Statistical description of magnetic
domains in the Ising model

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Abstract:
We use the Mandelbrot-Zipfs power law for the description of the inhomogeneity of the spin system. We describe the statistical distributions of the domain’s masses in the Ising model near the phase transition induced by the temperature. The statistical distribution near the critical point appears to be of the Pareto type.

We study in this paper for the description of the phase transition the Ising model [1]. This model is one of the simplest models which can be used to describe the phase transition in ferromagnet. Besides this model is known due to several applications: first of all the percolation [2], then trading activity [3], sociophysics [4] and others. A common feature in these problems is the presence of two choices of the variable \( S = \pm 1 \). However all the results one can generalize to the model containing many values of \( S \) for instance to the Potts model.

The Hamiltonian for the simplest Ising model is:

\[
H = -\frac{1}{2} \sum_{ij} J_{ij} S_i^z S_j^z
\]

with the sum over all neighbour pairs (z-th component) of spins. Usually it is assumed that the crystal lattice of ferromagnet is regular and in each site of a lattice the spin is localized with the value \( S^z = 1 \) or \( S^z = -1 \). Further

\[
J_{ij} = \begin{cases} 
J & \text{if } i, j \text{ are neighbour pairs of spins} \\
0 & \text{in oposite case.}
\end{cases}
\]

Two spins \( i \) and \( j \) interact with each other by an energy \(-J S_i^z S_j^z\) with \(-J\) if both spins are parallel and \(+J\) if they are opposite to each other. The energy needed for flipping of one spin is \(2J\).

For the simulation in this model we will use the Monte Carlo method with the Swendsen-Wang cluster algorithm [5].

In this algorithm clusters of spins are created by introducing bonds between neighboring spins with probability \( P(S_i^z, S_j^z) = 1 - \exp\left(-\frac{\Delta E}{k_B T}\right)\), where \( k_B \) is
Boltzmann constant, $\Delta E$ is the energy difference needed to transform a pair of equal spins to a pair of opposite spins. The probability is zero if spins are the same. All such clusters are generated and then updated by choosing a random new spin value for each cluster and assigning it to all spins in this cluster. The probability is zero if spins are the same. Starting with the simulation having the random distribution of half of the spins up and half down and using Swendsen-Wang algorithm with low temperature one sees growing domains, in which spins are parallel. We have two kind of domains: with spins up and with spins down. At last at the temperature Curie $T = T_c$ ($T_c = \frac{2J}{k_B \ln(1+\sqrt{2})}$) there appears an infinite domain in the limit $L \to \infty$ where $L$ denotes linear size of the system with one of the spin states being chosen.

The phase transition appears in the critical point $T = T_c$. The difference $M$ between a number of spins up and down is proportional to the magnetisation and near critical point vanishes as $(T - T_c)^{\beta}$, where for dimension $d = 2$, $\beta = \frac{1}{8}$. The correlation length $\xi \sim |T-T_c|^{\nu}$. The magnetisation is proportional to $\xi^{-\beta/\nu}$. In a finite system in critical temperature $T_c$ one can replace $\xi$ by $L$, hence $M \sim L^{d-\beta/\nu} = L^{D}$ with the fractal dimension $D = d - \beta/\nu$ ($d \leq 4$).

The simulation data were collected on square lattices of linear size $L = 500$ and 1000. A total of 10000 Monte Carlo (MC) time steps were used for equilibration. The value of MC time steps required for equilibration have been estimated from the energy time series, which is a common practice for cluster algorithms.

The main goal of this paper is the statistical description of the simple magnetic system when we approach the critical point of phase transition induced by the temperature.

For this purpose we will consider Mandelbrot-Zipf’s power law [6]:

$$x = k^{-\frac{1}{\mu}}$$

In our case $x$ is the number of spins up or down in the domain (domain mass), $k$ denotes the rank order of the domain mass $x$. (The greatest cluster has rank 1, smaller rank 2 and so on.)
Bouchaud [5] pointed out the strong correlation between Mandelbrot-Zipf’s power law and the inhomogeneity of the system: the slope (in log-log) of the straight line is determined by $-\frac{1}{\mu}$ and characterize the inhomogeneity of the physical structure of the system ($-\frac{1}{\mu} = \tan \alpha$, where $\alpha$ the angle of the slope). The inhomogeneity of the system means that its structures become fractal and more hierarchical.

In our considerations we shall concentrate on the sequence of random variables $x$ called here the $[\mu]$-variables. These variables are distributed according to the distribution of the appearance of a cluster with mass $x$, and probability $\rho(x)$ which decays as $\frac{x_0}{x^{1+\mu}}$, where $x_0$ is the typical scale. The index $\mu$ appearing in the tail of distribution $\rho(x)$ is a critical exponent. The main property of $[\mu]$ variable is that all its moments $m_q = \langle x^q \rangle$ with $q \geq \mu$ are infinite.

We are going to connect the statistics of domain masses with the process of approaching the critical point. When we start to advance from the paramagnetic phase to the critical point ($T \to T_c$) as we see on Fig (1)

Insert Fig (1)

Fig 1. Log-log distribution of the domain’s masses $x$ versus the rank order index $k$ ($L = 1000$).

\begin{align*}
1/T = 0.25: & \quad \ln(\ln(k)) = 4,01287428 - 0,148950538*\ln(k) \\
1/T = 0.35: & \quad \ln(\ln(k)) = 5,55609713 - 0,21268554*\ln(k) \\
1/T = 0.4: & \quad \ln(\ln(k)) = 6,89830264 - 0,261512561*\ln(k) \\
1/T = 0.44068: & \quad \ln(\ln(k)) = 10,4914484 - 1,014818*\ln(k)
\end{align*}

The angle between straight lines representing (in log-log) the Mandelbrot-Zipf’s inverse power law and the rank axis increase when $\mu > 1$ and $T \to T_c$.

For that case we observe the growing domains, their structure become more fractal (loss of an oval) and more hierarchical - the inhomogeneity of the system increase. The distribution of magnetisation of a whole system has the usual Gauss form. At high temperature correlations between spins are short
ranged in the whole high temperature region \( M \sim \sqrt{L^d} = L \) for \( d = 2 \), see Fig. 2.

Insert Fig (2)

Fig 2. Test \( M \sim L \) for \( T = 5T_c \).

At last at \( T = T_c \) and \( \mu \approx 1 \), when one domain covers the whole lattice and the other spin orientation is restricted to the small clusters or isolated single spins within the domain. In this case the distribution of the total magnetisation is centered at \( mL^d \) or \(-mL^d\), where \( m \) is the remnant magnetisation and it also has Gaussian form, as it is well known.

In the case when \( \mu \approx 1 \) (\( T = T_c \)) from fig (1) we see the straight line representing (when \( \alpha = \alpha_c \), \( \tan \alpha_c = \frac{1}{\mu_c} \approx 1 \)) the Mandelbrot law describing the phase transition. The highest point denotes in the critical temperature the domain which covers the whole lattice.

Fig (3) represents the histogram of the domain masses at critical point – the probability of the appearance of the cluster with mass \( x \).

Insert Fig (3)

Fig 3. Histogram of the domain’s masses at critical point \( T = T_c \) \((L = 500)\)

The distribution \( \rho(x) \) of domain mass is like has Pareto tail \( \frac{x_0^\mu}{x^{\mu+1}} \), where \( x_0 \) denotes a typical scale, with \( \mu \approx 1 \).

When \( \mu > 1 \) distribution \( \rho(x) \) is without power-law tail, which becomes truncated, see Fig 4.

Insert Fig (4)

Fig 4. The histogram of the domain’s masses \((\beta = \frac{1}{kT} = 0, 3; \ L = 500)\)

This results are in the agreement with standard percolation theory [2] and the paper of Janke and Schakel [6] because the distribution of domains with the mass \( x \) takes a general form

\[ \rho_x \sim x^{-\tau} \exp(-\Theta x) \]
where $\tau$ is the entropy factor ($\tau = \frac{d}{D} + 1$), $\Theta \sim (T - T_c)^{\frac{1}{2}}$, if $T \rightarrow T_c$. When $T = T_c$, we get $\Theta = 0$ and we have $\rho_x \sim x^{-\tau}$.

The statistical description of discontinuous metal films on dielectric substrates were analysed on the basis of experiments by Dobierzewska-Mozrzymas at all [8] and the distributions of local fields intensities in metal dielectric system was investigated by Liberman et al. [9]. Unfortunately the analogous experiments on ferromagnetics are not known.

The main result of our paper is to show the connection between the Mandelbrot-Zipf’s law and the statistics of the domain masses in the Ising model. The domain masses in the Ising model fulfil the Mandelbrot-Zipf’s inverse power law and when we approach the phase transition in this model the distribution of the domain masses appears to have the Pareto tail. This model is a such one which represents the system in which in the critical point the length scale diverges and leaves the system in self similar state. That feature denotes a finely-tuned criticality which should be contrasted with self-organized criticality. In such a case the system spontaneously evolves towards scale-invariant states and one raise the problem raises problem of the universality of the renormalized coupling constant at critical point. This conclusion is in agreement with the paper of Hilfer [10].

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References

[1] E. Ising, Physik 31, 253 (1925).

[2] D. Stauffer, Introduction to percolation theory Taylor and Francis, London and Philadelphia, 1985.

[3] R. Cont, J.P. Bouchand (2000), Macroeconomics Dynamics 4: 170.
[4] K. Sznajd-Weron and J. Sznajd (2000), Int. Mod. Phys. C11:1157.

[5] R. Swendsen and J. Wang, Phys. Rev. Leff. 58, 86 (1987).

[6] J.P. Bouchand, Proc. Int. Workshop on Lévy Flights and Related Topics in Physics (Nice, France, 27–30 June 1994) ed. by M.F. Shlesinger, G.M. Zaslavsky, V. Frisch, Berlin, Springer, 1995.

[7] W. Janke and A.M.J. Schakel, Phys. Rev. E 71 036703 (2005).

[8] E. Dobierzewska-Mozrzymas, P. Biegański, E. Pieciul and J. Wójcik, J. Phys. Condens. Matter 11 (1999) 5561.

[9] S. Liberman, F. Brouers, P. Gadenne, Physica B 279 (2000) 56.

[10] R. Hilfer and Z. Phys. B 96, (1994) 63–77.
