A Purification Scheme and Entanglement Distillations

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A purification scheme which utilizes the action of repeated measurements on a (part of a total) quantum system is briefly reviewed and is applied to a few simple systems to show how it enables us to extract an entangled state as a target pure state. The scheme is rather simple (e.g., we need not prepare a specific initial state) and is shown to have wide applicability and flexibility, and is able to accomplish both the maximal fidelity and non-vanishing yield.

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I. INTRODUCTION

It is well known that in quantum mechanics, the action of measurement affects the dynamics of the system just measured in an essential way. This has to be contrasted with the situation in classical mechanics, where the effect of measurement can be made as small as one wishes. Typical phenomenon reflecting such a peculiarity in quantum mechanics has been known under the name of “Quantum Zeno Effect (QZE)” and has been extensively studied recently. It states that if a system is frequently measured to confirm that it is in its initial state, the change of the state based on the Hamiltonian of the system is decelerated, or to state differently, the state, the change of the state based on the Hamiltonian frequently measured to confirm that it is in its initial state, the decay of an unstable state is hindered by frequent measurements. It is widely recognized, however, that there is no paradoxical point in such phenomena and the effect is solely understood quantum mechanically. Indeed, the projective measurement is not essential and is just replaced with the generalized spectral decompositions, and the effect is due to the peculiar short-time dynamics of quantum systems, known as the “flat derivative” of the condition of measurement affects the dynamics of the system in an essential way. This has to be contrasted with the situation in classical mechanics, where acceleration of decay when the measurements are not frequent enough. The latter effect is called the “Inverse (or Anti) Zeno Effect (IZE),” which has also been studied and discussed. The QZE (and/or IZE) has still been explored extensively in the hope of realizing a protection scheme for system’s coherence against possible decoherence by this (or its related) mechanism.

Here another interesting consequence of the peculiarity of quantum measurement shall be disclosed. We consider a total system that is composed of two (sub)systems A and B and measurements shall be repeatedly performed only on system A at regular intervals. Our interest lies in the (asymptotic) dynamics of system B, which is indirectly affected by the measurements on A through its interaction with A. It is shown that such indirect measurements can drive system B to a pure state (purification), irrespectively of its initial state that is mixed in general. We can say that the effect of measurement is far reaching and profound: It can control even the other parts of the system which are not touched directly. In the following sections, the mechanism of this purification is briefly reviewed (Sec. II) and is applied to a simple qubit system (Sec. III) to show how it works and how it can be optimized. Since the entangled state, which is one of the key elements in quantum technologies like quantum computation, information, teleportation, etc., is a pure state of a compound system, this method is used to extract entangled states in Secs. IV and V. A brief summary is presented in Sec. VI.

II. GENERAL FRAMEWORK OF PURIFICATION THROUGH REPEATED MEASUREMENTS

Let the total system consist of two parts, system A and system B, and the dynamics be described by the total Hamiltonian

\[ H = H_A + H_B + H_{\text{int}}, \]

where \( H_{\text{int}} \) stands for the interaction between the two (sub)systems. We initially prepare the system in a product state

\[ \rho_0 = |\phi\rangle\langle\phi| \otimes \rho_B(0) \]

at \( t = 0 \). Such a state can be realized, say, if the system A is found in the state \( |\phi\rangle \) after the zeroth measurement. Notice that system B can be in an arbitrary mixed state \( \rho_B(0) \). We perform measurements on A at regular intervals \( \tau \) to confirm that it is still in the state \( |\phi\rangle \), even though the total system A+B evolves unitarily in terms of the total Hamiltonian \( H \). Since the measurement is performed only on system A, the action of such a (projective, for simplicity) measurement can be conveniently described by the following projection operator

\[ \mathcal{O} \equiv |\phi\rangle\langle\phi| \otimes I_B. \]
Thus the state of system A is set back to $|\phi\rangle$ every after \(\tau\), while that of B just evolves dynamically on the basis of the total Hamiltonian \(H\). We repeat the same measurement, represented by \(B\), \(N\) times and collect only those events in which system A has been found in state $|\phi\rangle$ consecutively \(N\) times; other events are discarded. The state of system B is then described by the density matrix

$$\rho_B^{(\tau)}(N) = (V_\phi(\tau))^N \rho_B(0) (V_\phi^\dagger(\tau))^N / P^{(\tau)}(N),$$

where

$$V_\phi(\tau) \equiv \langle \phi | e^{-iH\tau} | \phi \rangle$$

is an operator acting on B and

$$P^{(\tau)}(N) = \text{Tr} \left[ (\mathcal{O} e^{-iH\tau} \mathcal{O})^N \rho_0 (\mathcal{O} e^{iH\tau} \mathcal{O})^N \right] = \text{Tr}_B \left[ (V_\phi(\tau))^N \rho_B(0) (V_\phi^\dagger(\tau))^N \right]$$

is the success probability for these events to occur (yield). This normalization factor in (4) reflects the fact that only right outcomes are collected in this process.

In order to examine the asymptotic state of system B, consider the spectral decomposition of the operator \(V_\phi(\tau)\), which is not hermitian, \(V_\phi(\tau) \neq V_\phi^\dagger(\tau)\). We therefore need to set up both the right- and left-eigenvalue problems

$$V_\phi(\tau)|u_n\rangle = \lambda_n |u_n\rangle,$$

$$\langle v_n | V_\phi(\tau) = \lambda_n \langle v_n |.$$  

The eigenvalue \(\lambda_n\) is complex valued in general, but its absolute value is bounded

$$0 \leq |\lambda_n| \leq 1,$$

which is a reflection of the unitarity of the time evolution operator \(e^{-iH\tau}\). These eigenvectors are assumed to form a complete orthonormal set in the following sense

$$\sum_n |u_n\rangle\langle v_n | = 1_B, \quad (v_n | u_m \rangle = \delta_{nm}.$$  

Then the operator \(V_\phi(\tau)\) itself is expanded in terms of these eigenvectors

$$V_\phi(\tau) = \sum_n \lambda_n |u_n\rangle \langle v_n |.$$  

It is now easy to see that the \(N\)th power of this operator is expressed as

$$(V_\phi(\tau))^N = \sum_n \lambda_n^N |u_n\rangle \langle v_n |$$  

and therefore it is dominated by a single term for large \(N\)

$$(V_\phi(\tau))^N \xrightarrow{\text{large } N} \lambda_0^N |u_0\rangle \langle v_0 |,$$

when the largest (in magnitude) eigenvalue \(\lambda_0\) is discrete, nondegenerate and unique. If these conditions are satisfied, the density operator of system B is driven to a pure state

$$\rho_B^{(\tau)}(N) \xrightarrow{\text{large } N} |u_0\rangle \langle u_0 | / (u_0 | u_0 \rangle$$

with the probability

$$P^{(\tau)}(N) \xrightarrow{\text{large } N} |\lambda_0|^2 N \langle u_0 | u_0 \rangle \langle v_0 | \rho_B(0) | v_0 \rangle.$$  

The pure state \(|u_0\rangle\), which is nothing but the right-eigenvector of the operator \(V_\phi(\tau)\) belonging to the largest (in magnitude) eigenvalue \(\lambda_0\), is thus distilled in system B. This is the purification scheme proposed in [9].

A few comments are in order. First, the final pure state \(|u_0\rangle\) toward which system B is to be driven depends on the choice of the state \(|\phi\rangle\) on which system A is projected every after measurement, the measurement interval \(\tau\) and the Hamiltonian \(H\), but does not depend on the initial state of system B at all. In this sense, the purification is accomplished irrespectively of the initial (mixed) state \(\rho_B(0)\). Second, as is clear in the above exposition, what is crucial in this purification scheme is the repetition of one and the same measurement (more appropriately, spectral decomposition) and the measurement interval \(\tau\) need not be very small. It remains to be an adjustable parameter. Third, if we can make other eigenvalues than \(\lambda_0\) much smaller in magnitude

$$|\lambda_n / \lambda_0| \ll 1 \quad \text{for } n \neq 0,$$

by adjusting parameters, we will need fewer steps (i.e., smaller \(N\)) to purify system B.

It is now evident that the purification can be made optimal, if the conditions

$$|\lambda_0| = 1$$

are satisfied. This condition assures that we can repeat as many measurements as we wish without running the risk of losing the yield (success probability) \(P^{(\tau)}(N)\) in order to make the fidelity to the target state \(|u_0\rangle\),

$$P^{(\tau)}(N) \equiv \text{Tr}_B \left[ \rho_B^{(\tau)}(N) |u_0\rangle \langle u_0 | / (u_0 | u_0 \rangle \right]$$

higher. Actually, the yield \(P^{(\tau)}(N)\) decays like

$$P^{(\tau)}(N) = \sum_{n,m} \lambda_n^N \lambda_m^N \langle v_n | \rho_B(0) | v_m \rangle \langle u_m | u_n \rangle$$

$$\xrightarrow{\text{large } N} |\lambda_0|^2 N \langle u_0 | u_0 \rangle \langle v_0 | \rho_B(0) | v_0 \rangle$$

and the condition can bring us with the non-vanishing yield \((u_0 | u_0 \rangle \langle v_0 | \rho_B(0) | v_0 \rangle)\) even in the \(N \to \infty\) limit. Therefore the condition makes the two (sometimes not compatible) demands, i.e., higher fidelity and non-vanishing yield, achievable, with fewer steps when the condition is met. In this sense, the purification is considered to be optimal.
It would be desirable if an optimal purification can be realized by an appropriate choice of the state $|\phi\rangle$ and/or tuning of the measurement interval $\tau$ and parameters in a given system. In the following sections, a few simple systems are examined to show how such optimal purifications are made possible.

III. PURIFICATION OF A QUBIT

As a simplest example, let us consider a total system of interacting two qubits. Systems A and B are represented by two qubits A and B, respectively, and their two levels can be described as the two degrees of freedom of spin-1/2 particle. We measure qubit A at regular intervals $\tau$ and examine the state of qubit B, which is in interaction with A. The measurement is conveniently parameterized as that of a spin-1/2 particle along a particular direction $\mathbf{n}$. After qubit A has been confirmed that its “spin” is up along $\mathbf{n}$, its state is projected to the eigenstate of the spin operator along $\mathbf{n}$, i.e., $|\phi\rangle = \mathbf{n} \cdot \mathbf{\sigma}|\phi\rangle$, with $\mathbf{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ being the Pauli matrices acting on qubit A. The state $|\phi\rangle$ is parameterized in terms of the two angles $\theta$ and $\varphi$ as

$$|\phi\rangle = \cos(\theta/2)e^{-i\varphi/2}|\uparrow\rangle + \sin(\theta/2)e^{i\varphi/2}|\downarrow\rangle. \quad (19)$$

It is an elementary task to write down the projected operator $V_\phi(\tau) \equiv (\hat{\mathbf{\sigma}} e^{-i\hat{H}_\sigma \tau}) |\phi\rangle$ in the following form

$$V_\phi(\tau) = c_0 + \mathbf{c} \cdot \mathbf{\sigma}, \quad (20)$$

where $\mathbf{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices acting on qubit B and parameters $c_0$ and $\mathbf{c} \equiv (c_1, c_2, c_3)$ are complex valued. Its eigenvalues $\lambda_\pm$ and the corresponding right-hand left-eigenvectors are easily found

$$\lambda_\pm = c_0 \pm c, \quad c \equiv \sqrt{c^2}, \quad c_\pm = c_1 \pm ic_2, \quad (21)$$

$$|u_+\rangle = \frac{1}{\sqrt{2(c - c_3)}}[c_-|\uparrow\rangle + (c - c_3)|\downarrow\rangle],$$

$$|u_-\rangle = \frac{1}{\sqrt{2(c - c_3)}}[(c - c_3)|\uparrow\rangle + c_+|\downarrow\rangle], \quad (22)$$

$$|v_+\rangle = \frac{1}{\sqrt{2(c - c_3)}}[c_+|\uparrow\rangle + (c - c_3)|\downarrow\rangle],$$

$$|v_-\rangle = \frac{1}{\sqrt{2(c - c_3)}}[(c - c_3)|\uparrow\rangle + c_-|\downarrow\rangle]. \quad (23)$$

Now let the total Hamiltonian of this system be given by

$$H = \frac{\omega_A}{2}(1 + \tau_3) + \frac{\omega_B}{2}(1 + \sigma_3)$$

$$+ g(\tau_+\sigma_- + \text{h.c.}) + h(\tau_+\sigma_+ + \text{h.c.}), \quad (24)$$

where real parameters $g$ and $h$ are responsible for the interaction between the two qubits, A and B. In this case, the parameters $c_0, \ldots, c_3$ in (20–23) are explicitly calculated to be

$$c_0 = \frac{1}{2} \left( \cos \frac{\tau \theta_h}{2} + \cos \frac{\tau \theta_g}{2} \right)$$

$$- \frac{i}{2} \left( \frac{\omega_+}{\theta_h} \sin \frac{\tau \theta_h}{2} + \frac{\omega_-}{\theta_g} \sin \frac{\tau \theta_g}{2} \right) \cos \theta, \quad (25)$$

$$c_1 = -i \left( \frac{h}{\theta_h} \sin \frac{\tau \theta_h}{2} + \frac{g}{\theta_g} \sin \frac{\tau \theta_g}{2} \right) \sin \theta \cos \varphi, \quad (26)$$

$$c_2 = i \left( \frac{h}{\theta_h} \sin \frac{\tau \theta_h}{2} - \frac{g}{\theta_g} \sin \frac{\tau \theta_g}{2} \right) \sin \theta \sin \varphi, \quad (27)$$

$$c_3 = \frac{1}{2} \left( \cos \frac{\tau \theta_h}{2} - \cos \frac{\tau \theta_g}{2} \right) \cos \theta$$

$$- \frac{i}{2} \left( \frac{\omega_+}{\theta_h} \sin \frac{\tau \theta_h}{2} - \frac{\omega_-}{\theta_g} \sin \frac{\tau \theta_g}{2} \right), \quad (28)$$

where we have introduced

$$\omega_\pm \equiv \omega_A \pm \omega_B, \quad \theta_h \equiv \sqrt{\omega_+^2 + 4h^2}, \quad \theta_g \equiv \sqrt{\omega_-^2 + 4g^2}. \quad (29)$$

In order to illustrate how an optimal purification can be achieved in this system, consider the case where we measure qubit A along the 3-direction, that is, we choose $\theta = 0$ and $|\phi\rangle = |\uparrow\rangle$. Then the eigenvalues $\lambda_{\pm}$ and the corresponding eigenvectors are

$$\lambda_+ = \cos \frac{\tau \theta_h}{2} - i \frac{\omega_+}{\theta_h} \sin \frac{\tau \theta_h}{2} \iff |u_+\rangle = |\uparrow\rangle, \quad (30)$$

$$\lambda_- = \cos \frac{\tau \theta_g}{2} - i \frac{\omega_-}{\theta_g} \sin \frac{\tau \theta_g}{2} \iff |u_-\rangle = |\downarrow\rangle. \quad (31)$$

Since we have

$$|\lambda_+|^2 = 1 - 4\frac{\omega_+^2}{\theta_h^2} \sin^2 \frac{\tau \theta_h}{2}, \quad |\lambda_-|^2 = 1 - 4\frac{\omega_-^2}{\theta_g^2} \sin^2 \frac{\tau \theta_g}{2} \quad (32)$$

the purification can be made optimal, e.g., when the parameters are adjusted so that $h \sin(\tau \theta_h/2) = 0$ is satisfied. In this case, $|\lambda_-| = 1$ and qubit B is driven to a pure state $|\uparrow\rangle$, more quickly for larger $g$ satisfying $\sin^2(\tau \theta_g/2) = 1$. A similar situation can happen; we can extract $|\downarrow\rangle$ in qubit B, more quickly for larger $h$ satisfying $\sin^2(\tau \theta_h/2) = 1$, if we adjust parameters so that $g \sin(\tau \theta_g/2) = 0$ holds.

Needless to say, there are cases where such purifications are not possible. For example, consider a case where we measure qubit A in the 1-2 plane, i.e., $\theta = \pi/2$. In this case, since the parameter $c_0$ is real, while all the other parameters $c_1, c_2$ and $c_3$ become pure imaginary, the eigenvalues $\lambda_{\pm} = c_0 \pm c$ are degenerated in magnitude

$$|\lambda_+| = \sqrt{c_0^2 + c^2} = |\lambda_-| \quad (33)$$

and no purification can occur in this particular case.
IV. ENTANGLEMENT DISTILLATION: I

As is mentioned in the Introduction, since entanglement is one of the key elements in quantum technologies, it would be useful if the present scheme of purification can be used to extract an entangled state as a target pure state. Notice that since the target system has never been measured directly in the present scheme, it is considered to be suited for extraction of such a fragile pure state as an entangled state. Actually any measurement on its subsystem that consists of entanglement would result in the destruction of the entanglement.

In order to see an entanglement distillation on the basis of the present idea of purification, we consider a total system composed of a compound system A+B, in which an entangled state is to be extracted, and another system C. Systems A and B interact with system C separately, but do not interact directly with each other. We measure system C repeatedly at regular intervals and endeavor to extract an entangled state as a pure state in the sense that one of the key elements in quantum technology is always one of the eigenstates of this operator and if the measurement of C projects its state on the operator reads

\[ H = \frac{\Omega}{2}(1 + \sigma_3^A) + \frac{\Omega}{2}(1 + \sigma_3^B) + \frac{\omega}{2}(1 + \tau_3) + g(\sigma_1^A\tau_- + \sigma_2^B\tau_- + \text{h.c.}) + h(\sigma_1^A\tau_+ + \sigma_2^B\tau_+ + \text{h.c.}), \]

(34)

where the Pauli matrices \( \tau_i \) act on system C. It is assumed here for simplicity that the two systems A and B are the same and the Hamiltonian is symmetric under the exchange A↔B.

In order to find the spectral decomposition of the projected operator \( \mathcal{P} \) of the total Hamiltonian \( H \) can be found after their classification according to the above mentioned A↔B symmetry and a “parity” \( \mathcal{P} \) of \( \sigma_3^A\sigma_3^B\tau_3 \): 1) A↔B symmetric and \( \mathcal{P} = + \)

\[ H = \begin{pmatrix} \Omega + \omega & g + h \\ \Omega & \Omega + \omega & -g + h \end{pmatrix} \begin{pmatrix} \Phi^+ \downarrow \\ \Phi^- \downarrow \\ \Psi^+ \downarrow \end{pmatrix}, \]

(36)

2) A↔B symmetric and \( \mathcal{P} = - \)

\[ H = \begin{pmatrix} \Omega + \omega & g + h \\ \Omega & \Omega + \omega & -g + h \end{pmatrix} \begin{pmatrix} \Phi^+ \downarrow \\ \Phi^- \downarrow \\ \Psi^+ \downarrow \end{pmatrix} \]

(37)

3) A↔B anti-symmetric and \( \mathcal{P} = - \)

\[ H |\Psi^+ \uparrow\rangle = (\Omega + \omega) |\Psi^+ \uparrow\rangle, \]

(38)

4) A↔B anti-symmetric and \( \mathcal{P} = + \)

\[ H |\Psi^- \downarrow\rangle = \omega |\Psi^- \downarrow\rangle. \]

(39)

Here \( |\Phi^+ \uparrow\rangle \equiv |\Phi^+ \rangle \otimes |\uparrow\rangle \), etc. Thus the time evolution of the projective operator \( e^{-iH\tau} \) is expressed as

\[ e^{-iH\tau} = \sum_s e^{-iE_s\tau} |s\rangle\langle s| + |\Psi^\prime (\Psi^-)\rangle \langle \Psi^- | \left[ e^{-i(\Omega + \omega)\tau} |\uparrow\rangle + e^{-iH\tau} |\downarrow\rangle \langle \downarrow| \right], \]

(40)

where the summation is taken over the six A↔B symmetric eigenstates of \( H \), denoted as \( |s\rangle \), that are given as linear combinations of the six states in (36) and (37). Owing to the A↔B symmetry of \( H \), the A↔B anti-symmetric state \( |\Psi^-\rangle \) does not mix with the other (A↔B symmetric) eigenstates.

We are now in a position to examine the spectrum of the projected operator \( V_\phi (\tau) \equiv \langle \phi | e^{-iH\tau} |\phi\rangle \). If the measurement of C projects its state on \( |\phi\rangle = \alpha |\uparrow\rangle + \beta |\downarrow\rangle \),

(41)

the operator reads

\[ V_\phi (\tau) = \langle \phi | e^{-iH\tau} |\phi\rangle = \sum_s e^{-iE_s\tau} \left[ |\alpha|^2 \langle \uparrow | s \rangle\langle s | \uparrow \rangle + |\beta|^2 \langle \downarrow | s \rangle\langle s | \downarrow \rangle \right. \]

\[ + \left. \alpha^* \beta \langle \uparrow | s \rangle\langle s | \downarrow \rangle + \text{h.c.} \right] + |\Psi^\prime (\Psi^-)\rangle \langle \Psi^- | \left[ |\alpha|^2 e^{-i(\Omega + \omega)\tau} + |\beta|^2 e^{-iH\tau} \right]. \]

(42)

From this expression, it is evident that the Bell state \( |\Psi^-\rangle \) is always one of the eigenstates of this operator and if the measurement interval \( \tau \) is so adjusted that the condition \( \omega \tau = 2\pi \) is met, its eigenvalue \( \lambda_{\Psi^-} \) becomes maximum in magnitude

\[ \omega \tau = 2\pi \quad \rightarrow \quad |\lambda_{\Psi^-}| = 1, \]

(43)

irrespective of the projected state \( |\phi\rangle \) of system C because \( |\alpha|^2 + |\beta|^2 = 1 \). This clearly demonstrates a possibility of entanglement distillation in this simple system, in the sense that one of the conditions for optimal purification can be realized by and the entangled state \( |\Psi^-\rangle \) would surely be extracted, only if the other eigenvalues of the operator than \( \lambda_{\Psi^-} \) can be made (much) smaller in magnitude. Further details of the analysis of such conditions for the entanglement distillation and its optimization are found in \([11, 12]\) in a slightly simplified case.
V. ENTANGLEMENT DISTILLATION: II

The example in the previous section explicitly demonstrates that we can distill an entangled state in the system A+B, through the repeated measurements on the other system C that separately interacts with A and B. The framework is rather simple and the distillation can be made optimal. There is, however, a kind of drawback in this scheme. As is clear in its exposition, it is assumed that system C, on which the measurement is performed, always and simultaneously interacts with both A and B and these interactions are crucial for the entanglement distillation. Stating differently, systems A and B (and C) are not (and/or will not be) able to be separated spatially, which implies that no entanglement between spatially separated systems is possible by the scheme presented in Sec. IV. It would not be suited to the situations where entanglements among spatially separated systems are required, as in quantum teleportation.

In this section, a resolution to this problem is presented. Since the two systems, A and B, an entanglement between which is to be driven, are considered to be placed at different places, let us consider, instead of system C which can no longer interact simultaneously with A and B, another quantum system, say X, which is assumed to interact with A and B, not simultaneously, but successively. System X plays the role of an “entanglement mediator.” After such successive interactions with A and then B, system X is measured to confirm that it is in a certain state. If system X is found in this particular state, X is again brought to interaction with A, that with B and measurement on X, will be repeated many (N) times and we are interested in the asymptotic state of system A+B in the hope of distilling an entangled state.

There are a couple of points to be mentioned here. First, it is clear that in spite of these modifications, the new scheme presented here shares essentially the same idea of purification with the previous ones: The dynamics of the system can be affected, in an essential way, by the action of measurement, even if its effect is not direct. Second, such a successive interaction would be conveniently treated in terms of a time-dependent (effective) Hamiltonian $H(t)$. We may thus avoid possible complications caused by the introduction of spatial degrees of freedom, still keeping the essential points.

In order to see how the new scheme works, consider again a three-qubit system, A+B+X, for definiteness and simplicity. We prepare system X, say in up state $|↑⟩$, while the system A+B can be in an arbitrary mixed state. It is assumed that systems A and B are spatially separated and have no contact with each other and that only system X can interact with them locally for definite time durations. Now consider the following process:

1. System X is first brought to interaction with system A for time duration $t_A$. The Hamiltonian here is given by $H(t) = H_0 + H_{XA}$. Then the interaction is switched off and the total system evolves freely with the free Hamiltonian $H_0$ for $\tau_A$.

2. System X then interacts with system B for time duration $t_B$, the dynamics of which is now described by another Hamiltonian $H(t) = H_0 + H_{XB}$. After that, the total system again evolves freely with the Hamiltonian $H_0$ for $\tau_B$.

3. A (projective) measurement is performed on system X to select only up state $|↑⟩$. Other states are discarded.

Then this process is repeated $N$ times: $1→2→3→1→ \cdots →1→2→3$.

It is shown below that the following choice of the Hamiltonians

$$H_0 = \frac{\omega}{2}(1 + \sigma_3^A) + \frac{\omega}{2}(1 + \sigma_3^B) + \frac{\omega}{2}(1 + \sigma_3^X),$$

$$H_{XA} = g_A\sigma_3^X\sigma_3^A, \quad H_{XB} = g_B\sigma_3^X\sigma_3^B$$

(44)

actually results in an entanglement distillation in system A+B. It is important to notice that the above choice of the interaction Hamiltonians is closely connected to the details of the process $1→2→3$ and another choice, e.g., $\sigma_3^X\sigma_3^A$ for $H_{XA(B)}$, would result not in an entangled state, but in a product state, in this particular process.

The next task is to find the spectral decomposition of the projected operator $V_τ$ defined, in this case, by

$$V_τ = (|↑⟩e^{-iH_0τ_B}e^{-i(H_0+H_{XB})τ_B} \times e^{-iH_0τ_A}e^{-i(H_0+H_{XA})τ_A}|↑⟩).$$

(45)

Since a parity defined by $P \equiv \sigma_3^A\sigma_3^B$ is conserved in this system, eigenstates of the operator $V_τ$ are easily found. Indeed, we can classify every state of system A+B into two sectors according to the parity $P = \pm$ and the action of the operator $V_τ$ is closed within each sector. For $P = +$ states, the action is represented by a matrix $M$

$$V_τ \begin{bmatrix} |↑↑⟩ \\ |↓↓⟩ \end{bmatrix} = e^{-iω(t_A+τ_A+t_B+τ_B)}M \begin{bmatrix} |↑↑⟩ \\ |↓↓⟩ \end{bmatrix},$$

(46)

where its matrix elements read

$$M_{11} = e^{-iω(t_A+2τ_A+t_B+2τ_B)}(\cos ζ_A - i \sin ζ_A \cos 2ξ_A) \times (\cos ζ_B - i \sin ζ_B \cos 2ξ_B),$$

$$M_{12} = -e^{-iωt_A} \sin ζ_A \sin 2ξ_A g_B t_B,$$

$$M_{21} = -e^{-iω(t_B+2τ_B)} \sin g_A t_A \sin ζ_B \sin 2ξ_B,$$

$$M_{22} = g_A t_A \cos g_B t_B,$$

(47)

while, for $P = -$ states, it is represented by another matrix $N$

$$V_τ \begin{bmatrix} |↑↓⟩ \\ |↓↑⟩ \end{bmatrix} = e^{-iω(t_A+τ_A+t_B+2τ_B)}N \begin{bmatrix} |↑↓⟩ \\ |↓↑⟩ \end{bmatrix},$$

(48)
with its matrix elements
\[ \mathcal{N}_{11} = e^{-i\omega(2\tau A + t_B)}(\cos \zeta_A - i \sin \zeta_A \cos 2\xi_A) \cos g_B t_B, \]
\[ \mathcal{N}_{12} = -\sin \zeta_A \sin 2\xi_A \sin \zeta_B \sin 2\xi_B, \]
\[ \mathcal{N}_{21} = -e^{-i\omega(t_A + 2\tau_A + t_B)} \sin g_A t_A \sin g_B t_B, \]
\[ \mathcal{N}_{22} = e^{-i\omega(t_A + 2\tau_A)} \cos g_A t_A (\cos \zeta_B - i \sin \zeta_B \cos 2\xi_B). \]

Here the angles are defined by
\[ \zeta_{A(B)} = t_{A(B)} \sqrt{\omega^2 + g_{A(B)}^2}, \quad \tan 2\xi_{A(B)} = \frac{g_{A(B)}}{\omega}. \]

In order to see the possibility of entanglement distillation in this framework, it is enough to consider a much simplified case. Let the two systems A and B be treated symmetrically, that is, all parameters are taken to be the same for A and B
\[ g_A = g_B \equiv g, \quad t_A = t_B \equiv t, \quad \tau_A = \tau_B \equiv \tau, \]
\[ (\zeta_{A(B)} \rightarrow \zeta, \quad \xi_{A(B)} \rightarrow \xi). \]

It is then easy to see that if the parameters satisfy
\[ \cos \zeta - i \sin \zeta \cos 2\xi = -e^{i\omega \tau} \cos gt, \]

an optimal purification of an entangled state |\Psi\rangle of the form
\[ |\Psi\rangle = \frac{1}{\sqrt{2}} \left[ |\uparrow\downarrow\rangle + e^{i\chi} |\downarrow\uparrow\rangle \right] \]

with \( \chi = \omega(t + \tau) \) is actually possible, provided
\[ \cos gt \sin gt \neq 0, \quad \omega(t + \tau) \neq 2n\pi \quad (n: \text{integer}). \]

In fact, one can show that |\Psi\rangle is an eigenstate of the operator \( V_\uparrow \)
\[ V_\uparrow |\Psi\rangle = \lambda_\Psi |\Psi\rangle. \]

The eigenvalue \( \lambda_\Psi \) is maximum in magnitude
\[ \lambda_\Psi = -e^{-3i\omega(t+\tau)}, \quad |\lambda_\Psi| = 1, \]

while all the other eigenvalues remain smaller than unity in magnitude, under the conditions on the conditions 52 and 54. Therefore, we can repeat the process 1→2→3 as many times as is required to achieve the desired (high) fidelity, without reducing the yield.

This is an example of (optimal) entanglement distillations, where the entanglement between two qubit systems that are (or can be) spatially separated, is extracted through their successive interactions with another qubit, on which one and the same measurement is repeated regularly. Further details of this model and applications to other quantum systems, e.g., extraction of entanglement between two cavity modes at a distance, will be reported elsewhere.

VI. SUMMARY

In this paper, a new purification scheme recently proposed is applied to a few simple qubit systems to explicitly show its ability of qubit purification (Sec. III) and entanglement distillations (Secs. IV and V) for two-qubit systems. The important and essential idea, on which these particular examples are based, is to utilize the effect caused by the action of measurement on quantum systems.

It should be stressed again that since the basic idea is so simple, that is, one has only to repeat one and the same measurement without being concerned about the preparation of a specific initial (pure) state, this purification scheme is considered to have wide applicability and flexibility. Furthermore, it enables us to make the two demands—the maximal fidelity and non-vanishing yield—compatible. The examples presented in this paper just show these characteristics and many variants can be devised according to the actual setups.

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