BEC Collapse, Particle Production and Squeezing of the Vacuum

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Phenomena associated with the controlled collapse of a Bose-Einstein condensate described in the experiment of Donley et al. [1] are explained here as a consequence of the squeezing and amplification of quantum fluctuations above the condensate by the condensate dynamics. In analyzing the changing amplitude and particle contents of these excitations, our simple physical picture provides excellent quantitative fits with experimental data on the scaling behavior of the collapse time and the amount of particles emitted in the jets.

In the experiment described by Donley et al. [1], a Bose-Einstein condensate (BEC) in a cold (3nK) gas of Rubidium atoms is rendered unstable by a sudden inversion of the sign of the interaction between atoms. This is done by altering the binding energy at Feshbach resonance with an external magnetic field. After a waiting time \(t_{\text{coll}}\), the condensate implodes (called Bose-Nova), and a fraction of the condensate is suddenly turned off. For a certain range of values of \(t_{\text{evolve}}\), new emissions of atoms from the condensate seem to corroborate this view.

The model is based on the Hamiltonian operator for \(N\) interacting atoms with mass \(M\) in a trap potential \(V (r) = \left( \omega_{z}^{2} r^{2} + \omega_{\rho}^{2} \rho^{2} / 2, \right)\), with radial \(\rho\) and longitudinal \(z\) coordinates measured in units \(a_{\text{ho}}\), where \(a_{\text{ho}}\) is a characteristic length of the trap, with associated (dimensionless) frequencies \(\omega_{z} = \omega_{\text{axial}} / \omega \sim 1 / 2\) and \(\omega_{\rho} = \omega_{\text{radial}} / \omega \sim \sqrt{2}\). The interaction is assumed to be short ranged. We introduce a dimensionless field operator \(\Psi (r) \equiv a_{\text{ho}}^{-3/2} \Psi (x)\), and a dimensionless coupling constant \(\kappa = (\hbar \omega a_{\text{ho}})^{-1} U = 4 \pi (a / a_{\text{ho}})^{-1}\).

\(\Psi\) obeys the equation of motion \(\dot{\Psi} = i \left[ \hat{H}, \Psi \right]\) and satisfies the equal time commutation relations \(\Psi (t, r), \Psi (t', r') = \delta^{(3)} (r - r')\). We decompose the Heisenberg operator \(\Psi = \Phi (r, t) + \psi (r, t)\) into a c-number condensate amplitude \(\Phi\) and a q-number non-condensate amplitude \(\psi\), consisting of the fluctuations or excitations. We obtain the equation of motion for the fluctuation field by subtracting from the full Heisenberg equation the Gross - Pitaevsky equation (GPE) governing its own expectation value under the self-consistent mean field approximation, \(\psi^{\dagger} \psi \simeq \text{resonance between the atoms and the molecules is expected to exist for all of these experiments, and has been shown to play an important role in the outcomes of some [3], we deem it unlikely that it plays a dominant role in this experiment other than renormalizing the scattering length (For details, see [4]). Indeed no oscillations are reported in the original experimental paper. Instead, as this letter shows, the primary mechanism for the Bose Nova phenomena is the parametric amplification of quantum fluctuations by the condensate dynamics, resulting in bursts and jets as particle production from the squeezing of the vacuum. Recent numerical simulations [4] and rigorous theoretical investigations [6] indicating the inadequacy of mean field theory seem to corroborate this view.

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\[^{1}\]We call attention to the distinction between the ‘Bose-Nova’ experiment studied here and other BEC collapse experiments [2, 3]. At magnetic fields around 160G, where the effective scattering length is of the order of 500\(a_{\text{ho}}\) (and positive)\((a_{\text{ho}} = 0.529 \times 10^{-10}\text{m} \) is the Bohr radius) it is possible to observe oscillations between the usual atomic condensate and the molecular state [3] with a frequency of oscillations of hundreds of KHz [4]. By contrast, in the ‘Bose-Nova’ experiment [1] typical fields were around 167G, the scattering length was only tens of Bohr radii (and negative) and the frequency of atom–molecule oscillations may be estimated as well over ten MHz [5]. While coherent

\[^{2}\]We use a sign convention such that the effective coupling constant is positive for an attractive interaction, and a system of units where the length \(a_{\text{ho}}\), time \(t_{\text{ho}}\) and energy scale \(E_{\text{ho}} = \hbar \omega = M a_{\text{ho}}^{2} \) are defined with reference to the average frequency \(\omega\). We work with units such that these three scales take the value 1.

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\[ \langle \psi^\dagger \psi \rangle = \bar{n}, \quad \psi^2 \simeq \langle \psi^2 \rangle = \bar{m} \quad \text{and} \quad \psi^\dagger \psi^2 \simeq 0. \]

We next parametrize the wave functions as \( \Phi = \Phi_0 e^{-i\Theta}, \quad \psi = \psi_0 e^{-i\Theta} \), where \( \Phi_0 \) and \( \Theta \) are real. During the early stages of evolution, we may regard the condensate density as time independent, and the condensate phase as homogeneous, \( \Phi_0 = \Phi_0(r), \quad \Theta = \Theta(t). \) We may then write the equation for the fluctuation field

\[
\left[ \frac{i}{\hbar} \frac{\partial}{\partial t} - H + E_0 \right] \psi_0 + u\Phi_0^2 \left( \psi_0 + \psi_0^\dagger \right) = 0 \quad (1)
\]

where \( E_0 = \frac{1}{2}(\omega_z + 2\omega_0) \). To solve this equation\(^3\) we decompose it into a self-adjoint and an anti-adjoint part \( \psi_0 = \xi + i\eta \), each part satisfying an equation

\[
\frac{\partial \xi}{\partial t} = [H - E_0] \eta \quad (2)
\]

\[
\frac{\partial \eta}{\partial t} + [H - E_0 - 2u\Phi_0^2] \xi = 0. \quad (3)
\]

Since the trap Hamiltonian is time - independent, we have

\[
\frac{\partial^2 \xi}{\partial t^2} + [H - E_0] H_{eff} \xi = 0. \quad (4)
\]

Here \( H_{eff} = H - E_0 - 2u\Phi_0^2 \). To have an unstable condensate it is necessary that at least one of the eigenvalues of \( H_{eff} \) is negative; the boundary of stability occurs when the lowest eigenvalue is exactly zero. One further consideration is that we are interested in the part of the fluctuation field which remains orthogonal to the condensate, since fluctuations along the condensate mode may be interpreted as condensate fluctuations rather than particle loss \([11]\). The ground state of \( H_{eff} \) is certainly not orthogonal to the condensate, since neither have nodes.

If we adopt the values \( \omega_z = 1/2, \quad \omega_0 = \sqrt{2} \), relevant to the JILA experiment, then instability occurs when \( \kappa = N_0 a_{crit}/a_{ho} = 0.51 \). This result compares remarkably well with the experimental value \( \kappa = 0.55 \) \([11, 12]\), as well as with the theoretical estimate presented in Ref. \([12]\). This agreement may be seen as natural, as the equations we postulate for the fluctuations may be obtained from the linearization of the GPE, discarding both \( \bar{n} \) and \( \bar{m} \). In both calculations, the geometry of the trap plays a fundamental role.

\(^3\)The squeezing of quantum unstable modes and its back reactions on the condensate has been considered before, e.g., as a damping mechanism for coherent condensate oscillations \([2]\), but the application to the description of condensate collapse has up to now been mostly qualitative \([11]\).

**Scaling of \( t_{\text{collapse}} \) and Critical Dynamics** As we have already noted, even for condensate densities above the stability limit, no particles are seen to be lost from the condensate during a waiting time \( t_{\text{collapse}} \). Experimentally, \( t_{\text{collapse}} \) is seen to get very large when the threshold of stability is approached from above, in a way which closely resembles the critical slowing down near the transition point characteristic of critical dynamics. In our problem, the quantity which plays the role of relaxation time is the characteristic time \( \varepsilon^{-1} \) of exponential growth for the first unstable mode. This quantity diverges at the stability threshold, which in our analogy corresponds to the critical point. By dimensional analysis, we are led to the estimate \( t_{\text{collapse}} \sim \varepsilon^{-1} \). Close to the critical point, we find

\[
t_{\text{collapse}} = t_{\text{crit}} \left( \frac{a}{a_{cr}} - 1 \right)^{-1/2}
\]

The power law Eq. \( (5) \) describes with great accuracy the way \( t_{\text{collapse}} \) scales with the scattering length; the best fit to the experimental data is obtained for \( t_{\text{crit}} \sim 5\text{ms} \).

In Fig. 1 we plot the scaling law \([5]\) (full line) derived here and compare it with the experimental data for \( N_0 = 6000 \) as reported in Refs. \([11, 13]\) (small black points), the \( t_{NL} \sim (aN_0)^{-1} \) prediction (suitably scaled) as given in \([10, 13]\) (dashed line) and the results of numerical simulations reported in \([14]\) (large grey dots). While all three theoretical predictions may be considered satisfactory, the \( t_{NL} \sim (aN_0)^{-1} \) behavior fails to describe the divergence of \( t_{\text{collapse}} \) as the critical point is approached, and the results of numerical simulations reported in \([14]\) based on an improved Gross-Pitaevskii equation tend to be systematically above the experimental results, which may be a further indication of
the quantum origin of this phenomenon.

Bursts and Jets as Amplified Quantum Fluctuations We now consider the evolution of quantum fluctuations, treated as a test field riding on the collapsing condensate whose dynamics is extracted from experiment. The initial state is defined by the condition that \( u = 0 \) for \( t < 0 \); we shall take it to be the particle vacuum \( |0\rangle \), defined by \( \psi_0(x,0)\langle 0| = 0 \) everywhere.

One can introduce a mode decomposition of the \( \xi \) operator based on the eigenfunctions of \( [H - E_0]H_{\text{eff}} \). For short wavelengths \( \lambda \), since \( H \sim \lambda^{-2} \gg 2u\Phi_0^2 \), we expect these eigenfunctions will approach the trap eigenmodes. The fact that particles in bursts are seen to oscillate with the trap frequencies \( \Omega \) also suggests that their dynamics is determined by the trap Hamiltonian. Based on these observations we can assume a homogeneous condensate \( 2u\Phi_0^2 \sim \kappa^{-1}a\omega_zN_0(t) \), where \( N_0(t) \) is the instantaneous total number of particles in the condensate. In practice, \( \kappa^{-1} \) is a measure of the overlap between the condensate and the excitation modes. Therefore, the approximation may be improved by adjusting \( \kappa \) according to the range of modes where it will be applied.

Let \( \bar{N}_0 \) be the initial number of particles in the condensate, and \( a_{cr} = \kappa/\bar{N}_0 \) the corresponding critical scattering length. Trap eigenfunctions \( \psi_\bar{n}(r) \) are labelled by a string of quantum numbers \( \bar{n} = (n_z, n_x, n_y) \). The eigenvalues of the trap Hamiltonian are (with the zero energy already subtracted) \( E_{\bar{n}} = \omega_z n_z + \omega_r(n_x + n_y) \). There are two kinds of modes, stable (oscillatory, or thawed) modes if \( E_{\bar{n}} > \left( \frac{a}{a_{cr}} \right) \omega_z \), and unstable (growing, or frozen) modes if not. In the former case we find that, although we assume vacuum initial conditions, these modes do not remain empty. Up to \( t_{\text{collapse}} \), when the number of particles in the condensate is constant, the density

\[
\dot{n}(r, t) = \frac{1}{8} \left( \frac{a}{a_{cr}} \right)^2 \omega_z^2 \sum_{\bar{n}} \psi_{\bar{n}}^2(r) \frac{\sin^2 \omega_{\bar{n}} t}{\omega_{\bar{n}}^2} \tag{6}
\]

(where \( \omega_{\bar{n}} = \sqrt{E_{\bar{n}} \left[ E_{\bar{n}} - \left( \frac{a}{a_{cr}} \right) \omega_z \right]} \) has a constant term and an oscillatory term. This oscillatory term is responsible for the appearance of ‘bursts’ of particles oscillating within the trap observed in the Bose-Nova experiment [1]. In the WKB limit it describes a swarm

of particles moving along classical trajectories in the trap potential.

In the opposite case \( E_{\bar{n}} \leq \left( \frac{a}{a_{cr}} \right) \omega_z \), the formulae for the density is obtained by the replacement of \( \omega_{\bar{n}} \) in (6) by \( i\sigma \), thus \( \omega_{\bar{n}}^{-1} \sin \omega_{\bar{n}} t \rightarrow \sigma_{\bar{n}}^{-1} \sin \sigma_{\bar{n}} t \). Physically their difference is immense. In the first place, the density is growing exponentially, but unlike the previous case, there is no oscillatory component, and these particles do not oscillate in the trap, in the sense described above. These modes come alive at \( \tau_{\text{evolve}} \) (as the scattering length is set to zero), whence they become ordinary trap modes which oscillate in the trap in the same way as the the burst modes . To the observer, they appear as a new ejection of particles from the core of the condensate, which makes up the so-called ‘jets’. The sudden activation of a frozen mode (we are borrowing the language and concept of cosmological structure formation) by turning off the particle - particle interaction may be described as a “thaw”.

Observe that in this picture several conspicuous features of jets become obvious. Jets may only appear if the turn - off time \( \tau_{\text{evolve}} \) is earlier than the formation time of the remnant. Once the condensate becomes stable again, there are no more frozen modes to thaw. On the other hand, jets will appear (as observed) for \( \tau_{\text{evolve}} < t_{\text{collapse}} \), when the condensate has not yet shed any particles. Also jets must be less energetic than bursts, since they are composed of lower modes.

Beyond \( t_{\text{collapse}} \) the number of particles in the condensate, and therefore the instantaneous frequency of the excited modes, becomes time dependent. If we confine ourselves to the early stages of collapse we may assume nevertheless that the condensate remains homogeneous. Shifting the origin of time to \( t_{\text{collapse}} \) for simplicity, we write \( N_0(t) = \bar{N}_0 \exp(-t/\tau) \) (see Fig. 2).

After expanding in trap eigenmodes we find the two kinds of behavior described above. If \( E_{\bar{n}} > \left( \frac{a}{a_{cr}} \right) \omega_z \), the
mode is always oscillatory. If $E_{\vec{n}} < (a\omega z)$, the mode is frozen at $t_{\text{collapse}}$, but thaws when $\exp(-t/\tau) \sim E_{\vec{n}}/a\omega z$. During the frozen period, the modes are amplified, but they only contribute to bursts after thawing. If the evolution is interrupted while still frozen, they appear as a jet. We therefore conclude that the number of particles $N_{\text{jet}}$ in a jet at time $\tau_{\text{evolve}}$ is essentially the total number of particles in all frozen modes at that time. This is plotted in Fig 3, for $N_0 = 16,000$, $\omega_{\text{radial}} = 110$ Hz, $\omega_{\text{axial}} = 42.7$ Hz, $a = 36a_0$, and $\kappa = 0.46$, and compared to the corresponding results as reported in [1].

![Figure 3:](image)

We see that the agreement is excellent at early times (up to about 6ms). For later times, this model overestimates the jet population. This is due to the fact that, by not considering the shrinking of the condensate, we are overestimating the overlap between the condensate and the fluctuations, thus delaying the thaw. It nevertheless reproduces the overall slope of particle number with $\tau_{\text{evolve}}$. It should also be remembered that we are computing the expected number of particles, but in the highly squeezed state which results from the frozen period, the fluctuations in particle number are comparable to the mean number itself.

In this letter, we have presented a new viewpoint towards understanding the salient features in the physics of controlled collapse of a Bose-Einstein condensate described in the experiment of [1], i.e., in terms of quantum vacuum fluctuations parametrically amplified by the condensate dynamics. Our way of thinking here is influenced by insights from the quantum field theory of particle creation and structure formation in cosmological spacetimes as well as theories of spinodal instability in phase transitions. One can conceivably design experiments with BEC dynamics to test out certain basic mechanisms and specific features of quantum processes in the early universe, thus opening a new venue for performing ‘laboratory cosmology’. Acknowledgement We are obliged to E. Donley and S. Kokkelmans for their prompt and informative responses to our queries on some details of these experiments and for communicating key unpublished data. EC also acknowledges discussions with Eric Bolda. This research and EC’s visits to UMD are supported in part by a NSF grant PHY98-00967, a NIST grant and by CONICET, UBA, Fundacion Antorchas and ANPCyT under grant PICT99 03-05229.

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