Surprising Pfaffian factorizations in random matrix theory with Dyson index $\beta = 2$

Mario Kieburg

Department of Physics and Astronomy, State University of New York at Stony Brook, NY 11794-3800, USA

E-mail: mario.kieburg@stonybrook.edu

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Abstract

In recent decades, determinants and Pfaffians were found for eigenvalue correlations of various random matrix ensembles. These structures simplify the average over a large number of ratios of characteristic polynomials to integrations over one and two characteristic polynomials only. Up to now it was thought that determinants occur for ensembles with Dyson index $\beta = 2$, whereas Pfaffians only for ensembles with $\beta = 1, 4$. We derive a non-trivial Pfaffian determinant for $\beta = 2$ random matrix ensembles which is similar to the one for $\beta = 1, 4$. Thus, it unveils a hidden universality of this structure. We also give a general relation between the orthogonal polynomials related to the determinantal structure and the skew-orthogonal polynomials corresponding to the Pfaffian. As a particular example, we consider the chiral unitary ensembles in great detail.

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1. Introduction

Random matrix ensembles serve as simple models in a wide range of applications [1–3, 4, 5] which can be found in number theory [6, 7], disordered systems [1], quantum chaos [8], empirical data analysis [9–11], information theory [12] and quantum chromodynamics (QCD) [13]. The complexity of most systems prevents derivations of correlation functions whereas analytic results are accessible for the corresponding random matrix model. The reason for the applicability of random matrix theory lies in the universality of spectral statistics on certain scales like the local mean level spacing [14–16] or on the global scale [17–20]. If the Lagrangian of the physical system drastically simplifies such that it is effectively described by global symmetries there might be a random matrix model fulfilling the same symmetries.
Already in the 1960s and 1970s [21–25], the \( k \)-point correlation functions of the Gaussian and circular ensembles for the three symmetries of orthogonal (\( \beta = 1; \text{GOE/COE} \)), unitary (\( \beta = 2; \text{GUE/CUE} \)) and unitary-symplectic (\( \beta = 4; \text{GSE/CSE} \)) invariance were derived. They can be expressed as a single determinant for the unitary case and a single Pfaffian for \( \beta \in \{1, 4\} \), where the integrals are pulled inside of these structures. Their matrix elements only depend on two eigenvalues which is a drastic simplification of the integrand. Since then many other random matrix ensembles were studied, e.g. the Ginibre ensembles [26–29] and the other two rotation groups: \( \text{O}(N) \) and \( \text{USp}(2N) \) [30]. The \( k \)-point correlation functions as well as the averages over ratios of characteristic polynomials for many of these ensembles are determinants and Pfaffians with relatively simple entries only depending on one or two eigenvalues [31–33]. For a long time it was thought that determinants appear for ensembles with \( \beta = 2 \) and Pfaffians for the other two cases. In [34, 35], the general conditions were derived to find these structures. Thus, all these particular random matrix ensembles were unified in one procedure to derive these structures.

Very recently, a random matrix model for the Wilson–Dirac operator was introduced [36] in lattice QCD. It generalizes the chiral GUE which was studied in a Hermitian version [36–39] and a non-Hermitian one [40]. The eigenvalue correlations exhibit Pfaffians for the Hermitian [39] as well as for the non-Hermitian case [41] reflecting the structure found in [35]. This structure has to be also valid in the continuum limit which is the chiral GUE. Hence, the question arises of whether the Pfaffian determinants obtained for the \( k \)-point correlation functions, and thus, for the averages over ratios of characteristic polynomials, are much more general than conjectured in the broad literature.

Also in other intermediate random matrix ensembles Pfaffians were found. For example, a similar situation arises in the transition from GUE to GOE or GSE [42, 34]. If the ensemble is purely a GUE then the eigenvalue correlations can be cast into determinants, whereas the smallest interaction with a GOE or a GSE yields a Pfaffian. It would be of theoretical, technical and numerical interest if all ensembles corresponding to \( \beta = 2 \) exhibit this phenomenon when coupling it to another random matrix ensemble. Such a property simplifies the spectral statistics of intermediate ensembles onto the behavior of the entries of the Pfaffian which are averages of one or two characteristic polynomials only.

Recently, Forrester and Sinclair introduced Pfaffians at \( \beta = 2 \). In [43], Sinclair extends the Pfaffian found for the partition function with \( \beta = 1, 4 \) to Hyperpfaffians with \( \beta = L^2, L^2 + 1 \) \((L \in \mathbb{N})\) which also comprises the \( \beta = 2 \) case. With the help of these results the authors of [44] studied a log-gas on a ring with two interacting species. One component of this gas is described by a \( \beta = 4 \) log-gas and the other one by a \( \beta = 1, 2 \) log-gas. The Pfaffian determinants found in [43, 44] are similar to, but not the same as, the one derived in section 4.

We derive Pfaffian determinants for averages over ratios of characteristic polynomials weighted by a joint probability density function factorizing in weights of the single eigenvalues apart from a squared Vandermonde determinant. This squared Vandermonde determinant can be cast into one determinant similar to the \( \beta = 4 \) case. Thus, it fulfils the same condition as presented in [35] which implies a Pfaffian. This unifies all ten symmetry classes in the Cartan classification [45, 46] and exhibits a hidden universal algebraic property in all of these ensembles.

An introduction of the main idea and of the important functions for the technique used here is given in section 2. In section 3, we recall some basics known about the determinantal structure obtained for averages over ratios of characteristic polynomials with respect to chiral unitary random matrix ensembles. In contrast to this structure, we derive Pfaffians for the same correlation functions in section 4. Thereby we discuss the Wilson–Dirac random matrix ensemble as a neat application and a good motivation of the derived Pfaffian determinant
at the end of this section. The skew-orthogonal polynomials corresponding to the Pfaffian
determinants are indeed closely related to the orthogonal polynomials which are found in
the determinantal structures. This relation is shown in section 5. In section 6, we discuss the
generalization of these results for chiral unitary ensembles to other random matrix ensembles
like GUE and CUE.

2. Preliminaries

Structures found in supersymmetry are the key ingredient for the technique used in the ensuing
sections. These structures allow us to derive determinants as well as Pfaffians of averaged ratios
of characteristic polynomials and, thus, \( k \)-point correlation functions for a large class of random
matrix ensembles in a direct way. The main idea is to recognize that these structures are a pure
algebraic property of the random matrix ensemble and not an analytic one. By an algebraic
rearrangement of the integrand one obtains the determinants and Pfaffians without explicitly
calculating any integrals. This idea was first proposed in [34, 35].

The requirements to obtain determinants were traced back to a factorization of the
probability density of the random matrix ensemble into densities for the single eigenvalues
times two Vandermonde determinants (see [34]), i.e. the measure for the single eigenvalues
has to be

\[
\mathrm{d}\mu(z) = \prod_{j=1}^{N} g_1(z_j) \mathrm{d}[z_j] \Delta_N(z)^2,
\]  

(2.1)

with the Vandermonde determinant

\[
\Delta_N(z) = \prod_{1 \leq a < b \leq N} (z_a - z_b) = (-1)^{N(N-1)/2} \det[z_a^{-1}]_{1 \leq a, b \leq N}.
\]  

(2.2)

The variables \( z \) can be complex which correspond to ensembles related to bi-orthogonal
polynomials [47]. For Pfaffians this requirement changes to a weight for pairs of eigenvalues
and a single Vandermonde determinant [35], i.e.

\[
\mathrm{d}\mu(z) = \prod_{j=1}^{N} g_2(z_{2j-1}, z_{2j}) \mathrm{d}[z_{2j-1}] \mathrm{d}[z_{2j}] \Delta_{2N}(z).
\]  

(2.3)

If one of these two conditions is fulfilled, then the technique presented in [34, 35] circumvents
the integration theorem by Dyson and Mehta [24, 25, 4, 48]. Moreover, the approach of
[34, 35] makes an integration theorem unnecessary at the end since it is automatically
fulfilled for random matrix ensembles traced back to measures of the form (2.1) or (2.3).
This can be readily seen by the combination of the determinantal and Pfaffian factorization for
averages over ratios of characteristic polynomials [34, 35], the representation of the orthogonal
and skew-orthogonal polynomials as averages of the corresponding ensemble [49, 50, 47,
4, 48, 51] and the expressions of the kernels of the determinants and Pfaffians in orthogonal
and skew-orthogonal polynomials [4, 48]. In sections 3 and 4, we derive the \( k \)-point correlation
function without using the integration theorem by Dyson and Mehta.

Although we do not explicitly need supersymmetry, in particular a superspace, some
functions are quite useful to write the algebraic expressions of the calculations in a very
compact, constructive and intuitive way. These functions have their origin in the theory of
supermatrices. For the interested reader, good introductions in supersymmetry are given in
[52] and in the appendix of [54]. Here, we only recall some of these useful algebraic functions
and notions.
A diagonal \((p/q) \times (p/q)\) supermatrix \(x\) consists of two blocks, \(x = \text{diag}(x_1, x_2)\). The \(p \times p\) matrix \(x_1\) and the \(q \times q\) matrix \(x_2\) are indeed diagonal, too. The supertrace ‘Str’ and the superdeterminant ‘Sdet’ of \(x\) is then defined by

\[
\text{Str}_x = \text{tr } x_1 - \text{tr } x_2 = \sum_{j=1}^{p} x_{j1} - \sum_{i=1}^{q} x_{i2},
\]

\[
\text{Sdet}_x = \frac{\text{det } x_1}{\text{det } x_2} = \prod_{j=1}^{p} x_{j1} / \prod_{i=1}^{q} x_{i2}.
\]

The crucial function of the method used here is

\[
B_{p/q}(x) = \frac{\Delta_p(x_1) \Delta_q(x_2)}{\prod_{a,b}(x_{a1} - x_{b2})}\]

\[
= (-1)^{q(q-1)/2+(q+1)p} \det \begin{pmatrix}
\frac{1}{x_{a1} - x_{b2}} & 1 \leq a \leq p \\
\frac{1}{q(x_{a1} - x_{b2})} & 1 \leq b \leq q
\end{pmatrix},
\]

for \(p \leq q\). It is the square root of a Berezinian,

\[
B_{p/q}^2(x) = \text{Ber}_{p/q}^2(x),
\]

which is the Jacobian in superspace when diagonalizing a Hermitian \((p/q) \times (p/q)\) supermatrix. The notation on the right-hand side of equation (2.6) refers to the one used in [34, 35].

All that we need for the method of [34, 35] are the functions ‘Sdet’ and ‘B’ embedded in an ordinary space like \(\mathbb{R}^{p\times q}\) or \(\mathbb{C}^{p\times q}\). Hence, those readers who are not accustomed to supersymmetry may consider these functions as ordinary, rational functions.

### 3. Review of chiral unitary random matrices

We consider the anti-Hermitian random matrix

\[
D = \begin{bmatrix}
0 & W \\
-W^\dagger & 0
\end{bmatrix},
\]

which is distributed by the density

\[
P(D) \, d[D] = \exp[-\alpha \text{ tr } V(WW^\dagger)] \prod_{a,b} \text{ d Re } W_{ab} \, \text{ d Im } W_{ab},
\]

with a non-zero normalization constant. In particular, it serves as a model for the Dirac operator in QCD [13]. The constant \(\alpha\) is proportional to \(n\). The matrix \(W\) is an \(n \times (n + \nu)\) rectangular matrix. Each of the \(n(n + \nu)\) entries of \(W\) is a complex number which might be statistically coupled by the arbitrary density \(P\). The parameter \(\nu\) with \(0 \leq \nu \leq n\) is the topological charge or also known as index of the Dirac operator such that \(D\) has \(\nu\) generic zero eigenmodes. The potential \(V\) is invariant under the group \(U(n)\), i.e.

\[
V(UWW^\dagger U^\dagger) = UV(WW^\dagger)U^\dagger,
\]

and is chosen such that all moments of the ensemble over \(\mathcal{D}^{n \times (n + \nu)}\) exist. In the simplest case \(P\) is Gaussian. Nevertheless, the arguments given here are also true for an arbitrary potential.

We only need the property

\[
P \left( \begin{bmatrix}
0 & \Lambda & 0 \\
-\Lambda & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \right) = \exp[-\alpha \text{ tr } V(\Lambda^2)] = \prod_{j=1}^{n} \exp \left[-\alpha V(\lambda_j^2) \right],
\]
for the matrix \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \) with the singular values \( 0 \leq \lambda_1 \leq \cdots \leq \lambda_n \) of \( W \), i.e. there are \( U \in U(n) \) and \( V \in U(n + v) \) with

\[
D = \text{diag}(U, V) \begin{bmatrix} 0 & \Lambda & 0 \\ \Lambda & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{diag}(U^\dagger, V^\dagger).
\]

(3.5)

In this basis, the measure (3.2) can be written as

\[
P(D) d[D] = \frac{\text{Vol}_n \text{Vol}_{n+v}}{\text{Vol}_n \text{Vol}_v} \Delta_n^2(\Lambda^2) \prod_{j=1}^n \exp \left[ -\alpha V(\lambda_j^2) \right] \lambda_j^{2v+1} d\lambda_j \\
\times d\mu_{U(n)/U(n)}(U) d\mu_{U(n+v)/U(v)}(V).
\]

(3.6)

The abbreviation of the constant

\[
\text{Vol}_l = \prod_{j=1}^l \frac{2\pi^j}{(j-1)!}
\]

refers to the volume of the unitary group \( U(l) \). Thus, the prefactor in equation (3.6) is the volume of the coset \([U(n) \times U(n+v)]/[U^v(1) \times U(v)]\). The measure \( d\mu_\Phi \) is the normalized Haar measure of the coset \( \Phi \).

An important quantity to analyze the eigenvalue statistics of this ensemble is the average over ratios of characteristic polynomials with respect to \( D \), i.e.

\[
Z_{\kappa_1, \kappa_2}^{(n,v)}(\kappa) = \int_{\mathbb{C}^{n \times (n+v)}} \prod_{l=1}^{2v} \det(D - i\kappa_{l2} I_{2n+v}) d[D] P(D) d[D],
\]

(3.8)

with the diagonal, non-degenerate \((k_1/k_2) \times (k_1/k_2)\) supermatrix \( \kappa = \text{diag}(\kappa_1, \kappa_2) = \text{diag}(\kappa_{11}, \ldots, \kappa_{k_11}, \kappa_{12}, \ldots, \kappa_{k_22}) \) and the \( 2n + v \) dimensional unit matrix \( I_{2n+v} \). This average is also known as the partition function with \( k_1 \) being bosonic and \( k_2 \) fermionic flavors in QCD [53, 55, 15]. The variables \( \kappa_{ij} \) are complex numbers with a non-vanishing imaginary part such that the integral is well defined. The partition function (3.8) is simply related to the matrix Green function and, thus, to the \( k \)-point correlation function by derivatives with respect to \( \kappa \).

The joint probability density (3.6) is of the class studied in [34] and can, therefore, be written as a determinant. This was derived in many articles before [56, 53, 55]. The crucial idea presented in [34] is the combination of the ratio of characteristic polynomials (3.8) with the two Vandermonde determinants (3.6) to square roots of Berezinians (2.5), i.e.

\[
\Delta^2(\Lambda^2) \prod_{l=1}^{2v} \det(\Lambda^2 - \kappa_{l2}^2 I_n) = \frac{B_{l_{11}+l_{21}+n}(\kappa_{11}, \Lambda^2) B_{l_{12}+l_{22}+n}(\kappa_{12}, \Lambda^2)}{B_{l_{11}/2}(\kappa_{11}^2) B_{l_{12}/2}(\kappa_{12}^2)}
\]

(3.9)

for any choice of natural numbers \( l_{11} + l_{22} = k_1 \) and \( l_{21} + l_{22} = k_2 \). In equation (3.9), we split the supermatrix \( \kappa \) into the two sets \( \tilde{\kappa}_1 = \text{diag}(\tilde{\kappa}_{11}, \tilde{\kappa}_{21}) = \text{diag}(\kappa_{11}, \ldots, \kappa_{l_{11}1}, \kappa_{12}, \ldots, \kappa_{l_{21}2}) \) and \( \tilde{\kappa}_2 = \text{diag}(\tilde{\kappa}_{12}, \tilde{\kappa}_{22}) = \text{diag}(\kappa_{l_{11}+1,1}, \ldots, \kappa_{11}, \kappa_{l_{12}+1,2}, \ldots, \kappa_{l_{22}2}) \). The choice of how we split this set is arbitrary and, thus, we get equivalent but not trivially related results. This was already recognized by the authors of [57] for products of characteristic polynomials. Let \( d_1 = n + l_{21} - l_{11} \) and \( d_2 = n + l_{22} - l_{12} \). The interesting case is \( d_1, d_2 \geq 0 \) because we want to discuss the limit \( n \to \infty \) and \( k_1, k_2 \) fixed, at the end of this section. The other cases are discussed in [34].
Without loss of generality we assume $d_1 \leq d_2$. We rearrange the integrand (3.8) with the help of equation (3.9) which yields

$$
Z_{k_1/k_2}^{(n,v)}(\kappa) \propto \text{Sdet}^{-v}\kappa \int \frac{\text{B}_{l_1/l_2}(\bar{\kappa}_1^2, \Lambda^2)\text{B}_{l_1/l_2}(\bar{\kappa}_2^2, \Lambda^2)}{\text{B}_{l_1/l_2}(\bar{\kappa}_1^2)\text{B}_{l_1/l_2}(\bar{\kappa}_2^2)} \prod_{j=1}^n \exp\left[-aV(\lambda_j^2)\right] \lambda_j^{2v+1} d\lambda_j,
$$

$$
\propto \text{Sdet}^{-v}\kappa \int \frac{1}{\text{B}_{l_1/l_2}(\bar{\kappa}_1^2)\text{B}_{l_1/l_2}(\bar{\kappa}_2^2)} \times \det \begin{bmatrix}
\frac{1}{\kappa_{a1}^2 - \kappa_{b2}^2} & 1_{l_1+1 \leq a, k_1} & 1_{l_2+1 \leq b, k_2} & \frac{1}{\kappa_{b2}^2} & 1_{l_1+1 \leq b, k_1} & 1_{l_2+1 \leq a, k_2} \\
\frac{1}{\kappa_{b1}^2 - \kappa_{b2}^2} & 1_{l_1+1 \leq a, k_1} & 1_{l_2+1 \leq b, k_2} & \frac{1}{\kappa_{b2}^2} & 1_{l_1+1 \leq b, k_1} & 1_{l_2+1 \leq a, k_2} \\
\frac{1}{\kappa_{a2}^2 - \kappa_{b2}^2} & 1_{l_1+1 \leq a, k_1} & 1_{l_2+1 \leq b, k_2} & \frac{1}{\kappa_{b2}^2} & 1_{l_1+1 \leq b, k_1} & 1_{l_2+1 \leq a, k_2} \\
\frac{1}{\kappa_{a2}^2 - \kappa_{b2}^2} & 1_{l_1+1 \leq a, k_1} & 1_{l_2+1 \leq b, k_2} & \frac{1}{\kappa_{b2}^2} & 1_{l_1+1 \leq b, k_1} & 1_{l_2+1 \leq a, k_2} \\
\end{bmatrix}.
\tag{3.10}
$$

Applying the generalized Andréief integration theorem \[58, 34\], we obtain

$$
Z_{k_1/k_2}^{(n,v)}(\kappa) \propto \frac{\text{Sdet}^{-v}\kappa}{\text{B}_{l_1/l_2}(\bar{\kappa}_1^2)\text{B}_{l_1/l_2}(\bar{\kappa}_2^2)} \times \det \begin{bmatrix}
0 & \frac{1}{\kappa_{b1}^2 - \kappa_{a2}^2} & \frac{1}{\kappa_{a2}^2 - \kappa_{b2}^2} & \frac{1}{\kappa_{b2}^2} & \frac{1}{\kappa_{a2}^2} & \frac{1}{\kappa_{b2}^2} \\
\frac{1}{\kappa_{a1}^2 - \kappa_{b2}^2} & \frac{F_1(\kappa_{a1}, \kappa_{b1})}{} & \frac{F_2(\kappa_{a1})}{} & \frac{F_3(\kappa_{b1})}{} & \frac{F_4(\kappa_{a1})}{} & \frac{F_5(\kappa_{b1})}{M_{ab}} \\
\end{bmatrix}.
\tag{3.11}
$$

Note that Andréief’s integration theorem, as well as its generalization, is only an algebraic rearrangement of the integrals without explicitly calculating any integral. The functions $F$ and $F_a$ are one-dimensional integrals and their explicit expressions are not so important as we will see in the discussion after equation (3.15). For the interested reader, we refer to [34] where the explicit integrals are given for general random matrix ensembles corresponding to determinants ($\beta = 2$). The constant $d_1 \times d_2$ matrix $M = [M_{ab}]$ is given by

$$
M_{ab} = \int_{\mathbb{R}} \lambda^{2(a+b-2)} \exp[-aV(\lambda^2)] \lambda^{2v+1} d\lambda,
$$

and, thus, generates the moments of the measure. In the next step, we use the identity

$$
\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \text{det} D \det [A - BD^{-1}C]
\tag{3.13}
$$

for arbitrary matrices $A$, $B$ and $C$ and an invertible matrix $D$. For the matrix $D$ we choose the $d_1 \times d_1$ matrix

$$
D = [M_{ab}]_{1 \leq a, b \leq d_1},
\tag{3.14}
$$
which is only a part of the full rectangular matrix $M$ appearing in equation (3.11). The determinant of $D$ is proportional to the normalisation constant of the ensemble (4.2) and $M$ is therefore invertible. Employing equation (3.13) we find

$$Z_{m,n}^{(v)}(\kappa) = \frac{1}{B_{i_1/l_1}(\kappa_1^2)B_{i_2/l_2}(\kappa_2^2)} \times \det \begin{bmatrix} Z_{m,n}^{(d,v)}(\kappa_{a_1}, \kappa_{b_1}) & G_{1,1}^{(d,v)}(\kappa_{b_1}, \kappa_{a_1}) \\ G_{1,1}^{(d,v)}(\kappa_{a_2}, \kappa_{b_2}) & Z_{m,n}^{(d,v)}(\kappa_{b_2}, \kappa_{a_2}) \\ H_{1,1}^{(d,v)}(\kappa_{a_1}) & H_{1,1}^{(d,v)}(\kappa_{b_1}) \\ H_{1,1}^{(d,v)}(\kappa_{a_2}) & H_{1,1}^{(d,v)}(\kappa_{b_2}) \end{bmatrix}. \quad (3.15)$$

In the last step, we identify the functions $G_1^{(d,v)}$, $G_2^{(d,v)}$, $G_3^{(d,v)}$, $H_1^{(a)}$ and $H_2^{(a)}$ by considering the particular choices $(l_1, l_2, l_1, l_2) \in \{(0, 0, 1, 1), (1, 0, 1, 0), (1, 1, 0, 0), (0, 0, 1, 1), (1, 0, 0, 1), (0, 0, 0, 1)\}$. In all of these cases the determinant reduces to one of the entries. Then, we obtain

$$Z_{m,n}^{(v)}(\kappa) = \frac{(-1)^{k(k_1-1)+l(l_1+1)}G_{m,n}^{(d,v)}}{B_{i_1/l_1}(\kappa_1^2)B_{i_2/l_2}(\kappa_2^2)} \times \det \begin{bmatrix} Z_{m,n}^{(d,v)}(\kappa_{a_1}, \kappa_{b_1}) & Z_{m,n}^{(d,v)}(\kappa_{a_1}, \kappa_{b_2}) & Z_{m,n}^{(d,v)}(\kappa_{a_2}, \kappa_{b_1}) & Z_{m,n}^{(d,v)}(\kappa_{a_2}, \kappa_{b_2}) \\ -Z_{m,n}^{(d,v)}(\kappa_{a_1}, \kappa_{b_1}) & Z_{m,n}^{(d,v)}(\kappa_{a_1}, \kappa_{b_2}) & Z_{m,n}^{(d,v)}(\kappa_{a_2}, \kappa_{b_1}) & Z_{m,n}^{(d,v)}(\kappa_{a_2}, \kappa_{b_2}) \\ -Z_{m,n}^{(d,v)}(\kappa_{a_1}, \kappa_{b_1}) & Z_{m,n}^{(d,v)}(\kappa_{a_1}, \kappa_{b_2}) & Z_{m,n}^{(d,v)}(\kappa_{a_2}, \kappa_{b_1}) & Z_{m,n}^{(d,v)}(\kappa_{a_2}, \kappa_{b_2}) \\ -Z_{m,n}^{(d,v)}(\kappa_{a_1}, \kappa_{b_1}) & Z_{m,n}^{(d,v)}(\kappa_{a_1}, \kappa_{b_2}) & Z_{m,n}^{(d,v)}(\kappa_{a_2}, \kappa_{b_1}) & Z_{m,n}^{(d,v)}(\kappa_{a_2}, \kappa_{b_2}) \end{bmatrix}. \quad (3.16)$$

for the partition function (3.8) which is a particular result of the general one derived in [34].

The determinant (3.16) interpolates between one-point and two-point kernels as the entries of the determinant. We emphasize again the choice of the numbers $0 < l_1 \leq k_1$ and $0 < l_2 \leq k_1$ and the splitting of $\kappa$ are arbitrary. The particular choices $l_1 = k_1$ and $l_2 = 0$ yield the $(k_1 + k_2)$-dimensional determinant with one-point kernels considered in [53, 15]. This choice is suitable for the microscopic limit in chiral random matrix theory. For bulk and soft edge correlations [15], the representation in two-point correlations is the better choice to make contact with other random matrix ensembles [17–19, 16]. This case relates to the choices $l_1 = k_1$ and $l_2 = k_2$ for $k_2 \leq k_1$, and $l_1 = 0$ and $l_2 = 0$ for $k_2 > k_1$.

The $k$-point correlation function at the $k$ variables $x = (x_1, \ldots, x_k)$ is given by

$$R_k^{(v)}(x) \propto \int_{\mathbb{R}^{k-1}} \Delta_n^2(\text{diag}(x^2, \Lambda^2)) \exp[-\alpha \text{ tr } V(x) - \alpha \text{ tr } V(\Lambda^2)] \text{det}^{2+1} \prod_{j=1}^{n-k} x_j^{2+1} \, d\lambda_j \propto \Delta_n^2(\text{det } x \exp[-\alpha \text{ tr } V(x^2)]) Z_{m,n}^{(v)}(\text{diag}(x, -x)) \quad (3.17)$$

Now, we employ formula (3.16) for $(l_1, l_2, l_1, l_2) = (0, 0, k, k)$ and find the result

$$R_k^{(v)}(x) \propto \text{det} \left[ \sqrt{x_{a,b} \over x_{a,b}} \exp \left[ -\alpha \left( V(x_a^2) + V(x_b^2) \right) / 2 \right] Z_{m,n}^{(v)}(x_a, -x_b) \right]_{1 \leq a,b \leq k}. \quad (3.18)$$
Since $Z_{0/2}^{(n-1,0)}(x_0, -x_0) = (-1)^n Z_{0/2}^{(n-1,0)}(x_0, x_0)$, this agrees with the general formula for $\beta = 2$ ensembles \cite{4}. Please note that we derived this formula without using the integration theorem by Dyson and Mehta \cite{24, 25, 4, 48}.

The constant $h_j^{(\nu)}$ in equation (3.16) is the normalization constant of the orthogonal polynomial

$$p_j^{(\nu)}(x^2) = \frac{(-1)^j Z_{0/1}^{(j,\nu)}(x)}{(-\pi)^{\nu+1}}. \tag{3.19}$$

These polynomials solve the orthogonality relation

$$\int_{0}^{\infty} p_j^{(\nu)}(x^2) p_{j'}^{(\nu)}(x^2) x^{2\nu+1} \exp[-\alpha V(x^2)] \, dx = h_j^{(\nu)} \delta_{jj'}. \tag{3.20}$$

The authors of \cite{59} have shown that these polynomials fulfill a recursion relation with respect to the topological charge $\nu$ by

$$\frac{p_j^{(\nu+1)}(x)}{p_j^{(\nu+1)}(0)} = \frac{1}{x} \frac{p_j^{(\nu)}(0) p_{j+1}^{(\nu)}(x) - p_j^{(\nu)}(0) p_{j+1}^{(\nu)}(x)}{p_{j+1}^{(\nu)}(0) - p_j^{(\nu)}(0) p_{j+1}^{(\nu)}(0)} \tag{3.21}$$

which is quite useful by taking the limit $n \to \infty$. This relation follows when setting $m = 0$ in equation (12) of \cite{59}. One can readily prove identity (3.21) by showing the orthogonality relation (3.20) for the right-hand side with respect to the $\nu + 1$ measure, i.e.

$$\int_{0}^{\infty} p_j^{(\nu)}(0) p_{j+1}^{(\nu)}(x^2) - p_j^{(\nu)}(0) p_{j+1}^{(\nu)}(x^2) \frac{p_j^{(\nu)}(0) p_{j+1}^{(\nu)}(x^2)}{x^2} x^{2\nu+3} e^{-\alpha V(x^2)} \, dx$$

$$\propto \int_{0}^{\infty} \sum_{a=0}^{\nu} \frac{p_a^{(\nu)}(0) p_{a+1}^{(\nu)}(x^2)}{h_a^{(\nu)}(x^2)} \left( p_j^{(\nu)}(0) p_{j+1}^{(\nu)}(x^2) - p_j^{(\nu)}(0) p_{j+1}^{(\nu)}(x^2) \right) x^{2\nu+3} e^{-\alpha V(x^2)} \, dx$$

$$\propto \delta_{jj'} \tag{3.22}$$

where we used the Christoffel–Darboux formula. The monic normalization of $p_j^{(\nu)}(x) = x^j + \cdots$ for all $j$ and $\nu$ explains the choice of the constants.

The Cauchy transform of $p_j^{(\nu)}$ is related to the partition function with one bosonic flavor by

$$\hat{p}_j^{(\nu)}(x^2) = \int_{0}^{\infty} \frac{p_j^{(\nu)}(\lambda^2)}{\hat{\lambda}^2} \frac{\hat{\lambda}^{2\nu+1} \exp[-\alpha V(\lambda^2)] \, d\lambda}{x^2} \tag{3.23}$$

In the result (3.16), we recognize that the choices $(l_{11}, l_{12}, l_{21}, l_{22}) = (0, 0, 1, 1), (1, 0, 1, 0), (1, 1, 0, 0)$ yield the same partition functions as the choices $(l_{11}, l_{12}, l_{21}, l_{22}) = (0, 0, 0, 2), (1, 0, 0, 1), (2, 0, 0, 0)$, respectively. Therefore, the two-flavor partition functions in equation (3.16) can also be expressed in the orthogonal polynomials (3.19) and their Cauchy transforms (3.23), i.e.

$$\frac{Z_{0/2}^{(1,0)}(\kappa_{a1}, \kappa_{b2})}{Z_{0/0}^{(1,0)}} = \frac{(-\kappa_{a1}\kappa_{b2})^2}{\kappa_{a1}^2 - \kappa_{b2}^2} \det \left[ \begin{array}{cc} p_{d_1}^{(1)}(\kappa_{a1}^2) & p_{d_1}^{(1)}(\kappa_{b2}^2) \\ p_{d_1}^{(1)}(\kappa_{a1}^2) & p_{d_1}^{(1)}(\kappa_{b2}^2) \end{array} \right], \tag{3.24}$$

$$\frac{Z_{0/2}^{(1,0)}(\kappa_{a1}, \kappa_{b1})}{Z_{0/0}^{(1,0)}} = \frac{1}{h_{d_1}^{(1)}(\kappa_{a1}^2) h_{d_1}^{(1)}(\kappa_{b1}^2)} \left( \frac{(-\kappa_{a1}\kappa_{b1})^2}{\kappa_{a1}^2 - \kappa_{b1}^2} \right) \det \left[ \begin{array}{cc} \hat{p}_{d_1}^{(1)}(\kappa_{a1}) & \hat{p}_{d_1}^{(1)}(\kappa_{b1}) \\ \hat{p}_{d_1}^{(1)}(\kappa_{a1}) & \hat{p}_{d_1}^{(1)}(\kappa_{b1}) \end{array} \right]. \tag{3.25}$$
The orthogonal polynomials and their Cauchy transforms become

\[ Z^{(d,\nu)}_{(k_1, k_2)}(\nu) = \frac{1}{h^{(d,\nu)}_{k_1} - k_2} \left( k_2 \right)_{\nu} \det \begin{bmatrix} \hat{p}^{(d,\nu)}_{d-1} \left( k_2 \right) & p^{(d,\nu)}_{d-1} \left( k_2 \right) \\ \hat{p}^{(d,\nu)}_{d} \left( k_2 \right) & \hat{p}^{(d,\nu)}_{d} \left( k_2 \right) \end{bmatrix} \].

(3.26)

These three relations are already well known [4, 48]. They can also be derived with the help of the Christoffel–Darboux formula.

The structure (3.16) is a general property of ensembles with a joint probability density including a squared Vandermonde determinant as considered in section 4.2 of [34], whereas the relations (3.24)–(3.26) have to be slightly modified for other ensembles.

In the microscopic limit, the authors of [14, 15] have shown that for a generic potential \( V \) the orthogonal polynomials and their Cauchy transforms become

\[ p^{(\nu)}(\frac{x^2}{(\nu)^2})_{d+1} \propto J_{2\nu}(x), \]

\[ \hat{p}^{(\nu)}(\frac{x^2}{(\nu)^2})_{d+1} \propto x \nu K_v(x), \]

where \( c \) is a constant depending on the potential \( V \). The functions \( J_{\nu} \) and \( K_{\nu} \) are the Bessel functions of the first kind and the modified one of the second kind, respectively. Hence, in the microscopic limit the partition function (3.8) is

\[ Z^{(d,\nu)}_{k_1, k_2} \left( \frac{K}{(cn)^2} \right)_{d+1} \propto \frac{1}{B_{l_1, l_2}(\hat{\beta}^2)} B_{l_1, l_2}(\hat{\beta}^2) \]

\[ \times \det \begin{bmatrix} \{ I^{(1)}(k_{a2}, k_{b2}) \}_{1 \leq a \leq l_1, 1 \leq b \leq l_2} & \{ I^{(2)}(k_{a1}, k_{b2}) \}_{1 \leq a \leq l_1, 1 \leq b \leq l_2} \\ \{ I^{(2)}(k_{b1}, k_{a2}) \}_{1 \leq a \leq l_1, 1 \leq b \leq l_2} & \{ I^{(3)}(k_{a1}, k_{b1}) \}_{1 \leq a \leq l_1, 1 \leq b \leq l_1} \end{bmatrix} \]

(3.29)

where

\[ I^{(1)}(k_{a2}, k_{b2}) = \begin{cases} \frac{k_{a2} J_{k_{b2}}(k_{a2})}{J_{k_{b2}}(k_{a2})} - \frac{k_{b2} J_{k_{a2}}(k_{b2})}{J_{k_{a2}}(k_{b2})}, & a \neq b, \\ \frac{k_{a2} J_{k_{b2}}(k_{a2})}{J_{k_{b2}}(k_{a2})} - \frac{k_{b2} J_{k_{a2}}(k_{b2})}{J_{k_{a2}}(k_{b2})}, & a = b, \end{cases} \]

(3.30)

\[ I^{(2)}(k_{a1}, k_{b2}) = \begin{cases} \frac{k_{a1} K_{k_{b2}}(k_{a1})}{K_{k_{b2}}(k_{a1})} - \frac{k_{b2} K_{k_{a2}}(k_{b2})}{K_{k_{a2}}(k_{b2})}, & a \neq b, \\ \frac{k_{a1} K_{k_{b2}}(k_{a1})}{K_{k_{b2}}(k_{a1})} - \frac{k_{b2} K_{k_{a2}}(k_{b2})}{K_{k_{a2}}(k_{b2})}, & a = b, \end{cases} \]

(3.31)

\[ I^{(3)}(k_{a1}, k_{b1}) = \begin{cases} \frac{k_{a1} K_{k_{b1}}(k_{a1})}{K_{k_{b1}}(k_{a1})} - \frac{k_{b1} K_{k_{a1}}(k_{b1})}{K_{k_{a1}}(k_{b1})}, & a \neq b, \\ \frac{k_{a1} K_{k_{b1}}(k_{a1})}{K_{k_{b1}}(k_{a1})} - \frac{k_{b1} K_{k_{a1}}(k_{b1})}{K_{k_{a1}}(k_{b1})}, & a = b, \end{cases} \]

(3.32)

This is the well-known result found in the literature [56, 53, 55].

**4. Derivation of the Pfaffian determinant**

In subsection 4.1, we derive a Pfaffian determinant for the same class of chiral random matrix ensembles discussed in section 3. A neat application of this Pfaffian is presented in subsection 4.2. This example is the random matrix model for the Wilson–Dirac operator in lattice QCD [36–40].
4.1. Pfaffian determinants in chiral random matrix theory

We show that the representations in determinants (3.16) are not the only existing ones for chiral unitary ensembles. A non-trivial Pfaffian can be derived for the partition function by noting that the square of the Vandermonde in the measure (3.6) can be rewritten as one Vandermonde determinant of the variables $±\lambda_j$, i.e.

$$
\Delta^2_{n}(\Lambda^2) = (-1)^{n(n-1)/2} \Delta_{2n}(\Lambda, -\Lambda)/2^n \det \Lambda.
$$

(4.1)

The determinant of $\Lambda$ will be put into the weight later on, cf equations (4.2) and (4.3) below. Considering the Wilson random matrix theory [36–40] such a splitting arises in a natural way for finite lattice spacing. Then an eigenvalue pair $±i\lambda_j$ becomes either a complex conjugated pair or two independent real eigenvalues corresponding to a pair of eigenvectors with positive and negative chiralities. Hence, the Pfaffian resulting from the single Vandermonde determinant (4.1) is the one which is generalized to non-zero lattice spacing and not the determinant [39, 41].

This allows us to define an anti-symmetric two-point measure on $\mathbb{R}^2$:

$$
g(x_1, x_2) = \frac{[x_1x_2]^\nu}{4} \exp \left[ -\alpha \frac{V(x_1^2) + V(x_2^2)}{2} \right] \delta(x_1 + x_2)[\Theta(x_1) - \Theta(x_2)],
$$

(4.2)

where $\Theta$ is the Heaviside distribution. Then, we consider the measure

$$
D[\lambda] = \frac{\text{Vol}_n \text{Vol}_{n+i}}{\text{Vol}_n^2 \text{Vol}_i} \Delta_{2n}(\lambda) \prod_{j=1}^{n} g(\lambda_{2j-1}, \lambda_{2j}) \, d\lambda_{2j} \, d\lambda_{2j-1}
$$

(4.3)

over $2n$ independent eigenvalues instead of the measure (3.6). This measure fulfills the general condition for finding a Pfaffian, cf [35] and see also equation (2.3).

The partition function (3.8) can be expressed in terms of this new measure,

$$
Z^{(n, v)}_{k_1/k_2}(\kappa) = \frac{(-1)^{n(k_1+k_2)}}{n!} \text{Sdet}^{-v} (-\kappa^2) \int_{\mathbb{R}^{2n}} \prod_{a=1}^{2n} \prod_{i=1}^{k_1} (\kappa_{2i}-\lambda_a) \prod_{j=1}^{k_2} (\kappa_{2j+1}-\lambda_a) D[\lambda].
$$

(4.4)

In the first step, we extend the Vandermonde determinant (4.1) with the characteristic polynomials

$$
Z^{(n, v)}_{k_1/k_2}(\kappa) = \frac{(-1)^{n(k_1+k_2)}}{n! \text{Vol}_n^{2n} \text{Vol}_i} \frac{\text{Vol}_n \text{Vol}_{n+i}}{\text{Vol}_n^2 \text{Vol}_i} \text{Sdet}^{-v} (-\kappa^2)
$$

$$
\times \int_{\mathbb{R}^{2n}} \frac{B_{k_1/k_2}(\kappa, \lambda)}{B_{k_1/k_2}(\kappa)} \prod_{j=1}^{n} g(\lambda_{2j}, \lambda_{2j-1}) \, d\lambda_{2j} \, d\lambda_{2j-1}.
$$

(4.5)

This representation is apart from the $z_{2n+1}$-integral of the form as that in equation (3.3) in [35]. Note that in this extension we do not have the same freedom as in the determinantal case (3.9), since there is only one Vandermonde determinant in the integrand (4.3). Let $d = 2n + k_2 - k_1 \geq 0$. Then, we employ the representation of the function ‘$B$’ as a determinant, see equation (2.5),

$$
Z^{(n, v)}_{k_1/k_2}(\kappa) \propto \frac{\text{Sdet}^{-v} \kappa}{B_{k_1/k_2}(\kappa)} \int_{\mathbb{R}^{2n}} \prod_{j=1}^{n} g(\lambda_{2j}, \lambda_{2j-1}) \, d\lambda_{2j} \, d\lambda_{2j-1}
$$

$$
\times \det \left[ \begin{array}{cccc}
\frac{1}{\kappa_{a1} - \kappa_{b2}} & 1 \leq a < b \leq k_1 \\
\frac{1}{\kappa_{a1} - \kappa_{b2}} & 1 \leq b \leq k_2 \\
\kappa_{a1} & 1 \leq a \leq d \\
\kappa_{b2}^{-1} & 1 \leq b \leq 2n
\end{array} \right].
$$

(4.6)
The generalized de Bruijn integration theorem \cite{60, 34} can be applied now which yields

\[
Z_{k_1/k_2}^{(n,v)}(\kappa) \propto \frac{\text{Sdet}^{-\nu} \kappa}{B_{k_1/k_2}(\kappa)} \times \text{PF} \left[ \begin{array}{ccc}
0 & \left\{ \frac{1}{\kappa_{b_1} - \kappa_{a_2}} \right\}_{1 \leq a < b \leq k_2} & \left\{ \kappa_{a_2}^{-1} \right\}_{1 \leq a < b \leq d} \\
- \frac{1}{\kappa_{a_1} - \kappa_{b_2}} & \left\{ \tilde{F}_1(\kappa_{a_1}, \kappa_{b_1}) \right\}_{1 \leq a, b \leq k_1} & \left\{ \tilde{F}_b(\kappa_{a_1}) \right\}_{1 \leq a \leq k_1} \\
- \kappa_{b_2}^{-1} & \left\{ -\tilde{F}_a(\kappa_{b_1}) \right\}_{1 \leq a \leq d} & \left\{ \tilde{M}_{ab} \right\}_{1 \leq a, b \leq d} \\
\end{array} \right].
\]

(4.7)

As Andréief’s integration theorem, the generalized de Bruijn integration theorem is only an algebraic rearrangement of the integrals without calculating any of them. The functions \(\tilde{F}\) and \(\tilde{F}_b\) are twofold integrals and are again not of much importance, see the discussion after equation (4.10). Explicit expressions of them are given in \cite{35} for general random matrix ensembles corresponding to Pfaffians comprising the measure (4.3), too.

The \(d \times d\) anti-symmetric matrix \(\tilde{M} = [\tilde{M}_{ab}]\) consists of the moments

\[
\tilde{M}_{ab} = \int \mathbb{R}^2 (\kappa_1^{-1} \kappa_2^{b-1} - \kappa_1^{b-1} \kappa_2^{-1}) g(\lambda_1, \lambda_2) \, d\lambda_1 \, d\lambda_2.
\]

(4.8)

Analogous to equation (3.13), we employ the identity

\[
\text{PF} \left[ \begin{array}{cc}
A & B \\
-B^T & C
\end{array} \right] = \text{PF} \text{PF}[A + BC^{-1}B^T],
\]

(4.9)

with an arbitrary matrix \(B\), an arbitrary antisymmetric matrix \(A\) and an arbitrary even-dimensional antisymmetric matrix \(C\) which has to be invertible. Let \(k_1 + k_2\) be even. Then \(d\) is also even and the Pfaffian of the matrix \(\tilde{M}\) is proportional to the normalization constant of the ensemble (3.2). Hence, the choice \(C = \tilde{M}\) is well defined. This yields

\[
Z_{k_1/k_2}^{(n,v)}(\kappa) \propto \frac{1}{B_{k_1/k_2}(\kappa)} \text{PF} \left[ \begin{array}{ccc}
\left\{ \tilde{G}_1^{(d)}(\kappa_{a_2}, \kappa_{b_2}) \right\}_{1 \leq a, b \leq k_2} & \left\{ \tilde{G}_2^{(d)}(\kappa_{b_1}, \kappa_{a_2}) \right\}_{1 \leq a \leq k_1} \\
- \tilde{G}_2^{(d)}(\kappa_{a_1}, \kappa_{b_2}) & \left\{ -\tilde{G}_1^{(d)}(\kappa_{a_1}, \kappa_{b_1}) \right\}_{1 \leq a, b \leq k_1} \\
\end{array} \right].
\]

(4.10)

The functions \(\tilde{G}_1^{(d)}\), \(\tilde{G}_2^{(d)}\) and \(\tilde{G}_3^{(d)}\) can be obtained by considering the cases \((k_1/k_2) = (0/2), (1/1), (2/0)\), respectively. In each of these cases, the Pfaffian (4.10) reduces to a single term. This leads to a particular case of the general result derived in \cite{35}. We find our main result of this paper

\[
\frac{Z_{k_1/k_2}^{(n,v)}(\kappa)}{Z_{0/0}^{(n,v)}} = \frac{1}{B_{k_1/k_2}(\kappa)} \prod_{l=0}^{d-1} h_{0}^{(l)} \prod_{l=0}^{d-1} h_{1}^{(l)} \times \text{PF} \left[ \begin{array}{ccc}
k_{b_1} - k_{a_2} & \frac{Z_{d/2-1,2}^{(d/2-1,v)}(\kappa_{a_2}, \kappa_{b_2})}{Z_{0/0}^{(d/2-1,v)}} & \frac{1}{Z_{d/2-1,2}^{(d/2-1,v)}} \frac{Z_{d/2+1,2}^{(d/2+1,v)}(\kappa_{a_1}, \kappa_{b_1})}{Z_{0/0}^{(d/2+1,v)}} \\
\frac{Z_{d/2+1,1}^{(d/2+1,v)}(\kappa_{a_1} - \kappa_{b_1})}{Z_{0/0}^{(d/2+1,v)}} & \frac{Z_{d/2+1,1}^{(d/2+1,v)}(\kappa_{a_1} - \kappa_{b_1})}{Z_{0/0}^{(d/2+1,v)}} & \frac{1}{Z_{d/2+1,1}^{(d/2+1,v)}} \frac{Z_{d/2-1,1}^{(d/2-1,v)}(\kappa_{a_1} - \kappa_{b_1})}{Z_{0/0}^{(d/2-1,v)}}
\end{array} \right].
\]

(4.11)
for even \(k_1 + k_2\). The indices \(a\) and \(b\) run from 1 to \(k_1\) in the first columns and the first rows, and from 1 to \(k_1\) in the last ones. The result for odd \(k_1 + k_2\) can be readily obtained by introducing an additional fermionic flavor and sending it to infinity. This shifts the parameter \(d\) to \(d + 1\) and adds a row and a column to the matrix in the Pfaffian (4.11) with the partition functions \(Z_{a/1}^{(d-1)/2} (κ_{k_2})\) and \(Z_{d/2}^{(d+1)/2} (κ_{k_1})\) which are apart from a factor \(κ^2\), an orthogonal polynomial, and its Cauchy transform, cf equations (3.19) and (3.23). Note that the matrix in the Pfaffian (4.11) is indeed antisymmetric because \(Z_{0/2}^{(d-1)/2} \) and \(Z_{2/0}^{(d+1)/2}\) are symmetric under a permutation of the entries.

Indeed, equation (4.11) cannot be traced back to the identity

\[
\text{Pr} \left[ \begin{array}{cc} 0 & X \\ -X^T & 0 \end{array} \right] = (-1)^{p(p-1)/2} \det X,
\]

(4.12)

with an arbitrary \(p \times p\) matrix \(X\). We refer to the relation (4.12) as a trivial Pfaffian extension of a determinant. The Pfaffian (4.11) seems to be the result of recursion relations of the orthogonal polynomials (3.19). It is difficult to see how these recursions have to be performed to map the Pfaffian (4.11) to the determinant (3.16). However, the construction of this structure seems to be the same for a broad class of ensembles. This is confirmed by the fact that the result (4.11) can be extended to all factorizing ensembles with a squared Vandermonde determinant in the joint probability density (2.1). This will be shown in section 6.

Again, one can consider the \(k\)-point correlation function (3.17) and what it looks like with the Pfaffian determinant. Using the result (4.11), we find for the \(k\)-point correlation function

\[
R_k^{(n,v)} (x) \propto \exp[-α \text{tr} V (x)]
\]

\[
\times \text{Pr} \left[ \begin{array}{c} (x_a - x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \\ -(x_a + x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \end{array} \right] \left( \begin{array}{c} (x_a + x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \\ -(x_a - x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \end{array} \right) \right]_{1 \leq a, b \leq k}
\]

\[
\times \exp[-α \text{tr} V (x)]
\]

\[
\times \text{Pr} \left[ \begin{array}{c} (x_a - x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \\ -(x_a + x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \end{array} \right] \left( \begin{array}{c} (x_a + x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \\ -(x_a - x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \end{array} \right) \right]_{1 \leq a, b \leq k}.
\]

(4.13)

Again, we have not employed the integration theorem by Dyson and Mehta [24, 25, 4, 48]. To see that equation (4.13) indeed agrees with the determinant (3.18) one can consider the square of the Pfaffian,

\[
\text{Pr}^2 \left[ \begin{array}{c} (x_a - x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \\ -(x_a + x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \end{array} \right] \left( \begin{array}{c} (x_a + x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \\ -(x_a - x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \end{array} \right) \right]_{1 \leq a, b \leq k}
\]

\[
= \det \left[ \begin{array}{c} (x_a - x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \\ -(x_a + x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \end{array} \right] \left( \begin{array}{c} (x_a + x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \\ -(x_a - x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \end{array} \right) \right]_{1 \leq a, b \leq k}
\]

\[
= 2^k \det \left[ \begin{array}{c} (x_a - x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \\ -(x_a + x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \end{array} \right] \left( \begin{array}{c} (x_a + x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \\ -(x_a - x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \end{array} \right) \right]_{1 \leq a, b \leq k}
\]

\[
= 2^k \det \left[ \begin{array}{c} (x_a - x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \\ -(x_a + x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \end{array} \right] \left( \begin{array}{c} (x_a + x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \\ -(x_a - x_b) Z_{0/2}^{(n-1,v)} (x_a, x_b) \end{array} \right) \right]_{1 \leq a, b \leq k}
\]

\[
= 2^k \det^2 x \det \left[ Z_{0/2}^{(n-1,v)} (x_a, x_b) \right]_{1 \leq a, b \leq k}.
\]

(4.14)

The square root of equation (4.14) yields equation (3.18).
In the large $n$ limit, we employ equations (3.24)–(3.28) and (3.30)–(3.32) and obtain

$$Z_{k_1/k_2}^{(n,v)} \left( \frac{\kappa}{\mathcal{C}} \right) \propto \frac{1}{\mathcal{B}_{k_1/k_2}(\kappa)} \times \text{Pf} \begin{bmatrix} (\kappa_{a_2} - \kappa_{b_2})I_v^{(1)}(\kappa_{a_2}, \kappa_{b_2}) & (\kappa_{b_1} + \kappa_{a_2})I_v^{(2)}(\kappa_{b_1}, \kappa_{a_2}) \\ - (\kappa_{a_1} + \kappa_{b_2})I_v^{(2)}(\kappa_{a_1}, \kappa_{b_2}) & (\kappa_{a_1} - \kappa_{b_1})I_v^{(3)}(\kappa_{a_1}, \kappa_{b_1}) \end{bmatrix}, \quad (4.15)$$

for even $k_1 + k_2$ and

$$Z_{k_1/k_2}^{(n,v)} \left( \frac{\kappa}{\mathcal{C}} \right) \propto \frac{1}{\mathcal{B}_{k_1/k_2}(\kappa)} \times \text{Pf} \begin{bmatrix} 0 & J_v(\kappa_{b_2}) & K_v(\kappa_{b_2}) \\ -J_v(\kappa_{a_2}) & (\kappa_{a_2} - \kappa_{b_2})I_v^{(1)}(\kappa_{a_2}, \kappa_{b_2}) & (\kappa_{b_1} + \kappa_{a_2})I_v^{(2)}(\kappa_{b_1}, \kappa_{a_2}) \\ -K_v(\kappa_{a_1}) & - (\kappa_{a_1} + \kappa_{b_2})I_v^{(2)}(\kappa_{a_1}, \kappa_{b_2}) & (\kappa_{a_1} - \kappa_{b_1})I_v^{(3)}(\kappa_{a_1}, \kappa_{b_1}) \end{bmatrix}, \quad (4.16)$$

for odd $k_1 + k_2$. These Pfaffians carry over to the Wilson–Dirac random matrix model [39, 41]. For small numbers of bosonic and fermionic flavors, these results were checked by the recursion relations of the Bessel functions [61].

Please note the difference in the prefactor of equations (3.29), (4.15) and (4.16). The entries of the Berezinian are the squares of the variables $\kappa$ for the determinantal structure (3.29), whereas it is only $\kappa$ for the Pfaffian. This yields a technical advantage when calculating eigenvalue correlations of the random matrix models for the Wilson–Dirac operator.

### 4.2. An application: Wilson–Dirac random matrix

The Wilson–Dirac operator is a modified Dirac operator on a lattice. In the infrared limit, this operator can be modeled by the Wilson–Dirac random matrix [36–38, 40] which is a $(2n + v) \times (2n + v)$ Hermitian matrix

$$D_W = \begin{bmatrix} aA & W \\ -W^\dagger & aB \end{bmatrix}, \quad (4.17)$$

distributed by the Gaussian

$$P(D_W) = \exp \left[ -\frac{n}{2} (\text{tr} A^2 + \text{tr} B^2) - n \text{tr} WW^\dagger \right]. \quad (4.18)$$

The variable $a$ plays the role of the lattice spacing. The chiral symmetry is explicitly broken by the Hermitian matrices $A$ and $B$, i.e.

$$\gamma_S D_W|_{m=0} \neq - D_W|_{m=0} \quad \text{with} \quad \gamma_S = \text{diag}(1_n, -1_{n+v}), \quad (4.19)$$

which have the dimensions $n \times n$ and $(n + v) \times (n + v)$, respectively. Hence, $A$ and $B$ model the Wilson term.

We consider the partition function with $N_f$ fermionic flavors,

$$Z_{N_f}^{(n,v)}(m, a) = \int \prod_{j=1}^{N_f} \det(D_W + m_j I_{2n+1}) P(D_W) \, d[D_W]. \quad (4.20)$$

The external variables $m = \text{diag}(m_1, \ldots, m_{N_f})$ play the role of the quark masses. Indeed, one can also consider bosonic flavors. However, we restrict ourselves to fermionic flavors to keep the example as simple as possible.
In the microscopic limit \((n \to \infty)\), \(\hat{\mu} = 2nm\), \(\hat{\alpha} = \sqrt{\alpha}n/2\) and \(v\) are kept fixed. This yields the integral

\[
Z_{\mathcal{N}}^{(n,\nu)}(\hat{\mu}, \hat{\alpha}) \equiv \int_{\text{U}(N_f)} \exp \left[ \frac{1}{2} \text{tr} \hat{\mu}(U + U^{-1}) - \hat{\alpha}^2 \text{tr} (U^2 + U^{-2}) \right] \det U \, d\mu(U). \tag{4.21}
\]

For a derivation of this result, we refer to [36, 37]. Exactly the integral (4.21) makes contact with lattice QCD [62–65].

At zero lattice spacing (\(\hat{\alpha} = 0\)) this partition function can be identified with the one considered in section 3,

\[
Z_{\mathcal{N}}^{(n,\nu)}(m, a = 0) \propto Z_{\mathcal{N}}^{(n,\nu)}(m). \tag{4.22}
\]

Considering again the microscopic limit (4.21), we trace the integral back to the \(a = 0\) result by introducing an \(N_f \times N_f\) Hermitian random matrix \(\sigma\) similar to the calculation in [38, 66],

\[
Z_{\mathcal{N}}^{(n,\nu)}(\hat{m}, \hat{\alpha}) \equiv \int_{\text{U}(N_f)} \exp \left[ -\frac{1}{4\hat{\alpha}^2} \text{tr}(\sigma - \hat{m})^2 - 2(\hat{\alpha}N_f)^2 \right] \times \exp[-i \text{tr} \sigma (U + U^{-1})] \det U \, d\mu(U) \, d[\sigma]
\]

\[
\propto \int_{\text{U}(N_f)} \exp \left[ -\frac{1}{4\hat{\alpha}^2} \text{tr}(\sigma - \hat{m})^2 - 2(\hat{\alpha}N_f)^2 \right] Z_{\mathcal{N}}^{(n,\nu)}(\sigma) \, d[\sigma]. \tag{4.23}
\]

Note that \(\sigma\) is an ordinary matrix and not a supermatrix because we consider fermionic flavors, only. The constant \(\exp[-2(\hat{\alpha}N_f)^2]\) can be shifted into the normalization constant and can, thus, be omitted in the ensuing calculations.

A diagonalization of \(\sigma = VsV^\dagger\) with \(V \in \text{U}(N_f)\) yields a Harish-Chandra–Itzykson–Zuber integral [67, 68] in the Gaussian term. The partition function \(Z_{\mathcal{N}}^{(n,\nu)}\) is invariant under \(U(\mathcal{N})\). We find

\[
Z_{\mathcal{N}}^{(n,\nu)}(\hat{m}, \hat{\alpha}) \propto \text{det}[\exp[-(s_j - \hat{m})^2/4\hat{\alpha}^2]]_{1 \leq j, i \leq N_f} Z_{\mathcal{N}}^{(n,\nu)}(s) \Delta_{\mathcal{N}}(s) \, d[s]. \tag{4.24}
\]

Employing the result as a determinant of the microscopic limit of \(\hat{\alpha} = 0\) partition function, cf equation (3.29), we end up with a complicated expression,

\[
Z_{\mathcal{N}}^{(n,\nu)}(\hat{m}, \hat{\alpha}) \equiv \int_{\text{U}(N_f)} \text{det}[\exp[-(s_j - \hat{m})^2/4\hat{\alpha}^2]]_{1 \leq j, i \leq N_f} \Delta_{\mathcal{N}}(\hat{m}) \times \text{det}[s_j^{i-1}J_{i-1+j}(s)]_{1 \leq j, i \leq N_f} \Delta_{\mathcal{N}}(s) \Delta_{\mathcal{N}}(s^2) \, d[s]. \tag{4.25}
\]

There is no obvious way to further simplify the integral (4.25) due to the factor \(\Delta_{\mathcal{N}}(s)/\Delta_{\mathcal{N}}(s^2)\). This was not much of a problem for the authors of [38, 66] because they only considered a small number of flavors. However, the problem is highly non-trivial for an arbitrary number of flavors.

This problem can be solved by using the Pfaffian expressions (4.15) and (4.16) instead of the determinant. Let \(\mathcal{N}\) be even to keep the expressions as simple as possible. Then, we have for the microscopic limit (4.21)

\[
Z_{\mathcal{N}}^{(n,\nu)}(\hat{m}, \hat{\alpha}) \equiv \int_{\text{U}(N_f)} \text{det}[\exp[-(s_j - \hat{m})^2/4\hat{\alpha}^2]]_{1 \leq j, i \leq N_f} \Delta_{\mathcal{N}}(\hat{m}) \times \text{Pf}\left[\frac{s_jJ_{i-1}(s_j)J_{s_j}(s_i) - s_iJ_{i-1}(s_j)J_{s_j}(s_i)}{s_j + s_i}\right]_{1 \leq j, i \leq N_f} \, d[s]. \tag{4.26}
\]

After expanding the determinant no term hinders us to pull the integrals into the Pfaffian. We obtain the compact result

\[
Z_{\mathcal{N}}^{(n,\nu)}(\hat{m}, \hat{\alpha}) \propto \frac{1}{\Delta_{\mathcal{N}}(\hat{m})} \text{Pf}\left[(\hat{m}_j - \hat{m}_i)Z_{\mathcal{N}}^{(n,\nu)}(\hat{m}_j, \hat{m}_i, \hat{\alpha})\right]_{1 \leq j, i \leq N_f}. \tag{4.26}
\]
with
\[
Z_{2}^{(n,\nu)}(\hat{m}_{1}, \hat{m}_{2}, \hat{a}) \propto \frac{1}{m_{1} - m_{2}} \int_{\mathbb{R}} \exp \left[ -\frac{(s_{1} - i\hat{m}_{1})^{2} + (s_{2} - i\hat{m}_{2})^{2}}{4\hat{a}} \right] \\
\times \frac{s_{1}J_{\nu - 1}(s_{1})J_{\nu}(s_{2}) - s_{2}J_{\nu}(s_{1})J_{\nu - 1}(s_{2})}{s_{1} + s_{2}} \, ds_{1} \, ds_{2}.
\]
(4.27)

This is a drastic simplification of the problem compared to equation (4.25).

5. Skew-orthogonal polynomials

What are the skew-orthogonal polynomials which correspond to the Pfaffian (4.11)? In order to solve this problem, we consider the two-point measure (4.2). The skew-orthogonal polynomials \( q_{j} \) are defined by
\[
\int_{\mathbb{R}^{2}} \det \begin{bmatrix} q_{2j-1}(x_{1}) & q_{2j-1}(x_{2}) \\ q_{2j-1}(x_{1}) & q_{2j-1}(x_{2}) \end{bmatrix} g(x_{1}, x_{2}) \, dx_{1} \, dx_{2} = 0,
\]
(5.1)
and
\[
\int_{\mathbb{R}^{2}} \det \begin{bmatrix} q_{2j+1}(x_{1}) & q_{2j+1}(x_{2}) \\ q_{2j+1}(x_{1}) & q_{2j+1}(x_{2}) \end{bmatrix} g(x_{1}, x_{2}) \, dx_{1} \, dx_{2} = \hat{h}_{j}^{(\nu)} \delta_{ij}.
\]
(5.2)

Moreover, one has to assume that \( q_{l} \) is a polynomial of order \( l \).

The integral over the measure (4.2) for two arbitrary and conveniently integrable functions \( f_{1} \) and \( f_{2} \) can be simplified to
\[
\int_{\mathbb{R}^{2}} \det \begin{bmatrix} f_{1}(x_{1}) & f_{1}(x_{2}) \\ f_{2}(x_{1}) & f_{2}(x_{2}) \end{bmatrix} g(x_{1}, x_{2}) \, dx_{1} \, dx_{2} = \frac{1}{2} \int_{0}^{\infty} \det \begin{bmatrix} f_{1}(x) & f_{1}(-x) \\ f_{2}(x) & f_{2}(-x) \end{bmatrix} x^{2} \exp[-nV(x^{2})] \, dx.
\]
(5.3)

Due to this identity the skew-orthogonal polynomials \( q_{l}^{(\nu)} \) are related by the orthogonal polynomials \( p_{l} \) in the following way:
\[
q_{2l}^{(\nu)}(x) = p_{l}^{(\nu)}(x^{2})
\]
(5.4)
for the even polynomials, and
\[
q_{2l+1}^{(\nu)}(x) = xp_{l}^{(\nu)}(x^{2}) + \text{const.} \, p_{l}^{(\nu)}(x^{2}),
\]
(5.5)
for the odd polynomials. Note that these skew-orthogonal polynomials for \( V(x) = x \) (the Laguerre ensemble) are similar to, but not completely the same as, the one for \( \beta = 1 \) and \( \beta = 4 \) shown in [4, 48] for the Laguerre ensemble. The reason is the two-point weight which is
\[
g_{\text{chGOE}}(x_{1}, x_{2}) = (x_{1}x_{2})^{\nu} \exp \left[ -\alpha \left( x_{1}^{2} + x_{2}^{2} \right) \right] \frac{x_{1} - x_{2}}{|x_{1} - x_{2}|},
\]
(5.6)
\[
g_{\text{chGSE}}(x_{1}, x_{2}) = (x_{1}x_{2})^{2\nu + 3/2} \left( x_{1}^{2} + x_{2}^{2} \right)^{\nu} \frac{x_{1} - x_{2}}{|x_{1} - x_{2}|},
\]
(5.7)
in comparison to \( \beta = 2 \), see equation (4.2). The labels ‘chGOE’ and ‘chGSE’ refer to the chiral Gaussian orthogonal ensemble (\( \beta = 1 \)) and to the chiral Gaussian symplectic ensemble (\( \beta = 4 \)), respectively. The sign function \((x_{1} - x_{2})/|x_{1} - x_{2}|\) generates the modulus of the Vandermonde determinant for \( \beta = 1 \). The distribution \( \delta' \) is the first derivative of the Dirac delta function and cancels with these terms of the Vandermonde determinant which are zero at
the support of the Dirac delta functions. This generates Cramers degeneracy in the quaternion case \( (\beta = 4) \).

The solution of equations (5.1) and (5.2) is not unique which is reflected by the arbitrary constant in the odd polynomials (5.5). One can readily confirm that this choice of the polynomials solves conditions (5.1) and (5.2) by recognizing the symmetry \( q_j(-x) = (-1)^j q_j(x) \) and the orthogonality relation (3.20) for \( p_j \). The normalization constant is

\[
\tilde{h}_i^{(\nu)} = h_i^{(\nu)}.
\] (5.8)

This relation between orthogonal and skew-orthogonal polynomials seems so trivial because of the particular and simple structure of the two-point weight (4.2).

6. A few more ensembles with Dyson index \( \beta = 2 \) and Pfaffians

The algebraic rearrangement for chiral unitary ensembles described in section 4 can be extended to other random matrix ensembles which have a squared Vandermonde determinant in the joint probability density function. By the same trick as in equation (4.1) we write

\[
\Delta_N^2(z) = (-1)^N (N-1)/2 \frac{\Delta_{2N} \sqrt{z} \sqrt{z}}{2^N \sqrt{\det z}},
\] (6.1)

where the variables \( z = \text{diag}(z_1, \ldots, z_N) \) might be complex. The square root is the positive one but this is without loss of generality since the right-hand side of equation (6.1) comprises both roots. Again the determinant of \( z \) will be put to the measure \( d\mu \) for a single eigenvalue.

We consider an average over ratios of characteristic polynomials for random ensembles like GUE and CUE, i.e.

\[
\bar{Z}_{k_1/k_2}^{(N)}(\kappa) = \int_{\mathbb{C}^N} \Delta_N^2(z) \prod_{i=1}^N \prod_{j=1}^{k_i} (z_i - \kappa_j) \prod_{j=1}^{k_i} (z_i + \kappa_j) \, d\mu(z_i),
\] (6.2)

where \( d\mu \) is a measure on \( \mathbb{C} \) and \( \kappa \) is chosen such that the integrals exist. Note that there is no modulus of the Vandermonde determinant which is a necessary property of the following discussion. A modulus of the Vandermonde is an obstacle to map equation (6.2) to the general joint probability density corresponding to the Pfaffian, see [35], which we have not managed yet. A modulus corresponds to the bi-orthogonal polynomials [47], whereas the choice without the modulus corresponds to the orthogonal polynomials, only. Apart from the modulus of the Vandermonde it is exactly the correlation function discussed in section 4.2 of [34].

With the help of the derivation in section 4, the integral (6.2) can be written as

\[
\bar{Z}_{k_1/k_2}^{(N)}(\kappa) \propto \frac{1}{\mathcal{B}_{k_1/k_2}(\sqrt{\kappa})} \times \text{Pf} \left[ \begin{array}{cc}
\frac{\sqrt{\kappa_{a1}^2 - \sqrt{\kappa_{a2}^2}}}{Z_{0/0}^{d_1}} & \frac{1}{Z_{0/0}^{d_1}} \\
\frac{\sqrt{\kappa_{b1}^2 - \sqrt{\kappa_{b2}^2}}}{Z_{0/0}^{d_2}} & \frac{1}{Z_{0/0}^{d_2}} \\
\frac{\sqrt{\kappa_{a1}^2 - \sqrt{\kappa_{a2}^2}}}{Z_{0/0}^{d_1}} & \frac{\sqrt{\kappa_{b1}^2 - \sqrt{\kappa_{b2}^2}}}{Z_{0/0}^{d_2}} \\
\end{array} \right],
\] (6.3)
for $k_1 + k_2$ even and

$$
\hat{Z}^{(N)}_{k_1,k_2}(\kappa) \propto \frac{1}{B_{k_1/k_2}(\sqrt{\kappa})}
$$

$$
\times \begin{vmatrix}
0 & -\tilde{d} \hat{Z}^{(-1)}_{a/0} (k_{b2}) & \frac{1}{\hat{Z}_{a/0}^{(d)}(\kappa_{b1})} \\
\tilde{d} \hat{Z}^{(-1)}_{a/0} (k_{a2}) & \frac{1}{\hat{Z}^{(d)}_{a/0} (\sqrt{\kappa_{b2}} - \sqrt{\kappa_{a2}})} & \tilde{d} \hat{Z}^{(-1)}_{a/0} (k_{b2}) \\
\frac{1}{\hat{Z}^{(d)}_{a/0} (\sqrt{\kappa_{a2}} - \sqrt{\kappa_{b2}})} & \tilde{d} \hat{Z}^{(-1)}_{a/0} (k_{b2}) & \frac{1}{\hat{Z}_{a/0}^{(d)} (\sqrt{\kappa_{a2}} - \sqrt{\kappa_{b2}})} \\
\end{vmatrix}
$$

(6.4)

for $k_1 + k_2$ odd. The variable $\tilde{d}$ is

$$
\tilde{d} = \begin{cases}
N + (k_2 - k_1)/2, & k_2 + k_1 \in 2\mathbb{N}, \\
N + (k_2 + k_1+1)/2, & k_2 + k_1+1 \in 2\mathbb{N}.
\end{cases}
$$

The indices $a$ and $b$ run from 1 to $k_1$ for $\kappa_1$ and from 1 to $k_2$ for $\kappa_2$. Apart from the square roots of the variables $\kappa$, these structures are exactly the same as those of random matrix ensembles with Dyson index $\beta \in [1, 4]$. Hence, it seems to be that the Pfaffian determinants (6.3) and (6.4) for the average over characteristic polynomials are more general than the determinant derived in [34].

Random matrix ensembles whose generating functions can be cast into the form (6.2) have this non-trivial expression as a Pfaffian. The Hermitian Gaussian unitary ensemble as well as its generalization with other potentials fulfill a priori this requirement without the joint probability density having a squared Vandermonde determinant without the modulus. More generally, our derivation applies to each ensemble with a real spectrum, a squared Vandermonde determinant and a factorizing probability distribution, cf. equation (3.4). Also, the CUE (unitary group) can be cast into the form (6.2). More ensembles can be found in tables 1 and 2 of [34]. The Ginibre ensemble as well as its chiral counterpart is not in this class. Their joint probability density incorporates a modulus of the Vandermonde determinant and is, thus, in the class for the bi-orthogonal polynomials. Therefore, it is possible that their eigenvalue correlation functions cannot be expressed in Pfaffians like equations (6.3) and (6.4).

The skew-orthogonal polynomials corresponding to the Pfaffians (6.3) and (6.4) have the same relation to the orthogonal polynomials as chiral unitary ensembles, see equations (5.4)–(5.8). By construction this relation is so simple.

### 7. Remarks and conclusions

We derived a non-trivial Pfaffian determinant for the average over ratios of characteristic polynomials of a large class of random matrix ensembles with Dyson index $\beta = 2$. This structure is similar to the one for $\beta \in [1, 4]$, cf [35]. Hence, it is universal and unifies most of the symmetry classes known in the literature, particularly the Cartan classification [45, 46]. It is unclear how far beyond this classification [69] this structure is applicable. It is only known that there are some of them which share the identity (4.11). For example, the real and quaternion Ginibre ensembles as well as their chiral counterpart fulfill an identity similar to equation (4.11).
For many random matrix ensembles like the GUE it seems an academical question whether one can derive a Pfaffian or not since there are no applications, yet. However, for the chiral GUE it is important to know this due to the new results obtained for the Wilson–Dirac random matrix ensemble discussed in [36–41, 66]. Pfaffians were found there for finite lattice spacing. On the level of the joint probability density the authors of [39] checked that the ensemble is the chiral GUE as well as the GUE at certain values of the lattice spacing. However, for the eigenvalue correlation functions the continuum limit has not yielded the known determinant (3.29). With this work we clarified this puzzle.

For intermediate ensembles, in general, our result might be helpful to understand the structure appearing by switching the interaction between the two ensembles on. It is numerically advantageous to think about spectral correlations of intermediate ensembles as kernels of Pfaffians since the integrand drastically simplifies. In combination with the supersymmetry method [70–73] also the number of integrals reduces a lot.

Moreover, we derived the relation between the orthogonal polynomials and the skew-orthogonal polynomials corresponding to the determinants and the Pfaffians, respectively. This relation, see equations (5.4)–(5.8), is not only quite simple but also universal since it applies to all random matrix ensembles discussed in this work. The relation between the orthogonal and skew-orthogonal polynomials for \( \beta = 2 \) slightly differs to those found in [48] for the cases \( \beta = 1 \) and \( \beta = 4 \). The difference in the two-point weight is the reason for this. Based on the representations (3.16), (6.3) and (6.4) shared by all random matrix ensembles with \( \beta = 2 \) as well as checks of these representations [61], we conjecture that the recursion relation of the orthogonal polynomials connects the determinant and the Pfaffian and this has to be done in a general way.

The Pfaffian found for the average over characteristic polynomials carries over to the \( k \)-point correlation functions. This structure is valid in the large matrix limit, too. It should not depend on which scaling limit is chosen since the Pfaffian is independent of the matrix size. Hence, the correlation functions appearing as kernels of this Pfaffian have non-trivial recursion relations mapping the determinant to the Pfaffian.

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