MODELLING OF NONLOCAL EFFECTS IN ELECTROMECHANICAL NANO-SWITCHES

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Abstract

Dielectric nano-switches made of the materials that exhibit piezoelectric and/or flexoelectric properties with significant electro-mechanical coupling are considered. In this case, a nonuniform strain field may locally break inversion symmetry and induce polarization even in nonpiezoelectrics. At reducing dimensions to the nanoscale, the flexoelectric effect demonstrates the nonlocality of the dielectric materials and plays more significant role than piezoelectric effect. The flexoelectric effect is included into consideration via additional term coupling strain gradient and polarization in the electric enthalpy density. The equations of motion of the improved Euler-Bernoulli and Timoshenko beam models, and 2-D plate theory have been obtained.

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I. INTRODUCTION

Electromechanical nano-switches as integral parts of nanoelectromechanical systems (NEMS) find numerous technological applications as, e.g., mass memory storage, high-frequency electrical switches, and mass or force sensors. Such high technological applications demand combined efforts of engineering and science through modelling and simulations. Hence, to design state-of-the-art nanotechnological devices with predetermined characteristics, engineers need in addition to experimental techniques not only new formulas and computational methods but also improved electromechanical models. The new models must account for the nonlocal properties of the materials and new physical phenomena.

The nonlocality appears due to the noticeable role of interatomic forces in nano-objects. In nonlocal theories, the constitutive equations take into account microstructure of real materials and microscopic interaction length between, e.g., molecules in a lattice. From the physical point of view, the nonlocality exhibits through special size-dependent effects such as, for example, flexoelectricity - induced polarization due to the strain gradient. It is well known that conventional continuum theories are size-independent and therefore cannot be applied automatically for the analysis of NEMS devices. The most suitable tools for their analysis are atomic and molecular models, but they are restricted by their computational capacity. One of the ways to resolve this contradiction consists in the employment of improved, size-dependent classical theories [1].

In the present work, we consider both piezoelectric and flexoelectric cases. In the latter case, a nonuniform strain field may locally break inversion symmetry and induce polarization. To take into account this effect, similar to [1], we assume that the electric enthalpy density depends not only on strain, electric field, and polarization but also on strain gradient. Then, we replace the polarization by its linear representations through the electric field and strain. Using Hamilton’s principle, we obtain the equations of motion for Euler-Bernoulli and Timoshenko beams as well as for 2-D plate. The higher order terms in the strain from the electric enthalpy density give additional contribution to the bending rigidity of the beam compared to classical solution. The analysis shows that taking into account the strain gradient increases the elastic characteristics of the nano-switch considered in this work. From the obtained formulas it is seen that at reducing dimensions to the nanoscale the flexoelectric effect plays more significant role than piezoelectric effect. In the dynamic
case, the Euler-Bernoulli beam model provides overestimated frequencies. Two-dimensional model allows us to evaluate the electric potential accumulated in the nano-switch due to induced nonuniform strain.

II. GOVERNING EQUATIONS

Electromechanical nano-switch may be simulated as a dielectric cantilever nanobeam (Fig. 1). The material of the beam may have tetragonal or cubic symmetry or in another words it may be piezo- or nonpiezoelectric.

First, let us consider flexoelectric properties of the beam. In [1]-[4] the fourth-order flexoelectric tensor is introduced in two different ways: as

$$P_l = \varepsilon_0 \eta_{lm} E_m + e_{ijk} S_{jk} + \mu_{ijkl} u_{i,jk}$$

(1)

or as $f_{ijkl}$ in the additional term in the thermodynamic potential [4]:

$$f_{ijkl} P_i u_{j,kl}.$$ 

(2)

Here, $S_{ij}$, $e_{ijk}$, $\eta_{lm}$, are the components of the strain, dielectric, and relative permittivity tensors, respectively, $P_i$, $u_i$, $E_m$, are the components of polarization, displacement, and electric field vectors, respectively, $\varepsilon_o$ is the permittivity of vacuum.

In [3] there is a structure of the flexoelectric coefficients for a cubic material (in matrix notation):

$$
\begin{pmatrix}
\mu_{11} & \mu_{12} & 0 & 0 & 0 \\
\mu_{12} & \mu_{11} & 0 & 0 & 0 \\
\mu_{12} & \mu_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & \mu_{44} & 0 \\
0 & 0 & 0 & 0 & \mu_{44} \\
0 & 0 & 0 & 0 & 0 & \mu_{44}
\end{pmatrix}.
$$

According to [1] and [6], the electric enthalpy may be presented in the form

$$H(S_{ij}, P_i, P_{i,k}, u_{i,jk}) = W(S_{ij}, P_i, P_{i,k}, u_{i,jk}) - \frac{1}{2}\varepsilon_0 \varphi.i \varphi.i + \varphi.i P_i,$$

(3)

$$W(S_{ij}, P_i, P_{i,k}, u_{i,jk}) = \frac{1}{2} a_{ij} P_i P_j + \frac{1}{2} b_{ijkl} P_{j,i} P_{l,k} + \frac{1}{2} c_{ijkl} S_{ij} S_{kl} + d_{ijk} S_{jk} P_i + e_{ijkl} P_{j,i} S_{kl} + f_{ijkl} P_i u_{j,kl} + g_{ijk} P_i P_{k,j},$$

(4)
where $W$ is an energy density of deformation and polarization, $a_{ij}$, $b_{ijkl}$, $c_{ijkl}$, $d_{ijk}$, $e_{ijkl}$, $f_{ijkl}$, $g_{ijk}$ are the components of reciprocal dielectric susceptibility, polarization gradient-polarization gradient coupling tensor, elastic tensor, piezoelectric tensor, polarization gradient-strain coupling tensor, flexoelectric tensor, and polarization-polarization gradient coupling tensor, respectively, $\varphi$ is the potential of electric field.

Since the components of the stress tensor and the electric field vector $E_i = -\varphi'_i$ are expressed as

$$T_{ij} = \frac{\partial W}{\partial S_{ij}}, \quad E_i = -\frac{\partial W}{\partial P_i},$$

we have

$$T_{ij} = c_{ijkl}S_{kl} + d_{ijk}P_k + e_{ijkl}P_{k,l}$$
$$-E_i = a_{ij}P_j + d_{ijk}S_{jk} + f_{ijkl}u_{i,jk} \quad (5)$$

Compare (1), (2), and the second equation of (5), we conclude that flexoelectric tensor $f_{ijkl}$...
has 21 nonzero components:

\[
\begin{align*}
    f_{1111} &= -\frac{\mu_{11}}{a_{11}}, & f_{2211} = f_{3311} &= -\frac{\mu_{12}}{a_{11}}, \\
    f_{1221} = f_{2121} = f_{1331} = f_{3131} &= -\frac{\mu_{44}}{a_{11}}, \\
    f_{2222} &= -\frac{\mu_{11}}{a_{22}}, & f_{1122} = f_{3322} &= -\frac{\mu_{12}}{a_{22}}, \\
    f_{1212} = f_{2112} = f_{2332} = f_{3232} &= -\frac{\mu_{44}}{a_{22}}, \\
    f_{3333} &= -\frac{\mu_{11}}{a_{33}}, & f_{1133} = f_{2233} &= -\frac{\mu_{12}}{a_{33}}, \\
    f_{1313} = f_{3113} = f_{2323} = f_{3223} &= -\frac{\mu_{44}}{a_{33}}.
\end{align*}
\]

In this work, we do not consider the terms with \(b_{ijkl}\), \(e_{ijkl}\), and \(g_{ijk}\) coefficients although it might be done without difficulties.

Using the equations (3)-(5) we derive the equations of motion for a dielectric nanoswitch. Here we consider a cantilever switch which may be presented as a Euler-Bernoulli or Timoshenko beam or 2-D narrow plate (Fig. 1). The load applied to the free edge of the cantilever simulates the case of nonuniform strain.

III. EULER-BERNOULLI BEAM MODEL

Within the framework of the Euler-Bernoulli model, the displacement can be written as

\[
\begin{align*}
    u_1(x, y, z, t) &\equiv u \approx -zw'(x, t), & u_2(x, y, z, t) &\approx 0, \\
    u_3(x, y, z, t) &\approx w(x, t),
\end{align*}
\]

Hence, based on the equations (6), the strains are equal to

\[
\begin{align*}
    S_{11} &\equiv \varepsilon_x = -zw'', & S_{22} &\equiv \varepsilon_y = 0, & S_{33} &\equiv \varepsilon_z = 0, \\
    S_{12} = S_{21} &\equiv \gamma_{xy} = S_{13} = S_{31} &\equiv \gamma_{xz} = S_{23} &\equiv S_{32} &\equiv \gamma_{yz} = 0.
\end{align*}
\]

We assume that the electric field acts only in \(z\)-direction. Then, from (5) we can express \(E_z\) as

\[
\begin{align*}
    -E_3 &\equiv -E_z = a_{33}P_3 + d_{31}\varepsilon_x + f_{1133}u_{,xx} + f_{1313}u_{,zz} + f_{3113}w_{,xx} = \\
    a_{33}P_z &- (d_{31}z - f_{1133})w''
\end{align*}
\]
or

\[ P_3 \equiv P_z = \frac{1}{a_{33}}(-E_z + (f_{1133} + d_{31} z)w''), \quad (8) \]

where the low letter indexes after a comma mean the derivation with respect to corresponding coordinates.

After substituting the equations (7), (8), into (3) and (4) and ignoring the polarization gradient, we have

\[
H = \frac{1}{2} c_{11} z^2 (w'')^2 + \frac{1}{2 a_{33}} (\varphi_z' + (f_{1133} + d_{31} z)w'')^2 - \]
\[
d_{31} w'' \frac{z}{a_{33}} (\varphi_z' + (f_{1133} + d_{31} z)w'') - f_{1133} \frac{w''}{a_{33}} (\varphi_z' + (f_{1133} + d_{31} z)w'') - \]
\[
\frac{\varepsilon_0}{2} E_z^2 \frac{\varphi_z'}{a_{33}} (\varphi_z' + (f_{1133} + d_{31} z)w'') \quad (9)\]

The equation of motion of the beam are derived via Hamilton’s principle [7]:

\[
\delta \int_{t_1}^{t_2} (T_k - H + W_d) dt = 0 \quad (10)\]

where \(\delta(\cdot)\) denotes the first variation, \(T_k\) is the kinetic energy, and \(W_d\) is the work done by the external forces and moments.

The kinetic energy of the beam by using the equation (6) can be written as

\[
T_k = \frac{1}{2} \int_{V_b} \varrho \{\dot{\vec{u}}\}^T \{\dot{\vec{u}}\} dV_b = \frac{1}{2} \int_{V_b} \varrho [\dot{\vec{w}}^2 + z^2 (\dot{\vec{w}}')^2] dV_b = \]
\[
\frac{1}{2} \varrho \int_0^l [(\dot{\vec{w}})^2 A + I (\dot{\vec{w}}')^2] dx, \quad (11)\]

where \(I = \int_A z^2 dA\), \(A\) is the area of the beam’s cross section, \(V_b\) is the volume of the beam, \(\varrho\) is the mass density of the nano-switch’s material, and the length of the beam is equal to \(l\).

If the beam is under the transverse force \(q\), the work done is equal to

\[
W_d = \int_0^l q \, dx. \quad (12)\]

Performing variation in (10) and using (9), (11), and (12), we obtain the equations describing the electromechanical behavior of the beam

\[
u^{IV}(c_{11} I - \frac{d_{31}^2 I}{a_{33}} - \frac{f_{1133}^2}{a_{33}} A) -\]
\[
\frac{1}{a_{33}} \frac{\partial^2}{\partial x^2} (f_{1133} \varphi_{z}^{el} + d_{31} M_{z}^{el}) + \varrho A \ddot{w} + \varrho I \frac{\partial^4 w}{\partial x^2 \partial t^2} = q, \quad (13)\]
\[
\left( 3 \frac{a_{33} - \varepsilon_0}{a_{33}} \right) \varphi''_{zz} + \frac{d_{31}}{a_{33}} w'' = 0 \tag{14}
\]

where \( Q_{el} \) and \( M_{el} \) are the electric transverse shear force and the electric bending moment, respectively:

\[
Q_{el} = \int_A E_z(x, z) dA
\]

\[
M_{el} = \int_A zE_z(x, z) dA
\]

with the associated boundary conditions
at \( x = 0 \)
\[
w = w' = 0, \tag{15}
\]

at \( x = l \)
\[
(c_{11} I - \frac{d_{31}^2 I}{a_{33}} - \frac{f_{1133}^2 A}{a_{33}}) w'' - \frac{d_{31}}{a_{33}} M_{el} - \frac{f_{1133}}{a_{33}} Q_{el} = 0,
\]

\[
(c_{11} I - \frac{d_{31}^2 I}{a_{33}} - \frac{f_{1133}^2 A}{a_{33}}) w'' - \frac{d_{31}}{a_{33}} \frac{\partial}{\partial x} M_{el} - \frac{f_{1133}}{a_{33}} \frac{\partial}{\partial x} Q_{el} = P_*, \tag{16}
\]

and at \( z = \pm h/2 \)
\[
\varphi'_{z} = 0, \tag{17}
\]

where \( P_* \) is a force applied to the right edge of the cantilever.

The equations (13), (14) and the boundary conditions (15) - (17) couple the mechanical displacement \( w \) and the electric potential \( \varphi \).

From the equation (13) it is seen that piezoelectric and flexoelectric effects increase the bending rigidity of the beam \((a_{33} < 0)\) and for the beams with large cross-section area and consequently large its moment of inertia, the term \( A f_{1133}^2 / a_{33} \) plays insignificant role. However, for small cross-sectional dimensions, when
\[
\frac{f_{1133}^2}{a_{33}} > (c_{11} - \frac{d_{31}^2}{a_{33}}) \frac{h^2}{12},
\]

where \( h \) is the thickness of the beam, the flexoelectric effect becomes noticeable.

The normalized Young modulus is
\[
Y = 1 - \frac{d_{31}^2}{c_{11} a_{33}} - \frac{12 f_{1133}^2}{c_{11} a_{33} h^2},
\]
It is seen that at $h \to 0$ the flexoelectric term plays the dominant role in the bending rigidity.

The classical formula for the maximum deflection of a cantilever beam $w_{\text{max}} = P_t l^3 / 3Y$ may be used to find the coefficient $f_{1133}$ from experimental results:

$$f_{1133} = \sqrt{\frac{1}{a_{33}} \left[ c_{11} \frac{h^2}{12} - \frac{P_t l^3}{3bhw_{\text{max}}} \right]}$$

It is obvious that the dispersive relation is

$$\omega^2 \rho(A - Ik^2) - k^4 \left( c_{11} I - \frac{d_{31} I}{a_{33}} - \frac{f_{1133}^2}{a_{33}} A \right) = 0 \quad (18)$$

where $\omega$ is a frequency and $k$ is a wavenumber.

IV. TIMOSHENKO BEAM MODEL

In this model we assume that the displacements are presented as

$$u_1(x, y, z, t) \approx -z\psi(x, t), \quad u_2(x, y, z, t) \approx 0,$$

$$u_3(x, y, z, t) \approx w(x, t), \quad (19)$$

where $w$ is the transverse displacement of the points of the centroidal axis ($y = z = 0$), and $\psi$ is the rotation of the beam cross-section about the positive $y$–axis.

From (19), the strains are equal to

$$\varepsilon_x = -z\psi', \quad \varepsilon_y = 0, \quad \varepsilon_z = 0,$$

$$\gamma_{xy} = \gamma_{yz} = 0, \quad \gamma_{xz} = w' - \psi. \quad (20)$$

Since the narrow beam deflects in $x-z$ plane, we consider only two components of the electric field $E_x$ and $E_z$. From (5) they can be expressed as

$$-E_x = a_{11} P_x + d_{15} \gamma_{xz} + f_{1111} u_{xx}$$

$$-E_z = a_{33} P_z + d_{31} \varepsilon_x + f_{1133} u_{xx} + f_{1313} u_{xz} + f_{3113} w_{xx},$$

or

$$-E_x = a_{11} P_x + d_{15} (w' - \psi) - f_{1111} z\psi''$$

$$-E_z = a_{33} P_z - d_{31} z\psi' - f_{1133} \psi' + f_{1313} (w'' - \psi'),$$
The work done is described by the equation (12).

\[ P_x = \frac{1}{a_{11}}(-E_x - d_{15}(w' - \psi) + f_{1111}z\psi''), \]
\[ P_z = \frac{1}{a_{33}}(-E_z + d_{31}z\psi' + f_{1133}\psi' - f_{1313}(w'' - \psi')), \] (21)

After substituting the equations (20), (21) into (3) and (4) and ignoring the polarization gradient, we have

\[
H = \frac{1}{2}a_{11}P_x^2 + \frac{1}{2}a_{33}P_z^2 + \frac{1}{2}c_{11}\varepsilon_x P_x^2 + \frac{1}{2}c_{55}\gamma_{xz} + d_{15}\gamma_{xz}P_x \\
+ d_{31}\varepsilon_x P_z + f_{1111}P_1u_{xx} + P_3(f_{1133}u_{xx} + f_{1313}u_{xx} + f_{3113}w_{xx}) - \\
\frac{1}{2}\varepsilon_0(\varphi_x')^2 + \frac{1}{2}\varepsilon_0(\varphi_z')^2 + \varphi_x'P_x + \varphi_z'P_z = \\
\frac{1}{2a_{11}}(\varphi_x' - d_{15}(w' - \psi) + f_{1111}z\psi'')^2 + \\
\frac{1}{2a_{33}}(\varphi_z' + d_{31}z\psi' + f_{1133}\psi' - f_{1313}(w'' - \psi'))^2 + \\
\frac{1}{2}c_{11}z^2(\psi')^2 + \frac{1}{2}c_{55}(w' - \psi)^2 + d_{15}(w' - \psi) \frac{1}{a_{11}}(\varphi_x' - d_{15}(w' - \psi) + f_{1111}z\psi'') - \\
d_{31}z\psi' \frac{1}{a_{33}}(\varphi_z' + d_{31}z\psi' + f_{1133}\psi' - f_{1313}(w'' - \psi')) - \\
f_{1111}z\psi'' \frac{1}{a_{33}}(\varphi_x' - d_{15}(w' - \psi) + f_{1111}z\varphi'') \\
f_{1133}\varphi_z' \frac{1}{a_{33}}(\varphi_z' + d_{31}z\psi' + f_{1133}\psi' - f_{1313}(w'' - \psi')) + \\
f_{1313}w'' - \psi' \frac{1}{a_{33}}(\varphi_z' + d_{31}z\psi' + f_{1133}\psi' - f_{1313}(w'' - \psi')) \\
-\frac{1}{2}\varepsilon_0(\varphi_z')^2 + \frac{1}{2}\varepsilon_0(\sigma_z')^2 + \varphi_x' \frac{1}{a_{11}}(\varphi_x' - d_{15}(w' - \psi) + f_{1111}z\psi'')+ \\
\frac{\varphi_z'}{a_{33}}(\varphi_z' + d_{31}z\psi' + f_{1133}\psi' - f_{1313}(w'' - \psi'))
\]

The kinetic energy is equal to

\[
T_k = \frac{1}{2} \int_{V_b} \rho\{\dot{\mathbf{u}}\}^T\{\dot{\mathbf{u}}\}dV_b = \frac{1}{2} \int_{V_b} \rho[z\dot{\psi}^2 + \dot{\psi}^2]dV_b = \frac{1}{2} \rho \int_0^t [I(\dot{\psi})^2 + A(\dot{\psi})^2]dt.
\]

The work done is described by the equation (12).

Applying the Hamilton's principle similar to previous section, we obtain the equations of
motion for the Timoshenko beam

\[
kA(c_{55} - \frac{d_{15}^2}{a_{11}})(w'' - \psi') + \frac{d_{15}}{a_{11}} \frac{\partial}{\partial x} Q_x^l + \frac{f_{1313}}{a_{33}} (-\frac{\partial^2}{\partial x^2} Q_x^l - kA(f_{1133} \psi''' - f_{1313}(w''' - \psi'''))) - q = gA\ddot{w}
\]

\[
(c_{11}I - \frac{d_{31}^2}{a_{33}} - \frac{(f_{1133} + f_{1313})^2}{a_{33}} kA)\psi'' + kA(c_{55} - \frac{d_{15}^2}{a_{11}})(w' - \psi) - \frac{d_{31}}{a_{33}} \frac{\partial}{\partial x} M_z^l + \frac{d_{15}}{a_{11}} \frac{\partial}{\partial x} Q_x^l + \frac{f_{1133} + f_{1313}}{a_{33}} (f_{1313} kA w''' - \frac{\partial}{\partial x} Q_x^l) +
\]

\[
\varphi_{xx} \left( \frac{3}{a_{11}} - \varepsilon_0 \right) + \varphi_{zz} \left( \frac{3}{a_{33}} - \varepsilon_0 \right) + \left( \frac{d_{15}}{a_{11}} + \frac{d_{31}}{a_{33}} \right) \psi' - \frac{d_{15}}{a_{11}} w'' = 0
\]

with boundary conditions:

at \( x = 0 \)

\[
w = w' = \psi = \psi' = \varphi = 0,
\]

and at \( x = l \):

\[
kA(c_{55} - \frac{d_{15}^2}{a_{11}})(w' - \psi) + \frac{d_{15}}{a_{11}} Q_x^l + \frac{f_{1313}}{a_{33}} (-\frac{\partial}{\partial x} Q_z^l - kA(f_{1133} \psi''' - f_{1313}(w''' - \psi'''))) = P_s
\]

\[
(c_{11}I - \frac{d_{31}^2}{a_{33}} - \frac{(f_{1133} + f_{1313})^2}{a_{33}} kA)\psi' - \frac{d_{31}}{a_{33}} M_z^l +
\]

\[
\frac{f_{1133} + f_{1313}}{a_{33}} (f_{1313} kA w''' - Q_x^l) + \frac{f_{1111}}{a_{11}} \frac{\partial}{\partial x} M_x^l + \frac{f_{1111}^2}{a_{11}} I \psi'' = 0,
\]

\[
kA[(f_{1133} + f_{1313})\psi' - f_{1313} w'''] + Q_z^l = 0,
\]

\[
f_{1111} I \psi'' + M_x^l = 0,
\]

\[
\varphi = 0,
\]

and at \( z = \pm h/2 \)

\[
\left( \frac{3}{a_{33}} - \varepsilon_0 \right) \varphi' + \frac{f_{1133} + f_{1313}}{a_{33}} \psi' - \frac{f_{1313}}{a_{33}} w'' = 0.
\]
Here $k = 5/6$ is the shear correction factor and

$$Q^e_x = \int_A E_x(x, z) dA.$$  

$$M^e_x = \int_A z E_x(x, z) dA.$$  

Now, let us introduce new quantities

$$c_1 = \sqrt{\frac{c_{11} - \frac{d_{44}^2}{a_{14}^2}}{\rho}}, \quad c_s = \sqrt{\frac{k(c_{55} - \frac{d_{15}^2}{a_{11}^2})}{\rho}},$$

$$r_0 = \sqrt{\frac{I}{A}}.$$  

Then, the dispersion relation can be written as

$$\omega^4 - \omega^2 \left( \frac{c_s^2}{r_0^2} + (c_s^2 + c_1^2)k^2 \right) = 0,$$

or

$$\omega = \sqrt{\frac{1}{2} \left( \frac{c_1^2 + c_s^2}{r_0^2}k^2 + \frac{c_s^2}{r_0^2} \right)} \pm \sqrt{\frac{1}{4} \left( \frac{c_1^2 + c_s^2}{r_0^2}k^2 + \frac{c_s^2}{r_0^2} \right)^2 - c_1^2 c_s^2 k^4} \quad (22)$$

The equation (22) presents the frequencies for two wave modes. The lower frequency relates to flexural wave and the higher frequency relates to shear wave.

From the formulas (22) and (18) it is seen that the eigenfrequencies increase if we take into account the piezoelectric and/or flexoelectric effects. The results of the comparison between the eigenfrequencies calculated within the framework of the Euler-Bernoulli and the Timoshenko beam models coincide with the conclusions related to single-walled nanotubes presented in [8].

V. TWO-DIMENSIONAL MODEL

In this model, we consider two components of the displacement $u = u_1$ and $w = u_3$ and only three components of strain tensor acting in $x$ and $z$ direction

$$\varepsilon_x = u_x, \quad \varepsilon_z = w_z, \quad \gamma_{xz} = u_z + w_x$$
The electric field may be written as

\[-E_x = a_{11}P_x + d_{15}\gamma_{xz} + f_{1111}u_{xx} + f_{1331}(u_{xx} + w_{xz}) + f_{3311}w_{31}\]
\[-E_z = a_{33}P_z + d_{31}u_x + f_{1333}u_{xz} + f_{1313}(u_{31} + w_{11}) + f_{3333}w_{zz}.\]

Now, the polarization may be expressed as

\[P_x = -\frac{1}{a_{11}}(\varphi'_x - d_{15}(u_z + w_x) - f_{1111}u_{xx} - f_{1331}(u_{xx} + w_{xz}) - f_{3311}w_{31})\]
\[P_z = \frac{1}{a_{33}}(\varphi'_z - d_{31}u_x - f_{1313}(u_{zz} + w_{xz}) - f_{3313}w_{3z}).\]

In this case, the equation (3) takes the form

\[H = \frac{1}{2}a_{11}P_x^2 + \frac{1}{2}a_{33}P_z^2 + \frac{1}{2}c_{11}\varepsilon_x^2 + \frac{1}{2}c_{33}\varepsilon_z^2 + \frac{1}{2}c_{55}\gamma_{xz}^2 + c_{13}\varepsilon_x\varepsilon_y + d_{15}\gamma_{xz}P_x + d_{31}\varepsilon_zP_z + P_1(f_{1111}u_{xx} + f_{1331}(u_{zz} + w_{xz}) + f_{3311}w_{3z}) + P_3(f_{1333}u_{xz} + f_{1313}(u_{zz} + w_{xz}) + f_{3333}w_{zz}) - \frac{1}{2}\varepsilon_0(\varphi'_x)^2 - \frac{1}{2}\varepsilon_0(\varphi'_z)^2 + \varphi'_xP_x + \varphi'_zP_z = \]
\[\frac{1}{2a_{11}}(\varphi'_x - d_{15}(u_z + w_x) - f_{1111}u_{xx} - f_{1331}(u_{xx} + w_{xz}) - f_{3311}w_{3z})^2 + \frac{1}{2a_{33}}(\varphi'_z - d_{31}u_x - f_{1313}(u_{zz} + w_{xz}) - f_{3313}w_{3z})^2 + \frac{1}{2c_{11}}u_x^2 + \frac{1}{2c_{33}}w_z^2 + \frac{1}{2c_{55}}(u_z + w_x)^2 + c_{13}u_xw_z + \frac{1}{a_{11}}(\varphi'_x - d_{15}(u_z + w_x) - f_{1111}u_{xx} - f_{1331}(u_{zz} + w_{xz}) - f_{3311}w_{3z}) \cdot (\varphi'_x + d_{15}(u_z + w_x) + f_{1111}u_{xx} + f_{1331}(u_{zz} + w_{xz}) + f_{3311}w_{3z}) + \frac{1}{2a_{33}}(\varphi'_z - d_{31}u_x - f_{1313}(u_{zz} + w_{xz}) - f_{3313}w_{3z}) \cdot (\varphi'_z + d_{31}u_x + f_{1333}u_{xz} + f_{1313}(u_{zz} + w_{xz}) + f_{3333}w_{zz}) - \frac{1}{2}\varepsilon_0(\varphi'_x)^2 - \frac{1}{2}\varepsilon_0(\varphi'_z)^2\]

The kinetic energy is

\[T_k = \frac{1}{2} \int_{V_b} \rho (\dot{u}^2 + \dot{w}^2) dV_b\]

The work done is

\[W_d = \int_0^l (pu + qw) dx,\]
where \( p \) is the longitudinal load and \( q \) is the transverse load. Via the Hamilton’s principle, we have the equations of motion

\[
u_{xx}(c_{11} - \frac{d_{31}^2}{a_{33}}) + u_{zz}(c_{55} - \frac{d_{15}^2}{a_{11}}) + w_{xx}(c_{13} + c_{55} - \frac{d_{15}^2}{a_{11}}) - \varphi_{xx}(\frac{d_{31}}{a_{33}} + \frac{d_{15}}{a_{11}}) + \\
\frac{d_{15}}{a_{11}} f_{1111} - \frac{d_{31}}{a_{33}} f_{1313} + \frac{f_{1331}}{a_{11}} d_{15} u_{zzz} - w_{xxx}(\frac{d_{31}}{a_{33}} f_{3333} + \frac{d_{15}}{a_{11}} f_{1331})
\]

\[
f_{1111}\left[\varphi_{xxx} + f_{1111} u_{xxxx} + f_{1331} (u_{xxxx} + w_{xxxx}) + f_{3331} w_{xxxx}\right] \\
f_{1331}\left[\varphi_{xxx} + f_{1111} u_{xxxx} + f_{1331} (u_{xxxx} + w_{xxxx}) + f_{3331} w_{xxxx}\right] \\
f_{1333} + f_{1331}\left[\varphi_{xxx} + f_{1113} u_{xxxx} + f_{1331} (u_{xxxx} + w_{xxxx}) + f_{3333} w_{xxxx}\right] \left. - p = \varrho \ddot{u}, \right.
\]

\[
w_{xx}(c_{55} - \frac{d_{15}^2}{a_{11}}) + w_{zz} c_{33} + u_{xx}(c_{13} + c_{55} - \frac{d_{15}^2}{a_{11}})
\]

\[
\frac{d_{15}}{a_{11}} \varphi_x - \frac{d_{15}}{a_{11}} f_{3331} w_{xx} + u_{xx}(\frac{d_{31}}{a_{33}} f_{1313} - \frac{d_{15}}{a_{11}} f_{1111}) - \\
\frac{f_{1331}}{a_{33}} [\varphi_{xxx} + f_{1113} u_{xxxx} + f_{1331} (u_{xxxx} + w_{xxxx}) + f_{3333} w_{xxxx}] \\
\frac{f_{1331}}{a_{11}} \left[\varphi_{xxx} + f_{1113} u_{xxxx} + f_{1331} (u_{xxxx} + w_{xxxx}) + f_{3331} w_{xxxx}\right] \\
\frac{f_{3333}}{a_{33}} \left[\varphi_{xxx} + f_{1113} u_{xxxx} + f_{1331} (u_{xxxx} + w_{xxxx}) + f_{3333} w_{xxxx}\right] \left. - q = \varrho \ddot{w}, \right.
\]

\[
\varphi_{xx}\left(\frac{3}{a_{11}} - \varepsilon_0\right) + \varphi_{zz}\left(\frac{3}{a_{33}} - \varepsilon_0\right) - u_{xx}(\frac{d_{15}}{a_{11}} + \frac{d_{31}}{a_{33}}) - \frac{d_{15}}{a_{11}} w_{xx} - \\
\frac{f_{1111}}{a_{11}} u_{xxx} - \frac{f_{3333}}{a_{33}} w_{zzz} - u_{xx}(\frac{f_{1331}}{a_{11}} + \frac{f_{1331}}{a_{33}}) + w_{xx}(\frac{f_{1331}}{a_{11}} + \frac{f_{3331}}{a_{11}} + \frac{f_{1331}}{a_{33}}) = 0
\]

with the boundary conditions presented for the brevity sake in the form of partial derivatives

of the electric enthalpy function as: at \( x = 0 \)

\[
u = u^\prime_x = w = w^\prime_x = \varphi = 0,
\]

at \( x = l \)

\[
\frac{\partial H}{\partial u_x} - \frac{\partial}{\partial x}\left( \frac{\partial H}{\partial u_{xx}} \right) - \frac{1}{2} \frac{\partial}{\partial z} \frac{\partial H}{\partial u_{zz}} = P_x, \\
\frac{\partial H}{\partial u_{xx}} = 0, \\
\frac{\partial H}{\partial w_x} - \frac{\partial}{\partial x}\left( \frac{\partial H}{\partial w_{xx}} \right) - \frac{1}{2} \frac{\partial}{\partial z} \frac{\partial H}{\partial w_{zz}} = 0, \\
\frac{\partial H}{\partial w_{xx}} = 0, \\
\frac{\partial H}{\partial \varphi_x} = 0,
\]
and at \( z = \pm h/2 \)

\[
\frac{\partial H}{\partial u_z} - \frac{\partial}{\partial z} \left( \frac{\partial H}{\partial u_{zz}} \right) - \frac{1}{2} \frac{\partial}{\partial x} \frac{\partial H}{\partial u_{xz}} = 0,
\]

\[
\frac{\partial H}{\partial u_{zz}} = 0,
\]

\[
\frac{\partial H}{\partial w_z} - \frac{\partial}{\partial z} \left( \frac{\partial H}{\partial w_{zz}} \right) - \frac{1}{2} \frac{\partial}{\partial x} \frac{\partial H}{\partial w_{xz}} = 0,
\]

\[
\frac{\partial H}{\partial w_{zz}} = 0,
\]

\[
\frac{\partial H}{\partial \varphi_z} = 0,
\]

(26)

If the flexoelectric coefficients are equal to zero, the equations (23) coincide with the equations for plates presented in [7].

Numerical solution to the system (23) with the boundary conditions (24) - (26) allow us to find the deflection of the nano-switch, the strains, the stresses, and the electric potential accumulated in the cantilever.

VI. CONCLUSIONS

In the present work, the nonlocal properties of the dielectric materials at nanoscale have been taken into account through the flexoelectric effect. Based on the Hamilton’s principle and the electric enthalpy density with additional term describing the coupling between the polarization and the strain gradient, the equations of motion for Euler-Bernoulli and Timoshenko beam models as well as 2-D plate model have been derived. These equations, may be used to analyse the static and dynamic behavior of cantilever nano-switch. The formula connected the flexoelectric coefficient and the maximum deflection of the cantilever has been presented.

Two dimensional model allows us to analyse the electric potential accumulated in the nano-switch due to the flexoelectric effect.

[1] M. S. Majdoub, P. Sharma, and T. Cagin, "Enhanced size-dependent piezoelectricity and elasticity in nanostructures due to the flexoelectric effect," 2008 Physical Review B., 77, pp. 1254241-1254249.
[2] W. Ma, "A study of flexoelectric coupling associated internal electric field and stress in thin film ferroelectrics", *Phys. Stat. Sol. (b)*, 245, 4, 761, 2008.

[3] R. Maranganti and P. Sharma, "Atomistic determination of flexoelectric properties of crystalline dielectrics", *Physical Review B*, 80, 054109, 2009.

[4] Sh. M. Kogan, *Sov. Phys.-Solid State*, 5, 2069, 1964.

[5] W. Ma, "Flexoelectricity: strain gradient effects in ferroelectrics", *Phys. Scr*, T129, 180, 2007.

[6] R. D. Mindlin, "Polarization gradient in elastic dielectrics," *1968 Int. J. Solids Structures*, 4, 637.

[7] H.F. Tiersten, Linear Piezoelectric Plate Vibrations, Plenum Press, New York, 1969.

[8] Pin Lu, H.P.Lee, C. Lu, P.Q.Ahang, "Application of nonlocal beam models for carbon nanotubes," *2007 Int. J. Solids Structures*, 44, 5289.