Relaxation of the entanglement spectrum in quench dynamics of topological systems

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Abstract. In this paper, we investigate how the entanglement spectrum relaxes to its steady-state values in one-dimensional quadratic systems after a quantum quench. In particular, we apply saddle-point expansion to dimerized chains and 1D $p$-wave superconductors. We found the entanglement spectrum to always exhibit a power-law relaxation superimposed with oscillations at certain characteristic angular frequencies. For dimerized chains, we found the exponent $\nu$ of the power-law decay to always be $3/2$. For 1D $p$-wave superconductors, however, we found that, depending on the initial and final Hamiltonian, the exponent $\nu$ can take its value from a limited list of values, the smallest possible of which is $\nu = 1/2$, which leads to a very slow convergence to its steady-state value.

Keywords: entanglement in topological phase, quantum quenches, topological insulators
1. Introduction

The experimental discovery of new topological phases such as topological insulators [1–3],
topological superconductors [4–6], and Weyl semimetals [7–12] symbolizes a new era in
physics. In contrast with conventional ordered phases, which are characterized by broken
symmetries, topological phases can be identified by topological invariants such as the
Chern number or the Berry phase [13, 14]. While topological phases cannot be character-
ized by a local order parameter, it is possible to detect topological phases by studying the
entanglement of an appropriate bipartition in the system [15–19]. One useful entanglement
measure is von Neumann entropy. If we consider a pure state \( |\Psi_{A\cup B}\rangle \) of a bipartite system
\( A \cup B \), we can obtain the reduced density matrix \( \rho_A \) of subsystem \( A \) by tracing out the
environment \( B \), i.e. \( \rho_A = \text{Tr}_B |\Psi_{A\cup B}\rangle\langle\Psi_{A\cup B}| \) and define the von Neumann entropy \( S_A \) as
\( S_A = -\text{Tr} \rho_A \log \rho_A \) [20, 21]. As in the literature, in the rest of this paper, we refer to \( S_A \) as
the entanglement entropy. Furthermore, we can define an entanglement Hamiltonian \( \mathcal{H}_E \)
via the relation \( \rho_A = e^{-\mathcal{H}_E}/\text{Tr}_B e^{-\mathcal{H}_E} \) [22]. For topological systems, if we cut out subsystem
\( A \), edge states will appear. Furthermore, the signatures of these edge states will appear
in the entanglement spectrum, which is defined as the eigenvalues of \( \rho_A \) (or, equivalently,
the eigenvalues of the entanglement Hamiltonian \( \mathcal{H}_E \)) [23, 24]. This phenomenon can be
viewed as the bulk-edge correspondence of the entanglement measure [25].

The thermodynamics of time-evolved many-body systems has also attracted much
attention due to a number of ground-breaking experiments with ultracold atoms
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Table 1. Definitions of the real and momentum space spinors \( c_n^\dagger \) and \( c_k^\dagger \), the number of sites \( N_n \), the first Brillouin zone \( BZ \) and its volume \( V_{BZ} \), and the pseudo magnetic field \( R(k) \) in equation (3) for dimerized chains and 1D \( p \)-wave superconductors. To simplify the notation, we use \( \delta_\pm = 1 \pm \delta \).

|              | Dimerized chains | 1D \( p \)-wave superconductors |
|--------------|-----------------|-------------------------------|
| \( c_n^\dagger \) | \((c_n^\dagger(n), c_n^\dagger(n))\) | \((c'(n), c(n))\) |
| \( c_k^\dagger \) | \((c_k^\dagger(k), c_k^\dagger(k))\) | \((c'(k), c(-k))\) |
| \( N_n \) | \(2N\) | \(N\) |
| \( BZ \) | \((-\pi/2, \pi/2]\) | \((-\pi, \pi]\) |
| \( V_{BZ} \) | \(\pi\) | \(2\pi\) |
| \( R(k) \) | \((\delta_+ + \delta_- \cos 2k, \delta_+ \sin 2k, 0)\) | \((0, -\Delta \sin k, \cos k + \mu/2]\) |

[26–28]. It was first conjectured by Rigol et al [29, 30] that, with respect to integrable models, the asymptotic steady state after a quantum quench can be described by the generalized Gibbs ensemble (GGE) [31]. To date, GGE remains a matter of debate. For general Gaussian initial states, Cazalilla, Iucci, and Chung [32, 33] proved that, through its relation to entanglement, GGE can serve as a general ensemble for integrable models after a quantum quench (for a review see [34]), while for nongaussian initial states, it has been pointed out that the resulting steady-state values of local observables can not be obtained from the GGE [35, 36].

This same concept can be applied to entanglement entropy and entanglement spectra [37–46]. The entanglement spectra of integrable models after a sudden quench will reach certain steady-state values at some infinite time [47, 48]. In this work, we consider dimerized chains and 1D \( p \)-wave superconductors [49]. In this work, we consider dimerized chains and 1D \( p \)-wave superconductors [49]. In this work, we consider dimerized chains and 1D \( p \)-wave superconductors [49]. In this work, we consider dimerized chains and 1D \( p \)-wave superconductors [49]. In this work, we consider dimerized chains and 1D \( p \)-wave superconductors [49].

In our previous studies [47, 48], we showed that the entanglement spectra of such systems from \( R(k) \) encircles the origin as \( k \) and runs through the whole Brillouin zone, the Berry phase of the system is \( \pi \), which indicates that the system is in a topological phase. In contrast, if \( R(k) \) does not encircle the origin, the Berry phase of the system is zero and the system is in a trivial phase. In this work, we are interested in the relaxation of the entanglement spectrum of topological systems after a sudden quench, i.e. \( R \) is suddenly tuned to \( R' \) at a certain time. In this paper, we consider a system with an infinitely long chain with periodic boundary conditions. Subsystem \( A \) is that part of the chain up to a cut in the middle at a length \( L \), and the rest of the chain is environment \( B \). The two models studied are both quadratic. We know that for quadratic models, we can calculate the reduced density matrix \( \rho_A \) from the block correlation function matrix. Specifically, \( \rho_A \) has the form \( \rho_A = \bigotimes_m \left[ \begin{array}{cc} \lambda_m & 0 \\ 0 & 1 - \lambda_m \end{array} \right] \), where the \( \lambda_m \) terms are eigenvalues of the correlation function matrix \( G_{n,m}(t) = \text{Tr} \rho_0(t) c_n \sigma \sigma_m = \rho_0(t) c_n \sigma \sigma_m \) with \( c_n \) defined in table 1 and \( n, m \) being the subsystem sites \( A \) and \( \rho_0(t) \) with the time evolution of the initial density matrix [22, 51, 52]. Here, we refer to \( \lambda_m \) as the one-particle entanglement spectrum (OPES). In our previous studies [47, 48], we showed that the entanglement spectra of such

\[^{4}\text{In the literature, the OPES may also refer to the one-particle eigenvalues \( \varepsilon_n \) of the entanglement Hamiltonian \( H_E \) [53]. These two definitions are equivalent, having a one-to-one correspondence: \( \varepsilon_n = \log \frac{1+\lambda_n}{1-\lambda_n} \).}\]
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systems will reach steady-state values. Furthermore, we can determine the steady-state values by an effective Hamiltonian $S_{\text{eff}}(k) \cdot \sigma$, where $S_{\text{eff}} = (\hat{R} \cdot \hat{R}')\hat{R}'$ with the unit vectors $\hat{R}$ and $\hat{R}'$. Consequently, whether the system at some infinite time is topological depends on the pseudomagnetic field $S_{\text{eff}}$. In [47, 48], we pointed out that the only way to obtain a topological phase at some infinite time is a quench within the same topological phase. For $p$-wave superconductors, a further constraint is needed: the gaps of the initial and final Hamiltonians should not be too far apart to allow for the probability that the edge states may revive through the dephasing process.

However, the dynamics of the edge states remain mysterious. We showed in our previous study that the OPES associated with two edge states remains at $1/2$ for a while, then breaks up at a coherent time proportional to the subsystem size, and then relaxes back to $1/2$ at some infinite time. However, for most numerical methods, such as matrix product state (MPS) algorithms, it is very difficult to probe infinite time behavior. As such, gaining a better understanding of how systems relax to their steady states can help theorists to fit their intermediate time-dependent results. In this paper, we use saddle-point expansion to analyze the power-law decay of the OPES of dimerized chains and 1D $p$-wave superconductors. We find the characteristic decay exponents of power-law decay to be model dependent. For dimerized chains, the exponents are always $3/2$, independent of the quench process. For 1D $p$-wave superconductors, however, the exponents can take different values depending on the quench parameters. The lowest such value is $1/2$, which leads to a very slow convergence to the steady-state values.

This paper is organized as follows: in section 2, we introduce dimerized chains and 1D $p$-wave superconductors as our initial models. In section 3, we introduce a method for obtaining the time evolution of entanglement spectra by the use of time-dependent correlation function matrices. In section 4, we present our main results, i.e. the obtained exponents of the power-law relaxation of entanglement spectra. Finally, we draw our conclusions and present the future outlook for our work in section 5. In appendix A, we illustrate in more detail the mathematic tools of our work, i.e. the steepest descent expansion.

2. Dimerized chains and 1D $p$-wave superconductors

In this paper, we consider two quadratic fermion systems-dimerized chains and 1D $p$-wave superconductors. Our main goal is to investigate how the OPES converges to steady-state values after a sudden quench.

For a dimerized chain, the Hamiltonian is as follows:

$$\mathcal{H} = - \sum_{n=1,3,5,\ldots}^{2N-1} [(1 - \delta)c_{\beta}^\dagger(n)c_{\alpha}(n) + \text{h.c.} + (1 + \delta)c_{\beta}^\dagger(n)c_{\alpha}(n + 2) + \text{h.c.}], \quad (1)$$

where $N$ is the number of unit cells containing two sites and $\delta \in [-1, 1]$ is the dimerized strength. $c_{\alpha}^\dagger(n) \equiv c_{\alpha}^\dagger(n)$ and $c_{\beta}^\dagger(n) \equiv c_{\beta}^\dagger(n + 1)$ are the fermionic creation operators at odd $n$ and even $n + 1$ sites, respectively.
The Hamiltonian of the 1D $p$-wave superconductor is given by the following:

$$
H = \sum_{n=1}^{N} \left[ -c^\dagger(n)c(n+1) + \text{h.c.} + \Delta c(n)c(n+1) + \text{h.c.} - \mu (c^\dagger(n)c(n) + 1/2) \right],
$$

(2)

where $N$ is the number of sites, $\mu$ is the chemical potential, $\Delta$ is the paring strength (or superconducting gap), and $c^\dagger(n)$ is the creation operator at site $n$.

By using the spinor representation $c^\dagger_n$ in table 1, imposing the periodic boundary condition $c^\dagger_n = c^\dagger_{n+N}$, and taking the Fourier transformation $c^\dagger_k = \frac{1}{\sqrt{N}} \sum_{k \in \text{BZ}} e^{-i k n} c^\dagger_n$, we can rewrite both Hamiltonians as follows:

$$
\mathcal{H}(R) = -\sum_{k \in \text{BZ}} c^\dagger_k [R(k) \cdot \sigma] c_k,
$$

(3)

where $R(k) \equiv (R_x(k), R_y(k), R_z(k))$ is the real pseudo magnetic field and $\sigma \equiv (\sigma_x, \sigma_y, \sigma_z)$ is the vector of the Pauli matrices. The specific range of the first Brillouin zone and the form of $R(k)$ are given in table 1.

For both models, the pseudo magnetic field $R$ is two-dimensional, which respects chiral symmetry. In this case, to characterize the topological properties of the system, we can use the closed loop $\ell_R$ formed by $R(k)$ as $k$ runs through the Brillouin zone [50]. If $\ell_R$ encloses the origin, the Berry phase of the occupied band is $\pm \pi$ and the system is in a topological phase. In contrast, if $\ell_R$ does not enclose the origin, the Berry phase is zero and the system is in a topologically trivial phase. Using this geometric perspective, we find the dimerized chain to be in the topologically trivial phase if $\delta \in [-1, 0)$ and the topological phase if $\delta \in (0, 1]$, as illustrated in figure 1. On the other hand, the $p$-wave superconductor is in a topological phase if $|\mu/2| < 1$ and in a topologically trivial phase if $|\mu/2| > 1$, as shown in figure 2. Furthermore, there are two distinct topological phases, depending on the clockwise or counterclockwise winding of the closed loop $\ell_R$. These topological phases are all known as symmetry-protected topological (SPT) phases. According to the classification of SPT phases [14], the topological phase of dimerized chains and 1D $p$-wave superconductors both belong to the class BDI.

Figure 1. Phase diagrams of equation (1) and the closed loops $\ell_R$ (circles). The red dot indicates the origin. (a) $\ell_R$ will not enclose the origin in the topological trivial phase. (b) $\ell_R$ will enclose the origin in the topological phase.
3. Time evolution of entanglement spectra

In the following, we illustrate how to obtain the OPES $\lambda_m(t)$ for one-dimensional quadratic fermionic systems after a sudden quench. Consider a pure state $|\Psi_{AB}\rangle$ in an infinite bipartite system $AB$ with a finite subsystem $A$ and an infinite environment $B$. According to [22, 51, 54], the OPES $\lambda_m$ between subsystem $A$ and environment $B$ are the eigenvalues of the correlation matrix $G$ with the matrix element $G_{n,m} \equiv \langle \Psi_{AB} | c_n c_m^\dagger | \Psi_{AB} \rangle$. Here, $n, m$ are unit cells or site indices for subsystem $A$ and the spinor $c_n^\dagger$ is defined in table 1.

Next, assume that when $t < 0$ the system is in the ground state $|\Psi_G\rangle$ of $H \equiv \mathcal{H}(\mathbf{R})$. At $t = 0$, the system is suddenly quenched to a new Hamiltonian $H' \equiv \mathcal{H}(\mathbf{R}')$. Starting from equation (3), we can show in a straightforward way that we can easily diagonalize the Hamiltonian $H$ using the eigenbasis $f_k = U(\mathbf{R}) c_k$ with the following unitary transformation: $U(\mathbf{R}) = [(R_z - R)\sigma_z + R_x\sigma_x + R_y\sigma_y]/[2R(R - R_z)]^{1/2}$. We then give the dispersion by $\pm R(k) = \pm (R_z^2 + R_y^2 + R_z^2)^{1/2}$. Similarly, for $H'$, we have the eigenbasis $f'_k = U(\mathbf{R}') c_k$.

Since at any $t > 0$, the system is at a pure state $|\Psi(t)\rangle = e^{-iH't}|\Psi_G\rangle$, we can obtain the OPES $\lambda_m(t)$ at time $t$ by diagonalizing the time-dependent correlation matrix $G(t)$ with the matrix element $G_{n,m}(t) \equiv \langle \Psi(t) | c_n c_m^\dagger | \Psi(t) \rangle$, where $G(t)|\lambda_m(t)\rangle = \lambda_m(t)|\lambda_m(t)\rangle$. Due to the translational invariance, we can apply the Fourier transformation to obtain the following:

$$G_{n,m}(t) = \frac{1}{N} \sum_{k \in \mathbb{BZ}} e^{ik(n-m)} G_k(t),$$

Figure 2. Phase diagrams of equation (2). The red lines are the phase boundaries. Each inset shows the closed loop $\ell_R$ with the parameters $(\mu/2, \Delta)$ indicated at the cross point with the same color and the red dot is the origin. Only in topological phases I and II, the closed loop $\ell_R$ encloses the origin (counterclockwise for phase I and clockwise for phase II).
where
\[ G_k(t) \equiv \langle \Psi_G \rangle U^\dagger_k e^{-iH't} U_k e^{iH't} |\Psi_G\rangle. \]  

Together with
\[ e^{iH't} U_k e^{-iH't} = U^\dagger_k e^{iH't} f_k \tilde{f}^\dagger_k e^{-iH't} = e^{-i\epsilon't} f_k \tilde{f}^\dagger_k, \]
where \( \epsilon' = R' \) for dimerized chains and \( \epsilon' = 2R' \) for 1D \( p \)-wave superconductors, and
\[ \langle \Psi_G | f_k \tilde{f}^\dagger_k | \Psi_G \rangle = \langle \Psi_G | (U(R') U^\dagger(R) f_k)(U(R') U^\dagger(R) \tilde{f}^\dagger_k) | \Psi_G \rangle, \]
we can show in a straightforward way that:
\[ G_k(t) = \frac{1}{2} - \frac{1}{2} \frac{S_{\text{eff}}(k) + \Delta S(k, t)}{2} \cdot \sigma, \]
where
\[ S_{\text{eff}}(k) \equiv \langle \hat{R} \cdot \hat{R}' \rangle \hat{R}'. \]
and
\[ \Delta S(k, t) \equiv \cos(2\epsilon't) \left[ \hat{R} - \langle \hat{R} \rangle \right] \hat{R}' + \sin(2\epsilon't) \langle \hat{R} \times \hat{R}' \rangle. \]

At the thermodynamic limit \( \frac{1}{N} \sum_{k \in BZ} \rightarrow \frac{1}{V_{BZ}} \int_{BZ} dk \), equation (4) becomes
\[ G_{n,m}(t) = G_{n,m}^\text{eff} + \Delta G_{n,m}(t), \]
with
\[ G_{n,m}^\text{eff} \equiv \delta_{n,m} \frac{1}{2} - \frac{1}{2V_{BZ}} \int_{BZ} e^{ik(n-m)} S_{\text{eff}}(k) \cdot \sigma dk, \]
and
\[ \Delta G_{n,m}(t) \equiv -\frac{1}{2V_{BZ}} \int_{BZ} e^{ik(n-m)} \Delta S(k, t) \cdot \sigma dk, \]
where \( V_{BZ} \) is the volume of the first Brillouin zone defined in table 1. According to the Riemann–Lebesgue lemma, we expect \( \Delta G_{n,m}(t) \rightarrow 0 \) as \( t \rightarrow \infty \), provided that the integrands are smooth functions of \( k \). Hence, we can obtain the steady-state values of OPES \( \lambda_m(\infty) \) by diagonalizing \( G_{n,m}^\text{eff} \).

4. Power-law relaxation of entanglement spectra

In our previous works [47, 48], we discussed the properties of the effective time-independent pseudo magnetic field \( S_{\text{eff}}(k) \) and how it affects the topology of the steady state after a quench. In this work, we are interested in how the OPES \( \lambda_m(t) \) approaches its steady-state value \( \lambda_m(\infty) \). To achieve this goal, we first rewrite \( \Delta G_{n,m} \) as \( \Delta G_{n,m}(t) = \mathcal{I}(t) \cdot \sigma \) with \( \mathcal{I} = (\mathcal{I}^x(t), \mathcal{I}^y(t), \mathcal{I}^z(t)) \). Furthermore, we can write \( \mathcal{I}^x(t), \mathcal{I}^y(t), \), and \( \mathcal{I}^z(t) \) in the form of \( \mathcal{I}_R(t) \) in equation (A.1). Using the steepest descent expansion, in appendix A,

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we show that $\mathcal{I}_R(t)$ behaves asymptotically as a power-law decay superimposed with an oscillating function associated with some characteristic angular frequencies. Consequently, $\Delta G_{n,m}$ behaves similarly. Due to its power-law decay, we can treat $\Delta G$ as a perturbation in a large time region, resulting in $\lambda_m(t) \approx \lambda_m(\infty) + \langle \lambda_m(\infty) | \Delta G(t) | \lambda_m(\infty) \rangle$. Because $|\lambda_m(\infty)\rangle$ is time independent, asymptotically, we have:

$$\lambda_m(t) \sim \lambda_m(\infty) + \frac{Q_m(t)}{t^\nu},$$

(14)

where $\nu$ is the smallest power-law decay exponent of all the $\mathcal{I}_R(t)$ integrals and $Q_m(t)$ is an oscillating function that contains all the characteristic angular frequencies associated with the exponent $\nu$. Later in this section, we present details for obtaining the power-law decay exponent $\nu$ and characteristic angular frequencies.

In this section we present our results regarding the asymptotic behavior of the OPES $\lambda_m(t)$ (equation (14)) for dimerized chains and 1D $p$-wave superconductors by performing a steepest descent expansion to equation (13). To have non-trivial dynamics, we assume that the initial and final Hamiltonians do not commute with each other and that the final Hamiltonian does not have a constant dispersion. Otherwise, the time-evolving state $|\Psi(t)\rangle$ remains the same up to a global phase difference. To confirm our theoretical results, we can numerically determine the time evolution of the OPES $\lambda(t)$ closest to $1/2$. Due to the particle-hole symmetry, $\lambda(t)$ and its particle-hole pair $1 - \lambda(t)$ behave in the same way. Hence, we only show $\lambda(t)$ values larger than $1/2$.

4.1. Dimerized chains

In this section we consider a sudden quench of a dimerized chain from $H = \mathcal{H}(\delta)$ to $H' = \mathcal{H}(\delta')$. To study how the OPES $\lambda_m(t)$ approaches its steady-state values, we first rewrite equation (13) as $\Delta G_{n,m}(t) = \mathcal{I}(t) \cdot \sigma$ and find the following:

$$\mathcal{I}^x(t) = \int_0^{\pi/2} \cos[k(n-m)] \left\{ \hat{R}_x(k) - \left[ \hat{R}(k) \cdot \hat{R}'(k) \right] \hat{R}'_x(k) \right\} \cos[2R'(k)t]dk,$$

(15)

$$\mathcal{I}^y(t) = i \int_0^{\pi/2} \sin[k(n-m)] \left\{ \hat{R}_y(k) - \left[ \hat{R}(k) \cdot \hat{R}'(k) \right] \hat{R}'_y(k) \right\} \cos[2R'(k)t]dk,$$

(16)

$$\mathcal{I}^z(t) = i \int_0^{\pi/2} \sin[k(n-m)] \left[ \hat{R}_z(k) \hat{R}'_y(k) - \hat{R}_y(k) \hat{R}'_z(k) \right] \sin[2R'(k)t]dk,$$

(17)

where $R = |R|$, $\hat{R} \equiv \hat{R}/R$, and $\hat{R}_{x,y,z} \equiv R_{x,y,z}/R$. During the derivation, we also use the properties $R_x(k) = R_x(-k)$, $R_y(k) = -R_y(-k)$ and $R_z(k) = 0$. We can obtain the asymptotic behavior of $\mathcal{I}(t)$ as $t \to \infty$ by applying the steepest descent expansion. To proceed, we rewrite $\mathcal{I}^x(t)$ as follows:

$$\mathcal{I}^x(t) = \text{Re} \left[ \int_{C_R} f(k) e^{i2R'(k)t} dk \right],$$

(18)

where $f(k) = \cos[k(n-m)](\hat{R} \cdot \hat{R}')\hat{R}'_x$ and $C_R = [0, \pi/2]$. For $\mathcal{I}^y(t)$ and $\mathcal{I}^z(t)$, we proceed in a similar fashion. Using the steepest decent method, as outlined in appendix.
A, we find there to be two order-2 saddle points, therefore \( n_1 = n_2 = 2 \), located at \( k_1 = 0 \) and \( k_2 = \pi/2 \), respectively. We can choose the steepest descent path \( C_S \) in the way indicated in figure 3. We define the power-law decay exponents \( \nu_j \) as \( \beta_j/n_j \), where \( \beta_j - 1 \) is the leading order of the Taylor expansion of \( f(k) \) around the saddle point (see equation (A.8)). As a result, \( \beta_{x,y,z} = 3 \) and \( \nu_{x,y,z} = 3/2 \). The OPES asymptotically behaves as follows:

\[
\lambda_m(t) \sim \lambda_m(\infty) + \frac{Q_m(t)}{t^{3/2}},
\]

where \( Q_m(t) \) contains two oscillating angular frequencies: \( \omega_1 = |w(k_1)| = 4 \) and \( \omega_2 = |w(k_2)| = 4|\delta'| \) (see equation (A.16)). In figure 3(a), we show our numerical results for the most important OPES \( \lambda(t) \), which is larger and closest to 1/2 for two different \( \delta' \). It is clear that \( at^{-3/2} + \lambda(\infty) \) can fit the envelopes of \( \lambda(t) \) well in both cases. In figure 3(b), we show the Fourier transform of \( \lambda(t) \), in which we can clearly observe two peaks around \( \omega_1 \) and \( \omega_2 \). This confirms that the characteristic frequencies of \( Q_m(t) \) are indeed \( \omega_1 \) and \( \omega_2 \). For quenching within the topological phase, however, the subsystem ultimately has a topological steady state, i.e. both the two most important OPES approaches to 1/2 degenerate as time goes to infinity. In this case finding the characteristic frequencies becomes numerically difficult due to the oscillation around the steady-state value 1/2. But as shown in figure 5(a) we can still obtain a nice fit of the envelopes to our predicted power-law decay with the exponent 3/2.

### 4.2. 1D p-wave superconductors

Next, we consider a sudden quench of a 1D p-wave superconductor from \( H = \mathcal{H}(\mu/2, \Delta) \) to \( H' = \mathcal{H}(\mu'/2, \Delta') \). Again, we rewrite equation (13) as \( \Delta G_{n,m}(t) = \mathbf{I}(t) \cdot \sigma \) and find the following:

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During the derivation we also used the properties \( R_z(k) = R_z(-k) \), \( R_y(k) = -R_y(-k) \) and \( R_x(k) = 0 \). Again, we can obtain the asymptotic behavior of \( \mathcal{I}(t) \) as \( t \to \infty \) by applying steepest descent expansion. To proceed, we rewrite \( \mathcal{I}^x(t) \) as follows:

\[
\mathcal{I}^x(t) = \mathrm{Re} \left[ \int_{C_R} f(k) e^{i4R'(k)t} dk \right],
\]

(23)

where \( f(k) = \sin [k(n - m)] \left[ \hat{R}_x - \left( \hat{R} \cdot \hat{R}' \right) \hat{R}'_z \right] \) and \( C_R = [0, \pi] \). For \( \mathcal{I}^y(t) \) and \( \mathcal{I}^z(t) \), we proceed similarly. The steepest descent analysis results identify three possible saddle points \( k_1 = 0 \), \( k_2 = \pi \), and \( k_3 = \cos^{-1} \left( \mu'/2 + \Delta' \right) \) near \( C_R \). However, in contrast to the dimerized chains, the orders of the saddle points depend on the value of \( k_3 \). To proceed, we must classify these situations into classes I, II, and III as cases

![Figure 4](https://example.com/fig4.png)

**Figure 4.** The numerical results for the quench from \( \delta = 0.5 \) to \( \delta' = -0.3 \) (left) and \( \delta' = -0.4 \) (right) with subsystem size \( L = 200 \). (a) The most important OPES \( \lambda(t) \) (red) and the power-law fitting \( at^{-3/2} + \lambda(\infty) \) (blue) to the envelopes of \( \lambda(t) \). (b) The Fourier transformation of (a) for a large time region in which we find two characteristic angular frequencies (open blue circles) at 4 and 4|\( \delta' \)| (1.2 for the left and 1.6 for the right case). Note that the peak at \( \omega = 0 \) is due to the steady OPES \( \lambda(\infty) \)
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Figure 6 pictorially illustrates the three classes in a phase diagram.

4.2.1. Class I: $|(\mu'/2)/(\Delta'2 - 1)| < 1$. In class I, $k_1$, $k_2$, and $k_3$ are all order-2 saddle points $n_j = 2$ and we can choose the steepest descent paths $C_S$, as illustrated in figure 7. The $\beta^{x,y,z}_j$ are as follows:

\[
\beta^{x,y,z}_1 = \begin{cases} 
5 & \text{if } \frac{\Delta'}{\Delta} = \frac{1+\mu'/2}{1+\mu'/2}, \\
3 & \text{otherwise} 
\end{cases},
\]

\[
\beta^{x,y,z}_2 = \begin{cases} 
5 & \text{if } \frac{\Delta'}{\Delta} = \frac{1-\mu'/2}{1-\mu'/2}, \\
3 & \text{otherwise} 
\end{cases},
\]

\[
\beta^{x,z}_3 = \beta^{y,z}_3 = \begin{cases} 
2 & \text{if } \frac{\mu'-\mu}{\Delta(\mu'\Delta' - \mu\Delta')} = 1, \\
1 & \text{otherwise} 
\end{cases},
\]

\[
\beta^{y}_3 = \begin{cases} 
3 & \text{if } \mu' = \mu = 0, \\
2 & \text{if } \frac{\mu'-\mu}{\Delta(\mu'\Delta' - \mu\Delta')} = 1, \\
1 & \text{otherwise} 
\end{cases}.
\]

When $\frac{\mu'-\mu}{\Delta(\mu'\Delta' - \mu\Delta')} \neq 1$, the smallest $\nu$ equals $1/2$, as shown in table 2(a), and the decay behavior is dominated by the saddle point $k_3$. Hence, we conclude that the asymptotic behavior of the OPES $\lambda_m(t)$ is a power-law decay with an exponent of $1/2$ and the following angular frequency:

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\[ \omega_3 = |w(k_3)| = 4|\Delta'| \sqrt{ (\Delta'2 + (\mu'/2)^2 - 1)/(\Delta'2 - 1)}. \]  

(27)
Table 2. Power-law decay exponents $\nu^{x,y,z}_j$ for (a) $\frac{\mu' - \mu}{\Delta(\mu'\Delta - \mu\Delta')} \neq 1$ and (b) $\frac{\mu' - \mu}{\Delta(\mu'\Delta - \mu\Delta')} = 1$. The dominant exponent $\nu^{x,y,z}$ for $I^{x,y,z}(t)$ is the smallest one in each column. Note that the exponent is defined as $\beta^{x,y,z}_j/n_j$, excepting the case of $j=3$ and $\frac{\mu' - \mu}{\Delta(\mu'\Delta - \mu\Delta')} = 1$, where the exponent is defined as $1 + 1/n_3$ because $\beta^{x,y,z}_3 = n_3$.

(a) $\begin{array}{ccc}
\hline
j & x & y & z \\
\hline
1 & 5/2 or 3/2 & 5/2 or 3/2 & 5/2 or 3/2 \\
2 & 5/2 or 3/2 & 5/2 or 3/2 & 5/2 or 3/2 \\
3 & 1/2 & 1/2 or 3/2 & 1/2 \\
\hline
\end{array}$

(b) $\begin{array}{ccc}
\hline
j & x & y & z \\
\hline
1 & 5/2 or 3/2 & 5/2 or 3/2 & 5/2 or 3/2 \\
2 & 5/2 or 3/2 & 5/2 or 3/2 & 5/2 or 3/2 \\
3 & 3/2 & 3/2 & 3/2 \\
\hline
\end{array}$

asymptotically like a power-law decay with the exponent $\nu = 3/2$ and at least one angular frequency $\omega_3$ equation (27).

We can summarize the above two cases as follows:

$$\lambda_m(t) \sim \lambda_m(\infty) + \begin{cases} Q_m(t) \nu^{x,y,z}_j & \text{if } \frac{\mu' - \mu}{\Delta(\mu'\Delta - \mu\Delta')} = 1 \\ Q_m(t) \nu^{x,y,z}_j & \text{otherwise} \end{cases}$$

(28)

where $Q_m(t)$ contains three possible oscillating angular frequencies: $\omega_1 = |w(k_1)| = 4|\mu'/2 + 1|$ (if $\beta^{x,y,z}_1 = 3$), $\omega_2 = |w(k_2)| = 4|\mu'/2 - 1|$ (if $\beta^{x,y,z}_2 = 3$) as well as $\omega_3$ equation (27), and $Q_m(t)$ has only one oscillating angular frequency $\omega_3$. As demonstrated in figure 8(a), our numerical calculations of the most important OPES $\lambda(t)$, which is larger and closest to $1/2$, show good agreement with the power-law fittings to the envelopes of $\lambda(t)$ in both cases of equation (28). In addition, the Fourier transformation of $\lambda(t)$ shown in figure 8(b) also confirms the existence of the predicted characteristic angular frequencies, at which we find peaks (blue circles).

4.2.2. Class II: $|\frac{\mu'/2}{\Delta' - 1}| = 1$. In class II, we have only two saddle points $k_1$ and $k_2$, since $k_3$ coincides with either $k_1$ or $k_2$. As a result, the order of the saddle points is as follows:

$$n_1 = \begin{cases} 4 & \text{if } \frac{\mu'/2}{\Delta' - 1} = 1 \\ 2 & \text{if } \frac{\mu'/2}{\Delta' - 1} = -1 \end{cases}$$

(29)

$$n_2 = \begin{cases} 2 & \text{if } \frac{\mu'/2}{\Delta' - 1} = 1 \\ 4 & \text{if } \frac{\mu'/2}{\Delta' - 1} = -1 \end{cases}$$

(30)

Since $n_1$ and $n_2$ cannot both be 4 or 2 simultaneously, we always have one order-2 saddle point and one order-4 saddle point in class II. We choose the required paths of steepest descent $C_5$ in the way shown in figure 9, whereas $\beta^{x,y,z}_1$ and $\beta^{x,y,z}_2$ are still given
Figure 8. The class I numerical calculations for the cases \( \frac{\mu' - \mu}{\Delta' - \mu \Delta} = 100 \) (left) and \( \frac{\mu' - \mu}{\Delta' - \mu \Delta} = 1 \) (right, together with the condition \( \beta_{1}^{x,y,z} = \beta_{2}^{x,y,z} = 3 \)) with subsystem size \( L = 100 \). (a) The most important OPES \( \lambda(t) \) (red) and the power-law fittings \( a t^{-\nu} + \lambda(\infty) \) to the envelopes of \( \lambda(t) \), where \( \nu = 1/2 \) for the left case and \( \nu = 3/2 \) for the right case. (b) The Fourier transformation of (a). We found the characteristic angular frequencies (open blue circle) \( \omega_{3} \approx 0.345826 \) for the left panel and \( \omega_{1}, \omega_{2}, \omega_{3} \approx 0.117, 1.209, 1.6 \), respectively, for the right panel. The peak at \( \omega = 0 \) is from the steady OPES \( \lambda(\infty) \).

Figure 9. The typical steepest descent paths \( C_{Sj} \) (red curves) starting from the saddle points \( k_{j} \) (blue dots) and the original integration path \( CR \) (blue line) in class II. We can choose \( C_{S} = C_{S1} - C_{S2} \) such that the path \( CR - C_{S} \) forms a closed loop. For this figure, we used \( (\mu'/2, \Delta') = (0.75, 0.5) \).
Relaxation of the entanglement spectrum in quench dynamics of topological systems

by equations (24) and (25). Since the exponent of the power-law decay of the order-4 saddle point is either 5/4 or 3/4, which is always smaller than the exponent of the order-2 saddle point (5/2 or 3/2), we can conclude that:

$$\lambda_m(t) \sim \lambda_m(\infty) + \begin{cases} 
\frac{Q_m(t)}{t^{3/4}} & \text{if } \Delta' = \frac{1+\mu'/2}{1+\mu/2} = 1 \text{ or } \Delta' = \frac{1-\mu'/2}{1-\mu/2} = -1, \\
\frac{Q_m(t)}{t^{5/4}} & \text{otherwise}
\end{cases}$$

(31)

where $Q_m(t)$ contains one oscillating angular frequency $\omega_1 = |w(k_1)| = 4|\mu'/2 + 1|$ when $(\mu'/2)/(\Delta' - 1) = 1$ or $\omega_2 = |w(k_2)| = 4|\mu'/2 - 1|$ when $(\mu'/2)/(\Delta' - 1) = -1$. We can confirm equation (31) by our numerical calculations shown in figure 10. In figure 10(a), the power-law behavior with the predicted exponents (equation (31)) fits relatively well to the envelopes of the most important OPES $\lambda(t)$ for the two cases. We also observe the associated characteristic angular frequencies in the Fourier transformation of $\lambda(t)$ shown in figure 10(b).

4.2.3. Class III: $|\mu'/2 \Delta' - 1| > 1$. For class III, in addition to the order-2 saddle points $k_1$, $k_2$, an extra conjugate complex pair of order-2 saddle point $k_3$ and $k_3^*$ appears where $k_3$ and $k_3^*$ are both solutions for $\cos^{-1}[(\mu'/2)/(\Delta' - 1)]$. These complex saddle points are irrelevant and can be ignored, which means we can choose the required paths of
steepest descent $C_S$ as shown in figure 11. As a result, we must consider only the two real saddle points $k_1$ and $k_2$.

$\beta_1^{x,y,z}$ and $\beta_2^{x,y,z}$ are the same as equations (24) and (25), respectively, with one extra condition: $\beta_1^i = \beta_2^i = \infty$ if $\Delta' = 0$. However, this extra constraint does not influence the dominant exponent $\nu$ since the exponent for the extra condition is maximal. In this case, $n_1 = n_2 = 2$. Therefore, we can conclude that:

$$\lambda_m(t) \sim \lambda_m(\infty) + \frac{Q_m(t)}{t^{3/2}}, \quad (32)$$

where $Q_m(t)$ contains the oscillating frequency $\omega_1$ when $\Delta'/\Delta = (1 - \mu'/2)/(1 - \mu/2)$, $\omega_2$ when $\Delta'/\Delta = (1 + \mu'/2)/(1 + \mu/2)$, or $\omega_1$ and $\omega_2$ when $\Delta'/\Delta \neq (1 \pm \mu'/2)/(1 \pm \mu/2)$. Figure 12 shows the numerical calculations for the $\Delta'/\Delta = (1 \pm \mu'/2)/(1 \pm \mu/2)$ cases. In figure 12(a), the power-law function $a/t^{3/2} + \lambda(\infty)$ fits the envelopes of the most important OPES $\lambda(t)$ very well. Furthermore, we also correctly find the corresponding characteristic angular frequencies by taking the Fourier transformation of figure 12(a), as shown in figure 12(b).

4.3. Discussion

It is instructive to connect the steepest descent method with the quasiparticle picture. According to the quasiparticle picture proposed by Calabrese and Cardy [55], after a sudden quench the initial state $|\Psi_G\rangle$ serves as a source of quasiparticle excitation. This is because, relative to the ground state of the new Hamiltonian $H'$, the energy of the initial state is very high. From the entanglement perspective, the most important process is the paired production of highly-entangled quasiparticles that move in opposition directions. In [55], the author showed that there is a time scale $t^*$ below which the entanglement between subsystem $A$ and the rest of the system grows linearly.

Figure 11. Typical steepest descent paths $C_S$ (red curves) starting from the saddle points $k_j$ (blue dots) and the original integration path $C_R$ (blue line) in class III. We choose the path $C_S = C_{S1} - C_{S2}$, such that it does not pass through $k_3$ and $k_3^*$, but remains a closed path in the complex plane. For this figure, we used $(\mu'/2, \Delta') = (-0.7, 0.8)$. 

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Qualitatively, this time scale is when an entangled pair of fastest quasiparticles created in the middle of the subsystem $A$ reach the left and right boundaries of $A$, respectively. After that, the entanglement continues to grow until it reaches the steady-state value. In this work, however, we are interested in the relaxation of the entanglement spectrum at late times, when the entanglement dynamics are dominated by entangled pairs of slowest quasiparticles. These pairs reach the left and right boundaries of $A$ at a very late time and, hence, they affect the asymptotic behavior of the relaxation dynamics.

On the other hand, a saddle point $k_j$ has the property $\frac{dH'(k)}{dk}|_{k=k_j} = 0$. Since $\omega(k) \propto R'(k)$, the saddle points $k_j$ can be interpreted as the momentum of those quasiparticles with the group velocity $v_g \equiv \frac{dR'(k)}{dk}$ equal to zero, i.e. the slowest quasiparticles. As mentioned above, the relaxation of entanglement at a late time is dominated by the slowest quasiparticles, which explains why the steepest descent method generally works very well, since it corresponds to the contributions of the slowest quasiparticles. However, the steepest descent expansion breaks down when the energy bands of $H'$ become extremely flat. The steepest descent method does not work well in this situation because the routes it chooses are not much better than any other routes due to the flat energy bands. In the extreme case, the energy bands are totally flat, wherein the states only evolve with a phase shift $e^{-i\omega t}$, and the steepest descent method totally fails. From the quasiparticle viewpoint, this happens when there is no clear distinction between fast and slow quasiparticles.

Figure 12. The class III numerical calculations when $\Delta'/\Delta = (1 - \mu'/2)/(1 - \mu/2)$ (left) and $\Delta'/\Delta = (1 + \mu'/2)/(1 + \mu/2)$ (right) with subsystem size $L = 100$. (a) The most important OPES $\lambda(t)$ (red) and power-law fitting $a t^{-3/2} + \lambda(\infty)$ to the envelopes of $\lambda(t)$ (blue). (b) The Fourier transform of (a), by which we find the characteristic angular frequencies (open blue circle) $\omega_1 \approx 0.116815$ for the left panel and $\omega_2 = 1.6$ for the right panel. The peak at $\omega = 0$ is from the steady OPES $\lambda(\infty)$.
We find the power-law decay exponents differ significantly in dimerized chains and 1D \( p \)-wave superconductors. For dimerized chains, \( \nu = 3/2 \), which was also found for the transverse Ising model by Fagotti \textit{et al} [56]. However, for 1D \( p \)-wave superconductors, the decay exponents are not universal. This characteristic essentially distinguishes dimerized chains from 1D \( p \)-wave superconducting chains.

For class I \( p \)-wave superconducting chains, in particular, the decay exponent can be 1/2, which is much slower than the decay exponent equal to 3/2. As shown in the left panel of figure 8(a), even at \( t = 30\,000 \), the system does not converge to its steady-state values. To investigate the infinite time behavior numerically, our finding suggests the choice of different class III parameters than those in class I. If we are interested in choosing the class I parameters due to some reality constraints, we should choose parameters for the final Hamiltonian such that \( (\mu' - \mu) = \Delta'(\mu'\Delta - \mu\Delta') \) to avoid a slowest power-law decay with the exponent 1/2. As we found in dimerized chains, the power-law decays to the steady-state values of the OPES are not limited to non-topologically steady-state values. As illustrated in figure 5(b) for the parameters satisfying the upper equation in equation (31), our fittings confirm the power-law decays with exponent \( \nu = 5/4 \) to the topologically steady-state values.

5. Conclusion and outlook

Starting from a one-dimensional topological system of infinite length of either a dimerized chain or a \( p \)-wave superconductor, and then measuring the entanglement spectra of a subsystem \( A \) with a length \( L \gg 1 \), we obtain two edge states (or Majorana fermions

---

**Figure 13.** By showing the full evolutions of the most important OPES on the right-hand side of figure 3(a), the left-hand side of figure 10(a), and figure 5, we can observe the evolutions of the edge states of the OPES for (a) dimerized chains and (b) \( p \)-wave superconductors.
for $p$-wave superconductors) with OPES $\lambda_m = 1/2$. These edge states characterize the topological systems according to the bulk-edge correspondence of the entanglement measure. After a sudden quench of the system at a certain time, we observe the evolution of those edge states (see figure 13(a)) for dimerized chains or of the Majorana fermions for $p$-wave superconductors (see figure 13(b)). As mentioned in section 4.3, a sudden quench will create entangled pairs of quasiparticles. Before an entangled pair of fastest quasiparticles created in the middle of the subsystem $A$ reach the left and the right boundaries of $A$, the edge states are protected and the two most important OPES retain their values as $1/2$. Hence, we can obtain the speed of the fastest quasiparticle from the first plateau of $\lambda_m = 1/2$. After that, they will relax back to their steady-state values, either to topological state values of $(\lambda_m = 1/2$, left-hand side) or to trivial state values $(\lambda_m \neq 1/2$, right-hand side), based on the effective pseudo-magnetic field $S_{\text{eff}}$.

We had already extracted some important information regarding the early life and final stage of the most important OPES. However, what happens in between was not at all clear prior to this study. Our findings presented herein complement details regarding the life of edge states and Majorana fermions between the plateau and the steady-state values. Using steepest descent expansion, we show that at a late time, the OPES decays with a power-law exponent $\nu$ along with an oscillation characterized by some special angular frequencies.

The power-law decay exponents are totally different in dimerized chains and $p$-wave superconductors. For dimerized chains, the decay exponents are always $3/2$, whereas for $p$-wave superconductors, the decay behavior is much richer, taking different values such as $1/2, 3/4, 5/4$, or $3/2$, depending on the chosen parameter regime. We also fit these analytical results to the numerics and obtained excellent agreement. We found that for the power-law decay exponent $\nu = 1/2$, the decay becomes very slow and takes a very long time to reach the steady-state values. This is the reason we must conduct these kinds of studies. It is typically very difficult to determine infinite time behavior from obtained data. Therefore, instead, we can use knowledge about how the time-evolution of the system decays to its steady-state values to fit those data. We can then obtain the infinite time behavior of the systems by extrapolating data from the intermediate stage of the time evolution. At this point, we must mention that other OPES behave in a similar way to the most important OPES, since the steepest decent method treats those OPES equally. We only discuss the most important OPES here because they could be a signal of and provide deep insights into topological systems.

The power-law decay exponents are totally model dependent and can be regarded as an ID card for the time evolutions of certain models. To date, only three models exist whose decay exponents $\nu$ have been deciphered: the transverse Ising model by Fagotti et al.\cite{56} and the dimerized chains and $p$-wave superconductors by the authors. For both transverse Ising models and dimerized chains, we find the exponents of both to be $\nu = 3/2$. In fact, there is an exact mapping between transverse Ising models and dimerized chains, so the exponents should be the same.\cite{57} Furthermore, we also find that for class III $p$-wave superconductors, the exponent is also $\nu = 3/2$. Since when $\Delta = 1$, the $p$-wave superconductors can be mapped to a transverse Ising model for any $\mu$\cite{58}, these results are also consistent with the findings for transverse Ising models and dimerized chains. They can also be transformed into each other by certain unitary transformations \cite{57}. Naturally, we must ask: can these exponents characterize the
universality of different models? to answer this question, further studies of different models will be necessary.

In this paper, we concentrated on the decay properties of OPES after a sudden quench. As noted in [56], the power-law convergence can be broken by other physical observables. To further investigate the robustness of the exponents and characteristic angular frequencies, it would be interesting to analyze how other physical observables converge. It would also be interesting to ask whether their interaction will change our findings. However, the former problem will require more advanced analytical techniques that are beyond the scope of this paper, whereas the latter would require that we overcome numerical difficulties due to the linear growth of the entanglement as time proceeds. We leave these two issues to future work.

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Appendix. Steepest descent expansion

In this work, we often encounter real integrals with the following form:

$$I_R(t) = \int_{C_R} f(k) e^{iw(k)t} dk,$$

where $C_R$ is the integration path along the real axis, $t$ is a real number, and $w(k)$ is a real function along the path $C_R$. Our interest is the asymptotic behavior of $I_R(t)$ as $t \to \infty$. In the following, we show how to obtain the asymptotic behavior by applying steepest descent expansion.

First, we sketch the whole procedure before addressing the details. The first step is to form a close contour $C_O = C_R + C_S$ and apply the residue theorem to obtain the following:

$$I_R(t) + I_S(t) = \sum \text{residue of } f \text{ inside } C_O,$$

where

$$I_S(t) = \int_{C_S} f(k) e^{iw(k)t} dk.$$  

Since the right-hand side of equation (A.2) is time independent, how $I_R(t)$ decays to its asymptotic value is solely determined by $I_S(t)$. To apply the steepest descent expansion method, we first identify all the saddle points $k_j$, $j = 1, 2, \cdots$, of $w(k)$, as well as their order $n_j$. For each saddle point $k_j$, we then find all the steepest descent paths $C_{S_j}$, $l = 0, 1, \cdots, n_j - 1$ that originate from $k_j$. Finally, we choose an appropriate set of steepest descent paths $\{C_{S_j}\}$ to form the path $C_S = \sum_{j,l} \pm C_{S_j}$ so that $I_S(t) = \sum_{j,l} \pm I_{S_j}(t)$, where:

$$I_{S_j}(t) = \int_{C_{S_j}} f(k) e^{iw(k)t} dk.$$
We determine the sign in front of each steepest descent path by whether or not \( C_{S_l j} \) must be chosen in opposite directions. For each \( I_{S_l j} \) we perform the integration and find the asymptotic behavior. We find that of the leading order is always a power-law decay accompanied by an oscillation. This implies that \( I_{R}(t) \) has the same asymptotic behavior, and the same exponent as the smallest exponent associated with \( I_{S_l j} \). Similarly, \( \Delta G_{n,m}(t) \) shows the same asymptotic behavior while its exponent is the same as the smallest exponent associated with \( I_{x}(t) \), \( I_{y}(t) \), or \( I_{z}(t) \), as defined in the main text.

Next, we discuss each step in detail. A saddle point is a point \( k_j \) at which some of the lower order derivatives of \( w(k) \) are zero at \( k_j \). Specifically, it is defined by the following:

\[
\begin{align*}
    w^{(l)}(k_j) &= \begin{cases} 
    \neq 0 & l = 0 \\
    0 & 1 \leq l \leq n_j - 1 \\
    \neq 0 & l = n_j
    \end{cases},
\end{align*}
\]

where

\[
    w^{(l)}(k_j) \equiv \frac{d^lw}{dk^l}igg|_{k=k_j}
\]

is the \( l \)th derivative of \( w(k) \) at \( k_j \). The order of the saddle point is defined as \( n_j \), the order of the first non-zero derivative of \( w(k) \) at \( k_j \). A steepest descent path \( C_{S_l j} \), starting from a saddle point \( k_j \), is a path with the property that \(-i(w(k) - w(k_j))\) is always real and positive along the path. In general, for a saddle point of order \( n \), there will be \( n \) steepest descent paths. To find the asymptotic behavior of \( I_{S_l j}(t) \) as \( t \to \infty \), we change the variable \( q \equiv -i(w(k) - w(k_j)) \) and \( I_{S_l j}(t) \) becomes as follows:

\[
    I_{S_l j}(t) \equiv \int_{C_{S_l j}} f(k)e^{iw(k)t}dk = e^{iw(k_j)t} \int_{0}^{\infty} \frac{-f(k)}{iw^{(1)}(k)} e^{-tq}dq.
\]
We set the upper limit to infinity since we are interested in the $t \to \infty$ behavior. We then expand $f(k)$ and $w^{(1)}(k)$ around $k_j$ to the next leading order and find the following:

\[
w^{(1)}(k) = \frac{w^{(n_j)}(k_j)}{(n_j - 1)!} \delta k^{n_j - 1} \left( 1 + \frac{w^{(n_j+1)}(k_j)}{w^{(n_j)}(k_j)n_j} \delta k + O(\delta k^2) \right)
\]

and

\[
f(k) = \frac{f^{(\beta_j-1)}(k_j)}{(\beta_j - 1)!} \delta k^{\beta_j - 1} + \frac{f^{(\beta_j)}(k_j)}{\beta_j!} \delta k^{\beta_j} + O(\delta k^{\beta_j+1}).
\]

We can see that $\beta_j - 1$, as shown in equation (A.8), is defined as the leading order of the Taylor expansion around the saddle point $k_j$. There is a need to go to next leading order when the leading order contribution from two $I_{\alpha_j}$ might cancel each other, as we show later. To finish the variable change, we must relate $\delta k$ and $q$. To do so, we expand $q$ around $k_j$ up to the first non-zero term and find the following:

\[
q \approx -\frac{|w^{(n_j)}(k_j)|}{n_j!} \delta k^{n_j} e^{i(\alpha_j + n_j \theta_j)},
\]

where $\alpha_j$ and $\theta_j$ are the phase of $iw^{(n_j)}(k_j)$ and $\delta k$, respectively. For a steepest descent path, $q \equiv -i(w(k) - w(k_j))$ must always be real and positive. It is evident that there are $n_j$ possible solutions to this requirement:

\[
\theta_j = \theta_j^l = \frac{(2l + 1)\pi - \alpha_j}{n_j},
\]

where $l = 0, 1, \cdots, n_j - 1$. As a result, there are $n_j$ steepest descent paths that originate from an order $n_j$ saddle point. Now, we can express $\delta k$ in terms of $q$, as follows:

\[
\delta k \approx \left( \frac{|w^{(n_j)}(k_j)|}{n_j!} q^{1/n_j} \right)^{1/n_j} e^{i\theta_j^l},
\]

and perform the $q$ integration to obtain the following:

\[
I_{\alpha_j}(t) = e^{iw(k_j)t} \left[ C_j^l t^{\frac{\beta_j}{n_j}} + D_j^l t^{\frac{\beta_j+1}{n_j}} + O \left( t^{\frac{\beta_j+2}{n_j}} \right) \right].
\]

In the equation above:

\[
C_j^l = \frac{f^{(\beta_j-1)}(k_j)}{n_j(\beta_j - 1)!} \left( \frac{n_j!}{|w^{(n_j)}(k_j)|} \right)^{\frac{\beta_j}{n_j}} \Gamma \left( \frac{\beta_j}{n_j} \right) e^{i\beta_j \theta_j^l},
\]

and

\[
D_j^l = \left( \frac{f^{(\beta_j)}(k_j)}{n_j \beta_j} - \frac{f^{(\beta_j-1)}(k_j)w^{(n_j+1)}(k_j)}{n_j^2(\beta_j - 1)!w^{(n_j)}(k_j)} \right) \left( \frac{n_j!}{|w^{(n_j)}(k_j)|} \right)^{\frac{\beta_j+1}{n_j}} \Gamma \left( \frac{\beta_j + 1}{n_j} \right) e^{i(\beta_j+1) \theta_j^l},
\]

where $\Gamma(x)$ is the gamma function.
All the integrals in this paper, however, have two additional properties: (i) the two boundary points are always saddle points, and (ii) we can always choose saddle points on the real axis and use their corresponding steepest descent paths to form the contour. Note that (ii) also means the time dependent factor $e^{i \omega(k) t}$ in equation (A.12) gives only the oscillation instead of the exponential decay or growth.

From equation (A.12), we find that to the leading order, $I_{S_j}(t)$ behaves asymptotically as follows:

$$\lim_{t \to \infty} I_{S_j}(t) \sim e^{i \omega(k_j) t} \frac{C_j}{t^{\beta_j/n_j}},$$

(A.15)

which is a power-law decay with exponent $\beta_j/n_j$, accompanied by an oscillation. The characteristic angular frequency of the oscillation is as follows:

$$\omega_j = |w(k_j)|.$$

(A.16)

Note that the angular frequency is independent of the paths, as we can see from equation (A.12). Moreover, the Riemann–Lebesgue lemma guarantees $I_R(t) \to 0$ as $t \to \infty$ as long as $f(k)$ and $w(k)$ are smooth and well-behaved functions along the path $C_R$. Hence, the time independent residue term in equation (A.2) must vanish at any time $t$ and we obtain the integral equation (A.1) as follows:

$$I_R(t) = \sum_{l,j} \pm I_{S_j}(t).$$

(A.17)

From equations (A.17) and (A.15), we can finally conclude that the asymptotic behavior of $I_R(t)$ as $t \to \infty$ is given by the following:

$$I_R(t) \sim \frac{1}{t^\nu} \sum_{l,j} a'_j e^{i \omega_j t},$$

(A.18)

where $\nu$ is the minimum of $\beta_j/n_j$, $a'_j$ is a constant and the summation $\sum_{l,j}$ sums only those $l, j$ with the constraint: $\nu = \beta_j/n_j$.

When $\beta_s = n_s$, however, equation (A.18) must be modified. This happens, for example, for a class I $p$-wave superconducting chains case. With this condition, we have $e^{i \beta_s \phi_s} = e^{(2 + 1) \pi^{-\alpha_s} - i \alpha_s} = -e^{-i \alpha_s}$ as seen from equation (A.10). When we choose two steepest descent paths originating from the same saddle point with $\beta_s = n_s$, the two coefficients of the leading term are the same and the leading order contributions cancel each other since one must choose a path in the opposite direction and pick up a minus sign. Therefore, the asymptotic of $-I_{S_j}(t) + I_{S_j'}(t)$ is dominated by sub-leading terms. With some algebra, we find the following:

$$-I_{S_j}(t) + I_{S_j'}(t) \sim \left( e^{i (2 \nu + 1) \pi^{-\alpha_s}} - e^{-i (2 \nu + 1) \pi^{-\alpha_s}} \right) \frac{E_s}{t^{\nu_s}} e^{i \omega_s t},$$

(A.19)

where $\nu_s = 1 + 1/n_s$ and

$$E_s = \frac{\beta_s}{n_s} \left( \frac{n_s!}{|w(n_s)(k_s)|} \right)^{1 + \frac{1}{n_s}} \Gamma \left( 1 + \frac{1}{n_s} \right) e^{-i \alpha_s (1 + 1/n_s)}.$$
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