Distributed Feature Selection for High-dimensional Additive Models

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Abstract

Distributed statistical learning is a common strategy for handling massive data where we divide the learning task into multiple local machines and aggregate the results afterward. However, most existing work considers the case where the samples are divided. In this work, we propose a new algorithm, DDAC-SpAM, that divides features under the high-dimensional sparse additive model. The new algorithm contains three steps: divide, decorrelate, and conquer. We show that after the decorrelation operation, every local estimator can recover the sparsity pattern for each additive component consistently without imposing strict constraints to the correlation structure among variables. Theoretical analysis of the aggregated estimator and empirical results on synthetic and real data illustrate that the DDAC-SpAM algorithm is effective and competitive in fitting sparse additive models.

Keywords: Divide, decorrelate and conquer; Feature space partition; Consistency; Variable selection; Nonparametric model.
1 Introduction

In modern statistics and computing practices, there exists a common bottleneck that complex data with unprecedented size cannot fit into memory nor be analyzed in a limited time on a single machine. One popular solution is distributed statistical learning which works by distributing the learning task to different machines and combining the estimates afterward (Boyd et al. 2011, Zhang et al. 2012, Lee et al. 2017). According to the way of partitioning the dataset, we call it observation-distributed or feature-distributed. Substantial progress has been made on the former type that partitions samples into subsets and then fits the same model using each subset with the same feature space in different machines. Most of the literature focuses on the massive linear or generalized linear models (Chen & Xie 2014, Battey et al. 2015, Zhang et al. 2015, Zeng & Lin 2015, Tang et al. 2016, Zhao et al. 2016, He et al. 2016, Lee et al. 2017, Shi et al. 2018). For nonparametric inference, the existing studies include a partial linear model (Zhao et al. 2016) and nonparametric regression model (Zhang et al. 2013). However, to the best of our knowledge, the feature-distributed statistical learning method, especially for high dimensional nonparametric regression, still remains to be developed.

In this paper, we propose a feature-distributed algorithm for sparse additive model with potentially massive number of covariates \((p \gg n)\). This problem can be formulated as follows: we are given \(n\) observations with response \(y_i \in \mathbb{R}\) and covariates \(\{x_{i1}, \ldots, x_{ip}\} \in \mathbb{R}^p\) for \(i = 1, \ldots, n\). The goal is to fit the additive model (Stone 1985)

\[
y_i = \sum_{j=1}^{p} f_j(x_{ij}) + \varepsilon_i,
\]

where the number of covariates \(p\) can grow much faster than the sample size \(n\) with \(\log(p) = n^v\) for some \(v \in (0, 1)\). What makes the high dimensional inference possible is the sparsity assumption where only a small subset of \(\{f_j, j = 1, \ldots, p\}\) are nonzero functions. Many sparsity-promoted estimators have been proposed for (1) (Aerts et al. 2002, Ravikumar et al. 2009, Meier
et al. (2009), Koltchinskii & Yuan (2010), Huang et al. (2010), Raskutti et al. (2012), Yuan & Zhou (2016), Petersen et al. (2016), Sadhanala & Tibshirani (2017). Since sparse additive models (SpAM) are essentially a functional version of the group-lasso, Ravikumar et al. (2009) borrowed ideas from the sparse linear model and proposed a corresponding algorithm for solving the problem

$$\min_{\beta_j \in \mathbb{R}^{d_n}, \beta_j = 1, \ldots, p} \frac{1}{2n} \left\| Y - \sum_{j=1}^{p} \Psi_j \beta_j \right\|^2_2 + \lambda \sum_{j=1}^{p} \sqrt{\frac{1}{n} \beta_j^T \Psi_j \beta_j}.$$

where $\beta_j = (\beta_{j1}, \cdots, \beta_{jd_n})^T$ is a length-$d_n$ vector of coefficients, and $\Psi_j = (\psi_{j1}, \cdots, \psi_{jd_n})$ is an $n \times d_n$ matrix of the truncated set of orthogonal basis functions for $f_j$ evaluated at the training data. Minimax optimal rates of convergence were established in Raskutti et al. (2012) and Yuan & Zhou (2016). Other extensions of the high dimensional additive model have been proposed by Lou et al. (2016), Sadhanala & Tibshirani (2017), Petersen & Witten (2019) and Haris et al. (2019).

Considering datasets of massive dimensions that the computational complexity or memory requirements cannot fit into one single computer, the aforementioned frameworks for additive models are not directly applicable. A distributed solution and the decentralized storage of datasets are necessary. Most works on distributed statistical inference assume that the data is partitioned by observations, because of the good theoretical properties of the averaged estimators. However, under the high-dimensional additive model, the spline basis functions are constructed from the whole sample. Besides, each component is represented by a group of basis functions which results in a much higher dimension of the design matrix than the number of observations. It is thus desirable to seek feature-distributed algorithms. But feature-distributed studies are scarce, partially due to the fact that the feature distribution process which ignores the correlation between covariates could lead to incorrect inference. Indeed, dividing feature space directly usually leads to misspecified models and ineradicable bias.

Next, we review several works on feature distributed methods. Inspired by group testing,
Zhou et al. (2014) proposed a parallelizable feature selection algorithm. They randomly section-alized features repeatedly for tests and then ranked the features by the test scores. This attempt can boost efficiency but its success heavily depends on the correlation structure of covariates. Song & Liang (2015) proposed a Bayesian variable selection approach for ultrahigh dimensional linear regression models based on splitting feature set into lower-dimensional subsets and screening relevant variables respectively with the marginal inclusion probability for final aggregation. Similar treatments can be found in the Yang et al. (2016), although in the final stage they utilized the sketch approach for further selection. The efficiency of this kind of algorithms will again be highly affected by the correlation structure among features. Thus the identifiability condition for controlling the degree of multicollinearity is necessary. Based on those key facts, Wang et al. (2016) relaxed the correlation requirements by preprocessing the data with a decorrelation operator (DECO) to lower the correlation in feature space under the linear regression model. In a related work, this decorrelation operator was proposed to satisfy the irrepresentable condition for lasso (Jia & Rohe 2015). With DECO, we can get consistent estimates of coefficients with misspecified submodels.

In this work, we consider decorrelating covariates and propose the feature-distributed algorithm DDAC-SpAM under the high-dimensional additive model. That is, we first divide the whole dataset by predictors, i.e. each local machine operates on only $p_i$ variables. Then local machines approximate each component in additive models with a truncated set of B-spline basis. After decorrelating the design matrix of the B-spline basis with the central machine, local machines can in parallel conduct group lasso fit efficiently. Finally, the central machine combines the discovered important predictors and refines the estimates. Our work can be regarded as a functional extension of the DECO procedure proposed by Wang et al. (2016).

The rest of this article is organized as follows. In Section 2 we reviewed the sparse additive
model problem. In Section 3 we introduce the distributed feature selection procedure for the additive models after decorrelation. We present the sparsistency property (i.e. sparsity pattern consistency) (Ravikumar et al., 2009) of our algorithm in Section 4. Our simulations and a real data analysis are presented in Section 5 and 6, showing the efficiency of our method. We conclude with a discussion in Section 7. All the technical details are relegated to the Appendix and supplementary material.

Some standard notation used throughout this paper is collected here. For number \(a\), \([a]\) represent the smallest integer larger than or equal to \(a\). For a square matrix \(A\), let \(\lambda_{\text{min}}(A)\) and \(\lambda_{\text{max}}(A)\) denote the minimum and maximum eigenvalues and we use the norms \(\|A\| = \sqrt{\lambda_{\text{max}}(A^\top A)}\) and \(\|A\|_\infty = \max_i \sum_{j=1}^n |A_{ij}|\). For vector \(v = (v_1, \cdots, v_k)^\top\), we use the norms \(\|v\| = \sqrt{\sum_{j=1}^k v_j^2}\) and \(\|v\|_\infty = \max_j |v_j|\).

2 Sparse Additive Model

Given a random sample \(\{(x_{i1}, \cdots, x_{ip}), y_i\}_{i=1}^n\), with \(x_{ij} \in [0, 1]\), we consider the nonparametric additive model

\[
y_i = \sum_{j=1}^p f_j(x_{ij}) + \varepsilon_i,
\]

where the error \(\varepsilon_i \overset{i.i.d.}{\sim} N(0, \sigma^2), i = 1, \cdots, n\). Let \(\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)^\top\). \(X = (X_1, \cdots, X_p)\) is the \(n \times p\) design matrix and \(Y = (y_1, \cdots, y_n)^\top\) is the response vector. To ensure identifiability of the \(\{f_j, j = 1, \cdots, p\}\), we assume \(Ef_j = 0\).

For function \(f_j\), let \(\{\psi_{jk}, k = 0, 1, \cdots\}\) denote the uniformly bounded basis functions with respect to Lebesgue measure on \([0, 1]\). Following Ravikumar et al. (2009), we assume the following smoothness condition.
Condition 1 For $j = 1, \cdots, p$, $f_j \in T_j$ where

$$T_j = \left\{ f_j \in \mathcal{H}_j : f_j(x) = \sum_{k=0}^{\infty} \beta_{jk} \psi_{jk}(x), \sum_{k=0}^{\infty} \beta_{jk}^2 k^4 \leq C^2 \right\}$$

for some $0 < C < \infty$, where $\mathcal{H}_j$ is a Hilbert space of square integrable functions with the inner product $\langle f_j, f'_j \rangle = E f_j f'_j$ and $\| f_j \|^2 = E f_j^2 < \infty$ and $\sup_x |\psi_{jk}(x)| \leq B$ for some $B$. \{\beta_{jk}, k = 0, 1, \cdots \} are the parameters corresponding to $f_j$.

The standard form of the penalized additive model optimization problem is

$$\min_{f_1 \in T_1, \cdots, f_p \in T_p} \sum_{i=1}^{n} \left\{ y_i - \sum_{j=1}^{p} f_j(x_{ij}) \right\}^2 + J(f_1, \cdots, f_p). \tag{2}$$

where $J$ is a sparsity-smoothness penalty. In this paper, we restrict our discussion to the sparsity-inducing penalty

$$J(f_1, \cdots, f_p) = \lambda_n \sum_{j=1}^{p} \sqrt{\sum_{i=1}^{n} f_j^2(x_{ij})}.$$

According to Meier et al. (2009), we approximate \{f_j, j = 1, \cdots, p\} by a cubic B-spline with a proper number of knots. The usual choice would be to place $d_n - 4 \simeq \lceil n^{1/5} \rceil$ interior knots at the empirical quantile of $X_j$, i.e.,

$$f_j(x) \approx f_{nj}(x) = \sum_{k=1}^{d_n} \beta_{jk} \psi_{jk}(x).$$

With Condition 1 we can bound the truncation bias by $\| f_j - f_{nj} \|^2 = O(1/d_n^4)$. Let $h = \sum_{j=1}^{p} f_j$ and $h_n = \sum_{j=1}^{p} f_{nj}$. Let $S = \{j : f_j \neq 0\}$. Assuming the sparsity condition $s = |S| = O(1)$, it follows that $\| h - h_n \|^2 = O(1/d_n^4)$.

Let $\Psi_j$ denote the $n \times d_n$ matrix with $\Psi_j(i, l) = \psi_{jl}(x_{ij})$. It is the block of B-spline basis for $f_j$. Then the optimization problem (2) can be reformulated as

$$\min_{\beta_1, \cdots, \beta_p} \left\| Y - \sum_{j=1}^{p} \Psi_j \beta_j \right\|^2 + \lambda \sum_{j=1}^{p} \frac{1}{\sqrt{n}} \| \Psi_j \beta_j \|. \tag{3}$$
This group-wise variable selection problem can be solved with the standardized group lasso technique. The algorithm for standardized group lasso can be regarded as a special group lasso procedure after orthogonalization within each group, in which group lasso is computationally more intensive than lasso. Since its solution paths are not piecewise linear, the least angle regression (LARS) algorithm is not applicable. Instead, the block coordinate-wise descent-type algorithms are common approaches. Computational complexity is somewhat tricky to quantify since it depends greatly on the number of iterations. Since each spline block costs $O(nd_n)$ operations, $O(npd_n)$ calculations are required for entire data in one pass. The number of back-fitting loops required for convergence is usually related to $p$. As for the noniterative components, orthogonalization within each block can be solved by QR decomposition which costs $O(npd_n^2)$ operations. Besides, compared with linear regression, memory footprint increases with the expanded spline basis functions $\Psi_j$ taking place of original $X_j$. All these manifest that we need distributed learning to relieve stress from computation time and memory cost.

Before introducing our method, some additional notation is needed. Let $\Psi = (\Psi_1, \cdots, \Psi_p)$ denote $n \times pd_n$ design matrix of the B-spline bases and $\beta = (\beta_1^T, \beta_2^T, \cdots, \beta_p^T)^T$ be the length-$pd_n$ coefficient vector. If $A \subset \{1, \cdots, p\}$, we denote the $n \times d_n |A|$ submatrix of $\Psi$ by $\Psi_A$ where for each $j \in A$, $\Psi_j$ appears as a submatrix in the corresponding order. Correspondingly, $\beta_A$ is the coefficients of $\Psi_A$. For parallel computing, assume $X$ has been column-wisely partitioned into $m$ parts. If $X_j$ is assigned to the $i$th group and it is the $k$th predictor in group $i$, we denote it by $X_k^{(i)}$. Its original index $j$ is one-to-one mapped to $k^{(i)}$. We denote the $i$th part of $X$ by $X^{(i)} = (X_1^{(i)}, X_2^{(i)}, \cdots, X_p^{(i)})$ which are stored in local machine $i$, $i = 1, \cdots, m$ and its spline basis matrix is denoted by $\Psi^{(i)}$. Excluding $\Psi^{(i)}$, we denote the remaining submatrix of $\Psi$ by $\Psi^{(-i)}$. 

7
Let $S^{(i)}$ denote the true set of important variables in the $i$th group, i.e., $S^{(i)} = \{ j^{(i)} : f_{j}^{(i)} \neq 0 \}$, with $s_i = |S^{(i)}|$, and let $S^{c(i)} = \{ j^{(i)} : f_{j}^{(i)} \equiv 0 \}$ denote its complement. Thus, $S$ is the union of $S^{(i)}, i = 1, \cdots, m$, and $S^{c}$ denotes its complement. $\Psi^{(i)}_S$ is the submatrix of $\Psi^{(i)}$ consisting of spline basis of important predictors in the $i$th group and $\Psi^{(i)}_{S^{c}}$ is the basis matrix for the noise predictors.

3 DDAC-SpAM Algorithm

Since $X$ has already been column-wisely partitioned, each local machine stores one subset of the predictors and $Y$. Before the parallel variable selection, let us begin with a decorrelation step for the additive model.

Reformulated as the linear combination of basis functions, we have

$$Y = \Psi \beta + Z + \varepsilon. \quad (4)$$

where $Z = (z_1, \cdots, z_n)^T$ with $z_i = \sum_{j=1}^{p} f_j(x_{ij}) - f_{nj}(x_{ij}), i = 1, \cdots, n$.

The most intuitive way to reduce correlation is orthogonalizing the basis matrix $\Psi$ to make its columns uncorrelated by left-multiplication. If $\Psi$ has full column rank with $n > pd_n$, we write $\Psi$ via singular value decomposition as $\Psi = UDV^T$, where $U$ is a $n \times pd_n$ tall matrix with orthonormal columns, $D$ is a $pd_n \times pd_n$ diagonal matrix and $V$ is a $pd_n \times pd_n$ orthogonal matrix. Then, we set $\tilde{\Psi} = F\Psi$. Here, $F = UD^{-1}U^T$. It is easy to see that the columns of $\tilde{\Psi}$ are orthogonal. Actually, $F$ can be calculated in the central machine by

$$\left( \sum_{i=1}^{m} \Psi^{(i)}\Psi^{(i)\top} \right)^{\frac{1}{2}},$$

where the $n \times n$ matrix $\Psi^{(i)}\Psi^{(i)\top}$ is transmitted from local machine and $A^+$ denotes the Penrose-Moore pseudo-inverse of $A$. 


Left multiplying $F$ on both sides of (4), we get

$$FY = F\Psi \beta + FZ + F\varepsilon.$$  

It can be denoted as

$$\tilde{Y} = \tilde{\Psi} \beta + \tilde{Z} + \tilde{\varepsilon}. \tag{5}$$

The spline basis matrix $\tilde{\Psi}$ satisfies $\tilde{\Psi}_i^T \tilde{\Psi}_j = 0$ for any $i \neq j$ and $\tilde{\Psi}_i^T \tilde{\Psi}_i = I_{d_n}$.

The group lasso working mechanism for the $i$th data subgroup $\{\tilde{Y}, \tilde{\Psi}^{(i)}\}$ can be shown as follow. Firstly, the optimization object (3) would be

$$L(\beta^{(i)}) = \left\| \tilde{Y} - \sum_{j=1}^{p_i} \tilde{\Psi}_j^{(i)} \beta_j^{(i)} \right\|^2_2 + \lambda_n \sum_{j=1}^{p_i} \frac{1}{\sqrt{n}} \|\tilde{\Psi}_j^{(i)} \beta_j^{(i)}\|. \tag{6}$$

As shown in Yuan & Lin (2006) and Ravikumar et al. (2009), a solution to (6) satisfies

$$\hat{\beta}_k^{(i)} = \left[1 - \frac{\lambda_n}{\|P_k^{(i)}\|}\right] P_k^{(i)} + \tilde{\beta}_k^{(i)}$$

where $P_k^{(i)} = \tilde{\Psi}_k^{(i)T} \tilde{Y}$.

Combining with (5), we can derive that

$$P_k^{(i)} = \tilde{\Psi}_k^{(i)T} \left( \tilde{\Psi}_k^{(i)} \beta_k^{(i)} + \tilde{\Psi}_{-k}^{(i)} \beta_{-k}^{(i)} + \tilde{\Psi}^{(-i)} \beta^{(-i)} + \tilde{Z} + \tilde{\varepsilon} \right)$$

$$= \beta_k^{(i)} + \tilde{\Psi}_k^{(i)T} \tilde{\Psi}_{-k}^{(i)} \beta_{-k}^{(i)} + \tilde{\Psi}_k^{(i)T} \tilde{\Psi}^{(-i)} \beta^{(-i)} + \tilde{\Psi}_k^{(i)T} \tilde{Z} + \tilde{\Psi}_k^{(i)T} \tilde{\varepsilon}$$

$$= \beta_k^{(i)} + \tilde{\Psi}_k^{(i)T} \tilde{Z} + \tilde{\Psi}_k^{(i)T} \tilde{\varepsilon} \tag{7}$$

where $\tilde{\Psi}_{-k} = (\tilde{\Psi}_1^{(i)}, \ldots, \tilde{\Psi}_{k-1}^{(i)}, \tilde{\Psi}_{k+1}^{(i)}, \ldots, \tilde{\Psi}_{p_i}^{(i)})$ and $\beta_{-k} = (\beta_{1}^{(i)T}, \ldots, \beta_{k-1}^{(i)T}, \beta_{k+1}^{(i)T}, \ldots, \beta_{p_i}^{(i)T})^T$.

Since the last two terms of (7) can be bounded by Condition [1] with a mild condition for $\tilde{\Psi}^{T}$ to be presented in Section 4, $P_k^{(i)}$ converges to $\beta_k^{(i)}$ almost at the same rate as that with the full data.

When $pd_n \geq n$, SVD of $\Psi$ generates a $n \times n$ orthogonal matrix $U$, a $pd_n \times n$ matrix $V$ with only orthonormal columns and $D$ is a $n \times n$ diagonal matrix. Then $F$ becomes $(\sum_{i=1}^{m} \Psi^{(i)T} \Psi^{(i)})^{-\frac{1}{2}}$. 

9
Although the columns of $\tilde{\Psi}$ are not exactly mutually orthogonal, i.e., for some $i \neq j$, $\tilde{\Psi}_i^T \tilde{\Psi}_j \neq 0$, according to Khatri & Pillai (1965), we have $E(\tilde{\Psi}^T \tilde{\Psi}) = (n/pd_n)I_{pd_n}$, which means that any two columns of $\tilde{\Psi}$ are orthogonal in expectation. Thus, we can still take the same decorrelation step to get a new response $\tilde{Y}$ and the design matrix $\tilde{\Psi}$.

The decorrelation operation mainly aims to lower the correlation between the basis functions in different blocks. To show it visually, we take a simple example with $n < pd_n$. We sample $X$ from zero mean normal distribution with covariance matrix $\Sigma = [\sigma_{ij}]$, where $\sigma_{ii} = 1$, $\sigma_{ij} = \rho$ with $i \neq j$ and $(n, p) = (500, 1000)$. A cubic B-spline with $d_n = 5$ is used. We focus on comparing $\|n^{-1} \tilde{\Psi}_i^T \tilde{\Psi}_j\|$ and $\|n^{-1} \Psi_i^T \Psi_j\|$, $1 \leq i < j \leq p$. The difference between these two terms is affected by the dependence between basis functions of different covariates. We call them quasi-correlation here. Figure 1 shows the boxplots of quasi-correlation before and after the decorrelation step when $\rho$ increases. It can be seen that while $\|n^{-1} \Psi_i^T \Psi_j\|$ increases with $\rho$, $\|n^{-1} \tilde{\Psi}_i^T \tilde{\Psi}_j\|$ is stable throughout the range of $\rho$, which means the decorrelation step reduces correlation between additive components in high-dimensional additive model significantly.

After the decorrelation step, since $\{\tilde{\Psi}_j^{(i)}, j = 1, \ldots, p_i\}$ is not exactly column-orthogonal, we cannot apply the SpAM backfitting algorithm (Ravikumar et al. 2009) directly with $\{\tilde{Y}, \tilde{\Psi}^{(i)}\}$ on the $i$th local machine. Following Simon & Tibshirani (2012), we use a similar method as the standardized Group Lasso to solve this problem.

Specifically, we apply QR decomposition for each block $\tilde{\Psi}_j^{(i)}$, $i = 1, \ldots, m$, $j = 1, \ldots, p_i$ into the product of an orthogonal matrix $\tilde{Q}_j^{(i)}$ and an upper triangular matrix $\tilde{R}_j^{(i)}$. Then, the $i$th local machine runs the SpAM backfitting algorithm with $\{\tilde{Y}, \tilde{Q}^{(i)}\}$ to solve the following problem

$$
\tilde{\theta}^{(i)} = \arg \min_{\theta^{(i)} = (\theta_1^{(i)^T}, \ldots, \theta_{p_i}^{(i)^T})^T} \frac{1}{n}\|\tilde{Y} - \tilde{Q}^{(i)} \theta^{(i)}\|^2 + \lambda_n \sum_{j=1}^{p_i} \|\tilde{Q}_j^{(i)} \theta_j^{(i)}\|, \quad i = 1, \ldots, m \quad (8)
$$

and select variables, where $\tilde{Q}^{(i)} = (\tilde{Q}_1^{(i)}, \ldots, \tilde{Q}_{p_i}^{(i)})$. 10
Figure 1: Comparison between $\Psi$ and $\widetilde{\Psi}$.

Although we can back-solve the original coordinates $\hat{\beta}^{(i)}$ by $\hat{\beta}^{(i)}_j = (\tilde{R}^{(i)}_j)^{-1}\tilde{\theta}^{(i)}_j$, this is unnecessary since $\hat{\theta}^{(i)}_j = 0$ implies $\hat{\beta}^{(i)}_j = 0$. The local machines only need to transfer the selected important variables and their basis functions to the central machine. The final estimator of $\hat{\beta}^{(i)}$ and $f_j(X_j)$ will be computed on the central machine.

Let $\hat{S}^{(i)} = \{j^{(i)} : \hat{\theta}^{(i)}_j \neq 0\}$ denote the $i$th estimated set of important variables, for $i = 1, \cdots, m$, and $\hat{S}$ be their union. The details of DDAC-SpAM are provided in Algorithm 1.

**Remark 1** Steps 1 and 2 are used for data initialization and division. Steps 3–5 are the decorrelation steps. Step 6 and 7 are distributed feature selection and final refinement steps, respectively.

In Step 4, we invert $\sum_{i=1}^{m} \psi^{(i)}(\psi^{(i)})^T + rI_n$ instead of $\sum_{i=1}^{m} \psi^{(i)}(\psi^{(i)})^T$ for robustness. Besides, using ridge regression in Step 7 instead of ordinary least squares is also for robustness.

Now, we analyze the computational complexity and memory consumption of DDAC-SpAM. For convenience, we assume that the $p$ features are evenly distributed to $m$ parts. Excluding spline
Algorithm 1: Divide, decorrelate and conquer SpAM (DDAC-SpAM)

Input: $Y, X, d_n$, the number of machines $m$, the ridge regularization parameter $r$.

1. On the central machine, store and standardize $Y$ to get $\bar{Y}$;

2. Randomly divide predictors into $m$ portions: $X^{(1)}, \ldots, X^{(m)}$ and allocate $(\bar{Y}, X^{(i)})$ to the $i$th local machine for $i = 1, \ldots, m$;

3. On the $i$th local machine, $i = 1, \ldots, m$, generate spline basis matrix $\Psi^{(i)}$ for $X^{(i)}$, standardize every column of $\Psi^{(i)}$ to get $\tilde{\Psi}^{(i)}$ and transmit $\tilde{\Psi}^{(i)}\tilde{\Psi}^{(i)T}$ to the central machine;

4. On the central machine, compute $F = (\sum_{i=1}^{m} \Psi^{(i)}\Psi^{(i)T} + rI_n)^{-1/2}$ and transmit it to the local machines;

5. On the $i$th local machine, $i = 1, \ldots, m$, compute $\bar{\Psi}^{(i)} = F\tilde{\Psi}^{(i)}$ and $\bar{Y} = FY$;

6. On the $i$th local machine, $i = 1, \ldots, m$, perform the QR factorization $\bar{\Psi}^{(i)} = \tilde{Q}^{(i)}\tilde{R}^{(i)}_j$, for $j = 1, \cdots, p_i$, run the SpAM backfitting algorithm to solve (8), push $\hat{S}^{(i)}$ and $\Psi^{(i)}_{S^{(i)}}$ to the central machine;

7. On the central machine, combine $\Psi_{S^{(i)}}, i = 1, \cdots, m$, to get $\hat{\Psi}_S$. Apply ridge regression on $(Y, \hat{\Psi}_S)$ and get $\hat{\beta}_{S}$;

Output: $\hat{S}$ and $\hat{f}_j, j \in \hat{S}$.

interpolation, the costs of the decorrelation operation and QR factorization are $O(n^3 + n^2pd_n/m + n^2m)$ and $O(npd_n^2/m)$ per local machine, respectively, so are $O(n^3 + n^2pd_n/m + n^2m)$ overall.

For parallel estimation, within each iteration of the SpAM backfitting algorithm, $O(npd_n/m)$ calculations are required. Assume the number of loops is $k$. So the total computational cost is $O(n^3 + n^2pd_n/m + n^2m + knpd_n/m)$ for DDAC-SpAM, compared with $O(knpd_n + npd_n^2)$ for SpAM on a single machine. Meanwhile, the memory consumption of every local machine through the entire algorithm is decreased by a factor of $m$. As shown above, DDAC-SpAM can
significantly speed up computation and relax memory requirements. This will be demonstrated in the numerical study section.

4 Theoretical Results

In this section, we provide the theoretical framework for DDAC-SpAM to show it is variable selection consistent (sparsistent) under mild conditions.

For results in this section, we will treat $X$ as random. When $pd_n \geq n$, let the spline interpolation $\Psi = UDV^T$, where $U$ is a $n \times n$ orthogonal, $D$ is a $n \times n$ diagonal matrix and $V$ satisfies $V^TV = I_n$. $\tilde{\Psi} = F\Psi = UV^T$ satisfies $\tilde{\Psi}\tilde{\Psi}^T = I_n$. All of these $n \times pd_n$ matrices whose rows are in orthogonal form $\mathcal{V}(n, pd_n)$, called the Stiefel manifold (Downs 1972).

For clarity, we review the definition of uniform distribution for a random matrix.

**Definition 1 (Chikuse 2012)** A random $n \times p$ matrix $H$ is uniformly distributed on $\mathcal{V}(n, p)$, written $H \sim \text{uniform} (\mathcal{V}(n, p))$, if $H$ has the same distribution as $HO$ for any fixed $p \times p$ orthogonal matrix $O$.

Besides Condition 1, we further make the following assumptions for $\Psi$.

**Condition 2** $V \sim \text{uniform} (\mathcal{V}(n, pd_n))$.

We allow the minimum eigenvalue of $\Psi^T\Psi$ decay with sample size at a certain rate.

**Condition 3** $\lambda_{\min}(\Psi^T\Psi/pd_n) > \delta n^{\alpha-1}$, for some $0 < \alpha \leq 1$ and $\delta > 0$.

Similar conditions as Condition 2 have been applied to the design matrix of linear model (Jia & Rohe 2015). Condition 3 is related to Theorem 2 in Ravikumar et al. (2009) in which they require the eigenvalues of $n^{-1}\Psi^T\Psi$ to be bounded by constants. When the number of covariates diverges with $n$, Condition 3 is more general than theirs.
We also assume that the truncation size \( d_n \), regularization parameter \( \lambda_n \) and the number of important variables \( s \) satisfy

\[ \textbf{Condition 4} \quad s d_n = o(n), \quad s^2 n^{1-\alpha} \lambda_n^{-2} d_n^{-3} \to 0, \quad s^2 n^{-\alpha} d_n^{-3} \to 0, \quad \text{and} \quad \lambda_n / \rho_n^* \to 0, \quad \text{where} \quad \rho_n^* = \min_{j \in S} \| \beta_j^* \|_{\infty}. \]

Similar conditions were assumed in many high-dimensional additive model variable selection literatures, such as condition (B2) for Theorem 3 in Huang et al. (2010).

Additionally, assume

\[ \textbf{Condition 5} \quad \log(pd_n) = o(\lambda_n^2 n^\alpha / d_n). \]

The breakdown point of diverging dimensionality \( p \) in Condition 5 decreases with \( \alpha \) and this rate is the same as Ravikumar et al. (2009) when \( \alpha = 1 \).

To clarify the implications of Conditions 4 and 5, assume the number of relevant variables is bounded, i.e., \( s = O(1) \). Then, in practice, we can set \( d_n = O(n^{1/5}) \). The order of dimension \( p_n \) can be as large as \( \exp(n^{1/5}) \). If \( 1 / \rho_n^* = o(n^{\alpha/2-1/5} / \log n) \), a suitable choice for the regularization parameter \( \lambda_n \) would be \( n^{1/5-\alpha/2} \log n \) for some \( 2/5 < \alpha \leq 1 \).

The key of our algorithm is to reduce the correlation between predictors which leads to a milder constraint for the correlation structure of variables or \( f(X_j) \) within Theorem 1 than before. It can be reflected in two aspects. First, there is no assumption for the correlation between important variables that are distributed to different local machines, since the bound of this kind of correlation is reduced to

\[ \Pr \left\{ \left\| \frac{pd_n \tilde{\Psi}_S^{(i)} T \tilde{\Psi}_S^{(-i)}}{n} \right\| \leq \frac{M_1 \rho_n^*}{\sqrt{s - s_i}} \right\} \to 1, \]

for some \( M_1 > 0 \), as shown in Lemma S.2 in the supplementary material of this paper. Second, there is no assumption for the correlation between important variables and irrelevant variables.
Specifically, we do not need a version of irrepresentable condition (Zhao & Yu 2006) for selection consistency of SpAM. In previous works, such as Ravikumar et al. (2009), this kind of condition can be formulated as an upper bound for

$$\max_{j \in S^c} \left\| \frac{1}{n} \Psi_j \Psi_S \right\| \left\| \left( \frac{1}{n} \Psi_S^T \Psi_S \right)^{-1} \right\|,$$

while this bound exists in our work by

$$\Pr \left\{ \max_{j \in S^{(i)}} \left\| \frac{p \tilde{\Psi}_S^{(i)} \tilde{\Psi}_j^{(i)}}{n} \right\| \leq \frac{M_2 \lambda_n}{\rho_n^*}, \left\| \frac{p \tilde{\Psi}_S^{(i)} \tilde{\Psi}_S^{(i)}}{n} \right\|^{-1} \leq M_3 \right\} \to 1,$$

for some $M_2, M_3 > 0$. The details and proof of these results can be found in Lemmas S.2, S.3, S.4 of the Supplementary Material.

**Theorem 1** Under Conditions 1-5, the local estimator on machine $i$ is sparsistent:

$$\Pr(\hat{S}^{(i)} = S^{(i)}) \geq 1 - 2s_i d_n \exp\left(-\frac{\rho_n^2 \delta}{1134 \sigma^2} \cdot n^\alpha\right) - 2(p_i - s_i) d_n \exp\left(-\frac{7\delta^2 \lambda_n^2}{20800 d_n \sigma^2} \cdot n^\alpha\right) \to 1.$$

The proof of Theorem 1 is provided in the Appendix. After aggregating the results from local machines, we can derive the following corollary.

**Corollary 1** Under Conditions 1-5, the central estimator is sparsistent:

$$\Pr(\hat{S} = S) \geq 1 - 2s d_n \exp\left(-\frac{\rho_n^2 \delta}{1134 \sigma^2} \cdot n^\alpha\right) - 2(p - s) d_n \exp\left(-\frac{7\delta^2 \lambda_n^2}{20800 d_n \sigma^2} \cdot n^\alpha\right) \to 1.$$

## 5 Simulation Studies

### 5.1 Performance Comparison Under Different Correlation Structures and Distributions

To study the performance of the DDAC-SpAM procedure on simulated data sets, we divide feature space evenly and randomly. The tuning parameter $r$ is fixed at 1 because the results are stable
for different choices. For each $f_j$, we use a cubic B-spline parameterization with $d_n = \lceil 2n^{1/5} \rceil$. For comparison, we also include the full data SpAM with a ridge refinement (SpAM) and SpAM with separated feature space without decorrelation (DAC-SpAM). We report the false positives (FP), false negatives (FN), the mean squared error $\|\hat{h} - h\|^2$ (MSE) and computational time (Runtime). We use gglasso (Yang & Zou 2017) and glmnet (Friedman et al. 2010) with five-fold cross-validation to fit group lasso and ridge regression, respectively. $\lambda_n$ varies from the smallest value for which all coefficients are zero to 0.1% of this maximal value and the number of $\lambda_n$ is 500.

We define the signal-to-noise ratio

$$\text{SNR} = \frac{\text{var}(h(X))}{\text{var}(\varepsilon)}.$$ 

Some of other simulation settings are adapted from Meier et al. (2009) and Fan et al. (2011).

**Independent Predictors.** We first consider the case with independent predictors. Two examples where predictors follow a normal distribution and uniform distribution are analyzed, respectively.

**Example 1** ($SNR \approx 15$). Following Example 1 in Meier et al. (2009), we generate the data from the following additive model:

$$y_i = 2g_1(x_{i1}) + 1.6g_2(x_{i2}) - 4g_3(x_{i3}, 2) + g_4(x_{i4}) + \varepsilon_i,$$

with

$$g_1(x) = x, \quad g_2(x) = x^2 - \frac{25}{12}, \quad g_3(x, \omega) = \sin(\omega x), \quad g_4(x) = e^{-x} - 2/5 \cdot \sinh(5/2).$$

The covariates $X = (X_1, \cdots, X_p)$ are simulated from independent Uniform (-2.5, 2.5) and $\varepsilon \sim N(0, 1)$.

**Example 2** ($SNR \approx 15$). In this example, the covariates $X = (X_1, \cdots, X_p)$ are simulated
from independent standard normal distribution. The model is

\[ y_i = 5g_1(x_{i1}) + 2.1g_5(x_{i2}) + 13.2g_6(x_{i3}, \frac{\pi}{4}) + 17.2g_7(x_{i4}, \frac{\pi}{4}) + 2.56\varepsilon_i, \]

with

\[ g_5(x) = (x - 1)^2, \quad g_6(x, \omega) = \frac{\sin(\omega x)}{2 - \sin(\omega x)}, \]

\[ g_7(x, \omega) = 0.1\sin(\omega x) + 0.2\cos(\omega x) + 0.3\sin^2(\omega x) + 0.4\cos^3(\omega x) + 0.5\sin^3(\omega x), \]

where \( \varepsilon \sim N(0, 1) \).

**Dependent Predictors.** For dependent predictors with different distributions, we investigate four different correlation structures.

**Example 3** (SNR \( \approx 6.7 \)). Following Example 3 in [Meier et al. (2009)](https://example.com), the covariates are generated with the random-effects model:

\[ X_j = \frac{W_j + tU_{\lceil j/20 \rceil}}{1 + t}, j = 1, \cdots, p, \]

where \( W_1, \cdots, W_p, U_1, \cdots, U_{\lceil p/20 \rceil} \) \( i.i.d. \sim \text{Uniform} \ (0, 1) \). By construction, the \( p \) predictors are partitioned into segments of size 20. Variables in different segments are independent while the variables in each segment are dependent through the shared \( U \) variable. As a result, the correlation strength within each segment is controlled by \( t \). Here, we set \( t = 1.5 \), leading to a correlation between \( X_i \) and \( X_j \) to be 0.6. The model is

\[ y_i = 2.5g_1(x_{i1}) + 2.6g_5(x_{i2}) + g_6(x_{i3}, 2\pi) + g_7(x_{i4}, 2\pi) + 0.3\varepsilon_i. \]

**Example 4** (SNR \( \approx 6.7 \)). The setting is the same as Example 3 but \( X_j = (W_j + tU_j)/(1 + t) \) and \( U_1 = U_2 = \cdots = U_p \sim \text{Uniform} \ (0, 1) \). We set \( t = 1.5 \) leading to the pairwise correlation of all covariates being 0.6.

**Example 5** (SNR \( \approx 15 \)). The covariates are generated according to a multivariate normal distribution with covariance matrix \( \Sigma_{ij} = 0.6|i-j|, i, j = 1, \cdots, p \) and mean 0. \( Y \) is generated
with
\[ y_i = 2.5g_1(x_{i1}) + g_5(x_{i2}) + 6.5g_6(x_{i3}, \pi \frac{\pi}{4}) + 8.5g_7(x_{i4}, \pi \frac{\pi}{4}) + 1.2\varepsilon_i. \]

The model dimension and the sample size are fixed at \( p = 10,000 \) and \( n = 500 \) respectively and the number of machines is fixed as \( m = 20 \). A total of 100 simulation runs are used for each setting. We report the average performance in Table 1. Several conclusions can be drawn from Table 1. In Examples 1 and 2 where all variables are independent, DDAC-SpAM performs similarly to DAC-SpAM but SpAM tends to include more irrelevant variables. This shows that for independent variables, distributed feature selection can enhance the selection accuracy. In Examples 3-5 where the variables are dependent, the performances of SpAM and DAC-SpAM deteriorate, partially due to the violation of the irrepresentable condition. On the contrary, DDAC-SpAM is far less affected and achieves the overall best performance. And in particular, it has a lower false-positive rate than the other two methods. This shows that the decorrelation step can handle such kinds of strong correlation structures. As for computation time, benefiting from distributed computing, DAC-SpAM and DDAC-SpAM take much less time than SpAM.

### 5.2 Performance Comparison with Various Number of Machines

It looks a bit surprising in Corollary 1 that the sparsistent rate of the aggregated result is irrelevant to the number of machines \( m \). This is mainly because this rate holds uniformly on all subsets of variables and aggregating the local selection results leads to the final sparsity pattern.

To verify this property, in this experiment, we keep the sample size \( n \) and the dimension \( p \) fixed, but vary the number of machines \( m \) from 1 to 200 (\( m = 1, 10, 20, 100, 200 \)) and use Example 4. Naturally, as \( m \) increases, each machine has a lower local dimension.

We summarize the results in Fig 2. First, we observe that all three methods capture nearly all important variables. Compared with DAC-SpAM and SpAM, DDAC-SpAM has the smallest
Table 1: Average false positive (FP), false negative (FN), mean squared error (MSE), time (in seconds) over 100 repetitions and their standard deviations (in parentheses).

| Model   | Method   | FP   | FN   | MSE     | Time   |
|---------|----------|------|------|---------|--------|
| Example 1 | DDAC-SpAM | 0.03 (0.17) | 0.00 (0.00) | 0.627 (0.10) | 16.22 (1.11) |
|         | DAC-SpAM  | 0.21 (1.71) | 0.00 (0.00) | 0.632 (0.11) | 13.56 (0.86) |
|         | SpAM      | 0.99 (1.99) | 0.00 (0.00) | 0.715 (0.18) | 212.39 (20.82) |
| Example 2 | DDAC-SpAM | 0.05 (0.26) | 0.00 (0.00) | 3.380 (0.84) | 24.10 (1.11) |
|         | DAC-SpAM  | 0.05 (0.32) | 0.00 (0.00) | 3.393 (0.85) | 24.10 (1.10) |
|         | SpAM      | 0.58 (1.26) | 0.00 (0.00) | 3.513 (0.84) | 349.94 (14.49) |
| Example 3 | DDAC-SpAM | 2.98 (2.60) | 0.17 (0.37) | 0.030 (0.03) | 16.24 (0.99) |
|         | DAC-SpAM  | 4.03 (3.83) | 0.16 (0.39) | 0.030 (0.03) | 13.62 (0.77) |
|         | SpAM      | 11.12 (8.97) | 0.00 (0.00) | 0.033 (0.01) | 214.00 (17.40) |
| Example 4 | DDAC-SpAM | 0.07 (0.43) | 0.02 (0.14) | 0.013 (0.01) | 16.41 (0.99) |
|         | DAC-SpAM  | 56.21 (18.31) | 0.00 (0.00) | 0.111 (0.07) | 25.60 (2.30) |
|         | SpAM      | 22.72 (11.30) | 0.00 (0.00) | 0.045 (0.01) | 213.51 (18.65) |
| Example 5 | DDAC-SpAM | 0.98 (0.51) | 0.01 (0.10) | 0.767 (0.30) | 24.44 (1.05) |
|         | DAC-SpAM  | 1.05 (0.47) | 0.01 (0.10) | 0.766 (0.29) | 23.93 (1.04) |
|         | SpAM      | 6.51 (8.72) | 0.00 (0.00) | 0.901 (0.23) | 348.79 (14.16) |
number of false positive variables. Due to the better variable selection performance, DDAC-SpAM also has a smaller estimation error. Besides, benefiting from the distributed framework, the time consumption of DDAC-SpAM and DAC-SpAM decreases as $m$ increases. In addition, although decorrelation increases the computational complexity which is evident when the data set is not partitioned, i.e. $m = 1$, DDAC-SpAM takes less time than DAC-SpAM as $m$ increases. The reason is that the reduced correlation between variables leads to less number of back-fitting loops required for convergence in the additive model fitting. Overall, DDAC-SpAM is a competitive distributed variable selection method for high-dimensional additive models.

Figure 2: Performance of DDAC-SpAM with different number of subsets.
6 An Application to Real Data

In this section, we compare the performances of DDAC-SpAM, SpAM, Deco-linear (Wang et al. 2016) and Lasso (Tibshirani 1996) on the meatspec data set analyzed by Meier et al. (2009) and Yang & Zou (2017). The data set was recorded by a Tecator near-infrared spectrometer which measured the spectrum of light transmitted through a sample of minced pork meat (Thodberg 1993, Borggaard & Thodberg 1992). It is available in the R package faraway. Our aim is to predict the fat content by absorbances which can be measured more easily. This original data set contains $n = 215$ observations with $p = 100$ predictors which are highly correlated (Meier et al. 2009). After all predictors are centered and scaled to have mean 0 and variance 1, we add 1900 independent and standard normal distributed variables as interaction terms. Then, DDAC-SpAM, SpAM, Deco-linear, and Lasso are applied to those 2000 features. We use 10 machines for the DDAC-SpAM algorithm, where the features are distributed randomly. To compare the performances of all methods, we randomly split the dataset into a training set of 172 observations and a test set of 43 observations. This procedure is repeated 100 times. We compute the number of predictors selected and the prediction errors on the test set. Table 2 includes the average values and their associated robust standard deviations over 100 replications.

From Table 2, all methods lead to a similar prediction error, but DDAC-SpAM selects significantly fewer predictors than competing methods. Considering the high correlation among predictors and to provide a more parsimonious list, DDAC-SpAM could be a very worthwhile method for distributed feature selection.
Table 2: Mean model size (MS) and prediction error (PE) over 100 repetitions and their robust standard deviations (in parentheses).

| Method     | PE   | MS    |
|------------|------|-------|
| DDAC-SpAM  | 0.357 (0.141) | 9.34 (2.17) |
| SpAM       | 0.339 (0.140) | 15.11 (2.91) |
| Deco-Linear| 0.377 (0.121) | 82.60 (4.51) |
| Lasso      | 0.401 (0.120) | 84.06 (13.60) |

7 Discussion

In this paper, we have studied a feature distributed learning framework named DDAC-SpAM for the high dimensional additive model. DDAC-SpAM makes predictors less correlated and more suitable for the further sparsistent variable selection. The experiments illustrate that this method not only reduces the computational cost substantially, but also outperforms the existing approach SpAM when covariates are highly correlated. This is the first paper to discuss how to combine the divide and conquer method with the high dimensional nonparametric model. The results demonstrate that DDAC-SpAM is attractive on theoretical analysis, empirical performance and its straightforward implement process.

Given that we specifically approximate the additive components by truncated B-spline bases and then impose the sparsity penalty only, DDAC-SpAM framework is readily available for other smoothing method with the additive models, for example, smoothing splines (Speckman 1985) and sparsity-smoothness penalized approaches (Meier et al. 2009). Besides, extension to the generalized additive model can be an interesting topic for future research. Although DDAC-SpAM is designed to solve large-\(p\)-small-\(n\) problems, it can be combined with a sample space partition step to deal with extremely scalable large-\(p\)-large-\(n\) problems. The details can be explored in
future work.

**Appendix: Proof of Theorem 1**

For the $i$-th group, we have

\[
\tilde{Y} = \tilde{\Psi}_S^{(i)} \beta^{(i)}_S + \tilde{Z} + W,
\]

where $W = \tilde{\Psi}_S^{(-i)} \beta^{(-i)}_S + \tilde{\epsilon} = W_1 + W_2$.

A vector $\hat{\beta}^{(i)} \in \mathbb{R}^{d_n p_i}$ is the minimizer of the objective function

\[
R_n(\beta^{(i)}) + \lambda_n \Omega(\beta^{(i)})
= \frac{1}{2n} \left\| \tilde{Y} - \sum_{j=1}^{p_i} \tilde{\Psi}_j^{(i)} \beta^{(i)}_j \right\|^2 + \lambda_n \sum_{j=1}^{p_i} \sqrt{\frac{1}{n} \beta^{(i)T}_j \tilde{\Psi}_j^{(i)} \beta^{(i)}_j} \tilde{\Psi}_j^{(i)} \beta^{(i)}_j
\]

\[
= \frac{1}{2n} \left\| \tilde{Y} - \sum_{j=1}^{p_i} \tilde{\Psi}_j^{(i)} \beta^{(i)}_j \right\|^2 + \lambda_n \sum_{j=1}^{p_i} \left\| \frac{1}{\sqrt{n}} \tilde{\Psi}_j^{(i)} \beta^{(i)}_j \right\|
\]

if and only if there exists a subgradient $\hat{g}^{(i)} \in \partial \Omega \left( \hat{\beta}^{(i)} \right)$, such that

\[
\frac{1}{n} \tilde{\Psi}^{(i)T} \left( \sum_{j=1}^{p_i} \tilde{\Psi}_j^{(i)} \hat{\beta}^{(i)}_j - \tilde{Y} \right) + \lambda_n \hat{g}^{(i)} = 0.
\]

The subdifferential $\partial \Omega(\beta^{(i)})$ is the set of vectors $g^{(i)} \in \mathbb{R}^{d_n}$ satisfying

\[
g^{(i)}_j = \frac{1}{n} \tilde{\Psi}^{(i)T} \tilde{\Psi}_j^{(i)} \beta^{(i)}_j \lambda_n \hat{g}^{(i)} \leq 1,
\]

if $\beta^{(i)}_j \neq 0$, and $g^{(i)}_j \leq 1$, if $\beta^{(i)}_j = 0$.

We use “witness” proof techniques (Wainwright 2009), i.e., set $\hat{\beta}^{(i)}_S = 0$ and $\hat{g}^{(i)}_S = \partial \Omega \left( \beta^{(i)}_S \right)$.

We then obtain $\hat{\beta}^{(i)}_S$ and $\hat{g}^{(i)}_S$ from the stationary condition in (A.3). By showing that, with high probability $\hat{\beta}^{(i)}_j \neq 0$ for $j \in S$ and $g^{(i)T}_j \left( \frac{1}{n} \tilde{\Psi}^{(i)T} \tilde{\Psi}_j^{(i)} \right)^{-1} g^{(i)}_j \leq 1$ for $j \in S^c$, we can then demonstrate that with high probability there exists a minimizer to the optimization problem in (A.2) that has the same sparsity pattern as the true model.
Setting $\beta_{S_s}^t = 0$ and $\hat{g}_j^{(i)} = \frac{\tilde{y}_j^{(i)}T\tilde{y}_j^{(i)}\beta_j^{(i)}}{\sqrt{\frac{1}{n}\beta_j^{(i)}W_j^{(i)T}\beta_j^{(i)}}}$ for $j \in S^{(i)}$, the stationary condition for $\beta_S^{(i)}$ is

$$
\frac{1}{n}\tilde{y}_s^{(i)T}(\tilde{y}_s^{(i)}\beta_s^{(i)} - \tilde{Y}) + \lambda_n \hat{g}_s^{(i)} = 0.
$$

With (A.1), it can be written as

$$
\frac{1}{n}\tilde{y}_s^{(i)T}\tilde{y}_s^{(i)}\left(\beta_s^{(i)} - \beta_s^{*^{(i)}}\right) - \frac{1}{n}\tilde{y}_s^{(i)T}W_1 - \frac{1}{n}\tilde{y}_s^{(i)T}W_2 - \frac{1}{n}\tilde{y}_s^{(i)T}\tilde{Z} + \lambda_n \hat{g}_s^{(i)} = 0
$$

or

$$
\beta_s^{(i)} - \beta_s^{*^{(i)}} = \left(\frac{1}{n}\tilde{y}_s^{(i)T}\tilde{y}_s^{(i)}\right)^{-1}\left(\frac{1}{n}\tilde{y}_s^{(i)T}W_1 + \frac{1}{n}\tilde{y}_s^{(i)T}W_2 + \frac{1}{n}\tilde{y}_s^{(i)T}\tilde{Z} + \lambda_n \hat{g}_s^{(i)}\right),
$$

assuming that $\frac{1}{n}\tilde{y}_s^{(i)T}\tilde{y}_s^{(i)}$ is nonsingular.

Recalling our definition $\rho_n = \min_{j \in S^{(i)}} \left\| \beta_j^{*^{(i)}} \right\|_\infty > 0$, it suffices to show that

$$
\left\| \beta_s^{(i)} - \beta_s^{*^{(i)}} \right\|_\infty < \frac{\rho_n}{2}
$$

in order to ensure that $\text{supp}(\beta_s^{(i)}) = \text{supp}(\beta_s^{*^{(i)}}) = \{ j : \left\| \beta_j^{*^{(i)}} \right\|_\infty \neq 0 \}$. Using $\Sigma_{SS}^{(i)} = \frac{1}{n}(\tilde{y}_s^{(i)T}\tilde{y}_s^{(i)})$ to simplify notation, we have the $l_\infty$ bound:

$$
\left\| \beta_s^{(i)} - \beta_s^{*^{(i)}} \right\|_\infty \leq \left\| \Sigma_{SS}^{(i)-1}(\frac{1}{n}\tilde{y}_s^{(i)T}W_1) \right\|_\infty + \left\| \Sigma_{SS}^{(i)-1}(\frac{1}{n}\tilde{y}_s^{(i)T}W_2) \right\|_\infty + \left\| \Sigma_{SS}^{(i)-1}(\frac{1}{n}\tilde{y}_s^{(i)T}\tilde{Z}) \right\|_\infty + \lambda_n \left\| \Sigma_{SS}^{(i)-1}\tilde{g}_s^{(i)} \right\|_\infty.
$$

Now, we proceed to bound the first term of (A.5). Notice that derived from Condition 1,

$$
\left\| \beta_s^{*-(-)} \right\| \leq \sqrt{s - s_1} C_1 \text{ for some } C_1 > 0,
$$

then

$$
\left\| \Sigma_{SS}^{(i)-1}(\frac{1}{n}\tilde{y}_s^{(i)T}W_1) \right\|_\infty \leq \left\| \Sigma_{SS}^{(i)-1}(\frac{1}{n}\tilde{y}_s^{(i)T}W_1) \right\|_\infty \leq \left\| \Sigma_{SS}^{(i)-1}(\frac{1}{n}\tilde{y}_s^{(i)T}\tilde{y}_s^{(i)}) \right\|_\infty \left\| \beta_s^{*-(-)} \right\|.
$$

$$
\leq \sqrt{s - s_1} C_1 \left[ \frac{3(\sqrt{\frac{s_1}{n}} + t_4) + 3(\sqrt{\frac{s_1}{p} + t_5})}{1 - 3(\sqrt{\frac{s_1}{p} + t_6})} \right]^{-1} \left[ \frac{3(\sqrt{\frac{s_1}{n}} + t_1) + 3(\sqrt{\frac{s_1}{p} + t_2})}{1 - 3(\sqrt{\frac{s_1}{p} + t_3})} \right],
$$

24
where the last inequality is derived with Lemma S.2 and Lemma S.3 in the supplemental material of this paper. \(t_1, t_2, t_3, t_4, t_5, t_6\) are some positive constants as shown in Lemma S.2 and S.3.

Then, consider the second term \(\left\| \Sigma_{SS}^{(i)} \left( \frac{1}{n} \tilde{\Psi}_S^{(i)} T W_2 \right) \right\|_\infty = \left\| \Sigma_{SS}^{(i)} \left( \frac{1}{n} \tilde{\Psi}_S^{(i)} T \tilde{\varepsilon} \right) \right\|_\infty \). Note that \(\tilde{\varepsilon} = F \tilde{\varepsilon}\) and \(\varepsilon \sim N(0, \sigma^2 I)\), so that \(Q := \Sigma_{SS}^{(i)} \left( \frac{1}{n} \tilde{\Psi}_S^{(i)} W_2 \right)\) is Gaussian as well, with zero mean. Consider its \(l\)-th component, \(Q_l = e_l^T Q\), where \(e_l\) is a unit vector with its \(l\)-th component equal to 1.

Then \(E[Q_l] = 0\), and
\[
\text{Var}(Q_l) = \frac{\sigma^2}{n^2} e_l^T \Sigma_{SS}^{(i)} \left( FF^T \right) \tilde{\Psi}_S^{(i)} \Sigma_{SS}^{(i)-1} e_l \leq \frac{\sigma^2}{n^2} \|FF^T\| \left\| \tilde{\Psi}_S^{(i)} \Sigma_{SS}^{(i)-1} e_l \right\|^2 \leq \frac{\sigma^2}{n} \|FF^T\| \left\| \Sigma_{SS}^{(i)-1} \right\|.
\]

So \(\|\text{Var}(Q)\|_\infty \leq \frac{\sigma^2}{n} \|FF^T\| \left\| \Sigma_{SS}^{(i)-1} \right\|\).

With Gaussian comparison results (Ledoux & Talagrand 2013), we have
\[
P\left(\|\Sigma_{SS}^{(i)-1} \left( \frac{1}{n} \tilde{\Psi}_S^{(i)} W_2 \right)\|_\infty \geq t_T\right) \leq 2s_id_n \exp \left\{ - \frac{t_T^2}{2 \|\text{Var}(Q)\|_\infty} \right\} \leq 2s_id_n \exp \left\{ - \frac{t_T^2}{2 \sigma^2 \left\| \Sigma_{SS}^{(i)-1} \right\| \|FF^T\|} \right\} \quad (A.7)
\]

with Lemma S.3. Denote the event \(\eta_3 = \left\{ \|\Sigma_{SS}^{(i)-1} \left( \frac{1}{n} \tilde{\Psi}_S^{(i)} W_2 \right)\|_\infty \leq t_T \right\}\).

Then, we need to bound \(\left\| \Sigma_{SS}^{(i)} \left( \frac{1}{n} \tilde{\Psi}_S^{(i)} T \tilde{Z} \right) \right\|_\infty\). Since,
\[
\tilde{Z} = FZ = F \sum_{j \in S} \sum_{k=d_n+1}^\infty \beta_{jk} \tilde{\Psi}_{jk},
\]
we can obtain that
\[
\left\| \Sigma_{SS}^{(i)-1} \left( \frac{1}{n} \tilde{\Psi}_S^{(i)} T \tilde{Z} \right) \right\|_\infty = \left\| \Sigma_{SS}^{(i)-1} \left( \frac{1}{n} \tilde{\Psi}_S^{(i)} T F \sum_{j \in S} \sum_{k=d_n+1}^\infty \beta_{jk} \tilde{\Psi}_{jk} \right) \right\|_\infty \leq \frac{1}{n} \left\| \Sigma_{SS}^{(i)-1} \tilde{\Psi}_S^{(i)} T F \right\| \left\| \sum_{j \in S} \sum_{k=d_n+1}^\infty \beta_{jk} \tilde{\Psi}_{jk} \right\|_\infty.
\]
Working over the Sobolev space of $\psi_{jk}$ (Condition 1),

$$|z_i| = \left| \sum_{j \in S} \sum_{k=d_n+1}^{\infty} \beta_{jk}^* \psi_{jk}(x_{ij}) \right| \leq B \sum_{j \in S} \sum_{k=d_n+1}^{\infty} |\beta_{jk}^*| \leq \frac{sC_2}{d_n^{3/2}}, \quad \text{(A.8)}$$

for some constant $C_2 > 0$. Thus, we have $\|Z\| \leq \frac{\sqrt{n}C_2}{d_n^{3/2}}$.

Combined with $\frac{1}{n} \left\| \Sigma_{SS}^{-1}(1 \tilde{\Psi}_S^T F) \right\| \leq \frac{1}{n} \left\| \Sigma_{SS}^{-1} \tilde{\Psi}_S^T \right\| \|F\| = \left( \frac{1}{n} \right) \left\| \Sigma_{SS}^{-1} \right\| \|F\|$, it can be derived that

$$\left\| \Sigma_{SS}^{-1}( \frac{1}{n} \tilde{\Psi}_S^T \tilde{Z} ) \right\| \leq C_2 \delta^{-\frac{1}{2}} \left[ 1 - \frac{3(\frac{sd_n}{n} + t_4) + 3(\sqrt{\frac{s}{p}} + t_5)}{1 - 3(\sqrt{\frac{s}{p}} + t_6)} \right] \frac{1}{2} \left( sn^{-\frac{1}{6}} d_n^{-\frac{3}{2}} \right), \quad \text{(A.9)}$$

following Lemma S.3.

Finally, we consider the term $\lambda_n \left\| \Sigma_{SS}^{-1} \hat{g}_S^{(i)} \right\|_\infty$. Note that for $j \in S^{(i)}$,

$$1 = \hat{g}_j^{(i)T} \left( \frac{1}{n} \tilde{\Psi}_j^{(i)T} \tilde{\Psi}_j^{(i)} \right)^{-1} \hat{g}_j^{(i)} \geq \frac{1}{\left\| \Sigma_{SS} \right\|} \left\| \hat{g}_j^{(i)} \right\|^2$$

and thus $\left\| \hat{g}_j^{(i)} \right\| \leq \sqrt{\left\| \Sigma_{SS} \right\|}$.

Therefore,

$$\lambda_n \left\| \Sigma_{SS}^{-1} \hat{g}_S^{(i)} \right\|_\infty \leq \lambda_n \left\| \Sigma_{SS}^{-1} \right\| \left\| \hat{g}_S^{(i)} \right\| \leq \lambda_n \left\| \Sigma_{SS}^{-1} \right\| \sqrt{\left\| \Sigma_{SS} \right\|}$$

It follows that

$$\lambda_n \left\| \Sigma_{SS}^{-1} \hat{g}_S^{(i)} \right\|_\infty \leq \lambda_n \left[ 1 - \frac{3(\frac{sd_n}{n} + t_4) + 3(\sqrt{\frac{s}{p}} + t_5)}{1 - 3(\sqrt{\frac{s}{p}} + t_6)} \right]^{-\frac{1}{2}} \left( sn^{-\frac{1}{6}} d_n^{-\frac{3}{2}} \right)^{\frac{1}{2}}, \quad \text{(A.10)}$$

with Lemma S.3.

Under Condition (C.2), we combine results (A.6), (A.7), (A.9), (A.10) with (A.5) and take

$$t_1 = \frac{\rho_n}{480c_1 \sqrt{s - s_i}}, \quad t_2 = \frac{\rho_n}{480c_1 \sqrt{s - s_i}} - \sqrt{\frac{s}{p}}, \quad t_3 = \frac{1}{10} - \sqrt{\frac{s}{p}}, \quad t_4 = \frac{1}{10} - \sqrt{\frac{s}{p}}, \quad t_5 = t_6 =$$
\[
\frac{1}{10} - \sqrt{\frac{s_n}{p}} \text{ and } t_7 = \frac{\rho_n^*}{g}.
\]
Therefore, we have

\[
\|\beta_S^{(i)} - \beta_S^{* (i)}\|_\infty \leq \sqrt{s - s_i} C_1 \left[ 1 - \frac{3(\sqrt{\frac{s_n d_n}{n}} + t_4) + 3(\sqrt{\frac{s_n}{p}} + t_5)}{1 - 3(\sqrt{\frac{s_n}{p}} + t_6)} \right]^{-1} \left[ \frac{3(\sqrt{\frac{s_n d_n}{n}} + t_1) + 3(\sqrt{\frac{s_n}{p}} + t_2)}{1 - 3(\sqrt{\frac{s_n}{p}} + t_3)} \right]^{\frac{1}{2}} s_n^{-\frac{1}{2}\alpha} d_n^{-\frac{1}{2}}
\]

\[
+ t_7 + C_2 \delta^{-\frac{1}{2}} \left[ 1 - \frac{3(\sqrt{\frac{s_n d_n}{n}} + t_4) + 3(\sqrt{\frac{s_n}{p}} + t_5)}{1 - 3(\sqrt{\frac{s_n}{p}} + t_6)} \right]^{-1} \left[ 1 + \frac{3(\sqrt{\frac{s_n d_n}{n}} + t_4) + 3(\sqrt{\frac{s_n}{p}} + t_5)}{1 - 3(\sqrt{\frac{s_n}{p}} + t_6)} \right]^{\frac{1}{2}}
\]

\[
= \rho_n^* + \frac{\rho_n^*}{9} + \sqrt{7} C_2 s \delta^{-1/2} n^{-\alpha/2} d_n^{-3/2} + \sqrt{91} \lambda_n
\]

\[
\leq \frac{35}{72} \rho_n^* \leq \frac{\rho_n^*}{2}.
\]

under the event \(\eta_3\) and events \(\eta_1\) and \(\eta_2\) which are defined in the supplemental materials.

Now, we analyze \(\hat{g}_{S^c}^{(i)}\). We require that

\[
\hat{g}_{j}^{(i)^T} \left( \frac{1}{n} \tilde{\Psi}_{j}^{(i)^T} \tilde{\Psi}_{j}^{(i)} \right)^{-1} \hat{g}_{j}^{(i)} \leq 1, \quad \text{for all } j \in S^{c(i)}.
\]

Since

\[
\hat{g}_{j}^{(i)^T} \left( \frac{1}{n} \tilde{\Psi}_{j}^{(i)^T} \tilde{\Psi}_{j}^{(i)} \right)^{-1} \hat{g}_{j}^{(i)} \leq \left\| \hat{g}_{j}^{(i)} \right\|^2 \left\| \left( \frac{1}{n} \tilde{\Psi}_{j}^{(i)^T} \tilde{\Psi}_{j}^{(i)} \right)^{-1} \right\|,
\]

it suffices to show that

\[
\max_{j \in S^{c(i)}} \left\| \hat{g}_{j}^{(i)} \right\|^2 \left\| \left( \frac{1}{n} \tilde{\Psi}_{j}^{(i)^T} \tilde{\Psi}_{j}^{(i)} \right)^{-1} \right\| \leq 1.
\]

Recall that we have set \(\hat{\beta}_{S^c}^{(i)} = \beta_{S^c}^{* (i)} = 0\). The stationary condition for \(j \in S^{c(i)}\) is thus given by

\[
\frac{1}{n} \tilde{\Psi}_{j}^{(i)^T} \left( \tilde{\Psi}_{S}^{(i)} \hat{\beta}_{S}^{(i)} - \tilde{\Psi}_{S}^{(i)} \beta_{S}^{* (i)} - \bar{Z} - W \right) + \lambda_n \hat{g}_{j}^{(i)} = 0.
\]
Therefore, for $j \in S^{c(i)}$,

$$
\hat{g}_j(i) = \frac{1}{\lambda_n} \cdot \frac{1}{n} \tilde{\Psi}_j(i) \mathbf{T}\left(\tilde{\Psi}_S(i)\beta_S - \tilde{\Psi}_S(i)\tilde{\beta}_S + \tilde{Z} + W\right)
= \frac{1}{\lambda_n} \cdot \frac{1}{n} \tilde{\Psi}_j(i) \mathbf{T}\tilde{\Psi}_S(i)\left(\beta_S - \tilde{\beta}_S\right) + \frac{1}{\lambda_n} \cdot \frac{1}{n} \tilde{\Psi}_j(i) \mathbf{T}\tilde{Z} + \frac{1}{\lambda_n} \cdot \frac{1}{n} \tilde{\Psi}_j(i) \mathbf{T}W_1 + \frac{1}{\lambda_n} \cdot \frac{1}{n} \tilde{\Psi}_j(i) \mathbf{T}W_2 \tag{A.11}
$$

$$
:= \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 + \mathcal{G}_4.
$$

Then, we obtain that

$$
\mu_1 = \mathbb{E}(\mathcal{G}_1) = \mathbb{E}\left(\frac{1}{\lambda_n} \cdot \frac{1}{n} \tilde{\Psi}_j(i) \mathbf{T}\tilde{\Psi}_S(i)\left(\beta_S - \tilde{\beta}_S\right)\right)
= -\frac{1}{\lambda_n} \cdot \frac{1}{n} \tilde{\Psi}_j(i) \mathbf{T}\tilde{\Psi}_S(i)\mathbf{S}_S^{-1}(\frac{1}{n} \tilde{\Psi}_S(i) \mathbf{T}W_1 + \frac{1}{n} \tilde{\Psi}_S(i) \mathbf{T}\tilde{Z} + \lambda_n \hat{g}_j(i)),
$$

where the last equation is derived from (A.4). Meanwhile, $\mu_2 = \mathbb{E}(\mathcal{G}_2) = \mathcal{G}_2$, $\mu_3 = \mathbb{E}(\mathcal{G}_3) = \mathcal{G}_3$, and $\mu_4 = \mathbb{E}(\mathcal{G}_4) = 0$.

Under events $\eta_4$, $\eta_5$, $\eta_6$ and $\eta_1$, $\eta_2$ (their definition and detailed description are included in the supplemental material), we have

$$
\|\mu_1\| \leq \frac{1}{\lambda_n} \left\|\frac{1}{n} \tilde{\Psi}_j(i) \mathbf{T}\tilde{\Psi}_S(i)\right\| \left(\left\|\frac{1}{n} \mathbf{S}_S^{-1}\tilde{\Psi}_S(i) \mathbf{T}W_1\right\| + \left\|\frac{1}{n} \mathbf{S}_S^{-1}\tilde{\Psi}_S(i) \mathbf{T}\tilde{Z}\right\| + \left\|\lambda_n \mathbf{S}_S^{-1}\hat{g}(i)\right\|\right)
\leq \frac{1}{pd_n\lambda_n} \cdot \frac{3\left(\sqrt{\frac{(s+1)d_\alpha}{n}} + t_{s} + 3\sqrt{\frac{2\alpha}{p} + t_{9}}\right)}{1 - 3\left(\frac{2\alpha}{p} + t_{10}\right)} \left(\frac{\rho^*}{32} + \sqrt{7}C_2s\delta^{-1/2n^{-1/2}}d^{-3/2} + \sqrt{91}\lambda_n\right) \tag{A.12}
$$

where the second inequality comes from (A.6), (A.9) and (A.10). Meanwhile,

$$
\|\mu_2\| = \left\|\frac{1}{\lambda_n} \cdot \frac{1}{n} \tilde{\Psi}_j(i) \mathbf{T}\tilde{Z}\right\|
= \left\|\frac{1}{\lambda_n} \cdot \frac{1}{n} \tilde{\Psi}_j(i) \mathbf{T}\Psi_S(-d_\alpha)\beta_S(-d_\alpha)\right\| \leq \left\|\frac{1}{\lambda_n} \cdot \frac{1}{n} \tilde{\Psi}_j(i) \mathbf{T}\right\| \left\|\Psi_S(-d_\alpha)\beta_S(-d_\alpha)\right\|
\leq C_2(pd_n)^{-1}\delta^{-1/2}\lambda_n^{-1}n^{1/2}d^{-3/2} \left[1 + \frac{3\left(\sqrt{\frac{d_\alpha}{n}} + t_{11} + 3\left(\sqrt{\frac{1}{p} + t_{12}}\right)\right)}{1 - 3\left(\sqrt{\frac{1}{p} + t_{13}}\right)}\right]^\frac{1}{2}, \tag{A.13}
$$

28
where the last inequality follows Lemma S.5 and (A.8), and

\[
\|\mu_3\| = \left\| \frac{1}{\lambda_n} \frac{1}{n} \widetilde{\Psi}^{(i)^T} W_1 \right\|
\]

\[
= \left\| \frac{1}{\lambda_n} \cdot \frac{1}{n} \widetilde{\Psi}^{(i)^T} \xi^{(-i)} \beta^{(-i)} \right\| \leq \frac{1}{\lambda_n} \left\| \frac{1}{n} \widetilde{\Psi}^{(i)^T} \xi^{(-i)} \right\| \left\| \beta^{(-i)} \right\|
\]

\[
\leq (pd_n)^{-1} \sqrt{s - s_i} C_1 \lambda_n^{-1} 3\left(\sqrt{s/d_n} - t_14\right) + 3\left(\sqrt{s/d_n} - t_15\right).
\]

where the last inequality follows Lemma S.6 and Condition 1.

With a simple calculation, we can get

\[
\mathcal{E}_j = \mathcal{E}_j^{(i)} - \mathbb{E}\mathcal{E}_j^{(i)} = \frac{1}{\lambda_n} \frac{1}{n} \widetilde{\Psi}^{(i)^T} (I - \frac{1}{n} \widetilde{\Psi}^{(i)} \Sigma^{(i)} \widetilde{\Psi}^T (i)) W_2.
\]

which follows Gaussian distribution of mean zero.

Then, following Lemma S.5,

\[
\mathbb{E}(\mathcal{E}_{jk}^2) = \mathbb{E}(\mathcal{E}_{jk}^T \mathcal{E}_{jk})
\]

\[
= \frac{\sigma^2}{\lambda_n^2 n^2} \left[ \mathcal{E}_{k}^{(i)^T} (I - \frac{1}{n} \widetilde{\Psi}^{(i)} \Sigma^{(i)} \widetilde{\Psi}^T (i)) \mathcal{E}_{k}^{(i)} \right]
\]

\[
\leq \sigma^2 \lambda_n^{-2} \left\| \frac{1}{n} \widetilde{\Psi}^{(i)^T} (I - \frac{1}{n} \widetilde{\Psi}^{(i)} \Sigma^{(i)} \widetilde{\Psi}^T (i)) \right\|^2 \left\| \mathcal{E}_{k}^{(i)} \right\|^2
\]

\[
\leq \sigma^2 \lambda_n^{-2} \left\| \frac{1}{n} \widetilde{\Psi}^{(i)} \right\|^2 \left\| (\frac{1}{n} \widetilde{\Psi}^{(i)} \Sigma^{(i)} \widetilde{\Psi}^T (i)) \right\|^2 \left\| \mathcal{E}_{k}^{(i)} \right\|^2
\]

\[
= \frac{2\sigma^2}{n} \lambda_n^{-2} \left\| \frac{1}{\sqrt{n}} \tilde{\Psi}_j \right\|^2 \left\| F \right\|^2
\]

\[
\leq \delta^{-1} \frac{2\sigma^2}{n^\alpha} \lambda_n^{-2} \left[ 1 + \frac{3(\sqrt{\frac{d_n}{n}} + t_{11}) + 3(\sqrt{\frac{1}{p}} + t_{12})}{1 - 3(\sqrt{\frac{1}{p}} + t_{13})} \right] (pd_n)^{-2}.
\]

Therefore, we have by Markov’s inequality and Gaussian comparison results (Ledoux & Tala-

\[
\text{grand 2013}) that
\]

\[
P\left( \max_{j \in S^{(i)}} \left\| \mathcal{E}_j \right\|_\infty \geq \frac{t_{17}(pd_n)}{\sqrt{d_n}} \right)
\]

\[
\leq \frac{\sqrt{d_n}}{t_{17}} \mathbb{E}\left( \max_{jk} \left| \mathcal{E}_{jk} \right| \right)
\]

\[
\leq 2(p_i - s_i) \frac{t_7 \delta \lambda_n^2}{4d_n \sigma^2} \left[ 1 + \frac{3(\sqrt{\frac{d_n}{n}} + t_{11}) + 3(\sqrt{\frac{1}{p}} + t_{12})}{1 - 3(\sqrt{\frac{1}{p}} + t_{13})} \right]^{-1} n^\alpha.
\]
Denote event \( \eta_t = \left\{ \max_{j \in S^t} \|E_j\|_\infty \leq \frac{t_{17}/(pd_n)}{\sqrt{d_n}} \right\} \).

Finally, considering

\[
\left\| \hat{g}_j^{(i)} \right\| \leq \left\| E\hat{g}_j^{(i)} \right\| + \left\| \hat{g}_j^{(i)} - E\hat{g}_j^{(i)} \right\|
\leq \left\| E\hat{g}_j^{(i)} \right\| + \sqrt{d_n} \left\| \hat{g}_j^{(i)} - E\hat{g}_j^{(i)} \right\|_\infty
\]

and

\[
\left\| E(\hat{g}_j^{(i)}) \right\| \leq \|\mu_1\| + \|\mu_2\| + \|\mu_3\|,
\]

thus setting \( t_8 = \frac{\lambda_n}{60n}, t_9 = \frac{\lambda_n}{60n} - \frac{\sqrt{(s_i+1)d_n}}{p}, t_{10} = \frac{10}{15} - \frac{\sqrt{s_i+1}}{p}, t_{11} = \frac{1}{10} - \frac{\sqrt{d_n}}{n}, \)

\( t_{12} = \frac{1}{15} - \frac{\sqrt{s_i+1}}{p}, t_{13} = \frac{1}{15} - \frac{1}{p}, t_{14} = \frac{\lambda_n}{100\sqrt{s_i-1}n^\gamma}, t_{15} = \frac{\lambda_n}{100\sqrt{s_i-1}n^\gamma} - \frac{\sqrt{s_i-1}}{p}, \)

\( t_{16} = \frac{1}{15} - \frac{\sqrt{s_i-1}}{p}, \) where \( \theta > 0 \) and \( t_{17} = \frac{1}{20}, \) we have

\[
\max_{j \in S^t} \left\| pd_n \hat{g}_j^{(i)} \right\| \leq \frac{1}{\lambda_n} \cdot \frac{3\left(\sqrt{\frac{(s_i+1)d_n}{n}} + t_8\right) + 3\left(\frac{\sqrt{\frac{s_i+1}{p}} + t_9}{1 - 3\left(\sqrt{\frac{\sqrt{s_i-1}}{p}} + t_{10}\right)}\right)}{1 + 3\left(\frac{\frac{\sqrt{\frac{d_n}{n}} + t_{11}}{1 - 3\left(\sqrt{\frac{\frac{1}{p} + t_{12}}{1 - 3\left(\sqrt{\frac{\frac{1}{p} + t_{13}}{1 - 3\left(\sqrt{\frac{\sqrt{s_i-1}}{p}} + t_{16}\right)}}}}\right)}\right)\right) + t_{17}}
\]

\[
+ C_2\delta^{-1/2}\lambda_n^{-1}n^{\frac{1}{2}(1-\alpha)}s d_n^{-\frac{3}{2}} + \frac{3\left(\sqrt{\frac{(s_i+1)d_n}{n}} + t_9\right)}{1 - 3\left(\sqrt{\frac{\sqrt{s_i-1}}{p}} + t_{10}\right)} + t_{17}
\]

\[
\leq \min_{j \in S^t} \left\| \left( pd_n \tilde{\Psi}_j^{(i)} + \tilde{\Psi}_j^{(i)}/n \right) \right\|,
\]

under events \( \eta_1, \eta_2, \cdots, \eta_7. \)

Considering the probability for the intersection of \( \eta_1, \cdots, \eta_7, \) we can conclude that the solution is sparsistent, i.e. \( \hat{S}^{(i)} = S^{(i)} \) with probability at least

\[
1 - 2s_i d_n \exp\left\{ - \frac{C_3\delta}{\sigma^2} \cdot n^\alpha \right\} - 2(p_i - s_i)d_n \exp\left\{ - \frac{C_4\delta \lambda_n^2}{d_n \sigma^2} \cdot n^\alpha \right\},
\]

where \( C_3 \) and \( C_4 \) are constants as shown in Theorem 1.

This completes the proof of Theorem 1.
Supplementary Materials

The supplementary material consists of Lemma S.1–S.6 and their proofs. Before the proof of Lemma S.2–S.6, we first restate Theorem C.1 of [Jia & Rohe (2015)] here for readers’ convenience.

**Lemma S.1** [Jia & Rohe (2015)] Suppose that \( V \in \mathbb{R}^{n \times p} \) comes uniformly from stiefel-manifold. Let \( V_B \in \mathbb{R}^{n \times b} \) be any \( b \) column of \( V \). Suppose that \( p - b \geq n \). For any \( v_1, v_2, v_3 > 0 \) with \( \sqrt{\frac{b}{n}} + v_1 < 1, \sqrt{\frac{b}{p}} + v_2 < 1 \) and \( \sqrt{\frac{b}{p}} + v_3 < \frac{1}{3} \), we have
\[
P \left[ \left\| \frac{p}{n} V_B^T V_B - I_b \right\| \geq \frac{3(\sqrt{\frac{b}{n}} + v_1) + 3(\sqrt{\frac{b}{p}} + v_2)}{1 - 3(\sqrt{\frac{b}{p}} + v_3)} \right] \leq 2 \exp\left\{ - \frac{nv_1^2}{2} \right\} + 2 \exp\left\{ - \frac{pv_2^2}{2} \right\} + \exp\left\{ - \frac{pv_3^2}{2} \right\}.
\]

**Lemma S.2** Suppose \((p-s)d_n \geq n\). Under Condition 2, for any \( t_1, t_2, t_3 > 0 \) with \( \sqrt{\frac{s}{n}} + t_1 < 1, \sqrt{\frac{s}{p}} + t_2 < 1 \) and \( \sqrt{\frac{s}{p}} + t_3 < \frac{1}{3} \), we have
\[
P \left\{ \left\| \frac{pd_n}{n} \bar{\Psi}^{(i)}_S \bar{\Psi}^{(-i)}_S \right\| n \geq \frac{3(\sqrt{\frac{sd_n}{n}} + t_1) + 3(\sqrt{\frac{s}{p}} + t_2)}{1 - 3(\sqrt{\frac{s}{p}} + t_3)} \right\} \leq 2 \exp\left\{ - \frac{nt_1^2}{2} \right\} + 2 \exp\left\{ - \frac{pd_n t_2^2}{2} \right\} + \exp\left\{ - \frac{pd_n t_3^2}{2} \right\}.
\]

**Proof of Lemma S.2.** Let \( H = [\bar{\Psi}^{(i)}_S, \bar{\Psi}^{(-i)}_S] \). From Lemma S.1, we have that
\[
P \left\{ \left\| \frac{pd_n}{n} H^T H - I_{sd_n} \right\| \geq \frac{3(\sqrt{\frac{sd_n}{n}} + t_1) + 3(\sqrt{\frac{s}{p}} + t_2)}{1 - 3(\sqrt{\frac{s}{p}} + t_3)} \right\} \leq 2 \exp\left\{ - \frac{nt_1^2}{2} \right\} + 2 \exp\left\{ - \frac{pd_n t_2^2}{2} \right\} + \exp\left\{ - \frac{pd_n t_3^2}{2} \right\},
\]
for any \( t_1, t_2, t_3 > 0 \) with \( \sqrt{\frac{sd_n}{n}} + t_1 < 1, \sqrt{\frac{s}{p}} + t_2 < 1 \) and \( \sqrt{\frac{s}{p}} + t_3 < \frac{1}{3} \).

Let
\[
\eta_1 = \left\{ \left\| \frac{pd_n}{n} H^T H - I_{sd_n} \right\| \leq \frac{3(\sqrt{\frac{sd_n}{n}} + t_1) + 3(\sqrt{\frac{s}{p}} + t_2)}{1 - 3(\sqrt{\frac{s}{p}} + t_3)} \right\}.
\]
Considering
\[ H^T H = \begin{pmatrix} \Psi_S^{(i)T} \Psi_S^{(i)} & \Psi_S^{(i)T} \Psi_S^{(-i)} \\ \Psi_S^{(-i)T} \Psi_S^{(i)} & \Psi_S^{(-i)T} \Psi_S^{(-i)} \end{pmatrix} \]
and
\[ \sim \Psi_S^{(i)T} \sim \Psi_S^{(-i)} = \begin{pmatrix} I_{sdn} & 0_{sdn \times (s-s_i)dn} \end{pmatrix} (H^T H - I_{sdn}) \begin{pmatrix} 0_{sdn \times (s-s_i)dn} \\ I_{(s-s_i)dn} \end{pmatrix}, \]
we have
\[ \left\| \frac{pd_n \sim \Psi_S^{(i)T} \sim \Psi_S^{(-i)}}{n} \right\| \leq \frac{3(\sqrt{s_{dn}/n} + t_1) + 3(\sqrt{s_i/n} + t_2)}{1 - 3(\sqrt{s_i/n} + t_3)}, \]
given the event \( \eta_1 \).

Taking \( t_1 = \frac{\rho_n}{480C_1 \sqrt{s-s_i}} - \sqrt{s_{dn}/n} > \frac{\rho_n^*}{960C_1 \sqrt{s-s_i}} \), \( t_2 = \frac{\rho_n}{480C_1 \sqrt{s-s_i}} - \sqrt{s_i/n} \), \( t_3 = \frac{1}{10} - \sqrt{s_i/n} \), we have
\[ P \left\{ \left\| \frac{pd_n \sim \Psi_S^{(i)T} \sim \Psi_S^{(-i)}}{n} \right\| \geq \frac{\rho_n^*}{56C_1 \sqrt{s-s_i}} \right\} \leq 5 \exp\left\{ -\frac{n\rho_n^2}{16200C_1^2(s-s_i)} \right\} \to 0. \]

**Lemma S.3** Suppose \((p-s_i)dn \geq n\). Under Condition 2, for any \( t_4, t_5, t_6 > 0 \) with \( \sqrt{s_{dn}/n} + t_4 < 1 \), \( \sqrt{s_i/n} + t_5 < 1 \) and \( \sqrt{s_i/n} + t_6 < \frac{1}{5} \), we have
\[ P \left\{ \left\| \frac{pd_n \sim \Psi_S^{(i)T} \sim \Psi_S^{(-i)}}{n} \right\| \right\} \leq 2 \exp\left\{ -\frac{nt_4^2}{2} \right\} + 2 \exp\left\{ -\frac{pd_nt_5^2}{2} \right\} + \exp\left\{ -\frac{pd_nt_6^2}{2} \right\}. \]

**Proof of Lemma S.3.** Lemma S.3 follows Lemma S.1 directly. Let
\[ \eta_2 = \left\| \frac{pd_n \sim \Psi_S^{(i)T} \sim \Psi_S^{(-i)}}{n} - I_{sdn} \right\| \leq \frac{3(\sqrt{s_{dn}/n} + t_4) + 3(\sqrt{s_i/n} + t_5)}{1 - 3(\sqrt{s_i/n} + t_6)}. \]

Then we have
\[ P(\eta_2) \geq 1 - \frac{3(\sqrt{s_{dn}/n} + t_4) + 3(\sqrt{s_i/n} + t_5)}{1 - 3(\sqrt{s_i/n} + t_6)}. \]
Taking $t_4 = \frac{1}{10} - \sqrt{\frac{s_ida}{n}} > \frac{1}{20}$, $t_5 = t_6 = \frac{1}{10} - \sqrt{\frac{s_i}{p}}$, we have

$$P \left\{ \left\| \frac{pd_n \tilde{\Psi}^{(i)}_S T \tilde{\Psi}^{(i)}_S}{n} \right\| \geq 7 \right\} \leq 5 \exp \left\{ - \frac{n}{800} \right\} \to 0.$$

**Lemma S.4** Suppose $(p - s_i - 1)d_n \geq n$. Under Condition 2, for any $t_8, t_9, t_{10} > 0$ with

$$\sqrt{\frac{(s_i+1)d_n}{n}} + t_8 < 1, \sqrt{\frac{s_i+1}{p}} + t_9 < 1 \text{ and } \sqrt{\frac{s_i+1}{p}} + t_{10} < \frac{1}{3},$$

we have

$$P \left\{ \max_{j \in S^{(i)}} \left\| \frac{pd_n \tilde{\Psi}^{(i)}_S T \tilde{\Psi}^{(i)}_j}{n} \right\| \geq \frac{3(\sqrt{\frac{(s_i+1)d_n}{n}} + t_8) + 3(\sqrt{\frac{s_i+1}{p}} + t_9)}{1 - 3(\sqrt{\frac{s_i+1}{p}} + t_{10})} \right\}$$

$$\leq (p_i - s_i) \left( 2 \exp \left\{ - \frac{nt_8^2}{2} \right\} + 2 \exp \left\{ - \frac{pd_n t_9^2}{2} \right\} + \exp \left\{ - \frac{pd_n t_{10}^2}{2} \right\} \right).$$

**Proof of Lemma S.4.** First, let $G = [\tilde{\Psi}^{(i)}_S, \tilde{\Psi}^{(i)}_j], j \in S^{(i)}$. From Lemma S.1, we have that

$$P \left\{ \left\| \frac{pd_n}{n} G^T G - I_{(s_i+1)d_n} \right\| \geq \frac{3(\sqrt{\frac{(s_i+1)d_n}{n}} + t_8) + 3(\sqrt{\frac{s_i+1}{p}} + t_9)}{1 - 3(\sqrt{\frac{s_i+1}{p}} + t_{10})} \right\}$$

$$\leq 2 \exp \left\{ - \frac{nt_8^2}{2} \right\} + 2 \exp \left\{ - \frac{pd_n t_9^2}{2} \right\} + \exp \left\{ - \frac{pd_n t_{10}^2}{2} \right\},$$

for any $t_8, t_9, t_{10} > 0$ with

$$\sqrt{\frac{(s_i+1)d_n}{n}} + t_8 < 1, \sqrt{\frac{s_i+1}{p}} + t_9 < 1 \text{ and } \sqrt{\frac{s_i+1}{p}} + t_{10} < \frac{1}{3}.$$

Then, considering

$$G^T G = \begin{pmatrix} \tilde{\Psi}^{(i)}_S T \tilde{\Psi}^{(i)}_S & \tilde{\Psi}^{(i)}_S T \tilde{\Psi}^{(i)}_j \\ \tilde{\Psi}^{(i)}_j T \tilde{\Psi}^{(i)}_S & \tilde{\Psi}^{(i)}_j T \tilde{\Psi}^{(i)}_j \end{pmatrix}$$

and

$$\tilde{\Psi}^{(i)}_S T \tilde{\Psi}^{(i)}_j = \begin{pmatrix} I_{s_i d_n} & 0_{s_id_n \times d_n} \end{pmatrix} \left( G^T G - I_{(s_i+1)d_n} \right) \begin{pmatrix} 0_{s_id_n \times d_n} \\ I_{d_n} \end{pmatrix},$$

and let

$$\eta_4 = \left\{ \left\| \frac{pd_n \tilde{\Psi}^{(i)}_S T \tilde{\Psi}^{(i)}_j}{n} \right\| \leq \frac{3(\sqrt{\frac{(s_i+1)d_n}{n}} + t_8) + 3(\sqrt{\frac{s_i+1}{p}} + t_9)}{1 - 3(\sqrt{\frac{s_i+1}{p}} + t_{10})} \right\},$$

we have

$$P(\eta_4) \geq 1 - 2 \exp \left\{ - \frac{nt_8^2}{2} \right\} - 2 \exp \left\{ - \frac{pd_n t_9^2}{2} \right\} - \exp \left\{ - \frac{pd_n t_{10}^2}{2} \right\}.$$
Taking \( t_8 = \frac{\lambda_n}{\overline{6} \rho_n} - \sqrt{\frac{(s_1+1)d_n}{n}} > \frac{\lambda_n}{120 \rho_n} \), \( t_9 = \frac{\lambda_n}{60 \rho_n} - \sqrt{\frac{s_1+1}{p}} \), \( t_{10} = \frac{1}{16} - \sqrt{\frac{s_1+1}{p}} \), we can derive that

\[
P\left\{ \max_{j \in S^{(i)}} \left\| \frac{p \widetilde{n}}{n} \widetilde{\Psi}_{S}^{(i)} \widetilde{\Psi}_{j}^{(i)} \right\| \geq \frac{\lambda_n}{t \rho_n} \right\} \leq 5(p_i - s_i) \exp\left\{ - \frac{n \lambda_n^2}{2880 \rho_n} \right\} \to 0.
\]

**Lemma S.5** Suppose \((p - 1)d_n \geq n\). Under Condition 2, for any \( t_{11}, t_{12}, t_{13} > 0 \) with \( \sqrt{\frac{d_n}{n}} + t_{11} < 1 \), \( \sqrt{\frac{1}{p}} + t_{12} < 1 \) and \( \sqrt{\frac{1}{p}} + t_{13} < \frac{1}{3} \), and \( j \in S^{(i)} \), we have

\[
P\left\{ \left\| \frac{p \widetilde{n}}{n} \widetilde{\Psi}_{j}^{(i)} - I_{d_n} \right\| \geq \frac{3\left( \sqrt{\frac{d_n}{n}} + t_{11} \right) + 3\left( \sqrt{\frac{1}{p}} + t_{12} \right)}{1 - 3\left( \sqrt{\frac{1}{p}} + t_{13} \right)} \right\} \leq 2 \exp\left\{ - \frac{nt_{11}^2}{2} \right\} + 2 \exp\left\{ - \frac{pd_nt_{12}^2}{2} \right\} + \exp\left\{ - \frac{pd_nt_{13}^2}{2} \right\}.
\]

**Proof of Lemma S.5.** Lemma S.5 follows Lemma S.1 straightly.

Then let

\[
\eta_5 = \left\{ \left\| \frac{p \widetilde{n}}{n} \widetilde{\Psi}_{j}^{(i)} \right\| \leq \left[ 1 + \frac{3\left( \sqrt{\frac{d_n}{n}} + t_{11} \right) + 3\left( \sqrt{\frac{1}{p}} + t_{12} \right)}{1 - 3\left( \sqrt{\frac{1}{p}} + t_{13} \right)} \right]^{\frac{1}{2}} \right\},
\]

we have

\[
P(\eta_5) \geq 1 - 2 \exp\left\{ - \frac{nt_{11}^2}{2} \right\} - 2 \exp\left\{ - \frac{pd_nt_{12}^2}{2} \right\} - \exp\left\{ - \frac{pd_nt_{13}^2}{2} \right\}.
\]

Taking \( t_{11} = \frac{1}{10} - \sqrt{\frac{d_n}{n}} \), \( t_{12} = \frac{1}{10} - \sqrt{\frac{1}{p}} \), \( t_{13} = \frac{1}{10} - \sqrt{\frac{1}{p}} \), it can be derived that \( P(\eta_5) \to 1 \).

**Lemma S.6** Suppose \((p - s + s_i - 1)d_n \geq n\). Under Condition 2, for any \( t_{14}, t_{15}, t_{16} > 0 \) with \( \sqrt{\frac{(s-s_1+1)d_n}{n}} + t_{14} < 1 \), \( \sqrt{\frac{s-s_1+1}{p}} + t_{15} < 1 \) and \( \sqrt{\frac{s-s_1+1}{p}} + t_{16} < \frac{1}{3} \), and \( j \in S^{(i)} \), we have

\[
P\left\{ \left\| \frac{p \widetilde{n}}{n} \widetilde{\Psi}_{S}^{(-i)} \widetilde{\Psi}_{j}^{(i)} \right\| \leq \frac{3\left( \sqrt{\frac{(s-s_1+1)d_n}{n}} + t_{14} \right) + 3\left( \sqrt{\frac{s-s_1+1}{p}} + t_{15} \right)}{1 - 3\left( \sqrt{\frac{s-s_1+1}{p}} + t_{16} \right)} \right\} \leq 2 \exp\left\{ - \frac{nt_{14}^2}{2} \right\} + 2 \exp\left\{ - \frac{pd_nt_{15}^2}{2} \right\} + \exp\left\{ - \frac{pd_nt_{16}^2}{2} \right\}.
\]

**Proof of Lemma S.6.** Let \( K = [\overline{\Psi}_{S}^{(-i)}, \overline{\Psi}_{j}^{(i)}] \). From Lemma S.1, we have that

\[
P\left\{ \left\| \frac{p \widetilde{n}}{n} \widetilde{K}^{T}K - I_{(s-s_1+1)d_n} \right\| \geq \frac{3\left( \sqrt{\frac{(s-s_1+1)d_n}{n}} + t_{14} \right) + 3\left( \sqrt{\frac{s-s_1+1}{p}} + t_{15} \right)}{1 - 3\left( \sqrt{\frac{s-s_1+1}{p}} + t_{16} \right)} \right\} \leq 2 \exp\left\{ - \frac{nt_{14}^2}{2} \right\} + 2 \exp\left\{ - \frac{pd_nt_{15}^2}{2} \right\} + \exp\left\{ - \frac{pd_nt_{16}^2}{2} \right\},
\]

34
for any $t_{14}, t_{15}, t_{16} > 0$ with $\sqrt{\frac{(s-s_1+1)d_n}{n}} + t_{14} < 1$, $\sqrt{\frac{s-s_1+1}{p}} + t_{15} < 1$ and $\sqrt{\frac{s-s_1+1}{p}} + t_{16} < \frac{1}{3}$.

Considering

$$K^T K = \left( \begin{array}{cc} \widetilde{\Psi}_S^{(-i)T} \widetilde{\Psi}_S^{(-i)} & \widetilde{\Psi}_j^{(-i)T} \widetilde{\Psi}_{j}^{(i)} \\ \widetilde{\Psi}_j^{(i)T} \widetilde{\Psi}_S^{(-i)} & \widetilde{\Psi}_j^{(i)T} \widetilde{\Psi}_{j}^{(i)} \end{array} \right),$$

and

$$\widetilde{\Psi}_S^{(-i)T} \widetilde{\Psi}_{j}^{(i)} = \left( I_{(s-s_i)d_n} \ 0_{(s-s_i)d_n \times d_n} \right) (K^T K - I_{(s-s_i)d_n \times d_n}) \left( \begin{array}{c} 0_{(s-s_i)d_n \times d_n} \\ I_{d_n} \end{array} \right),$$

and denote

$$\eta_6 = \left\{ \frac{\|pd_n \widetilde{\Psi}_S^{(-i)T} \widetilde{\Psi}_{j}^{(i)}\|}{n} \leq \frac{3(\sqrt{\frac{(s-s_1+1)d_n}{n}} + t_{14}) + 3(\sqrt{\frac{s-s_1+1}{p}} + t_{15})}{1 - 3(\sqrt{\frac{s-s_1+1}{p}} + t_{16})} \right\},$$

we have

$$P(\eta_6) \geq 1 - 2 \exp\left\{ -\frac{n t_{14}^2}{2} \right\} - 2 \exp\left\{ -\frac{pd_n t_{15}^2}{2} \right\} - \exp\left\{ -\frac{pd_n t_{16}^2}{2} \right\}.$$

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