Cofibrantly Generated Lax Orthogonal Factorisation Systems

Ignacio López Franco

Received: 13 August 2018 / Accepted: 22 January 2019 / Published online: 27 February 2019
© Springer Nature B.V. 2019

Abstract
The present note has three aims. First, to complement the theory of cofibrant generation of algebraic weak factorisation systems (AWFSS) to cover some important examples that are not locally presentable categories. Secondly, to prove that cofibrantly KZ-generated AWFSS (a notion we define) are always lax orthogonal. Thirdly, to show that the two known methods of building lax orthogonal AWFSS, namely cofibrantly KZ-generation and the method of “simple adjunctions”, construct different AWFSS. We study in some detail the example of cofibrant KZ-generation that yields representable multicategories, and a counterexample to cofibrant generation provided by continuous lattices.

Keywords  Algebraic weak factorization system · Weak factorization system · Lax factorization system · Multicategory · Continuous lattice · Orthogonal factorization system · Cofibrant generation

Mathematics Subject Classification  Primary 18A32; Secondary 55U35 · 18D20

1 Introduction
Lax orthogonal factorisation systems (LOFSS) are a type of algebraic weak factorisation systems (AWFSS) on 2-categories, introduced in [5], for which the diagonal fillers satisfy a universal property, similar to the property of a left Kan extension. Several examples of LOFSS were constructed in [5,6] using the method of “simple adjunctions.” The present note addresses an aspect of this theory that has not been touched upon: the cofibrant generation of LOFS. The cofibrant generation of AWFSS on locally presentable categories were studied

Communicated by M. M. Clementino.

The author gratefully acknowledges the support of the following institutions during the long gestation of this article: a Research Fellowship of Gonville and Caius College, Cambridge; the Department of Pure Mathematics and Mathematical Statistics of the University of Cambridge; SNI-ANII, PEDECIBA and Universidad de la República.

Ignacio López Franco
ilopez@cure.edu.uy

1 Departamento de Matemática y Aplicaciones, CURE, Universidad de la República, Tacuarembó s/n, Maldonado, Uruguay
I. López Franco

in [3], encompassing a large family of examples that, however, do not reach some important ones, as those based on the category of topological spaces. The present note is an attempt to fill this gap in the literature.

The article can be divided in two parts. The first, taking most of the article, that deals with cofibrant generation of AWFSS enriched over a base category \( \mathcal{V} \subseteq \textbf{Cat} \), and the second that looks at the case when \( \mathcal{V} \) is the category of preorders.

AWFSS were introduced by Grandis and Tholen [12], with latter contributions by Garner [11], and, as the name indicates, they are an algebraisation of the more classical notion of a weak factorisation system (WFS). A WFS consists of two classes of morphisms \( (\mathcal{L}, \mathcal{R}) \) with the property that each morphism \( f \) can be written as \( f = r \cdot \ell \), with \( \ell \in \mathcal{L} \) and \( r \in \mathcal{R} \), and each \( r \in \mathcal{R} \) precisely when it has the right lifting property with respect to each \( \ell \in \mathcal{L} \), i.e., for each commutative square as displayed, there exists a—non necessarily unique—diagonal filler

\[
\begin{array}{ccc}
\ell & & r \\
\downarrow & \swarrow & \downarrow \\
 & \exists! & \\
\end{array}
\]

(1.1)

and dually, \( \ell \in \mathcal{L} \) precisely when it has the left lifting property with respect to each \( r \in \mathcal{R} \). This kind of lifting situation was common in algebraic topology long before the importance of WFS was fully realised, for which Quillen’s definition of model category was central.

One of the features that distinguishes AWFSS from WFSs—on a category \( \mathcal{C} \), say—is that the factorisation of morphisms is functorial—as is also the case in many WFSs constructed by the so-called Quillen’s small object argument [26]. The left and right classes of morphisms are replaced, respectively, by the coalgebras and algebras for a certain comonad and monad on \( \mathcal{C}^2 \), and this extra algebraic structure ensures that diagonal fillers (1.1) not only exist but can be constructed from algebraic data.

Lax orthogonal factorisation systems, introduced in [5], are AWFSS on 2-categories for which the canonical diagonal, say \( d \), satisfies a certain universal property with respect to 2-cells: given any other diagonal filler \( w \) and 2-cells \( \alpha : h \Rightarrow w \cdot \ell \) and \( \beta : k \Rightarrow r \cdot w \), there exists a unique 2-cell \( \gamma : d \Rightarrow w \) such that \( \gamma \cdot \ell = \alpha \) and \( r \cdot \gamma = \beta \).

\[
\begin{array}{ccc}
h & & \\
\downarrow & \swarrow & \downarrow \\
\ell & & r \\
\downarrow & \swarrow & \downarrow \\
\exists! & & w \\
\end{array}
\]

See Sect. 4 or [5] for more on the definition of LOFS, along with the basic theory of LOFSs and a procedure to construct them via the so-called simple adjunctions. When the codomain of \( r \) is a terminal object, the above condition says that the identity 2-cell \( h = d \cdot \ell \) exhibits \( d \) as a left Kan extension of \( h \) along \( \ell \). Objects \( A \) with this property with respect to a family of morphisms \( \ell \) have been studied in the context of poset-enriched categories and called Kan objects [9] or Kan injective objects [1].

The notion of cofibrant generation adapted to AWFSS was introduced in [11], and later extended to generation by a double category in [3], where, in addition, enriched cofibrant generation is discussed.

There is a notion of cofibrant generation for LOFSs, that we call cofibrant KZ-generation. We show that cofibrantly KZ-generated AWFSS are always LOFS. Cofibrant KZ-generation can be seen as a case of the constructions in [3, §8]—even though [3] does not consider LOFSs and concentrates on locally presentable categories. Representable multicategories provide an
example that we study in some detail. There are important examples, however, that are not locally presentable categories, as the category of topological spaces. We study the case of categories enriched in posets and the cofibrant KZ-generation thereon, encompassing in this way the example of topological spaces.

We have mentioned two ways of constructing new LOFSS: via simple adjunctions and via cofibrant KZ-generation. It is natural to ask whether these two procedures construct the same LOFSS. We give a negative answer to this question, exhibiting a LOFS that can be constructed via simple adjunctions but are not cofibrantly KZ-generated, nor cofibrantly generated; furthermore, its underlying WFS is not cofibrantly generated, in the usual sense of the term. The LOFS in question is defined on the category of $T_0$ topological spaces, its left morphisms are the subspace embeddings, and its fibrant objects are the continuous lattices $[5, \S 12] [6,7]$.

We conclude this introduction with a description of the article's contents. Section 2 collects some of the constructions and results relative to AWFSS needed in the rest of the article. In Sect. 3 we adapt the notions of cofibrant generation introduced in $[3,11]$ to the case of categories enriched in $\mathcal{V} \subseteq \text{Cat}$. Lax orthogonal factorisation systems are recalled in Sect. 4 and the notion of cofibrant KZ-generation is introduced in Sect. 5. Before showing that cofibrantly KZ-generated AWFS are LOFS in Sect. 7, we show in Sect. 6 the existence of cofibrantly KZ-generated AWFS in the locally presentable case. Section 8 looks to the example of representable multicategories as arising from a cofibrantly KZ-generated LOFS on multicategories. Section 9 shows that cofibrantly generated and KZ-generated AWFS must satisfy certain accessibility, or colimit-creation property, to be used in a latter section. Section 10 proves an existence result for cofibrantly KZ-generated AWFS on preorder-enriched categories and compares it with $[1]$, while Sect. 11 exhibits an example of a LOFS on $\text{Top}_0$ that is not cofibrantly KZ-generated, nor cofibrantly generated, and whose underlying WFS is not cofibrantly generated in the usual meaning of the term. There is an Appendix A with background on lax idempotent 2-monads and our own results on reflections of lax idempotent 2-monads along functors.

2 Background on Algebraic Weak Factorisation Systems

As mentioned in the introduction, we are interested in 2-categories and locally ordered categories. One could choose to develop the exposition in the context of $\mathcal{V}$-enriched categories, for a fairly general symmetric monoidal closed category $\mathcal{V}$. On the other hand, one could treat only the case of 2-categories and, when necessary, argue that the relevant constructions and results restrict to locally ordered 2-categories. We will take a middle of the road approach and consider categories enriched in a full sub-2-category $\mathcal{V} \subseteq \text{Cat}$, closed under limits and exponentials, and that is cocomplete. Furthermore, it will be important for our applications that the arrow category $2$ should belong to $\mathcal{V}$. Our main examples of $\mathcal{V}$ will be the 2-categories $\text{Cat}$ of small categories and $\text{Ord}$ of posets. We do not assume that the inclusion $\mathcal{V} \subseteq \text{Cat}$ preserves colimits. This, together with $2 \in \mathcal{V}$, would imply that the inclusion is an equivalence. Many of the $\mathcal{V}$-enriched notions we discuss below hold for a much more general $\mathcal{V}$, and are not difficult to elaborate in that context. Since they do not add much to our main examples, we leave them to be developed elsewhere.
2.1 Functorial Factorisations

A \mathcal{V}\text{-functorial factorisation} on a \mathcal{V}\text{-category} \mathcal{C} is a \mathcal{V}\text{-functor} \mathcal{C}^2 \to \mathcal{C}^3 that is a section of the composition \mathcal{V}\text{-functor} \mathcal{C}^3 \to \mathcal{C}^2. Equivalently, it is a \mathcal{V}\text{-functor} \mathcal{K} : \mathcal{C}^2 \to \mathcal{C} with \mathcal{V}\text{-natural transformations} \text{dom} \Rightarrow \mathcal{K} \Rightarrow \text{cod} whose composition equals the canonical transformation \text{dom} \Rightarrow \text{cod} with \( f \)-component equal to \( f \). In other words, a \mathcal{V}\text{-functorial factorisation} is a functorial factorisation as defined in [12] that is compatible with the 2-cells of \( \mathcal{C} \).

\[
(A \xrightarrow{f} B) = (A \xrightarrow{Lf} Kf \xrightarrow{Rf} B)
\]

As in the case of ordinary categories, a functorial factorisation can be equivalently described by a copointed endo-\mathcal{V}\text{-functor} \( \Phi : L \Rightarrow 1 \) on \( \mathcal{C}^2 \) with \( \text{dom} \Phi = 1 \), or by a pointed endo-\mathcal{V}\text{-functor} \( \Lambda : 1 \Rightarrow R \) on \( \mathcal{C}^2 \) with \( \text{cod} \Lambda = 1 \).

Given a \( \mathcal{V}\)-functorial factorisation as in the previous paragraph, each coalgebra structure \((1, s) : f \to Lf\) for the copointed endo-\mathcal{V}\text{-functor} \( (L, \Phi) \) on \( f \in \mathcal{C}^2 \) and each algebra structure \((p, 1) : Rf \to f\) for the pointed endo-\mathcal{V}\text{-functor} \( (R, \Lambda) \) on \( g \in \mathcal{C}^2 \) induces a choice of diagonal fillers for morphisms \((h, k) : f \to g\) in \( \mathcal{C}^2 \), i.e., commutative squares in \( \mathcal{C} \). The diagonal filler for this square is the composite

\[
\text{diag}(h, k) : \text{cod} f = \text{dom} k \xrightarrow{s} Kf \xrightarrow{K(h, k)} Kg \xrightarrow{p} \text{cod} h = \text{dom} g.
\]

If \((\alpha, \beta) : (h, k) \Rightarrow (\bar{h}, \bar{k}) : f \to g\) is a 2-cell in \( \mathcal{C}^2 \), then there is a corresponding 2-cell \( \text{diag}(\alpha, \beta) : \text{diag}(h, k) \Rightarrow \text{diag}(\bar{h}, \bar{k}) \), given by \( \text{diag}(\alpha, \beta) = p \cdot K(\alpha, \beta) \cdot s \). See [11,12], and [5] for the 2-categorical case.

2.2 Categories Internal to \( \mathcal{V}\)-Categories

A category internal to \( \mathcal{CAT} \) is a double category, and can be described as having objects and two kinds of morphisms, horizontal and vertical ones. Objects together with, respectively, horizontal and vertical morphisms form a category. Furthermore, there are squares, each one of which has a vertical domain and codomain and a horizontal domain and codomain, and can be composed both vertically and horizontally. There are other details, like identity squares and various compatibilities between identities and composition, that can be found in [20].

If, instead of categories internal to \( \mathcal{CAT} \), we consider categories internal to \( \mathcal{V}\text{-CAT} \), the resulting structure is that of a double category except that now there are 2-cells between horizontal morphisms; objects, horizontal morphisms and these 2-cells form a \( \mathcal{V}\)-category. Furthermore, there are 2-cells between squares which, together with vertical morphisms as objects and squares as morphisms, form a \( \mathcal{V}\)-category. Each category internal to \( \mathcal{V}\text{-CAT} \) has an underlying double category obtained by disregarding all the 2-cells.

Most of the categories internal to \( \mathcal{V}\text{-CAT} \) we consider will be related to the one described below. The arrow category \( 2 \) has a cocategory structure, in the sense that it is an internal category in \( \text{Cat}^{op} \). Another way of putting this is to say that for each category \( \mathcal{A} \) the set \( \text{Cat}(2, \mathcal{A}) = \text{Mor} \mathcal{A} \) carries a category structure, namely, that of \( \mathcal{A} \).

As a consequence of the cocategory structure on \( 2 \), for any \( \mathcal{V}\)-category \( \mathcal{C} \) there is a category internal to \( \mathcal{V}\text{-Cat} \), called the internal category of squares in \( \mathcal{C} \), and depicted on the right below.

\[
\begin{array}{cccccccc}
3 & \leftrightarrow & 2 & \leftrightarrow & 1 & \Rightarrow & \mathcal{C} & \Rightarrow \& \mathcal{C}^2 & \leftrightarrow & \mathcal{C}
\end{array}
\]

The underlying double category of \( \text{Sq}(\mathcal{C}) \) has objects those of \( \mathcal{C} \) and both vertical and horizontal morphisms the morphisms of \( \mathcal{C} \). Squares are commutative squares in \( \mathcal{C} \) and 2-cells
between horizontal morphisms are just 2-cells in \( \mathcal{C} \). A 2-cell between squares is a pair of 2-cells as depicted, that satisfy \( g \cdot \alpha = \beta \cdot f \).

There is an obvious notion of internal functor between categories internal to \( \mathcal{V} \)-\textsc{cat}, the formulation of which is left to the reader.

### 2.3 Enriched AWFS

A \( \mathcal{V} \)-enriched AWFS on a \( \mathcal{V} \)-category \( \mathcal{C} \) is a \( \mathcal{V} \)-functorial factorisation on \( \mathcal{C} \) equipped with a comultiplication \( \Sigma : L \Rightarrow L^2 \) that makes \( L = (L, \Phi, \Sigma) \) a \( \mathcal{V} \)-enriched comonad, and a multiplication \( \Pi : R^2 \Rightarrow R \) that makes \( R = (R, \Lambda, \Pi) \) a \( \mathcal{V} \)-enriched monad. Furthermore, that the underlying ordinary comonad and monad on the underlying category of \( C^2 \) should form an AWFS as defined in [11]; in other words, the \( \mathcal{V} \)-natural transformation \( LR \Rightarrow RL \) with components \( (\sigma_f, \pi_f) : LRf \to RLf \) must be a mixed distributive law; here we have used the notation \( \Sigma_f = (1, \sigma_f) : Lf \to L^2 f \) and \( \Pi_f = (\pi_f, 1) : R^2 f \to Rf \). This distributivity condition, added in [11] to the original definition of AWFS—called natural wfs in [12]—is precisely what is needed in order to have an associative composition of \( R \)-algebras and of \( L \)-coalgebras. An associative composition of \( R \)-algebras chooses, for each pair of \( R \)-algebras \( f, g \) with \( \text{cod} f = \text{dom} g \), an \( R \)-algebra structure on the composition \( g \cdot f \); these assignments must be natural with respect to morphisms and 2-cells of \( R \)-algebras, and it must be associative in the sense that the \( R \)-algebra structures of \( (h \cdot g) \cdot f \) and \( h \cdot (g \cdot f) \) must coincide. A similar statement can be made about \( L \)-coalgebras.

The existence of the associative composition mentioned in the previous paragraph can be rephrased in a more concise way: if \( (L, R) \) is a \( \mathcal{V} \)-enriched AWFS, there are internal categories in \( \mathcal{V} \)-\textsc{cat}

\[
\begin{array}{ccc}
\text{L-Coalg} & \xrightarrow{\cong} & \mathcal{C} \\
\text{R-Alg} & \xleftarrow{\cong} & \mathcal{C}
\end{array}
\]

that we denote by \( \text{L-Coalg} \) and \( \text{R-Alg} \). These two internal categories come equipped with internal functors into \( \text{Sq}(\mathcal{C}) \) given by forgetting the (co)algebra structure. Furthermore, given a \( \mathcal{V} \)-monad \( R \) on \( \mathcal{C} \), there is a bijection between compositions that make \( \text{R-Alg} \Rightarrow \mathcal{C} \) into an internal category in \( \mathcal{V} \)-\textsc{cat} and \( \mathcal{V} \)-comonads \( L \) such that \( (L, R) \) is a \( \mathcal{V} \)-enriched AWFS. This is completely analogous to the case of ordinary AWFS [3, §2.8].

The internal categories in \( \mathcal{V} \)-\textsc{cat} that arise from an AWFS can be characterised as in [3, §3]. If \( \mathbb{D} = (D_1 \Rightarrow D_0) \) is an internal category and \( U : \mathbb{D} \to \text{Sq}(\mathcal{C}) \) an internal functor in \( \mathcal{V} \)-\textsc{cat}, then \( \mathbb{D} \) is isomorphic to \( \text{R-Alg} \) over \( \text{Sq}(\mathcal{C}) \), for an—essentially unique—\( \mathcal{V} \)-enriched AWFS \( (L, R) \) on \( \mathcal{C} \), if the following two conditions hold: (a) \( U_1 : D_1 \to C^2 \) is monadic (it has a \( \mathcal{V} \)-enriched left adjoint and the comparison \( \mathcal{V} \)-functor into the algebras of the associated \( \mathcal{V} \)-monad is an \emph{isomorphism}); (b) The induced \( \mathcal{V} \)-monad is isomorphic to a codomain-preserving \( \mathcal{V} \)-monad. A more elementary condition can be found in [3, Thm. 6].
2.4 Internal Categories of \textsc{Lari}s and \textsc{Ral}s

Later we will need to refer to certain internal \( \mathcal{V} \)-categories whose vertical morphisms are given by adjunctions.

A morphism \( f: A \to B \), in a \( \mathcal{V} \)-category \( A \), equipped with a right adjoint whose unit is an identity 2-cell may be called a \textit{left adjoint right inverse}, abbreviated LARI, following terminology used in [3,13]. Thus, a LARI is a triple \((f, r, \varepsilon)\), where \( f: A \to B \) and \( r: B \to A \) are morphisms and \( \varepsilon: f \circ r \Rightarrow 1_B \) is a 2-cell, satisfying \( r \circ f = 1_A \), \( \varepsilon \cdot f = 1 \), \( r \cdot \varepsilon = 1 \). A morphism of LARI\( \text{s} \) is a morphism between the underlying morphisms in \( C \) that commutes with the right adjoints and the corresponding counits; explicitly, a morphism \((f, r, \varepsilon) \to (f', r', \varepsilon')\) is a morphism \((h, k): f \to f'\) in \( \mathcal{A}^2 \) that satisfies \( r' \cdot k = h \cdot r \) and \( k \cdot \varepsilon = \varepsilon \cdot k \). Finally, a 2-cell between two morphisms \((h, k) \Rightarrow (\bar{h}, \bar{k})\) is simply a 2-cell in \( \mathcal{A}^2 \). We have, for any \( \mathcal{V} \)-category \( A \), a \( \mathcal{V} \)-category \textsc{Lari}(\( A \)) with objects LARI\( \text{s} \) in \( A \) and morphisms and 2-cells as described above.

Clearly, LARI\( \text{s} \) compose, in the sense that if \( f: A \to B \) and \( g: B \to C \) are morphisms with LARI structures \((f, r, \varepsilon)\) and \((g, s, \varphi)\), then \((g \cdot f, r \cdot s, \varphi(r \cdot \varepsilon \circ f))\) is a LARI. Furthermore, identity morphisms have an obvious LARI structure. As a consequence there is an internal category \textsc{Lari}(\( A \)) \Rightarrow \textsc{Lari}(\( A \)) in \( \mathcal{V} \)-\textsc{CAT} denoted by \textsc{Lari}(\( A \)), and an obvious internal functor \textsc{Lari}(\( A \)) \Rightarrow \textsc{Sq}(\( A \)).

Dually, a morphism \( f: A \to B \) in a \( \mathcal{V} \)-category \( A \) equipped with a left adjoint whose unit is an identity may be called a \textit{right adjoint left inverse}, or RALI. A morphism of RALI\( \text{s} \) is a morphism in \( \mathcal{A}^2 \) between the underlying morphisms that commutes with the left adjoints and the units. There is an internal category \textsc{Ral}(\( A \)) \Rightarrow \textsc{Lari}(\( A \)) in \( \mathcal{V} \)-\textsc{CAT} and a forgetful internal functor \textsc{Ral}(\( A \)) \Rightarrow \textsc{Sq}(\( A \)).

Each one of the internal categories \textsc{Lari}(\( A \)) and \textsc{Ral}(\( A \)) over \textsc{Sq}(\( A \)) are induced by an AWFS under weak assumptions on the \( \mathcal{V} \)-category \( A \). Recall that, as part of our assumptions, the category \( \mathcal{V} \subseteq \text{Cat} \) contains the arrow category \( 2 \) and is closed under limits so it makes sense to speak of comma-objects in \( \mathcal{V} \); these comma-objects are just comma-categories. The lax limit of a morphism \( f: X \to Y \) in \( \mathcal{V} \) is another name for the comma-category \( f/Y \). A \textit{lax limit} of a morphism \( f: A \to B \) in a \( \mathcal{V} \)-category \( A \) is a diagram as the one depicted on the left, that is sent by each representable \( A(C, -) \) to a lax limit in \( \mathcal{V} \). A \textit{lax colimit} of \( f \) is a diagram as shown on the right that is sent by each \( A(-, C) \) to a lax limit in \( \mathcal{V} \).

\[
\begin{array}{ccc}
\downarrow & & \downarrow \\
\text{col}_\ell f & \Rightarrow & f \\
\downarrow & & \downarrow \\
B & \to & A \\
\end{array}
\quad \begin{array}{ccc}
\downarrow & & \downarrow \\
\text{col}_\ell f & \Rightarrow & f \\
\downarrow & & \downarrow \\
B & \to & A \\
\end{array}
\]

If \( A \) has lax limits of morphisms, then \textsc{Lari}(\( A \)) \cong \text{l-Coalg}, where \((L, R)\) are given on a morphism \( f: A \to B \) in the following way: \( L(f): A \to f/B \) is the morphism that corresponds to the identity 2-cell \( f \Rightarrow f \), and \( R(f): f/B \to B \) is the projection; see [5, §3.3].

If \( A \) has lax colimits of morphisms, then there is a \( \mathcal{V} \)-enriched AWFS \((E, M)\) on \( A \) such that \textsc{Ral}(\( A \)) \cong \text{M-\textsc{Alg}}\,\text{g}, where \((E, M)\) is given on a morphism \( f: A \to B \) in the following way: \( E(f): A \to \text{col}_\ell f \) is the coprojection and \( M(f): \text{col}_\ell f \to B \) is the morphism corresponding to the identity 2-cell \( f \Rightarrow f \).
3 Categories of Lifting Operations

Classically, a wfs \((\mathcal{L}, \mathcal{R})\) on a category \(\mathcal{C}\) is cofibrantly generated by a set of morphisms \(\mathcal{J}\) if \(\mathcal{R}\) is the family of morphisms that have the right lifting operation with respect to each member of \(\mathcal{J}\). Quillen’s small object argument proves that each set of morphisms \(\mathcal{J}\) cofibrantly generates a wfs in categories that satisfy certain smallness and cocompleteness conditions. In the context of AWFSs, right lifting properties are replaced by lifting operations, whose definition we recall below, and the set of morphisms \(\mathcal{J}\) can be replaced by a category \(\mathcal{J}\) and a functor \(U : \mathcal{J} \to \mathcal{C}^2\).

A previous version of this manuscript used a slightly different exposition of enriched AWFSs from that of [3]. As our main concern here is to study the cofibrant \(\kappa\)-generation of AWFSs instead of providing an overarching exposition of enriched AWFSs, we shall follow [3, §8] and save space. The only difference in our, admittedly short, exposition is the fact that all the AWFSs will be \(\mathcal{V}\)-enriched.

A lifting operation against \(U\) on \(f\) can be described as a family

\[
\mathcal{C}^2(Uj, f) \to \mathcal{C}(\text{dom } Uj, \text{dom } f),
\]

natural in \(j \in \mathcal{J}\), each of which is a section to the dashed morphism into the pullback displayed below.

\[
\begin{array}{ccc}
\mathcal{C}(\text{dom } Uj, \text{dom } f) & \to & \mathcal{C}(\text{dom } Uj, \text{dom } f) \\
\downarrow & & \downarrow \text{pb} \\
\mathcal{C}(\text{cod } Uj, \text{cod } f) & \to & \mathcal{C}(\text{cod } Uj, \text{dom } f) \\
\end{array}
\]

Morphisms of \(\mathcal{C}\) equipped with a lifting operation against \(U\) form a category \(\mathcal{J}^{\kappa}\) that has an obvious forgetful functor \(U^{\kappa} : \mathcal{J}^{\kappa} \to \mathcal{C}^2\) that forgets the lifting operation.

In order to put these ideas in a setting that allows for an easy generalisation to the enriched case and takes account of the double category structures, we shall follow [3] in the utilisation of fibre squares. If \(\mathcal{A}\) is a \(\mathcal{V}\)-category with enriched pullbacks, there is a \(\mathcal{V}\)-functor \(C : \mathcal{A}^2 \times 2 \to \mathcal{A}^2\) that sends a square to the comparison morphism into the associated pullback square.

\[
\begin{array}{ccc}
h & \to & \mathcal{C}(f, g, h, k) \\
\downarrow & & \downarrow \text{pb} \\
g & \to & \mathcal{C}(f, g, h, k) \\
\end{array}
\]

(3.2)

If \(\mathcal{A}\) carries an enriched AWFS \((E, M)\), then the \(\mathcal{V}\)-category of fibre squares is defined by the following pullback.

\[
\begin{array}{ccc}
\text{Fibre}_M(\mathcal{A}) & \to & \text{M-Alg} \\
\downarrow & & \downarrow \text{C} \\
\mathcal{A}^2 \times 2 & \to & \mathcal{A}^2
\end{array}
\]

Its objects are squares as in (3.2) with an M-algebra structure on \(C(f, g, h, k)\), called fibre squares. It is not hard to show that if we “paste” two of these squares, or set them side
by side, the resulting outer square is also a fibre square (the proof depends on the fact that
the codomain functor \( M\text{-Alg} \to A \) is a discrete pullback-fibration). A symmetry argument
ensures that two fibre squares, one stacked on top of the other, yields a new fibre square. By
the aforementioned, there are two internal category structures on \( \text{Fibre}_M(A) \), one given by
pasting squares side by side, and the other by pasting squares vertically.

We now assume that \( \mathcal{V} \) is equipped with a \( \mathcal{V} \)-enriched AWFS \((E,M)\). Any \( \mathcal{V} \)-functor category
\( \mathcal{J}^{\text{op}}, \mathcal{V} \) has an induced AWFS, defined pointwise, that we still call \((E,M)\).

Given a \( \mathcal{V} \)-functor \( U : \mathcal{J} \to \mathcal{C}^2 \), there is an induced \( \mathcal{V} \)-functor \( \hat{U} : \mathcal{C}^2 \to [\mathcal{J}^{\text{op}}, \mathcal{V}]^{2 \times 2} \) that
sends \( f \) to the outer square in (3.1) (where \( j \) is a variable). This is in fact an internal functor
\( \hat{U} : \text{Sq}(C) \to \text{Sq}([\mathcal{J}^{\text{op}}, \mathcal{V}]^2) \). The \( \mathcal{V} \)-category \( \mathcal{J}_M^{\text{op}} \) is defined by the pullback diagram on
the left below. The \( \mathcal{V} \)-functor \( W_U = \hat{U} C \) sends \( f : A \to B \) to the dashed transformation
in (3.1). An object of \( \mathcal{J}_M^{\text{op}} \) is a morphism \( f : A \to B \) with an \( M \)-algebra structure on each
dashed arrow as in (3.1) that vary \( \mathcal{V} \)-naturally with \( j \in \mathcal{J} \).

\[
\begin{array}{ccc}
\mathcal{J}_M^{\text{op}} & \longrightarrow & \text{Fibre}_M([\mathcal{J}^{\text{op}}, \mathcal{V}]) \\
\downarrow U_M & & \downarrow M\text{-Alg} \\
\downarrow \hat{U} & \downarrow \downarrow & \downarrow [\mathcal{J}^{\text{op}}, \mathcal{V}]^2 \\
\downarrow \downarrow & \downarrow & \downarrow \text{Sq}(C) \\
\downarrow \text{Sq}(\mathcal{C}) & \longrightarrow & \text{Sq}([\mathcal{J}^{\text{op}}, \mathcal{V}]^2)
\end{array}
\]

(3.3)

Since \( \hat{U} \) is part of an internal functor \( \text{Sq}(C) \to \text{Sq}([\mathcal{J}^{\text{op}}, \mathcal{V}]^2) \), it is immediate that \( \mathcal{J}_M^{\text{op}} \) is
the arrow part of an internal category, with object part \( \mathcal{C} \), and that it fits in a pullback square
of internal functors displayed on the right above.

**Example 3.1** Suppose that \( \mathcal{V} = \text{Set} \) and that \( M \) is the free split epimorphism monad on \( \text{Set}^2 \),
which is explicitly given by sending \( f : A \to B \) to \( \left( \begin{array}{c} f \end{array} \right) : A + B \to B \). A lifting operation on
a morphism \( g : A \to B \) in a category \( \mathcal{C} \) against a functor \( U : \mathcal{J} \to \mathcal{C}^2 \) is a choice of section
\( \phi_j \) for each canonical \( \mathcal{C}(\text{dom} U_j, A) \to \mathcal{C}^2(U_j, g) \) in such a way that the family \( \phi_j \) is natural
in \( j \).

\[
\begin{array}{ccc}
dom U_j & \xrightarrow{h} & A \\
\downarrow U_j & & \downarrow \downarrow \\
\text{cod} U_j & \xrightarrow{\phi_j(h,k)} & B
\end{array}
\]

For this particular monad \( M \), we will suppress the suffix in the notation \( \mathcal{J}_M^{\text{op}} \) and simply write \( \mathcal{J}_M \), as to coincide with the notation used in [11].

Assume that \( \mathcal{J} \) is part of an internal category in \( \mathcal{V}\text{-Cat} \), say \( \mathcal{J} = (\mathcal{J} \Rightarrow \mathcal{J}_0) \), with an
internal functor \((U, U_0) : \mathcal{J} \to \text{Sq}(\mathcal{C})\). We shall now describe the internal category \( \mathcal{J}_M^{\text{op}} \) over
\( \text{Sq}(\mathcal{C}) \). There are two internal functors \( \mathcal{J}_M^{\text{op}} \to (\mathcal{J} \times \mathcal{J}_0 \mathcal{J})^{\text{op}} \), which at the level of object
of objects are the identity \( \mathcal{J}_0 \to \mathcal{J}_0 \). At the level of object of arrows, one of these internal
functors corresponds to the \( \mathcal{V} \)-functor \( \mathcal{J}_M^{\text{op}} \to \text{Fibre}_M([\mathcal{J} \times \mathcal{J}_0 \mathcal{J}]^{\text{op}}, \mathcal{V}) \) that sends \( f \) to the
pasted fibre square below. Here the fibre square structure is the one given by the pasting of
fibre squares. The second internal functor from \( \mathcal{J}_M^{\text{op}} \) to \((\mathcal{J} \times \mathcal{J}_0 \mathcal{J})^{\text{op}} \) corresponds to the \( \mathcal{V} \)
functor \( \mathcal{J}_M^{\text{op}} \to \text{Fibre}_M([\mathcal{J} \times \mathcal{J}_0 \mathcal{J}]^{\text{op}}, \mathcal{V}) \) that sends \( f \) to the fibre square of the outer rectangle.
Example 3.1. It is natural to ask for the following extra condition: if $M$ is the monad whose algebras are split epimorphisms.

Example 3.2. If $\mathcal{J}$ is a double category, with underlying graph $\mathcal{J} \Rightarrow \mathcal{J}_0$, and $U = (U, U_0): \mathcal{J} \to \text{Sq}(C)$ is a double functor, we may consider $\mathcal{J}^{\mathcal{J}_0}$, whose objects are morphisms $g$ of $C$ equipped with a lifting operation $\phi$ against the vertical composition of $i$ and $j$. This condition always holds in orthogonal factorisation systems, i.e., when the diagonal fillers are unique, but it is in general untrue.

Example 3.3. Suppose given a functorial factorisation on a category $\mathcal{C}$, with associated domain-preserving copointed endofunctor $(L, \Phi)$ and associated codomain-preserving pointed endofunctor $(R, \Lambda)$. Then, each $(R, \Lambda)$-algebra $(p, 1): Rg \to g$ induces an object $(g, \phi)$ of $(L, \Phi)$-$\text{Coalg}^{\mathcal{J}_0}$ in the following way. If $(1, s): f \to Lf$ is an $(L, \Phi)$-coalgebra, then

$$
\begin{array}{c}
\phi_{(g, s)}(h, k) \\
\downarrow k \\
\phi_{(h, k)}(g, s)
\end{array}
= 
\begin{array}{c}
f \\
\downarrow s \\
f
\end{array}
$$

This assignment can easily seen to be part of a functor over $\mathcal{C}^2$

$$(R, \Lambda)$-$\text{Alg} \to (L, \Phi)$-$\text{Coalg}^{\mathcal{J}_0}. \quad (3.6)$$

In the case when $(R, \Lambda)$ and $(L, \Phi)$ underlie an AWFS $(L, R)$, the inclusions $R$-$\text{Alg} \hookrightarrow (R, \Lambda)$-$\text{Alg}$ and $L$-$\text{Coalg} \hookrightarrow (L, \Phi)$-$\text{Coalg}$ induce, together with (3.6), a functor over $\mathcal{C}^2$ $R$-$\text{Alg} \hookrightarrow (R, \Lambda)$-$\text{Alg} \to (L, \Phi)$-$\text{Coalg}^{\mathcal{J}_0} \to L$-$\text{Coalg}^{\mathcal{J}_0}$ that is simply a restriction of the lifting operation described above.
One can say a few words regarding the behaviour of $U^{\mathfrak{H}M}$ and (co)limits.

**Lemma 3.4** Suppose given a pullback diagram of $\mathcal{V}$-functors as displayed.

$$
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{P} & \mathcal{B} \\
\downarrow & & \downarrow \\
\mathcal{E} & \xrightarrow{F} & \mathcal{F}
\end{array}
$$

Suppose that $D : \mathcal{D} \to \mathcal{P}$ is a $\mathcal{V}$-functor and that $PD$ has a (co)limit weighted by the weight $\varphi$. Then $P$ creates this (co)limit if $F$ preserves it and $G$ creates (co)limits of functors with domain $\mathcal{D}$ weighted by $\varphi$.

Since $U^{\mathfrak{H}M}$ is the pullback of the monadic $M$-$\text{Alg} \to [\mathcal{J}^{\text{op}}, \mathcal{V}]^2$ along $W_U : \mathcal{C}^2 \to [\mathcal{J}^{\text{op}}, \mathcal{V}]^2$, we readily obtain:

**Lemma 3.5** Let $U : \mathcal{J} \to \mathcal{C}^2$ be a $\mathcal{V}$-functor with $\mathcal{J}$ small. Then: (1) $W_U$ preserves any limit that may exist in $\mathcal{C}^2$ and it has a left adjoint if $\mathcal{C}$ is cocomplete. (2) $U^{\mathfrak{H}M}$ creates limits. (3) $U^{\mathfrak{H}M}$ creates $U^{\mathfrak{H}M}$-split coequalisers.

Since, up to isomorphism, the object part of the internal functor $U^{\mathfrak{H}M} : \mathcal{J}^{\mathfrak{H}M} \to \text{Sq}($ is the identity, we will use the same notation for the arrow part $U^{\mathfrak{H}M} : \mathcal{J}^{\mathfrak{H}M} \to \mathcal{C}^2$.

**Lemma 3.6** The $\mathcal{V}$-functor $U^{\mathfrak{H}M}$ enjoys the following properties.

1. It creates limits and $U^{\mathfrak{H}M}$-split coequalisers.
2. It is monadic if it has a left adjoint. If the base $\mathcal{V}$-category $\mathcal{C}$ has cotensor products with $2$, then it suffices that the underlying ordinary functor of $U^{\mathfrak{H}M}$ should have a left adjoint.

**Proof** The enriched version of Beck’s monadicity theorem together Lemma 3.6 mean that monadicity is guaranteed by the existence of a $\mathcal{V}$-enriched left adjoint. If $\mathcal{C}$ has a cotensor products with $2$, then $U^{\mathfrak{H}M}$ creates them, and therefore $\mathcal{J}^{\mathfrak{H}M}$ has and $U^{\mathfrak{H}M}$ preserves cotensor products with $2$. In these circumstances, a left adjoint for the underlying ordinary functor induces an enriched left adjoint; see [2, Prop. 3.1].

One of the observations of [3] is that, for any $\text{AWFS}$ $(L, R)$, the functor $R-$Alg $\to L$-$\text{Coalg}^{\mathfrak{H}}$ introduced in [11] that expresses the fact that each $R$ algebra has a canonical lifting operation against $L$-coalgebras, co-restricts to an isomorphism

$$R-$Alg $\cong L$-$\text{Coalg}^{\mathfrak{H}}. \quad (3.7)$$

The existence of this isomorphism can easily be extended from ordinary categories to 2-categories, or in our case, to categories enriched in our base of enrichment $\mathcal{V} \subseteq \text{Cat}$ satisfying our blanket hypotheses.

### 4 Lax Orthogonal Factorisation Systems

This section is a short exposition of the basic definitions of lax orthogonal $\text{AWFSS}$s of [5], beginning with lax idempotent 2-monads—which are closely related to Kock–Zöberlein doctrines [22,29].

Lax orthogonal $\text{AWFSS}$s were introduced in [5], in the context of 2-categories. The modification to our framework of categories enriched in a category $\mathcal{V} \subseteq \text{Cat}$ that satisfies our blanket conditions is trivial.
**Definition 4.1** A *lax orthogonal factorisation system* (LOFS) on \( C \) is a \( \mathcal{V} \)-enriched AWFS \((L, R)\) whose 2-comonad \( L \) and 2-monad \( R \) are lax idempotent. Equivalently, as shown in [5, §4], either \( L \) or \( R \) should be lax idempotent.

**Example 4.2** The two AWFSs of Sect. 2.4 are LOFSs.

**Assumption 4.3** For the rest of the section we equip \( \mathcal{V} \) with the LOFS \((E, M)\) described in Sect. 2.4—so \( M \) is the free RALI \( \mathcal{V} \)-monad.

Lax orthogonality is closely related to the notion of KZ-lifting operation [5, §5]. If \( f \) and \( g \) are morphisms in \( C \), a KZ-lifting operation from \( f \) to \( g \) is a RALI structure on the dashed morphism induced by the universal property of pullbacks. If a KZ-lifting operation from \( f \) to \( g \) exists we say that \( f \) and \( g \) are KZ-orthogonal.

\[
\begin{array}{ccc}
\mathcal{C}(f, 1) & \xrightarrow{\mathcal{C}(\text{cod } f, \text{dom } g)} & \mathcal{C}(\text{dom } f, \text{dom } g) \\
\downarrow & & \downarrow \\
\mathcal{C}(1, g) & \xrightarrow{\text{pb}} & \mathcal{C}(1, f)
\end{array}
\]

Slightly more generally, a KZ-lifting operation from a \( \mathcal{V} \)-functor \( U: A \to \mathcal{C}^2 \) to another \( V: B \to \mathcal{C}^2 \) is a RALI structure on the comparison \( \mathcal{V} \)-natural transformation \( \mathcal{C}(\text{cod } U, \text{dom } V) \Rightarrow \mathcal{C}^2(U, V) \) defined in a similar fashion to the previous paragraph. In terms of diagonal fillers, a KZ-lifting operation from \( U \) to \( V \) can be described as an assignment of a diagonal filler \( d(h, k) \) for each morphism \((h, k): Ua \to Vb \) in \( \mathcal{C}^2 \), that is, for each commutative square of the form depicted.

\[
\begin{array}{ccc}
Ua & \xrightarrow{d(h, k)} & Vb \\
\downarrow & & \downarrow \\
h & \xrightarrow{Vb \cdot e} & Vb
\end{array}
\]

These diagonals must be \( \mathcal{V} \)-natural in \( a \in A \) and \( b \in B \). Furthermore, if \( e \) is a morphism parallel to \( d(h, k) \), there must be a bijection between the following two sets of 2-cells:

- **2-Cells** \( \gamma: d(h, k) \Rightarrow e; \) and
- **Pairs of 2-cells** \( \alpha: h \Rightarrow e \cdot Ua \) and \( \beta: k \Rightarrow Vb \cdot e \) such that \( Vb \cdot \alpha = \beta \cdot Ua \).

The bijection must be given by \( \gamma \mapsto (\gamma \cdot Ua, Vb \cdot \gamma) \). More details can be found in [5]. Clearly, KZ-lifting operations are unique up to isomorphism.

**Theorem 4.4** [5, Thm. 6.6] A \( \mathcal{V} \)-enriched AWFS \((L, R)\) on \( C \) is a LOFS if and only if there exists a KZ-lifting operation from the forgetful \( U: L\text{-Coalg} \to \mathcal{C}^2 \) to the forgetful \( V: R\text{-Alg} \to \mathcal{C}^2 \).

**Notation 4.5** When \( \mathcal{V} \) is equipped with the AWFS \((E, M)\) whose \( M \)-algebras are the RALIs in \( \mathcal{V} \)—see Sect. 2.4—then \( \mathcal{J}^{\text{M}} \) and \( \mathcal{J}^{\text{M}} \) will be denoted by \( \mathcal{J}^{\text{lofs}} \) and \( \mathcal{J}^{\text{lofs}} \), respectively. The latter fits in the following pullback square, and is the universal \( \mathcal{V} \)-category over \( \mathcal{C}^2 \) equipped with a KZ-lifting operation from \( U \) to \( U^{\text{lofs}} \)—see [5, §6].

\[
\begin{array}{ccc}
\mathcal{J}^{\text{lofs}} & \xrightarrow{\text{Rali}} & \mathcal{J}^{\text{lofs}} \times \mathcal{V} \\
\downarrow & & \downarrow \\
U^{\text{lofs}} \times \mathcal{C}^2 & \xrightarrow{W_U} & [\mathcal{J}^{\text{op}}, \mathcal{V}]^2
\end{array}
\]
Theorem 4.6  A $\mathcal{V}$-enriched AWFS $(L, R)$ on a $\mathcal{V}$-category $C$ is lax orthogonal if and only if $R\text{-Alg} \cong L\text{-Coalg}^{\text{h}\text{KZ}}$ over $C^2$.

Proof  We use above notations. First observe that, for any $\mathcal{V}$-category $\mathcal{A}$ over $C^2$ the square displayed on the left is a pullback because RALIs are split epis. Moreover, the all four $\mathcal{V}$-functors are full and faithful and injective on objects.

$$
\begin{array}{ccc}
A^{\text{h}\text{KZ}} & \rightarrow & A^{\text{h}\text{KZ}} \\
\downarrow & & \downarrow \\
A^{\text{h}} & \rightarrow & A^{\text{h}}
\end{array}
\quad
\begin{array}{ccc}
R\text{-Alg} & \rightarrow & L\text{-Coalg}^{\text{h}\text{KZ}} \\
\cong & & \cong \\
L\text{-Coalg}^{\text{h}} & \rightarrow & L\text{-Coalg}^{\text{h}}
\end{array}
(4.1)
$$

Theorem 6.6 of [5] proves that our AWFS is lax orthogonal if and only if there exists a $\mathcal{V}$-functor $R\text{-Alg} \rightarrow L\text{-Coalg}^{\text{h}\text{KZ}}$, commuting with the forgetful functors into $C^2$; this $\mathcal{V}$-functor is, moreover, full and faithful. Then, if the AWFS is lax orthogonal, there is a commutative square displayed on the right above, where the isomorphism $R\text{-Alg} \cong L\text{-Coalg}^{\text{h}}$ over $C^2$ is that mentioned in (3.7). The fact that this diagram is a pullback square will follow from the following general argument.

A commutative square of full and faithful $\mathcal{V}$-functors is a pullback square if and only if it is a pullback at the level of objects. Any diagram of injective functions between sets

$$
\begin{array}{ccc}
& \rightarrow & \\
\rightarrow & \cdots & \rightarrow \\
\rightarrow & \cdots & \rightarrow 
\end{array}
$$

is a pullback square if the marked function is a bijection. These two facts show that the diagram on the right of (4.1) is a pullback.

For the converse, an isomorphism $R\text{-Alg} \cong L\text{-Coalg}^{\text{h}\text{KZ}}$ together with the inclusion $L\text{-Coalg}^{\text{h}\text{KZ}} \rightarrow L\text{-Coalg}^{\text{h}}$ gives a $\mathcal{V}$-functor $R\text{-Alg} \rightarrow L\text{-Coalg}^{\text{h}}$, so the AWFS is lax orthogonal by [5, Thm. 6.6], concluding the proof.

$\Box$

5 Cofibrant KZ-Generation

A WFS $(\mathcal{C}, \mathcal{R})$ on a category $C$ is said to be cofibrantly generated by a family of morphisms $J \subseteq \text{Mor} C$ if $\mathcal{R}$ consists of those morphisms with the right lifting property against all members of $J$. Certain cocompleteness and smallness conditions on $C$ guarantee that any small set of morphisms $J$ cofibrantly generates a—unique—WFS. This result is usually called Quillen’s small object argument. A notion of cofibrant generation for AWFSs and the corresponding small object argument appeared in [11] and was later built upon in [3]. The latter article also discussed cofibrant generation of AWFSs on enriched categories, of which we can now say a few words. Our base of enrichment $\mathcal{V} \subseteq \text{Cat}$ will be equipped with an AWFS $(E, M)$.

Definition 5.1  Let $C$ be a $\mathcal{V}$-category, $J$ an internal category in $\mathcal{V}\text{-Cat}$ and $U : J \rightarrow \text{Sq}(C)$ an internal functor. A $\mathcal{V}$-enriched AWFS $(L, R)$ on $C$ is cofibrantly $(E, M)$-generated by $J$ when there is an isomorphism of internal categories $R\text{-Alg} \cong J^{\text{h}M}$ over $\text{Sq}(C)$. This is a straightforward modification of [3, §6.2].
The \( \mathcal{V} \)-enriched AWFS cofibrantly generated by an internal category \( \mathbb{J} \) in \( \mathcal{V} \)-\textbf{Cat} exists if and only if \( \mathbb{J}^{\text{HM}} \rightarrow \mathcal{C}^2 \) is monadic. This can be shown by modifying [3, Thm. 6] to the case of \( \mathcal{V} \)-categories.

**Definition 5.2** We say that \((L, R)\) is cofibrantly generated by a \( \mathcal{V} \)-category \( \mathcal{J} \) over \( \mathcal{C}^2 \) if there is an isomorphism of internal categories \( R \text{-Alg} \cong \mathcal{J}^{\mathbb{M}} \) over \( \text{Sq}(\mathcal{C}) \).

**Example 5.3** When \( \mathbb{M} \) is the \( \mathcal{V} \)-monad on \( \mathcal{V}^2 \) given by \( \mathbb{M}(f) = 1_{\text{cod}(f)} \), then any AWFS that is cofibrantly \((E, M)\)-generated is orthogonal.

**Definition 5.4** The notion of cofibrant \( KZ \)-generation of a \( \mathcal{V} \)-enriched AWFS is a special case of Definition 5.1 when \( \mathcal{V} \) is equipped with the AWFS \((E, M)\), described in Sect. 2.4, for which \( M \)-algebras are RALIS.

**Example 5.5** The AWFS \((E, M)\) on \( \mathcal{V} \) whose \( M \)-algebras are RALIS is cofibrantly \( KZ \)-generated by the discrete \( \mathcal{V} \)-category \((0 \rightarrow 1) \subset \mathcal{V}^2 \). So, \( \mathbb{Rali}(\textbf{Cat}) \cong (0 \rightarrow 1)^{\text{\(KZ\)}} \). Observe that \( \mathbb{Rali}(\textbf{Cat}) \) is not of the form \( \mathbb{J}^{\text{\(KZ\)}} \) for a small category \( \mathcal{J} \) [3, Prop. 17] but can be obtained as \( \mathbb{J}^{\mathbb{J}} \) for a small double category \( \mathbb{J} \) [3, Prop. 19].

**Example 5.6** (Opfibrations) There is an isomorphism over \( \textbf{Cat} \)

\[(1 \rightarrow 2)^{\text{\(KZ\)}} \cong \text{Opfib}_s \quad (5.1)\]

with the 2-category of cloven opfibrations and morphisms in \( \mathcal{C}^2 \) that strictly preserve the cleavages. Indeed, an object of the left hand side of (5.1) is a functor \( g : A \rightarrow B \) for which

\[A^2 = \textbf{Cat}(2, A) \rightarrow \textbf{Cat}^2(0, g) = g \downarrow 1_B \quad (a \xrightarrow{\alpha} a') \mapsto (a, g(\alpha))\]

has a RALI structure. This structure is well-known to be equivalent to an opfibration structure on \( g \). The opcartesian lifting of a morphism \( \beta : g(a) \rightarrow b \) in \( B \) is the diagonal filler \( \phi(a, \beta) \) of the displayed square. The 2-monad of the induced AWFS on \( \textbf{Cat} \) is the free cloven opfibration 2-monad.

**Example 5.7** (Normal opfibrations) The 2-category of normal cloven opfibrations, by which we mean that the opcartesian lifting of an identity morphism is an identity morphism, can be cofibrantly \( KZ \)-generated by the subcategory of \( \textbf{Cat}^2 \) with objects \( 0 : 1 \rightarrow 2 \) and \( 1 : 1 \rightarrow 1 \), and a single non-identity morphism.

**Example 5.8** (Split opfibrations) The 2-category of split opfibrations can be obtained by adding to the previous example the \( KZ \)-generating cofibration

\[2 = (0 \rightarrow 1) \xrightarrow{\delta_2} (0 \rightarrow 1 \rightarrow 2) = 3\]
that is the inclusion that misses out the object \( 2 \in 3 \).

We require compatibility with the following squares, so the 2-category of split opfibrations is \( \mathcal{J}^\text{KZ} \), where \( \mathcal{J} \) is the internal subcategory of \( \text{Sq}(\mathbf{Cat}) \) generated by the square of Example 5.7 and the displayed squares—where \( \delta_i \) is the order-preserving inclusion that misses \( i \).

\[
\begin{array}{ccc}
1 & \xrightarrow{1=\delta_0} & 2 \\
\downarrow{0=\delta_1} \quad \downarrow{\delta_0} & \quad & \downarrow{\delta_2} \\
2 & \xrightarrow{\delta_2} & 3 \\
\end{array}
\quad \quad
\begin{array}{ccc}
1 & \xrightarrow{1} & 2 \\
\downarrow{\delta_1} \quad \downarrow{\delta_1} & \quad & \downarrow{\delta_2} \\
2 & \xrightarrow{\delta_2} & 3 \\
\end{array}
\]  

(5.2)

If \( g : A \to B \) is a functor, as seen in previous examples, a lifting operation against \( \delta_1 : 1 \to 2 \) is a choice of an opcartesian lifting \( \bar{\beta} \) for each \( \beta : g(\alpha) \to b \) in \( B \). A lifting operation against \( \delta_2 : 2 \to 3 \) is a choice, for each morphism \( \alpha \) in \( A \) and each morphism \( \beta' \) in \( B \) with \( \text{dom} \beta' = \text{cod} g(\alpha) \), of an opcartesian lifting for \( \beta' \). The compatibility with the square on the left above is the requirement that this opcartesian lifting should be equal to \( \bar{\beta} \).

The compatibility with the square on the right of (5.2) is the requirement that \( \bar{\beta'} : \bar{\beta} = \beta' \beta \). Together with the square of Example 5.7 we obtain all the conditions of a split opfibration.

6 Existence in the Locally Presentable Case

The existence of the cofibrantly KZ-generated AWFS over a small internal category \( \mathcal{J} \) over \( \text{Sq}(\mathcal{C}) \) in \( \mathcal{V}\text{-Cat} \) can be easily deduced in the case that the underlying category of \( \mathcal{C} \) is locally presentable using techniques from [3]. In Sect. 10 we shall study cofibrant KZ-generation in another context that encompasses examples which, as the category of \( T_0 \) topological spaces, are not locally presentable.

**Proposition 6.1** Suppose that our base of enrichment \( \mathcal{V} \subseteq \mathbf{Cat} \) is locally presentable as an ordinary category and it is equipped with a \( \mathcal{V} \)-enriched AWFS \((E, M)\) whose underlying ordinary AWFS is accessible. Then, the cofibrantly \((E, M)\)-generated AWFS on a small internal category \( \mathcal{J} \) over \( \text{Sq}(\mathcal{C}) \) in \( \mathcal{V}\text{-Cat} \) exists if \( \mathcal{C} \) has tensor products with \( 2 \) and the underlying category of \( \mathcal{C} \) is locally presentable.

**Proof** We use [3, §8.2] to get the result at the level of underlying ordinary categories, and then argue why this implies the \( \mathcal{V} \)-enriched version. By the mere definitions in Sect. 3, the underlying category of the \( \mathcal{V} \)-category \( \mathcal{J}^{\mathcal{V}M} \) is the category \((\mathcal{J}_0)^{\mathcal{V}M}\)—that appears implicitly in [3, §8.2]. Therefore \((U^{\mathcal{V}M})_0 : (\mathcal{J}^{\mathcal{V}M})_0 \to \mathcal{C}_0^2 \) has a left adjoint. The existence of a \( \mathcal{V} \)-enriched left adjoint would be guaranteed if we knew that \( U^{\mathcal{V}M} \) creates cotensor products. The \( \mathcal{V} \)-functor \( U^{\mathcal{V}M} : \mathcal{J}^{\mathcal{V}M} \to \mathcal{C}^2 \) automatically creates cotensors with \( 2 \). This because it is the pullback of a monadic \( \mathcal{V} \)-functor along a \( \mathcal{V} \)-functor \( \hat{U} \) that preserves any limit that exists in \( \mathcal{C}^2 \)—see Sect. 3. Recall that \( U^{\mathcal{V}M} \) is the composition of the equaliser \( \mathcal{J}^{\mathcal{V}M} \to \mathcal{J}^{\mathcal{V}M} \) with \( U^{\mathcal{V}M} : \mathcal{J}^{\mathcal{V}M} \to \mathcal{C}^2 \). Therefore, it will suffice to prove that \( \mathcal{J}^{\mathcal{V}M} \) is closed under cotensors with \( 2 \) in \( \mathcal{J}^{\mathcal{V}M} \). The pair of \( \mathcal{V} \)-functors \( \mathcal{J}^{\mathcal{V}M} \Rightarrow (\mathcal{J} \times_{\mathcal{J}_0} \mathcal{J})^{\mathcal{V}M} \) whose coequaliser is \( \mathcal{J}^{\mathcal{V}M} \) commute.

@Springer
with the respective \( \mathcal{V} \)-functors into \( \mathcal{C}^2 \), which create cotensors with \( \mathbf{2} \). It follows that \( \mathbb{R} \mathfrak{M} \) is closed under these, concluding the proof.  

When \( M \) is the free \( \text{rali} \ \mathcal{V} \)-monad, we obtain:

**Corollary 6.2** AWFSs on \( C \) cofibrantly \( \mathbb{KZ} \)-generated by small internal categories in \( \mathcal{V} \cdot \text{Cat} \) exist provided that \( C \) has cotensor products with \( \mathbf{2} \) and its underlying category is locally presentable.

In Sect. 7 we shall show that cofibrantly \((E, M)\)-generated AWFSs are LOFSs whenever \((E, M)\) is a LOFS. This applies to the corollary above. We next give an example in which \( \mathcal{V} = \text{Cat} \) and \( M \)-algebras are cloven opfibrations—see Example 5.6.

**Example 6.3** Let \((E, M)\) denote the LOFS on \( \text{Cat} \) whose \( M \)-algebras are cloven Grothendieck opfibrations. A pair of morphisms \( f : A \to B \) and \( g : C \to D \) in a 2-category \( C \) satisfies \( f \cdot \mathfrak{M} g \) when the comparison functor \( C(B, C) \to C(A, C) \times C(A, D) C(B, D) \) is an opfibration. This can be unpacked in the following way. Given a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
\downarrow{f} & \alpha & \downarrow{g} \\
B & \xrightarrow{k} & D
\end{array}
\]

with \( \beta \cdot f = g \cdot \alpha \), there exists a pre-opcartesian lifting, which is a 2-cell \( \gamma : d \Rightarrow \tilde{d} \) such that \( \gamma \cdot f = \alpha \) and \( g \cdot \gamma = \beta \). Its pre-opcartesian property means that any other \( \gamma' : d \Rightarrow d' \) with the same property as \( \gamma \) factors uniquely through \( \gamma \). Furthermore, pre-opcartesian morphisms should be closed under composition.

Therefore, any small set of functors \( \{f_i\} \) cofibrantly \((E, M)\)-generates an AWFS (in fact, a LOFS, as we shall see in Sect. 7) \((L, R)\) on \( \text{Cat} \). The \( R \)-algebras are those functors \( g : C \to D \) with the extra structure described in the previous paragraph, for each \( f_i \).

Let us look at the example of the family with one element \( 0 : 1 \to 2 \), the functor that picks out \( 0 \in 2 = (0 \to 1) \). This functor plays the role of \( f \) above. A functor \( h : 1 \to C \) is an object \( c \in C, k \) and \( d \) are morphisms in \( D \) and \( C \), respectively. The transformations \( \alpha \) and \( \beta \) are a morphism \( \text{dom}(d) \to c \) and a commutative square as depicted on the right below, respectively.

\[
\begin{array}{ccc}
\bullet & \xrightarrow{d} & \bullet \\
\alpha & & \downarrow{g} \\
\bullet & \xrightarrow{\beta} & \bullet \\
\end{array}
\quad
\begin{array}{ccc}
\bullet & \xrightarrow{g(d)} & \bullet \\
\downarrow{g(\alpha)} & & \downarrow{\beta_1} \\
\bullet & \xrightarrow{k} & \bullet
\end{array}
\]

The universal property of \( d \) asserts that the solid cospan on the left can be completed to a commutative square that is mapped by \( g \) to the square on the right. Furthermore, this dotted completion is universal, translating the pre-opcartesian property. This means that any other completion of the solid cospan to a commutative square that is mapped by \( g \) to the square on the right factors uniquely by the universal square. The fact that pre-opcartesian morphisms are closed under composition is, in this case, automatic, as it means that pushout squares are closed under pasting.

We may specialise to the case of an \( R \)-algebra of the form \( g : C \to 1 \), in other words, to fibrant objects. In this case all we have is a choice of a pushout for each cospan in \( C \). In other words, the fibrant replacement monad of \((L, R)\) is the monad on \( \text{Cat} \) whose algebras
are categories with chosen pullbacks. A general $R$-algebra can be regarded as a functor that has pushouts fibrewise.

7 Lax Orthogonality of KZ-Cofibrantly Generated AWFSs

When an AWFS $(L, R)$ is cofibrantly KZ-generated by $U : \mathcal{J} \to \mathcal{C}^2$, each $R$-algebra is KZ-orthogonal to each $Uj$. What is by no means obvious is that each $R$-algebra is KZ-orthogonal to all the $L$-coalgebras. I.e., the AWFS is lax orthogonal, or a LOFS. The proof of this fact is the subject of the present section.

Theorem 7.1 Assume that $\mathcal{V}$ is equipped with a LOFS $(E, M)$ and $C$ is a complete and cocomplete $V$-category. Any AWFS on $C$ that is cofibrantly $(E, M)$-generated by a small $\mathcal{V}$-category is lax orthogonal. In particular, any AWFS that is cofibrantly KZ-generated by a small $\mathcal{V}$-category is lax orthogonal.

Proof Suppose that the $\mathcal{V}$-enriched AWFS $(L, R)$ on $C$ is cofibrantly $(E, M)$-generated by a $\mathcal{V}$-category $\mathcal{J}$ over $\mathcal{C}^2$. Then we have that the forgetful $\mathcal{V}$-functor $R$-$Alg \to \mathcal{C}^2$ is the pullback of $M$-$Alg \to [\mathcal{J}^{\text{op}}, \mathcal{V}]^2$ along $W_U : \mathcal{C}^2 \to [\mathcal{J}^{\text{op}}, \mathcal{V}]^2$. Corollary A.3 will ensure that $R$ is lax idempotent once we have shown that the right Kan extensions $\text{Ran}_{W_U}$ exist. The latter condition is obvious since $W_U$ has a left adjoint—see Lemma 3.5. \(\square\)

The particular case of KZ-generation is obtained by setting $(E, M)$ the LOFS whose $M$-algebras are RALIS.

Theorem 7.2 Assume that $\mathcal{V}$ is equipped with a AWFS $(E, M)$. Suppose that $C$ is a complete $\mathcal{V}$-category with the property that any small $\mathcal{V}$-category over $\mathcal{C}^2$ cofibrantly $(E, M)$-generates a LOFS. Then, any AWFS cofibrantly $(E, M)$-generated by a small internal category in $\mathcal{V}$-$\text{Cat}$ is lax orthogonal. In particular, this applies to cofibrant KZ-generation.

Proof Suppose that $\mathcal{J}$ cofibrantly KZ-generates the AWFS $(L, R)$. By considering the equaliser (3.5), we can exhibit $\mathcal{J}^{\text{hM}}$ as the equaliser of a pair of $\mathcal{V}$-functors, as displayed in the left below, where $T$ and $S$ are the monad part of the lax orthogonal AWFSs generated by $\mathcal{J}$ and $\mathcal{J} \times_{\mathcal{J}_0} \mathcal{J}$, respectively. In fact, in what follows we only need to regard $R$, $T$ and $S$ as ordinary monads on the ordinary category $\mathcal{C}^2$. Since any functor between categories of algebras that commutes with the respective forgetful functors is induced by a unique morphism of monads, there is a commutative diagram in the category $\text{Mnd}(\mathcal{C}^2_o)$ as displayed on the right hand side.

\[
\begin{array}{ccc}
R\text{-Alg} & \cong & \mathcal{J}^{\text{hM}} \\
\downarrow & & \downarrow \\
T\text{-Alg} & \longrightarrow & \text{S-Alg} \\
\tau \downarrow & & \nu \downarrow \quad T_0 \quad \zeta \downarrow \\
S_0 & \longrightarrow & T_0 & \longrightarrow & R_0
\end{array}
\]

By definition, this diagram induces the equaliser diagram at the level of corresponding categories of algebras over $\mathcal{C}^2_o$, depicted on the left, so it exhibits the ordinary monad $R_o$ as the algebraic coequaliser of $\tau$ and $\nu$, and by [17, Prop. 26.2], as the coequaliser of this pair of morphisms in the category $\text{Mnd}(\mathcal{C}^2_o)$ of monads on $\mathcal{C}^2_o$.

Lax idempotent 2-monads on a 2-category $\mathcal{A}$ are characterised by being co-orthogonal to certain 2-monad morphisms $\sigma_f$, for each morphism $f$ in $\mathcal{A}$, as shown in [19] and recalled in Sect. A.3. Thus, we have to prove that $R_o$ is co-orthogonal to these morphisms $\sigma_{(h,k)}$, for $(h, k)$ a morphism in $\mathcal{C}^2$. As the full subcategory of objects co-orthogonal to each member of a family of morphisms is always closed under colimits, $R_o$ is co-orthogonal to each $\sigma_{(h,k)}$, completing the proof. \(\square\)
Under certain conditions, the hypotheses of Theorem 7.2 are always satisfied, as for example:

**Corollary 7.3** Suppose that \((E, M)\) is an accessible \(\mathcal{V}\)-enriched LOFS on the locally presentable \(\mathcal{V}\), and that the underlying category of the complete and cocomplete \(\mathcal{V}\)-category \(C\) is locally presentable. Then, any small internal category \(J\) in \(\mathcal{V}\)-\text{Cat} over \(\text{Sq}(C)\) cofibrantly \((E, M)\)-generates a \(\mathcal{V}\)-enriched AWFS which, moreover, is lax orthogonal. In particular, this applies to \(KZ\)-generation.

The proof of the corollary consists of a simple application of Proposition 6.1 and Theorem 7.2.

**Corollary 7.4** The AWFSs whose right morphisms are, respectively, opfibrations, normal opfibrations, and split opfibrations are lax orthogonal.

**Proof** The three AWFSs are of the form \(J^{\uparrow}_{KZ}\) for a small category \(J\) or \(J^{\uparrow\downarrow}_{KZ}\) for a small double category \(J\), as seen in Examples 5.6, 5.7 and 5.8. \(\Box\)

### 8 Representable Multicategories

As an application of the theory developed thus far, in this section we exhibit opfibrations of multicategories as the right part of a LOFS on the 2-category of multicategories that is cofibrantly \(KZ\)-generated by a double category of easy description. Opfibrations of multicategories were considered (and called covariant fibrations) in [15] as generalisations of the representable multicategories, in the sense that the latter are opfibrations over the terminal multicategory.

#### 8.1 Background on Multicategories

We assume the reader is acquainted with the notion of multicategory, or non-\(\Sigma\) coloured operad. The idea goes back to Lambek [23] and modern expositions can be found in [14] and [24, I.2].

We shall represent multicategories by blackboard bold letters, so \(\mathbb{A}\) will be a multicategory. The 2-category of multicategories will be \(\text{MCat}\). A module \(\phi: \mathbb{A} \to \mathbb{B}\) consists of sets \(\phi(b_1, \ldots, b_n; a)\) with actions of \(\mathbb{A}\) and \(\mathbb{B}\) on each side. See, for example, [24, I.2.3]. The collage for \(\phi\) is the multicategory \(\text{coll}(\phi)\) with objects the disjoint union of those of \(\mathbb{A}\) and \(\mathbb{B}\), which contains both multicategories as full sub-multicategories, and with \(\text{coll}(\phi)(b_1, \ldots, b_n, a) = \phi(b_1, \ldots, b_n; a)\). The remainder multihoms are empty. The composition of \(\text{coll}(\phi)\) uses the module structure of \(\phi\). We shall denote by \(j_\mathbb{B}: \mathbb{B} \to \text{coll}(\phi)\) the full inclusion. Collages enjoy a universal property that we shall need only in the following special case.

The \(n\)th cardinal \(\{0 < 1 < \cdots < n - 1\}\) we shall denote simply by \(n\), and regard it both as a set and as a discrete multicategory. Consider the module \(\phi_n: 1 \to n\) given by \(\phi_n(0, 1, \ldots, n - 1; 0) = 1\) and all the other possibilities equal to \(\emptyset\). Then \(C(n) := \text{coll}(\phi_n)\) has objects \(0, \ldots, n - 1\) and an extra object that we denote by \(*\). Aside from the multihoms that correspond to the identity maps, there is a single non-empty multihom \(C(n)(0, \ldots, n - 1; *) = 1\). There clearly is a bijection between morphisms of multicategories \(C(n) \to \mathbb{B}\) and multimorphisms of the form \(b_0, \ldots, b_{n-1} \to b\) in \(\mathbb{B}\).
8.2 Opfibrations of Multicategories

Let us now turn to opfibrations of multicategories, called covariant fibrations of multicategories in [15]. Given a multicategory morphism $P : E \to B$ and a multimap $f : P(e_1), \ldots, P(e_n) \to b$ in $B$, an opcartesian lifting of $f$ is a multimap $\bar{f} : e_1, \ldots, e_n \to e$ in $E$ with the property that: for any $g : e_1, \ldots, e_n \to x$ and $w : b \to P(x)$ such that $P(g) = w \cdot f$, there exists a unique $v : e \to x$ satisfying $v \cdot f = g$. One says that $P$ is an opfibration of multicategories if each multimap has an opcartesian lifting. This definition is completely analogous to that of an opfibration of categories, only with the appropriate modifications to allow for multimaps. The usual argument shows that the definition we use is equivalent to that of [15].

A cloven opfibration of multicategories is an opfibration with extra structure, a cleavage, that provides a choice of opcartesian lifting for each multimap $f$ as in the previous paragraph. It is easy to verify that to give a cleavage for $P$ is the same as giving, for each $n$, a rali structure to the functor

$$
\text{MCat}(C(n), E) \to \mathcal{E}^n \times_B \text{MCat}(C(n), B)
$$

(8.1)

(where $\mathcal{E}$ is the underlying category of $E$) whose first coordinate is induced by pre-composing with $n \hookrightarrow C(n)$, while the second coordinate is induced by post-composing with $P$.

If $P$ and $P'$ are two cloven opfibrations of multicategories, a strict morphism $P \to P'$ is a morphism of $\text{MCat}^2$:

$$
\begin{array}{ccc}
E & \xrightarrow{H} & E' \\
\downarrow P & & \downarrow P' \\
B & \xleftarrow{K} & B'
\end{array}
$$

where $H$ preserves the chosen opcartesian liftings. This means that if $f$ is an opcartesian multimap in $E$, then $H(f)$ is the chosen opcartesian lifting of $K(f)$. In this way we obtain a category $\text{OpFib}$ of cloven opfibrations of multicategories, together with a forgetful functor

$$
\text{OpFib} \longrightarrow \text{MCat}^2.
$$

(8.2)

We make $\text{OpFib}$ into a 2-category, and (8.2) into a 2-functor, by declaring that (8.2) is locally full and faithful.

Write $\text{ARepMCat}$ for the fibre of $\text{OpFib}$ (8.2) over 1. Its objects we call algebraic representable multicategories. A slight modification of the arguments of [14] shows that an algebraic representable multicategory structure on $\mathcal{A}$ amounts to an unbiased monoidal category structure on the underlying category of $\mathcal{A}$—see [24, I.3.1]. There is an isomorphism between $\text{ARepMCat}$ and the 2-category $\text{UMonCat}_s$ of unbiased monoidal categories and strict monoidal functors.

The last piece of background we shall need is the fact that $\text{MCat}$ is locally finitely presentable, as an ordinary category. This can be shown directly, by hand, as it were. It will be more convenient to appeal to [24, D.1], specifically Theorem 6.5.4 and Proposition 6.5.6. These deal with generalised multicategories, whose type, or shape, is defined by a certain monad $T$ on a category $\mathcal{E}$. In our case, these are the free monoid monad on $\text{Set}$. The said results tell us that the forgetful functor from $\text{MCat}$ to multigraphs is monadic with a finitary associated monad. The category of multigraphs is locally finitely presentable, being the slice of $\text{Set}$ over

$$
\text{Set} \xrightarrow{\Delta} \text{Set} \times \text{Set} \xrightarrow{T \times 1} \text{Set} \times \text{Set} \xrightarrow{\times} \text{Set}
$$

where $T$ is the free monoid monad. We deduce that $\text{MCat}$ is itself locally finitely presentable.
8.3 Cofibrant KZ-Generation

Let $\mathcal{F}$ be the set of finite cardinals (i.e., the set of natural numbers) and $U : \mathcal{F} \to \mathbf{MCat}^2$ the functor given by $U(n) = j_n : n \hookrightarrow C(n)$. An object of $\mathcal{F}^{\text{KZ}}$ consists of a morphism of multicategories $P : E \to B$ with a RALI structure on the comparison functor

$$\mathbf{MCat}(C(n), E) \to \mathbf{MCat}(n, E) \times \mathbf{MCat}(n, B) \times \mathbf{MCat}(C(n), B)$$

for each $n$. This is the same functor as (8.1), so we see that $\mathcal{F}^{\text{KZ}}$ has cloven opfibrations as objects. One easily verifies that its morphisms are morphisms of opfibrations that preserve cleavages.

**Proposition 8.1**

1. There is an isomorphism $\text{OpFib} \cong \mathcal{F}^{\text{KZ}}$.
2. There is an isomorphism between the fibre of $\mathcal{F}^{\text{KZ}}$ over $1$ and the 2-category of unbiased monoidal categories and strict monoidal functors.

An application of Corollary 6.2 and Theorem 7.1 yields:

**Proposition 8.2**

The category $\mathcal{F}$ over $\mathbf{MCat}^2$ cofibrantly KZ-generates a LOFS $(L, R)$ on $\mathbf{MCat}$. The $R$-algebras are the cloven opfibrations of multicategories.

Taking the fibre of $R$-Alg over the terminal multicategory, we easily obtain the following consequence.

**Corollary 8.3**

The 2-functor from $\mathbf{UMonCat}$ to $\mathbf{MCat}$ is strictly monadic. The associated monad is lax idempotent.

The corollary can be regarded as a version of the results in [14]. There are a couple of differences, though. First, the techniques in *ibid* are suitable for application to generalised multicategories, as done in [24] and [15]. Secondly, [14] compares strict monoidal categories with multicategories, while our corollary involves unbiased monoidal categories.

**Remark 8.4**

It is not hard to cofibrantly KZ-generate two LOFSS on $\mathbf{MCat}$ whose fibrant objects are normal unbiased monoidal categories, and strict monoidal categories, respectively. In order to do this, one should add some generating squares to $\mathcal{F}$, and employ the resulting double category.

9 Cofibrant Generation and Accessibility

If $\mathcal{J}$ is a small double category over $\mathbf{Sq}(\mathcal{C})$ and $\mathcal{C}$ is locally presentable, [3] showed that $\mathcal{J}^{\mathbf{Set}} \to \mathcal{C}^2$ creates $\kappa$-filtered colimits, for some $\kappa$. In this section, we look at the case when $\mathcal{C}$ is not necessarily locally presentable but, following [17], it is equipped with an OFS. Our main example will be the category of topological spaces. The section’s results will be used later on to give examples of LOFSS that are not cofibrantly (KZ-)generated.

Let $\mathcal{M}$ be a subcategory of a category $\mathcal{A}$. By an $\mathcal{M}$-diagram in $\mathcal{A}$ we will mean a functor $\mathcal{D} \to \mathcal{A}$ that factors through $\mathcal{M}$ and has small domain. This $\mathcal{M}$-diagram will be said to be $\kappa$-filtered, for a regular cardinal $\kappa$, if $\mathcal{D}$ is a $\kappa$-filtered category.

**Definition 9.1** Let $\mathcal{A}$ be a cocomplete category with a subcategory $\mathcal{M}$. A functor with domain $\mathcal{A}$ is $\mathcal{M}$-accessible if it preserves $\kappa$-filtered colimits of $\mathcal{M}$-diagrams, for some regular cardinal $\kappa$. 

\[ \text{Springer} \]
An object $A$ of $\mathcal{A}$ has $\mathcal{M}$-rank less or equal to a regular cardinal $\kappa$ if the representable functor $\mathcal{A}(A, -)$ is $\mathcal{M}$-accessible. The $\mathcal{M}$-rank of $A$ is the smallest regular cardinal $\kappa$ for which this happens. When there is no risk of confusion, we may omit $\mathcal{M}$ and simply say rank. The notion of rank of an object has been attributed by [10] to M. Barr. We say that $\mathcal{A}$ is locally $\mathcal{M}$-ranked if each of its objects has a rank. The subcategory $\mathcal{M}$ will most often be the right part of an OFS $(\mathcal{E}, \mathcal{M})$. Categories locally bounded with respect to an OFS $(\mathcal{E}, \mathcal{M})$—in the sense of [21, §6.1]—are locally ranked for $\mathcal{M}$, due to the argument given in [10, §3.2]. A number of examples are given in [10] and [21, §6.1]; later we will be interested in the example of the category of $T_0$ topological spaces equipped with the OFS $\mathcal{E} = \text{surjections}$ and $\mathcal{M} = \text{subspace inclusions}$.

Each subcategory $\mathcal{M}$ of $\mathcal{A}$ induces a subcategory of $\mathcal{A}^2$ component-wise, i.e., the subcategory consisting of those morphisms in $\mathcal{A}^2$ with both domain and codomain components in $\mathcal{M}$. We will still denote this subcategory by $\mathcal{M}$ when no confusion is likely. It is easy to show that a morphism $f : X \to Y$ has rank less or equal to $\kappa$ as an object of $\mathcal{A}^2$ if $X$ and $Y$ have rank less or equal to $\kappa$.

For an object $X$ of a $\mathcal{V}$-category $\mathcal{C}$ and a subcategory $\mathcal{M} \subseteq C_\circ$, one has two related notions. First, one can say that $\mathcal{C}(X, -) : \mathcal{C} \to \mathcal{V}$ preserves $\mathcal{V}$-enriched colimits of $\kappa$-filtered $\mathcal{M}$-diagrams. This is the same as asserting that the ordinary functor $\mathcal{C}(X, -) : C_\circ \to \mathcal{V}$ preserves these colimits. Secondly, one can say that $X$ has $\mathcal{M}$-rank less or equal to $\kappa$, or what is the same, that $C_\circ(X, -) : C_\circ \to \mathcal{Set}$ preserves $\kappa$-filtered colimits of $\mathcal{M}$-diagrams. The former implies the latter but not the other way around, not in general. Before stating a standard lemma looking at the relationship between these conditions, we recall a piece of terminology: an object $X \in \mathcal{V}$ is finitely presentable when $\mathcal{V}(X, -) : \mathcal{V} \to \mathcal{Set}$ preserves filtered colimits, i.e., when it has rank $\aleph_0$.

Lemma 9.2 Assume, in addition to our blanket hypotheses on $\mathcal{V}$, that the inclusion $\mathcal{V} \hookrightarrow \mathcal{Cat}$ is finitary. Let $\mathcal{C}$ be a cocomplete $\mathcal{V}$-category with a subcategory $\mathcal{M} \subseteq C_\circ$. Denote by $(C_\circ)_\kappa \subseteq C_\circ$ the full subcategory of objects of $\mathcal{M}$-rank less or equal to $\kappa$. For $X \in \mathcal{C}$, the functor $\mathcal{C}(X, -) : C_\circ \to \mathcal{V}$ preserves $\kappa$-filtered colimits of $\mathcal{M}$-diagrams if and only if $2 \times X \in (C_\circ)_\kappa$.

Proof The first observation is that $J : \mathcal{V} \hookrightarrow \mathcal{Cat}$ is finitary if and only if $2$ is a finitely presentable object of $\mathcal{V}$. Indeed, the arrow category $2$ is a strong generator in $\mathcal{Cat}$, so $\mathcal{Cat}(2, -) : \mathcal{Cat} \to \mathcal{Set}$ preserves and reflects filtered colimits. Then, $J$ is finitary if and only if its composition with $\mathcal{Cat}(2, -)$, which is isomorphic to $\mathcal{V}(2, -) : \mathcal{V} \to \mathcal{Set}$, is so.

The functor $\mathcal{C}(2 \times X, -) : C_\circ \to \mathcal{Set}$ is isomorphic to the composition of the representables $\mathcal{C}(X, -) : C_\circ \to \mathcal{V}$ and $\mathcal{V}(2, -) : \mathcal{V} \to \mathcal{Set}$. The latter preserves and reflects filtered colimits. It follows that $\mathcal{C}(2 \times X, -)$ and $\mathcal{C}(X, -)$ preserve the same filtered colimits. \(\square\)

Assumption 9.3 In addition to our blanket hypotheses on $\mathcal{V}$ (it is closed under limits and exponentials in $\mathcal{Cat}$, it is cocomplete and $2 \in \mathcal{V}$) we shall, for the rest of this section, assume that it is closed under filtered colimits in $\mathcal{Cat}$. As seen in the previous lemma’s proof, this new condition is equivalent to requiring that $2$ should be a finitely presentable object of $\mathcal{V}$. Furthermore, it makes $\mathcal{V}$ a finitely locally presentable category. Our main examples will be $\mathcal{V} = \mathcal{Cat}$ and $\mathcal{V} = \mathcal{Ord}$ the category of posets.

Lemma 9.4 Given a morphism $j : X \to Y$ in $\mathcal{C}$, the functor $\mathcal{C}(j, -) : C_\circ^2 \to \mathcal{V}$ preserves $\kappa$-filtered colimits of $\mathcal{M}$-diagrams if and only if $2 \times X$ and $2 \times Y$ lie in $(C_\circ)_\kappa$.

Proof The functor $\mathcal{C}(j, -)$ can be constructed as the pullback of the diagram

\[
\begin{array}{ccc}
C(X, \text{dom } -) & \longrightarrow & C(X, \text{cod } -) & \longleftarrow & C(Y, \text{cod } -)
\end{array}
\]
so it will preserve any filtered colimit that is preserved by \( C(X, -) \) and \( C(Y, -) \). The result now follows from Lemma 9.2.

**Definition 9.5** A \( \mathcal{V} \)-enriched AWFS \((E, M)\) on a cocomplete \( \mathcal{V} \)-category is \( M \)-accessible if one of the following equivalent conditions holds: the comonad \( E \) is \( M \)-accessible; the monad \( M \) is \( M \)-accessible; the \( \mathcal{V} \)-functor \( K = \text{dom} \ M = \text{cod} \ E : \mathcal{C}^2 \to \mathcal{C} \) is \( M \)-accessible. When \( M \) is the whole category, one says that the AWFS is accessible.

**Proposition 9.6** Suppose \( \mathcal{V} \) is equipped with an accessible AWFS \((E, M)\) and that it satisfies the hypotheses in Assumption 9.3. Suppose that \( \mathcal{C} \) is a cocomplete \( \mathcal{V} \)-category whose underlying category \( \mathcal{C}_0 \) is locally ranked with respect to the subcategory \( M \). If \( \mathcal{I} \) is a small internal category in \( \mathcal{V} \)-\textbf{Cat} over \( \mathcal{S}(\mathcal{C}) \), then \( \mathcal{I}^{\text{frm}} \to \mathcal{C}^2 \) creates \( \kappa \)-filtered colimits of \( \mathcal{M} \)-diagrams, for some regular cardinal \( \kappa \).

**Proof** We first prove the case of cofibrant generation by a small category \( \mathcal{J} \) over \( \mathcal{C}^2 \). By definition—see diagram (3.3)—this is the pullback of the forgetful \( V : \text{M-Alg} \to [\mathcal{J}^{\text{op}}, \mathcal{V}]^2 \) along \( W_U \). The enriched monad \( M \) is cocontinuous, so \( V \) creates all colimits and, by Lemma 3.4, it only remains to verify that \( W_U \) preserves \( \kappa \)-filtered colimits of \( \mathcal{M} \)-diagrams, for some \( \kappa \).

Since \( \mathcal{J} \) is small, there is a regular cardinal \( \kappa \) such that each \( Uj \in \mathcal{C}^2 \) has rank less or equal to \( \kappa \), i.e., \( \mathcal{C}^2(Uj, -) \) preserves \( \kappa \)-filtered colimits of \( \mathcal{M} \)-diagrams. This means that \( \tilde{U} : \mathcal{C}^2 \to [\mathcal{J}^{\text{op}}, \mathcal{V}] \), given by \( f \mapsto \mathcal{C}^2(U - , f) \), preserves \( \kappa \)-filtered colimits of \( \mathcal{M} \)-diagrams. Furthermore, the functor \( W_U \) sends \( f \) to the (dashed) comparison morphism depicted in (3.1), from where it is clear that \( W_U \) preserves \( \kappa \)-filtered colimits of \( \mathcal{M} \)-diagrams.

Now we prove the case of cofibrant generation by an internal category \( \mathcal{J} = (\mathcal{J} \Rightarrow \mathcal{J}_0) \) in \( \mathcal{V} \)-\textbf{Cat} and \( U = (U_1, U_0) : \mathcal{J} \to \mathcal{S}(\mathcal{C}) \) an internal functor. Denote by \( V \) the functor \( \mathcal{J} \times \mathcal{J}_0 \mathcal{J} \to \mathcal{C}^2 \) from the object of composable pairs. By definition, \( \mathcal{J}^{\text{frm}} \) is an equaliser (3.5) in \( \mathcal{V} \)-\textbf{Cat}/\( \mathcal{C}^2 \) of a pair of \( \mathcal{V} \)-functors of the form \( \mathcal{I}^{\text{frm}}_1 \Rightarrow \mathcal{I}^{\text{frm}}_2 \). Each of the two respective \( \mathcal{V} \)-functors into \( \mathcal{C}^2 \) creates \( \kappa \)-filtered colimits of \( \mathcal{M} \)-diagrams, for some \( \kappa \). We choose the largest of these two cardinals, so both \( \mathcal{V} \)-functors create \( \kappa \)-filtered colimits of \( \mathcal{M} \)-diagrams. It follows that the parallel \( \mathcal{V} \)-functors strictly preserve the created colimits and that \( \mathcal{J}^{\text{frm}} \) is closed under these in \( \mathcal{I}^{\text{frm}}_1 \), completing the proof.

**Corollary 9.7** Suppose that \( \mathcal{V} \) and \( \mathcal{C} \) are as in Proposition 9.6. Then, any cofibrantly \((E, M)\)-generated AWFS on \( \mathcal{C} \) is \( M \)-accessible.

The reader would remember that the notation \( \mathcal{J}^{\text{frm}} \), without mention of the \( \mathcal{V} \)-monad \( M \), means that we are taking as \( M \) the \( \mathcal{V} \)-monad whose algebras are split epimorphisms. The objects of \( \mathcal{J}^{\text{frm}} \) were described in Example 3.1.

**Lemma 9.8** Let \((L, R)\) be an AWFS on the \( \mathcal{V} \)-category \( \mathcal{C} \), with underlying wfs \((\mathcal{L}, \mathcal{R})\), and \( \mathcal{I} \subset \mathcal{L} \) a small set of morphisms. There exists a \( \mathcal{V} \)-functor over \( \mathcal{C}^2 \)

\[
R \text{-Alg} \longrightarrow \mathcal{I}^{\text{frm}}.
\]

**Proof** The class \( \mathcal{L} \) consists of those morphisms of \( \mathcal{C} \) that admit at least one coalgebra structure for the copointed endo-\( \mathcal{V} \)-functor \((L, \Phi)\). Since \( \mathcal{I} \) is small, we can choose a coalgebra structure for each \( i \in \mathcal{I} \), or equivalently, a \( \mathcal{V} \)-functor over \( \mathcal{C}^2 \)

\[
\mathcal{I} \longrightarrow (L, \Phi)\text{-Coalg}
\]
with domain the discrete \( \mathcal{V} \)-category associated to the set \( \mathcal{I} \). We can now consider the composition of functors

\[
\mathbf{R} \text{-Alg} \hookrightarrow (R, \Lambda)\text{-Alg} \rightarrow (L, \Phi)\text{-Coalg}^{\mathcal{V}} \rightarrow \mathcal{I}^{\mathcal{V}}
\]

where the first one is an inclusion, the second is the \( \mathcal{V} \)-functor of (3.6)—in fact, (3.6) describes an ordinary functor that can easily be reproduced in the \( \mathcal{V} \)-enriched context—and the last one is the \( \mathcal{V} \)-functor induced by (9.1).

The construction of the above lemma is valid only because \( \mathcal{I} \) is a set instead of a category, in which case we would not be able to guarantee the compatibility between the coalgebra structures chosen for each \( i \in \mathcal{I} \) and the morphisms of \( \mathcal{I} \).

Proposition 9.9 Suppose that \( \mathcal{V} \) satisfies Assumption 9.3 and that \( \mathcal{C} \) is a \( \mathcal{V} \)-category satisfying: (a) Its underlying category \( \mathcal{C}_o \) is cocomplete and is locally ranked with respect to the subcategory \( \mathcal{M} \). (b) It has a terminal object. (c) It is equipped with a \( \mathcal{V} \)-enriched awfs \( (\mathcal{L}, \mathcal{R}) \) whose underlying wfs \( (\mathcal{L}, \mathcal{R}) \) is cofibrantly generated by a set of morphisms with \( \mathcal{M} \)-rank. Then, there is a regular cardinal \( \kappa \) such that any \( \kappa \)-filtered colimit \( \operatorname{colim} X_j \) of an \( \mathcal{M} \)-diagram is a fibrant object for \( (\mathcal{L}, \mathcal{R}) \), provided that \( X_j \) is a diagram in \( \mathbf{R}_1 \text{-Alg} \).

Proof Suppose that \( \mathcal{I} \subset \mathcal{L} \) cofibrantly generates \( (\mathcal{L}, \mathcal{R}) \).

Consider a \( \mathcal{V} \)-functor \( \mathbf{R} \text{-Alg} \rightarrow \mathcal{I}^{\mathcal{V}} \) over \( \mathcal{C}^2 \) as provided by Lemma 9.8. Taking the fibre over the terminal object \( 1 \in \mathcal{C} \), we obtain the first arrow displayed below.

\[
\mathbf{R}_1 \text{-Alg} \rightarrow (\mathcal{I}^{\mathcal{V}})_1 \rightarrow \mathbf{Fib}
\] (9.2)

The second arrow is the inclusion into the full subcategory of fibrant objects, which exists since any object with a lifting operation against \( \mathcal{I} \) is certainly weakly orthogonal to \( \mathcal{I} \). Proposition 9.6 guarantees that the forgetful \( \mathcal{I}^{\mathcal{V}} \rightarrow \mathcal{C}^2 \) creates \( \kappa \)-filtered colimits of \( \mathcal{M} \)-diagrams for some regular cardinal \( \kappa \). If \( \{X_j\} \) is a \( \kappa \)-filtered diagram in \( \mathbf{R}_1 \text{-Alg} \) whose morphisms also lie in \( \mathcal{M} \), and if colim \( X_j \) exists in \( \mathcal{C} \), we can send \( \{X_j\} \) along (9.2) to a diagram in \( (\mathcal{I}^{\mathcal{V}})_1 \), and deduce that colim \( X_j \) is created by \( (\mathcal{I}^{\mathcal{V}})_1 \rightarrow \mathcal{C} \). In particular, colim \( X_j \) supports a structure of an object of \( (\mathcal{I}^{\mathcal{V}})_1 \), thus it is a fibrant object for \( (\mathcal{L}, \mathcal{R}) \). \( \square \)

10 Order-Enriched AWFSs

We now turn our attention to the case of order-enriched AWFSs and their cofibrant KZ-generation. The case of a locally presentable base category was dealt with in Sect. 6, so we now concentrate in the non-locally presentable case, as to include important examples as that of the category of topological spaces. The section ends with a short comparison with [1].

We will denote the category of posets by \( \mathbf{Ord} \). By poset we mean a set equipped with a partial ordering that is antisymmetric, or equivalently, a small category that has at most one morphism between any two objects and whose isomorphisms are identity morphisms.

Assumption 10.1 Let us assume that our category \( \mathcal{V} \subseteq \mathbf{Cat} \) is the Cartesian closed category \( \mathbf{Ord} \) of posets. We equip it with an accessible LOFS \( (\mathcal{E},\mathcal{M}) \).
Lemma 10.2 Let \( T = (T, \eta, \mu) \) be a lax idempotent monad on an \textit{Ord}-category \( A \). The forgetful \textit{Ord}-functor \( T\text{-Alg} \to (T, \eta)\text{-Alg} \) is an isomorphism.

This lemma is implicit in, for example, [9, Cor. 4.2.3]. To give a short proof, recall that the identity \( 1_{TA} \) is a left extension of \( \eta_A \) along \( \eta_A \), since \( T \) is lax idempotent. If \( a : TA \to A \) satisfies \( a \cdot \eta_A = 1_A \), it will be a \( T \)-algebra structure as soon as \( a \dashv \eta_A \). All that is left to show is that \( 1_{TA} \leq \eta_A \cdot a \), which is equivalent to \( \eta_A \leq \eta_A \cdot a \cdot \eta_A \), which holds true.

Lemma 10.3 If \( \mathbb{J} = (\mathcal{J} \dashv \mathcal{J}_0) \) is an internal category in \textit{Ord-Cat} over \textit{Sq}(C), then the inclusion \( \mathbb{J}^\text{hm} \subseteq \mathcal{J}^\text{hm} \) is an identity.

\textbf{Proof} By definition, the inclusion is fully faithful and injective on objects. It remains to show that it is surjective on objects. \( M \)-algebra structures are unique—see Appendix A. Inspecting the definition of fibre square in Sect. 3, we have that fibre square structures on objects of \( (\mathcal{J}^\text{op}, \text{Ord})^{2 \times 2} \) are unique. By definition, the objects of \( \mathbb{J}^\text{hm} \) are the objects of \( \mathcal{J}^\text{hm} \) such that a certain pair of fibre square structures on the square (3.4) coincide. This holds, by uniqueness, so \( \mathbb{J}^\text{hm} = \mathcal{J}^\text{hm} \).

An \textit{ofs} \( (E, \mathcal{M}) \) on a cocomplete category \( A \) is \textit{cocomplete} if wide pushouts of morphisms in \( E \) exist [17]. In other words, for each object \( A \in A \), the category \( A \downarrow E \) has arbitrary coproducts. We are now ready to prove the section’s main result.

Theorem 10.4 Let \( C \) be a cocomplete and finitely cocomplete \textit{Ord}-enriched category whose underlying category satisfies: (a) It has a cocomplete \textit{ofs} \( (E, \mathcal{M}) \). (b) It is locally \( \mathcal{M} \)-ranked. Then, the cofibrantly \( (E, M) \)-generated \textit{awfs} on any internal category in \textit{Ord-Cat} over \textit{Sq}(C) exists and is lax orthogonal.

\textbf{Proof} It suffices to consider the case of a small \textit{Ord}-category \( U : \mathcal{J} \to C^2 \), by Lemma 10.3. By definition, \( \mathcal{J}^\text{hm} \to C^2 \) is the pullback of \( M\text{-Alg} \to (\mathcal{J}^\text{op}, \text{Ord})^2 \) along \( W_U : C^2 \to (\mathcal{J}^\text{op}, \text{Ord})^2 \), or equivalently, by Lemma 10.2, the pullback of \( (M, \Lambda^M)\text{-Alg} \to (\mathcal{J}^\text{op}, \text{Ord})^2 \). Since \( W_U \) has a left adjoint \( G \dashv W_U \), the \textit{Ord}-category \( \mathcal{J}^\text{hm} \) is isomorphic to \( (R, \Lambda^R)\text{-Alg} \) for the pointed \textit{Ord}-functor \( (R, \Lambda^R) \) on \( C^2 \) given by the pushout depicted on the right.

\[ \begin{array}{ccc}
\mathcal{J}^\text{hm} & \to & (M, \Lambda^M)\text{-Alg} \\
\downarrow & & \downarrow \\
C^2 & \xrightarrow{W_U} & (\mathcal{J}^\text{op}, \text{Ord})^2 \\
\downarrow & & \downarrow \Lambda^R \\
GW_U & \xrightarrow{\text{counit}} & 1 \\
\downarrow & & \downarrow \\
GMW_U & \xrightarrow{\text{count}} & R
\end{array} \]

To conclude the proof, it suffices to show that the functor
\[
(R, \Lambda^R)\text{-Alg} \to C^2
\]
(10.1)
is monadic. In fact, it suffices to show that its underlying ordinary functor has a left adjoint, by Lemma 3.6. To do so, we can show that \( W_U \) is \( \mathcal{M} \)-accessible, in the same way as we did in the proof of Proposition 9.6. We then observe that \( GW_U \) and \( GMW_U \) are \( \mathcal{M} \)-accessible too, \( G \) is cocontinuous and \( M \) is accessible. Therefore, \( R \) is \( \mathcal{M} \)-accessible. We may now use [17, 14.3 & 15.6] to deduce that the ordinary functor (10.1) has a left adjoint.

\textbf{Remark 10.5} Theorem 10.4 applies to the case of the \textit{lofs} \( (E, M) \) for which \( M \) is the free \textit{rali} monad on \( \mathcal{V} \) of Sect. 2.4. In this case it guarantees that cofibrantly \( kz \)-generated \textit{lofs} exist on any \textit{Ord}-category whose underlying ordinary category is locally ranked with respect to a cocomplete \textit{ofs}.
Remark 10.6 The version of Theorem 10.4 mentioned in the previous remark is related to [1]. One of its main results, [1, Thm. 6.10], states that, if $C$ is locally ranked and $\mathcal{J}$ is a family of morphisms of $C$, the full subcategory of $C$ defined by the objects that are KZ-injective to each morphism of $\mathcal{J}$ is reflexive. Furthermore, the induced monad is lax-idempotent. The main component of the proof is the construction of the Kan-injective reflection chain in [1, §5]. When $C$ has a terminal object, this can be translated to our notation by saying that the fibre of the codomain functor $\mathcal{J}^{\text{ord}} \to C^2 \to C$ over $1 \in C$ is monadic over $C$, with lax-idempotent induced monad.

A point where [1] is more general than the present paper is that the class of morphism $\mathcal{J}$ need not be small, but only those which are not order-epimorphisms should form a small set.

On the other hand, our approach is more general in a couple of ways. The only part of this section that is specific to $\mathcal{V} = \text{Ord}$ is the existence part of Theorem 10.4. The lax orthogonality of the resulting AWFS is valid in more general cases, for example $\mathcal{V} = \text{Cat}$ (see Sect. 7). Another advantage of our approach is that it accommodates an $\text{Ord}$-category $\mathcal{J}$ over $C^2$, instead of just a family of morphisms of $C$. Finally, even though our requirements on $C$ and the OFS $(E, \mathcal{M})$ differ slightly from those of [1], this difference seems to be superfluous in the the examples of interest.

11 Continuous Lattices

The papers [7] and [6] constructed an LOFS on the category of $T_0$ topological spaces by means of the “method of the simple monad” introduced in [5]. In this section we shall show that this LOFS, which is closely related to complete lattices, is not cofibrantly KZ-generated, nor cofibrantly generated, nor is its underlying WFS cofibrantly generated. We begin with a brief summary of continuous lattices.

Let $\text{Top}_0$ be the category of $T_0$ topological spaces and continuous maps. Each $T_0$ space $X$ carries a posetal structure defined by $x \subseteq y$ if and only if each neighbourhood of $x$ is also a neighbourhood of $y$; equivalently, if $x \in \{ y \}$. This is sometimes called the specialisation order. A continuous function $f : X \to Y$ becomes an order-preserving function $f : (X, \subseteq) \to (Y, \subseteq)$, so we have a functor $\text{Top}_0 \to \text{Ord}$ into the category of posets. The Cartesian closed category $\text{Ord}$ can play the role of the base of enrichment $\mathcal{V}$ of the previous sections, and we can make $\text{Top}_0$ into a $\text{Ord}$-category by declaring $f \leq g$ if the associated morphisms of posets satisfy $f \leq g$. In elementary terms, $f \leq g$ if and only if $f(x) \subseteq g(x)$ for all $x$.

We now recall the definition of continuous lattices [27]. Suppose that $(L, \leq)$ is a complete poset. Given a pair of elements $x, y \in L$ we say that $x$ is way below $y$, written $x \ll y$, if for all directed subsets $D \subseteq L$, if $y \leq \vee D$ then there exists some $d \in D$ with $x \leq d$. A continuous lattice is a complete poset where every element is the supremum of the elements way below it: $x = \bigvee \{ y : y \ll x \}$. Equip $L$ with the Scott topology $\tau_L$, whose open sets are those subsets $U \subseteq L$ that satisfy: (1) If $x \in U$ and $x \leq y$, then $y \in U$; (2) If $D \subseteq L$ is a directed subset and $\bigvee D \in U$, then $D \cap U \neq \emptyset$. In this way we can regard any continuous lattice as a $T_0$ topological space, and the specialisation order for this topology coincides with the order of $L$. Conversely, if the poset $(X, \subseteq)$ of a $T_0$ space $X$ is a continuous lattice, the topology $\tau_{(X, \subseteq)}$ coincides with the original topology of $X$. In this way, continuous lattices can be identified with a class of $T_0$ topological spaces.

A function $f : L \to L'$ between continuous lattices is continuous for the associate topology if and only if it preserves directed suprema. Thus, the category of continuous lattices
and maps of posets that preserve directed suprema is isomorphic to a full subcategory \( \text{CL} \) of \( \text{Top}_0 \).

As part of his seminal work [27], D. Scott showed that a topological space is a continuous lattice if and only if it is injective with respect to all embeddings of topological spaces. Later, it was shown in [8] that the category of continuous lattices is isomorphic to the category of algebras of the filter monad \( F \) on \( \text{Top}_0 \), the monad that assigns to each space \( X \) its space of filters \( F X \) endowed with a suitable topology.

Before explaining how all this relates to \( \text{LOFS} \), let us make a point about the direction of the inequalities between continuous maps. The filter monad on \( \text{Top}_0 \) is colax idempotent, with the ordering between continuous maps induced by the specialisation order. When we want to apply the theory of \( \text{LOFS} \) and simple monads to this example, we are presented with two options: we either reverse the ordering on 2-cells (inequalities) in our statements, or else we reverse the specialisation ordering to make the filter monad lax idempotent. We choose the latter approach, used in [4,7], so \( f \leq g \) for a pair of parallel continuous maps shall mean \( f(x) \sqsubseteq g(x) \) for all \( x \) in their domain.

We are now ready to explain how to construct an \( \text{LOFS} \), as done in [6,7]. The filter monad \( F \) is lax idempotent—with the convention on the ordering of continuous maps of the previous paragraph. Furthermore, \( F \) is simple in the terminology of [5], thus inducing a \( \text{LOFS} \) \((L, R)\) on \( \text{Top}_0 \) such that: (1) The fibrant replacement \( \text{Ord}-\text{enriched} \) monad on \( \text{Top}_0 \) is the filter monad \( F \); (2) A continuous function \( f \) is an \( L \)-coalgebra if and only if \( f \) is a topological embedding. A detailed construction can be found in [7]. The underlying \( \text{WFS} \) of this \( \text{LOFS} \) was considered in [4].

**Theorem 11.1** The \( \text{LOFS} \) on \( \text{Top}_0 \) described is not cofibrantly \((E, M)\)-generated, for any cocontinuous \( \text{Ord}-\text{enriched} \) \( \text{AWFSs} \) \((E, M)\) on \( \text{Ord} \). In particular it is not \( \text{KZ} \)-generated nor cofibrantly generated. Furthermore, its underlying \( \text{WFS} \) is not cofibrantly generated.

**Proof** We first tackle the part of the statement that deals with the \( \text{AWFS} \). The proof is an application of Proposition 9.6. The category \( C \) will be that of \( T_0 \) topological spaces, \( \text{Top}_0 \), regarded as an \( \text{Ord} \)-category via the specialisation order, and \( M \) will be the the subcategory of \( \text{Top}_0 \) defined by continuous maps that are injective and homeomorphisms onto their image. It is well-known that each topological space has an \( M \)-rank.

Suppose that the \( \text{AWFS} \) \((L, R)\) described at the beginning of the section is cofibrantly generated with respect to a cocontinuous \( \text{AWFS} \) \((E, M)\) on \( \text{Ord} \). By Proposition 9.6, the forgetful functor \( \text{R-Alg} \to C^2 \) creates \( \kappa \)-filtered colimits of \( M \)-diagrams, for some regular cardinal \( \kappa \). In particular, if \( R_1 \) is the associated fibrant replacement monad, i.e., the restriction of \( R \) to \( \text{Top}_0 / 1 \cong \text{Top}_0 \), then the forgetful functor \( \text{CL} \cong R_1-\text{Alg} \to \text{Top}_0 \) creates the said colimits. This is a contradiction, as exhibited by the following example, which, furthermore, together with Proposition 9.9 shows that the underlying \( \text{WFS} \) of \((L, R)\) cannot be cofibrantly generated. \( \square \)

**Example 11.2** In the next few paragraphs we show that, for each regular cardinal \( \beta \), there is a \( \beta \)-filtered colimit of continuous lattices that is not created by the forgetful functor \( \text{CL} \to \text{Top}_0 \). As usual, \( \alpha < \nu \) will mean \( \alpha \in \nu \) for ordinals \( \alpha, \nu \).

The way-below relation \( \alpha \ll \nu \) on an ordinal \( \beta \) satisfies:

- if \( \alpha \in \nu \in \beta \), then \( \alpha \ll \nu \). For, if \( D \subseteq \beta \) verifies \( \nu \leq \text{sup} \, D \), then \( \alpha \in \text{sup} \, D \) and there must be a \( \delta \in D \) with \( \alpha \in \delta \).

It is now easy to verify that successor ordinals are continuous lattices, as they are complete and any \( \alpha \) is the supremum of the ordinals \( \gamma \in \alpha \). If \( \mu < \nu \), the inclusion \( \text{succ}(\mu) \subseteq \text{succ}(\nu) \) is continuous for the Scott topology, since it preserves suprema.
Now suppose that $\beta$ is a limit ordinal. Then $\beta$ is a filtered union of $\text{succ}(\mu) \subseteq \beta$, for $\mu \in \beta$. The cocone $\text{succ}(\mu) \leftrightarrow \beta$ exhibits $\beta$ as a colimit of the $\text{succ}(\mu)$, and we can make this a colimit in $\text{Top}_0$ by equipping $\beta$ with the colimit topology induced by the Scott topology of the continuous lattices $\text{succ}(\mu)$: a subset $U \subseteq \beta$ is open if $U \cap \text{succ}(\mu)$ is open in $\text{succ}(\mu)$, for all $\mu \in \beta$. More explicitly, $U \subseteq \beta$ is open if: (1) It is up-closed, and; (2) For any bounded $D \subseteq \beta$, sup $D \in U$ implies $U \cap D \neq \emptyset$. With this topology, $\text{succ}(\mu) \leftrightarrow \beta$ is an embedding of spaces.

Now assume further that $\beta$ is a regular cardinal, so $(\beta, \leq)$ is a $\beta$-filtered ordered set. By the above paragraph, $\beta$ is a $\beta$-filtered colimit of continuous lattices. But $\beta$ is not a continuous lattice, as it is not complete: it lacks a top element.

### A Lax Idempotent 2-Monads

A 2-monad $T = (T, i, m)$ is lax idempotent if one of the following equivalent conditions holds: (1) $Ti \dashv m$ with identity unit; (2) $m \dashv iT$ with identity counit; (3) There exists a modification $\delta : Ti \Rightarrow iT$ such that $m \cdot \delta = 1$ and $\delta \cdot i = 1$; (4) The forgetful 2-functor from the 2-category of $T$-algebras and lax morphisms is full and faithful on morphisms. Some of these conditions appear in [22] and [29] in the context of doctrines. See also [25]. A full list of equivalent conditions with the respective proofs can be found in [19].

A 2-comonad $G$ on $\mathcal{K}$ is lax idempotent when the corresponding monad $G^{\text{op}}$ on the 2-category $\mathcal{K}^{\text{op}}$, obtained by reversing the morphisms, is lax idempotent.

### A.1 Algebras and Morphisms as Monad Morphisms

There is a standard way of regarding algebras and algebra morphisms as monad maps, introduced in [16], that we proceed to describe.

Assume that $\mathcal{C}$ is a locally small complete $\mathcal{V}$-category; strictly speaking, all we need is finite limits and cotensor products. For each pair of objects $A$ and $B$ denote by $(A, B)$ the right Kan extension of the $\mathcal{V}$-functor $A : \mathbf{1} \to \mathcal{C}$ along $B : \mathbf{1} \to \mathcal{C}$.

$$\langle A, B \rangle = \text{Ran}_A B = \{\mathcal{C}(\_ , A), B\} : \mathcal{C} \to \mathcal{C}$$

In other words, $(A, B)(X) = \{C(X, A), B\}$, the cotensor product of $B \in \mathcal{C}$ by the category $\mathcal{C}(X, A)$. There is a morphism

$$\text{ev}_{A,B} : \langle A, B \rangle (A) = \{C(A, A), B\} \longrightarrow B \quad (A.1)$$

that is the counit of the cotensor product in $\mathcal{C}$, i.e., the morphism picked out by

$$\mathbf{1} \xrightarrow{1_A} C(A, A) \xrightarrow{\eta} C(\{C(A, A), B\}, B)$$

where $\eta$ is the unit of the cotensor product. The morphism $(A.1)$ satisfies the following universal property: for any $\mathcal{V}$-functor $S : \mathcal{C} \to \mathcal{C}$, the $\text{CAT}$-functor

$$[\mathcal{C}, \mathcal{C}](S, \langle A, B \rangle) \xrightarrow{\text{proj}_A} C(S(A), \langle A, B \rangle(A)) \xrightarrow{(A.1)} C(S(A), B)$$

is an isomorphism. The assignment $(A, B) \mapsto \langle A, B \rangle$ defines a $\text{CAT}$-functor $\langle - , - \rangle : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{End}(\mathcal{C})$.

As it is the case of any right Kan extension of a functor along itself, $\langle A, A \rangle = \text{Ran}_A A$ has a canonical structure of a $\mathcal{V}$-monad, sometimes called density monad. More explicitly, the multiplication $\langle A, A \rangle^2 \to \langle A, A \rangle$ corresponds to the morphism in $\mathcal{C}$.
while the unit 1 \rightarrow \langle A, A \rangle corresponds to the identity morphism 1 : A \rightarrow A.

The definition of the $\mathcal{V}$-monad $\langle A, A \rangle$ is such that the following statement holds. Suppose that $S$ is a 2-monad on $C$ and $\alpha : S \Rightarrow \langle A, A \rangle$ is a 2-natural transformation with corresponding morphism $a : S(A) \rightarrow A$ in $C$. Then, $\alpha$ is a—strict—morphism of 2-monads —i.e., it is compatible with the multiplications and units in the usual way—if and only if $a$ is an $S$-algebra structure on $A$.

For each morphism $f : A \rightarrow B$ in $C$, one may consider the following comma-object in the $\text{CAT}$-category $\text{End}(C)$.

\[
\langle f, f \rangle \quad \downarrow \quad \langle A, A \rangle \\
\langle f \cdot \alpha, f \cdot \beta \rangle 
\]

Thus, for any endo-2-functor $S$ of $C$ there is an isomorphism between the category of 2-natural transformations $S \Rightarrow \langle f, f \rangle$ and modifications between them, and the comma-category depicted.

\[
\text{C}(1, f) \downarrow \text{C}(Sf, 1) \quad \rightarrow \quad \text{C}(S(B), B) \\
\text{C}(S(A), A) \downarrow \quad \text{C}(S(A), B) \\
\text{C}(Sf, 1) \quad \text{C}(1, f) \quad \text{C}(Sf, 1)
\]

In particular, each 2-natural transformation $S \Rightarrow \langle f, f \rangle$ is defined by a unique triple formed by two morphisms $a : S(A) \rightarrow A$ and $b : S(B) \rightarrow B$ and a 2-cell $\varphi : f \cdot a \Rightarrow b \cdot Sf$.

The 2-functor $\langle f, f \rangle$ has a 2-monad structure that makes the projections $\langle A, A \rangle \leftarrow \langle f, f \rangle \rightarrow \langle B, B \rangle$—strict—morphisms of 2-monads. Furthermore, the bijection mentioned in the previous paragraph restricts to a bijection between morphisms of 2-monads $S \rightarrow \langle f, f \rangle$ and triples $(a, b, \varphi)$ where $a : S(A) \rightarrow A$ and $b : S(B) \rightarrow B$ are $S$-algebra structures, and $(f, \varphi)$ is a lax morphism of $S$-algebras from $(a, A)$ to $(B, b)$. This method of describing morphisms as monad maps can be found in the literature as early as in [16, §3].

### A.2 Reflections of Monads Through a Functor

The following lemma can be applied to the situation of a right adjoint functor $W : C \rightarrow B$.

**Lemma A.1** Suppose that $W : C \rightarrow B$ is a continuous $\mathcal{V}$-functor where $C$ is a complete $\mathcal{V}$-category and $B$ admits all right Kan extensions along $W$ (i.e., all limits weighted by $B(1, W)$).

1. If $A, B$ are objects of $C$, then the canonical $\mathcal{V}$-natural transformation

   \[
   \text{Ran}_W(W(A, B)) \rightarrow \text{Ran}_{W(A)}(W(B))
   \]

   is an isomorphism.

2. If $f, g$ are morphisms in $C$, then the canonical $\mathcal{V}$-natural transformation

   \[
   \text{Ran}_W(W\langle f, g \rangle) \rightarrow \langle Wf, Wg \rangle
   \]

   is an isomorphism.
Proof The proof is straightforward if one takes care of showing at each stage the existence of necessary limits in \( B \). The continuous \( \mathcal{V} \)-functor \( W \) preserves the cotensor products \( \langle A, B \rangle (X) = \{ C(X, A), B \} \), so \( W(A, B) \cong \text{Ran}_{\mathcal{W}}(W(B)) \). Thus, there is an isomorphism
\[
\tau : \text{Ran}_{\mathcal{W}}(W(A, B)) \cong \text{Ran}_{\mathcal{W}}(W(A)) \cong \text{Ran}_{\mathcal{W}}(W(B)) = \langle W(A), W(B) \rangle
\]
where the first object exists by the hypothesis on \( B \), the second exists by the reason mentioned above, and the third does by iterated Kan extensions [18, §4.4]

To prove the second part of the statement, suppose given \( f : A \to B \) and \( g : C \to D \). It is easy to see that the isomorphism of the first part of the statement is natural in \( A \) and \( B \), so we have a commutative diagram as follows.

\[
\begin{array}{cccc}
\text{Ran}_{\mathcal{W}}(W(B, C)) & \stackrel{\text{Ran}_{\mathcal{W}}(W(1, g))}{\longrightarrow} & \text{Ran}_{\mathcal{W}}(W(B, D)) & \stackrel{\text{Ran}_{\mathcal{W}}(W(f, 1))}{\longrightarrow} & \text{Ran}_{\mathcal{W}}(W(A, D)) \\
\langle W(B), W(C) \rangle & \underset{(1, Wg)}{\downarrow} & \langle W(B), W(D) \rangle & \underset{(Wf, 1)}{\downarrow} & \langle W(A), W(D) \rangle \]
\end{array}
\]

We shall show that \( \langle Wf, Wg \rangle_\ell \) exists and is isomorphic, in a canonical way, to \( \text{Ran}_{\mathcal{W}}(W(f, g)_\ell) \) in three stages represented by the following three isomorphisms.

\[
\begin{align*}
\text{Ran}_{\mathcal{W}}(W(f, g)_\ell) &= \text{Ran}_{\mathcal{W}}((W(f, 1) \downarrow (1, g))) \quad \longrightarrow \quad \text{Ran}_{\mathcal{W}}((W(f, 1) \downarrow (W(1, g)))) \\
\text{Ran}_{\mathcal{W}}((W(f, 1) \downarrow (W(1, g)))) &= \text{Ran}_{\mathcal{W}}(W(f, 1) \downarrow (W\text{Ran}_{\mathcal{W}}(W(1, g)))) \\
\text{Ran}_{\mathcal{W}}(W(f, 1) \downarrow (W\text{Ran}_{\mathcal{W}}(W(1, g)))) &= \approx \langle Wf, 1 \rangle \downarrow (1, Wg) = \langle Wf, Wg \rangle_\ell
\end{align*}
\]

By hypothesis, \( W \) is continuous, and, in particular, it preserves comma-objects, so the first of the above isomorphisms, and in particular, its codomain, exists.

Let us now show that the codomain of the second morphism, shown in (A.4), exists and is isomorphic to the domain. In doing so, we may substitute \( W(f, 1) \) and \( W(1, g) \) by two arbitrary \( \mathcal{V} \)-functors \( F, G : C \to B \). By hypothesis, the \( \mathcal{V} \)-functor \( [B, B] \to [C, B] \) given by restricting along \( W \) has a right adjoint, namely \( \text{Ran}_{\mathcal{W}} \). The \( \mathcal{V} \)-functor \( F \downarrow G : C \to B \) exists and is constructed pointwise by taking comma-objects in \( B \). This is preserved by right adjoint \( \text{Ran}_{\mathcal{W}} \), so \( \text{Ran}_{\mathcal{W}} F \downarrow \text{Ran}_{\mathcal{W}} G \) exists and is canonically isomorphic to \( \text{Ran}_{\mathcal{W}}(F \downarrow G) \).

The third transformation (A.5) is the one induced by the diagram (A.3). This completes the proof of the existence of \( \text{Ran}_{\mathcal{W}}(W(f, g)_\ell) = \langle Wf, Wg \rangle_\ell \). \( \Box \)

We shall denote the category of \( \mathcal{V} \)-monads on \( C \) by \( \text{Mnd}(C) \). This is the category of monoids in the monoidal category of endo \( \mathcal{V} \)-functors of \( C \) and monoid morphisms, by which we mean \( \mathcal{V} \)-natural transformations \( T \Rightarrow S \) that are compatible with the units and multiplications—morphism of \( \mathcal{V} \)-monads on \( C \).

If \( W : C \to B \) is a \( \mathcal{V} \)-functor and \( T = (T, i, m) \) a \( \mathcal{V} \)-monad on \( C \), then \( \text{Ran}_{\mathcal{W}}(WT) \) has a canonical structure of a \( \mathcal{V} \)-monad on \( B \), whenever this right Kan extension exists. The unit is the \( \mathcal{V} \)-natural transformation that corresponds to \( Wi : W \Rightarrow WT \), while the multiplication corresponds to the \( \mathcal{V} \)-natural transformation

\[
\text{Ran}_{\mathcal{W}}(WT) \text{Ran}_{\mathcal{W}}(WT)W \longrightarrow \text{Ran}_{\mathcal{W}}(WT)WT \longrightarrow WTT \xrightarrow{Wm} WT.
\]

If \( \text{Ran}_{\mathcal{W}} \) always exists, then \( E \mapsto \text{Ran}_{\mathcal{W}}(WE) \) is a (lax) monoidal functor from \( \text{End}(C) \) to \( \text{End}(B) \), so it induces a functor \( \text{Mnd}(C) \to \text{Mnd}(B) \). This is the case, for example, when \( W \) has a left adjoint.

In a moment we will need the more general notion of morphism between monads on different \( \mathcal{V} \)-categories. If \( (C, T) \) is a \( \mathcal{V} \)-monad on \( C \) and \( (B, S) \) a \( \mathcal{V} \)-monad on \( B \), a morphism \( (C, T) \to (B, S) \) is a \( \mathcal{V} \)-functor \( W : C \to B \) equipped with a \( \mathcal{V} \)-natural transformation

\[\]
\( \omega : SW \Rightarrow WT \) satisfying compatibility axioms with the unit and multiplication of the respective monads, that can be found, for example, in [28]. There is a bijection between \( \mathcal{V} \)-natural transformations \( \omega \) that make \( (W, \omega) \) a morphism of monads \( (C, T) \rightarrow (B, S) \) and liftings of \( W \) to the category of algebras, i.e., \( \mathcal{V} \)-functors, depicted by a dashed arrow, that make the square commutative.

\[
\begin{array}{ccc}
T-\text{Alg} & \rightarrow & S-\text{Alg} \\
\downarrow{\mathcal{V}}^{T} & & \downarrow{\mathcal{V}}^{S} \\
C & \rightarrow & B
\end{array}
\]

(A.6)

There is a bijection between morphisms of \( \mathcal{V} \)-monads \( (C, T) \rightarrow (B, S) \) over \( W : C \rightarrow B \) and morphisms \( S \rightarrow \text{Ran}_W(WT) \) in \( \text{Mnd}(C) \). Indeed, a morphism structure \( \omega : SW \Rightarrow WT \) on \( W \) corresponds to a monoid morphism \( \alpha : S \rightarrow \text{Ran}_W(WT) \) via the universal property of \( \text{Ran}_W \).

\[
\begin{array}{cccc}
C & \xrightarrow{W} & B & \xleftarrow{\alpha} \\
\downarrow{\text{Ran}_W} & & \downarrow{S} & \xleftarrow{\omega} \\
C & \xrightarrow{W} & B & \xleftarrow{S}
\end{array}
\]

Lemma A.2 Consider a commutative square of \( \mathcal{V} \)-functors (A.6), where \( T \) and \( S \) are \( \mathcal{V} \)-monads on \( C \) and \( B \), respectively, and assume that \( \text{Ran}_W \) of \( \mathcal{V} \)-functors into \( B \) always exists. Let \((W, w) : (C, T) \rightarrow (B, S)\) be the morphism of \( \mathcal{V} \)-monads associated to the square (A.6) and \( \alpha : S \rightarrow \text{Ran}_W(WT) \) the morphism in \( \text{Mnd}(B) \) associated to \((W, w)\). If the square (A.6) is a pullback, then \((W, w)\) exhibits \( T \) as a reflection of \( S \) along \( \text{Ran}_W(W-) : \text{Mnd}(C) \rightarrow \text{Mnd}(B) \).

Proof Let \( P \) be a \( \mathcal{V} \)-monad on \( C \) and consider the following sets: (a) \( \text{Mnd}(C)(T, P) \); (b) \( \mathcal{V} \)-functors \( P-\text{Alg} \rightarrow T-\text{Alg} \) that commute with the respective forgetful functors; (c) \( \text{Mnd}(B)(S, \text{Ran}_W(WP)) \); (d) \( \mathcal{V} \)-functors \( P-\text{Alg} \rightarrow S-\text{Alg} \) over \( W \). The sets (a) and (b) are bijective, as well as the sets (c) and (d), and the bijections are natural in \( P \), by the comments previous to this lemma. To say that \( T \) is the reflection of \( S \) is equally saying that (a) is naturally bijective to (c), while to say that (A.6) is a pullback implies that (b) and (d) are naturally bijective. \( \square \)

The existence of \( \text{Ran}_W \) in Lemma A.2 is satisfied, for example, when \( W \) has a left adjoint.

A.3 Reflections of Lax Idempotent \( \mathcal{V} \)-Monads

There is a characterisation of lax idempotent 2-monads, due to [19], which we will find useful. Given a morphism \( f : A \rightarrow B \) in \( C \), we denote by \( \sigma_f : \langle f, f \rangle_{\ell} \rightarrow \langle A, A \rangle \times \langle B, B \rangle \) the morphism of \( \mathcal{V} \)-monads induced by the projections of the comma-object (A.2). A \( \mathcal{V} \)-monad \( T \) is lax idempotent precisely when it is co-orthogonal to each \( \sigma_f \) in \( \text{Mnd}(C) \), i.e., when \( \text{Mnd}(C)(T, \sigma_f) \) is a bijection.

Corollary A.3 If the \( \mathcal{V} \)-monad \( S \) in Lemma A.2 is lax idempotent, \( C \) is cocomplete, \( B \) admits \( \text{Ran}_W \) and \( W \) is continuous, then \( T \) is lax idempotent.

Proof By the remarks previous to the present corollary, we must show that \( T \) is co-orthogonal to the morphism \( \sigma_f : \langle f, f \rangle_{\ell} \rightarrow \langle A, A \rangle \times \langle B, B \rangle \), for any morphism \( f : A \rightarrow B \) in \( C \). Equivalently, that \( S \) is co-orthogonal to \( \text{Ran}_W(W\sigma f) \) by Lemma A.2. This morphism is isomorphic to \( \sigma_{Wf} \), via the isomorphisms \( \text{Ran}_W(W(A, A)) \cong \langle(\text{W}(A), W(A)) \rangle \) and

\( \text{Springer} \)
\[ \text{Ran}_W(W(f, f)_\ell) \cong \langle W(f), W(f)_\ell \rangle \] of Lemma A.1, from where it is obvious that \( S \) is co-orthogonal to \( \text{Ran}_W(W\sigma) \). \hfill \Box

References

1. Adámek, J., Sousa, L., Velebil, J.: Kan injectivity in order-enriched categories. Math. Struct. Comput. Sci. 25(1), 6–45 (2015)
2. Blackwell, R., Kelly, G.M., Power, A.J.: Two-dimensional monad theory. J. Pure Appl. Algebra 59(1), 1–41 (1989)
3. Bourke, J., Garner, R.: Algebraic weak factorisation systems I: accessible AWFS. J. Pure Appl. Algebra 220(1), 108–147 (2016)
4. Cagliari, F., Clementino, M.M., Mantovani, S.: Fibrewise injectivity and Kock–Zöberlein monads. J. Pure Appl. Algebra 216(11), 2411–2424 (2012)
5. Clementino, M.M., Franco, I.López: Lax orthogonal factorisation systems. Adv. Math. 302, 458–528 (2016)
6. Clementino, M.M., Franco, I.L.: Lax orthogonal factorisations in monad-quantale-enriched categories. Log. Methods Comput. Sci. 13 (2017)
7. Clementino, M. M., Franco, I.L.: Lax orthogonal factorisations in ordered structures (2017). arXiv:1702.02602
8. Day, A.: Filter monads, continuous lattices and closure systems. Can. J. Math. 27, 50–59 (1975)
9. Escardó, M.H.: Properly injective spaces and function spaces. Topol. Appl. 89(1–2), 75–120 (1998)
10. Freyd, P., Kelly, G.M.: Categories of continuous functors. I. J. Pure Appl. Algebra 2, 169–191 (1972)
11. Garner, R.: Understanding the small object argument. Appl. Categ. Struct. 17(3), 247–285 (2009)
12. Grandis, M., Tholen, W.: Natural weak factorization systems. Arch. Math. (Brno) 42(4), 397–408 (2006)
13. Gray, J.: Fibred and cofibred categories. In: Proceedings Conference on Categorical Algebra (La Jolla, CA, 1965), pp. 21–83 (1966)
14. Hermida, C.: Representable multicategories. Adv. Math. 151(2), 164–225 (2000)
15. Hermida, C.: Fibrations for abstract multicategories. Galois Theory, Hopf Algebras, and Semiabelian Categories. Series in Fields Institute Communications, vol. 43, pp. 281–293. American Mathematical Society, Providence, RI (2004)
16. Kelly, G. M.: Coherence theorems for lax algebras and for distributive laws. In: Category Seminar (Proceeding Seminar, Sydney, 1972/1973). Lecture Notes in Mathematics, vol. 420, pp. 281–375. (1974)
17. Kelly, G.M.: A unified treatment of transfinte constructions for free algebras, free monoids, colimits, associated sheaves, and so on. Bull. Aust. Math. Soc. 22(1), 1–83 (1980)
18. Kelly, G.M.: Basic Concepts of Enriched Category Theory. Repr. Theory Appl. Categ. 10, 137 (2005)
19. Kelly, G.M., Lack, S.: On property-like structures. Theory Appl. Categ. 3(9), 213–250 (1997)
20. Kelly, G. M., Street, R.: Review of the elements of 2-categories. In: Category Seminar, Proceeding, Sydney 1972/1973. Lecture Notes in Mathematics, vol. 420, pp. 75–103 (1974)
21. Kelly, G.M.: Basic Concepts of Enriched Category Theory. London Mathematical Society Lecture Note Series, vol. 64. Cambridge University Press, Cambridge (1982)
22. Kock, A.: Monads for which structures are adjoint to units. J. Pure Appl. Algebra 104(1), 41–59 (1995)
23. Lambek, J.: Deductive systems and categories. II. Standard constructions and closed categories. In Category Theory, Homology Theory and Their Applications, Proceeding Conference on Seattle Research Center of Battelle Memorial Institute 1968, vol. 1, pp. 76–122 (1969)
24. Leinster, T. (ed.): Higher Operads, Higher Categories. London Mathematical Society Lecture Note Series, vol. 298. Cambridge University Press, Cambridge (2004)
25. Marmolejo, F.: Doctrines whose structure forms a fully faithful adjoint string. Theory Appl. Categ. 3(2), 24–44 (1997)
26. Quillen, D.G.: Homotopical Algebra. Lecture Notes in Mathematics, No. 43. Springer, Berlin (1967)
27. Scott, D.: Continuous Lattices. In: Toposes, Algebraic Geometry and Logic (Conference, Dalhousie University, Halifax, NS, 1971). Lecture Notes in Mathematics, vol. 274, pp. 97–136 (1972)
28. Street, R.: The formal theory of monads. J. Pure Appl. Algebra 2, 149–168 (1972)
29. Zöberlein, V.: Doctrines on 2-categories. Math. Z. 148(3), 267–279 (1976)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.