COLORFUL HYPERGRAPHS IN KNESER HYPERGRAPHS

FRÉDÉRIC MEUNIER

Abstract. Using a $Z_q$-generalization of a theorem of Ky Fan, we extend to Kneser hypergraphs a theorem of Simonyi and Tardos that ensures the existence of multicolored complete bipartite graphs in any proper coloring of a Kneser graph. It allows to derive a lower bound for the local chromatic number of Kneser hypergraphs (using a natural definition of what can be the local chromatic number of a hypergraph).

1. Introduction

1.1. Motivations and results. A hypergraph is a pair $H = (V(H), E(H))$, where $V(H)$ is a finite set and $E(H)$ a family of subsets of $V(H)$. The set $V(H)$ is called the vertex set and the set $E(H)$ is called the edge set. A graph is a hypergraph each edge of which is of cardinality two. A $q$-uniform hypergraph is a hypergraph each edge of which is of cardinality $q$. The notions of graphs and 2-uniform hypergraphs therefore coincide. If a hypergraph has its vertex set partitioned into subsets $V_1, \ldots, V_q$ so that each edge intersects each $V_i$ at exactly one vertex, then it is called a $q$-uniform $q$-partite hypergraph. The sets $V_1, \ldots, V_q$ are called the parts of the hypergraph. When $q = 2$, such a hypergraph is a graph and said to be bipartite. A $q$-uniform $q$-partite hypergraph is said to be complete if all possible edges exist.

A coloring of a hypergraph is a map $c : V(H) \rightarrow [t]$ for some positive integer $t$. A coloring is said to be proper if there in no monochromatic edge, i.e. no edge $e$ with $|c(e)| = 1$. The chromatic number of such a hypergraph, denoted $\chi(H)$, is the minimal value of $t$ for which a proper coloring exists. Given $X \subseteq V(H)$, the hypergraph with vertex set $X$ and with edge set $\{e \in E(H) : e \subseteq X\}$ is the subhypergraph of $H$ induced by $X$ and is denoted $H[X]$.

Given a hypergraph $H = (V(H), E(H))$, we define the Kneser graph $KG^2(H)$ by

$$
V(KG^2(H)) = E(H),
$$
$$
E(KG^2(H)) = \{\{e, f\} : e, f \in E(H), e \cap f = \emptyset\}.
$$

The “usual” Kneser graphs, which have been extensively studied – see [19, 20] among many references, some of them being given elsewhere in the present paper – are the special cases $H = ([n], \binom{[n]}{k})$ for some positive integers $n$ and $k$ with $n \geq 2k$. We denote them $KG^2(n, k)$. The main result for “usual” Kneser graphs is Lovász’s theorem [11].

Theorem (Lovász theorem). Given $n$ and $k$ two positive integers with $n \geq 2k$, we have $\chi(KG^2(n, k)) = n - 2k + 2$.

The 2-colorability defect $cd^2(H)$ of a hypergraph $H$ has been introduced by Dol’nikov [8] in 1988 for a generalization of Lovász’s theorem. It is defined as the minimum number of
Theorem (Alon-Frankl-Lovász theorem). Frankl, and Lovász [2].

The following generalization of Lovász’s theorem conjectured by Erdős and proved by Alon, graphs, i.e. $H$ Kneser hypergraphs, which are obtained with the same hypergraph $H$ uniformity is replaced by the $q$-uniformity for an integer $q$.

A Kneser hypergraph is thus the generalization of Kneser graphs obtained when the 2-vertex $H$ graph $\chi$ inequality $\chi$ to their natural order.

In 1976, Erdős [4] initiated the study of Kneser hypergraphs $KG^q(H)$ defined for a hypergraph $H = (V(H), E(H))$ and an integer $q \geq 2$ by

$$
V(KG^q(H)) = E(H)
$$
$$
E(KG^q(H)) = \{\{e_1, \ldots, e_q\} : e_1, \ldots, e_q \in E(H), e_i \cap e_j = \emptyset \text{ for all } i, j \text{ with } i \neq j\}.
$$

A Kneser hypergraph is thus the generalization of Kneser graphs obtained when the 2-uniformity is replaced by the $q$-uniformity for an integer $q \geq 2$. There are also “usual” Kneser hypergraphs, which are obtained with the same hypergraph $H$ as for “usual” Kneser graphs, i.e. $H = ([n], ([n]_k))$. They are denoted $KG^q(n, k)$. The main result for them is the following generalization of Lovász’s theorem conjectured by Erdős and proved by Alon, Frankl, and Lovász [2].

Theorem (Alon-Frankl-Lovász theorem). Given $n$, $k$, and $q$ three positive integers with $n \geq qk$, we have $\chi(KG^q(n, k)) = \left\lceil \frac{n-q(k-1)}{q-1} \right\rceil$.

There exists also a $q$-colorability defect $cd^q(H)$, introduced by Kříž, defined as the minimum number of vertices that must be removed from $H$ so that the hypergraph induced by the remaining vertices is of chromatic number at most $q$:

$$
cd^q(H) = \min\{|Y| : Y \subseteq V(H), \chi(H[V(H) \setminus Y]) \leq q\}.
$$

The following theorem, due to Kříž [9][10], generalizes Dol’nikov’s theorem. It also generalizes the Alon-Frankl-Lovász theorem since $cd^q([n], ([n]_k)) = n - q(k-1)$ and since again the inequality $\chi(KG^q(n, k)) \leq \left\lceil \frac{n-q(k-1)}{q-1} \right\rceil$ is the easy one.

Theorem (Kříž theorem). Let $H$ be a hypergraph and assume that $\emptyset$ is not an edge of $H$.

Then

$$
\chi(KG^q(H)) \geq \left\lceil \frac{cd^q(H)}{q-1} \right\rceil
$$

for any integer $q \geq 2$. 

2
Our main result is the following extension of Simonyi-Tardos’s theorem to Kneser hypergraphs.

**Theorem 1.** Let \( \mathcal{H} \) be a hypergraph and assume that \( \emptyset \) is not an edge of \( \mathcal{H} \). Let \( p \) be a prime number. Then any proper coloring \( c \) of \( KG^p(\mathcal{H}) \) with colors \( 1, \ldots, t \) (\( t \) arbitrary) must contain a complete \( p \)-uniform \( p \)-partite hypergraph with parts \( U_1, \ldots, U_p \) satisfying the following properties.

- It has \( cd^p(\mathcal{H}) \) vertices.
- The values of \( |U_j| \) for \( j = 1, \ldots, p \) differ by at most one.
- For any \( j \), the vertices of \( U_j \) get distinct colors.

We get that each \( U_j \) is of cardinality \( \lfloor cd^p(\mathcal{H})/p \rfloor \) or \( \lceil cd^p(\mathcal{H})/p \rceil \).

Note that Theorem 1 implies directly Kríž’s theorem when \( q \) is a prime number \( p \): each color may appear at most \( p - 1 \) times within the vertices and there are \( cd^p(\mathcal{H}) \) vertices.

There is a standard derivation of Kríž’s theorem for any \( q \) from the prime case, see [21, 22].

Theorem 1 is a generalization of Simonyi-Tardos’s theorem except for a slight loss: when \( p = 2 \), we do not recover the alternation of the colors between the two parts.

Whether Theorem 1 is true for non-prime \( p \) is an open question.

2. **Local chromatic number and Kneser hypergraphs**

In a graph \( G = (V,E) \), the closed neighborhood of a vertex \( u \), denoted \( N[u] \), is the set \( \{u\} \cup \{v : uv \in E\} \). The local chromatic number of a graph \( G = (V,E) \), denoted \( \chi_\ell(G) \), is the maximum number of colors appearing in the closed neighborhood of a vertex minimized over all proper colorings:

\[
\chi_\ell(G) = \min_c \max_{v \in V} |c(N[v])|,
\]

where the minimum is taken over all proper colorings \( c \) of \( G \). This number has been defined in 1986 by Erdős, Füredi, Hajnal, Komjáth, Rödl, and Seress [5]. For Kneser graphs, we have the following theorem, which is a consequence of the Simonyi-Tardos theorem: any vertex of the part with \( \lfloor r/2 \rfloor \) vertices in the completely multicolored complete bipartite subgraph has at least \( \lceil r/2 \rceil + 1 \) colors in its closed neighborhood (where \( r = cd^2(\mathcal{H}) \)).

**Theorem** (Simonyi-Tardos theorem for local chromatic number). Let \( \mathcal{H} \) be a hypergraph and assume that \( \emptyset \) is not an edge of \( \mathcal{H} \). If \( cd^2(\mathcal{H}) \geq 2 \), then

\[
\chi_\ell(KG^2(\mathcal{H})) \geq \left\lceil \frac{cd^2(\mathcal{H})}{2} \right\rceil + 1.
\]

Note that we can also see this theorem as a direct consequence of Theorem 1 in [17] (with the help of Theorem 1 in [13]).

We use the following natural definition for the local chromatic number \( \chi_\ell(\mathcal{H}) \) of a uniform hypergraph \( \mathcal{H} = (V,E) \). For a subset \( X \) of \( V \), we denote by \( \mathcal{N}(X) \) the set of vertices \( v \) such that \( v \) is the sole vertex outside \( X \) for some edge in \( E \):

\[
\mathcal{N}(X) = \{v : \exists e \in E \text{ s.t. } e \setminus X = \{v\}\}.
\]

We define furthermore \( \mathcal{N}[X] := X \cup \mathcal{N}(X) \). Note that if the hypergraph is a graph, \( \mathcal{N}[^v] = N[v] \) for any vertex \( v \). The definition of the local chromatic number of a hypergraph is then:

\[
\chi_\ell(\mathcal{H}) = \min_c \max_{e \in E, v \in e} |c(\mathcal{N}[e \setminus \{v\}])|,
\]
where the minimum is taken over all proper colorings \( c \) of \( \mathcal{H} \). When the hypergraph \( \mathcal{H} \) is a graph, we get the usual notion of local chromatic number for graphs.

The following theorem is a consequence of Theorem 1 and generalizes the Simonyi-Tardos theorem for local chromatic number to Kneser hypergraphs.

**Theorem 2.** Let \( \mathcal{H} \) be a hypergraph and assume that \( \emptyset \) is not an edge of \( \mathcal{H} \). Then

\[
\chi_{\ell}(KG^p(\mathcal{H})) \geq \min \left( \left\lceil \frac{cd^p(\mathcal{H})}{p} \right\rceil + 1, \left\lceil \frac{cd^p(\mathcal{H})}{p-1} \right\rceil \right)
\]

for any prime number \( p \).

**Proof.** Let \( c \) be any proper coloring of \( KG^p(\mathcal{H}) \). Consider the complete \( p \)-uniform \( p \)-partite hypergraph \( \mathcal{G} \) in \( KG^p(\mathcal{H}) \) whose existence is ensured by Theorem 1. Choose \( U_j \) of cardinality \( \left\lceil \frac{cd^p(\mathcal{H})}{p} \right\rceil \).

If \( \left\lceil \frac{cd^p(\mathcal{H})}{p} \right\rceil > \left\lceil \frac{cd^p(\mathcal{H})}{p-1} \right\rceil \), then there is a vertex \( v \) of \( \mathcal{G} \) not in \( U_j \) whose color is distinct of all colors used in \( U_j \). Choose any edge \( e \) of \( \mathcal{G} \) containing \( v \) and let \( u \) be the unique vertex of \( e \cap U_j \). We have then \( |c(N[e \setminus \{u\}]\setminus e)| \geq |U_j| + 1 = \left\lceil \frac{cd^p(\mathcal{H})}{p} \right\rceil + 1 \).

Otherwise, \( \left\lceil \frac{cd^p(\mathcal{H})}{p-1} \right\rceil = \left\lceil \frac{cd^p(\mathcal{H})}{p} \right\rceil \), and for any edge \( e \), we have \( |c(N[e \setminus \{u\}]\setminus e)| \geq \left\lceil \frac{cd^p(\mathcal{H})}{p} \right\rceil = \left\lceil \frac{cd^p(\mathcal{H})}{p-1} \right\rceil \), with \( u \) being again the unique vertex of \( e \cap U_j \).

As for Theorem 1, we do not know whether this theorem remains true for non-prime \( p \).

---

### 3. Combinatorial topology and proof of the main result

#### 3.1. Tools of combinatorial topology

**3.1.1. Basic definitions.** We use the cyclic and multiplicative group \( Z_q = \{\omega^j : j = 1, \ldots, q\} \) of the \( q \)th roots of unity. We emphasize that 0 is not considered as an element of \( Z_q \). For a vector \( X = (x_1, \ldots, x_n) \in (Z_q \cup \{0\})^n \), we define \( X^j \) to be the set \( \{i \in [n] : x_i = \omega^j\} \) and \( |X| \) to be the quantity \( |\{i \in [n] : x_i \neq 0\}| \).

An (abstract) simplicial complex \( K \) is a collection of subsets of a finite set \( V(K) \), called the *vertex set*, such that whenever \( \sigma \in K \) and \( \tau \subseteq \sigma \), we have \( \tau \in K \). Such a \( \tau \) is called a *face* of \( \sigma \). A simplicial complex is said to be *pure* if all maximal simplices for inclusion have same dimension. In the sequel, all simplicial complexes are abstract and we omit this specification from now on.

**3.1.2. Chains and chain maps.** Let \( K \) be a simplicial complex. We denote its chain complex by \( C(K) \). We always assume that the coefficients are taken in \( Z \).

**3.1.3. Special simplicial complexes.** For a simplicial complex \( K \), its first barycentric subdivision is denoted by \( sd(K) \). It is the simplicial complex whose vertices are the nonempty simplices of \( K \) and whose simplices are the collections of simplices of \( K \) that are pairwise comparable for \( \subseteq \) (these collections are usually called *chains* in the poset terminology, with a different meaning as the one used above in “chain complexes”).

As a simplicial complex, \( Z_q \) is seen as being 0-dimensional and with \( q \) vertices. \( Z_q^d \) is the join of \( d \) copies of \( Z_q \). It is a pure simplicial complex of dimension \( d - 1 \). A vertex \( v \) taken is the \( \mu \)th copy of \( Z_q \) in \( Z_q^d \) is also written \( (\epsilon, \mu) \) where \( \epsilon \in Z_q \) and \( \mu \in [d] \). Sometimes, \( \epsilon \) is called the *sign* of the vertex, and \( \mu \) its *absolute value*. This latter quantity is denoted \( |v| \).
The simplicial complex $sd(Z_q^{rd})$ plays a special role. We have $V\left(sd(Z_q^{rd})\right) \simeq (Z_q \cup \{0\})^d \setminus \{(0, \ldots, 0)\}$: a simplex $\sigma \in Z_q^{rd}$ corresponds to the vector $X = (x_1, \ldots, x_d) \in (Z_q \cup \{0\})^d$ with $x_\mu = \epsilon$ for all $(\epsilon, \mu) \in \sigma$ and $x_\mu = 0$ otherwise.

We denote by $\sigma_{q-2}^{q-1}$ the simplicial complex obtained from a $(q - 1)$-dimensional simplex and its faces by deleting the maximal face. It is hence a $(q - 2)$-dimensional pseudomanifold homeomorphic to the $(q - 2)$-sphere. We also identify its vertices with $Z_q$. A vertex of the simplicial complex $\left(\sigma_{q-2}^{q-1}\right)^{rd}$ is again denoted by $(\epsilon, \mu)$ where $\epsilon \in Z_q$ and $\mu \in [d]$. For $\epsilon \in Z_q$ and a simplex $\tau$ of $\left(\sigma_{p-2}^{p-1}\right)^{rd}$, we denote by $\tau^\epsilon$ the set of all vertices of $\tau$ having $\epsilon$ as sign, i.e. $\tau^\epsilon := \{ (\omega, \mu) \in \tau : \omega = \epsilon \}$. Note that if $q$ is a prime number, $Z_q$ acts freely on $\sigma_{q-2}^{q-1}$.

3.1.4. Barycentric subdivision operator. Let $K$ be a simplicial complex. There is a natural chain map $sd_\#: \mathcal{C}(K) \to \mathcal{C}(sd(K))$ which, when evaluated on a $d$-simplex $\sigma \in K$, returns the sum of all $d$-simplices in $sd(K)$ contained in $\sigma$, with the induced orientation. “Contained” is understood according to the geometric interpretation of the barycentric subdivision. If $K$ is a free $Z_q$-simplicial complex, $sd_\#$ is a $Z_q$-equivariant map.

3.1.5. The $Z_q$-Fan lemma. The following lemma plays a central role in the proof of Theorem 5.4. It is proved (implicitly and in a more general version) in [8][14].

Lemma 1 ($Z_q$-Fan lemma). Let $q \geq 2$ be a positive integer. Let $\lambda_\#: \mathcal{C}\left(sd(Z_q^{rn})\right) \to \mathcal{C}\left(Z_q^{rm}\right)$ be a $Z_q$-equivariant chain map. Then there is an $(n-1)$-dimensional simplex $\rho$ in the support of $\lambda_\#(\rho')$, for some $\rho' \in sd(Z_q^{rn})$, of the form $\{(\epsilon_1, \mu_1), (\epsilon_2, \mu_2), \ldots, (\epsilon_n, \mu_n)\}$, with $\mu_i < \mu_{i+1}$ and $\epsilon_i \neq \epsilon_{i+1}$ for $i = 1, \ldots, n$.

This $\rho'$ is an alternating simplex.

Proof. The proof is exactly the proof of Theorem 5.4 (p.415) of [8]. The complex $X$ in the statement of this Theorem 5.4 is our complex $sd(Z_q^{rn})$, the dimension $r$ is $n - 1$, and the generalized $r$-sphere $(x_i)$ is any generalized $(n - 1)$-sphere of $sd(Z_q^{rn})$ with $x_0$ reduced to a single point. The chain map $h^\#_\rho$ is induced by our chain map $\lambda_\#$, instead of being induced by the chain map $\ell_\#$ of [8] (itself induced by the labeling $\ell$). It does not change the proof since $h^\#_\rho$ only uses the fact that $\ell_\#$ is a $Z_q$-equivariant chain map. In the statement of Theorem 5.4 of [8], $\alpha_i$ is always a lower bound on the number of “alternating patterns” (i.e. simplices $\rho'$ as in the statement of the lemma) in $\ell_\#(x_i)$, even for odd $i$ since the map $f_i$ in Theorem 5.4 of [8] is zero on non-alternating elements. Since $\alpha_0 = 1$, we get that $\alpha_i \neq 0$ for all $0 \leq i \leq n - 1$.

In particular, for $q = 2$, it gives the Ky Fan theorem [6] used for instance in [7][15][17] to derive properties of Kneser graphs.

3.2. Proof of the main result.

Proof of Theorem 5.4. The proof goes as follows. We assume given a proper coloring $c$ of $KG^p(\mathcal{H})$. With the help of the coloring $c$, we build a $Z_p$-equivariant chain map $\psi_\#: \mathcal{C}(sd(Z_p^{rn})) \to \mathcal{C}(Z_p^{rm})$, where $m = n - cd^p(\mathcal{H}) + t(p - 1)$. We apply Lemma 1 to get the existence of some alternating simplex $\rho'$ in $sd(Z_p^{rn})$. Using properties of $\psi_\#$ (especially the fact that it is a composition of maps in which simplicial maps are involved), we show
that this alternating simplex provides a complete \( p \)-uniform \( p \)-partite hypergraph in \( \mathcal{H} \) with the required properties.

Let \( r = cd^p(\mathcal{H}) \). We denote \( L := Z_p^{(n-r)} \) and \( M := (\sigma_{p-2}^{p-1})^{\ast t} \). Following the ideas of \[12, 21\], we define \( f : (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\} \rightarrow Z_p \times [m] \) with \( m = n - r + t(p-1) \).

**If** \( X \in (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\} \) **is such that** \( |X| \leq n - r \). Then \( f(X) := (\epsilon, |X|) \) with \( \epsilon \) is the first nonzero component in \( X \).

**If** \( X \in (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\} \) **is such that** \( |X| \geq n-r+1 \). By definition of the colorability defect, at least one of the \( X^j \) with \( j \in [p] \), contains an edge of \( \mathcal{H} \). Choose \( j \in [p] \) such that there is \( S \subseteq X^j \) with \( S \in E(\mathcal{H}) \). Its defines \( F(X) := S \) and \( f(X) := (\epsilon, n-r+c(F(X))) \).

Note that \( f \) induces a \( Z_p \)-equivariant simplicial map \( f : \text{sd}(Z_p^* n) \rightarrow L \ast M \).

Let \( W_\ell \) be the set of simplices \( \tau \in M \) such that \( |\tau'| = 0 \) or \( |\tau'| = \ell \) for all \( \epsilon \in Z_p \). Let \( W = \bigcup_{\ell=1}^m W_\ell \). Choose an arbitrary equivariant map \( s : W \rightarrow Z_p \). Such a map can be easily built by choosing one representative in each orbit \( (Z_p \text{ acts freely on each } W_\ell) \). We build also an equivariant map \( s_0 : \sigma_{p-2}^{p-1} \rightarrow Z_p \), again by choosing one representative in each orbit of the action of \( Z_p \).

We define now a simplicial map \( g : \text{sd}(L \ast M) \rightarrow Z_p^{* m} \) as follows.

Take a vertex in \( \text{sd}(L \ast M) \). It is of the form \( \sigma \cup \tau \neq \emptyset \) where \( \sigma \in L \) and \( \tau \in M \).

**If** \( \tau \neq \emptyset \). Let \( \alpha := \min_{\epsilon \in Z_p} |\tau'| \).

- If \( \alpha = 0 \), define \( \tilde{\tau} := \{ \epsilon \in Z_p : \tau' = \emptyset \} \) and \( g(\sigma \cup \tau) = (s_0(\tilde{\tau}), n - r + |\tau|) \) (we have indeed \( \tilde{\tau} \in \sigma_{p-2}^{p-1} \).)
- If \( \alpha > 0 \), define \( \tilde{\tau} := \bigcup_{\epsilon : |\tau'| = \alpha} \tau' \) and \( g(\sigma \cup \tau) := (s(\tilde{\tau}), n - r + |\tau|) \).

The definition of \( \tilde{\tau} \) is illustrated on Figures 1 and 2.

**If** \( \tau = \emptyset \). Choose \( (\epsilon, \mu) \) in \( \sigma \) with maximal \( \mu \). Define \( g(\sigma \cup \tau) := (\epsilon, \mu) \). Note that \( L \) is such that there is only one \( \epsilon \) for which the maximum is attained.

We check now that \( g \) is a simplicial map. Assume for a contradiction that there are \( \sigma \subseteq \sigma', \tau \subseteq \tau' \) such that \( g(\sigma \cup \tau) = (\epsilon, \mu) \) and \( g(\sigma' \cup \tau') = (\epsilon', \mu) \) with \( \epsilon \neq \epsilon' \). If \( \tau = \emptyset \), then \( \mu \leq n - r \) and \( \tau' = \emptyset \). We should then have \( \epsilon = \epsilon' \), which is impossible. If \( \tau \neq \emptyset \), then \( |\tau'| = |\tau'| \), and thus \( \tau = \tau' \). We should again have \( \epsilon = \epsilon' \) which is impossible as well.

Moreover, \( g \) is increasing: for \( \sigma \subseteq \sigma' \) and \( \tau \subseteq \tau' \), we have \( |g(\sigma \cup \tau)| \leq |g(\sigma' \cup \tau')| \).

We get our map \( \psi_{\#} \) by defining: \( \psi_{\#} = g_{\#} \circ \text{sd}_{\#} \circ f_{\#} \). It is a \( Z_p \)-equivariant chain map from \( C(\text{sd}(Z_p^* n)) \) to \( C(Z_p^{* m}) \).

This chain map \( \psi_{\#} \) satisfies the condition of Lemma 1. Hence, there exists \( \rho \in Z_p^{* m} \) of the form \( \rho = \{(\epsilon_1, \mu_1), \ldots, (\epsilon_n, \mu_n)\} \) with \( \mu_i < \mu_{i+1} \) and \( \epsilon_i \neq \epsilon_{i+1} \) for \( i = 1, \ldots, n - 1 \) such that \( \rho \) is in the support of \( \psi_{\#}(\rho') \) for some \( \rho' \in \text{sd}(Z_p^* n) \).

Since \( g \) is a simplicial map, we know that there is a permutation \( \pi \) and a sequence \( \sigma_{\pi(1)} \cup \tau_{\pi(1)} \subseteq \ldots \subseteq \sigma_{\pi(n)} \cup \tau_{\pi(n)} \) of simplices of \( L \ast M \) such that \( g(\sigma_i \cup \tau_i) = (\epsilon_i, \mu_i) \) with \( \mu_i < \mu_{i+1} \)
and $\epsilon_i \neq \epsilon_{i+1}$ for $i = 1, \ldots, n - 1$. Since $g$ is increasing, we get that $\pi(i) = i$ for all $i$. Using the fact that $f$ is simplicial, we get moreover that $|\sigma_n \cup \tau_n| = n$, and then that $|\sigma_i \cup \tau_i| = i$.

The fact that all $\mu_i$ are distinct implies that $\tau_i = 0$ for $i = 1, \ldots, n - r$. Indeed, $\tau_i = \tau_{i+1}$ implies then that $\tau_i = \emptyset$. We have therefore $\tau_1 = \cdots = \tau_{n-r} = \emptyset$ and thus $|\sigma_{n-r}| = n - r$. It implies that $|\tau_{n-r+l}| = n - r + l - |\sigma_{n-r+l}| \leq l$. On the other hand, we have that $|\tau_{n-r+l}| = \mu_{n-r+l} - n + r \geq l$. Thus, $|\tau_{n-r+l}| = l$.

Consider the sequence $(\omega_1, \nu_1), \ldots, (\omega_{n-r}, \nu_{n-r})$, where $(\omega_1, \nu_1)$ is the unique vertex of $\tau_{r+1}$ and $(\omega_{l+1}, \nu_{l+1})$ the unique vertex of $\tau_{r+l+1} \setminus \tau_{r+l}$ for $l = 1, \ldots, n - 1 - r$. The sign $\omega_{l+1}$ is necessarily such that $\tau_{r+l}^{\omega_{l+1}}$ has a minimum cardinality among the $\tau_{r+l}^\epsilon$, otherwise the set
of $\epsilon$ for which $|\tau_{r+l+1}|$ is minimum would be the same as for $|\tau_{r+l}|$, and, according to the definition of the maps $s$ and $s_0$, we would have $\epsilon_{l+1} = \epsilon_l$.

We clearly have $||\tau_{r+1}| - |\tau_{r+l}|| \leq 1$ for all $\epsilon, \epsilon'$. Now assume that for $k \geq r + 1$ we have $||\tau_k| - |\tau_{r+l}|| \leq 1$ for all $\epsilon, \epsilon'$. Since the element added to $\tau_k$ to get $\tau_{k+1}$ is added to a $\tau_{r+l}$ with minimum cardinality, we have $||\tau_{k+1}| - |\tau_{r+l}|| \leq 1$ for all $\epsilon, \epsilon'$. By induction we have in particular

$$\left| |\tau_{n}| - |\tau_{n}'| \right| \leq 1 \text{ for all } \epsilon, \epsilon'.$$

Using the fact that $f$ is simplicial, we get a sequence $X_{n-r+1} \subseteq \ldots \subseteq X_n$ of signed vectors whose image by $f$ is $\tau_n$. Each $X_i$ provides a vertex $F(X_i)$ of $\text{KG}^p(H)$. For each $j$, define $U_j$ to be the set of $F(X_i)$ such that the sign of $f(X_i)$ is $\omega^j$. The $U_j$ are subsets of vertices of $\text{KG}^p(H)$ and pairwise disjoint. Moreover, for two distinct $j$ and $j'$, if $F(X_i) \in U_j$ and $F(X_{i'}) \in U_{j'}$, we have $F(X_i) \cap F(X_{i'}) = \emptyset$. Thus, the $U_j$ induces in $\text{KG}^p(H)$ a complete $p$-partite $p$-uniform hypergraph with $r = \text{cd}^p(H)$ vertices. Equation (1) indicates that the cardinalities of the $U_j$ differ by at most one. Since $|\tau_n| = n - r$, each $U_j$ has all its vertices of distinct colors.

4. Alternation number

4.1. Definition. Alishahi and Hajiabolhassan [1], going on with ideas introduced in [16], defined the $q$-alteration number of an hypergraph $\text{alt}_q(H)$ as an improvement of the $q$-colorability defect. It is defined as follows.

Let $q$ and $n$ be positive integers. An alternating sequence is a sequence $s_1, s_2, \ldots, s_n$ of elements of $\mathbb{Z}_q$ such that $s_i \neq s_{i+1}$ for all $i = 1, \ldots, n - 1$. For a vector $X = (x_1, \ldots, x_n) \in (\mathbb{Z}_q \cup \{0\})^n$ and a permutation $\pi \in S_n$, we denote $\text{alt}_\pi(X)$ the maximum length of an alternating subsequence of the sequence $x_{\pi(1)}, \ldots, x_{\pi(n)}$. Note that by definition this subsequence has no zero element.

Example. Let $n = 9$, $q = 3$, and $X = (\omega^2, \omega^2, 0, 0, \omega^1, \omega^3, 0, \omega^2, \omega^2)$, we have $\text{alt}_{\text{id}}(X) = 4$. If $\pi$ is a permutation acting only on the first four positions, then $\text{alt}_{\pi}(X) = \text{alt}_\pi(X)$. If $\pi$ exchanges the last two elements of $X$, we have $\text{alt}_\pi(X) = 5$.

Let $H = (V, E)$ be a hypergraph with $n$ vertices. We identify $V$ and $[n]$. The $q$-alteration number of an hypergraph $\text{alt}_q(H)$ with $n$ vertices is defined as:

$$\text{alt}_q(H) = \min_{\pi \in S_n} \max_{\pi \in S_n} \{\text{alt}_\pi(X) : X \in (\mathbb{Z}_q \cup \{0\})^n \text{ with } E(H[X^j]) = \emptyset \text{ for } j = 1, \ldots, q\}.$$ 

Note that this number does not depend on the way $V$ and $[n]$ have been identified.

4.2. Improving the results with the alternation number. Alishahi and Hajiabolhassan improved the Kríž theorem by the following theorem.

Theorem (Alishahi-Hajiabolhassan theorem). Let $H$ be a hypergraph and assume that $\emptyset$ is not an edge of $H$. Then

$$\chi(\text{KG}^q(H)) \geq \left\lceil \frac{|V(H)| - \text{alt}_q(H)}{q - 1} \right\rceil$$

for any integer $q \geq 2$. 

8
Theorem 4. Let be the permutation on which the minimum is attained in Equation (2). We replace \( r = \text{cd}^p(H) \) by \( r = |V(H)| - \text{alt}^p(H) \) in the both proofs and \( |X| \) in the definition of \( f \) by \( \text{alt}_\pi(X) \) in the proof of Theorem 1 without any other change. Since we have \( |V(H)| - \text{alt}^p(H) \geq \text{cd}^p(H) \), often with a strict inequality – see [1] – it improves these theorems.

Theorem 3. Let \( H \) be a hypergraph and assume that \( \emptyset \) is not an edge of \( H \). Let \( p \) be a prime number. Then any proper coloring \( c \) of \( KG^p(H) \) with colors \( 1, \ldots, t \) (\( t \) arbitrary) must contain a complete \( p \)-uniform \( p \)-partite hypergraph with parts \( U_1, \ldots, U_p \) satisfying the following properties.

- It has \( |V(H)| - \text{alt}^p(H) \) vertices.
- The values of \( |U_j| \) for \( j = 1, \ldots, p \) differ by at most one.
- For any \( j \), the vertices of \( U_j \) get distinct colors.

Theorem 4. Let \( H \) be a hypergraph and assume that \( \emptyset \) is not an edge of \( H \). Then

\[
\chi_{\ell}(KG^p(H)) \geq \min \left( \left[ \frac{|V(H)| - \text{alt}^p(H)}{p} \right] + 1, \left[ \frac{|V(H)| - \text{alt}^p(H)}{p - 1} \right] \right)
\]

for any prime number \( p \).

4.3. Complexity. It remains unclear whether the alternation number, or a good upper bound of it, can be computed efficiently. However, we can note that given a hypergraph \( H \), computing the alternation number for a fixed permutation is an NP-hard problem.

Proposition 1. Given a hypergraph \( H \), a permutation \( \pi \), and a number \( q \), computing

\[
\max\{\text{alt}_\pi(X) : X \in (Z_q \cup \{0\})^n \text{ with } E(H[X^j]) = \emptyset \text{ for } j = 1, \ldots, q\}
\]

is NP-hard.

Proof. The proof consists in proving that the problem of finding a maximum independent set in a graph can be polynomially reduced to our problem for \( q = 2 \), \( \pi = \text{id} \), and \( H \) being some special graph.

Let \( G \) be a graph. Define \( G' \) to be a copy of \( G \) and consider the join \( H \) of \( G \) and \( G' \). The join of two graphs is the disjoint union of the two graphs plus all edges \( vv' \) with \( v \) a vertex of \( G \) and \( v' \) a vertex of \( G' \). We number the vertices of \( G \) arbitrarily with a bijection \( \rho : V \to [|V|] \). It gives the following numbering for the vertices of \( H \). In \( H \), a vertex \( v \) receives number \( 2\rho(v) - 1 \) and its copy \( v' \) receives the number \( 2\rho(v) \). Let \( n = 2|V| \). As usual, we denote the maximum cardinality of an independent set of \( G \) by \( \alpha(G) \).

Let \( I \subseteq V \) be a independent set of \( G \). Define \( X \in (Z_2 \cup \{0\})^n \) as follows:

\[
X_{2\rho(v) - 1} = +1 \text{ and } X_{2\rho(v)} = -1 \text{ for all } v \in I, \text{ and } x_i = 0 \text{ for the other indices } i.
\]

By definition of the numbering, we have \( \text{alt}_{\text{id}}(X) = 2|I| \) and thus

\[
\max\{\text{alt}_{\text{id}}(X) : X \in (Z_2 \cup \{0\})^n \text{ with } E(H[X^j]) = \emptyset \text{ for } j = 1, 2\} \geq 2\alpha(G)
\]

Conversely, any \( X \in (Z_2 \cup \{0\})^n \) gives an independent set \( I \) in \( G \) and another \( I' \) in \( G' \): take a longest alternating subsequence in \( X \) and define the set \( I \) as the set of vertices \( v \) such that \( X_{2\rho(v) - 1} \neq 0 \) and the set \( I' \) as the set of vertices \( v \) such that \( X_{2\rho(v)} \neq 0 \). We have
alt\textsubscript{id}(X) = |I| + |I'| because two components of $X$ with distinct index parities cannot be of opposite signs: each vertex of $G$ is the neighbor of each vertex of $G'$. Thus

$$\max\{\text{alt}_{\text{id}}(X) : X \in (Z_2 \cup \{0\})^n \text{ with } E(H[X^j]) = \emptyset \text{ for } j = 1, 2\} \leq 2\alpha(G).$$

$\square$

The same proof gives also that computing the two-colorability defect $\text{cd}^2(H)$ of any hypergraph $H$ is an NP-hard problem.

**References**

1. M. Alishahi and H. Hajiabolhassan, *On chromatic number of Kneser hypergraphs*, preprint.
2. N. Alon, P. Frankl, and L. Lovász, *The chromatic number of Kneser hypergraphs*, Transactions Amer. Math. Soc. 298 (1986), 359–370.
3. V. L. Dolžnikov, *A certain combinatorial inequality*, Siberian Math. J. 29 (1988), 375–397.
4. P. Erdős, *Problems and results in combinatorial analysis*, Colloquio Internazionale sulle Teorie Combinatorie (Rome 1973), Vol. II, No. 17 in Atti dei Convegni Lincei, 1976, pp. 3–17.
5. P. Erdős, Z. Füredi, A. Hajnal, P. Komáth, V. Rödl, and Á. Seress, *Coloring graphs with locally few colors*, Discrete Mathematics 59 (1986), 21–34.
6. K. Fan, *A generalization of Tucker's combinatorial lemma with topological applications*, Annals Math., II Ser. 56 (1952), 431–437.
7. , *Evenly distributed subset of $S^n$ and a combinatorial application*, Pacific J. Math. 98 (1982), 323–325.
8. B. Hanke, R. Sanyal, C. Schultz, and G. Ziegler, *Combinatorial Stokes formulas via minimal resolutions*, Journal of Combinatorial Theory, Series A 116 (2009), 404–420.
9. I. Kríž, *Equivariant cohomology and lower bounds for chromatic numbers*, Transactions Amer. Math. Soc. 33 (1992), 567–577.
10. , *A correction to “Equivariant cohomology and lower bounds for chromatic numbers”*, Transactions Amer. Math. Soc. 352 (2000), 1951–1952.
11. L. Lovász, *Kneser’s conjecture, chromatic number and homotopy*, Journal of Combinatorial Theory, Series A 25 (1978), 319–324.
12. J. Matoušek, *A combinatorial proof of Kneser’s conjecture*, Combinatorica 24 (2004), 163–170.
13. J. Matoušek and G. Ziegler, *Topological lower bounds for the chromatic number: A hierarchy*, Jahresber. Deutsch. Math.-Verein. 106 (2004), 71–90.
14. F. Meunier, *A $Z_q$-Fan theorem*, Tech. report, Laboratoire Leibniz-IMAG, Grenoble, 2005.
15. , *A topological lower bound for the circular chromatic number of Schrijver graphs*, Journal of graph theory 49 (2005), 257–261.
16. , *The chromatic number of almost-stable Kneser hypergraphs*, Journal of Combinatorial Theory, Series A 118 (2011), 1820–1828.
17. G. Simonyi and G. Tardos, *Local chromatic number, Ky Fan’s theorem, and circular colorings*, Combinatorica 26 (2006), 587–626.
18. , *Colorful subgraphs of Kneser-like graphs*, European journal of Combinatorics 28 (2007), 2188–2200.
19. S. Stahl, *n-tuple colorings and associated graphs*, Journal of Combinatorial Theory, Series B 20 (1976), 185–203.
20. M. Valencia-Pabon and J. Vrecia, *On the diameter of Kneser graphs*, Discrete Mathematics 305 (2005), 383–385.
21. G. Ziegler, *Generalized Kneser coloring theorems with combinatorial proofs*, Invent. Math. 147 (2002), 671–691.
22. , *Erratum: Generalized Kneser coloring theorems with combinatorial proofs*, Invent. Math. 163 (2006), 227–228.
F. Meunier, Université Paris Est, CERMICS (ENPC), F-77455 Marne-la-Vallée
E-mail address: frederic.meunier@enpc.fr