Charged Anti-de Sitter BTZ black holes in Maxwell-$f(T)$ gravity

G. G. L. Nashed$^{1,2}$ and S. Capozziello$^{3,4,5}$

$^1$Centre for Theoretical Physics, The British University in Egypt, P.O. Box 43, El Sherouk City, Cairo 11837, Egypt
$^2$Department of Mathematics, Faculty of Science, Ain Shams University, Cairo 11566, Egypt
$^3$Dipartimento di Fisica “E. Pancini”, Università di Napoli “Federico II”, Complesso Universitario di Monte Sant’Angelo, Edificio G, Via Cinthia, I-80126, Napoli, Italy
$^4$Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Napoli, Complesso Universitario di Monte Sant’Angelo, Edificio G, Via Cinthia, I-80126, Napoli, Italy
$^5$Gran Sasso Science Institute, Viale F. Crispi, 7, I-67100, L’Aquila, Italy.

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Inspired by the BTZ formalism, we discuss the Maxwell-$f(T)$ gravity in (2+1)-dimensions. The main task is to derive exact solutions for a special form of $f(T) = T + \epsilon T^2$, with $T$ being the torsion scalar of Weitzenböck geometry. To this end, a triad field is applied to the equations of motion of charged $f(T)$ and sets of circularly symmetric non-charged and charged solutions have been derived. We show that, in the charged case, the monopole-like and the ln terms are linked by a correlative constant despite of known results in teleparallel geometry and its extensions [39]. Furthermore, it is possible to show that the event horizon is not identical with the Cauchy horizon due to such a constant. The singularities and the horizons of these black holes are examined: they are new and have no analogue in literature due to the fact that their curvature singularities are soft. We calculate the energy content of these solutions by using the general vector form of the energy-momentum within the framework of $f(T)$ gravity. Finally, some thermodynamical quantities, like entropy and Hawking temperature, are derived.

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I. Introduction

The explanation of the gravitational phenomena at large scales is difficult and it is considered as one of the main issues in physics. For example, the accelerated phase of the universe that is observationally probed cannot be investigated in the framework of General Relativity (GR) except by introducing a cosmic fluid possessing exotic characteristics, like dark energy, or inserting the cosmological constant that gives rise to other conceptual problems [1]-[6]. In the same manner, the rotation curves of galaxies appear to depart from the standard gravitational behavior asking for a large amount of dark matter [7].

Despite of these issues, GR has attained excellent achievements in describing the gravitational field in the last 100 years. The accuracy of GR is checked when theoretical predications and observations are challenged. However, the searching for a self-consistent gravitational theory at all scales is still an open question. A reliable gravitational theory should be able to trace gravitational fields in all domains, in addition, due to the fact that GR is consistent at Large
scales with observations, any new reliable gravitational theory must tend to GR in a suitable limit \cite{8}.

One of a possible way out to the above problems is to extend Einstein’s GR on a geometric background. Many theories of the gravitational field possessing variety of geometric formulations have been recently built up: for example, \( f(R) \) gravity that relies on arbitrary functions of the Ricci scalar \cite{9,10,11,12}. Assuming \( f(R) = R \), the Einstein-Hilbert Lagrangian, and thus GR, is recovered. Another reliable gravitation theory is the one that comes from the generalization of the Weitzenböck geometry, i.e., teleparallel equivalent of Einstein GR (TEGR) \cite{13,14,15,16}. The TEGR is built on the Riemann-Cartan geometry where a non-symmetric Weitzenböck connection is defined: it gives rise to a vanishing curvature and a non-vanishing torsion. In TEGR, we can deal with torsion tensor as the key ingredient instead of curvature, whilst the tetrad (4-dimension) field is considered as the dynamical quantity alternative to the metric one. It is interesting to mention that Einstein himself, in his attempt to unify gravitational and electromagnetic fields, used TEGR \cite{17,18,19,20,21}. Despite of the fact that GR and TEGR are gravitational theories having different geometric structures, they give rise to identical field equations and are invariant under local Lorentz transformations. Therefore, we can consider that any solution satisfying the field equations of GR is also a solution satisfying the field equations of TEGR. On the other hand, the straightforward extension of TEGR, the \( f(T) \) gravity, consists in Lagrangians depending on functions of the scalar torsion \( T \) \cite{22,23}. Such gravitational theories are attractive from many aspects. Firstly, they cannot be directly matched to GR \cite{24,25}. This means that \( f(T) \) is not a simple analogue of \( f(R) \) in the case of torsion \cite{23}. Secondly, they can be viewed as talent theories to solve several problems of GR as explained, for example, in \cite{26,27,28,29,30,31,32,33,34,35,36,37,38}. Due to this versatility, many studies on \( f(T) \) gravity have been done, ranging from exact cosmological solutions to stellar models \cite{39,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,56,57}. There is a price to pay in the approach of \( f(T) \) which is the fact that this theory is not invariant under local Lorentz transformations, and therefore different tetrads could arise different field equations \cite{58,59}. It is an important issue to note that \( f(T) \) theory is a frame-dependent because any solution of its equation of motion depends on the tetrad \cite{58,59,60}. However, we can forget this problem and discuss solutions in the special tetrad, this is like what happens in the electromagnetism when one study the special class of inertial frames \cite{53}.

At a fundamental level, the \((3+1)\) formalism, working in GR, has to be developed also in TEGR and its extension \( f(T) \), in view of achieving a consistent quantization approach. In fact, it is believed that the \((3+1)\)-dimensional formulation of GR is one of the best formulation of gravitational field, however, its quantization shows many problems. Due to these shortcomings, the \((2+1)\)-dimensional formulation of gravity has accomplished much interest, because classically it is easy to deal with it and one can explain in more efficiency a quantization procedure. Bañados, Teitelboim and Zanelli (BTZ), in (1992), showed that there is a solution corresponding to \(3\)-dimensional GR which has a negative value of the cosmological constant \cite{61}. BTZ solution shows several interesting characteristics ranging from classical to quantum levels; for example, some interesting contributes to the Kerr black-hole in \((3+1)\)-dimensions of GR have been developed starting from BTZ result \cite{62,63}.

Actually, among the motivations that make \((2+1)\)-dimension gravity a remarkable toy model, there is the existence of the BTZ solution. It has been proved that the BTZ black hole arises from collapsing matter \cite{64}. This kind of black hole requires a constant curvature in local spacetime \cite{65}. In fact, it has been shown that for a certain subset of Anti-de Sitter (AdS) spacetimes \cite{66}, there is a solution which one can consider as a black hole. Also, a charged BTZ solution, arising form AdS-Maxwell gravity in \((2+1)\)-dimensions, has been derived \cite{62,67,68}; \(3\)-dimensional dilatonic solutions, using a nonlinear electrodynamics, has been studied in \cite{69}. It is interesting to note that a \(3\)-dimensional charged black hole has been discussed using the quadratic form of \( f(T) \) \cite{42}.

Nevertheless the studies in \(3\)-dimensions, the final formulation of a self-consistent quantum gravity theory is still an open question. Thus, it is interesting to go deep in \(3\)-dimension outlines to check the features, as a preliminary step to investigate the \((3+1)\)-dimension gravity.

The main goal of the present paper is deriving rotating non-charged and charged black hole solutions in the \(3\)-dimensional Maxwell-\( f(T) \) gravity. This leads to solutions which asymptotically behave as AdS black holes for the special quadratic form of \( f(T) \). Among the advantages of these solutions, there is the fact that the electric potentials
has a monopole term, in addition to the logarithmic term, which are correlated by a constant. The second term exists despite of the fact that these solutions have a singularity when the radial coordinate is vanishing, i.e., \( r = 0 \); however, this singularity is much softer than any AdS charged or non-charged solutions derived in the framework of GR or TEGR \([73]\). Finally, besides the fact that these solutions behave asymptotically as AdS, they have distinct spatial and temporal components, i.e. \( g_{tt} \) and \( g^{rr} \) components are different and have different event and Killing horizons.

The outline of the paper is the following. In §2, the Maxwell-\( f(T) \) gravity is sketched. In §3, a triad field having 3 unknown functions is provided and applied to the non-charged and charged field equations of \( f(T) \) gravity. New exact non-charged and charged solutions are derived also in §3. In §4, the physics of the solutions is discussed by discussing the singularities of the scalars constructed from the Levi-Civita connection and from the Weitzenb"{o}ck. Furthermore, in §4, we derive the total energy related to each solution pointing out the physical meaning of the integration constants. In §5, some thermodynamical quantities are discussed. We show that the first law of thermodynamics is not satisfied for the charged black hole. Final section is devoted to some concluding remarks.

II. THE MAXWELL-\( f(T) \) GRAVITY

A. The Weitzenb"{o}ck geometry

The Weitzenb"{o}ck geometry is assigned by the couple \( \{ \mathcal{M}, h_i \} \), where \( \mathcal{M} \) is an N-dimensional manifold and \( h_i \) \((i = 1, 2, \ldots, N)\) are N vectors globally defined on the manifold \( \mathcal{M} \). The vectors \( h_i \) are called the parallelization fields. In N-dimensions, the covariant derivative of the covariant tetrad field is vanishing, that is

\[
h_i^{\mu} ; \nu \overset{\text{def}}{=} \partial_\nu h_i^\mu - \Gamma^\lambda_{\mu
u} h_i^\lambda = 0,
\]

where ”;” represents the covariant derivative and the ordinary derivative ”,” is defined as \( \partial_\nu \overset{\text{def}}{=} \frac{\partial}{\partial x^\nu} \). The connection \( \Gamma^\lambda_{\mu
u} \) is the Weitzenb"{o}ck non-symmetric connection \([70]\) has the form

\[
\Gamma^\lambda_{\mu
u} \overset{\text{def}}{=} h^\lambda_\iota \partial_\nu h_i^\mu.
\]

The tensor \( g_{\mu\nu} \) is defined as

\[
g_{\mu\nu} \overset{\text{def}}{=} \eta_{ij} h_i^\mu h_j^\nu,
\]

which is the metric tensor with \( \eta_{ij} = (+, -, -, \cdots) \) being the Minkowskian spacetime. The condition of metricity is satisfied as a consequence of Eq. (1). Eq. (2) has an interesting property that it gives a vanishing curvature tensor and a non-vanishing torsion tensor. We note that the tetrad field \( h_i^\mu \) fixes a unique metric \( g^{\mu\nu} \) while the inverse statement is not correct. The torsion and the contortion tensors are defined as

\[
T^\alpha_{\mu\nu} \overset{\text{def}}{=} \Gamma^\alpha_{\nu\mu} - \Gamma^\alpha_{\mu\nu} = h_i^\alpha \left( \partial_\mu h_i^\nu - \partial_\nu h_i^\mu \right),
\]

\[
K^\mu_{\alpha\nu} \overset{\text{def}}{=} -\frac{1}{2} \left( T^\mu_{\alpha\nu} - T^\nu_{\alpha\mu} - T^\alpha_{\mu\nu} \right).
\]

The teleparallel torsion scalar of TEGR theory is defined as

\[
T \overset{\text{def}}{=} T^\alpha_{\mu\nu} S_{\alpha}^{\mu\nu},
\]

where the tensor \( S_{\alpha}^{\mu\nu} \) is anti-symmetric in the last two pairs and has the form

\[
S_{\alpha}^{\mu\nu} \overset{\text{def}}{=} \frac{1}{2} \left( K^\mu_{\alpha\nu} + \delta^\mu_{\alpha} T^\beta_{\nu\beta} - \delta^\nu_{\alpha} T^\beta_{\mu\beta} \right).
\]

Using Eq. (4). Eq. (2) can be rewritten as

\[
\Gamma^\alpha_{\nu\rho} = \left\{ \nu_{\rho} \right\} + K^\mu_{\nu\rho},
\]

where \( \left\{ \nu_{\rho} \right\} \) is the Levi-Civita connection of GR theory, that depends on \( g_{\mu\nu} \) as well as its first derivatives, while \( K^\mu_{\nu\rho} \) is the contortion tensor that depends on the tetrad fields \( h_i^\mu \) as well as its first derivatives.
B. The Maxwell-$f(T)$ gravitational theory

Using the same approach as for $f(R)$ gravity, we define an arbitrary analytic function of the scalar torsion $T$, i.e., $f(T)$ gravitational theory in the 3-dimension action as:

$$\mathcal{L} = \frac{1}{2\kappa_3} \int |h|(f(T) - 2\Lambda) \, d^3x + \int |\mathcal{L}_{em}| \, d^3x,$$

(8)

where $\kappa_3$ is a three-dimensional constant and $\Lambda$ being the cosmological constant. In Eq. (8) $|h| = \sqrt{-g} = \det (h_{\mu}^a)$ and $\mathcal{L}_{em} = -\frac{1}{2} F \wedge^* F$ is the Maxwell Lagrangian with $F = dA$ and $A = A_{\mu} dx^\mu$ being the electromagnetic gauge potential [43]. Making the variation of Eq. (8) with respect to the triad $h_{i\mu}$ and the gauge potential gives [22, 43, 44]

$$S_{\mu}^{\rho\nu} \partial_{\rho} T_{TT} + \left[ h^{-1} h_{i\mu} \partial_{\rho} \left( h h_i^a S^{\alpha}_{a\rho\nu} - T^\alpha_{\lambda\mu} S^{\nu\lambda}_{a} \right) f_T - \frac{f - 2\Lambda}{4} \delta_{\mu}^{\nu} + \kappa_3 \Theta_{\mu}^{\nu} \right] = 0,$$

(9)

with $f \overset{\text{def}}{=} f(T)$, $f_T \overset{\text{def}}{=} \frac{\partial f(T)}{\partial T}$, $f_{TT} \overset{\text{def}}{=} \frac{\partial^2 f(T)}{\partial T^2}$. $\Theta_{\mu}^{\nu}$ is the energy-momentum tensor defined as

$$\Theta_{\mu}^{\nu} = F_{\mu\alpha} F^{\nu\alpha} - \frac{1}{4} \delta_{\mu}^{\nu} F_{\alpha\beta} F^{\alpha\beta}.$$

Now we are going to study two separate cases individually, the case of vacuum as well as the case of non-vacuum case respectively.

Eq. (9) can be has the form

$$\partial_{\alpha} \left[ h S^{b\beta\alpha} f_T \right] = \kappa_3 h h_{\mu}^b \left[ \tau_{b\mu} + \Theta_{b\mu} \right],$$

(11)

with $\tau^{\mu\nu}$ being defined as

$$\tau^{\mu\nu} \overset{\text{def}}{=} \frac{1}{\kappa_3^2} \left[ 4f(T) T S^{\alpha\nu\lambda} T_{\alpha\lambda}^{\mu} - g^{\mu\nu} f(T) \right].$$

(12)

Because of the skewness of $S^{\alpha\nu\lambda}$ we have

$$\partial_{\alpha} \partial_{\beta} \left[ h S^{\alpha\beta} f_T \right] = 0,$$

which leads to

$$\partial_{\beta} \left[ h \left( \tau^{\beta\mu} + \Theta^{\beta\mu} \right) \right] = 0.$$

(13)

From Eq. (13) we get

$$\frac{d}{dt} \int_\Sigma d^2x \, h a_{\mu} (\tau^{0\mu} + \Theta^{0\mu}) + \oint_C \left[ h a_{\mu} \left( \tau^{\mu\nu} + \Theta^{\mu\nu} \right) \right] \hat{n} \cdot dl = 0,$$

(14)

where $C$ is a contour enclosing the surface $\Sigma$, $\hat{n}$ is a unit normal vector to the closed contour $C$, and $dl$ is an infinitesimal length. Eq. (14) gives the conservation of the energy-momentum and of the quantity $\tau^{\beta\mu}$. Hence, the total energy-momentum of (2+1)-dimensional $f(T)$ theory contained in two-dimensional surface $\Sigma$ is defined as

$$P^b := \int_\Sigma d^2x \, h h_{\mu}^b \left( \tau^{0\alpha} + \Theta^{0\alpha} \right) = \frac{1}{\kappa_3} \int_\Sigma d^2x \partial_{\alpha} \left[ h h S^{b0\alpha} f(T) \right].$$

(15)

Eq. (15) is the generalization of the energy-momentum tensor for the $f(T)$ theory. The above equation can be used to carry out the calculation of energy and momentum and, as soon as $f(T) = T$, it returns to the well know form of (2+1)-dimensional TEGR [72].
It is important to mention here that Eq. (14) is valid only for solutions which behave asymptotically as a flat spacetime; however, for solutions which behave asymptotically as AdS/dS, Eq. (14) is not valid because the second term will not vanish asymptotically. Therefore, we must add a quantity which assures the vanishing of the second term asymptotically for any solution which behaves as AdS/dS. This expression has the form \( T^{\mu\nu} \) and, in that case, Eq. (14) takes the form

\[
\frac{d}{dt} \int_{\Sigma} d^2 x \ h_{\mu} \ (r^0_{\mu} + \Theta^0_{\mu}) + \oint_{C} [h_{\mu} \ (r^j_{\mu} + \Theta^j_{\mu} + T^j_{\mu})] \hat{n} \cdot dl = 0,
\]  

(16)

with \( T^j_{\mu} \) being the energy-momentum of pure AdS/dS spacetime.

III. Three-dimensional black holes in Maxwell-\( f(T) \) gravity

Using the coordinate \( \{t, r, \phi\} \), we write the triad that possesses three unknown functions in the form

\[
(h^I_{\mu}) = \begin{pmatrix}
N & 0 & 0 \\
0 & N_1 & 0 \\
0 & N_2 & r
\end{pmatrix},
\]  

(17)

where \( N(r), N_1(r) \) and \( N_2(r) \) are three unknown functions. The metric spacetime of triad (17) takes the form

\[
ds^2 = (N^2 - r^2 N_2^2) dt^2 - N_1^2 dr^2 - r^2 d\phi^2 - 2r^2 N_2 d\phi dr.
\]  

(18)

Using Eq. (17) in Eq. (5), we get

\[
T = \frac{4NN' + r^3 N_2^2}{2rN^2 N_1^2}, \quad \text{where} \quad N' = \frac{dN}{dr}.
\]  

(19)

Now we are going to study the two separate cases of the field Eqs. (9).

A. The vacuum (non-charged) case

Applying the triad (17) to Eq. (9), when \( \Theta^\nu_{\mu} = 0 \), we get

\[
I^I_t = \left( \frac{(2N^2 + r^3 N_2 N_2') fTT'}{rN^2 N_1^2} \right) + \frac{fT}{rN^3 N_1^3} \left( r^3 N N_1 N_2 N_2' + r^3 N N_1' N_2^2 - r^2 N_2 N_2' [rN N_1' + N_1 (r N' - 3 N)] 
\right.
- 2N^3 N_1' + 2N^2 N_1 N') - f + 2\Lambda = 0,
\]

\[
I^I_\phi = \left( \frac{2rN N_2 N' - r^3 N_2 N_2' - r N_2 N_2' - 2 N_2 N_2'}{rN^2 N_1^2} \right) fTT' - \frac{fT}{rN^3 N_1^3} \left( rN N_1 (r^2 N_2^2 + N^2) N_2' - 2r N^2 N_1 N_2 N'' 
\right.
+ 2r^3 N N_1 N_2 N_2' - N_2' (r N_2^2 + N^2) [r N N_1' + N_1 (r N' - 3 N)] + 2N^2 N_2' [r N' - N] \right) = 0,
\]
\[
I_r = \frac{f_T(4NN' + r^3N'')}{r^2N_1^2} - f + 2\Lambda = 0,
\]

\[
I_\phi = \frac{r^2N_0f_T T'}{N_1^2} + \frac{rf_T(rNN_1N_0'' - N_2[rNN_1' + N_1(rN' - 3N)])}{N^3N_1^3} = 0,
\]

\[
I_\phi = \frac{2NN' - 2rN_0^2N_2'}{N^2N_1^2} - \frac{rf_T}{N^3N_1^3} \left( r^3NN_1N_2N_0'' - 2rN^2N_1N'' + r^3NN_1N_2'' - 2rN_2N_1' \left[ rN_1N' + N(rN' - 3N) \right] + 2N^2N_1'[rN_1' - N] \right) - f + 2\Lambda = 0,
\]

where \( N' = \frac{dN}{dr}, N'_1 = \frac{dN_1}{dr}, N'_2 = \frac{dN_2}{dr} \). Using the quadratic form of \( f(T) \), i.e., \( f(T) = T + \epsilon T^2 \) in Eq. (20) we get

\[
I'_r = \frac{2\epsilon(2N^2 + r^3N_2N_2')T'}{rN^2N_1^2} + \frac{(1 + 2\epsilon T)}{rN^3N_1^3} \left( r^3NN_1N_2N_0'' + r^3NN_1N_2'' - 2rN^2N_1'[rNN_1' + N_1(rN' - 3N)] - 2N^3N_1' + 2N^2N_1N' \right) - (T + \epsilon T^2) + 2\Lambda = 0,
\]

\[
I'_\phi = \frac{2\epsilon(2rNN_2N' - r^3N_2N_0'' - N_2N_2' - 2N^2N_2)T'}{rN^2N_1^2} - \frac{(1 + 2\epsilon T)}{rN^3N_1^3} \left( rNN_1(r^2N_2^2 + N^2)N_0'' - 2rN^2N_1N_0'' + 2N^2N_0N_1'[rN' - N] \right) = 0,
\]

\[
I''_r = \frac{(1 + 2\epsilon T)(4NN' + r^3N'')}{rN^2N_1^2} - (T + \epsilon T^2) + 2\Lambda = 0,
\]

\[
I''_\phi = \frac{2\epsilon r^2N_0 T'}{N^2N_1^2} + \frac{r(1 + 2\epsilon T)(rNN_1N_0'' - N_2[rNN_1' + N_1(rN' - 3N)])}{N^3N_1^3} = 0,
\]

\[
I''_\phi = \frac{2\epsilon [2NN' - r^2N_2N_0']T'}{N^2N_1^2} - \frac{(1 + 2\epsilon T)}{rN^3N_1^3} \left( r^3NN_1N_2N_0'' - 2rN^2N_1N'' + r^3NN_1N_2'' - 2rN_2N_1' \left[ rN_1N' + N(rN' - 3N) \right] + 2N^2N_1'[rN_1' - N] \right) - (T + \epsilon T^2) + 2\Lambda = 0,
\]

where \( T' = \frac{dT}{dr} \). It is interesting to note that if the dimensional parameter \( \epsilon = 0 \) then Eq. (21) reduces to that derived in [73]. Now we are going to solve the above system of differential equations using the following constrains \( \Lambda = \frac{1}{24\epsilon} \) [42]

\[
i) \quad N = \frac{\sqrt{r^2 - 12c_1\epsilon}}{\sqrt{12\epsilon}}, \quad N_1 = \pm \frac{\sqrt{12\epsilon}}{\sqrt{12\epsilon} - r^2}, \quad N_2 = c_2,
\]

\[
ii) \quad N = \pm \frac{\sqrt{r^2 - 12c_1\epsilon r^2 + 12c_2^2\epsilon}}{r\sqrt{12\epsilon}}, \quad N_1 = \pm \frac{r\sqrt{12\epsilon}}{\sqrt{12\epsilon} - r^4 - 12c_2^2\epsilon}, \quad N_2 = c_2 + \frac{c_3}{r^2},
\]

where \( c_1, c_2 \) and \( c_3 \) are constants of integration. It is clear that when the constant \( c_3 = 0 \) the second set of solution (22) reduces to the first set of (22). All the above sets of solution (22) give constant torsion, i.e., \( T = \frac{1}{8|\gamma|} \) which coincides with [42]. It is important to mention here that solution (22) can not reduce to TEGR and therefore it has no analog in GR.
Applying the field Eq. (9) to triad (17) we get the following non-vanishing components when $\Theta_{\mu}^\nu \neq 0$

\[
I^t = \frac{(2N^2 + r^2N_2) f_{TT'}}{rN^2N_1^2} + \frac{f_T}{rN^3N_1^3} \left( r^3NN_1N_2N'_2 + 3NN_1N'_2^2 - 2N_2N_1[rNN'_1 + N_1(rN' - 3N)] \\
-2NN_1^2 + 2N^2N_1N' \right) - f + 2\Lambda - \frac{2q^2}{N^2N_1^2} = 0
\]

\[
I^\phi = \frac{(2rNN_2N' - r^2N_2^2N' - r_2N_2N'_2 - 2N_2N_1)f_{TT'}T'}{rN^2N_1^2} - \frac{f_{T}}{rN^3N_1^3} \left( rNN_1(r^2N_2^2 + N^2)N'_2 - 2rN^2N_2N_1N'' \\
+ 2r^3NN_1N_2N''^2 - N'_2(r^2N_2^2 + N^2)[rNN'_1 + N_1(rN' - 3N)] + 2N^2N_1N'_2[rN' - N] \right) + \frac{4N_2^2q^2}{N^2N_1^2} = 0,
\]

\[
I^r = \frac{f_T(4NN' + r^3N_2')}{rN^2N_1^2} - f + 2\Lambda - \frac{2q^2}{N^2N_1^2} = 0,
\]

\[
I^\phi = \frac{[2NN' - r^2N_2N_1']f_{TT'}T'}{N^2N_1^2} - \frac{f_{T}}{rN^3N_1^3} \left( r^3NN_1N_2N'_2 - 2N_2^2N_1N'' + r^3NN_1N_2N'_2 - 2N_2N'_2 \left[ rN_1N' \\
+ N(3N^2 - 3N_1) \right] + 2N^2N'[rN'_1 - N_1] \right) - f + 2\Lambda + \frac{2q^2}{N^2N_1^2} = 0,
\]

(23)

where the unknown $q(r)$ is the electric charge which is defined as

\[
A_\mu = q(r)\delta_\mu^t.
\]

We mention here that the above charged differential equation of $f(T)$ are different from those of [42] even when $N_2 = 0$. The difference raises due to the fact that the two field equations are different and become identical only when $f(T) = T$. Equation (23) reduces to (21) when the unknown function $q(r)$ vanishing. The general solutions of the above system of differential equations using the same constrain of the uncharged case, i.e., $f(T) = T + cT^2$ and $1 - 24c\Lambda = 0$, take the following form

\[
i) N = \frac{\sqrt{r^2 - 12c^2}}{\sqrt{12c}}, \quad N_1 = \pm \frac{\sqrt{12c}}{\sqrt{12c^2 - r^2}}, \quad N_2 = c_2, \quad q(r) = c_4,
\]

\[
ii) N = \pm \frac{\sqrt{r^2 - 12c^2}r^2 + 12c^2\epsilon}{r\sqrt{12c}}, \quad N_1 = \pm \frac{r\sqrt{12c}}{\sqrt{12c^2 - r^2} + 12c^2\epsilon}, \quad N_2 = c_2 + \frac{c_3}{r^2}, \quad q(r) = c_4.
\]

\[
iii) N = \pm \frac{c_4N_3}{\sqrt{2r}}, \quad N_1 = \pm \frac{2c_5(c_5r - 1)\sqrt{3\epsilon}}{N_3\sqrt{2r}}, \quad N_2 = c_6, \quad q(r) = c_4 + c_5^2\ln(r) + \frac{c_5}{r},
\]

with

\[
N_3 = \sqrt{3c_5 + 12c_6c_4 + 2r(1 + 3\ln(r))c_5^2 - c_5^4r^3},
\]

(24)

where $c_4$, $c_5$ and $c_6$ are integration constants. It is necessary to mention here that the above black hole solutions cannot reduce to that derived in [42] due to the appearance of the constant $c_5$. This constant cannot be equal to zero otherwise we get a travail charge and return to the non-charged case given by (22). This leads us to say that the charged solution derived in [42] is not a black hole solution of the present $f(T)$ theory because the charged term must have the logarithmic term as in (24) in addition to the monopole like one. A final remark about solution (24) is that the logarithmic term, which appears in the potential, is not standard in the Einstein-Maxwell (2+1)-dimensional
theory [74]. Therefore, solution (24) is a new analytic black hole solution in the frame of $f(T)$ gravitational theory whose field equations are given by (9) and when $f(T) = T + \epsilon T^2$. In the next section, we are going to extract the physics of the uncharged and charged solutions by calculating their metrics, singularities and their energies.

IV. Black hole physics

Now we are going to discuss the physical meaning of the above black hole solutions considering the main features of the related black holes.

A. The non-charged metric

The metric of the first set of solution (22) has the form

$$ds_1^2 = \frac{(r^2 - 12c_1 |\epsilon| - 12c_2^2 |\epsilon| r^2)}{12 |\epsilon|} dt^2 - \frac{12 |\epsilon| dr^2}{r^2 - 12c_1 |\epsilon|} - r^2 d\phi^2 - 2r^2 c_2 dt d\phi. \quad (25)$$

We can eliminate the cross term that appears in Eq. (25) using the following transformation

$$c_2 t + \phi \rightarrow \phi'. \quad (26)$$

Using Eq. (26) in (25) we get

$$ds_1^2 = \frac{(r^2 - 12c_1 |\epsilon|)}{12 |\epsilon|} dt^2 - \frac{12 |\epsilon| dr^2}{r^2 - 12c_1 |\epsilon|} - r^2 d\phi'^2. \quad (27)$$

Equation (27) can be rewritten as

$$ds_1^2 = (r^2 \Lambda_e - c_1) dt'^2 - \frac{dr^2}{r^2 \Lambda_e - c_1} - r^2 d\phi'^2, \quad (28)$$

where $\Lambda_e = \frac{1}{12 |\epsilon|}$. Equation (28) shows that the metric asymptotes to AdS/dS. For the second set of solution (22) the metric takes the form

$$ds_2^2 = \frac{r^2 - 12 |\epsilon| c_1 - 12c_2 c_3 [c_2^2 r^2 + 2c_3]}{12 |\epsilon|} dt^2 - \frac{12r^2 |\epsilon| dr^2}{r^4 - 12c_1 |\epsilon| r^2 + 12c_3^2 |\epsilon|} - r^2 d\phi^2 - 2r^2(c_2 + c_3) dt d\phi. \quad (29)$$

The cross term in Eq. (29) can not be removed by a coordinate transformation due to the appearance of the constant $c_3$. This constant is responsible for the rotating term which comes from the unknown function $N_3$. Eq. (29) can be rewritten as

$$ds_2^2 = (r^2 \Lambda_e - c_1) dt^2 - \frac{r^2 dr^2}{r^4 \Lambda_e - c_1 r^2 + c_3^2} - r^2 d\phi^2 - 2c_3 dt d\phi, \quad (30)$$

where we have put the constants $c_2 = 0$. Again Eq. (30) asymptotically goes to AdS/dS solution.
B. The metric of charged case

The first two sets of the charged solution (24) are the same as the two sets of the non-charged solution. The metric of the third set of Eq. (24) takes the form

\[ ds^2_3 = \frac{r^3(c_5^6 - 2c_6^2) - 4c_5^3 - 12rc_4c_5^2 - 2rc_5^4 - 6rc_5^4\ln r}{2r} dt^2 \]

\[ - \frac{12c_5^2(r_5 - 1)^2}{r(r^3c_5^4 - 4c_5 - 12rc_4 - 2rc_5^2 - 6rc_5^2\ln r)} dr^2 - r^2d\phi^2 - 2r^2c_6dtd\phi. \]  

(31)

Using the following transformation

\[ c_6t + \phi \rightarrow \phi', \]

we can eliminate the cross term that appears in Eq. (31) and get

\[ ds^2_3 = \frac{r^3c_5^6 - 4c_5^3 - 12r|\epsilon|c_4c_5^2 - 2rc_5^4 - 6rc_5^4\ln r}{2r} dt^2 - r^2d\phi'^2 \]

\[ - \frac{12c_5^2|\epsilon|(rc_5 - 1)^2}{r(r^3c_5^4 - 4c_5 - 12r|\epsilon|c_4 - 2rc_5^2 - 6rc_5^2\ln r)} dr^2. \]  

(32)

Eq. (32) can be rewritten as

\[ ds^2_3 = \left( r^2\Lambda_e - \frac{2}{r\sqrt{6|\epsilon|}} - \frac{1 + (6|\epsilon|^{1/3}c_4 + 3\ln r)}{\sqrt{36c_4^2}} \right) dt^2 - \frac{1}{f\left( r^2\Lambda_e - \frac{2}{r\sqrt{6|\epsilon|}} - \frac{1 + (6|\epsilon|^{1/3}c_4 + 3\ln r)}{\sqrt{36c_4^2}} \right)} dr^2 - r^2d\phi'^2, \]

(33)

where \( c_5 = \sqrt{2}\Lambda_e \) and \( f = \frac{1}{(1 - \frac{1}{r_5^2})^2} \). The metric of Eq. (33) asymptotes AdS/dS spacetime. It interesting to note that, from Eq. (33), we cannot recover Eq. (28). This is due to the fact that the third set of solutions (24) cannot return to the first set.

The torsion scalar of the non-charged case, given by solution (22), has the form

\[ T_1 = T_2 = \frac{1}{6|\epsilon|}, \]  

(34)

and, for the charged solution given by the third set of Eq. (24), has the form

\[ T_3 = \frac{c_5r + 2}{6rc_5|\epsilon|}. \]  

(35)

Now, let us discuss the singularities and the horizons of solution (22). The curvature scalars arise from the metric of first set of solution (22) have the form

\[ R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} = -R^{\mu\nu}R_{\mu\nu} = -\frac{1}{12\epsilon^2}, \]

\[ R = \frac{1}{2|\epsilon|}, \]

\[ T^{\mu\nu\lambda\nu}T_\mu\nu = \frac{r^4 - 12r^2|\epsilon|c_1 + 72\epsilon^2c_1^2}{3r^2|\epsilon|(r^2 - 12|\epsilon|c_1)} \sim \left( \frac{1}{r^4} \right), \]

\[ T^{\mu\nu}T_\mu = \frac{(r^2 - 6c_1|\epsilon|)^2}{3r^2|\epsilon|(r^2 - 12c_1|\epsilon|)} \sim \left( \frac{1}{r^4} \right). \]  

(36)
For the second set of solution (22), the invariants of curvature do not change from the first set however, the invariants of the torsion are given by

\[
T^{\mu\nu\lambda}T_{\mu\nu\lambda} = -\frac{72c^2c_1^2r^4 - 12|c|c_1r^6 + r^8 + 144c^2c_3^3r^2 - 24|c|c_2^2r^4 - 144c^2c_3^4}{3r^4|c|(12|c|c_1r^2 - c_2^2 - r^4)} \sim \left(\frac{1}{r^4}\right),
\]

\[
T^\mu T_\mu = -\frac{12|c|(c_1r^2 - 2c_3^2)}{r^4(12|c|c_1r^2 - c_2^2 - r^4)} \sim \left(\frac{1}{r^4}\right),
\]

(37)

It is worth mentioning here that the above invariants of the torsion behaves asymptotically as \(\left(\frac{1}{r}\right)\) in contrast to the asymptotic behavior in the TEGR case of the same triad which behaves as \(\left(\frac{1}{r^4}\right)\). This means that, in the \(f(T)\) theory, the invariants of torsion go to zero fast than those of TEGR as \(r \to \infty\). This means that the singularities of the invariants of \(f(T)\) are much softer than those of TEGR. It is worth to study if the solution of Eq. (22) is stable or not by studying its anti-evaporation \([75]\). All these issues need more investigation which will be clarified elsewhere.

The following scalars are satisfied for the two sets of solution (22).

\[
T(r) = \frac{1}{6|c|}, \quad \nabla_\alpha T^\alpha = \frac{1}{3|c|}, \quad \Rightarrow R = -T - 2\nabla_\alpha T^\alpha.
\]

(38)

For the charged case, the invariants of the first two sets are not change however for the third set of solution (24) we get the following invariants:

\[
R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} = \frac{1}{36r^2c^2c_3^4(c_3r - 1)^6} \left(3r^8c_5^{10} - 10r^7c_5^9 + 3r^6c_5^8 + 12r^5c_5^7 \ln r + 24r^5|c|c_5^5c_4
\]

\[
+ 5r^3c_5^3 - 24r^4c_5^6 \ln r - 48|c|r^4c_5^4 - 4r^2c_5^6 - 36r^3c_5^5 \ln r + 36r^2c_5^4(\ln r)^2 - 72r^3|c|c_5^3c_4
\]

\[-108r^3c_5^5 + 144r^2|c|c_5^3c_4 \ln r + 9r^2c_5^3c_4 \ln r + 144r^2c_5^2c_4^2 + 168r^2|c|c_5^2c_4 + 102r_2c_5^4
\]

\[+ 24rc_5^2 \ln r + 48r|c|c_5^3c_4 - 8rc_5^3 + 12c_5^2 \right) \sim \left(\frac{1}{r}\right),
\]

\[
R^{\mu\nu}R_{\mu\nu} = \frac{1}{72r^2c^2c_3^4(c_3r - 1)^6} \left(6r^8c_5^{10} - 20r^7c_5^9 + 5r^6c_5^8 + 24r^5c_5^7 \ln r + 48r^5|c|c_5^5c_4
\]

\[+ 50r^5c_5^7 - 42r^4c_5^6 \ln r - 84|c|r^4c_5^4 - 25r^4c_5^6 - 36r^3c_5^5 \ln r + 36r^2c_5^4(\ln r)^2 - 72r^3|c|c_5^3c_4
\]

\[-88r^3c_5^5 + 144r^2|c|c_5^3c_4 \ln r + 90r^2c_5^3c_4 \ln r + 144r^2c_5^3c_4c_2^2 + 180r^2|c|c_5^2c_4 + 77r^2c_5^4
\]

\[+ 24rc_5^2 \ln r + 48r|c|c_5^3c_4 + 12r_2c_5^3 + 8c_5^2 \right) \sim \left(\frac{1}{r}\right),
\]

\[
R = \frac{3r^4c_5^5 - 5r^3c_5^4 - 3r^2c_5^3 + 6rc_5^2 \ln r + 12r|c|c_4 + 8rc_5^2 + 2c_5}{6r|c|c_5^2(c_3r - 1)^3} \sim \left(\frac{1}{r}\right),
\]

\[
T^{\mu\nu\lambda}T_{\mu\nu\lambda} = -\frac{1}{6rc^2|c|(c_3r - 1)^2(4c_5 + 2c_5^2 + 12|c|r_4 + 6rc_5^2 \ln r - r^3c_5^4)} \left(2r_6c_5^8 - 12r^4c_5^6 \ln r
\]

\[-24r^4|c|c_5^4c_4 - 10r^4c_5^6 + 36r^2c_5^4(\ln r)^2 - 4r^3c_5^5 + 144r^2|c|c_5^3c_4 \ln r + 24r^2c_5^4 \ln r + 144r^2c_5^2c_4^2
\]

\[+ 48r^2|c|c_5^2c_4 + 13r^2c_5^4 + 48rc_5^3 \ln r + 96r|c|c_5c_4 + 4rc_5^3 + 20c_5^2 \right) \sim \left(\frac{1}{r}\right),
\]

\[
T^\mu T_\mu = -\frac{1}{12rc_5^3|c|(c_3r - 1)^2(4c_5 + 2c_5^2 + 12|c|r_4 + 6rc_5^2 \ln r - r^3c_5^4)} \sim \left(\frac{1}{r}\right)
\]

\[
T(r) = \frac{c_3r + 2}{6rc_5^2|c|} \sim \left(\frac{1}{r}\right), \quad \nabla_\alpha T^\alpha = \frac{13c_5^2 + 12|c|c_4 + 6c_5^2 \ln r - 6r^3c_5^3 - 6r^2c_5^4 + 4r^3c_5^5}{6c_5^2|c|(c_3r - 1)^3} \sim \left(\frac{1}{r}\right),
\]

\[
\Rightarrow R = -T - 2\nabla_\alpha T^\alpha.
\]

(39)
The above invariants (non-charged and charged) show:

a)- All scalars of curvature and torsion show irregular behaviors when $r \rightarrow 0$ that describes a real singularity, except the invariants of curvature of the first and second sets of solution (22).

b)- When the constant $c_1 = \frac{r^2}{12|\epsilon|}$, we get a singular metric for the first set of solution (22) in addition the invariants $T^{\mu\nu\lambda}T_{\mu\nu\lambda}$ and $T^{\mu}T_{\nu}$ have a singular behavior and when $c_1 = \frac{12c_3^2|\epsilon|+\epsilon^4}{12}\frac{1|\epsilon|}{r^3}$, we get a singular metric for the second set of solution (22). For the charged case, solution (24), we get a singular metric when $c_4 = \frac{r^3c_3^4-4c_5-2c_3^2-6rc_5^2}{12\frac{1}{r^3}}\ln r$. For the third set of solution (24), the scalars of curvature behave as $\frac{1}{r}$ in contrast to TEGR or GR which behave as $\frac{1}{r^2}$. However, the asymptotic behavior of the scalars of torsion does not change from the TEGR.

c)- For the third set of solution (24), the scalars of curvature behave as $\frac{1}{r}$ in contrast to TEGR or GR which behave as $\frac{1}{r^2}$. However, the asymptotic behavior of the scalars of torsion does not change from the TEGR.

It is worth mentioning that the above discussion shows that the dimensional parameter $\epsilon$ cannot be vanishing which ensures that all the above solutions have no analogue in GR.

C. The Energy

In this subsection, we are going to carry out the calculations of black hole energy solutions (22) and (24)\(^1\). From Eq. (15), using the non-charged solution (22), we get

$$S^{001} = -\frac{1}{2r}, \quad (40)$$

Using (40) in (15), we get

$$P^0 = E = -\frac{\pi(12c_1\epsilon - r^2)}{27\kappa_3\epsilon}, \quad (41)$$

which is not finite. Therefore, to obtain a finite value of Eq. (15), we use the following regularized expression

$$P^a := \frac{1}{\kappa_3} \int_{\Sigma} d^2x \partial_\nu \left[ hS^{\alpha\beta\gamma}f(T)_{\alpha\beta\gamma} \right] - \frac{1}{\kappa_3} \int_{\Sigma} d^2x \partial_\nu \left[ hS^{\alpha\beta\gamma}f(T)_{\alpha\beta\gamma} \right]. \quad (42)$$

Using (42) in (22), we get

$$E = M, \quad (43)$$

where $c_1 = -\frac{9\kappa_3M}{4\pi}$. The same above algorithm can be applied to the second set of Eq. (22) and the same value of Eq. (43) can be obtained. For the third set of solution (24) we get, after regularization, the energy in the form of (43) up to $\frac{1}{r}$.

V. THERMODYNAMICS

Hawking’s temperature of any solution can be derived by requiring the singularity to disappear at the horizon using the Euclidean continuation method. The temperature of the outer event horizon at $r = r_h$, for the first set of solution

\(^1\) We assume gravitational coupling to have the form $G_{eff} = G_{\text{Newtonian}} \frac{1+f_T}{1+f_T} [77]$. 

and the temperature of the second set of solution (24) is the same as that of Eq. (44). Finally, the temperature of the third set of solution (24) has the form

\[ T = \frac{r_h^3 - 3r_h \sqrt{36\epsilon^2 + 2\sqrt{6}\epsilon}}{24\pi |\epsilon| r_h^2}. \] (45)

Now we are going to carry on the calculations of the entropy of the black hole solutions (22) and (24). For this purpose, we use the terminology studied in [76]. The entropy associated with any solutions in the framework of \( f(T) \) gravitational has the form [76]

\[ S = \frac{A}{4} f_T \bigg|_{r=r_h}, \] (46)

where \( A \) is the horizon area. Using the first and second sets of solution (24) in (46) we get

\[ S = \pi r_h^2 \left[ 1 + 2 |\epsilon| T(r) \big|_{r=r_h} \right] = \frac{4\pi r_h^2}{3}, \] (47)

and for the third solution of Eq. (24) we get

\[ S = \pi r_h^2 \left[ 1 + \frac{r_h \sqrt{6|\epsilon| + 2}}{3r_h \sqrt{6|\epsilon|}} \right]. \] (48)

To investigate if the validity of the first law of the black hole solutions (24) is satisfied or not we are going to discuss the paper by Miao et al. [76]. They [76] rewrote the field equations (9), which are non-symmetric, into a symmetric part as well as a skew symmetric one as

\[ I_{(\mu\nu)} \overset{\text{def}}{=} S_{\mu\rho\nu} \partial^\rho T f_{TT} + f_T \left[ G_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right] + \frac{f - 2\Lambda}{2} g_{\mu\nu} = \kappa_3 T_{\mu\nu}, \]

\[ I_{[\mu\nu]} \overset{\text{def}}{=} S_{[\mu\nu]\rho} \partial^\rho T f_{TT} = 0. \] (49)

They have assumed a Killing vector field whose heat flux \( \delta Q \) has the form

\[ \delta Q = \frac{\kappa_3}{2\pi} \left[ \frac{f_T dA}{4} \right]_0^d\lambda + \frac{1}{\kappa_3} \int_H k^\nu f_{TT} T_{\mu} (\xi^\rho S_{\rho\mu}^\nu - \nabla_\nu \xi^\mu), \] (50)

where \( H \) refers to the black hole horizon.

In [76], it is proven that \( \left[ \frac{f_T dA}{4} \right]_0^d\lambda \) is equivalent to \( T \delta S \) [76]. Therefore, if \( \int_H k^\nu f_{TT} T_{\mu} (\xi^\rho S_{\rho\mu}^\nu - \nabla_\nu \xi^\mu) \neq 0 \), then the first law of thermodynamics is violated. It is shown that the first law is always violated in \( f(T) \) for non-trivial value of the scalar torsion [76]. In fact, the first and second sets of solution (24) have a trivial value of the scalar torsion and thus the first law is valid. However, the third set of black hole solution (24) has a non-trivial value of the torsion scalar in addition that this solution is reproduced from the quadratic form of \( f(T) \). Therefore, for this black hole solution, the third set of solution (24) violates the first law of thermodynamics.
VI. Concluding remarks

In this paper, we have studied 3-dimensional $f(T)$ and Maxwell-$f(T)$ gravity to check the existence of circularly symmetric solutions. To this end, we have applied off diagonal triad having three unknown functions of the radial coordinate, to the field equations of $f(T)$ theory (non-charged case). We have solved the field equations exactly for the quadratic form of $f(T)$ and have assumed the following relation between the cosmological constant and the dimension parameter $\epsilon$, i.e., $\Lambda = \frac{1}{24\epsilon}$ to simplify the form of the solution. We have obtained analytic solution having two sets which can be categorized as:

i) The first set makes the off-diagonal component has a constant value.

ii) The second one has a non-trivial value of the off-diagonal component.

All of these sets are new and have no analogues in standard GR because of the existence of the dimensional parameter $\epsilon$, coefficient of the quadratic term of the scalar torsion. Such a parameter cannot be equal to zero, otherwise, we get a singular form of the torsion scalar as well as of the metric. All these sets give constant torsion, i.e., $T = \frac{1}{6\epsilon}$. The singularities of these sets have been studied and we have indicated that all the scalars derived from curvature tensor as well as from torsion tensor show a singularity if the dimensional parameter $\epsilon$. The asymptotic behavior of the scalars, constructed out from the torsion behaves as $(\frac{1}{r^4})$ in contrast to what is derived both in GR and TEGR [61, 73]. Finally, we have calculated the energy of these sets and shown that it does not depend on the dimensional parameter $\epsilon$.

For the charged case we have applied the same triad to the equation of Maxwell-$f(T)$ gravitation theory. We have solved the resulted differential equations and obtained a solution which is a new one and completely different from that derived in [42]. This solution cannot reduce to that derived in [42] because of the difference of the field equations of the two theories. As it is clear from the potential vector-like term, i.e. $q(r) = c_4 + c_5^2 \ln(r) + \frac{c_5}{r}$, if the constant $c_5 = 0$, we return to the first set of solution (4) which is different the result presented in [42]. It is interesting to mention here that the 3-dimensional vector potential-like term derived in GR (TEGR) depends only on the logarithm, however our solution of higher order gravity (ultraviolet) depends on an additional monopole term. We may consider this term as a correction due to the higher order gravity.

For the charged solution we have shown that the torsion scalar is not constant. Therefore, this solution in higher-order torsion gravity is completely different from GR (TEGR).

Finally, we have calculated some of the thermodynamical quantities like the Hawking temperature and the entropy. For the non-charged sets, we have shown that the first law of thermodynamics is valid. However, the charged case shows that the first law is not satisfied. The violation of the first law of thermodynamics comes form the fact that the torsion scalar is not trivial and also $f_{TT} = 2\epsilon$, i.e., it is not TEGR where $f(T) = T$ [76]. This case needs more accurate studies because of the non-trivial value of the scalar torsion which is responsible for the deviation from TEGR due to the non-vanishing value of $f_{TT} = \epsilon$. In a forthcoming paper, a systematic discussion of thermodynamical properties of these solutions, in view of a (3+1) generalization, will be pursued.

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