Supersymmetric theories on squashed five-sphere

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Abstract

We construct supersymmetric theories on the $SU(3) \times U(1)$ symmetric squashed five-sphere with 2, 4, 6, and 12 supercharges. We first determine the Killing equation by dimensional reduction from 6d, and use Noether procedure to construct actions. The supersymmetric Yang-Mills action is straightforwardly obtained from the supersymmetric Chern-Simons action by using a supersymmetry preserving constant vector multiplet.

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1 Introduction

Recently five-dimensional (5d) supersymmetric (SUSY) gauge theories have attracted much interest. It is known that there exist 5d gauge theories with non-trivial fixed points\cite{1}, and their dynamics is closely related to the brane physics in string theory. The relation to six-dimensional $\mathcal{N} = (2, 0)$ theories provides another motivation to study 5d theories. There is an interesting proposal\cite{2, 3} that a $(2, 0)$ theory on a 6d manifold $\mathcal{M} \times S^1$ be equivalent to a 5d SUSY Yang-Mills theory on $\mathcal{M}$. Because we have no Lagrangian description of $(2, 0)$ theories and cannot directly analyze them, this relation provides an important access to $(2, 0)$ theories through 5d gauge theories.

The first step to analyze a theory is to construct the action. The action of the $\mathcal{N} = 1$ SUSY theory on the flat $\mathbf{R}^5$ is given in \cite{1}. SUSY gauge theories on various curved backgrounds are also used to obtain exact results by localization. Theories on the round $S^5$\cite{4, 5}, $S^4 \times S^1$\cite{6, 7}, and $S^3 \times \mathbf{R}^2$\cite{8} have been constructed. SUSY theories on contact manifolds are constructed in \cite{9}. The perturbative part of the $S^5$ partition function for the round $S^5$ is computed in \cite{9, 10, 5}, and used to confirm predictions of AdS/CFT correspondence\cite{5, 11, 12}. The superconformal index is computed in \cite{6} for $\mathcal{N} = 1$ SUSY gauge theories and the symmetry enhancement at the strong coupling limit is investigated.

The purpose of this paper is to give more examples of 5d SUSY theories on a curved background. We construct SUSY actions on the $SU(3) \times U(1)$ symmetric squashed $S^5$ with the metric

$$ds^2_{S^5} = ds^2_{CP^2} + \frac{1}{v^2}(d\psi + V)^2. \tag{1}$$

We treat $S^5$ as a Hopf fibration over $CP^2$. The first and the second terms in (1) are the metric of the base $CP^2$ and that of the Hopf fiber, respectively. They are normalized so that when $v = 1$ gives the round $S^5$ with radius $r$.

The eight supercharges of $\mathcal{N} = 1$ SUSY on the round $S^5$ belong to $4 + 4$ of the isometry group $SO(6)_{\text{iso}}$. The squashing breaks $SO(6)_{\text{iso}}$ to $SU(3) \times U(1)$. Correspondingly, the supercharges split to $3_{+} + \overline{3}_{-}$ and $1_{-} + 1_{+}$. We will show that only one of $3_{+} + \overline{3}_{-}$ or $1_{-} + 1_{+}$ can be preserved in the squashing. We call these two kinds of preserved SUSY $\mathcal{N} = 3/4$ and $\mathcal{N} = 1/4$. We also construct theories with the number of supercharges doubled, $\mathcal{N} = 3/2$ and $\mathcal{N} = 1/2$ theories, by combining a vector multiplet and an adjoint hypermultiplet with a critical value of the mass parameter.

The organization of this paper is as follows. In the next section, we summarize SUSY gauge theories on conformally flat backgrounds. We explain how we can obtain a SUSY Yang-Mills theory on the round $S^5$ from a SUSY Chern-Simons theory in the same background, which can be obtained from the theory on the flat $\mathbf{R}^5$ by using Weyl transformation. We also review the relation between SUSY in 5d and that in 6d following \cite{5}. In \S3 we construct SUSY theories.
on the squashed five-sphere. We first determine the Killing equation by using a twisted compactification of $S^5 \times R$, and construct SUSY actions by Noether procedure. We also construct $\mathcal{N} = 1/2$ and $\mathcal{N} = 3/2$ theories by combining a vector multiplet and an adjoint hypermultiplet. §4 is devoted to discussion. Conventions for $SU(2)$ and $SO(5)$ are summarized in the appendix.

We use $\mu, \nu, \ldots$ for 5d vector indices. We use local orthonormal frame unless otherwise noted. $I, J, \ldots$ and $a, b, \ldots$ are used for $SU(2)_R$-doublet and triplet indices, respectively, and $A, B, \ldots$ for $SU(2)_F$-doublet indices. For more details see the appendix.

\section{$\mathcal{N} = 1$ in conformally flat backgrounds}

\subsection{Conformal theories in 5d}

Let us first consider vector multiplets $V^\alpha$ of $\mathcal{N} = 1$ SUSY gauge theories in the flat $R^5$. A vector multiplet $V^\alpha$ consists of a gauge field $A^\alpha_\mu$, a real scalar field $\phi^\alpha$, a symplectic Majorana spinor $\lambda_I^\alpha$ ($I = 1, 2$), and three real auxiliary fields $D^a_\alpha$ ($a = 1, 2, 3$). $\lambda_I^\alpha$ and $D^a_\alpha$ form an $SU(2)_R$ doublet and a triplet, respectively. The action of vector multiplets is specified by the prepotential $F(\phi)$, a gauge invariant real function of $\phi^\alpha$. The Lagrangian density on the flat $R^5$ is

$$
L^\text{vector} = F_{\alpha\beta} \left( \frac{1}{4} F^\alpha_{\mu\nu} F^{\beta\mu\nu} + \frac{1}{2} D_\mu \phi^\alpha D^\mu \phi^\beta - \frac{1}{2} D^a_\alpha D^a_\beta + \frac{1}{2} \epsilon^{IJ} (\lambda_I^\alpha Q \lambda_J^\beta) - \frac{1}{2} \epsilon^{IJ} (\lambda_I^\alpha [\phi, \lambda_J^\beta]) \right) + F_{\alpha\beta\gamma} \left( \frac{i}{24} \epsilon^{\lambda_{\mu\nu\rho}} A^\alpha_\lambda F_{\mu\nu}^\beta F_{\rho}^\gamma + \cdots \right) - \frac{i}{4} \epsilon^{IJ} (\lambda_I^\alpha F^{\beta} \lambda_J^\gamma) + \frac{1}{4} \epsilon^{IK} (\tau_a)_K^J D^a_\alpha (\lambda_I^\beta \lambda_J^\gamma),
$$

where $F_{\alpha\beta}$ and $F_{\alpha\beta\gamma}$ are defined by

$$
F_{\alpha\beta} = \frac{\partial}{\partial \phi^\alpha} \frac{\partial}{\partial \phi^\beta} F(\phi), \quad F_{\alpha\beta\gamma} = \frac{\partial}{\partial \phi^\alpha} \frac{\partial}{\partial \phi^\beta} \frac{\partial}{\partial \phi^\gamma} F(\phi).
$$

The gauge covariant derivative and the field strength are defined by

$$
D = d - i[A, \cdot], \quad F = dA - iA \wedge A.
$$

When we consider a curved background $D_\mu$ also contain the spin connection. The terms in the parenthesis in the second line of (2) is the Chern-Simons action. The dots represent terms with one or no derivative, which exist when the gauge group is non-Abelian. The gauge invariance requires the coefficients $F_{\alpha\beta\gamma}$ of the Chern-Simons term to be constant. This means that the prepotential is at most cubic polynomial. The constant and linear terms in the prepotential do not affect the action in the flat spacetime, while we will see that the linear terms give the Fayet-Iliopoulos action in curved backgrounds.
A superconformal theory on a conformally flat background can be easily obtained by Weyl transformation from the theory on the flat $\mathbb{R}^5$. This is pointed out for 5d theories in [6]. If the prepotential is a cubic homogeneous polynomial, the action (2) is invariant under not only rigid SUSY transformation but also superconformal transformation

$$
\delta A_\mu = \epsilon^{IJ}(\epsilon_I \gamma_\mu \lambda_J),
$$
$$
\delta \phi = -i\epsilon^{IJ}(\epsilon_I \lambda_J),
$$
$$
\delta \lambda_I = -\frac{1}{2}\gamma^{\mu \nu} \epsilon_I F_{\mu \nu} + i\gamma^{\mu} \epsilon_I D_\mu \phi + iD_a(\tau_a) I^J \epsilon_J + 2i\kappa_I \phi,
$$
$$
\delta D_a = -i\epsilon^{IK}(\tau_a) K^J (\epsilon_I \gamma^\mu D_\mu \lambda_J) + i\epsilon^{IK}(\tau_a) K^J (\epsilon_I [\phi, \lambda_J]) + i\epsilon^{IK}(\tau_a) K^J (\epsilon_I \lambda_J),
$$

where the parameters $\epsilon_I$ and $\kappa_I$ are symplectic Majorana spinors satisfying the Killing equation

$$
D_\mu \epsilon_I = \gamma_\mu \kappa_I.
$$

Furthermore, we can make this action invariant under the local Weyl transformation

$$
g_{\mu \nu} = e^{-2 \alpha} g'_{\mu \nu}, \quad A = A', \quad \phi = e^{\alpha} \phi', \quad \lambda = e^{\frac{3}{2} \alpha} \lambda', \quad D = e^{2 \alpha} D',
$$

by introducing the curvature coupling of the scalar fields.

$$
\mathcal{L}_{\text{vector}} = \mathcal{L}_{\text{vector}}^0 + \frac{R}{4} \mathcal{F}(\phi).
$$

With the Weyl transformation (7), we can easily construct the $\mathcal{N} = 1$ SUSY Chern-Simons action on conformally flat backgrounds. The SUSY Yang-Mills action and the Fayet-Iliopoulos action are also easily constructed with the help of a constant vector multiplet as we will explain in the next subsection.

We use on-shell formalism for hypermultiplets. A hypermultiplet consists of four real scalar fields $q_i$ ($i = 1, 2, 3, 4$) and a symplectic Majorana spinor field $\psi_A$ ($A = 1, 2$). The largest symmetry of $k$ hypermultiplets is $SU(2)_R \times Sp(k)$, and an arbitrary subgroup of $Sp(k)$ can be gauged. We mainly focus only on the subgroup $SU(2)_F \times U(k) \subset Sp(k)$. We write down actions and transformation laws as if the gauge group $G$ is a subgroup of $U(k)$ and hypermultiplets belong to the adjoint representation of $G$. Extension to more general case is straightforward.

The kinetic action of hypermultiplets on the flat $\mathbb{R}^5$ is

$$
\mathcal{L}^{\text{hyper}}_0 = \frac{1}{2} D_\mu q_i D^\mu q_i - \frac{1}{2} \epsilon^{AB}(\psi_A D_\mu \psi_B) + \frac{1}{2}(\tau_a)_{ij} q_i [D_a, q_j] + \frac{1}{2}[q_i, \phi][\phi, q_i] + \epsilon^{AB}(\bar{\rho}_A) A^I \psi_B [\lambda_I, q_i] - \frac{1}{2} \epsilon^{AB}(\psi_A [\phi, \psi_B]),
$$

where $(\rho_i)^A$ and $(\bar{\rho}_A) A^I$ are $SU(2)_R \times SU(2)_F$ invariant tensors, and $(\tau_a)_{ij}$ is the 't Hooft symbol defined by $(\tau_a)_{ij} = -(1/2)(\tau_a)_{I^J} (\rho_i)^A (\bar{\rho}_j) A^I$. This is invariant.
under the superconformal transformation

\[
\delta q_i = -i \epsilon^{IJ} (\rho_i)_J^A (\epsilon_I \psi_A), \\
\delta \psi_A = i (\overline{\psi})_A I^I \gamma^\mu \epsilon_I D_\mu q_i + 3i (\overline{\psi})_A I^I \kappa_I q_i. 
\]

(10)

The Lagrangian

\[
\mathcal{L}_{\text{hyper}} = \mathcal{L}_{\text{hyper}}^0 + \frac{3R}{32} q_i q_i 
\]

improved by the curvature coupling of the scalar fields is invariant under the local Weyl transformation

\[
q_i = e^{-2\alpha} q_i', \quad \psi = e^{2\alpha} \psi'. 
\]

(12)

Note that the Weyl weights of fields in hypermultiplets are protected by the superconformal algebra. \(q_i\) and \(\psi_A\) have canonical weights \(3/2\) and \(2\), respectively. We can use (12) to obtain the action and the transformation laws for hypermultiplets in an arbitrary conformally flat background.

### 2.2 Round \(S^5\)

The quadratic term in the prepotential

\[
F_{\text{YM}} = \frac{1}{2g_{\text{YM}}^2} \text{tr} \phi^2 
\]

(13)

gives the Yang-Mills kinetic term

\[
\mathcal{L} = \frac{1}{4g_{\text{YM}}^2} \text{tr} (F_{\mu\nu} F^{\mu\nu}). 
\]

(14)

Although this is not conformal in 5d, we can easily construct the SUSY Yang-Mills action on a conformally flat background. For concreteness and as a preparation for the next section, let us consider the case of the round \(S^5\) with radius \(r\). On the round \(S^5\) the parameters \(\epsilon\) and \(\kappa\) (From this subsection we omit \(SU(2)\) indices. See Appendix for the rules.) belong to \(4 + 4\) of the isometry group \(SO(6)_{\text{iso}}\). The spinors in each irreducible representation satisfy

\[
\kappa^4 = -\frac{i}{2r} \epsilon^4, \quad \kappa^{\overline{4}} = \frac{i}{2r} \epsilon^{\overline{4}}. 
\]

(15)

It is convenient to define the chirality operator \(\Gamma_{\text{iso}}\) for \(SO(6)_{\text{iso}}\) which acts on \(4\) and \(\overline{4}\) as +1 and -1, respectively. We combine two equations in (15) into

\[
\kappa = -\frac{i}{2r} \Gamma_{\text{iso}} \epsilon. 
\]

(16)
With the relation (16) and the transformation laws in (5), we can show that the constant vector multiplet
\[ \mathcal{V}^{(1)} = (\phi^{(1)}, A^{(1)}, \lambda^{(1)}_I, D^{(1)}_a) = \left(1, 0, 0, \frac{i}{r} \delta_{a3}\right) \] (17)
preserves half of the supersymmetry whose parameter satisfies
\[ \tau_3 \epsilon = \Gamma_{\text{iso}} \epsilon. \] (18)
We can lift the prepotential \( \mathcal{F}_{\text{YM}} \) to a cubic polynomial by multiplying \( \phi^{(1)} = 1 \)
to it. Namely, we can obtain the SUSY Yang-Mills action as a special SUSY Chern-Simons action with the prepotential
\[ \mathcal{F} = \phi^{(1)} \mathcal{F}_{\text{YM}} = \frac{1}{2g_{\text{YM}}^2} \phi^{(1)} \text{tr} \phi^2. \] (19)

In 5d the constant \( 1/g_{\text{YM}}^2 \) has mass dimension 1, and we can regard (19) as a mass deformation to the Chern-Simons theory. The supersymmetry preserved after such a mass deformation is often called rigid supersymmetry. As in the case of \( S^4 \)[13, 14], the deformation breaks the R-symmetry \( SU(2)_R \) to \( U(1) \).

We can also construct the supersymmetric completion of the Fayet-Iliopoulos term \( \mathcal{L} = \zeta \text{tr} D_3 \) as a special SUSY Chern-Simons action with the prepotential
\[ \mathcal{F} = i r \zeta (\phi^{(1)})^2 \text{tr} \phi. \] (20)
Namely, we can regard the Fayet-Iliopoulos parameters as the coefficients of the linear terms in the prepotential.

Real mass parameters \( \mu_n \) for hypermultiplets, which are associated with global symmetries, are again introduced by using the constant vector multiplet (17). Let \( T_n \) be the generators of the global symmetries associated with the real mass parameters \( \mu_n \). We weakly gauge \( T_n \), and give the expectation values to the corresponding vector multiplets. This is realized by shifting the component fields of vector multiplets in the action (9) according to
\[ \mathcal{V} \rightarrow \mathcal{V} + \mu_n \mathcal{V}^{(1)} T_n. \] (21)

The vector multiplet \( \mathcal{V}^{(1)} \) is essentially the same as the central charge vector multiplet introduced in [15, 16]. The real mass parameters in (21) determine the central charges of hypermultiplets. The Yang-Mills kinetic term is also regarded as the coupling of the central charge vector multiplet to the instanton current \( j \propto \ast \text{tr}(F \wedge F) \). The Yang-Mills coupling constant is a kind of real mass parameters determining the central charge of instantons.

Let us consider a theory consisting of a vector multiplet and an adjoint hypermultiplet. In the flat background the global symmetry \( SU(2)_R \times SU(2)_F \) is
enhanced to $SO(5)_R$, and the theory is invariant under $\mathcal{N} = 2$ supersymmetry. A similar enhancement occurs in $S^5$. In this case, however, a non-trivial mass deformation is needed to obtain enhanced supersymmetry [5]. The mass parameter $\mu_F$ associated with the $SU(2)_F$ flavor symmetry is introduced by the shift

$$\mathcal{V} \rightarrow \mathcal{V} + \mu_F \mathcal{V}^{(1)} \tau'_3,$$

(22)

where $\tau'_3$ is the Cartan generator of $SU(2)_F$. This mass parameter is related to the deformation parameter $\Delta$ in [5] by $\Delta = 1/2 + i\mu_F r$. The supersymmetry enhancement to $\mathcal{N} = 2$ occurs at $\mu_F = \pm \mu_{\text{crit}}$ ($\mu_{\text{crit}} = i/(2r)$).

### 2.3 6d interpretation

As is argued in [5], supersymmetry on $S^5$ can be derived from that in six-dimensional manifold $S^5 \times R$, and the enhancement of supersymmetry at the critical points $\mu_F = \pm \mu_{\text{crit}}$ is clearly explained from this perspective. Let us look at this reduction in detail because this is quite useful when we consider squashing in the next section.

We take the following representation of 6d Dirac matrices.

$$\Gamma^\mu = \left( \begin{array}{c} \gamma^\mu \\ \gamma^\mu \end{array} \right), \quad (\mu = 1, \ldots, 5), \quad \Gamma^6 = \left( \begin{array}{cc} i & -i \\ i & -1 \end{array} \right), \quad \Gamma^7 = \left( \begin{array}{cc} 1 & -i \\ 1 & 1 \end{array} \right).$$

(23)

We use $M, N, \ldots = 1, \ldots, 6$ for 6d vector indices, and assign 12345 to $S^5$ and 6 to $R$. The $\mathcal{N} = (1, 0)$ superconformal symmetry in 6d is described by parameters $\epsilon^{(6)}$ and $\kappa^{(6)}$ which have positive and negative $\Gamma^7$ chirality, respectively. They satisfy the six-dimensional Killing equation

$$D_M \epsilon^{(6)} = \Gamma_M \kappa^{(6)}.$$

(24)

We take the ansatz for the spinors

$$\epsilon^{(6)} = \left( \begin{array}{c} \epsilon \\ 0 \end{array} \right), \quad \kappa^{(6)} = \left( \begin{array}{c} 0 \\ \kappa \end{array} \right),$$

(25)

where $\epsilon$ and $\kappa$ are the 5d spinors satisfying the 5d Killing equation (6). We have not yet fixed the normalization of $\epsilon$ and $\kappa$, which may depend on the coordinate $t \equiv x^6$ along $R$. The relation (16) gives

$$\kappa^{(6)} = -\frac{1}{2r} \Gamma_{\text{iso}} \Gamma^6 \epsilon^{(6)}.$$  

(26)

(24) is automatically satisfied by (25) for $M = 1, 2, 3, 4, 5$. Combining (24) with $M = 6$ and (26) we obtain

$$\partial_6 \epsilon^{(6)} = -\frac{1}{2r} \Gamma_{\text{iso}} \epsilon^{(6)}.$$  

(27)
This equation determines the \( t \) dependence of \( \epsilon \) and \( \kappa \). Because of this non-trivial \( t \) dependence we cannot impose the periodic boundary condition when we compactify \( R \) to \( S^1 \). Instead, we use the twisted boundary condition

\[
\Phi(t + \beta) = \exp \left( -\frac{\beta}{2r} \tau_3 \right) \Phi(t),
\]

(28)

where \( \Phi \) is an arbitrary field in the 6d theory, including \( \epsilon \) and \( \kappa \). \( \tau_3 \) is the Cartan generator of the \( SU(2)_R \) symmetry of the 6d \( \mathcal{N} = (1,0) \) theory. The Killing spinor satisfies this boundary condition only when \( \epsilon \) satisfies (18). This is an explanation for (18) in the context of compactification.

The symmetry enhancement in a theory with one adjoint hypermultiplet at the critical values of the mass parameter is explained as follows. Let us start from \( \mathcal{N} = (2,0) \) theory in 6d, which has \( SO(5)_R \) symmetry. The SUSY parameters \( \epsilon^{(6)} \) and \( \kappa^{(6)} \) belong to 4 of \( SO(5)_R \). The \( SU(2)_R \) symmetry of \( \mathcal{N} = (1,0) \) theory is a subgroup of this \( SO(5)_R \), and its centralizer is the flavor group \( SU(2)_F \). We denote the Cartan generators of \( SU(2)_R \) and \( SU(2)_F \) by \( \tau_3 \) and \( \tau'_3 \), respectively. We generalize the twisted boundary condition (28) by replacing \( \tau_3 \) by \( \tau_3 - 2i\mu_F \tau'_3 \). \( \mu_F \) is nothing but the mass parameter in (22). The condition (18) for preserved SUSY (18) is replaced by

\[
(\tau_3 - 2i\mu_F \tau'_3)\epsilon = \Gamma_{\text{iso}} \epsilon.
\]

(29)

For generic \( \mu_F \), this condition is satisfied by a quarter of \( \epsilon \), and the preserved SUSY in 5d is rigid \( \mathcal{N} = 1 \), while at the critical values \( \mu_F = \pm \mu_{\text{crit}} \), the number of preserved SUSY is doubled.

3 Squashing

3.1 Supersymmetry

The squashing of \( S^5 \) can be realized by a simple modification of the boundary condition (28). We consider the boundary condition

\[
\Phi(t + \beta) = \exp \left[ -\frac{\beta}{2r} ((1 + \alpha) \tau_3 + iuJ) \right] \Phi(t),
\]

(30)

where \( J \) is the shift along the Hopf fiber of \( S^5 \) normalized by \( e^{2\pi i J} = 1 \). As we will explicitly show shortly, this gives squashed sphere (1) after the dimensional reduction. The parameter \( u \) is related to \( v \) in (1) by

\[
v^2 = 1 + u^2.
\]

(31)

The parameter \( \alpha \) should be chosen so that there exist preserved SUSY.
The relation (18) is a condition for $\epsilon$ at each $t$-slice, and we assume that the change of the boundary condition does not affect this relation. Then, the boundary condition (30) implies

$$\alpha \epsilon = -iu \Gamma_{iso} J \epsilon.$$ 

(32)

The introduction of the generator $J$ in the boundary condition breaks $SO(6)_{iso}$ to $SU(3) \times U(1)$. Correspondingly, Killing spinors in $4 + \overline{4}$ split into $3_{+1} + \overline{3}_{-1}$ and $1_{-3} + 1_{+3}$. If we set $\alpha = -iu$, the condition (32) admits the parameters in the representation $3_{+1} + \overline{3}_{-1}$. We call this unbroken supersymmetry $\mathcal{N} = 3/4$. If we set $\alpha = 3iu$, only $1_{-3} + 1_{+3}$ are preserved, and we call this $\mathcal{N} = 1/4$.

### 3.2 Killing spinors

The metric of $S^5 \times R$ is

$$ds^2_{S^5 \times R} = ds^2_{CP^2} + (\overline{\sigma}^5)^2 + (\overline{\sigma}^6)^2.$$ 

(33)

The first two terms are the metric of the round $S^5$ with radius $r$ in the form of Hopf fibration, and the last term is the metric of $R$. We introduce coordinates $t$ and $\psi$ along $R$ and Hopf fibers, respectively, and use the local orthonormal frame

$$e^m \quad (m = 1, 2, 3, 4), \quad \overline{e}^5 = d\overline{\psi} + V, \quad \overline{e}^6 = dt.$$ 

(34)

$e^m$ are the vielbein in the base $CP^2$. In this subsection we use bars to mean the original coordinate system. We will later introduce a slanted coordinate system in the 56 plane, which is convenient for the dimensional reduction. $V$ is a differential on $CP^2$, which depends on the choice of the coordinate $\psi$. Its exterior derivative is proportional to the Kahler form $I$ on $CP^2$:

$$dV = -\frac{2}{r} I = -\frac{1}{r} J_{mn} e^m \wedge e^n = \frac{2}{r} (e^1 \wedge e^2 + e^3 \wedge e^4).$$ 

(35)

In the small radius limit $\beta \to 0$ almost all modes become infinitely massive. For modes remaining light we can replace the twisted boundary condition (30) by the differential equation

$$(\partial_t + u \partial_{\overline{\psi}}) \Phi = -\frac{1 + \alpha}{2r} \tau_3 \Phi,$$ 

(36)

where we used $J = -2ir \partial_{\overline{\psi}}$. Note that $\partial_t$ and $\partial_{\overline{\psi}}$ represents simple partial derivatives with respect to $t$ and $\overline{\psi}$, respectively, and they do not contain the vielbein unlike $D_\nu$, which has index of the orthonormal frame.

To perform the dimensional reduction by the condition (36), it is convenient to introduce the slanted coordinate

$$\psi = \overline{\psi} + ut.$$ 

(37)
With this coordinate, the compactification is simply represented by \((t, \psi) \sim (t + \beta, \psi)\). The metric in the new coordinate system is
\[
d s^2_{S^5 \times R} = d s^2_{\mathbb{C}P^2} + \frac{1}{v^2} (d \psi + V)^2 + \left( \frac{u}{v} (d \psi + V) \right)^2.
\]
(38)

If we neglect the last term in the metric, we obtain the squashed \(S^5\) in (1). We introduce the following 6d orthonormal frame for the slanted coordinate system:
\[
e_m, \quad e^5 = \frac{1}{v} (d \psi + V), \quad e^6 = v dt + u e^5.
\]
(39)

\(e^m\) are the same as before, but \(e^5\) and \(e^6\) are related to \(\tilde{e}^5\) and \(\tilde{e}^6\) by
\[
\begin{pmatrix} e^5 \\ e^6 \end{pmatrix} = \begin{pmatrix} \frac{1}{v} & -u \\ u & \frac{1}{v} \end{pmatrix} \begin{pmatrix} \tilde{e}^5 \\ \tilde{e}^6 \end{pmatrix}.
\]
(40)

In this new coordinate system, the constraint (36) becomes
\[
D_6 = \frac{1}{v} \partial_t = -\frac{1 + \alpha}{2rv} \tau_3.
\]
(41)

Note that \(\partial_t\) is the simple partial derivative with respect to \(t\), while \(D_6\) is the 6-th component of the covariant derivative in the orthonormal frame.

To derive the Killing equation on the squashed sphere, we need to rewrite the 6d Killing equation (24) in terms of 5d language. The 6d spin connection in the slanted frame has the components
\[
\Omega^{mn} = \omega_{\mathbb{C}P^2}^{mn} + \frac{1}{rv} I_{mn} (e^5 + u e^6), \quad \Omega^{m5} = \frac{1}{rv} I_{mn} e^n, \quad \Omega^{m6} = \frac{u}{rv} I_{mn} e^n,
\]
(42)

and the 6d covariant derivative becomes
\[
D^{(6)}_m = \partial_m + \frac{1}{2} \Omega_m^{\mu\nu} S_{\mu\nu} - \frac{u}{rv} I_{mn} S_{n6},
\]
(43)
\[
D^{(6)}_5 = v \partial_5 - \frac{u}{v} \partial_t + \frac{1}{2} \Omega_5^{\mu\nu} S_{\mu\nu},
\]
(44)
\[
D^{(6)}_6 = \frac{1}{v} \partial_t + \frac{u}{2rv} I^{\mu\nu} S_{\mu\nu},
\]
(45)

where \(S_{MN}\) are spin operators. For the spinor representation \(\tilde{S}_{MN} = (1/2) \Gamma_{MN}\). Among the components \(\Omega_L^{MN}\) \((L, M, N = 1, \ldots, 6)\) of 6d spin connection, \(\Omega_\lambda^{\mu\nu}\) \((\lambda, \mu, \nu = 1, \ldots, 5)\) are identified with the components of the spin connection on the squashed \(S^5\), and we include them in the definition of the 5d covariant derivative. The \(t\) derivatives are rewritten according to (41). Then the second term in (44) becomes
\[
\frac{u(1 + \alpha)}{2rv} \tau_3 e^5.
\]
(46)
This can be regarded as a background $SU(2)_R$ gauge field, and we also include this in the 5d covariant derivative. As the result, the explicit form of the 5d covariant derivative is

$$D^{(5)} = d - i[A, \ast] + \frac{1}{2} \Omega_{\mu\nu}[S_{\mu\nu}, \ast] + \frac{u(1 + \alpha)}{2rv} e^5[\tau_3, \ast],$$  \hspace{1cm} (47)$$

where we introduced gauge connection $A$, which has not been taken into account in the dimensional reduction. The 6d covariant derivatives are rewritten as

$$D^{(6)}_\mu = D^{(5)}_\mu - \frac{u}{rv} I_{\mu\nu} S_{\nu\delta},$$  \hspace{1cm} (48)$$

$$D^{(6)}_6 = -\frac{(1 + \alpha)}{2rv} \tau_3 + \frac{u}{2rv} I^{\mu\nu} S_{\mu\nu}. \hspace{1cm} (49)$$

With these relations and the explicit representation of the 6d Dirac matrices, we can rewrite the 6d Killing equation (24) as

$$D^{(5)}_\mu \epsilon - \frac{iu}{2rv} I_{\mu\nu} \gamma_\nu \epsilon = \gamma_\mu \kappa,$$  \hspace{1cm} (50)$$

$$-\frac{(1 + \alpha)}{2rv} \tau_3 \epsilon + \frac{u}{2rv} \chi \epsilon = -i \kappa,$$  \hspace{1cm} (51)$$

and (26) as

$$\kappa = -\frac{i}{2rv} \tau_3 (1 + iu \gamma_5) \epsilon.$$  \hspace{1cm} (52)$$

By eliminating $\kappa$ from these equations we obtain the differential equation

$$D^{(5)}_\mu \epsilon = -\frac{i(1 + \alpha)}{2rv} \tau_3 \gamma_\mu \epsilon + \frac{iu}{4rv} (3\gamma_3 \chi - \chi \gamma_3) \epsilon,$$  \hspace{1cm} (53)$$

and the algebraic equation

$$\alpha \epsilon = iu \gamma_5 \epsilon + u \tau_3 \chi \epsilon.$$  \hspace{1cm} (54)$$

The latter imposes a condition on the components of the spinor $\epsilon$ at every point. This reduces the number of independent components to six for $\mathcal{N} = 3/4$ and two for $\mathcal{N} = 1/4$.

### 3.3 Actions and transformation laws

Once we have obtained the equations that $\epsilon$ satisfies it is not difficult to obtain the SUSY actions and transformation laws by Noether procedure. We show only the results.
The transformation laws of vector multiplets are
\[
\begin{align*}
\delta \phi &= i(\epsilon \lambda), \\
\delta A_\mu &= - (\epsilon \gamma_\mu \lambda), \\
\delta \lambda &= - F \epsilon + i(D \phi) \epsilon + i D_a \tau_a \epsilon + \frac{(1 + \alpha)}{rv} \tau_3 \epsilon \phi - \frac{2u}{rv} \mathcal{I} \epsilon \phi, \\
\delta D_a &= i(\epsilon \tau_a \gamma^\mu D_\mu \lambda) - i(\epsilon \tau_a [\phi, \lambda]) + \frac{(1 + \alpha)}{2rv} (\epsilon \tau_3 \tau_a \lambda) - \frac{u}{2rv} (\epsilon \mathcal{I} \tau_a \lambda).
\end{align*}
\] (55)

The SUSY Chern-Simons action is
\[
\mathcal{L}_{CS} = F_{\alpha \beta \gamma} \left[ \left( \frac{i}{24} \epsilon^{\lambda \mu \nu \rho} A^\beta_\lambda F_{\mu \nu} F_{\rho \gamma} + \cdots \right) + \frac{i}{4} (\lambda^\alpha F_\beta^\gamma) + \frac{1}{4} \phi^\alpha F_{\mu \nu} F^{\gamma \mu \nu} + \frac{u}{2rv} \phi^\alpha \phi^\beta I_{\mu \nu} F^{\gamma \mu \nu} \right.
\]
\[
\left. - \frac{1}{4} D_a^\alpha (\lambda^\beta \tau_a \lambda^\gamma) - \frac{1}{2} \phi^\alpha (\lambda^\beta D_\lambda \lambda^\gamma) - \frac{1}{2} \phi^\alpha \lambda^\beta [\lambda, \phi]^\gamma + \frac{iu}{4rv} \phi^\alpha (\lambda^\beta D_\lambda \lambda^\gamma) + \frac{1}{2} \phi^\alpha D_\mu \phi^\beta D_\mu \phi^\gamma - \frac{1}{2} \phi^\alpha D_a^\beta D_a^\gamma + \frac{1}{2rv^2} \left( \frac{2}{3} + \frac{(1 + \alpha)^2}{6} + 2u^2 \right) \phi^\alpha \phi^\beta \phi^\gamma \right].
\] (56)

The SUSY Chern-Simons action is
\[
\mathcal{L}_{CS} = F_{\alpha \beta \gamma} \left[ \left( \frac{i}{24} \epsilon^{\lambda \mu \nu \rho} A^\beta_\lambda F_{\mu \nu} F_{\rho \gamma} + \cdots \right) + \frac{i}{4} (\lambda^\alpha F_\beta^\gamma) + \frac{1}{4} \phi^\alpha F_{\mu \nu} F^{\gamma \mu \nu} + \frac{u}{2rv} \phi^\alpha \phi^\beta I_{\mu \nu} F^{\gamma \mu \nu} \right.
\]
\[
\left. - \frac{1}{4} D_a^\alpha (\lambda^\beta \tau_a \lambda^\gamma) - \frac{1}{2} \phi^\alpha (\lambda^\beta D_\lambda \lambda^\gamma) - \frac{1}{2} \phi^\alpha \lambda^\beta [\lambda, \phi]^\gamma + \frac{iu}{4rv} \phi^\alpha (\lambda^\beta D_\lambda \lambda^\gamma) + \frac{1}{2} \phi^\alpha D_\mu \phi^\beta D_\mu \phi^\gamma - \frac{1}{2} \phi^\alpha D_a^\beta D_a^\gamma + \frac{1}{2rv^2} \left( \frac{2}{3} + \frac{(1 + \alpha)^2}{6} + 2u^2 \right) \phi^\alpha \phi^\beta \phi^\gamma \right].
\] (56)

The Yang-Mills action and the Fayet-Iliopoulos action are obtained by taking the prepotential
\[
\mathcal{F} = \frac{1}{2g_{YM}^2} \phi^{(1)} \epsilon (\phi^2) + \frac{irv}{1 + \alpha} \zeta (\phi^{(1)})^2 \epsilon \phi,
\] (57)

where the SUSY preserving constant vector multiplet on the squashed sphere is
\[
\mathcal{V}^{(1)} = (\phi^{(1)}, A^{(1)}, \lambda_1^{(1)}, D_a^{(1)}) = \left( 1, u e^5, 0, \frac{i(1 + \alpha)}{rv} \delta_{a3} \right).
\] (58)

The Yang-Mills action corresponding to the first term in (57) is
\[
\mathcal{L}_{YM} = \frac{1}{g_{YM}^2} \epsilon \left[ \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{iu}{8} \epsilon^{5 \mu \nu \rho \sigma} F_{\mu \nu} F_{\rho \sigma} - \frac{1}{2} D_a D_a - \frac{i(1 + \alpha)}{rv} \phi D_3 
\]
\[
- \frac{1}{2} \lambda D_\lambda - \frac{iu}{4rv} \lambda I \lambda - \frac{i(1 + \alpha)}{4rv} \lambda \tau_3 \lambda + \frac{1}{2} \lambda [\phi, \lambda] 
\]
\[
+ \frac{1}{2} D_\mu \phi D_\mu \phi + \left( \frac{2}{r^2} + \frac{(1 + \alpha)^2}{2r^2v^2} \right) \phi^2 \right].
\] (59)

The Fayet-Iliopoulos action corresponding to the second term in (57) is
\[
\mathcal{L}_{FI} = \zeta \epsilon \left[ D - \frac{iu}{2(1 + \alpha)} I_{\mu \nu} F^{\mu \nu} + \frac{4u^2}{rv(1 + \alpha)} A_5 
\]
\[
+ \frac{i}{rv(1 + \alpha)} \left( 2 + (1 + \alpha)^2 + 2u^2 \right) \phi \right].
\] (60)
The transformation laws for hypermultiplets are
\[
\begin{align*}
\delta q_i &= i(\epsilon \rho_i \psi), \\
\delta \psi &= i\bar{\rho}_i (\bar{D} q_i) \epsilon + \frac{3(1 + \alpha)}{2rv} \bar{\rho}_i \tau_3 \epsilon q_i - \frac{2u}{rv} \bar{\rho}_i \bar{\epsilon} q_i - i\bar{\rho}_i \epsilon [\phi, q_i] ,
\end{align*}
\] (61)

The kinetic action of hypermultiplets is
\[
\begin{align*}
\mathcal{L}\text{hyper} &= \frac{1}{2} \psi \bar{D} \psi + \frac{iu}{4vr} \psi \bar{\psi} + \frac{1}{2} \psi [\phi, \psi] + \psi \bar{\rho}_i [\lambda, q_i] \\
&\quad + \frac{1}{2} D_\mu q_i D^\mu q_i + \left( \frac{2}{r^2} - \frac{(1 + \alpha)^2}{8r^2v^2} \right) q_i q_i \\
&\quad + \frac{1}{2}(\tau_0)_{ij} q_i [D_a, q_j] - \frac{1}{2} [\phi, q_i] [\phi, q_i].
\end{align*}
\] (62)

By shifting the vector multiplet fields in (62) by \( V \rightarrow V + \mu \epsilon V^{(1)} \), the following real mass terms arise.
\[
\mathcal{L}\text{real mass}_{\text{hyper}} = \frac{1}{2} \psi [\mu \epsilon, \psi] - \frac{iu}{2} \psi \gamma_5 [\mu \epsilon, \psi] + \frac{1 + u^2}{2} [q_i, \mu \epsilon] [\mu \epsilon, q_i] + \frac{i(1 + \alpha)}{2rv} (\tau_3)_{ij} q_i [\mu \epsilon, q_j].
\] (63)

### 3.4 More actions in \( \mathcal{N} = 1/4 \)

When \( \alpha = 3iu \), the relation (54) implies \( \tau_3 \bar{\epsilon} = 2i\epsilon \), and there is another SUSY preserving constant vector multiplet in \( \mathcal{N} = 1/4 \) theory.
\[
\mathcal{V}^{(2)} = (\phi^{(2)}, A^{(2)}, \lambda_I^{(2)}, D_a^{(2)}) = \left( 0, e^5, 0, -\frac{4}{rv} \delta^{a3} \right).
\] (64)

With this multiplet we can construct the following quadratic action of vector multiplets corresponding to the prepotential \( \mathcal{F} = (1/2)\phi^{(2)} \text{tr} \phi^2 \).
\[
\mathcal{L}_1 = \text{tr} \left[ \frac{i}{8} \epsilon^{\mu
u\rho\sigma} F_{\mu\nu} F_{\rho\sigma} - \frac{i}{2rv} \lambda \bar{\epsilon} \lambda - \frac{1}{rv} \phi I^{\mu\nu} F_{\mu\nu} - \frac{4u}{r^2v^2} \phi \phi + \frac{1}{rv} \lambda \tau_3 \lambda + \frac{4}{rv} \phi D_3 \right].
\] (65)

This is not independent observable from the Yang-Mills action (59) in the sense that \( \mathcal{L}_1 \) and \( \mathcal{L}_{\text{YM}} \) are proportional to each other as \( Q \)-cohomology classes. Let us define \( Q_\pm \) by
\[
\delta(\eta \epsilon_\pm) = \eta Q_\pm ,
\] (66)

where the left hand side stands for the transformation with the Grassmann-odd parameter \( \eta \epsilon_\pm \). \( \eta \) is a constant Grassmann-odd number and \( \epsilon_\pm \) are the bosonic Killing spinors satisfying
\[
\tau_3 \epsilon_\pm = \pm \epsilon, \quad \epsilon_+^\dagger \epsilon_+ = \epsilon_-^\dagger \epsilon_- = 1 .
\] (67)
We can show
\[ \mathcal{L}_{\text{YM}} + \frac{i(1 + iu)}{g_{\text{YM}}^2} \mathcal{L}_1 = \frac{1}{g_{\text{YM}}^2} Q_+ \text{tr} \left[ \frac{1}{2} (Q_+ \lambda)^{\dagger} \lambda + \frac{2(1 - iu)}{rv} \epsilon^{\dagger} \lambda \phi \right]. \] (68)

Note that the bosonic part of (68) is not positive definite, and we cannot use this to localize the path integral.

We can introduce mass terms of hypermultiplets using (64). This can be regarded as the imaginary counterpart of the real mass terms (63). By shifting the vector multiplets in (62) by
\[ V \rightarrow V + \epsilon^{\dagger} \lambda \phi, \] (69)
we obtain the mass terms
\[ L_{\text{complex mass}}^{\text{hyper}} = -\frac{1}{2} [q_i, \mu_{re}][q_i, \mu_{re}] + \frac{1}{2} [q_i, \mu_{im}][q_i, \mu_{im}] + \frac{1}{2} \psi [\mu_{re}, \psi] - \frac{i}{2} \psi \gamma_5 [\mu_{im}, \psi] \]
\[ + \frac{i(1 + iu)}{2rv} (\tau_3)_{ij} q_i [\mu_{re}, q_i] - \frac{2}{rv} (\tau_3)_{ij} q_i [\mu_{im}, q_i]. \] (70)

### 3.5 \( \mathcal{N} = 1/2 \) and \( \mathcal{N} = 3/2 \)

In a gauge theory with a single adjoint hypermultiplet, the enhancement of supersymmetry occurs just as on the round \( S^5 \). We turn on the critical value of the \( SU(2)_F \) mass parameter by the shift
\[ \mathcal{V} \rightarrow \mathcal{V} + \mu_{re} \mathcal{V}^{(1)} + (\mu_{im} - u_{re}) \mathcal{V}^{(2)}, \] (69)
we obtain the mass terms

\[ L_{\text{hyper}} = -\frac{1}{2} [q_i, \mu_{re}][q_i, \mu_{re}] + \frac{1}{2} [q_i, \mu_{im}][q_i, \mu_{im}] + \frac{1}{2} \psi [\mu_{re}, \psi] - \frac{i}{2} \psi \gamma_5 [\mu_{im}, \psi] \]
\[ + \frac{i(1 + iu)}{2rv} (\tau_3)_{ij} q_i [\mu_{re}, q_i] - \frac{2}{rv} (\tau_3)_{ij} q_i [\mu_{im}, q_i]. \] (70)

This corresponds to the modification of the boundary condition (30) to
\[ \Phi(t + \beta) = \exp \left[ -\frac{\beta}{2r} ((1 + \alpha)(\tau_3 + \tau'_3) + iuJ) \right] \Phi(t). \] (72)

The Killing equation for the enhanced supersymmetry is obtained from the 6d \( \mathcal{N} = (2, 0) \) supersymmetry. \( \epsilon \) and \( \kappa \) are \( SO(5)_R \) quartet, and satisfy equations (53), and (54) with \( \tau_3 \) replaced by \( \tau_3 + \tau'_3 \). We can see that there are 12 supercharges for \( \alpha = -iu \) and 4 for \( \alpha = 3iu \). We call these supersymmetries \( \mathcal{N} = 3/2 \) and \( \mathcal{N} = 1/2 \), respectively. To write down the actions and the transformation laws in \( SO(5)_R \) covariant form, we embed fields and the SUSY parameters of \( \mathcal{N} = 1/4 \) or \( 3/4 \) theory into \( SO(5)_R \) multiplets as
\[ q_\alpha = (q_i, q_5) = (q_i, \phi), \quad \chi_a = \left( \begin{array}{c} \lambda_I \\ \psi_A \end{array} \right), \quad \epsilon_a = \left( \begin{array}{c} \epsilon_I \\ 0 \end{array} \right). \] (73)
After the elimination of the auxiliary fields $D_a$, we obtain the transformation laws

$$\delta q_a = i(\epsilon \tilde{\rho}_a \chi),$$
$$\delta A_\mu = - (\epsilon \gamma_\mu \chi),$$
$$\delta \psi = - \slashed{D}_\epsilon + i\tilde{\rho}_a (\slashed{D} q_a) \epsilon + \frac{1 + \alpha}{2rv} \tilde{\rho}_a \epsilon (\tau_3 + \tau'_3) q_a$$
$$+ \frac{2}{rv} \tilde{\rho}_a (\tau_3 + \tau'_3)(1 + \alpha) \epsilon q_a - \frac{2u}{rv} \tilde{\rho}_a \chi \epsilon q_a + i \frac{1}{2} \tilde{\rho}_{\alpha \beta} \epsilon [q_\alpha, q_\beta],$$

and the action

$$\mathcal{L} = tr \left[ \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{iu}{8} \epsilon_{\mu \nu \rho \sigma} F_{\mu \nu} F_{\rho \sigma} - \frac{1}{2} \chi D \chi - \frac{iu}{4rv} \chi \slashed{D} \chi - \frac{i(1 + \alpha)}{4rv} \chi (\tau_3 + \tau'_3) \chi - \frac{1}{2} \chi \tilde{\rho}_a [\chi, q_a] \right.$$
$$+ \frac{1}{2} D_\mu q_a D^\mu q_a - \frac{(1 + \alpha)^2}{2r^2 v^2} (q_1^2 + q_2^2) + \frac{2}{r^2} q_a q_a$$
$$- \frac{1}{4} [q_\alpha, q_\beta] [q_\alpha, q_\beta] - \frac{1 + \alpha}{3rv} \epsilon_{12\alpha \beta \gamma} [q_\alpha, q_\beta] q_\gamma \right],$$

where $SO(5)_R$ Dirac matrices $\tilde{\rho}_a$ are defined in the appendix.

4 Discussion

In this paper, we constructed SUSY transformation laws and SUSY actions in the $SU(3) \times U(1)$ symmetric squashed five-sphere. An important task we should try next is to compute the partition function. Although the instanton contribution has not yet been computed even for the round sphere, it should be possible to compute the perturbative sector of the partition function for the squashed $S^5$.

In the case of the round sphere, the saddle points in the perturbative sector are parameterized by the constant expectation values of the scalar fields $a^\alpha$ in the vector multiplets, and the on-shell action is obtained by substituting the constant vector multiplets $\mathcal{V}_a^\alpha = a^\alpha \mathcal{V}^{(1)}$ to the action. This is also the case for the squashed $S^5$. The classical Lagrangian density at the saddle point is

$$\mathcal{L} = \frac{1}{v^2 r^2} \left[ 4 + 4(1 + \alpha)^2 + 12u^2 - 8iu^3 \right] \mathcal{F}(a^\alpha).$$

If we multiply the volume of the squashed sphere $\pi^3 r^5 / v$ and set $\alpha = -iu$ and $\alpha = 3iu$, we obtain the classical action for $\mathcal{N} = 3/4$ and $\mathcal{N} = 1/4$

$$S_{\mathcal{N} = \frac{3}{4}} = (2\pi r)^3 \left( \frac{1 + iu}{v} \right)^{-1} \mathcal{F}(a^\alpha), \quad S_{\mathcal{N} = \frac{1}{4}} = (2\pi r)^3 \left( \frac{1 + iu}{v} \right)^{3} \mathcal{F}(a^\alpha).$$
Interestingly, in both cases the classical action depends on the squashing parameter through $(1 + iu)/v$. This is similar to the case of the $SU(2) \times U(1)$ invariant squashing of $S^3$[17]. Of course we cannot conclude whether the partition function depends on the squashing parameter until we compute the one-loop contribution because the dependence of the classical action may be absorbed in the normalization of the integration variables $a^\alpha$.

When we compute the partition function by localization, we need to choose one supercharge $Q$. Let us consider the $\mathcal{N} = 1/4$ case. In this case $Q$ is a linear combination of $Q_\pm$ defined by (66). If we choose $Q = aQ_+ + bQ_-$, its square is

$$Q^2 = 2abv \left[-\mathcal{L}_\psi - \frac{3i}{2r} \tau_3\right] + \text{gauge tr.},$$

(78)

where $\mathcal{L}_\psi$ is the Lie derivative along the vector field $\partial_\psi$. The squashing parameter dependence is factorized up to the field dependent gauge transformation term, and we can absorb it by the coefficients $a$ and $b$. This factorization strongly suggests that the partition function is independent of the squashing parameter. Indeed, in [9, 10] the partition function is computed based on the algebra (78), which is compatible with the contact structure of the manifold. It would be possible to apply the method in [9, 10] to $\mathcal{N} = 1/4$ theories on the squashed sphere.

Another way to obtain the partition function of $\mathcal{N} = 1/4$ theory is the direct calculation based on the harmonic expansion used in [5]. Because the $Q$-exact terms used in [5] breaks $SO(6)$ isometry of the round sphere to $SU(3) \times U(1)$, the computation in [5] does not rely on the full $SO(6)$ isometry, and the extension to the squashed sphere, which also has $SU(3) \times U(1)$ symmetry, is straightforward. For vector multiplets, we obtain

$$Z_{\text{vector}}^{1\text{-loop}} = \prod_{\alpha \in \text{root}} \prod_{k=1}^{\infty} \left(k + i \frac{1 + iu}{v} \alpha(a)\right)^{k^2+2}$$

(79)

up to a constant factor. Therefore, the squashing parameter dependence is absorbed by the rescaling

$$\frac{1 + iu}{v} a^\alpha \rightarrow a^\alpha$$

(80)

of the integration variables $a^\alpha$. This is nothing but the rescaling needed to absorb the squashing parameter dependence of the classical action $S_{\mathcal{N}=1/4}$ in (77). After the rescaling (80), the expression of the partition function becomes identical to that of the round $S^5$. Although we have not computed the partition function of hypermultiplets, it seems unlikely to depend on the squashing parameter.

On the other hand, in the case of $\mathcal{N} = 3/4$, there is no supercharges compatible with the contact structure. Namely, there is no supercharge $Q$ such that $Q^2$ generates shift along the Hopf fiber. In this case the partition function may depend on the squashing parameter.
The situation above is very similar to the 3d case. On the $SU(2) \times U(1)$ symmetric squashed $S^3$, there are two kinds of supersymmetry. One is $SU(2)$ singlet supersymmetry. In this case, the partition function does not depend on the squashing parameter[18]. See also [19, 20] for the analysis based on the contact structure. The other is $SU(2)$ doublet supersymmetry. In this case, the partition function depends on the squashing parameter[17] just as in the case of the ellipsoidal deformation[18].

It is an interesting problem whether the partition function of a 5d $\mathcal{N} = 3/4$ theory depends on the squashing parameter. We hope to return to this problem in the near future.

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A Appendix

A.1 Conventions for $SU(2)$ and $SO(5)$

$SU(2)_R$ generators $(\tau_a)_I^J$ and $SU(2)_F$ generators $(\tau'_a)_A^B$ ($a = 1, 2, 3$) are defined by

$$
\tau_1 = \tau'_1 = \sigma_x, \quad \tau_2 = \tau'_2 = \sigma_y, \quad \tau_3 = \tau'_3 = \sigma_z,
$$

where $\sigma_i$ are Pauli matrices:

$$
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

We follow the NW-SE rule for the contraction of $SU(2)$ indices, and when we need to raise an index, we use $\epsilon$ tensor with components $\epsilon^{12} = -\epsilon^{21} = 1$. The $SU(2)$ invariant product of two doublets are defined by

$$
XY = X^IY_I = (\epsilon^{IJ}X_J)Y_I.
$$

$SU(2)_R \times SU(2)_F$ invariant tensor $(\rho_i)_I^A$ and $(\overline{\rho}_i)_A^I$ ($i = 1, 2, 3, 4$) are defined by

$$
\rho_i = (\sigma_x, \sigma_y, \sigma_z, -i1_2), \quad \overline{\rho}_i = (\sigma_x, \sigma_y, \sigma_z, i1_2).
$$

$SO(5)_R$ Dirac matrices $\hat{\rho}_\alpha$ ($\alpha = 1, 2, 3, 4, 5$) are defined by

$$
\hat{\rho}_i = \begin{pmatrix} \rho_i \\ \overline{\rho}_i \end{pmatrix}, \quad \hat{\rho}_5 = \begin{pmatrix} \mathbf{1}_4 \\ -\mathbf{1}_4 \end{pmatrix}.
$$
Let \( \chi_a (a = 1, 2, 3, 4) \) be the \( SO(5)_R \) quartet consists of \( SU(2)_R \) doublet \( \lambda_I \) and \( SU(2)_F \) doublet \( \psi_A \),
\[
\chi_a = (\chi_1, \chi_2, \chi_3, \chi_4) = (\lambda_1, \lambda_2, \psi_1, \psi_2),
\]
and \( \chi'_a \) be defined from \( \lambda'_I \) and \( \psi'_A \) in the same way. The \( SO(5)_R \)-invariant product of these two \( SO(5)_R \) quartets is defined by
\[
\chi \chi' = \lambda \lambda' - \psi \psi' = \epsilon^{IJ} \lambda_I \lambda'_J - \epsilon^{IJ} \psi_I \psi'_J
\]
(87)

The spacetime Dirac matrices \( \gamma_\mu \) have the same components as \( \hat{\rho}_\alpha \)
\[
\gamma_1 = \hat{\rho}_1, \quad \gamma_2 = \hat{\rho}_2, \quad \gamma_3 = \hat{\rho}_3, \quad \gamma_4 = \hat{\rho}_4, \quad \gamma_5 = \hat{\rho}_5.
\]
(88)

\( \rho, \overline{\rho}, \hat{\rho}, \) and \( \gamma \) with multiple indices represent anti-symmetric products. For example,
\[
\gamma_{\mu \nu} = \frac{1}{2} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu).
\]
(89)

Backslashes represent the contraction with Dirac matrices. For example
\[
F = \frac{1}{2} \gamma^{\mu \nu} F_{\mu \nu}.
\]
(90)

The scalar product of two spinors are defined in the same way as \( SO(5)_R \)-invariant product (87).

\( SO(5)_R \) invariant antisymmetric tensor is defined by
\[
\hat{\rho}^{\alpha \beta \gamma \delta \epsilon} = \epsilon^{\alpha \beta \gamma \delta \epsilon} 1_4.
\]
(91)

With our representation of Dirac matrices, this has the component \( \epsilon^{12345} = -1 \).

The spacetime antisymmetric tensor has the same components with this.

References

[1] N. Seiberg, “Five-dimensional SUSY field theories, nontrivial fixed points and string dynamics,” Phys. Lett. B 388, 753 (1996) [hep-th/9608111].

[2] M. R. Douglas, “On D=5 super Yang-Mills theory and (2,0) theory,” JHEP 1102, 011 (2011) [arXiv:1012.2880 [hep-th]].

[3] N. Lambert, C. Papageorgakis and M. Schmidt-Sommerfeld, “M5-Branes, D4-Branes and Quantum 5D super-Yang-Mills,” JHEP 1101, 083 (2011) [arXiv:1012.2882 [hep-th]].

[4] K. Hosomichi, R. -K. Seong and S. Terashima, “Supersymmetric Gauge Theories on the Five-Sphere,” arXiv:1203.0371 [hep-th].
[5] H. -C. Kim and S. Kim, “M5-branes from gauge theories on the 5-sphere,” arXiv:1206.6339 [hep-th].

[6] H. -C. Kim, S. -S. Kim and K. Lee, “5-dim Superconformal Index with Enhanced En Global Symmetry,” arXiv:1206.6781 [hep-th].

[7] S. Terashima, “On Supersymmetric Gauge Theories on $S^4 \times S^1$,” arXiv:1207.2163 [hep-th].

[8] T. Kawano and N. Matsumiya, “5D SYM on 3D Sphere and 2D YM,” arXiv:1206.5966 [hep-th].

[9] J. Kallen and M. Zabzine, “Twisted supersymmetric 5D Yang-Mills theory and contact geometry,” JHEP 1205, 125 (2012) [arXiv:1202.1956 [hep-th]].

[10] J. Kallen, J. Qiu and M. Zabzine, “The perturbative partition function of supersymmetric 5D Yang-Mills theory with matter on the five-sphere,” arXiv:1206.6008 [hep-th].

[11] J. Kallen, J. A. Minahan, A. Nedelin and M. Zabzine, “$N^3$-behavior from 5D Yang-Mills theory,” arXiv:1207.3763 [hep-th].

[12] D. L. Jafferis and S. S. Pufu, “Exact results for five-dimensional superconformal field theories with gravity duals,” arXiv:1207.4359 [hep-th].

[13] V. Pestun, “Localization of gauge theory on a four-sphere and supersymmetric Wilson loops,” Commun. Math. Phys. 313, 71 (2012) [arXiv:0712.2824 [hep-th]].

[14] N. Hama and K. Hosomichi, “Seiberg-Witten Theories on Ellipsoids,” arXiv:1206.6359 [hep-th].

[15] T. Kugo and K. Ohashi, Prog. Theor. Phys. 104, 835 (2000) [hep-ph/0006231].

[16] T. Kugo and K. Ohashi, Prog. Theor. Phys. 105, 323 (2001) [hep-ph/0010288].

[17] Y. Imamura and D. Yokoyama, “N=2 supersymmetric theories on squashed three-sphere,” Phys. Rev. D 85, 025015 (2012) [arXiv:1109.4734 [hep-th]].

[18] N. Hama, K. Hosomichi and S. Lee, “SUSY Gauge Theories on Squashed Three-Spheres,” JHEP 1105, 014 (2011) [arXiv:1102.4716 [hep-th]].

[19] J. Kallen, “Cohomological localization of Chern-Simons theory,” JHEP 1108, 008 (2011) [arXiv:1104.5353 [hep-th]].
[20] K. Ohta and Y. Yoshida, “Non-Abelian Localization for Supersymmetric Yang-Mills-Chern-Simons Theories on Seifert Manifold,” arXiv:1205.0046 [hep-th].