BAIRE PROPERTY OF SOME FUNCTION SPACES

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(Received April 12, 2022; revised July 28, 2022; accepted August 3, 2022)

Abstract. A compact space $X$ is called $\pi$-monolithic if for any surjective continuous mapping $f : X \to K$ where $K$ is a metrizable compact space there exists a metrizable compact space $T \subseteq X$ such that $f(T) = K$. A topological space $X$ is Baire if the intersection of any sequence of open dense subsets of $X$ is dense in $X$. Let $C_p(X, Y)$ denote the space of all continuous $Y$-valued functions $C(X, Y)$ on a Tychonoff space $X$ with the topology of pointwise convergence.

In this paper we have proved that for a totally disconnected space $X$ the space $C_p(X, \{0, 1\})$ is Baire if, and only if, $C_p(X, K)$ is Baire for every $\pi$-monolithic compact space $K$.

For a Tychonoff space $X$ the space $C_p(X, \mathbb{R})$ is Baire if, and only if, $C_p(X, L)$ is Baire for each Fréchet space $L$.

We construct a totally disconnected Tychonoff space $T$ such that $C_p(T, M)$ is Baire for a separable metric space $M$ if, and only if, $M$ is a Peano continuum. Moreover, $C_p(T, [0, 1])$ is Baire but $C_p(T, \{0, 1\})$ is not.

1. Introduction

A topological space $X$ is Baire if the Baire Category Theorem holds for $X$, i.e., the intersection of any sequence of open dense subsets of $X$ is dense in $X$. A space is meager (or of the first Baire category) if it can be written as a countable union of closed sets with empty interior. Clearly, if $X$ is a Baire space, then $X$ is not meager. The opposite implication is in general not true. However, it holds for every homogeneous space $X$ (see [4,
Theorem 2.3]). Being a Baire space is an important topological property for a space and it is therefore natural to ask when function spaces are Baire. The Baire property for continuous mappings was first considered in [14]. Then a paper [4] appeared, where various aspects of this topic were considered. In [4], necessary and, in some cases, sufficient conditions on a space $X$ were obtained under which the space $C_p(X, \mathbb{R})$ is Baire.

In general, it is not an easy task to characterize when a function space has the Baire property. The problem for $C_p(X, \mathbb{R})$ was solved independently by Pytkeev [8], Tkachuk [11, 12] and van Douwen [5], and for a space $B_1(X, \mathbb{R})$ of Baire-one functions was solved by Osipov [6].

In this paper we present characterizations for the spaces $C_p(X, \{0, 1\})$ and $C_p(X, \mathbb{N})$ to be Baire. Moreover, we have proved that for a totally disconnected space $X$ the space $C_p(X, \{0, 1\})$ is Baire if, and only if, $C_p(X, K)$ is Baire for every $\pi$-monolithic compact space $K$. We also establish that, for a Tychonoff space $X$, the space $C_p(X, \mathbb{R})$ is Baire if, and only if, $C_p(X, L)$ is Baire for each Fréchet space $L$.

Finally, we construct a totally disconnected Tychonoff space $T$ such that $C_p(T, M)$ is Baire for a separable metric space $M$ if, and only if, $M$ is a Peano continuum. Moreover, $C_p(T, [0, 1])$ is Baire but $C_p(T, \{0, 1\})$ is not.

2. Main definitions and notation

Throughout this paper, all spaces are assumed to be Tychonoff, i.e., completely regular $T_1$-spaces. The set of positive integers is denoted by $\mathbb{N}$ and $\omega = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}$ be the real line, we put $D = \{0, 1\} \subset I = [0, 1] \subset \mathbb{R}$ and let $\mathbb{Q}$ be the rational numbers. Let $f: X \to \mathbb{R}$ be a real-valued function, then $\|f\| = \sup\{|f(x)| : x \in X\}$. Let $V = \{f \in \mathbb{R}^X : f(x_i) \in V_i, i = 1, \ldots, n\}$ where $x_i \in X$, $V_i \subseteq \mathbb{R}$ are bounded intervals for $i = 1, \ldots, n$, then $\text{supp}V = \{x_1, \ldots, x_n\}$, $V = \text{diam}V = \max\{\text{diam}V_i : 1 \leq i \leq n\}$.

Let $C_p(X, Y)$ be the space of continuous functions from $X$ to $Y$ with the topology of pointwise convergence. $C_p(X, \mathbb{R})$ is usually denoted by $C_p(X)$.

A mapping $f: X \to Y$ is said to be irreducible if the only closed subset $Z$ of $X$, with $f(Z) = Y$, is $Z = X$.

Recall that two sets $A$ and $B$ are functionally separated if there exists a continuous real-valued function $f$ on $X$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$.

Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of topological spaces. Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be the Cartesian product with the Tychonoff topology. Take a point $p = (p_\lambda)_{\lambda \in \Lambda} \in X$. For each $x = (x_\lambda)_{\lambda \in \Lambda} \in X$, let $\text{Supp}(x) = \{\lambda \in \Lambda : x_\lambda \neq p_\lambda\}$. Then the subspace $\sum(p) = \{x \in X : \text{Supp}(x) \text{ is countable}\}$ of $X$ is called a $\sum$-product of $\{X_\lambda : \lambda \in \Lambda\}$ about $p$ ($p$ is called the base point).

A topological space $X$ is a condensation of a topological space $Y$ if there is a continuous bijection from $Y$ onto $X$.

Acta Mathematica Hungarica 168, 2022
Recall that a Peano continuum is called a connected and locally connected metric compact space.

For the terms and symbols not defined here, please, consult [1,3].

3. Main results

Recall that a space $X$ is totally disconnected if the quasi-component of any point $x \in X$ consists of the point $x$ alone. Note that $X$ is totally disconnected if, and only if, $C_p(X,\mathbb{D})$ is dense in $\mathbb{D}^X$. Moreover, $C_p(X,Y)$ is dense in $Y^X$ for any space $Y$ and a totally disconnected space $X$.

**Definition 3.1.** We say that a space $X$ has the $B_D$-property if for every pairwise disjoint family $\{\Delta_n : n \in \mathbb{N}\}$ of non-empty finite sets $\Delta_n \subseteq X$, $\Delta_n = A_n \cup B_n$, $A_n \cap B_n = \emptyset$, $n \in \mathbb{N}$, there exists a subsequence $\{\Delta_{n_k} : k \in \mathbb{N}\}$ such that $\bigcup\{A_{n_k} : k \in \mathbb{N}\}$ and $\bigcup\{B_{n_k} : k \in \mathbb{N}\}$ are separated by a clopen set, i.e., there is a clopen set $W$ such that $\bigcup\{A_{n_k} : k \in \mathbb{N}\} \subseteq W \subseteq X \setminus \bigcup\{B_{n_k} : k \in \mathbb{N}\}$.

**Theorem 3.2.** Let $X$ be a totally disconnected space. Then the following assertions are equivalent:

(a) $C_p(X,\mathbb{D})$ is non-meager;
(b) $C_p(X,\mathbb{D})$ is Baire;
(c) $X$ has the $B_D$-property.

**Definition 3.3.** We say that a space $X$ has the $B_N$-property if for every pairwise disjoint family $\{\Delta_n : n \in \mathbb{N}\}$ of non-empty finite sets $\Delta_n \subseteq X$, $n \in \mathbb{N}$, there exists a subsequence $\{\Delta_{n_k} : k \in \mathbb{N}\}$ and a discrete family $\{W(\Delta_{n_k}) : W(\Delta_{n_k}) \supseteq \Delta_{n_k}, k \in \mathbb{N}\}$ of clopen subsets of $X$.

**Theorem 3.4.** Let $X$ be a totally disconnected space. Then the following assertions are equivalent:

(a) $C_p(X,\mathbb{N})$ is non-meager;
(b) $C_p(X,\mathbb{N})$ is Baire;
(c) $X$ has the $B_N$-property.

The proofs of Theorems 3.2 and 3.4 is similar to the proof of [7, Theorem 3.8] and is therefore omitted.

**Lemma 3.5** [4]. Let $C_p(X,\prod_{\alpha \in A}M_\alpha)$ be non-meager (Baire). Then $C_p(X,M_\alpha)$ is non-meager (Baire) for every $\alpha \in A$.

Recall that a mapping $\varphi : K \to M$ is called almost open if $\text{Int} \varphi(V) \neq \emptyset$ for any non-empty open subset $V$ of $K$. Note that irreducible mappings defined on compact spaces are almost open. Also note that if $\varphi_\alpha : K_\alpha \to M_\alpha$ ($\alpha \in A$) are surjective almost open mappings then the product mapping $\prod_{\alpha \in A} \varphi_\alpha : \prod_{\alpha \in A}K_\alpha \to \prod_{\alpha \in A}M_\alpha$ is also almost open.
Lemma 3.6 [7, Lemma 3.14]. Let $\psi: K \to M$ be a surjective continuous almost open mapping, and let $C_p(X,K)$ be a non-meager (Baire) dense subspace of $K^X$. Then $C_p(X,M)$ is non-meager (Baire).

Lemma 3.7 [7, Lemma 3.15]. Let $\psi: P \to L$ be a surjective continuous almost open mapping and let $E \subseteq P$ be a dense non-meager (Baire) subspace in $P$. Then $\psi(E)$ is non-meager (Baire).

Lemma 3.8 [2]. Let $L$ be a metrizable compact space without isolated points and let $T$ be the Cantor set. Then there exists a surjective continuous irreducible mapping $\psi: T \to L$.

Theorem 3.9. Let $X$ be a totally disconnected space. Then the following assertions are equivalent:

(a) $C_p(X,\mathbb{D})$ is Baire;
(b) $C_p(X,K)$ is Baire for some disconnected metrizable compact space $K$;
(c) $C_p(X,L)$ is Baire for any metrizable compact space $L$.

Proof. $(b) \Rightarrow (a)$. Let $K = K_1 \oplus K_0$ where $K_i \neq \emptyset$, $K_i$ is clopen for $i = 0,1$. Then $\varphi: K \to \mathbb{D}$ such that $\varphi(x) = i$ for $x \in K_i$ $(i = 0,1)$ is a continuous open surjection. By Lemma 3.7, $C_p(X,\mathbb{D})$ is Baire.

$(a) \Rightarrow (c)$. By Lemma 3.5, $C_p(X,\mathbb{D}^\omega) = (C_p(X,\mathbb{D}))^\omega$ is Baire. Let $L$ be a metrizable compact space and let $L_1$ be a metrizable compact space without isolated points. Then, by Lemma 3.8, there exists a surjective continuous irreducible mapping $\psi: K \to L \times L_1$ from the Cantor set $K$ onto $L \times L_1$. Then $\pi_\alpha \circ \psi: K \to L$ is an almost open continuous mapping. By Lemma 3.6, $C_p(X,L)$ is Baire.

$(c) \Rightarrow (b)$. It is trivial. □

Thus, if $C_p(X,\mathbb{D})$ is Baire for a totally disconnected space $X$ then $C_p(X,K)$ is Baire for any metrizable space $K$.

A natural question arises: to characterize the class $\mathcal{K}$ of compact spaces that $K \in \mathcal{K}$ provided that the condition “$C_p(X,\mathbb{D})$ is Baire” implies the condition “$C_p(X,K)$ is Baire”.

Definition 3.10. A compact space $X$ is called $\pi$-monolithic if for any surjective continuous mapping $f: X \to K$ where $K$ is a compact metrizable space there exists a compact metrizable subspace $T \subseteq X$ such that $f(T) = K$.

The class of $\pi$-monolithic compact spaces is multiplicative and it closed under continuous Hausdorff images. Hence, this class contains monolithic and dyadic compact spaces.

Lemma 3.11. A compact space $X$ is $\pi$-monolithic if, and only if, for any sequence $\{U_n : n \in \omega\}$ of non-empty open sets of $X$ there exists a compact metrizable subspace $K \subseteq X$ such that $K \cap U_n \neq \emptyset$ for every $n \in \omega$. 

Acta Mathematica Hungarica 168, 2022
Proof. Let $X$ be a $\pi$-monolithic compact space and let $\{U_n : n \in \omega\}$ be a sequence of non-empty open sets of $X$. Choose $V_n \subseteq U_n$ such that $V_n \neq \emptyset$, $V_n$ is a co-zero set of $X$ for every $n \in \omega$. Then, by [2], there are a surjective continuous mapping $f : X \to Y$ where $Y$ is a compact metrizable space and open sets $W_n \subseteq Y$ such that $f^{-1}(W_n) = V_n$ for each $n \in \mathbb{N}$. Since $X$ is $\pi$-monolithic, there exists a compact metrizable subspace $K \subseteq X$ such that $f(K) = Y$. Then $K \cap V_n \neq \emptyset$ for each $n \in \omega$.

Suppose that for any sequence $\{U_n : n \in \omega\}$ of non-empty open sets of $X$ there exists a compact metrizable subspace $K \subseteq X$ such that $K \cap U_n \neq \emptyset$ for every $n \in \omega$. Let $f : X \to Y$ be a continuous mapping $X$ onto a compact metrizable space $Y$. Choose a countable base $\{V_n : n \in \omega\}$ in $Y$. Then there exists a compact metrizable subspace $T \subseteq X$ such that $T \cap f^{-1}(V_n) \neq \emptyset$ for each $n \in \omega$. Clear that $f(T) = Y$. □

Theorem 3.12. The class $K$ coincides with the class of $\pi$-monolithic compact spaces.

Proof. Let $K \in K$ and $|K| \geq \omega$.

(*) Suppose that $X$ is an infinite totally disconnected space, $C_p(X,Y)$ is Baire for some space $Y$ and $\{V_m : m \in \omega\}$ is a sequence of non-empty open sets of $Y$. Then, there is $f \in C(X,Y)$ such that $f(X) \cap V_m \neq \emptyset$ for each $m \in \omega$.

Choose a disjoint family $\{\Delta_n : n \in \omega\}$ of finite subsets of $X$ such that $|\Delta_n| = n$ for $n \in \omega$. Let $M_n = \{f : f(\Delta_n) \cap V_i \neq \emptyset, i \leq n\}$. $F_m = \bigcap_{n \geq m}(C(X,Y) \setminus M_n)$, $n, m \in \omega$. Since $X$ is totally disconnected, $F_m$ is nowhere dense in $C(X,Y)$ for each $m \in \omega$. Hence there is $f \in C(X,Y)$ \ $\bigcup_{m \in \omega} F_m$. Then $f(X) \cap V_i \neq \emptyset$ for each $i \in \omega$.

By replacing $\mathbb{I}$ with $\mathbb{D}$ in [10], we construct $X_0$ such that

1. $X_0 \subseteq \mathbb{D}^c$ and $\beta(X) = \mathbb{D}^c$;
2. every countable subset $S$ of $X_0$ is closed and $\mathbb{D}$-embedded.

By the condition (2) and Theorem 3.2, $C_p(X_0,\mathbb{D})$ is Baire. Let $\{U_n : n \in \omega\}$ be a sequence of non-empty open subsets in $K$. By (*), there is $f \in C(X_0,K)$ such that $f(X_0) \cap U_i \neq \emptyset$ for $i \in \omega$. Then, by (1), there is a continuous extension $\tilde{f} : \mathbb{D}^c \to K$ of $f$. Then, $\tilde{f}(\mathbb{D}^c) \cap U_i \neq \emptyset$ for $i \in \omega$. Since $\tilde{f}(\mathbb{D}^c)$ is a dyadic compact space (hence, it is $\pi$-monolithic), there is a compact metrizable subspace $\tilde{K} \subseteq \tilde{f}(\mathbb{D}^c)$ such that $\tilde{K} \cap (\tilde{f}(\mathbb{D}^c) \cap U_i) \neq \emptyset$ for $i \in \omega$. It follows that $K$ is $\pi$-monolithic.

Suppose that $K$ is $\pi$-monolithic, $X$ is totally disconnected and $C_p(X,\mathbb{D})$ is Baire. We claim that $C_p(X,K)$ is Baire.

Assume the contrary. Then, there is a basis open set $P = \bigcap\{M(x_i,V_i) : i \leq k_0\}$ such that $P$ is meager, i.e., $P = \bigcup_{m \in \omega} F_m$ where $F_m$ is nowhere dense and $F_m \subseteq F_{m+1}$ for each $m \in \omega$. Let $\Delta_0 = \{x_i : i \leq k_0\}$, $\mu_0 = \{V_i : i \leq k_0\}$. Construct, by induction, finite disjoint sets $\Delta_n = \{x_{n_i}, i \leq k_n\} \subseteq X$.
Δ_n ∩ Δ_0 = ∅, n ∈ ω, finite families open (in K) sets μ_n, μ_n ⊆ μ_{n+1}, n ∈ ω,
and open (in K) sets V_n, i ∈ [k_n, n ≥ 2] such that
(3) for every basis set

\[ \bigcap \{ M(x, V(x)) : x \in \bigcup_{i=0}^{n} \Delta_i \}, \quad n \geq 1, \]

where V(x) ∈ μ_n, V(x_i) ⊆ V_i, i ≤ n_0 there are V'(x) ∈ μ_{n+1}, V'(x) ⊆ V(x),
x ∈ \bigcup_{i=0}^{n} \Delta_i such that

\[ \bigcap \{ M(x, V'(x)) : x \in \bigcup_{i=0}^{n} \Delta_i \} \bigcap \bigcap \{ M(x_{n+1,i}, V_{n+1,i}) : i \leq k_{n+1} \} \cap F_n = \emptyset, \]

(4) for every V ∈ μ_n there is V' ∈ μ_{n+1} such that \( \overline{V'} \subseteq V, n \geq 0 \).

Since K is π-monolithic, there is a compact metrizable subspace \( \tilde{K} \) such that
\( \tilde{K} \cap V \neq \emptyset \) for every V \( \in \bigcup_{n=0}^{\infty} \mu_n \).

We claim that there is a non-empty compact metrizable subspace
K* ⊆ \( \tilde{K} \) such that for every V \( \in \bigcup_{n=0}^{\infty} \mu_n, V \cap K^* \neq \emptyset \) and \{ V \cap K^* : V \in \bigcup_{n=0}^{\infty} \mu_n \} is a π-base of K*.

Let \( R = \{ F : F \) is a non-empty closed subset of \( \tilde{K} \) such that there is
a sequence \( V_m \in \bigcup_{n=0}^{\infty} \mu_n, \overline{V_{m+1}} \subseteq V_m, m \in \omega, \{ V_m \cap \tilde{K} : m \in \omega \} \) is a base
of F \}. Let K* = \( \bigcap \{ S : S \) is closed subset of \( \tilde{K} \) such that S \( \cap F \neq \emptyset \) for every
\( F \in R \} \). Then \( \{ F \cap K^* : F \in R \} \) is a π-network of K*. By (4), V \( \cap K^* \neq \emptyset \)
for every V \( \in \bigcup_{n=0}^{\infty} \mu_n \) and \{ V \cap K* : V \in \bigcup_{n=0}^{\infty} \mu_n \} is a π-base of K*.

Let

\[ M_{p+1} = \bigcup \{ \bigcap \{ M(x, V(x) \cap K^*) : x \in \bigcup_{i=0}^{n+1} \mu_i \text{ where } V(x_i) \subseteq V_i, i \leq k_0, \]

\[ n \geq p + 1, V(x_{n+1,i}) \subseteq V_{n+1,i}, i \leq k_{n+1}, V(x) \in \bigcup_{i=0}^{n+1} \mu_i \text{ and } \]

\[ \bigcap \{ M(x, V(x)) : x \in \bigcup_{i=0}^{n+1} \mu_i \} \cap F_n = \emptyset \} \}, \quad p \in \omega. \]

Then \( M_{n+1} \) is a non-empty open subset of \( C_p(X, K^*) \), n \( \geq 0 \).

We claim that \( M_{p+1} \) is dense in \( P^* = \bigcap \{ M(x_i, V_i \cap K^*) : i \leq k_0 \} \subseteq C_p(X, K^*) \). Let \( \varphi \in P^* \) and let \( O(\varphi) \) be a base neighborhood of \( \varphi \) in

Acta Mathematica Hungarica 168, 2022
$C(X, K^*)$. We can assume that

$$O(\varphi) = \bigcap \left\{ M(x, W(x)) : x \in \bigcup_{i=0}^{m} \mu_i \right\}$$

$$\cap \bigcap \left\{ M(y, W(y)) : y \in T \subseteq X \setminus \bigcup_{i=0}^{\infty} \Delta_i \right\}$$

where $W(x_i) \subseteq V_i \cap K^*$, $i \leq k_0$ and $W(x) \left( x \in \bigcup_{i=0}^{m} \Delta_i \right)$, $W(y) \left( y \in T \right)$ are non-empty open sets in $K^*$, $T$ is finite and $m \in \omega$.

Since $\left\{ V \cap K^* : V \in \bigcup_{n=0}^{\infty} \mu_n \right\}$ is a $\pi$-base of $K^*$, there are $V(x) \in \bigcup_{n=0}^{\infty} \mu_n$ for $x \in \bigcup_{i=0}^{m} \Delta_i$ such that $V(x) \cap K^* \subseteq W(x)$ for $x \in \bigcup_{i=0}^{m} \Delta_i$. Then, there is $k \in \omega$ such that $V(x) \in \mu_l$ for $x \in \bigcup_{i=0}^{m} \Delta_i$, $k \geq m$, $k \geq p+1$. By (3), there are sets $V'(x) \in \mu_{l+1}$, $x \in \bigcup_{i=0}^{l+1} \Delta_i$ such that $V'(x) \subseteq V(x)$, $x \in \bigcup_{i=0}^{m} \Delta_i$, $V'(x_{l+1,i}) \subseteq V_{l+1,i}$, $i \leq k_{l+1}$ and

$$\bigcap \left\{ M(x, V'(x)) : x \in \bigcup_{i=0}^{k+1} \Delta_i \right\} \cap F_k = \emptyset.$$

Choose

$$g \in \bigcap \left\{ M(x, V'(x) \cap K^*) : x \in \bigcup_{i=0}^{k+1} \Delta_i \right\} \cap \bigcap \left\{ M(y, W(y)) : y \in T \right\}.$$

Then $g \in O(\varphi) \cap P^* \cap M_{p+1}$. Hence, $M_{p+1}$ is a dense open set in $P^*$. By Theorem 3.2, $C_p(X, K^*)$ is Baire, hence, $\bigcap_{p=0}^{\infty} M_{p+1} \neq \emptyset$. Let $g \in \bigcap_{p=0}^{\infty} M_{p+1}$. So we proved that $g \notin \bigcup_{m=1}^{\infty} F_m = P$ which is a contradiction. □

**Lemma 3.13** [2]. For any Polish space $L$ there is a surjective continuous open mapping $\varphi : \omega^\omega \to L$.

**Theorem 3.14.** Let $X$ be a totally disconnected space. Then the following assertions are equivalent:

(a) $C_p(X, N)$ is Baire;

(b) $C_p(X, M)$ is Baire for some complete metric non-compact zero-dimensional space $M$;

(c) $C_p(X, K)$ is Baire for every complete metric space $K$.

**Proof.** (b) $\Rightarrow$ (a). Since $M$ is a non-compact zero-dimensional metric space, $M = \bigoplus \{ M_i : i \in \omega \}$ where $M_i$ is a non-empty clopen set for every $i \in \omega$. Consider $\psi : M \to \mathbb{N}$ such that $\psi(M_i) = i$ for each $i \in \omega$. Then $\psi$ is a continuous open mapping. By Lemma 3.6, $C_p(X, N)$ is Baire.

(a) $\Rightarrow$ (c). Note that, by Lemmas 3.5 and 3.13, the implication is true for a Polish space $K$.

*Acta Mathematica Hungarica 168, 2022*
Let $K$ be a complete metric space with a complete metric $\rho$.

Assume the contrary. Then, there is a basis open set $P = \bigcap \{ M(x_i, V_i) : i \leq k_0 \}$ such that $P$ is meager, i.e., $P = \bigcup_{m\in \omega} F_m$ where $F_m$ is nowhere dense and $F_m \subseteq F_{m+1}$ for each $m \in \omega$. Let $\Delta_0 = \{ x_i : i \leq k_0 \}$; $\mu_0 = \{ V_i : i \leq k_0 \}$. Construct, by induction, finite disjoint sets

$$
\Delta_n = \{ x_n, i \leq k_n \} \subseteq X, \Delta_n \cap \Delta_0 = \emptyset, \ n \in \omega,
$$

finite families open (in $K$) sets $\mu_n$, $\mu_n \subseteq \mu_{n+1}$, $n \in \omega$, and open (in $K$) sets $V_n, \in \mu_n$, $i \leq k_n$, $n \geq 2$ such that

(1) for every basis set $\bigcap \{ M(x, V(x)) : x \in \bigcup_{i=0}^n \Delta_i \}$, $n \geq 1$, where $V(x) \in \mu_n$, $V(x_i) \subseteq V_i$, $i \leq n_0$ there are $V'(x) \in \mu_{n+1}$, $V'(x) \subseteq V(x)$, $x \in \bigcup_{i=0}^n \Delta_i$ such that

$$
\bigcap \{ M(x, V'(x)) : x \in \bigcup_{i=0}^n \Delta_i \} \cap \bigcap \{ M(x_{n+1, i}, V_{n+1, i}) : i \leq k_{n+1} \} \cap F_n \neq \emptyset,
$$

(2) for every $V \in \mu_n$ there is $V' \in \mu_{n+1}$ such that $\overline{V'} \subseteq V$, $n \geq 0$;

(3) for every $V \in \mu_n$, diam $V < \frac{1}{n}$, $n \in \omega$.

Consider

$$
\tilde{K} = \left\{ x \in K : \text{there is a base } \left\{ V_n(x) : V_n(x) \in \bigcup_{k=0}^\infty \mu_k, \ n \in \omega \right\} \right\}
$$

of neighborhoods of $x$.

By (1) and (2), $V \cap K^* \neq \emptyset$ for every $V \in \bigcup_{k=0}^\infty \mu_k$ and $\{ V \cap \tilde{K} : V \in \bigcup_{k=0}^\infty \mu_k \}$ is a base in $\tilde{K}$. Then $K^* = \overline{\tilde{K}}$ is a Polish space and $\{ V \cap K^* : V \in \bigcup_{k=0}^\infty \mu_k \}$ is a $\pi$-base of $K^*$.

Let

$$
M_{p+1} = \bigcup \left\{ \bigcap \left\{ M(x, V(x) \cap K^*) : x \in \bigcup_{i=0}^{n+1} \mu_i \text{ where } V(x_i) \subseteq V_i, \ i \leq k_0, \right\} \cap F_n = \emptyset \right\}, \ p \in \omega.
$$

Then $M_{n+1}$ is a non-empty open subset of $C_p(X, K^*)$, $n \geq 0$.
We claim that $M_{p+1}$ is dense in

$$P^* = \bigcap \left\{ M(x_i, V_i \cap K^*): i \leq k_0 \right\} \subseteq C_p(X, K^*).$$

Let $\varphi \in P^*$ and let $O(\varphi)$ be a base neighborhood of $\varphi$ in $C(X, K^*)$. We can assume that

$$O(\varphi) = \bigcap \left\{ M(x, W(x)): x \in \bigcup_{i=0}^{m} \mu_i \right\} \cap \bigcap \left\{ M(y, W(y)): y \in T \subseteq X \setminus \bigcup_{i=0}^{\infty} \Delta_i \right\}$$

where $W(x_i) \subseteq V_i \cap K^*$, $i \leq k_0$ and $W(x)$ ($x \in \bigcup_{i=0}^{m} \mu_i$), $W(y)$ ($y \in T$) are non-empty open sets in $K^*$, $T$ is finite and $m \in \omega$.

Since $\left\{ V \cap K^*: V \in \bigcup_{n=0}^{\infty} \mu_n \right\}$ is a $\pi$-base of $K^*$, there are $V(x) \in \bigcup_{n=0}^{\infty} \mu_n$ for $x \in \bigcup_{i=0}^{m} \Delta_i$ such that $V(x) \cap K^* \subseteq W(x)$ for $x \in \bigcup_{i=0}^{m} \Delta_i$. Then, there is $k \in \omega$ such that $V(x) \in \mu_k$ for $x \in \bigcup_{i=0}^{m} \Delta_i$, $k \geq m$, $k \geq p+1$.

By (1), there are sets $V'(x) \in \mu_{l+1}$, $x \in \bigcup_{i=0}^{l+1} \Delta_i$ such that $V'(x) \subseteq V(x)$, $x \in \bigcup_{i=0}^{m} \Delta_i$, $V'(x_{l+1,i}) \subseteq V_{l+1,i}$, $i \leq k_{l+1}$ and

$$\bigcap \left\{ M(x, V'(x)): x \in \bigcup_{i=0}^{k+1} \Delta_i \right\} \cap F_k = \emptyset.$$

Choose

$$g \in \bigcap \left\{ M(x, V'(x) \cap K^*): x \in \bigcup_{i=0}^{k+1} \Delta_i \right\} \cap \bigcap \left\{ M(y, W(y)): y \in T \right\}.$$

Then $g \in O(\varphi) \cap P^* \cap M_{p+1}$. Hence, $M_{p+1}$ is a dense open set in $P^*$. By Theorem 3.2, $C_p(X, K^*)$ is Baire, hence, $\bigcap_{p=0}^{\infty} M_{p+1} \neq \emptyset$. Let $g \in \bigcap_{p=0}^{\infty} M_{p+1}$.

So we proved that $g \notin \bigcup_{m=1}^{\infty} F_m = P$ which is a contradiction.

(c) $\Rightarrow$ (b). It is trivial. $\square$

Recall that a topological vector space $X$ is a Fréchet space if $X$ is a locally convex complete metrizable space.

**THEOREM 3.15.** Let $X$ be a Tychonoff space. Then the following assertions are equivalent:

(a) $C_p(X)$ is Baire;

(b) $C_p(X, M)$ is Baire for some Fréchet space $M$, dim $M > 0$;

(c) $C_p(X, L)$ is Baire for any Fréchet space $L$.

**PROOF.** (b) $\Rightarrow$ (a). Let $\varphi$ be a continuous liner functional on $M$, $\varphi \neq 0$. By [9, Chapter III, Corollary 1], $\varphi: M \to \mathbb{R}$ is a surjective continuous open mapping. By Lemma 3.6, $C_p(X)$ is Baire.

*Acta Mathematica Hungarica* 168, 2022
(a) ⇒ (c). Assume the contrary. Then, there is a basis open set

$$P = \bigcap \{ M(x_i, V_i) : i \leq k_0 \} = \bigcup_{m=1}^{\infty} F_m$$

where $F_m$ is a nowhere dense set and $F_m \subseteq F_{m+1}$ for all $m \in \omega$. Let $\Delta_0 = \{ x_i : i \leq k_0 \}$, $\mu_0 = \{ V_i : i \leq k_0 \}$. Construct, by induction, finite sets $\Delta_n = \{ x_{n_i} : i \leq k_n \} \subseteq X$, $\Delta_n \cap \Delta_0 = \emptyset$, $n \in \omega$, finite families $\mu_n, \mu_n \subseteq \mu_{n+1}$, $n \in \omega$, of open sets in $L$, open sets $V_{n_i} \in \mu_n$ in $L$, $i \leq k_n$, $n \geq 2$, separable linear subspaces $L_n, L_n \subseteq L_{n+1}$, $n \in \omega$, families $B_n = \{ O_{n,m} : m \in \omega \}$ of open sets in $L$ for $n \in \omega$, such that

1. for every basic set $\bigcap \{ M(x, V(x)) : x \in \bigcup_{i=0}^{n} \Delta_i \}$, $n > 1$, where $V(x) \in \mu_n$, $V(x_i) \subseteq V_i$, $i \leq n_0$, there are $V'(x) \in \mu_{n+1}$, $V'(x) \subseteq V(x) \in \bigcup_{i=0}^{n} \Delta_i$

such that

$$\bigcap \{ M(x, V'(x)) : x \in \bigcup_{i=0}^{n} \Delta_i \} \cap \bigcap \{ M(x_{n+1,i}, V_{n+1,i}) : i \leq k_{n+1} \} \cap F_n = \emptyset,$$

2. for every $V \in \mu_n$ there is $V' \in \mu_{n+1}$ such that $V' \subseteq V$, $n \geq 0$,

3. $B_n = \{ O_{n,m} : m \in \omega \}$ such that $\{ O_{n,m} \cap L_n : m \in \omega \}$ is a base of $L_n$

and $\text{diam} O_{n,m} \to 0$ ($m \to \infty$), $n \in \omega$,

4. $\mu_n \subseteq \mu_{n+1}$ and $\{ O_{k,m} : k, m \leq n + 1 \}$, $n \in \omega$.

Let $L^* = \bigcup_{n=1}^{\infty} L_n$. Then $L^*$ is a separable Fréchet space. By (3) and (4), $V \cap L^* \neq \emptyset$ for all $V \in \bigcup_{n=0}^{\infty} \mu_n$ and $\{ V \cap L^* : V \in \bigcup_{n=0}^{\infty} \mu_n \}$ is a $\pi$-base in $L^*$. By [13], $L^*$ is homeomorphic to $\mathbb{R}^\alpha$, $\alpha \leq \aleph_0$. Hence, $C_p(X, L^*)$ is

Baire. Let

$$M_{p+1} = \bigcup \bigg\{ \bigcap \bigg\{ M(x, V(x) \cap K^*) : x \in \bigcup_{i=0}^{n+1} \mu_i \text{ where } V(x_i) \subseteq V_i, i \leq k_0, n \geq p + 1, V(x_{n+1,i}) \subseteq V_{n+1,i}, i \leq k_{n+1}, V(x) \in \bigcup_{i=0}^{n+1} \mu_i \text{ and} \bigg\} \cap F_n = \emptyset \bigg\}, p \in \omega.$$

Then $M_{n+1}$ is a non-empty open subset of $C_p(X, K^*)$ for $n \geq 0$.

We claim that $M_{p+1}$ is dense in

$$P^* = \bigcap \bigg\{ M(x_i, V_i \cap K^*) : i \leq k_0 \bigg\} \subseteq C_p(X, K^*).$$
Let $\varphi \in P^*$ and let $O(\varphi)$ be a base neighborhood of $\varphi$ in $C(X, K^*)$. We can assume that

$$O(\varphi) = \bigcap \left\{ M(x, W(x)) : x \in \bigcup_{i=0}^{m} \mu_i \right\} \cap \bigcap \left\{ M(y, W(y)) : y \in T \subseteq X \setminus \bigcup_{i=0}^{\infty} \Delta_i \right\}$$

where $W(x_i) \subseteq V_i \cap K^*$, $i \leq k_0$ and $W(x)$ ($x \in \bigcup_{i=0}^{m} \Delta_i$), $W(y)$ ($y \in T$) are non-empty open sets in $K^*$, $T$ is finite and $m \in \omega$.

Since $\{ V \cap K^* : V \in \bigcup_{n=0}^{\infty} \mu_n \}$ is a $\pi$-base of $K^*$ there are $V(x) \in \bigcup_{n=0}^{\infty} \mu_n$ for $x \in \bigcup_{i=0}^{m} \Delta_i$ such that $V(x) \cap K^* \subseteq W(x)$ for $x \in \bigcup_{i=0}^{m} \Delta_i$. Then, there is $k \in \omega$ such that $V(x) \in \mu_k$ for $x \in \bigcup_{i=0}^{m} \Delta_i$, $k \geq m$, $k \geq p + 1$.

By (3), there are sets $V'(x) \in \mu_{k+1}$, $x \in \bigcup_{i=0}^{l+1} \Delta_i$ such that $V'(x) \subseteq V(x)$, $x \in \bigcup_{i=0}^{m} \Delta_i$, $V'(x_{l+1,i}) \subseteq V_{l+1,i}$, $i \leq l$ and

$$\bigcap \left\{ M(x, V'(x)) : x \in \bigcup_{i=0}^{k+1} \Delta_i \right\} \cap F_k = \emptyset.$$ 

Choose

$$g \in \bigcap \left\{ M(x, V'(x) \cap K^*) : x \in \bigcup_{i=0}^{k+1} \Delta_i \right\} \cap \bigcap \left\{ M(y, W(y)) : y \in T \right\}.$$ 

Then $g \in O(\varphi) \cap P^* \cap M_{p+1}$. Hence, $M_{p+1}$ is a dense open set in $P^*$. By Theorem 3.2, $C_p(X, K^*)$ is Baire, hence, $\bigcap_{p=0}^{\infty} M_{p+1} \neq \emptyset$. Let $g \in \bigcap_{p=0}^{\infty} M_{p+1}$. So we proved that $g \not\in \bigcup_{m=1}^{\infty} F_m = P$ which is a contradiction.

(c) $\Rightarrow$ (b). It is trivial. $\Box$

We will next give an example of a totally disconnected space $T$ such that, for any second countable space $M$, the space $C_p(T, M)$ has the Baire property if and only if $M$ is a Peano continuum. In particular, the space $C_p(T, \mathbb{I})$ is Baire but $C_p(T, \mathbb{D})$ is not.

We use the following lemma in the construction of a space $T$.

**Lemma 3.16.** Let $X$ be a pseudocompact infinite space and let $C_p(X, Y)$ be a dense set in $Y^X$. Then, if $C_p(X, Y)$ is Baire then $Y$ is pseudocompact.

**Proof.** Assume the contrary. There is a unbounded continuous mapping $\varphi : Y \rightarrow \mathbb{R}$. Then, $C_p(X, Y) = \bigcup_{m=1}^{\infty} F_m$ where

$$F_m = \left\{ f \in C(X, Y) : \| \varphi \circ f \| \leq m \right\}, \quad m \in \omega.$$ 

Since $C_p(X, Y)$ is dense in $Y^X$ and $X$ is infinite, $F_m$ is nowhere dense which is a contradiction. $\Box$
Proposition 3.17. There exists a totally disconnected Tychonoff space $T$ such that $C_p(T, M)$ is Baire where $M$ is a separable metric space if, and only if, $M$ is a Peano continuum.

Proof. By [10], there is $Z \subseteq \mathbb{I}^A$ where $|A| = c$ such that
(1) $|Z| = c$;
(2) every countable subset $S \subseteq Z$ is closed and $C^*$-embedded;
(3) $Z$ is $\aleph_0$-dense in $\mathbb{I}^A$.

Let $\sum(0) \subseteq \mathbb{D}^A$ be a $\sum$-product about $0$ (0 is a function $f$ from $A$ into $\mathbb{D}$ such that $f(\alpha) = 0$ for each $\alpha \in A$), and let $\{A_\alpha \times B_\alpha : \alpha < c\}$ be a base of $G_\delta$-topology $\sum(0) \times Z$. It is clear that $|A_\alpha| = |B_\alpha| = c$ for $\alpha < c$. Let $\sum(0) = \{m_\alpha : \alpha < c\}$ and $Z = \{z_\alpha : \alpha < c\}$. By induction, we construct sets $\{T_\alpha : \alpha < c\}$ such that
(4) $T_\beta \supseteq T_\alpha$, $\beta > \alpha$, $|T_{\alpha+1} \setminus T_\alpha| < \aleph_0$, $T_\beta = \bigcup\{T_\alpha : \alpha < \beta\}$ for a limit ordinal $\beta$;
(5) $T_\alpha \cap (A_\beta \times B_\beta) \neq \emptyset$, $\alpha < c$, $\beta < \alpha$;
(6) $\pi_\sum(0)(T_\alpha) \supseteq \{m_\beta : \beta < \alpha\}$, $\pi_Z(T_\alpha) \supseteq \{z_\beta : \beta < \alpha\}$.

Let $T_0 = \{(m_0, z_0)\}$. Assume that $T_\alpha$, $\alpha < \beta$ are constructed. If $\beta$ is limit then $T_\beta = \bigcup\{T_\alpha : \alpha < \beta\}$. Let $\beta = \beta^* + 1$. If $m_\beta \not\in \pi_\sum(0)T_\beta^*$ then choose $z_\gamma \in Z \setminus \pi_ZT_\beta^*$. Let $T_\beta' = T_\beta^* \cup \{(m_\beta^*, z_\gamma)\}$. If $z_\beta \not\in \pi_ZT_\beta'$ then choose $m_\gamma' \in \sum(0) \setminus \pi_\sum(0)T_\beta'$. Let $T_\beta'' = T_\beta' \cup \{(m_\gamma', z_\beta^*)\}$.

Let $m \in A_\beta^* \setminus \pi_\sum(0)T_\beta''$, $z \in B_\beta^* \setminus \pi_ZT_\beta''$. Then $T_\beta = T_\beta'' \cup \{(m, z)\}$. It is clear that the conditions (4), (5) and (6) for $\{T_\alpha : \alpha \leq \beta\}$ hold.

We claim that $T = \bigcup\{T_\alpha : \alpha < c\}$ is as required. Note that
(7) by (4), (6), $\pi_\sum(0) \upharpoonright T : T \to \sum(0)$ and $\pi_Z \upharpoonright T : T \to Z$ are bijections. By (5),
(8) $T$ is $\aleph_0$-dense in $\sum(0) \times Z$, hence, in $\mathbb{D}^A \times \mathbb{I}^A$.

By (2), $C_p(Z, \mathbb{I})$ is Baire. Since $Z$ is a condensation of $T$, $C_p(T, \mathbb{I})$ is Baire. By Theorem 3.18 in [7], $C_p(T, K)$ is Baire if $K$ is a Peano continuum.

Since $\sum(0)$ is a condensation of $T$, $T$ is a totally disconnected. By (8), $T$ is pseudocompact. By Lemma 3.16, $M$ is compact.

(1) We claim that $M$ is connected. Let $U \cap V = \emptyset$ where $U$, $V$ are open sets of $M$. Choose $B \subseteq A$, $|B| = \aleph_0$ and a family $\{\Delta_n : n \in \omega\}$ of finite disjoint sets of $\mathbb{D}^B$ and $\Delta_n$ is a $\frac{1}{n}$-network in $\mathbb{D}^B$, $n \in \omega$. Let $\Delta'_n = \Delta_n \times 0 \subseteq \sum(0)$, $n \in \omega$. Since $\pi_\sum(0) \upharpoonright T : T \to \sum(0)$ is a condensation, there are finite disjoint sets $\Delta''_n$ such that $\pi_\sum(0)\Delta''_n = \Delta'_n$. Let $\Delta''_n = \Delta'_{n+1}$ and $\Delta'_n = \Delta''_{n+2}$, $n \in \omega$. Since $\sum_{n \in \omega})$ is disjoint and $T$ is totally disconnected, $F_m$ is nowhere dense. Since $C_p(T, M)$ is Baire, there is $\varphi \in C(T, M) \setminus \bigcup_{m=1}^\infty F_m$. It
follows that there is a sequence \( \{ n_k \} \), \( n_{k+1} > n_k \) such that \( \varphi \in M_{n_k} \), \( k \in \omega \). Since \( T \) is a \( \aleph_0 \)-dense in \( D^A \times I^A \), by [1] (see Lemma 0.2.3), there are a countable set \( B' \supseteq B \) and a continuous mapping \( \varphi' : D^{B'} \times I^{B'} \to M \) such that \( \varphi = \varphi'|_{D^{B'} \times I^{B'}} \). Let \( \pi_{D^{B'} \times I^{B'}}(\Delta''_n) = \nabla_n, n \in \omega \). Then \( \varphi'(\nabla_{2n_k}) \subseteq U, \varphi'(\nabla_{2n_{k+1}}) \subseteq V \).

Since \( \pi_{D^{B'}} \nabla_n = \Delta_n \times 0 \) where \( 0 \notin D^{B'} \setminus B \), there are open sets \( V(\nabla_n) = W(\Delta_n) \times O_n(0), n = 2n_k, 2n_k + 1, k \in \omega \) where \( W(\Delta_n) \) is open in \( D^B \), \( O_n(0) \) is open in \( D^{B'} \setminus B \) such that

\[
\varphi'(V(\nabla_{2n_k})) \subseteq U, \quad \varphi'(V(\nabla_{2n_{k+1}})) \subseteq V, \quad k \in \omega.
\]

Since \( \Delta_n \) is a \( \frac{1}{n} \)-network in \( D^B, n \in \omega, \)

\[
\bigcap_{k=1}^{\infty} \left( \bigcup_{m=k}^{\infty} W(\Delta_{2n_m}) \right) \cap \bigcap_{k=1}^{\infty} \left( \bigcup_{m=k}^{\infty} W(\Delta_{2n_{m+1}}) \right) \neq \emptyset,
\]

i.e., there are \( t, 2n_p, 2n_l + 1 \) such that \( t \in W(\Delta_{2n_p}) \cap W(\Delta_{2n_{l+1}}) \). Then \( \tilde{t} = t \times 0 \) where \( 0 \in \times D^{B'} \setminus B \) belongs \( V(\nabla_{2n_p}) \cap V(\nabla_{2n_{l+1}}) \). Hence,

\[
\varphi'(t \times I^{B'}) \cap U \neq \emptyset \quad \text{and} \quad \varphi'(t \times I^B) \cap V \neq \emptyset.
\]

Thus, for any non-empty open sets \( U, V \) in \( M \) there is the connected set \( S = \varphi'(t \times I^{B'}) \) such that \( S \cap U \neq \emptyset \) and \( S \cap V \neq \emptyset \). It follows that \( M \) is connected.

(II) We claim that for any sequence \( \{ U_n : n \in \omega \} \) of non-empty open sets in \( M \) there is a Peano continuum \( P \) such that \( |\{ n : U_n \cap P \neq \emptyset \}| = \aleph_0 \).

Let \( \Delta_n \subseteq D^B \), \( \Delta'_n, \Delta''_n \) be sequences from (I). Let

\[
\tilde{M}_n = \{ f \in C(T, M) : f(\Delta'_n) \subseteq U_n \}, \quad F_m = \bigcap_{n \geq m} (C(T, M) \setminus \tilde{M}_n),
\]

\( n, m \in \omega, \varphi \in C(T, M) \setminus \bigcup_{m=1}^{\infty} F_m \).

Then there is a sequence \( \{ n_k \}, n_{k+1} > n_k, k \in \omega, \) such that \( \varphi \in \tilde{M}_{n_k}, k \in \omega \).

Further, as in (I), there are a countable set \( \tilde{B} \supseteq B, \tilde{\varphi} : D^{\tilde{B}} \times I^{\tilde{B}} \to M, \)
open sets \( V(\nabla_{n_k}) = W(\Delta_{n_k}) \times O_{n_k}(0), k \in \omega \) such that \( \tilde{\varphi}(V(\nabla_{n_k})) \subseteq U_{n_k}, k \in \omega \). Let \( t \in \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} W(\Delta_{n_m}) \). Then \( \tilde{t} = t \times 0 \) where \( 0 \in D^{\tilde{B}} \setminus B \), belongs \( V(\nabla_{n_{m_l}}) \) for a subsequence \( n_{m_l}, l \in \omega \). Then, for the Peano continuum \( P = \tilde{\varphi}(t \times I^{\tilde{B}}) \) we have \( P \cap U_{n_{m_l}} \neq \emptyset \) for \( l \in \omega \).

By (II), we claim that \( M \) is a locally connected space. Assume the contrary. Then there is a non-empty open set \( V \subseteq M \) and some non-open component \( K \subseteq V \). Let \( t \in K \setminus \text{Int } K \). Since \( K \) is closed in \( V \), there is a sequence of non-empty open sets \( U_n \subseteq V \setminus K \) and \( U_n \to t \). By (II), there exists

\textbf{Acta Mathematica Hungarica 168, 2022}
a Peano continuum $P$ such that $P \cap U_n \neq \emptyset$ for $n \in N_1 \subseteq \omega$ ($N_1$ is infinite). Then $t \in P$ and there is a connected neighborhood $O(t) \subseteq V$ of $t$ in $P$. It follows that $O(t) \cap U_{n_0} \neq \emptyset$ for some $U_{n_0}$. Hence $K$ is not a component of $V$ which is a contradiction.

Since $M$ is locally connected, connected and compact, it is a Peano continuum. □

**Acknowledgement.** The authors are grateful to the referee for useful remarks and suggestions.

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