Unipotent Invariant Quadrics

Mahir Bilen Can,
Michael Joyce

October 12, 2010

Abstract

We describe the variety of fixed points of a unipotent operator acting on
the space of symmetric matrices or, equivalently, the corresponding space of
quadrics. We compute the determinant and the rank of a generic symmetric
matrix in the fixed variety, yielding information about the generic singular
locus of the corresponding quadrics.

1 Introduction.

The study of quadric hypersurfaces has long been of interest to algebraic geomet-
ters, dating back to the work of Chasles [1]. Of particular interest is a natural
compactification of the space of all quadric hypersurfaces, described by Schubert
[4]. This compactification, known as the variety of complete quadrics, is an exam-
ple of the more general construction of complete symmetric varieties discovered
by De Concini and Procesi [3]. For more on complete quadrics, we recommend:
[2], [10] and [9].

The space of complete quadrics is analogous in some ways to the more well
known flag varieties; in particular, there is a rich combinatorial structure in the
geometry of both. An important advance in the study of flag varieties is the anal-
ysis of Springer fibers. See the papers [6], [7] of Springer, as well as the papers of
Steinberg [8] and Spaltenstein [5]. Given a fixed unipotent element $u \in SL_n$, the
Springer fiber at $u$ consists of the flags that are fixed by $u$.

In this paper, we describe the space of unipotent-fixed quadrics. The cal-
culations are made in a naive compactification of quadrics, but they serve as a
crucial building block for the analogous computations on the variety of complete quadrics.

Given a unipotent element \( u \in SL_n \), we consider the natural action on the space of quadrics in \( \mathbb{P}^{n-1} \); there is a corresponding action on the space of symmetric \( n \)-by-\( n \) matrices. Our primary results are as follows. We explicitly describe the locus of symmetric matrices fixed by \( u \). In particular, the corresponding locus of \( u \)-fixed quadrics is a projective space whose dimension is given explicitly in terms of the Jordan type of \( u \). Given a generic symmetric matrix \( M \) fixed by \( u \), we give a formula for the determinant and the rank of \( M \); geometrically, this describes the singularity of a generic \( u \)-fixed quadric.

The organization of the paper is as follows: in Section 2 we set our notation. In Section 3 we present preliminary results. In Section 4 we describe our results in detail. Finally, in Section 5 we present proofs of the results.

2 Notation and Conventions.

Throughout the paper, \( \mathbb{K} \) denotes an algebraically closed field of characteristic 0 and \( V \) denotes a \( \mathbb{K} \)-vector space of dimension \( n \). We fix a basis \( e_1, e_2, \ldots e_n \) of \( V \) and let \( x_1, x_2, \ldots x_n \) be the corresponding dual basis in \( V^* \).

Let \( \mathcal{Q} = \mathbb{P}(\text{Sym}^2V^*) \) denote the space of quadric hypersurfaces in \( V \) and let \( \mathcal{Q}_0 \) denote the open subset of smooth (also called non-degenerate) quadrics in \( V \). With respect to our chosen basis, we may represent an element \( Q \in \mathcal{Q} \) as \( Q = \sum a_{ij}x_ix_j \) with \( a_{ij} = a_{ji} \). This representation is unique up to rescaling. Letting \( A = (a_{ij}) \), we may identify \( Q \) with \( [A] \in \mathbb{P}(\text{Sym}_{n \times n}) \), where \( \text{Sym}_{n \times n} \) denotes the vector space of \( n \)-by-\( n \) symmetric matrices with entries in \( \mathbb{K} \). Under this identification, elements of \( \mathcal{Q}_0 \) are represented by symmetric matrices \( A \) of rank \( n \), i.e. with \( \det(A) \neq 0 \). The group \( SL(V) \) acts on \( \text{Sym}_{n \times n} \) on the right; the action is given by \( A \cdot g = gAg^T \). This action descends to \( \mathbb{P}(\text{Sym}_{n \times n}) \) and hence to \( \mathcal{Q} \). (One can also consider the left action of \( SL(V) \) that is given by \( g \cdot A = (g^{-1})^TAg^{-1} \) and derive analogous results to those we present; our choice to use the right action is based on aesthetic considerations.) We denote by \( S^u, Q^u, Q_0^u \), respectively, the corresponding fixed-point loci of a unipotent element \( u \in SL(V) \) in the space \( \text{Sym}_{n \times n} \), \( \mathcal{Q}, \mathcal{Q}_0 \), respectively.

Let \( \lambda \) be a partition of \( n \). We use several notations to describe \( \lambda \). We may write \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1 \); in this case \( \lambda_1 + \lambda_2 + \cdots + \lambda_k = n \) and we say that \( \ell(\lambda) \), the length of \( \lambda \), is \( k \). We may also append an arbitrary number of zeroes to the end of the sequence, and this changes neither
\( \lambda \) nor its number of parts. Alternatively, we may write \( \lambda = (1^{\alpha_1}, 2^{\alpha_2}, \ldots, l^{\alpha_l}) \) to indicate that \( \lambda \) consists of \( \alpha_1 \) 1’s, \( \alpha_2 \) 2’s, and so on. Terms with zero exponent may be added and removed without altering \( \lambda \). For example, each of \((3, 3, 2)\), \((3, 3, 2, 0)\), \((1^0, 2^1, 3^2)\), and \((2^1, 3^2)\) represent the partition \(3 + 3 + 2\) of 8.

Given a partition \( \lambda \) of \( n \) with \( k \) parts, we introduce the notion of a \( \lambda \)-decomposition of an \( n \times n \) matrix. It is obtained by inserting horizontal lines after rows \( \lambda_1, \lambda_2, \ldots, \lambda_{k-1} \) and similarly inserting vertical lines after columns \( \lambda_1, \lambda_2, \ldots, \lambda_{k-1} \), thereby giving a block decomposition of the matrix. For example, here is the \( 5 \times 5 \) identity matrix \( I_5 \) with its \((2, 1, 1)\)-decomposition:

\[
I_5 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

3 Preliminaries.

**Lemma 3.1.** Let \( G \) be either the additive group \( \mathbb{G}_a = \mathbb{K}^+ \) or the multiplicative group \( \mathbb{G}_m = \mathbb{K}^\times \). Let \( X \) be a complete variety on which \( G \) acts, and let \( g \in G \) be an element of infinite order. Then \( X^g = X^G \).

**Remark 3.2.** We thank Joseph Silverman for showing us a counterexample to Lemma 3.1 in characteristic \( p \).

**Proof.** Clearly \( X^G \subset X^g \). Suppose \( x \in X^g \). Then \( x \in X^{g^n} \) for any \( n \in \mathbb{Z} \). Consider the map \( G \rightarrow X \) given by \( t \mapsto t \cdot x \). Since \( X \) is complete, this map extends to a morphism \( \phi : \mathbb{P}^1 \rightarrow X \). Since \( \text{char} \ \mathbb{K} = 0 \), \( \phi^{-1}(x) \) is infinite, and therefore the image of \( \phi \) is a point. Thus, \( x \in X^G \).

**Corollary 3.3.** Let \( N \) be a nilpotent matrix with entries in \( \mathbb{K} \), \( u = \exp N \), and \( U = \{\exp(tN) : t \in \mathbb{K}\} \). If \( X \) is any complete variety on which \( U \) acts, then \( X^u = X^U \).

**Proposition 3.4.** Consider a nilpotent endomorphism of \( V \) represented by the matrix \( N \) and let \( u = \exp(N) \). Let \( Q \) be a quadric in \( V \) defined by a symmetric \( n \times n \) matrix \( A \). Then the following are equivalent:

1. \( Q \) is fixed by \( u \);
2. $A$ is fixed by $u$;
3. $NA + AN^T = 0$.

**Proof.** Consider the one-dimensional unipotent subgroup of $SL_n(\mathbb{K})$ given by

$$U = \{ \exp(tN) : t \in \mathbb{K} \}.$$ 

By Corollary 3.3, $Q_u = Q_U$.

To find fixed points of the subgroup $U$, we seek solutions to the equations

$$\exp(tN)A\exp(tN)^T = A$$

(3.5)

for all $t \in \mathbb{K}$. Viewing this equation in the ring of $n$-by-$n$ matrices with coefficients in $\mathbb{K}[t]$, differentiating with respect to $t$ and then setting $t = 0$, we obtain

$$NA + AN^T = 0.$$ 

(3.6)

Conversely, assume that (3.6) holds. Then an easy induction shows that

$$N^kA = (-1)^kA(N^T)^k$$

(3.7)

for all $k \geq 0$. Expanding $\exp(tN)$ as a polynomial in $N$ and using (3.7) gives

$$\exp(tN)A = A\exp(-tN^T),$$

which is equivalent to (3.5). \qed

**Remark 3.8.** Note that the Jordan type of $u = \exp(N)$ is the same as the Jordan type of $N$, as a simple row reduction argument shows.

**Lemma 3.9.** Suppose that $N$ and $N'$ are two matrices representing nilpotent endomorphisms of $V$ that are conjugate in $SL(V)$, say $N' = SNS^{-1}$. Let $u = \exp(N)$, $u' = \exp(N')$. Then $S^u$ and $S^{u'}$ are isomorphic via $A \rightarrow SAS^T$. This isomorphism descends to an isomorphism between $Q_u$ and $Q_u'$ and further restricts to an isomorphism between $Q_0^u$ and $Q_0^{u'}$.

**Proof.** We use the criterion of Proposition 3.4 A simple calculation shows

$$NA + AN^T = 0 \iff S(NA + AN^T)S^T = 0$$

$$\iff N'SAS^T + SAS^TN'T = 0.$$ 

Moreover, $\det A \neq 0 \iff \det SAS^T \neq 0$. \qed
Consequently, the spaces $S^u$, $Q^u$, $Q^u_0$ depend only on the Jordan type of $u$, or equivalently, on the Jordan type of any $N$ for which $u = \exp(N)$. Recall that the Jordan classes of $n$-by-$n$ nilpotent matrices are in bijection with the partitions of $n$. Indeed, let $N_p$ be the $p$-by-$p$ matrix

$$N_p = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then the above correspondence associates a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, to the Jordan matrix $N_\lambda$ given in block form by

$$N_\lambda = \begin{pmatrix} N_{\lambda_1} & 0 & \cdots & 0 \\ 0 & N_{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_{\lambda_k} \end{pmatrix}.$$

Consequently, we can always choose a basis for $V$ in which our unipotent endomorphism $u$ is given by $u = \exp(N_\lambda)$. From now on, we assume that our chosen basis has this property and we write $S^\lambda$, $Q^\lambda$, $Q^\lambda_0$, respectively, for $S^u$, $Q^u$, $Q^u_0$, respectively. Of course, $Q^\lambda = \mathbb{P}(S^\lambda)$ and $Q^\lambda_0 = \{[A] \in Q^\lambda : \det A \neq 0\}$.

### 4 Statements of the Results.

We define two families of matrices that are used in our results. When $n = 2m - 1$ is odd,

$$A_n := \begin{pmatrix} a_1 & 0 & a_2 & \cdots & 0 & a_m \\ 0 & -a_2 & 0 & \cdots & -a_m & 0 \\ a_2 & 0 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -a_m & 0 & \cdots & 0 & 0 \\ a_m & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

(4.1)
When $n = 2m$ is even, $A_n$ is obtained from $A_{n-1}$ by adding a row of zeroes along the bottom and a column of zeroes at the end, i.e.,

$$A_n := \begin{pmatrix}
a_1 & 0 & a_2 & \cdots & a_m & 0 \\
0 & -a_2 & 0 & \cdots & 0 & 0 \\
a_2 & 0 & a_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_m & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}. \quad (4.2)$$

For $p \geq q$, we define the $p$-by-$q$ matrix

$$B_{p,q} := \begin{pmatrix}
a_1 & a_2 & a_3 & \cdots & a_{q-1} & a_q \\
-a_2 & -a_3 & -a_4 & \cdots & -a_{q-1} & 0 \\
a_3 & a_4 & a_5 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\pm a_q & 0 & 0 & \cdots & 0 & 0 \\
\mp a_q & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}. \quad (4.3)$$

The signs of each row alternate, so that the $(q, 1)$-entry is $a_q$ if $q$ is odd and $-a_q$ if $q$ is even. Note that there are $p - q$ rows of zeroes at the end of $B_{p,q}$.

**Proposition 4.4.** Suppose $\lambda = (n)$ is the partition of $n$ with just a single part. Let $m$ be defined either by $n = 2m - 1$, or by $n = 2m$. Then

$$\mathcal{S}^{(n)} \cong \{ A_n : a_1, a_2, \ldots, a_m \in \mathbb{K} \} \cong \mathbb{A}^{[n+1)/2]} = \mathbb{A}^m.$$  

From this description, we can immediately describe the smooth quadrics fixed by $u = N_n$.

**Corollary 4.5.** If $n$ is even, then $\mathcal{Q}^{(n)}_0 = \emptyset$.

**Corollary 4.6.** If $n = 2m - 1$ is odd, then

$$\mathcal{Q}^{(n)}_0 \cong \{ [A_n] : a_1, a_2, \ldots, a_m \in \mathbb{K}, a_m \neq 0 \} \cong \mathbb{A}^{m-1}.$$
For the rest of the section, let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be an arbitrary partition of $n$ of length $k$.

**Theorem 4.7.** $S^\lambda$ consists of matrices $M$ whose $\lambda$-decomposition have the form

$$M = \left( \begin{array}{ccc|ccc} A_{\lambda_1} & B_{\lambda_1, \lambda_2} & \cdots & B_{\lambda_1, \lambda_k} \\ B_{\lambda_1, \lambda_2}^{\top} & A_{\lambda_2} & \cdots & B_{\lambda_2, \lambda_k} \\ \vdots & \vdots & \ddots & \vdots \\ B_{\lambda_1, \lambda_k}^{\top} & B_{\lambda_2, \lambda_k}^{\top} & \cdots & A_{\lambda_k} \end{array} \right).$$

(4.8)

The matrices $A_{\lambda_i}$ have the form given by (4.1) or (4.2), and the matrices $B_{\lambda_i, \lambda_j}$ have the form given by (4.3). The variables occurring in the various $A_{\lambda_i}$’s and $B_{\lambda_i, \lambda_j}$’s are all distinct.

**Remark 4.9.** We can interpret (4.8) in two ways. Let $A$ be the set of variables that occur in the blocks on the right hand side. We can either think of (4.8) as an equation defining the elements of $S^\lambda$, or we can think of (4.8) as defining a particular matrix with entries in $\mathbb{K}(A)$. In the latter case, we say that $M$ is the generic element of $S^\lambda$.

**Example 4.10.** To illustrate Theorem 4.7, the generic element of $S^{(2,2,1,1)}$ is

$$M = \begin{pmatrix} a & 0 & b & c & e & h \\ 0 & 0 & -c & 0 & 0 & 0 \\ b & -c & d & 0 & f & i \\ c & 0 & 0 & 0 & 0 & 0 \\ e & 0 & f & 0 & g & j \\ h & 0 & i & 0 & j & k \end{pmatrix}.$$  

(4.11)

while the generic element of $S^{(3,2,1)}$ is

$$\widetilde{M} = \begin{pmatrix} a & 0 & b & c & d & f \\ 0 & -b & 0 & -d & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 \\ c & -d & 0 & e & 0 & g \\ d & 0 & 0 & 0 & 0 & 0 \\ f & 0 & 0 & g & 0 & h \end{pmatrix}.$$  

(4.12)

**Corollary 4.13.** The space $Q^\lambda$ is a projective space and

$$\dim Q^\lambda = \sum_{i=1}^{k} \left\lfloor \frac{\lambda_i + 1}{2} \right\rfloor + \sum_{i=1}^{k} (i - 1)\lambda_i - 1.$$
It is not immediately evident whether any of the $N_{\lambda}$-fixed quadrics are smooth or not. Our next result allows us to effectively determine this.

**Theorem 4.14.** Let $M$ be the generic element of $S^\lambda$. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ be the conjugate partition of $\lambda$, let $P$ be the matrix obtained by taking only the upper rightmost entry from each block in the $\lambda$-decomposition of $M$, and, for $1 \leq i \leq l$, let $P_i$ be the upper left $\mu_i$-by-$\mu_i$ submatrix of $P$. Then

$$\det M = \prod_{i=1}^{l} \det P_i.$$ 

**Example 4.15.** We return to Example 4.10. For $M$ given by (4.11),

$$P = \begin{pmatrix} 0 & c & e & h \\ -c & 0 & f & i \\ 0 & 0 & g & j \\ 0 & 0 & j & k \end{pmatrix}.$$ 

The conjugate partition of $\lambda = (2, 2, 1, 1)$ is $\mu = (4, 2)$. Theorem 4.14 gives

$$\det M = \det \begin{pmatrix} 0 & c & e & h \\ -c & 0 & f & i \\ 0 & 0 & g & j \\ 0 & 0 & j & k \end{pmatrix} \det \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix} = c^4 (gk - j^2).$$

Similarly, for $\tilde{M}$ given by (4.12),

$$P = \begin{pmatrix} b & d & f \\ 0 & 0 & g \\ 0 & 0 & h \end{pmatrix}$$

and hence $\det \tilde{M} = 0$.

When $Q^\lambda$ does not contain any smooth quadrics, we are able to specify the singular locus of a generic $N_{\lambda}$-fixed quadric.

**Theorem 4.16.** Let $M$ be the generic element of $S^\lambda$. Then the corank of $M$ is equal to the number of even parts which appear an odd number of times in $\lambda$.

**Corollary 4.17.** The determinant of the generic element of $S^\lambda$ is zero if and only if every even part which occurs in $\lambda$ occurs an even number of times.
Example 4.18. Looking back at Example 4.10 one more time, the generic element of $S^{(2,2,1,1)}$ has rank 6, as followed from the determinant calculation. The generic element of $S^{(3,2,1)}$ has rank 5, with a single column relation among the third and fifth columns. In Lemma 5.8 we show how to find such column relations, and then prove in Lemma 5.10 that there are no other relations.

5 Proofs.

5.1 Description of $S^\lambda$

In order to facilitate the proofs of Proposition 4.4 and Theorem 4.7, we introduce some notation. The motivation behind the notation is to characterize when a symmetric matrix $A$ has $N_\lambda A$ skew-symmetric.

Let $I = I_\lambda$ denote the set of zero columns of $N_\lambda$. Let $K = K_\lambda$ denote the set of zero rows of $N_\lambda$. Note that $1 \in I$, $n \in K$, and $i \in I \Leftrightarrow i - 1 \in K$ for $1 < i \leq n$.

We say that an entry index $(i, j)$ is an initial zero if either $i \notin I$ and $j = i - 1$ or $i \notin I$ and $j \notin K$. Entry indices with $i \notin I$ that are not initial zeroes are called asymmetric links. In the next figure, we illustrate these notions schematically by placing 0’s wherever initial zeros occur, *’s wherever asymmetric links occur, and •’s everywhere else (i.e., in the $i$-th rows for $i \in I$).

| $\lambda = (4), I_\lambda = \{1\}$ | $\lambda = (2, 2), I_\lambda = \{1, 3\}$ |
|---------------------------------|---------------------------------|
| $\begin{pmatrix} • & • & • & • \\ 0 & * & * & 0 \\ * & 0 & * & 0 \\ * & * & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} • & • & • & • \\ 0 & 0 & * & 0 \\ • & • & • & • \\ * & 0 & 0 & 0 \end{pmatrix}$ |

Fix $\lambda$ and write $N$ for $N_\lambda$. Let $A$ be a symmetric matrix such that $NA$ is skew-symmetric. We first determine the entries of $A$ that are moved to a diagonal in $NA$; since $NA$ is skew-symmetric, these variables are necessarily zero. If $(i, j)$ is the location of such an entry, then $j = i - 1$ and $i \notin I$. Next, observe that if the $i$-th row of $N$ is zero, then so is the $i$-th row of $NA$. Skew-symmetry forces variables appearing in the $i$-th column of $NA$ to be zero. These entries correspond to the condition $i \notin I$ and $j \in K$. These are precisely the initial zeros of $A$. 

9
We also have links between entries of $A$. Because $A$ is symmetric the entries $(i, j)$ and $(j, i)$ are symmetrically linked for all $i \neq j$. In other words, $a_{ij} = a_{ji}$. On the other hand, skew-symmetry of $NA$ causes certain pairs of entries in $A$ to be asymmetrically linked. The pair $(i, j)$ and $(j + 1, i - 1)$ are asymmetrically linked if both of the corresponding entries in $NA$ contain non-zero entries of $A$. This occurs precisely when $i \notin I$ and $(i, j)$ is not an initial zero, recovering our definition of asymmetric links. In this case, $a_{ij} = -a_{j+1,i-1}$.

We now prove Proposition 4.4, which states that the generic element of $S^{(n)}$ is a matrix of the form $A_n$ given by (4.1) and (4.2).

**Proof of Proposition 4.4.** Recall that, by Proposition 3.4,

$$S^{(n)} = \{ A \in \text{Sym}_{n \times n} : N_n A + A N_n^T = 0 \}.$$ 

The schematic representation described above for $N = N_n$ is

$$
\begin{pmatrix}
\bullet & \bullet & \bullet & \cdots & \bullet & \bullet \\
0 & \ast & \ast & \cdots & \ast & 0 \\
\ast & 0 & \ast & \cdots & \ast & 0 \\
\ast & \ast & 0 & \cdots & \ast & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\ast & \ast & \ast & \cdots & 0 & 0
\end{pmatrix}.
$$

(5.1)

Along every anti-diagonal any two entries can be connected by a sequence of symmetric and asymmetric links. Thus, each anti-diagonal consists entirely of zeroes or has the form

$$
\begin{array}{c}
a \\
-a \\
\ddots \\
a \\
-a \\
a
\end{array}
$$

for some $a \in \mathbb{K}$. The anti-diagonal will consist of zeroes if and only if it contains an initial zero. Looking at the schematic representation, we see that an anti-diagonal contains an initial zero precisely when the anti-diagonal is below the main anti-diagonal or the row and the column numbers of the anti-diagonal sum to an odd number. It follows that an arbitrary element of $S^{(n)}$ has the form (4.1) or (4.2), depending on the parity of $n$. \[\square\]
Next, we consider the case of a general partition $\lambda$ of $n$ with $k$ parts. Recall that Theorem 4.7 states that a matrix $M$ in $S^\lambda$ has the form

$$M = \begin{pmatrix}
A_{\lambda_1} & B_{\lambda_1, \lambda_2} & \cdots & B_{\lambda_1, \lambda_k} \\
B_{\lambda_1, \lambda_2}^T & A_{\lambda_2} & \cdots & B_{\lambda_2, \lambda_k} \\
& \ddots & \ddots & \ddots \\
B_{\lambda_1, \lambda_k}^T & B_{\lambda_2, \lambda_k}^T & \cdots & A_{\lambda_k}
\end{pmatrix}.$$  

The block decomposition is a $\lambda$-decomposition. The matrices $A_{\lambda_i}$ have the form given by (4.1) or (4.2) and the matrices $B_{\lambda_i, \lambda_j}$ have the form given by (4.3). The variables occurring in the various $A_{\lambda_i}$’s and $B_{\lambda_i, \lambda_j}$’s are all distinct.

**Proof of Theorem 4.7.** Let $\lambda$ be an arbitrary partition of $n$ and $M \in S^\lambda$. We begin by determining the sets $I_\lambda$ and $K_\lambda$ defined at the beginning of this section. It is easy to see that

$$I = \{1, \lambda_1 + 1, \lambda_1 + \lambda_2 + 1, \ldots, \lambda_1 + \lambda_2 + \cdots + \lambda_{k-1} + 1\}$$

and

$$K = \{\lambda_1, \lambda_1 + \lambda_2, \ldots, \lambda_1 + \lambda_2 + \cdots \lambda_k = n\}.$$  

We claim that there is no linking between the entries in different blocks of $M$, other than blocks which are reflections of each other along the main diagonal. To prove the claim, first note that there cannot be any symmetric links between different blocks unless one is the reflection of the other along the main diagonal. Additionally, the elements of $I$ label the top rows of the various blocks in the $\lambda$-decomposition of $M$, while the elements of $K$ label the right-most columns of the various blocks in the $\lambda$-decomposition of $M$. Therefore, there cannot be asymmetric links between different blocks that are not reflections of each other, as the sets $I$ and $K$ produce “walls” that prevent any such linking from occurring.

For a diagonal block, the schematic representation of the initial zeroes and asymmetric links is

$$\begin{pmatrix}
\bullet & \bullet & \cdots & \bullet \\
0 & * & \cdots & * & 0 \\
* & 0 & \cdots & * & 0 \\
* & * & 0 & \cdots & * & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
* & * & \cdots & 0 & 0
\end{pmatrix}.$$  

11
This is the exact same schema as (5.1) in the proof of Proposition 4.4, and so these blocks have the form $A_{\lambda_i}$.

To determine the upper right hand blocks, note that every initial zero in such a block is contained in the right most column, and the right most column consists of initial zeroes except for the top entry which is free. Schematically,

\[
\begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\ast & \ast & \ast & \cdots & \ast & 0 \\
\ast & \ast & \ast & \cdots & \ast & 0 \\
\ast & \ast & \ast & \cdots & \ast & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\ast & \ast & \ast & \cdots & \ast & 0
\end{pmatrix}.
\]

Since $A$ is symmetric, the schema for corresponding lower left block must be the transpose of (5.2). Using the linking rules, we see that each upper right block has the form $B_{\lambda_i, \lambda_j}$, and therefore that the corresponding lower left block has the form $B^T_{\lambda_i, \lambda_j}$.

\[5.2\]

**5.2 Determinant formula.**

Let us recall the determinant formula (Theorem 4.14) for an element $M \in S^\lambda$. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ be the conjugate partition of $\lambda$, let $P$ be the matrix obtained by taking the upper rightmost entry from each block of the $\lambda$-decomposition of $M$, and let $P_i$ be the upper left $\mu_i$-by-$\mu_i$ submatrix of $P$. Then

\[
\det M = \prod_{i=1}^l \det P_i.
\]

(5.3)

**Example 5.4.** Before proving the theorem, we illustrate the idea of the proof with an example. An element $M$ of $S^{(3,3)}$ with its $\lambda$-decomposition is given by

\[
M = \begin{pmatrix}
\begin{array}{ccc|ccc}
a & 0 & b & c & d & e \\
0 & -b & 0 & -d & -e & 0 \\
b & 0 & 0 & e & 0 & 0 \\
c & -d & 0 & f & 0 & g \\
d & -e & 0 & 0 & -g & 0 \\
e & 0 & 0 & g & 0 & 0 \\
\end{array}
\end{pmatrix}.
\]
where the boxed entries form the matrix $P$ given in Theorem 4.14. Since the unboxed entries in the rightmost column of each block are zero,

$$\det M = \det \begin{pmatrix} b & e \\ e & g \end{pmatrix} \det \begin{pmatrix} 0 & -b & -d & -e \\ b & 0 & e & 0 \\ d & -e & 0 & -g \\ e & 0 & g & 0 \end{pmatrix} = \det \begin{pmatrix} b & e \\ e & g \end{pmatrix}^3.$$

The boxed entries in the formula give rise to the further factorization in the same way as the initial factorization was obtained.

**Proof of Theorem 4.14.** Let $M \in S^\lambda$. With respect to its $\lambda$-decomposition, as in (4.8), place a box around the upper rightmost entry of each block of $M$. We define two submatrices of $M$. First, let $D_1(M)$ be the $k$-by-$k$ submatrix obtained from the boxed entries; let $D_2(M)$ be the $(n-k)$-by-$(n-k)$ submatrix obtained by removing the rows and columns of $M$ that contain boxed entries. In Example 4.15,

$$D_1(M) = \begin{pmatrix} b \\ e \\ g \end{pmatrix} \quad \text{and} \quad D_2(M) = \begin{pmatrix} 0 & -b & -d & -e \\ b & 0 & e & 0 \\ d & -e & 0 & -g \\ e & 0 & g & 0 \end{pmatrix}.$$

Because all of the unboxed entries of $M$ in columns containing a boxed entry are 0, it follows from the cofactor expansion that

$$\det M = (-1)^{n-k} \det D_1(M) \det D_2(M), \quad (5.5)$$

where $k = \ell(\lambda)$ is the length of $\lambda$.

Recall the definition of the family $A_m$ given by (4.1) and (4.2). Let $C_m$ be the $m$-by-$m$ matrix obtained by removing the top row and rightmost column from $A_{m+1}$. Note $C_m$ is skew symmetric and that removing the top row and rightmost column of $C_m$ produces $A_{m-1}$. If $M$ has the form given by (4.8), then $D_2(M)$ has the form

$$D_2(M) = \begin{pmatrix} C_{\lambda_1-1} & B_{\lambda_1-1,\lambda_2-1} & \cdots & B_{\lambda_1-1,\lambda_k-1} \\ B_{\lambda_1-1,\lambda_2-1}^T & C_{\lambda_2-1} & \cdots & B_{\lambda_2-1,\lambda_k-1} \\ \vdots & \vdots & \ddots & \vdots \\ B_{\lambda_1-1,\lambda_k-1}^T & B_{\lambda_2-1,\lambda_k-1}^T & \cdots & C_{\lambda_k-1} \end{pmatrix} \quad (5.6)$$

Moreover, a further application of $D_2$ produces a matrix of the form (4.8), with each $\lambda_i$ reduced by 2.
Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, we construct a finite sequence of $\lambda$ partitions inductively by setting $\lambda^{(1)} := \lambda$ and, for $1 \leq i < \lambda_1$, setting

$$
\lambda_j^{(i+1)} := \begin{cases}
\lambda_j^{(i)} - 1 & \text{if } \lambda_j^{(i)} \geq 1; \\
0 & \text{if } \lambda_j^{(i)} = 0.
\end{cases}
$$

For example, if $\lambda = (3, 3, 2)$, then $\lambda^{(1)} = (3, 3, 2)$, $\lambda^{(2)} = (2, 2, 1)$ and $\lambda^{(3)} = (1, 1)$. More geometrically, the Young diagram of $\lambda^{(i+1)}$ is obtained by removing the right-most box from each row of the Young diagram of $\lambda^{(i)}$. We also consider the series of conjugate partitions $\mu^{(i)}$ of the $\lambda^{(i)}$. Equivalently, the partition $\mu^{(i)}$ is obtained from $\mu = \mu^{(1)}$ by deleting the top $i - 1$ rows in the Young diagram of $\mu$. That is, $\mu^{(i)} = (\mu_i, \mu_{i+1}, \ldots, \mu_l)$.

In conjunction with the sequence of partitions constructed above, we construct a finite sequence of pairs of submatrices of $M$. First, $M^{(1)} = D_1(M)$ and $M_{\text{aux}}^{(1)} = D_2(M)$. For $1 \leq i < \lambda_1$, set $M^{(i+1)} = D_1(M_{\text{aux}}^{(i)})$ and $M_{\text{aux}}^{(i+1)} = D_2(M_{\text{aux}}^{(i)})$. Then, starting with $M = M^{(1)}$, successive application of the decomposition (5.5) gives us

$$
\det M = \prod_{i=1}^{l} (-1)^{|\lambda^{(i)}| - \ell(\lambda^{(i)})} \det M^{(i)}, \quad (5.7)
$$

where $|\lambda^{(i)}|$ is the size, and $\ell(\lambda^{(i)})$ the length, of the partition $\lambda^{(i)}$. Observe that $|\lambda^{(i)}| - \ell(\lambda^{(i)}) = |\lambda^{(i+1)}| = |\mu^{(i+1)}|$ and that the part $\mu_i$ occurs in $i - 1$ of $\mu^{(2)}, \mu^{(3)}, \ldots, \mu^{(l)}$. Therefore,

$$
\det M = \prod_{i=1}^{l} (-1)^{|\mu^{(i+1)}|} \det M^{(i)} = \prod_{i=1}^{l} (-1)^{(i-1)\mu_i} \det M^{(i)}.
$$

Note that $(i - 1)\mu_i$ is odd if and only if $i$ is even and $\mu_i$ is odd. At the same time, observe from (5.6) that $M^{(i)} = \pm P_i$, with $M^{(i)} = -P_i$ if and only if $i$ is even. It follows that $\det M^{(i)} = -\det P_i$ precisely when $i$ is even and $\mu_i$ is odd, yielding (5.3).

### 5.3 Rank of a unipotent fixed quadric

Recall our alternative notation for a partition $\lambda = (1^{\alpha_1}, 2^{\alpha_2}, \ldots, l^{\alpha_l})$, indicating that the parts of $\lambda$ are $\alpha_1$ 1’s, $\alpha_2$ 2’s, . . . , and $\alpha_l$ $l$’s. In this section we prove Theorem 4.16, that the corank of the generic element of $\mathfrak{S}^\lambda$ is equal to the number
of even parts which appear an odd number of times in $\lambda$. Theorem 4.16 follows immediately from Lemmas 5.8 and 5.10 below.

To facilitate the proof, we define the degeneracy number of $\lambda$, $d(\lambda)$, to be the number of even parts which appear an odd number of times in $\lambda$. For $1 \leq i \leq l$, define $\lambda[i] = (1^{\alpha_1}, 2^{\alpha_2}, \ldots, i^{\alpha_i})$ and $d_i(\lambda) = d(\lambda[i])$. For example, if $\lambda = (2^3, 4^1) = (4^2, 2^2, 2^2)$, then $d(\lambda) = 2$ and $d_1(\lambda) = 0$, $d_2(\lambda) = d_3(\lambda) = 1$, $d_4(\lambda) = 2$.

**Lemma 5.8.** Let $M$ be the generic element of $S^\lambda$. With respect to the vertical lines in its $\lambda$-decomposition, let $M'$ be the matrix obtained by taking, in each column block, only the last $d_i(\lambda)$ columns of $M$ if the block corresponds to the part $i$. Then the null space of $M'$ has dimension $d(\lambda)$. In particular, corank($M$) $\geq$ $d(\lambda)$.

**Remark 5.9.** The proof, while not difficult, is somewhat technical. To help facilitate the reader’s understanding, we have included Example 5.11, which illustrates many steps of the proof in a specific case. It may be useful to consult this example while reading the proof.

**Proof.** We argue by induction on $d(\lambda)$. In order to make the induction work, we must prove a more complicated statement. Use the horizontal lines in the $\lambda$-decomposition to form the matrix $M''$ obtained by taking, in each row block corresponding to the part $i$, only the first $d_i(\lambda)$ rows of $M'$. The statement we prove is:

The null spaces of $M'$ and $M''$ are the same, of dimension $d(\lambda)$. Moreover, there are $d(\lambda)$ linearly independent row relations in $M''$ that can be described explicitly in the following sense: for each even part $i$ that occurs an odd number of times, there is a relation among the $m_i^{th}$ rows in the blocks corresponding to the part $i$, where $m_i$ is the number of even parts $\leq i$ that occur an odd number of times.

The statement that the null space of $M'$ is the same as that of $M''$ is proven directly, without induction. Looking at (4.8) and noting that in each block there are no non-zero entries below the main antidiagonal, it is easily seen that the rows in $M''$ that are deleted in order to obtain $M'$ are all zero rows, thereby proving the claim.

Now we proceed with the inductive argument. Note that everything to be proved is now in terms of the smaller matrix $M''$. Let $i$ be the smallest even part that occurs an even number of times. We coarsen the block decomposition of $M''$ into simply

$$M'' = \begin{pmatrix} U'' & V'' \\ Z'' & W'' \end{pmatrix}.$$
The lines divide between the blocks corresponding to parts \( i \) and those corresponding to parts \( \leq i \).

We first note that \( Z'' = 0 \). To see this, note that every block in \( Z'' \) consists of the last \( d_j(\lambda) \) columns of a matrix \( B_{j,k}^T \) for some \( j > i \geq k \). The last \( j - k \) columns of \( B_{j,k}^T \) are zero. Moreover, \( d_j(\lambda) \leq j - i \leq j - k \), which establishes that \( Z'' = 0 \).

Let \( \lambda_U = ((i+1)^{a_i+1}, (i+2)^{a_i+2}, \ldots, l^{a_l}) \) and \( \lambda_W = \lambda [i] = (1^{a_1}, 2^{a_2}, \ldots, i^{a_i}) \). Then, letting \( U \) be the generic element of \( Q^{\lambda_U} \) and \( W \) the generic element of \( Q^{\lambda_W} \), \( U'' \) and \( W'' \) are formed from \( U \) and \( W \) in the same manner as \( M'' \) was formed from \( M \). Moreover, \( d(\lambda_U) = d(\lambda) - 1 \) and \( d(\lambda_W) = 1 \), so we may apply the inductive hypothesis to \( U'' \) and \( W'' \).

We first prove the claim about the row relations in \( M'' \). There is one relation among the first rows in \( W'' \) that come from the part \( i \). Since \( Z'' = 0 \), this gives a corresponding row relation in \( M'' \). For each even part \( j \) occurring an odd number of times, there are row relations in the \( (m_i - 1) \)th rows of the submatrix of \( U'' \) which contains only the last \( d_j(\lambda) - 1 \) columns in each block corresponding to \( j \). But, because of the form of the matrices \( A_n \) and \( B_{p,q} \) (c.f. (4.1) - (4.3)), the entire \( m_i \)th rows in \( U'' \) corresponding to such a \( j \) have the same row relations. Since \( m_i > 1 \), the corresponding rows in \( V'' \) are all zero (the only rows with non-zero entries in \( V'' \) are the top rows of each block), and hence \( d(\lambda) - 1 \) linearly independent row relations are obtained in \( M'' \) in the claimed locations. Together with the other relation found above, this gives \( d(\lambda) \) linearly independent row relations with the claimed form.

Now we prove the claim about the null space of \( M'' \). There are \( d(\lambda) - 1 \) independent column relations in \( U'' \) and since \( Z'' = 0 \), this produces \( d(\lambda) - 1 \) independent column relations in \( M'' \). There is also another column relation among the last columns in each block of \( W'' \) corresponding to the part \( i \). Let us call the columns occurring in this relation distinguished columns. However, the corresponding columns in \( V'' \) are not zero, so we cannot immediately extend this relation to one in \( M'' \). Instead, we show that we can use certain additional columns of \( M'' \) to obtain a column relation. To do this, it suffices to show that all of the distinguished columns of \( V'' \) lie in the column space of \( U'' \). For then we can add a linear combination of columns in \( U'' \) to the linear combination of distinguished columns in \( V'' \) to produce zero. The same linear combination of the full columns in \( M'' \) (obtained by simply adding 0’s at the bottom) plus the combination of distinguished columns in \( M'' \) will produce zero as well. This yields an independent column relation, giving a total of \( d(\lambda) \) linearly independent
column relations in $M''$, i.e. $d(\lambda)$ linearly independent vectors in the null space of $M''$.

To prove that the distinguished column space of $V''$ is contained in the column space of $U''$, it is enough to show that the distinguished column space of $V''$ is orthogonal to the kernel of $U''^T$, which we may interpret as the space of row relations in $U''$. Now since the row relations in $U''$ always involve the $m_i^{th}$ rows from the various blocks and each $m_i \geq 2$, while the nonzero entries in the distinguished columns of $V''$ are always located in the first rows of each block, the claim follows immediately.

\[\square\]

**Lemma 5.10.** Let $M$ be the generic element of $S^\lambda$. There exists a non-zero minor of size $(n - d(\lambda))$-by-$(n - d(\lambda))$. In particular, $\text{corank}(M) \leq d(\lambda)$.

**Proof.** We prove the existence of a non-zero minor of the specified size by finding a non-zero monomial term in the minor expansion that occurs only once, so that no cancellation can occur. To do this, we use a slightly weaker decomposition of $M$ than its $\lambda$-decomposition. In the $\lambda$-decomposition of $M$, remove any horizontal and vertical lines that divide two equal parts of $\lambda$. We then use the diagonal blocks of this decomposition to prove the result; since the variables in each block are distinct, it suffices to prove the corresponding result for a single such diagonal block.

If the part of $\lambda$ is odd, then it is easy to see that all of the main antidiagonal terms are nonzero and that their product is a desired monomial.

Similarly, if the part of $\lambda$ is even and occurs an even number of times, then all of the main antidiagonal terms are nonzero and their product is the desired monomial.

On the other hand, if the part of $\lambda$ is even and occurs an odd number of times, then all of the main antidiagonal terms are nonzero except those in the middle block. But the antidiagonal terms just above the main antidiagonal of the middle block are nonzero, so the product of all of these entries gives the desired monomial, proving that this matrix has corank at most 1.

\[\square\]

**Example 5.11.** We illustrate Theorem 4.16, as well as aspects of the proofs of Lemmas 5.8 and 5.10, in the case where $\lambda = (4, 2, 2)$. In this case, the generic
The element of $S^{(4,2,2,2)}$ is

$$M = \begin{pmatrix}
a & 0 & b & 0 & c & d & f & g & k & l \\
0 & -b & 0 & 0 & -d & 0 & -g & 0 & -l & 0 \\
b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c & -d & 0 & 0 & e & 0 & h & i & m & n \\
d & 0 & 0 & 0 & 0 & -i & 0 & -n & 0 & 0 \\
f & -g & 0 & 0 & h & -i & j & 0 & p & q \\
g & 0 & 0 & 0 & i & 0 & 0 & 0 & -q & 0 \\
k & -l & 0 & 0 & m & -n & p & -q & r & 0 \\
l & 0 & 0 & 0 & n & 0 & q & 0 & 0 & 0
\end{pmatrix}.$$}

We have

$$M' = \begin{pmatrix}
b & 0 & d & g & l \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -i & 0 & q \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -n & -q & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

and

$$M'' = \begin{pmatrix}
b & 0 & d & g & l \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -i & 0 & q \\
0 & 0 & -n & -q & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$}

The decomposition of $M'$ is induced from the $\lambda$-decomposition of $M$, while that of $M''$ is coarsened to show the matrices $U'', V'', Z'', Z''$ from Lemma 5.8.

This matrix $M$ has rank 8, with column relations

$$(qd - ng + il)C_3 - bqC_6 + bnC_8 - biC_{10} = 0 \quad (5.12)$$

$$C_4 = 0, \quad (5.13)$$

where $C_i$ denotes the $i^{th}$ column of $M$. 

18
In the proof of Lemma 5.8, we find the column relations in $U''$ and an additional column relation in $W''$. The second column of $U''$ being zero implies (5.13). There is a single relation among the columns of $W''$, namely $qC_{1}^{W''} - nC_{2}^{W''} + iC_{3}^{W''} = 0$. Moreover, in this example, the column space of $V''$ is the same as that of $U''$, therefore assuring a relation among $C_{3}''$, $C_{4}''$, and $C_{5}''$. Transporting that relation back to $M$ gives (5.12).

In the proof of Lemma 5.10, we consider the minor that uses rows and columns 1, 2, 3, 5, 6, 7, 9, 10. The relevant monomial is $-b^3jn^4$.

References

[1] M. Chasles. Détermination du nombre des sections coniques qui doivent toucher cinq courbes données d’ordre quelconque, ou satisfaire à diverses autres conditions. *C. R. Acad. Sci. Paris*, 58:222–226, 1864.

[2] C. De Concini, M. Goresky, R. MacPherson, and C. Procesi. On the geometry of quadrics and their degenerations. *Comment. Math. Helv.*, 63(3):337–413, 1988.

[3] C. De Concini and C. Procesi. Complete symmetric varieties. In *Invariant theory (Montecatini, 1982)*, volume 996 of *Lecture Notes in Math.*, pages 1–44. Springer, Berlin, 1983.

[4] Hermann Schubert. *Kalkül der abzählenden Geometrie*. Springer-Verlag, Berlin, 1979. Reprint of the 1879 original, With an introduction by Steven L. Kleiman.

[5] Nicolas Spaltenstein. On the fixed point set of a unipotent element on the variety of Borel subgroups. *Topology*, 16(2):203–204, 1977.

[6] T. A. Springer. Trigonometric sums, Green functions of finite groups and representations of Weyl groups. *Invent. Math.*, 36:173–207, 1976.

[7] T. A. Springer. A construction of representations of Weyl groups. *Invent. Math.*, 44(3):279–293, 1978.

[8] Robert Steinberg. On the desingularization of the unipotent variety. *Invent. Math.*, 36:209–224, 1976.
[9] Elisabetta Strickland. Schubert-type cells for complete quadrics. *Adv. in Math.*, 62(3):238–248, 1986.

[10] Israel Vainsencher. Schubert calculus for complete quadrics. In *Enumerative geometry and classical algebraic geometry (Nice, 1981)*, volume 24 of *Progr. Math.*, pages 199–235. Birkhäuser Boston, Mass., 1982.