Dynamics of order parameters for a population of globally coupled oscillators

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Abstract

Using an expansion in order parameters, the equation of motion for the centroid of globally coupled oscillators with natural frequencies taken from a distribution is obtained for the case of high coupling, low dispersion of natural frequencies and any number of oscillators. To the first order, the system can be approximated by a set of four equations, where the centroid is coupled with a second macroscopic variable, which describes the dynamics of the oscillators around their average. This gives rise to collective effects that suggest experiments aimed at measuring the parameters of the population.

Populations of coupled oscillators have been widely used as a framework for studying collective phenomena. Applications include chemical systems (e.g., reaction-diffusion systems in the oscillatory regime \([1]\)), electronic circuits \([2]\), and biological systems (e.g., spiking activity in neurons \([3]\), synchronization in β-cells \([4]\), and glycolytic oscillations \([5]\)).

Much of the interest has been devoted to properties arising at the population level, especially in relation with synchronization phenomena \([6]\). Borrowing from statistical mechanics, such properties may be quantified in terms of order parameters \([7]\), such as the mean value or centroid, for which phase diagrams can be constructed.

In this Letter, we consider a population of \(N\) Hopf normal forms with a mean-field coupling:

\[
\frac{dz_j}{dt} = (1 + i \omega_j - |z_j|^2)z_j + K \frac{1}{N} \sum_{i=1}^{N} (z_i - z_j), \tag{1}
\]

where the natural frequencies \(\omega_j\) are taken from a given distribution and \(K\) denotes the coupling strength.

Previous analyses \([8]\) of system (1) (or of its phase reduction) have been restricted mainly to properties arising in the asymptotic regimes. In this case,
an insight into the qualitative behavior of the system has been obtained by
observing the asymptotic properties of the centroid $Z = \langle z_j \rangle$ (using the notation
$\langle f_j \rangle = \frac{1}{N} \sum_{j=1}^{N} f_j$).

Our aim is to describe the dynamics of system $\mathbf{1}$ by expanding it in order
parameters or macroscopic variables (as they shall be called from now on in
order to avoid confusion between control parameters and order parameters).
The explicit deduction of their equations of motion shows how properties emerge
from the microscopic level, and allows us to describe the dynamics of the system
outside the asymptotic regime, a relevant issue in systems interacting with the
environment or in experiments using perturbation response techniques.

The macroscopic equations are derived through a change of coordinates from
the microscopic variables $z_j$ to macroscopic variables (like the centroid) that are
averaged over the population. The main advantage of this procedure is that the
macroscopic variables can be organized in a hierarchy, retaining, if the coupling
is strong and the natural frequency distribution is narrow, only the lowest
order terms of a series expansion. In this region of the parameter space, the
equation for the centroid has a functional form reminiscent of Eq. $\mathbf{1}$, but it
is also coupled with a second macroscopic variable that describes the dynamics
of the oscillators around their average. As a consequence, with respect to the
description of the system as a single macroscopic oscillator, a correction for the
collective oscillations amplitude appears and a critical macroscopic perturba-
tion (i.e., quenching, see later) can be identified. Analytical relations for such
quantities are given and suggest simple experiments for determining, by means
of purely macroscopic measures, the coupling strength and the variance of the
natural frequencies distribution in real systems. Our results are valid for any
number of oscillators, and for any (narrow) distribution of natural frequencies
and are compared with those obtained by numerically integrating the original
system $\mathbf{1}$.

**Deduction.** Let us start by writing the positions of the oscillators in terms
of their distance from the centroid: $z_j = Z + \epsilon_j$. This expansion is useful
when the coupling is strong, since in this case the dynamics leads the system to
collapse on a configuration, the so-called locked state $[9]$, that is peaked around
the centroid. In particular, the displacements from the centroid in the locked
state converge to zero when the coupling strength is increased$\dagger$. Moreover, when
the displacements are wider than the value at the equilibrium, the dynamics is
contracting, and thus the locked state furnishes an upper limit to the broadness
of the initial configuration required for the approximations to hold.

Eq. $\mathbf{1}$ now reads:

$$
\frac{dz_j}{dt} = \left( 1 - |Z|^2 + i\omega_j \right) Z + \epsilon_j \left( i\omega_j + 1 - K - 2|Z|^2 \right)
- Z^2\epsilon_j^* + o(|\epsilon_j|^2).
$$

$\dagger$

We then consider the time evolution of the centroid, differentiating its defi-
nition: $Z = \langle z_j \rangle$ and using Eq. $\mathbf{1}$:

$$
dZ/dt = d\langle z_j \rangle/dt = \langle dz_j/dt \rangle = \left( 1 - |Z|^2 + i\omega_j \right) Z + i\omega_j\epsilon_j + \left( 1 - K - 2|Z|^2 \right)\epsilon_j - Z^2\epsilon_j^* + o(|\epsilon_j|^2).
$$

By definition of a centroid, $\langle \epsilon_j \rangle = \langle \epsilon_j^* \rangle = 0$, and $dZ/dt$ reduces to:
\[ \frac{dZ}{dt} = (1 - |Z|^2 + i\omega_0)Z + i\langle \omega_j \epsilon_j \rangle + o(|\epsilon|^2) \]  

(3)

where \( \langle \omega_j \rangle = \omega_0 \) is the average natural frequency. A zeroth-order expansion thus leads to the equation:

\[ \frac{dZ}{dt} = (1 - |Z|^2 + i\omega_0)Z. \]  

(4)

This expansion has the same functional form as the individual, uncoupled elements. It exactly describes the case of a population of oscillators with the same natural frequency \( \omega_0 \) and with \( \epsilon_j = 0 \) \( \forall j \), the last condition being fulfilled if all the oscillators are assigned the same initial condition. In this (trivial) case, it shows the existence of a limit cycle of radius 1 and frequency \( \omega_0 \), and of an unstable focus in \( Z = 0 \). The equation is independent of the coupling strength, since the coincidence of the oscillators and the centroid is maintained in time.

For the description of the parameter mismatch to be included, the term of lower order that must be then taken into account is:

\[ W := \langle \omega_j \epsilon_j \rangle \].

Having in this way defined a new macroscopic variable, we now derive its equation of motion.

Since \( \frac{dW}{dt} = \frac{d\langle \omega_j \epsilon_j \rangle}{dt} = \langle \omega_j d\epsilon_j/dt \rangle - \langle \omega_j \rangle dZ/dt \), using the definition of \( W \) and Eq. (3), we obtain:

\[ \frac{dW}{dt} = i \left( \langle \omega_j^2 \rangle - \omega_0^2 \right) Z + \langle \omega_j \epsilon_j \rangle \right) + o(\langle |\epsilon|^2 \rangle) \].

(5)

The equation is closed with respect to \( Z \) and \( W \), if again the higher order terms in the displacements are discarded, when the dispersion of the natural frequencies is sufficiently small. Since we are dealing with narrow distributions, the second degree term in the frequencies will be neglected. Note that this approximation does not depend on specific symmetries or shapes for the frequency distribution. Calling \( \sigma^2 = \langle \omega_j^2 \rangle - \omega_0^2 \) its variance, and combining Eq. (5) with Eq. (3), we get the following description at first order:

\[ \begin{cases} 
\frac{dZ}{dt} = (1 - |Z|^2 + i\omega_0)Z + iW \\
\frac{dW}{dt} = i\sigma^2 Z + (1 - K - 2|Z|^2 + i\omega_0) W - Z^2 W^*.
\end{cases} \]  

(6)

This first-order expansion shows explicitly how the centroid is coupled with the dynamics of the oscillators around it. This “internal” dynamics is quantified in the \( W \) variable, by means of which the parameter mismatch (through the variance of natural frequencies) and the coupling strength affect the system. These equations describe the dynamics of the centroid from initial conditions with small \( \epsilon_j \) (and thus small \( W \)). Since in their derivation we have requested to have asymptotically a locked state, we will restrict our analysis to the case in which the system has an unstable focus surrounded by a limit cycle, which occurs if \( K > 1 + \sigma^2 \).

Although we stop at this order, the deduction can be carried further, and macroscopic descriptions at higher orders derived, applying the same method:
the terms that have been discarded in Eqs. (3) and (5) define new macroscopic variables, and their equations of motion can be derived by differentiating their definition.

**Analysis.** If $K > 1 + \sigma^2$ and $\epsilon_j$ are small, it is easy to see that the behavior of the system is essentially described by the dynamics on the invariant and attracting manifold on which the two macroscopic variables are orthogonal. This condition is fulfilled, and the description exact, when the natural frequency distribution and initial configuration are both symmetric. Setting $Z = Re^{i(\theta+\omega_0 t)}$ and $W = we^{i(\theta+\omega_0 t)}$, Eqs. (6) reduce on $\phi = \theta + \pi/2$ to the amplitude equations:

\[
\begin{align*}
\frac{dR}{dt} &= (1 - R^2) R - w \\
\frac{dw}{dt} &= \sigma^2 R + (1 - K - R^2) w.
\end{align*}
\]

System (7) can be analyzed with planar methods and furnishes both an estimate for the amplitude of the locked oscillations and a description of their transient behavior.

Let us first find the equilibrium values for $R = |Z|$ and $w = |W|$. Setting $\alpha = K/2 - \sqrt{(K/2)^2 - \sigma^2}$ and using Eq. (7), it is easy to show that $W$ and $Z$ display synchronous oscillations (the second with a phase delay of $\pi/2$) of amplitudes:

\[
R_1 = \sqrt{1 - \alpha}, \quad w_1 = \alpha \sqrt{1 - \alpha}.
\]

Taking into account that in our approximation the ratio $\sigma^2/K$ is small, this expression approximately results in:

\[
R_1 = 1 - \frac{\sigma^2}{2K}, \quad w_1 = \frac{\sigma^2}{K}.
\]

When these expressions are compared with the zeroth-order amplitudes $R_0 = 1$ and $w_0 = 0$, it appears that the term $\sigma^2/2K$ gives the first order correction to the amplitude of the oscillations due to the mismatch of the natural frequencies. It is worth remarking that previously derived formulae for the amplitude of $Z$ were only implicitly related to the parameters of the population, through a self-consistency integral [9], and in the limit of large $N$. Let us now address the structure of the phase space. This is particularly relevant when the perturbation response is addressed, since it qualitatively and quantitatively determines the features of the transient after the system is initialized in a configuration out of equilibrium. A fundamental example is the quenching technique [10], which consists in damping the oscillations of the system by displacing it onto the stable manifold of the unstable focus. This can be experimentally observed as a long-term vanishment of the oscillations amplitude. While in the case of the zeroth-order expansion the stable manifold reduces to the focus itself, in the case of the first-order expansion it consists of a two-dimensional variety, which on the invariant plane reduces to the stable eigenspace of the unstable focus:

\[
w - (K - \alpha)R = 0.
\]
In quenching experiments the system lies in the locked state and is then macroscopically perturbed through a rigid displacement of the whole population. Since only the amplitude of the collective oscillations is measured, the critical displacement is identified by means of the quenching radius, that indicates the amplitude of the initial state for which the oscillations are suppressed. According to Eq. (7), this radius is given by:

\[ R_q = \frac{\alpha^2 \sqrt{1 - \alpha}}{\sigma^2} \approx \frac{\sigma^2}{K^2}. \]  

(11)

Again, we notice that the expression reduces to the zeroth-order value, that is zero, in the limit of vanishing \( \sigma \) and large \( K \).

Moreover, as illustrated in Fig. 1, the structure of the phase space allows us to predict three qualitatively different kinds of transient amplitude increase. In one case (I), the system behaves like an ordinary oscillator, the impulsive perturbation being followed by a monotonic increase in amplitude. In the second case (II), the amplitude first decreases, while the centroid is approaching the origin, and then increases again, leading at the same asymptotic solution as in the previous case. Finally (III), if the initial configuration width is large enough, it will first reduce to zero and then increase again, but the asymptotic solution will be in phase opposition with respect to the initial oscillations.

**Comparison.** Let us now compare the behavior of the equations for the macroscopic variables Eq. (6) with the numerical simulations of the original system described by Eq. (1). Fig. 2 compares the transient behavior of the full system (initialized in three configurations having the same average \( Z \), but different \( W \)) with the one predicted according to Eq. (6). The transient behaviors shown in Fig. 2 are actually observed: the accuracy of the approximation holds along the whole trajectory and remains nearly unchanged for any population having the same \( \sigma^2 \) and \( K \). Fig. 3 compares the estimated and numerically computed values of the asymptotic amplitude of the collective oscillations and the quenching radius.

We considered a population of globally and strongly coupled oscillators with narrow natural frequency distribution and showed how the dynamics can be reduced, via an expansion in order parameters, to a low-order system containing all the essential information (for the first-order expansion, two complex variables instead of \( N \) and two parameters instead of \( N + 1 \)). Qualitative as well as quantitative results were then given for collective properties of experimental relevance. Although we restricted our attention to simple oscillators, the presented method can in principle be applied to other populations of globally and strongly coupled elements with small parameter mismatch. Moreover, the macroscopic description may furnish an instrument for studying collective behaviors outside the phase locking region. In particular, the number of the relevant terms in the expansion is expected to change as the system approaches the bifurcation boundaries, linking the onset of new regimes to a change in the dimensionality of the macroscopic system.

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† Calling $|z_j| = \rho_j$ and assuming a frame or reference rotating at $\omega_0 = \langle \omega_0 \rangle$, $|Z - z_j| = \frac{1}{K}[(1 - |z_j|^2 + i(\omega_j - \omega_0))z_j] \leq \frac{1}{K} \left[ (1 - |\rho_j|^2) \rho_j + |\omega_j - \omega_0| \rho_j \right] \leq \frac{1}{K} \left( (1 + |\omega_j - \omega_0|) \rho_j \right) \leq (1 + |\omega_j - \omega_0|)/K.$
Figure 1: Phase portrait of the reduced system Eq. (7). The equilibria \( E_1 \) and \( E_2 \) and the eigenspaces of the origin are indicated. The system is initialized in three points (circles) giving rise to qualitatively different trajectories.

Figure 2: The transient behavior predicted by Eq. (8) (solid line) is compared to that of the full system Eq. (1) (triangles) and of its zeroth-order approximation Eq. (4) (dotted line) for \( \sigma^2 = 0.5 \) and \( K = 3 \). Three initial states are chosen, having the same centroid’s position \(|Z|\), but different configuration, and thus different \(|W|\). Populations with different size and frequency distribution are considered: \( N = 800 \), Gaussian distribution (\( \triangle \)); \( N = 800 \) uniform distribution (\( \triangledown \)); \( N = 5 \), uniform distribution (\( \triangleleft \)); \( N = 2 \) (\( \triangleright \)).
Figure 3: The estimated values for the amplitude of the centroid’s oscillations (Eq. 9) and the quenching radius (Eq. 11) versus the coupling constant $K$ (solid lines) are compared to the numerically computed ones (triangles), for $\sigma^2 = 0.2$ (cases a and b) and $\sigma^2 = 0.5$ (cases c and d). The accordance is higher for more concentrated distributions, but is again almost unchanged when populations of different size and frequency distribution are considered (symbols as in Fig. 2). The accuracy increases with the coupling, approaching at the meantime the zeroth-order values $|Z| = 1$ and $R_q = 0$. 