Form-Factor Bootstrap and the Operator Content of Perturbed Minimal Models

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Abstract

The form-factor bootstrap approach is applied to the perturbed minimal models $M_{2,2n+3}$ in the direction of the primary field $\phi_{1,3}$. These theories are integrable and contain $n$ massive scalar particles, whose $S$–matrix is purely elastic. The form-factor equations do not refer to a specific operator. We use this fact to classify the operator content of these models. We show that the perturbed models contain the same number of primary fields as the conformal ones. Explicit solutions are constructed and conjectured to correspond to the off-critical primary fields $\phi_{1,k}$.

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1 Introduction

The identification of local operators in a given model and the computation of their multi-point correlators is one of the most important problems in Quantum Field Theory (QFT). For two-dimensional critical QFT, a successful approach is given by the conformal bootstrap analysis [1, 2, 3, 4, 5, 6]. The operator content of such theories consists of a set of conformal families which form irreducible representations of the conformal Virasoro algebra. The representatives of these families, organized in a tower of descendant operators of increasing anomalous dimensions and spin, are identified by their primary operators. The number of conformal families is finite for the so-called minimal models. Using the representation theory of the underlying Virasoro (or higher dimensional) algebra, it turns out that the correlation functions of conformal operators in the minimal models can be written as solutions of linear differential equations [3, 7].

Concerning the massive integrable QFT, the present understanding is incomplete, though much progress has been achieved recently. The on–shell behaviour of these theories is characterized by the factorizable $S$-matrix of the asymptotic states [8, 4, 10]. Several authors have pursued the idea that these QFT (or analogous off-critical lattice models) can be characterized by infinite dimensional algebras, which generalize the Virasoro algebra of the critical points and lead to the computation of correlation functions by exploiting this algebraic structure [11, 12, 13]. An alternative non–perturbative method in which the algebraic structure is less apparent is known as form factor bootstrap approach [14, 15, 16, 17, 18]. Using general properties of unitarity, analyticity and locality, this approach leads to a system of functional and recursive equations for the matrix elements of local operators between asymptotic states which allow their explicit determination. Using a parametrization of the external momenta in terms of the rapidity $\beta$

$$p_0^i = m_i \cosh \beta_i \, , \quad p_1^i = m_i \sinh \beta_i \, ,$$

and assuming CPT invariance, matrix elements of a generic local operator $\mathcal{O}$ can be cast in the following form

$$F^\mathcal{O}_{\epsilon_1, \epsilon_2, \ldots, \epsilon_n}(\beta_1, \beta_2, \ldots, \beta_n) = \langle 0 \mid \mathcal{O}(0, 0) \mid Z_{\epsilon_1}(\beta_1), Z_{\epsilon_2}(\beta_2), \ldots, Z_{\epsilon_n}(\beta_n) \rangle_{in} \, , \quad (1.1)$$
called the *form factors*. The important point is that the functional equations satisfied by the form factors do *not* refer to a specific operator. This physical arbitrariness is determined by the mathematical property of the existence of different solutions to the same set of functional equations. This fact opens the possibility to classify the operator content of the QFT under investigation by identifying the solution space of the form factor equations. In fact the space of the solutions is isomorphic to the space of local operators entering the QFT.

Besides the classification of the operator content the form–factor bootstrap approach gives a means of calculating correlation functions in the QFT under consideration. To see this, let us consider the two-point correlators

\[ G_i(x) = \langle O_i(x) O_i(0) \rangle , \]

of hermitian operators. It can be expressed as an infinite series over multi-particle intermediate states

\[
\langle O_i(x) O_i(0) \rangle = \sum_{n=0}^{\infty} \int \frac{d\beta_1 \ldots d\beta_n}{n!(2\pi)^n} < 0|O_i(x)|Z_{\epsilon_1}(\beta_1), \ldots, Z_{\epsilon_n}(\beta_n) >_{\text{in in}} < Z_{\epsilon_1}(\beta_1), \ldots, Z_{\epsilon_n}(\beta_n)|O_i(0)|0 > = \\
\sum_{n=0}^{\infty} \int \frac{d\beta_1 \ldots d\beta_n}{n!(2\pi)^n} |F_{\epsilon_1, \epsilon_2, \ldots, \epsilon_n}(\beta_1, \beta_2, \ldots, \beta_n)|^2 e^{-mr \sum_i \cosh \beta_i} \]

where \( r \) denotes the radial distance in the euclidean space, \( i.e. r = \sqrt{x_0^2 + x_1^2} \). It is evident from this expression that the form factor expansion plays the role of a partial wave decomposition of the correlators and therefore their knowledge provides the knowledge of the correlation–function in terms of a convergent series expansion.

We will examine the form–factor bootstrap approach for the perturbed minimal models \( M_{2,2p+3} + \Phi_{1,3} \). The Kac-table of these non–unitary conformal theories extends along a row, and the model \( M_{2,2p+3} \) contains exactly \( p \) primary fields. The perturbations of these models along the \( \Phi_{1,3} \) directions are integrable \[15\] and can be described as restrictions of the Sine-Gordon model \[15\].

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\[1\] Similar consideration can be easily extended to multi-point correlation functions.

\[2\] The form–factors for some special operators of these models have been calculated by different methods in \[14\] and the first model of this series, the perturbed Yang–Lee model has been analyzed in \[14\].
We choose this series of models because their $S$–matrix contains only scalar particles and they thus have a simple fusion algebra of massive particles. The on–shell $S$–matrices are given by \[ S_{ab} = f_{|a-b|/2}^{|a+b|/2} (\beta) \prod_{k=1}^{\min(a,b)-1} \left( f_{(a-b)+2k}^{|a-b|+2k} (\beta) \right)^2, \] (1.3)

where $\alpha = \frac{2\pi}{2p+1}$, and $a, b = 1, 2, \ldots p$ labels the particles of mass $m_a = \sin \left( \frac{a\alpha}{2} \right)$. The functions $f$ are given by

\[ f_{\alpha} (\beta) \equiv \frac{\tanh \frac{1}{2}(\beta + i\alpha)}{\tanh \frac{1}{2}(\beta - i\alpha)}. \] (1.4)

The particles obey the bootstrap fusion algebra

\[ a_i \times a_j \rightarrow a_{i+j} \quad \text{or} \quad a_i \times a_j \rightarrow a_{2p+1-i-j} \quad \text{and} \quad a_i \times a_j \rightarrow a_{i-j} \] (1.5)

where the choice between the first two is determined by which index lies in the range of physical particles $1, \ldots, p$. These fusion properties will play a key role in the resolution of the form–factor equations.

The paper is organized as follows: In the next section we give a short review of the form–factor axioms for models with only scalar particles. In section 3 we parametrize the form–factors in a way to reduce the resolution of the axioms to polynomial recursion relations. The Watson’s equations are fulfilled automatically by this parametrization and the kinematical pole equation (2.6) reduces to that encountered in the Sinh–Gordon model [18, 21]. In section 4 some examples are discussed, namely the theories $M_{2,5}$ (the Yang–Lee model) and the two-particle system $M_{2,7}$. We determine the operator–content of these models by examining the bound-state axiom (2.8). These methods are generalized in section 5 to the whole series $M_{2,2p+3}$. The key result of the paper is that the bound state axiom reduces for these models to a single polynomial recursion relation for the form–factors involving only particles of type one. This recursion relation is explicitly calculated and the dimensionality of the solution-space determined. This also gives the operator content of these models. We give the physical interpretation of these operators. An account on descendent operators is given in section 6 and finally in section 7 we present our conclusions.
2 Form–Factor Axioms for Systems with Scalar Particles

Let us review the properties that the form–factors must satisfy. We discuss them for the case that the spectrum consists of only scalar self-conjugated particles. They derive from crossing symmetry, CPT invariance and the properties of the Faddeev- Zamolodchikov operators \([14, 22, 23]\). The physical vacuum \(|0\rangle\) is annihilated by operators \(Z^*(\beta)\),

\[
Z^*(\beta)|0\rangle = 0
\]

and the physical states are created by

\[
|Z_{\epsilon_1}(\beta_1)\ldots Z_{\epsilon_n}(\beta_n)\rangle = Z^*_{\epsilon_1}(\beta_1)\ldots Z^*_{\epsilon_n}(\beta_n)|0\rangle.
\]

(2.1)

The importance of these operators lies in their commutation relations, which are governed by the \(S\)-matrix,

\[
Z^{\epsilon_1}(\beta_1)Z^{\epsilon_2}(\beta_2) = S_{\epsilon_1\epsilon_2}(\beta_1 - \beta_2)Z^{\epsilon_2}(\beta_2)Z^{\epsilon_1}(\beta_1),
\]

\[
Z^*_{\epsilon_1}(\beta_1)Z^*_{\epsilon_2}(\beta_2) = S_{\epsilon_1\epsilon_2}(\beta_1 - \beta_2)Z^*_{\epsilon_2}(\beta_2)Z^*_{\epsilon_1}(\beta_1),
\]

\[
Z^{\epsilon_1}(\beta_1)Z^*_{\epsilon_2}(\beta_2) = Z^*_{\epsilon_2}(\beta_2)S_{\epsilon_1\epsilon_2}(\beta_1 - \beta_2)Z^{\epsilon_1}(\beta_1) + 2\pi\delta^{\epsilon_1\epsilon_2}_2\delta(\beta_1 - \beta_2).
\]

(2.2)

A consequence of the commutation relations (2.2) is the following symmetry property of the form–factors,

\[
F_{\epsilon_1\ldots\epsilon_i\ldots\epsilon_n}^{O}(\beta_1, \ldots, \beta_i, \beta_{i+1}, \ldots, \beta_n) = S_{\epsilon_i\epsilon_{i+1}}(\beta_i - \beta_{i+1})F_{\epsilon_1\ldots\epsilon_{i+1}\ldots\epsilon_n}^{O}(\beta_1, \ldots, \beta_{i+1}, \beta_i, \ldots, \beta_n).
\]

(2.3)

Consider the analytic continuation \(\beta_1 \rightarrow \beta_1 + 2\pi i\), which from the kinematical point of view brings back to the initial configuration, but changes the ordering of the particles in the function \(F_{\epsilon_1\ldots\epsilon_n}^{O}(\beta_1, \ldots, \beta_n)\). This analytic continuation can be related to the original form–factor in an alternative way by scattering all other particles. The consequence for the form–factor is the constraint equation

\[
F_{\epsilon_1\epsilon_2\ldots\epsilon_n}^{O}(\beta_1 + 2\pi i, \beta_2, \ldots, \beta_n) = F_{\epsilon_2\ldots\epsilon_n\epsilon_1}^{O}(\beta_2, \ldots, \beta_n, \beta_1) =
\]

\[
S_{\epsilon_1\epsilon_2}S_{\epsilon_1\epsilon_3}\ldots S_{\epsilon_1\epsilon_n}F_{\epsilon_1\epsilon_2\ldots\epsilon_n}^{O}(\beta_1, \beta_2, \ldots, \beta_n).
\]

(2.4)
A further constraint is imposed by relativistic invariance. Assume that the operator \( \mathcal{O} \) has spin \( s \). Then

\[
F_{\epsilon_1 \ldots \epsilon_n}^\mathcal{O} (\beta_1 + \Lambda, \ldots, \beta_n + \Lambda) = e^{s\Lambda} F_{\epsilon_1 \ldots \epsilon_n} (\beta_1, \ldots, \beta_n) .
\]  

(2.5)

The form-factor equations discussed so far connect form-factors corresponding to a fixed particle number. \( n \) and \( m \) particle form-factors are still independent of each other. The following two constraint equations have a recursive structure and link form-factors of different particle numbers. They originate from the pole–structure of the form–factors.

The first type of poles has a kinematical origin and corresponds to zero–angle scattering,

\[
-i \lim_{\beta' \to \beta} (\beta' - \beta) F_{\epsilon_1 \ldots \epsilon_n}^\mathcal{O} (\beta' + i\pi, \beta, \beta_1, \ldots, \beta_n) = \left( 1 - \prod_{i=1}^{n} S_{\epsilon_i}(\beta - \beta_i) \right) F_{\epsilon_1 \ldots \epsilon_n} (\beta_1, \ldots, \beta_n) .
\]  

(2.6)

If particles \( A_i, A_j \) form a bound state \( A_k \), the corresponding two-particle scattering amplitude exhibits a pole with the residue

\[
-i \lim_{\beta' \to i\bar{\mu}_{\epsilon_i \epsilon_j}} (\beta - i\bar{\mu}_{\epsilon_i \epsilon_j} \Gamma_{\epsilon_i \epsilon_j}^k) S_{\epsilon_i \epsilon_j} (\beta) = (\Gamma_{\epsilon_i \epsilon_j}^k)^2 ;
\]  

(2.7)

\( \Gamma_{\epsilon_i \epsilon_j}^k \) is the three–particle on–shell vertex. Corresponding to this bound state the form–factor exhibits a pole at the point \( \beta_i - \beta_j = i\bar{\mu}_{\epsilon_i \epsilon_j} \) with the residue

\[
-i \lim_{\beta' \to \beta} (\beta' - \beta) F_{\epsilon_1 \ldots \epsilon_k \ldots \epsilon_n}^\mathcal{O} (\beta_1, \ldots, \beta' + i\bar{\mu}_{\epsilon_i \epsilon_j}, \beta - i\bar{\mu}_{\epsilon_i \epsilon_j}, \ldots, \beta_{n-1}) = \Gamma_{\epsilon_i \epsilon_j}^k F_{\epsilon_1 \ldots \epsilon_k \ldots \epsilon_n}^\mathcal{O} (\beta_1, \ldots, \beta, \ldots, \beta_{n-1}) .
\]  

(2.8)

The equations (2.3)–(2.5) together with the residue relations (2.8) and (2.6) can be used as a system of axioms for the form-factors. In order for the operators to satisfy proper locality relations their form-factors have to satisfy the following ultraviolet bound

\[
F_{\epsilon_1 \ldots \epsilon_n} (\beta_1 + \Lambda, \ldots, \beta_i + \Lambda, \beta_{i+1}, \ldots, \beta_n) = O(e^{S|\Lambda|}) \quad \text{for} \quad |\Lambda| \sim \infty ,
\]  

(2.9)

where \( S \) is a number common for all \( i \) and \( n \).
3 Parametrization of the $n$–Particle Form–Factor

In order to find solutions of the above discussed equations one needs to find a convenient parametrization of the form–factors. A solution process which has proved to be very useful \cite{18, 22, 16} is to start with the calculation of the two–particle form–factor and then to parametrize the $n$–particle form–factor in terms of it. Let us discuss these steps in detail.

The Watson’s equations for $n = 2$ read as

\[ F_{ab}^O(\beta) = S_{ab}(\beta)F_{ba}^O(-\beta), \quad F_{ab}^O(i\pi - \beta) = F_{ba}^O(i\pi + \beta). \tag{3.1} \]

This set of equations can be solved with the help of the following observation \cite{22}. If the $S$–matrix element $S_{ab}$ can be written in an integral representation of the form

\[ S_{ab}(\beta) = \exp \left\{ \int_0^\infty \frac{dx}{x} f(x) \sinh \left( \frac{x\beta}{i\pi} \right) \right\}, \tag{3.2} \]

then a solution of (3.1) is given by

\[ F_{ab}^O(\beta) = \exp \left\{ \int_0^\infty \frac{dx}{x} f(x) \sin^2 \left( \frac{x(i\pi - \beta)}{2\pi} \right) \frac{\sinh x}{\sinh x} \right\}. \tag{3.3} \]

Note that multiplying the expression (3.3) by an arbitrary function of $\cosh \beta$ we find another solution of equations (3.1). In order to determine the final form of $F_{ab}(\beta)$ it is necessary to consider a specific theory and to know the physical nature of the operator $O$. In the following we drop the index referring to the operator $O$ keeping this ambiguity in mind.

In order to select one specific solution we define the minimal two particle form–factor $F_{ab}^{\text{min}}$ as the solution of equations (3.1) with the additional property that it is analytic in $0 < \text{Im} \beta < \pi$ and has no zeros in this range.

We choose for simplicity the form–factor $F_{ab}^{\text{11}}$ where the indices indicate that we discuss the form–factor corresponding to the fundamental particle of the considered theory. This is just a technical simplification since other form–factors can be treated in a similar way. Moreover we will see that all other form–factors can be obtained from this one by means of the bound state equation (2.8). For this we need only $F_{11}^{\text{min}}$ \cite{22}

\[ F_{11}^{\text{min}}(\beta) = (-i) \sinh \frac{\beta}{2} \exp \left\{ \int_0^\infty \frac{dx}{x} \cosh x \left( \frac{1}{2} - \frac{2\pi}{x} \right) \sin^2 \frac{x(i\pi - \beta)}{2\pi} \frac{\sinh x}{\sinh x} \right\}. \tag{3.4} \]
It is useful to rewrite this expression in terms of $\Gamma$–functions as

$$F_{11}^{\text{min}}(\beta) = (-i) \sinh \frac{\beta}{2} \frac{\zeta(\beta)}{\zeta(i\pi)},$$

with

$$\zeta(\beta) = \prod_{k=0}^{\infty} \frac{\Gamma \left( k + \frac{1}{2} + \frac{\alpha}{2\pi} - \frac{i\beta}{2} \right) \Gamma \left( k + 1 - \frac{\alpha}{2\pi} - \frac{i\beta}{2} \right) \Gamma \left( k + \frac{3}{2} + \frac{\alpha}{2\pi} + \frac{i\beta}{2} \right) \Gamma \left( k + 2 - \frac{\alpha}{2\pi} + \frac{i\beta}{2} \right)}{\Gamma \left( k + \frac{1}{2} - \frac{\alpha}{2\pi} - \frac{i\beta}{2} \right) \Gamma \left( k + 1 - \frac{\alpha}{2\pi} + \frac{i\beta}{2} \right) \Gamma \left( k + 1 + \frac{\alpha}{2\pi} + \frac{i\beta}{2} \right) \Gamma \left( k + 1 + \frac{\alpha}{2\pi} + \frac{i\beta}{2} \right)}.$$

For later convenience we also give the explicit expression for the constant $\zeta(i\pi)$

$$\zeta(i\pi) = \prod_{k=0}^{\infty} \left( \frac{\Gamma \left( k + 1 + \frac{\alpha}{2\pi} \right) \Gamma \left( k + \frac{3}{2} - \frac{\alpha}{2\pi} \right)}{\Gamma \left( k + 1 - \frac{\alpha}{2\pi} \right) \Gamma \left( k + 1 + \frac{\alpha}{2\pi} \right)} \right)^2 = \frac{1}{\pi} \exp \left\{ 4 \int dt \frac{\sinh \frac{t}{2} \sinh \frac{\alpha}{2\pi} t \sinh \frac{1}{2} \left( 1 - \frac{\alpha}{\pi} \right) t}{t \sinh^2 t} \right\}$$

In the following we will drop the indices referring to particle 1 and denote it simply as

$$F_{1,1,\ldots,1}^{\text{n} \beta_1,\ldots,\beta_n} \equiv F_n(\beta_1,\ldots,\beta_n).$$

In general this form–factor can be parametrized as [22]

$$F_n(\beta_1,\ldots,\beta_n) = K_n(\beta_1,\ldots,\beta_n) \prod_{i<j} F_{\text{min}}^{\beta_{ij}},$$

where the function $K_n$ needs to satisfy Watson’s equations (2.3) and (2.4) with an $S$–matrix factor $S = 1$. Therefore it is a completely symmetric, $2\pi i$–periodic function of $\beta_i$. It must contain all expected kinematical and bound state poles. Finally it will contain the information on the operator $\mathcal{O}$.

Since we know the possible scattering processes we can split the function $K_n$ further in order to determine the pole structure. The kinematical poles are expected at the rapidity values $\beta_i \to \beta_j + i\pi$. These poles can be generated by the completely symmetric function $\prod_{i<j}(x_i + x_j)^{-1}$, where we have introduced the notation $x_i \equiv e^{\beta_i}$. Further, the $S$–matrix element $S_{11}$ exhibits a pole at $\beta = i\alpha$. Then according to (2.8) also the form–factor exhibits poles which can be generated by the function

$$\frac{1}{\sinh \frac{1}{2}(\beta_{ij} - i\alpha) \sinh \frac{1}{2}(\beta_{ij} + i\alpha)}.$$
The final parametrization of the $n$–particle form–factor reads as

$$F_n(\beta_1, \ldots, \beta_n) = \check{Q}_n(\beta_1, \ldots, \beta_n) \prod_{i<j} \frac{F_{\text{min}}(\beta_{ij})}{(x_i + x_j) \sinh \left( \frac{1}{2}(\beta_{ij} - i\alpha) \right) \sinh \left( \frac{1}{2}(\beta_{ij} + i\alpha) \right)}, \quad (3.9)$$

$\check{Q}_n$ is now a symmetric function free of singularities. The form factor equations have been reduced through this parametrization to a set of coupled recursive relations for $\check{Q}_n$.

Using the parametrization (3.9) it is straightforward to derive the recursion relation corresponding to the kinematical poles. Let us introduce $\omega = e^{i\alpha}$. We find

$$\check{Q}_{n+2}(-x, x, x_1, \ldots, x_n) = D_n(x, x_1, \ldots, x_n) \check{Q}_n(x_1, \ldots, x_n) \quad (3.10)$$

with

$$D_n(x, x_1, \ldots, x_n) = (-1)^n \left( \frac{\zeta(i\pi)}{\pi} \right)^2 \left( \frac{n}{2} \right)^{-\frac{n-1}{2}} i^n.$$  

It is useful to define

$$\check{Q}_n = H_n Q_n, \quad (3.11)$$

with

$$H_n = C_0 \left( 4 \cos^2 \frac{\alpha}{2} \sin \alpha \right)^{\frac{n}{2}} \left( \frac{\zeta(i\pi)}{\pi} \right)^{\frac{(n-1)^2-1}{2}} i^n \quad (3.12).$$

In this way the recursion relation for $Q_n$ coincides exactly with that of the Sinh–Gordon model [18, 21].

Before going on to the task of actually solving the form-factor equations let us discuss the structure of the functions $Q_n(x_1, \ldots, x_n)$. We require them to be fully symmetric homogenous functions, analytic apart from the origin and polynomially bounded because of (2.9). Further from Lorentz invariance it follows that $Q_1(x_1) = \text{const.}$. Therefore $Q_n$ has to take the structure

$$Q_n(x_1, \ldots, x_n) = \sum_{N} \frac{1}{\prod_{i=1}^{n} x_i^N} \mathcal{P}(x_1, \ldots, x_n) \quad , \quad (3.13)$$

where the sum needs to be finite and $\mathcal{P}$ are polynomials whose degree is fixed by Lorentz invariance. We choose as a basis in this space the single terms in this sum, determined by a fixed integer $N$. 

8
Let us discuss this ansatz. It results from a strict interpretation of the form-factor axioms. The problem is that in this way it seems difficult to obtain a direct correspondence with operators in the UV-limit conformal field theory. This is not the case for primary fields as we will see, but for descendent operators one cannot expect such a simple structure. This is due to the mixing of operators of different Verma modules in the perturbed theory. This occurs for example between operators of the Verma-module of the Identity and of the Verma module of the perturbing field, which constitute the conservation laws in the perturbed model \[9\].

This problem was analyzed in \[24\] for the example of the Yang-Lee model. In order to get a direct correspondance with the descendents of the field $\phi_{13}$ the admixture of the operators from the module of the identity had to be taken in account. This led to the introduction of further singularities.

In this article we investigate mainly the form–factors of what we will call primary operators. This notation is borrowed from the corresponding conformal field theories, where the operators are organized in a Verma module structure. The scalar operator with the lowest scaling dimension is identified as the primary field. Analogously we define as primary operators in the perturbed model those spinless fields which have the mildest ultraviolet behaviour. The scaling dimension increases with $N$ and therefore we define as primary operators those with $N = 0$.

In section 6 we will discuss the problem of the descendent operators using the ansatz (3.13). Inspite of the problems discussed we will see a remarkable coincidence with the structure of the conformal field theories.

### 3.1 Resolution of the kinematical Recursion Relation

Since $Q_n(x_1, \ldots, x_n)$ is a symmetric polynomial satisfying (3.10) it is useful to introduce as a basis in this space the elementary symmetric polynomials $\sigma_k^{(n)}(x_1, \ldots, x_n)$ which are generated by \[25\]

$$
\prod_{i=1}^{n} (x + x_i) = \sum_{k=0}^{n} x^{n-k} \sigma_k^{(n)}(x_1, x_2, \ldots, x_n).
$$

(3.14)
Conventionally the $\sigma_k^{(n)}$ with $k > n$ and with $n < 0$ are zero. The explicit expressions for the other cases are

\[
\begin{align*}
\sigma_0 &= 1, \\
\sigma_1 &= x_1 + x_2 + \ldots + x_n, \\
\sigma_2 &= x_1 x_2 + x_1 x_3 + \ldots x_{n-1} x_n, \\
& \vdots \\
\sigma_n &= x_1 x_2 \ldots x_n.
\end{align*}
\] (3.15)

The $\sigma_k^{(n)}$ are linear in each variable $x_i$ and their total degree is $k$.

In terms of this basis the recursive equations (3.10) take the form

\[
(-1)^n Q_{n+2}(-x, x, x_1, \ldots, x_n) = x D_n(x, x, x_1, \ldots, x_n) Q_n(x_1, x_2, \ldots, x_n)
\] (3.16)

where

\[
D_n(x, x_1, \ldots, x_n) = \sum_{k=1}^{n} \sum_{m=1, \text{odd}}^{k} [m] x^{2(n-k)+m} \sigma_k^{(n)} \sigma_{k-m}^{(n)} (-1)^{k+1}.
\] (3.17)

We have introduced the symbol $[l]$ defined by

\[
[l] = \frac{\sin(l\alpha)}{\sin \alpha}
\] (3.18)

The recursion relation (3.16) was solved in the space of polynomials in $[2 1]$. There a class of solutions was found, given by

\[
Q_n(k) = ||M_{ij}(k)||,
\] (3.19)

where $M_{ij}(k)$ is an $(n - 1) \times (n - 1)$ matrix with entries

\[
M_{ij}(k) = \sigma_{2i-j} [i - j + k].
\] (3.20)

In Sinh-Gordon theory the operators corresponding to these solutions are the exponents $e^{kg\phi}$, $g$ being the coupling constant and $\phi$ the elementary field appearing in the Lagrangian.

Important properties of the polynomials $Q_n$ can be obtained by analyzing the recursive equations (3.16) [21]. We are interested in the dimensionality of the solution space of this recursion relation. It is given by $\text{dim}(Q_{2n-1}) = \text{dim}(Q_{2n}) = n$. The proof is done by induction: Lorentz invariance (2.3) fixes the total degree of the polynomials $Q_n$ as
\[ \mathcal{O}^1 - - - \]
\[ \uparrow \]
\[ \mathcal{O}^1 \mathcal{O}^3 - - \]
\[ \uparrow \]
\[ \mathcal{O}^1 \mathcal{O}^3 \mathcal{O}^5 - \]
\[ \uparrow \]
\[ \mathcal{O}^1 \mathcal{O}^3 \mathcal{O}^5 \mathcal{O}^7 \]
\[ \vdots \]

Figure 1: Tower like structure of the operators \( \mathcal{O}^n \), determined by the independent solutions of the kinematical recursion relation

\[ \frac{1}{2}n(n - 1) \]. This implies that \( Q_1 = A_1 \) and \( Q_2 = A_2 \sigma_1 \), with \( A_1, A_2 \) arbitrary constants.

Let us examine the polynomials \( Q_n \) with odd index \( n \). Assume that \( \dim(Q_{2n-1}) = n \). Then the dimensionality of \( Q_{2n+1} \) is given by \( \dim(Q_{2n-1}) \) plus the dimension of the kernel of the recursion relation \( i.e. \) by

\[ Q_{2n+1}(-x, x, \ldots, x_{n+2}) = 0 \quad . \quad (3.21) \]

In the space of polynomials \( \mathcal{P} \) of total degree \( (2n + 1)n \), there is only one solution of this equation,

\[ Q_{2n+1} = \prod_{i<j}^{2n+1} (x_i + x_j) \quad . \quad (3.22) \]

This polynomial has partial degree \( 2n \) and coincides with the denominator in eq. \( (3.11) \).

Since the kernel is a one-dimensional manifold, the dimension of the space of solutions increases exactly by one at each step of the recursion, and we have proved our statement above. The proof goes analogously for the polynomials with even index.

In figure 1 we have exhibited the structure of the solution space. It has a tower-like structure growing linearly with every step of the kinematical recursion relation. It will be interesting to see how this tower gets truncated by the additional implementation of the bound state axiom \( (2.8) \).

Note that the polynomials \( Q_n(k) \), with \( k \) integer do not form a base in this space. This is because of their periodic dependence on the coupling \( \alpha \), so that for \( \mathcal{M}_{2,2p+3} \) only the first \( p \) polynomials are independent. We will see in the following how by enforcing the bound state recursion relation \( (2.8) \) the solution space gets restricted such that the
\( Q_n(k), \ k = 1, \ldots, p \) will form a base.

4 Examples

4.1 The Yang–Lee Model

The form–factors of the Yang–Lee model have been analyzed extensively \[24, 15, 16\]. We want to use it here as an example in order to introduce the techniques which we will generalize in the following sections.

For the Yang–Lee model, \( M_{2,5} \alpha = \frac{2}{3} \pi \) and the theory contains only one massive particle. It is for this reason not a typical example for the series \( M_{2,2p+3} \). Nevertheless let us see how the solution space of the recursive equations can be classified. The recursion relation for the bound state equation (2.8) reads as \[16\]

\[ Q_{n+1}(x \omega^{\frac{1}{2}}, x \omega^{-\frac{1}{2}}, x_1, \ldots, x_{n-1}) = x \prod_{i=1}^{n-1} (x + x_i) Q_n(x, x_1, \ldots, x_{n-1}) \] (4.1)

We saw that the solution space for the kinematical recursion relation was linearly growing. This is not the case if we impose both recursion relations, since the kernel of the kinematical recursion relation is not a solution of the bound state equation in a polynomial space.

Let us analyze the lowest polynomials \( Q_n \). From the solution of the kinematical recursion relation we know that the first polynomials \( Q_1 \) to \( Q_3 \) have the general form

\[ Q_1 = A_1 \quad , \quad Q_2 = A_2 \sigma_1 \quad , \quad Q_3 = A_3 \sigma_3 + (A_1 - A_3) \sigma_2 \sigma_1 \]

with arbitrary constants \( A_1, A_2, A_3 \). Now we additionally impose the bound state recursive equation (4.1) from which follows after a short calculation that the only consistent solution is

\[ Q_1 = A_1 \quad , \quad Q_2 = A_1 \sigma_1 \quad , \quad Q_3 = A_1 \sigma_2 \sigma_1 \]

In terms of the elementary solutions introduced in the last section this solution corresponds to \( Q_n(1) \).

In general we have to show that the kernel solutions of the kinematical recursive equation (3.22) are not solutions to the bound state equation. Suppose \( Q_i = 0 \), for
Then any non–zero solution for $Q_{n+1}$ must have zeros as well at locations of the kinematical poles as at those of the bound state poles in order to give a zero residue. This is, that

$$Q_{n+1} \sim K_{n+1} = \prod_{i<j}^{n+1} (x_i + x_j)(x_i + \omega x_j)(x_i + \omega^{-1} x_j).$$

(4.2)

On the other hand we know that because of relativistic invariance the total degree of $Q_{n+1}$ must be $\frac{(n+1)n}{2}$. The above expression (4.2) has total degree $\frac{3(n+1)n}{2}$, which shows that it cannot be a solution of the combined recursion relations. That is, the Kernel of the combined recursion relations (3.16) and (4.1) is zero-dimensional. Therefore no additional solutions exist to $Q_n(1)$, which was already found in [16]. The operator corresponding to this specific solution is the trace of the stress energy tensor.

This simple argument has a rather important physical consequence. It shows that also for the perturbed Yang–Lee model the space of primary operators (which in our case corresponds to polynomial solutions of the recursive equations) is one–dimensional, as it is the case for the conformal field theory.

4.2 The Model $M_{2,7}$

As a next step we examine the two–particle system, defined by the perturbed $M_{2,7}$ theory. The $S$–matrix of this model is given by

$$S_{11} = f_{\pi \pi}^2, \quad S_{12} = f_{\pi \pi} f_{\pi \pi}^4, \quad S_{22} = (f_{\pi \pi}^2)^2 f_{\pi \pi}^4.$$  

(4.3)

The corresponding fusion angles are

$$u^2_{11} = \frac{2\pi}{5}, \quad u^1_{12} = \frac{4\pi}{5}, \quad u^2_{12} = \frac{3\pi}{5}, \quad u^1_{22} = \frac{4\pi}{5}.$$  

(4.4)

Now we need to analyze all form–factors $F_{\epsilon_1,\epsilon_2,..\epsilon_n}$ where $\epsilon_i$ can take the values 1 or 2. By the bound state equation (2.8) any form–factor containing the particle 2 can be expressed in terms of residues of form–factors containing only particle 1. This reflects the situation in $S$–matrix theory where it suffices to know the $S$–matrix of the fundamental particle in order to calculate the full $S$–matrix. Therefore we concentrate on the form–factor with all indices corresponding to the particle 1. The difficulty is now that the bound–state residue equation not only links the different form–factors but gives also constraints on
Figure 2: Possible fusion processes leading back to particle 1

the form–factors of the particle 1. This since we can return by various fusions to the particle 1. Two examples are shown in figure 2.

Let us analyze the first of these processes. If one uses two times the bound state equation the corresponding recursion relation is

\[
(−1) \lim_{\tilde{\beta} \to \beta} \lim_{\beta' \to \beta} (\tilde{\beta} - \beta)(\tilde{\beta} - \beta) F_{n+2}(\tilde{\beta} + 2i\bar{u}_{12}, \tilde{\beta}, \beta - i\bar{u}_{11}, \beta_1, \ldots, \beta_{n-1}) = \\
= \Gamma_{11}^2 \Gamma_{21}^1 F_n(\beta, \beta_1, \ldots, \beta_{n-1}) .
\]  

Consider the difference between the first and third rapidity in this formula

\[
\tilde{\beta} + 2i\bar{u}_{12} - \beta + i\bar{u}_{11} = \tilde{\beta} - \beta + i\pi .
\]

The corresponding recursive equation for the \(Q_n\) will be a special case of the kinematical recursion relation, and therefore this process does not give any new constraint on the solutions.

The second process in figure 2 on the other hand gives an independent constraint. It consists of three fusion processes and the corresponding form–factor equation reads as

\[
(-i)^3 \lim_{\tilde{\beta} \to \beta} \lim_{\beta' \to \beta} \lim_{\beta'' \to \beta} (\tilde{\beta} - \beta)(\tilde{\beta} - \beta)(\tilde{\beta} - \beta) F_{n+3}(\tilde{\beta} + \frac{3i\pi}{5}, \tilde{\beta} + \frac{i\pi}{5}, \beta' - \frac{i\pi}{5}, \beta - \frac{3i\pi}{5}, \beta_1, \ldots, \beta_{n-1}) = \\
= (\Gamma_{11}^2)^2 \Gamma_{22}^1 F_n(\beta, \beta_1, \ldots, \beta_{n-1}) .
\]  

Note that in this relation no rapidity difference of \(i\pi\) appears and therefore this relation is independent from the kinematical recursion relation. All other possible fusion processes can be reduced either to (4.6) or to the kinematical residue equation (3.16).

For this model, containing only two particles this statement is easy to understand: In order to be sure that we have obtained all possible constraint equations, we need to...
consider all possible fusion processes which return to particle 1. Considering a specific fusion process it is not important which set of particles will be fused. The resulting relation for $Q_n$ will always be the same, because $Q_n$ is a symmetric polynomial.

Further one needs to consider only elementary fusion blocks, i.e. processes which come back to particle 1 only once, since other processes can be obtained by combining elementary ones.

Now in the model $\mathcal{M}_{2,7}$ we have only two particles and there are only few elementary fusion processes. In addition to the ones shown in fig. 3 the only other type of processes are those resulting from an multiple additional application of the fusion $1 \times 2 \rightarrow 2$. One example is shown in fig. 3. Analyzing the rapidity shifts entering the recursion relation, one will see immediately that any such process can be decomposed in applications of the recursion relations (2.6) and (4.6).

The next step is therefore to calculate the corresponding recursion relation for $Q_n$. It is given by

$$Q_{n+3}(x_\omega^{\frac{3}{2}}, x_\omega^{\frac{1}{2}}, x_\omega^{-\frac{1}{2}}, x_\omega^{-\frac{3}{2}}, x_1, \ldots, x_{n-1}) = U_n(x, x_1, \ldots, x_{n-1})Q_n(x, x_1, \ldots, x_{n-1}) \quad ,$$

(4.7)

with

$$U_n(x, x_1, \ldots, x_{n-1}) = x^6 \prod_{i=1}^{n-1} (x + x_i)(x - x_i\omega^2)(x - x_i\omega^{-2}) \quad .$$

(4.8)

As in the Yang–Lee model we have two polynomial equations to solve. The relation (4.7) again couples the odd and even form–factors but now the form–factor $F_{n+3}$ and $F_n$, whereas in the Yang–Lee model the recursion relation (4.1) related $F_{n+1}$ and $F_n$. 

Figure 3: Process of multiple application of the fusion $1 \times 2 \rightarrow 2$
Let us discuss now the dimension of the solution–space of the coupled recursive equations. First note that also in this case the kernel solutions of the kinematical recursive equation in general cannot be solutions to the bound state equation. The argument is similar as for the Yang–Lee model. We present it here in a slightly different form.

Suppose the solution–space has a certain dimensionality at level \( n \). Examine the kernel of the kinematical recursion relation

\[ K_{\text{kin}}^{n+2} = \prod_{i<j} (x_i + x_j) . \]

If this is a solution of the bound state equation then \( K_{\text{kin}}^{n+2} \to U_{n-1} Q_{n-1} \). That is, \( Q_{n+2}^{K} \) must factor into \( U_{n-1} \) times a polynomial. Now,

\[ K_{\text{kin}}^{n+2}(x, x_1, \ldots, x_{n-1}) = x^6 \prod_{i=1}^{n-2} (x \omega_{\frac{2}{n}} + x_i)(x \omega_{\frac{2}{n}} + x_i)(x \omega_{-\frac{2}{n}} + x_i)(x \omega_{-\frac{2}{n}} + x_i) \prod_{i<j} (x_i + x_j) \]

Factoring out \( U_{n-1} K_{\text{kin}}^{n+2} \) takes the form

\[ K_{\text{kin}}^{n+2}(x, x_1, \ldots, x_{n-1}) = U_{n-1}(x, x_1, \ldots, x_{n-2}) \frac{(-x \omega + x_i)(-x \omega + x_i)}{(x + x_i)} \prod_{i<j} (x_i + x_j) \]

Since the terms containing \( x \) cannot cancel for \( \omega = e^{2 \pi i 5} \) the right hand side is not of polynomial form, and therefore \( K_{\text{kin}}^{n+2} \) is not a solution of the bound state recursion relation, in the space we are considering.

This argument obviously works only for \( n > 4 \). This means that the dimensionality of the combined solution space is fixed by that of the first 4 polynomials \( Q_1, \ldots, Q_4 \). Let us analyze them. The first three of them \( Q_1, Q_2, Q_3 \) remain untouched by the bound state recursion relation and therefore are determined by the kinematical recursion relation and have dimensionality 1, 1, 2 respectively. The last one links the even and odd sectors of the polynomial, but for any solution \( Q_4 \) of the kinematical recursion relation we can choose a constant \( Q_1 \) such that the bound state recursion relation is fulfilled. Therefore the solution space of \( Q_4 \) is also two–dimensional.

With this analysis we have shown that the solution space for the recursion relations of the perturbed model \( M_{2,7} \) is two–dimensional. This implies that also the perturbed
model contains only two primary operators as the conformal one. A base in this space is given by the elementary solutions $Q(1)$ and $Q(2)$. A physical interpretation will be given in section 5.2.

5 The Models $\mathcal{M}_{2,2p+3}$

In the last section we discussed how in the two–particle system $\mathcal{M}_{2,7}$ the form–factor constraints have been reduced to only two polynomial recursion equations, the kinematical one (3.16) and one equation deriving from a multiple application of the bound state form factor axiom (4.6). We will now show, that this is true for all models of the series $\mathcal{M}_{2,2p+3}$, which contain $p$ particles.

Let us now describe the key point of how it is possible to resolve the form–factor axioms for a system containing more than one particle. The difficulty lies in the fact that in general form–factors corresponding to different particles will be linked by the bound state equations. Therefore one has not to resolve only one equation but a coupled system of them. Since this seems a hopeless goal usually only one–particle systems have been examined in the form–factor bootstrap approach.

It is more efficient to examine only the form–factors corresponding to one particle, and consider all possible constraints on them. That is one needs to consider all possible fusion processes which return to particle 1. At a first look it seems that no simplification of the problem has been obtained. Though we will see in this section that for all models $\mathcal{M}_{2,2p+3}$ only one additional constraint equation for the form-factors involving particle 1 appear.

Let us first examine the fusion rules of this model. They are given by [20]

$$u_{ab}^{(a-b)} = (1 - |a - b|^{\frac{\alpha}{2}}) , \quad u_{ab}^{\min(a+b,2p+1-a-b)} = (a + b)^{\frac{\alpha}{2}} , \quad (5.1)$$

$$\bar{u}_{ab}^{(a-b)} = |a - b|^{\frac{\alpha}{2}} , \quad \bar{u}_{ab}^{\min(a+b,2p+1-a-b)} = 1 - (a + b)^{\frac{\alpha}{2}} . \quad (5.2)$$

We have to find out which kind of fusion processes lead to independent recursion relations. In order to understand these techniques consider the simple fusion process
which is a generalization of the first process of fig. 4. It corresponds to applying twice the bound state equation (2.8). Analyzing the rapidity shifts, one finds that the corresponding recursion relation reads as

\begin{align*}
(-1) \lim_{\beta \to \tilde{\beta}} \lim_{\tilde{\beta} \to \beta} (\tilde{\beta} - \beta)(\tilde{\beta} - \beta)
\end{align*}

\begin{align*}
F_{n+2}(\tilde{\beta} + i\tilde{u}_{i,k}^j + i\tilde{u}_{i,j}^k, \tilde{\beta} - i\tilde{u}_{i,k}^j + i\tilde{u}_{i,j}^k, \beta - i\tilde{u}_{i,k}^j, \beta_1, \ldots, \beta_{n-1}) &=
\Gamma_{i,j}^{k} \Gamma_{k,i}^{j} F_{n}(\beta, \beta_1, \ldots, \beta_{n-1})
\end{align*}

(5.3)

Because of the relation

\begin{align*}
\tilde{u}_{i,k}^j + \tilde{u}_{i,j}^k + \tilde{u}_{j,k}^i = \pi,
\end{align*}

we see that as in the example of the \( \mathcal{M}_{2,7} \) model a shift of \( i\pi \) between the first and third rapidity-values appears and therefore the corresponding recursion relation will reduce to the kinematical one (3.16).

In order to realize the above process in the perturbed \( \mathcal{M}_{2,2p+3} \) models it is clear that we need to identify the particle \( k \) as the particle \( |i - j| \). This means we have considered a specific process where we have \textit{lowered} the particle number. It is therefore useful to introduce the notation of a \textit{minimal fusion process}. Such a process is defined as one which starts from a set of initial particles \( a_1, a_2, \ldots, a_k \) and leads to the particle \( \sum_{i=1}^{k} a_i \) using only the fusion rules \( a_i \times a_j \to a_i + a_j \). Such minimal fusion processes have the property that independent of how the particles are fused, the rapidity shifts entering the corresponding bound state recursion relation are the same. This is seen by analyzing the rapidity shifts in (2.8). In a minimal fusion process

\[
\begin{array}{ccc}
  k & \downarrow & j \\
  & \downarrow & \\
  k + j & & \\
\end{array}
\]
the rapidities get shifted as $\beta_1 \rightarrow \beta_1 + j \frac{i\alpha}{2}$ and $\beta_2 \rightarrow \beta_2 - k \frac{i\alpha}{2}$. Since the minimal fusions are additive the first rapidity in fusioning $a_1, a_2, \ldots a_k$ will have the shift $\beta_1 \rightarrow \beta_1 + (a_2 + a_3 + \ldots + a_k) \frac{i\alpha}{2}$, or in general the shift will be

$$\beta_j \rightarrow \beta_j + (-a_1 - a_2 \ldots - a_{j-1} + a_{j+1} + \ldots + a_k) \frac{i\alpha}{2}.$$  \hspace{1cm} (5.4)

The importance of the concept of the minimal fusion processes lies in the fact, that for the models under consideration one can exclude all non–minimal processes. That is, one can show that for any non–minimal fusion process the set of rapidity values contains a subset which is equal to the minimal one.

To prove this statement assume we have in our fusion process a subprocess of the type

$$\begin{array}{c}
\downarrow \\
 k-j \\
\end{array}$$

where we have for simplicity taken $k > j$. Further assume that we have formed these particles out of particles 1 in a minimal way, \textit{i.e.} the subprocess considered is the first in our fusion-graph of a non–minimal kind. Then, by (5.4) we know the rapidity shifts which are involved forming particles $k$ and $j$. They are

$$\beta_1 \rightarrow \beta_1 + (k - 1) \frac{i\alpha}{2}, \hspace{0.5cm} \beta_2 \rightarrow \beta_2 + (k - 3) \frac{i\alpha}{2} \hspace{0.5cm} \ldots \hspace{0.5cm} \beta_k \rightarrow \beta_k - (k - 1) \frac{i\alpha}{2}$$

and similar for the particle $j$. Now through the final fusion $k \times j \rightarrow k - j$ the first set of rapidities gets further shifted by $j \frac{i\alpha}{2}$ while the second set undergoes a shift of $-i\pi + k \frac{i\alpha}{2}$. This fusion process with its rapidities is shown in fig. 4, from which one can easily see that the final set of rapidity-shifts contains those ones which one obtains if one forms the particle $k - j$ in a minimal way. Further it is interesting to note that the excessive shifts pair up in rapidity differences of $i\pi$ and therefore all these fusion processes reduce to the kinematical recursion relation.

We have analyzed of how the rapidity-shifts behave if we form a particle $k$ out of particles 1. Our goal though is to return to particle 1. For this scope there are two possible processes, namely $a_k \times a_{k+1} \rightarrow 1$ and $a_p \times a_p \rightarrow 1$. The first one can be excluded as a special case of the above consideration, since it reduces to the kinematical
\[ \begin{align*} 
\beta_1 + (k-1)\frac{i\alpha}{2} & \quad \ldots \quad \beta_k - (k-1)\frac{i\alpha}{2} \\
+ j\frac{i\alpha}{2} & \\
\beta_1 + (j+k)\frac{i\alpha}{2} & \quad \ldots \quad \beta_k + j\frac{i\alpha}{2} - i\pi + (k-1)\frac{i\alpha}{2} \\
- (\pi - k\frac{i\alpha}{2}) & \\
\beta_1 + (k+j-1)\frac{i\alpha}{2} & \quad \ldots \quad \beta_k + (j-k+1)\frac{i\alpha}{2} - i\pi + (k+j-1)\frac{i\alpha}{2} 
\end{align*} \]

Figure 4: Rapidity shifts for non-minimal fusion process

\[ \begin{align*} 
\hat{\beta}_1 + i(2p-1)\frac{i\alpha}{2}, \\
\hat{\beta}_2 + i(2p-3)\frac{i\alpha}{2}, & \quad \ldots \quad \hat{\beta}_{2p-1} - i(2p-3)\frac{i\alpha}{2}, \\
\hat{\beta}_{2p} - i(2p-1)\frac{i\alpha}{2} & \quad \pi \quad \hat{\beta}_{2p+1} 
\end{align*} \]

Figure 5: Fusion-graph for the bound state recursive equation

The final recursion relation reads as

\[ (-i)^{(2p-1)} \lim_{\hat{\beta}_i \to \hat{\beta}_{i+1}} \lim_{\hat{\beta}_{i+1} \to \hat{\beta}_{i+2}} \lim_{\hat{\beta}_{i+2} \to \hat{\beta}_{2p}} \prod_{i=1}^{p-1} (\hat{\beta}_i - \hat{\beta}_{i+1}) \prod_{i=p+1}^{2p-1} (\hat{\beta}_i - \hat{\beta}_{i+1}) \times \\
F_{n+2p-1} \left( \hat{\beta}_1 + (2p-1)\frac{i\alpha}{2}, \hat{\beta}_2 + (2p-3)\frac{i\alpha}{2}, \ldots, \hat{\beta}_{2p} - (2p-1)\frac{i\alpha}{2}, \beta_1, \ldots, \beta_{n-1} \right) = \]
\[
= \prod_{k=1}^{p-1} (\Gamma_{1k}^{k+1})^2 \Gamma_{pp}^1 F_n(\beta_{2p}, \beta_1, \ldots, \beta_{n-1}) \quad .
\]

For consistency we have to check whether the product of on-shell vertices \( \Gamma_{ab}^c \) on the right hand side gives the same result for any minimal fusion process. Note that since minimal fusions map always \( a \times b \rightarrow a + b \) the on-shell vertex depends effectively only on two variables \( \Gamma_{ab} \). Using the expression for the \( S \)-matrix (1.3) and the definition for the on–shell vertices (2.7) we find (for simplicity of notation we put \( a > b \))

\[
(\Gamma_{ab})^2 = 2 \tan(a + b) \frac{\pi}{2p + 1} \tan a \frac{\pi}{2p + 1} \prod_{k=1}^{b-1} \left( \frac{\tan(a + k) \frac{\pi}{2p + 1}}{\tan(b - k) \frac{\pi}{2p + 1}} \right)^2 \quad .
\]

Using this expression one can show that the algebra formed by the on–shell vertices is associative, \textit{i.e.}

\[
\Gamma_{a_1,a_2} \Gamma_{a_1+a_2,a_3} = \Gamma_{a_1,a_2+a_3} \Gamma_{a_2,a_3} \quad ,
\]

and therefore we conclude that any minimal fusion–path gives the same constant on the right-hand side of equation (5.6).

We have therefore shown that all possible constraints arising from fusion processes can be encoded in exactly \textit{one} recursive equation for the form–factors of the particle 1. Note that this is a considerable simplification in confrontation with the coupled recursive equations for form–factors with indices corresponding to different particles with which we started with.

5.1 Polynomial recursive equation and operator content of perturbed \( \mathcal{M}_{2,2p+3} \) models

In section 3.1 we have classified the solutions of the kinematical recursion relation (3.16). We found a tower-like structure growing with the number of particles in the form-factor. In the examples of Yang–Lee and \( \mathcal{M}_{2,7} \) models the dimensionality was severely restricted by the bound state equations, which we imposed in addition to the kinematical recursion relation. We want to carry out this analysis now for the theories \( \mathcal{M}_{2,2p+3} \) in general. As a first step we need to calculate the recursion relation for \( Q_n \) corresponding to (5.6). In terms of the the parametrization (3.11) it is a tedious but nevertheless straightforward
calculation. The final result is given by
\[
Q_{n+2p-1}(x\omega^{2p-1}, x\omega^{2p-3}, \ldots, x\omega^{2p-3}, x\omega^{2p-1}, x_1, \ldots, x_{n-1}) =
= U^p_n(x, x_1, \ldots, x_{n-1})Q_n(x, x_1, \ldots, x_{n-1})
\]
where we have defined \( x \equiv e^{\hat{\beta}_{2p}} \). The function \( U^p_n \) is given by
\[
U^p_n(x, x_1, \ldots, x_{n-1}) = (-1)^{\frac{p(p+1)}{2}+1} \prod_{k=2}^{p-1} [k]^2 \times
\times x^{p(2p-1)} \prod_{i=1}^{n-1} (x + x_i) \prod_{k=2}^{p} \prod_{i=1}^{n-1} (x - x_i \omega^k)(x - x_i \omega^{-k})
\]
\( (5.10) \)

We want to determine the operator content of the theory. Note that the recursion relations \( (5.10) \) link the even and odd sectors of the form–factors. In fact they relate the form–factors \( F_{n+2p-1} \) to \( F_n \). We know from section 3.1 that without considering the bound–state axiom of the form–factors the solution space grows linearly which is a consequence of the kinematical recursion relation.

We want to determine the subspace which in addition fulfills the relation \( (5.10) \). With the recursion relation \( (5.10) \) at hand this is now a rather easy task. The argument for the truncation of the solution space carries over straight forward from our examples. The recursion relation \( (5.10) \) acts only on polynomials \( Q_n \) with \( n \geq 2p \). Therefore up to this level we have to consider only the kinematical recursion relation. At level 2p the even and odd sectors are linked which are independent by \( (3.16) \). Only for \( Q_n \) with \( n > 2p \) the bound state relation constrains the dimensionality of the solution space. Now consider \( Q_n, n > 2p \). Its dimensionality is determined by the kernel of the recursion relations, but now of both, \( i.e. \) \( (3.16) \) and \( (5.10) \). It is given by
\[
\mathcal{K}_n = \prod_{i<j}(x_i + x_j)(x + \omega x_i)(x + \omega^{-1} x_i)
\]
\( (5.11) \)
This function though is not of the degree \( n(n-1)/2 \) as required from Lorentz invariance. This means the kernel is zero-dimensional, and no additional solutions for the recursion relations exist at any level \( n > 2p \). Therefore the dimensionality of the solution space is determined by that of \( Q_{2p} \) which is \( p \). Similary the argument we have applied for \( \mathcal{M}_{2,7} \) equally carries over for all theories \( \mathcal{M}_{2,2p+3} \).
These arguments show that there exist only \( p \) independent scalar ‘primary’ operators in the perturbed minimal models \( \mathcal{M}_{2,2p+3} \). This is the same amount as there are in the conformally invariant theories. A base in this space is given by the elementary solutions \((3.20)\) \( Q_n(k) \) with \( k = 1, \ldots, p \). We denote the operators corresponding to these solutions as \( \Psi_k \).

5.2 Physical Interpretation

We conjecture that the set of solutions \( \Psi_k \) correspond to the off-critical fields which in the ultraviolet limit turn into the primary fields \( \phi_{1,2k+1} \).

There are several observations which support this hypothesis. The polynomial \( Q(1) \) factors as

\[
Q(1)(x_1, \ldots, x_n) = \sigma_1 \sigma_{n-1} \times Q'(x_1, \ldots, x_n),
\]

\( Q' \) being a polynomial, which is seen by analyzing the expression of the determinant \((3.20)\). Using the fact that

\[
\sum_{j} x_j^{-1} = \frac{\sigma_{n-1}}{\sigma_n},
\]

and the sum rule

\[
\sum_{k=1}^{n} \cos(2k-1)x = \frac{1}{2} \frac{\sin(2nx)}{\sin x},
\]

this implies that in general the form-factors corresponding to these solutions will factor as

\[
F_{\epsilon_1, \ldots, \epsilon_n} = \sum (m_{\epsilon_j} x_j) \sum (m_{\epsilon_j} x_j^{-1}) \times F'_{\epsilon_1, \ldots, \epsilon_n}.
\]

This is the requirement (together with appropriate normalization) for the form-factors to be interpreted as corresponding to the trace of the energy momentum tensor \( \Theta = \frac{1}{4} T_{\mu}^{\mu} \). But since in the perturbed models \( \Theta = \lambda (\Delta - 1) \phi_{1,3} \), we can identify these form-factors as belonging to the operator \( \phi_{1,3} \) up to some normalization. Further, Smirnov has identified two exponential operators in \( [13] \), namely \( Pe^{i\sqrt{\phi}} P \) and \( Pe^{i\sqrt{\phi}} P \). Even if we were not able to rewrite his integral representation in terms of our determinant form we have checked for several values of \( p \) and \( n \) that his form-factors...
are equivalent to ours for $\Psi_1$ and $\Psi_p$. In the scaling limit these two operators turn into the conformal fields $\phi_{1,3}$ and $\phi_{1,2}$ respectively.

The perturbed minimal models $M_{2,2p+3} + \phi_{1,3}$ correspond to the reduced Sine-Gordon model at the rational values of the coupling constant $\gamma$. Even if we did not use this correspondence in the derivation of the form-factors, it is useful to use this property in the physical interpretation of the operators $\Psi_k$. The operators in the reduced models correspond to those in Sine-Gordon theory when projected onto the soliton-free sector. That is, the fields $\phi_{1,k}$ can be identified with the operators $Pe^{ik\sqrt{\gamma}P}$, where $P$ denotes the projection and $\phi$ is the Sine-Gordon field [15]. Therefore for a consistent interpretation of our solutions we expect

$$Q_n(k) = Q_n(p - \frac{2k-1}{2}) .$$

That this is indeed the case follows directly from the symmetry properties of the symbols $[m]$ appearing in the expressions for the determinant $Q(k)$.

As a final point let us investigate the cluster property of the operators $\Psi_k$. By cluster transformation we generally mean the behaviour of a form factor under the shift of a subset of the rapidities, i.e.

$$F^O_n(\beta_1 + \Delta, \ldots, \beta_m + \Delta, \beta_{m+1}, \ldots, \beta_n) . \tag{5.12}$$

Taking the limit $\Lambda \to \infty$, $F^O_n$ can be decomposed into two functions of $m$ and $(n - m)$ variables respectively, where both functions satisfy all the set of axioms for the form-factors. Therefore they can be considered as FF of some operators $O_b$ and $O_c$

$$\lim_{\Lambda \to \infty} F^O_n(\beta_1 + \Delta, \ldots, \beta_m + \Delta, \beta_{m+1}, \ldots, \beta_n) = F^O_b(\beta_1, \ldots, \beta_m)F^O_c(\beta_{m+1}, \ldots, \beta_n) \tag{5.13}$$

We denote this operation as

$$O_a \to O_b \times O_c .$$

We will prove that the operators $\Psi_k$ are mapped onto themselves under the cluster transformation, i.e.

$$\Psi_k \to \Psi_k \times \Psi_k . \tag{5.14}$$
We define the cluster-operator $C_m$ (acting on the symmetric functions) by means of

$$C_m (f(x_1, \ldots, x_n)) \equiv f(x_1 e^{\Delta}, x_2 e^{\Delta}, \ldots, x_m e^{\Delta}, x_{m+1}, \ldots, x_n) \quad m < n.$$  

(5.15)

Further, using the notation

$$\check{\sigma}_i^{(n-k)} \equiv \sigma_i^{(n-k)}(x_{n-k+1}, x_{n-k+2}, \ldots, x_n).$$

one finds that

$$C_m(\sigma_k^{(n)}) = \sum_{i=1}^{k} \sigma_k^{(m)} e^{(k-i)\Delta} \check{\sigma}_i^{(n-m)}.$$  

(5.16)

Since the cluster properties are fixed by the leading term of this sum, we have

$$C_m(\sigma_k^{(n)}) \sim \sigma_k^{(m)} e^{m\Delta} \check{\sigma}_k^{(n-m)} \quad m \leq k,$$

$$C_m(\sigma_k^{(n)}) \sim \sigma_k^{(m)} e^{k\Delta} \quad m \geq k.$$  

(5.17)

Now let us consider separately the cluster property of each term entering their parametrization

$$F^k_n(\beta_1, \ldots, \beta_n) = H^k_n Q_n(k) \prod_{i<j} (x_i + x_j) \frac{F^{\min}(\beta_{ij})}{\sinh \frac{1}{2}(\beta_{ij} - i\pi\alpha) \sinh \frac{1}{2}(\beta_{ij} + i\pi\alpha)}.$$  

(5.18)

Since

$$F^{\min}(\beta) \xrightarrow{\beta \to \infty} -\frac{1}{4\zeta(i\pi)\pi} e^\beta$$

we have

$$\prod_{i<j}^{n} F^{\min}(\beta_{ij}) \to \prod_{i<j}^{m} F^{\min}(\beta_{ij}) \prod_{i<j=m+1}^{n} F^{\min}(\beta_{ij})(\frac{-1}{4\zeta(i\pi)\pi})^{m-n} \prod_{i=1}^{m} \prod_{j=1}^{m-n} e^{\beta_{ij} + \Delta}.$$  

(5.19)

Further, using eq. (5.17), the cluster property of the elementary solution $Q_n(k)$ is given by [21]

$$\mathcal{C}_m\left(\frac{Q_n(k)}{\prod_{i<j} (x_i + x_j)}\right) \sim [k] \frac{Q_m(k)}{\prod_{i<j}^{m} (x_i + x_j)} \frac{Q_{n-m}(k)}{\prod_{i<j}^{n-m} (x_i + x_j)}.$$  

(5.20)

Finally decomposing the factor

$$\prod_{i<j}^{n} \sinh \frac{1}{2}(\beta_{ij} - i\pi\alpha) \sinh \frac{1}{2}(\beta_{ij} + i\pi\alpha),$$

and using the values of the constants $H_n$ (3.12) we find

$$\mathcal{C}_m(F^k_n(\beta_1, \ldots, \beta_n)) = \frac{[k]}{C_0} F^k_m(\beta_1, \ldots, \beta_m) F^k_{n-m}(\beta_1, \ldots, \beta_{n-m})$$

we conclude that the FF of $\Psi_k$ are mapped onto themselves under the cluster transformation if $C_0 = [k]$. Since this is a distinguished property of exponential operators [15, 23], it is natural to identify the operators $\Psi_k$ with the fields $Pe^{k\sqrt{\gamma_\phi}}P$.  

25
6 Descendent Operators

A crucial point in our analysis was the strict interpretation of the form-factor axioms leading to the ansatz (3.13) for $Q_n$. In order to check the consistency one should also examine descendent operators in the theory. We will here only sketch the problem, a detailed account will be given elsewhere [26].

For simplicity we discuss the Yang-Lee model. We use the notation that $\bar{x} = 1/x$ and $\bar{\sigma}_i = \sigma_i(\bar{x}_1, \ldots, \bar{x}_n)$. Simple derivative operators can be obtained by multiplying $Q_n$ by $\sigma_1^m$ and $\bar{\sigma}_1^m$. Because of the identity

$$\bar{\sigma}_i(x_1, \ldots, x_n) = \frac{\sigma_{n-i}(x_1, \ldots, x_n)}{\sigma_n(x_1, \ldots, x_n)},$$

any $\bar{x}$ dependence can be rewritten in terms of $x$, retaining the form of $Q$ as in ansatz (3.13). We will therefore restrict this discussion to only ‘chiral’ descendents which we want to compare with the structure of the Verma module created by $L_{-n}$ leaving out the $\bar{L}_{-n}$ dependence.

As discussed in section 3 we cannot expect that ansatz (3.13) gives a direct correspondence between conformal and descendent states. Nevertheless let us examine the operator content of polynomial solutions for $Q_n$ for several spin values $s$. The strategy is the same as for the perturbed primary operators. We solve the relations (3.16) and (4.1) recursively using the property that the space of solutions at level $n$ is given by the number of solutions at level $n-1$ plus the dimension of the kernel of the combined recursion relations. In table 1 we have written down the degree for the Polynomials $Q_n$ required from Lorentz invariance in comparison with the dimensions of the kernel of the recursion relations. Let us now count the independent solutions for various spin levels. For $s = 1$ the only solution is that generated by $Q_1 = A_1 \sigma_1$, which corresponds exactly to the level 1 descendent of the field $\phi$ of the Yang-Lee model. For spin 2 we have two possible solutions, generated by the initial polynomials

$$Q_1 = A_1 \sigma_1^2 \quad Q_2 = A_2 K_2.$$

Similar higher spin values can be investigated. Confronting the dimensions in table 1 we see that the first time a Kernel solution for $n = 3$ occurs for spin $s = 6$. Let us examine
Table 1: Comparing the total degrees of the polynomials $Q_n$ and the Kernel of the recursion relations for spin $s$ and level $n$

the solution space for that particular spin value. The solutions are

\begin{align*}
Q_1 &= A_1 \sigma_1^6 \\
Q_2 &= (A_2 \sigma_1^4 + A_3 \sigma_2 \sigma_1^2 + A_4 \sigma_2^2) \mathcal{K}_2 \\
Q_3 &= A_5 \mathcal{K}_3
\end{align*}

that is we have 5 independent operators.

Now let us confront this situation with conformal field theory. The number of descendant operators is just given by the character expansions,

\begin{align*}
\chi_{1,1} &= 1 + q^2 + q^3 + q^4 + 2q^5 + 2q^6 + O(q^7) \\
\chi_{1,3} &= 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + O(q^7)
\end{align*}

Summing up the values we have in CFT 1 descendent operator at spin 1, 2 operators at spin 2 and 5 operators at spin 6. We confront these values with the ones in the perturbed model and find they coincide.

We have presented here this counting argument just for a few spin values. Using the Rogers-Ramanujan identities, one can prove \cite{20} that the result holds in general. The same procedure can be adopted for the other $\phi_{13}$ perturbed models discussed in this
paper. There the counting becomes more difficult since the bound state relation connects $Q_{n+2p}$ with $Q_n$ and therefore leaves the possibility of intermediate steps of the kinematical recursion relation giving an extra freedom of parameters. We do not yet have general results for these models, but in the cases we have investigated, the dimensions of the spaces of descendent operators in the perturbed and the conformal models coincide.

7 Conclusions

We have constructed and analyzed the form-factors for the class of models $M_{2,2p+3} + \phi_{1,3}$. We chose these models since they are the simplest multi-particle systems containing only scalar states. We constructed the full set of form-factors for ‘primary operators’ and found that the operator content of these perturbed models is isomorphic to that of the conformal ones. The proof uses the fact that the form–factor equations do not refer to a specific operator and therefore the classification of solutions gives a means of determining the operator content of the theory.

In this analysis it is interesting to note the role of the various equations. The Watson’s equations (2.3) and (2.4) can be solved by conveniently parametrizing the form-factors. The kinematical recursion equation (2.6) defines a constraint for the form-factors and its solutions have a tower like structure, but still admit an infinite number of ‘primary fields’, which we defined to be fields with the lowest scaling dimensions in the ultraviolet limit. As long as we do not enforce the bound state equation (2.8) the solution space of the form-factor equations is identical to that of the Sinh–Gordon model. The reduction of the solution space to finite dimensionality is achieved by the bound state axiom which in this phenomenological approach takes the role of the quantum group reduction mechanism. It would be interesting to understand whether this is a generic feature of the form-factor equations for perturbed minimal models.

We found a base in this space of solutions which we conjecture to correspond to the off-critical extension of the conformal primary fields $\phi_{1,2k+1}$. We have given several arguments in support of this identification. Since the form-factor expansion corresponds to a large distance expansion, the ultraviolet limit is not very tractable, since the whole
series should be resummed. We also carried out several numerical calculations in order
to determine the scaling dimensions of the operators $\Psi_k$ but did not obtain conclusive
answers, because of the complexity of the functional dependence of the form-factors on
the rapidity variables. A final verification of this identification should be done in an
algebraic framework.

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