UNIFIED STATISTICAL INFERENCE FOR A NOVEL NONLINEAR DYNAMIC FUNCTIONAL/LONGITUDINAL DATA MODEL

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In light of recent work studying massive functional/longitudinal data, such as the resulting data from the COVID-19 pandemic, we propose a novel functional/longitudinal data model which is a combination of the popular varying coefficient (VC) model and additive model. We call it Semi-VCAM in which the response could be a functional/longitudinal variable, and the explanatory variables could be a mixture of functional/longitudinal and scalar variables. Notably some of the scalar variables could be categorical variables as well. The Semi-VCAM simultaneously allows for both substantial flexibility and the maintaining of one-dimensional rates of convergence. A local linear smoothing with the aid of an initial B-spline series approximation is developed to estimate the unknown functional effects in the model. To avoid the subjective choice between the sparse and dense cases of the data, we establish the asymptotic theories of the resultant Pilot Estimation Based Local Linear Estimators (PE-BLLE) on a unified framework of sparse, dense and ultra-dense cases of the data. Moreover, we construct unified consistent tests to justify whether a parsimony submodel is sufficient or not. These test methods also avoid the subjective choice between the sparse, dense and ultra dense cases of the data. Extensive Monte Carlo simulation studies investigating the finite sample performance of the proposed methodologies confirm our asymptotic results. We further illustrate our methodologies via analyzing the COVID-19 data from China and the CD4 data.

1. Introduction. Increasingly, data is recorded continuously over an interval of time (spatial location, or wavelength and so on) or intermittently at several discrete points in time due to progress in modern computation technology. As a result, the data in which each individual has multiple observations becomes more and more common in almost all scientific, societal and economic fields. Obviously, a recent example of this kind of data is the COVID-19 data: the daily confirmed diagnoses, death toll and suspected

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cases of different countries are recorded and made available. When a variable is measured or observed at different times, the variable is usually treated as a function of time. As a result, the variable is called a functional variable, the data for the variable are called functional data and the related statistical analysis is called functional data analysis (FDA) ([38]). The functional data and corresponding FDA have been successfully applied to explore the interactions and co-movements among a group of temporally evolving subjects. Several monographs by [31], [32] and [9] provide comprehensive discussions on the methods and applications. More recent work about FDA could refer to [38].

According to [38], usually, the functional data could be divided into two cases: sparse and dense. Sparse functional data usually occurs in longitudinal studies where subjects are measured at different time points and the number of measurements for each subject is often bounded away from infinity. Inversely, in the dense functional data the number of measurements of each subject tends towards to infinity. In theory, the difference between sparse (longitudinal) and dense function data is clear. However, due to the limitations of humans, the observations in real data sets could not be infinite and are definitely finite. Therefore, the edge of sparse (longitudinal) and dense function data in practice is vague in some scenarios, especially when the number of measurement of each subject is moderate or different subjects have different numbers of measurements.

In many functional/longitudinal studies, repeated measurements within each subject are possibly correlated with each other, but different subjects can be supposed to be independent. One approach to take intra-subject variation into account is the mixed-effects model [42], which decomposes regression function into a fixed population mean and a subject-specific random trajectory with zero mean. For sparse and dense functional data, [19] considered a mixed-effects nonparametric regression model absence of covariates, and showed that the asymptotic distributions of kernel estimators are essentially different in these two situations. Therefore, a subjective choice between sparse and dense cases may lead to erroneous conclusion. To evade this problem, they proposed a self-normalized method, which can deal with sparse and dense functional data in a unified framework. Furthermore, [4] generalized the results of [19] to a mixed-effects VCM presence of covariates with sparse or dense functional data. Lately, [47] provided a comprehensive perspective that deals with a general weighing scheme on a unified platform for all types of sampling plan, including sparse, dense and ultra dense case. Motivated by a monotone relationship between gray matter volume and age in the older population, [5] considered sparse and dense cases on a unified
framework under monotone constraint of the mean function. The research work of [47, 5] has focused on the statistical inference about mean function of the underlying process. To the best of our knowledge, there exists no further development about unified inference parallel to [47] for nonparametric regression model presence of covariates, a common case in practice.

In the analysis of longitudinal data, a varying-coefficient models (VCM) enjoying flexibility, parsimony and interpretability, is a widely-used nonparametric regression method. One can refer to [8, 10, 12, 15, 16, 17, 29, 35]. An additive model (AM) is another popular nonparametric regression method, which has been studied by [1, 3, 25, 27, 28, 30, 43, 44, 34]. Recently, [13, 14, 46, 48] have investigated a novel nonparametric regression method, named the varying-coefficient additive model (VCAM), which can be viewed as a generalization of the VCM and AM. Let $T_{ij}$ be the observation time when the $j$th measurement of the $i$th subject is made, $Y_{ij}$ and $X_i(T_{ij}) := X_{ij}$ be the response and $p$-covariates for the $i$th subject at time $T_{ij}$, respectively. Then \{(Y_{ij}, X_{ij}, T_{ij}) ; i = 1, ..., n; j = 1, ..., m_i \} constitutes a longitudinal/functional sample from $n$ randomly selected subjects with $m_i$ repeated measurements of the $i$th subject. The VCAM for longitudinal/functional data is proposed by [14] as below

\[
Y_{ij} = \alpha_0 (T_{ij}) + \sum_{k=1}^{p} \alpha_k (T_{ij}) \beta_k (X_{ijk}) + \nu_i (T_{ij}) + \varepsilon_{ij},
\]

with the abuse of notations. Here $\nu_i (T_{ij})$ is the subject-specific random trajectory at observation time $T_{ij}$, and \{\varepsilon_{ij} \} are i.i.d. random measurement errors. The multiplicative factors $\alpha_k (k = 1, ..., p)$ and $\beta_k (k = 1, ..., p)$ are called to be varying-coefficient component functions and additive component functions, respectively, and $\alpha_0$ is a trend term. Obviously, the VCAM (1.1) reduces to an AM provided that each $\alpha_k (k = 0, ..., p)$ is time-invariant, whilst it becomes a VCM if each $\beta_k (k = 1, ..., p)$ has a simple linear form. Therefore, it can be said that the VCAM is a kind of hybird of an AM and a VCM, enjoying more flexibility, which can greatly decrease the bias of model misspecification. On the other hand, it is hard to address how to choose between an AM and a VCM in practice. The general type of a VCAM provides a data-driven method to decide which model may be more suitable for the real-life data at hand.

However, the product forms of $\alpha_k$ and $\beta_k$ in (1.1) exclude the discrete covariates from this model. It will vastly limit the scope of applications because categorical variables are often important influence factors in the practical fields. To accommodate both discrete and continuous covariates in regression model, in this paper we consider a mixed-effects semi varying-
coefficient additive model (Semi-VCAM) to analyze longitudinal data. Let $Z_i(T_{ij}) := (1, Z_{ij,1}, ..., Z_{ij,q})^\tau$ be a $(q + 1)$-vector of discrete covariates observed at time $T_{ij}$, and $\alpha_0(t) = (\alpha_{00}(t), \alpha_{01}(t), ..., \alpha_{0q}(t))^\tau$ is the vector of varying-coefficient functions for $Z$ that is i.i.d. with $Z_i$, and $\alpha_{00}$ denotes the trend function. Then, we generalize the VCAM (1.1) to a Semi-VCAM as below,

\begin{equation}
Y_{ij} = Z_{ij} \alpha_0(T_{ij}) + \sum_{k=1}^{p} \alpha_k(T_{ij}) \beta_k(X_{ijk}) + \nu_i(T_{ij}) + \sigma(T_{ij}) \varepsilon_{ij}, \label{semi-vcam}
\end{equation}

where the subject-specific random trajectory $\nu_i(t)$ satisfies $E[\nu_i(t)] = 0$ and covariance function $\gamma(t, t') = E[\nu_i(t) \nu_i(t')]$, $\{\varepsilon_{ij}\}$ are random errors such that $E(\varepsilon_{ij}) = 0$ and $E(\varepsilon_{ij}^2) = 1$, and $\sigma(t)$ is a smooth standard deviation function of process $\varepsilon(t)$. Note that (1.2) allows a mixture of functional/longitudinal predictors and scalar covariates, and it reduces to a partial linear additive model (PLAM), if each varying-coefficient function is time-invariant. Compared with the model (1.1) studied in [14], (1.2) allows categorical covariates and heteroscedasticity as time elapsed. Meanwhile, in this paper we also take into account intra-subject correlation, which was merged into random errors in [14]. Therefore, Semi-VCAM is a more refined nonparametric model than VCAM (1.1) in the analysis of longitudinal data.

As a global smoothing technique, spline method is widely used to fit a smooth nonparametric function because of its merit of cost saving. But it usually has no asymptotic distribution due to absence of decomposition of bias part and variance part, unless the asymptotic bias is smaller of high order than the asymptotic variance. All of the existing research literatures about VCAM are based upon a spline method, [46, 48] provide no asymptotic distributions of estimators, whilst [13, 14] obtain the asymptotic distributions under the condition that the asymptotic bias can be ignored. Alternatively, kernel method is a local smoothing tool, based upon which we can construct the involved asymptotic distribution presence of asymptotic bias, and make statistical inference on certain interested function. Specially, local linear smoothing is popular due to its nice properties, such as design adaptation, good boundary performance, and statistical efficiency in an asymptotic minimax sense, see [7] for more details.

In this paper, we build a pilot estimation based local linear estimator (PEBLLE) for varying-coefficient component functions and additive component functions, respectively. The proposed estimation method has wide applicability, including sparse data and dense data, and the data presence of functional/longitudinal covariates and scalar variables. We have shown the consistency of PEBLLE, and as a main contributor of this paper, we con-
struct the asymptotic distributions on a unified framework for sparse, dense and ultra dense data. For the convenience of concise presentation, we only consider the same weight to each subject (SUBJ), and our theoretical results can be viewed as a generalization of [47] to nonparametric regression model presence of covariates with SUBJ scheme. Another intriguing question is how to judge a general Semi-VCAM or a submodel is sufficient. To this end, we develop two hypothesis testing to decide whether each varying-coefficient component functions is time-invariant (i.e., a PLAM or especially, an AM if absence of $Z$ covariates), or whether each additive component function has linear form (i.e., a VCM). It has been shown that the proposed testing procedure is consistent on a unified framework of sparse, dense and ultra dense case of data.

In the empirical studies, we consider the new coronavirus disease (COVID-19) breaking out in December 2019, and apply our method to analyze the growth rate of cumulative confirmed (GRCC) cases in China except Hubei Province, Tibet, Macao, Taiwan and Hong Kong. We collect the data from https://github.com/CSSEGISandData/COVID-19, and take sample period from January 22th, 2020 and April 8th, 2020. To model GRCC, four function covariates and one scalar covariate (population size) are chosen. The testing procedures show that a Semi-VCAM is necessary for this dataset. Another example is CD4 data from the Multicenter AIDS Cohort Study (a data set in the R package “timereg”), which has been studied by [15, 45]. In this model, smoke status (1 for smoker and 0 for nonsmoker) is included. Employing Semi-VCAM, the testing procedure shows a VCM is sufficient, which verifies the rationality of the research results in [15].

The rest of this paper is organized as follows. Section 2 proposes a pilot estimation based local linear smoothing method and Section 3 presents a series of the asymptotic theories. In Section 4, we propose a testing procedure to justify whether a VCM or a PLAM is sufficient or not, and show its asymptotic properties. Section 5 speaks about the implementation of the proposed method. Extensive simulation studies investigating the finite-sample performance and real data applications illustrating our methodologies are considered in Section 6. Brief remarks are concluded in Section 7. The requirements for validity of the asymptotic theories are presented in the Appendix, and the main proofs are relegated to the Supplementary Material.

2. Estimation Method. We assume that observation time $\{T_{ij}\}$ are i.i.d. copies of $T$, which has a density function $f_T$ with a bounded support, say $[a, b]$. The vector of covariates $X_i = (X_{i1},...,X_{im})^T$ for the $i$-th subject is randomly drawn from a $p$-dimension stochastic process $X(T)$, of which
the $k$-th element $X_k(T)$ has a marginal density function $f_{X_k}$ with support $\mathbb{S}_k$. To identify the trend term and product terms in model (1.2), we impose the conditions $\mathbb{E}[\alpha_k(T)] = 1$ and $\mathbb{E}[\beta_k(X_k)] = 0$ ($k = 1, \ldots, p$), a similar practice with [48, 14].

In this section, we develop pilot estimation based local linear estimators (PEBLEs) for $\alpha_k$ and $\beta_k$. Suppose that $\beta_k$’s are known, then Semi-VCAM (1.2) become a VCM, and the LLE of $\alpha_k$’s are easily obtained. Let $a_{0l} = \{a_{00}(t), \ldots, a_{0q}(t)\}$, $a_{l} = \{a_{1}(t), \ldots, a_{p}(t)\}^\top$, $b_{0l} = \{b_{00}(t), \ldots, b_{0q}(t)\}$, $b_{l} = \{b_{1}(t), \ldots, b_{p}(t)\}$, where $t$ is any interior point on the interval $[a, b]$. We solve the optimization problem as below

$$Q_1(\hat{a}_l, \hat{b}_l) = \min_{t \in (a, b)} Q(a_l, b_l)$$

$$= \sum_{i=1}^{n} \frac{1}{m_i} \sum_{j=1}^{m_i} \left[ Y_{ij} - \sum_{l=0}^{q} Z_{ijl} \{ a_{0l}(t) + b_{0l}(t) (T_{ij} - t) \} \right. \left. - \sum_{k=1}^{p} \{ a_k(t) + b_k(t) (T_{ij} - t) \} \beta_k(X_{ijk}) \right]^2 k_{hc} (T_{ij} - t),$$

(2.1)

where $k_h(\cdot) = k(\cdot/h)/h$ for certain kernel function $k$. Then the LLE of varying-coefficient component functions are given by $\hat{\alpha}_{0l}(t) = \hat{\alpha}_{0l}$ for $l = 0, \ldots, q$ and $\hat{\alpha}_k(t) = \hat{\alpha}_k$ for $k = 1, \ldots, p$.

On the other hand, if $\alpha_k$’s are known, then Semi-VCAM reduces to an AM. Suppose that we have got estimation of additive component functions except $\beta_k$, denoted as $\hat{\beta}_l$ for $l \neq k$, and consider the following minimum problem

$$Q_2(\hat{a}_x, \hat{b}_x) = \min_{\mathbb{S}_k} Q_2(a_x, b_x)$$

$$= \sum_{i=1}^{n} \frac{1}{m_i} \sum_{j=1}^{m_i} \left[ \hat{Y}_{ij} - \alpha_k(T_{ij}) \{ a_x + b_x (X_{ijk} - x) \} \right]^2 k_{hc} (X_{ijk} - x),$$

where $x$ is any interior point of support $\mathbb{S}_k$ of $X_k$, and $\hat{Y}_{ij} = Y_{ij} - Z_{ij}^\top \alpha_0(T_{ij}) - \sum_{l \neq k} \alpha_l(T_{ij}) \hat{\beta}_l(X_{ijl})$. Then, the LLE of $\beta_k$ is given by $\hat{\beta}_k(x) = \hat{\beta}_x$.

However, both $\alpha_k$ and $\beta_k$ are unknown, implying the above-mentioned estimation methods are infeasible. To this end, we propose pilot estimations of additive component functions. Similar to [14], we view multiplicative term $\alpha_k(t) \beta_k(x)$ as a general bivariate function, say $g_k(t, x)$, and estimate it using tensor B-spline method. Specifically, for any given $t$ and $x$, the tensor
product is defined as \( T(t, x) = B_{k,A}(x) \otimes b_C(t) \), where \( \otimes \) means the Kronecker product of matrices or vectors, and \( b_C(t) \) and \( B_{k,A}(x) \) denote the B-spline basis approximating \( \alpha_k(t) \) and \( \beta_k(x) \), respectively.

Then, we approximate \( \alpha_0(t) \approx \gamma_0^\tau b_C(t) \) for \( l = 0, ..., q \), and \( g_k(t, x_k) \approx \gamma_k^\tau T_k(t, x_k) \) for \( k = 1, ..., p \). Solving the following optimization problem

\[
\min_\gamma \sum_{i=1}^n \frac{1}{m_i} \sum_{j=1}^{m_i} \left[ Y_{ij} - \gamma_0^\tau Z_{ij} \otimes b_C(T_{ij}) - \sum_{k=1}^p \gamma_k^\tau T_k(T_{ij}, X_{ijk}) \right]^2,
\]

we got the estimator of \( g_k \) as \( \hat{g}_k(t, x_k) = \hat{\gamma}_k^\tau T_k(t, x_k) \), where \( \hat{\gamma}_k \) is given by (2.2). Furthermore, the identification condition \( \mathbb{E}[\alpha_k(T_{ij})] = 1 \) implies \( \beta_k(x) = \int_a^b g_k(t, x) f(t) dt \). Hence, a pilot estimator of additive component function \( \beta_k \) can be given by

\[
\hat{\beta}_{k,P}(x) = \frac{1}{N} \sum_{i=1}^n \sum_{j=1}^{m_i} \hat{\gamma}_k^\tau T_k(t_{ij}, x), \quad k = 1, ..., p,
\]

where \( N = \sum_{i=1}^n m_i \) is total observation, and subscript ‘P’ means pilot estimator.

Now, we can define the PEBLLEs of varying-coefficient component functions and additive component functions.

- Substituting the pilot estimators \( \hat{\beta}_{k,P} \) \( (k = 1, ..., p) \) into the objective function \( Q_1 \), we obtain the PEBLLE of \( \alpha_0 \) for \( l = 0, ..., q \) and \( \alpha_k \) for \( k = 1, ..., p \), and still denote them as \( \hat{\alpha}_0 \) and \( \hat{\alpha}_k \), respectively.

- In the objective function \( Q_2 \), we take the PEBLLEs of varying-coefficient component functions as their pilot estimations, and \( \hat{\beta}_{l,P} \) \( (l \neq k) \) as the pilot estimators of additive component functions, and yield the PEBLLE of \( \beta_k \), still write as \( \hat{\beta}_k \).

**Remark 1.** Compared to the spline-based estimators of [14], the PEBLLE can provide asymptotic distribution with the specific expression of asymptotic bias, and make inference on the confidence interval of component functions. Meanwhile, our estimation methodologies adapt to both sparse and dense longitudinal/functional data, and have wide application in the real world.

### 3. Asymptotic Results.

In this section, we will present the asymptotic distribution and convergence rate of PEBLLE on a unified platform for different sampling plans.
3.1. Asymptotic Properties of Varying-coefficient Component Functions. Let \( \bar{N}_H = \left( \frac{1}{n} \sum_{i=1}^{n} \frac{1}{m_i} \right)^{-1} \) be the harmonic mean of \( \{m_1, \ldots, m_n\} \), and denote the interior knots number of B-spline basis \( b_C (t) \) and \( B_{k,A} (x) \) \( (k = 1, \ldots, p) \) as \( K_C \) and \( K_A \), respectively. Then, based upon the result of Proposition 1 presented in Supplement A, Theorem 3.1 shows the uniform convergence rates of PEBLLEs of varying-coefficient component functions.

**Theorem 3.1.** Under Assumption (A1) – (A6) and (A9), if \( K_C K_A = o (nN_{h_C}^2) \) and \( K_{A'}^{-r+1/2} + K_{C'}^{-r} = o (h_C^2) \), then we obtain that

\[
\sup_{t \in (a,b)} |\hat{\alpha}_l (t) - \alpha_l (t)| = O_p \left( h_C^2 + \sqrt{K_A} (K_{C'}^{-r} + K_{A'}^{-r}) + \frac{\log n}{n} \left( 1 + \frac{1}{N_{h_C}^2} \right) \right),
\]

\[
\sup_{t \in (a,b)} |\hat{\alpha}_k (t) - \alpha_k (t)| = O_p \left( h_C^2 + \sqrt{K_A} (K_{C'}^{-r} + K_{A'}^{-r}) + \frac{\log n}{n} \left( 1 + \frac{1}{N_{h_C}^2} \right) \right),
\]

where \( l = 0, \ldots, q \) and \( k = 1, \ldots, p \).

**Remark 2.** From Theorem 3.1, we notice that the variance term obtains a nonparametric rate of convergence \( \log n / (nN_{h_C}^2) \) provided that \( N_{h_C}^2 \rightarrow 0 \) and \( K_C \asymp (nN_{h_C}^2)^{-1/2} \), where “\( a \asymp b \)” means that \( a \) and \( b \) have the same order. On the other hand, a parametric rate of convergence is implied if \( N_{h_C}^2 \rightarrow C \) \( (0 < C < \infty) \) and \( K_C \asymp n^{1/2} \) or \( N_{h_C}^2 \rightarrow \infty \) and \( K_C = o (n^{1/2}) \).

**Remark 3.** Similar to [47], we split data into sparse, dense or ultra dense according to the ratio \( N_{h_C}^2 / n^{1/2} \) tends to 0, a nonzero constant or \( \infty \) as \( n \rightarrow \infty \). In fact, we give a more general method of partitioning data in the sense that the same split with [47] is used if \( r = 2 \).

Let \( \alpha (t) = \{\alpha_{00} (t), \alpha_{01} (t), \ldots, \alpha_{0q} (t), \alpha_1 (t), \ldots, \alpha_p (t)\}^\top \), and \( \hat{\alpha} (t) \) the PEBLLE of \( \alpha (t) \). Furthermore, we introduced the following symbols: \( F_{ij} = (Z_{ij}, \beta_{ij})^\top \) with \( \beta_{ij} = (\beta_1 (X_{ij1}), \ldots, \beta_p (X_{ijp}))^\top \), \( \Xi (t) = E[F_{ij} F_{ij}' | T_{ij} = t] := [v_1 (t) v_2 (t)] \) and \( G (t, t) = \lim_{t' \rightarrow t} G (t, t') \) with \( G (t, t') = E[F_{ij} F_{ij}' | T_{ij} = t, T_{ij'} = t'] \). In addition, we define \( \kappa = \int K^2 (v) \text{d}v, \kappa_2 = \int v^2 K (v) \text{d}v, \kappa_4 = \int v^4 K (v) \text{d}v, \) \( \kappa_2 = \int v^2 K^2 (v) \text{d}v \), and \( g'' \) denotes the second derivative of function \( g \).

Theorem 3.2 presents a unified asymptotic normality of \( \hat{\alpha} (t) \), which can be applied to sparse, dense and ultra dense cases of the data.
THEOREM 3.2. Under the assumption of (A1) - (A9), if

\[
\max \left\{ \frac{1}{n^2 h_C} \sum_{i=1}^{n} \frac{1}{m_i^2} \frac{1}{n^2 h_C} \sum_{i=1}^{n} \frac{1}{m_i^2} (m_i - 1), \frac{1}{n^2} \sum_{i=1}^{n} \left( 1 - \frac{1}{m_i} \right) \left( 1 - \frac{3}{m_i} \right) \right\} \left[ \frac{1}{n N h_C} + \frac{1}{n} \left( 1 - \frac{1}{N h} \right) \right]^{3/2}
\]

holds. Then, for any an interior \( t \) in \( (a, b) \), we obtain the asymptotic distribution of \( \hat{\alpha}(t) \) as below:

(3.1) \( \Gamma_{C}^{-1/2}(t) \left( \hat{\alpha}(t) - \alpha(t) - \frac{1}{2} h_C^2 \kappa_2 \Xi^{-1}(t) \rho_1(t) \right) \xrightarrow{D} N(0, I_{p+q+1}) \),

where \( \rho_1(t) = \sum_{l=0}^{q} \alpha_{l l}(t) v_1(t) + \sum_{k=0}^{p} \alpha_{k l}(t) v_2(t) \) with \( v_1(t) \) being the \( l \)th column of \( v_1(t) \) and \( v_2(t) \) being the \( k \)th column of \( v_2(t) \), and

\[
\Gamma_C(t) = \frac{\kappa}{n N h_C h(t)} \Sigma_{1,S}(t) + \frac{1}{n} \left( 1 - \frac{1}{N h} \right) \Sigma_{1,D}(t)
\]

with \( \Sigma_{1,S} = \Xi^{-1}(t) \left( \gamma(t, t) + \sigma^2(t) \right) \) and \( \Sigma_{1,D} = \Xi^{-1}(t) \gamma(t, t) G(t, t) \Xi^{-1}(t) \).

According to the method of partitioning data defined in Remark 3 and (3.1), Corollary 1 lists the asymptotic distributions for sparse, dense and ultra dense cases of the data as follows.

COROLLARY 1. Suppose that the conditions of Theorem 3.2 hold and \( t \) is a fixed interior point on the interval \( (a, b) \).

(i) Sparsity Case \( (N_h/n)^{1/2r} \to 0 \). If \( h_C \asymp (n N_h)^{-1/(2r+1)} \), then

(3.2) \( \sqrt{n N h C h(t)} \left( \hat{\alpha}(t) - \alpha(t) - \frac{1}{2} h_C^2 \kappa_2 \Xi^{-1}(t) \rho_1(t) \right) \xrightarrow{D} N(0, \kappa \Sigma_{1,S}) \).

(ii) Dense Case \( (N_h/n)^{1/2r} \to C_1 < \infty \). If \( h_C = O \left( n^{-1/(2r)} \right) \), then

(3.3) \( \sqrt{n} \left( \hat{\alpha}(t) - \alpha(t) - \frac{1}{2} h_C^2 \kappa_2 \Xi^{-1}(t) \rho_1(t) \right) \xrightarrow{D} N \left( 0, \frac{\kappa}{h_C(t) C_1} \Sigma_{1,S} + \Sigma_{1,D} \right) \).

(iii) Ultra Dense Case \( (N_h/n)^{1/2r} \to \infty \). If \( h_C = o \left( n^{-1/(2r)} \right) \), then

(3.4) \( \sqrt{n} \left( \hat{\alpha}(t) - \alpha(t) - \frac{1}{2} h_C^2 \kappa_2 \Xi^{-1}(t) \rho_1(t) \right) \xrightarrow{D} N (0, \Sigma_{1,D}) \).

Let \( \hat{\Xi}^{-1}(t), \hat{\rho}_1(t), \hat{\gamma}(t, t), \hat{\sigma}^2(t), \hat{f}_T(t), \hat{G}(t, t), \hat{v}_1(t) \) and \( \hat{v}_2(t) \) are kernel smoothing of \( \Xi^{-1}(t), \rho_1(t), \gamma(t, t), \sigma^2(t), f_T(t), G(t, t), v_1(t) \) and \( v_2(t) \). Then, the naive consistent estimators of asymptotic bias \( \rho_1(t) \)
and asymptotic variance $\Gamma_C(t)$ are given by $\hat{\rho}_1(t) = \sum_{l=0}^p \hat{\alpha}'_{\theta l}(t) \hat{\nu}_l(t) + \sum_{k=1}^p \hat{\alpha}'_{\beta k}(t) \hat{\nu}_k(t)$ and

$$\hat{\Gamma}_C(t) = \frac{k}{n Nh_C f_T(t)} \hat{\Sigma}_{1,S}(t) + \frac{1}{n} \left( 1 - \frac{1}{N_H} \right) \hat{\Sigma}_{1,D}(t),$$

where $\hat{\Sigma}_{1,S} = \hat{\Sigma}^{-1}(t) (\hat{\gamma}(t, t) + \hat{\sigma}^2(t))$ and $\hat{\Sigma}_{1,D} = \hat{\Sigma}^{-1}(t) \hat{\gamma}(t, t) \hat{G}(t, t) \hat{\Sigma}^{-1}(t)$.

Based upon (3.1), we can construct a $(1 - \alpha)\%$ confidence interval of varying-coefficient component functions as below

$$\hat{\alpha}'_{\theta l}(t) - \frac{1}{2} h_C^2 \kappa_2 \left( \hat{\Xi}^{-1}(t) \hat{\rho}_1(t) \right)_{l+1} \pm z_{1-\alpha/2} \left( \hat{\Gamma}_C^{1/2}(t) \right)_{l+1,l+1},$$

$$\hat{\alpha}'_{\beta k}(t) - \frac{1}{2} h_C^2 \kappa_2 \left( \hat{\Xi}^{-1}(t) \hat{\rho}_1(t) \right)_{q+k+1} \pm z_{1-\alpha/2} \left( \hat{\Gamma}_C^{1/2}(t) \right)_{q+k+1,q+k+1},$$

where $z_{1-\alpha/2}$ is the $1 - \alpha/2$ standard normal quantile, the subscript $k$ denotes the $k$-th element of involved vector, and the subscript $(k, k)$ means the $k$-th diagonal element of a given matrix. Note that (3.5) is a unified confidence interval suitable for sparse, dense and ultra dense cases of the data.

### 3.2. Asymptotic Properties of Additive Component Functions.

In this subsection, we focus on the asymptotic results of PEBLLE of additive component functions. Theorem 3.3 gives the uniform rates of convergence of $\hat{\beta}_K$.

**Theorem 3.3.** Suppose that (A1) – (A6) and (A9) hold. If $K_C K_A = o\left(n Nh_C^2\right)$ and $K_A^{-r} + K_C^{-r} = o\left(h_C^2\right)$, and $x$ is any interior in $S_k$, then sup$_{x \in S_k} |\hat{\beta}_K(x) - \beta_k(x)|$ is bounded by

$$O_p \left( h_A^2 + h_C^2 + \sqrt{K_A (K_C^{-r} + K_A^{-r})} + \sqrt{\frac{\log n}{n}} \left( 1 + \frac{1}{Nh_A} \right) \right).$$

Denote $\mu_k = E \left[ \alpha_k^2(T_{ij}) \right]$, $\psi_k,1 = E \left[ \alpha_k^2(T_{ij}) \{ \gamma(T_{ij}, T_{ij}) + \sigma^2(T_{ij}) \} \right]$ and $\psi_{k,2} = E \left[ \alpha_k(T_{ij}) \alpha_k(T_{ij'}) \gamma(T_{ij}, T_{ij'}) \right]$. Theorem 3.4 presents the asymptotic normality of $\hat{\beta}_K$ on a unified formwork for different types of data.

**Theorem 3.4.** Under the condition (A1) – (A9), if $h_C = o\left(h_A\right)$ and

$$\max \left\{ \frac{1}{h_A^2} \sum_{i=1}^n \frac{1}{m_i} \frac{1}{m_i} \sum_{i=1}^n \frac{1}{m_i} (m_i - 1), \frac{1}{h_A^2} \sum_{i=1}^n (1 - \frac{1}{m_i}) \right\} \left[ \frac{1}{n Nh_A} + \frac{1}{h_A} \left( 1 - \frac{1}{Nh} \right) \right]^{3/2},$$


hold. Then, for any an interior \( x \) in \( S_k \), we have

\[
\Gamma_{A,k}^{-1/2}(x) \left( \hat{\beta}_k(x) - \beta_k(x) - \frac{1}{2} \beta''_k(x) h_A^2 \kappa_2 / \mu_k \right) \xrightarrow{D} N(0, 1),
\]

where \( \Gamma_{A,k}(x) = \frac{\kappa \psi_{k,1}}{n \bar{N} h_A f_{X_k}(x)} + \frac{1}{n} \left( 1 - \frac{1}{N_H} \right) \psi_{k,2} / \mu_k^2 \).

As a corollary, we get different asymptotic results for sparse, dense and ultra dense data.

**Corollary 2.** Suppose that the conditions of Theorem 3.4 hold and \( x \) is a fixed interior point in \( S_k \).

- **(i) Sparsity Case.** If \( h_A \asymp (n \bar{N})^{-1/3} \), then it follows that

\[
\sqrt{n \bar{N}} h_A f_{X_k}(x) \left( \hat{\beta}_k(x) - \beta_k(x) - \frac{1}{2} \beta''_k(x) h_A^2 \kappa_2 / \mu_k \right) \xrightarrow{D} N(0, \kappa \psi_{k,1}).
\]

- **(ii) Dense Case.** If \( h_A = O \left( n^{-\frac{1}{3}} \right) \), then

\[
\sqrt{n} \left( \hat{\beta}_k(x) - \beta_k(x) - \frac{1}{2} \beta''_k(x) h_A^2 \kappa_2 / \mu_k \right) \xrightarrow{D} N \left( 0, \frac{\kappa \psi_{k,1}}{f_{X_k}(x) C_1} + \frac{\psi_{k,2}}{\mu_k^2} \right).
\]

- **(iii) Ultra Dense Case.** If \( h_A = o \left( n^{-\frac{1}{3}} \right) \), then

\[
\sqrt{n} \left( \hat{\beta}_k(x) - \beta_k(x) - \frac{1}{2} \beta''_k(x) h_A^2 \kappa_2 / \mu_k \right) \xrightarrow{D} N(0, \psi_{k,2} / \mu_k^2).
\]

Let \( \hat{\mu}_k, \hat{f}_{X_k}(x) \) and \( \hat{\psi}_{k,j}, j = 1, 2 \) be consistent estimators of \( \mu_k, f_{X_k}(x) \) and \( \psi_{k,j} \). Then, the asymptotic variance \( \Gamma_{A}(x) \) can be consistently estimated by

\[
\hat{\Gamma}_{A}(x) = \frac{\kappa \hat{\psi}_{k,1}}{n \bar{N} h_A \hat{f}_{X_k}(x)} + \frac{1}{n} \left( 1 - \frac{1}{N_H} \right) \hat{\psi}_{k,2} / \hat{\mu}_k^2,
\]

which gives a \((1 - \alpha)%\) pointwise confidence interval of \( \beta_k \) in a unified form for sparse, dense and ultra dense data. That is,

\[
\hat{\beta}_k(x) - \frac{1}{2} \beta''_k(x) h_A^2 h_k / \hat{\mu}_k \pm z_{1-\alpha/2} \hat{\Gamma}_{A}^{-1/2}(x).
\]

**4. Testing of Model Specification.** For the sake of parsimony, it is essential to test time-varying property of varying-coefficient component functions and to test linearity of additive component functions.
4.1. Time-varying Testing of Varying-coefficient Component Functions.
In this subsection, we propose a consistent testing to judge whether the varying-coefficient component functions are really time-varying or not. It is a problem of model selection between a general Semi-VCAM and a submodel PLAM or an AM in the practical applications.

We denote \( \delta_{ij} = \nu_i (T_{ij}) + \varepsilon_{ij} \) in Semi-VCAM (1.2), and consider a mixed-effect nonparametric model \( Y_{ij} = m (T_{ij}, Z_{ij}, X_{ij}) + \delta_{ij} \), where \( m (t, z, x) = E [Y_{ij} | T_{ij} = t, Z_{ij} = z, X_{ij} = x] \). The time-varying testing postulates \( m \) as

\[
m (t, z, x) = z^T a_0 + \sum_{k=1}^{p} a_k \beta_k (x_k) := g_0 (z, x; a, \beta (x))
\]

under null hypothesis \( H_{0,C} \), where \( a = (a_1^T, a_2, ..., a_p)^T \) is a unknown constant vector, and \( \beta (x) = (\beta_1 (x_1), ..., \beta_p (x_p))^T \). Whilst under alternative hypothesis \( H_{1,C} \), \( m \) is the regression function of Semi-VCAM (1.2), denoted as \( g (t, z, x; \alpha (t), \beta (x)) \). Then, the interested hypothesis is given as below

\[
H_{0,C} : m (t, z, x) = g_0 (t, z, x; \alpha, \beta (x)) \quad \text{a.s.}
\]
\[
\leftrightarrow H_{1,C} : m (t, z, x) = g (t, z, x; \alpha (t), \beta (x)) \quad \text{a.s.}
\]

Under \( H_{0,C} \), we replace \( \beta_k (x) \) with PEBLLE \( \hat{\beta}_k (x) \), and obtain the parametric estimator of vector \( a \) as follows

\[
a = \left( \sum_{i=1}^{n} \hat{S}_i \hat{S}_i^T \right)^{-1} \sum_{i=1}^{n} \hat{S}_i \hat{Y}_i,
\]

where \( \hat{S}_i = (\hat{S}_{i1}, ..., \hat{S}_{im_i})^T \) with \( \hat{S}_{ij} = (\hat{Z}_{ij}, \hat{\beta}_1 (X_{ij1}), ..., \hat{\beta}_p (X_{ijp}))^T \), and \( \hat{Y}_i = (Y_{i1}, ..., Y_{im_i})^T \).

For the \( i \)-th subject and the \( j \)-th subject, we introduce the weight matrix \( W_{ij} = (w_{ij}^{(t,v)})_{m_i \times m_j} \), where \( w_{ij}^{(t,v)} = k_{h_c} (T_{il}, T_{lj}) K_{h} (X_{il}, X_{lj}) \), with \( k_h (t, z) = \min \left( \frac{|t-z|}{\Delta}, 1 \right) \) and \( k_h (x_1, x_2) = \Pi_{i=1}^{p} k_h (x_{1i}, x_{2i}) \) for \( x_k = (x_{k1}, ..., x_{kp}) \), \( k = 1, 2 \). Let \( \hat{\varepsilon}_{ij} = Y_{ij} - g_0 (T_{ij}, Z_{ij}, X_{ij}; \hat{a}, \beta (X_{ij})) \), and we propose a testing statistic based upon the quadratic form of residuals as follows

\[
\hat{J}_n = \frac{1}{n^2 N_h^2 |H|} \sum_{i=1}^{n} \sum_{j \neq i} \hat{\varepsilon}_i^T W_{ij} \hat{\varepsilon}_j,
\]

where \( \hat{\varepsilon}_i = (\hat{\varepsilon}_{i1}, ..., \hat{\varepsilon}_{im_i})^T \) and \( |H| = h_c h_A^p \).

Furthermore, we assumes additional conditions as follows.
and 4.2 present the asymptotic distributions of \( \hat{\beta}_k \), and \( \| \|_F \) is the Frobenius norm of the involved matrix.

(T2) \( \tilde{N}_2 H \| H \| \to 0, n \tilde{N}_2 \sqrt{\| H \|} \to \infty \) and \( n \tilde{N}_2 \sqrt{\| H \|} h_A \to 0. \)

Let \( \tilde{N}_2 = \frac{1}{n} \sum_{i=1}^{n} m_i^2 \), Theorem 4.1 and 4.2 present the asymptotic distribution of the proposed test statistic \( \hat{\beta}_n \) under \( H_{0,C} \) and \( H_{1,C} \), respectively.

**Theorem 4.1.** Under Assumption (A1) – (A8) and (T1) – (T2), it holds that

\[
\sqrt{\frac{n^2 \tilde{N}_2^2}{N^2 - n \tilde{N}_2}} \| H \| \hat{\beta}_n / \hat{\sigma}_1 \overset{D}{\rightarrow} N(0, 1)
\]

under \( H_{0,C} \), where

\[
\hat{\sigma}_1^2 = \frac{1}{n^2 \tilde{N}_2 H \| H \|} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \sum_{l=1}^{m_i} \sum_{v=1}^{m_j} e_i l e_j v^2 \left( w_{ij} \right)^2
\]

is a consistent estimator of the asymptotic variance of \( \sqrt{\frac{n^2 \tilde{N}_2^2}{N^2 - n \tilde{N}_2}} \| H \| \hat{\beta}_n \), i.e.,

\[
\sigma_1^2 = \kappa^{p+1} E \left[ \left( \gamma (T, T) + \sigma^2 (T) \right)^2 f_T (T) \right] \Pi_{k=1}^{p} E \left[ f_k (X_k) \right].
\]

**Theorem 4.2.** Under the conditions of Theorem 4.1, If \( H_{1,C} \) holds, then \( \Pr \left( n \tilde{N}_2 \sqrt{\| H \|} \hat{\beta}_n / \hat{\sigma}_1 \geq M_n \right) \to 1 \) as \( n \to \infty \), where \( M_n \) is any non-stochastic positive sequence such that \( M_n = o \left( n \tilde{N}_2 \sqrt{\| H \|} \right) \).

4.2. Linearity Testing of Additive Component Functions. In this subsection, we check whether each additive component function in Semi-VCAM (1.2) reduces to a linear form, which yields a more parsimonious VCM.

Let \( h_0 (t, z, x; \alpha (t)) = z^T \alpha (t) + \sum_{k=1}^{p} \alpha_k (t) x_k \). It is expected to test

\[
H_{0,A} : m (t, z, x) = h_0 (t, z, x; \alpha (t)) \text{ a.s.}
\]

\( \leftrightarrow H_{1,A} : m (t, z, x) = g (t, z, x; \alpha (t), \beta (x)) \text{ a.s.} \)

Denote \( \hat{\alpha} (t) \) as the LLE of \( \alpha (t) \) under null hypothesis \( H_{0,A} \). Then, the testing statistics is given by

\[
\hat{I}_n = \frac{1}{n^2 \tilde{N}_2^2 \| H \|} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \hat{\xi}_i^T W_{ij} \hat{\xi}_j,
\]

where \( \hat{\xi}_i = (\hat{\xi}_{i1}, ..., \hat{\xi}_{im_i})^T \) with \( \hat{\xi}_{ij} = Y_{ij} - h_0 (T_{ij}, Z_{ij}, X_{ij}; \hat{\alpha} (T_{ij})) \). The asymptotic distributions of \( \hat{I}_n \) under \( H_{0,A} \) and \( H_{1,A} \) are presented in the following two theorems.
Theorem 4.3. Under the conditions of Theorem 4.1, it follows that
\[
\frac{n^2 \bar{N}_H^2}{\sqrt{N^2 - nN_2}} \sqrt{|H|} \hat{I}_n / \hat{\sigma}_1 \xrightarrow{D} N(0,1)
\]
under \(H_{0,A}\).

Theorem 4.4. Suppose that the conditions of Theorem 4.1 holds. Then under \(H_{1,A}\), we have \(\Pr(n \bar{N}_H \sqrt{|H|} / \hat{\sigma}_1 \geq E_n) \rightarrow 1\) as \(n \rightarrow \infty\), where \(E_n\) is any non-stochastic positive sequence such that \(E_n = o(n \bar{N}_H \sqrt{|H|})\).

5. Implementation. In this section, we address the practical issues that arise in the newly-proposed methodologies.

- B-spline method of pilot estimation

As a common practice in spline smoothing, we predetermine the order of the B-spline functions and then select optimal interior knots number through BIC criterion

\[
\text{BIC} (K_C, K_A) = \log (\text{RSS}) + N \log n / n,
\]
where \(\text{RSS} = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m_i} (Y_{ij} - \gamma_i^0 Z_{ij} \otimes b_C (T_{ij}) - \sum_{k=1}^{p} \gamma_k^0 T_k (T_{ij}, X_{ijk}))^2\) and \(N = (q + 1) J_C + p J_C J_A\), with \(J_C\) and \(J_A\) being the dimension of B-spline basis space \(b_C (t)\) and \(B_{k,A} (x_k)\). Then the optimal interior knots number is given by \((\hat{K}_C, \hat{K}_A) = \arg \min \text{BIC} (K_C, K_A)\).

- LLE based on pilot estimation

In local linear smoothing, we use Epanechnikov kernel function \(k(u) = 0.75 (1 - u^2) I_{|u| \leq 1}\), and select the optimal bandwidths using “leave-one-out” cross-validation procedure suggested by [33]. Define the subject-based cross-validation (CV) criterion as below

\[
\text{CV} (h_C, h_A) = \sum_{i=1}^{n} \frac{1}{m_i} \sum_{j=1}^{m_i} \left[ Y_{ij} - Z_{ij} \hat{\alpha}_{0,-i} (T_{ij}) - \sum_{k=1}^{p} \hat{\alpha}_{k,-i} (T_{ij}) \hat{\beta}_{k,-i} (X_{ijk}) \right]^2,
\]
where the subscript “\(-i\)” represents the estimator using the data with all repeated measurements except the \(i\)th subject. The optimal bandwidth is the unique minimizer of \(\text{CV}(h_C, h_A)\).
In simulation studies, we also can use the average squared error (ASE) as follows

\[
\text{ASE} (h_C, h_A) = \frac{1}{m} \sum_{i=1}^{n} \sum_{j=1}^{m_i} \left[ \mathbf{Z}_{ij}^\top (\mathbf{a}_0 (T_{ij}) - \hat{\mathbf{a}}_0 (T_{ij})) + \sum_{k=1}^{p} \alpha_k (T_{ij}) \beta_k (X_{ijk}) \right] - \sum_{k=1}^{p} \hat{\alpha}_k (T_{ij}) \hat{\beta}_k (X_{ijk}) \right] ^2.
\]

(5.1)

Similar to Remark 2.3 of [40], it is not difficult to show that the CV bandwidths approximately minimize ASE.

6. Numerical Studies.

6.1. Simulation Studies. In this subsection, we consider simulation examples to investigate the finite-sample performance of the proposed estimation method in Section 2 and the testing procedure in Section 4.

Example 1. Here we consider a mixed-effects Semi-VCAM. Let \(T_{ij}\) are uniformly distributed on \([0, 1]\), \(Z_{ij}\) are i.i.d. Bernoulli random variable with the probability of success \(p = 0.5\), and \(X_{ij} = U_i (1 + T_{ij}) + \vartheta_{ij}\), where \(U_i \sim U (-0.4, 0.4)\) and \(\vartheta_{ij} \sim N (0, 0.2^2)\). The response \(Y_{ij}\) is generated by a mixed-effects Semi-VCAM as below

\[
Y_{ij} = \alpha_{00} (T_{ij}) + \alpha_{01} (T_{ij}) Z_{ij} + \alpha_1 (T_{ij}) X_{ij} + \nu_i (T_{ij}) + \epsilon_{ij},
\]

for \(i = 1, ..., n\) and \(j = 1, ..., m\), where the measurement error \(\epsilon_{ij}\) are i.i.d from \(N (0, 1)\), and the subject-specific random trajectory \(\nu_i (T_{ij}) = \eta_{i1} + \sqrt{2} \eta_{i2} \sin (2 \pi T_{ij}) + \sqrt{2} \eta_{i3} \cos (2 \pi T_{ij})\) with \(\eta_{ij} \sim N (0, w_j)\) for \(j = 1, 2, 3\) and \((w_1, w_2, w_3) = (0.6, 0.2, 0.2)\). The univariate smooth component functions are given by \(\alpha_{00} (t) = 6t\), \(\alpha_{01} (t) = 2.5 \cos (2 \pi t)\), \(\alpha_1 (t) = \frac{t (1-t)}{\int_0^1 t (1-t) dt}\) and \(\beta (x) = 4.5 \sin (0.4 \pi x) - E [4.5 \sin (0.4 \pi X)]\).

We select 20 equally-spaced points on the range of \(T_{ij}\) and \(X_{ij}\), and define the mean prediction integrated squared error (MPISE) based on \(Q\) replications,

\[
\text{MPISE} (f) = \frac{1}{Q} \sum_{q=1}^{Q} \int \left[ \hat{f}_q (u) - f (u) \right]^2 du,
\]

where \(\hat{f}_q\) is the PEBLLE of the estimated function \(f\) in the \(q\)-th replication. Under different combinations of \(n\) and \(m\), based upon \(Q = 300\) Monte Carlo replications, Table 1 gives the MPISEs of PEBLLE of component functions,
and the standard deviation is shown in parentheses. We also list the optimal bandwidths according to (5.1). The result exhibits a good finite-sample performance whatever the data is sparse or dense. It is also found that MPISEs decrease markedly as the total observations increase.

Table 1
The MPISEs (standard deviation in parentheses) of component functions in Example 1.

| n   | m   | \( \hat{h}_C \) | \( \hat{h}_A \) | \( \hat{\alpha}_{00} \) | \( \hat{\alpha}_{01} \) | \( \hat{\alpha}_1 \) | \( \hat{\beta} \) |
|-----|-----|--------------|----------------|----------------|---------------|----------------|---------------|
| 5   | 0.1763 | 0.4132 | 0.0880 | (0.0605) | 0.1794 | (0.1129) | 0.0048 | (0.0028) | 0.1529 | (0.1122) |
| 10  | 0.1600 | 0.3950 | 0.0603 | (0.0400) | 0.0986 | (0.0585) | 0.0028 | (0.0016) | 0.0957 | (0.0646) |
| 50  | 30    | 0.1447 | 0.3500 | 0.0278 | (0.0194) | 0.0439 | (0.0245) | 0.0019 | (0.0012) | 0.0821 | (0.0574) |
| 50  | 0.1237 | 0.3395 | 0.0276 | (0.0193) | 0.0278 | (0.0137) | 0.0016 | (0.0010) | 0.0662 | (0.0516) |
| 100 | 0.1132 | 0.2921 | 0.0264 | (0.0263) | 0.0182 | (0.0092) | 0.0014 | (0.0010) | 0.0641 | (0.0441) |
| 10  | 0.1553 | 0.3553 | 0.0311 | (0.0200) | 0.0655 | (0.0369) | 0.0023 | (0.0013) | 0.0657 | (0.0395) |
| 30  | 0.1500 | 0.2831 | 0.0193 | (0.0153) | 0.0318 | (0.0129) | 0.0016 | (0.0009) | 0.0597 | (0.0405) |
| 100 | 60    | 0.1111 | 0.2278 | 0.0174 | (0.0147) | 0.0170 | (0.0075) | 0.0009 | (0.0008) | 0.0553 | (0.0443) |
| 100 | 0.0550 | 0.2200 | 0.0157 | (0.0110) | 0.0088 | (0.0033) | 0.0006 | (0.0004) | 0.0365 | (0.0284) |
| 150 | 0.0556 | 0.1778 | 0.0144 | (0.0013) | 0.0063 | (0.0026) | 0.0006 | (0.0006) | 0.0275 | (0.0180) |

Figure 1 visualizes the PEBLLE for \((n, m) = (100, 10)\). The solid curve plots true component function, the dashed line figures the PEBLLE, and the dash-dotted lines give 95% confidence bands based on the asymptotic distribution. The figure shows that our estimator is close to the true function even under the medium total observations \(N = 1000\).

We also investigate the performance of asymptotic distribution given in Theorems 3.2 and 3.4. After doing 300 Monte Carlo replications, we compare the average empirical coverage percentages (AECPs) based on four methods, that is, the unified method (U) given in (3.1) and (3.6), sparse method (S) in (3.2) and (3.7), dense method (D) in (3.3) and (3.8), and ultra dense method (UD) in (3.4) and (3.9). We take \(n = 50, 100\) and \(m = 5, 10, 30, 80, 200\).

Table 2 and 3 list the AECPs and the average empirical length (AEL) of confidence interval under the significance level 90% and 95%, respectively. From the resultant tables, we make a conclusion that:
Fig 1. Estimation of component functions in Example 1. The solid curve represents the true function, and the dashed line plots the PEBLLE, and the dash-dotted lines gives the 95% pointwise confidence bands based on (3.5) and (3.10).

(1) the AECPs of unified method (bold tags in tables) are superior to the other three methods, whatever the data is sparse, dense or ultra dense;
(2) the AECPs of sparse method decrease as $m$ grows, and they are inferior to the unified method even for sparse data;
(3) the AECPs of dense and ultra dense method increase as $m$ grows, and they are comparable to that of the unified method.

EXAMPLE 2. Now we investigate the performance of hypothesis testing constructed in Section 4. To this end, we consider the following two DGP:

- DGP I: In this case, we test the time-varying property of varying-coefficient component functions, that is to decide whether a PLAM is sufficient. We take the same settings with Example 1 for $T_{ij}, X_{ij}, Z_{ij}, \varepsilon_{ij}, \nu_i$ and $\beta_1(x)$. The time-varying testing of conditional mean function $m(t, z, x)$ is as follows

\[
H_0 : m(t, z, x) = g_0(z, x) \ a.s. \leftrightarrow H_1 : m(t, z, x) = g_1(t, z, x) \ a.s.,
\]
Table 2
The AECPs and AELs (in parentheses) of four methods with level 90% in Example 1.

|       | Fun   | n = 50 |              | U(%) | S(%) | D(%) | UD(%) |              | U(%) | S(%) | D(%) | UD(%) |
|-------|-------|--------|--------------|------|------|------|-------|--------------|------|------|------|-------|
|       |       |        |              |      |      |      |       |              |      |      |      |       |
| 5     |       |        |              |      |      |      |       |              |      |      |      |       |
| \(\alpha_0\) | 87.63 | 80.98  | 80.93        | 64.08|      |      |       |              |      |      |      |       |
|       | (0.9016) | (0.7539) | (0.7671) | (0.5309) |      |      |       |              |      |      |      |       |
| \(\alpha_0\) | 86.57 | 79.32  | 79.22        | 62.23|      |      |       |              |      |      |      |       |
|       | (1.2580) | (1.0501) | (1.0721) | (0.7443) |      |      |       |              |      |      |      |       |
| \(\alpha_1\) | 86.08 | 79.00  | 79.52        | 64.42|      |      |       |              |      |      |      |       |
|       | (0.3500) | (0.2909) | (0.2994) | (0.2092) |      |      |       |              |      |      |      |       |
| \(\beta_1\) | 87.53 | 86.67  | 81.68        | 44.95|      |      |       |              |      |      |      |       |
|       | (1.0994) | (1.0530) | (0.9188) | (0.2092) |      |      |       |              |      |      |      |       |
| 10    |       |        |              |      |      |      |       |              |      |      |      |       |
| \(\alpha_0\) | 88.40 | 72.55  | 85.88        | 77.98|      |      |       |              |      |      |      |       |
|       | (0.7374) | (0.4958) | (0.6884) | (0.5648) |      |      |       |              |      |      |      |       |
| \(\alpha_1\) | 86.63 | 68.90  | 83.28        | 73.93|      |      |       |              |      |      |      |       |
|       | (1.0573) | (0.7118) | (0.9868) | (0.8088) |      |      |       |              |      |      |      |       |
| \(\alpha_1\) | 88.00 | 71.70  | 85.35        | 76.75|      |      |       |              |      |      |      |       |
|       | (0.2770) | (0.1871) | (0.2584) | (0.2114) |      |      |       |              |      |      |      |       |
| \(\beta_1\) | 88.25 | 84.43  | 81.15        | 52.58|      |      |       |              |      |      |      |       |
|       | (0.8836) | (0.8200) | (0.7146) | (0.3007) |      |      |       |              |      |      |      |       |
| 30    |       |        |              |      |      |      |       |              |      |      |      |       |
| \(\alpha_0\) | 89.90 | 50.65  | 88.20        | 84.35|      |      |       |              |      |      |      |       |
|       | (0.6279) | (0.2601) | (0.6218) | (0.5762) |      |      |       |              |      |      |      |       |
| \(\alpha_1\) | 88.15 | 47.75  | 83.28        | 84.20|      |      |       |              |      |      |      |       |
|       | (0.8892) | (0.3691) | (0.8805) | (0.8156) |      |      |       |              |      |      |      |       |
| \(\alpha_1\) | 88.05 | 52.70  | 87.40        | 85.20|      |      |       |              |      |      |      |       |
|       | (0.2374) | (0.0984) | (0.2352) | (0.2180) |      |      |       |              |      |      |      |       |
| \(\beta_1\) | 88.50 | 71.65  | 85.60        | 77.85|      |      |       |              |      |      |      |       |
|       | (0.4139) | (0.2755) | (0.3775) | (0.3110) |      |      |       |              |      |      |      |       |
| 80    |       |        |              |      |      |      |       |              |      |      |      |       |
| \(\alpha_0\) | 89.01 | 39.67  | 88.42        | 87.16|      |      |       |              |      |      |      |       |
|       | (0.5937) | (0.1793) | (0.5851) | (0.5667) |      |      |       |              |      |      |      |       |
| \(\alpha_0\) | 89.49 | 38.71  | 89.07        | 87.93|      |      |       |              |      |      |      |       |
|       | (0.8367) | (0.2523) | (0.8248) | (0.7988) |      |      |       |              |      |      |      |       |
| \(\alpha_0\) | 88.71 | 41.89  | 88.13        | 87.07|      |      |       |              |      |      |      |       |
|       | (0.2198) | (0.0662) | (0.2167) | (0.2098) |      |      |       |              |      |      |      |       |
| \(\beta_1\) | 88.91 | 55.55  | 88.20        | 85.93|      |      |       |              |      |      |      |       |
|       | (0.3471) | (0.1583) | (0.3315) | (0.3105) |      |      |       |              |      |      |      |       |
| 200   |       |        |              |      |      |      |       |              |      |      |      |       |
| \(\alpha_0\) | 89.50 | 28.40  | 89.20        | 89.05|      |      |       |              |      |      |      |       |
|       | (0.5926) | (0.1166) | (0.5849) | (0.5618) |      |      |       |              |      |      |      |       |
| \(\alpha_0\) | 89.47 | 34.73  | 89.25        | 89.13|      |      |       |              |      |      |      |       |
|       | (0.8311) | (0.1636) | (0.8238) | (0.7959) |      |      |       |              |      |      |      |       |
| \(\alpha_0\) | 89.20 | 29.93  | 88.90        | 88.70|      |      |       |              |      |      |      |       |
|       | (0.2069) | (0.0406) | (0.2056) | (0.2032) |      |      |       |              |      |      |      |       |
| \(\beta_1\) | 89.60 | 38.60  | 89.13        | 89.04|      |      |       |              |      |      |      |       |
|       | (0.3219) | (0.0986) | (0.3163) | (0.3072) |      |      |       |              |      |      |      |       |
Table 3
The AECPs and AELs (in parentheses) of four methods with level 95% in Example 1. 

| m   | Fun | n = 50   |          |          |          |          | n = 100   |          |          |          |          |
|-----|-----|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
|     |     | U(%)     | S(%)     | D(%)     | UD(%)    | U(%)     | S(%)     | D(%)     | UD(%)    | U(%)     | S(%)     | D(%)     | UD(%)    |
| 5   | α₀₀ | 93.15    | 88.93    | 88.23    | 72.70    | 93.68    | 90.45    | 88.50    | 69.23    | 94.05    | 89.35    | 81.13    | 69.23    |
|     |     | 93.68    | 90.45    | 88.50    | 69.23    | 93.08    | 88.22    | 86.12    | 66.52    |          |          |          |          |
|     | α₀₁ | 93.60    | 87.12    | 85.92    | 71.92    | 93.12    | 90.10    | 87.65    | 70.23    |          |          |          |          |
|     |     | 93.30    | 92.25    | 90.18    | 43.37    | 93.30    | 92.25    | 90.18    | 43.37    |          |          |          |          |
|     | β₁  | 93.50    | 81.28    | 89.93    | 82.13    | 94.13    | 85.53    | 89.35    | 81.13    |          |          |          |          |
| 10  | α₀₀ | 93.50    | 82.28    | 89.98    | 82.40    | 94.05    | 86.90    | 90.65    | 82.18    |          |          |          |          |
|     |     | 93.90    | 86.60    | 90.18    | 66.52    | 94.05    | 86.90    | 90.65    | 82.18    |          |          |          |          |
|     | α₀₁ | 93.15    | 82.48    | 90.40    | 82.73    | 94.05    | 86.60    | 90.18    | 66.52    |          |          |          |          |
|     |     | 93.63    | 90.15    | 87.68    | 80.50    | 94.33    | 90.53    | 92.43    | 70.48    |          |          |          |          |
|     | β₁  | 93.63    | 90.15    | 87.68    | 80.50    | 94.33    | 90.53    | 92.43    | 70.48    |          |          |          |          |
| 30  | α₀₀ | 93.88    | 68.25    | 92.12    | 89.30    | 94.15    | 64.30    | 94.00    | 90.75    |          |          |          |          |
|     |     | 93.90    | 67.45    | 92.90    | 90.20    | 94.05    | 59.25    | 93.05    | 90.05    |          |          |          |          |
|     | α₀₁ | 93.80    | 72.90    | 92.05    | 89.75    | 94.20    | 68.75    | 93.15    | 89.95    |          |          |          |          |
|     |     | 94.10    | 81.00    | 92.20    | 86.00    | 94.55    | 82.05    | 92.45    | 83.85    |          |          |          |          |
|     | β₁  | 94.45    | 45.35    | 93.80    | 92.80    | 94.60    | 49.25    | 93.30    | 92.05    |          |          |          |          |
| 80  | α₀₀ | 94.25    | 52.80    | 94.20    | 93.80    | 94.75    | 51.40    | 94.35    | 93.70    |          |          |          |          |
|     |     | 94.50    | 58.15    | 94.00    | 92.45    | 94.80    | 65.40    | 94.00    | 93.15    |          |          |          |          |
|     | α₀₁ | 94.45    | 45.35    | 93.80    | 92.80    | 94.60    | 49.25    | 93.30    | 92.05    |          |          |          |          |
|     |     | 94.45    | 45.35    | 93.80    | 92.80    | 94.60    | 49.25    | 93.30    | 92.05    |          |          |          |          |
|     | α₀₁ | 94.50    | 39.65    | 94.40    | 94.25    | 94.85    | 47.35    | 94.65    | 94.35    |          |          |          |          |
|     |     | 94.50    | 39.65    | 94.40    | 94.25    | 94.85    | 47.35    | 94.65    | 94.35    |          |          |          |          |
|     | β₁  | 94.50    | 39.65    | 94.40    | 94.25    | 94.85    | 47.35    | 94.65    | 94.35    |          |          |          |          |
| 200 | α₀₀ | 94.35    | 35.20    | 94.15    | 94.00    | 94.75    | 41.45    | 94.55    | 94.30    |          |          |          |          |
|     |     | 94.75    | 47.40    | 94.30    | 94.15    | 94.85    | 47.35    | 94.60    | 94.35    |          |          |          |          |
|     | α₀₁ | 94.35    | 35.20    | 94.15    | 94.00    | 94.75    | 41.45    | 94.55    | 94.30    |          |          |          |          |
|     |     | 94.75    | 47.40    | 94.30    | 94.15    | 94.85    | 47.35    | 94.60    | 94.35    |          |          |          |          |
|     | β₁  | 94.75    | 47.40    | 94.30    | 94.15    | 94.85    | 47.35    | 94.60    | 94.35    |          |          |          |          |
where \( g_0(z, x) = 6 + 2.5z + \beta_1(x) \) and 
\[
g_1(t, z, x) = g_0(z, x) + \theta(t + z \cos(2\pi t) + t(1-t) \beta_1(x)),
\]
with \( \theta = 0.2, 0.4, 0.6, 0.8, 1.0 \).

**DGP II:** Here we consider the linearity testing of additive component functions to judge whether a VCM is sufficient. Let 
\[
T_{ij}, Z_{ij}, \alpha_{00}(t) \text{ and } \alpha_{01}(t) \text{ be given in Example 1, } \alpha_1(t) = \sin(\pi t)/\int_0^1 \sin(\pi t)dt, \text{ and } X_{ij} = U_i(1 + T_{ij}) + \vartheta_{ij}, \text{ where } U_i \sim U(-0.5, 0.5) \text{ and } \vartheta_{ij} \sim N(0,1).
\]
The interested hypothesis is given by
\[
(6.2) \quad H_0 : m(t, z, x) = h_0(t, z, x) \text{ a.s. } \leftrightarrow H_1 : m(t, z, x) = h_1(t, z, x) \text{ a.s.,}
\]
where 
\[
h_0(t, z, x) = \alpha_{00}(t) + \alpha_{01}(t) z + \alpha_1(t) x \text{ and } h_1(t, z, x) = h_0(t, z, x) + 1.5\theta\alpha_1(t) \sin(\pi x) \text{ with } \theta = 0.2, 0.4, 0.6, 0.8, 1.0.
\]

We consider different combinations of \( n = 30, 50, 100 \) and \( m = 5, 10, 30, 60, 100 \), and generate \( Q = 300 \) Monte Carlo replications and \( B = 300 \) bootstrap samples for each simulated data set. Under 5% and 10% significance levels, based upon bootstrap critical value, Table 4 and 5 present power of hypothesis (6.1) and (6.2) for different deviation parameters \( \theta \) ranging from 0 to 1 with the span of 0.2, respectively. The results show that the proposed testing procedure all performs well for sparse, dense and dense data. In fact, the power for \( \theta = 0 \) is size of hypothesis, which is close to the theoretical significance level 0.05 or 0.1. As expected, the power increases to one as \( \theta \) ascends whatever significance levels and sampling plans. Moreover, Figures 2 and 3 plot the rejection rates of testing (6.1) and (6.2) at the 5% and 10% significance levels for some combinations of \( n \) and \( m \), respectively.

### 6.2. Real Data Analysis.

**Example 3.** Now we apply our method to the new coronavirus disease (COVID-19) mentioned in Section 1. We collected the daily cumulative confirmed cases \((Z_{i,t})\) and the daily cumulative cured cases from https://github.com/CSSEGISandData/COVID-19, the daily movement population from Wuhan to other provinces (https://qianxi.baidu.com/), the maximum daily temperature (http://www.weather.com.cn) and the population data (https://zh.wikipedia.org/wiki/).

The response variable GRCC, denoted by \( Y_{i,t} \), is measured by \( \log(Z_{i,t}) - \log(Z_{i,t-1}) \), which is presented in Figure 4 (a) for 29 provinces in China from January 23th to April 8th. We notice that the big values of GRCC (above 0.5) mainly concentrate on the period from January 23th to February 3th.
Table 4

Power of testing (6.1) under confidence level $\alpha = 5\%$ and $10\%$.

| $\alpha$ | $\theta$ | $(n, m)$       |
|----------|----------|----------------|
|          |          | $(50, 5)$      |
| 5%       | 0        | 0.067          |
|          | 0.2      | 0.055          |
|          | 0.4      | 0.050          |
|          | 0.6      | 0.040          |
|          | 0.8      | 0.055          |
|          | 1.0      | 0.040          |
| 0        | 0.067    | 0.055          |
|          | 0.2      | 0.050          |
|          | 0.4      | 0.040          |
|          | 0.6      | 0.055          |
|          | 0.8      | 0.040          |
|          | 1.0      | 0.055          |
| 0.2      | 0.055    | 0.050          |
|          | 0.2      | 0.040          |
|          | 0.4      | 0.055          |
|          | 0.6      | 0.040          |
|          | 0.8      | 0.055          |
|          | 1.0      | 0.040          |
| 0.4      | 0.050    | 0.050          |
|          | 0.2      | 0.040          |
|          | 0.4      | 0.055          |
|          | 0.6      | 0.040          |
|          | 0.8      | 0.055          |
|          | 1.0      | 0.040          |
| 0.6      | 0.040    | 0.050          |
|          | 0.2      | 0.040          |
|          | 0.4      | 0.055          |
|          | 0.6      | 0.040          |
|          | 0.8      | 0.055          |
|          | 1.0      | 0.040          |
| 0.8      | 0.055    | 0.040          |
|          | 0.2      | 0.055          |
|          | 0.4      | 0.040          |
|          | 0.6      | 0.055          |
|          | 0.8      | 0.040          |
|          | 1.0      | 0.055          |
| 1.0      | 0.040    | 0.050          |
|          | 0.2      | 0.040          |
|          | 0.4      | 0.055          |
|          | 0.6      | 0.040          |
|          | 0.8      | 0.055          |
|          | 1.0      | 0.040          |

Table 5

Size and power of test (6.2) under confidence level $\alpha = 5\%$ and $10\%$.

| $\alpha$ | $\theta$ | $(n, m)$       |
|----------|----------|----------------|
|          |          | $(50, 5)$      |
| 5%       | 0        | 0.047          |
|          | 0.2      | 0.057          |
|          | 0.4      | 0.050          |
|          | 0.6      | 0.045          |
|          | 0.8      | 0.040          |
|          | 1.0      | 0.045          |
| 0        | 0.047    | 0.057          |
|          | 0.2      | 0.050          |
|          | 0.4      | 0.045          |
|          | 0.6      | 0.040          |
|          | 0.8      | 0.040          |
|          | 1.0      | 0.045          |
| 0.2      | 0.057    | 0.057          |
|          | 0.2      | 0.045          |
|          | 0.4      | 0.050          |
|          | 0.6      | 0.045          |
|          | 0.8      | 0.040          |
|          | 1.0      | 0.045          |
| 0.4      | 0.050    | 0.050          |
|          | 0.2      | 0.045          |
|          | 0.4      | 0.050          |
|          | 0.6      | 0.045          |
|          | 0.8      | 0.040          |
|          | 1.0      | 0.045          |
| 0.6      | 0.045    | 0.050          |
|          | 0.2      | 0.045          |
|          | 0.4      | 0.050          |
|          | 0.6      | 0.045          |
|          | 0.8      | 0.040          |
|          | 1.0      | 0.045          |
| 0.8      | 0.040    | 0.045          |
|          | 0.2      | 0.050          |
|          | 0.4      | 0.045          |
|          | 0.6      | 0.045          |
|          | 0.8      | 0.040          |
|          | 1.0      | 0.045          |
| 1.0      | 0.045    | 0.050          |
|          | 0.2      | 0.045          |
|          | 0.4      | 0.050          |
|          | 0.6      | 0.045          |
|          | 0.8      | 0.040          |
|          | 1.0      | 0.045          |
Fig 2. Power of time-varying testing (6.1) in Example 2. For three combinations of \( n \) and \( m \), (a) - (c) figure power at the level \( \alpha = 0.05 \) and 0.1; whist (a') - (c') give the simulated density of standardized test statistics (thick black) and five bootstrap approximations (thin).

Fig 3. Power of linearity testing (6.2) in Example 2. For three combinations of \( n \) and \( m \), (a) - (c) figure power at the level \( \alpha = 0.05 \) and 0.1; whist (a') - (c') give the simulated density of standardized test statistics (thick black) and five bootstrap approximations (thin).
It is a strong evidence that the intervention policy of China’s government plays a positive role in controlling the spread of Coronavirus disease.

To explore the influence factor of GRCC, we used five covariates: $X_{1,it}$ being the movement population from Wuhan (MPFW), which is measured by the proportion of the population moving from Wuhan to the $i$th province out of moving out population at day $t - 14$; $X_{2,it}$ the daily cumulative cured cases (CUCC) at day $t - 1$; $X_{3,it}$ the daily cumulative confirmed cases (CFCC) at day $t - 1$; $X_{4,it}$ the maximum daily temperature at day $t$, and $X_{5,i}$ the population of $i$th province.

We normalize the covariate $X_{1,it}$, and make the logarithm transformation for $X_{k,it}$, $k = 2, 3$ and $X_{5,i}$. Based on 500 bootstrap sampling, we do the time-varying testing (4.1) and linearity testing (4.4), obtaining the $p$ values 0.028 and 0.038, respectively. Therefore, we reject the AM and VCM at significant level 0.05, and adopt the general model as below:

$$Y_{it} = \alpha_0 (t/T) + \sum_{k=1}^{4} \alpha_k (t/T) \beta_k (X_{k,it}) + \alpha_5 (t/T) \beta_5 (X_{5,i}),$$

where $i = 1, ..., 29$, $t = 1, ..., T$ with $T = 77$. Figure 4 gives the PEBLLE of component functions, and 95% pointwise confidence bands according to (3.5) and (3.10).

From Figure 4, we conclude that the trend term $\alpha_0$, varying-coefficient function $\alpha_1$ for MPFW, $\alpha_3$ for CCFC and $\alpha_4$ for MDT have similar properties, i.e., they drop rapidly until about February 29th, and then maintain on the level close to zero; $\alpha_2$ for CUCC decreases until about February 22th, and increases until about March 9th, and thereafter levels near zero; $\alpha_5$ for POP decreases slowly until about February 9th, then increases until about February 29th, and decreases thereafter.

For the medium values of the normalized MPFW, the effect increases as MPFW grows, and some fluctuations appears for the large value (above 2), since large MPFW usually takes place in the early stage and the period of work resumption. The influence of CUCC increases as it grows, and the rate of increases become slower above 2; while the effect of CCFC ascends as it increases, and levels out above 3. The effect of MDT drops under 0℃, and increases until 10℃, and almost no influence between 10℃ and 20℃, then ascends rapidly above 20℃. The trend of effect of POP grows as the population size ascends, especially when log-POP is larger than 7.5.

**Example 4.** We revisit a CD4 data from the Multicenter AIDS Cohort Study, which contains 1817 observations from 283 homosexual men infected with HIV between 1984 and 1991. [4, 15] have analyzed this data set using a
VCM. Now, we apply our method to this dataset. The response variable $Y_{ij}$ is the $i$-th subject’s CD4 percentage at time $T_{ij}$. Following the covariates of [15], we let $X_{1i}$ be the $i$-th subject’s smoke status, a dichotomous variable, $X_{2i}$ the $i$-th subject’s centred age, and $X_{3i}$ the $i$-th subject’s centred pre-infection CD4 percentage. The relationship between response and covariates
are modeled by a Semi-VCAM as below

\[(6.4) \quad Y_{ij} = \alpha_0 (T_{ij}) + \alpha_1 (T_{ij}) X_{1i} + \alpha_2 (T_{ij}) \beta_1 (X_{2i}) + \alpha_3 (T_{ij}) \beta_2 (X_{3i}),\]

where the covariates are all time-invariant.

Based on 500 bootstrap sampling, we do the time-varying testing \((4.1)\) and the linearity testing \((4.4)\), obtaining the \(p\) values 0.028 and 0.457, respectively. That means, at significant level 0.05, VCM is a reasonable choice, which verifies that the model used in [15] is appropriate.

7. Concluding Remarks. In this paper, we have considered a Semi-VCAM for the functional/longitudinal data with different sampling plan. The Semi-VCAM is an extension of the existing VCAM. We have developed a pilot estimation based local linear estimation for the Semi-VCAM and have presented asymptotic distribution on a unified platform for sparse, dense and ultra dense cases of the data. The virtue of unified asymptotic results is to help us avoid deciding the types of data in advance, which is a subjective choice and may lead to wrong conclusions. From the viewpoint of model parsimony, we also have developed consistent testing procedures to justify whether a VCM or PLAM, especially an AM is sufficient for the real-life data. These test methods also avoid the subjective choice between the sparse, dense and ultra dense cases of the data.

Our model and inference methods may be extended in various directions. We close the paper by outlining some of them. In many application areas, data may be collected on a count or binary response. For example, daily death toll, suspected and confirmed cases of COVID-19. As a result, it is useful to extend our proposed model and inference to the generalized Semi-VCAM to accommodate the discrete functional/longitudinal responses. Data in the form of samples of densities or distributions are increasingly encountered in practice and same as [11] there is a need for flexible regression models that accommodate random densities as responses. We believe our proposed model could also be used to model the data in which the responses are random densities. In addition, due to the fact that the proposed test method in our paper is based on the local smoothing, it may suffer the curse of dimensionality, struggle to maintain the significance level and lose its power to an extent as the dimension of explanatory variables increases. Same as [20] and [22], we may use projection technique, or bridging between local smoothing and global smoothing methods to avoid this. Due to the complication of our model, extending the methods in [20] and [22] to our scenario is not simple.
APPENDIX A: APPENDIX SECTION

A.1. Appendix subsection. A function \( m \) defined on the interval \([a, b]\) is called to be Lipschitz-continuous, if there exists a fixed constant \( C > 0 \), such that \( |m(x) - m(x')| \leq C|x - x'| \) for any \( x, x' \in [a, b] \). Denote \( C_r[a, b] \) as the space of all functions \( m(x) \) defined on \([a, b]\), such that \( m \) is differentiable of \( r - 1 \) order, and \( m^{(r-1)} \) is Lipschitz-continuous, where \( m^{(l)} \) means the \( l \)-th order derivative of \( m \).

The necessary conditions to validate asymptotic properties are as follows.

(A1) The observation time points \( T_{ij} \)'s are drawn from an unknown distribution, which has a density \( f_T(t) \) with the support \( T \), and is continuously differentiable in a neighbourhood of \( t \) and is uniformly bounded away from 0 and infinity.

(A2) \( X_i \)'s are independent realizations of stochastic process \( X(t) \), and \( X_i \)'s are independent of \( T_{ij} \)'s. The marginal density function \( f_{X_k}(\cdot) \) of co-variates \( X_k \) is continuously differentiable in a neighbourhood of \( x \) and is uniformly bounded away from 0 and infinity.

(A3) \( Z_i \)'s are independent realizations of stochastic process \( Z(T) \), and \( Z_i \)'s are independent of \( T_{ij} \)'s. The eigenvalues of \( \text{E} [Z(T)Z(T)^\top] \) are bounded from 0 and infinity uniformly in \( T \in T \). In addition, there exists a positive constant \( M \) such that \( |Z_k(T)| \leq M \) uniformly for \( T \in T \) and \( k = 1, ..., q \), where \( Z_k \) i.i.d. with \( Z_{ij,k} \), a random sample of \( k \)-th covariate.

(A4) \( \alpha_k \in C_r[a, b] \) for \( k = 0, ..., p \) and \( \beta_k \in C_r[a_k, b_k] \) for \( k = 1, ..., p \).

(A5) \( \{\nu_i(\cdot)\}_{i}, \{T_{ij}\}_{ij}, \{\epsilon_{ij}\}_{ij} \) are independent and identically distributed and mutually independent. \( \{x_{ij}\}_i \) are independent and identically distributed. Moreover, \( \{\nu_i(\cdot)\}_i, \{x_{ij}\}_i \) and \( \{\epsilon_{ij}\}_{ij} \) are mutually independent.

(A6) \( \sigma^2(\cdot) < \infty \) is continuously differentiable. \( \gamma(t, t') \) is continuously differentiable and \( \gamma(t, t) = \lim_{t' \to t} \gamma(t, t') < \infty \).

(A7) \( \text{E}\{\nu_i(\cdot) + \sigma(\cdot)\epsilon_{ij} |\nu'\} \) is continuous and bounded from infinity for \( \nu \leq 4 \).

(A8) \( k(\cdot) \) is bounded and symmetric probability density function with a bounded support and a bounded derivative.

(A9) \( \sqrt{K}\{K_A^{-r} + K_C^{-r}\} = o(1) \) and \( K_A^2 K_C / n = o(1) \).

Remark 4. Assumptions A1 and A2 involve the distributions of time points \( T_{ij} \) and \( k \)-th covariate \( X_k \). Assumption A3 relates to co-variates \( Z \), a similar conditions with [4]. Assumption A4 specifies the degree of smoothness of varying-coefficient component functions and additive component functions. Assumptions A5–A7 are necessary for constructing asymptotic distribution, a common conditions with [4]. Assumption A8 is a standard
Proposition 1. Under Assumption (A1) – (A6) and (A9), it follows that
\[
\sup_{x \in [a_k, b_k]} |\hat{\beta}_{k,1}(x) - \beta_k(x)| = O_p \left( \sqrt{K_A K_p^{-r}} + K_C K_p^{r} + \frac{K_A K_p^2}{nN_H} + \frac{K_A}{n} \right).
\]

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