A Model for Tri-Bimaximal Mixing from a Completely Broken $A_4$

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Abstract

We propose a new $A_4$ model in which both the right-handed neutrinos and right-handed charged leptons transform as $A_4$ singlets. We reproduce tri-bimaximal mixing pattern exactly although the $A_4$ symmetry is broken completely at leading order in both the neutrino and charged lepton sectors. The charged lepton mass hierarchies are controlled by the spontaneous breaking of the flavor symmetry. The light neutrino spectrum is predicted to be of normal type and the lightest neutrino is massless at leading order. Although the reactor angle $\theta_{13}$ is expected to be of order $\lambda^2$ from the next to leading order corrections, this model cannot be ruled out by current experimental data including the latest T2K results. Leptogenesis is realized via the resonant leptogenesis of the second and the third heavy right-handed neutrinos which are degenerate at leading order. The phenomenological consequences for lepton flavor violation are discussed in detail.
1 Introduction

In the past years, considerable efforts have been devoted to discrete flavor symmetry and many discrete groups have been considered as family symmetry groups to derive some mass independent textures, see Refs. [1, 2] for reviews. In particular, it has been realized that tri-bimaximal (TB) mixing matrix [3], which is at least a good zeroth order approximation to the current neutrino oscillation data [4–7], can naturally arise as the result of a particular vacuum alignment of scalars that spontaneously break certain discrete flavor symmetries. Many discrete groups have been exploited to reproduce TB mixing so far and the $A_4$ group seems to be especially suitable to perform this task. It has been demonstrated, through group theoretical analysis, that the minimal flavor symmetry capable of yielding the TB mixing without fine tuning is $S_4$ [8]. However, from the model building point of view, $A_4$ appears to be the most economical and simplest realization which naturally produces the TB mixing pattern. There was great interest in $A_4$ as a family symmetry in the recent past and various $A_4$ models have been constructed. We can approximately categorize the $A_4$ models into three classes based on the neutrino mass generation mechanisms; there exist models in which neutrino masses arise from higher dimensional effective operators and models in which neutrino masses are generated via the see-saw mechanisms (being the canonical type I, type II, type III see-saw mechanisms or the combination of them or the inverse and linear see-saw mechanisms). The third class of $A_4$ models generates neutrino masses via one-loop or two-loop radiative corrections but they are rare. Some of the models also try to extend the $A_4$ flavor symmetry to the quark sector in the framework of the Standard Model (SM) and in Grand Unified Theories (GUT). Table I is an attempt to classify the large number of $A_4$ models on the market, based on an earlier classification done in Ref. [21]. We only list models where the focus was the study of flavor mixings; however, there are also papers based on $A_4$ discussing the dark matter [91–94] and the electroweak constraints/phenomenology [95–99]. From Table I, we can clearly see that the three lepton doublet fields are assigned as $A_4$ triplet in almost all the models and much the same happens for the right-handed neutrino $\nu^c_i$ in type I see-saw, for the field $\Delta$ in type II see-saw and for the $\Sigma$’s in type III (we will call these fields see-saw fields in the following). In this paper, we present an $A_4$ model for TB mixing where the right-handed neutrinos $\nu^c_i$ transform as $1$, $1'$ and $1''$ under $A_4$, all the right-handed charged leptons $e^c$, $\mu^c$ and $\tau^c$ are $A_4$ singlet $1$ and the three generations of left-handed lepton doublets $\ell_i$ are assigned to a triplet $3$. This assignment has not been considered so far, as it can be seen from Table I. This model is as simple as previous $A_4$ models but the phenomenological predictions are drastically different: the light neutrino mass spectrum is of normal hierarchy type and the lightest neutrino mass is exactly zero at LO, although three right-handed neutrinos are introduced. The first one does not contribute to the leptonic CP asymmetry even if NLO contributions are included and leptogenesis is realized via the resonant leptogenesis mechanism of the second and third right-handed heavy neutrinos, which are degenerate at LO. The resulting predictions for lepton flavor violation are distinct from existing $A_4$ models as well. The present model is a complete new variant of $A_4$ flavor models presented in the literature.

This paper is organized as follows. In Section 2 we discuss the structure of the model at leading order (LO) and show that the neutrino mass matrix is exactly diagonalized by TB matrix. In Section 3 we justify the vacuum alignment assumed in the previous section by minimizing the scalar potential of the model in the supersymmetric limit. The next to leading order (NLO) corrections induced by higher dimensional operators are analyzed in Section 4. We discuss the phenomenological predictions of the model for leptogenesis and lepton flavor violation in Section 5. Finally Section 6 is devoted to our discussions and conclusions. In order to make the paper self-contained we include Appendix A on the $A_4$.
2 The structure of the model

In this section, we present the model and discuss the LO results for lepton masses and flavor mixing. We formulate the model in the framework of type I seesaw mechanisms and supersymmetry (SUSY) is introduced to simplify the discussion of the vacuum alignment. The complete flavor symmetry of the model is $A_4 \times Z_4 \times Z_2$. The $Z_4$ symmetry distinguishes the neutrinos from the charged leptons, and it is responsible for the mass hierarchies of charged leptons; $Z_2$ further distinguishes the right-handed neutrinos $\nu^c_1$ from $\nu^c_2$ and $\nu^c_3$. Moreover, the $Z_4 \times Z_2$ symmetry plays an important role in ensuring the needed vacuum alignment. All the fields of the model with their transformation properties under the flavor symmetry group are shown in Table 2. We assign the three generations of left-handed lepton doublet $\ell_i$ as $A_4$ triplet $3$, while the right-handed charged lepton $e^c$, $\mu^c$ and $\tau^c$ are all invariant under $A_4$. Inspired by our previous work on $T_{13}$ flavor symmetry, the three right-handed neutrinos $\nu^c_1$, $\nu^c_2$ and $\nu^c_3$ transform as $1$, $1'$ and $1''$ respectively. It is remarkable that this transformation property is different from many previous $A_4$ models where the right-handed neutrinos generally form a triplet. In our model, all right-handed fields, being singlets of $A_4$, are treated democratically, a more symmetric situation than previous models. The flavor symmetry is spontaneously broken by four flavon fields $\varphi$, $\xi$, $\phi$ and $\chi$. At LO the flavons $\varphi$ and $\xi$ couple to the charged lepton sector, while $\phi$ and $\chi$ couple to the neutrino sector. For the time being, we assume that the scalar components of the flavon fields acquire vacuum expectation values (VEV) according to the following scheme:

$$
\langle \varphi \rangle = (0, v_{\varphi}, 0), \quad \langle \xi \rangle = v_{\xi} \\
\langle \chi \rangle = (v_{\chi}, v_{\chi}, v_{\chi}), \quad \langle \phi \rangle = (0, v_{\phi}, -v_{\phi}). \tag{1}
$$

We will demonstrate that this particular vacuum alignment is a natural solution of the scalar potential in Section 3. We note that if the auxiliary $Z_2$ symmetry was absent, it would be enough to introduce one flavon field only to generate the neutrino masses, thus giving a simpler model. However, all the resulting realizations would predict TB mixing in connection with $m_2 = 0$ or $m_1 = m_3$. This is obviously not allowed by neutrino oscillation data and the same remains true even after the NLO corrections are considered. One of the crucial points of our work is the observation that we need to introduce at least two flavon fields to break the $A_4$ symmetry in the neutrino sector at LO, if the right-handed neutrinos are assigned to $A_4$ singlets.

2.1 Charged lepton

The charged lepton masses are described by the following superpotential:

$$
w_{\ell} = \frac{y_{\tau}}{\Lambda} \tau^c(\ell \varphi)h_d + \frac{y_{\mu}}{\Lambda^2} \mu^c(\ell \varphi \varphi)h_d + \frac{y_{\mu_2}}{\Lambda^2} \mu^c(\ell \varphi)\xi h_d + \frac{y_{\mu_3}}{\Lambda^3} \mu^c(\ell \varphi)(\varphi \varphi)h_d + \frac{y_{e}}{\Lambda^3} e^c(\ell \varphi)(\varphi \varphi)\xi h_d \\
+ \frac{y_{e_1}}{\Lambda^3} e^c(\ell \varphi)\xi h_d + \frac{y_{e_2}}{\Lambda^4} e^c((\ell \varphi)_{3s}(\varphi \varphi)_{3s})h_d + \frac{y_{e_3}}{\Lambda^5} e^c((\ell \varphi)_{3A}(\varphi \varphi)_{3s})h_d + \frac{y_{e_4}}{\Lambda^6} e^c(\ell \varphi)\xi h_d \\
+ \frac{y_{e_5}}{\Lambda^7} e^c(\ell \varphi)\xi^2 h_d + \ldots \tag{2}
$$

where dots represent the higher dimensional operators which will be discussed later. Due to the constraint of the $Z_4$ symmetry, the electron, muon and tauon mass terms appear
|        | $\ell_i$ | $e_i^\ell$ | $\nu_i^\ell$ | $\Delta$ | $\Sigma$ | Quark | GUT | Refs. |
|--------|----------|-----------|--------------|---------|---------|-------|-----|-------|
| Effective | 3 | 1,1',1'' | — | — | — | ✔ | ✔ | 9 [21] |
|        | 3 | 3 | — | — | — | ✔ | ✔ | 10 |
|        | 3 | 1,1,1 | — | — | — | ✔ | ✔ | 22 |
|        | 3 | 3 | — | — | — | ✔ | ✔ | 23 |
|        | 3 | 1,1',1'' | — | — | — | ✔ | ✔ | 24 |
| Type I SS | 3 | 1,1',1'' | 3 | — | — | ✔ | ✔ | 31,32,33,34 |
|        | 3 | 3 | 1,1',1'' | — | — | ✔ | ✔ | 46,47 |
|        | 3 | 3 | 3 | — | — | ✔ | ✔ | 48 |
|        | 3 | 1,1',1'' | 1,1',1'' | — | — | ✔ | ✔ | 49 |
|        | 3 | 1,1,1 | 3 | — | — | ✔ | ✔ | 53,54,55,56 |
|        | 3 | 1,1',1'' | 3 | — | — | ✔ | ✔ | 52 |
|        | 3 | 1,1',1'' | 3 | — | — | ✔ | ✔ | 52 |
|        | 3 | 1,1,1 | 1,1 | — | — | ✔ | ✔ | 59 |
|        | 3 | 1,1,1 | 1,1 | — | — | ✔ | ✔ | 59 |
|        | 3 | 3 | 3 | — | — | ✔ | ✔ | 60,61 |
| Type II SS | 3 | 1,1',1'' | — | 3, 1,1',1'' | — | ✔ | ✔ | 67,69 |
|        | 3 | 3 | — | 3, 1,1',1'' | — | ✔ | ✔ | 70 |
|        | 3 | 3 | 3 | 1 | — | ✔ | ✔ | 71 |
|        | 3 | 3 | — | 3,1 | — | ✔ | ✔ | 71 |
|        | 3 | 3 | 3 | 1,1 | — | ✔ | ✔ | 72 |
|        | 3 | 3 | — | 3, 1,1',1'' | — | ✔ | ✔ | 74 |
|        | 3 | 3 | — | 3, 1', — | — | ✔ | ✔ | 75 |
| Type III SS | 3 | 1,1',1'' | — | — | — | ✔ | ✔ | 76 |
| Type I+II SS | 3 | 1,1',1'' | 3 | 3, 1,1',1'' | — | ✔ | ✔ | 77 |
|        | 3 | 3 | 3 | 1 | — | ✔ | ✔ | 78 |
|        | 3 | 3 | 1,1',1'' | 1 | — | ✔ | ✔ | 79 |
|        | 3 | 3 | 1,1',1'' | 3,1 | — | ✔ | ✔ | 80 |
|        | 3 | 3 | 3 | 1 | — | ✔ | ✔ | 81 |
| Type I+III SS | 1,1',1'' | 3 | 3 | 3 | — | ✔ | ✔ | 82 |
|        | 3 | 3 | 1,1',1'' | 1,1',1'' | — | ✔ | ✔ | 82 |
|        | 3 | 3 | 3 | 3 | — | ✔ | ✔ | 82 |
|        | 3 | 1,1,1 | 1 | 1 | — | ✔ | ✔ | 83 |
| Inverse SS | 3 | 3 | 3 | — | — | ✔ | ✔ | 84 |
| Linear SS | 3 | 3 | 3 | — | — | ✔ | ✔ | 84 |
| Radiative | 3 | 1,1',1'' | — | — | — | ✔ | ✔ | 86 |
| Only quark | — | — | — | — | — | ✔ | ✔ | 87,90 |

Table 1: Classification of $A_4$ models in terms of the neutrino mass generation mechanisms and the transformation properties of the matter fields presented in the literature. The notations $\ell_i$, $e_i^\ell$ and $\nu_i^\ell$ represent the left-handed lepton doublet, right-handed charged lepton and right-handed neutrinos, respectively. $\Delta$ denotes the Higgs triplet in type II see-saw mechanisms, $\Sigma$ denotes the fermion triplet in type III see-saw mechanisms. "Effective" means that the neutrino masses are generated via effective operators, the abbreviation "SS" denotes see-saw mechanisms and "Radiative" indicates that neutrino masses are induced as one-loop or two-loop radiative corrections. The symbol ✔ and ✗ in the Quark and GUT columns refers to whether $A_4$ has been extended to the quark sector and embedded into GUT theories, respectively. For the linear and inverse see-saw neutrino mass generation [87,88], an additional SM singlet transforming as 3 under $A_4$ is introduced.
at different orders in the expansion in terms of $1/\Lambda$. After electroweak and $A_4$ symmetry breaking, the superpotential $w_\ell$ gives rise to a diagonal charged lepton mass matrix:
\[
m_\ell = \begin{pmatrix}
(y_{e2} - 2y_{e4} + 2y_{e5})v^2_e + 2y_{e6}v^2_e + y_{e7}v^2_e & 0 & 0 \\
0 & 2y_{\mu1}v_\mu + y_{\mu2}v_\mu & 0 \\
0 & 0 & y_\tau
\end{pmatrix} \frac{v_\phi v_d}{\Lambda},
\]
where $\langle h_d \rangle = v_d$. We see that the electron, muon and tauon masses are controlled by the first, second and third power of $v_\phi/\Lambda$ and $v_\xi/\Lambda$. Therefore the mass hierarchies of the charged leptons are naturally recovered if $v_\phi/\Lambda$ and $v_\xi/\Lambda$ are of order $\lambda_c^2$ [22, 50], where $\lambda_c \simeq 0.22$ is the Cabibbo angle. We note that the charged lepton mass hierarchies are determined by the flavor symmetry itself without invoking a Froggatt-Nielsen mechanism. As it can be seen from Eq. (2), the $A_4$ group in the charged lepton sector is completely broken by the VEVs of the flavons $\varphi$ and $\xi$ at LO, since $T\langle \varphi \rangle = \omega^2 \langle \varphi \rangle$ and $T\langle \xi \rangle = \omega^2 \langle \xi \rangle$. However, the lepton flavor mixing is associated with the combination $\overline{m}_\ell = m_\ell m_\ell^T$, which is obviously invariant under $T$, i.e., $T^\dagger \overline{m}_\ell T = \overline{m}_\ell$. Consequently there is still a remnant $Z_3$ symmetry generated by $T$ in the charged lepton mass matrix.

### 2.2 Neutrino

Neutrino masses are generated by type I see-saw mechanism. The superpotential for the neutrino sector is:
\[
w_\nu = \frac{y_{\nu1}}{\Lambda} \nu^c_1 (\ell \phi) h_u + \frac{y_{\nu2}}{\Lambda} \nu^c_2 (\ell \chi)'' h_u + \frac{y_{\nu3}}{\Lambda} \nu^c_3 (\ell \chi) h_u + \frac{1}{2} M \nu^c_1 \nu^c_1 + \frac{1}{2} M' (\nu^c_2 \nu^c_3 + \nu^c_3 \nu^c_2).
\](4)

The first three terms contribute to the Dirac mass terms whereas the last two are the Majorana mass terms for the right-handed neutrinos. One can always set the masses $M$ and $M'$ to be real and positive by performing global phase transformations of the right-handed neutrino fields. After electroweak and $A_4$ symmetry breaking, we obtain the following LO contributions to the Dirac and Majorana mass matrices:
\[
m_D = \frac{v_\mu}{\Lambda} \begin{pmatrix}
0 & -y_{\nu1}v_\phi & y_{\nu1}v_\phi \\
y_{\nu1}v_\chi & y_{\nu2}v_\chi & y_{\nu2}v_\chi \\
y_{\nu3}v_\chi & y_{\nu3}v_\chi & y_{\nu3}v_\chi
\end{pmatrix},
\quad
m_M = \begin{pmatrix}
M & 0 & 0 \\
0 & 0 & M' \\
0 & M' & 0
\end{pmatrix}.
\](5)

The heavy right-handed neutrino mass matrix $m_M$ can be diagonalized by a unitary transformation
\[
U_R^T m_M U_R = \text{diag}(M, M', M').
\](6)

Two of the right-handed neutrinos are degenerate with mass equal to $M'$ so that the unitary matrix $U_R$ cannot be fixed uniquely; it can be expressed as:
\[
U_R = \begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i\theta}/\sqrt{2} & -ie^{i\theta}/\sqrt{2} \\
0 & ie^{-i\theta}/\sqrt{2} & ie^{-i\theta}/\sqrt{2}
\end{pmatrix},
\](7)
where $\vartheta$ is an arbitrary phase parameter. The light neutrino mass matrix is given by the see-saw formula:

$$m_\nu = -m_D^T M^{-1}_M m_D = \begin{pmatrix} a & a & a \\ a & a+b & a-b \\ a & a-b & a+b \end{pmatrix} \frac{v^2_\nu}{\Lambda},$$  \hspace{1cm} (8)

where

$$a = -2y_{\nu_2}y_{\nu_3} \frac{v^2_\chi}{\Lambda M'}, \quad b = -y^2_{\nu_1} \frac{v^2_\phi}{\Lambda M},$$  \hspace{1cm} (9)

and it is exactly diagonalized by the TB mixing matrix $U_{TB}$:

$$U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$  \hspace{1cm} (10)

In unit of $v^2_\nu/\Lambda$, the light neutrino masses are given by:

$$m_1 = 0, \quad m_2 = 3a, \quad m_3 = 2b.$$  \hspace{1cm} (11)

It is remarkable that the mass of the lightest neutrino is zero, although three (instead of two) right-handed neutrinos have been considered. As a result, the light neutrino mass spectrum is predicted to be of normal type. This is a special feature of the present model since generally a massless neutrino is realized with the help of minimal see-saw mechanisms [102]. Recalling that two observables related to the neutrino mass spectrum have been measured [4–6]:

$$\Delta m^2_{\text{sol}} \equiv m^2_2 - m^2_1 = (7.59^{+0.20}_{-0.18}) \times 10^{-5} \text{eV}^2,$$

$$\Delta m^2_{\text{atm}} \equiv m^2_3 - m^2_1 = (2.45 \pm 0.09) \times 10^{-3} \text{eV}^2,$$  \hspace{1cm} (12)

we have $m_2 \simeq 0.0087$ eV and $m_3 \simeq 0.049$ eV. Then the sum of the light neutrino masses is $m_1 + m_2 + m_3 \simeq 0.0577$ eV. Moreover, we can straightforwardly obtain the effective mass parameter $|m_{ee}|$ for the neutrinoless double-β decay:

$$|m_{ee}| \equiv \left| \sum_i (U_{PMNS})^2_{ei} m_i \right| = \frac{m_2}{3} \simeq 0.0029 \text{eV},$$  \hspace{1cm} (13)

where $U_{PMNS} \equiv U_{TB}$ in our case. Therefore the effective mass $|m_{ee}|$ is predicted to be far below the sensitivities of the planned neutrinoless double-β decay experiments. It is important to note that the alignment direction of $\langle \chi \rangle$ preserves a $Z_2$ subgroup generated by the element $S$. However, an overall negative sign appears if we act on $\langle \phi \rangle$ with $S$, thus also the residual $Z_2$ symmetry is broken. As a result, the $A_4$ group is broken to nothing also in the neutrino sector at LO. However, it can be checked that $m_\nu$ is invariant under $S$ and, therefore, an accidental $Z_2$ symmetry generated by $S$ is still preserved in the light neutrino mass matrix (in fact, $m_\nu$ is basically determined by $\langle \phi \rangle^2$ which leaves the $S$ generator invariant). In summary, the $A_4$ flavor symmetry is broken completely at LO in both charged lepton and neutrino sectors, nevertheless there is an accidental $Z_3 \times Z_2$ symmetry in the charged lepton and neutrino mass matrices, respectively, that ensures diagonal charged leptons and TB mixing matrix.

3 Vacuum alignment

The vacuum alignment problem can be solved by the supersymmetric driving field method introduced in Ref. [11]. This approach introduces a global continuous $U(1)_R$ symmetry which
contains the discrete $R$–parity as a subgroup. The flavon and Higgs fields are uncharged under $U(1)_R$, the matter fields have $R = 1$ and the so-called driving fields $\varphi^0$, $\chi^0$, $\Delta^0$ and $\rho^0$ carry two units of $R$ charge. The LO driving superpotential $w_d$, which is linear in the driving fields and invariant under the flavor symmetry $A_4 \times Z_4 \times Z_2$, is given by:

$$w_d = f_1(\varphi^0\varphi) + f_2(\varphi^0\varphi')\xi + M(\chi^0\chi) + g_1(\chi^0\chi) + g_2(\chi^0\phi\phi) + g_3(\phi\chi) + g_4(\phi\phi)' . \quad (14)$$

In the SUSY limit, the equations for the minimum of the scalar potential are obtained by deriving $w_d$ with respect to each component of the driving fields. The vacuum structure of the flavons $\varphi$ and $\xi$ is determined by

$$\frac{\partial w_d}{\partial \varphi_1^0} = 2f_1(\varphi_1^2 - \varphi_2\varphi_3) + f_2\varphi_3\xi = 0 \quad (15a)$$
$$\frac{\partial w_d}{\partial \varphi_2^0} = 2f_1(\varphi_2^2 - \varphi_1\varphi_3) + f_2\varphi_2\xi = 0 \quad (15b)$$
$$\frac{\partial w_d}{\partial \varphi_3^0} = 2f_1(\varphi_3^2 - \varphi_1\varphi_2) + f_2\varphi_1\xi = 0 . \quad (15c)$$

This set of equations admit two un-equivalent solutions, the first one is

$$\langle \varphi \rangle = (v_{\varphi}, v_{\varphi}, v_{\varphi}), \quad \langle \xi \rangle = 0 , \quad (16)$$

where $v_{\varphi}$ is undetermined. The second solution is:

$$\langle \varphi \rangle = (0, v_{\varphi}, 0), \quad \langle \xi \rangle = v_\xi , \quad (17)$$

with the condition:

$$v_{\varphi} = -\frac{f_2}{2f_1}v_\xi , \quad v_\xi \text{ undetermined} . \quad (18)$$

The VEVs $v_{\varphi}$ and $v_\xi$ are naturally of the same order of magnitude (without fine tuning among the parameters $f_1$ and $f_2$), this is consistent with the conclusions drew from the charged lepton mass hierarchies. Only the second alignment can provide the results of the previous section but we need of some soft masses in order to discriminate it as the lowest minimum of the scalar potential (not discussed here). The minimization equations for the vacuum configuration of $\phi$ and $\chi$ are given by:

$$\frac{\partial w_d}{\partial \chi_1} = M \chi_1 + 2g_1(\chi_1^2 - \chi_2\chi_3) + 2g_2(\phi_1^2 - \phi_2\phi_3) = 0 \quad (19a)$$
$$\frac{\partial w_d}{\partial \chi_2} = M \chi_2 + 2g_1(\chi_2^2 - \chi_1\chi_3) + 2g_2(\phi_2^2 - \phi_1\phi_3) = 0 \quad (19b)$$
$$\frac{\partial w_d}{\partial \chi_3} = M \chi_3 + 2g_1(\chi_3^2 - \chi_1\chi_2) + 2g_2(\phi_3^2 - \phi_1\phi_2) = 0 \quad (19c)$$
$$\frac{\partial w_d}{\partial \Delta^0} = g_3(\phi_1\chi_1 + \phi_2\chi_3 + \phi_3\chi_2) = 0 \quad (20)$$
$$\frac{\partial w_d}{\partial \rho^0} = g_4(\phi_3\phi_3 + \phi_1\phi_2 + \phi_2\phi_1) = 0 . \quad (21)$$

Taking into account the alignment of $\varphi$ in Eq.(17), we can infer from Eq.(21)

$$\langle \phi_1 \rangle = 0 . \quad (22)$$
Then Eqs. (19a, 19c), Eq. (20) and Eq. (21) admit the non-trivial vacuum configuration:

\[
\langle \chi \rangle = (v_\chi, v_\chi, v_\chi), \quad \langle \phi \rangle = (0, v_\phi, -v_\phi), \quad v_\phi^2 = -\frac{M v_\chi}{2 g_2},
\]

(23)

with \( v_\chi \) undetermined. As we will show in Section 4 all the three lepton mixing angles receive corrections of order \( v_\chi / \Lambda \) or \( v_\phi / \Lambda \). The solar neutrino angle \( \theta_{12} \) is the most precisely measured one, the experimental departure from its TB value being at most of order \( v_\phi / \Lambda \). Therefore we expect \( v_\chi / \Lambda \) and \( v_\phi / \Lambda \) of the same order of magnitude, \( \sim \lambda^2_c \) as well. As a consequence, the following relations among the VEVs hold:

\[
\frac{v_\phi}{\Lambda} \sim \frac{v_\chi}{\Lambda} \sim \frac{v_\phi}{\Lambda} \sim \lambda^2_c.
\]

(24)

Henceforth we will parameterize the ratio VEV/\( \Lambda \) by the parameter \( \varepsilon \). Given the symmetry of the superpotential \( w_d \), we can generate other minima of the scalar potential by acting on the configuration of Eq. (17) and Eq. (23) with the element of the flavor symmetry group \( A_4 \). However, these new minima are physically equivalent to the original one, they all lead to the same physics, i.e., lepton masses and flavor mixings, and the different scenarios are related by field redefinitions. Without loss of generality, we can analyze the model by choosing the vacuum in Eq. (17) and Eq. (23) as the local minimum.

4 Next to leading order corrections

It is important to check that the NLO contributions do not modify too much the successful LO predictions and that the deviations from TB mixing lie in the experimentally allowed range. The NLO corrections are indicated by the subleading higher dimensional operators in the 1/\( \Lambda \) expansion, which are compatible with all the symmetries of the model. In the following, we will study the NLO corrections to the vacuum alignment, to the charged lepton and to neutrino mass matrices.

4.1 NLO corrections to the vacuum alignment

After including the NLO operators the superpotential \( w_d \), depending on the driving fields \( \varphi^0, \chi^0, \Delta^0 \) and \( \rho^0 \), is modified to:

\[
w_d = w_d^0 + \delta w_d,
\]

(25)

where \( w_d^0 \) is given by Eq. (18) and \( \delta w_d \) denotes the NLO terms, suppressed by one additional power of \( 1/\Lambda \) with respect to \( w_d^0 \). The correction terms included in \( \delta w_d \) consist of the most general quartic, \( A_4 \times Z_4 \times Z_2 \) invariant polynomial linear in the driving fields, obtained inserting an additional flavon field in the LO terms. Concretely, \( \delta w_d \) is given by:

\[
\delta w_d = \frac{1}{\Lambda} \left( \sum_{i=1}^{8} v_i \mathcal{I}^0_i + \sum_{i=1}^{10} c_i \mathcal{I}^0_i + \sum_{i=1}^{2} d_i \mathcal{I}^{\Delta^0}_i + \sum_{i=1}^{3} r_i \mathcal{I}^{\rho^0}_i \right),
\]

(26)

where \( v_i, c_i, d_i \) and \( r_i \) are complex coefficients with absolute value of \( \mathcal{O}(1) \); \( \mathcal{I}^{\varphi}_i, \mathcal{I}^{\chi}_i, \mathcal{I}^{\Delta^0}_i \) and \( \mathcal{I}^{\rho^0}_i \) denote a basis of independent quartic invariants:

\[
\begin{align*}
\mathcal{I}^{\varphi}_i &= (\varphi^0 \chi)(\varphi \varphi), \quad \mathcal{I}^{\chi}_i = (\varphi^0 \chi)'(\varphi \varphi)'', \quad \mathcal{I}^{\Delta^0}_i = (\varphi^0 \chi)''(\varphi \varphi)'', \\
\mathcal{I}^{\varphi}_i &= ((\varphi^0 \chi)_3, \varphi \varphi)_3^s, \quad \mathcal{I}^{\rho^0}_i = ((\varphi^0 \chi)_3, \varphi \varphi)_3^s, \quad \mathcal{I}^{\chi}_i = (\varphi^0 \chi)_3^s(\varphi \varphi)_3^s, \quad \mathcal{I}^{\rho^0}_i = (\varphi^0 \chi)_3^s(\varphi \varphi)_3^s, \\
\mathcal{I}^{\varphi}_i &= (\varphi^0 \chi)(\varphi \varphi)_3^s, \quad \mathcal{I}^{\rho^0}_i = (\varphi^0 \chi)(\varphi \varphi)_3^s,
\end{align*}
\]

(27)
\[ D_1^c = (\chi^0\chi)(\chi\chi), \quad D_2^c = (\chi^0\chi)'(\chi\chi)'', \quad D_3^c = (\chi^0\chi)''(\chi\chi)'' \]
\[ D_1^\Delta = ((\chi^0\chi)^3_{A}(\chi\chi)_{A}), \quad D_2^\Delta = ((\chi^0\chi)^3_{A}(\chi\chi)_{A}), \quad D_3^\Delta = (\chi^0\chi)(\phi\phi), \]
\[ T_1^c = (\chi^0\chi)'(\phi\phi)'', \quad T_2^c = (\chi^0\chi)''(\phi\phi)', \quad T_3^c = ((\chi^0\chi)^3_{A}(\phi\phi)_{A}) \]
\[ T_1^\Delta = \Delta^0(\phi\chi\chi), \quad T_2^\Delta = \Delta^0(\phi\phi) \]
\[ T_1^\rho = \rho^0(\varphi(\phi)(\chi)^3_{A}), \quad T_2^\rho = \rho^0(\varphi(\phi)(\chi)^3_{A}), \quad T_3^\rho = \rho^0(\varphi(\chi)\xi). \]

The new vacuum configuration is obtained by imposing the vanishing of the first derivative of \( w_d + \delta w_d \) with respect to the driving fields \( \varphi^0, \chi^0, \Delta^0 \) and \( \rho^0 \). Denoting the general flavon field with \( \Phi \), we can write the new VEV as \( \langle \Phi_i \rangle = \langle \Phi_i \rangle_{LO} + \delta \nu_i \). By keeping only the terms linear in the shift \( \delta \nu \) and neglecting the terms proportional to \( \delta \nu / \Lambda \), the minimization equations become:

\[
\begin{align*}
-2f_1 v_\varphi + f_2 v_\chi \delta \nu_{\varphi 1} + a_3 v_\chi v_\varphi^2 / \Lambda &= 0, \\
(4f_1 v_\varphi + f_2 v_\chi) \delta \nu_{\varphi 2} + f_2 v_\varphi \delta \nu_{\xi} + a_2 v_\chi v_\varphi^2 / \Lambda &= 0, \\
-2f_1 v_\varphi + f_2 v_\chi \delta \nu_{\varphi 1} + a_1 v_\chi v_\varphi^2 / \Lambda &= 0, \\
(M + 4g_1 v_\chi) \delta \nu_{\chi 1} - 2g_1 v_\chi \delta \nu_{\chi 2} - 2g_1 v_\chi \delta \nu_{\chi 3} + 2g_2 v_\varphi (\delta \nu_{\phi 2} - \delta \nu_{\phi 3}) + a_4 v_\chi v_\varphi^2 / \Lambda &= 0, \\
-2g_1 v_\chi \delta \nu_{\chi 1} + 4g_1 v_\chi \delta \nu_{\chi 2} + (M - 2g_1 v_\chi) \delta \nu_{\chi 3} + 2g_2 v_\varphi (\delta \nu_{\phi 1} + 2\delta \nu_{\phi 2}) + a_4 v_\chi v_\varphi^2 / \Lambda &= 0, \\
-2g_1 v_\chi \delta \nu_{\chi 1} + (M - 2g_1 v_\chi) \delta \nu_{\chi 2} + 4g_1 v_\chi \delta \nu_{\chi 3} - 2g_2 v_\varphi (\delta \nu_{\phi 1} + 2\delta \nu_{\phi 3}) + a_4 v_\chi v_\varphi^2 / \Lambda &= 0, \\
v_\varphi (-\delta \nu_{\chi 2} + \delta \nu_{\chi 3}) + v_\chi (\delta \nu_{\phi 1} + \delta \nu_{\phi 2} + \delta \nu_{\phi 3}) &= 0, \\
g_4 [v_\varphi (\delta \nu_{\phi 1} - \delta \nu_{\phi 3}) + v_\varphi \delta \nu_{\phi 1}] + 2r_2 v_\varphi v_\varphi v_\chi / \Lambda &= 0,
\end{align*}
\]

where the parameters \( a_i (i = 1 - 4) \) are given by:

\[
\begin{align*}
a_1 &= v_2 + 4v_4 + 2f_1 (v_6 + v_7) / f_2 + 4f_1^2 v_8 / f_2^2, \\
a_2 &= v_2 - 2v_4 - 2v_5 + 2f_1 (v_6 - v_7) / f_2 + 4f_1^2 v_8 / f_2^2, \\
a_3 &= v_2 - 2v_4 + 2v_5 - 4f_1 v_6 / f_2 + 4f_1^2 v_8 / f_2^2, \\
a_4 &= 3(c_1 + c_2 + c_3) v_\chi^2 / v_\varphi^2 - 2c_6 + c_7 + c_8.
\end{align*}
\]

Eq. (31) is linear in the shift \( \delta \nu \) and can be straightforwardly solved, giving:

\[
\begin{align*}
\frac{\delta \nu_{\varphi 1}}{v_\varphi} &= \frac{a_1 v_\chi}{4f_1 \Lambda}, \\
\frac{\delta \nu_{\varphi 2}}{v_\varphi} &= -\frac{f_2}{2f_1} \frac{\delta \nu_{\xi}}{v_\varphi} - \frac{a_2 v_\chi}{2f_1 \Lambda}, \\
\frac{\delta \nu_{\varphi 1}}{v_\varphi} &= \frac{a_3 v_\chi}{4f_1 \Lambda}, \\
\delta \nu_{\chi 1} &= \delta \nu_{\chi 2} = \delta \nu_{\chi 3} \equiv \delta \nu_{\chi}, \\
\frac{\delta \nu_{\phi 1}}{v_\phi} &= \left( \frac{a_3 - a_1}{4f_1} - \frac{2r_2}{g_4} \right) \frac{v_\chi}{\Lambda}, \\
\frac{\delta \nu_{\phi 2}}{v_\phi} &= \frac{-M \delta v_{\chi}}{4g_2 v_\varphi} + \left( \frac{a_1 - a_3}{8f_1} + \frac{a_4}{4g_2} + \frac{r_2}{g_4} \right) \frac{v_\chi}{\Lambda}, \\
\frac{\delta \nu_{\phi 3}}{v_\phi} &= \frac{M \delta v_{\chi}}{4g_2 v_\varphi^2} + \left( \frac{a_1 - a_3}{8f_1} + \frac{a_4}{4g_2} + \frac{r_2}{g_4} \right) \frac{v_\chi}{\Lambda}.
\end{align*}
\]
We note that the corrections to \( \langle \chi \rangle \) are along the same direction of the LO alignment, all the components of \( \varphi \) acquire different corrections so that its alignment is tilted, and the shifts associated with the components of the flavon \( \phi \) are correlated with each other, i.e., \( \delta v_{\varphi_1} + \delta v_{\varphi_2} + \delta v_{\varphi_3} = 0 \). Recalling the LO relations \( v_\varphi = - f_2 v_\xi / (2 f_1) \) and \( v_\phi^2 = - M v_\chi / (2 g_2) \), the shifts \( \delta v_\xi \) and \( \delta v_\chi \) can be absorbed into the redefinition of the undetermined parameters \( v_\xi \) and \( v_\chi \) respectively. Therefore the LO vacuum configuration is modified as:

\[
\delta v_\varphi = (\delta v_{\varphi_1}, v_\varphi + \delta v_{\varphi_2}, \delta v_{\varphi_3}), \quad \delta v_\phi = (\delta v_{\phi_1}, \delta v_{\phi_2} + v_\phi, v_\phi + \delta v_{\phi_3}).
\]

The shifts are explicitly given in Eq.(33) and are all of \( O(\lambda^2) \).

### 4.2 NLO corrections to the mass matrices

The charged lepton and neutrino mass matrices are corrected by both the modified vacuum alignment and the subleading operators in the superpotentials \( w_\ell \) and \( w_\nu \). In this section, we present the corrections to the mass matrices and study the deviations from TB mixing.

#### 4.2.1 Charged lepton

The NLO operators contributing to the charged lepton masses can be obtained by inserting the flavon \( \chi \) in all possible ways into the LO operators and by extracting the \( A_4 \times Z_4 \times Z_2 \) invariants; the resulting NLO superpotential is given by:

\[
\delta w_\ell = \sum_{i=1}^{2} \frac{y_\ell^{(1)}}{A^2} \epsilon^c(\ell \chi \varphi)_i h_d + \frac{y_\ell^{(2)}}{A^2} \epsilon^c(\ell \chi \varphi)_i h_d + 2 \sum_{i=1}^{2} \frac{y_\mu^{(1)}}{A^3} \mu^c(\ell \chi \varphi)_i h_d + \sum_{i=1}^{2} \frac{y_\mu^{(2)}}{A^3} \mu^c(\ell \chi \varphi)_i h_d + \sum_{i=1}^{2} \frac{y_\mu^{(3)}}{A^3} \mu^c(\ell \chi \varphi)_i h_d
\]

where \( \epsilon \) represents different \( A_4 \) contractions. The corrected charged lepton mass matrix is obtained by adding the contributions of this new set of operators evaluated with the insertion of the LO VEVs of Eq.(14) and Eq.(22), to those of the LO superpotential in Eq.(2) evaluated with the NLO vacuum configuration in Eq.(31). After lengthy and tedious calculations, we find that every element of charged lepton mass matrix gets corrections from both the higher dimensional operators in \( \delta w_\ell \) and the shifted vacuum alignment. The off-diagonal elements become non-zero and are all suppressed by \( \epsilon \) with respect to diagonal ones. Consequently, the corrected charged lepton mass matrix has the following structure:

\[
m_\ell = \begin{pmatrix}
m_e & \varepsilon m_e & \varepsilon m_e \\
\varepsilon m_\mu & m_\mu & \varepsilon m_\mu \\
\varepsilon m_\tau & \varepsilon m_\tau & m_\tau
\end{pmatrix},
\]

where only the order of magnitude of each non-diagonal entry is reported. As a result, the unitary matrix \( U_\ell \) diagonalizing \( m_\ell^\dagger m_\ell \) is of the form:

\[
U_\ell \simeq \begin{pmatrix}
1 & (V_{12}^\ell \epsilon)^* & (V_{13}^\ell \epsilon)^* \\
-V_{12}^\ell \epsilon & 1 & (V_{13}^\ell \epsilon)^* \\
-V_{13}^\ell \epsilon & -V_{23}^\ell \epsilon & 1
\end{pmatrix},
\]

where \( V_{ij}^\ell \) are \( O(1) \) coefficients. We note that the charged lepton masses are corrected by terms of relative order \( \epsilon \), thus the LO mass hierarchies are not spoiled.
4.2.2 Neutrino

For the heavy right-handed neutrino mass matrix $m_M$, since no flavon field is involved in the LO Majorana mass terms of Eq. (1), $m_M$ does not receive corrections from the modified vacuum alignment. Due to the strong constraint of the flavor symmetry, the corrections to $m_M$ appear only at next to next to leading order (NNLO), the corresponding higher dimensional operators being:

\[
\frac{1}{\Lambda} \nu_1^c \nu_1^c(\phi\phi), \quad \frac{1}{\Lambda} \nu_1^c \nu_1^c(\chi\chi), \quad \frac{1}{\Lambda} \nu_1^c \nu_1^c(\chi\phi)', \quad \frac{1}{\Lambda} \nu_1^c \nu_1^c(\chi\phi)'', \quad \frac{1}{\Lambda} \nu_2^c \nu_2^c(\phi\phi), \quad \frac{1}{\Lambda} \nu_2^c \nu_2^c(\chi\chi), \quad \frac{1}{\Lambda} \nu_3^c \nu_3^c(\phi\phi)', \quad \frac{1}{\Lambda} \nu_3^c \nu_3^c(\chi\chi)''.
\]

(38)

Given the LO vacuum alignment in Eq. (23), we find that the 12, 13, 21, and 31 entries are still vanishing at NNLO. Taking into account the possibility of absorbing part of the corrections into the LO parameters $M$ and $M'$, the right-handed neutrino mass matrix can be parameterized as:

\[
m_M = \begin{pmatrix} M & 0 & 0 \\ 0 & c\varepsilon^2 M' & M' \\ 0 & M' & d\varepsilon^2 M' \end{pmatrix},
\]

(39)

where the parameters $c$ and $d$ are of order one, their specific values are not determined by the flavor symmetry. It is interesting to note that the mass degeneracy of the second and third right-handed neutrinos is lifted. Then we move to consider the corrections to the Dirac neutrino mass matrix; they are suppressed by $1/\Lambda^2$ compared to the LO and can be expressed as:

\[
d\nu = \frac{\bar{y}_{\nu} c}{\Lambda^2}(\ell\delta\phi)h_u + \frac{\bar{y}_{\nu} c}{\Lambda^2}(\ell\delta\chi)'h_u + \frac{\bar{y}_{\nu} c}{\Lambda^2}(\ell\delta\chi)'h_u + \frac{\bar{y}_{\nu} c}{\Lambda^2}(\ell(\chi\phi)_{3A})h_u
\]

\[
+ \frac{\bar{y}_{\nu} c}{\Lambda^2}(\ell(\chi\phi)^3_A)h_u + \frac{\bar{y}_{\nu} c}{\Lambda^2}(\ell(\chi\phi)^3_A)h_u + \frac{\bar{y}_{\nu} c}{\Lambda^2}(\ell(\chi\phi)^3_A)h_u + \frac{\bar{y}_{\nu} c}{\Lambda^2}(\ell(\chi\phi)^3_A)h_u,
\]

(40)

where $\delta\phi$ and $\delta\chi$ denote the shifted vacuum of the flavons $\phi$ and $\chi$, respectively. Since the shift $\delta\chi$ turns out to be proportional to the LO VEV and the symmetric triplet $(\phi\phi)_{3A} = 2(\phi^2_1 - \phi_2\phi_3, \phi^2_2 - \phi_1\phi_3, \phi^2_3 - \phi_1\phi_3)$ has a VEV in the same direction as $\langle\chi\rangle$, the contributions of the terms proportional to $\delta\chi$, $\bar{y}_{\nu 7}$ and $\bar{y}_{\nu 8}$ can be absorbed into the redefinition of the parameters $y_{\nu 2}$ and $y_{\nu 3}$. Moreover, the fourth term can be absorbed by a redefinition of $y_{\nu 1}$ whereas the operators with coefficients $\bar{y}_{\nu 0}$ and $\bar{y}_{\nu 8}$ give a vanishing contribution. Therefore the relevant correction to the Dirac mass matrix comes from the above terms proportional to $y_{\nu 1}$ and $\bar{y}_{\nu 9}$, giving:

\[
\delta m_D = \begin{pmatrix} \bar{y} & -\bar{y} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{v_\chi v_\phi}{\Lambda^2} v_u,
\]

(41)

where $\bar{y} = y_{\nu 1} \Lambda \delta \nu_{\phi 1}/(\nu_\chi v_\phi) - 2\bar{y}_{\nu 9}$. Then the NLO correction to the light neutrino mass matrix is given by:

\[
\delta m_{\nu} = -\delta m_D^T m_M^{-1} m_D - m_D^T m_M^{-1} \delta m_D =
\]

\[
= \bar{y} y_{\nu 1} \begin{pmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix} \frac{v_\phi^2 v_\chi}{\Lambda^2 M} \frac{v_u^2}{\Lambda \Lambda}.
\]

(42)
Diagonalizing the modified light neutrino mass matrix, we find that the first light neutrino is still massless and a non-zero mass only arises at NNLO. Combining the NLO corrections from the charged lepton and neutrino sectors, the parameters of the lepton mixing matrix are modified as:

\[
\sin \theta_{13} = \left| \frac{\bar{y} v_\chi}{\sqrt{2} y_{\nu_1} \Lambda} + \frac{1}{\sqrt{2}} (V_{12}^\ell - V_{13}^\ell) \varepsilon \right|
\]

\[
\sin^2 \theta_{12} = \frac{1}{3} - \frac{1}{3} [(V_{12}^\ell + V_{13}^\ell) \varepsilon + \text{h.c.}]
\]

\[
\sin^2 \theta_{23} = \frac{1}{2} + \frac{1}{4} \left[ \frac{\bar{y} v_\chi}{y_{\nu_1} \Lambda} + 2 V_{23}^\ell \varepsilon + \text{h.c.} \right].
\]

All the three mixing angles receive corrections of order \( \lambda^2_c \), the deviation of solar angle from its TB value is controlled by the flavor mixing in the charged lepton sector and the reactor angle \( \theta_{13} \) is expected to be of order \( \lambda^2_c \). Recently the T2K collaboration reported a relatively large value for \( \theta_{13} \) [103]. This result, combined with the world neutrino data, gives the 3\( \sigma \) ranges of \( \sin^2 \theta_{13} \) as [0.001, 0.044] and [0.005, 0.050], for the so-called ”old” and ”new” reactor neutrino flux, respectively [7]. Then values of \( \theta_{13} \sim \mathcal{O}(\lambda^2_c) \) lie within these ranges and our model cannot be ruled out, as is shown in Fig. 1. Precise measurement of \( \theta_{13} \) is an important test of this model: if a large \( \theta_{13} \) close to the present upper bound is confirmed by future data, then our construction would be ruled out. The same remark applies to a large class of recent discrete flavor symmetry models.

5 Phenomenological implications

In this section, we study the predictions for leptogenesis and lepton flavor violation both analytically and numerically.

5.1 Leptogenesis

It is interesting to estimate the order of magnitude of the right-handed neutrino masses. Recalling the light neutrino masses given in Eq.(11) and taking the couplings \( y_{\nu_1}, y_{\nu_2} \) and \( y_{\nu_3} \) to be of \( \mathcal{O}(1) \) and the VEVs \( v_\chi/\Lambda \) and \( v_\phi/\Lambda \) of \( \mathcal{O}(\lambda^2_c) \), we obtain:

\[
M \sim M' \sim 10^{12+13}\text{GeV}.
\]

It has been established that flavor effects may play an important role in leptogenesis [104]. If the right-handed neutrino masses are larger than \((1 + \tan^2 \beta) \times 10^{12} \text{GeV} \), with \( \tan \beta \equiv v_u/v_d \) being the ratio of the vacuum expectation values of the two Higgs doublets in the minimal supersymmetric standard model (MSSM), the three flavors \( e, \mu \) and \( \tau \) are indistinguishable and the so-called ”one-flavor” approximation can be safely used. For \((1 + \tan^2 \beta) \times 10^9 \text{GeV} \ll M(M') \ll (1 + \tan^2 \beta) \times 10^{12} \text{GeV} \), only the \( \tau \) Yukawa coupling is in equilibrium and should be treated separately in the Boltzmann equations, while the \( e \) and \( \mu \) flavors are still indistinguishable. On the other hand, for \((1 + \tan^2 \beta) \times 10^5 \text{GeV} \ll M(M') \ll (1 + \tan^2 \beta) \times 10^9 \text{GeV} \), the charged \( \mu \) and \( \tau \) Yukawa couplings are in thermal equilibrium and all flavors should be treated separately. For natural values of the parameters, e.g., \( \tan \beta < 30 \) and the neutrino Yukawa coupling \( y_{\nu_i} \) of \( \mathcal{O}(1) \), our model lies in the flavored regime where the \( \tau \) flavor should be considered separately from the others.

The implication of the \( A_4 \) group for leptogenesis has been discussed extensively [32,33,35,109,110]. In general, the leptonic CP asymmetries are predicted to be vanishing at LO, since
the combination $Y^{\nu}Y^{\nu\dagger}$, which is relevant for leptogenesis, is proportional to the unit matrix, where $Y^{\nu} = m_D/v_u$ is the neutrino Yukawa coupling matrix. Thus subleading operators, suppressed by additional powers of the cutoff $\Lambda$, are required to account for leptogenesis. It is well known that the leptonic CP asymmetry parameters $\epsilon_i^\alpha$ for the $i-$th heavy right-handed (s)neutrino $\nu_i^c (\bar{\nu}_i^c)$ decaying into $\alpha-$lepton ($\alpha = e, \mu, \tau$), provided the heavy neutrino masses are far from being almost degenerate, are given by \[111\]:

$$
\epsilon_i^\alpha = \frac{1}{8\pi} \sum_{j \neq i} \frac{\text{Im}[\hat{Y}^{\nu} \hat{Y}^{\nu\dagger}]_{ij} \hat{Y}^{\nu} \hat{Y}^{\nu\dagger}]}{(Y^{\nu}Y^{\nu\dagger})_{ii}} \left(\frac{M_i^2}{M_j^2}\right),
$$

with the loop function $g$ expressed as:

$$
g(x) = \sqrt{x} \left[\frac{2}{1-x} - \ln \left(\frac{1+x}{x}\right)\right],
$$

where the hat denotes the basis in which the mass matrices $M M$ and $M_\ell$ are diagonal with real and non-negative entries. On the other hand, for an almost degenerate heavy neutrino mass spectrum, leptogenesis can be naturally implemented through the so-called resonant leptogenesis mechanism \[112\]. In this case, the CP asymmetry generated by the decay of the $i-$th heavy right-handed (s)neutrino $\nu_i^c (\bar{\nu}_i^c)$ into a lepton flavor $\alpha$ is given by \[108\][112]:

$$
\epsilon_i^\alpha = -\frac{1}{8\pi} \sum_{j \neq i} \frac{M_i M_j \Delta M_{ij}^2}{(\Delta M_{ij}^2)^2 + M_i^2 \Gamma_j^2} \frac{\text{Im}[\hat{Y}^{\nu} \hat{Y}^{\nu\dagger}]_{ij} \hat{Y}^{\nu} \hat{Y}^{\nu\dagger}]}{(Y^{\nu}Y^{\nu\dagger})_{ii}},
$$

where $\Delta M_{ij}^2 = M_j^2 - M_i^2$ and $\Gamma_j = (\hat{Y}^{\nu} \hat{Y}^{\nu\dagger})_{jj} M_j / (8\pi)$ is the decay width of the $j-$th right-handed neutrino. In our model, the resonant leptogenesis mechanism is only applicable to the second and third heavy neutrinos, see Eq.(6).

Since we work in the hatted basis, we have to consider the diagonalization of right-handed neutrino mass matrix $m_M$ of Eq.(39); it is diagonalized as $\tilde{U}_R^T m_M \tilde{U}_R = \text{diag}(M_1, M_2, M_3)$, with the mass eigenvalues

$$
M_1 = M, \quad M_2 \simeq \left[1 + \frac{1}{2}(c + d)\varepsilon^2\right] M', \quad M_3 \simeq \left[1 - \frac{1}{2}(c + d)\varepsilon^2\right] M',
$$

where we take the parameters $c$ and $d$ to be real for simplicity, the complex case follows analogously. The matrix $\tilde{U}_R$ can be written as:

$$
\tilde{U}_R \simeq \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 + \frac{1}{4}(c - d)\varepsilon^2 & -i[1 - \frac{1}{4}(c - d)\varepsilon^2] \\ 9 & 1 - \frac{1}{4}(c - d)\varepsilon^2 & i[1 + \frac{1}{4}(c - d)\varepsilon^2] \end{pmatrix};
$$

consequently, in the hatted basis, the neutrino Yukawa coupling matrix is:

$$
\hat{Y}^{\nu} = \frac{1}{v_u} \tilde{U}_R^T (m_D + \delta m_D) U_\ell = \frac{1}{\sqrt{2}} \begin{pmatrix} y + z & -\sqrt{2} x & \sqrt{2} x \\ y + z & y + z & y + z \\ i(-y + z) & i(-y + z) & i(-y + z) \end{pmatrix} + \mathcal{O}(\varepsilon^2),
$$

where, for simplicity, we denoted $x \equiv y_{\alpha \beta} v_\phi / \Lambda, y \equiv y_{\alpha \beta} v_\chi / \Lambda$ and $z \equiv y_{\alpha \beta} v_\chi / \Lambda$. The leptogenesis is associated with both $\hat{Y}^{\nu}$ and the combination $\hat{Y}^{\nu} \hat{Y}^{\nu\dagger}$, which reads:

$$
\hat{Y}^{\nu} \hat{Y}^{\nu\dagger} = \begin{pmatrix} |x|^2 + |w|^2 + |x + w|^2 & 0 & 0 \\ 0 & 3/2|y + z|^2 & 3i/2(y + z)(y^* - z^*) \\ 0 & -3i/2(y - z)(y^* + z^*) & 3/2|y - z|^2 \end{pmatrix} + \mathcal{O}(\varepsilon^4).
$$
where \( w = \bar{y} v \chi v_0 / \Lambda^2 \) comes from the NLO correction \( \delta m_D \). It is remarkable that \((\hat{Y}^{\nu} \hat{Y}^{\nu^c})_{12} = \hat{Y}^{\nu} \hat{Y}^{\nu^c})_{13} = (\hat{Y}^{\nu} \hat{Y}^{\nu^c})_{21} = (\hat{Y}^{\nu} \hat{Y}^{\nu^c})_{31} \approx 0 \) even if the NLO corrections are taken into account. As a result, we have:

\[
\epsilon_1^\alpha \approx 0 . \tag{52}
\]

This implies that the heavy neutrino \( \nu^c_i \) decouples and the CP violating lepton asymmetry is produced in the out of equilibrium decays of the heavy neutrinos \( \nu^c_2 \) and \( \nu^c_3 \). Combining the expression in Eq. (17) with Eqs. (50) and (51), the flavor dependent CP asymmetry parameters are as follows:

\[
\begin{align*}
\epsilon_2^e & \approx \epsilon_2^\mu \approx \epsilon_2^\tau \approx \frac{1}{2} \frac{(c + d)\varepsilon^2}{2\pi (c + d)^2 \varepsilon^4 + \frac{9}{266\pi^2} |y - z|^4} \frac{|y|^2 - |z|^2}{y + z} \text{Im}(yz^*) \\
\epsilon_3^e & \approx \epsilon_3^\mu \approx \epsilon_3^\tau \approx \frac{1}{2} \frac{(c + d)\varepsilon^2}{2\pi (c + d)^2 \varepsilon^4 + \frac{9}{266\pi^2} |y + z|^4} \frac{|y|^2 - |z|^2}{y - z} \text{Im}(yz^*) . \tag{53}
\end{align*}
\]

It is interesting to note that all the parameters are proportional to the combination \((|y|^2 - |z|^2)\text{Im}(yz^*)\) so that the \( \epsilon_i^\alpha \)'s would be vanishing in the limit of \( |y| = |z| \) or \( \arg(y) = \arg(z) \). Besides the above \( \epsilon_i^\alpha \)'s, the baryon asymmetry depends on the so-called wash-out mass parameters \( \tilde{m}_i^\alpha \) associated with each lepton asymmetry:

\[
\tilde{m}_i^\alpha = \frac{|\hat{Y}_{i\alpha}^\nu|^2 v_u^2}{M} . \tag{54}
\]

Then we have:

\[
\begin{align*}
\tilde{m}_2^\alpha & \approx \tilde{m}_2^\mu \approx \tilde{m}_2^\tau \approx \frac{|y + z|^2 v_u^2}{2 + (c + d)\varepsilon^2 M}, \\
\tilde{m}_3^\alpha & \approx \tilde{m}_3^\mu \approx \tilde{m}_3^\tau \approx \frac{|y - z|^2 v_u^2}{2 - (c + d)\varepsilon^2 M}. \tag{55}
\end{align*}
\]

Once the values of the CP parameters \( \epsilon_i^\alpha \) is fixed, the final value of baryon asymmetry \( Y_B \) is governed by a set of flavor-dependent Boltzmann equations including the (inverse) decay and scattering process as well as the nonperturbative sphaleron interaction [113]. Here we will use simple analytical formulae to estimate baryon asymmetry [106,108,114]:

\[
Y_B \simeq \sum_{i=2}^{3} Y_{B_i} \simeq -\frac{10}{31 g_*} \sum_{i=2}^{3} \left[ \epsilon_i^{e+\mu} \eta \left( \frac{417}{589} \tilde{m}_{e+\mu} \right) + \epsilon_i^\tau \eta \left( \frac{390}{589} \tilde{m}^\tau \right) \right] , \tag{56}
\]

where \( \epsilon_i^{e+\mu} = \epsilon_i^e + \epsilon_i^\mu \), \( \tilde{m}_{e+\mu} = \tilde{m}_2^e + \tilde{m}_2^\mu + \tilde{m}_3^e + \tilde{m}_3^\mu \) and \( \tilde{m}^\tau = \tilde{m}_2^\tau + \tilde{m}_3^\tau \), \( g_* = 228.75 \) is the effective number of degrees of freedom in the MSSM. We note that the wash-out mass parameters are added up since the asymmetry generated in \( N_2 \) decays can be washed out by \( N_2 \) interactions and vice versa [114]. The wash-out factor \( \eta(\tilde{m}^\alpha) \) accounts for the washing out effect of the total baryon asymmetry due to the inverse decays and \( \Delta L = 1 \) scattering, its explicit expression depends on the magnitude of the various \( \tilde{m}^\alpha \). If all the flavors are in the strong wash-out regime, or some flavors are strongly washed out and others are either weakly or mildly washed out, it is given by [106,108]:

\[
\eta(\tilde{m}^\alpha) = \left[ \frac{8.25 \times 10^{-3} \text{eV}}{\tilde{m}^\alpha} + \left( \frac{\tilde{m}^\alpha}{0.2 \times 10^{-3} \text{eV}} \right)^{1.16} \right]^{-1} . \tag{57}
\]

While if all the flavors are in the weak wash-out regime, \( \eta(\tilde{m}^\alpha) \) is well approximated by [106]

\[
\eta(\tilde{m}^\alpha) = 1.5 \left( \frac{\tilde{m}}{3.3 \times 10^{-3} \text{eV}} \right) \left( \frac{\tilde{m}^\alpha}{3.3 \times 10^{-3} \text{eV}} \right) , \tag{58}
\]

where \( \tilde{m} = \sum_\alpha \tilde{m}^\alpha \).
5.2 Lepton flavor violation

We perform the analysis within the framework of the minimal supergravity (mSUGRA) scenario, which provides flavor universal boundary conditions at the scale of grand unification $M_G \simeq 2 \times 10^{16}$ GeV. It assumes that the slepton mass matrices are diagonal and universal in flavor and the trilinear couplings are proportional to the Yukawa couplings at the scale $M_G$. The branching ratio of the lepton flavor violation (LFV) radiative decay $\ell_i \rightarrow \ell_j + \gamma$ is approximately given by [115][116]:

$$\text{Br}(\ell_i \rightarrow \ell_j + \gamma) \simeq \frac{3 m_0^2 + A_0^2}{8 \pi^2} \left( \frac{m_{\ell_i}}{m_{\ell_j}} \right)^2 \left( \frac{3 m_0^2 + A_0^2}{8 \pi^2} \right)^2 \left| \tilde{Y}_{\ell_i} \tilde{Y}_{\ell_j} \right|^2 \tan^2 \beta,$$

(59)

where $G_F$ is the Fermi constant and $\alpha_{em}$ is the fine structure constant, $m_0$ is the common scalar mass, $A_0$ is the common trilinear parameter and the factor $L$ is given by:

$$L_{ij} = \ln \left( \frac{M_G}{M_i} \right) \delta_{ij}. \quad (60)$$

The parameter $m_S$ is the character mass scale of the SUSY particle and an excellent approximation to the exact result is given by [115]:

$$m_S^8 \simeq 0.5 m_0^2 m_{1/2}^2 (m_0^2 + 0.6 m_{1/2}^2)^2,$$

(61)

where $m_{1/2}$ is the universal gaugino mass. Recalling the neutrino Yukawa coupling matrix $\hat{Y}_\nu$ given in Eq. (50), we straightforwardly have:

$$\left( \hat{Y}_\nu \hat{L} \hat{Y}_\nu \right)_{12} = (\hat{Y}_\nu \hat{L} \hat{Y}_\nu)_{21} = (|y|^2 + |z|^2) \ln \left( \frac{M_G}{M_i} \right) + O(\varepsilon^3)$$

$$\left( \hat{Y}_\nu \hat{L} \hat{Y}_\nu \right)_{13} = (\hat{Y}_\nu \hat{L} \hat{Y}_\nu)_{31} = (|y|^2 + |z|^2) \ln \left( \frac{M_G}{M_i} \right) + O(\varepsilon^3)$$

$$\left( \hat{Y}_\nu \hat{L} \hat{Y}_\nu \right)_{23} = (\hat{Y}_\nu \hat{L} \hat{Y}_\nu)_{32} = -|x|^2 \ln \left( \frac{M_G}{M_i} \right) + (|y|^2 + |z|^2) \ln \left( \frac{M_G}{M_i} \right) + O(\varepsilon^3). \quad (62)$$

It is remarkable that the relation $(\hat{Y}_\nu \hat{L} \hat{Y}_\nu)_{21} = (\hat{Y}_\nu \hat{L} \hat{Y}_\nu)_{31}$ holds at LO, which is related to the $\mu - \tau$ symmetry of the light neutrino mass matrix. The same result has been obtained in previous $A_4$ and $S_4$ models [38][39]. As a consequence, the LFV branching ratios are as follows:

$$\text{Br}(\tau \rightarrow e\gamma) \simeq \text{Br}(\tau \rightarrow e\nu_e \bar{\nu}_e)\text{Br}(\mu \rightarrow e\gamma) \simeq 0.18\text{Br}(\mu \rightarrow e\gamma). \quad (63)$$

Given the latest experimental bound $\text{Br}(\mu \rightarrow e\gamma) < 2.4 \times 10^{-12}$ [121], the rate of $\tau \rightarrow e\gamma$ should be much below the present and future sensitivities [117]. Therefore, if the future experiments at SuperB factory will find $\text{Br}(\tau \rightarrow e\gamma) < 10^{-9}$, we would not constrain any further the present $A_4$ model. Otherwise the observation of $\tau \rightarrow e\gamma$ with branching ratio $\geq 10^{-9}$, combined with upper limit on $\text{Br}(\mu \rightarrow e\gamma)$, would rule out this model. Notice that, due to the $-|x|^2 \ln (M_G/M)$ term in the third equation of Eq. (62) and to the fact that the scale $M$ is generally smaller than the GUT scale $M_G$, the branching ratio $\text{Br}(\tau \rightarrow \mu\gamma)$ is not linearly related to $\text{Br}(\mu \rightarrow e\gamma)$ and should be smaller than $\text{Br}(\tau \rightarrow e\gamma)$.

Tripleton decays $\ell_i \rightarrow 3\ell_j$ and $\mu - e$ conversion in nuclei are generally related to the previous LFV radiative decays. In the mSUGRA scenario, the LFV processes are dominated by the contributions coming from the $\gamma$-penguin diagrams. As a consequence, the branching ratio for trilepton decays $\ell_i \rightarrow 3\ell_j$ is approximately given by [118]:

$$\text{Br}(\ell_i \rightarrow 3\ell_j) \simeq \frac{\alpha_{em}}{3\pi} \ln \left( \frac{m_{\ell_i}^2}{m_{\ell_j}^2} \right) - \frac{11}{4}\text{Br}(\ell_i \rightarrow \ell_j\gamma). \quad (64)$$

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Concretely, we have $Br(\mu \to 3e) \simeq 0.006 Br(\mu \to e\gamma)$, $Br(\tau \to 3e) \simeq 0.01 Br(\tau \to e\gamma)$ and $Br(\tau \to 3\mu) \simeq 0.002 Br(\tau \to \mu\gamma)$. For $\mu - e$ conversion in nuclei, the $\gamma$–penguin dominance implies [118]:

$$CR(\mu N \to eN) = \frac{\Gamma(\mu N \to eN)}{\Gamma_{cap}} \simeq \frac{\alpha_{em}^4 m_\mu^5 G_F^2}{12\pi^3 \Gamma_{cap}} Z Z_{\text{eff}} |F(q^2)|^2 Br(\mu \to e\gamma), \quad (65)$$

where $\Gamma_{cap}$ is the experimentally measured total muon capture rate, $Z$ is the proton number in the nucleus, $Z_{\text{eff}}$ is the effective atomic charge obtained by averaging the muon wavefunction over the nuclear density, and $F(q^2)$ denotes the nuclear form factor at momentum transfer $q$. For $^{48}_{22}$Ti, we have $Z_{\text{eff}} = 17.6$, $F(q^2) \simeq -m_\mu^2 \approx 0.54$ and $\Gamma_{cap} = 1.70422 \times 10^{-18}$ GeV [119]. In the case of $^{27}_{13}$Al, one finds $Z_{\text{eff}} = 11.5$, $F(q^2) \simeq -m_\mu^2 \approx 0.64$ and $\Gamma_{cap} = 4.64079 \times 10^{-19}$ GeV [120]. As a result, the $\mu - e$ conversion rates in $^{48}_{22}$Ti and $^{27}_{13}$Al are given by

$$CR(\mu^{48}_{22}\text{Ti} \to e^{48}_{22}\text{Ti}) \approx 0.0049 Br(\mu \to e\gamma), \quad CR(\mu^{27}_{13}\text{Al} \to e^{27}_{13}\text{Al}) \approx 0.0027 Br(\mu \to e\gamma). \quad (66)$$

The sensitivity of future $\mu - e$ conversion experiments in $^{48}_{22}$Ti and $^{27}_{13}$Al will be improved drastically to $10^{-18}$ and $10^{-16}$ respectively and this corresponds to the upper bounds on $Br(\mu \to e\gamma)$ of $2.04 \times 10^{-16}$ and $3.7 \times 10^{-14}$, smaller than the prospected sensitivity $Br(\mu \to e\gamma) < 10^{-13}$ in the MEG experiment [121]. Therefore the $\mu - e$ conversion experiments can further constrain the model if $\mu \to e\gamma$ is not observed by MEG.

### 5.3 Numerical results

In order to see more clearly the phenomenological implications of the model and to check the LO analytical results, we performed a numerical analysis. Since the parameters $x$, $y$ and $z$ of Eq. (50) are expected to be of order $\lambda^2$, they are treated as random complex numbers with absolute value between 0.01 and 0.1; the absolute value of the NLO parameter $w$ varies in the range of $[0.01 \lambda^2, 0.1 \lambda^2]$ and the corresponding phase between 0 and $2\pi$. The parameters $V_{12}^\ell$, $V_{13}^\ell$ and $V_{23}^\ell$ in the lepton mixing matrix $U_\ell$ of Eq. (37) and the parameters $c$ and $d$ in the NNLO Majorana neutrino mass matrix $m_M$ are taken to be complex numbers with absolute value in the interval $[1/3, 3]$, the heavy neutrino mass parameters $M$ and $M'$ are allowed to vary from $10^{11}$ GeV to $10^{14}$ GeV and the expansion parameter $\varepsilon$ is set to the indicative value 0.04. The correlations of $\sin^2 \theta_{12}$ and $\sin^2 \theta_{23}$ with respect to $\sin^2 \theta_{13}$ are shown in Fig. II. In these plots, we require that the corresponding $\Delta m^2_{\text{sol}}$ and $\Delta m^2_{\text{atm}}$ are within the $3\sigma$ ranges taken from Ref. [7]. We clearly see that most of the points fall in the region where $\sin^2 \theta_{12}$ and $\sin^2 \theta_{23}$ are in the $3\sigma$ interval; the values of $\theta_{13}$ are consistent with the global neutrino data analysis including the T2K results. However, if $\theta_{13}$ is measured to be near the present upper bound in future experiments, this model would be almost ruled out. Next we move to discuss the numerical results for leptogenesis parameters and LFV branching fractions. For definiteness, we shall present our results only for the mSUGRA point SPS3 [122]. The SPS3 point is in the co-annihilation region for the SUSY dark matter and the values of the universal soft SUSY breaking parameters are as follows:

$$m_0 = 90 \text{ GeV}, \quad m_{1/2} = 400 \text{ GeV}, \quad A_0 = 0 \text{ GeV}, \quad \tan \beta = 10. \quad (67)$$

We note that only the parameter $\tan \beta$ is relevant for leptogenesis. Our detailed numerical analysis shows that the observed baryon asymmetry can be obtained by requiring a moderate cancellation between $y$ and $z$ in the common factor $[(y)^2 - (z)^2] \text{Im}(yz^*)$ of the leptonic CP asymmetries given by Eq. (53). In the extreme case of $y = z$, the distribution of the predicted $Y_B$ is plotted in Fig. 2. The plot has been done by taking into account higher order
The scatter plot of $\sin^2 \theta_{12}$ and $\sin^2 \theta_{23}$ against $\sin^2 \theta_{13}$. The 1$\sigma$, 2$\sigma$ and 3$\sigma$ bounds on the mixing angles are taken from Ref. [7] with old reactor neutrino fluxes. It is interesting that the resulting baryon asymmetry $Y_B$ is rather small and a sizable part of points falls into the region where $Y_B$ is in the phenomenologically allowed interval of $[10^{-11}, 10^{-10}]$, while the predictions for mixing angles and LFV branching ratios are essentially the same as the general $y \neq z$ case. We note that the scenario $y = z$ could be realized by extending the $A_4$ flavor symmetry to $S_4$ and unifying the second and third right-handed neutrinos into an $S_4$ doublet. Considering the fact that $A_4$ is a normal subgroup of $S_4$ and the doublet representation of $S_4$ decomposes into $1'$ and $1''$ representations of $A_4$, the resulting model would be very similar to the present one.

The results for the LFV radiative decays $\mu \rightarrow e\gamma$, $\tau \rightarrow e\gamma$ and $\tau \rightarrow \mu\gamma$ are presented in Fig. 3 which shows the correlation between each two of the indicated branching ratios. In these plots, we require that the corresponding $\Delta m^2_{sol}$, $\Delta m^2_{atm}$ and the three lepton mixing

Figure 1: The scatter plot of $\sin^2 \theta_{12}$ and $\sin^2 \theta_{23}$ against $\sin^2 \theta_{13}$. The 1$\sigma$, 2$\sigma$ and 3$\sigma$ bounds on the mixing angles are taken from Ref. [7] with old reactor neutrino fluxes.

Figure 2: The distribution of the final baryon asymmetry $Y_B$ in the limit $y = z$. 

The results for the LFV radiative decays $\mu \rightarrow e\gamma$, $\tau \rightarrow e\gamma$ and $\tau \rightarrow \mu\gamma$ are presented in Fig. 3 which shows the correlation between each two of the indicated branching ratios. In these plots, we require that the corresponding $\Delta m^2_{sol}$, $\Delta m^2_{atm}$ and the three lepton mixing
angles lie in the 3σ ranges Ref. [7]. It is clear that \( \tau \to e\gamma \) and \( \tau \to \mu\gamma \) are far below from both the present and future experimental sensitivities in all the parameter space; although the \( Br(\mu \to e\gamma) \) is above the latest MEG upper bound in a relevant part of the parameter space, still a sizable amount of points are not excluded. The expected sensitivity of future experiments around \( 10^{-13} \) listed in PDG [117] will put severe constraints on the model. All LFV branching ratios are predicted to be rather small; this distinguishing feature of our construction is mainly due to their dependence on the fourth power of the neutrino Yukawa couplings which are of \( O(\lambda^2) \) at LO. In the left panel of Fig. 3, the densely populated straight band represents the ratio \( Br(\tau \to e\gamma)/Br(\mu \to e\gamma) \) around 0.18, in agreement with Eq.(63). The correlation between \( Br(\mu \to e\gamma) \) and \( Br(\tau \to e\gamma) \), which is shown in the right panel of Fig. 3, is somewhat involved and is due to the fact that they are non-linearly correlated, as it has been stated below Eq.(63). We have checked that the shape of the correlation follows exactly the LO prescription. Moreover, \( Br(\tau \to \mu\gamma) \) is always found to be smaller than \( Br(\tau \to e\gamma) \), in contrast with previous \( A_4 \) models where \( Br(\tau \to \mu\gamma) \) is approximately one order of magnitude larger than \( Br(\tau \to e\gamma) \) [38]. All these numerical results are consistent with our LO analysis. For the other LFV processes, we find that the trilepton decays \( \tau \to 3e \) and \( \tau \to 3\mu \) are predicted to be about six to eight orders of magnitude below the present and future sensitivities in all the allowed parameter space and \( \mu \to e \) conversion in \( ^{48}_{22}\)Ti and \( ^{17}_{13}\)Al are always below the present upper bound. However, the \( \mu \to e \) conversion processes are within the reach of next generation experiments in a considerable part of the parameter space; in particular, \( \mu \to e \) conversion in \( ^{48}_{22}\)Ti is expected to play an important role, due to the drastic improvement of the experimental sensitivity. The constraints imposed by future \( \mu \to e \) conversion experiments would be much stronger that those from the radiative decay \( \mu \to e\gamma \). If \( \mu \to e \) conversion in \( ^{48}_{22}\)Ti is not observed in the future, the present model would be almost ruled out.

Figure 3: Correlation between the LFV branching ratios \( Br(\mu \to e\gamma) \), \( Br(\tau \to e\gamma) \) and \( Br(\tau \to \mu\gamma) \). The horizontal dashed lines correspond to \( Br(\mu \to e\gamma) = 1.2 \times 10^{-11} \) and \( Br(\mu \to e\gamma) = 10^{-13} \), which are the present and future sensitivities on \( Br(\mu \to e\gamma) \) listed in PDG [117], respectively. The horizontal dotted line is the most recent experimental upper bound \( 2.4 \times 10^{-12} \) from the MEG collaboration [123].

\[^2\text{There are no plans to perform a new experimental searching for the } \mu \to 3e \text{ decay with higher precision.}\]
6 Conclusion and discussion

In this work we have constructed a new model for TB mixing with a global flavor symmetry $A_4 \times Z_4 \times Z_2$. All the right-handed matter fields are assigned to $A_4$ singlets: the three right-handed neutrinos $\nu^c_i$ transform as $1$, $1'$ and $1''$ and the right-handed charged leptons $e^c$, $\mu^c$ and $\tau^c$ are all invariant under $A_4$. The three generations of leptons form an $A_4$ triplet $3$ as usual. The easiest way to break the flavor symmetry is to introduce only one flavon in the neutrino sector but the resulting neutrino mass spectrum gives $m_2 = 0$ or $m_1 = m_3$ and then incompatible with the data. As a consequence, we should introduce at least two flavons in the neutrino sector to obtain a phenomenologically viable model. This statement generally holds if the right-handed neutrinos are assumed to be $A_4$ singlets. The model would appear more symmetric if we let the charged lepton fields $e^c$, $\mu^c$ and $\tau^c$ to transform as $1$, $1'$ and $1''$ as well. But in this case we would need to introduce a Froggatt-Nielsen $U(1)$ symmetry to generate the charged lepton mass hierarchies. It is remarkable that we can still produce TB mixing although the $A_4$ symmetry is broken completely at LO in both the neutrino and charged lepton sectors; the reason being an accidental $Z_2 \times Z_3$ symmetry preserved in the mass matrices. This is a special feature of present model.

The light neutrino mass spectrum is predicted to be normal hierarchy and the lightest neutrino mass is zero at LO although three right-handed neutrinos are introduced (generally massless neutrino are realized via the minimal see-saw mechanism). The effective mass of neutrinoless double-$\beta$ decay is predicted to be about 2.9 meV, which is much below the prospective sensitivities of future experiments. All three lepton mixing angles receive corrections of order $\lambda^2_{\ell}$ from the NLO corrections since no special dynamics is introduced to separate the corrections to $\theta_{13}$ from the other angles. Although the reactor angle $\theta_{13}$ is expected to be of order $\lambda^2_{\ell}$, we have shown that our model can accommodate values of $\theta_{13}$ consistent with the recent experimental results. Consequently the model cannot be ruled out by present data, although there is hint of relatively larger $\theta_{13}$ from the T2K collaboration. Precise measurement of $\theta_{13}$ is the most direct test of the model: if it is found to be near the present upper bound, a large class of discrete flavor symmetry models (including the present one) would be ruled out and the TB mixing may not be a good starting point for model building.

The predictions for leptogenesis and LFV branching ratios are analyzed in detail. We find that the first heavy right-handed neutrino does not contribute to the leptonic CP asymmetry even if NLO corrections are taken into account. The leptogenesis is realized via the so-called resonant leptogenesis of the second and third heavy neutrinos which are degenerate at LO. In order to account for the observed baryon asymmetry, moderate fine tuning among the neutrino Yukawa couplings is required. The LFV branching ratios are predicted to be rather small because they are proportional to the fourth power of the neutrino Yukawa couplings, of order $\lambda^2_{\ell}$ at LO. We find that the LFV processes $\mu \rightarrow e\gamma$ and $\mu - e$ conversion in $^{48}$Ti and $^{27}$Al are within the reach of next generation of experiments; in particular, $\mu - e$ conversion in $^{48}$Ti could impose extremely strong constraints on the model due to the considerable improvement of its sensitivity in near future, whereas the branching fractions of $\tau \rightarrow e\gamma$, $\tau \rightarrow \mu\gamma$ and trilepton decay $\ell_i \rightarrow 3\ell_j$ are far below the present and future sensitivities. The above theoretical predictions for neutrinoless double-$\beta$ decay and LFV processes are other important tests of the model.
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Appendix: The discrete group $A_4$

In this appendix, we briefly review some basic properties of the $A_4$ group. $A_4$ is the even permutation group of four objects, it has 12 elements. Geometrically, it is the symmetry group of a regular tetrahedron. The elements of $A_4$ can be generated by two generators $S$ and $T$ obeying the relation:

$$S^2 = T^3 = (ST)^3 = 1.$$  \hspace{1cm} (68)

The 12 elements of $A_4$ are obtained as 1, $S$, $T$, $ST$, $T^2$, $ST^2$, $STS$, $TST$, $T^2S$, $TST^2$ and $T^2ST$. Without loss of generality, we can choose

$$S = (14)(23), \quad T = (123),$$  \hspace{1cm} (69)

where the cycle (123) represents the permutation $(1, 2, 3, 4) \to (2, 3, 1, 4)$ and $(14)(23)$ means $(1, 2, 3, 4) \to (4, 3, 2, 1)$. The $A_4$ elements belong to 4 conjugate classes:

$$\begin{align*}
\mathcal{C}_1 : & \quad 1 \\
\mathcal{C}_2 : & \quad T = (123), \quad ST = (134), \quad TS = (142), \quad STS = (243) \\
\mathcal{C}_3 : & \quad T^2 = (132), \quad ST^2 = (124), \quad T^2S = (143), \quad ST^2S = (234) \\
\mathcal{C}_4 : & \quad S = (14)(23), \quad T^2ST = (12)(34), \quad TST^2 = (13)(24). \hspace{1cm} (70)
\end{align*}$$

There are 4 inequivalent irreducible representations of $A_4$: three singlet representation $1$, $1'$, $1''$ and one triplet representation $3$. For the one-dimensional representations, from the generator relation in Eq. (68), we can easily obtain that the representations are given by:

$$\begin{align*}
1 : & \quad S = 1, \quad T = 1 \\
1' : & \quad S = 1, \quad T = \omega^2 \\
1'' : & \quad S = 1, \quad T = \omega, \\
\end{align*}$$  \hspace{1cm} (71)

where $\omega = e^{2\pi i/3}$ is the cube root of unit. For the three-dimensional representation, in the basis where $T$ is diagonal, it is given by

$$3 : \quad S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}. \hspace{1cm} (72)$$

The multiplication rules between various irreducible representations are as follows:

$$\begin{align*}
1 \otimes R &= R, \quad 1' \otimes 1'' = 1, \quad 1' \otimes 1' = 1'', \quad 1'' \otimes 1'' = 1', \\
3 \otimes 3 &= 1 \oplus 1' \oplus 1'' \oplus 3_3 \oplus 3_A, \quad 3 \otimes 1' = 3, \quad 3 \otimes 1'' = 3, \\
\end{align*}$$  \hspace{1cm} (73)

where $R$ denotes any $A_4$ representation. From Eq. (71) and Eq. (72), we can straightforwardly obtain the decomposition of the product representations. For two $A_4$ triplets $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$, we have:

$$\begin{align*}
1 \equiv (\alpha\beta) &= \alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2 \\
1' \equiv (\alpha\beta)' &= \alpha_3\beta_3 + \alpha_1\beta_2 + \alpha_2\beta_1 \\
1'' \equiv (\alpha\beta)'' &= \alpha_2\beta_2 + \alpha_1\beta_3 + \alpha_3\beta_1 \\
3_A \equiv (\alpha\beta)_{3_A} &= (\alpha_2\beta_3 - \alpha_3\beta_2, \alpha_1\beta_2 - \alpha_2\beta_1, \alpha_3\beta_1 - \alpha_1\beta_3) \\
3_S \equiv (\alpha\beta)_{3_S} &= (2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2, 2\alpha_2\beta_2 - \alpha_1\beta_2 - \alpha_2\beta_1, 2\alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_3\beta_1 - \alpha_1\beta_3). \hspace{1cm} (74)
\end{align*}$$
Furthermore, if $\gamma$, $\gamma'$ and $\gamma''$ are $A_4$ singlets transforming as $1$, $1'$ and $1''$, then the products $\alpha \gamma$, $\alpha \gamma'$ and $\alpha \gamma''$ are triplets explicitly given by $(\alpha_1 \gamma, \alpha_2 \gamma, \alpha_3 \gamma)$ and $(\alpha_3 \gamma', \alpha_1 \gamma', \alpha_2 \gamma')$ and $(\alpha_2 \gamma'', \alpha_3 \gamma'', \alpha_1 \gamma'')$. It is interesting to note that if the $A_4$ flavor symmetry is broken down to the $Z_2$ subgroup generated by $S$ in the neutrino sector, then the neutrino mass matrix $m_\nu$ is invariant under the action of $S$, i.e., $S^T m_\nu S = m_\nu$; consequently $m_\nu$ has the general form:

$$
m_\nu = \begin{pmatrix}
A & B & C \\
B & D & A + C - D \\
C & A + C - D & B - C + D
\end{pmatrix}.
$$

(75)

It admits the eigenvector $(1,1,1)$ so that the trimaximal mixing can be reproduced naturally. With proper choice of flavon fields in the neutrino sector, e.g., in the absence of flavon transforming as $1'$ or $1''$, the $\mu - \tau$ symmetry arises accidentally. As a consequence, we have $B = C$ and the resulting neutrino mass matrix is exactly diagonalized by the TB mixing matrix.

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