Generalized quantum Rabi model with both one- and two-photon terms: A concise analytical study

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(Dated: June 14, 2018)

A generalized quantum Rabi Hamiltonian has emerged in the circuit quantum electrodynamics (QED) system where both one- and two-photon terms are included. The usual parity symmetry is broken naturally in the simultaneous presence of both couplings, which complicates its analytical treatments, even in the rotating wave approximations. In this paper, we propose an adiabatic approximation to this general model by using Bogoliubov operators, and obtain a very concise solution for both eigenvalues and eigenstates. In the wide parameter regime, the analytical results agree well with the numerical ones. In the rotating-wave approximations, we also derive the eigensolution analytically, which also works well in a wide parameter regime. Two dominate Rabi frequencies are found in the quantum Rabi oscillations of this generalized model. Our analytical results can be applicable to the recent QED system for one-photon coupling ranging from weak, ultra-strong, to deep-strong coupling regime at moderate two-photon coupling.

PACS numbers: 03.65.Yz, 03.65.Ud, 71.27.+a, 71.38.k

I. INTRODUCTION

The quantum Rabi model (QRM) describes the most simple and at the same time most important coupling between a continuous degree of freedom (a mode of the light field) and a discrete one (a two-level system or qubit) which is linear in the quadrature operators [1]. Here we study a natural generalization of the QRM which exhibits both linear and non-linear coupling between both constituents, i.e. the QRM with both one- and two-photon terms, with Hamiltonian

\[ H = \frac{\Delta}{2} \sigma_z + \omega a^\dagger a + \sigma_x \left( g_1 (a^\dagger + a) + g_2 \left( (a^\dagger)^2 + a^2 \right) \right) , \]

where \( \Delta \) and \( \omega \) are respectively frequencies of qubit and cavity, \( \sigma_x,z \) are Pauli matrices describing the two-level system and \( a, (a^\dagger) \) are the annihilation (creation) bosonic operators of the cavity mode, and \( g_1 (g_2) \) is the linear (nonlinear) qubit-cavity coupling constant.

The nonlinear coupling appears naturally as an effective model for a three-level system when the third (off-resonant) state can be eliminated. The two-photon model has been proposed to apply to certain Rydberg atoms in superconducting microwave cavities [2, 3]. Recently, a realistic implementation of the two-photon QRM using trapped ions has been proposed [4]. Usually, the two-photon term is the secondary effect, and thus limited to the weak-to-moderate coupling regime typical for experimental setups within cavity or circuit QED.

A realistic implementation of the generalized QRM [5] described by Eq. (1) has been proposed in circuit QED [6] with non-zero DC current biases. Using alternative methods, both linear and nonlinear interaction terms can be present in different circuit QED setup by Bertet et al. [7]. Most recently, Pedernales et al proposed that a background of a \((1+1)\)-dimensional black hole requires a QRM with one- and two-photon terms that can be implemented in a trapped ion for the quantum simulation of Dirac particles in curved spacetime [8].

Both the pure linear and nonlinear QRM have been studied extensively over a few decades (for a review, please refer to Refs. [1, 9, 10] for a review). The analytical exact solutions based on the well-defined G-function have been only found recently for one-photon [11] QRM and two-photon QRM [12]. These solutions have stimulated extensive research interests in the exact solutions to the QRM with both one-photon terms [13, 14] and two-photon terms [15, 16]. Many analytical approximate but still very accurate results have been also given [17, 18, 19, 20]. In some limits, the dynamics and quantum criticality have been also studied exactly [21, 22].

In the QRM with both linear and nonlinear couplings, the parity symmetry is broken naturally, and the analytical solution becomes more difficult, compared to the pure models. In this paper, we will propose concise analytical solutions to this general model, which may be helpful in the recent circuit QED experiments.

The paper is structured as follows: In section II, we propose adiabatic approximations to the nonlinear QRM. In section III, the results for the energy levels are calculated based on the concise analytic solutions. The dynamics for the population inverse are also studied. Section IV is devoted to an analytic analysis in the rotating wave approximations. The last section contains some concluding remarks.

II. OPERATOR TRANSFORMATIONS AND NUMERICALLY EXACT SCHEME

Associated with pure QRM are the conserved parity \( \Pi_{1p} = -\sigma_z \exp (i \pi a^\dagger a) \) for one-photon and \( \Pi_{2p} = -\sigma_z \exp (\frac{i \pi}{2} a^\dagger a) \) for two-photon qubit-cavity interaction, such that \( [H, \Pi] = 0 \). \( \Pi \) has two (four) eigenvalues for
one-photon (two-photon) QRM, depending on whether the excitation number is even or odd. So in the two pure models, we have $Z_2$-symmetry acting in the bosonic Hilbert space, which greatly facilitates the study even in the analytical analysis. However, in the present generalized model with both one- and two-photon interaction

\[
H = \left( \begin{array}{cc}
 a^\dagger a + g_1 (a^\dagger + a) + g_2 \left[ (a^\dagger)^2 + a^2 \right] \\
 -\frac{\Delta}{2}
\end{array} \right)
\]

with the qubit, no such a conserved parity is available, so the analytical study is more challenging.

First we introduce a scheme to pave the way to the solutions. For convenience, we can write a transformed Hamiltonian with a rotation around the $y$ axis by an angle $\pi/4$ in the matrix form in units of $\omega = 1$

Note that only the oscillator excitation operator $n$ emerges, so the Hilbert space can be decomposed into different $n$ manifolds spanned by the spin and oscillator basis of $|\uparrow\rangle |n\rangle_A$ and $|\downarrow\rangle |n\rangle_B$ where $|\uparrow\rangle$ ($|\downarrow\rangle$) denotes the upper (lower) states of the qubit. According to the transformation of the bosonic operators above, we can define $|n\rangle_A$ and $|n\rangle_B$ in terms of the number operator $|n\rangle_a$ in original bosonic operator $a$ as

\[
|n\rangle_A = S(r)D^\dagger(w)|n\rangle_a, \quad |n\rangle_B = S^\dagger(r)D(w')|n\rangle_a,
\]

where $S(r)$ is the squeezing operator and $D(\alpha)$ is the displaced operator

\[
S(r) = \frac{\alpha}{\sqrt{\alpha^2 + w^2}}\exp(\alpha w'),
\]

with $r = \text{arc} \cosh u, \alpha = w, w'$. Inserting Eq. (5) into the Schrödinger equation, we have

\[
\beta \left( m - \nu^2 - w^2 \right) c_{m+1} - \frac{\Delta}{2} \sum_{n=0}^{\min(m,n)} D_{mn} d_n = Ec_{m},
\]

\[
\beta \left( m - \nu^2 - w^2 \right) d_{m+1} - \frac{\Delta}{2} \sum_{n=0}^{\min(m,n)} D_{nm} c_n = Ed_{m},
\]

where

\[
D_{mn} = A(m|n\rangle_B
\]

\[
= \sqrt{n!m!\beta} (uv\beta)^{(m+n)/2} \exp \left( -\frac{2g_1^2}{\beta} \right) \times \sum_{i=0}^{\min(m,n)} \left( -1 \right)^{(m-i)/2} \frac{(uv)^{-i}}{i!(m-i)!(n-i)!} \frac{g_1}{\beta^{3/2}/\sqrt{-uv}} H_{m-i} \frac{g_1(u-v)}{\beta^{3/2}/\sqrt{-uv}} H_{n-i} \left( -\frac{g_1(u+v)}{\beta^{3/2}/\sqrt{-uv}} \right)
\]

where $H_n(x)$ stands for the Hermite polynomials.

Based on Eqs. (11) and (12), we can give numerically exact spectrum to the general qubit-oscillator system with both one- and two-photon process with the increase of the truncation of the summation.
III. ANALYTICAL APPROXIMATIONS

In the framework of Eqs. (11) and (12), analytical approximations can be performed systematically. As a zero-order approximation, we only consider transition between states in the same manifold spanned by $|\uparrow\rangle$ $|m\rangle_A$ and $|\downarrow\rangle$ $|m\rangle_B$, then we have

$$\beta (m - v^2 - w^2) c_m - \frac{\Delta}{2} d_m D_{mm} = E c_m,$$
$$\beta (m - v^2 - w^2) d_m - \frac{\Delta}{2} c_m D_{mm} = E d_m.$$  

Nonzero coefficients will give the following equation

$$[E - \beta (m - v^2 - w^2)] [E - \beta (m - v^2 - w^2)] \- \frac{\Delta^2}{4} D^2_{mm} = 0.$$  

The eigenvalues are then easily given by

$$E^\pm_m = \beta (m - v^2) - \frac{g_1^2}{\beta^2} \pm \frac{1}{2} \sqrt{\beta^2 (w^2 - w'^2)^2 + \Delta^2 D^2_{mm}}, \quad (13)$$

The corresponding eigenstate is

$$|m\rangle_\pm \propto \left( \frac{1}{\beta (m - v^2 - w^2)} |m\rangle_A - E^\pm_m |m\rangle_B \right). \quad (14)$$

This zero-order approximation is also called adiabatic approximation [34], which works best for small qubit frequency and strong coupling. For large qubit frequency, and weak coupling, the high order approximation should be preformed. Actually, we can straightforwardly consider the transitions between states belonging to the different $n$ manifolds, and more complicated analytical results would be obtained. In the recent circuit QED, since the qubit frequency is usually smaller than the frequency of oscillator and the oscillator-qubit interaction has entered the ultra-strong [35, 37], even deep-strong coupling regime [38], the adiabatic approximation should work well, and further high order corrections are not discussed here.

In the applications to the realistic circuit QED system, we also need to consider the full qubit Hamiltonian $H = - (\epsilon \sigma_z + \Delta \sigma_x)/2$, where $\epsilon$ is the static bias. The eigenenergy can be derived easily as

$$E_\pm = -E^\pm_m = \beta (m - v^2) - \frac{g_1^2}{\beta^2} \pm \frac{1}{2} \sqrt{\beta^2 (w^2 - w'^2)^2 + \Delta^2 D^2_{mm}}, \quad (15)$$

IV. ENERGY LEVELS AND DYNAMICS

With the recent progress in technology, the one-photon coupling term has reached the ultrastrong-coupling regime ($g_1/\omega \approx 0.12$) experimentally with superconducting flux qubits inductively coupled to superconducting resonators [35, 37]. More recently, it has accessed to the deep-strong coupling regime ($g_1/\omega \approx 1.34$) [38, 39]. The two-photon term emerges from the second process in cavity QED or the expansions in the second order in the circuit QED, so the two-photon terms should not be strong generally. Mathematically, it should be less than the interaction-induced spectral collapse point $g_2 = 0.5$ [4, 20, 40]. To show the validity of the present adiabatic approximation to the various physical phenomena in the presently experimentally accessible systems, we will examine the one-photon coupling strength $g_1$ ranging from weak to deep strong coupling, while the two-photon coupling strength $g_2$ is fixed to be a moderate value.

The energy levels by Eq. (13) as a function of $g_1$ for $g_2 = 0.05$ and 0.1, at the qubit frequency $\Delta = 0.5$ and 1 are displayed in Fig. 1. The exact results by the numerically exact diagonalization are also collected. For $\Delta = 0.5$, the present results are in excellent agreement with the exact ones, while for $\Delta = 1$ the adiabatic approximation still gives the good results. Actually, the present approach is basically a perturbation in the qubit frequency $\Delta$, so with increase of $\Delta$, the present results show a little bit deviation from the true spectrum. Practically, in the present experiments $\Delta/\omega$ is usually not larger than 1 [35, 38]. In the Table I of Ref. [38], the value of $\Delta/\omega$ is even in the order of magnitude of 0.01. Our results should be convincingly suited to the present superconducting flux qubit coupled to a circuit resonant.

We then examine the dynamics of the population difference $\langle \sigma_z(t) \rangle$ for different qubit frequencies $\Delta = 0.1, 0.2$, and 0.5 for the coupling strength $g_1 = 0.5, g_2 = 0.1$ in Fig. 2. For small value of $\Delta$, the adiabatic approximation can describe the dynamics almost exactly in
long time scale. For large $\Delta$, our analytical results can match the oscillation in phase for a long time. It only begins to get out of phase after many periods of oscillations. For all case studied, our theory can basically give right description of the dynamics. For the recent experimental parameters, set the moderate two-photon coupling strength, we have checked that our analytical theory works well compared with the numerical results.

V. ANALYTICAL ANALYSIS IN THE ROTATING-WAVE APPROXIMATIONS

The rotating-wave approximation (RWA) is made by neglecting the counter rotating terms, $g_1(a^\dagger \sigma_+ + a\sigma_-) + g_2((a^\dagger)^2 \sigma_+ + a^2 \sigma_-)$. The Hamiltonian of QRM with both one- and two-photon terms in the RWA is

$$H = a^\dagger a + \frac{\Delta}{2} \sigma_z + g_1(a^\dagger \sigma_- + a\sigma_+) + g_2((a^\dagger)^2 \sigma_+ + a^2 \sigma_+).$$

It takes the following matrix form in the basis of $\sigma_z$

$$H = \begin{pmatrix} a^\dagger a + \frac{\Delta}{2} & g_1a + g_2a^2 \\ g_1a^\dagger + g_2(a^\dagger)^2 & a^\dagger a - \frac{\Delta}{2} \end{pmatrix}.$$

For one-photon case, the energy level for the RQM in the RWA reads \[41\]

$$E_{n,1p}^{(k)} = n + \frac{1}{2} + (-1)^k \frac{1}{2}(\Delta - 1)^2 + 4g_1^2(n + 1), \quad k = 1, 2.$$ with eigenfunctions consisting of $|n\rangle |\uparrow\rangle$ and $|n + 1\rangle |\downarrow\rangle$. The RWA result for the eigenenergy for two-photon QRM \[42\] is given by

$$E_{n,2p}^{(k)} = n + 1 + (-1)^k \frac{1}{2}(\Delta - 2)^2 + 4g_2^2(n + 2)(n + 1),$$ with eigenfunctions consisting of $|n\rangle |\uparrow\rangle$ and $|n + 2\rangle |\downarrow\rangle$.

For both one- and two-photon interaction cases, we first propose

$$|n\rangle_1 = \left( \begin{array}{c} c_n |n\rangle \\ e_n |n + 1\rangle + f_n |n + 2\rangle \end{array} \right), \quad n = 0, 1, 2, \ldots \quad (18)$$

which includes the basic process of the pure models. The Hamiltonian cannot be decomposed cleanly into independent $n$ sub-space $R_n = \{ |n\rangle |\uparrow\rangle, |n + 1\rangle |\downarrow\rangle, |n + 2\rangle |\downarrow\rangle \}$, because the interaction also couples the states in different sub-space, unlike both pure models. So the true wavefunction should include the contribution from infinite bare states in all sub-spaces, however the wavefunction (15) defined in the $n$ sub-space $R_n$ would play the dominate role.

By the Schrödinger equation, we can get an univariate cubic equation, which is described in detail in Appendix A. Generally, there are three different real roots as listed in the end of Appendix A. We observed that only the first root $w_1$ from Eq. (14) is close to the numerical ones, which is denoted by $E_n^{(1)}$ for generalized model. We can easily find that $E_n^{(1)}$ for $g_2 = 0$ ($g_1 = 1$) here is reduced to $E_{n,1p}^{(1)} (E_{n,2p}^{(1)})$

The both processes in the pure models can also be realized in the following form of wavefunction

$$|n\rangle_2 = \left( \begin{array}{c} f_n |n - 1\rangle + e_n^* |n\rangle \\ c_n |n + 1\rangle \end{array} \right), \quad n = 0, 1, 2, \ldots \quad (19)$$

similarly, we can obtain the other univariate cubic equation, which is also given in Appendix A. We find that only the third root $w_3$ in Eq. (15) is more close to the numerical ones, which is denoted by $E_n^{(2)}$ for the generalized model. We can easily find that $E_n^{(2)}$ for $g_2 = 0$ ($g_1 = 1$) here is reduced to $E_{n,1p}^{(2)} (E_{n,2p}^{(2)})$

To show the validity of the approach with a few bare states, we compare the analytical results for the energy spectrum with the numerical ones. Since the two-photon process is of the second-order in the cavity QED, or the expansion part in the circuit QED, we study the effect of moderate two-photon terms. We calculate the energy spectra as a function of $g_1$ with fixed $g_2 = 0.1, 0.2$ for $\Delta = 0.5$ and 1.0, which are shown in Fig. 3. It is interesting to note that the analytical results agree well with the numerical ones. Our result based on a few Fock states deviates from the exact ones slightly around the avoiding crossing. In other words, in this area, the contributions from the neighbouring sub-spaces should be included.
we can get the photon states in the lower level

Then we have the population in the lower state

\[ |\psi(t)\rangle = |n\rangle_1 + d_2 |n\rangle_2 + d_3 |n + 1\rangle_2, \]

where

\[ d_1 = c_n, \quad d_2 = c'_n, \quad d_3 = f'_{n+1}, \]

the time-dependent wavefunction is then determined as

\[ |\psi(t)\rangle = c_n e^{-iE^{(1)}_n t} |n\rangle_1 + c'_n e^{-iE^{(2)}_n t} |n\rangle_2 + f'_{n+1} e^{-iE^{(2)}_{n+1} t} |n + 1\rangle_2, \]

we can get the photon states in the lower level

\[ |\psi(t)\rangle = \left(c_n e^{-iE^{(1)}_n t} e_n + c'_n e^{-iE^{(2)}_n t} e'_n\right) |n + 1\rangle_2 + \left(c_n e^{-iE^{(1)}_n t} f_n + c'_n e^{-iE^{(2)}_n t} f'_n\right) |n + 2\rangle_2. \]

Then we have the population in the lower state

\[ P_+ = c_n^2 \left(1 - c_n^2\right) + c_n^2 e_n^2 + f_{n+1}^2 e_n^2 + 2c_n c_n e_n c'_n e'_n \cos \left[\left(E^{(1)}_n - E^{(2)}_n\right) t\right] + 2c_n f_n f_{n+1} e'_n e_{n+1} \cos \left[\left(E^{(1)}_n - E^{(2)}_{n+1}\right) t\right]. \]

Finally we get atomic population inversion \( \langle \sigma_z(t) \rangle = 1 - 2P_+ \). Interestingly we obtain two Rabi frequencies:

\[ \omega^{(1)}_n = E^{(1)}_n - E^{(2)}_n, \quad \omega^{(2)}_n = E^{(1)}_n - E^{(2)}_{n+1}, \]

unlike the pure model where only one Rabi frequency in the quantum

Dynamics. - With the eigensolutions obtained above, the evolution from initial states can be analyzed. Initiated from the number state in the upper level \( |t = 0\rangle = |n\rangle \uparrow \rangle \), which can be expanded in terms of normalized eigenstates Eqs. (18) and (19)

\[ \langle \sigma_z \rangle = 1 - 2P_+ \]

Rabi oscillation is present. In the pure model with either one- or two-photon terms, \( \omega^{(2)}_n \) disappears, the evolution from a number state in the upper level oscillates sinusoidally.

We examine the dynamics of the population difference \( \langle \sigma_z(t) \rangle \) initiated from photonic vacuum states and atomic upper level at \( \Delta = 0.5 \) for \( g_1 \) ranging from ultra-strong to deep strong coupling, and small values of \( g_2 \). For better understanding, we turn to analysis of Fourier transform of \( \langle \sigma_z(t) \rangle \) for the long time window. The analytical results are shown in Fig. 4 with red lines. The numerical exact ones are also collected using black line for comparisons. The analytical results for \( \langle \sigma_z(t) \rangle \) match quite well with the numerical ones. If \( g_1 \) and \( g_2 \) are comparable, two Rabi frequencies are clearly shown with comparable peak height of the Fourier transform, as shown in \( g_1 = 0.1, g_2 = 0.05 \) and \( 0.1 \). If \( g_1 \) and \( g_2 \) differ considerably, e.g. \( g_1 = 0.5, 1 \), while \( g_2 = 0.05 \) and \( 0.1 \), Fourier transform indicates a dominate oscillation with single fre-
frequency $\omega_n^{(1)}$. However the population difference $\langle \sigma_z(t) \rangle$ shows an oscillation with varying amplitude, due to the weak oscillation with frequency $\omega_n^{(2)}$.

Without the nonlinear coupling i.e. $g_2 = 0$, it is known that the famous single Rabi oscillation should be present. With the presence of the non-linear coupling, the two dominate oscillations are present. With the interplay of the one- and two-photon terms, population difference $\langle \sigma_z(t) \rangle$ becomes more complicated. If one kind of the coupling term is relatively weak, and therefore can be omitted, only one of the two quantum Rabi oscillations is visible.

VI. CONCLUSION

In summary, the generalized QRM with both one- and two-photon terms is studied analytically. The adiabatic approximation where only the transition within the same manifold is considered produces the analytical eigenvalues and eigenstates completely. The obtained energy spectra are in good agreement with the numerically exact diagonalization ones in a wide range of coupling strength. The population dynamics obtained in the adiabatic approximation is also in quantitative agreement with the numerical ones, also indicating the validity of adiabatic approximation.

In the RWA, the mathematical simplicity of the eigen solution in the pure Rabi model with either one- or two-photon terms is lacking, because of the absence of the conserved excitation number. We propose an ansatz of the eigenfunctions including a few dominate Fock states. The corresponding analytical eigensolutions yield quite good energy levels compared to the numerical exact ones. The population dynamics can also match the oscillations for a long time. Two dominate Rabi frequencies are derived, which can be detected in the calculations.

The concise analytical solution in both full model and in the RWA can be easily applied in the circuit QED system with the one-photon coupling terms ranging from weak, ultra-strong, to deep strong coupling regime for moderate the two-photon coupling. Within the present theory, the two-photon coupling besides the one-photon coupling is possibly detected from the experimental data if both one- and two-photon interact with the oscillator simultaneously.

ACKNOWLEDGEMENTS This work is supported by the National Science Foundation of China (Nos. 11474256, 11674285), the National Key Research and Development Program of China (No. 2017YFA0303002).

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Appendix A: Solutions in the rotating-wave approximations by the univariate cubic equation

In terms of the wavefunction Eq. (15), the Schrödinger equation gives

$$\left( n + \frac{\Delta}{2} - E \right) c_n |n\rangle + g_1 e_n \sqrt{n+1} |n\rangle$$

$$+ f_n \sqrt{n+2} |n+1\rangle$$

$$+ g_2 \left( e_n \sqrt{n+1} |n+1\rangle + f_n \sqrt{n+2} |n+1\rangle \right) = 0$$

(A1)

$$g_1 \sqrt{n+1} (c_n |n+1\rangle) + g_2 \sqrt{(n+2)(n+1)} c_n |n+2\rangle$$

$$+ \left( n + 1 - \frac{\Delta}{2} - E \right) e_n |n+1\rangle$$

$$+ \left( n + 2 - \frac{\Delta}{2} - E \right) f_n |n+2\rangle = 0$$

(A2)

Note that the sub-space that wavefunction spanned is not closed, unlike the pure model.

Projecting Eq. (A1) onto $|n\rangle$, Eq. (A2) onto $|n+1\rangle$ and $|n+2\rangle$, we have three set equations

$$\left( n + \frac{\Delta}{2} - E \right) c_n + g_1 e_n \sqrt{n+1}$$

$$+ g_2 f_n \sqrt{(n+2)(n+1)} = 0$$

$$g_1 \sqrt{n+1} c_n + \left( n + 1 - \frac{\Delta}{2} - E \right) e_n = 0$$

$$g_2 \sqrt{(n+2)(n+1)} c_n + \left( n + 2 - \frac{\Delta}{2} - E \right) f_n = 0$$

Set

$$x = \left( n + 1 - \frac{\Delta}{2} \right)$$

$$y = g_1 \sqrt{n+1}$$

$$z = g_2 \sqrt{(n+2)(n+1)}$$

Nonzero coefficients yield an univariate cubic equation

$$E^3 + bE^2 + cE + d = 0.$$  \hspace{1cm} (A3)

where

$$b = -(3x + \Delta),$$

$$c = -(1 - 3x^2 + y^2 + z^2 - \Delta - 2x\Delta),$$

$$d = -(x^3 - y^2 - x(1 + y^2 + z^2 - \Delta) + x^2\Delta).$$

Similarly by the other form of wavefunction Eq. (19), we can also obtain three set equations as follows

$$\left( n + \frac{\Delta}{2} - E \right) f'_n + g_2 \sqrt{(n+1)} ne'_n = 0,$$

$$\left( n + \frac{\Delta}{2} - E \right) c'_n + g_1 \sqrt{n+1}e'_n = 0,$$
we can obtain another cubic equation
\[ E^3 + b'E^2 + c'E + d' = 0, \]
where
\begin{align*}
b' &= \Delta - 3x', \\
c' &= -y'^2 - z'^2 - 1 + 3x'^2 + \Delta - 2x\Delta, \\
d' &= -x'^3 - y'^2 + x' (1 + y'^2 + z'^2 - \Delta) + x^2\Delta,
\end{align*}
The solutions of an univariate cubic equation
\[ w^3 + bw^2 + cw + d = 0, \]
can be found in any Mathematics manual. If
\[ \Gamma = B^2 - 4AC < 0, \]
with
\[ A = b^2 - 3c, B = bc - 9d, C = c^2 - 3bd, \]
there are three different real roots
\begin{align}
w_1 &= \frac{-b - 2\sqrt{A} \cos \theta}{3} \\
w_2 &= \frac{-b + \sqrt{A} [\cos \theta - \sqrt{3}\sin \theta]}{3} \\
w_3 &= \frac{-b + \sqrt{A} [\cos \theta + \sqrt{3}\sin \theta]}{3},
\end{align}
where
\[ \theta = \frac{1}{3} \arccos \left( \frac{2Ab - 3B}{2\sqrt{A^3}} \right). \]
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