We have explicitly shown that the infrared structure of the full gluon propagator in QCD is
an infinite sum over all severe (i.e., more singular than $1/q^2$) infrared singularities. It reflects
the zero momentum modes enhancement effect in the true QCD vacuum. Its existence exhibits a
characteristic mass (the so-called mass gap), which is responsible for the scale of nonperturbative
dynamics in the QCD ground state. By an infrared renormalization of a mass gap only, the deep
infrared structure of the full gluon propagator is saturated by the simplest severe infrared singularity,
the famous $(q^2)^{-2}$. So, there is no smooth in the infrared limit the full gluon propagator. The main
dynamical source of severe infrared singularities in the gluon propagator is the two-loop skeleton
term of the corresponding equation of motion, which contains the four-gluon vertices only. Taking
into account the distribution nature of severe infrared singularities, the gluon confinement criterion
is formulated in a manifestly gauge-invariant way.

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I. INTRODUCTION

Quantum Chromodynamics (QCD) is widely accepted as a realistic, quantum field gauge theory of strong
interactions not only at the fundamental (microscopic) quark-gluon level, but at the hadronic (macroscopic) level
as well. The surprising fact, however, is that after more than thirty years of QCD, we still don’t know exactly the
interaction between quarks and gluons. To know it means that one knows exactly the full gluon propagator, the quark-
gluon proper vertex and the pure gluon proper vertices. In the weak coupling limit or in the case of heavy quarks only
this interaction is known. In the first case all the above-mentioned lower and higher Green’s functions (propagators and
vertices, respectively) become effectively free ones multiplied by the renormalization group corresponding perturbative
(PT) logarithm improvements. In the case of heavy quarks all the Green’s functions can be approximated by their
free counterparts from the very beginning. In general, the Green’s functions are essentially different from their free
counterparts (substantially modified) due to the response of the highly nontrivial large scale structure of the true
QCD vacuum. It is just this response which is taken into account by the full (”dressed”) propagators and vertices
(it can be neglected in the weak coupling limit or for heavy quarks). That is the main reason why they are still
unknown. In other words, it is not enough to know the Lagrangian of the theory. In QCD it is also necessary and
important to know the true nonperturbative (NP) structure of its ground state (also there might be symmetries of the
Lagrangian which do not coincide with symmetries of the vacuum). This knowledge can only come from the
investigation of a general system of the dynamical equations of motion, the so-called Schwinger-Dyson (SD) system
of equations, to which all the Green’s functions should satisfy.

The main purpose of this Letter is to fix the infrared (IR) structure of the full gluon propagator by analyzing the
structure and some general properties of the SD equation for the full gluon propagator. This is important and is of
broad interest, since it is closely related to the large scale structure of the true QCD vacuum as emphasized above.

II. GLUON PROPAGATOR

In order to investigate the problem of the true QCD ground state structure, let us begin with one of the main objects
in the Yang-Mills (YM) sector. The two-point Green’s function, describing the full gluon propagator, is (Euclidean
signature here and everywhere below)

$$D_{\mu\nu}(q) = \frac{1}{q^2} \left\{ T_{\mu\nu}(q) d(q^2, \xi) + \xi L_{\mu\nu}(q) \right\} \frac{1}{q^2}, \quad (2.1)$$
where $\xi$ is the gauge fixing parameter ($\xi = 0$ - Landau gauge and $\xi = 1$ - Feynman gauge) and $T_{\mu\nu}(q) = \delta_{\mu\nu} - q_\mu q_\nu/q^2 = \delta_{\mu\nu} - L_{\mu\nu}(q)$. Evidently, $T_{\mu\nu}(q)$ is the transverse (physical) component of the full gluon propagator, while $L_{\mu\nu}(q)$ is its longitudinal (unphysical) one. The free gluon propagator is obtained by setting simply the full gluon form factor $d(q^2, \xi) = 1$ in Eq. (2.1), i.e.,

$$D^0_{\mu\nu}(q) = i \{T_{\mu\nu}(q) + \xi L_{\mu\nu}(q)\} \frac{1}{q^2}.$$  \hspace{1cm} (2.2)

The solutions of the SD equation for the full gluon propagator (2.1) are supposed to reflect the complexity of the quantum structure of the QCD ground state. Just this determines one of the central roles of the full gluon propagator in the SD system of equations. The IR limit of these solutions is uniquely determined by asymptotic freedom (AF) [6]. The deep IR asymptotics of the full gluon propagator can be generally classified into the two different types: singular, which means that the zero momentum modes enhancement (ZMME) effect takes place in the NP QCD vacuum, or smooth, which means that the full gluon propagator is IR finite or even is IR vanishing. Formally, the full gluon propagator (2.1) has an exact power-type IR singularity, $1/q^2$, which is due to its longitudinal component. This is the IR singularity of the free gluon propagator, see Eq. (2.2). So by the ZMME effect we mean, in general, the IR singularities, which are more severe than $1/q^2$. Evidently, the singular asymptotics is possible at any value of the gauge fixing parameter. At the same time, the smooth behavior of the full gluon propagator (2.1) in the IR becomes formally possible either by choosing the Landau gauge $\xi = 0$ from the very beginning, or by removing the longitudinal (unphysical) component of the full gluon propagator with the help of ghost degrees of freedom [1] [7] [8] (for more detail discussion see below).

However, any deviation in the behavior of the full gluon propagator from the free one in the IR domain automatically assumes its dependence on a scale parameter (at least one) different, in general, from the QCD asymptotic scale parameter $\Lambda_{QCD}$. It can be considered as responsible for the NP dynamics (in the IR region) in the QCD vacuum. If QCD itself is a confining theory, then such a characteristic scale is very likely to exist. This is very similar to AF, which requires the above-mentioned asymptotic scale parameter $\Lambda_{QCD}$ associated with the nontrivial PT dynamics in the UV region (AF, scale violation, determining thus the deviation in the behavior of the full gluon propagator from the free one in the UV domain). In this connection it is worth emphasizing that, being numerically a few hundred MeV only, it cannot survive in the UV limit. This means that none of the finite scale parameters, in particular $\Lambda_{QCD}$, can be determined by PT QCD. It should come from the IR region, so it is NP by origin. How to establish a possible relation between these two independent scale parameters was shown in our paper [9]. Despite the fact that the PT vacuum cannot be the true QCD ground state [10], nevertheless, the existence of such kind of a relation is a manifestation that ”the problems encountered in perturbation theory are not mere mathematical artifacts but rather signify deep properties of the full theory” [11].

The message that we have tried to convey is that precisely AF clearly indicates the existence of the NP phase with its own characteristic scale parameter in the full QCD.

### III. GLUON SD EQUATION

The general structure of the SD equation for the full gluon propagator can be written down symbolically as follows (for our purposes it is more convenient to consider the SD equation for the full gluon propagator and not for its inverse):

$$D(q) = D^0(q) - D^0(q)T_{gh}(q)D(q) - D^0(q)T_{q}(q)D(q) + D^0(q)T_{gh}[D](q)D(q).$$ \hspace{1cm} (3.1)

Here and in some places below, we omit the dependence on the Dirac indices, for simplicity. $T_{gh}(q)$ and $T_{q}(q)$ describe the ghost and quark skeleton loop contributions into the gluon propagator. They do not contain the full gluon propagators by themselves. A pure gluon contribution $T_{gh}[D](q)$ is a sum of four pure gluon skeleton loops, and consequently they explicitly contain the full gluon propagators. Precisely this makes the gluon SD equation highly
nonlinear (NL), and this is one of the reasons why it cannot be solved exactly. However, its linear part, which contains only ghost and quark skeleton loops, can be summed up, so Eq. (3.1) becomes

\[ D(q) = \tilde{D}^0(q) + \tilde{D}^0(q)T_g[D](q)D(q) = \tilde{D}^0(q) + D^{NL}(q), \]  

(3.2)

with \( \tilde{D}^0(q) \) being a modified free gluon propagator as follows:

\[ \tilde{D}^0(q) = \frac{D^0(q)}{1 + [T_{gh}(q) + T_q(q)]D^0(q)}, \]  

(3.3)

where

\[ T_{gh}(q) = g^2 \int \frac{id^4k}{(2\pi)^4} k_{\nu} G(k) G(k - q) G_\mu (k - q, q), \]  

(3.4)

\[ T_q(q) = -g^2 \int \frac{id^4p}{(2\pi)^4} Tr[\gamma_\nu S(p - q) \Gamma_\mu (p - q, q) S(p)]. \]  

(3.5)

In general, these quantities can be decomposed as follows:

\[ T_{gh}(q) \equiv T_{\mu\nu}^{gh}(q) = \delta_{\mu\nu} q^2 T_{gh}^{(1)}(q^2) + q_\mu q_\nu T_{gh}^{(2)}(q^2), \]  

(3.6)

\[ T_q(q) \equiv T_{\mu\nu}^{q}(q) = \delta_{\mu\nu} q^2 T_q^{(1)}(q^2) + q_\mu q_\nu T_q^{(2)}(q^2), \]  

(3.7)

where all invariant functions \( T_{gh}^{(n)}(q^2) \) and \( T_q^{(n)}(q^2) \) at \( n = 1, 2 \) are dimensionless with a regular behavior at zero (they include the dependence on the coupling constant squared \( g^2 \)). In this connection a few remarks are in order. Due to the definition \( q_\mu q_\nu = q^2 L_{\mu\nu} \), instead of the independent structures \( \delta_{\mu\nu} \) and \( q_\mu q_\nu \) in Eqs. (3.6) and (3.7), one can use \( T_{\mu\nu} \) and \( L_{\mu\nu} \) as independent structures with their own invariant functions. For simplicity we assume here and everywhere below that all integrals are finite, and consequently all invariant functions are also finite at zero. Anyway, how to render them finite is well known procedure (see, for example Refs. [1, 7, 8, 12]).

From a technical point of view it is convenient to use the free gluon propagator (2.2) in the Feynman gauge (\( \xi = 1 \)), i.e., \( D^0_{\mu\nu}(q) = \delta_{\mu\nu}(i/q^2) \). Then from Eq. (3.3) it follows

\[ \tilde{D}^0(q) = D^0(q) A(q^2), \]  

(3.8)

where \( A(q^2) = 1/(1 + T(q^2)) \), and \( T(q^2) \) is regular at zero. Obviously, it is a combination of the previous ghost \( T_{gh}^{(n)}(q^2) \) and quark \( T_q^{(n)}(q^2) \) at \( n = 1, 2 \) invariant dimensionless functions (it includes the dependence on the coupling constant squared again and the gauge fixing parameter as well in the general case (i.e., when \( D^0(q) \) is given by Eq. (2.2)). Since \( A(q^2) \) is finite at zero, the IR singularity of the linear part of the full gluon propagator is completely determined by the power-type exact IR singularity of the free gluon propagator, as it follows from Eq. (3.8), i.e., \( \tilde{D}^0(q) = A(0) D^0(q), \quad q^2 \to 0 \). We are especially interested in the structure of the full gluon propagator in the IR region, so the exact result (3.8) will be used as an input in the direct iteration solution of the gluon SD equation (3.2). Evidently, this form of the gluon SD equation makes it possible to take into account automatically ghost and quark degrees of freedom in all orders of linear PT. On the other hand, it emphasizes the important role of the pure gluon contribution (i.e., YM one), which forms its NL part.

Let us present now explicitly the NL pure gluon part, which was symbolically denoted as \( T_g[D](q) \) in the gluon SD Eq. (3.2). As mentioned above, it is a sum of four terms, namely

\[ T_g[D](q) = \frac{1}{2} T_\xi + \frac{1}{2} T_1(q) + \frac{1}{2} T_2(q) + \frac{1}{6} T_2'(q), \]  

(3.9)
where the corresponding quantities are given explicitly below

\begin{equation}
T_t = g^2 \int \frac{id^4q_1}{(2\pi)^4} T_4^{(0)} D(q_1),
\end{equation}

\begin{equation}
T_1(q) = g^2 \int \frac{id^4q_1}{(2\pi)^4} T_3^{(0)}(q,-q_1,q_1) T_3(-q,q_1,q-q_1) D(q_1) D(q-q_1),
\end{equation}

\begin{equation}
T_2(q) = g^2 \int \frac{id^4q_1}{(2\pi)^4} \int \frac{id^4q_2}{(2\pi)^4} T_4^{(0)} T_3(-q_2,q_3, q_2 - q_3) T_3(-q, q_1, q_3 - q_2) D(q_1) D(-q_2) D(q_3) D(q_3 - q_2),
\end{equation}

\begin{equation}
T_2'(q) = g^4 \int \frac{id^4q_1}{(2\pi)^4} \int \frac{id^4q_2}{(2\pi)^4} T_4^{(0)} T_4(-q_1, q, q_1 - q, q_2) D(q_1) D(-q_2) D(q_3). \tag{3.13}
\end{equation}

In the last two equations \(q - q_1 + q_2 - q_3 = 0\) is assumed as usual. The \(T_t\) term, which is given in Eq. (3.10), is the so-called tadpole term contribution into the gluon propagator (gluon self-energy). The \(T_1(q)\) term describes the one-loop skeleton contribution, depending on the three-gluon vertices only. The \(T_2(q)\) term describes the two-loop skeleton contribution, depending on the three- and four-gluon vertices, while the \(T_2'(q)\) term describes the two-loop skeleton contribution, depending on the four-gluon vertices only.

The formal iteration solution of Eq. (3.2) looks like

\begin{equation}
D(q) = D^{(0)}(q) + \sum_{k=1}^{\infty} D^{(k)}(q) = D^{(0)}(q) + \sum_{k=1}^{\infty} T_g \left[ \sum_{m=0}^{k-1} D^{(m)}(q) \right] \left[ \sum_{m=0}^{k-1} D^{(m)}(q) \right] - \sum_{m=1}^{k-1} D^{(m)}(q), \tag{3.14}
\end{equation}

where, for example explicitly the first four terms are:

\begin{align*}
D^{(0)}(q) &= \tilde{D}^0(q), \\
D^{(1)}(q) &= \tilde{D}^0(q) T_g [\tilde{D}^0(q) \tilde{D}^0(q)], \\
D^{(2)}(q) &= \tilde{D}^0(q) T_g [\tilde{D}^0(q) + D^{(1)}(q)] (\tilde{D}^0(q) + D^{(1)}(q)) - D^{(1)}(q), \\
D^{(3)}(q) &= \tilde{D}^0(q) T_g [\tilde{D}^0(q) + D^{(1)}(q) + D^{(2)}(q)] (\tilde{D}^0(q) + D^{(1)}(q) + D^{(2)}(q)) - D^{(1)}(q) - D^{(2)}(q),
\end{align*}

and so on. It is worth mentioning that the order of iteration does not coincide with the order of PT in the coupling constant squared. For example, any iteration (even zero) in Eq. (3.14) contains ghost and quark degrees of freedom in all orders of PT, as underlined above. In other words, the iteration solution (3.14) is a general one, since the skeleton loop contributions (skeleton diagrams) are to be iterated (the so-called general iteration solution). In principle, it should be distinguished from the pure PT iteration solution, i.e., from the expansion in powers of the coupling constant squared. In this case a pure PT diagrams (with free propagators and point-like vertices) are to be iterated.

**IV. THE EXPLICIT FUNCTIONAL ESTIMATE**

Let us now establish a type of a possible functional dependence of the full gluon propagator in the IR region. For this purpose it is convenient to start with the gluon SD equation (3.2). Up to the first iteration it becomes

\begin{equation}
D(q) = \tilde{D}^0(q) + \tilde{D}^0(q) T_g [D(q)] D(q) = \tilde{D}^0(q) + \tilde{D}^0(q) T_g [\tilde{D}^0(q)] \tilde{D}^0(q) + \ldots., \tag{4.1}
\end{equation}

where we will use Eq. (3.8) for the modified free gluon propagator in the Feynman gauge in what follows.

In the first approximation of Eq. (3.13), i.e., at the order \(g^4\) we put \(T_4 = T_4^{(0)}\), and it becomes

\begin{equation}
T_2'(q) = T_2^{(2)}(q) = g^4 \int \frac{id^4q_1}{(2\pi)^4} \int \frac{id^4q_2}{(2\pi)^4} T_4^{(0)} T_4^{(0)} \tilde{D}^0_{\mu_1 \rho_1 \lambda_1 \sigma_1} \tilde{D}^0_{\rho_1 \lambda_1 \sigma_1} \tilde{D}^0_{\lambda_1 \sigma_1} (-q_2) \tilde{D}^0_{\sigma_1} (q - q_1 + q_2). \tag{4.2}
\end{equation}
where it is assumed that the summation over color group factors has been already done and is included into the coupling constant, since these numbers are not important. The summation over Dirac indices then yields

\[ T_{ν_1μ_1}^{(2)}(q) = -iδ_{ν_1μ_1}g^4 \int id^4q_1 \int id^4q_2 \frac{A(q_1^2)A(q_2^2)A((q - q_1 + q_2)^2)}{q_1^4 q_2^2 (q - q_1 + q_2)^2} = -iδ_{ν_1μ_1}g^4 F_2'(q^2), \]  

(4.3)

where we introduce

\[ F_2'(q^2) = \int id^4q_1 \int id^4q_2 \frac{A(q_1^2)A(q_2^2)A((q - q_1 + q_2)^2)}{q_1^4 q_2^2 (q - q_1 + q_2)^2}. \]  

(4.4)

In order to introduce a mass gap, which determines the deviation of the full gluon propagator from the free one in the IR region, at the level of the separate diagram (contribution), let us present the last integral as a sum of four terms, namely

\[ F_2'(q^2) = \sum_{n=1}^{n=4} F_2'^{(n)}(q^2), \]  

(4.5)

where

\[ F_2'^{(1)}(q^2) = \int_0^{Δ^2} id^4q_1 \int_0^{Δ^2} id^4q_2 \frac{A(q_1^2)A(q_2^2)A((q - q_1 + q_2)^2)}{q_1^4 q_2^2 (q - q_1 + q_2)^2}, \]  

(4.6)

\[ F_2'^{(2)}(q^2) = \int_0^{Δ^2} id^4q_1 \int_0^{Δ^2} id^4q_2 \frac{A(q_1^2)A(q_2^2)A((q - q_1 + q_2)^2)}{q_1^4 q_2^2 (q - q_1 + q_2)^2}, \]  

(4.7)

\[ F_3'(q^2) = \int_0^{Δ^2} id^4q_1 \int_0^{Δ^2} id^4q_2 \frac{A(q_1^2)A(q_2^2)A((q - q_1 + q_2)^2)}{q_1^4 q_2^2 (q - q_1 + q_2)^2}, \]  

(4.8)

\[ F_2'^{(4)}(q^2) = \int_0^{Δ^2} id^4q_1 \int_0^{Δ^2} id^4q_2 \frac{A(q_1^2)A(q_2^2)A((q - q_1 + q_2)^2)}{q_1^4 q_2^2 (q - q_1 + q_2)^2}, \]  

(4.9)

where not loosing generality we introduced the common mass gap squared $Δ^2$ for both loop variables $q_1^2$ and $q_2^2$. The integration over angular variables is assumed.

We are especially interested in the region of all the small gluon momenta involved $q ≈ q_1 ≈ q_2 ≈ 0$. However, in Eq. (4.6) we can formally consider the variables $q_1$ and $q_2$ as much smaller than the small gluon momentum $q$, i.e., to approximate $q_1 ≈ δ_1q$, $q_2 ≈ δ_2q$, so that $q - q_1 + q_2 ≈ q(1 + δ)$, where $δ = δ_2 - δ_1$. To leading order in $δ$, one obtains

\[ F_2'^{(1)}(q^2) = -\frac{A(q^2)}{q^2} \int_0^{Δ^2} dq_1^2 \int_0^{Δ^2} dq_2^2 A(q_1^2)A(q_2^2), \]  

(4.10)

where all the finite numbers after the trivial integration over angular variables will be included into the numerical factors below, for simplicity. Since $q^2$ is small, we can replace the dimensionless function $A(q^2)$ by its Taylor expansion as follows: $A(q^2) = A(0) + a_1(q^2/Δ^2) + O(q^4)$. Introducing further dimensionless variables $q_1^2 = x_1 Δ^2$ and $q_2^2 = x_2 Δ^2$, one finally obtains

\[ F_2'^{(1)}(q^2) = -\frac{Δ^4}{q^2} c_1 - Δ^2 c_1' + O(q^2), \]  

(4.11)

where

\[ c_1 = A(0) \int_0^1 dx_1 A(x_1) \int_0^1 dx_2 A(x_2), \]  

(4.12)
and \( c'_1 = a_1(c_1/A(0)) \).

In Eq. (4.7) it makes sense to approximate \( q_2 \approx \delta_5 q_1 \), \( q \approx \delta_2 q_1 \), so that \( q - q_1 + q_2 \approx q_1(1 + \bar{\delta}) \), where \( \bar{\delta} = \delta_4 - \delta_5 \). To leading order in \( \bar{\delta} \) and omitting some algebra, one finally obtains

\[
F_2^{(2)}(q^2) = -\Delta^2 c_2(\nu) + O(q^2),
\]

(4.13)

where

\[
c_2(\nu) = \int_1^{\nu} \frac{dx_1}{x_1} \int_0^1 dx_2 A(x_2),
\]

(4.14)

and \( \nu \) is the dimensionless auxiliary UV cut-off.

In Eq. (4.8) it makes sense to approximate \( q_1 \approx \delta_4 q_2 \), \( q \approx \delta_5 q_2 \), so that \( q - q_1 + q_2 \approx q_2(1 + \bar{\delta}) \), where \( \bar{\delta} = \delta_5 + \delta_6 \). To leading order in \( \delta \) and similar to the previous case, one obtains

\[
F_2^{(3)}(q^2) = -\Delta^2 c_3(\nu) + O(q^2),
\]

(4.15)

where

\[
c_3(\nu) = \int_1^{\nu} \frac{dx_2}{x_2} \int_0^1 dx_1 A(x_1).
\]

(4.16)

The last term (4.9) is left unchanged, since all loop variables are big. Conventionally, we will call it as the PT part of the contribution (diagram), i.e., denoting \( F_2^{(4)}(q^2) \) as \( T_2^{PT}(q^2) \). Since \( A(x) \) is regular at zero, the both integrals in Eqs. (4.14) and (4.16) are logarithmic divergent.

Summing up all terms, one obtains

\[
T_2(q) \equiv T_{\nu_1\mu_1}^{(2)}(q) = i \delta_{\nu_1 \mu_1} \left[ \frac{\Delta^4}{q^4} c_1 + \Delta^2 (c'_1 + c_2(\nu) + c_3(\nu)) \right] g^4 + O(q^2).
\]

(4.17)

The term \( T_2^{PT}(q^2) \) is hidden in terms \( O(q^2) \). Here the characteristic mass scale parameter \( \Delta^2 \) is responsible for the nontrivial dynamics in the IR domain. Let us also emphasize that the limit \( \nu \to \infty \) should be taken at the final stage.

In the same way can be decomposed the simplest one-loop contribution of the order \( g^2 \) into the quark self-energy provided by a three-gluon vertices insertion, see Eq. (3.11). Omitting all the algebra, it is instructive to present it in the similar to Eq. (4.17) form, namely

\[
T_1(q) \equiv T_{\nu_1\mu_1}^{(1)}(q) = -i \delta_{\nu_1 \mu_1} \Delta^2 c_4 g^2 + O(q^2),
\]

(4.18)

where \( c_4 \) is some finite number. The contribution of the \( T_2(q) \) term given in Eq. (3.12) can be given by the estimate similar to the estimate (4.18) with different finite coefficients, of course. Evidently, instead of \( g^2 \) it should be multiplied by \( g^4 \). Let us note here that in dimensional regularization the constant tadpole term (3.10) in the pure PT iteration solution (i.e., at the order \( g^2 \), which means \( D = D^0 + ... \)) of the gluon SD equation (3.1) can be generally discarded. Thus, it itself is not important at all.

Two principal distinction of the estimate (4.17) from the estimate (4.18) should be underlined. First, in the latter estimate there is only constant contribution, i.e., there is no singular with respect to \( q^2 \) term. Secondly, this contribution does not depend on the UV cut-off. This means that in the final limit when the UV cut-off will go to infinity it will be suppressed. This observation underlines the role of the four-gluon interactions in the IR structure of the true QCD vacuum (see below). The three-gluon proper vertex vanishes when all the gluon independent momenta involved go to zero, i.e., \( T_3(0,0) \to T_3^0(0,0) = 0 \). At the same time the four-gluon proper vertex survives in this limit, i.e., \( T_4(0,0,0) \to T_4^1(0,0,0) \neq 0 \). This is the main dynamical reason in the above-mentioned distinction between the functional estimates (4.17) and (4.18). Evidently, such kind of the auxiliary procedure, described in this Sec., is appropriate only for the establishing the most singular terms in the deep IR asymptotics of the full gluon propagator.
V. SEVERE IR STRUCTURE OF THE QCD VACUUM

At the NL two-loop level, i.e., at the order $g^4$, there is a number of the additional diagrams, which, however, contain the three-gluon vertices (plus the two-tadpole diagram) along with the four-gluon ones. As mentioned above, their contributions into the deep IR structure of the gluon propagator are given by the estimates similar to the estimate (4.18) with different coefficients. Summing up all these estimates, and omitting some really tedious algebra, the full gluon propagator up to the first iteration (including all the terms of the order $g^2$ and $g^4$) can be written as

$$D_{\mu\nu}(q) = i\delta_{\mu\nu}\left[\frac{\Delta^2}{(q^2)^2}a_1 + \frac{\Delta^4}{(q^2)^3}a_2 + \ldots\right] + D_{\mu\nu}^{PT}(q) = D_{\mu\nu}^{J\text{NP}}(q) + D_{\mu\nu}^{PT}(q),$$

(5.1)

where $a_1$, $a_2$ are, in general, the short-hand notations for a sums of the different coefficients, which include the coupling constant squared in the corresponding powers. Moreover, some of these coefficients contain the divergent integrals (see, for example Eqs. (4.14) and (4.16)). Here $D_{\mu\nu}^{PT}(q)$ denotes the contribution from the PT part of the full gluon propagator, since it is of the order $O(q^{-2})$ as $q^2 \to 0$. The superscript "INP" stands for the intrinsically NP part of the full gluon propagator (for the exact definition see below). Due to the above-mentioned distinction between the behavior of the tree- and four-gluon vertices in the deep IR domain, the coefficients $a_1$, $a_2$ are, in general, not zero. In other words, there is no way to cancel $D_{\mu\nu}^{J\text{NP}}(q)$ by performing the functional estimate at every order of the QCD coupling constant squared. In the deep IR region the quark and ghost degrees of freedom (the $A(q^2)$ function) are taken into account in all orders of linear PT numerically, i.e., they are simply numbers. As functions they can contribute into the PT part only of the full gluon propagator. So, in the first approximation the gluon propagates like Eq. (5.1) and not like the modified free one (3.3), though we just started from it.

The true QCD vacuum is really beset with severe (i.e., more singular than $1/q^2$ as $q^2 \to 0$) IR singularities if standard PT is applied. Moreover, each severe IR singularity is to be accompanied by the corresponding powers of the mass gap, responsible for the NP dynamics in the IR region. In more complicated cases of the multi-loop diagrams (i.e., the next iterations in Eq. (4.1)) more severe IR divergences will appear. The coefficients at each severe IR singularity become by themselves an infinite series in the coupling constant squared, and the coefficients of these expansions may depend on the gauge fixing parameter as well. These coefficients include numerically the information about quark and ghost degrees of freedom in all orders of linear PT, as underlined above.

It is worth emphasizing, however, that the ZMME effect in the QCD vacuum, which is explicitly shown in Eq. (5.1) in the Feynman gauge, can be demonstrated in any covariant gauge, for example in the Landau one $\xi = 0$. In other words, this effect itself is gauge-invariant, though the finite sum of all the relevant diagrams in the deep IR region at the same order of the coupling constant squared may be not. Let us also remind that this effect is not something new. It has been well known for a long time from the very beginning of QCD, and it was the basis for the proposed then IR slavery (IRS) mechanism of quark confinement. Just this violent IR behavior makes QCD as a whole an IR unstable theory, and therefore it has no IR stable fixed point, indeed.

The existence of a severe IR singularities automatically requires an introduction of a mass gap, responsible for the nontrivial dynamics in the IR region. This is important, since there is none explicitly present in the QCD Lagrangian (the current quark mass cannot be considered as a mass gap, since it is not renormalization group invariant). It precisely determines to what extent the full gluon propagator effectively changes its behavior from the behavior of the free one in the IR domain. The phenomenon of "dimensional transmutation" only supports our general conclusion that QCD may exhibit a mass, determining the characteristic scale of the NP dynamics in its ground state. Of course, such gluon field configurations, which are to be described by severely IR structure of the full gluon propagator, can be only of dynamical origin. The only dynamical mechanism in QCD which can produce such configurations in the vacuum, is the self-interaction of massless gluons – the main dynamical NL effect in QCD. Hence, the above-mentioned mass gap appears on dynamical ground. Let us remind that precisely this self-interaction in the UV limit leads to AF.

We have explicitly shown that the low-frequency components of the virtual fields in the true vacuum should have larger amplitudes than those of a PT ("bare") vacuum, indeed. "But it is to just this violent IR behavior that we must look for the key to the low energy and large distance hadron phenomena. In particular, the absence of quarks and other colored objects can only be understood in terms of the IR divergences in the self-energy of a color bearing objects." So, let us introduce the following definitions:

(i) The power-type IR singularity which is more severe than the exact power-type IR singularity of the free gluon propagator will be called a severe (or equivalently NP IR) singularity. In other words, the NP IR singularity is more severe than $1/q^2$ at $q^2 \to 0$.

(ii) At the same time, the IR singularity which is as much singular as the exact power-type IR singularity of the free gluon propagator, i.e., as much singular as $1/q^2$ at $q^2 \to 0$, will be called PT IR singularity.
It makes worth emphasizing in advance that the decomposition of the full gluon propagator into the INP and PT parts (5.1) can be made exact \cite{20, 21}. From the distribution theory point of view the NP IR singularities defined above present a rather broad and important class of functions with algebraic singularities \cite{22}. This explicit derivation shows how precisely the NP IR singularities, accompanied with a mass gap in the corresponding powers, may appear in the vacuum of QCD. Thus, the NP IR singularities should be summarized (accumulated) into the full gluon propagator and effectively correctly described by its structure in the deep IR domain, presented by its INP part. The second step is, of course, to assign a mathematical meaning to the integrals, where such kind of the NP IR singularities will finally survive.

One important thing should be made perfectly clear. In the exact calculation of a separate diagram the dependence on the characteristic masses (determining the deviation of the full gluon propagator from the free one in the IR and UV regions) is hidden. In other words, these masses cannot be "seen" by the calculation of the finite number of diagrams, which may be not even gauge-invariant. An infinite number of the corresponding diagrams should be summed up in order to trace such NP masses (i.e., to go beyond PT). The final result of such summation should, in principle, be gauge-invariant. In the weak coupling regime we know how to do this with the help of the renormalization group equations. As a result, the dependence on $\Lambda_{QCD} \equiv \Lambda_{PT}$ will finally appear. At the same time, we do not know how to solve these equations in the strong coupling regime. So, in order to avoid this problem, we decided to show the existence of a mass gap explicitly, by extracting the deep IR asymptotics of the gluon propagator within the separate relevant diagrams. The rest of the problem is to sum up an infinite number of the most singular terms (contributions) in order to see whether or not a mass gap will finally survive.

A. Gluon confinement criterion

Precisely this program has been carried out in Refs. \cite{20, 21}. For the sake of completeness, let us repeat it briefly here. We will show that a mass gap remains, indeed. Thus, on general ground one has

$$D^{INP}(q^2, \Delta^2) = \sum_{k=0}^{\infty} (q^2)^{-2-k}(\Delta^2)^{k+1} \sum_{m=0}^{\infty} a_{k,m}(\xi)g^{2m}, \quad (5.2)$$

which is the Laurent expansion with respect to the inverse powers of $q^2$ for the INP part of the full gluon propagator. The crucial observation is that the regularization of the NP IR singularities does not depend on their powers \cite{20, 21, 22}, namely

$$(q^2)^{-2-k} = \frac{1}{\epsilon} a(k)[\delta^4(q)]^{(k)} + f.t., \quad \epsilon \to 0^+; \quad (5.3)$$

where $a(k)$ is a finite constant depending only on $k$ and $[\delta^4(q)]^{(k)}$ represents the kth derivative of the $\delta$-function. Here $\epsilon$ is the IR regularization parameter, introduced within the dimensional regularization (DR) method \cite{23}, and which should go to zero at the end of the computations. In Ref. \cite{20} it has been proven that neither $g^2$ nor the gauge fixing parameter $\xi$ is to be IR renormalized, i.e., they are IR finite from the very beginning. So, in the Laurent expansion (5.2) the only quantity which should be IR renormalized is the mass gap itself. It is easy to show that it is IR renormalized as follows: $\Delta^2 = \epsilon \Delta^2$ as $\epsilon$ goes to zero. In this case, all the singularities with respect to $\epsilon$ will be cancelled in the Laurent expansion (5.2), indeed, and everything will be expressed in terms of the IR renormalized mass gap $\Delta^2$ only, which, by definition, exists as $\epsilon$ goes to zero (let us remind that $g^2 = \bar{g}^2$ and $\xi = \bar{\xi}$). Moreover, it is perfectly clear that only the simplest NP IR singularity will survive in the $\epsilon \to 0^+$ limit, namely

$$D^{INP}(q^2, \Delta^2) = \Delta^2 (q^2)^{-2} \sum_{m=0}^{\infty} a_{0,m}(\xi)g^{2m}, \quad (5.4)$$

and all other terms in the expansion (5.2) become terms of the order of $\epsilon$, at least, in this limit (they start from $(\Delta^2)^2 \sim \epsilon^2$, while $(q^2)^{-2-k}$ always scales as $1/\epsilon$). The so-called "f.t." terms in the dimensionally regularized Laurent expansion (5.3) after its substitution into expansion (5.2) become terms of the order of $\epsilon$, so here and everywhere they vanish in the $\epsilon \to 0^+$ limit.
Due to the distribution nature of the simplest NP IR singularity \((q^2)^{-2}\), which saturates the INP part of the full gluon propagator, the two different cases should be distinguished.

I. If there is an explicit integration over the gluon momentum (the so-called virtual gluon due to Mandelstam\(^{10}\)), then from Eq. (5.4), on account of Eq. (5.3) with \(a(0) = \pi^2\), it finally follows

\[
D^{INP}(q, \Delta^2) = \Delta^2 \pi^2 \delta^4(q).
\]

The \(\delta\)-type regularization is valid even for the multi-loop skeleton diagrams, where the number of independent loops is equal to the number of the gluon propagators. In the multi-loop skeleton diagrams, where these numbers do not coincide (for example, in the diagrams containing three or four-gluon proper vertices), the general regularization (5.3) should be used (i.e., derivatives of the \(\delta\)-functions). In Eq. (5.5) an infinite series over \(m\) is included into the IR renormalized mass gap \(\bar{\Delta}^2\) (retaining the same notation, for simplicity), making thus it the UV renormalized and gauge-invariant as well. Let us remind that some of these quantities are UV divergent.

II. If there is no explicit integration over the gluon momentum (the so-called actual gluon\(^{11}\)), then the function \((q^2)^{-2}\) in Eq. (5.4) cannot be treated as the distribution, and only the IR regularization of a mass gap comes out into the play. So the INP part of the full gluon propagator in this case disappears as \(\epsilon\) as \(\epsilon \to 0^+\), namely

\[
D^{INP}(q, \bar{\Delta}^2) \sim \epsilon, \quad \epsilon \to 0^+.
\]

This means that any amplitude for any number of soft-gluon emissions (no integration over their momenta) will vanish in the IR limit in our picture. In other words, there are no gluons in the IR, i.e., at large distances (small momenta) there is no possibility to observe gluons experimentally as free particles. So color gluons can never be isolated. This behavior can be treated as the gluon confinement criterion (see also Ref. \(^{24}\)). Evidently, this behavior does not explicitly depend on the gauge choice in the full gluon propagator, i.e., it is a manifestly gauge-invariant as it should be. It is worth mentioning that it coincides with the color confinement criterion proposed in Ref. \(^{25}\) (and references therein, see also our paper \(^{26}\)).

\section{VI. DISCUSSION}

\subsection{A. A possible generalization}

A tensor structure of each term, which contributes into the NL part of the full gluon propagator, is absolutely the same as the tensor structure of the linear contributions, i.e., the quark and ghost skeleton loops. These structures are given in Eqs. (3.6) and (3.7). However, the dynamical context of these decompositions, which is present in the corresponding invariant functions, may be distinguished. In analogy with the relations (3.6) and (3.7) and because of the Laurent expansion (5.1), the NL part \(T^g_{\mu\nu}[D](q)\) can be generally decomposed as follows:

\[
T^g_{\mu\nu}[D](q) = \delta_{\mu\nu} \left[ \frac{\Delta^4}{q^2} T^{(1)}_g(q^2) + \Delta^2 T^{(2)}_g(q^2) + q^2 T^{(3)}_g(q^2) \right] + q_\mu q_\nu \left[ \frac{\Delta^2}{q^2} T^{(4)}_g(q^2) + T^{(5)}_g(q^2) \right],
\]

where \(T^{(n)}_g(q^2)\) at \(n = 3, 5\) are invariant dimensionless functions. They are regular functions of \(q^2\), i.e., they can be present by the corresponding Taylor expansions, but possessing AF at infinity, and depending thus on \(\Lambda_{QCD}\) in this limit. At the same time, the invariant dimensionless functions \(L^{(n)}_g(q^2)\) at \(n = 1, 2, 4\) are to be present by the corresponding Laurent expansions, namely

\[
L^{(1,2,4)}_g(q^2) \equiv L^{(1,2,4)}_g(q^2, \Delta^2) = \sum_{k=0}^{\infty} (\Delta^2/q^2)^k a^{(1,2,4)}_k,
\]

where the numbers \(a^{(1,2,4)}_k\) by themselves are expansions in the coupling constant squared (see below). Let us emphasize the inevitable appearance of the mass gap \(\Delta^2\). It characterizes the nontrivial dynamics in the IR region. This precisely makes the difference between the linear and NL insertions into the gluon self-energy. Evidently, this difference is due to different dynamics: in the linear part there is no explicit direct interaction between massless gluons, while in the NL part there is. When the mass gap is zero then this decomposition takes the standard form. So, the generalization (6.1) makes the explicit dependence on the mass gap of the full gluon propagator perfectly clear.
In principle, the estimates like the estimates derived in the previous Sec. 4 can be extended to the quark and ghost skeleton loops as well. This means that we can formally generalize the decompositions (3.6) and (3.7) in a similar way as the decomposition of the NL part (6.1). However, this is not the case, since we know the sum of the linear contributions. If the invariant function $T(q^2)$ in Eq. (3.8) may contain singular with respect to $q^2$ terms (because of a possible formal generalization), then the IR singularity of the modified free gluon propagator (3.3) will be only softened in comparison with the IR singularity of the free gluon propagator (2.2). In general, such behavior will only compromise the role of ghosts to cancel unphysical degrees of freedom of gauge bosons (see explanation below). That is why we believe that all the dependence on the mass gap which possibly can come from the quark and ghost skeleton loops can be factorized into the separate block (the linear part), which itself cannot produce severe IR singularities in the full gluon propagator. At the same time, we don’t know the sum of the NL contributions. This also makes the difference between the linear and NL parts, but now from a mathematical point of view. Thus, it does not make any sense to generalize the ghost and quark decompositions (3.6) and (3.7), respectively, in the same way as the decomposition (6.1). They can contribute into the INP part of the full gluon propagator numerically only, and as a functions they contribute into its PT part. Neither ghost nor quark skeleton loops (3.4) and (3.5), which appear in Eq. (3.1), can cancel its severely singular behavior in the IR, which was demonstrated above, and which was due to the pure YM part (i.e., to its NL part) of the gluon SD equation (3.2).

**B. The role of ghosts**

It is well known that in order to maintain the unitarity of $S$-matrix in QCD the ghosts have to cancel unphysical degrees of freedom (longitudinal ones) of the gauge bosons $\frac{1}{i/k^2}$. Evidently, this is due to the general decomposition of the ghost skeleton loop (3.6), which shows that it always gives the contribution of the order $q^2$. In the iteration solution for the gluon propagator (2.4), $D(q) = D^0(q) - D^0(q)T_{gh}(q)D^0(q) + D^0(q)T_{gh}(q)D^0(q)T_{gh}(q)D^0(q) + \ldots$, it cancels one of $q^2$ in the denominator, which comes from the free gluon propagator. Thus, each term in this expansion becomes always as singular as $1/q^2$. Precisely this makes it possible for ghosts to cancel, in general, the longitudinal component of the full gluon propagator, which is, by definition, as singular as $1/q^2$. From a technical point of view the cancellation can be explicitly demonstrated in the lowest orders of PT in powers of the coupling constant squared (see, for example Ref. [7]). However, this is valid term by term in PT. In other words, in every order of PT the ghosts will cancel unphysical degrees of freedom of gauge bosons, making thus them always transverse. The above-mentioned cancellation in all orders of PT means that it goes beyond PT. It is a general feature, i.e., it does not depend on whether the solution, for example to the gluon SD equation is PT or NP, singular or regular at origin, etc. In other words, the general role of ghosts should not be spoiled by any truncation scheme (approach).

On the other hand, the general decomposition (3.6) of the ghost skeleton loop (3.4) takes place if and only if (iff) the full ghost propagator (Euclidean signature) $G(k) = -(i/k^2[1 + b(k^2)])$, where $b(k^2)$ is the ghost self-energy, is as singular as $1/k^2$ at $k^2 \to 0$. When the ghost self-energy is zero, i.e., $b(k^2) = 0$, then the full gluon propagator becomes the free one, i.e., $G(k) \to G_0(k) = -(i/k^2)$. Thus the IR singularity of the full ghost propagator cannot be more severe than the exact IR singularity of the free ghost propagator in order to maintain the cancellation role of unphysical degrees of freedom of gauge bosons by ghosts at any nonzero covariant gauge in all sectors of QCD. There is no way for ghosts to cancel severe IR singularities, which are of dynamical origin due to the self-interaction of massless gluons in the true QCD vacuum. There is no doubt left that the full gluon propagator is essentially severely modified in the IR because of the response of the NP QCD vacuum, which is not provided by the PT vacuum.

However, there exists one gap in these arguments. If one chooses by hand the Landau gauge $\xi = 0$ from the very beginning, then the unphysical longitudinal component of the full gluon propagator vanishes. Only the physical transverse component will contribute to the full gluon propagator, and it may become regular at zero in this case, indeed. Otherwise, it is always singular at the origin because the existence of the longitudinal component always produces, at least, the IR singularity $1/q^2$ (see Eq. (2.1)). In this case there is no restriction on the behavior of the ghost propagator in the IR, and it may become (depending on the truncation scheme) more singular in the IR than its free counterpart. In Ref. [24] (and references therein) precisely this type of the solution (regular gluon propagator and more singular than the free one ghost propagator) to the system of the SD equations in the Landau gauge has been found. However, this solution is due to the choice of the special Landau gauge, so it is a gauge artifact solution. Being thus a gauge artifact, it can be related to none of the physical phenomena such as quark and gluon confinement, SBCS, etc, which are, by definition, manifestly gauge-invariant. At the same time, gauge artifact solutions may exist as formal solutions to the SD system of equations. If a regular at zero gluon propagator will be found in a manifestly gauge-invariant way (i.e., in the way which does not explicitly depend on the particular covariant or non-covariant gauge choice), only then it should be taken seriously into the consideration. To our present knowledge a manifestly gauge-invariant solution for the smooth gluon propagator, which will not compromise the general role of ghosts, is not yet found. Moreover, there exists a serious doubt, in our opinion, that such kind of the solution can be found at
all. Thus, we are left with singular at the origin gluon propagator, which is possible in any covariant gauge.

VII. CONCLUSIONS

Emphasizing the importance of the IR structure of the full gluon propagator and its close relation to the highly nontrivial structure of the true QCD ground state, one can conclude:

1). The self-interaction of massless gluons (i.e., the NL gluodynamics) is responsible for the large scale structure of the true QCD vacuum.

2). The full gluon propagator in any gauge is inevitably more singular in the IR than its free counterpart.

3). This requires the existence of a mass gap, which is responsible for the NP dynamics in the QCD vacuum. It appears on dynamical ground due to the self-interaction of massless gluons only. It cannot be interpreted as the effective/dynamical gluon mass.

4). Though a mass gap gains contributions from all powers of the QCD coupling constant squared, it itself plays no any role in the presence of a mass gap.

5). We define the NP and the PT IR singularities as more severe than and as much singular as $1/q^2$, respectively, which is the power-type, exact IR singularity of the free gluon propagator.

6). On this basis we exactly decompose the full gluon propagator into the INP and PT parts.

7). An IR renormalization of a mass gap only is needed in order to fix uniquely and exactly the INP part of the full gluon propagator. It is saturated by the simplest NP IR singularity, the famous $(q^2)^{-2}$.

8). The main dynamical source of the NP (severe) IR singularities in the full gluon propagator is the two-loop skeleton term of the corresponding SD equation, which contains the four-gluon vertices only.

9). The regular at zero full gluon propagator is to be ruled out. Only its PT part can be rendered regular at zero due to the special gauge choice (Landau gauge).

10). As a functions ghost and quark degrees of freedom contribute only into the PT part of the full gluon propagator within our approach. As integrated out in all orders of linear PT (i.e., numerically) they contribute to its INP part as well.

11). Taking into account the distribution nature of the NP IR singularities, the gluon confinement criterion is formulated in a manifestly gauge-invariant way.

Our general conclusions can be formulated as follows: at the microscopic, dynamical level just the fundamental NL four-gluon interaction makes the full gluon propagator so singular in the IR. This requires the introduction of a mass gap, i.e., it arises mainly from the quartic gluon potential (see also Refs. [27, 28]).

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