Logic and Partially Ordered Abelian Groups

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Abstract

The unit interval in a partially ordered abelian group with order unit forms an interval effect algebra (IEA) and can be regarded as an algebraic model for the semantics of a formal deductive logic. There is a categorical equivalence between the category of IEA’s and the category of unigroups. In this article, we study the IEA-unigroup connection, focusing on the cases in which the IEA is a Boolean algebra, an MV-algebra, a Heyting MV-algebra, or a quantum logic.

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1. Introduction

Two competing (although not entirely unrelated) methods for providing the semantics of a formal deductive symbolic calculus \( \mathcal{L} \) are the Kripke many-worlds approach and the algebraic-logic approach via interpretations of \( \mathcal{L} \) in suitable mathematical models. Motivation for the developments in this article derives largely from the latter approach via algebraic logic.

Typically, a model \( L \) for a deductive calculus \( \mathcal{L} \) is a bounded partially-ordered set equipped with operations that qualify it as an algebra (in the

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general sense). An interpretation of $L$ in $L$ is a mapping $f \mapsto p$ from well-formed formulas $f$ of $L$ to elements $p$ in $L$ that relates the deductive structure of $L$ to the mathematical structure of $L$. Thus, an element $p \in L$ can be considered to be a logical proposition representing the equivalence class of all of its antecedent formulas $f$. The partial order structure on $L$ is understood in the sense that, for propositions $p, q \in L$, $p \leq q$ means that $p$ implies $q$ in a manner that is compatible with the rules of deduction in $L$ for the antecedents of $p$ and $q$. Likewise, the algebraic operations on $L$ are to be regarded as logical connectives.

Both $L$ and its models $L$ are logical systems or logical structures, and are often referred to, for short, as “logics.” Thus, in this article, we may refer to a partially ordered algebraic structure as a “logic” when we have in mind that it could be construed as a model for a deductive symbolic calculus.

In defining the mathematical structure of a logic $L$, one may treat the order and the algebraic structures on an equal footing, or one may award primacy to one of the structures and derive the other structure from the primary one. For instance, a Boolean algebra $L$ can be defined as a bounded, complemented, distributive lattice or it can be defined as an idempotent ring with unity. Starting with the lattice definition, the ring structure is derived by taking $x + y$ to be the symmetric difference and $xy$ to be the infimum of $x$ and $y$ in $L$. Starting with the ring definition, the order structure is derived by defining $x \leq y$ to mean that $x = xy$.

Boolean algebras serve as models for classical propositional calculus, polyadic Boolean algebras are models for first-order predicate calculus, and Heyting algebras are models for intuitionistic calculi. The appropriate models for the multi-valued logical calculi of Łukasiewicz are the MV-algebras defined in 1957 by C.C. Chang [6]. (See Section 5 below.)

In 1986, D. Mundici [28] discovered a remarkable connection between MV-algebras and lattice-ordered abelian groups with order units. (See Section 8 below.) The connection is as follows: If $G$ is a lattice ordered abelian group with positive cone $G^+$ and $u \in G^+$ is an order unit in $G$, then the interval $L = G^+[0,u] = \{ e \in G \mid 0 \leq e \leq u \}$ forms an MV-algebra $(L, 0, u, \bot, \hat{+})$, where $p\bot = u - p$ and the MV-sum is given by $p\hat{+}q = (p + q) \land u$, for all $p, q \in L$. Conversely, every MV-algebra $L$ can be realized as $G^+[0,u]$ for a lattice ordered abelian group $G$ with order unit $u$. The Mundici group $G$ is uniquely determined by $L$ up to an isomorphism of partially ordered abelian groups with order units.

It turns out that the connection $L \leftrightarrow G$ discovered by Mundici admits
a considerable generalization in which the MV-algebra $L$ is replaced by a
so-called interval effect algebra and $G$ is replaced by a so-called unigroup.
An interval effect algebra may be regarded as a “logic” in the sense indicated
above, whence—as per the title of this article—we have a connection between
a class of logics and a class of partially ordered abelian groups. As some logi-
cians may be unfamiliar with the theory of partially ordered abelian groups,
and some experts on partially ordered abelian groups my be uncomfortable
with algebraic logic, there is a need for an exposition of the logic-unigroup
connection. Our purpose in what follows is to explicate and study this con-
nection. Although this article is largely expository, a number of unpublished
results of N. Ritter [34] are cited, and Theorem 8.7, which characterizes the
unigroup associated with a Heyting effect algebra, is new.

2. Boole and the Logic-Algebra Connection

In laying the foundations for the algebra that now bears his name, George
Boole was strongly motivated by analogies between the logic of classes and
ordinary arithmetic. For classes $x$ and $y$, he took what we now call the
intersection as the proper interpretation of the “product” $xy$. However, his
interpretation of the “sum” $x + y$ differed from what we now call the union
in that he insisted that it be defined only when $xy = 0$. Thus, in his 1854
masterpiece, The Laws of Thought [4], Boole wrote,

“...The expression $x + y$ seems indeed uninterpretable, unless it be assumed that the
things represented by $x$ and the things represented by $y$ are entirely separate; that
they embrace no individuals in common.”

In The Laws of Thought, Boole had previously indicated that he was well
aware of what is now called the union (and even the symmetric difference)
of classes $x$ and $y$, so his decision to write $x + y$ only when $xy = 0$ might
seem puzzling. Indeed, Boole’s contemporary, W.S. Jevons, expressed strong
disagreement with Boole over his unwillingness to give $x + y$ an unrestricted
interpretation. Using mathematical tools unavailable to Boole and Jevons,
I. Hailperin has employed signed multisets to demonstrate a perfect har-
mony of Boole’s product $xy$ and restricted sum $x + y$ with the corresponding
operations of ordinary arithmetic [25, pp. 87-112], [26]. We prefer to re-
cast Hailperin’s signed multisets in mathematical terms more conducive to
the developments in this article. To begin with, we assume that the classes
1.1 Example Let \( B \) be a field of subsets of a nonempty set \( X \) and let \( \mathbb{Z} \) be the ordered ring of integers. Define \( \mathcal{F}(B, \mathbb{Z}) \) to be the commutative ring under pointwise operations of all bounded functions \( f : X \to \mathbb{Z} \) such that \( f^{-1}(n) \in B \) for all \( n \in \mathbb{Z} \). The function 1 that maps all elements in \( X \) to the integer 1 is a unity element for the ring \( \mathcal{F}(B, \mathbb{Z}) \). Under the pointwise partial order, \( \mathcal{F}(B, \mathbb{Z}) \) is a partially ordered (in fact, a lattice-ordered) commutative ring with unity. The interval \( E = \{ e \in \mathcal{F}(B, \mathbb{Z}) \mid 0 \leq e \leq 1 \} \) is in bijective correspondence with \( B \) under the mapping \( e \leftrightarrow e^{-1}(1) \).

Under this correspondence, the product \( ef \) in the ring \( \mathcal{F}(B, \mathbb{Z}) \) of elements \( e, f \in E \) corresponds to the intersection \( e^{-1}(1) \cap f^{-1}(1) \) and, if \( ef = 0 \), the sum \( e + f \) corresponds to the (disjoint) union \( e^{-1}(1) \cup f^{-1}(1) \). In this way, the restrictions to \( E \) of the product and sum in the commutative ring \( \mathcal{F}(B, \mathbb{Z}) \) match perfectly with Boole's product and sum of the corresponding elements of \( B \).

By the Stone representation theorem, a Boolean algebra \( B \) can be represented as the field \( B \) of compact open subsets of a compact, Hausdorff, totally-disconnected topological space \( X \). Let \( \mathcal{F}(B, \mathbb{Z}) \) be the partially ordered commutative ring with unity in Example 1.1. The Boolean algebra \( B \) forms an MV-algebra \( (B, 0, 1, \perp, \hat{+}) \), where \( x \mapsto x^\perp \) is the Boolean complementation and the MV-sum is given by \( x \hat{+} y = x \lor y \). As such, the Mundici group corresponding to \( B \) is in fact the partially ordered additive group of the ring \( \mathcal{F}(B, \mathbb{Z}) \) with 1 as the order unit.

Of course, a Boolean algebra \( B \) can be organized into a commutative idempotent ring with unity by using symmetric difference as the sum and the Boolean meet as the product. But, then, \( 2x = 0 \) holds for every element \( x \in B \), a radical departure from ordinary arithmetic with which Jevons might have been comfortable—but certainly not Boole. On the other hand, as in ordinary arithmetic, the additive group of the ring \( \mathcal{F}(B, \mathbb{Z}) \) is torsion free, i.e., if \( n \) is a nonzero integer, \( f \in \mathcal{F}(B, \mathbb{Z}) \), and \( nf = 0 \), then \( f = 0 \).

We note that the interval \( E \) in Example 1.1 is precisely the set of idempotents in the ring \( \mathcal{F}(B, \mathbb{Z}) \) and thus can be singled out without invoking the partial-order structure of the ring. This would be in keeping with the work of Hailperin alluded to above. However, it is the partial-order structure that best relates to the theme of this article.

\( x, y, \ldots \) that concerned Boole can be organized into field \( B \) of sets.
3. **Effect Algebras and Boolean Effect Algebras**

Boole’s pioneering work ultimately led to the conception of a Boolean algebra, either as a bounded, complemented, distributive lattice, or equivalently, as an idempotent ring with unity. However, neither of these formulations is based directly on Boole’s original notions of the product $xy$ and restricted sum $x + y$ for classes.

It is possible to formulate an alternative definition of a Boolean algebra involving nothing but the constants 0, 1 and the restricted sum $x + y$. (The product $xy$ then emerges as a derived concept.) In formulating this definition, (Definition 3.5 below), we write $\oplus$ rather than $+$ to emphasize that it is only a partially defined operation and also to avoid confusing it with addition in the abelian groups to be introduced later. Also, we write $u$ instead of 1 to avoid confusion with the numeral 1. We begin with a basic definition. Apart from the change of notation from $+ \to \oplus$, and from 1 to $u$, axioms (i)–(iv) in Definition 3.1 below are obviously consistent with Boole’s notion of a restricted sum.

**3.1 Definition** An **effect algebra** is a system $(E, 0, u, \oplus)$ consisting of a set $E$, special elements $0, u \in E$ called the **zero** and the **unit**, and a partially defined binary operation $\oplus$ on $E$, satisfying the following conditions for all $x, y, z \in E$:

(i) (**Commutativity of $\oplus$**) If $x \oplus y$ is defined, then $y \oplus x$ is defined and $x \oplus y = y \oplus x$.

(ii) (**Associativity of $\oplus$**) If $x \oplus y$ and $(x \oplus y) \oplus z$ are defined, then $y \oplus z$ and $x \oplus (y \oplus z)$ are defined and $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.

(iii) (**Supplementation property of $\oplus$**) For each $x \in E$, there is a uniquely determined $y \in E$ such that $x \oplus y$ is defined and $x \oplus y = u$.

(iv) (**Zero-unit property of $\oplus$**) If $x \oplus u$ is defined, then $x = 0$.

Effect algebras were introduced in 1994 [17] as abstractions of the algebra of Hilbert-space effect operators used in the study of the theory of measurement in quantum mechanics [5].
3.2 Example Let $R$ be a (not necessarily commutative) ring with unity $1$ and let $E$ be the set of idempotents in $R$. If $e, f \in E$, let $e \oplus f := e + f$ iff $ef = fe = 0$. (We use $:=$ to mean “equals by definition” and “iff” to mean “if and only if”.) Then $(E, 0, 1, \oplus)$ is an effect algebra. □

In accord with mathematical tradition, we often say that $E$ is an effect algebra when we really mean that $(E, 0, u, \oplus)$ is an effect algebra.

3.3 Definition Let $E$ be an effect algebra and let $x, y, z \in E$. Then:

(i) We say that $x$ and $y$ are orthogonal and write $x \perp y$ iff $x \oplus y$ is defined. If $x \perp y$, then $x \oplus y$ is called the orthogonal sum or for short, the orthosum of $x$ and $y$. If we assert that $x \oplus y = z$, we understand that, necessarily, $x \perp y$.

(ii) If there exists $z \in E$ such that $x \oplus z = y$, we say that $x$ is less than or equal to $y$ and write $x \leq y$.

(iii) $x^\perp$, called the supplement of $x$, denotes the unique element in $E$ such that $x \oplus x^\perp = u$.

For Boole’s classes $x$ and $y$, the relation $x \perp y$ in Definition 3.3 would correspond to the requirement that $xy = 0$, and the relation $x \leq y$ would correspond to the condition that the things represented by $x$ are among the things represented by $y$. Also, the “supplement” $x^\perp$ would be the complement $1 - x$ of $x$.

See [17] for proofs of the following properties of an effect algebra $E$: The relation $\leq$ is a partial order relation on $E$ and $x \in E \Rightarrow 0 \leq x \leq u$. Also, $\leq$ satisfies the following cancellation law: If $x, y, z \in E$, $x \perp z$, and $y \perp z$, then $x \oplus z \leq y \oplus z \Rightarrow x \leq y$. Furthermore, $x \perp y \Leftrightarrow x \leq y^\perp$, $x \leq y \Rightarrow y^\perp \leq x^\perp$, $(x^\perp)^\perp = x$, $x \perp 0$, $x \oplus 0 = x$, $0^\perp = u$, and $u^\perp = 0$.

If an effect algebra $E$ is regarded as a “logic” in the sense alluded to in Section 1 (i.e., as an algebraic model for a deductive logical calculus), then elements $x, y \in E$ can be thought of as “propositions,” $x \leq y$ means that $x$ “implies” $y$, and $0, u \in E$ are “anti-tautological” and “tautological” constants, respectively. The condition $x \perp y$ means that, in some sense, the propositions $x$ and $y$ “refute” each other. The supplementation mapping $x \mapsto x^\perp$ is a (perhaps attenuated) version of “logical negation,” and the “double negation law” $x = (x^\perp)^\perp$ holds. If $x \perp y$, then $x \oplus y$ is to be regarded as a sort of (perhaps rarefied) version of “logical disjunction” of the
mutually refuting propositions $x$ and $y$. Thus, in Definition 3.1, property (iii) may be considered to be a rendition of the "law of the excluded middle" and property (iv) may be regarded as a (very) weak "law of consistency" [9, Definition 5.1.1]. Parts (i) and (ii) of the following definition are motivated by the observation that, if $x \oplus y$ is a logical disjunction of $x$ and $y$ in more or less the classical sense, then $x, y \leq p \Rightarrow x \oplus y \leq p$ for each proposition $p \in E$. Part (iii) is suggested by a slightly more subtle property of classical logical disjunction.

3.4 Definition Let $E$ be an effect algebra. Then:

(i) An element $p \in E$ is principal iff, for all $x, y \in E$, the conditions $x \perp y$ with $x, y \leq p$ imply that $x \oplus y \leq p$.

(ii) $E$ is an orthomodular poset iff every element $p \in E$ is principal.

(iii) $E$ has the Riesz-decomposition property iff, for all $x, y, z \in E$, if $y \perp z$ and $x \leq y \oplus z$, there exist $x_1, x_2 \in E$ such that $x_1 \leq y$, $x_2 \leq z$, and $x = x_1 \oplus x_2$.

In Definition 3.4 (iii), note that it is not necessary to assume that $x_1 \perp x_2$ since the facts that $x_1 \leq y$, $x_2 \leq z$, and $y \perp z$ imply that $x_1 \leq y \leq x_2 \perp z = (x_2)\perp$, whence $x_1 \perp x_2$.

Let $(B, \leq, 0, u, \wedge, \vee)$ be a Boolean algebra, regarded as a bounded distributive lattice, and organize $B$ into an effect algebra $(B, 0, u, \oplus)$ with $x \oplus y := x \vee y$ iff $x \wedge y = 0$, for all $x, y \in B$. Then the effect-algebra inequality $\leq$ in Definition 3.3 (ii) coincides with the Boolean inequality and the effect-algebra supplement $x^\perp$ of $x \in B$ coincides with the Boolean complement of $x$. If $x, y, p \in B, x \perp y$, and $x, y \leq p$, then $x \oplus y = x \vee y \leq p$, so $p$ is principal, and therefore $B$ is an orthomodular poset. Also, if $x, y, z \in B$, with $y \perp z$ and $x \leq y \oplus z = y \vee z$, then with $x_1 := x \wedge y \leq y$ and $x_2 := x \wedge z \leq z$, we have $x = x_1 \vee x_2 = x_1 \oplus x_2$, so $B$ has the Riesz-decomposition property. Thus, every Boolean algebra is a "Boolean effect algebra" as per the following definition.

3.5 Definition A Boolean effect algebra is an orthomodular poset with the Riesz-decomposition property.

3.6 Theorem As a bounded partially ordered set, every Boolean effect algebra is a complemented distributive lattice, i.e., a Boolean algebra, in which the supplement of each element coincides with its Boolean complement.
As a consequence of Theorem 3.6 and the remarks preceding Definition 3.5, Boolean algebras are mathematically equivalent to Boolean effect algebras. But notice that, in Definitions 3.1, 3.3, 3.4, and 3.5, there is no direct reference to the meet and join operations $\land$ and $\lor$. The latter operations arise from the algebra of 0, $u$, and $\oplus$, rather than vice versa. A proof of Theorem 3.6 will emerge from the subsequent developments in this section.

Let $x$ and $y$ be elements of an effect algebra $E$. We write the meet (i.e., the infimum, or the greatest lower bound) of $x$ and $y$, if it exists in the partially ordered set $(E, \leq)$, as $x \land y$. Likewise, the join (i.e., the supremum, or the least upper bound) of $x$ and $y$, if it exists in $(E, \leq)$, is written as $x \lor y$. If we write an equation of the form $x \land y = z$, we mean that $x \land y$ exists and equals $z \in E$, and a similar convention holds for $x \lor y$. As $x \mapsto x^\perp$ is order inverting and of period two, we have the De Morgan laws—(for meet) if $x \land y$ exists, then $(x \land y)^\perp = x^\perp \lor y^\perp$, and (for join) if $x \lor y$ exists, then $(x \lor y)^\perp = x^\perp \land y^\perp$. (Caution: In the general case, if one regards $E$ as a logic, the question of whether $x \land y$ and $x \lor y$, when they exist, should be construed as the conjunction and disjunction of the propositions $x$ and $y$ presents subtleties.)

In the literature, an orthomodular poset is usually defined as a structure $(E, \leq, 0, u, \perp)$ consisting of a bounded partially ordered set $(E, \leq)$, as $x \land y$. Likewise, the join (i.e., the supremum, or the least upper bound) of $x$ and $y$, if it exists in $(E, \leq)$, is written as $x \lor y$. If we write an equation of the form $x \land y = z$, we mean that $x \land y$ exists and equals $z \in E$, and a similar convention holds for $x \lor y$. As $x \mapsto x^\perp$ is order inverting and of period two, we have the De Morgan laws—(for meet) if $x \land y$ exists, then $(x \land y)^\perp = x^\perp \lor y^\perp$, and (for join) if $x \lor y$ exists, then $(x \lor y)^\perp = x^\perp \land y^\perp$. (Caution: In the general case, if one regards $E$ as a logic, the question of whether $x \land y$ and $x \lor y$, when they exist, should be construed as the conjunction and disjunction of the propositions $x$ and $y$ presents subtleties.)

3.7 Definition The effect algebra $E$ is lattice ordered iff, as a bounded partially ordered set $(E, \leq, 0, u)$ together with an order-reversing mapping $\perp: E \to E$ of period two such that, (i) for all $x, y \in E$, $x \leq y^\perp \Rightarrow x \lor y$ exists in $E$, (ii) $x \lor x^\perp = u$, and (iii) $x \leq y \Rightarrow y = x \lor (x \lor y^\perp)^\perp$. Condition (iii) is called the orthomodular identity. To organize such a structure into an orthomodular poset according to Definition 3.4 (ii), one defines $x \oplus y = x \lor y$ iff $x \leq y^\perp$. Conversely, it is not difficult to verify that an orthomodular poset as per Definition 3.4 is an orthomodular poset according to the traditional definition.
A distributive effect algebra is not necessarily a Boolean effect algebra. For instance, the unit interval \([0, 1] \subseteq \mathbb{R}\) is organized into a distributive effect algebra with \(u = 1\) by defining \(x \oplus y := x + y\) iff \(x + y \in [0, 1]\) for \(x, y \in [0, 1]\). The resulting effect algebra has the Riesz-decomposition property, but it is not a Boolean effect algebra because the only principal elements in \([0, 1]\) are 0 and 1. In fact, \([0, 1]\) is a non-Boolean MV-effect algebra (see Section 5 below).

3.8 Definition An element \(z\) in an effect algebra \(E\) is said to be central in \(E\) iff (i) both \(z\) and \(z^\perp\) are principal in \(E\), and (ii) for every \(x \in E\) there are elements \(x_1, x_2 \in E\) such that \(x_1 \leq z, x_2 \leq z^\perp\), and \(x = x_1 \oplus x_2\). The set of all central elements of \(E\) is denoted by \(C(E)\) and called the center of \(E\) [21].

Clearly, if the effect algebra \(E\) has the Riesz-decomposition property, then condition (ii) in Definition 3.8 holds automatically. Also, every element \(z\) in an orthomodular poset satisfies condition (i) in Definition 3.8. Consequently, a Boolean effect algebra \(E\) is its own center, i.e., \(C(E) = E\).

3.9 Definition If \(E\) is an effect algebra, then a subset \(S \subseteq E\) is called a subeffect algebra of \(E\) iff \(0, u \in S, x \in S \Rightarrow x^\perp \in S\), and for all \(x, y \in S, x \perp y \Rightarrow x \oplus y \in S\).

If \(S\) is a subeffect algebra of the effect algebra \(E\), then \(S\) forms an effect algebra in its own right under the restriction to \(S\) of \(\oplus\). By [21], the center \(C(E)\) of an effect algebra \(E\) is a subeffect algebra of \(E\) and \(C(E)\) is a Boolean algebra (hence a Boolean effect algebra). The promised proof of Theorem 3.6 is now at hand, since if \(E\) is a Boolean effect algebra, then \(C(E) = E\), whence \(E\) is a Boolean algebra.

An alternative characterization of Boolean effect algebras can be formulated in terms of the notion of compatibility in the next definition.

3.10 Definition Let \(E\) be an effect algebra. We say that \(x, y, z \in E\) are jointly orthogonal iff \(x \perp y\) and \((x \oplus y) \perp z\), (whence \(y \perp z\) and \(x \perp (y \oplus z)\)). If \(x, y \in E\), then \(x\) and \(y\) are said to be compatible (or, Mackey compatible), in symbols \(xCy\), iff there are jointly orthogonal elements \(x_1, y_1, z \in E\) such that \(x = x_1 \oplus z\) and \(y = y_1 \oplus z\).

If \(E\) is an orthomodular poset, then \(C(E) = \{z \in E \mid zCx \text{ for all } x \in E\}\), hence a Boolean effect algebra is the same thing as an orthomodular poset in which every pair of elements is compatible.
The complete title of Boole’s 1854 classic is *An Investigation of the Laws of Thought on which are founded the Mathematical Theories of Logic and Probabilities*. We note that Boole’s restricted sum interacts perfectly with probability assignments $p$ in that $p(x + y) = p(x) + p(y)$ holds for $xy = 0$. This leads us to the following definition.

**3.11 Definition** Let $E$ be an effect algebra. If $K$ is an additive abelian group, then a *$K$-valued measure* on $E$ is a mapping $\phi : E \to K$ such that, for all $x, y \in E$, $x \perp y \Rightarrow \phi(x \oplus y) = \phi(x) + \phi(y)$. Regarding the ordered field $\mathbb{R}$ of real numbers as an additive abelian group, we define a *probability measure* on $E$ to be an $\mathbb{R}$-valued measure $\pi : E \to \mathbb{R}$ that is *positive* in the sense that $0 \leq \pi(x)$ for all $x \in E$ and *normalized* in the sense that $\pi(u) = 1$. Denote by $\Pi(E)$ the set of all probability measures on $E$. A subset $\Delta \subseteq \Pi(E)$ is *order-determining* iff, for $x, y \in E$, the condition $\pi(x) \leq \pi(y)$ for every $\pi \in \Delta$ implies that $x \leq y$.

The set $\Pi(E)$ is a convex subset of the real vector space under pointwise operations of all mappings $\rho : E \to \mathbb{R}$. We denote by $\partial_e \Pi(E)$ the set of all extreme points of $\Pi(E)$. If the elements of $E$ are regarded as “propositions,” then a probability measure $\pi \in \Pi(E)$, and especially a $\pi \in \partial_e \Pi(E)$, can be regarded as a (possibly multi-valued) “truth combination” assigning a “truth value” $\pi(x)$ on a scale from 0 (false) to 1 (true) for each proposition $x \in E$.

If $B$ is a Boolean effect algebra, then $\partial_e \Pi(B)$ is order-determining, elements of $\partial_e \Pi(B)$ are $\{0, 1\}$-valued, and $\partial_e \Pi(B)$ may be identified with the Stone space of $B$. That $\partial_e \Pi(B)$ is order determining accounts for the fact that truth tables provide an algorithmic decision procedure for classical propositional calculus.

**4. Quantum Logics**

Certain effect algebras $E$ can be considered to be algebraic models for the semantics of the “quantum logics,” that arise in the study of reasoning in quantum theory [9]. Rather than saying that such an $E$ is an algebraic model for a quantum logic, we shall say, for short, that $E$ is a quantum logic.

The genesis of quantum logic was von Neumann’s observation [29, p. 253], “.. the relation between the properties of a physical system on the one hand, and the projections on the other, makes possible a sort of logical calculus with these.”
The projections to which von Neumann referred are the bounded self-adjoint idempotent operators \( P = P^* = P^2 \) on a Hilbert space \( \mathcal{H} \), and these projections band together to form an orthomodular lattice \( \mathbb{P}(\mathcal{H}) \). In quantum mechanics, the question of whether two projections \( P \) and \( Q \) commute, i.e., whether \( PQ = QP \), is of considerable significance, and it is important to note that it can be settled strictly in terms the structure of \( \mathbb{P}(\mathcal{H}) \) as an effect algebra. In fact, two projections \( P \) and \( Q \) on the Hilbert space \( \mathcal{H} \) are compatible in the orthomodular lattice \( \mathbb{P}(\mathcal{H}) \) iff \( PQ = QP \).

Von Neumann’s observation that \( \mathbb{P}(\mathcal{H}) \) can be regarded as a logical calculus led to the study of more general orthomodular lattices as possible quantum logics. Indeed, S. Gudder and others were able to show that much of the theory of spectral measures and quantum probability carries over to the more general context of an orthomodular lattice that admits sufficiently many probability measures \[22\]. However, difficulties associated with the interpretation (as logical connectives) of the meet and join of noncommuting projections subsequently led to the consideration of more general orthomodular posets as quantum logics \[23\]. Further difficulties arising from the necessity of dealing with coupled quantum-mechanical systems led to the study of orthoalgebras as quantum logics \[11\].

An orthoalgebra is an effect algebra \( E \) such that \( x \wedge x^\perp = 0 \) for all \( x \in E \). Every orthomodular poset is an orthoalgebra, but not vice versa. By the De Morgan law, every element \( x \) in an orthoalgebra \( E \) satisfies both \( x \wedge x^\perp = 0 \) and \( x \vee x^\perp = u \), i.e., just as in a Boolean algebra, \( x^\perp \) is a complement of \( x \) in \( E \). Thus, in an orthoalgebra, we have a semantic version \( x \wedge x^\perp = 0 \) of the classical law of noncontradiction (ex contradictione quodlibet, or Duns Scotus’ law) and also the excluded middle law ( tertium non datur) \( x \vee x^\perp = u \).

If \( E \) is an orthoalgebra and \( x, y \in E \), then \( xCy \) iff there is a Boolean subeffect algebra \( B \) of \( E \) such that \( x, y \in B \). Also, for an orthoalgebra \( E \), the center \( C(E) \) is given by \( C(E) = \{ z \in E \mid zCx \text{ for all } x \in E \} \).

In the contemporary theory of quantum measurement \[5\] the projection-valued measures favored by von Neumann are replaced by more general measures defined on a \( \sigma \)-field of sets and taking on values in the set \( \mathbb{E}(\mathcal{H}) \) of effect operators on a Hilbert space \( \mathcal{H} \). An effect operator on \( \mathcal{H} \) is a bounded self-adjoint operator \( A \) on \( H \) such that \( 0 \leq A \leq 1 \), and the set \( \mathbb{E}(\mathcal{H}) \) can be organized into an effect algebra \( (\mathbb{E}(\mathcal{H}), 0, 1, \oplus) \), where, for \( A, B \in \mathbb{E}(\mathcal{H}) \), \( A \oplus B := A + B \) iff \( A + B \leq 1 \). As such, the effect-algebra partial order coincides with the restriction to \( \mathbb{E}(\mathcal{H}) \) of the usual partial order on bounded self-adjoint operators, and if \( A \in \mathbb{E}(\mathcal{H}) \), then \( A^\perp = 1 - A \). If \( \langle \cdot, \cdot \rangle \) is the inner
product on $\mathcal{H}$, then each unit vector $\psi \in \mathcal{H}$ determines a probability measure $\pi_\psi \in \partial_\mathcal{E}(\mathcal{E}(\mathcal{H}))$ according to $\pi_\psi(A) := \langle A\psi, \psi \rangle$ for every $A \in \mathcal{E}(\mathcal{H})$. Therefore, the effect algebra $\mathcal{E}(\mathcal{H})$ carries an order-determining set of probability measures.

The orthomodular lattice $\mathbb{P}(\mathcal{H})$ of projection operators on $\mathcal{H}$ is a subeffect algebra of $\mathcal{E}(\mathcal{H})$, and if $P \in \mathcal{E}(\mathcal{H})$, then $P \in \mathbb{P}(\mathcal{H}) \iff P$ is principal $\iff P \wedge P^\perp = 0$. In the passage from the orthomodular lattice $\mathbb{P}(\mathcal{H})$ to the larger effect algebra $\mathcal{E}(\mathcal{H})$, the supplementation mapping $A \mapsto A^\perp = 1 - A$ loses its character as a complementation, and $\mathcal{E}(\mathcal{H})$ becomes an algebraic model for a paraconsistent logic \cite{8}.

The Hilbert-space effect algebra $\mathcal{E}(\mathcal{H})$ is the prototypic effect algebra; both it and its subeffect algebra $\mathbb{P}(\mathcal{H})$ are the prototypic quantum logics. Nowadays, an effect algebra $E$ is regarded as a quantum logic only if it satisfies some of the special properties of the prototypes $\mathcal{E}(\mathcal{H})$ or $\mathbb{P}(\mathcal{H})$. Paramount among these properties are conditions relating to probability measures, especially the condition that $\Pi(\mathcal{E})$ is order determining. We shall resist the temptation to give a formal definition of a quantum logic. (For authoritative literature on the question of just what constitutes a quantum logic, see \cite{9} and \cite{30}.) However, if $\mathcal{A}$ is a unital $C^*$-algebra, we propose to regard $E := \{e \in \mathcal{A} \mid e = e^* \text{ and } 0 \leq e \leq 1\}$, as well as its subeffect algebra $P := \{p \in \mathcal{A} \mid p = p^* = p^2\}$ as bona fide quantum logics.

5. MV-Algebras

Material in this section is adopted from \cite{12}. The following definition is based on \cite{28} Lemma 2.6.

5.1 Definition An MV-algebra is a system $(E, 0, u, ^\perp, \hat{+})$ consisting of a set $E$, special elements $0, u \in E$ called the zero and the unit, a unary operation $p \mapsto p^\perp$ called supplementation on $E$, and a binary operation $\hat{+}$ called the MV-sum on $E$ that satisfies the following axioms for all $p, q, r \in E$:

\begin{align*}
(i) \quad & p \hat{+} (q \hat{+} r) = (p \hat{+} q) \hat{+} r & (ii) \quad & p \hat{+} q = q \hat{+} p & (iii) \quad & p \hat{+} 0 = p \\
(iv) \quad & p \hat{+} u = u & (v) \quad & p^{\perp \perp} = p & (vi) \quad & 0^{\perp} = u \\
(vii) \quad & p \hat{+} p^\perp = u & (viii) \quad & (p \hat{+} p^\perp)^\perp \hat{+} p = (q \hat{+} p^\perp)^\perp \hat{+} q.
\end{align*}

5.2 Definition An MV-effect algebra is a lattice-ordered effect algebra with the Riesz-decomposition property.
According to the following theorem, originally proved by Chovanec and Kôpka [7] and here translated into the language of effect algebras, MV-algebras and MV-effect algebras are mathematically equivalent notions.

5.3 Theorem An MV-algebra \((E, 0, u, \perp, \hat{+})\) forms an MV-effect algebra \((E, 0, u, \oplus)\) where, for \(p, q \in E\), \(p \oplus q := p \hat{+} q\) iff \(p \leq q\). Moreover, for \(p, q \in E\), \(p \oplus q = (p \hat{+} q)\) if and only if \(p \hat{+} q \leq \perp\). Conversely, an MV-effect algebra \((E, 0, u, \oplus)\) forms an MV-algebra \((E, 0, u, \perp, \hat{+})\) where \(p \mapsto p \hat{+}\) is the effect-algebra supplementation map and, for \(p, q \in E\), \(p \hat{+} q := p \oplus (p \hat{+} q)\).

5.4 Corollary An MV-effect algebra is a distributive effect algebra.

5.5 Theorem If \(E\) is a lattice-ordered effect algebra, then \(E\) is an MV-effect algebra iff, for all \(p, q \in E\), \(p \wedge q = 0 \Rightarrow p \perp q\).

Proof See [1, Theorem 3.11]. □

Clearly, every Boolean effect algebra is an MV-effect algebra. Furthermore, every MV-effect algebra is an extension of a Boolean subeffect algebra, namely its center.

5.6 Theorem Let \(E\) be an MV-effect algebra. Then

\[
C(E) = \{c \in E \mid c \wedge c^\perp = 0\} = \{c \in E \mid c \hat{+} c = c\}.
\]

Proof See [12, Theorem 6.1]. □

6. Heyting and Heyting Effect Algebras

6.1 Definition A Heyting algebra is a system \((H, \leq, 0, 1, \wedge, \vee, \supset)\) such that \((H, \leq, 0, 1, \wedge, \vee)\) is a bounded lattice and \(\supset\) is a binary operation on \(H\), called the Heyting conditional, such that for all \(p, q, r \in H\), \(p \wedge q \leq r \Leftrightarrow p \leq (q \supset r)\). If \(H\) is a Heyting algebra and \(p \in H\), then \(p^\prime := (p \supset 0)\) is called the Heyting negation of \(p\). A Heyting algebra \(H\) is called a Stone-Heyting algebra iff, for all \(p \in H\), \(p^\prime \vee (p^\prime)^\prime = 1\).

Every Boolean algebra is a Stone-Heyting algebra with the material conditional \(p \supset q := p^\perp \vee q\) as the Heyting conditional and \(p^\prime = p^\perp\) as the Heyting negation. If a Heyting algebra \(H\) is a model for the semantics of an intuitionistic logic, and if \(p, q \in H\), then \(p \supset q\) is supposed to be a proposition in \(H\) asserting that \(p\) implies \(q\). In this regard, we note that \((p \supset q) = 1 \Leftrightarrow p \leq q\).
Let \( H \) be a Heyting algebra. Then \((H, \leq, 0, 1, \land, \lor)\) is a bounded distributive lattice. Also, the Heyting negation mapping \( \prime : H \to H \) satisfies \( p \land q = 0 \iff q \leq p' \) for all \( p, q \in H \). In particular, \( p \land p' = 0 \), so the Heyting negation satisfies Duns Scotus’ law. However, tertium non datur does not necessarily hold, i.e., \( p \lor p' = 1 \) may fail. Also, although \( p \leq p'' := (p')' \) always holds, the condition \( p'' \leq p \) may fail. In fact, the set \( \{ c \in H \mid c = c'' \} \), which is the same as the set \( H' := \{ p' \mid p \in H \} \), forms a Boolean algebra under the restriction of the partial order on \( H \). If \( p, q \in H' \), then \( p \land q \in H' \) is the infimum of \( p \) and \( q \) in \( H' \); however, unless \( H \) is a Stone-Heyting algebra, \( p \lor q \) need not belong to \( H' \), and the supremum of \( p \) and \( q \) in \( H' \) is \( (p \lor q)'' = (p' \land q')' \). The restriction to \( H' \) of the Heyting negation is the Boolean complementation on \( H' \). We call the Boolean algebra \( H' \) the Heyting center of \( H \).

6.2 Definition A Heyting effect algebra is a lattice ordered effect algebra \( E \) equipped with a binary operation \( \supset \) such that \((E, \leq, 0, u, \land, \lor, \supset)\) is a Heyting algebra.

Although Boolean algebras are coextensive with Boolean effect algebras and MV-algebras are coextensive with MV-effect algebras, there are Heyting algebras, and even Stone-Heyting algebras, that cannot be organized into Heyting effect algebras.

6.3 Theorem Let \( E \) be a Heyting effect algebra and let \( e, f \in E \). Then:

(i) \( e' = (e \supset 0) \in C(E) \).

(ii) \( e' \leq e^\perp \) with equality iff \( e \in C(E) \).

(iii) The Heyting center of \( E \) coincides with the effect-algebra center \( C(E) \).

(iv) \( e \land f = 0 \Rightarrow e \perp f \).

(v) \( E \) is an MV-effect algebra.

(vi) \( E \) is a Stone-Heyting algebra.

Proof Part (i) follows from [34, Theorem 3.31], and in view of (i), parts (ii)–(vi) follow from [12, Theorems 8.3 and 8.5].

Since every Heyting effect algebra is an MV-effect algebra, we often refer to a Heyting effect algebra as an HMV-effect algebra, or simply as an HMV-algebra. Theorem 6.3 (vi) implies that, unlike Heyting algebras in general, the Heyting center of an HMV-algebra is closed under the formation of suprema. By [34, Corollary 3.34], every MV-algebra that is complete as a lattice is an HMV-algebra. By [12, Theorem 8.3], a Heyting effect algebra is the same thing as a lattice-ordered effect algebra \( E \) equipped with a mapping \( \prime : E \to C(E) \) such that, for all \( e, f \in E \), \( e \land f = 0 \iff e \leq f' \).
7. Interval Effect Algebras and Unigroups

An (additively-written) abelian group $G$ is called a partially ordered abelian group iff it is equipped with a partial order $\leq$ that is translation invariant in the sense that, for $g, h, k \in G$, $g \leq h \Rightarrow g + k \leq h + k$. If $G$ is a partially ordered abelian group, then the subset $G^+ := \{ g \in G \mid 0 \leq g \}$ is called the positive cone in $G$ (in spite of the fact that $0 \in G^+$). The positive cone $G^+$ satisfies the conditions (i) $0 \in G^+$, (ii) $g, h \in G^+ \Rightarrow g + h \in G^+$, and (iii) $g, -g \in G^+ \Rightarrow g = 0$. Conversely, if $G$ is an abelian group and $G^+ \subseteq G$ is a subset of $G$ satisfying conditions (i), (ii), and (iii), there is one and only one translation-invariant partial order $\leq$ on $G$ for which $G^+$ is the corresponding positive cone, and it is determined by $g \leq h \Leftrightarrow h - g \in G^+$ for all $g, h \in G$. A partially-ordered abelian group $G$ is said to be archimedean iff, for all $g, h \in G$, the condition $ng \leq h$ for all positive integers $n$ implies that $-g \in G^+$.

If $G$ is a partially ordered abelian group and $u \in G^+$, define the $u$-interval $G^+[0,u] := \{ e \in G \mid 0 \leq e \leq u \}$. Such a $u$-interval can be organized into an effect algebra $(G^+[0,u],0,u,\oplus)$ by defining $p \oplus q := p + q$ iff $p + q \leq u$, for all $p, q \in G^+[0,u]$. As such, the effect-algebra partial order is the restriction to $G^+[0,u]$ of the partial order on $G$, and for $p \in G^+[0,u]$, $p^\perp = u - p$.

A morphism from an effect algebra $E$ with unit $u$ into an effect algebra $F$ with unit $v$ is a mapping $\phi : E \to F$ such that: (i) $p, q \in E$ with $p \perp q$ implies that $\phi(p) \perp \phi(q)$ and $\phi(p \oplus q) = \phi(p) \oplus \phi(q)$, and (ii) $\phi(u) = v$. An isomorphism is a bijective morphism $\phi : E \to F$ such that $\phi^{-1} : F \to E$ is also a morphism. If there is an isomorphism $\phi : E \to F$, we say that $E$ and $F$ are isomorphic.

7.1 Definition An effect algebra $E$ is called an interval effect algebra (IEA) iff it can be realized as, or is isomorphic to, a $u$-interval $G^+[0,u]$ in a partially ordered abelian group $G$ with $u \in G^+$.

By Example 1.1 and the Stone representation theorem, every Boolean algebra is an IEA. By Mundici’s theorem, every MV-algebra is an IEA. Because a Heyting effect algebra is an MV-algebra, every Heyting effect algebra is an IEA. K. Ravindran [33] has generalized Mundici’s theorem by proving that every effect algebra with the Riesz-decomposition property is an IEA. If $\mathcal{H}$ is a Hilbert space and $\mathcal{G}(\mathcal{H})$, partially ordered in the usual way, is the additive group of bounded self-adjoint operators on $\mathcal{H}$, then by definition $\mathcal{E}(\mathcal{H}) = \mathcal{G}(\mathcal{H})^+[0,1]$, so the quantum logic $\mathcal{E}(\mathcal{H})$ is an interval effect
algebra.

Currently, it is not known how to give an intrinsic characterization of an IEA. However, by [2, Corollary 2.5], a subeffect algebra of an IEA is again an IEA, and by [2, Theorem 5.4], an effect algebra with an order-determining set of probability measures is in IEA. As a partial converse, it turns out that every IEA admits at least one probability measure [2, Theorem 5.5].

Let $G$ be a partially-ordered abelian group. If $G^+$ generates $G$ as a group, then $G$ is said to be directed. It is easy to see that $G$ is directed iff $G = G^+ - G^+$, i.e., iff every element $g \in G$ can be written as $g = g_1 - g_2$ with $g_1, g_2 \in G^+$. If, as a partially ordered set, $G$ forms a lattice, then $G$ is called a lattice-ordered abelian group. We say that $G$ has the interpolation property iff, given $a, b, c, d \in G$ with $a \leq c$, $a \leq d$, $b \leq c$, and $b \leq d$, there exists $t \in G$ such that $a \leq t$, $b \leq t$, $t \leq c$, and $t \leq d$ [20]. A partially-ordered abelian group with the interpolation property is called an interpolation group. If $G$ is lattice ordered, it is an interpolation group [20, p. 23].

Suppose that $G$ is a lattice-ordered abelian group and that $g, h, k \in G$. Then $(G, \land, \lor)$ is a distributive lattice, $-(g \land h) = (-g) \lor (-h)$, $-(g \lor h) = (-g) \land (-h)$, $(g \lor h) + k = (g + k) \land (h + k)$, $(g \land h) + k = (g + k) \lor (h + k)$, and $g + h = (g \lor h) + (g \land h)$ [20, Chapter 1]. Define $g^+ := g \lor 0 = g - (g \land 0)$ and $g^- := (-g)^+ = (-g) \lor 0 = -(g \land 0)$. Then $0 \leq g^+, g^-$ and $g = g^+ - g^-$. Hence $G$ is directed. Furthermore, $g^+ \land g^- = (g + g^-) \land g^- = (g \land 0) + g^- = (g \land 0) - (g \land 0) = 0$.

Let $u \in G^+$. We say that $u$ is an order unit for $G$ iff, for every $g \in G$, there is a positive integer $n$ such that $g \leq nu$. If every $g \in G^+$ can be written as a finite linear combination with positive integer coefficients of elements in the $u$-interval $G^+[0, u]$, i.e., if $G^+[0, u]$ generates $G^+$ as a semigroup, then $u$ is said to be generative [2, Definition 3.2]. If $G$ admits an order unit, then $G$ is directed [20, p. 4]. If $u$ is generative and $G$ is directed, then $u$ is an order unit for $G$ [2, Lemma 3.1]. As a consequence of [20, Proposition 2.2 (b)], if $G$ is an interpolation group, and $u$ is an order unit for $G$, then $u$ is generative.

**Definition 7.2** A unital group is a partially-ordered abelian group $G$ with a distinguished generative order unit $u \in G^+$, called the unit. If $G$ is a unital group with unit $u$, then the $u$-interval $E := G^+[0, u]$, regarded as an IIA, is called the unit interval in $G$. A unital homomorphism from a unital group $G$ with unit $u$ into a unital group $H$ with unit $v$ is a group homomorphism $\phi: G \to H$ such that $\phi(G^+) \subseteq H^+$ and $\phi(u) = v$. A unital isomorphism from $G$ onto $H$ is a bijective unital homomorphism $\phi: G \to H$ such that
\( \phi^{-1} : H \to G \) is also a unital homomorphism. Two unital groups \( G \) and \( H \) are \textit{isomorphic as unital groups} iff there is a unital isomorphism \( \phi : G \to H \).

Let \( E \) be the unit interval in a unital group \( G \). Then \( E \) generates \( G^+ \) as a semigroup, and (as \( G \) is necessarily directed) \( G^+ \) generates \( G \) as a group; hence, \( E \) generates \( G \) as a group. Therefore, if \( K \) is an abelian group, \( \Phi : G \to K \) is a group homomorphism, and \( \phi := \Phi|_E \) is the restriction of \( \Phi \) to \( E \), then \( \Phi \) is uniquely determined by the \( K \)-valued measure \( \phi : E \to K \).

7.3 Definition Let \( G \) be a unital group with unit interval \( E \). If \( K \) is an abelian group, we say that \( G \) is \textit{\( K \)-universal} iff every \( K \)-valued measure \( \phi : E \to K \) can be extended to a (necessarily unique) group homomorphism \( \Phi : G \to K \). If \( G \) is \( K \)-universal for every abelian group \( K \), then \( G \) is called a \textit{unigroup} \([19]\).

If \( G \) is an interpolation group with an order unit \( u \), then \( G \) is a unigroup with unit \( u \), and the unit interval \( E \) in \( G \) has the Riesz-decomposition property \([33]\). In particular, if \( G \) is a lattice-ordered abelian group with order unit \( u \), then \( G \) is a unigroup with unit \( u \), and the unit interval \( E \) in \( G \) is an MV-algebra. If \( V \) is a partially ordered vector space over any subfield of the real numbers and \( u \) is an order unit in \( V \), then, regarded as a partially-ordered additive abelian group, \( V \) is a unigroup with unit \( u \) \([2\,\text{Corollary 4.6}]\). In particular, with the identity operator \( 1 \) as order unit, the partially ordered additive group \( \mathbb{G}(\mathcal{H}) \) of bounded self-adjoint operators on a Hilbert space \( \mathcal{H} \) is a unigroup.

If an effect algebra \( E \) can be realized as (or is isomorphic to) the unit interval in a unigroup \( G \), then by definition \( E \) is an IEA. Conversely, by \([2\,\text{Corollary 4.2}]\) or by \([18\,\text{Theorem 5.4 and ff.}]\), \textit{every IEA \( E \) can be realized as the unit interval in a unigroup \( G \).} Furthermore, \( G \) is uniquely determined by \( E \) up to a unital isomorphism, hence (by a slight abuse of language) we shall refer to \( G \) as \textit{the unigroup for} \( E \). Thus, with \( E \) as the unit interval in \( G \), and \( G \) as the unigroup for \( E \), \textit{we have a correspondence} \( E \leftrightarrow G \) (up to isomorphism) \textit{between IEA’s} \( E \) \textit{and unigroups} \( G \). More formally, there is a categorical equivalence between the category of interval effect algebras and the category of unigroups \([31\,\text{Theorem 3}]\).

If \( G \neq \{0\} \) is a partially-ordered abelian group with order unit \( u \), then a \textit{state} on \( G \) is defined to be a homomorphism \( \omega : G \to \mathbb{R} \) from \( G \) to the additive group of real numbers such that \( \omega(G^+) \subseteq \mathbb{R}^+ \) and \( \omega(u) = 1 \) \([20\,\text{Chapter 4}]\). Denote by \( \Omega(G) \) the set of all states on \( G \). Then \( \Omega(G) \) is a subset of the locally convex linear topological space \( \mathbb{R}^G \) of all functions from
$G$ to $\mathbb{R}$ with pointwise operations and the topology of pointwise convergence. As such, $\Omega(G)$ is nonempty \cite{20, Corollary 4.4} and it is a compact convex subset of $\mathbb{R}^G$ \cite{20, Proposition 6.2}. Therefore, by the Krein-Milman theorem, $\Omega(G)$ is the closed convex hull of its own set $\partial \Omega(G)$ of extreme points. By \cite{20, Theorem 4.14}, $G$ is archimedean iff $\Omega(G)$ determines $G^+$ in the sense that $G^+ = \{ g \in G \mid 0 \leq \omega(g) \text{ for all } \omega \in \Omega(G) \}$.

Suppose that $E$ is the unit interval in a unigroup $G \neq \{0\}$. If $\omega \in \Omega(G)$, then the restriction $\pi := \omega|_E$ of $\omega$ to $E$ is a probability measure on $E$. Conversely, if $\pi \in \Pi(E)$, then $\pi : E \to \mathbb{R}$ is an $\mathbb{R}$-valued measure, hence it admits a unique extension to a homomorphism $\omega : G \to \mathbb{R}$ into the additive group of real numbers. Moreover, as $\pi(E) \subseteq \mathbb{R}^+$ and $E$ generates $G^+$, it follows that $\omega \in \Omega(G)$. Thus, we have an affine isomorphism $\pi \leftrightarrow \omega$ with $\pi = \omega|_E$ between the space $\Pi(E)$ of probability measures on $E$ and the state space $\Omega(G)$ of $G$. As a consequence, if $G$ is archimedean, then $\Pi(E)$ is an order-determining set of probability measures on $E$.

Let $E$ be an IEA and let $G$ be the unigroup with unit $u$ for $E$. By \cite{33}, $E$ has the Riesz-decomposition property iff $G$, the Ravindran group of $E$, is an interpolation group. By \cite{23}, $E$ is an MV-effect algebra iff $G$, the Mundici group of $E$, is lattice ordered. If $E$ is totally ordered, then so is its Mundici group $G$ \cite{2, Corollary 6.5}. By definition, $G$ is a Boolean unigroup iff $E$ is a Boolean effect algebra. By \cite{34, Theorem 4.26}, $G$ is a Boolean unigroup iff it is an interpolation group and $u$ is the smallest order unit in $G$. In Theorem 8.7 below, we characterize $G$ for the case in which $E$ is an HMV-effect algebra. In this connection, the following theorem of N. Ritter \cite{34, Theorem 4.21} is of interest (see Example 8.8 below).

**7.4 Theorem** Let $A$ be a lattice-ordered group with order unit $u$ and suppose that the $u$-interval in $A$ is an HMV-effect algebra. Then, if $0 \neq v \in A^+$, the $v$-interval in $A$ is also an HMV-effect algebra.

**8. Compressions**

Let $\mathcal{B}$ be a field of subsets of the nonempty set $X$ and let $\mathcal{F}(\mathcal{B}, \mathbb{Z})$ be the commutative lattice-ordered ring with unit defined in Example 1.1. Then, regarded as an additive lattice-ordered group with order unit 1, $G := \mathcal{F}(\mathcal{B}, \mathbb{Z})$ is a Boolean unigroup and $E := G^+[0,1]$ is a Boolean effect algebra. The Boolean sum $x \oplus y$ on $E$ extends to the addition operation $g + h$ on $G$. There are two natural options for extending the Boolean product $xy = x \wedge y$.
on $E$ to $G$, namely (1) to the product operation $gh$, or (2) to the infimum operation $g \land h$, for $g, h \in G$. Option (1), which Boole might have favored, can be generalized, but only to unigroups admitting a reasonable notion of a product (see [14]). Option (2), which Jevons might have preferred, can be generalized, but only to MV-algebras and their lattice-ordered unigroups.

With the notation of the last paragraph, the key to a more general solution of the extension problem for Boolean-type products is as follows: Rather than looking at the product $xy$ as a binary operation, we fix $x \in E$ and consider the unary operation $y \mapsto xy$ for all $y \in E$. This unary operation has a natural extension to a unary operation $J_x : G \to G$ defined by $J_x(g) := xg$ for all $g \in G$. (One could imagine that Boole would have been comfortable with $J_x$ because of his work with differential operators.) Evidently, $J_x$ is a "retraction" on $G$ with "focus" $x$ as per the following definition.

8.1 Definition Let $G$ be a unital group with unit $u$ and unit interval $E$. A mapping $J : G \to G$ is called a retraction with focus $p$ iff $J$ is an order-preserving endomorphism on $G$ such that $p := J(u) \leq u$ and, for all $e \in E$, $e \leq p \Rightarrow J(e) = e$. A compression on $G$ is a retraction $J$ on $G$ such that, if $p$ is the focus of $J$, $e \in E$, and $J(e) = 0$, then $e \leq p^\perp$. A retraction $J'$ on $G$ is a quasicomplement of the retraction $J$ on $G$ iff, for all $g \in G^+$, $J(g) = 0 \Leftrightarrow J'(g) = g$ and $J'(g) = 0 \Leftrightarrow J(g) = g$.

Every retraction $J$ on a unital group $G$ is idempotent, i.e., $J = J \circ J$ [14 Lemma 2.2]. If a retraction $J$ has a quasicomplement $J'$, then both $J$ and $J'$ are compressions [14 Lemma 3.2].

8.2 Definition A compressible group is a unital group $G$ such that every retraction on $G$ is determined by its focus and every retraction on $G$ has a quasicomplementary retraction on $G$. Let $G$ be a compressible group with unit interval $E$. An element $p \in E$ is called a projection iff it is the focus of a retraction (hence a compression) on $G$. The set of all projections in $E$ is denoted by $P(G)$, and if $p \in P(G)$, the unique compression on $G$ with focus $p$ is denoted by $J_p$.

See [13] [14] [15] for the basic theory of compressible groups. Let $G$ be a compressible group with unit interval $E$. Then $P(G)$ is a subeffect algebra of $E$ and, as such, $P(G)$ is an orthomodular poset. If $p, q \in P(G)$, then $p \ll q \iff J_p \circ J_q = J_q \circ J_p$. In fact, if $p \ll q$, then $p \land q$ exists in $E$, $p \land q \in P(G)$, and $J_p \circ J_q = J_q \circ J_p = J_{p \land q}$. Furthermore, the quasicomplement of $J_p$ is $J_{p^\perp}$.

If $E$ is an IEA and $G$ is the unigroup for $E$, then $G$ is a compressible group
iff $E$ is a compressible effect algebra in the sense of Gudder [24]. The notion of a compressible group enables a multiplicative characterization of a Boolean effect algebra. Indeed, if $E$ is the unit interval in a unital group $G$, then $E$ is a Boolean effect algebra iff $G$ is a compressible group with $E = P(G)$ [13, Theorem 5.5]. The following theorem applies to the Ravindran group $G$ of an effect algebra $E$ with the Riesz-decomposition property.

8.3 Theorem Let $G$ be an interpolation group with order unit $u$. Then (i) $G$ is a compressible unigroup with unit $u$ and unit interval $E := G^+[0,u]$, and $P(G) = \{ p \in E \mid p \wedge p^\perp = 0 \} = C(E)$ is a Boolean effect algebra. Let $p \in P(E) = C(E)$, $H := J_p(G)$, and $K := J_{p^+}(G)$. Then: (ii) $e \in E \Rightarrow J_p(e) = p \wedge e$. (iii) $g \in G \Rightarrow g = J_p(g) + J_{p^+}(g)$. (iv) $H$ and $K$ are subgroups of $G$, $H + K = G$, and $H \cap K = \{0\}$. (v) With the partial order induced from $G$, $H$ and $K$ are interpolation groups with order units $p$ and $p^+$, respectively. (vi) The mappings $J_p : G \to H$ and $J_{p^+} : G \to K$ provide a projective representation of $G$ as a direct product of $H$ and $K$ in the category of unigroups. (vii) If $G$ is lattice ordered, then $H$ and $K$ are sublattices of $G$, and for all $g, h \in G$, $J_p(g \vee h) = J_p(g) \vee J_p(h)$, and $J_p(g \wedge h) = J_p(g) \wedge J_p(h)$.

Proof For (i)–(vi), see [13, Theorem 3.5] and [20, pp.127–131]. Part (vii) follows from (vi). □

The compression operators on a compressible group provide a generalization—in the spirit of Boole—of the multiplicative structure of an MV-algebra. This generalization is applicable not only to MV-algebras, but also to a large class of quantum logics as per the following example.

Example 8.4 Let $\mathcal{A}$ be a unital $C^*$-algebra with unity 1. Then the additive group $G(\mathcal{A})$ of self-adjoint elements in $\mathcal{A}$, partially ordered with the positive cone $G(\mathcal{A})^+ = \{ aa^* \mid a \in \mathcal{A} \}$, is an archimedean compressible unigroup [13, Corollary 4.6]. The projections in $P(G(\mathcal{A}))$ are the idempotent elements $p = p^2 = p^* \in G(\mathcal{A})$, and for $a \in G(\mathcal{A})$, $J_p(a) = pap$. □

8.5 Definition Let $G$ be a compressible group, let $g \in G$, and $p \in P(G)$. Then (i) $C(p) := \{ g \in G \mid g = J_p(g) + J_{u-p}(g) \}$ and (ii) $CPC(g) := \bigcap \{ C(p) \mid p \in P(G) \text{ and } g \in C(p) \}$. Elements $g \in C(p)$ are said to be compatible with the projection $p$.

It is not difficult to verify that the notion of compatibility in Definition 8.5 (i) is consistent with the notion of compatibility in Definition 3.10, i.e., if $E$ is the unit interval in the compressible group $G$, then for $e \in E$ and
$p \in P(G)$, $e \in C(p) \Leftrightarrow eCp$. If $g \in G$, then $CPC(g)$ in Definition 8.5 (ii) is the set of all elements $h \in G$ that are compatible with every projection with which $g$ is compatible. By Theorem 8.3, if $G$ is an interpolation unigroup, then $G = C(p)$ for every $p \in P(G)$, hence $G = CPC(g)$ for every $g \in G$.

In Example 8.4, if $g \in G(A)$ and $p \in P(G(A))$, then $g \in C(p)$ iff $g_{pp} = pg$, i.e., iff $g$ commutes with $p$ in the $C^*$-algebra $A$. Hence, if $A$ is a von Neumann algebra and $g, h \in G(A)$, then $h \in CPC(g)$ iff $h$ “double commutes” with $g$, i.e., $h$ commutes with every element $a \in A$ that commutes with $g$.

### 8.6 Definition
Let $G$ be a compressible group.

(i) $G$ has the Rickart projection property iff there exists a mapping $\prime: G \to P(G)$, called the Rickart mapping, such that, for all $g \in G$ and all $p \in P(G)$, $p \leq g' \iff g \in C(p)$ with $J_p(g) = 0$.

(ii) $G$ has the general comparability (GC) property iff, for every $g \in G$, there exists $p \in P(G)$ such that $p \in CPC(g)$ and $J_{p^*}(g) \leq 0 \leq J_p(g)$.

(iii) An $RGC$-group (also called a Rickart comgroup [16]) is a compressible group with both the Rickart projection and general comparability properties.

With $1$ as the order unit, the group $G(A)$ of self-adjoint elements in a von Neumann algebra $A$ is an archimedean RGC-unigroup. In an RGC-group $G$, every element has a rational spectral resolution, which, if $G$ is archimedean, has the basic properties of the spectral resolution of a bounded self-adjoint operator on a Hilbert space [16].

Let $G$ be an interpolation group with order unit $u$. Then the conditions $g \in C(p)$ in Definition 8.6 (i) and $p \in CPC(g)$ in Definition 8.6 (ii) hold automatically, and $G$ satisfies the general comparability property iff it satisfies the condition with the same name in [20, Chapter 8]. Thus, by [20, Proposition 8.9], if $G$ satisfies the GC-property, then it is lattice ordered, and by [20, Proposition 9.9], if $G$ is lattice-ordered and Dedekind $\sigma$-complete, then $G$ has the GC-property. As a consequence of the following theorem, if $G$ is the Mundici group of an $MV$-algebra $E$, then $E$ is an HMV-algebra iff $G$ is an RGC-group.

### 8.7 Theorem
Let $G$ be a unital group with order unit $u$ and unit interval $E$. Then the following conditions are mutually equivalent: (i) $G$ is a unigroup
and \( E \) is an HMV-algebra. (ii) \( G \) is lattice ordered, has the Rickart projection property, and the Rickart mapping \( g \mapsto g' \) satisfies \( e \land f = 0 \Rightarrow e \leq f' \) for all \( e, f \in E \). (iii) \( G \) is an RGC-group and \( P(G) \subseteq C(E) \).

Proof (i) \( \Rightarrow \) (ii). Assume (i) and let \( ' : E \to E \) be the Heyting negation connective. As \( G \) is a unigroup, it is the Mundici group of the MV-algebra \( E \), hence \( G \) is lattice ordered, so it is an interpolation group. By Theorem 8.3, \( G \) is a compressible group and \( P(G) = C(E) \) is a Boolean algebra. Let \( g \in G^+ \). Then there are elements \( e_1, e_2, \ldots, e_n \in E \) such that \( g = \sum_{i=1}^{n} e_i \). Let \( q := (e_1)' \land (e_2)' \land \cdots \land (e_n)' \in P(G) = C(E) \) and let \( p \in P(G) \). Then \( J_p(g) = 0 \iff \sum_{i=1}^{n} J_p(e_i) = 0 \), and since \( 0 \leq J_p(e_i) \) for \( i = 1, 2, \ldots, n \), it follows that \( J_p(g) = 0 \iff J_p(e_i) = 0 \) for \( i = 1, 2, \ldots, n \). But, by Theorem 8.3, \( J_p(e_i) = 0 \iff p \land e_i = 0 \iff p \leq (e_i)' \) for \( i = 1, 2, \ldots, n \), and it follows that \( J_p(g) = 0 \iff p \leq q \). Thus, \( q \) depends only on \( g \) and is independent of the choice of \( e_1, e_2, \ldots, e_n \in E \), hence we can and do extend the Heyting negation \( ' \) from \( E \) to \( G^+ \) by defining \( g' := q \in P(G) \).

If \( g \in G \), then \( g \in G^+ \iff g = g^+ \) and \( g^- = 0 \). Therefore, we can and do extend \( ' \) from \( G^+ \) to \( G \) by defining \( g' := (g^+)' \land (g^-)' \) for all \( g \in G \). Let \( g \in G \) and let \( p \in P(G) = C(E) \). By Theorem 8.3 (vii), \( J_p(g^+) = J_p(g \lor 0) = J_p(g) \lor J_p(0) = J_p(g) \lor 0 = J_p(g)^+ \). Thus, \( J_p(g^-) = J_p((-g)^+) = (J_p(-g))^+ = (-J_p(g))^+ = (J_p(g))^- \), and it follows that \( J_p(g) = 0 \Rightarrow J_p(g^+) = J_p(g^-) = 0 \Rightarrow J_p(g) = J_p(g^+-g^-) = J_p(g^+)-J_p(g^-) = 0 \). Therefore, \( G \) has the Rickart projection property, and, since the Rickart mapping \( ' \) is an extension of the Heyting negation connective on \( E \), it satisfies \( e \land f = 0 \Rightarrow e \leq f' \) for all \( e, f \in E \).

(ii) \( \Rightarrow \) (iii). Assume (ii) and let \( ' : G \to P(G) \) be the Rickart mapping on \( G \). As \( G \) is lattice ordered, it has the interpolation property and is a compressible group with \( P(G) = C(E) \). By [15] Lemma 6.2 (iii), \( p' = p^+ \) for all \( p \in P(G) \).

Suppose \( a, b \in G^+ \) and \( a \land b = 0 \). Then there are elements \( e_i \in E \) for \( i = 1, 2, \ldots, n \) and \( f_j \in E \) for \( j = 1, 2, \ldots, m \) such that \( a = \sum_{i=1}^{n} e_i \) and \( b = \sum_{j=1}^{m} f_j \), and since \( e_i \land f_j \leq a \land b = 0 \), it follows from (ii) that \( e_i \leq (f_j)' \) for \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, m \). By [15] Lemma 6.2 (viii), it follows that \( (e_i)^n \leq (f_j)' \) for \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, m \). By [15] Theorem 6.4 (iv)], \( a^n = \bigvee_{i=1}^{n} (e_i)^n \) and \( b' = \bigwedge_{j=1}^{m} (f_j)' \), whence \( a^n \leq b' \). Therefore, if \( a, b \in G^+ \), then \( a \land b = 0 \Rightarrow a^n \leq b' \).

Let \( g \in G \) and define \( q := (g^+)' \) and \( p := (g^-)' \). Then, as \( g^+, g^- \in G^+ \)
and \( g^+ \land g^- = 0 \), it follows that \( p^\perp = p' = (g^-)^\prime \leq (g^+)^\prime = q. \) Since 
\( 0 = J_q(g^+) = J_q(g \lor 0) = J_q(g) \land 0 \), it follows that \( J_q(g) \leq 0. \) Likewise, \( J_p(-g) \leq 0 \), i.e., \( 0 \leq J_p(g). \) As \( p^\perp \leq q \), we have \( J_p^+(g) = J_p^+ \cap q(g) = J_p^+(J_q(g)) \leq 0. \) Therefore, \( J_p^+(g) \leq 0 \leq J_p(g) \), so \( G \) has the GC-property, and hence it is an RGC-group.

(iii) \( \Rightarrow \) (i). Assume (iii). Let \( p \in P(G) \subseteq C(E) \). Then, if \( e \in E \), we have \( pCe \), hence \( e \in C(p) \). If \( g \in G \), then \( g \) is finite linear combination with integer coefficients of elements of \( E \), hence \( g \in C(p) \). As \( G \) has the GC-property and \( g \in C(p) \) for all \( g \in G \), \( p \in P(G) \), it follows from \([13] \) Theorem 4.9] that \( G \) is lattice ordered. Therefore \( G \) is a unigroup, \( E \) is an MV-algebra, \( G \) is the Mundici group of \( E \), and \( P(G) = C(E) \).

Suppose \( e, f \in E \) with \( e \land f = 0 \). By the GC-property, there is a projection \( p \in P(G) = C(E) \) such that \( J_p^+(e - f) \leq 0 \leq J_p(e - f) \), i.e., \( p^\perp \land e = J_p^+(e) \leq J_p^+(e - f) = p^\perp \land f \) and \( p \land f = J_p(e) \leq J_p(e - f) = p \land e \). Thus, \( J_p^+(e) = p^\perp \land e = p^\perp \land f \land e = 0 \) and \( J_p(f) = p \land f = p \land f \land e = 0 \), and it follows that \( p' = p^\perp \leq e' \) and \( p \leq f' \). Therefore, \( e \leq e'' \leq p'' = p \leq f' \), and we conclude that, for \( e, f \in E, e \land f = 0 \Rightarrow e \leq f'. \) Conversely, suppose \( e, f \in E \) and \( e \leq f' \). Then, by \([13] \) Lemma 6.2 (vii)], \( e'' \leq e' \), so \( e'' \land f = J_{e''}(f) = 0 \), whence by \([13] \) Lemma 6.2 (vii)], \( 0 \leq e \land f \leq e'' \land f = 0 \). Consequently, the restriction to \( E \) of \( ' \) satisfies the condition \( e \land f = 0 \Rightarrow e \leq f' \) for all \( e, f \in E \). Therefore, by \([19] \) Theorem 6.13], \( E \) is a Heyting algebra under the Heyting conditional \( (e \supset f) := ((e \land f)') \lor f = ((e - f)^+)' \lor f \) for \( e, f \in E \).

The following example, which generalizes Example 1.1, provides a large class of archimedean lattice-ordered RGC-unigroups with HMV-algebras as their unit intervals.

8.8 Example Let \( B \) be a field of subsets of a set \( X \), let \( A \) be a lattice-ordered unigroup such that the unit interval in \( A \) is an HMV-effect algebra (e.g., \( A = \mathbb{Z} \) with unit 1). Define \( F(B, A) \) to be the partially ordered abelian group under pointwise addition and pointwise partial order of all functions \( f: X \rightarrow A \) such that (i) \( f^{-1}(a) \in B \) for all \( a \in A \) and (ii) \( \{f(x) \mid x \in X\} \) is a finite subset of \( A \). Then an element \( u \in F(B, A) \) is an order unit iff \( u(x) \) is an order unit in \( A \) for all \( x \in X \). If \( u \) is an order unit in \( F(B, A) \), then \( F(B, A) \) is a lattice-ordered RGC-unigroup with unit \( u \), hence the \( u \)-interval \( E \) in \( F(B, A) \) is an HMV-algebra. \( \square \)

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