A NOTE ON THE NO-THREE-IN-LINE PROBLEM ON A TORUS

ZOFIA STĘPIEŃ, ALICJA SZYMASZKIEWICZ, LUCJAN SZYMASZKIEWICZ, AND MACIEJ ZWIERZCHOWSKI

Abstract. In this paper we show that at most $2 \gcd(m, n)$ points can be placed with no three in a line on an $m \times n$ discrete torus. This limit is attained for infinitely many cases.

1. Introduction

The no-three-in-line problem originates from one of Dudeney’s problem in [1].

The no-three-in-line problem is the problem of choosing a subset of points from $n \times n$ square array of points in the plane so that no three of the chosen points are collinear. We want to choose the subset of maximum size.

The obvious upper limit is $2n$ since one can put at most two points in each row. This limit is attained for many small cases, for details see [3] and [4]. In [6] the authors give a probabilistic argument to support the conjecture that for a large $n$ this limit is unattainable.

As a lower bound, Erdős' construction (see [2]) shows that for $p$ prime one can select $p$ points. In [7] it is shown, that for $p$ prime one can select $3(p-1)$ points from $2p \times 2p$ grid.

In the literature we can find some extensions of the no-three-in-line problem, namely no-three-in-line on a torus (see [5]) and no-three-in-line-in-3D (see [8]).

In our article we cope with the no-three-in-line on a torus problem. We try to answer the following question:

How many points can be placed on an $m \times n$ discrete torus, such that no three points are in a line?

The authors of [5] gave the answer in some special cases. In our work we give a general upper bound which is attained in infinitely many cases. Details are shown below.

Let $m$ and $n$ be positive integers greater than 1. By a discrete torus $T_{m \times n}$ we mean $\mathbb{Z}_m \times \mathbb{Z}_n$. We define lines on $T_{m \times n}$ to be images of lines in the $\mathbb{Z} \times \mathbb{Z}$ under the covering projection $\pi$ defined as follows

$$\pi(a, b) := (a \mod m, b \mod n).$$

For convenience we also use notation

$$[a, b] := \pi(a, b)$$

and

$$[a] := a \mod \gcd(m, n),$$

$$[(a, b)] := ([a], [b]),$$

for any integers $a$ and $b$.

\textit{2000 Mathematics Subject Classification.} 05B99.
\textit{Key words and phrases.} Discrete torus; No-three-in-line problem; Chinese Remainder Theorem.

Corresponding author: Z. Stepień; e-mail: stepien@zut.edu.pl; Tel/fax:+48914494826.
We say that a set $X \subset T_{m \times n}$ satisfies the no-three-in-line condition if there are no three collinear points in $X$. Let $f(T_{m \times n})$ denote the size of the largest set $X$ satisfying the no-three-in-line condition.

In our paper we will prove the following theorems.

**Theorem 1.1.** We have

$$f(T_{m \times n}) \leq 2 \gcd(m, n).$$

**Theorem 1.2.** We have

1. For $\gcd(m, n) = 1$, $f(T_{m \times n}) = 2$.
2. For $\gcd(m, n) = 2$, $f(T_{m \times n}) = 4$.
3. Let $\gcd(m, n) = 3$.
   - If $\gcd(3m, n) = 9$ or $\gcd(m, 3n) = 9$, then $f(T_{m \times n}) = 6$.
   - If $\gcd(3m, n) = 3$ and $\gcd(m, 3n) = 3$, then $f(T_{m \times n}) = 4$.

2. **The determinant criterion and its consequences**

Let $X = (x_1, x_2) \in \mathbb{Z} \times \mathbb{Z}$, $Y = (y_1, y_2) \in \mathbb{Z} \times \mathbb{Z}$, $Z = (z_1, z_2) \in \mathbb{Z} \times \mathbb{Z}$. Denote by $D(X, Y, Z)$ the following determinant

$$\text{det} \begin{bmatrix} 1 & 1 & 1 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}.$$ 

Recall the determinant criterion for checking whether points are in a line:

**Lemma 2.1.** Three points $X, Y, Z \in \mathbb{Z} \times \mathbb{Z}$ are in line if and only if $D(X, Y, Z) = 0$.

Now we prove the determinant criterion on a torus.

**Lemma 2.2.** If three points $a, b$ and $c$ of $T_{m \times n}$ are in line, then $D(a, b, c) \equiv 0 \pmod{\gcd(m, n)}$.

**Proof.** Suppose that three points $a = (a_x, a_y)$, $b = (b_x, b_y)$ and $c = (c_x, c_y)$ are in line on $T_{m \times n}$. This means that there are $A, B, C \in \mathbb{Z} \times \mathbb{Z}$ such that $\pi(A) = a$, $\pi(B) = b$, $\pi(C) = c$ and $D(A, B, C) = 0$. More precisely

$$A = (a_x + \alpha x m, a_y + \alpha y n),$$

$$B = (b_x + \beta x m, b_y + \beta y n),$$

$$C = (c_x + \gamma x m, c_y + \gamma y n)$$

for some integer $\alpha_x, \alpha_y, \beta_x, \beta_y, \gamma_x, \gamma_y$.

We get

$$0 = D(A, B, C) = D(a, b, c) + nD((a_x, a_y), (b_x, \beta_y), (c_x, \gamma_y)) + mD((a_x, \alpha_y), (b_x, b_y), (c_x, \gamma_y)) + mnD((a_x, \alpha_y), (\beta_x, b_y), (\gamma_x, \gamma_y)).$$

Hence $D(a, b, c) \equiv 0 \pmod{\gcd(m, n)}$. 

**Lemma 2.3.** For $\gcd(m, n) \geq 2$, $f(T_{m \times n}) \geq 4$.

**Proof.** Let $\gcd(m, n) \geq 2$. Let

$$X = \{(0, 0), (0, 1), (1, 0), (1, 1)\}.$$

Since $m \geq 2$ and $n \geq 2$, it follows that $X \subset T_{m \times n}$. It is easy to check that $D(a, b, c) \in \{-1, 1\}$ for any $a, b, c \in X$. By Lemma 2.2, $X$ satisfies the no-three-in-line condition. Thus $f(T_{m \times n}) \geq 4$.

**Lemma 2.4.** Let $\gcd(m, n) = 3$. If $\gcd(3m, n) = 9$ or $\gcd(m, 3n) = 9$, then $f(T_{m \times n}) \geq 6$. 


To do this, we will show that three points 
\[ C = (2 + \gamma_1, 1, 0) \] 
are not in line on 
\[ T_{m \times n} \] 
and finishes the proof.

Proof. Let \( \gcd(m, n) = 3 \). First, suppose that \( \gcd(3m, n) = 9 \). Consequently \( m = 3l \) and \( n = 9k \) for some positive integer \( l, k \). Let 
\[ X = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 3), (2, 4)\} \].

Since \( m = 3l \geq 3 \) and \( n = 9k \geq 9 \), we have \( X \subset T_{m \times n} \).

Take any three points \( \{a, b, c\} \) from \( X \). A straightforward calculation shows that either
\[ D(a, b, c) \neq 0 \pmod{3} \]
or
\[ \{a, b, c\} = \{(0, 0), (1, 0), (2, 3)\} \text{ or } \{a, b, c\} = \{(0, 1), (1, 1), (2, 4)\} \].

We will show that neither \( \{0, 0\}, \{(0, 1), (2, 3)\} \) nor \( \{(0, 1), (1, 1), (2, 4)\} \) are in line. This together with Lemma 2.2 shows that \( X \) satisfies the no-three-in-line condition and finishes the proof.

First we will show that three points \( (0, 0), (1, 0), (2, 3) \) are not in line on \( T_{m \times n} \).

To do this, we will show that three points \( A = (m \alpha_1, n \alpha_2), B = (1 + m \beta_1, n \beta_2), C = (2 + m \gamma_1, 3 + n \gamma_2) \), where \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \) are arbitrary integers, are not in line on \( \mathbb{Z} \times \mathbb{Z} \).

We get
\[
D(A, B, C) = D((0, 0), (1, 0), (2, 3)) + nD((0, \alpha_2), (1, \beta_2), (2, \gamma_2))
+ mD((\alpha_1, 0), (\beta_1, 0), (\gamma_1, 3)) + mnD((\alpha_1, \alpha_2), (\beta_1, \beta_2), (\gamma_1, \gamma_2))
= 3 + 9k \cdot D((0, \alpha_2), (1, \beta_2), (2, \gamma_2))
+ 3l \cdot 3 \cdot D((\alpha_1, 0), (\beta_1, 0), (\gamma_1, 1)) + 3l \cdot 9k \cdot D((\alpha_1, \alpha_2), (\beta_1, \beta_2), (\gamma_1, \gamma_2))
= 3 + 9M \neq 0,
\]

since \( M \) is an integer.

Finally we will show that three points \( (0, 1), (1, 1), (2, 4) \) are not in line on \( T_{m \times n} \).

To do this, we will show that three points \( A' = (m \alpha_1', 1 + n \alpha_2'), B' = (1 + m \beta_1', 1 + n \beta_2'), C' = (2 + m \gamma_1', 4 + n \gamma_2') \), where \( \alpha_1', \alpha_2', \beta_1', \beta_2', \gamma_1', \gamma_2' \) are arbitrary integers, are not in line on \( \mathbb{Z} \times \mathbb{Z} \). Since \( D((\alpha_1', 1), (\beta_1', 1), (\gamma_1', 4)) = 3D((\alpha_1', 0), (\beta_1', 0), (\gamma_1', 1)) \), we get
\[
D(A', B', C') = D((0, 1), (1, 1), (2, 4)) + nD((0, \alpha_2'), (1, \beta_2'), (2, \gamma_2'))
+ mD((\alpha_1', 1), (\beta_1', 1), (\gamma_1', 4)) + mnD((\alpha_1', \alpha_2'), (\beta_1', \beta_2'), (\gamma_1', \gamma_2'))
= 3 + 9kD((0, \alpha_2'), (1, \beta_2'), (2, \gamma_2'))
+ 3lD((\alpha_1', 0), (\beta_1', 0), (\gamma_1', 1)) + 3l9kD((\alpha_1', \alpha_2'), (\beta_1', \beta_2'), (\gamma_1', \gamma_2')) \neq 0
\]

Now, for \( \gcd(m, 3m) = 9 \) the same argument as above shows that the set
\[ Y = \{(0, 0), (0, 1), (1, 0), (1, 1), (3, 2), (4, 2)\} \]
satisfies the no-three-in-line condition. \( \square \)

3. The Chinese Remainder Theorem and its consequences

In this paper we will use the following form of the Chinese Remainder Theorem.

**Theorem 3.1** (Chinese Remainder Theorem). Two simultaneous congruences
\[ x \equiv a \pmod{m}, \]
\[ x \equiv b \pmod{n} \]
are solvable if and only if \( a \equiv b \pmod{\gcd(m, n)} \). Moreover, the solution is unique modulo lcm\((m, n)\).
Now we define diagonal lines on $T_{m \times n}$. For any integer $s$, let
\[
L^+_s = \{[k, k-s] \in T_{m \times n} : k = 0, \pm 1, \pm 2, \ldots\},
\]
\[
L^-_s = \{[k, -k+s] \in T_{m \times n} : k = 0, \pm 1, \pm 2, \ldots\}.
\]

The following theorem is a consequence of Chinese Remainder Theorem.

**Theorem 3.2.** [see 9] Let $(i, j) \in T_{m \times n}$. The following holds:
1. $(i, j) \in L^+_{[i-j]}$. Moreover, we have $L^+_{s_1} \cap L^+_{s_2} = \emptyset$ for $s_1 \neq s_2$ and $s_1, s_2 \in \{0, 1, \ldots, \gcd(m, n) - 1\}$;
2. $(i, j) \in L^-_{[i+j]}$. Moreover, we have $L^-_{s_1} \cap L^-_{s_2} = \emptyset$ for $s_1 \neq s_2$ and $s_1, s_2 \in \{0, 1, \ldots, \gcd(m, n) - 1\}$.

**Proof.** (1) By Theorem 3.1 there exists an integer $k$ such that
\[
k \equiv i \pmod{m},
\]
\[
k \equiv j + [i-j] \pmod{n}.
\]
Consequently, $(i, j) = [k, k - [i-j]] \in L_{[i-j]}$. Suppose that $L^+_{s_1} \cap L^+_{s_2} \neq \emptyset$. This means that there are $k_1$ and $k_2$ such that $|k_1, k_1 - s_1| = |k_2, k_2 - s_2|$. In other words $k_1 - k_2$ is the solution of the following system
\[
k_1 - k_2 \equiv 0 \pmod{m},
\]
\[
k_1 - k_2 \equiv s_1 - s_2 \pmod{n}.
\]
By Theorem 3.1 again, we see that $|s_1 - s_2| = 0$ Hence $L_{s_1} \cap L_{s_2} = \emptyset$ for $s_1 \neq s_2$ and $s_1, s_2 \in \{0, 1, \ldots, \gcd(m, n) - 1\}$.
(2) The proof is similar to (1).

Let us define
\[
L^h_0 = \{[k, 3k+s] \in T_{m \times n} : k = 0, \pm 1, \pm 2, \ldots\},
\]
\[
L^v_0 = \{[3k+s, k] \in T_{m \times n} : k = 0, \pm 1, \pm 2, \ldots\}.
\]

**Lemma 3.1.** Let $\gcd(m, n) = 3$.
1. If $\gcd(3m, n) = 3$, then $(i, j) \in L^h_0$ for any $(i, j) \in T_{m \times n}$,
2. If $\gcd(m, 3n) = 3$, then $(i, j) \in L^v_0$ for any $(i, j) \in T_{m \times n}$.

**Proof.** (1) Assume $\gcd(m, n) = \gcd(3m, n) = 3$. By Theorem 3.1 there exists an integer $k_1$ such that
\[
k_1 \equiv 3i \pmod{3m},
\]
\[
k_1 \equiv j - [j] \pmod{n}.
\]
It is easy to see that $k_1 = 3k$ and we get
\[
3k \equiv 3i \pmod{3m},
\]
\[
3k \equiv j - [j] \pmod{n}.
\]
Hence
\[
k \equiv i \pmod{m},
\]
\[
3k \equiv j - [j] \pmod{n}
\]
and $(i, j) \in L^h_0$.
(2) The proof is similar to (1).

**Lemma 3.2.** Let $\gcd(m, n) = 3$ and $(i, j) \in T_{m \times n}$. If $\gcd(3m, n) = \gcd(m, 3n) = 3$, then
\[
(i, j) \in L^h_0 \cup L^v_0 \cup L^+_0 \cup L^-_0 \text{ for any } (i, j) \in T_{m \times n}.
\]
Moreover, $(0, 0) \in L^h_0 \cap L^v_0 \cap L^+_0 \cap L^-_0$. 

Proof. Let \((i, j) \in T_{m \times n}\). If \([j] = 0\) or \([i] = 0\) or \([i - j] = 0\), then by Lemma 5.1 and Theorem 5.2(1) we get 
\[(i, j) \in L_0^h \cup L_0^v \cup L_0^r.\]
Suppose that neither \([j] = 0\) nor \([i] = 0\) nor \([i - j] = 0\), this means that \([i, j]\) = \((1, 2)\) or \([i, j]\) = \((2, 1)\). By Theorem 5.2 (2) \((i, j) \in L_0^h\) for \([i + j] = 0\). The condition \((0, 0) \in L_0^h \cap L_0^v \cap L_0^r\) is obvious.

\[\blacksquare\]

**Theorem 3.3.** Let \(\gcd(m, n) = 3\). If \(\gcd(3m, n) = 3\) and \(\gcd(m, 3n) = 3\), then \(f(T_{m \times n}) \leq 4\).

**Proof.** Let \(X\) be a subset of \(T_{m \times n}\) satisfying the no-three-in-line condition. Suppose that \(|X| > 4\). We may assume without loss of generality that \((0, 0) \in X\). By Lemma 3.2, \(|X| = 5\) and
\[
|X \cap (L_0^h \setminus \{(0, 0)\})| = 1,
|X \cap (L_0^v \setminus \{(0, 0)\})| = 1,
|X \cap (L_0^- \setminus \{(0, 0)\})| = 1,
|X \cap (L_0^+ \setminus \{(0, 0)\})| = 1.
\]

Denote \(X = \{(0, 0), a, b, c, d\}\) where
\[
a \in L_0^h \setminus \{(0, 0)\}, \ b \in L_0^v \setminus \{(0, 0)\}, \ c \in L_0^- \setminus \{(0, 0)\}, \ d \in L_0^+ \setminus \{(0, 0)\}.
\]

Now, we consider four cases based on the choice of \(a\) and \(b\).

**Case 1.** Let \([a] = (1, 0), \ [b] = (0, 2)\).

We will show that since \(X\) satisfies the no-three-in-line condition, \(c = \{(2, 1)\}\). By the definition of \(L_0\), the possibilities for \(c\) are: \([c] = (0, 0), \ [c] = (2, 1), \ [c] = (1, 2)\).

For \([c] = (0, 0), \) by Lemma 5.1 (1), we have
\[
\{(0, 0), a, c\} \subset L_0^h.
\]
Moreover, by Lemma 5.1(2), we have
\[
\{(0, 0), b, c\} \subset L_0^v.
\]
For \([c] = (2, 1), \) by Theorem 5.2 (1), we have
\[
\{a, b, c\} \subset L_1^v.
\]
Therefore \([c] = (1, 2)\).

Finally, let \(d \in \{L_0^h \setminus \{(0, 0)\}\}\). By the definition of \(L_0^h\), the possibilities for \(d\) are \([d] = (0, 0), \ [d] = (1, 1), \ [d] = (2, 2)\). For \([d] = (0, 0), \) by Lemma 5.1 (1), we have
\[
\{(0, 0), a, d\} \subset L_0^h.
\]
Moreover, by Lemma 5.1(2), we have
\[
\{(0, 0), b, d\} \subset L_0^v.
\]
For \([d] = (1, 1), \) by Theorem 5.2 (2), we have
\[
\{a, c, d\} \subset L_1^v.
\]
For \([d] = (2, 2), \) by Theorem 5.2 (1), we have
\[
\{b, c, d\} \subset L_2^v.
\]
Thus for every possible value of \(d\) we get a contradiction with the assumption that \(X\) satisfies the no-three-in-line condition.

**Case 2.** Let \([a] = (2, 0), \ [b] = (0, 1)\). The proof of this case is similar to our analysis in Case 1.
**Case 3.** Let \([a] = (1, 0), \left[ b \right] = (0, 1)\). We will show that since \(X\) satisfies the no-three-in-line condition, \(d = \left[ (1, 1) \right]\). By the definition of \(L_0^+\), the possibilities for \(d\) are: \(\left[ d \right] = (0, 0), \left[ d \right] = (1, 1), \left[ d \right] = (2, 2)\).

For \(\left[ d \right] = (0, 0)\), by Lemma 3.1 (1), we have
\[
\{(0, 0), a, d\} \subset L_0^h.
\]
Moreover, by Lemma 3.1(2), we have
\[
\{(0, 0), b, d\} \subset L_0^v.
\]
For \(\left[ d \right] = (2, 2)\), by Theorem 3.2 (2), we have
\[
\{a, b, d\} \subset L_1^-.
\]
Therefore \(\left[ d \right] = (1, 1)\).

Finally, let \(c \in \left( L_0^- \setminus \{(0, 0)\} \right)\). By the definition of \(L_0^-\), the possibilities for \(c\) are \(\left[ c \right] = (0, 0), \left[ c \right] = (1, 2), \left[ c \right] = (2, 1)\). For \(\left[ c \right] = (0, 0)\), by Lemma 3.1 (1), we have
\[
\{(0, 0), a, c\} \subset L_0^h.
\]
Moreover, by Lemma 3.1(2), we have
\[
\{(0, 0), b, c\} \subset L_0^v.
\]
For \(\left[ c \right] = (1, 2)\), by Theorem 3.2 (2), we have
\[
\{a, c, d\} \subset L_1^v.
\]
For \(\left[ c \right] = (2, 1)\), by Theorem 3.2 (1), we have
\[
\{b, c, d\} \subset L_1^-.
\]
Thus for every possible value of \(c\) we get a contradiction with the assumption that \(X\) satisfies the no-three-in-line condition.

**Case 4.** Let \([a] = (2, 0), \left[ b \right] = (0, 2)\). The proof of this case is similar to our analysis in Case 3.

The proof is then complete. □

4. **Proof of main results**

Finally we can present the proofs of our main theorems.

**Proof of Theorem 1.1.** The Theorem follows from Theorem 3.2 (1) or Theorem 3.2 (2). □

**Proof of Theorem 1.2.** We will prove each case separately.

1. It is always true that \(f(T_{m,n}) \geq 2\). By Theorem 1.1 we get the statement.
2. Lemma 2.3 and Theorem 1.1 give the statement.
3. (a) Lemma 2.4 and Theorem 1.1 give the statement.
   (b) Lemma 2.5 together with Theorem 3.3 give the statement. □

**References**

[1] H. E. Dudeney, Amusements in Mathematics, Nelson, Edinburgh 1917, pp. 94, 222.
[2] P. Erdős. Appendix, in K.F. Roth, On a problem of Heilbronn, J. London Math. Soc. 26, 198–204, 1951.
[3] A. Flammenkamp, Progress in the no-three-in-line problem, J. Combin. Theory Ser. A, 60(2), 305–311, 1992.
[4] A. Flammenkamp, Progress in the no-three-in-line problem II, J. Combin. Theory Ser. A, 81(1), 108–113, 1998.
[5] J. Fowler, A. Groot, D. Pandya, B. Snapp, The no-three-in-line problem on a torus, arXiv:1203.6604v1.
[6] R. K. Guy P. A. Kelly, The No-Three-Line Problem, Math. Bull. Vol. 11, pp. 527-531, 1968.
A NOTE ON THE NO-THREE-IN-LINE PROBLEM ON A TORUS

[7] R. R. Hall, T. H. Jackson, A. Sudbery, and K. Wild, Some advances in the no-three-in-line problem, J. Combinatorial Theory Ser. A, 18:336–341, 1975.

[8] A. Por, D.R. Wood, No-Three-in-Line-in-3D, Algorithmica 47(4), 481–488, 2007.

[9] Z. Stępień, L. Szymaszkiewicz, M. Zwierzchowski, The Cartesian product of cycles with small 2-rainbow domination number, Journal of Combinatorial Optimization, doi:10.1007/s10878-013-9658-0.

School of Mathematics, West Pomeranian University of Technology, al. Piastów 48/49, 70-310 Szczecin, Poland
E-mail address: stepien@zut.edu.pl

School of Mathematics, West Pomeranian University of Technology, al. Piastów 48/49, 70-310 Szczecin, Poland
E-mail address: alicjasz@zut.edu.pl

Institute of Mathematics, Szczecin University, Wielkopolska 15, 70-451 Szczecin, Poland
E-mail address: lucjansz@wmf.univ.szczecin.pl

School of Mathematics, West Pomeranian University of Technology, al. Piastów 48/49, 70-310 Szczecin, Poland
E-mail address: mzwerz@zut.edu.pl