Analysis on Finite Element Grid Size and Discrete Error for Super-high Arch Dams

Yizhi Yan¹ᵃ*, Wanju Zhang¹ᵇ, Zhimin Shen¹ᵉ, Zimeng Li¹ᵈ, Yunpeng Wei¹ᵃ, Wenxiong Wang¹ᶠ

¹Electrical Power Engineering Building, Kunming University Of Science and Technology, Kunming, Yunnan Province, China.

ᵃ*Corresponding author’s email: 20040166@kust.edu.cn
ᵇemail: 1126900475@qq.com, ᶜemail: 2073244282@qq.com, ᵈemail: 1125150859@qq.com, ᵉemail: 15535108953@163.com, ᶠemail: 211962955@qq.com

Abstract: Compared with the world super-high arch dams with height more than 200 meters, the six super-high arch dams already built or under construction in China share the same special body shapes with higher ratios for both top/bottom width and thickness, and for top arch length with dam height. Also all of them are subjected to big top-level water pressures. In the paper, the computational formulation of thermal-coupled structure of hollow sphere similar to arch dam in shape and load-subjected is derived first and analysis of errors from the finite element method compared with the analytic solution is made. Then, a typical super-high arch dam is taken as the example of structure analysis including dam body mesh division, stress distribution, error estimation. Employing quadrilateral Coupled-field tetrahedral element with size decided by the bottom thickness multiple of arch crown beam, performance of mesh division and analysis for complex model is done conveniently and fast. Also, the paper investigates the error and convergence of results with different mesh density, discusses the special stress states of arch dam, and proposals acceptable dam body element size from calculation accuracy, convenient operation and elapsed time when performance is carried on with strong coupling on common personal computers. The research work indicates that errors of analysis results in most part of area are very small excepting those for the local singularity and concentration part when the dam body element size is decreased to 0.25 multiple of bottom thickness of arch crown beam, with dam body energy error norm less than 10% and consuming a couple of elapsed time. The conclusion should be common for the arch dams in China as they have similar shape.

1. Introduction

Some books call arch dam with more than 200m dam height for "super-high arch dam", and this is accepted by famous water resource experts[1]. In this paper, it is also the meaning of the super-high arch dam. At present, there are more than 20 super-high arch dams built and being built in the world. Among them there are 6 dams in China located in Ertan, Xiaowan, Goupitan, Xiluodu, Laxiwa and Jinping hydropower stations. And higher arch dams like Baihetan and Longpan will also be constructed. Compared with other super-high arch dams, the similar dams in China appear same special shape characteristics with higher ratios for both top/bottom width and thickness, and for top arch length with dam height. Also all of them are subjected to big top-level water pressures. They are all located on valleys of "V" shape or "U" shape. Their top-bottom thickness ratio is about 5 and arc
length-height ratio is about 2.6. The average top-level water pressure is 127.8 GN about 2.35 times of number for similar dams in other countries. In order to ensure the safety of such a huge structure, precise calculation and analysis of its mechanical behavior is very important.

When the structure mechanics behavior is analyzed using the quadratic coupled elements of 3D finite element method, general tetrahedron and hexahedron elements almost have the same element quality[2]. The computation cost for hexahedral element dividing computational domain is much less than that of tetrahedron element, as the element size is basically the same, or to say the calculation precision is almost equal[3]. However, when hexahedron element is used to mesh the special shape body like the super-high arch dam with big size differences in three directions, it is not easy to ensure the grid size rationality that is element edge ratio suitable or artificial cost is big and need to be supplemented by special element like pentahedron element. That may be difficulty for general finite element software. However, if tetrahedron element is employed, the mesh work and grid shape requirement are very easy to be done. Although the calculation elapsed time in that case may increase, this part of work is handed to the computer to do which can undergo it.

In this paper, the first work is that analytic formula for hollow sphere thermal-coupled structure with similar shape to super-high arch dam is derived, and the theoretical solution and the error of finite element solution are compared and analyzed. On the basis of these, then, a super-high arch dam in China is employed to be an example for mesh size and discrete error analysis. In the case, the scale size of crown cantilever bottom thickness of dam is used to be the indicator for mesh density of dam body divided by general tetrahedron quadratic coupled element. The operation of dam body meshing in different density is done in convenience and efficiency with that kind of element. The paper investigates discrete error and convergent information among different mesh density cases and analyses stress features of arch dam. Moreover, it put forward reasonable grid size of dam body for calculation precision, convenient operation and calculation cost to be accepted when the task is done to analyze mechanical behavior of super-high arch dam using tetrahedron quadratic element and strong-coupled method in personal computer.

2. Thermal-coupled Stress Equation of Hollow Sphere

Assuming that there is a hollow sphere with inside radius a and outside radius b. Its inner surface suffers from temperature \( T_a \). And its outside surface suffers from temperature \( T_b \) and radial stress \( \sigma \).

This moment, Sphere inner temperature and stress just change with radius \( r \). It is a one-dimensional problem. Under the spherical coordinates, radial heat conduction equation, force balance equation and boundary conditions are as follows[4-5]:

\[
\frac{d^2 T}{dr^2} + \frac{2}{r} \frac{dT}{dr} = 0 \quad (1)
\]

\[
\frac{d\sigma}{dr} + \frac{2}{r} (\sigma_r - \sigma) = 0 \quad (2)
\]

\[
T_{\text{a}} = T_a, \quad T_{\text{b}} = T_b \quad (3)
\]

\[
\sigma_{\text{r}} = 0, \quad \sigma_{\text{r-b}} = -\rho \quad (4)
\]

Substitute stress-strain and strain-displacement into the above expression and use the method of integral to get the sphere temperature-stress coupled field formula as follows:

\[
T = \frac{(T_a - T_b) ab}{b-a} + \frac{T_b - T_a}{b-a} \quad a \leq r \leq b \quad (5)
\]

\[
\sigma_r = \frac{2\alpha E}{1-\nu} \left[ \frac{(r^3 - a^3)}{(b^3 - a^3)} \frac{1}{r^2} \int_r^{b} dr \frac{1}{2} \int_2^{r} \frac{(r^3 - a^3)}{(b^3 - a^3)} \right] \quad (6)
\]

\[
\sigma_a = \frac{2\alpha E}{1-\nu} \left[ \frac{2(r^3 + a^3)}{(b^3 - a^3)} \frac{1}{2} \int_r^{b} dr \frac{1}{2} \int_2^{r} \frac{(r^3 + a^3)}{(b^3 - a^3)} \right] \quad (7)
\]

In the expressions:

\[
\int_r^{b} \frac{1}{2} \int_2^{r} \frac{(r^3 + a^3)}{(b^3 - a^3)} \sigma \quad (8)
\]
\[
\int T r^2 dr = \frac{1}{2} \left( \frac{T - T_0}{b - a} \right) ab \left( r^2 - a^2 \right) + \frac{1}{3} \left( \frac{T - T_0}{b - a} \right) ab \left( r^2 - a^2 \right)
\]  

(8)

\(T\) is the temperature. \(\sigma_r\) and \(\sigma_\theta\) are the radial stress and the tangential stress respectively with tension for positive and pressure for negative. \(E\), \(\nu\) and \(\alpha\) are elasticity moduli, poisson ratio and thermal expansion coefficient of sphere material respectively.

3. Stress Calculation and Error Evaluation of the Finite Element Method

3.1. Strong Coupling Method and Stress Calculation

The element adopted in the paper is 10 nodes tetrahedron quadratic coupled ordinary isoparametric element. Every node has 4 degree of freedom. They are temperature and displacement in three directions. The element stiffness matrix is calculated by 4 gauss integral points. After meshing the temperature and displacement field of computational domain by the element in common way, the strong coupling discrete equations can be gained as follow[6]:

\[
\begin{bmatrix}
K^s & K^u \\
0 & K^\nu
\end{bmatrix}
\begin{bmatrix}
\{u\} \\
\{T\}
\end{bmatrix} = \begin{bmatrix}
\{F^s\} \\
\{F^\nu\}
\end{bmatrix}
\]

(9)

The nodes temperature and displacement can be got by solving the equations. And the heat flow density and stresses at the gauss points also can be got. Then, the element node values of heat flow and stresses can be extrapolated from the follow liner formula:

\[
q = a + bs + ct + dr
\]

(10)

Where:

\(q\) is heat flow density or stresses of element nodes. The a, b, c, d are the coefficients confirmed by the values at four integral points. The s, t, r are bulk coordinates of nodes.

Finally, the heat flow density and stresses of node can be got from the element-node values around it by arithmetic average way.

3.2. Energy error norm

Energy error norm is a statistical parameter that can well reflect the global calculation accuracy and can be evaluated by follow formula:

\[
E = 100 \left( \frac{e}{U + e} \right)^{1/2}
\]

(11)

Where:

\(e = \sum_{j=1}^{N} e_j\), \(e_j = \frac{1}{2} \int_{V} \Delta \sigma^T D \Delta \sigma dV\), \(U = \sum_{j=1}^{N} U_j\), \(U_j = \frac{1}{2} \int_{V} \sigma^T D^{-1} \sigma dV\), \(\sigma = N \sigma_j, \Delta \sigma = N \Delta \sigma_{ij}, \Delta \sigma_{ij} = \sigma_i - \sigma_{ij}\)

\(m\) is the number of elements in computational domain, \(e\), \(U\), \(e_j\) and \(U_j\) are respectively strain energy error and strain energy of computational domain, and these of element, \(\sigma\) is the stress at any point within the element, \(N\) is the shape function matrix with the same case of displacement interpolation function, \(\sigma_i\) is node stress, \(\Delta \sigma\) is the stress error at any point within the unit, \(\Delta \sigma_{ij}\) is the error of node stress, \(\sigma_{ij}\) is the \(i\) node stress in element \(j\) from linear extrapolation.

| Entry | a(m) | b(m) | E(MPa) | \(\nu\) | k(MW/m·C\(^\circ\)) | \(\alpha\)/(C\(^\circ\)) | p(MPa) | T\(_a\)/(C\(^\circ\)) | T\(_b\)/(C\(^\circ\)) |
|-------|------|------|--------|--------|----------------|----------------|--------|----------------|----------------|
| Value | 300  | 360  | 2.0\times10^4 | 0.2    | 2.94\times10^{-6} | 1.0\times10^{-5} | 3      | 35             | 15             |
4. Analysis of Sphere Example and Super-high Arch Dam

4.1. Analysis of Sphere Example

Because of symmetry, one eighth of the sphere is taken as the finite element calculation domain. To study the super-high arch dam, geometry size and material parameters of computational domain are modeled on the super-high arch dam. Values of it are shown in table 1.

Taking the thickness of the sphere as the indicator of domain mesh size, twelve computational
schemes are performed respectively with different element size from 2, 1, 0.5, 0.3 and 0.25 times of the sphere thickness. Figure 1 presents the grid case that element size is 0.25 times of the thickness. Figure 2 is the comparison one of calculation results between the theoretical solution and finite element solution. Figure 3 shows the change of energy error norm as mesh density.

The calculation results show that the finite element solution is very close to the theoretical solution when grid element size is 0.25 times of the sphere thickness. The relative errors are very small as a whole and different from each point. Most stress relative errors are less than 1% and smaller for temperature. The energy error norm reaches 0.375%.

4.2. Example Analysis of Super-high Arch Dam

In this paper, a super-high arch dam in China is taken as a study case[7-8]. The dam body grid sizes that are 1, 0.5, 0.25 and 0.125 times of the bottom thickness of the dam are respectively calculated on the laptop whose memory is 4GB and CPU is 2GHz. The height of the dam is 278 meters. Its thickness-height ratio is 0.25 and arc-height ratio is 2.5. The material parameters are shown in table 2. And the calculation model is shown in figure 4. On the model, the left and right boundary suffers temperature of 15 degree and displacement is constrained in x direction. The bottom boundary suffers 15 degree and displacement is restricted in z direction. The displacement of downstream boundary is constrained in y direction, and the 35 degree temperature affects on the surface of downstream dam and river valley. At the same time, the upstream surfaces of dam and valley are subjected to the reservoir water temperature and pressure, and its temperature is evaluated by formula (12). The dam self-weight load is considered in the analysis. The calculation results summarized is shown in figure 5 and figure 6.

\[ T = \begin{cases} 
19 e^{-0.051 z} + 16 e^{0.031 z} \cos(-0.031 z) & 0 \leq z \leq 50m \\
9.002 & z > 50m 
\end{cases} \quad (12) 
\]

Where: \( T \) is the temperature of upstream dam surface. The \( z \) is the water depth in front of the dam.

The operation process and calculation results show that choosing the common tetrahedron quadratic mapped coupling element and taking the multiples of crown cantilever bottom thickness as element size to divide dam body grid can be very convenient and quick to implement on the ANSYS software. The dam body grid is uniform and boundary condition is easy to apply.

In operation, the element size on the outer surface of rock mass may be taken as large as possible to reduce amount of calculation. And this basically does not affect the calculation results of dam body.

Table 2. Material parameters of dam analysis model

| Parameter | Volume-weight \( \gamma \) (MN/m\(^3\)) | Elasticity modulus \( E \) (MPa) | Poisson ratio | Coefficient of linear expansion \( \alpha \) (1/°C) | Heat conductivity coefficient (W/m.C) |
|-----------|---------------------------------|-------------------------------|--------------|---------------------------------|-------------------------------|
| Concrete material | 2.4×10\(^2\) | 2.0×10\(^4\) | 0.167 | 1.0×10\(^{-5}\) | 2.94 |
| Bedrock materials | 2.2×10\(^2\) | 1.7×10\(^4\) | 0.2 | 1.0×10\(^{-5}\) | 2.94 |
Figure 5 presents the effect of dam element size on the calculation results including maximum displacement vector of dam, the Von Mises stress at the midpoint of crown cantilever bottom, dam strain energy error norm and the time consuming. As you can see from the figure, the displacement almost doesn’t change in spite of dam grid density multiple changes from 1 to 0.125. This indicates that even the coarse grid of 1 times the thickness of the dam bottom is enough if calculators only focus on displacements. The bottom midpoint stress is basically stable after the element size is less than 0.5 times of the base thickness which means that dam body element sizes need to be less than that size if concerning stress. Energy error norm reduces quickly with element size less and does not reach a stable value of convergence in the case like some other papers [10-11].

Elapsed time increases rapidly for the case of element size multiple less than 0.25, however the energy error norm reduces little. In the example, elapsed time jump from 6.5 minutes for multiple 0.25 to 32.4 minutes for multiple 0.125, while the energy error norm only decreases from 9.2% to 6.4%. That says dam element size with 0.25 times of bottom thickness is ok for the super-high arch dam analysis if the job is done on the laptop. In this example, that element sizes are equivalent to a layer of elements divided at the top of the dam and four layers of elements at the bottom. Analysis results in that grid of element size may produce a high calculation precision except stress singularity area around the corners of dam fringe. The reason that energy error norm dose not reduce to the extent like hollow sphere example may be the singular boundary area existed [12]. Figure 6 shows the Von Mises stress...
distribution of dam bottom under different element sizes where endpoint stresses increase rapidly with the decrease of the element sizes. That is conformed to the theory of elastic properties because the geometry and material properties in the region are changed suddenly and stresses will produce singular phenomenon but not converging. It is in keeping with the calculation results of gravity dam in some other papers[13-14].

5. Conclusion
According to the research situation in this paper, the quadratic tetrahedron element is used to calculate the mechanical behavior of heat-coupled structure of super-high arch dam, some conclusions can be drawn as follows:

(1) The effect is good for the element mentioned above to be used to analyze mechanical behavior of the super-high arch dam which owns special body shape with big thickness-height ratio and arc-height ratio. Mesh job is easy to be done quickly and conveniently. Grid density and element size are controlled well as you wish by using the bottom thickness multiple of crown cantilever to divide dam body. And the reasonable mesh is good for studying relationship between element size and discrete error.

(2) For finite element strong coupling method to analyze dam mechanical behavior on common laptop, as long as its element size about 0.25 times of dam bottom thickness is taken, the result error in most area of dam is very small except that in the stress singularity parts. The energy error norm of dam body is less than 10% and the elapsed time is only a few minutes. The influence of stress singularity parts decreases with mesh refinement.

Acknowledgments
This work was supported by the universities' key laboratory of water ecology and water flow structural engineering in Yunnan Province.

References
[1] Li Z., Chen F., Zhen J B. (2004) Analysis and Studies on Project and Major Technical Issues of Super-High Arch Dams. Beijing: China Electric Power Press.
[2] Erke Wang, Thomas Nelson, Rainer Rauch. (1995) A comparison of all-Hexahedra and all Tetrahedral Finite Element Meshes for elastic & elastoplastic analysis. Proceedings 4th International Meshing Round table Sandia National Labs, pp 179-181.
[3] Zhen P. (2004) Finite Element Analysis and Applications. Qinghua University Press.
[4] Yan S M. (2004) Heat Transfer Foundation. Higher Education Press.
[5] Wan L P. (1979) Elasticity Theory. Science Press.
[6] Theory Reference.ANSY14.0 Help. ANSYS Inc.,2011.
[7] Wan R K. (2003) Summary of Xiluodu Arch Dam Design. (11):17-19.
[8] Wan F, Su Z M., Jang C J, Jang N H.(2012) Arch Dam Modeling and Mess Division Based on ANSYS. J. South-to-North Water Diversion and Water Science & Technology. (3):157-158
[9] Design Specification for Concrete Arch Dams. SL282-2003, Water conservancy industry standard of the people's Republic of China, 2003
[10] Yan Q., Wu H., Zhou W H.(2005) Study on Determination of Dam Stress Based on h-version Adaptive FEM. J. Journal of Hydraulic Engineering. (3):83-87
[11] Li L Q. (2008) Energy Error Control Standard in Gravity Dam Stress Calculation. J. Water Resources and Power. (4):129-132.
[12] Zienkiewicz O C. and Taylor R L. (2000) The Finite Element Method. Vol. 1, 5th ed., Butterworth-Heinemann.
[13] Au L. (1984) Stress Distribution and Finite Element Solution nearby Gravity Dam Foundation. J. Journal of Hydraulic Engineering. (8):15-25.
[14] Yan Q P., Li J J. (2000) Investigation of Main Tensile Stress Distribution Law at Toe of Gravity Dam. J. Journal of Hydraulic Engineering. (4):64-68.