Euclidean and super Euclidean algebras and Localizations of $U_q(\mathfrak{sl}(2))$ and $U_q(\mathfrak{osp}(1|2))$

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Abstract. In Ref. 1 we described homomorphisms of $U_q(\mathfrak{sl}(2))$ and $U(\mathfrak{iso}(2))$ into localizations of $U(\mathfrak{iso}(2))$ and $U_q(\mathfrak{sl}(2))$, respectively, with $U(\mathfrak{iso}(2))$ being the universal enveloping algebra of the Lie algebra of $SO(2) \times_s T^2$, the Euclidean group in the plane, with $T^2$ denoting translations of the plane and $\times_s$ semidirect product. In particular, we obtained explicit expressions for the generators of $U_q(\mathfrak{sl}(2))$ and $U(\mathfrak{iso}(2))$, respectively as irrational functions of $U(\mathfrak{iso}(2))$ and $U_q(\mathfrak{sl}(2))$. Herewith we further develop our results extending them to $U_q(\mathfrak{osp}(1|2))$ and $U(\widetilde{\mathfrak{iso}}(2))$ with $\widetilde{\mathfrak{iso}}(2)$ being the super Euclidean algebra.

1. Introduction
Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$ and $H$ a closed subgroup with subalgebra $\mathfrak{h}$. Let $\mathfrak{h}$ have an $\text{Ad}_G(H)$ invariant complement $V$ in $\mathfrak{g}$ i.e. a subspace $V \subset \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{h} \oplus V \quad \text{and} \quad \text{Ad}_G(H)V \subset V.$$ 

In this case we can form the semidirect product $G_0 = H \times_s V$. An interesting problem is to try to relate representations of $G$ to representations of $G_0$, a problem which was studied by Mackey [2]. More recent developments along these lines are given in [3] and [4]. Inspired by the Gell-Mann formula, which is defined below, we have attacked the problem at the level of enveloping algebras. For certain physically important cases we have obtained a rather complete description, and it has led us to useful information on the relationship between representations of the Poincaré group and representations of $SO(1,4)$ and $SO(2,3)$ [5], [6], [7]. We have also extended this method to $q$ deformations, in particular, to $U_q(\mathfrak{sl}(2))$ in [1] and also to $U_q(\mathfrak{so}(4))$ and other low dimensional cases in [8] and [9], and in this present article to the $q$ deformed superalgebra $U_q(\mathfrak{osp}(1|2))$. We have also established some general results for $SO(p,q)$ groups and their associated motion groups [10]. For related results on the $SL(n)$ and $SU(n)$ cases see [11].

Since we work at the enveloping algebra level, our results, although being specific to a particular algebra, are quite general. We try to express the generators of $G$ as irrational functions of the generators of $G_0$. Specifically, for $P_0 \in V$, $\Sigma^2 \in Z(U(\mathfrak{g}_0))$ and $\Delta \in Z(U(\mathfrak{h}))$, we define

$$B_0 = \frac{1}{\sqrt{\Sigma^2}}[\Delta, P_0] + P_0,$$

Dedicated to Professor Miloslav Havlíček on the occasion of his 75th birthday.
which we call the Gell-Mann Formula [12]. ([·, ·] denotes commutator and Δ is usually a Casimir operator of h or some function of it.) For certain H and V and for a particular choice of P₀, Δ and Σ₂, B₀ together with the generators of H may close under the Lie bracket in G₀ to generate a new Lie algebra isomorphic to the Lie algebra g of G.

On the other hand, we may think of P₀ in the Gell-Mann formula as the variable and view the Gell-Mann formula as an algebraic equation for it with coefficients from a noncommutative quotient ring consisting of fractions made out of elements from the enveloping algebra of g or from an algebraic extension thereof. It is possible, for certain cases, to solve the Gell-Mann formula for P₀ exactly and to obtain a homomorphic image of G₀ inside this noncommutative quotient ring. The precise mathematical description of the Gell-Mann formula and its inversion, i.e. the solution P₀ to the Gell-Mann formula viewed as an algebraic equation in P₀, involves the notion of noncommutative localization to which we now turn.

Note on notation: except for elements of the Cartan subalgebras for which we always use plain faced letters, quantities made out of elements of the enveloping algebra of its localization.

2. Localization

R is a ring with unity and S ≠ ∅ a multiplicatively closed subset of R such that 0 ∉ S, Iₓ ∈ S where Iₓ(= I) the identity in R. A nonzero element a in a ring R is said to be a left [resp. right] zero divisor if there exists a nonzero b ∈ R such that ab = 0 [resp. ba = 0]. A zero divisor is an element of R which is both a left and a right zero divisor.

Definition 2.1 A ring Q is said to be a left quotient ring of R with respect to S if there exists a ring homomorphism ϕ : R → Q such that the following conditions are satisfied:

1) ϕ(s) is a unit in Q for all s ∈ S (This means that a = ϕ(s) is both left and right invertible i.e. ∃c ∈ Q(resp. b ∈ Q) such that ca = IQ resp. ab = IQ);

2) every element of Q is in the form (ϕ(s))⁻¹ϕ(r), for some r ∈ R, s ∈ S;

3) ker ϕ = {r ∈ R : sr = 0, for some s ∈ S}.

The left (resp. right) quotient ring of R w.r.t. S, if it exists, is called the left (resp. right) localization of R at S and it is denoted by S⁻¹R (resp. RS⁻¹). If S = R, the localization S⁻¹R is the left skew field of fractions of R i.e. the left quotient field of R.

Note that if rs = 0, for some r ∈ R, s ∈ S, then s′r = 0, for some s′ ∈ S. This is because 0 = ϕ(rs) = ϕ(r)ϕ(s) and thus ϕ(r) = 0, since, by condition 1), ϕ(s) is a unit of Q. We need to multiply fractions like (s⁻¹a)(s′⁻¹b) so we must be able to move s′⁻¹ to the other side of a. This leads to the Ore condition: Ra ∩ Rs ≠ ∅ for a ∈ R and s ∈ S. It is a necessary and sufficient condition for the existence of localizations [13].

Integral (no zero divisors) Noetherian rings satisfy the Ore condition, so that we can construct localizations [14]. Examples of such rings include enveloping algebras of finite dimensional Lie algebras and, for semisimple ones, their q-deformations [15].

The following important Lemma tells us when a given representation of a ring R lifts to a representation of its localization.

Lemma 1.1 Suppose f : R → R₁ is a ring homomorphism and Q = S⁻¹R (RS⁻¹) is a left (right) quotient ring of R with respect to S. If f(s) is a unit in R₁ for every s ∈ S, then there exists a (unique) ring homomorphism g : Q → R₁ which extends f.

Proof: If Q is a left quotient ring of R and ϕ is the map in Definition 2.1, then for all r ∈ R, s ∈ S we define g((ϕ(s))⁻¹ϕ(r)) := (f(s))⁻¹f(r) and similarly for right quotient rings. A proof that such defined map g is well defined, unique, and is a ring homomorphism is given in [16].
3. The Algebras $U_q(\mathfrak{sl}(2))$ and $U(\mathfrak{iso}(2))$

The q-deformation $U_q(\mathfrak{sl}(2)) \simeq U_q(\mathfrak{so}(3, \mathbb{C}))$ of the simple Lie algebra $\mathfrak{sl}(2)$ is defined as the unital associative $\mathbb{Q}(q)$ algebra with generators $E$, $F$, $K$, $K^{-1}$ and relations [17]

\[ KK^{-1} = K^{-1}K = I, KEK^{-1} = q^2 E, KFK^{-1} = q^{-2} F \]

Equivalently, with $K = q^H$ we have generators $H$, $X^\pm$ with relations:

\[ [H, X^\pm] = \pm 2X^\pm, \quad (1) \]
\[ [X^+, X^-] = [H]_q^2 \quad (2) \]

where $[x]_q = (q^{x/2} - q^{-x/2})/(q^{1/2} - q^{-1/2})$ and $E = X^+$, $F = X^-$. The Casimir operator is

\[ \Delta_q = X^+ X^- + \left( \frac{H-1}{2} \right) [q^2] - \frac{1}{4} \cdot I = X^-X^+ + \left( \frac{H+1}{2} \right) [q^2] - \frac{1}{4} \cdot I \quad (3) \]

Define a real form $U_q(\mathfrak{so}(2,1))$ with generators $L_{ij}$ specified by $X^\pm = L_{13} \pm iL_{32}$, $L_{21} = \frac{i}{2} H$. The $L_{ij}$ are preserved under the following antilinear, anti-involution (star structure) [17]:

\[ \omega(H) = H, \quad \omega(X^\pm) = -X^\mp. \]

A basis for the Euclidean Lie algebra $\mathfrak{iso}(2)$ is $L_{12}$ and $P_i$ ($i = 1, 2$). They satisfy the following commutation relations:

\[ [L_{12}, P_2] = -P_1, \quad [L_{12}, P_1] = P_2, \quad [P_1, P_2] = 0. \]

Complexified translations generators are $P^\pm = -P_1 \pm iP_2$. We also define as above $H = -2iL_{21}$. We have:

\[ [H, P^\pm] = \pm 2P^\pm, \quad [P^+, P^-] = 0. \quad (4) \]

The enveloping algebra of $\mathfrak{iso}(2)$ is $U(\mathfrak{iso}(2))$. The center $Z(U(\mathfrak{iso}(2)))$ of $U(\mathfrak{iso}(2))$ is generated by

\[ Y^2 = P^+ P^- = P^- P^+. \quad (5) \]

4. Localizations of $U(\mathfrak{iso}(2))$ and $U_q(\mathfrak{sl}(2))$

In what follows we make use of identities for $q$ numbers such as

\[ [x + 1]_q - [x - 1]_q = \frac{[2(x)]_q}{[x]_q}, \quad (6) \]

Proofs of this and all other $q$ identities which we require, such as $[\frac{x}{y}]_q = [\frac{x}{y}]_{q^2}$, can be found in Ref. [18]. We shall need quite often the following result and it is readily established by using Eqs. (4) and the Maclaurin series formula: let $f$ be any analytic function, then

\[ P^\pm f \left( \left[ \frac{H}{2} \right]_q \right) = f \left( \left[ \frac{H \pm 2}{2} \right]_q \right) P^\pm. \quad (7) \]
The Proposition 4.1 satisfies the equation \(X\) replacing \(\text{and (2), of the generators of}\). We refer the reader to Ref. \([1]\) to see how Eq. (8) can be related to the Gell-Mann formula for the real form of \(U(\text{iso}(2))\) introduced above.

**Proposition 4.1** The \(X^\pm\) defined by Eqs. (8) together with \(H\) satisfy the relations, Eqs. (1) and (2), of the generators of \(U_q(\text{sl}(2))\). Furthermore, let \(\Delta_q\) be defined by Eq. (3) but with \(X^\pm\) replacing \(X^\pm\). Then \(\Delta_q = Y^2 - \frac{1}{4} \cdot I\).

For the proof of this proposition we refer the reader to Ref. \([1]\). We now solve Eqs. (8) for the \(P^\pm\) and we obtain the following:

**Proposition 4.2** Let \(P^\pm = (D_L^\pm)^{-1}X^\pm = X^\pm(D_R^\pm)^{-1}\) with \(D_L^\pm = \left(\pm \frac{1}{Y} \left[\frac{H}{2} + 1\right] \cdot q^2 + I\right)\) and \(D_R^\pm = \left(\pm \frac{1}{Y} \left[\frac{H}{2} + 1\right] \cdot q^2 + I\right)\). If \(Y\) is such that it commutes with all elements of \(U_q(\text{sl}(2))\) and satisfies the equation

\[
Y^2 = \Delta_q + \frac{1}{4} \cdot I,
\]

then \(P^\pm\) and \(H\) satisfy the commutation relations, Eqs. (4), of \(\text{iso}(2)\) and \(P^+ P^- = Y^2\).

**Proof:** It suffices to show \([P^+, P^-] = 0\) and that \(P^+ P^- = Y^2\), the rest of the Proposition being trivial. We let \(\{\cdot, \cdot\}\) denote anticommutator and we have:

\[
\{P^+, P^-\} = \left(\frac{1}{Y} \left[\frac{H - 1}{2} \cdot q^2 + I\right] + 1\right)^{-1} X^+ X^- \left(-\frac{1}{Y} \left[\frac{H - 1}{2} \cdot q^2 + I\right] + 1\right)^{-1} + \\
+ \left(-\frac{1}{Y} \left[\frac{H + 1}{2} \cdot q^2 + I\right] + 1\right)^{-1} X^- X^+ \left(\frac{1}{Y} \left[\frac{H + 1}{2} \cdot q^2 + I\right] + 1\right)^{-1} = \\
= \frac{Y^2}{Y^2 - \left[\frac{H - 1}{2} \cdot q^2\right]^2} X^+ X^- + \frac{Y^2}{Y^2 - \left[\frac{H + 1}{2} \cdot q^2\right]^2} X^- X^+ = \\
= Y^2 \left(\Delta_q + \frac{1}{4} \cdot I - \left[\frac{H + 1}{2} \cdot q^2\right]^2\right) \left(\Delta_q + \frac{1}{4} \cdot I - \left[\frac{H - 1}{2} \cdot q^2\right]^2\right) + Y^2 \left(\Delta_q + \frac{1}{4} \cdot I - \left[\frac{H + 1}{2} \cdot q^2\right]^2\right). 
\]

In obtaining the last line we have used the expressions given in Eq. (3) for the \(q\)-Casimir operator, \(\Delta_q\). Using \(Y^2 = \Delta_q + \frac{1}{4} \cdot I\) gives \(\{P^+, P^-\} = 2Y^2\). Similar calculations show \([P^+, P^-] = 0\). From these two results the desired conclusions follow.
5. The Algebras $U_q(\mathfrak{osp}(1|2))$ and $U(\tilde{\mathfrak{iso}}(2))$

The $\tilde{q}$-deformation $U_q(\mathfrak{osp}(1|2)) = \mathfrak{osp}(1|2)_{\tilde{q}}$ of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$ is defined as the unital associative $\mathbb{Q}(\tilde{q})$ algebra with generators $e, f, k, k^{-1}$ and relations [17]:

$$kk^{-1} = k^{-1}k = I, kek^{-1} = \tilde{q}e, kfk^{-1} = \tilde{q}^{-1}f, ef + fe = \frac{K - K^{-1}}{\tilde{q} - \tilde{q}^{-1}}.$$ 

The $\mathbb{Z}_2$ grading on $U_q(\mathfrak{osp}(1|2))$ is $d(e) = d(f) = 1, d(k) = 0$ where $d(x)$ is the parity of $x$. Let $k = \tilde{q}^H, e = \tilde{X}^+, f = \tilde{X}^-$ and we obtain generators $\tilde{H}, \tilde{X}^\pm$ with relations:

$$[\tilde{H}, \tilde{X}^\pm] = \pm \tilde{X}^\pm, \quad (1^\text{bis})$$

$$\{\tilde{X}^+, \tilde{X}^\pm\} = [\tilde{H}]_{\tilde{q}^2}. \quad (2^\text{bis})$$

The Casimir operator of $U_q(\mathfrak{osp}(1|2))$ is $\tilde{\Delta}_{\tilde{q}} = \tilde{S}_{\tilde{q}}^2 + 2 \cdot I$ with [19]

$$\tilde{S}_{\tilde{q}} = \frac{\tilde{q}^{1/2}k - \tilde{q}^{-1/2}k^{-1}}{\tilde{q} - \tilde{q}^{-1}} - (\tilde{q}^{1/2} + \tilde{q}^{-1/2})fe = \frac{[\tilde{H} + 1/2]}{\tilde{q}^2} - 2\tilde{q}\tilde{X}^-\tilde{X}^+ = -\frac{\tilde{q}^{-1/2}k - \tilde{q}^{1/2}k^{-1}}{\tilde{q} - \tilde{q}^{-1}} + (\tilde{q}^{1/2} + \tilde{q}^{-1/2})ef = -[\tilde{H} - 1/2]_{\tilde{q}^2} + 2\tilde{q}\tilde{X}^+\tilde{X}^- \quad (3^\text{bis})$$

It is straightforward to show that $\tilde{S}_{\tilde{q}}$ anticommutes with $\tilde{X}^+$ and $\tilde{X}^-$ and commutes with $\tilde{H}$. A star structure (or real form) is specified as follows. Let $\tilde{\omega}$ be such that $\tilde{\omega}(H) = H$, $\tilde{\omega}(\tilde{X}^\pm) = -\tilde{X}^\mp$. $\tilde{\omega}$ is, as in the $\mathfrak{s}(2)$ case, an antilinear, anti-involution.

A basis for the three dimensional “super Euclidean Lie algebra” $\tilde{\mathfrak{iso}}(2)$ is given by $\tilde{L}_{12}$ and $\tilde{P}_i$ ($i = 1, 2$) with commutation relations

$$[\tilde{H}, \tilde{P}^\pm] = \pm \tilde{P}^\pm, \quad \{\tilde{P}^+, \tilde{P}^-\} = 0. \quad (4^\text{bis})$$

The universal enveloping algebra of $\tilde{\mathfrak{iso}}(2)$ is $U(\tilde{\mathfrak{iso}}(2))$. Let $\tilde{Y}^2 \in Z(U(\tilde{\mathfrak{iso}}(2)))$ be given by

$$\tilde{Y}^2 = -i(-1)^B \tilde{P}^+ \tilde{P}^- = i(-1)^B \tilde{P}^- \tilde{P}^+. \quad (5^\text{bis})$$

6. Localizations of $U(\tilde{\mathfrak{iso}}(2))$ and $U_q(\mathfrak{osp}(1|2))$

We start by establishing a direct relationship between $U(\tilde{\mathfrak{iso}}(2))$ and $U(\mathfrak{iso}(2))$.

Proposition 6.1 Let $H = -2\tilde{H}$, $P^+ = -\tilde{P}^-$, $P^- = e^{-i\pi\tilde{H}}\tilde{P}^+$, then $H, P^\pm$ satisfy Eqs. (4) $\iff$ $\tilde{H}, \tilde{P}^\pm$ satisfy Eqs. (4^bis).

We now describe homomorphisms of $U_q(\mathfrak{osp}(1|2))$ and $U(\tilde{\mathfrak{iso}}(2))$ into localizations of $U(\mathfrak{iso}(2))$ and $U_q(\mathfrak{osp}(1|2))$, respectively, i.e. the analogs of Propositions 4.1 and 4.2.

Proposition 6.2 Let

$$\tilde{X}^\pm = \left(\frac{1}{\tilde{Y}}\sqrt{-(-1)^{\tilde{H}}\tilde{H} \pm \frac{1}{2}}_{\tilde{q}^2} + \sqrt{\pm I} \right) \left(\frac{\tilde{P}^\pm}{\sqrt{2} \tilde{q}^2}\right) = \left(\frac{\pm i}{\tilde{Y}}\sqrt{-(-1)^{\tilde{H}}\tilde{H} \pm \frac{1}{2}}_{\tilde{q}^2} + \sqrt{\pm I} \right) \quad (6^\text{bis})$$

where $I$ is the identity in $U(\tilde{\mathfrak{iso}}(2))$ and by $(-1)^{\tilde{H}}$ we mean $e^{-i\pi\tilde{H}}$. $\tilde{Y}$ is such that it commutes with all elements of $U(\tilde{\mathfrak{iso}}(2))$ and satisfies Eq. (5^bis). Then the $\tilde{X}^\pm$ defined by Eqs. (6^bis) together with $H$ satisfy the relations, Eqs. (1^bis) and (2^bis), of the generators of $U_q(\mathfrak{osp}(1|2))$. Furthermore, let $\tilde{S}_{\tilde{q}}$ be defined by Eq. (3^bis) but with $\tilde{X}^\pm$ replacing $\tilde{X}^\pm$, then $\tilde{S}_{\tilde{q}} = (-1)^{-B}\tilde{Y}^2$. 

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The proof of this proposition is straightforward and is very similar to the proof of Proposition 4.1. In obtaining the last term of Eqs. (8\textsuperscript{bis}) we used the equations $\sqrt{|\tilde{H} + \frac{1}{2}| q^2} \tilde{P}^\pm = \tilde{P}^\pm \sqrt{|\tilde{H} + \frac{1}{2}| q^2}$ and $(-1)^{\frac{H}{2}} \tilde{P}^\pm = \pm \tilde{P}^\pm (-1)^{\frac{H}{2}}$ which equations follow easily from Eqs. (4\textsuperscript{bis}). (Note that quantities like $e^{-i\pi \tilde{H}}$ and $(-1)^{\frac{H}{2}} := e^{-i\pi \tilde{H}}$ require clarification in precisely defining them at the abstract algebra level in terms of power series in $\tilde{H}$ e.g. we should work in an extension of $U(\text{iso}(2))$ [20]. However, in representations where $\tilde{H}$ acts diagonally the actions of $e^{-i\pi \tilde{H}}$ and $e^{-i\pi \tilde{H}}$ have precise meanings.)

**Proposition 6.3** Let $\tilde{P}^\pm = (\tilde{D}^\pm L)^{-1} \tilde{X}^\pm = \tilde{X}^\pm (\tilde{D}^\pm R)^{-1}$ with $\sqrt{|2\tilde{q}| \tilde{D}^\pm L} = \frac{1}{\sqrt{2}} \sqrt{|-\tilde{H} + \frac{1}{2}| q^2 + i I}$ and $\sqrt{|2\tilde{q}| \tilde{D}^\pm R} = \frac{1}{\sqrt{2}} \sqrt{|-\tilde{H} - \frac{1}{2}| q^2 + i I}$ where now $I$ is the identity in $U_\tilde{q}(\text{osp}(1|2))$. If $\tilde{Y}$ is such that it commutes with all elements of $U_\tilde{q}(\text{osp}(1|2))$ and satisfies the equation

$$\tilde{Y}^2 + (-1)^{\tilde{H}} \tilde{S}_q = 0,$$

then $\tilde{P}^\pm$ and $\tilde{H}$ satisfy the commutation relations (4\textsuperscript{bis}) of $U(\text{iso}(2))$ and $\tilde{P}^+ \tilde{P}^- = i(-1)^{-\tilde{H}} \tilde{Y}^2$.

**Proof.** We shall establish that $\{\tilde{P}^+, \tilde{P}^-\} = 0$, since the rest of the proof is straightforward. From the proposition we have:

$$\tilde{Y}^2 \tilde{X}^\pm = \frac{1}{\sqrt{2}\tilde{q}} \frac{1}{\sqrt{|-\tilde{H} - \frac{1}{2}| q^2 + i I}} \tilde{X}^\pm \tilde{Y}^2 - \frac{1}{\sqrt{2}\tilde{q}} \frac{1}{\sqrt{|-\tilde{H} + \frac{1}{2}| q^2 + i I}} \tilde{Y}^2 \tilde{X}^\pm =$$

$$= i \tilde{Y}^2 \frac{\tilde{X}^\pm \tilde{X}^\pm}{(-\tilde{H} + \frac{1}{2}) q^2 - \tilde{Y}^2} - \frac{\tilde{X}^\pm \tilde{X}^\pm}{(-\tilde{H} - \frac{1}{2}) q^2 - \tilde{Y}^2} =$$

$$= i \tilde{Y}^2 \left\{ \frac{\tilde{X}^\pm \tilde{X}^\pm}{(-\tilde{H} - \frac{1}{2}) q^2 + (-\tilde{H}) \tilde{S}_q} - \frac{\tilde{X}^\pm \tilde{X}^\pm}{(-\tilde{H} + \frac{1}{2}) q^2 + (-\tilde{H}) \tilde{S}_q} \right\} =$$

$$= i \tilde{Y}^2 \left\{ \frac{\tilde{X}^\pm \tilde{X}^\pm}{(-\tilde{H} - \frac{1}{2}) q^2 + \tilde{S}_q} - \frac{\tilde{X}^\pm \tilde{X}^\pm}{(-\tilde{H} + \frac{1}{2}) q^2 - \tilde{S}_q} \right\}$$

where in obtaining the second to last line we used Eq. (9\textsuperscript{bis}). Now use Eq. (3\textsuperscript{bis}) twice to obtain the desired result.
7. Representations

Using Propositions 6.1 and 6.2 it is possible to construct representations of $\tilde{U}_q(\mathfrak{osp}(1|2))$ out of representations of $U(\mathfrak{iso}(2))$ and $U(\mathfrak{iso}(2))$. For example, let us start with the positive mass representations of the Euclidean Lie algebra. They are characterized by a real number $\rho$ and an integer $\epsilon$ which is either 0 or $\frac{1}{2}$. They are described as follows [21]. The representation space is $\mathcal{D}^{j_0,\epsilon} = \sum_m \oplus \mathcal{H}_{(m,\epsilon)}$ where $\mathcal{H}_{(m,\epsilon)}$ is the one dimensional vector space $\mathbb{C}e_m$ with $m = n + \epsilon, n = 0, \pm 1, \pm 2, \ldots$ fixed $\epsilon = 0$ or $\frac{1}{2}$. The actions of the generators of $U(\mathfrak{iso}(2))$ on $\mathcal{D}^{j_0,\epsilon}$ are given by

$$d\pi^{j_0,\epsilon}(\mathbf{P}^\pm) e_m = - (i \rho) e_{m\pm 1}$$

and

$$d\pi^{j_0,\epsilon}(\mathbf{H}) e_m = 2m e_m.$$  

Using Proposition 6.1 we readily obtain the following representation of $U(\mathfrak{iso}(2))$ on $\mathcal{D}^{j_0,\epsilon}$:

$$d\tilde{\pi}^{j_0,\epsilon}(\mathbf{P}^\pm) e_m = \mp (i \rho) e^{-i \frac{m+1}{2} \epsilon (m-1) \pm (m-1)} e_{m\mp 1}$$

and

$$d\tilde{\pi}^{j_0,\epsilon}(\mathbf{H}) e_m = -m e_m.$$  

Now use Proposition 6.2 together with Lemma 1.1 to obtain the representation of $U_q(\mathfrak{osp}(1|2))$ on $\mathcal{D}^{j_0,\epsilon}$. We claim that the conditions of the Lemma are satisfied if zero is in the resolvent set of $d\tilde{\pi}^{j_0,\epsilon}(\mathbf{Y})$. It is easy to see that this is always the case for any nonzero $\rho$ and any $\epsilon$, since from Eq. (5bis) we have $d\tilde{\pi}^{j_0,\epsilon}(\mathbf{Y}) e_m = -i(-1)^j d\tilde{\pi}^{j_0,\epsilon}(\mathbf{H}) d\tilde{\pi}^{j_0,\epsilon}(\mathbf{P}^+) d\tilde{\pi}^{j_0,\epsilon}(\mathbf{P}^-) e_m$, and using Eqs. (12) and (13) we easily obtain

$$d\tilde{\pi}^{j_0,\epsilon}(\mathbf{Y})^2 = -i(-1)^{2\epsilon} \rho^2 I$$

where $I$ is the identity operator on $\mathcal{D}^{j_0,\epsilon}$. Hence, for $\rho \neq 0$, $d\tilde{\pi}^{j_0,\epsilon}(\mathbf{Y})^2$ is invertible and so also its square root

$$d\tilde{\pi}^{j_0,\epsilon}(\mathbf{Y}) = i(-1)^{\epsilon} \rho I.$$  

Clearly the image of $\mathbf{Y}$ and its square root are units in the localization of the algebraic extension of $U(\mathfrak{iso}(2))$ obtained by adjoining the square root of $\mathbf{Y}$ and from Eqs. (14) and (15) we see that the conditions of Lemma 1.1 are satisfied. Using (8bis) together with Eqs. (12), (13) and (15) we can explicitly construct the representation of $U_q(\mathfrak{osp}(1|2))$ on the above representation space. We obtain:

$$\hat{X}^+ e_m = \frac{1}{\sqrt{2}}\frac{(-1)^{m/2} \sqrt{m - 1/2}}{\sqrt[4]{i(-1)^{\epsilon} \rho}} - 1\} (-1)^m e_{m-1}$$

$$\hat{X}^- e_m = \frac{1}{\sqrt{2}}\frac{(-1)^{m/2} \sqrt{m + 1/2}}{\sqrt[4]{i(-1)^{\epsilon} \rho}} + i\} e_{m+1}$$

with the action of $\mathbf{H}$ on the representation space being given by Eq. (13). To our knowledge these representations of $U_q(\mathfrak{osp}(1|2))$ are new.

On the other hand, we may use Propositions 4.2 and 6.3 to construct representations of $U(\mathfrak{iso}(2))$ and $\tilde{U}(\mathfrak{iso}(2))$ out of representations of $U_q(\mathfrak{sl}(2))$ and $U_q(\mathfrak{osp}(1|2))$, respectively. In [9] we constructed representations of $U(\mathfrak{iso}(2))$ out of irreducible star representations of $U_q(\mathfrak{sl}(2))$ with the star structure given by the involution $\omega$ defined in Sec. 3, i.e. out of representations of $U_q(\mathfrak{so}(2,1))$. We considered cases for $q$ generic and also at roots of unity. Using a slightly
on the basic vector \( \tilde{U} \) with \( \tilde{\text{adjoint}} \) operation is given by\( \ell \) For \( \lambda \) parameter \( \ell \) us now show that this indeed so. of \( U \) series representations of \( U \) different version of Proposition 4.2, we showed, in that paper, that only for the principal of \( U \) iso(2), we expect similar results for the supersymmetric case. Let us now show that this indeed so.

Finite dimensional irreducible representations of \( U(\tilde{\mathfrak{osp}}(1|2)) \) are given in [22]. They are characterized by a real number \( \ell \) which takes positive integer or half integer values and a parameter \( \lambda \) which takes the values 0 and 1. We shall consider here only the case of integer \( \ell \), and such representations are described as follows [22]: Let \( V^{(\ell)} = l.s., \{ e_m^{(\ell)}(\lambda) \} m = \ell, \ell - 1, \ldots, -\ell \). For \( \ell \) integer the action of the generators of \( U(\mathfrak{osp}(1|2)) \) on the basic vectors \( e_m^{(\ell)}(\lambda) \) are:

\[
\hat{H} e_m^{(\ell)}(\lambda) = -m e_m^{(\ell)}(\lambda)
\]

\[
\hat{X}^+ e_m^{(\ell)}(\lambda) = (-1)^{\ell-m-1} \left( \frac{1}{2} \right)_{\tilde{q}} \{ \ell + m \} \tilde{q} \{ \ell - m + 1 \} \tilde{q}^\frac{1}{2} e_{m-1}^{(\ell)}(\lambda)
\]

\[
\hat{X}^- e_m^{(\ell)}(\lambda) = (-1)^{\ell-m-1} \left( \frac{1}{2} \right)_{\tilde{q}} \{ \ell - m \} \tilde{q} \{ \ell + m + 1 \} \tilde{q}^\frac{1}{2} e_{m+1}^{(\ell)}(\lambda)
\]

where \( \{ m \} \tilde{q} = \tilde{q}^{-m^2/2} (1)^m \tilde{q}^{m^2/2} \). These representations are grade star for \( \tilde{q} \in \mathbb{R} \), where the grade adjoint operation is given by

\[
\hat{H}^* = H, \quad \hat{X}^{\pm*} = \pm (-1)^\ell \hat{X}^{\mp},
\]

with \( \tilde{\ell} = \lambda + 1(\text{mod 2}) \). To see that these representations do not lead to representations of \( U(\mathfrak{so}(2)) \) on \( V^{(\ell)} \), we take for the action of \( \hat{Y} \) in the representation the positive square root of \( \hat{Y}^2 \) as determined by Eq. (9ass), and it is sufficient to show that the action of \( \hat{D}_R^- \) vanishes on the basic vector \( e_0^{(\ell)}(\lambda) \), since then it is not an invertible operator and thus the conditions of Lemma 1.1 are not satisfied. Using Eqs. (14), (15) and (16) together with the definitions of \( \hat{Y} \) and \( \hat{D}_R \) given in Proposition 6.3 we obtain:

\[
\hat{D}_R^- e_0^{(\ell)}(\lambda) = \frac{1}{\sqrt{|2q|}} Y \left( -i \sqrt{(1)\hat{H} - \frac{1}{2}\tilde{q}} + i\hat{Y} \right) e_0^{(\ell)}(\lambda) = \frac{1}{\sqrt{|2q|}} Y \left( \sqrt{(1) - \ell} \left( \ell + \frac{1}{2} \right) \tilde{q} - (1)^{\ell/2} \sqrt{\left( \ell + \frac{1}{2} \tilde{q} \right)} \right) e_0^{(\ell)}(\lambda) = 0.
\]

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