Onsager vortex formation in two-component Bose–Einstein condensates in two-dimensional traps

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Abstract. We study numerically the dynamics of quantized vortices in two-dimensional one-component and two-component Bose–Einstein condensates (BECs) trapped by a harmonic and box potentials. In two-component miscible BECs, we confirmed the tendency of the formation of Onsager vortices in both traps. The vortices in one component separate spatially from those in the other component, which comes from their intercomponent-coupling. We also discuss the decay of the number of vortices.

1. Introduction

Turbulence shows strong dependence on the spatial dimension [1]. The three-dimensional (3D) classical turbulence sustains the direct cascade of energy transferred from large to small scales. Then, its energy spectrum obeys the Kolmogorov’s -5/3 power law. On the other hand, Onsager predicted that the spontaneous formation of large-scale, long-lived vortices in two-dimensional (2D) turbulence [2, 3]. Kraichnan predicted that 2D classical turbulence shows the inverse energy cascade from small to large scales [4].

These statistics are also studied in quantum turbulence. In Bose–Einstein condensate (BEC), the system can be described by the macroscopic wave function \( \psi \). As a result, a quantized vortex appear as a well-defined topological defect with the quantized circulation, being different from vortices in classical hydrodynamics. Various properties of quantum turbulence are attributed to dynamics of quantized vortices. In experiments of cold atoms, BECs are usually trapped by some potential like a harmonic one. Recently the box potential is realized and gives another important stage of quantum hydrodynamics [5, 6]. Controlling the potential enables us to realize 3D and 2D condensates. Three-dimensional quantum turbulence shows the same energy cascade and statistical Kolmogorov’s -5/3 power law as 3D classical turbulence [7]. If the confinement along one direction is much stronger than other directions, the condensate is highly oblate and becomes two-dimensional. Two-dimensional quantum turbulence is expected to show the same inverse energy cascade as 2D classical turbulence and the clustering of like-sign vortices into large-scale Onsager vortex structure.

In the experimental [8] and numerical [9] studies, highly oblate BECs trapped by harmonic potentials did not show the tendency to form Onsager vortices. On the other hand, BECs in the box trap show strong tendency to form Onsager vortices [10]. Groszek et al. studied numerically the dynamics of quantized vortices in BECs trapped by both harmonic and box potentials [11]. They concluded that the vortices can form Onsager vortices in the box trap but their formation is suppressed in the harmonic trap.
We study whether Onsager vortices form or not in 2C BECs trapped by the harmonic and the box potentials. We investigate the decay of the vortex number, the development of the dipole moment and the distribution of vortices in 2C BECs. The main interest of this work is to know the effect of intercomponent-coupling on the dynamics of vortices. We also do the same calculation in 1C BEC for the comparison with 2C BECs.

2. Model

The macroscopic wave function $\psi$ of 2D 1C BEC satisfies the Gross–Pitaevskii equation (GPE)

$$(i - \gamma)\frac{\partial}{\partial t} \psi(r, t) = \left\{ \frac{-\hbar^2}{2m} \nabla^2 + V_{\text{trap}}(r) + g|\psi(r, t)|^2 \right\} \psi(r, t),$$

where $\gamma$ is the dimensionless damping parameter [12], $m$ is the mass of an atom, $V_{\text{trap}}$ is the trapping potential and $g$ is the strength of interaction. We consider the harmonic potential

$$V_{\text{trap}}(r) = \frac{1}{2} m \omega_r^2 |r|^2,$$

with the radial harmonic trapping frequency $\omega_r$, and the box potential

$$V_{\text{trap}}(r) = \left\{ \begin{array}{ll} V_0 & (|r| > R_0) \\ 0 & (|r| < R_0) \end{array} \right.,$$

with an effective system radius $R_0$ and the potential height $V_0$. Two-dimensional BECs are described by the coupled GPEs for two macroscopic wave function $\psi_i$

$$\hbar \frac{\partial}{\partial t} \psi_i(r, t) = \left\{ \frac{-\hbar^2}{2m_i} \nabla^2 + V_{\text{trap}}(r) + \sum_{j=1,2} g_{ij} |\psi_j(r, t)|^2 \right\} \psi_i(r, t), \quad (i = 1, 2)$$

where $m_i$ is the mass of the $i$th component’s atom. Here, $g_{11}$ and $g_{22}$ are intracomponent-coupling constants, and $g_{12}$ is the intercomponent-coupling constant. In this study, we choose $m_1 = m_2 = m$, $g_{11} = g_{22} = g$ and $g_{12} = 0.9g$, where the two components are miscible.

We prepare the initial configuration of the vortices by the following method. For 1C BEC, we first imprint $N_v = 120$ vortices in the condensate by multiplying the ground state wave function by a phase factor $\Pi_i \exp(iq_i \phi_i)$, with $\phi_i(x, y) = s_i \arctan \{ (y - y_i)/(x - x_i) \}$. Here, the coordinate $(x_i, y_i)$ refers to the position of the $i$th vortex in the initial state and $s_i$ is its sign. We choose $x_i$ and $y_i$ randomly and imprint equal numbers of vortices ($s_i = 1$) and antivortices ($s_i = -1$). After the imprinting, the wave function is evolved in imaginary time to establish the structure of the vortex cores, and we solve the GPE in real time from this initial configuration. For 2C BECs system, we make vortices using the same method as 1C BEC system in each component. Then, we solve the GPE or the coupled GPEs by a fourth-order Fourier method on a $1024 \times 1024$ spatial grid.

The vortices are identified by measuring the phase singularities in the wave function. The vortex number is counted in a region $|r| < 0.9 R_{TF}$ with $R_{TF}$ is Thomas–Fermi radius in the harmonic trap, and $|r| < 0.9 R_0$ in the box trap in order to avoid counting ghost vortices[12]. We also calculate the dipole moment of the vortex distribution, which is defined as $d = |d| = |\Sigma_i q_i r_i|$, where $r_i$ is the position of the $i$th vortex and $q_i = s_i h/m$ is its charge with $s_i = \pm 1$. If the vortices are distributed uniformly, $d$ seldom grows. If like-sign vortices form an Onsager vortex, $d$ develops into some finite value.

In the previous studies [8, 9, 10, 11] and this work, the vortex number $N_v$ is found to decrease. Kwon et al. studied the decay of $N_v$ experimentally and suggested a phenomenological rate
equation for $N_v$ [8]. Groszek et al. also studied numerically the decay of $N_v$ by solving the GPE [11]. They found that the decay of $N_v$ was independent the geometry of the trapping potential. They extended the phenomenological rate equation introduced by Kwon et al. to
\[ \frac{dN_v}{dt} = -\Gamma_1 N_v - \Gamma_2 N_v^2 - \Gamma_3 N_v^3 - \Gamma_4 N_v^4 - \cdots , \tag{5} \]
in both systems of $\gamma = 0$ and $\gamma = 10^{-3}$. Here, the first term with $\Gamma_1$ means the drifting-out process, which means the vortices drift out of the condensate. The second term with $\Gamma_2$ refers to the vortex-antivortex pair annihilation [8]. The third and forth terms with $\Gamma_3$ and $\Gamma_4$ mean three-body and four-body loss, respectively. The three-body loss comes from the pair annihilation with a catalyst vortex and the four-body loss comes from the pair annihilation with two catalyst vortices. Groszek et al. fit their numerical results with Eq. (5) to study the effect of the thermal cloud [11]. From these simulation, they obtained $\Gamma_1 = 0.14 \, s^{-1}$, $\Gamma_2 = 0.044 \, s^{-1}$, $\Gamma_3 = \Gamma_4 = 0$ for the $\gamma = 10^{-3}$ case and found that the drifting-out and the pair annihilation were dominant. For the $\gamma = 0$ case, however, they obtained $\Gamma_1 = \Gamma_2 = 0$, $\Gamma_3 = 1.2 \times 10^{-4} \, s^{-1}$, $\Gamma_4 = 8.1 \times 10^{-7} \, s^{-1}$, and showed that the three-body and four-body losses were dominant.

3. Results

In this chapter, we first calculate $N_v$ and $d$. Then, we compare these between the harmonic and the box traps. We first address 1C BEC and then 2C BECs. We confine ourselves to the case of $\gamma = 0$.

The results in 1C BEC are shown by Figs. 1 and 2. Figure 1 (a) shows that the vortex number decays in both traps. We fit Eq. (5) with the two curves of $N_v$. In the harmonic trap, we obtain $\Gamma_1 = \Gamma_2 = 0$, $\Gamma_3 = 1.9 \times 10^{-5} \, s^{-1}$, $\Gamma_4 = 1.3 \times 10^{-7} \, s^{-1}$ (Fig. 1 (a) yellow (light) full thick line). In the box trap, we have $\Gamma_1 = \Gamma_2 = 0$, $\Gamma_3 = 2.3 \times 10^{-5} \, s^{-1}$, $\Gamma_4 = 2.2 \times 10^{-7} \, s^{-1}$ (Fig. 1 (a) purple (dark) full thick line). They are same order as the previously study [11]. The difference can come from the different sizes of the system. If the systems of us and Groszek et al. are normalized by each coherence length $\xi$, our condensate is larger than that of Groszek et al., while the initial number of imprinted vortices are same. Then, the density of vortices in our system is smaller than that of Groszek et al. This may make our values of $\Gamma_i$ smaller than those of Groszek et al.

Figure 1 (b) and (c) show the time development of the amplitude $d$ of the dipole moment and its $x$ component $d_x$. Figure 2 shows how the distribution of vortices changes as time goes by. In the harmonic trap, $d$ hardly increases, and $d_x$ fluctuates around zero (Fig. 1 (b),(c) black (dark) dotted line). Then, the vortices are distributed uniformly at $t = 0$ (Fig. 2 (a)), and they are still uniform at $t = 1335$ (Fig. 2 (b)) even after the decay. On the other hand, in the box trap, $d$ and $d_x$ increase into some finite values (Fig. 1 (b),(c) light blue (light) dashed line). The initially uniform vortices (Fig. 2 (c)) lead to two Onsager vortices, one consisting of vortices ($s_i = 1$) and the other consisting of antivortices ($s_i = -1$) at $t = 1335$ (Fig. 2 (d)). These results are consistent with the previous simulation [9, 10, 11] and experiment [8]. In Fig. 1 (c), $d_x$ in the box trap is almost constant and positive from $t \approx 400$ until $t \approx 1000$, and then $d_x$ changes the sign with keeping $d$, which means that the two Onsager vortices revolve inside the condensate.

In 2C BECs, both the decay of $N_v$ and the development of $d$ are different from those in 1C BEC. Figure 3 shows $N_v$ and $d$ in the harmonic and the box traps.
Figure 1. Time development of vortex number $N_v$ (a), the dipole moment $d$ (b) and its $x$ component $d_x$ (c) in 1C BEC. In (a), yellow (light) full thick line is fitted with decay in the harmonic trap and purple (dark) full thick line is fitted with decay in the box trap. Dipole moment is normalized by the effective system radius ($R_0$), the vortex charge ($\kappa$) and the vortex pair number ($N_v/2$).

Figure 2. The distribution of vortices in 1C BEC in the harmonic trap at $t = 0$ (a) and at $t = 1335$ (b), and in the box trap at $t = 0$ (c) and at $t = 1335$ (d). Vortices and antivortices are denoted by black (dark) points and white (light) points, respectively. White (light) circles represent the boundary of condensate. In (d), vortices cluster in the area surrounded by dashed black (dark) line and antivortices cluster in the area surrounded by dotted white (light) line.

Decay of $N_v$ in 2C BECs (Fig. 3 (a), (c)) is much faster than in 1C BEC. This difference of the decays can be attributed to the forces $F_{\text{intra}}$ between the vortices in the same component and $F_{\text{inter}}$ between the vortices in different components [13]. In 1C BEC and 2C BECs, $F_{\text{intra}}$ is given by

$$F_{\text{intra}}^{ij} = \frac{2\pi s_is_j\hbar^2n}{m} \frac{1}{R_{ij}},$$

where $2R_{ij}$ is the distance between the $i$th and the $j$th vortices [13]. Then, $F_{\text{intra}}$ is repulsive between the vortices with the same sign and attractive between the vortices with different signs.

In 2C BECs, Eto et al. found that the $F_{\text{inter}}^{ij'}$ can be written by [13]

$$F_{\text{inter}}^{ij'} = \frac{\pi\hbar^4g_{12}}{4m_1m_2(g_{11}g_{22} - g_{12}^2)} \frac{1}{R_{ij'}^3} \left( \ln \frac{R_{ij'}}{\xi} - \frac{1}{2} \right),$$

where $2R_{ij'}$ is the distance between the $i$th and $j'$th vortices belonging to different components. In this simulation, $g_{12} > 0$ and hence $F_{\text{inter}}^{ij'} > 0$ for $R_{ij'} > \xi\sqrt{\xi}$, which means that $F_{\text{inter}}$ is repulsive independently of the sign of vortices. When $R \gg \xi$, $F_{\text{intra}}^{ij}$ is superior to $F_{\text{inter}}^{ij'}$, while when $R \ll \xi$, $F_{\text{inter}}^{ij'}$ is superior to $F_{\text{intra}}^{ij}$. When $R_{ij'}$ is close to $\xi$, these two forces are comparable.
In 1C BEC, the total force between the $i$th vortex and all other vortices is written by

$$ F_{1C}^i = \sum_{j \neq i} F_{ij}^{\text{intra}}. $$  \hfill (8)

On the other hand, in 2C BECs, the total force between the $i$th vortex and all other vortices is written by

$$ F_{2C}^i = \sum_{j \neq i} F_{ij}^{\text{intra}} + \sum_{j' \neq i} F_{ij'}^{\text{inter}}, $$  \hfill (9)

where the first term is same as $F_{1C}^i$, and the second term could make $|F_{1C}^i| < |F_{2C}^i|$. In this condition, the average velocity of a vortex in 2C BECs becomes bigger than in 1C BEC. This makes the decay of $N_v$ in 2C BECs faster than in 1C BEC.

Although two components are equivalent in this study, the values of $d$ of two components differ after $t \simeq 100$ in the harmonic and box traps (Fig. 3 (b), (d)). Figure 4 shows the distribution of vortices after $t = 100$. In Fig. 4 (a), vortices cluster in the area surrounded by the dotted black ellipse in component 1 and in the areas surrounded by dashed black ellipses in component 2. We can see the similar tendency in Fig. 4 (b), (c), (d), which means that vortices in one component separate from those in the other component because of the repulsive force $F_{ij'}^{\text{inter}}$. We expect that this phenomenon can be treated as a kind of the phase separation about the distribution of vortices.

**Figure 3.** Time development of $N_v$ and $d$ in 2C BECs. Decay of $N_v$ of two components in harmonic trap and in box trap are shown in (a) and (c), and development of $d$ in (b) and (d), respectively.

**4. Conclusions and discussions**

We study the decay of $N_v$, the development of $d$ and the distribution of vortices in 1C BEC and 2C BECs. In 1C BEC, we obtain almost same results as the previously studies.

In 2C BECs, we can show the difference from in 1C BEC. In 2C BECs, the decay of $N_v$ is faster than in 1C BEC and vortices in one component are distributed separately from vortices in the other component, which comes from repulsive force $F_{ij'}^{\text{inter}}$ due to the positive $g_{12}$. Thus, $g_{12}$ affects greatly the dynamics of vortices and the decay of $N_v$. Thus, $d$ of one component increases, while the structure of Onsager vortex dose not appear obviously as in 1C BEC in the box trap. In this study, $g_{12}$ is relatively large, namely $g_{12} = 0.9g$, which suppresses the clear formation of Onsager vortices. If $g_{12}$ is smaller, the Onsager vortices should appear in the box trap.

This work is still just preliminary and it is necessary to study the dependence of the dynamics of vortices and the decay of $N_v$ on $g_{12}$. Further studies are also needed in order to investigate the phenomenological rate equation of the decay of $N_v$ in 2C BECs.
Figure 4. Top row shows the distribution of vortices in the component1 and the bottom row shows that in the component2. Vortices and antivortices are denoted by black (dark) points and white (light) points, respectively. White (light) circles represent the boundary of condensates. The distributions in the harmonic trap are shown in (a) \((t = 250)\) and (b) \((t = 350)\), in the box trap in (c) \((t = 150)\) and (d) \((t = 350)\).

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