Controversy on a dispersion relation for MHD waves

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Abstract

Kumar et al. (2006) obtained a fifth order polynomial in \(\omega\) for the dispersion relation and pointed out that the calculations performed by Porter et al. (1994) and by Dwivedi & Pandey (2003) seem to be in error, as they obtained a sixth order polynomial. The energy equation of Dwivedi & Pandey (2003) was dimensionally wrong. Dwivedi & Pandey (2006) corrected the energy equation and still claimed that the dispersion relation must be a sixth order polynomial. The equations (11) – (19) of Dwivedi & Pandey (2006) and the equations (24) – (32) Kumar et al. (2006) are the same. This fact has been expressed by Kumar et al. (2006) themselves. Even then they tried to show this set of equations on one side gives the sixth order polynomial as they got; on the other side, the same set of equations gives the fifth order polynomial as Kumar et al. (2006) obtained. The situation appears to be non-scientific, as the system of equations is a linear one. These are simple algebraic equations where the variables are to be eliminated. However, it is a matter of surprise that by solving these equations, two scientific groups are getting polynomials of different degrees. In the present discussion, we have attempted to short out this discrepancy.

1 Introduction

For application of magnetohydrodynamics (MHD) in solar physics as well as in plasma physics, dispersion relation, where \(\omega\) is expressed as a function of \(k\), plays a key role. A controversy for the degree of the polynomial in \(\omega\) for dispersion relation appeared when Kumar et al. (2006, henceforth KKS) raised a point about the degree of the polynomial obtained by Porter et al. (1994, henceforth PKS) and by Dwivedi & Pandey (2003). Consequently, the results of PKS as well as of Dwivedi & Pandey (2003) were kept before a question mark. Energy equations of Dwivedi & Pandey (2003) was found erroneous (Klimchuk et al., 2004). After making correction in their energy equation, Dwivedi & Pandey (2006, henceforth DP) made an attempt to show that the dispersion relation must be a sixth order polynomial. Since the results of an investigation involving MHD depend on the dispersion relation, it is important to resolve this controversy. This communication is an attempt to show that for the basic equations considered by DP, PKS and KKS, the dispersion relation comes out to be the same. In each case, it is a fifth order polynomial having the same coefficients.

2 Basic equations of DP

The basic equations used by DP as well as PKS are

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0
\]  

\[
\rho \frac{D \vec{v}}{Dt} = -\nabla p + \frac{1}{4\pi} (\nabla \times \vec{B}) \times \vec{B} - \nabla \Pi
\]  

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\[
\frac{DB}{Dt} = \nabla \times (\vec{v} \times \vec{B}) \tag{3}
\]
\[
\frac{Dp}{Dt} + \gamma p (\nabla \cdot \vec{v}) = (\gamma - 1)[Q_{th} + Q_{vis} - Q_{rad}] \tag{4}
\]
\[
p = \frac{2\rho k_B T}{m_p} \tag{5}
\]
with \(Q_{th} = \nabla \cdot \kappa \nabla T\) and \(\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{v} \cdot \nabla)\). These equations are, respectively, the equation of continuity, equation of momentum, induction equation, energy equation and the equation of state. Here, \(\rho, \vec{v}, k_B, m_p, p, \vec{B}, \gamma, T\) and \(\Pi\) are, respectively, the total mass density, velocity, Boltzmann constant, proton mass, total pressure, magnetic field, ratio of the specific heats, temperature and the viscous stress tensor.

For small perturbations from the equilibrium (PKS, KKS):
\[
\begin{align*}
\rho &= \rho_0 + \rho_1 & \vec{v} &= \vec{v}_1 & \vec{B} &= \vec{B}_0 + \vec{B}_1 \\
p &= p_0 + p_1 & T &= T_0 + T_1 & \Pi &= \Pi_0 + \Pi_1
\end{align*}
\]
where the equilibrium part is denoted by the subscript “0” and the perturbation part by the subscript “1”. For the magnetic field taken along the z-axis, \((i.e., \vec{B}_0 = B_0 \hat{z})\) and the propagation vector \(\vec{k} = k_x \hat{x} + k_z \hat{z}\), the equations (1) – (5) can be linearized in the following form.
\[
\frac{\partial \rho_1}{\partial t} + \rho_0 (\nabla \cdot \vec{v}_1) = 0 \tag{6}
\]
\[
\rho_0 \frac{\partial \vec{v}_1}{\partial t} = -\nabla p_1 + \frac{1}{4\pi} (\nabla \times \vec{B}_1) \times \vec{B}_0 - \nabla \Pi_0 \tag{7}
\]
\[
\frac{\partial \vec{B}_1}{\partial t} = \nabla \times (\vec{v}_1 \times \vec{B}_0) \tag{8}
\]
\[
\frac{\partial p_1}{\partial t} + \gamma p_0 (\nabla \cdot \vec{v}_1) + (\gamma - 1) \kappa || k^2 T_1 = 0 \tag{9}
\]
\[
\frac{p_1}{p_0} = \frac{\rho_1}{\rho_0} + \frac{T_1}{T_0} \tag{10}
\]
For the perturbations that are proportional to \(\exp[i(\vec{k} \cdot \vec{r} - \omega t)]\), equations (6) – (10) reduce to the following algebraic equations
\[
\begin{align*}
\omega \rho_1 - \rho_0 (k_x v_{1x} + k_z v_{1z}) &= 0 \tag{11} \\
\omega p_0 v_{1x} - k_x p_1 - \frac{B_0}{4\pi} (k_x B_{1z} - k_z B_{1x}) + \frac{i \eta_0}{3} (k^2 v_{1x} - 2 k_x k_z v_{1z}) &= 0 \tag{12} \\
\omega p_0 v_{1y} + k_z p_1 + \frac{B_0}{4\pi} (k_z B_{1y}) &= 0 \tag{13} \\
\omega p_0 v_{1z} - k_z p_1 - \frac{i \eta_0}{3} (4 k^2 v_{1z} - 2 k_x k_z v_{1z}) &= 0 \tag{14} \\
\omega B_{1x} + k_x B_0 v_{1x} &= 0 \tag{15} \\
\omega B_{1y} + k_z B_0 v_{1y} &= 0 \tag{16} \\
\omega B_{1z} - k_x B_0 v_{1x} &= 0 \tag{17}
\end{align*}
\]
where $c_s^2 = \gamma p_0/p_0$. Equations (13) and (16) for the variables $v_{1y}$ and $B_{1y}$ are decoupled from the rest and describe Alfven waves. The rest of the equations for $p_1$, $p_1$, $T_1$, $B_{1x}$, $B_{1z}$, $v_{1x}$ and $v_{1z}$ describe damped magnetoacoustic waves. For elimination of the variables $p_1$, $p_1$, $T_1$, $B_{1x}$, $B_{1z}$, $v_{1x}$ and $v_{1z}$, we have

\[
\begin{pmatrix}
0 & \omega & 0 & 0 & 0 & 0 & -\rho_0 k_x \\
-k_x & 0 & 0 & \frac{B_0}{4\pi} k_z & -\frac{B_0}{4\pi} k_x & (\omega \rho_0 + \frac{i\eta_0 k_x^2}{3}) & -\frac{2i\eta_0}{3} k_x k_z \\
-k_z & 0 & 0 & \omega & 0 & -\frac{2i\eta_0}{3} k_x k_z & (\omega \rho_0 + \frac{4i\eta_0}{3} k_z^2) \\
0 & 0 & 0 & -k_x & 0 & 0 & 0 \\
0 & 0 & 0 & \omega & 0 & k_z B_0 & 0 \\
i\omega & 0 & -\frac{1}{\rho_0} - \frac{1}{\rho} & 0 & 0 & -i\rho_0 c_s^2 k_x & -i\rho_0 c_s^2 k_z \\
\end{pmatrix} = 0
\]

### 3 Basic equations of KKS

The basic equations used by KKS are

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0
\]

\[
\rho \frac{D \vec{v}}{Dt} = -\nabla p + \frac{1}{4\pi} (\nabla \times \vec{B}) \times \vec{B} - \nabla \Pi
\]

\[
\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B})
\]

\[
\frac{Dp}{Dt} - \frac{\gamma p Dp}{\rho Dt} = (\gamma - 1) [Q_{th} + Q_{vis} - Q_{rad}]
\]

\[
p = \frac{2\rho k_B T}{m_p}
\]

Comparison of the two sets [equations (1) – (5) and equations (21) – (25)] show that there is difference on the left side in the induction and energy equations. For this set of equations (21) – (25), after going through the same procedure as discussed in the preceding section, we get the equations (KKS)

\[
\omega p_1 - \rho_0 (k_x v_{1x} + k_z v_{1z}) = 0
\]

\[
\omega p_0 v_{1x} - k_x p_1 - \frac{B_0}{4\pi} (k_x B_{1z} - k_z B_{1x}) + \frac{i\eta_0}{3} (k_x^2 v_{1x} - 2 k_x k_z v_{1z}) = 0
\]

\[
\omega p_0 v_{1y} + \frac{B_0}{4\pi} (k_z B_{1y}) = 0
\]

\[
\omega p_0 v_{1z} - k_z p_1 + \frac{i\eta_0}{3} (4k_z^2 v_{1z} - 2 k_x k_z v_{1z}) = 0
\]

\[
\omega B_{1x} + k_x B_0 v_{1x} = 0
\]

\[
\omega B_{1y} + k_y B_0 v_{1y} = 0
\]

\[
\omega B_{1z} - k_z B_0 v_{1z} = 0
\]

\[
i\omega p_1 - \rho_1 \omega c_s^2 - (\gamma - 1) \kappa || k_z^2 T_1 = 0
\]

\[
\frac{p_1}{p_0} - \frac{\rho_1}{\rho_0} - \frac{T_1}{T_0} = 0
\]
Obviously, equation (18) is different from (33). DP started from the set of equations (1)–(5), but the results reported by them are as given here in equations (26)–(34). It appears that DP did not derive their equations (11)–(19), but adopted directly from KKS. It is noticeable that the present equations (11)–(19) are not available in the paper of PKS. Equations (28) and (31) for the variables \( v_{1y} \) and \( B_{1y} \) are decoupled from the rest and describe Alfvén waves. The rest of the equations for \( p_1, \rho_1, T_1, B_{1x}, B_{1z}, v_{1x} \) and \( v_{1z} \) describe damped magnetoacoustic waves. For elimination of the variables \( p_1, \rho_1, T_1, B_{1x}, B_{1z}, v_{1x} \) and \( v_{1z} \), we have

\[
\begin{bmatrix}
0 & \omega & 0 & 0 & 0 & 0 & 0 \\
-k_x & 0 & 0 & \frac{\partial \rho}{\partial \eta} k_z & -\frac{\partial \rho_0}{\partial \eta} k_z & (\omega p_0 + \frac{4 i \eta}{3} k_x^2) & -\frac{2 i \eta}{3} k_x k_z \\
-k_z & 0 & 0 & 0 & 0 & -\frac{2 i \eta}{3} k_z k_x & (\omega p_0 + \frac{4 i \eta}{3} k_z^2) \\
0 & 0 & 0 & \omega & 0 & k_z B_0 & 0 \\
0 & 0 & 0 & 0 & \omega & -k_x B_0 & 0 \\
i\omega & -i \omega c_s^2 & -i(1 - 1) \kappa || k_z^2 & 0 & 0 & 0 & 0 \\
\frac{1}{p_0} & -\frac{1}{p_0} & -1 & 0 & 0 & 0 & 0
\end{bmatrix} = 0
\]

(35)

\section{Discussion and conclusion}

In order to resolve the controversy, now, we are left with two determinants (20) and (35), which correspond to the sets of equations used by DP (also PKS) and KKS, respectively. It is interesting to find out that both these determinants reduce to the following common determinant:

\[
\begin{bmatrix}
k_x \omega & \omega^2 \rho_0 + i \frac{\omega \eta}{3} k_x^2 - v_A^2 \rho_0 k_x^2 & -\frac{2 i \eta}{3} k_x k_z \\
k_z & 0 & \omega p_0 + \frac{4 i \eta}{3} k_z^2 \\
c_0 \omega - i \omega^2 & c_0 p_0 k_z - i \rho_0 c_s^2 k_x \omega & c_0 p_0 k_z - i \rho_0 c_s^2 k_z \omega
\end{bmatrix} = 0
\]

where \( c_0 = (\gamma - 1) \kappa || k_z^2 T_0 / p_0 \) and \( v_A = B_0 / \sqrt{4 \pi \rho_0} \). This determinant can be solved to get the following dispersion relation.

\[
\omega^5 + i A \omega^4 - B \omega^3 - i C \omega^2 + D \omega + i E = 0
\]

where

\[
A = c_0 + \frac{\eta}{3 \rho_0} (k_x^2 + 4 k_z^2) \\
B = \frac{c_0 \eta}{3 \rho_0} (k_x^2 + 4 k_z^2) + (c_s^2 + v_A^2) k_x^2 \\
C = \frac{3 \rho_0}{p_0} c_s^2 k_x^2 k_z^2 + \frac{c_0 p_0 k_x^2}{\rho_0} + v_A^2 c_0 k_x^2 + \frac{4 \eta \rho_0 v_A^2 k_x^2 k_z^2}{3 \rho_0} \\
D = \frac{3 c_0 \eta \rho_0 p_0 k_x^2 k_z^2}{\rho_0} + \frac{4 \eta \rho_0 v_A^2 k_x^2 k_z^2}{3 \rho_0} + v_A^2 c_s^2 k_z^2 k_x^2 \\
E = \frac{v_A^2 c_0 p_0 k_x^2 k_z^2}{\rho_0}
\]

Hence, the dispersion relation obtained from both sets of the basic equations of DP (also PKS) and KKS is a fifth order polynomial in \( \omega \). The coefficients obtained here are the same as obtained
by KKS. It may finally be resolved that the dispersion relations derived by KKS for the basic set of equations (21) – (25) is correct.

Though both the sets of basic equations produce a common dispersion relation, some points regarding the discrepancy between the induction and energy equations of the two sets can be noted as the following.

In the induction equation (3), the term $\vec{D} \vec{B} / Dt$ can be expressed as

$$\frac{\vec{D} \vec{B}}{Dt} = \frac{\partial \vec{B}}{\partial t} + (\vec{v} \cdot \nabla) \vec{B}$$

Linearization of this equation gives

$$\frac{\vec{D} \vec{B}_1}{Dt} = \frac{\partial \vec{B}_1}{\partial t} + (\vec{v}_1 \cdot \nabla) \vec{B}_1$$

The second term on right side can be dropped as it is a product of two perturbations. Thus, we have

$$\frac{\vec{D} \vec{B}_1}{Dt} = \frac{\partial \vec{B}_1}{\partial t}$$

and the induction equation in the two sets give the same final equations.

In the energy equation (4), the term $\gamma \rho (\nabla \cdot \vec{v})$ reduces to $- i \rho_0 c_s^2 (k_x v_{1x} + k_z v_{1z})$ whereas in the equation (24), the term $-(\gamma p/\rho)(D\rho/Dt)$ reduces to $-i \rho_1 \omega c_s^2$. However, after the calculations, no difference is found in the expression for dispersion relation.

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