Fast Commutative Matrix Algorithm

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Abstract

We show that the product of an $n \times 3$ matrix and a $3 \times 3$ matrix over a commutative ring can be computed using $6n + 3$ multiplications. For two $3 \times 3$ matrices this gives us an algorithm using 21 multiplications. This is an improvement with respect to Makarov algorithm using 22 multiplications\cite{10}. We generalize our result for $n \times 3$ and $3 \times 3$ matrices and present an algorithm for computing the product of an $l \times n$ matrix and an $n \times m$ matrix over a commutative ring for odd $n$ using $n(lm + l + m - 1)/2$ multiplications if $m$ is odd and using $(n(lm + l + m - 1) + l - 1)/2$ multiplications if $m$ is even. Waksman algorithm for odd $n$ needs $(n - 1)(lm + l + m - 1)/2 + lm$ multiplications\cite{16}, thus in both cases less multiplications are required by our algorithm.

1 Introduction

In 1969 Strassen showed that the product of two $n \times n$ matrices can be computed using $O(n^{\log_2(7)})$ arithmetic operations \cite{15}. This work opened a new field of research and over the years better upper bounds for the exponent of matrix multiplication were published. In 1990 Coppersmith and Winograd obtained an upper bound of 2.375477 for the exponent \cite{2}. For a long time this was the best result. Since 2010 further improvements were obtained in a series of papers \cite{7,8,14,17,18}. The best result so far was published in 2014 by Le Gall who obtained an upper bound of 2.3728639 for the exponent \cite{8}.

In this paper we first study the product of an $n \times 3$ matrix $A$ and a $3 \times 3$ matrix $B$ over a commutative ring and show that we can compute the product $AB$ using $6n + 3$ multiplications. The basic idea is to improve the computation of the product of a $1 \times 3$ vector $a$ and a $3 \times 3$ matrix $B$ over a commutative ring in the sense that we try to obtain as much as possible multiplications that contain only entries of the matrix $B$ but without using more than 9 multiplications overall. The multiplications which contain only entries of the matrix $B$ only need to be calculated once and can therefore be reused in the matrix multiplication. In the special case $n = 3$ we obtain an algorithm using 21 multiplications which improves the best result so far from Makarov using 22 multiplications \cite{10}. Our next step is to generalize this result to the computation of the product of an $l \times n$ matrix $A$ and an $n \times m$ matrix $B$ over a commutative ring for odd $n$. We show that the product $AB$ can be computed using $n(lm + l + m - 1)/2$ multiplications if $m$ is odd and $(n(lm + l + m - 1) + l - 1)/2$ multiplications if $m$ is even. This improves Waksman’s algorithm which requires $(n - 1)(lm + l + m - 1)/2 + lm$ multiplications for odd $n$ \cite{16}.

All algorithms we present in this paper do not make use of any additional multiplications with constants.

1.1 Related Work

In this section we present some related work. We start with presenting some results about multiplication of two square matrices. Note that a matrix multiplication algorithm can only be applied...
recursively if commutativity is not used. Since Strassen showed in 1969 that the product of two matrices can be computed using $O(n^{\log_2(7)})$ arithmetic operations \[15\] and since it is shown that for $2 \times 2$ matrices $7$ is the optimal number of multiplications \[5\] \[19\], it is interesting to study $n \times n$ matrices for $n \geq 3$, to obtain an even faster algorithm for matrix multiplication. For $3 \times 3$ matrices $21$ multiplications would be needed to obtain an even faster algorithm than Strassen’s since $\log_3(21) \approx 2.7712 < 2.807 \approx \log_2(7)$. In 1976 Laderman obtained a non-commutative $3 \times 3$ algorithm using only $23$ multiplications \[9\]. It is not known if there exists a non-commutative $3 \times 3$ algorithm that uses $22$ or less multiplications. For $5 \times 5$ matrices the best non-commutative result so far is $99$ multiplications by Sedoglavic \[12\] which is an improvement on Makarov’s algorithm for $5 \times 5$ matrices \[11\].

Hopcroft and Musinski showed in \[6\] that the number of multiplications to compute the product of an $l \times n$ matrix and an $n \times m$ matrix is the same number that is required to compute the product of an $n \times m$ matrix and an $m \times l$ matrix and of an $l \times m$ matrix and an $m \times n$ matrix etc. This means if one computes an algorithm for the product of an $l \times m$ matrix and an $m \times n$ matrix using $x$ multiplications there exists a matrix product algorithm for $lnm \times lnm$ matrices using $x^3$ multiplications overall. This algorithm for square matrices will then have an exponent of $\log_{lnm}(x^3)$.

We present some examples of non-square matrix multiplication algorithms. In \[4\] Hopcroft and Kerr showed that the product of a $p \times 2$ matrix and a $2 \times n$ matrix can be multiplied using $\lceil (3pn + \max\{n, p\})/2 \rceil$ multiplications without using commutativity. In the case $p = 3 = n$ this gives an algorithm using $15$ multiplications. Combined with the results of \[6\] this gives an algorithm for $18 \times 18$ matrices using $15^3 = 3375$ multiplications and an exponent of $\log_{18}(3375) \approx 2.811$. Smirnov obtained an algorithm for the product of a $3 \times 3$ matrix and a $3 \times 6$ matrix using $40$ multiplications\[13\]. By \[6\] this gives an algorithm for $54 \times 54$ matrices using $40^3 = 64000$ multiplications and an exponent of $\log_{54}(64000) \approx 2.7743$.

Cariow et al. developed a high-speed parallel $3 \times 3$ matrix multiplier structure based on the commutative $3 \times 3$ matrix algorithm using $22$ multiplications obtained by Makarov \[1\] \[10\]. We suppose that the structure could be improved by using our commutative $3 \times 3$ matrix algorithm using $21$ multiplications.

In \[3\] Drevet et al. optimized the number of required multiplications of small matrices up to $30 \times 30$ matrices. They considered non-commutative and commutative algorithms. Combined with our results for commutative rings we suppose that some results could be improved.

### 2 Matrix Product over a Commutative Ring

Let $R$ denote a commutative ring throughout this Section.

#### 2.1 Product of $n \times 3$ and $3 \times 3$ Matrices

Consider the vector-matrix product of an $1 \times 3$ vector $a$ and a $3 \times 3$ matrix $B$ over a commutative ring.

\[
a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}
\] (1)

In the usual way the vector-matrix product of $a$ and $B$ would be computed as:

\[
aB = [a_1b_{11} + a_2b_{21} + a_3b_{31}, \quad a_1b_{12} + a_2b_{22} + a_3b_{32}, \quad a_1b_{13} + a_2b_{23} + a_3b_{33}]\]

But it can also be computed by first computing these $9$ products:
Algorithm 1. Input: Vector \(a\) and Matrix \(B\) as in (1).

Let

\[
p_1 := (a_2 + b_{12})(a_1 + b_{21})
\]
\[
p_2 := (a_3 + b_{13})(a_1 + b_{31})
\]
\[
p_3 := (a_3 + b_{23})(a_2 + b_{32})
\]
\[
p_4 := a_1(b_{11} - b_{12} - b_{13} - a_2 - a_3)
\]
\[
p_5 := a_2(b_{22} - b_{21} - b_{23} - a_1 - a_3)
\]
\[
p_6 := a_3(b_{33} - b_{31} - b_{32} - a_1 - a_2)
\]
\[
p_7 := b_{12}b_{21}
\]
\[
p_8 := b_{13}b_{31}
\]
\[
p_9 := b_{23}b_{32}
\]

Output:

\[
aB = [p_4 + p_1 + p_2 - p_7 - p_8 \quad p_5 + p_1 + p_3 - p_7 - p_9 \quad p_6 + p_2 + p_3 - p_8 - p_9]
\]

Theorem 1. Let \(R\) be a commutative ring, let \(n \geq 1\), let \(A\) be an \(n \times 3\) matrix over \(R\) and let \(B\) be a \(3 \times 3\) matrix over \(R\). Then the product \(AB\) can be computed using \(6n + 3\) multiplications.

Proof. Consider Algorithm 1. The products \(p_7\), \(p_8\) and \(p_9\) contain only entries of the matrix \(B\). One can observe that for all \(n \geq 1\) the multiplications \(p_7\), \(p_8\) and \(p_9\) can be reused for the product \(AB\) and therefore \(3(n - 1)\) multiplications are saved.

We give an example. In the case \(n = 3\) we obtain an algorithm with 21 multiplications for the matrix-matrix product. This algorithm needs one multiplication less than Makarov’s [10].

Corollary 1. Let \(R\) be a commutative ring and let \(A\) and \(B\) be \(3 \times 3\) matrices over \(R\) as shown below. Then the product \(AB\) can be computed using 21 multiplications as follows:

\[
A = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix} \quad B = \begin{bmatrix}
    b_{11} & b_{12} & b_{13} \\
    b_{21} & b_{22} & b_{23} \\
    b_{31} & b_{32} & b_{33}
\end{bmatrix}
\]

\[
p_1 := (a_{12} + b_{12})(a_{11} + b_{21})
\]
\[
p_2 := (a_{13} + b_{13})(a_{11} + b_{31})
\]
\[
p_3 := (a_{13} + b_{23})(a_{12} + b_{32})
\]
\[
p_4 := a_{11}(b_{11} - b_{12} - b_{13} - a_{12} - a_{13})
\]
\[
p_5 := a_{12}(b_{22} - b_{21} - b_{23} - a_{11} - a_{13})
\]
\[
p_6 := a_{13}(b_{33} - b_{31} - b_{32} - a_{11} - a_{12})
\]
\[
p_7 := (a_{22} + b_{12})(a_{21} + b_{21})
\]
\[
p_8 := (a_{23} + b_{13})(a_{21} + b_{31})
\]
\[
p_9 := (a_{23} + b_{23})(a_{22} + b_{32})
\]
\[
p_{10} := a_{21}(b_{11} - b_{12} - b_{13} - a_{22} - a_{23})
\]
\[
p_{11} := a_{22}(b_{22} - b_{21} - b_{23} - a_{21} - a_{23})
\]
\[
p_{12} := a_{23}(b_{33} - b_{31} - b_{32} - a_{21} - a_{22})
\]
\begin{align*}
p_{13} & := (a_{32} + b_{12})(a_{41} + b_{21}) \\
p_{14} & := (a_{33} + b_{13})(a_{41} + b_{31}) \\
p_{15} & := (a_{33} + b_{23})(a_{42} + b_{32}) \\
p_{16} & := a_{31}(b_{11} - b_{12} - b_{13} - a_{32} - a_{33}) \\
p_{17} & := a_{32}(b_{22} - b_{21} - b_{23} - a_{31} - a_{33}) \\
p_{18} & := a_{33}(b_{33} - b_{31} - b_{32} - a_{31} - a_{32}) \\
p_{19} & := b_{12}b_{21} \\
p_{20} & := b_{13}b_{31} \\
p_{21} & := b_{23}b_{32}
\end{align*}

Hence,

\[
AB = \begin{bmatrix}
p_{4} + p_{1} + p_{2} - p_{19} - p_{20} & p_{5} + p_{1} + p_{3} - p_{19} - p_{21} & p_{6} + p_{2} + p_{3} - p_{20} - p_{21} \\
p_{10} + p_{7} + p_{8} - p_{19} - p_{20} & p_{11} + p_{7} + p_{9} - p_{19} - p_{21} & p_{12} + p_{8} + p_{9} - p_{20} - p_{21} \\
p_{16} + p_{13} + p_{14} - p_{19} - p_{20} & p_{17} + p_{13} + p_{15} - p_{19} - p_{21} & p_{18} + p_{14} + p_{15} - p_{20} - p_{21}
\end{bmatrix}
\]

### 2.2 General Matrix Product

Algorithm 1 from Section 2.1 is the basic idea of a general algorithm for the matrix-matrix product of \(l \times n\) and \(n \times m\) matrices over a commutative ring for odd \(n\). This general algorithm makes use of Waksman algorithm 16 for even \(n\). The algorithm we present below is split into two cases. In Case 1 \(m\) is odd and in Case 2 \(m\) is even. This leads us to the following:

**Theorem 2.** Let \(R\) be a commutative ring, let \(n \geq 3\) be odd, \(l \geq 1\), \(m \geq 3\) and let \(A \in R^{l \times n}\), \(B \in R^{n \times m}\) be matrices. Then the following holds:

- If \(m\) is odd the product \(AB\) can be computed using \(n(lm + l + m - 1)/2\) multiplications.
- If \(m\) is even the product \(AB\) can be computed using \((n(lm + l + m - 1) + l - 1)/2\) multiplications.

**Proof.** Let \(A\) and \(B\) be matrices as in the Theorem. Now split \(A\) and \(B\) in submatrices in the following way:

\[
A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}, \text{ with } A_1 \in R^{l \times 3} \text{ and } A_2 \in R^{l \times n - 3},
\]

\[
B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \text{ with } B_1 \in R^{3 \times m} \text{ and } B_2 \in R^{n - 3 \times m}.
\]

Then \(AB = A_1B_1 + A_2B_2\). With Waksman algorithm 16 mentioned before \(A_2B_2\) can be computed using \((n - 3)(lm + l + m - 1)/2\) multiplications. Let \(a_{ij}\) denote the entries of \(A_1\) and let \(b_{ij}\) denote the entries of \(B_1\) and let \(c_{ij}\) denote the entries of \(A_1B_1\). The matrix \(A_1B_1\) can be computed as follows.

**Case 1:** \(m\) is odd.

For \(i = 1, \ldots, l\) let

\[
c_{i1} = (a_{i1} + b_{21})(a_{i2} + b_{12}) + (a_{i1} + b_{31})(a_{i3} + b_{13}) + a_{i1}(b_{11} - b_{12} - b_{13} - a_{i2} - a_{i3}) - b_{12}b_{21} - b_{13}b_{31}
\]

\[
c_{i2} = (a_{i1} + b_{21})(a_{i2} + b_{12}) + (a_{i2} + b_{32})(a_{i3} + b_{23}) + a_{i2}(b_{22} - b_{21} - b_{23} - a_{i1} - a_{i3}) - b_{12}b_{21} - b_{23}b_{32}
\]

\[
c_{i3} = (a_{i1} + b_{31})(a_{i3} + b_{13}) + (a_{i2} + b_{32})(a_{i3} + b_{23}) + a_{i3}(b_{33} - b_{31} - b_{32} - a_{i1} - a_{i2}) - b_{13}b_{31} - b_{23}b_{32}
\]

and for \(i = 1, \ldots, l\) and \(j = 4, 6, 8, \ldots, m - 1\) let
\[ c_{ij} = (a_{i1} + b_{21})(a_{i2} + b_{12}) + (a_{i1} + b_{31})(a_{i3} + b_{13}) + (a_{i1} + b_{21} - b_{2j})(-a_{i2} - b_{12} + b_{1j} - b_{1(j+1)}) \\
+ (a_{i1} + b_{31} - b_{3j})(-a_{i3} - b_{13} + b_{1(j+1)}) - b_{12}b_{21} - b_{13}b_{31} - (b_{21} - b_{2j})(-b_{12} + b_{1j} - b_{1(j+1)}) \\
- (b_{31} - b_{3j})(-b_{13} + b_{1(j+1)}) \\
\]

\[ c_{i(j+1)} = (a_{i1} + b_{31})(a_{i3} + b_{13}) + (a_{i2} + b_{32})(a_{i3} + b_{23}) + (a_{i1} + b_{31} - b_{3j})(-a_{i3} - b_{13} + b_{1(j+1)}) \\
+ (a_{i2} + b_{32} + b_{3j} - b_{3(j+1)})(-a_{i3} - b_{23} + b_{2(j+1)}) - b_{13}b_{31} - b_{23}b_{32} - (b_{31} - b_{3j})(-b_{13} + b_{1(j+1)}) \\
- (b_{32} + b_{3j} - b_{3(j+1)})(-b_{23} + b_{2(j+1)}) \\
\]

It can easily be seen that \(6l + 3 + 3l(m - 3)/2 + 3(m - 3)/2 = 3(lm + l + m - 1)/2 \) multiplications are required to compute \(A_1B_1\).

Thus, \(AB\) can be computed using \(3(lm + l + m - 1)/2 + (n - 3)(lm + l + m - 1)/2 = n(lm + l + m - 1)/2 \) multiplications.

**Case 2:** \(m\) is even.

For \(i = 1, \ldots, l\) let

\[ c_{i1} = (a_{i1} + b_{21})(a_{i2} + b_{12}) + (a_{i1} + b_{31})(a_{i3} + b_{13}) + a_{i1}(b_{11} - b_{12} - b_{13} - a_{i2} - a_{i3}) - b_{12}b_{21} - b_{13}b_{31} \]

\[ c_{i2} = (a_{i1} + b_{21})(a_{i2} + b_{12}) + (a_{i2} + b_{32})(a_{i3} + b_{23}) + a_{i2}(b_{22} - b_{21} - b_{23} - a_{i1} - a_{i3}) - b_{12}b_{21} - b_{23}b_{32} \]

\[ c_{i3} = (a_{i1} + b_{31})(a_{i3} + b_{13}) + (a_{i2} + b_{32})(a_{i3} + b_{23}) + a_{i3}(b_{33} - b_{31} - b_{32} - a_{i1} - a_{i2}) - b_{13}b_{31} - b_{23}b_{32} \]

\[ c_{i4} = (a_{i1} + b_{21})(a_{i2} + b_{12}) + (a_{i1} + b_{21} - b_{24})(-a_{i2} - b_{12} + b_{14}) + a_{i3}b_{14} - b_{12}b_{21} - (b_{21} - b_{24})(-b_{12} + b_{14}) \]

and for \(i = 1, \ldots, l\) and \(j = 5, 7, 9, \ldots, m - 1\) let

\[ c_{ij} = (a_{i1} + b_{21})(a_{i2} + b_{12}) + (a_{i1} + b_{31})(a_{i3} + b_{13}) + (a_{i1} + b_{21} - b_{2j})(-a_{i2} - b_{12} + b_{1j} - b_{1(j+1)}) \\
+ (a_{i1} + b_{31} - b_{3j})(-a_{i3} - b_{13} + b_{1(j+1)}) - b_{12}b_{21} - b_{13}b_{31} - (b_{21} - b_{2j})(-b_{12} + b_{1j} - b_{1(j+1)}) \\
- (b_{31} - b_{3j})(-b_{13} + b_{1(j+1)}) \]

\[ c_{i(j+1)} = (a_{i1} + b_{31})(a_{i3} + b_{13}) + (a_{i2} + b_{32})(a_{i3} + b_{23}) + (a_{i1} + b_{31} - b_{3j})(-a_{i3} - b_{13} + b_{1(j+1)}) \\
+ (a_{i2} + b_{32} + b_{3j} - b_{3(j+1)})(-a_{i3} - b_{23} + b_{2(j+1)}) - b_{13}b_{31} - b_{23}b_{32} - (b_{31} - b_{3j})(-b_{13} + b_{1(j+1)}) \\
- (b_{32} + b_{3j} - b_{3(j+1)})(-b_{23} + b_{2(j+1)}) \]

One can easily verify that in this case \(8l + 4 + 3l(m - 4)/2 + 3(m - 4)/2 = 2(l - 1) + 3(\text{lm} + m)/2 \) multiplications are required to compute \(A_1B_1\).

Thus, \(AB\) can be computed using \(2(l - 1) + 3(\text{lm} + m)/2 + (n - 3)(\text{lm} + l + m - 1)/2 = n(lm + l + m - 1)/2 \) multiplications.

In both cases less multiplications are required to compute \(AB\) than Waksman algorithm \[10\] for odd \(n\) requires.

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