THE COMPLEXITY OF THE SPECHT MODULES CORRESPONDING TO HOOK PARTITIONS

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ABSTRACT. We show that the complexity of the Specht module corresponding to any hook partition is the \( p \)-weight of the partition. We calculate the variety and the complexity of the signed permutation modules. Let \( E_s \) be a representative of the conjugacy class containing an elementary abelian \( p \)-subgroup of a symmetric group generated by \( s \) disjoint \( p \)-cycles. We give formulae for the generic Jordan types of signed permutation modules restricted to \( E_s \) and of Specht modules corresponding to hook partitions \( \mu \) restricted to \( E_s \) where \( s \) is the \( p \)-weight of \( \mu \).

1. Introduction

Alperin and Evens \[1\] introduced the complexity of finitely generated modules over finite group algebras. Meanwhile, Carlson \[4\] had been studying some varieties for finitely generated modules over finite group algebras. Over elementary abelian \( p \)-groups, he showed that the complexity of a module is precisely the dimension of the cohomological variety and the dimension of the rank variety corresponding to the module. Avrumin and Scott \[2\] proved an analogue of Quillen’s Stratification. The complexity of a \( kG \)-module can be determined by looking at the restriction of the module to the elementary abelian \( p \)-subgroups of the group \( G \). The study of the varieties for modules over elementary abelian \( p \)-groups is closely related to the study of the notion of generic Jordan types of the modules introduced by Wheeler \[14\].

A partition \( \mu = (\mu_1^{a_1}, \ldots, \mu_s^{a_s}) \) is \( p \times p \) if for each \( 1 \leq i \leq s \), both \( \mu_i \) and \( a_i \) are multiples of \( p \). The VIGRE group in Georgia made the following conjecture.

**Conjecture 1.1** (VIGRE 2004). The complexity of the Specht module \( S\mu \) is the \( p \)-weight of the partition \( \mu \) if and only if \( \mu \) is not \( p \times p \).

In \[13\], we studied the support varieties and the complexities for Specht modules corresponding to some \( p \)-regular partitions and the partition \( (p^s) \). We showed that, for the case of abelian defects, Conjecture 1.1 is true. We also showed that a large class of Specht modules satisfy Conjecture 1.1. Recently \[11\], Hemmer shows one direction of Conjecture 1.1, i.e., if a partition \( \mu \) is \( p \times p \), then the complexity of the Specht module \( S\mu \) is strictly less than the \( p \)-weight of \( \mu \).

Let \( E \) be an elementary abelian \( p \)-group of rank \( m \). To show that a \( kE \)-module \( M \) has complexity \( m \), it suffices to show that the module \( M \) is not generically free (see Proposition 2.2). Let \( \mu \) be a hook partition of \( n \), \( s \) be the \( p \)-weight of \( \mu \) and \( E_s \) be the elementary abelian \( p \)-subgroup of \( S_n \) generated by the \( p \)-cycles \((i-1)p+1, (i-1)p+2, \ldots, ip\) with \( 1 \leq i \leq s \). We consider the restricted module \( S\mu \downarrow_{E_s} \) and show that the module is not generically free. This implies that
the complexity of the Specht module $S^\mu$ is bounded below by $s$. Since the value $s$ is also the $p$-rank of a defect group of the block containing $S^\mu$, we get the other half of Conjecture 1.1 for hook partitions.

**Remark 1.2.** Following 4.1 of [13], one may wonder if $V_{\mathfrak{E},n}(S^\mu) = \text{res}^*_n D^\mu D_n(k)$ whenever $\mu$ is not $p \times p$. A counterexample to the above statement is the partition $\mu = (7, 1^3)$ where the vertex of $S^\mu$ is $C_3 \times C_3 \times C_3$ [15] and $D^\mu \cong C_3 \wr C_3$.

**Theorem 1.3.** For any hook partition $\mu$, the complexity of the Specht module $S^\mu$ is exactly the $p$-weight of $\mu$.

2. Background materials and notations

Most of the basic materials about group cohomology and the representation theory of symmetric groups can be found in [3] and [10] respectively.

Let $G$ be a finite group, $k$ be an algebraically closed field of characteristic prime $p$ and $\text{Ext}^*_{kG}(M, M)$ be the cohomology ring of a finitely generated $kG$-module $M$. The cohomological variety $V_G(M)$ for the module $M$ is the set of maximal ideals spectrum of $\text{Ext}^*_{kG}(k, k)$ containing the kernel of the map $\Phi_M : \text{Ext}^*_{kG}(k, k) \xrightarrow{\otimes M} \text{Ext}^*(kG, M)$ where

$$\text{Ext}^*_{kG}(k, k) = \begin{cases} \text{Ext}^v_{kG}(k, k) & p \text{ is odd} \\ \text{Ext}^s_{kG}(k, k) & p = 2. \end{cases}$$

We have Theorem [2]

$$V_G(M) = \bigcup_{E \in \mathcal{E}(G)} \text{res}^*_{G, E} V_E(M)$$

where $\mathcal{E}(G)$ is a set of representatives for the conjugacy classes of elementary abelian $p$-subgroups of $G$ and $\text{res}^*_{G, E} : V_E(k) \to V_G(k)$ is the map induced by the restriction $\text{res}^*_{G, E} : \text{Ext}^*_{kG}(k, k) \to \text{Ext}^*_{kE}(k, k)$. So $\dim V_G(M) = \max_{E \in \mathcal{E}(G)} \{\dim V_E(M)\}$.

Let $E$ be an elementary abelian $p$-group of rank $n$ generated by the elements $g_1, g_2, \ldots, g_n$. The rank variety $V_E^r(M)$ of a finitely generated $kE$-module $M$ is the set

$$\{0 \neq \alpha \in k^n \mid M_{\downarrow k \langle u_\alpha \rangle} \text{ is not free} \} \cup \{0\}$$

where $u_\alpha = 1 + \sum_{i=1}^n \alpha_i (g_i - 1)$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in k^n$. Avrunin and Scott [2] showed that $V_E^r(M) \cong V_E(M)$.

Let $\alpha$ be a generic point in $k^n$. The generic Jordan type of a finitely generated $kE$-module $M$ is the Jordan type of the restricted module $M_{\downarrow \langle u_\alpha \rangle}$ where $\langle u_\alpha \rangle$ is the cyclic group $C_p$ of order $p$ described above [14]. In this case, we write $(1^{n_1}, \ldots, p^{n_p})$ for the generic Jordan type of $M$ if the number of Jordan type of size $1 \leq i \leq p$ is $n_i$. Suppose that $\beta$ is a multiple of $\alpha$. We have $M_{\downarrow \langle u_\beta \rangle} \cong M_{\downarrow \langle u_\alpha \rangle}$. We write $[\alpha]^*(M)$ for the isomorphism class of $kC_p$-modules containing $M_{\downarrow \langle u_\alpha \rangle}$. We now state a partial result of 4.7 [8] for our case.

**Proposition 2.1** (Part of 4.7 [8]). Let $E$ be an elementary abelian $p$-group, $M$ and $N$ be finitely generated $kE$-modules and $\alpha \in k^n$ be a generic point. We have $[\alpha]^*(M \oplus N) \cong [\alpha]^*(M) \oplus [\alpha]^*(N)$.
Proposition 2.2. Let $E$ be an elementary abelian $p$-group of rank $n$. A $kE$-module $M$ is not generically free if and only if $V^E_k(M) = V^E_k(k)$.

Proof. Suppose that $V^E_k(M)$ has dimension $m < n$. Let $α ∈ k^n$ be a generic point and $l_α ≲ k^n$ be the line containing the point $α$. Note that $V^E_k(M)$ is a closed homogeneous affine variety (see Theorem 4.3 of [5]). Since $m + 1 ≤ n$, we have $V^E_k(M) ∩ l_α = \{0\}$, i.e., $M$ is generically free. □

Let $n$ be a natural number. A partition $μ$ of $n$ is a non-increasing sequence of positive integers $(μ_1, \ldots, μ_s)$ such that $∑_{i=1}^s μ_i = n$. In this case, we write $n = |μ|$. The Young diagram $[μ]$ is the subset of $\mathbb{Z}^2$ consisting of all nodes $(i, j)$ satisfying $1 ≤ i ≤ s$ and $1 ≤ j ≤ μ_i$. The $p$-core $\tilde{μ}$ of $μ$ is the partition corresponding to the Young diagram obtained from $[μ]$ by removing as many skew $p$-hooks as possible. The number of skew $p$-hooks removed from $[μ]$ to get $[\tilde{μ}]$ is called the $p$-weight of $μ$. A $μ$-tableau is an assignment of the numbers 1, 2, ..., $n$ to the nodes of the Young diagram $[μ]$. For each $(i, j) ∈ [μ]$, we write $t_{ij}$ for the number assigned to $(i, j)$. For each number $1 ≤ m ≤ n$, we write $t_m$ for the node $(i, j)$ such that $t_{ij} = m$. We write $R_t(t)$ for the set consisting of the numbers in the $i$th row of $t$, i.e., $R_i(t) = \{t_{ij} | j ≥ 1\}$. Similarly, we write $C_j(t)$ for the set consisting of the numbers in the $j$th column of $t$. If numbers are increasing down each column and along each row of $t$, we say that $t$ is a standard $μ$-tableau.

Let $S_n$ denote the symmetric group on $n$ letters. The group acts on the set of all $μ$-tableaux $t$ by permuting the numbers assigned to $t$. Let $R_t$ and $C_t$ be the row stabilizer and column stabilizer of $t$ respectively, i.e., $R_t = S_{R_i(t)} × \ldots × S_{R_{s}(t)}$ and $C_t = S_{C_1(t)} × \ldots × S_{C_{s}(t)}$ where $S_Ω$ is the symmetric group corresponding to a given set $Ω$. We define an equivalence relation on the set of all $μ$-tableaux, $s ∼ t$ if and only if $s = σt$ for some $σ ∈ R_t$. A $μ$-tabloid $\{t\}$ is the equivalence class containing the $μ$-tableau $t$. The $μ$-polytabloid corresponding to the $μ$-tableau $t$ is

$$e_t = ∑_{σ ∈ C_t} sgn(σ)\{σt\}.$$  

The $k$-vector space spanned by all $μ$-polytabloids forms a $kS_n$-module. It is called the Specht module $S^μ$. If $t$ is a standard $μ$-tableau, the $μ$-polytabloid $e_t$ is called a standard $μ$-polytabloid. The set of all standard $μ$-polytabloids forms a basis of $S^μ$, the standard basis of $S^μ$. The Young subgroup $S_μ$ of $S_n$ corresponding to $μ$ is

$$S_{\{1, \ldots, μ_1\}} × S_{\{μ_1+1, \ldots, μ_1+μ_2\}} × \ldots × S_{\{μ_1+\ldots+μ_{s−1}+1, \ldots, μ_1+\ldots+μ_s\}}.$$  

The signed permutation module corresponding to a pair of partitions $(α, β)$ with $|α| = a$ and $|β| = b$ is the induced module

$$M(α|β) = (k ⊗ sgn)↑_{S_a × S_b} S_{a+b}$$  

where $⊗$ denotes the exterior tensor product of two modules [6]. It generalizes the notion of permutation modules, by taking $b = 0$. 

The stable generic Jordan type of $M$ is the generic Jordan type of $M$ modulo its projective summands. With the notation introduced earlier, the stable generic Jordan type of $M$ is $(1^{n_1}, \ldots, (p−1)^{n_{p−1}})$.
For each $1 \leq s \leq \lfloor n/p \rfloor$, we write $E_s$ for the elementary abelian $p$-subgroup of $S_n$ generated by the $p$-cycles $g_i = ((i-1)p + 1, (i-1)p + 2, \ldots, ip)$ with $1 \leq i \leq s$. For each positive integer $i$, we write $I_i$ for the set $\{(i-1)p + 1, (i-1)p + 2, \ldots, ip\}$.

**Proposition 2.3.** Let $\mu$ be a partition of $n = dp + r$ with $0 \leq r \leq p - 1$.

(i) [The Branching Theorem §9 of 9] Let $\Omega(\mu)$ be the set of partitions of $n - 1$ obtained from $\mu$ by removing a node. The module $S^\mu \downarrow_{\mathfrak{S}_{n-1}}$ has a Specht filtration with Specht factors $S^\lambda$ one for each $\lambda \in \Omega(\mu)$.

(ii) [Nakayama’s Conjecture] Let $\lambda$ be another partition of $n$. The Specht modules $S^\mu, S^\lambda$ lie in the same block if and only if the corresponding partitions $\mu, \lambda$ have the same $p$-cores.

(iii) If $|\bar{\mu}| > r$, then $S^\mu \downarrow_{E_d}$ is generically free.

**Proof.** For a proof of Nakayama’s Conjecture, see §6 of [10]. The proof of (iii) is similar to the proof of Proposition 2.2 (iv) [13]. Since the $p$-weights $m_\mu$ of $\mu$ is strictly less than $d$, a defect group $D_\mu$ of the block containing $S^\mu$ has $p$-rank $m_\mu$ and $V_{S_n}(S^\mu) \subseteq \text{res}_{S_n}^{S_n} V_{D_\mu}(S^\mu)$ (Proposition 2.1 (iv) of [13]), we have $\dim V_{E_d}(S^\mu) \leq m_\mu < d$. So $S^\mu \downarrow_{E_d}$ is generically free by Proposition 2.2. □

## 3. Signed permutation modules

Let $n_1, n_2, \ldots, n_u$ and $s$ be non-negative integers. We define the set $\Lambda(n_1, \ldots, n_u; s)$ to consist of all $u$-tuples $(c_1, \ldots, c_u) \in (\mathbb{N}_{\geq 0})^u$ such that $0 \leq c_i \leq n_i$ for each $1 \leq i \leq u$ and $c_1 + c_2 + \ldots + c_u = s$. If $s > \sum_{i=1}^{u} n_i$ or $s < 0$, then $\Lambda(n_1, \ldots, n_u; s) = \emptyset$.

**Theorem 3.1.** Let $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1, \ldots, \beta_n)$ and $|\alpha| + |\beta| = dp + r$ with $0 \leq r \leq p - 1$. Suppose that the $p$-residue of $\alpha_i$ is $s_i$ for each $1 \leq i \leq m$, the $p$-residue of $\beta_j$ is $s_{m+j}$ for each $1 \leq j \leq n$ and $\sum_{i=1}^{m+n} s_i = cp + r$. Let $1 \leq s \leq d$.

(i) The stable generic Jordan type of $M(\alpha|\beta)|_{E_s}$ is $(1^{N(\alpha,\beta,s)})$ where

$$N(\alpha,\beta,s) = \sum_{(c_1,\ldots,c_{m+n}) \in \Lambda} \left( \frac{s!}{\prod_{i=1}^{m+n} c_i!} \cdot \frac{(d-s)p + r)!}{\prod_{i=1}^{m} (\alpha_i - c_i p)! \prod_{j=1}^{n} (\beta_j - c_j + p)!} \right)$$

where $\Lambda = \left\{ \left( \frac{\alpha_1 - s_1}{p}, \ldots, \frac{\alpha_m - s_m}{p}, \frac{\beta_1 - s_{m+1}}{p}, \ldots, \frac{\beta_n - s_{m+n}}{p}; s \right) \right\}$.

(ii) We have $V_{S_{n+|\beta|}}(M(\alpha|\beta)) = \text{res}_{S_{n+|\beta|},S_n \times S_\beta} V_{S_n \times S_\beta}(k)$. In particular, the complexity of the signed permutation module $M(\alpha|\beta)$ is $d - c$.

**Proof.** We use the Mackey decomposition formula (see 3.3.4 [3]),

$$(k \boxtimes \text{sgn})|_{\mathfrak{S}_n \times \mathfrak{S}_\beta} \downarrow_{E_s} \cong \bigoplus_{E_s \cap g(\mathfrak{S}_n \times \mathfrak{S}_\beta)}^g (k \boxtimes \text{sgn})|_{E_s \cap g(\mathfrak{S}_n \times \mathfrak{S}_\beta)} \uparrow_{E_s}.$$

If $E_s \cap g(\mathfrak{S}_n \times \mathfrak{S}_\beta) \leq E_s$, then $g(k \boxtimes \text{sgn})|_{E_s \cap g(\mathfrak{S}_n \times \mathfrak{S}_\beta)} \uparrow_{E_s}$ is generically free. Suppose that $E_s \cap g(\mathfrak{S}_n \times \mathfrak{S}_\beta) = E_s$, i.e., $E_s g \subseteq g(\mathfrak{S}_n \times \mathfrak{S}_\beta)$. The double coset representatives of the subgroups $E_s, E_s \times \mathfrak{S}_\beta$ in $\mathfrak{S}_{n+|\beta|}$ correspond to the orbits of the $\mu$-tabloids under the action of $E_s$ where $\mu = (\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n)$. So the number of double coset representatives fixed by $E_s$ is precisely the number of $\mu$-tabloids fixed by $E_s$. 


A $\mu$-tabloid $\{ t \}$ is fixed by $E_s$ if and only if for each $1 \leq i \leq s$, we have $I_i \subseteq R_{j(i)}(t)$ for some $1 \leq j(i) \leq m + n$. In this case, it is necessary that

$$\sum_{i=1}^{m} (\alpha_i - s_i) + \sum_{j=1}^{n} (\beta_j - s_{m+j}) \geq sp$$

i.e., $s \leq d - c$. So $\Lambda = \emptyset$ if and only if $s > d - c$. Suppose that $s \leq d - c$. We fix an element $(c_1, c_2, \ldots, c_{m+n})$ in the set $\Lambda$. The number of choices assigning each $I_i$ with $1 \leq i \leq s$ into a row of the partition $(c_1p, c_2p, \ldots, c_{m+n}p)$ is

$$\frac{s!}{\prod_{i=1}^{m+n} c_i!}.$$ 

 Independently, the number of choices assigning the remaining $(d-s)p + r$ numbers $sp + 1, sp + 2, \ldots, dp + r$ into the remaining $(d-s)p + r$ nodes of $[\mu]$ with $\mu_i - c_i p$ nodes in each $i$th row is

$$\frac{(d-s)p + r)!}{\prod_{i=1}^{m} (\alpha_i - c_i p)! \prod_{j=1}^{n} (\beta_j - c_j m p)!}.$$ 

If we sum up over all elements of $\Lambda$, we get $N(\alpha, \beta, s)$. In these cases, the generic Jordan type of $g(k \boxtimes \text{sgn})_{E_s}$ is (1). This completes the proof for (i).

Let $G = \mathfrak{S}_{|\alpha|+|\beta|}$ and $H = \mathfrak{S}_\alpha \times \mathfrak{S}_\beta$. By Proposition 8.2.4 of [7] and Proposition 2.1 (iii) [13], we have $V_G(M(\alpha|\beta)) = \text{res}^*_G,H V_H(k \boxtimes \text{sgn}) = \text{res}^*_G,H V_H(k)$. Since the map $\text{res}^*$ is a finite map (4.2.5 of [3] II), we have

$$\dim V_G(M(\alpha|\beta)) = \dim V_H(k)$$

$$= p\text{-rank of } H$$

$$= \sum_{i=1}^{m} (\alpha_i - s_i)/p + \sum_{j=1}^{n} (\beta_j - s_{j+m})/p$$

$$= d - c \quad \square$$

**Remark 3.2.** Theorem 3.1 (ii) is an obvious generalization of 3.2.2 [12].

4. Proof of Theorem 1.3

The proof of Theorem 1.3 is a consequence of a more general statement, which we explicitly compute the stable generic Jordan type of $S^n|E_s$ where $s$ is the $p$-weight of a hook partition $\mu$ (Corollary 4.2 and Theorem 4.5). Let $\mu = (a, 1^b)$. Our aim is to show that $S^n|E_s$ is not generically free. We consider two cases, $p \not| a + b$ and $p \mid a + b$. We shall briefly describe the proofs of Theorem 4.1 and Theorem 4.5.

By the Littlewood-Richardson Rule (see 2.8.13 of [10]), the signed permutation module $M((a)|b))$ has a Specht filtration with Specht factors $S^{(a, 1^b)}$ and $S^{(a+1, 1^{b-1})}$. In the case where $a + b \not\equiv 0 \pmod p$, the sizes of $p$-cores are nonzero and $b \neq b + 1 \pmod p$, so $p$-cores of $(a, 1^b), (a + 1, 1^{b-1})$ are distinct. Using Nakayama’s Conjecture, we have a direct sum decomposition $M((a)|b)) \cong S^{(a, 1^b)} \oplus S^{(a+1, 1^{b-1})}$. We prove Theorem 4.1 by using induction on $b$. 
The case for \( a + b \equiv 0 \pmod{p} \) is slightly more complicated. We show that the short exact sequence
\[
0 \to S^{(a-1,1,1^b)} \to S^{(a,1^b+1)} \to S^{(a,1^b)} \to 0
\]
generically splits, i.e., \( S^{(a,1^b+1)} \downarrow \langle u, v \rangle \cong S^{\mu(1^b)} \oplus S^{\lambda(1^b)} \) for a generic point \( \alpha \in k^d \) where \( a + b = dp \). With Corollary 4.2, we prove Theorem 4.5 by using induction on \( b \).

4.1. Hook of size not a multiple of \( p \).

**Theorem 4.1.** Let \( \mu = (a, 1^b) \), \( a + b = dp + r \), \( a = up + a_0 \) and \( b = vp + b_0 \) with \( 0 \leq r, a_0, b_0 \leq p - 1 \) and \( r \neq 0 \). For any \( 1 \leq s \leq d \), the stable generic Jordan type of \( S^\mu \downarrow_{E_s} \) is \( (1^{N(\mu; s)}) \) where
\[
N(\mu; s) = \sum_{(c_1, c_2) \in \Lambda(\mu; s)} \binom{s}{c_2} \left( \frac{(d-s)p+r-1}{b-c_2p} \right).
\]

**Proof.** If \( d = 0 \), there is nothing to prove. For any hook \( (a, 1^b) \), we write \( \Lambda((a, 1^b); s) \) for the set \( \Lambda(u, v; s) \). We now fix the numbers \( a, b \). Let \( \mu = (a, 1^b) \) and \( \lambda = (a - 1, 1^b+1) \). We prove the formula by using induction on the number \( b \). If \( b = 0 \), then \( S^\mu \) is the trivial module and it has Jordan type \( 1 \). On the other hand, the set \( \Lambda(d, 0; s) \) contains precisely one element \((s, 0)\). So \( N(\mu; s) = \binom{s}{0} \left( \frac{(d-s)p+r-1}{0} \right) = 1 \) given that \( r \geq 1 \). Suppose that for some \( 0 \leq b \), the module \( S^\mu \downarrow_{E_s} \) has stable generic Jordan type as given by the formula. Since \( r \neq 0 \), we have a direct sum decomposition \( M((a-1)|(b+1)) \cong S^\mu \oplus S^\lambda \). Let \( a_1, b_1 \) be the \( p \)-residues of \( a-1, b+1 \) respectively. It is clear that \( b_1 \equiv b_0 + 1 \pmod{p} \) and \( a_0 \equiv a_1 + 1 \pmod{p} \). By Theorem 3.1 \( M((a-1)|(b+1)) \downarrow_{E_s} \) has stable generic Jordan type \( (1^{N((a-1),(b+1); s)}) \) where
\[
N((a-1), (b+1), s) = \sum_{(c_1, c_2) \in \Lambda(\lambda; s)} \binom{s}{c_2} \left( \frac{(d-s)p+r}{b+1-c_2p} \right).
\]
Our aim is to show that \( N(\lambda; s) = N((a-1), (b+1), s) - N(\mu; s) \). We consider 4 cases.

Case (i): Suppose that \( r \leq b_0 \leq b_1 \). If \( b_0 > r \), then \( \Lambda(\lambda; s) = \Lambda(\mu; s) \) and \( c = 1 \). The stable generic Jordan type of \( S^\lambda \downarrow_{E_s} \) is \( (1^w) \) where
\[
w = \sum_{(c_1, c_2) \in \Lambda(\lambda; s)} \binom{s}{c_2} \left( \frac{(d-s)p+r}{b+1-c_2p} \right) - \sum_{(c_1, c_2) \in \Lambda(\mu; s)} \binom{s}{c_2} \left( \frac{(d-s)p+r-1}{b-c_2p} \right)
\]
\[
= \sum_{(c_1, c_2) \in \Lambda(\lambda; s)} \binom{s}{c_2} \left( \left( \frac{(d-s)p+r}{b+1-c_2p} \right) - \left( \frac{(d-s)p+r-1}{b-c_2p} \right) \right)
\]
\[
= \sum_{(c_1, c_2) \in \Lambda(\lambda; s)} \binom{s}{c_2} \left( \frac{(d-s)p+r-1}{b+1-c_2p} \right) = N(\lambda; s)
\]
Suppose that \( b_0 = r \), we have \( \Lambda(\lambda; s) \cup \{(u, s-u)\} = \Lambda(\mu; s) \) if \( s < d \); otherwise, \( \Lambda(\lambda; s) = \emptyset \). In the first case, \( \binom{(d-s)p+r-1}{b-(s-u)p} = 0 \) given that \( b - (s-u)p = vp + r - (s-u)p = (d-s)p + r > (d-s)p + r - 1 \). So the stable generic Jordan type of \( S^\lambda \downarrow_{E_s} \) is \( (1^w) = (1^{N(\lambda; s)}) \). In the second case, the module \( M((a-1)|(b+1)) \downarrow_{E_s} \) is
generically free. As a direct summand of \( M((a - 1)(b + 1)) \downarrow_{E_d} \), the module \( S^\lambda \downarrow_{E_d} \) is generically free. This fits into the formula.

Case (ii): Suppose that \( r \leq b_0 \) and \( b_1 < r \), i.e., \( b_1 = 0 \) and \( b_0 = p - 1 \). Let \( a_0 > 0 \). We have \( \Lambda(\lambda; s) = \Lambda(\mu; s) \cup \{(s - v - 1, v + 1)\} \) if \( s \geq v + 1 \); otherwise, \( \Lambda(\lambda; s) = \Lambda(\mu; s) \). The second case is easy. For the first case, we have an extra term in \( N((a - 1)(b + 1), s) \) which is \( \binom{s}{v+1} \binom{(d-s)p+r}{b+1-(v+1)p} = \binom{s}{v+1} \binom{s}{v+1} \binom{(d-s)p+r}{b+1-(v+1)p} \). On the other hand, we also have an extra term in \( N(\lambda; s) \) which is \( \binom{s}{v+1} \binom{(d-s)p+r-1}{b+1-(v+1)p} = \binom{s}{v+1} \binom{s}{v+1} \binom{(d-s)p+r-1}{b+1-(v+1)p} \). This shows the desired formula for the stable generic Jordan type of \( S^\lambda \downarrow_{E_d} \). Let \( a_0 = 0 \). In this case \( r = p - 1 \). So we have \( \Lambda(\lambda; s) = (\Lambda(\mu; s) - \{(u, s - u)\}) \cup \{(s - v - 1, v + 1)\} \) if \( s \geq v + 1 \); otherwise, \( \Lambda(\lambda; s) = \Lambda(\mu; s) - \{(u, s - u)\} \). For the extra element \( (u, s - u) \), we have \( b - (s - u)p = vp + (p - 1) - sp + up = (d - s)p + (p - 1) > (d - s)p + r - 1 \). So the extra term corresponding to \( (u, s - u) \) is superfluous in \( N(\mu; s) \). Now the inductive argument follows similarly as the case where \( a_0 > 0 \).

Case (iii): Suppose that \( b_0 \leq b_1 < r \). In this case, \( \Lambda(\lambda; s) = \Lambda(\mu; s) \). So the inductive step is easy, the stable generic Jordan type of \( S^\lambda \downarrow_{E_d} \) is \( (1^{N(\lambda; s)}) \).

Case (iv): Suppose that \( b_0 < r \) and \( r \leq b_1 \), i.e., \( b_0 = r - 1 \) and \( b_1 = r \). In this case, \( \Lambda(\lambda; s) = \Lambda(\mu; s) \). So the stable generic Jordan type of \( S^\lambda \downarrow_{E_d} \) is \( (1^{N(\lambda; s)}) \).

\[ \text{Corollary 4.2.} \] Let \( \mu = (a, 1^b) \), \( a + b = dp + r \) with \( d \neq 0 \), \( 1 \leq r \leq p - 1 \) and \( b = vp + b_0 \) with \( 0 \leq b_0 \leq p - 1 \).

(i) If \( r \leq b_0 \), then \( V^\mu_{E_{d-1}} = V^\mu_{E_{d-1}} \) and the complexity of \( S^\mu \) is \( d - 1 \). In this case, the stable generic Jordan type of \( S^\mu \downarrow_{E_{d-1}} \) is \( (1^{N(\mu; d - 1)}) \) with

\[ N(\mu; d - 1) = \binom{d - 1}{v} \binom{p + r - 1}{b_0} \neq 0. \]

(ii) If \( b_0 < r \), then \( V^\mu_{E_d} = V^\mu_{E_d} \) and the complexity of \( S^\mu \) is \( d \). In this case, the stable generic Jordan type of \( S^\mu \downarrow_{E_d} \) is \( (1^{N(\mu; d)}) \) with

\[ N(\mu; d) = \binom{d}{v} \binom{r - 1}{b_0} \neq 0. \]

\[ \text{Proof.} \] Suppose that \( d \geq 1 \); otherwise, the result is trivial. In general, the complexity of an indecomposable \( kG \)-module is bounded above by the \( p \)-rank of a defect group of the block containing the module (see 2.1 (iv) of [13]). For \( b_0 \geq r \), a defect group of the block containing \( S^\mu \) is the Sylow \( p \)-subgroup of the symmetric group \( \mathfrak{S}_{(d-1)p} \), it has \( p \)-rank \( d - 1 \). Consider \( S^\mu \downarrow_{E_{d-1}} \) and apply the formula in Theorem 4.1. We have

\[ N(\mu; d - 1) = \sum_{(c_1, c_2) \in \Lambda(u, v; d - 1)} \binom{d - 1}{c_2} \binom{p + r - 1}{b - c_2 p} = \binom{d - 1}{v} \binom{p + r - 1}{b_0}. \]

Note that \( N(\mu; d - 1) \neq 0 \) unless \( v = d \) and \( b_0 = r \), i.e., \( a = 0 \). For \( b_0 < r \), a defect group of the block containing \( S^\mu \) is the Sylow \( p \)-subgroup of the symmetric group.
\( \mathcal{S}_{dp} \). Apply Theorem 4.11 with \( s = d \), we have 
\[
N(\mu; d) = \sum_{(c_1, c_2) \in A(u, v; d)} \binom{d}{c_2} \binom{r - 1}{b - c_2p} = \binom{d}{b_0} \binom{r - 1}{b_0} \neq 0. \tag{4.11}
\]

4.2. Hook of size a multiple of \( p \). Let \( \mu, \lambda \) be partitions of \( n \). We write \( \lambda \leq \mu \) if \( \mu \) dominates \( \lambda \) by the total ordering. In the case where \( \lambda \leq \mu \) and \( \lambda \neq \mu \), we write \( \lambda < \mu \). Let \( \mu \) be a hook partition with \( |\mu| \geq p \) and \( t \) be a \( \mu \)-tableau. We associate \( t \) to a partition \( \lambda(t) = (u, 1^{p - u}) \) of \( p \) where \( u = |R_1(t) \cap I_1| \).

**Lemma 4.3.** Let \( \mu \) be a hook partition with \( |\mu| \geq p \) and \( t \) be a standard \( \mu \)-tableau.

(i) The standard \( \mu \)-polytabloids \( e_s \) involved in \( g_1e_t \) satisfy the ordering \( \lambda(s) \leq \lambda(t) \). In the case \( \lambda(s) = \lambda(t) \), we have \( R_1(s) - I_1 = R_1(t) - I_1 \). In the case \( \lambda(s) < \lambda(t) \), we have \( (R_1(t) - I_1) \cup \{ m \} = R_1(s) - I_1 \) for some number \( m \in C_1(t) \).

(ii) For \( 2 \leq i \leq \lfloor |\mu|/p \rfloor \), \( g_i \) permutes the standard \( \mu \)-polytabloids up to a sign, \( R_1(g_i(t)) - I_i = R_1(t) - I_i \) and \( \lambda(g_i(t)) = \lambda(t) \). Furthermore, \( g_i e_t = \pm e_t \) if and only if \( I_i \subseteq R_1(t) \) or \( I_i \subseteq C_1(t) \). In this case, \( g_i e_t = e_t \).

**Proof.** All we need are the Garnir relations. Consider \( g_1 e_t = e_{g_1 t} \). Note that \( e_{g_1 t} = \varepsilon e_w \) where \( \varepsilon = \pm 1 \) and \( w \) is the \( \mu \)-tableau obtained from \( g_1 t \) by first rearranging numbers in the first row of \( g_1 t \) except 2 and then numbers in the first column of \( g_1 t \) such that numbers are increasing along the first row ignoring the first node and numbers are increasing down the first column. Note that \( R_1(w) - I_1 = R_1(t) - I_1 \). If \( 1 \in C_1(w) \), then \( w \) is standard. In this case, \( \lambda(w) = \lambda(t) \) and \( R_1(w) - I_1 = R_1(t) - I_1 \). Suppose that \( 1 \notin C_1(w) \), i.e., \( w_{12} = 1 \). We use the Garnir relation for the first two columns of \( w \). Consider the left coset representatives \((1, u) \) with \( u \in C_1(w) \cup \{ 1 \} \) of \( \mathcal{S}_{C_1(w)} \times \mathcal{S}_{\{1\}} \) in \( \mathcal{S}_{C_1(w) \cup \{1\}} \). So
\[
g_1 e_t = \varepsilon \sum_{m \in C_1(w)} e_{(1m)w}.
\]

Note that for each \( m \in C_1(w) \), \( e_{(1m)w} \) is standard up to a sign. If \( m \in C_1(w) \cap I_1 \), then \( \lambda((1m)w) = \lambda(t) \) and \( R_1((1m)w) - I_1 = R_1(w) - I_1 = R_1(t) - I_1 \). If \( m \in C_1(w) - I_1 \subseteq C_1(t) \), then \( \lambda((1m)w) < \lambda(t) \) and \( R_1((1m)w) - I_1 = (R_1(w) - I_1) \cup \{ m \} = (R_1(t) - I_1) \cup \{ m \} \). Case (i) is established.

Suppose that \( 2 \leq i \leq \lfloor |\mu|/p \rfloor \). Note that \( (g_i t)_{11} = 1 \). Let \( \sigma \in C_1(g_i t) \) such that \( \sigma(g_i t) \) has numbers increasing down the first column. So \( \text{sgn}(\sigma)e_{g_i t} \) is a standard polytabloid. Since \( g_i R_1(t) \cap I_1 = R_1(g_i t) \cap I_1 \), this gives the equivalent statement. If \( I_i \subseteq R_1(t) \), clearly \( g_i e_t = e_t \). In the case \( I_i \subseteq C_1(t) \), we have \( g_i e_t = \text{sgn}(g_i) e_t = e_t \).

Recall that for any point \( \alpha = (\alpha_1, \ldots, \alpha_s) \in k^s \) we define the element \( u_\alpha = 1 + \sum_{i=1}^s \alpha_i (g_i - 1) \) in the group algebra \( kE_s \).

**Lemma 4.4.** Let \( \mu = (a, 1^b) \), \( a + b = dp + r \) with \( 0 \leq r \leq p - 1 \). Let \( 1 \leq s \leq d \), \( \alpha \in k^s \) be a generic point and \( t \) be a standard \( \mu \)-tableau. Fix an integer \( 1 \leq m \leq p - 1 \). If \( (u_\alpha - 1)^{mp} e_t = 0 \), then for any \( 2 \leq j \leq s \) the set \( I_j \) lies entirely inside either the first column of \( \mu \) or the first row of \( \mu \).
Proof. Suppose that there is some \(2 \leq j \leq s\) such that \(I_j \not\subseteq R_1(t)\) or \(I_j \not\subseteq C_1(t)\). By Lemma 4.3 (ii), \(g_j\) permutes the set of \(\mu\)-standard polytabloids up to a sign. The size of the orbit \(O(e_t)\) under the action of \(g_j\) is \(p\), up to a sign. Note that \((u_\alpha - 1)^m\) is a linear combination of some products of not more than \(m\) copies of \(g_i\)’s with \(1 \leq i \leq s\) (may be repeated). Fix an \(m\)-string \(\beta = (\beta_1, \ldots, \beta_m)\) with each \(\beta_i \in \{0, 1, \ldots, s\}\), we write \(g_\beta = g_{\beta_1}g_{\beta_2} \cdots g_{\beta_m}\) assuming that \(g_{\beta_0} = 1\). Note that \(\lambda(g_\beta^m t) = \lambda(t)\). We claim that \(g_\beta e_t\) does not involve \(g_j^m e_t\) up to a sign unless and only unless \(\beta = j = (j, \ldots, j)\). Once we have proved this claim, since \(g_j\) occurs precisely once with coefficient \(\alpha_j^m\) in the expansion of \((u_\alpha - 1)^m\) and the point \(\alpha\) is generic, we conclude that \((u_\alpha - 1)^m e_t \neq 0\).

For each \(1 \leq i \leq d\), let \(R_{i,j}(t) = R_{i}(s) \cap I_i\) for a \(\mu\)-tableau \(s\). Let \(h, l\) be the multiplicities of \(g_1, g_j\) appearing in \(g_\beta\) respectively. Let \(e_s\) be a standard polytabloid involved in \(g_\beta e_t\) such that \(\lambda(s) = \lambda(t)\). By Lemma 4.3 (i) and (ii), we have \(R_{1,j}(s) - I_1 = (g_{\beta_i}g_i^{-h})(R_{1}(t) - I_1)\) and \(R_{i,j}(s) = g_j^l R_{1,j}(t)\). So \(R_{1,j}(g_j^m t) = R_{1,j}(s) = R_{1,j}(g_j^l t)\) if and only if \(l = m\), i.e., \(\beta = j\).

\[\square\]

Theorem 4.5. Let \(\mu = (a, b^1)\) with \(a + b = dp\) and \(b = sp + b_0\) with \(0 \leq b_0 \leq p - 1\). If \(b_0\) is even, then the stable generic Jordan type of \(S^\mu_{\downarrow E_d}\) is \(1^{(d-1)}\); otherwise, it is \((p-1)^{\left\lfloor \frac{d-1}{2} \right\rfloor}\).

Proof. We prove the result by induction on \(b\). If \(b = 0\), then \(S^\mu_{\downarrow E_d}\) is the trivial \(kE_d\)-module. So it has Jordan type (1). Suppose that we know the stable generic Jordan type of \(S^\mu_{\downarrow E_d}\) for some \(\mu = (a, b^1)\) with \(b = sp + b_0\) and \(0 \leq b_0 \leq p - 1\). Let \(\lambda = (a - 1, 1^{b-1})\). Consider two cases, \(b_0 < p - 1\) or \(b_0 = p - 1\). In the case \(b_0 < p - 1\), the \(p\)-residue of \(b + 1\) is \(b_0 + 1\). The \(kE_d\)-module

\[S^{(a-1, b^1)}_{\downarrow E_d} \otimes S_{\downarrow E_d}
\]

is generically free. On the other hand, using the Branching Theorem, the module has a filtration with factors \(S^{(a-1, 2, 1^{b-1})}_{\downarrow E_d}\) and \(S^{\lambda}_{\downarrow E_d}\). Since the \(p\)-residue of \(b\) is strictly less than \(p - 1\), the partition \((a - 1, 2, 1^{b-1})\) has non-empty \(p\)-core. So \(S^{(a-1, 2, 1^{b-1})}_{\downarrow E_d}\) is a direct summand of \(S^{(a-1, b^1)}_{\downarrow E_d} \otimes S_{\downarrow E_d}\) and it is generically free. So the stable generic Jordan types of \(S^\mu_{\downarrow E_d}\) and \(S^\lambda_{\downarrow E_d}\) are complementary. In the case \(b_0 = p - 1\), we have \(b + 1 \equiv 0 (\text{mod } p)\). By the Branching Theorem, \(S^{(a, 1^{b+1})}_{\downarrow E_d}\) has a filtration \(S^\mu_{\downarrow E_d}\) and \(S^\lambda_{\downarrow E_d}\) reading from the top. Let \(\alpha \in k^d\) be a generic point, we construct the short exact sequence

\[0 \longrightarrow S^{\lambda}_{\langle(u_\alpha)\rangle} \xrightarrow{f} S^{(a, 1^{b+1})}_{\langle(u_\alpha)\rangle} \longrightarrow S^\mu_{\langle(u_\alpha)\rangle} \longrightarrow 0\]

where \(f\) maps each standard \(\lambda\)-polytabloid \(e_\alpha\) to the standard \((a, 1^{b+1})\)-polytabloid \(e_{\phi(t)}\) where \(\phi(t)_{ij} = t_{ij}\) if \((i, j) \neq (1, a)\) and \(\phi(t)_{1a} = dp + 1\). Note that the set of standard tableaux of \((a, 1^{b+1})\) is the union of \(\Omega_1\) and \(\Omega_2\) with \(\Omega_1 \cap \Omega_2 = \emptyset\) and where \(\Omega_1\) is the set consisting of standard tableaux \(s\) such that \(s_{1a} = dp + 1\) and \(\Omega_2\) is the set consisting of standard tableaux \(s\) such that \(s_{b+2,1} = dp + 1\). We claim that \(f\) splits in the stable \(k\langle u_\alpha \rangle\)-module category. Once we have done this, we have \(S^{(a, 1^{b+1})}_{\langle(u_\alpha)\rangle} \cong S^\mu_{\langle(u_\alpha)\rangle} \otimes S^\lambda_{\langle(u_\alpha)\rangle}\). Using Theorem 4.4 with \(r = 1\) and \(b+1 = (s+1)p\),
the stable generic Jordan type of $S^{(a,1^{b+1})} \downarrow_{E_d}$ is $\left(1^{(d+1)}\right)$. By induction hypothesis, the stable generic Jordan type of $S^{a} \downarrow_{E_d}$ is $\left(1^{(d-1)}\right)$. So the stable generic Jordan type of $S^{x} \downarrow_{E_d}$ is

$$\left(1^{(d+1)}(d-1)\right) = \left(1^{(d-1)}\right).$$

This completes the inductive step.

We want to define a map $g : S^{(a,1^{b+1})} \downarrow_{(u_{\alpha})} \rightarrow S^{x} \downarrow_{(u_{\alpha})}$ in the stable module category such that $gf = \text{id}_{S^{(a,1^{b+1})}}$. For any $s \in \Omega_1$, we define $g(e_s) = e_t$ where $t$ is the unique standard $\mu$-tableau such that $\phi(t) = s$. Let $s \in \Omega_2$. If $(u_{\alpha} - 1)^{p-1}e_s = 0$, by Lemma 4.4 with $s = d$ (we have abused the notation $s$), for each $2 \leq j \leq d$ we have $I_j \subseteq R_1(s)$ or $I_j \subseteq C_1(s)$. Since $b_0 = p - 1$ and $s_{b+2,1} = dp + 1$, it follows that $I_1 \subseteq C_1(s)$. So $(u_{\alpha} - 1)e_s = 0$ if $(u_{\alpha} - 1)^{p-1}e_s = 0$. Since $e_s$ is fixed by $u_{\alpha}$, we may define $g(e_s) = 0$. If $(u_{\alpha} - 1)^{p-1}e_s \neq 0$, we claim that the set $\{e_s, (u_{\alpha} - 1)e_s, \ldots, (u_{\alpha} - 1)^{p-1}e_s\}$ is $k$-linearly independent. Suppose that we have a relation

$$a_0e_s + a_1(u_{\alpha} - 1)e_s + \ldots + a_{p-1}(u_{\alpha} - 1)^{p-1}e_s = 0$$

with $a_0, \ldots, a_{p-1} \in k$. Note that $(u_{\alpha} - 1)^p = 0$. Multiplying the equation by $(u_{\alpha} - 1)^{p-1}$, we get $a_0(u_{\alpha} - 1)^{p-1}e_s = 0$. Since $(u_{\alpha} - 1)^{p-1}e_s \neq 0$, we have $a_0 = 0$. Inductively, by multiplying $(u_{\alpha} - 1)^i$ for some suitable $i$ to the equation, we show that $a_0 = a_1 = \ldots = a_{p-1} = 0$. So $e_s$ lies inside a free summand of $S^{(a,1^{b+1})} \downarrow_{(u_{\alpha})}$. In this case, we may define $g(e_s) = 0$. The map $g$ gives a splitting for $f$ in the stable $k(u_{\alpha})$-module category. The proof is complete. \hfill \Box

Combining Corollary 4.2 and Theorem 4.5 we have our main result Theorem 1.3.

Acknowledgement. I would like to thank my supervisor Dave Benson for valuable discussions and Johannes Orlob for pointing out the obvious generalization of 3.2.2 [12] which leads to Theorem 3.4. (ii).

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