Correlation Functions of The Tri-critical 3-States Potts Model

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March 27, 2022

Abstract
We build the $\mathbb{Z}_3$ invariants fusion rules associated to the $(D_4,A_6)$ conformal algebra. This algebra is known to describe the tri-critical Potts model. The 4-pt correlation functions of critical fields are developed in the bootstrap approach, and in the other hand, they are written in term of integral representation of the conformal blocks. By comparing both the expressions, one can determine the structure constants of the operator algebra.

1 Introduction
The Conformal Field Theories on 2d euclidean spaces are of major importance, they allow through their mathematical tools the investigation of several topics in physics, such as statistical physics in condensed matter and String theory in high-energy physics. The minimal models of central charge $c = 1 - 6/m(m + 1)$ ($m = 6$ in our case) form a particular class of conformal theories, they describe the critical (or tricritical) points of known 2d statistical systems, they were introduced in the seminal work of Belavin and al [1]. Just after that discovery, Dotsenko calculated the linear differential equations obeyed by correlators containing some degenerate conformal fields, and he applied those results to the $Z_3$ Potts model [2]. This was done before the discovery of the modular ADE classification [3]. Next, the connection was been made in many ways such in [11].
Recently, the description of systems with tricritical point by non-minimal conformal theories was established in [5].

Our interest here is the determination of the structure constants of the tricritical Potts algebra from that of the \((D_4, A_6)\) one. For that we construct the \(Z_3\) invariant fusion rules and use a method which consists in comparing the integral representation of the correlation functions with their developments obtained in the bootstrap approach. This method was first used by McCabe in [6], [7] to determine the structure constants of the critical 3-states Potts model.

Because the studied algebra here contains an integer spin field \(W_5\) which generates an extended algebra, the model become classified in a diagonal series, and it is possible to study such models in the context of extended theory [12], [13].

As done for the associated conformal algebra of the critical 3-states Potts model, the identification in this work for the tri-critical 3-states Potts model is done by writing the operators of spin density (order parameters) as a certain complex function of the primary operators of the \((D_4, A_6)\) algebra studied in [4]. Knowing that the Hamiltonian of the critical model admits a \(Z_3\) symmetry, one can build the fusion rules from those established in [4] where the action of the transformation \(T_2\) was considered. It results from this fact that the obtained fusion rules are naturally \(Z_3\) invariants.

The paper is organized as follows, in section 1 we identify the operator algebra of the tri-critical 3-states Potts model from that of the \((D_4, A_6)\) model. In section 2, we establish the \(Z_3\) invariant fusion rules. The section 3 and 4 are devoted to write the conformal blocks of the correlation functions in their integral representation and the equations obtained from the bootstrap approach. Finally, we conclude by giving the numerical values of the structure constants.

2 The tricritical Potts model from the \((D_4, A_6)\) model

The \((D_4, A_6)\) model is the complementary series corresponding to the minimal model \(M(7, 6)\), it contains basic operators having conformal weights \(h_{r,s}\) which are given by Kac formula as follows
The physical Hilbert space of the \((D_4, A_6)\) conformal model is given by the modular invariant partition function hereafter \[3\]

\[
Z_{(D_4, A_6)} = \chi_{1,1} \chi_{1,1} + \chi_{1,3} \chi_{1,3} + \chi_{1,5} \chi_{1,5} + \chi_{1,1} \chi_{5,1} + \chi_{1,3} \chi_{5,3} + \chi_{1,5} \chi_{5,5} + \chi_{3,1} \chi_{3,1} + 2 \chi_{3,3} \chi_{3,3} + 2 \chi_{3,5} \chi_{3,5}
\]

(1)

where \(\chi_{rs}, (\chi_{rs})\) are characters of representations of the left (right) Virasoro algebra of central charge \(c = \frac{26}{3}\). We note that not all the fields of the \(\mathcal{M}(7,6)\) minimal model are present, the existence of spinless as well as spin left-right combinations and finally the presence of two copies for each of the spinless combinations \(\Phi_{(3,s)\mid 3,s}) = \phi_{(3,s)} \otimes \phi_{(3,s)}\) for \(s = 1, 2, 3\). The conformal anomaly \(c\) in the context of critical phenomena can be interpreted as the ground state energy of the statistical model, it is also related to the magnitude of the Casimir effect in the field theory \[14\].

Long time ago, Friedan et al \[15\] gave some conformal fields included in the tricritical Potts model. Some of those appear in (1). There are several manners to identify primary field living in the conformal model which correspond to the spin density operator (order parameter) of the critical model.

The identification is made by the knowledge of critical exponents \(x_\Phi\) (magnetic \(x_H\) and thermal \(x_T\)) given in statistical physics \[9\]. Those exponents are related to the conformal dimensions \(h\) of primaries by the equations \[18\]

\[
\begin{align*}
&x_\epsilon = x_T = d - y_T \\
&x_\sigma = x_H = d - y_H
\end{align*}
\]

(2)

with \(x_\Phi = h_\Phi + \bar{h}_\Phi\), and the exponents \(y_T\) and \(y_H\) are given in many references such the conjecture of Baxter \[8, 10\].
From where we obtain for the spin $\sigma$ and energy $\epsilon$ operators

\[ h_\epsilon = \frac{1}{7} \]
\[ h_\sigma = \frac{1}{21} \] (3)

Thus, the remaining operators are identified in the same way. Finally, the complete set of critical fields is shown in table 2,

| Primary fields | $\Delta = h + \bar{h}$ | Critical fields |
|----------------|------------------------|-----------------|
| $\Phi_{(33|33)}$ | 2/21                   | $\sigma$       |
| $\Phi_{(32|32)}$ | 20/21                  | $\sigma'$      |
| $\Phi_{(31|31)}$ | 8/3                    | $\sigma''$     |
| $\Phi_{(13|13)}$ | 10/7                   | $\epsilon$    |
| $\Phi_{(12|12)}$ | 2/7                    | $\epsilon'$    |
| $\Phi_{(51|11)}$ | 5                      | $\epsilon''$   |
| $\Phi_{(15|12)}$ | 23/7                   | $\Phi_{(15|12)}$ |
| $\Phi_{(14|13)}$ | 17/7                   | $\Phi_{(14|13)}$ |

Table 2: Critical fields and the corresponding primaries

At this stage, we would like to remark that the spin operator is again in the center of the conformal grid (Table 1), this fact was already predicted by Fridan [13] and used by Dotsenko [2] to obtain the conformal algebra of the Potts model.

Another way to identify the conformal field $\phi_{3,3}$ as the spin operator consists in using the relations between the exactly known critical exponents $\delta$ and $\eta$ (magnetic exponent) of the tricritical Potts model and the conformal weight of the primary field corresponding to the spin operator. The critical exponent $\delta$ is related to $\eta$, from the scaling laws, by

\[ \delta = \frac{d + 2 - \eta}{d - 2 + \eta} \] (4)

$(d = 2)$ in our case.

At the same time $\eta$ is the exponent appearing in spin 2-pt correlations in the statistical model and it is related to the conformal weight by

\[ \eta = 4h_{r,s} \] (5)

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using the results of Baxter \[8\] we have $\delta = 20$ and we obtain

$$\eta = \frac{4}{21} \quad (6)$$

Finally, the conformal weight of the spin operator is, as already shown, $h_{3.3} = h_{3.4} = \frac{1}{21}$.

3 The $Z_3$ invariants fusion rules

In the physical model, the Hamiltonian of the Potts model is invariant under the $Z_3$ symmetry. The spin density operator is sensitive to this action, whereas $\epsilon$ remains invariant. Therefore, if one would like to define the action of $Z_3$ on the operator with non vanishing spin ($s$), it is sufficient to observe that the left-right components of such operator come from the thermal algebra ($\epsilon$), and consequently we conclude by saying that those operators are $Z_3$ invariants.

Now, we are interesting in the construction of the fusion rules between the critical fields of the model. Before that, it is necessary to note that, for example, the field $\Phi_{\{33\mid33\}}$ representing the critical field $\sigma$ is doubly degenerate. This fact required the introduction of the $Z_2$ symmetry which separates between the two copies noted $\Phi^+_{\{33\mid33\}}$ and $\Phi^-_{\{33\mid33\}}$. For representing the operator $\sigma$, we realize the following change of basis

$$\sigma = \frac{1}{\sqrt{2}} \left( \Phi^+_{\{33\mid33\}} + i\Phi^-_{\{33\mid33\}} \right)$$
$$\sigma = \frac{1}{\sqrt{2}} \left( \Phi^+_{\{33\mid33\}} - i\Phi^-_{\{33\mid33\}} \right) \quad (7)$$

in this case, the action $Z_3$ is defined as follows

$$Z_3 : \sigma \rightarrow e^{i\frac{2\pi}{3}} \sigma$$
$$Z_3 : \sigma \rightarrow e^{-i\frac{2\pi}{3}} \sigma \quad (8)$$

Now to derive the $Z_3$ invariant fusion rules between the different fields of the model, we need to use the $Z_2$ invariant ones obtained in \[4\] with respect to the transformation of type $T_2$. This transformation acts as follows:

$$\Phi \quad T_2 \quad \left\{ \begin{array}{ll} \Phi & \text{if} \; s_\Phi = 0 \\ -\Phi & \text{if} \; s_\Phi \neq 0 \end{array} \right. \quad \Phi^+ \quad T_2 \rightarrow \pm \Phi^\pm$$

$$\Phi^\pm \quad T_2 \rightarrow \pm \Phi^\pm \quad (9)$$
where $s_{\Phi} = h - \overline{h}$, is the conformal spin of the field.

This leads to the following new fusion rules

\[
\begin{align*}
\sigma.\sigma & = \sigma + \sigma' + \sigma'' \\
\sigma.\overline{\sigma} & = 1 + \epsilon + \epsilon' + (51 | 51) + (14 | 14) + (15 | 15) + (51 | 11) + (11 | 51) \\
+ (13 | 14) + (14 | 13) + (15 | 12) + (12 | 15) \\
\overline{\sigma}.\overline{\sigma} & = \sigma + \sigma' + \sigma'' \\
\epsilon.\epsilon & = 1 + \epsilon' \\
\epsilon.\sigma & = \sigma + \sigma' \\
\epsilon.\overline{\sigma} & = \overline{\sigma} + \overline{\sigma}' \\
\sigma''.\sigma'' & = \sigma'' \\
\sigma''.\overline{\sigma}'' & = 1 + (51 | 51) + (11 | 51) + (51 | 11)
\end{align*}
\]

It should be noted here that the transformations $T_1$, $T_3$ and $T_4$ figuring in [4] with the basis change [7] do not lead to a $Z_3$ invariant fusion rules contrary to the case (10). From the obtained fusion rules, one can first note that the product spin density-spin density generates all the conformal algebra of the model and we see that only the operators of the first, the third and the fifth column of table (1) contribute (couple to $\sigma$) to the operator product of spins.

4 Correlation functions and conformal blocks

Let us first recall a fundamental property of the structure constants, which rises from the radial quantification of conformal theories. The Operator Product Expansion (OPE) of two conformal fields is defined by [11],

\[
\Phi_i(z, \overline{z}).\Phi_j(0, 0) = \sum_k \tilde{C}_{ijk} \Phi_k(0, 0) z^{h_i + h_j - h_k} \times \text{cc} \tag{11}
\]

Where $h, \overline{h}$ are the conformal weights. In the other hand, the 3-pt correlation function is fixed by conformal invariance and has the following form

\[
\langle \Phi_i(1)\Phi_j(2)\Phi_k(3) \rangle = \frac{C_{ijk}}{z_{12}^{h_i + h_j - h_k} z_{13}^{h_k + h_j - h_i} z_{23}^{h_j + h_k - h_1} \times \text{cc}} \tag{12}
\]
The $\tilde{C}^k_{ij}$ are the structure constants and they are related to the coefficients $C_{ijk}$ appearing in the (12) by

$$C_{ijk} = (-)^{S_k} \tilde{C}^k_{ij}$$  \hspace{1cm} (13)

The relation (13) becomes,

$$C_{abc} = (-)^{S_c} \tilde{C}^\sigma_{ab}$$  \hspace{1cm} (14)

when $a, b$ and $c$ indicate the critical fields constructed from the complex combination (8). $S_k = h - \bar{h}$ is the spin of the field $\Phi_k$. Our purpose here is the determination of these universal quantities in the case of the tricritical Potts algebra.

The 4-pt conformal correlations are composed of two chiral parts (holomorphic and anti-holomorphic), coupled with an unspecified non diagonal coupling constants. In our case, with the chosen correlation function, we write

$$\langle \Phi_{(kl,\bar{kl})} \sigma \Phi_{(kl,\bar{kl})} \rangle = \sum_{n,m} \gamma_{nm} \mathcal{F}_n \mathcal{F}_m$$  \hspace{1cm} (15)

where $\mathcal{F}_n$ are the conformal blocks, and $\gamma_{nm}$ are the coupling constants which are not diagonal in the case of $(D_4,A_6)$ conformal model, we will concentrate on the holomorphic part (all is true for the antiholomorphic one).

The conformal blocks are given by a correlation of vertex operators (integral representation) [19],

$$\mathcal{F}^{(33)}_{k1,k2} = \langle V_{\alpha_{k1}}(1)V_{\alpha_{33}}(2)V_{\alpha_{33}}(3)V_{2\alpha_{0}-\alpha_{k1}}(4)Q^N Q^M \rangle$$  \hspace{1cm} (16)

$V_{\alpha}$ are the vertex operator of charges $\alpha_{rs} = \frac{1}{2}[\alpha_{-} + (1 - s)\alpha_{+}]$, where $r, s$ belongs to the Kac indices. The screening operators $Q_\pm$ are injected into the correlation function so that the neutrality condition $\sum_i \alpha_i = 2\alpha_0$ is verified, we have

$$\langle \prod_{i=1}^K V_{\alpha_i}(z_i) \rangle = \begin{cases} \prod_{i<j}^K (z_i - z_j)^{2\alpha_i \alpha_j} & \text{if } \sum_{i=1}^K \alpha_i = 2\alpha_0 \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (17)

In the coulomb gas formalism [16], the operators $Q_\pm$ are integrals over closed contour of primary field of conformal weight $h = 1$.  

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1This is because the existence in the $D$ series of non diagonal primary fields with non vanishing spin, such as $\Phi_{(51|11)}$. 

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The neutrality condition applied to the conformal block in (16) gives
\[ N = M = 2, \] then it becomes
\[ \mathcal{F}_{k_1, k_2}^{(33)} = f(\eta)\lambda_{k_1}(\rho')\lambda_{k_2}(\rho)\mathfrak{S}_{k_1, k_2}^{(33)}(a, b, c; \rho; z) \]  
(18)

with the cross ration \( \eta = \frac{z_{12}z_{34}}{z_{13}z_{24}} \), and \( \lambda \) is an appropriate factor, \( f \) is function of \( \eta \). The \( \mathfrak{S} \)'s are integrals over closed contours with \( (k_1 \times k_2) \) independent solutions having monodromy properties, this means that they admit a development around \( \eta \to 0 \) and \( 1 - \eta \to 0 \). The integrals \( \mathfrak{S} \)'s depend of certain exponents \( a, b, c, \rho \), where their associated values in the case of \( (33, 33) \) being
\[ a = b = c = \frac{-1}{3}, \]
\[ a' = b' = c' = \frac{2}{7}, \]
\[ \rho' = \frac{1}{\rho} = \frac{6}{7} \]  
(19)

Now, in the limit \( \eta \to 0 \) \((s\text{-channel})\), the conformal block has the following form
\[ \mathfrak{S}_{k_1, k_2}^{(33)} = N_{k_1, k_2}^{(33)}(33)\eta^K [1 + \mathcal{O}(\eta)] \]  
(20)

with
\[ K = (k_1 - 1)[1 + a' + c' + \rho'(k_1 - 2)] + (k_2 - 1)[1 + a + c + \rho(k_2 - 2)] - 2(k_1 - 1)(k_2 - 1) \]

The \( N \)'s are the normalization constants of the integrals \( \mathfrak{S} \)'s [16]. So, there are nine normalized conformal blocks which can be written in a matrix form as
\[ \mathcal{F}_{k_1, k_2}^{(33)} \approx \begin{pmatrix} \eta^{1/21} & \eta^{8/21} & \eta^{64/21} \\ \eta^{34/21} & \eta^{-1/21} & \eta^{13/21} \\ \eta^{103/21} & \eta^{26/21} & \eta^{-2/21} \end{pmatrix} \]  
(21)

In the other hand, in the \( t \text{-channel} \) \((1 - \eta \to 0)\), the conformal blocks are written via a monodromy transformation as follow
\[ \mathfrak{S}_{k_1, k_2}^{(33)}(a, b, c; \rho; \eta) = \sum_{kk'} \Psi_{(k_1k_2, kk')} \mathfrak{S}_{kk'}^{(33)}(b, a; c; \rho; 1 - \eta) \]  
(22)

where the \( \Psi_{(k_1k_2, kk')} = \psi_{k_1k}^\dagger \times \psi_{k_2k'}^\dagger \) are the elements of the monodromy matrix.

Finally, the holomorphic and the anti-holomorphic conformal blocks are combined by the coupling \( \gamma \) to build the correlation function [15].
5 Bootstrap equations

The operator algebra (OPA) in (11) is associative, which induced a crossing symmetry of the 4-pt correlation functions, this fact allows to write the bootstrap equations [1]. For the correlation function

\[ G = \langle \Phi_1(z_1, \overline{z}_1) \Phi_2(z_2, \overline{z}_2) \Phi_3(z_3, \overline{z}_3) \Phi_4(z_4, \overline{z}_4) \rangle \] (23)

we have in the \( s \)-channel \((z_{12}, z_{34} \rightarrow 0)\)

\[ G(z_1, \ldots, \overline{z}_4) = \sum_m (-1)^{s_m} \tilde{C}_{12}^m \tilde{C}_{34}^m [1 + \mathcal{O}(z_{12}, \overline{z}_{34})] \frac{z_{12} h_{12} + z_{24} h_{24}}{z_{34} h_{34} + z_{24} h_{24}} \] (24)

and in the \( t \)-channel \((z_{14}, z_{23} \rightarrow 0)\)

\[ G(z_1, \ldots, \overline{z}_4) = \sum_n (-1)^{s_n} \tilde{C}_{14}^n \tilde{C}_{23}^n [1 + \mathcal{O}(z_{14}, \overline{z}_{23})] \frac{z_{14} h_{14} + z_{23} h_{23}}{z_{23} h_{23} + z_{14} h_{14}} \] (25)

Where \( m \) and \( n \) belong to the algebra (10). The appearance of the structure constants is realized in these two last expressions. To determine them, one has to choose the appropriate correlation function where the desired structure constant appear. Then one begin by writing the corresponding bootstrap equations and next write the correlation function in term of conformal blocks in their limits to compare them with the bootstraps.

For the correlation function

\[ G_1 = \langle \sigma \sigma \sigma \sigma \rangle \] (26)

where we calculated the corresponding conformal blocks in (18), it is easy to write the bootstrap equations using de fusion rules (10), and we have in the \( s \)- and the \( t \)-channel respectively

\[ G_1 \sim \frac{|C_{\sigma \sigma}|^2}{|z_{12} z_{34}|^{2/21} |z_{24}|^{4/21}} + \frac{|\tilde{C}_{\sigma \sigma}|^2}{|z_{12} z_{34}|^{16/21} |z_{24}|^{40/21}} + \frac{|\tilde{C}_{\sigma \sigma}'|^2}{|z_{12} z_{34}|^{52/21} |z_{24}|^{16/3}} \] (27a)

\[ G_1 \sim \frac{1}{|z_{14} z_{23}|^{4/21}} + \frac{|\tilde{C}_{\sigma \sigma}'|^2}{|z_{14} z_{23}|^{-2/21} |z_{13}|^{4/7}} + \frac{|\tilde{C}_{\sigma \sigma}'|^2}{|z_{14} z_{23}|^{26/21} |z_{13}|^{20/7}} + \ldots \] (27b)
Now, when comparing (27a) with the limit form of the conformal blocks (21), one obtains the following identities

\[
\gamma_{(22,22)} \left( N^{(33)}_{22} \right)^2 \approx \left| \tilde{C}_{\sigma \sigma} \right|^2
\]

\[
\gamma_{(12,12)} \left( N^{(33)}_{12} \right)^2 \approx \left| \tilde{C}_{\sigma \sigma} \right|^2
\]

\[
\gamma_{(32,32)} \left( N^{(33)}_{32} \right)^2 \approx \left| \tilde{C}_{\sigma \sigma} \right|^2
\]

and the remaining coupling constants \( \gamma \) are nulls.

The second expression (27b) will be compared with the development of the conformal blocks (22) when \( 1 - \eta \to 0 \). The correlation function written in this case has the form

\[
G_1 = \text{Tr} \tilde{F}^T \left( \Psi^T \gamma \Psi \right) \tilde{F}
\]

thus, when injecting (22) in the last expression, it leads to the following equations for the first three terms

\[
\left( \gamma_{(12,12)} \Psi^2_{(12,33)} + \gamma_{(22,22)} \Psi^2_{(22,33)} + \gamma_{(32,32)} \Psi^2_{(32,33)} \right) \lambda^2_3 \lambda^2_3 \tilde{N}^2_3 = 1
\]

\[
\left( \gamma_{(12,12)} \Psi^2_{(12,11)} + \gamma_{(22,22)} \Psi^2_{(22,11)} + \gamma_{(32,32)} \Psi^2_{(32,11)} \right) \lambda^2_1 \lambda^2_1 \tilde{N}^2_1 = \left| \tilde{C}_{\sigma \sigma} \right|^2
\]

\[
\left( \gamma_{(12,12)} \Psi^2_{(12,23)} + \gamma_{(22,22)} \Psi^2_{(22,23)} + \gamma_{(32,32)} \Psi^2_{(32,23)} \right) \lambda^2_2 \lambda^2_2 \tilde{N}^2_2 = \left| \tilde{C}_{\sigma \sigma} \right|^2
\]

It is noted here that before solving the system of equations (28) and (30), one can reduce the number of unknown coupling constants by calculating them from other correlation. For example, the constants in (30) can be obtained from the correlation \( \langle \sigma \epsilon \epsilon \sigma \rangle \).

In (29) there are three vanishing coupling \( \tilde{\gamma}_{ij,ij} \), this is because the presence of non-diagonal fields in the fusion \( \sigma \bar{\sigma} \), and more precisely it concerns the coupling \( \tilde{\gamma}_{12,12}, \tilde{\gamma}_{22,22} \) and \( \tilde{\gamma}_{32,32} \). Hence, we have the following algebraic equation

\[
\sum_{(\mu \nu)} \gamma_{\mu \nu,\mu \nu} \psi^l_{(\mu \nu)k} \psi^l_{(\mu \nu)l} = 0 \text{ if } k \neq l
\]

This equation allows to write the ratio between the constants \( \gamma \). These obtained ration are injected in the equations (28) and (30) which lead at the end to the determination of the values of the structure constants.
6 Conclusion

In this section, and after calculation the numerical values of the normalization constants \( N_{ij} \), the monodromy elements \( \Psi_{\alpha\beta,\mu\nu} \) and the pre-factors \( \lambda_k \), we resumed the obtained numerical values of the structure constants in table 3.

| Structure constant | Value |
|--------------------|-------|
| \( \tilde{C}_{\sigma\sigma} \) | \((B)_\sigma^{1/2} \frac{1}{\sqrt{3}} \frac{s(1)}{s(2)} \frac{275}{588} \sqrt{\pi} \Gamma(\frac{1}{2}) \Gamma(-\frac{1}{2}) \) |
| \( \tilde{C}_{\sigma\sigma} \) | \((A)_\sigma^{1/2} \frac{1}{\sqrt{3}} \frac{55}{4032} \sqrt{\pi} \Gamma(\frac{1}{2}) \Gamma(-\frac{1}{2}) \) |
| \( \tilde{C}_{\sigma\sigma} \) | \((1/\sigma) 1/2 \frac{s(2)}{s(1)} \frac{55}{81} \sqrt{\pi} \Gamma(\frac{1}{2}) \Gamma(-\frac{1}{2}) \) |
| \( \tilde{C}_{\sigma\sigma} \) | \((\theta_1)_\sigma^{1/2} \frac{1520640}{16807} \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{3}{2}) \Gamma(-\frac{1}{2}) \Gamma(-\frac{1}{2})}{\Gamma(\frac{3}{2}) \Gamma(\frac{3}{2}) \Gamma(-\frac{1}{2}) \Gamma(-\frac{1}{2})} \) |
| \( \tilde{C}_{\sigma\tau} \) | \((\theta_2)_\sigma^{1/2} \frac{4}{\pi(2)} \frac{1}{7} \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{3}{2}) \Gamma(-\frac{1}{2}) \Gamma(-\frac{1}{2})}{\Gamma(\frac{3}{2}) \Gamma(\frac{3}{2}) \Gamma(-\frac{1}{2}) \Gamma(-\frac{1}{2})} \) |

Table 3: Structure constants of the Tri-critical Potts model

With,

\[
\begin{align*}
A &= 2 \frac{s^2(3)}{s^2(1)} \frac{1}{s(1) s(2) - s(3)} ;
B &= \frac{s(3)}{s^2(1)} ;
\end{align*}
\]

\[
\begin{align*}
\frac{1}{\theta} &= 2 \frac{(s(1) s(2) - s(3))}{1 + \frac{1}{2} \frac{s^2(1)}{s(3)} (s(1) s(2) - s(3)) \left[ 1 + \frac{s(3)}{s(1)} \right]} \\
\frac{A}{\theta} &= 4 \frac{s^2(3)}{s(1)} \frac{1}{1 + \frac{1}{2} \frac{s^2(1)}{s(3)} (s(1) s(2) - s(3)) \left[ 1 + \frac{s(3)}{s(1)} \right]} \\
\frac{B}{\theta} &= 2 \frac{s(3)}{s^2(1)} \frac{1 + \frac{1}{2} \frac{s^2(1)}{s(3)} (s(1) s(2) - s(3)) \left[ 1 + \frac{s(3)}{s^2(1)} \right]} \\
\frac{\theta_1}{\theta} &= \frac{64}{9} \frac{s^2(3)}{s^2(1)} \frac{1 + \frac{1}{2} \frac{s^2(1)}{s^2(3)} (s(1) s(2) - s(3)) \left[ \frac{s^2(3)}{s^2(1) s^2(2)} + \frac{s(3)}{s^2(1)} \right]} \\
\frac{\theta_2}{\theta} &= \frac{s^2(2)}{s^2(1)} \frac{1 + \frac{1}{2} \frac{s^2(1)}{s^2(3)} (s(1) s(2) - s(3)) \left[ \frac{s^2(3)}{s^2(1) s^2(2)} + \frac{s(3)}{s^2(1)} \right]} \\
\end{align*}
\]
The calculated structure constants are universal quantities, the obtained results can be exploited in the study of materials classified in the universal class of the Tricritical 3-states Potts model. Also, one can realize a Monte Carlo simulation program or study the finite size scaling to compare with these results [17], but one may note that the operator content of the critical model contain several fields (15 parameters). Consequently, there are many correlators to measure and it will be probably difficult to do such a work, although there is the possibility to verify certain results.

Acknowledgement 1 The authors would like to thank especially Pr. John McCabe for a very useful discussions, for informing them about the existing simulation methods and for the carefully examination of the manuscript.

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