A new definition of analytic adjoint ideal sheaves via the residue functions of log-canonical measures I

TSZ ON MARIO CHAN

In the memory of Prof. Jean-Pierre Demailly

Abstract. A new definition of analytic adjoint ideal sheaves for quasi-plurisubharmonic (quasi-psh) functions with only neat analytic singularities is studied and shown to admit some residue short exact sequences which are obtained by restricting sections of the newly defined adjoint ideal sheaves to some unions of $\sigma$-log-canonical ($\sigma$-lc) centres. The newly defined adjoint ideal sheaves induce naturally some residue $L^2$ norms on the unions of $\sigma$-lc centres which are invariant under log-resolutions. They can also describe unions of $\sigma$-lc centres without the need of log-resolutions even if the quasi-psh functions in question have more general singularities. Furthermore, their relations between the algebraic adjoint ideal sheaves of Ein–Lazarsfeld as well as those of Hacon–McKernan are described in order to illustrate their role as a (potentially finer) measurement of singularities in the minimal model program. In the course of the study, a local $L^2$ extension theorem is proven, which shows that holomorphic sections on any unions of $\sigma$-lc centres can be extended holomorphically to some neighbourhood of the unions of $\sigma$-lc centres with some $L^2$ estimates. The proof does not rely on the techniques in the Ohsawa–Takegoshi-type $L^2$ extension theorems.

1. Introduction

1.1. A brief account on the background of adjoint ideal sheaves. The algebraic version of the adjoint ideal sheaf $\text{Adj}_{(X,S)}(D)$ arises when one tries to restrict the multiplier ideal sheaf $\mathcal{I}_X(D)$ of some $\mathbb{Q}$-divisor $D$ on an ambient projective manifold $X$ (or, more generally, normal variety) to a (possibly singular) hypersurface $S$, which does not lie in the support of $D$, and compares it with the multiplier ideal sheaf $\mathcal{I}_S(D|_S)$ of the restriction of that divisor $D|_S$ on $S$ (see [33, §9.3.E] and [41, Prop. 2.4]). According to Lazarsfeld ([33, §9.3.E]), it first appeared in the work of Vaquié ([42]) and was later rediscovered in the work of Ein and Lazarsfeld ([13]). That it fits into the short exact sequence induced by the restriction map, namely,

\[ 0 \to \mathcal{I}_X(D) \otimes \mathcal{O}_X(-S) \to \text{Adj}_{(X,S)}(D) \to \iota_*\nu_*\mathcal{I}_S(\nu^*\iota^*D) \to 0, \]

The author would like to thank Young-Jun Choi for his support and encouragement on publishing this work. His thanks also go to Chen-Yu Chi for drawing his attention to this topic during his stay at the NCTS, and to an anonymous referee for correcting a reference in an early version of this paper. The theoretical basis of this work started to take shape when the author visited Jean-Pierre Demailly in Institut Fourier, whom the author feels so indebted to and will never be able to repay. This work was partly supported by the National Research Foundation (NRF) of Korea grant funded by the Korea government (No. 2018R1C1B3005963 and No. 2023R1A2C1007227).
where \( \nu : \tilde{S} \to S \) is any resolution of singularities and \( \iota : S \hookrightarrow X \) is the natural inclusion, is originally a defining property of such kind of ideal sheaves. In [13], it is used as a measurement of the singularities on \( S \) (when \( D = 0 \)). Its another use can be found in the study of pluricanonical systems in the works of Takayama ([41]) and Hacon and M\"{e} Kernan ([25])\(^1\). In these works, its role is more or less an auxiliary object to facilitate the use of the associated long exact sequence and Nadel vanishing theorem to guarantee that global sections of \( \mathcal{S}_X(D) \) (after tensored with some line bundle with sufficient positivity) can be extended to some global sections of \( \mathcal{I}_X(D) \).

The adjoint ideal sheaves introduced in both [13] and [25] are defined also for \( S \) being a reduced divisor with simple normal crossings (snc)\(^2\) such that \( \text{supp } D \) contains no lc centres of \((X, S)\), i.e. no intersection of any number of irreducible components of \( S \) is contained in \( \text{supp } D \). While the two are the same when \( S \) is a disjoint union of smooth hypersurfaces (and \( X \) is smooth), they are different for more general \( S \) (see eq[6.1.3] and eq[6.1.2] for their definitions). Indeed, it follows from their respective definitions that the Ein–Lazarsfeld adjoint ideal sheaf, \( \text{Adj}^{(X,S)}_{\text{EL}}(D) \), is trivial if and only if \((X, S + D)\) is purely log-terminal (plt) with \( |D| = 0 \) (cf. [40, Remark 1.3 (2)]), while the Hacon–M\"{e} Kernan adjoint ideal sheaf, \( \text{Adj}^{(X,S)}_{\text{HM}}(D) \), is trivial if and only if \((X, S + D)\) is divisorially log-terminal (dlt) with \( |D| = 0 \) (see [25, Lemma 4.3 (1)]). (Readers are referred to [32, Def. 2.8] for various notions of singularities in the minimal model program. Note that the definition of “plt” here follows that in [32, Def. 2.8] (in which coefficients of \( D \) are assumed to be \( \leq 1 \) and only discrepancies of \textit{exceptional divisors} have to be \( > -1 \)) instead of [40, Def. 1.1 (ii)] (in which discrepancies of \textit{all} divisors not dominating components of \( S \) have to be \( > -1 \)). In this sense, \((X, S + D)\) being plt alone is not sufficient to guarantee \( \text{Adj}^{(X,S)}_{\text{EL}}(D) = \mathcal{O}_X \).

Further research and applications on these algebraic versions of adjoint ideal sheaves include works of Takagi ([40]) and Eisenstein ([12]), in which the generalisation of \( \text{Adj}^{(X,S)}_{\text{EL}}(D) \) with the reduced divisor \( S \) replaced by a reduced closed subscheme of pure codimension (possibly \( > 1 \)) is studied. Following the spirit of [41] and [25], Ein and Popa ([14]) consider \( \text{Adj}^{(X,S)}_{\text{HM}}(\sigma^c) \), where \( S \) is a reduced snc divisor and \( \sigma^c \) is the formally exponentiated ideal sheaf of a coherent ideal sheaf \( \sigma \) by a real number \( c \geq 0 \), and, among other things, improve the vanishing and extension statements in [25, Lemma 4.4 (2) and (3)] via the use of the corresponding short exact sequence of \( \text{Adj}^{(X,S)}_{\text{HM}}(\sigma^c) \) induced from a restriction map (in a slightly different form from the one shown above) and an induction on the number of components of \( S \) as well as the dimension (see [14, Thm. 2.9 and Prop. 4.1]).

For the analytic version, Guenancia proposes an analytic definition of the adjoint ideal sheaves in [23], which is later generalised by Dano Kim ([28]). In the case where the relevant subvariety \( S \) is a \textit{smooth hypersurface} whose associated line bundle has a canonical section \( \sigma \), for any plurisubharmonic (psh) function \( \varphi \) on a complex manifold \( X \) such that \( S \not\subset \varphi^{-1}(-\infty) \), their adjoint ideal sheaf can be given as

\[
\text{Adj}^{(X,S)}_{\text{EL}}(\varphi) = \bigcup_{\lambda > 1} \mathcal{I}_X(\lambda \varphi + \phi^s + \log |\psi^s|^\sigma) + 1 ,
\]

where \( \phi^s := \log |\sigma|^2 \) and \( \psi^s := \phi^s - \varphi^s_{\text{sm}} \) for some smooth potential \( \varphi^s_{\text{sm}} \) of \( S \) (see Notations 2.1.2 and 2.1.3) and \( \mathcal{I}_X(\varphi) \) denotes the multiplier ideal sheaf of a potential \( \varphi \) on \( X \) (see

\(^1\)The adjoint ideal sheaf in this paper is indeed named as “multiplier ideal sheaf” (see [25, Def.-Lemma 4.2]). It is eventually called “adjoint ideal sheaf” in [14, Def. 2.4].

\(^2\)Indeed Ein and Lazarsfeld do not require \( S \) to have only snc in [13, Prop. 3.1].
Section 2.2). It is proved that when $\varphi$ is obtained from some algebraic data (like the $\mathbb{Q}$-divisor $D$ or the formally exponentiated ideal sheaf $a^\varphi$ above), which has only neat analytic singularities (see Section 2.2 item (1) for the definition) such that the polar set does not contain $S$, their analytic version of adjoint ideal sheaves (with $\sigma \in (0,1]$) coincides with the algebraic version. (Note that the Ein–Lazarsfeld and Hacon–M^cKernan versions of adjoint ideal sheaves coincide when $X$ is smooth and $S$ is a disjoint union of smooth hypersurfaces.) Moreover, when $e^{\varphi}$ is locally Hölder continuous (thus including the case when $\varphi$ has only neat analytic singularities) and when the solution to the strong openness conjecture on multiplier ideal sheaves (see [22] or [26]) is taken into account, their adjoint ideal sheaves also fit into the corresponding short exact sequence induced from restriction as in the algebraic case (with $\mathcal{I}_X(D)$ replaced by $\mathcal{I}_X(\varphi)$ and $\mathcal{I}_S(\nu^*t^*D) = \mathcal{I}_S(D|_S)$ by $\mathcal{I}_S(\varphi|_S)$), which in turn implies the coherence of the analytic adjoint ideal sheaves. While the well-definedness of the residue morphism $\text{Adj}^\sigma(\varphi) \to \mathcal{I}_S(\varphi|_S)$ is guaranteed by the assumption that $e^{\varphi}$ being locally Hölder continuous, the surjectivity of the residue morphism relies on the $L^2$ estimates from the Ohsawa–Takegoshi–Manivel $L^2$ extension theorem (see [35] or [8]) or its variants. Even at the moment of writing, no Ohsawa–Takegoshi-type $L^2$ extension theorem is applicable for proving the surjectivity of the residue morphism when the subvariety $S$ is an snc divisor with intersecting components (due to the non-integrable singularities on the Ohsawa measure). That is why $S$ is mostly assumed to be a smooth hypersurface in the expositions in [23] and [28].

More recent research on the analytic adjoint ideal sheaves that the author is aware of includes the works of Qi’an Guan and Zhenqian Li ([20], which shows that existence of $\varphi$ such that the corresponding Guenancia’s version of adjoint ideal sheaf is not coherent), Zhenqian Li ([34], which generalises Guenancia’s adjoint ideal sheaf, in a way different from the current article, so that $S$ can be locally a complete intersection, and obtains a short exact sequence induced from restriction via the Ohsawa–Takegoshi–Manivel $L^2$ extension theorem) and Dano Kim and Hoseob Seo ([30], which considers Guenancia’s adjoint ideal sheaf with $S$ being a singular hypersurface).

1.2. The main results of this article. Let $X$ be a complex manifold, $L$ a holomorphic line bundle equipped with a singular metric $e^{-\varphi_L}$, and $\psi \leq -1$ a global function on $X$ such that $\varphi_L + m\psi$ are locally differences of quasi-psh functions with neat analytic singularities for all $m \in \mathbb{R}$. Let $m = m_1$ be the only jumping number of the family $\{\mathcal{I}_X(\varphi_L + m\psi)\}_{m \in [m_0, m_1]}$ on the interval $(m_0, m_1)$, where $0 \leq m_0 < m_1$ (be warned that $m_1$ can be any jumping number of the family in general and need not be the smallest one). After a suitable normalisation, assume that $m_1 = 1$ and, for most cases, $m_0 = 0$

\[3\] The index $\sigma$ in the current notation $\text{Adj}^\sigma(\varphi) \to \mathcal{I}_S(\varphi|_S)$ is off by 1 from the one given in [28] so that it is easier to compare with the new version of analytic adjoint ideal sheaves introduced in next section. For $S$ being a smooth hypersurface, Guenancia ([23]) gives the definition with $\sigma = 1$ while Dano Kim ([28]) suggests to allow any $\sigma > 0$. For $S$ being a reduced snc divisor, the adjoint ideal sheaf considered by Guenancia in [23], denoted by $\text{Adj}^G(\varphi)$, is $\text{Adj}^1(\varphi)$ but having the function $\psi_S$ replaced by $\Psi_G$, which is locally of the form $\Psi_G = \prod_{j=1}^n \log |z_j|^2$ (while $\psi_S$ is locally of the form $\sum_{j=1}^n \log |z_j|^2 + O(1)$). It is easy to check that $c|\psi_S|^2 \lesssim \log \Psi_G^2 \lesssim \log C|\psi_S|^{\sigma+1}$ (see Notation 2.1.5) on a small polydisc centred at the origin for some constants $c, C > 0$, so $\text{Adj}^G(\varphi) \subset \text{Adj}^G(\varphi) \subset \text{Adj}^G(\varphi)$.

4Guenancia also claims in [23, Prop. 2.11] that his analytic adjoint ideal sheaf coincides with the Ein–Lazarsfeld algebraic adjoint ideal sheaf even when $S$ is a reduced snc divisor. See Example 6.3.1 for a counter example to this claim. See also the Erratum [24] by Guenancia.
Assume also that \( \varphi_L + m_1 \psi = \varphi_L + \psi \) is quasi-psh on \( X \). The subvariety \( S \) playing the role in previous section is now given as the scheme-theoretic difference between the subvarieties associated to the multiplier ideal sheaves \( J_X(\varphi_L) \) and \( J_X(\varphi_L + \psi) \) (see Section 2.2 item (5)). The subvariety \( S \) such defined is reduced (see \([11, \text{Lemma } 4.2]\)). In \([5]\) and \([4]\), the author introduces a sequence of lc-measures \( \{dlc^\sigma_{\varphi_L}([\psi])\}_{\sigma \in \mathbb{N}} \) and their corresponding residue functions \( \Phi^\sigma_{\varphi_L} \) (see Section 2.4) which provide, at least in the case when the reduced subvariety \( S \) is an snc divisor, a natural way to define the residue morphism when restriction from \( X \) to \( lc^\sigma_X(S) \), the union of lc centres of \( (X, S) \) of codimension \( \sigma \) (or \( \sigma \)-lc centres for short; see Definition 5.2.4 and Remark 5.2.5 for the usage of the terminology when \( S \) is not an snc divisor), is considered for each natural number \( \sigma \geq 1 \) (see Theorem 4.1.2). This gives a good reason to consider the following new version of analytic adjoint ideal sheaves.

**Definition 1.2.1.** Given any integer \( \sigma \geq 0 \) and the family \( \{ J_X(\varphi_L + m\psi) \}_{m \in [0, 1]} \) with a jumping number at \( m = 1 \), the (analytic) adjoint ideal sheaf \( J(\varphi_L; \psi) := J_{X, \sigma}(\varphi_L; \psi) \) of index \( \sigma \) of \( (X, \varphi_L, \psi) \) is the sheaf associated to the presheaf over \( X \) given by

\[
\bigcap_{\varepsilon > 0} J(\varphi_L + \psi + \log(|\psi|^\sigma (\log|e^\psi|)^{1+\varepsilon}))(V)
\]

for every open set \( V \subset X \). Then, its stalk at each \( x \in X \) can be described as

\[
J_{x}(\varphi_L; \psi) = \left\{ f \in O_{X,x} \mid \exists \text{ open set } V_x \ni x \, , \, \varepsilon > 0 \, , \, \frac{|f|^2 e^{-\varphi_L - \psi}}{|\psi|^\sigma (\log|e^\psi|)^{1+\varepsilon}} \in L^1(V_x) \right\}.
\]

In studies involving restriction of a system (for example, \((X, \varphi)\) for some psh potential \( \varphi \), \((X, a^\sigma)\) for some ideal \( a \), or \((X, D)\) for some \( \mathbb{Q} \)-divisor \( D \)) to some lc centres, it is usual to assume that the support of the system (i.e. \( \varphi^{-1}(-\infty) \), the zero locus of \( a \) or \( \text{supp } D \) respectively in regard to the examples above) does not contain any of the lc centres (see, for example, \([15, \text{Thm. } 6.1]\) or \([14, \S 2.2]\)). In the current setup, thanks to the assumption on the jumping number, after passing to some log-resolution of \((X, \varphi_L, \psi)\) (possible under the assumption that \( \varphi_L \) and \( \psi \) having only neat analytic singularities), there is an effective (\( \mathbb{Z} \)-)divisor \( S_0 \) with \( \text{supp } S_0 \subset S \) such that

\[
\varphi_L + \psi = \varphi + \phi_{S_0} + \phi_S,
\]

where \( \phi_{S_0} := \log|s_0|^2 \) (\( \phi_S := \log|s|^2 \)), a potential defined from a canonical section \( s_0 \) of \( S_0 \) (\( s \) of \( S \)), and \( \varphi \) is a quasi-psh potential which is locally integrable around general points on \( S \) (see \( \text{eq 2.3.1} \) in Section 2.3). It follows automatically from the snc assumption on \( \varphi_L \) (hence \( \varphi \); it essentially means that the polar set of \( \varphi_L \) as well as \( \varphi \) is a divisor having snc with \( S \); see Snc assumption 2.3.7) that \( \varphi^{-1}(-\infty) \) contains no lc centres of \((X, S)\). The system \((X, \varphi_L, \psi)\) indeed describes a slightly more general setting than in other studies.

\(^5\)Note that the existence of jumping numbers is guaranteed when \( X \) is compact or is a relatively compact domain in some ambient manifold. Note also that the set of jumping numbers has no accumulation points when \( \varphi_L \) and \( \psi \) have only neat analytic singularities.

\(^6\)The notation \( J_{x}(\cdot) \) is chosen so as to be in line with the current choice of the typeface of the notation of multiplier ideal sheaves \( J(\cdot) \) and to signify that it denotes an adjoint ideal sheaf. The author also prefers to use a different notation from the algebraic and Guenancia’s versions of adjoint ideal sheaves because of the subtle difference indicated in Theorem 6.2.3.
One can compare the definitions of $\mathcal{J}_\sigma(\varphi_L; \psi_S)$ and $\text{Adj}^\sigma_{(X,S)}(\varphi_L)$, which are apparently similar, when $S$ is a disjoint union of smooth hypersurfaces. The union over the parameter $\lambda$ in the definition of $\text{Adj}^\sigma_{(X,S)}(\varphi_L)$ in (eq.1.1.1) is needed to compensate for the lack of the openness property for adjoint ideal sheaves, in the sense that, for any holomorphic function $f$, $\frac{|f|^2 e^{-|f|^2 \varphi_L - \psi_S}}{|\varphi_L + \psi_S|^{\sigma + 1}} \in L^1_{\text{loc}}$ does not imply that there exists some constant $\lambda > 1$ such that $\frac{|f|^2 e^{-\lambda |f|^2 \varphi_L - \psi_S}}{|\varphi_L + \psi_S|^{\sigma + 1}} \in L^1_{\text{loc}}$ for any $\sigma > 0$ (especially when $(X, \varphi_L, \psi_S)$ is not yet in an snc configuration; see [23, Counterexample on pp.1023]; also cf. Example 6.2.1). With the introduction of the parameter $\lambda$, Guenancia shows that his version of the adjoint ideal sheaves coincides with the algebraic version when $S$ is smooth (see [23, Prop. 2.11] and its erratum [24]; see also Example 6.3.1 and Remark 6.3.2).

Concerning the openness property under the Snc assumption 2.3.7, since $\varphi^{-1}(-\infty)$ contains no lc centres of $(X, S)$, it turns out that $\mathcal{J}_\sigma(\varphi_L; \psi)$ (one may put $\varphi_L = \varphi$ and $\psi = \psi_S$ for a fair comparison) retains the openness property of $\text{Adj}^\sigma_{(X,S)}(\varphi)$ (see Section 3.2) even without the direct limit over $\lambda > 1$ in the definition (although it is different from the openness property of $\text{Adj}^\sigma_{(X,S)}(\varphi_L)$ if $\varphi^{-1}_L(-\infty)$ contains some components of $S$).

Assume that $S$ given above is a reduced divisor (which need not have snc and may have more than one components). Without the parameter $\lambda$ in the definition, the new adjoint ideal sheaves introduced above coincide with the Ein–Lazarsfeld or Hacon–M–Kernan algebraic adjoint ideal sheaves according to the their indices $\sigma$ in “almost all” cases in the following sense.

**Theorem 1.2.2** (see Theorem 6.2.3). **Given an lc pair $(X, S)$ (in which $S$ is a reduced divisor but need not have snc), let $\varphi_a$ be a quasi-psh potential induced from $a^c$, where $a$ is a coherent ideal sheaf on $X$ whose zero locus contains no lc centres of $(X, S)$ in the sense of [32, Def. 4.15], and $c \geq 0$ is a real number. Set also $\psi := \psi_S$ (see Notation 2.1.3). Then, one has, for any $c \geq 0,$

$$\text{Adj}^{\text{EL}}_{(X,S)}(a^c) \subset \mathcal{J}_{X,1}(\varphi_a; \psi) \quad \text{and} \quad \text{Adj}^{\text{HM}}_{(X,S)}(a^c) \subset \mathcal{J}_{X,\sigma_{\text{mlc}}}(\varphi_a; \psi),$$

and there exists a countable discrete set $N := N(a, S) \subset \mathbb{R}_{<0}$ (excluding 0) such that equalities hold for both inclusions for any $c \in \mathbb{R}_{\geq 0} \setminus N$. Note that the integer $\sigma_{\text{mlc}}$ depends on $c$ and is the smallest integer such that $\mathcal{J}_{X,\sigma_{\text{mlc}}}(\varphi_a; \psi) = \mathcal{J}_{X}(\varphi_a + mc\psi),$ where $m^c \in [0, 1)$ such that $\mathcal{J}(\varphi_a + mc\psi)$ is unchanged as $m$ varies within $[m^c, 1].$

More specifically, if, for a given $c \geq 0$, the zero locus of $a$ contains no lc centres of $(X, \varphi_a, \psi)$ defined in Definition-Theorem 1.2.4, then $c \notin N$ and therefore, in such case,

$$\text{Adj}^{\text{EL}}_{(X,S)}(a^c) = \mathcal{J}_{X,1}(\varphi_a; \psi) \quad \text{and} \quad \text{Adj}^{\text{HM}}_{(X,S)}(a^c) = \mathcal{J}_{X,\sigma_{\text{mlc}}}(\varphi_a; \psi).$$

See Example 6.3.3 for an example of the different versions of adjoint ideal sheaves when $(X, S)$ is not an lc pair.

Despite being slightly different from the algebraic version of the adjoint ideal sheaves, the ease of analysis and nice properties of $\mathcal{J}_\sigma(\varphi_L; \psi)$ strongly suggest that the advantage has outweighed the drawback of adopting the new definition (see also Remark 6.2.2). Among those nice properties, the sheaves $\mathcal{J}_\sigma(\varphi_L; \psi)$ for various $\sigma$ give a finer structure via a filtration between $\mathcal{J}(\varphi_L) = \mathcal{J}(\varphi_L + m_0\psi)$ and $\mathcal{J}(\varphi_L + \psi)$ and fit into the expected residue short exact sequence, which are stated as follows (the statements are stated with the normalisation $m_1 = 1$ but for general $m_0 \geq 0$ for the ease of applications in practice).

**Theorem 1.2.3** (Theorem 4.1.2, Remark 4.1.5, Theorem 4.3.1 and Corollary 4.3.2). **Given the notation above (or as in Section 2.2) and under the Snc assumption 2.3.7**
(thus \(\varphi_L\) and \(\psi\) having only near analytic singularities with snc in particular), it follows that

\[(1)\] one has

\[\mathcal{J}_\sigma(\varphi_L; \psi) = \mathcal{J}_X(\varphi_L + m_0\psi) \cdot \mathcal{I}_{X}^{\sigma+1}(S)\]

for all integers \(\sigma \geq 0\), where \(\mathcal{I}_{X}^{\sigma+1}(S)\) is the defining ideal sheaf of the union of \((\sigma+1)\)-lc centres \(\mathcal{I}_{X}^{\sigma+1}(S)\) (with the reduced structure), and, therefore, obtains a filtration

\[\mathcal{J}(\varphi_L + \psi) = \mathcal{J}_0(\varphi_L; \psi) \subset \mathcal{J}_1(\varphi_L; \psi) \subset \cdots \subset \mathcal{J}_{\sigma_{mlc}}(\varphi_L; \psi) = \mathcal{J}(\varphi_L + m_0\psi),\]

where \(\sigma_{mlc}\) is the codimension of a minimal lc centre (mlc) of \((X, S)\);

\[(2)\] the ideal sheaves \(\mathcal{J}_\sigma(\varphi_L; \psi)\) fit into the residue short exact sequence

\[
\begin{array}{c}
0 \longrightarrow \mathcal{J}_{\sigma-1}(\varphi_L; \psi) \longrightarrow \mathcal{J}_\sigma(\varphi_L; \psi) \longrightarrow \text{Res} \mathcal{R}_\sigma(\varphi) \longrightarrow 0,
\end{array}
\]

where \(\mathcal{R}_\sigma(\varphi)\) is a coherent sheaf supported on \(\mathcal{I}_{X}^{\sigma}(S)\) such that, on an open set \(V\) with \(\mathcal{I}_{X}^{\sigma}(S) \cap V = \bigcup_{p \in \mathfrak{C}^\sigma} S_p\), where \(S_p\)'s are the \(\sigma\)-lc centres in \(V\) indexed by \(p \in \mathfrak{C}^\sigma\), one has

\[K_X \otimes \mathcal{R}_\sigma(\varphi)(V) = \prod_{p \in \mathfrak{C}^\sigma} K_{S_p} \otimes \left( S_0^{-1} \otimes S^{-1} \right) \big|_{S_p} \otimes \mathcal{J}_\sigma(\varphi)(S_p^\sigma); \]

\[(3)\] for any \((g_p)_{p \in \mathfrak{C}^\sigma} \in \mathcal{R}_\sigma(\varphi)(\overline{V})\) on a sufficiently small open set \(V\) with \(\mathcal{I}_{X}^{\sigma}(S) \cap V = \bigcup_{p \in \mathfrak{C}^\sigma} S_p^\sigma\) as above, there exists a section \(f \in \mathcal{J}_\sigma(\varphi_L; \psi)(\overline{V})\) such that \(\text{Res}(f) = (g_p)_{p \in \mathfrak{C}^\sigma}\), and, for any \(\varepsilon > 0\),

\[\varepsilon \int_V \frac{|f|^2 e^{-\varphi_L - \psi} \, d\text{vol}_V}{|\psi|^\sigma (\log|e\psi|)^{1+\varepsilon}} \leq C \sum_{p \in \mathfrak{C}^\sigma} \frac{\pi^\sigma}{(\sigma-1)!} \nu_p \int_{S_p^\sigma} |g_p|^2 e^{-\varphi} \, d\text{vol}_{S_p^\sigma},\]

where \(\nu_p\) is the product of the generic Lelong numbers \(\nu(\psi, D_i)\) of \(\psi\) along the irreducible components \(D_i\) of \(S\) containing \(S_p^\sigma\), and \(C > 0\) is a constant depending only "mildly" on \(\varphi\).

Readers are referred to Theorem 4.1.2 (see also Remark 4.1.5), Theorem 4.3.1 and Corollary 4.3.2 for the precise statements. The residue morphism \(\text{Res}\) in (2), if tensored by \(K_X\), is given by the Poincaré residue map (see (eq 4.2.1) or [32, §4.18]). The surjectivity of \(\text{Res}\) in Theorem 1.2.3 (2) (or Theorem 4.3.1) and the estimate in Theorem 1.2.3 (3) (or Corollary 4.3.2) can indeed be regarded as a local \(L^2\) extension theorem with estimates, which does not rely on any Ohsawa–Takegoshi-type \(L^2\) extension theorems. Note that both sides of the estimate in Theorem 1.2.3 (3) can be expressed as the residue functions (see Section 4.1) so that the estimate is read as

\[\mathfrak{F}(\varepsilon) \lesssim C \mathfrak{F}(0)\]

where \(\mathfrak{F}(\varepsilon)\) is the residue norm of \(f\) given respect to the \(\sigma\)-lc measure \(d\text{lc}^\sigma_{\varphi_L}[\psi]\) (see (eq 4.1.3) for a precise definition). The residue norm \(\mathfrak{F}(\varepsilon)\) indeed induces an \(L^2\) norm \(\|\cdot\|_{\mathfrak{F}(\varepsilon)}\) on \(\mathcal{R}_\sigma(\varphi)(\overline{V})\) for any open set \(V \subseteq X\). The latter norm is also referred to as the residue norm on \(\mathcal{R}_\sigma(\varphi)(\overline{V})\) for convenience.

The residue short exact sequence in Theorem 1.2.3 (2) can facilitate inductive arguments. One example can be found in the study of injectivity theorem in [6].
Further results show that the adjoint ideal sheaves \( \mathcal{I}_\sigma(\varphi_L; \psi) \) and the residue norms \( \mathfrak{S}_V^{|\cdot|^2} \) (hence \( \| \cdot \|_{\mathfrak{S}_V^{|\cdot|^2}} \)) exhibit good invariant properties with respect to log-resolutions such that it makes sense to define the residue morphisms \( \text{Res} \) and the residue norms even when \( (X, \varphi_L, \psi) \) is not in an snc configuration, i.e. not satisfying the Snc assumption 2.3.7*. To state the results, suppose \( m = 1 \) is again a jumping number of the family \( \{ \mathcal{I}_X(\varphi + m\psi) \}_{m \in [0,1]} \) and \( S \) is given as before (which is named as the lc locus\(^7\) of the family at jumping number \( m = 1 \) for convenience). Let \( \pi: \tilde{X} \to X \) be any log-resolution of \( (X, \varphi_L, \psi) \), which gives rise to the effective divisors \( E, \bar{R} \) (where \( K_{\tilde{X}/X} \sim E + R \)) and the potentials \( \pi^\ast \varphi_L \). Then let \( \tilde{S}, \tilde{S}_0 \) and \( \tilde{\varphi} \) (the counterparts of \( S, S_0 \) and \( \varphi \)) be defined from the family \( \{ \mathcal{I}_{\tilde{X}}(\pi^\ast \varphi_L + m\pi^\ast \psi) \}_{m \in [0,1]} \) at jumping number \( m = 1 \) (see Section 2.3, Proposition 2.3.1 and the beginning of Section 5). The lc locus \( \tilde{S} \) such defined is a reduced snc divisor in \( \tilde{X} \). Note that \( S \subset \pi(\tilde{S}) \) but the equality need not hold in general, as seen from Example 2.3.4. (An incorrect claim is made in [5, \S 2.1] and an earlier version of this writing. See Proposition 2.3.1 for the correct relation between \( S \) and \( \pi(\tilde{S}) \) and for some sufficient conditions for the two being equal.)

As \( \tilde{S} \) is reduced and has only snc, one can easily describe \( \text{lc}_X^\sigma(\tilde{S}) \), the union of lc centres of \( (\tilde{X}, \tilde{S}) \) of codimension \( \sigma \) in \( \tilde{X} \) in the sense of [32, Def. 4.15]. Notice the \( \mathcal{O}_X \)-isomorphism \( \mathcal{I}_{X,\sigma}(\varphi_L; \psi) \cong \pi_\ast \left( E \otimes \mathcal{I}_{\tilde{X},\sigma}(\pi^\ast \varphi_L; \pi^\ast \psi) \right) \) in (eq 5.1.1) (or \( K_X \otimes \mathcal{I}_{X,\sigma}(\varphi_L; \psi) \cong \pi_\ast \left( K_{\tilde{X}} \otimes R^{-1} \otimes \mathcal{I}_{\tilde{X},\sigma}(\pi^\ast \varphi_L; \pi^\ast \psi) \right) \); see Remark 5.1.7), and set \( \mathcal{R}_{X,\sigma}(\pi^\ast \varphi_L; \pi^\ast \psi) := \mathcal{R}_{\tilde{X},\sigma}(\tilde{\varphi}) \) for the consistency of notations on \( \tilde{X} \) and \( X \). The results on \( \mathcal{I}_{X,\sigma}(\varphi_L; \psi) \) with respect to a log-resolution \( \pi \) can then be stated.

**Definition-Theorem 1.2.4** (Theorem 5.1.2, Definition 5.1.5, Theorem 5.2.1, Definition 5.2.4 and Proposition 5.2.6). Suppose that \( (X, \varphi_L, \psi) \) is given as described in Section 2.2 (so, in particular, \( \varphi_L \) has only neat analytic singularities but \( (X, \varphi_L, \psi) \) may not satisfy the Snc assumption 2.3.7*).

(1) There exists a coherent \( \mathcal{O}_X \)-sheaf \( \mathcal{R}_{X,\sigma}(\varphi_L; \psi) \) and a residue morphism \( \text{Res} \), both are unique up to isomorphisms, such that the sequence
\[
0 \longrightarrow \mathcal{I}_{X,\sigma-1}(\varphi_L; \psi) \longrightarrow \mathcal{I}_{X,\sigma}(\varphi_L; \psi) \xrightarrow{\text{Res}} \mathcal{R}_{X,\sigma}(\varphi_L; \psi) \longrightarrow 0
\]

is exact for all integers \( \sigma \geq 1 \). The residue norm \( \mathfrak{S}_V^{|\cdot|^2} \) induces an \( L^2 \) norm \( \| \cdot \|_{\mathfrak{S}_V^{|\cdot|^2}} \) on \( \mathcal{R}_{X,\sigma}(\varphi_L; \psi)(V) \) for any open set \( V \subseteq X \) such that the monomorphism
\[
\tau: \mathcal{R}_{X,\sigma}(\varphi_L; \psi)(V) \hookrightarrow E \otimes \mathcal{R}_{\tilde{X},\sigma}(\pi^\ast \varphi_L; \pi^\ast \psi)(\pi^{-1}(V))
\]
induced via the residue morphisms is an isometric embedding, where the target space is equipped with the residue norm \( \| \cdot \|_{\mathfrak{S}_{\pi^{-1}(V)}(\tilde{S})} \).

(2) A \( \sigma \)-lc centre of \( (X, \varphi_L, \psi) \) is defined to be an irreducible component of the reduced closed analytic subset \( \text{lc}_X^\sigma(\varphi_L; \psi) \) of \( X \) given by the ideal sheaf
\[
\mathcal{I}_{\text{lc}_X^\sigma(\varphi_L; \psi)} := \operatorname{Ann}_X \left( \frac{\mathcal{I}_{X,\sigma}(\varphi_L; \psi)}{\mathcal{I}_{X,\sigma-1}(\varphi_L; \psi)} \right).
\]

\(^7\)The use of the term “lc locus” differs from the use in some literature. See item (5) in Section 2.2 as well as footnote 10.
It follows that
\[
\text{Ann}_{\sigma_X} \left( \frac{\mathcal{F}_{X,\sigma}(\varphi_L; \psi)}{\mathcal{F}_{X,\sigma-1}(\varphi_L; \psi)} \right) = \text{Ann}_{\sigma_X} \left( \pi_\ast \left( E \otimes \frac{\mathcal{F}_{X,\sigma}(\pi_\ast \varphi_L; \pi_\ast \psi)}{\mathcal{F}_{X,\sigma-1}(\pi_\ast \varphi_L; \pi_\ast \psi)} \right) \right),
\]
which implies that the sheaves $\mathcal{R}_{X,\sigma}(\varphi_L; \psi)$ and $\pi_\ast \left( E \otimes \mathcal{R}_{X,\sigma}(\pi_\ast \varphi_L; \pi_\ast \psi) \right)$ (being different in general) have the same support. Moreover, in general, one has
\[
\text{lc}_X^\sigma(\varphi_L; \psi) \subset \pi(\text{lc}_X^\sigma(\tilde{S})).
\]
If $\tilde{S}_p^\sigma$ is a $\sigma$-lc centre in $\text{lc}_X^\sigma(\tilde{S})$ such that $\pi(\tilde{S}_p^\sigma) \not\subset \text{lc}_X^\sigma(\varphi_L; \psi)$, then $\pi_\ast \frac{f}{s_0} \mid_{\tilde{S}_p^\sigma} = 0$ (or, equivalently, $\text{Res}(f)\mid_{\pi(\tilde{S}_p^\sigma)} = 0$) for all $f \in \mathcal{F}_{X,\sigma}(\varphi_L; \psi)(V)$ and open set $V \subset X$ such that $\pi^{-1}(V) \cap \tilde{S}_p^\sigma \neq \emptyset$, where $s_0$ is a holomorphic canonical section of $\tilde{S}_0$. A $\sigma$-lc centre $\tilde{S}_p^\sigma \subset \text{lc}_X^\sigma(\tilde{S})$ such that $\pi_\ast \frac{f}{s_0} \mid_{\tilde{S}_p^\sigma} \neq 0$ for some $f \in \mathcal{F}_{X,\sigma}(\varphi_L; \psi)(V)$ is said to be $\pi$-supportive.

(3) The index of the mlc of $(X, \varphi_L, \psi)$, denoted by $\sigma_{\text{mlc}} := \sigma_{\text{mlc}}(X, \varphi_L, \psi)$, is defined to be the smallest non-negative integer such that $\text{lc}_X^{\sigma+1}(\varphi_L; \psi) = \emptyset$ for all $\sigma \geq \sigma_{\text{mlc}}$. One then has $S = \text{lc}_X^1(\varphi_L; \psi) \cup \cdots \cup \text{lc}_X^{\sigma_{\text{mlc}}}(\varphi_L; \psi)$.

When $(X, \varphi_L, \psi)$ itself satisfies the Snc assumption 2.3.7, the sheaf $\mathcal{R}_{X,\sigma}(\varphi_L; \psi) = \mathcal{R}_{X,\sigma}(\varphi)$, the residue morphism $\text{Res}$ and the union of $\sigma$-lc centres $\text{lc}_X^\sigma(\varphi_L; \psi) = \text{lc}_X^\sigma(S)$ all reduce back to the corresponding entities considered in Theorem 1.2.3.

Under the definition of lc centres given above, even when the lc locus $S$ of $\{ \mathcal{F}_{X,\sigma}(\varphi_L + m\psi) \}_{m \in [0,1]}$ at jumping number $m = 1$ is a reduced snc divisor satisfying $S = \pi(\tilde{S})$, there may be more lc centres of $(X, \varphi_L, \psi)$ than those of $(X, S)$, as illustrated in Example 6.2.1. On the other hand, when $\Delta$ is an effective $\mathbb{Q}$-divisor on $X$ (which is $\mathbb{Q}$-Cartier as $X$ is smooth) such that $(\tilde{X}, \Delta)$ is an lc pair and $\psi_\Delta := \phi_\Delta - \varphi_\Delta^{sm} \leq -1$ a global function on $X$ given according to Notation 2.1.3, the irreducible components in the disjoint union $\bigsqcup_{\sigma \geq 1} \text{lc}_X^\sigma(0; \psi_\Delta)$ are exactly those lc centres of $(\tilde{X}, \Delta)$ on $X$ in the sense of [32, Def. 4.15 and Thm. 4.16] (see Theorem 1.2.6 and Proposition 5.2.7). These reflect the fact that the definition of lc centres of $(X, \varphi_L, \psi)$ via adjoint ideal sheaves given above incorporates not only the singularity information in $S (\subset \psi^{-1}(\infty))$, but also those in $\varphi_L$, which give some reasons for the difference between the algebraic and analytic versions of adjoint ideal sheaves in Theorem 1.2.2.

The residue exact sequence in Definition-Theorem 1.2.4 for $\sigma = 1$, together with the vanishing of the higher direct images of multiplier ideal sheaves via a log-resolution of $(X, \varphi_L, \psi)$, leads to the following statement.

**Theorem 1.2.5** (Theorem 5.2.8). Under the notation and assumptions in Definition-Theorem 1.2.4, given any 1-lc centre $\tilde{S}_p^1$ of $(X, \varphi_L, \psi)$ (which may not be a divisor), there is one and only one $\pi$-supportive 1-lc centre $\tilde{D} \subset \text{lc}_X^1(\tilde{S})$ of $(\tilde{X}, \tilde{S})$ such that $\tilde{D}$ is mapped into, and thus onto, $\tilde{S}_p^1$ by $\pi$.

This theorem is used in the proofs of Theorems 1.2.2 and 1.2.6. Although it is a different statement from Theorem 1.2.5, readers are referred to the connectedness lemma (see Shokurov [37, Connectedness Lemma 5.7] and Kollár [31, Thm. 17.4]) for a comparison. A brief survey and the recent research regarding the connectedness can be found in the article of Birkar ([3]).
Theorem 1.2.6 (Theorem 6.2.6). Let $\Delta$ be an effective $\mathbb{Q}$-divisor on a complex manifold $X$ and consider the function $\psi_\Delta := \phi_\Delta - \varphi_\Delta^{sm}$ as described in Notation 2.1.3. Then,

1. the pair $(X, \Delta)$ is klt if and only if $\mathcal{I}_{X,0}(0; \psi_\Delta) = \mathcal{O}_X$;
2. the pair $(X, \Delta)$ is plt if and only if $\mathcal{I}_{X,1}(0; \psi_\Delta) = \mathcal{O}_X$ and $\mathcal{I}_{X,0}(0; \psi_\Delta) = \mathcal{I}_{|\Delta|}$, the defining ideal sheaf of $|\Delta|$;
3. the pair $(X, \Delta)$ is lc if and only if $\mathcal{I}_{X,\sigma}(0; \psi_\Delta) = \mathcal{O}_X$ for some (sufficiently large) integer $\sigma \geq 0$ (and it suffices to consider only $\sigma \in [0, n]$).

The residue exact sequence together with Theorem 1.2.6 facilitates another proof of Kollár’s theorem on the inversion of adjunction for the case when the base space $X$ is smooth (see [31, Thm. 17.6]; also cf. [30, Thm. 1.7]).

Theorem 1.2.7 (Inversion of adjunction (Theorem 6.2.7); see [31, Thm. 17.6] and cf. [30, Thm. 1.7]). On a complex manifold $X$, let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$ such that $S := |\Delta|$ is a reduced divisor. Also let $\nu: S^\nu \to S$ be the normalisation of $S$ and $\text{Diff}^*_S \Delta$ the general different (see [32, §4.2]) such that $K_{S^\nu} + \text{Diff}^*_S \Delta \sim_\mathbb{Q} \nu^*(K_X + \Delta)|_S$ (where $\sim_\mathbb{Q}$ means “$\mathbb{Q}$-linearly equivalent to”). Then, $(X, \Delta)$ is plt near $S$ if and only if $(S^\nu, \text{Diff}^*_S \Delta)$ is klt.

The proof can be found in Theorem 6.2.7. This proof illustrates that the residue short exact sequences indeed provide information regarding to the inversion of adjunction in the more general situations.

1.3. Further questions. There are several follow-up questions which the author believes, worth further research.

1. It is natural to ask whether the results here can be carried over to the case where the potential $\varphi_L$ (or maybe even the function $\psi$) has more general singularities. If the result in Theorem 1.2.4 can be carried over, one can then discuss about the $\sigma$-lc centres of $(X, \varphi_L, \psi)$ without the need of any log-resolutions. The original adjoint ideal sheaf ([23] and [28]) in (eq 1.1.1) fits into the corresponding residue short exact sequence (and therefore coherent) when $e^{\varphi}$ is locally Hölder continuous, and [23, Remark 2.17] provides an example such that the residue map $\text{Res}$ is not well-defined. Although it is easy to follow [23, Remark 2.17] to construct examples of $\varphi_L$ such that the residue map $\text{Res}$ on $\mathcal{I}_\sigma(\varphi_L; \psi)$ is not well-defined, it is still legitimate to ask if $\mathcal{I}_\sigma(\varphi_L; \psi)$ could still fit in the residue short exact sequence for more general singularities on $\varphi_L$ than those restricted by the local Hölder continuity condition. This, among other things, will be discussed in the subsequent paper.

2. In [20], Guan and Li construct an example of a psh potential $\varphi$ such that the original adjoint ideal sheaf in (eq 1.1.1) is not coherent. By setting $\varphi_L := \varphi$ (the example in [20, Proof of Thm. 1.1]) and $\psi := \log|z|^2$, it is easy to check that $\{\mathcal{I}(\varphi_L + m\psi)\}_{m \in [0, 1]}$ has a sequence of jumping numbers in $[0, 1]$ accumulating
at \( m = 1 \) around the origin and the adjoint ideal sheaves \( \mathcal{J}_\sigma(\varphi_L; \psi) \) are also not coherent at the origin for all \( \sigma \geq 1 \). A natural follow-up question is whether the existence of accumulating jumping numbers has any relation with the incoherence of \( \mathcal{J}_\sigma(\varphi_L; \psi) \)'s.

In view of the isometric property of the residue norms \( \| \cdot \|_{\mathcal{K}_X^\gamma(\varphi_L; \psi)} \) and the results in Theorem 1.2.4, it is tempting to ask whether the adjoint ideal sheaves \( \mathcal{J}_\sigma(\varphi_L; \psi) \) and the residue norms \( \| \cdot \|_{\mathcal{K}_X^\gamma(\varphi_L; \psi)} \) (or even the residue function \( \delta(\varepsilon)_{X, \sigma} \)) can be defined on a complex space \( X \) which comes with singularities.

If one accepts the “mild” dependence of the constant \( C \) on the potential \( \varphi \) on the local \( L^2 \) extension theorem in Theorem 1.2.3 (3) (or Corollary 4.3.2), the subsequent pressing question is whether the local holomorphic extensions with \( L^2 \) estimates can be used to obtain the desired global extensions with estimates as stated in [4, Conj. 1.1.3].

This paper is organised as follows. Preliminaries are given in Section 2, in which Sections 2.1 and 2.2 explain some less commonly used notations as well as the basic setup and assumptions in this paper, while Section 2.3 describes the snc assumptions on \( X \) and \( \psi \) by passing to a suitable log-resolution. First properties of the new version of adjoint ideal sheaves are studied in Section 3. Section 4 exclusively discusses about the residue short exact sequences that the adjoint ideal sheaves satisfy, which leads to Theorem 1.2.3. Section 5 studies the adjoint ideal sheaves under non-snc scenarios which leads to Definition-Theorem 1.2.4. Theorem 1.2.5 is proved there as an application. Section 6 compares the current version of adjoint ideal sheaves with the algebraic versions and proofs of Theorems 1.2.2, 1.2.6 and 1.2.7 are given.

2. PRELIMINARIES

2.1. Notation. In this paper, the following notations are used throughout.

Notation 2.1.1. Set \( \hbar := \sqrt{-1} / 2\pi \). \(^8\)

Notation 2.1.2. Each potential \( \varphi \) (of the curvature of a metric) on a holomorphic line bundle \( L \) in the following represents a collection of local functions \( \{ \varphi_\gamma \} \) with respect to some fixed local coordinates and trivialisation of \( L \) on each open set \( V_\gamma \), in a fixed open cover \( \{ V_\gamma \} \) of \( X \). The functions are related by the rule \( \varphi_\gamma = \varphi_{\gamma'} + 2\text{Re} \ h_{\gamma\gamma'} \) on \( V_\gamma \cap V_{\gamma'} \), where \( e^{h_{\gamma\gamma'}} \) is a (holomorphic) transition function of \( L \) on \( V_\gamma \cap V_{\gamma'} \) (such that \( s_\gamma = s_{\gamma'} e^{h_{\gamma\gamma'}} \), where \( s_\gamma \) and \( s_{\gamma'} \) are the local representatives of a section \( s \) of \( L \) under the trivialisations on \( V_\gamma \) and \( V_{\gamma'} \) respectively). Inequalities between potentials is meant to be the inequalities under the chosen trivialisations over open sets in the fixed open cover \( \{ V_\gamma \} \).

Notation 2.1.3. For any prime (Cartier) divisor \( E \), let

- \( \varphi_E := \log |s_E|^2 \), representing the collection \( \{ \log |s_{E, \gamma}|^2 \} \), denote a potential (of the curvature of the metric) on the line bundle associated to \( E \) given by the collection of local representations \( \{ s_{E, \gamma} \} \) of some canonical section \( s_E \) (thus \( \varphi_E \) is uniquely defined up to an additive constant);
- \( \varphi_E^{\text{sm}} \) denote a smooth potential on the line bundle associated to \( E \);
- \( \psi_E := \varphi_E - \varphi_E^{\text{sm}} \), which is a global function on \( X \), when both \( \varphi_E \) and \( \varphi_E^{\text{sm}} \) are fixed.

\(^8\)The notation is chosen by mimicking the reduced Planck constant \( \hbar = \hbar / 2\pi \). It can be typeset with the code \{\raisebox{-0.9ex}{$\mathchar'26$}\mkern-6.7mu i\} \{\$\text{\textbackslash mathchar'26}$\} \{\text{mkern-6.7mu i}\}.
All the above definitions are extended to any \( \mathbb{R} \)-divisor \( E \) by linearity. For notational convenience, the notations for a \( \mathbb{R} \)-divisor and its associated \( \mathbb{R} \)-line bundle are used interchangeably. The notation of a line bundle is also abused to mean its associated invertible sheaf.

**Notation 2.1.4.** For any \((n,0)\)-form (or \( K_X \)-valued section) \( f \), define \( |f|^2 := c_n f \wedge \overline{f} \), where \( c_n := (-1)^{n(n-1)/2} (\pi i)^n \). For any hermitian metric \( \omega = \pi i \sum_{1 \leq j, k \leq n} h_{j,k} \, dz^j \wedge \overline{dz^k} \) on \( X \), set \( d \text{vol}_X,\omega := \frac{\omega^n}{n!} \). Set also \( |f|^2 \, d \text{vol}_X,\omega = |f|^2 \). When the integrand in question is supported in some complex coordinate chart of \( X \), \( d \text{vol}_X \) is used to mean the volume form on \( X \) given by the Euclidean metric in that coordinate chart.

**Notation 2.1.5.** For any two non-negative functions \( u \) and \( v \), write \( u \lesssim v \) (equivalently, \( v \gtrsim u \)) to mean that there exists some constant \( C > 0 \) such that \( u \leq Cv \), and \( u \sim v \) to mean that both \( u \lesssim v \) and \( v \gtrsim u \) hold true. For any functions \( \eta \) and \( \phi \), write \( \eta \lesssim_{\log} \phi \) if \( e^\eta \lesssim e^\phi \). Define \( \gtrsim_{\log} \) and \( \lesssim_{\log} \) accordingly.

### 2.2. Basic setup.
Let \((X,\omega)\) be a compact hermitian manifold or a relatively compact domain of some ambient hermitian manifold of complex dimension \( n \). This is served as the background manifold in this article. While only the local properties of the analytic adjoint ideal sheaves are considered in this paper, (relative) compactness is assumed here just to guarantee that the jumping number given below (see item (4) below) is well-defined and there are only finitely many lc centres of \((\varphi_L, \psi)\) described below to be considered.

Let \( \mathcal{I}(\varphi) := \mathcal{I}_X(\varphi) \) be the multiplier ideal sheaf of the potential \( \varphi \) on \( X \) given at each \( x \in X \) by

\[
\mathcal{I}(\varphi)_x := \mathcal{I}_X(\varphi)_x := \left\{ f \in \mathcal{O}_{X,x} \mid \begin{array}{c}
\text{\( f \) is defined on a coord. neighbourhood } V_x \ni x \\
\text{and } \int_{V_x} |f|^2 e^{-\varphi} \, d\text{vol}_{V_x} < +\infty
\end{array} \right\}.
\]

A potential \( \varphi \) is said to have Kawamata log-terminal (klt) singularities on \( X \) if \( \mathcal{I}(\varphi) = \mathcal{O}_X \) on \( X \), and log-canonical (lc) singularities on \( X \) if \( \mathcal{I}(1-\varepsilon \varphi) = \mathcal{O}_X \) for all \( \varepsilon > 0 \) on \( X \).

Throughout this paper, the following are assumed on \( X \):

1. \((L, e^{-\varphi_L})\) is a hermitian line bundle on \( X \) with a singular metric \( e^{-\varphi_L} \) such that \( \varphi_L \) is locally equal to \( \varphi_1 - \varphi_2 \), where each of the \( \varphi_i \)'s is a quasi-plurisubharmonic (quasi-psh) local function with neat analytic singularities, i.e. locally

\[
\varphi_i \equiv c_i \log \left( \sum_{j=1}^N |g_{ij}|^2 \right) \mod \mathcal{C}^\infty,
\]

where \( c_i \in \mathbb{R}_{\geq 0} \) is a constant and \( g_{ij} \in \mathcal{O}_X \) is a local holomorphic function (which comes with its polar ideal sheaf \( \mathcal{P}_{\varphi_L} \), the ideal sheaf generated by all \( g_{ij} \)'s) for each \( i = 1, 2 \) and \( j = 1, \ldots, N \);

2. \( \psi \) is a global function on \( X \) which is also locally a difference of quasi-psh functions with the associated polar ideal sheaf denoted by \( \mathcal{P}_\psi \);

3. \( \psi \leq -1 \) on \( X \) and \( \varphi_L + m\psi \) is locally bounded above on \( X \) for each \( m \in [0, 1] \) (which implies that \( \psi \) and \( \varphi_L + m\psi \) are quasi-psh after some blow-ups as they have only neat analytic singularities).
(4) $1$ is the only jumping number of the family $\{I(\varphi_L + m\psi)\}_{m \in [0,1]}$, i.e.
\[
I(\varphi_L) = I(\varphi_L + m\psi) \supseteq I(\varphi_L + \psi) \quad \text{for all } m \in [0,1)
\]
(the jumping numbers exist on (relatively) compact $X$ by the openness property of multiplier ideal sheaves of (quasi-)psh functions, and accumulation of them as in [21] and [29] does not occur since $\varphi_L$ and $\psi$ have only neat analytic singularities).

(5) $S \subset \psi^{-1}(-\infty)$ is a reduced subvariety defined by the annihilator
\[
I_S := \text{Ann}_{\mathcal{O}_X} \left( \frac{I(\varphi_L)}{I(\varphi_L + \psi)} \right)
\]
(see [11, Lemma 4.2] for the proof that $I_S$ is reduced). Call $S$ to be the lc locus of the family $\{I(\varphi_L + m\psi)\}_{m \in [0,1]}$ at jumping number $m = 1$ for easy reference.

Note that, since both $\varphi_L$ and $\psi$ are locally differences of quasi-psh functions with neat analytic singularities, it is easy to prove, via a log-resolution of the polar sets of $\varphi_L$ and $\psi$ and an application of Fubini’s theorem, that $I(\varphi_L + m\psi)$ is coherent for every $m \in \mathbb{R}$.

2.3. Effects of log-resolution and the snc assumption. When $\varphi_L$ and $\psi$ have only neat analytic singularities, the discussion in [5, §2.1] shows how the setting can be reduced to the snc scenario (the subvariety $S$ and the polar sets of $\varphi_L$ and $\psi$ have only simple normal crossings (snc) against one another) via some suitable log-resolution according to [27]. For the convenience of discussion in the subsequent papers, assume, only in this section and unless otherwise stated, that

- the potential $\varphi_L$ may have singularities worse than neat analytic singularities and
- $\varphi_L + m\psi$ is quasi-psh for each $m \in [\tilde{m}_0, 1]$ so that $\mathcal{I}_X(\varphi_L + m\psi)$ (as well as $\mathcal{I}_X(\pi^*\varphi_L + m\pi^*\psi)$ for some log-resolution $\pi$) is coherent.

While resolving the singularities of $\varphi_L$ as before is not possible when it has more general singularities, Lemma 2.3.5 below suggests an snc assumption on $\mathcal{I}_X(\varphi_L + \psi)$ which may be useful for the study in subsequent papers.

Let $\pi: \tilde{X} \to X$ be any modification (proper generically $1$-to-$1$ holomorphic map) of $X$ such that $\tilde{X}$ is smooth. Assume in what follows that the inverse image $\varphi_L \cdot \varphi_{\tilde{X}}$ of the polar ideal sheaf of $\psi$ is $\mathcal{O}_{\tilde{X}}(-P_{\psi})$ for some effective snc divisor $P_{\psi}$ on $\tilde{X}$. Moreover, the exceptional locus $\text{Exc}(\pi)$ of $\pi$ is an snc divisor and $\text{Exc}(\pi) + P_{\psi}$ also has only snc. Such modification is referred to as a log-resolution of $(X, \psi)$ in what follows. (When $\varphi_L$ has only neat analytic singularities, one can also consider a log-resolution of $(X, \varphi_L, \psi)$ such that $\text{Exc}(\pi) + P_{\varphi_L} + P_{\psi}$ has only snc.) Note that the function $\pi^*\psi$ is then quasi-psh on $\tilde{X}$.

---

9 Systems $(\varphi_L', \psi')$ with more general jumping numbers $0 \leq m_0 < m_1$ can be reduced to the current one by setting $(\varphi_L', \psi) := (\varphi_L' + m_0\psi', (m_1 - m_0)\psi')$. Sometimes it may also be convenient to set $(\varphi_L, \psi) := (\varphi_L' + (m_1 - 1)\psi', \psi')$, but then one should have $I(\varphi_L + m_0\psi) = I(\varphi_L + \psi)$ for all $m \in [m_0, 1]$ with $\tilde{m}_0 := \max\{m_0 - m_1 + 1, 0\}$, and “$I(\varphi_L')$” in most part of this paper should be replaced by “$I(\varphi_L + \tilde{m}_0\psi)$”.

10 The use of the terminology “lc locus” is different from its use in some literature like [1] (in which, when adapted to the current setup, the “lc locus” of $(X, \varphi_L, \psi)$ should mean the subset of $X$ on which $I((1 - \varepsilon)(\varphi_L + \psi)) = \mathcal{O}_X$ for all $\varepsilon > 0$). In more recent literatures, like [2] and [16], mostly only the complement “non-lc locus” or its sibling “non-klt locus” is considered. The author feels safe to use the term “lc locus” in the current context for a more intuitive description.
Let $E_{d\pi}$ be the exceptional divisor defined scheme-theoretically by the ideal sheaf generated by the holomorphic Jacobian of $\pi$ (which is linearly equivalent to the relative canonical divisor $K_{\bar{X}/X} := K_{\bar{X}} - \pi^*K_X$). Consider the decomposition

$$E_{d\pi} = E + R$$

of $E_{d\pi}$ into two effective $\mathbb{Z}$-divisors $E$ and $R$ (with the corresponding canonical holomorphic sections $s_E$ and $s_R$ fixed) such that $R$ is the maximal divisor satisfying

$$\pi^*\varphi_L - \phi_R + \pi^*\psi =: \pi^*\varphi_L + \pi^*\psi \quad \text{being quasi-psh},$$

where $\phi_R := \log|s_R|^2$ (see Notation 2.1.3; set also $\phi_E := \log|s_E|^2$). Notice that, with the divisor $R$ (hence $E$) such defined, the weight $e^{-\pi^*\varphi_L-\pi^*\psi}$ is locally integrable around general points of each component of $E$ \footnote{To see this, consider Siu’s decomposition for the closed positive current $i\partial\bar{\partial}(\pi^*\varphi_L+\pi^*\psi)$ (see, \cite{38, 6} or \cite{10, Ch. III, (8.17)]), which assures that $R$ (hence $E$) is chosen such that each $\nu(\pi^*\varphi_L+\pi^*\psi, D_i)$, the generic Lelong number of $\pi^*\varphi_L+\pi^*\psi$ along a component $D_i$ of $E$, has to be $< 1$. The claim then follows from Skoda’s lemma (see \cite{39, 7} or \cite{9, Lemma (5.6)]).} (so $E$ contains no components of $\bar{S}$ defined below).

The following lemma is an analogous result of (and a correction to\footnote{In \cite{5, \S2.1], it is incorrectly claimed that $S = \pi(\bar{S})$ (or, more precisely, $I_S = \pi_*I_{\bar{S}}$ with the sheaves given by the annihilators as in item (5) in Section 2.2 and in Proposition 2.3.1).}) the one in \cite[\S2.1]{5}.

**Proposition 2.3.1.** For any number $m \in [0, 1]$, one has

$$\mathcal{I}_X(\varphi_L + m\psi) \cdot \mathcal{O}_{\bar{X}} \hookrightarrow \mathcal{I}_X(\pi^*\varphi_L - \phi_E + m\pi^*\psi) \xrightarrow{\otimes s_E} E \otimes \mathcal{I}_X(\pi^*\varphi_L + m\pi^*\psi)$$

and

$$\mathcal{I}_X(\varphi_L + m\psi) \cong \pi_*\mathcal{I}_X(\pi^*\varphi_L - \phi_E + m\pi^*\psi) \cong \pi_*\left(E \otimes \mathcal{I}_X(\pi^*\varphi_L + m\pi^*\psi)\right),$$

where the maps are respectively globally defined $\mathcal{O}_{\bar{X}}$- and $\mathcal{O}_X$-homomorphisms and both maps on the far right-hand-side depend on the choice of $s_E$. The family $\{\mathcal{I}_X(\pi^*\varphi_L + m\pi^*\psi)\}_{m \in [\bar{m}_0, 1]}$ (as well as $\{\mathcal{I}_X(\pi^*\varphi_L - \phi_E + m\pi^*\psi)\}_{m \in [\bar{m}_0, 1]}$) has a jumping number at $m = 1$.

Moreover, when $\varphi_L$ has only neat analytic singularities, there exists a number $m_0 \in (\bar{m}_0, 1)$ sufficiently close to $1$ such that

$$\mathcal{I}_X(\pi^*\varphi_L + m\pi^*\psi) \quad \text{is coherent for all } m \in [m_0, 1]$$

and

$$\mathcal{I}_X(\pi^*\varphi_L + m_0\pi^*\psi) = \mathcal{I}_X(\pi^*\varphi_L + m\pi^*\psi)$$

$$\supseteq \mathcal{I}_X(\pi^*\varphi_L + \pi^*\psi) \quad \text{for all } m \in [m_0, 1].$$

Let $\bar{S}$ be the lc locus of the family $\{\mathcal{I}_X(\pi^*\varphi_L + m\pi^*\psi)\}_{m \in [\bar{m}_0, 1]}$ at jumping number $m = 1$, i.e. the reduced subvariety defined by the ideal sheaf

$$\mathcal{I}_{\bar{S}} := \text{Ann}_{\mathcal{O}_X}\left(\frac{\mathcal{I}_X(\pi^*\varphi_L + m_0\pi^*\psi)}{\mathcal{I}_X(\pi^*\varphi_L + \pi^*\psi)}\right).$$

Then, one has, in general,

$$S \subset \pi(\bar{S})$$

and the equality holds when, for example, either $\psi^{-1}(-\infty) = S$ or $\mathcal{I}_X(\varphi_L + m_0\psi) = \mathcal{O}_X$.\footnote{In \cite[\S2.1]{5}, it is incorrectly claimed that $S = \pi(\bar{S})$ (or, more precisely, $\mathcal{I}_S = \pi_*\mathcal{I}_{\bar{S}}$ with the sheaves given by the annihilators as in item (5) in Section 2.2 and in Proposition 2.3.1).}
**Proof.** For any $f \in \mathcal{O}_X(V)$ on any open set $V \subset X$ and for any $m \in [0,1]$, one has

\[
\int_V |f|^2 e^{-\varphi_L - m\psi} \, d\text{vol}_X \sim \int_{\pi^{-1}(V)} |\pi^* f \cdot s_E|^2 e^{-\pi^* \varphi_L - m\pi^* \psi} \, d\text{vol}_X.
\]

Recalling that $|s_E|^2 = e^{\phi_E}$, this immediately implies the monomorphisms $\mathcal{I}_X(\varphi_L + m\psi) : \mathcal{O}_X \hookrightarrow \mathcal{I}_X(\pi^* \varphi_L - \phi_E + m\pi^* \psi) \hookrightarrow \mathcal{I}_X(\pi^* \varphi_L + m\pi^* \psi)$ and $\mathcal{I}_X(\varphi_L + m\psi) \rightarrow \pi_* \mathcal{I}_X(\pi^* \varphi_L - \phi_E + m\pi^* \psi) \rightarrow \pi_* (E \otimes \mathcal{I}_X(\pi^* \varphi_L + m\pi^* \psi))$ given by $f \mapsto \pi^* f \mapsto \pi^* f \cdot s_E$. On the other hand, following the argument in [9, Prop. (5.8)], as $\pi$ is biholomorphic on the complement of $Z := \pi(\text{Exc}(\pi))$ in $X$ and as $Z \supset \pi(\text{supp} E)$, any section $\tilde{f} \in E \otimes \mathcal{I}_X(\pi^* \varphi_L + m\pi^* \psi) |_{\pi^{-1}(V)}$ gives rise to a holomorphic section $f \in \mathcal{I}_X(\varphi_L + m\psi) |_{V \setminus Z}$ such that $\pi^* f = \frac{\tilde{f}}{s_E}$ on $\pi^{-1}(V \setminus Z)$. It then follows from (1) (where $\pi^* f \cdot s_E$ is replaced by $\tilde{f}$ and $V$ by $V' \setminus Z$ for any $V' \subset V$), together with the fact that $\varphi_L + m\psi$ is locally bounded from above, that $f$ is in $L^2_{\text{loc}}(V)$ (in the unweighted norm). Thus $f$ can be analytically extended across $Z$ via the $L^2$ Riemann extension theorem ([7, Lemma 6.9]) and it follows that $\tilde{f} = \pi^* f \cdot s_E$ on $\pi^{-1}(V)$ by the identity theorem. (Alternatively, one can also use the fact that $Z$ has codimension $\geq 2$ in $X$ to see that $f$ can be extended to $X$; see [17, Satz 7] or [18, 10.6.2] for a generalization.) One therefore obtains the desired isomorphisms $\mathcal{I}_X(\varphi_L + m\psi) \cong \pi_* \mathcal{I}_X(\pi^* \varphi_L - \phi_E + m\pi^* \psi) \cong \pi_* (E \otimes \mathcal{I}_X(\pi^* \varphi_L + m\pi^* \psi))$.

If $\pi^* \varphi_L + m\pi^* \psi$ is quasi-psh on $\tilde{X}$, then its associated multiplier ideal sheaf is already known to be coherent. In the general situation, around every point $y \in \tilde{X}$, viewing $s_R$ as a local defining function of $R$, there is a local isomorphism

\[
\mathcal{I}_X(\pi^* \varphi_L - \phi_E + m\pi^* \psi) \cong \mathcal{I}_X(\pi^* \varphi_L + m\pi^* \psi) \cap \mathcal{O}_X(-R)
\]

As $\mathcal{I}_X(\pi^* \varphi_L + m\pi^* \psi)$ (hence $\mathcal{I}_X(\pi^* \varphi_L + m\pi^* \psi) \cap \mathcal{O}_X(-R)$) is coherent for all $m \in [\tilde{m}_0,1]$ and coherence is a local property, it follows that $\mathcal{I}_X(\pi^* \varphi_L + m\pi^* \psi)$ is coherent for all $m \in [\tilde{m}_0,1]$. (The same argument also verifies that $\mathcal{I}_X(\pi^* \varphi_L - \phi_E + m\pi^* \psi)$ is coherent.) Moreover, the local isomorphism also implies that the openness property of multiplier ideal sheaves of quasi-psh functions still applies so that one can talk about the jumping numbers of the family $\{\mathcal{I}_X(\pi^* \varphi_L + m\pi^* \psi)\}_{m \in [\tilde{m}_0,1]}$ (under the assumption that $X$ is compact or is a relatively compact domain in an ambient manifold). Therefore, as can be seen from (1), if

\[
f \in \mathcal{I}_X(\varphi_L + m\psi)_x \setminus \mathcal{I}_X(\varphi_L + \psi)_x
\]

at some $x \in X$ for all $m < 1$ sufficiently close to 1, then one has

\[
\pi^* f \cdot s_E \in \mathcal{I}_X(\pi^* \varphi_L + m\pi^* \psi)_y \setminus \mathcal{I}_X(\pi^* \varphi_L + \pi^* \psi)_y
\]

for some $y \in \pi^{-1}(x)$ and for all $m < 1$ sufficiently close to 1. That implies that $m = 1$ is a jumping number of the family $\{\mathcal{I}_X(\pi^* \varphi_L + m\pi^* \psi)\}_{m \in [\tilde{m}_0,1]}$, as it is a jumping number of the family $\{\mathcal{I}_X(\varphi_L + m\psi)\}_{m \in [\tilde{m}_0,1]}$. The same argument goes for the family $\{\mathcal{I}_X(\pi^* \varphi_L - \phi_E + m\pi^* \psi)\}_{m \in [\tilde{m}_0,1]}$.

When $\varphi_L$, as well as $\psi$, has only neat analytic singularities, the family $\{\mathcal{I}_X(\pi^* \varphi_L + m\pi^* \psi)\}_{m \in [\tilde{m}_0,1]}$ cannot have accumulating jumping numbers. The openness property of multiplier ideal sheaves of quasi-psh functions then guarantees that there exists a number
$m_0 \in \tilde{m}_0, 1)$ which satisfies the claim in this lemma. (Note that $m_0$ may not be 0 or $\tilde{m}_0$; see [5, footnote 4].)

Finally, the fact that $\tilde{S}$ is reduced follows from the argument in [11, Lemma (4.2)]. Note that one has the vanishing of the first direct image sheaf

$$R^1\pi_*(E \otimes \mathcal{I}_X(\pi_0^*\varphi_L + \pi^*\psi)) = R^1\pi_*(K_X \otimes R^{-1} \otimes \mathcal{I}_X(\pi_0^*\varphi_L + \pi^*\psi)) \otimes K^{-1}_X = 0$$

by the local vanishing theorem ([33, Thm. 9.4.1]) in the algebraic setting or the generalisation of the Grauert–Riemenschneider vanishing theorem by Matsumura ([36, Cor. 1.5]) in the analytic setting (one has to replace $\varphi_L$ in $\mathcal{I}_X(\pi_0^*\varphi_L + \pi^*\psi)$ by $\varphi_L + \varphi^{sm}$ on some local open set in $X$ for some smooth local function $\varphi^{sm}$ so that $\varphi_L + \varphi^{sm} + \psi$ is psh on the local set when applying Matsumura’s result). It then follows immediately from $\mathcal{I}_X(\varphi_L + m\psi) \cong \pi_*(E \otimes \mathcal{I}_X(\pi_0^*\varphi_L + m\pi^*\psi))$ and the exact sequence induced from the inclusion $\mathcal{I}_X(\pi_0^*\varphi_L + \pi^*\psi) \hookrightarrow \mathcal{I}_X(\pi_0^*\varphi_L + m_0\pi^*\psi)$ that

$$\mathcal{I}_S = \text{Ann}_{\mathcal{O}_X}\left(\frac{\mathcal{I}_X(\varphi_L + m_0\psi)}{\mathcal{I}_X(\varphi_L + \psi)}\right) = \text{Ann}_{\mathcal{O}_X}\left(\frac{E \otimes \mathcal{I}_X(\pi_0^*\varphi_L + m\pi^*\psi)}{\mathcal{I}_X(\pi_0^*\varphi_L + \pi^*\psi)}\right).$$

For any coherent sheaf $\mathcal{F}$ on $\tilde{X}$, the zero locus of $\text{Ann}_{\mathcal{O}_X}\mathcal{F}$ is indeed $\text{supp}_X \mathcal{F}$ (see, for example, [18, A.4.5]). Moreover, since $\pi$ is proper with $\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ (thus having connected fibres in particular), it is easy to check that $\pi_*\sqrt{\text{Ann}_{\mathcal{O}_X}\mathcal{F}} = \mathcal{I}_\pi(\text{supp}_X \mathcal{F})$ (the defining ideal sheaf of the analytic set $\pi(\text{supp}_X \mathcal{F})$ in $X$). It is also easy to see that $\text{Ann}_{\mathcal{O}_X}(\pi_*\mathcal{F}) \supset \pi_* \text{Ann}_{\mathcal{O}_{\tilde{X}}} \mathcal{F}$. Note that $\text{Ann}_{\mathcal{O}_{\tilde{X}}}\left(E \otimes \mathcal{J}_X(\pi_0^*\varphi_L + m\pi^*\psi)\right) = \text{Ann}_{\mathcal{O}_{\tilde{X}}}\left(\mathcal{J}_X(\pi_0^*\varphi_L + m\pi^*\psi)\right) = \mathcal{I}_{\tilde{S}}$ (since $E$, as a sheaf, is locally free) which is radical (this is why $\tilde{S}$ is reduced). It follows that $\mathcal{I}_S \supset \pi_*\mathcal{I}_{\tilde{S}} = \mathcal{I}_\pi(\tilde{S})$ and therefore $S \subset \pi(\tilde{S})$.

Since $\tilde{S} \subset \pi^{-1}\psi^{-1}(\tilde{S})$, the above result implies that $S = \pi(\tilde{S})$ when $\psi^{-1}(\tilde{S}) = S$. To see that $\mathcal{I}_X(\varphi_L + m_0\psi) = \mathcal{O}_X$ also implies $S = \pi(\tilde{S})$, notice that the assumption implies that $\mathcal{I}_X(\pi_0^*\varphi_L + m_0\pi^*\psi) = \mathcal{O}_X$, so (using the fact that $\text{Ann}_{\mathcal{O}_{\tilde{X}}}\left(E \otimes \mathcal{J}_X(\pi_0^*\varphi_L + \pi^*\psi)\right) = \mathcal{I}_{\tilde{S}}$)

$$\mathcal{I}_S = \mathcal{I}_X(\varphi_L + \psi) = \pi_*(E \otimes \mathcal{J}_X(\pi_0^*\varphi_L + \pi^*\psi)) = \pi_*\mathcal{I}_{\tilde{S}} = \mathcal{I}_\pi(\tilde{S}),$$

which gives $S = \pi(\tilde{S})$.

\[ \square \]

Remark 2.3.2. When $\varphi_L$ has only neat analytic singularities and $\pi: \tilde{X} \to X$ is a log-resolution of $(X, \varphi_L, \psi)$, it follows from the choice of $E$ (as well as $R$; see footnote 11 on page 13) and Fubini’s theorem that $\mathcal{J}_X(\pi_0^*\varphi_L - \phi_E + m\pi^*\psi) = \mathcal{J}_X(\pi_0^*\varphi_L + m\pi^*\psi)$ for $m \geq 0$. In this case, $\tilde{S}$ can also be defined by $\mathcal{I}_{\tilde{S}} := \text{Ann}_{\mathcal{O}_{\tilde{X}}}\left(\frac{\mathcal{J}_X(\pi_0^*\varphi_L - \phi_E + m\pi^*\psi)}{\mathcal{J}_X(\pi_0^*\varphi_L - \phi_E + \pi^*\psi)}\right)$.

Remark 2.3.3. One may follow [9, Prop. (5.8)] to see that $K_X \otimes \mathcal{J}_X(\varphi_L + m\psi) = \pi_*(K_{\tilde{X}} \otimes \mathcal{J}_X(\pi^*\varphi_L + m\pi^*\psi))$ and obtain

$$\text{Ann}_{\mathcal{O}_X}\left(\frac{\mathcal{J}_X(\varphi_L + m\psi)}{\mathcal{J}_X(\varphi_L + \psi)}\right) = \text{Ann}_{\mathcal{O}_X}\left(\frac{K_{\tilde{X}} \otimes \mathcal{J}_X(\pi^*\varphi_L + m\pi^*\psi)}{\mathcal{J}_X(\pi^*\varphi_L + \pi^*\psi)}\right).$$

However, the lc locus $\tilde{S}'$ of the family $\{\mathcal{J}_X(\pi^*\varphi_L + m\pi^*\psi)\}_{m \in [m_0, 1]}$ at $m = 1$ may contain more components than $\tilde{S}$ defined in Proposition 2.3.1 does. The advantage of considering

\[ ^{13} \text{The argument actually shows that the quotients inside } \text{Ann}_{\mathcal{O}_X}(\cdot) \text{ on both sides are equal.} \]
\((\tilde{X}, \tilde{S})\) over \((\tilde{X}, \tilde{S}')\) as the resolved model of \((X, \varphi_L, \psi)\) is that, when \(\mathcal{I}_X(\varphi_L + m_0 \psi) = \mathcal{O}_X\) and if the pair \((X, S)\) is lc and log-smooth, the codimension of the minimal lc centres (mlc) of \((X, S)\) is preserved by \((\tilde{X}, \tilde{S})\) but may not be so for \((\tilde{X}, \tilde{S}')\) (more generally, when \(\mathcal{I}_X(\varphi_L + m_0 \psi) = \mathcal{O}_X\), the index \(\sigma = \sigma_{\text{mlc}}\) at which the increasing sequence \(\{\mathcal{I}_{X, \sigma}(\varphi_L; \psi)\}_{\sigma \in \mathbb{N}}\) of adjoint ideal sheaves introduced in Definition 2.1 stabilises is equal to the codimension of the mlc of \((\tilde{X}, \tilde{S})\); see Proposition 5.2.7). For example, consider the unit 2-disc \(X := \Delta^2 \subset \mathbb{C}^2\) centred at the origin under the holomorphic coordinate system \((z_1, z_2)\) and take

\[
\varphi_L := 0 \quad \text{and} \quad \psi := \log|z_1|^2 - 1.
\]

Then, \(m = 1\) is a jumping number of the family \(\{\mathcal{I}_{X}(m\psi)\}_{m \in [0,1]}\) and the corresponding lc locus \(S = \{z_1 = 0\}\). Let \(\pi: \tilde{X} \to X\) be the blow-up of the origin with exceptional divisor \(R\). Then the subvariety \(\tilde{S}\) defined as in Proposition 2.3.1 is the proper transform \(\pi_*^{-1}S\) of \(S\), while the lc locus of \(\{\mathcal{I}_{\tilde{X}}(m\pi^*\psi)\}_{m \in [0,1]}\) at the jumping number \(m = 1\) is \(\pi_*^{-1}S + R\). Note that \(\pi_*^{-1}S\) and \(R\) have non-empty intersection. Note also that there exists \(f \in \mathcal{I}_{\tilde{X}}(m\psi)\) such that \(\pi_*f|_{\pi_*^{-1}S + R} \neq 0\) (thus also \(\pi_*f|_R \neq 0\)) for the comparison with Examples 2.3.4 and 5.2.3.

**Example 2.3.4.** This is an example where \(S \neq \pi(\tilde{S})\) (and the codimension of the mlc is not preserved in this case). Let \(X := \Delta^3 \subset \mathbb{C}^3\) be the unit 3-disc centred at the origin under the holomorphic coordinate system \((z_1, z_2, z_3)\). Take

\[
\varphi_L := 0 \quad \text{and} \quad \psi := \log|z_1|^2 + \frac{3}{2} \log|z_2|^2 + \frac{1}{2} \log(|z_2|^2 + |z_3|^2) - 1
\]

and consider the blow-up \(\pi: \tilde{X} \to X\) of the line \(\{z_2 = z_3 = 0\}\) which gives an exceptional divisor \(R\). It follows that

\[
K_{\tilde{X}} \sim \pi^*K_X + R.
\]

Set \(S_j := \{z_j = 0\}\) and let \(\tilde{S}_j\) be the proper transform of \(S_j\) for \(j = 1, 2, 3\). Using the notation in Notation 2.1.3, one then has

\[
\pi^*\psi \sim \log \phi_{\tilde{S}_1} + \frac{3}{2} \phi_{\tilde{S}_2} + 2 \phi_R \quad \text{and} \quad \pi_*^\varphi \varphi_L = -\phi_R.
\]

Set \(m_0 := \frac{2}{3}\). It can be seen that the family \(\{\mathcal{I}_{\tilde{X}}(\pi_*^\varphi \varphi_L + m\pi^*\psi)\}_{m \in [m_0,1]}\) has the only jumping number at \(m = 1\) with the lc locus \(\tilde{S} = \tilde{S}_1 + R\).

On the other hand, one has \(\mathcal{I}_{\tilde{X}}(\psi) = \langle z_1, z_2 \rangle\) since

\[
e^{-\psi} \sim \frac{1}{|z_1|^2 |z_2|^3 (|z_2|^2 + |z_3|^2)^\frac{1}{2}} \gtrsim \frac{1}{|z_1|^2 |z_2|^3} \not\in L^1_{\text{loc}}(X),
\]

which implies that \(\mathcal{I}_{\tilde{X}}(\psi) \subset \langle z_1, z_2 \rangle\), and, for any open set \(V \subset X\) (note that \(\frac{1}{2} \pi^* \log |z_2|^2 \sim \log \frac{1}{2} (\phi_{\tilde{S}_2} + \phi_R)\) and \(\frac{1}{2} \pi^* \log (|z_2|^2 + |z_3|^2) \sim \log \frac{1}{2} \phi_R\)),

\[
\int_V |z_1 z_2|^2 e^{-\psi} \, d\text{vol}_X \sim \int_V \frac{d\text{vol}_X}{|z_2|^2 (|z_2|^2 + |z_3|^2)^\frac{1}{2}} \sim \int_{\pi^{-1}(V)} e^{-\frac{1}{2} (\phi_{\tilde{S}_2} + \phi_R) - \frac{1}{2} \phi_R} e^{\phi_R} \, d\text{vol}_{\tilde{X}} = \int_{\pi^{-1}(V)} e^{-\frac{1}{2} \phi_{\tilde{S}_2}} \, d\text{vol}_{\tilde{X}} < +\infty,
\]

and

which implies \((z_1z_2) \subset \mathcal{I}_X(\psi)\). A similar analysis with 
\(e^{-mv} \sim \frac{1}{|z_1|^\frac{3}{2} |z_2|^2 (|z_2|^2 + |z_3|^2)^\frac{1}{2}}\)
yields \(\mathcal{I}_X(m\psi) = \langle z_2 \rangle \neq \mathcal{O}_X\). Therefore,
the lc locus \(S\) of the family \(\{\mathcal{I}_X(m\psi)\}_{m \in [m_0, 1]}\) at jumping
number \(m = 1\) is \(S = S_1 = \{z_1 = 0\}\), but \(\pi(\bar{S}) = S_1 \cup (S_2 \cap S_3) \supseteq S\).
Note that \(R\) is the “extra” component in \(\bar{S}\) which is not mapped into \(S\) and 
\(\pi^* f|_R \equiv 0\) for all \(f \in \mathcal{I}_X(m\psi)\). See Examples 5.2.2 and 5.2.3 for the similar phenomenon when \(\sigma\)-lc
centres are considered.

For the moment, it is not clear to the author whether there exists a log-resolution
\(\pi: \tilde{X} \to X\) such that the corresponding \(\tilde{S}\) has to be a divisor when \(\varphi_L\) has arbitrary
singularities. However, there is the following lemma.

**Lemma 2.3.5.** If the ideal sheaf \(\mathcal{I}_X(\varphi_L + \psi)\) is principal with an snc generator (i.e. the
generator is locally a monomial in some holomorphic coordinate system), then so is
\(\text{Ann}_{\mathcal{O}_X}(\mathcal{I}_X(\varphi_L + \psi))\).

Proof. This is a property of unique factorisation domains. Suppose, at some \(x \in X\),
that \(\mathcal{I}_X(\varphi_L + \psi)_x = \langle g \rangle\) and that \(\{f_j\}_{j \in J} \subset \mathcal{I}_X(\varphi_L + m\psi)_x\) is a generating set of
\(\mathcal{I}_X(\varphi_L + m\psi)_x\) (coherence assures that the index set \(J\) is finite). Assume that there exist
two distinct generators \(h_1\) and \(h_2\) in a minimal generating set of the annihilator at \(x\) (which
is finitely generated by coherence; see, for example, [18, Prop. A.4.5]). If \(\mu = \gcd(h_1, h_2)\)
(the greatest common divisor of \(h_1\) and \(h_2\), uniquely determined up to a multiple of unit
in \(\mathcal{O}_{X,x}\)) is an element of the annihilator in question, then \(h_1\) and \(h_2\) can be replaced by
\(\mu\) in the generating set of the annihilator, contradicting minimality. Therefore, \(\mu\) is not
in the annihilator. By definition of an annihilator, there exist \(\alpha_{ij} \in \mathcal{O}_{X,x}\) for \(i = 1, 2\) and
\(j \in J\) such that

\[
h_i f_j = g \alpha_{ij}.
\]

Writing \(h_i = h'_i \mu\) for \(i = 1, 2\), this implies that

\[
\frac{h'_1}{h'_2} = \frac{h_1}{h_2} = \frac{\alpha_{1j}}{\alpha_{2j}} \quad \text{for all } j \in J.
\]

Since \(h'_1\) and \(h'_2\) are relatively prime, one has \(h'_i | \alpha_{ij}\) for \(i = 1, 2\) and \(j \in J\), which implies
that \(\mu\) lies in the annihilator as

\[
\mu f_j = g \frac{\alpha_{ij}}{h'_i} \quad \text{for all } j \in J,
\]
giving a contradiction. Therefore, the minimal generating set of the annihilator can have
only one element \(h\), i.e. the annihilator is principal at \(x\).

To prove that \(h\) is an snc generator, write \(J = \{1, \ldots, r\}\) and

\[
h f_j = g \alpha_j \quad \text{for } j = 1, \ldots, r.
\]

Note that this generator \(h\) satisfies the relation \(\gcd(h, \alpha_1, \ldots, \alpha_r) = 1\). Let \(\mu_1 := \gcd(h, \alpha_1)\)
and write \(h = h_1 \mu_1\). One can see that \(h_1 | g\). Further set \(\mu_2 := \gcd(\mu_1, \alpha_2)\) and write
\(h = h_1 h_2 \mu_2\). It follows that \(h_2 | g\), thus \(h_1 h_2 | g\). Proceed inductively to obtain \(\mu_j := \gcd(\mu_{j-1}, \alpha_j)\) and \(h = h_1 \cdots h_j \mu_j\) for \(j = 2, \ldots, r\), where \(h_1 \cdots h_j | g\) for all \(j = 2, \ldots, r\).
Since \(\mu_r\) is a common factor of \(\alpha_1, \ldots, \alpha_r\) and \(h\), it is a unit. This means that \(h | g\). As \(g\)
is an snc generator, \(h\) is also an snc generator of the annihilator at \(x\). It is a generator on
a neighbourhood of \(x\) by the coherence of the annihilator. This completes the proof. \(\Box\)
Remark 2.3.6. The proof of Lemma 2.3.5 does not use the fact whether the family \( \{ \mathcal{I}_X(\varphi_L + m\psi) \}_{m \in [m_0, 1]} \) has any jumping numbers other than \( m = 1 \) or not. It uses only the coherence of \( \mathcal{I}_X(\varphi_L + m\psi) \) and \( \mathcal{I}_X(\varphi_L + \psi) \) (which implies the coherence of the annihilator), so the conclusion still holds true when \( m_0 \) in the statement is replaced by any \( m \in [m_0, 1) \).

In view of Lemma 2.3.5, the following assumption is made to assure that \( \tilde{S} \) is an snc divisor.

Snc assumption 2.3.7. Given \( \varphi_L \) and \( \psi \) on \( X \) described in Section 2.2 (except that \( \varphi_L \) may possibly have more general singularities) such that \( m = 1 \) is a jumping number of the family \( \{ \mathcal{I}_X(\varphi_L + m\psi) \}_{m \in [0, 1]} \), there exists a log-resolution \( \pi: \tilde{X} \to X \) of \( (X, \psi) \) such that
\[
\mathcal{I}_{\tilde{X}}(\pi^* \varphi_L + \pi^* \psi) \quad \text{is principal with an snc generator.}
\]

When \( \varphi_L \) has only neat analytic singularities, such assumption can be achieved via a log-resolution of \((X, \varphi_L, \psi)\), as in [5, §2.1]. Since most of the computations and constructions in this paper are done on \( \tilde{X} \), given the Snc assumption 2.3.7, it is assumed that the log-resolution \( \pi \) provided by the assumption is indeed \( \text{id} \), the identity map. Moreover, the number \( m_0 \) provided by Proposition 2.3.1 is assumed to be 0 by “renormalising” \( \varphi_L \) and \( \psi \) as in footnote 9. This assumption is summarised as follows.

Snc assumption 2.3.7*. Both \( \varphi_L \) and \( \psi \) have only neat analytic singularities such that the polar ideal sheaves \( \mathcal{P}_{\varphi_L} \) and \( \mathcal{P}_{\psi} \) as well as the product \( \mathcal{P}_{\varphi_L} \cdot \mathcal{P}_{\psi} \) are all principal with snc generators. In particular, one has
\[
\mathcal{I}_X(\varphi_L + \psi) \quad \text{being principal with an snc generator.}
\]

Moreover, \( m = 1 \) is the only jumping number of the family \( \{ \mathcal{I}(\varphi_L + m\psi) \}_{m \in [0, 1]} \).

For the sake of convenience, some consequences of the snc assumption are collected as follows.

Remark 2.3.8 (Consequences of Snc assumptions 2.3.7* or 2.3.7 with \( \pi = \text{id} \)).

- The function \( \psi \) is quasi-psh and its polar set \( \psi^{-1}(-\infty) \) is an snc divisor.
- According to Lemma 2.3.5, the reduced subvariety \( S \) is an snc divisor.
- When \((\tilde{X}, \varphi_L, \psi)\) satisfies the Snc assumption 2.3.7*, since \( S \) is an snc divisor, one can talk about the lc centres of \((X, \varphi_L, \psi)\) or \((X, S)\) as those defined in [32, Def. 4.15]. More explicitly, in this case, an lc centre of \((X, S)\) of codimension \( \sigma \) in \( X \) is an irreducible component of any intersections of \( \sigma \) irreducible components of \( S \) in \( X \). Define \( \text{lc}_X^\sigma(S) \) to be the union of all lc centres of \((X, S)\) of codimension \( \sigma \) in \( X \).
- There is an effective \( \mathbb{Z} \)-divisor \( F \) with snc such that \( \mathcal{I}(\varphi_L + \psi) = \mathcal{O}_X(-F) \). If \( S = \sum_{i \in I} D_i \), where \( D_i \)'s are the irreducible components of \( S \), then \( F \) is decomposed into
\[
F = G + \sum_{i \in I} \mu_i D_i + S,
\]
where \( G \) is an effective divisor containing no components of \( S \) and the coefficients \( \mu_i \geq 0 \) are integers (\( F \) must contain \( S \) as the quotient sheaf \( \frac{\mathcal{I}(\varphi_L)}{\mathcal{I}(\varphi_L + \psi)} \) is supported precisely on

\[\text{footnote 9:} \text{The concepts of lc centres of } (X, \varphi_L, \psi) \text{ and those of } (X, S) \text{ are used interchangeably only when } (X, \varphi_L, \psi) \text{ satisfies the Snc assumption 2.3.7*}. \text{ See Definition 5.2.4 for the definition of lc centres of } (X, \varphi_L, \psi) \text{ and Example 6.2.1 for an example on how the two concepts differ from each other.}\]
$S$ by item (5) in Section 2.2. Let $s_0$ be a canonical section of $S_0$ (which is set to 1 if $S_0$ is the zero divisor), which induces a potential on $S_0$ given by

$$\phi_{S_0} := \log|s_0|^2.$$  

Note that $s_0$ locally divides every $f \in \mathcal{I}(\varphi_L)$, because, for every $h \in \mathcal{I}_S = \mathcal{O}_X(-S)$, one has $hf \in \mathcal{I}(\varphi_L + \psi) \subset \mathcal{O}_X(-S_0 - S)$.

According to Siu's decomposition for closed positive currents (see, [38, §6] or [10, Ch. III, (8.17)]), given $S = \sum_{i \in I} D_i$ with $D_i$'s being the irreducible components, one has

$$i\partial\bar{i}\partial (\varphi_L + \psi) = \sum_{i \in I} \lambda_i [D_i] + \mathcal{R} = i\partial\bar{i}\partial \phi_{\lambda,S} + \mathcal{R},$$

where $\lambda_i = \nu(\varphi_L + \psi, D_i) := \inf_{x \in D_i} \nu(\varphi_L + \psi, x)$, the generic Lelong number of $\varphi_L + \psi$ (or $i\partial\bar{i}\partial (\varphi_L + \psi)$) along $D_i$, $D_i [D_i]$ is the current of integration along $D_i$ and $\mathcal{R}$ is a closed (1,1)-current such that $\mathcal{R} \geq i\partial\bar{i}\partial \varphi^{\text{sm}}$ for some smooth potential $\varphi^{\text{sm}}$ ($\mathcal{R} \geq 0$ when $\varphi_L + \psi$ is psh) and $\nu(\mathcal{R}, D_i) = 0$ for each $i \in I$. Note that the last equality follows from the Poincaré–Lelong formula $i\partial\bar{i}\partial \phi_{\lambda,S} = [\lambda \cdot S]$ (see Notation 2.1.3 for the definition of $\phi_{\lambda,S}$). ($S_0$ would have been denoted by $\mu \cdot S$ if it were not for notational convenience.) Therefore,

$$\varphi := \varphi_L + \psi - \phi_{\lambda,S}$$

is a quasi-psh potential (thus locally bounded from above). Let $s$ be a canonical section of $S$ such that $\phi_s = \log|s|^2$. The multiplier ideal sheaf $\mathcal{I}(\varphi_L + m\psi)$ having $m = 1$ as a jumping number implies the following lemma.

**Lemma 2.3.9.** $\lambda \cdot S = S_0 + S$.

**Proof.** It follows from the definition of multiplier ideal sheaves and the equality $\mathcal{I}(\varphi_L + \psi) = \mathcal{O}_X(-G - S_0 - S)$ that, for every $f \in \mathcal{I}(\varphi_L + \psi)$, written as $gs_0s$ for some $g \in \mathcal{O}_X(-G)$, one has

$$|g|^2 e^{\phi_{s_0} + \phi_s - \phi_{\lambda,S}} \lesssim |gs_0s|^2 e^{-\phi_{\lambda,S} - \varphi} = |f|^2 e^{-\varphi_L - \psi} \in L^1_{\text{loc}},$$

which, as there exists $g \in \mathcal{O}_X(-G)$ which does not vanish on any of the components of $S$, implies that

$$[\mu_i + 1 - \lambda_i] \geq 0 \quad \text{for all } i \in I,$$

where $[\cdot]$ is the ceiling function. Moreover, as $\nu(\varphi, D_i) = \nu(\mathcal{R}, D_i) = 0$ for all $i \in I$, $e^{-\varphi}$ is locally integrable at general points of $S$ (Skoda’s lemma, see [39, §7] or [9, Lemma (5.6)]), which forces the inequalities

$$\mu_i + 1 - \lambda_i \leq 0 \quad \text{for all } i \in I$$

(otherwise, $\mathcal{I}(\varphi_L + \psi)$ would be strictly containing $\mathcal{O}_X(-S_0 - S)$ at some point in $S$).

Since 1 is a jumping number of the family $\{\mathcal{I}(\varphi_L + m\psi)\}_{m \in \mathbb{R} \geq 0}$ and $S$ is the subvariety resulted from the jump by assumption, it follows that every $\lambda_i$ is a positive integer. Indeed, $\mu_i + 1 - \lambda_i \leq 0$ implies that all $\lambda_i$’s are positive. If some $\lambda_i$ is not an integer, since $\psi$ has logarithmic poles along $S$, there is then some $\varepsilon > 0$ such that $\mathcal{I}(\varphi_L + (1 - \varepsilon)\psi)_x = \mathcal{I}(\varphi_L + \psi)_x$ for some point $x \in D_i$, a contradiction.

As a result, it follows from (5) and (6) that $\lambda_i = \mu_i + 1$ for all $i \in I$, which concludes the proof.

$\square$
In summary, under the Snc assumption \( 2.3.7^* \) and the assumption that \( m = 1 \) is a jumping number of \( \{ J(\varphi_L + m\psi) \}_{m \in [0,1]} \), there is a quasi-psh potential \( \varphi \) on \( L \otimes S_0^{-1} \otimes S^{-1} \) (which is psh when \( \varphi_L + \psi \) is) such that

(eq 2.3.1) \[ \varphi + \phi_{S_0} + \phi_S = \varphi_L + \psi \quad \text{and} \quad \varphi < 0 \quad \text{locally on} \quad X, \]

which is defined for later use.

3. The new definition of adjoint ideal sheaves

From this section onward, \( J_\sigma(\varphi_L; \psi) \) denotes the analytic adjoint ideal sheaf given by Definition 1.2.1.

3.1. Local expression of germs in \( J_\sigma(\varphi_L; \psi) \). Suppose the Snc assumption \( 2.3.7^* \) holds in this section. Recall the canonical section \( s_0 \) on \( S_0 \) and the quasi-psh potential \( \varphi \) on \( L \otimes S_0^{-1} \otimes S^{-1} \) defined in Remark 2.3.8 and (eq 2.3.1). Let \( V \subset X \) be an open set sitting inside a coordinate chart with a holomorphic coordinate system \((z_1, \ldots, z_n)\), biholomorphic to a polydisc centred at the origin, such that \( L|_V \) is trivialised and

\[ S \cap V = \{ z_1 \cdots z_{\sigma V} = 0 \} \quad \text{for some integer} \quad \sigma V \leq n. \]

The goal of this section is to describe the sections of \( J_\sigma(\varphi_L; \psi)(V) \) for any integers \( \sigma = 1, \ldots, \sigma_V \) and to prove an openness property related to \( J_\sigma(\varphi_L; \psi) \).

Let \( \mathcal{S}_m \) be the group of permutations on a set of \( m \) elements for any \( m \in \mathbb{N} \) and set

\[ \mathcal{C}_\sigma^\sigma := \mathcal{S}_{\sigma V}/(\mathcal{S}_\sigma \times \mathcal{S}_{\sigma - \sigma}), \]

which is the set of choices of \( \sigma \) elements in a set of \( \sigma V \) elements. Any element \( p \in \mathcal{C}_\sigma^\sigma \) is abused to mean a permutation on the set of integer \( \{1, \ldots, \sigma_V\} \) such that, if \( p, p' \in \mathcal{C}_\sigma^\sigma \) and \( p \neq p' \), then \( p(\{1, \ldots, \sigma\}) \neq p'(\{1, \ldots, \sigma\}) \) (and one also has \( p(\{\sigma + 1, \ldots, \sigma_V\}) \neq p'(\{\sigma + 1, \ldots, \sigma_V\}) \)). Then, the set of subvarieties

\[ S_p := \{ z_{p(1)} = z_{p(2)} = \cdots = z_{p(\sigma)} = 0 \} \quad \text{for} \quad p \in \mathcal{C}_\sigma^\sigma, \]

are precisely the set of all of the \( \text{lc centres of codimension} \ \sigma \) (or \( \sigma \text{-lc centres for short} \)) in \( V \). Therefore,

\[ \text{lc}_\sigma^\sigma(S) = \text{lc}_\sigma^\sigma(S) \cap V = \bigcup_{p \in \mathcal{C}_\sigma^\sigma} S_p. \]

When orientations of the lc centres are in concern, assume also that \( p(1) < p(2) < \cdots < p(\sigma) \) and \( p(\sigma + 1) < p(\sigma + 2) < \cdots < p(\sigma_V) \) for each \( p \in \mathcal{C}_\sigma^\sigma \) for convenience.

For any \( \sigma = 1, \ldots, \sigma_V \) fixed, consider any \( f \in J_\sigma(\varphi_L; \psi)(V) \). It follows that

\[ \frac{f}{s_0} \leq \frac{e^{-\varphi_{S_0}}}{|\psi|^\sigma (\log|e\psi|)^{1+\varepsilon}} < \frac{|f|^2 e^{-\varphi_{S_0} - \varphi_S}}{|\psi|^\sigma (\log|e\psi|)^{1+\varepsilon}} \in L^1_{\text{loc}}(V) \]

for \( \varepsilon > 0 \). Recall from Section 2.3 that \( \frac{f}{s_0} \) is holomorphic. By considering the Taylor expansion of \( \frac{f}{s_0} \) about the origin, there is a minimal integer \( \sigma_f \in [0, \sigma_V] \) such that

\[ \frac{f}{s_0} = \sum_{p \in \mathcal{C}_\sigma^\sigma} g_p z_{p(\sigma_f + 1)} \cdots z_{p(\sigma_V)}, \quad \text{where} \quad g_p \in \mathcal{O}(V). \]
(Note that one has $\sigma_f = \sigma_V$ if $\frac{f}{s_0}$ does not vanish identically on $S_0^{\text{id}}$, where id is the identity permutation, while $\sigma_f = 0$ if $\frac{f}{s_0}|_S \equiv 0$. The integer $\sigma_f$ is the codimension of mlc of $(V, 0, \phi_S)$ with respect to $f$ defined in [5, Def. 2.2.5].) Since

$$\frac{1}{|\psi|^{|s_0|+\delta}} \leq e^{\delta \left(\frac{1+\varepsilon}{\delta e}\right)} \frac{1}{|\psi|^{\sigma(T)}}$$

for any $\delta > 0$ via the $x \log x$-inequality (see, for example, [4, §1.4.2]), the fact that $\left|\frac{f}{s_0}\right|^2 \frac{e^{-\phi_s}}{|\psi|^{1+\varepsilon}} \in L^1_{\text{loc}}(V)$, aided by the computation in [5, §2.2], implies that

$$\sigma_f \leq \sigma.$$

In summary, every $f \in \mathcal{F}_*(\varphi; \psi)(V)$ can be expressed as a finite sum

$$f = \sum_{p \in \mathcal{P}^*} g_p s_0 \cdot \frac{1}{z_{p(\sigma_{p})}}$$

for some $g_p \in \mathcal{O}_X(V)$. Note that the functions $g_p$ are not uniquely determined, but the restricted functions

$$g_p|_{s_0} = \left. \frac{f}{s_0} \right|_{s_0} \cdot \frac{1}{z_{p(\sigma_{p})}}$$

are (provided that the section $s_0$ and the coordinate system are fixed).

**3.2. An openness property.** Assume in this section the Snc assumption 2.3.7 in Section 2.3 and recall the notation for the canonical section $s_0$ on $S_0$ and the quasi-psh potential $\varphi$ defined there.

Let $s$ be the canonical section of $S$ such that $\phi_s = \log|s|^2$, which therefore belongs to $\mathcal{I}_S$ locally. For every $f \in \mathcal{F}_*(\varphi; \psi)(V) \subset \mathcal{F}(\varphi; \psi)(V)$ on some open set $V \subset X$, one has $s f \in \mathcal{F}(\varphi + \psi)$ by the definition of $S$ (see Section 2.2) and $g := \frac{f}{s_0} \in \mathcal{O}_X$ (see Section 2.3). It follows that

$$|g|^2 e^{-\varphi} = \left|\frac{s f}{s_0}\right|^2 e^{-\varphi - \phi_s} = |s f|^2 e^{-\varphi - \phi_s} \in L^1_{\text{loc}}(V).$$

After shrinking $V$ if necessary, there exists some number $\lambda > 1$ such that

$$|g|^2 e^{-\lambda \varphi} \in L^1_{\text{loc}}(V)$$

by the strong openness property. From the facts that

$$\frac{|f|^2 e^{-\varphi - \phi_s} \in L^1_{\text{loc}}(V)}{1+\varepsilon} \in L^1_{\text{loc}}(V)$$

and that $\varphi$ has only neat analytic singularities with snc singular loci which contain no components of $S$, the computation in [4, Prop. 2.2.1] shows that

$$|f|^2 e^{-\lambda \varphi - \phi_s} = \frac{|g|^2 e^{-\lambda \varphi - \phi_s}}{|\psi|^{\sigma(T)}} \in L^1_{\text{loc}}(V)$$

for all $\varepsilon > 0$.

The existence of such $\lambda > 1$ for each section $f$ is required in the definitions of analytic adjoint ideal sheaves by Guenancia ([23]) and Danilo Kim ([28]). Yet their definitions allow the involving quasi-psh function ($\varphi$ in the current notation) to have more general singularities, namely, $e^\varphi$ need only be locally Hölder continuous in their expositions. The
strong openness property of multiplier ideal sheaves of quasi-psh functions does not imply directly that such \( \lambda > 1 \) exists for each section \( f \).

4. The residue short exact sequences

In this section, assume that \( \varphi_L \) (as well as \( \psi \)) has as only neat analytic singularities and satisfies the Snc assumption 2.3.7* in Section 2.3. Recall that the potentials \( \phi_{S_0} \) and \( \varphi \) are defined such that \( \varphi \) is a quasi-psh potential on \( L \otimes S^{-1} \otimes S^{-1} \) satisfying

\[
\varphi + \phi_{S_0} + \phi_S = \varphi_L + \psi
\]

with the polar set of \( \varphi + \phi_S \) being an snc divisor and \( \varphi^{-1}(-\infty) \) containing no irreducible components of \( S \). Having only neat analytic singularities, this also implies that \( \varphi^{-1}(-\infty) \) contains no lc centres of \( (X, S) \) (see the local expression of \( \varphi \) in (eq 4.1.1) below).

4.1. Admissible open sets and results on residue functions in [4]. Consider an open set \( V \subset X \), sitting inside a coordinate chart with a holomorphic coordinate system \( (z_1, \ldots, z_n) \), biholomorphic to a polydisc centred at the origin such that \( L \) is trivialised on \( V \),

\[
S \cap V = \{z_1 \cdots z_{\sigma_V} = 0\} \quad \text{for some integer } \sigma_V \leq n,
\]

\[
\phi_S|_V = \sum_{j=1}^{\sigma_V} \log|z_j|^2, \quad \varphi|_V = \sum_{k=\sigma_V+1}^n b_k \log|z_k|^2 + \beta \quad \text{and}
\]

(eq 4.1.1)

\[
\psi|_V = \sum_{j=1}^{\sigma_V} \nu_j \log|z_j|^2 + \sum_{k=\sigma_V+1}^n a_k \log|z_k|^2 + \alpha,
\]

where, after shrinking \( V \) if necessary,

- \( \sup_V \log|z_j|^2 < 0 \) for \( j = 1, \ldots, n \),
- \( \alpha \) and \( \beta \) are smooth functions such that \( \sup_V \alpha \leq -1 \) and \( \sup_V \beta \leq 0 \),
- \( \nu_j \)'s are constants such that \( \nu_j > 0 \) for \( j = 1, \ldots, \sigma_V \),
- \( a_k \)'s and \( b_k \)'s are constants such that \( a_k, b_k \geq 0 \) for \( k = \sigma_V + 1, \ldots, n \), and
- \( \sup_V r_j \frac{\partial}{\partial \tau_j} \psi = 2\nu_j + \sup_V r_j \frac{\partial}{\partial \tau_j} \alpha > 0 \) for \( j = 1, \ldots, \sigma_V \), where \( r_j = |z_j| \) is the radial component of the polar coordinates.

Call such open set \( V \) as an admissible open set with respect to the data \( (\varphi_L, \psi) \) in the coordinate system \( (z_1, \ldots, z_n) \) for the ease of reference. It is simply said to be admissible if the data \( (\varphi_L, \psi) \) are understood and the above criteria are satisfied in some holomorphic coordinate system. Note that such admissible open set is the kind of open sets on which the computations in [4, §2.2] are valid. The family of all such admissible open sets forms a basis of the topology of \( X \).

For any \( f \in \mathcal{F}_\sigma(\varphi_L; \psi)(V) \), \( f \) can be decomposed into a finite sum (see Section 3.1)

(eq 4.1.2)

\[
f = \sum_{p \in \mathfrak{C}_C^\sigma} g_p s_0 z_{p(\sigma+1)} \cdots z_{p(\sigma_V)}.
\]

Note that, while the holomorphic functions \( g_p \) are not uniquely determined, \( g_p|_{\mathfrak{C}_C^\sigma} \) is uniquely determined (under the current coordinate system) by \( f \) and \( s_0 \) for each \( p \in \mathfrak{C}_C^\sigma \). An integrability property of the \( g_p \)'s is collected as follows.
Lemma 4.1.1. Given the Snc assumption 2.3.7, every \( f \in \mathcal{I}(\varphi_L) \) admits a decomposition of the form in (eq4.1.2) for some integer \( \sigma \geq 0 \) on an admissible open set \( V \) such that the \( g_p \)'s satisfy
\[
g_p \in \mathcal{I}(\varphi) \quad \text{and} \quad g_p s_0 \in \mathcal{I}(\varphi + \phi_{S_0} \mathcal{I}(\varphi_L) \quad \text{on} \ V \quad \text{for every} \ p \in \mathcal{C}_\sigma^\nu ,
\]
and thus
\[
f \in \mathcal{I}(\varphi + \phi_{S_0}) \cdot \mathcal{I}_{k^X}^{\sigma+1}(S) = \mathcal{I}(\varphi_L) \cdot \mathcal{I}_{k^X}^{\sigma+1}(S) \quad \text{on} \ V ,
\]
where \( \mathcal{I}_{k^X}^{\sigma+1}(S) \) is the defining ideal sheaf of \( \mathcal{I}_{k^X}^{\sigma+1}(S) \), which is generated as an ideal by the set \( \{ z_{p(\sigma+1)} \cdots z_{p(\sigma+1)} \mid p \in \mathcal{C}_\sigma^\nu \} \) on \( V \).

Proof. The discussion in Section 3.2 indeed shows that, for every \( f \in \mathcal{I}(\varphi_L) \), one has
\[
\left| \frac{f}{s_0} \right|^2 e^{-\varphi} \in L^1_{\text{loc}}(V) .
\]

Thanks to the Snc assumption 2.3.7 on \( \varphi_L + \psi \), one can see from Fubini’s theorem and the above integrability that \( \frac{f}{s_0} \) is divisible by some monomial \( \prod_{k=\sigma+1}^{n+1} m_k \) such that \( m_k - b_k > -1 \) for \( k = \sigma_1 + 1, \ldots, n \) (where \( b_k \)'s are the coefficients in the local expression of \( \varphi \mid V \) in (eq4.1.1)). Since \( \frac{f}{s_0} \) admits a decomposition of the form in (eq4.1.2) for some integer \( \sigma \in [0, \sigma_1] \), one can assume that \( f \) admits the decomposition (eq4.1.2) (with the same \( \sigma \)) such that \( g_p \) is divisible by \( \prod_{k=\sigma+1}^{n+1} z_k^m \) for each \( p \in \mathcal{C}_\sigma^\nu \), and thus
\[
|g_p|^2 e^{-\varphi} \in L^1_{\text{loc}}(V) .
\]

This also implies that \( g_p s_0 \in \mathcal{I}(\varphi + \phi_{S_0}) \) and \( f \in \mathcal{I}(\varphi + \phi_{S_0}) \cdot \mathcal{I}_{k^X}^{\sigma+1}(S) \), and therefore \( \mathcal{I}(\varphi_L) \subset \mathcal{I}(\varphi + \phi_{S_0}) \). Note that every germ in \( \mathcal{I}(\varphi + \phi_{S_0}) \) can be decomposed into the form \( g s_0 \) such that \( g \in \mathcal{I}(\varphi) \). Since
\[
\varphi_L = \varphi + \phi_{S_0} + \phi_L - \psi = \phi_{S_0} + \phi_L - \varphi_L + \sum_{k=\sigma+1}^{n+1} (b_k - a_k) \log |z_k|^2 + \beta - \alpha \quad \text{on} \ V
\]
and \( s_0 \) is locally \( L^2 \) with respect to \( e^{-\phi_{S_0} - \phi_L + \phi_S} \), while \( g \) is locally \( L^2 \) with respect to \( e^{-\varphi_L} \), one sees, in view of Fubini’s theorem, that
\[
|gs_0|^2 e^{-\varphi_L} \in L^1_{\text{loc}}(V) \quad \text{and thus} \quad \mathcal{I}(\varphi + \phi_{S_0}) \subset \mathcal{I}(\varphi_L) .
\]

This completes the proof. \( \square \)

Lemma 4.1.1 also implies that \( \mathcal{I}(\varphi_L; \psi) \subset \mathcal{I}(\varphi_L) \cdot \mathcal{I}_{k^X}^{\sigma+1}(S) \) under the Snc assumption 2.3.7.

Recall that the residue function \( \varepsilon \mapsto \mathfrak{I}^G(\varepsilon)_{\nu, \sigma} \) defined in [4] is given by
\[
\mathfrak{I}^G(\varepsilon)_{\nu, \sigma} := \mathfrak{I}^G(\varepsilon) := \varepsilon \int_V G e^{-\varphi_L - \psi} \, d\mu_{\nu} \quad \text{for any} \ G \in \mathcal{C}_\nu^\nu(V) \text{and} \ \varepsilon > 0 .
\]

The computations in [4, Prop. 2.2.1, Thm. 2.3.1 and Cor. 2.3.3] assure the following statements.

Theorem 4.1.2. Under the Snc assumption 2.3.7, for every \( f \in \mathcal{I}(\varphi_L)(V'') \) and for every admissible open set \( V \subset V'' \) described above, there exists a unique integer \( \sigma_f \in [0, \sigma_V] \) such that
(1) one has \( f \in \mathcal{I}(\varphi + \phi_S) \cdot I_{k_X^{\sigma} + 1}(S) = \mathcal{I}(\varphi_L) \cdot I_{k_X^{\sigma} + 1}(S) \) on \( V \) and, for every integer \( \sigma \) and real number \( \varepsilon > 0 \),

\[
\mathfrak{F}^{|f|^2}(\varepsilon)_{V,\sigma} = \begin{cases} < +\infty & \text{when } \sigma \geq \sigma_f, \\
+\infty & \text{when } \sigma < \sigma_f,
\end{cases}
\]

and it therefore follows that

\[ J_\sigma(\varphi_L; \psi) = J_\sigma(\varphi_L) \cdot I_{k_X^{\sigma} + 1}(S) \quad \text{for all integers } \sigma \geq 0, \]

which implies, in particular, that

\[
J_0(\varphi_L; \psi) = J_0(\varphi_L) = J_0(\varphi_L + \psi) \quad \text{and} \quad J_\sigma(\varphi_L; \psi) = J_\sigma(\varphi_L) \cdot I_0 = J_\sigma(\varphi_L) \quad \text{for all } \sigma \geq \sigma_{\text{mcl}},
\]

where \( \sigma_{\text{mcl}} \) is the codimension of the minimal lc centres (mlc) of \( (X, S) \);

(2) for any smooth cut-off function \( \rho: V' \to [0, 1] \) which is compactly supported in an admissible open set \( V' \) in the same coordinate system on \( V \) such that \( V \in V' \in V'' \) and that \( \rho \equiv 1 \) on \( V \), the function \( \varepsilon \mapsto \mathfrak{F}^{|\rho f|^2}(\varepsilon)_{V',\sigma} \) can be analytically continued across \( \varepsilon = 0 \) when \( \sigma \geq \sigma_f \) and one has the residue ( squared) norm given by

\[
\mathfrak{F}^{|\rho f|^2}(0)_{V',\sigma} = \lim_{\varepsilon \to 0^+} \mathfrak{F}^{|\rho f|^2}(\varepsilon)_{V',\sigma} = \begin{cases} \int_{k_X^{\sigma} + 1}(S) \rho|f|^2 \, d\text{lcv}_{\varphi_L}^\sigma[\psi] < +\infty & \text{when } \sigma \geq \sigma_f, \\
+\infty & \text{when } \sigma < \sigma_f,
\end{cases}
\]

in which, when \( f \) admits the decomposition in (eq4.1.2), the integral with respect to the \( \sigma\)-lc-measure is given by

\[
\int_{k_X^{\sigma} + 1}(S) \rho|f|^2 \, d\text{lcv}_{\varphi_L}^\sigma[\psi] = \sum_{p \in \mathcal{S}_p \cap V'_{\sigma}} \frac{\pi^{\sigma}_{p}}{(\sigma - 1)!} \int_{\mathcal{S}_p} \rho|y_p|^2 e^{-\varphi} \, d\text{vol}_{\mathcal{S}_p},
\]

where \( \nu_{p} := \prod_{j=1}^{\sigma} \nu_{\rho_{p(j)}} \) (product of the coefficients in the local expression of \( \psi \) in (eq4.1.1), or, equivalently, product of the generic Lelong numbers \( \nu(\psi, D_i) \) of \( \psi \) along the irreducible components \( D_i \) of \( S \) which contain \( \mathcal{S}_p \), and it thus follows that

\[
\mathfrak{F}^{|\rho f|^2}(0)_{V',\sigma} = 0 \quad \text{when } \sigma > \sigma_f.
\]

Proof. Take \( \sigma_f \in [0, \sigma_V] \) to be the minimal integer such that the conclusion of Lemma 4.1.1, namely, \( f \in \mathcal{I}(\varphi + \phi_S) \cdot I_{k_X^{\sigma} + 1}(S) = \mathcal{I}(\varphi_L) \cdot I_{k_X^{\sigma} + 1}(S) \) holds true on \( V \) (then \( y_p \) in the decomposition is non-trivial on \( \mathcal{S}_p \) for some \( p \in \mathcal{C}_\sigma(S) \)). The inequality (eq3.1.1) and the computation in [5, §2.2] implies that \( \mathfrak{F}(\varepsilon)_{\psi,\sigma} = +\infty \) for all \( \varepsilon > 0 \) when \( \sigma < \sigma_f \). On the other hand, given a cut-off function \( \rho \) described in (2), the computation in [4, Prop. 2.2.1] shows that

\[
\mathfrak{F}(\varepsilon)_{\sigma,\psi} \leq \mathfrak{F}(\varepsilon)_{\sigma,\psi} < +\infty \quad \text{for all } \varepsilon > 0 \text{ when } \sigma \geq \sigma_f.
\]

Therefore, \( f \in J_\sigma(\varphi_L; \psi) \) on \( V \) for \( \sigma \geq \sigma_f \). This gives (1).

The claims in (2) follow from the computations in [4, Thm. 2.3.1 and Cor. 2.3.3]. The expression of the integral under the \( \sigma\)-lc-measure follows from the computation in [5, §2.2].
Remark 4.1.3. The integer \( \sigma_f \) is the codimension of the mlc of \((V, \varphi_L, \psi)\) (or of \((V, S)\)) with respect to \( f \) defined in [5, Def. 2.2.5]. The proof of Theorem 4.1.2 also shows that it is the same as the codimension of the mlc of \((V, 0, \phi_S)\) with respect to \( f \) stated in Section 3.1.

Remark 4.1.4. For the convenience of computations in practice, the essence of the convergence result for the integral \( \int_{f^{[\ell]}_x}^j \) in Theorem 4.1.2 (1) is stated in terms of local coordinates. Suppose \( \psi \) is given as in (eq 4.1.1) on \( V \), in which \( \nu_j > 0 \) for \( j = 1, \ldots, \sigma_V \).

For the purpose of illustration, assume that \( \sigma_V < n \) while \( a_{n-1} > 0 \) and \( a_n = 0 \) (\( a_k \) is the coefficient of \( \log |z_k|^2 \) in \( \psi \)). The computation in [4, Prop. 2.2.1] as well as [5, Prop. 2.2.1] (hence Theorem 4.1.2 (1)) implies that, for any \( \varepsilon > 0 \) and \( \sigma_f \leq \sigma_V \) and \( \sigma_f < n - 1 \), one has

\[
\int_V \frac{d\text{vol}_V}{|z_1 \cdots z_{\sigma_f}|^2 |z_{n-1}|^{2n-1} |z_n|^{2n} |\psi|^\sigma (\log |e\psi|)^{1+\varepsilon}} \begin{cases} +\infty & \text{when } \sigma > \sigma_f, \ell_{n-1} \leq 1, \ell_n < 1, \\ +\infty & \text{when } \sigma = \sigma_f \text{ and } \ell_{n-1}, \ell_n < 1, \\ +\infty & \text{when } \sigma = \sigma_f \text{ and } \ell_{n-1} \geq 1, \\ +\infty & \text{when } \sigma < \sigma_f, \\ +\infty & \text{when } \ell_{n-1} > 1 \text{ or } \ell_n \geq 1. 
\end{cases}
\]

Remark 4.1.5. When the family \( \{ \mathcal{I}(\varphi_L + m\psi) \}_{m \in [0, 1]} \) has another jumping number \( m_0 \in (0, 1) \) besides \( m = 1 \) such that \( \mathcal{I}(\varphi_L) \supseteq \mathcal{I}(\varphi_L + m_0 \psi) = \mathcal{I}(\varphi_L + m \psi) \) for all \( m \in [m_0, 1) \), all the multiplier ideal sheaves \( \mathcal{I}(\varphi_L) \) appearing in the statement of Theorem 4.1.2 have to be replaced by \( \mathcal{I}(\varphi_L + m_0 \psi) \). In particular, Theorem 4.1.2 (1) should conclude that \( \mathcal{I}_f(\varphi_L; \psi) = \mathcal{I}(\varphi_L + m_0 \psi) \cdot C_{X_{k+1}}(S) \) for all integers \( \sigma \geq 0 \). Indeed, assuming the existence of \( m_0 \in (0, 1) \), one has \( \int_{f^{[\ell]}_x}^j = +\infty \) for any \( f \in \mathcal{I}(\varphi_L)_x \), \( \mathcal{I}(\varphi_L + m_0 \psi)_x \), \( \sigma \geq 0 \) and \( \varepsilon > 0 \) and for any admissible open neighbourhood \( V \) of \( x \) in \( X \) on which \( f \) is defined.

Remark 4.1.6. The computation of \( \int_{f^{[\ell]}_x}^j \) invokes only Fubini’s theorem and integration by parts on each variables in the admissible open set \( V \) (see [4, Prop. 2.2.1, Thm. 2.3.1 and Cor. 2.3.3], also cf. [5, Prop. 2.2.1]). The holomorphicity of \( f \) plays little role in the computation. Indeed, it suffices to have \( f \in \mathcal{I}(\varphi_L) \cdot C_{X_{\infty}}^\infty \) such that \( f \) can be written in the form (eq 4.1.2) via Taylor’s theorem (in which case \( g_\rho \)’s are smooth local functions on \( V \)) and the computation of \( \int_{f^{[\ell]}_x}^j \) is still valid. It turns out that both Lemma 4.1.1 and Theorem 4.1.2 remain true when the ideal sheaves \( \mathcal{I}(\varphi_L) \) and \( \mathcal{I}_f(\varphi_L; \psi) \) are extended to \( \mathcal{I}(\varphi_L) \cdot C_{X_{\infty}}^\infty \) and \( \mathcal{I}_f(\varphi_L; \psi) \cdot C_{X_{\infty}}^\infty \) respectively (via the injection \( \mathcal{O}_X \hookrightarrow C_{X_{\infty}}^\infty \)) and the involving functions \( f \) and \( g_\rho \)’s are only smooth.

Theorem 4.1.2 (1) shows, in particular, that, for any admissible open set \( V \subseteq X \), the maps \( f \mapsto \int_{f^{[\ell]}_x}^j \) for \( \varepsilon > 0 \) define a family of \( L^2 \) norms on \( \mathcal{I}_f(\varphi_L; \psi)(V) \) and that

\[
\mathcal{I}_f(\varphi_L; \psi)(V) = \left\{ f \in \mathcal{I}(\varphi_L)(V) \mid \exists \varepsilon > 0 \text{ such that } \int_{f^{[\ell]}_x}^j < +\infty \right\}
\]

For the sake of notational convenience, for any \( f \in C_{X_{\infty}}^\infty \cdot \mathcal{I}(\varphi_L; \psi)(V) \) on some open set \( V \subseteq X \) such that \( f \) does not have compact support in \( V \), define

\[
(\text{eq 4.1.3}) \quad \mathcal{I}^{[j]}_f(0)_V, \sigma := \int_{C_{X_{\infty}}(S)} |f|^j \, d\text{lcv}_{\varphi_L}^V[\psi] := \lim_{\rho \downarrow 0} \mathcal{I}^{[j]}_f(0)_V, \sigma,
\]
where $\rho: V' \to [0, 1]$ is a compactly supported smooth cut-off function on an open set $V' \supset V$ with $\rho|_V \equiv 1$ such that $f$ is defined on $V'$ and $\lim_{\rho \searrow 1_V}$ indicates taking limit by considering a sequence of the functions $\rho$ which descends pointwisely to the characteristic function $1_V$ of $V$ on $X$. Then, Theorem 4.1.2 (2) implies, in particular, that, for any admissible open set $V \subseteq X$, 

$$\tilde{\mathcal{F}}(0)_{V,\sigma} = \sum_{p \in \mathcal{C}_\sigma^V} \frac{\pi^\sigma}{(\sigma - 1)!} \mathcal{L}_p \int_{\mathbb{S}_p^\sigma} |g_p|^2 e^{-\varphi} d\text{vol}_{\mathbb{S}_p^\sigma},$$

given the decomposition (eq 4.1.2) of $f$, and thus

$$\mathcal{J}_{\sigma-1}(\varphi_L; \psi)(V) = \left\{ f \in \mathcal{J}_\sigma(\varphi_L; \psi)(V) \mid \tilde{\mathcal{F}}(0)_{V,\sigma} = 0 \right\}.$$

4.2. The residue morphism. For a given integer $\sigma \geq 1$, let $\nu: \lceil \mathcal{C}_X^\sigma(S) \rceil \to \mathcal{C}_X^\sigma(S)$ be the normalisation of $\mathcal{C}_X^\sigma(S)$ (thus $\lceil \mathcal{C}_X^\sigma(S) \rceil$ is the disjoint union of all $\sigma$-lc centres of $(X, S)$) and $\nu: \mathcal{C}_X^\sigma(S) \hookrightarrow X$ the natural inclusion. By a slight abuse of notation, write temporarily that $\mathcal{C}_X^\sigma(S)$ is an snc divisor by assumption, in view of the adjunction formula, there is a divisor $\text{Diff}_q^\sigma S$ on $\mathcal{S}_q^\sigma$ (the general different of $S$ on $\mathcal{S}_q^\sigma$; see [32, §4.2]) such that (in terms of their associated line bundles)

$$K_{\mathcal{S}_q^\sigma} \otimes S^{-1}|_{\mathcal{S}_q^\sigma} = K_X|_{\mathcal{S}_q^\sigma} \otimes (\text{Diff}_q^\sigma S)^{-1}.$$

Indeed, on any admissible open set $V$ such that $\mathcal{S}_q^\sigma \cap V = \mathcal{S}_p^\sigma$ for some $p \in \mathcal{C}_\sigma^V$ (following the notation in Section 3.1), the line bundle $\text{Diff}_q^\sigma S$ is isomorphic to the line bundle associated to the divisor $\sum_{k=\sigma+1}^{\nu} \{ z_p(k) = 0 \}|_{\mathcal{S}_p^\sigma}$. Again by an abuse of notation, let $(\text{Diff}_q^\sigma S)^{-1}$ to mean the extension to $\lceil \mathcal{C}_X^\sigma(S) \rceil$ by zero of its associated invertible sheaf.

Now set

$$\mathcal{R}_\sigma(\varphi) := \mathcal{R}_{X,\sigma}(\varphi) := S_0^{-1} \otimes \nu_* \left( \bigoplus_{q \in I_q^\sigma} \left( \text{Diff}_q^\sigma S \right)^{-1} \otimes \mathcal{J}_{\mathcal{C}_X^\sigma(\nu^*t^*\varphi)} \right)$$

$$= \bigoplus_{q \in I_q^\sigma} \nu_* \left( S_0^{-1}|_{\mathcal{S}_q^\sigma} \otimes (\text{Diff}_q^\sigma S)^{-1} \otimes \mathcal{J}_{\mathcal{S}_q^\sigma}(\nu^*t^*\varphi) \right),$$

which is a coherent sheaf on $X$ supported on $\mathcal{C}_X^\sigma(S)$ (note that the supports of $(\text{Diff}_q^\sigma S)^{-1}$ for $q \in I_q^\sigma$ are mutually disjoint in $\lceil \mathcal{C}_X^\sigma(S) \rceil$). Let $V$ be an admissible open set and let $\mathcal{S}_p^\sigma \subseteq \mathcal{C}_\sigma^V$ be the set of $\sigma$-lc centres of $(V, S \cap V)$ as described in Section 3.1. One then has

$$\nu^{-1}(\mathcal{C}_X^\sigma(S) \cap V) = \bigcup_{p \in \mathcal{C}_\sigma^V} \mathcal{S}_p^\sigma$$

and, by writing $\text{Diff}_p^\sigma S := \text{Diff}_q^\sigma S$ when $\mathcal{S}_q^\sigma \cap V = \mathcal{S}_p^\sigma$,

$$\mathcal{R}_\sigma(\varphi)(V) = \prod_{p \in \mathcal{C}_\sigma^V} S_0^{-1}|_{\mathcal{S}_p^\sigma} \otimes (\text{Diff}_p^\sigma S)^{-1} \otimes \mathcal{J}_{\mathcal{S}_p^\sigma}(\varphi)(\mathcal{S}_p^\sigma).$$
It therefore follows from Theorem 4.1.2 that the residue map
\[
\mathcal{J}_\sigma(\varphi_L; \psi)(V) \xrightarrow{\text{Res}} \mathcal{R}_\sigma(\varphi)(V)
\]
is well-defined on every admissible open set \( V \), where \( f \) and the \( g_p \)'s are related by the decomposition (eq 4.1.2) and \( g_p|_{\mathcal{S}^p} \) is viewed as a section of \( S_0^{-1} \otimes (\text{Diff}^+ \mathcal{S})^{-1} \) (see (eq 3.1.2)).

It can also be seen that the map is independent of the choice of the coordinate system and induces a morphism of sheaves (which depends on the choice of the canonical sections \( s_0 \) and \( z_p(\sigma+1) \cdots z_p(\sigma) \) of \( S_0 \) and of \( \text{Diff}^+ \mathcal{S} \) respectively).

Sometimes it may be convenient to describe the residue morphism in terms of holomorphic \( n \)-forms. With the adjunction formula in mind, one has

\[
K_X \otimes \mathcal{R}_\sigma(\varphi)(V) = \bigoplus_{p \in \mathcal{C}_V^\sigma} K_{\mathcal{S}^p} \otimes \left( (S_0^{-1} \otimes S^{-1})|_{\mathcal{S}^p} \right) \otimes \mathcal{J}_\sigma(\varphi)(\mathcal{S}^p). 
\]

Let \( s \) be a canonical section of \( S \) such that \( s|_V = z_1 \cdots z_{\sigma_V} \). Identifying \( f \) with the \((n,0)\)-form \( f dz_1 \wedge \cdots \wedge dz_n \) on \( V \) and each \( g_p \) with the \((n-\sigma,0)\)-form \( g_p \, \text{sgn}(p) \, dz_{p(\sigma+1)} \wedge \cdots \wedge dz_{p(\sigma)} \wedge dz_{\sigma_V+1} \wedge \cdots \wedge dz_n \) on \( V \) (where \( \text{sgn}(p) \) is the sign of the permutation representing \( p \) described in Section 3.1), one then has

\[
f = \sum_{p \in \mathcal{C}_V^\sigma} u_p \wedge g_p \, s_0 \, z_{p(\sigma+1)} \cdots z_{p(\sigma)} = \sum_{p \in \mathcal{C}_V^\sigma} \frac{u_p}{z_{p(1)} \cdots z_{p(\sigma)} \wedge g_p \, s_0} 
\]

(cf. (eq 4.1.2)). With a suitable choice of orientation on the normal bundle of \( \mathcal{S}^p \) in \( X \) which determines the sign on the Poincaré residue map \( \mathcal{R}_{\mathcal{S}^p} \) (see [32, §4.18]), it follows that

(eq 4.2.1)
\[
g_p|_{\mathcal{S}^p} = \mathcal{R}_{\mathcal{S}^p} \left( \frac{f}{s_0} \right)
\]
for all \( p \in \mathcal{C}_V^\sigma \) (cf. (eq 3.1.2)). The residue morphism is then given by

\[
K_X \otimes \mathcal{J}_\sigma(\varphi_L; \psi)(V) \xrightarrow{\text{Res}} K_X \otimes \mathcal{R}_\sigma(\varphi)(V)
\]
on each admissible open set \( V \), which depends on the choice of \( s_0 \) and \( s \). It is well-defined as assured by Theorem 4.1.2.

4.3. A local \( L^2 \) extension theorem. In the versions of analytic adjoint ideal sheaves of Guenancia ([23]) and Dano Kim ([28]), surjectivity of the residue morphism is obtained via an application of the Ohsawa–Takegoshi–Manivel extension theorem (see [35] and [8]), which is not sufficient for proving the surjectivity of the residue morphism in the present setting. In this section, a local extension which belongs to \( \mathcal{J}_\sigma(\varphi_L; \psi) \) (i.e. a preimage of a germ in \( \mathcal{R}_\sigma(\varphi) \) under \( \text{Res} \)) is constructed directly instead.

Given an admissible open set \( V \subset X \), for any fixed integer \( \sigma \in [1, \sigma_V] \) and any \( \sigma \)-lc centre \( \mathcal{S}^p \subset \text{lc}_{\sigma_V}^V(S) \) where \( p \in \mathcal{C}_V^\sigma \) (see Section 3.1), consider the maps given in the
diagram
\[ S^\sigma_p \cong \{0\} \times S^\sigma_p \xrightarrow{\iota_p} U_p \times S^\sigma_p = U_p \times W_p \cong V \]
where \(U_p \subset \mathbb{C}^\sigma\) is an \(\sigma\)-disc centred at the origin with coordinates given by \(z_{p(1)}, \ldots, z_{p(\sigma)}\), and the maps \(\iota_p\) and \(pr_p\) are, respectively, the natural inclusion and the projection to the second factor. \(S^\sigma_p\) in the product \(U_p \times S^\sigma_p\) is written as \(W_p\) so that the symbol \(S^\sigma_p\) is reserved for the subvariety \(\{0\} \times W_p\). Set
\[ \varphi|_p := pr_p^* \iota_p^* \varphi \quad \text{(and \(\varphi|_p := \varphi\) when \(\sigma = 0\)} \]
and notice that, from the smoothness of \(\beta\) in the local expression of \(\varphi|_V\) in (eq 4.1.1) and that \(\varphi^{-1}(-\infty)\) is in snc with \(S\) but contains no components of \(S\), one has, for any integer \(\sigma \geq 0\),
\[ \varphi \sim_{\log} \varphi|_p \quad \text{on} \quad V \quad \text{for all} \quad p \in \mathcal{C}^\sigma_{\nu} \).

With the above preparation, the following key property of adjoint ideal sheaves can be proved.

**Theorem 4.3.1.** Given the Snc assumption 2.3.7, the sequence
\[ 0 \xrightarrow{} J_{\sigma-1}(\varphi_L; \psi) \xrightarrow{} J_{\sigma}(\varphi_L; \psi) \xrightarrow{\text{Res}} \mathcal{R}_\sigma(\varphi) \xrightarrow{} 0 \]
where the map between \(J_{\sigma-1}(\varphi_L; \psi)\) and \(J_{\sigma}(\varphi_L; \psi)\) is the natural inclusion, is exact.

**Proof.** To see that \(J_{\sigma-1}(\varphi_L; \psi)\) is the kernel of \(\text{Res}\), note that, according to Theorem 4.1.2, for any \(f \in J_{\sigma}(\varphi_L; \psi)\), one has \(\delta_{\psi}(0)_{\sigma} = 0\) (for some smooth cut-off function \(\rho\) which is \(\equiv 1\) on some admissible open set \(V\) as in Theorem 4.1.2 (2)) if and only if \(g_p|_{S^\sigma_p} \equiv 0\) on \(S^\sigma_p \subset \text{lc}^\nu(S)\) for all \(p \in \mathcal{C}^\sigma_{\nu}\), which is equivalent to \(f \in J(\varphi_L) \cdot \mathcal{I}_{\text{lc}^\nu(S)} = J_{\sigma-1}(\varphi_L; \psi)\) according to (eq 4.1.2) and Lemma 4.1.1.

For the surjectivity of the residue morphism \(\text{Res}\), first note that \(\mathcal{R}_\sigma(\varphi)_x = 0\) for any \(x \in X \setminus \text{lc}^\nu(S)\) and thus \(\text{Res}_x\) is automatically surjective for such \(x\). Next, consider any \(x \in \text{lc}^\nu(S) \subset X\) and any germ \((g_p)_{p \in \mathcal{C}^\nu_x} \in \mathcal{R}_\sigma(\varphi)_x\) such that \(V\) is an admissible open set in \(X\) centred at \(x\) and
\[ \sum_{p \in \mathcal{C}^\nu_x} \int_{S^\sigma_p} |g_p|^2 e^{-\psi} d\text{vol}_{S^\sigma_p} < +\infty \quad \text{(where} \quad S^\sigma_p \subset \text{lc}^\nu(S) \subset V) .\]

Note that
\[ |\psi_p| := \left| \sum_{j=1}^\sigma \nu_{p(j)} \log |z_{p(j)}| \right|^2 + 1 \leq |\psi| \quad \text{on} \quad V \]
as seen from (eq 4.1.1). Abusing the notation \(g_p\) to mean also its pullback \(pr_p^* g_p\) to \(V\), it then follows that, for each \(p \in \mathcal{C}^\nu_x\) and for any \(\varepsilon > 0\),
\[ \int_V \left| g_p z_{p(\sigma+1)} \cdots z_{p(\sigma+\nu)} \right|^2 e^{-\psi} d\text{vol}_V \leq \int_V \frac{|g_p z_{p(\sigma+1)} \cdots z_{p(\sigma+\nu)}|^2 e^{-\psi} d\text{vol}_V}{|\psi|^2 \log |e\psi|^2} \leq \int_V \frac{|g_p z_{p(\sigma+1)} \cdots z_{p(\sigma+\nu)}|^2 e^{-\psi} d\text{vol}_V}{|\psi|^2 \log |e\psi|^2} \]
\[
\begin{align*}
&= \int_{U_p \times W_p} \frac{|g_p|^2 e^{-\varphi_p} \, d\text{vol}_{U_p} \, d\text{vol}_{W_p}}{|z_{p(1)} \cdots z_{p(\sigma)}|^2 |\psi_p|^\sigma (\log |e^{\psi_p}|)^{1+\varepsilon}} \\
&= \int_{U_p} \frac{d\text{vol}_{U_p}}{|z_{p(1)} \cdots z_{p(\sigma)}|^2 |\psi_p|^\sigma (\log |e^{\psi_p}|)^{1+\varepsilon}} \cdot \int_{\mathbb{P}_p^\sigma} |g_p|^2 e^{-\varphi} \, d\text{vol}_{\mathbb{P}_p^\sigma} < +\infty.
\end{align*}
\]

Therefore, the holomorphic function
\[
f := \sum_{p \in \mathbb{C}^\sigma V} g_p \, z_{p(\sigma+1)} \cdots z_{p(\sigma_V)} \quad \text{on } V
\]
satisfies \(\mathfrak{F}(\varepsilon)^2 \) < +\infty for all \(\varepsilon > 0\), and thus \(f \in \mathcal{A}(\varphi_L; \psi)_x\). One has \(\text{Res}_x(f) = (g_p)_{p \in \mathbb{C}^\sigma V}\) by construction. This completes the proof. \(\square\)

The following can be considered as a local \(L^2\) extension theorem for extending holomorphic sections on \(\sigma\)-lc centres locally with estimates. The constant of the estimate depends only “mildly” on the given metric.

**Corollary 4.3.2.** For any given \((g_p)_{p \in \mathbb{C}^\sigma V} \in \mathcal{A}_\sigma(\varphi)(V)\) on an admissible open set \(V\), the extension \(f \in \mathcal{A}_\sigma(\varphi_L; \psi)(V)\) constructed in the proof of Theorem 4.3.1 satisfies the estimate
\[
\mathfrak{F}(\varepsilon)^2 V,\sigma \leq C \sum_{p \in \mathbb{C}^\sigma V} \frac{\pi^\sigma}{(\sigma - 1)! \, \mu_p} \int_{\mathbb{P}_p^\sigma} |g_p|^2 e^{-\varphi} \, d\text{vol}_{\mathbb{P}_p^\sigma} = C \mathfrak{F}(0)^2 V,\sigma
\]
for all \(\varepsilon > 0\) and for some constant \(C > 0\) which depends only on the constants involved in \(\lesssim_\log\) in the inequality \(\varphi|_p \lesssim_\log \varphi\) for every \(p \in \mathbb{C}^\sigma V\) given in (eq 4.3.1). \(\text{(See (eq 4.1.3) for the definition of \(\mathfrak{F}(0)^2 V,\sigma\) when \(f\) is not compactly supported in \(V\) and see Theorem 4.1.2 (2) for the definition of \(\mu_p\).)}\)

**Proof.** This essentially follows from the estimates of integrals in the proof of Theorem 4.3.1 and the computation in [4, Proof of Prop. 2.2.1]. Following the notation in the proof of Theorem 4.3.1, one has
\[
\mathfrak{F}(\varepsilon)^2 V,\sigma \lesssim \sum_{p \in \mathbb{C}^\sigma V} \varepsilon \int_V |g_p \, z_{p(\sigma+1)} \cdots z_{p(\sigma_V)}|^2 e^{-\varphi - \psi} \, d\text{vol} \\
\text{pf. of Thm. 4.3.1} \lesssim \sum_{p \in \mathbb{C}^\sigma V} \varepsilon \int_{U_p} \frac{d\text{vol}_{U_p}}{|z_{p(1)} \cdots z_{p(\sigma)}|^2 |\psi_p|^\sigma (\log |e^{\psi_p}|)^{1+\varepsilon}} \cdot \int_{\mathbb{P}_p^\sigma} |g_p|^2 e^{-\varphi} \, d\text{vol}_{\mathbb{P}_p^\sigma},
\]
where the constant involved in \(\lesssim\) depends only on the constants involved in \(\lesssim_\log\) in the inequalities \(\varphi|_p \lesssim_\log \varphi\) for all \(p \in \mathbb{C}^\sigma V\) given in (eq 4.3.1).

---

\(^{15}\text{In Theorem 1.2.3 (3), the constant } C \text{ is described as depending only “mildly” on } \varphi \text{ because such kind of constants is “robust” under the approximation of quasi-psh functions. Indeed, if } \varphi \text{ has more general singularities and if } \varphi|_p \leq \varphi + C \text{ for some constant } C \geq 0, \text{ then the Bergman kernel approximations } \varphi|_p^{(k)} \text{ and } \varphi^{(k)} \text{ (with neat analytic singularities) of the quasi-psh potentials } \varphi|_p \text{ and } \varphi, \text{ respectively, satisfy } \varphi|_p^{(k)} \leq \varphi^{(k)} + C \text{ for the same } C. \text{ The variables of } \varphi|_p^{(k)} \text{ can be separated relatively easily on admissible open sets in favour of the computation using Fubini’s theorem as in the proof of Theorem 4.3.1. These will be discussed in details in the subsequent paper.}\)
It remains to estimate, for each $p \in \mathcal{C}^{\sigma \nu}_{\sigma}$, the integral

$$\mathcal{J}^1(\varepsilon) := \varepsilon \int_{U_p} \frac{d \text{vol}_{U_p}}{|z_{p(1)} \cdots z_{p(\sigma)}|^2} |\psi_p|^\sigma (\log|e\psi_p|)^{1+\varepsilon}.$$  

(Note that the symbol $\mathcal{J}$ is chosen to match with the one in [4, Proof of Prop. 2.2.1] so that [4, (eq 2.2.4)] can be applied directly.) For simplicity, assume that $U_p$ is a $\sigma$-disc with a uniform poly-radius $R$ (adjust the function $\rho$ defined below suitably when the poly-radius is not uniform). For any $\delta \in (0, R)$, let $\tilde{\rho} := \tilde{\rho}_\delta : \mathbb{R}_{\geq 0} \to [0, 1]$ be a non-increasing smooth function such that

$$\tilde{\rho}(t) = \begin{cases} 1 & \text{for } t \in [0, R - \delta], \\ 0 & \text{for } t \geq R. \end{cases}$$

Then, set

$$\rho := \prod_{j=1}^\sigma \rho_{p(j)} := \prod_{j=1}^\sigma \tilde{\rho} \circ |z_{p(j)}|^2.$$

Notice that one has

$$\mathcal{J}^\rho(\varepsilon) = \varepsilon \int_{U_p} \frac{\rho \ d \text{vol}_{U_p}}{|z_{p(1)} \cdots z_{p(\sigma)}|^2} |\psi_p|^\sigma (\log|e\psi_p|)^{1+\varepsilon} \to \mathcal{J}^1(\varepsilon) \quad \text{as} \quad \delta \to 0^+$$

for each $\varepsilon > 0$ by the dominated convergence theorem.

Let $(r_j, \theta_j)$ be the polar coordinate system of the $z_j$-plane and write $\frac{\partial}{\partial r}$ as $\partial_{r_j}$ for convenience. Now, applying [4, (eq 2.2.4)] to $\mathcal{J}^\rho(\varepsilon)$ and noting the deliberate choice of $\psi_p$ in the denominator of the integrand of $\mathcal{J}^\rho(\varepsilon)$, one obtains

$$\mathcal{J}^\rho(\varepsilon) \leq \frac{(-1)^\sigma}{(\sigma - 1)!} \int_{U_p} \frac{\partial_{r_{p(\sigma)}} \cdots \partial_{r_{p(1)}} \rho \prod_{j=1}^\sigma dr_{p(j)} d\theta_{p(j)}}{(\log|e\psi_p|)^\nu} \cdot \prod_{j=1}^\sigma \frac{\partial (-\rho_{p(j)})}{\partial r_{p(\sigma)}} dr_{p(j)}$$

$$= \frac{\pi^\sigma}{(\sigma - 1)!} \mathcal{L}_p \int_{[0, R]^{\sigma}} \frac{1}{(\log|e\psi_p|)^\nu} \prod_{j=1}^\sigma \frac{\partial (-\rho_{p(j)})}{\partial r_{p(\sigma)}} dr_{p(j)}$$

$$\leq \frac{\pi^\sigma}{(\sigma - 1)!} \mathcal{L}_p \int_{[0, R]^{\sigma}} \prod_{j=1}^\sigma \frac{\partial (-\rho_{p(j)})}{\partial r_{p(\sigma)}} dr_{p(j)} = \frac{\pi^\sigma}{(\sigma - 1)!} \mathcal{L}_p$$

for all $\varepsilon > 0$. Note that the last inequality above makes use of the facts that $\log|e\psi_p| \geq 1$ and $\frac{\partial (-\rho_{p(j)})}{\partial r_{p(j)}} \geq 0$ on $U_p$ for $j = 1, \ldots, \sigma$. Finally, letting $\delta \to 0^+$ yields

$$\mathcal{J}^1(\varepsilon) \leq \frac{\pi^\sigma}{(\sigma - 1)!} \mathcal{L}_p$$

for all $\varepsilon > 0$. This results in the desired estimate for $\mathcal{F}_V^{\sigma, \nu}(\varepsilon)$.

\[\square\]

5. The non-snc scenarios

Suppose in this section that $\varphi_L$ and $\psi$ are given as in Section 2.2 (thus having only neat analytic singularities in particular) and satisfy the Snc assumption 2.3.7 (but need not satisfy 2.3.7**). Let $\pi : \tilde{X} \to X$ be the log-resolution of $(X, \varphi_L, \psi)$ given in Snc assumption 2.3.7. Recall from Section 2.3 the decomposition $E_{d\pi} = E + R$ and the notation of the canonical sections $s_E$ and $s_R$ and the potential $\pi^* \varphi_L := \pi^* \varphi_L - \phi_R$. Let $\tilde{S}$ be the lc locus
of \( \{ \mathcal{I}_X(\pi \otimes \varphi L + m \pi^* \psi) \}_{m \in [0,1]} \) at the jumping number \( m = 1 \), which is a reduced snc divisor. Recall also that \( E \) and \( \tilde{S} \) have no common irreducible components.

Let \( \tilde{S}_0 \) be the divisor on \( \tilde{X} \) corresponding to the divisor \( S_0 \) described in Remark 2.3.8 with \( (\tilde{X}, \pi^* \varphi L, \pi^* \psi) \) in place of \( (X, \varphi L, \psi) \). Also let \( \tilde{\varphi} \) be the quasi-psh potential on \( \pi^* L \otimes R^{-1} \otimes \tilde{S}_0^{-1} \otimes \tilde{S}^{-1} \) corresponding to the potential \( \varphi \) in (eq2.3.1) such that

\[
\tilde{\varphi} + \phi_{\tilde{S}_0} + \phi_{\tilde{S}} := \pi^* \varphi L + \pi^* \psi,
\]
where \( \phi_{\tilde{S}_0} = \log|\tilde{s}_0|^2 \) and \( \phi_{\tilde{S}} = \log|\tilde{s}|^2 \), in which \( \tilde{s}_0 \) and \( \tilde{s} \) are respectively fixed canonical holomorphic sections of \( \tilde{S}_0 \) and \( \tilde{S} \) on \( \tilde{X} \). Moreover, let \( m_0 \in (0,1) \) be the number provided by Proposition 2.3.1 such that

\[
\mathcal{I}_{\tilde{X}}(\pi^* \varphi L + m_0 \pi^* \psi) = \mathcal{I}_{\tilde{X}}(\pi^* \varphi L + m \pi^* \psi) \supseteq \mathcal{I}_{\tilde{X}}(\pi^* \varphi L + \pi^* \psi) \quad \text{for all } m \in [m_0, 1).
\]

For definiteness, define the volume form \( d\text{vol}_{\tilde{X},(\pi^* \omega)_{>0}} \) on \( \tilde{X} \) given by

(eq5.0.1) \[
\pi^* d\text{vol}_{X,\omega} = \frac{\pi^* \omega^n}{n!} =: |S_E|^2 e^{\phi_E} d\text{vol}_{\tilde{X},(\pi^* \omega)_{>0}}.
\]

In this section, for the sake of notational convenience, the subscripts "\( \omega \)" and "\( (\pi^* \omega)_{>0} \)" are made implicit in the notation for volume forms, including those induced on the lc centres. (One can also avoid considering the volume form induced from \( \omega \) if \( (n,0) \)-forms are considered in the following discussion; see Remark 5.1.7.)

5.1. The direct image of the residue short exact sequence. Following the arguments in the proof of Proposition 2.3.1, since

\[
\int_V \left| f \right|^2 e^{-\varphi L - \psi} d\text{vol}_X = \int_{\pi^{-1}(V)} \left| \pi^* f \cdot S_E \right|^2 e^{-\pi^* \varphi L - \pi^* \psi} d\text{vol}_{\tilde{X}}
\]

for any \( f \in \mathcal{O}_X(V) \) on any open set \( V \subset X \) and for any \( \sigma \geq 0 \) and \( \varepsilon > 0 \) (notice also that \( \frac{e^{-\varphi L - \psi}}{|\psi|^{(\log|\psi|)^{1+\varepsilon}}} \geq C > 0 \) locally in \( X \) for some constant \( C \)), one immediately obtains

\[
\mathcal{I}_{X,\sigma}(\varphi L; \psi) \cdot \mathcal{O}_{\tilde{X}} \hookrightarrow \mathcal{I}_{\tilde{X},\sigma}(\pi^* \varphi L - \phi_E; \pi^* \psi) \xrightarrow{\otimes S_E} E \otimes \mathcal{I}_{\tilde{X},\sigma}(\pi^* \varphi L; \pi^* \psi) \quad \text{and}
\]

(eq5.1.1) \[
\mathcal{I}_{X,\sigma}(\varphi L; \psi) \cong \pi_* \mathcal{I}_{\tilde{X},\sigma}(\pi^* \varphi L - \phi_E; \pi^* \psi) \cong \pi_* \left( E \otimes \mathcal{I}_{\tilde{X},\sigma}(\pi^* \varphi L; \pi^* \psi) \right)
\]

for any integer \( \sigma \geq 0 \), in which both globally defined sheaf-homomorphisms on the far right-hand-side depend on the choice of \( S_E \). Recall that \( \text{lcm}_{\tilde{X}}(S) \) is defined via the definition of lc centres of \( (\tilde{X}, \tilde{S}) \) in [32, Def. 4.15] according to Remark 2.3.8.

Theorem 4.3.1 provides the residue short exact sequences

\[
0 \longrightarrow \mathcal{I}_{\tilde{X},\sigma-1}(\pi^* \varphi L - \phi_E; \pi^* \psi) \longrightarrow \mathcal{I}_{\tilde{X},\sigma}(\pi^* \varphi L - \phi_E; \pi^* \psi) \xrightarrow{\text{Res}} \mathcal{R}_{\tilde{X},\sigma}(\tilde{\varphi} - \phi_E) \longrightarrow 0
\]

\[
0 \longrightarrow E \otimes \mathcal{I}_{\tilde{X},\sigma-1}(\pi^* \varphi L; \pi^* \psi) \longrightarrow E \otimes \mathcal{I}_{\tilde{X},\sigma}(\pi^* \varphi L; \pi^* \psi) \xrightarrow{\text{Res}} E \otimes \mathcal{R}_{\tilde{X},\sigma}(\tilde{\varphi}) \longrightarrow 0
\]

for any integer \( \sigma \geq 1 \). From the facts that all involving divisors are in the snc configuration and that \( E \) contains no lc centres of \( (\tilde{X}, \tilde{S}) \) and no components of \( \tilde{\varphi}^{-1}(-\infty) \), one sees
that the map \( t_E \) is injective. Applying \( \pi_* \) to the above diagram and taking (eq 5.1.1) into account, a diagram-chasing argument then yields the exact sequence

\[
\begin{array}{cccccccc}
0 & \longrightarrow & \mathcal{I}_{X,\sigma-1}(\varphi_L; \psi) & \longrightarrow & \mathcal{I}_{X,\sigma}(\varphi_L; \psi) & \longrightarrow & \mathcal{I}_{X,\sigma}(\varphi_L; \psi) & \longrightarrow & 0 \\
& & \downarrow \cong & \downarrow \pi_* \Res & \downarrow \pi_* \Res & \downarrow \pi_* \Res & \downarrow \pi_* \Res & \downarrow 0 \\
& & \pi_* \mathcal{R}_{X,\sigma}(\overline{\varphi} - \Phi_E) & \longrightarrow & & & & & \\
\end{array}
\]

(eq 5.1.2)

In what follows, the sheaf \( E \otimes \mathcal{I}_{X,\sigma}(\pi_*\varphi_L; \pi^*\psi) \) is used as a model of \( \mathcal{I}_{X,\sigma}(\varphi_L; \psi) \) on \( \widetilde{X} \) via the log-resolution, but all results still hold true when \( \mathcal{I}_{X,\sigma}(\pi_*\varphi_L - \Phi_E; \pi^*\psi) \) is used as the model instead.

The homomorphism \( \pi_* \Res \) need not be surjective (so \( R^1\pi_*(E \otimes \mathcal{I}_{X,\sigma-1}(\pi_*\varphi_L; \pi^*\psi)) \neq 0 \) in general), as can be seen in the following example.

**Example 5.1.1.** Let \( X := \Delta^2 \subset \mathbb{C}^2 \) be the unit 2-disc centred at the origin \( \textcircled{0} = (0, 0) \) with holomorphic coordinates \( z_1 \) and \( z_2 \). Let also

\[
\varphi_L := 0 \quad \text{and} \quad \psi = \log|z_1|^2 + \log|z_2|^2 - 1.
\]

Then the family \( \{ \mathcal{I}_X(m\psi) \}_{m \in [0, 1]} \) has only \( m = 1 \) as a jumping number, with \( \mathcal{I}_X(m\psi) = \mathcal{O}_X \) for \( m \in [0, 1) \) and \( \mathcal{I}_X(\psi) = (z_1z_2) \). Therefore, \( S = S_1 + S_2 \), where \( S_j := \{ z_j = 0 \} \) for \( j = 1, 2 \). It can also be seen that

\[
lc^1_X(S) = S \quad \text{and} \quad lc^2_X(S) = S_1 \cap S_2 = \{ \textcircled{0} \}.
\]

Let \( \pi: \widetilde{X} \to X \) be the blow-up at the origin with an exceptional divisor \( R \) (and \( E = 0 \)). Write \( \widetilde{S}_j := \pi^{-1}S_j \) as the proper transform of \( S_j \) on \( \widetilde{X} \). Note that

\[
\pi^*\psi \sim_{\log} \phi_{\widetilde{S}_1} + \phi_{\widetilde{S}_2} + 2\phi_R
\]

(see Notation 2.1.3), so the family \( \{ \mathcal{I}_{\widetilde{X}}(-\phi_R + m\pi^*\psi) \}_{m \in [0, 1]} \) jumps only at \( m = 1 \) and \( \mathcal{I}_{\widetilde{X}}(-\phi_R + m\pi^*\psi) = \mathcal{O}_{\widetilde{X}} \) for all \( m \in (0, 1) \). The subvariety corresponding to \( \mathcal{I}_{\widetilde{X}}(-\phi_R + \pi^*\psi) \) is \( \widetilde{S}_1 + \widetilde{S}_2 + R \), so \( \widetilde{S} = \widetilde{S}_1 + \widetilde{S}_2 + R \) and thus

\[
lc^1_{\widetilde{X}}(\widetilde{S}) = \widetilde{S} \quad \text{and} \quad lc^2_{\widetilde{X}}(\widetilde{S}) = (\widetilde{S}_1 \cap R) \cup (\widetilde{S}_2 \cap R) = \{ p_1, p_2 \},
\]

where \( \{ p_j \} := \widetilde{S}_j \cap R \) for \( j = 1, 2 \) and \( lc^2_{\widetilde{X}}(\widetilde{S}) \) is the union of two distinct points. Note also that \( \pi(\{ p_1, p_2 \}) = \{ \textcircled{0} \} \). Moreover, \( \widetilde{S}_i = 0 \). From a local computation, \( \overline{\varphi} \) can be chosen to be the constant \(-1\).

The sheaf \( \mathcal{R}_{\widetilde{X},2}(\overline{\varphi}) \cong \mathcal{C}_{p_1} \oplus \mathcal{C}_{p_2} \) is thus a skyscraper sheaf supported at \( p_1 \) and \( p_2 \). For any 2-disc \( V = \Delta^2 \subset X \) of radius \( r \in (0, 1) \) centred at \( \textcircled{0} \) and for any section \( f \in \mathcal{I}_{\widetilde{X},2}(0; \psi)(V) = \pi_*\mathcal{I}_{\widetilde{X},2}(-\phi_R; \pi^*\psi)(V) \), the image \( \pi_*\Res(f) = \Res(\pi^*f) \) has to lie inside the diagonal of \( \pi_*\mathcal{R}_{\widetilde{X},2}(\overline{\varphi})(V) \cong \mathbb{C} \times \mathbb{C} \), so \( \pi_*\Res \) cannot be surjective at the origin \( \textcircled{0} \).

The following Theorem implies that the image of \( \pi_*\Res \) is indeed independent of the log-resolution \( \pi \) up to isomorphisms.

**Theorem 5.1.2.** Suppose that \( (X, \varphi_L, \psi) \) itself satisfies the Snc assumption 2.3.7*; and let \( \Res \) and \( \mathcal{R}_{X,\sigma}(\varphi) \) be the residue morphism and the target sheaf of \( \Res \) given in the
residue short exact sequence in Theorem 4.3.1. Then, there is an isomorphism $\tau$ between $\mathcal{R}_{X,\sigma}(\varphi)$ and $\im \pi_* \Res \subset \pi_* \left( E \otimes \mathcal{R}_{X,\sigma}(\varphi) \right)$ such that the diagram

\[
\begin{array}{c}
\mathcal{R}_{X,\sigma}(\varphi) \xrightarrow{\tau} \im \pi_* \Res \\
\end{array}
\]

commutes. Moreover, $\tau$ is locally an isometry in the sense that, for any admissible open set $V \in X$ and for any $(g_p)_{p \in \mathbb{C}^V} \in \mathcal{R}_{X,\sigma}(\varphi)(V)$ and any $f \in \mathcal{F}_{X,\sigma}(\varphi_L; \psi)(V)$ such that $(g_p)_{p \in \mathbb{C}^V} = \Res(f)$, one has $\tau(g_p)_{p \in \mathbb{C}^V} = \pi_* \Res(f) = \Res(\pi^* f \cdot \mathcal{S}_E) = (g_q \cdot \mathcal{S}_E)_{q \in \mathcal{I}_q}^\sigma$ (where $\mathcal{S}_q^\sigma$ for $q \in \mathcal{I}_q^\sigma$ are the $\sigma$-lc centres of $\left( \pi^{-1}(\mathcal{S}), \mathcal{S} \cap \pi^{-1}(V) \right)$ such that $\mathcal{S} \subset \bigcup_{q \in \mathcal{I}_q^\sigma} \mathcal{S}_q^\sigma$ and

\[
\int_{\mathcal{L}_{\varphi}(\mathbb{S})} |f|^2 d\lc_{\varphi, \sigma}^\psi = \int_{\mathcal{L}_{\pi^{-1}(\mathbb{S})}} |\pi^* f \cdot \mathcal{S}_E|^2 d\lc_{\pi^* \psi, \pi, \mathcal{S}_E}^\psi = \sum_{q \in \mathcal{I}_q^\sigma} (\sigma - 1)! \nu_q \int_{\mathbb{S}_q^\sigma} |g_q \cdot \mathcal{S}_E|^2 e^{-\varphi} d\delta_{\mathcal{S}_q^\sigma},
\]

(see (eq4.1)). or, more explicitly (according to Theorem 4.1.2 (2)),

\[
\sum_{p \in \mathbb{C}^V} \frac{\pi^\sigma}{(\sigma - 1)!} \nu_p \int_{\mathbb{S}_p^\sigma} |g_p|^2 e^{-\varphi} d\delta_{\mathcal{S}_p^\sigma} = \sum_{q \in \mathcal{I}_q^\sigma} \frac{\pi^\sigma}{(\sigma - 1)!} \nu_q \int_{\mathbb{S}_q^\sigma} |g_q \cdot \mathcal{S}_E|^2 e^{-\varphi} d\delta_{\mathbb{S}_q^\sigma},
\]

where $\nu_p$’s (resp. $\nu_q$’s) are the products of the generic Lelong numbers $\nu(\varphi, D_i)$ of $\psi$ (resp. $\nu(\pi^* \psi, D_i)$ of $\pi^* \psi$) along irreducible components $D_i$ of $S$ (resp. $D_i$ of $\mathcal{S}$) which contain $\mathcal{S}_p^\sigma$ (resp. $\mathcal{S}_q^\sigma$).

**Proof.** The well-definedness and bijectivity of the homomorphism $\tau$ follow immediately from the residue short exact sequence on $X$ (see Theorem 4.3.1) and the direct image of the one on $\mathbb{X}$ (see the beginning of Section 5.1).

To see that $\tau$ is an isometry, suppose that $V' \supset V$ is an admissible open set in the same coordinate system on $V$ and that $(g_p)_{p \in \mathbb{C}^{V'}} \in \mathcal{R}_{X,\sigma}(\varphi)(V')$. Let $\rho: V' \rightarrow [0, 1]$ be any compactly supported smooth cut-off function with $\rho|_V \equiv 1$. For any $f \in \mathcal{F}_{X,\sigma}(\varphi_L; \psi)(V')$ such that $(g_p)_{p \in \mathbb{C}^{V'}} = \Res(f)$ (which exists by the proof of Theorem 4.3.1, or from the fact that $V'$ is Stein and $\mathcal{F}_{X,\sigma^{-1}}(\varphi_L; \psi)$ is coherent such that $H^1(V' \setminus \mathcal{F}_{X,\sigma^{-1}}(\varphi_L; \psi)) = 0$) and $\pi_* \Res(f) = \tau(g_p)_{p \in \mathbb{C}^{V'}} = (g_q \cdot \mathcal{S}_E)_{q \in \mathcal{I}_q^{\sigma}}$, one has

\[
\delta^\rho(\epsilon)_{V', \sigma}^\psi = \epsilon \int_{V'} \frac{\rho |f|^2 e^{-\varphi - \psi} d\vol_X}{|\psi|^\epsilon (\log |e^\psi|)^{1+\epsilon}}
\]

\[
= \epsilon \int_{\pi^{-1}(V')} \frac{\pi^\sigma \rho |\pi^* f \cdot \mathcal{S}_E| e^{-\pi^\sigma \varphi - \pi^* \psi} d\vol_X}{|\pi^* \psi|^\epsilon (\log |e^{\pi^* \psi}|)^{1+\epsilon}} =: \delta^\rho(\epsilon)_{\pi^{-1}(V'), \sigma}
\]

for any $\epsilon > 0$. By Theorem 4.1.2 (2) (one may need to apply an argument with a partition of unity on $\pi^{-1}(V')$ when handling the integral on the right-hand-side), the above functions in $\epsilon$ can be continued analytically across $\epsilon = 0$ and thus it follows that

\[
\int_{\mathcal{L}_{\varphi}(\mathbb{S})} \rho |f|^2 d\lc_{\varphi, \sigma}^\psi = \delta^\rho(0)_{V', \sigma}
\]
Theorem 5.2.1 below implies that

Moreover, an

Definition 5.1.5.

even for the system

Remark 5.1.4

Theorem 4.1.2 (2) is still valid for smooth

σ

for all integers

Remark 5.1.4

Theorem 5.1.2 (2) is still valid for smooth

f

obtain a smooth extension

5.1.3

Remark

34 MARIO CHAN

V

J

sheaves

extended to general relatively compact open sets

5.1.3

Remark

Corollary 5.1.6.

log-resolution

setting, for any

The result of this section is summarised as follows.

For

Theorem 5.2.1 below implies that

Moreover, an

Definition 5.1.5.

even for the system

Remark 5.1.4

Theorem 4.1.2 (2) is still valid for smooth

f

obtain a smooth extension

5.1.3

Remark

34 MARIO CHAN

V

J

sheaves

extended to general relatively compact open sets

5.1.3

Remark

Corollary 5.1.6.

log-resolution

setting, for any

The result of this section is summarised as follows.

For

Theorem 5.2.1 below implies that

Moreover, an

Definition 5.1.5.

even for the system

Remark 5.1.4

Theorem 4.1.2 (2) is still valid for smooth

f

obtain a smooth extension

5.1.3

Remark

34 MARIO CHAN

V

J

sheaves

extended to general relatively compact open sets

5.1.3

Remark

Corollary 5.1.6.

log-resolution

setting, for any

The result of this section is summarised as follows.

For

Theorem 5.2.1 below implies that

Moreover, an

Definition 5.1.5.

even for the system

Remark 5.1.4

Theorem 4.1.2 (2) is still valid for smooth

f

obtain a smooth extension

5.1.3

Remark

34 MARIO CHAN

V

J

sheaves

extended to general relatively compact open sets

5.1.3

Remark

Corollary 5.1.6.

log-resolution

setting, for any

The result of this section is summarised as follows.

For

Theorem 5.2.1 below implies that

Moreover, an

Definition 5.1.5.

even for the system

Remark 5.1.4

Theorem 4.1.2 (2) is still valid for smooth

f

obtain a smooth extension

5.1.3

Remark

34 MARIO CHAN

V

J

sheaves

extended to general relatively compact open sets

5.1.3

Remark

Corollary 5.1.6.

log-resolution

setting, for any

The result of this section is summarised as follows.

For
is equipped with the $L^2$ residue norm $\|\cdot\|_{k^\sigma_X(\varphi_L,\psi)}$ with respect to the $\sigma$-lc measure while $E \otimes \mathcal{B}_X(\pi_\otimes \varphi_L; \pi^*\psi) (\pi^{-1}(V))$ (where $\mathcal{B}_X(\pi_\otimes \varphi_L; \pi^*\psi) := \mathcal{B}_X(\varphi)$) is equipped with $\|\cdot\|_{k^\sigma_{\pi^{-1}(V)}(\tilde{S})}$ with respect to the corresponding $\sigma$-lc measure (which is induced from the volume form on $\tilde{X}$ described in (eq 5.0.1)), then the monomorphism
\[
\tau: \mathcal{B}_X(\varphi) (V) \hookrightarrow E \otimes \mathcal{B}_X(\pi_\otimes \varphi_L; \pi^*\psi) (\pi^{-1}(V))
\]
is an isometric embedding.

**Remark 5.1.7.** Using the isomorphism
\[
K_X \otimes \mathcal{I}_X(\varphi_L; \psi) \cong \pi_* \left( K_X \otimes R^{-1} \otimes \mathcal{I}_X^\sigma(\pi_\otimes \varphi_L; \pi^*\psi) \right)
\]
given by $f \mapsto \pi(f)$ in place of (eq 5.1.1), Theorem 5.1.2 and Corollary 5.1.6 can be reformulated for $K_X \otimes \mathcal{I}_X(\varphi_L; \psi)$. The volume form on $\tilde{X}$ described in (eq 5.0.1) is not needed and the residue norm $\|\cdot\|_{k^\sigma_X(\varphi_L; \psi)}$ thus constructed on $K_X \otimes \mathcal{B}_X(\varphi_L; \psi) (V)$ is independent of $\omega$. The isometry between $K_X \otimes \mathcal{B}_X(\varphi_L; \psi)$ and its image in $\pi_* \left( K_X \otimes R^{-1} \otimes \mathcal{B}_X(\varphi) \right)$ is therefore a more intrinsic property.

**Remark 5.1.8.** Using the vanishing of the higher direct image $R^1\pi_* \left( E \otimes \mathcal{I}_X(\pi_\otimes \varphi_L + \pi^*\psi) \right) = 0$ (see [33, Thm. 9.4.1] in the algebraic setting and [36, Cor. 1.5] in the analytic setting), one indeed always has $\mathcal{B}_X(\varphi_L; \psi) \cong \pi_* \left( E \otimes \mathcal{B}_X(\varphi) \right)$.

### 5.2. Definition of $\sigma$-lc centres via adjoint ideal sheaves.

Under the assumption that both $\varphi_L$ and $\psi$ have only neat analytic singularities and according to Theorem 4.1.2 (1) and Remark 4.1.5, one has
\[
\mathcal{I}_X(\pi_\otimes \varphi_L; \pi^*\psi) = \mathcal{I}_X(\pi_\otimes \varphi_L + m_0\pi^*\psi) \cdot I_{k^\sigma_X(\tilde{S})} = \mathcal{I}_X(\varphi + \phi_{S_0}) \cdot I_{k^\sigma_X(\tilde{S})} \quad \text{for all integers } \sigma \geq 0.
\]
Recall again that $\text{lc}_X^\sigma(\tilde{S})$ is defined via the definition of lc centres of $(\tilde{X}, \tilde{S})$ in [32, Def. 4.15] according to Remark 2.3.8, so every irreducible component of $\text{lc}_X^\sigma(\tilde{S})$ is precisely a connected component of the intersection of some choice of $\sigma$ irreducible components of $\tilde{S}$, as $\tilde{S}$ is a reduced snc divisor. Moreover, it is easy to see from a direct computation that
\[
\text{Ann}_{\mathcal{I}_X} \left( \frac{\mathcal{I}_X(\varphi_L; \psi)}{\mathcal{I}_X(\varphi_L; \psi)} \right) = \text{Ann}_{\mathcal{I}_X} \left( \frac{\mathcal{I}_X(\varphi + \phi_{S_0}) \cdot I_{k^\sigma_X(\tilde{S})}}{\mathcal{I}_X(\varphi + \phi_{S_0}) \cdot I_{k^\sigma_X(\tilde{S})}} \right) = I_{k^\sigma_X(\tilde{S})}.
\]
This indeed makes sense as the residue short exact sequence implies that
\[
\mathcal{I}_X(\pi_\otimes \varphi_L; \pi^*\psi) \cong \mathcal{I}_X(\pi_\otimes \varphi_L; \pi^*\psi) \quad \text{and} \quad \text{Ann}_{\mathcal{I}_X} \left( \mathcal{I}_X(\pi_\otimes \varphi_L; \pi^*\psi) \right)
\]
defines the support of $\mathcal{I}_X(\pi_\otimes \varphi_L; \pi^*\psi)$ (at least set-theoretically), which is exactly $\text{lc}_X^\sigma(\tilde{S})$ (note that the zero locus of $\text{Ann} \mathcal{F}$ of a coherent sheaf $\mathcal{F}$ is exactly the support of $\mathcal{F}$; see, for example, [18, A.4.5]). Notice also that
\[
\text{Ann}_{\mathcal{I}_X} \left( \frac{\mathcal{I}_X(\varphi_L; \psi)}{\mathcal{I}_X(\varphi_L; \psi)} \right) = \text{Ann}_{\mathcal{I}_X} \left( \mathcal{I}_X(\varphi_L; \psi) \right)
\]
is a radical ideal sheaf since so is $I_{k^\sigma_X(\tilde{S})}$.

The following proposition essentially shows that the union of $\sigma$-lc centres of $(X, \varphi_L, \psi)$ can be defined independent of log-resolutions.
Theorem 5.2.1. Under the assumption that $\varphi_L$ has only neat analytic singularities such that the Snc assumption 2.3.7 is satisfied (but $(X, \varphi_L, \psi)$ may not satisfy the Snc assumption 2.3.7), for any log-resolution $\pi: \tilde{X} \to X$ of $(X, \varphi_L, \psi)$, one has

$$\text{Ann}_{\mathcal{O}_X} \left( \frac{\mathcal{J}_{X,\sigma}(\varphi_L; \psi)}{\mathcal{J}_{X,\sigma-1}(\varphi_L; \psi)} \right) = \pi_* \left( E \otimes \mathcal{R}_{X,\sigma}(\varphi_L; \psi) \right)$$

and the above ideal sheaf is a radical ideal sheaf. This implies that the sheaves $\mathcal{R}_{X,\sigma}(\varphi_L; \psi)$ and $\pi_* \left( E \otimes \mathcal{R}_{X,\sigma}(\varphi_L; \psi) \right)$ have the same support in particular. Let $\text{lc}_X(\varphi_L; \psi)$ be the analytic subset defined by the above ideal sheaf. Then one has, in general,

$$\text{lc}_X(\varphi_L; \psi) \subset \pi(\text{lc}_X(\tilde{S})) .$$

If $\mathcal{S}_q$ is a $\sigma$-lc centre in $\text{lc}_X(\tilde{S})$ such that $\pi(\mathcal{S}_q) \not\subset \text{lc}_X(\varphi_L; \psi)$, then $\pi^* f \bigg|_{\mathcal{S}_q} \equiv 0$ on $\mathcal{S}_q \cap \pi^{-1}(V)$ (or, equivalently, the component of $\text{Res}(\pi^* f)$ on $\mathcal{S}_q \cap \pi^{-1}(V)$ vanishes identically) for all $f \in \mathcal{J}_{X,\sigma}(\varphi_L; \psi)(V)$ and any open set $V \Subset X$ such that $\mathcal{S}_q \cap \pi^{-1}(V) \neq \emptyset$.

Note that if $(X, \varphi_L, \psi)$ already satisfies the Snc assumption 2.3.7, the above ideal sheaf is indeed $\mathcal{I}_{\text{lc}_X(\tilde{S})}$ and $\text{lc}_X(\varphi_L; \psi) = \text{lc}_X(\tilde{S})$.

Proof. It follows from the residue short exact sequence (see Theorem 4.3.1 or Corollary 5.1.6) that the question under consideration is reduced to proving the claim

$$\text{Ann}_{\mathcal{O}_X} \left( \mathcal{R}_{X,\sigma}(\varphi_L; \psi) \right) = \text{Ann}_{\mathcal{O}_X} \left( \pi_* \mathcal{R}_{X,\sigma}(\varphi_L; \psi) \right) .$$

It follows immediately from Remark 5.1.8 (a consequence of the vanishing of the higher direct images) that the above equality (on the left) always holds true when $\sigma = 1$. The proof below deals with the cases where $\sigma \geq 2$ (although it is also applicable to the case $\sigma = 1$).

Recall that the sheaf $\mathcal{R}_{X,\sigma}(\varphi_L; \psi)$ is given as a direct sum $\bigoplus_{q \in \mathcal{I}_X} \mathcal{F}_{\mathcal{S}_q}$, where each

$$\mathcal{F}_{\mathcal{S}_q} = (i_q)_* \left( \mathcal{S}_q^{-1} \|_{\mathcal{S}_q} \otimes (\text{Diff}_{\mathcal{S}_q}(\tilde{S}))^{-1} \otimes \mathcal{F}_{\mathcal{S}_q}(\varphi_L; \psi) \right)$$

is, before taking the direct image $(i_q)_*$ via the inclusion $i_q: \mathcal{S}_q \to \tilde{X}$, a torsion-free $\mathcal{O}_{\mathcal{S}_q}$-sheaf (so its support is precisely $\mathcal{S}_q$). By setting

$$\text{lc}_{\tilde{X}}(\tilde{S})' := \bigcup_{q \in \mathcal{I}_X : \pi_* \mathcal{F}_{\mathcal{S}_q} \neq 0} \mathcal{S}_q \subset \bigcup_{q \in \mathcal{I}_X} \mathcal{S}_q = \text{lc}_{\tilde{X}}(\tilde{S}) ,$$

which is the union of $\sigma$-lc centres in $\text{lc}_{\tilde{X}}(\tilde{S})$ which contribute to the zero locus of the annihilator $\text{Ann}_{\mathcal{O}_X} \left( \pi_* \mathcal{R}_{X,\sigma}(\varphi_L; \psi) \right)$ of $\pi(\text{lc}_{\tilde{X}}(\tilde{S})')$ in $X$ (it is shown below that the annihilator is indeed radical). If the equality between the annihilators in the claim holds true, it then implies that $\text{lc}_{X}(\varphi_L; \psi) = \pi(\text{lc}_{\tilde{X}}(\tilde{S})') \subset \text{lc}_{X}(\tilde{S})$. This also implies that, if $\mathcal{S}_q'$ is a $\sigma$-lc centre in $\text{lc}_{X}(\tilde{S})$ such that $\pi(\mathcal{S}_q') \not\subset \text{lc}_{X}(\varphi_L; \psi)$, one then has $\mathcal{S}_q' \not\subset \text{lc}_{X}(\tilde{S})'$ and thus $\pi_* \mathcal{F}_{\mathcal{S}_q} = 0$.

The claim on the component of $\text{Res}(\pi^* f)$ on $\mathcal{S}_q \cap \pi^{-1}(V)$ for any $f \in \mathcal{J}_{X,\sigma}(\varphi_L; \psi)(V)$ then follows.

Back to the proof of the equality between the annihilators. Recall from Definition 5.1.5 (together with the isomorphism in (eq 5.1.2)) that $\mathcal{R}_{X,\sigma}(\varphi_L; \psi) := \text{im} \pi_* \mathcal{R}_{\tilde{X},\sigma}(\varphi_L; \psi) \otimes \mathcal{O}_X$.
\( \phi_E \). Therefore, one immediately has

\[
\text{Ann}_{\sigma} (\mathcal{R}_{X, \sigma}(\varphi_L; \psi)) \supset \text{Ann}_{\sigma} \left( \pi_* \mathcal{R}_{\mathcal{X}, \sigma}(\varphi - \phi_E) \right).
\]

If \( \text{Ann}_{\sigma} \left( \pi_* \mathcal{R}_{\mathcal{X}, \sigma}(\varphi - \phi_E) \right) = \mathcal{O}_X \), the reverse inclusion follows automatically. It remains to prove the reverse inclusion under the assumption that \( \text{Ann}_{\sigma} \left( \pi_* \mathcal{R}_{\mathcal{X}, \sigma}(\varphi - \phi_E) \right) \neq \mathcal{O}_X \) (so \( \text{lc}_{X}^\sigma (\mathcal{S})' \neq \emptyset \)). For that, take any open polydisc \( V \subseteq X \) (in some coordinate chart) and any \( h \in \text{Ann}_{\sigma} (\mathcal{R}_{X, \sigma}(\varphi_L; \psi))(\mathcal{V}) \). Suppose \( \text{lc}_{X}^\sigma (\mathcal{S}) \cap \pi^{-1}(V) = \bigcup_{q \in \mathcal{I}_V' \cap \mathcal{B}_{V,q}} \mathcal{S}_{V,q} \) is the union of the \( \sigma \)-lc centres in \( \pi^{-1}(V) \) and let \( g = \tau^{-1} g = \tau^{-1} (\bar{g}_q)_{q \in \mathcal{I}_V'} \in \mathcal{R}_{X, \sigma}(\varphi_L; \psi)(\mathcal{V}) = \text{im} \pi_* \text{Res}(\mathcal{V}) \subset \mathcal{R}_{\mathcal{X}, \sigma}(\varphi - \phi_E)(\pi^{-1}(V)) \) (in the notation in Theorem 5.1.2 and Corollary 5.1.6) be an element such that each \( \bar{g}_q \) is defined on the \( \sigma \)-lc centre \( \mathcal{S}_{V,q}^\sigma \). Considering the residue norm (given in Corollary 5.1.6) of \( h \cdot g = \tau^{-1}(\pi^* h \bar{g}_q)_{q \in \mathcal{I}_V'} \), it yields

\[
0 = \| h \cdot g \|_{L^1(\pi^*(\varphi_L;\psi); \mathcal{V})}^2 = \sum_{q \in \mathcal{I}_V'} \frac{\pi^\sigma}{(\sigma - 1)!} \left| \int_{\mathcal{S}_{V,q}^\sigma} |\pi^* h \bar{g}_q \cdot \mathcal{S}_E| e^{-\varphi} d \text{vol}_{\mathcal{S}_{V,q}^\sigma} \right|^2 \tag{\star}
\]

It thus follows that, if \( \bar{g}_q \neq 0 \) on \( \mathcal{S}_{V,q}^\sigma \), then \( \pi^* h \equiv 0 \) on \( \mathcal{S}_{V,q}^\sigma \) and if this holds true for all \( \sigma \)-lc centres \( \mathcal{S}_{V,q}^\sigma \subset \text{lc}_{X}^\sigma (\mathcal{S})' \cap \pi^{-1}(V) \), then \( h \in \text{Ann}_{\sigma} \left( \pi_* \mathcal{R}_{\mathcal{X}, \sigma}(\varphi - \phi_E) \right)(V) \) and the desired inclusion follows.

(The formula (\star) also shows that both of the annihilators \( \text{Ann}_{\sigma} \left( \pi_* \mathcal{R}_{\mathcal{X}, \sigma}(\varphi - \phi_E) \right) \) and \( \text{Ann}_{\sigma} (\mathcal{R}_{X, \sigma}(\varphi_L; \psi)) \) are radical ideal sheaves. One sees this by replacing \( h \) by \( h^r \) for some integer \( r > 0 \) in (\star) and assuming \( \bar{g}_q \) to be an element in \( \pi_* \mathcal{R}_{\mathcal{X}, \sigma}(\varphi - \phi_E) \) or \( \mathcal{R}_{X, \sigma}(\varphi_L; \psi) \) (and \( h^r \) being an element in the corresponding annihilator). Since \( \pi^* h^r \bar{g}_q \equiv 0 \) on \( \mathcal{S}_{V,q}^\sigma \) implies that \( \pi^* h \bar{g}_q \equiv 0 \) on \( \mathcal{S}_{V,q}^\sigma \), the equality \( \| h^r \cdot g \|_{L^1(\pi^*(\varphi_L;\psi); \mathcal{V})}^2 = 0 \) then implies that \( \| h \cdot g \|_{L^1(\pi^*(\varphi_L;\psi); \mathcal{V})}^2 = 0 \), and thus the corresponding annihilator is a radical ideal sheaf.)

Write \( \mathcal{J}_\sigma \) for \( \mathcal{J}_{X, \sigma}(\varphi_L; \psi) \) and \( \tilde{\mathcal{J}}_\sigma \) for \( \pi_* \mathcal{R}_{\mathcal{X}, \sigma}(\varphi - \phi_E; \pi^* \psi) \) for convenience, and suppose that \( \xi_1, \ldots, \xi_r \in \mathcal{J}_{X, \sigma}(\varphi_L; \psi)(\mathcal{V}) \) are generators of \( \mathcal{J}_\sigma \) on a neighbourhood of \( \mathcal{V} \subseteq X \). Note that \( \pi^* \xi_1, \ldots, \pi^* \xi_r \) are then generators of \( \tilde{\mathcal{J}}_\sigma \) on a neighbourhood of \( \pi^{-1}(\mathcal{V}) \). To complete the proof, it suffices to show that, for any \( q' \in \mathcal{I}_\sigma' \) such that \( \mathcal{S}_{V,q'}^\sigma \subset \text{lc}_{X}^\sigma (\mathcal{S})' \cap \pi^{-1}(V) \), there exists \( \xi \) among the generators \( \xi_1, \ldots, \xi_r \) of \( \mathcal{J}_\sigma \) on a neighbourhood of \( \mathcal{V} \) such that, by setting \( g_i := \text{Res}(\xi_i) \) (thus Res(\( \pi^r \xi_i \)) = \( \bar{g}_i = (\bar{g}_{i,q})_{q \in \mathcal{I}_\sigma'} \)), one has \( \bar{g}_{i,q'} \neq 0 \) on \( \mathcal{S}_{V,q'}^\sigma \).

Suppose, on the contrary, that \( \bar{g}_{i,q'} \equiv 0 \) on \( \mathcal{S}_{V,q'}^\sigma \) for all \( i = 1, \ldots, r \). Recall the direct sum decomposition \( \pi_* \mathcal{R}_{\mathcal{X}, \sigma}(\varphi - \phi_E)(\mathcal{V}) = \bigoplus_{q \in \mathcal{I}_\sigma'} \pi_* \mathcal{J}_{\mathcal{S}_{V,q}^\sigma}(\mathcal{V}) \) (such that each \( \pi_* \mathcal{J}_{\mathcal{S}_{V,q}^\sigma} \) is supported on \( \mathcal{S}_{V,q}^\sigma \)). Note that \( \pi_* \mathcal{J}_{\mathcal{S}_{V,q}^\sigma}(\mathcal{V}) \) is non-trivial as \( \mathcal{S}_{V,q}^\sigma \subset \text{lc}_{X}^\sigma (\mathcal{S})' \cap \pi^{-1}(V) \). Take any non-trivial holomorphic section \( \bar{g} = (\bar{g}_q)_{q \in \mathcal{I}_\sigma'} \) in \( \bigoplus_{q \in \mathcal{I}_\sigma'} \pi_* \mathcal{J}_{\mathcal{S}_{V,q}^\sigma}(\mathcal{V}) \) whose support contains \( \mathcal{S}_{V,q'}^\sigma \) (one may conveniently take \( \bar{g}_q \equiv 0 \) on \( \mathcal{S}_{V,q}^\sigma \) for all \( q \in \mathcal{I}_\sigma' \setminus \{q'\} \) and pick any non-trivial \( \bar{g}_{q'} \in \mathcal{J}_{\mathcal{S}_{V,q'}^\sigma}(\pi^{-1}(\mathcal{V})) \)). Let \( \{U_\alpha \}_{\alpha \in \Gamma} \) be an open cover of \( \pi^{-1}(\mathcal{V}) \) and \( \{\rho_\alpha \}_{\alpha \in \Gamma} \) be a partition of unity subordinate to this cover. The cover is assumed to be
sufficiently fine such that every section of $F_{U_\alpha}$ on each $U_\alpha$ has a lifting to $\widetilde{F}_\sigma$ on $U_\alpha$. The section $\widetilde{g}$ can then be expressed as a Čech cocycle $\left\{ \left[ \tilde{f}_\alpha \right] \right\}_{\alpha \in \Gamma}$, where $\tilde{f}_\alpha \in \widetilde{\mathcal{F}}_\sigma(U_\alpha)$ and $\left[ \tilde{f}_\alpha \right] := \tilde{f}_\alpha \text{ mod } \widetilde{\mathcal{F}}_{\sigma-1}(U_\alpha)$ for each $\alpha \in \Gamma$ such that $\tilde{f}_\beta - \tilde{f}_\alpha \in \widetilde{\mathcal{F}}_{\sigma-1}(U_\alpha \cap U_\beta)$ for any $\alpha, \beta \in \Gamma$ with $U_\alpha \cap U_\beta \neq \emptyset$. Set also $\tilde{F} := \sum_{\alpha \in \Gamma} \rho_\alpha \tilde{f}_\alpha \in \mathcal{C}_X^\infty \cdot \mathcal{F}_\sigma(\pi^{-1}(\overline{V}))$.

Note that $\tilde{g} = (\tilde{g}_\alpha)_{\alpha \in \Gamma} = \overline{\text{Res}}(\tilde{F})$ (after extending $\overline{\text{Res}}$ to a $\mathcal{C}_X^\infty$-homomorphism). Pick any admissible open set $U \subseteq \pi^{-1}(V)$ with respect to $(\pi_\sigma^* \varphi_L, \pi^* \psi)$ such that $U \cap \text{lc}_X^\sigma(\overline{S}) = U \cap \overline{\mathcal{S}}_{V_{V'}}^\sigma$ (i.e. $U$ intersects no $\sigma$-lc centres other than $\overline{\mathcal{S}}_{V_{V'}}^\sigma$). The computation of residue norm (see Theorem 4.1.2 (2), Remark 4.1.6 and (eq 4.1.3)) yields

$$\left( |F(0)|_{U,\sigma} \right)^2 = \lim_{\rho \to 1} \lim_{\varepsilon \to 0^+} \frac{\left| \overline{\text{Res}}(\tilde{F}) \right|^2}{\mathcal{S}(\varepsilon)|\mathcal{S}(\varepsilon)|} \mathcal{S}(\varepsilon)^{-1} |\mathcal{S}(\varepsilon)|^2 \mathcal{S}(\varepsilon)^{-1}$$

where $U' \supseteq U$ is some relatively compact subset in $\pi^{-1}(V)$ and $\rho$ runs through a decreasing sequence of compactly supported, non-negative, smooth cut-off functions on $U'$ which are $\equiv 1$ on $U$. It suffices to show that $\tilde{g}_\alpha \equiv 0$ on $U \cap \overline{\mathcal{S}}_{V_{V'}}^\sigma$, which implies that $\tilde{g}_\alpha \equiv 0$ on the whole of $\overline{\mathcal{S}}_{V_{V'}}^\sigma$ by the identity theorem, to obtain a contradiction.

The function $\tilde{F}$ can be written as a sum $\sum_{i=1}^r a_i \pi^* \xi_i$ with $a_i$'s being smooth functions on a neighbourhood of $\pi^{-1}(\overline{V})$ thanks to the facts that $\pi^* \xi_1, \ldots, \pi^* \xi_r$ generate $\mathcal{F}_\sigma$ (which induce epimorphisms $\mathcal{O}_X^\infty \to \mathcal{F}_\sigma \to 0$ and $\mathcal{C}_X^\infty \to \mathcal{F}_\sigma \to 0$) on the neighbourhood and all $\mathcal{C}_X^\infty$-modules are soft (see, for example, [19, A.4.1 and A.4.4]; the neighbourhood of $\pi^{-1}(\overline{V})$ can, of course, be taken to be paracompact). As all coefficients $a_i$ in the sum $\tilde{F} = \sum_{i=1}^r a_i \pi^* \xi_i$ are bounded on a neighbourhood $U'$ of $\overline{U}$, it follows that

$$\left| \frac{\tilde{g}_\alpha (\tilde{F})}{\mathcal{S}(\varepsilon)_{U',\sigma}} \right| \lesssim \sum_{i=1}^r a_i \frac{|\pi^* \xi_i|}{\mathcal{S}(\varepsilon)_{U',\sigma}} \xrightarrow{\varepsilon \to 0^+} 0$$

by the assumption on $\xi_i$'s that $\tilde{g}_\alpha \equiv 0$. This gives the contradiction and thus concludes the proof.

The sets $\text{lc}_X^\sigma(\varphi_L; \psi)$ and $\pi(\text{lc}_X^\sigma(\overline{S}))$ are different in general, as is illustrated in the following examples.

**Example 5.2.2.** Example 2.3.4 (which is an example having $S \subsetneq \pi(\overline{S})$) also provides an example with $\text{lc}_X^\sigma(\varphi_L; \psi) \subsetneq \pi(\text{lc}_X^\sigma(\overline{S}))$ for $\sigma = 1, 2$. Following the notation and the computation there, one already has $\text{lc}_X^1(\overline{S}) = \widetilde{S} = \widetilde{S}_1 + R$, $\text{lc}_X^2(\overline{S}) = \widetilde{S}_1 \cap R$, $\mathcal{F}_X(\psi) = \langle z_1 z_2 \rangle$ and $\mathcal{F}_{X,2}(0; \psi) = \mathcal{F}_X(0; \psi) = \langle z_2 \rangle$. Furthermore, it can also be seen that

$$\text{div}(\pi^* z_1) = \widetilde{S}_1, \quad \text{div}(\pi^* z_2) = \widetilde{S}_2 + R, \quad \text{div}(\pi^* z_3) = \widetilde{S}_3 + R$$

and

$$\mathcal{F}_{X,1}(0; \psi) = \pi_\sigma \mathcal{F}_{X,1}(-\phi_R; \pi^* \psi) \quad \text{Thm. 4.1.2 (1)} = \pi_\sigma \left( \mathcal{F}_X(3 \phi_{\mathcal{S}_1}) \cdot \mathcal{I}_{\mathcal{S}_1 \cap R} \right) = \pi_\sigma \left( \mathcal{O}_X(\mathcal{S}_1) \cdot \mathcal{I}_{\mathcal{S}_1 \cap R} \right) = \langle z_1 z_2, z_2 \rangle.$$

As a result, $\mathcal{I}_{\text{lc}_X^1(0; \psi)} = \text{Ann}_X \mathcal{I}_{\mathcal{F}_{X,1}(0; \psi)} = \mathcal{O}_X$ and $\mathcal{I}_{\text{lc}_X^2(0; \psi)} = \text{Ann}_X \mathcal{I}_{\mathcal{F}_{X,2}(0; \psi)} = \langle z_1 \rangle$, i.e. $\text{lc}_X^1(0; \psi) = \emptyset \subsetneq \pi(\widetilde{S}_1 \cap R) = \pi(\text{lc}_X^1(\overline{S}))$ and $\text{lc}_X^2(0; \psi) = S_1 \subsetneq \pi(\widetilde{S}_1 \cup R) = \pi(\text{lc}_X^1(\overline{S}))$. 


Note that $\pi(R) = \{z_2 = z_3 = 0\}$ is lying in the “base locus” of $\mathcal{I}_{X,1}(0;\psi)$, i.e. $\pi^*f|_R \equiv 0$ for all $f \in \mathcal{I}_{X,1}(0;\psi)$.

**Example 5.2.3.** This example provides a simple instance which has $S = \pi(\tilde{S})$ but $\text{lc}_X^0(\mathcal{I}(\mathcal{I}_{X,1}(0;\psi)) \subseteq \pi(\text{lc}_X^0(\tilde{S}))$. Let $X := \Delta^2 \subset \mathbb{C}^2$ be the unit 2-disc centred at the origin under the holomorphic coordinate system $(z_1, z_2)$. Take

$$\varphi_L := \log|z_1|^2 + \log|z_2|^2 \quad \text{and} \quad \psi := \log(|z_1|^2 + |z_2|^2),$$

and consider the modification $\pi: \tilde{X} \to X$ which is the blow-up at the origin followed by another blow-up at a general point on the exceptional divisor (i.e. a point away from the proper transforms of $S_1 := \{z_1 = 0\}$ and $S_2 := \{z_2 = 0\}$). Let $R_1$ be the proper transform of the exceptional divisor of the first blow-up and $R_2$ be the exceptional divisor of the second blow-up, which gives

$$K_{\tilde{X}} \sim \pi^*K_X + R_1 + 2R_2.$$ 

Also let $\tilde{S}_1$ and $\tilde{S}_2$ be the proper transforms of $S_1$ and $S_2$ respectively. One then has

$$\text{div}(\pi^*z_1) = \tilde{S}_1 + R_1 + R_2 \quad \text{and} \quad \text{div}(\pi^*z_2) = \tilde{S}_2 + R_1 + R_2$$

and

$$\pi^*\varphi_L \sim_{\log} \phi_{\tilde{S}_1} + \phi_{\tilde{S}_2} + \phi_{R_1} \quad \text{and} \quad \pi^*\psi \sim_{\log} \phi_{R_1} + \phi_{R_2}$$

in the notation in Notation 2.1.3. Note also that $E = 0$. This shows that the family $\{\mathcal{I}_{X}(\varphi_L + m\varphi_L^\mu(\psi))\}_{m \in \mathbb{R}_{\geq 0}}$ has jumping numbers $m = \mu \in \mathbb{N}$ (which need not be the jumping numbers of $\{\mathcal{I}_{X}(\varphi_L + m\psi)\}_{m \in \mathbb{R}_{\geq 0}}$) with lc locus

$$\tilde{S} = R_1 + R_2$$

for every jumping number $\mu$. Using Theorem 4.1.2 (1) together with the equality $\mathcal{I}_{X,\sigma}(\varphi_L;\mu;\psi) = \pi_*\mathcal{I}_{\tilde{X},\sigma}(\varphi_L;\mu;\pi^*\psi)$ and the fact that the ideal sheaves in question are toric (thus generated by monomials), the adjoint ideal sheaves for every jumping number $m = \mu$ are computed as follows:

$$\begin{align*}
\mathcal{I}_{X,2}(\varphi_L;\mu;\psi) &= \pi_*\mathcal{O}_{\tilde{X}}\left(-\tilde{S}_1 - \tilde{S}_2 - \mu R_1 - (\mu - 1)R_2\right) \\
&= (z_1^p z_2^q \mid p \geq 1, q \geq 1, p + q \geq \mu), \\
\mathcal{I}_{X,1}(\varphi_L;\mu;\psi) &= \pi_*\left(\mathcal{O}_{\tilde{X}}\left(-\tilde{S}_1 - \tilde{S}_2 - \mu R_1 - (\mu - 1)R_2\right) \cdot \mathcal{I}_{R_1 \cap R_2}\right) \\
&= (z_1^p z_2^q \mid p \geq 1, q \geq 1, p + q \geq \mu), \\
\mathcal{I}_{X,0}(\varphi_L;\mu;\psi) &= \pi_*\mathcal{O}_{\tilde{X}}\left(-\tilde{S}_1 - \tilde{S}_2 - (\mu + 1)R_1 - \mu R_2\right) \\
&= (z_1^p z_2^q \mid p \geq 1, q \geq 1, p + q \geq \mu + 1).
\end{align*}$$

It follows that $m = \mu$ is a jumping number of $\{\mathcal{I}_{X}(\varphi_L + m\psi)\}_{m \in \mathbb{R}_{\geq 0}}$ if and only if $\mu \geq 2$ (note that $\mathcal{I}_{X}(\varphi_L + \psi) = \mathcal{I}_{X,0}(\varphi_L;\psi) = \mathcal{I}_{X,2}(\varphi_L;\psi) = \mathcal{I}_{X}(\varphi_L)$). Moreover, for $\mu \geq 2$, one has $I_{S} = \text{Ann}_{\mathcal{O}_{\tilde{X}}}(\mathcal{I}_{X,2}(\varphi_L;\mu;\psi)) = (z_1, z_2)$ (which is also equal to $I_{\text{lc}_{\tilde{X}}(\varphi_L;\mu;\psi)} = \text{Ann}_{\mathcal{O}_{\tilde{X}}}(\mathcal{I}_{X,2}(\varphi_L;\mu;\psi))$) and $I_{\text{lc}_{\tilde{X}}(\varphi_L;\mu;\psi)} = \mathcal{O}_{\tilde{X}}$, thus $S = \{(0,0)\} = \pi(R_1 \cup R_2) = \pi(\tilde{S})$ and $\text{lc}_{\tilde{X}}(\varphi_L;\mu;\psi) = 0 \subseteq \{(0,0)\} = \pi(R_1 \cap R_2) = \pi(\text{lc}_{\tilde{X}}(\tilde{S}))$. Note also that $\tilde{S}_0 = \tilde{S}_0^\mu = \mu R_1 + (\mu - 1)R_2$ for every jumping number $\mu$. Writing $\tilde{s}_0^\mu$ as a canonical section of $\tilde{S}_0^\mu$, one has $\mathcal{I}_{f|_{R_1 \cap R_2}} \equiv 0$ for all $f \in \mathcal{I}_{X,2}(\varphi_L;\mu;\psi)$.\"
With the above understanding, now it makes sense to give the following definition.

**Definition 5.2.4** (σ-lc centres). Suppose that \( \varphi_L \) has only neat analytic singularities and that \((X, \varphi_L, \psi)\) satisfies the Snc assumption 2.3.7 (but need not satisfy 2.3.7). A σ-lc centre \( S_p^\sigma \) of \((X, \varphi_L, \psi)\) is an irreducible component of the (reduced) closed analytic subset \( \text{lc}^\sigma_X(\varphi_L; \psi) \) of \( X \) defined by the ideal sheaf

\[
\mathcal{I}_{\text{lc}^\sigma_X(\varphi_L; \psi)} := \text{Ann}_{\mathcal{O}_X} \left( \mathcal{I}_{X, \varphi}(\varphi_L; \psi) / \mathcal{I}_{X, \sigma-1}(\varphi_L; \psi) \right)
\]

For any log-resolution \( \pi: \tilde{X} \to X \) of \((X, \varphi_L, \psi)\) with \( \tilde{S} \) being the corresponding snc lc locus on \( \tilde{X} \), a σ-lc centre \( \tilde{S}_q^\sigma \subset \text{lc}^\sigma_\tilde{X}(\tilde{S}) \) of \((\tilde{X}, \tilde{S})\) is said to be \( \pi\)-supportive\(^{16}\) if it is a subset of the closed analytic subset \( \text{lc}^\sigma_\tilde{X}(\tilde{S})' \) defined as in (eq 5.2.1), i.e. there exists \( f \in \mathcal{I}_{X, \varphi}(\varphi_L; \psi) \) such that \( \text{Res}(\pi^*f) \mid_{\tilde{S}_q^\sigma} \not= 0 \), or equivalently, \( \pi^* f \not= 0 \) on \( \tilde{S}_q^\sigma \). Theorem 5.2.1 asserts that

\[
\text{lc}^\sigma_X(\varphi_L; \psi) = \pi(\text{lc}^\sigma_\tilde{X}(\tilde{S})') \subset \pi(\text{lc}^\sigma_\tilde{X}(\tilde{S}))
\]

and \( \pi^* f / \pi^* \not= 0 \) on the closure of \( \text{lc}^\sigma_X(\tilde{S})' \setminus \text{lc}^\sigma_X(\tilde{S})' \) for all \( f \in \mathcal{I}_{X, \varphi}(\varphi_L; \psi) \). Note that each σ-lc centre \( S_p^\sigma \subset \text{lc}^\sigma_X(\varphi_L; \psi) \) must have codim \( \text{lc}^\sigma_X(\varphi_L; \psi) \geq \sigma \). Define also the *index of the mlc* of \((X, \varphi_L, \psi)\), denoted by \( \sigma_{\text{mlc}} := \sigma_{\text{mlc}}(X, \varphi_L, \psi) \), to be the smallest non-negative integer such that \( \text{lc}^{\sigma + 1}_X(\varphi_L; \psi) = 0 \) for all \( \sigma \geq \sigma_{\text{mlc}} \), which is upper-bounded by, but can possibly be different from, the codimension of the mlc of \((\tilde{X}, \tilde{S})\). (The latter is denoted by \( \tilde{\sigma}_{\text{mlc}} \) or \( \tilde{s}_{\text{mlc}}(\tilde{X}, \tilde{S}) \) in the rest of this section.)

**Remark 5.2.5.** It is deliberate to call it “σ-lc centre” instead of “lc centre of codimension σ”, as a σ-lc centre in \( X \) may not necessarily be of codimension \( \sigma \) in \( X \), as indicated in [5, Remark 1.4.2]. See also [5, Example 3.5.1]. Note also that the above definition is different from (yet a refined version of) the ad hoc definition given in [5, Def. 1.4.1] as well as [4, Def. 1.3.3].

Theorem 5.2.1 shows that the sequence

\[
\mathcal{I}_{X, \varphi}(\varphi_L + \psi) = \mathcal{I}_{X, \varphi}(\varphi_L; \psi) \subset \mathcal{I}_{X, 1}(\varphi_L; \psi) \subset \cdots \subset \mathcal{I}_{X, \sigma_{\text{mlc}}}(\varphi_L; \psi) = \mathcal{I}_{X, \varphi_L} \]

contains bimeromorphic information of the σ-lc centres of \((X, \varphi_L, \psi)\) for various integers \( \sigma \geq 0 \). It is tempting to use this definition to define σ-lc centres in the more general situations (for example, when the singularities of \( \varphi_L \) are worse than neat analytic singularities). However, the coherence of \( \mathcal{I}_\varphi(\varphi_L; \psi) \) and the existence of the residue short exact sequence will then be in question. This will be studied in subsequent papers.

With such definition of σ-lc centres, their relation with the lc locus \( S \) of \((X, \varphi_L, \psi)\) is the expected one.

**Proposition 5.2.6.** Under the notation and assumptions in Definition 5.2.4 and given the lc locus \( S \) of \((X, \varphi_L, \psi)\) defined by the ideal sheaf \( \mathcal{I}_S = \text{Ann}_{\mathcal{O}_X} \left( \mathcal{I}_{X, \varphi}(\varphi_L; \psi) \right) \) as in item (5) in Section 2.2, one has

\[
\mathcal{I}_S = \mathcal{I}_{\text{lc}^1_X(\varphi_L; \psi)} \cap \mathcal{I}_{\text{lc}^2_X(\varphi_L; \psi)} \cap \cdots \cap \mathcal{I}_{\text{lc}^{\sigma_{\text{mlc}}}_X(\varphi_L; \psi)}
\]

which is translated to \( S = \text{lc}^1_X(\varphi_L; \psi) \cup \text{lc}^2_X(\varphi_L; \psi) \cup \cdots \cup \text{lc}^{\sigma_{\text{mlc}}}_X(\varphi_L; \psi) \).

\(^{16}\)Such σ-lc centre in \( \text{lc}^\sigma_X(\tilde{S}) \) is “supporting” \( \mathcal{I}_{X, \sigma}(\varphi_L; \psi) \) via \( \pi \).
Proof. Write $\mathcal{J}_\sigma$ for $\mathcal{J}_{X,\sigma}(\varphi_L; \psi)$ and $\text{le}^\sigma$ for $\text{le}_X^\sigma(\varphi_L; \psi)$ for convenience. Consider the short exact sequence

$$0 \rightarrow \mathcal{J}_\sigma \rightarrow \mathcal{J}_{\sigma-1} \rightarrow \mathcal{J}_\sigma \rightarrow 0$$

for any $\sigma' \geq \sigma \geq 1$, which is induced from the inclusions of the adjoint ideal sheaves. As all the maps in the exact sequence are $\mathcal{O}_X$-homomorphisms, it can be checked readily that

$$\text{Ann}_{\mathcal{O}_X} \left( \frac{\mathcal{J}_{\sigma'}}{\mathcal{J}_{\sigma-1}} \right) \subset \text{Ann}_{\mathcal{O}_X} \left( \frac{\mathcal{J}_\sigma}{\mathcal{J}_{\sigma-1}} \right) \cap \text{Ann}_{\mathcal{O}_X} \left( \frac{\mathcal{J}_{\sigma'}}{\mathcal{J}_\sigma} \right) = \mathcal{I}_{\text{le}^\sigma} \cap \text{Ann}_{\mathcal{O}_X} \left( \frac{\mathcal{J}_{\sigma'}}{\mathcal{J}_\sigma} \right).$$

By induction, one also sees that

$$\text{Ann}_{\mathcal{O}_X} \left( \frac{\mathcal{J}_{\sigma'}}{\mathcal{J}_{\sigma-1}} \right) \subset \mathcal{I}_{\text{le}^\sigma} \cap \mathcal{I}_{\text{le}^{\sigma+1}} \cap \cdots \cap \mathcal{I}_{\text{le}^{\sigma'}}.$$

Putting $\sigma' = \sigma_{\text{mle}}$ and $\sigma = 1$ yields $\mathcal{I}_S \subset \mathcal{I}_{\text{le}^1} \cap \mathcal{I}_{\text{le}^{\sigma_{\text{mle}}}}$, which means $h \in \mathcal{I}_{\text{le}^\sigma} = \text{Ann}_{\mathcal{O}_X} \left( \frac{\mathcal{J}_{\sigma}}{\mathcal{J}_{\sigma-1}} \right)$ for each $\sigma = 1, \ldots, \sigma_{\text{mle}}$. Therefore, $h_{\text{mle}} \in \text{Ann}_{\mathcal{O}_X} \left( \frac{\mathcal{J}_{\text{mle}}}{\mathcal{J}_{\text{le}}} \right) = \mathcal{I}_S$. As $\mathcal{I}_S$ is radical by [11, Lemma 4.2], it follows that $h \in \mathcal{I}_S$. This completes the proof.

Here is a sufficient condition for the equalities $\sigma_{\text{mle}} = \tilde{\sigma}_{\text{mle}}$ and $\text{le}_X^\sigma(\varphi_L; \psi) = \pi \left( \text{le}_X^\sigma(S) \right)$ to hold true, which may be useful in practice.

**Proposition 5.2.7.** Under the notation and assumptions in Definition 5.2.4, suppose that $\mathcal{J}_{X,\sigma}(\varphi_L; \psi) = \mathcal{O}_X$ for some $\sigma \geq 0$, or equivalently, $\mathcal{J}_{X,\sigma_{\text{mle}}}(\varphi_L; \psi) = \mathcal{O}_X$. Then, for any log-resolution $\pi: \tilde{X} \to X$ of $(X, \varphi_L, \psi)$, one has $\sigma_{\text{mle}}(X, \varphi_L, \psi) = \tilde{\sigma}_{\text{mle}}(\tilde{X}, \tilde{S})$. Moreover, one also has $\text{le}_X^\sigma(\varphi_L; \psi) \setminus \pi \left( \text{le}_X^{\sigma+1}(\tilde{S}) \right) = \pi \left( \text{le}_X^\sigma(\tilde{S}) \setminus \pi \left( \text{le}_X^{\sigma+1}(\tilde{S}) \right) \right)$, and thus $\pi \left( \text{le}_X^\sigma(S) \right) = \bigcup_{\sigma' = \sigma}^\sigma \text{le}_X^{\sigma'}(\varphi_L; \psi)$ for all integers $\sigma \geq 1$.

**Proof.** Write $\mathcal{J}$ for $\mathcal{J}_{X,\sigma}(\varphi_L; \psi)$, $\mathcal{J}$ for $\mathcal{J}_{\tilde{X},\sigma}(\varphi_L; \psi)$, $\text{le}^\sigma$ for $\text{le}_X^\sigma(\varphi_L; \psi)$ and $\text{le}^\sigma(\tilde{S})$ for convenience. Since $\mathcal{O}_X = \mathcal{J}_{\sigma_{\text{mle}}} = \pi_{\ast} \mathcal{J}_{\sigma_{\text{mle}}} = \mathcal{J}_{\text{mle}}$, this forces the equality $\sigma_{\text{mle}} = \tilde{\sigma}_{\text{mle}}$ to hold true (notice that one must have $\mathcal{J}_{\sigma-1} \subseteq \mathcal{J}_\sigma$ for all $\sigma = 1, \ldots, \sigma_{\text{mle}}$). Moreover, it follows that

$$\mathcal{I}_{\text{le}^\sigma} = \text{Ann}_{\mathcal{O}_X} \left( \frac{\mathcal{J}_{\text{mle}}}{\mathcal{J}_{\sigma_{\text{mle}}}} \right) = \mathcal{J}_{\sigma_{\text{mle}}},$$

and therefore $\text{le}^{\sigma}_{\text{mle}} = \pi \left( \text{le}^\sigma \right)$.

The rest is proceeded via induction. Suppose it is known that

$$\pi \left( \text{le}^{\sigma'} \right) = \text{le}^\sigma \cup \text{le}^{\sigma+1} \cup \cdots \cup \text{le}^{\sigma_{\text{mle}}} \quad \text{for } \sigma' = \sigma + 1, \sigma + 2, \ldots, \sigma_{\text{mle}}.$$

To prove the equality $\text{le}^\sigma \setminus \pi \left( \text{le}^{\sigma+1} \right) = \pi \left( \text{le}^\sigma \setminus \pi \left( \text{le}^{\sigma+1} \right) \right)$ (which will then imply $\pi \left( \text{le}^\sigma \right) = \bigcup_{\tau = \sigma}^\sigma \text{le}^{\tau}$), consider the short exact sequence

$$0 \rightarrow \mathcal{J}_{\sigma-1} \rightarrow \mathcal{J}_\sigma \rightarrow \mathcal{J}_{\sigma-1} \rightarrow 0.$$
for any $\sigma' \geq \sigma$ (which is induced from the inclusions of the adjoint ideal sheaves). Take any point $x \in X \setminus \pi(\hat{\mathcal{I}}_{\sigma+1})$, which implies that $x \notin \sigma'$ for all $\sigma' \geq \sigma$ by the inductive assumption. Since $\left(\frac{f_{\sigma+1}}{f_{\sigma'}}\right)_x = 0$ for all $\sigma' \geq \sigma$ by the choice of $x$, the short exact sequence yields $\left(\frac{f_{\sigma+1}}{f_{\sigma'-1}}\right)_x = \left(\frac{f_{\sigma+1}}{f_{\sigma'-1}}\right)_x$ for all $\sigma' \geq \sigma$, which results in $\left(\frac{f_{\sigma}}{f_{\sigma'-1}}\right)_x = \left(\frac{f_{\sigma}}{f_{\sigma'-1}}\right)_x$.

Therefore, following the computation for $\mathcal{I}_{\sigma+1}$ above (which makes use of the assumption $\mathcal{J}_{\sigma+1} = \mathcal{O}_X$), one has

$$\mathcal{I}_{\sigma'} = \mathcal{I}_{\sigma+1} \big|_{X \setminus \pi(\hat{\mathcal{I}}_{\sigma+1})},$$

which is translated to $\mathcal{I}_{\sigma'} = \pi(\hat{\mathcal{I}}_{\sigma+1}) \cap \pi(\mathcal{I}_{\sigma+1})$, as desired. The claim is thus proved by induction.

With the terminology of $\sigma$-lc centres, the following theorem can be stated, which is needed in Theorem 6.2.3. Set $\hat{S'} := \mathcal{I}_{\mathcal{I}}(\hat{S})'$, defined as in (eq 5.2.1), to be the union of all $\pi$-supportive $1$-lc centres in $\mathcal{I}_{\mathcal{I}}(\hat{S})$ in what follows.

**Theorem 5.2.8.** Under the notation and assumptions in Definition 5.2.4, given any $1$-lc centre $\mathcal{J}_p$ of $(X, \varphi_L, \psi)$ (which may not be a divisor), the divisor $\pi^{-1}(\mathcal{J}_p) \cap \hat{S'}$ is irreducible, that is, there is one and only one $\pi$-supportive $1$-lc centre $\hat{D}$ of $(\hat{X}, \hat{S})$ such that $\hat{D}$ is mapped into, and thus onto, $\mathcal{J}_p$ by $\pi$.

**Proof.** According to Remark 5.1.8 and (eq 5.1.2), one has $\mathcal{R}_{X,1}(\varphi_L; \psi) \cong \pi_\# \mathcal{R}_{X,1}(\varphi - \varphi_E)$ (which is a consequence of the local vanishing theorem ([33, Thm. 4.1.4]) in the algebraic situation or the generalisation of the Grauert–Riemenschneider vanishing theorem by Matsumura ([36, Cor. 1.5]) in the analytic situation). Let $\hat{D} \subset \hat{S}'$ be a $\pi$-supportive $1$-lc centre of $(\hat{X}, \hat{S})$ such that $\pi(\hat{D}) = \mathcal{J}_p$. Suppose that $\tilde{\mathcal{J}}_1 \subset \hat{S}'$ is another $\pi$-supportive $1$-lc centre such that $\pi(\tilde{\mathcal{J}}_1) \subset \mathcal{J}_p$. Since $\mathcal{R}_{X,1}(\varphi - \varphi_E)$ is a direct sum where each of its summands is supported on an irreducible component of $\hat{S}'$, there exists a decomposition

$$\mathcal{R}_{X,1}(\varphi_L; \psi) \cong \pi_\# \mathcal{R}_{X,1}(\varphi - \varphi_E) = \pi_\# \mathcal{J}_1 \oplus \pi_\# \mathcal{J},$$

where $\mathcal{J}_1$ is supported on $\tilde{\mathcal{J}}_1$ and $\mathcal{J}$ is supported on the components of $\hat{S}' - \tilde{\mathcal{J}}_1$ (which include $\hat{D}$). Note that both summands $\pi_\# \mathcal{J}_1$ and $\pi_\# \mathcal{J}$ are non-trivial as both $\tilde{\mathcal{J}}_1$ and $\hat{D}$ are $\pi$-supportive. By restricting attention to a local open set in $X$, one can assume that $X$ is Stein. Let $\tilde{g} \in \mathcal{J}_1(\pi^{-1}(X))$ be a non-trivial holomorphic section and $\tilde{h}$ be the zero section in $\mathcal{J}(\pi^{-1}(X))$. Take also $\tilde{h} \in \mathcal{J}(\pi^{-1}(X))$ such that $\tilde{h}$ is non-trivial on every irreducible component of $\hat{S}' - \tilde{\mathcal{J}}_1$.

It follows from the surjectivity of the residue morphism on $X$ (see Corollary 5.1.6) that there exist $f_0, f_\tilde{h} \in \mathcal{J}_{X,1}(\varphi_L; \psi)(X)$ such that $\text{Res}(\pi^*f_0) = (\tilde{g}, \tilde{0})$ and $\text{Res}(\pi^*f_{\tilde{h}}) = (\tilde{g}, \tilde{h})$. As seen from the formula for Res, these functions satisfy

$$\left.\frac{\pi^*f_0}{\hat{s}_0}\right|_{\tilde{\mathcal{J}}_1} \neq 0 \quad (\text{as } \tilde{g} \neq 0), \quad \left.\frac{\pi^*f_{\tilde{h}}}{\hat{s}_0}\right|_{\tilde{\mathcal{J}}_1} \neq 0 \quad (\text{as } \tilde{h} \neq 0) \quad \text{and} \quad \left.\frac{\pi^*f_0}{\hat{s}_0}\right|_{\hat{D}} \equiv 0 \quad \left.\frac{\pi^*f_{\tilde{h}}}{\hat{s}_0}\right|_{\hat{D}} \equiv 0.$$

Simultaneously, which in turn yield

$$\pi^*\left(\frac{f_0}{f_{\tilde{h}}}\right)_{\tilde{\mathcal{J}}_1} \equiv 1 \quad \text{and} \quad \pi^*\left(\frac{f_0}{f_{\tilde{h}}}\right)_{\hat{D}} \equiv 0.$$
As \( \frac{f_1}{f_2} \) is a meromorphic function on \( X \), the equalities above contradict the inclusion \( \pi(D_1) \subset \pi(D) = S_p^1 \). This completes the proof. \( \square \)

6. Comparison between various algebraic and analytic adjoint ideal sheaves

6.1. Ein–Lazarsfeld and Hacon–McKernan adjoint ideal sheaves. Let \( S \) be a reduced divisor (which need not be snc) on a complex manifold \( X \). Let \( \pi : \tilde{X} \to X \) be any log-resolution of \((X,S)\) such that \( \pi^*S + \text{Exc}(\pi) \) is snc. There exist a reduced snc divisor \( \Gamma \) and an exceptional snc divisor \( E^0 \), which have no common irreducible components, such that \( E^0 \) has no coefficients \( -1 \) and

(eq 6.1.1) \[
K_{\tilde{X}} + \Gamma = K_{\tilde{X}} + \pi^{-1}_*S + \Gamma_{\text{Exc}} \sim \pi^*(K_X + S) + E^0,
\]

where \( \sim \) stands for the linear equivalence relation between divisors and \( \Gamma_{\text{Exc}} \) is an effective reduced exceptional divisor such that it shares no irreducible components with the proper transform \( \pi^{-1}_*S \) of \( S \). Readers are referred to [32, Def. 2.8] for the definitions of \( (X,S) \) being klt, plt, dlt and lc. Note that \( (X,S) \) being plt (resp. lc) is equivalent to \( \phi_S := \varphi_S^m \) (see Notation 2.1.3) having klt (resp. lc) singularities as described in Section 2.2, and \( (X,S) \) being plt implies that \( \Gamma = \pi^{-1}_*S \) (i.e. \( \Gamma_{\text{Exc}} = 0 \)). Note also that, when \((X,S)\) is lc, since all the divisors involved in (eq 6.1.1) are integral, one has \( E^0 \geq 0 \).

Let \( a \) be a coherent ideal sheaf on \( X \) and assume that \( \pi \) is chosen such that \( a \cdot \mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-F) \) for some effective divisors \( F \) where \( F + \pi^*S + \text{Exc}(\pi) \) is snc. The Hacon–McKernan adjoint ideal sheaf \( \text{Adj}_{\text{HM}}(X,S)(a^\ast) \) for any given number \( c \geq 0 \) (following the definition in [25, Def.-Lemma 4.2] as well as [14, Def. 2.4]) is given by

(eq 6.1.2) \[
\text{Adj}_{\text{HM}}(X,S)(a^\ast) := \pi_\ast \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} - \pi^*(K_X + S) + \Gamma - \lfloor cF \rfloor)
= \pi_\ast \mathcal{O}_{\tilde{X}}(E^0 - \lfloor cF \rfloor),
\]

where \( \lfloor \cdot \rfloor \) is the floor function. When \( a \) is already a principal ideal \( \mathcal{O}_X(-D) \), the adjoint ideal sheaf \( \text{Adj}_{\text{HM}}(X,S)(a^\ast) \) is written as \( \text{Adj}_{\text{HM}}(X,S)(cD) \). It is independent of the choice of log-resolution \( \pi \) of \((X,S,a)\) as shown in [25, Def.-Lemma 4.2].

Assume further that \( \pi \) is chosen such that \( \pi^{-1}_*S \) is smooth (so all irreducible components are disjoint). The Ein–Lazarsfeld adjoint ideal sheaf \( \text{Adj}_{\text{EL}}(X,S)(a^\ast) \) for any given \( c \geq 0 \) (see [13, Prop. 3.1], [33, Def. 9.3.47] as well as [40, Def. 1.2 (ii)]) is given by

(eq 6.1.3) \[
\text{Adj}_{\text{EL}}(X,S)(a^\ast) := \pi_\ast \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} - \pi^*(K_X + S) + \pi^{-1}_*S - \lfloor cF \rfloor)
= \pi_\ast \mathcal{O}_{\tilde{X}}(E^0 - \Gamma_{\text{Exc}} - \lfloor cF \rfloor).
\]

It is independent of the choice of log-resolution \( \pi \) of \((X,S,a)\) such that \( \pi^{-1}_*S \) is smooth (see, for example, [33, Thm. 9.2.18] or [40, Lemma 1.7 (1)] for a proof).

It can be seen that \( \text{Adj}_{\text{EL}}(X,S)(a^\ast) \subset \text{Adj}_{\text{HM}}(X,S)(a^\ast) \) in general. The two adjoint ideal sheaves coincide when \((X,S)\) is plt as seen from the definitions. One can construct simple examples to see that the two adjoint ideal sheaves are different in more general setting. For example, take \( X = \mathbb{C}^2 \), \( a = \mathcal{O}_X \) and \( S = \{z_2^2 = z_1^2(z_1+1)\} \) a nodal curve with a node at the origin (note that \( S \) has only 1 component, yet \((X,S)\) is lc but not plt). Take \( \pi : \tilde{X} \to X \) to be the blow-up of \( X \) at the origin and let \( E \) be the exceptional divisor. One can check that \( K_{\tilde{X}} = \pi^*K_X + E \) and \( \pi^*S = \pi^{-1}_*S + 2E \), thus \( \text{Adj}_{\text{EL}}(X,S)(\mathcal{O}_X) = \pi_\ast \mathcal{O}_{\tilde{X}}(-E) = \mathfrak{m}_{X,0} \) (the defining ideal sheaf of the origin in \( X \)) while \( \text{Adj}_{\text{HM}}(X,S)(\mathcal{O}_X) = \pi_\ast \mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}} \).
Indeed, for any effective \( \mathbb{Q} \)-divisor \( D \) whose support does not contain any lc centre of \( (X, S) \), \( Adj_{(X,S)}^{\mathbb{Q}}(D) \) is trivial if and only if \( (X, S + D) \) is plt with \( |D| = 0 \) (cf. [40, Remark 1.3 (2)] and see [32, Def. 2.8] instead of [40, Def. 1.1 (ii)] for the definition of plt). When \( S \) is snc (thus \( (X, S) \) is lc and log-smooth), \( Adj_{(X,S)}^{\mathbb{Q}}(D) \) is trivial if and only if \( (X, S + D) \) is dlt with \( |D| = 0 \) (see [25, Lemma 4.3 (1)]).

6.2. Analytic and algebraic adjoint ideal sheaves in the lc case. To compare the algebraic adjoint ideal sheaves with the current version of analytic adjoint ideal sheaves, now suppose that \( (X, S) \) is lc (but need not be log-smooth, i.e. \( S \) may not have snc) and that the zero locus of \( a \) contains no lc centres of \( (X, S) \) in the sense of [32, Def. 4.15]. Take \( I = \mathcal{O}_X \) and let \( \varphi_L := \varphi_{\mathbb{R}} \) be a quasi-psh function on \( X \) associated to \( a \) such that, given an open cover \( \{V_\gamma\}_\gamma \) of \( X \), if \( a \) is generated on an open set \( V_\gamma \) by \( g_1, \ldots, g_N \in \mathcal{O}_X(V_\gamma) \), one then has

\[
\varphi_{\mathbb{R}}|_{V_\gamma} = c \varphi_a|_{V_\gamma} = c \log(|g_1|^2 + \cdots + |g_N|^2) + c\beta
\]

for some \( \beta \in \mathcal{O}_X(V_\gamma) \) (so \( \varphi_{\mathbb{R}} \) has neat analytic singularities for all \( c \geq 0 \)). Set also

\[
\psi := \psi_S := \phi_S - \varphi_{\mathbb{R}}^{\mathbb{Q}} \leq -1
\]

(see Notation 2.1.3). As \( \varphi_{\mathbb{R}}^{-1}(-\infty) \) contains no lc centres of \( (X, S) \) by the assumption on \( a \), it follows that, for every given constant \( c \geq 0 \), the number \( m = 1 \) is a jumping number of the family \( \{\mathcal{I}_X(\varphi_{\mathbb{R}} + m\psi)\}_{m \in [0,1]} \). Moreover, there is a number \( m^c \in [0, 1) \) for each \( c \geq 0 \) such that

\[
\mathcal{I}_X(\varphi_{\mathbb{R}} + m^c\psi) = \mathcal{I}_X(\varphi_{\mathbb{R}} + m\psi) \supseteq \mathcal{I}_X(\varphi_{\mathbb{R}} + \psi) \quad \text{for all } m \in [m^c, 1)
\]

and that

\[
\text{Ann}_{\mathcal{O}_X}(\mathcal{I}_X(\varphi_{\mathbb{R}} + m^c\psi)) = \mathcal{I}_S.
\]

To see the inclusion \( \text{Ann}_{\mathcal{O}_X}(\cdots) \subseteq \mathcal{I}_S \) (the reverse inclusion is easy to see), on any sufficiently small open set \( V \subset X \), take any \( h \in \text{Ann}_{\mathcal{O}_X}(\cdots)(V) \) and take for each irreducible component \( D \) of \( S \cap V \) a section \( f_D \in \mathcal{I}_X(\varphi_{\mathbb{R}} + m^c\psi)(V) \) such that \( f_D \) does not vanish on \( D \) (the existence follows from assumption on \( a \) and the fact \( m^c < 1 \)). Since \( \varphi_{\mathbb{R}} \) is locally bounded from above, one then has

\[
\int_V |h f_D|^2 e^{-\psi_S} \, \text{vol}_X \lesssim \int_V |h f_D|^2 e^{-\varphi_{\mathbb{R}} - \psi} \, \text{vol}_X < +\infty,
\]

which implies \( h \in \mathcal{I}_S \) after letting \( D \) run through all components in \( S \cap V \).

Given the log-resolution \( \pi: \tilde{X} \to X \) of \( (X, S, \varphi_{\mathbb{R}}, \psi) \) (also of \( (X, S, a) \)), which comes with the exceptional \((\mathbb{Z}_r)\) divisors \( E^c \) \(^{17}\) and \( R^c \) defined as in Section 2.3 (with \( \varphi_{\mathbb{R}} \) in place of \( \varphi_{\mathbb{R}} \)), and assuming that \( m^c \) is sufficiently close to 1 in view of Proposition 2.3.1, let \( \tilde{S}^c \) be the reduced subvariety defined by

\[
\mathcal{I}_{\tilde{S}^c} := \text{Ann}_{\mathcal{O}_{\tilde{X}}}(\mathcal{I}_X(\pi^*_{\tilde{X}}\varphi_{\mathbb{R}} + m^c\pi^*\psi)) \quad \text{(where } \pi^*_{\tilde{X}}\varphi_{\mathbb{R}} := \pi^*\varphi_{\mathbb{R}} - \phi_{R^c})\),
\]

which is an snc divisor as

\[
(\text{eq6.2.1}) \quad \mathcal{I}_X(\pi^*_{\tilde{X}}\varphi_{\mathbb{R}} + \pi^*\psi) = \mathcal{O}_{\tilde{X}}(-[c F - R^c + \pi^* S])
\]

is a principal ideal sheaf with an snc generator and Lemma 2.3.5 applies.

\(^ {17}\)Coincidentally, \( E^c \) with \( c = 0 \) is exactly \( E^0 \) defined in (eq6.1.1).
The divisors $\Gamma$ and $\tilde{S}^c$ are equal for all $c \geq 0$ on the complement of a discrete subset of $\mathbb{R}_{>0}$ (so $\Gamma = \tilde{S}^0$ in particular), which can be illustrated in the following example.

**Example 6.2.1.** Let $X := \Delta^3 \subset \mathbb{C}^3$ be the unit 3-disc centred at the origin in the holomorphic coordinate system $(z_1, z_2, z_3)$. Take

$$a := (z_1, z_2, z_3), \quad \varphi_{a^c} := c \log(|z_1|^2 + |z_2|^2 + |z_3|^2) \quad \text{and} \quad \psi := \log |z_1|^2 + \log |z_2|^2 - 1$$

such that $m = 1$ is a jumping number for the family $\{\mathcal{I}(\varphi_{a^c} + m\psi)\}_{m \in [0,1]}$ for any $c \geq 0$ and the corresponding lc locus $S$ is given by $\{z_1z_2 = 0\} =: S_1 + S_2$. Let $\pi : \tilde{X} \to X$ be a log-resolution of $(X, S, a)$ obtained by first blowing up the line $\{z_1 = z_2 = 0\}$ in $X$ (with exceptional divisor $E_1$) and then blowing up the intersection of $E_1$ and the proper transform of $\{z_3 = 0\} =: S_3$ (with exceptional divisor $E_2$). Abusing $E_1$ to mean also its proper transform via the second blow-up, it follows from a simple computation that

$$K_{\tilde{X}} \sim \pi^* K_X + E_1 + 2E_2, \quad \pi^* S = \pi_s^{-1} S + 2E_1 + 2E_2,$$

$$\pi^* \varphi_{a^c} \sim \log c \phi_{E_2} \quad \text{and} \quad \pi^* \psi \sim \log \phi_{\pi_s^{-1} S} + 2\phi_{E_1} + 2\phi_{E_2},$$

in which the notations $\phi_{\pi_s^{-1} S}, \phi_{E_1}$ and $\phi_{E_2}$ are explained in Notation 2.1.3, and thus

$$R^c = R^0 = E_1 + 2E_2, \quad E^0 = E^c = 0 \quad \text{and} \quad \Gamma = \pi_s^{-1} S + E_1 \quad \text{(thus $\Gamma_{\text{Exc}} = E_1$),}$$

where $\Gamma, \Gamma_{\text{Exc}}, E^0, E^c$ and $R^c$ are the divisors described above. Note also that the two irreducible components of $\pi_s^{-1} S = S'_1 + S'_2$ (where $S'_i$ is the proper transform of $S_i = \{z_i = 0\}$ for $i = 1, 2, 3$) are disjoint in $\tilde{X}$. The weight in the $L^2$ norm associated to $\mathcal{I}_{\tilde{X}}(\pi_{\tilde{c}} \varphi_{a^c} + m\pi^* \psi)$ is then of the form

$$e^{-\pi_{\tilde{c}} \varphi_{a^c} - m\pi^* \psi} \, d\text{vol}_{\tilde{X}} \sim e^{-(c+2m-2)\phi_{E_2} - (2m-1)\phi_{E_1} - m\phi_{\pi_s^{-1} S}} \, d\text{vol}_{\tilde{X}}.$$

It can then be seen that, for $c \geq 0$, by setting

$$m^c := \begin{cases} 
1 - \frac{c - \lfloor c \rfloor}{2} & \text{if } c \notin \mathbb{N} \cup [0, 1], \\
\frac{1}{2} & \text{if } c \in \mathbb{N} \cup [0, 1], 
\end{cases}$$

one has

$$\mathcal{I}_{\tilde{X}}(\pi_{\tilde{c}} \varphi_{a^c} + m^c \pi^* \psi) = \mathcal{I}_{\tilde{X}}(\pi_{\tilde{c}} \varphi_{a^c} + m \pi^* \psi) \supset \mathcal{I}_{\tilde{X}}(\pi_{\tilde{c}} \varphi_{a^c} + \pi^* \psi) \quad \text{for all } m \in [m^c, 1).$$

Recall that $\mathcal{I}_{\tilde{S}^c} = \text{Ann} \phi_X \left( \mathcal{I}_{\tilde{X}}(\pi_{\tilde{c}} \varphi_{a^c} + m^c \pi^* \psi) \right) / \mathcal{I}_{\tilde{X}}(\pi_{\tilde{c}} \varphi_{a^c} + \pi^* \psi)$. Therefore, one also sees that

$$\tilde{S}^c = \begin{cases} 
\Gamma & \text{if } c \notin \mathbb{N} = \mathbb{N}_{\geq 1}, \\
\Gamma + E_2 & \text{if } c \in \mathbb{N}. 
\end{cases}$$

The various adjoint ideal sheaves can now be computed. Notice that the ideal sheaves in question are toric, so they are generated by monomials. A direct use of (eq.6.1.2), (eq.6.1.3) and (eq.6.2.1), together with the equalities

$$\text{div}(\pi^* z_1) = S'_1 + E_1 + E_2, \quad \text{div}(\pi^* z_2) = S'_2 + E_1 + E_2 \quad \text{and} \quad \text{div}(\pi^* z_3) = S'_3 + E_2,$$

shows that

$$\text{Adj}^{\text{HM}}_{(X,S)}(a^c) = \pi_\ast \mathcal{O}_{\tilde{X}}(\sim [c] E_2) = \langle z_1^p z_2^q z_3^r \mid p, q, r \geq 0, \ p + q + r \geq |c| \rangle,$$

$$\text{Adj}^{\text{EL}}_{(X,S)}(a^c) = \pi_\ast \mathcal{O}_{\tilde{X}}(\sim E_1 - [c] E_2) = \langle z_1^p z_2^q z_3^r \mid p, q, r \geq 0, \ p + q \geq 1, \ p + q + r \geq |c| \rangle.$$
\[ \mathcal{J}(\varphi^e) = \pi_*(\mathcal{I}(\varphi^e + m^e\pi^e)) = \begin{cases} \pi_*\mathcal{O}_X(-|c|E_2) & \text{if } c \not\in \mathbb{N}, \\ \pi_*\mathcal{O}_X(-(c-1)E_2) & \text{if } c \in \mathbb{N}, \end{cases} \]

As both \( \Gamma \) and \( \Gamma + E_2 \) are reduced snc divisors in \( \tilde{X} \), their lc centres (in the sense of [32, Def. 4.15], also in the sense of Definition 5.2.4 when \( (\tilde{X}, 0, \phi_\tau) \) or \( (\tilde{X}, 0, \phi_{\Gamma+E_2}) \) is considered) can be obtained directly from the intersections of their irreducible components. Recall that \( \pi_*^{-1}S = S_1' + S_2' \). One sees that \( \text{lc}^1_X(\Gamma) = \emptyset \) while \( \text{lc}^3_X(\Gamma + E_2) \) consists of two distinct points \( (S_1' \cap E_1 \cap E_2 \text{ and } S_2' \cap E_1 \cap E_2) \), and \( \text{lc}^2_X(\Gamma) \) consists of two disjoint lines \( (S_1' \cap E_1 \text{ and } S_2' \cap E_1) \) while \( \text{lc}^2_X(\Gamma + E_2) \) is the union of five lines (the extra three lines are obtained from the intersections of \( E_2 \) with \( S_1', S_2' \) and \( E_1 \)). Using Theorem 4.1.2 and (eq5.1.1) (which says \( \mathcal{J}_{X,0}(\varphi^e; \psi) = \pi_*(\mathcal{J}_X(\varphi^e + m^e\pi^e) \cdot \mathcal{I}_{\text{lc}_{X}^c(\tilde{S}^c)}) \), one obtains

\[ \mathcal{J}_{X,3}(\varphi^e; \psi) = \begin{cases} \pi_*\mathcal{O}_X(-|c|E_2) = (z^p_1 z^q_2 z^r_3 | p, q, r \geq 0, p + q + r \geq |c|) & \text{if } c \not\in \mathbb{N}, \\ \pi_*\mathcal{O}_X(-(c-1)E_2) = (z_1^p z_2^q z_3^r | p, q, r \geq 0, p + q + r \geq c - 1) & \text{if } c \in \mathbb{N}, \end{cases} \]

\[ \mathcal{J}_{X,2}(\varphi^e; \psi) = \begin{cases} \pi_*\mathcal{O}_X(-|c|E_2) \cdot \mathcal{I}_{\text{lc}_{X}^c(\Gamma + E_2)} & \text{if } c \in \mathbb{N}, \end{cases} \]

\[ \mathcal{J}_{X,1}(\varphi^e; \psi) = \begin{cases} \pi_*\mathcal{O}_X(-|c|E_2) \cdot \mathcal{I}_{\text{lc}_{X}^c(\Gamma + E_2)} & \text{if } c \not\in \mathbb{N}, \\ \pi_*\mathcal{O}_X(-(c-1)E_2) \cdot \mathcal{I}_{\text{lc}_{X}^c(\Gamma + E_2)} & \text{if } c \in \mathbb{N}, \end{cases} \]

\[ \mathcal{J}_{X,0}(\varphi^e; \psi) = \pi_*\mathcal{O}_X(-|c|E_2) = (z_1^p z_2^q z_3^r | p \geq 1, q \geq 1, r \geq 0, p + q + r \geq |c|). \]

It follows directly from Definition 5.2.4 that

\[ \mathcal{I}_{\text{lc}_{X}^c(\varphi^e; \psi)} = \begin{cases} \mathcal{O}_X & \text{if } c \not\in \mathbb{N}, \\ (z_1, z_2, z_3) & \text{if } c \in \mathbb{N}, \end{cases} \]

and

\[ \mathcal{I}_{\text{lc}_{X}^c(\varphi^e; \psi)} = (z_1, z_2) \quad \text{and} \quad \mathcal{I}_{\text{lc}_{X}^c(\varphi^e; \psi)} = (z_1 z_2) \quad \text{for any } c \geq 0. \]

Computing directly the images \( \pi(\text{lc}_{X}^c(\tilde{S}^c)) \), one sees that

\[ \text{lc}_{X}^1(\varphi^e; \psi) = \pi(\text{lc}_{X}^1(\tilde{S}^c)) = S \]

\[ \text{lc}_{X}^2(\varphi^e; \psi) = \pi(\text{lc}_{X}^2(\tilde{S}^c)) = \{z_1 = z_2 = 0\} \quad \text{for any } c \geq 0 \quad \text{and} \]

\[ \text{lc}_{X}^3(\varphi^e; \psi) = \pi(\text{lc}_{X}^3(\tilde{S}^c)) = \{0\} \quad \text{if } c \not\in \mathbb{N}, \]

\[ \text{lc}_{X}^3(\varphi^e; \psi) = \pi(\text{lc}_{X}^3(\tilde{S}^c)) = \{(0, 0, 0)\} \quad \text{if } c \in \mathbb{N}. \]
This shows the difference between lc centres of \((X, \varphi^a, \psi)\) in the sense of Definition 5.2.4 and those of \((X, S)\) in the sense of [32, Def. 4.15]. Notice that the zero locus of \(a\) contains the 3-lc centre of \((X, \varphi^a, \psi)\) when \(c \in \mathbb{N}\).

Observe that

\[
\text{Adj}_{(X,S)}^{\text{EL}}(a^c) = \begin{cases} \mathcal{J}_{X,1}(\varphi^a; \psi) & \text{if } c \not\in \mathbb{N} \text{ or } c = 1, 2, \\ \subset \mathcal{J}_{X,1}(\varphi^a; \psi) & \text{if } c \in \mathbb{N} \setminus \{1, 2\}, \end{cases}
\]

and

\[
\text{Adj}_{(X,S)}^{\text{HM}}(a^c) = \begin{cases} \mathcal{J}_{X,2}(\varphi^a; \psi) = \mathcal{J}_{X,3}(\varphi^a; \psi) & \text{if } c \not\in \mathbb{N}, \\ \subset \mathcal{J}_{X,2}(\varphi^a; \psi) \subset \mathcal{J}_{X,3}(\varphi^a; \psi) & \text{if } c = 1, \\ \subset \mathcal{J}_{X,2}(\varphi^a; \psi) \subset \mathcal{J}_{X,3}(\varphi^a; \psi) & \text{if } c \in \mathbb{N} \setminus \{1\}. \end{cases}
\]

Notice that \(\sigma_{\text{mlc}}(X, \varphi^a, \psi) = \tilde{\sigma}_{\text{mlc}}(\tilde{X}, \tilde{S}^c) = \begin{cases} 2 & \text{if } c \in \mathbb{R}_{\geq 0} \setminus \mathbb{N}, \\ 3 & \text{if } c \in \mathbb{N}, \end{cases}\) in this example (see Definition 5.2.4 for the notation).

Remark 6.2.2. In the version of analytic adjoint ideal sheaves given by Guenancia in [23] (see (eq1.1.1)), the parameter \(\lambda > 1\) (in (eq1.1.1)) is introduced precisely to force both the analytic and algebraic adjoint ideal sheaves to coincide for all \(c \geq 0\). However, the description of the germs of sections of the current version of adjoint ideal sheaves relies on the jumping numbers (see Sections 2.3 and 3.1), while including \(\lambda\) in the definition would complicate the description. Complication from the parameter \(\lambda\) also arises when one wants to involve log-resolutions in order to handle the non-snc case as in Section 5, especially in cases where the polar set of an adjoint ideal sheaf already coincides in almost all cases under consideration, the author takes the liberty to remove the parameter \(\lambda\) from the definition of Guenancia in order to simplify the analysis.

It turns out that the example is reflecting the relation between the analytic and algebraic versions of adjoint ideal sheaves in general.

Theorem 6.2.3. Given an lc pair \((X, S)\) and a coherent ideal sheaf \(a\) such that the zero locus of \(a\) contains no lc centres of \((X, S)\) in the sense of [32, Def. 4.15] and considering \(\varphi^a\) and \(\psi = \psi_S\) as described at the beginning of Section 6.2, one has, for any \(c \geq 0\),

\[
\text{Adj}_{(X,S)}^{\text{EL}}(a^c) \subset \mathcal{J}_{X,1}(\varphi^a; \psi) \quad \text{and} \quad \text{Adj}_{(X,S)}^{\text{HM}}(a^c) \subset \mathcal{J}_{X,\sigma_{\text{mlc}}}(\varphi^a; \psi),
\]

and there exists a countable discrete subset \(N := N(a, S) \subset \mathbb{R}_{\geq 0}\) (excluding 0) such that equalities hold for both inclusions for any \(c \in \mathbb{R}_{\geq 0} \setminus N\). Note that the integer \(\sigma_{\text{mlc}}\) depends on \(c\) and is the smallest integer such that \(\mathcal{J}_{X,\sigma_{\text{mlc}}}(\varphi^a; \psi) = \mathcal{J}_{X}(\varphi^a + mc\psi)\).

More specifically, if, for a given \(c \geq 0\), the zero locus of \(a\) contains no lc centres of \((X, \varphi^a, \psi)\) in the sense of Definition 5.2.4, then \(c \not\in N\) and therefore, in such case,

\[
\text{Adj}_{(X,S)}^{\text{EL}}(a^c) = \mathcal{J}_{X,1}(\varphi^a; \psi) \quad \text{and} \quad \text{Adj}_{(X,S)}^{\text{HM}}(a^c) = \mathcal{J}_{X,\sigma_{\text{mlc}}}(\varphi^a; \psi).
\]

Proof. Taking the log-resolution \(\pi: \tilde{X} \to X \) of \((X, S, a)\) described at the beginning of this section, it follows from the equalities (eq6.1.1) and \(a \cdot \mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-F)\) and the linear equivalence \(K_{\tilde{X}/X} \sim E_{\text{det}} = E^c + R^c\) (for any \(c \geq 0\)) that

\[
\pi^*\varphi^a \sim_{\text{log}} c \phi_F \quad \text{and} \quad \pi^*\psi = \pi^*\psi_S \sim_{\text{log}} \phi_T - \phi_{E^0} + \phi_{E^c} + \phi_{R^c}.
\]
(see Notation 2.1.3). Therefore, for any $m \in \mathbb{R}$,
\[
\pi^*\varphi_{\alpha^c} + m\pi^*\psi \sim_{\log} c \phi_F - (1 - m)\phi_{R^c} + m(\phi_T - \phi_{E^0} + \phi_{E^c}).
\]
In particular, one has
\[
\pi^*\varphi_{\alpha^c} + \pi^*\psi \sim_{\log} c \phi_F + \phi_T - \phi_{E^0} + \phi_{E^c}.
\]

To determine the divisor $\widetilde{S}^c$, note that $R^0 = E^c + R^c - E^0 \geq 0$ and that $\widetilde{S}^c$ is contained inside the polar set of $\pi^*\psi$. Therefore, $\widetilde{S}^c \subset \Gamma \cup \text{supp}(E^c + R^c - E^0)$. Recall that $R^c$ is, by definition, the maximal sub-divisor of $E_{d\pi} = E^c + R^c$ such that $\pi^*\varphi_{\alpha^c} + \pi^*\psi$ is quasi-psh, so $\widetilde{S}^c$ contains no components of $E^c$ (see footnote 11 on page 13). Recall also that the zero locus of $a$ contains no lc centres of $E^c$ for general $\pi$ and $\Gamma$ have no common irreducible components. Moreover, $E^0$ also contains no irreducible components of $\Gamma$ by definition. Since a prime divisor $D_i$ in $\widetilde{X}$ is contained in $\widetilde{S}^c$ if and only if $D_i \subset \Gamma \cup \text{supp}(E^c + R^c - E^0)$ and $\nu(\pi^*\varphi_{\alpha^c} + \pi^*\psi, D_i) = 0 \in \mathbb{N}$ (as can be seen from ($\ast$)), it follows that $\Gamma \leq \widetilde{S}^c$ and every prime divisor $D_i$ contained in $\widetilde{S}^c - \Gamma$ must satisfy
\[
\nu(\pi^*\varphi_{\alpha^c} + \pi^*\psi, D_i) = 0 \quad \text{and} \quad \nu(c \phi_F + \phi_T - \phi_{E^0} + \phi_{E^c}, D_i) = \text{coef}_{D_i}(cF + \Gamma - \phi_{E^0} + \phi_{E^c}) = cF - \phi_{E^0} \in \mathbb{N}.
\]

As $\text{coef}_{D_i}(cF - \phi_{E^0})$, the coefficient of $D_i$ in the divisor $cF - \phi_{E^0}$, is continuous in $c$, the above conditions hold only for those $c$ sitting inside some countable discrete subset $N \subset \mathbb{R}_{>0}$. Such $N$ is chosen such that $\widetilde{S}^c = \Gamma$ for $c \in \mathbb{R}_{>0} \setminus N$ and $\widetilde{S}^c \geq \Gamma$ for $c \in N$. In either case, one can check at this point that
\[
\pi(\widetilde{S}^c) = S
\]
as $\pi(\widetilde{S}^c) \subset \psi^{-1}(-\infty) = S \subset \pi(\Gamma)$.

If the zero locus of $a$ contains no lc centres of $(X, \varphi_{\alpha^c}, \psi)$, then $F$ and $\widetilde{S}^c$ have no common components. In this case, $\text{coef}_{D_i}(cF) = 0$ for any component $D_i$ of $\widetilde{S}^c$. Thus $c \notin N$ according to the above criteria (and $\widetilde{S}^c = \Gamma$ in this case).

Recall that all the relevant divisors on $\widetilde{X}$ are in snc. In view of the isomorphism $\mathcal{J}_{X,\sigma}(\varphi_{\alpha^c}; \psi) \cong \pi_*\left(E^c \otimes \mathcal{J}_{\widetilde{X},\sigma}(\pi^*\varphi_{\alpha^c}; \pi^*\psi)\right)$, for any $f \in \mathcal{O}_X$, one is led to consider
\[
(\dagger) \quad \left|\pi^*f \cdot s_{E^c}\right|^2 e^{-\pi^*\varphi_{\alpha^c} - \pi^*\psi} \sim \left|\pi^*f \cdot s_{E^c}\right|^2 e^{-c\phi_F + \phi_T + \phi_{E^0} - \phi_{E^c}} \left|\pi^*\psi\right|^\sigma (\log|\pi^*\psi|)^{1+\varepsilon} = \left|\pi^*f \cdot s_{E^c}\right|^2 e^{-c\phi_F - \phi_T} \left|\pi^*\psi\right|^{\sigma}(\log|\pi^*\psi|)^{1+\varepsilon}.
\]
This expression is in $L^1_{\text{loc}}(\widetilde{X})$ for all $\varepsilon > 0$ if and only if $f \in \mathcal{J}_{X,\sigma}(\varphi_{\alpha^c}; \psi)$.

For the claim concerning the Hacon–McKernan adjoint ideal sheaf, let $\sigma_{\text{mlc}} := \sigma_{\text{mlc}}(\widetilde{X}, \widetilde{S}^c)$ and $\sigma_{\text{mlc}} := \sigma_{\text{mlc}}(X, \varphi_{\alpha^c}, \psi)$ (see Definition 5.2.4 for the notation; note also that $\sigma_{\text{mlc}} \geq \sigma_{\text{mlc}}$ and $\sigma_{\text{mlc}}(\widetilde{X}, \widetilde{S}^c) \geq \sigma_{\text{mlc}}(X, \Gamma)$ for general $c \geq 0$). Take $\sigma := \sigma_{\text{mlc}}$. As $F$ and $\Gamma$ have no common irreducible components and $\sigma_{\text{mlc}}(\widetilde{X}, \widetilde{S}^c)$ is in $L^1_{\text{loc}}(\widetilde{X})$ for all $\varepsilon > 0$, it follows from the definition of $\text{Adj}^\text{HM}_{(X,S)}(\alpha^c)$ (in eq 6.1.2) and the computation in [4, Prop. 2.2.1] (see also Remark 4.1.4) that $f \in \text{Adj}^\text{HM}_{(X,S)}(\alpha^c)$ implies that the expression $(\dagger)$ is in $L^1_{\text{loc}}(\widetilde{X})$ for all $\varepsilon > 0$, thus
\[
\text{Adj}^\text{HM}_{(X,S)}(\alpha^c) \subset \mathcal{J}_{X,\sigma_{\text{mlc}}}(\varphi_{\alpha^c}; \psi) = \mathcal{J}_{X,\sigma_{\text{mlc}}}(\varphi_{\alpha^c}; \psi) \quad \text{for any } c \geq 0.
\]
Suppose \( c \in \mathbb{R}_{\geq 0} \setminus N \) and \( f \in \mathcal{J}_{X,\sigma_{\text{mlc}}} (\varphi_\sigma; \psi) = \mathcal{J}_{X,\tau_{\text{mlc}}} (\varphi_\tau; \psi) \). Since \( \Gamma = \tilde{S}_c \) in this case, the ideal sheaf \( \text{Ann}_{\mathcal{O}_X} \left( \mathcal{J}_{X,\tau_{\text{mlc}}} (\varphi_\tau; \psi) \right) \) defines \( \Gamma \) (Remark 5.2.3) and thus
\[
|\pi^* f \cdot s_{E^0}|^2 e^{-\varphi_\tau f} \sim |s_{\Gamma} \cdot \pi^* f|^2 e^{-\varphi_\tau f + \varphi_{E^c} - \pi^* \psi} \in L^1_{\text{loc}}(\tilde{X}),
\]
where \( s_{\Gamma} \) is a canonical section of \( \Gamma \) and \( \phi_{\Gamma} = \log |s_{\Gamma}|^2 \). Therefore, one obtains \( f \in \text{Adj}^{\text{HM}}_{\Gamma_\sigma \sigma} (\mathfrak{a}) \) according to (eq 6.1.2). Combining with the previous result, it follows that
\[
\text{Adj}^{\text{HM}}_{\Gamma_\sigma \sigma} (\mathfrak{a}) = \mathcal{J}_{X,\sigma_{\text{mlc}}} (\varphi_\sigma; \psi) \quad \text{for any } c \in \mathbb{R}_{\geq 0} \setminus N.
\]

For the claim concerning the Ein–Lazarsfeld adjoint ideal sheaf, take \( \sigma = 1 \) and assume further that \( \pi^{-1}_c S \) is smooth (thus having disjoint irreducible components). Following the previous arguments, it can be checked directly from (eq 6.1.3), the expression in (\dag) and the computation in [4, Prop. 2.2.1] that
\[
\text{Adj}^{\text{EL}}_{\Gamma_\sigma \sigma} (\mathfrak{a}) \subset \mathcal{J}_{\text{X},1} (\varphi_\sigma; \psi) \quad \text{for any } c \geq 0.
\]

It remains to show the equality for any \( c \in \mathbb{R}_{\geq 0} \setminus N \) and \( c \neq 0 \). Take any \( f \in \mathcal{J}_{\text{X},1} (\varphi_\sigma; \psi) \subset \mathcal{J}_{X,\sigma_{\text{mlc}}} (\varphi_\sigma; \psi) \). As \( f \in \text{Adj}^{\text{HM}}_{\Gamma_\sigma \sigma} (\mathfrak{a}) \) and \( \Gamma = \tilde{S}_c \) has no common components with \( F \) and \( E^0 \). Since \( S \) is a divisor, Proposition 5.2.6, together with the fact that \( \text{codim}_X \mathcal{L}^\xi_{\sigma} (\varphi_\sigma; \psi) \geq \sigma \) for all \( \sigma \geq 1 \), implies that the irreducible components of \( S \) are all \( 1 \)-lc centres of \( (X, \varphi_\sigma, \psi) \) in the sense of Definition 5.2.4. Being exceptional, each component of \( \Gamma_{\text{Exc}} \) must have its image under \( \pi \) properly sitting inside a component of \( S \). According to Theorem 5.2.8, components of \( \Gamma_{\text{Exc}} \) cannot be \( \pi \)-supportive \( 1 \)-lc centres (see Definition 5.2.4), which implies that \( \pi^* f \) vanishes on \( \Gamma_{\text{Exc}} \), as desired.

Remark 6.2.4. When \( S \) is of higher pure codimension in \( X \), one can still construct the corresponding global function \( \psi \) such that \( S \subset \psi^{-1}(-\infty) \) and such that \( S \) is the lc locus of the family \( \{ \mathcal{J}(m\psi) \}_{m \in [0,1]} \) at jumping number \( m = 1 \). Assuming the zero locus of the coherent ideal sheaf \( \mathfrak{a} \) contains no lc centres of \( (X, 0, \psi) \) (in the sense of Definition 5.2.4), one can carry out the same analysis to compare \( \mathcal{J}_{X,\sigma} (\varphi_\sigma; \psi) \) with the adjoint ideal sheaf introduced by Takagi in [40], which is a generalisation of the Ein–Lazarsfeld adjoint ideal sheaf.

Corollary 6.2.5. Given an lc pair \( (X, S) \) and a \( \mathbb{Q} \)-divisor \( D \) in \( X \) such that \( \text{supp} \, D \) contains no lc centres of \( (X, S) \) in the sense of [32, Def. 4.15] and considering the potential \( \phi_D \) (see Notation 2.1.3) and the function \( \psi = \psi_S \) as described at the beginning of Section 6.2, the pair \( (X, S + D) \) is a plt pair with \( [D] = 0 \) if and only if \( \mathcal{J}_{X,1} (c\phi_D; \psi) = \mathcal{O}_X \) for some \( c > 1 \).

If \( \text{supp} \, D \) contains no lc centres of \( (X, \phi_D, \psi) \) in the sense of Definition 5.2.4, then \( (X, S + D) \) is a plt pair with \( [D] = 0 \) if and only if \( \mathcal{J}_{X,1} (\phi_D; \psi) = \mathcal{O}_X \).

Proof. It is known that \( (X, S + D) \) is a plt pair with \( [D] = 0 \) if and only if \( \text{Adj}^{\text{EL}}_{\Gamma_D \sigma} (\mathfrak{a}) = \mathcal{O}_X \) (cf. [40, Remark 1.3 (2)]) and notice the difference between the definitions of “plt” in [32, Def. 2.8] and in [40, Def. 1.1 (ii)]). When \( \text{supp} \, D \) contains no lc centres of \( (X, \phi_D, \psi) \) (in the sense of Definition 5.2.4), Theorem 6.2.3 implies that \( \text{Adj}^{\text{EL}}_{\Gamma_D \sigma} (\mathfrak{a}) = \mathcal{J}_{X,1} (\phi_D; \psi) \) and thus the last claim follows.

Now consider the case when \( \text{supp} \, D \) is only assumed to contain no lc centres of \( (X, S) \) in the sense of [32, Def. 4.15]. Notice that \( (X, S + D) \) is plt with \( [D] = 0 \) if and only if there
exists some constant $c > 1$ such that $(X, S+c'D)$ is plt with $[c'D] = 0$ for all $c' \in [1, c]$. By Theorem 6.2.3, one can also choose $c > 1$ sufficiently close to 1 such that $A d_{\pi(X)}(cD) = \mathcal{I}_{X,1}(c\phi_D; \psi)$ for all $c' \in (1, c]$. Note also that $\mathcal{I}_{X,1}(c\phi_D; \psi) \subset \mathcal{I}_{X,1}(c'\phi_D; \psi)$ for any $c' \in (1, c]$ (in particular, $\mathcal{I}_{X,1}(c\phi_D; \psi) = \mathcal{O}_X$ when $\mathcal{I}_{X,1}(c\phi_D; \psi) = \mathcal{O}_X$). The remaining claim then follows immediately.

By altering the formulation a little bit, one can obtain a more direct result for determining whether a pair is plt or lc.

**Theorem 6.2.6.** Let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$ and consider the function $\psi_\Delta$ as described in Notation 2.1.3. Then,

1. the pair $(X, \Delta)$ is klt if and only if $\mathcal{I}_{X,0}(0; \psi_\Delta) = \mathcal{O}_X$;
2. the pair $(X, \Delta)$ is plt if and only if $\mathcal{I}_{X,0}(0; \psi_\Delta) = \mathcal{I}_{X,1}(0; \psi_\Delta) = \mathcal{I}_{X,2}(0; \psi_\Delta)$;
3. the pair $(X, \Delta)$ is lc if and only if $\mathcal{I}_{X,0}(0; \psi_\Delta) = \mathcal{O}_X$ for some (sufficiently large) integer $\sigma \geq 0$ (and it suffices to consider only $\sigma \in [0, n]$).

**Proof.** The first claim is trivial as $\mathcal{I}_{X,0}(0; \psi_\Delta) = \mathcal{I}_{X}(\psi_\Delta)$.

For the rest of the claims, take any log-resolution $\pi : \tilde{X} \to X$ of $(X, \Delta)$ and the corresponding effective divisors $E$ and $R$ as in Section 2.3 such that $K_{\tilde{X}/X} \equiv E + R$ and $R$ is the maximal divisor with $\pi^*\psi_\Delta - \phi_R$ being quasi-psh. Following the notation at the beginning of Section 5 and also Notation 2.1.3, one has

$$\pi^*\psi_\Delta - \phi_R = \tilde{\varphi} + \phi_{\tilde{S}_0} + \phi_{\tilde{S}} ,$$

where $\tilde{S}$ is the lc locus of the family $\{\mathcal{I}_{\tilde{X}}(m\pi^*\psi_\Delta - \phi_R)\}_{m \in [0, 1]}$ at the jumping number $m = 1$ (which is a reduced snc divisor, and $\tilde{S} = 0$ if $m = 1$ is not a jumping number), and $\tilde{S}_0$ is a divisor with supp $\tilde{S}_0 \subset \tilde{S}$ such that $e^{-\tilde{\varphi}}$ is locally integrable at general points of $\tilde{S}$. Indeed, it can be seen that, if $\pi^*\Delta - R = \sum_{i \in I} c_i D_i$ where $D_i$’s are distinct prime divisors and $c_i \in \mathbb{Q}_{>0}$ for all $i \in I$, one has

(eq 6.2.2) $$\tilde{S}_0 + \tilde{S} = \sum_{i \in I; c_i \in \mathbb{N}} c_i D_i .$$

Writing $\tilde{B} := \pi^*\Delta - R - \tilde{S}_0 - \tilde{S}$ (which is an effective $\mathbb{Q}$-divisor whose coefficients are all non-integral), it follows that

$$(\star) \quad K_{\tilde{X}} + \tilde{S}_0 + \tilde{S} + \tilde{B} \sim_{\mathbb{Q}} \pi^*(K_X + \Delta) + E \quad \text{and} \quad \tilde{\varphi} \sim_{\log} \phi_{\tilde{B}} .$$

Note that $\tilde{S}$ and $E$ have no common irreducible components. If $\tilde{B}$ and $E$ have a common component $D_i$, then $\mathrm{coef}_{D_i}(\tilde{B}) \in (0, 1)$ by the definition of $E$ (and hence $R$). Let $m_0 \in [0, 1)$ be a number such that $\mathcal{I}_{\tilde{X}}(m\pi^*\psi_\Delta - \phi_R) = \mathcal{I}_{\tilde{X}}(m_0\pi^*\psi_\Delta - \phi_R) \supset \mathcal{I}_{\tilde{X}}(\pi^*\psi_\Delta - \phi_R)$ for all $m \in [m_0, 1]$.

Suppose $(X, \Delta)$ is plt (but not klt). From $(\star)$ and the definition of plt in [32, Def. 2.8], it follows that $\tilde{S}_0 = 0$ and $|\tilde{B}| = 0$, which also implies that $\tilde{S} := |\Delta|$ is a reduced divisor and $\tilde{S} = \pi^{-1}_*S$ (which is $\neq 0$, as $(X, \Delta)$ would then be klt otherwise). Assume that $\pi^{-1}_*S$ is smooth (possibly a union of disjoint components) by choosing $\pi$ suitably. It follows easily from Theorem 4.1.2 (and Remark 4.1.5) that

$$\mathcal{I}_{\tilde{X},1}(-\phi_R; \pi^*\psi_\Delta) = \mathcal{I}_{\tilde{X}}(m_0\pi^*\psi_\Delta - \phi_R) \cdot \mathcal{I}_{\tilde{X}}(\tilde{S}) = \mathcal{I}_{\tilde{X}}(m_0(\phi_\tilde{B} + \phi_\tilde{S}) - (1-m_0)\phi_R) = \mathcal{O}_{\tilde{X}} ,$$

where $\mathcal{I}_{\tilde{X}}(\tilde{S}) = \emptyset$.
which implies that $\mathcal{I}_{X,1}(0; \psi) \cong \pi_* \left( E \otimes \mathcal{I}_{X,1}(-\phi_R; \pi^* \psi) \right) \cong \mathcal{O}_X$. Since $\pi(\mathcal{S}) = S$, Proposition 5.2.7 guarantees that $S = \text{lcm}_X(0; \psi)$ and, therefore, $\mathcal{I}_{X,0}(0; \psi) = \text{Ann}_{\mathcal{O}_X} \left( \mathcal{I}_{X,1}(0; \psi) \right) = \mathcal{I}_S = \mathcal{I}_{\{\Delta\}}$.

Conversely, suppose $\mathcal{I}_{X,1}(0; \psi) = \mathcal{O}_X$ and $\mathcal{I}_{X,0}(0; \psi) = \mathcal{I}_{\{\Delta\}}$, which implies immediately that $\sigma_{\text{mck}} = \sigma_{\text{mck}}(X, 0, \psi) \leq 1$ (and $= 1$ if and only if $\mathcal{I}_{\{\Delta\}} \neq \mathcal{O}_X$). Furthermore, one has

$$\mathcal{O}_X = \mathcal{I}_{X,1}(-\phi_R; \pi^* \psi) = \mathcal{I}_m \left( \phi_B + \phi_{\mathcal{S}_0} + \phi_{\mathcal{S}} - (1 - m)\phi_R \right) \cdot \mathcal{I}_{\mathcal{S}}^{2} \left( \mathcal{S} \right)$$

for any $m \in \mathbb{N}$, thus $\mathcal{I}_m \left( \phi_B + \phi_{\mathcal{S}_0} + \phi_{\mathcal{S}} - (1 - m)\phi_R \right) \cdot \mathcal{I}_{\mathcal{S}}^{2} \left( \mathcal{S} \right)$, which is reduced. One then uses the construction (see (eq 6.2.2)), and $\pi(\bar{S}) = S$ thanks to the assumption $\mathcal{I}_{X,0}(0; \psi) = \mathcal{I}_S$. Proposition 5.2.7 guarantees that $S = \text{lcm}_X(0; \psi)$. Since the constant function 1, which is nowhere vanishing, belongs to $\mathcal{I}_{X,1}(0; \psi) = \mathcal{O}_X$, every irreducible component of $S$ are $\pi$-supportive (see Definition 5.2.4). Theorem 5.2.8 then guarantees that $\bar{S} = \pi^{-1} S$. Collecting these results, one observes from $(\ast)$ that $(X, \Delta)$ is plt.

When $(X, \Delta)$ is lc, one also has $\bar{S}_0 = 0$ and $\mathcal{B} = 0$. Choosing an integer $\sigma \in [0, n]$ such that $\text{lcm}_X^{\sigma+1}(\bar{S}) = 0$, the above computation yields $\mathcal{I}_{X,\sigma}(0; \psi) = \mathcal{O}_X$. Conversely, when $\mathcal{I}_{X,\sigma}(0; \psi) = \mathcal{O}_X$ for some $\sigma \geq 0$, the above argument shows again that $\bar{S}_0 = 0$ and $\mathcal{B} = 0$, which implies that $(X, \Delta)$ is lc as observed from $(\ast)$. This completes the proof.

Another proof of Kollár’s theorem on the inversion of adjunction for the case when the base space $X$ is smooth via the use of the residue exact sequences can now be given.

**Theorem 6.2.7** (Inversion of adjunction (Theorem 1.2.7); see [31, Thm. 17.6] and cf. [30, Thm. 1.7]). On a complex manifold $X$, let $\Delta$ be an effective $Q$-divisor on $X$ such that $S := |\Delta|$ is a reduced divisor. Also let $\nu: S'' \to S$ be the normalisation of $S$ and $\text{Diff}_{S''}^* \Delta$ the general different such that $K_{S''} + \text{Diff}_{S''}^* \Delta \sim Q \nu^*(K_X + \Delta)|_S$. Then, $(X, \Delta)$ is plt near $S$ if and only if $(S'', \text{Diff}_{S''}^* \Delta)$ is klt.

**Proof.** Write $S = \sum_{i \in I} D_i$ and $S'' = \sum_{i \in I} D_i''$ such that each $D_i$ is an irreducible divisor and $\nu|_{D_i''}: D_i'' \to D_i$ is the normalisation of $D_i$. One then has $\text{Diff}_{D_i''}^* \Delta = (\text{Diff}_{S''}^* \Delta)|_{D_i''}$ for each $i \in I$. Take any log-resolution $\pi: \tilde{X} \to X$ of $(X, \psi)$. Following the notation in the proof of Theorem 6.2.6, one has

$$\pi^* \psi - \phi_R = \varphi + \phi_{\mathcal{S}_0} + \phi_{\mathcal{S}} \sim \text{log} \phi_B + \phi_{\mathcal{S}_0} + \phi_{\mathcal{S}}$$

and

$$K_{\tilde{X}} + \tilde{S}_0 + \tilde{S} + \tilde{B} \sim Q \pi^*(K_X + \Delta) + E.$$

Recall that $\tilde{B}$ is effective and all of its coefficients are non-integral, and supp $\tilde{S}_0 \subset \tilde{S}$. Since $S = |\Delta|$ is reduced, the divisor $\bar{S}_0$ is $\pi$-exceptional, and $\bar{S} \supset \pi^{-1} S$ can be observed from (eq 6.2.2). Therefore, by writing $\pi^{-1} S = \sum_{i \in I} D_i$, one has

$$\text{Diff}_{D_i''}^* \Delta = \pi'_i \left( \text{Diff}_{D_i}^* (\bar{S}_0 + \bar{S} + \bar{B}) \right) = \pi'_i \left( \text{Diff}_{\tilde{D}_i}^* (\tilde{S}_0 + \tilde{S} + \tilde{B}) \right)$$

and

$$K_{\tilde{D}_i} + \text{Diff}_{\tilde{D}_i}^* \tilde{S} + \tilde{S}_0|_{\tilde{D}_i} + \tilde{B}|_{\tilde{D}_i} \sim Q \pi'' \left( K_{D_i''} + \text{Diff}_{D_i''}^* \Delta \right) + E|_{\tilde{D}_i},$$
where $\pi'$ is a log-resolution of $(S^\nu, \Diff_{S^\nu}^\nu \Delta)$ obtained from the factorisations $\pi|_{\tilde{D}_i} = \nu|_{D^\nu_i} \circ \pi'|_{\tilde{D}_i}$. Moreover, consider the residue short exact sequence in Corollary 5.1.6 with $\sigma := 1$, $\varphi_L := 0$ and $\psi := \psi_\Delta$ given as in Notation 2.1.3. Write $\tilde{S} = \lc_X^1(\tilde{S}) = \sum_{p \in I} \tilde{S}_p^1$.

According to Remark 5.1.8, (eq 5.1.2) and discussion in Section 4.2, one has

$$\mathcal{R}_{X,1}(0; \psi_\Delta) = \pi_* \mathcal{R}_{\tilde{X},1}(\tilde{\varphi} - \phi_E) = \bigoplus_{p \in I} \pi_* \mathcal{I}_{\tilde{S}_p^1}$$

$$= \bigoplus_{p \in I} \left( \pi|_{\tilde{S}_p^1} \right)_* \left( \tilde{S}_0^{-1}|_{\tilde{S}_p^1} \otimes \left( \Diff_{\tilde{S}_p^1}^* \tilde{S} \right)^{-1} \otimes \mathcal{I}_{\tilde{S}_p^1}(\tilde{\varphi} - \phi_E) \right)$$

$$= \bigoplus_{p \in I} \left( \pi|_{\tilde{S}_p^1} \right)_* \left( \tilde{S}_0^{-1}|_{\tilde{S}_p^1} \otimes \left( \Diff_{\tilde{S}_p^1}^* \tilde{S} \right)^{-1} \otimes \mathcal{I}_{\tilde{S}_p^1}(\phi_E - \phi_E) \right).$$

Suppose that $(X, \Delta)$ is plt near $S$. By shrinking $X$ to a neighbourhood of $S$, one can assume that $(X, \Delta)$ is plt everywhere. The proof of Theorem 6.2.6 shows that $\tilde{S} = \pi_*^{-1}S$ (thus $\overline{\{\tilde{D}_i \mid i \in I\}} = \overline{\{\tilde{S}_p \mid p \in I\}}$), $\tilde{S}_0 = 0$, $[\tilde{B}] = 0$ and $\Diff^*_{\tilde{S}_p} \tilde{S} = 0$ (as $\lc_X^2(\tilde{S}) = 0$, i.e. irreducible components of $\tilde{S}$ are mutually disjoint). Therefore, $(S^\nu, \Diff_{S^\nu}^\nu \Delta)$ is klt, as can be seen from (**).

Now suppose that $(S^\nu, \Diff_{S^\nu}^\nu \Delta)$ is klt. It follows from (**) that $\tilde{S}_0|_{\tilde{D}_i} = 0$, $\Diff^*_{\tilde{D}_i} \tilde{S} = 0$ (note that $\tilde{S}_0$ and $\tilde{S}$ are effective $\mathbb{Z}$-divisors) and $[\tilde{B}] = 0$ for all $i \in I$. The general different $\Diff^*_{\tilde{D}_i} \tilde{S}$ being trivial implies that $\tilde{D}_i$ is disjoint from any other components of $\tilde{S}$. Recall that $\tilde{D}_i \in \{\tilde{S}_p^1 \mid p \in I\}$ for all $i \in I$. It follows that the summand of $\mathcal{R}_{X,1}(0; \psi_\Delta)$ supported on $\tilde{D}_i$ is given by $\pi_* \mathcal{I}_{\tilde{D}_i} = (\pi|_{\tilde{D}_i})_* \mathcal{O}_{\tilde{D}_i} = 0$, so $\tilde{D}_i$ is $\pi$-supportive for each $i \in I$ (see Definition 5.2.4) and therefore $\lc_X^1(0; \psi_\Delta) \supset S$. On any open polydisc $V \subset X$ (in some coordinate chart) such that $V \cap \Delta \neq \emptyset$, take any $\tilde{g} = (\tilde{g}_p)_{p \in I} \in \mathcal{R}_{\tilde{X},1}(\tilde{\varphi} - \phi_E)(\pi^{-1}(V))$ such that the component $\tilde{g}_{\tilde{D}_i}$ of $\tilde{g}$ on $\tilde{D}_i$ for each $i \in I$ (such that $D_i \cap V \neq \emptyset$) is a nowhere vanishing function (for example, a constant function), while the other components $\tilde{g}_{\tilde{S}_p}$ are chosen to be 0 on $\tilde{S}_p^\nu \subset \tilde{S} - \pi_*^{-1}S$. Surjectivity of the residue morphism guarantees there exists $f \in \mathcal{I}_{X,1}(0; \psi_\Delta)(V)$ such that $\Res(\pi^*f) = \tilde{g}$. Note that, given $\tilde{S}_0|_{\tilde{D}_i} = 0$ and $\Diff^*_{\tilde{D}_i} \tilde{S} = 0$, the image of the residue morphism $\Res$ on $\tilde{D}_i$ is given simply by restriction $\pi^*f \mapsto \pi^*f|_{\tilde{D}_i} = u_i \tilde{g}_{\tilde{D}_i} \neq 0$ up to a multiple factor of a nowhere vanishing function $u_i$ on $\tilde{D}_i$. This indeed implies that the components $D_i = \pi(\tilde{D}_i)$ of $S$ has to be disjoint from one another as well as from each $\pi(\tilde{S}_p)$ for $\tilde{S}_p^1 \subset \tilde{S} - \pi_*^{-1}S$ (as one has $\pi^*|_{\tilde{S}_p^1} = 0$ on 1-lc centres $\tilde{S}_p^1 \subset \tilde{S} - \pi_*^{-1}S$ by the choice of $\tilde{g}$, regardless of whether $\tilde{S}_0|_{\tilde{S}_p^1}$ and $\Diff^*_{\tilde{S}_p^1} \tilde{S}$ are trivial or not). This implies that there exists an open neighbourhood $W \subset X$ of $S$ disjoint from $\pi(\tilde{S}) \setminus S$ such that $f$ is nowhere vanishing on $W \cap V$, and hence $\mathcal{I}_{X,1}(0; \psi_\Delta) = \mathcal{O}_X$ on such neighbourhood $W$.

With such choice of $W$, one has $\pi(\tilde{S})|_W = S$ and that every 1-lc centre in $\tilde{S}|_{\pi^{-1}(W)}$ is $\pi$-supportive (since $\mathcal{I}_{X,1}(0; \psi_\Delta) = \mathcal{O}_X$ on $W$). In view of Theorem 5.2.8, one has $\tilde{S}|_{\pi^{-1}(W)} = \pi_*^{-1}S$. It thus follows that $\mathcal{I}_{X,0}(0; \psi_\Delta) = \mathcal{I}_S$ on $W$. Theorem 6.2.6 then assures that $(X, \Delta)$ is plt on that neighbourhood. □
6.3. Further examples. The first example is to show that the adjoint ideal sheaf introduced by Guenancia is different from the Ein–Lazarsfeld adjoint ideal sheaf in general.

Example 6.3.1 (Guenancia’s adjoint ideal sheaf). Guenancia’s adjoint ideal sheaf in its original form (see [23, Def. 2.7 and Def. 2.10] or footnote 3 on page 3) may not coincide with the Ein–Lazarsfeld adjoint ideal sheaf as claimed (note that [23, Def. 2.1], which is corrected in [24], is the definition of $\text{Adj}^{\text{EL}}_{X,S}(a^c)$) when the reduced snc divisor $S$ has more than one component, as can be illustrated from the example in Example 5.1.1. Recall that $X = \Delta^2 \subset \mathbb{C}^2$ is the unit 2-disc centred at the origin and $S = \{z_1z_2 = 0\}$ (thus one can set $\psi := \log|z_1|^2 + \log|z_2|^2 - 1$). Take $a = \mathcal{O}_X$ and $\varphi_{c^i} = 0$ for any $c \geq 0$. Taking $\pi$ to be the blow-up at the origin of $X$ with exceptional divisor $E$, it follows easily that $\text{Adj}^{\text{EL}}_{X,S}(\mathcal{O}_X) = \pi_*\mathcal{O}_X(-E) = m\mathcal{O}_X$, the defining ideal sheaf of the origin $\mathcal{O}$ in $X$. However, the integral

$$\int_{\Delta^2} \frac{d\text{vol}_{\Delta^2}}{|z_1z_2|^2} \left(\log\frac{|z_1|^2}{e} - \log\frac{|z_2|^2}{e}\right)^2 = \left(\int_{\Delta} \frac{\pi^*id\zeta \wedge d\bar{\zeta}}{|\zeta|^2} \left(\log\frac{|\zeta|^2}{e}\right)^2\right)^2$$

being convergent implies that Guenancia’s adjoint ideal sheaf is simply $\mathcal{O}_X$ in this case.

Remark 6.3.2. The flaw in the claim [23, Prop. 2.11] comes from the proof of [23, Lemma 2.12]. The argument that there exists $0 < \epsilon' \leq \epsilon$ such that $\lambda_{\tilde{S}}(\epsilon') > -1$ for all $k > p$ (in the notation in [23, Lemma 2.12]) can be seen false easily by considering the situation following the argument in the last part of the proof of Theorem 6.2.3. See the Erratum [24] by Guenancia for the corrected proof of the claim when $S$ is smooth. While Guenancia’s adjoint ideal sheaf does not coincide with the Ein–Lazarsfeld adjoint ideal sheaf in general when $S$ has more than one intersecting components, the two ideal sheaves coincide when $S$ has only one irreducible component, even if it is not smooth (a setup studied in [34] and [30]). A proof can be obtained by following the argument in the last part of the proof of Theorem 6.2.3.

The second example gives an instance of the algebraic and analytic adjoint ideal sheaves when $(X, S)$ is no longer lc.

Example 6.3.3 (Non-lc case). Let $X = \Delta^2 \subset \mathbb{C}^2$ again be the unit 2-disc and $S = \{z_2^2 = z_1^3\}$ be the cuspidal curve. Let $\pi: \tilde{X} \to X$ be the standard resolution of the singularity at the origin $\mathcal{O} = (0, 0)$ such that

$$K_{\tilde{X}} \sim \pi^*K_X + E_1 + 2E_2 + 4E_3 \quad \text{and} \quad \pi^*S = \pi_{-1}^*S + 2E_1 + 3E_2 + 6E_3,$$

where $E_1$, $E_2$ and $E_3$ are exceptional (prime) divisors. The pair $(X, S)$ is apparently not lc. It follows immediately from (eq 6.1.2) and (eq 6.1.3) that

$$\text{Adj}^{\text{EL}}_{X,S}(\mathcal{O}_X) = \pi_*\mathcal{O}_{\tilde{X}}(-E_1 - E_2 - 2E_3) \quad \text{and} \quad \text{Adj}^{\text{HM}}_{X,S}(\mathcal{O}_X) = \pi_*\mathcal{O}_{\tilde{X}}(-2E_3).$$

Let $\psi = \psi_S = \log|z_1^3 - z_2^2|^2 - 1$. Notice that $S$ is the lc locus of the family $\{\mathcal{I}_X(m\psi)\}_{m \in [0, 1]}$ at jumping number $m = 1$. Since, for $m \in [0, 1],$

$$m\pi_{\tilde{X}}^*\psi \sim \log m\phi_{\pi_{-1}^*S} + (2m - 1)\phi_{E_1} + (3m - 2)\phi_{E_2} + (6m - 4)\phi_{E_3},$$

one sees that $\tilde{S} = \pi_{-1}^*S + E_1 + E_2 + E_3$ and $m = \frac{5}{6}$ is the jumping number preceding $m = 1$. Note that $\pi_{-1}^*S$, $E_1$ and $E_2$ are mutually disjoint, while $E_3$ intersects all 3 other lines exactly once. Therefore, $\text{lc}^1_{\tilde{X}}(\tilde{S}) = \tilde{S}$ and $\text{lc}^2_{\tilde{X}}(\tilde{S}) = \{p_0, p_1, p_2\}$, a set of 3 distinct
points. A computation as before shows that
\[ \mathcal{I}_X \left( \frac{5}{6} \psi \right) = \mathcal{J}_{X,2}(0; \psi) = \pi_* \mathcal{O}_{\bar{X}}(-E_3), \]
\[ \mathcal{J}_{X,1}(0; \psi) = \pi_* \left( \mathcal{O}_{\bar{X}}(-E_3) \cdot \mathcal{I}_{(p_0,p_1,p_2)} \right), \]
\[ \mathcal{J}_X(\psi) = \mathcal{J}_{X,0}(0; \psi) = \pi_* \mathcal{O}_{\bar{X}}(-\pi_*^{-1}S - E_1 - E_2 - 2E_3), \]
where \( \mathcal{I}_{(p_0,p_1,p_2)} \) is the defining ideal sheaf of the 3 points in \( \bar{X} \). Notice that one has \( \text{Adj}^{\text{HM}}(O_X) \subset \mathcal{J}_1(0; \psi) \) a priori (although the two sheaves are indeed equal by the computation below). Since
\[ \text{div}(\pi^*z) = S'_1 + E_1 + E_2 + 2E_3 \quad \text{and} \quad \text{div}(\pi^*z_2) = S'_2 + E_1 + 2E_2 + 3E_3, \]
where \( S'_i \) is the proper transform of \( \{z_i = 0\} \), it can be checked directly that
\[ \text{Adj}^{\text{EL}}(O_X) = \text{Adj}^{\text{HM}}(O_X) = \mathcal{J}_{X,1}(0; \psi) = \mathcal{J}_{X,2}(0; \psi) = \langle z_1, z_2 \rangle \]
and \( \mathcal{J}_{X,0}(0; \psi) = \mathcal{J}_X(\psi) = \mathcal{I}_S \).

When the family \( \{\mathcal{J}(m\psi)\}_{m \geq 0} \) at jumping number \( m = \frac{5}{6} \) is considered, one can find that, using the same calculation, \( S = E_3 \) and the corresponding adjoint ideal sheaves are given by
\[ \mathcal{J}_{X,0} \left( \frac{5}{6} \psi \right) = \mathcal{J}_X \left( \frac{5}{6} \psi \right) = \langle z_1, z_2 \rangle \quad \text{and} \quad \mathcal{J}_{X,1} \left( \frac{5}{6} \psi \right) = \mathcal{J}_X \left( \frac{2}{3} \psi \right) = O_X. \]

References

[1] F. Ambro, Basic properties of log canonical centers, Classification of algebraic varieties, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011, pp. 39–48, DOI 10.4171/007-1/2. MR2779468
[2] , An injectivity theorem, Compos. Math. 150 (2014), no. 6, 999–1023, DOI 10.1112/S0010437X13007768. MR3223880
[3] C. Birkar, On connectedness of non-klt loci of singularities of pairs (2022), arXiv version at arXiv:2010.08226v2 [math.AG]. to appear in J. Differential Geom.
[4] T. O. M. Chan, On an \( L^2 \) extension theorem from log-canonical centres with log-canonical measures, Math. Z. 301 (2022), no. 2, 1695–1717, DOI 10.1007/s00209-021-02890-9, available at https://rdcu.be/cFDPA, arXiv version at arXiv:2008.03019 [math.CV]. Numbering of cited sections and theorems follows the arXiv version. MR4418335
[5] T. O. M. Chan and Y.-J. Choi, Extension with log-canonical measures and an improvement to the plt extension of Demailly-Hacon-Păun, Math. Ann. 383 (2022), no. 3-4, 943–997, DOI 10.1007/s00208-021-02152-3, available at https://rdcu.be/cn586, arXiv version at arXiv:1912.08076 [math.CV]. MR4458394
[6] , On an injectivity theorem for log-canonical pairs with analytic adjoint ideal sheaves (2022), arXiv version at arXiv:2205.06954 [math.CV]. accepted by Trans. Amer. Math. Soc.
[7] J.-P. Demailly, Estimations \( L^2 \) pour l’opérateur \( \bar{\partial} \) d’un fibré vectoriel holomorphe semi-positif au-dessus d’une variété kählérienne complète, Ann. Sci. École Norm. Sup. (4) 15 (1982), no. 3, 457–511 (French). MR606050
[8] , On the Ohsawa-Takegoshi-Manivel \( L^2 \) extension theorem, Complex analysis and geometry (Paris, 1997), Progr. Math., vol. 188, Birkhäuser, Basel, 2000, pp. 47–82 (English, with English and French summaries). MR1782659
[9] , Multiplier ideal sheaves and analytic methods in algebraic geometry, School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000), ICTP Lect. Notes, vol. 6, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001, pp. 1–148. MR1919457
[10] , Complex analytic and differential geometry (2012), https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf. OpenContent Book.
[11] ____, Extension of holomorphic functions defined on non reduced analytic subvarieties, The legacy of Bernhard Riemann after one hundred and fifty years. Vol. 1, Adv. Lect. Math. (ALM), vol. 35, Int. Press, Somerville, MA, 2016, pp. 191–222. MR3525916

[12] E. Eisenstein, Inversion of adjunction in high codimension, Ph.D. Thesis, University of Michigan, 2011.

[13] L. Ein and R. Lazarsfeld, Singularities of theta divisors and the birational geometry of irregular varieties, J. Amer. Math. Soc. 10 (1997), no. 1, 243–258, DOI 10.1090/S0894-0347-97-00223-3. MR1396893

[14] L. Ein and M. Popa, Extension of sections via adjoint ideals, Math. Ann. 352 (2012), no. 2, 373–408, DOI 10.1007/s00208-011-0639-2. MR2874961

[15] O. Fujino, Fundamental theorems for the log minimal model program, Publ. Res. Inst. Math. Sci. 47 (2011), no. 3, 727–789, DOI 10.2977/PRIMS/50. MR2823205

[16] O. Fujino and Y. Gongyo, Log pluricanonical representations and the abundance conjecture, Compos. Math. 150 (2014), no. 4, 593–620, DOI 10.1112/S0010437X13007495. MR3200670

[17] H. Guenancia, A proof of Demailly’s strong openness conjecture, Ann. of Math. (2) 182 (2015), no. 4, 1011–1035. MR3418526

[18] H. Grauert and R. Remmert, Zur Theorie der Modifikationen. I. Stetige und eigentliche Modifikationen komplexer Räume, Math. Ann. 129 (1955), 274–296, DOI 10.1007/BF01362372 (German). MR71085

[19] , Coherent analytic sheaves, Grundbegriffe der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 265, Springer-Verlag, Berlin, 1984. MR755331

[20] , Theory of Stein spaces, Classics in Mathematics, Springer-Verlag, Berlin, 2004. Translated from the German by Alan Huckleberry; Reprint of the 1979 translation. MR2029201

[21] , Cluster points of jumping coefficients and equisingularities of plurisubharmonic functions, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 18 (2018), no. 1, 391–395. MR3799407

[22] Q. Guan and Z. Li, Analytic adjoint ideal sheaves associated to plurisubharmonic functions, Ann. Math. 153 (2001), no. 3, 593–620, DOI 10.1090/S0002-9947-01-02746-6. MR1832073

[23] Q. Guan and Z. Li, Erratum for the article “Toric plurisubharmonic functions and analytic adjoint ideal sheaves”, Ann. Math. 167 (2008), no. 2, 755–757. DOI 10.4007/annals.2008.167.755. MR2423854

[24] Q. Guan and Z. Li, Coherence of analytic adjoint ideal sheaves on non reduced analytic manifolds, J. Reine Angew. Math. 670 (2012), 69–84. DOI 10.1515/crelle-2011-0045. MR2945594

[25] Q. Guan and Z. Li, Jumping numbers of analytic multiplier ideals, Math. Ann. 362 (2015), no. 2, 605–616, DOI 10.1007/s00208-014-1137-x. MR3416751

[26] J. Kollár, Flips and abundance for algebraic threefolds - A summer seminar at the University of Utah (Salt Lake City, 1991), Astérisque, Société mathématique de France, 1992 (English). Zbl 0782.00075

[27] J. Kollár, Singularities of the minimal model program, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge, 2013. With a collaboration of Sándor Kovács. MR3057950

[28] R. Lazarsfeld, Positivity in algebraic geometry. II, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 49, Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals. MR2095472

[29] Z. Li, Analytic adjoint ideal sheaves associated to plurisubharmonic functions. II, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 22 (2021), no. 1, 183–193. MR4288652
[35] L. Manivel, *Un théorème de prolongement \( L^2 \) de sections holomorphes d’un fibré hermitien*, Math. Z. 212 (1993), no. 1, 107–122, DOI 10.1007/BF02571643 (French). MR1200166

[36] S. Matsumura, *Injectivity theorems with multiplier ideal sheaves for higher direct images under Kähler morphisms*, Algebr. Geom. 9 (2022), no. 2, 122–158, DOI 10.14231/ag-2022-005, arXiv version at arXiv:1607.05554v2 [math.CV]. MR4429015

[37] V. V. Shokurov, *3-fold log flips*, Russian Acad. Sci. Izv. Math. 40 (1993), no. 1, 95–202.

[38] Y. T. Siu, *Analyticity of sets associated to Lelong numbers and the extension of closed positive currents*, Invent. Math. 27 (1974), 53–156, DOI 10.1007/BF01389965. MR352516

[39] H. Skoda, *Sous-ensembles analytiques d’ordre fini ou infini dans \( \mathbb{C}^n \)*, Fonctions analytiques de plusieurs variables et analyse complexe (Colloq. Internat. CNRS, No. 208, Paris, 1972), Gauthier-Villars, Paris, 1974, pp. 235. “Agora Mathematica”, No. 1 (French). MR0460705

[40] S. Takagi, *Adjoint ideals along closed subvarieties of higher codimension*, J. Reine Angew. Math. 641 (2010), 145–162, DOI 10.1515/CRELLE.2010.031. MR2643928

[41] S. Takayama, *Pluricanonical systems on algebraic varieties of general type*, Invent. Math. 165 (2006), no. 3, 551–587, DOI 10.1007/s00222-006-0503-2. MR2242627

[42] M. Vaquié, *Irrégularité des revêtements cycliques*, Singularities (Lille, 1991), London Math. Soc. Lecture Note Ser., vol. 201, Cambridge Univ. Press, Cambridge, 1994, pp. 383–419 (French). MR1295085

Email address: mariochan@pusan.ac.kr

Dept. of Mathematics, Pusan National University, Busan 46241, South Korea