Nonlinear waves in solar plasmas - a review

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Abstract. Nonlinearity is a direct consequence of large scale dynamics in the solar plasmas. When nonlinear steepening of waves is balanced by dispersion, solitary waves are generated. In the vicinity of resonances, waves can steepen into nonlinear waves influencing the efficiency of energy deposition. Here we review recent theoretical breakthroughs that have lead to a greater understanding of many aspects of nonlinear waves arising in homogeneous and inhomogeneous solar plasmas.

1. Introduction

One of the most interesting processes in solar and astrophysical plasmas is the complicated interaction of plasma motions with magnetic fields. These media are highly non-uniform and as a consequence are a natural environment for magnetohydrodynamic (MHD) waves. Waves can transport energy and momentum. When part of their energy or momentum is transferred to the plasma they can heat and accelerate the plasma (e.g. resonant absorption). Waves can carry information about the medium in which they propagate, therefore they can provide a unique tool for plasma diagnostics.

In the present contribution we review two important nonlinear waves arising in inhomogeneous solar plasmas. Firstly, solitary waves arising in structured plasmas (i.e. waveguides) are discussed in different structures and for different dispersions. Secondly, nonlinear waves generated in the vicinity of resonant positions (slow resonance) are revisited and we show how nonlinearity will influence the efficiency of heat deposition.

2. Nonlinear waves in waveguides

One of the basic properties of solar plasma is that is structured, the magnetic field is not distributed smoothly over the surface of the Sun, but it tends to accumulate in entities called magnetic loops, the building blocks of the solar corona. These structures can support, e.g. longitudinal wave propagation over long distances. The effect of the structuring is that it introduces dispersion, i.e. a modification in the propagation characteristic of the wave.

Solitons are finite-amplitude waves of permanent shape which owe their existence to the balance between nonlinear wave-steepening and wave dispersion. Nonlinearity appears for waves of finite amplitude and generally is a consequence of large scale dynamics. Dispersion could arise due to two different effects. Geometrical dispersion appears for waves propagating in a magnetic guide (flux tube or sheet). This dispersion does not depend on the reaction of the external media and its value is defined by the transversal geometrical scale of the duct. Alternatively, waves in open ducts could have dispersion due to the reaction of the external media. It is
not always simple to separate these two sources of dispersion despite their different behaviour. Furthermore, physical dispersion appears due to plasma (magnetic) effects (generalized Ohm’s law or Finite Larmor Radius (FLR) effects). In general, these two dispersive effects give rise to different dispersive behaviour but they have the same result: creation of a new length scale in addition to the natural length scale of the waves, i.e. their wavelength.

Guided waves in solar and space plasmas are investigated in two cases: magnetic slab (Cartesian geometry) and magnetic tube (cylindrical geometry). The dynamics of solitary waves are best described in the so-called thin flux tube approximation. For a motion \( v(z,t) \) along a tube (slab or cylinder) of cross-sectional area \( A(z,t) \), the one-dimensional equations of continuity, longitudinal momentum, isentropic energy and flux conservation are

\[
\frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial z} \rho v A = 0, \quad \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z},
\]

\[
\frac{\partial}{\partial t} \left( \frac{p}{\rho^2} \right) + v \frac{\partial p}{\partial z} \left( \frac{p}{\rho^2} \right) = 0, \quad BA = \text{const},
\]

where the quantities \( p(z,t) \), \( \rho(z,t) \), \( B(z,t) \) and \( v(z,t) \) are supposed uniform across the tube.

In a magnetic slab of width \( 2a \) with longitudinal magnetic field, the dispersion relation of long wavelength slow sausage modes is (Roberts 1981)

\[
\omega/k = c_T - \alpha_1 |k|, \quad \alpha_1 = \frac{1}{2} \frac{\rho_e}{\rho} \left( \frac{c_T}{v_A} \right)^3 acT,
\]

(1)

where \( c_T \) is the tube speed (the propagation speed of slow magnetoacoustic waves in an unbounded medium). The \( \alpha_1 |k| \) term in Equation (1) arises due to dispersion and in general is a small quantity. If the amplitude of slow waves becomes large enough, the nonlinear evolution of these waves is described by the Benjamin-Ono (BO) equation written for the \( z \)-component of the velocity perturbation (Roberts and Mangeney 1982, Edwin and Roberts 1986, Ballai et al. 2002)

\[
\frac{\partial v}{\partial t} + c_T \frac{\partial v}{\partial z} + \tau v \frac{\partial v}{\partial z} + \frac{\alpha_1}{\pi} \frac{\partial^3}{\partial z^3} \int_{-\infty}^{\infty} v(z',t) \left[ \lambda^2 + (z' - z)^2 \right]^{1/2} \, dz' = 0,
\]

(2)

where \( \tau \) is a coefficient which depends on the characteristic speeds (sound, Alfvén and cusp speeds). The single-soliton solution of this equation is the algebraic soliton,

\[
v(z,t) = \frac{A}{1 + [(z - st)/L]^2},
\]

(3)

where the velocity amplitude \( A \), the speed \( s \) and scale of the soliton \( L \) are related by

\[
s = c_T + \frac{\beta A}{4}, \quad L = \frac{4\alpha_1}{\lambda^2}. \quad (4)
\]

In a magnetic cylinder the dispersion relation of long wavelength slow surface sausage modes is

\[
\omega/k = c_T - \alpha_2 k^2 K_0(|k|\lambda),
\]

(5)

where \( K_0(x) \) is the modified Bessel function of the zeroth-order. The quantities \( \alpha_2 \) and \( \lambda \) depend on characteristic speeds and the radius of the tube and \( \lambda^2 \) can be both negative or positive quantity. If these waves steepen into nonlinear waves, their evolution is described by the Leibovich-Roberts (LR) equation (Roberts 1985),

\[
\frac{\partial v}{\partial t} + c_T \frac{\partial v}{\partial z} + \tau v \frac{\partial v}{\partial z} + \frac{\alpha_2}{\pi} \frac{\partial^3}{\partial z^3} \int_{-\infty}^{\infty} \frac{v(z',t) \, dz'}{[\lambda^2 + (z' - z)^2]^{1/2}} = 0.
\]

(6)
Although this equation was derived 20 years ago, there is no known analytical solution, however, numerical investigations showed that it has a solitary-like solution. If the propagation speed of the slow waves inside and outside the tube are approaching each other, the LR equation reduces to a nonlinear wave equation without dispersion which describes shock waves with zero-width. If the internal cusp speed approaches the external sound or Alfvén speed (supposing a magnetized environment), the LR equation reduces to the Leibovich equation describing nonlinear waves on a cylindrical vortex core. The LR equation is valid provided $\lambda^2 > 0$. If $\lambda^2 < 0$, then slow leaky sausage modes will propagate in the tube draining energy away from the structure. In this case, the LR equation can be modified to describe slow leaky sausage modes as (Ballai and Zhugzhdza 2002)

$$\frac{\partial v}{\partial t} + c_T \frac{\partial v}{\partial z} + \tau v \frac{\partial v}{\partial z} + \alpha_1 \frac{\partial^3 v}{\partial z^3} \left[ \int_{-\infty}^{\lambda|z|} + \int_{\lambda|z|}^{\infty} \frac{v(s, t) \, ds}{\sqrt{\lambda^2 + (z - s)^2}} \right] = 0. \quad (7)$$

One of the limitations of these equations is that the solitary wave solution appears only up to some critical amplitude. This amplitude threshold appears because the dispersion relation has a maximum, i.e. the maximum value of the dispersion is not enough to smooth out the front of the waves if the amplitude of the waves exceeds a critical threshold value. Thus, the (LR) equation describes the nonlinear behaviour of weakly nonlinear slow sausage modes whose phase velocity in the linear limit has an extremum.

There is one aspect which so far has been neglected, and this is related to the dissipative character of the plasma. In fact, the right choice for a dissipative mechanism depends on the location where physical processes are to be studied and also on the physical mechanism itself. When dissipation is taken into account, solitary waves will exhibit a slow damping, i.e. the energy and momentum of solitary waves are not conserved any longer. The most important dissipative mechanisms are viscosity, thermal and electrical conduction and radiation. If we take into account the first three mechanisms, the solitary wave equations must be supplemented by an extra term proportional to $\frac{\partial^2 v}{\partial z^2}$ which results in an algebraic decay of the soliton. If radiation is considered, the nonlinear equations will have an extra term proportional to $v$ which leads to a slow exponential decay of the solution.

Dispersion can also arise due to the presence of the magnetic field, through, e.g. the Hall term in the generalised Ohm’s law. Strictly speaking, Hall MHD is relevant to plasma dynamics occurring on length scales shorter than the ion inertial length, $c/\omega_i$, where $c$ is the speed of light and $\omega_i$ is the ion plasma frequency. Inclusion of the Hall term in the magnetohydrodynamic induction equation is known to affect the polarisation of waves because it includes the dispersion of Alfvén waves near the ion cyclotron frequency.

The nonlinear wave evolution in the presence of a Hall effect in a viscous plasma has been studied in connection to the acceleration of the solar wind. When the nonlinear steepening of compressional waves is balanced by the broadening of the wavefront caused by the Hall effect, the dynamics of solitary waves propagating in a super-radial magnetic field is described by the Korteweg-de Vries-Burgers (KdV-B) equation

$$\frac{\partial v}{\partial t} + c_{ph} \frac{\partial v}{\partial z} + \alpha_1 v \frac{\partial v}{\partial z} - \alpha_2 \frac{\partial^3 v}{\partial z^3} - \alpha_3 \frac{\partial^2 v}{\partial z^2} = 0. \quad (8)$$

where $c_{ph}$ is the phase speed of linear waves and the coefficients $\alpha_i$ depend on characteristic speeds and the angle of propagation with respect to the magnetic field. Choosing a nearly-parallel propagation, we obtain that solitons arising from the nonlinear steepening of compressional slow waves are able to accelerate the plasma, while solitons which are generated by the nonlinear steepening of fast waves will decelerate the plasma.

Solitary waves have unique properties which make them special for mathematics and their applications to other fields: (i) Integrability: Before the discovery of solitons, mathematicians
were under the impression that nonlinear PDEs could not be solved analytically. However, solitons showed us that it is possible to solve PDEs (at least the solitary wave equations) exactly, which gives us a tremendous “window” into what is possible in nonlinear mathematics. (ii) Nonlinear superposition: In linear theory, there is a simple way to generate a new solution from known ones, just by multiplying them with a scalar and adding them together. This is known as superposition. Before the discovery of solitons, there was no analogue of this construction for nonlinear equations, but the way that a 2-soliton solution can be viewed as a combination, although not a simple linear one, of two 1-soliton solutions leads us to the recognition that (at least for solitons) there is a nonlinear superposition principle, as well. (iii) The particle-like behaviour of solitons leads to a large number of applications. This is true to some extent: there are soliton models for nuclei and the technique known as bosonization allows to view fermions as solitons in appropriate situations. Solitons have also a series of other applications in fields like oceanography, fibre optics, telecommunications and geophysics, biology.

3. Nonlinear resonant waves
Resonances are ubiquitous every time MHD (magnetohydrodynamic) waves are driven in inhomogeneous plasmas. In weakly dissipative plasmas (as in the case of solar plasma) driven MHD waves show nearly resonant behaviour, which deviates from the resonant behaviour in ideal plasmas only in thin dissipative layers surrounding the ideal resonant positions.

A very important property of nearly resonant waves is that their damping rate is almost independent on the values of dissipative coefficients. As a result, the damping rate of nearly resonant MHD waves can be many orders of magnitudes larger than the damping rate of MHD waves with the same frequencies in homogeneous plasmas. This property of resonant waves being strongly damped in weakly dissipative plasmas has attracted ample attention from plasma physicists since the transferred energy can be converted into heat or it might give valuable information about the density of the plasma and the characteristic scale of inhomogeneity.

Resonant absorption can be considered as an effective process of generating small length scales comparable to the dissipation length scales. The local oscillation modes of an inhomogeneous plasma are represented by continuous spectra for slow MHD and Alfvén waves and a discrete spectrum for fast MHD waves. The resonant absorption occurs when the frequency of a laterally driven oscillation matches the local slow and/or Alfvén wave frequency and a resonant field line is created which transfers energy from the surface disturbance to its environment.

Usually, the importance of the dissipation is characterized by the viscous and magnetic Reynolds numbers (if viscosity and magnetic diffusion are considered as dissipative effects) and we denote by $R$ the total Reynolds number which, under solar conditions, is a very large ($10^6$ in the photosphere and up to $10^{12}$ in the corona).

Linear theory of resonant absorption has shown that in the vicinity of a resonant position the perturbations have steep gradients and large amplitudes and therefore the linear theory in this region can break down and nonlinear theory has to be considered. Nonlinearity in the dissipative layer was first taken into account in the theory of resonant absorption by Ruderman et al. (1997a) and Ballai et al. (1998a) where they studied the nonlinear evolution of slow resonant MHD waves in the isotropic and anisotropic dissipative layer using a Cartesian geometry. These theories were applied to study the resonant absorption of sound and fast magneto-acoustic waves in solar structures (Ruderman et al. 1997b, Ballai et al. 1998b, Erdélyi and Ballai 1999). One of their main results was that in contrast to the linear theory, the coefficient of wave energy absorption was dependent on the particular type of dissipation. They have also found that in the long wavelength and weak nonlinearity limit the general tendency of nonlinearity is to decrease the absolute value of the coefficient of wave energy absorption.

Characteristic quantities used to scale the problem are $\epsilon$ (the dimensionless amplitude of perturbations away from the dissipative layer) and the total Reynolds number. One way to
determine the importance of nonlinearity is to calculate the ratio (the nonlinearity parameter)
\[ M = \frac{f}{\nu} \left( \frac{\partial f}{\partial \theta} \right) / \left( \frac{\partial^2 f}{\partial x^2} \right) = \epsilon R^{2/3}, \tag{9} \]
where \( f \) is any large variable, i.e. the most singular perturbations (e.g. for slow wave resonance, the most singular are the parallel component of the velocity and magnetic field perturbation) and \( \nu \) is the coefficient of kinematic viscosity. Linear theory works as long as \( M \ll 1 \), i.e. \( \epsilon R^{2/3} \ll 1 \). For a typical value of \( \epsilon \approx 10^{-2} \) to have resonant absorption described by linear theory, we need \( R \ll 10^3 \) which is in contrast to previously accepted values. Based on these scalings, it is obvious that resonant absorption is a nonlinear phenomenon.

In nonlinear theories perturbations cannot be Fourier analysed. However, we suppose that waves are plane periodic propagating modes with permanent shape, i.e. all perturbations depend only on \( \theta = z - Vt \) so they are periodic with respect to \( \theta \).

Outside the dissipative layer, the plasma dynamics can be described by the linear ideal MHD system of equations which can be reduced to two coupled first order PDE
\[ \frac{\partial u}{\partial x} = \frac{V}{D} \frac{\partial P}{\partial \theta}, \quad \frac{\partial P}{\partial x} = \frac{\rho_0 D_A \partial u}{V} \frac{\partial \theta}{\partial \theta}, \tag{10} \]
where
\[ D = \frac{\rho_0 D_A D_T}{V^4 - V^2 (v_A^2 + c_S^2) + v_A^2 c_S^2 \cos^2 \alpha}, \]
\[ D_A = V^2 - v_A^2 \cos^2 \alpha, \quad D_T = (v_A^2 + c_S^2) (V^2 - c_T^2 \cos^2 \alpha), \tag{11} \]
and the indices \( A \) and \( T \) refer to Alfvén and slow (or tube, or cusp) waves, respectively. In the case of cylindrical tube when the equilibrium magnetic field is such that \( \mathbf{B}_0 = (0, B_{0\phi}(r), B_{0z}(r)) \) and the wave-vector now has a helical component, therefore the running variable is \( \theta = m \phi + k z - \omega t \). The governing equations outside the dissipative layer are
\[ D \frac{\partial u r}{\partial r} = C_1 u r + \omega C_2 r \frac{\partial P}{\partial r}, \]
\[ \omega r D \frac{\partial^2 P}{\partial r \partial \theta} = C_3 u r - \omega r C_2, \tag{12} \]
where the coefficient functions are given by
\[ C_1 = 2 \omega^4 \frac{B_{0\phi}}{\mu r} - 2 \frac{m f_B B_{0\phi}}{\mu r^2} D_C, \quad C_2 = \omega^4 - \left( \frac{m^2}{r^2} + k^2 \right) D_F, \]
\[ C_3 = D \left( \rho_0 D_A \frac{\partial}{\partial \theta^2} + \frac{2 B_{0\phi}}{r} \frac{d}{dr} \left( \frac{B_{0\phi}}{r} \right) \right) + 4 \omega^4 \left( \frac{B_{0\phi}}{\mu r} \right)^2 + 4 \rho_0 D_T \omega_A^2 B_{0\phi}^2, \]
\[ f_B = \frac{m}{r} B_{0\phi} + kB_{0z}, \]
and the coefficients \( D, D_A \) and \( D_T \) are similar to the equations given by Equation (11) with \( V \) replaced by \( \omega \) and speeds are multiplied by \( k^2 \).

In the present study we only focus on the slow resonance given by the condition \( V^2 = c_T^2 (x) \) or \( \omega^2 = \omega_C^2 (r) \). The resonant position (\( x = x_{res} \) in Cartesian geometry and \( r = r_{res} \) in cylindrical geometry) is a regular singular point of the system of Eqs. (11)-(12) and as a consequence, the solutions are obtained in form of Fröbenius series. The equilibrium quantities have a slight change across the dissipative layer and they are approximated by the first non-vanishing term.
in their Taylor expansion. These expansions are valid in a layer wider than the dissipative layer since the characteristic scale of the inhomogeneity is larger than the scale of dissipation.

Inside the dissipative layer, the solutions are obtained in form of asymptotic expansions. In order to connect the solutions in the two regions (inside and outside the dissipative layer) we use the so-called matched asymptotic expansions developed by Nayfeh (1981). Both the internal and external solutions have to coincide in the overlap regions.

The dynamics of resonant slow waves in the vicinity of the resonance propagating along the magnetic field is given in cartesian geometry (with isotropic anisotropy) by

\[(x - x_{res}) \frac{\partial v_{\parallel}}{\partial \theta} + \Phi_{1} v_{\parallel} \frac{\partial v_{\parallel}}{\partial \theta} + k \frac{\partial^{2} v_{\parallel}}{\partial x^{2}} = \Phi_{2} P(\theta),\]  

(13)

and in cylindrical geometry

\[(r - r_{res}) \frac{\partial v_{\parallel}}{\partial \theta} + \Psi_{1} v_{\parallel} \frac{\partial v_{\parallel}}{\partial \theta} + k \frac{\partial^{2} v_{\parallel}}{\partial r^{2}} = \Psi_{2} C(\theta),\]  

(14)

where the function \(C(\theta)\) is a sum of the \(\theta\)-derivative of the total pressure and a function containing the \(\varphi\)-component of the magnetic field. In anisotropic plasmas, the nonlinear governing equation is modified in the dissipative term (the third term in the LHS) and instead of having a 2\(^{nd}\) order derivative with respect to the transversal coordinate, we have a 2\(^{nd}\) order derivative with respect to \(\theta\).

When solving the MHD equations for the entire domain, the resonances are considered as singularities, therefore the evolution of physical quantities in the vicinity of resonances are given as jumps (connection formulae), exactly as the Rankine-Hugeniot relations for shock waves. The jump in a quantity \(Q\) across the dissipative layer can be calculated as \([Q] = \lim_{x \to x_{res}} \{Q(x) - Q(-x)\}\). When connecting the solutions, the jump conditions serve as boundary conditions. In the case of Cartesian geometry, the jumps in the total pressure and the normal component of the velocity are given by

\[[P] = 0, \quad [u] = \Omega_{1} P \int_{-\infty}^{\infty} \frac{\partial v_{\parallel}}{\partial \theta} \, dx,\]  

(15)

and in cylindrical geometry by

\[[P] = \Lambda_{1} P \int_{0}^{\infty} v_{\parallel} \, dr, \quad [u] = \Lambda_{2} P \int_{0}^{\infty} \frac{\partial v_{\parallel}}{\partial \theta} \, dr.\]  

(16)

Here \(P\) is used for the Cauchy principal part because the integrals are divergent at infinity.

When calculating the efficiency of the resonant absorption (coefficient of wave energy absorption) it is found that the effect of nonlinearity is to decrease the net coefficient of wave absorption, i.e. the large amount of energy stored in nonlinear waves does not go into increasing the absorption rate but into generating a mean flow outside the dissipative layer. This turbulent flow is generated by the absorption of wave momentum in the dissipative layer and its amplitude is determined by the balance of forces created by resonant absorption and shear viscosity. The mean shear flow is a piecewise continuous function of \(r\) (e.g. in cylindrical geometry) but its vorticity has a jump given by

\[[v_{\varphi}] = A_{1} \int_{0}^{\infty} \left\langle \left( \frac{\partial v_{\parallel}}{\partial r} \right)^{2} \right\rangle \, dr, \quad [v_{z}] = A_{2} \int_{0}^{\infty} \left\langle \left( \frac{\partial v_{\parallel}}{\partial r} \right)^{2} \right\rangle \, dr,\]  

(17)

where the coefficients \(A_{1}\) and \(A_{2}\) depend on characteristic speeds, the location, \(r_{res}\), of the resonance, the dissipative coefficients and \(<, >\) is the mean value of a quantity over a period.
Estimates of this mean shear flow give us speeds of the order of $0.1 \text{km} \text{s}^{-1}$ in the solar photosphere and a few $\text{km} \text{s}^{-1}$ in the solar corona. Observation of this flow might be a first indirect evidence for resonant absorption in solar plasmas.

The results presented here considered that the equilibrium is static; in reality the plasma is very dynamic, showing motion on all time and space scales. The inclusion of an equilibrium steady flow will change the governing equations inside and outside the dissipative layer, as well as the jump conditions across the singularity. The following subsection is devoted to study the absorption of acoustic waves by inhomogeneous plasmas taking into account a dynamical and steady equilibrium.

3.1. Interaction of sound waves with slow wave dissipative layers

As an application we study the nonlinear interaction of incident sound waves with one-dimensional steady plasmas, i.e., the nonlinear coupling of sound waves and slow MHD waves. Monochromatic sound waves are impinging from the $x < 0$ unmagnetised half-space (Region I) and penetrate into the inhomogeneous region, $0 < x < x_0$ (Region II). This inhomogeneous region is bounded on its right by a semi-infinite subspace containing a homogeneously magnetised plasma (Region III). The magnetic field is parallel to the $z$ direction and a field-aligned steady flow $v_0$ is present in the magnetic regions. In what follows we use the subscripts 'e', '0' and 'i' to indicate equilibrium quantities in the three regions (regions I, II and III respectively). All equilibrium quantities are continuous at the boundaries of region II, and they must satisfy the equation of total pressure balance. In order to make analytical progress, we assume from the very beginning that the inhomogeneous region is thin or that the wavelength of perturbations is long, i.e. $kx_0 \ll 1$. A schematic picture of the three-layer model is shown in Figure 1.

### 3.1.1. Weakly nonlinear limit

Weak nonlinearity means that the nonlinearity can be considered as a perturbation of the linear problem and the effects of nonlinearity are obtained as corrections to the linear results. Let us introduce some new dimensionless variables defined by

$$\sigma = \frac{x - x_{\text{res}}}{\delta_c}, \quad q = \frac{k\nu \delta_c}{v_A} v_{||}, \quad \delta_c = \left[ \frac{|V|}{k|\Delta|} \left( \nu + \frac{c_T^2}{v_A^2} \eta \right) \right]^{1/3}, \quad V = V - kv_0, \quad \Delta = \frac{d}{dx}(V - c_T)$$
where $\sigma$ is the scaled dimensionless distance, $q$ is the dimensionless velocity parallel to the magnetic field lines, and $\delta_e$ is the thickness of the dissipative layer, $\nu$ and $\eta$ are the coefficients of viscosity and resistivity. With these new variables, the equation (13) governing the wave motion in the dissipative layer becomes

$$
\text{sgn}(\Delta)\sigma \frac{\partial q}{\partial \theta} - \Lambda' q \frac{\partial q}{\partial \theta} + k \text{sgn}(V) \frac{\partial^2 q}{\partial \sigma^2} = \frac{kV^4}{\rho_0 v_A^4 |\Delta|} \frac{dP}{d\theta},
$$

where $\Lambda'$ is an expression of the equilibrium quantities. The plasma motion outside the dissipative layer is described by the set of equations (10). The solution of these equation can be obtained in form of a sum of incoming and outgoing waves. Due to nonlinearity higher harmonics are generated in the dissipative layer in addition to the fundamental mode. The solution in the magnetic free region is written in the form

$$
P = \epsilon P e \text{Re} \left[ e^{i(k(\theta + \chi_e x))} + \sum_{n \geq 1} A_n e^{i k (\theta - \chi_e x)} \right],
$$

$$
u = \frac{\epsilon \chi e P e}{k V \rho e} \text{Re} \left[ e^{i(k(\theta + \chi_e x))} - \sum_{n \geq 1} A_n e^{i k (\theta - \chi_e x)} \right],
$$

where $\chi_e = k_x / k$, $k = (k_y^2 + k_z^2)^{1/2}$ and since we assume that $V > v_{Se}$, the waves in this region are propagating.

In the homogeneous region (quantities with subscript 'i') the solution of the system (10) is

$$
P = \epsilon P_i \text{Re} \left[ \sum_{n \geq 1} D_n e^{i k (\theta - \lambda_i (x-x_0))} \right],
$$

$$
u = \frac{\epsilon V \lambda_i}{k \rho_i (V^2 - v_A^2) \text{Re} \left[ \sum_{n \geq 1} D_n e^{i k (\theta - \lambda_i (x-x_0))} \right]},
$$

where

$$
\lambda_i^2 = -\frac{(V^2 - v_A^2)(V^2 - c_{0i}^2)}{(c_{0i}^2 + v_A^2)(V^2 - c_{0i}^2)},
$$

Since $\lambda_i^2 > 0$, the waves are evanescent in the homogeneous region.

In the inhomogeneous region (quantities with subscript '0') the solution to the system (10) can be written as

$$
\frac{\partial}{\partial x} \frac{1}{\rho_0 (V^2 - v_A^2)} \frac{\partial P}{\partial x} = O(k^2 x_0^2),
$$

and

$$
\frac{\partial}{\partial x} F(x) \frac{\partial P}{\partial \theta},
$$

$$
F(x) = \frac{V (V^2 - v_A^2)}{\rho_0 (v_A^2 + v_0^2)(V^2 - v_A^2)},
$$

In general it is impossible to solve analytically the nonlinear equation that govern the waves dynamics in the dissipative layer, but if we assume that the nonlinear term is small in Equation (18) in comparison to the dissipative term, we can look for the solution in the form of an expansion in the small parameter $\delta = V \delta$, i.e.,

$$
f = \epsilon \sum_{n \geq 1} \delta^{n-1} f_n,
$$

where $f$ represents any of the quantities $P$, $u$, and $q$.

The system of MHD equations has to solved subject to the conditions that that $P$ and $u$ have to be continuous at $x = 0$ and $x = x_0$, and in addition that $P$ has to be continuous at $x = x_{res}$. To give a quantitative description of the efficiency of the weak resonant interaction we calculate the coefficient of wave absorption. Retaining only the monochromatic overtones we obtain the following approximative relation for the coefficient of energy absorption

$$
\alpha \approx \alpha_L + \delta^2 \alpha_{NL}
$$
Taking into account the assumptions (weak nonlinearity and long wavelength approximation) we obtain for the coefficient of energy absorption

\[ \alpha_L = -\frac{4\xi \zeta}{\zeta^2 + \phi^2}, \quad \alpha_{NL} = \frac{4\rho_e^2 \xi^3 \zeta^3 I}{\pi^2 \nu^2 (\zeta^2 + \phi^2)^2}, \]  

where \( I \) is an integral that can be found in Erdélyi and Ballai (1999). The integral \( I \) was estimated to be of the order of unity, and

\[ \xi = \frac{\pi k \nu^5}{\rho_0 v_A^2 \Delta}, \quad \zeta = \frac{\chi e}{\rho_e \nu}, \quad \phi = \frac{\chi \nu^{2}}{\rho_1 (\nu^2 - v_A^2)}, \quad \xi = \frac{\pi k P}{\int_0^{2\phi} F(x) \, dx} \]  

In contrast to the static case, where nonlinearity decreases the coefficient of energy absorption, in steady plasmas nonlinearity can increase the absolute value of this coefficient. Moreover, the linear part of this coefficient can be negative for some specific values of the flow strength. These negative values are related to over-reflection or resonant instability.

3.1.2. Strongly nonlinear limit

Since in the weak nonlinear limit we used a regular perturbation method the solution is just a correction to the linear results. In a more realistic situation, we should consider that the nonlinearity is strong, i.e., \( \epsilon R^{2/3} \gg 1 \).

Let us introduce some new dimensionless variables

\[ \vartheta = k \theta, \quad \sigma = \frac{-(x - x_{res}) \text{sgn}(\Delta)}{\delta_c (2M)^{3/2}}, \quad q = \frac{N' v_{\parallel}}{\delta_c (2M)^{3/2}}, \quad Q = \frac{2
u^3 \Phi_1 P_c}{\rho_0 v_A^2 \delta_c (2M)^{3/2}}. \]

With these variables the jump condition for the normal component of the velocity and the governing equation (13) become

\[ [u] = \frac{k \nu^2 \delta_c^2 \text{sgn}(\Delta)}{N' v_{Ac}} (2M)^{3/2} \frac{d}{d\vartheta} P \int_{-\infty}^{\infty} q(\sigma, \vartheta) \, d\vartheta, \]

\[ 2\sigma \frac{\partial q}{\partial \vartheta} + 2q \frac{\partial q}{\partial \vartheta} - \frac{\text{sgn}(\nu)}{M} \frac{\partial^2 q}{\partial \sigma^2} = -\frac{dQ}{d\vartheta}. \]

In order to obtain the asymptotic behaviour of equation (30) we neglect the last term in the left-hand side which contains the large parameter \( M \). Cumbersome derivation of this problem was carried out by Ruderman (2000) for a static equilibrium. We apply his method of derivation to study the effect of a mass flow on the resonant absorption of sound waves in the slow wave dissipative layer.

The solution in the magnetic free region (Region I) is a sum of the incoming and outgoing waves. We assume that the incoming wave contains only the fundamental harmonic, so the perturbation of the total pressure can be written as \( P_{in} = \epsilon \text{Re} \left[ P_e e^{ik(\theta + \chi e x)} \right] \). For the outgoing wave we have \( P_{out} = \epsilon p_e A(\theta - \chi e x) \), where \( A \) is a periodic function with zero mean value over a period. The normal component of the velocity can be found using the system describing the wave motion outside the dissipative layer, namely,

\[ P = \epsilon p_e \{ \cos[k(\theta + \chi e x)] + A(\theta - \chi e x) \}, \quad u = \frac{\chi e p_e}{\rho_e \nu} \{ \cos[k(\theta + \chi e x)] - A(\theta - \chi e x) \}. \]

The solutions in the internal region (Region III) are found in the form of Fourier series

\[ P = \sum_{n=-\infty}^{\infty} P_n(x) e^{ink\theta}, \quad u = \sum_{n=-\infty}^{\infty} u_n(x) e^{ink\theta} \]
where
\[ P_n = P_n(x_0) \exp[-\chi ikn|(x-x_0)|], \quad u_n = \frac{iV \chi \text{sgn}(n)}{\rho_i(V^2 - v_A^2)} P_n(x_0) \exp[-\chi ikn|(x-x_0)|]. \] (33)

Applying the same boundary conditions as in the case of weak nonlinearity, we obtain that the coefficient of wave energy absorption
\[ \alpha_{NL} = -\frac{32 \chi \zeta}{\chi^2 + \xi^2} + O(k^2 x_0^2). \] (34)

where now
\[ \xi = \frac{\chi \rho_i V^2}{\rho_i(v_A^2 - V^2)}, \quad \zeta = \frac{k \rho_i V^4 V^2}{\pi \rho_0 v_A^2 \Delta} \] (35)

For the ratio of the nonlinear to the linear coefficient of energy absorption we obtain
\[ \frac{\alpha_{NL}}{\alpha_L} = \frac{8V}{\pi^2 V^2} \text{sgn}(V \Delta) + O(kx_0) \] (36)

This is a surprising result which tells us that the nonlinear coefficient is proportional to its counterpart obtained in linear theory. The factor of proportionality is of the order of unity for subsonic mass flows. This result also justifies the efforts made in the studies of linear resonant MHD waves, and their applicability to solar physics.

The model described here considered a simplified atmosphere. Possible further investigations could be performed for a more realistic equilibrium (e.g. equilibrium quantities vary not only across the field but also along the field, inclusion of gravity, etc.), and for intermediate values of the wavelengths (here the governing equations were obtained in the limit of long wavelength approximation).

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