Harmonics on the factored three-sphere and the Hopf map

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Laplacian eigenmodes on homogeneous Clifford–Klein factors of the three–sphere are obtained as pullbacks of harmonics on the orbifolded two–sphere using the Hopf map. A method of obtaining these polyhedral, or crystal, harmonics using binary invariants is presented which has computational advantages over those based on projection techniques, or those using invariants constructed in terms of Cartesian coordinates. In addition, modes transforming according to the irreps of the deck group are found in easy fashion using the covariants already conveniently calculated by Desmier and Sharp and by Bellon. Agreement is found with existing results.

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1. Introduction.

The Hopf map gives the structure of the three–sphere as a fibering, with a two sphere as base and a circle as fibre. The projection $S^3 \to S^2$ can be expressed in terms of the Cartesian coordinates on the corresponding embedding $\mathbb{R}^4$ and $\mathbb{R}^3$ by the non–linear mapping (Hopf [1]),

$$
\begin{align*}
    y_1^\pm &= 2(x^2 x^0 \mp x^1 x^3) \\
    y_2^\pm &= 2(x^2 x^3 \pm x^1 x^0) \\
    y_3^\pm &= (x^3)^2 + (x^0)^2 - (x^1)^2 - (x^2)^2.
\end{align*}
$$

The pullback of a harmonic polynomial in a set of $y$s is a harmonic polynomial in the $x$s. In this paper I wish to expand on this fact, which is useful because it relates eigenfunctions on $S^2$ and $S^3$. These two facts are really the same.

Kibler et al, [2], have used the Hopf mapping to relate modes. Hage Hassan and Kibler, [3], have elaborated on this aspect using the Fock–Bargmann–Schwinger approach to angular momentum theory. Some of the material I give in the next section will be found in these works, and elsewhere.

A fibre bundle treatment is given by Boiteaux, [4], (see also Gilkey et al, [5]) but my discussion will be more simple minded.

In later sections I extend the discussion to include homogeneous factors of $S^3$ and will give a practical means of obtaining the modes. In this regard, the Hopf approach has also been expounded by Weeks, [6], and Lachi`ez e-Rey and Weeks, [7], in an astrophysical context and I refer to these works for references to some other mode calculations. I should also mention the interesting work of Bellon, [8], [9].

An important step on the road to modes on the factored three–sphere is the construction of modes on the orbifolded two–sphere. These polyhedral harmonics, also known as crystal, or lattice, harmonics, have been investigated for many years; early papers being by Elert, [10], and Bethe, [11]. In particular, more recently, icosahedral harmonics have found use in the study of viruses, fullerenes and quasi–crystals. The method presented here gives a relatively simple way of finding them and information in [8] allows the tensor icosahedral harmonics to be found too.
2. Harmonic polynomials and eigenmodes

Working with polynomials has its attractions, but the pullback result is most easily appreciated in angular coordinates. The Euler angle expression for the $x^i$, going back at least to Klein and Sommerfeld, is (I repeat some standard relations),\(^2\)

\[
\begin{align*}
x^1 &= -R \sin \theta/2 \sin(\psi - \phi)/2 \\
x^2 &= R \sin \theta/2 \cos(\psi - \phi)/2 \\
x^3 &= R \cos \theta/2 \sin(\psi + \phi)/2 \\
x^0 &= R \cos \theta/2 \cos(\psi + \phi)/2 ,
\end{align*}
\]

whence, from (1), the $\psi$–independent, and $\phi$–independent, combinations,

\[
\begin{align*}
y_+^1 &= r \sin \theta \cos \phi , & y_-^1 &= r \sin \theta \cos \psi \\
y_+^2 &= r \sin \theta \sin \phi , & y_-^2 &= r \sin \theta \sin \psi \\
y_+^3 &= r \cos \theta , & y_-^3 &= r \cos \theta
\end{align*}
\]

with $r = R^2$, implying the map $S^3 \to S^2$. In fact, $S^2$ lifts to a quartic surface in $\mathbb{R}^4$ which is the product of two (identical) three- spheres.

I have chosen the factors so that the SU(2) element, $U$, agrees in form with that in [12], eqn.(2.21), \textit{viz},

\[
U = \mathcal{D}^{1/2} = x^0 1 - i \mathbf{x} \cdot \mathbf{\sigma}
\]

in terms of Pauli matrices.

Each set of $y$s in (3) produces a system of spherical polar coordinates on a corresponding $\mathbb{R}^3$. I will use just the $y_+$, and denote these by $y^i$. A complete set of Laplacian eigenfunctions on $S^2$ is provided by the spherical harmonics, $C_{l}^{m} (\theta, \phi)$. The relation between these and the SU(2) rep matrices, $\mathcal{D}_{m}^{j} (\theta, \phi, \psi)$, is (\textit{e.g.} Brink and Satchler, [13], Vilenkin, [14], Talman, [15]),

\[
C_{l}^{m} (\theta, \phi) = (-1)^{m} \mathcal{D}_{-m}^{l} (\theta, \phi, \psi) , \quad l \in \mathbb{Z} ,
\]

the right–hand side being independent of the fibre angle, $\psi$. Since the $\mathcal{D}$s are Laplacian eigenfunctions on $S^3 \sim SU(2)$ the result follows. It is standard (\textit{e.g.} Hill,

\(^2\) My Euler angles correspond to the ‘z-y-z’ rotation convention, adopted by [12], [13]. Note that Vilenkin, [14], uses the other popular choice, z-x-z.
[16, [12]), that the solid quantity, $R^{2l} D^l$, is a harmonic polynomial in the $x^i$ of degree $2l$ and the corresponding $S^3$ eigenvalues equal $4l(l + 1)$ agreeing with the $S^2$ eigenvalues from (5) taking into account the scaling relation between the relevant Laplacians (see below).

This can be made more explicit by introducing the relevant differential operators such as the right–generators of the SO(4) action on $S^3$,

$$Y_1 = \frac{i}{\sqrt{2}} e^{i\psi} \left( i \frac{\partial}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} - \cot \theta \frac{\partial}{\partial \psi} \right),$$

$$Y_{-1} = Y_1^*,$$

$$Y_0 = i \frac{\partial}{\partial \psi},$$

the first two of which allow the right index in (5) to be raised and lowered by the standard action,

$$-i Y_m D^j(g) = D^j(g) J_m,$$

where $J_m$ are the spin–j angular momentum matrices.

The Laplacian, $\Delta_3$, on the unit three–sphere is the Casimir operator,

$$\Delta_3 = 4 Y^2 = 4(Y_1 Y_{-1} + Y_{-1} Y_1 - Y_0^2)$$

$$= 4 \left( \partial^2_\theta + \csc^2 \theta \left( \partial^2_\phi + \partial^2_\psi \right) - 2 \cos \theta \csc \theta \partial_\phi \partial_\psi + \cot \theta \partial_\theta \right).$$

This corresponds to the metric,$^3$

$$ds^2 = \frac{1}{4} \left( d\theta^2 + d\phi^2 + d\psi^2 + 2 \cos \theta d\phi d\psi \right)$$

$$= \frac{1}{4} \left( d\theta^2 + \sin^2 \theta d\phi^2 + (d\psi + \cos \theta d\phi)^2 \right),$$

which occurs as the spatial part of the metric on $\mathbb{R}^4$,

$$d\tau^2 = dR^2 + R^2 ds^2,$$

giving the Dalembertian (see e.g. Hund, [18]),

$$\Box_4 = \frac{1}{R^3} \partial_R \left( R^3 \partial_R \right) + \Delta_3.$$

$^3$ The numerical factor can be checked by computing the volume of the three–sphere using the conventional ranges $0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi, -2\pi \leq \psi \leq 2\pi$ (but see [12], [17]), and the metric determinant $\sqrt{g} = \sin \theta/8$. The volume is $2\pi^2 a^3$ for radius $a$. cf [5].
Now let $\Box_4$ act on a pullback, i.e. on a $\psi$–independent function, $f_0(\theta, \phi)$, e.g. $C_1^m(\theta, \phi)$. Then from (8), and using $r = R^2$ and $Y_0 f_0 = 0$,

$$\Box_4 f_0 = \left( \frac{1}{R^3} \partial_R (R^3 \partial_R) + \frac{1}{R^2} \Delta_3 \right) f_0 = 4r \left( \frac{1}{r^2} \partial_r (r^2 \partial_r) + \frac{1}{r^2} \Delta_2 \right) f_0 = 4r \Box_3 f_0,$$

(9)

where $\Delta_2$ is the unit two–sphere Laplacian, in $\mathbb{R}^3$ with coordinates $y^\alpha$,

$$\Delta_2 = \partial^2_\theta + \cot \theta \partial_\theta + \cosec^2 \theta \partial^2_\phi.$$

Equation (9) shows that the Hopf mapping pulls harmonic functions back to harmonic functions. Most of the above is well known, in one form or another.

3. Harmonic projection

It is relevant to consider, briefly, the relation between harmonic projections in the light of the Hopf map.

A standard result is that, in an $\mathbb{R}^d$, a rational integral polynomial of degree $n$, $f_n(\mathbf{r})$, can be decomposed as

$$f_n(\mathbf{r}) = Y_n(\mathbf{r}) + \mathbf{r}^2 f_{n-2}(\mathbf{r}).$$

(10)

$Y_n(\mathbf{r})$ is a harmonic polynomial (solid spherical harmonic). For $d = 3$ this was proved by Gauss and his method easily extends to $d$–dimensions.

The polynomial $Y_n$ is the harmonic projection of $f_n$ and there is an explicit expression for it, (e.g. Hobson, [19], Vilenkin, [14]),

$$Y_n = H(f_n) = \left[ 1 - \frac{\mathbf{r}^2 \Box^2_d}{2(2n + d - 4)} + \frac{\mathbf{r}^4 \Box^4_d}{2.4(2n + d - 4)(2n + d - 6)} - \cdots \right] f_n,$$

(11)

obtained from the iteration of (10) and applications of $\Box_d$ (cf Clebsch, [20], for $d = 3$).

I apply this formula for $d = 3$ and $d = 4$. In the former case, to accord with my previous notation, I use $y^\alpha$ as Cartesian coordinates. I also write $\mathbf{y.}\mathbf{y} = r^2$ and, for $d = 4$, set $\mathbf{r}^2 = R^2$. 

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For \( d = 3 \)
\[
f_n(y) = H(f_n(y)) + r^2 f_{n-2}(y),
\]
and
\[
H(f_n) = \left[ 1 - \frac{r^2 \Box_3}{2(2n-1)} + \frac{r^4 \Box_3^2}{2.4(2n-1)(2n-3)} - \cdots \right] f_n,
\]
while for \( d = 4 \)
\[
F_n(x) = H(F_n(x)) + R^2 F_{n-2}(x),
\]
and
\[
H(F_n) = \left[ 1 - \frac{R^2 \Box_4}{2.2n} + \frac{R^4 \Box_4^2}{2.4.2n(2n-2)} - \cdots \right] F_n,
\]
In the special case of the Hopf mapping, harmonic projection commutes with lifting. If \( f_n \) lifts to \( F_{2n} \), then \( H(f_n) \) lifts to \( H(F_{2n}) \) and, if \( f_{n-2} \) lifts to \( F_{2n-4} \), one has \( F_{2n-2} = R^2 F_{2n-4} \) by comparing (12) and (14). Therefore, in this instance, (14) becomes
\[
F_n(x) = H(F_n(x)) + R^4 F_{n-4}(x),
\]
and also one sees that (13) lifts to (15). A direct proof, using the relation between Laplacians, (9), is not obvious.

The extension to higher dimensions, \( d \), of Thomson and Tait’s approach to spherical harmonics is straightforward on noting that \( 1/r^{d-2} \) is harmonic, \( r^2 \neq 0 \), with \( r^2 = (x^1)^2 + (x^2)^2 + \ldots + (x^d)^2 \). This is an ancient fact, and another expression for the harmonic projection is, \[19\],
\[
H(f_n) = (-1)^n \frac{1}{(d-2)d(d+2)\ldots(d+2n-4)} r^{2n+d-2} f_n(\nabla) \frac{1}{r^{d-2}}.
\]
In particular, in three dimensions, Maxwell’s multipole expression of a general solid harmonic is,
\[
Y_n(y, p_{(k)}) = C r^{2n+1} \prod_{k=1}^{n} (p_{(k)}, \nabla) \frac{1}{r},
\]
which depends on the \( n \) 3–vectors, \( p_{(k)} \), \( k = 1, \ldots, n \).

\[4\] To include the point \( r = 0 \) formally, it would be best to adopt a distributional, Green function approach. See e.g. Rowe, [21].
4. Spherical factors

As described in section 2, lifting the modes from $S^2$ produces only the integer
spin modes on $S^3$. A complete set would also include the half–odd integer $D_{s}$. These
can be eliminated by dividing $S^3$ by a $\mathbb{Z}_2$ antipodal action (to give the projective
three–sphere) and requiring periodicity. (Relatedly, the Hopf map is unchanged
under parity, $r \rightarrow -r$.) Alternatively, the integer spin modes could be eliminated
by choosing anti–periodicity, as allowed by the topology, $[22], [23], [24]$.

The total set of integer–spin modes, $D^l_{m'}$, is obtained, as mentioned, by acting
with the raising and lowering right operators on the pullbacks of the $S^2$ modes, (5),
in familiar angular momentum fashion. Weeks, $[6]$, see also $[7]$, describes this process
using the polynomial approach and refers to the index $m'$ as the ‘twist’. A more
conventional name is weight, coming from invariant theory, via Lie–group theory.
It is measured by the vertical operator, $Y_0$.

Since the answer is already known, there is no especial calculational merit
in finding the modes on $S^3/\mathbb{Z}_2$ in this way but it does give them an $\mathbb{R}^3$, SO(3),
egative character, arising originally from the isomorphisms $SO(4) \sim SU(2) \times
SU(2)/\mathbb{Z}_2$ and $SO(3) \sim SU(2)/\mathbb{Z}_2$.

It is a particular example of the relation between the spectral problems on
the free action $S^3/\Gamma'$ and the orbifolded $S^2/\Gamma$ where $\Gamma'$ is the double of $\Gamma$, noting
that $\Gamma' = \mathbb{Z}_2$ when $\Gamma = 1$. The eigenvalue aspects of this relation have already
been expounded and used in $[25]$ and now the geometrical underpinnings are more
apparent, $cf$ $[6], [7], [9]$.

It is only possible to find the modes in this pullback way for a general (say left)
symmetry action, $S^3/\Gamma'$, if $\Gamma'$ contains an antipodal $\mathbb{Z}_2$. This applies to even lens
spaces, and when $\Gamma' = O', T', Y'$.

The modes on $S^3/\Gamma'$ can be obtained by symmetry adaptation. I define, to
begin with, the left group average, or projection,

$$
\phi^l_{m} (g) \equiv \left[ \frac{2l + 1}{2\pi^2 a^3 |\Gamma'|} \right]^{1/2} \sum_{\gamma' \in \Gamma'} D^l_{m} (\gamma' g)
= C \sum_{\gamma' \in \Gamma'} D^l_{m} (\gamma') D^l_{n} (\theta, \phi, \psi)
= 2C \sum_{\gamma \in \Gamma'} D^l_{m} (\gamma) D^l_{n} (\theta, \phi, \psi)
= \left[ \frac{2l + 1}{2\pi^2 a^3 |\Gamma'|} \right]^{1/2} \sum_{\gamma \in \Gamma} D^l_{m} (\gamma) C^m_l (\theta, \phi).
$$

(18)
The last line is now the preliminary symmetry adaptation of the modes on the Hopf 
\((\theta, \phi)\)-sphere base.\(^5\) I do not pursue this approach to compute these modes but will 
present a better method in the next section.

In the derivation of the relation (18) between the two projections, I have used 
the fact that \(\Gamma = \Gamma'/\mathbb{Z}_2\) where \(\mathbb{Z}_2\) is an antipodal action, the non–trivial element of 
which corresponds to a rotation through \(2\pi\) and is the ‘central’ element, introduced 
by Bethe to generate the binary group from the pure rotation one. For \(l\) integral, 
it is equivalent to the identity. Hence the factor of two.

As before, the complete set of (preliminary) modes on \(S^3/\Gamma'\) is obtained by 
acting on the right with the raising and lowering operators. For a particular eigen-
value (depending on only the label \(l\)), the degeneracy is the product of the right 
degeneracy, which is the range of the right index, \textit{i.e.} \((2l + 1)\), and the left degener-
acy, \(d(l; \Gamma)\), evaluated by cutting down the overcomplete set of the \(\phi^l m^0\) modes to 
the minimal number, usually by diagonalisation. The results were given in [23] and 
derived using the results of Polya and Meyer, [26], who employed Molien’s theorem 
rather than constructing the group average directly.

Most easily, the (left) degeneracies can be obtained from those on cyclic groups 
by making use of the cyclic decomposition of a spectral quantity \(S\),

\[
S(\Gamma) = \frac{1}{|\Gamma|} \left( \sum_q q n_q S(\mathbb{Z}_q) - (\sum_q n_q - 1)S(\mathbb{Z}_1) \right),
\]

(19)

where the group \(\Gamma\) has \(n_q\) axes of order \(q\). For those groups satisfying the orbit–
stabiliser relation, \(|\Gamma| = 2qn_q\),

\[
S(\Gamma) = \frac{1}{2} \left( \sum_q S(\mathbb{Z}_q) - S(\mathbb{Z}_1) \right).
\]

(20)

Relation (19) is given by Meyer, [26], and proved in [27]. It is the simplest 
way of deriving the degeneracies, and also their generating functions, which satisfy 
(20) and are given in [26] (see also Laporte, [28]). For example, the cyclic \((\mathbb{Z}_q)\) and 
octahedral \((O)\) groups, the generating functions, defined by

\[
g(\sigma; \Gamma) = \sum_{l=0}^{\infty} d(l; \Gamma) \sigma^l,
\]

\(^5\) I make \(\Gamma'\) act on the left so that the coordinates on the sphere are the conventional \(\theta\) and \(\phi\). 
The coordinate ranges and boundary identifications are detailed by Jonker and De Vries, [17],
and exhibit the twisted product structure of \(S^3\).
equal
\[ g(\sigma; Z_q) = \frac{1}{1 - \sigma^q} \frac{1 + \sigma^q}{1 - \sigma^q}, \quad g(\sigma; O) = \frac{1 + \sigma^9}{(1 - \sigma^4)(1 - \sigma^6)}. \] (21)

The construction of the modes themselves is not so straightforward.

5. Scalar polyhedral modes

The easiest part of determining the independent set of symmetry adapted modes on \( S^3/\Gamma' \) is the right raising and lowering, which is routine. The difficulty lies in the construction of the symmetry adapted spherical harmonics on \( S^2/\Gamma \), a topic of great interest to physicists, chemists and biologists over a long period. A lot of expository and technical material exists which it is impossible to survey. There are various approaches. In some, the group average is performed element by element. In others, it is bypassed, or camouflaged.

I will spend a little time developing a method of calculating the symmetry adapted modes after which the pullback modes on \( S^3/\Gamma' \) can be considered to be known by the process outlined earlier.

Perhaps the method that is most readily automated is one described first by Hodgkinson, [29], and based on the expression (13). The same technique was developed later, independently by e.g. [30], [31].

Hodgkinson presented a complete process, if somewhat sketchily,\(^6\) and complained at the ‘very heavy labour’ involved in the harmonic projection last step, (13). Nowadays this can be alleviated using symbolic manipulation, and the algorithm has been detailed, independently, with this in mind by Ronveaux and Saint–Aubin, [32].

The method depends on the existence of an invariant polynomial integrity basis whose evaluation is classic, going back, in this setting, to Klein and with much subsequent work.

The process yields a basis for invariant harmonic polynomials in the Cartesian coordinates of an embedding \( \mathbb{R}^3 \). In the present notation, these coordinates are the \( y^1, y^2, y^3 \) of (3). These polynomials can then be converted to polynomials on \( S^3/\Gamma' \) via the relation (1) and the right raising and lowering operators applied to complete the computation of a full basis, if it is desired to be so explicit. However, the results would not be illuminating. For large angular momentum, Cartesian coordinates

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\(^6\) Hodgkinson considers only those modes even under reflections in the polyhedral symmetry planes. He also does not give any specific computations.
become unwieldy. Better for the purpose if the harmonic basis on $S^2/\Gamma$ could be expanded in (standard) spherical harmonics, $C_l^m$ (for fixed $l$), for then, using the relation (5) with the irrep matrices, the right raising and lowering is immediate and consists simply of replacing the right 0 index by $m'$, running from $l$ to $-l$. This is because left and right are completely independent for homogeneous factorings.

I will now elaborate on the computational method outlined by Hodgkinson based on the form (17) rather than on (13).

The tesseral harmonics can be defined in the Thomson and Tait, [33], way, (e.g. Maxwell, [34], Hobson, [35], Hönl, [36]),

$$Y_{m}^l \equiv P_{m}^l \cos \theta e^{im\phi} = N_{lm} r^{l+1} (-1)^m \frac{\partial^l}{\partial y_1^m} \frac{1}{r} \frac{1}{y_0^{l-m}},$$

$$Y_{-m}^l \equiv P_{l}^m \cos \theta e^{-im\phi} = N_{lm} r^{l+1} \frac{\partial^l}{\partial y_1^m} \frac{1}{r} \frac{1}{y_0^{l-m}}$$

for $l \geq m \geq 0$, $r^2 = y.y$, and

$$N_{lm} = (-i)^l (\frac{2\sqrt{2}}{(l-m)!}).$$

Instead of the Cartesian components, I use spherical ones, defined by

$$y_1 \equiv -\frac{y^1 - iy^2}{i\sqrt{2}}, \quad y_{-1} \equiv \frac{y^1 + iy^2}{i\sqrt{2}}, \quad y_0 = -iy^3,$$

for the ‘standard’ components, and

$$y^1 = y_{-1}, \quad y^{-1} = y_1, \quad y^0 = -y_0,$$

for the contrastandard ones. Also $y_1^* = y_{-1}$ and $y_0^* = -y_0$.

If the Cartesian polynomials are real then one will need the combinations, 

\textit{(unnormalised spherical harmonics)},

$$\mathcal{Y}_{c}^m \equiv P_{l}^m (\cos \theta) \cos m\phi = \frac{1}{2i} N_{lm} r^{l+1} \frac{\partial^{l-m}}{\partial y_{0}^{l-m}} \left( \frac{\partial^m}{\partial y_1^m} + (-1)^m \frac{\partial^m}{\partial y_{-1}^m} \right) \frac{1}{r},$$

$$\mathcal{Y}_{s}^m \equiv P_{l}^m (\cos \theta) \sin m\phi = \frac{1}{2} N_{lm} r^{l+1} \frac{\partial^{l-m}}{\partial y_{0}^{l-m}} \left( \frac{\partial^m}{\partial y_1^m} - (-1)^m \frac{\partial^m}{\partial y_{-1}^m} \right) \frac{1}{r}.$$

These are the functions in terms of which the existing expressions for the polyhedral harmonics are written. They are not, formally, the most convenient. A neater organisation is given at the end of this section.
In (22), the $P_{lm}^m$ are the usual Legendre polynomials, in terms of which the usual surface spherical harmonics are, [13],

$$C_{lm}^m(\theta, \phi) = (-1)^m \left[\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_{lm}^m(\cos \theta) e^{im\phi}, \; m \geq 0,$$

and so,

$$2 \left[\frac{(l-m)!}{(l+m)!}\right]^{1/2} Y_{c l}^m(\theta, \phi) = (-1)^m C_{lm}^m + C_{l-m}^m, \; m \geq 0$$

$$2i \left[\frac{(l-m)!}{(l+m)!}\right]^{1/2} Y_{s l}^m(\theta, \phi) = (-1)^m C_{lm}^m - C_{l-m}^m, \; m \geq 0$$

corresponding to (24).

The principle now to be employed, founded on (17), is that a basis for invariant harmonic polynomials is provided by the action of the set of independent invariant polynomial operators, $Q(\partial_{y_1}, \partial_{y_2}, \partial_{y_3}) = Q(\nabla_y)$, on $1/r$, (Poole, [37], Meyer, [26], and Laporte, [28]). This set is built algebraically from the invariant polynomial integrity basis.

This technique is no different from the one mentioned above, only that, in using (22) the harmonic projection, (17), has really already been performed. It yields the polynomial series for the $P_{lm}^m$, [35]. §85.

In $Q(\nabla_y)$, the three–vector $\nabla_y$ is effectively null,

$$\Delta = \nabla_y \cdot \nabla_y = 2 \frac{\partial^2}{\partial y_1 \partial y_{-1}} - \frac{\partial^2}{\partial y_0^2} = 2 \frac{\partial^2}{\partial y_{-1} \partial y_1} - \frac{\partial^2}{\partial y_0^2} = 0.$$

Hodgkinson and Poole derived the required integrity basis directly from the rotational 3–geometry of the polyhedra (the icosahedron in their case) and then imposed the null–vector condition (27). Klein, [38], p.238, calculates the polynomials as higher polars of binary forms (found from geometry) to which they return, up to a factor, on enforcing the null condition, (27), or, rather, its dual. It is, therefore, logically more satisfactory to start from these binary forms.\textsuperscript{7}

Following Hönl, [36], define, in a symbolic way, the binary pseudo–operators,

$$\lambda_{1/2} = \left( \frac{\partial}{\partial y_1} \right)^{1/2} \left( \frac{\partial}{\partial y_{-1}} \right)^{1/2}, \quad \lambda_{-1/2} = \left( \frac{\partial}{\partial y_{-1}} \right)^{1/2} \left( \frac{\partial}{\partial y_1} \right)^{1/2} \quad (28)$$

\textsuperscript{7} A detailed account of the construction of the integrity bases using 3–vectors is given in Appendix C of [39].
so that, in accord with (27),
\[
\frac{\partial}{\partial y^0} = \pm \sqrt{2} \lambda_{1/2} \lambda_{-1/2}.
\] (29)

With these correspondances, and the upper sign in (29),\(^8\) it can easily be checked that \(\lambda_{1/2}\) and \(\lambda_{-1/2}\) transform as standard spin–1/2 spinors under SU(2) (cf the interesting paper by Hönl, [36]) and hence invariant polynomial operators can be obtained from Klein’s invariant binary forms on making the replacements \(z_1 \rightarrow \lambda_{1/2}\) and \(z_2 \rightarrow \lambda_{-1/2}\) for his binary variables. For notational, and comparison, convenience, I will use \(z_1\) and \(z_2\) for the operators, as well.

Klein’s fundamental icosahedral binary invariant ground form is the 12–ic (spin–6),
\[
f = z_1 z_2 (z_1^{10} - z_2^{10}) + 11 z_1^6 z_2^6
\] (30)
in canonical form.

This has the associated Hessian (spin–10),
\[
H = -(z_1^{20} + z_2^{20}) + 228 (z_1 z_2)^5 (z_1^{10} - z_2^{10}) - 494 z_1^{10} z_2^{10}.
\] (31)

and the Jacobian transvectant (spin–15),
\[
T = (z_1^{30} + z_2^{30}) + 522 (z_1 z_2)^5 (z_1^{20} - z_2^{20}) - 10005 (z_1 z_2)^{10} (z_1^{20} + z_2^{20}).
\]

The tesseral harmonics (24) can be rewritten, with the operator replacements,
\[
N_l(l - m)! Y^{m}_{l l} = r^{l+1} (z_1 z_2)^{l-m} \left( z_2^{2m} + (-1)^m z_1^{2m} \right) \frac{1}{r},
\]
\[
i N_l(l - m)! Y^{m}_{s l} = r^{l+1} (z_1 z_2)^{l-m} \left( z_2^{2m} - (-1)^m z_1^{2m} \right) \frac{1}{r},
\] (32)

where \(N_l = i^{l} 2^{l/2} / 2\) is an irrelevant \(i.e.\) overall, constant.

As examples, one has the invariant harmonic forms \(r^7 f 1/r, \ r^{11} H 1/r\) and

---

\(^8\) In this case, for consistency because of (23), it is necessary to make the negative sign correspond to the complex conjugate, \((\partial/\partial y_0)^* = -\sqrt{2} \lambda_{1/2} \lambda_{-1/2}.\)
where the $\sim$ sign indicates equality up to an (irrelevant) overall constant factor.

The expressions (33) agree with those listed by Zheng and Doerschuk, [40], who obtained them by a complicated method from first principles using projection. Making use of Klein’s work in deriving his forms, many evaluations can be done by hand.

The modes, (33), pull back to vertical modes on $S^3/\Gamma'$. The remaining modes are obtained, in accordance with previous remarks, simply by replacing the right 0 index on the $D$s by $m$, with $-l \leq m \leq l$, for each appropriate $l$.

A somewhat similar technique is given by Kramer, [41], although he starts with the highest weight modes on $S^3$, $D_{l l}^m$ (in my conventions), and lowers the $l$ index. This means he does not use the pullback description and works only on the three-sphere.

The general method for the construction of all polyhedral harmonics is given by Hodgkinson; see also Laporte [28] and Meyer, [26]. The number of independent invariant harmonics is the number of terms of the form $f^a H^b T^c$ in the general polynomial, homogeneous of degree $2l$ as $z_1$ and $z_2$ are equally scaled. However, because of the necessary syzygy,

$$T^2 = 1728 f^5 - H^3;$$

(34)

c is either zero (even, Neumann modes) or one (odd, Dirichlet modes), [28]. The number of modes, $d_l$, is encoded in the icosahedral generating function, [28], [26],

$$g(\sigma; Y) = \frac{1 + \sigma^{15}}{(1 - \sigma^6)(1 - \sigma^{10})} = \sum_l d(l; Y) \sigma^l;$$

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the numerator, $\sigma^{15}$, corresponding to the odd modes $^9$.

In order to expand the harmonic in any particular case, set

$$r^{l+1} f^a H^b T^c \left| \frac{1}{r} = \sum_{m=0}^{l-1} (l-m)! A^{cl}_m Y^m_{l} \right.$$ (35)

with the upper limit $l_c = 6a + 10b + 15c$, $c = 0, 1$. It can be seen that the problem reduces to the algebraic one of organising the polynomial $f^a H^b T^c$ into a sum of the expressions appearing on the right-hand side of (32), [29].

For this purpose, it is sufficient to set $z_1 = 1$, so that, from (32), we seek

$$f^a H^b T^c \bigg|_{z_1 = 1} \sim \sum_{m=0}^{l-1} A^{cl}_m (z^{l+m} + (-1)^{m+c} z^{l-m})$$ (36)

and, as indicated, I am working up to an irrelevant overall constant.

Evaluating the left-hand side using (30) and (31), the coefficients $A^{cl}_m$ can be read off yielding the expansion coefficients in (35). In more detail,

$$\left(11 - 11 z^6 - z\right)^a \left(-z^{20} - 228 z^{15} - 494 z^{10} + 228 z^5 - 1\right)^b \times \left(z^{30} - 522 z^{25} - 10005 z^{20} - 10005 z^{10} + 522 z^5 + 1\right)^c = \sum_{m=0,5,10,\ldots} A^{cl}_m (z^{l+m} + (-1)^{m+c} z^{l-m}) \right.$$ (37)

Equation (37) for $c = 0$ agrees with the general form given by Poole, [43], equn.(11).

Machine algebra allows one to proceed as far as required. For a given $l$ it is only necessary to determine the possible values of $a$, and $b$, an easy task, and then extract some polynomial coefficients by commonplace routine. An example is $l = 30$ for which $c = 0$ and there are two values for $(a,b)$, namely $(5,0)$ and $(0,3)$. Correspondingly, one obtains two modes from (37). I find for these,

$$\Psi_{30}^{(5,0)} \sim 5! Y_{c\,30}^{25} - 10! 55 Y_{c\,30}^{20} + 15! 1205 Y_{c\,30}^{15} - 20! 13090 Y_{c\,30}^{10} + 25! 69585 Y_{c\,30}^{5} - 30! 134761 Y_{c\,30}^{0}$$

$^9$ For the purely rotational situation, there is no obligation to refer to even and odd modes. The numerator simply reflects the possible existence of a single factor of $T$, which is related to the syzygy. Also, the generating function is being derived here from the integrity basis. Often, the generating function is derived by other means (e.g. by cyclic decomposition) and information about the basis obtained therefrom, e.g. Sloane, [42], [30].
and

$$\Psi_{30}^{(0,3)} \sim \mathcal{Y}_{c_{30}}^{30} + 5! \cdot 684 \mathcal{Y}_{c_{30}}^{25} + 10! \cdot 157434 \mathcal{Y}_{c_{30}}^{20} + 15! \cdot 12527460 \mathcal{Y}_{c_{30}}^{15} + 20! \cdot 77460495 \mathcal{Y}_{c_{30}}^{10} + 25! \cdot 130689144 \mathcal{Y}_{c_{30}}^5 - 30! \cdot 33211924 \mathcal{Y}_{c_{30}}^0.$$  

The modes listed in [40] are linear combinations of these. Thus

$$T_{30,1} \sim 12251 \Psi_{30}^{(0,3)} - 33211924/11 \Psi_{30}^{(5,0)}$$

$$T_{30,0} \sim \Psi_{30}^{(0,3)} - 45750 \Psi_{30}^{(5,0)}.$$  

The procedure for the cubic groups is similar. The fundamental ground form can be taken to be the special sextic,

$$f = z_1 z_2 (z_1^4 - z_2^4) = a_z^6$$  \hspace{1cm} (38)

(corresponding to the Cartesian, $y^1 y^2 y^3$) from which the complete form system can be derived, geometrically, e.g. [44], or algebraically by invariant theory, [45].

The Hessian, $H$, and the Jacobian, $T$, of $f$ and $H$ are

$$H = (f, f)^2 = (ab)^2 a_x^4 b_x^4 = -\frac{1}{18} (z_1^8 + 14 z_1^4 z_2^4 + z_2^8)$$

$$T = (f, H) = -\frac{1}{108} (z_1^{12} - 33 z_1^8 z_2^4 - 33 z_1^4 z_2^8 + z_2^{12}) ,$$

where $(f, g)$ is the transvectant of the forms $f$ and $g$, e.g. [46].

The form $f$ is an absolute invariant for the tetrahedral group and so a complete set of absolute invariants is $f$, $H$, and $T$, corresponding to spins 3, 4 and 6, respectively.

The standard syzygy,

$$T^2 = -\frac{1}{108} f^4 - \frac{1}{2} H^3 ,$$  \hspace{1cm} (39)

means that the general term can be written $f^a H^b T^c$ where $c = 0, 1$ and $l = 3a + 4b + 6c$ so that the generating function is,

$$g(\sigma; T) = \frac{1 + \sigma^6}{(1 - \sigma^3)(1 - \sigma^4)} = \sum_l d(l; T) \sigma^l .$$

By contrast, the octahedral group replaces $f$ by $\pm f$ which means that the Hessian, $H$, is an absolute invariant, but that $T$ is replaced by $\pm T$. Therefore a basis set of absolute invariants in this case is provided by $f^2$, $H$ and $fT$, with spins
6, 4 and 9, respectively. e.g. [38], p.69. Comparing notations with Laporte, [28], 
\( f \sim \phi_3, H \sim \phi_4, T \sim \Phi_6, fT \sim \Phi_9, \) as can be checked algebraically.

The syzygy, (39), is recast as,

\[
(fT)^2 = -f^2 \left( \frac{1}{108} f^4 + \frac{1}{2} H^3 \right),
\]

so that the term in the general polynomial this time is \( f^{2a'} H^b (fT)^c \) with \( c = 0, 1 \) and \( l = 6a' + 4b + 9c. \) The octahedral generating function is therefore,

\[
g(\sigma; O) = \frac{1 + \sigma^9}{(1 - \sigma^4)(1 - \sigma^6)} = \sum_l d(l; O) \sigma^l,
\]
given previously.

As a typical example set \( l = 12 \) so that \( c = 0 \) and \( (a', b) = (2, 0) \) or \( (0, 3). \) The general definitions (35) and (36) still apply with the upper limits \( l_c = 3a + 4b + 9c, \) where \( a = 2a' \) and, in place of (37), one has

\[
(z - z^5)^a (1 + 14z^4 + z^8)^b (1 - 33z^4 - 33z^8 + z^{12})^c
\]

\[
= \sum_{m=0,4,8,...} A_m^{c,l} (z^{l+m} + (-1)^{m+c} z^{l-m}),
\]

which readily yields the invariant modes, \( (c = 0), \)

\[
\Psi_{12}^{(2,0)} \sim 4! \mathcal{Y}_{c\,12}^8 - 8! \mathcal{Y}_{c\,12}^4 + 12! 3 \mathcal{Y}_{c\,12}^0
\]

and

\[
\Psi_{12}^{(0,3)} \sim \mathcal{Y}_{c\,12}^{12} + 4! 42 \mathcal{Y}_{c\,12}^8 + 8! 591 \mathcal{Y}_{c\,12}^4 + 12! 1414 \mathcal{Y}_{c\,12}^0.
\]

These quantities agree with the corresponding ones in Table 8 in Altmann and Bradley, [47], after a linear combination which orthogonalises them. I have not checked them against the Cartesian expressions given by Ronveaux and Saint–Aubin, [32], equn.(34).

Incidentally, it is interesting to note that the fourth transvectant of the octahedral form \( f \) with itself is zero. In fact, this characterises the form, (38), Klein, [44], Cayley, [48], Gordan, [45]. In angular momentum terms this means that the vanishing of the Clebsch–Gordan spin–two combination of two spin–three quantities, \( f, \)

\[
f^l f^m \left( \begin{array}{c} 3 & 3 & 2 \\ l & m & n \end{array} \right) = 0,
\]

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implies that \( f \) is equivalent to the special octahedral form, (38).

Similar remarks hold for the icosahedral case and are related to Fuch’s notion of *Primformen*.

A further ancillary point concerns the nature of the syzygies. These can be proved either geometrically, [38], or algebraically, [45]. Another approach uses the action under the reflections of the *extended* groups. Taking the icosahedral syzygy, (34) as an example, \( T \) is odd under reflection while \( H \) and \( f \) are even. Hence \( T^2 \), being even, can be expressed algebraically in terms of \( f \) and \( H \), e.g. [28]. In fact, all the left–hand sides of the syzygies, (34), (39) and (40) are squares of odd quantities (Jacobians) and the corresponding modes thus are counted by the numerator in the generating functions given above. This is a well known behaviour. The denominator basis elements sometimes are referred to as ‘free’, and the numerator ones as ‘constrained’ and occur only once in the construction of the algebraic basis, e.g. Cummins and Patera, [49], Patera *et al*, [30], McLellan, [50].

Of course the entire scheme is a classic example of the invariants of finite reflection groups and can be treated from this point of view *ab initio*.

For the cyclic and dihedral groups, the construction of invariant bases in terms of the Legendre functions proceeds simply by a process of selection and is given in, e.g., Meyer, [26] and, earlier, in the classic, Pockels, [51]. See also Altmann and Bradley, [47].

In order to make comparison with existing results easier, I have used the tesseral harmonics. However it is more elegant to use the usual spherical harmonics, (25). Then, instead of (32), some rearrangement yields the neater relation,

\[
(-1)^l N_l C_m^l(\theta, \phi) = r^{l+1} Z_m^{(l)} \frac{1}{r},
\]

with the familiar monomial (or null \( l \)–spinor),

\[
Z_m^{(l)} \equiv \frac{z_1^{l+m} z_2^{l-m}}{((l-m)!(l+m)!)^{1/2}},
\]

constructed from the two–spinor \( \left( \frac{z_1}{z_2} \right) \), cf [36], equn.(28).

However, we have now come full circle, because (42) can be rewritten

\[
\frac{(2l)!}{2^l l!} C_m^l(\theta, \phi) = r^{l+1} C_m^l(-\nabla_y) \frac{1}{r},
\]

where \( C_m^l(a) \) is the solid spherical harmonic and \( \nabla_y \) is a null vector according to (27). Equation (43) is an example of Nivens’ general theorem, [52], [35], p.127, and
could be taken conveniently as the starting point of the analysis, rather than the specific (22). It also follows from (17).

6. Tensor polyhedral modes

The modes found in the last section are invariant, i.e. they transform as the identity rep, 1, of \( \Gamma \). In order to find those that transform equivariantly according to the other irreps (some are listed in [47] for example) I use the results of Patera et al, [30], and, especially, of Desmier and Sharp, [53] and of Bellon [8]. I restrict myself to the octahedral and icosahedral groups.

To start with, the mode (which could be called a twisted scalar mode) transforming as the one–dimensional rep, \( 1' \), of the octahedral group has the general form

\[
\Psi_l(1') = r^{l+1} (A f + B T) \frac{1}{r}
\]

where \( A \) and \( B \) are invariants constructed as in the preceding section from \( f^2 \) and \( H \). For example, for \( l = 12 \), \( A = 0 \) and \( B \sim f^2 \) and the mode is easily calculated to be,

\[
\Psi_{12}(1') \sim 10! 34 \nu_{c12}^2 - 6! 35 \nu_{c12}^6 + 2! \nu_{c12}^{10},
\]

gain in agreement with [47] Table 8.

The next most complicated modes transform according to the irrep 2 and have the general structure

\[
\Psi_l(2) = r^{l+1} \Psi_l(2) \frac{1}{r}
\]

where, [53] (18),

\[
\Psi_l(2) = A \left( \sqrt{3} (z_1^4 + 6z_1^2z_2^2 + z_2^4) -3(z_1^2 - z_2^2)^2 \right) + B \left( \sqrt{3} (z_1^8 + 4z_1^6z_2^2 - 10z_1^4z_2^4 + 4z_1^2z_2^6 + z_2^8) \right)
\]

with \( A \) and \( B \) as before. Choosing again \( l = 12 \), and writing

\[
A \sim f^{2a} H^b, \quad B \sim f^{2a'} H^{b'},
\]

the only possibilities are \((a, b) = (1, 1)\) and \((a', b') = (0, 2)\) giving two independent modes. I write them out just to show how the method proceeds.
Applying the technique explained in the last section directly to the 2–tensor integrity basis given by Desmier and Sharp yields the two modes, (I write \( \mathcal{Y} \) for \( \mathcal{Y}_c \) for space reasons),

\[
\Psi_{12}^{1}(2) = \left( \frac{\sqrt{3}(-12!78\mathcal{Y}_c^0_{12} - 10!14\mathcal{Y}_c^2_{12} + 8!72\mathcal{Y}_c^4_{12} + 6!13\mathcal{Y}_c^6_{12} + 4!6\mathcal{Y}_c^8_{12} + 2!\mathcal{Y}_c^{10}_{12})}{12!78\mathcal{Y}_c^0_{12} + 10!42\mathcal{Y}_c^2_{12} + 8!72\mathcal{Y}_c^4_{12} - 6!39\mathcal{Y}_c^6_{12} + 4!6\mathcal{Y}_c^8_{12} - 2!3\mathcal{Y}_c^{10}_{12}} \right)
\]

and

\[
\Psi_{12}^{2}(2) = \left( \frac{\sqrt{3}(-12!962\mathcal{Y}_c^0_{12} + 10!2712\mathcal{Y}_c^2_{12} - 8!81\mathcal{Y}_c^4_{12} - 6!348\mathcal{Y}_c^6_{12} + 4!18\mathcal{Y}_c^8_{12} + 2!12\mathcal{Y}_c^{10}_{12} + \mathcal{Y}_c^{12}_{12})}{12!962\mathcal{Y}_c^0_{12} + 10!142\mathcal{Y}_c^2_{12} - 8!81\mathcal{Y}_c^4_{12} + 6!13\mathcal{Y}_c^6_{12} + 4!6\mathcal{Y}_c^8_{12} - 2!3\mathcal{Y}_c^{10}_{12}} \right)
\]

which are, in appearance, more complicated than those in [47] Table 10. The reason is that the basis in 2–irrep space used in [53] is different to that in [47].

In order to convert the modes (44) and (45) to Altmann and Bradley’s, firstly multiply them by the SO(2) rotation,

\[
R = \begin{pmatrix} \cos \pi/3 & -\sin \pi/3 \\ \sin \pi/3 & \cos \pi/3 \end{pmatrix}
\]

to convert the 2 basis, and then take linear combinations in order to get precise agreement. I find

\[
\Psi_{12}^{AB,1}(2) \sim 37 R\Psi_{12}^{1}(2) - 3 R\Psi_{12}^{2}(2)
\]

\[
\Psi_{12}^{AB,2}(2) \sim 8 R\Psi_{12}^{1}(2) + R\Psi_{12}^{2}(2)
\]

The same considerations apply to the other irreps of the octahedral (and the binary octahedral) group but I do not need to give any further examples. I comment that the 3–irrep of O (\( \Gamma_4 \) in [53]) requires no rotation of basis. For example the fundamental \( l = 1 \) mode is,

\[
E_{4,6}^{(2)} \sim \begin{pmatrix} x \\ y \\ z \end{pmatrix}
\]

with the notation of [53], and so has a Cartesian basis. This agrees as it stands with the entry in [47], Table 11. \( \Gamma_6 \) of [53] is the spinor, quaternion irrep, \( 2_s \).

The general structure of the bases is outlined in Section 4 of [53] and it is clear that any desired mode can be constructed without too much bother. A basic symbolic manipulator (I used DERIVE) is all that is required. If one were to pursue

\[\text{[10] There appears to be a misprint in equn.}(56)\] of [30]. The irrep for O should be \( \Gamma_4 \).
this extensively, it would be advisable to rotate the 2-octahedral expressions given in [53] by $R$ at the start.

The generating functions for the icosahedral group are given in Table VII of [53] but not the corresponding reps. However, Bellon, [8], gives a list of those based on the fundamental quaternion irrep, $2_s$, which can be used in exactly the same way as above to give the corresponding tensor icosahedral harmonics.

A technical point is that the reps in [8] are taken with respect to a spherical basis and it is easiest to use standard spherical functions and the basic relation (42).

As an illustration, the $l = 1$ and $l = 5$ modes for the vector 3-irrep are rapidly found to be proportional to,

$$
\begin{pmatrix}
\sqrt{2} C_1^1 \\
C_0^1 \\
\sqrt{2} C_{-1}^1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\sqrt{30} C_1^5 - \sqrt{70} C_{-4}^5 \\
\sqrt{7} C_{-5}^5 - 6 C_0^5 - \sqrt{7} C_5^5 \\
\sqrt{30} C_{-1}^5 + \sqrt{70} C_4^5
\end{pmatrix},
$$

respectively. In this form, the lifting to dodecahedron space is easy.

It would need a certain motivation to compute further cases or to check orthogonality for degenerate modes, although there is no particular difficulty.

The modes which transform according to the spinor irreps of the binary group, $\Gamma'$, will involve what might be termed polyhedral spinor harmonics leading to polyhedral spinor hyperharmonics on $S^3/\Gamma'$. These will be dealt with at another time using fractional derivatives.

7. Concluding remarks

I have employed the Hopf map to lift modes on the two-sphere to modes on the three-sphere, a known procedure, and have divided by a polyhedral symmetry. I used this to motivate the construction of the required symmetry adapted modes on the orbifolded two-sphere which are computed by a binary method and agree with those obtained many years ago by the more standard, and in my view more involved, technique of projection. This has been extended to tensor modes under the action of the deck group and some specific cases have been evaluated using the results of Desmier and Sharp and of Bellon.

Expressing the symmetry adapted modes in the traditional way as sums of spherical harmonics allows the complete set of modes on the factored three-sphere to be found from the pullbacks using raising and lowering with no extra work.
An aspect that deserves exploration is the significance of the degeneracies, $d(l; \Gamma)$, as the dimensions of the irreps of the symmetry group of $S^2/\Gamma$ which is the centraliser of $\Gamma$ in SO(3), or for $S^3/\Gamma'$, the centraliser of the binary group, $\Gamma'$, in SU(2). This is the ‘missing label’ question.

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