Frames, their relatives and reproducing kernel Hilbert spaces

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Abstract
This paper considers different facets of the interplay between reproducing kernel Hilbert spaces (RKHS) and stable analysis/synthesis processes: first, we analyze the structure of the reproducing kernel of a RKHS using frames and reproducing pairs. Second, we present a new approach to prove the result that finite redundancy of a continuous frame implies atomic structure of the underlying measure space. Our proof uses the RKHS structure of the range of the analysis operator. This in turn implies that all the attempts to extend the notion of Riesz basis to general measure spaces are fruitless since every such family can be identified with a discrete Riesz basis. Finally, we show how the range of the analysis operators of a reproducing pair can be equipped with a RKHS structure.

Keywords: continuous frames, reproducing pairs, reproducing kernel Hilbert spaces, redundancy, atomic measures

1. Introduction

Reproducing kernel Hilbert spaces (RKHS) were introduced by Zaremba [37] and Mercer [26] and systematically studied by Aronszajn [8] in 1950. These spaces play an important role in many diverse branches of mathematics such as complex analysis [20], and learning theory [30]. Another field with manifold connections to RKHS, which is the main focus of this paper, is frame theory and related concepts.
A mapping \( x \mapsto \Psi_x \in \mathcal{H} \) is called a \textit{continuous frame} if there exist constants \( m, M > 0 \) such that
\[
m\|f\|^2 \leq \int_X |\langle f, \Psi_x \rangle|^2 d\mu(x) \leq M\|f\|^2, \quad \forall f \in \mathcal{H},
\]
(1)

Frames have proven to be a viable tool in many different fields such as signal processing [12] acoustics [13] or mathematical physics [2, 11]. In particular, the theory of coherent states [2, 31] motivated the introduction of continuous frame as a generalization and the two concepts have been in a fruitful symbiosis ever since. It is however not always possible to satisfy both inequalities, which is why new concepts like semi-frames [3, 4] and reproducing pairs [5, 34, 35] were introduced recently. An upper (resp. lower) semi-frame is a complete system that only satisfies the upper (resp. lower) frame inequality in (1). A reproducing pair is a pair of mappings \((\Psi, \Phi)\) that generates a bounded and boundedly invertible analysis/synthesis process, i.e. the operator \( S_{\Psi, \Phi} \)
\[
S_{\Psi, \Phi} f = \int_X \langle f, \Psi_x \rangle \Phi_x d\mu(x), \quad f \in \mathcal{H},
\]
is an element of \( GL(\mathcal{H}) \). Note that neither \( \Psi \) nor \( \Phi \) is assumed to satisfy any of the two frame inequalities.

This paper is divided into three main parts, portraying different facets of the interplay between frames, reproducing pairs, and RKHS.

In section 3, we study systems taking values in RKHS. In particular, we give an explicit expression for the reproducing kernel in terms of a reproducing pair in theorem 4 that extends the results from [27, 29] and introduce a necessary condition for a family of vectors to form a frame, see proposition 6.

In section 4 we study the dependence of the redundancy of (semi-)frames on the structure of the measure space. In the discrete case, the redundancy of a frame measures, loosely speaking, how much a Hilbert space is oversampled by that frame, see for example [14, 15]. It is however impossible to directly translate this concept to continuous (semi-)frames. In [9] the authors therefore used a property of Riesz bases to define redundancy. A Riesz basis is a discrete, non-redundant frame, i.e. its analysis operator \( C_{\Psi} : \mathcal{H} \to \ell^2 \) is surjective. We therefore define redundancy of a (semi-)frame \( \Psi \) by
\[
R(\Psi) := \dim(\text{Ran } C_{\Psi}^\perp).
\]
It has been observed in several articles [9, 24, 25] that \( R(\Psi) \) depends on the underlying measure space \((X, \mu)\). In particular, if a (lower semi-)frame has finite redundancy, then it follows that \((X, \mu)\) is atomic. The proofs in the aforementioned papers all rely in one way or the other on the following argument: if the redundancy of a frame is zero (finite), then
\[
\inf \left\{ \mu(A) : A \text{ measurable and } \mu(A) > 0 \right\} = C > 0,
\]
which implies that \((X, \mu)\) is atomic. We present a new proof in section 4.1 using that \( \text{Ran } C_{\Psi} \) is a RKHS, which, in our opinion, better explains the underlying structure of the problem. On the other hand, if \((X, \mu)\) is non-atomic (as it is the case for coherent states systems), then \( R(\Psi) = \infty \).

If \( \Psi \) is a family of coherent states [2], then \( |\langle f, \Psi_x \rangle|^2 \) is the probability density of a system in the state \( f \) being in the pure state \( \Psi_x \). If \( X \) is an-atomic, then \( R(\Psi) = \infty \) implies that there is always an infinite dimensional subspace of probability densities that do not describe any system in any state \( f \in \mathcal{H} \), see remark 15.
Moreover, the result shows that all efforts to generalize the notion of Riesz bases \( R(\Psi) = 0 \), see [7, 21], to the setting of measure spaces that are not atomic are essentially an attempt to square the circle. The measure space of a continuous Riesz basis is always atomic and every such system can therefore be written as a discrete family of vectors, see corollary 19.

For upper semi-frames on the other hand, there is no connection between the structure of the measure space and the redundancy. In particular, we give a sufficient condition for the existence of upper semi-frames indexed by a non-atomic measure space with redundancy zero, see proposition 20.

Finally, section 5 is concerned with characterizing the ranges of the analysis operators of a reproducing pair. The omission of the frame inequalities causes the inconvenience that \( \text{Ran} \, C_\Psi \) and \( \text{Ran} \, C_\Phi \) need no longer be contained in \( L^2(X, \mu) \). Therefore, in [5] a pair of Hilbert spaces, intrinsically generated by a reproducing pair, was introduced to study this problem. We demonstrate that these spaces are actually RKHS and calculate the reproducing kernels.

2. Preliminaries

2.1. Atomic and non-atomic measures

Throughout this paper we assume that \((X, \mu)\) is a measure space, where \(\mu\) is a non-trivial, \(\sigma\)-finite, and non-negative measure. A measurable set \(A \subset X\) is called an \textit{atom} if \(\mu(A) > 0\), and for any measurable subset \(B \subset A\), with \(\mu(B) < \mu(A)\), it holds \(\mu(B) = 0\). A measure space is called \textit{atomic} if there exists a partition \(\{A_n\}_{n \in \mathbb{N}}\) of \(X\) consisting of atoms and null sets. The space \((X, \mu)\) is called \textit{non-atomic} if there are no atoms in \((X, \mu)\). To our knowledge there is no term to denote a measure space that is not atomic. In order to avoid any confusion with non-atomic spaces, we therefore call a measure space \textit{an-atomic} if it is not atomic.

A well-known result by Sierpiński states that non-atomic measures take a continuity of values.

\textbf{Theorem 1 (Sierpiński [32]).} Let \((X, \mu)\) be non-atomic and let \(A \subset X\) be measurable with \(\mu(A) > 0\). For every \(0 \leq b \leq \mu(A)\), there exists \(B \subset A\) such that \(\mu(B) = b\).

Since we could not find any reference for the second part of the following lemma, we provide a proof in the appendix.

\textbf{Lemma 2.} Let \((X, \mu)\) be a \(\sigma\)-finite measure space.

(i) There exists \(\mu_a\) atomic and \(\mu_c\) non-atomic such that \(\mu = \mu_a + \mu_c\).

(ii) If \((X, \mu)\) is an-atomic, then there exists \(A \subset X\) with \(\mu(A) > 0\) and \((A, \mu)\) non-atomic.

2.2. Continuous frames, semi-frames and reproducing pairs

Let \(H\) be a separable Hilbert space. We denote by \(GL(H)\) the space of bounded linear operators on \(H\) with bounded inverse.

\textbf{Definition 1.} A mapping \(\Psi : X \to H\) is called \textit{complete}, if \(\Psi\) is weakly measurable, i.e. \(x \mapsto \langle f, \Psi_x \rangle\) is a measurable function for every \(f \in H\), and

\[ 0 < \int_X |\langle f, \Psi_x \rangle|^2 \, d\mu(x), \quad \forall f \in H \setminus \{0\}. \]
A complete mapping \( \Psi \) is called a **continuous frame** if there exist positive constants \( m, M > 0 \) such that
\[
m \|f\|^2 \leq \int_X |\langle f, \Psi_x \rangle|^2 \, d\mu(x) \leq M \|f\|^2, \quad \forall f \in \mathcal{H}.
\]  
(4)

The constants \( m, M \) are called the frame bounds and \( \Psi \) is called Bessel if the second inequality in (4) is satisfied. If \( m = M = 1 \), then \( \Psi \) is called a Parseval frame. If \( (X, \mu) \) is a countable set equipped with the counting measure, then one recovers the classical definition of a discrete frame, see for example [16]. For a self-contained introduction to continuous frames, we refer the reader to [28].

The fundamental operators in frame theory are given by the **analysis operator** \( C_{\Psi} : \mathcal{H} \to L^2(X, \mu), [C_{\Psi}f](x) := \langle f, \Psi_x \rangle \), and the **synthesis operator** \( D_{\Psi} : L^2(X, \mu) \to \mathcal{H}, D_{\Psi}f := \int_X F(x) \Psi_x \, d\mu(x). \)

where the integral is defined weakly. If the upper frame bound is satisfied, then \( D_{\Psi} = C_{\Psi}^* \) and one has \( L^2(X, \mu) = \text{Ran} \, C_{\Psi} \oplus \text{Ker} \, D_{\Psi} \). The **frame operator** \( S_{\Psi} \in GL(\mathcal{H}) \) is defined as the composition of \( C_{\Psi} \) and \( D_{\Psi} \)
\[
S_{\Psi} : \mathcal{H} \to \mathcal{H}, \quad S_{\Psi}f := D_{\Psi}C_{\Psi}f = \int_X \langle f, \Psi_x \rangle \Psi_x \, d\mu(x).
\]

Every frame \( \tilde{\Psi} \) satisfying
\[
f := D_{\tilde{\Psi}}C_{\tilde{\Psi}}f = D_{\Psi}C_{\Psi}f, \quad \forall f \in \mathcal{H},
\]
is called a **dual frame** for \( \Psi \).

There exists a variety of interesting complete systems that do not meet both frame conditions, see section 4.3. Note that these mappings are not artificially constructed, but they appear naturally for example in the context of coherent states [3, 18, 36]. Several concepts to generalize the frame property were thus introduced and studied. An **upper semi-frame** is a complete Bessel system, i.e.
\[
0 < \int_X |\langle f, \Psi_x \rangle|^2 \, d\mu(x) \leq M \|f\|^2, \quad \forall f \in \mathcal{H}\{0\},
\]
whereas a **lower semi-frame** satisfies
\[
m \|f\|^2 \leq \int_X |\langle f, \Psi_x \rangle|^2 \, d\mu(x),
\]
see [3, 4].

Another generalization is the concept of reproducing pairs, defined in [34] and further investigated in [5, 6, 35]. Here, one considers a pair of mappings instead of a single one without any assumption on frame inequalities.

**Definition 2.** Let \( \Psi, \Phi : X \to \mathcal{H} \) be weakly measurable. The pair \( (\Psi, \Phi) \) is called a **reproducing pair** for \( \mathcal{H} \) if the operator \( S_{\Psi, \Phi} : \mathcal{H} \to \mathcal{H}, \) defined by
\[
\langle S_{\Psi, \Phi}f, g \rangle := \int_X \langle f, \Psi_x \rangle \langle \Phi_x, g \rangle \, d\mu(x),
\]
(5)
is an element of \( GL(\mathcal{H}) \).
Note that the definition of reproducing pairs is symmetric, i.e. \((\Psi, \Phi)\) is a reproducing pair if and only if \((\Phi, \Psi)\) is a reproducing pair. Therefore, we have \(S_{\Phi, \Psi} = S_{\Psi, \Phi}^*\). Moreover, we have for every \(f, g \in H\)

\[
\langle f, g \rangle = \int_X \langle f, S_{\Phi, \Psi}^{-1} \Psi \rangle \langle \Phi, g \rangle \, d\mu(x) = \int_X \langle f, \Psi \rangle \langle S_{\Phi, \Psi}^{-1} \Psi, g \rangle \, d\mu(x). \tag{6}
\]

**Examples:** As a trivial example consider \(\mu\) to be the counting measure on \(\mathbb{N}\) and let \(\{e_n\}_{n \in \mathbb{N}}\) be an orthonormal basis of \(H\). The two families of vectors \(\Psi = (e_1, 2e_2, 3e_3, 4e_4, \ldots)\) and \(\Phi = (e_1, \frac{1}{2}e_2, 3e_3, \frac{1}{4}e_4, \ldots)\) form a reproducing pair even though both families violate both frame inequalities.

Examples on non-atomic measure spaces can be constructed from (sections of) square-integrable group representations. Take two vectors that are not admissible but satisfy a joint admissibility condition then these two generalized wavelet systems form a reproducing pair but neither of them forms a continuous frame, see [34, example 5.1].

### 2.3. Reproducing kernel Hilbert spaces (RKHS)

Let \(F(\Omega, \mathbb{C})\) denote the vector space of all functions \(f : \Omega \to \mathbb{C}\).

**Definition 3.** A Hilbert space \(H_K \subset F(\Omega, \mathbb{C})\) is called a reproducing kernel Hilbert space (RKHS), if the point evaluation functional \(\delta_z : H_K \to \mathbb{C}\), \(\delta_z(f) := f(z)\) is bounded for every \(z \in \Omega\), that is, if there exists \(C_z > 0\) such that \(|\delta_z(f)| \leq C_z \|f\|\) for every \(f \in H_K\).

As \(\delta_z\) is bounded, there exists a unique vector \(k_z \in H_K\) such that \(f(z) = \langle f, k_z \rangle\), for every \(f \in H_K\). The function \(K(z, w) := k_z(w) = \langle k_z, k_w \rangle\) is called the reproducing kernel for \(H_K\).

The reproducing kernel is unique, \(K(z, w) = \overline{K(w, z)}\) and its diagonal reads

\[
K(z, z) = \langle k_z, k_z \rangle = \|k_z\|^2 = \sup \{ |f(z)|^2 : f \in H_K, \|f\| = 1 \}.
\]

The following result can be found in [1, theorems 3.1 and 3.2].

**Theorem 3.** If \(H_K\) is a RKHS and \(\{\phi_i\}_{i \in I} \subset H_K\) an orthonormal basis, then

\[
K(z, w) = \sum_{i \in I} \phi_i(z) \overline{\phi_i(w)}, \tag{7}
\]

with pointwise convergence of the series. In particular,

\[
0 < \sum_{i \in I} |\phi_i(z)|^2 = K(z, z) < \infty, \quad \forall z \in \Omega. \tag{8}
\]

Conversely, if there exists an orthonormal basis for a Hilbert space \(H_K \subset F(\Omega, \mathbb{C})\) that satisfies (8), then \(H_K\) can be identified with a RKHS consisting of functions \(f : \Omega \to \mathbb{C}\).

For a thorough introduction to RKHS we refer the reader to [8, 27].

### 3. Frames and reproducing pairs taking values in a RKHS

In this section, we investigate the pointwise behavior of frames in RKHS, characterize the reproducing kernel and introduce sufficient conditions on a frame that ensures the existence of a reproducing kernel.
The following result adapts the arguments of [27, theorem 3.12] to reproducing pairs.

**Theorem 4.** Let \( \mathcal{H}_K \) be a RKHS and \( \Psi = \{ \psi_i \}_{i \in I} \). \( \Phi = \{ \phi_i \}_{i \in I} \subset \mathcal{H}_K \). The pair \( (\Psi, \Phi) \) is a reproducing pair for \( \mathcal{H}_K \) if and only if there exists \( A \in GL(\mathcal{H}_K) \) such that

\[
K(z, w) = \sum_{i \in I} (A\phi_i)(z)\bar{\psi}_i(w) = \sum_{i \in I} (A^*\psi_i)(z)\bar{\phi}_i(w), \quad \forall z, w \in \Omega, \tag{9}
\]

where the series converges pointwise. In particular, \( A \) is uniquely given by \( S_{\Psi, \Phi}^{-1} \).

**Proof.** If \( (\Psi, \Phi) \) is a reproducing pair, then by (6)

\[
K(z, w) = \langle k_w, k_z \rangle = \sum_{i \in I} \langle k_w, \psi_i \rangle \langle S_{\Psi, \Phi}^{-1} \phi_i, k_z \rangle = \sum_{i \in I} \overline{\psi}_i(w)\langle S_{\Psi, \Phi}^{-1} \phi_i(z) \rangle.
\]

Conversely, assume that \( K \) is given by (9). If \( f, g \in \text{span}\{ k_z : z \in \Omega \} \), that is, there exist \( \alpha_n, \beta_m \in \mathbb{C} \), and \( z_n, w_m \in \Omega \), such that \( f = \sum_n^{N} \alpha_n k_{z_n} \), and \( g = \sum_m^{M} \beta_m k_{w_m} \), then

\[
\langle f, g \rangle = \sum_{n,m=1}^{N,M} \overline{\alpha_n \beta_m} \langle k_{z_n}, k_{w_m} \rangle = \sum_{n,m=1}^{N,M} \alpha_n \overline{\beta_m} K(w_m, z_n)
\]

\[
= \sum_{n,m=1}^{N,M} \overline{\alpha_n \beta_m} \sum_{i \in I} \langle A\phi_i \rangle(w_m)\overline{\psi_i(z_n)}
\]

\[
= \sum_{n,m=1}^{N,M} \overline{\alpha_n \beta_m} \sum_{i \in I} \langle k_{z_n}, \psi_i \rangle \langle A\phi_i, k_{w_m} \rangle
\]

\[
= \sum_{i \in I} \left( \sum_{n=1}^{N} \alpha_n k_{z_n}, \psi_i \right) \left( A\phi_i, \sum_{m=1}^{M} \beta_m k_{w_m} \right)
\]

\[
= \sum_{i \in I} \langle f, \psi_i \rangle \langle A\phi_i, g \rangle = \langle AS_{\Psi, \Phi} f, g \rangle.
\]

In [27, proposition 3.1] it is shown that \( \text{span}\{ k_z : z \in \Omega \} \) is dense in \( \mathcal{H}_K \). Therefore, it follows that \( AS_{\Psi, \Phi} = I \). As \( A \in GL(\mathcal{H}_K) \) we may conclude that \( S_{\Psi, \Phi} \in GL(\mathcal{H}_K) \), that is, \( (\Psi, \Phi) \) is a reproducing pair.

To show the second equality in (9), we need to recall that the notion of a reproducing pair is symmetric. Then all previous arguments carry through exactly the same way with the roles of \( \Psi \) and \( \Phi \) interchanged. \( \square \)

**Remark 5.** For certain special cases, theorem 4 is already known. In particular, the result can be found in [29, theorem 7] if \( \Psi \) and \( \Phi \) are dual frames, and in [27, theorem 3.12] if \( \Psi = \Phi \) is a Parseval frame.

**Proposition 6.** Let \( \mathcal{H}_K \) be a RKHS and \( \{ \psi_i \}_{i \in I} \subset \mathcal{H}_K \).

(i) If the family \( \{ \psi_i \}_{i \in I} \) is Bessel with bound \( M \), then

\[
\sum_{i \in I} \left| \psi_i(z) \right|^2 \leq M, \quad \forall z \in \Omega. \tag{10}
\]
(ii) If \( \{\psi_i\}_{i \in I} \) satisfies the lower frame inequality with bound \( m \), then
\[
0 < m \leq \frac{\sum_{i \in I} |\psi_i(z)|^2}{K(z, z)}, \quad \forall z \in \Omega.
\] (11)

Proof. If \( \{\psi_i\}_{i \in I} \) is Bessel, then, for every \( z \in \Omega \), it holds
\[
\sum_{i \in I} |\psi_i(z)|^2 = \sum_{i \in I} |(k_z, \psi_i)|^2 \leq M\|k_z\|^2 = MK(z, z).
\]
The same argument shows the lower bound in (11) if \( \{\psi_i\}_{i \in I} \) is a lower semi-frame.

Remark 7.

(i) The converse statements of proposition 6 are in general false. First, consider the sequence
\[
\psi_1 = (1, 0, 0, \ldots), \quad \psi_2 = (0, 1, 1, 0, \ldots), \quad \psi_3 = (0, 0, 0, 1, 1, 1, \ldots), \ldots \in \mathcal{H}_K = \ell(\mathbb{N}),
\]
i.e. \( \psi_n(l) = 1 \) if \( l \in n(n-1)/2 + 1, \ldots, n(n+1)/2 \), and zero otherwise. This system satisfies (10) with \( M = 1 \) but \( \{\psi_i\}_{i \in \mathbb{N}} \) is not Bessel. To see this, recall that a sequence is Bessel if and only if its synthesis operator \( D_\psi \) is bounded. Taking \( a = \{1/n\}_{n \in \mathbb{N}} \) gives
\[
\|D_\psi a\|_{\ell^2(\mathbb{N})}^2 = \left\| \left( \frac{1}{2}, \frac{1}{2} \right) \right\|_{\ell^2(\mathbb{N})}^2 = \frac{1}{k} \rightarrow \infty.
\]
To see that the reverse of (11) is false, consider \( \mathcal{H}_K = \mathbb{C}^2 \), \( \Omega = \{1, 2\} \) and \( \psi = (1, 1) \). Then \( K(z, z) = 1 \) and \( |\psi(1)|^2 = |\psi(2)|^2 = 1 \), but \( \psi \) is not spanning.

The reverse statement remains false if one additionally assumes that \( \{\psi_i\}_{i \in I} \) is complete. Let us construct a counterexample from the system of integer time-frequency shifts of the Gaussian window \( \varphi(t) = 2^{1/2}e^{-\pi t^2} \), which is a complete Bessel sequence (i.e. an upper semi-frame) but not a frame in \( L^2(\mathbb{R}) \), see [22, 35]. Let \( z = (x, \omega) \in \mathbb{R}^2 \), and \( \pi(z)f(t) := f(t-x)e^{2\pi i wt} \). The short-time Fourier transform with window \( \varphi \) is defined as \( V_\varphi f(z) = (f, \pi(z)\varphi) \), and the space
\[
\mathcal{H}_K := \{V_\varphi f : f \in L^2(\mathbb{R})\} \subset L^2(\mathbb{R}^2),
\]
forms a RKHS with kernel \( K(z, w) = \langle \pi(\omega)\varphi, \pi(z)\varphi \rangle \). If \( F = V_\varphi f \in \mathcal{H}_K \) for some \( f \in L^2(\mathbb{R}) \), then
\[
\sum_{\lambda \in \mathbb{Z}} |\langle F, K(\cdot, \lambda) \rangle|^2 = \sum_{\lambda \in \mathbb{Z}} |\langle f, \pi(\lambda)\varphi \rangle|^2,
\]
and the isometry property \( \|F\| = \|f\| \) implies that \( \{\psi_\lambda\}_{\lambda \in \mathbb{Z}^2} := \{K(\cdot, \lambda)\}_{\lambda \in \mathbb{Z}^2} \) is a complete Bessel sequence for \( \mathcal{H}_K \), but not a frame. Moreover, we have \( |K(z, w)| = e^{-\pi |z-w|^2}/2 \) by [22, lemma 1.5.2], which gives that
\[
z \mapsto \sum_{\lambda \in \mathbb{Z}^2} |\psi_\lambda(z)|^2 = \sum_{\lambda \in \mathbb{Z}^2} e^{-\pi |z-\lambda|^2} > e^{-\pi/2},
\]
is a strictly positive, continuous, and \( \mathbb{Z}^2 \)-periodic function. Therefore, as \( |K(z, z)| = 1 \), (11) is satisfied with \( m \geq e^{-\pi/2} \).

\[7\]
(ii) Another consequence of proposition 6 is that, if \( \{ \psi_i \} \in I \subset \mathcal{H} \) is a Bessel sequence, then
\[
\left\| \sum_{i \in I} \psi_i(z) \psi_i \right\| \leq M \sqrt{K(z, z)} < \infty, \quad \forall z \in \Omega.
\] (12)

Let us now consider the converse part of theorem 3, which gives a sufficient condition on orthonormal bases that ensures that the space is a RKHS. We generalize the result to a condition on discrete frames.

**Theorem 8.** Let \( \mathcal{H} \subset \mathcal{F}(X, \mathbb{C}) \) be a Hilbert space of functions. If there exists a discrete frame \( \Psi = \{ \psi_i \} \in I \subset \mathcal{H} \) such that
\[
0 < \sum_{i \in I} |\psi_i(z)|^2 = C_z < \infty, \quad \forall z \in \Omega,
\] (13)
then \( \mathcal{H} \) is a RKHS.

**Proof.** If \( \Psi \) is a frame such that (13) holds, then
\[
\sum_{i \in I} \psi_i(z) S^{-1}_\Psi \psi_i
\] (14)
is well-defined in \( \mathcal{H} \) for every \( z \in \Omega \). Moreover, for \( f \in \mathcal{H} \), we have
\[
|f(z)| = \left| \sum_{i \in I} \langle f, S^{-1}_\Psi \psi_i \rangle \psi_i(z) \right| = \left| \sum_{i \in I} \psi_i(z) S^{-1}_\Psi \psi_i \right| \leq \|f\| \left| \sum_{i \in I} \psi_i(z) \psi_i \right| \leq \frac{1}{m} \|f\| \left| \sum_{i \in I} \psi_i(z) \psi_i \right| \leq \frac{\sqrt{M} \left( \sum_{i \in I} |\psi_i(z)|^2 \right)^{1/2}}{m} \|f\| = \frac{\sqrt{CzM}}{m} \|f\|.
\]
Hence, point evaluation is continuous. \( \square \)

Note that all results of this section can be reformulated for continuous frames and reproducing pairs, if one replaces the index set \( I \) by \( (X, \mu) \).

### 4. Continuous (semi-)frames and their redundancy

Redundancy of a discrete frame measures, roughly speaking, how much the frame is oversampling the Hilbert space \( \mathcal{H} \). A non-redundant discrete frame is in fact a basis. Those so-called Riesz bases are frames with the additional property that \( \text{Ran} \, C_\Psi = (\text{Ker} \, D_\Psi) = \ell^2(\mathcal{I}) \). This justifies defining the redundancy for general families \( \Psi \) by
\[
R(\Psi) := \dim(\text{Ran} \, C_\Psi) = \dim(\text{Ran} \, D_\Psi) = \text{ind}(C_\Psi) = \text{ind}(D_\Psi),
\] (15)

Note that \( C_\Psi \) is a Fredholm operator if \( \Psi \) is Bessel and has finite redundancy. In that case, \( R(\Psi) = -\text{ind}(C_\Psi) = \text{ind}(D_\Psi) \), where \( \text{ind}(A) \) denotes the Fredholm index of a bounded operator \( A \).
Example 1. Let \( \{ e_n \}_{n \in \mathbb{N}} \) be an orthonormal basis. If we define \( \Psi = \{ e_1, e_1, e_2, e_3, e_4, \ldots \} \), and \( \Phi = \{ e_1, e_2, e_2, e_3, e_3, e_4, e_4, \ldots \} \), then \( R(\Psi) = 1 \) and \( R(\Phi) = \infty \).

If \( \Psi = \{ f_n \}_{n=1}^{N} \) is a finite frame for \( C^d \), then \( R(\Psi) = N - d \), whereas the classical definition of redundancy for finite frames gives \( N/d \).

4.1. Frames and lower semi-frames

The main goal of this section is to give a new proof for theorem 9, which connects finite redundancy with the structure of the measure space, a result that has already been stated in [24, theorem 2], [9, theorem 2.2] and [25, proposition 3.3]. We thereby use the property that \( \text{Ran} \, C_{\Psi} \) forms a RKHS. In our opinion, this proof does a better job explaining the inherent structure of continuous frames.

Theorem 9. If a (lower semi-)frame \( \Psi \) has finite redundancy \( R(\Psi) < \infty \), then the measure space \( (X, \mu) \) is atomic.

The converse is obviously not true, take for example the discrete family \( \Phi \) in example 1.

Corollary 10. If \( (X, \mu) \) is an-atomic and \( \Psi \) a frame, then \( R(\Psi) = \infty \) and there exist infinitely many dual frames for \( \Psi \).

Proof. The first part is theorem 9. For the second part, let \( \Phi \) be any Bessel mapping with \( \text{Ran} \, C_{\Phi} \subset \text{Ran} \, C_{\Psi}^{*} \). If we define \( \tilde{\Psi} := S_{\Psi}^{-1} \Psi + \Phi \), where \( \tilde{\Psi}_x = S_{\Psi}^{-1} \Psi_x + \Phi_x \), we have that \( \tilde{\Psi} \) is a dual frame for \( \Psi \). An example for such a system \( \Phi \) is for example \( \Phi_x = g \cdot G(x) \), where \( g \in \mathcal{H} \) arbitrary and \( G \) a particular representative of an element in \( \text{Ran} \, C_{\Psi}^{*} \).

We need to collect some auxiliary results in order to prove theorem 9.

Proposition 11 ([21], corollary 2.9). If \( \Psi \) satisfies the lower frame inequality, then \( (\text{Ran} \, C_{\Psi}, \| \cdot \|_2) \) is a RKHS. Moreover, for any subspace \( \mathcal{H}_K \) of \( L^2(X, \mu) \), the following are equivalent:

(i) There exists a continuous frame \( \Psi \) such that \( \text{Ran} \, (C_{\Psi}) = \mathcal{H}_K \).
(ii) \( \mathcal{H}_K \) is a RKHS.

By definition, \( L^2(X, \mu) \) is a space of equivalence classes of functions and does therefore not allow for pointwise evaluation. However, if we take an orthonormal basis \( \{ \phi_i \}_{i \in \mathcal{I}} \) for \( L^2(X, \mu) \) and fix one particular representative \( \phi_i(x) \) of the equivalence class \( \phi \) for every \( i \in \mathcal{I} \), then every vector \( F \in L^2(X, \mu) \) can formally be written as

\[
F(x) = \sum_{i \in \mathcal{I}} \langle F, \phi_i \rangle \phi_i(x),
\]

where the series converges for almost every \( x \in X \). In particular, \( F(x) \) is well defined, whenever the series converges. If \( L^2(X, \mu) \) were a RKHS, then theorem 3 would imply that (16) holds not only formally but for every \( x \in X \). But \( L^2(X, \mu) \) can never be considered a RKHS if \( (X, \mu) \) is an-atomic, even if the pitfalls of the equivalence classes connected to pointwise evaluation as presented above are avoided. In particular, we show that point evaluation cannot be continuous.
Proposition 12. If $(X, \mu)$ is an an-atomic measure space, then $L^2(X, \mu)$ is not a reproducing kernel Hilbert space.

Proof. Every an-atomic measure space contains a non-atomic subspace by lemma 2(ii). Without loss of generality we may therefore assume that $(X, \mu)$ is non-atomic. Take $A \subset X$ with $\mu(A) > 0$. We may assume without loss of generality that $\mu(A) = 1$ as $(X, \mu)$ is $\sigma$-finite. Let $\{A_m\}_{m=1}^\infty$ be a partition of $A$ satisfying $\mu(A_m) = 1/n$, for every $m = 1, \ldots, n$. Such a partition exists by theorem 1. Now choose representatives $\phi_i(x)$ of an orthonormal basis $\{\phi_i\}_{i \in I}$ and take $\chi_{A_m}$ to be the characteristic function of $A_m$. In the spirit of (16), $\chi_{A_m}$ has the pointwise expression

$$\chi_{A_m}(x) = \sum_{i \in I} (\chi_{A_m}, \phi_i) \phi_i(x),$$

whenever this series converges. Define $B^n \subset A$ by

$$B^n := \{ x \in A : \chi_{A_m}(x) = 1, \text{ for some } m \in \{1, \ldots, n\} \}.$$ 

Clearly, $\mu(B^n) = \mu(A)$ for every $n \in \mathbb{N}$. If we assume that $L^2(X, \mu)$ is a RKHS, then, for every $x \in B^n$, and some $m \in \{1, \ldots, n\}$, one has

$$|\chi_{A_m}(x)|^2 = 1 = |(\chi_{A_m}, k_n)|^2 \leq \|k_n\|^2 / n.$$ 

In particular, $\|k_n\|^2 \geq n$ for every $x \in B^n$. Setting $B := \bigcap_{n \in \mathbb{N}} B^n$, one gets that $\mu(B) = \mu(A)$. Consequently, $\|k_n\|^2 = K(x, x) = \infty$ for almost every $x \in A$, a contradiction to (8). □

Corollary 13. Let $(X, \mu)$ be an-atomic. There is no orthonormal basis $\{\phi_i\}_{i \in I} \subset L^2(X, \mu)$ and representatives $\phi_i(x)$ of $\phi_i$ satisfying

$$\sum_{i \in I} |\phi_i(x)|^2 < \infty, \quad \forall x \in X.$$ 

In particular, for every orthonormal basis $\{\phi_i\}_{i \in I} \subset L^2(X, \mu)$ and every choice of representatives $\phi_i(x)$, there exists a set $A \subset X$ of positive measure, such that

$$\sum_{i \in I} |\phi_i(x)|^2 = \infty, \quad \forall x \in A.$$ 

Proposition 14. If $(X, \mu)$ is an-atomic and $H_K \subset L^2(X, \mu)$ is a RKHS, then $\dim(H_K^\perp) = \infty$.

Proof. If we assume that $\dim(H_K^\perp) = N < \infty$, then any orthonormal basis $\{\phi_i\}_{i \in I}$ of $H_K$ can be complemented to a orthonormal basis of $L^2(X, \mu)$ using $N$ vectors $\{u_n\}_{n=1}^N$. In particular, we may chose representatives of $u_n$ such that $|u_n(x)| < \infty$ for every $x \in X$. We thus have by theorem 3 that

$$\sum_{i \in I} |\phi_i(x)|^2 + \sum_{n=1}^N |u_n(x)|^2 < \infty, \quad \forall x \in X,$$

a contradiction to corollary 13. □
Proof of theorem 9: The range of $C_\Psi$ is a RKHS by propositions 11. By proposition 14 it thus follows that either $R(\Psi) = \infty$ or $(X, \mu)$ is atomic.

Remark 15. The results of this section yield an interesting observation about coherent states in quantum mechanics. Let us assume that $\Psi : X \to H$ is a system of coherent states, see [2]. The probability density of a system in the state $f$ being in the pure state $\Psi_x$ is given by $|\langle f, \Psi_x \rangle|^2$. Hence, in light of theorem 9, it follows that there is always an infinite dimensional subspace of probability densities that does not describe the behavior of any system in any state $f \in H$.

It was Pauli who first noted that not every self-adjoint operator is an observable. We can now add that not every probability density describes the probability distribution of a physical system with respect to a system of pure states.

4.2. Strictly continuous mappings

In the following, we show that the discrete components of a continuous frame can be separated and we call the remaining mapping a strictly continuous mapping.

Definition 4. A mapping $\Psi : X \to H$ is called strictly continuous if $(X, \mu)$ is non-atomic and there exists no set $A \subset X$, $\mu(A) > 0$, such that $C_\Psi f|_A$ is constant for every $f \in H$.

Example 2. Let $G$ be a locally compact group, $\pi : G \to U(H)$ a square-integrable group representation for $G$, and $\psi \in H$ an admissible vector, see e.g. [23] and [2, 31] for further reading. If the left Haar measure of $G$ is non-atomic, then $\Psi = \{\pi(g)\psi\}_{g \in G}$ is a strictly continuous frame, as $\langle f, \pi(g_1)\psi \rangle = \langle f, \pi(g_2)\psi \rangle$ for every $f \in H$, implies $g_1 = g_2$. Short-time Fourier systems or continuous wavelet systems, see [22], are just two instances from this large class of strictly continuous mappings.

Throughout the rest of this section we show that every continuous frame can be decomposed into a discrete and a strictly continuous Bessel system.

Lemma 16 ([33], theorem 3.8.1). If $A \subset X$ is an atom, and $F : X \to \mathbb{C}$ is measurable, then $F$ is constant almost everywhere on $A$.

Lemma 17. If $\Psi$ is Bessel, $A \subset X$ such that $0 < \mu(A) < \infty$, and $\langle f, \Psi(\cdot) \rangle$ is constant on $A$ for every $f \in H$, then there exists a unique $\psi \in H$ such that
\[
\|C_\Psi f\|^2 = \|C_\Psi f|_{X\setminus A}\|^2 + |\langle f, \psi \rangle|^2, \quad \forall f \in H.
\]

In particular, $\psi$ is weakly given by
\[
\langle f, \psi \rangle := \mu(A)^{-1/2} \int_A \langle f, \Psi_x \rangle d\mu(x), \quad \forall f \in H. \tag{17}
\]

Proof. First, observe that $\psi$ defined by (17) is unique for every $n \in \mathbb{N}$ by the Riesz representation theorem [17, theorem 3.4]
\[
|\langle f, \psi \rangle| \leq \mu(A)^{-1/2} \int_A |\langle f, \Psi_x \rangle| d\mu(x) \leq \left( \int_A |\langle f, \Psi_x \rangle|^2 d\mu(x) \right)^{1/2} \leq M \|f\|,
\]
where $M$ is the upper frame bound of $\Psi$. Moreover,
\[
\int_X |\langle f, \Psi_x \rangle|^2 d\mu(x) = \int_{X \setminus A} |\langle f, \Psi_x \rangle|^2 d\mu(x) + \int_A |\langle f, \Psi_x \rangle|^2 d\mu(x) \]
\[
= \int_{X \setminus A} |\langle f, \Psi_x \rangle|^2 d\mu(x) + |\langle f, \psi \rangle|^2
\]
where we used (17) and the fact that $\langle f, \Psi(\cdot) \rangle$ is almost everywhere constant on $A$. Note that the restriction to $\mu(A) < \infty$ follows from the Bessel assumption.

**Theorem 18.** Every Bessel mapping $\Psi$ can be written as $\Psi = \Psi_d \cup \Psi_c$, where $\Psi_d$ is a discrete Bessel system and $\Psi_c : (X_c, \mu_c) \to \mathcal{H}$ is a strictly continuous Bessel mapping with $X_c \subset X$. In particular, if $(X, \mu)$ is atomic, then $\Psi$ can be written as a discrete Bessel sequence.

**Proof.** By lemma 2(i), every measure $\mu$ can be written as $\mu = \mu_a + \mu_c$, where $\mu_a$ is atomic and $\mu_c$ is non-atomic. By lemmas 16 and 17 we deduce that $\Psi$ defined on $(X, \mu_a)$ can be identified with a discrete Bessel system $\Psi_a$. Let $X_d = \bigcup X_i \subset X$ be the disjoint union of all sets $X_i \subset X$, such that $\mu_c(X_i) > 0$, and $\mathcal{C}(f|_{X_i})$ is constant for all $f \in \mathcal{H}$. Setting $A = X_i$ in (17) for every $i$, then defines a family of vectors $\{\psi_i\}_{i \in I}$. By definition $\Psi_c := \Psi|_{X \setminus X_d}$ is a strictly continuous Bessel mapping. It therefore remains to show that $I$ is countable. This, however, is a direct consequence from the fact that $\sigma$-finite measure spaces can only be partitioned into countably many sets of positive measure. Hence, $\Psi_d := \Psi_a \cup \{\psi_i\}_{i \in I}$ is a discrete Bessel sequence, and the result follows.

In an attempt to generalize the concept of Riesz bases, continuous Riesz bases [7] and Riesz-type mappings [21] were introduced. It turns out that theses notions are equivalent and characterized as frames with redundancy zero [7, proposition 2.5 & theorem 2.6].

**Corollary 19.** Every continuous Riesz basis (Riesz-type mapping) can be written as a discrete Riesz basis.

**Proof.** If $\Psi$ is a continuous Riesz basis, then $R(\Psi) = 0$ by definition. By theorem 9, $(X, \mu)$ is atomic. Consequently, $\Psi$ corresponds to a discrete Riesz basis by theorem 18.

### 4.3. Upper semi-frames

Let us now illustrate how upper semi-frames behave fundamentally different than (lower semi-)frames. In particular, the closure of the range of the analysis operator is not necessarily a reproducing kernel Hilbert space and there exist upper semi-frames on non-atomic measure spaces with redundancy zero (compare to proposition 11 and theorem 9).

**Example 3.** In [3, 5] the following upper semi-frame has been studied. Take $\mathcal{H}_n := L^2(\mathbb{R}^+, r^{n-1} dr)$, where $n \in \mathbb{N}$, and $(X, \mu) = (\mathbb{R}, dx)$. For $f \in L^1(\mathbb{R})$, the Fourier transform of $f$ is defined as follows
\[
\hat{f}(\omega) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \omega} dx, \quad \omega \in \mathbb{R}.
\]

By a standard argument, the Fourier transform extends to a unitary operator on $L^2(\mathbb{R})$. For $\psi \in \mathcal{H}_n$, we define the affine coherent state $\Psi_x$ by
\[ \Psi_x(r) := e^{-2\pi inr} \psi(r), \quad r \in \mathbb{R}^+, \ x \in \mathbb{R}. \]

The mapping \( \Psi \) forms an upper semi-frame if \( \text{ess sup}_{r \in \mathbb{R}^+} r^{n-1} |\psi(r)|^2 < \infty \), and \( |\psi(r)| \neq 0 \) for a.e. \( r \in \mathbb{R}^+ \). The frame operator is then given by a multiplication operator on \( \mathcal{H}_n \)

\[ (S_\Phi f)(r) = r^{n-1} |\psi(r)|^2 f(r), \quad \text{for a.e. } r \in \mathbb{R}^+. \]

It is thus easy to see that \( \Psi \) cannot form a frame since for every \( \psi \in \mathcal{H}_n \), \( \text{ess inf}_{r \in \mathbb{R}^+} r^{n-1} |\psi(r)|^2 = 0 \).

In [5, section 5.2] it is shown that \( \ker D_\Phi = \mathcal{F}_+ \), where

\[ \mathcal{H}_+ := \{ f \in L^2(\mathbb{R}) : \hat{f}(\omega) = 0 \text{ for a.e. } \omega \geq 0 \}. \]

Clearly, \( \overline{\text{Ran} C_\Psi} = (\ker D_\Phi)^\perp = \mathcal{F}_+^\perp = \mathcal{H}_- \), where

\[ \mathcal{H}_- := \{ f \in L^2(\mathbb{R}) : \hat{f}(\omega) = 0 \text{ for a.e. } \omega \leq 0 \}. \]

Therefore, \( \Psi \) has infinite redundancy and a short argument shows that \( \mathcal{H}_- \) is not a RKHS:

The dilation operator \( D_\alpha \), defined by \( D_\alpha f(x) := a^{-1/2} f(x/a), a \in \mathbb{R}^+ \), acts isometrically on \( \mathcal{H}_- \). Take \( f \in \mathcal{H}_- \) with \( \|f\| = 1 \) and \( f(0) \neq 0 \), then \( |D_\alpha f(0)| = \|a^{-1/2} f(0)\| \to \infty \), as \( a \to 0 \).

Consequently, point evaluation cannot be continuous and \( \overline{\text{Ran} C_\Psi} = \mathcal{H}_- \) is not a RKHS.

The mapping \( \Psi \) possesses several other interesting properties, see [5]. For instance, it forms a total Bessel system with no dual, i.e. there is no mapping \( \Phi \) such that \( (\Psi, \Phi) \) generates a reproducing pair.

**Proposition 20.** Let \( (X, \mu) \) be a measure space, and \( \{\psi_n\}_{n \in \mathbb{N}} \) be an orthonormal basis of \( L^2(X, \mu) \). If it is possible to fix one representative \( \psi_n(x) \) of the equivalence class \( \psi_n \) for every \( n \in \mathbb{N} \), such that

\[ \sup_{n \in \mathbb{N}} \sup_{x \in X} |\psi_n(x)| = C < \infty, \quad (18) \]

then there exists an upper semi-frame \( \Psi \) for \( \mathcal{H} \) such that \( \overline{\text{Ran} C_\Psi} = L^2(X, \mu) \). In particular, \( R(\Psi) = 0 \).

**Proof.** Take an arbitrary orthonormal basis \( \{e_n\}_{n \in \mathbb{N}} \) of \( \mathcal{H} \), and define

\[ \Psi_x := \sum_{n \in \mathbb{N}} n^{-1} e_n \psi_n(x). \]

The series converges absolutely in every point, and \( \Psi \) is an upper semi-frame with the desired properties. To see this, we first observe that \( \Psi : X \to \mathcal{H} \) is well-defined as, for \( x \in X \) fixed,

\[
|\langle f, \Psi_x \rangle| \leq \sum_{n \in \mathbb{N}} |\langle f, e_n \rangle n^{-1} \psi_n(x)| \leq \|f\| \left( \sum_{n \in \mathbb{N}} n^{-2} |\psi_n(x)|^2 \right)^{1/2} \\
\leq C \|f\| \left( \sum_{n \in \mathbb{N}} n^{-2} \right)^{1/2} = \frac{\pi}{\sqrt{6}} C \|f\| ,
\]

\[ \|\Psi_x\|^2 = C \|f\|^2 \left( \sum_{n \in \mathbb{N}} n^{-2} \right) . \]
where we used (18) and Cauchy–Schwarz inequality. Moreover,
\[
\int_X |\langle f, \Psi \rangle|^2 \, d\mu(x) \leq \int_X \|f\|^2 \sum_{n \in \mathbb{N}} n^{-2} |\psi_n(x)|^2 \, d\mu(x)
\]
\[
= \|f\|^2 \sum_{n \in \mathbb{N}} n^{-2} \int_X |\psi_n(x)|^2 \, d\mu(x) = \|f\|^2 \sum_{n \in \mathbb{N}} n^{-2} = \frac{\pi^2}{6} \|f\|^2.
\]
Since \(\{\psi_n\}_{n \in \mathbb{N}}\) is an orthonormal basis of \(L^2(X, \mu)\), it follows that \(\Psi\) is total in \(\mathcal{H}\), as
\[
\int_X |\langle f, \Psi \rangle|^2 \, d\mu(x) = \int_X \sum_{n,k \in \mathbb{N}} \langle f, e_n \rangle \langle e_k, f \rangle (nk)^{-1} |\psi_n(x)|^2 |\psi_k(x)|^2 \, d\mu(x)
\]
\[
= \sum_{n,k \in \mathbb{N}} \langle f, e_n \rangle \langle e_k, f \rangle (nk)^{-1} \delta_{n,k} = \sum_{n \in \mathbb{N}} |\langle f, e_n \rangle|^2 n^{-2} > 0,
\]
for every \(f \neq 0\). Finally, the range of the analysis operator of the system \(\{n^{-1} e_n\}_{n \in \mathbb{N}}\) is dense in \(L^2(\mathbb{N})\), which implies that \(\text{Ran} \ C_\Psi\) is dense in \(L^2(X, \mu)\).

**Example 4.** Take the non-atomic measure space \((X, \mu) = (\mathbb{T}, dx)\), where \(\mathbb{T}\) denotes the torus, and the orthonormal Fourier basis \(\psi_n(x) = e^{2\pi i n x}, n \in \mathbb{Z}\), then
\[
\sup_{n \in \mathbb{Z}} \sup_{x \in \mathbb{T}} |\psi_n(x)| = 1.
\]
Hence, there exists an upper semi-frame \(\Psi\) with the closure of \(\text{Ran} \ C_\Psi\) being \(L^2(\mathbb{T}, dx)\), i.e. \(R(\Psi) = 0\).

### 4.4. Existence of duals for lower semi-frames

In this section, we fill a gap in the arguments of the proof of [3, proposition 2.6], which states that for every lower semi-frame \(\Psi\) there exists a dual Bessel mapping \(\Phi\) such that \(S_{\Psi, \Phi} = I\) on \(\text{Dom} \ C_\Psi\). While the result itself is correct, the construction of the dual system \(\Phi\) in [3] is in general not well-defined. In particular, \(\Phi\) is defined there by
\[
\Phi_x := \sum_{n \in \mathbb{N}} \phi_n(x) V \phi_n = V \left( \sum_{n \in \mathbb{N}} \phi_n(x) \phi_n \right),
\]
where \(V : L^2(X, \mu) \to \mathcal{H}\) is a bounded operator depending on \(\Psi\) only and \(\{\phi_n\}_{n \in \mathbb{N}}\) is an orthonormal basis for \(L^2(X, \mu)\). However, if \((X, \mu)\) is an-atomic, then by corollary 13 there exists a set of positive measure \(A\) such that \(\sum |\phi_n(x)|^2 = \infty\), for all \(x \in A\). Thus, \(\Phi\) may not be well-defined on a set of positive measure.

**Proposition 21 ([3], proposition 2.6).** If \(\Psi\) is a lower semi-frame in \(\mathcal{H}\), then there exists a Bessel mapping \(\Phi\) such that
\[
f = \int_X \langle f, \Psi \rangle \, d\mu(x), \quad \forall f \in \text{Dom} \ C_\Psi. \tag{19}
\]
Moreover, if \(\text{Dom} \ C_\Psi\) is dense in \(\mathcal{H}\), then
\[
f = \int_X \langle f, \Phi \rangle \, d\mu(x), \quad \forall f \in \mathcal{H}. \tag{20}
\]
Proof. If $\Psi$ is a lower semi-frame, then $\mathrm{Ran} \ C_\Psi$ is a RKHS in $L^2(X, \mu)$ by proposition 11. Moreover, let $P$ denote the orthogonal projection from $L^2(X, \mu)$ onto $\mathrm{Ran} \ C_\Psi$, and $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. Define the linear operator $V : L^2(X, \mu) \to \mathcal{H}$ by $V := C_\Psi^{-1}$ on $\mathrm{Ran} \ C_\Psi$ and $V := 0$ on $(\mathrm{Ran} \ C_\Psi)^\perp$. Then $V$ is bounded and for all $f \in \text{Dom} \ C_\Psi$, $g \in \mathcal{H}$, it holds
\[
\langle f, g \rangle = \langle V C_\Psi f, g \rangle = \langle C_\Psi f, V^* g \rangle_2 = \langle C_\Psi f, \sum_{n \in \mathbb{N}} (g, e_n) e_n \rangle_2 = \langle C_\Psi f, \sum_{n \in \mathbb{N}} (g, e_n) V^* e_n \rangle_2 = \langle C_\Psi f, \sum_{n \in \mathbb{N}} (g, e_n) P V^* e_n \rangle_2 = \langle C_\Psi f, C_\Psi g \rangle_2,
\]
where $\Phi_x := \sum_{n \in \mathbb{N}} (P V^* e_n)(x) e_n$. It remains to show that $\Phi_x$ is well-defined for every $x \in X$. Since $\{e_n\}_{n \in \mathbb{N}}$ is an orthonormal basis, if follows that $\Phi_x$ is well-defined if and only if
\[
\sum_{n \in \mathbb{N}} \| (P V^* e_n)(x) \|^2 < \infty, \quad \forall \ x \in X.
\]
By Proposition 6, it is sufficient to show that $\{P V^* e_n\}_{n \in \mathbb{N}}$ is a Bessel sequence on $\mathrm{Ran} \ C_\Psi$. If $F \in \mathrm{Ran} \ C_\Psi$, then
\[
\sum_{n \in \mathbb{N}} \| (F, P V^* e_n) \|^2_2 = \sum_{n \in \mathbb{N}} \| (V P F, e_n) \|^2_2 = \| V F \|^2 \leq C \| F \|_2^2,
\]
as $PF = F$ and $V$ is bounded. Clearly, $\{P V^* e_n\}_{n \in \mathbb{N}}$ is therefore Bessel for the whole space $L^2(X, \mu)$. Finally, it remains to show that $\Phi$ is Bessel. If $f \in \mathcal{H}$, then
\[
\int_X |\langle f, \Phi_x \rangle|^2 \, d\mu(x) = \int_X \left| \sum_{n \in \mathbb{N}} \langle f, e_n \rangle P V^* e_n(x) \right|^2 \, d\mu(x)
\]
\[
= \| D_{P V^* e_n}((f, e_n))_{n \in \mathbb{N}} \|^2_2 \leq C \sum_{n \in \mathbb{N}} |\langle f, e_n \rangle|^2 = C \| f \|^2.
\]
Let $f \in \mathcal{H}$, and $g \in \text{Dom} \ C_\Psi$. As (19) holds weakly, we have
\[
\langle f, g \rangle = \int_X \langle f, \Phi_x \rangle \langle \Psi_x, g \rangle \, d\mu(x),
\]
which implies (20) if $\text{Dom} \ C_\Psi$ is dense. \hfill \square

Remark 22. There is no analogue result of proposition 21 if $\Psi$ is an upper semi-frame. In [5] it is shown that the affine coherent state system presented in section 4.3 is a complete Bessel mapping with no dual.

5. Reproducing pairs and RKHSs

The absence of frame bounds causes trouble analyzing $\mathrm{Ran} \ C_\Psi$ and $\mathrm{Ran} \ C_\Phi$ of a reproducing pair $(\Psi, \Phi)$. Without an upper frame bound it is no longer guaranteed that $\mathrm{Ran} \ C_\Psi$ is a subspace of $L^2(X, \mu)$. The lower frame inequality, on the other hand, ensures that $\mathrm{Ran} \ C_\Psi$ is a
RKHS. In [5], a construction of two mutually dual Hilbert spaces that are intrinsically generated by the pair \((\Psi, \Phi)\) is given. Let us first recall some of the results before explaining how RKHS enter the picture.

Let \(V_{\Phi}(X, \mu) \subset \mathcal{F}(X, \mathbb{C})\) be the space of all measurable functions \(F : X \to \mathbb{C}\) such that

\[
\left| \int_X F(x)\langle \Phi_x, g \rangle d\mu(x) \right| \leq M \|g\|, \quad \forall g \in \mathcal{H}.
\]

Note that in general neither \(V_{\Phi}(X, \mu) \subset L^2(X, \mu)\) nor \(L^2(X, \mu) \subset V_{\Phi}(X, \mu)\). The linear map \(T_{\Phi} : V_{\Phi}(X, \mu) \to \mathcal{H}\) given weakly by

\[
\langle T_{\Phi}F, g \rangle = \int_X F(x)\langle \Phi_x, g \rangle d\mu(x), \quad g \in \mathcal{H},
\]

is thus well-defined by Riesz representation theorem, and can be seen as the natural extension of the synthesis operator \(D_{\Phi}\) (defined on \(\text{Dom}D_{\Phi} \subseteq L^2(X, \mu)\)) to \(V_{\Phi}(X, \mu)\).

Let \((\Psi, \Phi)\) be a reproducing pair. According to [5], it then holds that

\[
V_{\Phi}(X, \mu) = \text{Ran} \ C_{\Psi} \oplus \text{Ker} \ T_{\Phi}.
\]

This observation, together with the fact that \(T_{\Phi}\) is in general not one-to-one, motivates us to define the redundancy for arbitrary complete mappings via

\[
R(\Phi) := \dim(\text{Ker} \ T_{\Phi}).
\]

We expect that similar results as in section 4.1 hold for this definition.

**Conjecture 1.** Let \((\Psi, \Phi)\) be a reproducing pair. If \(R(\Phi) < \infty\), or \(R(\Psi) < \infty\), then \((X, \mu)\) is atomic.

The main difficulty is that there is no characterization of \(V_{\Phi}(X, \mu)\) that would allow to treat the problem in a similar manner as in section 4.1 using (22). It is in particular not even clear if \(V_{\Phi}(X, \mu)\) is normable.

Let us introduce the following vector space

\[
V_{\Phi}(X, \mu) = V_{\Phi}(X, \mu)/\text{Ker} \ T_{\Phi},
\]

equipped with the inner product

\[
\langle F, G \rangle_{\Phi} := \langle T_{\Phi}F, T_{\Phi}G \rangle, \quad F, G \in V_{\Phi}(X, \mu).
\]

This is indeed an inner product as \(\|F\|_{\Phi} := \langle F, F \rangle_{\Phi} = 0\) if and only if \(F \in \text{Ker} \ T_{\Phi}\). Hence, \(V_{\Phi}(X, \mu)\) forms a pre-Hilbert space and \(T_{\Phi} : V_{\Phi}(X, \mu) \to \mathcal{H}\) is an isometry. By (21) \(\langle \cdot, \cdot \rangle_{\Phi}\) can be written explicitly as

\[
\langle F, G \rangle_{\Phi} = \int_X \int_X F(x)\langle \Phi_x, \Phi_y \rangle G(y)d\mu(x)d\mu(y).
\]

The following result answers the question if, given a mapping \(\Phi\), there exist another mapping \(\Psi\) such that \((\Psi, \Phi)\) forms a reproducing pair.

**Theorem 23 ([5], theorem 4.1).** Let \(\Phi : X \to \mathcal{H}\) be a weakly measurable mapping and \(\{e_i\}_{i \in \mathcal{I}}\) an orthonormal basis of \(\mathcal{H}\). There exists another family \(\Psi\), such that \((\Psi, \Phi)\) is a reproducing pair if and only if

(i) \(\text{Ran} T_{\Phi} = \mathcal{H}\),

(ii) there exists \(\{\mathcal{E}_i\}_{i \in \mathcal{I}} \subset V_{\Phi}(X, \mu)\) satisfying \(T_{\Phi}\mathcal{E}_i = e_i, \ \forall \ i \in \mathcal{I}, \text{ and} \)

...
A reproducing partner $\Psi$ is then given by
\[
\Psi_x := \sum_{i \in I} \xi_i(x) e_i.
\]  

(26)

Theorem 23 is a powerful tool for the study of complete systems. It has for example been used to construct a reproducing partner for the Gabor system of integer time-frequency shifts of the Gaussian window [35] and to prove the non-existence of a dual for the system of affine coherent states in example 3.

Let us briefly discuss the conditions (i) and (ii) and some conceptual interpretations. For a complete system one can show that (under mild conditions [3, lemma 2.2]) $\text{Ran} D_\phi = \mathcal{H}$. It might therefore seem that (i) is trivially satisfied for complete systems since $T_\phi$ extends $D_\phi$ to its domain $\mathcal{V}_\phi(X, \mu)$. The complete upper semi-frame from example 3 however does not satisfy (i), see [5, section 6.2.3].

**Coefficient map interpretation:** Property (i) ensures the existence of a linear coefficient map $A : \mathcal{H} \to \mathcal{V}_\phi(X, \mu)$ satisfying $f = T_\phi A(f)$ for every $f \in \mathcal{H}$. Property (ii) then guarantees that $A(f)$ can be calculated taking inner products of $f$ with a second mapping $\Psi : X \to \mathcal{H}$.

**RKHS interpretation:** If (i) and (ii) are satisfied, then it follows that $\{\xi_i\}_{i \in I}$ forms an orthonormal family with respect to $\langle \cdot, \cdot \rangle_\phi$, since (ii) implies
\[
\langle \xi_i, \xi_k \rangle_\phi = \langle T_\phi \xi_i, T_\phi \xi_k \rangle = \langle e_i, e_k \rangle = \delta_{ik}.
\]

Hence, $\{\xi_i\}_{i \in I}$ forms an orthonormal basis for
\[
\mathcal{H}_K^\phi := \text{span}\{\xi_i : i \in I\}^{1 \cdot \phi}.
\]

Theorem 3 together with (25) thus ensure that $\mathcal{H}_K^\phi$ is a RKHS. Moreover, the definition of the reproducing partner $\Psi$ in (26) yields that
\[
\mathcal{H}_K^\phi \simeq \mathcal{V}_\phi(X, \mu) \simeq (\text{Ran} C_\phi, \| \cdot \|_\phi).
\]

(27)

To put it another way, (i) and (ii) guarantee the existence of a RKHS $\mathcal{H}_K^\phi \subset \mathcal{V}_\phi(X, \mu)$ reproducing $T_\phi$ in the sense that $T_\phi(\mathcal{H}_K^\phi) = \mathcal{H}$.

Let us assume that $(\Psi, \Phi)$ is a reproducing pair. There is a natural way to generate frames on $\mathcal{H}$ and $\mathcal{H}_K^\phi$ via the analysis and synthesis operators.

**Proposition 24.** Let $(\Psi, \Phi)$ be a reproducing pair for $\mathcal{H}$, $\{g_i\}_{i \in I}$ a frame for $\mathcal{H}$, and $\{G_i\}_{i \in I}$ a frame for $\mathcal{H}_K^\phi$. If $H(x) := \langle g_i, \Psi \rangle$ and $h_i := T_\phi G_i$, then $\{H_i\}_{i \in I}$ is a frame for $\mathcal{H}_K^\phi$ and $\{h_i\}_{i \in I}$ is a frame for $\mathcal{H}$.

**Proof.** If $F \in \mathcal{H}_K^\phi$, then
\[
\sum_{i \in I} |\langle F, H_i \rangle|^2 = \sum_{i \in I} |\langle T_\phi F, T_\phi H_i \rangle|^2 = \sum_{i \in I} |\langle T_\phi F, S_\phi g_i \rangle|^2
\]
\[
= \sum_{i \in I} |\langle (S_\phi g_i)^* T_\phi F, g_i \rangle|^2 \leq M \| (S_\phi g_i)^* T_\phi F \|^2
\]
\[
\leq M \| S_\phi g_i \|^2 \| T_\phi F \|^2 = \tilde{M} \| F \|^2_\phi.
\]
The lower bound follows from the same argument as \( (S_{\Psi,\Phi})^* \) is boundedly invertible. Hence, \( \{H_i\}_{i \in \mathcal{I}} \) is a frame for \( \mathcal{H}_K^\Phi \).

For \( f \in \mathcal{H} \), we have
\[
\|f\| = \|T_\Phi C_\Psi S_{\Psi,\Phi}^{-1}f\| = \|C_\Phi S_{\Psi,\Phi}^{-1}f\|_\Phi.
\]
which, together with
\[
\sum_{i \in \mathcal{I}} |\langle f, h_i \rangle|^2 = \sum_{i \in \mathcal{I}} |\langle T_\Phi C_\Psi S_{\Psi,\Phi}^{-1}f, T_\Phi G_i \rangle|^2 = \sum_{i \in \mathcal{I}} |\langle C_\Phi S_{\Psi,\Phi}^{-1}f, G_i \rangle|^2,
\]
yields that \( \{h_i\}_{i \in \mathcal{I}} \) is a frame for \( \mathcal{H} \).

The rest of this section is concerned with the explicit calculation of the reproducing kernel for \( \mathcal{H}_K^\Phi \). For a reproducing pair \( (\Psi, \Phi) \), there exists a similar characterization of the range of the analysis operators as for frames. In particular, if \( R_{\Psi,\Phi}(x,y) := \langle S_{\Psi,\Phi}^{-1}\Phi_x, \Psi_y \rangle \) defines the integral operator
\[
\mathcal{R}_{\Psi,\Phi}(F)(x) := \int_X F(y) \langle \Phi_y, (S_{\Psi,\Phi})^{-1}\Psi_x \rangle d\mu(y), \quad F \in \mathcal{V}(X, \mu),
\]
then by [34, proposition 2] \( \mathcal{R}_{\Psi,\Phi}(F)(x) = F(x) \) if and only if there exists \( f \in \mathcal{H} \) such that \( F(x) = \langle f, \Psi_x \rangle \), for all \( x \in X \). However, \( R_{\Psi,\Phi} \) is not the reproducing kernel for \( \mathcal{H}_K^\Phi \) since the reproducing formula is based on the inner product of \( L^2(X, \mu) \) and not on \( \langle \cdot, \cdot \rangle_\phi \).

If \( F \in \text{Ran} \, C_\Psi \), then (6), (24) and the identity \( f = T_\Phi C_\Psi S_{\Psi,\Phi}^{-1}f \) yield
\[
F(x) = \mathcal{R}_{\Psi,\Phi}(F)(x) = \int_X F(y) \langle \Phi_y, (S_{\Psi,\Phi})^{-1}\Phi_x \rangle d\mu(y)
= \int_X \int_X F(y) \langle \Phi_y, \Phi_z \rangle \langle (S_{\Psi,\Phi})^{-1}\Phi_x, (S_{\Psi,\Phi})^{-1}\Phi_z \rangle d\mu(z) d\mu(y)
= \left\langle F, \left( \langle (S_{\Psi,\Phi})^{-1}\Phi_x, (S_{\Psi,\Phi})^{-1}\Phi_z \rangle \right)_\phi \right\rangle_{\phi}.
\]
Hence, using \( (S_{\Psi,\Phi})^* = S_{\phi,\Psi}^{-1} \), we finally obtain that
\[
K_{\phi}(x,y) = \langle S_{\phi,\Psi}^{-1}\Phi_x, S_{\phi,\Psi}^{-1}\Psi_y \rangle
\]
is the reproducing kernel for \( \mathcal{H}_K^\Phi \).

6. Conclusion

With the results of section 4 in mind, we suggest that one uses the term continuous frame only in the case of a strictly continuous frame, and semi-continuous or semi-discrete frame if it can be decomposed into nontrivial strictly continuous and discrete parts. Moreover, the notion of a continuous Riesz basis (Riesz type mapping) should not be used any further as every such system can be written as a discrete Riesz basis.

An interesting topic for future research is to find and study alternative notions of redundancy for continuous frames. A promising approach that may be adapted can be found in [10]. Exploring the dependence of any notion of redundancy on the underlying measure space should remain a key task.
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Appendix

Proof of lemma 2. Ad (i): see [19].

Ad (ii): let \((X, \mu)\) be an-atomic. Let us assume to the contrary that for every measurable set \(A \subset X\) with \(\mu(A) > 0\) there exists an atom \(B \subset A\), and take \(\{A_n\}_{n \in \mathbb{N}} \subset X\) to be a countable partition of \(X\) by sets of finite measure. We show that each \(A_n\) can be partitioned into atoms and null sets, a contradiction. Assume without loss of generality that \(\mu(A_1) > 0\). By assumption, there exists an atom \(B_1 \subset A_1\). If \(\mu(B_1) = \mu(A_1)\), then \(A_1\) is an atom. If \(0 < \mu(B_1) < \mu(A_1)\), then \(\mu(A_1 \setminus B_1) > 0\). Hence, there exists an atom \(B_2 \subset A_1 \setminus B_1\) and the preceding argument can be repeated. If one has \(\mu(A_1 \setminus \left( \bigcup_{k=1}^{K} B_k \right)) > 0\) for all iteration steps \(K\), then \(\mu_K := \mu\left( \bigcup_{k=1}^{K} B_k \right)\) defines a strictly increasing sequence, bounded by \(\mu(A_1)\). Hence, \(\mu_K\) is convergent to some \(\mu^*\) and the limit equals \(\mu(A_1)\). Indeed, if \(\mu^* < \mu(A_1)\) then, by assumption, there exists an atom \(B^* \subset A_1 \setminus \bigcup_{k \in \mathbb{N}} B_k\) and

\[
\mu\left( \bigcup_{k \in \mathbb{N}} B_k \cup B^* \right) > \mu^*,
\]

a contradiction. Consequently, \(A_1 = \bigcup_{k \in \mathbb{N}} B_k \cup N\), where \(N = A_1 \setminus \bigcup_{k \in \mathbb{N}} B_k\) is of measure zero. In particular, we constructed a partition of \(A_1\) consisting of atoms and null sets. Repeating this argument for every \(A_n, n \in \mathcal{I}\), with \(\mu(A_n) > 0\) shows that \((X, \mu)\) is atomic, a contradiction. □

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