LOCAL NONCOLLAPSING FOR COMPLEX MONGE-AMPÈRE EQUATIONS

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Abstract. We prove a local volume noncollapsing estimate for Kähler metrics induced from a family of complex Monge-Ampère equations, assuming a local Ricci curvature lower bound. This local volume estimate can be applied to establish various diameter and gradient estimate.

1. Introduction

The non-local collapsing of Ricci flow on Riemannian manifolds is a fundamental theorem for the compactness of Ricci flows [20], which states that along the flow the volume of the geodesic balls is uniformly bounded below by a positive constant, if the scalar curvature along the flow is bounded. In Kähler geometry, it has generated great interest to study degenerating families of Kähler metrics, satisfying certain complex Monge-Ampère (MA) equations [22]. Analogous to Ricci flow, it is tempting to study the non-local collapsing of Kähler metrics along a degenerating family.

Complex Monge-Ampère equations have been an important tool in the construction of canonical Kähler metrics, ever since Yau’s solution to the Calabi conjecture [26]. A priori estimates on the Kähler potentials are the keys to study the geometry of the Kähler metrics. Using pluripotential theory, Kołodziej [16] first proved the sharp $C^0$ estimate for complex MA equations, assuming the right-side belongs to some Orlicz space. Kołodziej’s approach was generalized to a family of complex MA equations, allowing the Kähler classes to degenerate in [6, 4, 5, 27]. Recently, a PDE-based method [10] was utilized to give a new and uniform proof of $C^0$ estimates for a class of fully nonlinear partial differential equations, which include in particular the complex MA and Hessian equations. In [7], X. Fu and the authors first revealed the relationship between the $C^0$ estimate of the Kähler potentials with the global diameter bound of the Kähler metrics, generalizing the distance estimate in [22]. The current paper aims to study how the $C^0$ estimates of Kähler potentials affect the geometry locally.

Let $(X, \omega_X)$ be a compact Kähler manifold of complex dimension $n$. Suppose $\chi$ is a smooth closed $(1, 1)$-form such that the class $[\chi]$ is nef, which means the class $[\chi]$ lies in the closure of the Kähler cone of $X$. For $t \in (0, 1]$, we consider a family of smooth closed forms

$$\hat{\omega}_t = \chi + t \omega_X.$$ 

Though $\hat{\omega}_t$ may not be Kähler forms, each $[\hat{\omega}_t]$ is a Kähler class. We consider the following complex Monge-Ampère equation

$$(\hat{\omega}_t + i\partial \overline{\partial} \phi_t)^n = c_t e^{F \omega_X^n}, \quad \sup_X \phi_t = 0.$$ 

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where $F$ is a smooth function normalized such that
\[ \int_X e^F \omega_X^n = \int_X \omega_X^n = V, \quad c_t = \frac{V_t}{V} = O(t^{n-\nu}), \quad V_t = \int_X \hat{\omega}_t^n \]
and $\nu \in \{0, \ldots, n\}$ is the numerical dimension of the class $[\chi]$.

We denote $\omega_t = \hat{\omega}_t + i\partial \bar{\partial} \varphi_t$ to be the Kähler metric satisfying (1.1) and write $g_t$ for the associated Riemannian metric. The following is our main theorem.

**Theorem 1.1.** For any $p > n$ and $R_0 \in (0, 1]$, if $\omega_t$ solves (1.1) for $t \in (0, 1]$ and the Ricci curvature $\text{Ric}(g_{t_0})$ satisfies
\[ \text{Ric}(g_{t_0}) \geq -\frac{K^2}{R_0^2} \text{ on } B_{g_{t_0}}(z_0, 2R_0) \]
for some $K \geq 0$, $t_0 \in (0, 1]$ and $z_0 \in X$, then
\[ \frac{\text{Vol}_{g_{t_0}}(B_{g_{t_0}}(z_0, R_0))}{\text{Vol}_{g_{t_0}}(X)} \geq CR_0^{\frac{2np}{p-n}}, \]
for some constant $C > 0$ depending only on $n, p, K, \omega_X, \chi$ and $\|e^F\|_{L^1(\log L)^p}$.

We remark that if $p \to \infty$, then the power exponent of $R_0$ in (1.2) tends to $2n$, which is natural since the metrics $\omega_t$ are close to Euclidean ones when $R_0$ is small. However, an example on Riemann surfaces (i.e. $n = 1$) shows that this power exponent $\frac{2np}{p-n}$ is sharp in some sense (see Example 3.1 in section 3). If $e^F \in L^q(X, \omega_X^q)$ for some $q > 1$, the right side of (1.2) can be made as $C\epsilon R_0^{2n+\epsilon}$ for any $\epsilon > 0$ with the constant $C\epsilon > 0$ depending additionally on $\epsilon$.

The estimate (1.2) can be viewed as an analogue of Perelman’s $\kappa$-noncollapsing theorem [20] as the following. Let $g(t)$ be the smooth solution of the Ricci flow on a real $n$-dimensional compact Riemannian $M$ for $t \in [0, T]$ with the initial metric $g_0$. Let $(x_0, t_0) \in M \times [0, T)$ and $r_0 \in (0, (t_0)^{\frac{2}{n}})$. If the scalar curvature satisfies
\[ R < r_0^{-2}, \text{ on } B_{g(t_0)}(x_0, r_0), \]
then
\[ \text{Vol}_{g(t_0)}(B(x_0, t_0, r_0)) \geq \kappa r_0^n, \]
for some $\kappa > 0$ that only depends on $n$ and $\nu[g_0, 2T]$, Perelman’s $\mu$-functional at the initial time.

We define the **radius of Ricci curvature lower bound** at $z$ for the metric $g_t$ by
\[ \bar{r}_{g_t}(z) = \sup\{r > 0 \mid \text{Ric}(g_t) \geq -\frac{1}{r^2} \text{ in } B_{g_t}(z, r)\} \]
as an analogue of the curvature radius introduced in [1]. An equivalent way to state Theorem 1.1 is that under the same setup
\[ \frac{\text{Vol}_{g_t}(B_{g_t}(z, \bar{r}_{g_t}(z)))}{\text{Vol}_{g_t}(X)} \geq C\bar{r}_{g_t}(z)^{\frac{2np}{p-n}}, \text{ for any } z \in X, \]
for some constant $C > 0$ depending only on $n, p, \omega_X, \chi$ and $\|e^F\|_{L^1(\log L)^p}$.
The main idea of the proof of Theorem 1.1 is motivated by that in [10, 11], that is, we compare the Kähler metric $\omega_t$ to some auxiliary complex Monge-Ampère equation, with the function on the right-side being truncated squared distance function of $g_t$. Additionally we also need the Riemannian geometric tools like the Laplacian comparison and volume comparison theorems.

Theorem 1.1 immediately implies the following diameter bound established in [7], by choosing $R_0 = 1$ in Theorem 1.1.

**Corollary 1.1.** For any $p > n$, if $\omega_t$ solves (1.1) for $t \in (0, 1]$ and the Ricci curvature $\text{Ric}(g_t)$ is bounded below by

$$\text{Ric}(g_t) \geq -K^2$$

for some $K \geq 0$, then there exists $C > 0$ depending only on $n, p, K, \omega_X, \chi$ and $\|e^F\|_{L^1(\log L)^p}$ such that

$$\text{Diam}(X, g_t) \leq C$$

for all $t \in (0, 1]$.

We also establish a local gradient estimate as an application of Theorem 1.1. We consider the following complex Monge-Ampère equation

$$(\omega_X + i\partial \bar{\partial} \varphi)^n = e^F \omega^n_X, \sup_X \varphi = 0$$

on a compact Kähler manifold $(X, \omega_X)$ of complex dimension $n$ with $F \in C^\infty(X)$ satisfying the normalization condition

$$\int_X e^F \omega^n_X = \int_X \omega^n.$$

**Theorem 1.2.** Let $\varphi$ be the solution of equation (4.1) and $g$ be the Kähler metric corresponding to the Kähler form $\omega = \omega_X + i\partial \bar{\partial} \varphi$. Given $p > n$ and $R_0 \in (0, 1]$, if

$$\text{Ric}(g) \geq -\frac{K^2}{R^2_0}, \text{ on } B_g(z_0, 2R_0)$$

for some $K \geq 0$ and $z_0 \in X$, then

$$|\nabla \varphi|^2_g \leq C, \text{ on } B_g(z_0, R_0)$$

for some constant $C > 0$ depending on $n, p, \omega_X, \|e^F\|_{L^1(\log L)^p}$, $K$ and $R_0$.

We will prove Theorem 1.1 in section 2. In section 3, we construct an example and show that the exponent in (1.2) is sharp. In section 4, we will prove Theorem 1.2 and discuss some other applications of Theorem 1.1 and Corollary 1.1 on the diameter bound of Kähler metrics satisfying certain complex Monge-Ampère equations.

2. **Proof of Theorem 1.1**

Since $\hat{\omega}_t$ may not be Kähler, we define the “envelope” associated to $\hat{\omega}_t$:

$$V_t = \sup\{v \mid v \in PSH(X, \hat{\omega}_t), v \leq 0\}.$$

It is known that for each $t \in (0, 1]$, $V_t$ is a $C^{1,1}$ function. We recall the following uniform $L^\infty$-estimates on $\varphi_t$, the solution to (1.1).
Lemma 2.1 ([7, 12]). There is a constant $C_0 > 0$ depending only on $n, p, \omega_X, \chi$ and $\|e^F\|_{L^1(\log L)^p}$ such that
\[
\sup_X |\varphi_t - \mathcal{V}_t| \leq C_0, \quad \forall t \in (0, 1].
\]

We are now ready to prove Theorem 1.1.

Proof. We break the proof into four steps.

Step 1. We fix a family of smooth positive functions $\eta_k : \mathbb{R} \to \mathbb{R}_+$ such that $\eta_k(x)$ converges uniformly and decreasingly to the function $x \cdot \chi_{\mathbb{R}_+}(x)$ as $k \to \infty$. We solve the auxiliary complex Monge-Ampère equations
\[
(2.2) \quad (\omega_t + i\partial\bar{\partial}\psi_{t,k})^n = c_t \frac{\eta_k(R_0^2 - d_t^2)}{A_{k,t}} e^{F \omega^n_X}, \quad \sup_X \psi_{t,k} = 0,
\]
where $0 < A_{k,t} = \frac{c_t}{V_t} \int_X \eta_k(R_0^2 - d_t^2) e^{F \omega^n_X}$ is the normalizing constant making the equation (2.2) solvable by Yau’s theorem [26]. Here we write $d_t(x) = d_{g_t}(x, z_0)$ to be the geodesic distance of $x$ to the fixed point $z_0$ under the varying metric $g_t$. We note that although the positive function on the right-hand side of (2.2) is only Lipschitz but not necessarily smooth, the solution $\psi_{t,k}$ is still in $C^{2,\alpha}(X)$ for some $\alpha > 0$, which follows from the regularity theory of complex Monge-Ampère equations (see e.g. [9, 2]). We observe that by dominated convergence theorem
\[
A_{k,t} \to A_t := \frac{c_t}{V_t} \int_{B_{g_t}(z_0, R_0)} (R_0^2 - d_t^2) e^{F \omega_X^n}, \quad \text{as } k \to \infty.
\]

Step 2. We aim to compare $\psi_{t,k}$ with the solution $\varphi_t$. As in [10, 11], we look at the test function
\[
(2.3) \quad \Phi := -\varepsilon (\psi_{t,k} - \varphi_t + C_1) + (R_0^2 - d_t^2),
\]
where $C_1 = C_0 + 1$ and $C_0 > 0$ is the constant in Lemma 2.1, and $\varepsilon > 0$ is chosen as
\[
(2.4) \quad \varepsilon = \left(\frac{n^2 + (n + 1)(4n + 2K)}{2n} \frac{1}{A_{k,t}^{1+n}} \right)^{\frac{1}{n+1}} =: C_2 A_{k,t}^{\frac{1}{n+1}},
\]
where we fix the constant $C_2 > 0$ which depends on $n$ and $K$. As an initial observation we note that on $X$
\[
-\psi_{t,k} + \varphi_t + C_1 = -(\psi_{t,k} - \mathcal{V}_t) + (\varphi_t - \mathcal{V}_t) + C_0 + 1 \geq 1,
\]
by Lemma 2.1 and the fact $\psi_{t,k} \leq \mathcal{V}_t$ for each $k$. Therefore the function $\Phi < 0$ on $X \setminus B_{g_t}(z_0, R_0)$.

We claim that $\Phi \leq 0$ on $X$. Let $x_{\text{max}} \in X$ be a maximum point of $\Phi$. If $x_{\text{max}} \notin B_{g_t}(z_0, R_0)$, we are done. So we may assume $x_{\text{max}} \in B_{g_t}(z_0, R_0)$. Applying Calabi’s trick if necessary [21], we may assume $d_t^2$ is smooth at $x_{\text{max}}$. Then at $x_{\text{max}}$ we have (write $\Delta = \Delta_{\omega_t}$)
\[
0 \geq \Delta_{\omega_t} \Phi(x_{\text{max}}) = \frac{n\varepsilon}{n+1} \left( -\psi_{t,k} + \varphi_t + C_1 \right)^{\frac{n+1}{n}} (\Delta \psi_{t,k} - \Delta \varphi_t)
\]
\[
+ \frac{n\varepsilon}{(n+1)^2} \left( -\psi_{t,k} + \varphi_t + C_1 \right)^{-\frac{n+1}{n}} |\nabla(\psi_{t,k} - \varphi_t)|^2_{\omega_t} - \Delta d_t^2
\]

\[
0 \geq \Delta_{\omega_t} \Phi(x_{\text{max}}) = \frac{n\varepsilon}{n+1} \left( -\psi_{t,k} + \varphi_t + C_1 \right)^{\frac{n+1}{n}} (\Delta \psi_{t,k} - \Delta \varphi_t)
\]
\[
+ \frac{n\varepsilon}{(n+1)^2} \left( -\psi_{t,k} + \varphi_t + C_1 \right)^{-\frac{n+1}{n}} |\nabla(\psi_{t,k} - \varphi_t)|^2_{\omega_t} - \Delta d_t^2
\]
From Φ
Step 3.
invariant of (2.6) for some uniform constant $C > 0$. Hence at $x_{\max}$ we have

$$R_0^2 - d_t^2 \leq A_{k,t}(n + \frac{(n+1)(4n+2K)}{n^2 \varepsilon})^n (-\psi_{t,k} + \varphi_t + C_1)^{\frac{n}{n+1}} < \varepsilon (-\psi_{t,k} + \varphi_t + C_1)^{\frac{n}{n+1}},$$

by the choice the $\varepsilon$ in (2.4). This finishes the proof of the claim that $\Phi \leq 0$ on $X$.

**Step 3.** From $\Phi \leq 0$ we infer that on $B_{r_t}(z_0, R_0)$

$$\frac{(R_0^2 - d_t^2)^{\frac{n+1}{n}}}{A_{k,t}^{1/n}} \leq C_2^{\frac{n+1}{n}} (-\psi_{t,k} + \varphi_t + C_1).$$

We can view $\psi_{t,k}$ as a $C_3 \omega_X$-PSH function for some $C_3 > 0$ depending only on $\chi$ and $\omega_X$. So the Hörrmander-Tian estimate [24, 15] holds for each $\psi_{t,k}$. We thus have a small constant $\alpha = \alpha(\omega_X, \chi, K) > 0$ such that the following Trudinger-type inequality holds

$$\int_{B_{r_t}(z_0, R_0)} e^{\alpha \frac{(R_0^2 - d_t^2)^{\frac{n+1}{n}}}{A_{k,t}^{1/n}}} \omega_X^n \leq \int_X e^{\alpha^{\frac{n+1}{n}} C_2^{\frac{n+1}{n}} (-\psi_{t,k} + \varphi_t + C_1)} \omega_X^n \leq C,$$

for some uniform constant $C > 0$, where we have chosen $\alpha > 0$ small so that $\alpha C_2^{\frac{n+1}{n}} < \alpha$-invariant of $(X, C_3 \omega_X)$.

**Step 4.** It then follows from a generalized Young’s inequality that

$$v^p e^F \leq e^F (1 + |F|^p) + C_p e^{2v}$$

which applied to $v = \alpha \frac{(R_0^2 - d_t^2)^{\frac{n+1}{n}}}{2A_{k,t}^{1/n}}$ yields by (2.5) that

$$\int_{B_{r_t}(z_0, R_0)} (R_0^2 - d_t^2)^{\frac{n+1}{n}} e^{F \omega_X^n} \leq CA_{k,t}^{p/n},$$
for some $C > 0$ depending on $n, p, \omega_X, \chi, K$ and $\|e^F\|_{L^1(\log L)^p}$. Letting $k \to \infty$, (2.6) implies that

$$\int_{B_{g_t}(z_0, R_0)} (R_0^2 - d_t^2)^{\frac{n+1}{n} p} e^F \omega_X^n \leq C A_t^{\frac{p}{n}}. \tag{2.7}$$

On the other hand, by H"older inequality we have

$$A_t = \frac{c_t}{V_t} \int_{B_{g_t}(z_0, R_0)} (R_0^2 - d_t^2) e^F \leq \left( \int_{B_{g_t}(z_0, R_0)} (R_0^2 - d_t^2)^{\frac{n+1}{n} p} e^F \omega_X^n \right)^{\frac{1}{p(n+1)}} \cdot \left( \int_{B_{g_t}(z_0, R_0)} e^F \omega_X^n \right)^{1/q}$$

$$\leq C A_t^{1/(n+1)} \left( \int_{B_{g_t}(z_0, R_0)} e^F \omega_X^n \right)^{1/q}$$

where $q = \frac{p(n+1)}{p(n+1)-n}$ is the conjugate of $p$. We thus conclude that

$$A_t \leq C \left( \int_{B_{g_t}(z_0, R_0)} e^F \omega_X^n \right)^{\frac{n+1}{nq}} = C \left( \int_{B_{g_t}(z_0, R_0)} e^F \omega_X^n \right)^{1+\frac{\alpha_0}{np}}. \tag{2.8}$$

Note that $c_t/V_t$ is uniformly bounded, so (2.8) shows that there exists a uniform constant $C > 0$ such that

$$\int_{B_{g_t}(z_0, R_0)} (R_0^2 - d_t^2) e^F \omega_X^n \leq C \left( \int_{B_{g_t}(z_0, R_0)} e^F \omega_X^n \right)^{1+\frac{\alpha_0}{np}}. \tag{2.9}$$

In particular, we obtain

$$R_0^2 \int_{B_{g_t}(z_0, R_0/2)} e^F \omega_X^n \leq C \left( \int_{B_{g_t}(z_0, R_0)} e^F \omega_X^n \right)^{1+\frac{\alpha_0}{np}}. \tag{2.10}$$

Multiplying $c_t$ on both sides of (2.9) we get

$$R_0^2 \text{Vol}_{g_t}(B_{g_t}(z_0, R_0/2)) \leq \frac{C}{c_t^{\frac{\alpha_0}{np}}} \left( \text{Vol}_{g_t}(B_{g_t}(z_0, R_0)) \right)^{1+\frac{\alpha_0}{np}}. \tag{2.11}$$

By volume comparison [21], the function $(0, 2R_0) \ni r \mapsto r^{-2n} e^{-\frac{K}{2} r} \text{Vol}_{g_t}(B_{g_t}(z_0, r))$ is non-increasing, which implies

$$\text{Vol}_{g_t}(B_{g_t}(z_0, R_0)) \leq C(n, K) \text{Vol}_{g_t}(B_{g_t}(z_0, R_0/2)).$$

Combined with (2.10), this implies that

$$R_0^{\frac{2n}{n-n}} V_t \leq C \text{Vol}_{g_t}(B_{g_t}(z_0, R_0)),$$

where as usual $V_t = \int_X \omega_t^n$ and $C > 0$ is a uniform constant. This finishes the proof of Theorem 1.1.

$\square$
3. An example

In this section, we will construct an example and demonstrate that the exponent $\frac{2np}{p-n}$ is sharp in the estimate of Theorem 1.1. In other words, the estimate in Theorem 1.1 may fail with the exponent replaced by $\frac{2np}{p-n} - \epsilon$ for any $\epsilon > 0$.

**Example 3.1.** Let $D \subset \mathbb{C} \subset \mathbb{C}P^1$ be the disk with radius 1/2. Consider the function $\varphi(z) = (-\log |z|^2)^{-a}$ for some $a > 0$. We calculate the “metric” defined by $i\partial\bar\partial \varphi$:

$$\omega = i\partial\bar\partial \varphi = a(a+1)\frac{idz \wedge d\bar{z}}{|z|^2(-\log |z|^2)^{a+2}} = e^\varphi idz \wedge d\bar{z}.$$ 

Straightforward calculations show that $\|e^\varphi\|_{L^1(\log L)^p(D)}$ is bounded for any $1 < p < a + 1$ and is unbounded when $p \geq a + 1$. Fix a point $z_0 \in D \setminus \{0\}$ close to 0, and $w \in D$ with $\arg z_0 = \arg w$. By the rotational symmetry of $\omega$, we see that

$$d_\omega(z_0, w) = \sqrt{a(a+1)} \int_{\min\{|z_0|, |w|\}}^{\max\{|z_0|, |w|\}} \frac{dr}{r(-\log r^2)^{1+a/2}} = \frac{a+1}{2a} \left(\frac{1}{(-\log \max\{|z_0|, |w|\})^{a/2}} - \frac{1}{(-\log \min\{|z_0|, |w|\})^{a/2}}\right).$$

In particular, letting $w \to 0$ we see that

$$d_\omega(z_0, 0) = \sqrt{\frac{a+1}{2a}} \left(\frac{1}{(-\log |z_0|)^{a/2}}\right) = 6R_0 > 0.$$ 

Take $z_0^\pm \in D$ with the same arguments as $z_0$ and $|z_0^+| = |z_0|^2/3 < |z_0|$ and $|z_0^-| = |z_0|(3/2)^a > |z_0|$ so that $(-\log |z_0^+|)^{a/2} = 2(-\log |z_0|)^{a/2}$ and $(-\log |z_0^-|)^{a/2} = \frac{2}{3}(-\log |z_0|)^{a/2}$. Then it follows that

$$d_\omega(z_0, z_0^+) = \frac{1}{2} \sqrt{\frac{a+1}{2a}} \left(\frac{1}{(-\log |z_0|)^{a/2}}\right) = 3R_0.$$ 

On the other hand, the length of the circles around 0 in the annulus $\{|z_0^+| \leq |z| \leq |z_0^-|\}$ is given by (again by the rotational symmetry of $\omega$ these circles are $\omega$-geodesics)

$$L_{\omega}(\text{circle}) = \sqrt{a(a+1)} \int_0^{2\pi} \frac{d\theta}{(-\log |z|^2)^{1+a/2}} = \frac{2\pi \sqrt{a(a+1)}}{(-\log |z|^2)^{1+a/2}} < \frac{R_0}{10}$$

if $|z_0| > 0$ small enough. This implies that

$$B_\omega(z_0, 2R_0) \subset \{|z_0^+| \leq |z| \leq |z_0^-|\} \subset B_\omega(z_0, 4R_0) \subset D \setminus \{0\}.$$ 

By straightforward calculations, we have

$$\text{Ric}(\omega) = i\partial\bar\partial \log \left(|z|^2(-\log |z|^2)^{a+2}\right) = -\frac{a+2}{|z|^2(-\log |z|^2)^a} idz \wedge d\bar{z} = -\frac{a+2}{a(a+1)}(-\log |z|^2)^a \cdot \omega$$

$$\geq -\frac{K^2}{R_0^2} \omega,$$ 

in the annulus $\{|z_0^+| \leq |z| \leq |z_0^-|\} = A,$
where \( K > 0 \) is a constant depending only on \( a > 0 \). Finally we calculate the \( \omega \)-volume of the above annulus as
\[
\operatorname{Vol}_\omega(A) = \int_{|z_0|}^{2\pi r dr} \frac{2r dr}{r^2 (-\log r^2)^{a+2}} = \frac{C_a}{(-\log |z_0|)^{a+1}} = C'_a R_0^{2(a+1)}.
\]
Note that \( e^F \in L^1(\log L)^p \) for any \( 1 < p < a + 1 \) and \( \operatorname{Vol}_\omega(A) \geq \operatorname{Vol}_\omega(B_\omega(z_0, 2R_0)) \). We see that the exponent \( 2ap \) of \( R_0 \) in Theorem 1.1 is sharp. Moreover, the Ricci curvature \( \operatorname{Ric}(\omega) \) is not bounded below on the whole \( D \) since it decays to \(-\infty\) near 0.

Though \( \omega \) in this example is a ‘singular’ Kähler metric on a local domain, we can regularize it near 0 to make it a genuine Kähler metric, and glue it to \( \mathbb{CP}^1 \) to get an example on compact Kähler manifolds (cf. e.g. [23]). One can naturally generalize the above example to higher dimensions and indeed the exponent \( 2ap \) is sharp.

4. Applications of Theorem 1.1

We discuss some geometric applications of the noncollapsing result in Theorem 1.1. We will show a local gradient estimate of the Kähler potential, if the Ricci curvature is bounded below locally. Under certain assumption on the Ricci curvature lower bound, we will prove the diameter bound and a local noncollapsing result for Kähler metrics along the normalized Kähler-Ricci flow on minimal Kähler manifolds.

4.1. MA equations with a fixed background Kähler metric. As the first application of Theorem 1.1, we prove Theorem 1.2 as generalization and a new proof of the global gradient estimate in [7].

Let \((X, \omega_X)\) be a given compact Kähler manifold. We consider the following complex Monge-Ampère equation
\[
(\omega_X + i\partial \bar{\partial} \varphi)^n = e^F \omega_X^n, \quad \sup_X \varphi = 0.
\]
Under the assumption of locally Ricci curvature lower bound, we prove the following local gradient estimate on \( \varphi \). We write \( \omega = \omega_X + i\partial \bar{\partial} \varphi \), which satisfies (4.1) and denote \( g \) the associated Riemannian metric of \( \omega \). We recall the statement of Theorem 1.2 below.

**Theorem 4.1.** Given \( p > n \) and \( R_0 \in (0, 1] \), suppose \( \operatorname{Ric}(g) \geq -K^2/R_0^2 \) on the geodesic ball \( B_g(z_0, 2R_0) \), then
\[
|\nabla \varphi|^2_g \leq C, \quad \text{on } B_g(z_0, R_0)
\]
for some constant \( C > 0 \) depending on \( n, p, \omega_X, \|e^F\|_{L^1(\log L)^p}, K \) and \( R_0 \).

We first recall the \( L^\infty \) estimate on \( \varphi \) in [16, 10]
\[
\|\varphi\|_{L^\infty} \leq C(n, p, \omega_X, \|e^F\|_{L^1(\log L)^p}).
\]
(4.2)
To prove Theorem 4.1, we need the following mean value inequality [21].
Lemma 4.1. Let \((M, g)\) be a Riemannian manifold such that \(B_g(p, 2R_0)\) is relatively compact in \(M\) with \(R_0 \in (0, 1]\). Suppose \(u \geq 0\) is a nonnegative function on \(B_g(p, 2R_0)\) satisfying \(\Delta_g u \geq -A^2/R_0^2\) and \(\text{Ric}(g) \geq -K^2/R_0^2\) on \(B_g(p, 2R_0)\), then the following mean value inequality holds

\[
\sup_{B_g(p, \tau R_0)} u^2 \leq \frac{C}{\text{Vol}_g(B_g(p, 2R_0))} \int_{B_g(p, 2R_0)} (u^2 + 1) dV_g,
\]

for some \(C > 0\) depending on \(\tau \in [1, 4/3]\), \(K\) and \(A\).

Proof. We consider \(v = u + 1 \geq 1\), and \(v\) satisfies \(\Delta_g v \geq -(A^2/R_0^2)v\). We look at the product manifold \(B_g(p, 2R_0) \times \mathbb{R}\) with the metric \(\hat{g} = g + ds^2\), where \(s \in \mathbb{R}\) is the natural coordinate. Define \(\hat{v} = e^{A^2/R_0}v\) to be function on \(B_g(p, 2R_0) \times \mathbb{R}\), and it satisfies \(\Delta_{\hat{g}} \hat{v} \geq 0\). Clearly \(\text{Ric}(\hat{g}) \geq -K^2/R_0^2\) and we can then apply the standard mean value inequality (cf. Theorem 6.2, Ch. 2, [21]) to conclude that

\[
\sup_{B_g(p, \tau R_0) \times (-\tau R_0, \tau R_0)} \hat{v}^2 \leq \frac{C}{\text{Vol}_g(B_g(p, 2R_0))} \int_{-\tau R_0}^{\tau R_0} \int_{B_g(p, 2R_0)} \hat{v}^2 dV_g ds
\]

\[
\leq \frac{C}{\text{Vol}_g(B_g(p, 2R_0))} \int_{B_g(p, 2R_0)} v^2 dV_g ds
\]

from which the lemma follows. \(\square\)

Lemma 4.2. Under the same assumptions as in Theorem 4.1, we have

\[
\text{tr}_\omega \omega_X \leq \exp \left( CR_0^{-np/(p-n)} \right), \quad \text{on } B_g(z_0, 3R_0/2),
\]

for some \(C > 0\) depending on \(n, p, \omega_X, \|e^f\|_{L^1(\log L)^p}\) and \(K \geq 0\).

Proof. It follows from the Schwarz-lemma type inequality that on \(B_g(z_0, 2R_0)\)

\[
\Delta_\omega \text{tr}_\omega \omega_X \geq -\frac{K^2}{R_0^2} \text{tr}_\omega \omega_X - C_0 (\text{tr}_\omega \omega_X)^2 + \frac{|\nabla \text{tr}_\omega \omega_X|^2}{\text{tr}_\omega \omega_X},
\]

where \(C_0 > 0\) is an upper bound of the bisectional curvature of \(\omega_X\) on \(B_g(z_0, 2R_0)\). Straightforward calculations show that

\[
\Delta_\omega \log \text{tr}_\omega \omega_X \geq -\frac{K^2}{R_0^2} - C_0 \text{tr}_\omega \omega_X.
\]

Hence the function \(u := (\log \text{tr}_\omega \omega_X)_+ - C_0\varphi\) satisfies

\[
\Delta u \geq -\frac{K^2}{R_0^2} - C_0 n = -\frac{A^2}{R_0^2}, \quad \text{with } A^2 = K^2 + C_0 n R_0^2.
\]

We apply Lemma 4.1 to conclude that (we denote \(B_R = B_g(z_0, R)\) for simplicity)

\[
\sup_{B_{3R_0/2}} [(\log \text{tr}_\omega \omega_X)_+]^2 \leq 2 \sup_{B_{3R_0/2}} u^2 + 2(C_0 C)^2
\]

\[
\leq \frac{C}{\text{Vol}_g(B_{2R_0})} \int_{B_{2R_0}} [(\log \text{tr}_\omega \omega_X)_+]^2 dV_g + C
\]
\[
\leq \frac{C}{R_0^{2np/(p-n)}} \int_X \text{tr}_\omega \omega^n + C \leq \frac{C}{R_0^{2np/(p-n)}},
\]
where in the third line we use the calculus inequality \([(\log x)_+]^2 \leq x\) for any \(x > 0\) and Theorem 1.1. Hence we have

\[
\sup_{B_{3R_0/2}} \text{tr}_\omega \omega_X \leq \exp \left( C R_0^{-np/(p-n)} \right).
\]

\[ \square \]

**Proof of Theorem 4.1.** By Bochner formula, we have on \(B_g(z_0, 2R_0)\)

\[
\Delta_\omega |\nabla \varphi|_\omega^2 = |\nabla \nabla \varphi|_\omega^2 + |\nabla \varphi|_\omega^2 + \text{Ric}_g(\nabla \varphi, \nabla \varphi) - 2Re\langle \nabla \varphi, \nabla \text{tr}_\omega \omega_X \rangle_\omega
\]

\[
\geq |\nabla \nabla \varphi|_\omega^2 + |\nabla \varphi|_\omega^2 - K^2 R_0^2 |\nabla \varphi|_\omega^2 - 2|\nabla \varphi||\nabla \text{tr}_\omega \omega_X|_\omega,
\]

which by the Kato’s inequality\(^1\) implies that

\[
\Delta_\omega |\nabla \varphi|_\omega \geq -K^2 R_0^{-2} |\nabla \varphi|_\omega - |\nabla \text{tr}_\omega \omega_X|_\omega.
\]

Define \(u = |\nabla \varphi| + \varphi^2 + (\text{tr}_\omega \omega_X)^2\). We calculate using (4.3)

\[
\Delta_\omega u \geq -K^2 R_0^{-2} |\nabla \varphi|_\omega - |\nabla \text{tr}_\omega \omega_X|_\omega + 2|\nabla \varphi|_\omega^2 + 2n \varphi - 2 \varphi \text{tr}_\omega \omega_X
\]

\[
+ 2|\nabla \text{tr}_\omega \omega_X|_\omega^2 - 2K^2 R_0^{-2} (\text{tr}_\omega \omega_X)^2 - 2C_0 (\text{tr}_\omega \omega_X)^3
\]

\[
\geq -C(R_0), \quad \text{in } B_g(z_0, 3R_0/2),
\]

where in the last inequality the Schwarz inequality, (4.2) and Lemma 4.2. It then follows from Lemma 4.1 that

\[
\sup_{B_{R_0}} u^2 \leq \frac{C(R_0)}{\text{Vol}_g(B_g(z_0, \frac{3R_0}{2}))} \int_{B_{3R_0/2}} (1 + u^2) \omega^n \leq \frac{C(R_0)}{R_0^{2np/(p-n)}} \int_X |\nabla \varphi|_\omega^2 \omega^n + C(R_0).
\]

The proof is complete by observing that

\[
\int_X |\nabla \varphi|_\omega^2 \omega^n = \int_X (-\varphi)(\omega - \omega_X) \wedge \omega^{n-1} \leq \int_X (-\varphi) \omega^n
\]

and the latter integral is bounded due to (4.2).

\[ \square \]

\(^1\)The Kato’s inequality states that \(2|\nabla|\nabla \varphi|_\omega^2 \leq |\nabla \nabla \varphi|_\omega^2 + |\nabla \varphi|_\omega^2\), which can be verified by the Cauchy-Schwarz inequality. It suffices to verify this inequality when \(|\nabla \varphi| > 0\). In normal coordinates of \(\omega\), we calculate

\[
4|\nabla \varphi|^2 |\nabla \varphi| = |\nabla \nabla \varphi|^2
\]

\[
= (\varphi_i \varphi_j)(\varphi_k \varphi_k)
\]

\[
= \varphi_i \varphi_i \varphi_k \varphi_k + \varphi_j \varphi_j \varphi_k \varphi_k + \varphi_i \varphi_j \varphi_k \varphi_k + \varphi_i \varphi_i \varphi_k \varphi_k
\]

\[
\leq 2|\nabla \varphi|^2 |\nabla \nabla \varphi|_\omega + |\nabla \varphi|^2 |\nabla \varphi| + |\nabla \varphi|^2 |\nabla \nabla \varphi|^2
\]

\[
= |\nabla \varphi|^2 (|\nabla \nabla \varphi| + |\nabla \varphi|) \leq 2|\nabla \varphi|^2 (|\nabla \nabla \varphi|^2 + |\nabla \varphi|^2).
\]

Cancelling \(2|\nabla \varphi|^2\) on both sides gives the desired Kato’s inequality.
We remark that if the Ricci curvature $\text{Ric}(\omega)$ is bounded below on $X$, then Proposition 4.1 implies a global gradient estimate of $\varphi$, which was proved in [7] using maximum principle. Note that in [7], the assumption $i\partial\bar{\partial}F \leq \Lambda\omega_X$ was made to ensure the Ricci curvature being bounded below. With this extra assumption, the bound on $\|e^F\|_{L^1(\log L)^p}$ with $p > n$ together with Corollary 1.1 imply the diameter bound of $(X, \omega)$, where $\omega = \omega_\varphi$ satisfies (4.1).

Finally we mention that when the function $e^F$ on the right-side of (4.1) belongs to $L^q$ for some $q > 1$, the diameter bound of $\omega_\varphi$ has been proved in [19], using the Hölder continuity of the solution $\varphi$ established in [18, 3] (see [13]). When $\|e^F\|_{L^1(\log L)^p} \leq A$ for some $A > 0$ and $p > 3n$, the diameter bound of $\omega_\varphi$ has been obtained in [13] after they establish the modulus of continuity of the Kähler potentials.

4.2. Normalized Kähler-Ricci flow. As in the previous subsection, we assume $X$ is a minimal Kähler manifold (i.e. $K_X$ is nef, and we do not assume $K_X$ is semi-ample) with $\omega_X$ a given Kähler metric. Let $\Omega$ be a smooth volume form such that $\chi = i\partial\bar{\partial}\log \Omega \in |K_X|$ is a representative of the canonical class of $X$. Multiplying a constant to $\omega_X$ if necessary we may assume that $\chi \leq \omega_X$. We consider the following normalized Kähler-Ricci flow

\begin{equation}
\frac{\partial \omega_t}{\partial t} = -\text{Ric}(\omega_t) - \omega_t, \quad \omega_t|_{t=0} = \omega_X.
\end{equation}

It is well-known [25] that (4.4) exists for all $t \in [0, \infty)$ and (4.4) is equivalent to the following parabolic complex Monge-Ampère equation

\begin{equation}
\frac{\partial \varphi_t}{\partial t} = \log \left( \frac{\chi + e^{-t}(\omega_X - \chi) + i\partial\bar{\partial}\varphi_t}{e^{-(n-\nu)t}\Omega} \right) - \varphi_t, \quad \varphi_t|_{t=0} = 0,
\end{equation}

with $\omega_t = \chi + e^{-t}(\omega_X - \chi) + i\partial\bar{\partial}\varphi_t > 0$.

Lemma 4.3. There is a uniform constant $C > 0$ such that

$$\sup_X \dot{\varphi}_t = \sup_X \frac{\partial \varphi_t}{\partial t} \leq C \quad \text{and} \quad \sup_X \varphi_t \leq C.$$  

Proof. Taking $\frac{\partial}{\partial t}$ on both sides of (4.5), we get

$$\frac{\partial \dot{\varphi}_t}{\partial t} = \Delta_{\omega_t} \dot{\varphi}_t - e^{-t} \text{tr}_{\omega_t}(\omega_X - \chi) - \dot{\varphi}_t + (n - \nu), \quad \dot{\varphi}_t|_{t=0} = 0,$$

so by maximum principle and $\omega_X - \chi \geq 0$ it follows that for $\varphi_{t,\text{max}} = \max_X \varphi_t$

$$\frac{d}{dt} \dot{\varphi}_{t,\text{max}} \leq -\dot{\varphi}_{t,\text{max}} + (n - \nu)$$

multiplying both sides by $e^t$ and integrating over $t$ it yields that $\dot{\varphi}_{t,\text{max}} \leq (n - \nu)(1 - e^{-t}) \leq n - \nu$, which gives the upper bound of $\varphi_t$. To see the upper bound of $\varphi_t$, we calculate (denote $V = \int_X \Omega$)

\begin{align*}
\frac{d}{dt} \frac{1}{V} \int_X \varphi_t \Omega &= \frac{1}{V} \int_X \log \frac{\omega^n_t}{e^{-(n-\nu)t}\Omega} - \frac{1}{V} \int_X \varphi_t \Omega \\
&= (n - \nu)t + \frac{1}{V} \int_X \log \frac{\omega^n_t}{\Omega} - \frac{1}{V} \int_X \varphi_t \Omega
\end{align*}
\[
\leq (n - \nu)t + \log \left( \frac{1}{V} \int_X \omega_t^n \right) - \frac{1}{V} \int_X \varphi_t \Omega
\]
\[
\leq (n - \nu)t + \log \left( \frac{C}{\sqrt{e^{-(n-\nu)t}}} \right) - \frac{1}{V} \int_X \varphi_t \Omega
\]
\[
\leq C - \frac{1}{V} \int_X \varphi_t \Omega,
\]
where in the third line above we applied Jensen’s inequality. Integrating both sides of the inequality above it follows that \( \frac{1}{V} \int_X \varphi_t \Omega \leq C \) for some uniform \( C > 0 \). The desired upper bound of \( \varphi_t \) follows from mean value theorem. \( \square \)

We re-write the equation (4.5) as
\[
(\chi + e^{-t}(\omega_X - \chi) + i\partial \bar{\partial} \varphi_t)^n = e^{-t(n-\nu)}e^{\varphi_t + \varphi_t} \Omega.
\]
Integrating both sides of (4.6) and applying Lemma 4.1 we get \( \int_X e^{\varphi_t} \Omega \geq c > 0 \) for some uniform \( c > 0 \), which implies that \( \sup_X \varphi_t \geq -C \) where \( C > 0 \) is independent of \( t \). We can now apply the \( L^\infty \)-estimate of \( \varphi_t \) as the solution to the “elliptic” complex MA equation (4.6) with \( e^F := e^{\varphi_t + \varphi_t} \leq C \) from Lemma 4.3, and it follows that
\[
\sup_X |(\varphi_t - \sup X \varphi_t) - \mathcal{V}_t| \leq C \quad \Rightarrow \quad \sup_X |\varphi_t - \mathcal{V}_t| \leq C,
\]
in which we use \( |\sup_X \varphi_t| \leq C \) and \( \mathcal{V}_t \) is the envelope associated to \( \chi + e^{-t}(\omega_X - \chi) \). With (4.7) we are ready to state the local noncollapsing along the normalized Kähler-Ricci flow (4.4).

**Proposition 4.1.** Let \( X \) be a minimal Kähler manifold and \( \omega_t \) be the solution to the Kähler-Ricci flow (4.4). Let the assumptions be as above. Then the following holds. For any \( \epsilon > 0 \), \( t \in [0, \infty) \), \( z_0 \in X \), and \( R_0 \in (0, 1] \), if \( \text{Ric}(\omega_t) \geq -K^2/R_0^2 \) on \( B_{\omega_t}(z_0, 2R_0) \), then there exists a constant \( c_\epsilon = c_\epsilon(n, \chi, \omega_X, K, \Omega, \epsilon) > 0 \) such that
\[
\frac{\text{Vol}_{\omega_t}(B_{\omega_t}(z_0, R_0))}{\mathcal{V}_t} \geq c_\epsilon R_0^{2n+\epsilon}.
\]

The proof of Proposition 4.1 is almost the same as that of Theorem 1.1, given the \( L^\infty \)-estimate (4.7) of \( \varphi_t \). In fact, the proof is even simpler since the function \( e^F = e^{\varphi_t + \varphi_t} \) is bounded in \( L^\infty \)-norm. We leave the details to interested readers.

If the Ricci curvature \( \omega_t \) is uniformly bounded below for \( t \in [0, \infty) \), the local noncollapsing in Proposition 4.1 will imply the uniform diameter bound of \( (X, \omega_t) \), thus providing a new proof of one of the results in [8], which studied the Kähler-Ricci flow on minimal manifolds with numerical dimension \( \nu = n \). By volume comparison, it will follow that \( \text{Vol}_{\omega_t}(B_{\omega_t}(z_0, r)) \geq cV_t r^{2n} \) for any \( r \in (0, 1] \). This volume decay property appears in the assumptions of the main theorems in [14]. It suggests that this extra assumption may be superfluous for Kähler-Ricci flow.

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