A CENTRAL LIMIT THEOREM FOR THE DEGREE OF A RANDOM PRODUCT OF RATIONAL SURFACE MAPS

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Abstract. We prove a central limit theorem for the algebraic and dynamical degrees of a random composition of dominant rational maps of $\mathbb{P}^2$.

1. Introduction

A rational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, defined over the field of complex numbers $\mathbb{C}$, is a function which is given in homogeneous coordinates by

$$f : [x : y : z] \mapsto [P_0(x, y, z) : P_1(x, y, z) : P_2(x, y, z)],$$

where $P_0, P_1$ and $P_2$ are homogeneous polynomials in $\mathbb{C}[x, y, z]$ of the same degree $d \in \mathbb{N}$ with no common factor. We say that $f$ is dominant if its image is not contained in an algebraic curve. The integer $d$, denoted $\deg(f)$, is called the degree (or algebraic degree) of $f$ and is in general distinct from the topological degree of $f$, denoted $d_{\text{top}}(f)$, which is the number of preimages by $f$ of a general point.

Unlike the situation on $\mathbb{P}^1$, the algebraic degree of $f \circ g$ where $f$ and $g$ are two rational maps on $\mathbb{P}^2$ is not equal in general to the product $\deg(f) \deg(g)$, but it satisfies a submultiplicative property:

$$\deg(f \circ g) \leq \deg(f) \deg(g).$$

Using this fact, one can define the (first) dynamical degree of $f$ [RS97], denoted $\lambda_1(f)$, given by

$$\lambda_1(f) := \lim_{n \to +\infty} \deg(f^n)^{1/n}.$$ 

This dynamical quantity is invariant under birational conjugacy [DS05, Tru16, Dan17] and measures the growth rate of the preimages of a generic hyperplane on $\mathbb{P}^2$.

The degree and the dynamical degree of an arbitrary composition are quite difficult to predict in general and this is due to the presence of points where the rational maps are not defined (called indeterminacy points) and their behavior under iteration (see [Sib99, Proposition 1.4.3]).

As a result, the growth of the sequence $\deg(f^n)$ and the dynamical degree of a given rational map $f$ has been the subject of much research, and is known only in certain cases: for endomorphisms of a projective variety, monomial maps [Lin12, FW12], birational surfaces maps [DF01, BC16], polynomial automorphisms and endomorphisms of the affine plane [FM89, Fur99, PJ04, PJ07, PJ11], meromorphic surface maps (under certain assumptions) [BFJ08], birational transformations of hyperkähler manifolds [LB19], certain automorphisms of the affine 3-space [BvS19a, BvS19b] and certain rational maps associated to matrix inversions [AADBM99, AdMV06, AAB+99, BK08, BT10]. Starting from dimension 3, the degree sequences are partially known for birational transformations with very slow degree growth [CX18] or for specific examples [D18], for a specific group of automorphisms on $\text{SL}_2(\mathbb{C})$ [Dan18], while a lower bound on unbounded degree sequences was recently obtained for a large class of birational transformations in [LU20].

Note, however, that when $f, g$ are generic rational maps (i.e. belong to suitable Zariski open subsets of the space of rational maps of degree $d$), the product satisfies $\deg(f \circ g) = \deg(f) \deg(g)$. In other words,
if \( f, g \) are chosen “randomly” enough, then \( \lambda_1(f) = \deg(f) \) and the degree of a product behaves well. This simple, but natural observation motivates the current paper.

Let us fix a probability measure \( \mu \) on the space of dominant maps on \( \mathbb{P}^2 \) whose support is countable. We consider a random product of rational maps

\[
f_n := g_1 g_2 \cdots g_n
\]

where \((g_n)\) is a sequence of i.i.d. random elements of \( G \) chosen with distribution \( \mu \); thus, the sequence \((f_n)\) describes a random walk in the space of dominant rational maps of \( \mathbb{P}^2 \).

Our aim is to understand the distribution of the sequences of algebraic degrees \((\deg(f_n))\) and dynamical degrees \((\lambda_1(f_n))\). Heuristically, one expects the sequence \( \log \deg(f_n) \) to be close to the sum \( \log \deg(g_1) + \cdots + \log \deg(g_n) \). Since the logarithmic sum \( \sum \log \deg(g_i) \) satisfies the classical law of large numbers and a central limit theorem, the sequence \( \log \deg(f_n) \) should also display similar features.

For the law of large numbers, observe that because of (1) the sequence \( (\log \deg(f_n)) \) is subadditive and Kingman’s theorem asserts that under a finite moment condition

\[
\int_G \log \deg(f) d\mu(f) < +\infty,
\]

there exists a constant \( \ell_\mu \geq 0 \) such that the sequence

\[
\frac{1}{n} \log \deg(f_n)
\]

converges almost surely to \( \ell_\mu \). In other words, the sequence \( \log \deg(f_n) \) follows a law of large numbers. This was shown in the case of rational maps on \( \mathbb{P}^2 \) in [Lin19], where the quantity \( \ell_\mu \) is referred to as the random dynamical degree. For birational maps of \( \mathbb{P}^2 \), it was shown in [MT18b] that the limit \( \ell_\mu \) is positive if the semigroup generated by the support of \( \mu \) is non-elementary. Moreover, in that case the limit

\[
\lim_{n \to +\infty} \frac{1}{n} \log \lambda_1(f_n)
\]

equals almost surely (even though \( (\log \lambda_1(f_n)) \) is not subadditive) and also equals \( \ell_\mu \) whenever the support of \( \mu \) is bounded.

The various quantities described above have an explicit description when one restricts to monomial maps ([Lin12 FW12]). A monomial map on \( \mathbb{P}^k \) is a transformation of the form:

\[
g_M : [x_0 : \ldots : x_k] \mapsto [x_0^{m_{00}} x_1^{m_{01}} \cdots x_k^{m_{0k}} : \ldots : x_0^{m_{k0}} x_1^{m_{k1}} \cdots x_k^{m_{kk}}],
\]

where \( M := (m_{ij})_{i,j \leq k} \) are the entries of a \((k+1) \times (k+1)\) matrix with integer coefficients. Using the fact that \( g_{MN} = g_M \circ g_N \) for any two matrices \( M, N \), a random product of monomial maps is induced by a random product of matrices. Since the degree of \( g_M \) is a multiple of the largest coefficient of \( M \) and the dynamical degree is the spectral radius of \( M \), the convergence of \( \frac{1}{n} \log \deg(f_n) \) follows from Oseledets’ theorem [Ose68], while the central limit theorem for \( \log \deg(f_n) \) can be deduced using Furstenberg-Kesten’s results [FK60]. Apart from the above special situation, very little is known for the behavior of the degree of a random product of dominant transformations of \( \mathbb{P}^k \) for \( k \geq 3 \).

The main result of our paper shows that \( \log \deg(f_n) \) for maps on \( \mathbb{P}^2 \) follows a central limit theorem in the sense that the difference \( \frac{\log \deg(f_n) - n \ell_\mu}{\sqrt{n}} \) converges to a Gaussian or a “folded” Gaussian.

Let \( \mu_n \) be the distribution of \( f_n \), and let \( C_b(\mathbb{R}) \) be the space of bounded, continuous functions \( \varphi: \mathbb{R} \to \mathbb{R} \). Given \( \sigma \geq 0 \), we denote as \( \mathcal{N}_\sigma \) the Gaussian measure of variance \( \sigma \) and mean 0, i.e. the probability measure

\[
d\mathcal{N}_\sigma(t) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{t^2}{2\sigma^2}} dt \text{ if } \sigma > 0, \text{ and the } \delta\text{-mass at } 0 \text{ if } \sigma = 0.
\]

Given a symmetric, 2 \times 2 matrix \( \Sigma \), we define the folded Gaussian measure centered at 0 and of variance \( \Sigma \) as the probability measure \( \mathcal{F}_\Sigma \) on \( \mathbb{R} \) defined as the pushforward of a Gaussian measure of mean 0 and covariance \( \Sigma \) under the map \( (x, y) \mapsto x + |y| \).
Theorem A. Let $G$ be a countable semigroup of dominant rational maps on $\mathbb{P}^2$ and let $\mu$ be a measure whose support generates $G$ satisfying
\begin{equation}
\int_G (\log \deg(f))^2 \, d\mu(f) < +\infty
\end{equation}
and
\begin{equation}
\int_G \frac{\deg(f)}{\log(f)} \, d\mu(f) < +\infty.
\end{equation}

Then there exists $\ell \geq 0$ such that the following two properties hold.

(i) (CLT for algebraic degree) Either, there exists $\sigma \geq 0$ such that
\[ \lim_{n \to +\infty} \int_G \varphi \left( \frac{\log \deg(f) - n\ell}{\sqrt{n}} \right) \, d\mu_n(f) = \int_{\mathbb{R}} \varphi(t) \, dN_{\sigma}(t) \]
for any function $\varphi \in C_b(\mathbb{R})$, or there exists $\Sigma$ such that
\[ \lim_{n \to +\infty} \int_G \varphi \left( \frac{\log \deg(f) - n\ell}{\sqrt{n}} \right) \, d\mu_n(f) = \int_{\mathbb{R}} \varphi(t) \, d\mathcal{F}N_{\Sigma}(t) \]
for any $\varphi \in C_b(\mathbb{R})$.

(ii) (CLT for dynamical degree) A similar limit law holds for the dynamical degree.

Indeed, either there exists $\sigma \geq 0$ such that
\[ \lim_{n \to +\infty} \int_G \varphi \left( \frac{\log \lambda_1(f) - n\ell}{\sqrt{n}} \right) \, d\mu_n(f) = \int_{\mathbb{R}} \varphi(t) \, dN_{\sigma}(t) \]
for any $\varphi \in C_b(\mathbb{R})$, or there exists $\Sigma$ such that
\[ \lim_{n \to +\infty} \int_G \varphi \left( \frac{\log \lambda_1(f) - n\ell}{\sqrt{n}} \right) \, d\mu_n(f) = \int_{\mathbb{R}} \varphi(t) \, d\mathcal{F}N_{\Sigma}(t) \]
for any $\varphi \in C_b(\mathbb{R})$.

Let us remark that condition (i) is only needed if the semigroup generated by the support of $\mu$ is focal or parabolic. A more refined version of this theorem, with a classification of all cases, formulas for $\ell$ and $\sigma$, as well as a characterization of the cases where $\sigma = 0$ will be given in Theorem 4.4.

Our result is reminiscent of similar statements, proved in other contexts. The central limit theorem is known for the norm of a random product of $n \times n$ matrices [FK60, GLPR83, GR85, GR86, BQ16a], for the translation length and the escape rate for a random composition of isometries on a tree [NW02] or more generally on a Gromov-hyperbolic space [BQ16b, MS14], for quasimorphisms on a random product of elements in a countable hyperbolic group [CP10, BH11], for the distance in the Teichmüller metric of a random product of mapping classes [Hor18].

Let us now observe that the second case of Theorem A (with the folded normal law) actually occurs. Take a Hénon map $h : \mathbb{C}^2 \to \mathbb{C}^2$ of degree $d$, of the form:
\[ h(x, y) = (y + P(x), x), \]
where $P(x) \in \mathbb{C}[x]$ is a polynomial of degree $d$ and let us take $\mu = \frac{1}{d} \delta_h + \frac{1}{d} \delta_{h^{-1}}$, putting uniform mass on $h$ and $h^{-1}$. Since $\log \deg(h^p) = \log \lambda_1(h^p) = |p| \log(d)$ for all $p \in \mathbb{Z}$, the classical central limit theorem yields the convergence:
\[ \lim_{n \to +\infty} \frac{1}{\sqrt{n}} \int_G \phi(\log \deg(f)) \, d\mu_n(f) = \lim_{n \to +\infty} \frac{1}{\sqrt{n}} \int_G \phi(\log \lambda_1(f)) \, d\mu_n(f) = \int_{\mathbb{R}} \phi(|t|) e^{-t^2/2} \, dt, \]
for any bounded continuous function $\phi \in L^1(\mathbb{R})$. In this situation, the logarithm of the degree of a random product does not converge to a normal law but to a folded normal law and satisfies the second assertion of Theorem A. This example is an analogue in this setting of Furstenberg-Kesten’s [FK60 Example 2] for products of random matrices, where the folded normal law already appears. Further concrete examples of the different asymptotic behaviours are given in Section 2.4.
Our proof exploits in a crucial way the relationship between rational maps and a suitable isometric action on an infinite dimensional Gromov-hyperbolic space or Hilbert space, developed by Cantat, Boucksom-Favre-Jonsson, Blanc-Cantat [Can11, BFJ08, BC16]. The construction of this Gromov-hyperbolic space, denoted $\mathbb{H}^\infty$, is of algebraic nature: it is obtained by considering a subspace of divisors on the space of infinite blow-ups of $\mathbb{P}^2$ and taking its completion with respect to a norm induced by the intersection product. The Hodge index theorem guarantees that the intersection product on $\mathbb{H}^\infty$ defines a Lorentzian metric:

$$d(\alpha, \beta) := \cosh^{-1}(\alpha \cdot \beta),$$

where $\alpha, \beta \in \mathbb{H}^\infty$ and $(\alpha \cdot \beta)$ denotes the intersection product of $\alpha$ and $\beta$. One advantage in working on the space of divisors over all blow-ups of $\mathbb{P}^2$ is that the pullback action by a rational map $f$ becomes functorial. Namely, if $\alpha \in \mathbb{H}^\infty$, and $f, g$ are rational maps, then:

$$(f \circ g)^* \alpha = g^* f^* \alpha,$$

as if we were working with the action of an endomorphism on the Néron-Severi group of a surface. When $f$ is an invertible map of $\mathbb{P}^2$, Cantat exploited the fact that the pullback action on $\mathbb{H}^\infty$ is an isometry to show that the Tits alternative holds for the Cremona group of $\mathbb{P}^2$ [Can11]. In our case, $f$ can be non-invertible so the pullback is not exactly an isometry. But the fact that:

$$(f^* \alpha \cdot f^* \beta) = d_{\text{top}}(f)(\alpha \cdot \beta),$$

for all $\alpha, \beta \in \mathbb{H}^\infty$ shows that $\sqrt{d_{\text{top}}(f)} f^*$ gives an isometry of $\mathbb{H}^\infty$. We thus obtain a representation

$$\rho_f := \frac{1}{\sqrt{d_{\text{top}}(f)}} f^*$$

from the semigroup of dominant rational maps of $\mathbb{P}^2$ to the group of isometries of $\mathbb{H}^\infty$. Taking as $L$ the class of a line in $\mathbb{P}^2$, a random walk $(f_n)$ on the space of rational maps induces a sample path $\rho_{f_n}(L)$ in the hyperbolic space $\mathbb{H}^\infty$ and we relate the degree to the distance on this space:

$$\cosh d(\rho_{f_n}(L), L) = \frac{\deg(f_n)}{\sqrt{d_{\text{top}}(f_n)}}.$$

Denote by $G$ the semigroup of rational maps generated by the support of $\mu$. According to the classification [Gro87] of semigroups of isometries of a hyperbolic space, $\rho(G)$ is either non-elementary or elementary, which is further subdivided into elliptic, parabolic, focal, or lineal (see Section 2.4).

In the non-elementary case, $\rho(G)$ contains two loxodromic elements with different axis; here, we import into our setting the known central limit theorems for the translation length and for the escape rate [Bjo10, MS14, BQ16b, Gou17, Hor18]; we remark that, since the maps we consider are not necessarily birational, the algebraic degree is a “sum” of the translation distance and the topological degree, and these two are not independent, hence additional care is needed.

Otherwise, $\rho(G)$ either contains no loxodromic elements (elliptic or parabolic case) or every loxodromic element has a common fixed point on the (Gromov) boundary of $\mathbb{H}^\infty$ (focal or lineal case). Here, we show that $G$ is particularly rigid and we conclude using some techniques from birational geometry by showing that the logarithm of a random product can be reduced to a random walk on the line or on the plane $\mathbb{R}^2$.

The possible limit distributions in the various cases are summarized in the following table. Note that the presence of a folded Gaussian implies that $G$ is either focal or lineal.

| Type of isometry group | Limit law for $\log \deg$ | Limit law for $\log \lambda_1$ |
|------------------------|--------------------------|-------------------------------|
| elliptic               | Gaussian (possibly trivial) | Gaussian (possibly trivial) |
| parabolic              | Gaussian (possibly trivial) | Gaussian (possibly trivial) |
| focal                  | Gaussian                 | Gaussian or Folded Gaussian   |
| lineal                 | Gaussian or Folded Gaussian | Gaussian or Folded Gaussian |
| non-elementary         | Gaussian ($\ell > 0$)    | Gaussian ($\ell > 0$)         |
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2. Rational maps, degrees and isometric actions

2.1. Topological, algebraic and dynamical degrees. Let $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be a rational map. The map $f$ can be expressed in homogeneous coordinates as:

$$f([x : y : z]) = [P_0(x, y, z) : P_1(x, y, z) : P_2(x, y, z)],$$

where $P_0, P_1, P_2 \in \mathbb{C}[x, y, z]$ are homogeneous polynomials of the same degree $d \in \mathbb{N}$ with no common factor. We call $f$ dominant if the image of $\mathbb{P}^2$ is not contained in an algebraic curve.

The integer $d$ is called the degree or the algebraic degree of the rational map $f$ and is denoted $\deg(f)$. One can show ([RS97]) for any dominant rational map $f, g : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ that:

$$\deg(f \circ g) \leq \deg(f) \deg(g),$$

so the sequence $(\deg(f^n))_{n \geq 1}$ for a given rational map $f$ is submultiplicative. We thus define the (first) dynamical degree of $f$ as

$$\lambda_1(f) := \lim_{n \to \infty} \deg(f^n)^{1/n}.$$

Recall that given a dominant rational map $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$, the topological degree of $f$, denoted $d_{\text{top}}(f)$, is the number of preimages counted with multiplicity of a generic point of $\mathbb{P}^2$. When the topological degree of $f$ is equal to 1, one says that $f$ is birational and its inverse is a rational map which we denote by $f^{-1}$. In this paper, we shall adopt the following convention. Denote by $L$ the divisor on $\mathbb{P}^2$ given by the line at infinity. One can express the topological degree and the degree of $f$ by computing the following intersection products.

$$d_{\text{top}}(f) = (f^*L \cdot f^*L),$$

$$\deg(f) = (f^*L \cdot L).$$

The dynamical degree and the topological degree are dynamical invariants and their properties are stated in the following result.

**Theorem 2.1.** Let $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be a dominant rational map. Then the following properties hold.

(i) ([RS97], [Tru16 Theorem 1.1], [Dan17 Theorem 1]) For any birational map $g : \mathbb{P}^2 \rightarrow \mathbb{P}^2$,

$$\lambda_1(f) = \lambda_1(g \circ f \circ g^{-1}).$$

(ii) (Khovanskii-Teissier’s relation [Laz04 Corollary 1.6.3])

$$d_{\text{top}}(f) \leq \deg(f)^2.$$

(iii) One has:

$$d_{\text{top}}(f) \leq \lambda_1(f)^2.$$

Observe that assertion $(iii)$ is a direct consequence of assertion $(ii)$ in the previous statement using the definition of the first dynamical degree.
2.2. The construction of the hyperbolic space. In this section, we recall the construction of the Picard-Manin space of divisors, following closely the presentation in [BC16]. We start with $X_0 = \mathbb{P}^2$. If $\pi : X \to X_0$ is a birational morphism, we say that $X$ is a **birational model of** $X_0$. When this happens, the morphism $\pi$ induces a pullback in the Nérö-Severi group

$$\pi^* : \text{NS}(X_0) \to \text{NS}(X).$$

Moreover, for any two birational models $X, Y$ over $X_0$, there exists a third birational model $Z$ over both $X$ and $Y$. We thus define the Picard-Manin space as the inductive limit:

$$\mathcal{Z} := \varinjlim \text{NS}(X),$$

where $X$ describes all birational models of $X_0$. If $X$ is a blow-up of $X_0$ at one point, we denote by $E$ the exceptional divisor on $X$ and $\text{NS}(X) \simeq \text{NS}(X_0) \oplus \mathbb{Z}E$. If one takes an arbitrary sequence of blow-ups of $\mathbb{P}^2$, we obtain finitely many exceptional divisors which are all inside $\mathcal{Z}$. The Picard-Manin space can be described as:

$$\mathcal{Z} = \text{NS}(\mathbb{P}^2) \oplus \bigoplus E_i \simeq \mathcal{Z} \oplus \bigoplus E_i,$$

where $E_i$ describes all the exceptional divisors on a birational model of $\mathbb{P}^2$ and where $L$ denotes the class of a line in $\mathbb{P}^2$.

The intersection product on each birational model of $\mathbb{P}^2$ induces a scalar product on $\mathcal{Z}$, denoted $(\alpha, \beta)$, and a norm on $\mathcal{Z} \otimes \mathbb{R}$. We denote by $\overline{\mathcal{Z}}$ the completion of $\mathcal{Z}$ with respect to this norm. Observe that the Hodge index theorem on each birational model of $\mathbb{P}^2$ shows that the metric induced by the intersection product is hyperbolic; as a result, the space $\overline{\mathcal{Z}}$ endowed with the metric induced by the intersection product has the structure of an infinite-dimensional hyperbolic space. For more details on this construction, we shall refer to [Can11].

**Definition 2.2.** The hyperbolic space $\mathbb{H}^\infty$ is the set

$$\mathbb{H}^\infty := \{ \alpha \in \overline{\mathcal{Z}} : (\alpha \cdot \alpha) = 1, (\alpha \cdot L > 0) \}.$$

It is endowed with a hyperbolic metric $d : \mathbb{H}^\infty \times \mathbb{H}^\infty \to \mathbb{R}^+$ given by the formula:

$$d(\alpha, \beta) := \cosh^{-1}(\alpha \cdot \beta),$$

for any $\alpha, \beta \in \mathbb{H}^\infty$. Its boundary, denoted $\partial \mathbb{H}^\infty$, is the set

$$\partial \mathbb{H}^\infty := \{ \alpha \in \overline{\mathcal{Z}} : (\alpha \cdot \alpha) = 0, (\alpha \cdot L > 0) \}.$$

This space corresponds to the choice of a “positive” hyperboloid in the vector space $\overline{\mathcal{Z}}$.

2.3. Isometric action of rational maps on the hyperbolic Picard-Manin space. Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a dominant rational map. Its graph $\Gamma_f$ in $\mathbb{P}^2 \times \mathbb{P}^2$ is a natural birational model of $\mathbb{P}^2$ and the maps $\pi_1, \pi_2$ induced by the projection onto the first and second factor, respectively, are regular. If $\alpha$ is a divisor in $\mathbb{P}^2$, then we can take its pullback by $\pi_2$, denoted $f^*\alpha$. More generally, we can do the same if $\alpha$ is a class in a birational model $X$ of $\mathbb{P}^2$ by pulling back on the corresponding graph. The latter definition is compatible with the inductive definition and induces a continuous pullback map $f^* : \mathbb{H}^\infty \to \mathbb{H}^\infty$.

We now define a contravariant action by rational maps on $\mathbb{H}^\infty$, namely for any $\alpha \in \mathbb{H}^\infty$ and any dominant rational map $f$, the element $\rho_f(\alpha)$ is given by the formula

$$\rho_f(\alpha) := \frac{1}{\sqrt{d_{\text{top}}(f)}} f^*\alpha.$$

Note that since $(f \circ g)^* = g^* \circ f^*$, this action reverses the order. Using the fact that $(f^*\alpha \cdot f^*\beta) = d_{\text{top}}(f)(\alpha, \beta)$ for all $\alpha, \beta \in \mathbb{H}^\infty$, one verifies that the above action induces an isometry of $(\mathbb{H}^\infty, d)$.

As a consequence, if the associated isometry is loxodromic on $\mathbb{H}^\infty$ then we relate the dynamical degree of $f$ with the translation distance as follows.

**Lemma 2.3.** For any dominant rational map $f$ on $\mathbb{P}^2$, we have

$$\log \left( \frac{(f^*L \cdot L)}{\sqrt{d_{\text{top}}(f)}} \right) \leq d(f(L), L) \leq \log \left( \frac{2(f^*L \cdot L)}{\sqrt{d_{\text{top}}(f)}} \right).$$
Proof. By definition of the hyperbolic metric,
\[
\cosh d(\rho_f(L), L) = \frac{(f^*L \cdot L)}{\sqrt{d_{\text{top}}(f)}}
\]
hence, since \(\frac{1}{2}e^x \leq \cosh x \leq e^x\),
\[
\frac{e^{d(\rho_f(L), L)}}{2} \leq \frac{(f^*L \cdot L)}{\sqrt{d_{\text{top}}(f)}} \leq e^{d(\rho_f(L), L)}
\]
which immediately yields the claim. \(\square\)

We shall use frequently the following observation.

Lemma 2.4. Let \(G\) be a semigroup of dominant rational maps of \(\mathbb{P}^2\). Suppose that there exists an element \(\alpha \in \mathbb{H}^\infty \cup \partial \mathbb{H}^\infty\) which is an eigenvector for every element of \(G\). Then the map \(\pi : G \to (\mathbb{R},+)
\)
\[
\pi(f) := \log(f^*\alpha \cdot L) - \log(\alpha \cdot L)
\]
is a morphism of semigroups.

In the following sections, we will use many times the following result. Recall that a class is nef if it intersects non-negatively any curve class and that a nef class is also big if its self-intersection is positive.

Lemma 2.5. If \(\alpha \in \mathbb{H}^\infty\) is big and nef then there exists a constant \(C\) such that for all \(f \in G\), one has
\[
|\log(f^*L \cdot L) - \log(f^*\alpha \cdot L)| \leq C.
\]
Proof. Since \(\alpha\) and \(L\) are big and nef, Siu’s inequality (see [Laz95, Laz04] Theorem 2.2.13) yields:
\[
\frac{(L^2)}{2(\alpha \cdot L)} f^*\alpha \leq f^*L \leq 2\left(\frac{\alpha \cdot L}{\alpha^2}\right) f^*\alpha,
\]
where \(\alpha \leq \beta\) means that the difference \(\beta - \alpha\) lies in the closure of the cone generated by effective curves. Intersecting with the nef class \(L\) thus yields:
\[
\frac{(L^2)(f^*\alpha \cdot L)}{2(\alpha \cdot L)} \leq \deg(f) = (f^*L \cdot L) \leq 2\left(\frac{\alpha \cdot L}{\alpha^2}\right)(f^*\alpha \cdot L),
\]
which implies the claim. \(\square\)

2.4. Classification of semigroups of isometries. By the classification of semigroups of isometries of hyperbolic spaces (see [DSU17] Theorem 6.2.3 and Proposition 6.2.14), the semigroup \(G\) satisfies one of the following properties:

(i) \(G\) is elliptic, i.e. there exists a class \(\alpha \in \mathbb{H}^\infty\) globally fixed by \(G\).

(ii) \(G\) is parabolic, i.e. there exists a class \(\alpha \in \partial \mathbb{H}^\infty\) globally fixed by \(G\) and every element of \(G\) is parabolic.

(iii) \(G\) is focal, i.e. it globally fixes a class \(\alpha \in \partial \mathbb{H}^\infty\) and contains a hyperbolic element.

(iv) \(G\) is non-elementary, i.e. there exists two hyperbolic elements whose fixed sets at infinity do not intersect.

(v) \(G\) is linear, i.e. it contains a hyperbolic element and any other hyperbolic element fixes the same points at infinity.

We call a semigroup \(G\) elementary if it satisfies condition (i), (ii), (iii) or (v) in the above characterization.

We now give some concrete examples of subgroups in each of these classes and discuss the central limit theorem.

Example 2.6. If \(G \subset \text{PGL}_2(\mathbb{C})\) is a discrete subgroup acting linearly on \(\mathbb{P}^2\), then the semigroup induced by \(G\) on \(\mathbb{H}^\infty\) is elliptic. In this case, the degrees and dynamical degrees are always 1 and the sequence \(\log \deg(f_n)\) is the constant random variable equal to zero.

We then give an example of elliptic semigroup containing non-invertible elements.
Example 2.7. Take \( f, g \) two rational transformations of the form:
\[
  f : (x, y) \mapsto (P(x), P(y)),
\]
\[
  g : (x, y) \mapsto (Q(x), Q(y)),
\]
where \( P, Q \in \mathbb{C}[x] \) are polynomials in one variable. We consider the measure \( \mu = \frac{1}{2} \delta_f + \frac{1}{2} \delta_g \). The semigroup \( G \) is elliptic even though \( f \) and \( g \) are not necessarily invertible. Since \( f \) and \( g \) commute and \( \text{deg}(f^n g^q) = \text{deg}(f)^p \text{deg}(g)^q \), we obtain that \( \log \text{deg}(f_n) \) induces a random walk on a line and the central limit theorem holds.

We now discuss the relationship with the non-random situation.

Example 2.8. Suppose that \( f \) is a dominant rational map on \( \mathbb{P}^2 \) such that \( \lambda_1(f)^2 > \text{d}_{\text{top}}(f) \). We take \( \mu = \delta_f \). Then a theorem of Boucksom-Favre-Jonsson [BFJ08] states that
\[
\frac{\text{deg}(f^n)}{\lambda_1(f)^n} = C + O \left( \left( \frac{\text{d}_{\text{top}}(f)}{\lambda_1(f)^2} \right)^{n/2} \right),
\]
and so the sequence:
\[
|\log \text{deg}(f^n) - n \log \lambda_1(f)| = \log \left( C + O \left( \left( \frac{\text{d}_{\text{top}}(f)}{\lambda_1(f)^2} \right)^{n/2} \right) \right) \leq M,
\]
where \( M > 0 \) is a constant. As a result the sequence
\[
\frac{\log \text{deg}(f^n) - n \log \lambda_1(f)}{\sqrt{n}}
\]
converges to zero. Thus, the central limit theorem holds with \( \sigma = 0 \) and the limit distribution is the Dirac mass at 0.

Example 2.9. If \( G \) is a family of non-trivial Jonquières transformations, i.e. of the form:
\[
(x, y) \mapsto \left( ax + b, \frac{\alpha(x)y + \beta(x)}{\gamma(x)y + \delta(x)} \right),
\]
where \( a \in \mathbb{C}^*, b \in \mathbb{C} \) and \( \alpha, \beta, \gamma, \delta \in \mathbb{C}(x) \) are non-constant rational functions on \( x \) and \( \alpha \delta - \beta \gamma \) is a non-zero function. Then the subgroup \( G \) induces a parabolic action on \( \mathbb{H}^\infty \).

In the same spirit, we construct a non-invertible parabolic semigroup.

Example 2.10. Take \( f, g \) some transformations of the form:
\[
  f : (x, y) \mapsto \left( P(x), \frac{\alpha(x)y + \beta(x)}{\gamma(x)y + \delta(x)} \right),
\]
\[
  g : (x, y) \mapsto \left( Q(x), \frac{\alpha'(x)y + \beta'(x)}{\gamma'(x)y + \delta'(x)} \right),
\]
where \( P, Q \in \mathbb{C}[x] \) are polynomials of degree \( p \) and \( q \), respectively, and such that \( \alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta' \) are non-trivial affine functions satisfying \( \alpha \delta - \beta \gamma \neq 0, \alpha' \delta' - \beta' \gamma' \neq 0 \). Observe that \( \text{d}_{\text{top}}(f) = p^2 \) and \( \text{d}_{\text{top}}(g) = q^2 \) and both \( f, g \) preserve the fibration induced by the projection onto the first factor. If \( C \) is the class of a fiber of this projection, then \( f^*C = pC, g^*C = qC \) and \( C^2 = 0 \). As a result, the semigroup generated by \( f, g \) induces a parabolic semigroup on \( \mathbb{H}^\infty \) (by Proposition 2.13). Let \( \mu = \frac{1}{2} \delta_f + \frac{1}{2} \delta_g \), then the degree of a product is the product of the degrees and the central limit theorem follows.

We now construct a focal semigroup using the same procedure.

Example 2.11. Under the same assumptions of the previous example, let us consider \( f, g \) defined as
\[
  f : (x, y) \mapsto \left( P(x), \frac{\alpha(x)y^{p-1} + \beta(x)}{\gamma(x)y^{p-1} + \delta(x)} \right),
\]
\[
  g : (x, y) \mapsto \left( Q(x), \frac{\alpha'(x)y^{q-1} + \beta'(x)}{\gamma'(x)y^{q-1} + \delta'(x)} \right),
\]
Observe that $d_{\text{top}}(f) = p(p-1) < p^2$, $d_{\text{top}}(g) = q(q-1) < q^2$ and if $C$ is the class of a fiber of the projection onto the first factor, $f^*C = pC, g^*C = qC$. Using Proposition 2.13 both $f$ and $g$ are loxodromic and the semigroup generated by $f, g$ is focal; if we again set $\mu = \frac{1}{2}\delta_f + \frac{1}{2}\delta_g$, then the degree of a product is the product of the degrees and the central limit theorem follows.

**Example 2.12.** Take a Hénon map, i.e. of the form:

$$h : (x, y) \mapsto (y + P(x), x),$$

where $P(x) \in \mathbb{C}[x]$ is a polynomial of degree $d \geq 2$. Consider the measure $\mu = \frac{1}{2}\delta_h + \frac{1}{2}\delta_h^{-1}$. The subgroup generated by $h$ and its inverse induces a lineal subgroup of isometries on $\mathbb{H}^\infty$. Since $\deg(h^p) = d^{|p|}$ for all $p \in \mathbb{Z}$, we have that $\frac{1}{\sqrt{n}} \log \deg(f^n)$ follows a folded normal law.

2.5. **Characterization of semigroups having a global fixed point on the boundary.** In this section, we study the semigroup of rational transformations whose action on the Picard-Manin space has a global fixed point on the boundary. In many cases, we shall use the following result.

Recall that the *translation length* of an isometry $f$ of a hyperbolic metric space $(X, d)$ is

$$\tau(f) := \lim_{n \to \infty} \frac{d(o, f^n o)}{n},$$

where $o \in X$ is any base point. Moreover the isometry $f$ is loxodromic if $\tau(f) > 0$.

**Proposition 2.13.** Take a semigroup $G$ and suppose that there exists a class $\alpha \in \partial \mathbb{H}^\infty$ which is fixed by $G$ and such that $f^*\alpha = \lambda(f)\alpha$ for $\lambda(f) \in \mathbb{R}^\ast$. Then the following properties are equivalent.

1. One has $\lambda(f) = \sqrt{d_{\text{top}}(f)}$.
2. The action of $f$ on $\mathbb{H}^\infty$ is not loxodromic.
3. One has $\lambda_1(f) = \sqrt{d_{\text{top}}(f)}$.

This proposition yields the following corollary:

**Corollary 2.14.** Take a semigroup $G$ and suppose that there exists a class $\alpha \in \partial \mathbb{H}^\infty$ which is fixed by $G$ and that $f^*\alpha = \lambda(f)\alpha$ for $\lambda(f) \in \mathbb{R}^\ast$. Then

$$\lambda_1(f) = \max(\lambda(f), d_{\text{top}}(f)\lambda(f)^{-1})$$

for all $f \in G$.

Before proving the above statement we will need the following lemma.

**Lemma 2.15.** For any $f$ in $G$, one has

$$\max \left( \frac{\lambda(f)}{\sqrt{d_{\text{top}}(f)}}, \frac{\sqrt{d_{\text{top}}(f)}}{\lambda(f)} \right) \leq 2 \frac{\deg(f)}{\sqrt{d_{\text{top}}(f)}}.$$

**Proof.** Let us prove the inequality $\frac{\lambda(f)}{\sqrt{d_{\text{top}}(f)}} \leq 2 \frac{\deg(f)}{\sqrt{d_{\text{top}}(f)}}$. Since $\alpha$ is nef, we have by Siu’s inequality

$$(f^*\alpha \cdot L) \leq 2 \frac{(\alpha \cdot L)}{(L^2)} (f^*L \cdot L).$$

Hence, since $f^*\alpha = \lambda(f)\alpha$, we obtain $\lambda(f) \leq 2 \deg(f)$ and the inequality follows by dividing by $\sqrt{d_{\text{top}}(f)}$ as required.

For the second inequality, we compute the intersection product $(f_\ast \alpha \cdot L)$ and obtain:

$$(f_\ast \alpha \cdot L) = \frac{1}{\lambda(f)} (f_\ast f^* \alpha \cdot L) = \frac{d_{\text{top}}(f)}{\lambda(f)} (\alpha \cdot L).$$

Moreover, the projection formula shows that:

$$(f_\ast \alpha \cdot L) = (\alpha \cdot f^*L).$$
By Siu’s inequality, we have:
\[ f^* L \leq 2 \frac{(f^* L \cdot L)}{L^2} L, \]
and using the fact that \( \alpha \) is nef, we obtain:
\[ \frac{d_{\text{top}}(f)}{\lambda(f)}(\alpha \cdot L) = (\alpha \cdot f^* L) \leq 2 \deg(f)(\alpha \cdot L). \]
Dividing by \( (\alpha \cdot L) \) and by \( \sqrt{d_{\text{top}}(f)} \) yields the second inequality.

One important result is the following lemma, which provides good estimates on the degree.

**Lemma 2.16.** For any \( f, g \in G \), one has:
\[
\sqrt{\frac{\deg(f \circ g)}{\lambda(f \circ g)}} \leq \sqrt{\frac{\deg(f)}{\lambda(f)}} + \sqrt{\frac{d_{\text{top}}(f)}{\lambda(f)}} \sqrt{\frac{\deg(g)}{\lambda(g)}}.
\]

**Proof of Lemma 2.16.** Take \( f, g \in G \), we write:
\[ f^* L = \deg(f)\alpha + v_1, \]
and
\[ g_* L = \deg(g)\alpha + v_2 \]
where \( v_1, v_2 \in \mathbb{H}^\infty \) and \( (v_i \cdot L) = 0 \). Using the projection formula, the fact that \( (\alpha^2) = 0 \) and our decomposition, we have:
\[
1 = (\alpha \cdot L) = \frac{1}{\lambda(g)}(g^* \alpha \cdot L) = \frac{1}{\lambda(g)}(\alpha \cdot g_* L) = \frac{1}{\lambda(g)}(\alpha \cdot v_2).
\]
Similarly since \( f_* \alpha = \frac{d_{\text{top}}(f)}{\lambda(f)} \alpha \), we also have:
\[
1 = (\alpha \cdot L) = \frac{\lambda(f)}{d_{\text{top}}(f)}(f_* \alpha \cdot L) = \frac{\lambda(f)}{d_{\text{top}}(f)}(\alpha \cdot f^* L) = \frac{\lambda(f)}{d_{\text{top}}(f)}(\alpha \cdot v_1).
\]
Let us also compute \( (f^* L^2) \) and \( (g_* L \cdot g_* L) \),
\[
d_{\text{top}}(f) = (f^* L^2) = 2 \deg(f)(\alpha \cdot v_1) + (v_1^2),
\]
and
\[
(g_* L \cdot g_* L) = 2 \deg(g)(\alpha \cdot v_2) + (v_2^2).
\]
Since \( L \) is nef and nef classes in \( \mathbb{H}^\infty \) are stable by pushforward, we have \( (g_* L \cdot g_* L) \geq 0 \), hence:
\[
(\alpha \cdot v_2) \leq 2 \deg(g)(\alpha \cdot v_2).
\]
We now compute \( \deg(f \circ g) \):
\[
\deg(f \circ g) = (g^* f^* L \cdot L) = (f^* L \cdot g_* L) = \deg(f)(\alpha \cdot v_2) + \deg(g)(\alpha \cdot v_1) + (v_1 \cdot v_2).
\]
Since the intersection form is negative definite on \( \{ v \in \mathbb{H}^\infty \mid (v \cdot L) = 0 \} \), the Cauchy-Schwartz inequality implies that:
\[
|(v_1 \cdot v_2)| \leq \sqrt{(v_1^2)(v_2^2)}.
\]
Applying the above inequality to (12), we get:
\[
\deg(f \circ g) \leq \deg(f)(\alpha \cdot v_2) + \deg(g)(\alpha \cdot v_1) + \sqrt{2 \deg(g)(\alpha \cdot v_2)|d_{\text{top}}(f) - 2 \deg(f)(\alpha \cdot v_1)|}.
\]
We now apply (10) and (11):
\[
\deg(f \circ g) \leq \deg(f)(\alpha \cdot v_2) + \deg(g)(\alpha \cdot v_1) + \sqrt{2 \deg(g)(\alpha \cdot v_2)|d_{\text{top}}(f) - 2 \deg(f)(\alpha \cdot v_1)|}.
\]
This last inequality together with (8) and (9) gives:

$$\deg(f \circ g) \leq \deg(f) \lambda(g) + \deg(g) \frac{d_{\text{top}}(f)}{\lambda(f)} + (2d_{\text{top}}(f) \deg(g) \lambda(g))^{1/2} \sqrt{1 - 2 \frac{\deg(f)}{\lambda(f)}}.$$ 

By Lemma 2.15, we have\( \sqrt{2 \deg(f)/\lambda(f)} - 1 \leq \sqrt{2 \deg(f)/\lambda(f)} \), hence:

$$\deg(f \circ g) \leq \left( \sqrt{\deg(f) \lambda(g)} + \sqrt{\deg(g) d_{\text{top}}(f)/\lambda(f)} \right)^2.$$ 

We conclude that:

$$\sqrt{\frac{\deg(f \circ g)}{\lambda(f \circ g)}} \leq \sqrt{\frac{\deg(f)}{\lambda(f)}} + \sqrt{\frac{\deg(g) d_{\text{top}}(f)}{\lambda(g) \lambda(f)^2}},$$

as required. \( \square \)

**Proof of Proposition 2.13** (ii) \( \Leftrightarrow \) (iii). We claim that for any \( f \in G \)

(13) \( \tau(\rho_f) = \log \lambda_1(f) - \frac{1}{2} \log d_{\text{top}}(f) \).

This is because

$$\tau(\rho_f) = \lim_{n \to \infty} d(\rho_{f^n}(L), L)$$

$$= \lim_{n \to \infty} \frac{1}{n} \cosh^{-1} \left( \frac{\deg(f^n)}{d_{\text{top}}(f)^n/2} \right)$$

$$= \lim_{n \to \infty} \left( \log \deg(f^n) - \frac{n}{2} \log d_{\text{top}}(f) + O(1) \right)$$

$$= \log \lambda_1(f) - \frac{1}{2} \log d_{\text{top}}(f),$$

where we used \( \log x \leq \cosh^{-1}(x) \leq \log x + \log 2 \). Hence, the isometry \( \rho_f \) is not loxodromic if and only if \( \lambda_1(f) = \sqrt{d_{\text{top}}(f)} \).

We now prove the implication (i) \( \Rightarrow \) (ii), by showing that if \( f \) is loxodromic, then \( \lambda(f) \neq \sqrt{d_{\text{top}}(f)} \). If \( f \) is loxodromic, there exists a nef class \( \beta \in \partial \mathbb{H}^\infty \) fixed by \( f \) and which is not proportional to \( \alpha \). Suppose that \( f^* \beta = \mu(f) \beta \) where \( \mu \in \mathbb{R}^* \). By the Hodge index theorem, the product \( (\alpha \cdot \beta) \) is non-zero and using the projection formula, we obtain:

$$d_{\text{top}}(f)(\alpha \cdot \beta) = (f^* \alpha \cdot f^* \beta) = \lambda(f) \mu(f)(\alpha \cdot \beta).$$

We thus obtain that \( \mu(f) = \frac{d_{\text{top}}(f)}{\lambda(f)} \). We now compute:

(14) \( ((f^n)^*(\alpha + \beta) \cdot L) = \left( \lambda(f)^n(\alpha \cdot L) + \frac{d_{\text{top}}(f)^n}{\lambda(f)^n}(\beta \cdot L) \right) \)

hence

$$\frac{1}{\sqrt{d_{\text{top}}(f^n)}}((f^n)^*(\alpha + \beta) \cdot L) = \left( \frac{\lambda(f)}{\sqrt{d_{\text{top}}(f)}} \right)^n(\alpha \cdot L) + \left( \frac{\sqrt{d_{\text{top}}(f)}}{\lambda(f)} \right)^n(\beta \cdot L).$$

Since \( f \) is loxodromic and since \( \alpha + \beta \) is big and nef, Lemma 2.15 shows that the above sequence must diverge to infinity, hence \( \lambda(f) \neq \sqrt{d_{\text{top}}(f)} \).

We finally show that (ii) \( \Rightarrow \) (i). By contradiction, suppose that \( \lambda(f) \neq \sqrt{d_{\text{top}}(f)} \); then Lemma 2.15 implies

$$2 \frac{\deg(f^n)}{\sqrt{d_{\text{top}}(f^n)}} \geq \max \left( \left( \frac{\lambda(f)}{\sqrt{d_{\text{top}}(f)}} \right)^n, \left( \frac{\sqrt{d_{\text{top}}(f)}}{\lambda(f)} \right)^n \right).$$

We thus conclude that \( \lim_{n \to +\infty} \frac{1}{n} \log \frac{\deg(f^n)}{\sqrt{d_{\text{top}}(f^n)}} = \tau(\rho_f) > 0 \), hence \( f \) is loxodromic, which contradicts our assumption. \( \square \)
Proof of Corollary 2.14. If \( f \in G \) is not loxodromic, then \( \lambda_1(f) = \lambda(f) = \sqrt{d_{\text{top}}(f)} \) by the previous proposition. Otherwise, \( f \) is loxodromic, and by Lemma 2.5 and equation (14) we have

\[
\log \deg(f^n) = \log ((f^n)^*(\alpha + \beta) \cdot L) + O(1) = \log (\lambda(f)^{n}(\alpha \cdot L) + d_{\text{top}}(f)^n \lambda(f)^{-n}(\beta \cdot L)) + O(1)
\]

Hence, \( \lambda_1(f) = \max(\lambda(f), d_{\text{top}}(f) \lambda(f)^{-1}) \), as required. \( \square \)

3. General facts on random products

3.1. Random products of dominant rational surface maps. We fix a countable semigroup \( G \) of dominant rational maps on \( \mathbb{P}^2 \) and consider a random walk with transition law \( \mu \) on \( G \). We assume that the support of the measure \( \mu \) generates \( G \) and that the following integral is finite:

\[
\int_G \log \deg(g) \ d\mu(g) < +\infty.
\]

Recall that the topological degree is multiplicative and the algebraic degree is submultiplicative, i.e.

\[
d_{\text{top}}(f \circ g) = d_{\text{top}}(f) d_{\text{top}}(g) \quad \deg(f \circ g) \leq \deg(f) \deg(g)
\]

hence Kingman’s subadditive ergodic theorem shows the existence of the limit

\[
\ell_\mu := \lim_{n \to +\infty} \frac{1}{n} \int_G \log \deg(f) \ d\mu_n(f).
\]

Now, \( d_{\text{top}}(f) \leq (\deg(f))^2 \) by Theorem 2.1, hence by (15) the function \( f \mapsto \log d_{\text{top}}(f) \) is integrable, hence we also have the limit

\[
D_\mu := \lim_{n \to +\infty} \frac{1}{n} \int_G \frac{1}{2} \log d_{\text{top}}(f) \ d\mu_n(f).
\]

3.2. Projection of two dimensional random walks. Let us recall the classical central limit theorem in two variables.

**Theorem 3.1.** Let \((Z_n)\) be a sequence of i.i.d. random variables with values in \( \mathbb{R}^2 \), with \( \mathbb{E}[Z_1] = m \in \mathbb{R}^2 \) and covariance matrix \( \Sigma \). Then

\[
\frac{\sum_{i=1}^n Z_i - nm}{\sqrt{n}} \to \mathcal{N}(0, \Sigma)
\]

where \( \mathcal{N}(0, \Sigma) \) is a two-dimensional Gaussian distribution.

For our result, we shall need a random variable which is closely related to a normal law.

**Definition 3.2.** Fix \( \Sigma \) a \( 2 \times 2 \) symmetric matrix. The folded Gaussian distribution parametrized by \( \Sigma \), denoted \( \mathcal{F}\mathcal{N}(0, \Sigma) \) is the pushforward of the normal distribution \( \mathcal{N}(0, \Sigma) \) by the map \( \varphi(x, y) = x + |y| \).

We will then apply it in the following situation.

**Proposition 3.3.** In the above situation, let \( \varphi : \mathbb{R}^2 \to \mathbb{R} \) be defined as \( \varphi(x, y) := x + |y| \), and let \( m = (m_x, m_y) \in \mathbb{R}^2 \). Then:

1. If \( m_y \neq 0 \), then

\[
\frac{\varphi(\sum Z_i) - n(m_x + |m_y|)}{\sqrt{n}} \to \mathcal{N}(0, \sigma).
\]

2. If \( m_y = 0 \), then

\[
\frac{\varphi(\sum Z_i) - nm_x}{\sqrt{n}} \to \mathcal{F}\mathcal{N}(0, \Sigma)
\]

where \( \mathcal{F}\mathcal{N}(0, \Sigma) \) is the pushforward of the normal distribution \( \mathcal{N}(0, \Sigma) \) by the map \( \varphi(x, y) = x + |y| \).
Proof. Let $S_n := \sum_{i=1}^n Z_i$, let $S_n = (Z_{x,n}, Z_{y,n})$, the two coordinates of $S_n$, and let $m = (m_x, m_y)$. If $m_y > 0$, then
\[
P(Y_n < 0) \rightarrow 0
\]
since $Z_{y,n}$ satisfies a CLT with positive drift $m_y > 0$. Note that, by applying the linear map $(x, y) \mapsto x + y$ to the above Theorem,
\[
P\left( \frac{Z_{x,n} + Z_{y,n} - n(m_x + m_y)}{\sqrt{n}} \in [a, b] \right) \rightarrow \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^b e^{-t^2/2\sigma^2} dt
\]
Moreover, note that
\[
\left| P\left( \frac{\varphi(S_n) - n(m_x + m_y)}{\sqrt{n}} \in [a, b] \right) - P\left( \frac{Z_{x,n} + Z_{y,n} - n(m_x + m_y)}{\sqrt{n}} \in [a, b] \right) \right| \leq 2P(Y_n < 0)
\]
hence, since the last term tends to 0,
\[
P\left( \frac{\varphi(S_n) - n(m_x + m_y)}{\sqrt{n}} \in [a, b] \right) \rightarrow \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^b e^{-t^2/2\sigma^2} dt
\]
as required. If $m_y < 0$, the same argument yields
\[
P\left( \frac{\varphi(S_n) - n(m_x - m_y)}{\sqrt{n}} \in [a, b] \right) \rightarrow \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^b e^{-t^2/2\sigma^2} dt
\]
The case $m_y = 0$ is more subtle. By replacing $Z_n$ by $Z_n - (m_x, 0)$, we can assume $m_x = 0$. Then by the CLT we have $\left( \frac{Z_{x,n}}{\sqrt{n}}, \frac{Z_{y,n}}{\sqrt{n}} \right) \rightarrow N(0, \Sigma)$, hence
\[
\frac{Z_{x,n} + |Z_{y,n}|}{\sqrt{n}} \rightarrow \mathcal{FN}(0, \Sigma)
\]
where $\mathcal{FN}(0, \Sigma)$ is the pushforward of the normal distribution $N(0, \Sigma)$ by the map $\varphi(x, y) = x + |y|$. \qed

4. Proof of Theorem A for elementary semigroups

4.1. Central limit theorem for the algebraic degree.

4.1.1. Degree for elliptic semigroups. Suppose that $G$ induces an elliptic semigroup action on $\mathbb{H}^\infty$. Observe that we can write
\[
\log \deg(f_n) = \frac{1}{2} \log d_{\text{top}}(f_n) + \log \frac{f_n^* L \cdot L}{d_{\text{top}}(f_n)}
\]
Since the semigroup $G$ is elliptic and $d(L, \rho_{f_n}(L)) = \log \frac{(f_n^* L \cdot L)}{d_{\text{top}}(f_n)} + O(1)$ by Lemma 2.3, the second term above is bounded. Now, the sequence $\left( \frac{1}{2} \log d_{\text{top}}(f_n) \right)$ follows a central limit theorem since it is the sum of i.i.d. random variables, hence also the sequence
\[
\frac{\log \deg(f_n) - n \ell_\mu}{\sqrt{n}}
\]
converges to the normal law $N(0, \sigma)$, where $\sigma$ is given by $\sigma^2 = \int_G \left( \frac{1}{2} \log d_{\text{top}}(f) - \ell_\mu \right)^2 d\mu(f)$.

4.1.2. Lineal semigroups. Suppose the action of $G$ on the hyperbolic space is lineal. Let $\theta_+, \theta_-$, be the two invariant nef classes on the boundary which are either globally $G$-invariant or swapped, normalized so that $(\theta_+ \cdot L) = (\theta_- \cdot L) = 1$. Let $\lambda : G \rightarrow \mathbb{R}$ be defined so that $f^* (\theta_+) = \lambda(f) \theta_+$. Since the classes $\theta_+$ and $\theta_-$ are invariant classes on the boundary, the random variables
\[
\log (f_n^* \theta_+ \cdot L) \quad \text{and} \quad \log (f_n^* \theta_- \cdot L)
\]
describe a random walk on the real line by Lemma 2.4. We prove the following central limit theorem.
Theorem 4.1. Suppose that the averages given by:

\[ D_\mu := \frac{1}{2} \int_G \log d_{\text{top}}(f) \, d\mu(f) \quad \Lambda_\mu := \int_G \log \lambda(f) \, d\mu(f) \]

are finite, and moreover the variances

\[ \sigma_1^2 := \int_G \left( \frac{1}{2} \log d_{\text{top}}(f) - D_\mu \right)^2 \, d\mu(f) \quad \sigma_2^2 := \int_G (\log \lambda(f) - \Lambda_\mu)^2 \, d\mu(f) \]

are also finite. Then:

1. If \( D_\mu \neq \Lambda_\mu \), then there exists \( \sigma \geq 0 \) such that

\[ \frac{\log \deg(f_n) - n(D_\mu + |D_\mu - \Lambda_\mu|)}{\sqrt{n}} \to \mathcal{N}(0, \sigma). \]

2. If \( D_\mu = \Lambda_\mu \), then

\[ \frac{\log \deg(f_n) - nD_\mu}{\sqrt{n}} \to \mathcal{FN}(0, \Sigma) \]

where \( \mathcal{FN}(0, \Sigma) \) is the pushforward of the two-dimensional normal distribution \( \mathcal{N}(0, \Sigma) \) by the map \( \phi(x, y) = x + |y| \).

To prove Theorem 4.1, we first compute the degree as follows.

Lemma 4.2. For each \( f \in G \), we have

\[ \log \deg(f) = \frac{1}{2} \log d_{\text{top}}(f) + \log \lambda(f) - \frac{1}{2} \log d_{\text{top}}(f) + O(1) \]

where \( O(1) \) is a universal constant.

Proof. Take \( \theta_+, \theta_- \) the two invariant nef classes on the boundary which are either globally \( G \)-invariant or swapped, normalized so that \( (\theta_+ \cdot L) = (\theta_- \cdot L) = 1 \). Observe that the Hodge index theorem implies that \( \theta_+ + \theta_- \) is big and nef. By Lemma [2.5] we have:

\[ |\log \deg(f) - \log((f^*\theta_+ \cdot L) + (f^*\theta_- \cdot L))| \leq C. \]

Since \( f^*\theta_+ = \lambda(f)\theta_+ \) and \( f^*\theta_- = \frac{d_{\text{top}}(f)}{\lambda(f)}\theta_- \), we have

\[ \log((f^*\theta_+ \cdot L) + (f^*\theta_- \cdot L)) = \log \left( \lambda(f) + \frac{d_{\text{top}}(f)}{\lambda(f)} \right) = \log \left( \lambda(f) + d_{\text{top}}(f) \right) + \log(2 \cosh u(f)) \]

where \( u(f) = \log \lambda(f) - \frac{1}{2} \log d_{\text{top}}(f) \) and \( \cosh \) is the hyperbolic cosine. Hence, we rewrite (16) as follows:

\[ \left| \log \deg(f) \sqrt{d_{\text{top}}(f)} - \log \cosh u(f) \right| \leq C' \]

with \( C' = C + \log 2 \). Observe that the following inequality is satisfied for all \( x \in \mathbb{R} \):

\[ \frac{e^{\frac{|x|}{2}}}{2} \leq \cosh(x) \leq e^{|x|}, \]

hence we get:

\[ |\log \cosh u(f) - |u(f)|| \leq \log 2. \]

In particular, using the above equation and (18), we obtain:

\[ \left| \log \frac{\deg(f)}{\sqrt{d_{\text{top}}(f)}} - |u(f)| \right| \leq C'', \]
where $C'' = C + 2\log 2 > 0$. Hence, we decompose $\log \deg(f)$ as follows:

$$
\log \deg(f) = \frac{1}{2} \log d_{\text{top}}(f) + \log \frac{\deg(f)}{\sqrt{d_{\text{top}}(f)}}
$$

(20)

$$
= \frac{1}{2} \log d_{\text{top}}(f) + \left| \log \lambda(f) - \frac{1}{2} \log d_{\text{top}}(f) \right| + O(1),
$$

completing the proof of the lemma.  

Proof of Theorem 4.1. Let us now consider the map $\Phi : G \to \mathbb{R} \times \mathbb{R}$ defined as

$$
\Phi(f) := \left( \frac{1}{2} \log d_{\text{top}}(f), \log \lambda(f) - \frac{1}{2} \log d_{\text{top}}(f) \right)
$$

which defines a semigroup homomorphism. Then, the random walk on $G$ pushes forward to a random walk on $\mathbb{R} \times \mathbb{R}$.

More precisely, take $\nu$ the pushforward of the measure $\mu$ by $\phi$ and take $Z_i$ a sequence of i.i.d random variables with distribution law $\nu$. Take $\varphi : (x, y) \mapsto x + |y|$, then the logarithm of the degree of a random products rewrites as:

$$
\log \deg(f_n) = \varphi \left( \sum_{i=1}^{n} Z_i \right).
$$

Using Proposition 3.3, we conclude that Theorem 4.1 holds.  

4.1.3. Parabolic semigroups. We now suppose that the action of $G$ on $\mathbb{H}^{\infty}$ is parabolic. Take $\alpha \in \partial \mathbb{H}^{\infty}$ a globally $G$-invariant class, we shall choose $\alpha$ so that $(\alpha \cdot L) = 1$. For each $f \in G$, we denote by $\lambda : G \to \mathbb{R}^*$ the factor $\lambda(f) = (f^* \alpha \cdot L)$.

Note that by (7),

$$
\frac{d_{\text{top}}(f)}{\lambda(f)} \leq \lambda_1(f)
$$

hence, using $\lambda_1(f) \leq \deg(f)$,

$$
\frac{\deg(f)}{\lambda(f)} \leq \frac{\deg(f) \lambda_1(f)}{d_{\text{top}}(f)} \leq \frac{(\deg(f))^2}{d_{\text{top}}(f)}
$$

and, since

$$
\int_{G} \frac{\deg(f)}{\sqrt{d_{\text{top}}(f)}} \, d\mu(f) < +\infty
$$

(21)

holds, we also have

$$
\int_{G} \left( \frac{\deg(f)}{\lambda(f)} \right)^{1/2} \, d\mu(f) < +\infty.
$$

(22)

By multiplying by a factor $\frac{\sqrt{d_{\text{top}}(h)}}{\lambda(h)}$ in Lemma 2.16 we get that for all $f, g, h \in G$, the following inequality holds:

$$
\sqrt{\frac{\deg(f \circ g)}{\lambda(f \circ g)}} \cdot \sqrt{\frac{d_{\text{top}}(h)}{\lambda(h)}} \leq \sqrt{\frac{\deg(f)}{\lambda(f)}} \cdot \sqrt{\frac{d_{\text{top}}(h)}{\lambda(h)}} + \sqrt{\frac{\deg(g)}{\lambda(g)}} \cdot \sqrt{\frac{d_{\text{top}}(f \circ h)}{\lambda(f \circ h)}}.
$$

(23)

Observe that since $G$ is parabolic, it has no loxodromic element and Proposition 2.13 shows that $\lambda(f) = \sqrt{d_{\text{top}}(f)}$ for all $f \in G$. As a result, we obtain:

$$
\sqrt{\frac{\deg(f \circ g)}{\lambda(f \circ g)}} \leq \sqrt{\frac{\deg(f)}{\lambda(f)}} + \sqrt{\frac{\deg(g)}{\lambda(g)}},
$$

where $C'' = C + 2\log 2 > 0$. Hence, we decompose $\log \deg(f)$ as follows:
for all $f, g \in G$. Moreover, equation (22) proves that the cocycle $\sqrt{\deg(\cdot)/\lambda(\cdot)}$ belongs to $L^1(\mu)$, hence Kingman’s subadditive ergodic theorem yields the almost sure convergence

$$\lim_{n \to +\infty} \frac{1}{n} \sqrt{\frac{\deg(f_n)}{\lambda(f_n)}} = C,$$

where $C \in [0, \infty)$. This proves that almost surely

$$\limsup_{n \to \infty} \frac{1}{\sqrt{n}} \log \frac{\deg(f_n)}{\lambda(f_n)} \leq 0.$$

Moreover, Lemma 2.13 implies

$$\liminf_{n \to \infty} \frac{1}{\sqrt{n}} \log \frac{\deg(f_n)}{\lambda(f_n)} \geq 0,$$

hence the sequence of random variables

$$\frac{1}{\sqrt{n}} (\log \lambda(f_n) - \log \deg(f_n))$$

converges to zero almost surely, hence in probability. Since $\log \lambda(f_n)$ describes a random walk on the real line, hence follows a central limit theorem, we deduce that the central limit theorem also holds for $\log \deg(f_n)$.

4.1.4. Focal semigroups. Recall from Proposition 2.13 that the product $f_n$ is not loxodromic if and only if $\lambda(f_n) = \sqrt{\operatorname{d}_{\text{top}}(f_n)}$.

Let us look at the map $\varphi : G \to \mathbb{R}$ defined as $\varphi(f) := \log \sqrt{\frac{\operatorname{d}_{\text{top}}(f)}{\lambda(f)}}$. Since both the topological degree and $\lambda$ are multiplicative, the image of the random walk under $\varphi(f)$ describes a random walk on the real line. Consider:

$$D_\mu = \frac{1}{2} \int \log \operatorname{d}_{\text{top}}(f) d\mu,$$

and

$$\Lambda_\mu = \int \log \lambda(f) d\mu,$$

so that that $D_\mu - \Lambda_\mu = \int_G \log \sqrt{\frac{\operatorname{d}_{\text{top}}(f)}{\lambda(f)}} d\mu(f)$ is the drift of such random walk. There are two cases:

**Case 1.** $D_\mu \neq \Lambda_\mu$. In this case, the random walk induced by $\varphi$ on the line is transient. Hence, by Lemma 2.13 almost surely the element $f_n$ will be loxodromic except for finitely many values of $n$. Moreover, $\mathbb{P}(f_n \text{ is loxodromic}) \to 1$ as $n \to \infty$. If $f_n$ is loxodromic, then by Lemma 4.2

$$\log \deg(f) = \frac{1}{2} \log \operatorname{d}_{\text{top}}(f) + \left| \log \lambda(f) - \frac{1}{2} \log \operatorname{d}_{\text{top}}(f) \right| + O(1).$$

Thus, by the same argument as in the proof of Theorem 4.1, the variable $\log \deg(f_n)$ follows a central limit theorem.

**Case 2.** $D_\mu = \Lambda_\mu$. Consider the map $\tau : (G^n, \mu^n) \to \mathbb{N}$ defined as $\tau(\omega) := \min\{k \geq 1 : f_k \text{ is not loxodromic}\}$. Since the random walk induced by $\varphi$ is recurrent, then $\tau$ is defined almost everywhere. This defines a new measure $\mu_\tau$ on $G$ by $\mu_\tau(g) := \mathbb{P}(\omega \in G^n : f_{\tau(\omega)} = g)$.

Denote by $n_k = \tau_1 + \ldots + \tau_k$. Let us now note by Lemma 2.16 the functions $a_k(\omega) := \sqrt{\frac{\deg(f_{n_k})}{\lambda(f_{n_k})}}$ are a subadditive cocycle. Hence, by Kingman’s theorem, almost surely we have

$$\lim_{k \to \infty} \frac{1}{k} \sqrt{k} \sqrt{\frac{\deg(f_{n_k})}{\lambda(f_{n_k})}} = C$$

which implies almost surely

$$\limsup_{k \to \infty} \frac{1}{\sqrt{k}} \log \frac{\deg(f_{n_k})}{\lambda(f_{n_k})} \leq 0$$

(24)
Now, given a typical path $\omega$ and $n$, there exists $k$ such that $n_k \leq n < k_{k+1}$. Then
\[\frac{\deg(f_n)}{\lambda(f_n)} \leq \frac{\deg(f_{n_k})}{\lambda(f_{n_k})} \cdot \frac{\deg(g_{n_{k+1}} \cdots g_n)}{\lambda(g_{n_{k+1}} \cdots g_n)}\]
Thus,
\[
\frac{1}{\sqrt{n}} \log \frac{\deg(f_n)}{\lambda(f_n)} \leq \frac{1}{\sqrt{n}} \log \frac{\deg(f_{n_k})}{\lambda(f_{n_k})} + \frac{1}{\sqrt{n}} \log \frac{\deg(g_{n_{k+1}} \cdots g_n)}{\lambda(g_{n_{k+1}} \cdots g_n)}
\]
and, using that $k \leq n$ and subadditivity,
\[
\frac{1}{\sqrt{n}} \log \frac{\deg(f_n)}{\lambda(f_n)} \leq \frac{1}{\sqrt{n}} \log \frac{\deg(f_{n_k})}{\lambda(f_{n_k})} + \frac{1}{\sqrt{n}} \sum_{i=n_{k+1}}^{n} \log \frac{\deg(g_i)}{\lambda(g_i)}
\]
Now, since
\[
\int \log^+ \frac{\deg(f)}{\lambda(f)} \, d\mu(f) < +\infty
\]
we have by Lemma 4.3
\[\limsup_{n \to \infty} \frac{1}{\sqrt{n}} \sum_{i=n_{k+1}}^{n} \log \frac{\deg(g_i)}{\lambda(g_i)} \leq 0\]
in probability. Hence, combining (24), (25), and (26), we obtain
\[\limsup_{n \to \infty} \frac{1}{\sqrt{n}} \log \frac{\deg(f_n)}{\lambda(f_n)} \leq 0\]
in probability. Finally, we have as before by Lemma 2.13
\[\liminf_{n \to \infty} \frac{1}{\sqrt{n}} \log \frac{\deg(f_n)}{\lambda(f_n)} \geq 0\]
almost surely and in probability. Hence, $\frac{\log \deg(f_n) - \log \lambda(f_n)}{\sqrt{n}}$ converges to 0 in probability, and $\log \lambda(f_n)$ follows a CLT, hence also $\log \deg(f_n)$ follows a central limit theorem.

\[\square\]

**Lemma 4.3.** Let $T : (\Omega, \lambda)$ be a measure-preserving system, and let $f : \Omega \to \mathbb{R}$ be in $L^1(\Omega, \lambda)$. Let $\tau : \Omega \to \mathbb{N}$ be integrable, and let $S(\omega) := T^{\tau(\omega)}(\omega)$. Let $\tau_k(\omega) := \sum_{i=1}^{k-1} \tau \circ T^i(\omega)$ and $k_n(\omega) := \sup \{k : \tau_k(\omega) \leq n\}$. Then
\[\limsup_{n \to \infty} \frac{1}{\sqrt{n}} \sum_{i=k_n(\omega)+1}^{n} f \circ T^i(\omega) \leq 0\]
in probability.

**Proof.** Since $f$ is integrable, then also $f^+ := \max \{0, f\}$ is integrable. Moreover, since $\tau$ is also integrable, then the function $g(\omega) := \sum_{i=1}^{\tau(\omega)} f^+(T^i(\omega))$ also lies in $L^1$. Now, we have
\[\sum_{i=k_n+1}^{n} f \circ T^i(\omega) \leq \sum_{i=n_{k+1}+1}^{n} f^+ \circ T^i(\omega) = g(S^k(\omega))\]
and since $g \in L^1$, we have
\[\lim_{k \to \infty} \frac{g(S^k(\omega))}{\sqrt{k}} = 0\]
in probability, since the law of $g \circ S^k$ is equal to the law of $g$. Hence, since $k \leq n$, we have
\[\frac{1}{\sqrt{n}} \sum_{i=k_n+1}^{n} f^+ \circ T^i(\omega) \to 0\]
in probability, which implies the claim.
\[\square\]
4.2. Central limit theorem for the first dynamical degree. We now prove that $\log \lambda_1(f_n)$ satisfies a central limit theorem. Since there is an invariant class on the boundary, Corollary 2.14 holds and $\lambda_1(f) = \max(\lambda(f), d_{\text{top}}(f)\lambda(f)^{-1})$ for all $f$ in $G$. This proves that

$$\log \lambda_1(f_n) = \frac{1}{2} \log d_{\text{top}}(f_n) + \log \left| \frac{\lambda(f_n)}{\sqrt{d_{\text{top}}(f_n)}} \right| = \frac{1}{2} \log d_{\text{top}}(f_n) + |u(f_n)|,$$

where $u(f) = \log \lambda(f) - \frac{1}{2} \log d_{\text{top}}(f)$.

Thus, as in the proof of Theorem 4.1, we obtain that the sequence

$$\frac{\log \lambda_1(f_n) - n\ell_{\mu}}{\sqrt{n}}$$

converges to $\mathcal{N}(0, \sigma)$ if $D_\mu \neq \Lambda_\mu$ and to $\mathcal{FN}(0, \Sigma)$ if $D_\mu = \Lambda_\mu$. This completes the proof of Theorem A for elementary semigroups.

4.2.1. Remarks on the folded normal law. Let $X, Y : \Omega \to \mathbb{R}$ be two random variables. Their covariance matrix is

$$\Sigma := \begin{pmatrix} \sigma^2_x & \sigma_{xy} \\ \sigma_{yx} & \sigma^2_y \end{pmatrix}$$

with

$$\sigma^2_x = \mathbb{E}[(X - \mathbb{E}[X])^2], \quad \sigma_{xy} = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])], \quad \sigma^2_y = \mathbb{E}[(Y - \mathbb{E}[Y])^2].$$

Let us note that if $\sigma^2_y = 0$, then the folded normal law $\mathcal{FN}(0, \Sigma)$ just coincides with the normal law $\mathcal{N}(0, \sigma_x)$. Indeed, by Cauchy-Schwarz we also have $\sigma_{xy} = 0$ and the matrix $\Sigma$ has only one non-zero entry.

In our setting, we have $X = \frac{1}{2} \log d_{\text{top}}(f)$ and $Y = \log \frac{\sqrt{d_{\text{top}}(f)}}{\lambda(f)}$. Suppose that $\Lambda_\mu = D_\mu$. Then the "vertical variance" equals

$$\sigma^2_y = \int \left( \log \frac{\sqrt{d_{\text{top}}(f)}}{\lambda(f)} \right)^2 d\mu(f).$$

Thus, clearly $\sigma_y = 0$ if and only if $\sqrt{d_{\text{top}}(f)} = \lambda(f)$ for any $f$ in the support of $\mu$, which by Proposition 2.13 is equivalent to all elements in the support of $\mu$ (equivalently, all elements of $G$) being not loxodromic. We have thus the following refinement of the main theorem in the elementary case.

**Theorem 4.4.** Suppose that $G$ is elementary and $\mu$ satisfies the assumptions of Theorem A. We have the following cases for the distribution of $\log \deg(f_n)$ and $\log \lambda_1(f_n)$.

1. If $G$ does not contain any loxodromic element (i.e. it is elliptic or parabolic), then $D_\mu = \Lambda_\mu$, $\sigma_y = 0$ and the limit law for both $\log \deg(f_n)$ and $\log \lambda_1(f_n)$ is the Gaussian $\mathcal{N}(D_\mu, \sigma)$ with

$$\sigma^2 = \int \left( \frac{1}{2} \log d_{\text{top}}(f) - D_\mu \right)^2 d\mu(f);$$

moreover, $\sigma = 0$ if and only if every element in the support of $\mu$ has the same topological degree.

2. If $G$ contains a loxodromic element and $D_\mu = \Lambda_\mu$, then the limit law for $\log \lambda_1(f_n)$ is a non-trivial folded Gaussian; moreover, for $\log \deg(f_n)$ the limit law is the same folded Gaussian if $G$ is lineal, while it is a Gaussian of variance

$$\sigma^2 = \int (\log \lambda(f) - \Lambda_\mu)^2 d\mu(f)$$

if $G$ is focal; thus, $\sigma = 0$ if and only if $\lambda(f)$ is constant on the support of $\mu$.

3. If $G$ contains a loxodromic element and $D_\mu \neq \Lambda_\mu$, then the limit law for both $\log \deg(f_n)$ and $\log \lambda_1(f_n)$ is Gaussian with mean $\ell := D_\mu + |\Lambda_\mu - D_\mu|$ and variance $\sigma^2$. Moreover:
• if $D_\mu > \Lambda_\mu$, then
  \[ \sigma^2 = \int \left( \log \frac{d_{\text{top}}(f)}{\lambda(f)} - 2D_\mu + \Lambda_\mu \right)^2 \, d\mu(f) \]
  hence $\sigma = 0$ if and only if $\frac{d_{\text{top}}(f)}{\lambda(f)}$ is constant on the support of $\mu$;
• if $D_\mu < \Lambda_\mu$, then
  \[ \sigma^2 = \int (\log \lambda(f) - \Lambda_\mu)^2 \, d\mu(f) \]
  hence $\sigma = 0$ if and only if $\lambda(f)$ is constant on the support of $\mu$.

The following table summarizes all possible limit behaviours.

| Type of group | Mean | Limit law for $\log \deg$ | Limit law for $\log \lambda_1$ |
|---------------|------|---------------------------|-------------------------------|
| elliptic      | $\ell = D_\mu = \Lambda_\mu$ | Gaussian | Gaussian |
|               | $\sigma^2 = \int \left( \frac{1}{2} \log d_{\text{top}}(f) - D_\mu \right)^2 \, d\mu$ | $\sigma^2 = \int \left( \frac{1}{2} \log d_{\text{top}}(f) - D_\mu \right)^2 \, d\mu$ |
| parabolic     | $\ell = D_\mu = \Lambda_\mu$ | Gaussian | Gaussian |
|               | $\sigma^2 = \int \left( \frac{1}{2} \log d_{\text{top}}(f) - D_\mu \right)^2 \, d\mu$ | $\sigma^2 = \int \left( \frac{1}{2} \log d_{\text{top}}(f) - D_\mu \right)^2 \, d\mu$ |
| focal, $D_\mu = \Lambda_\mu$ | $\ell = D_\mu = \Lambda_\mu$ | Gaussian | Folded Gaussian |
|               | $\sigma^2 = \int (\log \lambda(f) - \Lambda_\mu)^2 \, d\mu(f)$ | $\sigma^2 = \int (\log \lambda(f) - \Lambda_\mu)^2 \, d\mu(f)$ |
| focal, $D_\mu > \Lambda_\mu$ | $\ell = 2D_\mu - \Lambda_\mu$ | Gaussian | Gaussian |
|               | $\sigma^2 = \int \left( \log \frac{d_{\text{top}}(f)}{\lambda(f)} - 2D_\mu + \Lambda_\mu \right)^2 \, d\mu$ | $\sigma^2 = \int \left( \log \frac{d_{\text{top}}(f)}{\lambda(f)} - 2D_\mu + \Lambda_\mu \right)^2 \, d\mu$ |
| focal, $D_\mu < \Lambda_\mu$ | $\ell = \Lambda_\mu$ | Gaussian | Gaussian |
|               | $\sigma^2 = \int (\log \lambda(f) - \Lambda_\mu)^2 \, d\mu(f)$ | $\sigma^2 = \int (\log \lambda(f) - \Lambda_\mu)^2 \, d\mu(f)$ |
| lineal, $D_\mu = \Lambda_\mu$ | $\ell = D_\mu = \Lambda_\mu$ | Folded Gaussian | Folded Gaussian |
|               | $\sigma^2 = \int (\log \lambda(f) - \Lambda_\mu)^2 \, d\mu(f)$ | $\sigma^2 = \int (\log \lambda(f) - \Lambda_\mu)^2 \, d\mu(f)$ |
| lineal, $D_\mu > \Lambda_\mu$ | $\ell = 2D_\mu - \Lambda_\mu$ | Gaussian | Gaussian |
|               | $\sigma^2 = \int \left( \log \frac{d_{\text{top}}(f)}{\lambda(f)} - 2D_\mu + \Lambda_\mu \right)^2 \, d\mu$ | $\sigma^2 = \int \left( \log \frac{d_{\text{top}}(f)}{\lambda(f)} - 2D_\mu + \Lambda_\mu \right)^2 \, d\mu$ |
| lineal, $D_\mu < \Lambda_\mu$ | $\ell = \Lambda_\mu$ | Gaussian | Gaussian |
|               | $\sigma^2 = \int (\log \lambda(f) - \Lambda_\mu)^2 \, d\mu(f)$ | $\sigma^2 = \int (\log \lambda(f) - \Lambda_\mu)^2 \, d\mu(f)$ |
| non-elementary | $\ell > 0$ | Gaussian | Gaussian |

where
\[
\Sigma = \left( \begin{array}{c}
\int \left( \frac{1}{2} \log d_{\text{top}}(f) - D_\mu \right)^2 \, d\mu \\
\int \left( \frac{1}{2} \log d_{\text{top}}(f) - D_\mu \right) \left( \log \frac{d_{\text{top}}(f)}{\lambda(f)} - D_\mu + \Lambda_\mu \right) \, d\mu \\
\int \left( \frac{1}{2} \log d_{\text{top}}(f) - D_\mu \right)^2 \, d\mu
\end{array} \right).
\]

5. Proof of Theorem A for non-elementary semigroups

Let us now assume that the semigroup $G$ generated by the support of $\mu$ is non-elementary. We recall the following results due to Maher-Tiozzo.

Theorem 5.1 (Maher-Tiozzo [MT18a]). Let $G$ be a non-elementary, countable semigroup of isometries of a $\delta$-hyperbolic space $X$, with hyperbolic (Gromov) boundary $\partial X$. Let $\mu$ be a measure whose support generates $G$, and let $o \in X$ be a base point. Then for almost every sample path $f_n = g_1 \cdot \ldots \cdot g_n$, the sequence $(f_n o)$ converges to a point $\xi \in \partial X$. Moreover, the resulting hitting measure is non-atomic and is the unique $\mu$-stationary measure on the boundary.

Recall that a measure $\nu$ on a $G$-space $M$ is $\mu$-stationary if $\int_G g_* \nu \, d\mu(g) = \nu$. Moreover, it is $\mu$-ergodic if it is not a non-trivial convex combination of $\mu$-stationary probability measures on $M$.

Theorem 5.2 (Maher-Tiozzo [MT18b]). Let $\mu$ be an atomic non-elementary probability measure on Bir($\mathbb{P}^2$) with finite first moment. Then there exists $\ell_\mu > 0$ such that for a.e. random product $f_n = g_1 \cdot \ldots \cdot g_n$, we have:
\[
\lim_{n \to +\infty} \frac{1}{n} \log \deg(f_n) = \ell_\mu.
\]
Moreover, if $\mu$ is bounded then for almost every sample path, one has:

$$\lim_{n \to +\infty} \frac{1}{n} \log \lambda_1(f_n) = \ell_\mu.$$ 

5.1. The horofunction boundary. Let us recall the construction of the horofunction compactification of a non-proper hyperbolic space, as developed in [MT18a].

Let $(X, d)$ be a metric space and let $o \in X$ be a base point. Then we define for each $x \in X$ the map $\rho_x : X \to \mathbb{R}$

$$\rho_x(z) := d(x, z) - d(x, o) \quad \text{for } z \in X.$$ 

The function $\rho_x$ is 1-Lipschitz, and $\rho_x(o) = 0$. The assignment $x \mapsto \rho_x$ defines a map $\Phi : X \to \text{Lip}^1(X)$ into the space of 1-Lipschitz functions on $X$. The horofunction compactification $X_h$ of $X$ is defined as the closure of $\Phi(X)$ in $\text{Lip}^1(X)$, with respect to the topology of pointwise convergence. If $X$ is separable, then $X_h$ is compact and metrizable. Elements of $X_h$ are called horofunctions, and there are two types of them: finite horofunctions, if $\inf_{x \in X} h(x) \in \mathbb{R}$, and infinite horofunctions if $\inf_{x \in X} h(x) = -\infty$. We denote as $X_h^\infty$ the space of infinite horofunctions.

Moreover, there is a local minimum map $\pi : X_h^\infty \to X \cup \partial X$ defined as follows. If $h \in X_h^\infty$, then there exists a sequence $(x_n) \subseteq X$ such that $h(x_n) \to -\infty$. It turns out that such a sequence must converge in the Gromov topology to a point in the Gromov boundary $\partial X$, and the limit point does not depend on the particular choice of $(x_n)$. Hence, one defines a $G$-equivariant map $\pi : X_h^\infty \to \partial X$ as

$$\pi(h_x) := \lim_{n \to +\infty} x_n \in \partial X.$$ 

In fact, the local minimum map can also be defined for finite horofunctions, but we do not need it here. By [MT18a, Proposition 4.4], any $\mu$-stationary probability measure $\nu$ on $X_h^\infty$ only charges infinite horofunctions, i.e. $\nu(X_h^\infty) = 1$.

5.2. Central limit theorems for cocycles. Fix $G$ a semigroup of dominant rational maps of $\mathbb{P}^2$, and let $M$ be a compact $G$-space. Recall that a cocycle is a function $\sigma : G \times M \to \mathbb{R}$ such that

$$\sigma(gh, x) = \sigma(g, hx) + \sigma(h, x) \quad \forall g, h \in G, \forall x \in M.$$ 

A cocycle $\sigma : G \times M \to \mathbb{R}$ has constant drift $\lambda$ if there exists $\lambda \in \mathbb{R}$ such that

$$\int_G \sigma(g, x) \, d\mu(g) = \lambda$$

for any $x \in M$. A cocycle $\sigma : G \times M \to \mathbb{R}$ is centerable if it can be written as

$$\sigma(g, x) = \sigma_0(g, x) + \psi(x) - \psi(g \cdot x)$$

where $\sigma_0$ is a cocycle with constant drift and where $\psi : M \to \mathbb{R}$ is a continuous function. Given a cocycle, we denote by $\sigma_{\sup}(g) := \sup_{x \in M} |\sigma(g, x)|$. Finally, a cocycle has unique covariance $\nu$ if

$$\nu^2 = \int_{G \times M} (\sigma(g, x) - \lambda)^2 \, d\mu(g) \, d\nu(x)$$

for any $\mu$-stationary measure $\nu$. Recall the key ingredient in Benoist-Quint’s central limit theorem for cocycles ([BQ16a, Theorem 3.4]).

**Theorem 5.3** (Central limit theorem for cocycles, I). Let $G$ be a discrete group, $M$ be a compact metrizable $G$-space and $\mu$ an atomic measure on $G$. Assume $\sigma : G \times M \to \mathbb{R}$ is a centerable cocycle with drift $\lambda$ and unique covariance $\nu \geq 0$ and such that

$$\int_G \sigma_{\sup}^2(g) \, d\mu(g) < +\infty.$$ 

Then for any bounded continuous function $F$ on $\mathbb{R}$, uniformly in $x \in M$, one has:

$$\lim_{n \to +\infty} \int_G F\left(\frac{\sigma(g, x) - n\lambda}{\sqrt{n}}\right) \, d\mu_n(g) = \frac{1}{\sqrt{2\pi}v} \int_{\mathbb{R}} F(t)e^{-\frac{t^2}{2\nu^2}} \, dt.$$
However, a more general version of this theorem does not require the cocycle to have unique covariance. Indeed we have the following. As remarked in [Hor18, Remark 1.7], the proof is exactly the same as the proof of [BQ16b, Theorem 4.7].

**Theorem 5.4 (Central limit theorem for cocycles, II).** Let $G$ be a discrete group, $M$ be a compact metrizable $G$-space and $\mu$ an atomic measure on $G$. Let $\nu$ be a $\mu$-ergodic, $\mu$-stationary probability measure on $M$, and let $\sigma : G \times M \to \mathbb{R}$ be a centerable cocycle with drift $\lambda$. Then there exists $\nu \geq 0$ such that for $\nu$-a.e. $x \in M$ we have, for any bounded, continuous function $f$,

$$\lim_{n \to \infty} \int_{G} F \left( \frac{\sigma(g,x) - n\lambda}{\sqrt{n}} \right) \, d\mu_n(g) = \frac{1}{\sqrt{2\pi\nu}} \int_{\mathbb{R}} F(t) e^{-\frac{t^2}{2\nu}} \, dt.$$ 

**Remark 5.5.** Let us note that [MT18a] define the random walk as $f_n = g_1 \ldots g_n$, while [BQ16b, Hor18] use the definition $f_n = g_n \ldots g_1$. In this paper, we define the random walk as $f_n = g_1 \ldots g_n$ on the semigroup of rational maps, which, since the pullback is contravariant, induces the random walk $\rho_{f_n} = \rho_{g_n} \ldots \rho_{g_1}$ on the space of isometries. Thus, we can use the results of [BQ16b, Hor18] verbatim. Note finally that the $n$-step distributions $\mu_n$ of the left and right random walk are equal, hence, as far as convergence in probability is concerned, results on one and the other are equivalent. On the other hand, results on almost sure convergence do not automatically translate, but we do not directly use them here.

### 5.3. The Busemann cocycle

Let us now define $V$ as the subset of the Picard-Manin space given by $V := \text{Span}_{f \in G}(f^* L)$ and $X := V \cap \mathbb{H}^\infty$. Then $X$, with the metric $d$ induced by $\mathbb{H}^\infty$, is a geodesic, $\delta$-hyperbolic, and separable (since $G$ is countable) metric space, hence we can construct its horofunction compactification $M := \overline{X}^\delta$, which is metrizable. Moreover, $G$ acts by isometries on $X$ and by homeomorphisms on $M$.

Let us define the **Busemann cocycle**, denoted $\beta : G \times \overline{X}^\delta \to \mathbb{R}$, as

$$\beta(g,x) := h_x(d_{\text{top}}(g)^{-1/2} g_x \cdot L),$$

where $h_x$ is the horofunction associated to $x$. We use the following properties of the Busemann cocycle.

**Proposition 5.6.** Let $\beta : G \times \overline{X}^\delta \to \mathbb{R}$ be the Busemann cocycle, and let $\mu$ be an atomic probability measure on the group of isometries of $(X,d)$ with finite second moment. Then there exists $\lambda \in \mathbb{R}$ such that:

1. ([Hor18, Corollary 2.7]) For any $\mu$-stationary measure $\nu$ on $\overline{X}^\delta$,

$$\int \beta(g,x) \, d\nu(g)d\nu(x) = \lambda.$$

2. ([Hor18, Proposition 2.8]) For all $\epsilon > 0$ there exists a sequence $(C_n) \in \ell^1(\mathbb{N})$ such that

$$\mu_n(g \in G : |\beta(g,x) - n\lambda| \geq \epsilon n) \leq C_n$$

for any $x \in \overline{X}^\delta$.

Now, let us consider the cocycle

$$\eta(g,x) := \beta(g,x) + \frac{1}{2} \log d_{\text{top}}(g).$$

Let us recall that there is a $G$-equivariant map $\pi : X^h_\infty \to \partial X$ from the set of infinite horofunctions to the Gromov boundary.

**Lemma 5.7.** For all $x, y \in X^h_\infty$ such that $\pi(x) \neq \pi(y)$, there exists $C > 0$ such that for all $g \in G$

$$d(o,go) + \frac{1}{2} \log d_{\text{top}}(g) - C \leq \max\{\eta(g,x), \eta(g,y)\} \leq d(o,go) + \frac{1}{2} \log d_{\text{top}}(g)$$

**Proof.** By [Hor18, Corollary 2.3], for all $x, y \in X^h_\infty$ such that $\pi(x) \neq \pi(y)$, there exists $C > 0$ such that for all $g \in G$

$$d(o,go) - C \leq \max\{\beta(g,x), \beta(g,y)\} \leq d(o,go).$$

Now, the desired inequality follows by adding $\frac{1}{2} \log d_{\text{top}}(g)$ on all sides. \[\square\]
Recall that the Gromov product between $y$ and $z$ based at $x$ is $\langle y, z \rangle_x := \frac{d(x, y) + d(x, z) - d(y, z)}{2}$. We use the following basic fact about the Gromov product (see e.g. [MT18a Proposition 5.8]).

**Lemma 5.8.** Let $(X, d)$ be a $\delta$-hyperbolic space, and let $o \in X$ be a base point. Then there exists a constant $C > 0$ such that, for any isometry $f$ of $X$,

$$|\tau(f) - d(o, fo) + 2\langle fo, f^{-1}o \rangle_o| \leq C.$$ 

Moreover, we use the fact that the Gromov product decays faster than any given function:

**Lemma 5.9** (Taylor-Tiozzo [TT16], Lemma 3.4). Let $\mu$ be a non-elementary probability measure on a countable group $G$ of isometries of a $\delta$-hyperbolic space $X$, let $o \in X$ be a base point and let $(f_n)$ be a random walk driven by $\mu$. Then for any function $\varphi : \mathbb{N} \to \mathbb{R}$ with $\limsup_{n \to \infty} \frac{\varphi(n)}{n} = 0$, we have

$$\mathbb{P} \left( \langle f_n o, f_n^{-1} o \rangle_o \geq \varphi(n) \right) \to 0.$$

We are now ready to complete the proof of our main theorem.

**Proof of Theorem A; non-elementary case.** Let us first prove the CLT for $\log \deg(f_n)$. By [Hor18 Proposition 1.5] the cocycle $\beta(g, x)$ is centerable, as a consequence of Proposition 5.6. Let us note that $\beta(g, x)$ is centerable if and only if $\eta(g, x)$ is centerable. Hence, $\eta(g, x)$ is also centerable. Now, since $M$ is compact, there exists a $\mu$-stationary measure $\nu$ on $M$, and by taking its ergodic components we can assume that $\nu$ is $\mu$-ergodic. By [MT18a Proposition 4.4], stationarity implies $\nu(X^\infty) = 1$.

Moreover, by Theorem 5.4 we obtain that for $\nu$-almost every $x \in M$, the sequence $(\eta(f_n, x))$ satisfies a central limit theorem. Then, by taking a generic $x \in X^\delta_\infty$ and applying Lemma 5.7 we obtain a central limit theorem for

$$d(L, \rho_{f_n}(L)) + \frac{1}{2} \log d_{\text{top}}(f_n).$$

We have by definition

$$\log \deg(f_n) = \frac{1}{2} \log d_{\text{top}}(f_n) + \log \left( \frac{f_n^* L}{d_{\text{top}}(f_n)} \cdot L \right)$$

thus, since

$$\left( \frac{f_n^* L}{d_{\text{top}}(f_n)} \cdot L \right) \geq 1,$$

we get:

$$\log \deg(f_n) = \frac{1}{2} \log d_{\text{top}}(f_n) + \cosh^{-1} \left( \frac{f_n^* L}{d_{\text{top}}(f_n)} \cdot L \right) + O(1)$$

$$= \frac{1}{2} \log d_{\text{top}}(f_n) + d(L, \rho_{f_n}(L)) + O(1),$$

so we also obtain the central limit theorem for $(\log \deg f_n)$.

Let us now prove it for $(\log \lambda_1(f_n))$. As we just proved, the sequence

$$d(L, \rho_{f_n}(L)) + \frac{1}{2} \log d_{\text{top}}(f_n) - n \ell$$

converges in distribution to a Gaussian. Note that

$$\log \lambda_1(f) = \frac{1}{2} \log d_{\text{top}}(f) + \tau(f)$$

and by Lemma 5.8

$$\log \lambda_1(f) = \frac{1}{2} \log d_{\text{top}}(f) + d(L, \rho_f(L)) - 2\langle \rho_f(L), \rho_{f^{-1}}(L) \rangle_L + O(1)$$

Now, by Lemma 5.9 applied to $\varphi(n) = \sqrt{n}$,

$$\mathbb{P} \left( \frac{\langle \rho_{f_n}(L), (\rho_{f_n})^{-1}(L) \rangle_L}{\sqrt{n}} \geq \epsilon \right) \to 0.$$
Hence, by combining (28) and (29), we get a CLT for \( \log \lambda_1(f_n) \), as required.

\[ \square \]

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