Abstract

We consider the large deviations associated with the empirical mean of independent and identically distributed random variables under a subexponential moment condition. We show that non-trivial deviations are observable at a subexponential scale in the number of variables, and we provide the associated rate function, which is non-convex and is not derived from a Legendre–Fenchel transform. The proof adapts the one of Cramer’s theorem to the case where the fluctuation is generated by a single variable. In particular, we develop a new tilting strategy for the lower bound, which leads us to introduce a condition on the second derivative of the moment generating function. Our results are illustrated by a couple of simple examples.

1 Introduction

In most cases, the empirical mean of independent and identically distributed random variables converges to the expectation of the variable, according to the law of large number. The central limit theorem (CLT) describes small fluctuations around the mean, which are gaussian and scale in the square root of the number of observations. A spectacular feature of the CLT is its universality: all variables with the same variance have the same gaussian small fluctuations. Moreover, convergence rates towards the CLT are available, for instance through Berry-Esseen type bounds [13].

It is often interesting to control fluctuations far away from the CLT regime, for both theoretical and practical reasons. This is the concern of large deviations theory, which provides asymptotic control typically at exponential scale [6, 5, 14]. In a standard situation where the random variables have some finite exponential moment, probabilities of fluctuations are indeed exponentially small with the number of observations, the rate of smallness being controlled by a function called rate function, which is in general quadratic around the mean. As a result, a large deviations principle generalizes the strong law of large numbers (by Borel-Cantelli) and the CLT (by expanding the rate function around the mean). Contrarily to the CLT, the rate function is not universal and depends a priori on the entire distribution of the variable.

However, the exponential fluctuation scaling does not always hold true with a non-trivial rate function. Actually, the empirical mean can be controlled at an exponential scale if and only if the variable has some exponential moment [15]. When the variable does not have any exponential moment, the situation is much more complicated. Fluctuation theorems for subexponential variables have been investigated by Nagaev [11, 12] and Borovkov [2] before being recently revisited [9, 3]. Although these works provide useful subexponential estimates, they are not precise enough to provide full large deviations principles with amenable rate function as we could expect from the modern theory [5], in particular concerning the lower bound.

In this paper, we prove a full large deviations principle for a class of subexponential variables. Contrarily to previous works on the topic (see [9, 3] and references therein), we do not assume any form for the cumulative distribution of the random variable, but rather work with a scaled version of the cumulant function that encompasses subexponentiality. This allows in particular to consider cases where the distribution at hand is not known. For proving the upper bound, we rely on the very insightful work [9] that we adapt to our moment assumption. For the lower bound, we develop a new tilting strategy by using a subexponential transform on one
variable. Since only one variable is tilted, we cannot use standard concentration estimates, and rather control the deviations of the tilted variable through an assumption on the second derivative of the moment generating function, which strengthens the Gartner–Ellis condition. Although the upper bound sheds some light on the large deviations mechanism, the proof of the lower bound is the most instructive and original part of the paper. As a side product, it also provides the optimal sampling strategy for reducing variance of a large deviations estimator.

The work is organized as follows. Section 2 presents our assumptions and the associated large deviations result. Some examples of application are proposed in Section 3, while the proofs of lower and upper bounds are postponed to Section 4. Some perspectives are finally discussed in Section 5.

2 Large deviations at subexponential scale

We consider i.i.d. samples \((X_i)_{i \in \mathbb{N}}\) of a random variable \(X\) with law \(\mu\) on \(\mathbb{R}\), satisfying the following assumption.

**Assumption 1.** The random variable \(X\) is symmetric and has finite polynomial moments of any order.

An immediate consequence of this assumption is that \(E[X] = 0\). These conditions are not restrictive for the problem we are considering, but simplify the presentation of the results. We associate the samples \((X_i)_{i \in \mathbb{N}}\) the empirical mean

\[
S_n = \frac{1}{n} \sum_{i=1}^{n} X_i. \tag{1}
\]

Our goal is to derive a large deviations principle for \(S_n\) under a subexponential moment condition on \(X\). For this, we introduce the following scaling functions.

**Definition 1.** For \(\alpha \in (0, 1)\), we define a scaling function by

\[
\forall x \in \mathbb{R}, \quad \phi_\alpha(x) = \text{sign}(x)|x|^\alpha.
\]

The scaling function \(\phi_\alpha\) is a natural tool to introduce a subexponential moment condition through a scaled free energy \(^1\), which we define, for any \(\alpha \in (0, 1)\), by

\[
\forall \eta \in \mathbb{R}, \quad \lambda_\alpha(\eta) = \log E \left[e^{\eta \phi_\alpha(X)}\right]. \tag{2}
\]

Just like a standard free energy, \(\lambda_\alpha\) is a convex function from \(\mathbb{R}\) into \((-\infty, +\infty]\) with domain

\[
D_{\lambda_\alpha} = \{\eta \in \mathbb{R}, \lambda_\alpha(\eta) < +\infty\}. \tag{3}
\]

Since \(\lambda_\alpha\) is convex, its domain is convex, hence it is a segment. Since \(X\) is symmetric so is \(\lambda_\alpha\). We thus have \(D_{\lambda_\alpha} = (-\xi, \xi)\) or \(D_{\lambda_\alpha} = [-\xi, \xi]\) for some \(\xi \in [0, +\infty]\). Moreover, we know that \(\lambda_\alpha\) is (infinitely) differentiable on the interior of its domain by standard dominated convergence results \(^2\) Lemma 2.2.5. We now propose a generalization of the essential smoothness condition of the Gartner–Ellis theorem (see \(^2\) Definition 2.3.5).

**Assumption 2 (Second order essential smoothness).** There exists \(\alpha \in (0, 1)\) such that the function \(\lambda_\alpha : \mathbb{R} \to (\infty, +\infty] \) satisfies:

- **Non-trivial bounded domain:** \(D_{\lambda_\alpha} = (-\xi, \xi)\) for some \(\xi \in (0, +\infty)\).
- **Steepness:** for any sequence \((\eta_n)_{n \in \mathbb{N}}\) converging to \(\pm \xi\), it holds

\[
|\lambda'_\alpha(\eta_n)| \xrightarrow{n \to +\infty} +\infty. \tag{4}
\]

\(^1\)We prefer the naming free energy to cumulant generating function because scaled cumulant generating function refers to another concept related to the Gartner–Ellis theorem \(^2\).
Bounded relative variance: define
\[ V(\eta) = \frac{\lambda''(\eta)}{\lambda'(\eta)^2}. \] (5)
There exist \( \xi_0 \in (0, \xi) \) and \( \omega \in (0, +\infty) \) such that \( \lambda''(\eta) \) (resp. \( V(\eta) \)) is non-decreasing (resp. non-increasing) on \( [\xi_0, \xi) \) and
\[ \forall \eta \in (\xi_0, \xi), \quad V(\eta) \leq \omega. \] (6)

We are now in position to state our main theorem. The definition of a large deviations principle is recalled (with some technical details) in Appendix A.

**Theorem 1.** Let Assumptions 1 and 2 hold. Then the empirical mean \( S_n \) defined in (1) satisfies a large deviations principle at speed \( n^\alpha \) with rate function \( I_\alpha \) defined by:
\[ I_\alpha(x) = \xi |x|^\alpha. \] (7)
The proof of Theorem 1 is postponed to Section 4. We propose some remarks on this result before presenting in Section 3 a couple of situations where it applies.

**Remark 1.**
- Although, for i.i.d. variables, \( \lambda_\alpha \) is differentiable on the interior of its domain (so we do not need to make a smoothness assumption on \( \lambda_\alpha \)), there exist non-steep such free energies [5, exercise 2.3.17].
- One may be confused by the reference to Cramér’s theorem, whereas we use a generalization of the Gartner–Ellis steepness condition. We refer here to the range of applications of Theorem 1, which concerns independent variables, rather than the assumption. Moreover, the steepness condition is not used for deriving the lower bound from the upper bound through convex analysis arguments, but to perform an arbitrarily large exponential tilting on one variable, which is quite different.
- If \( X \) is a variable with density \( p \) whose tail scales like \( e^{-\xi_\alpha |x|^{\alpha}} \) at infinity, then the scaled variable \( Y = \phi_\alpha(X) \) mostly scales like \( e^{-\xi_\alpha |y|^{\alpha}} \) at infinity, up to slowly varying functions at exponential scale, like in [9]. One can thus understand Assumption 8 as a way to find the right scaling to bring the subexponential tail back to an exponential one (or in other words to find the correct subexponential decay scale). The speed of the LDP is given by the exponent \( \alpha \) while the rate function is fully determined by the factor \( \xi_\alpha \).
- One can consider the scaled random variable \( Y = \phi_\alpha(X) \). Since this variable has an exponential moment, its empirical mean satisfies a LDP with a good rate function defined by the Fenchel transform
\[ J_\alpha(x) = \sup_{\eta} \{ \eta x - \lambda_\alpha(\eta) \}. \]
We can check by convex analysis that actually
\[ \xi = \lim_{n \to +\infty} \frac{J_\alpha(n)}{n}, \]
and thus, \( \forall x \in \mathbb{R} \),
\[ I_\alpha(x) = \lim_{n \to +\infty} \frac{J_\alpha(n^\alpha |x|^{\alpha})}{n^\alpha}. \]
One could actually expect the subexponential rate function \( I_\alpha \) to be equal to \( J_\alpha(\cdot |^\alpha) \) because \( Y \) is equal to \( X \) transported by the mapping \( \phi_\alpha \). However, the fact that the fluctuation is most typically generated by one variable makes only the tail of \( J_\alpha \) asymptotically visible. This is one interpretation for the coefficient \( \xi_\alpha \) in (7).
- In general, (6) is not a necessary condition for Theorem 1 to hold. From the proof of the lower bound, a closer-to-optimal condition might be:
\[ \forall x > 0, \quad V(\eta^\alpha) = o(n^\alpha), \quad \text{with} \quad \eta^\alpha = (\lambda_\alpha)^{-1}(nx^\alpha). \]
However it does not seem necessary to reach such a precision in the cases we are interested in.
• The monotonicity assumptions on \( \lambda' \) and \( V \) are set for pure convenience in order to simplify the last step of the proof of the lower bound. The important part of the assumption is the boundedness of \( V \).

• We assume that \( \xi \in (0, +\infty) \) for simplicity but Theorem 2 also holds when \( \xi = 0 \) or \( \xi = +\infty \). In such a situation the LDP at scale \( n^{\alpha} \) is trivial. We therefore avoid distinguishing cases and focus on the non-trivial situation. In particular, Theorem 2 is consistent with the standard Cramér’s theorem at exponential scale \([5, Theorem 2.2.3]\), since one scaling of LDP only provides a non-trivial result.

3 Simple applications

Before diving into the proof, we propose a couple of illustrative applications that actually motivated this study. The baseline is to consider a simple random variable and to raise it to some power \( p > 0 \). In general, for \( p \) small enough, a standard large deviations principle holds, while for \( p \) large we can use Theorem 1, which uncovers a phase transition. The physical idea behind these examples is to observe a function of interest of a simple system.

Let us start with the exponential case. Let \( Y \) be a two-sided exponential random variable with distribution on \( \mathbb{R} \) given by
\[
\nu(dy) = \frac{e^{-|y|}}{2} dy,
\]
and consider \( X = \phi^p(Y) \) for \( p \geq 0 \). In this case, the large deviations of \( X \) for \( p \in (0,1] \) are covered by Cramer’s theorem. This result however fails to provide a useful information for \( p > 1 \), since then \( X \) does not have any exponential moment any more. Theorem 1 allows to get the full picture on this situation.

**Proposition 1 (Powers of exponential variables).** For \( p \in (0,1] \), the empirical mean of \( X \) satisfies a large deviations principle at speed \( n \) with rate function given by
\[
J(x) = \sup_{\eta} \{ \eta x - \lambda(\eta) \} \quad \text{where} \quad \lambda(\eta) = \log \mathbb{E}[e^{\eta X}],
\]
while for \( p > 1 \) a large deviations principle holds at speed \( n^{1/p} \) with rate function
\[
I_{1/p}(x) = |x|^{1/p}.
\]

A similar result can easily be obtained for powers of symmetrized Gamma random variables. We consider instead \( Z = \phi^p(G) \) where \( G \) is a standard Gaussian random variable for \( p > 0 \).

**Proposition 2 (Powers of Gaussian variables).** For any \( p \in (0,2] \), the empirical mean of \( Z \) satisfies a large deviations principle at speed \( n \) with rate function given by
\[
J(x) = \sup_{\eta} \{ \eta x - \lambda(\eta) \} \quad \text{where} \quad \lambda(\eta) = \log \mathbb{E}[e^{\eta Z}],
\]
while for \( p > 2 \), a large deviations principle holds at speed \( n^{2/p} \) with rate function
\[
I_{2/p}(x) = \frac{|x|^{2/p}}{2}.
\]

**Proof of Propositions 1 and 2.** When \( X = \phi^p(Y) \) with \( Y \) defined by (8), the variable \( X \) has exponential moments for any \( p \leq 1 \), so we focus on \( p > 1 \). In this case, setting \( \alpha = 1/p < 1 \), a simple computation shows that
\[
\lambda_{\alpha}(\eta) = \log \mathbb{E}[e^{\eta \phi^\alpha(X)}] = \log \mathbb{E}[e^{\eta Y}] = -\log \left( 1 - \frac{\eta^2}{2} \right).
\]

We can then check the criteria of Assumption B by first noting that \( D_{\lambda_{\alpha}} = (-1,1) \) and
\[
\forall \eta \in D_{\lambda_{\alpha}}, \quad \lambda'_{\alpha}(\eta) = \frac{2\eta}{1-\eta^2}, \quad \lambda''_{\alpha}(\eta) = 2 \frac{1+\eta^2}{(1-\eta^2)^2}.
\]
As a results, (4) holds and
\[
\mathcal{V}(\eta) = \frac{1+\eta^2}{2\eta^2} \xrightarrow{\eta \to 1} 1.
\]
As a result, we get

\[
\lambda_n(\eta) = \log \mathbb{E}[e^{\eta \phi_n(x)}] = \log \mathbb{E}[e^{\eta \phi_n(C)^2}] = \log \left( \frac{1}{\sqrt{1 + 2\eta}} + \frac{1}{\sqrt{1 - 2\eta}} \right) - \log \left( \frac{2}{\sqrt{2}} \right).
\]

We thus have \( D\lambda_n = (-1/2, 1/2) \). Introducing

\[
f(\eta) = (1 + 2\eta)^{-1/2} + (1 - 2\eta)^{-1/2},
\]

we compute

\[
f'(\eta) = -(1 + 2\eta)^{-3/2} + (1 - 2\eta)^{-3/2}, \quad f''(\eta) = 3(1 + 2\eta)^{-5/2} + 3(1 - 2\eta)^{-5/2}.
\]

As a result, we get

\[
\lambda_n'(\eta) = \frac{f'(\eta)}{f(\eta)} \sim (1 - 2\eta)^{-1} \frac{1}{\eta - 1/2} + \infty,
\]

and a symmetric conclusion holds for \( \eta \to -1/2 \), so \( \lambda_n'' \) holds. In a similar fashion,

\[
\lambda_n''(\eta) = \frac{f(\eta)f''(\eta) - f'(\eta)^2}{f(\eta)^2},
\]

so

\[
\mathcal{V}(\eta) = \frac{f(\eta)f''(\eta)}{f'(\eta)^2} - 1 \sim \frac{1}{\eta - 1/2} \frac{(1 - 2\eta)^{-5/2}(1 - 2\eta)^{-1/2}}{(1 - 2\eta)^{-6/2}} - 1 \frac{1}{\eta - 1/2} + 2.
\]

This entails that \( \mathcal{V} \) again holds with \( \omega = 2 + \varepsilon \) for any \( \varepsilon > 0 \). The monotonicity conditions on \( \lambda_n'' \) and \( \mathcal{V} \) are also easily checked so all the conditions of Assumption \( \mathcal{A} \) hold and Proposition \( \mathcal{B} \) is a consequence of Theorem \( \mathcal{1} \). \( \square \)

4 Proof of Theorem \( \mathcal{1} \)

We now present the proof of Theorem \( \mathcal{1} \), which is separated between lower and upper bounds.

4.1 Proof of the lower bound

It is standard for the following condition to be sufficient to prove the lower bound:

\[
\forall x \in \mathbb{R}, \quad \lim_{x \to 0} \lim_{\delta \to +\infty} \mathbb{P}(S_n \in B(x, \delta)) \geq I_n(x),
\]

where \( B(x, \delta) = (x - \delta, x + \delta) \). By symmetry we can restrict ourselves to \( x > 0 \). Let then \( x, \delta, \varepsilon > 0 \) be arbitrary and compute

\[
\mathbb{P}(S_n \in B(x, \delta)) = \mathbb{P} \left( x - \delta \leq \frac{X_1}{n} + \frac{1}{n} \sum_{i=2}^{n} X_i \leq x + \delta \right)
\]

\[
\geq \mathbb{P} \left( x - \delta - \varepsilon \leq \frac{X_1}{n} \leq x + \delta + \varepsilon, -\varepsilon \leq \frac{1}{n} \sum_{i=2}^{n} X_i \leq \varepsilon \right)
\]

\[
= \mathbb{P}(n \varepsilon \leq X_1 \leq n \varepsilon) \mathbb{P} \left( n \varepsilon \leq \frac{1}{n} \sum_{i=2}^{n} X_i \leq \varepsilon \right).
\]

We introduced in the last line the notation \( x_{\pm} = x \pm (\delta + \varepsilon) \), which we shall use again below. By the law of large numbers and because \( \mathbb{E}[X] = 0 \) while \( X \) has a finite second moment, the second probability in the last line converges to one, so we can focus on the first probability: the one of fluctuation.

We revisit the exponential (Escher) transform used in the theorems of Cramér and Gartner–Ellis by modifying several of its main features. The plan of the proof below is as follows:

5
• Perform a subexponential transform on one variable;
• Find the optimal tilting parameter and the rate function;
• Derive concentration estimates to ensure that the tilted variable asymptotically has the correct mean with controlled variance (this last part is itself split in two steps).

**Single variable Esscher transform at subexponential scale**

We start by tilting the variable $X_1$ at a subexponential scale. Let $\eta \in (0, \xi)$ be arbitrary and write

$$
P\left( nx_- \leq X_1 \leq nx_+ \right) = \int_{nx_- \leq y \leq nx_+} e^{-\eta \phi_{\alpha}(y) + \eta \phi_{\alpha}(y)} \mu(dy)$$

$$\geq e^{-\eta \alpha x_+^n} \int_{nx_- \leq y \leq nx_+} e^{\eta \phi_{\alpha}(y)} \mu(dy)$$

$$= e^{-\eta \alpha x_+^n + \lambda_\alpha(n)} P\left( nx_- \leq X_\eta \leq nx_+ \right),$$

where we used the scaled free energy \[\phi\] to define the tilted random variable $\tilde{X}_\eta$ with law

$$\tilde{\mu}_\eta(dy) = e^{\eta \phi_{\alpha}(y) - \lambda_\alpha(n)} \mu(dy). \tag{9}$$

We recall that $\mu$ is the law of $X$. This resembles the standard tilting technique but on one variable, and with the terms inside the exponential scaled by $\phi_{\alpha}$.

By also applying the increasing function $\phi_{\alpha}$ in the probability, we thus reach

$$P\left( nx_- \leq X_1 \leq nx_+ \right) \geq e^{-\eta \alpha x_+^n + \lambda_\alpha(n)} P\left( n^\alpha \phi_{\alpha}(x_-) \leq \phi_{\alpha}(\tilde{X}_\eta) \leq n^\alpha \phi_{\alpha}(x_+) \right). \tag{10}$$

In the following, we use that, for $\delta, \varepsilon$ small enough, it holds $x_- > 0$ and so $\phi_{\alpha}(x_-) = x_-^\alpha$. We now have to choose a sequence $\eta_n$ such that $\eta_n \to \xi$ and the last probability has an appropriate lower bound. In other words, we have to find the parameter $\eta$ that makes the fluctuation $x$ most likely for $X_\eta$ at minimal cost.

**Optimal tilting**

It is natural in such a proof to look for a critical point of $e^{-\eta \alpha x_+^n + \lambda_\alpha(n)}$. Heuristically, such a critical point depends on $n$ and should satisfy

$$\eta_n \in \arg\max_{\eta \in (-\xi, \xi)} \left\{ \eta (nx)^\alpha - \lambda_\alpha(\eta) \right\},$$

which is a scaled Legendre–Fenchel transform. Actually, by Assumption \[\phi\] the function $\lambda_\alpha$ is differentiable and its derivative is one-to-one from $(-\xi, \xi)$ into $\mathbb{R}$. Therefore, the unique critical point within $(-\xi, \xi)$ of the function inside brackets above is well-defined by:

$$\eta_n = (\lambda'_\alpha)^{-1}\left((nx)^\alpha\right). \tag{11}$$

Since $nx \to +\infty$, we see that $\eta_n \to \xi$. In order to make \[\phi\] more explicit, we use a standard dominated convergence theorem together with \[\phi\] to obtain

$$\forall \eta \in (-\xi, \xi), \quad \lambda'_\alpha(\eta) = \frac{\mathbb{E} \left[ \phi_{\alpha}(X) e^{\eta \phi_{\alpha}(X)} \right]}{\mathbb{E} e^{\eta \phi_{\alpha}(X)}} = \mathbb{E} \left[ \phi_{\alpha}(\tilde{X}_\eta) \right].$$

The choice \[\phi\] thus ensures that

$$\mathbb{E} \left[ \phi_{\alpha}(\tilde{X}_{\eta_n}) \right] = \lambda'_\alpha(\eta_n) = (nx)^\alpha.$$

It is an intriguing feature that the tilting does not make the average of the tilted variable $\tilde{X}_{\eta_n}$ to be equal to $nx$ but rather works with $\phi_{\alpha}(\tilde{X}_{\eta_n})$, which is why we write the last probability in \[\phi\] in this way. In this situation, since $\eta_n \to \xi$ and $\lambda_\alpha > 0$ by symmetry, the lower bound \[\phi\] actually becomes

$$\lim_{n \to +\infty} \frac{1}{n^\alpha} \log \mathbb{P}\left( nx_- \leq X_1 \leq nx_+ \right) \geq -\xi x_+^\alpha + \lim_{n \to +\infty} \frac{1}{n^\alpha} \log \mathbb{P}\left( n^\alpha \phi_{\alpha}(x_-) \leq \phi_{\alpha}(\tilde{X}_{\eta_n}) \leq n^\alpha \phi_{\alpha}(x_+) \right). \tag{12}$$
Thus the lower bound holds (by letting $\varepsilon, \delta \to 0$) provided we can control the last probability in the above inequality.

However, although the average of $\phi_\alpha(\tilde{X}_n)$ is correct, we cannot obtain a straightforward control of $P(n^\alpha \phi_\alpha(x_-) \leq \phi_\alpha(\tilde{X}_n) \leq n^\alpha \phi_\alpha(x_+))$ because there is only one variable, so using a law of large numbers as is usual for Cramér-like results is not an option. Applying the upper bound to control this term like in the Gartner–Ellis theorem also looks difficult (again because we do not manipulate an average). This is why we control the standard deviation of $\phi_\alpha(\tilde{X}_n)$ with explicit bounds.

**Concentration through second derivative**

For the sake of simplicity we introduce $\gamma > 0$ defined by

$$\gamma^\alpha = \min(x_+^\alpha - x_-^\alpha, x_+^\alpha - x_-^\alpha).$$

This number goes to zero as $\varepsilon + \delta$ goes to zero (we don’t write explicitly the dependency to avoid overloading notation). We can thus write

$$P((nx_-)^\alpha \leq \phi_\alpha(\tilde{X}_n) \leq (nx_+)^\alpha) \geq P(0 - (n\gamma)^\alpha \leq \phi_\alpha(\tilde{X}_n) - (nx)^\alpha \leq (n\gamma)^\alpha)$$

$$= P\left|\phi_\alpha(\tilde{X}_n) - \mathbb{E}[\phi(\tilde{X}_n)]\right| \leq (n\gamma)^\alpha$$

$$= 1 - P\left(\left|\phi_\alpha(\tilde{X}_n) - \mathbb{E}[\phi(\tilde{X}_n)]\right| > (n\gamma)^\alpha\right).$$

Our goal is now to control the last probability. For this we rely on a (easily proved) symmetrized Tchebychev inequality: for any random variable $Z$ and $a, k > 0$ it holds

$$P(|Z - \mathbb{E}[Z]| > a) \leq \max\left(\mathbb{E}\left[e^{k(Z-\mathbb{E}[Z]-a)}\right], \mathbb{E}\left[e^{k(\mathbb{E}[Z]-a)}\right]\right). \quad (13)$$

By symmetry we can consider one case only. Let us focus on the first one and choose an arbitrary $k_a \in (0, \xi - \eta_a)$.

Since $Z = \phi_\alpha(\tilde{X}_n)$, recalling that here $\mathbb{E}[Z] = (nx)^\alpha$ and $a = (n\gamma)^\alpha$, we have

$$\mathbb{E}\left[e^{k_a(Z-\mathbb{E}[Z]-a)}\right] = e^{-k_a n^\alpha (x_-^\alpha + \gamma^\alpha)} \frac{e^{k_a n^\alpha \phi_\alpha(\tilde{X}_n)}}{e^{k_a n^\alpha \phi_\alpha(\tilde{X}_n)}} = e^{-k_a n^\alpha x_-^\alpha + k_a n^\alpha (k_a + \eta_a) - \lambda_a(\eta_a)},$$

where we introduced $\tilde{x}_+ = (x_+^\alpha + \gamma^\alpha) > x$. Since $k_a \in (0, \xi - \eta_a)$, we set $y_a = k_a + \eta_a \in (\eta_a, \xi)$, which leads to

$$-k_a (n\tilde{x}_+^\alpha + \lambda_a(k_a + \eta_a) - \lambda_a(\eta_a)) = \eta_a (n\tilde{x}_+^\alpha - \lambda_a(\eta_a)) - (y_a (n\tilde{x}_+^\alpha - \lambda_a(y_a))).$$

When optimizing over $y_a \in (\eta_a, \xi)$, the infimum of the above quantity is attained inside $(\eta_a, \xi)$ (easily proved) at the value

$$y_a = \tilde{\eta}_a = (\lambda_a')^{-1}(n\tilde{x}_+^\alpha).$$

By steepness of $\lambda_a$ the above quantity is well-defined, and by convexity (hence monotonicity of $\lambda_a'$), the inequality $\tilde{x}_+ > x$ implies that $\tilde{\eta}_a \geq \eta_a$.

Since the second case in (13) is symmetric, we obtain

$$\log P(|Z - \mathbb{E}[Z]| > a) \leq \lambda_a(\tilde{\eta}_a) + (\eta_a - \tilde{\eta}_a)\lambda_a'(\tilde{\eta}_a) - \lambda_a(\eta_a).$$

Markov’s inequality implies that

$$P\left(\left|\phi_\alpha(\tilde{X}_n) - \mathbb{E}[\phi(\tilde{X}_n)]\right| > (n\gamma)^\alpha\right) \leq \frac{\mathbb{E}\left[(\phi_\alpha(\tilde{X}_n) - \mathbb{E}[\phi(\tilde{X}_n)])^2\right]}{(n\gamma)^{2\alpha}} = \left(\frac{x}{\gamma}\right)^{2\alpha} \frac{\lambda_a''(\eta_a)}{\lambda_a'(\eta_a)^2} = \left(\frac{x}{\gamma}\right)^{2\alpha} \Phi(\eta_a).$$

Therefore, if $\Phi(\eta_a) \to 0$ for any sequence $\eta_a \to \xi$, we reach the desired result (recall that $x$ is fixed and $\gamma$ is small). However, this assumption does not seem to be applicable in practical cases, which is why we have to consider the weaker condition (13) and compute more precise estimates at exponential scale.
We see that the above quantity looks like a Taylor expansion of \( \lambda_n \) at first order. We thus expand the cumulant function (backward) as follows: there exists \( \bar{\eta} \in [\eta_n, \bar{\eta}_n] \) such that

\[
\lambda_n(\eta_n) = \lambda_n(\bar{\eta}_n) + (\eta_n - \bar{\eta}_n). \lambda_n'(\bar{\eta}_n) + \frac{1}{2} \lambda_n''(\bar{\eta}) (\eta_n - \bar{\eta}_n)^2.
\]

As a result, since \( \lambda''_n \) is increasing close enough to \( \xi \) and \( \eta_n \leq \bar{\eta} \), we have

\[
\log \mathbb{P}(|Z - \mathbb{E}[Z]| > a) \leq -\frac{1}{2} \lambda''_n(\eta_n)(\eta_n - \bar{\eta}_n)^2.
\]

In order to control (14), we see at this stage a competition between \( \lambda''_n(\eta_n) \) that typically diverges to infinity and \((\eta_n - \bar{\eta}_n)^2\), which goes to zero. Let us derive the estimates on this second term to reach the desired conclusion.

**Convex analysis for variance control**

We first introduce the Legendre transform of \( \lambda_n \):

\[
J_n(x) = \sup_\eta \{ \eta x - \lambda_n(\eta) \}.
\]

Standard convex analysis [7, Chapter VI] shows that

\[
J_n(x) = (\lambda_n')^{-1}(x).
\]

Therefore

\[
\bar{\eta}_n - \eta_n = J_n'((n\bar{\eta} + \eta_n)) - J_n'((\eta_n)^n).
\]

We then perform another expansion but at order one: there is \( b_n \in [(nx)^\alpha, (n\bar{\eta} + \eta_n)^\alpha] \) such that

\[
\bar{\eta}_n - \eta_n = n^\alpha(\bar{\eta}^\alpha + x^\alpha)J_n''(b_n) \geq (n\gamma)^\alpha J_n''(b_n).
\]

We now use [10] to relate the second derivative of \( J_n \) to the one of \( \lambda_n \):

\[
J_n''(b_n) = \frac{1}{\lambda_n''((\lambda_n')^{-1}(b_n))} \geq \frac{1}{\lambda_n''((\lambda_n')^{-1}((nx)^\alpha))} = \frac{1}{\lambda_n''(\eta_n)},
\]

where we used the monotonicity of \( \lambda'_n \) and \( \lambda''_n \) for \( n \) large enough to obtain the inequality above. Combining (15) with (16), we can turn (14) into

\[
\log \mathbb{P}(|Z - \mathbb{E}[Z]| > a) \leq -\frac{1}{2} \lambda''_n(\eta_n)\left(\frac{(n\gamma)^\alpha}{\lambda''_n(\eta_n)}\right)^2.
\]

Introducing

\[
c_x = \frac{(x\gamma)^2n}{2(n\gamma)^2} > 0,
\]

then (17) may be arranged as (recalling that \((nx)^\alpha = \lambda'_n(\eta_n)\) and similarly \((n\bar{\eta} + \eta_n)^\alpha = \lambda'_n(\bar{\eta}_n)\)):

\[
\log \mathbb{P}(|Z - \mathbb{E}[Z]| > a) \leq -c_x \frac{\lambda''_n(\eta_n)}{(nx)^{2\alpha}}\left(\frac{(n\bar{\eta} + \eta_n)^2}{\lambda''_n(\eta_n)}\right)^2
\]

\[
= -c_x \lambda''_n(\eta_n) \left(\frac{\lambda'_n(\bar{\eta}_n)^2}{\lambda''_n(\eta_n)}\right)^2
\]

\[
= -c_x \frac{\mathcal{V}(\eta_n)}{(\eta_n)^2} \leq \frac{c_x}{\mathcal{V}(\bar{\eta}_n)} \leq \frac{c_x}{\omega}.
\]

where, in Assumption [8] we used monotonicity of \( \mathcal{V} \) to get \(-\mathcal{V}(\eta_n) \leq -\mathcal{V}(\bar{\eta}_n)\) for \( n \) large enough, and as well as [6] for the last inequality. Note that when \( \mathcal{V}(\bar{\eta}_n) \rightarrow 0\), the probability of deviating from the mean converges to zero, but in general it is just smaller than one. We thus obtain by (13) that there exists \( c_{x,\omega} > 0 \) (depending also on the fixed parameters \( \varepsilon, \delta > 0 \)) such that, for \( n \) large enough, it holds

\[
\mathbb{P}(n^\alpha \phi_n(x_-) \leq \phi_n(\bar{X}_{\eta_n}) \leq n^\alpha \phi_n(x_+)) \geq c_{x,\omega}.
\]

Plugging this estimate in (12) allows to conclude the proof of the lower bound.
4.2 Upper bound

We now turn to the upper bound, for which it is sufficient [5] Theorem 2.2.3] to study the probability $P(S_n \geq x)$ for $x > 0$. We thus fix $x > 0$, and first write

$$P(S_n \geq x) \leq P\left(\max_{1 \leq i \leq n} X_i \geq nx\right) + P\left(\max_{1 \leq i \leq n} X_i < nx, \frac{1}{n} \sum_{i=1}^{n} X_i \geq x\right) = A_n^1 + A_n^2. \quad (18)$$

We recall that, from [5] Lemma 1.2.15], we have

$$\lim_{n \to +\infty} \frac{1}{n^{\alpha}} \log(A_n^1 + A_n^2) = \max\left(\lim_{n \to +\infty} \frac{1}{n^{\alpha}} \log A_n^1, \lim_{n \to +\infty} \frac{1}{n^{\alpha}} \log A_n^2\right). \quad (19)$$

We can therefore study the sequences $A_n^1$ and $A_n^2$ separately and take the maximum of the two when going at logarithmic scale. We closely follow the path of [9] by generalizing some elements along Assumption 2.

Large deviations for the heavy tail term $A_n^1$.

For the first term we use the union’s bound together with Tchebychev’s inequality at subexponential scale to obtain, for any $\eta \in (0, \xi)$:

$$P\left(\max_{1 \leq i \leq n} X_i \geq nx\right) \leq nP(X_1 \geq nx) \leq ne^{-\eta(nx)^\theta} e^{\lambda_\alpha(\eta)}.$$

Therefore,

$$\frac{1}{n^{\alpha}} \log P\left(\max_{1 \leq i \leq n} X_i \geq nx\right) \leq -\eta|x|^\theta + \frac{\lambda_\alpha(\eta)}{n^{\alpha}} + \frac{\log(n)}{n^{\alpha}}.$$

Since $\eta \in (0, \xi)$ is fixed, Assumption 2 implies that $\lambda_\alpha(\eta) < +\infty$, so

$$\lim_{n \to +\infty} \frac{1}{n^{\alpha}} \log P\left(\max_{1 \leq i \leq n} X_i \geq nx\right) \leq -\frac{\eta|x|^\theta}{n^{\alpha}}.$$

We now can pass to the limit $\eta \to \xi$ to obtain

$$\lim_{n \to +\infty} \frac{1}{n^{\alpha}} \log A_n^1 \leq -\frac{\xi|x|^\theta}{n^{\alpha}}.$$

Controlling the light tail term $A_n^2$.

Let us turn to the second term in (18). The idea now is to use Tchebychev’s inequality at exponential scale but with a parameter $\beta_n > 0$ depending on $n$:

$$A_n^2 \leq e^{-\beta_n x} E\left[I_{\{X_i < nx\}} e^{\alpha_n \sum_{i=1}^{n} X_i}\right] \leq e^{-\beta_n x} \prod_{i=1}^{n} E\left[I_{\{X_i < nx\}} e^{\alpha_n X_i}\right].$$

It is natural to choose $\beta_n = n^{\alpha}\theta$ for some $\theta > 0$, since we then obtain

$$\frac{1}{n^{\alpha}} \log A_n^2 \leq -\theta x + n^{1-\alpha} \log E\left[I_{\{X_i < nx\}} e^{\alpha_n X_i}\right]. \quad (20)$$

It is tempting to set $\theta = \xi x^{\alpha-1}$, however this will be a limit case. We actually need a precise control of the remaining term, in which the bound on $X_i < nx$ going to infinity comes in competition with the factor $\theta n^{\alpha-1}$ inside the exponential, which goes to zero. Following [5], we then prove the following lemma.

Lemma 1. For any $\theta < \xi x^{\alpha-1}$ it holds

$$\lim_{n \to +\infty} n^{1-\alpha} \log E\left[I_{\{X_i < nx\}} e^{\theta n^{\alpha-1} X_i}\right] \leq 0.$$

If we prove Lemma 1 then we can take the limit $\theta \to \xi x^{\alpha-1}$ in (20) and (19) allows to conclude the proof of the upper bound, and therefore the one of Theorem 1.
Proof of Lemma 1

We follow the strategy of [7] by first noting that \( \log y \leq y - 1 \) for any \( y > 0 \). Noting that the exponential is increasing and using a Taylor expansion, we also get \( e^y - 1 \leq y + y^2/2 + \ldots + e^y y^{k+1}/(k+1)! \) where for now \( k \) is an arbitrary large integer. We thus obtain

\[
\log \mathbb{E} \left[ I_{\{X_1 < nx\}} e^{\theta n^{-1} X_1} \right] \leq n^{1-\alpha} \sum_{j=1}^{k} \mathbb{E} \left[ \left( \theta n^{-1} X_1 \right)^j \right] I_{\{X_1 < nx\}} + \frac{R_n}{(k+1)!}, \tag{21}
\]

where

\[
R_n = n^{1-\alpha} \theta n^{-1} (k+1)! \mathbb{E} \left[ X_1^{k+1} e^{\theta n^{-1} X_1} \right].
\]

For the sum on the right hand side of (21), the term for \( j = 1 \) is equal to zero because \( \mathbb{E}[X_1] = 0 \). For \( j > 1 \) each term is bounded by

\[
n^{1-\alpha} \mathbb{E} \left[ |X_1|^j \right] (n^{-1})^j = n^{(\alpha-1)(j-1)} \mathbb{E} \left[ |X_1|^j \right],
\]

which goes to zero as \( n \to +\infty \) since \( X_1 \) has finite moments of any order by Assumption 1.

It thus only remains to show that \( \lim R_n = 0 \). For this we use Holder’s inequality for some \( p, q > 1 \) with \( 1/p + 1/q = 1 \) to separate the exponential and polynomial moment parts:

\[
R_n \leq n(\theta n^{-1}) (k+1)! \mathbb{E} \left[ |X_1|^{(k+1)p} I_{\{X_1 < nx\}} \right]^{1/p} \left( \frac{1}{n^\alpha} \mathbb{E} \left[ e^{\theta n^{-1} X_1} \right] \right)^{1/q}, \tag{22}
\]

For the first term we have

\[
n(\theta n^{-1}) (k+1)! \mathbb{E} \left[ |X_1|^{(k+1)p} I_{\{X_1 < nx\}} \right]^{1/p} \leq \theta^{k+1} n^{(\alpha-1)(k+1)+1} \mathbb{E} \left[ |X_1|^{(k+1)p} \right]^{1/p}, \tag{23}
\]

which goes to zero for any \( p > 1 \) as soon as \( k \) is large enough for the following condition to hold:

\[
\alpha < \frac{k}{k+1}. \tag{24}
\]

Since \( \alpha < 1 \), we can then choose \( k \) such that

\[
k > \frac{\alpha}{1-\alpha},
\]

in which case (24) is satisfied and the right hand side of (23) goes to zero for any \( p > 1 \).

The last step is to prove that there is some \( q > 1 \) such that the second term on the right hand side of (22) satisfies

\[
\lim_{n \to \infty} \frac{1}{n^\alpha} \mathbb{E} \left[ I_{\{X_1 < nx\}} e^{\theta n^{-1} X_1} \right]^{1/q} < +\infty. \tag{25}
\]

Lemma 2 in Appendix A implies that (since the first boundary term is zero and the second is negative):

\[
\mathbb{E} \left[ I_{\{X_1 < nx\}} e^{\theta n^{-1} X_1} \right] \leq q \theta n^{-1} \int_{-\infty}^{nx} e^{q\theta n^{-1} z} \mathbb{P}(X_1 \geq z) \, dz.
\]

We have

\[
\int_{-\infty}^{nx} e^{q\theta n^{-1} z} \mathbb{P}(X_1 \geq z) \, dz = \int_{-\infty}^{0} e^{q\theta n^{-1} z} \mathbb{P}(X_1 \geq z) \, dz + \int_{0}^{nx} e^{q\theta n^{-1} z} \mathbb{P}(X_1 \geq z) \, dz \leq C_+ + \int_{0}^{nx} e^{q\theta n^{-1} z} \mathbb{P}(X_1 \geq z) \, dz,
\]

where

\[
C_+ = \int_{-\infty}^{0} e^{q\theta z} \, dz < +\infty.
\]
We therefore focus on the behavior of the integral on \([0, nx]\) as \(n \to +\infty\). Using again Tchebychev’s inequality at subexponential scale for some \(\eta \in (0, \xi)\) together with the change of variable \(z = nxy\) (recall \(x > 0\) is fixed) we have
\[
\int_0^{nx} e^{\theta n^{-1} z - x^\alpha y} \, dz \leq \int_0^{nx} e^{\theta n^{-1} z - x^\alpha + \lambda_\alpha(y)} \, dz = nx e^{\lambda_\alpha(y)} \int_0^1 e^{x^\alpha y} \, dy,
\]
where we introduced the function \(g\) defined by
\[
\forall y \in [0, 1], \quad g(y) = q \theta xy - \eta x^\alpha y^\alpha.
\]
Recall that for now \(\theta < \xi x^\alpha - 1\) is fixed. Therefore, for \(\varepsilon > 0\) small enough, it holds
\[
\theta x < (1 - \varepsilon)^2 \xi x^\alpha.
\]
We can then choose \(q = 1/(1 - \varepsilon) > 1\) and \(\eta = (1 - \varepsilon) \xi < \xi\). In this case (28) becomes
\[
q \theta x < \eta x^\alpha.
\]
The above condition implies in particular that
\[
\forall y \in [0, 1], \quad g(z) \leq 0,
\]
so
\[
\forall n \geq 1, \quad \int_0^1 e^{n \alpha g(y)} \, dy \leq 1.
\]
Finally, we gather the above estimates to reach
\[
\frac{1}{n^\alpha} \mathbb{E} \left[ I_{(X_1 < nx)} e^{\frac{z}{n} X_1} \right]^{1/q} \leq \frac{1}{n^\alpha} \left( \frac{\theta n^{-1} - \lambda_\alpha(\eta)}{1 - \varepsilon} \left( C_+ + nx e^{\lambda_\alpha(\eta)} \right) \right)^{1 - \varepsilon} = \frac{1}{n^\alpha} \left( \frac{\theta}{1 - \varepsilon} \left( C_+ + nx e^{\lambda_\alpha(\eta)} \right) \right)^{1 - \varepsilon} \to 0 \quad \text{as} \quad n \to +\infty.
\]
This shows that (25) is satisfied, so Lemma 1 holds and the theorem is proved.

5 Discussion

In this work we investigated large deviations principles for empirical averages of i.i.d. subexponential random variables. Under a subexponential moment condition, we showed that a LDP holds at a subexponential rate with an explicit rate function. This result generalizes earlier works by providing a full LDP that includes a lower bound, and by avoiding assumptions on the cumulative distribution of the variable.

In this subexponential regime, the rate function is always singular at zero, as it does not even admit a first derivative. Although we could expect non-existence of a second derivative (because this would contradict the central limit theorem), the phase transition is still more abrupt. Indeed, we illustrated in Section 3 that a smooth rate function can become non-differentiable by raising the underlying random variable to a power arbitrarily close to one. Moreover, in this new regime, the rate function does not depend on the full probability distribution of the random variable but only on its coefficient in the tail, so it is universal. On the contrary the rate of decay is distribution specific, while it is universally exponential for variables with exponential moments (although they may have very different tails, like Gaussian and exponential variables).

Concerning the proof, although the upper bound part (which develops the techniques used in [9]) is interesting and helps understanding the problem, the most original part of the paper is the proof of the lower bound. For this we design a new tilting strategy, which requires an assumption on the second derivative on the free energy. This condition has the attractive interpretation of a control on the relative variance of the unique tilted random variable.

An exciting outcome of this proof is to provide the optimal sampling strategy for numerically estimating large deviations probabilities of subexponential variables. It is generally believed that a good tilting scheme in the heavy
tail scenario is to replace \( X_1 \) by \( X_1 + nx \). We show on the contrary that the optimal scheme is to replace \( X_1 \) by the variable \( \tilde{X}_\eta_n \) defined by (9)-(11). As is usual for this kind of tilting, the optimal value \( \eta_n \) depends on the inverse of the derivative of the free energy. However, in the exponential scaling case, it does not depend on \( n \) and is applied to all variables. Here, one variable only is tilted with a parameter that depends on the full sample size.

Finally, our motivation to replace assumptions on the cumulative distribution by a moment condition is to move forward to correlated systems instead of independent variables. In particular, if one studies empirical averages of stochastic differential equations in the long time limit, it is hard to define a cumulative distribution to make an assumption on. However, in this situation, it is natural to work under an exponential moment condition, for instance through the Gartner–Ellis theorem [10]. This is why we hope to extend the results in [8], that describe the limiting situation in the exponential moment case, to the subexponential regime. Recent results [1] suggest that the rate function is again singular in this case, and we believe our strategy can be useful to improve our understanding of such problems.

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A A couple of technical results

We recall the definition of a large deviations principle.

**Definition 2.** A sequence of random variables \((Z_n)_{n \geq 1}\) taking values in a topological space \(Z\) equipped with its Borel \(\sigma\)-field satisfies a large deviations principle at speed \(v_n\) and with rate function \(I: Z \to [0, +\infty]\) if \(I\) is lower semicontinuous and for any measurable set \(B \subset Z\) it holds

\[
-\inf_B I \leq \lim_{n \to +\infty} \frac{1}{v_n} \log P(Z_n \in B) \leq \lim_{n \to +\infty} \frac{1}{v_n} \log P(Z_n \in B) \leq -\inf_B I,
\]

where \(\text{int}\) and \(\overline{\text{int}}\) denote respectively the interior and the closure of \(B\) for the topology of \(Z\). Moreover we say that \(I\) is a good rate function if it has compact level sets, and that \(I\) is trivial if it is equal to 0 everywhere or equal to +\(\infty\) everywhere except at \(\mathbb{E}[Z]\).

We regularly use Tchebychev type inequalities in the paper. In our terminology, the inequality at exponential scale is:

\[
\forall z, \eta \geq 0, \quad P(X \geq z) \leq e^{\eta z} E[e^{\eta X}].
\]

In order to prove our results, we also need to scale the various quantities at hand by \(\phi_n\). We can then use Tchebychev’s inequality at subexponential scale, which reads

\[
\forall z, \eta \geq 0, \quad P(X \geq z) \leq e^{-\eta n \phi_n} E[e^{\eta \phi_n(X)}].
\]

It is a simple corollary of the first inequality.

For the proof of the upper bound, we recall the following useful integration by part formula [9] Lemma 5).

**Lemma 2.** For any real-valued random variable \(X\) on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), for any \(a > 0\) and real numbers \(r_1 < r_2\), it holds

\[
E[e^{aX}1_{(r_1 \leq X \leq r_2)}] = a \int_{r_1}^{r_2} e^{az}P(X \geq z)dz + e^{a r_1}P(X \geq r_1) - e^{a r_2}P(X \geq r_2).
\]
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