Classification of poset-block spaces admitting MacWilliams-type identity

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Abstract—In this work we prove that a poset-block space admits a MacWilliams-type identity if and only if the poset is hierarchical and at any level of the poset, all the blocks have the same dimension. When the poset-block admits the MacWilliams-type identity we explicit the relation between the weight enumerators of a code and its dual.

Index Terms—Poset-block codes, MacWilliams identity, weight distribution, MacWilliams-type identity.

I. INTRODUCTION

Due to both the interest in generalizing classic problems in coding theory and to applications in cryptography, experimental designs and high-dimensional numerical integration (see for example [1] and [2]), by the mid 1990s researches began to study codes considering metrics others than the usual Hamming metric over $\mathbb{F}_q^n$. Among those families of metrics are the poset metrics [3] and the block metrics [2]. Much of the classical theory has been generalized to codes in spaces endowed with a poset metric, as can be seen, for example, in [4], [5], [6] and [7].

In 2008 Firer et al [8] presented the family of metrics called poset-block that generalizes all the previous ones. In this work we generalize to poset-block spaces the characterization given in [3] for poset-metric spaces of poset-block metrics admitting MacWilliams-type identity.

Let $[m] := \{1, 2, \ldots, m\}$ be a finite set. If $\preceq$ is a partial order relation in $[m]$, we say $P := ([m], \preceq)$ is a poset and denote by $\preceq_P$ the order in $P$. An ideal in a poset is a nonempty subset $I \subset [m]$ such that, for $i \in I$ and $j \in [m]$, if $j \preceq_P i$ then $j \in I$. Given $A \subset [m]$, we denote by $(A)_P$ the smaller ideal of $P$ containing $A$. If $A = \{i\}$, we will denote by $(i)_P$ the ideal $(\{i\})_P$. A chain in a poset $P$ is a subset of $[m]$ such that no two elements are comparable.

Let $\mathbb{F}_q$ be a finite field and $\mathbb{F}_q^n$ the vector space of $n$-tuples over $\mathbb{F}_q$. Given $m \in [n]$, $P$ a poset over $[m]$ and $\pi : [m] \to \mathbb{N}$ a map such that $n = \sum_{i=1}^{m} \pi(i)$, we say that $\pi$ is a labeling of the poset $P$ and that the pair $(P, \pi)$ is a poset-block structure over $[m]$.

We denote $k_i = \pi(i)$, and consider the vector space over $\mathbb{F}_q$

$$V := \mathbb{F}_q^{k_1} \times \mathbb{F}_q^{k_2} \times \cdots \times \mathbb{F}_q^{k_m},$$

isomorphic to $\mathbb{F}_q^n$. Given $u \in \mathbb{F}_q^n$, there is a unique decomposition $u = (u_1, \ldots, u_m)$ with $u_i \in \mathbb{F}_q^{k_i}$, $i \in [m]$. The $\pi$-support and the $(P, \pi)$-weight of $u$ are defined respectively as

$$\text{supp}_\pi(u) := \{ i \in [m] : u_i \neq 0 \in \mathbb{F}_q^{k_i} \}$$

and

$$w_{(P, \pi)}(u) := |(\text{supp}_\pi(u))_P|,$$

where $|.|$ denotes the cardinality of the given set. For $u, v \in \mathbb{F}_q^n$,

$$d_{(P, \pi)}(u, v) := w_{(P, \pi)}(u - v)$$

defines a metric over $\mathbb{F}_q^n$ called poset-block metric, or just $(P, \pi)$-distance between $u$ and $v$.

We note that when $\pi(i) = 1$ for every $i \in [m]$ the $(P, \pi)$-distance is usual poset distance introduced in [3], while imposing $P$ to be a trivial poset ($i \not\preceq j \iff i = j$) turns the $(P, \pi)$-distance into the block distance defined in [2]. Interweaving the poset and the block structures opens a wide range of possibilities for searching for codes with interesting metric characteristics, such as perfect codes, since poset and block metrics have opposite effects on distances: while enlarging the relations on a poset enlarges the distances (hence “shrinks” metric balls), enlarging the blocks diminishes distances (hence “blows” metric balls).

Concerned with MacWilliams-type identities, dual posets play a crucial role:

Definition 1: Given a poset $P$ over $[m]$, the dual poset $\overline{P}$ defined by the relations

$$i \preceq_P j \iff j \preceq_P i$$

for every $i, j \in [m]$. The pair $(\overline{P}, \pi)$ is called the dual poset-block.

Given $j \in [m]$, the rank of $j$, denoted by $h_P(j)$, is

$$h_P(j) := \max\{|C| : C \subset (j)_P \text{ and } C \text{ is a chain}\}.$$

The height $h(P)$ of $P$ is the maximal rank of the elements of $[m]$. The $i$-level of $P$ is $\Gamma_i^P := \{ j \in [m] : h_P(j) = i \}$. We define $b_i = \sum_{j \in \Gamma_i^P} k_j$ as the sum of the dimensions of the blocks associated by $\pi$ to the $i$-level of $P$, and we call it the dimension of $\Gamma_i^P$.

A poset-block $(P, \pi)$ is said to be hierarchical if given $j_1 \in \Gamma_i^P$ we have that $j_1 \preceq_P j$ for all $j \in \Gamma_{i+1}^P$. Defining a hierarchical poset on $[m]$ is equivalent to choosing an ordered partition of $[m]$ (the partition defined by the different levels), thus it is a quite large set of posets (or poset metrics) including, as a particular case, the block structures presented in [2] when the poset structure is trivial ($h(P) = 1$), the Niederreiter-Rosenbloom-Tsfasman metric (see [9]) with a unique chain when $h(P) = m$ and the block structure is trivial ($k_i = 1$ for
every $i \in [m]$) and the usual Hamming structure when both the poset and the block structures are trivial.

Given a poset-block $(P, \pi)$ over $[m]$ such that $|\Gamma_P^i| = m_i$, let $\sigma$ be a permutation of $[m]$ such that $\{\sigma^{-1}(r_i + 1), \ldots, \sigma^{-1}(r_i + m_i)\} = \Gamma_P^i$, where $r_i = m_1 + \cdots + m_{i-1}$ and $m_0 = 0$. We let $P_1$ be the poset induced by $\sigma$, i.e., the poset in which $\sigma(j_1) \preceq \sigma(j_2)$ if $j_1 \preceq j_2$. Obviously, $P_1$ and $P$ are isomorphic posets. If we put $\pi_1(i) = \pi(\sigma^{-1}(i)) = k_i'$, then the map

$$g : \begin{pmatrix} \mathbb{F}_q^{k_1} \times \cdots \times \mathbb{F}_q^{k_m}, d(P, \pi_1) \end{pmatrix} \rightarrow \begin{pmatrix} \mathbb{F}_q^{k_1'} \times \cdots \times \mathbb{F}_q^{k_m'}, d(P, \pi_1) \end{pmatrix}$$

$$(v_1, \ldots, v_m) \mapsto (v_{\sigma(1)}, \ldots, v_{\sigma(m)})$$

is, by construction, a linear isometry. Hence, up to a linear isometry, we can and will assume that $\Gamma_P^i = \{r_i + 1, \ldots, r_i + m_i\}$, and in this case we say $(P, \pi)$ has a natural labeling. Hence, given $u \in \mathbb{F}_q^n$ we may decompose it as

$$u = \sum_{i=1}^{h(P)} \sum_{j=1}^{k(r_i+j)} \sum_{l=1}^{r_i+j-1} u_{i+j}^l e_{s(i,j,l)}$$

where $u_{i+j}^l \in \mathbb{F}_q$ are scalars and

$$\{e_{s(i,j,l)} : 1 \leq l \leq k_{r_i+j}, 1 \leq j \leq m_i, 1 \leq i \leq h(P)\}$$

is the usual basis of $\mathbb{F}_q^n$, with $s(i,j,l) = l + \sum_{t=0}^{r_i+j-1} k_l$ and $k_0 = 0$.

A $[n, k, \delta]_q$ linear $(P, \pi)$-code is a $k$-dimensional linear code $C \subset \mathbb{F}_q^n$ where $\mathbb{F}_q^n$ is equipped with the poset-block metric $d_{(P, \pi)}$ and

$$\delta = \min\{w_{(P, \pi)}(v) : v \neq 0, v \in C\}$$

is the $(P, \pi)$-minimum distance of $C$.

Definition 2: Let $C$ be a linear $(P, \pi)$-code. Its dual code is defined as

$$C^\perp = \{x \in \mathbb{F}_q^n : x \cdot u = 0 \quad \forall u \in C\}$$

where $x \cdot u$ is the usual formal inner product. We remark that $C^\perp$ is an $(n-k)$-dimensional linear code. Along this work, $C^\perp$ is considered to be a linear $(P', \pi')$-code with parameters $[n-n-k]_q$ and we denote by $\delta^{\perp}$ its minimal distance (according to the $(P', \pi')$-metric).

Given a linear $(P, \pi)$-code $C$, the $(P, \pi)$-weight enumerator of $C$ is the polynomial

$$W_{C,(P, \pi)}(x) = \sum_{u \in C} x^{w_{(P, \pi)}(u)} = \sum_{i=0}^{m} A_i(P, \pi)(C)x^i,$$

where $A_i(P, \pi)(C) = |\{u \in C : w_{(P, \pi)}(u) = i\}|$. When no confusion may arise, we will use a simplified notation for those coefficients: $A_i = A_i(P, \pi)(C)$ and $\overline{A}_i = A_i(\Gamma_P^i)(C^\perp)$.

Note that

$$V := \mathbb{F}_q^{b_1} \times \cdots \times \mathbb{F}_q^{b_l}$$

is a vector space over $\mathbb{F}_q$ isomorphic to $\mathbb{F}_q^n$, so that given $u \in \mathbb{F}_q^n$ we can write $u = (u^1, \ldots, u^l)$ where $u^i \in \mathbb{F}_q^{b_i}$ and $u^i = (u_{r_i+1}, \ldots, u_{r_i+m_i})$ is such that $u_{r_i+j} \in \mathbb{F}_q^{k_{r_i+j}}$.

If $P$ is a poset with $t$ levels, the leveled $(P, \pi)$-weight enumerator of $C$ is the formal expression

$$W_{C,(P, \pi)}(x; y_0, \ldots, y_t) := \sum_{u \in C} x^{w_{(P, \pi)}(u)y_{s_P(u)}},$$

where $s_P(u) = \max\{i : u^i \not\in \mathbb{F}_q^{|\Gamma_P^i|}\}$ and $s_P(0) = 0$. This definition is similar to the one used in [5] in the classification of poset metrics that admits MacWilliams-type identity, i.e., the case where the block structure is trivial. It is clear that $W_{(P, \pi)}(x) = W_{C,(P, \pi)}(x; 1, \ldots, 1)$.

Definition 3: We say that a poset-block $(P, \pi)$ admits a MacWilliams-type identity (MW-I) if the $(\overline{P}, \pi')$-weight enumerator of $C^\perp$ is uniquely determined by the $(P, \pi)$-weight enumerator of $C$ for every linear $(P, \pi)$-code $C$.

MacWilliams-type identities in the context of poset codes have interested researchers (see [4], [10] and [11]) since they establish a relation between important invariants of a high information rate code with those of a low dimension code, that are much easier to compute. In 2005, Kim and Oh [5] proved that a poset space admits a MW-I if and only if the poset is hierarchical. In this work we extend this result to the instances that remained open: the instance of poset-block (and block metrics as a particular case).

II. MacWilliams-type identity in $(P, \pi)$ spaces

The example below shows that the condition established in [5] is not sufficient to ensure MacWilliams-type identity in $(P, \pi)$ spaces.

Example 1: Let $P = \{1, 2, 3\}$ be the hierarchical poset with partial order defined by the relations $1 \preceq_P 2$ and $1 \preceq_P 3$ so that the dual poset $\overline{P}$ is defined by the relations $2 \preceq_P 1$ and $3 \preceq_P 1$. Define $\pi : [3] \rightarrow \mathbb{N}$ by $\pi(1) = 1$, $\pi(2) = 1$ and $\pi(3) = 2$. Then, direct computations show that the linear codes

$$C_1 = \{(0, 0, 0, 0), (0, 0, 1, 0)\}$$

and

$$C_2 = \{(0, 0, 0, 0), (0, 1, 0, 0)\}$$

over $\mathbb{F}_2^4$ has the same $(P, \pi)$-weight enumerator:

$$W_{C_1,(P, \pi)}(x) = 1 + x^2 = W_{C_2,(P, \pi)}(x).$$

However,

$$W_{C_1^\perp,(\overline{P}, \pi')}(x) = 1 + 2x + x^2 + 4x^3$$

and

$$W_{C_2^\perp,(\overline{P}, \pi')}^2(x) = 1 + 3x + 4x^3,$$

so that MW-I does not hold.

A. Necessary condition for MacWilliams-type identity

Let $(P, \pi)$ be a poset-block in $[m_t]$ with $t$ levels such that $|\Gamma_P^i| = m_i$ for $i \in [t]$. The three lemmas below are the equivalent, for the poset-block case, of Lemmas (2.1)–(2.4) in [5]. Despite the fact their proofs for poset-block being more delicate than in the case of posets (where the blocks are trivial), they are quite similar.
Lemma 1: Given \( u \in \mathbb{F}_q^n \) then \( w(\overline{\mathcal{P}}, \pi)(u) = m \iff \operatorname{supp}_\pi(u) \supseteq \Gamma_1^\pi \). Furthermore, if \( u \) satisfies \( \operatorname{supp}_\pi(u) \subseteq \Gamma_1^\pi \), we have that

\[
q^{n-b_1} \cdot \left| \{ v \in \mathbb{F}_q^n : u \cdot v = 0 \text{ and } w(\overline{\mathcal{P}}, \pi)(v) = m \} \right| = q^{n-b_1} |A| \prod_{j=1}^{m_1} (q^{k_j} - 1),
\]

where \( a|b \) means \( a \) divides \( b \) and \( b_1 \) is the dimension of \( \Gamma_1^\pi \).

Proof: The first affirmation is evident. Let \( u \in \mathbb{F}_q^n \) such that \( \operatorname{supp}_\pi(u) \subseteq \Gamma_1^\pi \). Without loss of generality we can assume that \( \Gamma_1^\pi = [m_1] \) and \( u = (u_1, \ldots, u_i, 0, \ldots, 0) \) where \( i \leq m_1 \) and \( u_j \in \mathbb{F}_q^j \{0\} \) for all \( j \in [i] \). Set

\[
A := \{ (v_1, \ldots, v_i) : v_j \in \mathbb{F}_q^j \{0\} \forall j \in [i] \text{ and } u_1 \cdot v_1 + \cdots + u_i \cdot v_i = 0 \}.
\]

In each \( \mathbb{F}_q^j \) space we have \( q^{k_j} - 1 \) non null vectors, then we have \( \prod_{j=i+1}^{m_1} (q^{k_j} - 1) \) possibilities of vectors in the blocks associated to elements of the subset \( \{ i+1, \ldots, m_1 \} \) of \( [m] \), since we do not impose restrictions in the \( m-1 \) remaining blocks, by first claim it follows that

\[
\left| \{ v \in \mathbb{F}_q^n : u \cdot v = 0 \text{ and } w(\overline{\mathcal{P}}, \pi)(v) = m \} \right| = q^{n-b_1} |A| \prod_{j=1}^{m_1} (q^{k_j} - 1),
\]

hence, by Lemma 1 and by Equations (1) and (2) it follows that

\[
q \mid \prod_{j=1}^{m_1} (q^{k_j} - 1),
\]

a contradiction because \( q \) is power of a prime. Therefore \( |(i)\| = 1 + |\Gamma_1^\pi| \), ie, \( j \leq \pi \) for all \( j \in \Gamma_1^\pi \).

Let \( P^1 = P \setminus \bigcup_{i=1}^{k} \Gamma_1^\pi \). Consider on \( P^1 \) the order induced by \( P \) and let \( \pi^1 = \pi|_{[m]\setminus \bigcup_{i=1}^{k} \Gamma_1^\pi} \) be the restriction of \( \pi \) to \( [m]\setminus \bigcup_{i=1}^{k} \Gamma_1^\pi \).

Lemma 3: If a poset-block \((P, \pi)\) admits the MW-I, then the poset-block \((P^1, \pi^1)\) also admits.

Proof: If \( m = m_1 \) we have that \([m]\setminus \Gamma_1^\pi = \emptyset\) and there is nothing to be proved. Let us assume that \( m > m_1 \) and let \( C_1^\pi \) and \( C_2^\pi \) be linear \((P^1, \pi^1)\)-codes with length \( n - b_1 \) and same \((P^1, \pi^1)\)-weight enumerator. For \( i = 1, 2 \), let

\[
C_i := \mathbb{F}_q^{b_1} \oplus C_i^\pi = \{ (u, v) : u \in \mathbb{F}_q^{b_1} \text{ and } v \in C_i^\pi \}
\]

be linear \((P, \pi)\)-codes with length \( n \) and same \((P, \pi)\)-weight enumerator. Since \((P, \pi)\) admits MW-I, \( C_1^\pi \) and \( C_2^\pi \) have the same \((P, \pi)\)-weight enumerator. Furthermore, the dual codes \( C_1^\perp \) and \( C_2^\perp \) can be described as

\[
C_i^\perp = \{ (u, v) \in \mathbb{F}_q^{b_1} \times \mathbb{F}_q^{n-b_1} : (u, v) \cdot (a, b) = 0 \forall a \in \mathbb{F}_q^{b_1} \text{ and } b \in C_i^\pi \}.
\]

Being \( b \in C_i^\pi \) the null code-word of \( C_i^\pi \), by definition of \( C_i^\perp \) it follows that \( u \) is the null element of \( \mathbb{F}_q^{b_1} \), hence

\[
C_i^\perp = \{ (u, v) : u = 0 \in \mathbb{F}_q^{b_1} \text{ and } v \in C_i^\pi \}.
\]

Therefore, by puncturing the codes \( C_1^\perp \) and \( C_2^\perp \) in the first \( b_1 \) coordinates, it follows that \( C_1^\perp \) and \( C_2^\perp \) have the same \((P^1, \pi^1)\)-weight enumerator.

By induction, using Lemmas 2 and 3 we have the following necessary condition for a poset-block \((P, \pi)\) to admit a MW-I.

Proposition 1: If \((P, \pi)\) admits the MW-I, then \( P \) is a hierarchical poset.

By Example 1 we can conclude that the previous condition is not sufficient to assure an MW-I and the following is also necessary:

Proposition 2: Suppose that \((P, \pi)\) admits a MW-I. Then, \( \pi(j_1) = \pi(j_2) \) for all \( j_1, j_2 \in \Gamma_1^\pi \) and every \( 1 \leq i \leq h(P) \), ie, blocks at the same level have the same dimension.

Proof: Given \( i \in [h(P)] \) consider \( j_1, j_2 \in \Gamma_1^\pi \) and assume \( \pi(j_1) \leq \pi(j_2) \). Let \( C_u \) and \( C_v \) be the one-dimensional linear \((P, \pi)\)-codes with length \( n \) generated by \( u = e_{s(i,j_1-r_1,1)} \) and \( v = e_{s(i,j_2-r_1,1)} \), respectively, where \( r_1 = m_1 + \cdots + m_{i-1} \). By Proposition 1 the poset \( P \) is hierarchical, and since there are

\[
(q^{k_{i-1}} - 1) + \sum_{j \neq e_{s(i,j_1-r_1,1)}} (q^{k_j} - 1)
\]
elements in $C_u^\perp$ with support contained in a unique block at the $i$-level of $P$, then
\[
A_{m_{i+1},\ldots,m_t+1,\mathcal{P},\pi}(C_u^\perp) = (q^{k_ji - 1} - 1) \prod_{j \in \mathcal{P}} q^{k_ji} + \sum_{j \in \mathcal{P}} (q^{k_ji - 1} - 1) \prod_{j \in \mathcal{P}} q^{k_ji}.
\]

We note that $n = b_1 + \cdots + b_t$ and $m = m_1 + \cdots + m_t$.

Given $i \in \{0, 1, \ldots, t\}$, set
- $\hat{b}_i = n - (b_1 + \cdots + b_i)$;
- $\hat{m}_i = m - (m_1 + \cdots + m_i)$ and
- $u^{i+1} = (u^{i+1}, \ldots, u^t) \in F_q^{\otimes t}$.

With these definitions we have that $w_{\calP,\pi}(u) = \hat{m}_i + w_{\pi}(u^i)$ where $w_{\pi}(u^i)$ is the $(\Gamma_p, \pi_{\calP^i})$-weight of $u^i$, the block weight as introduced in \cite{1}. Given a linear $(P, \pi)$-code $C$, the set
\[
C_i = \{ u \in C : u^{i+1} = 0 \}
\]
is a subcode of $C$ that can be decomposed as $C_i = C_0^i \cup C_1^i$ where
\[
C_0^i = \{ u \in C_i : u^i = 0 \} \quad \text{and} \quad C_1^i = \{ u \in C_i : u^i \neq 0 \}.
\]

Given $i \in [\ell]$, the weight enumerator of the $i$-level of $P$ is defined as
\[
LW^{(i)}_{C, (P, \pi)}(x) := \sum_{j=1}^{m_i} A_{r_j + j} x^{r_j + j}.
\]

The coefficients of this polynomial represent the weight distribution of code-words such that its support contains elements in the $i$-level and do not contain elements that are above the $i$-level. If we define $LW^{(0)}_{C, (P, \pi)}(x) = A_0$, it is clear that
\[
W_{C, (P, \pi)}(x; y_0, y_1, \ldots, y_t) = \sum_{i=0}^{t} LW^{(i)}_{C, (P, \pi)}(x) y_i. \tag{4}
\]

If for each $i \in [\ell]$ we have that $y_j = 1$ for $j \leq i$ and $y_j = 0$ for all $j > i$, then the leveled $(P, \pi)$-weight enumerator of $C$ coincides with the $(P, \pi)$-weight enumerator of $C_i$, hence,
\[
W_{C_i, (P, \pi)}(x) - W_{C_{i-1}, (P, \pi)}(x) = \sum_{u \in C_i^1} x^{w_{(P, \pi)}(u)}. \tag{5}
\]

We introduce now some concepts related to additive characters, that will be used in the proof in a way similar to what was done first by MacWilliams \cite{12} in the classical case and later in the poset case (see \cite{13, 14} and \cite{15}).

**Definition 4:** An additive character $\chi$ in $\mathbb{F}_q$ is an homomorphism of the additive group $\mathbb{F}_q$ into the multiplicative group of complex numbers with norm 1. If $\chi \equiv 1$, we say that $\chi$ is the trivial additive character.

**Lemma 4:** Let $\chi$ be a non trivial additive character of $\mathbb{F}_q$ and $\alpha$ a fixed element of $\mathbb{F}_q^\ast$. Then
\[
\sum_{\beta \in \mathbb{F}_q^\ast} \chi(\alpha \cdot \beta) = \begin{cases} q, & \text{if } \alpha \text{ is null} \\ 0, & \text{otherwise} \end{cases}
\]

**Lemma 5:** Let $\chi$ be a non trivial additive character of $\mathbb{F}_q$. For any linear code $C \subset \mathbb{F}_q^m$
\[
\sum_{u \in C} \chi(u \cdot v) = \begin{cases} 0, & \text{if } u \in \mathbb{F}_q^m \setminus C^\perp, C \subset \mathbb{F}_q^m \setminus C^\perp \end{cases}
\]

**Definition 5:** (Hadamard Transform) Let $f$ be a complex function defined in $\mathbb{F}_q^m$. The Hadamard transform of $f$ is
\[
\hat{f}(u) = \sum_{v \in \mathbb{F}_q^m} \chi(u \cdot v) f(v).
\]

The proof of the following lemma may be found in \cite{13}.
**Lemma 6:** (Discrete Poisson Summation Formula) Let $\mathcal{C} \subset \mathbb{F}_q^n$ be a linear code and $f$ a complex function defined on $\mathbb{F}_q^n$. Then

$$
\sum_{v \in \mathcal{C}} f(v) = \frac{1}{|\mathcal{C}|} \sum_{u \in \mathcal{C}} \widehat{f}(u).
$$

(6)

In case both the block and the poset structures are trivial (the Hamming case), the use of the discrete Poisson summation formula to establish the MacWilliams identity is simple: just consider $f(u) = x^{w(u)}$ and apply the discrete Poisson summation formula to the Hadamard transform

$$
\widehat{f}(u) = (1 + (q-1)x)^{n-w(u)} (1-x)^{w(u)} \text{ as in (12)}.
$$

If

$$
f(u) = x^u \beta_i(u) z_{\pi(u)}, \text{ where } s_{\pi}(u) = \min\{i : u^i \in \mathbb{F}_q^t \setminus \{0\}\} \text{ and } s_{\pi}(0) = t+1,
$$

then

$$
\sum_{u \in \mathcal{C}} f(u) = W_{\mathcal{C},\pi}(x; z_{t+1}, \ldots, z_1).
$$

(7)

Therefore we will extend this result determining the Hadamard transform of the function $f(u) = x^{w(u)} \beta_i(u) z_{\pi(u)}$.

Given $i \in \{0, \ldots, t\}$, we set

$$
B_i = \{u \in \mathbb{F}_q^n : u^i = 0 \forall 1 \leq j \leq i \text{ and } u^{i+1} \neq 0\}
$$

and then

$$
\widehat{f}(u) = \sum_{v \in \mathbb{F}_q^n} \chi(u \cdot v) f(v)
$$

$$
= \sum_{i=0}^{t} \sum_{v \in B_i} \chi(u \cdot v) f(v)
$$

$$
= \sum_{i=0}^{t} \sum_{v \in B_i} \chi(u \cdot v) x^{w(u)} \beta_i(v) z_{\pi(v)}.
$$

Defining $S_i(u) = \sum_{v \in B_i} \chi(u \cdot v) x^{w(u)} \beta_i(v) z_{\pi(v)}$, since $B_t = \{0\}$ it follows that

$$
\widehat{f}(u) = z_{t+1} + \sum_{i=1}^{t} S_{i-1}(u).
$$

(8)

The proof of the sufficiency condition will be done with the aid of four lemmas that allow us to determine $\sum_{u \in \mathcal{C}} \widehat{f}(u)$ as a function of the leveled weight enumerator of $\mathcal{C}$. From Equation (3) and assuming that the poset is hierarchical and that blocks at the same level have the same dimension, we get the following four lemmas.

**Lemma 7:** To $i \in [t]$, denote $\gamma_i = (q^d_i - 1)$, then for all $u \in \mathbb{F}_q^n$ we have that

$$
S_{i-1}(u) = z_i x^{\bar{m}_i} q^{\gamma_i} \left[ (1-x) z_{\pi_i}(u^i) (1+\gamma_i x) m_i - w_{\pi_i}(u^i) - 1 \right]
$$

if $u^{i+1}$ is a null vector and $S_{i-1}(u) = 0$ if $u^{i+1}$ is not a null vector.

**Proof:** Since $P$ is a hierarchical poset, if $v \in B_{i-1}$, then $w_{\pi_i}(v) = \bar{m}_i + w_{\pi}(v^i)$, and we denote $v = (v^1, \ldots, v^{i-1}, v^i, v^{i+1})$. By definition of $S_{i-1}(u)$ and since a character is an additive homomorphism, we have that

$$
S_{i-1}(u) = z_i x^{\bar{m}_i} \sum_{u^{i+1} \in \mathbb{F}_q^t} \chi(u^{i+1} \cdot v^{i+1}) \times \sum_{v^i \in \mathbb{F}_q^t \setminus \{0\}} \chi(u^i \cdot v^i) x^{w_{\pi_i}(v^i)}.
$$

(9)

By Lemma 4

$$
\sum_{u^{i+1} \in \mathbb{F}_q^t} \chi(u^{i+1} \cdot v^{i+1}) = \begin{cases} q^{\gamma_i}, \text{ if } u^{i+1} \text{ is null} \\ 0, \text{ otherwise} \end{cases}
$$

(10)

Being $r_i = m_1 + \cdots + m_{i-1}$ and $\chi$ a non trivial additive character, since $\nu_{r_i+j} \in \mathbb{F}_q^{d_i}$ for every $j \in \{1, \ldots, m_i\}$ and $w_{\pi}(v^i) = \sum_{j=1}^{m_i} \delta(v_{r_i+j})$ where $\delta(u)$ is the Kronecker function (it returns 1 if $u$ is not null and 0 otherwise), it follows that

$$
\sum_{v^i \in \mathbb{F}_q^t \setminus \{0\}} \chi(u^i \cdot v^i) x^{w_{\pi}(v^i)} = \left\{ \begin{array}{ll}
1 + x, & \text{if } u^{i+1} \text{ is null} \\
1 + x \sum_{v_{r_i+j} \in \mathbb{F}_q^{d_i}} \chi(u_{r_i+j} \cdot v_{r_i+j}) x^{\delta(v_{r_i+j})} = 1 + x,
\end{array} \right.
$$

hence

$$
\sum_{v^i \in \mathbb{F}_q^t \setminus \{0\}} \chi(u^i \cdot v^i) x^{w_{\pi}(v^i)} = (1-x) z_{\pi_i}(u^i) (1+\gamma_i x) m_i - w_{\pi_i}(u^i) - 1.
$$

(11)

The result follows from Equations (9), (10) and (11).

**Lemma 8:** Given $i \in [t]$, define

$$
Q_i(x) := \frac{1-x}{1+\gamma_i x},
$$

$$
a_i(x) := q^{\gamma_i} \left( \frac{1+\gamma_i x}{x} \right)^{m_i} - (1-x)^{m_i-1}
$$

and

$$
c_i(x) := x^{\bar{m}_i} q^{\gamma_i} \left( \frac{1-x}{Q_i(x)} \right)^{m_i},
$$

where $\gamma_i = q^{d_i} - 1$.

Then

$$
\sum_{u \in \mathcal{C}} \widehat{f}(u) = \sum_{u \in \mathcal{C}} |\mathcal{C}| z_{t+1}
$$

$$
+ \sum_{i=1}^{t} a_i(x) z_i LW_{\mathcal{C},\pi}(Q_i(x))
$$

$$
+ \sum_{i=1}^{t} c_i(x) |\mathcal{C}_{i-1}| - \sum_{i=1}^{t} z_i x^{\bar{m}_i} q^{\gamma_i} |\mathcal{C}_i|.
$$

(12)
Proof: If \( u \notin C \), then \( \widehat{u}^{(i)} \) is not a null vector and by Lemma 7 we find that
\[
\sum_{u \in C} S_{i-1}(u) = \sum_{u \in C_i} S_{i-1}(u) + \sum_{u \in C \setminus C_i} S_{i-1}(u) = z_t x^{\widehat{m}} q^j \left[ \left( \frac{1 - x}{Q_i(x)} \right)^{m_i} \sum_{u \in C_i} Q_i(x)^{w_{\pi_i}(u)} - |C| \right]. \tag{13}
\]
If \( u \in C^1 \), then \( w_i(P_{\pi_i}(u)) = w_{\pi_i}(u) + (m - \widehat{m}_i - 1) \), and if \( u \in C^0 \) we have \( w_{\pi_i}(u') = 0 \). Since \( C_{i-1} = C_i^0 \), then \( |C_{i-1}| = |C^0_i| \) and hence
\[
\sum_{u \in C^i} Q_i(x)^{w_{\pi_i}(u')} = \sum_{u \in C^i} Q_i(x)^{w_{\pi_i}(u')} + \sum_{u \in C^0_i} Q_i(x)^{w_{\pi_i}(u')} = \frac{1}{Q_i(x)^{m_i}} \sum_{u \in C^i} Q_i(x)^{w_{\pi_i}(u')} + |C_{i-1}|. \tag{14}
\]
Since \( m - \widehat{m}_i + 1 + m_i = m - \widehat{m}_i \) and by Equation 3 we have that \( \sum_{u \in C} Q_i(x)^{w_{(\pi^i)}(u)} = LW_{C_i(P, \pi)}(Q_i(x)) \), by replacing Equation (14) into (13) it follows that
\[
\sum_{u \in C} S_{i-1}(u) = \left( \frac{x}{1 - x} \right)^m \widehat{m}_i \left( 1 + \gamma_i x \right)^{m - \widehat{m}_i} (1 - x)^{- \widehat{m}_i - 1} z_t LW_{C_i(P, \pi)}(Q_i(x)) + z_t x^{\widehat{m}} q^j \left[ \left( \frac{1 - x}{Q_i(x)} \right)^{m_i} |C_{i-1}| - |C_i| \right]. \tag{15}
\]
By Identity 8, \( \widehat{f}(u) = z_{t+1} + \sum_{i=1}^t S_{i-1}(u) \), then by Equation (15)
\[
\sum_{u \in C} \widehat{f}(u) = |C| z_{t+1} + \sum_{i=1}^t \sum_{u \in C} S_{i-1}(u) = |C| z_{t+1} + \left( \frac{x}{1 - x} \right)^m \sum_{i=1}^t \alpha_i(x) z_i LW_{C_i(P, \pi)}(Q_i(x)) + \sum_{i=1}^t z_i C_i(x) |C_{i-1}| - \sum_{i=1}^t z_i x^{\widehat{m}_i} q^j |C_i|.
\]
In the definition of \( W_{C_i(P, \pi)}(x; y_0, \ldots, y_t) \), the \( y_j \)'s were considered as formal symbols. In two next lemmas we consider specific situations that will determine the weight enumerator in the stated conditions.

**Lemma 9:** Let \( g_j = \left\{ \begin{array}{ll} \sum_{i=j+1}^t c_i(x) z_i, & \text{if } 0 \leq j \leq t - 1, \\ 0, & \text{if } j = t \end{array} \right. \)

Then
\[
\sum_{i=1}^t z_i c_i(x) |C_{i-1}| = W_{C_i(P, \pi)}(1; g_0, \ldots, g_t).
\]

Proof: Since \( r_i = m_1 + \cdots + m_i - 1 \) and
\[
|C_i| = A_0 + A_1 + \cdots + A_{r_i + m_i} = \sum_{j=0}^{i} LW_{C_i(P, \pi)}^{(j)}(1), \tag{16}
\]
then
\[
\sum_{i=1}^t z_i c_i(x) |C_{i-1}| = \sum_{i=1}^t \left( A_0(c_1(x) z_1 + c_i(x) z_2 + \cdots + c_i(x) z_t) + (A_1 + \cdots + A_{m_i}) c_2(x) z_2 + \cdots + c_i(x) z_t + \cdots + (A_{m_1 + \cdots + m_{i-2} + 2} + \cdots + A_{m_1 + \cdots + m_{i-1} - 1}) c_i(x) z_t \right)
\]
\[
= \sum_{i=0}^t LW_{C_i(P, \pi)}(1) g_i.
\]

hence the result follows from Identity 4.

The proof of the next lemma is omitted since it follows the same steps as in the proof of Lemma 9.

**Lemma 10:** Let
\[
h_j = \left\{ \begin{array}{ll} \sum_{i=j}^t z_i x^{\widehat{m}_i} q^j, & \text{if } 1 \leq j \leq t, \\ \sum_{i=1}^t z_i x^{\widehat{m}_i} q^j, & \text{if } j = 0 \end{array} \right. \]

Then
\[
\sum_{i=1}^t z_i x^{\widehat{m}_i} q^j |C_i| = W_{C_i(P, \pi)}(1; h_0, \ldots, h_t).
\]

Before we proceed to prove the next theorem we recall we are assuming the following collection of conditions and notations:
- \((P, \pi)\) a poset-block over \([m]\) with \(t\) levels;
- \(P\) is hierarchical;
- \(r_i = m_1 + \cdots + m_i - 1\);
- \(\Gamma_P = \{r_1 + 1, \ldots, r_i + m_i\}\);
- \(d_j = \pi(r_i + j)\) for every \(j \in \{1, \ldots, m_i\}\);
- \(b_i = m_i d_i\) is such that \(\sum_{i}^t b_i = n\).

Now we can prove that necessary conditions stated in Theorem 1 are also sufficient to have a MW-I.

**Theorem 2:** Under the conditions above stated, the poset-block \((P, \pi)\) admits a MacWilliams-type identity.

**Proof:** By (6) and (7) we have that
\[
W_{\widehat{C}_i(P, \pi)}(x; z_{t+1}, \ldots, z_1) = \frac{1}{|C|} \sum_{u \in C} \widehat{f}(u). \tag{17}
\]

Considering Equation 4 we have that
\[
a_i(x) z_i LW_{C_i(P, \pi)}^{(i)}(Q_i(x)) = W_{C_i(P, \pi)}(Q_i(x); y_0, \ldots, y_t),
\]
for every \(i \in \{1, \ldots, t\}\), where \(a_i(x) z_i = y_i\) and \(y_j = 0\) for every \(j \neq i\). Substituting the identities obtained in Lemma 9.
and Lemma 10 into Equation 12, it follows that
\[|C|W_{C^{+},(F,P)}(x; z_{t+1}, \ldots, z_{1}) = |C|z_{t+1} \]
\[+ \left( \frac{x}{1-x} \right)^{m} W_{C,(P,\pi)}(Q_{1}(x); 0, a_{1}(x)z_{1}, 0, \ldots, 0) \]
\[+ \left( \frac{x}{1-x} \right)^{m} W_{C,(P,\pi)}(Q_{2}(x); 0, 0, a_{2}(x)z_{2}, 0, \ldots, 0) \]
\[+ \cdots + \left( \frac{x}{1-x} \right)^{m} W_{C,(P,\pi)}(Q_{l}(x); 0, \ldots, 0, a_{l}(x)z_{l}) \]
\[+ W_{C,(P,\pi)}(1; g_{0}, \ldots, g_{t}) - W_{C,(P,\pi)}(1; h_{0}, \ldots, h_{t}). \]

On the left side of the above equality we have the leveled weight enumerator of \(C^{+}\) (the dual code of \(C\)). On the right side we have an expression that depends not on the code itself but only on the leveled weight enumerator of \(C\). Hence, if \(C_{1}\) is a linear \((P, \pi)\)-code that has the same \((P, \pi)\)-polynomial as \(C\), since \(W_{C_{1}^{+},(F,P)}(x; 1, \ldots, 1)\) is the \((F, \pi)\)-polynomial of \(C_{1}^{+}\), it follows that
\[W_{C_{1}^{+},(F,P)}(x; 1, \ldots, 1) = W_{C^{+},(F,P)}(x; 1, \ldots, 1), \]
ie, the \((F, \pi)\)-polynomial of \(C^{+}\) is uniquely determined by \((P, \pi)\)-polynomial of \(C\) for every code \(C\), hence the poset-block structure admits a MW-I.

\[\text{Lemma 11:} \quad \text{Let} \quad (P, \pi) \quad \text{be a poset-block over} \quad [m] \quad \text{that admits MW-I and} \quad C \quad \text{a linear} \quad (P, \pi)\)-code with length \(n\). \quad \text{Then} \]
\[|C|W_{C^{+},(F,P)}(x) = |C| + \sum_{i=1}^{t} q_{i} \sum_{k=1}^{m} \left( a_{k}(j : m_{i}) + \left( \frac{m_{i}}{k} \right) \gamma_{i}^{k}|C_{i-1}| \right) x^{k} \]
where \(a_{k}(j : m_{i}) = \sum_{j=1}^{m_{i}} A_{r_{i}+j} P_{k}^{\gamma_{i}}(j : m_{i}). \)

\[\text{Proof:} \quad \text{Set} \]
\[E_{1}(x) = \sum_{i=1}^{t} q_{i} \left( 1 + \gamma_{i}x \right)^{n} \left( 1 - x \right)^{m_{i}} LW_{C,(P,\pi)}^{(i)}(Q_{i}(x)) \]
and
\[E_{2}(x) = \sum_{i=1}^{t} x^{-m_{i}} q_{i} \left( 1 + \gamma_{i}x \right)^{m_{i}} \left( 1 - x \right)^{m_{i}} \left| C_{i-1} \right| \left| C_{i} \right| \]

Putting \(z_{1} = \cdots = z_{t+1} = 1\) and replacing (12) in (17), it follows that
\[W_{C^{+},(F,P)}(x; 1, \ldots, 1) = 1 + F_{1}(x) + \frac{1}{|C|} E_{2}(x) \]
where
\[F_{1}(x) = \frac{1}{|C|} \sum_{i=1}^{t} q_{i} \left( 1 + \gamma_{i}x \right)^{m_{i}} \left( 1 - x \right)^{m_{i}} \left( 1 - x \right)^{j} \]
and
\[E_{2}(x) = \sum_{i=1}^{t} x^{-m_{i}} q_{i} \left( 1 + \gamma_{i}x \right)^{m_{i}} \left( 1 - x \right)^{m_{i}} \left| C_{i-1} \right| \left| C_{i} \right| \]
the result follows from (20), (22) and (23).}
In the conditions stated in Lemma 11 we have that
\[
W_{C^+}(P, \pi)(x) = \sum_{i=1}^t x_{m_i} \prod_{k=1}^{m_i} A_{m_i+k}\]

therefore from (19) and (24) follows the next theorem, that characterizes the weight distribution of \(C^+\) in terms of the distribution of \(C\).

**Theorem 3:** Let \((P, \pi)\) be a hierarchical poset-block over \([m]\) with \(t\) levels satisfying MW-I and \(C\) a linear \((P, \pi)\)-code with length \(n\) over \(\mathbb{F}_q\). Being \(\gamma_i = (q^{d_i} - 1)\) and \(b_j\) the dimension of \(\Gamma_j\), for any given \(i \in [t]\) and \(k \in [m_i]\) we have that
\[
\prod_{k=1}^{m_i} A_{m_i+k} = \frac{q^{d_i}}{|C|} \sum_{j=1}^{m_i} \left( A_{r_i+j} P_k^{(j : m_i)} \right) + \frac{q^{d_i}}{|C|} \sum_{j=0}^{r_i} \sum_{j=0}^{r_i} A_j.
\]

We remark that when we consider a trivial structure of blocks, \(b_j = m_j\) and \(d_j = 1\) for all \(j \in [t]\), then we have the result obtained in Theorem 4.4 from [5]. On the other hand, when considering a trivial poset structure (an antichain poset where none of elements are comparable), then \(t = 1\) and \(m = m_1\), hence given \(k \in [m_1]\) we have that
\[
\prod_{k=1}^{m_1} A_k = \frac{1}{|C|} \sum_{j=0}^{m_1} A_j P_k^{(j : m)}.
\]

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