Relativistic three-body effects in black hole coalescence

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Three-body interactions are expected to be common in globular clusters and in galactic cores hosting supermassive black holes. Here we consider an equal-mass binary-black-hole system in the presence of a third black hole. Using numerically generated binary-black-hole initial-data sets, and first- and second-order post-Newtonian (1PN and 2PN) techniques, we find that the presence of the third black hole has non-negligible relativistic effects on the location of the innermost stable circular orbit (ISCO), and that these effects arise at 2PN order. In particular, we study the more astrophysically realistic situation of a stellar-mass black-hole binary in orbit about a third supermassive black hole. In general, the proximity of the massive black hole has stabilizing effects on the orbiting binary, leading to an increase in merger time and a decrease of the terminal orbital frequency, and an amplification of the gravitational radiation emitted from the binary system by up to 6% percent.

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INTRODUCTION

Gravitational radiation, as predicted by general relativity, plays an important observable role in the relativistic dynamics of astrophysical systems. Collisions of compact objects, such as neutron stars and black holes, produce characteristic gravitational wave signals that are observable up to very high redshifts. In particular, binary-black-hole systems are the prime scientific target source of gravitational waves for both the current generation of ground-based detectors, such as LIGO [1], and for the next generation of space-based detectors, such as LISA [2].

Theoretically accurate models of black hole coalescences based on the theory of general relativity are expected to provide crucial information for the interpretation of these gravitational-wave observations. So far, all numerical models have focused on isolated binary-black-hole systems. However, in a realistic scenario it is reasonable to expect that some of these black-hole binaries will, at some point in their lifetime, interact gravitationally with a third compact object. The presence of the third body may influence the evolution and gravitational radiation emission of the black-hole binary during the inspiral and merger phase. This may produce some important observational effects on the gravitational waveforms.

Relativistic three-body interactions are expected to play an important role in astrophysical scenarios including (1) multibody interactions in high-density cores of globular clusters [3], (2) stellar-mass binary-black-hole systems interacting with a central supermassive black hole, and (3) hierarchical triples of massive black holes that might be formed in the nuclei of galaxies undergoing sequential mergers [4, 5]. In the globular cluster scenario, a possible mechanism to produce hierarchical triple systems is through binary-binary interactions [4, 6, 7, 8]. At least 20-50% of binary-binary encounters may actually result in a stable hierarchical three-body system [4]. Estimates of the lifetime of a triple system are made in [3], where it is noted that a three-body system can survive several hundred times longer than the orbital period of one of the original binaries [3]. Further astrophysically motivated studies of three-body systems have been considered in Refs. [10, 11, 12].

In this paper we consider the effect of the presence of a third black hole on the location of the innermost stable circular orbit (ISCO) of a binary non-spinning black-hole system. We use numerical and post-Newtonian techniques to study two three-body scenarios: (1) a test case, in which the third black hole has comparable mass to each black hole in the binary, and for simplicity is stationary, and (2) the more astrophysically realistic case of a stellar-mass black-hole binary in orbit about a supermassive black hole. Scenario (2) could be treated using only post-Newtonian methods, but in scenario (1) we find that calculations at 2PN accuracy give results consistent with the fully general-relativistic numerical approach.

THREE BLACK-HOLE CONFIGURATION

We consider the following configuration, illustrated in Fig. 1. Two black holes are separated by a distance d, and are assumed to be in quasi-circular orbit. A third black hole is located on the axis of rotation of the binary, a distance l from the center of rotation. Each black hole in the binary has equal and opposite momentum P, and the total angular momentum of the binary is $J = Pd$. In the first set of results we will treat the third black hole as being instantaneously stationary; we will later relax this simplifying assumption.

Each black hole in the binary has mass $m$. For the remainder of this paper, when referring to numerical calculations, $m$ will correspond to the bare mass of each
black hole, as defined later; in analytic post-Newtonian calculations $m$ will be the Newtonian mass of the body. The third body has mass $m_3$. The total mass of the three black holes will be $M = 2m + m_3$. In numerical calculations, the total ADM mass of the spacetime (which will differ from $M$), will be denoted by $M_{ADM}$.

![Diagram of three black holes](image)

FIG. 1: Initial three black hole configuration: the third body sits motionless directly above the orbital plane of the binary.

We will model this configuration using two approaches. The first is to consider numerically generated initial-data sets, i.e., solutions of the initial-value equations of general relativity. We will identify circular orbits (and the ISCO) of the binary by searching for minima in an effective-potential of the spacetime \[13\]. As such, we are making a number of assumptions and approximations. We are assuming that the effective-potential method reliably yields parameters consistent with quasi-circular orbits, and that the initial data we use are a reasonable approximation to the “correct” initial data for two black holes in orbit. In this numerical approach we are ignoring entirely any motion of the third black hole: we consider only the effect of a stationary third black hole on the instantaneous binding energy of the binary system, and make the loose assumption that its effect on the locations of quasi-circular orbits will be indicative of the effect of a third black hole passing near the binary. The purpose of this work is to determine the qualitative effect of the presence of a third body on the ISCO of a binary.

The second approach is to use post-Newtonian calculations. Quasi-circular orbits and the ISCO have been identified for binary systems (see, for example, \[14, 15\]), and it is possible to extend this procedure to a three-body system. The results, obtained to 1PN and 2PN accuracy, are compared with the numerical initial-data results. The post-Newtonian approach has the advantage that it is straightforward to generalize to cases where the binary is in circular orbit about the third black hole.

**METHOD**

**Initial-data approach**

We first outline the numerical initial-data approach. Black-hole initial data consist of the solutions to the four initial-value equations that result from the (3+1) decomposition of the field equations of general relativity \[16, 17\]. These equations are the Hamiltonian and momentum constraints for the spatial metric $\gamma_{ij}$ and extrinsic curvature $K_{ij}$ on one time-slice \[16\], which, in vacuum, are

\[
\nabla_j \left( K_{ij} - \gamma_{ij} K \right) = 0, \tag{1}
\]

\[
R + K^2 - K_{ij} K^{ij} = 0, \tag{2}
\]

where $K = \gamma^{ij} K_{ij}$ is the trace of the extrinsic curvature, $R$ is the Ricci scalar, and all quantities, including the covariant derivative $\nabla_i$, are defined with respect to the time-slice.

One way to solve the constraint equations is through the conformal transverse-traceless decomposition \[16\]. In this decomposition there exist analytic solutions of the momentum constraint, the Bowen-York solutions, that can describe any number of boosted black holes. In particular, we can write down a solution for two boosted black holes, plus a single stationary black hole (which is a trivial solution). The Hamiltonian constraint remains to be solved numerically, and we do this using the puncture approach of Brandt and Brügmann \[18\].

Given the freedom in specifying the orbital parameters of the initial-data sets for black-hole spacetimes, we would like to identify which of those sets represents two black holes in quasi-circular orbits. To this end we implement the effective-potential method of Cook \[13\] in which minima are located in sequences of total ADM energy, $E_{ADM}$, versus coordinate separation, $d$, for constant values of the orbital angular momentum of the binary, $J$, and individual black-hole masses $m$ and $m_3$. In this work the ADM mass calculated at each puncture is used to denote the bare mass of each black hole \[19\]. This procedure is carried out for many sequences until the lowest value of $J$ that produces a sequence with a minimum is located. We denote this last minimum as the innermost stable circular orbit (ISCO) of the binary. Once the ISCO is located, its orbital angular frequency, $\Omega$, is calculated via

\[
\Omega = \left. \frac{\partial E_{ADM}}{\partial J} \right|_{m_3}. \tag{3}
\]

The application of this approach for equal-mass binary-black-hole systems can be found in \[12, 20\].

**Post-Newtonian approach**

We now turn to the procedure to find quasi-circular orbits and the ISCO in post-Newtonian theory. The ISCO for two-body systems has been calculated using post-Newtonian methods by a number of authors \[14, 15\]. The approach can be generalized to three bodies and compared with our numerical results. This method will not
tell us anything about the stability of the “orbits” that have been identified, and indeed in general we do not expect a general relativistic three-body system to be any more stable than its Newtonian counterpart. However, we can still identify parameters that meet the requirements of quasi-circular orbits, as described below, and suggest configurations that we reasonably expect to be stable. In particular, when a stellar-mass black-hole binary orbits a third supermassive black hole, which is the astrophysically relevant scenario we ultimately wish to study.

In the two-body case, the post-Newtonian and numerical (Bowen-York data) results differ, but in this work we are interested in the differential effect of the presence of a third body, and can test whether that effect is comparable when calculated using numerical or post-Newtonian methods.

We locate the post-Newtonian ISCO using the procedure outlined in [15]. We start with the Hamiltonian for a three-body system in the configuration described above, which is given by [21, 22, 23]. The three-body Hamiltonian is a function of the locations, masses and momenta of the three bodies; in total, 21 parameters. In the configuration described above, these parameters are reduced to five: \( m \), the mass of each body in the equal-mass binary, \( m_3 \), the mass of the third body, \( d \), the binary separation, \( l \), the distance of the third body from the center of mass of the binary, and \( p \), the magnitude of the linear momentum of each body in the binary. In all cases we consider, the third body is either stationary, or its motion is such that its angular momentum about the origin of the system is perpendicular to the angular momentum of the binary, and so the magnitude of the binary’s total angular momentum may be clearly given as \( J = pd \).

To identify circular orbits, one first locates minima order by order in the post-Newtonian expansion of the Hamiltonian. This procedure was performed semi-analytically with Mathematica, to yield the separation \( d \) of a circular orbit to 2PN accuracy in terms of the total binary angular momentum, \( J \). This value of \( d \) was inserted into the post-Newtonian Hamiltonian to get the energy, \( E_{\text{circ}} \), of circular orbits to 1PN or 2PN accuracy and to calculate the angular velocity of the orbit via (8), with \( E_{\text{ADM}} \) now replaced by \( E_{\text{circ}} \). This procedure was performed for different values of \( J \), to produce a plot of \( E_{\text{circ}} \) versus \( \Omega \); the minimum in this plot corresponds to the ISCO.

**RESULTS**

In the first configuration, the masses are \( m = 1; m_3 = 0.5 \), and the third body is stationary. The results of the numerical and post-Newtonian approaches are compared in Fig. 2 where shows the percentage change in the binding energy of the system, \( E_b \), and the total angular momentum of the binary, \( J \), at the binary’s ISCO, as a function of the coordinate distance of a third body from the center of the binary is shown. The percentage differences are with respect to the corresponding two-body ISCO for numerical, 1PN and 2PN calculations. Note that the numerical, 1PN and 2PN results are in different gauges, and the coordinate distance of the third black hole, \( l/M \), may correspond to (slightly) different physical distances in each of the three gauges. It is therefore not appropriate to make a quantitative comparison of these results. Note also that we are considering the binding energy of the entire three-body system, not only the binary. However, we can make the qualitative observations that (1) the binding energy of the system increases dramatically as the third black hole is placed closer to the binary, (2) there is better agreement between 2PN and numerical data than between 1PN and numerical data, and (3) most significantly, we need at least 2PN accuracy to see the qualitatively correct effect on the binary’s angular momentum, \( J \). For this reason we will use 2PN data in the subsequent calculations for a better motivated astrophysical scenario.

![Figure 2](image.png)

**FIG. 2:** The percentage change in the binding energy of the three-body system, \( E_b \), and the total angular momentum of the binary, \( J \), at the binary’s ISCO, as a function of the coordinate distance of a third body from the center of the binary. \( m = 1; m_3 = 0.5 \), and the third body is stationary.

Having seen that the presence of a third body has a significant effect on the binary’s ISCO, we wish to consider the effect in an astrophysically more realistic situation: the third body is far larger than each of the bodies in
the binary (by a factor of, for example, $10^5$), and the binary is in orbit about the third body. It is not practical to consider this scenario numerically: in particular, we cannot achieve suitable accuracy for such extreme mass ratios with our finite-difference code. However, the previous results show that 2PN calculations are adequate to describe the problem.

Fig. 4 shows the percentage change in the binding energy of the binary due to the presence of a third body with a mass $10^5$ times larger than that of each body in the binary. This configuration models a stellar-mass binary orbiting a supermassive black hole in a galactic core. All results were calculated to 2PN accuracy. The effect of the location of the third body was calculated when the third body was stationary (starred points), and when it had momentum (circles) consistent with the 2PN circular orbit of a two-body system of masses $2m$ and $m_3$ respectively. We can see that the effect is practically the same in both cases. Note that we are now considering the binding energy of the binary only (using the two-body Hamiltonian with the binary parameters found in the orbit search procedure), not that of the entire system, as we did in Fig. 2, and the effect of the third body is much smaller. However, it is still appreciable, around 6%, when the third body is close to its own ISCO with the binary.

![Figure 3: The percentage change in the binding energy, $E_b$, and orbital frequency, $\Omega$, of the binary at the ISCO, as a function of the distance of a third body from the center of the binary. The masses are $m_1 = m_2 = 1; m_3 = 10^5$.](image)

To examine the generality of this effect, we placed the third body at different orientations to the binary: in the plane of the binary, at 45 degrees to that plane, and we changed the direction of the third body’s momentum. The results were unaffected by these changes to within the accuracy of the method.

In addition to being an interesting purely relativistic effect, the displacement of the ISCO due to the presence of the third body has observational consequences on the emission of gravitational waves. A larger negative value of the binding energy at the ISCO can be associated with a larger loss of energy radiated by the system during the inspiral phase (comparable to that during the plunge and ring-down phase [24, 25].) We thus expect (1) an increase in the terminal amplitude of the inspiral gravitational waveform; (2) an increase in the duration of the pre-plunge phase; and (3) since the orbital frequency of the ISCO decreases due to the presence of the third body (see Fig. 3), the corresponding waveform will display a lower pre-plunge frequency. Similarly, the rotation parameter of the post-plunge remnant black hole will be smaller, due to a decrease in $J_{ISCO}$. This, in turn, will reduce the value of the least damped quasi-normal frequency of the final Kerr black hole.

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