Accessory parameters for Liouville theory on the torus

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Abstract

We give an implicit equation for the accessory parameter on the torus which is the necessary and sufficient condition to obtain the monodromy of the conformal factor. It is shown that the perturbative series for the accessory parameter in the coupling constant converges in a finite disk and give a rigorous lower bound for the radius of convergence. We work out explicitly the perturbative result to second order in the coupling for the accessory parameter and to third order for the one-point function. Modular invariance is discussed and exploited. At the non perturbative level it is shown that the accessory parameter is a continuous function of the coupling in the whole physical region and that it is analytic except at most a finite number of points. We also prove that the accessory parameter as a function of the modulus of the torus is continuous and real-analytic except at most for a zero measure set. Three soluble cases in which the solution can be expressed in terms of hypergeometric functions are explicitly treated.
1 Introduction

Liouville theory plays an important role in several fields both at the classical and quantum level \([1, 2, 3, 4, 5, 6, 7, 8, 9]\). Recently a renewed interest has developed due to a conjecture \([10, 11]\) that Liouville theory on a Riemann surface of genus \(g\) is related to a certain class of \(N = 2, 4\)-dimensional gauge theories and the conjecture has been supported by extensive tests on genera 0 and 1 \([10, 12, 13]\) and proven in a class of cases \([14, 15]\). At the classical level the key point in solving the theory is the determination of the accessory parameters which on the sphere are related to the semiclassical limit of the operator product expansion via the Polyakov relation.

The determination of the accessory parameters turns out to be a highly transcendental problem. The mathematical literature is concentrated mainly on the limit case of parabolic singularities i.e. punctures. On the other hand in quantum Liouville theory, elliptic singularities which exhibit a continuum spectrum are of most interest.

In the three point problem on the sphere the accessory parameters are algebraically fixed by the Fuchs relations. On the other hand the four point problem on the sphere \([4, 16, 17, 18]\) and the one point problem on the torus \([19, 20, 21, 15]\) lead to differential equations with four regular singularities which are special cases of the Heun equation. Higher number of point lead to still more complex equations.

The Heun accessory parameter \(\beta\) depends on three quantities: the coupling \(\eta\) whose physical range is \(0 < \eta \leq 1/2\), the modulus \(\tau\) and a scale parameter. The dependence of \(\beta\) on the scale parameter is trivial while \(\beta\) turns out to be a weight two modular form with some simple conjugation and inversion properties. This allows to predict through invariance argument the value of \(\beta\) in two special cases \([19, 20, 21]\): 1) The so called harmonic case i.e. the square; 2) The equianharmonic case i.e. the rhombus with opening angle \(\pi/6\). In both cases the value of \(\beta\) is zero. In these two cases the Heun equation reduces through respectively a quadratic and a cubic transformation to an hypergeometric equation and thus the conformal factor can be explicitly given in terms of hypergeometric functions for any value of the source strength in the physical region \([20, 21]\). Such a reduction is possible due to a symmetry in the parameters \(e_1, e_2, e_3\) which together with infinity give the position of the singularities in a two sheet plane which describes the torus.

In order to connect the Heun equation to more familiar cases Maier \([22]\) examined all rational substitution of the independent variable which transform the Heun equation into an hypergeometric equation. He found that this transformation occurs only for certain polynomial of degree \(2, 3, 4, 5, 6\). The harmonic case corresponds to the order two and the equianharmonic case to an order three case. The other transformations introduce in
addition to the physical source some additional “kinematical sources” which correspond to spurious sources. Thus despite the interest of the transformation, as far as the single source problem on the torus is concerned, no new physically interesting case is reached. The nature of the dependence of the accessory parameters on the moduli and the source strengths is not completely known. The reason is that while the proof of the uniqueness of the solution is relatively simple [23, 24] the existence of the solution relies on a variational method i.e. on the minimization of a certain functional [25, 26]. In [25] this was achieved by expanding the conformal factor in terms of a complete set of functions while the more modern treatment of [26] exploits the techniques of Sobolev spaces proving first the existence of a weak solution and then the existence of the solution. The outcome is that it is very difficult at the end to follow the nature of the dependence of the solution on the coupling and the moduli.

An exception is the case of parabolic singularities (punctures) where general properties of fuchsian mappings can be applied. Using such a technique Keen, Rauch and Vaquez [19] found that the accessory parameter for the torus with one parabolic singularity (puncture) is a real-analytic functions of the modulus; in addition in [19] some numerical investigation of the accessory parameter was performed. Zograf and Takhtajan [27] treated the case of parabolic singularities on a Riemann surface of genus 0. The result of [19, 27] is that the accessory parameters are real-analytic functions of the moduli. Kra [28] gave an extension of such a result to the case of a collection of parabolic and a special class of elliptic singularities i.e. finite order elliptic singularities where the strength of the source can assume only the values $\eta = 1/2(1 - 1/n), \ n \in \mathbb{Z}^+$. This is a discrete set which accumulates to the parabolic point. On the other hand in quantum Liouville theory, elliptic singularities which exhibit a continuum spectrum are of most interest.

In the general case of elliptic singularities and parabolic singularities it was proved in [29, 30] that the accessory parameters are real-analytic in the couplings and in the moduli in an everywhere dense open set: given a value of the coupling, if the accessory parameter is not analytic at that point there is an open set as near as we like to the given point, on which the accessory parameter is analytic.

Here we shall prove a much stronger result i.e. that the accessory parameter for the torus is an analytic function of the coupling in the whole physical region except at most a finite number of points and it is a real-analytic function of the modulus in the whole fundamental region except a zero measure set.

In proving such results we shall rely on some properties of the solution which are extracted using potential-theory techniques which were used in the solution of the uniformization problem, combined with some results on analytic varieties [31]. The first is the existence
and uniqueness property of the solution, a result which goes back to Picard himself. The second is the boundedness property of the solution \( \phi \) of the Liouville equation and its first and second derivatives with respect to the argument, in any region which excludes finite disks around the singularities and which was proved in [30].

The accessory parameter \( \beta \) obeys an implicit equation. From such implicit equation a power series expansion for \( \beta \) in the coupling \( \eta \) can be extracted and we prove such expansion to be rigorously convergent in a finite disk. We also compute a rigorous lower bound on the convergence radius. We compute also explicitly the expansion of \( \beta \) in \( \eta \) up to second order in terms of integrals of elliptic and related functions.

For general couplings i.e. couplings not necessarily small, exploiting the uniqueness theorem and some results on complex analytic varieties [31] we are able to prove that the \( \beta \) which solves the monodromy problem is analytic in the whole physical range of the coupling except at most for a finite number of points.

The nature of the dependence of the accessory parameters on the moduli of a punctured Riemann surface is important in several respects; e.g. the \( C^1 \) nature of such a dependence is an essential input in proving Polyakov relation on the sphere [29, 30, 27, 32]. Here we prove that both for elliptic and parabolic singularity the dependence of \( \beta \) on the modulus is real-analytic except for a zero measure set thus extending the results of [29, 30]. The technique developed here can be applied to the four or higher point functions on the sphere and also to higher genus surfaces.

Within the AGT [10, 11] correspondence Ferrari and Piatek [18] exploited the relation between the semiclassical limit of quantum Liouville theory and the Nekrasov-Shatashvili limit of the \( N = 2, U(2) \) super Yang-Mills theory to give an expression of the accessory parameter for the 4-point function on the sphere in terms of a contour integral containing the ratio of the column length of critical Young diagrams.

It should be possible to extend such technology to the case of the torus. On the other hand once this is accomplished, a direct comparison with the result obtained here will not be straightforward as they are based on different expansions. In the present paper, the accessory parameter has been considered as a function of the source strength and an expansion in the source strength given. Instead in the approach of [18] an expansion of the accessory parameter in the position \( x \) of the fourth singularity w.r.t. the position of the first \( z = 0 \) singularity appears. A similar approach which computes the accessory parameter expanding in \( x \) is found in [4]. As for the torus the modulus is related to the positions of the singularities in the \( u \)-plane, it appears that the such expansion should correspond to a perturbation around the degenerate case in which two singularities coincide i.e. the infinite strip which we treat in section 6. In [20] a general perturbation
technique under the variation of the moduli has been developed, and one could apply it
to the present situation allowing a direct comparison.
The paper is organized as follows: In section 2 we write down the differential equation
in the cut-\(u\) plane and derive the monodromicity condition. The fulfillment of a single
complex implicit equation is necessary and sufficient to assure the monodromic behavior at
all singularities. In section 3 we give the explicit expression of the monodromy matrices. In
section 4 we discuss modular invariance and the consequent determination of the accessory
parameter in two soluble cases. Section 5 is devoted to the relation of the action in
different coordinate systems. Section 6 gives the exact expression of the conformal factor
and of the one point function for three soluble cases, one of which is the limit case of
the infinite strip. In section 7 we develop perturbation theory in the coupling constant
up to the second order and give lower bounds for its convergence radius. In section 8 we
give a different approach to perturbation theory by working directly with the conformal
factor. Here we are able to go easily to third order even if the control of the convergence
property of the series relies on the results of section 7. In section 9 we derive the general
analytic properties of the accessory parameter both in the coupling and in the modulus.
In section 10 we give some concluding remark and point to some open problems.

2 The differential equation and the monodromy conditions

The equation

\[- \partial_z \partial_{\bar z} \phi + e^\phi = 2\pi \eta \delta^2(z - z_t)\] (1)
does not contain information about the torus. They have to be put in through periodic
boundary conditions.

To have a faithful representation of the torus we have to use the two-sheet representation
of the torus in the variable \(u = \wp(z)\). For simplicity and without losing generality due
to translational invariance we shall set in this section \(z_t = 0\).

We recall [20, 21] that the problem of finding a solution to eq. (1) can be reduced to
finding the value of the accessory parameter \(\beta\) and of a real parameter \(\kappa\), such that
the expression

\[e^{-\varphi/2} = \frac{1}{\sqrt{2|w_{12}|}} \left[ \kappa^{-2} y_1(u)y_1(u) - \kappa^2 y_2(u)y_2(u) \right]\] (2)
is monodromic. \(y_1, y_2\) are two solutions of an ordinary differential equation which contains
the parameter \(\beta\) and \(w_{12} = y_1y'_2 - y'_1 y_2\) is the constant Wronskian.
Such equation in $u$ is given by

$$y'' + Qy = y'' + (Q_0 + q)y = 0$$

\label{eq:3}

where

$$Q_0(u) = \frac{3}{16} \left( \frac{1}{(u-e_1)^2} + \frac{1}{(u-e_2)^2} + \frac{1}{(u-e_3)^2} + \frac{2e_1}{(e_2-e_3)(e_1-e_2)(u-e_1)} \right) + \frac{2e_2}{(e_2-e_3)(e_1-e_2)(u-e_2)} + \frac{2e_3}{(e_3-e_1)(e_2-e_3)(u-e_3)}$$

and

$$q(u) = \frac{1 - \lambda^2}{16} - \frac{u + \beta}{(u-e_1)(u-e_2)(u-e_3)} = \frac{1 - \lambda^2}{4} \frac{u + \beta}{[\varphi'(z)]^2} \equiv \epsilon \frac{u + \beta}{[\varphi'(z)]^2} \tag{4}$$

where $e_k = \varphi(\omega_k)$.

We recall that $\lambda = 1 - 2\eta$ and that $\eta$ has to satisfy the Picard condition $0 < \eta \leq \frac{1}{2}$, the lower limit being due to the negative nature of the curvature in the bulk and the upper to the local finiteness of the area. The upper limit correspond to a puncture, or parabolic singularity. The range of $\epsilon$ is $0 < \epsilon \leq \frac{1}{4}$.

We know two independent solution to eq.

$$y'' + Q_0y = 0 \tag{5}$$

They are

$$y_1^{(0)} = [4(u-e_1)(u-e_2)(u-e_1)]^{1/4} = [\varphi'(z)]^{1/2} \equiv \Pi(u), \tag{6}$$

$$y_2^{(0)} = (z - \omega_3)[4(u-e_1)(u-e_2)(u-e_1)]^{1/4} \equiv Z \, [\varphi'(z)]^{1/2} = Z \, \Pi(u) \tag{7}.$$

Defining

$$K(u, u') = y_1^{(0)}(u)\theta(u, u')\frac{q(u')}{w_{12}}y_2^{(0)}(u') - y_2^{(0)}(u)\theta(u, u')\frac{q(u')}{w_{12}}y_1^{(0)}(u') \tag{8}$$

we can solve the full equation \ref{eq:3} by the convergent expansion

$$y_j = (1 + K + KK + KKK + \ldots)y_j^{(0)} \tag{9}.$$

It will be useful to write

$$f_j(z) = \frac{y_j(u)}{\Pi(u)} \tag{10}$$

and now the $f_j(z)$ are given by

$$f_1(z) = 1 + \epsilon \int_{\omega_3}^{z}(z' - z)(\beta + \varphi(z'))dz' + \ldots \tag{11}$$
\[ +\epsilon^2 \int_{\omega_3}^z (z' - z)(\beta + \varphi(z'))dz' \int_{\omega_3}^{z'} (z'' - z')(\beta + \varphi(z''))dz'' + \ldots \]

and

\[ f_2(z) = Z + \epsilon \int_{\omega_3}^z (z' - z)(\beta + \varphi(z'))Z' dz' + \]

\[ +\epsilon^2 \int_{\omega_3}^z (z' - z)(\beta + \varphi(z'))dz' \int_{\omega_3}^{z'} (z'' - z')(\beta + \varphi(z''))Z'' dz'' + \ldots \]

with \( \omega_3 = \omega_1 + \omega_2 \). We shall be interested in the values of \( f_j \) and their derivatives \( g_j \equiv f'_j \) at the points \( \omega_1, \omega_2 \) and \( \omega_2 - \omega_1, \omega_1 - \omega_2 \). Due to the triangular structure of the multiple integrals and the boundedness of \( \varphi(z) \) along the integration paths \( \omega_3 - r\omega_1 \) and \( \omega_3 - r\omega_2 \) we see that such \( f_j \) and \( g_j \) are analytic function of \( \epsilon \) and \( \beta \) as the series converges absolutely for any given \( \epsilon \) and \( \beta \). This is a well known fact.

We have

\[ g_1(\omega_k) = -\epsilon \int_{\omega_3}^{\omega_k} (\beta + \varphi(z'))f_1(z')dz' \] (13)

and

\[ g_2(\omega_k) = 1 - \epsilon \int_{\omega_3}^{\omega_k} (\beta + \varphi(z'))f_2(z')dz' \] (14)

and similarly for the other values of the argument. The \( f_j(\omega_k) \) and the \( g_j(\omega_k) \) are holomorphic functions of \( \epsilon \) and \( \beta \).

There are two ways to impose monodromy. The first \[20\] is to exploit the symmetry for \( z_t = 0 \) of the equation (1) and of its solution under the inversion \( z \rightarrow -z \). As proven in \[20\] it is sufficient to impose monodromy around the three kinematical singularities \( e_j = \varphi(\omega_j) \) under a full turn in \( u \) which corresponds to half turn in \( z \). A more general method is to impose monodromy under two independent cycles of the torus \[21\]. While the first method requires the solution of the differential equation on the tracts \( (\omega_3, \omega_1) \), \( (\omega_3, \omega_2) \) the second method requires the solution on the longer tracts \( (\omega_1, \omega_3 - 2\omega_1) \), \( (\omega_3, \omega_3 - 2\omega_2) \), i.e. along the full cycles.

In order to derive non perturbative results we shall need some results from the potential theory approach to the Liouville equation. The input we shall use is Picard result \[23\] about the existence and uniqueness of the solution of the uniformization problem of which eq.(1) is a particular case. Picard was concerned only with elliptic singularities i.e. \( \eta < \frac{1}{2} \). Later the treatment was extended to elliptic and/or parabolic singularities in \[24, 25\]. For a more modern treatment using Sobolev spaces see \[26\].

In \[20\] after choosing the canonical pair of solutions \( y_l \) it was proven in the first approach that the monodromies at \( e_1, e_2 \) have the form

\[ M(\omega_j) = \begin{pmatrix} i(a_j d_j + b_j c_j) & -2ia_j b_j \\ 2i c_j d_j & -i(a_j d_j + b_j c_j) \end{pmatrix} \] (15)
with $a_j, b_j, c_j, d_j$ elements of a $SL(2, C)$ matrix. We have still at our disposal a scale transformation on the canonical solutions under which the matrices go over to
\[
\begin{pmatrix}
  i(a_j d_j + b_j c_j) & -2i a_j b_j \kappa^{-2} \\
  2i c_j d_j \kappa^2 & -i(a_j d_j + b_j c_j)
\end{pmatrix}.
\] (16)

Picard existence theorem tells us that for each $j$, $a_j b_j$ and $c_j d_j$ are either both zero or both different from zero. From this remark it follows that necessary and sufficient condition for the existence of a $\kappa$ which renders
\[
a_j b_j \kappa^{-2} = \bar{c}_j \bar{d}_j \kappa^2
\] (17)
is that in [15]
\[
M_{12}(\omega_1)M_{21}(\omega_2) = M_{12}(\omega_2)M_{21}(\omega_1).
\] (18)

It is not difficult to prove [20] that once relation (17) is satisfied it follows that all the monodromies become $SU(1,1)$ i.e. the conformal factor $\varphi$ becomes single valued and regular. Thus $\beta$ has to be chosen as to satisfy (18).

Similar considerations hold in the cycle approach [21] where the necessary and sufficient condition for the realization of the monodromic solution takes the form
\[
M_{12}(C_1)M_{21}(C_2) = M_{12}(C_2)M_{21}(C_1).
\] (19)

### 3 Computation of the monodromies

Given the complex
\[
Y(u) = \begin{pmatrix} y_1(u) \\ y_2(u) \end{pmatrix}
\] (20)
the monodromy matrices are defined by
\[
\tilde{Y}(u) = MY(u)
\] (21)
where $\tilde{Y}(u)$ denotes the complex after a complete turn in $u$ at $e_1$ or $e_2$, in the first approach, or after a cycle, in the second approach. We have also
\[
\tilde{Y}'(u) = MY'(u)
\] (22)
and thus
\[
\begin{pmatrix}
  \tilde{y}_1 & \tilde{y}_1' \\
  \tilde{y}_2 & \tilde{y}_2'
\end{pmatrix} = M \begin{pmatrix}
  y_1 & y_1' \\
  y_2 & y_2'
\end{pmatrix}
\] (23)
from which
\[ M = \begin{pmatrix} \ddot{y}_1 & \ddot{y}_1' \\ \ddot{y}_2 & \ddot{y}_2' \end{pmatrix} \begin{pmatrix} y_2' & -y_1' \\ -y_2 & y_1 \end{pmatrix} \]  
(24)
due to \( w_{12} = y_1y_2' - y_1'y_2 = 1 \). Application of eq. (24) to eq. (17) gives
\[ M_{12}(\omega_k) = -\ddot{y}_1y_1' + \ddot{y}_1'y_1 = -2e^{i\frac{\pi}{2}} f_1(\omega_k) g_1(\omega_k) \]  
(25)
and
\[ M_{21}(\omega_k) = \ddot{y}_2y_2' - \ddot{y}_2'y_2 = 2e^{i\frac{\pi}{2}} f_2(\omega_k) g_2(\omega_k). \]
Thus equation (18) becomes in the first approach
\[ f_1(\omega_1)g_1(\omega_1)f_2(\omega_2)g_2(\omega_2) = f_1(\omega_2)g_1(\omega_2)f_2(\omega_1)g_2(\omega_1). \]  
(26)
Using the cycle method the monodromy equation (19) becomes
\[ g_1(\omega_3 - 2\omega_1)f_2(\omega_3 - 2\omega_2) = g_1(\omega_3 - 2\omega_2)f_2(\omega_3 - 2\omega_1). \]  
(27)
Due to the uniqueness of the Picard solution the two equations are equivalent. We shall come back to this property in section 7.

4 Modular invariance

We recall some simple properties of the accessory parameter \( \beta \) which are derived from the differential equation and the uniqueness theorem.

The equation
\[ f''(z) + \epsilon(\wp(z) + \beta)f(z) = 0 \]  
(28)
has invariance properties related to the transformation properties of \( \wp \) under dilatations, conjugation and modular transformations \[19\]. From
\[ \wp(\mu z | \mu \omega_1, \mu \omega_2) = \frac{1}{\mu^2} \wp(z | \omega_1, \omega_2) \]  
(29)
with \( \mu \in C \) we have \( \mu^2 \beta(\mu \omega_1, \mu \omega_2) = \beta(\omega_1, \omega_2) \). From \( \wp(z | \bar{\omega}_1, \bar{\omega}_2) = \overline{\wp(z | \omega_1, \omega_2)} \) one obtains \( \beta(\bar{\omega}_1, \bar{\omega}_2) = \overline{\beta(\omega_1, \omega_2)} \). Moreover as the lattice is left invariant under \( \omega_1 \to -\omega_1 \) and \( \omega_2 \to -\omega_2 \) and under \( \omega_1 \leftrightarrow \omega_2 \), \( \wp(z) \) is unchanged and also \( \beta \) is unchanged. Similarly \( \beta \) is unchanged under \( \omega_1 \to \omega_1, \omega_2 \to \omega_2 + \omega_1 \). The two transformations \( \omega_1 \to \omega_1, \omega_2 \to \omega_2 + \omega_1 \) and \( \omega_1 \to \omega_2, \omega_2 \to -\omega_1 \) are, apart for a dilatation, the generators \( T \) and \( S \) of the
modular group \([42]\). Thus defining \(\beta[\omega_2] = \beta(1, \omega_2)\) we can synthesize the transformation properties of \(\beta\) as

\[
\beta\left[ \frac{a\tau + b}{c\tau + d} \right] = (c\tau + d)^2 \beta[\tau] \tag{30}
\]

telling us that \(\beta\) is a modular form of weight 2 and

\[
\overline{\beta[\tau]} = \beta[-\tau]. \tag{31}
\]

From such transformation properties a few simple facts follow \([19, 20, 21]\).

1) For \(\text{Re}\ \tau = 0\), \(\beta\) is real; this describes the rectangle.

2) For \(\text{Re}\ \tau = \pm \frac{1}{2}\) we have \(\beta = \text{real}\).

From the fact that the stabilizer in the fundamental region is \(\pm I\) except \(S\) for \(\tau = i\), \(ST\) for \(\tau = e^{2\pi i/3}\) and \(TS\) for \(\tau = e^{\pi i/3}\) \([42]\) we have

3) For \(\tau = i\) we have \(\beta = 0\); this describes the square.

4) For \(\tau = e^{2\pi i/3}\) we also have \(\beta = 0\); this describes the so called equianharmonic case where the fundamental region is a rhombus with opening angle \(2\pi/6\).

5) For \(\tau = e^{\pi i/3}\) we also have \(\beta = 0\); this again describes the equianharmonic case where the rhombus with opening angle \(2\pi/6\) has a different orientation so it does not differ from case 4.

In \([20, 21]\) the explicit form of the conformal factor in terms of hypergeometric functions was given for the cases 3 and 4,5.

From the viewpoint of the differential equation in \(u\), modular transformations boil down to a simple permutation of the \(e_k\) and a scale transformation. Thus in studying the monodromies in the \(u\) cut-plane with the first method we have a simple interchange of \(e_k\) in the basic equations. If instead we exploit the cycle approach, modular invariance is due to the group composition properties for the transfer matrices, when we add to a given cycle one or more cycles. This will be relevant in discussing the modular invariance of the perturbation theory results.

### 5 The action in different coordinates

It is well known \([4, 41]\) that also the classical action has to be regularized due to the logarithmic divergences which arises from the kinetic term at the singularities.

In this section we shall write the relation among the two regularized action \(S_z\) and \(S_u\) related to the \(z\)- and \(u\)-representation of the torus.
In the $z$-representation the action is given by

$$\frac{S_z}{2\pi} = \frac{1}{2\pi} \int_{D_\varepsilon} \left(\frac{1}{2}d\phi \wedge d\phi + e^\phi dz \wedge d\bar{z}\right) i - \frac{\eta}{4\pi i} \oint_{\partial D_\varepsilon} \phi \left(\frac{dz}{z - z_t} - \frac{d\bar{z}}{\bar{z} - \bar{z}_t}\right) - \eta^2 \log \varepsilon^2$$

(32)

where $D_\varepsilon$ is exterior of a circle of radius $\varepsilon$ around the source at $z_t$. Writing

$$\phi = -2\eta \log |z - z_t|^2 + X + o(z - z_t)$$

(33)

we have also

$$\frac{S_z}{2\pi} = \frac{1}{2\pi} \int_{D_\varepsilon} \left(\frac{1}{2}d\phi \wedge d\phi + e^\phi dz \wedge d\bar{z}\right) i - \eta X + \eta^2 \log \varepsilon^2$$

(34)

and the important relation

$$\frac{1}{2\pi} \frac{\partial S_z}{\partial \eta} = -X.$$  

(35)

In order to compute the action explicitly in the soluble cases it is however better to put the source at the origin $z_t = 0$, which gives rise to a singularity in $u$ at infinity. The transition to the $u$-representation is given by $u = \wp(z)$

$$e^\phi dz \wedge d\bar{z} = e^\varphi du \wedge d\bar{u}$$

(36)

i.e.

$$\phi = \varphi - \log J \bar{J}$$

(37)

with

$$J = \frac{dz}{du} = \frac{1}{\wp(z)'}.$$  

(38)

Thus

$$\phi = \varphi + \log |4(u - e_1)(u - e_2)(u - e_3)|$$

(39)

We have

$$\phi = -2\eta \log |z|^2 + X + o(z)$$

(40)

and taking into account that $u = \wp(z) = \frac{1}{z^2} + o(z)$ we have at infinity in $u$

$$\varphi = (\eta - \frac{3}{2}) \log u\bar{u} + X_u$$

(41)

with

$$X_u \propto X - \log 4.$$  

(42)
The regularized $S_u$ action now takes the form

$$\frac{1}{2\pi} S_u = \frac{1}{2\pi} \int_D \left( \frac{1}{2} d\varphi \wedge d\bar{\varphi} + e^\varphi du \wedge d\bar{u} \right) \frac{i}{2}$$

$$- \frac{1}{8\pi i} (\eta - \frac{3}{2}) \oint_{R_u} \varphi \left( \frac{du}{u} - \frac{d\bar{u}}{\bar{u}} \right) - \frac{1}{16\pi i} \oint_{e_k} \varphi \left( \frac{du}{u-e_k} - \frac{d\bar{u}}{\bar{u}-\bar{e}_k} \right)$$

$$+ \frac{1}{2} (\eta - \frac{3}{2})^2 \log R_u^2 - \frac{1}{8} \log \varepsilon_k^2 =$$

$$= \frac{1}{2\pi} \int_D \left( \frac{1}{2} d\varphi \wedge d\bar{\varphi} + e^\varphi du \wedge d\bar{u} \right) \frac{i}{2}$$

$$- \frac{1}{2} (\eta - \frac{3}{2})^2 \log R_u^2 - (\eta - \frac{3}{2}) X_\infty^u + \frac{1}{8} \log \varepsilon_k^2 - \frac{1}{2} X_k^u$$

(43)

where the integration in $u$ is extended to the two sheets which describe the torus. $D$ excludes disks of radius $\varepsilon_k$ around $e_k$ and on both sheets the exterior of a circle of radius $R_u$.

We also have

$$\frac{1}{2\pi} \frac{\partial S_u}{\partial \eta} = - X_\infty^u.$$  

(44)

Using eqs. (34, 36, 43) we find for the relation between the two actions

$$S_z = S_u - \frac{1}{4} \log \left| (e_1 - e_2)(e_2 - e_3)(e_3 - e_1) \right|^2 - 2\eta \log 2.$$  

(45)

consistent with eq. (42).

6 Soluble cases

In this section we shall give the explicit value of the action, i.e. of the semiclassical 1-point function, for three soluble cases i.e. the square, the equianharmonic case i.e. a rhombus with opening angle $2\pi/6$ and the limit case of the infinite strip for any coupling in the physical region.

1. The harmonic case: the square

In [20, 21] the Liouville field for the harmonic case i.e. the square was computed in terms of hypergeometric functions. The result was with $\lambda = 1 - 2\eta$

$$- \frac{\phi(z)}{2} = - \log (\sqrt{2} |\kappa|^2)$$

$$+ \log \left[ \left( \frac{1 - \lambda}{8}, \frac{1 + \lambda}{8} ; \frac{3}{4} ; u^2(z) \right) \right] - |\kappa|^4 |u(z)| \left( \frac{3 - \lambda}{8}, \frac{3 + \lambda}{8} ; \frac{5}{4} ; u^2(z) \right)$$

$$\right]^2$$

(46)
with

$$|\kappa|^4 = \frac{8}{\gamma^2(\frac{1}{4}) \gamma(\frac{1-\lambda}{4}) \gamma(\frac{1+\lambda}{4})}$$  
(47)

where as usual

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}.$$  
(48)

From (46) we obtain

$$X = (3 + 4\eta) \log 2 - 2 \log \gamma(\frac{1-2\eta}{4}) + \log \gamma(1-\eta)$$  
(49)

and using (35) we have with

$$F(x) = \int_\frac{x}{2}^x \log \gamma(x') \, dx'$$  
(50)

$$S_z(\text{square}) = -(3\eta + 2\eta^2) \log 2 - 4F(\frac{1-2\eta}{4}) + F(1-\eta) + 4F(\frac{1}{4}) - F(1).$$  
(51)

2. The equianharmonic case

We have [21]

$$- \frac{\phi(z)}{2} = - \log(2\sqrt{2}|\kappa|^2)$$  
(52)

$$+ \log \left[ 2F_1\left(\frac{1-\lambda}{12}, \frac{1+\lambda}{12}; \frac{2}{3}; u^3(z)\right) - |\kappa|^4 |u(z)|^2 \right] \left[ 2F_1\left(\frac{5-\lambda}{12}, \frac{5+\lambda}{12}; \frac{4}{3}; u^3(z)\right) \right]^2$$

with

$$|\kappa|^4 = 9 \frac{\pi \Gamma(\frac{2-\lambda}{6}) \Gamma(\frac{2+\lambda}{6})}{\Gamma^2(\frac{1-\lambda}{6}) \Gamma(\frac{1+\lambda}{6})}$$  
(53)

which gives

$$X = 2 \log 3 + \frac{7\eta}{3} \log 2 + \frac{4}{3} \eta \log 2 - \log \gamma(\frac{2}{3} + \frac{\eta}{3}) - 2 \log \gamma(\frac{1-2\eta}{6}) + \log \gamma(1-\frac{\eta}{3})$$  
(54)

Integrating

$$S_z(\text{equianharmonic}) = - \left[ 2\eta \log 3 + \frac{7\eta}{3} \log 2 + \frac{2\eta^2}{3} \log 2 - 3F\left(\frac{2+\eta}{3}\right) + 6F\left(\frac{1-2\eta}{6}\right) 
- 3F(1-\frac{\eta}{3}) + 3F\left(\frac{2}{3}\right) - 6F\left(\frac{1}{6}\right) + 3F(1) \right].$$  
(55)

3. The infinite strip

\[\text{We correct for a factor 2 in the argument of the first logarithm}\]
We discuss here a limit case of the torus topology which is soluble i.e. the infinite strip.
The vertical infinite strip is reached with the parameters \(e_1 = 2a, e_2 = e_3 = -a\). For \(a = 1\) we have
\[
Q(u) = \frac{1 - \lambda^2}{16} \frac{u + \beta}{(u + 1)^2(u - 2)} + \frac{3}{16} \frac{u^2 - 2u + 9}{(u - 2)^2(u + 1)^2}.
\] (56)
The accessory parameter \(\beta\) has to be fixed to 1 otherwise the pole of order 2 at \(u = -1\) would not have the correct kinematical value \(1/4\) as it is required for the limit of an infinite rectangle. It is of interest that the value 1 is already given by first order perturbation theory
\[
\beta = \frac{\zeta(\omega_2)\bar{\omega}_1 - \zeta(\omega_1)\bar{\omega}_2}{\omega_2\bar{\omega}_1 - \omega_1\bar{\omega}_2}.
\] (57)
In fact for the case at hand i.e. \(e_1 = 2a, e_2 = e_3 = -a\) the Weierstrass \(\wp\) and \(\zeta\) functions become
\[
u = \wp(z) = -a + \frac{3a}{\sin^2(\sqrt{3a}z)}, \quad \zeta(z) = az + \sqrt{3a} \frac{\cos(\sqrt{3a}z)}{\sin(\sqrt{3a}z)}
\] (58)
with
\[
\omega_1 = \frac{\pi}{2\sqrt{3a}}
\] (59)
and thus
\[
\lim_{\omega_2 \to i\infty} \beta = a = 1.
\] (60)
Two independent solutions of the differential equation (3) with \(Q\) given by (56), canonical at \(u = 2\) are
\[
y_1 = (u - 2)^{1/4}(u + 1)^{1/2} \, _2F_1\left(\frac{1 - \lambda}{4}, \frac{1 + \lambda}{4}, \frac{1}{2}; \frac{2 - u}{3}\right)
\] (61)
\[
y_2 = (u - 2)^{3/4}(u + 1)^{1/2} \, _2F_1\left(\frac{3 - \lambda}{4}, \frac{3 + \lambda}{4}, \frac{3}{2}; \frac{2 - u}{3}\right)
\] (62)
giving for the \(\phi\)
\[
- \frac{\phi}{2} = -\log(3\sqrt{2}) - \log \kappa^2
\] (63)
\[
+ \log \left[ \frac{2}{3}F_1\left(\frac{1 - \lambda}{4}, \frac{1 + \lambda}{4}, \frac{1}{2}; \frac{2 - u}{3}\right) \right]^2 - \kappa^4 \left| u - 2 \right| \frac{2}{3}F_1\left(\frac{3 - \lambda}{4}, \frac{3 + \lambda}{4}, \frac{3}{2}; \frac{2 - u}{3}\right)^2
\]
with
\[
\kappa^4 = \frac{1}{3} \left[ 2 \, \gamma(3 - \lambda) \gamma(3 + \lambda) \right]^2.
\] (64)
We find
\[
X = 2\eta \log \frac{4}{3} + \log 6 + 2 \log \gamma(1 - \eta) - 2 \log \gamma\left(\frac{1}{2} - \eta\right)
\] (65)
from which
\[
S_z(\text{strip}) = -\left[ \eta^2 \log \frac{4}{3} + \eta \log 6 - 2F(1 - \eta) + 2F\left(\frac{1}{2} - \eta\right) + 2F(1) - 2F\left(\frac{1}{2}\right) \right].
\] (66)
7 Second order perturbation theory and convergence radius

In this section we shall develop the perturbation theory around $\epsilon = 0$. We shall show that the perturbative series is convergent in a neighborhood of $\epsilon = 0$ and give a rigorous lower bound on the convergence radius. We shall also give the explicit expression of the first and second order term for the accessory parameter $\beta$. We stress that the treatment of this section requires neither Picard’s existence and uniqueness theorem nor other results from the potential theory approach to the problem.

The perturbative series for $\beta$ is obtained by solving the implicit equation (26) or (27). We shall use (27)

$$\epsilon F(\beta, \bar{\beta}, \epsilon) = g_1(\omega_3 - 2\omega_1)f_2(\omega_3 - 2\omega_2) - g_1(\omega_3 - 2\omega_2)f_2(\omega_3 - 2\omega_1) = 0 .$$

(67)

We notice that

$$g_1(\omega_3 - 2\omega_k) = 2\epsilon (\omega_k\beta - \zeta(\omega_k)) + O(\epsilon^2)$$

(68)

being $\zeta(z)$ the Weierstrass zeta-function, while

$$f_2(\omega_3 - 2\omega_k) = -2\omega_k + O(\epsilon) .$$

(69)

Thus after dividing (27) by $\epsilon$ we have for the Jacobian, at $\epsilon = 0$

$$J = \frac{\partial(F, \bar{F})}{\partial(\beta, \bar{\beta})} \bigg|_{\epsilon=0} = 16|\bar{\omega}_2\omega_1 - \omega_2\bar{\omega}_1|^2 \neq 0 .$$

(70)

Due to the analyticity of $F$ in $\beta, \bar{\beta}$ and $\epsilon$ we have that $\beta$ will be an analytic function of $\epsilon$ in an open neighborhood of $\epsilon = 0$ [35]. We can therefore develop a perturbative series around $\epsilon = 0$. From (68,69) we have [20]

$$\beta_1 = \frac{\bar{\omega}_2\zeta(\omega_1) - \omega_1\zeta(\omega_2)}{\bar{\omega}_2\omega_1 - \omega_1\bar{\omega}_2} .$$

(71)

On equation (71) we can check already the following properties: 1. For $\omega_2 = i\omega_1$ i.e. the square $\beta_1 = 0$. 2. For $\omega_2 = e^{2\pi i/3}\omega_1$ i.e. the equianharmonic case $\beta_1 = 0$. For general $\omega_1, \omega_2$ we verify modular invariance, i.e. invariance under the two generating transformations

$$\omega_1 \rightarrow \omega_1, \quad \omega_2 \rightarrow \omega_2 + \omega_1 \quad (T)$$

$$\omega_1 \rightarrow -\omega_2, \quad \omega_2 \rightarrow \omega_1 \quad (S) .$$

(72)

All these properties are proven easily using

$$\zeta(\omega_1 + \omega_2) = \zeta(\omega_1) + \zeta(\omega_2).$$

(73)
Moreover in the limit of the infinite strip we have $\beta \to 1$ (see eq.(60)).

We come now to the second order. Developing eq.(67) we obtain for the accessory parameter $\beta$ to the second order

$$
\begin{align*}
\beta &= \beta_1 - \frac{\epsilon}{\omega_1 \omega_2 - \bar{\omega}_1 \omega_2} \left[ \bar{\omega}_2^2 \zeta(\omega_1) - \zeta^2(\omega_1) (\bar{\omega}_1 \omega_2 + \omega_1 \bar{\omega}_2) \right] \\
&\quad - 2 \bar{\omega}_1 \zeta(\omega_1) \zeta_2(\omega_1, \omega_3) - 2 \zeta_3(\omega_1, \omega_3) \zeta(\omega_1) + \bar{\omega}_1 I(\omega_1, \omega_3) + \\
&\quad \beta_1 \left( - \bar{\omega}_2^2 \omega_1 \zeta(\omega_1) + 2 \bar{\omega}_1 \zeta_3(\omega_1, \omega_3) + 2 \zeta_3(\omega_1, \omega_3) + \bar{\omega}_2 \omega_1^2 (2 \zeta(\omega_1) + \zeta(\omega_2)) \right) \\
&\quad + \frac{2}{3} \beta_1 \omega_1^2 \zeta(\omega_1) + \frac{2}{3} \beta_1 \bar{\omega}_2 \omega_2 \omega_2 + \frac{2}{3} \beta_1 \bar{\beta}_1 \bar{\omega}_1 \omega_2 \omega_2 - \{\omega_1 \leftrightarrow \omega_2\} \\
&\quad + \frac{2}{3} \beta_1 \bar{\omega}_2^3 \zeta(\omega_1) + \frac{2}{3} \beta_1 \bar{\omega}_2 \omega_2 \omega_3 + \frac{2}{3} \beta_1 \bar{\omega}_1 \omega_1 \omega_2 \omega_2 - \{\omega_1 \leftrightarrow \omega_2\}
\end{align*}
$$

(74)

where in the above expression $\beta_1$ is the first order result (71) and

$$
\begin{align*}
\zeta_2(z, \omega_3) &= \int_{\omega_3}^{z} \zeta(z')dz' \log \frac{\sigma(z)}{\sigma(\omega_3)}, \\
\zeta_3(z, \omega_3) &= \int_{\omega_3}^{z} \zeta_2(z', \omega_3)dz' \\
I(z, \omega_3) &= \int_{\omega_3}^{z} \zeta_2(z')dz'.
\end{align*}
$$

(75)

Some comments are in order about such a result. $\zeta(z)$ is an entire function and $I(z, \omega_3)$ is a single-valued function of $z$ in the fundamental parallelogram due to the absence of the constant term in the expansion of $\zeta(z)$

$$
\zeta(z) = \frac{1}{z} - \frac{g_2 z^3}{2^2 \cdot 3 \cdot 5} + \ldots
$$

(76)

On the other hand $\zeta_2(z, \omega_3)$ and $\zeta_3(z, \omega_3)$ are not single-valued functions; nonetheless the combinations

$$
\begin{align*}
\bar{z} \zeta_2(z, \omega_3) + \zeta_3(z, \omega_3) \\
\bar{z} \zeta_3(z, \omega_3) + \bar{z} \zeta_3(z, \omega_3)
\end{align*}
$$

(77)

are single valued in the fundamental parallelogram. In fact we have

$$
\zeta_2(z, \omega_3) = \log z + \text{regular terms}
$$

(78)

and

$$
\zeta_3(z, \omega_3) = z \log z + \text{regular terms}
$$

(79)

from which it follows that the terms $\bar{\omega}_1 \zeta_2(\omega_1, \omega_3) + \zeta_3(\omega_1, \omega_3)$ and $\omega_1 \zeta_3(\omega_1, \omega_3) + \bar{\omega}_1 \zeta_3(\omega_1, \omega_3)$ in (74) are well defined.

It is very important that the paths chosen in evaluating $\zeta(\omega_1, \omega_3)$ and $\zeta_3(\omega_1, \omega_3)$ are the same even if there is no preferred path.
One can easily verify that, as expected, for the square and the equianharmonic case such second order contribution vanishes. On the other hand is very cumbersome to verify directly the modular invariance of eq. (74). Modular invariance of eq. (74) is assured at the exact and also perturbative level by the group composition properties of the monodromies over cycles. Obviously the same result (74) is obtained using the first approach to the monodromy problem i.e. starting from eq. (26). Iterating the process in eq. (67) one can go to higher orders.

We come now to the convergence radius of the perturbative series in $\epsilon$. A rigorous lower bound on the convergence radius can be obtained applying Rouché theorem [43]. Equation (67) can be rewritten as

$$\beta - \beta_1 + \epsilon G(\beta, \bar{\beta}, \epsilon) = 0$$

(80)

with $\beta_1$ given by (71). It will be useful to exploit the polarization technique [36] i.e. to introduce in addition to $\beta$ an other independent complex variable $\beta_c$ and consider the system

$$\beta - \beta_1 + \epsilon G(\beta, \beta_c, \epsilon) = 0$$

(81)

$$\beta_c - \bar{\beta}_1 + \epsilon \bar{G}(\beta_c, \beta, \epsilon) = 0$$

(82)

where $\bar{G}$ is the analytic function obtained by conjugating in the power expansion the coefficients of $G$. Obviously if $\beta, \beta_c$ for real $\epsilon$ is a solution of the above system, also $\bar{\beta}_c, \bar{\beta}$ is a solution. If, always for real $\epsilon$, the solution is unique then we have $\beta_c = \bar{\beta}$ and such solution is the solution of the monodromy problem.

Given a positive constant $B$ we can always find a $\delta$ such that for $|\epsilon| < \delta$

$$B > |\epsilon \bar{G}(\beta_c, \beta, \epsilon)|$$

(83)

for all $\beta, \beta_c$ with $|\beta - \beta_1| \leq B$, $|\beta_c - \bar{\beta}_1| \leq B$. Then due to the analyticity of $G$ we can apply Rouché theorem to conclude that (82) for $|\epsilon| < \delta$ has one and only one solution $\beta_c = \beta_c(\beta, \epsilon)$, with $|\beta_c - \bar{\beta}_1| \leq B$. Moreover such $\beta_c$ will be an analytic function of $\beta$ and $\epsilon$.

We substitute now such $\beta_c(\beta, \epsilon)$ into (81) where we can again apply Rouché theorem and thus find a unique solution $\beta = \beta(\epsilon)$. For real $\epsilon$, $\beta(\epsilon)$, $\beta_c = \bar{\beta}(\epsilon)$ is the unique solution of the system and being self conjugate it is the solution of the monodromy problem and $\delta$ will be a rigorous lower bound for the convergence of the perturbative expansion. Obviously if we want to optimize the outcome, we have to choose $B$ as to render $\delta$ as large as possible.

As we shall see in the following choosing a too large $B$ makes the bounds on $G$ increase faster than $B$ and thus $\delta$ has to decrease to satisfy (83). On the other hand it is obvious that $B$ small requires $\delta$ small.
As already mentioned the first approach of eq. (26) requires the integration along a shorter path in the $z$-plane. As the simple bounds on $f_j$ and $g_j$ we shall give below behave exponentially in the length of the integration path, the first approach, even if eq. (26) is more complicated than eq. (27), is more apt to give a larger lower bound on the convergence radius.

To compute such lower bound on the convergence radius we use the following simple rigorous inequalities

\[
|f_1(\omega_1) - 1| \leq \cosh(\sqrt{\epsilon(|\beta| + m_1)} \ |\omega_2|) - 1
\]

\[
|f_2(\omega_1) + \omega_2| \leq \frac{\sinh(\sqrt{\epsilon(|\beta| + m_1)} \ |\omega_2|)}{\sqrt{|\epsilon|(|\beta| + m_1)}} - |\omega_2|
\]

\[
|g_1(\omega_1) - \epsilon(\beta \omega_2 - \zeta(\omega_2))| \leq \epsilon \sqrt{|\epsilon|(|\beta| + m_1)} \ |\omega_2| \ [\sinh(\sqrt{\epsilon(|\beta| + m_1)} \ |\omega_2|) - \sqrt{|\epsilon|(|\beta| + m_1)} \ |\omega_2|]
\]

\[
|g_2(\omega_1) - 1| \leq \cosh(\sqrt{\epsilon(|\beta| + m_1)} \ |\omega_2|) - 1
\]

where $m_1$ is the maximum of the modulus of $\wp(z)$ along the segment $[\omega_3, \omega_1]$, and similar inequalities for the functions $f_k$ and $g_k$ with argument $\omega_2$. More elaborate inequalities can provide a larger lower bound for the convergence radius. We report in Table 1 the lower bounds on the convergence radius for a few values of the modulus $\tau$ obtained with the above described method. As expected the square (for which we know that $\beta$ is zero) gives the largest lower bound. Due to the exponential behavior of the inequalities (84-87) the bound shrinks to zero in the highly asymmetric configurations. The method applies for any $\tau$.

| $\tau$ | $i$ | $2i$ | $3i$ | $4i$ | $5i$ |
|---|---|---|---|---|---|
| $\epsilon_c$ | 0.1202 | 0.05244 | 0.02581 | 0.01512 | 0.00988 |
| $B$ | 2.1667 | 3.80021 | 5.64317 | 7.435 | 9.19403 |

Table 1: $\epsilon_c$ is the rigorous lower bound on the convergence radius. The physical region for the coupling is $0 < \epsilon \leq 1/4$. $B$ is the parameter appearing in eq. (83).

### 8 $\phi$-perturbation theory

In this section we shall develop perturbation theory directly from the Liouville equation. The analytic nature of the perturbative expansion has to be borrowed form the rigorous
treatment of the previous section; on the other hand one can easily obtain in this way the value of the action to third order in $\eta$.

The Green function on the torus of half-periods $\omega_1 = \text{real}$, $\omega_2$, $\tau = \omega_2/\omega_1$ is given by

$$G(z) = \frac{1}{4\pi} \log \left| \frac{\theta_1(\frac{\pi z}{2\omega_1})\tau}{\eta_D} \right|^2 + \frac{i(z - \bar{z})^2}{16\omega_1(\omega_2 - \bar{\omega}_2)} \quad (88)$$

satisfying

$$\Delta G(z) = \delta^2(z) - \frac{1}{a} \quad (89)$$

with $a = 4\omega_1^2 \tau_I$. The arbitrary additive constant in $G$ has been chosen in (88) as to have

$$\int G(z) dz \wedge d\bar{z} = 0. \quad (90)$$

Then we expand $\phi = \phi_0 + \phi_1 + \phi_2 + \ldots$ to have

$$e^{\phi_0} = \frac{2\pi \eta}{a} \quad (91)$$

$$- \frac{1}{4} \Delta \phi_1(z) = 2\pi \eta (\delta^2(z) - \frac{1}{a}) \quad (92)$$

from which

$$\phi_1(z) = -8\pi \eta G(z). \quad (93)$$

This is true if the constant term in $G$ is chosen as in (88, 90). Next we have

$$- \frac{1}{4} \Delta \phi_2(z) = \frac{16\pi^2 \eta^2}{a} G(z) \quad (94)$$

and then

$$\phi_2(z) = -\frac{64\pi^2 \eta^2}{a} \int G(z - z')G(z')dz' \wedge d\bar{z}' \frac{i}{2}. \quad (95)$$

Using (91,93) and $\theta_1'(0) = 2\eta_D^3$, being $\eta_D$ the Dedekind modular function, we obtain for the $X$ of section 5

$$X = \log \frac{2\pi \eta}{4\omega_1^2 \tau_I} - 4\eta \log \left| \frac{\pi \eta_D^2}{\omega_1} \right| \quad (96)$$

from which

$$\frac{1}{2\pi} S_z = \eta - \eta \log \frac{2\pi \eta}{4\omega_1^2 \tau_I} + 2\eta^2 \log \left| \frac{\pi \eta_D^2}{\omega_1} \right| + O(\eta^3). \quad (97)$$

Eq. (97) can be compared with the exact result for the square (51) where with $e_1 = -e_2 = 1$, $e_3 = 0$ we have

$$\omega_1 = -i\omega_2 = \frac{\sqrt{\pi} \Gamma(\frac{1}{4})}{4\Gamma(\frac{3}{4})}. \quad (98)$$

---

2 Here we use for $\theta_1$ the convention of [34] not the one of [33]
Using
\[ \eta_D(i) = \frac{\Gamma(\frac{1}{4})}{2\pi^{3/4}} , \quad \frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})}{\pi} = \sqrt{2} \]  \hspace{1cm} (99)
we obtain
\[ \frac{1}{2\pi} S_z = -\eta \log \eta + \eta \left( 1 - 2 \log \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} - 3 \log 2 \right) + \eta^2 \log 2 \]  \hspace{1cm} (100)
which agrees with the expansion to second order of (51). Similarly one compares eq. (97) with the exact result (55) for the equianharmonic case finding agreement.

Using the exact relation
\[ \frac{1}{2} \partial_z^2 \phi - \frac{1}{4} (\partial_z \phi)^2 = \epsilon (\beta + \wp(z)) \]  \hspace{1cm} (101)
and the expression of \( \phi_1 \) (93) one can also retrieve the accessory parameter \( \beta \) to first order i.e. eq. (71) using
\[ \zeta(z) = \frac{\zeta(\omega_1)z}{\omega_1} + \frac{\pi}{2\omega_1} \frac{\theta_1(\frac{\tau z}{\omega_1})}{\theta_1(\frac{\tau \omega_1 z}{\omega_1})} \]  \hspace{1cm} (102)
and the Legendre relation [33].

It is easy in this framework to obtain \( \phi_2(0) \) i.e. \( X \) to second order which integrated provides the action to third order in \( \eta \) for any \( \tau \). This would be very tedious to obtain in the approach described in section 7 which however provides the value of the accessory parameter to second order.

We must add to eq. (96) \( \phi_2(0) \) given by
\[ \phi_2(0) = -\frac{16\pi^2 \eta^2}{\pi^2 \omega_1^2} \int G(z')G(-z')dz \wedge d\bar{z} \frac{i}{2} = -\frac{4\eta^2 \tau^2}{\pi^2} \sum_{mn} \frac{1}{(m-\tau n)^2(m-\bar{\tau} n)^2} \sum_{mn} \]  \hspace{1cm} (103)
where the prime means \( m = n = 0 \) excluded. Such a sum, using standard resummation formulas [37], can be rewritten in terms of two simple sums which converge rapidly due to the presence of the imaginary part of \( \tau \)
\[ s = \sum_{mn} \frac{1}{(m-\tau n)^2(m-\bar{\tau} n)^2} = \frac{\pi^4}{45} + 2\pi \sum_{n} \frac{1}{n^3(\tau - \bar{\tau})^3} \left( \cot \pi \tau n - \cot \pi \bar{\tau} n \right) \]  \hspace{1cm} (104)

If we integrate in \( \eta \) according to eq. (35) we obtain the third order contribution to the one-point function
\[ \frac{1}{2\pi} S_z^{(3)} = \frac{4\eta^3 \tau^2}{3\pi^2} \eta^3 s . \]  \hspace{1cm} (105)
9 General analytic properties

In this section we shall examine the general analytic properties of the accessory parameter as a function of the coupling and of the modulus at the non-perturbative level.

We start with the remark that the uniqueness of Picard solution implies also the uniqueness of the accessory parameter as

$$e^{\varphi/2} \partial_u^2 e^{-\varphi/2} = -Q(u)$$

which identifies uniquely $\beta$. Actually $\beta$ can be obtained from

$$\frac{1}{2\pi i} \oint_{e_1} e^{\varphi/2} \partial_u^2 e^{-\varphi/2} du = \frac{\epsilon}{4(e_1 - e_2)(e_3 - e_1)} + \frac{3e_1}{8(e_1 - e_2)(e_1 - e_3)}.$$  

In [30] it was proven using Green function technique that when $\epsilon$ varies in the physical interval $0 < \epsilon \leq 1/4$, the functions $\varphi$, $\partial_u \varphi$, $\partial_u^2 \varphi$ are uniformly bounded functions of $u$ in any region of the $u$ plane, obtained by excluding finite disks around the singularities, with bounds which depend continuously on $\epsilon$.

Thus taking the contour of the integral (107) at a finite distance from $e_1$ we have that $\beta$ is a bounded function of $\epsilon$ when it varies in the physical region. Such a result combined with the uniqueness of the solution implies that $\beta$ is a continuous function of $\epsilon$. In fact if $\epsilon_n$ is a sequence of values converging to $\epsilon_0$, due to the boundedness the corresponding sequence $\beta_n$ must have at least one limit point. However a limit point due to the continuity of the basic relations (18,19) is a solution of the monodromy problem and being such solution unique there must be only one limit point. Continuity plays an important role in the following as in most of the procedures related to the zeros of analytic functions [31].

Starting from the relation (19) we recall that if at a point $\epsilon_0$ in the physical range $M_{12}(C_2) \neq 0$ we have also $M_{21}(C_2) \neq 0$ as explained in section 2. On the other hand if $M_{12}(C_2) = 0$ we have also $M_{21}(C_2) = 0$. We cannot have at the same time $M_{12}(C_1) = 0$ and $M_{12}(C_2) = 0$ otherwise the parameter $\kappa$ would be left undetermined against Picard’s uniqueness theorem. Thus given any value $\epsilon_0$ in the physical region dividing either by $M_{12}(C_2)M_{21}(C_2)$ or by $M_{12}(C_1)M_{21}(C_1)$ we reach in an open interval around $\epsilon_0$ the structure

$$A(\beta, \epsilon) = \bar{B}(\bar{\beta}, \epsilon)$$

with $A$ analytic function of $\beta$ and $\epsilon$ and $\bar{B}$ analytic function of $\bar{\beta}$ and $\epsilon$.

As done in section 7 it will be useful to employ the polarization technique [36] introducing in addition to $\beta$ an other independent complex variable $\beta_c$. 

20
We consider now the system

\[ E_1 = A(\beta, \epsilon) - \bar{B}(\beta_c, \epsilon) = 0 \]
\[ E_2 = B(\beta, \epsilon) - \bar{A}(\beta_c, \epsilon) = 0. \]  \tag{109}

We look for solutions of the above system for \(0 < \epsilon \leq 1/4\). Obviously if \((\beta, \beta_c)\) is a solution also \((\bar{\beta}, \bar{\beta}_c)\) is a solution but we shall be particularly interested in self-conjugate solutions i.e. those for which \(\beta_c = \bar{\beta}\) insofar they are the solution of the monodromy problem. Actually from the existence and uniqueness result of the monodromic solution we know that for \(0 < \epsilon \leq 1/4\) there is always one and only one self-conjugate solution, in addition, possibly, to other non self-conjugate solutions. In the following we shall denote such unique self-conjugate solution as \(\beta(\epsilon)\).

For \(\epsilon = \epsilon_0\) we have

\[ A(\beta(\epsilon_0), \epsilon_0) = B(\beta(\epsilon_0), \epsilon_0). \]  \tag{110}

The Weierstrass preparation theorem [35, 31] can be applied to \(A(\beta, \epsilon) - A(\beta(\epsilon_0), \epsilon_0)\) if

\[ A(\beta, \epsilon) - A(\beta(\epsilon_0), \epsilon_0) \]  \tag{111}

is not identically zero in \(\beta\). This can happen only at a finite number of points in the open interval otherwise \(\frac{\partial A}{\partial \beta} \equiv 0\) i.e. \(A\) would be a function only of \(\epsilon\) which from the structure of the \(M_{jk}(C_l)\) of section 3 is not true. We exclude such a finite number of points.

Thus except at most a finite number of points we can apply Weierstrass preparation theorem [35, 31]

\[ A(\beta, \epsilon) - A(\beta(\epsilon_0), \epsilon_0) = u(\beta, \epsilon)(\beta - \beta(\epsilon_0))^m + c_{m-1}(\epsilon)(\beta - \beta(\epsilon_0))^{m-1} + ... + c_0(\epsilon) \]  \tag{112}
\[ \bar{B}(\beta_c, \epsilon) - \bar{B}(\beta(\epsilon_0), \epsilon_0) = v(\beta_c, \epsilon)(\beta_c - \beta(\epsilon_0))^n + g_{n-1}(\epsilon)(\beta_c - \beta(\epsilon_0))^{n-1} + ... + g_0(\epsilon) \]  \tag{113}

with \(u(\beta, \epsilon), v(\beta_c, \epsilon)\) units and \(c_k, g_k\) analytic functions of \(\epsilon\), vanishing at \(\epsilon_0\).

We consider first the case: \(m = n = 1\).

At \(\epsilon_0, \beta(\epsilon_0), \bar{\beta}(\epsilon_0)\) we have for the system (109) the Jacobian

\[ J = \frac{\partial(E_1, E_2)}{\partial(\beta, \beta_c)} = -|u(\beta(\epsilon_0), \epsilon_0)|^2 + |v(\beta(\epsilon_0), \epsilon_0)|^2. \]  \tag{114}

If \(J \neq 0\) we can apply the implicit function theorem according to which the solution \(\beta\) and \(\beta_c\) is unique (and thus self-conjugate for real \(\epsilon\)) and \(\beta\) is an analytic function in an open interval around \(\epsilon_0\) and thus we have local analyticity.

If \(J = 0\) then we look at the equation

\[ u(re^{i\alpha} + \beta(\epsilon_0), \epsilon_0) re^{i\alpha} - v(re^{-i\alpha} + \bar{\beta}(\epsilon_0), \epsilon_0) re^{-i\alpha} = 0. \]  \tag{115}
For \( r \neq 0 \) divide (115) by \( r \) and call it \( F(r, \alpha) \).

\[
F(r, \alpha) = u(re^{i\alpha} + \beta(\epsilon_0), \epsilon_0)e^{i\alpha} - v(re^{-i\alpha} + \overline{\beta(\epsilon_0)}, \epsilon_0)e^{-i\alpha} = 0 .
\] (116)

Consider a solution \( \alpha_0 \) of \( F(0, \alpha) = 0 \)

\[
u(\beta(\epsilon_0), \epsilon_0)e^{i\alpha_0} = v(\overline{\beta(\epsilon_0)}, \epsilon_0)e^{-i\alpha_0}
\] (117)

which is soluble because \( J = 0 \). Then in the product of the open intervals \( \alpha_0 - \delta < \alpha < \alpha_0 + \delta, -\delta_r < r < \delta_r \) we have that \( F(r, \alpha) \) is a \( C^1 \) function of \( \alpha \) and \( r \), with \( F(0, \alpha_0) = 0 \) and \( F_\alpha(0, \alpha_0) \neq 0 \). Then for small \( r \) we have one solution \( \alpha(r) \) for \( \alpha \) [38], thus a self-conjugate solution with \( r \neq 0 \) (any \( r \) in the above interval) in addition to the \( \beta = \beta(\epsilon_0), \beta_c = \overline{\beta(\epsilon_0)} \). This however violates Picard’s uniqueness result. The conclusion is that either \( J \neq 0 \) or the Weierstrass polynomials (112,113) cannot be both first order. In the same way one excludes Weierstrass polynomials (112) and (113) with the same order, \( m = n > 1 \) and \( |u(\beta(\epsilon_0), \epsilon_0)| = |v(\overline{\beta(\epsilon_0)}, \epsilon_0)| \).

We go back now to the system (109).

Given \( \epsilon_0 \) we have

\[
\bar{B}(\beta(\epsilon_0), \epsilon_0) - A(\beta(\epsilon_0), \epsilon_0) = 0
\]

\[
\bar{A}(\beta(\epsilon_0), \epsilon_0) - B(\beta(\epsilon_0), \epsilon_0) = 0
\] (118)

and in a neighborhood \( \Delta_0 \) of \( \epsilon_0 \), \( \Delta_\beta \) of \( \beta(\epsilon_0) \), \( \Delta_{\beta c} \) of \( \overline{\beta(\epsilon_0)} \) using Weierstrass preparation theorem we can write system (109) as

\[
U(\beta, \beta_c, \epsilon)P_1(\beta_c - \overline{\beta(\epsilon_0)}; \beta, \epsilon) = 0
\]

\[
V(\beta, \beta_c, \epsilon)P_2(\beta_c - \overline{\beta(\epsilon_0)}; \beta, \epsilon) = 0
\] (119)

which, as \( U \) and \( V \) are units, is equivalent to

\[
P_1(\beta_c - \overline{\beta(\epsilon_0)}; \beta, \epsilon) = 0
\]

\[
P_2(\beta_c - \overline{\beta(\epsilon_0)}; \beta, \epsilon) = 0
\] (120)

Necessary and sufficient condition for the two polynomials in (120) to have a common solution in \( \beta_c \) is that the resultant [39][31][40] of the two polynomials \( P_1 \) and \( P_2 \) is zero

\[
R(P_1, P_2) \equiv f(\beta, \epsilon) = 0 .
\] (121)

In particular we know from the existence result that

\[
f(\beta(\epsilon_0), \epsilon_0) = 0 .
\] (122)
Exploiting again Weierstrass preparation theorem eq. (121) can be written for \( \epsilon \in \Delta_1 \subset \Delta_0, \beta \in \Delta_{\beta 1} \subset \Delta_{\beta} \) as

\[
u(\beta, \epsilon) P(\beta - \beta(\epsilon_0); \epsilon) = 0 \tag{123}
\]

with

\[
P(\beta - \beta(\epsilon_0); \epsilon) = (\beta - \beta(\epsilon_0))^m + (\beta - \beta(\epsilon_0))^{m-1}a_{m-1}(\epsilon) + \cdots + a_0(\epsilon). \tag{124}
\]

In order to apply Weierstrass preparation theorem to \( f(\beta, \epsilon) \) we need that \( f(\beta, \epsilon_0) \) does not vanish identically in \( \beta \). The vanishing of \( f(\beta, \epsilon_0) \) would mean that the system (109) at \( \epsilon_0 \) has solution for all \( \beta \) near \( \beta(\epsilon_0) \). This means, using the Weierstrass-polynomial expression for \( A \) and \( \bar{B} \), that \( m = n \) and \(|u(\beta(\epsilon_0), \epsilon_0)| = |v(\beta(\epsilon_0), \epsilon_0)| \) and this implies the existence of infinite self-conjugate solutions with \( \beta \neq \beta(\epsilon_0) \) at \( \epsilon_0 \) and this goes against the uniqueness theorem.

We start now by computing the resultant \( R(P, P') \) i.e. the discriminant of \( P \). If it is not identically zero it can vanish in the interval around \( \epsilon_0 \) included in the Weierstrass set at most at a finite number of points, otherwise it would be identically zero. Thus except at those finite number of points we can apply the analytic implicit function theorem [35] to have analyticity of \( \beta(\epsilon) \) in a open interval around \( \epsilon_0 \).

The general case can be treated by computing the reduced Gram determinants \( D_n \) of the power-vectors of the roots [31]

\[
D_n = \begin{vmatrix}
s_0 & s_1 & \cdots & s_{n-1} \\
s_1 & s_2 & \cdots & s_n \\
\vdots & \vdots & \ddots & \vdots \\
s_{n-1} & s_n & \cdots & s_{2n-2}
\end{vmatrix} \tag{125}
\]

where

\[
s_i = \xi_1^i + \xi_2^i + \cdots + \xi_m^i \tag{126}
\]

being \( \xi_k \) the \( m \) roots of \( P \). Being \( D_n \) a symmetric polynomial of the roots it is a polynomial in the coefficients \( a_k(\epsilon) \) and as such an analytic function of \( \epsilon \). If \( R(P, P') \equiv D_m \) vanishes identically it means that we have at each \( \epsilon \) a double or higher order root. Then compute \( D_{m-1} \). If it is not identically zero it means that the maximum number of distinct roots is \( m - 1 \) and the set where they are \( m - 1 \) is open and given by subtracting from the initial open set the zeros of \( D_{m-1} \). These are isolated points [31] and thus finite in number. Moreover in the region where the maximum number of distinct roots is reached all the solutions of \( P(\beta - \beta(\epsilon_0); \epsilon) = 0 \) (the so called local sheets) are analytic [31], and in particular Picard solution is analytic.
Suppose now that $D_m = D_{m-1} \equiv 0$. Then compute $D_{m-2}$ and proceed as above. If $D_{m-2}$ is not identically zero it means that the maximum number of distinct roots is $m - 2$; it can vanish only at a finite number of point and except at those points all solutions of (121) are analytic.

The procedure ends due to the fact that $D_1 = s_0 \equiv m$. The vanishing of all $D_n$, $n > 1$ corresponds to the situation where we have only one $m$-time degenerate solution i.e.

$$P(\beta - \beta_0; \epsilon) = (\beta - \beta(\epsilon))^m$$

from which $\beta(\epsilon) - \beta(\epsilon_0) = -\frac{1}{m}a_{m-1}(\epsilon)$ which is analytic.

Removing the described finite number of points we have that given any $\epsilon_0$ there is an open disk around $\epsilon_0$ where all the solutions of (121) and in particular the unique self-conjugate Picard solution, are analytic except for at most a finite number of points.

We saw in section 7 that a finite interval around the origin $0 < \epsilon \leq \delta$ is covered by the convergent perturbation theory treatment. For the remainder $\delta \leq \epsilon \leq 1/4$ we can associate to each $\epsilon$ an open set with the above properties and then as $\delta \leq \epsilon \leq 1/4$ is compact we can extract a finite covering. We conclude that the unique self-conjugate Picard solution is analytic on the whole physical region except at most at a finite number of points.

Similarly one treats the dependence of $\beta$ on the modulus $\tau$.

Choose any $\tau_0$ belonging to the fundamental region and $\epsilon_0$ with $0 < \epsilon_0 \leq 1/4$.

From now on we shall neglect in the notation $\epsilon_0$ i.e. we shall work at fixed $\epsilon$.

We start again from the equation

$$A(\beta, \tau) - \bar{B}(\bar{\beta}, \bar{\tau}) = 0.$$  \hspace{1cm} (128)

As done above it is useful to apply the polarization technique to $\beta$ by introducing an other independent complex variable $\beta_c$, but this time we apply the polarization technique also to the variable $\tau$, the modulus, by introducing in addition to $\tau$ an independent complex variable $\tau_c$. We remark that in the previous treatment of the dependence of $\beta$ on $\epsilon$ we could have applied the polarization technique also to the variable $\epsilon$ but being the physical values of $\epsilon$ real we would have reached the same results. Here instead the physical values of $\tau$ are in the complex.

We consider the system

$$A(\beta, \tau) - \bar{B}(\beta_c, \tau_c) = 0$$

$$B(\beta, \tau) - \bar{A}(\beta_c, \tau_c) = 0.$$  \hspace{1cm} (129)
We look for solutions of the above system for \( \tau \) in the fundamental region and \( \tau_c = \overline{\tau} \).

Obviously if \( \beta, \beta_c \) is a solution also \( \overline{\beta_c}, \overline{\beta} \) is a solution but we shall be interested in self-conjugate solutions i.e. those for which \( \beta_c = \overline{\beta} \) insofar they are the solution of the monodromy problem. Actually from the existence and uniqueness result of the monodromic solution we know that for \( 0 < \epsilon_0 \leq 1/4 \) and \( \tau_c = \overline{\tau} \) there is always one and only one self-conjugate solution, in addition, possibly, to other non self-conjugate solutions. In the following we shall denote the unique self-conjugate solution as \( \beta(\tau) \).

Chosen \( \tau_0 \) in the fundamental region we have

\[
A(\beta(\tau_0), \tau_0) = \overline{B(\beta(\tau_0), \overline{\tau_0})}.
\]

Applying the Weierstrass preparation theorem to \( A(\beta, \tau) \) and \( \overline{B(\beta_c, \tau_c)} \) we have

\[
A(\beta, \tau) - A(\beta(\tau_0), \tau_0) = u(\beta, \tau)((\beta - \beta(\tau_0))^m + c_{m-1}(\tau)(\beta - \beta(\tau_0))^{m-1} + \ldots + c_0(\tau)) \tag{131}
\]

\[
\overline{B(\beta_c, \tau_c)} - \overline{B(\beta(\tau_0), \overline{\tau_0})} = v(\beta_c, \tau_c)((\beta_c - \beta(\tau_0))^n + g_{n-1}(\tau_c)(\beta_c - \beta(\tau_0))^{n-1} + \ldots + g_0(\tau_c)) \tag{132}
\]

with \( c_k \) analytic functions of \( \tau \) vanishing at \( \tau_0 \) and \( g_k \) analytic functions of \( \tau_c \) vanishing at \( \overline{\tau_0} \) and \( u \) and \( v \) units.

As done for the dependence on \( \epsilon \), for \( m = n = 1 \) if \( J(\tau_0, \overline{\tau_0}) \neq 0 \) we are in the analytic situation while \( J(\tau_0, \overline{\tau_0}) = 0 \) is excluded by the uniqueness result. In the same way one excludes Weierstrass polynomials \([131]\) and \([132]\) with the same order \( m = n > 1 \) and \( |u(\beta(\tau_0), \tau_0)| = |v(\beta(\tau_0), \overline{\tau_0})| \).

Given \( \tau_0 \) as

\[
\overline{B(\beta(\tau_0), \overline{\tau_0})} - A(\beta(\tau_0), \tau_0) = 0 \\
A(\overline{\beta(\tau_0)}, \overline{\tau_0}) - B(\beta(\tau_0), \tau_0) = 0 \tag{133}
\]

in a neighborhood \( \Delta \) of \( \tau_0, \Delta_c \) of \( \overline{\tau_0}, \Delta_\beta \) of \( \beta(\tau_0) \), \( \Delta_{\beta c} \) of \( \overline{\beta(\tau_0)} \) we can write system \([129]\) as

\[
U(\beta, \beta_c, \tau, \tau_c)P_1(\beta_c - \overline{\beta(\tau_0)}; \beta, \tau, \tau_c) = 0 \\
V(\beta, \beta_c, \tau, \tau_c)P_2(\beta_c - \overline{\beta(\tau_0)}; \beta, \tau, \tau_c) = 0 \tag{134}
\]

which as \( U \) and \( V \) are units is equivalent to

\[
P_1(\beta_c - \overline{\beta(\tau_0)}; \beta, \tau, \tau_c) = 0 \\
P_2(\beta_c - \overline{\beta(\tau_0)}; \beta, \tau, \tau_c) = 0. \tag{135}
\]
A common solution of (135) in $\beta_c$ implies (necessary and sufficient condition) the resultant of $P_1, P_2$ to be zero

$$R(P_1, P_2) \equiv f(\beta, \tau, \tau_c) = 0.$$  

(136)

In particular we know from the existence result that

$$f(\beta(\tau_0), \tau_0, \bar{\tau}_0) = 0.$$  

(137)

$f(\beta, \tau_0, \bar{\tau}_0)$ cannot be identically zero in $\beta$ for the same reasoning as the one performed after eq.(124); thus we can apply Weierstrass preparation theorem and write

$$f(\beta, \tau, \tau_c) = u(\beta, \tau, \tau_c)P(\beta - \beta(\tau_0); \tau, \tau_c)$$  

(138)

with

$$P(\beta - \beta(\tau_0); \tau, \tau_c) = (\beta - \beta(\tau_0))^m + a_{m-1}(\tau, \tau_c)(\beta - \beta(\tau_0))^{m-1} + \cdots + a_0(\tau, \tau_c)$$  

(139)

and the coefficients $a_n(\tau, \tau_c)$ analytic in $\tau, \tau_c$ and vanishing at $\tau_0, \bar{\tau}_0$.

Thus the equation has become

$$P(\beta - \beta(\tau_0); \tau, \tau_c) = 0.$$  

(140)

$\beta$ is analytic in $\tau, \tau_c$ at all points $\tau, \bar{\tau}$ except those at which $P'(\beta(\tau); \tau, \bar{\tau}) = 0$. These $\tau$ satisfy the discriminant equation

$$R(P, P') \equiv D_m(\tau, \bar{\tau}) = 0$$  

(141)

with $R(P, P')$ analytic in $\tau, \tau_c$ being a polynomial in the $a_n(\tau, \tau_c)$.

We distinguish two cases

1. $D_m(\tau, \bar{\tau})$ is identically zero. Then due to a theorem on polarization [36] $D_m(\tau, \tau_c)$ is identically zero.

2. Otherwise $D_m(\tau, \tau_c)$ can vanish only on a thin set [35, 31], of which the points $\tau$ such that $D_m(\tau, \bar{\tau}) = 0$ are a subset. Thin set have zero measure [35]. Outside such thin set the equation is invertible and thus $\beta$ analytic function of $\tau, \tau_c$, i.e. $\beta(\tau, \bar{\tau})$ a real-analytic function of $\tau$.

In case 1 i.e.

$$D_m(\tau, \tau_c) \equiv 0$$  

(142)

we compute $D_{m-1}$. If it is not identically zero it means that the maximum number of distinct roots is $m - 1$ and the set where they are $m - 1$ is open and given by subtracting
from the initial open set the zeros of $D_{m-1}$ which is a thin set and as such of zero measure. In the region where the maximum number of distinct roots is reached all the solutions of (136) (local sheet) are analytic [31], and in particular Picard solution is analytic. Suppose now that

$$D_m = D_{m-1} \equiv 0.$$  \hspace{1cm} (143)

Then we compute $D_{m-2}$ and proceed as above.
The procedure ends due to the fact that $D_1 \equiv m$. It corresponds to the situation where we have only one $m$-times degenerate solution i.e.

$$P(\beta - \beta(\tau_0); \tau, \tau_c) = (\beta - \beta(\tau, \tau_c))^m = 0$$ \hspace{1cm} (144)

from which we have $\beta(\tau, \tau_c) - \beta(\tau_0) = -\frac{1}{m} a_{m-1}(\tau, \tau_c)$ which is analytic in $\tau, \tau_c$ and thus $\beta(\tau, \bar{\tau})$ real-analytic in $\tau$.

We can divide the fundamental region of $\tau$ in a denumerable set of horizontal strips which are compact. We have a zero-measure set of possible non real-analyticity points in each strip and the union of such infinite zero measure set has zero measure.

We conclude that for each $\epsilon$ in the physical region the accessory parameter $\beta$ is a real-analytic function of $\tau$ in the whole fundamental region except at most for a zero measure set.

10 Conclusions

We have considered the problem of accessory parameters on the torus. The specific case we dealt with is that of a single source which corresponds to a special cases of the Heun equation. We proved that necessary and sufficient condition to obtain monodromy at all singularities is the fulfillment of a single implicit equation. Several features of the accessory parameter can be extracted from such an equation. A perturbative series was developed and rigorous lower bound on the radius of convergence of the perturbative series has been given. The second order result for the accessory parameter and third order result for the one point function was explicitly computed.

Modular invariance is useful to find the value of the accessory parameter in some special cases and it is satisfied by the perturbative solution. General analytic properties of the dependence of the accessory parameter on the source strength and on the modulus have been proved. The real-analyticity of the dependence of the accessory parameters on the moduli is an essential step in proving Polyakov relation on the sphere which has the meaning of determining the response of the on-shell action on the position of the
singularities. We shall devote a separate paper to the structure and meaning of the Polyakov relation on the torus.

The described technique can be extended to treat the four-point case or higher number of points on the sphere, higher point function on the torus or higher genus surfaces. In [18] an integral expression has been given for the accessory parameter for the four point function on the sphere. Such a procedure should be extensible to the one point function on the torus. However comparison of that result e.g. with the second and third order result of sections 7, 8 will not be immediate as our result is an expansion in the source strength while the results of [18] are nearer to an expansion in the position of the singularities which in the case of the torus correspond to the value of the modulus.

On the other hand in [20, 21] analytical technique for dealing with the deformation of the torus have been developed. These could allow the comparison of the result obtained along the lines of [18].

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