On Memoryless Quantitative Objectives

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Abstract. In two-player games on graph, the players construct an infinite path through the game graph and get a reward computed by a payoff function over infinite paths. Over weighted graphs, the typical and most studied payoff functions compute the limit-average or the discounted sum of the rewards along the path. Besides their simple definition, these two payoff functions enjoy the property that memoryless optimal strategies always exist.

In an attempt to construct other simple payoff functions, we define a class of payoff functions which compute an (infinite) weighted average of the rewards. This new class contains both the limit-average and the discounted sum functions, and we show that they are the only members of this class which induce memoryless optimal strategies, showing that there is essentially no other simple payoff functions.

1 Introduction

Two-player games on graphs have many applications in computer science, such as the synthesis problem \[7\], and the model-checking of open reactive systems \[1\]. Games are also fundamental in logics, topology, and automata theory \[17\,14\,20\]. Games with quantitative objectives have been used to design resource-constrained systems \[27\,9\,3\,4\], and to support quantitative model-checking and robustness \[5\,6\,26\].

In a two-player game on a graph, a token is moved by the players along the edges of the graph. The set of states is partitioned into player-1 states from which player 1 moves the token, and player-2 states from which player 2 moves the token. The interaction of the two players results in a play, an infinite path through the game graph. In qualitative zero-sum games, each play is winning exactly for one of the two players; in quantitative games, a payoff function assigns a value to every play, which is paid by player 2 to player 1. Therefore, player 1 tries to maximize the payoff while player 2 tries to minimize it. Typically, the edges of the graph carry a reward, and the payoff is computed as a function of the infinite sequences of rewards on the play.

Two payoff functions have received most of the attention in literature: the mean-payoff function (for example, see \[11\,27\,15\,19\,12\,21\]) and the discounted-sum function (for example, see \[24\,12\,22\,23\,9\]). The mean-payoff value is the long-run average of

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the rewards. The discounted sum is the infinite sum of the rewards under a discount factor $0 < \lambda < 1$. For an infinite sequence of rewards $w = w_0 w_1 \ldots$, we have:

$$\text{MeanPayoff}(w) = \lim \inf_{n \to \infty} \frac{1}{n} \cdot \sum_{i=0}^{n-1} w_i \quad \text{DiscSum}_\lambda(w) = (1 - \lambda) \cdot \sum_{i=0}^{\infty} \lambda^i \cdot w_i$$

While these payoff functions have a simple, intuitive, and mathematically elegant definition, it is natural to ask why they are playing such a central role in the study of quantitative games. One answer is perhaps that memoryless optimal strategies exist for these objectives. A strategy is memoryless if it is independent of the history of the play and depends only on the current state. Related to this property is the fact that the problem of deciding the winner in such games is in $\text{NP} \cap \text{coNP}$, while no polynomial time algorithm is known for this problem. The situation is similar to the case of parity games in the setting of qualitative games where it was proved that the parity objective is the only prefix-independent objective to admit memoryless winning strategies [8], and the parity condition is known as a canonical way to express $\omega$-regular languages [25].

In this paper, we prove a similar result in the setting of quantitative games. We consider a general class of payoff functions which compute an infinite weighted average of the rewards. The payoff functions are parameterized by an infinite sequence of rational coefficients $\{c_i\}_{n \geq 0}$, and defined as follows:

$$\text{WeightedAvg}(w) = \lim \inf_{n \to \infty} \frac{1}{\sum_{i=0}^{n} c_i} \cdot \sum_{i=0}^{n} c_i \cdot w_i.$$ 

We consider this class of functions for its simple and natural definition, and because it generalizes both mean-payoff and discounted-sum which can be obtained as special cases, namely for $c_i = 1$ for all $i \geq 0$, and $c_i = \lambda^i$ respectively. We study the problem of characterizing which payoff functions in this class admit memoryless optimal strategies for both players. Our results are as follows:

1. If the series $\sum_{i=0}^{\infty} c_i$ converges (and is finite), then the discounted sum is the only payoff function that admits memoryless optimal strategies for both players.
2. If the series $\sum_{i=0}^{\infty} c_i$ does not converge, but the sequence $\{c_n\}_{n \geq 0}$ is bounded, then for memoryless optimal strategies the payoff function is equivalent to the mean-payoff function (equivalent for the optimal value and optimal strategies of both players).

Thus our results show that the discounted sum and mean-payoff functions, besides their elegant and intuitive definition, are the only members from a large class of natural payoff functions such that both players have memoryless optimal strategies. In other words, there is essentially no other simple payoff functions in the class of weighted infinite average payoff functions. This further establishes the canonicity of the mean-payoff and discounted-sum functions, and suggests that they should play a central role in the emerging theory of quantitative automata and languages [10,16,25].

In the study of games on graphs, characterizing the classes of payoff functions that admit memoryless strategies is a research direction that has been investigated in [13].

\footnote{Note that other sequences also define the mean-payoff function, such as $c_i = 1 + 1/2^i$.}