SINGULAR HEAT AND WAVE EQUATIONS ON THE EUCLIDIEN SPACE $\mathbb{R}^n$

Mohamed Vall Ould Moustapha

Abstract

In this paper we give the explicit formulas for the solution of the singular generalized heat and wave equations on the Euclidian space $\mathbb{R}^n$:

Math. Subj. Classification 2010 : 35J05, 35J08, 35K08.

1 Introduction

In this paper we discuss the explicit formulas for the solution of the following singular generalized heat and wave equations on the Euclidian space $\mathbb{R}^n$:

\[
\begin{cases}
\left(\frac{\partial}{\partial t} + \frac{k}{t}\right) u(t, X) = \Delta u(t, X); (t, X) \in \mathbb{R}_+^n \times \mathbb{R}^n \\
u(0, X) = f(X); f \in C^\infty(\mathbb{R})
\end{cases}
\]

\[
\begin{cases}
\left(\frac{\partial}{\partial t} + \frac{k}{t}\right) \left(\frac{\partial}{\partial t} - \frac{k}{t}\right) w(t, X) = \Delta w(t, X); (t, X) \in \mathbb{R}_+^n \times \mathbb{R} \\
w(0, X) = 0, w_t(0, X) = g(X), g \in C^\infty(\mathbb{R})
\end{cases}
\]

where

\[
\Delta = \frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2} + \ldots + \frac{\partial^2}{\partial X_n^2}
\]

is the usual n-dimensional Euclidian Laplacian on $\mathbb{R}^n$ and $k$ is a real number.

The mathematical interest in these equations, however, comes mainly from the fact that the time inverse potential $\frac{k}{t}$ (resp. the time inverse square $\frac{k(1-k)}{t^2}$) is homogeneous of degree -1 (resp. -2) and therefore scales exactly the same as $\partial / \partial t$ (resp. $\partial^2 / \partial t^2$).

An inconvenient of the time dependent potential is the absence of the relation between the semi-groups of the Schrödinger equation and the spectral properties of the operator. The space inverse potential $k/x$ is called Coulomb potential and is widely studied in physical and mathematical literature[1].

The space inverse square potential $k(1-k)/x^2$ arises in several contexts, one of them is the Schrödinger equation in non relativistic quantum mechanics (Reed and Simon [7]). For example, the Hamiltonian for a spinzero particle in Coulomb field gives rise to a Schrödinger operator involving the space inverse square potential (Case [2]). The Cauchy problem for the wave equation with the space inverse square potential in Euclidean space $\mathbb{R}^n$ is extensively studied (Cheeger and Taylor [3]), (Planchon et al[6]). The cases considered frequently are $k = 0$, the equations in (1.1) and (1.2) then turn into the classical heat and wave equation on the Euclidean spaces $\mathbb{R}^n$ and these equations appear in several branches of mathematics and physics (Folland [4], p.143, 171).

Our main objective of this paper is to solve the Cauchy problems (1.1) and (1.2).
2 Singular heat equation

**Theorem 2.1** The generalized singular heat equation in (1.1) has the following general solution

\[ \varphi(t, X) = At^{-n/2} {}_1F_1 \left( \frac{n}{2} - k, \frac{n}{2}, \frac{|X|^2}{4t} \right) + \\
Bt^{-n/2} \exp \left( -\frac{|X|^2}{4t} \right) U \left( k, \frac{n}{2}, \frac{|X|^2}{4t} \right) \] (2.1)

with \( A \) and \( B \) are complex constants and \( {}_1F_1(a, c, z) \) and \( U(a, c, z) \) are the confluent hypergeometric functions of the first and the second kind given respectively by ([5], p.263):

\[ {}_1F_1(a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k k!} z^k \quad c \neq 0, -1, -2, \ldots \] (2.2)

\[ U(a, c, z) = \frac{\pi \Gamma(c)}{\sin \pi c} \left[ {}_1F_1(a; c; z) - z^{1-c} {}_1F_1(a+1-c; 2-c; z) \right] \] (2.3)

where as usual \((a)_n\) is the Pochhammer symbol defined by

\[ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \] (2.4)

and \( \Gamma \) is the classical Euler function.

**Proof** Using the geodesic polar coordinates centred at \( X, Y = X+r\omega, r > 0; \omega \in S^{n-1} \) with \( S^{n-1} \) is the sphere of dimension \( n-1 \), and setting \( y = r^2 \) in the generalized singular heat equation in (1.1) we obtain

\[ \left[ 4y(\partial^2/\partial y^2) + 2n(\partial/\partial y) \right] \Psi(t, y) = \left[ (\partial/\partial t) + (k/t) \right] \Psi(t, y) \] (2.5)

by the change of function and the change of the variable below.

\[ \Psi(t, y) = t^{-n/2} \Phi(t, y); z = -y/4t \] (2.6)

the equation (2.5) is transformed into the following confluent hypergeometric equation

\[ z(d^2/dz^2) + ((n/2) - z)(d/dz) \Phi(z) - ((n/2) - k) \Phi(z) = 0 \] (2.7)

with parameters \( a = n/2 - k; c = n/2, ([5] p.268) \). An appropriate independent solutions of this equation are: ([5] p.270) \( {}_1F_1(a, c, z) \) and \( \exp(z)U(c-a, c, -z) \). From the formulas (2.5), (2.6) and (2.7) we conclude that the function \( \varphi \) in (2.1) is the general solution of the generalized singular heat equation in (1.1).

**Theorem 2.2** For \( n \geq 2 \) and \( k \neq 0, -1, -2, \ldots \), the Cauchy problem for the singular generalized heat equation (1.1) has the unique solution given by

\[ u(t, X) = \int_{R^n} H^k_n(t, X, Y) f(Y) dm(Y) \] (2.8)
where
\[ H_n^k(t, X, Y) = \Gamma(k)(4\pi t)^{-n/2} \exp \left( -\frac{|X - Y|^2}{4t} \right) U \left( k, \frac{n}{2}, \frac{|X - Y|^2}{4t} \right) \] (2.9)

and \( U(a, c, z) \) is the confluent hypergeometric function of the second kind given in (2.3).  

**Proof.** In view of the proposition 2.1, to finish the proof of the theorem it remains to show the limit condition in (1.1), for this we recall the asymptotic behavior of the degenerate confluent hypergeometric function \( U(a, c, z) \), ([5] p.288 – 289), for \( z \to +\infty \)

\[
U(a, c, z) = z^{-a} + O(|z|^{-a-1})
\] (2.10)

and for \( z \to 0 \)

\[
U(a, c, z) = (1/\Gamma(a)) [\log z + \psi(a) - 2\gamma] + O(|z \log z|), c = 1
\] (2.11)

\[
U(a, c, z) = (\Gamma(c - 1)/\Gamma(a)) z^{1-c} + O(1), 1 < \Re c < 2
\] (2.12)

\[
U(a, c, z) = (\Gamma(c - 1)/\Gamma(a)) z^{1-c} + O(|z \log z|), c = 2
\] (2.13)

\[
U(a, c, z) = (\Gamma(c - 1)/\Gamma(a)) z^{1-c} + O(|z|^{\Re c - 2}), \Re c \geq 2, c \neq 2
\] (2.14)

Using the geodesic polar coordinates centred at \( X \), and by setting \( y = r^2; z = y/4t \) in (1.1) we get

\[
u(t, X) = (\Gamma(k)/2\pi^{n/2}) \int_0^\infty \exp(-z) U(k, n/2, z) z^{(n/2) - 1} f_X^\#(\sqrt{4tz}) dz
\] (2.15)

with

\[
f_X^\#(r) = \int_{S^{n-1}} f(X + r\omega) d\omega
\] (2.16)

Taking the limit in (2.15) in view of the formulas (2.10) – (2.14) we can reverse the limit and the integral and we obtain

\[
\lim_{t \to 0} u(t, X) = c_n f_X^\#(0) \int_0^\infty \exp(-z) U(k, n/2, z) z^{(n/2) - 1} dz
\] (2.17)

and by the formula ([1] p.266):

\[
\int_0^\infty \exp(-z) U(a, c, z) z^{c-1} dz = -\exp(-z) z^c U(a, c + 1, z)
\] (2.18)

\[
\lim_{t \to 0} u(t, X) = \left[ -\frac{\Gamma(k)}{\pi^{n/2}} f_X^\#(0) \exp(z) U(k, n/2, z) z^{(n/2)} \right]_0^\infty
\] (2.19)

using again the formulas (2.10) and (2.14) we have

\[
\lim_{t \to 0} w(t, X) = \frac{\Gamma(k)}{\pi^{n/2}} \frac{2\pi^{n/2}}{\Gamma(n/2)} \frac{\Gamma(n/2)}{\Gamma(k)} f_X^\#(0) = f(X)
\] (2.20)

The unequeness is clear from the properties of the confluent hypergeometric equation ([5] p.268-270).
3 The generalized singular wave equation on $\mathbb{R}^n$

**Theorem 3.1** The generalized singular wave equation in (1.2) has the following general solution

$$w(t, X, Y) = A t^{1-n} F_1 \left( \frac{n-k}{2}, \frac{n-1+k}{2}, \frac{n+1}{2}, 1 - \frac{|X-Y|^2}{t^2} \right) +$$

$$B \left( t^2 - |X-Y|^2 \right)^{(1-n)/2} \times F_1 \left( \frac{1-k}{2}, \frac{k}{2}, \frac{3-n}{2}, 1 - \frac{|X-Y|^2}{t^2} \right)$$ (3.1)

with $A$ and $B$ are complex constants and $F_1(a,b,c,z)$ is the Gauss hypergeometric function given by

$$F(a, b, c; z) = \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n n!} z^n, \quad |z| < 1$$ (3.2)

**Proof** Using the geodesic polar coordinates centred at $X$, $Y = X + r\omega$, $r > 0; \omega \in S^{n-1}$, and setting $y = r^2$ and $x = t^2$ in the generalized singular wave equation in (1.2) we obtain

$$[4y(\partial^2/\partial y^2) + 2n(\partial/\partial y)] \Psi(x, y) =$$

$$\left[ 4x(\partial^2/\partial x^2) + 2(\partial/\partial x) + (k(1-k)/x) \right] \Psi(x, y)$$ (3.3)

setting

$$\Psi(x, y) = x^{-(n-1)/2} \Phi(x, y); \quad z = y/x$$ (3.4)

we obtain the following Gauss hypergeometric equation

$$z(1-z) \frac{d^2}{dz^2} \Phi(z) + \left[ n/2 - (n+1/2)z \right] \frac{d}{dz} \Phi(z) - (n-k)(n-1+k)/4\Phi(z) = 0$$ (3.5)

with parameters: $a = (n-k)/2; b = (n-1+k)/2; c = n/2$.

The hypergeometric equation (3.5) has the following system of solutions ([5], p.42–43)

$$\Phi_1(z) = F((n-k)/2, (n-1+k)/2; (n+1)/2, 1-z)$$ (3.6)

and

$$\Phi_2(z) = (1-z)^{(1-n)/2} F((1-k)/2, k/2; (3-n)/2, 1-z)$$ (3.7)

hence the following functions satisfy the generalized singular wave equation in (1.2)

$$\varphi^k_1(t, X, Y) = t^{1-n} \times$$

$$F \left( \frac{(n-k)/2, n-1+k)/2; \frac{n+1}{2}, 1 - \frac{|X-Y|^2}{t^2} \right)$$ (3.8)

$$\varphi^k_2(t, X, Y) = (t^2 - |X-Y|^2)^{(1-n)/2} \times$$

$$F \left( \frac{(1-k)/2, k/2; (3-n)/2, 1 - \frac{|X-Y|^2}{t^2} \right)$$ (3.9)
and the proof of the theorem 3.1 is finished.
In the remainder of this section we present several lemmas.

**Lemma 3.2** For $X, Y \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$ set

$$W_2(t, X, Y) = c_2 \left( t^2 - |X - Y|^2 \right)^{-1/2} \binom{1 - k/2}{k/2} \binom{1/2}{1/2} 1 - \frac{|X - Y|^2}{t^2} \right)$$

(3.10)

and

$$c_2 = \frac{\Gamma(1 + k/2)\Gamma(3 - k/2)}{\pi^{3/2}}$$

(3.11)

for $n$ even $n \geq 4$

$$W_n(t, X, Y) = c_n \left( t^2 - |X - Y|^2 \right)^{(1-n)/2} \binom{1 - k/2}{k/2} \binom{3 - n}{2} \binom{1}{2} 1 - \frac{|X - Y|^2}{t^2} \right)$$

(3.12)

and

$$c_n = \frac{2^{n/2-1}(n-3)!!\Gamma(n/2)}{((n-2)/2)!\pi^{(n-1)/2}} c_2$$

(3.13)

then for

$$A_x = (a|X - Y|^2)^{-1} x^{1-a} \frac{\partial}{\partial x} x^a = (a|X - Y|^2)^{-1} \left( x \frac{\partial}{\partial x} + a \right)$$

(3.14)

the following formulas hold

i) $$A_{4/2}^{(n-3)/2} W_n^k(t, X, Y) = W_{n+2}^k(t, X, Y)$$

(3.15)

ii) $$W_n^k(t, X, Y) = c_n A_{4/2}^{n-3} A_{4/2}^{n-5} ... A_{4/2}^{1/2} W_2^k(t, X, Y)$$

(3.16)

iii) For $g \in C_0^\infty(\mathbb{R}^n)$ we have

$$A_x A_x^{n-3} A_x^{n-5} ... A_x^{1/2} \left[ x^{1/2} g(\sqrt{xyz}) \right] =$$

$$\frac{y}{2} (n-2)/2 \binom{n-2}{n-3}!! x^{1/2} g(\sqrt{xyz}) + x \sum_{i=1}^{(n-2)/2} b_i x^{(i-1)/2} z^{i/2} g^{(i)}(\sqrt{xyz})$$

(3.17)

with $b_i ; i = 1, 2, ..., (n-2)/2$ are real constants.

**Proof**: To show i) we use the formula ([5] p.41)

$$\frac{d}{dz} z^{c-1} F(a, b; c, z) = (c - 1) z^{c-2} F(a, b; c - 1, z)$$

(3.18)

ii) comes from i).

iii) we can demonstrate iii) by induction over even $n \geq 4$.

**Lemma 3.3** For $z \rightarrow 0$ we have:

i) $F((1 + k)/2, 1 - k/2; (n + 1)/2, 1 - z) =

$$\frac{\Gamma((n+1)/2)\Gamma((2-n)/2)}{\Gamma((1+k)/2)\Gamma((2-k)/2)} z^{(n-2)/2} + O(1)$$

(3.19)
ii) \( F((1 - k)/2, k/2; 1/2, 1 - z) = \)
\[
\frac{\Gamma(1/2)}{\Gamma(k/2)\Gamma((1 - k)/2)} [1 + o(\log z)]
\] (3.20)

\( k \neq 0, -2, -4, \ldots \)

**Proof** i) is easily seen from the formula ([5], p.47)
\[
F(a, b, c, z) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} F(a, b + c + 1, 1 - z) +
(1 - z)^{c - a - b} \frac{\Gamma(c)(a + b - c)}{\Gamma(b)\Gamma(a)} F(c - a, c - b, c - a - b + 1, 1 - z) \] (3.21)

ii) is a consequence of the formula ([5] p.44)
\[
F(a, b; a + b, z) = (\Gamma(a + b)/\Gamma(a)\Gamma(b)) \sum_{n=0}^{\infty} \times
((a)_n(b)_n/(n!)^2)[2\psi(n + 1) - \psi(a + n) - \psi(b + n) - \log(1 - z)](1 - z)^n \] (3.22)

\( \arg(1 - z) < \pi; |1 - z| < 1 \)

### 4 Cauchy problem for the singular wave equation on \( \mathbb{R}^n \), \( n \) odd

**Theorem 4.1** Suppose \( n \) is odd and

\[
W_n^k(t, X, Y) = C_n t^{1 - n} F_1 \left( \frac{n - k}{2}, \frac{n - 1 + k}{2}, \frac{n + 1}{2}, 1 - \frac{|X - Y|^2}{t^2} \right) \] (4.1)

with
\[
C_n = \frac{\Gamma(n/2)k^{k-1}}{2^{n+2}(n-1)!} \times \frac{\Gamma((n - k)/2)\Gamma((n - 1 - k)/2)}{\Gamma((n - k)/2)\Gamma((n - 1 - k)/2) - \Gamma(n/2)\Gamma((n - 1)/2)} \] (4.2)

If \( g \in C_0^\infty(\mathbb{R}^n) \), the function
\[
w(t, X) = \int_{|X - Y| < t} W_n^k(t, X, Y)g(Y) dY \] (4.3)

solves the Cauchy problem for the generalized singular wave equation (1.2)

**Proof** In view of the theorem 3.1, we see that the kernel in (4.1) satisfies the generalized singular wave equation in (1.2) and hence the function \( w(t, X) \) in (4.3) satisfies the same equation. To complete the proof of the theorem 4.1 it remains to show the limit conditions. Using the geodesic polar coordinates and setting \( y = r^2; x = t^2, z = y/x \) in (4.3) we have
\[
w(t, X) = C_n \int_{0}^{1} \frac{1}{2} y X(t, \sqrt{z}) z^{(n-2)/2} \times
x^{1/2} F((n - k)/2, (n - 1 + k)/2; (n + 1)/2, 1 - z) dz \] (4.4)
with $g_X^X(r)$ is as in (2.16). By the formula ([5] p.47).

$$F(a,b;c,z) = (1-z)^cF(c-a,c-b;c,z)$$  \tag{4.5}

we can write

$$w(t,X) = C_n \frac{t}{2} \int_0^1 F((1+k)/2,1-k/2;(n+1)/2,1-z)g_X^X(t\sqrt{z})dz$$  \tag{4.6}

hence by taking the limit in (4.6) using the formula (3.19) we can reverse the integral and the limit to obtain

$$\lim_{t\to 0} w(t,X) = 0.$$  \tag{4.7}

For the second condition we derive the expression in (4.6), using again (3.19) we can derive under the integral sign to obtain

$$\frac{\partial}{\partial t}w(t,X) =$$

$$C_n \frac{1}{2} \int_0^1 g_X^X(t\sqrt{z})F((1+k)/2,1-k/2;(n+1)/2,1-z)dz + tO(1)$$  \tag{4.8}

Hence by taking the limit of (4.8) in view of (3.19) we can reverse the limit and the integral to write

$$\lim_{t\to 0} \frac{\partial}{\partial t}w(t,X) = C_n \frac{1}{2} g_X^X(0) \int_0^1 F((1+k)/2,1-k/2;(n+1)/2,1-z)dz$$  \tag{4.9}

$$\lim_{t\to 0} \frac{\partial}{\partial t}w(t,X) = C_n \frac{1}{2} g_X^X(0) \int_0^1 F((1+k)/2,1-k/2;(n+1)/2,1-z)dz$$  \tag{4.10}

In view of the formula ([5] p.41)

$$\frac{d}{dz}F(a,b;c,z) = \frac{ab}{c}F(a+1,b+1;c+1,z)$$  \tag{4.11}

we obtain

$$\lim_{t\to 0} \frac{\partial}{\partial t}w(t,X) = -C_n (1/2) g_X^X(0) \frac{n-1}{x(k-1)} \times$$

$$\left[F((k-1)/2,-k/2;(n-1)/2,1)\right]_0^1$$  \tag{4.12}

that is

$$\lim_{t\to 0} \frac{\partial}{\partial t}w(t,X) = -C_n (1/2) g_X^X(0) \frac{n-1}{x(k-1)} \times$$

$$\left[F((k-1)/2,-k/2;(n-1)/2,1)-1\right]$$  \tag{4.13}

And by the formula (4.2) and ([5] p.40):

$$F(a,b;c,1) = (\Gamma(c)\Gamma(c-a-b)/\Gamma(c-a)\Gamma(c-b)) ; \Re(a+b-c) < 0$$  \tag{4.14}

we obtain

$$\lim_{t\to 0} \frac{\partial}{\partial t}w(t,X) = g(X).$$
5 Cauchy problem for the singular wave equation on the Euclidean plane \( IR^2 \)

**Theorem 5.1** Suppose \( n = 2 \) and

\[
W^k_2(t, X, Y) = c_2 \left( t^2 - |X - Y|^2 \right)^{-\frac{1}{2}} F \left( \frac{k}{2}, \frac{1 - k}{2}, \frac{1}{2}; 1 - \frac{|X - Y|^2}{t^2} \right) \tag{5.1}
\]

with

\[
c_2 = \frac{\Gamma(1 + k/2)\Gamma(3 - k/2)}{\pi^{3/2}} \tag{5.2}
\]

If \( g \in C_0^\infty(R^2) \), the function

\[
w(t, X) = \int_{|X - Y| < t} W^k_2(t, X, Y) g(Y) dY \tag{5.3}
\]

solves the Cauchy problem (1.2).

**Proof** From the theorem 3.1 we see that the functions \( w(t, X) \) in (5.3) satisfies the generalized singular wave equation in (1.2).

Now to show the limit conditions, by the geodesic polar coordinates and the change of variables \( y = r^2; x = t^2, z = y/x \) in (5.3), we have for \( n = 2 \)

\[
w(t, X) = c_2 (t/2) \int_0^1 (1 - z)^{-1/2} F((1 - k)/2, k/2, 1/2, 1 - z) g^#_X(t\sqrt{z}) dz \tag{5.4}
\]

By taking the limit in (5.4) we can use the formula (3.20) to reverse the limit and the integral and to obtain

\[
\lim_{t \to 0} w(t, X) = 0 \tag{5.5}
\]

Now to show the second condition we derive the expression (5.4) and in view of the formula (3.20) we can derive under the integral sign to obtain

\[
\frac{\partial}{\partial t} w(t, X) = c_2 \frac{1}{2} \int_0^1 (1 - z)^{-1/2} \times
\]

\[
F((1 - k)/2, k/2, 1/2, 1 - z) g^#_X(t\sqrt{z}) dz + tO(1))g^#_X(t\sqrt{z}) dz \tag{5.6}
\]

Using again the formula (3.20) we can reverse the limit and the integral and we have

\[
\lim_{t \to 0} \frac{\partial}{\partial t} w(t, X) = c_2 \frac{1}{2} g^#_X(0) \int_0^1 (1 - z)^{-1/2} F((1 - k)/2, k/2, 1/2, 1 - z) dz \tag{5.7}
\]

\[
\lim_{t \to 0} \frac{\partial}{\partial t} w(t, X) = c_2 \frac{1}{2} g^#_X(0) \int_0^1 z^{-1/2} F((1 - k)/2, k/2, 1/2, z) dz \tag{5.8}
\]

we have by the formula (3.18)

\[
\lim_{t \to 0} \frac{\partial}{\partial t} w(t, X) = c_2 \frac{1}{2} g^#_X(0) 2z^{1/2} [F((1 - k)/2, k/2, 3/2, z)]_0^1 \tag{5.9}
\]

and from (4.14)

\[
\lim_{t \to 0} \frac{\partial}{\partial t} w(t, X) = c_2 \pi^{3/2}/\Gamma((3 - k)/2))\Gamma((1 + k/2)g(X) = g(X) \tag{5.10}
\]
6 Cauchy problem for the singular wave equation on \( \mathbb{R}^n \), \( n \geq 4 \) even

**Theorem 6.1** Suppose \( n \) is even and \( n \geq 4 \), let \( W^k_w(t, X, Y) \) is as in theorem 5.1 and \( A^a_x \) is as in (3.11) and

\[
c_n = \frac{(n - 3)!! \Gamma(n/2)}{2^{1-n/2}((n - 2)/2)!\pi^{(n-1)/2}}
\]  

(6.1)

If \( g \in C^\infty_0(\mathbb{R}^n) \), the function

\[
w(t, X) = c_n A^{a-3}_t A^{a-5}_t ... A^{1}_t \int_{|X-Y|<t} W^k_w(t, X, Y)g(Y)dY
\]

(6.2)

solves the Cauchy problem (1.2).

**Proof** In view of the theorem 3.1, we see that the functions \( w(t, X) \) in (6.2) satisfies the generalized singular wave equation in (1.2).

To finish the proof of the theorem we show the limit condition in the even case \( n \geq 4 \): using the geodesic polar coordinates and setting \( y = r^2; x = t^2; z = y/x \) in (6.2); we have:

for \( n \) even \( n \geq 4 \):

\[
w(t, X) = c_n c_2 B_t \left[ (t/2) \int_0^1 (1 - z)^{-1/2} F((1 - k)/2, k/2; 1/2, 1 - z)g^n_X(t\sqrt{z})dz \right]
\]

(6.3)

with

\[
B_t = A^{a-3}_t A^{a-5}_t ... A^{1}_t
\]

Using the formula iii) of lemma 3.2 we have

\[
w(t, X) = C_n B_t \int_0^1 (1 - z)^{-1/2} F((1 - k)/2, k/2; 1/2, 1 - z)g^n_X(t\sqrt{z})dz +
\]

\[
t^{2}\sum_{i=0}^{(n-2)/2} b_i t^{i-1} \int_0^1 (1 - z)^{-1/2} F((1 - k)/2, k/2; 1/2, 1 - z)z^{i/2-\frac{1}{2}}g^n_i_X(t\sqrt{z})dz
\]

(6.4)

with

\[
C_n = c_n c_2 2^{-(n-2)/2} \frac{(n - 2)/2!}{(n - 3)!!}
\]

Taking the limit of the expression (6.3) and using the formula (3.20) we can reverse the integral and the limit to obtain

\[
\lim_{t \to 0} w(t, X) = 0
\]

(6.5)

For the second condition we derive the expression (6.4) and by (3.17) we can derive under the integral sign

\[
\frac{\partial}{\partial t} w(t, X) = c_n c_2 2^{-(n-2)/2} \frac{(n-2)/2!}{(n-3)!!} \frac{1}{2} \int_0^1 (1 - z)^{-1/2} \times
\]

\[
F((1 - k)/2, k/2; 1/2, 1 - z)g_X^n(t\sqrt{z})dz + tO(1))g_X^n(t\sqrt{z})dz
\]

(6.6)
Taking now the limit in (6.6) and using again (3.20) we can reverse the limit and the integral sign
\[
\lim_{t \to 0} \frac{\partial}{\partial t} w(t, X) = c_n c_2 \frac{2}{(n-3)!} \frac{1}{2!} g_X(0) \times \\
\int_0^1 (1 - z)^{-1/2} F((1 - k) / 2, k / 2; 1 / 2, 1 - z)dz
\]
(6.7)

\[
\lim_{t \to 0} \frac{\partial}{\partial t} w(t, X) = \\
= c_n 2^{-(n-2)/2} \frac{(n-2)/2!}{(n-3)!} \frac{1}{2} g_X(0) \int_0^1 z^{-1/2} F((1 - k) / 2, k / 2; 1 / 2, z)dz
\]
(6.8)

by the formula (3.18) we have
\[
\lim_{t \to 0} \frac{\partial}{\partial t} w(t, X) = c_n c_2 2^{-(n-2)/2} \frac{(n-2)/2!}{(n-3)!} \frac{1}{2} \times \\
\tilde{g}_X(0) 2z^{1/2} (F((1 - k) / 2, k / 2; 3/2, z))^1_0
\]
(6.9)

using the formula (4.14) we have
\[
\lim_{t \to 0} \frac{\partial}{\partial t} w(t, X) = c_n c_2 2^{-(n-2)/2} \frac{(n-2)/2!}{(n-3)!} \times \\
\frac{\pi^{(n+2)/2}}{\Gamma((3-k)/2) \Gamma((1+k/2) \Gamma(n/2))} g(X) = g(X)
\]
(6.10)

7 Applications

Remark 7.1: we have
\[
\lim_{k \to 0} [\Gamma(k)]^{-1} H^k_n(t, X, Y) = K_n(t, X, Y)
\]
(7.1)

where
\[
K_n(t, X, Y) = (4\pi t)^{-n/2} \exp \left(-|X - Y|^2 / 4t\right)
\]
(7.2)

is the classical heat kernel on \( \mathbb{R}^n \).

Corollary 7.2 The generalized Cauchy problem for the heat equation on \( \mathbb{R}^n \):
\[
\left\{ \begin{array}{l}
\left( \frac{\partial}{\partial t} \right) v(t, X) = 0 \\
\lim_{k \to 0} v(t, X) = v_0(X) ; (t, X) \in \mathbb{R}_+^* \times \mathbb{R}^n
\end{array} \right.
\]
(7.3)

has the unique solution given by
\[
v(t, X) = \int_{\mathbb{R}^n} K^k_n(t, X, Y) f(Y) dm(Y)
\]
(7.4)

where
\[
K^k_n(t, X, Y) = \Gamma(k) t^k (4\pi t)^{-n/2} \exp \left(-|X - Y|^2 / 4t\right) U \left(k, \frac{n}{2}, \frac{|X - Y|^2}{4t}\right)
\]
(7.5)
Proof The proof of this corollary is simple and is omitted.

Corollary 7.3 We have

$$\lim_{k \to 0} W_k^n(t, X, Y) = W_n(t, X, Y)$$

with

$$W_n(t, X, Y) = (2\pi)^{-n/2} \left( t^2 - |X - Y|^2 \right)^{(1-n)/2}$$

is the classical wave kernel on $\mathbb{R}^n$ [4]

Proof The proof of this corollary is simple and is left to the reader.

References

[1] Blinder S. M. On green functions, Propagators, and Strumians for non relativistic Coulomb problem, International Journal of Chemistry: Quantum Chemistry Symposium 18, 293-307 (1984)

[2] Case, J. K.M. Singular potential, Phys. Rev. 80 797 – 806(1950).

[3] Cheeger, J.,Taylor, M. On the diffraction of waves by canonical singularites I, Comm. Pure Appl. Math. 35(3) : 275 – 331, 1982.

[4] Folland G. B., Introduction to partial differential equations, Princeton university press, Princeton N. J. 1976.

[5] Magnus F., Oberhettinger and Soni R. P., Formulas and Theorems for special functions of Mathematical Physics, Third enlarged edition, Springer-Verlag Berlin Heidelberg New York (1966).

[6] Planchon F., Stalker J. and Shadi Tahvildar-Zadeh A., Dispersive estimate for the wave equation with the inverse square potential, Discrete contin. Dynam. Systems, Vol. 9, No 6 2003, 1337 – 1400.

[7] Reed M. and Simon B., Methods of moderne mathematical physics vol. II, Academic press, New-York, 1979.

Al Jouf University
College of Sciences and Arts
Al-Qurayyat Saoudi Arabia.

and

Université de Nouakchott Al-asriya
Faculté des Sciences et Techniques
Unité de Recherche: Analyse, EDP et Modélisation: (AEDPM)
B.P: 5026, Nouakchott-Mauritanie
E-mail adresse:mohamedvall.ouldmoustapha230@gmail.com