Forward invariance and Wong–Zakai approximation for stochastic moving boundary problems

MARTIN KELLER-RESSEL and MARVIN S. MÜLLER

Abstract. We discuss a class of stochastic second-order PDEs in one space-dimension with an inner boundary moving according to a possibly nonlinear, Stefan-type condition. We show that proper separation of phases is attained, i.e., the solution remains negative on one side and positive on the other side of the moving interface, when started with the appropriate initial conditions. To extend results from deterministic settings to the stochastic case, we establish a Wong–Zakai-type approximation. After a coordinate transformation, the problems are reformulated and analyzed in terms of stochastic evolution equations on domains of fractional powers of linear operators.

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Introduction

Moving boundary problems allow for modeling of multi-phase systems with separating boundaries evolving in time. The classical model for the evolution of the temperature $v(t, x)$ at time $t$ and (one-dimensional) space position $x$ in a system of water and ice is the so-called Stefan problem [34] and reads as

$$
\begin{align*}
&\frac{dv(t, x)}{dt} = \eta_+ \frac{\partial^2}{\partial x^2} v(t, x) \ dt, \quad x > x_*(t), \\
&\frac{dv(t, x)}{dt} = \eta_- \frac{\partial^2}{\partial x^2} v(t, x) \ dt, \quad x < x_*(t), \\
&dx_*(t) = \rho \cdot \left( \frac{\partial}{\partial x} v(t, x_*(t)) - \frac{\partial}{\partial x} v(t, x_*(t) +) \right) \ dt, \\
v(t, x_*(t)) &= 0.
\end{align*}
$$

Here, $x_*$ describes the spatial position of the layer between water and ice and its derivative is proportional to the local imbalance of heat flux. Recently, stochastic and nonlinear extensions attracted attention in modeling of demand and supply in electronic financial markets, see for instance [3, 13, 25, 43] and references therein. In such a setup, $v$ describes the density of buy or sell orders at price level $x$ and $x_*$ is the market price which defines an inner interface, separating buy and sell sides of the density. The Stefan condition then tells that the price dynamics are driven by local imbalances of the density. We will study a class of stochastic semilinear moving boundary problems in two directions: On the one hand, we show a Wong–Zakai-type approximation result for stochastic moving boundary problems. This gives an understanding of the stochastic problems based on deterministic extensions of the Stefan problem, which have been widely studied in the second half of the 20th century. On the other hand, note that proper separation of the two phases is attained only if $v$ remains negative on one side, and positive on the other side of the moving interface $x_*$, i.e.,

$$
v(t, x) \geq 0, \quad \text{if} \quad x > x_*(t), \quad \text{and} \quad v(t, x) \leq 0, \quad \text{if} \quad x < x_*(t).
$$

In many applications such properties of the system are expected to be fulfilled, e.g., in temperature modeling or, under the convention that buy orders take negative and sell
orders positive signs in demand and supply modeling. Using the Wong–Zakai-type approximation, we show that under reasonable “inward-pointing drift” and “parallel-to-the-boundary-diffusion” conditions on the coefficients, separation of phases is indeed maintained for the solutions.

The analysis builds on a framework of stochastic evolution equations on domains of fractional power, as set up in [18,25] to prove existence and uniqueness for semilinear stochastic moving boundary problems.

More detailed, by a change of coordinates, these problems are linked with the so-called forward invariance of closed sets for stochastic evolution equations,

\[ dX_t = [AX_t + B(X_t)] \, dt + C(X_t) \, dW_t, \quad t \geq 0, \]

(0.1)

and their mild formulation

\[ X_t = X_0 + \int_0^t e^{(t-s)A} B(X_s) \, ds + \int_0^t e^{(t-s)A} C(X_s) \, dW_s, \quad t \geq 0, \]

(0.2)

on a separable Hilbert space \( E \). Here, \( W \) is a cylindrical Wiener process on another separable Hilbert space \( U \). The coefficients are \( A : \mathcal{D}(A) \to E \), a generator of a \( C_0 \) semigroup \( (e^{tA})_t \) on \( E \), \( B : \mathcal{D}(B) \to E \) and \( C : E \to L^2(U; E) \). A subset \( M \subset E \) is called forward invariant for (0.2), if for every local solution \((X, \tau)\) of (0.2) with initial data \( X_0 \in M \) it holds that \( X_t \in M \) on the stochastic interval \([0, \tau]\). A priori, we should assume that \( M \) is invariant under \((S_t) := (e^{tA})\), that is, \( S_tM \subset M \) for all \( t \geq 0 \).

A typical application is when (0.2) describes an SPDE on \( E = L^2 \) and \( M := L^2_+ \) is the closed convex cone of nonnegative functions in \( L^2 \). The main motivation in the literature for forward invariance in the framework of mild solutions seems to come from the question of positivity of solutions for HJM interest rate models; see for example [11,28,42].

Milian [23] used Yosida approximations to extend inward-pointing and parallel-to-the-boundary conditions from finite-dimensional equations to prove a comparison result for stochastic evolution equations, under Lipschitz conditions of \( B \) and \( C \) on \( E \). These results have been extended by Filipovic et al. [11], to show positivity for HJM equations provided that point-wise versions of inward-pointing and parallel-to-the-boundary conditions are satisfied.

Forward invariance for deterministic evolution equations (\( C = 0 \)) was extensively studied in the 1970s and 1980s. For mild and strong solutions of deterministic evolution equations, Pavel [29] and Jachimiak [16] have shown that under Lipschitz assumptions on \( B : E \to E \), forward invariance is equivalent to the Nagumo condition in the form

\[ \text{dist}_E (S_\epsilon u_0 + \epsilon B(u_0); \bar{M}) = o(\epsilon), \quad \text{as } \epsilon \searrow 0. \]

(0.3)

Zabczyk [42] extended this result to stochastic evolution equations with additive noise. For multiplicative noise, Nakayama [28] used a support theorem to extend the Nagumo condition (0.3) to stochastic evolution equations. Following this approach, we extend
the Wong–Zakai approximation theorem in [27] in two directions. On the one hand, to the situation when $-A$ is generator of an analytic semigroup of negative type, but $B: \mathcal{D}((-A)^\alpha) \to E$ for some $\alpha \in [0, 1)$ and $C: \mathcal{D}((-A)^\alpha) \to \mathcal{L}_2(U; \mathcal{D}((-A)^\alpha))$ and, on the other hand, that the coefficients need to be Lipschitz continuous only on bounded sets and the solution might explode in finite time. For an overview on Wong–Zakai approximations in infinite dimensions, see also [36,37] and references therein.

For the corresponding deterministic equations, general existence and invariance results are given in [1] for compact semigroups. Since we are interested in SPDEs on unbounded domains, we do not have compact semigroups and will make use of the very general result in [32] to show that the Nagumo condition is a sufficient criterion for the forward invariance of this class of stochastic evolution equation. Applying the results to a class of stochastic moving boundary problems, we derive sufficient point-wise criteria on the coefficients.

Notation

For a stopping time $\tau$, we denote the closed stochastic interval by $[0, \tau] := \{(t, \omega) \in [0, \infty) \times \Omega \mid t \leq \tau(\omega)\}$. Respectively, we define $[0, \tau[. 0, \tau[ and ]0, \tau]$. For stochastic processes $X$ and $Y$, we say $X(t) = Y(t)$ on $[0, \tau[$, if equality holds for almost all $\omega \in \Omega$ and all $t \geq 0$ such that $(t, \omega) \in [0, \tau[$. Given Hilbert spaces $E$ and $H$, we write $E \hookrightarrow H$ when $E$ is continuously and densely embedded into $H$. As usual, we denote by $L^q$ the Lebesgue space, $q \geq 1$, and with $H^s$, $s > 0$, the Sobolev spaces of order $s > 0$, for $k \in \mathbb{N}$, $C^k$ will be the space of $k$-times continuously differentiable functions, $C^k_b$ the subspace of $C^k$ where the elements and all derivatives up to order $k$ are bounded and $BUC^k$ will be the subspace of all elements which together with their derivatives up to order $k$ are bounded and uniformly continuous. Moreover, for separable Hilbert spaces $U$ and $E$, $\mathcal{L}(U, E)$ is the space of linear continuous operators from $U$ to $E$ and $\mathcal{L}_2(U; E)$ is the space of Hilbert–Schmidt operators from $U$ into $E$. The scalar product on $E$ will be denoted by $(.,.)_E$. We will work only with real separable Hilbert spaces and implicitly use their complexification when necessary to apply results from the literature.

1. Phase separation and approximation for SMBPs

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with the usual conditions on which a $\text{Id}_U$-cylindrical Wiener process $W$ taking values in the Hilbert space $U = L^2(\mathbb{R})$ lives. For a CONS $(e_k)$ of $U$ and a series $(\beta_k)$ of independent real Brownian motions, we can represent $W$ by the formal series

$$W_t = \sum_{k=1}^{\infty} e_k \beta_k(t), \quad t \in [0, T].$$
for a finite time horizon \( T > 0 \), where the series converges in \( L^2(\Omega; U_1) \). Here, \( U_1 \) is a separable Hilbert space such that there exists an Hilbert–Schmidt embedding from \( L^2(\mathbb{R}) \) into \( U_1 \), cf. Sect. 3 and, e.g., [6].

Let \( \zeta : \mathbb{R}^2 \to \mathbb{R} \) be a measurable integral kernel satisfying \( \zeta(x, \cdot) \in L^2(\mathbb{R}) \), for all \( x \in \mathbb{R} \). We denote by \( \xi \) the spatially colored noise which is defined for \( x \in \mathbb{R} \) as

\[
\xi_t(x) := T_\zeta W_t(x) := \sum_{k=1}^\infty \beta_k(t) T_\zeta \varepsilon_k(x), \quad T_\zeta w(x) := \int_\mathbb{R} \zeta(x, y) w(y) \, dy.
\]

By assumption, for each \( x \in \mathbb{R} \), the mapping \( w \mapsto T_\zeta w(x) \) is Hilbert–Schmidt from \( L^2(\mathbb{R}) \) into \( \mathbb{R} \) which yields that \( \xi_t(x), t \geq 0 \), exist as \( L^2(\Omega; \mathbb{R}) \) limits and define for each \( x \in \mathbb{R} \) a real-valued Brownian motion.

In that setting, we consider the following class of stochastic 2-phase systems in one space dimension,

\[
\begin{cases}
\mathrm{dv}(t, x) = \left[ \eta_+ \varfrac{\partial^2}{\partial x^2} v + \mu_+ (x - x_*(t), v, \varfrac{\partial}{\partial x} v) \right] \, dt \\
+ \sigma_+ (x - x_*(t), v) \, d\xi_t(x), \quad x > x_*(t), \\
\mathrm{dv}(t, x) = \left[ \eta_- \varfrac{\partial^2}{\partial x^2} v - \mu_- (x - x_*(t), v, \varfrac{\partial}{\partial x} v) \right] \, dt \\
- \sigma_- (x - x_*(t), v) \, d\xi_t(x), \quad x < x_*(t), \\
v(0, x) = v_0(x),
\end{cases}
\]

with inner boundary conditions

\[
\begin{cases}
\varfrac{\partial}{\partial x} v(t, x_*(t)+) = \kappa_+ v(t, x_*(t)+), \\
\varfrac{\partial}{\partial x} v(t, x_*(t)-) = -\kappa_- v(t, x_*(t)-),
\end{cases}
\]

for \( \kappa_+, \kappa_- \in (0, \infty) \) or \( \kappa_+ = \kappa_- = \infty \) and dynamics of the interface \( x_* \) governed by

\[
\begin{cases}
\varfrac{\partial}{\partial t} x_*(t) = \varphi \left( \varfrac{\partial}{\partial x} v(t, x_*(t)+), \varfrac{\partial}{\partial x} v(t, x_*(t)-) \right), \\
x_*(0) = x_0,
\end{cases}
\]

when \( \kappa_+ = \kappa_- = \infty \) or, else,

\[
\begin{cases}
\varfrac{\partial}{\partial t} x_*(t) = \varphi \left( v(t, x_*(t)+), v(t, x_*(t)-) \right), \\
x_*(0) = x_0,
\end{cases}
\]

where \( t \in [0, T], T \in (0, \infty), \) and \( \mu_+, \mu_- : \mathbb{R}^3 \to \mathbb{R}, \sigma_+, \sigma_- : \mathbb{R}^2 \to \mathbb{R}, \varphi : \mathbb{R}^2 \to \mathbb{R}, \) and \( \eta_+, \eta_- > 0. \)

While \( v \) follows the dynamics of the respective second-order SPDE inside of the respective phase, the phase transition at the inner interface \( x_* \) comes with Dirichlet or Robin boundary conditions. The dynamics of \( x_* \) are then driven by local imbalances of the system.
Here, the case where \( \kappa_+ = \kappa_- = \infty \) is interpreted as imposing Dirichlet boundary conditions on \( v \) which means we ask for continuity at \( x_* \). For this case, existence and uniqueness of solutions in an analytically strong framework have been shown in [18]. Neumann or Robin boundary conditions, corresponding to \( \kappa_+ < \infty \), appeared recently in finance where \( v \) represents demand and supply which is allowed to be discontinuous at the price level \( x_* \). In this case, changes in \( x_* \) depend on the generally different demand and supply levels \( v(t, x_*(t)+) \) and \( v(t, x_*(t)-) \). This case was investigated in [25] from application and mathematical point of view. We will refer to these as “first-order” boundary conditions. In both cases, under sufficient assumptions on the coefficients and initial data, there exists a solution, in the sense that there exists a maximal predictable strictly positive stopping time \( \tau \leq T \), an \( L^2(\mathbb{R}) \otimes \mathbb{R} \) predictable stochastic process \((v, x_*)\) taking values in

\[
\bigcup_{x \in \mathbb{R}} (\Gamma(x) \times \{x\}),
\]

where

\[
\Gamma(x) := \{ v \in H^2(\mathbb{R} \setminus \{x\}) \mid \frac{\partial}{\partial x} v(x) = \kappa_+ v(x), \quad \frac{\partial}{\partial x} v(x) = -\kappa_- v(x) \}.
\]

and a such that for all \( \phi \in C^\infty_0(\mathbb{R}) \), on \([0, \tau]\),

\[
\langle v(t, .) - v_0, \phi \rangle = \int_0^t \langle \bar{\mu}(., v(s, .), \nabla v(s, .), \Delta v(s, .), x_*(s)), \phi \rangle \, ds
\]

\[
+ \int_0^t \langle \bar{\sigma}(., v(s, .)), \phi \rangle \, d\xi_s
\]

\[
+ \int_0^t (v(s, x_*(s)-) - v(s, x_*(s)+)) \phi(x_*(s)) \, dx_*(s), \quad (1.6)
\]

and either (1.4) or (1.5) being satisfied. Moreover, uniqueness holds true under sufficient regularity constraints on the solution, cf. [18, Theorem 2.11], [25, Theorem 1.15].

This describes a two-phase system with phase change at the moving interface \( x_* \), which itself is driven by local imbalances of the system between both phases. In applications as the two-phase Stefan problem or modeling of limit order books or demand and supply in financial mathematics, cf. [25], one often expects that a proper separation of both phases is preserved, i.e., that \( dx \)-a.e.

\[
v(t, x) \leq 0, \quad x < x_*(t) \quad \text{and} \quad v(t, x) \geq 0, \quad x > x_*(t), \quad (1.7)
\]

holds on \([0, \tau]\), provided that it holds true for \( t = 0 \).

In relative coordinates, namely

\[
u_1(t, x) := v(t, x_*(t) + x), \quad \text{and} \quad u_2(t, x) := v(t, x_*(t) - x), \quad x > 0, \quad (1.8)
\]
the moving boundary problem becomes the coupled system of stochastic equations on $\mathbb{R}_+$,

$$
du_1(t, x) = \left[ \eta_+ \frac{\partial^2}{\partial x^2} u_1 + \mu_+ \left( x, u_1, \frac{\partial}{\partial x} u_1 \right) + \frac{\partial}{\partial \tau} x_*(t) \frac{\partial}{\partial x} u_1(t, x) \right] dt, \\
+ \sigma_+ \left( x, u_1 \right) d\xi_t(x_*(t) + x), \\
du_2(t, x) = \left[ \eta_- \frac{\partial^2}{\partial x^2} u + \mu_- \left( -x, u_2, -\frac{\partial}{\partial x} u_2 \right) - \frac{\partial}{\partial \tau} x_*(t) \frac{\partial}{\partial x} u_2(t, x) \right] dt, \\
u_1(0, x) = u_{1,0}(x), \quad u_2(0, x) = u_{2,0}(x), \quad x_*(0) = x_0,
$$

with boundary conditions at 0, for $t \in (0, T]$,

$$
u_1(t, 0) = \kappa_+ \frac{\partial}{\partial x} u_1(t, 0), \quad \nu_2(t, 0) = \kappa_- \frac{\partial}{\partial x} u_2(t, 0).
$$

(1.10)

Here,

$$
u_{1,0}(x) = v_0(x_0 + x), \quad \nu_{2,0}(x_0 - x).
$$

(1.11)

The interface conditions becomes

$$
\frac{\partial}{\partial \tau} x_*(t) = \varrho \left( \frac{\partial}{\partial x} u_1(t, 0+), \frac{\partial}{\partial x} u_2(t, 0+) \right),
$$

(1.12)

and else, when $\kappa_+, \kappa_- < \infty$,

$$
\frac{\partial}{\partial \tau} x_*(t) = \varrho \left( u_1(t, 0+), u_2(t, 0+) \right).
$$

(1.13)

Let us shortly summarize the existence results for the centered equations which we derived in [18, 25], respectively. In both cases, there exists a unique maximal strong solution $(u_1, u_2, x_*)$, up to a predictable stopping time $\tau_*$, such that (1.9) is satisfied in the sense of $L^2(\mathbb{R}_+)$ integral equations and the boundary conditions (1.10) and either (1.12) or (1.13) hold true $dt \otimes \mathbb{P}$ almost everywhere, on $[0, \tau_]$. For Dirichlet boundary conditions, $u_1, u_2$ take values in $C([0, \tau); H^2 \cap H^1_0(\mathbb{R}_+))$ provided that $u_{1,0}, u_{2,0} \in H^2 \cap H^1_0(\mathbb{R}_+)$. For first-order boundary conditions and when $u_{1,0}, u_{2,0} \in H^1(\mathbb{R}_+)$, we get a unique strong solution with almost surely $u_1, u_2 \in C([0, \tau); H^1(\mathbb{R}_+) \cap L^2([0, \tau); H^2(\mathbb{R}_+))$, and $u_1, u_2$ fulfill (1.10). Moreover, note that the moving boundary problem (1.2) can be characterized completely by the centered equations (1.9).

Translated into the notion of the centered equations (1.9), the condition for phase separation (1.7) becomes

$$
u_1(t, x) \geq 0, \quad \text{and} \quad \nu_2(t, x) \leq 0, \quad \text{for almost all} \ x \in \mathbb{R}_+.
$$

(1.14)

A well-known criterion also from theory of finite-dimensional equations are the so-called inward-pointing-drift and parallel-to-the-boundary-diffusion conditions. Formulated point-wise, they read as follows.
**Assumption (PIP).** For all $x \geq 0$, it holds that
\[
\mu_+(x, 0, 0) \geq 0, \quad \mu_-(x, 0, 0) \leq 0, \quad \text{and} \quad \sigma_+(x, 0) = \sigma_-(x, 0) = 0.
\]

To formulate the result for (1.9), we introduce the following closed convex cone,
\[
\mathcal{M} := \{(u_1, u_2, x) \in L^2 \times L^2 \times \mathbb{R} | u_1 \geq 0, u_2 \leq 0, \, \text{dx-a.e.}\}.
\]

Phase separation in the sense of (1.14) is now equivalent to so-called forward invariance of $\mathcal{M}$, supposed that the coefficients and initial data are sufficiently regular, see assumptions below.

**Theorem 1.1.** (Forward Invariance) Assume that Assumption (PIP) and one of the following hold true.

(a) $\kappa_+, \kappa_- < \infty$ and Assumptions (Interface), (Correlation), (Drift$_0$), (Noise$_1$) and (Initial$_1$), or;

(b) Dirichlet boundary conditions at 0 and Assumptions (Interface), (Correlation), (Drift$_0$), (Noise$_0$) and (Initial$_0$)

Then, the set $\mathcal{M}$ is forward invariant for (1.9) in the sense that $(u_{0,1}, u_{0,2}, x_0) \in \mathcal{M}$ yields that $(u_1(t, .), u_2(t, .), x_s(t)) \in \mathcal{M}$ on $[0, \tau]$.

**Corollary 1.2.** (Phase separation) Let the assumptions of Theorem 1.1 hold true and assume that $\text{dx}$-almost everywhere,
\[
v_0(x) \leq 0, \quad x < x_0, \quad v_0(x) \geq 0, \quad x > x_0.
\]

Then, on $[0, \tau]$,
\[
v_t(x) \leq 0, \quad x < x_s(t), \quad v_t(x) \geq 0, \quad x > x_s(t).
\]

We now list the assumptions on the coefficients, starting with the interface dynamics and the correlation structure of $\xi$.

**Assumption (Interface).** $\varrho : \mathbb{R}^2 \to \mathbb{R}$ is locally Lipschitz continuous. More precisely, for all $N \in \mathbb{N}$ there exists an $L_{\varrho, N}$ such that for all $y, \tilde{y} \in \mathbb{R}^2$ with $|y|, |\tilde{y}| \leq N$ holds
\[
|\varrho(y) - \varrho(\tilde{y})| \leq L_{\varrho, N} |y - \tilde{y}|.
\]

**Assumption (Correlation).** $\zeta(., y) \in C^4(\mathbb{R})$ for all $y \in \mathbb{R}$ and $\frac{\partial^i}{\partial x^i} \zeta(x, .) \in L^2(\mathbb{R})$ for all $x \in \mathbb{R}$, $i \in \{0, 1, \ldots, 4\}$. Moreover,
\[
\sup_{x \in \mathbb{R}} \left\| \frac{\partial^i}{\partial x^i} \zeta(x, .) \right\|_{L^2(\mathbb{R})} < \infty, \quad i = 0, 1, \ldots, 4.
\]

For the remainder of this paper, we use the notation $\zeta^{(i)} := \frac{\partial^i}{\partial x^i} \zeta$.

**Remark 1.3.** When $\kappa_+, \kappa_- < \infty$, it suffices to assume that (Correlation) holds for $i \in \{0, 1, 2, 3\}$.

**Example 1.4.** (Convolution) Let $\xi$ be a convolution kernel, i.e., $\zeta(x, y) := \zeta(x - y)$, $x, y \in \mathbb{R}$. If $\xi \in C^4(\mathbb{R}) \cap H^4(\mathbb{R})$, then Assumption (Correlation) is satisfied. In this case, one can write $T_\xi = \xi * (.)$. 

1.1. Dirichlet boundary conditions

In comparison with the assumptions one requires to obtain existence from [18], we can keep the assumption on the initial data and the drift coefficient but need additional regularity for the noise coefficient.

Assumption. (Drift\textsubscript{0}) For $\mu := \mu_+$, resp. $\mu := \mu_-$, it holds that $\mu \in C^1(\mathbb{R} \times \mathbb{R}^2; \mathbb{R})$, and

(i) there exist $a \in L^2(\mathbb{R})$, $b, \tilde{b} \in L^\infty_{loc}(\mathbb{R}^2; \mathbb{R})$ such that for all $x \in \mathbb{R}$, $y, z \in \mathbb{R}$

$$|\mu(x, y, z)|, \quad \left| \frac{\partial}{\partial x} \mu(x, y, z) \right| \leq b(y, z) (a(x) + |y| + |z|),$$

and

$$\left| \frac{\partial}{\partial y} \mu(x, y, z) \right|, \quad \left| \frac{\partial}{\partial z} \mu(x, y, z) \right| \leq \tilde{b}(y, z),$$

(ii) $\mu$ and its partial derivatives are locally Lipschitz continuous, and the local Lipschitz constants are uniformly bounded in $x \in \mathbb{R}$.

Assumption. (Noise\textsubscript{0}) For $\sigma := \sigma_+$, resp. $\sigma := \sigma_-$, it holds that $\sigma \in C^4(\mathbb{R} \times \mathbb{R}; \mathbb{R})$, and

(i) there exist $a_{i,j} \in L^2(\mathbb{R})$ and $b_{i,j} \in L^\infty_{loc}(\mathbb{R}, \mathbb{R}_+), i, j \in \mathbb{N}_0, i + j \leq 4$ such that

$$\left| \frac{\partial^{i+j}}{\partial x^i \partial y^j} \sigma(x, y) \right| \leq \begin{cases} b_{i,0}(y)(a_{i,0}(x) + |y|), & j = 0, \\ b_{i,j}(y), & j \neq 0. \end{cases}$$

(ii) $\sigma$ and its partial derivatives are locally Lipschitz with Lipschitz constants independent of $x \in \mathbb{R}$.

(iii) $\sigma$ fulfills the boundary conditions

$$\sigma(0, 0) = 0. \quad (1.18)$$

Assumption. (Initial\textsubscript{0}) Assume that $x_0 \in \mathbb{R}$ and $v_0 \in \Gamma(x_0)$, i.e., $u_{1,0}, u_{2,0} \in H^2(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+)$.

1.2. Neumann and Robin boundary conditions

For first-order boundary conditions, we omit the $\frac{\partial}{\partial x}$-terms in the dynamics of $x_\ast$, see (1.13), and so it suffices to work on $H^1$ instead of $H^2$. Consequently, we can relax the spatial regularity assumptions on the coefficients compared with the situation of Dirichlet boundary problems.

Assumption. (Drift\textsubscript{1}) For $\mu \in \{\mu_+, \mu_\}$, it holds that $\mu : \mathbb{R}^3 \to \mathbb{R}$ and

(i) there exist $a \in L^2(\mathbb{R})$, $b \in L^\infty_{loc}(\mathbb{R}; \mathbb{R})$ such that for all $x \in \mathbb{R}$, $y, z \in \mathbb{R}$

$$|\mu(x, y, z)| \leq b(y) (a(x) + |y| + |z|).$$
(ii) For all \( x \in \mathbb{R}, \mu(x, ..) \) is locally Lipschitz continuous and the local Lipschitz constants are uniformly bounded in \( x \in \mathbb{R} \).

**Assumption.** (Noise1) For \( \sigma := \sigma_+, \text{resp. } \sigma := \sigma_- (-..) \) holds \( \sigma \in C^3(\mathbb{R}_{\geq 0} \times \mathbb{R}; \mathbb{R}) \), and

(i) there exist \( a_{i,j} \in L^2(\mathbb{R}) \) and \( b_{i,j} \in L^\infty_{loc}(\mathbb{R}, \mathbb{R}^+), i, j \in \mathbb{N}_0, i + j \leq 3 \) such that

\[
\frac{\partial^{i+j}}{\partial x^i \partial y^j} \sigma(x, y) \leq \begin{cases} 
\beta_{i,j}(y)(a_{i,j}(x) + |y|), & j = 0, \\
\beta_{i,j}(y), & j \neq 0.
\end{cases}
\]

(ii) \( \sigma \) and its partial derivatives are locally Lipschitz with Lipschitz constants independent of \( x \in \mathbb{R} \).

**Assumption.** (Initial1) Assume that \( x_0 \in \mathbb{R} \) and \( v_0 \in H^1(\mathbb{R} \setminus \{x_0\}) \), i.e., \( u_{1,0}, u_{2,0} \in H^1(\mathbb{R}^+) \).

### 1.3. Wong–Zakai approximations

As a first step, we will study an approximation technique reducing the forward invariance question to deterministic equations. To this end, we fix a finite time horizon \( T > 0 \) and denote by \( P_m \) the partitions

\[
P_m := \left\{ \left[ \frac{k}{m} T, \frac{k+1}{m} T \right] \mid k = 0, \ldots, m - 1 \right\}.
\]

To map onto the time grid, we use the notation \( [t]_m := \frac{k}{m} T \), where \( k \in \{0, \ldots, m\} \) is chosen such that \( t \in \left[ \frac{k}{m} T, \frac{k+1}{m} T \right) \). In order to approximate \( \xi_t(x) = \sum_{k=1}^\infty T \zeta(t) \beta_k(t) \), we interpolate the Brownian motions linearly,

\[
\beta_k^m(t) := \beta_k([t]_m) + (t - [t]_m)(\beta_k([t + \frac{1}{m} T]_m) - \beta_k([t]_m)), \quad t < T,
\]

and \( \beta_k^m(T) := \beta_k(T) \). Then, we consider \( \omega \)-wise the partial differential equations,

\[
\begin{aligned}
\frac{\partial}{\partial t} w_{1}^{m,n}(t, x) &= \eta \frac{\partial^2}{\partial x^2} w_{1}^{m,n} + \mu_+(x, w_{1}^{m,n}, \frac{\partial}{\partial x} w_{1}^{m,n}) + \frac{\partial}{\partial t} \gamma x_+^m(t) \frac{\partial}{\partial x} w_{1}^{m,n}(t, x) \\
&\quad - \frac{1}{2} \sigma_+(x, w_{1}^{m,n}) \frac{\partial}{\partial y} \sigma_+(x, w_{1}^{m,n}) \| \zeta(x, w_{1}^{m,n} + \frac{1}{m} T) \|_L^2, \\
&\quad + \sigma_+(x, w_{1}^{m,n}(t, x)) \sum_{k=1}^n T \zeta(t) \beta_k(t) \beta_k^m([t]_m),
\end{aligned}
\]

\[
\begin{aligned}
\frac{\partial}{\partial t} w_{2}^{m,n}(t, x) &= \eta \frac{\partial^2}{\partial x^2} w_{2}^{m,n} + \mu_-(-x, w_{2}^{m,n}, -\frac{\partial}{\partial x} w_{2}^{m,n}) - \frac{\partial}{\partial t} \gamma x_-^m(t) \frac{\partial}{\partial x} w_{2}^{m,n}(t, x) \\
&\quad - \frac{1}{2} \sigma_-(-x, w_{1}^{m,n}) \frac{\partial}{\partial y} \sigma_-(x, w_{1}^{m,n}) \| \zeta(x, w_{1}^{m,n} + \frac{1}{m} T) \|_L^2, \\
&\quad + \sigma_-^k(-x, w_{2}^{m,n}(t, x)) \sum_{k=1}^n T \zeta(t) \beta_k(t) \beta_k^m([t]_m),
\end{aligned}
\]

\[
w_{1}^{m,n}(0, x) = u_{1,0}(x), \quad w_{2}^{m,n}(0, x) = u_{2,0}(x),
\]

\[\text{(1.19)}\]
For $t \geq 0, x \in \mathbb{R}_+$, with interface condition $x_+(0) = x_0$ and either
\[
\frac{\partial}{\partial t} x_{m,n}(t) = \varrho \left( w_1^{m,n}(t, 0), \frac{\partial}{\partial x} w_1^{m,n}(t, 0), w_2^{m,n}(t, 0), \frac{\partial}{\partial x} w_2^{m,n}(t, 0) \right),
\] (1.22)
for the case $\kappa_+ = \kappa_- = \infty$ or, else,
\[
\frac{\partial}{\partial t} x_{m,n}(t) = \varrho \left( w_1^{m,n}(t, 0), \frac{\partial}{\partial x} w_1^{m,n}(t, 0), w_2^{m,n}(t, 0), \frac{\partial}{\partial x} w_2^{m,n}(t, 0) \right),
\] (1.23)
and with boundary conditions, for $t \in (0, T]$,
\[
\frac{\partial}{\partial x} w_1^{m,n}(t, 0) = \kappa_+ w_1^{m,n}(t, 0), \quad \frac{\partial}{\partial x} w_2^{m,n}(t, 0) = \kappa_- w_2^{m,n}(t, 0).
\] (1.24)

Since explosion of the solutions might happen in finite time, let us introduce the exit times,
\[
\tau_k^{(r)} := \inf \{ t \geq 0 | t < \tau, \| u_1(t) \|_{H^k(\mathbb{R}_+)} + \| u_2(t) \|_{H^k(\mathbb{R}_+)} + | x_+(t) | > r \},
\] (1.25)
for $r > 0, k \in \mathbb{N}$. The appearance of $\frac{\partial}{\partial x} \sigma$ in the dynamics of (1.21) indicates already why we need to assume existence of higher-order derivatives in Assumption (Noise0) and (Noise1), compared to the assumption for the existence results in [18,25].

**Theorem 1.5.** (Approximation 1) Assume that $\kappa_+ = \kappa_- = \infty$ and that Assumptions (Interface), (Correlation), (Drift0) and (Noise0) are fulfilled and denote by $(u_1, u_2, x_+)$ the unique solution of (1.9) with (1.12) on the maximal interval $[0, \tau]$. Respectively, denote by $(w_1^{m,n}, w_2^{m,n}, x_{m,n}^*)$ the unique solutions of (1.21) with (1.22) and Dirichlet conditions at $0 \pm$. Then, it holds for $i \in \{1, 2\}, r > 0$, that
\[
\lim_{n \to \infty} \lim_{m \to \infty} w_i^{m,n} = u_i,
\]
and
\[
\lim_{n \to \infty} \lim_{m \to \infty} x_{m,n}^* = x_*,
\]
with uniform convergence on $[0, \tau^{(r)}_2]$ in $L^2 p(\Omega; H^2 \oplus H^2 \oplus \mathbb{R})$.

**Theorem 1.6.** (Approximation 2) Assume that $\kappa_+, \kappa_- < \infty$ and that Assumptions (Interface), (Correlation), (Drift1) and (Noise1) are fulfilled and denote by $(u_1, u_2, x_+)$ the unique solution of (1.9) with (1.13) on the maximal interval $[0, \tau]$. Respectively, denote by $(w_1^{m,n}, w_2^{m,n}, x_{m,n}^*)$ the unique solutions of (1.21) with (1.23) and first-order boundary conditions at $0 \pm$. Then, it holds for $i \in \{1, 2\}, r > 0$,
\[
\lim_{n \to \infty} \lim_{m \to \infty} w_i^{m,n} = u_i,
\]
and
\[
\lim_{n \to \infty} \lim_{m \to \infty} x_{m,n}^* = x_*,
\]
with uniform convergence on $[0, \tau^{(r)}_1]$ in $L^2 p(\Omega; H^1 \oplus H^1 \oplus \mathbb{R})$.

A more clean and precise formulation of the convergence in terms of the corresponding evolution equations is provided in (4.10) and (4.11), respectively.
1.4. Outline of the proof

To prove the results, we will consider the equations as stochastic evolution equations and proceed as follows:

– In Sect. 2, we will first recall concepts of a class of interpolation spaces from analysis, which will be used for the analytic setups for the equations. Then, we discuss forward invariance and viability results for the deterministic evolution equations.

– In Sect. 3, we switch to the abstract framework of stochastic evolution equations and consider approximations of Wong–Zakai type. This will provide the basis for extensions of properties of deterministic equations.

– In Sect. 4, we reformulate the centered equations as (stochastic) evolution equations. We show that the assumptions stated above are sufficient to apply the results from the abstract setting and finish the proofs of Theorem 1.1, 1.5 and 1.6. The convergence statements from the latter two statements are stated explicitly in (4.10) and (4.11).

– Some results on Fréchet differentiability of Nemytskii operators and of the noise coefficients appearing here are delayed to Appendix A and Appendix B, respectively.

2. Semigroups and forward invariance

In this section, we first discuss some preliminary results on analytic semigroups on a Banach space $E$ and the domains of fractional powers of their generators. In the second part of this section, we then focus on evolution equations and the question of forward invariance in the deterministic setup. Whenever the space $E$ is real and it becomes necessary to consider a complex Banach space, we will implicitly work with its complexification.

2.1. Analytic semigroups

**Assumption 2.1.** $A$ is a densely defined and sectorial operator with domain $\mathcal{D}(A) \subset E$. Moreover, the resolvent set of $A$ contains $[0, \infty)$ and there exists an $M > 0$ such that the resolvent $R(\lambda, A)$ satisfies

$$\|R(\lambda, A)\| \leq \frac{M}{1 + \lambda}, \quad \text{for all } \lambda > 0. \quad (2.1)$$

**Remark 2.2.** This assumption is equivalent to each of the following statements

– Equation (2.1) holds and the resolvent set of $A$ contains $0$ and a sector

$$\{\lambda \in \mathbb{C} : |\arg \lambda| < \theta\}$$

for some $\theta \in (\pi/2, \pi)$,
The operator $A$ is sectorial and $-A$ is positive in the sense of [22],

$A$ is the generator of an analytic $C_0$-semigroup $(S_t)_{t \geq 0}$ of negative type.

In particular, there exist $\delta, M > 0$ such that $\|S_t\| \leq Me^{-\delta t}$. Assumption 2.1 also ensures that fractional powers of $-A$ are well defined.

**Notation.** For $\alpha \geq 0$ we write

$$E_\alpha := \mathcal{D}((-A)^\alpha), \quad \|h\|_\alpha := \|(-A)^\alpha h\|_E, \; h \in E_\alpha.$$  

(2.2)

It is known that also $E_\alpha$ with the induced scalar product is a separable Hilbert space.

In particular, $\|\cdot\|_1$ is equivalent to the graph norm of $A$ and the following continuous embedding relations hold for $\alpha \in [0, 1]$: \[ D(A) = E_1 \hookrightarrow E_\alpha \hookrightarrow E_0 = E, \] (2.3)

Note that the restriction of $A$ to any $E_\alpha$, $\alpha \in [0, 1]$ is again a densely defined and closed operator on $E_\alpha$. Moreover, it is the infinitesimal generator of the restriction of $S_t$ to $E_\alpha$, which is again an analytic (contraction) semigroup; see, e.g., [9, Ch. II.5]. We in particular have the following property.

**Proposition 2.3.** For $\alpha, \beta \in \mathbb{R}$, and $u \in \mathcal{D}((-A)^{\min\{\alpha, \beta, \alpha+\beta\}})$, it holds that

$$(-A)^\alpha((-A)^\beta u) = (-A)^\beta((-A)^\alpha u) = (-A)^{\alpha+\beta}u.$$

**Remark 2.4.** The part of $A$ in $E_\alpha, \alpha > 0$, is again a densely defined and closed operator on $E_\alpha$. Moreover, it is the infinitesimal generator of the restriction of $S_t$ to $E_\alpha$, which is again an analytic and strongly continuous semigroup. The same holds true on the spaces $D(\alpha, p)$ and $D(\alpha)$, with $\alpha \in (0, 1)$, $p < \infty$ and for the extension of $S_t$ to $E_\alpha$, when $\alpha < 0$; see, e.g., [9, Ch. II.5].

The following regularity property of $S_t$ between different interpolation spaces $E_\alpha$, $\alpha \in [0, 1]$ will be crucial in the proofs that follow. We derive it from results in [22] on interpolation spaces.

**Lemma 2.5.** Let $\alpha \in [0, 1], \beta \in [0, \alpha)$. Then, for all $t > 0$ and $h \in E_\beta$,

$$\|S_t h\|_\alpha \leq K_{\alpha, \beta} t^{\beta-\alpha} e^{-\delta t} \|h\|_\beta.$$  

Note that the factor in front of $\|h\|_\beta$ is integrable at time $t = 0$, which is the key property used in the following sections. On the other hand, to deal with the singularity in 0, we will use an extended version of Gronwall’s lemma, see [21, Lem 7.0.3] or, for a proof, [14, p. 188].

**Lemma 2.6.** (Extended Gronwall’s lemma) Let $\alpha > 0$, $a, b \geq 0$, $T \geq 0$, and $u : [0, T] \rightarrow \mathbb{R}$ be nonnegative and integrable. If, for all $t \in [0, T]$,

$$u(t) \leq a + b \int_0^t u(s)(t-s)^{\alpha-1} \, ds,$$  

(2.4)

then exists a constant $K_{\alpha, b, T}$, depending only on $\alpha, b$ and $T$, such that,

$$u(t) \leq aK_{\alpha, b, T}, \quad t \in [0, T].$$  

(2.5)
2.2. Evolution equations and forward invariance

We now assume that $E$ is real and separable, and consider the semilinear (deterministic) evolution equation

$$\frac{\partial}{\partial t} u(t) = A u(t) + B(u(t)), \quad t \in [0, T], \quad u(0) = u_0. \quad (2.6)$$

where $A : \mathcal{D}(A) \subset E \to E$ is a linear operator on $E$ and $B : E \to E$ is Borel measurable. We want to discuss conditions under which a closed set $M \subset E$ is forward invariant for (2.6), i.e., $u_0 \in M$ yields that the (local) solution $u$ takes values in $M$.

Now, assume that $u : [0, T] \to E$ is differentiable at 0 with $\dot{u}(0) = g$ and $u(0) = u_0$, i.e.,

$$u(t) = u_0 + tg + o(t), \quad t > 0.$$  

Hence, a necessary condition to get that there exists an $\epsilon > 0$ such that $u(t) \in M$ for all $t \in [0, \epsilon)$, is given by,

$$\text{dist}_E(u_0 + tg; M) = o(t). \quad (2.7)$$

Conversely, when $\dot{u} = F \circ u$ for a function $F : E \to E$ the condition

$$\forall u_0 \in M : \lim_{t \searrow 0} \frac{1}{t} \text{dist}_E(u_0 + tF(u_0); M) = 0, \quad (2.8)$$

is called Nagumo- or tangency-condition and is well known to be also sufficient in many cases. Moreover, if $M$ is closed convex and replacing lim by lim inf then (2.8) becomes equivalent to the condition that for all $\phi \in E^*$ such that $\phi(h) = \inf_{f \in M} \phi(f)$, it holds that $\phi(g) \geq 0$, see [7, Lemma 4.1]. For a detailed discussion also in connection to the geometry behind the Nagumo condition (2.8), we refer to [30].

We go back to the theory of evolution equations, where

$$F(u) := Au + B(u).$$

We additionally know that $A$ generates a strongly continuous semigroup $(S_t)$ in the applications we are interested in. From [30, Section 4.1], we extract the following.

**Proposition 2.7.** Let $A$ be the generator of a strongly continuous semigroup $(S_t)$ on a Banach space $E$. For a closed subset $M \subset E$, $u_0 \in M \cap \mathcal{D}(A)$ and an element $v \in E$, the following tangential conditions are equivalent,

(i) $\lim_{t \searrow 0} \frac{1}{t} \text{dist}_E(u_0 + t(Au_0 + v); M) = 0,$

(ii) $\lim_{t \searrow 0} \frac{1}{t} \text{dist}_E(S_t u_0 + tv; M) = 0.$

Moreover, the so-called tangential points, which are the points for which (i) or (ii) are satisfied, can be identified in the following way.
Proposition 2.8. ([30, Prop 4.1.4]) Let $A$ be the generator of a strongly continuous semigroup $(S_t)$ on a Banach space $E$ and let $\mathcal{M} \subset E$ be closed. Moreover, assume that $S_t(\mathcal{M}) \subset \mathcal{M}$. If $v \in E$ fulfills
$$\lim_{t \searrow 0} \frac{1}{t} \operatorname{dist}_E(u_0 + tv; \mathcal{M}) = 0, \quad \forall u_0 \in \mathcal{M},$$
for some $u_0 \in \mathcal{M}$, then $v$ also satisfies the second statement in Proposition 2.7.

This allows to separate the tangential conditions for $A$ and $B$. In fact, if $B$ satisfies (2.8) for $u_0 \in \mathcal{M}$ and $S_t \mathcal{M} \subset \mathcal{M}$, then it also holds that
$$\lim_{t \searrow 0} \frac{1}{t} \operatorname{dist}_E(S_t u_0 + tB(u_0); \mathcal{M}) = 0.$$  

The converse direction does not hold, in general. However, in some special situation as when $B : E \to E$ is Lipschitz continuous, the latter conditions is known to be necessary and sufficient for forward invariance for evolution equations. We refer to [30, Chapter 4] for a detailed discussion and proofs.

The equations discussed in the previous sections are beyond the scope of these results. As in the previous sections, $B$ will be only continuous on a certain subspace of $E$.

Recall from Sect. 2.1 that when $A$ is the generator of an analytic $C_0$-semigroup of negative type, then we have defined the inter- and extrapolation spaces $E_\alpha$, $\alpha \in \mathbb{R}$ and for $\alpha \in (0, 1)$ we have
$$\mathcal{D}(A) = E_1 \hookrightarrow E_\alpha \hookrightarrow E_0 = E,$$
Thus, Kuratowski’s theorem yields that $E_\alpha$ is a Borel subset of $E$ for $\alpha > 0$. In the sequel, when $\mathcal{M} \subset E$ we will use the notation $\mathcal{M}_\alpha := \mathcal{M} \cap E_\alpha$, $\alpha \geq 0$.

Assumption 2.9. (A) $A$ is the generator of an analytic $C_0$-semigroup of negative type, denoted by $(S_t)$,
(B) $B : \mathcal{M}_\alpha \to E$ is Lipschitz continuous, for some $\alpha < 1$,
(M) $\mathcal{M} \subset E$ is closed,
(N) Assume that $S_t(\mathcal{M}) \subset \mathcal{M}$ and the so-called Nagumo condition is satisfied, that is
$$\lim_{\epsilon \searrow 0} \frac{1}{\epsilon} \operatorname{dist}_E(u + \epsilon B(t, u); \mathcal{M}) = 0,$$
for all $u \in \mathcal{M}_\alpha$ and $t \in [0, T]$.

Note that also $\mathcal{M}_\alpha$ is closed as a subset of $E_\alpha$ under Assumption 2.9.

An established way to prove existence and forward invariance results in such a setting, but under weaker constraints on $B$, is the concept of $\epsilon$-approximate solutions, see [30] or [32] for instance. In order to construct these approximations and the solution, one often uses compactness of the semigroup $(S_t)$, which we will not have in the situation of Sect. 1. Instead, we use a result of Pruess [32] in this direction relying estimates of the non-compactness of $B$. 

Lemma 2.10. Assume that $B : \mathbb{M}_\alpha \to E$ is Lipschitz continuous with Lipschitz constant $L > 0$, then for all Borel sets $G \subset \mathbb{M}_\alpha$ it holds that
\[\nu_E(B(G)) \leq 2L\nu_{E_\alpha}(G),\]
where $\nu$ is the Hausdorff measure of non-compactness defined for $G \subset E$ as
\[\nu_E(G) := \inf \{r > 0 \mid G \text{ admits a finite covering of balls (in } E\text{) with radius } r\}.
\]

Proof. It is easy to see that for $K_\alpha(u, r)$, the $E_\alpha$-ball of radius $r > 0$, centered at $u \in \mathbb{M}_\alpha$ it holds that
\[B(K_\alpha(u, r) \cap \mathbb{M}_\alpha) \subset K_0(B(u), Jr), \tag{2.9}\]
where $L$ is the Lipschitz constant of $B$. Let $G \subset \mathbb{M}_\alpha$ be a Borel set with $\nu_{E_\alpha}(G) = r < \infty$. For $r' > r$ let $n \in \mathbb{N}$ and $u_1, \ldots, u_n \in E_\alpha$ such that
\[G \subset \bigcup_{k=1}^n K_\alpha(u_k, r').\]
Now, choose arbitrary $\tilde{u}_k \in \mathbb{M}_\alpha \cap K_\alpha(u_k, r')$, $k = 1, \ldots, n$, and thus $G \subset \bigcup_{k=1}^n K_\alpha(\tilde{u}_k, 2r')$. With (2.9) we get
\[B(G) \subset B\left(\bigcup_{k=1}^n K_\alpha(\tilde{u}_k, 2r') \cap \mathbb{M}_\alpha\right) = \bigcup_{k=1}^n B(K_\alpha(\tilde{u}_k, 2r') \cap \mathbb{M}_\alpha) \subset \bigcup_{k=1}^n K_0(B(\tilde{u}_k), 2Lr').\]

\[\Box\]

Theorem 2.11. (Pruess [32]) Assume that Assumption 2.9 holds true for some $\alpha \in [0, 1)$ and let $\gamma \in (\alpha, 1)$. For all $u_0 \in \mathbb{M}_\gamma$, there exists a unique mild solution $u : [0, T] \to \mathbb{M}$ of the evolution equation
\[\dot{u}(t) = Au(t) + B(u(t)), \quad u(0) = u_0. \tag{2.10}\]
Moreover, $u$ is $E_\gamma$-continuous on $[0, T]$ and $E$-continuously differentiable on $(0, T]$.

Let us collect the respective results from [32]. First note that Assumption 2.9 is sufficient for Assumptions (A), ($\Omega$), (Y), (F), (S) and (L) in [32]. Thanks to the estimate from Lemma 2.10, we can apply [32, Theorem 2] which yields local existence. Moreover, for any local solution $u$ it holds that
\[
\|u(t)\|_\alpha \leq K\|u_0\|_\alpha + K_\alpha \int_0^t \|B(u(s))\|_E \frac{ds}{(t-s)^\gamma} \leq K\|u_0\|_\alpha + KT \left(1 + \int_0^t \|u(s)\|_\alpha \frac{ds}{(t-s)^\gamma}\right). \tag{2.11}
\]
Indeed, by Lipschitz continuity of $B$ there exists $M > 0$ such that
\[
\| S_t B(x) \|_\alpha \leq K_{E, \alpha} t^{-\alpha} M (1 + \| x \|_\alpha).
\]

Hence, by Lemma 2.6
\[
\| u(t) \|_\alpha \leq K (1 + \| u_0 \|_\alpha),
\]
for all $t \in [0, T]$ such that $u(t)$ solves (2.10). Assuming that $u$ would be a non-continuable mild solution would now contradict [32, Theorem 4], and thus, $u$ is a global mild solution. The last statement is then a consequence of the regularity theorem in [32]. Finally, by global Lipschitz assumptions on $B$ we get uniqueness of the solution.

**Remark 2.12.** In fact, $u$ is even a mild solution in $E_\gamma$, since $u_0 \in E_\gamma$ and by Lemma 2.5,
\[
\int_0^t \| (\varrho)^{\gamma} S_t B(u(s)) \| \, ds \leq M \int_0^t (t - s)^{- (\gamma - \alpha)} (1 + \| u(s) \|_\alpha) \, ds < \infty.
\]

**Remark 2.13.** By concatenation, the existence result extends to $[0, \infty)$ without further effort.

We now replace the global Lipschitz assumption by the local one
\[
B : \mathbb{M}_\alpha \rightarrow E \text{ is Lipschitz continuous on bounded sets.} \quad (B_{loc})
\]

**Theorem 2.14.** Assume that Assumption 2.9 holds true for some $\alpha \in [0, 1)$ with (B) replaced by $(B_{loc})$, and fix $\gamma \in (\alpha, 1)$. For all $u_0 \in \mathbb{M}_\gamma$, there exists a maximal $t_\infty > 0$ and a unique local mild solution on $[0, t_\infty)$ of the evolution equation (2.10). In particular, $u(t) \in \mathbb{M}_\gamma$ for all $t \in [0, t_\infty)$. Moreover, $u$ is $E_\gamma$-continuous on $[0, t_\infty)$, $E$-continuously differentiable on $(0, t_\infty)$ and it holds that $t_\infty = \infty$ or
\[
\lim_{t \uparrow t_\infty} \| u(t) \|_\alpha = \infty.
\]

**Proof.** For $N \in \mathbb{N}$ let $h_N \in C^\infty([0, \infty); \mathbb{R})$ be a non-increasing function such that $h = 1$ on $[0, N]$ and $h = 0$ on $[N + 1, \infty)$. Moreover, assume that
\[
\sup_{N \in \mathbb{N}} \| h'_N \|_\infty < \infty.
\]

We now get that $B_N(u) := h_N(\| u \|_\alpha) B(u)$ is globally Lipschitz continuous. To verify the Nagumo condition, let $\tilde{u} \in \mathbb{M}_\alpha$. Without loss of generality assume that $\tilde{c} := h_N(\| \tilde{u} \|_\alpha) > 0$. Writing $\epsilon' := \tilde{c} \epsilon$ we get
\[
\frac{1}{\epsilon} \text{dist}_E(\tilde{u} + \epsilon h_N(\| \tilde{u} \|_\alpha) B(\tilde{u}); \mathbb{M}) \leq \frac{\tilde{c}}{\epsilon'} \text{dist}_E(\tilde{u} + \epsilon' B(\tilde{u}); \mathbb{M}) \rightarrow 0, \quad (2.12)
\]
as $\epsilon \searrow 0$. Hence, the assumptions of Theorem 2.11 are satisfied for the localized equation
\[ \frac{\partial}{\partial t} u = Au + B_N(u(t)), \quad u(0) = u_0, \]
and we get for each $N \in \mathbb{N}$ a unique global mild solution, say $u_N$. Set
\[ t_N := \inf\{ t \geq 0 \mid \|u_N(t)\|_\alpha > N \} \land T, \]
then by uniqueness claim of Theorem 2.11 it holds that $u_N = u_{N+1} = u_{N+k}$ on $[0, t_N]$ and $(t_N)_{N \in \mathbb{N}}$ is an increasing sequence. Set $t_\infty := \lim_{N \to \infty} t_N$ and
\[ u(t) := u_N(t), \quad \text{on } [0, t_N] \]
which is well defined. For $t < t_\infty$, then exists an $N \in \mathbb{N}$ such that $t < t_N$, and thus,
\[ u(t) = u_N(t) = S_t u_0 + \int_0^t S_{t-s} B(u_N(s)) \, ds = S_t u_0 + \int_0^t S_{t-s} B(u(s)) \, ds. \]
In fact, either $t_\infty = \infty$ and $u$ exists even on all of $[0, \infty)$ or $t_\infty < \infty$ and
\[ \lim_{t \nearrow t_\infty} \|u(t)\|_\alpha = \infty. \]
By approximation, the existence results extend to all initial values in $\mathbb{M}_\alpha$, instead of $\mathbb{M}_\gamma$ only:

Corollary 2.15. Assume that Assumption 2.9 holds true but instead of (B) assume that there exists an open set $O$ in $E_\alpha$ such that $\mathbb{M}_\alpha \subset O$ and $B : O \to E$ is Lipschitz continuous on bounded sets. Then, the statement of Theorem 2.14 also holds true with $\gamma = \alpha$.

Proof. First, by standard existence results [14, Theorem 3.3.3] there exists a unique mild solution $u$ in $E_\alpha$, for all initial data $u_0 \in O$. By approximation, we show that $u$ stays in $\mathbb{M}$ when started there. Let $\gamma' \in (\alpha, 1)$.

Given $u_0 \in \mathbb{M}_\alpha$, set $u_0^{(n)} := S_{1/n} u_0$, which converges to $u_0$ in $E_\alpha$. Since $(S_t)$ is analytic and $\mathbb{M}$ is $(S_t)$-invariant, it holds that $u_0^n \in \mathbb{M}_{\gamma'}$, and thus, Theorem 2.14 yields unique maximal mild solutions $u^n$ of (2.10) for initial data $u_0^n$. We denote the explosion times by $t^n_\infty$ for $u^n$ and by $t_\infty$ for $u$.

By continuity in initial data, it holds that $u^n \to u$ in $E_\alpha$, uniformly on compact subintervals of $[0, t_\infty)$, see [14, Theorem 3.4.1]. Since $\mathbb{M}_\alpha$ is closed in $E_\alpha$, this finishes the proof.

We close this section by proving an easy-to-check condition sufficient for the Nagumo condition (2.8).
Lemma 2.16. Let $I \subseteq \mathbb{R}$ be a not necessarily bounded interval, $E = L^2(I)$ and consider the closed set

$$
\mathbb{M} := \{ h \in H \mid h(\xi) \geq 0 \text{ d}\xi - a.e. \}.
$$

Assume that $V \hookrightarrow E$ is an arbitrary Banach space and $F : V \to E$ Lipschitz continuous, satisfying the point-wise inward-pointing property $\text{d}\xi$-a.e.

$$
F = \mathbb{M}_V := \mathbb{M} \cap V, \text{ and } \xi \in I : h(\xi) = 0 \implies F(h)(\xi) \geq 0. \quad (2.13)
$$

Then, $F$ satisfies the Nagumo condition (2.8), i.e.,

$$
\lim_{\epsilon \downarrow 0} \epsilon^{-1} \text{dist}_E (g + \epsilon F(g), \mathbb{M}) = 0, \quad \forall g \in \mathbb{M}_V.
$$

Proof. Fix $g \in \mathbb{M}_V$, $\epsilon > 0$ and set

$$
\underline{F} := \text{ess inf} \{ F(g)(\xi) \mid \xi \in I \} \geq -\infty
$$

If $\underline{F} \geq 0$ then obviously (2.8) holds true, so it suffices to consider the case $\underline{F} < 0$. Define

$$
h_\epsilon(\xi) := \max\{g(\xi) + \epsilon F(g)(\xi), 0\}, \xi \in I,
$$

which is an element of $\mathbb{M}$ by definition. Note that (2.13) implies that for $\text{d}\xi$ almost all $\xi \in I$

$$
-\epsilon F(g)(\xi) > g(\xi) \implies g(\xi) > 0.
$$

Here, recall that $g \geq 0$ a.e. Moreover,

$$
\int_I (g(\xi) + \epsilon F(g)(\xi) - h_\epsilon(\xi))^2 \text{ d}\xi = \int_I (g(\xi) + \epsilon F(g)(\xi))^2 1_{(g(\xi) + \epsilon F(g)(\xi))>0}(\xi) \text{ d}\xi
$$

$$
= \int_I (-\epsilon F(g)(\xi) - g(\xi))^2 1_{-\epsilon F(g)(\xi) - g(\xi) > 0}(\xi) \text{ d}\xi
$$

$$
\leq \epsilon^2 \int_I F(g)(\xi)^2 1_{\epsilon F(g)(\xi) > g(\xi) > 0}(\xi) \text{ d}\xi
$$

$$
= o(\epsilon^2).
$$

The latter estimate holds because (2.13) yields $\text{d}\xi$-a.e.

$$
F(g)(\xi) 1_{-\epsilon F(g)(\xi) - g(\xi) > 0}(\xi) \to 0,
$$

and by dominated convergence theorem the convergence is also true in $L^2$. Indeed, $h_\epsilon$ is the minimal projection of $g + \epsilon F(g)$, $\epsilon > 0$, onto $\mathbb{M}$ in the sense that

$$
\text{dist}_E (g + \epsilon F(g); \mathbb{M}) = \| g + \epsilon F(g) - h_\epsilon \|_E,
$$

which then finishes the proof. $\square$
3. Wong–Zakai approximation for stochastic evolution equations

In this section, we discuss an approximation method of Wong–Zakai type for a class of semilinear stochastic evolution equations in the mild framework. We extend the proof of Nakayama [27] to the situation where the linearity generates an analytic semigroup but the drift can be controlled only on a smaller subspace, and, in addition, consider the case when the coefficients are Lipschitz continuous only on bounded sets. The latter seems to be new even for the classical situation of Heath–Jarrow–Morton-type equations.

Let $U$ and $E$ be separable Hilbert spaces and $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space on which an $U$-valued cylindrical Wiener process with covariance operator $\text{Id}_U$ lives. Recall that there exist independent real $(\mathcal{F}_t)$-Brownian motions $\beta^k$, $k \in \mathbb{N}$, such that for a CONS $(e_k)_{k\in\mathbb{N}}$ of $U$, $t \in [0, T]$,

$$W_t = \sum_{k=1}^{\infty} e_k \beta^k(t).$$

Note that $W$ diverges in $U$ but the formal series is to be read in the following sense: There exists a separable Hilbert space $U_1$ and a Hilbert–Schmidt embedding $L : U \hookrightarrow U_1$. $W$ exists as the limit in $L^2(\Omega; U)$ of the series

$$W_t = \sum_{k=1}^{\infty} L e_k \beta^k(t).$$

In particular, $W$ is independent of $L$ and $U_1$, cf. [6]. We keep $(e_k)_{k\in\mathbb{N}}$ fixed for the remainder of this section.

We now consider the stochastic evolution equation

$$\begin{cases} 
\mathrm{d}X(t) = [AX(t) + B(X(t))] \, \mathrm{d}t + C(X(t)) \, \mathrm{d}W_t, & t \geq 0, \\
X(0) = X_0,
\end{cases} \tag{3.1}$$

where $A$ is a linear operator on $E$ with domain $\mathcal{D}(A)$, and $C : E \to \mathcal{L}_2(U; E)$ the noise coefficient and $B : E \to E$ the nonlinear part of the drift term, are assumed to be Borel measurable functions. Recall that a mild solution of (3.1) on a Hilbert space $H \hookrightarrow E$ is a predictable $H$-valued process which satisfies the $H$-integral equation

$$X(t) = S_t X_0 + \int_0^t S_{t-s} B(X(s)) \, \mathrm{d}s + \int_0^t S_{t-s} C(X(s)) \, \mathrm{d}W_s, \tag{3.2}$$

on $[0, \tau]$ for a strictly positive predictable stopping time $\tau > 0$. The stopping time $\tau$ is called maximal if there does not exist a solution on a strictly larger stochastic interval and the solution is global, if $\tau = T$ almost surely. In particular, the first integral is assumed to be well defined as a Bochner integral on $H$, and the second integral as the stochastic convolution. Recall that when $A$ generates a $C_0$ semigroup, then for each predictable and square-integrable $\Psi : [0, T] \times \Omega \to \mathcal{L}_2(U; H)$, there exists a
stochastic process $W_{A, \psi}$ called stochastic convolution, which satisfies for all $t \geq 0$ that almost surely
\[ W_{A, \psi}(t) = \int_0^t S_{t-s} \psi(s) \, dW_s. \]

$W_{A, \psi}$ admits a continuous modification, cf [5], which we will work with in the following without further mentioning.

We will focus now on the situation where $H$ will be the domain of a fractional power of $-A$. In order for them to be well defined, we need the following.

**Assumption 3.1.** Let $A$ be the generator of an analytic $C_0$-semigroup of negative type on $E$.

As in (2.2), we set $E_\alpha := D((-A)^\alpha)$ for $\alpha \in \mathbb{R}$, and write $\|u\|_\alpha := \|(-A)^\alpha u\|_E$, $\alpha \in \mathbb{R}$. It is worth to recall that $A$ fulfills Assumption 3.1 on $E_\alpha$ for any $\alpha \in \mathbb{R}$, once it is fulfilled for $\alpha = 0$. From now on, we keep $\alpha \in [0, 1)$ fixed. Throughout this section, we will assume $X_0 \in E_\alpha$ is deterministic. The following assumptions ensure that there exists a unique global mild solution of (3.1) on $E_\alpha$, cf. [18, Theorem 3.9].

**Assumption 3.2.**

(i) $B : E_\alpha \to E_0$ is Lipschitz continuous with Lipschitz constant $L_B$.

(ii) For some constant $M_B$ and all $u \in E_\alpha$, it holds that $\|B(u)\|_0 \leq M_B$.

Introduce the shorthand,
\[ \sigma_k(u) := C(u)e_k, \ u \in E_\alpha, \ k \in \mathbb{N}. \]

Because $(e_k)$ is an CONS of $U$, we can decompose $C$ as
\[ C(u)w = \sum_{k=1}^{\infty} \langle w, e_k \rangle_U \sigma_k(u), \quad (3.3) \]

for all $w \in U, u \in E_\alpha$. Keeping this mind, we define the Stratonovich or Wong–Zakai correction term for the projection of $C$ on the linear span span \{ $e_1, \ldots, e_n$ \},
\[ \Sigma_n(u) := \frac{1}{2} \sum_{k=1}^{n} D\sigma_k(u)\sigma_k(u), \ u \in E_\alpha. \]

**Assumption 3.3.**

(i) $C : E_\alpha \to \mathcal{L}_2(U; E_\alpha)$ is Lipschitz continuous with Lipschitz constant $L_C$.

(ii) There exists a constant $M_C$ such that $\|C(u)\|_{\mathcal{L}_2(U; E_\alpha)} \leq M_C, \forall u \in E_\alpha$.

(iii) $\sigma_k : E_\alpha \to E_\alpha, \ k \in \mathbb{N}$ are twice Fréchet differentiable and the derivatives are bounded.

(iv) There exists an $L_\Sigma > 0$ such that $\|\Sigma_n(u) - \Sigma_n(v)\|_{E_\alpha} \leq L_\Sigma \|u - v\|_{E_\alpha}, \forall u, v \in E_\alpha, \forall n \in \mathbb{N}$. 
(v) There exists $\Sigma_\infty : E_\alpha \to E_\alpha$ such that

$$\lim_{n \to \infty} \Sigma_n(u) = \Sigma_\infty(u), \quad \forall u \in E_\alpha.$$ 

Remark 3.4. The conditions on $\Sigma_n$ imply that $\Sigma$ is Lipschitz continuous on $E_\alpha$ with Lipschitz constant $L_\Sigma$. Moreover, $\Sigma_n \to \Sigma$ as $n \to \infty$ uniformly on compact sets in $E_\alpha$. Indeed, let $K \subset E_\alpha$ be compact. For $\epsilon > 0$ let $\bigcup_{k=1}^{N} K(u_k, \delta)$ be the covering by open balls of radius $\delta := \epsilon/(4L_\Sigma)$. Moreover, let $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ it holds that

$$\sup_{k=1, \ldots, N} \| \Sigma_n(u_k) - \Sigma_\infty(u_k) \|_\alpha < \epsilon/2.$$ 

Hence, for all $n \geq n_0$,

$$\sup_{u \in K} \| \Sigma_n(u) - \Sigma_\infty(u) \|_\alpha \leq \sup_{k=1, \ldots, N} \sup_{u \in K(u_k, \delta)} \left[ \| \Sigma_n(u) - \Sigma_n(u_k) \|_\alpha + \| \Sigma_\infty(u) - \Sigma_\infty(u_k) \|_\alpha + \| \Sigma_n(u_k) - \Sigma_\infty(u_k) \|_\alpha \right] < 2L_\Sigma \delta + \frac{\epsilon}{2} = \epsilon.$$ 

Remark 3.5. Even in the finite-dimensional case with $A = 0$, it is often assumed that $B$ is globally bounded and $C$ of class $C^2_b$, see [41] for the scalar case $E = \mathbb{R}$ and [35], [15, Theorem 7.2] for Wong–Zakai approximations for stochastic differential equations on $\mathbb{R}^d$. We will pass over to the more general case by truncation in Sect. 3.3.

Note that $B$ and $C$ can be trivially extended to Borel functions on $E$, since $E_\alpha$ is an $E$-Borel set by continuity of the imbedding $E_\alpha \hookrightarrow E$ and Kuratowski’s theorem. Moreover, the Lipschitz conditions on $B$ and $C$ yield existence of a unique mild solution $X$ of (3.1) on $E_\alpha$, cf. [18, Theorem 3.9]. Moreover, $X \in C([0, T]; E_\alpha)$ a.s., and for all $p > 1$

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} \| X(t) \|_\alpha^{2p} \right] \leq K_{p, T} \left( 1 + \mathbb{E} \left[ \| X_0 \|_\alpha^{2p} \right] \right). \quad (3.4)$$

We now use the linearly interpolated Brownian motions $\hat{\beta}^k_m$ and the time grid $i\delta_m, i = 1, \ldots, m$, $\delta_m := \tau/m$ which we have defined already in (1.20). By $\hat{\beta}^k_m$ we denote the time-derivatives which exist piece-wise on $[0, T]$. The approximating Wong–Zakai equations are then the random evolution equations,

$$\frac{\hat{\delta}}{m} Z_{m,n}(t) = A Z_{m,n}(t) + B(Z_{m,n}(t)) - \Sigma_\infty(Z_{m,n}(t))$$

$$+ \sum_{k=1}^{n} \sigma_k(Z_{m,n}(t)) \hat{\beta}^k_m(t), \quad t > 0, \quad (3.5)$$

$$Z_{m,n}(0) = X_0.$$
**Theorem 3.6.** Let Assumptions 3.1–3.3 be satisfied and denote, respectively, by $X$ and $Z_{m,n}$ the unique mild solutions of (3.1) and (3.5) on $E_\alpha$ for initial data $X_0 \in E_\alpha$. Then, for all $p > 1$ it holds that

$$\lim_{n \to \infty} \lim_{m \to \infty} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \| X(t) - Z_{m,n}(t) \|_{2p}^{2p} \right] = 0.$$ 

The proof is split into two main steps. First, we project the noise coefficient onto $\text{span}\{e_1, \ldots, e_n\}, n \in \mathbb{N}$ and obtain convergence.

To this end, set $C_n := \sum_{k=1}^{n} \langle \cdot, e_k \rangle U \sigma_k$ and $B_n := B + \Sigma_n - \Sigma_\infty$ for $u \in E_\alpha$. Since $C_n$ is the projection of $C$ to $\text{span}\{e_1, \ldots, e_n\}$ we get that $C_n : E_\alpha \to \mathcal{L}_2(U; E_\alpha)$ is Lipschitz continuous, uniformly in $n \in \mathbb{N}$. By assumption, also $B_n$ is Lipschitz continuous with uniform Lipschitz constant. We denote by $X_n$ the unique global mild solutions of

$$dX_n(t) = [AX_n(t) + B_n(X_n(t))] \, dt + C_n(X_t) \, dW_t \quad (3.6)$$

The following result is now a direct application of the continuity of the solution map in the coefficients.

**Proposition 3.7.** Assume that Assumptions 3.1–3.3 hold true. Then, for the unique global mild solutions $X_n$ of (3.6) and the unique mild solution $X$ of (3.1) on $E_\alpha$, for all $p > 1$, it holds that

$$\lim_{n \to \infty} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \| X_n(t) - X(t) \|_{2p}^{2p} \right] = 0.$$ 

**Proof.** The convergence follows from the continuity property of the solution map as shown in [26, Theorem 2.8] or [19, Prop. 3.2].

This reduces the problem to the situation where $\sigma_k = 0$ for all but finitely many $k \in \mathbb{N}$. To keep a level of generality, we treat this case in a separate framework. Proposition 3.7 together with Theorem 3.11 then yields Theorem 3.6.

### 3.1. Finite-dimensional noise

To prove the convergence in the case of finite-dimensional noise, we extend several estimates in [27] using the tools from interpolation theory, see Sect. 2. We now consider the stochastic evolution equation, with finite-dimensional noise,

$$\begin{cases}
    dX(t) = [AX(t) + B(X(t)) + \Sigma_n(X(t))] \, dt + \sum_{k=1}^{n} \sigma_k(X(t)) \, d\beta_t^k, \\
    X(0) = X_0.
\end{cases} \quad (3.7)$$

Here, $X_0 \in E_\alpha$ is deterministic, as above. For this subsection, $\sigma_k$ are functions on $E_\alpha$ satisfying the following conditions.
**Assumption 3.8.** For \( k = 1, \ldots, n \) assume that \( \sigma_k : E_\alpha \to E_\alpha \) is twice Fréchet differentiable, and that \( \sigma_k, D\sigma_k \) and \( D^2\sigma_k \) are bounded in \( E_\alpha \).

We define, similar to the situation above, 
\[
\Sigma_n(u) := \frac{1}{2} \sum_{k=1}^{n} D\sigma_k(u)\sigma_k(u), \quad u \in E_\alpha.
\]

**Lemma 3.9.** \( \Sigma_n : E_\alpha \to E_\alpha \) is bounded and Lipschitz continuous.

**Proof.** By assumption, for all \( k = 1, \ldots, n \), \( \sigma_k \) and \( D\sigma_k \) are differentiable with bounded derivative. For \( u, v \in E_\alpha \), it holds that 
\[
\| D\sigma_k(u)\sigma_k(u) - D\sigma_k(v)\sigma_k(v) \|_\alpha 
\leq \| D\sigma_k(u) \|_{\mathcal{L}(E_\alpha)} \| u - v \|_\alpha + \| D\sigma_k(u) - D\sigma_k(v) \|_{\mathcal{L}(E_\alpha)} \| v \|_{E_\alpha}.
\]
Hence, application of the mean-value theorem to \( D\sigma_k \) yields Lipschitz continuity of the maps \( u \mapsto D\sigma_k(u)\sigma_k(u), k \in \mathbb{N} \). Since the finite sum of Lipschitz functions is Lipschitz again the lemma is proven. \( \square \)

We define \( \beta^k_m, k = 1, \ldots, n \), in the same way as in (1.20) so that the Wong–Zakai approximation of (3.7) will be the solution of the random evolution equation

\[
\begin{align*}
\frac{d}{dt} Z_m(t) &= AZ_m(t) + B(Z_m(t)) + \sum_{k=1}^{n} \sigma_k(Z_m(t))\beta^k_m(t), \quad t \in [0, T] \\
Z_m(0) &= X_0.
\end{align*}
\]

(3.8)

On \( [i\delta_m, (i+1)\delta_m) \), \( i = 0, \ldots, m-1 \), the coefficients of (3.8) are time-homogeneous and satisfy global Lipschitz assumptions. Standard existence results for deterministic equations, e.g., [31, Theorem 6.3.3] yield existence and uniqueness of a global mild solution of the random equation, \( \omega \)-wise. By concatenation, we observe existence and uniqueness of a global continuous mild solution on \( E_\alpha \).

**Remark 3.10.** The correction term \( \Sigma_n \) can be removed from (3.7) but then has to appear in (3.8) with a negative sign. Recall that the occurrence of the Stratonovich correction term was quite surprising and an important step in the understanding of stochastic differential equations in terms of physical systems [41].

**Theorem 3.11.** Let Assumptions 3.1, 3.2 and 3.8 hold true and denote by \( X \) and \( Z_m \), respectively, the unique mild solution of (3.7) and (3.8), for \( m \in \mathbb{N} \). Then for any \( p > 1 \),

\[
\lim_{m \to \infty} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \| X(t) - Z_m(t) \|_{E_\alpha}^p \right] = 0.
\]

(3.9)
This theorem is an extension of [27, Prop 2.1]. We roughly follow its proof from [27, Section 2] but perform the necessary changes. Since the noise terms $\sigma_k$ and the linear operator $A$ fulfill the assumptions in the reference, on $E_\alpha$, Lemmas 2.9, 2.11 and 2.12 in [27] also apply in our setting. These parts are the basis for [27, Lemma 2.13], which reads in our framework as follows.

**Lemma 3.12.** For $p > 1$ exists a constant $K_{T,p}$ and a sequence $(\epsilon_m)_{m \in \mathbb{N}}$ with
\[
\lim_{m \to \infty} \epsilon_m = 0,
\]
such that
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left\| \int_0^t S_{t-s} \sigma_k(Z_m(s)) \dot{\beta}_m^k(s) \, ds - \int_0^t S_{t-s} \sigma_k(X(s)) \, d\beta^k(s) \right\|^{2p} \right]^{\frac{1}{2p}}
\leq \frac{1}{2} \int_0^T \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|Z_m(s) - X(s)\|^{2p} \right] ds + \epsilon_m.
\]

The proof of this lemma is based on the da Prato–Kwapień–Zabczyk factorization method which was introduced in [4] and has also been used, for instance, in [5] and [6] to show continuity of the stochastic convolution. Since the noise operator $C$ satisfies the “standard” assumptions, there are no modifications necessary in the proof. However, we have to adapt the parts involving also the drift term $B$, which does not fulfill the assumptions in reference [27]. We now restrict the solutions onto the time grid $\{i \delta_m \mid i = 0, \ldots, m\}$.

For $m \in \mathbb{N}$, we define
\[
\bar{Z}_m(t) := S_{t-[t]_m} Z_m([t]_m), \quad \text{and} \quad \bar{X}_m(t) := S_{t-[t]_m} X([t]_m).
\]

At this point, recall that we have set $[t]_m := i \delta_m$ for $i < m$ so that $t \in [i \delta_m, (i+1) \delta_m)$.

**Lemma 3.13.** There exists a constant $K_{n,T,p,\alpha} > 0$ such that for all $m \in \mathbb{N}$
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|Z_m(t) - \bar{Z}_m(t)\|^{2p} \right]^{\frac{1}{2p}} \leq K_{n,T,p,\alpha} (m^{-2p(1-\alpha)} + m^{1-p}).
\]

**Proof.** We first decompose, for $t \in [0, T]$,
\[
Z_m(t) - \bar{Z}_m(t) = \int_{[t]_m}^t S_{t-s} B(Z_m(s)) \, ds + \sum_{k=1}^n \int_{[t]_m}^t S_{t-s} \sigma_k(Z_m(s)) \dot{\beta}_m^k(s) \, ds.
\]

From boundedness of $B$ and Lemma 2.5, the first integral can be controlled by
\[
\left\| \int_{[t]_m}^t S_{t-s} B(Z_m(s)) \, ds \right\|_{2p} \leq \int_{[t]_m}^t K_\alpha M_B(t-s)^{-\alpha} \, ds \leq \frac{K_\alpha M_B}{1-\alpha} \delta_m^{1-\alpha},
\]

(3.11)
To bound the second term, first note that
\[
\left\| \int_{[t]_m}^{t} S_{t-s} \sigma_k(\xi_m(s)) \hat{\beta}_m^k(s) \, ds \right\|_\alpha \leq M_k \left| \beta^k([t + \delta_m]_m) - \beta^k([t]_m) \right|
\]
and then, following a standard procedure,
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \beta^k([t + \delta_m]_m) - \beta^k([t]_m) \right|^{2p} \right] \leq \delta_m^p \mathbb{E} \left[ |\beta_k(1)|^{2p} \right] =: K_{T,p} m^{1-p}.
\]

We put everything together,
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \| Z_m(t) - \bar{Z}_m(t) \|_{\alpha}^{2p} \right] \leq K_2 p M_B^2 \delta_m^{2p(1-\alpha)} + n M_k^2 \tilde{K}_{T,p} m^{1-p} \leq K_{T,p,\alpha} (m^{-2p(1-\alpha)} + nm^{1-p}).
\]

Lemma 3.14. There exists a constant $K_{n,T,p} > 0$ such that for all $m \in \mathbb{N}$
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \| X(t) - \tilde{X}_m(t) \|_{\alpha}^{2p} \right] \leq K_{n,T,p,\alpha} \left( m^{-2p(1-\alpha)} + m^{1-p} \right).
\]

Proof. First, we write
\[
X(t) - \tilde{X}_m(t) = \int_{[t]_m}^{t} S_{t-s} B(X(s)) \, ds + \frac{1}{2} \sum_{k=1}^{n} \int_{[t]_m}^{t} S_{t-s} D \sigma_k(X(s)) \sigma_k(X(s)) \, ds + \sum_{k=1}^{n} \int_{[t]_m}^{t} S_{t-s} \sigma_k(X(s)) \, d\beta_k(s), \quad t \in [0, T].
\]

The first term can be controlled by $m^{-2p(1-\alpha)}$ in the same way as in the proof of Lemma 3.13. For the Stratonovich correction, we get
\[
\left\| \int_{[t]_m}^{t} S_{t-s} D \sigma_k(X(s)) \sigma_k(X(s)) \, ds \right\|_{\alpha} \leq K_1 M_k \delta_m,
\]
using that $(S_t)$ is strongly continuous and that Assumption 3.8 holds true. The stochastic integrals can be bounded by [27, Lemma 2.4] which says, applied to $t \mapsto \sigma_k(X(t)) \in E_\alpha$,
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left\| \int_{[t]_m}^{t} S_{t-s} \sigma_k(X(s)) \, d\beta_k(s) \right\|_{\alpha}^{2p} \right] \leq K_{T,p,k} \delta_m^{p-1}.
\]

Finally,
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \| X(t) - \tilde{X}_m(t) \|_{\alpha}^{2p} \right] \leq K_{T,\alpha,p} \left( m^{-2p(1-\alpha)} + nm^{-2p} + nm^{1-p} \right).
\]
Remark 3.15. Here, however, we directly see how the modifications of the assumptions can be covered by the tools provided by interpolation theory in Sect. 2.1. Likewise, one could directly apply results on space-time regularity of \( X(t) - S_t X_0 \). For instance, by [40, Prop. 4.2] the stochastic convolution is Hölder continuous with exponent < \( \frac{1}{2} - \frac{1}{2p} \) which yields an estimate of type \( \delta_m^\lambda \) with \( 0 < \lambda < p - 1 \).

Remark 3.16. The last two proofs particularly show why we have to consider first the limits for \( m \to \infty \), and then \( n \to \infty \).

We now collect the arguments to finish the proof of Theorem 3.11.

Proof of Theorem 3.11. Applying Lemmas 3.13 and 3.14, we get a constant \( K \), depending on \( n, p, T, \) and \( \alpha \), such that

\[
\mathbb{E} \sup_{0 \leq t \leq T} \| X(t) - Z_m(t) \|_{2p}^2 \leq K \left( \mathbb{E} \sup_{0 \leq t \leq T} \| \tilde{X}_m(t) - \tilde{Z}_m(t) \|_{2p}^2 + m^{-2p\alpha} + m^{1-p} \right).
\]

Inserting the mild integral formulae, respectively, for \( Z_m \) and \( X \) yields the decomposition

\[
\tilde{X}_m(t) - \tilde{Z}_m(t) = \int_0^t S_t - s \left( B(X(s)) - B(Z_m(s)) \right) \, ds + \sum_{k=1}^n \int_0^t S_t - s \sigma_k(X(s)) \, d\beta^k(s) - \int_0^t S_t - s \sigma_k(Z_m(s)) \hat{\beta}_m^k(s) \, ds \ni \frac{1}{2} \int_0^t S_t - s \, D\sigma_k(X(s)) \sigma_k(X(s)) \, ds.
\]

For the first term on the right-hand side, note that Assumption 3.2.(i) implies

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left\| \int_0^s S_t - s \left[ B(Z_m(s)) - B(X(s)) \right] \, ds \right\|_{2p}^2 \right] \leq \tilde{K}_{T,\alpha,p} \int_0^T \mathbb{E} \left[ \sup_{0 \leq s \leq t} \| Z_m(s) - X(s) \|_{2p}^2 \right] \frac{dt}{(T - t)\alpha}. \tag{3.13}
\]

The remaining summands are covered by Lemma 3.12, so that

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \| \tilde{X}_m(t) - \tilde{Z}_m(t) \|_{2p}^2 \right] \leq K_{T,p,\alpha} \int_0^T \mathbb{E} \left[ \sup_{0 \leq s \leq t} \| Z_m(s) - X(s) \|_{2p}^2 \right] \frac{dt}{(T - t)\alpha} + K_{T,p,\alpha} \int_0^T \mathbb{E} \left[ \sup_{0 \leq s \leq t} \| Z_m(s) - X(s) \|_{2p}^2 \right] dt + \epsilon_m,
\]

where

\[
K_{T,p,\alpha} = \mathbb{E} \left[ \sup_{0 \leq s \leq t} \| Z_m(s) - X(s) \|_{2p}^2 \right] \frac{dt}{(T - t)\alpha}.
\]
where \((\epsilon_m)_{m \in \mathbb{N}}\) is a sequence with \(\lim_{m \to \infty} \epsilon_m = 0\). Note that on any time interval \((0, t)\) it holds that
\[
1 = s^{\alpha-1} s^{1-\alpha} \leq t^{1-\alpha} s^{\alpha-1}, \ s \in (0, t)
\]
and thus
\[
\int_0^T \mathbb{E} \left[ \sup_{0 \leq s \leq t} \| Z_m(s) - X(s) \|_2^p \right] \, dt 
\leq K_{T, \alpha} \int_0^T \mathbb{E} \left[ \sup_{0 \leq s \leq t} \| Z_m(s) - X(s) \|_2^p \right] \frac{dt}{(T-t)^\alpha} + K_{T, \alpha} \epsilon_m.
\]
To obtain the convergence, we use Gronwall’s lemma in its extended version given in Lemma 2.6. □

3.2. Convergence of local solutions

Before we go into more detail for the approximation of local solutions of stochastic evolution equations, we discuss preliminary results on explosion times for stochastic processes.

Let \(Y_n, n \in \tilde{\mathbb{N}} := \mathbb{N} \cup \{ \infty \}\) be continuous stochastic processes on a Banach space \(V\), with explosion times denoted by \(\sigma_n\), \(n \in \tilde{\mathbb{N}}\). For \(r > 0\) denote by \(\varsigma_n(r)\) and \(\tau_n(r)\), the first exit times of \(Y_n\) of the, respectively, open and closed balls of radius \(r\), for \(n \in \tilde{\mathbb{N}}\).

More precisely
\[
\tau_n(r) := \inf \{ t \geq 0 \mid \|Y_n(t)\|_V > r \}, \quad \varsigma_n(r) := \inf \{ t \geq 0 \mid \|Y_n(t)\|_V \geq r \}.
\]

**Assumption 3.17.** Assume that for all \(n \in \tilde{\mathbb{N}}\) there exist stochastic processes \(Y_n^{(r)}\), which are \(\mathcal{F}_T\)-measurable have almost surely paths in \(C([0, T]; V)\) and satisfy \(Y_n^{(r)} = Y_n\) on [0, \(\tau_n(r)\)).

**Assumption 3.18.** Assume that for all \(r > 0\) it holds that
\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T} \left\| Y_\infty^{(r)}(t) - Y_n^{(r)}(t) \right\|_V = 0, \text{ in probability.}
\]

**Remark 3.19.** Note that \(\varsigma_n^{(r)}\) are also the exit times of the processes \(Y_n^{(r)}\). In particular, Assumption 3.17 yields that for all \(n \in \tilde{\mathbb{N}}\) and all \(r' \geq r > 0\),
\[
Y_n^{(r)} = Y_n^{(r')} \text{, on } [0, \tau_n^{(r')}].
\]

We are interested in the convergence of \(Y_n\) as \(n \to \infty\). This has also been discussed in [19, Thm 2.1], but was stated under the assumption that the stochastic processes are adapted. Note that this is not a restriction since we can pass over to the filtration generated by \(Y_n, n \in \tilde{\mathbb{N}}\), without any problems. This would allow us to derive Proposition 3.23. Instead, we will provide a path-wise and more direct proof below. Our proof will reduce the problem to the deterministic situation, which we will discuss now a priori.
Lemma 3.20. Let $V$ be a real Banach space, $T > 0$ and $f_n, f_\infty \in C([0, T]; V)$ for $n \in \mathbb{N}$, such that $f_n \to f_\infty$ uniformly, as $n \to \infty$. Fix $r > 0$ and define

$$t_n := \inf \{ t \geq 0 \mid \| f_n(t) \|_V > r \}, \quad s_n := \inf \{ t \geq 0 \mid \| f_n(t) \|_V \geq r \}, \quad n \in \mathbb{N} \cup \{ \infty \}.$$  

Here, we set $\inf \emptyset := T$. Then,

$$\liminf_{n \to \infty} s_n \geq s_\infty, \quad t_\infty \geq \limsup_{n \to \infty} t_n.$$  

Moreover, if $s_\infty = t_\infty$, then

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} t_n = t_\infty.$$  

Remark 3.21. Denote by $t_n^{(r)}, s_n^{(r)}$ the exit times of the balls of radius $r > 0$, for $f_n$ as in the lemma. Then, Lemma 3.20 yields for all $\epsilon > 0$

$$\limsup_{n \to \infty} s_n^{(r)} \leq \limsup_{n \to \infty} t_n^{(r)} \leq t_\infty \leq s_\infty \leq \liminf_{n \to \infty} s_n^{(r+\epsilon)} \leq \liminf_{n \to \infty} t_n^{(r+\epsilon)}.$$  

(3.14)

Proof. Without loss of generality assume $t_\infty < T$. Else, the estimate holds true trivially. Now, for all $\epsilon > 0$ there exists $\delta > 0$ such that $t_\infty + \epsilon \leq T$ and

$$\sup_{0 \leq s \leq t_\infty + \epsilon} \| f_\infty(s) \|_V > r + \delta.$$  

Let $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\sup_{0 \leq t \leq T} \| f_\infty(t) - f_n(t) \|_V < \delta.$$  

By triangle inequality, it holds that

$$\sup_{0 \leq s \leq t_\infty + \epsilon} \| f_n(t) \|_V > r,$$

and thus $t_n < t_\infty + \epsilon$.

The proof for $s_n$ in the opposite direction works quite similar. Without loss of generality, we assume $s_\infty > 0$. Then, for all $\epsilon > 0$ there exists $\delta > 0$ such that $s_\infty - \epsilon > 0$ and

$$\sup_{0 < s < s_\infty - \epsilon} \| f_\infty(s) \|_V < r - \delta.$$  

Choosing $N \in \mathbb{N}$ as above, we get that for all $n \geq N$,

$$\sup_{0 < s < s_\infty - \epsilon} \| f_\infty(s) \|_V < r,$$

and thus, $s_n > s_\infty - \epsilon$. Since $\epsilon > 0$ was chosen arbitrarily, the first result follows.
To prove the last statement in the lemma, we assume additionally \( s_\infty = t_\infty \), and observe
\[
\liminf_{n \to \infty} t_n \geq \liminf_{n \to \infty} s_n \geq s_\infty = t_\infty \geq \limsup_{n \to \infty} t_n \geq \limsup_{n \to \infty} s_n.
\]
\( \Box \)

We now also obtain the following convergence result.

Lemma 3.22. In the setting and notation of Lemma 3.20, it holds for all \( \epsilon > 0 \), \( r > 0 \),
\[
\lim_{n \to \infty} f_n(s_n^{(r+\epsilon)} \wedge .) = f_\infty, \quad \text{uniformly on } [0, t_\infty^{(r)}].
\]

Proof. Without loss of generality assume \( t_\infty^{(r)} > 0 \), else, the claim holds true trivially. Now, let \( N \in \mathbb{N} \) such that for all \( n \geq N \),
\[
\sup_{0 \leq t \leq T} \| f_n(t) - f_\infty(t) \|_V < \epsilon.
\]
Thus,
\[
\sup_{0 \leq t \leq t_\infty^{(r)}} \| f_n \|_V \leq r + \epsilon,
\]
which implies \( t_\infty^{(r)} \leq s_n^{(r+\epsilon)} \), for all \( n \geq N \). Finally,
\[
\lim_{n \to \infty} \sup_{0 \leq t \leq t_\infty^{(r)}} \left\| f_n(t \wedge s_n^{(r+\epsilon)}) - f_\infty(t) \right\|_V \leq \lim_{n \to \infty} \sup_{0 \leq t \leq T} \| f_n(t) - f_\infty(t) \|_V = 0.
\]
\( \Box \)

The extension to the stochastic situation will heavily rely on the subsequence criterion, which says that a sequence of random variables in a metric space converges in probability if and only if every subsequence has an almost surely convergent subsequence with the same limit, see, e.g., [17, Lemma 4.2].

Proposition 3.23. Let Assumptions 3.17 and 3.18 hold true.

(i) For all \( r > 0 \), \( \epsilon > 0 \), it holds almost surely that,
\[
\liminf_{n \to \infty} \tau_n^{(r)} \leq \tau_\infty^{(r)} \leq \varsigma_n^{(r+\epsilon)} \leq \limsup_{n \to \infty} \varsigma_n^{(r+\epsilon)}.
\]
Moreover, for all \( r > 0 \), \( \epsilon_0 > 0 \) and \( (n_k) \subset \mathbb{N} \), with \( n_k \to \infty \) as \( k \to \infty \), there exists a subsequence \( (n_{k_j}) \subset (n_k) \) such that for all \( \epsilon \in (0, \epsilon_0) \), almost surely,
\[
\limsup_{j \to \infty} \tau_{n_{k_j}}^{(r)} \leq \tau_\infty^{(r)} \leq \varsigma_\infty^{(r+\epsilon)} \leq \liminf_{j \to \infty} \varsigma_{n_{k_j}}^{(r+\epsilon)}.
\] (3.15)
(ii) Almost surely,
\[
\lim_{Q \ni r \to \infty} \liminf_{n \to \infty} \tau_n^{(r)} \leq \sigma_\infty \leq \lim_{Q \ni r \to \infty} \limsup_{n \to \infty} \zeta_n^{(r)}.
\]
Moreover, for all \((n_k) \subset \mathbb{N}\), with \(n_k \to \infty\) as \(k \to \infty\), and \(r > 0\) denote by \(\bar{\tau}^{(r)}\) and \(\underline{\zeta}^{(r)}\), respectively, left- and right-hand sides of (3.15). Then, it holds with probability one,
\[
\lim_{Q \ni r \to \infty} \bar{\tau}^{(r)} = \lim_{Q \ni r \to \infty} \underline{\zeta}^{(r)} = \sigma_\infty.
\]

(iii) For all \(r > 0\) and \(\epsilon > 0\), it holds that
\[
\lim_{n \to \infty} Y_n(\cdot \wedge \tau_\infty \wedge \zeta_n^{(r+\epsilon)}) = Y_\infty(\cdot \wedge \tau_\infty), \quad \text{u. c. p.}
\]
Moreover, for all \(r > 0\)
\[
\lim_{n \to \infty} Y_n(\cdot \wedge \tau^{(r)}) = Y_\infty(\cdot \wedge \tau_\infty), \quad \text{u. c. p.}
\]

Remark 3.24. Part (iii) includes that \(\tau_\infty^{(r)} < \sigma_n\), for large \(n\) at least along subsequences, where the meaning of “large” will typically depend on \(\omega\).

Proof. By u. c. p. convergence of \(Y_n\) and subsequence criterion, each \((n_k)\) admits a subsequence \((n_{k_j})\) such that almost surely,
\[
\lim_{j \to \infty} \sup_{0 \leq t \leq T} \left\| Y_n^{(r)}(t \wedge \tau^{(r)} \wedge \zeta_n^{(r+\epsilon)}) - Y_n^{(r)}(t \wedge \tau_\infty) \right\|_V = 0.
\]
Let \((n_{k_j})\) be a subsequence of \((n_k)\) and \(\Omega' \subset \Omega\) be a set of full measure such that the convergence holds for \(r' > r > 0\) fixed, and such that the paths of all the processes considered in the following are continuous. We emphasize that these are at most countably many. Recall Remark 3.19, which states that for all \(\epsilon \in (0, r' - r]\), \(\zeta_n^{(r+\epsilon)}\) and \(\tau_n^{(r)}\) are the exit times of \(Y_n^{(r')}\). The second claim of (i) follows from Lemma 3.20 and (3.14), applied to \(Y_n^{(r')}\), \(n \in \mathbb{N}\), for all \(\omega \in \Omega'\). To verify the first claim, recall that we have just shown that for \(\delta > 0\) and all \(\omega \in \Omega'\), there exists a \(J > 0\) such that
\[
\sup_{j \geq J} \tau_{n_{k_j}}^{(r)} < \tau_\infty^{(r)} + \delta.
\]
Hence, for all \(N \in \mathbb{N}\)
\[
\inf_{n \geq N} \tau_n^{(r)} \leq \inf_{n \geq (n_{k_j} \vee N)} \tau_n^{(r)} \leq \sup_{n \geq (n_{k_j} \vee N)} \tau_n^{(r)} \leq \sup_{j \geq J} \tau_{n_{k_j}}^{(r)} < \tau_\infty^{(r)} + \delta.
\]
We let \(N \to \infty\) and since \(\delta > 0\) was chosen arbitrarily, the first claim in (i) holds on all of \(\Omega'\). Note that the estimates for \(\zeta_n^{(r+\epsilon)}\) hold true due to the same arguments.

We now directly apply Lemma 3.22 to \(Y_n^{(r')}(\omega, \cdot)\), for all \(\omega \in \Omega'\), and obtain
\[
\lim_{j \to \infty} \sup_{0 \leq t \leq T} \left\| Y_{n_{k_j}}^{(r')}(t \wedge \zeta_n^{(r+\epsilon)} \wedge \tau_n^{(r)}) - Y_\infty^{(r')}(t \wedge \tau_\infty) \right\|_V = 0.
\]
Recall that $Y_n^{(r)} = Y_n$ on $[0, \zeta_n^{(r+\epsilon)}]$, for all $n \in \mathbb{N}$, $\epsilon \in [0, r'-r]$ by assumption and monotonicity of the exit times. In fact, this finishes the proof of (iii).

Part (ii) is a direct application of the first. Let $\Omega' \subset \Omega$ be a set of full measure such that the first statement holds true for all $r \in (0, \infty) \cap \mathbb{Q}$. Hence, part (i) yields

$$
\limsup_{n \to \infty} \zeta_n^{(r)}(\omega) \geq \zeta_\infty^{(r)}(\omega), \quad \forall r > 0, r \in \mathbb{Q},
$$

(3.17)

for all $\omega \in \Omega'$. Note that $\zeta_n^{(r)}$ is increasing in $r > 0$ and bounded by $T$ so that its limit exists. This yields the right-hand side of the first claim and for the left-hand side, we argue in the same way.

For fixed $r > 0$ denote by $\Omega_r \subset \Omega$ the set such that all the statements up to now hold true for fixed $r > 0$ and set $\Omega' := \bigcap_{r > 0, r \in \mathbb{Q}} \Omega_r$. Now, taking limits only along $\mathbb{Q}$,

$$
\sigma_\infty = \lim_{r \to \infty} \tau_\infty^{(r)} \geq \limsup_{r \to \infty} \tau_\infty^{(r)} \geq \liminf_{r \to \infty} \tau_\infty^{(r)} \geq \sigma_\infty.
$$

For the latter estimate, we just used that $\tau_\infty^{(r)} \geq \tau_\infty^{(r-\epsilon)}$ for $\epsilon \in (0, r)$, (3.15). We can use the same argumentation for $\zeta_\infty^{(r)}$. \qed

By iterative construction of subsequences, Kunze and van Neerven prove the following additional results, see [19, Theorem 2.1.(3) and Cor 2.5]. As above, we can drop the adaption assumption without any problems.

**Proposition 3.25.** In the situation of the previous proposition, for all $t > 0$ it holds that

$$
\lim_{n \to \infty} Y_n(t)1_{[0, \sigma_\infty \land \sigma_n]}(t, \cdot) = Y_\infty(t)1_{[0, \sigma_\infty]}(t, \cdot) \quad \text{in probability.}
$$

If, in addition, $\sigma_n = T$ almost surely for all $n \in \mathbb{N}$, then

$$
\lim_{n \to \infty} \sup_{0 \leq t \leq T} \|Y_\infty(t) - Y_n(t)\|_V = 0, \quad \text{in probability.}
$$

In the special situation where $\tau_\infty^{(r)} = \zeta_\infty^{(r)} = 1$, we can get rid of the “$\epsilon$” in Proposition 3.23, as a consequence of Lemma 3.20.

**Proposition 3.26.** If in addition to Assumptions 3.17 and 3.18, it holds that

$$
P \left[ \tau_\infty^{(r)} = \zeta_\infty^{(r)} \right] = 1,
$$

(3.18)

for some $r > 0$, then, as limits in probability,

$$
\lim_{n \to \infty} \tau_n^{(r)} = \lim_{n \to \infty} \zeta_n^{(r)} = \tau_\infty^{(r)}.
$$

In particular, uniformly on compacts in probability,

$$
\lim_{n \to \infty} Y_n(\cdot \land \tau_n^{(r)}) = \lim_{n \to \infty} Y_n(\cdot \land \zeta_n^{(r)}) = Y_\infty(\cdot \land \tau_\infty^{(r)}).
$$
Remark 3.27. Condition (3.18) is essential since one can construct counterexamples in the deterministic case even when \( V = \mathbb{R} \).

On the other hand, it is well known that (3.18) holds true for all \( r > 0 \) when \( Y_\infty \) is a real-valued Brownian motion. However, the situation becomes more delicate, e.g., when \( V \) is an infinite-dimensional Hilbert space and \( Y_\infty \) is the mild solution of a stochastic evolution equation which is not a strong solution on \( V \).

Proof. As in the proof of Proposition 3.23, fix \( r' > r > 0 \) and let \((n_k) \subset \mathbb{N}\) such that

\[
\lim_{k \to \infty} \sup_{0 \leq t \leq T} \left\| Y_{n_k}^{(r)}(t) - Y_\infty^{(r)}(t) \right\| + \sup_{0 \leq t \leq T} \left\| Y_{n_k}^{(r')}(t) - Y_\infty^{(r')}(t) \right\| = 0,
\]

almost surely. Let \( \Omega' \subset \Omega \) be a set of full measure such that \( \tau^{(r)}_\infty = \varsigma^{(r)}_\infty \) and the convergence and continuity of the involved processes hold true on all of \( \Omega' \). Again, since for each \( \epsilon < r' - r \), \( \tau_n^{(r+\epsilon)} \) and \( \varsigma_n^{(r+\epsilon)} \) are the exit times of \( Y_n, n \in \mathbb{N} \), we can now apply Lemma 3.20 to \( Y^{(r')} \) to get the convergence for the exit times.

Moreover, recall that \( Y_n(t \wedge \tau_n^{(r)}) = Y_n^{(r)}(t \wedge \tau_n^{(r)}) \) so that

\[
\sup_{0 \leq t \leq T} \left\| Y_\infty(t \wedge \tau_\infty^{(r)}) - Y_n(t \wedge \tau_n^{(r)}) \right\| \leq \sup_{0 \leq t \leq T} \left\| Y_\infty^{(r)}(t) - Y_n^{(r)}(t) \right\| + \sup_{0 \leq t \leq T} \left\| Y_\infty^{(r)}(t \wedge \tau_\infty^{(r)}) - Y_\infty^{(r)}(t \wedge \tau_n^{(r)}) \right\|.
\]

The first term vanishes as \( n \to \infty \), almost surely, by Assumption 3.18. Switching to subsequences the latter one also vanishes almost surely, since \( Y_\infty^{(r)} \) is uniformly continuous on the compact set \([0, T]\). We can apply the whole procedure to an arbitrary subsequence, and thus, by subsequence criterion we obtain the convergence in probability. \( \square \)

3.3. Localized Wong–Zakai approximation

We are now prepared to extend the Wong–Zakai approximation to the case where the solution of (3.1) might explode in finite time. More precisely, we have the following assumptions on the coefficients.

Assumption 3.28. \( B : E_\alpha \to E_0 \) is Lipschitz continuous on bounded sets.

Assumption 3.29. (i) \( C : E_\alpha \to \mathcal{L}_2(U; E_\alpha) \) is Lipschitz continuous on bounded sets

(ii) \( \sigma_k : E_\alpha \to E_\alpha, k \in \mathbb{N} \) are twice Fréchet differentiable. \( \sigma_k \) and its derivatives map bounded sets into bounded sets, for each \( k \in \mathbb{N} \).

(iii) For all \( N \in \mathbb{N} \) there exists an \( L^{(N)}_\Sigma > 0 \) such that for all \( u, v \in E_\alpha \) with \( \|u\|_\alpha, \|v\|_\alpha \leq N \),

\[
\| \Sigma_n(u) - \Sigma_n(v) \|_{E_\alpha} \leq L_\Sigma \| u - v \|_{E_\alpha}, \quad \forall u, v \in E_\alpha, \forall n \in \mathbb{N}.
\]
(iv) There exists \( \Sigma_\infty : E_\alpha \to E_\alpha \) such that for all \( u \in E_\alpha \),
\[
\lim_{n \to \infty} \Sigma_n(u) = \Sigma_\infty(u).
\]

For the localization procedure, we will consider smooth truncation functions \( h_r \), \( r > 0 \). We assume that \( h_r \in C^\infty(\mathbb{R}_{\geq 0}) \) is non-increasing, constant to 1 on \([0, r^2]\) and constant to 0 on \([(r + 1)^2, \infty)\). Moreover, we assume for the first two derivatives \( h'_r \) and \( h''_r \) that
\[
\sup_{r > 0} (\|h'_r\|_\infty + \|h''_r\|_\infty) \leq c < \infty.
\]
For the truncation of the coefficients, we will need the following two lemmas.

**Lemma 3.30.** The map \( \Xi_r : u \mapsto h_r(\|u\|_\alpha^2) \) is of class \( C^2 \) from \( E_\alpha \) into \( \mathbb{R} \). In addition, for \( i \in \{1, 2\} \),
\[
\text{supp}(D^i \Xi_r) \subset \{ u \in E_\alpha | r \leq \|u\|_\alpha \leq r + 1 \}.
\]
In particular, \( \Xi_r \) has global Lipschitz constant smaller than \( 2c(r + 1) \).

Here, we denote by \( D^i \) the \( i \)-th Fréchet derivative and will write \( D = D^1 \) in the following.

**Proof.** Recall that \( E_\alpha \) is a real Hilbert space, hence
\[
D(\|\cdot\|_\alpha^2)(u)v = 2 \langle u, v \rangle_\alpha
\]
Indeed, for \( u, v \in E_\alpha \) and \( \epsilon > 0 \),
\[
\frac{1}{\epsilon} \left[ \langle u + \epsilon v, u + \epsilon v \rangle_\alpha - \langle u, u \rangle_\alpha \right] = \langle v, u \rangle_\alpha + \langle u, v \rangle_\alpha + \epsilon \langle v, v \rangle_\alpha \to 2 \langle u, v \rangle_\alpha.
\]
Moreover, by bilinearity of the scalar product,
\[
D^2(\|\cdot\|_\alpha^2)(u)[v, w] = 2 \langle v, w \rangle_\alpha, \quad u, v, w \in E_\alpha
\]
Using chain rule, this yields
\[
D \Xi_r(u)v = 2h'_r(\|u\|_\alpha^2) \langle v, u \rangle_\alpha,
\]
\[
D^2 \Xi_r(u)[v, w] = 2h'_r(\|u\|_\alpha^2) \langle v, w \rangle_\alpha + 4h''_r(\|u\|_\alpha^2) \langle v, u \rangle_\alpha \langle w, u \rangle_\alpha.
\]
The statement on the support follows directly from definition of \( h_r \). Mean-value theorem for Fréchet derivatives yields
\[
\|\Xi_r\|_{\text{Lip}} \leq \sup_{u \in E_\alpha} \|D \Xi_r(u)\|_{\mathcal{L}(E_\alpha; \mathbb{R})} \leq 2c(r + 1).
\]
\( \square \)
Lemma 3.31. Let $V_0$ and $V$ be Banach spaces, $\Phi : V \to V_0$ Lipschitz continuous on bounded sets, and $h : V \to \mathbb{R}$ Lipschitz continuous with Lipschitz constant $L_h$ and

$$\text{supp}(h) \subset \{ u \in V \mid \| u \|_V \leq r \} =: \bar{K}_r,$$

for some $r \in \mathbb{N}$. Denote by $L_r$ the Lipschitz constant of $\Phi$ on $\bar{K}_r$. The map $u \mapsto h(u)\Phi(u)$ is globally bounded and Lipschitz continuous from $V$ into $V_0$ with Lipschitz constant

$$\| h \Phi \|_{\text{Lip}(V; V_0)} \leq L_r \sup_{w \in \bar{K}_r} \| h(w) \| + L_h \sup_{w \in \bar{K}_r} \| \Phi(w) \|_V.$$

(3.20)

Proof. Let $u, v \in V$. W.l.o.g. assume that $\| v \|_V \leq \| u \|_V$ and write

$$h(u)\Phi(u) - h(u)\Phi(v) = h(u)(\Phi(u) - \Phi(v)) + \Phi(v)(h(u) - h(v)).$$

The first term vanishes when $\| u \|_V \geq r$. Else, we know that $\| v \|_V \leq \| u \|_V < r$ by assumption, so that

$$\| h(u)\Phi(u) - h(v)\Phi(v) \|_{V_0} \leq \sup_{w \in \bar{K}_r} |h(w)| L_r \| u - v \|_V.$$

(3.21)

The second summand vanishes when $\| v \|_V \geq r$. Else, Lipschitz continuity of $h$ yields

$$\| \Phi(v) \|_{V_0} |h(u) - h(v)| \leq \| h \|_{\text{Lip}(V; \mathbb{R})} \sup_{w \in \bar{K}_r} \| \Phi(w) \|_{V_0} \| u - v \|_V.$$

□

We consider now again (3.1) with the assumptions introduced above. Recall that we have $\alpha \in [0, 1)$ fixed so that $E_\alpha$ and $E_0$ will take, respectively, the part of $V$ and $V_0$ in the previous lemma.

We define the truncated coefficients for $r > 0$ as

$$B^{(r)}(u) := h_r(\| u \|_\alpha^2) B(u), \quad C^{(r)}(u) := h_r(\| u \|_\alpha^2) C(u), \quad u \in E_\alpha.$$ 

We now again use the notation introduced in the beginning of Sect. 3, in particular,

$$\sigma_{k}^{(r)}(u) := C^{(r)}(u) e_k, \quad \Sigma_{n}^{(r)}(u) := \frac{1}{2} \sum_{k=1}^{n} D\sigma_{k}^{(r)}(u) \sigma_{k}^{(r)}(u).$$

for $u \in E_\alpha$. Indeed, $B^{(r)}$ and $C^{(r)}$ do fulfill the assumptions we have imposed formerly.

Lemma 3.32. For all $r > 0$, $B^{(r)}$ and $C^{(r)}$ fulfill Assumption 3.2 and 3.3, respectively.

Proof. First, let us emphasize that Lemma 3.31 yields global boundedness and Lipschitz continuity of $B^{(r)}$ and $C^{(r)}$. The Fréchet differentiability of $\sigma_{k}^{(r)}$ and its derivative
is inherited from differentiability of \( \sigma_k \), \( D\sigma_k \) and by application of Lemma 3.30. This also yields boundedness of \( \sigma_k \) and its first two derivatives.

It remains to verify the properties of \( \Sigma_n^{(r)} \). For all \( u \in E_\alpha \),

\[
\Sigma_n^{(r)}(u) = \sum_{k=1}^{n} (D \Xi_r(u) \sigma_k^{(r)}(u)) \sigma_k(u) + \sum_{k=1}^{n} \Xi_r(u) D\sigma_k(u) \sigma_k^{(r)}(u) \\
= 2h_r(\|u\|_\alpha^2) h'_r(\|u\|_\alpha^2) \sum_{k=1}^{n} \langle \sigma_k(u), u \rangle_\alpha \sigma_k(u) + h_r(\|u\|_\alpha^2)^2 \Sigma_n(u).
\]

(3.22)

To see that the first summation in the last equation converges as \( n \to \infty \), we apply triangle and Cauchy–Schwarz inequality,

\[
\left\| \sum_{k=1}^{n} \langle \sigma_k(u), u \rangle_\alpha \sigma_k(u) \right\|_\alpha \leq \|u\|_\alpha \sum_{k=1}^{n} \|\sigma_k(u)\|_\alpha^2.
\]

(3.23)

In fact, since \( C(u) \in \mathcal{L}_2(U; E_\alpha) \), the sequence \( (\sigma_k(u))_k \) is square summable in \( E_\alpha \), and we can define

\[
\Sigma_\infty^{(r)}(u) := 2h_r(\|u\|_\alpha^2) h'_r(\|u\|_\alpha^2) \sum_{k=1}^{\infty} \langle \sigma_k(u), u \rangle_\alpha \sigma_k(u) + h_r(\|u\|_\alpha^2)^2 \Sigma_\infty(u).
\]

(3.24)

We obtain \( \Sigma_n^{(r)} \to \Sigma_\infty^{(r)} \) strongly in \( E_\alpha \) by assumptions on \( \Sigma_n \) and the same estimates as in (3.23), more detailed

\[
\left\| \Sigma_\infty^{(r)}(u) - \Sigma_n^{(r)}(u) \right\|_\alpha \leq \|\Sigma_\infty - \Sigma_n(u)\|_\alpha + \sum_{k=n+1}^{\infty} |\langle \sigma_k(u), u \rangle_\alpha| \|\sigma_k(u)\|_\alpha \\
\leq \|\Sigma_\infty(u) - \Sigma_n(u)\|_\alpha + \|u\|_\alpha \sum_{k=n+1}^{\infty} \|\sigma_k(u)\|_\alpha^2 \to 0.
\]

(3.25)

To prove Lipschitz continuity of \( \Sigma_n^{(r)} \), we go back into (3.22) and first apply Lemma 3.31 to the last summand. For the remaining part, we decompose for \( u, v \in E_\alpha \) and for \( n \in \mathbb{N} \)

\[
\langle \sigma_k(u), u \rangle_\alpha \sigma_k(u) - \langle \sigma_k(v), v \rangle_\alpha \sigma_k(v) = \langle \sigma_k(u), u - v \rangle_\alpha \sigma_k(u) + \langle \sigma_k(u) - \sigma_k(v), v \rangle_\alpha \sigma_k(u) \\
+ \langle \sigma_k(v), v \rangle_\alpha (\sigma_k(u) - \sigma_k(v)) \\
=: R_1^k + R_2^k + R_3^k.
\]

With the same estimates as for (3.23), we get

\[
\sum_{k=1}^{n} \left\| R_1^k \right\|_\alpha \leq \|u - v\|_\alpha \|C(u)\|_{\mathcal{L}_2(U; E_\alpha)}^2,
\]
which is independent of \( n \in \mathbb{N} \). For the remaining part, we apply Cauchy–Schwarz inequality on \( E_\alpha \) and on \( \mathbb{R}^n \), to get
\[
\sum_{k=1}^{n} \left\| R_k^x \right\|_\alpha + \left\| R_k^3 \right\|_\alpha \leq \| v \|_\alpha \sum_{k=1}^{n} \| \sigma_k(u) - \sigma_k(v) \|_\alpha (\| \sigma_k(u) \|_\alpha + \| \sigma_k(v) \|_\alpha )
\]
\[
\leq \| v \|_\alpha \sqrt{\sum_{k=1}^{n} \| \sigma_k(u) - \sigma_k(v) \|_\alpha^2} \left( \sqrt{\sum_{k=1}^{n} \| \sigma_k(u) \|_\alpha^2} + \sqrt{\sum_{k=1}^{n} \| \sigma_k(v) \|_\alpha^2} \right)
\]
\[
\leq \| v \|_\alpha \left( \| C(u) \|_{\mathcal{L}_2(U; E_\alpha)} + \| C(v) \|_{\mathcal{L}_2(U; E_\alpha)} \right) \| C(u) - C(v) \|_{\mathcal{L}_2(U; E_\alpha)}.
\]
In other words, we have Lipschitz continuity on bounded sets, with local Lipschitz constants independent of \( n \in \mathbb{N} \). By Lemma 3.31 and (3.22), this yields that \( \Sigma_n^{(r)} \), \( n \in \mathbb{N} \), are globally Lipschitz with uniform Lipschitz constant. \( \square \)

We denote by \( X^{(r)} \) the unique global mild solution on \( E_\alpha \), of the truncated equation,
\[
dX^{(r)}(t) = \left[ AX^{(r)}(t) + B^{(r)}(X^{(r)}(t)) \right] \, dt + C^{(r)}(X^{(r)}(t)) \, dW_t,
\]
with initial condition \( X^{(r)}(0) = X_0 \). Respectively, \( X_n^{(r)} \) and \( Z_m^{(r)} \), for \( n, m \in \mathbb{N}, r > 0 \), denote the solutions of the localized approximating equations
\[
\begin{cases}
\begin{align*}
&dX_n^{(r)}(t) = \left[ AX_n^{(r)}(t) + B_n^{(r)}(X^{(r)}(t)) \right] \, dt + \sum_{k=1}^{n} \sigma_k^{(r)}(X_n^{(r)}(t)) \, d\beta_k(t), \\
&X_n^{(r)}(0) = X_0,
\end{align*}
\end{cases}
\]
and \( \omega \)-wise,
\[
\begin{cases}
\begin{align*}
&\frac{\partial}{\partial t} Z_{m,n}^{(r)}(t) = A Z_{m,n}^{(r)}(t) + B^{(r)}(Z_{m,n}^{(r)}(t)) - \Sigma_\infty^{(r)}(Z_{m,n}^{(r)}(t)) \\
&\quad + \sum_{k=1}^{n} \sigma_k^{(r)}(Z_{m,n}^{(r)}(t)) \beta_k(t), \quad t > 0,
\end{align*}
\end{cases}
\]
\[
Z_{m,n}^{(r)}(0) = X_0.
\]
Here \( B_n^{(r)} := B^{(r)} - \Sigma_n^{(r)} + \Sigma_\infty^{(r)} \). The latter one has been defined in (4.4).

With \( X, X_n \) and \( Z_{m,n} \), for \( m, n \in \mathbb{N} \) we denote the unique maximal solutions of the non-truncated equations (3.1), (3.6) and (3.5), respectively. Moreover, by \( \tau, \tau_n \) and \( \tau_{m,n} \) we denote their \( E_\alpha \)-explosion times, and for \( r > 0 \),
\[
\begin{align*}
\tau^{(r)} &:= \inf \{ t \geq 0 \mid t < \tau, \| X(t) \|_\alpha > r \}, \\
\tau_n^{(r)} &:= \inf \{ t \geq 0 \mid t < \tau_n, \| X_n(t) \|_\alpha > r \}, \\
\tau_{m,n}^{(r)} &:= \inf \{ t \geq 0 \mid t < \tau_{m,n}, \| Z_{m,n}(t) \|_\alpha > r \}.
\end{align*}
\]
In the same way, we define the exit times of the open balls

\[ s^{(r)} := \inf \{ t \geq 0 \mid \|X^{(r)}(t)\|_{\alpha} \geq r \} = \inf \{ t \geq 0 \mid t < \tau, \|X(t)\|_{\alpha} \geq r \}, \]

\[ s_{n}^{(r)} := \inf \{ t \geq 0 \mid \|X_{n}^{(r)}(t)\|_{\alpha} \geq r \} = \inf \{ t \geq 0 \mid t < \tau_{n}, \|X_{n}(t)\|_{\alpha} \geq r \}, \]

\[ s_{m,n}^{(r)} := \inf \{ t \geq 0 \mid \|Z_{m,n}^{(r)}(t)\|_{\alpha} \geq r \} = \inf \{ t \geq 0 \mid t < \tau_{m,n}, \|Z_{m,n}(t)\|_{\alpha} \geq r \}. \]

Remark 3.33. \( Z_{m,n}^{(r)} \) and \( \tau_{m,n}^{(r)} \) can be seen as \( \mathcal{F}_{t} \)-measurable functions of \( \omega \in \Omega \), but \( Z_{m,n}^{(r)} \) is not adapted and hence, \( \tau_{m,n}^{(r)} \) is not a stopping time.

The following result extends [27, Theorem 2.1 and Prop 7.3] and seems to be new even when \( \alpha = 0 \).

Theorem 3.34. Let Assumptions 3.1, 3.28 and 3.29 hold true. For all \( r > 0, \epsilon > 0 \), it holds that

\[ \lim_{n \to \infty} \lim_{m \to \infty} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| X(t \wedge \tau^{(r)}) - Z_{m,n}(t \wedge s_{m,n}^{(r+\epsilon)} \wedge \tau^{(r)}) \right|^{2/p} \right] = 0 \]

Proof. Thanks to Lemma 3.32 we can apply Theorem 3.6 and observe that the unique solution \( X^{(r)} \) of the localized equation can be approximated first by the solutions of equations with finite-dimensional noise \( X_{n}^{(r)} \) and then by Wong–Zakai approximations \( Z_{m,n}^{(r)} \). By uniqueness claims of the existence results, we get

\[ X^{(r)} = X^{(r')}, \text{ on } [0, \tau^{(r)}], \quad X_{n}^{(r)} = X_{n}^{(r')}, \text{ on } [0, \tau_{n}^{(r)}], \quad Z_{m,n}^{(r)} = Z_{m,n}^{(r')}, \text{ on } [0, \tau_{m,n}^{(r)}], \]

for \( r' > r \). Now, the assumptions of Proposition 3.23 are fulfilled, respectively, for \( Z_{m,n} \) and \( X_{n} \), and for \( X_{n} \) and \( X \). For \( \epsilon' \in (0, \epsilon) \), triangle inequality yields

\[
\sup_{0 \leq t \leq T} \left| X(t \wedge \tau^{(r)}) - Z_{m,n}(t \wedge s_{m,n}^{(r+\epsilon)} \wedge \tau^{(r)}) \right|_{\alpha} \\
\leq \sup_{0 \leq t \leq T} \left| X(t \wedge \tau^{(r)}) - X_{n}(t \wedge \tau_{n}^{(r+\epsilon')} \wedge \tau^{(r)}) \right|_{\alpha} \\
+ \sup_{0 \leq t \leq T} \left| X_{n}(t \wedge \tau^{(r)} \wedge \tau_{m,n}^{(r+\epsilon} \wedge \tau^{(r)}) - Z_{m,n}(t \wedge s_{m,n}^{(r+\epsilon)} \wedge \tau^{(r)}) \right|_{\alpha}. \tag{3.29}
\]

For every subsequence in \( n \), there exists a further subsubsequence such that the first summand on the right-hand side vanishes as \( n \to \infty \) by Proposition 3.23. Applying Lebesgue’s dominated convergence theorem, the convergence holds true also in \( L^{2p} \), along the subsequences.

For the second term, we take converging subsequences \( (X_{n_{k}}^{(r)}) \) and \( (u_{m_{l}^{k},n_{k}}^{(r)}) \) such that on a set of full measure \( \Omega' \subset \Omega \), uniformly in \( [0, T] \),

\[ X_{n_{k}}^{(r)} \to X^{(r)}, \quad X_{n_{k}}^{(r')} \to X^{(r')}, \quad \text{as } k \to \infty \]

\[ u_{m_{l}^{k},n_{k}}^{(r)} \to X_{n_{k}}^{(r)}, \quad u_{m_{l}^{k},n_{k}}^{(r')} \to X_{n_{k}}^{(r')}, \quad \text{as } l \to \infty. \]
Applying the proof of Lemma 3.22 to $X_n^{(r)}$, there exists $k_0 \in \mathbb{N}$ such that for all $k > k_0$ it holds that
\[
\tau_{n_k}^{(r+\epsilon)} \geq \tau^{(r)} .
\] (3.30)

Therefore, the second term on the right-hand side of (3.29) can be estimated by
\[
\sup_{0 \leq t \leq T} \left| X_{n_k}(t \wedge \tau_{n_k}^{(r+\epsilon)}) - u_{m_l^k,n_k}(t \wedge \zeta_{m_l^k,n_k}^{(r+\epsilon)} \wedge \tau_{n_k}^{(r+\epsilon)}) \right|_\alpha , \quad \forall k > k_0.
\]

This term goes to 0 as $l \to \infty$, and dominated convergence yields $L^{2p}$-convergence.

Let us be more precise,
\[
\mathbb{E} \sup_{0 \leq t \leq T} \left( \left| X_{n_k}(t \wedge \tau_{n_k}^{(r+\epsilon)} \wedge \tau^{(r)}) - u_{m_l^k,n_k}(t \wedge \zeta_{m_l^k,n_k}^{(r+\epsilon)} \wedge \tau_{n_k}^{(r)} \wedge \tau^{(r)}) \right|^{2p}_\alpha \right.
\]
\[
\leq \mathbb{E} \sup_{0 \leq t \leq T} \left( \left| X_{n_k}(t \wedge \tau_{n_k}^{(r+\epsilon)}) - u_{m_l^k,n_k}(t \wedge \zeta_{m_l^k,n_k}^{(r+\epsilon)} \wedge \tau_{n_k}^{(r+\epsilon)}) \right|^{2p}_\alpha 1_{\tau_{n_k}^{(r+\epsilon)} \geq \tau^{(r)}} \right)
\]
\[
+ K_{\alpha,p} (2r + \epsilon' + \epsilon)^{2p} \mathbb{P} \left( \tau_{n_k}^{(r+\epsilon)} < \tau^{(r)} \right).
\]

The first term goes to 0, for $l \to \infty$, as we have discussed above. The second one does, as $k \to \infty$ owing to (3.30). For each subsequence, we can find again such subsequences so that $L^{2p}$-convergence holds true. But since $L^{2p}$-convergence is metrizable, convergence is equivalent to the statement that every subsequence admits a subsequence converging to the same limit, which we have just shown. \hfill \Box

We close this section with a result for the case where linear growth of $B$ and $C$ holds true, but not necessarily global Lipschitz continuity and boundedness. In this case, the solutions still exist globally, cf. [18, Theorem 3.20].

**Theorem 3.35.** Assume that Assumptions 3.1, 3.28 and 3.29 hold true and, in addition, that there exists an $M > 0$ such that
\[
\|B(u)\|_0 + \|C(u)\|_{L^2([0,T];E_\alpha)} \leq M (1 + \|u\|_\alpha) , \quad \forall u \in E_\alpha ,
\] (3.31)

and $D\sigma_k$ is globally bounded. Then, uniformly on compacts in probability,
\[
\lim_{n \to \infty} \lim_{m \to \infty} Z_{m,n} = X.
\]

**Proof.** First note that under the given constraints the solutions of (3.26)–(3.28) exist globally and take values almost surely in $C([0,T];E_\alpha)$, cf. [18, Theorem 3.20]. We have shown already that the assumptions of Proposition 3.25 are fulfilled which finally yields the convergence claim. \hfill \Box
4. Phase separation and approximation: proofs

We now apply the previous two sections to (1.2). Using the notation from Sect. 1, we will work on the spaces,

\[ \Omega^2 := L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+) \oplus \mathbb{R}, \quad \mathcal{S}^k(\mathbb{R}_+) := H^k(\mathbb{R}_+) \oplus H^k(\mathbb{R}_+) \oplus \mathbb{R}. \]

In order to reformulate the coupled systems of S(P)DEs (1.9), we define the coefficients

\[ A = \begin{pmatrix} \eta_+ \Delta_+ & 0 & 0 \\ 0 & \eta_- \Delta_- & 0 \\ 0 & 0 & 0 \end{pmatrix} - c \text{Id}, \]

\[ B(u)(x) = \begin{pmatrix} \mu_1(x, u_1(x), u_1'(x)) + \frac{\partial}{\partial x} u_1(x) \cdot \varrho (\mathcal{I}(u)) \\ \mu_2(x, u_2(x), u_2'(x)) - \frac{\partial}{\partial x} u_2(x) \cdot \varrho (\mathcal{I}(u)) \end{pmatrix} + cu(x), \]

\[ \Sigma_n(u)(x) = \frac{1}{2} \sum_{k=1}^{n} \left( \frac{\partial}{\partial y} \sigma_1(x, u_1(x)) \sigma_1(x, u_1(x)) (T_\xi e_k (u_3 + x))^2 \\ \frac{\partial}{\partial y} \sigma_2(x, u_2(x)) \sigma_2(x, u_2(x)) (T_\xi e_k (u_3 - x))^2 \right), \]

\[ C(w)(x) = \begin{pmatrix} \sigma_1(x, u_1(x)) T_\xi w(x_1 + x) \\ \sigma_2(x, u_2(x)) T_\xi w(x_2 - x) \end{pmatrix}, \]

where

\[ \mathcal{I}(u) := \left\{ \left( \frac{\partial}{\partial x} u_1, \frac{\partial}{\partial x} u_2 \right), \quad \kappa_+ = \kappa_- = \infty, \right. \]

\[ \left. \left( u_1, u_2 \right), \quad \kappa_+ < \kappa_- < \infty, \right\}, \]

for \( u = (u_1, u_2, x_*) \in \mathcal{S}^2, \ w \in U, \ x \geq 0. \) Here, we write \( \mu_1 = \mu_+, \mu_2(x, y, z) := -\mu(-x, y, -z), \) and \( \sigma_1 := \sigma_+, \sigma_- := \sigma_-(x, y), \) \( x, y, z \in \mathbb{R}. \)

As we will see below, \( \Sigma_n \) converges under sufficient assumptions on \( \sigma_1/2 \) and \( \xi \) strongly to

\[ \Sigma_\infty(u)(x) = \frac{1}{2} \begin{pmatrix} \frac{\partial}{\partial y} \sigma_1(x, u_1(x)) \sigma_1(x, u_1(x)) \| \xi (u_3 + x, .) \|_{L^2(\mathbb{R})}^2 \\ \frac{\partial}{\partial y} \sigma_2(x, u_2(x)) \sigma_2(x, u_2(x)) \| \xi (u_3 - x, .) \|_{L^2(\mathbb{R})}^2 \end{pmatrix}, \quad (4.1) \]

The domain of the diagonal operator \( A \) is then given by

\[ \mathcal{D}(A) = \mathcal{D}(-\mathcal{A}_+ \times \mathcal{D}(-\mathcal{A}_-) \times \mathbb{R}, \]

where \( \mathcal{A}_+ \) and \( \mathcal{A}_- \) denote the Laplacian on \( \mathbb{R}_+ \) with respective boundary conditions defined in (1.10). The constant \( c > 0 \) is arbitrary and used to shift the spectrum of \( -\mathcal{A} \) to the positive half-line, so that \( -\mathcal{A} \) is positive self-adjoint. Hence, its fractional powers \( (-\mathcal{A})^\alpha, \alpha \in \mathbb{R}, \) are well defined and we set \( E := \mathcal{L}^2. \) Let us shortly note that \( E_\alpha \subset \mathcal{S}^{2\alpha} \) for all \( \alpha > 0 \) and \( E_1 = \mathcal{D}(A) \) with equivalence of norms.
Writing \( X = (u_1, u_2, x) \), (1.9) becomes the stochastic evolution equation

\[
\mathrm{d}X(t) = [AX(t) + B(X(t))] \, \mathrm{d}t + C(X(t)) \, \mathrm{d}W_t,
\]

with initial conditions \( X(0) = X_0 \in \mathcal{D}(A) \). The approximating equations (1.21) then become random evolution equations, with initial data \( Z_{m,n}(0) = X_0, m, n \in \mathbb{N} \), which read (piece-wise where \( \hat{\beta}^k_m \) is well defined) as

\[
\frac{\partial}{\partial t} Z_{m,n}(t) = A Z_{m,n}(t) + B(Z_{m,n}(t)) - \sum_{k=1}^{\infty} (C(Z_{m,n}(t)) e_k) \hat{\beta}^k_m(r).
\]

(4.3)

**Definition 4.1.** A set \( \mathcal{M} \subset L^2 \) is called forward invariant for the stochastic evolution equation (4.2) with initial conditions \( X(0) = X_0 \), if \( X_0 \in \mathcal{M} \) yields \( X \in \mathcal{M} \) on \([0, \tau] \). where \((X, \tau)\) is the unique maximal mild solution of (4.2).

**Remark 4.2.** As we have intensively discussed in the previous section, the forward invariance property is defined in the same way for deterministic or random evolution equations.

For the following discussion, set

\[
\mathcal{M} := \{(u_1, u_2, x) \in L^2 \mid u_1 \geq 0, u_2 \leq 0 \text{ a.e.}\} = L^2_+ \times L^2_- \times \mathbb{R},
\]

\[
L^2_+ := \{u \in L^2(\mathbb{R}_+^+) \mid u \geq 0 \text{ a.e.}\}, \quad L^2_- := \{u \in L^2(\mathbb{R}_+^+) \mid u \leq 0 \text{ a.e.}\}.
\]

The following example illustrates why we make the detour using the geometric criterions from Sect. 2.2, instead of the direct, infinite-dimensional formulation of “inward-pointing” and “parallel-to-the-boundary” constraints.

**Example 4.3.** Set \( E := L^2(\mathbb{R}_+^+) \), \( A := \Delta \) with Dirichlet boundary conditions, say, and \( C(u)w := \sigma(x, u(x))T_x w(x) \), where \( \sigma \) is as in Assumption (Noise0). A classical approach to prove forward invariance for the cone of nonnegative functions in \( L^2 \) is to show

\[
\|X(t) - \| E^2 \leq 0, \quad \forall t \geq 0,
\]

for \( u^- := \min\{0, u\} \), see [33, Lemma 3.6] for a mathematically rigorous procedure. Here, \( X \) is the solution of the stochastic evolution equation

\[
\mathrm{d}X(t) = AX(t) \, \mathrm{d}t + C(X(t)) \, \mathrm{d}W_t.
\]

(4.4)

At least formally, on that way one ends up with the “parallel-to-the-boundary” condition

\[
\langle C(u)w, u^- \rangle = 0, \quad \forall u \in H^1(\mathbb{R}_+).
\]
With Fubini theorem, this can be rewritten as

\[
0 = \int_0^\infty \sigma(x, u(x)) u^-(x) T_\zeta w(x) \, dx \\
= \int_\mathbb{R} \int_0^\infty \sigma(x, u(x)) u^-(x) \zeta(x, y) \, dx \, w(y) \, dy.
\]

Since the equality has to be true for all \( w \in L^2(\mathbb{R}) \), this requires the inner integral to be 0 for all \( y \in \mathbb{R} \), and thus

\[
\int_0^\infty \sigma(x, u(x)) u(x) 1_{u(x) \leq 0} \zeta(x, y) \, dy = 0.
\]

Except for degenerate choices of \( \zeta \) and due to differentiability constraints on \( \sigma \), this excludes the case \( \sigma(x, u(x)) = \sigma \cdot u(x) \) for a constant \( \sigma \in \mathbb{R} \) so that the condition would be too restrictive.

We observe already that due to structure of \( \mathcal{B} \), we cannot expect to get global boundedness of \( \mathcal{B} \) which means that Assumption 3.2 will not be satisfied. Instead, we have to rely on the results which we obtained with help of the truncation procedure in Sect. 3.3. Having this in mind, we now discuss Dirichlet and first-order boundary conditions separately. Since we will be able to reuse many calculations for Dirichlet boundary conditions, we start with the first-order case.

4.1. First-order boundary conditions

Let \( \kappa_1, \kappa_2 < \infty \), then

\[
\mathcal{D}(\Delta_+) = \{ u \in H^2(\mathbb{R}_+) \mid \frac{\partial}{\partial x} u(0) = \kappa_1 u(0) \}, \\
\mathcal{D}(\Delta_-) = \{ u \in H^2(\mathbb{R}_+) \mid \frac{\partial}{\partial x} u(0) = \kappa_2 u(0) \},
\]

and \( \mathcal{D}(\mathcal{A}) = \mathcal{D}(\Delta_+) \times \mathcal{D}(\Delta_-) \times \mathbb{R} \). Recall that up to equivalences of norms \( E = \mathcal{L}^2 \), \( E_1 = \mathcal{D}(\mathcal{A}) \) and \( E_\frac{1}{2} = \mathcal{H}^1 \), cf. [12].

**Proposition 4.4.** Let Assumptions (Interface), (Correlation), (Drift1) and (Noise1) hold true. Then, \( \mathcal{B} : E_{\frac{1}{2}} \to E \) and \( \mathcal{C} : E_{\frac{1}{2}} \to \mathcal{L}^2(U; E_{\frac{1}{2}}) \) are Lipschitz continuous on bounded sets.

**Proof.** This is shown in [25, Lemma 3.6 and 3.11(i)]. \( \square \)

Moreover, we observe from Appendix B, Theorem B.9, that under Assumptions (Correlation) and (Noise1), \( \mathcal{C} : E_{\frac{1}{2}} \to \mathcal{L}^2(U; E_{\frac{1}{2}}) \) of class \( C^2 \) and its derivatives map bounded sets into bounded sets. It remains to show that the \( \Sigma_n \) admit Lipschitz constants uniformly in \( n \in \mathbb{N} \).

**Lemma 4.5.** Assume that (Correlation) and (Noise1) are satisfied. For all \( N \in \mathbb{N} \), there exists \( L_{\Sigma}^{(N)} > 0 \) such that for all \( u, v \in \mathcal{H}^1 \) with norm smaller than \( N \) it holds that

\[
\| \Sigma_n(u) - \Sigma_n(v) \|_{\mathcal{H}^1} \leq L_{\Sigma}^{(N)} \| u - v \|_{\mathcal{H}^1}, \quad \forall n \in \mathbb{N}.
\]
Proof. First recall that for all \(x \in \mathbb{R}\), by Parseval’s identity

\[
\sum_{k=1}^{\infty} |T_\varepsilon e_k(x)|^2 = \sum_{k=1}^{\infty} |(\zeta(x, .), e_k)_{L^2(\mathbb{R})}|^2 = \|\zeta(x, .)\|_{L^2}^2. \tag{4.6}
\]

Lemma B.4 tells us that \(T_\varepsilon e_k\) takes values in the Banach algebra \(BUC^1(\mathbb{R}_+)\) and so does \((T_\varepsilon e_k)^2\). Hence, with \(N_{D\sigma\sigma}(u) := \frac{\partial}{\partial y}\sigma(. , u)\sigma(. , u)\), and Lemma B.5,

\[
\begin{align*}
\|2\Sigma_n^1(u) - 2\Sigma_n^1(v)\|_{H^1} \leq & \ K \|N_{D\sigma\sigma}(u) - N_{D\sigma\sigma}(v)\|_{H^1} \sup_{z \in \mathbb{R}} \sum_{i=0}^{1} \sum_{k=1}^{n} |T_{\zeta(i)} e_k(z)|^2 \\
+ & \ K \|N_{D\sigma\sigma}(u)\|_{H^1(\mathbb{R}_+)} \sup_{z \in \mathbb{R}} \sum_{i=0}^{1} \sum_{k=1}^{n} |T_{\zeta_{u,v}} e_k(z)|^2 \\
\leq & \ K \|N_{D\sigma\sigma}(u) - N_{D\sigma\sigma}(v)\|_{H^1} \sup_{z \in \mathbb{R}} \sum_{i=0}^{1} \|\zeta(i)(z, .)\|_{L^2} \\
+ & \ K \|N_{D\sigma\sigma}(u)\|_{H^1(\mathbb{R}_+)} \sup_{z \in \mathbb{R}} \|\zeta(i)(z - u_3, .)\|_{L^2} \\
- & \zeta(i)(z - v_3, .)L^2 \tag{4.7}
\end{align*}
\]

where we set \(\zeta_{u,v}(x, y) := \zeta(x - u_3, y) - \zeta(x - v_3, y)\). Similar to (B.5) we get by application of fundamental theorem of calculus and Fubini theorem

\[
\begin{align*}
\int_{\mathbb{R}} \left|\zeta(i)(z - x, y) - \zeta(i)(z - \tilde{x}, y)\right|^2 \, dy \\
\leq & \int_{\mathbb{R}} \int_{0}^{1} \left|\zeta^{(i+1)}(z + \epsilon(x - \tilde{x}), y)\right|^2 |x - \tilde{x}|^2 \, dy \, d\epsilon \\
\leq & \ |x - \tilde{x}|^2 \sup_{z \in \mathbb{R}} \|\zeta^{(i+1)}(z, .)\|_{L^2}^2, \tag{4.8}
\end{align*}
\]

which is finite by Assumption (Correlation). Note that, since \(\sigma_1\) and \(\sigma_2\) fulfill Assumption A.2 for \(m = 2\), it holds that \(\sigma = \partial_y \sigma\) fulfills this assumption for \(m = 1\), and thus, by Theorem A.7 the Nemytskii operator \(N_{D\sigma\sigma}\) is Lipschitz on bounded sets on \(H^1(\mathbb{R}_+)\), for \(\sigma = \sigma_1\) and \(\sigma = \sigma_2\). Hence, the local Lipschitz constants of \(\Sigma_n\) depend on \(\sigma_1\), \(\sigma_2\) and \(\zeta\) only, but are particularly independent of \(n \in \mathbb{N}\). \(\blacksquare\)

Lemma 4.6. Let Assumption (Correlation) and (Noise1) be satisfied. Then, for \(\Sigma_\infty\) defined in (4.1), it holds that \(\lim_{n \to \infty} \Sigma_n(u) = \Sigma_\infty(u) \in \mathcal{F}^1\), for all \(u \in \mathcal{F}^1\).

Proof. Again, let \(\sigma = \sigma_1\) or \(\sigma = \sigma_2\) and set \(N_{D\sigma\sigma}(u) := \frac{\partial}{\partial y}\sigma(x, u(x))\sigma(x, u(x))\) and fix \(u \in H^1(\mathbb{R}_+)\) and \(z \in \mathbb{R}\). By Parseval’s identity, for all \(x \in \mathbb{R}\),

\[
\sum_{k=1}^{n} \frac{\partial}{\partial y}\sigma(x, u(x))\sigma(x, u(x)) T_\varepsilon e_k(x - z)^2 \\
\rightarrow & \frac{\partial}{\partial y}\sigma(x, u(x))\sigma(x, u(x)) \|\zeta(x - z, .)\|_{L^2}^2,
\]
as \( n \to \infty \). Moreover, the sequence is bounded by the square-integrable function

\[
x \mapsto N_{D\sigma\sigma}(u)(x) \sup_{z \in \mathbb{R}} \| \xi(z,.) \|_{L^2}^2,
\]

and hence, Lebesgue’s dominated convergence theorem yields that \( \Sigma_n \to \Sigma_\infty \) in \( L^2(\mathbb{R}_+) \). The first weak derivative is given by

\[
\frac{d}{dx} N_{D\sigma\sigma}(u)(x) \sum_{k=1}^n \left| T_\xi e_k(x-z) \right|^2 = \left( \frac{d}{dx} N_{D\sigma\sigma}(u)(x) \right) \sum_{k=1}^n \left| T_\xi e_k(x-z) \right|^2 + 2 N_{D\sigma\sigma}(u)(x) \sum_{k=1}^n T_\xi' e_k(x-z) T_\xi e_k(x-z).
\]

(4.9)

More detailed computations concerning the derivatives of \( T_\xi \) will be given in Appendix B. Note that the second summand converges, again by Parseval’s identity, to

\[
N_{D\sigma\sigma}(u)(x) \left[ \xi'(x-z, .), \xi(x-z, .) \right]_{L^2(\mathbb{R})}.
\]

By dominated convergence theorem, the series converges in \( L^2(\mathbb{R}_+) \) and in the same way we can treat the first summand in (4.9). Summarizing, \( \Sigma_n(u) \) is \( \mathcal{F}_1 \)-convergent and thus the limit \( \Sigma_\infty(u) \) is an element of \( \mathcal{F}_1 \), too.

Collecting the latter two results, the assumptions of Theorem 3.34 are satisfied and we observe for all \( r > 0, \epsilon > 0, \)

\[
\lim_{n \to \infty} \lim_{m \to \infty} \mathbb{E} \sup_{0 \leq t \leq T} \left\| X(t \wedge \tau^{(r)}) - Z_{m,n}(t \wedge \tau_{m,n}^{(r+\epsilon)} \wedge \tau^{(r)}) \right\|_{\mathcal{F}_1}^{2p} = 0. \quad (4.10)
\]

This already finishes the proof of Theorem 1.6. Part (b) of Theorem 1.1 is covered by the following theorem.

**Theorem 4.7.** Let Assumptions (Interface), (Correlation), (Drift1), (Noise1) and in addition the point-wise inward-pointing assumption (PIP) hold true. Then, the set \( \mathbb{M} \cap \mathcal{F}_1 \) is forward invariant for the stochastic evolution equation (4.2).

**Proof.** Let \( X = (u_1, u_2, x_\ast) \) and \( Z := (w_1, w_2, y_\ast) \) be the unique mild solutions, respectively, of (4.2) and (4.3) on \( \mathcal{F}_1 \) and write

\[
\Phi_{m,n}(t, X) := B(X) - \Sigma_\infty + \sum_{k=1}^n C(X) e_k \hat{\beta}_k^m(t).
\]

**Step I.** First, we show that the nonlinearity \( \Phi_{m,n} \) fulfills the Nagumo condition on \( [0, \frac{1}{m} T] \). Note that \( \Phi_{m,n} \) is constant in time now and so we write

\[
\Phi_{m,n}(t, X) =: (\Phi_{m,n}^1(X), \Phi_{m,n}^2(X), \Phi_{m,n}^3(X))
\]
on \([0, \frac{1}{m}T]\). According to Lemma 2.16, it now suffices to prove that \(\Phi_{m,n}^1\), satisfy the condition (2.13)—note that the problem for \(\Phi_{m,n}^2\) becomes the same after reflection. Let \(u \in H^1(\mathbb{R}_+)\) with \(u \geq 0\). Recall that there exists a set \(G\) such that \(\mathbb{R}_+ \setminus \{G\}\) is a null set and \(u\) is differentiable on \(G\), see, e.g., [10, Theorem 5.8.5]. For all \(x \in G\) with \(u(x) = 0\), it thus holds by non-negativity of \(u\) that also \(\frac{\partial}{\partial x} u(x) = 0\). In particular, it holds for all \(x \in G\) with \(u(x) = 0\), that

\[
\mu_1(x, u(x), \frac{\partial}{\partial x} u(x)) + \varrho(a, b) \frac{\partial}{\partial x} u(x) = \mu_1(x, u(x), \frac{\partial}{\partial x} u(x)) \geq 0.
\]

Here, \(a, b \in \mathbb{R}\) are arbitrary. Even more straightforward, we get that the noise and correction term vanish for all \(x \in \mathbb{R}_+\) such that \(u(x) = 0\), and thus, \(\Phi_{m,n}^1(u) \geq 0\). As we have seen in Lemma 2.16, this yields

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \text{dist}_{L^2}(u_0 + \epsilon \Phi_{m,n}^1(u_0); L^2_+) = 0, \quad \forall u_0 \in L^2_+,\n\]

and the respective result holds true for \(\Phi_{m,n}^2\) and \(L^2_-\).

**Step II.** We apply Corollary 2.15 iteratively on \(\left[\frac{k}{m} T, \frac{k+1}{m} T\right]\), as long as the solution is continuabale. The uniqueness and maximality claim in Theorem 2.14 shows that the unique mild solution of (4.3) stays in \(\mathbb{M} \cap \mathbb{S}_1\) up to the explosion time \(\tau_{m,n}\), for all \(\omega \in \Omega\), and for all \(m, n \in \mathbb{N}\).

**Step III.** By (4.10), we find a subsequence such that the convergence holds true almost surely, and thus, \(X(t \wedge \tau^{(r)}) \in \mathbb{M} \) almost surely, for all \(r > 0, r \in \mathbb{Q}\). With \(r \to \infty\), along the countable set \(\mathbb{Q}\), we get that \(X(t) \in \mathbb{M}\) on \([0, \tau]\).

\[ \square \]

### 4.2. Dirichlet boundary conditions

We consider the case \(\kappa_+ = \kappa_+ = \infty\) in which

\[
\mathcal{D}(\Delta_+) = \mathcal{D}(\Delta_-) = H^2(\mathbb{R}_+) \cap H^1_0(\mathbb{R}_+).
\]

In particular, we have \(E = L^2\) and \(E_\alpha = \mathbb{S}^{2\alpha}\), for all \(\alpha < \frac{1}{4}\), see [18, Lemma 4.1]. From [18, Lemma 4.3] we get that the assumption imposed in the beginning yield Lipschitz continuity on bounded sets.

**Proposition 4.8.** Assume that Assumptions (Interface), (Correlation), (Drift\(_0\)), and (Noise\(_0\)) hold true. Then, for \(\alpha \in [0, \frac{1}{2})\),

\[
\mathcal{B} : \mathcal{D}(A) \to \mathbb{S}^{2\alpha}, \quad \text{and} \quad \mathcal{C} : \mathcal{D}(A) \to \mathcal{L}_2(U; \mathcal{D}(A))
\]

are Lipschitz continuous on bounded sets.

In particular, for all initial data \(X_0 \in \mathcal{D}(A)\), there exist unique maximal mild solution \((X, \tau)\) and \((Z_{m,n}, \tau_{m,n})\) on \(\mathcal{D}(A)\), resp. of (4.2) and of (4.3). Moreover, \(X\) and \(Z_{m,n}\) have almost surely continuous paths in \(\mathcal{D}(A)\) and \(\tau, \tau_{m,n} > 0\).
Proof. See [18, Lemma 4.3 and Theorem 3.17].

The following lemma can be shown exactly in the same way as Lemma 4.5, but replacing $\tilde{\mathcal{H}}^1$ by $\tilde{\mathcal{H}}^2$. Note that, due to Assumption (Noise$_0$),(iii), $\Sigma_n(u)$ fulfills Dirichlet boundary conditions at 0 when $u$ does.

Lemma 4.9. Let Assumptions (Correlation) and (Noise$_0$) be satisfied. For all $N \in \mathbb{N}$ there exists $L_{\Sigma}^{(N)} > 0$ such that for all $u, v \in \tilde{\mathcal{H}}^2$ with norm smaller than $N$ it holds that

$$\|\Sigma_n(u) - \Sigma_n(v)\|_{\tilde{\mathcal{H}}^2} \leq L_{\Sigma}^{(N)} \|u - v\|_{\tilde{\mathcal{H}}^2}, \quad \forall n \in \mathbb{N}.$$  

Moreover, $\Sigma_n$ maps $\mathcal{D}(A)$ into $\mathcal{D}(A)$.

By Theorem B.9, $\mathcal{C}$ is of class $C^2$ and its derivatives map bounded sets into bounded sets. Hence, the definition of $\Sigma_n$ is consistent with the definition in Sect. 3. Applying Lemma 4.6, we get $\Sigma_n(u) \to \Sigma_{\infty}(u)$ in $\tilde{\mathcal{H}}^2$, for all $u \in \tilde{\mathcal{H}}^1$. With the same arguments, also the second weak derivatives converge in $\mathcal{L}_2^2$. Indeed, this works iteratively by applying chain rule and the same arguments as in the proof of Lemma 4.6. We will not go into more details but note that, as we will show in the proof of Lemma B.4,

$$\frac{d^2}{dx^2}(T_\xi e_k)^2(x) = T_{\xi''}e_k(x)T_\xi e_k(x) + (T_\xi e_k)^2(x).$$

Lemma 4.10. Let Assumption (Correlation) and (Noise$_0$) be satisfied. Then, for all $u \in \tilde{\mathcal{H}}^2$ it holds that $\lim_{n \to \infty} \|\Sigma_{\infty}(u) - \Sigma_n(u)\|_{\tilde{\mathcal{H}}^2} = 0$.

Recall that $\mathcal{D}(A)$ is a closed subset of $\tilde{\mathcal{H}}^2$ so that $\Sigma_{\infty}$ maps $\mathcal{D}(A)$ into $\mathcal{D}(A)$, since $\Sigma_n$ does for all $n \in \mathbb{N}$. Hence, we can apply Theorem 3.34 to finish the proof of Theorem 1.5 and obtain,

$$\lim_{n \to \infty} \lim_{m \to \infty} \mathbb{E} \sup_{0 \leq t \leq T} \|X(t \cap \tau^{(r)}) - Z_{m,n}(t \cap \tau^{(r)}(m,n) \cap \tau^{(r)})\|_{\tilde{\mathcal{H}}^2}^{2p} = 0. \quad (4.11)$$

Remark 4.11. Technically, we have to choose $\eta \in (0, 1)$, we set $\tilde{E} := \mathcal{D}((-A)^{\eta})$. Then, the restriction of $A$ to $\tilde{E}$ fulfills again Assumption 3.1 and, moreover, $\tilde{E}_{\theta} = E_1$, for $\theta := 1 - \eta < 1$, so that we fit into the notation of Sect. 3; see also Proposition 2.3 and Remark 2.4.

In order to apply the forward invariance results from Sect. 2.2 to the approximating solutions $Z_{m,n}$, we need to assure that $\mathcal{B}$, $\Sigma_{\alpha}^n$ and $\sigma_k$ are also Lipschitz on bounded sets as mapping from $E_{\alpha}$ into $E$, for some $\alpha < 1$, whereas the Nagumo conditions need to be satisfied on $E$ itself. Here, as above, we set $\tilde{E} := \mathcal{L}_2^2$ so that $E_1 = \mathcal{D}(A)$, $E_\alpha \subset \tilde{\mathcal{H}}^{2\alpha}$, $\alpha \in (0, 1)$ and $E_\alpha = H^{2\alpha}$ for all $\alpha \in [0, 1/4]$.

Lemma 4.12. Let Assumptions (Interface), (Correlation), (Drift$_0$), and (Noise$_0$) hold true. Then, $\mathcal{B}$, $\Sigma_n$, $k \in \mathbb{N}$, $n \in \mathbb{N}$ are Lipschitz continuous on bounded sets from $E_\alpha$ into $E$, for all $\alpha > \gamma/4$. 

**Proof.** First, note that $\mathcal{I}$ is continuous from $E_\alpha$ into $\mathbb{R}^2$ for all $\alpha > \frac{3}{4}$ by continuity of the trace operator on Sobolev spaces, cf. [20, Theorem 9.4]. We now keep $\alpha \in \left(\frac{3}{4}, 1\right)$ fixed. Since $\varrho$ is locally Lipschitz and the gradient is Lipschitz from $H^{1}_{0}$ into $L^{2}$, it holds that

$$u \mapsto \varrho(\mathcal{I}(u)) \left( \frac{\partial}{\partial x} u_1 - \frac{\partial}{\partial x} u_2 \right)$$

is Lipschitz continuous on bounded sets from $E_\alpha$ into $E$. For $\mu := \mu_1$ or $\mu := \mu_2$, we now need to prove that the Nemytskii operator $N_\mu(u) := \mu(\cdot, u(\cdot), \frac{\partial}{\partial x} u(\cdot))$ is Lipschitz from $H^{2\alpha}(\mathbb{R}_+)$ into $L^{2}(\mathbb{R}_+)$. Let $u, v \in H^{2\alpha}(\mathbb{R}_+)$ with $\|u\|_{H^{2\alpha}}, \|v\|_{H^{2\alpha}} \leq N_1$ for some $N_1 \in \mathbb{N}$. By Sobolev imbeddings, we can assume that $u, v \in \text{BUC}^1(\mathbb{R}_+)$ and $\|u\|_{C^1}, \|v\|_{C^1} \leq N_2$ for some $N_2 \geq N_1$. Denote by $L$ the Lipschitz constant of $(y, z) \mapsto \mu(x, y, z)$ on the $\mathbb{R}^2$-ball of radius $N_2$. Indeed, $L$ can be chosen independently of $x \in \mathbb{R}$ by Assumption $(\text{Drift}_0)$. Then,

$$\int_{0}^{\infty} \left| \mu(x, u(x), \frac{\partial}{\partial x} u(x)) - \mu(x, v(x), \frac{\partial}{\partial x} v(x)) \right|^2 \, dx \leq L^2 \int_{0}^{\infty} |u(x) - v(x)|^2 + \left| \frac{\partial}{\partial x} u(x) - \frac{\partial}{\partial x} v(x) \right|^2 \, dx. \quad (4.12)$$

Since $E_\alpha \subset \mathcal{F}^{2\alpha}(\mathbb{R}_+)$, this finishes the proof for $B$.

For $\mathcal{C}(\cdot)e_k$, recall that Assumption $(\text{Noise}_0)$ is stronger than Assumption $(\text{Noise}_1)$ and as a consequence of Proposition 4.4, $\mathcal{C}(\cdot)e_k$ is Lipschitz continuous on bounded sets on $E_{1/2}$. The same follows from Lemma 4.5 for $\Sigma_n$. Finally, recall the imbedding relation

$$E_\alpha \hookrightarrow E_{1/2} \hookrightarrow E,$$

which yields that $\mathcal{C}(\cdot)e_n$ and $\Sigma_n$ are also Lipschitz on bounded sets from $E_\alpha$ into $E$, for all $n \in \mathbb{N}$. \hfill $\Box$

We close this section by the following theorem, which is the remaining part, (a), of Theorem 1.1.

**Theorem 4.13.** Let Assumption (Interface), (Correlation), (Drift$_0$), (Noise$_0$) and in addition the point-wise inward-pointing assumption (PIP) hold true. Then, the set $\mathcal{M} \cap \mathcal{D}(A)$ is forward invariant for the stochastic evolution equation (4.2).

**Proof.** Similar to the proof Theorem 4.7, let $X = (u_1, u_2, x_*)$ and $Z_{m,n}$ be the unique mild solutions, respectively, of (4.2) and (4.3) on $\mathcal{D}(A)$ and write

$$\Phi_{m,n}(t, X) := \mathcal{B}(X) - \Sigma_{\infty} + \sum_{k=1}^{n} \mathcal{C}(X)e_k \hat{\beta}^{m}_{k}(t).$$
In the same way as in Step I of the proof of Theorem 4.7, we get that \( \varPhi_{m,n} \) fulfills the Nagumo condition on \( L^2 \). We consider \( \varPhi_{m,n} \), restricted to each interval \([\frac{k}{m}T, \frac{k+1}{m}T)\), \( k \leq m \), as a map from \( E_\alpha \) to \( E \), for \( \alpha \in (\frac{3}{4}, 1) \). Its Lipschitz continuity on bounded is covered by Lemma 4.12. Since \( (-A, D(A)) \) is positive self-adjoint, the assumptions of Theorem 2.14 yield existence of a unique maximal mild solution \( \tilde{Z}_{m,n} \) on \( E_\alpha \) up to the explosion time \( \tilde{\tau}_{m,n} \), taking values in \( \mathbb{M} \cap E_\alpha \). When \( \tilde{\tau}_{m,n} \geq \frac{1}{m}T \), we construct the solution on \( [\frac{1}{m}T, \frac{2}{m}T) \) and concatenate the solutions. We iterate this argument as long as we find \( k \leq m \) with \( \tilde{\tau}_{m,n} \geq \frac{k}{m}T \) or \( \tilde{\tau}_{m,n} = T \). Recall that this works \( \omega \)-wise.

On the other hand, by Proposition 4.8, \( \varPhi_{m,n} \) is also Lipschitz continuous on bounded sets from \( E_1 \) into \( E_{\alpha'} \), for any \( \alpha' < \frac{1}{4} \), and thus, there exists a unique maximal mild solution \( Z_{m,n} \) of (4.3) on \( E_1 \), with explosion time \( \tau_{m,n} \). By the continuous imbedding \( E_1 \hookrightarrow E_\alpha \), \( Z_{m,n} \) is also a mild solution on \( E_\alpha \), and thus, the uniqueness and maximality claim yields \( \tilde{\tau}_{m,n} \geq \tau_{m,n} \) and \( Z_{m,n} = \tilde{Z}_{m,n} \in \mathbb{M} \cap E_1 \) on \([0, \tau_{m,n})\) for all \( \omega \in \Omega \). Hence, the set \( \mathbb{M} \cap E_1 \) is forward invariant for (4.3). By switching to subsequences, the convergence (4.11) implies \( X(t) \in \mathbb{M} \) on \([0, \tau]\). \( \square \)

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### A Nemytskii operators on Sobolev spaces

We continue with the analysis on the Sobolev spaces \( H^k(\mathbb{R}_+) \), \( k \in \mathbb{N} \). In this section we prove some regularity results on the nonlinear Nemytskii operator

\[
 u \mapsto N(u)(x) = \mu(x, u(x)),
\]

where, \( \mu : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R} \) and \( x \in \mathbb{R}_+ \). Note that these operators are well understood but most of the literature focuses on bounded domains, see, e.g., [2,38,39]. However, in the case of unbounded domains several additional conditions on \( \mu \) are necessary to make them work. First, we state a result on the spaces \( H^k \) which guarantees that, under certain assumptions on \( \mu \), \( N \) will map \( H^k \) into \( H^k \). For a proof, we refer to [39, Theorem 1], of which it is a special case.

**Lemma A.1.** For each integer \( k \geq 1 \), the space \( H^k(\mathbb{R}_+) \) is a Banach algebra. In particular, there exists a constant \( c \) such that for all \( u, v \in H^k(\mathbb{R}_+) \) it holds that \( uv \in H^k(\mathbb{R}_+) \) and

\[
 \|uv\|_{H^k} \leq c \|u\|_{H^k} \|v\|_{H^k}.
\]

**A.1 Continuity**

We now adapt the proof of the continuity result [39, Theorem 2] to our setting, but with some corrections in the proof. For notational reasons, we also introduce the
Nemytskii operators

\[ \begin{align*}
N_x(u)(x) &:= \left( \frac{\partial}{\partial x} \mu \right)(x, u(x)), \\
N_{y_j}(u)(x) &:= \left( \frac{\partial}{\partial y_j} \mu \right)(x, u(x)), \quad j = 1, \ldots, d,
\end{align*} \]

(A.1)

for \( u \in H^k(\mathbb{R}_+; \mathbb{R}^d), \quad x \in \mathbb{R}_+ \). In order for \( N \) to map \( H^k \) into \( H^k \) again, we need certain growth restrictions, which is not the case on bounded domains. For a multi-index \( \alpha \), we denote by \( D^\alpha \) the respective partial derivative operator.

**Assumption A.2.** Assume \( \mu \in C^m(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}) \) and

(a) For each integer \( l, 0 \leq l \leq m \), there exists an \( a_l \in L^2(\mathbb{R}_+) \) and some \( b_l : \mathbb{R}^d \to \mathbb{R}_+ \) locally bounded, such that

\[ \left| D^{(l,0,\ldots,0)} \mu(x, y) \right| \leq b_l(y) (a_l(x) + |y|), \quad \forall x \in \mathbb{R}_+, \ y \in \mathbb{R}^d \]

(b) For each multi-index \( \alpha \) with \( \alpha_1 < |\alpha| \leq m \), the family of functions \( (D^\alpha \mu(x, .))_{x \in \mathbb{R}_+} \) is equicontinuous and \( \sup_{x \in \mathbb{R}_+} |D^\alpha \mu(x, .)| \) is locally bounded.

**Assumption A.3.** Assume that \( \mu \in C^m(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}) \) and \( D^\alpha \mu(x, .) \) is locally Lipschitz for all multi-indices \( \alpha, |\alpha| \leq m \) with Lipschitz constants uniform in \( x \in \mathbb{R}_+ \), i.e., we assume that for all \( r \geq 0 \) there exists \( L_r \geq 0 \) such that

\[ \left| D^\alpha \mu(x, y) - D^\alpha \mu(x, z) \right| \leq L_r |y - z|. \]

holds for all \( x, y, z \in \mathbb{R}^d \) with \( |y|, |z| \leq r \) and \( \alpha, |\alpha| \leq m \).

**Remark A.4.** If \( \mu \) satisfies Assumption A.2 for some integer \( m \geq 1 \), then \( \mu \) satisfies Assumption A.3 for \( m - 1 \).

**Remark A.5.** Recall the Sobolev imbeddings

\[ H^{m+1}(\mathbb{R}_+) \hookrightarrow BUC^m(\mathbb{R}_+). \]

As usual \( BUC^m(\mathbb{R}_+) \) is equipped with the \( C^m_b \)-norm. In the following, we will work with the \( BUC^m \) representative of the elements in \( H^{m+1} \) without further comment.

Note that Assumption A.2 is stronger than [18, Assumption 6.2] so that we get the following two results from [18, Appendix 1].

**Theorem A.6.** If Assumption A.2 holds for some integer \( m \geq 1 \), then the operator \( N \) is continuous from \( (H^m(\mathbb{R}_+))^d \) into \( H^m(\mathbb{R}_+) \).

**Theorem A.7.** Let \( \mu \) satisfy Assumptions A.2 and A.3 for some positive integer \( m \). Then, \( N \) is Lipschitz continuous from bounded subsets of \( (H^m(\mathbb{R}_+))^d \) into \( H^m(\mathbb{R}_+) \).
A.2 Differentiability

We now discuss differentiability of $N$ in Fréchet sense. Here we run into the following problem compared with the literature. To get continuity of the Fréchet derivatives, Valent [38] uses that $H^m$ is a Banach algebra and the Nemytskii operator corresponding to $(\frac{\partial}{\partial y_j} \mu)$ maps into $H^m$. On unbounded domains, this would exclude the linear case $\mu(x, y) := y$ which is of particular interest for applications in this work. We resolve this problem in Lemma A.9. Note that multiplication is not only bilinear continuous on $H^k$, but also from $C^k_b \times H^k$ into $H^k$. More precisely, see Lemma B.1, for all $k \geq 0$ there exists $c > 0$ such that for all $g \in C^k_b(R_+), u \in H^k(R_+)$ it holds that

$$\|gu\|_{H^k} \leq c \|g\|_{C^k_b} \|u\|_{H^k}.$$  \hfill (A.2)

We start with a result on continuity of the nonlinear operators, adapting [39, Theorem 2]. We now write shortly $H^m(R_+; \mathbb{R}^d)$ for $H^m(R_+; \mathbb{R}^d)$. Theorem A.8. If Assumption A.2.(b) holds for some integer $m \geq 1$, then the operator $N y_j$ is continuous from $H^m(R_+; \mathbb{R}^d)$ into $C^{m-1}_b(R_+)$ and maps bounded sets into bounded sets.

Proof. We prove the continuity in a similar way as done for Theorem A.6. First, let $m = 1$, and $(u_n) \subset H^1(R_+; \mathbb{R}^d)$ converging to some $u \in H^1(R_+; \mathbb{R}^d)$. Sobolev imbeddings imply that $u_n, u \in BUC(R_+; \mathbb{R}^d), n \in \mathbb{N}$ and $u_n \to u$ uniformly, as $n \to \infty$. Thus, $x \mapsto \frac{\partial}{\partial y_j} \mu(x, u(x))$ is continuous and bounded.

Define $R := \sup_{n \in \mathbb{N}} \|u_n\|_{\infty} < \infty$ and observe that the family of functions

$$y \mapsto \frac{\partial}{\partial y_j} \mu(x, y), \quad x \in R_+$$

is uniformly equicontinuous on the $\mathbb{R}^d$-ball of radius $R$.

Let $\epsilon > 0$ be arbitrary and $\delta = \delta_{\epsilon, R} > 0$ such that for all $y, \tilde{y} \in \mathbb{R}^d$ with $|y|, |\tilde{y}| \leq R$ and $|\tilde{y} - y| < \delta$ it holds that

$$\sup_{x \in R_+} \left( \frac{\partial}{\partial y_i} \mu(x, y) - \frac{\partial}{\partial y_i} \mu(x, \tilde{y}) \right) < \epsilon.$$

Now, let $N_\delta \in \mathbb{N}$ such that for all $n \geq N$ it holds that

$$\|u_n - u\|_{\infty} < \delta.$$

Hence, $\sup_{x \in R_+} \left( \frac{\partial}{\partial y_i} \mu(x, u(x)) - \frac{\partial}{\partial y_i} \mu(x, u_n(x)) \right)_{\infty} < \epsilon$ for all $n \geq N_\delta$, and thus

$$\|N y_j (u_n) - N y_j (u)\|_{\infty} \to 0, \quad n \to \infty.$$

Let $M \subset H^1(R_+; \mathbb{R}^d)$ be bounded and $R > 0$ so that $M$ is contained in the radius $R$ ball of $C_b(R_+)$. Then, for all $u \in M$,

$$\|N y_j (u)\|_{\infty} \leq \sup_{|y| < R} \sup_{x \in R_+} \left( \frac{\partial}{\partial y_j} \mu(x, y) \right) < \infty.$$
We finish the proof by induction, so assume the claim holds true for $m \in \mathbb{N}$. By induction hypothesis, $N_{y_j}$ is continuous from $H^{m+1}$ into $C_b^{m-1}$, so it remains to show that the same holds true for $\frac{d}{dx} N_{y_j}$. Chain rule yields

$$\frac{d}{dx} N_{y_j}(u)(x) = \frac{\partial^2}{\partial x \partial y_j} \mu(x, u(x)) + \sum_{i=1}^{d} \frac{\partial^2}{\partial y_i \partial y_j} \mu(x, u(x)) \nabla u_i(x), \quad (A.3)$$

for $u \in H^{m+1}(\mathbb{R}^d) \hookrightarrow BUC^m(\mathbb{R}^d)$. The function $\tilde{\mu}$ defined as

$$\tilde{\mu}(x, y, z) := \frac{\partial}{\partial x} \mu(x, y) + \sum_{i=1}^{d} \frac{\partial}{\partial y_i} \mu(x, y) z_i,$$

for $x \in \mathbb{R}_{\geq 0}$, $(y, z) \in \mathbb{R}^{d+d}$, satisfies Assumption A.2.(b) for $m$. Hence, by induction hypothesis, the Nemytskii operators corresponding to the $y_j$ (and $z_i$)-derivatives of $\tilde{\mu}$ are continuous and map bounded sets into bounded sets, from $H^m(\mathbb{R}^d) \rightarrow C_b^{m-1}(\mathbb{R}^{d+d})$. Since the map $u \mapsto \nabla u_i$ is linear continuous from $H^{m+1}(\mathbb{R}^d)$ into $H^m(\mathbb{R}^d)$, we get the properties for $\frac{d}{dx} N_{y_j}$. \hfill $\square$

In the following, we write for $j = 1, \ldots, d$, $u \in H^m(\mathbb{R}^d)$, $v \in H^m(\mathbb{R}^d)$,

$$\tilde{N}_{y_j}(u, v) := N_{y_j}(u)v = (\frac{\partial}{\partial y_j} \mu)(., u(\cdot))v(\cdot).$$

**Lemma A.9.** Let $m \geq 1$ and $\mu$ fulfilling Assumption A.2.(b) for $m+1$, then, the mapping

$$\Phi_j : u \mapsto \tilde{N}_{y_j}(u, \cdot)$$

is continuous from $H^m(\mathbb{R}^d)$ into $\mathcal{L}(H^m(\mathbb{R}^d))$, for all $j = 1, \ldots, d$. Moreover, $\Phi_i$ maps bounded sets into bounded sets.

**Proof.** Note that $\tilde{\mu}(x, y, z) := \mu(x, y) z$ fulfills Assumption A.2.(a) for $m$ so that Theorem A.6 yields continuity of

$$H^m(\mathbb{R}^d) \ni (u, v) \mapsto \tilde{N}_{y_j}(u, v) \in H^m(\mathbb{R}^d).$$

Of course, $\tilde{N}_{y_j}$ is linear in its second argument so that $\tilde{N}_{y_j}(u, \cdot) \in \mathcal{L}(H^m(\mathbb{R}^d))$, for each $u \in H^m(\mathbb{R}^d)^d$. It remains to prove continuity in the uniform operator topology for which we proceed by induction, again.

**Step I.** With $m = 1$ let $(u^{(n)}) \subset H^1(\mathbb{R}^d)$, $u \in H^1(\mathbb{R}^d)^d$ and $v \in H^1(\mathbb{R}^d)$. First note that by Theorem A.8, $N_{y_j}$ is continuous from $H^1$ into $C_b$, so that

$$\left\| \tilde{N}_{y_j}(u^{(n)}) - \tilde{N}_{y_j}(u, v) \right\|_{L^2} \leq c \left\| N_{y_j}(u^{(n)}) - N_{y_j}(u^{(n)}) \right\|_{C_b} \|v\|_{L^2} ,$$

converges to 0, as $n \to \infty$, uniformly in $v$. Similar we get uniform $L^2$-convergence for $N_{y_j}(u^{(n)}) \frac{\partial}{\partial x} v$ and for the operator

$$(u, v) \mapsto (\frac{\partial^2}{\partial x \partial y_j} \mu)(x, u(x))v(x).$$
Moreover, for all $i, j \in \{1, \ldots, d\}$, by Sobolev imbeddings
\[
\int_{\mathbb{R}^+} \left| \frac{\partial^2}{\partial y_j \partial y_i} \mu(x, u^{(n)}(x)) \nabla u_i^{(n)}(x) - \frac{\partial^2}{\partial y_j \partial y_i} \mu(x, u(x)) \nabla u_i(x) \right|^2 |v(x)|^2 \, dx \leq
\leq K \|v\|_{H^1}^2 \int_{\mathbb{R}^+} \left| \frac{\partial}{\partial y_j} \mu(x, u^{(n)}(x)) \nabla u_i^{(n)}(x) - \frac{\partial}{\partial y_j} \mu(x, u(x)) \nabla u_i(x) \right|^2 \, dx.
\]
(A.4)

Note that $\frac{\partial}{\partial y_j} \mu$ fulfills Assumption A.2.(b) and recall that multiplication is continuous from $C_b \times L^2$ into $L^2$. Hence, the integral converges to 0, as $n \to \infty$ by application of Theorem A.8 to the corresponding Nemytskii operator, and so we conclude continuity of $\Phi$.

Using almost the same estimates and applying the corresponding of part Theorem A.8, we get that $\Phi_i$ maps bounded sets into bounded sets again.

**Step II:** By induction hypothesis, the Lemma holds true for $m \in \mathbb{N}$ fixed, so assume that $\mu$ fulfills Assumption A.2.(b) for $m + 2$. Then, $\Phi_j$ is continuous from $H^{m+1}(\mathbb{R}_+)^d$ into $L(H^m(\mathbb{R}_+))$. Let $(u^{(n)}) \subset H^{m+1}(\mathbb{R}_+)^d$ converging to $u \in H^{m+1}(\mathbb{R}_+)^d$. Note that
\[
\left\| \Phi(u^{(n)}) - \Phi(u) \right\|_{L(H^{m+1})}^2 \leq \sup_{w \in H^m} \left\| \Phi(u^{(n)})w - \Phi(u)w \right\|_{H^m}^2 + \sup_{v \in H^{m+1}} \left\| \frac{d^{m+1}}{dx^{m+1}} (\Phi(u^{(n)})v - \Phi(u)v) \right\|_{L^2}^2.
\]
(A.5)

The first term vanishes as $n \to \infty$ by induction hypothesis. For all $v \in H^{m+1}$, we can write the latter one can be estimated by
\[
\frac{d^{m+1}}{dx^{m+1}} (\Phi(u^{(n)})v - \Phi(u)v) = \frac{d^m}{dx^m} \left[ (\frac{\partial}{\partial x} N_{y_j}(u^{(n)}) - \frac{\partial}{\partial x} N_{y_j}(u))v \right] + \frac{d^m}{dx^m} \left[ (N_{y_j}(u^{(n)}) - N_{y_j}(u))w \right],
\]
for $w := \frac{\partial}{\partial x} v \in H^m$. By induction hypothesis, the latter term converges to 0 in $L^2$, uniformly over all $w \in H^m(\mathbb{R}_+)$.

For the first summand, we observe that $D_x \mu(x, y)$ fulfills Assumption A.2.(b) for $m + 1$ so that the induction hypothesis applied on
\[
\Psi(u)v := (\frac{\partial^2}{\partial x \partial y_j} \mu)(, u(.))v
\]
yields $L^2$ convergence. Plugging in into (A.5) finally yields the convergence uniform in $L(H^{m+1})$. With the same decomposition, we deduce from induction hypothesis that $\Phi_j$ maps bounded sets from $H^{m+1}$ into bounded sets of $L(H^{m+1})$. □
Based on the continuity in the uniform topology, we are now able to prove Fréchet differentiability.

**Theorem A.10.** If Assumption A.2 holds for some integer \( m + 1, m \geq 1 \), then the operator \( N \) defined above is in \( C^1 \left( H^m(\mathbb{R}_+)^d, H^m(\mathbb{R}_+) \right) \) with derivative

\[
DN(u)v = \sum_{j=1}^{d} N_{y_j}(u)v_j, \quad u, v \in H^m(\mathbb{R}_+)^d.
\]

**Proof.** From Theorem A.6, we already know that \( N \) maps \( H^m(\mathbb{R}_+)^d \) continuously into \( H^m(\mathbb{R}_+) \). Moreover, Lemma A.9 tells us that \( DN \), defined as above, is continuous from \( H^m(\mathbb{R}_+)^d \) into \( L(H^m(\mathbb{R}_+)) \). Thus, it remains to verify that \( DN \) is at least the Gâteaux derivative of \( N \), i.e., that for any \( u, v \in H^{m+1}(\mathbb{R}_+)^d \),

\[
\frac{1}{\epsilon} \left\| N(u + \epsilon v) - N(u) - \epsilon \sum_{j=1}^{d} N_{y_j}(u)v_j \right\|_{H^m} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0. \quad (A.6)
\]

By fundamental theorem of calculus, we get for fixed \( u, v \in H^m \), and all \( x \in \mathbb{R}_+ \),

\[
N(u + \epsilon v)(x) - N(u)(x) = \int_{0}^{1} \sum_{j=1}^{d} \frac{\partial}{\partial y_j} \mu(x, u(x) + t\epsilon v(x)) \epsilon v_j(x) \, dt
\]

\[
= \epsilon \sum_{j=1}^{d} \int_{0}^{1} \tilde{N}_{y_j}(u + t\epsilon v, v_j)(x) \, dt \quad (A.7)
\]

The map \( t \mapsto \tilde{N}_{y_i}(u + t\epsilon v, v_i) \) is continuous from \([0, 1]\) into \( H^m(\mathbb{R}_+) \) by Theorem A.6. Therefore, the integral in Eq. (A.7) can be considered as an Bochner integral and (A.6) follows from Lemma A.9 and the estimate

\[
\left\| N(u + \epsilon v) - N(u) - \epsilon \sum_{j=1}^{d} N_{y_j}(u)v_j \right\|_{H^m} \leq |\epsilon| \sum_{j=1}^{d} \int_{0}^{1} \left\| (N_{y_j}(u + t\epsilon v) - N_{y_j}(u))v_j \right\|_{H^m} \, dt.
\]

\[
\Box
\]

If \( \mu \in C^2 \), then we define for \( u \in H^m(\mathbb{R}_+)^d \), \( v, w \in H^m(\mathbb{R}_+) \), \( x \in \mathbb{R}_+ \),

\[
N_{y_i,y_j}(u)(x) := \frac{\partial^2}{\partial y_i \partial y_j} \mu(x, u(x)).
\]

**Theorem A.11.** Assume that \( \mu \) satisfies Assumption A.2 for \( m + 2, m \geq 1 \), then the Nemytskii operator \( N : H^m(\mathbb{R}_+)^d \rightarrow H^m(\mathbb{R}_+) \) is of class \( C^2 \) with second derivative

\[
D^2 N(u)[v, w] = \sum_{i=1}^{d} \sum_{j=1}^{d} N_{y_i,y_j}(u)v_j w_i.
\]
for \( u, v, w \in H^m(\mathbb{R}_+)^d \).

**Proof.** By the previous theorem, \( N \) is of class \( C^1 \), so we have to show the same for the map

\[
DN : H^m(\mathbb{R}_+) \to \mathcal{L}(H^m(\mathbb{R}_+)^d; H^m(\mathbb{R}_+)), \quad DN(u) := \left( v \mapsto \sum_{j=1}^d \tilde{N}_{y_j}(u, v_j) \right)
\]

Since \( H^m \) is a Banach algebra, we get for \( u, \bar{u} \in H^m(\mathbb{R}_+)^d \),

\[
\left\| D^2 N(u) - D^2 N(\bar{u}) \right\|_{\mathcal{L}(H^m(\mathbb{R}_+)^d; H^m(\mathbb{R}_+))} \leq \sum_{i,j=1}^d \sup_{\|v\| = 1} \| (N_{y_i, y_j}(u) - N_{y_i, y_j}(\bar{u})) v_i w_j \|_{H^m} \leq c \sum_{i,j=1}^d \sup_{\|v\| = 1} \| N_{y_i, y_j}(u) v - N_{y_i, y_j}(\bar{u}) v \|_{H^m}.
\]

Now, we apply Lemma A.9 to \( \frac{\partial}{\partial y_i} \mu(x, u(x)), i = 1, \ldots, d \), which indeed fulfill Assumption A.2.(b) for \( m + 1 \). This yields continuity of \( D^2 N \).

To finish the proof, it again suffices to show differentiability in Gâteaux sense. Fix \( u, w \in H^m(\mathbb{R}_+)^d \) and let \( \epsilon > 0 \). As in the proof of Theorem A.10, cf. (A.7), we get by fundamental theorem of calculus, for all \( v \in H^m(\mathbb{R}_+)^d \)

\[
DN(u + \epsilon w)v - DN(u)v - D^2 N(u)(v, w) = \epsilon \sum_{i,j=1}^d \int_0^1 (N_{y_i, y_j}(u + t\epsilon w) - N_{y_i, y_j}(u)) w_i v_j \, dt
\]

\[
= \epsilon \int_0^1 D^2 N(u + t\epsilon w)(v, w) - D^2 N(u)(v, w) \, dt.
\]

From the first part of this proof, we know that \( \epsilon \mapsto D^2 N(u + \epsilon w) \) is uniformly continuous from \([0, 1]\) into the space of continuous bilinear operators. Hence, the right-hand side is in \( o(\epsilon) \), uniformly in \( v \in H^m \).

The following conclusion is a combination of Theorem A.8 and the representations of \( DN \) and \( D^2 N \).

**Corollary A.12.** Under the assumptions of, respectively, Theorems A.10 and A.11, the maps \( DN : H^m(\mathbb{R}_+)^d \to \mathcal{L}(H^m(\mathbb{R}_+)^d; H^m(\mathbb{R}_+)) \)

and

\( D^2 N : H^m(\mathbb{R}_+)^d \to \mathcal{L}(H^m(\mathbb{R}_+)^d, H^m(\mathbb{R}_+)^d; H^m(\mathbb{R}_+)) \)

map bounded sets into bounded sets.
B The noise operator

We will now study the operator-valued map $\mathcal{C}$, defined previously by

\[
(\mathcal{C}(u)w)(x) = \begin{pmatrix}
\sigma_+(x, u_1(x))(T_\zeta w)(u_3 + x) \\
\sigma_-(-x, u_2(x))(T_\zeta w)(u_3 - x) \\
0
\end{pmatrix}
\]

for $u \in \mathcal{D}(\mathcal{C}) \subset L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+) \oplus \mathbb{R}$, $w \in L^2(\mathbb{R}) =: U$ and $x \in \mathbb{R}$. We can reduce the problem to the operator

$\Psi : (u, x_*) \mapsto \sigma(., u(\cdot)) T_\zeta (. + x_*)$

for $\sigma : \mathbb{R}^2 \to \mathbb{R}$, and $\zeta$ and an integral kernel $\zeta : \mathbb{R}^2 \to \mathbb{R}$, which we aim to take values in spaces of Hilbert–Schmidt operators like $L^2_2(U; L^2(\mathbb{R}_+))$. As above, we write

$T_\zeta : w \mapsto \int_{\mathbb{R}} \zeta(x, y)w(y) \, dy$

and define the Nemytskii operator

$N_\sigma : u \mapsto \sigma(., u(\cdot))$.

Naturally, it will make sense to separate the study of $\Psi$ into the operators $N_\sigma$ and $T_\zeta$. Recall that we have discussed the Nemytskii operators $N_\sigma$ in Appendix A.

B.1 The Hilbert–Schmidt property

Note that on $L^2(D)$, for a domain $D \subset \mathbb{R}^d$, $d \in \mathbb{N}$, every Hilbert–Schmidt operator is of the form $T_\kappa$, for an integral kernel $\kappa$ satisfying

\[
\int_D \int_D |\kappa(x, y)|^2 \, dx \, dy < \infty,
\]

see, e.g., [8, Section XI.6]. When $D$ has infinite Lebesgue measure, this condition is obviously violated for convolution kernels $\kappa(x, y) = \kappa(x - y)$, in which have been interested in Example 1.4 for instance. Hence, $T_\zeta$ itself will typically not be Hilbert–Schmidt on the spaces of interest. We skip the proofs in the following three lemmas since they will be the same as the proofs of, respectively, Lemmas 7.1, 7.2 and 7.4 in [18].

**Lemma B.1.** For any integer $n \geq 0$, multiplication is bilinear continuous from $H^n(\mathbb{R}_+) \times C^n_b(\mathbb{R}_{\geq 0})$ into $H^n(\mathbb{R}_+)$. 

The lemma is the first step in the direction to separate our discussion of $\Psi$ into the operators $N_\sigma$ and $T_\zeta$. Provided that $\zeta$ is sufficiently nice, $T_\zeta$ will indeed map into the space of bounded and uniformly continuous functions.
**Assumption B.2.** Let $n \geq 1, \zeta(., y) \in C^{n+1}(\mathbb{R})$ for all $y \in \mathbb{R}$ and $\frac{\partial^i}{\partial x^i} \zeta(x, .) \in L^2(\mathbb{R})$ for all $x \in \mathbb{R}, i \in \{0, \ldots, n + 1\}$. Moreover,

$$\sup_{x \in \mathbb{R}} \left\| \frac{\partial^i}{\partial x^i} \zeta(x, .) \right\|_{L^2(\mathbb{R})} < \infty, \quad i = 0, 1, \ldots, n + 1. \quad (B.3)$$

In the following, we use the notation $\zeta^{(i)} := \frac{\partial^i}{\partial x^i} \zeta$.

**Remark B.3.** For convolution kernels $\zeta(x, y) := \zeta(x - y)$, this assumption is satisfied when $\zeta \in H^{n+1}(\mathbb{R}) \cap C^{n+1}(\mathbb{R})$.

**Lemma B.4.** Let Assumption B.2 be fulfilled for $n \in \mathbb{N}$. Then, $T_\zeta$ maps $U$ into $BUC_n(\mathbb{R})$. Moreover, $T_\zeta w$ and its first $n$ derivatives are Lipschitz continuous for all $w \in U$ and it holds that $T_\zeta \in \mathcal{L}(U; BUC^2(\mathbb{R}))$.

**Lemma B.5.** Let $n \in \mathbb{N}$ and Assumption B.2 be satisfied. For $u \in H^n(\mathbb{R}^+) \cap H^1_{0}(\mathbb{R}^+)$ and $x^* \in \mathbb{R}$ it holds that

$$\left\| u \cdot T_\zeta(. + x^*) \right\|_{L^2(U; H^n(\mathbb{R}^+))} \leq K \| u \|_{H^n(\mathbb{R}^+)} \sup_{x \in \mathbb{R}} \sum_{i=0}^{n} \left\| \zeta^{(i)}(x, .) \right\|_{L^2(\mathbb{R})}$$

For application in Sect. 4, we need to deal $C$ on the domain of the Dirichlet Laplacian. In fact, Assumption (Noise0) and Lemma B.1 ensure $N_\sigma(u) \in H^2(\mathbb{R}^+) \cap H^1_{0}(\mathbb{R}^+)$ for all $u \in H^2(\mathbb{R}^+) \cap H^1_{0}(\mathbb{R}^+)$.

**B.2 Lipschitz continuity and differentiability**

In order to apply the results, let us introduce the translation group $(\theta_x)_{x \in \mathbb{R}}$ which is strongly continuous on $BUC(\mathbb{R})$.

**Remark B.6.** By the structure of the direct sum of Hilbert spaces, the following two results directly extend to $C$ as a mapping from $H^n(\mathbb{R}^+) \oplus H^n(\mathbb{R}^+) \oplus \mathbb{R}$ into $L^2(U; H^n(\mathbb{R}^+) \oplus H^n(\mathbb{R}^+) \oplus H^1_{0}(\mathbb{R}^+) \oplus \mathbb{R})$.

**Remark B.7.** Note that for $x \in \mathbb{R}$

$$\theta_x \circ T_\zeta = T_{\zeta_x},$$

where $\zeta_x := \zeta(x + ., .)$ satisfies Assumption B.2, whenever $\zeta$ does.

We impose the following conditions on $\sigma$.

**Assumption B.8.** Let $n \geq 1$ and assume that $\sigma \in C^n(\mathbb{R}^2; \mathbb{R})$ satisfies

(i) For every multi-index $I = (i, j) \in \mathbb{N}^2$ with $|I| \leq n$, there exist $a_I \in L^2(\mathbb{R}^+)$ and $b_I \in L^\infty_{loc}(\mathbb{R}, \mathbb{R}^+)$ such that

$$\left| \frac{\partial^{|I|}}{\partial x^i \partial y^j} \sigma(x, y) \right| \leq \begin{cases} b_I(y) (a_I(|x|) + |y|), & j = 0, \\ b_I(y), & j \neq 0. \end{cases}$$
(ii) $\sigma$ and its partial derivatives (in $x$ and $y$) are locally Lipschitz with Lipschitz constants independent of $x \in \mathbb{R}$.

**Theorem B.9.** Let $n \in \mathbb{N}$ and assume that Assumption B.2 is fulfilled for $n + 1$ and, respectively, B.8 for $n + 2$. Then, $\Psi$ is of class $C^2$ from $H^n(\mathbb{R}^+) \oplus \mathbb{R}$ into $\mathcal{L}_2 := \mathcal{L}_2(U; H^n(\mathbb{R}^+))$, with derivatives

$$D\Psi(u, x)(v, y) = DN_\sigma(u)v \cdot \theta_x T_\epsilon + yN_\sigma(u) \cdot \theta_x T_\epsilon'$$

$$= \left( w \mapsto \frac{\partial}{\partial y}\sigma(\cdot, u)v T_\epsilon w(\cdot + x) + y\sigma(\cdot, u)T_\epsilon' w(\cdot + x) \right),$$

$$D^2\Psi(u)(v, (\bar{v}, \bar{y})) = D^2N_\sigma(u)[v, \bar{v}] \cdot \theta_x T_\epsilon + yDN_\sigma(u)\bar{v} \cdot \theta_x T_\epsilon' + \bar{y}DN_\sigma(u)v \cdot \theta_x T_\epsilon'' + y\bar{y}N_\sigma(u) \cdot \theta_x T_\epsilon''.$$

Moreover, $\Psi$, $D\Psi$, and $D^2\Psi$ map bounded sets into bounded sets.

**Remark B.10.** For $n = 2$ and under the additional assumption that $\sigma(0, 0) = 0$, it holds that $\Psi(u, x) \in H^2 \cap H^1_0(\mathbb{R}^+)$, when $u \in H^2 \cap H^1_0(\mathbb{R}^+)$. This even translates to $D\Psi$ and $D^2\Psi$.

**Proof.** Let $u, v \in H^n(\mathbb{R}^+)$, $x, y \in \mathbb{R}$ and $\epsilon > 0$, then with Lemma B.5,

$$\|\Psi((u, x) + \epsilon(v, y)) - \Psi(u, x) - \epsilon D\Psi(u, x)(v, y)\|_{\mathcal{L}_2(U; H^n)}$$

$$\leq K_\epsilon \|N_\sigma(u + v) - N_\sigma(u) - DN_\sigma(u)\epsilon v\|_{H^n}$$

$$+ K \|N_\sigma(u) + \epsilon DN_\sigma(u)v\|_{H^n} \sup_{z \in \mathbb{R}} \sum_{i=0}^{n} \left\| \zeta^{(i)}_{\epsilon y}(z, .) \right\|_{L^2(\mathbb{R})} \quad (B.4)$$

In fact, the first term is in $o(\epsilon)$ because of differentiability of $N_\sigma$ we get from Theorem A.10. For the second summand, we have defined

$$\zeta(z, y) := \zeta(z + x, y) - \zeta(x, y) - z \frac{\partial}{\partial x} \zeta(x, y), \quad x, y, z \in \mathbb{R}.$$

Thus,

$$\sup_{z \in \mathbb{R}} \left\| \zeta_{\epsilon y}(z, .) \right\|_{L^2} \leq \sup_{z \in \mathbb{R}} \int_{\mathbb{R}} \left| \zeta(z + \epsilon y, \xi) - \zeta(z, \xi) - \epsilon y \zeta'(z, \xi) \right|^2 d\xi$$

$$\leq |\epsilon y|^2 \sup_{z \in \mathbb{R}} \int_{\mathbb{R}} \int_0^1 \left| \zeta'(z + \alpha \epsilon y, \xi) - \zeta'(z, \xi) \right|^2 d\alpha \, d\xi. \quad (B.5)$$

Using fundamental theorem of calculus again, we obtain

$$\sup_{z \in \mathbb{R}} \int_{\mathbb{R}} \int_0^1 \left| \zeta'(z + \alpha \epsilon y, \xi) - \zeta'(z, \xi) \right|^2 d\alpha \, d\xi \leq |\epsilon y|^2 \sup_{z \in \mathbb{R}} \left\| \zeta''(z, .) \right\|_{L^2},$$

which goes to 0, as $\epsilon \to 0$. Using that (B.3) holds for $i = 0, \ldots, n+2$, the same calculation can be done for $\bar{\zeta}^{(i)}$, $i = 1, \ldots, n$ which then shows that $D\Psi$ is at least the Gâteaux derivative of $\Psi$. To finish the proof, it is now enough to show that

$$D\Psi : H^n \oplus \mathbb{R} \to \mathcal{L}(H^n \oplus \mathbb{R}; \mathcal{L}_2(U; H^n)).$$
is Gâteaux differentiable, and

$$D^2\Psi : H^2 \oplus \mathbb{R} \to \mathcal{L}(H^n \oplus \mathbb{R}, H^n \oplus \mathbb{R}; \mathcal{L}_2(U; H^n))$$

is continuous. Let us start with the latter claim and show continuity of each summand separately. To this end, we first decompose as above

$$D^2\Psi(u, x)[(v, y), (\tilde{v}, \tilde{y})] = \sum_{k=1}^4 R_k(u, v, \tilde{v}, x, y, \tilde{y}).$$

Consider $u, \tilde{u}, v, \tilde{v} \in H^n, x, \tilde{x}, y, \tilde{y} \in \mathbb{R}$. Because $N_\sigma \in C^2$ by Theorem A.11, we get

$$\| R_1(u, v, \tilde{v}, x, y, \tilde{y}) - R_1(\tilde{u}, \tilde{x}, v, \tilde{v})\|_{\mathcal{L}_2(U; H^n)}$$

$$\leq \left\| \left( D^2N_\sigma(u)[v, \tilde{v}] - D^2N_\sigma(\tilde{u})[v, \tilde{v}] \right) \cdot \theta_x(T_\xi(.)) \right\|_{\mathcal{L}_2}$$

$$+ \left\| D^2N_\sigma(\tilde{u})[v, \tilde{v}] \cdot (\theta_x(T_\xi(.)) - \tilde{\theta}_x(T_\xi(.))) \right\|_{\mathcal{L}_2}. \quad (B.6)$$

Applying Lemma B.5, we see that both terms go to 0, as $\|u - \tilde{u}\|_{H^n} + |x - \tilde{x}|$ does, and that the convergence is uniformly in $v, \tilde{v} \in H^n$ with norm smaller than 1. Indeed, for the first term this is continuity of $D^2N$, the second term can be estimated by

$$\left\| D^2N_\sigma(\tilde{u})[v, \tilde{v}] \cdot (\theta_x(T_\xi(.)) - \tilde{\theta}_x(T_\xi(.))) \right\|_{\mathcal{L}_2(U; H^n)}$$

$$\leq K \left\| D^2N_\sigma(\tilde{u}) \right\|_{\mathcal{L}(H^n)} \|v\|_{H^n} \sup_{z \in \mathbb{R}} \sum_{i=0}^n \left\| \xi^{(i)}_x(z, .) - \xi^{(i)}_{\tilde{x}}(z, .) \right\|_{L^2(\mathbb{R})}. \quad (B.7)$$

Convergence of the right-hand side follows with the same procedure as in (B.5). For $R_2$ and $R_3$, we use continuity of $DN_\sigma$, for $R_4$ continuity of $N_\sigma$ itself. With almost the same estimates, we observe that $D^2\Psi$ maps bounded sets into bounded sets. In fact, this property is inherited by $N, DN$ and $D^2N$, see Corollary A.12.

It remains to show that $D^2\Psi$ is indeed the derivative of $D\Psi$. The derivative of the second summand can be computed in the same way as $D\Psi$ itself has been computed. For the first summand, we have to be slightly more careful, but note that by Lemma B.5

$$\sup_{\|v\| \leq 1} \left\| D^2N_\sigma(u)v \cdot (\theta_x+\epsilon y T_\xi - \theta_x T_\xi - \epsilon y T_{\xi^r}) \right\|_{\mathcal{L}_2(U; H^n)}$$

$$\leq K \left\| D^2N_\sigma(u) \right\|_{\mathcal{L}(H^n(\mathbb{R}))} \sup_{z \in \mathbb{R}} \sum_{i=0}^n \left\| \xi^{(i)}_{\epsilon y}(z, .) \right\|_{L^2(\mathbb{R})}$$

which is in $o(\epsilon)$ thanks to (B.5). The remaining estimates follow in the same way: First apply Lemma B.5, but then use that $N_\sigma$ is of class $C^2$. Hence, $D^2\Psi$ is the Gâteaux derivative of $D\Psi$. By continuity of $D^2\Psi$, the differentiability also holds true in Fréchet sense. \hfill \Box
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