POLYNOMIAL MAPS AND POLYNOMIAL SEQUENCES IN GROUPS

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Abstract. This paper develops a theory of polynomial maps from commutative semigroups to arbitrary groups and proves that it has desirable formal properties when the target group is locally nilpotent. We will apply this theory to solve Waring’s problem for general discrete Heisenberg groups in a sequel to this paper.

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1. Introduction

Motivation. The motivation of this work is the following question of Michael Larsen:

Question. Find good notions of “polynomial sequence” and “generalized cone” so that if \( G \) is a finitely generated nilpotent group and \( g_0, g_1, g_2, \ldots \) is a polynomial sequence in \( G \) such that no coset of any infinite index subgroup of \( G \) contains the whole sequence, then there exists a positive integer \( M \), a generalized cone \( C \subset G \), and a subgroup \( H \) of finite index in \( G \) such that every element of \( C \cap H \) is a product of \( M \) elements of the sequence.

A polynomial sequence in \( \mathbb{Z} \) should be given by a polynomial \( \mathbb{N}_0 \to \mathbb{Z} \) and a generalized cone should be the set of all integers in an unbounded open interval. Thus, the classical Waring’s problem with integer-valued polynomials of degree \( \geq 2 \) solved by Kamke [Kam21] should be a trivial consequence of the above question for \( G = \mathbb{Z} \). The goal is to find a uniform notion of polynomial
maps from nonempty semigroups to groups that would allow us to define both polynomial sequences \( \mathbb{N}_0 \to \mathcal{U}_n(\mathbb{Z}) \) and generalized cones as the image of continuous polynomial maps \( \mathbb{R}_{\geq 0}^N \to \mathcal{U}_n(\mathbb{R}) \) with nonempty interiors, where \( \mathcal{U}_n(\mathbb{R}) \) is the group of \( n \times n \) unipotent matrices over \( \mathbb{R} \).

**Work of Leibman.** Leibman developed a theory of polynomial sequences \( \mathbb{Z} \to G \) in any group \( G \) [Lei98] and polynomial mappings \( G \to F \) between two groups [Lei02]. Let \( G = G_1 \supseteq G_2 \supseteq G_2 \supseteq \cdots \) be the lower central series of \( G \). In [Lei98], he defines a difference operator \( Dg(n) = g(n)^{-1}g(n+1) \) on the sequence \( g : \mathbb{Z} \to G \) and calls \( g \) a polynomial if for any \( k \) there exists \( d \) such that the sequence obtained from \( g \) by applying this difference operator \( d \) times takes its values in \( G_k \), i.e., \( D^d g(n) \in G_k \) for all \( n \in \mathbb{Z} \). Then, he introduces the notion of the degree of a polynomial sequence and proves that polynomial sequences of degrees not exceeding a fixed superadditive sequence form a group with group law defined elementwise.

In [Lei02], given any \( h \in G \), he defines the left (resp. right) \( h \)-derivative of a mapping \( \varphi : G \to F \) between two groups by \( D_h^L \varphi(g) = \varphi(hg)\varphi(g)^{-1} \) (resp. \( D_h^R \varphi(g) = \varphi(g)^{-1}\varphi(gh) \)), and calls \( \varphi \) a left-polynomial (resp. right-polynomial) mapping of degree \( \leq d \) if for all \( h_1, \ldots, h_{d+1} \in G \), \( D_{h_1}^L \cdots D_{h_{d+1}}^L \varphi \equiv 1_F \) (resp. \( D_{h_1}^R \cdots D_{h_{d+1}}^R \varphi \equiv 1_F \)). Then, he proves that if \( F \) is nilpotent, right-polynomial mappings \( G \to F \) form a group with group law defined elementwise, and \( \varphi : G \to F \) is a right-polynomial if and only if it is a left-polynomial. However, the degree of polynomial map \( f \) as a right-polynomial is not necessarily the same as the degree of \( f \) as a left-polynomial.

**Our generalization.** To meet our own needs, we modify and generalize Leibman’s theory. In particular, a polynomial of degree \( d \) should be killed by any sequence of \( d + 1 \) difference operators, left, right, or a combination of the two. The difficulty which this definition is intended to meet is the unavailability of inverses in general semigroups. The most important quantity of a polynomial map is its lc-degree, which is a vector formed by the degree of the induced polynomial map modulo lower central series of the target group and conveys more information than the degree. A polynomial map is its lc-degree, which is a vector formed by the degree of the induced polynomial map modulo lower central series of the target group and conveys more information than the degree. A polynomial sequence obtained from \( g \) by applying this difference operator \( d \) times takes its values in \( G_k \), i.e., \( D^d g(n) \in G_k \) for all \( n \in \mathbb{Z} \). Then, he introduces the notion of the degree of a polynomial sequence and proves that polynomial sequences of degrees not exceeding a fixed superadditive sequence form a group with group law defined elementwise.

**Main results.** Just as a polynomial \( \mathbb{R} \to \mathbb{R} \) of degree \( d \) is in general determined by \( d + 1 \) polynomial values, a polynomial map is uniquely determined by certain special values and \( \langle f(S) \rangle \) is finitely generated if \( S \) is finitely generated. The set of polynomial maps from \( S \) to \( G \) is invariant under conjugations in \( G \), translations in both \( S \) and \( G \), and taking elementwise inverse. In particular, the fact that the elementwise inverse \( f^{-1} : S \to G \) is also a polynomial map of the same degree resembles the fact that the additive inverse \(-f \) of a polynomial \( f \in R[x] \) is a polynomial of the same degree with \( f \). (Leibman’s left or right polynomial does not have such a nice property.)

Elementwise product of two polynomial maps \( S \to G \) may not be a polynomial map. The simplest example might be provided by the multiplicative functions \( f_1(n) = x^n \) and \( f_2(n) = y^n \) from \( \mathbb{N} \) to the free group \( F_2 \) generated by two generators \( x, y \). Even in metabelian (let alone solvable) groups, the situation is still unpleasant; cf. Example 1 and the remark following it. The first main result is about elementwise product of polynomial maps:
Theorem 1. Let $S$ be any nonempty commutative semigroup, $G$ be any group and $f, f' : S \to G$ be polynomial maps of degree $\leq d$ and respectively $\leq d'$. If the subgroup $\langle f, f' \rangle$ generated by $f(S)$ and $f'(S)$ is nilpotent, then the (elementwise) product

$$ff' : S \to G; \quad t \mapsto f(t)f'(t)$$

is a polynomial map.

It follows that if $G$ is nilpotent of class $n$, then all polynomial maps from $S$ to $G$ form a nilpotent group of class $n$, cf. Corollary 6. Leibman proves something similar: if $F$ is nilpotent, right-polynomial mappings $G \to F$ form a group, cf. [Lei02, Theorem 3.2]. Both of these proofs are elementary and essentially done by induction on the (lc-)degree of the polynomial map and the nilpotency class of the target group. Moreover, Leibman proves that if $F$ is nilpotent of class $c$, then $\varphi : G \to F$ is a right-polynomial if and only if $\varphi$ is a left-polynomial, cf. [Lei02, Proposition 3.16]. In fact, he proves if $\varphi$ is a right-polynomial of degree $\leq d$, then $\varphi$ is a left-polynomial of degree $\leq dc^2$. But if the target group is locally nilpotent but not nilpotent, this statement does not necessarily hold; see Example 2 and Remark 1 for more detail. It is not the purpose of this paper to distinguish the slight difference between polynomial maps and left or right polynomial mappings.

Polynomial maps $\mathbb{R}_{\geq 0} \to \mathbb{R}$ in our sense are not necessarily given by polynomials in the usual sense. Discontinuous additive functions $\mathbb{R}_{\geq 0} \to \mathbb{R}$ provide such pathological examples of polynomial maps of degree 1; cf. Remark 2. This can be avoided if the continuity is required. Indeed, Theorem 4 shows that every continuous polynomial map $f : \mathbb{R}_{\geq 0}^N \to \mathbb{R}$ is the usual polynomial. Theorem 5 shows that for $1 \leq i < j \leq n$ each entry $f_{i,j}$ of the matrix form of any continuous polynomial map $f : \mathbb{R}_{\geq 0}^N \to \mathcal{U}_n(\mathbb{R})$ is a polynomial $f_{i,j} : \mathbb{R}_{\geq 0}^N \to \mathbb{R}$.

The most important quantity of a polynomial map is its (lc-)degree. So we are very interested in the lower and upper bounds of the (lc-)degree, in particular, of polynomial maps of the form $\mathbb{R}_{\geq 0}^N \to \mathcal{U}_n(\mathbb{R})$. Theorems 8 and 9 gives lower and upper bounds of the (lc-)degree of $f : \mathbb{R}_{\geq 0}^N \to \mathcal{U}_n(\mathbb{R})$ via the degree of $f_{i,j}$.

Theorem 12 states that a nonconstant polynomial sequence $g : \mathbb{N}_0 \to G$ in a finitely generated torsion-free nilpotent group $G$ cannot repeat the same value infinitely many times. Theorem 13 states that every infinite subsequence (not necessarily corresponding to any arithmetic progression) in any nilpotent group generates a finite index subgroup of the group generated by the whole sequence.

Denote the direct sum of $N$ copies of a commutative semigroup $S$ by $S^N$ and the set of all polynomial maps from $S^N$ to a group $G$ by $G_{p}^{S^N}$, on which the symmetric group $S_N$ naturally acts. Then, we call a polynomial map $f : S^N \to G$ symmetric with respect to this $S_N$-action, if it is invariant under this action. The strategy that we call the \textit{iterated symmetrization} enables us to prove Theorem 15, which guarantees that any polynomial map $f : S^N \to G$, where $G$ is nilpotent of class $n$, can be turned into a symmetric polynomial map $\tilde{f} = \sigma_1(f)\sigma_2(f)\cdots\sigma_M(f)$, where $\sigma_1, \sigma_2, \ldots, \sigma_M \in S_N$. Results in this section will lay the foundation for our work on Waring’s problem in discrete Heisenberg groups.

Organization. In Section 2, we generalize the usual polynomials to polynomial maps from a nonempty commutative semigroup $S$ to a group $G$. Proposition 1 and Corollary 2 characterize
polynomial maps of degree 1. Two possible ways to construct induced polynomial maps via homomorphisms of semigroups or groups are given in Propositions 3 and 4 respectively. Then, we give an elementary proof of Theorem 1. We also prove that polynomial maps from a commutative semigroup to a nilpotent group with lc-degree bounded by a fixed superadditive vector form a nilpotent subgroup; cf. Corollary 7. Apart from elementwise product, we also talk about ordered product $f \odot f' : S \times S' \to G$ of two polynomial maps $f : S \to G$ and $f' : S' \to G$ given by $f \odot f'(s, s') = f(s)f'(s')$ and prove that the ordered product $\bigodot_{i=1}^{k} f$ of polynomial map $f : S \to G$ with lc-degree bounded by a fixed superadditive vector is a polynomial map with lc-degree bounded by the same superadditive vector.

Section 3 is devoted to a characterization of (continuous) polynomial maps in several variables, such as $f : \mathbb{R}^N_{\geq 0} \to \mathbb{R}$ and $f : \mathbb{R}^N_{\geq 0} \to \mathcal{U}_n(\mathbb{R})$.

Section 4 provides estimations of lower and upper bounds for the (lc-)degree of polynomial maps $f : \mathbb{R}^N_{\geq 0} \to \mathcal{U}_n(\mathbb{R})$.

Section 5 consists of basic results about polynomial sequences and subsequences in a group $G$, which are polynomial maps $\mathbb{N}_0 \to G$.

Section 6 proves some technical results about symmetric polynomial maps with the help of some 1-cocycles of non-abelian group cohomology.

Section 7 introduces the concept of polynomial sets in nilpotent groups and finds a proper polynomial set inside any Kamke domain, an open subset in $\mathbb{R}^n$ played an important role in Kamke’s work and our future work.

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2. Polynomial Maps

A semigroup $S$ is a set $S$ together with a binary operation $\cdot : S \times S \to S$ that satisfies the associative property: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, $\forall a, b, c \in S$. A monoid $S$ is a semigroup with the identity $e \in S$ such that $e \cdot a = a \cdot e = a$, $\forall a \in S$. If the binary operation is commutative, i.e., $a \cdot b = b \cdot a$, $\forall a, b \in S$, then the semigroup (or monoid) $S$ is called commutative, and the binary operation is denoted by $+$ and the identity is denoted by 0.

Remark. Vacuously, the empty set with the empty function as the binary operation forms an empty semigroup. All semigroups mentioned in this paper are assumed to be nonempty.

The rank of a semigroup $S$ is the smallest cardinality of a generating set for the semigroup, i.e.,

$$\text{rank}(S) = \min\{|X| : X \subseteq S, \langle X \rangle = S \text{ or } \langle X \rangle = S \setminus \{\text{the unity element}\}\}.$$ 

Thus, the rank of a finitely generated semigroup is the minimal number of elements generating this semigroup either by themselves or after the addition of the unity element. A commutative semigroup $(S, +)$ is called divisible, if for any $s \in S$ and any $n \in \mathbb{N}$, there exists $t \in S$ such that $nt = s$, and is called uniquely divisible, if $t$ is unique.

We adopt the following conventions on commutators and conjugations in any group $G$: The commutator of the elements $x, y$ in a group $G$ is defined by $[x, y] := xyx^{-1}y^{-1}$, the $y$-conjugate of
x in G by \( x^y := yxy^{-1} \), the \( n \)-fold left-commutator of \( x_1, x_2, \ldots, x_n \) in G by
\[
[x_1, x_2, \ldots, x_n] := [[\ldots[x_1, x_2], \ldots, x_{n-1}], x_n],
\]
and the 1-fold left-commutator of \( x \) simply by \([x] := x.\)

The following commutator identities will greatly facilitate calculations related to commutators.

**Lemma 1.** Let \( G \) be a group and \( x, y, z, x_1, \ldots, x_n \in G \). Then, the following identities hold:

1. \( x^y = [y, x]x \);
2. \( [x, y]^{-1} = [y, x] \);
3. \( [x^{-1}, y] = [x^{-1}, [y, x]]/[y, x] = [y, x]^x \);
4. \( [x, yz] = [x, y][y, [x, z]]/[x, z] = [x, y][x, z]^y \);
5. \( [xy, z] = [x, [y, z]][y, z][x, z] = [y, z]^x[x, z] \);
6. \( [x_1, \ldots, x_n]^y = [x_1^y, \ldots, x_n^y] \);
7. \( [x^{-1}, y, z]^x[y^{-1}, z, x]^y = 1 \) and \( [y, x, z]^x[x, z, y]^y = 1 \).

The Identity (7) is also known as the Hall-Witt identity.

**Proof.** The verification of these statements is routine. \( \square \)

Let \( X \) and \( Y \) be subsets of a group \( G \). The commutator subgroup of \( X \) and \( Y \) is defined to be \([X, Y] := \{[[x, y] | x \in X, y \in Y]\} \). In particular, the derived or commutator subgroup of \( G \) is defined to be \( G^{(1)} = G' = [G, G] \). Identity (2) implies that the commutator subgroup is symmetric: \([X, Y] = [Y, X] \).

**Definition 1.** We say that a map \( f : S \rightarrow G \) is a polynomial map of degree \( -\infty \) \(^1\) if \( f \) maps \( S \) to the identity \( 1_G \) of \( G \), and \( f \) is a polynomial map of degree 0 if it is a constant but not the identity. Inductively, we say that \( f \) is a polynomial map of degree \( \leq d + 1 \), if for all \( s \in S \) the following left and right forward finite differences
\[
L_s(f) : S \rightarrow G; \; t \mapsto f(s + t)f(t)^{-1} \quad R_s(f) : S \rightarrow G; \; t \mapsto f(t)^{-1}f(s + t)
\]
are polynomial maps of degree \( \leq d \).

We call the minimal \( d \) with this property the degree of the polynomial map. We call \( L \) (resp. \( R \)) the left (resp. right) difference operator. If \( G \) is abelian, then there is no need to distinguish \( L \) from \( R \), so \( D \) is used to denote either one of them.

If \( f : S \rightarrow G \) has degree \( \leq 0 \), then we may abuse notations and simply denote its image by \( f \) and thus any element \( g \in G \) is also viewed as a constant polynomial map \( g : S \rightarrow G \). Let \( Z_s = \mathbb{N}_0 \cup \{-\infty\} \) and adopt the following convention
\[
-\infty < n \quad \text{and} \quad -\infty + n = -\infty = (-\infty) + (-\infty), \forall n \in \mathbb{Z}
\]
to extend the addition in \( \mathbb{N}_0 \) to \( Z_s \), and the following convention
\[
a - b = \begin{cases} a - b, & \text{if } a \geq b, \\ -\infty, & \text{if } a < b, \end{cases} \forall a \in \mathbb{Z}_s, \forall b \in \mathbb{N}_0
\]
to partially extend the subtraction in \( \mathbb{N}_0 \) to \( \mathbb{Z}_s \) and leave \( a - (-\infty) \) undefined.

\(^1\)To be compatible with the definition of the lc-degree and superadditive vectors later, \(-\infty \) turns out to be a better choice for the degree of the zero map than \(-1 \).
Remark. With this convention, the definition of polynomial maps can be summarized as follows: 
if \( f : S \to G \) is a polynomial map of degree \( \leq d \), then for any \( s_1, s_2, \ldots, s_{d+1} \in S \),
\[
D_{s_1} D_{s_2} \cdots D_{s_{d+1}} f \equiv 1_G,
\]
where each \( D \) is arbitrarily taken to be \( L \) or \( R \).

Any nonconstant multiplicative function from \( S \) to \( G \) is a polynomial map of degree 1. Moreover,

**Proposition 1.** If \( S \) is a commutative monoid, then any polynomial map \( f : S \to G \) of degree 1 is a nonconstant affine multiplicative function, i.e., a multiplicative function multiplied by a constant in \( G \) either on the left or on the right.

**Proof.** A nonconstant map \( f : S \to G \) is a polynomial map of degree \( \leq 1 \), if for each \( s \in S \),
\[
L_s(f) : S \to G; \; t \mapsto f(s+ t) f(t)^{-1} \quad R_s(f) : S \to G; \; t \mapsto f(t)^{-1} f(s + t)
\]
are polynomial maps of degree \( \leq 0 \). Thus, we have \( f(s+ t) = l_s f(t) = f(t) r_s \), where \( l_s := L_s(f)(t) \) and \( r_s := R_s(f)(t) \) are constants (\( \neq 1_G \)) for each \( s \in S \). In particular, we have
\[
f(0)^{-1} f(s + t) = f(0)^{-1} f(s) f(s)^{-1} f(t + s) = f(0)^{-1} f(s) f(0)^{-1} f(t),
\]
\[
f(s + t) f(0)^{-1} = f(s + t) f(t)^{-1} f(t) f(0)^{-1} = f(s) f(0)^{-1} f(t) f(0)^{-1}.
\]
Thus, \( f(0)^{-1} f \) and \( f f(0)^{-1} \) are both multiplicative functions. \( \square \)

Just as a polynomial of degree \( d \) is in general determined by \( d + 1 \) polynomial values, certain special values of a polynomial map will suffice to determine it.

**Proposition 2.** Let \( S_0 \) be a set of generators of a commutative monoid \( S \) and \( f : S \to G \) be a polynomial map of degree \( d \). Then, \( f \) is uniquely determined by its values on \( \{0\} \cup S_0^{\leq d} \), where \( S_0^{\leq d} = \{s_1 + s_2 + \ldots + s_i \mid 1 \leq i \leq d, s_1, s_2, \ldots, s_d \in S \} \).

If \( S \) is a commutative semigroup without 0 and is generated by \( S_0 \), then \( f \) is uniquely determined by its values on \( S_0^{\leq d+1} \).

Furthermore, if \( S \) is finitely generated commutative monoid or semigroup, then the subgroup generated by the image of a polynomial map \( f : S \to G \) is also finitely generated.

**Proof.** The proof is by induction on the degree. If \( d = -\infty, 0 \), then \( f(s) \) is a constant for all \( s \in S \). Suppose we have shown this for polynomial maps of degree \( < d \). Then, for any \( s \in S_0 \) and \( t \in S_0^{\leq d-1} \), we have \( s + t \in S_0^{\leq d} \) and
\[
L_s(f)(t) = f(s + t) f(t)^{-1}, \quad R_s(f)(t) = f(t)^{-1} f(s + t)
\]
are polynomial maps of degree \( \leq d - 1 \). Since the values \( f \mid S_0^{\leq d} \) are given, the values \( L_s(f) \mid S_0^{\leq d-1} \) and \( R_s(f) \mid S_0^{\leq d-1} \) are known. The induction hypothesis implies that \( L_s(f) \) and \( R_s(f) \) are uniquely determined for any \( s \in S_0 \). By the lemma below, they are also uniquely determined for any \( s \in S \). Hence, \( f(s) = L_s(f)(0) f(0) = f(0) R_s(f)(0) \) is uniquely determined.

A similar argument applies if \( S \) is a commutative semigroup without 0, except in the last step. We write \( S = \bigcup_{m=1}^{\infty} S_0^{\leq m} \) and proceeds by induction on \( m \). In this case, the values of \( f \) on \( S_0^{\leq d+1} \)

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\(^2\)It is understood that the empty word (i.e., \( i = 0 \)) denotes the identity element 0.
Lemma 2. For any map $f : S \to G$ and any $s_1, s_2, t \in S$, we have
\[
L_{s_1+s_2}(f)(t) = L_{s_1}(f)(s_2+t)L_{s_2}(f)(t) = L_{s_2}L_{s_1}(f)(t)L_{s_1}(f)(t)L_{s_2}(f)(t),
\]
\[
R_{s_1+s_2}(f)(t) = R_{s_1}(f)(t)R_{s_2}(f)(s_1+t) = R_{s_1}(f)(t)R_{s_2}f(t)R_{s_1}R_{s_2}(f)(t).\]

Proof. By direct calculations.

We can construct induced polynomial maps via homomorphism of either semigroups or groups.

Proposition 3. Let $\phi : S_0 \to S_1$ be a homomorphism of commutative semigroups and $f : S_1 \to G$ be a polynomial map of degree $d$. Then, the induced function $f^* = f \circ \phi : S_0 \overset{\phi}{\to} S_1 \overset{f}{\to} G$ is a polynomial map of degree $\leq d$.

In particular, if $\phi : S_1 \to S_1$ is an automorphism of commutative semigroups, then the induced function $f^* = f \circ \phi$ has the same degree as $f$.

Proof. By induction on the degree $d$. In particular, if $\phi : S_1 \to S_1$ is an automorphism, then the degree of $f = f^* \circ \phi^{-1}$ is not larger than the degree of $f^* = f \circ \phi$.

Proposition 4. Let $\phi : G \to H$ be a homomorphism (or antihomomorphism) of groups and $f : S \to G$ be a polynomial map of degree $d$. Then, the induced function $f_* = \phi \circ f : S \overset{f}{\to} G \overset{\phi}{\to} H$ is a polynomial map of degree $\leq d$.

In particular, if $\phi : G \to H$ is an isomorphism (or antiisomorphism) of groups, then $f : S \to G$ is a polynomial map of degree $d$, if and only if $f_*$ is.

Proof. By induction on the degree $d$. In particular, if $\phi : G \to H$ is an (anti)isomorphism, then the degree of $f = \phi^{-1} \circ f_*$ is not larger than the degree of $f_* = \phi \circ f$.

Let $G$ be any group. Set $C^1G = G$ and inductively define $C^{i+1}G$ by $[C^iG, G]$ for all $i \geq 1$. Then, the descending series
\[
G = C^1G \geq C^2G \geq \cdots \geq C^nG \geq \cdots
\]
is the lower central series of $G$. Each $C^nG$ is normal in $G$ and $C^nG/C^{n+1}G$ is contained in the center of $G/C^{n+1}G$. Moreover, the lower central series of a group is graded with respect to commutators, i.e., for every $i, j \geq 1$, we have $[C^iG, C^jG] \leq C^{i+j}G$.

Set $Z_0G = \{1_G\}$ and inductively define $Z_{i+1}G$ to be the subgroup of $G$ such that $Z_{i+1}G/Z_iG = Z(G/Z_iG)$ for all $i \geq 1$. Then, the ascending series
\[
\cdots \geq Z_nG \geq Z_{n-1}G \geq \cdots \geq Z_1G \geq Z_0G = \{1_G\},
\]
is the upper central series of $G$. Each $Z_nG$ is normal in $G$ and $Z_1G$ is the center of $G$.

A group $G$ is said to be nilpotent if $G$ has a lower/upper central series of finite length. The smallest $n$ such that $G$ has a lower/upper central series of length $n$ is called the nilpotency class of $G$. 


Definition 2. A function \( f : S \to G \) from a semigroup \( S \) to a group \( G \) is said to have lc-height \( \geq k \) (relative to \( G \)), if the image of \( f \) lies in \( C^k G \), and is said to have uc-height \( \leq k \) (relative to \( G \)), if the image of \( f \) lies in \( Z_k G \).

Remark. Since \( C^k G \cdot C^{k'} G \subseteq C^{\min\{k,k'\}} G \), the product \( f f' : S \to G \) of a function \( f : S \to G \) of lc-height \( \geq k \) and a function \( f' : S \to G \) of lc-height \( \geq k' \) has lc-height \( \geq \min\{k,k'\} \). Since the lower central series of \( G \) is graded with respect to commutators, the commutator \([f,f']: S \to G\) is a function of lc-height \( \geq k + k' \).

Next, we state and prove a few corollaries of Proposition 4. Since conjugation by a group element is an inner automorphism of the group, we have

**Corollary 1.** For any \( f \in G \), the \( f \)-conjugate of a polynomial map \( f' \) of lc-height \( k' \) and degree \( d' \)

\[
ff'f^{-1} : S \to G; \quad t \mapsto f f'(t)f^{-1}
\]

is a polynomial map of lc-height \( k' \) and degree \( d' \).

Notice that any translation \( T_s(f)(t) := f(t + s) \) of a polynomial map \( f \) of lc-height \( \geq k \) and degree \( \leq d \) by \( s \in S \) is a polynomial map of lc-height \( \geq k \) and degree \( \leq d \).

**Corollary 2.** For any \( f \in G \setminus \{1_G\} \) of lc-height \( \geq k \), the left \( f \)-translation \( ff' \) (resp. the right \( f \)-translation \( f'f \)) of a polynomial map \( f' \) of lc-height \( \geq k' \) and degree \( \leq d' \) is a polynomial map of lc-height \( \geq \min\{k,k'\} \) and degree \( \leq \max\{0,d'\} \).

**Proof.** By induction on the degree \( d' \), we prove this for \( ff' \) and the proof for \( f'f \) is similar. The assertion clearly holds for \( d' = -\infty, 0 \). If \( d' > 0 \), then

\[
L_s(f f')(t) = (f f'(s + t))(f f'(t))^{-1} = f f'(s + t)f'(t)^{-1}f^{-1} = f L_s(f')(t)f^{-1}
\]

is the \( f \)-conjugate of a polynomial map \( L_s(f') \) of degree \( \leq d' - 1 \), and

\[
R_s(f f')(t) = (f f'(t))^{-1}(f f'(s + t)) = f'(t)^{-1}f^{-1}f f'(s + t) = R_s(f')(t)
\]

is a polynomial map of degree \( \leq d' - 1 \). By Corollary 1, \( f L_s(f')f^{-1} \) is a polynomial map of degree \( \leq d' \). The assertion about the lc-height is an easy consequence of the fact \( C^k G \cdot C^{k'} G \subseteq C^{\min\{k,k'\}} G \).

**Remark.** The above corollary shows that the converse of Proposition 1 also holds.

**Corollary 3.** The composition \( \iota \circ f : S \overset{f}{\to} G \overset{\iota}{\to} G \) of a polynomial map \( f \) of lc-height \( k \) and degree \( d \) with the inverse function \( \iota : G \to G \) is a polynomial map of lc-height \( k \) and degree \( d \).

**Corollary 4.** Let \( H \) be a normal subgroup of \( G \) and \( f : S \to G \) be a polynomial map of lc-height \( \geq k \) and degree \( \leq d \). Then, the induced function \( f \mod H : S \overset{f}{\to} G \overset{\pi}{\to} G/H \) is also a polynomial map of lc-height \( \geq k \) and degree \( \leq d \).

If the induced polynomial map \( f \mod H \) is of degree \( \leq 0 \), then \( f \) is a \( g \)-translation of a polynomial map \( h : S \to H \) of degree \( \leq d \), for some \( g \in G \). (In particular, if \( f \mod H \) has degree \(-\infty\), then we can take \( f = h \) and \( g = 1_G \).)

Footnote: Here, lc is short for lower central, and uc is short for upper central.
**Proof.** The first assertion is a trivial consequence of Proposition 4. If the induced polynomial map $f \mod H$ is of degree 0, then the image of $f$ lies in a left coset of $H$, say, $gH$ for some $g \in G \setminus H$. Then, we can write $f = gh$, where $h : S \to H$ is some function; If $f \mod H$ has degree $-\infty$, then we take $f = h$ and $g = 1_G$. If $d = -\infty$, then $f \mod H$ has degree $-\infty$, and if $d \geq 0$, then Corollary 2 implies that $h = g^{-1}f : S \to H$ is a polynomial map of degree $\leq \max\{0, d\} = d$. Hence, $f$ is a left $g$-translation of a polynomial map $h$ of degree $\leq d$. Similarly, $f$ can be written as a right $g$-translation of a polynomial map of degree $\leq d$, for some other $g \in G$. □

**Remark.** Suppose that $f : S \to G$ is a function and the induced function $f_\star = \phi \circ f : S \overset{\phi}{\to} H$ is a polynomial map of degree $\leq d$. Then, what can we say about the function $f : S \to G$? The answer is not much. Even when $d = -\infty$, i.e., $f_\star$ is a constant, we only know that $f : S \to G$ is a function such that its image lies in some fiber $\phi^{-1}(h)$ for some $h \in H$.

A natural question that one may ask is whether the (elementwise) product

$$f_1f_2 : S \to G; t \mapsto f_1(t)f_2(t)$$

of two polynomial maps $f_1 : S \to G$ and $f_2 : S \to G$ a polynomial map? The answer, in general, is no. The simplest example might be provided by the multiplicative functions $f_1(n) = x^n$ and $f_2(n) = y^n$ from $\mathbb{N}$ to the free group $F_2$ generated by two generators $x, y$.

**Example 1.** Inspired by [Lei02, Example in Section 3.1], we provide another beautiful example, which is related to the Fibonacci sequence $F_n$ with $F_1 = F_2 = 1$. Consider the group

$$G = \langle x, y, z \mid [x, y] = 1_G, zxz^{-1} = yx, zyz^{-1} = x \rangle.$$  

Then, $f_1(n) = z^n x$ and $f_2(n) = z^{-n}$ are two polynomial maps from $\mathbb{N}$ to $G$ of degree 1. Let $f$ be the elementwise product of $f_1$ and $f_2$, i.e., $f(n) = f_1(n)f_2(n) = z^n x z^{-n}$. One can show by induction that $f(n) = x^{F_n+1}y^{F_n}$ for all $n$. Therefore, the following identity

$$D_1(f(n)) = f(n + 1)f(n)^{-1} = x^{F_{n+2}y^{F_{n+1}}x^{F_n+1}y^{-F_n}} = x^{F_n}y^{F_n-1} = f(n - 1),$$

implies that $f$ is not a polynomial map from $\mathbb{N}$ to either $G$ or the normal abelian subgroup of $G$ generated by $x$ and $y$. Roughly speaking, $f$ is more like an “exponential map”.

**Remark.** The group $G$ given in the above example has a normal abelian subgroup $H$ generated by $x$ and $y$, such that $G/H$ is a cyclic group generated by the image of $z$, and thus is metabelian, i.e., solvable of derived length 2. Example 1 implies that the product of two polynomial maps in metabelian (let alone solvable) groups may not be a polynomial map.

**Proposition 5.** Let $\Lambda$ be a finite index set. For $\lambda \in \Lambda$, let $S_\lambda$ be a commutative semigroup and $G_\lambda$ a nilpotent group of class $n_\lambda$. Then, the direct sum $\bigoplus_{\lambda \in \Lambda} S_\lambda$ is a commutative semigroup, the direct sum $\bigoplus_{\lambda \in \Lambda} G_\lambda$ is a nilpotent group of class $\max_{\lambda \in \Lambda} \{n_\lambda\}$, and the direct sum

$$\bigoplus_{\lambda \in \Lambda} f_\lambda : \bigoplus_{\lambda \in \Lambda} S_\lambda \to \bigoplus_{\lambda \in \Lambda} G_\lambda$$

of a finite family of polynomial maps $f_\lambda : S_\lambda \to G_\lambda$ of lc-height $\geq k_\lambda$ and degree $d_\lambda$ is a polynomial map of lc-height $\geq \min_{\lambda \in \Lambda} \{k_\lambda\}$ and degree $\max_{\lambda \in \Lambda} \{d_\lambda\}$.

**Proof.** The proof is trivial and thus omitted. □
Given two polynomial maps \( f : S \to G \) of degree \( \leq d \) and \( f' : S \to G \) of degree \( \leq d' \), their product \( ff' : S \to G \) is given by the following composition

\[
ff' : S \xrightarrow{\Delta} S \times S \xrightarrow{f \times f'} G \times G \xrightarrow{m} G,
\]

where \( \Delta : S \to S \times S \) is the diagonal map and \( m : G \times G \to G \) is the multiplication map of \( G \).

Proposition 5 implies that \( f \times f' \) is a polynomial of degree \( \leq \max\{d,d'\} \). From Proposition 4, we see that if \( m : G \times G \to G \) or even \( m : (f \times f') \to G \) is a homomorphism of groups, then the product \( ff' \) will be a polynomial map of degree \( \leq \max\{d,d'\} \). This simple idea allows us to prove the following corollary.

**Corollary 5.** Let \( G \) be a nilpotent group of class \( n \). Given two polynomial maps \( f : S \to G \) of degree \( \leq d \) and \( f' : S \to G \) of degree \( \leq d' \), if \( f \) has lc-height \( \geq k \) and \( f' \) has lc-height \( \geq k' \) with \( k + k' \geq n + 1 \), then the product \( ff' \) has lc-height \( \geq \min\{k,k'\} \) and degree \( \leq \max\{d,d'\} \).

**Proof.** Indeed, for any \( a, b, c, d \in G \), we have \( m((a,b))m((c,d)) = abcd \) and

\[
m((a,b)(c,d)) = m((ac, bd)) = acbd = abc[c^{-1}, b^{-1}]d
\]

\[
= abcd[c^{-1}, b^{-1}][c^{-1}, b^{-1}-1, d^{-1}].
\]

Since \( [C^kG, C^{k'}G] = C^{n+1}G = \{1_G\} \), \( m : C^kG \times C^{k'}G \to G \) is a homomorphism of groups. \( \square \)

Now, we are ready to prove Theorem 1.

**Proof of Theorem 1.** Replacing \( G \) by the subgroup \( \langle f, f' \rangle \), we may assume that \( G \) is nilpotent of class \( n \). The proof is given by double induction with the outer descending induction on \( k + k' \), where \( k, k' \) are lc-heights of the polynomial maps \( f \) and \( f' \) relative to \( G = \langle f, f' \rangle \) respectively, and inner ordinary induction on \( d_f + d_{f'} \), where \( d_f \) and \( d_{f'} \) are respectively the induced degrees of the polynomial maps \( f \mod C^{k+1}G \) and \( f' \mod C^{k'+1}G \).

The idea is to apply the definition of polynomial maps and commutator identities to create polynomial maps of either lower degrees or larger lc-heights and move them to the correct place. Doing so creates extra commutators of larger lc-heights, which is easy to deal with by the induction hypothesis.

To facilitate the proof, the following statements will be proved simultaneously:

1. The product \( ff' : S \to G \) is a polynomial map;
2. The commutator \( [f, f'] : S \to G \) is a polynomial map.

Although it suffices to show (1), the proof is easier if the induction hypothesis contains several claims, each of which depends on the other for larger lc-heights or smaller induced degrees.

Clearly, the theorem holds when \( n \leq 1 \), i.e., when \( G \) is abelian. Indeed, the commutator of two polynomial maps is always a polynomial map of degree \(-\infty\); the product \( ff' \) is a polynomial map of degree \( \leq \max\{d,d'\} \) by Corollary 5. So we may assume that \( n \geq 2 \).

By Corollary 5, if \( k + k' \geq n + 1 \), then the product \( ff' \) is a polynomial map of lc-height \( \geq \min\{k,k'\} \) and degree \( \leq \max\{d,d'\} \), and the commutator \( [f, f'] \) is a polynomial map of lc-height \( n+1 \) and degree \(-\infty\). This gives the outer induction base on the sum \( k + k' \) of lc-heights \( k \) and \( k' \).

Suppose that we have shown this for \( k + k' > m \) with \( 2 \leq m \leq n \). The goal of the outer induction step is to prove the theorem claim holds for \( k + k' = m \). We proceed by the ordinary induction
on $(d_f, d_{f'})$. If either $d_f$ or $d_{f'}$ is $-\infty$, then we are in the case when $k + k' = m + 1$, which has been proved by induction hypothesis. Therefore, we assume that $d_f, d_{f'} \geq 0$.

Then, for the product case, we have

\[
(2.1) \quad (f(s + t)f'(s + t))(f(t)f'(t))^{-1} = f(s + t)f'(s + t)f'(t)^{-1}f(t)^{-1} = f(s + t)L_s(f')(t)f(t)^{-1} = f(s + t)f(t)^{-1}L_s(f')(t)[L_s(f')(t)^{-1}, f(t)] = L_s(f)(t)L_s(f')(t)[L_s(f')(t)^{-1}, f(t)],
\]

and

\[
(2.2) \quad (f(t)f'(t))^{-1}(f(s + t)f'(s + t)) = f'(t)^{-1}f(t)^{-1}f(s + t)f'(s + t) = f'(t)^{-1}R_s(f)(t)f'(s + t) = \left[f'(t)^{-1}, R_s(f)(t)\right]R_s(f)(t)f'(s + t) = \left[f'(t)^{-1}, R_s(f)(t)\right]R_s(f)(t)R_s(f')(t).
\]

By the inner induction hypothesis, $[L_s(f')^{-1}, f]$ and $[f'^{-1}, R_s(f)]$ are polynomial maps of lc-height $\geq k + k' = m$, and $L_s(f)L_s(f')$ and $R_s(f)R_s(f')$ are polynomial maps of lc-heights $\geq \min\{k, k'\} \geq 1$, and thus $L_s(f)L_s(f')[L_s(f')^{-1}, f]$ and $[f'^{-1}, R_s(f)]R_s(f)(R_s(f'))$ are polynomial maps of lc-height $\geq \min\{k, k'\}$. It follows that $ff'$ is a polynomial map of lc-height $\geq \min\{k, k'\}$.

Next, we deal with the commutator case. We have

\[
(2.3) \quad \left[f(s + t), f'(s + t)\right][f(t), f'(t)]^{-1} = f(s + t)f'(s + t)f(s + t)^{-1}f'(s + t)^{-1}f(t)^{-1}f'(t)^{-1}f(t) = f(s + t)f'(s + t)f(s + t)^{-1}R_s(f')(t)^{-1}f'(t)^{-1}f(t) = f(s + t)f'(s + t)f(s + t)^{-1}R_s(f')(t)^{-1}R_s(f')(t)^{-1}C_1(t)f'(t)^{-1}f(t) = f(s + t)f'(s + t)R_s(f)(t)^{-1}R_s(f')(t)^{-1}C_1(t)f'(t)^{-1}f(t) = f(s + t)f'(s + t)R_s(f)(t)^{-1}R_s(f')(t)^{-1}C_1(t)f'(t)^{-1}f(t) = f(s + t)f'(s + t)R_s(f)(t)^{-1}f'(t)^{-1}C_2(t)f(t) = f(s + t)f'(s + t)R_s(f)(t)^{-1}f'(s + t)^{-1}C_2(t)f(t) = f(s + t)f'(s + t)^{-1}R_s(f)(t)^{-1}C_3(t)f(t) = f(t)C_3(t)f(t)^{-1} = C_4(t),
\]

where by induction hypothesis

- $C_1(t) = [R_s(f')(t), f(t)^{-1}]$, $C_2(t) = C_1(t)[C_1(t)^{-1}, f'(t)]$,
- $C_3(t) = [R_s(f)(t), f'(s + t)]C_2(t)$, $C_4(t) = [f(t), C_3(t)]C_3(t)$

are polynomial maps of lc-height $\geq k + k'$, and

\[
(2.4) \quad [f(t), f'(t)]^{-1}[f(s + t), f'(s + t)] = f'(t)f(t)f'(t)^{-1}f(t)^{-1}f(s + t)f'(s + t)^{-1}f'(s + t)^{-1} = f'(t)f(t)f'(t)^{-1}R_s(f)(t)f'(s + t)f(s + t)^{-1}f'(s + t)^{-1}
\]
= f'(t)f(t)f'(t)^{-1}f'(s + t)^{-1}R_a(f)(t)D_1(t)f(s + t)^{-1}f'(s + t)^{-1}
= f'(t)f(t)R_a(f')(t)R_a(f)(t)D_1(t)f(s + t)^{-1}f'(s + t)^{-1}
= f'(t)f(t)R_a(f')(t)R_a(f)(t)f(s + t)^{-1}D_2(t)f'(s + t)^{-1}
= f'(t)f(t)f(t)^{-1}R_a(f')(t)D_3(t)f'(s + t)^{-1}
= f'(s + t)D_3(t)f'(s + t)^{-1} = D_4(t),

where by induction hypothesis

\[ D_1(t) = [R_a(f)(t)^{-1}, f'(s + t)^{-1}], \quad D_2(t) = D_1(t)[D_1(t)^{-1}, f(s + t)], \]
\[ D_3(t) = [R_a(f')(t)^{-1}, f(t)]D_2(t), \quad D_4(t) = [f'(s + t), D_3(t)]D_3(t). \]

are polynomial maps of lc-height \( \geq k + k' \). So \([f, f']\) is a polynomial map of lc-height \( \geq k + k' \).

\( \square \)

**Remark.** Since each subgroup of a nilpotent group is nilpotent, Theorem 1 holds when \( G \) is nilpotent.

**Corollary 6.** Let \( S \) be a commutative semigroup and \( G \) a nilpotent group of class \( n \). The set \( G^S_p \) of all polynomial maps \( f : S \to G \) forms a nilpotent group of class \( n \), with group law given by elementwise multiplication. In particular, \( G \) can be viewed as a subgroup of \( G^S_p \).

**Proof.** It is an immediate consequence of Theorem 1 that \( G^S_p \) is a nilpotent group of class \( \leq n \). The subset of all constant polynomial maps is easily seen to be a subgroup, which is isomorphic to \( G \). Thus, the nilpotency class of \( G^S_p \) is at least \( n \) and hence must be exactly \( n \).

\( \square \)

**Example.** Up to isomorphism, there is only one semigroup \( S \) with one element, i.e., the singleton \{s\} with operation \( s \cdot s = s \). Then, any polynomial \( f : S \to G \) has degree \( \leq 0 \). If one identifies \( f \) with its image \( f(s) \), then the nilpotent group \( G^S_p \) is seen to be isomorphic to \( G \).

Recall that a group is \( G \) is said to be locally nilpotent, if every finitely generated subgroup of \( G \) is nilpotent. Then, subgroups and quotient groups of a locally nilpotent group are locally nilpotent and the external product of two locally nilpotent groups is locally nilpotent. Since finitely generated subgroups of a nilpotent group are nilpotent, nilpotent groups are locally nilpotent. Here is an example of a locally nilpotent group which is not nilpotent.

**Example.** Let \( p \) be a prime number. The Prüfer \( p \)-group \( \mathbb{Q}_p/\mathbb{Z}_p \) can be viewed as the direct limit

\[ \mathbb{Q}_p/\mathbb{Z}_p = \lim_{k \to \infty} \mathbb{Z}/p^k = \lim_{k \to \infty} \left( \mathbb{Z}/p \leftarrow \mathbb{Z}/p^2 \leftarrow \mathbb{Z}/p^3 \leftarrow \cdots \leftarrow \mathbb{Z}/p^k \leftarrow \cdots \right) \), \]

where the embedding \( \mathbb{Z}/p^k \to \mathbb{Z}/p^{k+1} \) is induced by multiplication by \( p \). A presentation of \( \mathbb{Q}_p/\mathbb{Z}_p \) is given by \( \langle g_1, g_2, g_3, \ldots \mid g_1^p = 1, g_2^p = g_1, g_3^p = g_2, \ldots \rangle \), where the group operation is written as multiplication. Then, each element of \( \mathbb{Q}_p/\mathbb{Z}_p \) has \( p \) different \( p \)-th roots.

For any abelian group \( H \), the generalized dihedral group corresponding to \( H \)

\[ \text{Dih}(H) = \langle H, s \mid s^2 = (sh)^2 = 1, \forall h \in H \rangle, \]
can be viewed as the semidirect product $H \rtimes \varphi \mathbb{Z}/2$, where $\varphi : \mathbb{Z}/2 = \langle s \mid s^2 = 1 \rangle \to \text{Aut}(H)$ is a homomorphism given by $\varphi(s)(h) = shs^{-1} = shs = h^{-1}, \forall h \in H$. The dihedral groups $D_{2n} = \text{Dih}(\mathbb{Z}/n) = \langle r, s \mid r^n = s^2 = (sr)^2 = 1 \rangle$ are special cases of generalized dihedral groups and $D_{2n}$ is nilpotent if and only if it has order $2n = 2^k$ for some $k \in \mathbb{N}$, and $D_{2,1} = \mathbb{Z}/2$ and $D_{2,2} = \mathbb{Z}/2 \times \mathbb{Z}/2$ are the only abelian ones. Moreover, $D_{2,2^k}$ is of nilpotency class $k$ for $k \geq 1$.

Then, the generalized dihedral group $\text{Dih}(\mathbb{Q}_2/\mathbb{Z}_2) = \mathbb{Q}_2/\mathbb{Z}_2 \rtimes \mathbb{Z}/2$ corresponding to the Prüfer 2-group is an example of a locally nilpotent group which is not nilpotent. Then, one sees that

$$\lim_{k \to \infty} D_{2,2^k} = \lim_{k \to \infty} (\mathbb{Z}/2^k \rtimes \mathbb{Z}/2) \leq (\lim_{k \to \infty} \mathbb{Z}/2^k) \rtimes \mathbb{Z}/2 = \text{Dih}(\mathbb{Q}_2/\mathbb{Z}_2).$$

It is locally nilpotent as every finitely generated subgroup must be contained in some finite nilpotent subgroup $D_{2,2^k} = \mathbb{Z}/2^k \rtimes \mathbb{Z}/2$ and thus is nilpotent. Since one can find nilpotent subgroups $D_{2,2^k}$ of nilpotency class $k$ for arbitrarily large $k$, $\text{Dih}(\mathbb{Q}_2/\mathbb{Z}_2)$ cannot be nilpotent.

A trivial consequence of Theorem 1 is that the product of two polynomial maps $f_1, f_2 : S \to G$ in a locally nilpotent group $G$ is a polynomial map, if the subgroup $\langle f_1, f_2 \rangle$ generated by $f_1(S)$ and $f_2(S)$ is finitely generated. Moreover, we have

**Proposition 6.** If $g : S \to G$ is a polynomial map from a finitely generated commutative semigroup $S$ to a locally nilpotent group $G$, then the subgroup $\langle g \rangle$ is finitely generated and nilpotent.

**Proof.** This follows easily from the Proposition 2 and the definition of a locally nilpotent group. $\square$

But if the commutative semigroup is not finitely generated, we can construct an example such that the product of two polynomial maps in locally nilpotent groups may not be a polynomial map. The motivation is that the alternating sequence $f : \mathbb{N} \to \mathbb{Z}$ sending $n$ to $(-1)^n$ cannot be a polynomial map. Indeed, $D_1 f(n) = 2 \cdot (-1)^{n-1}$, $D_1 D_1 f(n) = 2^2 \cdot (-1)^n$, etc.

**Example 2.** Let $F_\mathbb{N}$ be the free abelian group on the generators $x_1, x_2, \ldots$. Then, each element $x = \sum_{i=1}^{\infty} n_i x_i \in F_\mathbb{N}$ can be uniquely written as an infinite dimensional vector $(n_1, n_2, \ldots)$ with only finitely many nonzero $n_i$. Let $\varepsilon : F_\mathbb{N} \to \mathbb{Z}$ be the augmentation given by $\varepsilon(\sum_{i=1}^{\infty} n_i x_i) = \sum_{i=1}^{\infty} n_i$.

Let $\langle g_1, g_2, g_3, \ldots \mid g_1^2 = 1, g_2^2 = g_1, g_3^2 = g_2, \ldots \rangle$ be a presentation of the Prüfer 2-group $\mathbb{Q}_2/\mathbb{Z}_2$. Consider the homomorphism $\varphi_0 : F_\mathbb{N} \to \text{Dih}(\mathbb{Q}_2/\mathbb{Z}_2) = \mathbb{Q}_2/\mathbb{Z}_2 \rtimes \mathbb{Z}/2$ such that $\varphi_0(x_i) = (g_i, 0)$ and the following composition of homomorphisms

$$\varphi_1 : F_\mathbb{N} \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{2} \mathbb{Z}/2 \hookrightarrow \mathbb{Q}_2/\mathbb{Z}_2 \rtimes \mathbb{Z}/2,$$

where the last arrow embeds $\mathbb{Z}/2$ as the second factor of $\mathbb{Q}_2/\mathbb{Z}_2 \rtimes \mathbb{Z}/2$, i.e., $\varphi_1(x) = (0, \varepsilon(x) \mod 2)$. Then, $\varphi_0$ and $\varphi_1$ are polynomial maps of degree 1. The subgroup $\langle \varphi_0 \rangle$ is isomorphic to $\mathbb{Q}_2/\mathbb{Z}_2$, which is not finitely generated, and the subgroup $\langle \varphi_1 \rangle$ is $\{0\} \rtimes \mathbb{Z}/2$.

Then, the product $\varphi = \varphi_0 \varphi_1 : F_\mathbb{N} \to \text{Dih}(\mathbb{Q}_2/\mathbb{Z}_2)$ is not a polynomial map. Indeed, for any $s = \sum_{i=1}^{\infty} s_i x_i \in F_\mathbb{N}$, we have $\varphi(s) = \varphi_0(s) \varphi_1(s) = (\prod g_i^{s_i}, \varepsilon(s) \mod 2)$. Then, for any $t = \sum_{i=1}^{\infty} t_i x_i$, we have the following:

$$L_s(\varphi)(t) = \varphi(s + t)\varphi(t)^{-1} = \begin{cases} \left( \prod g_i^{s_i}, \varepsilon(s) \mod 2 \right), & \text{if } \varepsilon(s) \equiv 0 \mod 2, \\ \left( \prod g_i^{s_i+2t}, \varepsilon(s) \mod 2 \right), & \text{if } \varepsilon(s) \equiv 1 \mod 2, \end{cases}$$

$\text{In the sequel, } \varphi \text{ will be understood and omitted.}$
\[ R_s(\varphi)(t) = \varphi(t)^{-1} \varphi(s + t) = \begin{cases} 
(\prod g_i^{s_i}, 0), & \text{if } n \equiv 0 \mod 2, \\
(\prod g_i^{-s_i}, 0), & \text{if } n \equiv 1 \mod 2, 
\end{cases} \]

In particular, fixing \( i \in \mathbb{N} \) and \( s \in F_{\mathbb{N}} \) such that \( \varepsilon(s) = 0 \mod 2 \), choosing \( t \) among the sequence \( x_i, 2x_i, 3x_i, \ldots, nx_i, \ldots \), we have

\[ R_s(\varphi)(nx_i) = \begin{cases} 
(\prod g_i^{2^m s_i}, 0), & \text{if } n \equiv 0 \mod 2, \\
(\prod g_i^{2^m s_i}, 0), & \text{if } n \equiv 1 \mod 2, 
\end{cases} \]

where \( D \) is either \( L \) or \( R \). Then, repeating this for \( 2m \) times, we see that

\[ D_{x_1} \cdots D_{x_1} R_s(\varphi)(nx_i) = (1, 0), \text{ for all sufficiently large } m, \]

because \( \varphi_0(s) = \prod g_i^{s_i} \) and its inverse have finite order. However, \( \varphi_0(s) = (\prod g_i^{s_i}, 0) \) may have arbitrarily large finite order. This implies that \( \varphi(t) \) is not a polynomial map.

**Remark 1.** Notice that \( L_s(\varphi) \) behaves in a completely different way when we apply difference operators to it. In fact, \( L_s(\varphi) \) is a polynomial of degree \( \leq 1 \) for any \( s \in F_{\mathbb{N}} \), and thus \( \varphi \) is a left polynomial of degree \( \leq 2 \). If \( \varepsilon(s) \equiv 0 \mod 2 \), we have \( L_s(\varphi)(t) = (\prod g_i^{s_i}, 0) \) and thus \( D_u L_s(\varphi)(t) = (1, 0) \) for any \( u \in F_{\mathbb{N}} \). If \( \varepsilon(s) \equiv 1 \mod 2 \), we have \( L_s(\varphi)(t) = (\prod g_i^{s_i + 2u}, 1) \) and

\[ L_u L_s(\varphi(t)) = \left( \prod g_i^{2u_i}, 0 \right), \quad R_u L_s(\varphi(t)) = \left( \prod g_i^{-2u_i}, 0 \right). \]

Thus, for any \( v \in F_{\mathbb{N}} \), we have \( D_v L_u L_s(\varphi(t)) = (1, 0) \) and \( D_v R_u L_s(\varphi(t)) = (1, 0) \).

Following Leibman’s definition, we may also define the left-polynomial and right-polynomial from a commutative semigroup \( S \) to any group \( G \) in the same manner. These definitions coincide if \( S \) is an abelian group. Then, the above example shows that \( \varphi \) is a left-polynomial of degree \( \leq 2 \), but not a right-polynomial.

Notice that in Theorem 1, we have no estimate of the degree of the product of two general polynomial maps. But with the following concept of lc-degree, we may strengthen Corollary 6.

**Definition 3.** Let \( G \) be a nilpotent group of class \( n \) and \( f : S \to G \) be a polynomial map of degree \( d \). By Corollary 4, \( f \mod C_i G \) is a polynomial map of degree \( \leq d \) for each \( i = 1, 2, \ldots, n \). Let \( d_i \) be the degree of \( f \mod C_i G \), i.e., \( d_i \) is least number in \( Z_s \) such that for any \( s_1, s_2, \ldots, s_{d_i+1} \in S \),

\[ D_{s_1} D_{s_2} \cdots D_{s_{d_i+1}} f(S) \subseteq C^{d_i+1} G. \]

Then, \( \hat{d} = (d_1, \ldots, d_n) \in \mathbb{Z}_+^n \) will be called the lc-degree of \( f \).

Leibman [Lei02] has a slightly different definition for lc-degree using the superadditive vectors:
Definition. A vector $\bar{d} = (d_1, \ldots, d_n) \in \mathbb{Z}_n^+$ is said to be superadditive, if $d_i \leq d_j$ for all $i \leq j$, and $d_i + d_j \leq d_{i+j}$, for all $i, j$ with $i + j \leq n$.

With the obvious lexicographical order on $\mathbb{Z}_n^+$, there exists a unique smallest superadditive vector $\bar{d} \geq \bar{d} = (d_1, \ldots, d_n)$ and Leibman calls $\bar{d}$ the lc-degree of $f$. To distinguish them, our lc-degree $\bar{d}$ does not have to be superadditive.

Then, for any $c \in \mathbb{N}_0$, we see that $(a - b) - c = a - (b + c)$. For any superadditive vector $\bar{d} = (d_1, \ldots, d_n)$ in $\mathbb{Z}_n^*$ and any $b \in \mathbb{N}_0$, we define $\bar{d} - b = (d_1 - b, \ldots, d_n - b)$. Then, we see that if $\bar{d}$ is superadditive, then $\bar{d} - b$ is also superadditive, and $(\bar{d} - b) - c = \bar{d} - (b + c)$. However, if $\bar{d}$ is superadditive, $\bar{d} + a = (d_1 + a, \ldots, d_n + a)$ may not be superadditive.

Now let us focus on our definition of lc-degree. Notice that $d_i \leq d_j$ for all $i \leq j$ and that $d_i = -\infty$, if and only if $f(S) \subseteq C^{i+1} G$, if and only if the lc-height of $f$ is $\geq i + 1$. If the lc-degree of $f$ is $\bar{d} = (d_1, \ldots, d_n)$, then the degree of $f$ is $d_n$; conversely, if the degree of $f$ is $\bar{d}$, then $d_i \leq (d, d, \ldots, d)$. Also, if $f : S \to G$ is a polynomial map of lc-degree $\leq \bar{d}$, then $L_s(f)$ and $R_s(f)$ are polynomial maps of lc-degree $\leq \bar{d} - 1$ for any $s \in S$. By Corollary 3, if $f$ mod $C^{i+1} G$ has degree $\leq d_i$, then $f^{-1}$ mod $C^{i+1} G$ has degree $\leq d_i$ and vice versa. So $f$ and $f^{-1}$ have the same lc-degree.

Moreover, for all $s \in S$ if both $L_s(f)$ mod $C^{i+1} G$ and $R_s(f)$ mod $C^{i+1} G$ have degree $\leq d_i$, then what can we say about the degree of $f$ mod $C^{i+1} G$? Well, if $d_i \geq 0$, then the degree of $f$ mod $C^{i+1} G$ must be $\leq d_i + 1$ and equality holds if at least one of $L_s(f)$ mod $C^{i+1} G$ and $R_s(f)$ mod $C^{i+1} G$ has degree $d_i$ for some $s \in S$; if $d_i = -\infty$, which means that $L_s(f)(t)$ and $R_s(f)(t)$ all lie in $C^{i+1} G$ and thus $f(s + t) \equiv f(t)$ mod $C^{i+1} G$ for all $s, t \in S$, then the degree of $f$ mod $C^{i+1} G$ must be $\leq 0$. So if $L_s(f)$ and $R_s(f)$ are polynomial maps of lc-degree $\leq \bar{d}' = (d_1', \ldots, d_n')$ for all $s \in S$, then $f$ is a polynomial map of lc-degree $\leq \bar{d} = (d_1, \ldots, d_n)$, where

$$d_i = \begin{cases} d_i' + 1, & \text{if } d_i' \geq 0, \\ 0 & \text{if } d_i' = -\infty. \end{cases}$$

So even if one starts with a superadditive vector $\bar{d}'$, there is no reason to expect $\bar{d}$ to be superadditive, because it may happen that $d_i + d_j = d_i' + 1 + d_j' + 1 \leq d_i' + 1 = d_i + j$.

Furthermore, if $f$ has lc-degree $\bar{d} = (d_1, \ldots, d_n)$ and $f'$ has lc-degree $\bar{d}' = (d_1', \ldots, d_n')$, then what can we say about $[f, f']$? Let $i$ (resp. $j$) be the largest index such that $d_i \geq 0$ (resp. $d_j' \geq 0$). Then, we see that $f$ has lc-height $i$ and $f'$ has lc-height $j$. Hence, $[f, f']$ has lc-height $i + j$.

Now we are ready to strengthen Corollary 6, whose proof is similar to the one of Theorem 1 and to the one of [Lei02, Proposition 3.4].

Corollary 7. Let $S$ be a commutative semigroup and $G$ a nilpotent group of class $n$. The set of all polynomial maps $f : S \to G$ with lc-degree $\bar{d}$ bounded above by some fixed superadditive vector $\bar{d} = (d_1, \ldots, d_n)$, i.e., $\bar{d} \leq \bar{d}$, forms a nilpotent subgroup of $G^S_n$.

Proof. Let $b, c, c' \in \mathbb{N}_0$. We will prove the following statements by descending induction on $b$:

1. If $f, f'$ are polynomial maps of lc-degree $\leq \bar{d} - b$, then $f f'$ is a polynomial map of lc-degree $\leq \bar{d} - b$;
2. If $f, f'$ are polynomial maps of lc-degree $\bar{d} \leq \bar{d} - c$ and $\bar{d}' \leq \bar{d} - c'$, then $[f, f']$ is a polynomial map of lc-degree $\leq \bar{d} - (c + c')$. 


(3) If $f$ is a polynomial map of lc-degree $\hat{d} \leq \bar{d} - b$, then $f^{-1}$ is a polynomial of lc-degree $\hat{d} \leq \bar{d} - b$.

Notice that the last statement has been proved, since $f$ and $f^{-1}$ have the same lc-degree. Also, if $b$ is large enough ($b > d_n$), then a polynomial map has lc-degree $\leq \bar{d} - b = (\infty, \ldots, \infty)$ implies that it has degree $-\infty$. Then, assertion (1) is trivially satisfied, while by the same reason assertion (2) is trivially satisfied if $b + b' > 2d_n$, as in this case either $b > d_n$ or $b' > d_n$. So this gives the induction base.

Suppose that we have shown that assertions (1) and (2) hold for $b > m$ and $c + c' > m$ for some $m \in \mathbb{N}_0$. The goal of the induction step is to show that they also hold for $b = m$ and $c + c' = m$.

Then, for the product case, we take a closer look at equations (2.1) and (2.2). By the induction hypothesis, $L_s(f), L_s(f'), R_s(f)$ and $R_s(f')$ are polynomial maps of lc-degree $\leq \bar{d} - (m + 1)$, and $[L_s(f')^{-1}, f]$ and $[f'^{-1}, R_s(f)]$ are polynomial maps of lc-degree $\leq \bar{d} - (2m + 1) \leq \bar{d} - (m + 1)$. So $f f'$ is a polynomial map of lc-degree $\leq (a_1, \ldots, a_n)$, where $a_i = d_i - m$ if $d_i \geq m$. If there exists $i$ such that $d_i < m$, then the assumption that $f \bmod C^{i+1}G$ and $f' \bmod C^{i+1}G$ have degree $\leq d_i - m = -\infty$ implies that $f(t) \in C^{i+1}G$ and $f'(t) \in C^{i+1}G$, thus $f(t)f'(t) \in C^{i+1}G$, i.e., $f f'$ $\bmod C^{i+1}G$ has degree $-\infty$, so $a_i = -\infty = d_i - m$. Hence, it follows that $f f'$ is a polynomial map of lc-degree $\leq \bar{d} - m$.

For the commutator case, we check equations (2.3) and (2.4). With the induction hypothesis, it is easy to see that $C_1, C_2, C_3, D_1, D_2, D_3, L_s([f, f']) = C_4$ and $R_s([f, f']) = D_4$ are polynomial maps of lc-degree $\leq \bar{d} - (c + c' + 1) = \bar{d} - (m + 1)$. Therefore, $[f, f']$ is a polynomial map of lc-degree $\leq (a_1, \ldots, a_n)$, where $a_i = d_i - m$ if $d_i \geq m$. If there exists $i$ such that $d_i < m$, we let $j$ (resp. $j'$) be the smallest index such that $f \bmod C^{j+1}G$ (resp. $f' \bmod C^{j'+1}G$) has nonnegative degrees. Thus, this implies that $f(t) \in C^{j+1}G$ and $f'(t) \in C^{j}G$ for all $t \in S$, and that $d_j - c \geq 0$ and $d_{j'} - c' \geq 0$. If $j + j' \geq n + 1$, then $[f(t), f'(t)] \in C^{j+j'}G = \{1_G\}$, so there is nothing to prove. If $j + j' \leq n$, then we have $^5$

$$d_j + j' \geq d_j + j' \geq c + c' = m > d_i,$$

and thus $j + j' > i$. So it follows that $[f(t), f'(t)] \in C^{j+j'}G \subseteq C^{i+1}G$, i.e., $a_i = -\infty = d_i - m$. Hence, it follows that $[f, f']$ is a polynomial map of lc-degree $\leq \bar{d} - m$. $\square$

Let $S$ and $S'$ be commutative semigroups. Then, any polynomial map $f : S \to G$ can be viewed as a polynomial map $\tilde{f} : S \times S' \to G$ by sending $(s, s')$ to $f(s)$. One checks that $\tilde{f}$ is a polynomial map and shares many of the same quantities with $f$, such as degree, lc-height, lc-degree, etc.

**Definition 4.** We say that $\tilde{f} : S \times S' \to G$ defined above is a polynomial map lifted from $f : S \to G$ to the commutative semigroup $S \times S'$. If $\tilde{f}' : S \times S' \to G$ is lifted from another polynomial map $f' : S' \to G$, then we can define an ordered product of $f$ and $f'$ by the following formula:

$$f \circ f' : S \times S' \to G; \quad (s, s') \mapsto \tilde{f}(s, s')f'(s').$$

The ordered product of two polynomial maps will be a polynomial map in some good cases, for example when $G$ is nilpotent. But the most interesting result will be the following

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5This is where we need the superadditivity.
Corollary 8. Let $S$ be a commutative semigroup and $G$ a nilpotent group of class $n$. Let $f : S \to G$ be any polynomial map, whose lc-degree is bounded by a fixed superadditive vector $d$. Then, for any $k \in \mathbb{N}$, the ordered product $\bigoplus_{i=1}^{k} f : \bigoplus_{i=1}^{k} S \to G$ is a polynomial map whose lc-degree is bounded by the same superadditive vector $d$.

Moreover, if the group $\langle f \rangle$ generated by $f(S)$ is finitely generated, then the subgroup $\langle \bigoplus_{i=1}^{k} f \rangle$ generated by $\bigoplus_{i=1}^{k} f(\bigoplus_{i=1}^{k} S)$ is of finite index in $\langle f \rangle$.

Proof. By induction on $k$, the first assertion follows from Corollary 7. The subgroup $\langle \bigoplus_{i=1}^{k} f \rangle$ contains the subgroup generated by the image of $\bigoplus_{i=1}^{k} f$ on the diagonal $S$ of $\bigoplus_{i=1}^{k} S$, which is nothing but $\langle f^k \rangle$. Since $\langle f \rangle$ is finitely generated and nilpotent, by a result due to Mal’tsev (cf. [CMZ17, Thm 2.23]), $\langle f^k \rangle$ has finite index in $\langle f \rangle$ and so is $\langle \bigoplus_{i=1}^{k} f \rangle$.

3. Continuous Polynomial Maps

Any usual polynomial $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ of degree $\leq d$ $^6$ is a polynomial map of degree $\leq d$ in our sense. The converse also holds, provided that the polynomial map $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is assumed to be continuous. But before proving this, we need a few definitions and lemmas.

Definition 5. Let $S$ be a commutative semigroup and $G$ be an abelian group. A function $f : S \to G$ is called additive, if it satisfies Cauchy’s functional equation:

$$f(s + t) = f(s) + f(t), \quad \forall s, t \in S.$$  

Remark. If $\{f_i : S \to G \mid 1 \leq i \leq n\}$ is a finite family of additive functions, then so is their linear combination $\sum_{i=1}^{n} r_i f_i$, where $r_i \in \mathbb{Z}$.

The following property is fundamental for additive functions $f : S \to G$:

Lemma 3. Let $f : S \to G$ be an additive function.

1. The function $f$ is always $\mathbb{N}$-linear, i.e., for all $n \in \mathbb{N}$ and all $s_1, \ldots, s_n \in S$, we have $f(\sum_{i=1}^{n} s_i) = \sum_{i=1}^{n} f(s_i)$ and $f(ns) = nf(s)$, $\forall n \in \mathbb{N}$ and all $s \in S$.

2. If $S$ is a commutative monoid, then $f$ is $\mathbb{N}_0$-linear and in particular $f(0) = 0$.

3. If $S$ is an abelian group, then $f$ is $\mathbb{Z}$-linear.

For the following three assertions, we need to assume that $G$ is torsion-free.

1'. If $S$ is a uniquely divisible commutative semigroup, i.e., for each $s \in S$ and each $n \in \mathbb{N}$, there exists a unique $t \in S$, such that $s = nt$, then $f$ is $\mathbb{Q}_{\geq 0}$-linear.

2'. If $S$ is a uniquely divisible commutative monoid, then $f$ is $\mathbb{Q}_{\geq 0}$-linear.

3'. If $S$ is a divisible abelian group, then $f$ is $\mathbb{Q}$-linear.

Proof. The proofs are all very similar, cf. [Kuc09, Thm 5.2.1]. Here we only show the last one. By induction, for all $n \in \mathbb{N}$ and all $s_1, \ldots, s_n \in S$, we have $f(\sum_{i=1}^{n} s_i) = \sum_{i=1}^{n} f(s_i)$. Letting $s_1 = s_2 = \cdots = s_n = s$, we see that $f(ns) = nf(s)$ for all $n \in \mathbb{N}$, and thus $f$ is $\mathbb{N}$-linear. Since $f(0) = f(0 + 0) = f(0) + f(0)$, we see that $f(0) = 0$, and thus $f$ is $\mathbb{N}_0$-linear. Since $0 = f(0) = f(s - s) = f(s) + f(-s)$.

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$^6$The degree of the zero polynomial is either left undefined, or defined to be negative (usually $-1$ or $-\infty$). But to be compatible with our definition of the degree of polynomial maps, it will be defined to be $-\infty$ as well.
we see that \( f(s) = -f(s) \), i.e., \( f \) is an odd function, and thus is \( \mathbb{Z} \)-linear. Since any \( \lambda \in \mathbb{Q} \) can be written as \( \lambda = n/m \), where \( n \in \mathbb{Z} \) and \( m \in \mathbb{N} \), and \( S \) is divisible, for all \( s \in S \), there exists a unique \( t \in S \) such that \( mt = ns \), we write \( t = ns/m = \lambda s \) and thus have
\[
mf(\lambda s) = mf(t) = f(mt) = f(ns) = nf(s),
\]
for all \( s \in S \). Since \( G \) is torsion-free, we have \( f(\lambda s) = \lambda f(s) \) for all \( s \in S \), and thus \( f \) is \( \mathbb{Q} \)-linear. \( \square \)

**Remark.** Uniqueness is needed here to make sense of \( \lambda s \) for \( \lambda \in \mathbb{Q} \).

**Theorem 2.** Every continuous polynomial map \( f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R} \) of is the usual polynomial.

**Proof.** By the continuity of \( f \), it suffices to prove that the restriction \( f : \mathbb{Q}_{\geq 0} \rightarrow \mathbb{R} \) is a polynomial. The proof is by induction on the degree \( d \). It is clear for \( d \leq 0 \). If \( f \) is a polynomial map of degree \( \leq 1 \), then for all \( s \in \mathbb{Q}_{\geq 0} \), \( f(s + t) - f(t) \) is a polynomial map of degree \( \leq 0 \), i.e., a constant, say \( C_s \). For any \( s_1, s_2 \in \mathbb{Q}_{\geq 0} \), we have
\[
C_{s_1 + s_2} = f(s_1 + s_2 + t) = f(s_1 + s_2) - f(s_2 + t) + f(s_2 + t) - f(t) = C_{s_1} + C_{s_2}.
\]
This implies that \( C_s \) is additive with respect to \( s \). By Lemma 3, \( C_{\lambda s} = \lambda C_s \) for all \( \lambda \in \mathbb{Q}_{\geq 0} \). Setting \( k = C_1 \), we obtain \( C_s = sC_1 = ks \) for all \( s \in \mathbb{Q}_{\geq 0} \). Then, \( f(s) = C_s + f(0) = ks + f(0) \) for all \( s \in \mathbb{Q}_{\geq 0} \). Hence, \( f(t) = kt + f(0) \) is a polynomial of degree \( \leq 1 \).

Suppose that the assertion holds when \( d \leq n - 1 \) with \( n \geq 2 \). Let \( f \) be a polynomial map of degree \( \leq n \). By Corollary 5, for any \( A_n \in \mathbb{R} \), \( f(t) - A_n t^n \) is a polynomial map of degree \( \leq n \). We claim that there exists some \( A_n \in \mathbb{R} \), such that \( f(t) - A_n t^n \) is a polynomial map of degree \( \leq n - 1 \). By the induction hypothesis, \( f(t) - A_n t^n \) is a polynomial of degree \( \leq n - 1 \), and therefore \( f(t) \) is a polynomial of degree \( \leq n \). So it suffices to find such an \( A_n \). By definition, for all \( s \in \mathbb{Q}_{\geq 0} \), \( P_s(t) := f(s + t) - f(t) \) is a polynomial map of degree \( \leq n - 1 \). Then, by the induction hypothesis,
\[
P_s(t) = \alpha_0(s) + \sum_{i=1}^{n-1} \alpha_i(s)t^i
\]
is a polynomial in variable \( t \) of degree \( \leq n - 1 \), where \( \alpha_i(s) \) are functions from \( \mathbb{Q}_{\geq 0} \) to \( \mathbb{R} \). Then, it is a routine check that \( A_n := \frac{1}{n} \alpha_{n-1}(1) \) is the desired number. \( \square \)

**Remark 2.** It was G. Hamel who first succeeded in proving the existence of discontinuous additive functions. In fact, he proved the following theorem in the case when \( N = 1 \):

**Theorem ([Kuc09, Thm 5.2.2]).** Any function \( g : H \rightarrow \mathbb{R} \) from an arbitrary Hamel basis \( H \) of the \( \mathbb{Q} \)-vector space \( \mathbb{R}^N \) to \( \mathbb{R} \) extends to a unique additive function \( f : \mathbb{R}^N \rightarrow \mathbb{R} \) such that \( f \mid_H = g \).

In fact, every (discontinuous) additive function \( f : \mathbb{R}^N \rightarrow \mathbb{R} \) may be obtained in such a way. Hence, there exist pathological polynomial maps \( \mathbb{R}^N_{\geq 0} \rightarrow \mathbb{R} \) if one does not insist on the continuity.

**Corollary 9.** The image of any nonconstant continuous polynomial map \( f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R} \) is an unbounded interval.

**Proof.** By Theorem 2, \( f \) must be a polynomial of degree \( \geq 1 \), whose image \( f(\mathbb{R}_{\geq 0}) \) is certainly an unbounded interval in \( \mathbb{R} \). \( \square \)
Corollary 10. Every continuous polynomial map \( f : \mathbb{R}_{\geq 0} \to \mathbb{R}^M \) of degree \( \leq d \) is a vector of polynomials of degree \( \leq d \).

Proof. By Theorem 2, the induced map \( f_i := \pi_i \circ f \), where \( \pi_i : \mathbb{R}^M \to \mathbb{R} \) be the projection map of the \( i \)th coordinates, is a continuous polynomial map of degree \( \leq d \). \( \square \)

Before stating another theorem, we record some properties about the group \( U_n(\mathbb{R}) \) of upper unitriangular \( n \times n \) matrices over \( \mathbb{R} \) in the lemma below. Let \( E_{i,j} \) be the \( n \times n \) matrix with all entries 0 except the \((i,j)\)-entry 1, and \( I \) be the identity \( n \times n \) matrix, and \( T_{i,j}(a) \) be the unitriangular matrix of the form \( I + aE_{i,j} \) for \( i \neq j \) and \( a \in \mathbb{R} \).

Lemma 4. For any \( T \in U_n(\mathbb{R}) \), \( T \) can be written as \( I + T_u \), where \( T_u \) is strictly upper triangular and thus nilpotent with index \( \leq n \), i.e., \( T_u^i = 0 \) for all \( i \geq n \).

1. The inverse of \( T \) is given by \( T^{-1} = I + \sum_{i=1}^{n-1} (-T_u)^i \).
2. For distinct \( i, j \in \{1, \ldots, n\} \) and \( a \in \mathbb{R} \), one has \( T_{i,j}(a)^{-1} = T_{i,j}(-a) \).
3. For distinct \( i, j, l \in \{1, \ldots, n\} \) and \( a, b \in \mathbb{R} \), one has \( [T_{i,j}(a), T_{j,l}(b)] = T_{i,l}(ab) \).
4. For distinct \( i, j, k, l \in \{1, \ldots, n\} \) and \( a, b \in \mathbb{R} \), one has \( [T_{i,j}(a), T_{k,l}(b)] = I \).

Proof. We have the following equation of power series,

\[
(I + T_u)^{-1} = \sum_{i=0}^{\infty} (-T_u)^i,
\]

from which (1) follows easily, since the right hand side ends in finitely many steps, while the other statements follow easily from (1) and the formula of matrix products

\[
E_{i,j}E_{k,l} = \delta_{j,k}E_{i,l} = \begin{cases} E_{i,l}, & \text{if } j = k, \\ 0, & \text{otherwise}. \end{cases}
\]

For all \( 1 \leq k \leq n \), let \( U_{n,k}(\mathbb{R}) \) be the subset of \( U_n(\mathbb{R}) \) consisting of matrices \((t_{i,j})\) such that \( t_{i,j} = \delta_{i,j} \) for \( j < i + k \), and for all \( 1 \leq k \leq n - 1 \), define maps \( \phi_k \) in the following way

\[
\phi_k : U_{n,k}(\mathbb{R}) \to (\mathbb{R}^{n-k}, +); \quad T = (t_{i,j}) \mapsto (t_{1,1+k}, t_{2,2+k}, \ldots, t_{n-k,n}).
\]

For convenience sake, we call \((t_{1,1+k}, t_{2,2+k}, \ldots, t_{n-k,n})\) the \( k \)th diagonal entries of \( T \). Note that \( U_{n,k}(\mathbb{R}) \) is a subgroup of \( U_n(\mathbb{R}) \) with \( U_{n,1}(\mathbb{R}) = U_n(\mathbb{R}) \) and \( U_{n,n}(\mathbb{R}) = \{I\} \).

Lemma 5. The following properties hold:

1. The map \( \phi_k \) is a homomorphism of groups with kernel \( U_{n,k+1}(\mathbb{R}) \).
2. The derived group \( C^2U_{n,k}(\mathbb{R}) \) is a subgroup of \( U_{n,k+1}(\mathbb{R}) \).
3. The subgroup \( U_{n,k+1}(\mathbb{R}) \) is a normal subgroup of \( U_{n,k}(\mathbb{R}) \) for all \( k \geq 1 \).
4. The subgroup \( U_{n,k}(\mathbb{R}) \) is generated by \( \{T_{i,j}(a) \mid a \in \mathbb{R}, j \geq i + k\} \).
5. For all \( k \geq 1 \), \( C^kU_{n}(\mathbb{R}) \) is a subgroup of \( U_{n,k}(\mathbb{R}) \) and \( U_{n}(\mathbb{R}) \) is nilpotent of class \( \leq n - 1 \).

Proof. See [DK18, Exercise 13.38]. \( \square \)

The last statement in the previous lemma can be strengthened as follows:
Corollary 11. For all $k \geq 1$, $C^k U_n(\mathbb{R})$ is the group $U_{n,k}(\mathbb{R})$ and $U_n(\mathbb{R})$ is nilpotent of class $n-1$.

Proof. One proves by induction on $k$ that $T_{i,j}(a) \in C^k U_n(\mathbb{R})$ for $j \geq i+k$ and $a \in \mathbb{R}$. $\Box$

Theorem 3. Let $f_{i,j} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ with $1 \leq i < j \leq n$ be continuous polynomial maps of degree $d_{i,j}$ and $f : \mathbb{R}_{\geq 0} \rightarrow U_n(\mathbb{R})$ be a function whose matrix form is

$$
\begin{pmatrix}
1 & f_{1,2} & f_{1,3} & \cdots & f_{1,n} \\
1 & f_{2,3} & \cdots & f_{2,n} \\
1 & \ddots & \ddots & \ddots \\
& & \ddots & f_{n-1,n} \\
& & & 1
\end{pmatrix}
$$

(3.2)

Then, the function $f$ is a continuous polynomial map. Conversely, every continuous polynomial map $f : \mathbb{R}_{\geq 0} \rightarrow U_n(\mathbb{R})$ is of this form.

Proof. Notice that $f$ is given by the following elementwise product with a particular order

$$
f = \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} (I + f_{i,j} E_{i,j}).
$$

By Theorem 1, it suffices to show that each $I + f_{i,j} E_{i,j}$ is a polynomial map. This follows easily from the assumption that each $f_{i,j}$ is a continuous polynomial map. By Theorem 2, $f_{i,j} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ are polynomials of degree $d_{i,j}$ for all $1 \leq i < j \leq n$. Hence, it follows that $f$ is continuous.

Conversely, suppose that $f$ has lc-height $k$, i.e., the image of $f$ lies in the subgroup $C^k U_n(\mathbb{R}) = U_{n,k}(\mathbb{R})$ for $1 \leq k \leq n$. The proof for the converse statement is by downward induction on $k$. If $k = n$, then $U_{n,k}(\mathbb{R}) = \{I\}$ and $f_{i,j}$ are zero polynomials. If $k = n-1$, then all $f_{i,j}$ are zero polynomials except $f_{1,n}$. By (1) in Lemma 5, $\phi_{n-1}$ is a homomorphism (in fact an isomorphism). Since $f$ is a continuous polynomial map, so is $f_{1,n}$. By Theorem 2, $f_{1,n}$ is a polynomial.

Suppose we have shown this for continuous polynomial maps of lc-height $> 2$ and $f$ is a continuous polynomial map of lc-height $1$. Then, we apply the group homomorphism $\phi_1$ to $f(t)$ to pick out the first diagonal entries $(f_{1,1}, f_{2,2}, \ldots, f_{n-1,n})$. Then, $(f_{1,2}, f_{2,3}, \ldots, f_{n-1,n}) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n-1}$ is a continuous polynomial map. By Corollary 10, $f_{i+1,i} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ are polynomials for all $1 \leq i \leq n-1$.

Next, we multiply $h = I - \sum_{i=1}^{n-1} f_{i,i+1}(t) E_{i,i+1}$ on the left hand side of $f$ and obtain:

$$
h f = \begin{pmatrix}
1 & -f_{1,2} & -f_{1,3} & \cdots & -f_{1,n} \\
1 & 0 & \ddots & \ddots & \ddots \\
& & \ddots & 0 & \ddots \\
& & & 1 & \ddots \\
& & & & 1
\end{pmatrix}
\begin{pmatrix}
1 & f_{1,2} & f_{1,3} & \cdots & f_{1,n} \\
1 & f_{2,3} & \cdots & f_{2,n} \\
1 & \ddots & \ddots & \ddots \\
& & \ddots & f_{n-1,n} \\
& & & 1
\end{pmatrix}
$$

\footnote{The real Lie group $U_n(\mathbb{R})$ is given the usual metric topology.}
Then, $hf$ is a continuous polynomial map of lc-height $\geq 2$. By the induction hypotheses, each entry of $hf$ is a polynomial. Then, the second diagonal entries

$$f_{i,i+2} = (f_{i,i+2} - f_{i,i+1}f_{i+1,i+2} + f_{i,i+1}f_{i+1,i+2}$$

are polynomials, since $f_{i,i+1}f_{i+1,i+2}$ are already known to be polynomials. Inductively, we can show that all $j$th diagonal entries $f_{i,j} = (f_{i,j} - f_{i,i+1}f_{i+1,j}) + f_{i+1}f_{i+1,j}$ are polynomials. $\square$

Next, we will talk about polynomial maps in several variables. Any usual polynomial $f : \mathbb{R}^N \to \mathbb{R}$ in $N$ variables of degree $\leq d$ is a polynomial map of degree $\leq d$ in our sense. The converse also holds, provided that the polynomial map $f : \mathbb{R}^N \to \mathbb{R}$ is assumed to be continuous.

**Theorem 4.** Every continuous polynomial map $f : \mathbb{R}^N_\geq 0 \to \mathbb{R}$ of degree $d$ is a polynomial in $N$ variables of degree $d$.

**Proof.** We proceed by induction on $N$ with the base case $N = 1$ being Theorem 2. For any $a \in \mathbb{R}^{N-1}_\geq 0$ define $f_a(t) = f(a,t)$. Then, $f_a(t)$ is a continuous polynomial map of degree $\leq d$ in the variable $t$. By Theorem 2, we can write $f_a(t) = \sum_{i=0}^{d} c_i(a) t^i$, where $c_i : \mathbb{R}^{N-1}_\geq 0 \to \mathbb{R}$ are continuous functions. Applying the finite forward difference operator of the form $D_{(x_1,\ldots,x_{N-1},0)}^d + 1$ times to

$$f(t_1,\ldots,t_{N-1},t_N) = \sum_{i=0}^{d} c_i(t_1,\ldots,t_{N-1}) t^i,$$

we have $0 = \sum_{i=0}^{d} D_{(x_1,\ldots,x_{N-1},0)}^{d+1} c_i(t_1,\ldots,t_{N-1}) t_N^i$. By the linear independence of geometric progressions $t_1, t_1^n, t_1^n, \ldots, t_N, c_i(t_1,\ldots,t_{N-1})$ are continuous polynomial maps of degree $\leq d$. By the induction hypothesis, they are given by polynomials in the usual sense, so the same is true for $f$. $\square$

**Corollary 12.** The image of any nonconstant continuous polynomial map $f : \mathbb{R}^N_\geq 0 \to \mathbb{R}$ is an unbounded interval.

**Proof.** By Theorem 4, $f$ must be a polynomial of degree $\geq 1$, whose image $f(\mathbb{R}^N_\geq 0)$ is certainly an unbounded interval in $\mathbb{R}$. Alternatively, one may argue with the help of the induced continuous polynomial map $\mathbb{R}_\geq 0 \hookrightarrow \mathbb{R}^N_\geq 0 \to \mathbb{R}$ and Theorem 2. $\square$

**Corollary 13.** Every continuous polynomial map $f : \mathbb{R}^N_\geq 0 \to \mathbb{R}^M$ of degree $\leq d$ is vector of polynomials in $N$ variables of degree $\leq d$.

**Proof.** The proof is the same as the one of Corollary 10. $\square$
Theorem 5. Let \( f_{i,j} : \mathbb{R}^n_{\geq 0} \to \mathbb{R} \) with \( 1 \leq i < j \leq n \) be continuous polynomial maps of degree \( \leq d_{i,j} \) and \( f : \mathbb{R}^n_{\geq 0} \to \mathcal{U}_n(\mathbb{R}) \) be a function with matrix form given by (3.2). Then, the function \( f \) is a continuous polynomial map.

Conversely, every continuous polynomial map \( f : \mathbb{R}^n_{\geq 0} \to \mathcal{U}_n(\mathbb{R}) \) is of this form.

Proof. The proof is the same as the one of Theorem 3, except that one needs to replace Theorem 2 by Theorem 4, and Corollary 10 by Corollary 13. \( \square \)

4. Estimation of the Degree

The most important quantity of a polynomial map is its (lc-)degree. The first attempt has been given in Corollary 7 via superadditive vectors, but it is not good enough. So we will try to estimate them by working out a formula for the lower and upper bounds of the (lc-)degree, in particular, of polynomial maps of the form \( \mathbb{R}^n_{\geq 0} \to \mathcal{U}_n(\mathbb{R}) \).

Notice that in the first part of Theorem 3, we do not give any information about the degree \( d \) of the polynomial map \( f \), which should be closely related to the degrees \( d_{i,j} \) for all \( 1 \leq i < j \leq n \). Notice that \( I + f_{i,i+k}E_{i,i+k} \) is a polynomial map of lc-degree

\[
(-\infty, \ldots, -\infty, d_{i,i+k}, \ldots, d_{i,i+k}) \in \mathbb{Z}^{n-1},
\]

where the first \( d_{i,i+k} \) appears in the \( k \)th entry, as its image lies in \( \mathcal{U}_{n,k}(\mathbb{R}) = C^k\mathcal{U}_n(\mathbb{R}) \). Then, there exists at least superadditive vector above all of the lc-degree, which should be an upper bound of the (lc)-degree of \( f \) in view of Corollary 7. But this estimation is quite coarse.

For a better understanding, two motivating examples are provided. They are the continuous Heisenberg group \( H_3(\mathbb{R}) = \mathcal{U}_3(\mathbb{R}) \) and the nilpotent Lie group \( \mathcal{U}_4(\mathbb{R}) \).

Example 3. Let \( f_{i,j} : \mathbb{R}_{\geq 0} \to \mathbb{R} \) be continuous polynomial maps of degree \( d_{i,j} \) and set

\[
\begin{pmatrix}
1 & f_{1,2} & f_{1,3} \\
0 & 1 & f_{2,3} \\
0 & 0 & 1
\end{pmatrix} : \mathbb{R}_{\geq 0} \to H_3(\mathbb{R}),
\]

\[
\begin{pmatrix}
1 & f_{1,2} & f_{1,3} & f_{1,4} \\
0 & 1 & f_{2,3} & f_{2,4} \\
0 & 0 & 1 & f_{3,4} \\
0 & 0 & 0 & 1
\end{pmatrix} : \mathbb{R}_{\geq 0} \to \mathcal{U}_4(\mathbb{R}).
\]

Then, \( f_3 \) is a continuous polynomial map of degree \( \leq \max\{d_{1,3}, d_{1,2} + d_{2,3}\} \) and \( \geq \max\{f_{1,2}, f_{2,3}\} \), and \( f_4 \) is a continuous polynomial map of degree \( \leq \max\{d_{1,4}, d_{1,2} + d_{2,4}, d_{1,3} + d_{3,4}, d_{1,2} + d_{2,3} + d_{3,4}\} \) and \( \geq \max\{d_{1,2}, d_{2,3}, d_{3,4}\} \).

Theorem 6. Let \( f \) be as in Theorem 3. Then, \( f \) is a polynomial map of degree bounded below by

\[
\max\{d_{k,k+1} \mid 1 \leq k \leq n-1\},
\]

and bounded above by

\[
\max \left\{ d_{k_1,k_2} + \cdots + d_{k_{n-1},k_n} \mid 1 = k_1 \leq k_2 \leq \cdots \leq k_{n-1} \leq k_n = n \right\},
\]

where \( d_{i,j} \) is defined to be 0 if \( i = j \).

Proof. Denote \( f_{i,j}(s+t) \) by \( s_{i,j} \) and \( f_{i,j}(t) \) by \( t_{i,j} \), set \( s_{i,i} = t_{i,i} = 1 \), and write \( S = I + S_u = f(s+t) \) and \( T = I + T_u = f(t) \), where \( S_u \) (resp. \( T_u \)) has entries given by \( s_{i,j} \) (resp. \( t_{i,j} \)).
Suppose that \( f \) is a polynomial map of degree \( d \). By (1) in Lemma 5, \( \phi_1 \) is a homomorphism. By Proposition 5, the induced polynomial map \( \phi_1 \circ f = (f_1, \ldots, f_{n-1,n}) : \mathbb{R}_2 \to (\mathbb{R}^{n-1}, +) \) is a polynomial of degree \( \max \{ d_k, k+1 \mid 1 \leq k \leq n-1 \} \), which by Proposition 4 is at most \( d \).

The proof for the upper bound is given by induction on \( d \). The case when \( d \leq 0 \) is trivial. For the induction step, in view of (1) in Lemma 5, we see that \( L := f(s + t)f^{-1}(t) \) is given by

\[
(I + S_t)(I + T_u)^{-1} = I + \sum_{i=1}^{n} (-1)^i (T_u - S_u) T_u^{i-1},
\]

and that \( R := f(t)^{-1}f(s + t) \) is given by

\[
(I + T_u)^{-1}(I + S_t) = I + \sum_{i=1}^{n} (-1)^i T_u^{i-1}(T_u - S_u).
\]

Since each entry of the matrix \( L \) (resp. \( R \)) is a polynomial, the crux of the proof is to find its expression and estimate the upper bound of its degree.

Notice that these two equations are similar to the inverse of \( T \) as in Lemma 5, except that one replaces the first (resp. last) \( T_u \) by \( T_u - S_u \). Since \( T_u \) is strictly upper triangular and nilpotent with index \( \leq n \), one could easily write down the expression of each entry of \( T^{-1} \). Indeed, for example, the first diagonal entries of \( T^{-1} \) are given by \( (-t_1, t_2 - t_3, \ldots, t_{n-1} - t_n) \), and the second diagonal entries of \( T^{-1} \) are given by \( (-t_1 + t_2t_3, t_2 + t_3t_4, \ldots, t_{n-1} + t_n - t_{n-1}t_{n-2} - t_{n-2}t_{n-3}, \ldots, -t_n - t_{n-2} - t_{n-3} - \cdots - t_1) \). In general, the \((i,j)\)-entry in the \((j-i)\)th diagonal entries of \( T^{-1} \) is given by

\[
-t_{i,j} + \sum_{i<k<j} t_{i,k} t_{k,j} - \sum_{i<k_1<k_2<j} t_{i,k_1} t_{k_1,k_2} t_{k_2,j} + \cdots.
\]

Similarly, the first diagonal entries of \( L \) and \( R \) are given by \( s_{i,i+1} - t_{i,i+1} \), and the second diagonal entries of \( L \) (resp. \( R \)) are given by

\[
(-(t_{1,3} - s_{1,3}) + (t_{1,2} - s_{1,2})t_{2,3} + \cdots, s_{n-2,n} - t_{n-2,n} + (t_{n-2,n-1} - s_{n-2,n-1})t_{n-1,n})
\]

(resp. \( -(t_{1,3} - s_{1,3}) + t_{1,2}(t_{2,3} - s_{2,3}) + \cdots, s_{n-2,n} - t_{n-2,n} + t_{n-2,n-1}(t_{n-1,n} - s_{n-1,n}) \)).

In general, the \((i,j)\)-entry in the \((j-i)\)th diagonal entries \( L \) (resp. \( R \)) is given by

\[
-(t_{i,j} - s_{i,j}) + \sum_{i<k<j} (t_{i,k} - s_{i,k}) t_{k,j} - \sum_{i<k_1<k_2<j} (t_{i,k_1} - s_{i,k_1}) t_{k_1,k_2} t_{k_2,j} + \cdots,
\]

(resp. \( -(t_{i,j} - s_{i,j}) + \sum_{i<k<j} t_{i,k}(t_{k,j} - s_{k,j}) - \sum_{i<k_1<k_2<j} t_{i,k_1} t_{k_1,k_2}(t_{k_2,j} - s_{k_2,j}) + \cdots \)).

Then, the degree of the \((i,j)\)-entry in the \((j-i)\)th diagonal entries of \( L \) and \( R \) is

\[
\leq e_{i,j} := \max_{i = k_1 \leq k_2 \leq \cdots k_{j-i} \leq k_{j-i+1}} \left\{ d_{k_1,k_2} + \cdots + d_{k_{j-i},k_{j-i+1}} \right\} - 1,
\]

which is 1 less than the degree of the \((i,j)\)-entry in the \((j-i)\)th diagonal entries of \( T^{-1} \).

Since \( L \) and \( R \) are polynomial maps of degree \( d - 1 \), by the induction hypothesis, we can apply the upper bounds (4.2) to \( L \) and \( R \), and obtain that

\[
d - 1 \leq \max \{ e_{j_1,j_2} + \cdots + e_{j_{n-1},j_n} \mid 1 = j_1 \leq j_2 \leq \cdots \leq j_{n-1} \leq j_n = n \}
\]
where the last inequality holds for the following reason. For each possible choice of $\phi$ the coefficients of these polynomials satisfy a certain system of nontrivial polynomial equations, so that

Generically, the degree $d$ should achieve this upper bound. But as one can see from the proof, there is a rare possibility that some polynomials might cancel with each other when the coefficients of these polynomials satisfy a certain system of nontrivial polynomial equations, so that is why inequality (4.2) only gives an upper bound.

Here is an example in which the actual degree of the polynomial map can be much smaller than this upper bound. A one-parameter subgroup in a topological group $G$ is a continuous group homomorphism $\varphi : (\mathbb{R},+) \to G$. If $\varphi$ is injective, then the image $\varphi(\mathbb{R})$ will be a subgroup isomorphic to $\mathbb{R}$ as an additive group. Typically, trivial homomorphisms are not considered to be one-parameter subgroups. So one-parameter subgroups are polynomial maps of degree 1.

**Example.** Then, $f(t) = e^{tA} : \mathbb{R} \to U_n(\mathbb{R})$, where $A$ is a strictly upper triangular $n \times n$ matrix in $M_n(\mathbb{R})$, is a one-parameter subgroup in $U_n(\mathbb{R})$. For each $1 \leq k \leq n - 1$, the $k$th diagonal entries $f_{i,i+k}$ are polynomials of degree $\leq d_{i,i+k} = k$. Hence, the inequality (4.2) yields an upper bound

$$\max \{d_{k_1,k_2} + \cdots + d_{k_{n-1},k_n} \mid 1 = k_1 \leq k_2 \leq \cdots \leq k_{n-1} \leq k_n = n\} = n - 1.$$

As mentioned before, the degree of the product of two polynomial maps is quite mysterious in Theorem 1. But with the help of Theorem 6, we can say more about this.

**Corollary 14.** Let $f, f' : \mathbb{R}_{\geq 0} \to U_n(\mathbb{R})$ be two polynomial maps of degree $\leq d$ and $\leq d'$ respectively, where $d$ and $d'$ are the upper bounds given by (4.2). Then, the product $ff' : \mathbb{R}_{\geq 0} \to U_n(\mathbb{R})$ is a polynomial map of degree $\leq d + d'$.

**Proof.** The $(i,j)$-entry of $f(t)f'(t)$ is given by $\sum_{i \leq k \leq j} f_{i,k}(t)f'_{k,j}(t)$, which has degree

$$\leq e_{i,j} := \max\{d_{i,k} + d'_{k,j} \mid i \leq k \leq j\}.$$

Hence, by Theorem 6, $ff'$ is a polynomial map of degree $\leq$

$$\max\{d_{k_1,k_2} + \cdots + d_{k_{n-1},k_n} \mid 1 = k_1 \leq k_2 \leq \cdots \leq k_{n-1} \leq k_n = n\}$$

$$\leq \max_{1 \leq k_1 \leq k_2 \leq \cdots \leq k_{n-1} \leq k_n = n} \left\{\max_{k_1 \leq l_1 \leq k_2} \{d_{k_1,l_1} + d'_{l_1,k_2}\} + \cdots + \max_{k_{n-1} \leq l_{n-1} \leq k_n} \{d_{k_{n-1},l_{n-1}} + d'_{l_{n-1},k_n}\}\right\}$$

$$\leq \max_{1 \leq k_1 \leq k_2 \leq \cdots \leq k_{n-1} \leq k_n = n} \left\{d_{k_1,k_2} + \cdots + d_{k_{n-1},k_n}\right\} + \max_{1 \leq k_1 \leq k_2 \leq \cdots \leq k_{n-1} \leq k_n = n} \left\{d'_{k_1,k_2} + \cdots + d'_{k_{n-1},k_n}\right\},$$

where the last inequality holds for the following reason. For each possible choice of $l_1, \ldots, l_{n-1}$, such that $1 = k_1 \leq l_1 \leq k_2 \leq \cdots \leq k_{n-1} \leq l_{n-1} \leq k_n = n$, we have

$$d_{k_1,l_1} + d'_{l_1,k_2} + \cdots + d_{k_{n-1},l_{n-1}} + d'_{l_{n-1},k_n} = d_{k_1,l_1} + \cdots + d_{k_{n-1},l_{n-1}} + d'_{l_{n-1},k_n}.$$
Theorem 7. Let \( f \) be as in Theorem 3 and \( \hat{d} = (d_1, d_2, \ldots, d_{n-1}) \) be the lc-degree of \( f \). Then,

\[
\begin{align*}
\leq &\; d_{k_1,i_1} + d_{i_2,k_2} + \cdots + d_{k_{n-1},i_{n-1}} + d_{i_n,k_n} + d'_{k_1,i_1} + d'_{i_2,k_2} + \cdots + d'_{k_{n-1},i_{n-1}} + d'_{i_n,k_n} \\
\leq &\; \max_{1=k_1 \leq k_2 \leq \cdots \leq k_{n-1} \leq k_n = n} \{ d_{k_1,k_2} + \cdots + d_{k_{n-1},k_n} \} + \max_{1=k_1 \leq k_2 \leq \cdots \leq k_{n-1} \leq k_n = n} \{ d'_{k_1,k_2} + \cdots + d'_{k_{n-1},k_n} \}.
\end{align*}
\]

Hence, the proof is complete, since the last row is nothing but \( d + d' \). \( \square \)

In the same manner, we can talk more about the lc-degree of \( f \).

Theorem 8. Let \( f \) be as in Theorem 5. Then, \( f \) is a polynomial map of degree bounded below by

\[
\max\{d_{k,k+1} \mid 1 \leq k \leq n - 1\},
\]

and bounded above by

\[
\leq \max\{d_{k_1,k_2} + \cdots + d_{k_{n-1},k_n} \mid 1 = k_1 \leq k_2 \leq \cdots \leq k_{n-1} \leq k_n = n\},
\]

where \( d_{i,j} \) is defined to be 0 if \( i = j \). In particular, when \( i = n - 1 \), we obtain the same upper bound for the degree of \( f \) as in Theorem 6.

Proof. By definition, \( d_1 \) is the degree of \( f \mod U_n(\mathbb{R}) \), which by Corollary 11 is the same as the degree of \( f \mod U_{n,2}(\mathbb{R}) \), and the same as the degree of

\[
\phi_1 \circ f = (f_{1,2}, \ldots, f_{n-1,n}) : \mathbb{R}_{\geq 0} \to (\mathbb{R}^{n-1}, +).
\]

We know that \( \phi_1 \circ f \) is a polynomial of degree \( \max\{d_{k,k+1} \mid 1 \leq k \leq n - 1\} \).

For \( 2 \leq i \leq n - 1 \), the proof is almost the same as the one of Theorem 6, except that instead of calculating the degree of \( f \), one calculates the degree of \( f \mod C^{i+1}U_n(\mathbb{R}) = f \mod U_{n,i+1}(\mathbb{R}) \). It is easy to see why only the \( \leq i \)th diagonal terms are involved in the inequalities (4.3).

Remark. The lc-degree \( \hat{d} \) should generically achieve this upper bound, but there is a rare possibility that it is strictly less than that. The one-parameter subgroups in \( U_n(\mathbb{R}) \) provide such examples.

Of course, the above results generalize to polynomial maps in multivariate cases.

Corollary 15. Let \( f, f' : \mathbb{R}_{\geq 0}^N \to U_n(\mathbb{R}) \) be two polynomial maps of degree \( \leq d \) and \( \leq d' \) respectively, where \( d \) and \( d' \) are the upper bounds given by (4.5). Then, the product \( ff' : \mathbb{R}_{\geq 0}^N \to U_n(\mathbb{R}) \) is a polynomial map of degree \( \leq d + d' \).

Proof. The proof is the same as the one of Corollary 14. \( \square \)
Theorem 9. Let $f$ be as in Theorem 5 and $\hat{d} = (d_1, d_2, \ldots, d_{n-1})$ be the lc-degree of $f$. Then,

\[
\begin{cases}
-\infty \leq d_1 = \max\{d_{k,k+1} | 1 \leq k \leq n - 1\}, \\
d_{i-1} \leq d_i \leq \max\{d_{k_1,k_2} + \cdots + d_{k_i,k_{i+1}} \mid 1 \leq k_1 \leq k_2 \leq \cdots \leq k_i \leq k_{i+1} = k_1 + i \leq n\}, \quad 2 \leq i \leq n - 1,
\end{cases}
\]

where $d_{i,j}$ is defined to be 0 if $i = j$. In particular, when $i = n - 1$, we obtain the same upper bound for the degree of $f$ as in Theorem 8.

Proof. The proof is the same as the one of Theorem 8. $\square$

5. POLYNOMIAL SEQUENCES

In this section, we will concentrate on special polynomial maps of the form $N_0 \to G$.

Definition 6. A polynomial map $g$ from $N_0$ to a group $G$ will be called a polynomial sequence. By abuse of terminology, we often call $g_0, g_1, g_2, \ldots$ a polynomial sequence in $G$, where $g_i := g(i)$, $\forall i \in N_0$, and denote by $(g)$ the subgroup of $G$ generated by the polynomial sequence $g_0, g_1, g_2, \ldots$.

Remark. We can talk about polynomial subsequence of $g_0, g_1, g_2, \ldots$ in the following sense:

(1) Since the translation $T_s(g)(t) := g(t+s)$ of a polynomial sequence $g$ by $s \in N_0$ is a polynomial sequence, $g_s, g_{1+s}, g_{2+s}, \ldots$ can be viewed as a polynomial subsequence.

(2) We have the polynomial sequence $N_0 \xrightarrow{\hat{k}} N_0 \xrightarrow{g} G$ induced by a homomorphism $\hat{k} : N_0 \to N_0$; $t \mapsto kt$, where $k \in N_0$. Then, $g_k, g_{2k}, \ldots$ can be viewed as a polynomial subsequence.

Notice that a polynomial subgroup has degree no larger than the original degree.

By Proposition 6, the subgroup generated by a polynomial sequence is always finitely generated. Then, Theorem 1 can be slightly generalized in the case of polynomial sequences.

Corollary 16. The product of two polynomial sequences $f, f' : N_0 \to G$ in a locally nilpotent group $G$ is a polynomial sequence.

Definition 7. A sequence $g$ is called periodic, if there exists $P \in N$ such that $g_{i+P} = g_i$, $\forall i \in N_0$.

Proposition 7. A polynomial sequence in a finite group is always periodic.

Proof. Let $G$ be a finite group and $g : N_0 \to G$ be any polynomial sequence. Let $|G|$ be the order of $G$. The proof is by induction on the degree $d$ of the polynomial sequence. If $d = 1$, then $g$ is constant, and thus periodic. If $d = 1$, then we have

\[g_{i+P} = l_1g_{i+P-1} = \cdots = P_1 g_{i}, \quad g_{i+P} = g_{i+P-1}r_1 = \cdots = g_r g_{P}.
\]

A suitable $P$ (for example $|G|$) can be chosen so that $l_1^P = r_1^P$ is the identity of $G$.

Suppose that we have proved this for all polynomial sequences of degree $< d$ and $g$ is a polynomial sequence of degree $\leq d$. Then, we have

\[g_{i+P} = L_1(g)(i+P-1)g_{i+P-1} = \cdots = L_1(g)(i+P-1) \cdots L_1(g)(i)g_{i}, \]

\[g_{i+P} = g_{i+P-1}R_1(g)(i+P-1) = \cdots = g_i R_1(g)(i) \cdots R_1(g)(i+P-1).
\]
Since $L_1(g)$ and $R_1(g)$ are polynomials of degree $\leq d - 1$, they are periodic polynomial sequences, say, of periods $L$ and $R$ respectively. Thus, for a certain natural number $P$ (for example, $\text{lcm}(L, R)$), $L_1(g)(i+P-1)\cdots L_1(g)(i)$ and $R_1(g)(i)\cdots R_1(g)(i+P-1)$ are constant for all $i \in \mathbb{N}_0$. If necessary, one replaces $P$ by $P^{[d]}$ in order to assure that

$$1_G = L_1(g)(i + P - 1)\cdots L_1(g)(i) = R_1(g)(i)\cdots R_1(g)(i + P - 1).$$

Hence, $g_{i+P} = g_i$ for all $i \in \mathbb{N}_0$, i.e., $g$ has period $P$.

**Theorem 10.** Every polynomial sequence $f : \mathbb{N}_0 \to R$, where $R$ can be the ring $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$, of degree $\leq d$ is a polynomial of degree $\leq d$.

**Proof.** The proof is the same as the proof of Theorem 2. One simply ignores the continuity part and replaces $\mathbb{Q}_{\geq 0}$ by $\mathbb{N}_0$ everywhere.

**Corollary 17.** Every polynomial sequence $f : \mathbb{N}_0 \to R^M$, where $R$ could be the ring $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$, of degree $\leq d$ is a vector of polynomials of degree $\leq d$.

**Proof.** Let $\pi_i : \mathbb{R}^M \to \mathbb{R}$ be the projection map of the $i$th coordinates. Then, $f_i := \pi_i \circ f$ is a polynomial sequence of degree $\leq d$. Then, by Theorem 10, $f = (f_1, \ldots, f_M)$ is a vector of polynomials of degree $\leq d$.

**Theorem 11.** Let $f_{i,j} : \mathbb{N}_0 \to R$, where $R$ could be the ring $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$, be polynomial sequences of degree $\leq d_{i,j}$ for all $1 \leq i < j \leq n$ and $f : \mathbb{N}_0 \to \mathcal{U}_n(R)$ be a function with matrix form given by (3.2). Then, $f$ is a polynomial sequence.

Conversely, every polynomial sequence $f : \mathbb{N}_0 \to \mathcal{U}_n(R)$ is of this form.

**Proof.** The proof is the same as the proof of Theorem 4. One simply ignores the continuity part and replaces $\mathbb{Q}_{\geq 0}$ by $\mathbb{N}_0$ everywhere.

The combination of the following theorems by Mal’tsev and Ado shows that each finitely generated torsion-free nilpotent group embeds in $\mathcal{U}_n(\mathbb{R})$ for some $n$:

**Theorem** (A. I. Mal’tsev [Mal49]). Every finitely generated torsion-free nilpotent group $\Gamma$ of class $k$ embeds as a uniform lattice in a simply-connected nilpotent Lie group $N$ of class $k$. Furthermore, the group $N$ and the embedding $\Gamma \to N$ are unique up to an isomorphism.

**Theorem** (Ado-Engel theorem). Every simply-connected nilpotent Lie group $N$ embeds into $\mathcal{U}_n(\mathbb{R})$ for some $n$.

**Theorem 12.** In a finitely generated torsion-free nilpotent group $G$, a polynomial sequence $g_0, g_1, g_2, \ldots$ can repeat a value infinitely many times if and only if the sequence is constant.

**Proof.** By theorems of Mal’tsev and Ado, there exists $n \in \mathbb{N}$ such that $G$ embeds into $\mathcal{U}_n(\mathbb{R})$. Then, consider the induced polynomial sequence $f : \mathbb{N}_0 \xrightarrow{\pi} G \hookrightarrow \mathcal{U}_n(\mathbb{R})$. By Theorem 11, $f$ can be written as an upper unitriangular matrix form with polynomials $f_{i,j} : \mathbb{N}_0 \to \mathbb{R}$ in each entry. Then, each $f_{i,j}$ can repeat a value infinitely many times if and only if $f_{i,j}$ is a constant. Hence, the same assertion holds for $f$ and thus for $g$. 

\[ Q.E.D. \]
Corollary 18. If a coset of an infinite index subgroup $H$ of a finitely generated nilpotent group $G$ contains infinitely many elements of a fixed nonzero power of a polynomial sequence $g : \mathbb{N}_0 \to G$, then there exists a normal subgroup $N$ of infinite index in $G$ such that a coset of $N$ contains the whole sequence.

Proof. Suppose that a coset of $H$ contains infinitely many of elements in the sequence $g_0^n, g_1^n, g_2^n, \ldots$ for some $0 \neq n \in \mathbb{Z}$. We may assume that $n > 0$.

Since $H$ is of infinite index in $G$, by a technical Lemma 6 proved later, there exists a normal subgroup $N$ of infinite index in $G$ containing $H$. Then, a coset of $N$ contains infinitely many elements of the sequence $g_0^n, g_1^n, g_2^n, \ldots$, i.e., the induced sequence $\bar{g}^n : \mathbb{N}_0 \to G \to G/N$ repeats a value infinitely many times.

If $G/N$ is torsion-free, then by Theorem 12 $\bar{g}^n$ is a constant. If $G/N$ is not torsion-free, then the torsion subgroup $\text{Tor}(G/N)$ is a finite normal subgroup of infinite index in $G/N$ and thus there exists a normal subgroup $N'$ of infinite index in $G$ such that $G/N' \cong (G/N)/\text{Tor}(G/N)$ is a finitely generated torsion-free nilpotent group. Since $g^n \mod N' = \bar{g}^n \mod \text{Tor}(G/N)$ repeats a value infinitely many times, it is constant. So we may assume that $G/N$ is finitely generated torsion-free nilpotent and $\bar{g}^n$ is a constant.

By a classical result on extraction of roots in torsion-free nilpotent groups (cf. [CMZ17, Thm 2.7]), $\bar{g}$ can only take values from a finite set of at most $n$ elements, which are all $n$th roots of the constant $\bar{g}$, and thus must repeat one of the value infinitely many times. Hence, $\bar{g}$ itself is constant, i.e., a coset of $N$ contains the whole sequence $g_0, g_1, g_2, \ldots$. \square

Alternatively, we may use the following theorem of P. Hall to embed a finitely generated torsion-free nilpotent group $G$ into $U_n(\mathbb{Z})$ for some $n$ and to prove Theorem 12.

Theorem ([Hal57, Thm 7.5], [CMZ17, Thm 6.5]). Every finitely generated torsion-free nilpotent group $G$ is isomorphic to a subgroup of $U_n(\mathbb{Z})$ for some $n = n(G)$.

Remark. If one wishes to avoid using Theorem 11 and the canonical Mal’tsev embedding or the theorem of Hall to prove Theorem 12, one could also argue by induction on the nilpotency class of $G$ via upper central series $G = Z_n \triangleright Z_{n-1} \triangleright \cdots \triangleright Z_1 \triangleright Z_0 = \{1\}$. Without loss of generality, suppose that the polynomial sequence $g$ repeats the value $g_0 \in G$ infinitely many times. We may assume that $g_0$ is the identity element $1_G$ of $G$; otherwise, replace $g$ by the polynomial sequence $g_0^{-1}g$. Let $\pi_i : Z_i \to Z_i/Z_{i-1}$ be the quotient map. Since $G$ is torsion-free, each quotient $Z_{i+1}/Z_i$ is torsion-free abelian, cf. [DK18, Lem 13.69] or [LR04, Thm 1.2.20]. Consider the induced polynomial maps $\mathbb{N}_0 \xrightarrow{g} G \xrightarrow{\pi_n} G/Z_{n-1}$. Then, $G/Z_{n-1}$ is a direct sum of finitely many copies of $\mathbb{Z}$, since $G$ is finitely generated. By Corollary 17, $\pi_n \circ g$ is a vector of polynomials. Since $\pi_n \circ g$ vanishes for infinitely many values $n$, it is identically zero, i.e., $g$ is a polynomial sequence in $Z_{n-1}$, i.e., $g$ has uc-height $\leq n-1$. By induction, one proves that $g$ is a polynomial sequence in $Z_i$, or has uc-height $\leq i$, for all $i = n, \ldots, 1, 0$. Hence, $g$ is constant.

For the proof given in the above remark, one has to argue with the upper central series, because for a torsion-free nilpotent group $G$, the quotients $C^iG/C^{i+1}G$ may not be torsion-free.

Next, we state the technical lemma, whose proof is postponed later.
Lemma 6. If $H$ is a subgroup of infinite index in a finitely generated nilpotent group $G$, there exists a normal subgroup $N$ of $G$ containing $H$ and having infinite index in $G$.

With the help of the above lemma, we can prove the following theorem:

**Theorem 13.** Let $G$ be any nilpotent group and $g: \mathbb{N}_0 \to G$ be any polynomial sequence such that $G = \langle g \rangle$. Then, every infinite subsequence (not necessarily corresponding to any arithmetic progression) generates a finite index subgroup of $G$.

**Proof.** The assertion is trivial if $G$ is a torsion group. So we may assume that $G$ has elements of infinite order. Suppose there exists an infinite subsequence of $g$, which generates an infinite index subgroup $H$ of $G$. Then, by Lemma 6, there exists a normal subgroup $N$ in $G$ containing $H$ and having infinite index in $G$. Consider the induced polynomial sequence

$$
\bar{g}: \mathbb{N}_0 \xrightarrow{\text{def}} G \to G/N \to (G/N)/\text{Tor}(G/N).
$$

Since $G/N$ is a finitely generated nilpotent group, the torsion elements of $G/N$ form a finite normal subgroup $\text{Tor}(G/N)$. Since $G/N$ is infinite, $(G/N)/\text{Tor}(G/N)$ is infinite and thus finitely generated torsion-free nilpotent. Then, $\bar{g}$ repeat the identity element in $(G/N)/\text{Tor}(G/N)$ infinitely many times. By Theorem 12, $\bar{g}$ must be the constant identity map. This is a contradiction to the assumption that $G = \langle g \rangle$, which implies $(G/N)/\text{Tor}(G/N) = \langle \bar{g} \rangle$. \qed

The technique we use to prove Lemma 6 is the theory of nearly maximal subgroups. Recall that a proper subgroup $M$ of a group $G$ is called maximal if it is not properly contained in any other proper subgroups of $G$. Similarly, a subgroup $M$ of a group $G$ is said to be nearly maximal if it has infinite index, but any subgroup properly containing $M$ has finite index in $G$.

It is known that every proper subgroup in a finitely generated group is contained in a maximal one. Similarly, with the assumption of Zorn’s lemma, we have the following result:

**Lemma 7.** Every proper subgroup of infinite index in a finitely generated infinite group is contained in a nearly maximal subgroup.

**Proof.** Let $G$ be any finitely generated infinite group and $H$ be a subgroup of infinite index. Let

$$
\Omega_H = \{K \leq G \mid H \leq K, (G : K) = \infty\}
$$

be the partially ordered set of all infinite index subgroups of $G$ containing $H$ with the partial order given by the inclusion of subgroups. Let $\mathcal{C}$ be a chain (i.e., a totally ordered subset) in $\Omega_H$. Let $J = \bigcup_{K \in \mathcal{C}} K$ be the union of all elements in $\mathcal{C}$. Then, $J$ is easily seen to be a subgroup of $G$. By Schreier’s lemma, any finite index subgroup of a finitely generated group is finitely generated. So if $J$ were of finite index in $G$, then $J$ would be a finitely generated infinite group and its finitely many generators would lie in $\bigcup_{K \in \mathcal{C}} K$ and thus in some $K \in \mathcal{C}$, since $\mathcal{C}$ is totally ordered. Hence, $K = J$ has finite index in $G$, which is a contradiction. Hence, $J$ must have infinite index in $G$. So far, the hypothesis of Zorn’s lemma has been checked. Then, by Zorn’s Lemma, $\Omega_H$ contains a maximal element $M$. Then, every subgroup properly containing $M$ has finite index in $G$. Hence, $M$ is a nearly maximal subgroup of $G$. \qed

Then, Lemma 6 is an easy consequence of the following theorem of Lennox and Robison [LR82, Thm C] or [LR04, Thm 10.4.5], which establishes a criterion for all nearly maximal subgroups being normal in finitely generated virtually solvable groups.
**Theorem.** Let $G$ be a finitely generated virtually solvable group. Then, the following conditions are equivalent:

1. each nearly maximal subgroup has finitely many conjugates in $G$;
2. each nearly maximal subgroup is normal in $G$;
3. $G$ is finite-by-nilpotent.

Here, a group $G$ is said to be finite-by-nilpotent, if it has a normal subgroup $N$ which is finite such that the quotient $G/N$ is nilpotent.

6. **Symmetric Polynomial Maps**

Let $S$ be a commutative semigroup and $S^N$ be the direct sum of $N$ copies of $S$. Let $G$ be any group and denote by $G^{S^N}$ the set of all polynomial maps $S^N \to G$. We want to define an action of the symmetric group $S_N$ on the set $G^{S^N}$ by the following manner:

For each $\sigma \in S_N$ and each polynomial map

$$f : S^N \to G; \quad (s_1, s_2, \ldots, s_N) \mapsto f(s_1, s_2, \ldots, s_N),$$

of degree $d$, we define the function

$$\sigma(f) : S^N \to S^N \quad \sigma \to G$$

$$(s_1, s_2, \ldots, s_N) \mapsto (s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(N)}) \mapsto f(s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(N)}).$$

Since $\sigma : S^N \to S^N$ is an isomorphism of commutative semigroups, by Proposition 3, $\sigma(f)$ is a polynomial map of degree $d$. Also, we have $e(f) = f$, where $e \in S_N$ is the identity element, and $\sigma \tau(f) = \sigma(\tau(f))$ for all $f \in G^{S^N}$ and $\sigma, \tau \in S_N$. Therefore, this is indeed an action.

**Definition 8.** A polynomial map $f : S^N \to G$ is called symmetric with respect to this $S_N$-action, if $\sigma(f) = f$ holds for all $\sigma \in S_N$.

Moreover, if $G^{S^N}$ is a group (for example, when $G$ is nilpotent of class $n$), then $\sigma$ fixes the identity of $G^{S^N}$, and $\sigma(fg) = \sigma(f)\sigma(g)$ for all $f, g \in G^{S^N}$ and $\sigma \in S_N$. This implies that each $\sigma$ induces a homomorphism from $S_N$ to the automorphism group of $G^{S^N}$. In this sense, we may say that $G^{S^N}$ is a (left) **non-abelian** $S_N$-module. Thus, all symmetric polynomial maps form an $S_N$-invariant subgroup $G^{S^N}_{S_N}$ of the group $G^{S^N}$.

The goal of this section is to prove that every polynomial map from $S^N$ to any nilpotent group $G$ admits an iterated symmetrization. But let us first start the discussion in a concrete case when $S = \mathbb{R}_{\geq 0}$ and $G = \mathcal{U}_n(\mathbb{R})$ to illustrate how this is done.

Any polynomial $f_{i,j} : \mathbb{R}^N_{\geq 0} \to \mathbb{R}$ admits a symmetric polynomial $\hat{f}_{i,j} = \sum_{\sigma \in S_N} \sigma(f)$. This simple fact can be generalized to the following result, which says that one can symmetrize any continuous polynomial map $f : \mathbb{R}^N_{\geq 0} \to \mathcal{U}_n(\mathbb{R})$ within a finite number of steps.

**Theorem 14.** Let $f : \mathbb{R}^N_{\geq 0} \to \mathcal{U}_n(\mathbb{R})$ be a continuous polynomial map as in Theorem 5. Then there is a natural number $M$, only dependent on $N$ and $n$, and a sequence $\sigma_1, \sigma_2, \ldots, \sigma_M \in S_N$, such that
Proof. For \( n = 2 \), we have \( U_2(\mathbb{R}) \cong \mathbb{R} \) and \( M \) can be taken to be \( N! \). So we may assume that \( n > 2 \).

The proof is given by induction on the \( k \)th diagonal entries. Clearly, the first diagonal entries of \( f^{(1)} := \prod_{\sigma \in S_N} \sigma(f) \) are given by the symmetric polynomials \( \prod_{\sigma \in S_N} \sigma(f_{i,i+1}), \forall 1 \leq i \leq n \). If one sets \( M_1 = N! \), then this gives the basis step.

Suppose that there is a finite sequence \( \sigma_1, \sigma_2, \ldots, \sigma_{M_{k-1}} \) for some \( M_{k-1} \), such that the first, \( \ldots, (k-1)\)th diagonal entries of \( f^{(k-1)} := \prod_{i=1}^{M_{k-1}} \sigma_i(f) \) are all symmetric polynomials. The goal is to show that the first, \( \ldots, k \)th diagonal entries of \( f^{(k)} := \prod_{\sigma \in S_N} \sigma(f^{(k-1)}) \) are all symmetric polynomials. Clearly, the first, \( \ldots, (k-1)\)th diagonal entries of \( f^{(k)} := \prod_{\sigma \in S_N} \sigma(f^{(k-1)}) \) remain symmetric, because they are linear combinations of finite products of first, \( \ldots, (k-1)\)th diagonal entries of \( f^{(k-1)} \), which are symmetric by induction. Similarly, the \( k \)th diagonal entries of \( f^{(k)} \) are given by the summation of

\[
\sum_{\sigma \in S_N} \sigma(f^{(k-1)}_{i,i+k})
\]

and linear combinations of finite products of first, \( \ldots, (k-1)\)th diagonal entries of \( f^{(k-1)} \), both of which are symmetric. This proves the inductive step. The induction method implies that \( M \) can be taken to be \( (N!)^{n-1} \) and hence the proof is complete. \( \square \)

The idea given in previous proof suggests the following more general result:

**Theorem 15.** Let \( S \) be a commutative semigroup, \( G \) be a nilpotent group of class \( n \) and \( f : S^N \to G \) be a polynomial map. Then there is a natural number \( M \), only dependent on \( N \) and \( n \), and a sequence \( \sigma_1, \sigma_2, \ldots, \sigma_M \in S_N \), such that the product

\[
\tilde{f} = \prod_{i=1}^{M} \sigma_i(f) = \sigma_1(f)\sigma_2(f) \cdots \sigma_M(f) : S^N \to G
\]

is a symmetric polynomial map.

Moreover, if the group \( \langle f \rangle \) generated by \( f(S^N) \) is finitely generated and the subgroup \( \langle f \mid_S \rangle \) generated by the image of the restriction of \( f \) on the diagonal \( S \) of \( S^N \) has finite index in \( \langle f \rangle \), then the subgroup \( \langle \tilde{f} \rangle \) generated by \( \tilde{f}(S^N) \) is of finite index in \( \langle f \rangle \).

**Proof.** If \( n = 0 \), then the theorem is trivial. If \( n = 1 \), then setting \( M = N! \) and \( \{\sigma_1, \sigma_2, \ldots, \sigma_M\} = S_N \), one finds that the product

\[
\tilde{f} = \prod_{i=1}^{M} \sigma_i(f) = \prod_{\sigma \in S_N} \sigma(f)
\]

satisfies that \( \tau(\tilde{f}) = \tilde{f} \) for all \( \tau \in S_N \), since \( G \) is abelian. Thus, \( \tilde{f} \) is a symmetric polynomial map.
So we may assume that \( n \geq 2 \). If one defines \( f_1 = \prod_{\sigma \in S_N} \sigma(f) \), then for any \( \tau \in S_N \), one has
\[
\tau(f_1) = \prod_{\sigma \in S_N} \tau \sigma(f) \equiv f_1 \mod C^2G,
\]i.e., \( f_1 \) is symmetric modulo \( C^2G \). Write \( \tau(f_1) = f_1 \alpha_\tau \), where \( \alpha_\tau = f_1^{-1} \tau(f_1) \) is a polynomial map from \( S^N \) to \( C^2G \). Then, \( \alpha_e = f_1^{-1} f_1 \) is the identity of \( G^S_p \), where \( e \in S_N \) is the identity, and
\[
f_1 \alpha_\tau = \sigma \tau(f_1) = \sigma(f_1 \alpha_\tau) = \sigma(f_1) \sigma(\alpha_\tau) = f_1 \alpha_\sigma \sigma(\alpha_\tau)
\]and thus \( \alpha_\sigma = \alpha_\sigma \sigma(\alpha_\tau) \). Then, we obtain a 1-cocycle \( \alpha : S_N \to C^2(G^S_p) \).

If one defines \( f_2 = \prod_{\sigma \in S_N} \sigma(f_1) \), then for any \( e \neq \tau \in S_N \), one has
\[
\tau(f_2) = \prod_{\sigma \in S_N} \tau \sigma(f_1) = \prod_{\sigma \in \{S_N \}^{-1} \tau} \sigma \sigma^{-1} \tau \sigma(f_1)
= \prod_{\sigma \in \{S_N \}^{-1} \tau} \sigma(f_1) \sigma(\alpha_{\sigma^{-1} \tau}) = \prod_{\sigma \in S_N} \sigma(f_1) \sigma(\alpha_{\sigma^{-1} \tau}),
\]Notice that \( \sigma(\alpha_{\sigma^{-1} \tau}) : S^N \to C^2G \) and \( \sigma(f_1) : S^N \to G \) are polynomial maps for all \( \sigma \in S^N \) and commutators of such terms are certainly polynomial maps from \( S^N \) to \( C^3G \). Pushing \( \sigma(\alpha_{\sigma^{-1} \tau}) \) to the rightmost, we see that
\[
\tau(f_2) \equiv f_2 \mod C^3G,
\]i.e., \( f_2 \) is symmetric modulo \( C^3G \). Similarly, we write \( \tau(f_2) = f_2 \beta_\tau \), where \( \beta_\tau = f_2^{-1} \tau(f_2) \) is a polynomial map from \( S^N \) to \( C^3G \). Then, \( \beta_e = f_2^{-1} f_2 \) is the constant map to the identity of \( G \), and
\[
f_2 \beta_\tau = \sigma \tau(f_2) = \sigma(f_2 \beta_\tau) = \sigma(f_2) \sigma(\beta_\tau) = f_2 \beta_\sigma \sigma(\beta_\tau)
\]and thus \( \beta_\tau = \beta_\sigma \sigma(\beta_\tau) \). Then, we obtain another 1-cocycle \( \beta : S_N \to C^3(G^S_p) \).

It is clear that the procedure described above continues and ends in finitely many steps, since \( G^S_p \) is nilpotent of class \( n \).
Hence, there is a finite sequence \( \sigma_1, \sigma_2, \ldots, \sigma_M \in S_N \), where \( M = (N!)^n \) such that the product
\[
\tilde{f} = \prod_{i=1}^M \sigma_i(f) = \sigma_1(f) \sigma_2(f) \cdots \sigma_M(f) : S^N \to G
\]is a symmetric polynomial map.

We have the following relations of inclusion of subgroups in \( \langle f \rangle \):
\[
\langle \langle f \rangle \rangle^M \subset \langle f \rangle \subset \langle \langle f \rangle \rangle^M = \langle \langle f \rangle \rangle^M \subset \langle \tilde{f} \rangle \subset \langle f \rangle.
\]Since \( \langle f \rangle \) is finitely generated and nilpotent, \( \langle f \rangle \) is finitely generated and nilpotent. By a result due to Mal’tsev (cf. [CMZ17, Thm 2.23]), \( \langle \langle f \rangle \rangle^M \) has finite index in \( \langle f \rangle \) and since \( \langle f \rangle \) has finite index in \( \langle f \rangle \), it also has finite index in \( \langle f \rangle \). Hence, \( \langle \tilde{f} \rangle \) has finite index in \( \langle f \rangle \). \( \Box \)

Remark. The strategy given in proofs above will be called iterated symmetrization, which turns out to be very useful in the sequel.
7. Polynomial Sets

Definition 9. A subset $U$ of a path-connected nilpotent Lie group $N$ is said to be parameterized by some continuous polynomial map $f : \mathbb{R}^n_{\geq 0} \to N$, if it is the image of $f$. In this case, we denote $(U | f : \mathbb{R}^n_{\geq 0} \to N)$ and call it a polynomial set in $N$; but sometimes we will abbreviate $f$ and simply call $U$ a polynomial set in $N$ for short. If $U$ is open (resp. is closed, resp. has nonempty interior) in $N$, then we call $U$ an open (resp. a closed, resp. a proper) polynomial set in $N$.

A nonempty subset $V$ of a nilpotent group $G$ is called a polynomial set, if it is the inverse image $\phi^{-1}(U)$ of a polynomial set $(U | f : \mathbb{R}^n_{\geq 0} \to N)$ of a nilpotent Lie group $N$ along some group homomorphism $\phi : G \to N$. In this case, for completeness, we denote the polynomial set by $(V | \phi : G \to N, (U | f : \mathbb{R}^n_{\geq 0} \to N))$.

We call $V = \phi^{-1}(U)$ an open (resp. a closed, resp. a proper) polynomial set in $G$, if $U$ has the same property in $N$. In particular, the generalized cones in $G$ is given by proper polynomial set.

Remark 3. One may wonder why we require the nilpotent Lie group $N$ to be path-connected. For one reason, continuous image of a path-connected set is path-connected; for the other, if we do not assume this, then in the trivial sense any group having at most countably many elements can be viewed as a 0-dimensional Lie group with the discrete topology. For example, the group $\mathcal{U}_n(\mathbb{Z})$ of upper unitriangular $n \times n$ matrices in integers, has at most countably many elements. Then, any singleton of $\mathcal{U}_n(\mathbb{Z})$ viewed as a 0-dimensional Lie group with the discrete topology is open.

Remark. Corollary 12 implies that the only polynomial sets in $\mathbb{R}$ are singletons, unbounded closed intervals and the whole $\mathbb{R}$. Hence, the only open polynomial set in $\mathbb{R}$ is $\mathbb{R}$ itself, and the only proper polynomial sets in $\mathbb{R}$ are either unbounded closed intervals or the whole $\mathbb{R}$.

The following lemma proves the existence of a proper polynomial set inside any Kamke domains.

Lemma 8. For any $B \geq 2$, a Kamke domain

$$U(B, N) = \{(l_1, \ldots, l_B) \in \mathbb{R}^B_{\geq 0} | k_1 < l_1, k_\nu l_1^\nu < l_\nu < K_\nu l_1^\nu, \nu = 2, 3, \ldots, B\}$$

always contains a proper polynomial set.

Proof. Let $y_\kappa = x_\kappa + k_1/n + \varepsilon/n$ and consider the following polynomial map

$$q : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^B; \ (x_1, \ldots, x_n) \mapsto (l_1, \ldots, l_B),$$

where

$$\begin{align*}
l_1 &= \sum_{\kappa=1}^n y_\kappa > k_1, \\
l_2 &= C_2 \sum_{\kappa=1}^n y_\kappa^2 + D_2 l_1^2 + \varepsilon, \\
&\vdots \\
l_B &= C_B \sum_{\kappa=1}^n y_\kappa^B + D_B l_1^B + \varepsilon,
\end{align*}$$

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and \( \varepsilon, C_2, \ldots, C_B, D_2, \ldots, D_B \) are some positive numbers. By the multinomial formula,

\[
l_1^{\nu} = \left( \sum_{\kappa=1}^{n} y_\kappa \right)^{\nu} = \sum_{j_1+j_2+\cdots+j_n=\nu} \left( \sum_{\kappa=1}^{n} y_\kappa \right)^{\nu} \prod_{\kappa=1}^{n} y_\kappa^{j_\kappa}
\]

\[
= \sum_{\kappa=1}^{n} y_\kappa^{\nu} + \sum_{j_1+j_2+\cdots+j_n=\nu} \left( \sum_{\kappa=1}^{n} y_\kappa \right)^{\nu} \prod_{\kappa=1}^{n} y_\kappa^{j_\kappa}
\]

\[
\geq \sum_{\kappa=1}^{n} y_\kappa^{\nu} + \sum_{j_1+j_2+\cdots+j_n=\nu} \left( \sum_{\kappa=1}^{n} y_\kappa \right)^{\nu} \prod_{\kappa=1}^{n} (k_1/n + \varepsilon/n)^{j_\kappa}
\]

\[
= \sum_{\kappa=1}^{n} y_\kappa^{\nu} + (n^{\nu} - n) (k_1/n + \varepsilon/n)^{\nu}
\]

\[
= \sum_{\kappa=1}^{n} y_\kappa^{\nu} + (1 - n^{1-\nu}) (k_1 + \varepsilon)^{\nu},
\]

where equality holds if all \( x_\kappa \) are 0, and by the generalized mean inequality,

\[
\left( \frac{l_1^{\nu}}{n} \right)^{\nu} = \left( \frac{1}{n} \sum_{\kappa=1}^{n} y_\kappa \right)^{\nu} \leq \frac{1}{n} \sum_{\kappa=1}^{n} y_\kappa^{\nu},
\]

where equality holds if all \( x_\kappa \) are the same. Then, for all \( \nu = 2, 3, \ldots, B \), we must have

\[
(C_\nu n^{1-\nu} + D_\nu) l_1^{\nu} + \varepsilon \leq C_\nu \sum_{\kappa=1}^{n} y_\kappa^{\nu} + D_\nu l_1^{\nu} + \varepsilon = l_\nu
\]

\[
\leq C_\nu \left( l_1^{\nu} - (1 - n^{1-\nu}) (k_1 + \varepsilon)^{\nu} \right) + D_\nu l_1^{\nu} + \varepsilon
\]

\[
= (C_\nu + D_\nu) l_1^{\nu} - (C_\nu (1 - n^{1-\nu}) (k_1 + \varepsilon)^{\nu} - \varepsilon).
\]

For \( k_\nu l_1^{\nu} < l_\nu < K_\nu l_1^{\nu} \) to be true, it suffices to take some sufficiently large \( n > 1 \), \( C_\nu, D_\nu \) satisfying

\[
0 < k_\nu - C_\nu n^{1-\nu} = D_\nu = K_\nu - C_\nu, \quad 0 < C_\nu = \frac{K_\nu - k_\nu}{1 - n^{1-\nu}},
\]

and some sufficiently small \( \varepsilon > 0 \) such that

\[
C_\nu (1 - n^{1-\nu}) (k_1 + \varepsilon)^{\nu} - \varepsilon = \frac{K_\nu - k_\nu}{1 - n^{1-\nu}} (1 - n^{1-\nu}) (k_1 + \varepsilon)^{\nu} - \varepsilon
\]

\[
= (K_\nu - k_\nu) (k_1 + \varepsilon)^{\nu} - \varepsilon
\]

\[
> (K_\nu - k_\nu) k_\nu^{\nu} - \varepsilon > 0.
\]

To show that \( q(\mathbb{R}^n_{\geq0}) \) has a nonempty interior inside \( U(B, N) \), one check that the rank of the Jacobian matrix of \( q \) is \( B \), which basically follows from the linearly independence of geometric series. \( \square \)
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