Zurek claims to have derived Born’s rule noncircularly in the context of an ontological no-collapse interpretation of quantum states, without any “deus ex machina imposition of the symptoms of classicality.” After a brief review of Zurek’s derivation it is argued that this claim is exaggerated if not wholly unjustified. In order to demonstrate that Born’s rule arises noncircularly from deterministically evolving quantum states, it is not sufficient to assume that quantum states are somehow associated with probabilities and then prove that these probabilities are given by Born’s rule. One has to show how irreducible probabilities can arise in the context of an ontological no-collapse interpretation of quantum states. It is argued that the reason why all attempts to do this have so far failed is that quantum states are fundamentally algorithms for computing correlations between possible measurement outcomes, rather than evolving ontological states.

Keywords: Born’s rule; envariance; interpretation of quantum mechanics; probability.

1 Introduction

In any metaphysical framework that treats quantum states as deterministically evolving ontological states, such as Everett’s many-worlds interpretation, Born’s rule has to be postulated. Repeated attempts to derive Born’s rule within the many-worlds framework have proved circular, as was noted even by its proponents. (For references see [1].) In the context of environment-induced decoherence, Born’s rule emerges naturally [2] but decoherence is based on reduced density matrices, and the partial tracing that leads to reduced density matrices is predicated on Born’s rule. No noncircular derivation of Born’s rule has ever been put forward in the context of an ontological no-collapse interpretation of quantum states.

Recently, however, Wojciech H. Zurek has claimed to have shown how Born’s rule arises noncircularly “in a purely quantum setting, i.e., without appeals to ‘collapse,’ ‘measurement,’ or any other such deus ex machina imposition of the symptoms of classicality that violate the unitary spirit of quantum theory” [3, 4]. After a brief review
of Zurek’s derivation in Sec. 2, it is argued that this claim is exaggerated if not wholly unjustified. In order to demonstrate that Born’s rule arises noncircularly from deterministically evolving ontological quantum states (DEOQS), it is not sufficient to assume that quantum states are somehow associated with probabilities and then prove that these probabilities are given by Born’s rule, as Zurek has done.

Section 3 offers an instructive attempt to complete Zurek’s derivation by demonstrating how irreducible probabilities—probabilities associated not with averaged states but with microstates of individual systems—can arise in the context of an ontological no-collapse interpretation of quantum states. The attempt fails because it tacitly relies on decoherence and thus makes implicit use of Born’s rule. It is argued in Sec. 4 that this is not a failing of only this particular attempt but a failing of the notion that quantum states are evolving ontological states, rather than being fundamentally algorithms for computing correlations between possible measurement outcomes. It is further argued that Gleason’s theorem and more recent Gleason-like derivations of Born’s rule offer a deeper understanding of the axioms of standard quantum theory than Zurek’s does. (See also the comment by Schlosshauer and Fine on Zurek’s derivation of Born’s rule.)

2 Zurek’s derivation of Born’s rule

The following assumptions are made:

(i) The universe consists of systems.

(ii) The states of a system \( S \) are (associated with) the normalized elements \( |\psi\rangle \) of a Hilbert space that describes \( S \).

(iii) A composite system is described by the tensor product of the Hilbert spaces of the constituent systems.

(iv) States evolve according to \( i\hbar \dot{\psi} = H\psi \) where \( H \) is Hermitean.

Envariance (environment–assisted invariance) is defined as follows: If for a vector \( |\psi_{S\&E}\rangle \) associated with the composite system \( S\&E \), where \( E \) is a dynamically decoupled environment of \( S \), and for a transformation \( U_S = u_S \otimes 1_E \) there exists a transformation \( U_E = 1_S \otimes u_E \) such that

\[
U_E U_S |\psi_{S\&E}\rangle = |\psi_{S\&E}\rangle
\]

then \( |\psi_{S\&E}\rangle \) is envariant under \( u_S \).

Suppose that

\[
\sum_{k=1}^{N} a_k |s_k\rangle \otimes |\varepsilon_k\rangle
\]
is a biorthonormal (Schmidt) decomposition of $|\psi_{S&E}\rangle$, and that

$$u_S |s_k\rangle = e^{i\phi_k} |s_k\rangle, \quad k = 1, \ldots, N.$$  \hfill (3)

For any set of coefficients $a_k$ and any set of integers $l_k$ the effect of $u_S$ on $|\psi_{S&E}\rangle$ can be undone by

$$u_E |\varepsilon_k\rangle = e^{-i(\phi_k + 2\pi l_k)} |\varepsilon_k\rangle.$$  \hfill (4)

Thus $|\psi_{S&E}\rangle$ is invariant under $u_S$. Since the properties of $S&E$, as well as the respective properties of $S$ and $E$, are fully determined by $|\psi_{S&E}\rangle$, no feature of $|\psi_{S&E}\rangle$ that is affected by $u_S$ can represent a property that is possessed by $S$ while the joint state of $S&E$ is $|\psi_{S&E}\rangle$. It follows that the phases of the coefficients $a_k$ in a biorthogonal decomposition of $|\psi_{S&E}\rangle$ cannot represent properties of (or contain information about) $S$.

If any two of the coefficients in the decomposition (2) have equal norms—say, $a_1 = |a|e^{-i\phi_1}$ and $a_2 = |a|e^{-i\phi_2}$—then $|\psi_{S&E}\rangle$ is also invariant under the swap

$$u_S(1\leftrightarrow 2) = e^{i\phi_{12}} |s_1\rangle \langle s_2| + e^{-i\phi_{12}} |s_2\rangle \langle s_1|,$$  \hfill (5)

which can be undone by the “counterswap”

$$u_E(1\leftrightarrow 2) = e^{-i(\phi_{12} + \phi_1 - \phi_2 + 2\pi l_{12})} |\varepsilon_1\rangle \langle \varepsilon_2| + e^{i(\phi_{12} - \phi_1 + \phi_2 - 2\pi l_{12})} |\varepsilon_2\rangle \langle \varepsilon_1|.$$  \hfill (6)

It follows that the information about $S$ contained in $|\psi_{S&E}\rangle$ is invariant under $u_S(1\leftrightarrow 2)$.

To complete his derivation of Born’s rule, Zurek considers the case in which the coefficients $a_k$ in the decomposition (2) are proportional to $\sqrt{m_k}$ with natural numbers $m_k$, so that the norms $|a_k|^2$ are commensurate. (The assumption that all coefficients have the same phase is warranted by the fact, as shown, that the phases of the Schmidt coefficients are irrelevant as far as the properties of $S$ concerned.) He then extracts from $E$ a “counterweight” $A$ such that

$$|\psi_{A&S&E}\rangle \propto \sum_{k=1}^{N} \sqrt{m_k} |A_k\rangle \otimes |s_k\rangle \otimes |\varepsilon_k\rangle.$$  \hfill (7)

By “increasing the resolution of $A$”,

$$|A_k\rangle = \sum_{j_k=1}^{m_k} |a_{jk}\rangle / \sqrt{m_k},$$  \hfill (8)
and letting $A$ interact with $E$ so that

$$|a_{jk}\rangle \otimes |\epsilon_k\rangle \rightarrow |a_{jk}\rangle \otimes |e_{jk}\rangle$$  (9)

with orthonormal vectors $|e_{jk}\rangle$, he reduces this case to the case of coefficients with equal norms. The upshot: Born’s rule $p(s_k) = m_k/M$, where $M = \sum_{k=1}^{N} m_k$. The generalization to coefficients with incommensurate norms is straightforward.

3 Whence the probabilities?

As said, in order to demonstrate that Born’s rule arises noncircularly from DEOQS, it is not sufficient to assume that quantum states are somehow associated with probabilities and then prove that these probabilities are given by Born’s rule. What is thereby proved is that if quantum states are associated with probabilities then Born’s rule holds. But how do quantum states come to be associated with probabilities? As long as this question remains unanswered, one has not elucidated the origin of probabilities in quantum physics, as Zurek claims to have done [3].

Zurek abruptly concludes that “the probabilities for any two swappable $|s_k\rangle$ are equal.” If these states are associated with probabilities then it is certainly the case that “swappable” implies “equiprobable.” But the idea that states are associated with probabilities comes out of the blue, without justification. What invariance under $u_S(1\leftrightarrow 2)$ does imply is a symmetry with regard to the properties $s_1$ and $s_2$. If these properties are mutually exclusive, $S$ cannot simultaneously possess both, but it may possess neither of them: the propositions “$S$ is/has $s_i$” ($i=1,2$) may both be false, or else they may both lack truth values (i.e., the physical situation may be such that these propositions are neither true nor false but meaningless).

Can Zurek’s incomplete derivation of Born’s rule from DEOQS be completed by demonstrating how irreducible probabilities—probabilities associated not with averaged states but with microstates of individual systems—can arise in the context of an ontological no-collapse interpretation of quantum states? Here is how one might attempt to do this. If all of the coefficients in the decomposition (2) have equal norms, and if the properties $s_k$ are mutually exclusive and jointly exhaustive (i.e., if the normalized $|s_k\rangle$ form an orthonormal basis) then the propositions “$S$ is/has $s_k$” cannot all be false. We already know that they cannot all be true, and that if they are in possession of truth values, they must be in possession of identical truth values. We conclude that they cannot be in possession of truth values. If all Schmidt coefficients have equal norms and the $|s_k\rangle$ form an orthonormal basis, these propositions are neither true nor false. But if predicative propositions can lack truth values, a criterion has to exist: under what conditions does such a proposition possess a truth value?

In the spirit of Zurek’s “existential interpretation” [1, 2] (in the context of which Zurek’s claims ought to be evaluated, if only to be fair) we may require the existence
of a record. A truth value exists if and only if one is indicated by, or inferable from, a predictably evolving pointer state—a record. Once we have reached this point, we can with sufficient justification assign probabilities to all possible value-indicating states or records (“measurement outcomes”) inasmuch as we cannot predict the actual outcome. (If we could, we could assign truth values to the propositions “S is/has $s_k$” in advance of the outcome.) And if all coefficients in the decomposition (2) have equal norms, we can invoke the principle of indifference and deduce from the nonexistence of truth values that all of those propositions are equally likely to come out true in the event that a record of their truth values is created.

Is this attempt to complete Zurek’s derivation of Born’s rule noncircular? I don’t think so. Probabilities are associated with possibilities, and the relevant possibilities are possible measurement outcomes, or outcome-indicating states, or records. How do records enter the picture? By Zurek’s account, through correlations. The recorded properties are those that are entangled with the most predictable states of the environment or of parts thereof. And the most predictable states are the most abundantly monitored ones—the ones entangled with the largest number of subsystems of the environment. The information recorded in them is so abundantly replicated that it is for all practical purposes indelible. And why are the most abundantly monitored states the most predictable ones? The chief ingredient in Zurek’s answer to this question is decoherence, and decoherence involves in an essential way reduced density matrices, partial tracing, and thus, ultimately, an implicit use of Born’s rule.

Is there any other way to complete Zurek’s derivation noncircularly? I don’t think so. The symbols $s_k$ and $\varepsilon_k$ in (2) stand for possible properties of $S$ and $E$. Predicative propositions are needed to relate these properties to their respective systems. Since these propositions are not necessarily in possession of truth values, a necessary condition for the existence of a truth value is required. There is broad consensus that, in the context of standard quantum mechanics (unadulterated with, e.g., Bohmian trajectories [10] or nonlinear modifications of the “dynamical” equations [11]) this involves environment-induced decoherence in an essential way.

There are of course ways to cloud the issue, such as Zurek’s suggestion [3] that the uniqueness of a Schmidt decomposition (in case none of the coefficients have equal norms) is sufficient for the existence of a unique set of pointer states. States selected by a unique Schmidt decomposition are not necessarily pointer states. To be indicators of measurement outcomes, states must retain information about outcomes, and decoherence arguments are essential for establishing which states are capable of retaining such information. Information about the (relative) values of an observable (relative as in “relative state” [11]) can be contained (via a unique Schmidt decomposition) in a system’s nonlocal properties (or in the values of a nonlocal observable) in which case the system or observable is exceedingly unlikely to retain this information, owing to environment-induced decoherence.
4 Moral

By simply assuming that quantum states are somehow associated with probabilities, Zurek has glossed over the fact that the reason why quantum states are associated with probabilities involves decoherence and hence Born’s rule. Probabilities are associated with possibilities, and the relevant possibilities are measurement outcomes or records thereof. I don’t see a way of introducing probabilities without introducing measurement outcomes, no matter what other names we invent for them. My failed attempt to complete Zurek’s derivation noncircularly is a case in point. To be able to justify the introduction of probabilities, I had to assume that the properties $s_k$ are mutually exclusive and jointly exhaustive. This assumption owes its meaning to measurements. The physical meaning of “mutually exclusive” (as against its mathematical implementation through orthogonality) is that if the propositions “$S$ is/has $s_k$” are in possession of measured truth values then the truth of one implies the falsity of the others. The physical meaning of “jointly exhaustive” is that if these propositions have measured truth values then at least one of them is true. Zurek’s claim to have explained “how Born’s rule arises... without appeals to ‘collapse,’ ‘measurement,’ or any other such deus ex machina imposition of the symptoms of classicality” is therefore unjustified.

Zurek emphasizes that, in contrast with other derivations of Born’s rule, his “relies on the most quantum of foundations—the incompatibility of the knowledge about the whole and about the parts, mandated by entanglement.” “Entanglement,” too, owes its meaning to measurements. Its physical meaning is that one can measure (at any rate, obtain information about) the value of an observable $V$ by measuring the value of another observable $W$. Besides, what has knowledge to do with quantum foundations in the context of an ontological interpretation of quantum states? The statement that knowledge about the whole implies ignorance of the parts has two parts, one that conveys a lack of factuality and one that is counterfactual. Saying that a system composed of two spin-1/2 particles is in the singlet state is the same as saying (i) that the components of the individual spins lack values, and (ii) that if a spin component of each particle were measured then the results would be correlated as specified by the singlet state. There is therefore nothing factual about this “state,” nothing that would warrant an ontological interpretation. If there is an ontological state that is responsible for the correlations encapsulated by the singlet state, it is not the singlet state, and that’s all we know about it.

The rather mystical-sounding statement that knowledge about the whole implies ignorance of the parts is thus largely a statement about correlated probability distributions over measurement outcomes. Given its implicit reference to probabilities, it does not elucidate the “origin of probabilities” but rather shows that probabilities are present from the start, however cleverly they may be concealed by mystical language. If the
quantum formalism is (i) the fundamental theoretical framework of physics and (ii) fundamentally what somehow it obviously is (namely, a probability algorithm) then there is nothing "more fundamental"—a contradiction in terms since what is “less fundamental” just isn’t fundamental—from which probabilities could emerge.

To my mind, the conclusion to be drawn from the past failures (including Zurek’s) to derive probabilities noncircularly from DEOQS, is that quantum states are probability measures and should not be construed as evolving ontological states. Theorists ought to think of them the way experimentalists use them, namely as algorithms for computing the probabilities of possible measurement outcomes on the basis of actual measurement outcomes. No end of pseudophysics is generated by the notion that quantum states are evolving ontological states. Then one is presented with two modes of evolution, one unitary and one projective, and a host of ensuing pseudoproblems. The way to get rid of these pseudoproblems is not to reject one mode of evolution but to reject them both. The lawlike features of the world are encapsulated in correlations between value-indicating events, not in an evolving ontological state. (The time dependence of quantum states is a dependence on the time of a measurement, relative to the time of another measurement, not the dependence of a state of affairs that persists and evolves.)

As long as we believe in DEOQS, the assumptions listed at the beginning of Sec. 2 are nothing short of baffling. They can be rendered comparatively transparent by preceding them with another assumption:

- The mathematical formalism of quantum mechanics is an algorithm for assigning probabilities to possible measurement outcomes on the basis of actual measurement outcomes.

As the following will show, this assumption elucidates assumptions (i)–(iv) provided that one does not invoke assumption (iii) to derive Born’s rule, as Zurek does.

(i) Why does the universe consist of systems? Because our fundamental theoretical framework is an algorithm that correlates possible outcomes of measurements that may be performed either on the same system at different times or on different systems in spacelike relation.

(ii) Why are the states of $\mathcal{S}$ normalized elements $|\psi\rangle$ of a Hilbert space?

(a) Hilbert space because we want nontrivial probabilities (probabilities greater than 0 and less than 1) to be assigned to the possible outcomes of maximal measurements (measurements yielding the greatest possible amount of information) even if they are assigned on the basis of outcomes of maximal measurements. Nontrivial probabilities because otherwise we would have no reason to treat quantum states as probability measures.
Classically, a property is represented by a subset $W$ of some phase space, and the probability measure determined by a maximal measurement is represented by a (0-dimensional) point $P$ in that space. All probabilities are therefore trivial: $1$ if $P \subset W$ and $0$ otherwise. This makes it possible to interpret the classical probability measure as an evolving ontological state. If a property is represented by a subspace $W$ of some vector space and the probability measure determined by a maximal measurement is represented by a (1-dimensional) ray $R$ in that space, probability $1$ goes with $R \subset W$ and probability $0$ goes with $R \perp W$, which makes room for nontrivial probabilities [13, 14].

(b) Normalized because as a possible outcome to which a probability is assigned, $|\psi\rangle$ is a shorthand notation for a projector $|\psi\rangle\langle\psi|$, and because the probabilities it assigns when it is an actual outcome add up to $1$.

(iii) Why is a composite system “described” by the tensor product of the Hilbert spaces of the constituent systems? Given Born’s rule, because

$$p(a'|a) p(b'|b) = |\langle a'|a\rangle \langle b'|b\rangle|^2 = |(|a'| \otimes |b'|)[|a\rangle \otimes |b\rangle]|^2.$$  

(10)

(iv) Why do states “evolve” according to $i\hbar|\dot{\psi}\rangle = H|\psi\rangle$ with a Hermitian $H$? Because according to Born’s rule the probability of outcome $|\psi_2\rangle$ at $t_2$ based on outcome $|\psi_1\rangle$ at $t_1$ is given by

$$|\langle \psi_2(t_2)|U(t_2,t_1)|\psi_1(t_1)\rangle|^2,$$  

(11)

where $U$ is unitary. And why is $U$ unitary? Because probability is conserved whenever the system in question persists (is stable).

Given that quantum states are probability measures, it stands to reason that they are $\sigma$-additive positive functionals on the projection operators in a Hilbert space [15]. The usual choice of projection operators allows one to derive the trace rule, [5] and hence Born’s rule, for Hilbert spaces of dimension $d \geq 3$, without invoking (iii), which makes it possible to understand the probabilistic origin of (iii). By a perfectly justifiable generalization from projection valued measures to positive operator valued measures, the trace rule can be shown to hold for $d = 2$ as well [6, 7, 8]. The reason why I prefer the Gleason-like derivations of Born’s rule (given that quantum states are probability measures) to Zurek’s is that the latter makes it so much harder (if not impossible) to justify assumptions (iii) and (iv).

A final word: I do not wish to give the impression that the rejection of DEOQS is a cure-all. Rather, I believe that once our vision is no longer clouded by pseudoquestions entailed by the notion that quantum states are evolving ontological states, we will have a clearer view of the genuine issues, and a better chance of finding satisfactory solutions. For discussions of these issues see Refs. [12]–[14] and [16]–[18].
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