zeta \in CP^1 \) the form

\[ a = \frac{1}{2(1 + \zeta \bar{\zeta})} (\bar{\zeta} \, d\zeta - \zeta \, d\bar{\zeta}). \]

(4.19)

Consider a local section of the bundle (4.18) given by the matrix

\[ g = \frac{1}{(1 + \zeta \bar{\zeta})^2} \begin{pmatrix} 1 & -\bar{\zeta} \\ \zeta & 1 \end{pmatrix} \in SU(2) \cong S^3 \]  

(4.20)

8
and introduce the $su(2)$-valued one-form (flat connection)

$$\ g^{-1}dg =: \left( \begin{array}{cc} a & -\frac{1}{2\pi} \theta^3 \\ \frac{1}{2\pi} \theta^3 & -a \end{array} \right), \quad (4.21)$$

where

$$\theta^3 = \frac{2Rd\zeta}{1 + \zeta \bar{\zeta}} \quad \text{and} \quad \theta^3 = \frac{2Rd\bar{\zeta}}{1 + \zeta \bar{\zeta}} \quad (4.22)$$

are the forms of type $(1,0)$ and $(0,1)$ on $\mathbb{C}P^1$, $a$ is the one-monopole gauge potential (4.19) and $R$ is the radius of the Riemann sphere $\mathbb{C}P^1$ with the metric

$$\ ds^2_{\mathbb{C}P^1} = \theta^3 \bar{\theta}^3 = \frac{4R^2d\zeta d\bar{\zeta}}{(1 + \zeta \bar{\zeta})^2}. \quad (4.23)$$

The Kähler form on $\mathbb{C}P^1$ is

$$\omega_{\mathbb{C}P^1} = \frac{i}{2} \theta^3 \wedge \bar{\theta}^3. \quad (4.24)$$

Let us introduce $4 \times 4$ matrices

$$G = \left( \begin{array}{cc} 1 & 0 \\ 0 & g \end{array} \right) \quad \text{and} \quad \hat{Q} = QG \in \left\{ \begin{array}{ll} \text{Sp}(2) \subset \text{SU}(4) & \text{for} \quad \varepsilon = 1 \\ \text{Sp}(1,1) \subset \text{SU}(2,2) & \text{for} \quad \varepsilon = -1 \end{array} \right. \quad (4.25)$$

where $Q$ and $g$ are given in (4.4) and (4.20). The matrix $\hat{Q}$ is a local section of the bundle

$$\text{Sp}(2) \to \text{Sp}(2)/\text{Sp}(1) \times \text{U}(1) := T(S^4) \quad (4.26)$$

or

$$\text{Sp}(1,1) \to \text{Sp}(1,1)/\text{Sp}(1) \times \text{U}(1) := T(B_4), \quad (4.27)$$

depending on the choice $\varepsilon = 1$ or $\varepsilon = -1$. In (4.26) and (4.27) the twistor spaces $T(S^4)$ and $T(B_4)$ appear as coset spaces and the matrices $\hat{Q}(\varepsilon = \pm 1)$ from (4.25) are representatives for the cosets $T(S^4)$ and $T(B_4)$ which both are fibred,

$$\pi : T(M^4) \to M^4, \quad (4.28)$$

over $M^4 = S^4$ or $B_4$ with $\mathbb{C}P^1 = \text{SU}(2)/\text{U}(1)$ as a typical fibre.

**Kähler structure on twistor spaces $T(S^4)$ and $T(B_4)$**. Let us consider a trivial complex vector bundle $T(M^4) \times \mathbb{C}^4$ with the flat connection

$$\hat{A} = \hat{Q}^{-1}d\hat{Q} = G^{-1}AG + G^{-1}dG =: \left( \begin{array}{cc} \hat{A}^- & -\varepsilon \hat{\phi} \\ \hat{\phi}^\dagger & \hat{A}^+ \end{array} \right), \quad (4.29)$$

where

$$\hat{\phi} = \phi g =: -\frac{i}{2\lambda} \left( \begin{array}{cc} \omega^2 & \omega^1 \\ \omega^1 & -\omega^2 \end{array} \right), \quad \hat{A}^- = A^- = \left( \begin{array}{cc} \alpha^- & -\beta^- \\ \beta^- & -\alpha^- \end{array} \right), \quad \hat{A}^+ =: \left( \begin{array}{cc} \hat{\alpha}^+ & -\frac{1}{2R} \omega^3 \\ \frac{1}{2R} \omega^3 & -\hat{\alpha}^+ \end{array} \right), \quad (4.30)$$

with $\alpha^-, \beta^-$ given in (4.12) and

$$\hat{\alpha}^+ := \frac{1}{1 + \zeta \bar{\zeta}} \left\{ (1 - \zeta \bar{\zeta}) \alpha^+ + \bar{\zeta} \beta^+ - \zeta \bar{\beta}^+ + \frac{1}{2} (\bar{\zeta} d\zeta - \zeta d\bar{\zeta}) \right\}, \quad (4.31)$$
\[
\begin{align*}
\omega^1 &= \frac{1}{(1 + \zeta \bar{\zeta})^{1/2}} (\theta^1 - \zeta \theta^2), \\
\omega^2 &= \frac{1}{(1 + \zeta \bar{\zeta})^{1/2}} (\theta^2 + \zeta \bar{\theta}^1), \\
\omega^3 &= \frac{2R}{(1 + \zeta \bar{\zeta})^{1/2}} (d\zeta + \beta_+ - 2\zeta \alpha_+ + \zeta^2 \beta_+) .
\end{align*}
\] (4.32)

Note that forms (4.32) and (4.33) define on \( T(M^4) \) an integrable almost complex structure \([45]\)
\[ J = J_+ \] such that
\[ J \omega^i = i \omega^i \] (4.34)
with \( i = 1, 2, 3 \). In other words, \( \omega^i \)'s are (1,0)-forms w.r.t. \( J \).

From flatness of the connection (4.29), \( d\hat{A} + \hat{A} \wedge \hat{A} = 0 \), we obtain the equations
\[
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3
\end{pmatrix}
= \begin{pmatrix}
\hat{\alpha}_+ + \alpha_- & -\beta_- & \frac{1}{2R} \omega^3 \\
\beta_- & \hat{\alpha}_+ - \alpha_- & \frac{2R}{\bar{\alpha}_+} \omega^1 \\
\frac{\bar{\alpha}_+}{\bar{\alpha}_-} \omega^2 & \frac{\alpha_-}{\alpha_+} \omega^1 & \omega^2
\end{pmatrix}
\wedge
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3
\end{pmatrix}
\] (4.35)
defining the connection on \( T(M^4) \) with \( M^4 = S^4 \) or \( B_4 \). For both cases the metric on \( T(M^4) \) has the form
\[
d\xi^2 = \omega^1 \bar{\omega}^1 + \omega^2 \bar{\omega}^2 + \varepsilon \omega^3 \bar{\omega}^3
\] (4.36)
and the almost Kähler 2-form \( \omega \) reads\(^7\)
\[
\omega_\varepsilon = \frac{1}{2} (\omega^1 \wedge \bar{\omega}^1 + \omega^2 \wedge \bar{\omega}^2 + \varepsilon \omega^3 \wedge \bar{\omega}^3)
\] (4.37)
From (4.35) one obtains that the 2-form (4.37) is Kähler, i.e. \( d\omega_\varepsilon = 0 \), if and only if \( R^2 = \Lambda^2 \). In this case (4.35) defines for \( \varepsilon = 1 \) the Levi-Civita connection with \( U(3) \) holonomy group on \( T(S^4) = \text{Sp}(2)/\text{Sp}(1) \times \text{U}(1) \equiv \text{SU}(4)/\text{U}(3) \equiv \mathbb{C}P^3 \) [54]. Similarly, for \( \varepsilon = -1 \) the structure equations (4.35) define on \( T(B_4) = \text{Sp}(1,1)/\text{Sp}(1) \times \text{U}(1) \) the Levi-Civita connection for the metric (4.36) with the holonomy group \( U(2,1) \).

5 Kähler geometry on twistor spaces of \( \mathbb{C}P^2 \) and \( \mathbb{C}B_2 \)

**Manifolds \( \mathbb{C}P^2 \) and \( \mathbb{C}B_2 \) as coset spaces.** We introduce \( \mathbb{C}P^2 \) and \( \mathbb{C}B_2 \) as coset spaces
\[
\mathbb{C}P^2 = \text{SU}(3)/\text{S(U(1) \times U(2))} \quad \text{and} \quad \mathbb{C}B_2 = \text{SU}(1,2)/\text{S(U(1) \times U(2))}
\] (5.1)
and denote both of them as \( M^4 \equiv \mathbb{C}P^2 \) or \( \mathbb{C}B_2 \) with local complex coordinates \( y^\alpha, \alpha = 1, 2 \). Consider now the principal bundles
\[
\text{SU}(3) \to \mathbb{C}P^2 \] (5.2)
and
\[
\text{SU}(1,2) \to \mathbb{C}B_2 \] , (5.3)
both having the structure group \( \text{S(U(1) \times U(2))} \equiv \text{U(1) \times SU}(2) \). Local sections of the fibrations (5.2) and (5.3) are given by \( 3 \times 3 \) matrices
\[
V = \gamma^{-1} \begin{pmatrix} 1 & -\varepsilon T^1 \\ T & W \end{pmatrix} \in \begin{cases} \text{SU}(3) & \text{for} \; \varepsilon = 1 \\ \text{SU}(1,2) & \text{for} \; \varepsilon = -1 \end{cases}
\] (5.4)

\(^7\)For \( \varepsilon = -1 \) the metric (4.36) is not positive definite and one can say about pseudo-Hermitian metric, pseudo-Kähler 2-form etc. but we will avoid this.
where
\[
T := \left( \begin{array}{c} y^2 \\ y^1 \end{array} \right), \quad W := \gamma \cdot 1_2 - \frac{\varepsilon}{\gamma + 1} T T^\dagger \quad \text{and} \quad \gamma = \left( 1 + \varepsilon T T^\dagger \right)^{\frac{1}{2}} = \left( 1 + \varepsilon y^a y^a \right)^{\frac{1}{2}} > 0
\] (5.5)
ober
\[
W^1 = W, \quad WT = T \quad \text{and} \quad W^2 = \gamma^2 \cdot 1_2 - \varepsilon T T^\dagger
\] (5.6)and therefore
\[
V^\dagger \eta V = V \eta V^\dagger = \eta \quad \text{with} \quad \eta = \text{diag}(1, \varepsilon, \varepsilon).
\] (5.7)Matrices (5.4) with \( \varepsilon = \pm 1 \) are representative elements for cosets (5.2) and (5.3) encoding information about their geometry.

(Anti-)self-dual gauge fields on \( CP^2 \) and \( CB_2 \). Let us introduce a flat connection on the trivial vector bundle \( M^4 \times \mathbb{C}^4 \) given by the formula
\[
\mathcal{A} = V^{-1} dV =: \left( \begin{array}{cc} 2b & -\frac{\varepsilon}{\gamma} \theta^1 \\ \frac{2\Lambda}{\gamma^2} & \frac{\varepsilon}{\gamma} B \end{array} \right)
\] (5.8)with \( b \in u(1) \) and \( B \in u(2) \) on \( M^4 \cong CP^2 \) or \( CB_2 \), where from (4.4) we obtain
\[
b = \frac{\varepsilon}{4\gamma^2} (T^\dagger dT - dT^\dagger T) \quad \text{and} \quad B = \frac{1}{\gamma^2} (W dW - T dT^\dagger - \frac{\varepsilon}{2} dT^\dagger T - \frac{\varepsilon}{2} T^\dagger dT),
\] (5.9)
\[
\theta = \frac{2\Lambda}{\gamma^2} W dT = \left( \begin{array}{c} \theta^2 \\ \theta^1 \end{array} \right) = \frac{2\Lambda}{\gamma} \left( \begin{array}{c} dy^2 \\ -\frac{\varepsilon}{2 \gamma^2} \gamma^2 + 1 \end{array} \right) y^1 dy^1 + y^2 dy^2 
\] (5.10)Here, \( \theta^1 \) and \( \theta^2 \) are local orthonormal basis of \((1,0)\)-forms on \( CP^2 \) for \( \varepsilon = 1 \) and \( CB_2 \) for \( \varepsilon = -1 \). The real parameter \( \Lambda \) characterizes "size" of these cosets.

The flatness condition, \( d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0 \), leads to the following component equations
\[
f^- := dB = \frac{\varepsilon}{8\Lambda^2} \theta^1 \wedge \theta^1 = -\frac{\varepsilon}{8\Lambda^2} (\theta^1 \wedge \theta^1 - \theta^2 \wedge \theta^2),
\] (5.11)
\[
B := B^+ - b \cdot 1_2 = \left( \begin{array}{c} a_+ \\ b_+ \end{array} \right) - b \cdot 1_2,
\] (5.12)
\[
F = dB + B \wedge B = \frac{\varepsilon}{4\Lambda^2} \theta^1 \wedge \theta^1 = -\frac{\varepsilon}{4\Lambda^2} \left( \theta^1 \wedge \theta^1 - \theta^2 \wedge \theta^2 \right) =: F^+ - f^- \cdot 1_2,
\] (5.13)where
\[
F^+ = dB^+ + B^+ \wedge B^+ = -\frac{\varepsilon}{8\Lambda^2} \left( \begin{array}{cc} \theta^1 \wedge \theta^1 + \theta^2 \wedge \theta^2 \\ -2\theta^1 \wedge \theta^2 \end{array} \right) = -\frac{2\theta^1 \wedge \theta^2}{2\Lambda^2} = -\theta^1 \wedge \theta^1 - \theta^2 \wedge \theta^2.
\] (5.14)From (5.11) and (5.14) it follows that \( *f^- = -f^- \) and \( *F^+ = F^+ \), i.e. \( b \) is an anti-self-dual \( u(1) \)-connection on a complex line bundle over \( M^4 \) and \( B^+ \) is a self-dual \( su(2) \)-connection on a rank-2 complex vector bundle over \( M^4 \cong CP^2 \) or \( CB_2 \). Note that the field \( B^+ - b \cdot 1_2 \) can be identified with the \( u(2) \)-valued Levi-Civita connection on \( M^4 \) and then the curvature \( F^+ \) and \( f^- \cdot 1_2 \) will be the self-dual \((su(2))\)-valued) and anti-self-dual \((u(1))\)-valued) parts of the Riemannian curvature tensor of the metric \( ds^2 = \theta^1 \theta^1 + \theta^2 \theta^2 \) for \( \theta^a \) given in (5.10).
Homogeneous twistor spaces of $\mathbb{CP}^2$ and $\mathbb{CB}_2$. Twistor spaces of $\mathbb{CP}^2$ and $\mathbb{CB}_2$ are the following nonsymmetric coset spaces:

$$T(\mathbb{CP}^2) = \text{SU}(3)/\text{U}(1) \times \text{U}(1) \quad \text{and} \quad T(\mathbb{CB}_2) = \text{SU}(1, 2)/\text{U}(1) \times \text{U}(1).$$  \hspace{1cm} (5.15)

They can be described via representative matrices similar to the twistor spaces $T(S^4)$ and $T(B_4)$ discussed before. Namely, we consider again the $2 \times 2$ matrix (4.20) and $3 \times 3$ matrices

$$\hat{G} = \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix} \quad \text{and} \quad \hat{V} = V \hat{G} \in \begin{cases} \text{SU}(3) & \text{for } \varepsilon = 1 \\ \text{SU}(1, 2) & \text{for } \varepsilon = -1 \end{cases}$$  \hspace{1cm} (5.16)

for $V$ given in (5.4). The matrix $\hat{V}$ defines a local section of the bundle

$$\text{SU}(3) \to T(\mathbb{CP}^2) \quad \text{or} \quad \text{SU}(1, 2) \to T(\mathbb{CB}_2)$$  \hspace{1cm} (5.17)

depending on $\varepsilon = \pm 1$. Both fibrations (5.17) have the group $\text{U}(1) \times \text{U}(1)$ as a typical fibre. Thus, the matrices (5.16) represent the twistor coset spaces (5.15). We again have fibrations (4.28) but with $M^4 \cong \mathbb{CP}^2$ or $M^4 \cong \mathbb{CB}_2$.

Kähler structure on twistor spaces $T(\mathbb{CP}^2)$ and $T(\mathbb{CB}_2)$. Consider a trivial complex vector bundle $T(M^4) \times \mathbb{C}^4 \to T(M^4)$ with $M^4 \cong \mathbb{CP}^2$ or $\mathbb{CB}_2$. A flat connection on this bundle is defined by formula

$$\hat{A} = \hat{V}^{-1} d\hat{V} = \hat{G}^{-1} A \hat{G} + \hat{G}^{-1} d\hat{G} = \begin{pmatrix} \frac{2b}{2A} \hat{\theta} & -\frac{\hat{\varepsilon}}{2A} \hat{\theta}^\dagger \\ \frac{1}{2A} \left( \theta^2 + \bar{\theta} \bar{\theta} \right) & \frac{1}{2A} \left( \theta^1 - \bar{\theta} \bar{\theta} \right) \end{pmatrix},$$  \hspace{1cm} (5.18)

where

$$\hat{\theta} = g^\dagger \theta = \frac{1}{(1 + \zeta \bar{\zeta})} \left( \theta^2 + \bar{\zeta} \bar{\theta}^\dagger \right) = \begin{pmatrix} \omega^3 \\ \omega^1 \end{pmatrix}, \quad \hat{\theta}^\dagger = \theta^\dagger g = (\omega^2 \omega^1),$$  \hspace{1cm} (5.19)

$$\hat{B} = g^\dagger B g + g^\dagger d g = \begin{pmatrix} \hat{a}^+ + \frac{1}{2R} \omega^3 \\ \frac{1}{2A} \left( \theta^2 + \bar{\theta} \bar{\theta} \right) \end{pmatrix} - b \cdot 1_2$$  \hspace{1cm} (5.20)

with

$$\hat{a}^+ = \frac{1}{1 + \zeta \bar{\zeta}} \left( (1 - \zeta \bar{\zeta}) a^+ + \bar{\zeta} b^+ - \bar{\theta} b^+ + \frac{1}{2} (\bar{\zeta} d\zeta - d\bar{\zeta}) \right),$$  \hspace{1cm} (5.21)

$$\omega^3 = \frac{2R}{1 + \zeta \bar{\zeta}} \left( d\zeta + b^+ - 2\zeta a^+ + \bar{\theta} b^+ \right),$$  \hspace{1cm} (5.22)

and $b, a^+, b^+$ are given in (5.9)-(5.12).

From flatness of $\hat{A}$ we obtain

$$\hat{\mathcal{F}} = d\hat{A} + \hat{A} \wedge \hat{A} = \begin{pmatrix} 2db - \frac{\hat{\varepsilon}}{4A} (d\hat{\theta}^\dagger + \hat{\theta} \wedge \hat{B} - 2\hat{\theta} \wedge b) \\ \frac{1}{2A} (d\hat{\theta} + \hat{B} \wedge \hat{\theta} - 2b \wedge \hat{\theta}) \end{pmatrix} = 0$$  \hspace{1cm} (5.23)

and therefore

$$f^- = db = -\frac{\hat{\varepsilon}}{8A^2} (\omega^1 \wedge \omega^2 \wedge \omega^3) = -\frac{\hat{\varepsilon}}{8A^2} (\theta^1 \wedge \theta^2 \wedge \theta^3),$$  \hspace{1cm} (5.24)

$$\hat{f}^+ = -\frac{\hat{\varepsilon}}{8A^2} \begin{pmatrix} \omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^3 \\ -2\omega^1 \wedge \omega^2 \wedge \omega^2 \wedge \omega^3 \end{pmatrix}$$  \hspace{1cm} (5.25)
along with
\[ d\theta + (\dot{B} - 2b \cdot \mathbf{1}_2) \wedge \theta = 0. \] (5.26)

The metric and an almost Kähler structure on \( T(CP^2) \) and \( T(CB_2) \) read
\[ ds^2 = \omega^1 \omega^1 + \omega^2 \omega^2 + \varepsilon \omega^3 \omega^3 \quad \text{and} \quad \omega = \frac{i}{2} (\omega^1 \wedge \omega^1 + \omega^2 \wedge \omega^2 + \varepsilon \omega^3 \wedge \omega^3), \] (5.27)
where \( \omega^i \)'s are given in (5.19) and (5.22). From (5.23)-(5.26) it follows that \( \omega \) is Kähler, i.e. \( d\omega = 0 \), if\( R^2 = 2\Lambda^2 \). Furthermore, from (5.23)-(5.26) we obtain the structure equations
\[ d\begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} = \begin{pmatrix} \dot{a}_+ + 3b & 0 & \frac{1}{2\pi} \omega^2 \\ 0 & \dot{a}_+ - 3b & -\frac{1}{2\pi} \omega^1 \\ -\frac{1}{2\pi} \omega^2 & \frac{1}{2\pi} \omega^1 & 2\dot{a}_+ \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} \] (5.28)
which define the Levi-Civita \( U(3) \) connection on \( T(CP^2) \) and the Levi-Civita \( U(1,2) \) connection on \( T(CB_2) \).

6 Nearly Kähler and nearly Calabi-Yau twistor spaces

Definitions. Let us consider an oriented 6-dimensional manifold \( X^6 \) with a Riemannian metric \( g \) and an almost complex structure \( J \) (\( U(3) \)-structure). We may choose a local orthonormal basis \( \{ e^a \} \) of \( T^*X^6 \) with \( a = 1, ..., 6 \) such that the metric and the fundamental 2-form \( \omega \) read
\[ ds^2 = \delta_{ab} e^a e^b, \] (6.1)
\[ \omega = e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6 \] (6.2)
and
\[ J e^1 = -e^2, \quad J e^3 = -e^4 \quad \text{and} \quad J e^5 = -e^6. \] (6.3)
Then forms \( \Theta^i \) of type \( (1,0) \) w.r.t. \( J \) read
\[ \Theta^1 = e^1 + ie^2, \quad \Theta^2 = e^3 + ie^4 \quad \text{and} \quad \Theta^3 = e^5 + ie^6, \] (6.4)
so that
\[ J \Theta^i = i\Theta^i \] (6.5)
and
\[ ds^2 = \Theta^1 \Theta^1 + \Theta^2 \Theta^2 + \Theta^3 \Theta^3 \quad \text{and} \quad \omega = \frac{i}{2} (\Theta^1 \wedge \Theta^1 + \Theta^2 \wedge \Theta^2 + \Theta^3 \wedge \Theta^3). \] (6.6)

We assume that \( c_1(X^6) = 0 \) and introduce a \( (3,0) \)-form
\[ \Omega := \Theta^1 \wedge \Theta^2 \wedge \Theta^3 = \Re \Omega + i \Im \Omega = e^{135} + e^{145} + e^{245} + e^{146} + e^{246} + i(e^{136} + e^{146} + e^{156} + e^{234}). \] (6.7)

So, our manifold \( X^6 \) has an \( SU(3) \) structure defined by nowhere vanishing forms \( \omega \) and \( \Omega \). Such a manifold is called \textit{nearly Kähler} if \( \omega \) and \( \Omega \) satisfy
\[ d\omega = 3c \Im \Omega \quad \text{and} \quad d\Omega = 2c \omega \wedge \omega \] (6.8)
with a constant $c \in \mathbb{R}$. A manifold $(X^6, J, \omega, \Omega)$ is called nearly Calabi-Yau manifold [22] if
\[ \text{d} \omega = 0 \quad \text{and} \quad \text{d} \text{Im} \Omega = 0. \] (6.9)

For more details see [19]-[22], [38], [55]-[62].

In two previous sections we have described Kähler structures on the twistor spaces $\mathcal{T}(S^4)$, $\mathcal{T}(\mathbb{C}P^2)$, $\mathcal{T}(B_4)$ and $\mathcal{T}(\mathbb{C}B_2)$ endowed with integrable almost complex structures. In this section we provide these spaces with never integrable almost complex structures and introduce on them nearly Kähler or nearly Calabi-Yau structure.

**Nearly Kähler structure on** $\mathcal{T}(S^4)$. Consider the almost Kähler twistor space $\mathcal{T}(S^4)$ with the complex structure $\mathcal{J} = \mathcal{J}_+ [45]$ such that $\mathcal{J}_+ \omega^i = i \omega^i$ with $(1,0)$-form $\omega^i$ given in (4.32), (4.33). Let us introduce the forms
\[ \Theta^1 := \omega^1, \quad \Theta^2 := \omega^2 \quad \text{and} \quad \Theta^3 := \omega^3, \] (6.10)
which are forms of type $(1,0)$ w.r.t. an almost complex structure $\mathcal{J} = \mathcal{J}_+$ [46], $\mathcal{J} \Theta^i = i \Theta^i$, defined in (6.5). Note that in terms of $\{e^a\}$ we have
\[ \mathcal{J}_\pm e^1 = -e^2, \quad \mathcal{J}_\pm e^3 = -e^4 \quad \text{and} \quad \mathcal{J}_\pm e^5 = \pm e^6. \] (6.11)

Here and in the following we consider $\mathcal{J} = \mathcal{J}_+$ which is never integrable almost complex structure.

From (4.35) with $\varepsilon = 1$ we get
\[ \text{d} \begin{pmatrix} \Theta^1 \\ \Theta^2 \\ \Theta^3 \end{pmatrix} = \begin{pmatrix} \hat{\alpha}_+ + \alpha_- & \beta_- & -\beta_- \\ \beta_- & \hat{\alpha}_+ - \alpha_- & 0 \\ 0 & 0 & -2\hat{\alpha}_+ \end{pmatrix} \wedge \begin{pmatrix} \Theta^1 \\ \Theta^2 \\ \Theta^3 \end{pmatrix} + \frac{1}{2R} \begin{pmatrix} \Theta^2 \wedge \Theta^3 \\ \Theta^3 \wedge \Theta^1 \\ \frac{2\Theta^2 \wedge \Theta^1 \wedge \Theta^2} {\Lambda^2} \end{pmatrix}, \] (6.12)
where the first term defines the $su(2) \oplus u(1)$ (torsionful) connection and the last term defines the Nijenhuis tensor (torsion) with components $N^i_{jk}$ and their complex conjugate. Namely, we have
\[ N^2_{23} = N^3_{31} = \frac{1}{2R} \quad \text{and} \quad N^3_{12} = \frac{R}{\Lambda^2}. \] (6.13)

From (6.12) it follows that the manifold $(\mathcal{T}(S^4), \mathcal{J}, \omega, \Omega)$ is nearly Kähler, i.e. $\omega$ and $\Omega$ from (6.6) and (6.7) satisfy the equations (6.8), if $R^2 = \frac{1}{2} \Lambda^2$ and $c = \frac{1}{2R}$. In this case we have $N^3_{12} = \frac{1}{2R}$ and therefore the components
\[ N_{ijk} = \delta_{il} N^l_{jk} = \frac{1}{2R} \varepsilon_{ijk} \quad \text{and} \quad N_{ijk} = \frac{1}{2R} \varepsilon_{ijk} \] (6.14)
are totally antisymmetric. The connection with torsion $T = \frac{1}{4} N$ has holonomy contained in SU(3). Recall that the $(3,0)$-form $\Omega$ from (6.7) is a nowhere vanishing global section of the canonical bundle of $\mathcal{T}(S^4)$ which is a trivial bundle since the first Chern class of $\mathcal{T}(S^4)$ vanishes, $c_1(\mathcal{T}(S^4)) = 0$.

**Nearly Kähler structure on** $\mathcal{T}(\mathbb{C}P^2)$. For the manifold $\mathcal{T}(\mathbb{C}P^2)$ we use the same redefinition (6.10) but with $\omega^i$ given by (5.18)-(5.22). This endows $\mathcal{T}(\mathbb{C}P^2)$ with a nonintegrable almost complex
structure defined by (6.3)-(6.5). Then from (5.23)-(5.26) with \( \varepsilon = 1 \) we obtain the structure equations

\[
d \begin{pmatrix} \Theta^1 \\ \Theta^2 \\ \Theta^3 \end{pmatrix} = \begin{pmatrix} \hat{a}_+ + 3b & 0 & 0 \\ 0 & \hat{a}_+ - 3b & 0 \\ 0 & 0 & -2\hat{a}_+ \end{pmatrix} \wedge \begin{pmatrix} \Theta^1 \\ \Theta^2 \\ \Theta^3 \end{pmatrix} + \frac{1}{2R} \begin{pmatrix} \Theta^2 \wedge \Theta^3 \\ \Theta^3 \wedge \Theta^1 \\ \Theta^1 \wedge \Theta^2 \end{pmatrix},
\]

(6.15)

where the first term defines \( u(1) \oplus u(1) \) connection and the last term defines torsion with \( N_{12}^1 = N_{31}^2 = \frac{1}{2R} \) and \( N_{13}^3 = \frac{R}{2R^2} \). For \( T(\mathbb{C}P^2) \), the conditions (6.8) for a manifold to be nearly Kähler yield \( R^2 = \Lambda^2 \) that follows from (6.15). Furthermore, for \( R^2 = \Lambda^2 \) one has

\[
N_{ijk} = \frac{1}{2R} \varepsilon_{ijk} \quad \text{and} \quad N_{ij} = \frac{1}{2R} \varepsilon_{ij} ,
\]

(6.16)

so that \( T = \frac{1}{N} \) is a totally antisymmetric torsion.

**Nearly Calabi-Yau structure on** \( T(B_4) \). On \( T(B_4) \) we consider the redefinition (6.10) with \( \omega^i \) from (4.32), (4.33) and \( \alpha_+, \beta_+, \bar{\theta}_1, \bar{\theta}_2 \) given by (4.13), (4.14) with \( \varepsilon = -1 \). This redefinition again corresponds to the choice of the nonintegrable almost complex structure (6.3)-(6.5) and \( c_1(T(B_4)) = 0 \). Then from (4.35) with \( \varepsilon = -1 \) one obtains the equations

\[
d \begin{pmatrix} \Theta^1 \\ \Theta^2 \\ \Theta^3 \end{pmatrix} = \begin{pmatrix} \hat{a}_+ + \alpha_- - \beta_- \\ \hat{\beta}_- + \alpha_- - \beta_- \\ 0 \end{pmatrix} \wedge \begin{pmatrix} \Theta^1 \\ \Theta^2 \\ \Theta^3 \end{pmatrix} + \frac{1}{2R} \begin{pmatrix} \Theta^2 \wedge \Theta^3 \\ \Theta^3 \wedge \Theta^1 \\ -\frac{2R^2}{\Lambda^2} \Theta^1 \wedge \Theta^2 \end{pmatrix},
\]

(6.17)

with the \( u(2) \) torsional connection defined by the first term and the Nijenhuis tensor \( N_{jk}^i \) defined by the second term. From (6.17) one readily derives that \( (\omega, \Omega) \) on \( T(B_4) \) satisfy the nearly Calabi-Yau requirements (6.9) if and only if \( R^2 = \Lambda^2 \). Note also that in this case

\[
d \omega = \frac{1}{\Lambda R} (R^2 - \Lambda^2) \, \text{Im} \, \Omega = 0 \quad \text{for} \quad R^2 = \Lambda^2 ,
\]

(6.18)

\[
d \Omega = -\frac{1}{2R} (2\Theta^1 \wedge \Theta^2 \wedge \Theta^1 \wedge \Theta^2 - \Theta^1 \wedge \Theta^3 \wedge \Theta^1 \wedge \Theta^3 - \Theta^2 \wedge \Theta^3 \wedge \Theta^2 \wedge \Theta^3) \in \Lambda^{2,2}(T(B_4)),
\]

(6.19)

and therefore\(^8\)

\[
\bar{\partial} \Omega = 0.
\]

(6.20)

Thus, we again obtain a manifold with vanishing first Chern class and \( \text{SU}(3) \) structure. The manifold \( T(B_4) \) has negative scalar curvature and can, in principle, be used in string compactifications to the de Sitter space-time [1]. Compact twistor spaces with negative scalar curvature can be obtained from \( T(B_4) \) via the quotients of \( B_4 \) by a discrete isometry group.

**Nearly Calabi-Yau space** \( T(\mathbb{C}B_2) \). In this case we consider the redefinition (6.10) with \( \omega^i \) from (5.19), (5.22) and \( \theta^a \) given by (5.10) with \( \varepsilon = -1 \). From (5.28) with \( \varepsilon = -1 \) we obtain the structure equations

\[
d \begin{pmatrix} \Theta^1 \\ \Theta^2 \\ \Theta^3 \end{pmatrix} = \begin{pmatrix} \hat{a}_+ + 3b & 0 & 0 \\ 0 & \hat{a}_+ - 3b & 0 \\ 0 & 0 & -2\hat{a}_+ \end{pmatrix} \wedge \begin{pmatrix} \Theta^1 \\ \Theta^2 \\ \Theta^3 \end{pmatrix} + \frac{1}{2R} \begin{pmatrix} \Theta^2 \wedge \Theta^3 \\ \Theta^3 \wedge \Theta^1 \\ -\frac{2R^2}{\Lambda^3} \Theta^1 \wedge \Theta^2 \end{pmatrix},
\]

(6.21)

\(^8\)Recall that \( \Omega \equiv \Omega^{1,0} \) and \( d\Omega = (d^{1,0} + d^{0,1} + d^{-1,2} + d^{1,-1}) \Omega \), where \( d^{1,0} = \bar{\partial} \) and \( d^{0,1} = \partial \). On nearly Kähler manifolds (6.20) is also satisfied due to (6.8).
defining the \( u(1) \oplus u(1) \) connection and the Nijenhuis torsion \( N^i_{jk} \) on \( T(\mathbb{C}B_2) \). From (6.21) we obtain
\[
d\omega = \frac{1}{2\Lambda R} (2\Lambda^2 - R^2) \text{Im} \Omega \quad \text{and} \quad d\text{Im} \Omega = 0 ,
\] so that \( T(\mathbb{C}B_2) \) is a nearly Calabi-Yau space iff \( R^2 = 2\Lambda^2 \). Compact analogues of this manifold with an SU(3) structure can be obtained via quotients of \( \mathbb{C}B_2 \) by a discrete isometry group.

7 Hermitian-Yang-Mills gauge fields on twistor spaces of \( S^4 \), \( \mathbb{C}P^2 \), \( B_4 \) and \( \mathbb{C}B_2 \)

We have described Kähler, nearly Kähler and nearly Calabi-Yau structures on the twistor spaces \( T(S^4) \), \( T(\mathbb{C}P^2) \), \( T(B_4) \) and \( T(\mathbb{C}B_2) \). Now we will discuss in more details some explicit solutions of the Hermitian-Yang-Mills equations defined on bundles \( \hat{E} \) over these manifolds.

Kähler \( T(S^4) \) and \( T(B_4) \). Let us consider forms \( \omega^i \) of type (1,0) w.r.t. \( J = J_+ \) given in (4.32), (4.33), the metric (4.36) and the (almost) Kähler (1,1)-form (4.37). Consider again the flat connection (4.29) for which we have
\[
\hat{F} = d\hat{A} + \hat{A} \wedge \hat{A} = \begin{pmatrix} \hat{F}^- - \varepsilon \hat{\phi} \wedge \hat{\phi}^\dagger & -\varepsilon (d\hat{\phi} + \hat{\phi} \wedge \hat{A}^\dagger + \hat{A} \wedge \hat{\phi}) \\ d\hat{\phi}^\dagger + \hat{A}^\dagger \wedge \hat{\phi}^\dagger + \hat{\phi} \wedge \hat{A}^- & \hat{F}^+ - \varepsilon \hat{\phi}^\dagger \wedge \hat{\phi} \end{pmatrix} = 0 ,
\] where \( \hat{\phi} \) and \( \hat{A}^\pm \) are given in (4.30). From (7.1) it follows that
\[
\hat{F}^- = \varepsilon \hat{\phi} \wedge \hat{\phi}^\dagger = -\frac{\varepsilon}{4\Lambda^2} \begin{pmatrix} \omega^1 \wedge \omega^1 - \omega^2 \wedge \omega^2 & 2\omega^1 \wedge \omega^2 \\ -2\omega^1 \wedge \omega^2 & -\omega^1 \wedge \omega^1 + \omega^2 \wedge \omega^2 \end{pmatrix} = \frac{\varepsilon}{4\Lambda^2} \begin{pmatrix} \theta^1 \wedge \theta^1 - \theta^2 \wedge \theta^2 & 2\theta^1 \wedge \theta^2 \\ -2\theta^1 \wedge \theta^2 & -\theta^1 \wedge \theta^1 + \theta^2 \wedge \theta^2 \end{pmatrix} = \varepsilon \phi \wedge \phi^\dagger = F^- ,
\]
\[
\hat{F}^+ = \varepsilon \phi^\dagger \wedge \phi = g^\dagger F^+ g = -\frac{\varepsilon}{4\Lambda^2} \begin{pmatrix} \omega^1 \wedge \omega^1 + \omega^2 \wedge \omega^2 & 2\omega^1 \wedge \omega^2 \\ -2\omega^1 \wedge \omega^2 & -\omega^1 \wedge \omega^1 - \omega^2 \wedge \omega^2 \end{pmatrix} = -\frac{\varepsilon}{2\Lambda^2 (1+\zeta)} \begin{pmatrix} \frac{1}{2}(1-\zeta \bar{\zeta})(\theta^{11}+\theta^{22})+\zeta \bar{\theta}^{12} & \theta^{12} \bar{\zeta}^2 \theta^{12} \\ -[\theta^{12}+\zeta (\theta^{11}+\theta^{22})+\zeta^2 \theta^{12}] & -\frac{1}{2}(1-\zeta \bar{\zeta})(\theta^{11}+\theta^{22}) - \zeta \theta^{12} \bar{\zeta}^{12} \end{pmatrix}.
\] Recall that we use hats for fields on \( T(M^4) \) and denote fields on \( M^4 \) by letters without hats.

From (7.2) it follows that
\[
(\hat{F}^-)^{0,2} = 0 \quad \text{and} \quad \omega \wedge \hat{F}^- = 0 ,
\] i.e. the \( su(2) \)-valued gauge field \( \hat{F}^- \) satisfies the HYM equations on \( T(M^4) \) with \( M^4 = S^4 \) or \( B_4 \). This solution is a pull-back to \( T(M^4) \) of the ASD gauge field \( F = F^- \) on \( M^4 = S^4 \) or \( B_4 \). However, there are solutions of the HYM equations on \( T(M^4) \) which are not lifted from instantons on \( M^4 \). To give an example, we rewrite the flat connection (4.29) in the form
\[
\hat{A} = \begin{pmatrix} \hat{A} \\ \hat{T} \\ \hat{A}_+ \end{pmatrix} \quad \text{with} \quad \hat{T} = \frac{i}{2\Lambda} (\omega^1, -\omega^2, -i\omega^3) \quad \text{and} \quad \hat{T}_c = -\frac{i}{2\Lambda} \begin{pmatrix} \omega^1 \\ -\omega^2 \\ i\omega^3 \end{pmatrix} .
\]
Then from the flatness condition
\[
\hat{F} = d\hat{A} + \hat{A} \wedge \hat{A} = \left( \begin{array}{cc}
\hat{F} - \varepsilon \hat{T}_c \wedge \hat{T} & -\varepsilon [d\hat{T}_c + (\hat{A} + \hat{\alpha}_+) \wedge \hat{T}_c] \\
\hat{d}\hat{T} + \hat{T} \wedge (\hat{A} + \hat{\alpha}_+) & -(d\hat{\alpha}_+ + \varepsilon \hat{T} \wedge \hat{T}_c)
\end{array} \right) = 0
\] (7.6)
it follows that the Yang-Mills field
\[
\hat{F} = d\hat{A} + \hat{A} \wedge \hat{A} \equiv \frac{\varepsilon}{4\Lambda^2} \left( \begin{array}{ccc}
-\omega^{11} & \omega^{21} & i\omega^{31} \\
\omega^{12} & -\omega^{22} & -i\omega^{32} \\
-i\varepsilon \omega^{13} & i\varepsilon \omega^{23} & -\varepsilon \omega^{33}
\end{array} \right)
\] (7.7)
satisfies the equations
\[
\hat{F}^{0,2} = 0 \quad \text{and} \quad \omega \cdot \hat{F} = -\frac{\varepsilon}{4\Lambda^2} \cdot 1_3.
\] (7.8)
Therefore the connection \( \hat{A} = \hat{A} - \frac{1}{3} (\text{tr} \hat{A}) \cdot 1_3 \) with the curvature \( \hat{F} = \hat{F} - \frac{1}{3} (\text{tr} \hat{F}) \cdot 1_3 \) satisfies the HYM equations
\[
\hat{F}^{0,2} = 0 \quad \text{and} \quad \omega \cdot \hat{F} = 0.
\] (7.9)
From (7.7) one sees that \( \hat{F} \) and \( \tilde{F} \) have nonvanishing components along \( \mathbb{CP}^1 \hookrightarrow T(M^4) \) and hence they cannot be obtained by the pull-back of an ASD gauge field on \( M^4 = S^4 \) or \( B_4 \).

**Kähler** \( T(\mathbb{CP}^2) \) and \( T(\mathbb{CB}_2) \). In this case from (5.18) with \( \check{B} = \check{B} + b \cdot 1_2 \) and (5.23) it follows that the Abelian gauge potential
\[
\check{B}^* := \text{diag}(b, b)
\] (7.10)
satisfies the HYM equations for \( \check{F}^* := d\check{B}^* \),
\[
(\check{F}^*)^{0,2} = 0 \quad \text{and} \quad \omega \cdot \check{F}^* = 0
\] (7.11)
since
\[
db = -\frac{\varepsilon}{8\Lambda^2} (\omega^1 \wedge \omega^5 - \omega^2 \wedge \omega^6) \quad \Leftrightarrow \quad \omega \cdot db = 0.
\] (7.12)

**Nearly Kähler** \( T(S^4) \) and \( T(\mathbb{CP}^2) \). Recall that an SU(3)-structure \( (\mathcal{T}(S^4), \omega, \Omega) \) is nearly Kähler if \( R^2 = \frac{1}{2} \Lambda^2 \) and an SU(3)-structure \( (\mathcal{T}(\mathbb{CP}^2), \omega, \Omega) \) is nearly Kähler if \( R^2 = \Lambda^2 \). Assuming this and substituting (6.10) into (7.2), we obtain that
\[
\tilde{F}^* = -\frac{1}{4\Lambda^2} \left( \begin{array}{ccc}
\Theta^1 \wedge \Theta^1 - \Theta^2 \wedge \Theta^2 & 2\Theta^1 \wedge \Theta^2 & -\Theta^1 \wedge \Theta^1 + \Theta^2 \wedge \Theta^2 \\
-2\Theta^1 \wedge \Theta^2 & 2\Theta^1 \wedge \Theta^2 & -\Theta^1 \wedge \Theta^1 + \Theta^2 \wedge \Theta^2
\end{array} \right).
\] (7.13)
is a solution of the HYM equations on \( \mathcal{T}(S^4) = \text{Sp}(2)/\text{Sp}(1) \times U(1) \) which is essentially the same as (7.2). At the same time, the analogue of 3×3 matrix \( \tilde{F} \) from (7.7) does not satisfy the HYM equations on the nearly Kähler space \( \mathcal{T}(S^4) \) contrary to the Kähler case. On the other hand, on the nearly Kähler space \( \mathcal{T}(\mathbb{CP}^2) \) we have two canonical Abelian connections satisfying the HYM equations on \( \mathcal{T}(\mathbb{CP}^2) \),
\[
\check{B}_1^- = \text{diag}(b, b) \quad \text{with} \quad db = -\frac{1}{8\Lambda^2} (\Theta^1 \wedge \Theta^1 - \Theta^2 \wedge \Theta^2)
\] (7.14)
and
\[
\check{B}_2^- = \text{diag}(\hat{a}_+, -\hat{a}_+) \quad \text{with} \quad d\hat{a}_+ = -\frac{1}{8\Lambda^2} (\Theta^1 \wedge \Theta^1 + \Theta^2 \wedge \Theta^2 - 2\Theta^3 \wedge \Theta^3),
\] (7.15)
where \( b \) and \( \hat{a}_+ \) are introduced in (5.18)-(5.21). Note that the Abelian gauge potential \( b \) is pulled back from \( \mathbb{C}P^2 \) but \( \hat{a}_+ \) is not.

**Nearly Calabi-Yau \( T(B_4) \) and \( T(\mathbb{C}B_2) \).** Recall that forms \( \omega \) and \( \Omega \) define a nearly Calabi-Yau structure on an almost complex manifold \( X^6 \) if they obey equations (6.9). For the twistor space \( T(B_4) \) this yields \( R^2 = \Lambda^2 \) and the twistor space \( T(\mathbb{C}B_2) \) is a nearly Calabi-Yau manifold if \( R^2 = 2\Lambda^2 \). Assuming this and substituting (6.10) into (7.1) with \( \varepsilon = -1 \), we obtain that the gauge field

\[
\hat{F}^\pm = d\hat{A}^\pm + \hat{A}^- \wedge \hat{A}^+ = \frac{1}{4\Lambda^2} \left( \Theta^1 \wedge \Theta^1 - \Theta^2 \wedge \Theta^2 \right) + \frac{1}{2\Lambda^2} \left( \Theta^1 \wedge \Theta^2 + \Theta^2 \wedge \Theta^1 \right)
\]

(7.16)

satisfies the HYM equations (7.4) on \( T(B_4) \). Note that (7.16) differs by sign from (7.13).

Similarly, on nearly Calabi-Yau space \( T(\mathbb{C}B_2) \) we have the Abelian Hermitian-Yang-Mills connection

\[
\hat{B}^- = \text{diag}(b, b) \quad \text{with} \quad db = \frac{1}{8\Lambda^2} (\Theta^1 \wedge \Theta^1 - \Theta^2 \wedge \Theta^2)
\]

(7.17)

which is the pull-back of an Abelian anti-self-dual gauge potential on \( \mathbb{C}B_2 \).

**Lifted ASD gauge fields.** In section 3 we have shown that anti-self-dual gauge fields \( F = F^- \) on any oriented Riemannian 4-manifolds \( M^4 \) are pulled back to Hermitian-Yang-Mills gauge fields on the twistor space \( T(M^4) \) of \( M^4 \) with an almost complex structure \( \mathcal{J} = \mathcal{J}_+ \). The same is true for the twistor spaces \( T(S^4) \), \( T(\mathbb{C}P^2) \), \( T(B_4) \) and \( T(\mathbb{C}B_2) \) with the never integrable almost complex structure \( \mathcal{J} = \mathcal{J}_- \) since \( \hat{F} = \pi^* F \) has no components along \( \mathbb{C}P^1 \rightarrow T(M^4) \). Using \( \Theta^i \) from (6.10) on all above-mentioned twistor spaces, we obtain

\[
\hat{F}^+ = \pi^* F^+ = \frac{1}{2} (\hat{F}_{11} + \hat{F}_{22}) (\Theta^{11} + \Theta^{22}) + \hat{F}_{12} \Theta^{12} + \hat{F}_{12} \Theta^{12}^\dagger,
\]

(7.18)

\[
\hat{F}^- = \pi^* F^- = \frac{1}{2} (\hat{F}_{11} - \hat{F}_{22}) (\Theta^{11} - \Theta^{22}) + \hat{F}_{12} \Theta^{12} + \hat{F}_{21} \Theta^{21},
\]

(7.19)

where \( \Theta^{11} = \Theta^1 \wedge \Theta^1, \Theta^{22} = \Theta^2 \wedge \Theta^2 \) etc. Furthermore, for the components \( \hat{F}_{ij} \) we have the same formulae (3.13) and (3.14) as for the case of an almost complex structure \( \mathcal{J}_+ \). Thus, any anti-self-dual gauge field \( F = F^- \) on a vector bundle \( E \) over \( M^4 = S^4, \mathbb{C}P^2, B_4 \) or \( \mathbb{C}B_2 \) lifted to the twistor space \( (T(M^4), \mathcal{J}_+) \) satisfies the Hermitian-Yang-Mills equations on the pulled-back bundle \( \hat{E} = \pi^* E \) over \( T(M^4) \). Some particular examples of such solutions to the HYM equations on \( T(M^4) \) were described in this section. A lot of explicit solutions of the HYM equations on \( T(S^4) \) can be obtained by lifting multi-instanton solutions on \( S^4 \). Their moduli space is known from the ADHM construction [63]. Note that for \( B_4 \) families of solutions to the ASDYM equations were described in [64]. These ASD gauge fields are lifted to the HYM gauge fields on nearly Calabi-Yau space \( T(B_4) \). Furthermore, all HYM gauge fields on the nearly Calabi-Yau spaces \( T(M^4) \) are obtainable from ASD gauge fields on \( M^4 \) lifted to \( T(M^4) \). This follows from the constraint equation \( d\Omega \wedge \mathcal{F} = 0 \) which along with \( \omega \wedge \mathcal{F} = 0 \) yields \( \mathcal{F}_{33} = 0 \).
8 Twistor action for bosonic and supersymmetric Yang-Mills theories

In the previous sections we considered the spaces $M^4 = S^4$, $\mathbb{CP}^2$, $B_4$ and $\mathbb{CB}_2$ with the nearly Kähler twistor spaces $\mathcal{T}(S^4)$, $\mathcal{T}(\mathbb{CP}^2)$ and the nearly Calabi-Yau twistor spaces $\mathcal{T}(B_4)$, $\mathcal{T}(\mathbb{CB}_2)$. For all these cases $c_1(\mathcal{T}(M^4)) = 0$ and on $\mathcal{T}(M^4)$ we have a nonintegrable almost complex structure $\mathcal{J}$, an almost Hermitian $(1,1)$-form $\omega$ and a $(3,0)$-form $\Omega$ satisfying (6.8) or (6.9) and defining an SU(3)-structure on $\mathcal{T}(M^4)$. Furthermore,

$$\bar{\partial} \Omega := d^{0,1} \Omega = 0 \quad \text{and} \quad \bar{\partial} \omega = 0$$

(8.1)
on the nearly Kähler spaces $\mathcal{T}(S^4)$, $\mathcal{T}(\mathbb{CP}^2)$ and

$$\bar{\partial} \Omega = 0 \quad \text{and} \quad d \omega = 0$$

(8.2)
on the nearly Calabi-Yau spaces $\mathcal{T}(B_4)$, $\mathcal{T}(\mathbb{CB}_2)$ and their compact quotients. The SU(3)-structure on the above-mentioned twistor spaces allows us to introduce analogues of holomorphic Chern-Simons (hCS) theory on Calabi-Yau (super)spaces. We briefly recall the hCS theory.

**Holomorphic Chern-Simons theory on Calabi-Yau manifolds.** Let $Z (\cong X^6)$ be a complex three-dimensional Calabi-Yau manifold, $\mathcal{E}$ a rank $k$ complex vector bundle over $Z$ and $\mathcal{A}$ a connection one-form on $\mathcal{E}$. Consider the action [65]

$$S = \int_Z \Omega \wedge \text{tr}(\mathcal{A}^{0,1} \wedge \bar{\partial} \mathcal{A}^{0,1} + \frac{2}{3} \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1})$$

(8.3)
where $\Omega$ is a nowhere vanishing holomorphic $(3,0)$-form on $Z$ and $\mathcal{A}^{0,1}$ is the $(0,1)$-component of the connection one-form $\mathcal{A}$. This action functional was obtained by Witten [65] as a full target space action of the open topological $B$-model on a complex three-dimensional target space, on which the Calabi-Yau restriction arises from $N = 2$ supersymmetry of the corresponding topological sigma model and an anomaly cancellation condition.

The field equations following from the action functional (8.3) read

$$\mathcal{F}^{0,2} = \bar{\partial} \mathcal{A}^{0,1} + \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1} = 0$$

(8.4)
Thus, the hCS theory (8.3), (8.4) describes inequivalent holomorphic structures $\bar{\partial} \mathcal{A} = \bar{\partial} + \mathcal{A}^{0,1}$ on the bundle $\mathcal{E} \to Z$.

**Holomorphic Chern-Simons theory on Calabi-Yau supermanifolds.** In [66] it was observed that the Calabi-Yau restriction on the manifold $Z$ can be relaxed by considering a topological B-model (twistor string theory) whose target spaces are Calabi-Yau supermanifolds. For them, fermionic directions also make a contribution to $c_1(Z)$ yielding more freedom to have an overall vanishing first Chern class. As a main example of $Z$, Witten considered the supertwistor space $\mathcal{P}^{3|4} := \mathbb{CP}^{3|4} \setminus \mathbb{CP}^{1|4}$ with embedded projective lines $\mathbb{CP}_{x,\theta}^1$ parametrized by the chiral superspace $\mathcal{R}^{4|8} \supset (x^\mu, \theta^\alpha A)$, where $\mu = 1, \ldots, 4$, $\alpha = 1, 2$, $A = 1, \ldots, 4$. Under some assumptions, including triviality of the bundle $\mathcal{E} \to \mathcal{P}^{3|4}$ after restriction to each $\mathbb{CP}_{x,\theta}^1 \hookrightarrow \mathcal{P}^{3|4}$, it was shown that hCS

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9This condition is equivalent to vanishing of a part of the curvature $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ having components along subspaces $\mathbb{CP}^{1|0}_{x,\theta} \hookrightarrow \mathcal{P}^{3|4}$. Without this assumption the hCS theory is not equivalent to the anti-self-dual $N = 4$ SYM theory.
theory on the supertwistor space $\mathcal{P}^{3|4}$ is equivalent to anti-self-dual $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory in four dimensions.\(^{10}\)

As equations of motion for hCS theory on $\mathcal{P}^{3|4}$ and $\mathbb{C}P^{3|4}$ one has (8.4) but with $A^{0,1}$ holomorphically depending on fermionic coordinates. The spectrum of physical states contained in $A^{0,1}$ is the same as that of $\mathcal{N} = 4$ SYM theory but the interactions of both theories differ. It was also shown that the perturbative amplitudes of the full $\mathcal{N} = 4$ SYM theory are recovered by adding to the hCS action a nonlocal term interpreted as D-instantons wrapping holomorphic curves in $\mathcal{P}^{3|4}$. Another option is to construct an action on the super-ambitwistor space [66, 69, 70] but this was not entirely successful.

**Pseudo-holomorphic Chern-Simons theory.** Recall that twistor string theory establish a connection with $\mathcal{N} = 4$ SYM theory in four dimensions but, contrary to the standard topological string theory on Calabi-Yau 3-folds, lost the connection with superstring theory. For restoring such a connection one should consider not the superspace $\mathbb{C}P^{3|4}$ but an ordinary 6-manifold as a target space for twistor strings. In fact, the complex twistor space $\mathbb{C}P^3$ was used for some proposals on a possible twistor action for nonsupersymmetric $d = 4$ Yang-Mills theory [71]. However, nearly Kähler and/or nearly Calabi-Yau twistor spaces $T(M^4)$ may be more suitable for this purpose since all these twistor spaces carry an SU(3) structure defined by forms $\omega$ and $\Omega$. Thus, we can consider the action functional (8.3) with $Z = T(M^4)$ and $M^4 = S^4$, $\mathbb{C}P^2$ or $B_4$, $\mathbb{C}B_2$ (or their compact quotients) for almost complex $Z$ with $c_1(Z) = 0$. In this case, $A^{0,1}$ will be a (0,1)-form w.r.t. the nonintegrable almost complex structure $\mathcal{J} = \mathcal{J}_-$ introduced in (6.3)-(6.5) and (6.10). The field equations (8.4) of this pseudo-holomorphic Chern-Simons (pshCS) theory describe inequivalent pseudo-holomorphic structures $\bar{\partial} A = \bar{\partial} + A^{0,1}$ on the bundle $\mathcal{E} \to Z$. In its turn, pshCS theory on the almost complex twistor space $T(M^4)$ is equivalent to the (bosonic) anti-self-dual Yang-Mills theory on $M^4 = S^4$, $\mathbb{C}P^2$, $B_4$, $\mathbb{C}B_2$ or $\mathbb{R}^4$. Thus, one may consider (8.3) as a candidate to a twistor action for bosonic ASDYM theory and consider nearly Kähler & nearly Calabi-Yau twistor spaces as candidates for a target space for twistor string theory, which is close to the standard topological string theory.

**Action functionals on nearly Kähler twistor spaces.** As it was shown in section 3, for any anti-self-dual gauge field $F$ on $M^4$, its pull-back $\hat{F} := \pi^* F$ to the twistor space\(^{11}\) $T(M^4)$ satisfies not only (8.4) but also the equation $\omega \cdot \hat{F} = 0$, where $\omega$ is an almost Hermitian (1,1)-form on $T(M^4)$. Thus, $\hat{F}$ is a solution of the Hermitian-Yang-Mills equations on $T(M^4)$ which are the BPS equations for Yang-Mills theory in $d = 6$.

It is of interest that on nearly Kähler manifolds $X^6$ not only (8.4) but the full HYM equations can be obtained from the action functional [22]

$$S = \int_{X^6} \text{Im} \Omega \wedge \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \ ,$$

where $A$ is a connection one-form on a complex vector bundle $\mathcal{E}$ over $X^6$ and $\Omega$ is a (3,0)-form on $X^6$. Note that $d\omega = 3c \text{Im} \Omega$ and therefore in (8.5) one can use $d\omega$ instead of $\text{Im} \Omega$.

The field equations following from (8.5) read

$$\text{Im} \Omega \wedge \mathcal{F} = 0 \ ,$$

\(^{10}\)For reductions of this model to $d = 3$ and $d = 2$ see [67, 68].

\(^{11}\)Recall that $\pi : T(M^4) \to M^4$ is the canonical projection.
where $\mathcal{F} = dA + A \wedge A$ is the curvature of $A$. It is easy to show [22] that on nearly Kähler manifolds from (8.6) it follows
\begin{equation}
\text{Re} \Omega \wedge \mathcal{F} = 0 ,
\end{equation}
and differentiating (8.7) we obtain
\begin{equation}
\omega \wedge \omega \wedge \mathcal{F} = 0 \iff \omega \hookleftarrow \mathcal{F} = 0 \quad (8.8)
\end{equation}
after using (6.8) and the Yang-Mills Bianchi identities. In fact, on nearly Kähler manifolds eq.(8.8) follows from (8.4) due to (6.8).

The above observations allow us to propose (8.5) as a twistor action on $X^6 \cong \mathbb{C}P^3$ (or $SU(3)/U(1) \times U(1)$) for the bosonic ASDYM theory on $S^4$ (or $\mathbb{C}P^2$) after assuming, as in hCS and pshCS theories, that components of $\mathcal{F}$ along $\mathbb{C}P^1 \hookrightarrow X^6$ vanish. Such $\mathcal{F}$ can be identified with the gauge field $\hat{F}$ pulled back from $S^4$ (or $\mathbb{C}P^2$) to $X^6$ with the components defined in (7.18), (7.19) and (3.13), (3.14). Furthermore, for the full $d = 4$ (bosonic) Yang-Mills theory one can use the $d = 6$ Yang-Mills action functional
\begin{equation}
S = - \int_{\mathcal{P}^3} \text{vol}_6 \text{tr}(\hat{F}_{ab} \hat{F}_{ab}) .
\end{equation}
Integrating (8.9) over $\mathbb{C}P^1 \hookrightarrow \mathcal{P}^3$, we obtain the standard Yang-Mills action on $\mathbb{R}^4$ (on $S^4$ for $X^6 \cong \mathbb{C}P^3$). This action functional is a natural part of the low-energy heterotic string theory.

On the other hand, anti-self-dual Yang-Mills theory on $S^4$ and $\mathbb{C}P^2$ is related with the Hermitian-Yang-Mills model on the twistor spaces $\mathbb{C}P^3$ and $SU(3)/U(1) \times U(1)$, respectively, and with heterotic string theory compactified on these nearly Kähler spaces. It would be of interest to study open topological string theories (both A and B types) with such target spaces. According to [72], A-model on $T(M^4)$ can be a holographic dual to topological M-theory on a $d = 7$ $G_2$-manifold naturally associated with any nearly Kähler space $T(M^4)$ [73].

**Hermitian-Yang-Mills equations on supermanifolds.** Our observation on relation between ASDYM theory on $M^4$ and HYM theory on the twistor space $T(M^4)$ of $M^4$ can be useful also in $\mathcal{N} = 4$ supersymmetric case. Namely, consider the complex supertwistor space $\mathcal{P}^3|4 = \mathbb{C}P^3|4 \setminus \mathbb{C}P^1|4$ [69] with holomorphic fermionic coordinates
\begin{equation}
\theta^A := \theta^{2A} - \zeta \theta^{1A} ,
\end{equation}
where $\zeta \in U \subset \mathbb{C}P^1$ is a local coordinate on $\mathbb{C}P^1$ and $\theta^{1A}, \theta^{2A}$ are Grassmann variables. Introduce local fermionic (1,0)-forms
\begin{equation}
\omega^A = \frac{1}{(1 + \bar{\zeta} \zeta)^2} (d\theta^{2A} - \zeta d\theta^{1A})
\end{equation}
taking values in the Hermitian line bundle $\mathcal{L}_{+1}$ over $\mathbb{C}P^1$ associated with the Hopf bundle (4.18) and the corresponding (0,1)-forms
\begin{equation}
\omega^\bar{A} = \frac{1}{(1 + \zeta \bar{\zeta})^2} (d\theta^{1\bar{A}} + \bar{\zeta} d\theta^{2\bar{A}}) = T^{\bar{A}\bar{B}} \omega^{\bar{B}} \quad \text{with} \quad T^{12} = -T^{21} = T^{34} = -T^{43} = -1
\end{equation}
taking values in the dual line bundle $\mathcal{L}_{-1} \rightarrow \mathbb{C}P^1$. Thus, holomorphic fermionic “volume form” $vol_4 \omega$ takes values in $\mathcal{L}_{-4}$ and antiholomorphic fermionic “volume form” $vol_4 \omega$ takes values in $\mathcal{L}_{+4}$. 21
We also introduce odd (local) vector fields

\[ V_A = \frac{1}{(1 + \zeta \bar{\zeta})^2} \left( \frac{\partial}{\partial \theta^{2A}} - \bar{\zeta} \frac{\partial}{\partial \theta^{A}} \right) \quad \text{and} \quad V_{\bar{A}} = \frac{1}{(1 + \zeta \bar{\zeta})^2} \left( \frac{\partial}{\partial \theta^{1A}} + \zeta \frac{\partial}{\partial \theta^{2A}} \right) \]  

(8.13)

of type (1,0) and (0,1) which are dual to the forms (8.11) and (8.12), respectively.\(^{12}\) For discussion of reality conditions for odd variables \(\theta^{A}\) and more details see e.g. [69].

Let us consider a holomorphic vector bundle \(E\) over Calabi-Yau supermanifold \(P^{3|4}\) [66] and a connection one-form

\[ A = A^i_B \omega^i + A^i_A \omega^i + A^i_B \omega^i =: A^{1,0} + A^{0,1}, \]

(8.14)

where \(\omega^i\) are (1,0)-forms on \(P^3\) (see (4.32), (4.33) with \(\theta^1 = dy, \theta^2 = dz\) and \(\alpha_+ = \beta_+ = 0\)) and by "b" and "f" we denote even and odd components of \(A\). Here, \(A^{1,0}\) are given by the first two terms in (8.14). On \(P^{3|4}\) we introduce the (1,1)-form

\[ \omega = \frac{1}{2} (\delta_{ij} \omega^i \wedge \omega^j + \delta_{AB} \omega^A \omega^B), \]

(8.15)

where \(i, j = 1, 2, 3\) and \(A, B = 1, \ldots, 4\).

The Hermitian-Yang-Mills equations on the supertwistor space \(P^{3|4}\) can be written as follows:

\[ \mathcal{F}^{0,2} = 0 \iff \mathcal{F}_{ij} = 0, \quad \mathcal{F}_{iA} = 0 \quad \text{and} \quad \mathcal{F}_{AB} = 0, \]

(8.16)

\[ \omega \wedge \mathcal{F} = 0 \iff \delta^{ij} \mathcal{F}_{ij} + \delta^{AB} \mathcal{F}_{AB} = 0, \]

(8.17)

\[ \mathcal{F}^{2,0} = 0 \iff \mathcal{F}_{ij} = 0, \quad \mathcal{F}_{iA} = 0 \quad \text{and} \quad \mathcal{F}_{AB} = 0. \]

(8.18)

Here,

\[ \mathcal{F}_{ij} = [V_i + A^i_B, V_j + A^j_B], \quad \mathcal{F}_{iA} = [V_i + A^i_B, V_A + A_A^i], \quad \mathcal{F}_{AB} = \left\{ V_A + A_A^i, V_B + A_B^i \right\}, \]

(8.19)

and similar for other components of \(\mathcal{F}\).

**Twistor action for \(\mathcal{N} = 4\) SYM theory.** Let us introduce

\[ (x^{\alpha A}) = \left( \begin{array}{cc} x^{11} & x^{12} \\ x^{21} & x^{22} \end{array} \right), \quad (\theta^{\alpha A}) = (\theta^{\alpha A}, \theta^{\alpha A'}) = \left( \begin{array}{cccc} \theta^{11} & \bar{\theta}^{21} & \theta^{11'} & \bar{\theta}^{21'} \\ \bar{\theta}^{21} & \theta^{21} & \bar{\theta}^{11} & \theta^{11'} \\ \bar{\theta}^{11} & \bar{\theta}^{21} & \theta^{11'} & \bar{\theta}^{21} \end{array} \right) \]

(8.20)

and

\[ (\zeta_\alpha) := \rho \left( \begin{array}{c} -\zeta \\ 1 \end{array} \right), \quad (\zeta^\alpha) := \rho \left( \begin{array}{c} 1 \\ \zeta \end{array} \right), \quad (\bar{\zeta}_\alpha) := \rho \left( \begin{array}{c} 1 \\ \bar{\zeta} \end{array} \right), \quad (\bar{\zeta}^\alpha) := \rho \left( \begin{array}{c} \bar{\zeta} \\ -1 \end{array} \right) \quad \text{with} \quad \rho := \frac{1}{(1 + \zeta \bar{\zeta})^{\frac{1}{2}}}, \]

(8.21)

where in (8.20) we used the Euclidean reality conditions [69] for \(x^{\alpha A}\) and \(\theta^{\alpha A}\). Using (8.21), we can rewrite (8.10)-(8.13) as

\[ \omega^A = \zeta_\alpha d\theta^{\alpha A}, \quad V_A = -\bar{\zeta}_\alpha \frac{\partial}{\partial \theta^{\alpha A}}, \quad \omega_{\bar{A}} = \bar{\zeta}_\alpha d\theta^{\alpha A} \quad \text{and} \quad V_{\bar{A}} = \bar{\zeta}^\alpha \frac{\partial}{\partial \theta^{\alpha A}}. \]

(8.22)

\(^{12}\)Note that one can use a “nonsymmetric” formulation by erasing \((1 + \zeta \bar{\zeta})^{-\frac{1}{2}}\) in (8.13) and using \((1 + \zeta \bar{\zeta})^{-1}\) in (8.11), (8.12). Then \(V_A\) will take values in the holomorphic line bundle \(O(1) \rightarrow CP^1\), \(\omega^A\) will be a smooth section of the bundle \(O(-1), V_A \in O(1)\) and \(\omega^A \in O(-1)\).
The standard $\mathcal{N} = 4$ anti-self-dual Yang-Mills equations [74, 75] can be written in terms of gauge potential components $A_{\alpha\dot{\alpha}}(x, \theta)$ and $A_{\alpha A}(x, \theta)$ and after introducing

$$V_\alpha := \zeta^\alpha \frac{\partial}{\partial x^{\alpha}}, \quad \tilde{A}_\alpha^b := \zeta^\alpha A_{\alpha\dot{\alpha}}, \quad A^b_3 = 0 \quad \text{and} \quad \tilde{A}^f_A := \zeta^\alpha A_{\alpha A}, \quad (8.23)$$

they are equivalent to eqs.(8.16) (see e.g. [69]) with

$$V_1 + A^b_1 = \tilde{V}_2 + \tilde{A}^b_2 \quad \text{and} \quad V_2 + \tilde{A}^b_2 = \tilde{V}_1 + \tilde{A}^b_1 \quad (8.24)$$
due to our definition of spinor and vector indices, and eqs.(8.18) are Hermitian conjugate to (8.16)\(^1\). Substituting (8.27) into the rest equations (8.18), we obtain three group of equations (cf. [75]):

$$\delta^{\alpha\dot{\alpha}} V_\alpha \tilde{A}_\alpha^b + \delta^{\dot{A}\dot{A}} V_\tilde{A} \tilde{A}^f_A = 0 \quad (8.28)$$

which are solved as

$$\tilde{A}^b_1 = -V_2 \Upsilon, \quad \tilde{A}^b_2 = V_1 \Upsilon \quad \text{and} \quad \tilde{A}^f_A = V_4 \Upsilon, \quad (8.29)$$

where the $sl(k, \mathbb{C})$-valued prepotential $\Upsilon$ has weight $-2$, i.e. takes value in the bundle $\mathcal{L}-2$ over $\mathbb{C}P^1$. Substituting (8.27) into the rest equations (8.18), we obtain three group of equations (cf. [75]): one equation without the Grassmann derivatives and two groups with $V_A$ entering linearly. The equations with the derivatives $V_A$ simply fix the dependence of $\Upsilon$ on $\theta^A$ in terms of the “physical” field

$$\Phi(x, \zeta, \bar{\zeta}, \theta^A) = \Upsilon(x, \zeta, \bar{\zeta}, \theta^A, \theta^4) \mid_{\theta^4=0} \quad (8.30)$$

and its derivatives.

We omit the details here\(^1\) and write out only final formulae. Namely, (8.18) reduce to the one equation

$$(V_1 V_1 + V_2 V_2)\Phi + [V_1 \Phi, V_2 \Phi] = 0 \quad (8.29)$$
on the matrix-valued prepotential $\Phi$ of weight $-2$ encoding all information about $\mathcal{N} = 4$ ASDYM theory. Expanding the $\mathcal{N} = 4$ $sl(k, \mathbb{C})$-valued prepotential $\Phi$ in $\tilde{\theta}^A := \zeta^\alpha \theta_{\alpha A}$, we obtain

$$\Phi = \phi_{\beta\beta} \zeta^\alpha \bar{\zeta}^\beta + \phi_{\alpha A} \zeta^\alpha \tilde{\theta}^A + \phi_{AB} \tilde{\theta}^A \tilde{\theta}^B + \chi^A_{\alpha} \zeta^\alpha \varepsilon_{ABC\overline{D}} \tilde{\theta}^A \tilde{\theta}^B \tilde{\theta}^C + G_{\alpha\beta} \zeta^\alpha \zeta^\beta \tilde{\theta}^1 \tilde{\theta}^2 \tilde{\theta}^3 \tilde{\theta}^4. \quad (8.30)$$

\(^1\)By the pull-back construction $A^b_3 = 0$ but in general $A^b_3 \neq 0$. However, $\tilde{F}_{43} = g^{-1} F_{43} g = 0$.

\(^1\)The details will be published elsewhere.
where \((\phi_{AB}(x), \chi^A_\alpha(x), G_{\alpha\beta}(x))\) are space-time fields of helicities \((0, +\frac{1}{2}, +1)\) while \(\phi_{\alpha A}(x)\) and \(\phi_{\alpha\beta}(x)\) are prepotentials for fields \(\chi_\alpha A\) and \(f_{\alpha\beta}\) which have helicities \(-\frac{1}{2}\) and \(-1\). Finally, the action, whose equations of motion are (8.29), have the form

\[
S = \int d^4x \frac{d\zeta d\bar{\zeta}}{(1 + \zeta \bar{\zeta})^2} \text{vol}_4 \bar{\omega} \text{tr} \left\{ \Phi \Box \Phi + \frac{2}{3} \Phi [V_1 \Phi, V_2 \Phi] \right\},
\]

where \(\Box := V_1 V_1 + V_2 V_2 = \partial_x \partial_y \partial_y + \partial_z \partial_z\). Note that the Lagrangian in (8.31) has weight \(-4\) and \(\text{vol}_4 \bar{\omega}\) has weight \(+4\) as it should be. The functional (8.31) is the twistor action describing \(N = 4\) ASDYM theory in terms of a single prepotential \(\Phi\). Furthermore, one can introduce a twistor action for the full \(N = 4\) SYM theory by adding terms of 2nd, 3rd and 4th degree in \(\Phi\), \(\Phi^\dagger\) and their derivative w.r.t. \(\bar{\theta}^A\) and integrating them with the full Grassmann measure \(\text{vol}_4 \omega \text{vol}_4 \bar{\omega}\). These terms are

\[
\Phi^\dagger \Phi, \quad \bar{\theta}^A \Phi \theta^B \Phi^\dagger \frac{\partial}{\partial \theta^A} \frac{\partial}{\partial \theta^B} \Phi,
\]

\[
\left( \theta^{A_1} \bar{\theta}^{B_1} \frac{\partial}{\partial \theta^{A_1}} \frac{\partial}{\partial \theta^{B_1}} \Phi \right) \left( \theta^{A_2} \bar{\theta}^{B_2} \frac{\partial}{\partial \theta^{A_2}} \frac{\partial}{\partial \theta^{B_2}} \Phi \right) \left( \theta^{C_1} \theta^{D_1} \frac{\partial}{\partial \theta^{C_1}} \frac{\partial}{\partial \theta^{D_1}} \Phi \right) \left( \bar{\theta}^{C_2} \bar{\theta}^{D_2} \frac{\partial}{\partial \theta^{C_2}} \frac{\partial}{\partial \theta^{D_2}} \Phi \right).
\]

Details will be published elsewhere.

9 Conclusions

In this paper we considered the twistor space \(X^6 = T(M^4)\) of an oriented Riemannian manifold \(M^4\) and explored solvability properties of the first-order Hermitian-Yang-Mills equations for gauge fields on pseudo-holomorphic bundles \(E\) over \(X^6\). It was shown that the anti-self-dual gauge fields on \(M^4\) lifted to \(T(M^4)\) satisfy the Hermitian-Yang-Mills equations on \(T(M^4)\). Specializing to the cases \(M^4 = S^4, \mathbb{C}P^2, B_4\) or \(CB_2\), we discussed the nearly Kähler and nearly Calabi-Yau structures on their 6-dimensional twistor spaces \(T(M^4)\) and wrote down some explicit solutions of the Hermitian-Yang-Mills equations on \(T(M^4)\). Note that for all these twistor spaces \(X^6\) the first Chern class vanishes, \(c_1(X^6) = 0\), and these spaces carry an SU(3) structure. We hope that the described Yang-Mills instanton solutions on the nearly Kähler spaces \(T(S^4)\) and \(T(\mathbb{C}P^2)\) can be used in the flux compactification of heterotic supergravity to AdS\(_4\) and HYM gauge fields on the nearly Calabi-Yau spaces \(T(B_4)\), \(T(\mathbb{C}B_2)\) and their compact quotients can be used in the compactifications to the de Sitter space dS\(_4\) [1, 2]. These possibilities will be explored and described elsewhere.

We have introduced an analogue of holomorphic Chern-Simons theory on nearly Kähler twistor spaces \(T(M^4)\) and shown that under some restrictions it is equivalent to the anti-self-dual Yang-Mills theory on \(M^4 = S^4\) or \(\mathbb{C}P^2\). A twistor action for non-self-dual Yang-Mills theory is also proposed. Considering Yang-Mills theory on the supertwistor space \(\mathbb{C}P^{3|4}\) and its open subset \(\mathcal{P}^{3|4}\), we have shown that the HYM equations encoding the \(\mathcal{N} = 4\) supersymmetric ASDYM equations reduce to the equation on a single scalar superfield defined on the supertwistor space. An expansion of this superfield in Grassmann variables contains all fields from the \(\mathcal{N} = 4\) Yang-Mills supermultiplet or prepotentials for these fields. All terms for a proper twistor action for full \(\mathcal{N} = 4\) SYM theory are written down.

A natural direction for further study would be to solve explicitly the supersymmetry constraint equations of heterotic supergravity by using solutions to the HYM equations described in this paper. Further study of twistor actions for the \(\mathcal{N} \leq 4\) SYM theories may constitute another direction.
It is also of interest to extend the techniques of the equivariant dimensional reduction for Kähler coset spaces [76]-[78] to heterotic supergravity compactified on six-dimensional nearly Kähler and nearly Calabi-Yau coset spaces.

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