NON-SEPARABLY VALUED ORLICZ SPACES

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Abstract. For a measure space \( \Omega \) we extend the theory of Orlicz spaces generated by an even convex integrand \( \varphi: \Omega \times X \to [0, \infty] \) to the case when the range Banach space \( X \) is arbitrary. Besides settling fundamental structural properties such as completeness, we characterize separability, reflexivity and represent the dual space. This representation includes for the first time the case when \( X' \) has no Radon-Nikodym property. Moreover, we characterize relatively sequentially compact sets in weak topologies that subspaces of the Orlicz space and subspaces of its dual induce on each other. We apply our theory to represent convex conjugates and subdifferentials of general integral functionals, leading to the first such result on function spaces with non-separable range space. For this, we prove a new general interchange criterion between infimum and integral for non-separable range spaces, which we consider of independent interest.

1. Introduction

We initiate a theory of non-separably vector valued Orlicz spaces \( L_{\varphi}(\mu) \) generated by an even convex integrand \( \varphi: \Omega \times X \to [0, \infty] \) when the range Banach space \( X \) is arbitrary. Requiring \( \varphi \) to satisfy

\[
\lim_{x \to 0} \varphi(\omega, x) = 0, \quad \lim_{|x| \to \infty} \varphi(\omega, x) = \infty \quad \text{for } \mu\text{-a.e. } \omega \in \Omega,
\]

our \( L_{\varphi}(\mu) \) consists of those strongly measurable functions having finite Luxemburg norm

\[
\|u\|_{\varphi} = \inf \left\{ \alpha > 0 \left| \int \varphi(\omega, \alpha^{-1} u(\omega)) \, d\mu(\omega) \leq 1 \right. \right\}.
\]  (1)

We settle basic properties such as completeness, characterize separability and reflexivity in terms of the parameters \( \varphi, X \) and \( \mu \), represent the dual space and study sequential compactness in various weak topologies that \( L_{\varphi}(\mu) \) and its dual space or subsets of them induce on each other, which yields substitutes in absence of reflexivity.

The first important novelty in our approach to vector valued Orlicz spaces consists of extending the theory to arbitrary range spaces, overcoming the need for separability of \( X \) or the Radon-Nikodym property of the dual \( X' \). The second one consists of yet allowing an extensive class of \( \omega \)-dependent generating integrands that contains all normal ones, overcoming the pertaining measurability problems on non-separable range spaces. As a major new application of the theory, we obtain for an integrand \( f: \Omega \times X \to [-\infty, \infty] \) the first general representation result of the convex conjugate and the Fenchel-Moreau subdifferential of an integral functional

\[
I_f(u) = \int f(\omega, u(\omega)) \, d\mu(\omega)
\]  (2)
on a non-separably valued function space. This contrasts with all prior such results, which were either restricted to separable range spaces or had to assume that the measure space carries some topological structure in which the integrand or its subdifferential enjoy at least a sort of semicontinuity, cf., e.g., [6, Ch. VII, §3] or [7, §2.7]. As our strategy of proof for treating the non-separable range space will remain valid in more general situations, this idea opens the door to extending various types of results to non-separably valued function spaces, e.g. representing different kinds of subdifferentials for integral functionals as in [15] or advancing the theory of further non-separably valued function spaces as in [19].

The basic theory of vector valued Orlicz spaces for $\dim X = \infty$ and an $\omega$-dependent generating integrand was initiated by A. Kozek [24, 25], who addressed duality and the representation of Fenchel-Moreau subdifferentials. It was further developed by E. Giner [13], who studied duality and compact subsets. The main results in these fundamental papers were restricted to the case when the dual space $X'$ is separable. Equivalently, one may state that $X$ was separable and $X'$ had the Radon-Nikodym property. Both conditions entered crucially in the duality theory to identify the adjoint Orlicz space $L_{\varphi'}(\mu)$ with the function component of the dual space of $L_{\varphi}(\mu)$. First, unless $X'$ has the Radon-Nikodym property, there is no hope of obtaining each element of the function component as a strongly measurable one. Second, even if this is the case, to equate this function space with $L_{\varphi}(\mu)$, one needs to prove that the Luxemburg norm of $I_{\varphi}$ defines an equivalent norm on it. This is tantamount to proving that the convex conjugate $I_{\varphi}^*$ with respect to the standard integral pairing

$$ (u, v) \mapsto \int \langle v(\omega), u(\omega) \rangle \, d\mu(\omega) \quad (3) $$

is given by $I_{\varphi^*}$ on the function component. If $X$ is separable, then this holds, a result usually proved by appealing to an interchange criterion between integral and infimum of the form

$$ \inf_u I_f(u) = \int \inf_{x \in X} f(\omega, x) \, d\mu(\omega). \quad (4) $$

The proof of (4) rests crucially on the Kuratowski measurable selection theorem and the theory of normal integrands, both of which are not known to admit counterparts in non-separable metric spaces. Therefore, to surmount the limitation of a separable range space, we prove for $S(X)$ the separable subspaces of $X$ a new and general interchange criterion between infimum and integral of the form

$$ \inf_u I_f(u) = \int \text{ess-inf}_{W \in S(X)} \inf_{x \in W} f(\omega, x) \, d\mu(\omega). \quad (5) $$

This formula is a remarkable generalization of (4), as it remains valid if $X$ is not separable while still relating the infimum to a pointwise infimum of the integrand, though generally not in the simplest possible way. For in fact, there always exists a $W$ for which the essential infimum function is attained if the common value in (5) is finite. This opens the door to calculating on a pair of function spaces, whose elements are strongly measurable functions $u: \Omega \to X$ and weak* measurable ones $v: \Omega \to X'$, the convex conjugate of $I_f$ with respect to the pairing (3) in a form from which the subdifferential can be read. In contrast, the insight that computing the conjugate this way yields for the duality theory is negative: the convex conjugate $I_{\varphi^*}$ is in general not given by $I_{\varphi^*}$ so that we can not simply copy the separably valued
duality theory. Nevertheless, a characterization of those generating integrands \( \varphi \) for which this is possible to some extent can be read from our conjugate formula. For the general case, some new ideas are in order: by an argument exploiting essential infimum functions indexed by the separable subspaces of \( X \), we can prove the following almost embedding: if \( \mu \) is finite, given \( \varepsilon > 0 \), there exists \( \Omega_\varepsilon \) with \( \mu(\Omega, \Omega_\varepsilon) < \varepsilon \) such that there hold the continuous embeddings

\[
L_\infty(\Omega_\varepsilon; X) \to L_\varphi(\Omega_\varepsilon) \to L_1(\Omega_\varepsilon; X).
\]

Thus, properties of \( L_\varphi(\mu) \) can be systematically derived from those of the better understood Bochner-Lebesgue spaces. We will represent the function component of the dual space this way and characterize compactness in weak topologies. Together with (5), the inconspicuous almost embedding (6) is enough of a basis to build a significant duality theory, including convex conjugacy of a general integral functional on \( L_\varphi(\mu) \).

A secondary yet nice improvement we achieve besides arbitrary range spaces is avoiding uniformity assumptions on the behaviour of \( \varphi \) at the origin or infinity. Even in the finite-dimensional setting, the previous theory of Orlicz spaces [16, 27] imposes such conditions to ensure that sufficiently many functions are contained in the Orlicz space, i.e. it is a kind of decomposability condition. We circumvent such uniformity assumptions by introducing the notion of an almost decomposable space, Definition 2.1, and recognizing \( L_\varphi(\mu) \) and some of its most important subspaces as such spaces.

Our final choice of topic is to study relatively sequentially compact sets in weak topologies that our Orlicz space \( L_\varphi(\mu) \) and the function component of its dual space or subspaces of them induce on each other. This is motivated by applications to the existence for abstract doubly non-linear evolution equations in non-separable, non-reflexive Banach spaces, which we currently prepare [32]. Such equations are interesting also but not only because Orlicz and Orlicz-Sobolev spaces provide important examples of state spaces that need neither be separable nor reflexive if their generating integrand grows rapidly at infinity. Yet, these and many other non-reflexive spaces possess benign weak compactness properties permitting an existence proof. Our results allow passing to a limit in these proofs once a priori estimates have been obtained.

Open and follow-up questions: We assume the generating integrand \( \varphi \) to vanish continuously at the origin. However, the Orlicz space remains a Banach space if this assumption is dropped. What then is the dual space of \( L_\varphi(\mu) \)? At least for \( \varphi(\omega, x) = \varphi(x) \) we show in Sec. 3 that a discontinuity at the origin can always be avoided by adapting the range space, but in general we do not know. The assumption of an even generating integrand restricts. Can analogues to our results be proved for Orlicz cones generated by a non-even convex generating integrand \( \phi: \Omega \times X \to [0, \infty] \)? Can one devise an associated subdifferential calculus for functionals on locally convex cones that, together with the Orlicz cones \( L_\phi(\mu) \), provides a systematic, more powerful approach to treating unilateral constraints such as obstacle problems on spaces of smooth functions?

Structure of the paper: we prove in Sec. 2 our interchange criterion Theorem 2.1 implying conditions on the function space and the integrand \( f \) under which (5)
holds. From this follows our representation Theorem 2.2 for the convex conjugate $I^*_f$ w.r.t. the pairing (3), implying that the subdifferential $\partial I_f$ are those $v$ whose pointwise restriction to any $W \in \mathcal{S}(X)$ belongs to the subdifferential of the restricted integrand $f_W$ almost everywhere.

In Sec. 3 we define and discuss our notion of a generating integrand and introduce the Orlicz spaces $L_\varphi(\mu)$. Their completeness for a general measure $\mu$ is proved in Theorem 3.1. We prove (6) and that $L_\varphi(\mu)$ is almost decomposable.

Sec. 4 introduces the space $E_\varphi(\mu)$ of the closure of simple functions in $L_\varphi(\mu)$ and demonstrates that its elements can approximate all of $L_\varphi(\mu)$ from below, i.e., there exists for $u \in L_\varphi(\mu)$ an isotonic exhausting sequence of measurable sets $\Omega_j$ such that $u\chi_{\Omega_j} \in E_\varphi(\mu)$. This is an ancillary chapter that in the duality theory will help to identify the function component of the dual space of $L_\varphi(\mu)$.

In Sec. 5 we study the subspace $C_\varphi(\mu)$ of those elements having absolutely continuous norm, that is, which satisfy the implication

$$\mu \left( \lim_n E_n \right) = 0 \implies \lim_n \|u\chi_{E_n}\|_\varphi = 0.$$  

It turns out that those elements of $C_\varphi(\mu)$ vanishing outside a $\sigma$-finite set form the maximal linear subspace of $\text{dom } L_\varphi$ if $\varphi$ is real-valued on atoms, see Theorem 5.1. We characterize the separability of $C_\varphi(\mu)$ in Theorem 5.2 as equivalent to $\mu$ and $X$ being separable. Thereby we also fully understand separability of $L_\varphi(\mu)$ because Lemma 5.6 shows that the identity $C_\varphi(\mu) = L_\varphi(\mu)$ is necessary for $L_\varphi(\mu)$ to be separable. Thus $L_\varphi(\mu)$ is separable iff $\mu$ and $X$ are separable and $\text{dom } L_\varphi$ is linear.

Finally, we characterize when $C_\varphi(\mu)$ is an Asplund space in Lemma 5.7.

Sec. 6 addresses duality theory of $L_\varphi(\mu)$ and its integral functionals: by Theorems 6.1 and 6.2 the dual space of $L_\varphi(\mu)$ is a direct sum consisting of a component of weak* measurable functions $V_\varphi^*(\mu)$ and a singular component of strictly finitely additive vector measures $S_\varphi^*(\mu)$. Moreover, if $X'$ has the Radon-Nikodym property and there holds the identity $I^*_\varphi = L^*_\varphi$ on $V_\varphi^*(\mu)$, which can be checked by means of the results from Sec. 2, then we recover the identity $V_\varphi^*(\mu) = L_\varphi^*(\mu)$ known from the separably valued theory. Theorem 6.3 characterizes that the Orlicz space is reflexive iff $X$ is reflexive and both $\text{dom } L_\varphi$ and $\text{dom } I^*_\varphi$ are linear. For non-atomic, $\sigma$-finite measures, reflexivity also is equivalent to $\varphi$ and $\varphi^*$ satisfying the $\Delta_2$ growth condition

$$\forall W \in \mathcal{S}(X) \exists k \geq 1, f \in L_1(\mu) : \varphi(\omega, 2x) \leq \varphi(\omega, x) + f(\omega) \quad \forall x \in W \mu\text{-a.e.}$$

The section culminates in the convex duality of integral functionals on the Orlicz space, whose conjugate and subdifferential we compute in Theorem 6.4. Conjugates and subdifferentials turn out to behave additively with respect to the direct sum decomposition of the dual, i.e., $I^*_\varphi(\ell) = I^*_{\varphi_a}(\ell_a) + I^*_{\varphi_s}(\ell_s)$ for $\ell = \ell_a + \ell_s$ with $\ell_a \in V_\varphi^*(\mu)$ and $\ell_s \in S_\varphi^*(\mu)$. The function component of the subdifferential behaves as in Theorem 2.2.

In Sec. 7 we characterize relatively sequentially compact (r.s.c.) subsets in the weak topology $\tau$ induced by $V_\varphi^*(\mu)$ on $L_\varphi(\mu)$. Actually, we even consider topologies induced by subsets of $V_\varphi^*(\mu)$ having certain properties. A family of functions $\mathcal{F}$ in $L_\varphi(\mu)$ is r.s.c. in $\tau$ iff (i) it is norm bounded, (ii) pairing $\mathcal{F}$ with any $\tau$-continuous functional yields an equi-integrable set in $L_1(\mu; \mathbb{R})$, (iii) it is weakly compact in $L_1(\Omega_j; X)$ on an exhausting sequence of sets $\Omega_j$. Of course, the last condition can be equivalently replaced by any condition that together with norm boundedness
and equi-integrability will yield weak compactness in the Bochner-Lebesgue space $L_1(F; X)$ on a finite measure space $F$, Theorem 7.1. We discuss three such conditions. Mirroring the previous considerations, we also study $V_{\varphi\mu}(\mu)$ in the topologies induced by $C_\varphi(\mu)$ or $L_\varphi(\mu)$. A set $G$ turns out to be r.s.c. only if (i) it is norm bounded, (ii) pairing $G$ with any continuous functional yields an equi-integrable set in $L_1(\mu; \mathbb{R})$ and (iii) the integrals $(\int_A v \, d\mu)_{v \in G}$ are weak$^*$ r.s.c. in $X'$ for any measurable set such that $x\chi_A$, $x \in X$, belongs to $C_\varphi(\mu)$ or $L_\varphi(\mu)$. If in addition (iv) any sequence in $G$ generates an initial $\sigma$-algebra via its weak$^*$ measurability on which $\mu$ is separable, then this necessary condition is also sufficient, Theorem 7.2. Analogous to Theorem 7.1, this result can be interpreted to say that sets are r.s.c. iff they are bounded, weak$^*$ equi-integrable and r.s.c. in $\sigma(V_1(\Omega_j; X'); L_\infty(\Omega_j; X))$.

**Remark on notation:** $(\Omega, A, \mu)$ is a measure space with non-trivial positive measure. $A_\mu$ is the completion of $A$ w.r.t. $\mu$ and $\overline{\mu}$ is the completion of $\mu$. The ring of sets having finite measure is $A_f$, the $\sigma$-ring of sets having $\sigma$-finite measure is $A_\sigma$, the $\sigma$-ring of countable unions of atoms and $\sigma$-finite sets is $A_{\text{atom}}$. If $A \subset \Omega$, then $A(A)$ denotes the trace $\sigma$-algebra of $A$ on $\Omega$. The indicator of a set $S$ is $\chi_S$ with $\chi_S(a) = 1$ if $a \in S$ and $\chi_S(a) = 0$ otherwise. For a sequence $A_n \in A$ we write $\lim_n A_n = A$ to mean $\chi_{A_n} \to \chi_A$ almost everywhere. $T$ is a topological space, $(M, d)$ a metric space and $X$ is a real Banach space with dual space $X'$. A ball in $M$ with centre $x$ and radius $r > 0$ is denoted by $B_r(x)$. If no radius is specified, then $r = 1$. If no centre is specified, then $x = 0$ if $M = X$. We write $B_W = B \cap W$ for $W \subset M$. The system $S(T)$ are the separable subsets of $T$, $CL(T)$ are the closed subsets of $T$, $LS(T)$ are the lower semicontinuous, proper functions $f : T \to (-\infty, \infty]$ and $\Gamma(X)$ are the closed, convex, proper functions $g : X \to (-\infty, \infty]$. $L_0(\Omega; M)$ are the strongly measurable functions $u : \Omega \to M$. The function $u$ is strongly measurable if there exists a sequence $u_n : \Omega \to M$ of measurable functions taking finitely many values such that $u_n \to u$ pointwise. This is equivalent to $u$ being the uniform limit of a sequence of measurable functions taking countably many values; or to $u$ being measurable and having a separable range [5, Prop. 1.9]. A function $u : \Omega \to X'$ is weak$^*$ measurable if for any $x \in X$ the function $\langle v(\omega), x \rangle$ is measurable.

2. **AN INF-INT INTERCHANGE CRITERION AND CONVEX CONJUGACY**

We prove in this section the interchange criterion (5) and compute with it the convex conjugate of a general integral functional $I_f$. Besides representing the subdifferential, we conclude from the conjugate formula a characterization of those integrands for which integration and convex conjugacy continue to commute as if $X$ were separable. To make our criterion applicable, we propose two sufficient conditions: if $\Omega$ is a separable metric Borel space, $X$ a reflexive Banach space and $f$ a normal convex integrand, then $I_f^*(v) = I_{f^*}(v)$ for any strongly measurable function $v$ admissible for the pairing (3). If $f$ is strongly measurable in the Attouch-Wets topology, e.g. independent of $\omega \in \Omega$, then the same is true without further assumptions on $\Omega$ or $X$. Even if $X$ is separable, our result is more general than previous ones since the measure $\mu$ may be arbitrary. We shall briefly relate our result to similar criteria after the proof.
2.1. The interchange criterion. Before we can state and prove our interchange criterion, we define necessary notions and provide measure theoretic background material. We work with a metric range space \( M \) as this adds no complications.

**Definition 2.1** (almost decomposable space). A space \( S \) of (strongly) measurable functions \( u : \Omega \to M \) is almost decomposable with respect to \( \mu \) if for every \( u_0 \in S \), every \( F \in \mathcal{A}_f \), every \( \varepsilon > 0 \) and every bounded (strongly) measurable function \( u_1 : F \to M \) there exists \( F_\varepsilon \subset F \) with \( \mu (F \setminus F_\varepsilon) < \varepsilon \) such that the function

\[
    u (\omega) = \begin{cases} 
    u_0 (\omega) & \text{for } \omega \in \Omega \setminus F_\varepsilon, \\
    u_1 (\omega) & \text{for } \omega \in F_\varepsilon 
    \end{cases}
\]

belongs to \( S \). The space \( S \) is decomposable if \( F_\varepsilon = F \) may be chosen. \( S \) is weakly (almost) decomposable if only \( u_1 \in S \) are allowed.

Equivalently, \( u_1 \) may be unbounded in the definition of almost decomposability. However, the same is not possible for decomposability. If two function spaces defined over the same measure space \( \Omega \) and the same range space \( M \) are almost decomposable and weakly decomposable, then their intersection retains both properties. If \( S \) is a weakly decomposable vector space of \( X \)-valued functions, then its weak decomposability is equivalent to closedness under multiplication by indicators of sets having finite or co-finite measure.

As we aim to prove our interchange criterion for general measures, we need a proposition about divergent integrals.

**Proposition 2.1.** Let \( \alpha : \Omega \to [0, \infty] \) be a measurable function with \( \int \alpha \, d\mu = \infty \).

There either exists \( A \in \mathcal{A}_\sigma \) or an atom \( A \) with \( \mu (A) = \infty \) such that \( \int_A \alpha \, d\mu = \infty \).

**Proof.** Employing [12, Prop. 1.22] and its terminology we find a pair of measures \( \mu_i \) with \( \mu_1 \) purely atomic, \( \mu_2 \) non-atomic and \( \mu = \mu_1 + \mu_2 \). Setting \( d\nu_1 = ad\mu_1 \), either \( \nu_1 \) or \( \nu_2 \) is infinite. If \( \nu_1 \) is infinite, let \( \mathcal{A}_n \) be the system of countable unions of atoms and \( A_n \in \mathcal{A}_n \) an isotonic sequence with \( \lim_n \nu_1 (A_n) = \sup_{A \in \mathcal{A}_n} \nu_1 (A) \). There is nothing left to prove if the supremum is infinite. Otherwise, we set \( A = \lim_n A_n \).

As \( \nu_1 (A^n) = \infty \) and \( \mu_1 \) is purely atomic, the set \( A^n \cap \{ \alpha > 0 \} \) has positive measure whence it contains an atom \( A_{\nu_1} \). In particular \( \nu_1 (A_{\nu_1}) > 0 \) so that \( \nu_1 (A_{\nu_1} \cup A) \) surpasses the supremum, a contradiction; the countable union of atoms \( A \) either is \( \sigma \)-finite or contains an atom of infinite measure.

In the remaining case if \( \nu_2 \) is infinite, we consider an isotonic sequence \( B_n \in \mathcal{A}_\sigma \) with \( \lim_n \nu_2 (B_n) = \sup_{A \in \mathcal{A}_\sigma} \nu_2 (A) \). Again, we are finished if this supremum is infinite. Otherwise, set \( B = \lim_n B_n \).

Since \( \nu_2 (B^n) = \infty \) and \( \mu_2 \) is non-atomic, there exists \( F \subset B^n \cap \{ \alpha > 0 \} \) with \( 0 < \mu_2 (F) < \infty \). We may assume \( \nu_1 (F) < \infty \) without loss of generality so that \( F \in \mathcal{A}_\sigma \). But then \( B \cup F \in \mathcal{A}_\sigma \) and \( \nu_2 (B \cup F) \) surpasses the supremum, yielding a contradiction.

Proposition 2.1 prompts us to define a notion of integral that will be apt for stating our interchange criterion very concisely. Denoting by \( \mathcal{A}_{a\sigma} \) the \( \sigma \)-ring of sets arising as a union of countably many \( \mu \)-atoms and a \( \sigma \)-finite set, we call a function \( \alpha : \Omega \to [-\infty, \infty] \) such that the restriction of \( \alpha \) to any \( A \in \mathcal{A}_{a\sigma} \) is measurable integrally measurable. Similarly, we shall say that some measurability property holds integrally if it holds on any atom and every \( \sigma \)-finite set. In particular, we consider integrally negligible sets, which are defined as sets whose intersection with any atom or \( \sigma \)-finite set is null. We say that a property holds integrally almost everywhere
if it holds except on an integral null set and abbreviate this by i.a.e. A moment’s reflection together with [12, Prop. 1.22] shows that a measurable set is integrally null if and only if it is null. We define the integral of the integrally measurable positive part \( \alpha^+ \) as

\[
\int \alpha^+ \, d\mu = \sup_{A \in \mathcal{A}_{\sigma}} \int_A \alpha^+ \, d\mu.
\]

As usual, we then define \( \int \alpha \, d\mu = \int \alpha^+ \, d\mu - \int \alpha^- \, d\mu \) if one of these integrals is finite. Finally, we set \( \int \alpha \, d\mu = \infty \) if neither the positive part \( \alpha^+ \) nor the negative part \( \alpha^- \) is thus integrable. If \( \alpha \) is measurable, this corresponds to the convention of interpreting \( \int \alpha \, d\mu \) as an (extended) Lebesgue integral if \( \alpha^+ \) or \( \alpha^- \) is integrable and setting \( +\infty \) if both parts fail to be so. If \( \int \alpha \, d\mu \) is finite, then the integrally measurable function \( \alpha \) equals a measurable function a.e. since it vanishes outside of a \( \sigma \)-finite set, on which it is measurable. We call this an exhausting integral. This integral is monotone, i.e. if \( \alpha \leq \beta \) i.a.e. then \( \int \alpha \, d\mu \leq \int \beta \, d\mu \). Let \( \alpha_n \) be a sequence of integrally measurable functions converging to a limit function \( \alpha \) locally in \( \mu \) and a.e. on every atom. Then there holds the Fatou lemma

\[
\int \alpha^+ \, d\mu \leq \liminf_n \int \alpha^+_n \, d\mu.
\]

Indeed, if \( A \) is an atom or a set of finite measure, then

\[
\int_A \alpha^+ \, d\mu \leq \liminf_n \int_A \alpha^+_n \, d\mu \leq \liminf_n \int \alpha^+_n \, d\mu
\]

by the classical Fatou lemma. Taking the supremum over all such \( A \) on the left-hand side then yields the claim. More generally, let \( \alpha_A : A \to [-\infty, \infty] \) be a family of measurable functions indexed by \( A \in \mathcal{A}_{\sigma} \) such that \( \alpha_A = \alpha_B \) a.e. on \( A \cap B \). Then we define the exhausting integral of the family \( \alpha_A \) by means of

\[
\int \alpha^+_A \, d\mu = \sup_{A \in \mathcal{A}_{\sigma}} \int_A \alpha^+_A \, d\mu.
\]

This renders the integral of an essential infimum function of an arbitrary family \( v : \Omega \to [-\infty, \infty] \) of measurable functions meaningful, even though it need only exist on any \( \sigma \)-finite set by [12, Lem. 1.108] and on any atom by an elementary consideration. Indeed, in the last case, since any extended real-valued function is constant a.e. on an atom, we may define the essential infimum function as the infimum of these constants. If the integral of such a family is finite, then it derives from an \( \tilde{\mu} \)-integrable function \( \alpha \) by \( \alpha_A = \alpha \) a.e. on each \( A \in \mathcal{A}_{\sigma} \). To see this, pick \( A \in \mathcal{A}_{\sigma} \) where the supremum of the exhausting integral is obtained and argue by contradiction that any member of \( \alpha_A \) vanishes a.e. outside \( A \) as in the proof of Proposition 2.1. Monotonicity and the Fatou lemma continue to hold for this type of integral. When we consider integral functionals in the following, we interpret all integrals in this sense. It is worth mentioning that this is reduces to the extended Lebesgue integral if \( \Omega \) is \( \sigma \)-finite.

We briefly recapitulate technical background on the measurability of integrands. A set-valued multifunction \( \Gamma : \Omega \to \mathcal{P}(T) \) is (Effros) measurable if for every open set \( O \subset T \) the set \( \Gamma^{-1}(O) = \{ \omega \in \Omega \mid \Gamma(\omega) \cap O \neq \emptyset \} \) is measurable. A pre-normal integrand is defined to be a function \( f : \Omega \times T \to [-\infty, \infty] \) such that the epigraphical mapping \( S_f : \Omega \to \mathcal{P}(T) : \omega \mapsto \text{epi} \, f_{\omega} \) is (Effros) measurable. A pre-normal integrand is normal iff \( S_f \) is closed-valued. By Lemma C.7 the normality of an integrand \( f : \Omega \times M \to (-\infty, \infty] \) on a separable metric space \( M \) is equivalent to lower
semicontinuity and $\mathcal{A} \otimes \mathcal{B}(M)$-measurability if the measure $\mu$ is complete. In the following, a subscript $f_W$ denotes the restriction of an integrand $f: \Omega \times M \to [-\infty, \infty]$ in its second component to a subset $W \subset M$.

**Definition 2.2** (separable measurability). An integrand $f: \Omega \times M \to [-\infty, \infty]$ is said to be separably measurable if for any $W \in \mathcal{S}(M)$ the restriction $f_W$ is $\mathcal{A} \otimes \mathcal{B}(W)$-measurable.

The composition of a separably measurable integrand with a strongly measurable (hence separably valued) function $u: \Omega \to M$ is measurable as a composition of measurable functions.

**Theorem 2.1.** Let $M$ be complete and $R$ a space of integrally strongly measurable functions $u: \Omega \to M$ that is almost decomposable with respect to $\mu$. Let $f: \Omega \times M \to [-\infty, \infty]$ be an integrally separably measurable integrand. Suppose that for any atom $A \in \mathcal{A}$ with $\mu(A) = \infty$ and every $W \in \mathcal{S}(M)$ there holds $\inf_W f_\omega \geq 0$ for a.e. $\omega \in A$. Then, if

$$I_f \neq \infty \text{ on } R \text{ where } I_f(u) = \int f[\omega, u(\omega)] \, d\mu(\omega),$$

one has

$$\inf_{u \in R} I_f(u) = \int \operatorname{ess}-\inf_{W \in \mathcal{S}(M)} \inf_{x \in W} f(\omega, x) \, d\bar{\mu}(\omega). \quad (8)$$

Moreover, if the common value in (8) is not $-\infty$, then the essential infimum function $\bar{m}$ exists on all of $\Omega$ and is attained by a $W \in \mathcal{S}(M)$. In this case, for $\bar{u} \in R$, one has

$$\bar{u} \in \operatorname{Argmin}_{u \in R} I_f(u) \iff f[\omega, \bar{u}(\omega)] = \bar{m}(\omega) \quad \mu\text{-a.e.} \quad (9)$$

We consider essential infimum functions for families of functions $v_i: \Omega \to [-\infty, \infty]$, $i \in I$ an index, such that there exists a family of measurable functions $u_i$ with $u_i = v_i$ a.e. for any $i \in I$. It is elementary to check by [12, Def. 1.106] that the essential infimum functions of the families $v_i$ and $u_i$ agree in this situation. Any such family $v_i$ admits an essential infimum function on any $A \in \mathcal{A}_{\mu\sigma}$ as explained before. The essential infimum function in (8) reduces to the pointwise infimum $\inf_M f_\omega$ if $M$ itself is separable. We may take all integrals in the ordinary extended Lebesgue sense obeying the convention $+\infty - \infty = +\infty$ if $\mu$ is $\sigma$-finite so that Theorem 2.1 is a genuine generalization of the classical infimum-integral interchange criterion [33, Thm. 14.60] from $\sigma$-finite $\mu$ and $M = \mathbb{R}^d$ to arbitrary measures and non-separable metric range spaces.

**Proof.** Generalizing [33, Thm. 14.60], we follow its basic strategy of proof whenever no adaption is necessary. Remember that a strongly measurable function is separably valued. For $A \in \mathcal{A}_{\mu\sigma}$ we set

$$m_A(\omega) = \operatorname{ess}-\inf_{W \in \mathcal{S}(M)} \inf_{x \in W} f(\omega, x), \quad \omega \in A.$$ 

Hence, for any $u \in R$ with $I_f(u) < \infty$ we may apply Proposition 2.1 to find $A_0 \in \mathcal{A}_{\sigma}$ for which

$$\int_{\Omega \setminus A_0} f(u)^+ \, d\mu = 0, \quad \int_{A_0} f(u)^- \, d\mu = \int f(u)^- \, d\mu. \quad (10)$$

We used the assumption $\inf_W f_\omega \geq 0$ on atoms of infinite measure together with $f(u)^- \leq (\inf_W f_\omega)^-$ i.a.e. for $W \in \mathcal{S}(M)$ containing the range of $u$. Thus, for any
sequence \( v_n \in \text{dom } I_f(v_n) \cap R \) with \( \inf_R I_f = \lim_n I_f(v_n) \) there exists \( A_0 \) satisfying (10) simultaneously for all \( v_n \) hence \( \lim_n I_f(v_n) = \lim_n \int_A f(v_n) \, d\mu \geq \int_A m_A \, d\mu \) whenever \( A \in A_r \) with \( A_0 \subset A \). Taking the supremum over \( A \in A_r \), we find \( \inf_R I_f \geq \int m_A \, d\mu \) with the last integral being the exhausting one of the family \( m_A \). It remains to prove the opposite inequality when \( \inf_R I_f > -\infty \). Since \( f(u)^+ \geq (\inf_{W^+} f) \) i.e. for any \( W \in S(M) \) containing the range of \( u \), it suffices to show that for any \( A \in A_r \) and \( \alpha > \int_A m_A \, d\mu \) there exists \( u \in R \) with \( I_f(u) < \alpha \). To simplify notation, we write \( m \) instead of \( m_A \). We may enlarge the subspace \( W \) so that \( m = \inf_W f \) a.e. on \( A \) and \( W \) is closed. We restrict our consideration to the subspace of \( W \)-valued functions in \( R \), so that we may assume \( M \) itself to be separable. Since \( \int_A m \, d\mu < \infty \), the positive part \( m^+ \) is integrable on \( A \) so that

\[
m_\varepsilon(\omega) = \max \{ m(\omega), -\varepsilon^{-1} \}, \quad \lim_{\varepsilon \downarrow 0} \int_A m_\varepsilon \, d\mu = \int_A m \, d\mu
\]

by monotone convergence. The set \( A \) being \( \sigma \)-finite, there exists a non-negative integrable function \( p: \Omega \to \mathbb{R}^+ \) that is positive on \( A \). Setting \( q_\varepsilon(\omega) = \varepsilon p(\omega) + m_\varepsilon(\omega) \), we have \( \int_A q_\varepsilon \, d\mu \to \int_A m \, d\mu < \alpha \) as \( \varepsilon \downarrow 0 \). Since \( q_\varepsilon > m \) on \( A \), the sets

\[
L_\varepsilon(\omega) = \{ x \in M \mid f_w(x) < q_\varepsilon(\omega) \}, \quad \omega \in A
\]

are non-empty. Choose \( \varepsilon \) small enough that \( \int_A q_\varepsilon \, d\mu < \alpha \). Let \( A' \) be the trace \( \sigma \)-algebra of \( A \) on \( \Omega \). By assumption, the integrand \( g := f - q_\varepsilon \) is \( A' \otimes B(M) \)-measurable so that the separably valued multifunction \( L_\varepsilon: A \to \mathcal{P}(M) \setminus \{ \emptyset \} \) has the measurable graph \( \text{gph} \, L_\varepsilon = \{(\omega, x) \in A \times M \mid g_w(x) < 0 \} \), whence there exists a \( A'_\varepsilon \)-measurable selection \( u_1 \) by [12, Thm. 6.10]: an \( A'_\varepsilon \)-measurable function \( u_1: A \to M \) with \( u_1(\omega) \in L_\varepsilon(\omega) \) for all \( \omega \in A \), i.e.

\[
f[\omega, u_1(\omega)] < q_\varepsilon(\omega), \quad \omega \in A.
\]

As \( M \) is separable, Lemma B.3 yields a strongly \( A' \)-measurable function \( u_2 \) with \( u_1 = u_2 \) a.e. We have \( \int_A f[\omega, u_2(\omega)] \, d\mu(\omega) < \alpha \). The set \( A \) being \( \sigma \)-finite, we can express \( A \) as a union of an isotonic sequence of sets \( \Omega_n \) with \( \mu(\Omega_n) < \infty \). Fix \( x \in M \) and let \( A_n = \{ \omega \in \Omega_n \mid |d[x, u_2(\omega)]| \leq n \} \in \mathcal{A} \). Note that \( A_n \uparrow A \). The space \( R \) being almost decomposable, there exists an isotonic sequence \( A'_n \subset A_n \) with \( \mu(A_n \setminus A'_n) < n^{-1} \) such that the function \( w_n: \Omega \to M \) agreeing with \( v_0 \) on \( \Omega \setminus A'_n \) and with \( u_2 \) on \( A'_n \) belongs to \( R \). Since \( A'_n \uparrow A \), we have

\[
\int_{A \setminus A'_n} f[\omega, v_0(\omega)] \, d\mu \to 0, \quad \int_{A'_n} f[\omega, u_2(\omega)] \, d\mu \to \int_A f[\omega, u_2(\omega)] \, d\mu \quad (11)
\]
as \( n \to \infty \) by the theorems of dominated and monotone convergence. Since

\[
I_f(w_n) = \int_{A \setminus A'_n} f[\omega, v_0(\omega)] \, d\mu(\omega) + \int_{A'_n} f[\omega, u_2(\omega)] \, d\mu(\omega),
\]

we have \( I_f(w_n) \to \int A f[\omega, u_2(\omega)] \, d\mu(\omega) < \alpha \) by (11) hence \( I_f(w_n) < \alpha \) if \( n \) is sufficiently large.

Regarding the second part of the claim, we start by showing that the \( A'_\mu \)-measurable function \( m \) induced by the integrable family \( m_A \) indeed defines the essential infimum function in (8) on \( \Omega \). Otherwise there were \( W \in S(M) \) such that the set \( \{ m > \inf_W f \} \) is not contained in a negligible set.

Assume first that \( [m - \inf_W f]^+ \) is \( A'_\mu \)-measurable so that not being contained in a null set is equivalent to having positive measure. No atom \( A \) with \( \mu(A) = \infty \) may contribute to the positive measure since \( m'^+_A \) is integrable as the common value (8)
is not $\infty$. Here, we have used the assumption $\inf_W f \geq 0$ a.e. on atoms of infinite measure. Hence some $A \in \mathcal{A}_\sigma$ contributes to the positive measure by \cite[Prop. 1.22]{12}. But then $m_A > \inf_W f$ a.e. on $A$ is contradictory.

If second the function $[m - \inf_W f]^+$ is only known to be integrally measurable, attempt its integration w.r.t. the completion $\hat{\mu}$ in the exhausting sense. If the integral is finite, then $[m - \inf_W f]^+$ is integrable and integrally measurable hence equals an $\mathcal{A}$-measurable function a.e. We are back to first the case. If the integral is not finite, then the subintegral over an atom of infinite measure or a $\sigma$-finite set is infinite, on which $[m - \inf_W f]^+$ is $A_\mu$-measurable. Proceed as in the first case, arriving at a contradiction; The subintegral hence the integral is finite. We are back to first the case. If the integral is finite, then $m_{\mathcal{A}} - \mu = \infty$ obtains. Therefore some $A \in \mathcal{A}_\sigma$ contributes to the positive measure. Setting $\bar{n} = \max\{n, m\}$ yields the contradiction

$$
\inf_{u \in R} I_f(u) = \lim_n I_f(v_n) \geq \int_{A_\sigma \cup A} \bar{n} d\mu > \int_{A_\sigma \cup A} m d\mu = \int_{A_\sigma} m_{A_\sigma} d\mu = \inf_{u \in R} I_f(u).
$$

To see that the essential infimum function $\bar{m}$ is attained by some $W \in S(M)$ if it is integrable, consider again the sequence $v_n$ with $\int R I_f = \lim_n I_f(v_n)$. Choose $W \in S(M)$ containing the range of $v_n$ and observe that $W$ provides the desired subspace as

$$
\int \bar{m} d\mu = \lim_n I_f(v_n) \geq \inf_W I_f(v_n) \geq \int \bar{m} d\mu.
$$

The addendum (9) is equivalent to $\mu(\{\omega \mid f(\omega, \bar{u}(\omega)) > \bar{m}(\omega)\}) = 0$ if $\int \bar{m} d\mu$ is finite, whence it follows.

We know of no previous interchange result for a function space $S$ with a non-separable range space except \cite[Thm. 6.1]{26}. There it is proved in the particular case of convex conjugacy that if the function space $S$ is weakly decomposable and $M = X$, then the infimum may be computed by taking the $L_1$-infimum under the integral sign. While this formulation appeals by its elegance, it does not satisfy our need to relate the infimum function under the integral sign to the pointwise infimum of the integrand. Under the mere assumption of weak decomposability, no analogue of our result can be expected in this respect, a property like our almost decomposability is indispensable for it. Our criterion could be generalized to the effect that one could compute the infimum function under the integral in $L_1$ on an (almost) weakly decomposable function space $S$ and then derive our representation of this infimum function in the special case when the space has the stronger property of being almost decomposable.

More recently, necessary and sufficient interchange criteria for separable range spaces were discussed in \cite{14}, including an overview of previous results. We note that, at least for $\sigma$-finite measures, an alternative proof of Theorem 2.1 could be devised by appealing to results of \cite{14}. However, since we are interested in bringing the pointwise infimum of the integrand into play, no generalization would result
directly from this, even though [14] provides conditions that are both necessary and sufficient for essential infima to be interchanged with an integral.

2.2. Convex conjugacy. We can now represent the convex conjugate of a general integral functional along with its Fenchel-Moreau subdifferential on a space of strongly measurable functions in duality with a space of weak* measurable ones. Though this result will not apply directly to Orlicz spaces, as their dual space may contain elements that are no functions, it is fundamental in analysing convex conjugacy for the function component of the dual.

**Theorem 2.2.** Let $R$ be a linear space of integrally strongly measurable functions $u: \Omega \to X$ that is almost decomposable with respect to $\mu$. Let $S$ be a linear space of integrally weak* measurable functions $u': \Omega \to X'$ such that the bilinear form

$$ R \times S \to \mathbb{R}: (u, u') \mapsto \int \langle u' (\omega), u (\omega) \rangle \, d\mu (\omega) $$

(12)

is well-defined. Let $f: \Omega \times X \to [-\infty, \infty]$ be an integrally separably measurable integrand. Suppose that for $v \in S$, any atom $A \in \mathcal{A}$ with $\mu (A) = \infty$ and any $W \in \mathcal{S} (X)$ there holds $\sup_{x \in W} \langle v (\omega), x \rangle - f_\omega (x) \leq 0$ for a.e. $\omega \in A$. Then, if

$I_f \neq \infty$ on $R$ where $I_f (u) = \int f (\omega, u (\omega)) \, d\mu (\omega),$

the convex conjugate $I_f^*$ of $I_f$ at $v$ with respect to the pairing (12) is given by

$$ I_f^* (v) = \int \text{ess-sup}_{W \in \mathcal{S}_a (X)} \sup_{x \in W} \langle v (\omega), x \rangle - f_\omega (x) \, d\mu (\omega). $$

(13)

Denoting by $\mathcal{S}_a (X)$ the separable subsets almost containing the range of $u$, the Fenchel-Moreau subdifferential of $I_f$ on $\text{dom} \, I_f$ is given by

$$ \partial I_f (u) = \bigcap_{W \in \mathcal{S}_a (X)} \{ v \in S \, | \, v^*_W (\omega) \in \partial f_W [\omega, u (\omega)] \text{ a.e.} \}. $$

(14)

Moreover, if $v \in \text{dom} I_f^*$, then the following two are equivalent: the mapping $\omega \mapsto f^* [\omega, v (\omega)]$ is $\mathcal{A}_\mu$-measurable and there holds

$$ I_f^* (v) = I_f^* (v) = \int f^* [\omega, v (\omega)] \, d\mu (\omega); $$

(15)

There exists $W \in \mathcal{S} (X)$ such that

$$ f^* [\omega, v (\omega)] = \sup_{x \in W} \langle v (\omega), x \rangle - f_\omega (x) \quad \mu\text{-a.e.} $$

(16)

The intersection in (14) over $W \in \mathcal{S}_a (X)$ may then be replaced by $W = X$.

**Proof.** By Theorem 2.1 our task of proving (13) reduces to checking that the integrand $f_\omega - \langle v (\omega), \cdot \rangle$ is integrally separably measurable. Since the integrand $\langle v (\omega), \cdot \rangle$ is integrally separably Carathéodory, it is integrally separably measurable by Lemma C.4 so that the sum retains integral separable measurability.

By the Fenchel-Young identity there holds

$$ v \in \partial I_f (u) \iff I_f (u) + I_f^* (v) = \langle v, u \rangle. $$

This is equivalent to $v$ fulfilling $\forall W (\omega) \in \partial f_W [\omega, u (\omega)]$ a.e. for $W \in \mathcal{S}_a (X)$. Indeed, the addendum (9) together with Fenchel-Young identity yields

$$ f_W [\omega, u (\omega)] + \text{ess-sup}_{W \in \mathcal{S} (X)} \sup_{x \in W} \langle v (\omega), x \rangle - f_\omega (x) = \langle v (\omega), u (\omega) \rangle \quad \mu\text{-a.e.} $$

\[ \]
This together with the a.e. estimate
\[ \langle v(\omega), u(\omega) \rangle \leq f_W [\omega, u(\omega)] + f^*_W [\omega, v(\omega)] \]
\[ \leq f_W [\omega, u(\omega)] + \operatorname{ess-sup}_{W \in S(X)} \sup_{x \in W} \langle v(\omega), x \rangle - f_\omega(x) \]
implies \( v_W \in \partial f_W [\omega, u(\omega)] \) a.e. Conversely, if
\[ v \in \bigcap_{W \in S_0(X)} \{ u^* \in S \mid u^*_W(\omega) \in \partial f_W [\omega, u(\omega)] \text{ a.e.} \}, \]
then, by Theorem 2.1, we find \( W_0 \in S_0(X) \) with
\[ \operatorname{ess-sup}_{W \in S(X)} \sup_{x \in W} \langle v(\omega), x \rangle - f_\omega(x) = \sup_{x \in W_0} \langle v(\omega), x \rangle - f_\omega(x) \quad \mu\text{-a.e.} \]
Consequently
\[ \langle v, u \rangle = \int_\Sigma f_{W_0}[\omega, u(\omega)] + \sup_{x \in W_0} \langle v(\omega), x \rangle - f_\omega(x) \, d\mu(\omega) = I_f(u) + I^*_f(v) \]
whence \( v \in \partial I_f(u) \). Regarding the addendum on the conjugate, observe that
\[ f^*_W[\omega, u(\omega)] \geq \operatorname{ess-sup}_{W \in S(X)} \sup_{x \in W} \langle v(\omega), x \rangle - f_\omega(x) \geq \operatorname{ess-sup} \sup_{x \in W} \langle v(\omega), x \rangle - f_\omega(x) \quad \mu\text{-a.e.} \]
Consequently, if \( f^*_W(v) \) is \( A^*_\mu \)-measurable and (15) holds as an identity of real numbers, then (16) obtains since Theorem 2.1 guarantees attainment of the essential supremum function. Conversely, if (16) holds, then the integrals in (15) and (13) agree. The function \( f^*_W(v) \) then equals an \( A^*_\mu \)-measurable function a.e. hence is \( A^*_\mu \)-measurable.

The addendum on the subdifferential again follows by Fenchel-Young considerations analogous to the case of (14) if \( I^*_f(v) = I^*_f(v) \). \( \square \)

Theorem 2.2 suggests to introduce the following notion:

**Definition 2.3 (dualizable integrand).** An integrand \( f : \Omega \times X \to [-\infty, \infty] \) that is separably measurable and such that for a weak* measurable function \( v : \Omega \to X' \) there exists \( W \in S(X) \) with

\[ f^*_W[\omega, v(\omega)] = \sup_{x \in W} \langle v(\omega), x \rangle - f_\omega(x) \quad \forall \omega \in \Omega \quad (17) \]

is called dualizable at \( v \). We say that \( f \) is dualizable for a space \( S \) of such functions if it is dualizable at each \( v \in S \).

We shall also consider integrands that are dualizable a.e. or i.a.e. This is meaningful if \( f \) and \( v \) are merely integrally measurable. If \( f \) is dualizable for \( v \) and \( W \in S(X) \), then the integrand \( f_\omega(x) - \langle v(\omega), x \rangle - f^*_W[\omega, v(\omega)] \) is \( A^*_\mu \otimes B(W) \)-measurable and thus an \( A^*_\mu \)-pre-normal integrand on \( W \) by Lemma C.7. As such it is infimally measurable by Lemma C.2, its strict sublevel multifunctions are measurable by Lemma C.1 and non-empty for positive level values. Hence, we find from them (strongly) \( A^*_\mu \)-measurable selections by the Aumann theorem [12, Thm. 6.10] if \( W \) is closed. Conversely, if the integrand \( f_\omega(x) - \langle v(\omega), x \rangle - f^*_W[\omega, v(\omega)] \) admits such selections, then it is obvious that it dualizable for \( v \). We apply this characterizing observation to discuss our first of two sufficient conditions for dualizability at all strongly measurable functions.
Lemma 2.1. Let $\Omega$ be a separable metric Borel space, $X$ a reflexive Banach space and $f: \Omega \times X \to (\mathbb{R}, \mathbb{R}^\infty)$ a normal convex integrand. Then $f$ is dualizable for any strongly measurable function $v: \Omega \to X'$.

Proof. Any Borel measurable map on $\Omega$ into another metric space has a separable range by [5, Prop. 1.11]. The integrand $f_\omega(x) - \langle v(\omega), x \rangle$ is finitely measurable in the sense of Definition C.1 by Lemma C.3 and an easy limiting argument that approximates $v$ pointwise by a sequence of simple functions. Thus the function $f^* [\omega, v(\omega)] = -\inf_{x \in X} f_\omega(x) - \langle v(\omega), x \rangle$ is measurable by Lemma C.1 whereby we recognize $f_\omega(x) - \langle v(\omega), x \rangle + f^* [\omega, v(\omega)]$ as finitely measurable. Consequently, its (strict) sublevel multifunctions are measurable by Lemma C.1 and non-empty for positive level values. We now want to apply [3, Cor. 5.19] to obtain Borel-measurable selections from the sublevels and thus conclude dualizability by our initial comment and the observation before this lemma. Note in this regard that $X$ is locally uniformly rotund by reflexivity [36]. Literally, the result [3, Cor. 5.19] requires a finite measure space and weakly compact convex values of the epigraphical multifunction. However, an extended inspection of the proof reveals that the statement holds on any measurable space and only the intersection of any value of $\text{epi} (f_\omega - \langle v(\omega), \cdot \rangle + f^* [\omega, v(\omega)])$ with any closed ball centred at the origin needs to be weakly compact and convex. To see this, check in [3, Lem. 5.3] that the cardinality $\gamma$ may be countably infinite on any measurable space and observe in [3, Lem. 5.11] that the proof still works if the sublevel sets of the function $\gamma$ therein have compact intersections with the values of the multifunction therein. Finally, by a limiting argument approximating any bounded closed convex (hence weakly compact) set $K$ by the open sets $K_\varepsilon = K + B_\varepsilon$ for $\varepsilon > 0$ and then approximating any closed convex set by bounded closed convex sets, it is easy to check that the $\mathcal{M}^{\infty}$-measurability required in [3, Cor. 5.19] is implied by Effros measurability in a reflexive space so that in total our adapted application of [3, Cor. 5.19] has been warranted and the proof is complete.

Assuming the continuum hypothesis, the above argument still works for any $\sigma$-algebra $\mathcal{A}$ whose cardinality is at most $|\mathbb{R}|$, cf. the remarks after [5, Prop. 1.11]. In particular, this covers the case of any countably generated $\sigma$-algebra $\mathcal{A}$, see [5].

We now present our second sufficient condition for dualizability. The proof and formulation of this condition requires some background information and technical results about hyperspace topologies on the lower semicontinuous proper functions $\text{LS}(X)$. We defer the technical results to the appendix but repeat the basic definitions here. For proofs and further information, we refer to [1]. The facts about the Wijsman topology will be needed later. The Attouch-Wets topology $\tau_{AW}$ on the closed subsets $\text{CL}(M)$ of the metric space $(M, d)$ is obtained by identifying $A \in \text{CL}(M)$ with the distance function $d_A(x) = \inf_{a \in A} d(x, a)$ and considering the topology of their uniform convergence on bounded sets, i.e.

$$\tau_{AW} \text{-lim } A_\alpha \iff \lim_{x \in B} \sup_{\alpha} |d_A(A_\alpha) - d_A(A)| = 0 \quad \forall B \subset M \text{ bounded.} \quad (18)$$

Similarly, the Wijsman topology $\tau_W$ is defined by considering pointwise convergence in (18). We omit the dependence of $\tau_{AW}$ and $\tau_W$ on the metric $d$ to ease notation. One defines on $\text{LS}(M)$ the Attouch-Wets and Wijsman topologies, denoted again by $\tau_{AW}$ and $\tau_W$, via the identification of $f$ with $\text{epi} f \subset M \times \mathbb{R}$, where $M \times \mathbb{R}$ carries the box metric $\rho([(x_0, \alpha_0), (x_1, \alpha_1)]) = \max \{d(x_0, x_1), |\alpha_0 - \alpha_1|\}$. The topology
\[ \tau_{AW} \text{ is metrizable and complete w.r.t. the metric} \]
\[ d_{AW}(A, B) = \sum_{n=1}^{\infty} 2^{-n} \min\{1, \sup_{x \in B_n(\tau_0)} |d_x(A) - d_x(B)|\} \quad (19) \]
if \((M, d)\) is complete. The topology \(\tau_W\) is metrizable and separable if and only if \(M\) is separable. A important feature of \(\tau_W\) is that for separable \(M\) an integrand \(f\) is normal if and only if it is \(\tau_W\)-measurable as a \(LS^pM\)-valued mapping by the Hess theorem [1, Thm. 6.5.14]. As \(\tau_{AW}\) is metrizable, we may consider strongly \(\tau_{AW}\)-measurable integrands \(f\), those for which there exists a sequence of simple integrand mappings \(f_n\) with \(\tau_{AW}\)-lim \(f_{\omega,n} = f_\omega\) for \(\omega \in \Omega\).

**Lemma 2.2.** Let \(f: \Omega \to LS(X)\) be an integrand. If \(f\) is strongly \(\tau_{AW}\)-measurable, then it is dualizable for any strongly measurable function \(v: \Omega \to X'\). In particular, any autonomous integrand \(f \in LS(X)\) is thus dualizable.

**Proof.** By Lemma A.3 it suffices to show that if \(v\) is strongly measurable, then the mapping \(f_\omega - \langle v(\omega), \cdot \rangle\) is strongly \(\tau_{AW}\)-measurable. Let \(v_n\) be a sequence of simple functions and \(f_n\) a sequence of simple integrands with
\[
\lim_n v_n(\omega) = v(\omega), \quad \tau_{AW}\lim_n f_{\omega,n} = f_\omega \quad \forall \omega \in \Omega.
\]
Then Lemma A.2 implies
\[
\tau_{AW}\lim_n f_{\omega,n} - \langle v_n(\omega), \cdot \rangle = f_\omega - \langle v(\omega), \cdot \rangle \quad \forall \omega \in \Omega.
\]
One might try coarser hyperspace topologies to establish more general criteria similar to Lemma 2.2, thereby placing dualizable integrands within the theoretical framework of measurable multifunctions in a deeper way. The slice topology seems most apt.

**3. ORLICZ SPACES**

In this section we define and discuss the notion of a generating integrand or generator \(\varphi\) and show how it induces the Banach spaces \(L_\varphi(\mu)\) of vector-valued functions called Orlicz spaces, whose basic properties like completeness, decomposability and embedding properties we study. As \(L_\varphi(\mu)\) enjoys better properties when each of its elements vanishes outside a \(\sigma\)-finite set, we characterize this behaviour in terms of the generator. Similar spaces can be found in the literature under various names, such as Fenchel-Orlicz, generalized Orlicz, or Musielak-Orlicz spaces.

**3.1. Generator integrands.** As mentioned in the introduction, we never impose any kind of uniform behaviour w.r.t. \(\omega \in \Omega\) on our generators. In fact, all our theory requires to work are the following assumptions.

**Definition 3.1** (generator). An even function \(\varphi \in \Gamma(X)\) satisfying \(\lim_{x \to 0} \varphi(x) = 0\) and \(\lim_{|x| \to \infty} \varphi(x) = \infty\) is an autonomous generator. An integrand \(\varphi: \Omega \times X \to [0, \infty]\) is a generator if

a) the function \(x \mapsto \varphi_\omega(x)\) is a generator for a.e. \(\omega \in \Omega\);

b) the integrand \(\varphi\) is integrally separably measurable.

By convexity and evenness, a generator assumes a global minimum at the origin hence is non-negative. We shall achieve stronger results resembling the separably valued theory for dualizable generators. However, all our main results are also available in a form that remains valid without dualizability. In many cases, it will
not be necessary at all. Remember that the request of dualizability is trivially satisfied if \( X \) is separable.

In the literature we find divergent names and definitions for autonomous generators, which are sometimes equivalent to Definition 3.1 or whose apparently greater generality is to some extent spurious. For example, some authors \([35, 37]\) weaken Definition 3.1 by requiring \( \varphi(0) = 0 \) instead of continuous vanishing, or dispense with the bounded sublevel sets. At least for autonomous generators, we will below provide a construction to subsume these cases to our seemingly narrower setting.

Another reason for divergent definitions is that the convexity of a generator allows to express the continuity at the origin as well as the bounded sublevel sets equivalently by formally stronger or weaker properties:

**Lemma 3.1.** Let \( \varphi \in \Gamma(X) \) be even. The function \( \varphi \) vanishes continuously at the origin iff \( \lim_{r \to 0} \varphi(rx) = 0 \) for each fixed \( x \in X \).

**Proof.** It suffices to deduce the first from the second statement. The sublevel sets

\[
C_n = \{ x \in B_X \mid \phi(n^{-1}x) \leq 1 \}
\]

are closed and \( B_X = \bigcup_{n \in \mathbb{N}} C_n \) hence \( \text{int} \, C_n \neq \emptyset \) for some \( n \in \mathbb{N} \) by the Baire category theorem. Since \( \text{int} \, C_n \) is symmetric as \( C_n \) is, we conclude \( 0 \in \text{int} \, C_n \) hence \( \phi \) is continuous at the origin by [20, §3.2, Thm. 1]. \( \Box \)

**Lemma 3.2.** For a convex function \( \phi : X \to (-\infty, \infty) \) with \( \phi(0) = 0 \) there holds

\[
\lim_{|x| \to \infty} \phi(x) = \infty \iff \exists r > 0 : \inf_{|x|=r} \phi(x) > 0 \iff \liminf_{|x| \to \infty} \frac{\phi(x)}{|x|} > 0. \quad (20)
\]

**Proof.** The first statement implies the second, the second implies the third as convexity renders the quotient non-decreasing in \( |x| \), and the third implies the first. \( \Box \)

The following lemma reveals why our notion of a generator is apt for duality theory:

**Lemma 3.3.** \( \varphi \in \Gamma(X) \) is an autonomous generator iff \( \varphi^* \in \Gamma(X') \) is one.

**Proof.** It suffices to prove that \( \varphi^* \) is a generator if \( \varphi \) is one since \( \varphi \) is conjugate to \( \varphi^* \) for the duality between \( X \) and \( X' \). The function \( \varphi^* \) is even. As \( \varphi \) has bounded sublevel sets, we see that \( \varphi^* \) vanishes continuously at the origin and since \( \varphi \) vanishes continuously at the origin, we see that \( \varphi^* \) has bounded sublevel sets. More precisely

\[
\exists r, s > 0 : |x| > r \implies \varphi(x) > s|x|
\]

by Lemma 3.2 hence there holds

\[
|x'| < s \implies \varphi^* (x') \leq \sup_{|x|<r} \langle x', x \rangle \leq r|x'|
\]

For the second claim, note

\[
\forall \varepsilon > 0 \exists \delta > 0 : |x| < \delta \implies \varphi(x) < \varepsilon.
\]

Therefore

\[
\varphi^* (x') \geq \sup_{|x|<\delta} \langle x', x \rangle - \sup_{|x|<\delta} \varphi(x) \geq \delta|x'| - \varepsilon
\]

so that \( \varphi^* \) has bounded sublevel sets. \( \Box \)

Lemma 3.3 implies that the conjugate integrand \( \varphi^*_\omega (x') = \sup_{x \in X} \langle x', x \rangle - \varphi_\omega (x) \) retains the generator property iff it is integrally separably measurable. For dualizable generators, this is the case:
**Lemma 3.4.** Let the generator $\varphi$ be dualizable for a decomposable space $S$ of strongly measurable functions and let $\mu$ have no atom of infinite measure. Then the integrand $\varphi^*$ is integrally separably measurable. If $\mu$ is complete, then it suffices if $S$ is almost decomposable.

**Proof.** Since $\mu$ has no atom of infinite measure, we are left to demonstrate that the restriction of $\varphi^*$ to a $\sigma$-finite set in the first component and a separable set $V \in S(X')$ in the other component is measurable. It suffices therefore to assume that $\mu$ is $\sigma$-finite. Given $F_n \in \mathcal{A}_f$ with $\Omega = \bigcup_n F_n$, it suffices if given $\varepsilon > 0$, we obtain $F \subset F_n$ with $\mu(F \setminus F_n) < \varepsilon$ and such that $\varphi^*$ is $\mathcal{A}(F) \otimes B(V)$-measurable if $\mu$ is complete. If $\mu$ is incomplete, it suffices if the same holds with $F = \Omega$. Because then $\varphi^*$ equals a measurable function $\mu$-a.e. in the first case and everywhere in the second case hence is measurable.

Let $v_n \in V$ be a dense sequence. Using the almost decomposability of $S$, we find

$$G_n \in \mathcal{A}(F), \quad v_n \chi_{G_n} \in S, \quad \mu(F \setminus G_n) \leq 2^{-n} \varepsilon.$$ 

If $S$ is decomposable, we may pick $G_n = F$ instead. We find $W_n \in S(X)$ with

$$\varphi^*_\omega(v_n) = \sup_{x \in W_n} \langle v_n, x \rangle - \varphi_\omega(x) \quad \text{for a.e. } \omega \in G_n.$$ 

Consequently, there holds for $W = \bigcup_{n \geq 1} W_n \in S(X)$ and $E_\varepsilon = \bigcup_{n \geq 1} G_n$ that

$$\mu(F \setminus E_\varepsilon) \leq \varepsilon, \quad \varphi^*_\omega(v_n) = \sup_{x \in W} \langle v_n, x \rangle - \varphi_\omega(x) \quad \text{for a.e. } \omega \in E_\varepsilon, \quad \forall n \in \mathbb{N}.$$ 

Setting $g(v) = \sup_{x \in W} \langle v, x \rangle - \varphi_\omega(x)$ for $v \in V$ we have $\varphi^*_\omega \geq g_\omega$ and $\varphi^*_\omega(v_n) = g_\omega(v_n)$ for $n \in \mathbb{N}$ hence $\varphi^*_\omega$ and $g_\omega$ agree on int dom $g_\omega \neq \emptyset$ for all $\omega \in E_\varepsilon$ by convex continuity in the interior by [20, §3.2, Thm. 1]. By lower semicontinuity and since both $r \mapsto \varphi^*_\omega(rv)$ and $r \mapsto g_\omega(rv)$ for $r \geq 0$ are non-decreasing for all $v \in V$, we deduce $\varphi^* = g$ globally. As $g$ is $\mathcal{A}(E) \otimes B(V)$-measurable, our claim obtains. \qed

For later reference we record another simple observation about generators.

**Proposition 3.1.** Let $\varphi \in \Gamma(X)$ be a generator. A sequence $x_n \in X$ converges iff

$$\forall \epsilon \in \mathbb{N} \exists N \in \mathbb{N}: m, n \geq N \implies \varphi[k(x_m - x_n)] < k^{-1}.$$ 

We close this section demonstrating how autonomous generators that are discontinuous or lack bounded sublevel sets may be adapted to match our definition of a generator without essentially altering their Orlicz space. This discussion is significant for the scope of our theory, but is not logically necessary for its understanding so that with this regard there is no harm in skipping ahead to Subsection 3.2.

**What we demonstrate:** let $\varphi \in \Gamma(X)$ be an even function with $\varphi(0) = 0$ for the rest of this subsection. One can always introduce a finer norm on dom($\varphi$) to make $\varphi$ continuous at the origin. Furthermore, one can arrange coercivity of $\varphi$ in two steps: if directions $u \in X$ with $\lim_{r \to \infty} \varphi(ru) < \infty$ occur, the Orlicz space will not be a Hausdorff space when equipped with the Luxemburg seminorm $\| \cdot \|_\varphi$ in (1). If one is interested in obtaining normed spaces and therefore applies the standard procedure of factoring out the kernel of $\| \cdot \|_\varphi$, it will be seen that this corresponds to a transition to the quotient $X/U$ on the domain of $\varphi$, where $U$ is the linear subspace of those directions satisfying $\lim_{r \to \infty} \varphi(ru) < \infty$. Then, in a second step, the coercivity in rays $\lim_{r \to \infty} \varphi(rx) = \infty$ on $X/U$ can be strengthened by introducing a suitable norm to yield bounded sublevel sets. One possibly still needs to complete the resulting space and can then recognize that in the result one
finds an Orlicz space $L_\phi(\mu)$ (of equivalence classes of functions) whose generator $\phi$ satisfies our axioms and in which the original set of functions $L_\phi(\mu)$ sits as a subspace that is dense if $L_\phi(\mu) = L_\phi^*(\mu)$.

**Details of the construction:** as a consequence of Lemma 3.1, we may arrange $\phi$ to be continuous at the origin by refining the topology on $\text{dom}(\phi)$ without excluding a part of its domain by considering the Minkowski functional $p_\phi(x) = \inf \left\{ \alpha > 0 \mid \phi(\alpha^{-1}x) \leq 1 \right\}$, which is absolutely homogeneous and sublinear by evenness and convexity of $\phi$. It is lower semicontinuous since $\phi$ is so. Consequently, its linear domain $Y$ becomes a Banach space when equipped with the norm $\| \cdot \|_Y = \|\cdot\|_X + p_\phi$, and $\phi$ is bounded around the origin hence continuous at zero on $(Y, \| \cdot \|_Y)$. Lemma 3.1 guarantees that the domain of $\phi$ is contained in $Y$ so that this procedure won’t change the Orlicz space generated by $\phi$ via the Minkowski functional (1). Now, the set

$$U = \bigcap_{\alpha \in \mathbb{R}} \{ x \in Y \mid \phi(\alpha x) = 0 \}$$

is a closed linear subspace since $\phi \in \Gamma(Y)$ is even and vanishes at zero. There holds

$$\phi(x + u) = \phi(x) \quad \forall u \in U, x \in X. \quad (21)$$

Indeed, if $\phi(x) = \alpha \in [0, \infty]$, then $\phi(x + u) \leq \alpha$ for any $u \in U$, whence (21) will follow. To see the last claim, note for $\lambda \in (0, 1)$ and $\mu \in \mathbb{R}$ that $\phi(\lambda x + (1 - \lambda)\mu u) \leq \lambda \phi(x) + (1 - \lambda)\phi(\mu u) = \lambda \phi(x)$ hence $\phi(\lambda x + u) \leq \lambda \phi(x)$ wherefore sending $\lambda \uparrow 1$ proves the claim by lower semicontinuity. Thus, upon passing to the quotient space $Z = Y/U$, we may assume that

$$\lim_{r \to \infty} \phi(rz) = \infty \quad \forall x \in Z \setminus \{0\}. \quad (22)$$

How does this passage to the quotient affect the Orlicz space $L_\phi$? First, if the subspace $U$ is non-trivial, the space $L_\phi$ will be a seminormed non-Hausdorff space of $Y$-valued functions when equipped with the Luxemburg norm $\| \cdot \|_\phi$. Applying the standard procedure of factoring out the kernel of $\| \cdot \|_\phi$, which consists of all a.e. equivalence classes of strongly measurable functions $u: \Omega \to U$, will then yield a normed space that may be seen to be isometrically isomorphic to the Orlicz space in the Luxemburg norm that arises if $\phi$ is considered on the quotient space $Z$ since for any strongly measurable function $y: \Omega \to Y$ there holds

$$\int \inf_{y \sim y(\omega)} \phi(y) \, d\mu(\omega) = \int \phi(y(\omega)) \, d\mu(\omega) = \inf_{u \in \ker y \mid 1} \int \phi(y(\omega) + u(\omega)) \, d\mu(\omega).$$

In conclusion, if one is interested in obtaining $L_\phi(\mu)$ as a normed space, then (22) may always be assumed to hold. To guarantee uniform coercivity, we equip $Z$ with the norm of the Minkowski-functional $q_\phi(z) = \{ \alpha > 0 \mid \phi(\alpha^{-1}z) \leq 1 \}$. Indeed, $q_\phi$ is positive definite by (22). To obtain a Banach space, we pass to the completion $C$ of $(Z, q_\phi)$ and set $\phi(x) = \liminf_{z \to x} \phi(x)$ for $x \in C$. Clearly, the lower closure $\phi \in \Gamma(C)$ is even. We claim that $\phi$ agrees with $\phi$ on the subspace $Z$. Note that $x_n \to x$ in $C$ iff $\limsup_{n \to \infty} \phi(\alpha(x - x_n)) \leq 1$ for every $\alpha > 0$. Estimating

$$\phi(x_n) \leq \lambda \phi(\lambda^{-1}x) + (1 - \lambda)\phi \left( \frac{x_n - x}{1 - \lambda} \right)$$

so that $\limsup_{n \to \infty} \phi(x_n) \leq \lambda \phi(\lambda^{-1}x) + (1 - \lambda)$ and sending $\lambda \downarrow 1$ proves $\phi(x) = \phi(x)$ for $x \in Z$ such that there exists $\mu > 1$ with $\phi(\mu x) < \infty$. But then from $\phi(x) \geq \phi(\lambda x)$ for $0 \leq \lambda \leq 1$ follows $\phi(x) = \phi(x)$ for all $x \in Z$ by lower semicontinuity of $\phi$. 

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Moreover, since $\phi = \varphi$ on $Z$, the function $\phi$ is bounded above by $1$ on $B_C$ hence vanishes continuously at the origin. Now, we see that $\phi$ has bounded sublevels by Lemma 3.2 since $\inf_{\delta(z) > 1} \phi(z) \geq 1$. Thus $\phi$ is a generator in our sense and we may consider $L_\phi(\mu)$ as a subspace of $L_\varphi(\mu)$, that is dense if $\mu$ is $\sigma$-finite or more generally if $L_\phi(\mu) = L^*_\phi(\mu)$. To see the denseness, consider first the case of a finite measure. A function $u \in L_\phi(\mu)$ being strongly measurable, we may approximate it uniformly by a sequence of countably valued measurable functions $u_n$. Uniform convergence is finer than convergence in $L_\phi(\mu)$ since $\phi$ is autonomous. Adapting the approximating sequence to take values in $Z$, we obtain the claim. In the general case if $L_\phi(\mu) = L^*_\phi(\mu)$, we may reduce to a $\sigma$-finite measure by restricting $\mu$ to a $\sigma$-finite set off which $u$ vanishes. Let
\[ \Omega_j \in \mathcal{A}_f, \quad \Omega_j \cap \Omega_k = \emptyset, \quad \Omega = \bigcup_j \Omega_j. \]
Invoking for given $\varepsilon > 0$ the case of a finite measure space, we find $u_j \in L_{\varphi}(\mu)$ with
\[ \| (u - u_j) \chi_{\Omega_j} \|_\phi \leq 2^{-j}\varepsilon \implies \| u - \sum_{j=1}^\infty u_j \chi_{\Omega_j} \|_\phi \leq \varepsilon \]
so the $\sigma$-finite case obtains. In conclusion, we have obtained a generator $\phi$ on a new Banach space $C$ which will serve to define an Orlicz space $L_\phi$ which contains $L_{\varphi}$ in a natural way while $\phi$ vanishes continuously at the origin and has bounded sublevels.

3.2. Definition and basic properties. Having defined the notion of a generator, we are ready to introduce our Orlicz spaces. For a generator $\varphi$ and $u \in L_0(\Omega; X)$ we set
\[ I_\varphi(u) = \int \varphi[\omega, u(\omega)] \, d\mu(\omega), \quad \| u \|_\varphi = \inf \{ \alpha > 0 \mid I_\varphi(\alpha u) \leq 1 \}. \]
The Minkowski functional $\| \cdot \|_\varphi$ is a seminorm on its domain $L_{\varphi}(\mu)$, for which we write $L_{\varphi}$ if no other measure is involved. By Proposition 3.1 the kernel of $\| \cdot \|_\varphi$ is characterized as the functions that vanish a.e. Factoring out the kernel, we arrive at the Orlicz space $L_{\varphi}(\mu)$ on which $\| \cdot \|_\varphi$ is called the Luxemburg norm. An equivalent norm is given by the Orlicz norm
\[ \| u \|_{\varphi} = \inf_{\alpha > 0} \alpha^{-1} \left[ 1 + I_{\varphi}(\alpha u) \right] \]
with
\[ \| u \|_\varphi \leq \| u \|_{\varphi} \leq 2\| u \|_\varphi \quad \forall u \in L_{\varphi}(\mu) \]
according to [27, Thm. 1.10]. Similarly, we define the dual Luxemburg norm and the dual Orlicz norm on the dual space $L_{\varphi}(\mu)^*$ as
\[ \| v \|_{\varphi} = \inf \{ \alpha > 0 \mid I_{\varphi}^*(\alpha^{-1} v) \leq 1 \}, \quad \| v \|_{\varphi}^* = \inf_{\alpha > 0} \alpha \left[ 1 + I_{\varphi}^*(\alpha^{-1} v) \right]. \]
Again
\[ \| v \|_{\varphi}^* \leq \| v \|_{\varphi} \leq 2\| v \|_{\varphi}^* \quad \forall v \in L_{\varphi}(\mu)^*. \]
One may easily check that the dual Orlicz norm agrees with the canonical operator norm induced by the Luxemburg norm. In the same way, the Orlicz norm agrees with the operator norm that $L_{\phi}(\mu)$ carries as a subset of its bidual space. For frequent later use, we record the following useful inequalities relating in particular $I_{\varphi}$ and $I_{\varphi}^*$ with their Luxemburg norms.
Lemma 3.5. Let $V$ be a real vector space, $f: V \to [0, \infty]$ a convex function with $f(0) = 0$ and left-continuous, i.e. \( \lim_{x \to 0^+} f(x) = f(0) \) for $v \in V$. Let $\mu: \nu \to [0, \infty]$ be the Minkowski functional of the sublevel set \( \{ f \leq 1 \} \). Then there hold the following inequalities:

a) $p(v) \leq 1 \implies f(v) \leq p(v)$,

b) $1 < p(v) \implies f(v) \geq p(v)$,

c) $p(v) \leq 1 + f(v)$.

Proof. This follows from the proof of [9, Cor. 2.1.15], where the same assertion is made for $f$ a semimodular, but only the assumptions above are actually used. \( \square \)

To prove that $L_\varphi(\mu)$ is complete, we first record a simple observation that will frequently be used to reduce considerations for $\sigma$-finite measures to finite ones.

Proposition 3.2. Let $\mu$ be $\sigma$-finite and $f: \Omega \to \mathbb{R}$ a positive integrable function. For the finite measure $d\nu = fd\mu$ and the generator $\phi = \frac{\varphi}{f}$ there holds $L_\varphi(\mu) = L_\phi(\omega)$.

The existence of such a function $f$ is equivalent to the $\sigma$-finiteness of $\mu$. We need to get one last measure theoretic generality out of our way: the space $L_\varphi(\mu)$ does not change if $\mu$ is replaced by its completion $\bar{\mu}$. More precisely, the total set of a.e. equivalence classes of strongly measurable functions w.r.t. $\mu$ does not change under completion as can be seen by appealing to Lemma B.3. In this sense, there exists a canonical isometric isomorphism between $L_\varphi(\mu)$ and $L_\phi(\bar{\mu})$. We now prove the completeness of $L_\varphi(\mu)$ for an arbitrary measure $\mu$. The adaptation of the usual proof for Lebesgue spaces is not completely trivial in the case of a non-$\sigma$-finite measure due to the $\omega$-dependence of the generator $\varphi$.

Theorem 3.1. $L_\varphi(\mu)$ in the Luxemburg-norm $\| \cdot \|_\varphi$ is a Banach space. Each convergent sequence in $L_\varphi(\mu)$ has a subsequence that converges a.e. to its limit.

The following proof remains valid if the generator $\varphi$ has no point of continuity on a set of positive measure.

Proof. Since completeness is preserved under isometry, we may assume $\mu$ complete without loss of generality. It is standard to check that $L_\varphi(\mu)$ is a normed linear space. We extend the completeness proof of [24, Thm. 2.4] to the non-$\sigma$-finite case. It suffices to prove that any Cauchy sequence $u_n$ has a norm convergent subsequence that converges a.e. We claim that it is enough to supply a subsequence $u_{n_k}$ that converges a.e. Because then the a.e. limit of $u_{n_k}$ agrees with some strongly measurable function $u$ a.e. so that the Fatou lemma implies

\[
\int \varphi[\omega, \lambda(u - u_{n_k})] \, d\mu \leq \liminf_k \int \varphi[\omega, \lambda(u_{n_k} - u_{n_l})] \, d\mu \leq \liminf_k \int \varphi[\omega, \lambda(u_{n_k} - u_{n_l})] \, d\mu
\]

whence $\| u - u_{n_k} \|_\varphi \leq \liminf_k \| u_{n_k} - u_{n_l} \|_\varphi$ follows. Consequently $u \in L_\varphi$ and $u_n \to u$ thus obtaining completeness. Since each member of the sequence $u_n$ is almost separably valued, we may assume $X$ to be separable without loss of generality. As $\mu$ is complete, we may then also assume that $\varphi$ is an integrally normal integrand on $X$ by Lemma C.7.

\[
\forall m \in \mathbb{N} \exists M \in \mathbb{N}: n_1, n_2 \geq M \implies \| u_n_1 - u_n_2 \| < \frac{1}{m}.
\]
The set

\[
A = \bigcup_{m \geq 1} \bigcup_{n_1, n_2 \geq M(m)} \{ \varphi(\omega, m [u_{n_1}(\omega) - u_{n_2}(\omega)]) > 0 \}
\]

is a countable union of sets permitting positive integrable functions hence \(\sigma\)-finite. We claim that

\[
\exists \lim_n u_n(\omega) \in X \quad \text{for a.e. } \omega \in \Omega \setminus A.
\]  

(23)

Indeed

\[
n_1, n_2 \geq M(m) \implies \varphi(\omega, m [u_{n_1}(\omega) - u_{n_2}(\omega)]) = 0 \quad \forall \omega \in \Omega \setminus A.
\]

Therefore (23) follows from Proposition 3.1 by Definition 3.1. We have reduced to the problem of extracting from \(u_n\) a subsequence that converges a.e. on the \(\sigma\)-finite set \(A\). Thus we may from now on assume that \(\mu\) is \(\sigma\)-finite without loss of generality hence we may take \(\mu\) finite by possibly modifying the integrand and measure as in Proposition 3.2. We argue by contradiction that \(u_n\) converges in measure: suppose that there exists \(\varepsilon > 0\) and \(\delta > 0\) such that for any subsequence of \(n\) there exists a subsubsequence \(n_k\) with

\[
C_\varepsilon := \{|u_{n_k} - u_{n_k}| > \varepsilon\}, \quad \mu(C_\varepsilon) > \delta.
\]

Note that the sets

\[
\left\{ \inf_{\|x\| > \varepsilon} \frac{\varphi(\omega, rx)}{|x|} > 1 \right\}
\]

are measurable by normality of \(\varphi\) and Lemma C.1. The measure \(\mu\) being finite, we find \(r > 0\) so large that

\[
\mu \left( \omega \in C_\varepsilon : \inf_{\|x\| > \varepsilon} \frac{\varphi(\omega, rx)}{|x|} > 1 \right) > \mu(C_\varepsilon) - \frac{\delta}{2}
\]

hence by Definition of \(C_\varepsilon\) there follows

\[
\mu \left( \omega \in C_\varepsilon : \varphi(\omega, r [u_{n_k} - u_{n_k}]) > \varepsilon \right) > \frac{\delta}{2}.
\]

However, by Markov’s inequality, we have

\[
\mu \left( \omega \in C_\varepsilon : \varphi(\omega, r [u_{n_k} - u_{n_k}]) > \varepsilon \right) \leq \frac{1}{\varepsilon} \int \varphi(\omega, r [u_{n_k} - u_{n_k}]) \, d\mu \xrightarrow{k, l \to \infty} 0.
\]

We have arrived at a contradiction; \(u_n\) converges in measure hence admits an a.e. convergent subsequence on \(A\) thus on \(\Omega\). □

An important difference between the well-known Bochner-Lebesgue spaces \(L_p(\mu; X)\) for \(1 \leq p < \infty\) and a general Orlicz space is the possibility that an element of \(L_\varphi(\mu)\) need not vanish outside a \(\sigma\)-finite set. Many results about \(L_p(\mu; X)\) are easy to prove for \(\sigma\)-finite measures and may then be transferred to the case of an arbitrary measure by using this observation. Also, functions vanishing off a \(\sigma\)-finite set appear naturally when one characterizes the maximal linear subspace of \(\text{dom} \, I_\varphi\), cf. Theorem 5.1. In order to capture this behaviour in our theory, we introduce

**Definition 3.2** (\(\sigma\)-finite concentration). A function \(u: \Omega \to X\) is \(\sigma\)-finitely concentrated iff it vanishes outside a \(\sigma\)-finite set. For \(L \subset L_\varphi(\mu)\) we denote by \(L^\sigma\) the subset of \(\sigma\)-finitely concentrated elements in \(L\).

**Lemma 3.6.** The space \(L_\varphi^\sigma(\mu)\) is a closed linear subspace of \(L_\varphi(\mu)\).
Hence, for any given \( n \) \( \nu \) whence for this assume superspace \( \mathcal{W} \). This set admits a positive integrable function. As is strict for a.e. \( \omega \) \( \varphi \) subspace \( F \) or separably measurable generators, Theorem 3.2 characterizes precisely when the converse is true as well.

We want to show that any assumption implies that \( \text{assumption} \Rightarrow \text{conclusion} \). Regarding the converse, suppose \( W_0 \in \mathcal{S}(X) \) with \( W_0 \subset W \) such that \( A_W \in \mathcal{A}_\nu \) and \( A_W \) is \( \sigma \)-finite, then \( L_{\varphi}(\mu) = L_{\varphi}^*(\mu) \). If \( \varphi \) is separately measurable, then the converse is true as well.

For separably measurable generators, Theorem 3.2 characterizes precisely when the subspace \( L_{\nu}^*(\mu) \) is not proper. This applies in particular if the minimum of \( \varphi \) at zero is strict for a.e. \( \omega \in \Omega \) as happens for the Bochner-Lebesgue spaces with \( 1 \leq p < \infty \).

**Proof.** We want to show that any \( u \in L_{\varphi}(\mu) \) is \( \sigma \)-finitely concentrated and may for this assume \( \|u\|_{\varphi} = 1 \) hence \( I_{\varphi}(u) \leq 1 \). We have \( \{ \varphi [\omega, u(\omega)] > 0 \} \in \mathcal{A}_\nu \) since this set admits a positive integrable function. As \( u \) is almost separably valued, our assumption implies that

\[
\{ u(\omega) \neq 0 \} \setminus \{ \varphi [\omega, u(\omega)] > 0 \} \in \mathcal{A}_\nu
\]

whence \( u \in L_{\nu}^*(\mu) \) follows as \( \mathcal{A}_\nu \) is a \( \sigma \)-ring.

Regarding the converse, suppose \( W_0 \in \mathcal{S}(X) \) with \( A_W \notin \mathcal{A}_\nu \) for every superspace \( W \in \mathcal{S}(X) \) of \( W_0 \). We shall show that \( L_{\varphi}(\mu) \setminus L_{\nu}^*(\mu) \) is non-empty by constructing an element. The generator \( \varphi \) being separately measurable, there exists a closed superspace \( W \in \mathcal{S}(X) \) such that the restriction of \( \varphi \) is \( A \otimes B(W) \)-measurable. Hence, for any given \( n \in \mathbb{N} \), the multifunction

\[
\omega \mapsto \Gamma_n(\omega) := \begin{cases} \{ x \in W \mid \|x\| \geq \frac{1}{n}, \varphi(\omega, x) = 0 \} & \text{if this set is non-empty}, \\ \{0\} & \text{else} \end{cases}
\]

has a \( A_\nu \otimes B(W) \)-measurable graph and thus permits an \( A_\nu \)-measurable selection \( u_n: \Omega \to W \) by the Aumann theorem in the form of [12, Thm. 6.10]. The same theorem together with [12, Thm. 6.5] shows that \( \Gamma_n \) is Effros \( A_\nu \)-measurable. Note that the \( A_\nu \)-measurable sets \( \Omega \setminus \Gamma_n(\{0\}) \) converge to \( A_W \notin \mathcal{A}_\nu \) as \( n \to \infty \) hence are non-\( \sigma \)-finite eventually. Therefore \( u_n \) eventually differs from zero on a set of non-\( \sigma \)-finite measure. In conclusion \( u_n \in \{ L_{\varphi}(\mu), L_{\nu}^*(\mu) \} \).

### 3.3. Embeddings and almost embeddings

We close the section by proving (6), which will be instrumental in deducing properties of \( L_{\varphi}(\mu) \) from those of the better understood Bochner-Lebesgue spaces. We prepare this result with a simple embedding lemma providing continuous inclusions between Orlicz spaces in terms of their generators.

**Lemma 3.7.** Let \( \varphi \) and \( \phi \) be generators such that

\[
\exists L > 0, f \in L_1(\mu): \varphi(\omega, x) \leq \phi(\omega, Lx) + f(\omega).
\]

Then

\[
\|u\|_{\varphi} \leq L (1 + \|f\|_{L_1}) \|u\|_{\phi} \quad \forall u \in L_{\phi}(\mu).
\]

**Proof.** For \( u \in L_{\phi}(\mu) \) there holds

\[
\|u\|_{\varphi} \leq \inf_{\alpha > 0} \alpha^{-1} [1 + \|f\|_{L_1} + I_{\phi}(L\alpha u)] \leq (1 + \|f\|_{L_1}) \|Lu\|_{\phi} = L (1 + \|f\|_{L_1}) \|u\|_{\phi}.
\]
Lemma 3.8. Let \( \mu \) be finite. Then there exists an isotonic family \( \Omega_\varepsilon \in \mathcal{A} \) with \( \lim_{\varepsilon \to 0} \mu(\Omega_\varepsilon) = 0 \) such that there hold the continuous embeddings \( L_\infty(\Omega_\varepsilon; X) \to L_\varphi(\Omega_\varepsilon) \to L_1(\Omega_\varepsilon; X) \) via identical inclusion.

Proof. Consider for \( W \in \mathcal{S}(X) \) the \( \mathcal{A}_\mu \)-measurable sets

\[
E'_\varepsilon(W) = \{ \sup \varphi_\omega(B_{\varepsilon, W}) \leq 1 \}, \quad E''_\varepsilon(W) = \{ \inf \varphi_\omega(W \setminus B_{\varepsilon, W}) \geq 1 \}.
\]

Measurability follows from separable measurability of \( \varphi \). More precisely, we may for the first set use by Lemma C.7 and the Hess theorem that it is the pre-image of the \( \mathcal{A}_\mu \)-Wijsman-measurable epigraphical multifunction under the Wijsman-closed set

\[
\bigcap_{x \in B_{\varepsilon, W} \times [1, \infty)} \{ F \in \text{CL}(X \times \mathbb{R}) \mid d_x(F) = 0 \}.
\]

For the second set, this follows from the infimal measurability of normal integrands by Lemma C.1. We may by [12, Thm. 1.108] define the essential intersections

\[
\Omega'_\varepsilon = \text{ess-} \bigcap_{W \in \mathcal{S}(X)} E', \quad \Omega''_\varepsilon = \text{ess-} \bigcap_{W \in \mathcal{S}(X)} E''.
\]

By the same theorem and since \( E' \) and \( E'' \) are decreasing w.r.t. \( W \) there exist \( W'_\varepsilon, W''_\varepsilon \in \mathcal{S}(X) \) with \( \Omega' = E'(W'_\varepsilon) \) and \( \Omega'' = E''(W''_\varepsilon) \) a.e. so that for any null sequence \( \varepsilon_n \) we find \( W'_n \in \mathcal{S}(X) \) and \( W''_n \in \mathcal{S}(X) \) independent of \( n \) with \( \Omega'_{\varepsilon_n} = E'(W'_n) \) and \( \Omega''_{\varepsilon_n} = E''(W''_n) \) a.e. hence

\[
\lim_{\varepsilon \to 0} \mu(\Omega \setminus \Omega'_{\varepsilon}) = \lim_{\varepsilon \to 0} \mu(\Omega \setminus \Omega''_{\varepsilon}) = 0
\]

because \( \varphi \) is a generator thus vanishing continuously at the origin with bounded sublevels. Setting \( \Omega_\varepsilon = \Omega' \cap \Omega'' \), we have \( \lim_{\varepsilon \to 0} \mu(\Omega \setminus \Omega_\varepsilon) = 0 \). Denoting by \( I_{B_X} \) the indicator in the sense of convex analysis of the unit ball \( B_X \) we have \( \varphi_\omega(x) \leq I_{B_X}(\frac{x}{\varepsilon}) + 1 \) a.e. on \( \Omega' \) and \( \varepsilon \| x \| \leq \varphi_\omega(x) + 1 \) a.e. on \( \Omega'' \) for all \( x \in W \) for any \( W \in \mathcal{S}(X) \) hence

\[
\| u \|_\varphi \leq \frac{1}{\varepsilon} \left( 1 + \mu(\Omega) \right) \| u \|_\infty, \quad \varepsilon \| u \|_1 \leq \left( 1 + \frac{\mu(\Omega)}{\varepsilon} \right) \| u \|_\varphi
\]

by Lemma 3.7 as any \( u \in L_\varphi(\mu) \) is almost separably valued. \( \square \)

We found the idea for Lemma 3.8 in [4, Thm. 3.2], where the corresponding statement for separable range spaces is attributed to [13]. We would like to relate our technique of proof to that of our predecessors but are currently unable to do so since [4] contains no proof and [13] had been unobtainable for us as of this writing. In view of Sec. 2 it becomes important to understand almost decomposability of \( L_\varphi(\mu) \) and its subspaces. Obviously, \( L_\varphi(\mu) \) and \( L_\varphi^I(\mu) \) are weakly decomposable. We also have

Corollary 3.1. \( L_\varphi(\mu) \) and \( L_\varphi^I(\mu) \) are almost decomposable.

Proof. Let \( F \in \mathcal{A}_I \) and \( v \in L_\infty \left( F; X \right) \). Since \( L_\varphi(\mu) \) and \( L_\varphi^I(\mu) \) are weakly decomposable and linear, it suffices to prove that for \( \varepsilon > 0 \) there exists \( F_\varepsilon \subset F \) with \( \mu(\left( F \setminus F_\varepsilon \right) \subset \varepsilon \) and \( v\chi_{ F_\varepsilon } \in L_\varphi(\mu) \), which follows from Lemma 3.8. \( \square \)
4. The closure of simple functions

We compile in this ancillary section basic facts about the space $E_\phi(\mu)$ of the closure of simple functions in $L_\phi(\mu)$. Even though the simple functions are in general not dense in $L_\phi(\mu)$, their closure $E_\phi(\mu)$ can still be used to approximate all of $L_\phi(\mu)$ in a suitable sense, at least on $\sigma$-finite sets.

**Definition 4.1** (convergence from below). A sequence $u_n : \Omega \to X$ of measurable functions converges from below to $u$ iff there exists a sequence $\Omega_n \in A$ with $\mu(\lim_n \Omega_n) = 0$ and $u_n = u\chi_{\Omega_n}$. We write $u_n \uparrow u$ if $u_n$ converges from below to $u$. We say that a convergence from below is monotonic if the sequence $\Omega_n$ increases.

We shall define the class of absolutely continuous functionals as those enjoying continuity from below. This class will be seen to agree with the function component in the duality theory. The space $E_\phi(\mu)$ is obviously weakly decomposable. We also have

**Lemma 4.1.** $E_\phi(\mu)$ and $E^{\sigma}_{\phi}(\mu)$ are almost decomposable.

**Proof.** Remembering the remark below Definition 2.1 on intersections of almost decomposable spaces, we need only consider $E_\phi(\mu)$ since $E^{\sigma}_{\phi}(\mu) = E_\phi(\mu) \cap L^\sigma_{\phi}(\mu)$ and these spaces are weakly decomposable in addition to being almost decomposable by Corollary 3.1. Let $F \in A$ and $v \in L^\sigma_{\phi}$. Since $E_\phi(\mu)$ is weakly decomposable and linear, it suffices to prove that for every $\varepsilon > 0$ there exists $F_\varepsilon \subset F$ with $\mu(F \setminus F_\varepsilon) < \varepsilon$ and $v\chi_{F_\varepsilon} \in E_\phi(\mu)$. We find by Egorov’s theorem a subset $E_\varepsilon \subset F$ with $\mu(F \setminus E_\varepsilon) < \frac{\varepsilon}{2}$ and $\lim_{\lambda \to 0} \phi(\lambda v(\omega)) = 0$ uniformly on $E_\varepsilon$ hence $v\chi_{E_\varepsilon} \in L_{\phi}(\mu)$. Pick $v_n$ a sequence of simple functions with $v_n \to v$ a.e. By Egorov’s theorem we find a sequence $F_{k,\varepsilon} \subset E_\varepsilon$ with $\mu(E_\varepsilon \setminus F_{k,\varepsilon}) < 2^{-k-1}\varepsilon$ and $\lim_{n} \phi(\lambda v_n(\omega)) = 0$ uniformly on $F_{k,\varepsilon}$ for fixed $k$. Consequently the same holds on $F_{\varepsilon} = \bigcap_{k \geq 1} F_{k,\varepsilon}$ for all $k \geq 1$ so that $v_n\chi_{F_{\varepsilon}} \to v\chi_{F_{\varepsilon}}$ in $L_{\phi}(\mu)$ by definition of the Luxemburg norm. In conclusion $v\chi_{F_{\varepsilon}} \in E_{\phi}(\mu)$ and $\mu(F \setminus F_{\varepsilon}) = \mu(F \setminus E_\varepsilon) + \mu(E_\varepsilon \setminus F_{\varepsilon}) = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

**Lemma 4.2.** Given $u \in E^{\sigma}_{\phi}(\mu)$ and an almost decomposable subspace $L \subset L_{\phi}(\mu)$, there exists a sequence $u_n \in L$ with $u_n \uparrow u$ monotonically.

**Proof.** Let $u$ vanish outside $\Sigma \in A_\sigma$ with $\Sigma = \bigcup_n F_n$ for an isotonic sequence $F_n \in A$. By the first remark below Definition 2.1 it is immaterial that $u$ might be unbounded so that there exists an increasing sequence of sets $G_n \subset F_n$ with $\lim_n \mu(F_n \setminus G_n) = 0$ and $u\chi_{G_n} \in L$ hence $u\chi_{G_n} \uparrow u$.

5. Absolutely continuous norms

In this section we study properties of the space $C_{\phi}(\mu)$ of the elements in $L_{\phi}(\mu)$ whose norm is absolutely continuous, i.e. for which $\lim_n \|u\chi_{\Omega_n}\|_{\phi} = 0$ whenever $E_n \in A$ is a sequence with $\mu(\lim_n E_n) = 0$. In the scalar theory $X = \mathbb{R}$, this space is important because $C_{\phi}(\mu)^* = L_{\phi^*}(\mu)$ if $\phi$ is real-valued, inducing a weak* topology on $L_{\phi^*}(\mu)$ that can serve to compensate if $L_{\phi^*}(\mu)$ lacks reflexivity. The situation is similar, yet somewhat more complicated for the vector valued case. Nevertheless, our main interest in $C_{\phi}(\mu)$ lies in its role of inducing a weak* topology...
on the function component of the dual space of \( L_\varphi (\mu) \), for which the dual ball often is sequentially compact by the results in Sec. 7. Besides this, the space \( C_\varphi (\mu) \) is key to understanding separability and reflexivity of its superspace \( L_\varphi (\mu) \), as mentioned in the introduction. Indeed, the linearity of \( \text{dom } I_\varphi \) is necessary for both these properties to occur as well shall see. Since \( C_\varphi (\mu) \) turns out to be the maximal linear subspace of \( \text{dom } I_\varphi \), so that the linearity of this domain is under mild conditions equivalent to \( C_\varphi (\mu) = L_\varphi (\mu) \), settling these matters for \( C_\varphi (\mu) \) will solve the actual questions.

5.1. Basic properties. We start by proving the basic characterization of \( C_\varphi (\mu) \) as the maximal linear Banach subspace of \( \text{dom } I_\varphi \).

**Lemma 5.1.** \( C_\varphi (\mu) \) and \( C_\varphi^\sigma (\mu) \) are closed linear subspace of \( L_\varphi (\mu) \) and \( L_\varphi^\sigma (\mu) \).

**Proof.** Linearity is clear. Closedness of \( C_\varphi (\mu) \) follows by an obvious \( \mathfrak{F} \)-argument. The case of \( C_\varphi^\sigma (\mu) \) then obtains by Lemma 3.6. \( \square \)

**Theorem 5.1.** For \( A_\lambda = \{ u \in L_\varphi \mid \lambda u \in \text{dom } I_\varphi \} \) there holds

\[
\bigcap_{\lambda > 0} A_\lambda = \bigcap_{n \in \mathbb{N}} A_n \subset C_\varphi^\sigma (\mu). \tag{24}
\]

If \( \varphi \) is real-valued on atoms of finite measure, the inclusion in (24) is an equality. It is proper if \( \varphi \) is not real-valued on an atom of finite measure.

**Proof.** The first identity in (24) holds since \( A_\lambda \) decreases as \( \lambda \) increases. Ad inclusion: for \( n \in \mathbb{N}, \ u \in \bigcap_{\lambda > 0} A_\lambda \) and an evanescent sequence \( E_j \subset A \) there holds

\[
\lim_j I_\varphi \left( nu \chi_{E_j} \right) = \lim_j \int_{E_j} \varphi \left[ \omega, u(\omega) \right] d\mu(\omega) = 0
\]

by absolute continuity of the integral hence \( u \in C_\varphi^\sigma (\mu) \). As each set in the union \( \{ u \neq 0 \} = \bigcup_{n \in \mathbb{N}} \{ \varphi \left( nu \right) > 0 \} \) permits a positive integrable function hence is \( \sigma \)-finite, we conclude \( u \in C_\varphi^\sigma (\mu) \).

Ad addendum: fixing \( u \in C_\varphi^\sigma (\mu) \) we may assume \( \mu \) to be \( \sigma \)-finite. We claim that each set \( R = R_n = \{ \varphi \left( nu \right) = \infty \} \) and hence their union is null. Since \( \varphi \) is real on atoms with finite measure, \( R \) contains no atom hence \( \mu \) is non-atomic on \( R \) thus has the finite subset property there [12, Def. 1.16, Rem. 1.19]. If \( \mu (R) > 0 \), there exists a sequence \( Q_m \subset R \) with \( \mu (Q_m) \searrow 0 \) so that the contradiction \( 0 = \lim \inf_{m \to \infty} \left\| u \chi_{Q_m} \right\|_\varphi \geq \frac{1}{m} > 0 \) obtains and the claim follows. Let \( f : \Omega \to \mathbb{R} \) be an integrable positive function and consider the sets \( A_{\lambda,n} = \{ \varphi \left( nu \right) \leq \lambda f \} \) for \( \lambda > 0 \). By \( \mu (\bigcup_{n \geq 1} R_n) = 0 \) there holds \( \mu (\lim_{\lambda \to \infty} \Omega \setminus A_{\lambda,n}) = 0 \) so that for \( \lambda \) sufficiently large we have \( \| u \chi_{\Omega \setminus A_{\lambda,n}} \|_\varphi < \frac{1}{\lambda} \) hence

\[
I_\varphi (nu) = \int_{\Omega \setminus A_{\lambda,n}} \varphi \left[ \omega, nu(\omega) \right] d\mu(\omega) + \int_{A_{\lambda,n}} \varphi \left[ \omega, nu(\omega) \right] d\mu(\omega)
\]

\[\leq 1 + \lambda \int f d\mu < \infty\]

whence \( u \in \bigcap_{\lambda > 0} A_\lambda \) follows. Regarding the necessity of \( \varphi \) being real-valued on each atom \( A \in A_f \), consider \( x \) such that \( \varphi (\omega, x) = \infty \) a.e. on \( A \). Then

\[x \chi_A \in C_\varphi \setminus \bigcap_{\lambda > 0} A_\lambda. \]
It seems hard to imagine that an element of $C_\varphi(\mu)$ could not vanish outside a $\sigma$-finite set, so that Theorem 5.1 should in fact already contain the claimed identification between $C_\varphi(\mu)$ and the maximal linear subspace of dom $I_\varphi$. Indeed, this is the case except for some pathological measures:

**Corollary 5.1.** Let $\varphi$ be real-valued on atoms of finite measure and let $\mu$ have no atom of infinite measure. Then $C_\varphi(\mu) = C^\sigma_\varphi(\mu)$.

**Proof.** Suppose there were $u \in C_\varphi \setminus C^\sigma_\varphi$. Then there exists $n \in \mathbb{N}$ with $\int \varphi(nu) \, d\mu = \infty$ by Theorem 5.1. Since we assume that no atom of infinite measure exists, Proposition 2.1 yields a set $\Sigma \in \mathcal{A}_\sigma$, $\int_{\Sigma} \varphi(nu) \, d\mu = \infty$, which is impossible by Theorem 5.1 because $u\chi_{\Sigma} \in C^\sigma_\varphi(\mu)$.

**Corollary 5.2.** If dom $I_\varphi$ is linear, then $L_\varphi(\mu) = C^\sigma_\varphi(\mu)$. Conversely, if $L_\varphi(\mu) = C^\sigma_\varphi(\mu)$ and $\varphi$ is real-valued on atoms of finite measure, then dom $I_\varphi$ is linear.

**Proof.** Observing that $\text{lin dom } I_\varphi = L_\varphi$ by definition of the Luxemburg norm, this is immediate by Theorem 5.1.

Theorem 5.1 allows a simple characterization of generators for which all elements of $L_\varphi(\mu)$ have absolutely continuous norms in terms of a growth condition often dubbed $\Delta_2$ or doubling condition. Similar conditions and their role in the theory of Orlicz spaces are well-known in the scalar and vector valued cases, cf. [29, 25].

**Definition 5.1 ($\Delta_2$-condition).** We say the generator $\varphi$ satisfies the $\Delta_2$-condition and write $\varphi \in \Delta_2$ iff

$$\forall A \in \mathcal{A}_\sigma \forall S \in \mathcal{S}(X) \exists k \geq 1, f \in L_1(\mu): \varphi(x, 2x) \leq k \varphi(x) + f(x) \quad \forall x \in S, \ a.e. \ \omega \in A.$$ 

**Lemma 5.2.** There holds $L_\varphi(\mu) = C^\sigma_\varphi(\mu)$ if $\varphi \in \Delta_2$. If $\mu$ is non-atomic, then $\varphi \in \Delta_2$ is also necessary for $L_\varphi(\mu) = C^\sigma_\varphi(\mu)$ to hold.

**Proof.** The first claim will follow by Theorem 5.1 once we prove that dom $I_\varphi$ is linear if $\varphi \in \Delta_2$. As dom $I_\varphi$ is an absolutely convex set, its linearity is equivalent to the implication

$$I_\varphi(u) < \infty \implies I_\varphi(2u) < \infty.$$

Arguing by contradiction, we assume $I_\varphi(2u) = \infty$. Proposition 2.1 yields $A \in \mathcal{A}_\sigma$ with $I_\varphi(2u\chi_A) = \infty$. As $u$ is almost separably valued, the assumption $\varphi \in \Delta_2$ yields

$$I_\varphi(2u\chi_A) \leq kI_\varphi(u\chi_A) + \int_{\mathcal{A}} f \, d\mu < \infty$$

whence we have arrived at a contradiction. Regarding the necessity, let $\Sigma \in \mathcal{A}_\sigma$ and $S \in \mathcal{S}(X)$ as in Definition 5.1. Since $\varphi$ is integrally separably measurable, there exists a closed subspace $W \in \mathcal{S}(X)$ with $S \subset W$ such that the restriction $\varphi|_{\Sigma \times W}$ is $\mathcal{A}(\Sigma) \otimes \mathcal{B}(W)$-measurable. Hence we may via restriction assume that $\mu$ is $\sigma$-finite and $\varphi$ is $\mathcal{A} \otimes \mathcal{B}$-measurable on a separable space. Let $\phi_\varphi(x) = \varphi_\omega(2x)$ so that our assumption $L_\varphi(\mu) = C_\varphi(\mu)$ implies dom $I_\varphi \subset$ dom $I_\varphi$ by Theorem 5.1 as $\mu$ is non-atomic. Hence $\varphi \in \Delta_2$ follows by [25, Thm. 1.7].
If \( \mu \) has an atom, then \( L_\varphi(\mu) = C_\varphi^\sigma(\mu) \) may hold even if \( \varphi \not\in \Delta_2 \). For example, consider \( \mathbb{R}^n \) as an Orlicz space of real valued functions on the uniform measure space \( \{1, \ldots, n\} \) and take any real-valued function \( \varphi \in \Gamma(\mathbb{R}) \) with \( \varphi \not\in \Delta_2 \) as a generator.

### 5.2. Decomposability

As \( C_\varphi(\mu) \) will be seen the predual space to the function component of \( L_\varphi(\mu) \) if \( \varphi \) is real-valued, it is interesting to understand convex duality also on \( C_\varphi(\mu) \) as this implies, for example, weak* lower semicontinuity for functionals that arise as convex conjugates w.r.t. this pairing. \( C_\varphi(\mu) \) is weakly decomposable. We also have

**Lemma 5.3.** If \( \varphi \) is real-valued, then \( C_\varphi(\mu) \) and \( C_\varphi^\sigma(\mu) \) are almost decomposable.

**Proof.** Let \( F \in \mathcal{A}_f \) and \( v \in L_\infty(F; X) \). Since \( C_\varphi(\mu) \) and \( C^\sigma_\varphi(\mu) \) are weakly decomposable linear spaces, it suffices by a remark below Definition 2.1 to prove that for \( \varepsilon > 0 \) there exists a set \( F_\varepsilon \subset F \) with \( \mu(F\setminus F_\varepsilon) < \varepsilon \) and \( v\chi_{F_\varepsilon} \in C_\varphi(\mu) \). Since \( \varphi \) is real-valued, we find for \( k \in \mathbb{N} \) a set \( F_{k,\varepsilon} \subset F \) with \( \mu(F\setminus F_{k,\varepsilon}) < 2^{-k \varepsilon} \) and \( \sup_{\omega \in F_{k,\varepsilon}} \varphi(\omega, k\varepsilon(\omega)) < \infty \). Thus for \( F_\varepsilon = \bigcap_k F_{k,\varepsilon} \) there holds \( \mu(F\setminus F_\varepsilon) < \varepsilon \) and \( v\chi_{F_\varepsilon} \in C_\varphi(\mu) \) by Theorem 5.1. \( \Box \)

If \( \varphi \) is not real-valued, then the maximal linear subspace of \( \text{dom } I_\varphi \) may be trivial hence Lemma 5.3 ceases to hold. Consider the example \( L_\infty([0,1]; X) \) with the generator \( I_{B_\infty} \).

For every countable family in \( L_\infty^\sigma(\mu) \), there exists an evanescent sequence of sets outside which each element has absolutely continuous norm. This observation will yield insights into the dual spaces of \( \varphi \) and \( \varphi^\sigma \).

**Lemma 5.4.** If \( \varphi \) is real-valued, then for any sequence \( u_k \in L_\infty^\sigma(\mu) \) there exists a decreasing sequence \( E_\ell \in \mathcal{A} \) with \( \mu(\text{lim}_\ell E_\ell) = 0 \) and \( u_k \chi_{\Omega\setminus E_\ell} \in C_\varphi(\mu) \).

**Proof.** We may assume \( \mu \) to be \( \sigma \)-finite. Hence we may take \( \mu \) to be finite by Proposition 3.2. Since \( C_\varphi^\sigma(\mu) \) is almost decomposable by Lemma 5.3, we find by Lemma 4.2 sequences of sets \( D_{k,\ell} \in \mathcal{A} \) decreasing in \( \ell \) with

\[
\mu(D) \leq 2^{-k-\ell}, \quad u_k \chi_{\Omega\setminus D} \in C_\varphi^\sigma(\mu),
\]

so that for \( E_\ell = \bigcup_{k \geq 1} D \) we have

\[
\mu(E_\ell) \leq \sum_{k \geq 1} \mu(D) \leq 2^{-\ell}, \quad u_k \chi_{\Omega\setminus E_\ell} \in C_\varphi^\sigma(\mu) \quad \forall k \in \mathbb{N}. \quad \Box
\]

As our last fundamental fact on \( C_\varphi(\mu) \) and a first step towards investigating separability, we prove the denseness of simple functions.

**Lemma 5.5.** There holds \( C_\varphi^\sigma(\mu) \subset E_\varphi^\sigma(\mu) \). If \( \varphi \) is real-valued, then integrable simple functions are dense in \( C_\varphi^\sigma(\mu) \).

**Proof.** It suffices to consider \( \sigma \)-finite measures \( \mu \) hence we may equivalently consider a modified generator \( \phi \) and a finite measure \( \nu \) as in Proposition 3.2. Note however that we want to obtain a density set of \( \mu \)-integrable simple functions. Pick a sequence of simple functions with \( u_n \to u \) a.e. so that for \( m \in \mathbb{N} \) there holds

\[
\lim_n \phi(\omega, m [u(\omega) - u_n(\omega)]) = 0 \quad \text{for a.e. } \omega \in \Omega \tag{25}
\]

since \( \phi \) vanishes continuously at the origin. For \( k \in \mathbb{N} \) we find by Egorov’s theorem a set \( B = B_{k, m} \in \mathcal{A} \) with \( \nu(\Omega \setminus B) < 2^{-m k^{-1}} \) and such that (25) uniformly on \( B \).
Hence, setting $C_k = \bigcap_{m \geq 1} B_{k,m}$, we have $\nu(\Omega \setminus C_k) < k^{-1}$ and the convergence (25) holds uniformly on $C_k$. In particular, the simple functions $u_n \chi_{C_k}$ eventually belong to $L_\varphi(\mu)$ hence to $L_\varphi(\mu)$. Lemma 5.4 yields a sequence $D_j \in A$ with $\mu(\lim_j D_j) = 0$ such that for all $n \in \mathbb{N}$ sufficiently large there holds $u_n \chi_{C_k \cap D_j} \in C_\varphi(\mu)$ for all $j \in \mathbb{N}$ if $\varphi$ is real-valued. Otherwise we set $D_j = \emptyset$. In the former case, pick $F_j \in A_f$ with $\Omega = \bigcup_j F_j$ and possibly replace $D_j$ by $D_j \cup \Omega \setminus F_j$ so that $\mu(\Omega \setminus D_j) < \infty$ hence each $u_n \chi_{C_k \cap D_j}$ is a $\mu$-integrable simple function. Since $u$ has absolutely continuous norm, there exists for any given $\varepsilon > 0$ a $\delta > 0$ such that there holds $\|u \chi_{C_k \cap D_j}\varphi < \varepsilon$ whenever $j, k > \delta^{-1}$. Therefore

$$\|u - u_n \chi_{C_k \cap D_j}\varphi \leq \|u \chi_{C_k \cap D_j}\varphi + \| (u - u_n) \chi_{C_k \cap D_j}\varphi < \varepsilon + \| (u - u_n) \chi_{C_k \cap D_j}\varphi \xrightarrow{n \to \infty} \varepsilon.$$ 

Since $\varepsilon > 0$ is arbitrary, the proof is complete. \hfill \Box

5.3. Separability. Before we can characterize separability of $C_\varphi(\mu)$, hence of $L_\varphi(\mu)$ if the spaces agree, we recall the notion of a separable measure \cite[§3.5]{29}.

**Definition 5.2** (separable measure). The measure $\mu$ is called separable if $(A, d_\mu)$ is a separable space for the pseudometric $d_\mu(A, B) = \arctan\mu(A \Delta B)$.

This notion relates to a separable measurable spaces $(\Omega, A)$: if $\mathcal{A} = \sigma(A_n : n \geq 1)$ and the measure $\mu$ is $\sigma$-finite, then $\mu$ is separable and the countable algebra $\mathcal{A}_n$ generated by the sequence $A_n$ is dense in $(\mathcal{A}, d_\mu)$. This is implicit in the proof of \cite[Thm. 2.16]{12}. A measure is separable iff its completion is separable.

**Theorem 5.2.** Let $\varphi$ be real-valued. If $\mu$ and $X$ are separable, then there exists a dense sequence of integrable simple functions in $C_\varphi(\mu)$ hence the space is separable. Conversely, if $C_\varphi(\mu)$ is separable, then $X$ is separable. If in addition $\mu$ has no atom of infinite measure, then $\mu$ also is separable.

**Proof.** $\implies$ : we shall pass to several subsequences in the proof none of which we relabel. Lemma 5.5 reduces our task to constructing a sequence whose closure includes each function $x \chi_A \in C_\varphi(\mu)$ with $x \in X$ and $A \in \mathcal{A}_f$. Let $x \in X$ and $E_\ell \in A$ be dense sequences. We pass to the subsequence of members with $A_\ell \in \mathcal{A}_f$. Observe that $A_\ell$ still is dense in $\mathcal{A}_f$ as $d_\mu$ is continuous. In particular, each $A \in \mathcal{A}_f$ is contained in the $\sigma$-finite set $\bigcup_{\ell \geq 1} A_\ell$ except for a null set. Therefore we may assume $\mu$ to be $\sigma$-finite hence finite by Proposition 3.2. Lemma 5.3 yields sets $B = B(k, m) \in \mathcal{A}$ such that $\mu(\Omega \setminus B) < \frac{1}{k}$ and $x \chi_B \in C_\varphi(\mu)$. We claim the countable family $x_k \chi_{B(k, m)} \chi_{A_\ell}$ for $k, \ell, m \in \mathbb{N}$ to yield the required sequence. Indeed, each function of the form $x_k \chi_B \chi_{A_\ell}$ belongs to its closure since there exists a subsequence with $d_\mu(A, A_\ell) \to 0$ as $\ell \to \infty$. Pick a subsequence with $x_k \to x$ so that Egorov’s theorem yields for $\delta > 0$ a set $C_\delta \in \mathcal{A}$ such that $\mu(C_\delta) < \delta$ and for any given $n \in \mathbb{N}$ there holds $\varphi[n(x - x_k)] \to 0$ uniformly on $\Omega \setminus C_\delta$ as $k \to \infty$. Because $x \chi_A$ has absolutely continuous norm, we find for any given $\varepsilon > 0$ a $\delta_1 > 0$ such that $\|x \chi_A \chi_B \varphi\varphi < \varepsilon$ whenever $\delta, m^{-1} \leq \delta_1$. Choosing $\delta, m^{-1}$ sufficiently small and combining the last two statements yields

$$\|x \chi_A - x_k \chi_B \chi_{C_\delta}\varphi \leq \|x \chi_A \chi_B \varphi\varphi + \| (x - x_k) \chi_{A \cap B \cap C_\delta}\varphi < \varepsilon + \| (x - x_k) \chi_{A \cap B \cap C_\delta}\varphi \xrightarrow{k \to \infty} \varepsilon$$

Sending $\varepsilon \to 0^+$ completes the first part of the proof.
The proof requires results about weak topologies on Lemma 5.6.

Secs 6 and 7.

Proof. Theorem 5.2 settles the separability of which contradicts density of this sequence. □

Let \( u_n \in C^\sigma_\varphi(\mu) \) be a dense sequence. The set \( \bigcup_{n \geq 1} u_n(\Omega) \) is almost separably valued hence there exists a null set \( N \) such that

\[
\bigcup_{n \geq 1} u_n(\Omega \setminus N) =: S \in \mathcal{S}(X).
\]

But then every element of \( C^\sigma_\varphi(\mu) \) is \( S \)-valued a.e. since convergence in \( C^\sigma_\varphi(\mu) \) implies convergence a.e. up to subsequences. In conclusion \( S = X \in \mathcal{S}(X) \) because \( C^\sigma_\varphi(\mu) \) is almost decomposable by Lemma 5.3. Now, consider the separable \( \sigma \)-algebra \( \mathcal{A'} = \sigma(u_n; n \geq 1) \). To see that \( \mathcal{A'} \) is separable, note that it is generated by the sets \( u_n^{-1}(G_m) \) for \( G_m \) a sequence generating the Borel \( \sigma \)-algebra \( \mathcal{B}(X) \). As \( u_n \) is dense, each element \( u \in C^\sigma_\varphi(\mu) \) is \( \mathcal{A'} \)-measurable so that since \( C^\sigma_\varphi(\mu) \) is almost decomposable, we deduce \( \mathcal{A}_f \subset \mathcal{A'}_1 \) hence \( \mathcal{A}_f \subset \mathcal{A'}_1 \). Therefore our proof will be finished if we prove that \( \mu \) is \( \sigma \)-finite. As each member of the dense sequence \( u_n \) is \( \sigma \)-finitely concentrated, all elements of \( C^\sigma_\varphi(\mu) \) vanish outside some \( A \in \mathcal{A}_\sigma \) that is independent of the element under consideration. Suppose \( \mu(\Omega \setminus A) > 0 \).

Since \( \mu \) has no atom of infinite measure, we find \( B \in \mathcal{A}_f \) with \( B \subset \Omega \setminus A \) and \( \mu(B) > 0 \). As \( C^\sigma_\varphi(\mu) \) is almost decomposable, there exists a non-trivial element \( v \in C^\sigma_\varphi(\mu) \) vanishing outside of \( B \) hence \( v \) does not belong to the closure of \( u_n \), which contradicts density of this sequence.

Theorem 5.2 settles the separability of \( L_\varphi(\mu) \) if it happens to coincide with its subspace \( C^\sigma_\varphi(\mu) \). Indeed, this coincidence is also necessary for \( L_\varphi(\mu) \) to be separable. The proof requires results about weak topologies on \( L_\varphi(\mu) \) that will be proved in Secs 6 and 7.

Lemma 5.6. If \( L_\varphi(\mu) \) is separable and \( \mu \) has no atom of infinite measure, then \( L_\varphi(\mu) = C_\varphi(\mu) \). The same is true for \( L^\sigma_\varphi(\mu) \) and \( C^\sigma_\varphi(\mu) \) without restriction on the measure.

Proof. Let \( u \in L_\varphi(\mu) \setminus C_\varphi(\mu) \) so that we may pick \( E_n \in \mathcal{A} \) with

\[
\mu\left( \lim_n E_n \right) = 0, \quad \|u\chi_{E_n}\|_\varphi \geq \delta > 0.
\]

By Lemma 7.2 we find \( v_n \in V^\varphi(\mu) \) with

\[
\|v_n\|_\varphi \leq 1, \quad \lim_n \int_{E_n} \langle v_n, u \rangle \, d\mu \geq \frac{\delta}{2}.
\]

These integrals being finite, it is not restrictive to assume that \( \mu \) is \( \sigma \)-finite by restricting it to a set outside which the sequence of integrands \( \langle v_n, u \rangle \) vanishes. Separability of \( L_\varphi(\mu) \) yields a weak* convergent subsequence (not relabelled) of \( v_n \) that is weak* equi-integrable on \( L_\varphi(\mu) \) since by Lemma 6.2 and Theorem 6.1 the space \( V^\varphi(\mu) \) is sequentially weak* closed and by Theorem 7.2 the sequence \( v_n \) is weak* equi-integrable in the sense of Definition 7.1 hence we arrive at the contradiction

\[
0 < \frac{\delta}{2} \leq \lim_n \int_{E_n} \langle v_n, u \rangle \, d\mu = 0.
\]

The addendum follows by the first part and restriction of the measure. □

The above results on separability seem to be new for \( \omega \)-dependently generated Orlicz spaces even if \( X = \mathbb{R} \), though then known sufficient conditions [27, Thm. 7.10] and [16, Thm. 3.52] come close to ours. The autonomous, scalar case of our result can be found in [29, §3.5, Thm. 1].
To conclude our section on separability, we characterize the Asplund property of \( C_{\varphi}(\mu) \). A Banach space is called an Asplund space if each of its separable subspaces has a separable dual. This is equivalent to \( X' \) having the Radon-Nikodym property, which is relevant in the duality theory Sec. 6. From this stems our interest in view of applications where \( X \) itself is an Orlicz space.

**Lemma 5.7.** Let \( \varphi \) be real-valued. If \( X \) is an Asplund space and \( I_{\varphi}^* \) is real-valued, then \( C_{\varphi}^*(\mu) \) is an Asplund space. Conversely, if \( C_{\varphi}^*(\mu) \) is Asplund and \( \mu \) has no atom of infinite measure, then \( X \) is Asplund and \( I_{\varphi}^* \) real-valued.

*Proof.* Regarding the first claim, it suffices to prove that any separable subspace of \( C_{\varphi}^*(\mu) \) has a superspace with separable dual as then the subspace itself will have a separable dual via restriction of the density set. Here, the Hahn-Banach extension theorem enters. Therefore it suffices if any sequence \( u_n \in C_{\varphi}^*(\mu) \) is contained in a superspace with separable dual. We may assume \( \mu \) to be complete. Pick \( W \in S(X) \) such that the sequence \( u_n \) is almost \( W \)-valued and take \( \Sigma \in A_\sigma \) a set off which all \( u_n \) vanish. We denote by \( \phi \) the restriction of \( \varphi \) to \( \Sigma \times W \). As \( \phi \) is a normal integrand hence strongly measurable in the Wijsman topology by the Hess theorem, the \( \sigma \)-algebra \( A' \) generated by \( u_n \) and \( \phi \) on \( \Sigma \) is separable. Consequently, the restriction \( \nu \) of \( \mu \) to \( A' \) is separable if we arrange \( \nu \) to be \( \sigma \)-finite, which is possible without restriction by enlarging \( A' \) by a sequence \( F_n \in A \) such that \( \Sigma = \bigcup F_n \). There holds \( C_{\varphi}(\nu)^* = L_{\varphi}^*(\nu) \) by Corollary 6.2 since \( X \) is an Asplund space iff \( X' \) has the Radon-Nikodym property. Therefore we are finished if we prove that \( L_{\varphi}^*(\nu) \) is separable. Denoting by \( \mathcal{E} : C_{\varphi}(\nu) \to C_{\varphi}(\mu) \) the identical embedding, we have \( I_{\phi}^* = E^* I_{\varphi}^* \) by [20, §3.4, Thm. 3] hence \( I_{\varphi}^* = I_{\varphi}^* \) is real-valued so that the dual space \( L_{\varphi}^*(\nu) \) equals \( C_{\varphi}(\nu)^* \) by Theorem 5.1. In conclusion, this space is separable by Theorem 5.2 as \( W' \) and \( \nu \) are.

For the converse, since for any sequence \( u_n \in C_{\varphi}(\mu) \) the measure \( \nu \) defined as above is separable, the space \( C_{\varphi}(\nu) \) is separable by Theorem 5.2 because we ruled out atoms of infinite measure. Thus \( C_{\varphi}(\mu)^* = L_{\varphi}^*(\nu) \) is separable by the Asplund property. Consequently, Theorem 5.2 yields that \( W' \) is separable hence \( X \) is an Asplund space.

Suppose now that for \( v \in C_{\varphi}(\mu)^* = L_{\varphi}(\mu) \) the sequence \( u_n \) is such that

\[
I_{\varphi}^*(v) = \lim_n \langle v, u_n \rangle - I_{\varphi}(u_n).
\]

Since the separable dual space \( L_{\varphi}^*(\nu) \) coincides with \( C_{\varphi}(\nu)^* \) by Lemma 5.6, there holds \( I_{\varphi}^*(v) = I_{\varphi}(v) < \infty \) by Theorem 5.1 hence \( I_{\varphi}^* \) is real-valued.

If \( \mu \) is non-atomic and \( \varphi \notin \Delta_2 \), then \( L_{\varphi}(\mu) \) contains an isometric copy of \( \ell_\infty (\mathbb{N}; \mathbb{R}) \) by [18, Thm. 1.1] hence fails the Asplund property since closed subspaces inherit this property but \( \ell_\infty (\mathbb{N}; \mathbb{R}) \) does not have it. To this extent, the condition \( L_{\varphi}(\mu) = C_{\varphi}^*(\mu) \) is necessary for \( L_{\varphi}(\mu) \) to be an Asplund space.

### 6. Duality theory

In this section, we represent the dual spaces of \( C_{\varphi}(\mu) \) and \( L_{\varphi}(\mu) \), characterize their reflexivity, and represent the convex conjugate and the subdifferential of a general integral functional (2) on \( L_{\varphi}(\mu) \). We frequently make the request that all elements of \( L_{\varphi}(\mu) \) be \( \sigma \)-finitely concentrated. However, wherever we do so, it were possible to work instead on the space \( L_{\varphi}^*(\mu) \) regardless whether it agrees with its superspace \( L_{\varphi}(\mu) \).
6.1. **Types of functionals.** We start by introducing the important classes of absolutely continuous and singular functionals. These will turn out to induce a decomposition of the dual space as two complemented subspaces, i.e. each element of the dual space has a unique representation as a sum of an absolutely continuous functional and a singular one. Furthermore, we will be able to obtain the duality theory for $C_\varphi(\mu)$ as a corollary to the one for $L_\varphi(\mu)$ by identifying conditions under which its annihilator in $L_\varphi(\mu)$ coincides with the singular functionals.

**Definition 6.1** (absolutely continuous and singular functionals). The space of absolutely continuous functionals on $L_\varphi(\mu)$ is defined as $A_{\varphi*}(\mu) = \{ \ell \in L_\varphi(\mu)^* \mid u_n \uparrow u \implies \lim_{n} \ell(u_n) = \ell(u) \}.$

We denote by $c_0(\mathcal{A})$ the collection of sequences $C_n \in \mathcal{A}$ with $\mu(\lim_n C_n) = 0$. The space of singular functionals on $L_\varphi(\mu)$ is defined as

$$S_{\varphi*}(\mu) = \{ \ell \in L_\varphi(\mu)^* \mid \exists C_n \in c_0(\mathcal{A}) : \ell(u_{\chi_{\Omega \setminus C_n}}) = 0 \quad \forall u \in L_\varphi(\mu) \quad \forall n \in \mathbb{N} \}.$$ 

Such a sequence $C_n$ is called a concentration sequence for the functional $\ell$. We introduce the annihilator

$$N_{\varphi*}(\mu) = C_\varphi(\mu)^\perp = \{ \ell \in L_\varphi(\mu)^* \mid \ell(u) = 0 \quad \forall u \in C_\varphi(\mu) \}.$$ 

If a concentration sequence $C_n$ does not decrease, a decreasing one $D_n$ may be obtained by setting

$$D_1 = C_1, \quad D_n = \bigcap_{i=1}^{n} C_i \quad n \geq 2.$$ 

A consequence of a functional $\ell$ being absolutely continuous is that if $A_i \in \mathcal{A}$ is a measurable partition of $\Omega$ and $u \in L_\varphi$, then

$$\ell(u) = \sum_{i=1}^{\infty} \ell(u_{\chi_{A_i}}).$$

Combining this with the existence of decreasing concentration sequences, we see $A_{\varphi*}(\mu) \cap S_{\varphi*}(\mu) = \{0\}$. We say that $\ell \in L_\varphi(\mu)^*$ is $\sigma$-finitely concentrated if there exists $\Sigma \in \mathcal{A}_\varphi$ such that $\ell(u) = 0$ whenever $u$ vanishes on $\Sigma$. The property of all elements being $\sigma$-finitely concentrated carries over from $L_\varphi$ to elements of $S_{\varphi*}$ and, if $\varphi$ is real-valued, also to $N_{\varphi*}$:

**Proposition 6.1.** Let $L_\varphi(\mu) = L_{\varphi\sigma}(\mu)$. If a functional $\ell \in L_{\varphi}(\mu)^\sigma$ admits for each $\Sigma \in \mathcal{A}_\varphi$ the decomposition

$$\ell(u_{\chi_{\Sigma}}) = \ell_a(u_{\chi_{\Sigma}}) + \ell_s(u_{\chi_{\Sigma}}), \quad u \in L_\varphi(\mu)$$

for some $\ell_a \in A_{\varphi*}$ and $\ell_s \in S_{\varphi*}$, then it admits a decomposition

$$\ell(u) = L_a(u) + L_s(u), \quad u \in L_\varphi(\mu)$$

with $L_a \in A_{\varphi*}(\mu)$ and $L_s \in S_{\varphi*}(\mu)$ independent of $\Sigma$. In particular $S_{\varphi*}(\mu) = S_{\varphi\sigma*}(\mu)$.

If $\varphi$ is real-valued, $S_{\varphi*}(\mu)$ may be replaced by $N_{\varphi*}(\mu)$ throughout the statement.
Proof. Pick a sequence \( u_n \in L^\varphi_\sigma(\mu) \) with
\[
\|u_n\|_\varphi < 1, \quad \ell (u_n) > \|\ell\|^*_\varphi - \frac{1}{n}
\] (28)
and let \( \Sigma \in A_\sigma \) be a set outside of which all \( u_n \) vanish. We claim that any singular component \( \ell_s \) arising in a decomposition (26) vanishes outside \( \Sigma \). Otherwise there would exist \( v \in L^\varphi(\mu) \) vanishing on \( \Sigma \) but
\[
I_\varphi (v) \leq 1, \quad \ell_s (v) = a > 0.
\] (29)
Let \( \ell \) admit the decomposition (26) on some \( \sigma \)-finite set outside of which \( v \) and all \( u_n \) vanish; let \( C_m \in A \) be a concentration sequence for the singular functional \( \ell_s \).

If \( \varphi \) is real-valued, pick instead \( C_m \) as a decreasing sequence such that
\[
\mu \left( \lim_m C_m \right) = 0, \quad v\chi_{C_m} \in C_\varphi(\mu)
\] (30)
by Lemma 5.4. Since \( I_\varphi (u_n) \leq \|u_n\|_\varphi < 1 \) by Lemma 3.5, absolute continuity yields
\[
\forall n \in \mathbb{N} \exists m \in \mathbb{N}: I_\varphi (v\chi_{C_m}) < 1 - I_\varphi (u_n), \quad \ell_a (v\chi_{C_m}) > -\frac{1}{n}.
\] (31)
Setting \( \bar{u} = v\chi_{C_m(n)} + u_n \), we have \( I_\varphi (\bar{u}) = I_\varphi (v\chi_{C_m}) + I_\varphi (u_n) < 1 \) hence \( \|\bar{u}\|_\varphi \leq 1 \).

Now, picking \( n \) so large that \( \frac{2}{n} < a \) and combining (31), (29), (30) and (28), we obtain the contradiction
\[
\ell (\bar{u}) = \ell_a (v\chi_{C_m}) + \ell_s (v) - \ell_s \left( v\chi_{C_m} \right) + \ell (u_n) \geq a + \|\ell\|^*_\varphi - \frac{2}{n} > \|\ell\|^*_\varphi.
\]

Consequently, any singular component \( \ell_s \) is concentrated on the \( \sigma \)-finite set \( \Sigma \). In particular \( \ell_s \in S^\sigma_\sigma(\mu) \) and \( \ell - \ell_s \) is an absolutely continuous functional so that setting \( L_a = \ell - \ell_s \) and \( L_s = \ell_s \) obtains the decomposition (27).

\[\square\]

**Lemma 6.1.** There holds \( S^\sigma_\varphi(\mu) \subset N^\sigma_\varphi(\mu) \), with equality if \( \varphi \) is real-valued and \( L^\varphi(\mu) = L^\varphi_\sigma(\mu) \).

**Proof.** If \( C_j \in A \) is a concentration sequence for \( \ell \in S^\sigma_\varphi(\mu) \) so that \( \mu (\lim_j C_j) = 0 \), then there holds
\[
\ell (u) = \lim_j \ell \left( u\chi_{C_j} \right) + \lim_j \ell \left( u\chi_{\Omega \setminus C_j} \right) = \lim_j \ell \left( u\chi_{C_j} \right) = 0 \quad \forall u \in C_\varphi(\mu).
\]

Regarding equality, let \( \ell \in N^\sigma_\varphi(\mu) \). We shall construct a concentration sequence for \( \ell \). Proposition 6.1 yields \( N^\sigma_\varphi(\mu) = N^\sigma_\varphi(\mu) \) so that we may assume \( \mu \) to be \( \sigma \)-finite without loss of generality hence we may take \( \mu \) finite by Proposition 3.2. Let \( u_i \in L^\varphi(\mu) \) be a sequence with
\[
\|u_i\|_\varphi < 1, \quad \|\ell\|^*_\varphi < 2^{-i} + \ell (u_i).
\] (32)
We set \( C_i,0 = \Omega \) and employ Lemma 5.4 together with the absolute continuity of the integral to inductively construct a sequence of sets \( C_{i,n} \subset C_{i,n-1} \) such that
\[
u_i \chi_{\Omega \setminus C_{i,n}} \in C_\varphi(\mu), \quad I_\varphi \left( u_i \chi_{C_{i,n}} \right) + \mu (C_{i,n}) \leq 2^{-i-n}.
\] (33)
We aim to prove that the decreasing evanescent sequence
\[
C_n = \bigcup_{i \geq 1} C_{i,n}, \quad \mu (C_n) \leq \sum_{i \geq 1} \mu (C_{i,n}) \leq 2^{-n},
\]
provides the sought concentration sequence. For this, it remains to prove \( \ell (u) = 0 \) if \( u \) vanishes on some \( C_n \). Suppose \( u \in L^\varphi(\mu) \) vanishes on \( C_n \) but \( \ell (u) = a > 0 \) and
We have \( \|\bar{u}\|_\varphi \leq 1 \) since by (33) there holds
\[
I_\varphi(\bar{u}) \leq 2^{-k} < 1.
\]
Now by (32) there follows
\[
\ell(\bar{u}) = \ell(u) \geq a + \|\ell\|_\varphi^* - 2^{-k} > \|\ell\|_\varphi^*,
\]
which contradicts \( a > 0 \). \( \square \)

Absolutely continuous and singular functionals are rather stable classes:

**Lemma 6.2.** If \( L_\varphi(\mu) = L_\varphi^*(\mu) \), then \( A_\varphi(\mu) \) and \( S_\varphi^*(\mu) \) are sequentially weak* closed.

**Proof.** For \( \ell_n \in A_\varphi^*(\mu) \) with \( \ell_n \rightharpoonup^* \ell \) and \( u \in L_\varphi(\mu) \), we define a signed finite measure \( \lambda_n(E) = \ell_n(u\chi_E) \) that is absolutely continuous w.r.t. \( \mu \). Set \( \lambda(E) = \ell(u\chi_E) \). Restricting to a relevant set \( \Sigma \in \mathcal{A} \) outside of which \( u \) vanishes, we find \( \nu \) a finite measure as in Proposition 3.2 w.r.t. which each \( \lambda_n \) is absolutely continuous. As \( \lambda(E) = \lim_n \lambda_n(E) \) for all \( E \in \mathcal{A} \), we conclude by the Vitali-Hahn-Saks theorem [12, Thm. 2.53] that the sequence \( \lambda_n \) hence its limit \( \lambda \) is uniformly absolutely continuous w.r.t. \( \nu \) and \( \mu \) so that \( \ell \) is absolutely continuous as a functional.

Let now \( \ell_n \in S_\varphi^*(\mu) \). We find by Proposition 6.1 a set \( \Sigma \in \mathcal{A} \) outside of which all \( \ell_n \) vanish. Pick a double sequence \( E = E(m, n) \) such that \( \nu(E) < 2^{-m-n} \) and \( m \to E(m, n) \) is a concentration sequence for \( \ell_n \). Setting \( C = C(m) = \bigcup_{n \geq 1} E \) we have \( \nu(C) < 2^{-m} \) and \( C(m) \) is a joint concentration sequence for all \( \ell_n \), i.e. each \( \ell_n(u) \) vanishes if \( u \) vanishes on some \( C \). Then \( C \) is a concentration sequence for the limit functional \( \ell \). \( \square \)

### 6.2. Representation results.

Throughout this subsection, we assume that \( \mu \) has no atom of infinite measure. We introduce the space
\[
\mathcal{V}_\varphi^*(\mu) = \{v \mid v : \Omega \to X' \text{ is integrally weak* measurable and } \|v\|_\varphi^* < \infty\}
\]
along with its operator seminorm
\[
\|v\|_\varphi^* = \sup_{\|u\|_\varphi \leq 1} \int \langle v(\omega), u(\omega) \rangle \, d\mu(\omega).
\]
The integral is exhausting as in Sec. 2. Applying the standard procedure of identifying elements whose difference belongs to the kernel of \( \|\cdot\|_\varphi^* \), we obtain a normed space \( V_\varphi^* \) of continuous linear functionals on \( L_\varphi \). It is insightful to describe this kernel more explicitly. There holds \( \|v\|_\varphi^* = 0 \) iff \( \sup_{u \in L_\varphi} \int \langle v, u \rangle \, d\mu = 0 \). To interchange this supremum with the integral, we want to apply Theorem 2.1 and need to check its assumptions. First, the space \( L_\varphi \) is almost decomposable. Second, the integrand is integrally Carathéodory hence integrally separably measurable. Third, \( \mu \) has no atom of infinite measure. Fourth, the value of the supremum is not \( +\infty \) by assumption. Therefore we conclude
\[
\int \text{ess-sup}_{W \in \mathcal{S}(X)} \sup_{x \in W} \langle v(\omega), x \rangle \, d\mu(\omega) = \sup_{u \in L_\varphi} \int \langle v, u \rangle \, d\mu = 0
\]
hence \( \text{ess-sup}_{W \in \mathcal{S}(X)} \sup_{x \in W} \langle v(\omega), x \rangle = 0 \) a.e. This implies that (the equivalence class of) \( v \) vanishes a.e. if \( X \) is separable. In general, it suggests one should think
We shall prove that the absolutely continuous functionals $A$ of equivalence classes in Hölder and reverse Hölder inequalities to determine if a given measurable function $f$ satisfies

$$\int |\langle f(\omega), u(\omega) \rangle| \, d\mu(\omega) \leq 2\|v\|_{\varphi^*}^* \|u\|_{\varphi}.$$  \hfill (34)

Proof. Combine the Fenchel-Young inequality with Lemma 3.5. \hfill \square

**Lemma 6.3.** For $u \in L_{\varphi}(\mu)$ and $v \in L_{\varphi^*}(\mu)$ there holds

$$\int |\langle v(\omega), u(\omega) \rangle| \, d\mu(\omega) \leq 2\|v\|_{\varphi^*}^* \|u\|_{\varphi}.$$  \hfill (35)

Then $v \in V_{\varphi}(\mu)$ with $\|v\|_{\varphi^*}^* \leq 1$. In particular, there holds $\|v\|_{\varphi^*}^* \leq 1$ if $L_{\varphi^*} = I_{\varphi^*}$ on $\text{lin}(v)$. If moreover $v$ is strongly measurable, then $v \in L_{\varphi^*}(\mu)$.

The lemma applies if $L$ are the simple functions in $L_{\varphi}^r(\mu)$ or if $\varphi$ is real-valued and $L$ the simple functions in $C_{\varphi}(\mu)$. These assertions follow from Lemmas 4.1 and 5.3 together with 5.5.

Proof. We start by proving that (35) holds for $u \in \text{cl}\, L$ if $L_f = L$ for $A \in \mathcal{A}$, thereby reducing to the case when $L$ is linear and almost decomposable. Pick by Theorem 3.1 a sequence $u_n \in L$ with $u_n \to u$ in $L_{\varphi}(\mu)$ and a.e. Let $\Omega^+ = \{\langle v, u \rangle > 0\}$ and $\Omega_n^+ = \{\langle v, u_n \rangle > 0\}$. We have $u_n \chi_{\Omega_n^+} \in \varphi$ by assumption. The Fatou lemma yields

$$\int_{\Omega_n^+} \langle v, u \rangle \, d\mu \leq \liminf_n \int_{\Omega_n^+} \langle v, u_n \rangle \, d\mu \leq \lim_n \|u_n\|_{\varphi} = \|u\|_{\varphi}.$$  

Since $\text{cl}\, L$ is linear, we may argue analogously for the negative part $\langle v, u \rangle^-$ so that $\langle v, u \rangle$ is integrable and (35) holds for $u \in \text{cl}\, L$. We may from now on assume $L$ linear and almost decomposable.

We claim that (35) holds for all $u \in L_{\varphi}(\mu)$. Otherwise there were $u \in L_{\varphi^*}(\mu)$ with $\langle v, u \rangle \geq 0$ a.e. and $\int \langle v, u \rangle \, d\mu = \infty$. According to Proposition 2.1 there is $\Sigma \in \mathcal{A}_{\varphi}$ with $\int_{\Sigma} \langle v, u \rangle \, d\mu = \infty$ since we ruled out atoms with infinite measure. Choose $u_n \in L$ with $u_n \uparrow u$ on $\Sigma$ by Lemma 4.2 and recognize the contradiction

$$\infty = \lim_n \int_{\Sigma} \langle v, u_n \rangle \, d\mu \leq \lim_n \|u_n \chi_{\Sigma}\|_{\varphi} = \|u \chi_{\Sigma}\|_{\varphi} \leq \|u\|_{\varphi} < \infty.$$  

Consequently, the integral $\int \langle v, u \rangle \, d\mu$ exists for every $u \in L_{\varphi}(\mu)$. In particular, there exists $\Sigma \in \mathcal{A}_{\varphi}$ outside of which $\langle v, u \rangle$ vanishes, so that we find

$$\int \langle v, u \rangle \, d\mu = \lim_n \int_{\Sigma} \langle v, u_n \rangle \, d\mu \leq \lim_n \|u_n \chi_{\Sigma}\|_{\varphi} \leq \|u\|_{\varphi}.$$  

---

1Whenever the space $L_{\varphi^*}(\mu)$ appears, we tacitly assume $\varphi^*$ to be a generator.
Therefore \( v \in V_{\varphi}^*(\mu) \) with \( \| v \|_\varphi^* \leq 1 \). The addenda are obvious by definition of the dual Orlicz norm.

\[ \Box \]

**Corollary 6.1.** Let \( L \subset L_{\varphi}(\mu) \) be an almost decomposable linear subspace. If \( \varphi \) is dualizable i.e. for every element of \( L_{\varphi}^*(\mu) \), then

\[ \| v \|_{\varphi^*} = \sup_{w \in B_L} \langle v, u \rangle \quad \forall v \in L_{\varphi}^*(\mu). \]  

**Proof.** By Theorem 2.2 there holds \( I_{\varphi}^* = I_{\varphi^*} \) on \( L_{\varphi}^*(\mu) \) since we ruled out atoms of infinite measure. Therefore the Orlicz norm \( \| \cdot \|_{\varphi^*} \) and the dual Orlicz norm \( \| \cdot \|_{\varphi^*}^* \) coincide there. Because \( L \) is almost decomposable and the functional induced by \( v \) is absolutely continuous, the right-hand side in (36) coincides with the operator norm, that is, the dual Orlicz norm \( \| v \|_{\varphi^*}^* \) according to Lemma 4.2.

We start by recasting a known characterization of the absolutely continuous functionals on \( L_\infty(\mu; X) \) to match our setting. This will be the foundation on which we build the general case by means of the almost embedding Lemma 3.8.

**Proposition 6.2.** Let \( \mu \) be \( \sigma \)-finite and \( \ell \in A_1(\mu; X') \) an absolutely continuous element of \( L_\infty(\mu; X)^* \). Then there exists a weak* measurable function \( v: \Omega \to X' \) such that

\[ \ell(u) = \int \langle v(\omega), u(\omega) \rangle \, d\mu(\omega) \quad \forall u \in L_\infty(\mu; X). \]

In particular, there holds \( A_1(\mu; X') = V_1(\mu; X') \) via this identification.

**Proof.** Observe that for \( v \in V_1(\mu; X') \) there holds

\[ \| v \|_{\varphi}^* = \sup_{|u|_{\infty} \leq 1} \int \langle v(\omega), u(\omega) \rangle \, d\mu(\omega) = \int \text{ess-sup} \sup_{x \in B_X} \langle v(\omega), x \rangle \, d\mu(\omega) \]

according to Theorem 2.1 as can be seen by absorbing the pointwise a.e. restriction \( |u|_{\infty} \leq 1 \) into the integrand as an indicator of the ball \( B_X \). Therefore \( V_1(\mu; X') \) is isometric to the space \( L_{\varphi}(X) \) defined in [6, VIII] through the identification remarked below [6, Lem. VIII.3]. Moreover, the definition [6, VIII, Def. 5] of a singular functional agrees with ours in the current situation since for any family of measurable sets on a \( \sigma \)-finite measure space there exists a countable subfamily whose intersection returns the essential intersection of the entire family by [12, Thm. 1.108]. Therefore [6, Thm. VIII.5] yields

\[ L_{\infty}(\mu; X)^* = V_1(\mu; X') \oplus S_1(\mu; X') \]

whence any \( \ell \in A_1(\mu; X') \) is induced by some \( v \in V_1(\mu; X') \) as \( A_1(\mu; X') \cap S_1(\mu; X') = \{0\} \).

We shall reduce the matter of representing absolutely continuous functionals by measurable functions to a finite measure space on which we can apply Lemma 3.8 to invoke Proposition 6.2. We need then piece together the representations found on an exhausting sequence. The next proposition shall serve for both these steps. In the following, we consider functions defined on a set \( A \in \mathcal{A} \) as trivially extended to all of \( \Omega \).
Proposition 6.3. Let \( L_\varphi(\mu) = L_\varphi^*(\mu) \) and \( \ell \in A_\varphi^*(\mu) \) such that for any \( \Sigma \in A_\sigma \) there exists a sequence of sets \( \Sigma_n \in A \) with \( \Sigma = \bigcup_{n \geq 1} \Sigma_n \) and elements \( v_n \in V_\varphi^*(\Sigma_n) \) such that

\[
\ell (u \chi_{\Sigma_n}) = \int \langle v_n, u \rangle \, d\mu \quad \forall u \in L_\varphi(\mu).
\]  (37)

Then there exists a sequence \( w_n \in V_\varphi(\Sigma) \) with \( w_n = v_n \) on \( \Sigma_n \) and a unique \( v \in V_\varphi^*(\mu) \) with \( w_n \uparrow v \) on \( \Sigma \) and

\[
\ell (u) = \int \langle v, u \rangle \, d\mu \quad \forall u \in L_\varphi(\mu).
\]  (38)

Moreover, if each \( v_n \) is (integrally) strongly measurable, then so is \( v \). If in addition \( \varphi \) is dualizable i.e. for every element of \( L_\varphi^*(\mu) \) the minimum of \( \varphi^* \) at the origin is strict outside a \( \sigma \)-finite set, then \( v \) is uniquely determined as an element of \( L_\varphi^*(\mu) \).

Proof. By (37) there holds

\[
\ell (u \chi_{\Sigma_n} \chi_{\Sigma_m}) = \int_{\Sigma_m} \langle v_n, u \rangle \, d\mu = \int_{\Sigma_n} \langle v_m, u \rangle \, d\mu \quad \forall u \in L_\varphi(\mu) \quad \forall n, m \in \mathbb{N}
\]

hence \( v_n \chi_{\Sigma_m} = v_m \chi_{\Sigma_n} \) in \( V_\varphi^*(\mu) \). If \( \varphi \) is dualizable, then \( \| \chi_{\Sigma_n} \varphi^* = \| \chi_{\Sigma_n} \| \varphi^* \) holds by Corollary 6.1 so that if each \( v_n \) is (integrally) strongly measurable, then the identity obtains even as one of strongly measurable functions in \( L_\varphi^*(\mu) \) hence a.e. in \( X' \). We see from this that the linear integrand

\[
w_n (\omega) = \begin{cases} v_m (\omega) & \text{if } \omega \in \Sigma_m \text{ for } 1 \leq m \leq n \\ 0 & \text{else.} \end{cases}
\]

is well-defined a.e. and converges from below to a weak* \( A_\mu \)-measurable function \( v_\Sigma \in V_\varphi^*(\Sigma) \) as \( n \to \infty \) while \( w_n \chi_{\Sigma_n} = v_n \). In the dualizable case, if each \( w_n \) is strongly \( A_\mu \)-measurable, the limit \( v_\Sigma \) will inherit this property as well. Now, the mapping

\[
A \to \mathbb{R}: A \mapsto \int_A \langle v_\Sigma, u \rangle \, d\mu
\]

is continuous w.r.t. a.e. convergence of indicators hence the norm of linear integrands is lower semicontinuous w.r.t. convergence from below. Also \( \| w_n \| \varphi^* \leq \| \ell \| \varphi^* \) is obvious so that in total

\[
\| v_\Sigma \| \varphi^* = \liminf_{n \to \infty} \| w_n \| \varphi^* \leq \limsup_{n \to \infty} \| w_n \| \varphi^* \leq \| \ell \| \varphi^*
\]

whence we conclude \( v_\sigma \in V_\varphi^*(\mu) \). In the dualizable case, we invoke Corollary 6.1 to find \( \| v_\Sigma \| \varphi^* = \| v_\Sigma \| \varphi^* \) hence \( v_\Sigma \in L_\varphi^*(\mu) \). As \( \ell \) is absolutely continuous, we have

\[
\ell (u \chi_{\Sigma}) = \lim_n \ell (u \chi_{\Sigma_n}) = \lim_n \int \langle w_n, u \rangle \, d\mu = \int \langle v_\Sigma, u \rangle \, d\mu
\]

by the dominated convergence theorem. Since \( \Sigma \in A_\sigma \) is arbitrary, this yields a well-defined equivalence class of integrally weak* measurable functions \( v \) defined to agree with \( v_\Sigma \) on \( \Sigma \) hence \( v \in V_\varphi^*(\mu) \). Moreover, we have seen that \( v \) will even be integrally strongly measurable in the dualizable case. Because \( L_\varphi(\mu) = L_\varphi^*(\mu) \) is assumed, we may conclude that (38) holds.
Ad addendum: for \( \Sigma \in \mathcal{A}_\sigma \), there exists a unique \( v_\Sigma \in \mathcal{L}^* (\Sigma) \) representing \( \ell \) on \( \Sigma \). We have
\[
\sup_{\Sigma \in \mathcal{A}_\sigma} \|v_\Sigma\|_{\mathcal{L}^*} \leq \sup_{\Sigma \in \mathcal{A}_\sigma} \|v_\Sigma\|_{\mathcal{L}^*} \leq \|\ell\|_{\mathcal{L}^*} \leq 1
\]
so there exists \( \Sigma_0 \in \mathcal{A}_\sigma \) with
\[
\int_{\Sigma_0} \varphi^* [\omega, v_{\Sigma_0} (\omega)] \, d\mu (\omega) = \sup_{\Sigma \in \mathcal{A}_\sigma} \int_{\Sigma} \varphi^* [\omega, v_\Sigma (\omega)] \, d\mu (\omega) \leq 1. \tag{39}
\]
We may assume \( \Sigma_0 \) to contain the \( \sigma \)-finite set off which the minimizer of \( \varphi^* \) the origin is isolated. If \( v_{\Sigma_0} \) could be extended outside of \( \Sigma_0 \) in a non-trivial way to still represent \( \ell \), then there were \( u \in \mathcal{L}^* (\mu) \) concentrated on \( \Omega \setminus \Sigma_0 \) with \( \ell (u) = b > 0 \). Since \( u \) is concentrated on a \( \sigma \)-finite set, we can extend \( v_{\Sigma_0} \) to this set, which contradicts the definition of \( \Sigma_0 \) as the extension would surpass the supremum in (39) if the minimizer of \( \varphi^* \) at the origin is isolated. \( \square \)

**Theorem 6.1.** Let \( \mathcal{L}_\varphi (\mu) = \mathcal{L}^* (\mu) \). If \( v \in \mathcal{V}_{\varphi^*} (\mu) \) is identified with the continuous linear functional
\[
\mathcal{L}_\varphi (\mu) \rightarrow \mathbb{R} : u \mapsto \langle v (\omega), u (\omega) \rangle \, d\mu (\omega), \tag{40}
\]
then one has
\[
\mathcal{A}_{\varphi^*} (\mu) = \mathcal{V}_{\varphi^*} (\mu). \tag{41}
\]
If moreover \( X' \) has the Radon-Nikodym property, then elements of \( \mathcal{V}_{\varphi^*} (\mu) \) are integrally strongly measurable. If in addition \( \varphi \) is dualizable i.e. for every \( v \in \mathcal{L}^* (\mu) \), the minimum of \( \varphi^* \) at the origin is strict outside a \( \sigma \)-finite set and \( v \) is identified with the continuous linear function (40), then one has
\[
\mathcal{A}_{\varphi^*} (\mu) = \mathcal{L}^* (\mu). \tag{42}
\]
Remember that the identity \( \mathcal{L}_\varphi (\mu) = \mathcal{L}^* (\mu) \) was characterized in Theorem 3.2. If \( \mu \) is \( \sigma \)-finite, then this identity holds and the condition on \( \varphi^* \) is trivially satisfied. One may relax the Radon-Nikodym property by requiring instead that \( X' \) have the Radon-Nikodym property w.r.t. every restriction of the particular measure \( \mu \) to any set of finite measure. This always holds if \( \mu \) is purely atomic, cf. [11, p. 62].

**Proof.** Ad (41): The identification (40) defines an isometric embedding by the dominated convergence theorem so it remains to represent a given \( \ell \in \mathcal{A}_{\varphi^*} (\mu) \) by some \( v \in \mathcal{V}_{\varphi^*} (\mu) \) this way. Let \( F \in \mathcal{A}_f \) and consider the functional \( \ell_F (u) = \ell (u \chi_F) \). If each \( \ell_F \) permits a representation via \( v_F \in \mathcal{V}_{\varphi^*} (F) \) by (40), then (41) will follow by Proposition 6.3.

Let \( F_\varepsilon \) be an isotonic family with \( F_\varepsilon \uparrow F \) as in Lemma 3.8. Consider the mapping
\[
v_\varepsilon : \mathcal{L}_\varphi (\mu; X) \rightarrow \mathbb{R} : u \mapsto \ell (u \chi_{F_\varepsilon}).
\]
It is linear continuous since \( \|v \chi_{F_\varepsilon}\|_X \leq C_\varepsilon \|v\|_X \). By Proposition 6.2 there exists a unique \( v_\varepsilon \in \mathcal{V}_1 (\mu; X') \) with
\[
v_\varepsilon (u) = \langle v_\varepsilon, u \rangle \, d\mu \quad \forall u \in \mathcal{L}_\varphi (\mu; X). \tag{43}
\]
As \( \mathcal{L}_\varphi (\mu; X) \) is linear and almost decomposable, we may invoke Lemma 6.4 to find that (43) defines an element \( v_\varepsilon \in \mathcal{V}_{\varphi^*} (F_\varepsilon) \). Lemma 4.2 together with the absolute
continuity of $\ell$ then implies that (43) agrees with $\nu_\varepsilon$ on all of $L_\varphi(\mu)$ hence we conclude existence of a representing function $\nu_F \in V_{\varphi^*}(F)$ as required by Proposition 6.3. The first claim has been proved.

Ad first addendum and (42): arguing as in the first step, we may reduce the problem to the set $F$ by the addendum in Proposition 6.3. Consider the restriction of the mapping $\nu_\varepsilon$ (not relabelled)

$$
\nu_\varepsilon : A \times X \to \mathbb{R} : (A, x) \mapsto \ell(x\chi_{AF_\varepsilon}.
$$

We may regard $A \to \nu_\varepsilon(A)$ as an $X'$-valued vector measure because $\nu_\varepsilon(A, \cdot) \in X'$ by $\|\nu X_{AF_\varepsilon}\|_{X'} \leq C\|v\|_{X'}$. Since $\ell$ is absolutely continuous, the vector measure $\nu_\varepsilon$ is weak* $\sigma$-additive. Let $\Omega_j \in A$ be a measurable partition of $\Omega$ and pick for $\nu_\varepsilon(\Omega_j)$ an $x_j \in B_X$ with $\|\nu_\varepsilon(\Omega_j)\|_{X'} < 2^{-j} + \nu_\varepsilon(\Omega_j, x_j)$ so that absolute continuity of $\ell$ yields the estimate

$$
\sum_{j \geq 1} \|\nu_\varepsilon(\Omega_j)\|_{X'} \leq 1 + \ell \left( \sum_{j \geq 1} x_j \chi_{\Omega_j} \chi_{AF_\varepsilon} \right) \leq 1 + C\|\ell\|_{\varphi^*} < \infty.
$$

Consequently, $\nu_\varepsilon$ is $\sigma$-additive in norm convergence and its total variation

$$
\|\nu_\varepsilon\|_{(F)} := \sup \left\{ \sum_{i=1}^n \|\nu_\varepsilon(A_i)\|_{X'} : A_i \in A \text{ a finite partition of } F \right\}
$$

is finite. Thus, applying the Radon-Nikodym theorem, we deduce existence of a density $\nu_\varepsilon \in L_1(\mu; X')$ with $\nu_\varepsilon(A) = \int_A \nu_\varepsilon \, d\mu$ for all $A \in A$. We claim that

$$
\ell(u\chi_{AF_\varepsilon}) = \int_{F_\varepsilon} \langle \nu_\varepsilon, u \rangle \, d\mu \quad \forall u \in L_\varphi(\mu).
$$

Since this identity holds if $u$ is a simple function, Lemma 6.4 and the remark below it imply $\|v_\varepsilon\|_{p^*} \leq 1$ hence, if $\varphi$ is dualizable, $\nu_\varepsilon \in L_{\varphi^*}(\mu)$ with $\|\nu_\varepsilon\|_{\varphi^*} \leq 1$. Lemma 4.2 and the absolute continuity of $\ell$ then imply that $\nu_\varepsilon$ induces an integral representation on $F_\varepsilon$ for all $u \in L_\varphi(\mu)$. The claim follows by Proposition 6.3 since $F_\varepsilon \uparrow F$. 

\[ \Box \]

**Theorem 6.2.** Let $L_\varphi(\mu) = L_\varphi^\circ(\mu)$. Then there holds

$$
L_\varphi(\mu)^* = A_{\varphi^*}(\mu) \oplus S_{\varphi^*}(\mu) = V_{\varphi^*}(\mu) \oplus S_{\varphi^*}(\mu),
$$

that is, these subspaces are closed and for every $\ell \in L_\varphi(\mu)^*$ there exists an absolutely continuous functional $\ell_a \in A_{\varphi^*}(\mu)$ and a singular functional $\ell_s \in S_{\varphi^*}(\mu)$ achieving the unique decomposition

$$
\ell = \ell_a + \ell_s.
$$

Moreover

$$
\|\ell\|_{\varphi^*} = \|\ell_a\|_{\varphi^*} + \|\ell_s\|_{\varphi^*} = \sup_{\|u\|_{L_\varphi} \leq 1} \ell(u) + \sup_{\|v\|_{\varphi} \leq 1} \limsup_{E \to \partial \Omega} |\ell_s(u\chi_E)|,
$$

where $L$ is any almost decomposable linear subspace of $L_\varphi(\mu)$.

**Proof.** We have $A_{\varphi^*}(\mu) = V_{\varphi^*}(\mu)$ by Theorem 6.1. It remains to prove the decomposition (44). Ad existence: Proposition 6.1 makes it sufficient to treat the case when $\mu$ is $\sigma$-finite. Let $\nu$ be an equivalent finite measure as in Proposition 3.2. We define the seminorm

$$
[\ell]_{\varphi^*} = \sup_{\|u\|_{L_\varphi} \leq 1} \limsup_{\nu(E) \to 0} |\ell(u\chi_E)| = \sup_{\|u\|_{L_\varphi} \leq 1} \inf_{\varepsilon > 0, \nu(E) \leq \varepsilon} |\ell(u\chi_E)|.
$$

...
We find functions $u_n \in L_\varphi(\mu)$ and sets $A_1(n, m) \in \mathcal{A}$ such that
\[
\| u_n \|_\varphi < 1, \quad \left( [\ell]_{\varphi^*} - \ell(u_n \chi_{A_1}) \right) + I_\varphi(u_n \chi_{A_1}) + \mu(A_1) \leq 2^{-n-m}
\]
because the integral is absolutely continuous. Setting
\[
A_2(n) = \bigcup_{m \geq 1} A_1(n, m), \quad \ell_n(u) = \ell(u \chi_{\Omega \setminus A_2}),
\]
we are going to prove that $\ell_n$ is an absolutely continuous functional. Otherwise there were a function $u \in L_\varphi(\mu)$ and a family of measurable sets $E(n, m) \subset \Omega \setminus A_2(n)$ such that
\[
\lim_{m \to \infty} \nu(E) = 0, \quad \lim_{m \to \infty} \ell_n(u \chi_E) = a > 0, \quad I_\varphi(u) = b < 1.
\]
Let $2^{-n} < \min \{a, 1 - b\}$ and $\bar{u}_m = u \chi_E + u_n \chi_{A_1}$. Since $E(n, m)$ and $A_1(n, m)$ are disjoint, we obtain by (47) the estimate
\[
I_\varphi(\bar{u}_m) = I_\varphi(u \chi_E) + I_\varphi(u_n \chi_{A_1}) \leq b + 2^{-n} < 1
\]
so that $\| \bar{u}_m \|_\varphi \leq 1$. This together with $\lim_n \nu(A_1) + \lim_n \nu(E) = 0$ yields the contradiction
\[
\lim_{m \to \infty} \ell(\bar{u}_m) = \lim_{m \to \infty} \ell_n(u \chi_E) + \lim_{m \to \infty} \ell(u_n \chi_{A_1}) \geq a + [\ell]_{\varphi^*} - 2^{-n} > [\ell]_{\varphi^*}
\]
so $\ell_n$ is absolutely continuous. Therefore Theorem 6.1 yields $v_n \in V_{\varphi^*}(\mu)$ representing $\ell_n$ whence Proposition 6.3 yields $v \in V_{\varphi^*}(\mu)$ with $v_n = v \chi_{\Omega \setminus A_2}$. Hence, denoting by $\ell_a$ the functional induced by $v$, there holds the convergence
\[
\ell_n \to \ell_a \text{ as } n \to \infty.
\]
The functional $\ell_s = \ell - \ell_a$ is singular as the sets
\[
C_k = \bigcup_{n=k}^\infty A_2(n), \quad \mu(C_k) \leq \sum_{n=k}^\infty \mu(A_2) \leq \sum_{n=k}^\infty \sum_{m=1}^\infty \mu(A_1) \leq \sum_{n=k}^\infty 2^{-n} = 2^{1-k}
\]
provide a concentration sequence by definition of $\ell_n$. Since $A_{\varphi^*}(\mu) \cap S_{\varphi^*}(\mu) = \{0\}$ by the observation below Definition 6.1, uniqueness of the decomposition $\ell = \ell_a + \ell_s$ is immediate.

Ad (46): Note that this will prove a fortiori that the subspaces are closed. We pick $u_n$ and $v_n$ such that
\[
\| u_n \|_\varphi < 1, \quad \| v_n \|_\varphi < 1, \quad \| \ell_a \|_{\varphi^*} = \lim_n \ell_a(u_n), \quad \| \ell_s \|_{\varphi^*} = \lim_n \ell_s(v_n).
\]
Let $C_m$ be a concentration sequence for $\ell_s$. Since $I_\varphi(u_n) \leq \| u_n \|_\varphi < 1$ by Lemma 3.5, we find by absolute continuity a sequence of sufficiently large numbers $m = m(n)$ such that for $w_n = u_n \chi_{\Omega \setminus C_m}$ and $x_n = v_n \chi_{C_m}$ there holds
\[
\lim_n \ell_a(w_n) = \| \ell_a \|_{\varphi^*}, \quad \lim_n \ell_a(x_n) = 0, \quad I_\varphi(w_n + x_n) = I_\varphi(w_n) + I_\varphi(x_n) \leq 1.
\]
Therefore $\| w_n + x_n \|_\varphi \leq 1$ and consequently
\[
\| \ell \|_{\varphi^*} \geq \lim_n \ell(w_n + x_n) = \lim_n \ell_a(w_n) + \ell_s(x_n) = \lim_n \ell_a(w_n) = \ell_a \|_{\varphi^*} \geq \| \ell \|_{\varphi^*}.
\]
The identity \( \| \ell_s \|_{\varphi}^\ast = \sup_{u \in B_1} \ell(u) \) obtains by Lemma 4.2. It remains to prove \( \| \ell_s \|_{\varphi}^\ast \leq [\ell]_{\varphi^\ast} \) as the converse inequality is trivial. There holds
\[
\| \ell_s \|_{\varphi}^\ast = \lim_{n} \ell_s(v_n) = \lim_{n} \ell_s(v_n \chi C_m) \leq \sup_{|u|_\varphi \leq 1} \limsup_{E \to \varnothing} |\ell_s(u \chi E)| = [\ell]_{\varphi^\ast}. \]

**Corollary 6.2.** Let \( L_\varphi(\mu) = L_\varphi^\ast(\mu) \) and \( \varphi \) real-valued. Then
\[
C_\varphi(\mu)^\ast = V_\varphi^\ast(\mu)
\]
as an isometric isomorphism via the identification of \( v \in V_\varphi^\ast(\mu) \) with the functional
\[
\ell(u) = \int \langle v(\omega), u(\omega) \rangle d\mu(\omega), \quad u \in C_\varphi(\mu).
\]

**Proof.** We use that for a Banach space \( X \) and a closed subspace \( U \) there holds \( U^\ast = X'/U^\perp \) through the isometric isomorphism induced by \( x' + U^\perp \to x'|_U \). In the situation at hand, we have by Theorem 6.2 and Lemma 6.1
\[
C_\varphi(\mu)^\ast = [V_\varphi^\ast(\mu) \ominus S_\varphi^\ast(\mu)]/N_\varphi^\ast(\mu) = [V_\varphi^\ast(\mu) \ominus N_\varphi^\ast(\mu)]/N_\varphi^\ast(\mu) = V_\varphi^\ast(\mu)
\]
where the action of a functional is described by (49). The norm of this quotient space is the operator norm by the decomposition (46) so that the isomorphism induced by (49) indeed is isometric.

**Corollary 6.3.** If all elements of \( V_\varphi^\ast(\mu) \) are integrally strongly measurable, then \( X' \) has the Radon-Nikodym property w.r.t. the restriction of \( \mu \) to any set of finite measure. In particular, this is necessary for \( V_\varphi^\ast(\mu) = L_\varphi^\ast(\mu) \) to hold.

**Proof.** Arguing by contradiction, we suppose there were \( F \in \mathcal{A}_f \) on which the restriction of \( \mu \) fails the Radon-Nikodym property. Theorem 6.2 yields \( L_1(F;X)^\ast = V_\infty(F;X') \) and by [11, §4.1, Thm. 1] we know then that there exists
\[
v \in V_\infty(F;X') \setminus L_\infty(F;X').
\]
Pick by Lemma 3.8 an isotonic exhausting sequence \( F_n \) with \( \lim_n \mu(F \setminus F_n) = 0 \) such that
\[
L_\infty(F_n;X) \to L_\varphi(F_n) \to L_1(F_n;X).
\]
As the second embedding in this chain is dense, its adjoint operator is an embedding, too, so that \( v \chi F_n \in V_\varphi^\ast(\mu) \). If each \( v \chi F_n \) were strongly measurable, then its limit from below \( v \) were likewise, which would yield \( v \in L_\infty(F;X') \) by the reverse Hölder inequality Lemma 6.4 and since autonomous generators are dualizable by Lemma 2.2.

As a consequence of the duality theory obtained so far, we may give a precise characterization of reflexivity for the Orlicz space \( L_\varphi(\mu) \) if the pair of generators \( \varphi \) and \( \varphi^\ast \) is dualizable. Implicitly, this also supplies us with a characterization in the general case since a Banach space is reflexive if and only if each sequence is contained in a reflexive subspace. Because the range of any sequence in \( L_\varphi(\mu) \) is almost contained in some closed linear \( W \in \mathcal{S}(X) \), one may apply the following criterion to the restriction \( \varphi_W \), which together with its convex conjugate is dualizable if \( X \) is reflexive.

**Theorem 6.3.** Let the range space \( X \) be reflexive and let the conjugate generators \( \varphi \) and \( \varphi^\ast \) be real-valued, dualizable i.a.e. for every element of \( L_\varphi^\ast(\mu) \) and \( L_\varphi(\mu) \),
integral pairing, while this dual space has \( L_\Delta \) reflexivity. The addendum on the pairing so that the canonical embedding of \( C \) analysis [20, §4.2, Thm. 2] if \( \text{dom} X \) the convex conjugate of integral functionals on a vector valued Orlicz space, which

An interesting application of Theorem 6.2 is the following representation result for \( L \)-smooth functions that embed continuously into its Fenchel-Moreau subdifferential may be transferred to functionals on spaces of \( C \)-More precisely, the space \( L \)-finite. Then \( \int \phi \) and \( \phi^* \) of the range space \( X \), respectively, and such that the set where their minimum at the origin is not strict is \( \sigma \)-finite. Then \( L_\phi(\mu) \) is reflexive if and only if

\[ C_\phi(\mu) = L_\phi(\mu) \quad \text{and} \quad C_{\phi^*}(\mu) = L_{\phi^*}(\mu). \]  

(50)

Hence, if \( \phi \in \Delta_2 \) and \( \phi^* \in \Delta_2 \), then \( L_\phi(\mu) \) is reflexive. If \( \mu \) is non-atomic, then the \( \Delta_2 \)-conditions are also necessary for \( L_\phi(\mu) \) to be reflexive.

By the almost embedding result Lemma 3.8, we know that \( L_\phi(\mu) \) contains a copy of the range space \( X \) whence reflexivity of \( X \) is clearly a necessary assumption unless in the trivial case when no set of positive measure exists, which we ruled out in Sec. 1. This contrasts with [37], where a reflexive Orlicz space with an autonomous generator on a non-reflexive range space is presented. The catch is that the author of [37] does not require continuity of a generator at the origin, which makes such pathologies possible. At least for autonomous generators, there is no need to consider such situations by our construction in Sec. 3.

Proof. The assumptions on \( \phi^* \) being the same as on \( \phi \), all considerations hold mutatis mutandis for the spaces that \( \phi^* \) generates. We have \( L_\phi(\mu) = L_{\phi^*}(\mu) \) by Theorem 3.2 so \( L_\phi(\mu) = L_{\phi^*}(\mu) \oplus S_\phi(\mu) \) and \( C_\phi(\mu)^* = L_{\phi^*}(\mu) \) by Theorem 6.2 and Corollary 6.2 as \( \phi \) is real-valued. \( \implies \) : if \( L_\phi(\mu) \) is reflexive, then \( C_\phi(\mu) \) is reflexive as a closed subspace by Lemma 5.1 thus

\[ C_\phi(\mu) = C_\phi(\mu)^* = L_{\phi^*}(\mu)^* = L_\phi(\mu) \oplus S_\phi(\mu). \]

More precisely, the dual space of \( C_\phi(\mu) \) is \( L_{\phi^*}(\mu) \) by means of the standard integral pairing, while the dual of \( L_{\phi^*}(\mu) \) is \( L_\phi(\mu) \oplus S_\phi(\mu) \) with the absolutely continuous functionals \( L_\phi(\mu) \) acting again through the integral pairing. Since the canonical embedding of \( C_\phi(\mu) \) into the bidual \( L_\phi(\mu) \oplus S_\phi(\mu) \) via the integral pairing is surjective by reflexivity, we deduce \( S_\phi(\mu) = \{0\} \) and \( C_\phi(\mu) = L_\phi(\mu) \). The argument for the reflexive space \( L_{\phi^*}(\mu) = C_\phi(\mu)^* \) is the same. \( \iff \) : the space \( L_\phi(\mu) \) is reflexive as

\[ L_\phi(\mu)^* = C_\phi(\mu)^* = L_{\phi^*}(\mu)^* = C_{\phi^*}(\mu)^* = L_\phi(\mu). \]

More precisely, the space \( C_\phi(\mu) \) has \( L_{\phi^*}(\mu) = C_{\phi^*}(\mu) \) as a dual space via the integral pairing, while this dual space has \( L_\phi(\mu) = C_\phi(\mu) \) as a bidual via the same pairing so that the canonical embedding of \( C_\phi(\mu) \) into its bidual is surjective, i.e. reflexivity. The addendum on the \( \Delta_2 \)-conditions obtains by Lemma 5.2. \( \square \)

An interesting application of Theorem 6.2 is the following representation result for the convex conjugate of integral functionals on a vector valued Orlicz space, which is a new result if \( X \) is non-separable. If \( I_f \) is convex, the ensuing representation of its Fenchel-Moreau subdifferential may be transferred to functionals on spaces of smooth functions that embed continuously into \( L_\phi(\mu) \) by the chain rule of convex analysis [20, §4.2, Thm. 2] if \( \text{dom} I_f \) has a point of continuity. This happens by [20, §3.2, Thm. 1] if \( f \) satisfies the \( \phi \)-growth condition

\[ \exists k \geq 1, g \in L_1(\mu): |f(\omega, x)| \leq k \phi(\omega, kx) + g(\omega) \quad \forall \omega \in \Omega, x \in X \]

since then \( I_f \) is bounded above on a neighbourhood of the origin. Remember that the exhausting integral of an essential infimum function always exists, even if the infimum function does not exist globally.
Theorem 6.4. Let \( L_\varphi(\mu) = L_\varphi^s(\mu) \) and \( f: \Omega \times X \to (-\infty, \infty] \) an integrally separably measurable integrand. Then, if \( I_f \not\equiv \infty \) on \( L_\varphi \) where \( I_f(u) = \int f[\omega, u(\omega)] \, d\mu(\omega) \),
the convex conjugate \( I_f^*(\ell) \) of \( I_f \) with respect to the norm topology is given by
\[
I_f^*(\ell) = I_f^*(\ell_a) + s_{\text{dom } I_f}(\ell_s)
\]
(51)
wherever \( I_f^* \) is finite. Let \( S_\alpha(X) \) be the separable subspaces of \( X \) almost containing the range of \( u \). The Fenchel-Moreau subdifferential of \( I_f \) on \( \text{dom } I_f \) is given by
\[
\partial I_f(u) = \partial_a I_f(u) + \partial_s I_f(u)
\]
(52)
Moreover, if \( f \) is a convex integrand, denoting by \( p(\omega, u) = f'(\omega, u(\omega); v) \) the radial derivative, the closure of the radial derivative \( v \mapsto I_f'(u; v) \) at a point \( u \in \text{dom } (\partial I_f) \) is given by
\[
v \mapsto \overline{cl}_{I_f'}(u; v) = \int \text{ess-inf } W \in S_\alpha(\Omega) \left\{ u^* \in V_\varphi^* : \left| u^*_w(\omega) \in \partial f_w[\omega, u(\omega)] \right| \text{ i.e.} \right\} \bigcup_{W \in S_\alpha(X)} \left\{ \ell \in S_\varphi^* : \left| \langle \ell, v - u \rangle \leq 0 \right| \quad \forall v \in \overline{\text{dom } I_f} \right\}.
\]
(53)
Finally, if \( f \) is dualizable i.a.e. for \( \ell_a \in V_\varphi^* (\mu) \), then \( I_f^*(\ell_a) = I_f^*(\ell_a) \) with \( I_f^*(\ell_a) = \int f^*[\omega, \ell_a(\omega)] \, d\mu(\omega) \), the intersection in (52) over \( W \in S_\alpha(X) \).

Proof. Ad (51): clearly \( I_f^*(\ell) \leq I_f^*(\ell_a) + s_{\text{dom } I_f}(\ell_s) \) so that it remains to prove the converse inequality. Let \( C_n \in \mathcal{A} \) be a concentration sequence for the singular part \( \ell_s \). Let \( u_0, u_1 \in \text{dom } I_f \). Setting \( \tilde{u} = u_0 \chi_{C_n} + u_1 \chi_{C_n} \) we have \( \tilde{u} \in \text{dom } I_f \) so that
\[
I_f^*(\ell) \geq \ell(\tilde{u}) - \int f[\omega, \tilde{u}(\omega)] \, d\mu(\omega)
\]
\[
= \ell_a(u_0 \chi_{\Omega \setminus C_n}) - \int_{\Omega \setminus C_n} f[\omega, u_0(\omega)] \, d\mu(\omega)
\]
\[
+ \ell_s(u_1) - \ell_s(u_1 \chi_{C_n}) - \int_{C_n} f[\omega, u_1(\omega)] \, d\mu(\omega)
\]
\[
\to \ell_a(u_0) - \int f[\omega, u_0(\omega)] \, d\mu(\omega) + \ell_s(u_1) \quad \text{as } n \to \infty.
\]
Taking the supremum over all \( u_0, u_1 \) concludes the proof by Theorem 2.2. Ad (52): by the Fenchel-Young equality and (51) there holds
\[
\ell \in \partial I_f(u) \iff I_f(u) + I_f^*(\ell_a) + s_{\text{dom } I_f}(\ell_s) = \langle \ell, u \rangle
\]
\[
\iff I_f(u) + I_f^*(\ell_a) = \langle \ell_a, u \rangle \quad \text{and } s_{\text{dom } I_f}(\ell_s) = \langle \ell_s, u \rangle
\]
which is equivalent to \( \ell \) fulfilling \( \ell_a \in \partial f_w[\omega, u(\omega)] \) a.e. and \( \langle \ell_s, v - u \rangle \leq 0 \) for all \( v \in \text{dom } I_f \). The former follows from Theorem 2.2. Ad (53): since \( \partial_s I_f(u) \) is a non-empty cone, the function
\[
v \mapsto s_{\partial_s I_f(u)}(v) = \sup_{\ell_s \in \partial_s I_f(u)} \langle \ell_s, v \rangle
\]
takes values in \([0, +\infty)\). Let \(v \in \text{dom } I_f\). Remember that the subdifferential of a convex function at a point \(u\) consists of those continuous linear functionals that are dominated by the radial derivative of the function at \(u\). In particular, the closure of the sublinear derivative functional is the supremum of the subgradients. One-sided difference quotients of convex functions being monotone decreasing, we have

\[
I' (u; v) \geq \int p(\omega, v(\omega)) \, d\mu(\omega) + s_{\partial I(u)}(v)
\]

\[
\geq \int \operatorname{ess}
\inf_{W \in S(X)} \operatorname{cl}_v p_W (\omega, v(\omega)) \, d\mu(\omega) + s_{\partial I(u)}(v)
\]

\[
\geq \langle \ell_a (\omega), v(\omega) \rangle d\mu(\omega) + s_{\partial I(u)}(v)
\]

for any \(\ell_a \in \partial I_f (u)\) by (52). Taking the supremum over all such \(\ell_a\) obtains

\[
I' (u; v) \geq \int \operatorname{ess}
\inf_{W \in S(X)} \operatorname{cl}_v p_W (\omega, v(\omega)) \, d\mu(\omega) + s_{\partial I(u)}(v) \geq \operatorname{cl}_v I'(u; v).
\]

Therefore the claim will obtain if we prove that the function (53) is lower semicontinuous. For this, let \(v_n \to v \in L_\varphi\). It suffices to extract a subsequence \(n_k\) such that

\[
\liminf_k \int \operatorname{ess}
\inf_{W \in S(X)} \operatorname{cl}_v p_W (\omega, v_{n_k}(\omega)) \, d\mu(\omega) \geq \int \operatorname{ess}
\inf_{W \in S(X)} \operatorname{cl}_v p_W (\omega, v(\omega)) \, d\mu(\omega).
\]

(54)

By extracting a subsequences, we may assume the left-hand integrals in (54) to be finite. Hence we find a set \(\Sigma \in A_\sigma\) outside of which the pertaining integrands vanish. As the right-hand integral is exhausting and the integrand has an integrable minorant \(\langle \ell_a (\omega), v(\omega) \rangle\), we find \(A \in A_{\sigma_{\varphi}}\) with \(\Sigma \subset A\) over which it attains its value. By restricting \(f\) on \(A\) to a suitable separable subspace \(W_0\), we may replace the essential infimum functions in both sides of (54) by \(\operatorname{cl}_v p_{W_0}\), attaining the essential infimum function as explained below Proposition 2.1. For \(\ell \in \partial I_f(u)\), choose a subsequence \(n_k\) with \(v_{n_k} \to v\) a.e. and such that the \(L_1(\mu)\)-convergent sequence \(\langle \ell_a, v_{n_k} \rangle\) has an integrable minorant \(m\). This implies

\[
\operatorname{cl}_v p_{W_0} (\omega, v_{n_k}(\omega)) \geq \langle \ell_a (\omega), v_{n_k}(\omega) \rangle \geq m(\omega) \text{ a.e.}
\]

so that Fatou’s lemma yields (54). The addendum on (52) follows by the corresponding addendum in Theorem 2.2.

\[\square\]

7. Weak topologies and compactness

We obtain in this section our characterization results Theorems 7.1 and 7.2 on sequential compactness for various weak topologies that \(L_\varphi(\mu)\) and \(V_{\varphi \ast}(\mu)\) or subsets of them induce on each other through the standard integral pairing (3). For a \(\sigma\)-finite measure, both theorems provide sufficient conditions that are also necessary if the inducing subspace is rich enough in terms of decomposability properties, as happens for example if all of \(V_{\ast \varphi}(\mu)\) or \(L_{\varphi}(\mu)\) induce the topology. Our approach to the matter is the same in both theorems: starting with the prototypical cases of the almost superspaces \(L_1(\mu; X)\) and \(V_1(\mu; X')\), into which \(L_\varphi(\mu)\) and \(V_{\ast \varphi}(\mu)\) embed on an exhausting sequence of sets \(\Omega_j\) by Lemma 3.8 if \(\mu\) is \(\sigma\)-finite, we seek additional conditions under which a sequence whose images under one of the
sequences of embeddings
\[ L_\varphi(\mu) \to L_1(\Omega_j; X), \quad V_\varphi(\mu) \to V_1(\Omega_j; X) \]
is (weakly) compact has a compact pre-image. Thus we reduce the general matter to these prototypes. While characterizations of weak compactness in \( L_1(\mu; X) \) are abundant and will be discussed to some extent below, we need to establish a corresponding result for \( \sigma(V_1(\mu; X'); L_\varphi(\mu; X)) \) ourselves in Lemma 7.3. Throughout this section, we assume that \( \mu \) has no atom of infinite measure.

7.1. **Weak topologies.** Before we study weak compactness, we need to obtain a pair of auxiliary results that will serve to prove that \( V_\varphi(\mu) \) is a norming subspace of the dual space, which we require of the subspaces that induce topologies. The first of these results is of interest for its own sake, as it implies in particular that the convex functional \( I_\varphi \) is lower semicontinuous in the weak topology induced on \( L_\varphi(\mu) \) by \( V_\varphi(\mu) \) if \( \mu \) is \( \sigma \)-finite, as the proof of Lemma 7.2 will show.

**Lemma 7.1.** The Mackey topology on \( L_\varphi(\mu) \) induced by its duality with \( V_\varphi(\mu) \) implies local convergence in \( \mu \).

**Proof.** We may assume \( \mu \) finite by definition of local convergence in measure. Clearly, it suffices to obtain an isotonic sequence \( \Omega_j \in A \) with \( \lim_{j \to 0} \mu(\Omega_j) = 0 \) on which convergence in \( \mu \) obtains. Lemma 3.8 yields an isotonic sequence \( \Omega_j \) with
\[ L_\infty(\Omega_j; X) \to L_\varphi(\Omega_j) \to L_1(\Omega_j; X). \]
As the last embedding in this chain is dense, its adjoint operator induces an embedding of \( L_1(\Omega_j; X)' = V_\varphi(\Omega_j; X) \) into \( V_\varphi(\Omega_j) \). Since the ball of \( V_\varphi(\Omega_j; X) \) is weak* compact by the Alaoglu theorem, it is weak* compact in \( V_\varphi(\Omega_j) \). Therefore the \( L_j(\Omega_j; X) \)-norm is Mackey continuous as a supremum over a convex set that is weakly compact for this duality. Hence, since norm convergence in \( L_1 \) implies convergence in measure, Mackey convergence implies convergence in measure on any \( \Omega_j \).

**Lemma 7.2.** The space \( V_\varphi(\mu) \) is norming, i.e. the support functional of its ball
\[ S(u) = \sup_{\|v\|_\varphi \leq 1} \int \langle v(\omega), u(\omega) \rangle d\mu(\omega) \]
defines an equivalent norm on \( L_\varphi(\mu) \). There holds
\[ \|u\|_\varphi \leq S(u) \leq 2\|u\|_\varphi \quad \forall u \in L_\varphi(\mu). \]

**Proof.** It suffices to show that the support functional controls the Luxemburg norm. We first assume \( \mu \) to be \( \sigma \)-finite. The functional \( I_\varphi \) is closed w.r.t. local convergence in \( \mu \) by the Fatou lemma. More precisely, since the Fatou lemma holds for sequences but not for nets, we may argue as follows: Given \( u_i \in L_\varphi(\mu) \) a net with \( \lim_i u_i = u \) in the Mackey topology, there holds \( \lim_i u_i = u \) locally in measure by Lemma 7.1 so we may conclude \( I_\varphi(u) \leq \liminf_i I_\varphi(u_i) \) once we prove lower semicontinuity of \( I_\varphi \) for the local convergence in \( \mu \). As the latter is metrizable on \( L_0(\Omega; X) \) by [22, Satz 6.7] together with the separable valuedness of strongly measurable functions and the \( \sigma \)-finiteness of \( \mu \), we may reduce to checking lower semicontinuity for sequences \( w_n \in L_0(\Omega; X) \) with \( \lim_n w_n = w \) locally in \( \mu \). Applying the Fatou lemma in the version discussed below Proposition 2.1, we conclude \( I_\varphi(w) \leq \liminf_n I_\varphi(w_n) \).
Hence, \( I_\varphi \) is lower semicontinuous for the Mackey topology. Consequently, \( I_\varphi = I_\varphi^* \).
w.r.t. this pairing by [12, Thm. 4.92] so that if \( S(u) \leq \frac{1}{2} \), then Lemma 3.5 yields the estimate

\[
I_\varphi(u) = \sup_{v \in V_{\varphi}} \langle v, u \rangle - I_{\varphi}^*(v) \leq \sup_{v \in V_{\varphi}} \frac{1}{2} \|v\|_{\varphi}^* - I_{\varphi}^*(v) \leq \sup_{v \in V_{\varphi}} \|v\|_{\varphi}^* - I_{\varphi}^*(v) \leq 1
\]

hence \( \|u\|_{\varphi} \leq 1 \) whence \( \| \cdot \|_{\varphi} \leq 2S \) follows. It remains to remove the restriction of \( \sigma \)-finiteness. Let \( \|u\|_{\varphi} > \alpha \) so that \( I_{\varphi}(\alpha^{-1}u) > 1 \). Because we assume \( \mu \) to have no atom of infinite measure, we may invoke Proposition 2.1 to find a set \( \Sigma \in \mathcal{A}_\varphi \) such that \( I_{\varphi}(\alpha^{-1}u\chi_\Sigma) > 1 \) hence \( \|u\chi_\Sigma\|_{\varphi} > \alpha \). Consequently, we find a sequence \( \Sigma_n \in \mathcal{A}_\varphi \) with

\[
\|u\|_{\varphi} = \lim_n \|u\chi_{\Sigma_n}\|_{\varphi} \leq \limsup_n 2S(u\chi_{\Sigma_n}) \leq 2S(u)
\]

whence the restriction has been lifted. \( \square \)

7.2. **Weak compactness.** Before addressing weak compactness, let us briefly settle the matter of characterizing strong compactness in \( L_{\varphi}(\mu) \), thereby explaining why we feature no section on strong compactness. We know from Theorem 3.1 that up to subsequences convergence a.e. is necessary for strong convergence in \( L_{\varphi}(\mu) \). By definition of the Luxemburg norm, the Vitali convergence theorem implies that a sequence \( u_n \) converging to a limit \( u \) a.e. up to subsequences will converge strongly in \( L_{\varphi}(\mu) \) if and only if for every \( \lambda > 0 \) the extended real-valued function \( \varphi(\omega, \lambda \{u(\omega) - u_n(\omega)\}) \) is equi-integrable and has non-escaping mass in \( L_1(\mu) \). This is equivalent to the existence of integrable majorants up to subsequences by the Lebesgue dominated convergence theorem. These conditions become considerably easier to establish if \( \varphi \) satisfies a \( \Delta_2 \)-type condition on all of \( \Omega \) since then equi-integrability and non-escaping mass hold for all \( \lambda > 0 \) if they hold for some \( \lambda > 0 \). Notably, these considerations are completely analogous to the scalar case without requiring any noteworthy adaption. This is not to say that more apt characterizations of strongly compact sets cannot be given in situations with additional structure on the measure space \( (\Omega, \mathcal{A}, \mu) \), cf., e.g., [4] for a result in this direction.

7.2.1. **Compactness in \( L_{\varphi}(\mu) \).** The following notions mimic the definition of equi-integrability as known in the theory of Lebesgue spaces in a way adapted to weak topologies. Their role in the theory of weakly compact subsets in scalar valued Orlicz spaces is well-established, cf., e.g., [29, §4.5, Thm. 1].

**Definition 7.1** (weak and weak* equi-integrability). A subset \( \mathcal{F} \subset L_{\varphi}(\mu) \) is weakly equi-integrable on \( \mathcal{G} \subset V_{\varphi}^*(\mu) \) if for each evanescent sequence of sets \( E_n \in \mathcal{A} \), \( \mu(\lim_n E_n) = 0 \), and every \( v \in \mathcal{G} \) there holds

\[
\limsup_n \sup_{u \in \mathcal{F}} \left| \int_{E_n} \langle v, u \rangle \, d\mu \right| = 0 . \tag{55}
\]

Similarly, the subset \( \mathcal{G} \) is called weak* equi-integrable on \( \mathcal{F} \) if for \( E_n \) and \( u \in \mathcal{F} \) there holds

\[
\limsup_n \sup_{v \in \mathcal{G}} \left| \int_{E_n} \langle v, u \rangle \, d\mu \right| = 0 . \tag{56}
\]

All of \( L_{\varphi}(\mu) \) is weakly equi-integrable on \( C_{\varphi}^*(\mu) \) and the entire space \( V_{\varphi}^*(\mu) \) is weak* equi-integrable on \( C_{\varphi}(\mu) \). The sets \( \mathcal{F} \) and \( \mathcal{G} \) may always be taken linear by passing to the linear hull. Weak equi-integrability agrees with the formally stronger
notion of absolute weak equi-integrability, i.e. the absolute value in (55) may be equivalently placed inside the integral. Indeed, suppose \( F \) is weakly equi-integrable but fails to be absolutely weakly equi-integrable. Then we find a sequence \( E_n \) as above and \( u_n \in F \) for which there obtains the contradiction
\[
\lim_n \int_{E_n} |\langle v, u_n \rangle| \, d\mu = \lim_n \int_{E_n} \langle v, u_n \rangle \, d\mu \geq \varepsilon > 0.
\]
The same is true for weak* equi-integrability. To localize weak sequential compactness, we need several preparatory results.

**Proposition 7.1.** Let \( u_n \in L_p(\mu) \) be weakly equi-integrable at \( v \in V_p(\mu) \) and \( v_m \uparrow v \). Then
\[
\lim_n \lim_m \langle u_n, v_m \rangle = \lim_m \lim_n \langle u_n, v_m \rangle
\]
whenever one of these iterated limits exists. Mutatis mutandis, the same is true for weak* equi-integrability.

**Proof.** This follows easily from the definition of weak equi-integrability and a straightforward \( \varepsilon \)-argument. \( \square \)

The next result reduces convergence considerations to a finite measure space.

**Proposition 7.2.** Let \( F \subset L_p(\mu) \) be a norm bounded set that is weakly equi-integrable on a subset \( G \subset V_p(\mu) \) inducing on \( G \) an equivalent norm via
\[
\|u\|_G = \sup_{v \in B_G} |\langle v, u \rangle|.
\]
(57)

Then the sequential relative compactness in \( \sigma(L_p, G) \) of the following sets is equivalent: (i) \( F \) (ii) \( F \chi_A \) for each \( A \in A \) (iii) \( F \chi_{A_k} \) for each member \( A_k \) of an isotonic sequence with \( \lim_k A_k = \Omega \).

We call \( G \) a norming subset for \( F \) if (57) defines an equivalent norm.

**Proof.** It suffices to deduce (i) from (iii). Let \( B_k \) be the partition of \( \Omega \) defined by
\[
B_1 = A_1, \quad B_k = A_k \setminus A_{k-1} \quad k \geq 2.
\]
Pick a sequence \( u_n \in F \) and extract a diagonal sequence (not relabelled) such that \( u_n \chi_{B_k} \) converges for every \( k \). Define \( u \) to agree with the limit \( \lim_n u_n \chi_{B_k} \) on \( B_k \). Then \( u \in L_p(\mu) \) since
\[
c\|u\|_G = c \lim_k \|u \chi_{A_k}\|_p \leq \limsup_k \|u \chi_{A_k}\|_G \leq \limsup_k \liminf_n \|u_n \chi_{A_k}\|_G \leq C_F < \infty.
\]
We used that \( \|\cdot\|_G \) is lower semicontinuous as a supremum of continuous functions. By Proposition 7.1 we conclude
\[
\lim_n \langle u_n, v \rangle = \lim_k \langle u_n, v \chi_{B_k} \rangle = \lim_k \langle u_n, v \chi_{B_k} \rangle = \lim_k \langle u, v \chi_{B_k} \rangle = \langle u, v \rangle
\]
for any \( v \in \text{lin} G \) so that a convergent subsequence has been extracted. \( \square \)

An analogous statement and proof holds for subsets of \( V_p(\mu) \):

**Proposition 7.3.** Let \( G \subset V_p(\mu) \) be a norm bounded set that is weak* equi-integrable on \( F \subset V_p(\mu) \) inducing on \( G \) an equivalent norm via
\[
\|v\|_F = \sup_{u \in B_F} |\langle u, v \rangle|.
\]
(58)
Then the sequential relative compactness in $\sigma (V_\varphi^+; F)$ of the following sets is equivalent:

(i) $\mathcal{G}$

(ii) $\mathcal{G}_{X_A}$ for each $A \in \mathcal{A}$

(iii) $\mathcal{G}_{X_k}$ for each member $A_k$ of an isotonic sequence with $\lim_k A_k = \Omega$.

We call $F$ a norming subset for $\mathcal{G}$ if (58) defines an equivalent norm. Before we can state our first compactness theorem, we need further terminology.

**Definition 7.2** (weak tightness). A family $F$ of strongly measurable functions $u: \Omega \to X$ is weakly tight if there exists a separably measurable integrand $h: \Omega \times X \to [0, \infty]$ with weakly compact sublevels $\{ x \in X \mid h(\omega, x) \leq \alpha \}$, $\alpha \geq 0$, such that

$$\sup_{u \in F} \int h [\omega, u(\omega)] \, d\mu(\omega) < \infty.$$ 

If $F$ is weakly tight on every set of finite measure, then $F$ is called locally weakly tight.

**Definition 7.3** (weak biting convergence). Given a sequence of strongly measurable functions $u_n: \Omega \to X$ we say that $u_n$ converges to $u$ in the weak or $\sigma (X, X')$-biting sense if there exists an exhausting sequence $\Omega_j \in \mathcal{A}$, $\Omega = \bigcup_j \Omega_j$, such that $u_n \rightharpoonup u$ in $L_1 (\Omega_j; X)$ for every $j \in \mathbb{N}$.

Remember that $L_1 (\Omega_j; X)^* = V_\varphi (\Omega_j; X')$ by Corollary 6.2. In particular, the limit function is almost separably valued and $\mathcal{A}_w$-measurable hence agrees with a strongly measurable function a.e. by Lemma B.3 so that it is unique up to null sets and strongly measurable up to modification on a null set. In view of the role that biting convergence plays in the upcoming compactness theorem, we are interested in approximating elements of $V_\varphi^- (\mu)$ by sequences in $V_\varphi^- (\mu; X')$ in convergence from below.

**Proposition 7.4.** Let $\mu$ be $\sigma$-finite. For a weak* measurable function $v: \Omega \to X'$ with $v \in V_\varphi^- (\mu)$ there exists a sequence $v_k \in V_\varphi^- (\mu; X')$ such that $v_m \rightharpoonup v$.

**Proof.** By Lemma 3.8 we find an isotonic sequence $F_n \in \mathcal{A}_f$ with $\Omega = \bigcup_n F_n$ and $L_\infty (F_n; X) \to L_\varphi (F_n)$. We claim that the non-negative function

$$M(\omega) = \text{ess-sup} \sup_{w \in \mathcal{S}(X)} \langle v(\omega), x \rangle$$

is integrable on $F_n$. To see this, note that

$$\int_{F_n} M \, d\mu = \sup_{u \in L_\infty} \int_{F_n} \langle v, u \rangle - I_{B_X} (u) \, d\mu = \sup_{|u| \leq 1} \int_{F_n} \langle v, u \rangle \, d\mu < \infty$$

by Theorem 2.1, where $I_{B_X}$ is the indicator in the sense of convex analysis of $B_X$. In particular, $M$ is finite a.e. on all $F_n$ hence on $\Omega$ thus $v_k = \chi_{\{ M \leq k \}} v \uparrow v$ as $k \uparrow \infty$ on $\Omega$ where $v_k \in V_\varphi^- (\Omega; X')$. 

Bearing in mind the Diestel/Ruess/Schachermayer characterization [10, Thm. 2.1] of weak compactness in $L_1 (\mu; X)$, the next proposition implies that for bounded sequences in $L_1 (\mu; X)$, sequential compactness in the weak biting sense is actually a condition on the range of (the convex hull of) the sequence. By Lemma 3.8 and a standard diagonal argument the same is true for sequences that are bounded in $L_\varphi (\mu)$ if $\mu$ is $\sigma$-finite.

**Proposition 7.5.** Let $u_n$ be a sequence bounded in $L_1 (\mu; X)$. Given $\varepsilon > 0$ there exists $\Omega_\varepsilon \in \mathcal{A}$ such that $\mu (\Omega \setminus \Omega_\varepsilon) < \varepsilon$ and a subsequence $u_{n_k}$ that is equi-integrable in $L_1 (\Omega_\varepsilon; X)$. 

Proof. It suffices to consider scalar functions. \cite[Lem. 2.31]{12} yields a subsequence of $u_n$ (not relabelled) and a sequence of sets $\Omega_n$ with $\lim_n \mu(\Omega \setminus \Omega_n) = 0$ such that $u_n \chi_{\Omega_n}$ is equi-integrable. Passing to a subsequence $u_{nk}$ with $\mu(\Omega \setminus \Omega_{nk}) < 2^{-k} \varepsilon$ with find that $\Omega_\varepsilon = \bigcup_{k \geq 1} \Omega_{nk}$ and $u_{nk}$ work.

**Theorem 7.1.** Let $\mu$ be $\sigma$-finite, $\mathcal{G} \subset V_{p*}(\mu)$ a norming subset and $A \subset L_\varphi(\mu)$. Then $A$ is relatively sequentially compact if (i) $A$ is norm bounded (ii) $A$ is weakly equi-integrable on $\mathcal{G}$ (iii) $A$ is relatively sequentially compact in the local convergence of the $\sigma(X, X')$-biting sense. Conversely, if $A$ is relatively sequentially compact, then (i) holds if $\lim \mathcal{G}$ is closed, (ii) holds if $\lim \mathcal{G}$ is closed under multiplication with indicators of measurable sets (iii) holds if in addition to the latter closedness, the sequential closure of $\lim \mathcal{G}$ w.r.t. convergence from below contains $V_\varphi(\mu; X')$. Moreover, condition (iii) may be equivalently replaced by any of the following conditions: (iii\textsubscript{a}) Given any sequence in $A$ there exists a subsequence $v_k \in \text{co} \{u_n | n \geq k\}$ such that $v_k(\omega)$ is norm convergent for a.e. $\omega \in \Omega$ (iii\textsubscript{b}) $v_k(\omega)$ is weakly convergent for a.e. $\omega \in \Omega$ (iii\textsubscript{c}) $v_k$ is locally weakly tight.

The condition involving $V_\varphi(\mu; X')$ holds if $\lim \mathcal{G}$ agrees with all of $V_{p*}(\mu)$, as can be seen by exhausting the $\sigma$-finite measure space $\Omega$ with an isotonic sequence $F_j \in \mathcal{A}_f$ such that $\Omega = \bigcup_j \Omega_j$ and $L_\varphi(F_j; X) \rightarrow L_\varphi(F_j)$ to $L_\varphi(F_j; X)$ by Lemma 3.8 so that $V_\varphi(F_j; X') \rightarrow V_{p*}(F_j)$ for all $j$.

**Proof.** Ad sufficiency: we may assume $\mu$ finite by Proposition 7.2 and the $\sigma$-finiteness of $\mu$. Let $u_n \in \mathcal{G}$ be a sequence. We extract a subsequence (not relabelled) such that for $\Omega_j$ an isotonic sequence with $\Omega = \bigcup \Omega_j$ there holds convergence to a function $u$ weakly in $L_1(\Omega_j; X)$ for all $j \in \mathbb{N}$. We claim that $u \in L_\varphi(\mu)$. Indeed, if $B$ denotes the ball of $V_{p*}(\mu)$, then the support functional $S(u)$ of the ball $B$ in $V_{p*}(\mu)$ induces an equivalent norm on $L_\varphi(\mu)$ by Corollary 6.1. We may restrict to taking this supremum over $B \cap \bigcup_j V_\varphi(\Omega_j; X')$ by Proposition 7.4 hence $S$ is lower semicontinuous w.r.t. the $\sigma(X, X')$-biting convergence of $u_n$, so that $\|u\|_{\varphi} \leq S(u) \leq \liminf S(u_n) < \infty$. As $u$ is strongly measurable by the remark below Definition 7.3, we conclude $u \in L_\varphi(\mu)$. Consequently, by Proposition 7.2, it suffices if for any given $\Omega_j$ the sequence $u_n \chi_{\Omega_j}$ converges to $u \chi_{\Omega_j}$ in $\sigma(L_\varphi, \lim \mathcal{G})$. Since we can obtain any $v \in \mathcal{G}$ as a limit from below of elements belonging to $V_\varphi(\mu; X')$ by Proposition 7.4, the sufficiency has been proved due to Proposition 7.1. More precisely, for $v \in \mathcal{G}$ pick $v_m \in V_\varphi(\mu; X')$ with $v_m \uparrow v$. Then $\langle u, v \rangle = \lim_m \langle u, v_m \rangle = \lim_n \lim_m \langle u_n, v_m \rangle = \lim_n \lim_m \langle u_n, v \rangle = \lim_m \langle u_n, v \rangle$.

Ad necessity: (i) it suffices to prove that any convergent sequence $u_n$ is bounded. The Banach Steinhaus theorem yields a bound for $u_n$ in the dual of $\mathcal{G}$, as this is a Banach space by assumption. Since $\mathcal{G}$ is norming, we have obtained a bound in $L_\varphi(\mu)$. (ii) Again, it suffices to prove that $u_n$ is weakly equi-integrable on $\mathcal{G}$. Otherwise there were $v \in \mathcal{G}$ and $E_n \in \mathcal{A}$ with $\mu(\lim_n E_n) = 0$ such that

$$\inf_{n \in \mathbb{N}} \left| \int_{E_n} \langle v, u_n \rangle d\mu \right| \geq \varepsilon > 0. \quad (59)$$

Let $\lambda_n(E) = \int_E \langle u_n, v \rangle d\mu$, a signed finite measure for which the limit $\lambda(E) = \lim_n \lambda_n(E)$ exists for every $E \in \mathcal{A}$ since $v \chi_E \in \lim \mathcal{G}$ by assumption. We shall invoke the Vitali-Hahn-Saks theorem \cite[Thm. 2.53]{12} in order to conclude that $\lim_n \lambda_n(E_n) = 0$ thus contradicting (59). To justify this, let $\Sigma \in \mathcal{A}_\sigma$ such that every $\langle u_n, v \rangle$ vanishes outside $\Sigma$. Then $\mu$ is $\sigma$-finite on $\Sigma$ and hence equivalent to
some finite measure $\nu$ on $\Sigma$. Therefore each $\lambda_n$ is absolutely continuous w.r.t. the finite measure $\nu$ and Vitali-Hahn-Saks has been justified. (iii) By Lemma 3.8 there exists $F_k \in \mathcal{A}_f$ an isotonic sequence with $\Omega = \bigcup_k F_n$ such that for $\Omega_n = \Omega \cap F_n$ there holds $L_\chi(\Omega_n; X) \to L_\varphi(\Omega_n) \to L_1(\Omega_n; X)$ hence $L_\varphi(\Omega_n; X') \to V_{\varphi^*}(\Omega_n)$. For $v \in V_{\chi}(\Omega_n; X')$ pick $v_m \in \mathcal{G}$ a sequence with $v_m \uparrow v$. By Proposition 7.1 we have $\langle u, v \rangle = \lim_m \langle u, v_m \rangle = \lim_m \lim_n \langle u_n, v_m \rangle = \lim_m \lim_n \langle u_n, v \rangle \rangle$ hence necessity obtains.

Ad addendum: assuming (i) and (ii), we shall prove that (iii) may be replaced by the other (iii) in a circular fashion. We start with (iii) to obtain (iiia). If $u_n \in A$ is a sequence converging locally in the $\sigma(X', X)$-biting sense, then there exists an exhausting sequence $\Omega_j \in \mathcal{A}_f$, $\Omega = \bigcup_j \Omega_j$, such that $u_n \to u$ in $L_1(\Omega_j; X)$. By [10, Thm. 2.1] this yields $v_{k,j} \in \cos n_k j \geq k \rangle$ such that $v_{k,j} \chi_{\Omega_j}(\omega)$ is norm convergent for a.e. $\omega \in \Omega$. Inductively, we find $v_{k,j+1} \in \cos n_k j \geq k \rangle$ such that $v_{k,j+1} \chi_{\Omega_{j+1}}(\omega)$ is norm convergent for a.e. $\omega \in \Omega_j$ hence a.e. on $\Omega$. Using (iiia) to obtain (iiib) is trivial. From (iiib) to (iii): let $u_n \in A$. There exists an isotonic sequence $F_j \in \mathcal{A}_f$ such that $\Omega = \bigcup_j F_j$ and $L_\chi(F_j; X) \to L_\varphi(F_j) \to L_1(F_j; X)$ by Lemma 3.8. Invoking Proposition 7.5, after possibly decreasing each $F_j$ and extracting a diagonal subsequence from $u_n$ (not relabelled), we may assume that $u_n$ is equi-integrable in each $L_1(F_j; X)$. The sequence $u_n$ is weakly relatively compact in $L_1(F_j; X)$ by [10, Thm. 2.1] so that we may pass to a diagonal sequence (not relabelled) that converges weakly in $L_1(F_j; X)$ for all $j$. From this, we deduce by [34, Thm. 8] existence of $v_{k,j} \in \cos n_k j \geq k \rangle$ such that $v_{k,j}$ is weakly tight on $F_j$. Inductively, we find $v_{k,j+1} \in \cos n_k j \geq k \rangle$ that is weakly tight on $F_{j+1}$. Passing to the diagonal sequence, we have found $v_{k,k} \in \cos n_k \geq k \rangle$ that is locally weakly tight, i.e. (iiia) holds. From (iiia) to (iii): it suffices if for any sequence $u_n \in A$ we furnish a sequence of sets $\Omega_j \in \mathcal{A}_f$, $\Omega = \bigcup_j \Omega_j$, and a subsequence of $u_n$ that is weakly relatively compact in $L_1(\Omega_j; X)$ for $j \in \mathbb{N}$ since then $u_n$ will have a diagonal subsubsequence that converges locally in the $\sigma(X', X)$-biting sense. Again by Lemma 3.8 and the $\sigma$-finiteness of $\mu$ we find $F_j \in \mathcal{A}_f$ with $\Omega = \bigcup_j F_j$ such that $u_n$ is bounded in $L_1(F_j; X)$. Applying Proposition 7.5, we find a measurable set $\Omega_j \subset F_j$ with $\lim_j \mu(F_j \setminus \Omega_j) = 0$ and a subsequence of $u_n$ (not relabelled) that is equi-integrable in a fixed $L_1(\Omega_j; X)$. A standard diagonal argument allows then to extract a subsequence $u_n$ that is equi-integrable in all $L_1(\Omega_j; X)$. Now, in order to conclude that $u_n$ is weakly relatively compact in $L_1(\Omega_j; X)$, it suffices by [23, §24.3(8)] if $u_n$ is weakly relatively compact, i.e. if we may extract from the decreasing convex hull $\cos n_k j \geq k \rangle$ of any subsequence of $u_n$ (not relabelled) a weakly convergent subsequence $v_k$. By (iiia) we find $v_k \in \cos n_k \geq k \rangle$ that is locally weakly tight. Therefore our proof is finished if we show that a bounded, equi-integrable and locally weakly tight sequence has a weakly convergent subsequence in $L_1(\Omega_j; X)$. Pick $$\Omega'_j \subset \Omega_j, \quad \lim_j \mu(\Omega_j \setminus \Omega'_j) = 0$$ such that each $v_k \chi_{\Omega'_j}$ is weakly tight hence has a weakly convergent subsequence by [34, Thm. 8]. Extract a diagonal sequence (not relabelled) such that $v_k \chi_{\Omega_j}$ converges weakly for all $j \in \mathbb{N}$. Then, as $\lim_j \sup_k \|v_k \chi_{\Omega_j} \| = 0$ by equi-integrability, the sequence $v_k$ converges weakly, too. The proof is finished. \qed
Corollary 7.1. Theorem 7.1 fully applies in the following cases: (i) $\mathcal{G}$ is the unit ball of $V_{\varphi^*}(\mu)$ (ii) $X'$ has the Radon-Nikodym property, $\mathcal{G}$ is the unit ball of $L_{\varphi^*}(\mu)$ and $\varphi$ is dualizable i.a.e. for every element of $L_{\varphi^*}(\mu)$.

Proof. (i) The set $\mathcal{G}$ is norming by Lemma 7.2. The closedness under multiplication with indicators is obvious from the definition of $V_{\varphi^*}(\mu)$. To see that any $v \in V_{\mathcal{L}}(\mu; X')$ admits a sequence $v_n \in V_{\varphi^*}(\mu)$ with $v_n \uparrow v$, use Lemma 3.8 to find an exhausting sequence $F_n \in \mathcal{A}$ with $\Omega = \bigcup_n F_n$ such that $v_n = v\chi_{F_n} \in V_{\varphi^*}(\mu)$. (ii) This is a particular case of the former by Theorem 6.1. □

7.2.2. Compactness in $V_{\varphi^*}(\mu)$. Our second main theorem on compactness primarily applies to the weak* topology of $V_{\varphi^*}(\mu)$ hence of $L_{\varphi^*}(\mu)$ should these spaces coincide. As before, we state our result in a more general abstract setting. We start by analysing the particular case of the space $V_1(\mu; X')$ to obtain a preparatory result for the general case, thereby characterizing relatively sequentially compact sets if $\mu$ is a finite separable measure.

Lemma 7.3. Let $\mu$ be a finite measure. A subset $\mathcal{F} \subset V_1(\mu; X')$ is relatively sequentially compact in $\sigma(V_1(\mu; X'); L_{\mathcal{L}}(\mu; X))$ if (i) $\mathcal{F}$ is norm bounded (ii) $\mathcal{F}$ is weak* equi-integrable on $L_{\mathcal{L}}(\mu; X)$ (iii) For any $E \in \mathcal{A}$ the set $\left(\int_E u \, d\mu\right)_x \in \mathcal{F}$ is relatively sequentially compact in $\sigma(X', X)$ (vi) Given a sequence in $\mathcal{F}$, the initial $\sigma$-algebra $\mathcal{A}'$ generated by the sequence via its weak* measurability such that the restriction of $\mu$ to $\mathcal{A}'$ is separable. The conditions (i), (ii) and (iii) are also necessary.

The integral of $u$ is to be understood in the sense of Pettis for the duality of $\sigma(X', X)$, i.e. it is defined as the unique element of $X'$ for which

$$\left(\int_E u \, d\mu, x\right)_{X', X} = \int_E \langle u(\omega), x\rangle_{X', X} \, d\mu(\omega) \quad \forall x \in X.$$ 

Remember that a $\sigma$-finite measure on a separable $\sigma$-algebra is itself separable. The $\sigma$-algebra $\mathcal{A}'$ generated by $u_n \in \mathcal{F}$ via its weak* measurability is the smallest one for which any $u_n$ is weak* measurable, i.e. it is generated by the family of scalar functions $\omega \rightarrow \langle u_n(\omega), x\rangle$ for $x \in X$. Equivalently, we may restrict to a dense subset of $X$, which shows in particular that $\mathcal{A}'$ is separable if $X$ is. Another sufficient condition for separability of $\mathcal{A}'$ is if $\mathcal{F}$ consists of strongly measurable functions, as is implicit in the proof of Lemma B.3. Finally, another condition is separability of the superalgebra $\mathcal{A}$. Indeed, since the pseudometric in Definition 5.2 becomes a metric upon passing to a.e. equivalence classes of sets, this follows from the fact that subspaces of metric spaces retain separability.

Proof. Ad sufficiency: let $u_n \in \mathcal{F}$ with $\mathcal{A}'$ the separable $\sigma$-algebra generated by the sequence. Conditional expectations are nonexpansive on $L_p(\mu; X)$ for $1 \leq p < \infty$ by [11, Ch. 5.1, Thm. 4]. We may use $\lim_{p \to \infty} \|u\|_p = \|u\|_\infty$ for $u \in L_{\mathcal{L}}(\mu; X)$ to find that the conditional expectation operators $E(\cdot | \mathcal{A}')$ are also nonexpansive on $L_{\mathcal{L}}(\mu; X)$. Hence if $v_n \to v$ a.e. and $\sup_n \|v_n\|_\infty < \infty$, then $E(v_n | \mathcal{A}') \to E(v | \mathcal{A}')$ a.e. and bounded in $L_{\mathcal{L}}(\mu; X)$. Consequently, elementary properties of conditional expectations and approximation by simple functions yield the identity

$$\int \langle u_n(\omega), v(\omega) \rangle \, d\mu(\omega) = \int \langle u_n(\omega), E(v | \mathcal{A}')(\omega) \rangle \, d\mu(\omega) \quad \forall v \in L_{\mathcal{L}}(\mu; X)$$
so that we have reduced to considering pairings of \( u_n \) with \( \mathcal{A}' \)-measurable elements of \( \mathcal{L}_\infty (\mu; X) \). Let \( \Omega_j \) be a dense sequence for \( \mu \). By a diagonal argument and the boundedness of \( \mathcal{F} \), we may pass to a subsequence (not relabelled) such that \( \sigma (X', X') \lim \mu \int_{\Omega_j} u_n \, d\mu \) exists for all \( j \). Consequently, this limit exists for any \( \Omega' \in \mathcal{A}' \) by weak* equi-integrability of \( \mathcal{F} \) and since for \( \Omega' \) we find \( \Omega_j \) with 
\[
\lim_k \mu (\Omega' \Delta \Omega_j) = 0. 
\]
Hence, the pairing \( \langle u_n, v \rangle \) with any simple function \( v \) converges. By strong measurability we may approximate a general \( v \in \mathcal{L}_\infty (\mu; X) \) with a sequence \( v_t = \sum_{j=1}^{\infty} x_j \chi_{A_j} \) of countably valued functions converging uniformly to \( v \), where \( v_t \) in turn is approximated by its simple partial sums. Using the weak* equi-integrability of \( u_n \) to invoke Proposition 7.1, we deduce that \( \langle u_n, v \rangle \) converges for any \( v \in \mathcal{L}_\infty (\mu; X) \). In particular, there exists \( u \) belonging to the dual space of \( \mathcal{L}_\infty (\mu; X) \) such that \( w^* \lim u_n = u \). Lemma 6.2 then yields \( u \in V_1 (\mu; X') \).

Ad necessity: \( \mathcal{F} \) is bounded since \( \mathcal{L}_\infty (\mu; X')^\ast = V_1 (\mu; X') \oplus S_1 (\mu; X') \) by Theorem 6.2. As, for any \( v \in \mathcal{L}_\infty (\mu; X) \), the set \( \{ \langle u, v \rangle \}_{u \in \mathcal{F}} \) is weakly relatively compact in \( L_1 (\mu) \), the set \( \mathcal{F} \) is weak* equi-integrable on \( \mathcal{L}_\infty (\mu; X) \). The necessity of (iii) follows by considering functions \( x \chi_E \in \mathcal{L}_\infty^p (\mu; X) \) for \( x \in X \) and \( E \in \mathcal{A} \).

**Definition 7.4** (weak* biting convergence). Given integrally weak* measurable functions \( v_n : \Omega \to X' \), we say that \( v_n \) converges locally to \( v \) in the weak* or \( \sigma (X', X') \)-biting sense if there exists an exhausting increasing sequence \( \Omega_j \in \mathcal{A} \), \( \Omega = \bigcup_j \Omega_j \), such that \( v_n \to v \) in \( \sigma (V_1 (\Omega_j; X'); \mathcal{L}_\infty (\Omega_j; X)) \) for every \( j \in \mathbb{N} \).

Weak* biting convergence implies the limit to be integrally measurable in the weak* sense. Let \( v_n \in V_1 (\Omega; X') \) be a bounded sequence. We may extract from \( v_n \) a subsequence (not relabelled) such that for the integrable function \( M_n \) defined in the proof of Proposition 7.4 there exists a sequence \( \Omega_n \) with 
\[
\lim_n \mu (\Omega_j \Delta \Omega_n) = 0
\]
and \( M_n \chi_{\Omega_j} \) is equi-integrable by Proposition 7.5 hence \( \chi_{\Omega_j; X'} \) is weak* equi-integrable on \( \mathcal{L}_\infty (\mu; X) \). The measure in Proposition 7.4 was \( \sigma \)-finite, however, we only used that \( \mu \) has no atom of infinite measure to obtain the function \( M_n \) there. Therefore, by Lemma 7.3, the question of convergence in the weak* biting sense reduces to obtaining (iii) and (iv) in Lemma 7.3, which as for weak biting convergence can be interpreted as sequential weak* biting convergence being about the behaviour of the range for bounded sequences in \( V_1 (\mu; X') \) if \( \mu \) is \( \sigma \)-finite. The same is true for sequences bounded in \( V_{\sigma \ast} (\mu) \) because this space embeds into \( V_1 (\mu; X') \) on an exhausting sequence by Lemma 3.8.

**Theorem 7.2.** Let \( \mu \) be \( \sigma \)-finite, \( \mathcal{F} \subset \mathcal{L}_\infty (\mu) \) a norming subset for \( V_{\sigma \ast} (\mu) \). Then \( A \) is relatively sequentially compact in \( \sigma (V_{\sigma \ast} (\mu), \text{lin} \mathcal{F}) \) if (i) \( A \) is norm bounded (ii) \( A \) is weak* equi-integrable on \( \mathcal{F} \) (iii) \( A \) is relatively sequentially compact in the \( \sigma (X', X') \)-biting sense. Conversely, if \( A \) relatively sequentially compact, then (i) holds if \( \text{lin} \mathcal{F} \) is closed, (ii) holds if \( \text{lin} \mathcal{F} \) is closed under multiplication with indicators of measurable sets (iii) holds if in addition to the latter closedness, the closure of \( \text{lin} \mathcal{F} \) w.r.t. the convergence from below contains \( \mathcal{L}_\infty (\mu; X) \).

The condition involving \( \mathcal{L}_\infty (\mu; X) \) is satisfied if \( \mu \) is \( \sigma \)-finite and \( \text{lin} \mathcal{F} \) almost decomposable.

**Proof.** Ad sufficiency: let \( v_n \in A \) be a sequence. We may assume \( \mu \) to be finite since then the \( \sigma \)-finite case will follow by Proposition 7.3. Let \( \Omega_j \) be an exhausting sequence for which \( v_n \) is sequentially relatively compact in \( \sigma (V_1 (\Omega_j; X'); \mathcal{L}_\infty (\Omega_j; X)) \) for \( j \in \mathbb{N} \). Invoking again Proposition 7.3, we may reduce to extracting from \( v_n \).
a subsequence converging in $V_{\varphi^*}(\Omega_j)$ for any given $j$. We shall extract several subsequences, none of which we relabel. Select $v_n$ with

$$\lim_n v_n = v$$

Then for $u \in L_{\varphi^*}(\Omega_j)$ and simple functions $u_m$ with $\lim_m \|u - u_m\|_\varphi = 0$ there holds

$$|\langle v, u \rangle - \langle v_n, u \rangle| \leq |\langle v, u - u_m \rangle| + |\langle v - v_n, u_m \rangle| + |\langle v_n, u - u_m \rangle|$$

by the boundedness of $A$ so that sending $n \to \infty$ and then $m \to \infty$ yields

$$\lim_n \langle v_n, u \rangle = \langle v, u \rangle \quad \forall u \in L_{\varphi^*}(\Omega_j).$$

Proposition 7.1 together with Lemma 4.2 then yields the same convergence for $u \in F$ since $A$ is weak* equi-integrable on $F$.

Ad necessity: (i) Given a sequence $v_n \in A$, the Banach-Steinhaus theorem yields a bound on $v_n$ in the dual space of the Banach space $F$ so that we obtain a bound in $V_{\varphi^*}(\mu)$ because $F$ is norming. (ii) If $A$ were not weak* equi-integrable on $u \in F$, then we would find sequences $v_n \in A$ and $E_n \in A$ with $\mu \langle \lim_n E_n \rangle = 0$ and

$$\inf_{n \in \mathbb{N}} \left| \int_{E_n} \langle v_n, u \rangle \, d\mu \right| \geq \varepsilon > 0. \quad (60)$$

Let $\lambda_n (E) = \int_E \langle u_n, v \rangle \, d\mu$, a signed finite measure for which the limit $\nu (E) = \lim_n \nu_n (E)$ exists for every $E \in A$ since $\nu E \in \text{lin } F$ by the closedness under multiplication with indicators. We shall invoke the Vitali-Hahn-Saks theorem [12, Thm. 2.53] in order to conclude that $\lim_n \nu_n (E_n) = 0$ thus contradicting (60).

To justify this, note that $\mu$ is $\sigma$-finite hence equivalent to some finite measure $\nu$ on $\Sigma$. Therefore each $\lambda_n$ is absolutely continuous w.r.t. the finite measure $\nu$ and Vitali-Hahn-Saks has been justified. (iii) By Lemma 3.8 there exists $F_n \in A_f$ an isometric sequence with $\Omega = \bigcup_n F_n$ such that $L_{\varphi^*}(F_n; X) \to L_{\varphi^*}(F_n)$ hence the adjoint mapping of this embedding is itself an embedding $V_{\varphi^*}(F_n) \to V_{\varphi^*}(F_n; X')$ as $L_{\varphi^*}(F_n; X')$ is dense from below in $L_{\varphi^*}(F_n)$ by Lemma 4.2. For $v \in L_{\varphi^*}(F_n; X)$ pick $v_m \in F$ a sequence with $v_n \uparrow v$. By Proposition 7.1 we have

$$\langle u, v \rangle = \lim_n \langle u, v_m \rangle = \lim_n \langle u_n, v_m \rangle = \lim_n \langle u_n, v \rangle = \lim_n \langle u_n, v \rangle$$

hence necessity obtains.

\[ \square \]

**Corollary 7.2.** Theorem 7.2 fully applies if $F$ is the unit ball of $L_{\varphi^*}(\mu)$.

**Proof.** Clearly, the space $L_{\varphi^*}(\mu)$ being predual to $V_{\varphi^*}(\mu) \oplus S_{\varphi^*}(\mu)$, it is norming. It is closed under multiplication with indicators. As $L_{\varphi^*}(\mu)$ is almost decomposable by Corollary 3.1, its sequential closure from below contains $L_{\varphi^*}(\mu; X)$ if $\mu$ is $\sigma$-finite.

\[ \square \]

**Corollary 7.3.** If $\varphi$ is real-valued so that $C_{\varphi^*}(\mu) = V_{\varphi^*}(\mu)$, then Theorem 7.2 characterizes weak* convergent sequences in $V_{\varphi^*}(\mu)$. The weak* equi-integrability is always satisfied in this case. If moreover $X'$ has the Radon-Nikodym property, then the ball of $V_{\varphi^*}(\mu)$ is relatively sequentially compact.

**Proof.** We have $C_{\varphi^*}(\mu) = V_{\varphi^*}(\mu)$ by Corollary 6.2. Theorem 7.2 fully applies since the sequential closure of $C_{\varphi^*}(\mu)$ from below contains $L_{\varphi^*}(\mu; X)$ as $C_{\varphi^*}(\mu)$ is almost decomposable by Lemma 5.3 if $\mu$ is $\sigma$-finite. The closedness under multiplication
with measurable indicators is obvious for \( C_\varphi(\mu) \). Clearly, weak* equi-integrability w.r.t. \( C_\varphi(\mu) \) is a vacuous assumption.

If \( X' \) has the Radon-Nikodym property, then elements of \( V_{\varphi^*}(\mu) \) are strongly measurable by Theorem 6.1 if \( \mu \) is \( \sigma \)-finite as in Theorem 7.2. Let \( F_j \in A_f \) be a sequence with \( \Omega = \bigcup_j F_j \) such that

\[
V_{\varphi^*}(F_j) \to V_1(F_j; X')
\]

according to Lemma 3.8. By strong measurability any sequence in \( V_{\varphi^*}(F_j) \) generates a countable \( \sigma \)-algebra \( \mathcal{A}' \) hence \( \mu \) is separable on \( \mathcal{A}' \). Now, by the remark below Lemma 7.3, we have reduced the matter to checking (iii) in Lemma 7.3 for \( E \in \mathcal{A}' \). This obtains since the ball \( B_X \) is sequentially weak* compact by [8, XIII, Thm. 6] upon remembering that \( X' \) has the Radon-Nikodym property iff separable subspaces of \( X \) have separable duals by [11, VII.2, Cor. 8].

\[ \square \]

**Appendix A. Hyperspace Topologies**

We prove here results about the Attouch-Wets \( \tau_{AW} \) topology.

**Lemma A.1.** Let \( f_n \in \text{LS}(M) \) be a sequence with \( \tau_{AW} \)-lim \( f_n = f \). For each bounded set \( B \subset M \) holds

\[
\liminf_{n} \inf_{B} f_n \geq \inf_{B} f.
\]  

(61)

For each open set \( O \subset M \) holds

\[
\limsup_{n} \inf_{O} f_n \leq \inf_{O} f.
\]  

(62)

**Proof.** Ad (61): suppose \( n_k \) were a subsequence with \( \lim_k \inf_B f_{n_k} < \alpha < \inf_B f \). Pick \( x_k \in B \) with \( \lim_k f_{n_k}(x_k) < \alpha \) and a sequence \( \alpha_k \in [f_{n_k}(x_k), \alpha] \) bounded below by \( \beta \in \mathbb{R} \). We have \( y_k = (x_k, \alpha_k) \in \text{epi} f_{n_k} \) so that \( \tau_{AW} \)-lim \( f_n = f \) by definition of the box metric on \( M \times \mathbb{R} \) yields the contradiction

\[
0 < |\inf_B f - \alpha| \leq \lim_k |\inf_B f - \alpha_k| \leq \lim_k |d_{y_k}(\text{epi} f) - 0| \\
\leq \lim_k \sup_{y \in B \times [\beta, \alpha]} |d_y(\text{epi} f) - d_{y_k}(\text{epi} f_{n_k})| = 0.
\]

Ad (62): as \( \tau_{AW} \) implies epi-convergence, this obtains by [2, Prop. 1.18].

**Lemma A.2.** For \( \tau_{AW} \) the mapping

\[
\text{LS}(X) \times X' \times \mathbb{R} \to \text{LS}(X) : (f, x', \alpha) \mapsto f + x' + \alpha
\]

is continuous.

**Proof.** If \( \lim_n x'_n = x' \) and \( \lim_n \alpha_n = \alpha \), then \( \lim_n x'_n + \alpha_n = x' + \alpha \) uniformly on bounded subsets whence the claim follows by [1, Thm. 7.1.5].

**Proposition A.1.** Given a sequence of subsets \( S_k \subset M \) and \( W_0 \in \mathcal{S}(M) \), there exists a closed set \( W \in \mathcal{S}(M) \) such that \( W_0 \subset W \) and

\[
d_x(S_k) = d_x(S_k \cap W) \quad \forall x \in W \quad \forall k \in \mathbb{N}.
\]

If moreover \( M = U \times V \) with \( V \) separable, then \( W \) may be chosen of the form \( Y \times V \).
Finally, since $\tau$ $\tau$ $\\overline{\operatorname{epi}}$ $\operatorname{epigraph}$ $G$ $\exists$ $\operatorname{invoking}$ Proposition A.1 we find a closed set $W_m$ in $W_\ell$, pick $x_n(k,m)$ a sequence in $S_k$ with $d(w_m,x) \leq d(w_m,S_k) + \frac{1}{n}$ and set

$$W_{\ell+1} = \operatorname{cl} \bigcup_{k,m,n} \{x\} \cup \{w\}.$$  \hspace{1cm} (64)

Then $W_\ell \subset W_{\ell+1} \in S(M)$ and $d_w(S_k) = \inf_n d_w(x) = d_w(S_k \cap W_{\ell+1})$. The functions $d(S_k)$ and $d(S_k \cap W_{\ell+1})$ coincide on $W_\ell$ as they are continuous and equal on the dense sequence $w_m$. This completes the inductive construction. Finally, we set $W = \operatorname{cl} \bigcup_{\ell \in \mathbb{N}} W_\ell$

and observe that $d_x(S) = d_x(S_k \cap W_{\ell+1}) \geq d_x(S_k \cap W) \geq d_x(S)$ for $x \in W_\ell$ so that the first part follows by density. Regarding the addendum, note that the construction still works if we increase $W_{\ell+1}$ to be any separable superset of the right-hand side in (64). In particular, we may take $W_{\ell+1} = P_U(W_{\ell+1}) \times X$ if $P_U$ denotes the projection of $M$ onto $U$. By definition, $W$ will then be of the required form. \hfill $\Box$

Lemma A.3. If $f : \Omega \to \operatorname{LS}(M)$ is strongly $\tau_{AW}$-measurable, then there exists $W = \operatorname{cl} W \in S(M)$ such that $f$ is normal on $W$ and $\inf_W f_\omega = \inf_M f_\omega$ for all $\omega \in \Omega$.

Proof. Pick $B_m$ an isotonic sequence of bounded open balls with $M = \bigcup_{m \geq 1} B_m$. As $f_n$ takes finitely many values

$$\exists W_0 \in S(M) : \inf_{B_m} f_{\omega,n} = \inf_{W_0 \cap B_m} f_{\omega,n} \ \forall m \in \mathbb{N}, \ \omega \in \Omega.$$  \hspace{1cm}

Invoking Proposition A.1 we find a closed set $W \in S(M)$ with $W_0 \subset W$ and

$$d_x(\operatorname{epi} f_{\omega,n}) = d_x(\operatorname{epi} f_{\omega,n} \cap W \times \mathbb{R}) \ \forall x \in W \times \mathbb{R}, \ \omega \in \Omega, \ n \in \mathbb{N}.$$  \hspace{1cm}

Thus the restriction $f_{\omega,n,W}$ is a Cauchy sequence in $(\operatorname{LS}(W), \tau_{AW})$. Hence, there exist $G_\omega = \tau_{AW}$-lim$_n$ $\operatorname{epi} f_{\omega,n,W} \in \operatorname{CL}(W \times \mathbb{R})$ which agrees with the (relative) epigraph $\operatorname{epi} f_\omega$ since for $x \in W \times \mathbb{R}$ there holds

$$d_x(\operatorname{epi} f_\omega) = \lim_n d_x(\operatorname{epi} f_{\omega,n}) = \lim_n d_x(\operatorname{epi} f_{\omega,n} \cap W \times \mathbb{R}) = d_x(G_\omega).$$  \hspace{1cm}

Lemma A.1 implies $\lim_n \inf_B f_{\omega,n} = \inf_B f_\omega$ for any bounded relatively open set $B \subset W$ and so

$$\inf_M f_\omega = \inf_M \inf_f f_\omega = \inf f_{\omega,n} = \inf \lim_n \inf f_{\omega,n} = \inf \lim_{m \to \infty} \inf_{W \cap B_m} f_{\omega,n}$$  \hspace{1cm}

$$= \inf_{W \cap B_m} f_\omega$$  \hspace{1cm}

$$= \inf_W f_\omega.$$  \hspace{1cm}

Finally, since $\tau_{AW}$ is finer than $\tau_W$, we conclude that the limiting integrand is $\tau_W$-measurable hence normal by the Hess theorem [1, Thm. 6.5.14]. \hfill $\Box$
APPENDIX B. MULTIFUNCTIONS

We compile here auxiliary results about (Effros) measurable multifunctions. Throughout this section, the metric space $M$ is separable unless stated otherwise and $\Gamma : \Omega \to \mathcal{P}(M)$ is a multifunction.

**Lemma B.1.** Let the multifunction $\Gamma$ be closed and measurable. Then

a) its graph $\text{gph} \Gamma$ is $\mathcal{A} \otimes \mathcal{B}(M)$-measurable.

b) The multifunction $\partial \Gamma : \Omega \to \mathcal{P}(M)$ is measurable.

**Proof.** Ad a): the proof for $M = \mathbb{R}^d$ is contained in [33, Thm. 14.8] and may be adapted without further ado by replacing $\mathbb{Q}^d$ with a dense sequence in $M$. Ad b): let $O \subset M$ be open. The set

$$V(O) = \{A \in \text{CL}(M) \mid O \subset A\} = \bigcap_{x \in O} \{A \in \text{CL}(M) \mid d_x(A) = 0\}$$

is closed in the Wijsman topology $\tau_W$. By Hess’ theorem [1, Thm. 6.5.14] the multifunction $\Gamma$ is $\tau_W$-measurable as a single-valued mapping to $\text{CL}(M)$. Hence

\begin{align*}
\Gamma^-(O) &= \{\omega \mid O \subset \Gamma(\omega)\} \cup \{\omega \mid \partial \Gamma(\omega) \cap O \neq \emptyset\} \\
&= \Gamma^{-1}[V(O)] \cup \{\omega \mid \partial \Gamma(\omega) \cap O \neq \emptyset\} \\
&= \Gamma^{-1}[V(O)] \cup (\partial \Gamma)^-(O)
\end{align*}

so that $(\partial \Gamma)^-(O)$ is measurable as a difference of measurable sets. □

**Corollary B.1.** If $\Gamma$ is open and measurable, then $\text{gph} \Gamma \in \mathcal{A} \otimes \mathcal{B}(M)$.

**Proof.** The multifunctions $\text{cl}\Gamma$ and $\partial \text{cl}\Gamma$ are measurable with measurable graphs by Lemma B.1. As $\Gamma$ is open, we have $\Gamma = \text{cl}\Gamma \setminus \partial \text{cl}\Gamma$ so that $\text{gph} \Gamma = \text{gph} \text{cl}\Gamma \setminus \text{gph} \partial \text{cl}\Gamma$ belongs to $\mathcal{A} \otimes \mathcal{B}(M)$. □

**Lemma B.2.** Let $\text{cl}\Gamma = \text{cl}\text{int}\Gamma$. Then $\Gamma$ is measurable iff $\Gamma^-(\{x\})$ are measurable for $x \in M$.

**Proof.** $\implies$ : pre-images of compact sets under measurable multifunctions are measurable. $\Leftarrow$ : let $O \subset M$ be open and $\{x_n\} \subset O$ a dense sequence. From $\text{cl}\Gamma = \text{cl}\text{int}\Gamma$ follows

\begin{align*}
\Gamma^-(O) &= \{\omega \mid \Gamma(\omega) \cap O \neq \emptyset\} \\
&= \bigcup_{n \geq 1} \{\omega \mid \text{int}\Gamma(\omega) \cap \{x_n\} \neq \emptyset\}.
\end{align*}

As measurability of the multifunctions $\text{int}\Gamma$ and $\text{cl}\text{int}\Gamma$ is equivalent, the last set is measurable and our claim obtains. □

**Lemma B.3.** Let $M$ be an arbitrary metric space and $\mathcal{A}_\mu$ the completion of $\mathcal{A}$ w.r.t. $\mu$. A function $u: \Omega \to M$ is $\mathcal{A}_\mu \otimes \mathcal{B}(M)$-measurable and almost separably valued iff there exists a strongly $\mathcal{A} \otimes \mathcal{B}(M)$-measurable function $v: \Omega \to M$ with $u = v$ a.e.

**Proof.** $\implies$ : modify $u$ on a null set to obtain a separably valued $\mathcal{A}_\mu$-measurable function $w$ and take a sequence $B_n$ of balls generating the topology of $w(\Omega)$. Express $w^{-1}(B_n)$ as a disjoint union of two sets $\Omega_n \in \mathcal{A}$ and $M_n$ such that $M_n \subset N_n$ for a null set $N_n \in \mathcal{A}$. For $N = \bigcup_{n \geq 1} N_n$ define $w$ to agree with $w$ on $\Omega \setminus N$ and assign any constant value on $N$. Any $v^{-1}(B_n)$ is $\mathcal{A}$-measurable whence $v$ is strongly $\mathcal{A}$-measurable and $u = v$ a.e.
Hence \( u \) required form, we give proofs.

We compile here auxiliary results about measurability of integrands. Since none of the standard references \([1, 6, 17, 31]\) contain these statements directly in the required form, we give proofs.

**Definition C.1** (infimally measurable). An integrand \( f : \Omega \times T \to [-\infty, \infty] \) is called infimally measurable iff the sets \( S_f^-(O \times I) = \{ \omega \mid \text{epi} f_\omega \cap O \times I \neq \emptyset \} \) for \( O \subset T \) open and \( I \subset \mathbb{R} \) an open interval are measurable.

**Lemma C.1.** For an integrand \( f : \Omega \times T \to [-\infty, \infty] \) the following are equivalent:

a) \( f \) is infimally measurable;

b) For \( O \subset T \) open the functions \( \inf_{O} f_{\omega} \) are measurable;

c) For \( \alpha \in [-\infty, \infty] \) the strict sublevel multifunctions
\[
L_\alpha : \Omega \to \mathcal{P}(T) : \omega \mapsto \text{lev}_{\leq \alpha} f_{\omega} = \{ x \mid f_{\omega}(x) < \alpha \}
\]
are Effros measurable.

**Proof.** For an open subset \( O \subset T \) and an interval \( I = (\alpha, \beta) \) there holds
\[
S_f^- (O \times I) = \{ \text{epi} f_{\omega} \cap O \times I \neq \emptyset \} = \left\{ \inf_{O} f_{\omega} < \beta \right\} = L_\beta^{-}(O). \]

**Lemma C.2.** If the integrand \( f : \Omega \times T \to [-\infty, \infty] \) is pre-normal, then \( f \) is infimally measurable. If \( T \) is second countable, the converse is true as well.

**Proof.** \( \implies \) : recall Definition C.1. \( \iff \) : let \( O_n \subset T \) be a base sequence of open sets. By definition of the product topology, every open set \( U \subset T \times \mathbb{R} \) may be written as \( U = \bigcup_{k \in \mathbb{N}} O_{n_k} \times (\alpha_k, \beta_k) \) with \( \alpha_k, \beta_k \in \mathbb{Q} \) whence there follows measurability of the set
\[
S_f^- (U) = \bigcup_{k \in \mathbb{N}} S_f^- (O_{n_k} \times (\alpha_k, \beta_k)).
\]

We call a map \( F : T \to [-\infty, \infty] \) upper semicontinuous if its hypograph is closed. When \( F \) is \([\infty, \infty] \)-valued, this coincides with other known characterizations of upper semicontinuity such as open sublevel sets.

**Lemma C.3.** If \( f : \Omega \times T \to [-\infty, \infty] \) is pre-normal and \( g : T \to \mathbb{R} \) is upper semicontinuous, then \( h_\omega(x) = f_\omega(x) + g(x) \) is infimally measurable.

**Proof.** By upper semicontinuity of \( g \) the set \( V = \{(x,r) \mid x \in O, r < \beta - g(x) \} \) is open. The claim obtains if we show that
\[
S_h^- [O \times (\alpha, \beta)] = \{ \text{epi} h_\omega \cap O \times (\alpha, \beta) \neq \emptyset \} = \{ \text{epi} f_{\omega} \cap V \neq \emptyset \} = S_f^- (V). \]

We check the set identity: let \( (x,r) \in \text{epi} h_\omega \cap O \times (\alpha, \beta) \) so that \( f_{\omega}(x) + g(x) \leq r \) while \( x \in O \) and \( r \in (\alpha, \beta) \). Then \( f_{\omega}(x) < \beta - g(x) \) so that \( \text{epi} f_{\omega} \cap V \) is non-empty. Conversely, if \( (x,r) \in \text{epi} f_{\omega} \cap V \), then \( f_{\omega}(x) < r + g(x) < \beta \) so that the intersection \( \text{epi} h_\omega \cap O \times (\alpha, \beta) \) is non-empty. \( \square \)
Lemma C.4. Let $M$ be separable. Suppose $f : \Omega \times M \to [-\infty, \infty]$ is such that

a) For all $\omega \in \Omega$, $x \mapsto f_\omega(x)$ is upper semicontinuous;

b) For all $x \in M$, $\omega \mapsto f_\omega(x)$ is measurable.

Then $f$ is a pre-normal integrand.

Proof. By a) holds $\text{cl} \, \text{epi} f_\omega = \text{cl} \, \text{epi} f_\omega$ so that Lemma B.2 makes it sufficient to observe that for all $(x, \alpha) \in M \times \mathbb{R}$ the set $S_f^{-} (\{(x, \alpha)\}) = \{ \omega \mid f_\omega(x) \leq \alpha \}$ is measurable by b).

Lemma C.5. Let $f : M \to [-\infty, \infty]$ be a function and $\lambda > 0$. If for some $x_0 \in M$ the Lipschitz regularization

\[ f_\lambda(x) = \inf_{y \in M} f(y) + \lambda d(x,y) \]

is finite, then $f_\lambda$ is finite-valued and Lipschitz continuous with constant $\lambda$.

Proof. For $x_1, x_2, y \in M$ holds $f(y) + \lambda d(x_1, y) \leq f(y) + \lambda d(x_2, y) + \lambda d(x_1, x_2)$ so that $f_\lambda(x_1) \leq f_\lambda(x_2) + \lambda d(x_1, x_2)$. In particular, if $f_\lambda(x_1) \in \mathbb{R}$ for some $x_1 \in M$, then $f_\lambda$ is Lipschitz continuous with constant $\lambda$. Also, if $f_\lambda(x_1)$ is infinite, then $f_\lambda \equiv -\infty$ or $f_\lambda \equiv \infty$.

Lemma C.6. Let $M$ be separable. If $f : \Omega \times M \to [-\infty, \infty]$ is a pre-normal integrand, the Lipschitz regularization

\[ f_{\omega, \lambda}(x) = \inf_{y \in M} h_\omega(y) + \lambda d(x,y), \quad \lambda > 0 \]

also is a pre-normal integrand. Moreover, for all $\omega \in \Omega$, the partial map $x \mapsto f_{\omega, \lambda}(x)$ is upper semicontinuous.

Proof. By Lemma C.3 the integrand $h_\omega(y) + \lambda d(x,y)$ is infimally measurable for $x \in M$ and $\lambda > 0$ so that $f_{\omega, \lambda}(x)$ is measurable in $\omega$ and Lipschitz continuous or assumes a constant value $\{-\infty, \infty\}$ in $x$. Either way, the partial map is upper semicontinuous, hence Proposition C.4 obtains the claim.

Lemma C.7. Let $M$ be separable and $f : \Omega \times M \to (-\infty, \infty]$ an integrand.

a) If $f$ is normal, then it is $\mathcal{A} \mathcal{B}(M)$-measurable.

b) If $f$ is $\mathcal{A}_\mu \mathcal{B}(M)$-measurable, then it is pre-normal w.r.t. $\mathcal{A}_\mu$.

Proof. Ad a): Lemmas C.1 and C.2 guarantee that truncation of an integrand retains pre-normality hence we may reduce to the case when $f$ is bounded below. Since $f = \lim_{\lambda \to \infty} f_\lambda$ pointwise as a monotone limit for the Lipschitz regularization $f_\lambda$ of $f$ according to [2, Prop. 1.33], we may reduce to considering $f_\lambda$. Lemma C.6 shows that for all $\omega \in \Omega$, the partial map $x \mapsto f_{\omega, \lambda}(x)$ is upper semicontinuous. Consequently, for $\alpha \in [-\infty, \infty]$, the strict sublevel multifunction

\[ \Omega \to \mathcal{P}(T) : \omega \mapsto L_{f,\alpha}(\omega) := \{ x \in T \mid f_\omega(x) < \alpha \} \]

is open and measurable hence by Corollary B.1 its graph

\[ \text{gph} L_{f,\alpha} = \{ (\omega, x) \mid f_\omega(x) < \alpha \} \]

belongs to $\mathcal{A} \mathcal{B}(M)$ whence $f$ is $\mathcal{A} \mathcal{B}(M)$-measurable.

Ad b): the Aumann theorem [12, Thm. 6.10] yields an $\mathcal{A}_\mu$-measurable Castaing representation for the closure of the epigraphical multifunction $\omega \mapsto \text{epi} f_\omega$ hence its measurability follows [6, Thm. III.9]. As the Effros measurability of $\text{cl} S_f$ and $S_f$ is equivalent, the claim obtains.
NON-SEPARABLY VALUED ORLICZ SPACES

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