Vertical angular momentum constraint on lunar formation and orbital history

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The Moon likely formed in a giant impact that left behind a fast-rotating Earth, but the details are still uncertain. Here, we examine the implications of a constraint that has not been fully exploited: the component of the Earth–Moon system’s angular momentum that is perpendicular to the Earth’s orbital plane is nearly conserved in Earth–Moon history, except for possible intervals when the lunar orbit is in resonance with the Earth’s motion about the Sun. This condition sharply constrains the postimpact Earth orientation and the subsequent lunar orbital history. In particular, the scenario involving an initial high-obliquity Earth cannot produce the present Earth–Moon system. A low-obliquity postimpact Earth followed by the evection limit cycle in orbital evolution remains a possible pathway for producing the present angular momentum and observed lunar composition.

Significance

There are various scenarios for the formation of the Moon and subsequent dynamical evolution of the Earth–Moon system, all of which are subject to a constraint that has not previously been fully exploited. Using this constraint, we demonstrate that the recently proposed high-obliquity scenario is not consistent with the present Earth–Moon system. This constraint will have to be taken into account in all future investigations of the formation and evolution of the Moon.

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Data deposition: The computer codes we used for the simulations in this paper are available at GitHub, https://github.com/zhenliangtian/em3d.

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(1, 10–12) can modify \( L_z \). Such resonances or near-resonances are absent in scenarios like the LPT instability, which we term as nonresonant scenarios. The demonstration of the conservation of \( L_z \) for nonresonant scenarios exactly follows the derivation for the Lidov–Kozai problem (SI Appendix, SI Text). The Hamiltonian is averaged over the orbital period of the Moon and the orbital period of the motion of the Earth about the Sun. The average over the lunar period leads to the conservation of the semimajor axis of the Moon. The average over the motion of the Earth about the Sun leads to the conservation of \( L_z \). Tides between the Earth and Moon conserve \( L_z \), but induce long-term changes in \( a \) and the other system parameters.

The AM of the Earth–Moon system \( \vec{L} \) is approximately \( \vec{L}_{\oplus} + \vec{L}_{\odot} \), where \( \vec{L}_{\oplus} \) is the rotational AM of the Earth and \( \vec{L}_{\odot} \) is the orbital AM of the Moon. Here, we are neglecting the small rotational AM of the Moon. Taking the ecliptic as the reference plane, the component of AM perpendicular to this plane is

\[
L_z = L_{\oplus} \cos I_e + L_{\odot} \cos i,
\]

where \( L_{\oplus} \) is the magnitude of \( \vec{L}_{\oplus} \), and \( L_{\odot} \) is the magnitude of \( \vec{L}_{\odot} \). We denote the scalar sum \( L_{\oplus} + L_{\odot} \) by \( L_z \).

In our model, which includes the full rotational and orbital dynamics of the Moon, interacting with an oblate, precessing Earth on a near circular orbit about the Sun, with tidal interactions between the Moon and Earth, we find that \( L_z \) is conserved to a part in a thousand, as the Moon evolves from \( 5R_e \) to \( 50R_e \) (where we terminate the integrations; \( R_e = 6,371 \) km). This is the case, even though the system passes through the instability associated with the LPT and the Cassini state (lunar spin) transitions. Touma and Wisdom (20) investigated the evolution in a model that was, in a number of ways, more complete than that used here. That model included a fully evolving eccentric and inclined Earth orbit, perturbed by all of the chaotically evolving planets, with not only tidal interactions between the Earth and Moon, but also direct tides from the Sun on the Earth, and cross-tidal interactions from tides raised on the Earth by the Sun that affect the Moon. (But it did not include tides raised on the Moon and the rotational dynamics of the Moon, which are included here.) A reexamination of those results shows that, even with all these additional effects, \( L_z \) was still conserved to about 3.5% (Fig. 1). (Note that refs. 16, 20, and 21 already recognized that \( L_z \) is conserved in the averaged \( e = 0 \) nontidal case, and refs. 16 and 20, in addition, stated that \( L_z \) is conserved if only tides between the Earth and Moon are considered. Our derivation is more general.) We show in SI Appendix, SI Text that even without averaging, when solar tides are ignored, the amplitude of oscillation of \( L_z \) is at most of order \( 10^{-3} L_z \) in Earth–Moon history (SI Appendix, Eq. S11 and Fig. S2) \((L_z = C_{\oplus} \sqrt{GM}R_{\odot}^2)\) is the reference AM, where \( C_{\oplus} \) is Earth’s largest principal moment, \( C_{\odot} = 0.3308 M_eR_e^2 \) is the present-day value, \( M_e \) is Earth’s mass; \( L_{\oplus} = 0.345 L_z \) and \( L_{\odot} = 0.309 L_z \).

A dynamical evolution, \( L_z \) is also susceptible to small changes due to collisional processes after the Moon-forming giant impact, e.g., the stochastic late-accretional impacts on Earth that are proposed to explain the presence of highly siderophile elements in the terrestrial mantle (22–26). These impacts can change the Earth’s rotational AM by up to 4%, with the Earth rotating with a period between 6 and 8 h (22). Taking \( C_\odot \approx C_{\oplus} \), the change in \( L_z \) is \( \Delta L_z \approx C_{\odot} \cdot 2\pi/(72 h) \cdot 4\% \times 10^{-3} L_z \), i.e., \( 2.3\% \) of \( L_z \). (Because of the variation of \( L_z \) in the Moon, these effects may be multiplied by this factor, and the effects on the system can be ignored. Even if the change in \( L_{\odot} \) is perfectly aligned to the vertical direction, this will only cause changes to \( L_z \) by up to \( 2.4\% \) of \( L_z \) (either increase or decrease), so we ignore these possible late stochastic variations in \( L_z \).

Therefore, we can take \( L_z \) conservation as a strong constraint on the evolution of the Earth–Moon system. For nonresonant scenarios of Earth–Moon formation and evolution (e.g., ref. 13), the postimpact \( L_z \) value should be near \( L_z^e \) (at most a few percent different). For models of Earth–Moon history that involve resonances related to Earth’s motion about the Sun (e.g., refs. 1 and 10–12), the postresonance \( L_z \) should be near \( L_z^e \) in the same way.

**L_z** Constraint on the High-Obliquity Scenario

Čuk et al. (13) argued that following a high-AM, high-obliquity postimpact Earth, the present-day AM, \( I_e \) and \( i \) can be produced through nonresonant orbital evolution. However, the values of \( L_z \) for their initial conditions were much lower than \( L_z^e \) (SI Appendix, Table S1 and Figs. S1, S3, and S4) and therefore inconsistent with the present Earth–Moon system.* We investigate the high-obliquity Moon-forming scenario with the \( L_z \) constraint taken into account. Our numerical model is exactly the same as used in ref. 10. It is based on the N-body symplectic mapping algorithm (27) and the conventional Darwin–Kaula constant \( Q \) model (28). We provide a detailed comparison of our algorithm to that of ref. 13 in the SI Appendix, Model Comparison. As a check, we did calculations with the same initial conditions as used in ref. 13. The main features of ref. 13 are reproduced, but some differences are found. We find smaller final \( L_z \), with larger final \( I_e \) and \( i \) (SI Appendix, Figs. S1, S3, and S4).

We sample the \( L_z \)-consistent (i.e., \( L_z = L_z^e \)) initial conditions (postimpact states) in the whole range of successful high-AM giant impact simulations; \( I_e \), \( i \), and \( L_z \) are of key interest. We assume that the Moon accreted on the Earth’s equatorial plane, i.e., initially \( i = I_e \). We take the initial \( a = 3.5 R_e \), just outside the Earth’s Roche limit. So \( (I_e, L_z) \) adequately represents an initial state. In ref. 1, for successful impacts, \( L_z \) ranges in 1.94 to 2.84 \( L_z^e \) or 0.67 to 0.98 \( L_z^e \). Candidate impacts in ref. 2 produce \( L_z \) from 1.77 to 2.71 \( L_z^e \) or 0.61 to 0.94 \( L_z^e \). With the constraint of \( L_z = L_z^e \),

*Even though Čuk et al. (13) tried to match the present-day AM, \( I_e \), and \( i \), the \( L_z \) values were not \( L_z^e \). They simulated the Earth–Moon history in two steps, first for \( a \leq 25 R_e \), then for \( 25 R_e < a \leq 60 R_e \). After the first step, AM and \( i \) get close to the present values, but \( i \) is still much larger than \( \pi/2 \). Then, in the second step, they concentrated on reducing \( i \) to \( \pi \), but did not track \( L_z \). Actually, \( L_z \) will increase in the second step, so the final state will not match the present values of AM, \( I_e \), and \( i \).
we take four sampling points: (57.6°, 0.63Lr), (61°, 0.7Lr), (65°, 0.8Lr), and (70°, 0.98Lr). A larger initial \( I_0 \) corresponds to a larger initial \( L_z \). We take \( Q_e/k_{2e} = 100 \) and \( Q_m/k_{2m} = 100 \), the values used in ref. 13, where \( k_2 \) is the potential Love number, and \( Q \) is the tidal quality factor.

In the case (57.6°, 0.63Lr), \( L_z \) does not get large enough for the instability during the LPT (17), so the AM is not decreased.

Results for the (61°, 0.7Lr), (65°, 0.8Lr), and (70°, 0.98Lr) cases are shown in Fig. 2. In the (70°, 0.98Lr) case, there are a lot of sudden, large excursions in eccentricity, and it ends with an unbound orbit. All of the cases end with a high \( L_z \sim 0.5Lr \), 45% larger than the present value of 0.345Lr. The Earth’s obliquity after the LPT, around 50°, is too large to produce the present \( L_z \) of 23.4° in the later evolution (SI Appendix, Low-e Phases of Evolution). The post-LPT inclination being high (~34°) may not be a problem, since it can be damped during the subsequent Cassini-state transition, provided that the lunar magma ocean has not solidified by that point (29). However, conservation of \( L_z \) implies that if the inclination is damped, then the obliquity must increase, making it even more difficult to produce the present \( L_z \).

Since these initial conditions are representative of all possible combinations of post-giant-impact \( I_0 \) and \( L_z \), these results show that, with \( L_{LPT} = L_z \), the high-obliquity scenario does not work to produce the present-day AM and Earth’s obliquity, at least for the tidal parameters used (\( Q_e/k_{2e} = 100 \) and \( Q_m/k_{2m} = 100 \)). Next, we show that this is the case regardless of the tidal model and tidal parameters.

### Characteristics of the High-\( L_z \) Scenario

The decrease in \( L_z \) (AM scalar sum) occurs predominantly during the LPT instability, during which the lunar eccentricity is nonzero and the semimajor axis, \( a \), stalls. The rate of change of \( a \) (\( da/dt \)) is a competition between tides on Earth (which tend to increase \( a \)) and tides on the Moon (which tend to decrease \( a \)). At zero \( e \), \( da/dt \) is positive. But \( da/dt \) decreases as \( e \) is increased (e.g., ref. 10). There is an \( e \) at which \( da/dt \) is zero. The value of \( e \) at this point depends on a ratio of the tidal parameters of the Earth and Moon and the tidal model. Though the expressions in ref. 10 need to be generalized, they give a rough estimate of the value of \( e \) for \( da/dt = 0 \) that is consistent with our simulations and tidal parameters.

During the phase in which \( da/dt \approx 0 \), \( L_z \) declines. We can calculate the rate of decline if we assume \( da/dt = 0 \). In this case, the changing part of \( L_z \) is predominantly the rotational AM of the Earth. The rate of change of AM is the component of the tidal torque on the spin axis of the Earth. Though the tidal torque depends on many factors, the leading term, \( T_0 \), sets the timescale and depends only on parameters and the semimajor axis. This term is the same for the constant \( Q \) tidal model and the constant \( \Delta t \) tidal model (20):

\[
T_0 = \frac{3k_{2e}GM_e^3R_e^5}{2Q_eM_e^2a^6} n,
\]

where \( M_m \) is the mass of the Moon. Using the expression for the reference AM \( L_{r} \), we find

\[
\frac{dL_z}{dt} = -\frac{3k_{2e}}{2} \left( \frac{M_m}{M_e} \right)^2 \left( \frac{R_e}{a} \right)^{9/2} n,
\]

where \( \lambda = \frac{3308}{9} \) is the present moment of inertia of the Earth divided by \( M_eR_e^2 \) and \( n \) is the mean motion of the lunar orbit. Evaluating this expression for \( n = 18R_e \), we find, independent of tidal models, a decline of about \( 7.9 \times 10^{-3} \) per million years (My). The decline found in the simulations is comparable to this, about \( 7.8 \times 10^{-3} \) per My. The agreement is excellent. This success allows us to generalize our simulation results to other tidal parameters. The rate of decline of \( L_z/L_r \) is simply proportional to \( k_{2e}/Q_e \). With a larger \( Q_e \), we can expect that it would take longer to leave the LPT, but that the ending value of \( L_z/L_r \) would be roughly the same (see below).

Termination of the LPT is marked by a change in the behavior of the angle between the ascending node of the lunar orbit on the ecliptic (\( \Omega \)) and the ascending node of the Earth’s equator on the ecliptic (\( h_0 \)); we denote this angle by \( h := \Omega - h_0 \). During the LPT, \( h \) oscillates about 0; after exit from the LPT, \( h \) circulates through all angles (Fig. 2). The point of transition from oscillation to circulation of \( h \) is well defined. At this point,
the semimajor axis resumes its outward evolution. Though there is a brief interval in which \( e \) is still nonzero after this point, the decline of \( L_a \) is small. Once \( e \) decays to zero, \( L_a \) changes very little.

We can determine the lowest value of \( L_a \) that can be obtained during the LPT instability by systematically exploring the behavior of the averaged, nontidal Earth–Moon system. For \( e \) to be nonzero, the circular orbit must be unstable; otherwise, the orbit will stay circular (\( e = 0 \)) and not get elliptical (\( 0 < e < 1 \)). To be in the LPT instability, \( h \) must oscillate. So we can determine the minimum \( L_a \) (or \( L_e \)). The minimum \( L_a \) that can be obtained by systematically finding all \( L_a \)-consistent states that satisfy two conditions: 1) \( e = 0 \) is unstable, and 2) \( h \) oscillates. If the minimum \( L_a \) determined in this way is much larger than \( L_e^p \), then the high-\( I_e \) scenario is not consistent with the present Earth–Moon system. The conclusion is independent of tidal models and tidal parameters.

The Hamiltonian describing the evolution of the nontidal Earth–Moon system, averaged over the orbital timescales of the Earth and Moon, denoted as \( \mathcal{H}_{EM} \), is shown in SI Appendix, SI Text. \( \mathcal{H}_{EM} \) is very similar to the Hamiltonian derived in Touma and Wisdom (20), but is generalized to arbitrary nonzero eccentricity. For the averaged system, three quantities are conserved: \( L_a \), \( \omega \), and Earth’s rotation rate. Then, \( \mathcal{H}_{EM} \) has two degrees of freedom (or a four-dimensional phase space).

Since the Hamiltonian has two degrees of freedom, it is natural to study the evolution with surfaces of section, which reveal the phase-space structure and determine the stability of fixed points (30). The values of the three conserved quantities must be specified for each section. We set \( L_s = L_e^p \) for all sections. We make a stability diagram for each \( a \). On this stability diagram, initial values of \( L_e \) and \( i \) are chosen, for \( h = 0 \) and \( e = 0 \). From these, we determine the Earth’s rotation rate and the value of the Hamiltonian. Since \( \mathcal{H}_{EM} \) has no time dependence, it is conserved. All points on a section share the same \( \mathcal{H}_{EM} \) value.

We take the axes of the surface of section to be \( x = e \cos \omega \) and \( y = e \sin \omega \), where \( \omega \) is the argument of pericenter of the lunar orbit. The section condition is \( h = 0 \) (restricted to the \( h < 0 \) case). The value of the momentum conjugate to \( h \) is determined by requiring that \( \mathcal{H}_{EM} \) has the chosen value. The return map is obtained by integrating the evolution until the section condition \( h = 0 \) (\( h < 0 \)) is again satisfied. The map from the pair \((x, y)\) to the next \((x', y')\) defines the return map \( P \).

Linear stability analysis of the map \( P \) determines the stability of the fixed point \((0, 0)\) (at \( e = 0 \)) (30). We make a stability diagram for a specified \( a \) (e.g., Fig. 3). For the initial values of \( L_e \) and \( i \), if the fixed point \((0, 0)\) is unstable, then a color corresponding to the value of \( L_s / L_e \) is plotted. A black line marks the boundary between \( h \) oscillating and \( h \) circulating. For \( L_s / L_e \) to decline substantially, \( e = 0 \) must be unstable, and \( h \) must be oscillating.

The stability diagram for \( a = 18 \sqrt{R} \) is shown in Fig. 3. The minimum value of \( L_s / L_e \) satisfying both conditions is 0.47, which is significantly larger than the present value of 0.34. The minimum Earth obliquity reached is about 38°, much larger than the 20° post-LPT obliquity that is required for subsequent evolution to reach the present 23.4° (SI Appendix, Low-e Phases of Evolution). The results for other values of \( a \) are shown in Table 1. \( L_{min}^e \) and \( L_{min}^h \) are the minimum \( L_e \) and obliquity in the unstable \( e \), oscillating \( h \) regions, such as the colored region in Fig. 3. Notice that the predicted \( L_{min}^e \) agrees well with the minimum \( L_e \) obtained in our simulations (Fig. 2). We see that with \( L_e^p \) set at \( L_e^p \), the high-\( I_e \) scenario is not able to produce the present Earth–Moon system, regardless of tidal parameters and tidal models. The present-day lunar inclination \( i^p \) remains puzzling.

Even though the stability diagrams are for the nontidal averaged system, they suggest what the tidal evolution through the LPT instability would look like on the \( I_e - i \) plane. The system begins with large \( I_e \) and \( i \). As it evolves into the region of the LPT instability, the system enters the colored tongues of instability and develops nonzero eccentricity. At this point, the system begins to undergo large variations in \( I_e \) and \( i \), while maintaining roughly constant \( a \) (the system roughly stays on the stability diagram). These large oscillations in \( I_e \) and \( i \) are reminiscent of those found by Atobe and Ida (16) in the \( e = 0 \) case. Tidal torques reduce \( L_s / L_e \), and the system proceeds down the tongue of instability (diagonally toward the lower left). But once the boundary between \( h \) oscillation and circulation is reached, the system changes course and soon leaves the LPT instability. There is no further significant reduction in \( L_s / L_e \).

Examination of the simulations confirms this picture (Fig. 4). Whenever \( |h| < 0.03 \) radians, with \( h < 0 \), we plot a point on the \( i - I_e \) plane. The semimajor axis is not constant in the simulations, so we indicate the value of \( a \) by a color. The evolution begins in the upper right. Once the system enters the LPT instability, the semimajor axis \( a \) is roughly constant. During this phase, the colors are orange to yellow. The system evolves down diagonally to the left until the boundary between \( h \) oscillation and circulation is reached. At this point, the trajectory on the plot changes direction. The semimajor axis then resumes its outward evolution, as indicated by the change of \( a \).

Table 1. Minimum \( L_s / L_e \) obtained while the system is undergoing the LPT instability, at different semimajor axis values

| \( a / \sqrt{R} \) | \( L_{min}^e / L_e \) | \( L_{min}^h / L_e \) |
|---|---|---|
| 16 | 0.437 | 0.29 |
| 17 | 0.452 | 0.32 |
| 18 | 0.472 | 0.38 |
| 19 | 0.498 | 0.43 |
| 20 | 0.496 | 0.47 |

Fig. 4. \( i \) versus \( L_e \) whenever \( |h| < 0.03 \) radians and \( h < 0 \) for the simulations shown in Fig. 2, with initial conditions (in \((L_s, L_e)\): (61°, 0.7L_e [circle], (65°, 0.8L_e [square], and (70°, 0.98L_e [diamond]). The color indicates the value of \( a \). Note that the three simulations, though started with different initial conditions, merge onto a common track. The black line marks the boundary between \( h \) oscillating (to the right) and \( h \) circulating for \( a = 18 \sqrt{R} \). For the simulation with initial \( L_s = 61° \), the arrows show the direction of time evolution, and the marks a-d correspond to the points in Fig. 2 with the same labels.
color to red. In this final phase, the stability diagram at fixed $a$ no longer applies. The eccentricity damps to zero, and $L_z$ no longer declines substantially.

$L_z$ Constraint on Resonant Scenarios

The evection resonance and the evection-limit cycle involve a resonance between the precession of the lunar pericenter and the motion of the Earth around the Sun. Substantial AM can be lost in the form of a decrease in $L_z$. The evection limit cycle (near-resonance) described in refs. 10 and 11 both involve high lunar eccentricities ($>0.5$). Such high eccentricities will lead to severe heating in the Moon and cause impacts to only modify AM conserved in the postresonance evolution. The late-accretional formation of the Earth-Moon system inferred from W isotopes in lunar metals.

The evection resonance (1) and the near-resonance described in ref. 10 involve high lunar eccentricities ($>0.5$). However, the post-limit cycle minimum $L_{z}$ is $0.393 L_z$ in ref. 10 and $0.404 L_z$ in ref. 12, both with $Q_z = 400$. It was then thought that the subsequent evolution would decrease $L_z$, but the $L_z$ constraint rules out this possibility.

It was found in ref. 12 that a smaller $L_z$ can be produced with a larger $Q_z$ (0.436, 0.404, and 0.389 $L_z$ for $Q_z = 300, 400$, and $500$). With a large $Q_z$ ($10^3$ to $10^5$) in the early history of the Earth (31), the evection limit cycle remains a possible mechanism to drain the excess AM from a fast-spinning Earth.

Code Availability

The computer codes we used for the simulations in this paper are available at GitHub, https://github.com/zhenliangtian/em3d.

Conclusion

The $L_z$ constraint places limits on the possible orbital histories of the Earth–Moon system and thus limits the details of the Moon-forming giant impact. For a high-AM impact (1–3), which is able to produce a Moon with an Earth-like composition, the impact geometry is constrained to cases where the postimpact Earth has a small to medium obliquity.

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