Quantaloidal Completions of Order-enriched Categories and Their Applications

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Abstract

By introducing the concept of quantaloidal completions for an order-enriched category, relationships between the category of quantaloids and the category of order-enriched categories are studied. It is proved that quantaloidal completions for an order-enriched category can be fully characterized as compatible quotients of the power-set completion. As applications, we show that a special type of injective hull of an order-enriched category is the MacNeille completion; the free quantaloid over an order-enriched category is the Down-set completion.

Keywords: Quantaloid, order-enriched category, completion; injective hull, free quantaloid

1 Introduction

An order-enriched category is a locally small category such that the hom-sets are partially ordered sets and composition of morphisms preserve order in both variables. An order-enriched category with only one object can be viewed as a partially ordered semigroup. Thus order-enriched categories can be viewed as a generalization of partially ordered semigroups. Several works devoted to this subject are from computer science\([15,33]\), especially with strong background of the study of programming languages. In 1979, M. Wand studied fixed-point constructions in order-enriched categories, which extended Scott’s result based on continuous lattices. Note that an order-enriched category in the sense of\([33]\) means a category with hom-sets not only ordered but also with certain completeness. Later, M. Smyth and G. Plotkin considered solving recursive domain equations in this framework\([26]\). In 2007, this ideal was further extended to the framework of bicategories\([5]\). In 1991, C. E. Martin, C. A. R. Hoare and He Jifeng studied pre-adjunctions in order enriched-categories\([15]\). In\([15]\), the concepts of lax functors, natural transformations and pre-adjunctions are studied with the purpose to explain their understanding of programming languages. We also note that an order-enriched category in the sense of\([15]\) means a...
category with hom-sets preordered. These works are all devoted to study special kind of order-enriched
categories. There are little works devoted to study on them systematically.

A quantaloid $Q$ \cite{22, 17} is a category enriched in the symmetric monoidal closed category $\text{Sup}$ of
complete lattices and morphisms that preserve arbitrary sups. Just as every complete lattice is a special
partially ordered set, every quantaloid is a special order-enriched categories. A quantaloid with only
one object is a quantale \cite{19}, thus quantaloids are naturally viewed as quantales with many objects.
Quantaloids were studied by Pitts \cite{17} in investigating distributive categories of relations and topos theory
under the name of sup-lattice enriched categories. In \cite{1} quantaloids are studied in order to include a notion
of type on the processes. Quantaloids and their applications were further developed in the monograph \cite{22}
In recent years, Quantaloid-enriched categories received considerable attention \cite{6, 8, 11, 13, 16, 23–25, 27–32}.

The process of completion is a classic approach to study ordered structures. Various completion methods
for ordered structures are developed with different characteristics \cite{3, 4, 7, 9, 14, 18, 34}. Relationships
between order-enriched categories and quantaloids have not received enough attention, though they have
similar backgrounds and close relations. Inspired by research on completion methods for ordered semi-
groups and their applications \cite{9, 12, 20, 34}, this paper is devoted to study quantaloidal completions of
order-enriched categories and their applications.

The contents of the paper are arranged as follows. Section 2 lists some preliminary notions and results
about order-enriched categories and quantaloids. In Section 3, based on compatible nuclei on quantaloids,
quantaloidal completions for an order-enriched category are fully characterized as compatible quotients
of the power-set completion. In Section 4, two aspects of applications of quantaloidal completions are
given. It is proved that the injective hull of an order-enriched category with respect to a special kind
of morphisms is the MacNeille completion; the free quantaloid over an order-enriched category is the
Down-set completion.

2 Preliminaries on order-enriched categories and quantaloids

For category theory, we refer to \cite{2, 10}. Let $C_0$ be the class of objects of a category $C$. $C(a, b)$ denotes the
hom-set for $a, b \in C_0$. For $a \in C_0$, $1_a$ denotes the identity on $a$.

**Definition 2.1** (\cite{15}) An order-enriched category is a locally small category $\mathcal{A}$ such that:

1. for $a, b \in A_0$, the hom-set $\mathcal{A}(a, b)$ is a poset,
2. composition of morphisms of $\mathcal{A}$ preserves order in both variables.

**Definition 2.2** (\cite{35}) Let $\mathcal{C}, \mathcal{D}$ be order-enriched categories. A lax semifunctor $F : \mathcal{C} \to \mathcal{D}$ is given by
functions $F : C_0 \to D_0$ and $F_{a,b} : \mathcal{C}(a, b) \to \mathcal{D}(F a, F b)$ for all $a, b \in C_0$ such that $F_{a,b}$ is order-preserving and $(F g) \circ (F f) \leq F (g \circ f)$ for all $a, b, c \in C_0$, $f \in \mathcal{C}(a, b)$, $g \in \mathcal{C}(b, c)$. A lax functor $F : \mathcal{C} \to \mathcal{D}$ is a lax semifunctor such that $1_{F a} \leq F (1_a)$ for all $a \in C_0$. A 2-functor $F : \mathcal{C} \to \mathcal{D}$ is a functor such that

$$F_{a,b} : \mathcal{C}(a, b) \to \mathcal{D}(F a, F b)$$

is order-preserving for all $a, b \in C_0$.

A quantaloid $Q$ \cite{22} is a category enriched in the symmetric monoidal closed category $\text{Sup}$ of complete
lattices and morphisms that preserve arbitrary sups. In elementary terms:

**Definition 2.3** (\cite{20}) A quantaloid is a locally small category $Q$ such that:

1. for $a, b \in Q_0$, the hom-set $Q(a, b)$ is a complete lattice,
2. composition of morphisms of $Q$ preserves sups in both variables.

In this paper, $Q$ always denotes a small quantaloid, and $Q_0$ denotes the set of its objects. The identity
$Q$-arrow on $q \in Q_0$ will be denoted by $1_q$. The greatest element of the complete lattice $Q(p, q)$ will be
denoted by $\top_{p,q}$. For a $Q$-arrow $u : p \to q$, we denote the domain and the codomain of $u$ by $\text{dom}(u)$ and
$\text{cod}(u)$, respectively. Given $Q$-arrows $u : p \to q$, $v : q \to r$, the corresponding adjoints induced by the
compositions \( - \circ u : \mathcal{Q}(q, r) \to \mathcal{Q}(p, r) \) and \( v \circ - : \mathcal{Q}(p, q) \to \mathcal{Q}(p, r) \) are denoted by \( u \to_l - \) and \( v \to_r - \) respectively.

For more details on quantaloids, we refer to [20, 22].

**Definition 2.4** ([20]) Let \( \mathcal{Q}, \mathcal{S} \) be quantaloids. A quantaloidal homomorphism \( F : \mathcal{Q} \to \mathcal{S} \) is a functor such that
\[
F : \mathcal{Q}(X, Y) \to \mathcal{S}(FX, FY)
\]
is sup-preserving for all \( X, Y \in \mathcal{Q}_0 \).

A quantaloidal isomorphism is a quantaloidal homomorphism such that it is bijective on objects and hom-sets.

**Example 2.5** Let \( \mathcal{A} \) be an order-enriched category.

1. \( \mathcal{P}(\mathcal{A}) \) is a quantaloid [20]. The objects of \( \mathcal{P}(\mathcal{A}) \) are those of \( \mathcal{A} \). For \( a, b \in \mathcal{A} \), the hom-set \( \mathcal{P}(\mathcal{A})(a, b)=\mathcal{P}(\mathcal{A}(a, b)) \), the power set of the hom-set \( \mathcal{A}(a, b) \). For \( S \in \mathcal{P}(\mathcal{A})(a, b), T \in \mathcal{P}(\mathcal{A})(b, c) \), \( T \circ S = \{ g \circ f \mid g \in T, f \in S \} \).

2. \( \mathcal{D}(\mathcal{A}) \) is a quantaloid. The objects of \( \mathcal{D}(\mathcal{A}) \) are those of \( \mathcal{A} \). For \( a, b \in \mathcal{A} \), the hom-set \( \mathcal{D}(\mathcal{A})(a, b)=\mathcal{D}(\mathcal{A}(a, b)) \), the set of down sets\(^4\) of the hom-set \( \mathcal{A}(a, b) \). For \( S \in \mathcal{D}(\mathcal{A})(a, b), T \in \mathcal{D}(\mathcal{A})(b, c) \), \( T \circ S = \{ g \circ f \mid g \in T, f \in S \} \). We note that \( \downarrow 1_a \in \mathcal{D}(\mathcal{A}(a, a)) \) is the identity morphism.

**Definition 2.6** ([20]) Let \( \mathcal{Q} \) be a quantaloid. A quantaloidal nucleus is a lax functor \( j : \mathcal{Q} \to \mathcal{Q} \), which is the identity on the objects of \( \mathcal{Q} \) and such that the maps \( j_{a,b} : \mathcal{Q}(a,b) \to \mathcal{Q}(a,b) \) satisfy:

1. \( f \leq j_{a,b}(f) \) for all \( f \in \mathcal{Q}(a,b) \),
2. \( j_{a,b}(j_{a,b}(f)) = j_{a,b}(f) \) for all \( f \in \mathcal{Q}(a,b) \),
3. \( j_{a,c}(g) \circ j_{a,b}(f) \leq j_{a,c}(g \circ f) \) for all \( g \in \mathcal{Q}(b,c) \) \( f \in \mathcal{Q}(a,b) \).

For a quantaloidal nucleus \( j \) on a quantaloid \( \mathcal{Q} \), let \( \mathcal{Q}_j \) be the bicategory with the same objects as \( \mathcal{Q} \) and \( \mathcal{Q}_j(a,b) = \{ f \in \mathcal{Q}(a,b) \mid j_{a,b}(f) = f \} \) for \( a,b \in \mathcal{Q}_0 \). Composition in \( \mathcal{Q}_j \) is defined as follows: \( g \circ_j f = j_{a,c}(g \circ f) \) for \( f \in \mathcal{Q}_j(a,b), g \in \mathcal{Q}_j(b,c) \).

**Proposition 2.7** ([20]) If \( j \) is a quantaloidal nucleus on a quantaloid \( \mathcal{Q} \), then \( \mathcal{Q}_j \) is a quantaloid and \( j : \mathcal{Q} \to \mathcal{Q}_j \) is a quantaloidal homomorphism.

**Proposition 2.8** ([20]) Let \( \mathcal{S} \) be a subcategory of a quantaloid \( \mathcal{Q} \), which contains all the objects of \( \mathcal{Q} \). Then, \( \mathcal{S} \) is a quotient quantaloid of the form \( \mathcal{Q}_j \) for some quantaloidal nucleus \( j \) iff

1. each hom-set \( \mathcal{S}(a,b) \) is closed under infs, and
2. if \( f \in \mathcal{S}(a,c) \), then \( g \to_l f \in \mathcal{S}(b,c) \) for all \( g \in \mathcal{Q}(a,b) \) and \( h \to_r g \in \mathcal{S}(a,b) \) for all \( h \in \mathcal{Q}(b,c) \).

### 3 Quantaloidal completions of order-enriched categories

In order to study quantaloidal completions of order-enriched categories, let us begin with the concept of a compatible nucleus on a quantaloid.

**Definition 3.1** Let \( \mathcal{A} \) be an order-enriched category, \( j : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A}) \) a quantaloidal nucleus. \( j \) is said to be compatible if for \( a,b \in \mathcal{A}_0, f \in \mathcal{A}(a,b) \), we have \( j_{a,b}(\{ f \}) = \downarrow f \).

**Definition 3.2** Let \( \mathcal{A} \) be an order-enriched category, \( \mathcal{Q} \) a quantaloid, \( F : \mathcal{A} \to \mathcal{Q} \) a 2-functor. The pair \((F, \mathcal{Q})\) is said to be a quantaloidal completion of \( \mathcal{A} \), if the following conditions are satisfied:

1. \( F : \mathcal{A}_0 \to \mathcal{Q}_0 \) is bijective,
2. \( F_{a,b} : \mathcal{A}(a,b) \to \mathcal{Q}(Fa,Fb) \) is an order embedding for all \( a,b \in \mathcal{A}_0 \),

\(^4\) A set \( D \) in a poset \( P \) is a down set, if \( D = \downarrow D \), where \( \downarrow D = \{ x \mid \exists d \in D, \text{ s. t. } x \leq d \} \).
Lemma 3.7 Let \( \mathcal{L} \) denote the set of all quantaloidal completions of \( \mathcal{A} \) on \( \mathcal{P}(\mathcal{A}) \). Then \( \mathcal{L} \) corresponds to the quantaloidal nucleus and is compatible.

Theorem 3.3 If \( j \) is a compatible nucleus on an order-enriched category \( \mathcal{A} \), then \((F_j, \mathcal{P}(\mathcal{A}))_j\) is a quantaloidal completion of \( \mathcal{A} \), where \( F_j : \mathcal{A} \to \mathcal{P}(\mathcal{A}) \) is defined as follows:

1. \( F_j : \mathcal{A}_0 \to (\mathcal{P}(\mathcal{A}))_0 \) is the identity map,
2. \( F_j(f) = \downarrow f \) for every \( f \in \mathcal{A}(a,b) \), \( a, b \in \mathcal{A}_0 \).

**Proof.** By definition, \( F_j : \mathcal{A}_0 \to (\mathcal{P}(\mathcal{A}))_0 \) is bijective, and \( F_j : \mathcal{A}(a,b) \to \mathcal{P}(\mathcal{A})(a,b) \) is an order-embedding. For \( S \in \mathcal{P}(\mathcal{A})(a,b) \), we have \( S = j_{a,b}(S) = \bigvee_{f \in S} \{f\} = \bigvee_{f \in S} j_{a,b}(\{f\}) = \bigvee_{f \in S} F_j(f) \). This completes the proof. \( \square \)

Example 3.4 (Down-set completion) Let \( \mathcal{A} \) be an order-enriched category. Define a lax functor \( \downarrow : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A}) \) as follows:

1. \( \downarrow : \mathcal{P}(\mathcal{A})_0 \to \mathcal{P}(\mathcal{A})_0 \) is the identity map,
2. \( \downarrow_{a,b} S = \downarrow S \) for \( S \in \mathcal{P}(\mathcal{A})(a,b) \), \( a, b \in \mathcal{P}(\mathcal{A})_0 \).

Then \( \downarrow \) is a compatible nucleus. The quotient corresponding to \( \downarrow \) is \( \mathcal{D}(\mathcal{A}) \).

Example 3.5 (MacNeille completion) Let \( \mathcal{A} \) be an order-enriched category. Define a lax functor \( \downarrow : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A}) \) as follows:

1. \( \downarrow : \mathcal{P}(\mathcal{A})_0 \to \mathcal{P}(\mathcal{A})_0 \) is the identity map,
2. \( \downarrow_{a,b}(S) = \{f \in \mathcal{P}(\mathcal{A})(a,b) \mid \forall g \in \mathcal{P}(\mathcal{A})(a',a), h \in \mathcal{P}(\mathcal{A})(b,b'), k \in \mathcal{P}(\mathcal{A})(a,b), h \circ S \circ g \subseteq \downarrow k \text{ implies } h \circ f \circ g \leq k\} \) for \( S \in \mathcal{P}(\mathcal{A})(a,b) \), \( a, b \in \mathcal{P}(\mathcal{A})_0 \).

Then \( \downarrow \) is a compatible nucleus.

Example 3.6 (Equivariant completion) Let \( \mathcal{A} \) be an order-enriched category. Suppose \( S \subseteq \mathcal{P}(\mathcal{A})(a,b) \). If the join of \( S \) exists and is preserved by composition, i.e., \( f \circ (\bigvee S) = \bigvee (f \circ S), (\bigvee S) \circ g = \bigvee (S \circ g) \) whenever the composition is well-defined, then \( \bigvee S \) is said to be an equivariant join with respect to \( S \). Clearly, every \( f \in \mathcal{P}(\mathcal{A})(a,b) \) is an equivariant join with respect to \( f \). If \( k \) is an equivariant join with respect to \( S \), then \( g \circ k \) (resp., \( k \circ h \)) is an equivariant join with respect to \( g \circ S \) (resp., \( S \circ h \)), whenever the composition is well-defined. For \( S \subseteq \mathcal{P}(\mathcal{A})(a,b) \), let

\[
S^{EJ} = \{f \in \mathcal{P}(\mathcal{A})(a,b) \mid \exists T \subseteq S, \text{ s.t. } f = \bigvee T \text{ is an equivariant join with respect to } T\}.
\]

Let \( EJ(\mathcal{A}) \) be the subcategory of \( \mathcal{A} \), which contains all the objects of \( \mathcal{A} \). The hom-sets

\[
EJ(\mathcal{A})(a,b) = \{S \in \mathcal{D}(\mathcal{A})(a,b) \mid S = S^{EJ}\}.
\]

Then \( EJ(\mathcal{A}) \) is a quotient of \( \mathcal{A} \) such that \( \downarrow f \in EJ(\mathcal{A})(a,b) \) for every \( f \in \mathcal{P}(\mathcal{A})(a,b) \). Consequently, the corresponding quantaloidal nucleus is compatible.

For an order-enriched category \( \mathcal{A} \), \( CN(\mathcal{A}) \) denotes the class of all compatible nuclei on \( \mathcal{P}(\mathcal{A}) \), \( QC(\mathcal{A}) \) denotes the set of all quantaloidal completions of \( \mathcal{A} \).

Let \( \mathcal{A} \) be an order-enriched category, \((F, Q) \in QC(\mathcal{A}) \). Define \( j_{(F, Q)} : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A}) \) as follows:

1. \( j_{(F, Q)} : \mathcal{P}(\mathcal{A})_0 \to \mathcal{P}(\mathcal{A})_0 \) is the identity map,
2. \( j_{(F, Q)}(S) = \{f \in \mathcal{A}(a,b) \mid F(f) \leq \bigvee_{g \in S} F(g)\} \) for every \( S \in \mathcal{P}(\mathcal{A})(a,b), a, b \in \mathcal{A}_0 \).

**Lemma 3.8** Let \( \mathcal{A} \) be an order-enriched category, \((F, Q) \in QC(\mathcal{A}) \). Then \( j_{(F, Q)} \) is a compatible nucleus on \( \mathcal{P}(\mathcal{A}) \).
Proof. By definition, \( j_{(F, Q)} : \mathcal{P}(\mathcal{A})_0 \to \mathcal{P}(\mathcal{A})_0 \) is bijective, \( j_{(F, Q)} : \mathcal{P}(\mathcal{A})(a, b) \to \mathcal{P}(\mathcal{A})(a, b) \) is order preserving and increasing for all \( a, b \in \mathcal{A}_0 \). Suppose \( S \in \mathcal{P}(\mathcal{A})(a, b), f \in j_{(F, Q)}(j_{(F, Q)}(S)) \). Then, \( F(f) \leq \bigvee_{g \in j_{(F, Q)}(S)} F(g) \). For every \( g \in j_{(F, Q)}(S) \), we have \( F(g) \leq \bigvee_{k \in S} F(k) \). Thus, \( F(f) \leq \bigvee_{k \in S} F(k) \). Consequently, \( f \in j_{(F, Q)}(S) \). So we can conclude that \( j_{(F, Q)} \circ j_{(F, Q)} = j_{(F, Q)} \). Thus, \( j_{(F, Q)} : \mathcal{P}(\mathcal{A})(a, b) \to \mathcal{P}(\mathcal{A})(a, b) \) is a closure operator for every \( a, b \in \mathcal{A}_0 \).

Suppose \( K \in \mathcal{P}(\mathcal{A})(b, c), S \in \mathcal{P}(\mathcal{A})(a, b) \). Then \( j_{(F, Q)}(K) \circ j_{(F, Q)}(S) = \{ g \circ f \mid g \in \mathcal{A}(b, c), f \in \mathcal{A}(a, b), F(g) \leq \bigvee_{k \in K} F(k), F(f) \leq \bigvee_{t \in S} F(t) \} \). If \( F(g) \leq \bigvee_{k \in K} F(k), F(f) \leq \bigvee_{t \in S} F(t) \), then \( F(g \circ f) \leq \bigvee_{k \in K, t \in S} F(k \circ t) \leq \bigvee_{p \in K \circ S} F(p) \). Thus, \( j_{(F, Q)}(K) \circ j_{(F, Q)}(S) \subseteq j_{(F, Q)}(K \circ S) \).

For \( f_0 \in \mathcal{A}(a, b) \), by the fact that \( F : \mathcal{A}(a, b) \to Q(F(a), F(b)) \) is an order embedding, we have \( j_{(F, Q)}(\{ f_0 \}) = \{ f \in \mathcal{A}(a, b) \mid F(f) \leq F(f_0) \} = \downarrow f_0 \).

So we can conclude that \( j_{(F, Q)} \) is a compatible nucleus on \( \mathcal{P}(\mathcal{A}) \).

\( \square \)

**Theorem 3.8** Let \( \mathcal{A} \) be an order-enriched category, \( (F, Q) \in QC(\mathcal{A}) \). Then \( Q \) is quantaloidal isomorphism to \( \mathcal{P}(\mathcal{A})_{j_{(F, Q)}} \).

**Proof.** Let \( F^{-1} : Q_0 \to \mathcal{A}_0 \) be the inverse of the map \( F : \mathcal{A}_0 \to Q_0 \). Define \( G : Q \to \mathcal{P}(\mathcal{A})_{j_{(F, Q)}} \) as follows:

1. \( G(a) = F^{-1}(a) \) for every \( a \in Q_0 \).
2. \( G(p) = \{ f \in \mathcal{A}(F^{-1}(c), F^{-1}(d)) \mid F(f) \leq p \} \) for every \( p \in Q(c, d) \).

Then \( G : Q_0 \to (\mathcal{P}(\mathcal{A})_{j_{(F, Q)}})_0 \) is bijective. For \( f \in j_{(F, Q)}(G(p)) \), we have \( F(f) \leq \bigvee_{g \in G(p)} F(g) \leq p \), thus \( f \in G(p) \). Thus, \( j_{(F, Q)}(G(p)) \subseteq G(p) \). Consequently, \( G(p) = j_{(F, Q)}(G(p)) \in \mathcal{P}(\mathcal{A})_{j_{(F, Q)}} \). Thus, \( G \) is well-defined.

Suppose \( a, b \in Q_0, S \subseteq Q(a, b) \). Then \( G(\bigvee S) = \{ f \in \mathcal{A}(F^{-1}(a), F^{-1}(b)) \mid F(f) \leq \bigvee S \} \),

\[
\bigvee_{t \in S} G(t) = j_{(F, Q)} \left( \bigcup_{t \in S} G(t) \right) = j_{(F, Q)} \left( \bigcup_{t \in S} \{ g \in \mathcal{A}(F^{-1}(a), F^{-1}(b)) \mid F(g) \leq t \} \right) = j_{(F, Q)} \{ g \in \mathcal{A}(F^{-1}(a), F^{-1}(b)) \mid \exists t \in S, \text{ s.t. } F(g) \leq t \} = \{ f \in \mathcal{A}(F^{-1}(a), F^{-1}(b)) \mid F(f) \leq \bigvee \{ F(g) \mid g \in \mathcal{A}(F^{-1}(a), F^{-1}(b)), \exists t \in S, \text{ s.t. } F(g) \leq t \} \}.
\]

For \( s_0 \in S \), we have

\[
s_0 = \bigvee \{ F(g) \mid g \in \mathcal{A}(F^{-1}(a), F^{-1}(b)), F(g) \leq s_0 \} \leq \bigvee \{ F(g) \mid g \in \mathcal{A}(F^{-1}(a), F^{-1}(b)), \exists t \in S, \text{ s.t. } F(g) \leq t \}.
\]

Thus, \( \bigvee S \leq \bigvee \{ F(g) \mid g \in \mathcal{A}(F^{-1}(a), F^{-1}(b)), \exists t \in S, \text{ s.t. } F(g) \leq t \} \), whence \( G(\bigvee S) \leq \bigvee_{t \in S} \mathcal{P}(\mathcal{A})_{j_{(F, Q)}} G(t) \).

The inverse inequality holds trivially. Therefore, \( G(\bigvee S) = \bigvee_{t \in S} \mathcal{P}(\mathcal{A})_{j_{(F, Q)}} G(t) \).

For \( a \in Q_0 \), we have \( G(1_a) = \{ f \in \mathcal{A}(F^{-1}(a), F^{-1}(a)) \mid F(f) \leq 1_a \} = \downarrow 1_{G(a)} \), which is the identity in \( \mathcal{P}(\mathcal{A})_{j_{(F, Q)}} \).

Suppose \( f \in Q(a, b), g \in Q(b, c) \). Then

\[
G(g) \circ j_{(F, Q)} G(f) = j_{(F, Q)}(G(g) \circ G(f)) = \{ t \in \mathcal{A}(F^{-1}(a), F^{-1}(c)) \mid F(t) \leq \bigvee \{ F(h) \mid h \in G(g) \circ G(f) \} \}.
\]
Since,
\[ \sqrt{\{F(h) \mid h \in G(g) \circ G(f)\}} = \sqrt{\{F(t_2 \circ t_1) \mid t_1 \in A(F^{-1}(a), F^{-1}(b)), t_2 \in A(F^{-1}(b), F^{-1}(c)), F(t_1) \leq f, F(t_2) \leq g\} = (\sqrt{\{F(t_2) \mid t_2 \in A(F^{-1}(b), F^{-1}(c)), F(t_2) \leq g\}) \circ \sqrt{\{F(t_1) \mid t_1 \in A(F^{-1}(a), F^{-1}(b)), F(t_1) \leq f\}\}} = g \circ f, \]
we have \( G(g) \circ j_{(F, Q)} G(f) = \{t \in A(F^{-1}(a), F^{-1}(c)) \mid F(t) \leq g \circ f\} = G(g \circ f) \).
So we can conclude that \( G \) is a quantaloidal homomorphism.
Suppose \( p_1, p_2 \in Q(c, d) \) with \( G(p_1) = G(p_2) \). Then \( p_1 = \sqrt{F(G(p_1))} = \sqrt{F(G(p_2))} = p_2 \). Thus, \( G : Q(c, d) \rightarrow (\mathcal{P}(A)_{j_{(F, Q)}})(F^{-1}(c), F^{-1}(d)) \) is injective for all \( c, d \in Q_0 \).
Suppose \( S \in \mathcal{P}(A)_{j_{(F, Q)}}(a, b) \). Then \( S \subseteq A(a, b) \). For every \( f \in A(a, b) \), we have \( G(f) = \{g \in A(F^{-1}(a), F^{-1}(b)) \mid F(g) \leq F(f)\} = \{g \in A(F^{-1}(a), F^{-1}(b)) \mid g \leq f\} = F(f) \). Thus, \( S = j_{(F, Q)}(S) = j_{(F, Q)} \left( \bigcup_{f \in S} \{f\} \right) = \bigvee_{f \in S} j_{(F, Q)}(\{f\}) = \bigvee_{f \in S} \mathcal{P}(A)_{j_{(F, Q)}}(a, b) \downarrow f = \bigvee_{f \in S} \mathcal{P}(A)_{j_{(F, Q)}}(a, b) G(f) = G \left( \bigvee_{f \in S} Q(F(a), F(b)) F(f) \right) \). Thus, \( G : (a, b) \rightarrow (\mathcal{P}(A)_{j_{(F, Q)}})(F^{-1}(a), F^{-1}(b)) \) is surjective. Therefore, \( G \) is a quantaloidal isomorphism.

As a combination of the above results, we obtain that quantaloidal completions of an order-enriched category \( A \) are completely determined by compatible quantaloidal nuclei on \( \mathcal{P}(A) \).

**Theorem 3.9** Let \( A \) be an order-enriched category. Then \((F, Q)\) is a quantaloidal completion of \( A \) if and only if there is a compatible nucleus \( j \) on \( \mathcal{P}(A) \) such that \( Q \) is quantaloidal isomorphism to \( \mathcal{P}(A)_j \).

### 4 Applications

In this section, we shall give two kinds of applications for the quantaloidal completions of order-enriched categories.

#### 4.1 Injective constructs of order-enriched categories

Let \( \mathbf{O-Cat}_l \) be the category of order-enriched categories and lax functors. Let \( \mathcal{E}_{\leq}^{ls} \) be the class of all lax functors in \( \mathbf{O-Cat}_l \) satisfying the following conditions:

1. \( F : C_0 \rightarrow D_0 \) is bijective;
2. \( F(f_1) \circ F(f_2) \circ \cdots \circ F(f_n) \leq F(f) \) implies \( f_1 \circ f_2 \circ \cdots \circ f_n \leq f \) for \( f_1 \circ f_2 \circ \cdots \circ f_n, f \in C(a, b), a, b \in C_0 \).

**Lemma 4.1** In the category \( \mathbf{O-Cat}_l \), every retract of a quantaloid is a quantaloid.

**Proof.** Let \( S \) be a retract of a quantaloid \( Q \). Then there exist lax functors \( I : S \rightarrow Q \) and \( F : Q \rightarrow S \) such that \( F \circ I = \text{id}_S \). Suppose \( S, T \in S_0 \). Then \( S(X, T) \) is a retract of \( Q(I(X), Y) \). By the fact that \( Q(I(X), Y) \) is a complete lattice, we can deduce that \( S(X, Y) \) is a complete lattice and \( F(\bigvee I(A)) \) is the least upper bound of \( A \) in \( S(X, Y) \). Suppose \( A \subseteq S(X, Y) \), \( g \in S(Y, Y'), t \in S(X', X) \). Then \( g \circ (\bigvee A) \) is an upper bound of \( g \circ A \). If \( h \) is an upper bound of \( g \circ A \), then \( I(g) \circ \bigvee_{f \in A} I(f) = \bigvee_{f \in A} (I(g) \circ I(f)) \leq \bigvee_{f \in A} (I(g \circ f)) \leq I(h) \). Thus, \( h = FI(h) \geq F \left( I(g) \circ \bigvee_{f \in A} I(f) \right) \geq FI(g) \circ F \left( \bigvee_{f \in A} I(f) \right) = g \circ (\bigvee A) \). Thus, \( g \circ (\bigvee A) = \bigvee(g \circ A) \). Similarly, we have \((\bigvee A) \circ t = \bigvee(A \circ t) \). Therefore, \( S \) is a quantaloid. \( \Box \)

**Theorem 4.2** Let \( A \) be an order-enriched category. Then \( A \) is \( \mathcal{E}_{\leq}^{ls} \)-injective in \( \mathbf{O-Cat}_l \) if and only if \( A \) is a quantaloid.
Define $G : T \to Q$ as follows:

1. $GX = FH^{-1}(X)$, $\forall X \in T_0$;
2. $G(g) = \bigvee \{ F(f_1) \circ F(f_2) \circ \cdots \circ F(f_n) \mid H(f_1) \circ H(f_2) \circ \cdots \circ H(f_n) \leq g, f_1 \circ f_2 \circ \cdots \circ f_n \in S(H^{-1}X, H^{-1}Y) \}$ for $g \in T(X, Y)$, $X, Y \in T_0$.

Then $G : T(X, Y) \to Q(GX, GY)$ is order-preserving for $X, Y \in T_0$. Suppose $g_1 \in T(X, Y), g_2 \in T(Y, Z)$. Since composition in a quantaloid distribute over arbitrary joins, we can deduce that $G(g_2) \circ G(g_1) \leq G(g_2 \circ g_1)$. Thus $G : T \to Q$ is a lax semifunctor. For $X \in S$, we have $GH(X) = FH^{-1}H(X) = F(X)$.

Thus, it is routine to check that $Q$ is a quantaloid, as it is a retract of the quantaloid $Q_{\leq s}$. So we can conclude that $Q$ is $Q_{\leq s}$-injective.

Conversely, suppose $A$ is $Q_{\leq s}$-injective in $O\text{-Cat}_t$. Define $F : A \to D(A)$ as follows:

1. $F : A_0 \to D(A)_0$ is the identity map;
2. $F(f) = \downarrow f$ for $f \in A(a, b)$, $a, b \in A_0$.

Then its routine to check that $F \in Q_{\leq s}$. Thus, for the identity functor $id_A : A \to A$, there is a lax semifunctor $G : D(A) \to A$ such that $G\overline{F} = id_A$. So, $A$ is a quantaloid, as it is a retract of the quantaloid $D(A)$.

Let $A$ be an order-enriched category. Define $\eta : A \to P(A)_{cl}$ as follows:

1. $\eta : A_0 \to (P(A)_0)_0$ is the identity map;
2. $\eta(f) = \downarrow f$ for $f \in A(a, b)$, $a, b \in A_0$.

Then it is routine to check that $\eta$ is a lax semifunctor and it is $Q_{\leq s}$-essential. As the proof is quite similar to that of Theorem 5.8 in [12], we leave it to the reader.

**Theorem 4.3** Let $A$ be an order-enriched category. Then $P(A)_{cl}$ is an $Q_{\leq s}$-injective hull of $A$ in $O\text{-Cat}_t$.

### 4.2 Free quantaloids

Let $LocSm$ be the category of locally small categories and functors between them. Let $O\text{-Cat}$ be the category of order-enriched categories and 2-functors. Let $Qtlds$ be the category of quantaloids and quantaloidal homomorphisms.

**Theorem 4.4** The functor $D : O\text{-Cat} \to Qtlds$ is left adjoint to the forgetful functor $Qtlds \to O\text{-Cat}$.

**Proof.** Let $A$ be an order-enriched category. Define $\eta : A \to D(A)$ as follows:

1. $\eta : A_0 \to (D(A)_0)_0$ is the identity map;
2. $\eta(f) = \downarrow f$ for $f \in A(a, b)$, $a, b \in A_0$.

Then $\eta$ is a 2-functor in $O\text{-Cat}$.

Suppose that $Q$ is a quantaloid and that $F : A \to Q$ is a 2-functor in $O\text{-Cat}$. Define $\overline{F} : D(A) \to Q$ as follows:

1. $\overline{F}(a) = F(a)$ for every $a \in (D(A)_0)_0$;
2. $\overline{F}(S) = \bigvee \{ F(f) \mid f \in S \}$ for every $S \in D(A)(a, b)$.

For $a \in (D(A)_0)_0$, we have $\overline{F}(\downarrow 1_a) = \bigvee \{ F(f) \mid f \in \downarrow 1_a \} = \bigvee \{ F(f) \mid f \leq 1_a \} = F(1_a) = 1_F(a)$.

For $T \in D(A)(b, c), S \in D(A)(a, b)$, we have $\overline{F}(T \circ S) = \bigvee \{ F(h) \mid h \in T \circ S \} = \bigvee \{ F(g \circ f) \mid g \in T, f \in S \} = \bigvee \{ F(g) \mid g \in T \} \circ \bigvee \{ F(f) \mid f \in S \} = F(T) \circ \overline{F}(S)$. For $S_i \in D(A)(a, b), i \in I$, we have $\overline{F}(\bigcup_{i \in I} S_i) = \bigvee \{ F(f) \mid f \in \bigcup_{i \in I} S_i \} = \bigvee_{i \in I} \bigvee \{ F(f) \mid f \in S_i \} = \bigvee_{i \in I} \overline{F}(S_i)$.

Thus, $\overline{F}$ is a quantaloidal homomorphism. Furthermore, we can check that $\overline{F} \circ \eta = F$.

Suppose $G : D(A) \to Q$ is a quantaloidal homomorphism with $G \circ \eta = F$. Then we have

1. $\forall a \in (D(A)_0)_0$, $F(a) = \overline{F}(\eta(a)) = F(\eta(a)) = (G \circ \eta)(a) = G(a)$;
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(2) \( \forall S \in \mathcal{D}(A)(a, b), \bar{F}(S) = \bar{F}(\bigcup \{ \downarrow f \mid f \in S \}) = \bigvee_{f \in S} \bar{F}(\downarrow f) = \bigvee_{f \in S} \bar{F}(\eta(f)) = \bigvee_{f \in S} F(f) = \bigvee_{f \in S}(G \circ \eta)(f) = \bigvee_{f \in S} G(\downarrow f) = G\left( \bigvee_{f \in S} \downarrow f \right) = G(S). \) Thus, \( \bar{F} : \mathcal{D}(A) \to Q \) is the unique quantaloidal homomorphism such that \( \bar{F} \circ \eta = F. \)

Every locally small category can be viewed as an order-enriched category with the discrete order on hom-sets. We know \( \mathcal{D}(A) = \mathcal{P}(A) \) for every locally small category with discrete order on hom-sets. Thus, we can recover the following results [20].

**Corollary 4.5** The functor \( \mathcal{P} : \text{LocSm} \to \text{Qtlds} \) is left adjoint to the forgetful functor \( \text{Qtlds} \to \text{LocSm}. \)

5 Conclusion and some further work

In this paper, we only considered quantaloidal completions for order-enriched categories. As order-enriched category with other completeness have deep applications in domain theory [15, 26, 33], other types of completions and applications deserve to be developed further.

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