A GRAVITATIONAL INSTABILITY-DRIVEN VISCOSITY IN SELF-GRAVITATING ACCRETION DISKS

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ABSTRACT

We derive a viscosity from gravitational instability in self-gravitating accretion disks, which has the required properties to account for the observed fast formation of the first supermassive black holes in highly redshifted quasars and for the cosmological evolution of the black hole mass distribution.

Subject headings: accretion, accretion disks — galaxies: active — hydrodynamics — quasars: general — turbulence

1. INTRODUCTION

Viscous accretion through disks and the ensuing dissipation are known to be very efficient processes of converting gravitational energy into radiation. In particular, accretion into black holes allows the liberation of a sizeable fraction of the accreted matter’s rest energy. For (stationary) accretion at a rate \( \dot{M} \) this amounts to an accretion luminosity \( L_{\text{acc}} = \eta \dot{M} c^2 \). Here \( c \) is the speed of light and \( \eta \) a parameter that takes care of the spin of the black hole, i.e., the metrics in the vicinity of the horizon and the corresponding radius of the innermost stable orbit \( (r_{\text{ins}}) \), the decoupling of the material from the disk near \( r_{\text{ins}} \) and the radiation efficiency. For standard accretion disks, i.e., those that do not belong to the class of radiation-inefficient accretion flows (Rees et al. 1982), \( \eta \) is of order \( 10^{-1} \).

The evolution of the disk and its timescale are governed by the value of the viscosity parameter \( \nu \). The viscous timescale \( \tau_{\text{visc}} \) is given by

\[ \tau_{\text{visc}} = \frac{R^2}{\nu}. \]

Here \( R \) is the radial coordinate in a cylindrical system \( (R, \varphi, z) \). In the following, we assume rotational symmetry about the \( \varphi \)-axis and make use of the approximation of vertically geometrically thin accretion disks. For exhaustive descriptions of details of the theory of this class of disks, we refer the reader to the pertinent literature (e.g., Frank et al. 2002).

It is not disputed that molecular viscosity is too small by many orders of magnitude and leads in almost all relevant situations to timescales surpassing the Hubble time. Far less clear, however, is what viscosity to use instead. This impasse was solved originally by Shakura & Sunyaev (1973). Their Ansatz is based on the insight that molecularly viscous accretion disks are prone to exceedingly large Reynolds numbers, indicative of the onset of turbulence. In their ad hoc prescription, they parameterized the viscous length scale by \( h_{\text{visc}} \), the disk’s thickness and the velocity by that of the sound speed \( c_s \). The entire unknown physics was subsumed in a parameter \( \alpha \) that was assumed to be (more or less) a constant. The assumptions of isotropic subsonic turbulence require \( \alpha \approx 1 \).

This parameterization, \( \nu = \alpha h c_s \), is often referred to as “\( \alpha \) viscosity.” It proved to be very successful for non–self-gravitating disks, i.e., disks in which the gravitational potential is solely given by the central accretion body, like in close binaries, late phases of star formation, etc. The observed (or derived) evolutionary timescales in these systems allowed the obtaining of at least the order of magnitude of \( \alpha \) by assuming that the evolutionary timescales can be reasonably estimated by \( \tau_{\text{visc}} \). While practically all derived values are compatible with the requirement of \( \alpha \) being smaller than unity, it also turned out that in the vast majority of cases values not too much smaller than this limit were required. Typical values were \log \alpha = -1 \pm 1.

Despite a number of successful applications of the \( \alpha \) parameterization (e.g., in explaining the dwarf nova phenomenon as a disk instability, which is compatible with a functional dependence of viscosity on the physical parameters as present in \( \alpha \) viscosity), this parameterization suffers from a number of shortcomings:

1. It was introduced in a pure ad hoc fashion and, in its original form, is not based on an instability that could drive the turbulence. While the otherwise exceedingly large Reynolds numbers in these flows strongly point toward the occurrence of turbulence, the Rayleigh criterion indicates that—at least in the linear regime—the radial angular momentum stratification of disks is stable against the onset of turbulence.

2. A naive extrapolation into the regime where the mass of the disk is no longer negligible leads to the rather unphysical result of a radially constant effective temperature (Duschl et al. 2000).

3. While the energy release from disks is held responsible for the output from galactic centers, in particular active ones (active galactic nuclei), even for an \( \alpha \) approaching unity, the timescales in these disks are far to long. This problem was stepped up with the detection of supermassive black holes (SMBHs) in highly redshifted quasars. If one assumes that these black holes gained their observed masses by accretion, in the most extreme cases, less than \( 10^7 \) yr are available for amassing more than \( 10^8 M_\odot \).

Balbus & Hawley (1991) rediscovered a magnetorotational instability (MRI), originally described by Velikhov (1959) and Chandrasekhar (1960). It could serve as an explanation for the onset of turbulence, even in only weakly magnetized disks, and led to a viscosity of the type and amount proposed by Shakura & Sunyaev. The question whether purely hydrodynamic insta-
ilities are also possible in massless disks is not settled so far. While Balbus & Hawley’s work solved the seeming contradiction between the large Reynolds numbers and Rayleigh stability, in particular the importance of nonlinear hydrodynamic instabilities is far from clear. MRI, however, suffers from the same problems when going into the self-gravitating domain.

The problem with any viscosity prescription of the functional form of $\alpha$ viscosity is that there the factor $c_s$, a local quantity, while $b$, through the vertical hydrostatic equilibrium $h/R = c_s/v_{\phi}$, is actually a quantity that contains global information about the disk’s (radial) structure: $v_{\phi}$ is the azimuthal velocity, which in this case is given by Kepler’s third law (or its relativistic version). Thus, in non–self-gravitating disks, the viscosity prescription contains both information about the local and global disk structure. In the self-gravitating case, however, the vertical hydrostatic equilibrium depends (almost) only on the local mass distribution, i.e., $v_{\phi}$ becomes a purely local quantity.

The third issue of the far too long viscous timescales in galactic center disks has finally been tried to be overcome by appealing to non-axisymmetric effects, like bars in the disks, which are meant to speed up the transfer of angular momentum to larger, and of mass to smaller radii by orders of magnitudes. While this, in principle, is a very attractive proposal to solve the problem in situations where bars are present, it seems that bars are not present in all galactic centers. This then rules it out as a general solution of the problem, although bars may very well play an important role in some systems.

Duschl et al. (1998), Duschl et al. (2000), and independently Richard & Zahn (1999) proposed a generalization of the Shakura-Sunyaev parameterization, which solved the latter two of the above mentioned three major problems of $\alpha$ viscosity, namely, the unphysical transition into the self-gravitating regime and the exceedingly long viscous timescales of galactic center disks. Their Ansatz, however, is still a parameterization and not an encompassing solution to the turbulence problem. Based on laboratory experiments and on theoretical considerations (Wendt 1933; Taylor 1923, 1936), the viscosity $\nu$ is written as

$$\nu = \frac{Rv_{\phi}}{Re_{\text{crit}}} = \beta Rv_{\phi},$$

where, in analogy with the $\alpha$ parameterization, a scaling quantity $\beta = Re_{\text{crit}}^{-1}$ has been defined. In this prescription, however, the scaling quantity $\beta$ is not arbitrarily chosen but is rather the inverse of the critical Reynolds number $Re_{\text{crit}}$, which, in turn, is thought to be of order $10^2, \ldots, 10^3$. In addition to the prescription of equation (2), it is required that the corresponding turbulent velocity scale $v_{\text{turb}}$ is smaller than or equal to the sound velocity: $v_{\text{turb}} \leq c_s$. This is the so-called dissipation limit.

The ensuing viscous timescale, $\tau_{\text{visc}} = (\beta\omega)^{-1}$ with the azimuthal angular frequency $\omega = v_{\phi}/R^2$, is sufficiently short to allow for efficient disk accretion in galactic centers, thus reconciling the observed luminosities and the derived viscous timescales. They are even sufficiently short to allow for the rapid formation of the SMBHs in the highest redshift quasars. Finally, the transition to non–self-gravitating, dissipation-limited accretion disks (i.e., the regime in which Shakura & Sunyaev’s original parameterization is so successful) not only recovers the $\alpha$ prescription as the limiting case, but $\beta = Re_{\text{crit}}^{-1} \approx 10^{-2}, \ldots, 10^{-1}$ also yields corresponding values of $\alpha$ that are compatible with the above discussed range of values.

In this contribution, we discuss gravitational instability as a possible origin for turbulence, in particular in self-gravitating accretion disks, and its relation to $\beta$ viscosity parameterization. In § 2 we describe the properties of the instability and in § 3 its connection with viscosity. In § 4 we summarize and discuss our results.

2. GRAVITATIONAL INSTABILITY IN SELF-GRAVITATING ACCRETION DISKS

For a geometrically thin stationary accretion disk (Frank et al. 2002), neglecting boundary terms, we have

$$\nu \Sigma = -\frac{\dot{M}}{2\pi R} \Omega \left( \frac{\partial \Omega}{\partial R} \right)^{-1}$$

$$\dot{M} = -2\pi R \Sigma v_{\phi},$$

where $\dot{M}$ is the (constant) mass accretion rate, $R, \varphi$ are the radial and azimuthal coordinates, $v_{\phi}$ is the corresponding velocity component, and $\nu$ is the kinematic viscosity. Using the monopole approximation for the gravitational potential (Mihalas & Numemura 1997), approximating the disk mass enclosed within a radius $R$ by $M_{\text{disk}}(R) = R^2 \pi \Sigma(R)$, neglecting the central mass in comparison to the disk mass, and assuming the $\beta$-parameterization of the viscosity, one can express all quantities as explicit functions of $M$, $\beta$, and $R$. For the surface density (integrated in vertical direction), for instance, one gets $\Sigma = (M/2\pi \beta[1/(G\pi)^{1/2}])^{3/2} R^{-1}$. The mass flow rate $\dot{M}$, in turn, relates directly to the surface density at the disk’s outer radius, $\dot{M}_{\text{out}} = (2\pi)^{3/2} (G/2)^{1/2} \beta R_0^{3/2} \Sigma_0^{3/2}$.

It is well known (e.g., Mishurov et al. 1976; Kato & Kumar 1960; Kumar 1960; Stephenson 1961, and in more detail Hunter & Schweiker 1981; Hunter & Horak 1983) that an infinite rotating viscous medium is Jeans-unstable; i.e., the Jeans criterion for instability

$$k < \frac{2\pi G \Sigma}{c_s^2} = k_{\text{j}}$$

is valid, where $k$ is the wavenumber, $G$ is the gravitational constant, $k_{\text{j}}$ is the Jeans wavenumber, and $\lambda_{\text{j}} = 2\pi/k_{\text{j}}$ is the corresponding Jeans wavelength.

The same is true for uniformly rotating viscous disks (Lynden-Bell & Pringle 1974). A somewhat more elaborate discussion on viscous disks can be found in Gammie (1996) and Antonov & Kondratyev (1999). All this implies that the Toomre criterion for stability of an inviscid disk

$$Q = \frac{c_s \kappa}{\pi G \Sigma} > 1$$

(6)

(where $Q$ is called the Toomre parameter and $\kappa$ is the epicyclic frequency, Toomre 1964; Goldreich & Lynden-Bell 1965) is not a valid criterion for stability in viscous disks. In the above mentioned papers, in the Navier-Stokes equation the term $\partial \eta/\partial R = (\partial \eta/\partial \Sigma)(\partial \Sigma/\partial R)$ is missing, which, however, can be

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4 In the framework of our approximations, $\kappa = \sqrt[3]{3\Omega}$.
of importance (see, e.g., Schmit & Tscharnuter 1995; Fridman & Polyachenko 1984).

Using the Toomre parameter $Q$ as defined in equation (6), solving for $c_s$, and putting this into the hydrostatic equilibrium for fully self-gravitating disks, leads to

$$Q^2 = \frac{\hbar k^3}{\pi G \Sigma}.$$  

(7)

In the framework of our approximations we get

$$Q^2 = 3 \frac{h}{R}.$$  

(8)

This means that, neglecting viscosity, any geometrically thin $(h \ll R)$ fully self-gravitating accretion disk is Toomre-unstable $(Q < 1)$.

In the following we derive the full dispersion relation for axisymmetric $(\partial/\partial \phi = 0)$ waves in a thin viscous disk starting from the Navier-Stokes equations in cylindrical coordinates with a full handling of the $\mathbf{V} \cdot \mathbf{a}$ term, where $\mathbf{a}$ is the viscous stress tensor (see, e.g., Landau & Lifshitz 1963).

We linearize the disturbances about a stationary value for all quantities $X \in \{ \Sigma, v_r, v_\phi, \Phi \}$ ($\Phi$ is the gravitational potential)

$$X = X_0 + \delta X,$$

$$\delta X = X_i \exp \{i(\omega + kR)\},$$  

(9)

$$\delta X \ll X_0,$$

and omit quadratic and higher order terms of small values. Derivatives of small values are assumed to be also small.

In the limit $k \gg (1/R)$, the Poisson equation $\Delta \delta \Phi = 4\pi G \Sigma \delta(\zeta)$, where $\delta(\zeta)$ is the Kronecker symbol, is solved by (Binney & Tremaine 1987)

$$\delta \Phi = -2\pi G \frac{\delta \Sigma}{|k|}.$$  

(10)

A linearized combination of the continuity equation for geometrically thin disks (eq. [3]), the Poisson equation, and of the axisymmetric version of the Navier-Stokes equation leads to the following dispersion relation:

$$s^3 + \frac{2}{3} \nu k^2 s^2 + \left(\frac{4}{3} \nu^2 k^4 + 3\Omega^2 + c_s^2 k^2 - 2\pi G \Sigma k\right)s$$

$$+ \nu k^2 \left(c_s^2 k^2 + \Omega^2 - 2\pi G \Sigma k - \nu^2 k^2\right) = 0,$$  

(11)

where $\text{Re}(s) = -\text{Im}(\omega)$. Using the Routh-Hurwitz theorem (see, e.g., Gradshteyn & Ryzhik 2000), we determine the sign of the (real) root of 11 and derive as a necessary criterion for stability:

$$a_1(k) = c_s^2 k^2 + \Omega^2 - 3 \frac{\nu^2}{R^2} k^2 - 2\pi G \Sigma k > 0.$$  

(12)

The only extremum is $k_0 = \pi G \Sigma/[c_s^2 - 3(\nu^2/R^2)]$ with the two cases

$$\frac{\partial^2 a_1}{\partial k^2} = 2(c_s^2 - 3\beta^2 v_\phi^2) > 0 \Rightarrow c_s > 3\beta v_\phi$$

$$\text{or} \quad \frac{\partial^2 a_1}{\partial k^2} < 0 \Rightarrow c_s < 3\beta v_\phi.$$  

(13)

For $c_s > 3\beta v_\phi$, $k_0$ is a minimum and the condition for stability can be rewritten as

$$\frac{1}{3} Q^2 - \frac{(\nu^2 \Omega^2)}{(\pi G \Sigma)^2} > 1.$$  

(14)

This condition, however, cannot be fulfilled, since the second term is negative and $3^{-1} Q^2 \ll 1$ by virtue of equation (8). On the other hand, if $c_s < 3\beta v_\phi$, then obviously $a_1(k \to \infty) < 0$, and all large $k$, i.e., small $\lambda$, are unstable. Thus all geometrically thin FSG $\beta$-disks are unstable.

A typical dispersion relation is shown in Figure 1. The imaginary part of $\omega$ (the second solution) is plotted here for the parameters $R = 1 \times 10^{18}$, $R_o = 3 \times 10^{14}$ m, $\Sigma_o = 30$ kg m$^{-2}$, $c_s = 1000$ m s$^{-1}$, and $\beta = 5 \times 10^{-3}$. Solutions 1 and 3 seem to be always stable, i.e., $\text{Im}(\omega) > 0$.

3. THE LINK BETWEEN GRAVITATIONAL INSTABILITY AND VISCOSITY IN SELF-GRAVITATING ACCRETION DISKS

We propose that the gravitational instability of the disks, instead of a hydrodynamic instability, is the main driver of turbulence. Neglecting pressure and shear, the wavelength where the timescale of viscosity equals the timescale of gravity defines a natural length scale. This requires $\nu k^2 = (2\pi G \Sigma k)^{1/2}$ and leads again in the framework of our approximations to

$$\lambda_{\text{min}} = (4\pi^{1/4})^{\frac{2}{5}} \beta^{2/5} R.$$  

(15)

This characteristic wavelength corresponds, up to a factor of 1.5, to the minimum of the imaginary part of $\omega$—which is rather independent of all the other parameters—and thus is the predominant size of structure in the system.
If we now identify $\lambda_{\text{min}}$ with the characteristic length scale of turbulence, we get

$$l_{\text{turb}} = \sqrt{\beta} R = \lambda_{\text{min}} = (4\pi^{3})^{1/3} \beta^{2/3} R.$$  \hfill (16)

This, in turn, allows us to derive a value for $\beta$,

$$\beta = \frac{1}{16\pi^n} = 6.5 \times 10^{-5}. \hfill (17)$$

Given our approximations, this is a rather remarkable agreement with the value, proposed for a hydrodynamically driven turbulence, $\beta = \operatorname{Re}^{-1} = 10^{-2}, \ldots, 10^{-3}$. Moreover, one has to note that the determination of minimum of the dispersion relation, which of course suffers from our approximations, enters by the sixth power.

### 4. SUMMARY

We have shown (eq. [7]) that geometrically thin, self-gravitating accretion disks are gravitationally unstable (which, in itself, is of rather little surprise). This instability may lead to turbulence and thus viscosity in the disk. This viscosity does not require the presence of a (magneto-)hydrodynamic instability. However, although based on different physical considerations, the functional form of this gravitationally driven viscosity is the same as that of the $\beta$ parameterization. For the scaling parameter $\beta$ we derive a value of order $10^{-4}$. Thus, this viscosity has the required properties to account for the observed fast formation of the first supermassive black holes in highly redshifted quasars and for the cosmological evolution of the black hole mass distribution (Duschl & Strittmatter 2005).

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