INTERACTING RELATIVISTIC PARTICLE: TIME-SPACE NONCOMMUTATIVITY AND SYMMETRIES

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Abstract: We discuss the symmetry properties of the reparametrization invariant model of an interacting relativistic particle where the electromagnetic field is taken as the constant background field. The direct coupling between the relativistic particle and the electromagnetic gauge potential is a special case of the above with a specific set of subtleties involved in it. For the above model, we demonstrate the existence of a time-space noncommutativity (NC) in the spacetime structure from the symmetry considerations alone. We further show that the NC and commutativity properties of this model are different aspects of a unique continuous gauge symmetry that is derived from the non-standard gauge-type symmetry transformations by requiring their consistency with (i) the equations of motion, and (ii) the expressions for the canonical momenta, derived from the Lagrangians. We provide a detailed discussion on the noncommutative deformation of the Poincaré algebra.

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1 Introduction

In various branches of physics and mathematics, the noncommutative spaces and corresponding algebras have appeared in a consistent and cogent manner [1]. The recent upsurge of interest in the study of field theories, based on the above noncommutative spaces, stems from the fact that the existence of noncommutativity (NC) of spacetime has been found in the context of the string theories, $D$-branes and $M$-theories which are deemed to be the forefront areas of research in theoretical high energy physics. To be more precise, the end points of the open strings, trapped on the $D$-branes, turn out to be noncommutative in the presence of a 2-form (i.e. $B = (1/2!) (dx^\mu \wedge dx^\nu) B_{\mu\nu}$) background gauge field $B_{\mu\nu}$ [2,3]. Furthermore, it has been argued that the string dynamics could be shown to be equivalent to the minimally coupled gauge field theory on a noncommutative space [4]. The study of the black hole physics and quantum gravity is yet another source of the NC in the spacetime structure [5,6]. Some attempts have already been made to gain an insight into the perturbative and non-perturbative aspects of the noncommutative field theories and a few nice results have been obtained (see, e.g., [5-10] and references therein).

The understanding of the reparametrization invariant models have played some notable roles in the developments of the modern theoretical high energy physics. In particular, the symmetries, constraints, dynamics, etc., associated with the free as well as interacting relativistic (super)particles, have enriched our understanding of the more complicated reparametrization invariant (super)string and (super)gravity theories. In this context, it is pertinent to point out that, in a couple of papers [11,12], the free as well as interacting particle mechanics has been studied in the framework of Dirac brackets formalism and the existence of the NC has been shown to owe its origin to the reparametrization invariance in the theory. To be more precise, it has been argued that, for the above models, the commutativity and NC of spacetime are equivalent in the sense that they correspond to different choices of gauge conditions. These gauge conditions, in turn, have been shown to be connected to each-other by a gauge type of transformation (see, e.g. [12]). The deformation of the Poincaré (and related) algebras for the massive free relativistic particle has been studied in detail in the untransformed frames [11,12]. This is because of the fact that the linear momentum and angular momentum generators for this model remain invariant under the gauge transformations for the spacetime variables. As a result, there is no need to consider the deformation of the above algebras in the gauge transformed frames.

A different source of the NC in spacetime structure has been shown to exist in the mechanical description of the free massless relativistic particle [13]. To be more accurate, the existence of a very specific kind of local scale type symmetry (which is distinctly different from the usual global scale symmetry of the conformal group of transformations) has been shown [13] for the free massless relativistic particle. This new scale type symmetry leads to the existence of the NC in spacetime structure which, in turn, enforces the extension of the conformal algebra for the above model [13]. A thorough discussion on the dynamical
implications of the above NC has been performed in [14] where the emphasis is laid on the symplectic structures associated with the Poisson bracket formalism of dynamics. In a recent couple of papers [15,16], the toy model of a reparametrization invariant system of a non-relativistic free particle and a physically interesting model of the reparametrization invariant free massive relativistic have been studied where the NC of the spacetime emerges from the consideration of the non-standard gauge type continuous symmetries. As it turns out, for these models, the mass parameter becomes noncommutative in nature.

The purpose of the present paper is to study, in detail, the interacting reparametrization invariant model of the massive relativistic particle where the interaction is present through a constant electromagnetic background field. We demonstrate the existence of a time-space NC in the spacetime structure by tapping the potential and power of the continuous gauge symmetry transformations. The emphasis, in our present work, has been laid on the standard gauge symmetry transformations for the spacetime (that correspond to a commutative geometry) and the non-standard gauge type of symmetry transformations for the spacetime (that correspond to a noncommutative geometry). We also demonstrate, in the language of the continuous gauge symmetry transformations, the absence of the space-space NC in the theory. The time-space NC is physically very interesting because a whole lot of studies, connected with the developments of the unitary quantum mechanics and their possible physical consequences, have been performed in [17-19]. In our present discussion, this time-space NC emerges very naturally. The trick, to obtain such a kind of NC, is the same as in our earlier works [15,16] where one begins with a non-standard gauge type of transformations for the spacetime (as well as other) variables of a given Lagrangian and, ultimately, enforces these transformations to reduce to the standard continuous gauge transformations. In the process, one obtains a specific set of restrictions on the noncommutative parameter as well as the momenta variables (see, e.g. (3.6),(3.8) and (3.9) below). For our present interacting model, these restrictions lead to, at least, a triplet of key consequences. First, they establish, in a new way, the equivalence of the commutativity and NC in the language of the continuous symmetry properties which turns out to be consistent with such an observation made in the language of the Dirac bracket formalism [11,12]. Second, they enforce a connection between the electric field and magnetic field of the theory (see, e.g., (3.10) below). Finally, they lead to the deformation of the Poincaré (and related) algebras in the (un)transformed frames (see, Sec. 5 for details). We would like to emphasize that, to the best of our knowledge, a detailed discussion on the deformation of the Poincaré (and related) algebras in the (un)transformed frames, for this interacting reparametrization invariant model, has not been performed in the literature (see, e.g., [11,12] and references therein). Thus, the results of Sec. 5 are the central part of our present paper. The logical explanation for the choice of the non-standard gauge-type transformations (cf. (3.1) below) has been provided in the language of the Becchi-Rouet-Stora-Tyutin (BRST) cohomology connected with the spacetime transformations (cf. Sec. 6 below).

Our present study is essential on five counts. First and foremost, it is important to
generalize the ideas of our earlier works [15,16] which were valid for the (non-)relativistic free particle to an interacting relativistic particle. The latter is, of course, more general than the previous ones. Second, for the model under consideration, the values of the components of the momenta \( p_0 \) and \( p_i \) are not fixed because they transform under the gauge transformations as well as non-standard gauge type transformations. This is distinctly different from the free particle case where the momenta \( p_\mu \) is a gauge-invariant quantity. As a consequence, the components of \( p_\mu \) can be fixed to a constant quantity in terms of the mass parameter (while still satisfying the mass-shell condition \( p_0^2 - p_i^2 = m^2 \)) (see, e.g. [16] for details). However, one pays a price for this fixed values of the components of momenta in the sense that the mass parameter of the model becomes noncommutative in nature. In our present model, we do not end up with the mass parameter being noncommutative in nature. Third, the deformation of the Poincaré algebra in the (un)transformed frames emerges very naturally for the model under discussion because of the fact that the components of momenta are found to be related to one-another. This, in turn, implies the deformation of the canonical brackets which, ultimately, leads to the deformation of the Poincaré algebra. Fourth, the connection between the components of momenta (cf. (3.6) and (3.8) below) enforces a connection between the electric and magnetic fields through the noncommutative parameter (cf. (3.10) below). Finally, the model under discussion, possesses richer theoretical (as well as mathematical) structures and is certainly more general than its free counterparts [15,16].

The contents of our present paper are organized as follows. In Sec. 2, we recapitulate the bare essentials of the Lagrangian formulation of the interacting relativistic particle where the electromagnetic field is a constant (i.e. \( F_{\mu\nu} \neq F_{\mu\nu}(\tau) \)) background field. We provide a detailed discussion on the Poincaré (and related) algebras for our present model in the untransformed frame as well as in the gauge transformed frames. Sec. 3 is devoted to a thorough discussion on the time-space NC from the point of view of the continuous gauge symmetry transformations alone. In Sec. 4, we deal with the more general NC of spacetime and show that the space-space NC is ruled out (i.e. \( \theta_{ij} = 0 \)) from the continuous symmetry considerations. Sec. 5 focuses on the deformation of the Poincaré algebra in the untransformed frames as well as in the gauge-transformed frames due to time-space NC. In Sec. 6, we show the cohomological equivalence of the commutativity and the NC within the framework of BRST formalism. Finally, we make some concluding remarks and point out a few future directions for further investigations in Sec. 7.

2 Preliminary: Standard Continuous Symmetries and Commutativity

Let us begin with the reparametrization invariant Lagrangians for the relativistic particle in interaction with the constant background electromagnetic field \( F_{\mu\nu} \) (with \( F_{\mu\nu} = -F_{\nu\mu}, F_{0i} = E_i, F_{ij} = \epsilon_{ijk}B_k \)) which is independent of the parameter \( \tau \) that characterizes the trajectory of the particle. This system is embedded in the (3+1)-dimensional Minkowskian flat target
space*. The three equivalent Lagrangians for the system are (see, e.g., [11,12])

\[ L_0 = m \left( \dot{x}^2 \right)^{1/2} - \frac{1}{2} F_{\mu\nu} x^\mu \dot{x}^\nu, \quad L_f = p_\mu \dot{x}^\mu - \frac{1}{2} F_{\mu\nu} x^\mu \dot{x}^\nu - \frac{1}{2} e (p^2 - m^2), \]
\[ L_s = \frac{1}{2} \dot{x}^2 - \frac{1}{2} F_{\mu\nu} x^\mu \dot{x}^\nu + \frac{1}{2} e m^2; \]

where \( L_0, L_f \) and \( L_s \) are the Lagrangian with the square root, the first-order Lagrangian and the second order Lagrangian, respectively. In the above, \( e(\tau) \) is the einbein field and the canonical momenta \( \pi_\mu \) for the Lagrangians \( L_0 \) and \( L_f \) is \( \pi_\mu = p_\mu + \frac{1}{2} F_{\mu\nu} x^\nu \). The explicit form of \( p_\mu(\tau) \) (derived from \( L_0 \)) and \( e(\tau) \) (derived from \( L_s \)), that would be useful for our later discussions, are

\[ p_\mu = \frac{m \dot{x}_\mu}{(\dot{x}^2)^{1/2}} \equiv \frac{m \dot{x}_\mu}{[\dot{x}_0^2 - \dot{x}_i^2]^{1/2}}, \quad e = \frac{(\dot{x}^2)^{1/2}}{m} \equiv \frac{[\dot{x}_0^2 - \dot{x}_i^2]^{1/2}}{m}. \]  

(2.1)

It should be re-emphasized that (a) the mass \( m \) (i.e. the analogue of the cosmological constant term), and (b) the constant background field \( F_{\mu\nu} \) are independent of the monotonically increasing evolution parameter \( \tau \) that characterizes the trajectory of the particle. The following canonical Poisson brackets between the canonical variables \( x_\mu \) and \( \pi_\mu \):

\[ \{x_\mu, x_\nu\}_{(PB)} = 0, \quad \{x_\mu, \pi_\nu\}_{(PB)} = \eta_{\mu\nu}, \quad \{\pi_\mu, \pi_\nu\}_{(PB)} = 0, \]

(2.2)

imply that the Poisson brackets \( \{x_\mu, p_\nu\}_{(PB)} = \eta_{\mu\nu}, \{p_\mu, p_\nu\}_{(PB)} = -F_{\mu\nu} \) are true where the latter has been derived from the requirement \( \{\pi_\mu, \pi_\nu\}_{(PB)} = 0 \) by exploiting the definition \( \pi_\mu = p_\mu + (1/2) F_{\mu\nu} x^\nu \) and the bracket \( \{x^\mu, p_\nu\}_{(PB)} = \delta_\nu^\mu \). These brackets demonstrate that the operators \( p_\mu \) are noncommutative and their NC owes its origin to the non-zero constant background field \( F_{\mu\nu} \). However, \( x_\mu \) are still commutative implying that the spacetime geometry is commutative too. Let us now focus on the symmetry properties of the first-order Lagrangian \( L_f \) which is (i) equivalent to the other Lagrangians \( L_0 \) and \( L_s \), (ii) devoid of the square root as well as the presence of a field in the denominator, and (iii) endowed with the maximum number of dynamical variables (i.e. \( x_\mu, \dot{x}_\mu, p_\mu, e \)) allowing it to provide more freedom for theoretical discussions compared to the other two Lagrangians \( L_0 \) and \( L_s \). Under the infinitesimal version of the reparametrization transformation \( \tau \rightarrow \tau' = \tau - \epsilon(\tau) \) (where \( \epsilon(\tau) \) is an infinitesimal transformation parameter), the variables of the first-order Lagrangian \( L_f \) undergo the following change

\[ \delta_\tau x_\mu = \epsilon \dot{x}_\mu, \quad \delta_\tau p_\mu = \epsilon \dot{p}_\mu, \quad \delta_\tau F_{\mu\nu} = 0, \quad \delta_\tau e = \frac{d}{d\tau} [\epsilon e], \]

(2.4)

*We adopt here the conventions and notations such that the flat metric \( \eta_{\mu\nu} \), characterizing the Minkowskian spacetime manifold, is diagonal (i.e. \( \eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1) \)) so that \( A \cdot B = \eta_{\mu\nu} A^\mu B^\nu = \eta^{\mu\nu} A_\mu B_\nu = A_0 B_0 - A_i B_i \) is the definition of the dot product between two four vectors. The totally antisymmetric four dimensional (4D) Levi-Civita tensor \( \varepsilon_{\mu\nu\lambda\sigma} \) is chosen to satisfy \( \varepsilon_0123 = +1 = -\varepsilon^{0123}, \)
\( \varepsilon_{\mu\nu\lambda\sigma} \varepsilon^{\mu\nu\lambda\sigma} = -4!, \varepsilon_{\mu\nu\lambda\sigma} \varepsilon_{\mu\nu\lambda\sigma} = -3 \delta^2_\varepsilon \), etc., and \( \varepsilon_{ijk} = \epsilon_{ijk} = -\varepsilon^{0ijk} \) corresponds to the totally antisymmetric 3D Levi-Civita tensor. Here the Greek indices \( \mu, \nu, \lambda \ldots = 0, 1, 2, 3 \) stand for the spacetime directions on the manifold and Latin indices \( i, j, k \ldots = 1, 2, 3 \) correspond to the space directions only.
where \( \delta_r \Psi(\tau) = \Psi'(\tau) - \Psi(\tau) \) for any generic field \( \Psi(\tau) \equiv x_{\mu}, p_{\mu}, e \) of the first-order Lagrangian \( L_f \). There exists a gauge symmetry transformation \( \delta_g \) for the above system which is generated by the first-class constraints \( \Pi_e \approx 0, (p^2 - m^2) \approx 0 \) of the theory [20,21] where \( \Pi_e \) is the conjugate momentum corresponding to \( e(\tau) \). These continuous transformations, with the infinitesimal parameter \( \xi(\tau) \), are

\[
\delta_g x_\mu = \xi p_\mu, \quad \delta_g p_\mu = -\xi F_{\mu\nu} p^\nu, \quad \delta_g F_{\mu\nu} = 0, \quad \delta_g e = \dot{\xi}.
\]

It is clear that the infinitesimal gauge symmetry transformations (2.5) and the infinitesimal reparametrization transformations (2.4) are equivalent for (i) the identification \( \xi = \epsilon e \), and (ii) the validity of the equations of motion \( \dot{x}_\mu = e p_\mu, \dot{p}_\mu = -e F_{\mu\nu} p^\nu, p^2 - m^2 = 0 \) written for the first-order Lagrangian \( L_f \). In fact, all the equations of motion, emerging from \( L_0, L_f \) and \( L_s \) (that will be useful for our later discussions) are listed below:

\[
\begin{align*}
L_0 : & \quad \frac{m}{(\dot{x}^2)^{3/2}} [\dot{x}_\mu (\dot{x}^\nu) - \dot{x}_\mu (\dot{x} \cdot \dot{x})] + F_{\mu\nu} \dot{x}^\nu = 0, \\
L_f : & \quad \dot{x}_\mu = e p_\mu, \quad p^2 - m^2 = 0, \quad \dot{p}_\mu + F_{\mu\nu} \dot{x}^\nu \equiv \dot{\mathcal{P}}_\mu = 0, \\
L_s : & \quad \frac{1}{\epsilon^2} (\ddot{x}_\mu e - \ddot{x} e) + F_{\mu\nu} \dot{x}^\nu = 0, \quad \epsilon^2 = \frac{\dot{x}^2}{m^2}.
\end{align*}
\]

The above equations, corresponding to \( L_f \), imply (i) \( (\dot{\mathcal{P}} \cdot \dot{x}) = 0 \) and/or \( (\dot{\mathcal{P}} \cdot p) = 0 \), and (ii) \( \mathcal{P}_\mu = p_\mu + F_{\mu\nu} x^\nu \) is a conserved (i.e. \( \dot{\mathcal{P}}_\mu = 0 \))- and gauge-invariant (i.e. \( \delta_g \mathcal{P}_\mu = 0 \)) quantity.

The gauge transformations (2.5) for \( x_\mu, p_\mu \) (and \( \pi_\mu \)) lead to the following

\[
\begin{align*}
x_0 & \rightarrow X_0 = x_0 + \xi p_0, & \quad p_0 & \rightarrow P_0 = p_0 - \xi F_{00} p^1, \\
x_i & \rightarrow X_i = x_i + \xi p_i, & \quad p_i & \rightarrow P_i = p_i - \xi (F_{0i} p^0 + F_{ij} p^j), \\
\pi_0 & \rightarrow \Pi_0 = \pi_0 - \frac{\xi}{2} F_{00} p^1, & \quad \pi_i & \rightarrow \Pi_i = \pi_i - \frac{\xi}{2} (F_{0i} p^0 + F_{ij} p^j).
\end{align*}
\]

A few comments are in order as far as the gauge transformations (2.7) are concerned. First, the above equations are valid up to linear in the gauge parameter \( \xi \) (i.e. \( \sim \xi \)). Second, the above gauge transformations, together with the gauge transformation for the angular momentum operator \( M_{\mu\nu} = x_\mu \pi_\nu - x_\nu \pi_\mu \), can be concisely expressed as

\[
\begin{align*}
x_\mu & \rightarrow X_\mu = x_\mu + \xi p_\mu, & \quad p_\mu & \rightarrow P_\mu = p_\mu - \xi F_{\mu\nu} p^\nu, & \quad \pi_\mu & \rightarrow \Pi_\mu = \pi_\mu - \frac{\xi}{2} F_{\mu\nu} p^\nu, \\
M_{\mu\nu} & \rightarrow M_{\mu\nu} = M_{\mu\nu} + \xi (x_\nu F_{\mu\rho} - x_\rho F_{\nu\mu}) p^\rho + \xi (p_\mu \pi_\nu - p_\nu \pi_\mu).
\end{align*}
\]

It will be noted that, in the limit \( F_{\mu\nu} \rightarrow 0 \), all the three quantities \( p_\mu, \pi_\mu, M_{\mu\nu} \) remain gauge invariant. Fourth, the basic brackets \( \{x_\mu, x_\nu\}_{(PB)} = 0, \{x_\mu, p_\nu\}_{(PB)} = \eta_{\mu\nu}, \{p_\mu, p_\nu\}_{(PB)} = -F_{\mu\nu}, \{x_\mu, \pi_\nu\}_{(PB)} = \eta_{\mu\nu}, \{\pi_\mu, \pi_\nu\}_{(PB)} = 0 \) remain invariant (i.e. \( \{X_\mu, X_\nu\}_{(PB)} = 0, \{X_\mu, P_\nu\}_{(PB)} = \eta_{\mu\nu}, \{P_\mu, P_\nu\}_{(PB)} = -F_{\mu\nu}, \{X_\mu, \Pi_\nu\}_{(PB)} = \eta_{\mu\nu}, \{\Pi_\mu, \Pi_\nu\}_{(PB)} = 0 \) under the gauge transformations (2.8) up to linear in order \( \xi \). Fifth, the usual Poincaré algebra

\[
\begin{align*}
\{\pi_\mu, \pi_\nu\}_{(PB)} = 0, & \quad \{M_{\mu\nu}, \pi_\lambda\}_{(PB)} = \eta_{\mu\lambda} \pi_\nu - \eta_{\nu\lambda} \pi_\mu, \\
\{M_{\mu\nu}, M_{\lambda\xi}\}_{(PB)} = \eta_{\mu\lambda} M_{\nu\xi} + \eta_{\nu\xi} M_{\mu\lambda} - \eta_{\mu\xi} M_{\nu\lambda} - \eta_{\nu\lambda} M_{\mu\xi},
\end{align*}
\]
remains form-invariant, up to linear in order $\xi$. In other words, we have exactly the same algebra in the gauge-transformed frames as illustrated below:

\begin{align}
\{\Pi_\mu, \Pi_\nu\}_{(PB)} &= 0, \quad \{\mathcal{M}_{\mu\nu}, \Pi_\lambda\}_{(PB)} = \eta_{\mu\lambda} \Pi_\nu - \eta_{\nu\lambda} \Pi_\mu, \\
\{\mathcal{M}_{\mu\nu}, \mathcal{M}_{\lambda\zeta}\}_{(PB)} &= \eta_{\mu\lambda} \mathcal{M}_{\nu\zeta} + \eta_{\nu\zeta} \mathcal{M}_{\mu\lambda} - \eta_{\nu\lambda} \mathcal{M}_{\mu\zeta}.
\end{align}

(2.10)

The following algebra between the angular momentum $M_{\mu\nu}$ and the spacetime variable $x_\lambda$ also remains form-invariant in the (un)transformed frames, namely;

\begin{align}
\{M_{\mu\nu}, x_\lambda\}_{(PB)} &= \eta_{\mu\lambda} x_\nu - \eta_{\nu\lambda} x_\mu \quad \Rightarrow \quad \{M_{\mu\nu}, X_\lambda\}_{(PB)} = \eta_{\mu\lambda} X_\nu - \eta_{\nu\lambda} X_\mu.
\end{align}

(2.11)

Sixth, the useful Poisson Brackets, that have been used in the above computation, are:

\begin{align}
\{p_\mu, \pi_\nu\}_{(PB)} &= -\frac{1}{2} F_{\mu\nu}, \quad \{M_{\mu\nu}, p_\lambda\}_{(PB)} = \eta_{\mu\lambda} \pi_\nu - \eta_{\nu\lambda} \pi_\mu + \frac{1}{2} (x_\mu F_{\lambda\nu} - x_\nu F_{\lambda\mu}).
\end{align}

(2.12)

In a nut-shell, we observe that the basic Poisson brackets between the canonical variables $x_\mu$ and $\pi_\mu$ (as well as their off-shoot brackets between $x_\mu$ and $p_\mu$) remain invariant up to linear in $\xi$. On the other hand, the Poincaré algebra remains form-invariant in the untransformed- and gauge transformed frames up to linear in gauge parameter $\xi$. The key point to be emphasized is the fact that the spacetime retains its commutative nature in the (un)transformed frames because $\{x_\mu, x_\nu\}_{(PB)} = 0$ and $\{X_\mu, X_\nu\}_{(PB)} = 0$ up to linear in $\xi$.

Now we dwell a bit on the direct interaction of the relativistic particle with an arbitrary electromagnetic gauge field $A_\mu(\tau)$, keeping the reparametrization invariance intact. The analogue of the Lagrangians in (2.1) can be written as: $L^{(1)}_0 = m (\dot{x}^2)^{1/2} - A_\mu \dot{x}^\mu, L^{(1)}_f = p_\mu \dot{x}^\mu - A_\mu \dot{x}^\mu - \frac{\xi}{2} (p^2 - m^2), L^{(1)}_s = \frac{\dot{x}^2}{2m} - A_\mu \dot{x}^\mu + \frac{1}{2} m^2 e$ which can be obtained from (2.1) by the substitution $A_\mu = -\frac{1}{2} F_{\mu\nu} \dot{x}^\nu$. It will be noted that (i) the electromagnetic field (i.e. the curvature tensor) $F_{\mu\nu}(\tau) = \partial_\mu A_\nu(\tau) - \partial_\nu A_\mu(\tau)$ is no longer a constant background field, and (ii) the canonical Hamiltonian $(H^{(1)}_c = \pi_\mu \dot{x}^\mu - L^{(1)}_0 = 0)$ derived from the Lagrangian $L^{(1)}_0$ is zero where the canonical momentum $\pi^{(1)}_\mu = p_\mu - A_\mu$ and $p_\mu$ is given by (2.2). The analogue of the continuous reparametrization transformations (2.4) and the gauge transformations (2.5) can be defined for the first-order Lagrangian $L^{(1)}_f$ too. However, for our further elaborate discussions, we shall focus on only the first-order Lagrangian of (2.1) and, in the rest of our discussions, we shall not take into account $L^{(1)}_0, L^{(1)}_f$ and $L^{(1)}_s$.

Let us concentrate on the derivation of the gauge transformations (2.5) for $L_f$ by requiring the consistency among (i) the equations of motion (2.6), (ii) the definitions (2.2) for $p_\mu$ and $e$, and (iii) the basic gauge symmetry transformations on the spacetime variables $x_0$ and $x_i$ in (2.7). In other words, given the basic gauge symmetry transformations for the spacetime variables, we wish to deduce all the rest of the gauge transformations of (2.5) by taking the help from the equations of motion (2.6) and the definition (2.2). It is straightforward to check that $\delta_g e = (1/m) \delta_g [\dot{x}_0^2 - \dot{x}_i^2]^{1/2}$ (cf. (2.2)) leads to the derivation $\delta_g e = \xi$ if we use the basic transformations $\delta_g x_0 = \xi p_0, \delta_g x_i = \xi p_i$, the definition of $p_\mu$ in (2.2) and the equation of motion $\dot{p}_\mu + \frac{1}{2} F_{\mu\nu} \dot{x}^\nu = 0$ which implies $\dot{p} \cdot \dot{x} \equiv \dot{p}_0 \dot{x}_0 - \dot{p}_i \dot{x}_i = 0$. Now, taking
\( \delta_g e = \dot{\xi}, \ \delta_g x_0 = \xi p_0 \) as inputs, it can be seen, from the application of the gauge transformations on the equation of motion \( \dot{x}_0 = e p_0 \) (i.e. \( \delta_g x_0 = \delta_g[e p_0] \)), that \( \delta_g p_0 = -\xi F_{0 \mu}p^\mu \) if we use the equations of motion \( \dot{p}_0 + F_{0 i} \dot{x}^i = 0, \ \dot{x}^i = e p^i \). In an exactly similar fashion, we can derive \( \delta_g p_i = -\xi (F_{0 i p}^0 + F_{ij} p^j) \). It is clear that, combined together, the above transformations for \( p_0 \) and \( p_i \) imply that: \( \delta_g p_\mu = -\xi F_{\mu \nu}p^\nu \). It is essential to check the consistency of the above transformations with remaining equations of motion \( p^2 - m^2 = 0, \dot{p}_\mu + F_{\mu \nu} \dot{x}^\nu = 0 \) that are obtained from \( L_f \). The application of the gauge transformation on the l.h.s. of the mass-shell condition \( p^2 - m^2 = 0 \) leads to \( 2p^\mu \delta g p_\mu = -2\xi F_{\mu \nu}p^\mu p^\nu \) which is automatically equal to zero. The consistency check between the gauge transformations and the equation of motion \( \dot{p}_\mu + F_{\mu \nu} \dot{x}^\nu = 0 \) leads to \( (d/d\tau)[\delta g p_\mu + F_{\mu \nu} \delta g x^\nu] = 0 \). The above requirement is very easily satisfied with \( \delta g x_\mu = \xi p_\mu \) and \( \delta g p_\mu = -\xi F_{\mu \nu}x^\nu \) which were derived earlier in our present discussion. We shall exploit, in the next section, the above trick of deriving the symmetry transformations for the rest of the dynamical variables of \( L_f \) when a specific kind of transformations for the basic spacetime variables \( (x_\mu) \) are given to us.

### 3 Noncommutativity and Non-Standard Gauge-Type Symmetries

Analogous to the gauge transformations (2.7) on the time and space variables \( x_0 \) and \( x_i \) (which lead to the commutative spacetime structure \( \{X_0, X_i\}_{PB} = 0, \{X_i, X_j\}_{PB} = 0 \) etc.), let us consider the following gauge-type transformations on \( x_0 \) and \( x_i \) variables †

\[
\begin{align*}
x_0 &\rightarrow X_0 = x_0 + \zeta \theta_{0i} p_i \Rightarrow \delta_g x_0 = \zeta \theta_{0i} p_i, \\
x_i &\rightarrow X_i = x_i + \zeta \theta_{0i} p_0 \Rightarrow \delta_g x_i = \zeta \theta_{0i} p_0,
\end{align*}
\]

(3.1)

where \( \zeta(\tau) \) is an infinitesimal parameter and here we obtain a time-space NC because the nontrivial Poisson bracket for the transformed spacetime variables turns out to be non-zero (i.e. \( \{X_0, X_i\}_{PB} = -2\zeta \theta_{0i} \)). In the above derivation, we have (i) treated the antisymmetric (i.e. \( \theta_{0i} = -\theta_{i0} \)) parameter \( \theta_{0i} \) to be a constant (i.e. independent of the parameter \( \tau \) as well as the phase space variables), (ii) exploited the brackets \( \{x_\mu, x_\nu\}_{PB} = 0, \{p_\mu, p_\nu\}_{PB} = -F_{\mu \nu}, \{x_\mu, p_\nu\}_{PB} = \delta_{\mu \nu} \) which are the off-shoots of the canonical brackets (2.3), and (iii) computed the Poisson brackets up to linear in transformation parameter \( \zeta(\tau) \). It can be readily checked that \( \{X_0, X_0\}_{PB} = \{X_i, X_j\}_{PB} = 0 \) up to linear in the above infinitesimal transformation parameter \( \zeta(\tau) \) of the non-standard gauge-type transformation (3.1).

One can treat the above NC to be a special case of the general NC defined through \( \{X_\mu(\tau), X_\nu(\tau)\}_{PB} = \Theta_{\mu \nu}(\tau) \) on the spacetime target manifold where \( \Theta_{\mu i}(\tau) = -2\zeta(\tau) \theta_{0i}, \Theta_{ij}(\tau) = -2\zeta(\tau) \theta_{ij} = 0 \). In fact, such a kind of NC has been discussed extensively in [17-19]. The special type of transformations (3.1) have been taken into account primarily for three reasons. First, they lead to the time-space NC (i.e. \( \theta_{0i} \neq 0, \theta_{ij} = 0 \)) in the transformed spacetime manifold which has been used, in detail, for the development of

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†We shall be following, in Secs. 3, 4 and 5, the Euclidean notations with lower (i.e. covariant) indices only so that the analogue of (2.3) now becomes \( \{x_\mu, x_\nu\}_{PB} = \{\pi_\mu, \pi_\nu\}_{PB} = 0, \{x_\mu, \pi_\nu\}_{PB} = \delta_{\mu \nu} \) etc. However, we shall be careful and consistent with our notations used in the previous section.
a unitary quantum mechanics [17-19]. Second, they are relevant in the context of BRST symmetry transformations and corresponding cohomology (see, e.g., Sec. 6 below for a detailed discussion). Finally, they are still “gauge-type”, in the sense that, these transformations can be guessed from the usual standard gauge transformations (2.7). To be precise, in the non-standard case, the standard increments of (2.7) have been exchanged by exploiting the antisymmetric \( \theta_{0i} \) so that \( \delta_y x_0 = \theta_{0i} (\delta_g x_i), \delta_y x_i = \theta_{i0} (\delta_g x_0) \). This trick works for the reparametrization invariant theories as can be seen in our earlier works on the free (non-)relativistic particle [15,16].

Considering the basic non-standard transformations (3.1) for the spacetime variables and demanding their consistency with some of the equations of motion and the expressions for the canonical momenta (derived from the set of Lagrangians (2.1) for the model under consideration), we obtain (using the trick discussed earlier in connection with the derivation of the standard gauge transformations), the following non-standard transformations for the rest of the dynamical variables of the Lagrangian \( L_f \) of (2.1), namely;

\[
\begin{align*}
\tilde{\delta}_g e &= \frac{2}{m^2} \frac{\dot{\zeta} \theta_{0i}}{e} \left( p_0 \dot{p}_i + p_i \dot{p}_0 \right) + \frac{\theta_{0i} \zeta}{m^2 e} \left( p_0 \dot{p}_i + p_i \dot{p}_0 \right) \equiv \frac{\dot{\zeta} \theta_{0i}}{m^2} p_0 \dot{p}_i + \frac{d}{d\tau} \left[ \frac{\theta_{0i}}{m^2} p_0 \dot{p}_i \right], \\
\tilde{\delta}_g p_0 &= \frac{\dot{\zeta} \theta_{0i}}{m^2} \left[ 1 + \frac{-m^2}{p_0^2} \right] - \frac{\theta_{0i} \zeta}{m^2 e} \left( \frac{p_0}{e} \right) \frac{d}{d\tau} \left[ \frac{\theta_{0i} p_0 p_i}{m^2} \right], \\
\tilde{\delta}_g p_i &= -\frac{\theta_{0j} \zeta p_0}{e} \left[ \delta_{ij} + \frac{p_i p_j}{m^2} \right] - \frac{\theta_{0i} \zeta}{m^2 e} \left( \frac{p_i}{e} \right) \frac{d}{d\tau} \left[ \frac{\theta_{0i} p_0 p_j}{m^2} \right].
\end{align*}
\]  

(3.2)

At this juncture, a few comments are in order. First, it can be seen that the above transformations are different from the gauge transformations (2.5) that are obtained for the first-order Lagrangian of (2.1). Second, it can be checked that the above non-standard transformations are consistent with the equation of motion \( \delta_g p_0 = (1/p_0)[p_i \delta_g p_i] \) is satisfied without any restriction on any parameters. To see it explicitly, the following straightforward expressions

\[
\begin{align*}
\tilde{\delta}_g p_0 &= \theta_{0i} \frac{\zeta}{e} \left[ 1 - \frac{p_0^2}{m^2} \right] \dot{p}_i - \frac{\theta_{0i} \zeta}{m^2 e} \left( \frac{p_0}{e} \right) \dot{p}_i + \frac{\theta_{0i} \zeta}{m^2 e} \left[ 1 - \frac{2 p_0^2}{m^2} \right] \dot{\zeta}, \\
p_i \tilde{\delta}_g p_i &= -\left[ \frac{\zeta \theta_{0i} p_0 \mathbf{p}^2}{m^2 e} \right] \dot{p}_i - \frac{\dot{\zeta} \theta_{0i} p_i}{m^2 e} \left[ 1 + \frac{\mathbf{p}^2}{m^2} \right] \dot{p}_0 - \frac{\theta_{0i} \theta_{0j} p_0 p_i}{e} \left[ 1 + \frac{2 \mathbf{p}^2}{m^2} \right] \dot{\zeta},
\end{align*}
\]  

(3.3)

where \( \mathbf{p}^2 \equiv p_ip_i - m^2 \) and its consequence \( (p_0^2/m^2) = 1 + (\mathbf{p}^2/m^2) \) have to be exploited for the proof that the relation \( \delta_g p_0 = (1/p_0)[p_i \delta_g p_i] \) is really correct. Finally, it should be noted that the transformations (3.2) are more general than the standard continuous gauge transformations (2.5) because the latter turns out to be a limiting case of the former. To see it clearly, let us first concentrate on the transformation for the einbein field \( e(\tau) \) which happens to be the gauge field of the theory. Under the following restrictions:

\[
\frac{\theta_{0i} p_0 p_i}{m^2} = \frac{m^2}{2}, \quad \zeta(\tau) = \dot{\zeta}(\tau),
\]  

(3.4)

the non-standard gauge-type transformation \( \tilde{\delta}_g e \) reduces to the standard continuous gauge transformation \( \delta_g e \). We started off concentrating on the transformation for the einbein
field \( e(\tau) \) because this is the “gauge” field of the model under consideration and the transformation (2.5) for it (i.e. \( \delta_1e = \xi \)) is generated due to the first-class constraints. Exploiting the basic inputs from (3.4), it can be seen that the non-standard transformation (i.e. \( \tilde{\delta}_g p_0 \)) on the variable \( p_0 \) becomes

\[
\tilde{\delta}_g p_0 = \frac{\xi}{e} \theta_{0i} \dot{p}_i + \frac{\dot{\xi}}{e} \left( \theta_{0i} p_i - p_0 \right).
\]  

(3.5)

It is very clear now that the non-standard transformation for \( p_0 \) (i.e. \( \tilde{\delta}_g p_0 \)) becomes the standard gauge transformation (i.e. \( \delta_g p_0 \)) for \( p_0 \) in the following manner

\[
\theta_{0i} p_i = p_0, \quad \theta_{0i} \dot{p}_i = \dot{p}_0 \Rightarrow \tilde{\delta}_g p_0 = \delta_g p_0 = \frac{\xi}{e} \dot{p}_0 \equiv -\xi F_{0i} p^i.
\]  

(3.6)

Similarly, the inputs from (3.4) lead to the following transformation

\[
\tilde{\delta}_g p_i = -\frac{\xi}{e} \theta_{0i} \dot{p}_0 - \frac{\dot{\xi}}{e} \left( \theta_{0i} p_0 + p_i \right),
\]  

(3.7)

which, ultimately, leads to the consequences as given below

\[
\theta_{i0} p_0 = p_i, \quad \theta_{i0} \dot{p}_0 = \dot{p}_i \Rightarrow \tilde{\delta}_g p_i = \delta_g p_i = \frac{\xi}{e} \dot{p}_i \equiv -\xi \left( F_{i0} p^0 + F_{ij} p^j \right).
\]  

(3.8)

It will be noted that, purposely, we have explicitly expressed the equations (3.6) and (3.8) in the upper and lower indices so that they could be compared with the gauge transformations (2.5). The derivation of the above equations establishes clearly that the non-standard transformations (3.1) and (3.2) (i.e. \( \tilde{\delta}_g \)) for the variables of the first-order Lagrangian \( L_f \) reduce to the standard continuous gauge transformation (2.5) (i.e. \( \delta_g \)) under the following more conditions on the antisymmetric \( \theta_{0i} \) and the momenta \( p_0 \) and \( p_i \):

\[
\theta_{0i} \theta_{0j} = -\delta_{ij} \equiv \theta_{i0} \theta_{j0}, \quad \theta_{0i} \theta_{0i} = \theta_{i0} \theta_{i0} = -1, \quad p_0^2 = \frac{m^2}{2}, \quad p_i p_j = -\frac{m^2}{2} \delta_{ij},
\]  

(3.9)

which are the consequences of the restrictions listed in (3.4), (3.6) and (3.8). The off-shoot of the last entry in (3.9) implies that \( p^2 \equiv p_i p_i = -(m^2/2) \). It will be noted that (i) the constraint condition \( p_0^2 - p_i^2 = m^2 \) is satisfied with the above solutions, (ii) the values of the \( p_0 \) and \( p_i \) are not fixed as is the case in our earlier works \([15,16]\) on the description of NC for the (non-)relativistic free particle, (iii) the conditions \( \dot{p}_0 = \theta_{0i} \dot{p}_i \) and \( \dot{p}_i = \theta_{i0} \dot{p}_0 \) imply the following relationship between the electric field \( E \) and magnetic field \( B \)

\[
F_{0i} = \frac{1}{2} \theta_{0i} F_{ij} \Rightarrow E = \frac{1}{2} \left( B \times \theta \right),
\]  

(3.10)

where \( \theta = \theta_{0i}, \quad E = F_{0i}, \quad B_i = \frac{1}{2} \epsilon_{ijk} F_{jk} \equiv B \) and the equations of motion (2.6) for the first-order Lagrangian \( L_f \) have been used, (iv) the solutions in (3.9) imply that \( p_0 \dot{p}_0 = 0 \) and \( p_i \dot{p}_i = 0 \) which are satisfied \( \dagger \) if we take into account the results of (3.10), (v) the results of

\footnote{\( \dagger \)It is straightforward to check that \( p_0 \dot{p}_0 = eF_{0i} p_0 p_i \equiv -(e/2)F_{ij} p_i p_j = 0 \) and \( p_i \dot{p}_i = e p_i (F_{0i} p_0 + F_{ij} p_j) = -(e/2)F_{ij} p_i p_j = 0 \) where we have made use of \( p_i = \theta_{i0} p_0 \) and the equation (3.10).}
\[(3.9)\) do not imply that \(\delta g p_0 = 0\) and \(\delta g p_i = 0\). Rather, they imply that \(p_0 \delta g p_0 = 0, p_i \delta g p_i = 0\) which are readily satisfied if we take into account the transformations from (2.5) and, supplement that, with (3.10), (vi) the equation of motion \(\dot{p}_\mu + F_{\mu\nu}\dot{x}^\nu = 0\) is automatically consistent with the transformations in (3.2) with the conditions listed in (3.4), (3.6), (3.8) (3.9) etc., and (vii) the NC parameter \(\theta = \theta_{0i}\) also appears in the Poynting vector \(\mathbf{P}\) which measures the flux density. The explicit expression for this vector is
\[
\mathbf{P} = \frac{1}{2} (\mathbf{E} \times \mathbf{B}) = \frac{1}{4} \left[ \theta (\mathbf{B} \cdot \mathbf{B}) - \mathbf{B} (\theta \cdot \mathbf{B}) \right].
\]
It is evident that the existence of the above time-space NC enforces the electric and magnetic fields of the 4D target space to be connected with each other (cf. (3.10)). As a consequence, even the flux density (i.e. Poynting vector) turns out to be dependent on the noncommutative parameter \(\theta = \theta_{0i}\) (cf. (3.11)).

4 More General Noncommutativity and Gauge-Type Symmetries

Let us begin with more general transformations than the ones given in (3.1). These can be written, with the infinitesimal transformation parameter \(\zeta^{(1)}(\tau)\), as follows:
\[
\begin{align*}
x_0 &\to X_0 = x_0 + \zeta^{(1)} \theta_{0i} p_i, \\
x_i &\to X_i = x_i + \zeta^{(1)} (\theta_{0i} p_0 + \theta_{ij} p_j), \\
\delta g, x_0 &= \zeta^{(1)} \theta_{0i} p_i, \\
\delta g, x_i &= \zeta^{(1)} (\theta_{0i} p_0 + \theta_{ij} p_j).
\end{align*}
\]
(4.1)
It is straightforward to check that, in the transformed frames, we have \(\{X_0, X_i\}_{PB} = -2\zeta^{(1)} \theta_{0i} \equiv \Theta_{0i}\) and \(\{X_i, X_j\}_{PB} = -2\zeta^{(1)} \theta_{ij} \equiv \Theta_{ij}\). This demonstrates that we have now more general NC than the earlier case of spacetime transformations (3.1). Exploiting the same trick as discussed earlier, we obtain the following transformations for the einbein field \(e(\tau)\) and momenta variables \(p_0(\tau)\) and \(p_i(\tau)\):
\[
\begin{align*}
\delta g, e &\equiv \frac{2 \zeta^{(1)} \theta_{0i} p_0 p_i}{m^2} + \frac{\theta_{0i} \zeta^{(1)}}{m^2} (p_0 \dot{p}_i + p_i \dot{p}_0) - \frac{\zeta^{(1)}}{m^2} (\theta_{ij} \dot{p}_i \dot{p}_j) \\
&\equiv \frac{\zeta^{(1)}}{m^2} \theta_{0i} p_0 p_i + \frac{d}{d\tau} \left[ \frac{\zeta^{(1)} \theta_{0i} p_0 p_i}{m^2} \right] - \frac{\zeta^{(1)}}{m^2} (\theta_{ij} \dot{p}_i \dot{p}_j), \\
\delta g, p_0 &\equiv \frac{\zeta^{(1)}}{m^2} \theta_{0i} p_i \left[ 1 - \frac{p_0^2}{m^2} \right] + \frac{1}{m^2} \frac{d}{d\tau} \zeta^{(1)} \dot{p}_i \\
&\equiv \left( \frac{p_0}{m^2} \right) \frac{d}{d\tau} \left[ \frac{\zeta^{(1)} \theta_{0i} p_0 p_i}{m^2} \right] + \left( \frac{\zeta^{(1)}}{m^2} \right) \left( \frac{p_0}{m^2} \right) (\theta_{ij} \dot{p}_i \dot{p}_j), \\
\delta g, p_i &\equiv -\frac{\theta_{0i} p_0 \zeta^{(1)}}{m^2} \left[ \delta_{ij} + \frac{p_i p_j}{m^2} \right] - \frac{\theta_{0i} \zeta^{(1)}}{m^2} \dot{p}_0 \\
&\equiv -\left( \frac{p_i}{m^2} \right) \frac{d}{d\tau} \left[ \frac{\zeta^{(1)} \theta_{0j} p_0 p_j}{m^2} \right] + \left( \frac{\zeta^{(1)}}{m^2} \right) \left( \frac{p_i}{m^2} \right) (\theta_{jk} \dot{p}_j \dot{p}_k).
\end{align*}
\]
Let us focus on the transformations for the einbein field \(e(\tau)\) which happens to be the gauge field of the theory. It is very clear that the following conditions
\[
\theta_{0i} p_0 p_i = \frac{m^2}{2}, \quad \theta_{ij} p_i \dot{p}_j = 0, \quad \zeta^{(1)}(\tau) = \xi(\tau),
\]
(4.3)
lead to the derivation of the usual gauge transformation \( \delta g e = \dot{\xi} \) for the einbein field (cf. (2.5)). Exploiting the above basic conditions, we obtain the following transformations for the momenta variables from the most general expressions (4.2):

\[
\begin{align*}
\delta_{g_1} p_0 &= \frac{\xi}{e} \theta_{0i} \dot{p}_i + \frac{\dot{\xi}}{e} \left( \theta_{0i} p_i - p_0 \right), \\
\delta_{g_1} p_i &= +\frac{\xi}{e} \left( \theta_{0i} \dot{p}_0 + \theta_{ij} \dot{p}_j \right) + \frac{\dot{\xi}}{e} \left( \theta_{0i} p_0 - p_i + \theta_{ij} p_j \right).
\end{align*}
\] (4.4)

It is now straightforward to claim that the following conditions

\[
\begin{align*}
\theta_{0i} p_i &= p_0, \quad \theta_{0i} \dot{p}_i = \dot{p}_0, \quad p_i &= \theta_{0i} p_0 + \theta_{ij} p_j, \quad \dot{p}_i = \theta_{0i} \dot{p}_0 + \theta_{ij} \dot{p}_j,
\end{align*}
\] (4.5)

reduce the continuous transformations \( \delta_{g_1} \) of (4.4) to the ordinary gauge transformations \( \delta_g \) of (2.5). It is clear from the above relationship \( \dot{p}_0 = \theta_{0i} \dot{p}_i \) that the last entry in (4.5) leads to the following connection between \( \theta_{ij} \) and \( \theta_{0i} \), namely;

\[
\theta_{ij} = \delta_{ij} + \theta_{0i} \theta_{0j}.
\] (4.6)

Furthermore, the combination of relationships in (4.3) and (4.5) yields

\[
p_0^2 = \frac{m^2}{2}, \quad \theta_{0i} \theta_{0j} p_i p_j = \frac{m^2}{2}.
\] (4.7)

However, the validity of the mass-shell condition \( p_0^2 - \mathbf{p}^2 = m^2 \) (which happens to be the secondary first-class constraint for the first-order Lagrangian \( L_f \)) implies that \( \mathbf{p}^2 \equiv p_i p_i = -(m^2/2) \). The requirement of the consistency between this result and the last relationship of (4.7) lead to the following interesting consequences:

\[
\theta_{0i} \theta_{0j} \equiv \theta_{00} \theta_{0j} = -\delta_{ij} \quad \rightarrow \quad p_i p_j = -\frac{m^2}{2} \delta_{ij}.
\] (4.8)

The substitution of the first expression of (4.8) into (4.6) establishes the fact that, for the derivation of the continuous gauge symmetry (2.5) from the non-standard gauge-type symmetry transformations (4.2), the NC parameter \( \Theta_{ij}(\tau) = -2\zeta^{(1)}(\tau) \theta_{ij} \) is zero because of the fact that \( \theta_{ij} = 0 \) (cf. (4.8) and (4.6)). This demonstrates that, for the model under discussion, we are allowed to have only the time-space NC and space-space NC is zero (i.e. \( \{X_i, X_j\}_{PB} = 0 \), because \( \theta_{ij} = 0 \)). This also establishes that the transformations (3.1) and (3.2) are allowed and they are the limiting cases of (4.1) and (4.2) when \( \theta_{ij} = 0 \).

\section*{5 Deformations of the Algebras}

It is clear from the relationships \( p_0 = \theta_{0i} p_i \) and \( p_i = \theta_{0i} p_0 \) that we have the following Poisson brackets between the spacetime variables \((x_0, x_i)\) and their conjugate momenta \((\pi_0, \pi_i)\) in phase space (where the Hamiltonian dynamics is defined):

\[
\begin{align*}
\{x_0, p_0\}_{(PB)} &= 1 \iff \{x_0, \pi_0\}_{(PB)} = 1, \quad \{x_0, p_i\}_{(PB)} = -\theta_{0i} \iff \{x_0, \pi_i\}_{(PB)} = -\theta_{0i}, \\
\{x_i, p_j\}_{(PB)} &= \delta_{ij} \iff \{x_i, \pi_j\}_{(PB)} = \delta_{ij}, \quad \{x_i, p_0\}_{(PB)} = -\theta_{i0} \iff \{x_i, \pi_0\}_{(PB)} = -\theta_{i0},
\end{align*}
\] (5.1)
where \((p_0, p_i)\) are the momenta for the free relativistic particle defined through the equation (2.2). A few comments are in order. First, it will be noted that, in the above, we have canonical Poisson brackets as well as nontrivial Poisson brackets that include the time-space noncommutative parameter \(\theta_{0i}\). Second, it is clear that the above non-triviality of the brackets leads to the modification of the Poincaré algebra and connected algebras (which are illustrated in Sec. 2). Third, it is interesting to point out that, under the transformations (3.1), the time-space NC retains its original form §, namely;

\[
\{X_0, X_i\}_{(PB)} = -2 \zeta(\tau) \theta_{0i} \equiv \Theta_{0i}(\tau), \quad \{X_i, X_j\}_{(PB)} = 0, \quad (5.2)
\]

up to linear in transformation parameter \(\zeta(\tau)\) even if we use the Poisson brackets (5.1) in the above computation. Fourth, the Poisson brackets among the components of \(p_\mu\) are computed from the requirement that \(\{\pi_\mu, \pi_\nu\}_{(PB)} = 0\) where, in the Euclidean notation, \(\pi_\mu = p_\mu + (1/2)F_{\mu\nu}x_\nu\) implies \(\pi_0 = p_0 + (1/2)F_{0i}x_i\) and \(\pi_i = p_i - (1/2)F_{0i}x_0 + (1/2)F_{ij}x_j\). The resulting brackets (with \(\{x_0, p_0\}_{(PB)} = 1, \{x_i, p_j\}_{(PB)} = \delta_{ij}\) etc.) are

\[
\begin{align*}
\{\pi_0, \pi_0\}_{(PB)} &= 0 \Rightarrow \{p_0, p_0\}_{(PB)} = 0, \quad \{\pi_i, \pi_i\}_{(PB)} = 0 \Rightarrow \{p_i, p_i\}_{(PB)} = 0, \\
\{\pi_0, \pi_i\}_{(PB)} &= 0 \Rightarrow \{p_0, p_i\}_{(PB)} = -F_{0i} + \frac{1}{2} \theta_{0k} F_{ki}, \\
\{\pi_i, \pi_j\}_{(PB)} &= 0 \Rightarrow \{p_i, p_j\}_{(PB)} = -F_{ij} + \frac{1}{2} (\theta_{0i} F_{0j} - \theta_{0j} F_{0i}),
\end{align*}
\]

(5.3)

where the basic brackets of (5.1) have been used for the explicit computation. It is clear that, in the limit \(\theta_{0i} \rightarrow 0\), we get back our original brackets \(\{p_\mu, p_\nu\}_{(PB)} = -F_{\mu\nu}\).

To observe the impact of the NC on the algebra (2.11) in the untransformed frame, we obtain the following deformed Poisson brackets:

\[
\begin{align*}
\{M_{0i}, x_j\}_{(PB)} &= -\delta_{ij} x_0 - x_i \theta_{0j}, \quad \{M_{0i}, x_0\}_{(PB)} = x_i + x_0 \theta_{0i}, \\
\{M_{ij}, x_0\}_{(PB)} &= -x_j \theta_{0i} + x_i \theta_{0j}, \quad \{M_{ij}, x_k\}_{(PB)} = \delta_{ik} x_j - \delta_{jk} x_i,
\end{align*}
\]

(5.4)

where the boost generator \(M_{0i} = x_0 \pi_i - x_i \pi_0\) and the rotation generator \(M_{ij} = x_i \pi_j - x_j \pi_i\). In fact, in the above, the non-vanishing components \(M_{0i}\) and \(M_{ij}\) of the angular momentum generator \(M_{\mu\nu} = x_\mu \pi_\nu - x_\nu \pi_\mu\) have been taken into account and the basic algebraic relations (5.1) have been exploited for the explicit computation. It is clear that, in the \(\theta_{0i} \rightarrow 0\) limit, the above deformed algebra in (5.4) reduces to the explicit form of such an algebra in the untransformed frame (cf. (2.11)) as given below

\[
\begin{align*}
\{M_{0i}, x_j\}_{(PB)} &= -\delta_{ij} x_0, \quad \{M_{0i}, x_0\}_{(PB)} = x_i, \\
\{M_{ij}, x_0\}_{(PB)} &= 0, \quad \{M_{ij}, x_k\}_{(PB)} = \delta_{ik} x_j - \delta_{jk} x_i.
\end{align*}
\]

(5.5)

---

§It should be noted that the direct substitution of \(p_i = \theta_{0i} p_0, p_0 = \theta_{0i} p_i\) in (3.1) leads to the gauge transformations for \(x_0\) and \(x_i\) which entails upon the spacetime structure to become commutative in nature. However, the above relations between \(p_0\) and \(p_i\) should be treated like a set of constraint equations and should be imposed only after the computation of the relevant Poisson brackets is over.
Now let us focus on the deformation of the Poincaré algebra (2.9) due to the time-space NC (i.e. $\theta_{0i} \neq 0$) first in the untransformed frames. It is evident, from equation (5.3), that the canonical brackets $\{\pi_\mu, \pi_\nu\}_{PB} = 0$ which lead to the deformation of the algebra between $p_\mu$ and $p_\nu$ (cf. (5.3)). However, there are some modifications of the algebra between various components of the momentum generator $\pi_\mu$ and the angular momentum generator $M_\mu = x_\mu \pi_\nu - x_\nu \pi_\mu$. In explicit form, these are as given below

\[
\begin{align*}
\{M_{0i}, \pi_j\}_{PB} &= -\delta_{ij} \pi_0 - \theta_{0j} \pi_i, \\
\{M_{ij}, \pi_0\}_{PB} &= -\theta_{i0} \pi_j + \theta_{j0} \pi_i, \\
\{M_{ij}, \pi_k\}_{PB} &= \delta_{ik} \pi_j - \delta_{jk} \pi_i.
\end{align*}
\]

(5.6)

It is straightforward to check that the analogue of (5.6) in the commutative spacetime can be readily derived from (2.9). These undeformed part of the Poincaré algebra are as follows

\[
\begin{align*}
\{M_{0i}, \pi_j\}_{PB} &= -\delta_{ij} \pi_0, \\
\{M_{0i}, \pi_0\}_{PB} &= \pi_i, \\
\{M_{ij}, \pi_0\}_{PB} &= 0, \\
\{M_{ij}, \pi_k\}_{PB} &= \delta_{ik} \pi_j - \delta_{jk} \pi_i,
\end{align*}
\]

(5.7)

which are the limiting ($\theta_{0i} \to 0$) case of (5.6). There are a triplet of Poisson brackets between the boost generator $M_{0i}$ and the rotation generator $M_{ij}$. These can be explicitly expressed, in our notation of the Euclidean space, as listed below

\[
\begin{align*}
\{M_{ij}, M_{kl}\}_{PB} &= \delta_{ik} M_{jl} + \delta_{jl} M_{ik} - \delta_{il} M_{jk} - \delta_{jk} M_{il}, \\
\{M_{ij}, M_{0k}\}_{PB} &= \delta_{ik} M_{0j} - \delta_{jk} M_{0i}, \\
\{M_{ij}, M_{0j}\}_{PB} &= M_{ij}.
\end{align*}
\]

(5.8)

Let us now concentrate on the deformation of the above algebra due to the time-space NC. It is very clear that the first of the above brackets will \emph{not} get modified at all. However, the second and third ones will get contributions from the time-space NC. The exact form of the modified brackets, with NC parameter $\theta_{0i}$, are as follows

\[
\begin{align*}
\{M_{ij}, M_{kl}\}_{PB} &= \delta_{ik} M_{jl} + \delta_{jl} M_{ik} - \delta_{il} M_{jk} - \delta_{jk} M_{il}, \\
\{M_{0i}, M_{0j}\}_{PB} &= M_{ij} + \theta_{0i} \left( x_0 \pi_j + x_j \pi_0 \right) - \theta_{0j} \left( x_0 \pi_i + x_i \pi_0 \right), \\
\{M_{ij}, M_{0k}\}_{PB} &= \delta_{ik} M_{0j} - \delta_{jk} M_{0i} + \theta_{0j} \left( x_i \pi_k + x_k \pi_i \right) - \theta_{0i} \left( x_j \pi_k + x_k \pi_j \right).
\end{align*}
\]

(5.9)

It is straightforward to see that, in the limit $\theta_{0i} \to 0$, the algebraic relations (5.9) reduce to their undeformed counterpart (5.8) derived from the usual Poincaré algebra (2.9) in the Euclidean space where $\eta_{\mu\nu} \to \delta_{\mu\nu}$ (i.e. $\{x_\mu, \pi_\nu\}_{PB} = \eta_{\mu\nu} \to \{x_\mu, \pi_\nu\}_{PB} = \delta_{\mu\nu}$).

Let us pay our attention to the NC deformations of the algebras (2.10) and (2.11) in the gauge-transformed frames where the change of variables is governed by the equation (2.8). First of all, let us concentrate on the gauge transformed form of the momenta in the Euclidean space where $\Pi_\mu = \pi_\mu - \frac{\xi}{2} F_{\mu\nu} p_\nu$. The time and space components of this generator can be explicitly expressed as (cf. (2.8))

\[
\begin{align*}
\Pi_0 &= \pi_0 - \frac{\xi}{2} F_{0i} p_i, \\
\Pi_i &= \pi_i - \frac{\xi}{2} F_{i0} p_0 - \frac{\xi}{2} F_{ij} p_j.
\end{align*}
\]

(5.10)
The algebra obeyed by the above generators is not like the ones (i.e. $\{\Pi_\mu, \Pi_\nu\}_{PB} = 0$) given in (2.10) where the spacetime geometry is commutative. Rather, we obtain the deformation of this algebra due to the time-space NC. The resulting algebra, up to linear in $\xi$, is

$$\begin{align*}
\{\Pi_0, \Pi_0\}_{PB} &= 0, & \{\Pi_i, \Pi_i\}_{PB} &= 0, \\
\{\Pi_0, \Pi_i\}_{PB} &= \frac{\xi}{4} \theta_{0i} F_{0j} F_{0j} + \frac{\xi}{4} \theta_{ij} \left[ F_{0i} F_{0j} + F_{jk} F_{ik} \right], \\
\{\Pi_i, \Pi_j\}_{PB} &= \frac{\xi}{4} \left[ \theta_{0i} F_{ik} - \theta_{0j} F_{jk} \right] F_{0k} - \frac{\xi}{2} \theta_{0j} \left[ F_{i0} F_{j1} - F_{0j} F_{i1} \right] \theta_{i0}.
\end{align*}$$

(5.11)

It is straightforward to note that the above algebra, in the limit $\theta_{0i} \to 0$, goes over to the algebra in the commutative spacetime where $\{\Pi_\mu, \Pi_\nu\}_{PB} = 0$ (cf. (2.10)). Furthermore, in the computation of (5.11), we have used the deformed algebra (5.3) and the following additional algebra that is computed directly, namely;

$$\begin{align*}
\{p_0, \pi_0\}_{PB} &= \frac{1}{2} \theta_{0j} F_{i0}, & \{p_0, \pi_i\}_{PB} &= -\frac{1}{2} F_{0i} + F_{ij} \theta_{j0}, \\
\{p_i, \pi_0\}_{PB} &= \frac{1}{2} \left[ F_{0i} + \theta_{0j} F_{ij} \right], & \{p_i, \pi_j\}_{PB} &= -\frac{1}{2} \left[ F_{ij} + F_{i0} \theta_{j0} \right].
\end{align*}$$

(5.12)

The stage is now set for the computation of the deformed algebra between the gauge transformed momenta (5.10) and the antisymmetric angular momentum generator $\mathcal{M}_{\mu\nu}$. The expression for the latter in the Euclidean space and its non-vanishing components are

$$\begin{align*}
\mathcal{M}_{\mu\nu} &= M_{\mu\nu} + \frac{\xi}{2} \left( x_\nu F_{\mu\rho} - x_\mu F_{\nu\rho} \right) p_\rho + \xi \left( p_\mu \pi_\nu - p_\nu \pi_\mu \right), \\
\mathcal{M}_{0i} &= M_{0i} + \frac{\xi}{2} \left( x_i F_{0j} p_j - x_0 F_{i0} p_0 - x_0 F_{ij} p_j \right) + \xi \left( p_0 \pi_i - p_i \pi_0 \right), \\
\mathcal{M}_{ij} &= M_{ij} + \frac{\xi}{2} \left( x_j F_{0i} p_0 + x_j F_{ik} p_k - x_i F_{j0} p_0 - x_i F_{jk} p_k \right) + \xi \left( p_i \pi_j - p_j \pi_i \right).
\end{align*}$$

(5.13)

The deformed algebra between the component $\mathcal{M}_{0i}$ with the gauge transformed momenta generators $\Pi_0$ and $\Pi_i$ (cf. (5.10)), up to linear in parameter $\xi$, are as follows

$$\begin{align*}
\{\mathcal{M}_{0i}, \Pi_0\}_{PB} &= \Pi_i + \theta_{0j} \left( \delta_{ij} - \frac{\xi}{2} F_{ij} \right) \pi_0 + \frac{\xi}{2} \theta_{0i} \left( F_{0j} p_j - \frac{1}{2} x_0 F_{0j} F_{0j} \right) \\
&\quad - \frac{\xi}{4} x_0 \theta_{0k} \left( F_{0i} F_{0k} - F_{ij} F_{jk} \right), \\
\{\mathcal{M}_{0i}, \Pi_j\}_{PB} &= -\delta_{ij} \Pi_0 - \theta_{0j} \Pi_i + \frac{\xi}{2} \theta_{0i} \left[ F_{j0} \pi_0 - \frac{1}{2} x_0 F_{jk} F_{0k} \right] \\
&\quad + \frac{\xi}{2} \theta_{0j} \left[ F_{0i} \pi_0 - \frac{1}{2} x_0 F_{0k} F_{0k} + \frac{1}{2} x_0 F_{ik} F_{0k} \right] \\
&\quad - \frac{\xi}{4} x_i \theta_{0k} \left[ F_{j0} F_{k0} - F_{jl} F_{lk} \right] + \frac{\xi}{2} x_0 \theta_{k0} \left[ F_{j0} F_{ik} - F_{jk} F_{i0} \right].
\end{align*}$$

(5.14)
where the following brackets have played the key roles in the exact computation

\[
\{M_{0i}, p_0\}_{(PB)} = \pi_i + \theta_{i0} \pi_0 + \frac{1}{2} x_0 F_{0i} - x_0 F_{ij} \theta_{j0} + \frac{1}{2} x_i \theta_{0j} F_{j0},
\]

\[
\{M_{0i}, p_j\}_{(PB)} = \theta_{j0} \pi_i - \delta_{ij} \pi_0 + \frac{1}{2} \left[ x_0 F_{ji} + x_i F_{0j} \right] + \frac{1}{2} \left[ x_i \theta_{0k} F_{kj} + x_0 \theta_{0i} F_{0j} \right].
\]

(5.15)

A couple of more algebras between the gauge transformed components of the angular momentum (i.e. \(M_{ij}\)) and the components of the transformed linear momenta (i.e. \(\Pi_0\) and \(\Pi_i\)), up to linear in the gauge parameter \(\xi\), are

\[
\{M_{ij}, \Pi_0\}_{(PB)} = \theta_{j0} \Pi_i - \theta_{i0} \Pi_j + \frac{\xi}{4} \left[ \theta_{0i} x_j - \theta_{0j} x_i \right] F_{0k} F_{0k}
\]

\[
\{M_{ij}, \Pi_k\}_{(PB)} = \delta_{ik} \Pi_j - \delta_{jk} \Pi_i + \frac{\xi}{2} \left[ \theta_{i0} \pi_j - \theta_{j0} \pi_i \right] F_{k0}
\]

\[
+ \frac{\xi}{4} \theta_{0l} \left[ F_{ik} x_j - F_{jk} x_i \right] F_{lk},
\]

\[
\{M_{ij}, p_k\}_{(PB)} = \delta_{ik} \pi_j - \delta_{jk} \pi_i + \frac{1}{2} x_i \left[ F_{kj} + \theta_{0j} F_{0k} \right] - \frac{1}{2} x_j \left[ F_{ki} + \theta_{0i} F_{0k} \right],
\]

\[
\{M_{ij}, p_0\}_{(PB)} = -\theta_{i0} \pi_j + \theta_{j0} \pi_i + \theta_{k0} \left[ F_{ik} x_j - F_{jk} x_i \right] + \frac{1}{2} \left[ x_i F_{0j} - x_j F_{0i} \right].
\]

(5.16)

(5.17)

It is straightforward to note that the algebra (5.16) reduces to the algebra (2.9), in the notations of the Euclidean space, when we take the limit \(\theta_{0i} \to 0\). Thus, it is crystal clear that the algebras (5.14) and (5.16) are the noncommutative generalization of the algebra in (2.9) which corresponds to the commutative geometry.

Let us discuss the algebra (2.11) in the gauge transformed frame where the time-space NC is present (i.e. \(\theta_{0i} \neq 0\)). The deformed Euclidean version of the algebra (2.11), in the
gauge transformed frame (up to linear in order \(\xi\)), are as follows

\[
\begin{align*}
\{ \mathcal{M}_{0i}, X_0 \}^{(PB)} &= X_i + X_0 \theta_{0i} - 2 \xi \theta_{0i} \pi_0, \\
\{ \mathcal{M}_{0i}, X_j \}^{(PB)} &= -\delta_{ij} X_0 - X_i \theta_{j0} + 2 \xi \pi_i \theta_{j0}, \\
\{ \mathcal{M}_{ij}, X_0 \}^{(PB)} &= X_i \theta_{0j} - X_j \theta_{0i} - \frac{\xi}{2} \left[ x_i F_{kj} - x_j F_{ik} \right] \theta_{k0}, \\
\{ \mathcal{M}_{ij}, X_k \}^{(PB)} &= \delta_{ik} X_j - \delta_{jk} X_i + \frac{\xi}{2} \left[ x_i \theta_{0j} - x_j \theta_{0i} \right] F_{0k} \\
&+ \frac{\xi}{2} \left[ x_i F_{j0} - x_j F_{i0} \right] \theta_{0k},
\end{align*}
\]

(5.18)

where the transformed versions of the angular momentum \(\mathcal{M}_{\mu\nu}\) and spacetime variable \(X_\mu\) have been taken from (2.8). The explicit expressions for the former in the component forms are given in (5.13). It should be noted that the above algebra is true for the transformations (3.1) if we exploit the conditions (3.4), (3.6), (3.8-3.10), etc., and consequences thereof. It is interesting to point out that, in the limit \(\theta_{0i} \to 0\), we do recover the algebra (2.11).

Ultimately, we focus on the algebra among the gauge transformed components of the rotation generator \(\mathcal{M}_{ij} = X_i \Pi_j - X_j \Pi_i\) and the boost generator \(\mathcal{M}_{0i} = X_0 \Pi_i - X_i \Pi_0\) up to linear in the gauge parameter \(\xi\). It is clear that the following expression is true, namely;

\[
\{ \mathcal{M}_{0i}, \mathcal{M}_{0j} \}^{(PB)} = \{ \mathcal{M}_{0i}, X_0 \Pi_j - X_j \Pi_0 \}^{(PB)} = \{ \mathcal{M}_{0i}, X_0 \Pi_j \}^{(PB)} - \{ \mathcal{M}_{0i}, X_j \Pi_0 \}^{(PB)}.
\]

(5.19)

Exploiting the results of (5.14) and (5.18) in the Leibnitz rule applied to the above Poisson brackets, we obtain the following algebra between the two of the boost generators

\[
\begin{align*}
\{ \mathcal{M}_{0i}, \mathcal{M}_{0j} \}^{(PB)} &= \mathcal{M}_{ij} + X_0 \left( \theta_{0i} \Pi_j - \theta_{0j} \Pi_i \right) + X_i \theta_{j0} \Pi_0 - \theta_{0i} X_j \pi_0 \\
&- 2 \xi \left( \theta_{0i} \pi_0 \Pi_j + \pi_i \theta_{j0} \Pi_0 \right) - \frac{\xi}{2} \left[ x_0 \left( \theta_{0i} F_{0j} - \theta_{0j} F_{0i} \right) + x_i \theta_{0k} F_{kj} \right] \Pi_0 \\
&+ \frac{\xi}{2} X_j \left[ \theta_{0k} F_{ik} \pi_0 - \theta_{0i} \left( F_{0k} p_k - \frac{x_0}{2} F_{0k} F_{0k} \right) + \frac{1}{2} x_0 \theta_{0k} \left( F_{0i} F_{0k} - F_{il} F_{kl} \right) \right] \\
&+ \frac{\xi}{2} X_0 \left[ x_0 \theta_{k0} \left( F_{j0} F_{ik} - F_{jk} F_{i0} \right) + \theta_{0j} \left( F_{0i} \pi_0 - \frac{1}{2} x_i F_{0k} F_{0k} + \frac{1}{2} x_0 F_{ik} F_{0k} \right) \right] \\
&- \frac{1}{2} x_i \theta_{0k} \left( F_{j0} F_{k0} - F_{jk} F_{i0} \right) - \theta_{0i} \left( F_{j0} \pi_0 - \frac{1}{2} x_0 F_{jk} F_{0k} \right). 
\end{align*}
\]

(5.20)

It can be readily seen that, in the limit \(\theta_{0i} \to 0\), we recover the earlier relation (2.10) from (5.20) where \(\{ \mathcal{M}_{0i}, \mathcal{M}_{0j} \}^{(PB)} = \mathcal{M}_{ij}\). Applying the above trick and exploiting the algebras given in (5.14), (5.16) and (5.18), we derive the following angular momentum algebras
between a rotation generator and a boost generator (in the gauge transformed frames):

\[
\{ \mathcal{M}_{ij}, \mathcal{M}_{kl} \}_{(PB)} = \delta_{ik} \mathcal{M}_{0j} - \delta_{jk} \mathcal{M}_{0i} + \left[ X_i \theta_{0j} - X_j \theta_{0i} \right] \Pi_k
\]

\[
- i \frac{\xi}{2} \left[ x_i F_{jl} \theta_{0l} - x_j F_{il} \theta_{0l} \right] \Pi_k + X_k \left[ \theta_{0i} \Pi_j - \theta_{0j} \Pi_i \right]
\]

\[
- X_k \left[ \left( \theta_{0i} x_j - \theta_{0j} x_i \right) F_{0l} - \theta_{0l} \left( F_{im} x_j - F_{jm} x_i \right) F_{lm} \right]
\]

\[
- X_k \theta_{0l} \left[ F_{il} \pi_j - F_{ij} \pi_i + \frac{1}{2} \left( x_j F_{i0} - x_i F_{j0} \right) F_{i0} \right]
\]

\[
- \frac{i}{2} \left[ \left( x_i \theta_{0j} - x_j \theta_{0i} \right) F_{0k} + \left( x_i F_{j0} - x_j F_{i0} \right) \theta_{0k} \right] \Pi_0
\]

\[
+ \frac{i}{2} X_0 \left[ \left( \theta_{i0} \pi_j - \theta_{j0} \pi_i \right) F_{k0} - \frac{1}{2} \left( \theta_{i0} x_j - \theta_{j0} x_i \right) F_{kl} F_{0l} \right]
\]

\[
+ \frac{i}{2} X_0 \left[ \left( x_i F_{j0} - x_j F_{i0} \right) F_{k0} - \left( x_i F_{j0} - x_j F_{i0} \right) F_{kl} \right] \theta_{0l}
\]

\[
+ \frac{i}{2} X_0 \theta_{0k} \left[ F_{0i} \pi_j - F_{0j} \pi_i + \frac{1}{2} \left( x_i F_{j0} - x_j F_{i0} \right) F_{0l} \right].
\]

(5.21)

It is very transparent from the above that the algebra (2.9), in the commutative spacetime, can be obtained from (5.21) as the limiting case where \( \theta_{0i} \to 0 \). The deformed algebra between two rotation operators, in the gauge transformed frames, is as follows:

\[
\{ \mathcal{M}_{ij}, \mathcal{M}_{kl} \}_{(PB)} = \delta_{ik} \mathcal{M}_{jl} + \delta_{jl} \mathcal{M}_{ik} - \delta_{il} \mathcal{M}_{jk} - \delta_{jk} \mathcal{M}_{il}
\]

\[
+ \frac{i}{2} \left[ X_k F_{0l} - X_l F_{k0} \right] \left[ \theta_{0i} \pi_j - \theta_{0j} \pi_i \right]
\]

\[
+ \left[ X_k F_{0l} - X_l F_{k0} \right] \left[ x_i F_{jm} - x_j F_{im} \right] \theta_{0m}
\]

\[
- \left[ X_k F_{lm} - X_l F_{km} \right] \left[ x_i F_{j0} - x_j F_{i0} \right] \theta_{0m}
\]

\[
+ \frac{i}{2} \left[ X_k \theta_{0l} - X_l \theta_{0k} \right] \left[ F_{0i} \pi_j - F_{0j} \pi_i + \frac{1}{2} \left( x_i F_{j0} - x_j F_{i0} \right) F_{0m} \right]
\]

\[
- \frac{i}{2} \left[ X_k F_{lm} - X_l F_{km} \right] \left[ \theta_{i0} x_j - \theta_{j0} x_i \right] F_{0m}
\]

\[
+ \frac{i}{2} \left[ x_i \theta_{0j} - x_j \theta_{0i} \right] \left[ F_{k0} \Pi_l - F_{l0} \Pi_k \right]
\]

\[
+ \frac{i}{2} \left[ x_i F_{j0} - x_j F_{i0} \right] \left[ \theta_{0k} \Pi_l - \theta_{0l} \Pi_k \right].
\]

(5.22)

It is evident that, in the limit \( \theta_{0i} \to 0 \), we do obtain the algebra (2.9) valid in the commutative spacetime. Thus, in the above, we have systematically derived the noncommutative deformation of the Poincaré algebra up to linear in the gauge transformation parameter \( \xi \).

6 (Anti-)BRST Symmetries and Noncommutativity

In this section, we demonstrate the cohomological equivalence of the gauge transformations (2.5) (that correspond to the commutative geometry) and the non-standard gauge-type symmetry transformations in (3.1) (that correspond to the noncommutative geometry). To this end in mind, let us begin with the (anti-)BRST invariant Lagrangian corresponding to
the first-order Lagrangian in (2.1). In its full blaze of glory, this Lagrangian is

$$L_b = p_\mu \dot{x}^\mu - \frac{1}{2} F_{\mu\nu} x^{\mu} \dot{x}^\nu - \frac{1}{2} e (p^2 - m^2) + B \dot{e} + \frac{1}{2} B^2 - i \dot{C} \dot{C},$$

(6.1)

where $B$ is the Nakanishi-Lautrup auxiliary field and $(\dot{C})C$ are the anticommuting (i.e. $C^2 = \dot{C}^2 = 0, CC + CC = 0$) (anti-)ghost fields which are required in the theory to maintain the unitarity (see, e.g., [22] for details on non-Abelian gauge theories). The above Lagrangian $L_b$ remains quasi-invariant under the following off-shell nilpotent ($s^2_{(a)b} = 0$) and anticommuting ($s_b s_{ab} + s_{ab} s_b = 0$) (anti-)BRST transformations $s_{(a)b}$

$$s_b x_\mu = C p_\mu, \quad s_b p_\mu = -C F_{\mu\nu} p^\nu, \quad s_b C = 0,$$

$$s_b e = \dot{C}, \quad s_b B = 0,$$

(6.2)

$$s_{ab} x_\mu = \bar{C} p_\mu, \quad s_{ab} p_\mu = -\bar{C} F_{\mu\nu} p^\nu, \quad s_{ab} \bar{C} = 0,$$

$$s_{ab} e = \dot{\bar{C}}, \quad s_{ab} B = 0,$$

(6.3)

which are the “quantum” generalization of the “classical” local gauge transformations (2.5). To be precise, under the above off-shell nilpotent (anti-)BRST transformations, the Lagrangian $L_b$ undergoes the following change

$$s_b L_b = \frac{d}{d\tau} \left[ \frac{C}{2} \left( p^2 + m^2 - \frac{1}{2} F_{\mu\nu} x^{\mu} \dot{x}^\nu + B \dot{e} \right) + B \dot{C} \right],$$

$$s_{ab} L_b = \frac{d}{d\tau} \left[ \frac{\bar{C}}{2} \left( p^2 + m^2 - \frac{1}{2} F_{\mu\nu} x^{\mu} \dot{x}^\nu + B \dot{e} \right) + B \dot{\bar{C}} \right].$$

(6.4)

The above (anti-)BRST transformations (cf. (6.2) and (6.3)) are generated by the conserved and off-shell nilpotent ($Q^2_{(a)b} = 0$) (anti-)BRST charges $Q_{(a)b}$ as given below:

$$Q_b = B \dot{C} + \frac{C}{2} (p^2 - m^2), \quad Q_{ab} = B \dot{\bar{C}} + \frac{\bar{C}}{2} (p^2 - m^2),$$

(6.5)

because $s_{(a)b} \Psi = -i [\Psi, Q_{(a)b}]_{\pm}$ is true for the generic field $\Psi = x_\mu, p_\mu, e, C, \dot{C}, B$ of the theory. The subscripts $(\pm)$ on the square bracket correspond to the (anti-)commutators for the generic field $\Psi$ being (fermionic) bosonic in nature.

Since the BRST transformations $s_b$ (i.e. $s_b \Psi = -i [\Psi, Q_b]_{\pm}$ for the generic field $\Psi$) imbibe the nilpotency property of $Q_b$, the cohomologically equivalent transformations can be defined in terms of the nilpotent $s^2_b = 0$ BRST transformations. For instance, the BRST transformed spacetime variables in (6.2) can be re-expressed in the following form:

$$x_0 \rightarrow X_0 = x_0 + C p_0 \Rightarrow x_0 \rightarrow X_0 = x_0 + s_b [x_0],$$

$$x_i \rightarrow X_i = x_i + C p_i \Rightarrow x_i \rightarrow X_i = x_i + s_b [x_i].$$

(6.6)

We follow here the notations and conventions adopted by Weinberg [23]. In fact, in its totality, the nilpotent ($s^2_{(a)b} = 0$) (anti-)BRST transformations $s_{(a)b}$ are product of an anticommuting ($\eta C + C \eta = 0$, etc.) spacetime independent parameter $\eta$ and $s_{(a)b}$ with $s^2_{(a)b} = 0$. The (anti-)BRST prescription is to replace the local gauge parameter $\xi$ of the gauge transformation (2.5) by $\eta$ and the (anti-)ghost fields $(\dot{C})C$. 

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This shows that the untransformed spacetime physical variables \((x_i, x_0)\) and the transformed spacetime variables \((X_i, X_0)\) belong to the same cohomology class w.r.t. the nilpotent transformations \(s_b\) as they differ, with each other, by a BRST exact transformation. It should be noted that the above transformations do not lead to any NC in the spacetime structure because the non-trivial brackets (i.e. \(\{X_0, X_i\}_{PB} = 0, \{X_i, X_j\}_{PB} = 0\)), in the transformed frames and the corresponding brackets (i.e. \(\{x_\mu, x_\nu\}_{PB} = 0\)) in the untransformed frames, are found to be zero.

Let us concentrate now on the basic transformations (3.1) and argue their consequences in the language of the BRST cohomology. The BRST versions of these transformations imply the presence of a time-space NC in the spacetime structure. With the identification \(\zeta(\tau) = \xi(\tau)\) and the application of the BRST prescription, the transformations (3.1) can be written in the language of the BRST transformations, as

\[
\begin{align*}
x_0 &\to X_0 = x_0 + \theta_{0i} C p_i \equiv x_0 + s_b [\theta_{0i} x_i], \\
x_i &\to X_i = x_i + \theta_{0i} C p_0 \equiv x_i + s_b [\theta_{0i} x_0].
\end{align*}
\]

The above transformations lead to the NC in the spacetime structure because the non-trivial bracket (i.e. \(\{X_0, X_i\}_{PB} = -2C\theta_{0i}\)) is non-zero. Here we have used the basic canonical brackets \(\{x_0, p_0\}_{PB} = 1, \{x_i, p_j\}_{PB} = \delta_{ij}, \text{ etc.}\), and as before, the antisymmetric (i.e. \(\theta_{0i} = -\theta_{i0}\)) NC parameter is treated as a constant tensor. It is elementary to note that, once again, the spacetime untransformed variables \((x_i, x_0)\) and the transformed variables \((X_0, X_i)\) belong to the same cohomology class w.r.t. the BRST transformations \(s_b\). Thus, it is clear that the NC and commutativity for the reparametrization invariant model for the interacting massive relativistic particle belong to the same cohomology class w.r.t. the nilpotent BRST transformation \(s_b\). All the above arguments could be repeated with the nilpotent anti-BRST transformations \(s_{ab}\) (and the nilpotent charge \(Q_{ab}\)), too.

We wrap up this section with a note of caution. In fact, the consideration of the BRST cohomology allows a whole range of transformations (e.g. analogues of (3.1) (3.2), (4.1), (4.2), etc.). However, the BRST transformations corresponding to these transformations are not the symmetry transformations for the Lagrangian (6.1) of the theory. In fact, ultimately, it is the gauge-symmetry transformations (2.5) and its analogues (6.2) and/or (6.3) that are the real symmetry transformations. In the process of starting out from (3.1) and going over once again to the gauge transformations (2.5), we obtain certain specific restrictions on the NC parameter \(\theta_{0i}\) and momenta (cf. (3.6), (3.8), etc.). These restrictions, in one way, imply commutativity of the spacetime because of the presence of the gauge transformations. However, in another way, we do end up with the time-space NC of the spacetime and obtain the deformation of the Poincaré (and related) algebras.

7 Conclusions

In our present investigation, we have concentrated on the continuous symmetry transformations of the first-order Lagrangian for the interacting massive relativistic particle where
the interaction is brought in through the constant electromagnetic background field. The NC of spacetime structure emerges merely due to the continuous non-standard gauge type transformations (3.1) which, ultimately, lead to the derivation of the corresponding transformations for the einbein field and the components of momenta in (3.2). The equivalence of the commutativity (corresponding to the standard gauge transformations (2.5)) and the NC (corresponding to the non-standard gauge type transformations (3.1)) is proven through a set of restrictions on the noncommutative parameter $\theta_{0i}$ and the components of momenta $p_{\mu}$ listed in the equations (3.6), (3.8) and (3.9). For instance, if we substitute $\theta_{0i} = p_0$ and $\theta_{ai} = p_i$ directly in the transformations (3.1), they convert themselves to the gauge transformations (2.5), and thereby, lead to the commutativity of spacetime. On the other hand, if we treat the above relations between $p_0$ and $p_i$ as constraints, the explicit computation of the Poisson bracket between the transformed time $(X_0)$ and space $(X_i)$ variables leads to the time-space NC (because $\{X_0, X_i\}_{PB} = -2\zeta \theta_{0i} \equiv \Theta_{0i}$). Thus, the continuous gauge transformations for the spacetime variables can be looked upon in two different ways where one interpretation leads to the commutativity of the gauge transformed spacetime and the other interpretation leads to the NC of the gauge transformed spacetime.

One of the interesting features of our present reparametrization invariant interacting model is the fact that the mass parameter of this system does not become noncommutative in nature. This feature is drastically different from our earlier works [15,16] on the reparametrization invariant systems of the free (non-)relativistic particles where the mass parameter turns out to be noncommutative in nature. For the present interacting model, the components $(p_0, p_i)$ of momenta $p_{\mu}$ have noncommutative behaviour with both the space $(x_i)$ as well as time $(x_0)$ variables (cf. (5.1)). In this context, it should be noted that, for the interacting as well as free relativistic particle, the restrictions $p_0^2 = (m^2/2)$ and $p_i p_i = -(m^2/2)$ are valid so that the mass-shell condition $p_0^2 - p_i^2 = m^2$ could be satisfied. However, for the free relativistic particle, it turns out that one can choose $p_0 = (m/\sqrt{2})$ and $p_i = \theta_{ai}(m/\sqrt{2})$ to satisfy $\dot{p}_0 = 0, \dot{p}_i = 0, \delta_g p_0 = 0, \delta_g p_i = 0$ and $p_0^2 - p_i^2 = m^2$. On the contrary, for the interacting particle, these choices are not allowed because $\dot{p}_0 \neq 0, \dot{p}_i \neq 0, \delta_g p_0 \neq 0, \delta_g p_i \neq 0$ but the mass-shell condition $p_0^2 - p_i^2 = m^2$ has to be satisfied. Thus, for the model under consideration, the components $p_0$ and $p_i$ are not individually fixed but their squares are. This is the basic reason that, in the former case, the mass parameter becomes noncommutative in nature but, in the latter case, there is no such unusual property associated with the mass parameter. It is not out of place to mention that the NC of the mass parameter has already appeared in the context of the application of quantum groups to some (non-)relativistic systems [24,25].

The central result of our investigation is Sec. 5 where the noncommutative deformation of the Poicaré (and related) algebras is explicitly obtained for the untransformed frames as well as for the gauge transformed frames. This derivation, to the best of our knowledge, is a new one. It should be noted that the deformation of these algebras is such that, in the limit $\theta_{0i} \to 0$, we do get back the results of Sec. 2 where there is no spacetime NC.
The basic reason behind the above deformation is hidden in the relations $p_i = \theta_{i0} p_0$ and $p_0 = \theta_{0i} p_i$ which lead to the deformation of the basic canonical Poisson brackets (cf (5.1)). This, in turn, enforces the Poincaré (and related) algebras to modify.

As claimed earlier, our approach to obtain the time-space NC, is quite general in the sense that it can be applied to any reparametrization invariant model. In fact, the logical origin for our trick comes from the BRST cohomology (cf. (6.6),(6.7)) related to the spacetime BRST transformations. It would be very nice endeavour to apply our trick to the reparametrization invariant model of a superparticle which has already been studied in the framework of quantum group [26]. In fact, it would be interesting to find a common ground for (i) the discussions of the NC of spacetime associated with the quantum groups, and (ii) the discussions connected with the Snyder’s idea of the NC of spacetime. In this connection, it is worthwhile to mention that, we have already made some modest attempts in this direction [14,27]. We have pointed out here a few problems that are under investigation and our results would be reported in our future publications [28].

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