Interval Valued Intuitionistic Fuzzy Line Graphs

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Abstract

Objectives: In the field of graph theory, an intuitionistic fuzzy set becomes a useful tool to handle problems related to uncertainty and impreciseness. We introduced the interval-valued intuitionistic fuzzy line graphs (IVIFLG) and explored the results related to IVIFLG.

Result: Some propositions and theorems related to IVIFLG are proposed and proved, which are originated from intuitionistic fuzzy graphs (IVIG). Furthermore, Isomorphism between two IVIFLGs toward their IVIFGs was determined and verified.

Keywords: Fuzzy set, Interval-valued intuitionistic fuzzy line graph, Interval-valued intuitionistic fuzzy graph, Isomorphism

Mathematics Subject Classification: Primary 05C72, Secondary 03B20

Introduction

After Euler was presented with the impression of Königsberg bridge problem, Graph Theory has become recognized in different academic fields like engineering, social science in medical science, and natural science. A few operations of graphs like line graph, wiener index of graph, cluster and corona operations of graph, total graph, semi-total line and edge join of graphs have been valuable in graph theory and chemical graph theory to consider the properties of boiling point, heat of evaporation, surface tension, vapor pressure, total electron energy of polymers, partition coefficients, ultrasonic sound velocity and internal energy [1–4]. The degree sequence of a graph and algebraic structure of different graphs operations were determined and its result is to the join and corona products of any number of graphs [5]. These operations are not only in classical graphs, they are more useful in fuzzy and generalizations of fuzzy graphs. The real-world problems are often full of uncertainty and impreciseness, Zadeh introduced fuzzy sets and membership degree [6]. Based on Zedeh’s work, Kaufman introduced the notion of fuzzy relations [7]. Then, Rosenfeld [8] followed the Kaufman work and he introduced fuzzy graphs.

Later, Atanassov witnessed that many problems with uncertainty and imprecision were not handled by fuzzy sets (FS) [9]. Then considering this, he added the falsehood degree to membership degree and presented intuitionistic fuzzy sets (IFS) with relations and IFG which is a generalization of FS and their applications [9–11]. In 1993, Mordeson examined the idea of fuzzy line graphs (FLG) for the first time by proving both sufficient and necessary conditions for FLG to be bijective homomorphism to its FG. And also some theorems and propositions are developed [12].

In 2011 IVFG and its properties were discussed by Akram and Dudek [13]. After that, Akram innovated IVFLG [14]. Afterward, Akram and Davvaz introduced ideas of intuitionistic fuzzy line graphs (IFLG) [15]. Moreover, IFLG and its properties are investigated in [16].

As far as, there exists no research work on the IVIFLG until now. So that, we put forward a new idea and
definitions of IVIFLG. The novelty of our works are given as follows: (1) IVIFLG is presented and depicted with an example, (2) many propositions and Theorems on properties of IVIFLG is developed and proved, (3) further, interval-valued intuitionistic weak vertex homomorphism and interval-valued intuitionistic weak line isomorphism are proposed. For the notations not declared in this manuscript, to understand well we recommend the readers to refer [10, 12, 14, 18, 19].

**Main text**

We start the section with basic definitions related to IVIFLG. So that, the definitions [1-12] are the well-known definitions used to discuss the main result of this work.

**Definition 1** [17] The graph of the form \( G = (V, E) \) is an intuitionistic fuzzy graph (IFG) such that

\[
\begin{align*}
(i) & \quad \sigma_1, \gamma_1 : V \rightarrow \{0, 1\} are membership and nonmembership value of vertex set of G respectively and \\
& \quad 0 \leq \sigma_1(v) + \gamma_1(v) \leq 1 \quad \forall v \in V,
(ii) & \quad \sigma_2, \gamma_2 : V \times V \rightarrow \{0, 1\} are membership and nonmembership with \\
& \quad \sigma_2(v_1, v_2) \leq \sigma_1(v_1) \wedge \sigma_1(v_2) \quad and \\
& \quad \gamma_2(v_1, v_2) \leq \gamma_1(v_1) \vee \gamma_1(v_2) \quad and \quad 0 \leq \sigma_2(v_1, v_2) + \gamma_2(v_1, v_2) \leq 1, \quad \forall v_1, v_2 \in E.
\end{align*}
\]

**Definition 2** [20] The line graph \( L(G) \) of graph \( G \) is defined as

i. Every vertex in \( L(G) \) corresponds to an edge in \( G \),
ii. Pair of nodes in \( L(G) \) are adjacent iff their correspondence edges \( e_i, e_j \in G \) have a common vertex \( v \in G \).

**Definition 3** For \( G = (V, E) \) is a graph with \( |V| = n \) and \( S_i = \{v_i, e_{i1}, \ldots, e_{ip_i}\} \) where \( 1 \leq i \leq n, 1 \leq j \leq p_i \) and \( e_j \in E \) has \( v_i \) as a vertex. Then \((S, T)\) is called intersection graph where \( S = \{S_i\} \) is the vertex set of \((S, T)\) and \( T = \{S_i \cap S_j : S_i, S_j \in S; S_i \cap S_j \neq \emptyset, i \neq j\} \) is an edge set of \((S, T)\).

**Remark** The given simple graph \( G \) and its intersection graph \((S, T)\) are isomorphic to each other\((G \cong (S, T))\).

**Definition 4** [14] The line(edge) graph \( L(G) = (H, J) \) is where \( H = \{e \cup [u_{e}, v_{e}] : e \in E, u_{e}, v_{e} \in V, e = u_{e}v_{e}\} \) and \( J = \{[S_{i}, S_{j}] : e, f \in E, e \neq f, S_{i} \cap S_{j} \neq \emptyset\} \) with \( S_e = \{e \cup [u_{e}, v_{e}, e \in E]\} \).

**Definition 5** [16] Let \( I = (A_1, B_1) \) is an IFG with \( A_1 = (\sigma_{A_1}, \gamma_{A_1}) \) and \( B_1 = (\sigma_{B_1}, \gamma_{B_1}) \) be IFS on \( V \) and \( E \) respectively. Then \((S, T) = (A_2, B_2)\) is an intuitionistic fuzzy intersection graph of \( I \) whose membership and nonmembership functions are defined as

\[
\begin{align*}
(i) & \quad \sigma_{A_2}(S_i) = \sigma_{A_1}(v_i), \quad \gamma_{A_2}(S_i) = \gamma_{A_1}(v_i), \quad \forall S_i \in S
(ii) & \quad \sigma_{B_2}(S_i, S_j) = \sigma_{B_1}(v_i, v_j), \quad \gamma_{B_2}(S_i, S_j) = \gamma_{B_1}(v_i, v_j), \quad \forall S_i, S_j \in T.
\end{align*}
\]

where \( A_2 = (\sigma_{A_2}, \gamma_{A_2}) \) and \( B_2 = (\sigma_{B_2}, \gamma_{B_2}) \) on \( S \) and \( T \) respectively. So, IFG of the intersection graph \((S, T)\) is isomorphic to \( I(\text{means, } (S, T) \cong I)\).

**Definition 6** Consider \( L(I^*) = (H, J) \) be line graph of \( I^* = (V, E) \). Let \( I = (A_1, B_1) \) be IFG of \( I^* \) with \( A_1 = (\sigma_{A_1}, \gamma_{A_1}) \) and \( B_1 = (\sigma_{B_1}, \gamma_{B_1}) \) be IFS on \( X \) and \( E \) receptively. Then we define the intuitionistic fuzzy line graph \( L(I) = (A_2, B_2) \) of \( I \) as

\[
\begin{align*}
(i) & \quad \sigma_{A_2}(S_e) = \sigma_{B_1}(e) = \sigma_{B_1}(u_{e}v_{e}), \quad \gamma_{A_2}(S_e) = \gamma_{B_1}(e) = \gamma_{B_1}(u_{e}v_{e}), \quad \forall S_e \in H
(ii) & \quad \sigma_{B_2}(S_e S_f) = \sigma_{B_1}(e) \wedge \sigma_{B_1}(f), \quad \gamma_{B_2}(S_e S_f) = \gamma_{B_1}(e) \vee \gamma_{B_1}(f), \quad \forall S_e S_f \in J
\end{align*}
\]

where \( A_2 = (\sigma_{A_2}, \gamma_{A_2}) \) and \( B_2 = (\sigma_{B_2}, \gamma_{B_2}) \) are IFS on \( H \) and \( J \) respectively.

The \((L(I)) = (A_2, B_2)\) of IFG \( I \) is always IFG.

**Definition 7** [16] Let \( I_1 = (A_1, B_1) \) and \( I_2 = (A_2, B_2) \) be two IFGs. The homomorphism of \( \psi : I_1 \rightarrow I_2 \) is mapping \( \psi : V_1 \rightarrow V_2 \) such that

\[
\begin{align*}
(i) & \quad \sigma_{A_1}(v_i) \leq \sigma_{A_2}(\psi(v_i)), \quad \gamma_{A_1}(v_i) \leq \gamma_{A_2}(\psi(v_i))
(ii) & \quad \sigma_{B_1}(v_i, v_j) \leq \sigma_{B_2}(\psi(v_i), \psi(v_j)), \quad \gamma_{B_1}(v_i, v_j) \leq \gamma_{B_2}(\psi(v_i), \psi(v_j)) \quad \forall v_i \in V_1, v_i v_j \in E_1.
\end{align*}
\]

**Definition 8** [13] The interval valued FS \( A \) is characterized by

\[
A = \{v_i, [\sigma_A^{-}(v_i), \sigma_A^{+}(v_i)] : v_i \in X\}.
\]

Here, \( \sigma_A^{-}(v_i) \) and \( \sigma_A^{+}(v_i) \) are lower and upper interval of fuzzy subsets \( A \) of \( X \) respectively, such that \( \sigma_A^{-}(v_i) \leq \sigma_A^{+}(v_i) \forall v_i \in V \).

For simplicity, we used IVFS for interval valued fuzzy set.

**Definition 9** Let \( A = \{[\sigma_A^{-}(v), \sigma_A^{+}(v)] : v \in X\} \) be IVFS. Then, the graph \( G^* = (V, E) \) is called IVFG if the following conditions are satisfied;

\[
\begin{align*}
\sigma_B^{-}(v_i v_j) & \leq (\sigma_A^{-}(v_i) \wedge \sigma_A^{+}(v_i))
\sigma_B^{+}(v_i v_j) & \leq (\sigma_A^{-}(v_i) \wedge \sigma_A^{+}(v_i))
\end{align*}
\]
\[ \forall v_i, v_j \in V, \ \forall v_i v_j \in E \text{ and where } A = [\sigma_A^-, \sigma_A^+], B = [\sigma_B^-, \sigma_B^+] \text{ is IVFS on } V \text{ and } E \text{ respectively.} \]

**Definition 10** Let \( G = (A_1, B_1) \) be simple IVFG. Then we define IVF intersection graph \((S, T) = (A_2, B_2)\) as follows:

1. \( A_2 \) and \( B_2 \) are IFS of \( S \) and \( T \) respectively,
2. \( \sigma_{A_2}^-(S_i) = \sigma_{A_2}^+(S_i) = \sigma_{A_2}^+(v_i), \forall S_i, S_j \in S \) and \( \sigma_B^-(S_i) = \sigma_B^+(S_i) = \sigma_B^+(v_i), \forall S_i, S_j \in T \).

**Remark** The given IVFG \( G \) and its intersection graph \((S, T)\) are always isomorphic to each other.

**Definition 11** [14] An interval valued fuzzy line graph (IVFLG) \( L(G) = (A_2, B_2) \) of IVFG \( G = (A_1, B_1) \) is defined as follows:

- \( A_2 \) and \( B_2 \) are IVFS of \( H \) and \( J \) respectively, where \( L(G^*) = (H, J) \)
- \( \sigma_{A_2}^-(v_i) = \sigma_{A_2}^-(v_i) = \sigma_{A_2}^-(v_i), \sigma_{A_2}^+(v_i) = \sigma_{A_2}^+(v_i), \sigma_{B_2}^+(v_i) = \sigma_{B_2}^+(v_i), \sigma_{B_2}^-(v_i) = \sigma_{B_2}^-(v_i) \)
- \( \sigma_{B_2}^+(v_i) = \sigma_{B_2}^+(v_i) \) for all \( S_i, S_j \in H, S_e S_f \in J \).

**Definition 12** A graph \( I = (A, B) \) with underlying fuzzy set \( V \) is IVIFG if

(i) The map \( \sigma_{A_1}, \gamma_{A_1} : V \rightarrow [0, 1] \) where \( \sigma_{A_1} = [\sigma_{A_1}^-(v_i), \sigma_{A_1}^+(v_i)] \) denote a membership degree and non membership degree of vertex \( v_i \in V \), receptively such that \( 0 \leq \sigma_{A_1}^+(v_i) + \gamma_{A_1}^+(v_i) \leq 1 \forall v_i \in V \),

(ii) The map \( \sigma_B, \gamma_B : V \times V \subseteq E \rightarrow [0, 1] \) where \( \sigma_B(v_i v_j) = [\sigma_B^-(v_i v_j), \sigma_B^+(v_i v_j)] \) and \( \gamma_B(v_i v_j) = [\gamma_B^-(v_i v_j), \gamma_B^+(v_i v_j)] \) such that

\[
\begin{align*}
\sigma_B^-(v_i v_j) &\leq \sigma_A^-(v_i) \land \sigma_A^-(v_j) \\
\sigma_B^+(v_i v_j) &\leq \sigma_A^+(v_i) \land \sigma_A^+(v_j) \\
\gamma_B^-(v_i v_j) &\leq \gamma_A^-(v_i) \lor \gamma_A^-(v_j) \\
\gamma_B^+(v_i v_j) &\leq \gamma_A^+(v_i) \lor \gamma_A^+(v_j)
\end{align*}
\]

where \( 0 \leq \sigma_B^+(v_i v_j) + \gamma_B^+(v_i v_j) \leq 1 \) and \( \forall v_i v_j \in E \).

Now we start the main results of this work by introducing Interval-valued Intuitionistic Fuzzy Line Graph (IVIFLG) and providing examples.

**Definition 13** An interval valued intuitionistic fuzzy line graphs \( L(I) = (H, J) \) of IVIFG \( I = (A_1, B_1) \) is denoted by \( L(I) = (A_2, B_2) \) and whose functions of membership and non membership defined as

(i) \( A_2 \) and \( B_2 \) are IVFS of \( H \) and \( J \) respectively, such that \( \sigma_{A_2}^-(S_e) = \sigma_{B_1}^-(e) = \sigma_{B_1}(u_e v_e) \)
\( \sigma_{A_2}^+(S_e) = \sigma_{B_1}^+(e) = \sigma_{B_1}(u_e v_e) \)
\( \gamma_{A_2}^-(S_e) = \gamma_{B_1}^-(e) = \gamma_{B_1}(u_e v_e) \)
\( \gamma_{A_2}^+(S_e) = \gamma_{B_1}^+(e) = \gamma_{B_1}(u_e v_e) \) for all \( S_e \in H \).

(ii) The edge set of \( L(I) \) is
\( \sigma_{B_2}^-(S_e S_f) = \sigma_{B_1}(e) \land \sigma_{B_1}(f) \)
\( \sigma_{B_2}^+(S_e S_f) = \sigma_{B_1}(e) \land \sigma_{B_1}(f) \)
\( \gamma_{B_2}^-(S_e S_f) = \gamma_{B_1}(e) \lor \gamma_{B_1}(f) \)
\( \gamma_{B_2}^+(S_e S_f) = \gamma_{B_1}(e) \lor \gamma_{B_1}(f) \)

for all \( S_e S_f \in J \).

**Example 14**

Given IVIFG \( I = (A_1, A_2) \) as shown in Fig. 1.

![Fig. 1 MIFG](image-url)
From the given IVIFS we have

\[
\begin{align*}
\sigma_A(v_1) &= \left[\sigma_{A_1}^-(v_1), \sigma_{A_1}^+(v_1)\right] = [0.3, 0.6] \\
\sigma_A(v_2) &= \left[\sigma_{A_1}^-(v_2), \sigma_{A_1}^+(v_2)\right] = [0.2, 0.7] \\
\sigma_A(v_3) &= \left[\sigma_{A_1}^-(v_3), \sigma_{A_1}^+(v_3)\right] = [0.1, 0.3] \\
\sigma_A(v_4) &= \left[\sigma_{A_1}^-(v_4), \sigma_{A_1}^+(v_4)\right] = [0.3, 0.4] \\
\gamma_A(v_1) &= \left[\gamma_{A_1}^-(v_1), \gamma_{A_1}^+(v_1)\right] = [0.1, 0.4] \\
\gamma_A(v_2) &= \left[\gamma_{A_1}^-(v_2), \gamma_{A_1}^+(v_2)\right] = [0.1, 0.2] \\
\gamma_A(v_3) &= \left[\gamma_{A_1}^-(v_3), \gamma_{A_1}^+(v_3)\right] = [0.4, 0.5] \\
\gamma_A(v_4) &= \left[\gamma_{A_1}^-(v_4), \gamma_{A_1}^+(v_4)\right] = [0.4, 0.5]
\end{align*}
\]

\[
\begin{align*}
\sigma_B(v_1 v_2) &= \left[\sigma_{B_1}^-(v_1 v_2), \sigma_{B_1}^+(v_1 v_2)\right] = [0.2, 0.5] \\
\sigma_B(v_2 v_3) &= \left[\sigma_{B_2}^-(v_2 v_3), \sigma_{B_2}^+(v_2 v_3)\right] = [0.1, 0.2] \\
\sigma_B(v_3 v_4) &= \left[\sigma_{B_3}^-(v_3 v_4), \sigma_{B_3}^+(v_3 v_4)\right] = [0.1, 0.1] \\
\sigma_B(v_4 v_1) &= \left[\sigma_{B_4}^-(v_4 v_1), \sigma_{B_4}^+(v_4 v_1)\right] = [0.2, 0.4] \\
\gamma_B(v_1 v_2) &= \left[\gamma_{B_1}^-(v_1 v_2), \gamma_{B_1}^+(v_1 v_2)\right] = [0.1, 0.3] \\
\gamma_B(v_2 v_3) &= \left[\gamma_{B_2}^-(v_2 v_3), \gamma_{B_2}^+(v_2 v_3)\right] = [0.3, 0.4] \\
\gamma_B(v_3 v_4) &= \left[\gamma_{B_3}^-(v_3 v_4), \gamma_{B_3}^+(v_3 v_4)\right] = [0.3, 0.4] \\
\gamma_B(v_4 v_1) &= \left[\gamma_{B_4}^-(v_4 v_1), \gamma_{B_4}^+(v_4 v_1)\right] = [0.2, 0.3]
\end{align*}
\]

To find IVIFLG \(L(I) = (H, J)\) of I such that

\[
H = \{v_1v_2 = S_{e_1}, v_2v_3 = S_{e_2}, v_3v_4 = S_{e_3}, v_4v_1 = S_{e_4}\} \quad \text{and} \quad J = \{S_{e_1}S_{e_2}, S_{e_2}S_{e_3}, S_{e_3}S_{e_4}, S_{e_4}S_{e_1}\}.
\]

Now, consider \(A_2 = [\sigma_{A_2}^-, \sigma_{A_2}^+]\) and \(B_2 = [\sigma_{B_2}^-, \sigma_{B_2}^+]\) are IVFS of H and J respectively. Then we have

\[
\begin{align*}
\sigma_{A_2}(S_{e_1}) &= [\sigma_{A_2}^-(e_1), \sigma_{A_2}^+(e_1)] = [0.2, 0.5] \\
\sigma_{A_2}(S_{e_2}) &= [\sigma_{A_2}^-(e_2), \sigma_{A_2}^+(e_2)] = [0.1, 0.2] \\
\sigma_{A_2}(S_{e_3}) &= [\sigma_{A_2}^-(e_3), \sigma_{A_2}^+(e_3)] = [0.1, 0.1] \\
\sigma_{A_2}(S_{e_4}) &= [\sigma_{A_2}^-(e_4), \sigma_{A_2}^+(e_4)] = [0.2, 0.4] \\
\gamma_{A_2}(S_{e_1}) &= [\gamma_{A_2}^-(e_1), \gamma_{A_2}^+(e_1)] = [0.1, 0.3] \\
\gamma_{A_2}(S_{e_2}) &= [\gamma_{A_2}^-(e_2), \gamma_{A_2}^+(e_2)] = [0.3, 0.4] \\
\gamma_{A_2}(S_{e_3}) &= [\gamma_{A_2}^-(e_3), \gamma_{A_2}^+(e_3)] = [0.3, 0.4] \\
\gamma_{A_2}(S_{e_4}) &= [\gamma_{A_2}^-(e_4), \gamma_{A_2}^+(e_4)] = [0.2, 0.3]
\end{align*}
\]

Then \(L(I)\) of IVIFG I is shown in Fig. 2.

**Proposition 15** \(L(I) = (A_2, B_2)\) is IVIFLG corresponding to IVIFG \(I = (A_1, B_1)\).

**Definition 16** A homomorphism mapping \(\psi : I_1 \to I_2\) of two IVIFG \(I_1 = (M_1, N_1)\) and \(I_2 = (M_2, N_2)\), \(\psi : V_1 \to V_2\) is defined as

(i)

\[
\begin{align*}
\sigma_{M_2}^-(v_i) &\leq \sigma_{M_2}^+(\psi(v_i)) \\
\sigma_{M_2}^-(v_i) &\leq \sigma_{M_2}^+(\psi(v_i)) \\
\gamma_{M_2}^-(v_i) &\leq \gamma_{M_2}^+(\psi(v_i)) \\
\gamma_{M_2}^-(v_i) &\leq \gamma_{M_2}^+(\psi(v_i))
\end{align*}
\]

for all \(v_i \in V_1\).

(ii)

\[
\begin{align*}
\sigma_{N_2}^-(v_i v_j) &\leq \sigma_{N_2}^+(\psi(v_i) \psi(v_j)) \\
\sigma_{N_2}^-(v_i v_j) &\leq \sigma_{N_2}^+(\psi(v_i) \psi(v_j)) \\
\gamma_{N_2}^-(v_i v_j) &\leq \gamma_{N_2}^+(\psi(v_i) \psi(v_j)) \\
\gamma_{N_2}^-(v_i v_j) &\leq \gamma_{N_2}^+(\psi(v_i) \psi(v_j))
\end{align*}
\]

for all \(v_i v_j \in E_1\).
**Definition 17** A bijective homomorphism \( \psi : I_1 \rightarrow I_2 \) of IVIFG is said to be a weak vertex isomorphism, if

\[
\begin{align*}
\sigma_{\psi^{-1}}(v_i) &= [\sigma_{\psi^{-1}}(v_{i_1}), \sigma_{\psi^{-1}}(v_{i_2})] \\
\gamma_{\psi^{-1}}(v_i) &= [\gamma_{\psi^{-1}}(v_{i_1}), \gamma_{\psi^{-1}}(v_{i_2})] \quad \forall v_i \in V_1.
\end{align*}
\]

A bijective homomorphism \( \psi : I_1 \rightarrow I_2 \) of IVIFG is said to be a weak line isomorphism if

\[
\begin{align*}
\sigma_{\psi^{-1}}(v_i v_j) &= [\sigma_{\psi^{-1}}(v_{i_1} v_{j_1}), \sigma_{\psi^{-1}}(v_{i_2} v_{j_2})] \\
\gamma_{\psi^{-1}}(v_i v_j) &= [\gamma_{\psi^{-1}}(v_{i_1} v_{j_1}), \gamma_{\psi^{-1}}(v_{i_2} v_{j_2})] \quad \forall v_i v_j \in E_1.
\end{align*}
\]

If \( \psi : I_1 \rightarrow I_2 \) is a bijective homomorphism and satisfies Definition 17, \( \psi \) is said to be a weak isomorphism of IVIFGs \( I_1 \) and \( I_2 \).

**Proposition 18** The IVIFLG \( L(I) \) is connected if IVIFG \( I \) is connected graph.

**Proof**

Suppose \( L(I) \) be connected IVIFLG of \( I \). We need to show that necessary condition. Consider \( I \) is disconnected IVIFG. Then there exist at least two nodes of \( I \) which are not linked by path, say \( v_1 \) and \( v_j \). If we take one edge \( e \) in the first component of the edge set of \( I \), then it does not have any edges adjacent to \( e \) in other components. Then the IVIFLG of \( I \) is disconnected and contradicts. Therefore, \( I \) is connected.

Conversely, suppose that \( I \) is connected IVIFG. Then, there is a path among every pair of nodes. Thus by IVIFLG definition, edges which are adjacent in \( I \) are adjacent nodes in IVIFLG. Therefore, each pair of nodes in IVIFLG of \( I \) are connected by a path. Hence the proof. \( \square \)

**Proposition 19** The IVIFLG of IVIFG \( K_{1,n} \) is \( K_n \) which is complete IVIFG with \( n \) nodes.

**Proof**

For IVIFG \( K_{1,n} \), let us take \( v \in V(K_{1,n}) \) which is adjacent to every \( u_i \in V(K_{1,n}) \) where \( i = 1, 2, \ldots, n \). Implies that \( v \) is adjacent with every \( u_i \). Thus, in IVIFLG of \( K_{1,n} \), all the vertices are adjacent. This implies that it is complete. Hence the proof. \( \square \)

**Example 20**

Consider the IVIFG \( K_{1,3} \) with vertex sets of \( V = \{v, v_1, v_2, v_3\} \) and edge sets \( E = \{v_1 v_2, v_2 v_3\} \) where

\[
\begin{align*}
v &= ([0.3, 0.5], [0.1, 0.4]), \\
v_1 &= ([0.3, 0.4], [0.2, 0.5]), \\
v_2 &= ([0.5, 0.8], [0.1, 0.2]), \\
v_3 &= ([0.1, 0.3], [0.5, 0.7])
\end{align*}
\]

\[
\begin{align*}
e_1 &= v_1 v_2 = ([0.2, 0.3], [0.3, 0.5]), \\
e_2 &= v_2 v_3 = ([0.2, 0.5], [0.0, 0.3]) \\
e_3 &= v_3 v_3 = ([0.1, 0.2], [0.3, 0.6])
\end{align*}
\]
Then by definition of IVIFLG, the vertex sets of $L(K_{1,3})$ is $V = \{S_1, S_2, S_3\}$ and \{S_1, S_2, S_3, S_4, S_5\} edge sets where

\[
S_1 = ([0.2, 0.3], [0.3, 0.5]), \\
S_2 = ([0.2, 0.5], [0.3, 0.3]), \\
S_3 = ([0.1, 0.2], [0.2, 0.6]), \\
S_4, S_5 = ([0.2, 0.3], [0.3, 0.5]), \\
S_2, S_3 = ([0.1, 0.2], [0.2, 0.6]).
\]

Here $L(K_{1,3})$ is complete graph $K_3$.

The Fig. 3 depicts the example 20

**Proposition 21** Let $L(I)$ be IVIFLG of IVIFG of $I$. Then $L(I^*)$ is a line graph of $I^*$ where $I^* = (V, E)$ with underlying set $V$.

**Proof**

Given $I = (A_1, B_1)$ is IVIFG of $I^*$ and $L(I) = (A_2, B_2)$ is IVIFG of $L(I^*)$ then

\[
\sigma_{A_2}(S_e) = [\sigma_{A_2}^{-}(S_e), \sigma_{A_2}^{+}(S_e)] = [\sigma_{B_1}^{-}(e), \sigma_{B_1}^{+}(e)], \\
\gamma_{A_2}(S_e) = [\gamma_{A_2}^{-}(S_e), \gamma_{A_2}^{+}(S_e)] = [\gamma_{B_1}^{-}(e), \gamma_{B_1}^{+}(e)] \quad \forall e \in E.
\]

This implies, $S_e \in H = \{e \cup \{u_e, v_e\} : e \in E, u_e, v_e \in V \land e = u_ev_e\}$ if and only if $e \in E$.

\[
\sigma_{B_2}(S_e) = [\sigma_{B_2}^{-}(S_e), \sigma_{B_2}^{+}(S_e)] = [\sigma_{B_2}^{-}(e), \sigma_{B_2}^{+}(e)] \\
\gamma_{B_2}(S_e) = [\gamma_{B_2}^{-}(S_e), \gamma_{B_2}^{+}(S_e)] = [\gamma_{B_2}^{-}(e), \gamma_{B_2}^{+}(e)]
\]

where \( J = \{S_e \mid S_e \in E \land e \in E \land e \neq e\} \). Hence, $L(I^*)$ is a line graph of $I^*$. \hfill \Box

**Proposition 22** Let $L(I) = (A_2, B_2)$ be IVIFLG of $L(I^*)$. Then $L(I)$ is also IVIFLG of some IVIFG $I = (A_1, B_1)$ iff

\[
(i) \quad \sigma_{B_2}(S_e) = [\sigma_{B_2}^{-}(S_e), \sigma_{B_2}^{+}(S_e)] = [\sigma_{A_2}^{-}(S_e), \sigma_{A_2}^{+}(S_e)] \\
(ii) \quad \gamma_{B_2}(S_e) = [\gamma_{B_2}^{-}(S_e), \gamma_{B_2}^{+}(S_e)] = [\gamma_{A_2}^{-}(S_e), \gamma_{A_2}^{+}(S_e)]
\]

We know that IVIFS $A_1 = ([\sigma_{A_1}^{+}, \sigma_{A_1}^{-}], [\gamma_{A_1}^{-}, \gamma_{A_1}^{+}])$ yields the properties

\[
\sigma_{A_1}^{-}(v_i v_j) \leq \sigma_{A_1}^{+}(v_i) \land \sigma_{A_1}^{+}(v_j) \\
\sigma_{A_1}^{+}(v_i v_j) \leq \sigma_{A_1}^{-}(v_i) \land \sigma_{A_1}^{-}(v_j) \\
\gamma_{A_1}^{-}(v_i v_j) \leq \gamma_{A_1}^{-}(v_i) \lor \gamma_{A_1}^{-}(v_j) \\
\gamma_{A_1}^{+}(v_i v_j) \leq \gamma_{A_1}^{+}(v_i) \lor \gamma_{A_1}^{+}(v_j)
\]

will suffice. From definition of IVIFLG the converse of this statement is well known. \Box

**Proposition 23** An IVIFLG is always a strong IVIFG.

**Proof**

It is straightforward from the definition, therefore it is omitted. \hfill \Box

**Proposition 24** Let $I_1$ and $I_2$ IVIFGs of $I_1^*$ and $I_2^*$ respectively. If the mapping $\psi : I_1 \rightarrow I_2$ is a weak isomorphism, then $\psi : I_1^* \rightarrow I_2^*$ is isomorphism map.

**Proof**

Suppose $\psi : I_1 \rightarrow I_2$ is a weak isomorphism. Then

\[
\psi \in V_1 \Leftrightarrow (\psi(v) \in V_2 \land u \in E_1 \Leftrightarrow (\psi(u) \psi(v) \in E_2).
\]

Hence the proof. \hfill \Box

**Theorem 25** Let $I^* = (V, E)$ is connected graph and consider that $L(I) = (A_2, B_2)$ is IVIFLG corresponding to IVIFG $I = (A_1, B_1)$. Then
1. There is a map $\psi : I \to L(I)$ which is a weak isomorphism if $I^*$ a cyclic graph such that

$$\sigma_{A_1}(v) = [\sigma_{A_1}^-(v), \sigma_{A_1}^+(v)] = [\sigma_{B_1}^-(e), \sigma_{B_1}^+(e)]$$

$$\gamma_{A_1}(v) = [\gamma_{A_1}^-(v), \gamma_{A_1}^+(v)] = [\gamma_{B_1}^-(e), \gamma_{B_1}^+(e)]$$

$$\forall v \in V, e \in E$$

where $A_1 = (\sigma_{A_1}, \sigma_{A_1}^+), B_1 = (\gamma_{B_1}, \gamma_{B_1}^+)$.

2. The map $\psi$ is isomorphism if $\psi : I \to L(I)$ is a weak isomorphism.

Proof

Consider $\psi : I \to L(I)$ is a weak isomorphism. Then we have

$$\sigma_{A_1}(v) = [\sigma_{A_1}^-(v), \sigma_{A_1}^+(v)] = [\sigma_{A_1}^-(v), \sigma_{A_1}^+(v)]$$

$$\gamma_{A_1}(v) = [\gamma_{A_1}^-(v), \gamma_{A_1}^+(v)] = [\gamma_{A_1}^-(v), \gamma_{A_1}^+(v)]$$

$$\forall v \in V, e \in E$$

This follows that $I^* = (V, E)$ is a cyclic from proposition 24.

Now let $v_1v_2v_3 \cdots v_nv_1$ be a cycle of $I^*$ where vertices set $V = \{v_1, v_2, \cdots, v_n\}$ and edges set $E = \{v_1v_2, v_2v_3, \cdots, v_nv_1\}$. Then we have IVIFS

$$\sigma_{A_1}(v) = [\sigma_{A_1}^-(v), \sigma_{A_1}^+(v)] = [t_i^-, t_i^+]$$

$$\gamma_{A_1}(v) = [\gamma_{A_1}^-(v), \gamma_{A_1}^+(v)] = [\gamma_i^-, \gamma_i^+]$$

and

$$\sigma_{B_1}(v) = [\sigma_{B_1}^-(v), \sigma_{B_1}^+(v)] = [\sigma_{B_1}^-(v), \sigma_{B_1}^+(v)]$$

$$\gamma_{B_1}(v) = [\gamma_{B_1}^-(v), \gamma_{B_1}^+(v)] = [\gamma_{B_1}^-(v), \gamma_{B_1}^+(v)]$$

$$\forall v \in V$$

where $i = 1, 2, \cdots, n$ and $v_{n+1} = v_1$. Thus, for $t_i^- = t_{i+1}^-$, $t_i^+ = t_{i+1}^+$, $f_i^+ = f_{i+1}^+$, $f_i^- = f_{i+1}^-$.

Now

$$H = \{S_e : i = 1, 2, \cdots, n\}$$

and

$$J = \{S_eS_{e+1} : i = 1, 2, \cdots, n - 1\}$$

And also,

$$\sigma_{A_2}(S_e) = [\sigma_{A_2}^-(S_e), \sigma_{A_2}^+(S_e)]$$

$$= [\sigma_{B_1}^-(v_{i+1}), \sigma_{B_1}^+(v_{i+1})]$$

$$\gamma_{A_2}(S_e) = [\gamma_{A_2}^-(S_e), \gamma_{A_2}^+(S_e)]$$

$$= [\gamma_{B_1}^-(v_{i+1}), \gamma_{B_1}^+(v_{i+1})]$$

$$\forall v \in V$$

where $v_{n+1} = v_1, v_{n+2} = v_2, \cdots, v_{n+1} = v_1$. Thus, for $t_i^- = t_{i+1}^-$, $t_i^+ = t_{i+1}^+$, $f_i^- = f_{i+1}^-$, $f_i^+ = f_{i+1}^+$.

The map $\psi : V \to H$ is bijective map since $\psi : I^* \to L(I^*)$ is isomorphism. And also, $\psi$ preserves adjacency. So that $\psi$ persuades an alternative $r$ of $\{1, 2, \cdots, n\}$ which

$$\psi(v_i) = S_{e_{ri}}$$

and for $e_i = v_{i+1}$ then $\psi(v_i) = S_{e_{ri}}S_{e_{ri+1}}$, $i = 1, 2, \cdots, n - 1$. 


Now
\[
\begin{align*}
t_i^- &= \sigma_{A_1}(v_i) \leq \sigma_{A_2}(\psi(v_i)) = \sigma_{A_2}(S_{e(t(i)}) = t_{\tau(i)}, \\
t_i^+ &= \sigma_{A_1}(v_i) \leq \sigma_{A_2}(\psi(v_i)) = \sigma_{A_2}(S_{e(t(i)}) = t_{\tau(i)}, \\
f_i^- &= \gamma_{A_1}(v_i) \leq \gamma_{A_2}(\psi(v_i)) = \gamma_{A_2}(S_{e(t(i)}) = q_{\tau(i)}, \\
f_i^+ &= \gamma_{A_1}(v_i) \leq \gamma_{A_2}(\psi(v_i)) = \gamma_{A_2}(S_{e(t(i)}) = q_{\tau(i)}.
\end{align*}
\]

And let \( v_i = v_i v_{i+1}. \)
\[
\begin{align*}
t_i^- &= \sigma_{B_1}(v_{i1}) + \sigma_{B_2}(\psi(v_i)) = \sigma_{B_2}(S_{e(t(i))}) = \min(\sigma_{B_1}(e(t(i))), \sigma_{B_2}(e(t(i)))) \\
t_i^+ &= \sigma_{B_1}(v_{i1}) + \sigma_{B_2}(\psi(v_i)) = \sigma_{B_2}(S_{e(t(i))}) = \min(\sigma_{B_1}(e(t(i))), \sigma_{B_2}(e(t(i)))) \\
q_i^- &= \gamma_{B_1}(v_{i1}) + \gamma_{B_2}(\psi(v_i)) = \gamma_{B_2}(S_{e(t(i))}) = \max(\gamma_{B_1}(e(t(i))), \gamma_{B_2}(e(t(i)))) \\
q_i^+ &= \gamma_{B_1}(v_{i1}) + \gamma_{B_2}(\psi(v_i)) = \gamma_{B_2}(S_{e(t(i))}) = \max(\gamma_{B_1}(e(t(i))), \gamma_{B_2}(e(t(i))))
\end{align*}
\]

Which implies,
\[
\begin{align*}
t_i^- &\leq t_{\tau(i)}^- \leq t_{\tau(i)}^+, \\
f_i^- &\leq q_{\tau(i)}^- \leq q_{\tau(i)}^+
\end{align*}
\]
(2)

and
\[
\begin{align*}
t_i^- &\leq \min(t_{\tau(i)}^-, t_{\tau(i)}^+), \\
q_i^- &\leq \max(q_{\tau(i)}^-, q_{\tau(i)}^+)
\end{align*}
\]
(3)

Thus from the above equations, we obtain
\[
\begin{align*}
t_i^- &\leq t_{\tau(i)}^- \leq t_{\tau(i)}^+ \leq q_{\tau(i)}^- \leq q_{\tau(i)}^+ \\
t_i^- &\leq q_{\tau(i)}^- \leq q_{\tau(i)}^+ \leq q_{\tau(i)}^- \leq q_{\tau(i)}^+
\end{align*}
\]

where \( t^{k+1} \) is the identity function. It follows
\[
\begin{align*}
t_i^- &\leq t_{\tau(i)}^- \leq \cdots \leq t_{\tau(i)}^+ \leq t_i^- \\
q_i^- &\leq q_{\tau(i)}^- \leq \cdots \leq q_{\tau(i)}^+ \leq q_i^+
\end{align*}
\]

This implies for all \( i = 1, 2, \cdots, n, v_i^- = v_{i1}, v_i^+ = v_{i1}, q_i^- = q_{i1} \) and \( q_i^+ = q_{i1}^+ \). Thus, from Eqs. 1 and 2 we obtain
\[
\begin{align*}
t_i^- &\leq t_{\tau(i)}^- \leq \cdots \leq t_n^- \leq t_n^+ \leq t_i^- \\
q_i^- &\leq q_{\tau(i)}^- \leq \cdots \leq q_n^- \leq q_n^+ \leq q_i^+
\end{align*}
\]

Hence the proof. \( \square \)

**Theorem 26** The IVILG of connected simple IVIFG I is a path graph if I is a path graph.

**Proof**

Consider a path IVIFG I of with \( |V(I)| = k \). This implies I is \( P_k \) path graph and \( |E(I)| = k - 1 \). Since the edge sets of I is the vertices set of IVILG L(I), clearly L(I) is a path graph with number of vertices \( |V(L(I))| = k - 1 \) and \( |E(L(I))| = k - 2 \) edges. Then it’s a path graph.

Conversely, suppose L(I) is a path graph. This implies that each degree of vertex v_i \( \in I \) is greater than two. If one of the degrees of vertex v_i \( \in I \) is greater than two, then the edges e which incident to v_i \( \in I \) would form a complete subgraph of IVILG L(I) of more than two vertices. Therefore, the IVIFG I must be either cyclic or path graph. But, it can’t be the cyclic graph since a line graph of the cyclic graph is the cyclic graph. Hence the proof. \( \square \)

**Limitations**

- This paper introduces only the new concept of IVIFLGS which is the extension of IFLG.
• We focused only on some properties of IVFLG and not all properties are mentioned.
• Due to uncertainty and imprecise many real-world problems like networks communication, machine learning, data organization, traffic light control, computational devices, medical diagnosis, decision making, and the flow of computation is difficult to solve without using IFS, IVIF models it has become rapidly useful in the world. But, in this paper the application part is not included.

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VNSR involved in formal analysis, methodology, writing and supervising the work. KAT and MAA contributed in the conceptualization, methodology, writing and editing the article. All authors read and approved the final manuscript.

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References
1. Goyal S, Garg P, Mishra VN. New corona and new cluster of graphs and their wiener index. Electron J Math Anal Appl. 2020;8(1):100–8.
2. Goyal S, Jain D, Mishra VN. Wiener index of sum of shadow graphs. Discret Math Algorithms Appl. 2022. https://doi.org/10.1142/S1793830922500689.
3. Praveena K, Venkataraman M, Rokini A, Mishra VN. Equitable coloring on subdivision-vertex join and subdivision-edge join of graphs. Ital J Pure Appl Math. 2021;46:836–49.
4. Goyal S, Garg P, Mishra VN. New composition of graphs and their wiener indices. Appl Math Nonlinear Sci. 2019;175–80. DOI: 10.2478/AMNS.2019.1.00016
5. Mishra VN, Delen S, Cangul IN. Algebraic structure of graph operations in terms of degree sequences. Int J Anal Appl. 2018;16(6):809–21. DOI: 10.28924/2291-8639-162018-809.
6. Zadeh AL. Information and control. Fuzzy Sets. 1965;8(3):338–53.
7. Kaufmann A. Introduction theory of fuzzy sets. New York: Academic Press; 1975. p. 4.
8. Rosenfeld A. Fuzzy graphs, fuzzy sets and their applications. New York: Academic Press; 1975. p. 77–95.
9. Atanassov K. Review and new results on intuitionistic fuzzy sets. Int J Bioautomation. 2016;20:17–26.
10. Atanassov K. Intuitionistic fuzzy sets. Theory and applications. New York: Physica-Verlag; 1999. https://doi.org/10.1007/978-3-7908-1870-3.
11. Atanassov KT. On intuitionistic fuzzy sets theory. Heidelberg: Springer; 2012. p. 283. https://doi.org/10.1007/978-3-642-29127-2.
12. Mordeson J. Fuzzy line graphs. Pattern Recognit Lett. 1993;14(5):381–4. https://doi.org/10.1016/0167-8655(93)90015-T.
13. Akram M, Dudek W. Interval-valued fuzzy sets. Comput Math Appl. 2011;61:289–99. https://doi.org/10.1016/j.camwa.2010.11.004.
14. Akram M. Interval-valued fuzzy set theory. Neural Comput Appl. 2012;21:1–6. https://doi.org/10.1007/s00521-011-0733-0.
15. Akram M, Davvaz B. Strong intuitionistic fuzzy graphs. Filomat. 2012;26(1):177–96. https://doi.org/10.2298/FIL1201177A.
16. Akram M, Parvathi R. Properties of intuitionistic fuzzy line graphs. Notes Intuitionistic Fuzzy Sets. 2012;18(3):52–60.
17. Parvathi R, Karunambigai MG, Atanassov KT. Operations on intuitionistic fuzzy graphs. Jeju Island: 2009 IEEE International Conference on Fuzzy Systems, 2009. p. 1396–401. https://doi.org/10.1109/FUZZY.2009.5277067.
18. Mishra VN, Delen S, Cangul IN. Degree sequences of join and corona products of graphs. Electron J Math Anal Appl. 2019;7(1):5–13.
19. Mishra VN. Some problems on approximations of functions in banach spaces. PhD thesis. Roorkee: Indian Institute of Technology; 2007. p. 247–667.
20. Gowri S, Venkatachalam M, Mishra VN, Mishra LN. On r-dynamic coloring of double star graph families. Palest J Math. 2021;10(1):53–62.

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