Linear Spinor Fields in Relativistic Dynamics

James Lindesay
Stanford Institute for Theoretical Physics

Abstract

Linear spinor fields are a generalization of the Dirac field that have transparent cluster decomposability properties needed for classical correspondence of relativistic quantum systems. The algebra of these fields directly incorporate gravitation within a group that unifies the dynamics of the same number of additional hermitian carriers of quantum numbers as there are gauge fields in SU(3)×SU(2)×U(1). They also provide a mechanism for the dynamic mixing of massless neutrinos using a “transverse mass” conjugate to the affine parameter labeling translations along its light-like trajectory, consistent with those in the standard model.

1 Introduction

The Dirac equation utilizes a matrix algebra to construct a linear relationship between the quantum operators for energy and momentum in the equations of motion. The properties of evolution dynamics described using such linear operations on quantum states are straightforward, and have direct interpretations. In particular, the cluster decomposability properties necessary for classical correspondence of relativistic quantum systems is most directly realized using linear quantum operations[1][2][3].

It is therefore advantageous to extend the Dirac formulation to include operators whose matrix elements reduce to the Dirac matrices for spin \( \frac{1}{2} \) systems, but generally require that the form \( \tilde{\Gamma}^\mu \tilde{\hat{P}}_\mu \) be a Lorentz scalar operation. The finite dimensional representations of the resulting extended Poincare group of transformations can be constructed using the little group of operations \( D^{\lambda'}_\lambda \) on the standard state vectors, defining general transformations of the form

\[
\hat{U}(\mathbf{b}) \left| \psi_\lambda \bar{a} \right\rangle = \sum_{\lambda'} \left| \psi'_{\lambda'} \bar{z}(\mathbf{b}; \bar{a}) \right\rangle \ D^{\lambda'}_{\lambda}(\mathbf{b}; \bar{a}).
\]

A spinor field equation will be demonstrated for configuration space eigenstates of the operator \( \hat{\tilde{\Gamma}}^\mu \hat{\tilde{\hat{P}}}_\mu \) of the form

\[
\tilde{\Gamma}^\beta \cdot \frac{\hbar}{i} \frac{\partial}{\partial x^\beta} \hat{\Psi}^{(\Gamma)}(\vec{x}) = -(\gamma)m\c \hat{\Psi}^{(\Gamma)}(\vec{x}).
\]

For the \( \Gamma = \frac{1}{2} \) representation, the matrix representations of the operators \( \tilde{\Gamma}^\beta \) are just one half of the Dirac matrices, and the particle type label takes values \( \gamma = \pm \frac{1}{2} \).

*Permanent address, Computational Physics Lab, Department of Physics, Howard University, Washington, DC 20059
An Extension of the Lorentz Group

The finite dimensional representations of an extension of the Lorentz group will be constructed by developing a spinor representation of the algebra. The group elements will include 3 parameters representing angles, 3 boost parameters, and 4 group parameters associated with four operators $\hat{\Gamma}^\mu$.

2.1 Extended Lorentz Group Commutation Relations

For the extended Lorentz group, the commutation relations for angular momentum and boost generators remain unchanged from those of the standard Lorentz group. The additional extended group commutation relations will be chosen to be consistent with the Dirac matrices as follows:

\[
\begin{align*}
[\Gamma^0, \Gamma^k] &= i K_k, & (2.3) \\
[\Gamma^0, J_k] &= 0, & (2.4) \\
[\Gamma^0, K_k] &= -i \Gamma^k, & (2.5) \\
[\Gamma^j, \Gamma^k] &= -i \epsilon_{jkm} J_m, & (2.6) \\
[\Gamma^j, J_k] &= i \epsilon_{jkm} \Gamma^m, & (2.7) \\
[\Gamma^j, K_k] &= -i \delta_{jk} \Gamma^0. & (2.8)
\end{align*}
\]

A Casimir operator can be constructed for the extended Lorentz (EL) group in the form

\[
C = J \cdot J - K \cdot K + \Gamma^0 \Gamma^0 - \Gamma \cdot \Gamma. 
\]

This operator can directly be verified to commute with all generators of the group. The operators $C$, $\Gamma^0$, and $J_z$ will be chosen as the set of mutually commuting operators for the construction of the finite dimensional representations.

2.2 Group metric of the extended Lorentz group

The group metric for the algebra represented by

\[
[\hat{G}_r, \hat{G}_s] = -i (c_s)_{r}^{m} \hat{G}_m 
\]

can be developed from the adjoint representation in terms of the structure constants:

\[
\eta_{ab} \equiv (c_a)_{r}^{s} (c_b)_{s}^{r}. 
\]

The non-vanishing components of the extended Lorentz group metric are given by

\[
\eta^{(EL)}_{J_m J_n} = -6 \delta_{mn}, \quad \eta^{(EL)}_{K_n K_n} = +6 \delta_{mn}, \quad \eta^{(EL)}_{\Gamma^{\mu} \Gamma^{\nu}} = +6 \eta_{\mu\nu}. 
\]
It is important to note that the group structure of the extended Lorentz group generates the Minkowski metric $\eta_{\mu\nu}$ as a direct consequence of the group structure. Neither the group structure of the usual Lorentz group nor that of the Poincare group can *generate* the Minkowski metric as a metric describing an invariance due to the abelian nature of the generators for infinitesimal space-time translations.

2.3 Symmetry Behavior of Spinor Forms

The substitution of the angular momenta $J_k$ and $\Gamma^0$ with barred operators identified as

$$J \leftrightarrow \bar{J}$$
$$\Gamma^0 \leftrightarrow -\bar{\Gamma}^0$$

will preserve the commutation relations [2.8]. For the Dirac case, this is seen to represent a "particle-antiparticle" symmetry of the system, and it represents a general symmetry under negation of the eigenvalues of the operator $\Gamma^0$.

2.4 Finite dimensional spinor representations

Spinor representations of this extension of the Lorentz group will next be constructed.

2.4.1 Number of States

The order of the spinor polynomial of the finite dimensional state of the $\Gamma = J_{\text{max}}$ representation can be determined by examining the minimal state from which other states can be constructed using the raising operators and orthonormality. This minimal state takes the form

$$\psi^{(\Gamma)}_{-\Gamma, -\Gamma} = A^{(\Gamma J)} \chi^{(-)2\Gamma},$$

(2.15)

where the $\chi^{(\pm)}$ represent the four spinor states. The lower index $\pm$ labels the angular momentum basis, while the upper index labels eigenvalues of $\Gamma^0$. The general state involves spinor products of the type

$$\chi^a_+ \chi^b_+ \chi^c_- \chi^d_-.$$

(2.16)

A complete basis of states requires then that $a + b + c + d = 2\Gamma$. By direct counting this yields the number of states for a complete basis:

$$N_\Gamma = \frac{1}{3}(\Gamma + 1)(2\Gamma + 1)(2\Gamma + 3).$$

(2.17)

For instance, $N_0 = 1, N_{\frac{1}{2}} = 4, N_1 = 10, N_{\frac{3}{2}} = 20$, and so on.

A single $J$ basis with $(2J + 1)^2$ states does not cover this space of spinors. However, one can directly verify that

$$N_{J_{\text{max}}} = \sum_{J=J_{\text{min}}}^{J_{\text{max}}} (2J + 1)^2,$$

(2.18)
where $J_{\min}$ is zero for integral systems and $\frac{1}{2}$ for half integral systems. Thus one can conclude that $\Gamma$ represents the maximal angular momentum state of the system:

$$J \leq \Gamma = J_{\text{max}}.$$  \hspace{1cm} (2.19)

Higher order representations will include states of differing angular momenta with common quantum statistics.

### 2.4.2 Spinor metrics

Invariant amplitudes are usually defined using dual spinors so that the inner product is a scalar under the transformation rule for the spinors $D$:

$$<\bar{\psi}|\phi> = <\bar{\psi}'|\phi'>, \text{ or } \psi^\dagger_a g_{ab} \phi_b = (D_{ca} \psi_a)^\dagger g_{cd} (D_{db} \psi_b).$$  \hspace{1cm} (2.20)

This means the spinor metric $g$ should satisfy

$$g = D^\dagger g D. \hspace{1cm} (2.21)$$

The Dirac conjugate spinor $\bar{\psi} \equiv \psi^\dagger g$ includes this spinor metric.

The eigenvalues of the hermitian angular momentum operators $\underline{J}$ and $\Gamma^0$ are given by real numbers

$$\underline{J}^\dagger = \underline{J} \hspace{1cm} \Gamma^0 = \Gamma^0,$$  \hspace{1cm} (2.22)

which requires that the finite dimensional representations satisfy

$$g \Gamma^0 = \Gamma^0 g \hspace{1cm} g \underline{J} = \underline{J} g.$$  \hspace{1cm} (2.23)

The spinor metric therefore takes the general form

$$g_{a'a'}^{(\Gamma J)} = (-)^{\Gamma-\gamma} \delta_{\gamma\gamma'} \delta_{s_z s'_z}.$$  \hspace{1cm} (2.24)

using the quantum number shorthand $a : \{\gamma, s_z\}$. One can show that the general form of this spinor metric anti-commutes with the boost generator and spatial components of the $\underline{\Gamma}$ matrices:

$$g \underline{\Gamma} = -\underline{\Gamma} g \hspace{1cm} g K = -K g.$$  \hspace{1cm} (2.25)

For the Dirac representation, the spinor metric takes the form of the Dirac matrix $\gamma^0$. However, the spinor metric is not related to $\Gamma^0$ for higher spin representations.

### 2.4.3 Representation of $\Gamma = \frac{1}{2}$ systems

The forms of the matrices corresponding to $\Gamma = \frac{1}{2}$ are expected to have dimensionality $N_{\frac{1}{2}} = 4$, and can be expressed in terms of the Pauli spin matrices $\sigma_j$ as shown below:

$$\Gamma^0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2} g \hspace{1cm} \underline{J}_j = \frac{1}{2} \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}$$

$$\Gamma^j = \frac{1}{2} \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \hspace{1cm} K_j = -\frac{1}{2} \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}.$$  \hspace{1cm} (2.26)
The $\Gamma^\mu$ matrices can directly be seen to be proportional to a representation of the Dirac matrices$^4$$^5$. A representation for $\Gamma = 1$ can be found in reference $^1$.

3 An Extension of the Poincare Group

Once space-time translations are included in the group algebra, these additional commutation relations must result in a self-consistent set of generators$^1$. The extended Lorentz group structure can be minimally expanded to include space-time translations as long as all operators continue to satisfy the algebraic Jacobi identities,

$$[\hat{A}, [\hat{B}, \hat{C}]] + [\hat{C}, [\hat{A}, \hat{B}]] + [\hat{B}, [\hat{C}, \hat{A}]] = 0. \quad (3.27)$$

An attempt to only include the 4-momentum operators in addition to the extended Lorentz group operators does not produce a closed group structure, due to Jacobi relations of the type $[\hat{P}_j, [\hat{\Gamma}^0, \hat{\Gamma}^k]]$. The non-vanishing of this commutator in the Jacobi identity implies a non-vanishing commutator between operators $\Gamma^\mu$ and $P_\nu$, and that this commutator must connect to an operator which then has a commutation relation with $\Gamma^\mu$ that yields a 4-momentum operator $P_\beta$. Since the 4-momentum operators self-commute, at least one additional operator, which will be referred to as $\hat{M}_T$, must be introduced.

The additional non-vanishing commutators involving Jacobi consistent operators $\hat{P}_\mu$ and $\hat{M}_T$ are given by

\begin{align*}
[J_j, P_k] &= i\hbar \epsilon_{jkm} P_m, \quad (3.28) \\
[K_j, P_0] &= -i\hbar P_j, \quad (3.29) \\
[K_j, P_k] &= -i\hbar \delta_{jk} P_0, \quad (3.30) \\
[\Gamma^\mu, P_\nu] &= i \delta^\mu_\nu \hat{M}_T c, \quad (3.31) \\
[\Gamma^\mu, \hat{M}_T] &= \frac{i}{c} \eta^{\mu\nu} P_\nu, \quad (3.32)
\end{align*}

The first three of these relations are identical to those of the Poincare group. The final two relations consistently incorporate the additional operator $\hat{M}_T$ needed to close the algebra.

3.1 Invariants and group metric

As demonstrated in section 2.2, given the structure constants defining the commutation relationships of the generators, a metric for the complete group $\eta_{\alpha\beta}$ can be developed using (2.11). The non-vanishing group metric elements generated by the structure constants of this extended Poincare group are given by

\begin{align*}
\eta^{(EP)}_{J_m J_n} &= -8 \delta_{m,n} \\
\eta^{(EP)}_{K_m K_n} &= +8 \delta_{m,n} \\
\eta^{(EP)}_{\Gamma^\mu \Gamma^\nu} &= 8 \eta_{\mu\nu} \quad (3.33) \\
\eta^{(EP)}_{\Gamma^\mu \hat{M}_T} &= 8 \eta_{\mu\nu} \quad (3.34)
\end{align*}
where $\eta_{\mu\nu}$ is the usual Minkowski metric of the Lorentz group. The Minkowski metric is non-trivially generated by the extended Lorentz group algebra. This group theoretic metric can be used to develop Lorentz invariants using the operators $\Gamma^\mu$. Since $\Gamma^\mu P_\mu$ is also Lorentz invariant, the group transformation properties of the generators $P_\mu$, as well as their canonically conjugate translations $x^\mu$, are direct consequences of the group properties of the extended Poincare group. The standard Poincare group has no non-commuting operators that can be used to connect the group structure to the metric properties of space-time translations.

### 3.2 Unitary quantum states

A Casimir operator for the complete group can be constructed using the Lorentz invariants given by

$$C_m \equiv M^2_c - \eta^{\beta\nu} P_\beta P_\nu.$$  \hfill (3.35)

The label $m$ in the Casimir operator $C_m$ parameterizes the eigenstates that can be developed to construct a finite dimensional representation. The form of this group invariant suggests that the hermitian operator $M_2$ is a transverse mass parameter of the state, which can have a non-vanishing value for massless states $\eta^{\beta\nu} P_\beta P_\nu = 0$. A set of quantum state vectors that are labeled by mutually commuting operators are given by

$$\hat{C}_m |m, \Gamma, \gamma, J, s_z\rangle = m^2 c^2 |m, \Gamma, \gamma, J, s_z\rangle,$$

$$\hat{C}_\Gamma |m, \Gamma, \gamma, J, s_z\rangle = 2\Gamma(\Gamma + 2) |m, \Gamma, \gamma, J, s_z\rangle,$$

$$\hat{J}^2 |m, \Gamma, \gamma, J, s_z\rangle = J(J + 1)\hbar^2 |m, \Gamma, \gamma, J, s_z\rangle,$$

$$\hat{J}_z |m, \Gamma, \gamma, J, s_z\rangle = s_z \hbar |m, \Gamma, \gamma, J, s_z\rangle,$$

where $m^2$ is generally a continuous real parameter, and all other parameters are discrete. $\Gamma$ is an integral or half-integral label of the representation of the extended Lorentz group, $J$ labels the internal angular momentum representation of the state, and $J$ has the same integral signature as $\Gamma$.

Eigenvalues of the group Casimir $\hat{C}_m$ and $\hat{J}^2$ will be used to label an arbitrary standard state vector. An additional invariant can be constructed from the pseudo-vector

$$\hat{W}_\alpha \equiv i\epsilon_{\alpha\beta\mu\nu} \hat{\Gamma}^\beta \hat{\Gamma}^\mu \eta^{\nu\lambda} \hat{P}_\lambda,$$

$$\hat{W}_0 = \hat{J} \cdot \hat{P},$$

$$\hat{W} = \hat{K} \times \hat{P} + \hat{J} \hat{P}_0,$$  \hfill (3.37)

where the antisymmetric tensor $\epsilon_{\alpha\beta\mu\nu}$ is defined by

$$\epsilon_{\alpha\beta\mu\nu} \equiv \begin{cases} +1 & \text{for } (\alpha\beta\mu\nu) \text{ an even permutation of } (0,1,2,3), \\ -1 & \text{for } (\alpha\beta\mu\nu) \text{ an odd permutation of } (0,1,2,3), \\ 0 & \text{for any two indexes equal.} \end{cases}$$  \hfill (3.38)

The Lorentz invariant $\hat{W}^2 \equiv \hat{W}_\alpha \eta^{\alpha\beta} \hat{W}_\beta$ commutes with $\hat{P}_\beta$ and $\hat{M}_T$, since $[\hat{W}_\beta, \hat{P}_\mu] = 0 = [\hat{W}_\beta, \hat{M}_T]$, The covariant 4-vector $\hat{W}_\alpha$ is orthogonal to the 4-momentum operator, $\hat{P}_\mu \eta^{\mu\nu} \hat{W}_\nu = 0$, due to the antisymmetric form defining $\hat{W}_\alpha$. When acting upon a massive particle state at rest (which has a time-like 4-momentum),
the 4-vector $W_\alpha$ is seen to be a space-like vector whose invariant length is related to the particle's spin times its mass.

Unitary representations of general momentum states are obtained by boosting standard states satisfying Eq. 3.36. Standard massive states have covariant 4-momentum components $\vec{p}^{(s)} = (-mc, 0, 0, 0)$ and vanishing eigenvalue of transverse mass $\hat{M}_T$ labeled $m_T = 0$. Standard massless states have covariant 4-momentum components $\vec{p}^{(s)} = (-1, 0, 0, 1)$ with an eigenvalue of transverse mass operator given by $m_T$ that need not vanish. General Lorentz transformations on the 4-momentum eigenstates satisfy

$$U(\Lambda^{(L)}) |\vec{p}, (m_T), m, J, \kappa\rangle = \sum_{\kappa'} |(\Lambda^{(L)}) \vec{p}, (m_T), m, J, \kappa'\rangle Q^{(s)}_{\kappa'\kappa}(\Lambda^{(L)}, \vec{p}),$$

where the matrices $Q^{(s)}_{\kappa'\kappa}$ with discrete indices $\kappa$ describe the finite dimensional, unitary transformations on the standard momentum state of the system. The Casimir label $m$ satisfies $m^2 c^2 = m_T^2 c^2 - \vec{p} \cdot \vec{p}$ for a state with 4-momentum $\vec{p}$.

### 3.3 Linear Wave Equation for Single Particle States

Eigenstates of the operator $\Gamma^\mu P_\mu$ will give linear operator dispersion relations for energy and momenta in a spinor wave equation. The commutators of the various group generators with this Lorentz invariant operator are given by

$$[J_k, \Gamma^\mu P_\mu] = 0$$

$$[K_k, \Gamma^\mu P_\mu] = 0$$

$$[P_\beta, \Gamma^\mu P_\mu] = -i\hat{M}_T P_\beta$$

$$[\hat{M}_T, \Gamma^\mu P_\mu] = -i\eta_{\beta\nu} P_\beta P_\nu$$

One should note that, from Eq. 3.43, the transverse mass only commutes with $\Gamma^\mu P_\mu$ for massless particles. Similarly, from Eq. 3.42 the 4-momentum operator only commutes with $\Gamma^\mu P_\mu$ if the transverse mass vanishes. Therefore, only massless states can have non-vanishing transverse mass values $m_T \neq 0$. The transverse mass operator $\hat{M}_T$ is the generator for translations of the affine parameter labeling the trajectory of a massless particle.

Spinor forms of the quantum state vectors and matrix representations of the operators can be developed in the usual manner:

$$\langle \chi_a | \Gamma^\mu | \chi_b \rangle \equiv (\Gamma^\mu)_{ab} , \quad \Psi^{(T)}_a (\vec{p}, J, \kappa) \equiv \langle \chi_a | \vec{p}, m, J, \kappa \rangle.$$  

This results in a momentum-space form of the spinor equation given by

$$\Gamma^\mu \hat{P}_\mu \Psi^{(T)}_a (\vec{p}, J, \kappa) = (\gamma) mc \Psi^{(T)}_a (\vec{p}, J, \kappa),$$

where $(\gamma)$ is the eigenvalue of $\hat{P}^0$ for massive particle states. It is worth noting that this linear spinor formulation does not have negative energy solutions. Rather, there is a sign is associated with the particle
type eigenvalue $\gamma$. Therefore, straightforward interpretations of the energetics of particles can be made without introducing any filled Dirac sea of fermions to prevent transitions from positive energy states. There is no need to introduce any additional degrees of freedom to stabilize the ground state once radiative coupling is included.

The configuration-space representation of (3.45) takes the form

$$\Gamma^\mu \frac{\hbar}{i} \frac{\partial}{\partial x^\mu} \Psi^{(\gamma)}_{(\gamma)}(\vec{\gamma}) = (\gamma) mc \Psi^{(\gamma)}_{(\gamma)}(\vec{\gamma}).$$

The spinor fields satisfy microscopic causality as long as their quantum statistics is fermionic for $\Gamma$ half-integral, and bosonic for $\Gamma$ integral. The transformation properties of the fields under the improper Lorentz transformations of parity and time reversal, as well as under charge conjugation, can be developed in a straightforward manner[1].

A form of a Lagrangian for gravitating linear spinor fields with local gauge symmetries will next be displayed. One can define spinor-valued geometric matrices of the form $U_{\beta}(x) \equiv \Gamma^\hat{\mu} \frac{\partial}{\partial \xi^\hat{\mu}} \hat{\beta}$, where the $\xi^\hat{\mu}$ represent locally flat coordinates. A gauge covariant Lagrangian density for a particle with Casimir label $m$ is given by

$$L_m = \frac{1}{2\Gamma} \frac{h_c}{i} \left[ \Psi^{(\gamma)}_{(\gamma)} U^\beta \left( \partial_\beta - \frac{q}{h_c} A_\beta^G \right) \Psi^{(\gamma)}_{(\gamma)} - (\text{c. c.}) \right] + \frac{(\gamma)}{\Gamma} mc^2 \Psi^{(\gamma)}_{(\gamma)} \Psi^{(\gamma)}_{(\gamma)},$$

where (c. c.) specifies the complex conjugate of the previous expression, and the matrices $G_r$ are hermitian generators of the local gauge group of symmetries for the linear spinor field $\Psi^{(\gamma)}_{(\gamma)}$. Such a Lagrangian form has straightforward cluster decomposition properties when systems of mixed entanglements are being described.

### 3.4 Spinor Lie transformation algebra and the principle of equivalence

Linear spinor fields are useful for describing the micro-physics of gravitating systems for several reasons. One useful property is their cluster decomposition properties that allow straightforward combinations of systems with arbitrary degrees of quantum entanglements at varying times. Another property is that the group metric generated by the operators $\hat{\Gamma}^\hat{\mu} \hat{\beta}$ constructs the Minkowski metric. Because operators like $\hat{\Gamma}^\hat{\mu} \hat{P}_\mu$ are invariant under the Lorentz subgroup of transformations, the components of the momenta likewise transform in a manner consistent with this metric defining subgroup invariants. There is no analogous Lorentz subgroup metric for the Poincare group, since there are no non-abelian operators in that group to generate this metric. Since the 4-momentum operators $\hat{P}_\mu$ transform like basis vectors, the invariance of $\hat{P}_\mu \eta^{\hat{\mu} \hat{\nu}} \hat{P}_\nu$ will have significance defining the space-time metric. The relevant group properties of the extended Poincare group will be further explored in this section.

The general extended Poincare group transformation can be characterized by the 15 parameters $\mathcal{P}_X \equiv \{ \omega, U, \Theta, a, \alpha \}$ conjugate to generators $\{ \hat{\Gamma}^\mu, \hat{K}_j, \hat{J}_k, \hat{P}_\nu, \hat{M}_T \}$ representing “Dirac boosts”, Lorentz boosts,
rotations, space-time translations, and lightlike translations. For brevity, the 5 mutually commuting extended translations will be labeled by a barred parameter \( \bar{a} \equiv \{ \alpha, \bar{a} \} \), while the set of 10 extended Lorentz group parameters will be underlined \( \underline{a} = \{ \Theta, U, \underline{\omega} \} \). The overall group structure will be examined for the product transformation of pure translations with pure extended Lorentz group transformation defined by the convention \( \hat{U}(\mathcal{P}_X) \equiv \hat{X}(\bar{a})\hat{W}(a) \). A reversal of this order results in a representation that is a similarity transformation on the elements.

The individual subgroups have well-defined group operations within each subgroup, while the overall group will have group operations based upon the convention:

\[
\begin{align*}
\hat{X}(\bar{b})\hat{X}(\bar{a}) & = \hat{X}(\bar{\phi}_x(\bar{b}, \bar{a})), \\
\hat{W}(\bar{b})\hat{W}(a) & = \hat{W}(\bar{\phi}_x(\bar{b}, \underline{\omega}a)), \\
\hat{U}(\mathcal{P}_X')\hat{U}(\mathcal{P}_X) & = \hat{U}(\Phi(\mathcal{P}_X'; \mathcal{P}_X)).
\end{align*}
\]

Using group properties, one can generally show that the translation group operation is independent of the initial extended Lorentz group parameter \( a \). \( \Phi_x(\bar{b}, \bar{a} ; \bar{a}) = \bar{\phi}_x(\bar{b}, \Phi_x(I, \underline{\omega} ; \bar{a})) \), where \( I \) is the identity element. Likewise, the extended Lorentz group operation is independent of the final translation \( \bar{b} \) using this convention, \( \Phi_x(\bar{b}; \bar{a}, \bar{a}) = \bar{\phi}_w(\Phi_x(\underline{\omega}a; \bar{b}, b^{-1}); \underline{\phi}_w(\underline{\omega}a)) \). The inverse group element to \( \mathcal{P}_X = \{ a, \bar{a} \} \), which can be calculated using \( W^{-1}X^{-1} = W^{-1}X^{-1}W W^{-1} \), is given by \( \mathcal{P}_X^{-1} = \{ \Phi_w(a^{-1}; \bar{a}^{-1}, I); \Phi_x(I, \underline{\omega}^{-1}; \bar{a}^{-1}) \} \).

Group associativity for operations within the subgroups is expressed in the relationships

\[
\begin{align*}
\bar{\phi}_x(\bar{c}, \bar{\theta} ; \bar{\phi}_x(\bar{b}, \bar{a} ; \bar{a})) & = \bar{\phi}_x(\bar{c}, \bar{\theta} ; \bar{\phi}_x(\bar{b}, \bar{a} ; \bar{a})), \\
\bar{\phi}_w(\underline{\phi}_x(\bar{b}, \bar{a} ; \bar{a}) ; \underline{\phi}_w(\underline{b}, \underline{a} ; \underline{a})) & = \bar{\phi}_w(\underline{\phi}_w(\underline{b}, \underline{a} ; \underline{a}) ; \underline{\phi}_x(\bar{b}, \bar{a} ; \bar{a})).
\end{align*}
\]

In particular, the translationally independent group of transformations

\[
\bar{\phi}_x(I, \underline{\theta} ; \bar{\phi}_x(I, \underline{\omega} ; \bar{\theta})) = \bar{\phi}_x(I, \underline{\phi}_w(\underline{\omega}I, \underline{\theta} ; \bar{\theta})
\]

forms a Lie transformation group \( \hat{x}' \equiv \hat{x}(\bar{a} ; \bar{\theta}) = \bar{\phi}_x(I, \underline{\omega} ; \bar{\theta}) \). However, one should note that the full group of transformations is generally beyond those of a traditional Lie transformation group. The transformations in gauge symmetries are typically represented by traditional Lie transformation groups.

Given the group operation, the complete set of group parameters (like structure constants, Lie structure matrices, transformation matrices, etc.) can be constructed. The matrices that define how the group transformations mix the generators of the group \( U(\mathcal{P}^{-1}) G_r U(\mathcal{P}) = \oplus_r^s(\mathcal{P}) G_r \) are given by

\[
\oplus_r^s(\mathcal{P}) = \frac{\partial}{\partial \mathcal{P}'} \Phi^s(\mathcal{P}^{-1}; \Phi(\mathcal{P}'; \mathcal{P})) \bigg|_{\mathcal{P}'=I} = \frac{\partial \Phi^m(\mathcal{P}'; \mathcal{P})}{\partial \mathcal{P}'} \bigg|_{\mathcal{P}'=I} \frac{\partial \Phi^s(\mathcal{P}^{-1}; \mathcal{P}')}{\partial \mathcal{P}'} \bigg|_{\mathcal{P}'=\mathcal{P}},
\]

where it is convenient to define the Lie transformation matrices \( \Theta_m^s(\mathcal{P}) \equiv \frac{\partial \Phi^m(\mathcal{P}^{-1}; \mathcal{P}')}{{\partial \mathcal{P}'}} \bigg|_{\mathcal{P}'=\mathcal{P}} \), and \( \Theta_m^m(\mathcal{P}) \equiv \frac{\partial \Phi^m(\mathcal{P}'; \mathcal{P})}{{\partial \mathcal{P}'}} \bigg|_{\mathcal{P}'=I} \). With these definitions, the transformation matrices for the generators satisfy

\[
\oplus_r^s(\mathcal{P}) = \Theta_r^m(\mathcal{P}) \Theta_m^s(\mathcal{P}).
\]
The transformations \( \bar{\phi}(\vec{b}; \vec{a}) = \bar{\Phi}(\vec{b}, I; \vec{a}) \) form an abelian subgroup of translations. The extended translations all commute with each other, but they are mixed amongst each other by the extended Lorentz transformations. This means that a general operation of the form \( F(\xi^\Lambda \bar{P}_\Lambda) \) will transform under extended Lorentz transformations according to

\[
\hat{U}(\{\vec{b}, \vec{I}\}) F(\xi^\Lambda \bar{P}_\Lambda) \hat{U}^{-1}(\{\vec{b}, \vec{I}\}) = F(\xi^\Lambda \oplus \Lambda^\Delta(\vec{u}^{-1}) \bar{P}_\Delta),
\]

where the capital Greek indices sum over the five parameters including the space-time coordinates and the affine coordinate conjugate to the transverse mass. Thus, the \( \oplus \Lambda^\Delta(\vec{u}) \) define the extended Lorentz transformation matrices on the momenta. The fact that the translations are abelian allows a very useful choice of the translation group parameters associated with space-time coordinates. If one utilizes the factor

\[
O^\Lambda_\Delta \Delta(\vec{a}) \equiv \partial \xi^\Lambda \bar{\Phi}_x(\vec{a}) = \partial \Phi^\Lambda_x(\vec{a}^{-1}; \vec{a}),
\]

one finds that the associativity condition (3.49) implies that

\[
\xi^\Lambda(\vec{a}) \oplus \Lambda^\Delta(\vec{a}) \oplus \Lambda^\Lambda(\vec{a}) = \xi^\Lambda(\vec{a}) \oplus \Lambda^\Lambda(\vec{a}) \oplus \Lambda^\Lambda(\vec{a}) + \xi^\Lambda(\vec{a}).
\]

Therefore, if one defines the special set of coordinates \( \xi^\Lambda \) by

\[
\partial \xi^\Lambda \partial \bar{x} = O^\Lambda_\Delta \Delta(\vec{x}),
\]

then these coordinates have the property that

\[
\xi^\Lambda(\vec{a}) = \int_\Delta \partial \Phi^\Lambda_x(\vec{a}) \partial \Phi^\Lambda_x(\vec{a}); \vec{a}) d\phi^\Lambda_x(\vec{a}),
\]

or, in a more suggestive form

\[
\bar{\phi}(\vec{x}_2, \vec{a}_2; \vec{x}_1) = \bar{\phi}(\vec{x}_2) + \xi^\Lambda(\vec{x}_1) \oplus \Lambda^\Lambda(\vec{a}_2),
\]

This means that these coordinates satisfy \( \xi^\Lambda(\bar{\phi}(\vec{x}_1; \vec{a})) = \xi^\Lambda(\bar{\phi}(\vec{x}_1; \vec{a})) + \xi^\Lambda(\vec{a}) \). The coordinate transformation is related to the group operation via

\[
\frac{\partial \xi^\Lambda(\vec{x})}{\partial x^\Lambda} = \mathcal{V}^\Lambda(\vec{x}) = \left. \frac{\partial \Phi^\Lambda_x(\vec{x}_1; \vec{I}; \vec{x})}{\partial x^\Lambda} \right|_{\vec{x}_1 = \vec{x}}.
\]

This equation directly relates the tetrads \( \mathcal{V}^\Lambda_\mu_\beta \) to the extended Poincare group operation.

More generally, the special coordinates satisfy

\[
\xi^\Lambda(\bar{\phi}(\vec{x}_2; \vec{a}_2; \vec{x}_1)) = \xi^\Lambda(\vec{x}_2) + \xi^\Lambda(\vec{x}_1) \oplus \Lambda^\Lambda(\vec{a}_2^{-1}),
\]

or, in a more suggestive form

\[
\bar{\phi}(\vec{x}_2; \vec{a}_2; \vec{x}_1) = \bar{\phi}(\vec{x}_2) + \xi^\Lambda(\vec{x}_2) \oplus \Lambda^\Lambda(\vec{a}_2^{-1}); \vec{x}_1)
\]

The expression (3.59) demonstrates a direct mapping of locally flat coordinates into curvilinear coordinates, consistent with the principle of equivalence.
4 Dynamic mixing of massless states

The transverse mass operator $\hat{M}_T$ of a massless particle is the generator for affine parameter translations $\Delta \lambda$ along the particle’s light cone trajectory, and its non-vanishing eigenvalue propagates a stationary particle with the usual quantum phase $e^{-i\bar{\hbar}m_j c \Delta \lambda}$. Since all massless particles share the same phase for space-time propagation $e^{i\bar{\hbar} \vec{p} \cdot \vec{x}}$, the usual manner for differentiating particles for dynamic mixing requires the introduction of small masses for the particles[6]. However, massless particles of differing transverse mass can be mixed in a straightforward manner. In particular, a mechanism for mixing massless neutrinos of fixed helicity $\pm \frac{1}{2}$ will be developed.

Suppose that massless neutrinos of finite transverse mass mix to form flavor eigenstates in a manner consistent with the single helicity states giving V-A couplings in weak interactions, and analogous to the quark mixing that suppresses neutral, strangeness-changing currents. The transverse mass eigenstates will be labeled by $|m_j\rangle$, while the eigenstates of flavor that define generation $a$ will be labeled $|f_a\rangle$. Any mixing due to the dynamics is expected to relate the states in a manner that preserves unitarity,

$$|f_a\rangle = \sum_j |m_j\rangle U_{ja},$$

requiring that the components $U_{ja}$ define a unitary matrix. Finite translations for massless particles take the form of a simple exponential in terms of the affine parameter along the trajectory $\lambda$, and the generator of infinitesimal affine translations $\hat{M}_T$, i.e. $T_\lambda = e^{-i\bar{\hbar} \hat{M}_T c \lambda}$. This means that the transition amplitude for mixing massless flavor eigenstates $f_a \rightarrow f_b$ is of the form

$$A(f_b \leftarrow f_a) = \sum_j U^*_{jb} e^{-i\bar{\hbar}m_j c \Delta \lambda} U_{ja}.$$  

The scale of the affine parameter is given by the spatial/temporal distance of the null particle trajectory $\Delta \lambda = L = cT$.

The transition probability for the mixing $P(f_b \leftarrow f_a) = |A(f_b \leftarrow f_a)|^2$ satisfies

$$P(f_b \leftarrow f_a) = \delta_{ba} +$$

$$-2 \sum_{j<k} \left\{ 2Re[\Upsilon_{jk}(b,a)] \sin^2 \left( \frac{\delta m_{jk} c L}{2\bar{\hbar}} \right) + Im[\Upsilon_{jk}(b,a)] \sin \left( \frac{\delta m_{jk} c L}{\bar{\hbar}} \right) \right\},$$

where $\Upsilon_{jk}(b,a) \equiv U_{jb} U^*_{ja} U^*_{kb} U_{ka}$. Thus, massless particles with differing transverse mass eigenvalues $\delta m_{jk} = m_j - m_k$ can indeed allow dynamical mixing of flavor eigenstates.

5 Additional Hermitian generators

The fundamental representation of the extended Lorentz group can be developed in terms of $4 \times 4$ matrices, with a particular representation given in (2.26). The three angular momentum generators, along
with $\Gamma^0$, make up the 4 Hermitian generators of this group. It is of interest to examine the other Hermitian
generators in the group GL(4).

There are 16 Hermitian generators whose representations are $4 \times 4$ matrices. This means that there
are 12 additional $4 \times 4$ Hermitian generators beyond those of the extended Lorentz group. One of these
generators is proportional to the identity matrix, and thus commutes with all other generators. Thus, this
generator forms a U(1) internal symmetry group that defines a conserved hypercharge on the algebra.

Three additional generators $\tau_j$ form a closed representation of SU(2) on the lower components of a
spinor:

$$\tau_j = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & \sigma_j \end{pmatrix}.$$  \hfill (5.63)

These generators transform as components of a 3-vector under the little group of transformations for a
massive particle. In a Lagrangian of the form (3.47), transformations involving these generators vanish
on upper component standard state vectors.

Six additional generators are given by the Hermitian forms of the anti-Hermitian generators $\Gamma^j$ and
$K_j$ given by $T_j = i \Gamma^j$ and $T_{j+3} = i K_j$. The final two generators are given by

$$T_7 = \frac{i}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T_8 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  \hfill (5.64)

The set of 8 Hermitian generators $T_s$ do not form a closed algebra independent of the other Hermitian
generators. The spinor metric transforms these generators according to $gT_s g = -T_s$. Transformations
involving at most 5 re-combinations of these generators will vanish on upper component standard state
vectors for a Lagrangian of the form (3.47). However, the remaining three combinations necessarily mix
particle states defined by the $\Gamma^\mu P_\mu$ form of the Lagrangian.

6 Conclusion

Physical models that are unitary, maintain quantum linearity, have positive definite energies, and have
straightforward cluster decomposition properties, can be constructed in a straightforward manner using
linear spinor fields. The piece of the group algebra that connects the group structure to metric gravitation
necessitates the inclusion of an additional group operator that generates affine parameter translations
for massless particles. This allows dynamic mixing of massless particles in a manner not allowed by the
standard formulations of Dirac or Majorana.

Just as classical mechanics emerges from the expectation values of quantum processes, space-time
geometry can be assumed to emerge from expectation valued measurements of quantum energies and
momenta via Einstein’s equation. Using this interpretation, classical geometrodynamics is emergent from
the behaviors of ensembles of mixed quantum states as they independently decohere. Linear spinor fields
then maintain their linearity by describing coherence using coordinate descriptions transformed from the
proper coordinates of the gravitating fields.
Conserved particle type can be shown to be a consequence of the spinor field equation. Internal gauge symmetries can be incorporated into the formulation in the usual manner. In addition, the fundamental representation of the linear spinor fields unifies a set of micro-physical interactions involving quanta exchanging 12 Hermitian degrees of freedom, with the geometrodynamics of general relativity through a single unified group of transformations. At least some of these interactions necessarily mix fundamental standard states of representations. It is intriguing that the number of additional Hermitian degrees of freedom beyond those defining the extended Lorentz group is the same as the number of generators in the standard model of fundamental interactions.

7 Acknowledgements

The author wishes to acknowledge the support of Elnora Herod and Penelope Brown during the intermediate periods prior to and after his Peace Corps service (1984-1988), during which time the bulk of this work was accomplished. In addition, the author wishes to recognize the hospitality of the Department of Physics at the University of Dar Es Salaam during the three years from 1985-1987 in which a substantial portion of this work was done. Finally, the author wishes to express his appreciation of the hospitality of the Stanford Institute for Theoretical Physics during his year of sabbatical leave.

References

[1] J. Lindesay, Foundations of Quantum Gravity, Cambridge University Press (2013), ISBN 978-1-107-00840-3.

[2] J.V. Lindesay, A.J. Markevich, H.P. Noyes, and G. Pastrana, Phys. Rev. D33, 2339 (1986).

[3] M. Alfred, P. Kwizera, J.V. Lindesay, and H.P. Noyes, A Non-Perturbative, Finite Particle Number Approach to Relativistic Scattering Theory, SLAC-PUB-8821, hep-th/0105241 (2001).

[4] P.A.M. Dirac, Proc. Roy. Soc. (London), A117, 610 (1928); ibid, A118, 351 (1928).

[5] J.D. Bjorken and S.D. Drell, Relativistic Quantum Mechanics, McGraw-Hill, New York, 1964.

[6] Particle Data Group, as posted (2009).