An Existence Result for Impulsive Stochastic Functional Differential Equations with Multiple Delays

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Abstract
In this paper we consider Impulsive stochastic neutral functional differential equations with multiple delays. By using Schaefer’s fixed point theorem, we prove the existence of solutions for stochastic differential equations with impulses.

Keywords: Impulsive differential equations; Stochastic differential equations; Multiple delay; Fixed point theorem

AMS Mathematical Subject Classification: 34A37, 34K50, 34K45, 60H10.

Introduction
The theory of impulsive differential equations is an important area of scientific activity. Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. These short term perturbations act instantaneously, that is in the form of impulses. For example, that many biological phenomena involving thresholds, optimal control models in economics and frequency modulated systems, do exhibit impulsive effects. So the impulsive differential equations appear as a natural description of observed evolution phenomena of several real world problems. Existence of solutions of impulsive differential equations has been studied by many authors. If the impulses are random the solution becomes a stochastic process. Existence of solutions of differential equations with random impulses have been studied by many authors [1-3].

Furthermore, besides impulsive effects, stochastic effects likewise exist in real systems. There is a wide range of interesting process in robotics, economics and biology that can be described as differential equations with non-deterministic dynamics such phenomena are described by stochastic differential equations. The solution of stochastic differential equation is a stochastic process. However the solution of differential equation with random impulses is different from the solution of stochastic differential equations. Existence, Uniqueness and qualitative analysis of solutions of stochastic differential equations have been discussed by several authors [4,5].

Since both impulsive and stochastic effects exist it is very difficult to investigate the existence of solution of impulsive stochastic differential equations. In [6] Anguraj and Vinodkumar discussed the existence, uniqueness and stability of impulsive stochastic semi linear functional differential with infinite delays. Lakrib [7] discussed about the existence results for impulsive neutral functional differential equations with multiple delays. Based on the existing literature, stochastic impulsive differential equations involved mainly on controllability and stability. To the best of our knowledge, there is no work reported on impulsive stochastic differential equations with multiple delays. The purpose of this paper is to discuss about the existence results of impulsive stochastic neutral functional differential equations with multiple delays. Our approach is based on Schaefer’s fixed point theorem.

In this paper we study the existence results for stochastic impulsive differential equations with multiple delays

\[ d[x(t) - f(t, x(t))] = [g(t, x(t)) + \sum_{k=1}^{m} \Delta x(t_k)] dt + a(t, x(t)) dB(t), \]

where \( f: [0,1] \times \mathbb{R}^{n} \to \mathbb{R}^{n}, g: [0,1] \times \mathbb{R}^{n} \to \mathbb{R}^{n} \) and \( a: [0,1] \times \mathbb{R}^{n} \to \mathbb{R}^{n} \) are Borel Measurable functions, \( I_k = [-r, 0] \), and \( \phi: [-r, 0] \to \mathbb{R}^{n} \). Further-more the fixed moments of time \( t_k \) satisfy \( 0< t_{k+1} <t_k<...<t_1<0 \), where \( x(t_k^+) \) and \( x(t_k^-) \) represent the right and left limits of \( x(t) \) at \( t=t_k \), respectively. And \( \Delta x(t_k) = x(t_k^+) - x(t_k^-) \), \( k=1,2,...,m \) represent the jump in the state at time \( t_k \) with \( I_k \) determining the size of the jump.

Preliminaries
Let \( \Omega,F,\{F_t\}_{t \geq 0},P \) be a complete probability space with a filtration \( \{F_t\}_{t \geq 0} \) satisfying the conditions that it is right continuous and \( F_t \) contains all \( P \)-null sets and \( \omega(t)=(\omega(t_1),...\omega(t_n)) \) is an m-dimensional Brownian motion defined on \( (\Omega,F,\{F_t\}_{t \geq 0},P) \). Let \( C=C([-r,0],\mathbb{R}^{n}) \) denote the family of all continuous \( \mathbb{R}^{n} \)-valued function \( f \) on \([-r,0] \) with the norm

\[ \|f\| = \text{sup}_{t \in [-r,0]} \|f(t)\| \]

where \( | \cdot | \) is Euclidean norm of \( \mathbb{R}^{n} \). Denote by \( C_{F_t}([-r,0],\mathbb{R}^{n}) \) the family of all bounded \( F_t \)-measurable, \( c([-r,0],\mathbb{R}^{n}) \) -valued random variables \( \phi \), satisfying \( \|\phi\|_F = \text{sup}_{t \in [-r,0]} E[|\phi(t)|] < \infty \), where \( E \) denotes the expectation of stochastic process [8-10]. The initial condition \( \phi \in C_{F_t}([-r,0],\mathbb{R}^{n}) \).

Let \( PC(J,\mathbb{R}^{n}) \) the space of piecewise continuous functions \( x: J \to \mathbb{R}^{n} \) such that \( x \) is continuous everywhere except for \( t=t_k \) at which \( x(t_k^+) \) and \( x(t_k^-) \) exist and \( x(t_k^+) = x(t_k^-) \), \( k=1,2,...,m \). If we set \( \hat{\Omega} = \{x: J \to \mathbb{R}^{n}, x \in \mathbb{R}^{n} \cap PC(J,\mathbb{R}^{n}) \} \) where \( J=[-r,1] \) then \( \hat{\Omega} \) is a Banach space normed by

\[ \|x\| = \text{sup}_{t \in J}\{ |x(t)|; t \in J \}, x \in \hat{\Omega}. \]

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Thus we have, for all $x(t)$ measurable for each $x \in X$ for all $x \in [-\tau,1]$, is called a solution of equations (1.1) if

(i) $x(t)$ is $F_t$ adapted;

(ii) $x(t)$ satisfies the integral equation

$$x(t) = \begin{cases} \phi(t) & \text{for } t \in J, \\ \phi(0) - f(0,\phi(0)) + f(t,x_t) + \int_0^t g(s,x_s)ds + \sum_{i \in J_c} I_i(x(t_i^-)) & \text{for } t \in J \end{cases} \tag{2.1}$$

Definition 2.1: A stochastic process $\{x(t) \in C_{\mathbb{F}}([-\tau,0],[0,\infty]), t \in [-\tau,1] \}$ is called a solution of equations (1.1) if

Step 1: $\Gamma$ has bounded values for bounded sets in $\Omega$.

Let $B$ be a bounded set in $\Omega$. Then there exists a real number $\rho > 0$ such that $\|x(t)\| \leq \rho$, for all $x \in B$.

Let $x \in B$ and $t \in J$, we have

$$\Gamma(x(t)) = \begin{cases} \phi(t) & \text{for } t \in J, \\ \phi(0) - f(0,\phi(0)) + f(t,x_t) + \int_0^t g(s,x_s)ds + \sum_{i \in J_c} I_i(x(t_i^-)) + \sum_{i \in J_c} a(s,x_s)dB(s) + \sum_{i \in J_c} I_i(x(t_i^-)) & \text{for } t \in J \end{cases}$$

Step 2: $\Gamma$ maps bounded sets into equicontinuous sets.

Let $B$ be as in Step 1 and $x \in B$. Let $t$ and $h \neq 0$ be such that $t + h \in \mathbb{J}(t_1, t_2, ..., t_n)$.

Now

$$\begin{align*}
\|\Gamma(x(t + h)) - \Gamma(x(t))\| &\leq \|\phi(t + h) - \phi(t)\| + \|f(t,x_t) + f(t,x_{t+h})\| + \int_0^t \|g(s,x_s)ds\| + \sum_{i \in J_c} \|I_i(x(t_i^-))\| + \|a(s,x_s)dB(s)\| + \sum_{i \in J_c} \|I_i(x(t_i^-))\|
\end{align*}$$

as $h \to 0$, the right hand side of the above inequality tends to zero. This implies the equicontinuity on $\mathbb{J}(t_1, t_2, ..., t_n)$.

It remains to examine $t = t_i, i = 1, 2, ..., m$. Let $h \neq 0$ be such that $\{t_i : k \neq i\} \cap [t_i - |h|, t_i + |h|] = \emptyset$. Thus we have

$$\|\Gamma(x(t_i)) - \Gamma(x(t_i - h))\| \leq \|f(t,x(t_i - h)) - f(t,x(t_i))\|$$
The equicontinuity on $J_0$ follows from the uniform continuity of $\varphi$ on this interval.

**Step 3:** Now we show that $\Gamma$ is continuous

Let $\{x_n\} \subset \Omega$ be a sequence such that $x_n \to x$. We will show that $\Gamma x_n \to \Gamma x$.

For $t \in J$

$$
\begin{align*}
E[\|x_t - \Gamma_t\|] &= E[\|\int_0^t (g(t, x_s) - g(t, x_s))ds + p\int_0^t \varphi(t)\|] \\
&+ pE[\|\int_0^t (g(t, x_s) - a(t, x_s))dB(s) + \sum_{k=1}^m (I_k(x_t, t)) - I_k(x_t))\|] \\
&\leq 3E[\|f(t, x_t) - f(t, x_t)\|] + 3E[\|g(t, x_t) - g(t, x_t)\|] \\
&+ 3pE[\|\int_0^t g(t, x_s) - a(t, x_s)\|] \\
&+ 3mE[\|I_k(x_t, t) - I_k(x_t))\|] \\
&\leq 3E[\|f(t, x_t) - f(t, x_t)\|] + 3E[\|g(t, x_t) - g(t, x_t)\|] \\
&+ 3pE[\|\int_0^t g(t, x_s) - a(t, x_s)\|] \\
&+ 3mE[\|I_k(x_t, t) - I_k(x_t))\|] \quad (3.2)
\end{align*}
$$

Using $H_2$, it can be easily shown that the function $t \mapsto g(t, x_t) - g(t, x_t)$ is Lebesgue integrable. By the continuity of $f$ and $H_k$, $k = 1, 2, \ldots, m$ and the dominated convergence theorem, the right hand side of inequality (3.2) tends to zero as $n \to \infty$, which completes the proof that $\Gamma$ is continuous [15-18].

As a sequence of steps 1 to 3, together with the Arzela-Ascoli theorem, we conclude that $\Gamma$ is completely continuous.

To complete the proof of the theorem, it suffices to prove the following step.

**Step 4:**

There exists a priori bound of the set

$$
\mathcal{C} = \{x \in \Omega : \lambda x \in \Gamma x \text{ for some } \lambda > 1\}
$$

Let $x \in \mathcal{C}$ and $\lambda > 1$ be such that $\lambda x = \Gamma x$. Then $x \in [-r, t_1]$ satisfies for each $t \in [0, t_1]$,

$$
x(t) = \lambda^{-1} [\varphi(0) - f(0, \varphi(0))] + f(t, x_t) + \int_0^t g(t, x_s)ds + \int_0^t a(s, x_s)dB(s)
$$

It is easy to verify that

$$
E[\|x(t)\|] \leq E[\|\varphi(0)\| + 1 + c_1 + \sum_{k=1}^m r_k] + 3c_2 + 3E[\|\int_0^t g(t, x_s)ds + \int_0^t a(s, x_s)dB(s)\|] \\
\quad + 3m E[\|I_k(x_t, t)\|] \quad (3.3)
$$

Consider the function $v_t = \sup \left\{ E[\|x(s)\|] : s \in [-r, t]\right\}$, for $t \in [0, t_1]$. We have $\|x(t)\| \leq v_t$ for all $t \in [0, t_1]$ and there is a point $t^* \in [-r, t]$ such that $v_t = \|x(t^*)\|$. If $t^* < 0$, we have $v_{t^*} \leq \|x(0)\|$ for all $t \in [0, t_1]$. Now, if $t^* \geq 0$ from (3.3) it follows that,

$$
v_{t^*} \leq E[\|\varphi(0)\| + 1 + c_1 + \sum_{k=1}^m r_k] + 2c_2 + 3E[\|\int_0^t g(t, x_s)\|] + \|x(t^*)\|.
$$

And hence

$$
v_{t^*} \leq C_1^* + E[\|\int_0^t g(t, x_s)\|] + C_2^* + 3c_2 + \|x(t^*)\|.
$$

Where

$$
C_1^* = C_1^* + E[\|\varphi(0)\| + 1 + c_1 + \sum_{k=1}^m r_k] + 3c_2,
$$

And $Q = \max \{q(t), p\}, \forall t \in [0, t_1]$.

Set,

$$
w_t = C_1^* + \int_0^t Q(s)\|\varphi(v_t(s)) + v_t(s)\|ds\quad \forall t \in [0, t_1]
$$

Then we have $v_t \leq w_t$ for all $t \in [0, t_1]$.

A direct differentiation of $w_t$ yields

$$
w_t(t) = C_1^* + \int_0^t (Q(s)\|\varphi(v_t(s)) + v_t(s)\| + Q(s))ds \quad \forall t \in [0, t_1]
$$

By integration, this gives

$$
\int_0^t w_t(s)ds \leq \int_0^t Q(s)ds \leq \|w_t\| \quad t \in [0, t_1]
$$

By a change of variables, inequality (3.4) becomes,

$$
\int_0^t w_t(s)ds \leq \int_0^t Q(s)ds \leq \|w_t\| \quad t \in [0, t_1]
$$

By (3.1) and the mean value theorem, there is a constant $M_t = M_t > 0$ such that $w_t(t) \leq M_t$ for all $t \in [0, t_1]$.

That is $v_t(t) \leq M_t$ for all $t \in [0, t_1]$.

At last we choose $M_t$, such that $\|x(t)\| \leq M_t$ to get

$$
\sup \left\{ E[\|x(t)\|] : t \in [-r, t]\right\} = v_t(t) \leq M_t
$$

Now, consider $x \in \mathcal{C} = \{x \in \Omega : \lambda x \in \Gamma x \text{ for some } \lambda > 1\}$.

Let $x(t) = \lambda^{-1} [\varphi(0) - f(0, \varphi(0))] + f(t, x_t) + \int_0^t g(t, x_s)ds + \int_0^t a(s, x_s)dB(s)

+ \sum_{k=1}^m \int_0^t \int_0^s (g(s, x_r)\varphi(s) + a(r, x_r))dr ds + \int_0^t (1 + c_1 + \sum_{k=1}^m r_k) + 2c_2 + 3c_2

+ \sum_{k=1}^m \int_0^t \int_0^s (g(s, x_r)\varphi(s) + a(r, x_r))dr ds + \int_0^t (1 + c_1 + \sum_{k=1}^m r_k) + 2c_2 + 3c_2

$$
$$
Denote $v_t = \sup \left\{ E[\|x(s)\|] : s \in [-r, t]\right\}$, for $t \in [0, t_1]$. Then for each $t \in [0, t_1]$, we have $E[\|x(t)\|] \leq v_t$, such that $v_t \leq v_t(t)$.

Let $t^* \in [-r, t]$ be such that $v_t = \|x(t^*)\|$. If $t^* < 0$, we have $v_{t^*} \leq \|x(0)\|$ for all $t \in [0, t_1]$.

Now if $t^* \geq 0$, then by (3.5) we have for $t \in [0, t_1]$

$$
v_t \leq E[\|\varphi(0)\| + 1 + c_1 + \sum_{k=1}^m r_k] + 2c_2 + 3c_2 + \|x(t^*)\| + \sum_{k=1}^m \int_0^t (g(s, x_r)\varphi(s) + a(r, x_r))dr ds + \|x(t^*)\| + \sum_{k=1}^m \int_0^t (g(s, x_r)\varphi(s) + a(r, x_r))dr ds + \|x(t^*)\|
$$

$$
C_1^* + C_2^* \int_0^t Q(s)\|\varphi(v_t(s)) + v_t(s)\|ds
$$

Where

$$
C_1^* + C_2^* \int_0^t Q(s)\|\varphi(v_t(s)) + v_t(s)\|ds + \|x(t^*)\| \leq M_t
$$
\[ C^2_t = \frac{1}{1 - 2c_t} \]

And \( Q(t) = \max \{ q(t), p \} \) for \( t \in [0, t_1] \).

If we set,
\[
w_1(t) = C^2_t + C^2_t \int_0^t Q(s) \phi(\psi(s)) + \psi(s)ds \text{ for } t \in [0, t_1] \]

Then we get \( v_1(t) \leq w_1(t) \) for all \( t \in [0, t_1] \) and
\[
\begin{align*}
& w_2(t) \leq Q(t) [\phi(\psi(w_1(t))) + \psi(w_1(t))], \ a.e. t \in [0, t_1] \\
& w_2(0) = C^2_t
\end{align*}
\]

These yields,
\[
\begin{align*}
& \int_0^t \psi(w(s)) + p w(s) ds \leq \int_0^t Q(s) ds \leq \| Q \|_t \text{ for } t \in [0, t_1] \\
& \int_0^t \psi(s) + s ds \leq \| Q \|_t \leq (0, t_1)
\end{align*}
\]

Again by (3.1) and the mean value theorem, there is a constant \( M_1 = M_1(t, t_n) > 0 \) such that \( w(t) \leq M_1 \) for all \( t \in [0, t_1] \), and then \( v_1(t) \leq M_1 \) for all \( t \in [0, t_1] \).

Finally, we choose \( M_1 \) such that \( \varphi \leq M_1 \), we get,
\[
\begin{align*}
& \sup \{ E \| x(t) \| : t \in [-r, t_n] \} = v_1(t_1) \leq M_2 \\
& \text{Continue this process for } x, \ldots, x, \text{ we obtain that there exists a constant } M = M(t_1, \ldots, t_n) > 0 \text{ such that } x \leq M.
\end{align*}
\]

This finish to show that the \( \zeta \) is bounded in \( \Omega \).

As a result the conclusion of theorem holds and consequently the problem (1.1) has a solution \( x \) on \( J \). This completes the proof [19].

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