Abstract

This paper studies the spatial coalescent on $\mathbb{Z}^2$. In our setting, the partition elements are located at the sites of $\mathbb{Z}^2$ and undergo local delayed coalescence and migration. That is, pairs of partition elements located at the same site coalesce into one partition element after exponential waiting times. In addition, the partition elements perform independent random walks. The system starts in either locally finite configurations or in configurations containing countably many partition elements per site. These two situations are relevant if the coalescent is used to study the scaling limits for genealogies in Moran models respectively interacting Fisher-Wright diffusions (or Fleming-Viot processes), which is the key application of the present work.

Our goal is to determine the longtime behavior with an initial population of countably many individuals per site restricted to a box $[-t^{\alpha/2}, t^{\alpha/2}]^2 \cap \mathbb{Z}^2$ and observed at time $t^\beta$ with $1 \geq \beta \geq \alpha \geq 0$. We study both asymptotics, as $t \to \infty$, for a fixed value of $\alpha$ as the parameter $\beta \in [\alpha, 1]$ varies, and for a fixed $\beta$, as the parameter $\alpha \in [0, \beta]$ varies. This exhibits the genealogical structure of the mono-type clusters arising in 2-dimensional Moran and Fisher-Wright systems.

A new random object, the so-called coalescent with rebirth, is constructed via look-down and shown to arise in the limit. For sake of completeness, and in view of future applications we introduce the spatial coalescent with rebirth and study its longtime asymptotics as well.

The present paper is the basis for forthcoming works [20] and [23], where the genealogies in interacting Moran models and Fisher-Wright diffusions on $\mathbb{Z}^2$ are studied, and where the spatial continuum limit of the Moran model on $\mathbb{Z}^1$ (Brownian web) is developed, respectively. There the coalescent with rebirth is needed to describe the “complete” genealogical forests, i.e., the genealogical structures which include also the “fossils”.

Keywords: spatial coalescent, Kingman coalescent, coalescent with rebirth, two-dimensional random walk asymptotics, Erdös-Taylor formula, asymptotic exchangeability, look-down construction

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1 Introduction

The spatial (delayed) coalescent processes on $\mathbb{Z}^d$ and their space-time scale behavior are the key mathematical tools for the analysis of the asymptotic behavior of a certain class of neutral population models, namely of interacting particle models, known as the interacting Moran models, and their diffusion limit, the interacting Fisher-Wright diffusions, respectively Fleming-Viot diffusions. These models describe populations in which individuals have a type and a geographic location evolving by resampling and migration. (See Shiga [30] and Durrett [14]). The coalescent process allows to construct the genealogies of the current population in these models explicitly. In particular if one attempts to understand the scaling behavior of the genealogical trees generated by the population in critical dimension the spatial coalescent process is the key tool.

We believe, however, that a number of the results on the spatial coalescent are of independent interest and have possible applications outside of the context of Moran and Fleming-Viot models. For this reason we present and prove them here separately, and refer the reader for example to [19, 20, 21] for population model applications. In this paper the results are formulated for individuals, types and locations only and do not involve continuum constructions using $\mathbb{R}$-trees etc., which will be necessary in forthcoming work [20, 21] that builds on the results presented here and in fact motivates many constructions in the form given here.

A class of spatial stochastic systems on $\mathbb{Z}^d$ that combine migration between the sites and a stochastic mechanism acting at each site (including the voter model, branching random walks or interacting diffusions, see, for example, Liggett [26], Dawson [10], Shiga [30] and Cox and Greven [8]) exhibit a dichotomy between low (typically $d \leq 2$) and high dimensions (typically $d > 2$) in their longtime behavior. In high dimensions non-trivial equilibria exist, while in low dimension such systems approach laws which are concentrated on the “traps” of the stochastic evolution, i.e. on the configurations which the system cannot ever leave with probability 1.

A special rôle is played by the critical dimension $d = 2$, which is characterized by the fact that the underlying (symmetrized) migration random walk is recurrent, while its Green’s function $\sum_{k=1}^{n} P\{X_k = 0\}$ grows only logarithmical in n. There (as in general for the recurrent setting) the above processes converge weakly to a law concentrated on mono-type configurations as time evolves from 0 to infinity. Somewhat surprisingly, as first explored for the voter model by Cox and Griffeath in 1986 [9], the order of magnitude of the regions where the system looks mono-type is not asymptotically deterministic (unlike in the $d = 1$ setting where we get $\sqrt{t}$ as order of magnitude for the size of the mono-type regions). In fact, the mono-type cluster containing the origin has an area of the order $t^\alpha$, as $t \to \infty$, where the random exponent $\alpha$ takes values in $[0, 1]$ and its distribution can be specified as follows: take a Fisher-Wright diffusion $(Z_t, t \geq 0)$, and define

$$T := \inf \{ t \geq 0 : Z_t \in \{0, 1\} \},$$

(1.1)

then $\alpha = e^{-T}$ (see [20] for details). This phenomenon is called the diffusive clustering.

Another interesting question concerns the “age” of a cluster. More precisely, in particle systems language, suppose the configuration at some large time $t$ contains a monochrome cluster around the origin of area $t^\alpha$. Then its age is, informally, the amount of time during which this cluster has already persisted in the spatial volume of volume $t^{\beta}$. It turns out that this age is of the order $t^{\beta}$, for some random $\beta \in (\alpha, 1)$. To obtain more detailed results on cluster formation one needs to consider the time-space configuration of the process providing the information on which types populated a specific site in space at a specific time. This type of analysis for the time-space configuration as a function of $\alpha$ and $\beta$ has been carried out by Fleischmann and Greven [17, 18] for interacting diffusions with components in $[0, 1]$ on the hierarchical group with a symmetric critically recurrent migration kernel, by Klenke [25] in the $[0, \infty)$-valued (branching) component case on the hierarchical group, and by Winter [32] on $\mathbb{Z}^2$.

The behavior of the above particle systems and their diffusion limit is reflected in the behavior of their dual processes, the spatial coalescent, and the “time-space” dual processes, which we introduce.
here and which we call the spatial coalescent with rebirth. These dual processes generate the genealogies of the population of Moran and Fisher-Wright systems.

In the present paper we systematically explore the longtime behavior of the spatial coalescent with and without rebirth in the geographic space $\mathbb{Z}^2$. One of the interesting new features concerning the spatial coalescent with rebirth is that it enables a description of the whole genealogical structure (including “fossils”) rather than only that of the current population at a reference time of the corresponding population models. The results we shall prove will replace and extend the earlier ad hoc constructions via spatial or time-space moment dualities used in previous work by various authors. The full potential of this genealogical viewpoint will become apparent in future applications. For example, in [20] and [23] we shall prove convergence theorems for the complete genealogical structure of the coalescent with rebirth in order to describe the genealogy in the interacting Moran models and the interacting Fisher-Wright diffusions including “fossils”.

We decided to devote a separate paper solely to coalescent processes since we believe that the coalescent process with rebirth constructed in Subsection 2.2 is likely to appear in the scaling limit for a whole class of similar mathematical population genealogy (particle and diffusion) models. In particular four points are important and different from previous work:

1. the universality of the scaling results in the sense that the migration mechanism belongs to a large class of random walks,

2. initial configuration may contain sites with countably infinite number of individuals,

3. the concept of the coalescent with rebirth allows for further applications to the study of the genealogies for the underlying population models via a weighted $\mathbb{R}$-tree-valued

4. an analytical characterization of the coalescent with rebirth as well as its construction via a look-down procedure.

At the end of Subsection 3 after the outline, we comment in detail on earlier work by Cox and Griffeath ([19]) and Bramson, Cox and Griffeath ([14]) who considered the spatial instantaneous coalescent with a simple random walk migration mechanism.

2 Models

The coalescent processes considered in the present paper are describing the genealogies of neutral population models involving resampling between any two individuals where two individuals are replaced by descendants of one of them (the one at the end of the arrow in Figure 1). In the time-reversed evolution these time points correspond to the times at which the ancestral lines of the two individuals coalesce to a common ancestral line (compare with Figure 1).

We shall define in this section the spatial coalescent, the spatial coalescent with rebirth and finally the so-called look-down process, which allows for a graphical representation of our coalescent processes which give straightforward explicit constructions for a version of these processes.

2.1 The spatial coalescent on $\mathbb{Z}^2$

Processes describing the dynamics of finitely many moving and coalescing particles appeared already in the 1980’s (see, for example, [5, 4, 9] and compare Liggett [26] for more detailed references). For coalescents representing genealogies of diffusion processes, it is essential to allow configurations with countably many particles per site on a countable geographic space. Moreover, while the above papers were only recording occupation numbers at various sites, we will provide a set-up which also exhibits the partition structure.
The spatial coalescent that we analyze in the current paper was introduced on a class of Abelian groups in [19]. For the benefit of the reader we briefly recall in three steps the relevant notation, appropriate topologies and its construction. We restrict the setting to $\mathbb{Z}^2$.

Step 1 (Migration) Let $a(\cdot, \cdot)$ be an irreducible random walk kernel which has finite exponential moments, i.e.,

$$a(x, y) = a(0, y - x),$$

for all $x, y \in \mathbb{Z}^2$, and

$$\sum_{(z_1, z_2) \in \mathbb{Z}^2} e^{\lambda_1 z_1 + \lambda_2 z_2} a(0, (z_1, z_2)) < \infty,$$

for all $\lambda_1, \lambda_2 \in \mathbb{R}$. We consider the continuous time random walk with jump rate 1 and transition probability $a(\cdot, \cdot)$.

We next present the standard way to construct particle systems that possibly start in configurations with countably many particles at some or all sites and which involve migration as mechaism. These particle systems are constructed as extensions of particle systems which start in specific locally finite states and the dynamics are such that they guarantee local finiteness of the particle process at all times $t > 0$ (compare also with Remark 2.2). To construct such locally finite systems we follow an approach due to Liggett and Spitzer [27].

Fix a finite measure $\alpha$ on $\mathbb{Z}^2$ with $\alpha(\{x\}) > 0$, for all $x \in \mathbb{Z}^2$, such that for a constant $\Gamma$

$$\sum_{y \in \mathbb{Z}^2} a(x, y) \alpha(\{y\}) \leq \Gamma \cdot \alpha(\{x\}),$$

for all $x \in \mathbb{Z}^2$. Denote by $\mathcal{N}(\mathbb{Z}^2)$ the set of all locally finite $\mathbb{N}_0$-valued measures on $\mathbb{Z}^2$. Then

$$\mathcal{E}_\alpha \equiv \mathcal{E} := \{ \eta \in \mathcal{N}(\mathbb{Z}^2) : \sum_{x \in \mathbb{Z}^2} \eta(x) \alpha(\{x\}) < \infty \}$$

is the Liggett-Spitzer space (corresponding to $\alpha$).
Remark 2.1 ($\mathcal{E}$ is a state space). Let $\{(X_t^i)_{t \geq 0} : i \in \mathcal{I}\}$ be a countable collection of independent random walks, and put for all $t \geq 0$, $\eta_0 := \sum_{i \in \mathcal{I}} \delta_{X_t^i} \in \mathcal{N}(\mathbb{Z}^2)$. If
\[
\eta_0 \in \mathcal{E}, \quad \mathbf{P}\text{-a.s.,} \tag{2.5}
\]
then an easy calculation shows that the process $(e^{-t \Gamma_1} \sum_{i \in \mathcal{I}} \alpha((\{X_t^i\}))_{t \geq 0})$ is a super-martingale.

In particular, under $(2.5)$, for all $t \geq 0$,
\[
\mathbf{P}\{\eta_t \in \mathcal{E}\} = 1, \tag{2.6}
\]
that is, $\mathcal{E}$ is a state space for $\{\eta_t, t \geq 0\}$. Note furthermore that $(2.6)$ implies $\eta_t \in \mathcal{N}(\mathbb{Z}^2)$, for all $t \geq 0$, almost surely. As topology on $\mathcal{N}(\mathbb{Z}^2)$ choose the vague topology, then $\mathcal{E}_\alpha$ is a (not closed) subset of the polish space $\mathcal{M}(\mathbb{Z}^2)$, the locally finite measures on $\mathbb{Z}^2$. $\square$

To build in countably many individuals per site we shall make use of the coalescence mechanism introduced next.

Step 2 (Coalescence) Recall that a partition of a set $\mathcal{I}$ is a collection $\{\pi_\lambda\}$ of pairwise disjoint subsets of $\mathcal{I}$ such that $\mathcal{I} = \cup_\lambda \pi_\lambda$. We refer to the elements of a partition as partition elements. Let us denote by
\[
\Pi^\mathcal{I} := \text{collection of all partitions of } \mathcal{I}. \tag{2.7}
\]

For all $\mathcal{I}' \subseteq \mathcal{I}$, write $\rho_{\mathcal{I}'}$ for the restriction map from $\Pi^\mathcal{I}$ to $\Pi^{\mathcal{I}'}$ and hence for any $\mathcal{P} \in \Pi^\mathcal{I}$, the induced partition
\[
\rho_{\mathcal{I}'} \circ \mathcal{P} := \{\rho_{\mathcal{I}'}(\pi); \pi \in \mathcal{P}\}. \tag{2.8}
\]

We say that a sequence $(\mathcal{P}_n)_{n \in \mathbb{N}}$ converges in $\Pi^\mathcal{I}$ if for all finite subsets $\mathcal{I}' \subseteq \mathcal{I}$, the sequence $(\rho_{\mathcal{I}'} \circ \mathcal{P}_n)_{n \in \mathbb{N}}$ converges in $\Pi^{\mathcal{I}'}$ equipped with the discrete topology. In particular, a function $f : \Pi^\mathcal{I} \to \mathbb{R}$ that depends on $\Pi^\mathcal{I}$ only through $\Pi^{\mathcal{I}'}$, for some finite subset $\mathcal{I}' \subseteq \mathcal{I}$, is continuous. Note that $\Pi^\mathcal{I}$ equipped with this topology is a Polish space.

Definition 2.1 (The $\mathcal{I}$-Kingman coalescent). The $\mathcal{I}$-Kingman coalescent, or short the Kingman-coalescent,
\[
K := (K_t)_{t \geq 0}, \tag{2.9}
\]
is the unique strong Markov process such that for all finite $\mathcal{I}' \subseteq \mathcal{I}$, the restricted process
\[
K^{\mathcal{I}'} := \rho_{\mathcal{I}'} \circ K \tag{2.10}
\]
is a $\Pi^{\mathcal{I}'}$-valued Markov chain that starts in some $\mathcal{P} \in \Pi^\mathcal{I}$, and given $K^{\mathcal{I}'}$, each pair of partition elements is merging to form a single partition element after an exponential waiting time with rate $\gamma_{\text{King}} > 0$.

Step 3 (Migration and coalescence combined) We next combine migration and coalescence. For that purpose, fix a site space $\mathcal{M}$ which in the present paper is $\mathbb{Z}^2$ unless stated otherwise. Then from any $\mathcal{P} \in \Pi^\mathcal{I}$ one can form a marked partition
\[
\mathcal{P}^\mathcal{M} := \{(\pi, L(\pi)); \pi \in \mathcal{P}\}, \tag{2.11}
\]
by assigning to each partition element $\pi \in \mathcal{P}$, its location $L(\pi) \in \mathcal{M}$. Put
\[
\Pi^{\mathcal{I}, \mathcal{M}} := \text{set of all marked partitions}. \tag{2.12}
\]

Note that $\Pi^{\mathcal{I}, \mathcal{M}}$ is a Polish space if we introduce the topology as follows. For all $\mathcal{I}' \subseteq \mathcal{I}$ and $\mathcal{P} \in \Pi^{\mathcal{I}, \mathcal{M}}$, we extend the restriction operator as
\[
\rho_{\mathcal{I}'} \circ \mathcal{P} := \{(\rho_{\mathcal{I}'}(\pi), L(\pi)); \pi \in \mathcal{P}\}, \tag{2.13}
\]
and say that a sequence \((P_n)_{n \in \mathbb{N}}\) converges in \(\Pi^M\) if and only if for all finite subsets \(I' \subseteq I\), the sequence \((\rho_{I',M} \circ P_n)_{n \in \mathbb{N}}\) converges in \(\Pi^{I',M}\), equipped with the discrete topology.

We are now ready to define the spatial \(I\)-coalescent.

**Definition 2.2 (The spatial \(I\)-coalescent).** The spatial \(I\)-coalescent on \(\mathbb{Z}^2\),

\[
(C, L) := (C_t, L_t)_{t \geq 0} = \left\{ (\pi, L_t(\pi)) ; \; \pi \in C_t \right\}_{t \geq 0},
\]  

is a strong \(\Pi^{I,\mathbb{Z}^2}\)-valued Markov process with càdlàg paths such that for all subsets \(I' \subseteq I\) with

\[
\sum_{\pi \in C_0, \rho_{I'}(\pi) \neq \emptyset} \delta_{L_0(\pi)} \in \mathcal{E},
\]

the restricted process is a \(\Pi^{I',\mathbb{Z}^2}\)-valued strong Markov particle system which undergoes the following two independent mechanisms:

- **Migration** The marks of the partition elements perform independent continuous time random walks with rate \(1\) and transition kernel \(a(\cdot, \cdot)\).
- **Coalescence** Each pair of partition elements whose locations are equal merges into one partition element independently after exponential waiting times with rate \(\gamma\).

**Remark 2.2 (Spatial coalescent is well-defined).**

(i) Note that for the marked \(I\)-coalescent process above there is a natural coupling with the migration random walks \(\{(X^i_t)_{t \geq 0} : i \in I\}\) such that

\[
\sum_{\pi \in C_t} \delta_{L_t(\pi)}(B) \leq \sum_{i \in I} \delta_{X^i_t}(B), \quad a.s.,
\]

for all \(B \subseteq \mathbb{Z}^2\). Therefore, by Remark 2.1, the spatial \(I\)-coalescent is locally finite and in particular well-defined if \(I\) is already such that (2.15) holds.

(ii) By Proposition 3.4 in [19], the spatial \(I\)-coalescent is well-defined for all initial mark configurations. Specifically, it is even well-defined if started in a configuration which contains countable infinitely many partition elements at each site in \(\mathbb{Z}^2\). In all cases, we have that \(\sum_{\pi \in C_t} \delta_{L_t(\pi)} \in \mathcal{E}\), almost surely, for all \(t > 0\).

**Remark 2.3 (Consistency Property).** In all of our constructions of concrete realizations of coalescents below we use the following important consistency property: if \((C, L)\) is the \(I\)-coalescent and \(I' \subseteq I\) then \(\rho_{I'} \circ (C, L)\) is the \(I'\)-coalescent started in \(\rho_{I'} \circ (C_0, L_0)\).

**Remark 2.4 (Instantaneous coalescent; \(\gamma = \infty\)).** Note that if the finite parameter \(\gamma\) is replaced by \(\infty\), the spatial coalescent changes into the system of *instantaneously* coalescing random walks which can be obtained from particle systems as they are considered in [4, 9]. This observation will become important in Section 5.
2 MODELS

2.2 The coalescent with rebirth

In neutral population models coalescent processes arise in the study of genealogical relationships between individuals currently alive by looking in reversed time. Each coalescent event corresponds to a splitting of an ancestral line and a simultaneous death of another ancestral line in a forward population model. However, if one considers genealogies which include also the the individuals alive at earlier times (commonly referred to as “fossils”), then a richer object than the spatial coalescent is needed. We call this new object the coalescent with rebirth. The coalescent with rebirth accounts in the forward model for the descendant lines which died before the current time. More precisely, whenever an individual dies and gets replaced by a descendent of another individual in the forward model, in the time-reversed model the coalescent dynamics with rebirth generates a new individual at the corresponding time.

Example 2.1. Assume $\mathcal{I} := \{1, 2, 3\}$ and consider the initial configuration $\{\{1\}, \{2\}, \{3\}\}$ at time $s$, and the transitions (without rebirth) at times $t_1$ and $t_2$, respectively,

$$\{\{1\}, \{2\}, \{3\}\} \xrightarrow{\text{at time } t_1} \{\{1\}, \{2, 3\}\} \xrightarrow{\text{at time } t_2} \{\{1, 2\}, \{3\}\}$$

(compare, for example, Figure 2). Then the corresponding coalescent with rebirth would start at time $s$ from $\{\{(1, s)\}, \{(2, s)\}, \{(3, s)\}\}$, and at time $t_1$ the state would change to $\{\{(1, s)\}, \{(2, s), (3, s)\}, \{(3, t_1)\}\}$. In particular $(3, t_1)$ corresponds to the “reborn” individual $(3, s)$. Next after time $t_2$ the new individual $(2, t_2)$ is born, etc. All new born partition elements also undergo resampling and migration. Assume, for example, that in addition to the above mentioned resampling events, there would be one at time $u \in (t_1, t_2)$ between the ancestral lines of 2 and 3 then we would observe, for example, the following transitions

$$\{\{(1, s)\}, \{(2, s)\}, \{(3, s)\}\} \xrightarrow{\text{at time } t_1} \{\{(1, s)\}, \{(2, s), (3, s)\}, \{(3, t_1)\}\} \xrightarrow{\text{at time } u} \{\{(1, s)\}, \{(2, s), (3, s), (3, t_1)\}, \{(3, u)\}\} \xrightarrow{\text{at time } t_2} \{\{(1, s), (2, s), (3, s), (3, t_1)\}, \{(2, t_2)\}, \{(3, u)\}\}.$$ 

Notice that the transition at time $u$ is not observable in the original coalescent.

The goal of this subsection is to introduce the coalescent with rebirth first in the non-spatial setting and then in the spatial setting.

Step 1 (Coalescence with rebirth). As before let $\mathcal{I}$ be a countable set.

To define the state space of the coalescent with rebirth consider first a subset

$$\mathbb{S}(\mathcal{I}) \subseteq \mathcal{I} \times \mathbb{R}^+$$

such that for each $i \in \mathcal{I}$,

$$\# \{t \in \mathbb{R} : (i, t) \in \mathbb{S}(\mathcal{I})\} < \infty.$$  

We refer to the elements $(i, t) \in \mathbb{S}(\mathcal{I})$ as individuals, and call $t$ the birth time of the individual $(i, t)$. Let $\text{pr}_{\text{index}}$ and $\text{pr}_{\text{time}}$ be the projection maps of individuals to their indices and birth times, i.e.,

$$(i, t) = (\text{pr}_{\text{index}}(i, t), \text{pr}_{\text{time}}(i, t)),$$ 

for all $(i, t) \in \mathcal{I} \times \mathbb{R}.$
Recall from (2.7) the collection $\Pi^S$ of all partitions of a set $S$. Call $\mathcal{P}$ a sub-partition of $S(\mathcal{I})$, if it is a partition of a subset of $S(\mathcal{I})$, or equivalently; a collection $\{\pi_{\lambda}\}$ of pairwise disjoint subsets of $S(\mathcal{I})$. With a slight abuse of notation denote by

$$\Pi^{S(\mathcal{I})} := \text{set of all sub-partitions of } S(\mathcal{I}).$$

(2.22)

Notice that the coalescent was defined in a symmetric manner. To define the coalescence dynamics with rebirth we need to break the involved symmetry and declare which of the patches is getting “lost” and simultaneously reborn. For that recall that since $\mathcal{I}$ is countable we can fix an order relation $\preceq$ such that for all $i_0 \in \mathcal{I}$, $\#\{i \preceq i_0\} < \infty$. This extends to the lexicographic order relation on $\mathcal{I} \times \mathbb{R}$, that is, for $(i, s), (j, t) \in \mathcal{I} \times \mathbb{R},$

$$(i, s) \preceq (j, t) \text{ if and only if } i \preceq j \text{ or } i = j \text{ and } s < t.$$  

(2.23)

Given $\mathcal{P} \in \Pi^{S(\mathcal{I})}$, define the label $\kappa(\pi)$ of a partition element $\pi \in \mathcal{P}$ as its smallest element with respect to $\preceq$, i.e.,

$$\kappa(\pi) := \min \{v; v \in \pi\}. \quad \text{(2.24)}$$

As illustrated in Example 2.1 the coalescent with rebirth dynamics relies on the rule that if two partition elements coalesce it is always the one with the bigger label that gets “lost and reborn”. We therefore need our process to take values in the following subset of $\Pi^{S(\mathcal{I})}$:

$$\hat{\Pi}^{S(\mathcal{I})} := \{\mathcal{P} \in \Pi^{S(\mathcal{I})} : \exists \text{ bijection } \iota : \mathcal{I} \rightarrow \mathcal{P} \text{ s.t. } \forall i \in \mathcal{I} \exists t \in \mathbb{R} \text{ with } \kappa(\iota(i)) = (i, t)\}.$$  

(2.25)

We equip $\mathbb{R} \times \hat{\Pi}^S$ with a topology that takes both the partition structure and the birth times into account. Notice that in contrast to the original coalescent where the set of individuals is fixed, in the coalescent with rebirth the set of individuals increases as time increases. Let therefore for all $\mathcal{P} \in \hat{\Pi}^{S(\mathcal{I})}$,

$$S(\mathcal{P}) := \bigcup_{\pi \in \mathcal{P}} \pi$$  

(2.26)

be the basic set of $\mathcal{P}$.

Recall from (2.8) the restriction map, and abbreviate for $\mathcal{I}' \subseteq \mathcal{I}$ and a subpartition $\mathcal{P} \in \hat{\Pi}^{S(\mathcal{I}')}$, \begin{equation} \rho_{\mathcal{I}'} \circ \mathcal{P} := \rho_{(\mathcal{I}' \times \mathbb{R}) \cap S(\mathcal{P})} \circ \mathcal{P}. \end{equation}

(2.27)
Example 2.2. Take for example \( I := \{1, 2, 3\} \) and \( P := \{(1,0),(2,0),(3,0),(3,t_1),(2,t_2),(3,s)\}\). Then
\[
\rho_{\{1,2\}} \circ P := \{(1,0),(2,0),(2,t_2)\}.
\]  

We now introduce a topology on the state space \( S(I) \) that accounts for the differences in both the indices and the birth times. Loosely speaking, we say that a sequence
\[
(P_n)_{n \in \mathbb{N}} \text{ converges in } \hat{\Pi}^{S(I)},
\]
if and only if for each finite subset \( I' \subseteq I \), the projections to the index component of the restricted partitions \( \rho_{I'} \circ P_n \) converge in the discrete topology and the corresponding birth times converge with respect to the Euclidian distance. More precisely, we consider the topology generated by a metric satisfying the properties (2.46) through (2.47).

We will need some further notation. For a finite subset \( S' \subset S(I) \), denote the restriction map from \( \hat{\Pi}^{S(I)} \) to \( \hat{\Pi}^{S'} \) by \( \rho_{S'} \).

Since the coalescent with rebirth is keeping track of the birth time of an individual we need in addition (to obtain a time-homogeneous mechanism) to encode explicitly the time in the state. That is, we finally choose
\[
\mathbb{R} \times \hat{\Pi}^{S(I)}
\]
as the state space. We also write \( \rho_{I'}(t, P) := (t, \rho_{I'} \circ P) \).

We are now ready to define the coalescent with rebirth.

Definition 2.3 (Kingman-type coalescent with rebirth). Fix \( t_0 \in \mathbb{R} \). The Kingman-type coalescent with rebirth is a strong \( \mathbb{R} \times \hat{\Pi}^{S(I)} \)-valued Markov process
\[
K^{\text{birth}} = (K^{\text{birth}}_u)_{u \geq t_0}, \quad t_0 \in \mathbb{R},
\]
whose initial condition \( K^{\text{birth}}_0 := (t_0, P_0) \) satisfies for all \( i \in I \),
\[
\# \{ t \in \mathbb{R} : (i,t) \in P_0 \} < \infty,
\]
and such that for all finite subsets \( I' \subseteq I \), the restricted process \( \rho_{I'} \circ K^{\text{birth}} \) is a \( \mathbb{R} \times \hat{\Pi}^{S'} \)-valued Markov chain which starts in \( (t_0, \rho_{I'} \circ K^{\text{birth}}_0) \) for some \( K^{\text{birth}}_0 \in \hat{\Pi}^{S(I)} \) such that

- the time coordinate grows at a deterministic speed one, and
- given the current state \( (t_0, P) \in \mathbb{R} \times \hat{\Pi}^{S'} \) at time \( t \), each pair of partition elements \( \pi_1, \pi_2 \in P \) merges into \( \pi_1 \cup \pi_2 \) after an exponential waiting time with rate \( \gamma_{\text{King}} > 0 \), and at this time \( t' > t \), instantaneously a new partition element \( \{(\text{pr}_{\text{index}}(\kappa(\pi_1) \lor \kappa(\pi_2)), t')\} \) is born. (\( \lor \) is the maximum taken in the sense of relation (2.22)).

Proposition 2.1 (Existence and uniqueness in law).

(a) The Kingman-type coalescent with rebirth is a well-defined pure jump process for every initial state with finitely many partition elements at time \( t_0 \).

(b) For every initial point in \( \mathbb{R} \times \hat{\Pi}^{S(I)} \) of the form \( (t_0, \{(i,t_0) : i \in \mathbb{N}\}) \), \( t_0 \in \mathbb{R} \), there exists a unique càdlàg process satisfying the requirements of Definition 2.3.

Proof. Proposition 2.1 is a special case of Proposition 2.2. We therefore omit the proof at this point.\( \Box \)
Remark 2.5. \( (K_{n}^{\text{birth}})_{n \in [a,b]} \) has the property that at each time \( u > b \), for each \( i \in \mathbb{N} \), there is exactly one (partition) element \( \pi \) in \( K_{n}^{\text{birth}} \) with \( \kappa(\pi) = (i, s) \), for some \( s \). Indeed the new individual \( (i, s) \) will be born/introduced at time \( s \) only if a partition element \( \pi \) with label \( \kappa(\pi) = (i, u) \), for some \( u < s \), coalesces at time \( s \) with a partition element \( \pi' \) such that \( \kappa(\pi') < \kappa(\pi) \). \( \square \)

**Step 2** (Migration and coalescence with rebirth combined) In the case of the spatial coalescent with rebirth all partition elements have in addition to an index and a birth-time also a current location that changes according to a random walk independently over partition elements. Fix again a countable site space \( M \). By assigning to each partition element \( \pi \in \Pi^{\mathcal{I}} \) we consider the topology generated by the metric (2.46) through (2.47).

**Definition 2.4** (The spatial \( \mathcal{I} \)-coalescent with rebirth). The spatial \( \mathcal{I} \)-coalescent with rebirth,

\[
(C^{\text{birth}}, L^{\text{birth}}) := (t_{0} + t, (C^{\text{birth}}_{t}, L^{\text{birth}}_{t}))_{t \geq 0},
\]

is a strong \( \mathbb{R} \times \hat{\Pi}^{\mathcal{I}},M \)-valued Markov process with c\‘adl\‘ag paths such that for all subsets \( \mathcal{I}' \subseteq \mathcal{I} \) with

\[
\sum_{\pi \in C_{0} \cap \mathcal{E}} \delta_{i,0}^{\text{birth}}(\pi) \in \mathcal{E} ,
\]

and all initial birth times less than or equal to \( t_{0} \), the restricted process \( \rho_{\mathcal{I}'} \circ (C^{\text{birth}}, L^{\text{birth}}) \) is a \( \mathbb{R} \times \hat{\Pi}^{\mathcal{I}'},M \)-valued strong Markov particle system which undergoes the following three independent transition mechanisms:

- **Time growth** The time coordinate grows at deterministic rate one.
- **Migration** The marks of the partition elements perform independent random walks.
- **Coalescence with rebirth** Given the current state \( (t, \{\pi, L^{\text{birth}}(\pi); \pi \in \mathcal{P}\}) \in \mathbb{R} \times \hat{\Pi}^{\mathcal{I}},\mathbb{Z}^{2} \), each pair of partition elements \( \pi_{1}, \pi_{2} \in \mathcal{P} \) merges into \( \pi_{1} \cup \pi_{2} \) after an exponentially distributed waiting time with hazard function given by the density \( \gamma \cdot 1\{L^{\text{birth}}(\pi_{1}) = L^{\text{birth}}(\pi_{2})\} \), and at this random time \( t' > t - t_{0} \), instantaneously the marked partition element \( \{(\text{pr}_\text{index}(\kappa(\pi_{1}) \vee \kappa(\pi_{2})), t' + t_{0}) \}, L^{\text{birth}}_{t' - t_{0}}(\pi_{1}) \} \) is created.

**Proposition 2.2** (The spatial \( \mathcal{I} \)-coalescent rebirth is well-defined).

(a) The spatial \( \mathcal{I}' \)-coalescent with is a well-defined particle system, for all \( \mathcal{I}' \) satisfying (2.37).

(b) For every initial point in \( \mathbb{R} \times \hat{\Pi}^{\mathcal{I}},M \) of the form \( (t_{0}, \{(i, t_{0}), L^{\text{birth}}((i, t_{0}))\}; i \in \mathbb{N}\}) \) \( t_{0} \in \mathbb{R} \), there exists a unique c\‘adl\‘ag process satisfying the requirements of Definition 2.4.

The proof will be given in the following subsection.
2 MODELS

Figure 3: (a) a realization of the Poisson processes $M$, (b) the set of all descendants up to time $s$ of the individual labeled 1 at time $t$ is indicated in bold, (c) the ancestral line of the individual 4 alive at time $s$ is indicated in bold.

2.3 The look-down construction (Proof of Propositions 2.1 and 2.2)

In this subsection we give the explicit construction of a version of the coalescent and the coalescent with rebirth. For that purpose we will rely on the graphical representation of the look-down process introduced first by Donnelly and Kurtz in [18] and generalized to the spatial setting in [19]. In the look-down construction we can link both the population model of locally infinite population size in forward time and the coalescent starting with locally infinitely many patches in reversed time.

In order to give the explicit construction based on a random graph we proceed as follows. Fix a rate $\gamma > 0$, and a non-empty countable set $\mathcal{I}$ referred to as the set of all individuals. Assume we are given a collection

$$\{(L_i^t)_{t \geq 0}, i \in \mathcal{I}\}$$

of the independent continuous time irreducible random walks on an Abelian group $G$. Then we can choose a total order $\preceq$ on $\mathcal{I}$ such that for all $i \in \mathcal{I}$ and $x \in \mathbb{Z}^2$,

$$\#\{i' : L_{t_0}^i = x\} < \infty, \quad \text{a.s.}$$

Let

$$M := \{M_{i,j}^t : i, j \in \mathcal{I}; i \preceq j\}$$

be a family of independent Poisson point processes on $\mathbb{R}^+$ with intensity measure $\gamma \, dt$. The random collections in (2.38) and (2.40) are independent. This specifies our probability space. Starting from $(L_{t_0}^i, i \in \mathcal{I})$, we follow the random walk $(L_i^t)_{t \geq 0}$ and draw an arrow from $i$ to $j$ at time $t$ if $t$ is a point of $M_{i,j}^t$ and $L_i^t = L_j^t$. This defines a random graph embedded in $\mathbb{R} \times \mathcal{I}$ with (random) marks in $\mathbb{Z}^2$, which is defined on our probability space.

For $s < t$ we say that $(i, s)$ and $(j, t)$ are connected by a path if in the $\mathbb{R} \times \mathcal{I}$ diagram we can move vertically without crossing the tip of an arrow or horizontally along arrows from $(i, s)$ to $(j, t)$. This means that in forward time we think of the points in $M_{i,j}$ as the times at which individual $i$ is pushing out an individual $j$ from the population in order to replace it by a new individual of its type. We therefore call such a path a line of descent and $(j, t)$ a descendent of $(i, s)$.

On the other hand we can reverse this path and say that the reverse path is an ancestral line associating with $(j, t)$ its ancestor $(i, s)$ at time $s$. In this path $(A_{t,s}^i)_{t \geq 0}$ time now runs backward, so that $A_{t,s}^i = j$. In reversed time we therefore interpret the points in $M_{i,j}$ as the times at which the ancestral lines of the individuals $i$ and $j$ may coalesce to a common ancestral line. For example, in
Figure 3(c) the ancestor back at time $t$ of the individual which lives at time $s$ and corresponds to the fourth ancestral line is the individual which corresponds to the first ancestral line. If we define for each $j \in \mathcal{I}$ and $t \geq s \geq 0$,

$$\pi^s_j(t) := \{i \in \mathcal{I} : A^i_{s,t} = j\},$$  \hfill (2.41)

we obtain that the partition element $\pi^s_j(t)$ as the set of all descendants at time $s$ of the individual $j$ which lived at time $t$ in the past. For example, in Figure 3(b) the individual which lives at time $t$ and corresponds to the first ancestral line has 4 descendants at time $s$ which correspond to the first four ancestral lines.

By condition (2.39) the forward construction is automatically well-defined and hence the following key result holds:

**Lemma 2.1** (Ancestors are well-defined.). For each $i \in \mathcal{I}$ and $s \geq 0$, there exists a unique function $(A^i_{s,t})_{t \geq 0}$ from $[s, \infty)$ into $\mathcal{I}$ with càdlàg paths such that $A^i_{s,s} = i$ and

$$A^i_{s,t-} \neq A^i_{s,t}, \quad \text{if and only if} \quad t \in M^{-i}\pi; \pi^{-i}_{s,t-} \text{ and } L_{t-}^{A^i_{s,t-}} = L_{t-}^{A^i_{s,t}}. \hfill (2.42)$$

**Remark 2.6** (The look-down process and the spatial $\mathcal{I}$-coalescent). Construct the infinitely old population for the forward model in times $(-\infty, s)$.

Recall from the look-down construction from Subsection 2.3 the notion $A^i_{t,0}$ of the ancestor at time $t$ in the past of the individual $i$ which lives at time $s$ and the notion $\pi_j^s(t)$ of the set of all descendants which are alive at time $s$ of the individual which lived in the past at time $t$. Put $s = 0$ and set

$$C_t := \{\pi^0_j(t) : j \in \mathcal{I}, \pi_j(t) \neq \emptyset\} \in \mathfrak{P}^\mathcal{I}. \hfill (2.43)$$

Notice that if $\pi \in C_t$ then $A^i_{t,0} = A^i_{t,0}$ for all $i, i' \in \pi$. Write therefore $A^\pi_{t,0}$ for common ancestor of all individuals in $\pi$ at time $t$ in the past, and put

$$L_t(\pi) := L_t^{A^\pi_{t,0}}. \hfill (2.44)$$

Then the process $(C_t, L_t)_{t \geq 0}$ is the spatial $\mathcal{I}$-coalescent. Notice that the càdlàg path property follows immediately from the choice of topology.

**Proof of Proposition 2.2** The proof of Assertion (a) is obvious and the proof of Part (b) will be given with the look-down process we define next.

(b) The uniqueness of the process is a direct consequence of the fact that all finite sub-coalescents are uniquely determined and hence if the desired object exists it must be unique.

In order to get the existence of the process starting from a state containing countably many individuals, we use once more the look-down construction.

Recall Lemma 2.1 the notion $A^i_{t,0}$ of the ancestor at time $t$ back in the past of the individual $i$ which lives at time $s$.

We here let for each $t \geq 0$ and $j \in \mathcal{I}$,

$$\hat{\pi}_{(j,t)} := \{i \in \mathcal{I} : \exists s < t \text{ such that } A^j_{s,t} = i\},$$

and denote by

$$u^j_t(i) := \inf\{s \leq t : A^j_{s,t} = j\}$$

the birth time of the descendent $i \in \hat{\pi}_{(j,t)}$ of individual $j$ which lives at time $s$, and put

$$C_t := \{(i, u^j_t(i)) ; i \in \hat{\pi}_{(j,t)} ; j \in \mathcal{I}\}.$$
3 Main results

We study the asymptotic behavior of the spatial coalescent with initial configurations concentrated on bounded regions as the region and the time of observation both become large. Our parameter tending to infinity will be \( t \), size in the geographic space will be measured on the scale \( t^{\alpha/2} \) and the time at which we observe the process is on the scale \( t^{\beta} \) with \( \alpha \) and \( \beta \) being the corresponding macroscopic space parameter and time parameter, respectively. More precisely, set for \( \alpha \in (0,1] \) and \( t \geq 0 \),

\[
\Lambda^{\alpha,t} := \left[-t^{\frac{2\alpha}{2}}, t^{\frac{2\alpha}{2}}\right]^2 \cap \mathbb{Z}^2,
\]

(3.1)

to define the region where all the individuals will be placed initially and then observe this process at time \( t^{\beta} \). Note that we are interested in \( \Lambda^{\alpha,t} \uparrow \mathbb{Z}^2 \) by letting \( t \to \infty \).

We consider three settings, (1) the spatial coalescent (without rebirth) as process in the macroscopic time parameter \( \beta \) for fixed space parameter \( \alpha \), (2) the spatial coalescent as process in the macroscopic space parameter \( \alpha \) for fixed time parameter \( \beta \), and (3) the spatial coalescent with rebirth. In all settings we state that certain functionals of the spatial coalescent started from a configuration which contains particles at each site of \( \Lambda^{\alpha,t} \), and which is observed at times \( t^{\beta} \), for a \( \beta \geq \alpha \), converge to corresponding functionals of the Kingman coalescent with or without rebirth.
3.1 The spatial coalescent in the macroscopic time parameter

We are now in our setting (1) and we consider the various functionals of the coalescent in different subsections containing each a theorem.

3.1.1 The number of partition elements as a process indexed by the time parameter

Recall from Definition 2.2 the spatial coalescent \((C, L)\) on \(\mathbb{Z}^2\), and let \(K\) be the Kingman coalescent. Denote by \(C_{\alpha, t, \rho}\) \((3.2)\), \(\alpha \in (0, 1]\) and \(t \geq 0\), the spatial coalescent that starts in a Poisson configuration with either intensity \(\rho \in (0, \infty)\) or with intensity \(\rho = \infty\), i.e. with initially countable infinitely many particles at each site of \(\Lambda_{\alpha, t}\). We refer to these processes as to the \(\alpha\)-coalescent.

Remark 3.1. The case \(\rho < \infty\) is used in the study of the so-called interacting Moran models, while the case \(\rho = \infty\) is needed to analyze its diffusion limit, the so-called Fisher-Wright diffusions, or in the setting of infinitely many types, the so-called interacting Fleming-Viot processes. See [7] and [11] for more on these processes.

The following result states the convergence of the number of partition elements of the \(\alpha\)-coalescent observed at time \(t\) in the space \(D((\alpha', \infty), N)\) of càdlàg functions on \([\alpha', \infty)\) with values in \(N\), equipped with the Skorohod topology, where \(\alpha' > \alpha\). Here and in the remainder of the paper, for any (marked) partition \(\mathcal{P}\), we denote by \(\#\mathcal{P}\) the number of equivalence classes in \(\mathcal{P}\).

Theorem 1 (Number of partition elements as processes in \(\beta; \rho < \infty\)). Fix \(0 < \rho < \infty\) and consider the spatial coalescent and the Kingman coalescent for the same coalescence parameter \(\gamma > 0\). Then for all \(\alpha' > \alpha > 0\),

\[
\mathcal{L}[(\#C_{\alpha, t, \rho})_{\beta \in [\alpha', \infty)}] \xrightarrow{t \to \infty} \mathcal{L}[(\#K_{\log(\beta/\alpha)})_{\beta \in [\alpha', \infty)}].
\]

If \((C_0, L_0) \in \hat{\Pi}^{\mathbb{Z}^2}\) is such that in addition to (2.15) the following generalization holds:

\[
\sup_{z \in \mathbb{Z}^2} E\left[\#\{\pi \in C_0, L_0(\pi) = z\}\right] < \infty,
\]

and

\[
\#\{\pi \in C_0, L_0(\pi) \in \Lambda^{1, t}\} \to \infty \text{ in probability as } t \to \infty
\]

then (3.3) holds.

More generally a careful reader will note that the r.h.s. in (3.3) does not involve the parameter \(\rho\) and hence the scaling limit does not depend on \(\rho\). Indeed the more general statement shows that there is very little dependence between the initial state and the scaling limit.

3.1.2 The number of partition elements as a process indexed by the time parameter; infinite intensity

We next turn to \(\rho = \infty\). This case arises if one studies the genealogies in a model corresponding to interacting measure-valued Fleming-Viot diffusions. These models are limits of the spatial Moran model as the number of individual per site tends to \(\infty\). The genealogy of the limiting model can be represented by the spatial coalescent starting with countable many particles at each site, see [17].

In this situation the total number of initial individuals (partition elements) does not come down from infinity in positive time (compare [3]) since partition elements can escape into empty space. However we will show that the fraction of the partition elements which do escape quickly is small and its relative frequency in the total population is in fact 0 and therefore they can be neglected.
Remark 3.2. The frequency of a certain property in the population is here defined by taking the "n-smallest" in the order individuals and counting how many of them have the property in question. Then normalizing by $n$ and letting $n \to \infty$ gives the frequency of the property. The limit exists using de Finetti if our property is a function of the individual which generates an exchangeable array if we observe occurrence or non-occurrence of the property.

To make our approach to the case $\rho = +\infty$ precise, assume we are given a realization of $(C_s, L_s)_{s \geq 0}$ with $C_0 := \{(i); \ i \in I\}$ which is constructed from collections of independent random walks for migration $(L_s^i)_{s \geq 0}, i \in I$ and Poisson point processes $(M_{i,j}; i < j)$. We construct now an increasing collection of sub-coalescents of this spatial coalescent, namely we remove every individual in the original configuration for which $L_{i,j}$ jumps before a given time $\delta > 0$. Then we start the process in this new sub-configuration. This gives the sub-coalescent of $(C_{\alpha,t,\infty}^s)_{s \geq 0}$ denoted by

$$(C_{\alpha,t,\infty,\delta}^s)_{s \geq 0}. \quad (3.6)$$

Note that $C_{\alpha,t,\infty,\delta}^s \uparrow C_{\alpha,t,\infty}^s$ a.s. $\delta \downarrow 0, s \geq 0$ in our topology. The following result is the analogue of Theorem 1 for $\rho = \infty$.

Theorem 2 (Number of partition elements as processes in $\beta; \rho = \infty$). Let $\rho = \infty$, and $\delta > 0$. Then for all $\alpha' > \alpha > 0$,

$$\mathcal{L}((\#C_{\alpha,t,\infty,\delta}^s)_{\beta \in [\alpha', \infty)}) \Rightarrow \mathcal{L}((\#K_{\log(\beta/\alpha)})_{\beta \in [\alpha', \infty]}). \quad (3.7)$$

Remark 3.3. Notice the following:

(i) The right hand side of (3.7) does not depend on $\delta > 0$.

(ii) The frequency of the individuals in $C_{\alpha,\infty}^s$ that are not contained in $C_{\alpha,\infty,\delta}^s$ tends to zero, as $\delta \to 0$. Hence the theorem describes the behavior of the coalescent’s initial population of individuals (partition elements) with exception of a subset of frequency 0.

Remark 3.4. Proving results for the system with the exception of a set of frequency 0 of initial individuals important if one anticipates describing the genealogy of the Fleming-Viot process by a weighted $\mathbb{R}$-tree since then one gets convergence in the canonical Gromov-weak topology, as in [22].

3.1.3 The number of partition elements as a process indexed by the time parameter; refinement

The next goal is to extend the results in Theorems 1 and 2 to the case where $\alpha' = \alpha$. Let $\bar{N}$ be equipped with the discrete topology and denote by

$$\bar{N} := \mathbb{N} \cup \{\infty\} \quad (3.8)$$

its one point compactification. This means that a sequence $(n_k)_{k \in \mathbb{N}}$ with values in $\bar{N}$ converges in $\bar{N}$ if either $(n_k)_{k \in \mathbb{N}}$ is a convergent sequence in $\mathbb{N}$, or $(n_k)_{k \to \infty}$ diverges to infinity.

Now we can consider the processes

$$(\#C_{\beta}^s, \rho)_{\beta \geq \alpha}, (\#C_{\beta}^{s,\infty,\delta})_{\beta \geq \alpha}, \text{ and } (\#K_{\log(\beta/\alpha)})_{\beta \geq \alpha} \quad (3.9)$$

in the Skorokhod space $D([\alpha, \infty), \bar{N})$. For brevity, and in mind of future applications (see [20] [21]), we will consider only particular initial configurations.
**Theorem 3** (Convergence to the entrance law). Fix $\alpha > 0$.

(i) Assume that the initial configuration is either a Poisson process with intensity $\rho$, or a Bernoulli field with success probability $p \in (0, 1]$, for both choices we write $(C^{\alpha,t}_s, L^{\alpha,t}_s)_{s \geq 0}$ for the corresponding coalescent. Then

\[
\mathcal{L}\left([\#C^{\alpha,t}_s]_{s \in [\alpha, \infty)}\right) \xrightarrow{t \to \infty} \mathcal{L}\left([\#K_{\log(\beta/\alpha)}]_{\beta \in [\alpha, \infty)}\right) .
\]  

(ii) For each $\delta > 0$,

\[
\mathcal{L}\left([\#C^{\alpha,t,\infty,\delta}_s]_{s \in [\alpha, \infty)}\right) \xrightarrow{t \to \infty} \mathcal{L}\left([\#K_{\log(\beta/\alpha)}]_{\beta \in [\alpha, \infty)}\right) .
\]

### 3.2 Spatial coalescent as a function of macroscopic spatial parameter $\alpha$

We are now in our setting 2 and now the coalescent with rebirth occurs as limit object. Fix $\alpha \in (0, 1]$. Consider $C^{\alpha,t}$ the spatial coalescent on $\mathbb{Z}^2$ but restricted to individuals initially in $\Lambda^{\alpha,t}$. Let $I^{\alpha,t}$ be the set of individuals initially placed in $\Lambda^{\alpha,t}$. Then we can consider for every $t$ the collection of sub-coalescents

\[
\rho_{I^\alpha} \diamond C_{1,t}^{\alpha} .
\]

Notice (equality in distribution)

\[
C^{\alpha,t} \overset{d}{=} \rho_{I^\alpha} \diamond C_{1,t}^{\alpha} .
\]

and, of course, in (3.12) the objects for different $\alpha$ all live on one probability space and they are coupled as sub-coalescents of $C_{1,t}^{\alpha}$. Next we give a limiting object for partition element numbers. Fix $0 \leq \alpha_l < \alpha_u \leq 1$.

Recall from Definition 2.3 the Kingman-type coalescent $K^{\text{birth}}$ with rebirth during the time interval $[\log \alpha_l, \log \alpha_u]$ only. We start the Kingman coalescent at time $\log \alpha$ (with $I = \mathbb{N}$) and we are interested in the latter, evaluated at time 0.

**Remark 3.5.** Recall the order relation (2.23) that was used in the construction of a particular realization of the process $K^{\text{birth}}$. Moreover, one could naturally order partition elements within a partition according to their leading indices. In the Definition (3.15) below, for reasons that will become apparent later, we are introducing implicitly a “reordering according to age”. Note that, formally speaking, it is not a priori clear that the earliest born element of a partition element exists.

Recall from (2.24) the label $\kappa(\pi)$ of a partition element $\pi$ and from (2.21) the projection maps which send the individual $(i, t)$ to its index and birth time. For $\alpha \in [\alpha_l, \alpha_u]$, define:

\[
N_\alpha := \#\{ \pi \in K^{\text{birth}}_0[\log \alpha_l, \log \alpha_u] : \text{there exists } (s, i) \in \pi \text{ such that } s \leq \log(\alpha) \} .
\]

and refer it to as the number of partition elements of $K^{\text{birth}}_0[\log \alpha_l, \log \alpha_u]$ that are born before time $\log \alpha$.

The following result describes the asymptotic joint law of the sizes of sub-coalescents $(\#\rho_{I^\alpha} C_{1,t}^{\alpha})_{\alpha \in [\alpha_l, \alpha_u]}$ observed at time $t$, as $t \to \infty$. 

Theorem 4 (Convergence as processes in $\alpha$). Fix $0 \leq \alpha_l < \alpha_u < 1$.

(i) For all $\rho \in [0, \infty)$,
\[
\mathcal{L} \left( \# \rho_{\mathbb{Z}^2} \circ C_{t,\rho}^{1,\alpha,\rho} \right)_{\alpha \in [\alpha_l, \alpha_u]} \Rightarrow \mathcal{L} \left( (N_\alpha)_{\alpha \in [\alpha_l, \alpha_u]} \right).
\] (3.16)

(ii) For all $\delta > 0$,
\[
\mathcal{L} \left( \# \rho_{\mathbb{Z}^2} \circ C_{t,\rho}^{1,\alpha,\rho,\delta} \right)_{\alpha \in [\alpha_l, \alpha_u]} \Rightarrow \mathcal{L} \left( (N_\alpha)_{\alpha \in [\alpha_l, \alpha_u]} \right).
\] (3.17)

Remark 3.6. Notice that since $N_\alpha \to \infty$ as $\alpha \to 1$, and $\# C_{t,\rho}^{1,\alpha,\rho} \to \infty$ as $t \to \infty$, the result holds also for $\alpha_u = 1$. However, in order to rigorously include $\alpha_u = 1$ in the statement we would again have to consider the one point compactification of $\overline{\mathbb{N}}$ and apply similar techniques as in the proof of Theorem 3. \qed

3.3 Rescaling the spatial coalescent with rebirth

We are now in the setting 3. Recall from Definition 2.34 the spatial coalescent $(C^\text{birth}, L^\text{birth})$ with rebirth. Fix $\alpha \in (0,1)$ and $t > 1$. At time $t$ we observe the spatial coalescent with rebirth which started in Poisson configuration on $\mathbb{Z}^2$ with intensity $\rho \in (0, \infty]$ at time 0. In particular, even if $\rho < \infty$, the total number of initial partition elements is infinity. Note that if $\rho < \infty$, all the partition elements which “die” due to coalescence get replaced. Hence during $[0, \infty)$ the configuration of locations on $\mathbb{Z}^2$ of partition elements remains Poisson.

Observe in the spatial coalescent with rebirth at a late time $t$ the partition elements which are observed in a box $\Lambda^{\alpha,t}$ at some times $s_1, \ldots, s_m \leq t$, where $m \in \mathbb{N}$, which forms a sub-coalescent. How many partition elements has this sub-coalescent currently in the limit as $t \to \infty$? This question is also of interest since this sub-coalescent arises as a dual object in resampling models if one considers the configuration in macroscopic time-space windows, and is explained in Remark 3.7 below. In view of the previous scaling results we look at the system in times $t^u$, with $u \in (0,1)$ the macroscopic time parameter and then let $t \to \infty$. Since the times $t^u, t^u'$ for $u' \neq u$ separate, we cannot use a continuous macroscopic time parameter in our analysis. We have to discretize.

For $m \in \mathbb{N}$, fix parameters $\alpha < u_1 < u_2 < \ldots < u_m < 1$. We are now interested in the asymptotic behavior, as $t \to \infty$, of the number of those partition elements observed in the population at time $t$ which were located in $\Lambda^{\alpha,t}$ at an $m$-tuple of time points of the form $t^u$, where $u \in \{u_1, \ldots, u_m\}$. For $u \in (\alpha, 1]$, we therefore put
\[
N_{u,\alpha,t,u}^{\alpha,t,u} := \# \{ \pi \in C^\text{birth}_t : \exists i \in \{1, \ldots, m\} \text{ s.t. } u_i \leq u, \text{pr}(\kappa(\pi)) \leq t^u, L^\text{birth}_t(\pi) \in \Lambda^{\alpha,t} \}.
\] (3.18)

The dependence on $u_1, \ldots, u_m$ in the above definition is recorded by the third superscript $\vec{u}$. We chose to form the vector $\vec{u} = (u_1, \ldots, u_m)$ out of notational convenience. Similarly we will write below $\log \vec{u}$ for the vector $(\log u_1, \ldots, \log u_m)$. We keep the dependence on $\vec{u}$ in mind, yet we omit it sometimes from the notation by setting $N_{\alpha,t,u}^{\alpha,t,u} \equiv N_{\alpha,t,\vec{u}}^{\alpha,t,\vec{u}}$. The parameter $u$ (in the subscript) will play the rôle of the new time index running in $(\alpha, 1]$.

As before, if we consider $\rho = \infty$, we need to observe $\delta$-thinnings of our spatial coalescent with rebirth. We will denote the corresponding functionals by
\[
N_{\alpha,t,\infty,\delta}^{\alpha,t,\infty,\delta} = N_{\alpha,t,\vec{u},\infty,\delta}^{\alpha,t,\vec{u},\infty,\delta}.
\] (3.19)

In order to study the asymptotic behavior of a suitable rescaling of $N_{\alpha,t,u}^{\alpha,t,u}$, we introduce a limit object, which we call the family of merging coalescents, a collection of coalescents which start at specified times to interact by coalescence, and which we denote by
\[
(K_{s^{\text{mer,log} \vec{u}}})_{s \geq \log u_1}.
\]
HERE 1 < \log u_1 < \ldots < \log u_m < \infty \text{ is a given sequence of merging times, at which the “inter-coalescing” of partitions belonging to two or more different coalescents is enabled, as described precisely below. Note that the coalescent structure is of Kingman-type, that is, only pairs of partition elements (for which the coalescence is enabled) coalesce at a constant rate. The process } (K^\text{mer,log} \vec{u})_{s \geq \log u_1} \text{ is } \mathbb{N}^N \text{-valued and evolves informally as follows.}

We consider } m \text{ copies of the Kingman coalescent } \{K^i, i \in \{1, \ldots, m\}\}, \text{ where the } i^{th} \text{ copy is initially started in the configuration } \{\{km + i - 1\} : k \in \mathbb{N}\} \text{ and runs from time } \log u_1 \text{ until time } \log u_i \text{ independently from the others, but after time } \log u_i \text{ its partition elements coalesce mutually as well as with the partition elements of } \{K^j, j \in \{1, \ldots, i - 1\}\}.

A realization is constructed as follows. The family of merging coalescents process starts at time } \log u_1 \text{ in } \{\{n\} : n \in \mathbb{N}\}, \text{ and given a time } s \geq \log u_1 \text{ two partition elements } \pi_1, \pi_2 \in K^\text{mer,log} \vec{u} \text{ with } \kappa(\pi_1) = m k_l + n_l, \text{ for some } k_l \in \mathbb{Z} \text{ and } n_l \in \{0, 1, \ldots, m - 1\}, \text{ then we write for } n_l := [\kappa(\pi_1)]_{\mod m}, \text{ coalesce at rate } 1\{n_1 = n_2\} + 1\{n_1 \neq n_2\} s \geq \log u_{n_1} \lor \log u_{n_2}. \text{ Upon coalescing the new partition inherits, as usual, the smaller label where we define that for } \kappa(\pi_1), l = 1, 2, \text{ of the form (3.20)}

\[ \kappa(\pi_1) \leq \kappa(\pi_2) \text{ iff } n_1 \leq n_2 \text{ or } n_1 = n_2, k_1 \leq k_2. \] (3.21)

Note that } (K^\text{mer,log} \vec{u})_{s \geq \log u_1} \text{ can be coupled with the coalescent with rebirth on } \mathbb{Z}^2, \text{ where in the latter at each time countably many individuals are reborn (immigrate back into the system), so that both can be constructed in such a way that the number of partition elements in } (K^\text{mer,log} \vec{u})_{s \geq \log u_1} \text{ is almost surely smaller than the number of partition elements in the coalescent with rebirth. In particular, } (K^\text{mer,log} \vec{u})_{s \geq \log u_1} \text{ is well-defined, and its number of partition elements is finite at all times } s > \log u_1, \text{ almost surely.}

Put (note } 0 < \log(u_1/\alpha) < \ldots < \log(u_m/\alpha) < \log(1/\alpha):)

\[ N^\text{mer,log} \vec{u}/\alpha := \#\{\pi \in K^\text{mer,log}(\vec{u}/\alpha) : [\kappa(\pi)]_{\mod (m)} \leq i - 1\}. \] (3.22)

**Theorem 5 (Asymptotics of coalescent with rebirth).** Fix } 0 < \alpha < \beta < 1.

(a) If } \rho < \infty, \text{ then for all } m \in \mathbb{N} \text{ and } \alpha \leq u_1, \ldots, u_m \leq \beta, \text{ we have}

\[ \mathcal{L}\left[\left(N_{u_1}^{t,\rho}, \ldots, N_{u_m}^{t,\rho}\right)\right] \Rightarrow \mathcal{L}\left[\left(N^\text{mer,log} \vec{u}/\alpha, \ldots, N^\text{mer,log} \vec{u}/\alpha\right)\right] \text{ as } t \to \infty. \] (3.23)

(b) If } \rho = \infty, \text{ then for all } \delta > 0, m \in \mathbb{N} \text{ and } \alpha \leq u_1, \ldots, u_m \leq \beta, \text{ we have}

\[ \mathcal{L}\left[\left(N_{u_1}^{t,\infty,\delta}, \ldots, N_{u_m}^{t,\infty,\delta}\right)\right] \Rightarrow \mathcal{L}\left[\left(N^\text{mer,log} \vec{u}/\alpha, \ldots, N^\text{mer,log} \vec{u}/\alpha\right)\right] \text{ as } t \to \infty. \] (3.24)

**Remark 3.7 (Space-time cluster formation).** As already indicated, the spatial coalescent with rebirth describes the space-time genealogy of the interacting Moran models. To make this more precise, let us fix some large } t \text{ and introduce the reversed time } s^{-}\equiv s^{-}(s) := t - s. \text{ Then, provided that the original configuration of particles is Poisson, and that the particles evolve according to the enriched interacting Moran models in forward time (where the types that die due to resampling are kept as fossils), then their paths observed in reversed time evolve according to the spatial coalescent with rebirth. Moreover, a resampling event that occurs at time } t - s \text{ corresponds to a unique rebirth event occurring at time } s^{-}(s).

In this way, Theorem 5 plays an important rôle in the study of the space-time cluster formation of the interacting Moran models on } \mathbb{Z}^2. \text{ Namely, assume that at the initial time } 0 \text{ each individual (particle) carries its own type. The following questions arise naturally in this context: if we fix a time...}
t > 0 and a large window W of observation, how far back in time do we have to look so that most of the population present in W at time t has a single ancestor and hence carries a single type (color)?; how does this information change if the population is sampled at several time instances from the same window W?

The natural time scale for answering these questions is the logarithmic time scale. Fix \( \alpha \in (0, 1] \) and choose, out of convenience, the \( \alpha \)-box \( \Lambda^{t, \alpha} \) as the window of observation. Fix \( u \in (0, 1] \), and observe the subpopulation, located in the \( \alpha \)-box at time \( t \), during the time interval \([t - t^u, t]\). It turns out that for large \( t \) and \( u \leq \alpha \), with overwhelming probability we find in this subpopulation a certain non-trivial (\( \geq 2 \)) number of types at time \( t - t^u \), and a non-trivial number (\( \geq 2 \)) of these types are still visible in the \( \alpha \)-box at time \( t \). However, if \( u > \alpha \), the event that only one of the types observed at time \( t - t^u \) is visible in the \( \alpha \)-box at time \( t \), has positive probability, for \( t \) large. This is equivalent to saying that with positive probability, all the individuals observed in the \( \alpha \)-box at time \( t \) have a common ancestor among the particles observed at time \( t - t^u \), for \( t \) large. If this happens, we say that the age (on logarithmic scale) of our chosen subpopulation is at most \( u \), since one type is carried by a substantial fraction of the subpopulation, and its original carrier can therefore be considered as the ancestor.

More generally, fix \( m \geq 1 \) and \( u_1, \ldots, u_m \) such that \( 0 < u_1 < \ldots < u_m \leq 1 \). During the time interval \([t - t^{u_m}, t]\), consider the joint evolution of \( m \) different Moran model subpopulations where the “0th” subpopulation consists of particles present in the \( \alpha \)-box at time \( t \), and for \( i = 1, \ldots, m - 1 \), the \( i \)th subpopulation consists of particles present in the \( \alpha \)-box at time \( t - t^{u_i} \). By reversing time all the interesting information about their joint genealogy is expressed precisely in terms of the quantities \( (N_{t, \alpha}^{u_1, (1, \ldots, u_m)}, \ldots, N_{t, \alpha}^{u_m, (u_1, \ldots, u_m)}) \) as defined in (6.18). For example, the event on which the latter vector takes value \((1, \ldots, 1)\) is precisely the event that all the individuals (in \( m \) subpopulations combined) have a common ancestor at time \( t - t^{u_m} \).

Outline. The rest of the paper is organized as follows: In Section 4 we recall and extend some basic facts on coalescents on \( \mathbb{Z}^2 \), and in Sections 5 and 6 we provide the asymptotic analysis of coalescents which allows us to prove Theorems 1 and 2 in Section 7, and Theorems 4 and 5 in Section 8. Section 9 contains the proof of a moment estimate on the number of partition elements.

Result and Problem History. Here we give some information concerning the history of the problems treated in this paper. In the setting of instantaneous coalescence for simple random walks on \( \mathbb{Z}^2 \), i.e., two partition elements coalesce immediately when the hit the same site, Lemma 6.2 was proved in [9], and Proposition 6.1 in [4]. Propositions 5.1 and 6.2 and Lemma 7.1 are to the best of our knowledge novel in the setting of any spatial coalescent model on \( \mathbb{Z}^2 \). Due to the applications we have in mind (using duality with the IMM and IFWD) in the subsequent papers, we are primarily interested in the spatial (and delayed) coalescents, and therefore the results are phrased and proved in the current setting. However, it is important to note that the arguments, and therefore statements, in Section 3 remain to hold in the setting of [4] and [9].

4 Preliminaries

In this section we present several basic techniques on coalescents and present the key properties of random walks which we will need for our subsequent arguments. We first state in Subsection 4.1 some notational conventions which will be used throughout the rest of the paper. In Subsection 4.2 we recall a famous result by Erdős-Taylor which gives the asymptotics of the hitting time of a planar random walk. In Subsection 4.3 we state the asymptotic exchangeability for the spatial coalescent on \( \mathbb{Z}^2 \). In Subsection 4.4 we recall some consequences of monotonicity properties.
4 PRELIMINARIES

4.1 Notational conventions

In the rest of the paper we often use the following convention concerning notation.

- For functions $g, h : [0, \infty) \to \mathbb{R}$, we write $g(t) = O(h(t))$ or $g(t) = o(h(t))$ if and only if $\limsup_{t \to \infty} \frac{g(t)}{h(t)} < \infty$ or $\lim_{t \to \infty} \frac{g(t)}{h(t)} = 0$, respectively.
- For a set $A$, we denote by $A^c$ its complement (with respect to the natural superset, determined by the context).
- Recall $\mathbb{N}$ from §3.8, and denote for a finite or countable set $A$ by $\#A \in \mathbb{N}$ the number of elements in $A$.
- If $a, b \in \mathbb{R}$, let $a \wedge b$ denote the minimal, and $a \vee b$ the maximal element of $\{a, b\}$.
- Poisson($\rho$) random variable (or distribution) has intensity (rate, expectation) $\rho$.
- For a partition $\mathcal{P}$, recall that $\#\mathcal{P}$ denotes the number of partition elements of $\mathcal{P}$.
- If $\mathcal{P}$ is a partition then we write $i \sim_{\mathcal{P}} j$ if $i$ and $j$ belong to the same partition element of $\mathcal{P}$. If $(\mathcal{P}_t, t \geq 0)$ is a partition-valued process then $i \sim_{\mathcal{P}_t} j$ will be sometimes abbreviated as $i \sim^t j$.

4.2 Erdős-Taylor formula

Recall a well-known result by Erdős and Taylor [15] for planar random walks with finite variance: if $\tau$ is the first hitting time of the origin of a two-dimensional random walk, then

$$\lim_{t \to \infty} \mathbb{P}^{x^{\alpha/2}/2} \left\{ \tau > t^\beta \right\} = \frac{\alpha}{\beta} \wedge 1,$$

(4.1)

for all $\alpha, \beta \in [0, 1]$, and all $x \in \mathbb{R}^2 \setminus \{(0, 0)\}$ (see, for example, Proposition 1 in [9]). In particular, the right hand side of (4.1) does not depend on $x \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Due to this peculiar (specific to $d = 2$) property, the behavior of the spatial coalescent started in $\Lambda^{\alpha, t}$ and observed at time $t^\beta$, asymptotically as $t \to \infty$, depends only on the logarithmic scales $\alpha$ and $\beta$, while all the finer distinctions are washed out.

For $c \in (0, \infty)$, define

$$I_\alpha(c, t) := [A_\alpha^-(t), A_\alpha^+(t)] := [(c \log t)^{-1} \cdot t^{\frac{\alpha}{2}}, (c \log t) \cdot t^{\frac{\alpha}{2}}].$$

(4.2)

Say that a set of locations (marks) $\{x_1, \ldots, x_n\}$ is contained in $I_\alpha(c, t)$ if and only if $\|x_i - x_j\| \in I_\alpha(c, t)$, for all $1 \leq i < j \leq n$.

From (4.1) one sees immediately that if $\{x_1, \ldots, x_n\}$ is contained in $I_\alpha(c, t)$, then for the corresponding random walks $\{(X_i^j)_{t \geq 0}, j = 1, \ldots, n\}$ with $X_0^j := x_j$,

$$\mathbb{P}\{X_i^j \neq X_j^i, \forall i \neq j, \forall s \in [0, g(t)]\} \xrightarrow{t \to \infty} 1,$$

(4.3)

whenever $g(t)$ is a function satisfying $g(t) = O(t^{\alpha+\varepsilon})$ for all $\varepsilon > 0$.

4.3 Asymptotic exchangeability

In this subsection we perform some preliminary calculations implying “asymptotic exchangeability” that will be useful in the sequel. The main result is Proposition 4.1 below.

Let $\alpha \in (0, 1]$, and set

$$g_\alpha(t) := t^\alpha \log^3 t.$$  

(4.4)
Remark 4.1. In fact, any function $g_\alpha(t)$ with $t^\alpha \log^2 t = o(g_\alpha(t))$ could be used instead of $t^\alpha \log^3 t$.

For $k \in \mathbb{N}$, let $\zeta$ be a permutation on $\{1, \ldots, k\}$, given $\{x_1(t), \ldots, x_k(t)\} \subset \mathbb{Z}^2$, we denote by $(C_{s,t}^\zeta, L_{s,t}^\zeta)_{s \geq 0}$ the spatial coalescent that starts from $C_0^\zeta = \{(1), \ldots, (k)\}$, $L_0^\zeta(\{i\}) = x_i(t)$, $i = 1, \ldots, k$, and by $(C_{s,t}, L_{s,t})_{s \geq 0}$ the spatial coalescent that starts from $C_0^\zeta = C_0^\alpha$, $L_0(\{i\}) = x_i(t)$, $i = 1, \ldots, k$.

Proposition 4.1 (Asymptotic exchangeability for the spatial coalescent). Fix $\alpha \in (0,1]$ and $k \in \mathbb{N}$, and assume that $\{x_1(t), \ldots, x_k(t)\} \subset \mathbb{Z}^2$ is contained in $I_s(c,t)$. If the spatial coalescent $(C_{s,t}^\zeta, L_{s,t}^\zeta)_{s \geq 0}$ starts in the marked partition $\{\{(1), x_1(t)\}, \ldots, \{(k), x_k(t)\}\}$, then for all $M \in \mathcal{B}(D([0,\infty), \Pi^2))$,

$$
\lim_{t \to \infty} \left| \mathbb{P}\{(C_{s,t}^\zeta)_{s \geq g_\alpha(t)} \in M\} - \mathbb{P}\{(C_{s,t}^\zeta)_{s \geq g_\alpha(t)} \in M\} \right| = 0. \tag{4.5}
$$

We prepare the proof by stating the corresponding result for the underlying random walks.

Lemma 4.1 (Asymptotic exchangeability for random walks). Fix $\alpha \in (0,1]$, $k \in \mathbb{N}$, and let $\zeta$ be a permutation on $\{1, \ldots, k\}$. Let $(Y_s)_{s \geq 0}$ be the $2k \times 2$ dimensional random walk

$$
Y_s := (X_{1,s}^1, X_{1,s}^2, X_{2,s}^1, X_{2,s}^2, \ldots, X_{k,s}^1, X_{k,s}^2), \tag{4.6}
$$

where, for each $i \in \{1, \ldots, k\}$, $(X_i^s)_{s \geq 0} = (X_{1,s}^i, X_{2,s}^i)_{s \geq 0}$ is the two dimensional random walk with transition kernel $a(x,y)$, and the $k$ random walks are taken to be independent. Moreover, let

$$
(Y_s^\zeta)_{s \geq 0} := (X_{1,s}^{\zeta_1}, X_{2,s}^{\zeta_1}, X_{1,s}^{\zeta_2}, X_{2,s}^{\zeta_2}, \ldots, X_{k,s}^{\zeta_k}, X_{k,s}^{\zeta_k}). \tag{4.7}
$$

Then for all $M \in \mathcal{B}(D([0,\infty), \Pi^{2k}))$,

$$
\left| \mathbb{P}\{(Y_s)_{s \geq g_\alpha(t)} \in M\} - \mathbb{P}\{(Y_s^\zeta)_{s \geq g_\alpha(t)} \in M\} \right| \to 0. \tag{4.8}
$$

Proof. The proof relies on a consequence of the local central limit theorem for continuous time random walks that we recall next: if $(Z_s)_{s \geq 0}$ is a random walk in $\mathbb{Z}^d$ (here no moment assumption is needed), then there exists a finite constant $c$ (see, for example for our setting [28]) that depends on the dimension and the transition mechanism only, such that for all $y \in \mathbb{Z}^2$,

$$
\sum_{z \in \mathbb{Z}^2} \left| \mathbb{P}(Z_s = z|Z_0 = 0) - \mathbb{P}(Z_s = z|Z_0 = y) \right| \leq c \|y\| s^{1/2}. \tag{4.9}
$$

We will apply the above difference estimate (4.9) to $(Y_s)_{s \geq 0}$ and $(Y_s^\zeta)_{s \geq 0}$.

Let $M \in \mathcal{B}(D([0,\infty), \Pi^{2k}))$. For each $2k$-tuple $(z_1, z_2, z_3, \ldots, z_k) \in \mathbb{Z}^{2k}$, set

$$
q(z_1, z_2, z_3, \ldots, z_k) := \mathbb{P}\{(Y_s)_{s \geq 0} \in M|Y_0 = (z_1, z_2, z_3, \ldots, z_k, z_1, z_2)\}. \tag{4.10}
$$

Denote by $B(r)$ the ball in $\mathbb{R}^2$ of radius $r$ centered at 0. Suppose $x_1^1, \ldots, x_k^k \in \mathbb{Z}^2 \cap B(c \log(t) t^{\alpha/2})$, and let $X^1, \ldots, X^k$ be $k$ independent random walks with transition kernel $a(\cdot, \cdot)$ started at locations $x_1^1, x_2^2, \ldots, x_k^k$, respectively. Let $Y$ be the walk formed as in (4.7) using $X^1, X^2, \ldots, X^k$ as input, instead. Then clearly $Y$ and $Y^\zeta$ have the same transition mechanism, and the difference $Y_0 - Y_0^\zeta$ of their starting locations is a vector with norm bounded by $O(t^{\alpha/2} \log t)$. Therefore, by (4.9), for all $u \in [0,1]$,

$$
\left| \mathbb{P}\{q(Y_{g_\alpha(t)}) \geq u\} - \mathbb{P}\{q(Y_{g_\alpha(t)}^\zeta) \geq u\} \right| \leq \frac{O(t^{\alpha/2} \log t)}{t^{\alpha/2} \log t^{1/2}} \to 0. \tag{4.11}
$$
That is, the $[0, 1]$-valued random variables $q(Y_{g_a}(t))$ and $q(Y_{g_a}^\zeta(t))$ are asymptotically equal in distribution. In particular,

$$\left| \mathbb{E}[q(Y_{g_a}(t))] - \mathbb{E}[q(Y_{g_a}^\zeta(t))] \right| = \left| \mathbb{P}\{(Y_s)_{s \geq g_a(t)} \in M\} - \mathbb{P}\{(Y_s^\zeta)_{s \geq g_a(t)} \in M\} \right| \to 0,$$

and we are done. 

**Proof of Proposition 4.1** Let the $2k$-dimensional processes $Y^{sc}$ and $Y^{sc, \zeta}$ ("sc" stands for semi-coalescent) be formed as in (4.6) and (4.7), however the input random processes $X^1, \ldots, X^k$ are changed so that $X's$ are independent continuous-time random walks with kernel $a(\cdot, \cdot)$ until time $g_a(t)$, and after time $g_a(t)$ their joint evolution is the evolution of the location process of the spatial coalescent with initial configuration $(X^1_{g_a(t)}, \ldots, X^k_{g_a(t)})$. Moreover, let $Y^c$ and $Y^{c, \zeta}$ ("c" stands for coalescent) be the $2k$-dimensional processes whose joint evolution is the evolution of the location process of the spatial coalescent with initial configuration $(X^1_{g_a(t)}, \ldots, X^k_{g_a(t)})$.

It is obvious how to construct couplings $(Y^{sc}, Y^c)$ and $(Y^{sc, \zeta}, Y^{c, \zeta})$, so that on the event $\{\text{no coalescence up to time } g_a(t)\}$ the two processes, the coalescent and the corresponding semi-coalescent, in both couplings agree for all times. Hence,

$$\mathbb{P}\{(Y^c_s)_{s \geq g_a(t)} \in M\} = \mathbb{P}\{(Y^{c, \zeta}_s)_{s \geq g_a(t)} \in M\}$$

The claim follows immediately from the previous observations and from the fact

$$\mathbb{P}\{\text{coalescence occurs before time } g_a(t)\} \to 0,$$

which is a direct consequence of (4.3).

**4.4 Monotonicity and consequences**

Recall the set of marked partitions $\Pi^{\mathbb{Z}^2}$ from (2.12). It is convenient to introduce a partial order "$\leq"$ on $\Pi^{\mathbb{Z}^2}$. Let for $\mathcal{P}_1, \mathcal{P}_2 \in \Pi^{\mathbb{Z}^2}$,

$$\mathcal{P}_1 \leq \mathcal{P}_2$$

iff for each $g \in \mathbb{Z}^2$ the number of partition elements in $\mathcal{P}_1$ with mark $g$ is bounded above by the number of partition elements in $\mathcal{P}_2$ with mark $g$. For brevity reasons, we will often omit from (1.15) the dependence on the location processes when evident from the context, so we will write

$$C_1 \leq C_2$$

to mean $(C_1, L_1) \leq (C_2, L_2)$.

**Remark 4.2.** Note that if $\mathcal{P}_1 \leq \mathcal{P}_2$, one can easily construct a coupling $((C^1_s, L^1_s), (C^2_s, L^2_s))_{s \geq 0}$ of the spatial coalescents where $(C^1_0, L^1_0) = \mathcal{P}_j$, $j = 1, 2$, such that $C^1_s \leq C^2_s$, for all $s \geq 0$, almost surely.
Suppose that \( f : \Pi^{I, Z^2} \to \mathbb{R} \) is non-decreasing, and let \( g : [0, \infty) \to (0, \infty) \). For \( a, b \in [-\infty, \infty] \) consider asymptotic behavior(s) of the type

\[
\limsup (\liminf)_{t \to \infty} \frac{f(C_t, L_t)}{g(t)} = a, \quad \limsup (\liminf)_{t \to \infty} \frac{E(f(C_t, L_t))}{g(t)} = b. \tag{4.17}
\]

An important observation is the next easy consequence of monotonicity and Remark 4.2. Namely, if any of the four types of asymptotic behavior (4.17) holds for both spatial coalescents \((C^j_t, t \geq 0)\), \(j = 1, 3\), and if

\[
P\{C^1_0 \leq C^2_0 \leq C^3_0\} = 1, \tag{4.18}
\]

then the same asymptotic behavior holds for the spatial coalescent \((C^2_t, t \geq 0)\).

Moreover, let \(A \subseteq \mathbb{R}\), and suppose we are given three coalescent families

\[
\{(C^j_s, \alpha)_{s \geq 0}; \alpha \in A, j \in \{1, 2, 3\}\}, \tag{4.19}
\]

with initial states such that

\[
P\{C^{1, \alpha}_0 \leq C^{2, \alpha}_0 \leq C^{3, \alpha}_0, \forall \alpha \in A\} = 1. \tag{4.20}
\]

In addition, assume that càdlàg path such that

\[
\lim_{\alpha \to \alpha_0} (C^{1, \alpha}_s)_{s \geq 0} = \lim_{\alpha \to \alpha_0} (C^{3, \alpha}_s)_{s \geq 0}, \tag{4.21}
\]

where the above convergence is weak convergence on \(D([0, \infty), \Pi^Z)\) equipped with the Skorokhod topology.

**Lemma 4.2.** If (4.20) and (4.21) hold, then \((C^2_s, \alpha)_{s \geq 0}\) also converges in law as \(\alpha \to \alpha_0\), and

\[
\lim_{\alpha \to \alpha_0} (C^{2, \alpha}_s)_{s \geq 0} = \lim_{\alpha \to \alpha_0} (C^{1, \alpha}_s)_{s \geq 0}. \tag{4.22}
\]

The next result will, together with the above consequences of monotonicity, eventually be used for deducing various asymptotics for the spatial coalescent started from infinite configurations, given the results for the spatial coalescents started from finite configurations.

Let \((K_s)_{s \geq 0}\) be the Kingman coalescent.

**Lemma 4.3.** For each \(\delta > 0\) there exists \(\rho = \rho(\delta) \in (0, \infty)\) such that

\[
P\{\# K_\delta \geq n\} \leq P\{X_\rho \geq n\}, \tag{4.23}
\]

where \(X_\rho \overset{d}{=} 1 + \text{Poisson}(\rho)\). That is, \(P\{X_\rho = k\} = e^{-\rho} \rho^{k-1} / (k-1)!\), for all \(k \geq 1\).

**Remark 4.3.** The shift by one unit is necessary here since \(P(K_\delta \geq 1) = 1\). \(\square\)

**Proof.** Let \(\{\Upsilon_n; n \geq 1\}\) be the family of independent exponential random variables where \(\Upsilon_n\) has rate \(n(n+1)/2\). Then by construction of Kingman’s coalescent (see, for example, [24, 2]),

\[
P\{\# K_\delta > n\} = P\{\sum_{k \geq n} \Upsilon_k > \delta\} \leq e^{-\theta \delta} \mathbb{E}[e^{\theta \sum_{k \geq n} \Upsilon_k}], \tag{4.24}
\]

for all \(\theta \in \mathbb{R}\). Assume that \(\theta < n(n+1)/2\), and consequently that \(\mathbb{E}[e^{\theta \sum_{k \geq n} \Upsilon_k}] < \infty\).
Since
\[ E[e^\theta \sum_{k \geq n} \mathbb{Y}_k] = \prod_{k=n}^{\infty} \frac{(k+1)k}{2} - \theta \]
\[ = \exp \left[ \sum_{k=n}^{\infty} \log \left( 1 + \frac{\theta}{(k+1)k} - \theta \right) \right] \]
\[ \leq \exp \left[ \sum_{k=n}^{\infty} \left( \frac{\theta}{(k+1)k} - \theta + O\left( \frac{\theta^2}{(k+1)k} - \theta \right)^2 \right) \right], \]
by (4.24)
\[ P\{\# K_\delta > n\} \leq \exp \left[ -\delta \theta + \sum_{k=n}^{\infty} \frac{\theta}{(k+1)k} - \theta \right]. \] (4.26)

Plugging in, for example, \( \theta = n \log^2 n \) gives
\[ P\{\# K_\delta > n\} = O(e^{-\delta n \log n}), \] (4.27)
which is of a smaller order than
\[ P\{\mathrm{Poisson}(\rho) + 1 > n\} \approx C(\rho) e^{-n \log n + O(1)} \], (4.28)
for all large \( n \), where \( O(1) \) indicates a term that stays bounded as \( n \to \infty \). Since the sum of independent Poisson random variables is another Poisson random variable, we can choose \( \rho \) appropriately large so that \( P\{\# K_\delta > n\} \leq P\{\mathrm{Poisson}(\rho) > n - 1\} \), for all \( n \geq 1 \). \( \square \)

## 5 Asymptotics for sparse particles

Fix throughout this section \( \alpha \in (0, 1] \). Our goal in this section is to analyze the behavior of a finite coalescent with particles spaced at distance \( t^{\alpha/2} \) and observed at time \( t^{\beta} \), \( \beta > \alpha \), as \( t \to \infty \).

Recall the instantaneous coalescent that corresponds to the spatial coalescent with resampling rate \( \gamma = \infty \). In our setting \( \gamma \in (0, \infty) \) is fixed. Nevertheless, we still can rely on the “loss of the spatial structure” property of the coalescent on time scales \( t^{\beta} \) for the instantaneous coalescent with partition elements situated initially at mutual distances of order \( t^{\alpha/2} \) that was exploited in \( [9] \).

Recall \( \Lambda^{\alpha,t} \) from (3.1). We denote by
\[ (C^{\alpha}_s)_{s \geq 0}, \quad \text{and} \quad (IC^{\alpha}_s)_{s \geq 0} \] (5.1)
the spatial coalescent and the instantaneous coalescent starting from initial configuration \( C^0 \) with marks contained in \( \Lambda^{\alpha,t} \). Notice that \( t \) is suppressed from the notation, but this should not cause confusion.

There are classical results on \( (IC^{\alpha}_s)_{s \geq 0} \) with initially \( N \) individuals spread out in \( \Lambda^{\alpha,t} \), and observed at time \( t^{\beta} \), where \( \beta > \alpha \), which we wish to recall first. Let \( c > 0 \) and recall \( I_\alpha(c,t) \) from (4.2).

The following result was proved in a beautiful paper by Cox and Griffeath \( [9] \) under the additional assumption that the underlying random walks are simple random walks: for fixed \( N \in \mathbb{N} \), the initial locations \( \{x_1(t), \ldots, x_N(t)\} \) contained in \( I_\alpha(c,t) \) and for each \( \beta > \alpha \),
\[ \mathcal{L}\left[\# IC^{\alpha}_{t^{\beta}}\right] \overset{i \to \infty}{\Rightarrow} \mathcal{L}\left[\# K_{t^{\alpha/2}}\right]. \] (5.2)

We next consider the spatial (delayed) coalescent, and show the stronger form of weak convergence in two ways: (i) in the sense of path-valued random variables where \( \beta \) is the “time”-parameter, and (ii) accounting for the partition structure. Note that the weak convergence is done in the sense of the discrete topology.
Remark 5.1. In the setting of [9] the walks are simple symmetric walks, but the proof of the cor-
responding lemma is more general, depending solely on the uniform bound \( \tau \) for all \( \alpha \in \{0, 1\} \) random walks with \( X_t \) and assume that \( \{x_1(t), \ldots, x_N(t)\} \subset \mathbb{Z}^2 \) contained in \( I_\alpha(c, t) \). Let then \( \text{Lemma 5.2} \)

\( \text{Proposition 5.1} \)

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Then clearly \( C_{t, \alpha} \) and \( \alpha \in \{0, 1\} \). Let the spatial coalescent \( (C^\alpha_{s})_{s \geq 0} \) start in \( \{(1), x_1(t)\}, \ldots, \{(N), x_N(t)\} \). Then

\[ \mathcal{L}[(C^\alpha_{t, \beta})_{\beta \in [\alpha, \infty)}] \rightarrow \mathcal{L}[(K^N_{t, \beta})_{\beta \in [\alpha, \infty)}], \quad \forall \alpha \geq 0, \quad \forall t \rightarrow \infty \]  

(5.3)

where \( (K^N_{t})_{t \geq 0} \) is the Kingman coalescent started in \( \{(1), \ldots, (N)\} \).

The proof of this result is given in the next two subsections.

5.1 Convergence of marginal distributions

A key element of the proof is the following fact which we state for future reference.

\( \text{Lemma 5.1 (Lemma 1 from [10])} \). Fix \( \alpha_0 > 0, \) and \( \alpha > 0 \). Let \( \{(X^i_s)_{s \geq 0}; i = 1, \ldots, 4\} \) be a family of independent random walks with \( X^i_0 = x_i \), for \( i = 1, \ldots, 4 \). Then uniformly in \( \alpha \in [\alpha_0, \infty) \) and \( \{x_1, \ldots, x_4\} \subset \mathbb{Z}^2 \) contained in \( I_\alpha(c, t) \), we have

\[ \int_{t}^{\infty} ds \mathbb{P}([X^1_s = X^2_s] \cap [X^1_s, X^3_s, X^4_s \text{ not contained in } I_1(4c, s)]) \rightarrow 0. \quad \forall \alpha \geq 0, \quad \forall t \rightarrow \infty \]  

(5.4)

**Remark 5.1.** In the setting of [9] the walks are simple symmetric walks, but the proof of the corresponding lemma is more general, depending solely on the uniform bound

\[ \mathbb{P}(X_s = y) \leq \frac{c}{s}, \quad \forall y \in \mathbb{Z}^2, \quad s \geq 0 \]  

(5.5)

for all \( y \in \mathbb{Z}^2 \), and \( s \geq 0 \). It therefore applies to our setting (see [11]).

The first major step to prove Proposition 5.1 is to show:

\( \text{Lemma 5.2 (Finite sparse coalescents: convergence of marginals)} \). Fix \( N \in \mathbb{N}, \alpha \in (0, 1] \) and \( c > 0 \), and assume that \( \{x_1(t), \ldots, x_N(t)\} \subset \mathbb{Z}^2 \) is contained in \( I_\alpha(c, t) \). Let the spatial coalescent \( (C^\alpha_{s})_{s \geq 0} \) start in \( \{(1), x_1(t)\}, \ldots, \{(N), x_N(t)\} \). Then for all \( \beta > \alpha \),

\[ \mathcal{L}[(\#C^\alpha_{s})_{s \geq 0}] \rightarrow \mathcal{L}[(\#K^N_{s, \beta})_{s \geq 0}], \quad \forall \alpha \geq 0, \quad \forall t \rightarrow \infty \]  

(5.6)

where \( (K^N_{s})_{s \geq 0} \) is the Kingman coalescent started in \( \{(1), \ldots, (N)\} \).

**Proof.** The argument makes use of an obvious coupling of \( (C^\alpha, L^\alpha) \) and \( (IC^\alpha, IL^\alpha) \) where \( IC^\alpha_0 := C^\alpha_0 \). We proceed by induction on \( N \in \mathbb{N} \).

We start with \( N = 2 \). Put

\[ \tau'_1(t) := \inf \{s > 0 : \#IC^\alpha_s = 1\}, \quad \forall \alpha \geq 0, \quad \forall t \rightarrow \infty \]  

(5.7)

and set \( C^\alpha_s := IC^\alpha_s \), for all \( s \in [0, \tau'_1(t)] \). Define \( (C^\alpha_{s})_{s > \tau'_1(t)} \) in a standard way, using additional (independent) randomness. Let then

\[ \tau_1(t) := \tau'_1 \cap \tau_1(t) := \inf \{s > 0 : \#C^\alpha_s = 1\}, \quad \forall \alpha \geq 0, \quad \forall t \rightarrow \infty \]  

(5.8)

so that \( \tau'_1(t) \) and \( \tau_1(t) \) are the coalescence times of the two particles in \( IC^\alpha \), and \( C^\alpha \), respectively. Then clearly

\[ \tau'_1(t) \leq \tau_1(t) \leq \tau'_1(t) + \sum_{i=0}^{G} \tau'_i \]  

(5.9)
where $G$ has shifted geometric distribution with success probability $\gamma/(2 + \gamma)$, i.e., $P\{G \geq m\} = (2/(2 + \gamma))^{m-1}$, for all $m \geq 1$. $\tau_i^0$, $i \geq 1$, is distributed as the length of the (almost surely finite) excursion away from 0 for the underlying migration walk, and where the family $\{\tau_i^t(\cdot), G, \{\tau_i^0, i \geq 0\}\}$ is an independent family of random variables.

The result of Cox and Griffeath discussed above is based on the Erdős-Taylor asymptotics (4.1) and stronger estimates of a similar type. In particular, we rewrite (4.1) in the current setting, where $\beta > \alpha$ and the random walk is twice as fast as the simple one, as

$$P\{\tau_i^t(t) > t^\beta/2\} \xrightarrow{t \to \infty} \alpha/\beta.$$  \hfill (5.10)

Note that (5.10) can be restated as the following convergence in distribution: for all $u \geq 0$,

$$\lim_{t \to \infty} P\left\{ \log \left( \frac{\log \tau_i^t(t)}{\alpha \log t} \right) < u \right\} = 1 - e^{-u}. \hfill (5.11)$$

We would like to show the same convergence holds with $\tau_1(t)$ in place of $\tau_i^t(t)$. Due to (5.11) it suffices to show that, as $t \to \infty$, with overwhelming probability,

$$\sum_{i=0}^{G} \tau_i^0 \leq \tau_1(t), \hfill (5.12)$$

since then $\log(\tau_i^t(t) + \sum_{i=0}^{G} \tau_i^0) \leq \log \tau_1(t) + \log 2$, and $\log 2/\log t$ becomes negligible in the limit. Since $\sum_{i=0}^{G} \tau_i^0 < \infty$, almost surely, and $\tau_1(t) \to \infty$, as $t \to \infty$, in probability, (5.12) trivially follows, and we have

$$\lim_{t \to \infty} P\left\{ \log \left( \frac{\log \tau_1(t)}{\alpha \log t} \right) < u \right\} = 1 - e^{-u}, \hfill (5.13)$$

for all $u \geq 0$.

Now note that for $N > 2$ and for $\beta \geq \alpha$, using analogous coupling of $(C^\alpha, L^o)$ and $(IC^\alpha, IL^o)$ up to the first coalescence time $\tau_{N-1}^l(t)$ in $IC^\alpha$,

$$\lim_{t \to \infty} P\{#C_{\beta}^\alpha = N\} = \lim_{t \to \infty} P\{#IC_{\beta}^o = N\} = \left( \frac{\alpha}{\beta} \right)^{\left( \frac{\beta}{2} \right)}. \hfill (5.14)$$

where the second limit above was evaluated in Proposition 2 of [9]. Moreover, if

$$\tau_{N-1}(t) := \inf \{ s > 0 : \#C_s^\alpha = N - 1 \}, \hfill (5.15)$$

due to the fact that $|\log \tau_{N-1}(t) - \log \tau_{N-1}(t)| \to 0$, as $t \to \infty$, almost surely (argue as for above), the induction step in the proof of [9] Theorem 3 can be carried out verbatim. The details are tedious, so we omit them, and state instead that

$$p_{N,k}(\alpha/\beta) := \lim_{t \to \infty} P\{#C_{\beta}^\alpha = k\} \hfill (5.16)$$

satisfies the recursion of [9] Theorem 3,

$$p_{N+1,k}(\frac{1}{\alpha}) \frac{1}{s} = \binom{N+1}{2} \int_1^s dy y^{-\left(\frac{N+1}{2}\right)-1} p_{N,k}(y/s), \hfill (5.17)$$

for all $s \geq 1$ and $1 \leq k \leq N + 1$.

Since the initial conditions (5.13), (5.14) to the recursion are identical to those in Theorem 3 in [9], as argued above, the solution is the same, and so we have verified that for each $\beta > \alpha$ and each $k \in \{1, \ldots, N\}$,

$$\lim_{t \to \infty} P\{#C_{\beta}^\alpha = k\} = \lim_{t \to \infty} P\{#IC_{\beta}^o = k\} = P\{#K_{\log(\frac{\alpha}{\beta})}^N = k\}, \hfill (5.18)$$

where the last identity was again obtained in [9].
5.2 Convergence in path space

In order to show path convergence of \((\#C^\alpha_{\beta})_{\beta \geq \alpha}\) to \((\#K_{\log \beta/\alpha})_{\beta \geq \alpha}\) one defines a sequence of random times \(\{\tau^\alpha_k(t); 1 \leq k \leq N\}\), where for each \(k \geq 1\),

\[
\tau^\alpha_k(t) := \inf \left\{ s \geq 0 : \#C^\alpha_s \leq k \right\},
\]

(5.19)

where as usual \(\tau^\alpha_k(t) = \infty\) if \(\inf_{s \geq 0} \#C^\alpha_s > k\). That is, \(\tau^\alpha_0 = 0\), and \(\tau^\alpha_{N-1}(t)\) is the first coalescence time, (also denoted by \(\tau_{N-1}(t)\) in the proof of Lemma 5.2), \(\tau^\alpha_{N-2}(t)\) is the second coalescence time, etc. It is not difficult to see that the arguments of the proof of Theorem 3 in [9] extend to showing that, with probability one \(\#C^\alpha_{\tau^\alpha_k}(t) = k\), for all \(k = N - 1, \ldots, 1\) (see also Lemma 5.1), and that with respect to convergence in probability,

\[
\lim_{t \to \infty} \frac{\tau^\alpha_k(t)}{\tau^\alpha_{k-1}(t)} = 0,
\]

(5.20)

for each \(k \geq 2\). (Note here that the remaining \(k\) partition elements are spread out). Moreover, the following joint convergence in distribution holds

\[
\left( \log \left( \frac{\log(\tau^\alpha_{N-1}(t))}{\alpha \log t} \right), \log \left( \frac{\log(\tau^\alpha_{N-2}(t))}{\log(\tau^\alpha_{N-1}(t))} \right), \ldots, \log \left( \frac{\log(\tau^\alpha_{i}(t))}{\log(\tau^\alpha_{i+1}(t))} \right) \right) \quad \overset{\text{d}}{\longrightarrow} \quad (U_{N-1}, U_{N-2}, \ldots, U_1),
\]

(5.21)

where \(\{U_i; i = 1, \ldots, N-1\}\) is a family of independent random variables such that for all \(i = 1, \ldots, N-1\), \(U_i\) has the rate \((\frac{1}{2} + 1)\) exponential distribution. Now (5.20) and (5.21) imply the convergence of random vectors

\[
\left( \log \left( \frac{\log(\tau^\alpha_{N-1}(t))}{\alpha \log t} \right), \log \left( \frac{\log(\tau^\alpha_{N-2}(t))}{\alpha \log t} \right) - \log \left( \frac{\log(\tau^\alpha_{N-1}(t))}{\alpha \log t} \right), \ldots, \log \left( \frac{\log(\tau^\alpha_{i}(t))}{\alpha \log t} \right) - \log \left( \frac{\log(\tau^\alpha_{i+1}(t))}{\alpha \log t} \right) \right) \quad \overset{\text{d}}{\longrightarrow} \quad (U_{N-1}, U_{N-2}, \ldots, U_1).
\]

(5.22)

Since

\[
\#C^\alpha_{\beta} = \left[ \frac{\log(\tau^\alpha_{N-1}(t))}{\alpha \log t} \right] \left( \frac{\beta}{\alpha} \right) \quad + \quad \sum_{k=2}^{N} k \left[ \frac{\log(\tau^\alpha_{N-1}(t))}{\alpha \log t} - \frac{\log(\tau^\alpha_{N-k}(t))}{\alpha \log t} \right] \left( \frac{\beta}{\alpha} \right),
\]

(5.23)

and with \(\bar{U}_k = \exp(U_{N} + \ldots + U_k)\),

\[
\#K^N_{\log(\beta/\alpha)} = \left[ \bar{U}_{1, \infty} \right] \left( \frac{\beta}{\alpha} \right) \quad + \quad \sum_{k=2}^{N} k \left[ \bar{U}_{k} \bar{U}_{k-1} \right] \left( \frac{\beta}{\alpha} \right),
\]

(5.24)

it immediately follows that the process \((\#C^\alpha_{\beta})_{\beta \geq \alpha}\) converges in the sense of Skorokhod topology to the process \((\#K^N_{\log(\beta/\alpha)})_{\beta \geq \alpha}\) as \(t \to \infty\).

In order to upgrade the above convergence to the one on the level of partitions, as stated in Proposition 5.1 we need to make sure that for any fixed \(N\) and any choice of initial locations \(x_1, \ldots, x_N\) contained in \(I_{c}(c, t)\), asymptotically as \(t \to \infty\), any two current partitions elements coalesce equally likely and independently of the coalescent time. That is,

\[
P(i, j \text{ coalesces at time } \tau^\alpha_{N-1}(t)|\tau^\alpha_{N-2}(t)) \overset{\text{a.s.}}{\longrightarrow} \binom{N}{2}^{-1}.
\]

(5.25)

Assume without loss of generality that \(i < j\). Fix \(\beta > \alpha\), and let (with \(C^\alpha_0 := \{\{1\}, \ldots, \{N\}\}\) and \(g_\alpha\) as in (4.1))

\[
M^\beta_{i,j}(t) := \bigcup_{s \in [0, t^\beta - g_\alpha(t)]} \{C^\alpha_s = C^\alpha_0, C^\alpha_s = \{i, j\} \cup C^\alpha_s \setminus \{\{i\}, \{j\}\}\},
\]

(5.26)
and put
\[ N^\beta(t) := \bigcup_{1 \leq i < j \leq N} M^\beta_{i,j}(t). \] (5.27)

Note that the events \( \{ M^\beta_{i,j}(t); 1 \leq i < j < \infty \} \) are disjoint.

Recall from the proof of Proposition 4.1 that \( (Y^c_s)_{s \geq 0} \) denotes the \( 2N \)-dimensional process (i.e. \( \mathbb{Z}^{2N} \)-valued), whose joint evolution is the evolution of the location processes of \( (C^\alpha_s, L^\alpha_s)_{s \geq 0} \) but started at time 0 in \( (X^1_{g_\alpha(t)}, \ldots, X^N_{g_\alpha(t)}) \) where the latter are \( N \)-independent \( a(\cdot, \cdot) \)-random walks on \( \mathbb{Z}^2 \). We consider the path of \( Y^c \) after time \( g_\alpha(t) \) up to time \( t \) and ask whether the coalescent with these paths in the time interval \([0, t^\beta - g_\alpha(t)]\) would have a first coalescence event, we write \( (Y^c_s)_{s \geq g_\alpha(t)} \in M^\beta(t) \) for this event.

Then
\[ \left| \mathbb{P}\left\{ (Y^c_s)_{s \geq g_\alpha(t)} \in N^\beta(t) \right\} - \mathbb{P}\left\{ \tau^\alpha_{N-1}(t) \leq t^\beta \right\} \right| \leq \mathbb{P}\left\{ \tau^\alpha_{N-1}(t) \leq g_\alpha(t) \right\}, \] (5.28)

and similarly for each \( i < j \),
\[ \left| \mathbb{P}\left\{ (Y^c_s)_{s \geq g_\alpha(t)} \in M^\beta_{i,j}(t) \right\} - \mathbb{P}\left\{ \tau^\alpha_{N-1}(t) \leq t^\beta, i \sim \tau^\alpha_{N-1}(t) j \right\} \right| \leq \mathbb{P}\left\{ \tau^\alpha_{N-1}(t) \leq g_\alpha(t) \right\}. \] (5.29)

Proposition 4.1 together with Lemma 5.2 and (5.29) imply
\[ \left| \mathbb{P}\left\{ \tau^\alpha_{N-1}(t) \leq t^\beta, 1 \sim \tau^\alpha_{N-1}(t) 2 \right\} - \mathbb{P}\left\{ \tau^\alpha_{N-1}(t) \leq t^\beta, i \sim \tau^\alpha_{N-1}(t) j \right\} \right| \underset{t \rightarrow \infty}{\longrightarrow} 0, \] (5.30)

and again due to (5.26), (5.27), and (5.28),
\[ \left| \mathbb{P}\left\{ \tau^\alpha_{N-1}(t) \leq t^\beta, 1 \sim \tau^\alpha_{N-1}(t) 2 \right\} - \left( \frac{N}{2} \right)^{-1} \mathbb{P}\left\{ \tau^\alpha_{N-1}(t) \leq t^\beta \right\} \right| \underset{t \rightarrow \infty}{\longrightarrow} 0, \] (5.31)

which proves (5.25).

Due to the asymptotic exchangeability given by Proposition 4.1 and uniform estimates (5.24) on locations of partition elements at each coalescence time, it is easy to extend (for example by induction) (5.26) to an analogous statement at any future coalescence time. This indeed confirms that the limiting object \( K^N \) is the Kingman coalescent, since the right hand sides of (5.22) and (5.25) characterizes its law completely.

6 Asymptotics for dense particles at small times

This section concentrates on the behavior of the system for fixed \( \alpha \in [0, \infty) \) at times of order only slightly larger than the area of the rectangle on which the initial configuration is supported. More precisely, we set
\[ \Lambda(r) := [-r, r]^2 \cap \mathbb{Z}^2. \] (6.1)

and study the corresponding restricted spatial coalescent.

6.1 Coupled spatial coalescents and moment bound

Here and at many other occasions it is useful to couple coalescents starting in different but comparable initial configurations. We next describe a formal setting that will be used in Sections 6, 7, and 8.

Let
\[ F := \{ F_z, z \in \mathbb{Z}^2 \} \] (6.2)
be a family of $\tilde{N}$-valued valued random variables. We think of $F_z$ as the number of partition elements (particles) present at site $z \in \mathbb{Z}^2$ in the coalescent at time 0. In symbols,

$$F_z := \#\{ \pi \in C_0 : L_0(\pi) = z \}. \quad (6.3)$$

Typically we will choose the collection $F$ such that $\sum_{z \in \mathbb{Z}^2} F_z \delta_z \in \mathcal{E}$, almost surely. In addition, for the applications we have, we often assume $F$ to be a family of independent random variables with the same Poisson (rate $\rho \in (0, \infty)$) distribution.

Fix a countable (or finite) set $\mathcal{I}$, and recall for all $\mathcal{I}' \subseteq \mathcal{I}$ satisfying (6.10) the restricted process $(C_{s,t}', L_{s,t}')_{s \geq 0}$.

Sometimes we are interested in restricting $(C_s, L_s)_{s \geq 0}$ to geographical information. That is, for $A \subseteq \mathbb{Z}^2$, let

$$\mathcal{I}_A := \{ i \in \mathcal{I} : L_0(\{ j : j \sim C_0, i \}) \in A \}. \quad (6.4)$$

In this particular case, we write

$$C(A) := C_{s,t}. \quad (6.5)$$

In particular, if $F^A$ gives the number of partition elements of the restricted coalescent $C(A)$, then

$$F^A_z := \begin{cases} F_z, & z \in A, \\ 0, & z \notin A. \end{cases} \quad (6.6)$$

Moreover, if $A \subseteq B \subseteq \mathbb{Z}^2$ then $C^A_{s,t} \leq C^B_{s,t}$ and due to the comment following (6.10), the two coalescents $(C^A_{s,t}, L^A_{s,t})$ and $(C^B_{s,t}, L^B_{s,t})$ can be coupled so that at any point in time and space, the number of partition elements in $(C^A_{s,t}, L^A_{s,t})$ dominates from above the number of partition elements in $(C^B_{s,t}, L^B_{s,t})$.

Assume we are given the coupled spatial coalescents from above and recall $\{ F_z : z \in \mathbb{Z}^2 \}$ from (6.2). Assume that

$$\mathbb{E}[F_z] > 0, \quad \text{and} \quad \text{Var}[F_z] < \infty, \quad (6.7)$$

for all $z \in \mathbb{Z}^2$.

Our goal is to show next that the sparse initial configurations necessary for the results of the previous section arise if the coalescent is started in the torus $\Lambda^{\alpha,t} = \Lambda(t^{\alpha/2})$, and observed at time $t^{\alpha'}$ for $\alpha' > \alpha$ and $\alpha'$ approaching $\alpha$.

We will rely on the following tightness result for $C^{\alpha,t}_{i,t}$ started in $\Lambda^{\alpha,t}$, whose somewhat technical proof is given in Section 8. Denote by

$$\{ C^{\alpha,t}_i : \alpha \in (0, 1) \}, \quad (6.8)$$

the collection of coupled coalescent processes constructed in (3.12), (3.13), where we abbreviate

$$(C_{s,t}^{\alpha,t}, L_{s,t}^{\alpha,t})_{s \geq 0} := (C_{s,t}^{\Lambda^{\alpha,t}}, L_{s,t}^{\Lambda^{\alpha,t}})_{s \geq 0}. \quad (6.9)$$

**Proposition 6.1** (Uniformly bounded expectation on logarithmic scale). There are finite constants $M$ and $t_0$ such that for all $t \geq t_0$, satisfying $\alpha \in (0, \infty)$, and $\beta \in (\alpha, 3\alpha/2)$,

$$\mathbb{E}[\#C^{\alpha,t}_2] \leq M \left\{ \frac{\alpha}{2(\beta - \alpha)} \mathbb{V}[\#C^{\alpha,t}_2] \right\} \quad (6.10)$$

**Remark 6.1.** The $C^{\alpha,t}_2$ in (6.10) denotes the coalescent partition evaluated at time 2, any finite positive time could be taken instead of 2 here, and the two constants $t_0$ and $M$ would change accordingly. Our special choice of the time point 2 is convenient from the perspective of the time discretization used in the proof of Proposition 6.1 (compare with (9.3)).
6.2 Consequences of the expectation bound: Tightness

Recall notation (6.8) and in addition assume that

$$\rho := \limsup_{t \to \infty} \sup_{z \in \Lambda^{1,t}} E[F_z] < \infty. \quad (6.11)$$

The next result states that as $t \to \infty$ the coalescents in $\Lambda^{1,t}$ remain finite and localized in certain boxes.

**Proposition 6.2** (The asymptotically infinite spatial case: small time scales). Consider the coalescent restricted to $\Lambda^{1,t}$. Let to be as specified in Proposition 6.1. Then the following holds.

(a) For each fixed $\alpha' > \alpha$, there exists a sequence $(a_N)_{N \in \mathbb{N}} \uparrow 1$ such that for all $N \in \mathbb{N},$

$$\inf_{t \geq t_0} P\{ \#C_{\alpha,t}^{\alpha,t} \leq N \} \geq a_N, \quad (6.12)$$

and ($\sim t$ denoting the equivalence relation w.r.t. time $t$ partition)

$$\liminf_{t \to \infty} P\{ \max_i \| L_{\alpha,t}^{\alpha,t}(\{i \sim t^\alpha\}) \| \leq t^{\alpha/2} \log t \} \geq a_N. \quad (6.13)$$

(b) For each fixed $\alpha' > \alpha$ and each $N \in \mathbb{N}$, $L_{\alpha,t}^{\alpha,t}$, the set of all marks at time $t^\alpha$ and $I_{\alpha}(1,t)$ as in (4.2) we have:

$$P(\{L_{\alpha,t}^{\alpha,t} \text{ is contained in } I_{\alpha}(1,t)\} | \#C_{\alpha,t}^{\alpha,t} \leq N) \xrightarrow{t \to \infty} 1. \quad (6.14)$$

(c) For each $N \in \mathbb{N}$,

$$\lim_{\alpha' \uparrow \alpha} \liminf_{t \to \infty} P\{ \#C_{\alpha,t}^{\alpha,t} \geq N \} = 1. \quad (6.15)$$

**Proof.** Assertion (6.12) is now an immediate consequence of Proposition 6.1 and the Markov inequality.

Assertion (6.13) follows from a large deviation estimate. It will be convenient here and below to set

$$\alpha^* = \alpha^*(\alpha, \alpha') := (\alpha + \alpha')/2. \quad (6.16)$$

Let $\{(X^i_s)_{s \geq 0}; i \geq 1\}$ be an infinite collection of independent random walks with kernel $a(\cdot, \cdot)$ such that the initial locations $\{X^i_0; i \geq 1\}$ are distributed as the location process $L^{\alpha, t}_{\alpha,t}$ of the coalescent restricted to the box $\Lambda^{1,t}$. Take $\varepsilon < (\alpha' - \alpha)/2$ so that $\alpha^* + \varepsilon < \alpha'$. Since (6.11) holds, we have that $\#C_{\alpha,t}^{\alpha,t}$ is bounded by $2p t^\alpha$ with overwhelming probability. Due to a large deviation estimate (for example, (9.3) is more than needed here)

$$\lim_{t \to \infty} P\{ \max_i \| X^i_{t^\alpha} \| > t^{(\alpha^* + \varepsilon)/2} : i \in \{1, \cdots, 2p t^\alpha\} \} = 0, \quad (6.17)$$

and hence

$$\lim_{t \to \infty} P\{ \max_i \| X^i_{t^\alpha} \| > t^{(\alpha^* + \varepsilon)/2} : i \in \{1, \cdots, \#C_{\alpha,t}^{\alpha,t}, t_0\} \} = 0. \quad (6.18)$$

Therefore

$$\lim_{t \to \infty} P\{ \max_i \| L_{\alpha,t}^{\alpha,t}(\{i \sim t^\alpha\}) \| > t^{(\alpha^* + \varepsilon)/2} \} = 0. \quad (6.19)$$

In order to get (6.13) from (6.19) we use (6.12) together with the fact that during the remaining time $t^\alpha - t^{\alpha^*}$ none of the finitely many partition classes reaches distance larger than $t^{\alpha^*} t / 2 \log t$, with overwhelming probability.

In order to prove (6.14) fix $\alpha' > \alpha$. 

7 LARGE TIME-SPACE SCALE ASYMPTOTICS OF COALESCENT

Fix $N \geq 1$, and note that (6.12) implies the uniform lower bound $\bar{p}$ on the probability of $\{#C^\alpha_{t^\alpha} \leq N\}$. So (6.14) will follow provided we show that for any $\varepsilon > 0$ we have

$$
P \{L^\alpha_{t^\alpha} \text{ is contained in } I_\alpha(t,1) \} \geq 1 - \varepsilon \bar{p}. \tag{6.20}
$$

Again due to part (a), it is possible to pick $M_t$ so that $C^\alpha_{t^\alpha}$ contains at most $M_t$ equivalence classes, with probability higher than $1 - \bar{p}\varepsilon/3$, and such that any pair of them is at mutual distance smaller than $2t^{\alpha^*/2}\log t$ with probability higher than $1 - \bar{p}\varepsilon/3$. During the remaining time interval $(t^{\alpha^*}, t^\alpha]$ of length $t^\alpha - t^{\alpha^*}$, which is of order $t^{\alpha^*}$, each pair of non-coalescing walks (out of at most $(M_t)^2$ many pairs) achieves, with overwhelming probability, a mutual distance of order $(1 - \varepsilon)\log(t)$. Moreover, for a random variable $n$ contained in $I_\alpha(t,1)$, we have by convergence of the first component in (5.14) that

$$
\Pr \{\max_i \max_{t' \in [0, t]} L^\alpha_{t^\alpha}(\{i \sim t'\}) \leq 2t^{\alpha^*/2}/\log t\} \geq 1 - \varepsilon \bar{p}/3,
$$

so (6.20), and therefore (6.14) holds.

It still remains to prove (6.15). Fix $\alpha' > \alpha > 0$. Note that for any $N$ particles started at locations $x_1, \ldots, x_N$ contained in $I_{\alpha}(1,t)$, we have by convergence of the first component in (6.14) that

$$
P \{\text{no coalescence by time } t^{\alpha'}\} \xrightarrow{t \to \infty} (\alpha' \downarrow \alpha)(\frac{N}{2}). \tag{6.22}
$$

For fixed $N$ first choose large $t$ so that it is possible to find $N$ particles from the initial configuration at time 0 with locations contained in $I_\alpha(1,t)$, and then note that as $\alpha' \downarrow \alpha$ the right hand side above converges to 1.

7 Large time-space scale asymptotics of coalescent

In this section we combine the results of Sections 4, 5 and 6 to prove Theorems 1 through 3.

7.1 Proof of Theorem 1

Fix $1 \geq \alpha' > \alpha > 0$ and $\varepsilon \in (0, \alpha' - \alpha)$.

By (6.15), for all $N \in \mathbb{N}$ there exists $\alpha^* \in (\alpha, \alpha + \varepsilon) \subset (\alpha, \alpha')$ and $t_1 = t_1(N)$ such that for all $t \geq t_1$,

$$
P \{#C^\alpha_{t^\alpha} \geq N\} \geq 1 - \varepsilon. \tag{7.1}
$$

From now on assume that $t \geq t_0$ where $t_0$ is specified as in Proposition 6.1. Proposition 6.2 implies that with probability tending to 1 as $t \to \infty$, the configuration $C^\alpha_{t^\alpha}$ has finitely many particles in locations contained in $I_{\alpha}(t,1)$.

Put

$$
n^{\alpha^{\star}, t} := #C^\alpha_{t^\alpha}. \tag{7.2}
$$

Then Proposition 5.1 joint with Proposition 6.2 (a) and (b), yield

$$
d_{W_1} \left(\mathcal{L}(\{#C^\alpha_{t^\alpha} \in [\alpha^{\star}, \infty]\}, \mathcal{L}(\{#K^{n^{\alpha^{\star}, t}}_{\log(\beta/\alpha^{\star})} \in [\alpha^{\star}, \infty]\}) \right) \xrightarrow{t \to \infty} 0, \tag{7.3}
$$

where $d_{W_1}$ is the Prohorov metric which is known to metrize the weak topology (see, for example, [16]). Moreover, for a random variable $n$ and $s \geq 0$, $#K_{s}^{n}$ is a random variable which, given $n = k$, is distributed as the Kingman coalescent started in $\{1, 2, \ldots, k\}$ and evaluated at time $s$. 


Recall that we denote by $K$, the Kingman coalescent started from the trivial infinite partition $\{i\} : i \in \mathbb{N}$. Easy properties of the Kingman coalescent guarantee that for all $\delta > 0$,

\[
(\#K^s_n)_{s \geq \delta} \xrightarrow{n \to \infty} (\#K_s)_{s \geq \delta},
\]

and

\[
(\#K^\infty_{s+n})_{s \geq \delta} \xrightarrow{n \to \infty} (\#K^\infty_s)_{s \geq \delta}.
\]

Note that Proposition 6.2(a) insures that the family \{\(n^{\alpha,t}; t \geq t_0\)\} is tight. Choose \((t_m) \to \infty\) and \(n^{\alpha'}\) such that \(n^{\alpha'}\cdot t_m \to n^{\alpha'}\), as \(m \to \infty\). Then \(n^{\alpha'}\) is a finite random variable and

\[
(\#C^{n^{\alpha',t_m}}_{t_m})_{\beta \in [\alpha',\infty)} \xrightarrow{m \to \infty} (\#K^{n^{\alpha'}}_{\log(\beta/\alpha^*)})_{\beta \in [\alpha',\infty)}.
\]

The left hand side of (7.6) does not depend on \(\varepsilon\). By (7.1), (7.4) and (7.5) we have, after letting \(\varepsilon \to 0\),

\[
(\#C^{\alpha,t_m}_{t_m})_{\beta \in [\alpha',\infty)} \xrightarrow{m \to \infty} (\#K^{\alpha}_{\log(\beta/\alpha)})_{\beta \in [\alpha',\infty)}.
\]

Since one obtains the same limit regardless of the choice of the subsequence \((t_m)\), the statement of the theorem follows.

### 7.2 Proof of Theorem 2

Recall from \[3.2\] that \{\((C^{\alpha,t,\rho}_s, L^{\alpha,t,\rho}_s)_{s \geq 0}; \alpha \in (0,1)\)\} denotes the family of spatial coalescents on \(\Lambda^{\alpha,t}\) corresponding to the parameter \(\rho \in (0,\infty]\).

Recall the initial states \(\{F_z^A; z \in A\}\) from \[6.6\]. In this section we assume that \(\{F_z^\rho; z \in \Lambda^{1,t,\rho}, \rho \geq 1\}\) is for fixed \(\rho\) a family of independent identically distributed random variables with \(\text{Poisson}(\rho)\) distribution. In fact, due to thinning and superposition properties of the Poisson process on the line we can consider a coupling such of the families for different \(\rho\) that if \(\rho_1 \leq \rho_2\) then

\[
F^\rho_{z_1} \leq F^\rho_{z_2},
\]

for all \(z \in \Lambda^{1,t}\).

Due to this coupling and the monotonicity properties collected in Subsection \[1.4\]

\[
(C^{\alpha,t,\rho}_s, L^{\alpha,t,\rho}_s)_{s \geq 0} \xrightarrow{\rho \to \infty} (C^{\alpha,t,\infty}_s, L^{\alpha,t,\infty}_s)_{s \geq 0},
\]

here convergence is meant in the sense of convergence defined \[2.13\].

The goal of this subsection is to show that the results obtained in Subsection \[6.2\] hold in the limit \(\rho \to \infty\).

Fix \(\delta > 0\) and recall from \[3.6\] the spatial coalescent \((C^{\alpha,t,\infty,\delta}_s)_{s \geq 0}\), thinned out by those particles which were attempted to jump in the time period \([0,\delta]\).

**Lemma 7.1** (The limit of infinite density). For each \(\delta > 0\) fixed,

\[
\lim_{N \to \infty} \lim_{t \to \infty} \inf P\{\#C^{\alpha,t,\infty,\delta}_s \leq N\} = 1.
\]

**Proof.** Recall from Lemma \[4.3\] that the number of partition elements of a Kingman coalescent can be dominated by a Poisson variable with suitably large parameter \(\rho_0\). By monotonicity we can construct a coupling

\[
\left((C^{\alpha,t,\infty,\delta}_s, L^{\alpha,t,\infty,\delta}_s), (C^{\text{Poisson}(\rho_0)+1}_s, L^{\text{Poisson}(\rho_0)+1}_s)\right)_{s \geq \delta},
\]

where \((C^{\text{Poisson}(\rho_0)+1}_s, L^{\text{Poisson}(\rho_0)+1}_s)_{s \geq 0}\) is started from the initial configuration where \(\{F_z^A; z \in \Lambda^{1,t}\}\) is a family of independent random variables which equal in distribution one plus a rate \(\rho_0\) Poisson distributed random variable such that \(C^{\alpha,t,\infty,\delta}_s \leq C^{\text{Poisson}(\rho_0)+1}_s\), for all \(s \geq \delta\), almost surely. The statement now follows from Proposition \[6.2(a)\] applied to \((C^{\text{Poisson}(\rho_0)+1}_s)_{s \geq 0}\). \[\square\]
In addition, notice that \( P\{\#C^{\alpha,d,\infty}_{\delta} \geq 1\} = 1 \), so if \( (C^1, L^1)_{s \geq d} \) is the family of spatial coalescents started with 1 particle at each site of \( \Lambda(t^{1/2}) \), we have
\[
C^1_{s} \leq C^{\alpha,t,\infty}_{\delta} \leq C^{\text{Poisson}(\rho) + 1}. \tag{7.12}
\]

The extension of Theorem 3 in Proposition 2.2 proved in Subsection 7.1 clearly applies to both the left-most and the right-most family of coalescents. Therefore, by Lemma 4.2 for fixed \( \alpha' > \alpha > 0 \),
\[
(\#C^{\alpha',t,\infty}_{\delta})_{\beta \in [\alpha',\infty)} \Rightarrow (\#K_{\log(\beta/\alpha)})_{\beta \in [\alpha',\infty)}, \tag{7.13}
\]
and Theorem 2 follows.

### 7.3 Proof of Theorem 3

The proof of Theorem 3 makes use of a convergence result stated in Theorem 1 in [12], which applies in a much more general setting than ours. For the benefit of the reader, we will rephrase it in our setting.

**Lemma 7.2** (Donnelly, 1991). Suppose \( \{(B^N_N)_{s \geq 0}; N \geq 1\} \) is a family of \( D([0,\infty),\mathbb{N}) \)-valued random variables which satisfy the following three assumptions:

(A1) For all \( N \in \mathbb{N}, l \geq n \in \mathbb{N}, s \geq \alpha \) and \( y \geq 1 \),
\[
P\left( \inf_{u \in [\alpha,s]} B^N_{\alpha} \leq y | B^N_{\alpha} = l \right) \leq P\left( \inf_{u \in [\alpha,s]} B^N_{\alpha} \leq y | B^N_{\alpha} = n \right). \tag{7.14}
\]

(A2) For all \( n \in \mathbb{N} \),
\[
\mathcal{L}\left[(B^N_{u})_{u \geq \alpha} | B^N_{\alpha} = n \right] \Longrightarrow \mathcal{L}\left[(\#K^N_{\log(u/\alpha)})_{u \geq \alpha} \right], \tag{7.15}
\]

(A3) Suppose we have a sequence \( (n_M) \rightarrow \infty \), such that for each \( u > \alpha \),
\[
\lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} P\left( B^N_{u} \leq M | B^N_{\alpha} = n_M \right) = 1. \tag{7.16}
\]

Then
\[
\mathcal{L}\left[(B^N_{u})_{u \geq \alpha} | B^N_{\alpha} = n_N \right] \Longrightarrow \mathcal{L}\left[(\#K_{\log(u/\alpha)})_{u \geq \alpha} \right]. \tag{7.17}
\]

**Proof of Theorem 3** (i) Take a subsequence \( (t_N) \uparrow \infty \), and let
\[
s_N := \#\Lambda^{\alpha,t_N}. \tag{7.18}
\]

We consider first a special case. Draw \( \text{Bin}_N \) according to the Binomial distribution with parameters \( s_N \) and \( p \in (0,1] \) or the Poisson distribution with parameter \( s_N \cdot \rho \). Given \( \text{Bin}_N = k \), place \( k \) particles uniformly without and with replacement at \( k \) positions in \( \Lambda^{\alpha,t_N} \). Notice that the random configurations obtained this way will equal in law to \( C^{\alpha,t_N l}_0 \) under the assumption that \( \{F^z; z \in \Lambda^{\alpha,t_N}\} \) are independent and identically distributed random variables with the Bernoulli (parameter \( p \)) or with the Poisson(\( \rho \)) distribution, respectively.

Put for all \( u \geq \alpha \),
\[
B^N_{u} := \#C^{\alpha,t_N}_{(t_N)^u}. \tag{7.19}
\]

By Lemma 5.2 given that \( \text{Bin}_N = k \), with probability tending to 1, \( B^N_{\alpha} = k \). The advantage of the above construction(s) is that the assumption (A1) is automatically satisfied provided we keep the same algorithm for “positioning the \( k \) particles in \( \Lambda^{\alpha,t_N} \), for all \( k \in \mathbb{N} \), i.e., provided that for each \( k \in \mathbb{N} \) and all \( l < k \), the first \( l \) points in \( C^{\alpha,t_N}_0 \) given \( \text{Bin}_N = l \) match those in \( C^{\alpha,t_N}_0 \) given \( \text{Bin}_N = k \).
The assumptions (A2) and (A3) (provided that \( n_N = O(s_N) \)) are implied by Lemma 5.2 and 6.12, respectively. Therefore, (7.17) holds in the (special) Binomial case for any \( p \in (0, 1) \) and any sequence \( n_N \leq s_N \) going to \( \infty \). Similarly, (7.17) holds in the (special) Poisson case for any \( \rho \in (0, \infty) \) and any sequence \( n_N = O(s_N) \).

In particular, if \( p = 1 \) and \( n_N = s_N \) (almost surely) then

\[
\left( \#C_{tN}^\beta \right)_{\beta \geq \alpha} \xrightarrow{\text{N} \to \infty} \left( \#K_{\log(\beta/\alpha)} \right)_{\beta \geq \alpha}.
\]

(7.20)

Since the limit is uniform in the choice of the subsequence \( t_N \to \infty \), we conclude the statement of the theorem in this case.

The general Bernoulli(\( p \)) case can be dealt with similarly as the general Poisson(\( \rho \)) case, as we explain next. Fix \( \rho \in (0, \infty) \) and note that (7.17) holds both with \( n_N := \lfloor \rho s_N / 2 \rfloor \) and with \( n_N := \lfloor 2 \rho s_N \rfloor \). Since with probability tending to \( 1 \), the Poisson (\( \rho s_N \)) distributed random variable Bin\(_N\) satisfies

\[
\lfloor \rho s_N / 2 \rfloor \leq \text{Bin}_N \leq \lfloor 2 \rho s_N \rfloor,
\]

we can apply Lemma 4.2 to conclude the needed statement as done before.

(ii) Note that due to part (i), the family of processes \( (t_N) \) is a sequence with \( t_N \to \infty \) as \( N \to \infty \)

\[
\left( \#C_{tN}^\beta \right)_{\beta \in [\alpha, \infty)},
\]

(7.22)

where the family \( \{F_z; z \in \Lambda^{1,t}\} \) is drawn from the “Poisson(\( \rho \)) + 1” distribution is tight in \( D([\alpha, \infty), \bar{N}) \) since we can sandwich it between from below the case where we start with exactly 1 particle per site (Bernoulli with \( p = 1 \)) and from above with the independent sum of two spatial coalescent processes one started in Poisson(\( \rho \)) and the other one with exactly 1 particle per site (Bernoulli with \( p = 1 \)). Here we use monotonicity in \( \beta \) for every \( N \). Moreover, the process \( \left( \#K_{\log(\beta/\alpha)} \right)_{\beta \in [\alpha, \infty)} \) is the only possible (subsequential) limit due to Theorem 1. Therefore, applying monotonicity and using (7.12) as in the proof of Theorem 2 implies the statement.

\[ \square \]

8 Convergence on the spatial scale (Proof of Theorems 4 and 5)

In this section we prove results which involve the coalescent with rebirth using the results established in Sections 5 and 6.

8.1 Proof of Theorem 4

Consider the family \( \{\rho_{I^0} \circ C_{1,t}^1; \alpha \in (0, 1)\} \) from (3.12). Fix \( \rho \in (0, \infty) \), and assume (6.11). By Theorem 1 (with \( \beta = 1 \)) and (6.11) we already know that, for a fixed \( \alpha \in (0, 1) \),

\[
\#\rho_{I^0} \circ C_{t}^{1}\xrightarrow{t \to \infty} \#K_{-\log \alpha}.
\]

(8.1)

Our first and key goal is to extend (8.1) to the f.d.d. convergence of \( (\#\rho_{I^0} \circ C_{1,t}^1)_{\alpha \in [\alpha_l, \alpha_u]} \) to \( (N_{\alpha})_{\alpha \in [\alpha_l, \alpha_u]} \), where \( \alpha_l, \alpha_u \in (0, 1) \), stated below in Proposition 8.1. In particular, here \( N_{\alpha} \) is the number of partition elements of \( K_{\log \alpha_l, \log \alpha_u} \) born before time \( \log \alpha \), as defined in (3.15). As a second (small) step we derive at the end the convergence on path space as stated in Theorem 4.

**Proposition 8.1** (Partition number f.d.d. convergence).

(a) Fix \( \rho \in [0, \infty) \).
(i) For all \( m \in \mathbb{N} \) and \( \alpha_l \leq \alpha_1 < \ldots < \alpha_m \leq \alpha_u \),
\[
(\#\rho_{T_1} \diamond C_{t_1}^{1,t}, \ldots, \#\rho_{T_m} \diamond C_{t_1}^{1,t})_{t \to \infty} \rightarrow (N_{\alpha_1}, \ldots, N_{\alpha_m}). \tag{8.2}
\]

(ii) For any \( \varepsilon > 0 \), the family
\[
\Gamma := \left\{ L \left[ \#\rho_{T_1} \diamond C_{t_1}^{1,t} \right]; \alpha \in [0, 1 - \varepsilon], t > 0 \right\} \tag{8.3}
\]
is tight.

(b) The statements of (a) remain valid if \( \rho = \infty \) and, for a fixed \( \delta > 0 \), \( C_{t_1}^{1,t} \) is replaced by \( C_{t_1}^{1,t,\infty,\delta} \).

Remark 8.1. Note that the generalization of the proposition in terms of the corresponding convergence of the partition structure could be formulated in the setting of finite (and sparse) initial configurations considered in the proof of (a.i) below, and proved by applying the technique of Section 5.2 (see also Lemma 7.3 in [19] or Proposition 14 in [29]). □

Before giving the argument we present a key tool. For \( m \in \mathbb{N} \), fix parameters \( 0 < \alpha_l \leq \alpha_1 < \alpha_2 < \ldots < \alpha_m \leq \alpha_u \). To make the argument more transparent we introduce the coalescent with rebirth at finitely many prescribed times \( \{ \log \alpha_1, \ldots, \log \alpha_m \} \) only and call this process:
\[
(K_s^\vec{\alpha})_{s \geq \log \alpha_l} := \left( K_s \left[ \{ \log \alpha_1, \ldots, \log \alpha_m \} \right] \right)_{s \geq \log \alpha_l}. \tag{8.4}
\]

In words, the process \((K_s^\vec{\alpha})_{s \geq \log \alpha_l}\) behaves as follows: it starts in the configuration \( \{ (n, \log \alpha_1); n \in \mathbb{N} \} \) and during each interval of the form \( [\log \alpha_{k-1}, \log \alpha_k), k = 1, \ldots, m \) only and call this process:

To define the new process formally replace in \((K_s^\vec{\alpha})_{s \geq \log \alpha_l}\) every \((i, t)\) by \((i, \log \alpha_k)\) if \( t \in [\log \alpha_{(k-1)}, \log \alpha_k), k = 1, \ldots, m \), where \( \alpha_0 := 0 \).

We are particularly interested in the state of \( K^\vec{\alpha} \) at time 0. We shall show that it agrees with the coalescent with (continuous) rebirth \((K_s[\log \alpha_l, \log \alpha_u])_{s \geq \log \alpha_l}\) with respect to the functional of interest. This observation will then imply the statement once we have handled the case of finitely many rebirth times introduced above.

Denote by \( \hat{N}_k^\vec{\alpha} \) the total number of partition elements of \( K_s^\vec{\alpha} \) with birth time \textit{equal to or smaller than} \( \log \alpha_k \). Note that \( 1 \leq \hat{N}_1^\vec{\alpha} \leq \hat{N}_2^\vec{\alpha} \leq \ldots \leq \hat{N}_m^\vec{\alpha} \), almost surely. By construction, it is not difficult to verify that the following key identity holds,
\[
\hat{N}^\vec{\alpha} = (N_{\alpha_1}, \ldots, N_{\alpha_m}). \tag{8.5}
\]
Figure 4 illustrates the evolution of the process $K^N_{\mathbf{\alpha}}[\{\log \alpha_1, \ldots, \log \alpha_3\}]$ to the left of time 0, where partition elements with birth time $\log \alpha_1$ are colored black, those with birth time $\log \alpha_2$ are colored gray, etc. For this realization we see that $N^{\mathbf{\bar{\alpha}}}_1 \geq 1$, $N^{\mathbf{\bar{\alpha}}}_2 \geq 3$, and $N^{\mathbf{\bar{\alpha}}}_3 \geq 4$.

The scenario of Figure 4 corresponds in the spatial set-up to the following. For $\alpha \in (0,1]$ we refer to the set $\Lambda^{\alpha,t}$ as the $\alpha$-box. We observe the coalescents corresponding to the $\alpha_1, \ldots, \alpha_m$-boxes at times $t^{\alpha_1}, \ldots, t^{\alpha_m}$ and finally at time $t$, and apply here our results from Sections 5 and 6. The structure of the arising coalescents is depicted in Figure 5.

Proof of Proposition 8.1 (a.i) Theorem 6 of [9] gives some information for the case of the sparse particles and for instantaneous coalescence in terms of the convergence in the sense of finite dimensional distributions.

The gap between the instantaneous coalescent f.d.d. convergence case of [9] and our delayed coalescent path space convergence case is bridged as in Section 5. It would be tedious to write out (again) all the details, yet we encourage the reader to verify the steps of the argument outlined below.

Step 1 (Sparse individuals). We first treat finitely many sparse particles as initial state, where we can use some techniques from [9]. As above, fix $\alpha_1, \ldots, \alpha_m$, where $0 < \alpha_i \leq \alpha_1 < \alpha_2 < \ldots \alpha_{m-1} < \alpha_m \leq \alpha_u$. Initially consider finitely many particles in the box $\Lambda^{\alpha_1,t}$, $i = 1, \ldots, m$, such that, in analogy to the statement of Proposition 5.1 the initial positions of particles in the box $\Lambda^{\alpha_i,t}$ are contained in $I_{\alpha_i}(c,t)$ and moreover that, for each $i = 2, \ldots, m$:

$$\text{the positions of all particles initially in } \Lambda^{\alpha_i,t} \setminus \Lambda^{\alpha_{i-1},t}, \text{ is in } I_{\alpha_i}(c,t).$$

(8.6)

For concreteness, assume that there are initially :

$$n_1 \text{ particles in } \Lambda^{\alpha_1,t}, \text{ and } n_i \text{ particles in } \Lambda^{\alpha_i,t} \setminus \Lambda^{\alpha_{i-1},t}, i = 2, \ldots, m.$$  

(8.7)

We write for this spatial coalescent

$$(C_t^\mathbf{\bar{\alpha}}, L_t^\mathbf{\bar{\alpha}})_{t \geq 0}. \quad (8.8)$$
Figure 5 illustrates the occurrence of $\tilde{N}^{\alpha} = (\tilde{N}^{\alpha}_{\alpha_1}, \ldots, \tilde{N}^{\alpha}_{\alpha_m})$ in the limit of the spatial coalescent asymptotics. Notice that the colors of the particles in Figures 4 and 5 match on purpose to emphasize the correspondence between space (for the spatial coalescent) and time (for the Kingman-type coalescent with rebirth).

In [9], Theorem 6, it is proved that if the coalescent starts with $n_1, n_2, \ldots, n_m$ particles in $I_{\alpha_1}(c, t), \ldots, I_{\alpha_m}(c, t)$, then

$$P(\#C_{\alpha}^n = m) \Rightarrow P_{n_1, \ldots, n_m, m}(\alpha_1, \ldots, \alpha_m, 1) \quad (8.9)$$

and in (5.3) in [9] the r.h.s. is defined by the following recursive equation (here $t^\beta$ instead of $t$ is considered in (8.9))

$$p_{n_1, \ldots, n_k; m}(\alpha_1, \ldots, \alpha_m; \beta) = \sum_{i_1, \ldots, i_{m-1}} p_{n_1, i_1}(\alpha_1/\alpha_2) \cdots p_{n_m + i_{m-1}, m}(\alpha_m/\beta), \quad (8.10)$$

with (3.10) in [9] defining the input of the recursion for $m = 1$ as

$$p_{n, i}(\alpha) = P(|K^n_{\log(1/\alpha)}| = i). \quad (8.11)$$

It is straightforward to see that the Theorem 6 in [9] now implies (with a reinterpretation of formula (5.3) and (3.10) in [9]) that the following convergence in distribution holds for instantaneous coalescence:

$$(\#\rho_{I_{\alpha_1}} \circ C_{\alpha}^n, \cdots, \#\rho_{I_{\alpha_m}} \circ C_{\alpha}^n, t) \Rightarrow (N_{\alpha_1}^n, \cdots, N_{\alpha_m}^n), \text{ as } t \to \infty. \quad (8.12)$$

where $N_{\alpha}^n$ on the r.h.s. is the number of partition elements added to the system before time $\alpha$ in the following Kingman coalescent with immigration evaluated at time 0. We start in with $n_1$-individuals
at time \( \log \alpha_1 \) and evolve until time \( \log \alpha_2 \) where \( n_2 \) new individuals are added, then continue evolving until time \( \log \alpha_3 \) where \( n_3 \) new individuals are added \( \cdots \), and continue until time \( \log \alpha_n \) where the last immigration takes place. Then the coalescent runs until time 0, without further immigration.

The point here is that the above assumptions ensure that with overwhelming probability, for each \( i = 2, \ldots, m \), none of the particles initially in \( \Lambda(t^{\alpha_n}) \setminus \Lambda(t^{\alpha_{i-1}}) \) coalesce with any other particle during the time interval \([0, g_\alpha(t)]\) (see [8.3] for the definition of \( g_\alpha(t) \)), while during the same time interval, on the appropriate time scale, the evolution of the partitions containing particles with initial positions \( j < i \), a population of size \( n_j \) is adjoined to the existing configuration. By Lemma 5.1, the partitions stay sparse with overwhelming probability, so that the asymptotic exchangeability applies, and an easy inductive argument yields the convergence in this finite setting, where the limit is the described coalescent with immigration (which is different from the limit on the r.h.s. of (8.2), since here we only have sparse individuals). Our arguments give hence a convergence statement for instantaneous coalescence in the sparse case.

As in Lemma 5.2, the convergence of [9] Theorem 6, extends to the convergence in the delayed coalescent setting. Moreover, using the asymptotic exchangeability as in Subsection 5.2, this can be extended to the convergence in path space.

**Step 2** In the previous step we had finitely many sparse particles, even as \( t \to \infty \), in our problem we have in fact a growing number of particles as \( t \to \infty \) and this will lead to the actual limit in (8.1).

The above mentioned “immigration” becomes infinite in the limit as \( n_i \to \infty, i = 2, \ldots, m \). Indeed, the reasoning of Section 7, in particular that of the proof of Theorem 1, based on the estimates of Proposition 6.1 and Proposition 5.2 in Section 6 will extend to the current setting and yield (8.2). The proof is by induction on \( m \). We start with \( m = 2 \).

Let \( \rho < \infty \) and consider the joint asymptotics of \( \# \rho_{t^\alpha_1} \circ C_{2}^{1,t,\rho} \) and \( \# \rho_{t^\alpha_2} \circ C_{2}^{1,t,\rho} \). We know that \( \# \rho_{t^\alpha_1} \circ C_{2}^{1,t,\rho} \) follows approximately the law of \( K_{\log(\alpha_2/\alpha_1)}^\infty \), where \( K^\infty \) is the Kingman coalescent started with infinitely many particles. In particular, \( \{ \# \rho_{t^\alpha_1} \circ C_{2}^{1,t,\rho}, t \geq t_0 \} \) is a tight family of random variables. Moreover, for any \( \varepsilon > 0 \), due to Proposition 6.1 \( \{ \# \rho_{t^\alpha_2} \circ C_{2}^{1,t,\rho}, t \geq t_0 \} \) is a tight family as well.

Due to (6.14), we have that for each \( \varepsilon > 0 \), the total collection of partition elements \( \rho_{t^\alpha_2} \circ C_{2}^{1,t,\rho} \) has positions in \( I_{\alpha_2+\varepsilon}(1, t) \) with overwhelming probability, as \( t \to \infty \). Hence the sparse particle convergence of Proposition 5.1 applies. By letting \( \varepsilon \to 0 \), and using (6.13) and (6.15) as in the proof of Theorem 1 we obtain the statement (a.i) in the case \( m = 2 \). The induction step is standard now.

Note that, in view of the proof of part (b), one should verify the estimates analogous to those of Proposition 6.2, as well as the extension of (a.i), in the slightly more general setting of the coupled spatial coalescents satisfying (6.11).

(a.ii) To prove (8.3) note that by the construction in Subsection 6.1 \( \# \rho_{t^\alpha} \circ C_{1}^{1,t} \) has monotone non-decreasing and càdlàg (or càglàd) paths in \( \alpha \), for all \( t > 0 \), almost surely. Furthermore by Theorem 1 we know that the family \( \{ \# \rho_{t^\alpha} \circ C_{1}^{1,t}; t \geq 0 \} \) is tight, for each \( \alpha < 1 \). Therefore we obtain (8.3).

(b) Again the statements can be easily extended to \( \rho = \infty \), for all \( \delta > 0 \) fixed, using monotonicity and the coupling (7.11).

**Proof of Theorem 4.** So far we have shown with Proposition 8.1 the f.d.d. convergence. It remains to show the tightness in path space. This is now a direct consequence of the monotonicity of the process \( (N_\alpha)_{\alpha \in (0,1]} \), as well as of all the processes \( \{ \# \rho_{t^\alpha} \circ C_{1}^{1,t} \}_{\alpha \in (0,1]} \) and \( \{ \# \rho_{t^\alpha} \circ C_{1}^{1,\infty,\delta} \}_{\alpha \in (0,1]} \) in \( \alpha \), more precisely, of the fact that their paths are non-decreasing and bounded from below (by identity 0), almost surely and from above by (8.3).

\[ \square \]
8.2 Proof of Theorem 5

Fix $\alpha \in (0,1)$. For $m \in \mathbb{N}$, consider the parameters $\alpha < u_1 < u_2 < \ldots < u_m < 1$. Recall the definitions \ref[(3.18)]{8} and \ref[(3.22)]{8}, and as before denote by $\vec{u}/\alpha$ the vector $(u_1/\alpha, \ldots, u_m/\alpha)$.

**Proof of Theorem 5.** Note that the case $m = 1$ is covered by Theorem 1, hence we will assume $m \geq 2$. The key is to understand the case $m = 2$, since then we can conclude the argument easily by making the induction step from $m$ to $m + 1$. We will concentrate on \ref[(3.23)]{8}, and we comment at the very end on the extension \ref[(8.24)]{8}.

Fix a finite $\rho$ and $t \geq t_0$, where as usual $t_0$ is taken from Proposition \ref[(6.1)]{8}. For $i = 1, \ldots, m$, define

$$\bar{C}_i := \{ \pi \in C_{1}^{\mathrm{birth}} : I_{t_i}^{\mathrm{birth}} \in \Lambda_{\alpha,t} \}.$$  \hfill (8.13)

We consider the joint evolution of partition elements $\bar{C}_i$, $i = 1, \ldots, m$. As mentioned before, there are Poisson($\rho$) many partition elements present at each site of the $\alpha$-box, at all times $s \geq 0$, almost surely. In particular, $\# \bar{C}_i$ has Poisson($\rho \cdot \# \Lambda_{\alpha,t}$) distribution.

Note that, for $t$ large, due to \ref[(6.14)]{8} we will have that, with overwhelming probability,

$$\pi^1 \not\subseteq \pi^2, \quad \forall i < j \in \{1, \ldots, m\}, \pi^1 \in \bar{C}_i, \pi^2 \in \bar{C}_j.$$ \hfill (8.14)

In words, it is highly unlikely to have any equivalence class of $C_{1_{i}}^{\mathrm{birth}} \cap \bar{C}_i$ reappear (as a subclass) in the $\alpha$-box at any of the later times $t^u$, $l \in \{i + 1, \ldots, m\}$. We will henceforth consider our realization on the event \ref[(8.14)]{8} in the rest of the argument.

Fix $\delta \in (0,1) \alpha$ a small quantity, which will be sent to 0, eventually. For each $i \in \{1, \ldots, m\}$ and $s \geq t^u_i$, denote by $\bar{N}_s^i$ the number of equivalence classes of $C_{1_{i}}^{\mathrm{birth}}$ containing at least one element of $\bar{C}_i$. By Theorem 1, $\bar{N}_t^{1_{i}}$ follows approximately the law of $\# K_{\log(u_2 / \alpha)}$. By \ref[(6.14)]{8}, the corresponding equivalence classes have locations in $I_{u_2}(1,t)$ at time $t^u_2$, and stay in $I_{u_2}(2+\delta, t)$ during the time interval $[t^u_2, 2t^u_2]$, with overwhelming probability. Note that, similarly, $\bar{N}_t^{1_{i}}$ with overwhelming probability, as $t \to \infty$.

Next consider during the time interval $[t^u_2, 2t^u_2]$ the process counting the number of equivalence classes process for the coalescent ($C_{1_{i}}^{\mathrm{birth}}, I_{t_i}^{\mathrm{birth}}$) restricted to the equivalence classes in $\bar{C}_i$. Due to Theorem 1, the law of the above counting process is (after appropriate rescaling) approximately that of $(\# K_{\log(u_2 / \alpha)}$, $s \in [0, \log(u_3 / u_2)])$, as $t \to \infty$.

Also note that, on $A_{\delta}^{1_{2}}$, the positions of the $\bar{N}_t^{1_{2}}$ acquire $\bar{N}_t^{2_{2}}$ equivalence classes in $C_{1_{2}}^{\mathrm{birth}}$, that contain at least one element either of $\bar{C}_1$ or of $\bar{C}_2$ are contained in $I_{u_2}(2+\delta, t)$. Therefore the joint evolution of these equivalence classes during the time interval $[2t^u_2, t^u_3]$ (by Lemma \ref[(5.2)]{8} and Section 5.2) is again well approximated, on the appropriate scale, by that of $(K_{\log(u_3 / u_2)}[\log(u_3 / u_2)], s \in [0, \log(u_3 / u_2)])$, where the last coalescent process depends on $\bar{N}_t^{1_{2}}, \bar{N}_t^{2_{2}}$ solely through its initial configuration.

Denote by $\bar{u}^i/\alpha$ the vector $(u_1/\alpha, \ldots, u_i/\alpha) \in \mathbb{R}^i$. It is now clear by the above argument that $(\bar{N}_{t}^{1_{3}}, \bar{N}_{t}^{2_{3}})$ converges in law as $t \to \infty$ as follows

$$\begin{align*}
\lim_{t \to \infty} \left( \left\{ \pi \in K_{\log(u_3 / \alpha)}^{\mathrm{mer}, \log(u^2 / \alpha)} : [(\kappa(\pi))]_{\mathrm{mod}(m)} = 0 \right\}, \left\{ \pi \in K_{\log(u_3 / \alpha)}^{\mathrm{mer}, \log(u^2 / \alpha)} : [(\kappa(\pi))]_{\mathrm{mod}(m)} \leq 1 \right\} \right) \end{align*}$$ \hfill (8.16)

By setting $u_3 = 1$, one obtains the result for $m = 2$. 
Moreover, one can use (8.16) in the induction step for the argument where \( m \geq 3 \). In fact, using induction one first obtains for each \( i \), \( 3 \leq i \leq m \) a generalization of (8.16):

\[
(\tilde{N}_{t,v_i}^1, \tilde{N}_{t,v_i}^2, \ldots, \tilde{N}_{t,v_i}^i, \ldots, \tilde{N}_{t,v_i}^{i-1}) \Rightarrow_{i \to \infty} \left( \# \{ \pi \in K_{\log(u_i/\alpha)}^{\text{mer}, \log(\tilde{u}_i^{i-1}/\alpha)} : (\kappa(\pi))_{\text{mod}(m)} \leq 0 \}, \ldots, \right.
\]
\[
\left. \# \{ \pi \in K_{\log(u_i/\alpha)}^{\text{mer}, \log(\tilde{u}_i^{i-1}/\alpha)} : (\kappa(\pi))_{\text{mod}(m)} \leq i - 1 \} \right). \tag{8.17}
\]

and from here easily the general statement of part (a).

Note that part (b) will follow as usual from (3.23) by monotonicity. Here it suffices to extend the result of (a) to the two additional settings where: (i) the initial configuration has precisely one particle at each site, and (ii) the initial configuration has \( 1 + \text{Poisson}(\rho) \) particles at each site, i.i.d. over sites. All the reasoning above carries through provided that for each \( i = 1, \ldots, m \), the configuration \((\tilde{C}_{t,v_i}^{\text{birth}}, \tilde{L}_{t,v_i}^{\text{birth}})\) satisfies an analogue of (6.11). This property is trivially satisfied in the Poisson case, due to stationarity, as mentioned already. In the above more general settings one can verify, by approximating the infinite system by the systems on large finite tori, that the expected number of particles at any particular site at any particular time is bounded from above by a fixed constant (1 in the first setting, and \( 1 + \rho \) in the second one).

\[\square\]

9 Proof of the moment bound

In this section we present the proof of Proposition 6.1 which follows the proof of a similar statement for the instantaneous coalescent stated in the proposition on page 615 in [4]. In [4] the particles move according to the nearest neighbor random walks, while here the partition elements move according to more general random walks. Moreover, coalescence happens with a rate \( \gamma \) delay, and it is therefore possible (often likely) to have more than 1 (up to countable many) partition elements per site.

Proof of Proposition 6.1. Recall the box \( \Lambda(r) \) from (6.1), and let for \( A, B \subseteq \mathbb{Z}^2 \),

\[
(C_s^A, L_s^A)_{s \geq 0}, \tag{9.1}
\]

be the coalescent started from the configuration \((C_0^A, L_0^A)\) restricted to locations in \( A \). This coalescent was denoted by \( C_s^A \) in Subsection 6.1. If \( A = \Lambda(t) \) we will in most cases omit the superscript from the notation. For \( A, B \subseteq \mathbb{Z}^2 \) and \( s \geq 0 \), let

\[
\# C_s^A(B) := \# \{ \pi \in C_s^A : L_s(\pi) \in B \}. \tag{9.2}
\]

As done before, if \( B = \mathbb{Z}^2 \) we simply write \( \# C_s^A := \# C_s^A(\mathbb{Z}^2) \).

Following the lines of Section 3 in [4], we introduce an auxiliary spatial coalescing system \((\tilde{C}, \tilde{L})\) which follows the spatial coalescent dynamics over the time interval \([0, 2]\), then keeps coalescing as long as the number of partition elements is not decreasing too quickly, while otherwise the “coalescence is switched off for a while”. More precisely, we discretize the time on a logarithmic scale, i.e., set for \( T \geq 0 \),

\[
m(T) := \begin{cases} 0, & \text{if } T \leq 1, \\ 2^{\log_2 T}, & \text{if } T > 1. \end{cases} \tag{9.3}
\]

In this way we have \( T \in [m(T), m(2T) \lor 1], T \geq 0 \).

Now, let \((\tilde{C}_0, \tilde{L}_0) := (C_0^{\Lambda(t)}, L_0^{\Lambda(t)})\), and run the coalescent until time \( T = 2 \). To define \((\tilde{C}_t, \tilde{L}_t)\), we proceed by induction. Put

\[
\tau^{\log_2 T} := m(2T) \land \inf \left\{ s \in [m(T), m(2T)] : E[\# \tilde{C}_s] \leq \frac{1}{2} E[\# \tilde{C}_{m(T)}] \right\}, \tag{9.4}
\]
and start $\tilde{C}$ at time $m(T)$ in the spatial configuration given by $\tilde{C}_{m(T)}$. The coalescent $(\tilde{C}, \tilde{L})$ follows the same dynamics as the spatial coalescent on $[m(T), \tau^{[\log_2 T]}]$, while its partition elements perform independent random walks with kernel $a(x,y)$ on $[\tau^{[\log_2 T]}, m(2T)]$ yielding the random configuration $(\tilde{C}_{m(2T)}, \tilde{L}_{m(2T)})$. Now reset $T := 2T$ and repeat the induction step starting at (9.3). Obviously, $E[\#C_t] \geq E[\#\tilde{C}_t]$, for all $t \geq 0$. In fact, one can easily construct a coupling in such a way that the corresponding inequality for processes holds for all times, almost surely. Hence it suffices to prove Proposition 6.1 with $(C, L)$ replaced by $(\tilde{C}, \tilde{L})$.

Set
\[
Y_T := E[\#\tilde{C}_T] = E[\#\tilde{C}_{\Lambda(t)}(Z^2)],
\]
and note that $Y_T$ also depends on $t$ through the initial configuration (6.1), although this is suppressed from the notation.

We will need a few preliminary lemmas. We start with a basic fact estimating the "speed" of escape from large balls centered at the origin for a zero mean random walk with finite exponential moments.

**Lemma 9.1.** Let $(\xi(t))_{t \geq 0}$ be the unit rate continuous time random walk on $\mathbb{Z}$ with transition kernel $b_t(x,y)$. If $\sum_{x \in \mathbb{Z}} b_1(0,x) = 0$ and $\varphi(\lambda) := \sum_{x \in \mathbb{Z}} e^{\lambda x} b_1(0,x) < \infty$, for all $\lambda > 0$, then there exists a finite constant $c_0 = c_0(\xi)$ such that
\[
P\{\xi_t > u\sqrt{t}\} \leq e^{-c_0 u}
\]
for all $u, t \geq 1$.

**Proof.** The argument is based on standard large deviation techniques. For all $s, t, \lambda > 0$,
\[
P\{\xi_t > st\} = P\{e^{\lambda \xi_t} > e^{\lambda st}\} \leq e^{-\lambda st} e^{t(\varphi(\lambda) - 1)}.
\]
In particular, if $I(s) := \sup_{\lambda > 0} \{s \lambda - (\varphi(\lambda) - 1)\}$, then
\[
P\{|\xi_t| > st\} \leq e^{-I(s)t}.
\]
Note that $I(s) : [0, \infty) \to [0, \infty)$ is a convex function, such that $I(s) = 0$ if and only if $s = 0$. Therefore, there exists a positive constant $c_0^1$ such that
\[
I(s) \geq c_0^1 s^2, \quad \text{if } s \geq 1.
\]

Moreover, under our assumptions on exponential moments, there exists a finite constant $c_0^2$ (without loss of generality can assume that $c_0^2 \geq 1$) such that $\varphi(\lambda) \leq 1 + c_0^2 \lambda^2$, for all $\lambda \in [0, 1]$. Thus, for all $s \leq 1$,
\[
I(s) \geq \sup_{\lambda \in [0, 1]} \{s \lambda - c_0^2 \lambda^2\} \geq \frac{1}{4c_0^2} s^2,
\]
where we have used the fact that if $s \leq 1$ then $\lambda^* := \frac{s}{4c_0^2} \leq 1$.

Now set $c_0 := \min\{c_0^1, (4c_0^2)^{-1}\}$, and take $u, t \geq 1$. If $u \geq \sqrt{t}$ we obtain (9.6) from (9.8) by substituting $s = u/\sqrt{t}$ into (9.9). Similarly, if $1 \leq u \leq \sqrt{t}$ we obtain (9.6) by substituting $s = u/\sqrt{t}$ into (9.10).}

The next result states that if the spatial coalescent starts in $\Lambda(t)$, then at time $T$ the fraction of partition elements which lie outside of $\Lambda(t + u\sqrt{T})$ decreases at least exponentially fast, as $u \to \infty$. 


Lemma 9.2. Fix $t > 0$. Let $\tilde{R} := (\tilde{R}^1, \tilde{R}^2)$ be the random walk on $\mathbb{Z}^2$ with kernel $a(x, y)$. Fix $c_0 = c_0(\tilde{R})$ such that (9.6) holds. Put $c_1 := 2(\sqrt{2} - 1) \land e^{c_2}$ where $c_2 := c_2(\tilde{R}) := \sqrt{2}(c_0(\tilde{R}^1) \land c_0(\tilde{R}^2))$. Then

$$E\left[ \#\tilde{C}_T(\Lambda^c(t + u\sqrt{T})) \right] \leq c_1 e^{-c_2 u} Y_T,$$  
(9.11)

for all $u \geq 0$ and $T \geq 1$.

Choosing $a$ large enough so that $c_1 e^{-c_2 a} \leq 1/3$ we obtain the following:

Corollary 9.1. For sufficiently large $a \geq 1$,

$$E\left[ \#\tilde{C}_T(\Lambda^c(t + a\sqrt{T})) \right] \leq \frac{1}{3} Y_T,$$  
(9.12)

for all $T \geq 1$.

Proof of Lemma 9.2. The proof is by induction over $|\log_2 T|$. First, suppose that $2 \leq T \leq 2^4$ and $u \geq 1$. By comparison with the independent random walks equal in law to $\bar{R} := (\bar{R}^1, \bar{R}^2)$ on $\mathbb{Z}^2$, we obtain (with $\| \cdot \|$ the maximum norm)

$$E\left[ \#\tilde{C}_T(\Lambda^c(t + u\sqrt{T})) \right] \leq E\left[ \#C_0^{\Lambda(t)} \right] P^{(0,0)}(\|\tilde{R}_T\| \geq u\sqrt{T}) \leq E\left[ \#C_0^{\Lambda(t)} \right] \left( P^0(\|\bar{R}_T\| \geq u\sqrt{T}) + P^0(\|\bar{R}_T^2\| \geq u\sqrt{T}) \right) \leq 4 \cdot E\left[ \#C_0^{\Lambda(t)} \right] e^{-(c_0(\bar{R}^1) \land c_0(\bar{R}^2))u}.$$  
(9.13)

By definition, $Y_T \geq Y_{2^2} \geq \frac{1}{2} Y_{2^3} \geq \ldots \geq 2^{-4} \#C_0$. Moreover the map $s \mapsto E\left[ \#C_s \right]$ is continuous, and therefore

$$E\left[ \#\tilde{C}_T(\Lambda^c(t + u\sqrt{T})) \right] \leq 2^6 \cdot e^{-(c_0(\bar{R}^1) \land c_0(\bar{R}^2))u} \cdot Y_T,$$  
(9.14)

as required. So (9.11) holds in the case $2 \leq T \leq 2^4$, for all $u \geq 1$, and for $u \in [0, 1]$, (9.11) holds trivially due to the fact that $c_1 e^{-c_2} \geq 1$.

Suppose now that for some $m \geq 1$, (9.11) holds for all $2 \leq T \leq 2^{m+3}$. Then for $T \in (2^{m+3}, 2^{m+4}]$,

$$E\left[ \#\tilde{C}_T(\Lambda^c(t + u\sqrt{T})) \right] \leq E\left[ \#\tilde{C}_{2^m}(\Lambda^c(t + \frac{u}{2} \sqrt{T})) \right] + Y_{2^m} P^{(0,0)}(\|\tilde{R}_{T-2^m}\| \geq \frac{u}{2} \sqrt{T}) \leq 2 \cdot E\left[ \#\tilde{C}_{2^m}(\Lambda^c(t + \sqrt{2}u\sqrt{T})) \right] + Y_{2^m} P^{(0,0)}(\|\tilde{R}_{T-2^m}\| \geq \frac{u}{2} \sqrt{(T-2^m)} \sqrt{(T-2^m)}).$$  
(9.15)

The first inequality above is obtained by the following observation: each partition element in $\Lambda^c(t + \sqrt{2u})$ at time $T$ corresponds to some partition element, located either in $\Lambda(t + \frac{u}{2} \sqrt{T})$ or its complement, at time $2^m$. Applying the induction hypotheses to the first term, and Lemma 9.1 to the second term on the right hand side of (9.15), we obtain that

$$E\left[ \#\tilde{C}_T(\Lambda^c(t + u\sqrt{T})) \right] \leq Y_{2^m} \left( c_1 e^{-c_2 \sqrt{2}u} + 2e^{-c_2 u} \right) \leq Y_T c_1 e^{-c_2 u} (2^4 e^{-c_2 (\sqrt{2} - 1)u} + 2^6 \frac{e^{c_2 u}}{c_1}),$$  
(9.16)

where we have used the facts that $\sqrt{T/(T-2^m)} \leq \sqrt{8/7}$, for all $T \in (2^{m+3}, 2^{m+4}]$, and $Y_T \geq 2^{-4} Y_{2^m}$.
Define \( u_0 := \frac{5 \log 2}{c_2(\sqrt{2} - 1)} \). Then an elementary calculation shows that for all \( u \geq u_0 \),

\[
2^4 e^{-c_2(\sqrt{2} - 1)u} + \frac{2^6}{c_1} \leq 2^4 e^{-c_2(\sqrt{2} - 1)u_0} + \frac{1}{2} \leq 2^4 2^{-5} + \frac{1}{2} \leq 1,
\]
while for all \( u \in [0, u_0] \), \( c_1 e^{-c_2 u} \geq c_1 2^{-\frac{\sqrt{2} - 1}{2}} \geq 1 \), so (9.11) trivially holds for all \( u \in [0, u_0] \). This completes the induction step and the proof.

We next provide an estimate of the rate of decrease for the number of partition elements during an interval of time, provided that the coalescence dynamics is switched on.

For two partition elements \( \{i\}, \{j\} \in C_0 \), put

\[
\sigma^{(i,j)} := \min \{ u \geq 0 : L_u(\{i\}) = L_u(\{j\}) \}
\]

as the waiting time until these particles share the same location, and set

\[
h^u_s(A) := \inf_{i,j \in \mathbb{Z}^4} P\{\sigma^{(i,j)} \leq s\}.
\]

One can verify using a last-exit-time decomposition and the assumption (2.2) (compare Lemma 5 in [5]) that for fixed \( b > 0 \),

\[
h^u_s(\Lambda(br)) \geq M(b) \frac{1}{\log(r)},
\]

for some \( M(b) > 0 \).

Similarly, define \( \tau^{(i,j)} := \min \{ u \geq 0 : i \sim_u j \} \), and set for \( A \subseteq \mathbb{Z}^2 \),

\[
H^u_s(A) := \inf_{i,j \in \mathbb{Z}^4} P\{\tau^{(i,j)} \leq s\}.
\]

We are particularly interested in bounding from below the quantity

\[
H^u_{sT}(\Lambda(\sqrt{2} R_T)),
\]

where

\[
R_T = R^a_{T, t} := 7(1 + a) \sqrt{\frac{t^2 + 2T}{t_T}},
\]

with \( a \geq 1 \) chosen according to Corollary 9.1 such that (9.12) holds. We will henceforth assume that \( T \leq t^3 \) (as in (9.39) below). Then, if

\[
s_T := 4R^2_T,
\]

inequality (9.20) implies that

\[
h_{sT/2}(\Lambda(\sqrt{2} R_T)) \geq \frac{M(1)}{\log R_T} \geq \frac{M_1}{\log t},
\]

where \( M_1 \in (0, 2M(1)/3) \subset (0, \infty) \) is chosen depending on \( a \). Recalling inequality (7.48) from [19], we obtain that

\[
H^u_{sT}(\Lambda(\sqrt{2} R_T)) \geq \frac{\gamma}{2 + \gamma} \left( 1 - \exp\left( - \frac{2 + \gamma}{2} s_T \right) \right) h_{sT/2}(\Lambda(\sqrt{2} R_T)) \geq \frac{M_2}{\log t},
\]

for some \( M_2 \in (0, \infty) \), for all \( t \geq 2 \), where we use \( s_T \geq 4 \cdot 49 \cdot (1 + a)^2 \cdot \frac{t^2}{2y_0} > 0 \), since \( t \geq 2 \).

**Lemma 9.3** (Rate of decay for the auxiliary coalescent). Let \( 2 \leq T \leq r < r + s \leq 2T \). Suppose that \( Y_T \geq 49 \), and that \( \mathcal{C} \) is coalescing during the entire time interval \([T, r + s] \). Then

\[
Y_{r+s} \leq Y_r \exp \left[ - \frac{1}{3} H^u_s(\Lambda(\sqrt{2} R_T)) \right].
\]
9 PROOF OF THE MOMENT BOUND

Proof. Write \( C_s^C \) for the spatial coalescent started in the random partition \( C \) at time 0, and evaluated at time \( s \). For all \( T \leq r < r+s \leq 2T \),

\[
Y_{r+s} \leq E[\# C_s^{C_r}(\Lambda(t+a\sqrt{T}))] + E[\# \tilde{C}_r(\Lambda^c(t+a\sqrt{r}))].
\] (9.28)

Choose a covering of \( \Lambda(t+a\sqrt{T}) \) by

\[
n_T := \left[ 1 + \frac{[\text{Area}(\Lambda(t+a\sqrt{T}))]^{1/2}}{R_T} \right]^2
\] (9.29)
disjoint boxes \( \{\Lambda_{i,r}, i = 1, \ldots, n_T\} \) of side length

\[
l_T := \left( \frac{\text{Area}(\Lambda(t+a\sqrt{T}))}{n_T} \right)^{1/2} \leq \sqrt{2}R_T.
\] (9.30)

The last inequality holds since \( r \in [T, 2T] \).

After ignoring coalescing events between partition elements that are located in different sub-boxes \( \Lambda_{r,i} \cap \Lambda_{r,j} = \emptyset \) at time \( r \), one can bound from above the first term on the right hand side of (9.28) by

\[
\sum_{i=1}^{n_T} \sum_{C:C(\Lambda_{i,r})=\emptyset} P\{\tilde{C}_r(\Lambda(t+a\sqrt{T}) \cap \Lambda_{i,r}) = C(\Lambda(t+a\sqrt{T}) \cap \Lambda_{i,r})\}E[\# C_s^C].
\] (9.31)

It is straightforward to conclude, as in (7.44)–(7.46) in [19], that for \( C \) as above

\[
E[\# C_s^C] \leq \# C - (\# C - 1)H^*_\lambda(\Lambda(\sqrt{2}R_T)).
\] (9.32)

Insert (9.32) into (9.31) to get

\[
E[\# C_s^{C_r}(\Lambda(t+a\sqrt{T}))]
\leq \sum_{i=1}^{n_T} \sum_{C:C(\Lambda_{i,r})=\emptyset} P\{\tilde{C}_r(\Lambda(t+a\sqrt{T}) \cap \Lambda_{i,r}) = C(\Lambda(t+a\sqrt{T}) \cap \Lambda_{i,r})\}
\cdot \left( \# C(\Lambda(t+a\sqrt{T}) \cap \Lambda_{i,r}) - (\# C(\Lambda(t+a\sqrt{T}) \cap \Lambda_{i,r}) - 1)H^*_\lambda(\Lambda(\sqrt{2}R_T)) \right)
= E[\# \tilde{C}_r(\Lambda(t+a\sqrt{T}))] - E[\# \tilde{C}_r(\Lambda(t+a\sqrt{T}))] - n_T)H^*_\lambda(\Lambda(\sqrt{2}R_T))
\leq E[\# \tilde{C}_r(\Lambda(t+a\sqrt{T}))] \left( 1 - \frac{1}{2}H^*_\lambda(\Lambda(\sqrt{2}R_T)) \right).
\] (9.33)

For the last inequality in (9.33) we use (9.29) and the following observations

(a) \( Y_u \geq Y_T/2 \), for all \( u \in [T, r+s) \), and therefore in particular, \( Y_r \geq Y_T/2 \), since otherwise the coalescing would not last during the entire interval \([T, r+s)\),

(b) for any \( r \geq 1 \),

\[
Y_r = E[\# \tilde{C}_r(\Lambda(t+a\sqrt{T}))] + E[\# \tilde{C}_r(\Lambda^c(t+a\sqrt{T}))]
\leq E[\# \tilde{C}_r(\Lambda(t+a\sqrt{T}))] + \frac{Y_r}{3},
\] (9.34)

by Corollary 9.41 and

\[
n_T \leq \frac{2}{7} \sqrt{Y_T}^2 \leq \frac{4}{49} \cdot 4Y_r
\leq \frac{4 \cdot 4}{49} \cdot \frac{3}{2} E[\# \tilde{C}_r(\Lambda(t+a\sqrt{T}))] \leq \frac{1}{2} E[\# \tilde{C}_r(\Lambda(t+a\sqrt{T}))].
\] (9.35)
Now by (9.28), (9.33), (9.34) and (9.12), we have
\[
\begin{align*}
Y_{t+s} & \leq E\left[\#\tilde{C}_r(\Lambda(t+\sqrt{2}t))\left(1 - \frac{1}{2}H_s^\gamma(\Lambda(\sqrt{2}R_T))\right) + E\left[\#\tilde{C}_r(\Lambda^c(t+\sqrt{2}t))\right]\right] \\
& = Y_r\left(1 - \frac{1}{2}H_s^\gamma(\Lambda(\sqrt{2}R_T))\right) + \frac{1}{6}Y_r H_s^\gamma(\Lambda(\sqrt{2}R_T)) \\
& \leq Y_r\left(1 - \frac{1}{3}H_s^\gamma(\Lambda(\sqrt{2}R_T))\right) \\
& \leq Y_r \exp\left(-\frac{1}{3}H_s^\gamma(\Lambda(\sqrt{2}R_T))\right),
\end{align*}
\]
as required. \(\Box\)

**Lemma 9.4** (Upper bound for the decay rate of partition elements). Fix \(t \geq 2\), and let for \(T \geq 2\),
\[
g(T) := \frac{\log(1 + T)}{\log t} \cdot Y_T \cdot \left(1 \lor \frac{E[\#C_2^\Lambda(t)]}{t^2}\right)^{-1}, \quad T \geq 2.
\]
Then there exists a finite constant \(M\) such that
\[
g(T) \leq M, \quad 2 \leq T \leq 4,
\]
and
\[
g(2T) \leq g(T) \lor M, \quad 2 \leq T \leq t^3.
\]

**Proof.** Recall \(M_2\) from (9.20), and fix \(a \geq 1 \lor \frac{M_2 \log_2 5}{48}\) suitably large such that (9.12) holds. Put
\[
M := \frac{3 \cdot 16 \cdot 49 \cdot a(1+a)^2}{M_2},
\]
and notice that \(M \geq 49 \cdot \log_2 5\).

Assume first that \(2 \leq T \leq 4\). In this case, since \(Y_T/t^2 \leq Y_2/t^2 \leq 1 \lor E[C_2^\Lambda(t)]/t^2\), and since \(\log(1+x) \leq x\), for all \(x > -1\),
\[
g(T) \leq \frac{Tt^2}{t^2 \log t} \leq \frac{4}{\log 2} \leq M.
\]
Next assume that \(2 \leq T \leq t^3\) and \(Y_T \leq 49\). Then since \(Y_{2T} \leq Y_T \leq 49\), we get
\[
g(2T) \leq 49 \frac{\log(1+2t)}{\log t} \leq 49 \log_2 5 \leq M.
\]
It therefore remains to prove (9.39) for \(Y_T > 49\). Without loss of generality we may assume that
\(\tilde{C}\) is coalescing during the entire interval \((T, \frac{3}{2}T)\).

Indeed otherwise we could find an \(m \in \mathbb{N}\) such that \(\tau^m \in (T, \frac{3}{2}T)\) (recall 9.3) and therefore since \(Y_{2T} \leq Y_{2m} \leq Y_T/2\), we get
\[
\frac{g(2T)}{g(T)} \leq \frac{1}{2} \cdot \frac{\log(1+\frac{2T}{\tau})}{\log(1+\frac{T}{\tau})} \leq 1.
\]
However, under (9.33), Lemma 9.3 applies with any \((r, r + s) \subset (T, \frac{3}{2}T)\), so that \(\lfloor \frac{T}{2sT} \rfloor\) iterations of (9.27) yield that
\[
Y_{2T} \leq Y_T \exp\left[-\frac{1}{3}\left(\frac{T}{2sT}\right)H_{sT}^\gamma(\Lambda(\sqrt{2}R_T))\right].
\]
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By (9.24),

\[
\left| \frac{T}{2s_T} \right| \geq \frac{T}{4s_T} = \frac{Y_T T}{16 \cdot 49 \cdot (1 + a)^2 (t^2 + aT)} \geq g(T) \frac{T \log t}{16 \cdot 49 \cdot (1 + a)^2 (t^2 + aT) \log (1 + \frac{t}{T})}.
\]

(9.45)

Finally, inserting (9.45) and (9.44) into (9.37), and recalling (9.26), yields

\[
\frac{g(2T)}{g(T)} \leq \frac{\log (1 + \frac{2T}{T})}{\log (1 + \frac{1}{T})} \exp \left[ -\frac{1}{3} \left| \frac{T}{2s_T} \right| H_{s_T}^T (\Lambda(\sqrt{2}R_T)) \right] \\
\leq \exp \left[ \frac{T}{(t^2 + T) \log (1 + \frac{1}{T})} - \frac{1}{3} \left| \frac{T}{2s_T} \right| H_{s_T}^T (\Lambda(\sqrt{2}R_T)) \right] \\
\leq \exp \left[ \frac{T}{(t^2 + T) \log (1 + \frac{1}{T})} (1 - M^{-1}g(T)) \right].
\]

(9.46)

We therefore find that either \( g(T) \leq M \) or if \( g(T) > M \) then \( g(2T) \leq g(T) \), which proves (9.39). \( \square \)

To finish off the proof of the proposition, note that Lemma 9.4 readily implies \( g(T) \leq M \), for all \( t \geq 2, 0 \leq T \leq t^3 \). Therefore,

\[
Y_T \leq M \frac{\log t}{\log (1 + \frac{1}{T})} \left( 1 \lor \frac{\mathbb{E}[\#C^N(0)]}{t^2} \right), \quad 2 \leq T \leq t^3,
\]

and after replacing \( t \) with \( t^{\alpha/2} \) and \( T \) with \( t^\beta \) where \( \beta \in (\alpha, 3\alpha/2] \),

\[
Y_{t^{\beta}} \leq M \frac{\log t^{\alpha/2}}{\log (1 + \frac{1}{T})} \left( 1 \lor \frac{\mathbb{E}[\#C^N(t^{\alpha/2})]}{t^\alpha} \right) \leq M \left( 1 \lor \frac{\alpha}{2(\beta - \alpha)} \lor \frac{\mathbb{E}[\#C^N(t^{\alpha/2})]}{t^\alpha} \right).
\]

(9.48)

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