CRITICAL POINTS OF RANDOM POLYNOMIALS
WITH INDEPENDENT IDENTICALLY
DISTRIBUTED ROOTS

ZAKHAR KABLUCHKO

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Abstract. Let $X_1, X_2, \ldots$ be independent identically distributed random variables with values in $\mathbb{C}$. Denote by $\mu$ the probability distribution of $X_1$. Consider a random polynomial $P_n(z) = (z - X_1) \ldots (z - X_n)$. We prove a conjecture of Pemantle and Rivin [in: I. Kotsireas and E. V. Zima, eds., Advances in Combinatorics, Waterloo Workshop in Computer Algebra, 2011] that the empirical measure

$$\mu_n := \frac{1}{n-1} \sum_{z \in \mathbb{C}: P_n'(z) = 0} \delta_z$$

counting the complex zeros of the derivative $P_n'$ converges in probability to $\mu$, as $n \to \infty$.

1. Statement of the result

A critical point of a polynomial $P$ is a root of its derivative $P'$. There are many results on the location of critical points of polynomials whose roots are known; see, e.g., [10]. One of the most famous examples is the Gauss–Lucas theorem stating that the complex critical points of any polynomial are located inside the convex hull of the complex zeros of this polynomial. Pemantle and Rivin [8] initiated the study of the probabilistic version of the problem. Let $X_1, X_2, \ldots$ be independent identically distributed (i.i.d.) random variables with values in $\mathbb{C}$. Denote by $\mu$ the probability distribution of $X_1$. Consider a random polynomial

$$P_n(z) = (z - X_1) \ldots (z - X_n).$$

Let $\mu_n$ be a probability measure which assigns to each critical point of $P_n$ the same weight, that is,

$$\mu_n = \frac{1}{n-1} \sum_{z \in \mathbb{C}: P_n'(z) = 0} \delta_z.$$

We agree that the roots are always counted with multiplicities. Pemantle and Rivin [8] conjectured that the distribution of roots of $P_n'$ should be stochastically close to the distribution of roots of $P_n$, for large $n$. In terms of logarithmic potentials, this means that the distribution of the equilibrium points of a two-dimensional electrostatic field generated by a large number of unit charges with i.i.d. locations should be close to the distribution of the charges themselves.
Theorem 1.1. Let $\mu$ be any probability measure on $\mathbb{C}$. Then, the sequence $\mu_n$ converges as $n \to \infty$ to $\mu$ in probability.

Pemantle and Rivin [8] proved Theorem 1.1 for all measures $\mu$ having a finite 1-energy. Later, Subramanian [11] gave a proof in the case when $\mu$ is concentrated on the unit circle. We refer to these two papers for more background information and motivation. Our aim is to prove Theorem 1.1 in full generality.

We need to be more specific about the mode of convergence in Theorem 1.1. Let $\mathcal{M}$ be the set of probability measures on $\mathbb{C}$. Endowed with the topology of weak convergence, $\mathcal{M}$ becomes a Polish space. We view $\mu_n$ as a random element with values in $\mathcal{M}$ and $\mu$ as a deterministic point in $\mathcal{M}$. With this convention, Theorem 1.1 states that for every open set $U \subset \mathcal{M}$ containing $\mu$,

$$
\lim_{n \to \infty} \mathbb{P}[\mu_n \notin U] = 0.
$$

Since convergence in distribution and convergence in probability are equivalent if the limit is a.s. constant (see Lemma 3.7 in [6]), we can state Theorem 1.1 as follows: the law of $\mu_n$ (viewed as a probability measure on $\mathcal{M}$) converges weakly to the unit point mass at $\mu$.

Our proof is based on the connection with the logarithmic potential theory (and does not follow the methods of [8] and [11]). The basic idea is to use the following formula (see, e.g., §2.4.1 in [3]): for every analytic function $f$ (which does not vanish identically),

$$
\frac{1}{2\pi} \Delta \log |f(z)| = \sum_{z \in \mathbb{C}: f(z) = 0} \delta_z.
$$

Here, $\Delta$ is the Laplace operator which should be understood in the sense of generalized functions [2]. A similar method appeared in the study of roots of polynomials whose coefficients (not roots) are independent random variables (see [5]), and in the random matrix theory (see [12]). We expect that there should be numerous further applications of the method. On the heuristic level, we learned the idea to use formula (1) from [1]; see also [4].

2. Proof

2.1. Method of proof. Consider the logarithmic derivative of $P_n$:

$$
L_n(z) := \frac{P_n'(z)}{P_n(z)} = \frac{1}{z - X_1} + \ldots + \frac{1}{z - X_n}.
$$

The main steps of the proof of Theorem 1.1 are collected in the following two lemmas.

Lemma 2.1. There is a set $F \subset \mathbb{C}$ of Lebesgue measure 0 such that for every $z \in \mathbb{C}\setminus F$ we have

$$
\frac{1}{n} \log |L_n(z)| \xrightarrow{P_{n \to \infty}} 0.
$$

Lemma 2.2. Let $\lambda$ be the Lebesgue measure on $\mathbb{C}$ and $\psi : \mathbb{C} \to \mathbb{R}$ any compactly supported continuous function. Then,

$$
\frac{1}{n} \int_{\mathbb{C}} (\log |L_n(z)|) \psi(z) d\lambda(z) \xrightarrow{P_{n \to \infty}} 0.
$$
After the lemmas have been established, the proof of Theorem 2.1 can be completed as follows. It suffices to show that for every infinitely differentiable, compactly supported function \( \varphi : \mathbb{C} \to \mathbb{R} \),

\[
\frac{1}{n} \sum_{z \in \mathbb{C} : P_n'(z) = 0} \varphi(z) \frac{P_n}{n \to \infty} \int_{\mathbb{C}} \varphi \, d\mu.
\]

(5)

Indeed, (5) implies that the law of \( \mu_n \) converges weakly (as a probability measure on \( \mathbb{M} \)) to the unit point mass at \( \mu \); see Theorem 14.16 in [6]. This implies convergence in probability since the limit is constant a.s.; see Lemma 3.7 in [6]. To prove (5) we use the following equality of generalized functions:

\[
\frac{1}{2\pi n} \Delta \log |L_n(z)| = \frac{1}{n} \sum_{z \in \mathbb{C} : P_n'(z) = 0} \delta_z - \frac{1}{n} \sum_{z \in \mathbb{C} : P_n(z) = 0} \delta_z.
\]

(6)

This equality follows from (1) with \( f = P_n' \) and \( f = P_n \) after subtraction and division by \( n \). Evaluating both sides of (6) on the test function \( \varphi \), we obtain

\[
\frac{1}{2\pi n} \int_{\mathbb{C}} (\log |L_n(z)|) \Delta \varphi(z) \, d\lambda(z) = \frac{1}{n} \sum_{z \in \mathbb{C} : P_n'(z) = 0} \varphi(z) - \frac{1}{n} \sum_{z \in \mathbb{C} : P_n(z) = 0} \varphi(z).
\]

(7)

As \( n \to \infty \), the left-hand side of (7) tends to 0 in probability by Lemma 2.2. Since the zeros of \( P_n \) are i.i.d. random variables, the second term on the right-hand side of (7) tends to \( \int \varphi \, d\mu \) in probability (and even a.s.) by the law of large numbers. This proves (5). In the rest of the paper we are occupied with the proofs of Lemmas 2.1 and 2.2.

2.2. Notation. Let \( \mathbb{D}_r(z) = \{ x \in \mathbb{C} : |x - z| < r \} \) be the open disk of radius \( r > 0 \) centered at \( z \in \mathbb{C} \), and let \( \mathbb{D}_r(z) \) be its closure. We also write \( \mathbb{D}_r = \mathbb{D}_r(0) \) and \( \mathbb{D}_r = \mathbb{D}_r(0) \). Let

\[
\log_- z = \begin{cases} 
\log |z|, & 0 \leq z \leq 1, \\
0, & z \geq 1,
\end{cases}
\]

\[
\log_+ z = \begin{cases} 
0, & 0 \leq z \leq 1, \\
\log z, & z \geq 1.
\end{cases}
\]

Note that \( \log_0 = +\infty \). The Lebesgue measure on \( \mathbb{C} \) is denoted by \( \lambda \).

2.3. Proof of Lemma 2.1. First of all, let us stress that, in general, (3) does not hold for every \( z \in \mathbb{C} \) since it evidently fails if \( z \) is an atom of \( \mu \). We need to introduce an exceptional set \( F \). It consists of points at which \( \mu \) has bad regularity properties.

Lemma 2.3. Consider the set

\[
F = \left\{ z \in \mathbb{C} : \int_{\mathbb{C}} \frac{d\mu(y)}{|y - z|} = +\infty \right\}.
\]

Then, the Lebesgue measure of \( F \) is zero.

Proof. Write \( K_1(y, z) = \frac{1}{|y - z|} 1_{|y - z| < 1} \) and \( K_2(y, z) = \frac{1}{|y - z|} 1_{|y - z| \geq 1} \). Since \( K_2(y, z) \) is bounded above by 1, we have \( \int_{\mathbb{C}} K_2(y, z) \, d\mu(y) \leq 1 \) for all \( z \in \mathbb{C} \). By Fubini’s theorem,

\[
\int_{\mathbb{C}} \left( \int_{\mathbb{C}} K_1(y, z) \, d\mu(y) \right) \, d\lambda(z) = \int_{\mathbb{C}} \left( \int_{\mathbb{C}} K_1(y, z) \, d\lambda(z) \right) \, d\mu(y) = 2\pi,
\]

\(
\)
where the second equality holds since the integral in the brackets is $2\pi$ for every $y \in \mathbb{C}$ and $\mu$ is a probability measure. Hence, $\int_{\mathbb{C}} K_1(y, z) d\mu(y) < \infty$ for $\lambda$-a.e. $z \in \mathbb{C}$. It follows that $\lambda(F) = 0$.

Proof of Lemma 2.1 First we show that for every $z \in \mathbb{C} \setminus F$ and every $\varepsilon > 0$,

$$
\lim_{n \to \infty} \mathbb{P}[|L_n(z)| \geq e^{\varepsilon n}] = 0. \tag{8}
$$

By the law of large numbers, we have

$$
\lim_{n \to \infty} \frac{L_n(z)}{n} = \int_{\mathbb{C}} \frac{d\mu(y)}{z - y} =: V(z) \quad \text{a.s.}
$$

Note that the integral on the right-hand side is finite since $z \notin F$. This immediately implies (8). It is possible that $V(z) = 0$ on a set of positive Lebesgue measure. For example, if $\mu$ is the uniform distribution on the unit circle, then $V(z) = 0$ on the unit disk. This is why the above argument gives just an upper bound on $|L_n(z)|$.

To obtain a lower bound, we have to use a different method. We now prove that for every $z \in \mathbb{C}$ which is not an atom of $\mu$ and every $\varepsilon > 0$,

$$
\lim_{n \to \infty} \mathbb{P}[|L_n(z)| \leq e^{-\varepsilon n}] = 0. \tag{9}
$$

If $X_i = c$ a.s., then $L_n(z) = n/(z - c)$ and (9) holds trivially. Assume therefore that the $X_i$’s are non-degenerate. Given a real-valued random variable $\xi$ we denote by

$$
Q(\xi; \delta) = \sup_{t \in \mathbb{R}} \mathbb{P}[t \leq \xi \leq t + \delta], \quad \delta > 0,
$$

the concentration function of $\xi$. We will use the fact that the concentration function of the sum of $n$ i.i.d. random variables decays like $O(1/\sqrt{n})$. More precisely, by Theorem 2.2 on p. 76 in [5], for every sequence of non-degenerate i.i.d. real-valued random variables $\xi_1, \xi_2, \ldots$ there is a constant $C$ such that for all $n \in \mathbb{N}$, $\delta > 0$, we have

$$
Q(\xi_1 + \ldots + \xi_n; \delta) \leq C \frac{1 + \delta}{\sqrt{n}}. \tag{10}
$$

The constant $C$ depends on the distribution of $\xi_1$ but does not depend on $\delta$ and $n$. Note that no moment requirements on the $\xi_i$’s are imposed. Fix $z \in \mathbb{C}$ which is not an atom of $\mu$. Consider the complex-valued random variables $Y_i = \frac{1}{z - X_i}, \quad i \in \mathbb{N}$. Since we assume that the $X_i$’s are non-degenerate, at least one of the random variables $\Re Y_1$ or $\Im Y_1$ is non-degenerate. Suppose for concreteness that $\Re Y_1$ is non-degenerate. Then,

$$
\mathbb{P}[|L_n(z)| \leq e^{-\varepsilon n}] \leq \mathbb{P}\left[\sum_{k=1}^{n} \Re Y_k \leq e^{-\varepsilon n}\right] \leq Q\left(\sum_{k=1}^{n} \Re Y_k, 2e^{-\varepsilon n}\right) \leq \frac{2C}{\sqrt{n}}.
$$

The last inequality follows from (10) for $n$ large. This completes the proof of (9). □

2.4. Proof of Lemma 2.2. We already know from Lemma 2.1 that $\frac{1}{n} \log |L_n(z)|$ converges to 0 in probability for Lebesgue almost all $z \in \mathbb{C}$. To prove Lemma 2.2 we need to interchange the limit and the integral in (4). This is done by means of the following lemma whose proof can be found in [12].

Lemma 2.4 (Lemma 3.1 in [12]). Let $(X, \mathcal{A}, \nu)$ be a finite measure space and $f_1, f_2, \ldots : X \to \mathbb{R}$ random functions which are defined over a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and are jointly measurable with respect to $\mathcal{A} \otimes \mathcal{B}$. Assume that:
In the next three lemmas we estimate
\[ (14) \]
For every \( n \in \mathbb{N} \), the sequence \( \int_X |f_n(x)|^{1+\delta} d\nu(x) \) is tight.

Then, \( \int_X f_n(x) d\nu(x) \) converges in probability to 0, as \( n \to \infty \).

Recall that \( \psi \) is a continuous function with compact support. Let \( r \) be such that the support of \( \psi \) is contained in the disk \( \mathbb{D}_r \). The first condition of Lemma 2.4 with \( f_n(z) = \frac{1}{n} \log |L_n(z)| \psi(z) \), \( X = \mathbb{D}_r \), and \( \nu = \lambda \) has been already verified in Lemma 2.1. The second condition with \( \delta = 1 \) follows from the next lemma.

**Lemma 2.5.** The sequence \( \frac{1}{n} \int_{\mathbb{D}_r} \log^2 |L_n(z)| d\lambda(z) \) is tight.

To prove Lemma 2.5 we need to control the zeros and the poles of \( L_n \) since at these points \( \log |L_n(z)| \) becomes infinite. We will use the Poisson–Jensen formula. Take some \( R > r \) such that with probability 1 there are no roots of \( P_n \) or \( P'_n \) on the circle \( |z| = R \), for every \( n \in \mathbb{N} \). This is possible for the following reason. Define a finite measure \( \rho_n \) on \( [0, \infty) \) by

\[
\rho_n(B) = \mathbb{E} \left[ \sum_{z \in \mathbb{C} : P_n(z) P'_n(z) = 0} 1_{|z| \in B} \right], \quad B \subset \mathbb{R} \text{ Borel.}
\]

The measure \( \rho_n \) has at most countably many atoms. Hence, we can find \( R > r \) which is not an atom of \( \rho_n \), for every \( n \in \mathbb{N} \).

Denote by \( x_{1,n}, \ldots, x_{k_n,n} \) those zeros of \( P_n \) which are located in the disk \( \mathbb{D}_R \). They form a subset of \( X_1, \ldots, X_n \). Let also \( y_{1,n}, \ldots, y_{n,n} \) be the zeros of \( P'_n \) located in the disk \( \mathbb{D}_R \). Note that \( k_n \leq n \) and \( l_n < n \). By the choice of \( R \), we can assume that no zero of \( P_n \) or \( P'_n \) is on the boundary of \( \mathbb{D}_R \). By the Poisson–Jensen formula (see [7], Chapter 8), we have for any \( z \in \mathbb{D}_R \) which is not a zero or pole of \( L_n \),

\[
(11) \quad \log |L_n(z)| = I_n(z; R) + \sum_{l=1}^{l_n} \log \left| \frac{R(z - y_{l,n})}{R^2 - \bar{y}_{l,n}z} \right| - \sum_{k=1}^{k_n} \log \left| \frac{R(z - x_{k,n})}{R^2 - \bar{x}_{k,n}z} \right|
\]

where we use the notation

\[
(12) \quad I_n(z; R) = \frac{1}{2\pi} \int_0^{2\pi} \log |L_n(Re^{i\theta})| |P_R(|z|, \theta - \arg z)| d\theta
\]

and \( P_R \) is the Poisson kernel

\[
(13) \quad P_R(\rho, \varphi) = \frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho \cos \varphi}, \quad \rho \in [0, R], \quad \varphi \in [0, 2\pi].
\]

In the next three lemmas we estimate \( I_n(z; R) \). The following lemma is valid for every deterministic choice of \( X_1, \ldots, X_n \).

**Lemma 2.6.** For every \( 0 < r < R \) there is a constant \( B_1 = B_1(r, R) \) such that for every \( n \in \mathbb{N} \) and \( X_1, \ldots, X_n \in \mathbb{C} \),

\[
(14) \quad \frac{1}{n} \sup_{z \in \mathbb{D}_r} I_n(z; R) \leq B_1.
\]

**Proof.** It follows from (13) that there is \( M = M(r, R) > 1 \) such that

\[
(15) \quad \frac{1}{M} < P_R(|z|, \theta) < M \quad \text{for all } z \in \mathbb{D}_r, \theta \in [0, 2\pi].
\]
Let us estimate \( \log |L_n(Re^{i\theta})| \). For every \( w_1, \ldots, w_n \in \mathbb{C} \) we have an elementary inequality
\[
\log_+ \left| \sum_{k=1}^{n} w_k \right| \leq \sum_{k=1}^{n} \log_+ |w_k| + \log n.
\]
Taking in this inequality \( w_k = 1/(Re^{i\theta} - X_k) \), \( k = 1, \ldots, n \), we obtain
\[
(16) \quad \log_+ |L_n(Re^{i\theta})| \leq \sum_{k=1}^{n} \log_+ |Re^{i\theta} - X_k| + \log n.
\]
There is a constant \( C = C(R) \) such that we have
\[
(17) \quad \sup_{x \in \mathbb{C}} \int_{0}^{2\pi} \log_+ |Re^{i\theta} - x|d\theta < C.
\]
To see this, note that for every \( t > 0 \) the Lebesgue measure of those \( \theta \in [0, 2\pi] \) for which \( \log_+ |Re^{i\theta} - x| > t \) does not exceed \( c_0 e^{-t} \), where \( c_0 \) does not depend on \( x \in \mathbb{C} \). Using (12), (15), (16), (17), we obtain that for every \( z \in D_r \),
\[
(18) \quad I_n(z; R) \leq M \int_{0}^{2\pi} \log_+ |L_n(Re^{i\theta})|d\theta \leq CMn + 2\pi M \log n.
\]
It follows that (14) holds.

In the next two lemmas we establish a lower bound on \( I_n(z; R) \) which is uniform in \( z \in D_r \). First we consider the case \( z = 0 \). Recall that \( F \) is an exceptional set defined in Lemma 2.3.

**Lemma 2.7.** Assume that \( 0 \notin F \). There is a constant \( A = A(R) \) such that
\[
\lim_{n \to \infty} \mathbb{P} \left[ \frac{1}{n} I_n(0; R) \leq -A \right] = 0.
\]

**Proof.** In the special case \( z = 0 \) the Poisson–Jensen formula (11) takes the form
\[
(19) \quad \frac{1}{n} I_n(0; R) = \frac{1}{n} \log |L_n(0)| - \frac{1}{n} \sum_{l=1}^{l_n} \log \left| \frac{y_{l,n}}{R} \right| + \frac{1}{n} \sum_{k=1}^{k_n} \log \left| \frac{x_{k,n}}{R} \right|.
\]
Recall that \( x_{1,n}, \ldots, x_{k_n,n} \) are those of the points \( X_1, \ldots, X_n \) which belong to the disk \( D_r \). By the law of large numbers,
\[
(20) \quad \frac{1}{n} \sum_{k=1}^{k_n} \log \left| \frac{x_{k,n}}{R} \right| \xrightarrow{a.s.} -\mathbb{E} \left[ \log_+ \left| \frac{X_1}{R} \right| \right].
\]
The expectation on the right-hand side is finite. To see this note that \( z \mapsto \log_+ |z/R| - \log_+ |z| \) is a bounded function with compact support and observe that \( \mathbb{E} \log_+ |X_1| < \infty \), a consequence of the assumption \( 0 \notin F \). It follows from (9) and (20) that there is \( A_1 = A_1(R) \) such that
\[
\lim_{n \to \infty} \mathbb{P} \left[ \frac{1}{n} \log |L_n(0)| \leq -1 \quad \text{or} \quad \frac{1}{n} \sum_{k=1}^{k_n} \log \left| \frac{x_{k,n}}{R} \right| \leq -A_1 \right] = 0.
\]
For the second term on the right-hand side of (19) we have trivially
\[
\frac{1}{n} \sum_{l=1}^{l_n} \log \left| \frac{y_{l,n}}{R} \right| \leq 0.
\]
The statement of the lemma follows with $A = A_1 + 1$. □

In the sequel we assume that $0 \notin F$. This is not a restriction of generality since in the case $0 \in F$ we can choose any $a \notin F$ (which exists by Lemma 2.3) and prove Theorem 1.1 for the random variables $Y_i = X_i - a$ instead of $X_i$.

**Lemma 2.8.** For every $0 < r < R$ there is a constant $B_2 = B_2(r, R)$ such that

$$\lim_{n \to \infty} \PP \left[ \frac{1}{n} \inf_{z \in \mathbb{D}_r} I_n(z; R) \leq -B_2 \right] = 0.$$  

Proof. Write $q_n^+(\theta) = \frac{1}{n} \log |L_n(Re^{i\theta})|$ and $q_n^-(\theta) = \frac{1}{n} \log |L_n(Re^{i\theta})|$, where $\theta \in [0, 2\pi]$. Then, $\frac{1}{n} \log |L_n(Re^{i\theta})| = q_n^+(\theta) - q_n^-(\theta)$. Note that $q_n^+(\theta) \geq 0$ and $q_n^-(\theta) \geq 0$. It follows from (12) and (15) that for all $z \in \mathbb{D}_r$,

$$\frac{2\pi}{n} I_n(z; R) = \int_0^{2\pi} q_n^+(\theta) P_R(|z|, \theta - \arg z) d\theta - \int_0^{2\pi} q_n^-(\theta) P_R(|z|, \theta - \arg z) d\theta \geq \frac{1}{M} \int_0^{2\pi} q_n^+(\theta) d\theta - M \int_0^{2\pi} q_n^-(\theta) d\theta = \frac{2\pi M}{n} I_n(0; R) - \left( M - \frac{1}{M} \right) \int_0^{2\pi} q_n^+(\theta) d\theta.$$

We have used the identity $I_n(0; R) = \frac{1}{2\pi} \int_0^{2\pi} \log |L_n(Re^{i\theta})| d\theta$. By Lemma 2.7 and the second inequality in (18) we have

$$\lim_{n \to \infty} \PP \left[ \frac{1}{n} I_n(0; R) \leq -A \right] = 0, \quad \int_0^{2\pi} q_n^+(\theta) d\theta \leq C + 2\pi.$$

The statement of the lemma follows. □

We are in position to complete the proof of Lemma 2.5. Applying the inequality $(\sum_{j=1}^m w_j)^2 \leq m \sum_{j=1}^m w_j^2$ three times to the Poisson–Jensen formula (11) and dividing by $n^2$ we obtain

$$\frac{1}{n^2} \log^2 |L_n(z)| \leq \frac{3}{n^2} I_n^2(z; R) + \frac{3}{n^2} \left( \sum_{i=1}^{l_n} \log \left| \frac{R(z - y_{1,i,n})}{R^2 - \bar{y}_{1,i,n} z} \right| \right)^2 + \frac{3}{n^2} \left( \sum_{k=1}^{k_n} \log \left| \frac{R(z - x_{k,n})}{R^2 - \bar{x}_{k,n} z} \right| \right)^2 \leq \frac{3}{n^2} I_n^2(z; R) + \frac{3l_n}{n^2} \sum_{i=1}^{l_n} \log^2 \left| \frac{R(z - y_{1,i,n})}{R^2 - \bar{y}_{1,i,n} z} \right| + \frac{3k_n}{n^2} \sum_{k=1}^{k_n} \log^2 \left| \frac{R(z - x_{k,n})}{R^2 - \bar{x}_{k,n} z} \right|.$$

It follows from Lemma 2.6 and Lemma 2.8 that the sequence $\frac{3}{n^2} \int_{\mathbb{D}_r} I_n^2(z; R) d\lambda(z)$ is tight. We estimate the remaining two terms on the right-hand side. We have, for some finite $C_1 = C_1(r, R),

$$\sup_{y \in \mathbb{D}_r} \int_{\mathbb{D}_r} \log^2 \left| \frac{R(z - y)}{R^2 - \bar{y} z} \right| d\lambda(z) < C_1.$$  

To see this, note that $|R^2 - \bar{y} z|$ remains bounded below as long as $z \in \mathbb{D}_r$, $y \in \mathbb{D}_R$, and use the integrability of the squared logarithm. Recall also that $k_n$ (resp., $l_n$) is the number of roots of $P_n$ (resp., $P'_n$) in the disk $\mathbb{D}_R$. Hence, both numbers do
not exceed $n$. It follows that there is a deterministic constant $C_2 = C_2(r, R)$ such that for every $n \in \mathbb{N}$,
\[
\frac{3n}{n^2} \sum_{l=1}^{k_n} \int_{D_r} \log^2 \left| \frac{R(z - y_{l,n})}{R^2 - y_{l,n}z} \right| d\lambda(z) + \frac{3k_n}{n^2} \sum_{k=1}^{k_n} \int_{D_r} \log^2 \left| \frac{R(z - x_{k,n})}{R^2 - x_{k,n}z} \right| d\lambda(z) \leq C_2.
\]

The sum of a tight sequence and an a.s. bounded sequence is tight. Hence, the sequence $\frac{1}{n^2} \int_{D_r} \log^2 |L_n(z)| d\lambda(z)$ is tight. The proof of Lemma 2.5 is complete.

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**Institute of Stochastics, Ulm University, Helmholtzstr. 18, 89069 Ulm, Germany**

**E-mail address**: zakhar.kabluchko@uni-ulm.de