Topography

On the commutator length of a Dehn twist

Sur la longueur des commutateurs d’un twist de Dehn

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1. Introduction

Let $S$ be a closed surface of genus $g$. If $S$ is nonorientable, then $g$ is the number of projective planes in a connected sum decomposition. The mapping class group $\mathcal{M}(S)$ of $S$ is the group of isotopy classes of all, orientation preserving if $S$ is orientable, self-homeomorphisms of $S$. For orientable $S$, the extended mapping class group $\mathcal{M}^\circ(S)$ is the group of isotopy classes of all self-homeomorphisms of $S$, including those reversing orientation. For notational convenience we define $\mathcal{M}^\circ(S)$ to equal $\mathcal{M}(S)$ for nonorientable $S$.

For a two-sided simple closed curve $c$ on $S$ we denote by $t_c$ a Dehn twist about $c$. For a Dehn twist $t_c$ we always assume that $c$ does not bound a disc or a Möbius band, so that $t_c$ is a nontrivial element of $\mathcal{M}(S)$. It is well known that $\mathcal{M}(S)$ is generated by Dehn twists if $S$ is orientable. If $S$ is nonorientable, then the twist subgroup $T(S)$ generated by all Dehn twists has index 2 in $\mathcal{M}(S)$ (cf. [9]).

For a group $G$ let $[G, G]$ denote the commutator subgroup generated by all commutators $[a, b] = aba^{-1}b^{-1}$. For $x \in [G, G]$ the commutator length $cl_G(x)$ is the minimum number of factors needed to express $x$ as a product of commutators. The stable commutator length is the limit

$$scl_G(x) = \lim_{n \to \infty} \frac{cl_G(x^n)}{n}.$$ 

Recall that the first homology group $H_1(G; \mathbb{Z})$ of $G$ is isomorphic to the quotient $G/[G, G]$. 

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For orientable \( S \) and \( g \geq 3 \) it is well known that \( \mathcal{M}(S) \) is perfect, i.e. \( \mathcal{M}(S) = [\mathcal{M}(S), \mathcal{M}(S)] \) (cf. [10]), and for any Dehn twist \( t_c \) we have \( cl_{\mathcal{M}(S)}(t_c) = 2 \) and \( scl_{\mathcal{M}(S)}(t_c) > 0 \) [1,2,5,6]. For nonorientable \( S \) the groups \( H_1(\mathcal{M}(S); \mathbb{Z}) \) and \( H_1(T(S); \mathbb{Z}) \) were computed by Korkmaz [4] and Stukow [11]. In particular, if \( g \geq 7 \) then we have \([\mathcal{M}(S), \mathcal{M}(S)] = [T(S), T(S)] = T(S)\).

In this Note we prove the following:

**Theorem 1.1.** Let \( S \) be a closed orientable surface of genus \( g \geq 3 \) or a closed nonorientable surface of genus \( g \geq 7 \). Then for every two-sided simple closed curve \( c \) on \( S \) and every \( n \in \mathbb{Z} \), \( t_c^n \) is equal to a single commutator of elements of \( \mathcal{M}^\circ(S) \).

For even \( n \) Theorem 1.1 and hence also Corollary 1.3 below, follow immediately from the fact that \( t_c \) is conjugate to its inverse in \( \mathcal{M}^\circ(S) \) (see Remark below), and were known already, at least for orientable \( S \) [7, Remark 12].

**Theorem 1.2.** Let \( c \) be a two-sided simple closed curve on a closed nonorientable surface \( S \) satisfying one of the following assumptions:

- \( c \) is separating and \( g \geq 7 \), or
- \( S \setminus c \) is connected and nonorientable and \( g \geq 8 \), or
- \( S \setminus c \) is connected and orientable, \( g \geq 6 \) and \( g \equiv 2 \mod 4 \).

Then for any \( n \in \mathbb{Z} \), \( t_c^n \) is equal to a single commutator of elements of \( T(S) \).

The following corollary is an immediate consequence of Theorems 1.1 and 1.2 and the definition of the stable commutator length:

**Corollary 1.3.** For \( S \) and \( c \) as in Theorem 1.1 or Theorem 1.2 we have respectively \( scl_{\mathcal{M}^\circ(S)}(t_c) = 0 \) or \( scl_{T(S)}(t_c) = 0 \).

Our proof of Theorem 1.2 fails when \( c \) is nonseparating and \( g = 7 \), or \( g = 4k \) for \( k \geq 2 \) and \( S \setminus c \) is orientable. We conjecture that also in these cases we have \( cl_{T(S)}(t_c^n) = 1 \) for any \( n \in \mathbb{Z} \).

2. Proofs

Consider a torus with three holes \( T \). Let \( c_1, c_2, c_3 \) be its boundary curves and let \( b, a_1, a_2, a_3 \) be nonseparating simple closed curves in the interior of \( T \), such that \( a_1, a_2, a_3 \) are pairwise disjoint, and \( b \) intersects \( a_i \) transversally at one point for \( i = 1, 2, 3 \) (Fig. 1). The right Dehn twists about these curves satisfy the following relations in the mapping class group of \( T \):

- Twists about disjoint curves commute,
- \( t_b t_0 t_b = t_b t_0 t_b \) for \( i = 1, 2, 3 \),
- \( (t_b t_0 t_b)^3 = t_c t_c t_c \).

The first two are the well known braid relations, the third is the star relation discovered by Gervais [3]. By using the braid relations we can rewrite the star relation in the following way.

\[
t_{c_1} t_{c_2} t_{c_3} = (t_b t_{a_1} t_{a_2} t_{a_3})(t_b t_{a_1} t_{a_2} t_{a_3})(t_b t_{a_1} t_{a_2} t_{a_3}) = t_b t_{a_2} t_{a_3}(t_a B t_{a_1}) t_{a_2} (t_{a_3} t_b t_{a_1}) t_{a_1} t_{a_2} = t_b t_{a_2} t_{a_3} t_b t_{a_1} t_{a_2} = (t_b t_{a_2} t_{a_3} t_b t_{a_1} t_{a_2}) (t_b t_{a_2} t_{a_3} t_b t_{a_1} t_{a_2}).
\]

Since \( t_{c_i} \) commute with all twists, for every \( n \in \mathbb{Z} \) we have

\[
t_{c_1}^n = (t_b t_{a_2} t_{a_3} t_b t_{a_1} t_{a_2} t_{c_2}^{-1}) (t_{c_1}^{-1} t_b t_{a_2} t_{a_3} t_b t_{a_1} t_{a_2})^n.
\]

(1)
There is a reflectional symmetry \( r : T \to T \) such that \( r(b) = b \), \( r(a_1) = a_1 \), \( r(a_2) = a_3 \), \( r(c_1) = c_1 \), \( r(c_2) = c_3 \). Since \( r \) is orientation reversing it conjugates right twists to left twists, and so we have

\[
r(t_b t_d t_a t_b^{-1} t_d^{-1}) = (t_c^{-1} t_a t_b t_a t_b^{-1} t_c^{-1})^n.
\]

By the braid relations we have

\[
t_c^{-1} t_a t_b t_a t_b^{-1} t_a t_b = t_c^{-1} t_a t_b t_a t_b t_c t_d t_a t_b t_d t_c = t_a t_b t_c t_a t_b t_d t_c t_a t_b t_d t_c.
\]

Thus

\[
(t_c^{-1} t_b t_d t_a t_b^{-1} t_d^{-1})^n = t_a^{-1} r(t_b t_d t_a t_b)^n t_a^{-1} r t_a.
\]

and by (1)

\[
t_c^n = \left[ (t_b t_d t_a t_b t_a t_b^{-1} t_d^{-1})^n, t_a^{-1} r \right].
\]

Proof of Theorem 1.1. Let \( S \) be a closed orientable surface of genus \( g \geq 3 \) or a closed nonorientable surface of genus \( g \geq 7 \). Consider the torus \( T \) as embedded in \( S \) in such a way that the reflectional symmetry \( r \) extends to \( r : S \to S \). Then (2) holds in \( \mathcal{M}^n(S) \). The embedding of \( T \) in \( S \) can be arranged in such a way that \( c_1 \) is nonseparating in \( S \) or separating, bounding a subsurface of arbitrary topological type. Moreover, if \( S \) is nonorientable of even genus and \( c_1 \) is nonseparating then \( S \setminus c_1 \) may be orientable or not (the former case is shown in Fig. 1). It follows that for every simple closed curve \( c \) on \( S \) there is a homeomorphism \( h : S \to S \) such that \( h(c) = c_1 \), for appropriate embedding of \( T \) in \( S \). Thus \( t_c^n = h^{-1} t_c^n h \) for \( n \in \mathbb{Z} \) and, since a conjugate of a commutator is also a commutator, we have proved Theorem 1.1. \( \square \)

Recall that for a closed nonorientable surface \( S \) of genus \( g \), \( H_1(S; \mathbb{R}) \) is a real vector space of dimension \( g - 1 \). For \( f \in \mathcal{M}(S) \) let \( f_* : H_1(S; \mathbb{R}) \to H_1(S; \mathbb{R}) \) be the induced automorphism. It turns out that the determinant homomorphism \( f \mapsto \det f_* \) takes values in the group \( \{-1, 1\} \) and its kernel is the twist subgroup \( \mathcal{T}(S) \) (cf. [8] and [11, Corollary 6.3]).

Proof of Theorem 1.2. The idea of the proof is the same as for Theorem 1.1. The only problem is that the involution \( r \) may not be an element of \( \mathcal{T}(S) \), in which case it has to be replaced by a different mapping class.

Suppose that \( S \) and \( c \) satisfy one of the assumptions of the theorem. Consider \( T \) as embedded in \( S \) in such a way that \( c_1 = c \), as in the proof of Theorem 1.1. If \( c \) is separating, then we may arrange that the component of \( S \setminus c \) which does not contain \( T \) is nonorientable of genus at least 2. If \( c \) is separating or nonseparating with \( \partial S \setminus c \) nonorientable, then \( N = S \setminus T \) is a nonorientable surface of genus at least 2 and hence it supports a homeomorphism \( h \) which is not a product of Dehn twists (we may take \( h \) to be a crosscap slide or \( \phi \)-homeomorphism introduced by Lickorish [8]). Since \( h \) is equal to the identity on \( T \), thus it commutes with twists about \( b, a_i, c_1 \) for \( i = 1, 2, 3 \). Now if the involution \( r \), which is an extension of the reflectional symmetry of \( T \), is in \( \mathcal{T}(S) \), then \( ct_{c_1} \in \mathcal{T}(S) \), and \( ct_{c_1} t_c = 1 \) by (2). If \( r \notin \mathcal{T}(S) \) then \( r h \in \mathcal{T}(S) \), and \( ct_{c_1} t_c = 1 \) by (2) with \( r h \) in the place of \( r \).

Now suppose that \( S \setminus c_1 \) is orientable. Fig. 1 shows \( S \) as being obtained from an orientable surface \( S' \) by identifying two boundary components. The homology classes of the curves \( a_1, b, c_2, d, h, e_1, f_1 \) for \( 1 \leq i \leq k \), where \( g = 2(k + 3) \), form a basis of \( H_1(S; \mathbb{R}) \) (note that \( [a_1] = 0 \) in \( H_1(S; \mathbb{R}) \)). Now we may take \( r \) as being induced by a reflection of \( S' \), so that \( r_*[a_1] = [a_1], r_*[b] = -[b], r_*[c_2] = [c_2], r_*[d] = -[d], r_*[h] = [h] - [d], r_*[e_1] = -[e_1], r_*[f_1] = [f_1] \) for \( 1 \leq i \leq k \). We see that \( \det r_* = (-1)^k \), which means that \( r \in \mathcal{T}(S) \) if and only if \( g = 2 \mod 4 \). \( \square \)

Remark. Let \( c \) be any two-sided simple closed curve on a surface \( S \). There is a homeomorphism \( h : S \to S \) preserving \( c \) and reversing orientation of its neighborhood. We have \( t_c h t_c^{-1} h^{-1} = t_c^2 \)

\[
t_c^{2n} = t_c^n t_c^n = t_c^2 h t_c^{-n} h^{-1} = t_c^n h
\]

for any \( n \in \mathbb{Z} \). Thus any even power of any Dehn twist on any surface \( S \) is equal to a single commutator of elements of \( \mathcal{M}(S) \). Moreover, if \( S \setminus c \) is nonorientable of genus at least 2, then we may take \( h \in \mathcal{T}(S) \) by composing it if necessary with a homeomorphism fixing \( c \) which is not a product of Dehn twists. In particular, if \( c \) is a nonseparating two-sided curve on a closed nonorientable surface of genus 7 then \( scl_{T,S}(t_c) = 0 \), which slightly improves Corollary 1.3.

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