Propagator of the interacting Rarita-Schwinger field

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We obtain in analytical form the dressed propagator of the massive Rarita-Schwinger field taking into account all spin components and discuss shortly its properties.

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Introduction

The covariant description of the spin 3/2 particles is usually based on the Rarita-Schwinger formalism [1] where the main object is the spin-vector field $\Psi^\mu$. However in addition to spin 3/2 this field contains extra spin 1/2 components and this circumstance generates the main difficulties in its description [2, 3]. The problem has a long history, we mention here only relatively recent works [4, 5, 6, 7, 8, 9] (see for older references therein) which contain some discussion of the problem.

The most general lagrangian for free Rarita-Schwinger field has the following form (see e.g. [10, 11, 12]):

$$\mathcal{L} = \bar{\Psi}^\mu \Lambda^{\mu\nu} \Psi^\nu,$$

$$\Lambda^{\mu\nu} = (\hat{p} - M) g^{\mu\nu} + A(\gamma^\mu p^\nu + \gamma^\nu p^\mu) + \frac{1}{2}(3A^2 + 2A + 1)\gamma^\mu \hat{p}\gamma^\nu + M(3A^2 + 3A + 1)\gamma^\mu \gamma^\nu.$$ (1)

Here $M$ is the mass of Rarita-Schwinger field, $A$ is an arbitrary parameter, $p_\mu = i\partial_\mu$.

This lagrangian is invariant under the point transformation:

$$\Psi^\mu \to \Psi'^\mu = (g^{\mu\nu} + \alpha \gamma^\mu \gamma^\nu) \Psi^\nu, \quad A \to A' = \frac{A - 2\alpha}{1 + 4\alpha},$$

with parameter $\alpha \neq -1/4$. 

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The Lagrangian (1) leads to the following equations of motion:

\[ \Lambda^{\mu\nu} \Psi_{\nu} = 0. \]  

(2)

The free propagator of the Rarita-Schwinger field in a momentum space obeys the equation:

\[ \Lambda^{\mu\nu} G^{\nu\rho}_0 = g^{\mu\rho}. \]  

(3)

The expression for the free propagator \( G^0_{\mu\nu} \) is well known (see references in above), thus we do not present it here.

As concerned for the dressed propagator, its construction is a more complicated issue and its total expression is unknown up to now. Thus a practical use of \( G^{\mu\nu} \) (in particular for the case of \( \Delta(1232) \) production) needs some approximation in its description. The standard approximation [13, 14] consist in a dressing of the spin 3/2 components only while the rest components of \( G^{\mu\nu} \) can be neglected or considered as bare. Another way to take into account the spin 1/2 components is a numerical solution of the appearing system of equations [4, 9]. In Ref. [13] it was noticed that the spin 1/2 components are necessary to reproduce the experimental data on the \( \Delta(1232) \) production in Compton scattering. So the correct account of the extra spin 1/2 component in the \( \Psi^\mu \) field has also a practical meaning.

In this paper we derive an analytical expression for the interacting Rarita-Schwinger field’s propagator with accounting all spin components and shortly discuss its properties. It turned out that the spin 1/2 part of the dressed propagator has rather compact form, and a crucial point for its deriving is the choosing of a suitable basis.

**Dyson-Schwinger equation and its solution**

The Dyson-Schwinger equation for the propagator of the Rarita-Schwinger field has the following form

\[ G^{\mu\nu} = G^0_{\mu\nu} + G^{\alpha\beta} J^{\alpha\beta} G^0_{\beta\nu}. \]  

(4)

Here \( G^0_{\mu\nu} \) and \( G^{\mu\nu} \) are the free and full propagators respectively, \( J^{\mu\nu} \) is a self-energy contribution. The equation may be rewritten for inverse propagators as

\[ (G^{-1})^{\mu\nu} = (G^{-1})^0_{\mu\nu} - J^{\mu\nu}. \]  

(5)

If we consider the self-energy \( J^{\mu\nu} \) as a known value \(^1\), than the problem is reduced to reversing of relation (5). This procedure is not technically evident and it needs some preparations. First

\(^1\) That is the widely used in the resonance physics "rainbow approximation", see e.g. recent review [15].
of all it is useful to have a basis for both propagators and self-energy.

1. The most natural basis for the spin-tensor $S^{\mu\nu}(p)$ decomposition is the $\gamma$-matrix one:

$$S^{\mu\nu}(p) = g^{\mu\nu} \cdot s_1 + p^\mu p^\nu \cdot s_2 +$$
$$+ \hat{p} p^\mu p^\nu \cdot s_3 + \hat{p} g^{\mu\nu} \cdot s_4 + p^\mu \gamma^\nu \cdot s_5 + \gamma^\mu p^\nu \cdot s_6 +$$
$$+ \sigma^{\mu\nu} \cdot s_7 + \sigma^{\mu\lambda} p^\lambda p^\nu \cdot s_8 + \sigma^{\nu\lambda} p^\mu p^\rho \cdot s_9 + \gamma^\lambda \gamma^\mu \varepsilon^{\lambda\mu\rho\sigma} p^\rho \cdot s_{10}. \quad (6)$$

Here $S^{\mu\nu}$ is an arbitrary spin-tensor depending on the momentum $p$, $s_i(p^2)$ are the Lorentz invariant coefficients, and $\sigma^{\mu\nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu]$. Altogether there are ten independent components in the decomposition of $S^{\mu\nu}(p)$.

It is known that the $\gamma$-matrix decomposition is complete, the coefficients $s_i$ are free of kinematical singularities and constraints, and their calculation is rather simple. However this basis is inconvenient at multiplication and reversing of the spin-tensor $S^{\mu\nu}(p)$ because the basis elements are not orthogonal to each other. As a result the reversing of the spin-tensor $S^{\mu\nu}(p)$ leads to a system of 10 equations for the coefficients.

2. There is another basis used in consideration of the dressed propagator $G^{\mu\nu}$. It is constructed from the following set of operators\(^2\)\(^4\)\(^9\)\(^13\) $G^{\mu\nu}$. It is constructed from the following set of operators\(^2\)\(^4\)\(^9\)\(^13\)\(^16\)

$$\mathcal{P}_{3/2}^{\mu\nu} = g^{\mu\nu} - \frac{2}{3} \frac{p^\mu p^\nu}{p^2} - \frac{1}{3} \gamma^\mu \gamma^\nu + \frac{1}{3p^2} (\gamma^\mu p^\nu - \gamma^\nu p^\mu) \hat{p},$$
$$\mathcal{P}_{11}^{\mu\nu} = \frac{1}{3} \gamma^\mu \gamma^\nu - \frac{1}{3} \frac{p^\mu p^\nu}{p^2} - \frac{1}{3p^2} (\gamma^\mu p^\nu - \gamma^\nu p^\mu) \hat{p},$$
$$\mathcal{P}_{22}^{\mu\nu} = \frac{p^\mu p^\nu}{p^2},$$
$$\mathcal{P}_{21}^{\mu\nu} = \sqrt{\frac{3}{p^2}} \cdot \frac{1}{3p^2} (-p^\mu + \gamma^\mu \hat{p}) \hat{p} p^\nu,$$
$$\mathcal{P}_{12}^{\mu\nu} = \sqrt{\frac{3}{p^2}} \cdot \frac{1}{3p^2} p^\mu (-p^\nu + \gamma^\nu \hat{p}) \hat{p}. \quad (7)$$

Here $\mathcal{P}_{3/2}, \mathcal{P}_{11}, \mathcal{P}_{22}$ are the projection operators while $\mathcal{P}_{21}, \mathcal{P}_{12}$ are nilpotent. As for their physical meaning, it is clear that $\mathcal{P}_{3/2}$ corresponds to spin 3/2. The remaining operators should describe two spin 1/2 representations and transitions between them.

Let us rewrite the operators\(^4\)\(^9\)\(^13\) to make their properties more obvious:

$$\mathcal{P}_{3/2}^{\mu\nu} = g^{\mu\nu} - (\mathcal{P}_{11}^{\mu\nu}) - (\mathcal{P}_{22}^{\mu\nu}),$$

\(^2\) We changed here the normalization of $\mathcal{P}_{21}, \mathcal{P}_{12}$ for convenience.
\[(P_{11}^{1/2})^\mu_\nu = 3\pi^\mu \pi^\nu,\]
\[(P_{22}^{1/2})^\mu_\nu = \frac{p^\mu p^\nu}{p^2},\]
\[(P_{21}^{1/2})^\mu_\nu = \sqrt{\frac{3}{p^2}} \cdot \pi^\mu p^\nu,\]
\[(P_{12}^{1/2})^\mu_\nu = \sqrt{\frac{3}{p^2}} \cdot p^\mu \pi^\nu.\]  

(8)

Here we introduced the vector
\[\pi^\mu = \frac{1}{3p^2}(-p^\mu + \gamma^\mu \hat{p}),\]  

with the following properties:
\[(\pi p) = 0, \quad (\gamma \pi) = (\pi \gamma) = 1, \quad (\pi \pi) = \frac{1}{3}, \quad \hat{p} \pi^\mu = -\pi^\mu \hat{p}.\]  

(9)

The set of operators (7) can be used to decompose the considered spin-tensor as following

\[S^\mu_\nu(p) = (S_1 + S_2 \hat{p})(P^{3/2})^\mu_\nu + (S_3 + S_4 \hat{p})(P_{11}^{1/2})^\mu_\nu + (S_5 + S_6 \hat{p})(P_{22}^{1/2})^\mu_\nu + (S_7 + S_8 \hat{p})(P_{21}^{1/2})^\mu_\nu + (S_9 + S_{10} \hat{p})(P_{12}^{1/2})^\mu_\nu.\]  

(11)

Let us call this basis as \(\hat{p}\)-basis. It is more convenient at multiplication since the spin 3/2 components \(P^{3/2}\) have been separated from spin 1/2 ones. However, the spin 1/2 components as before are not orthogonal between themselves and we come to a system of 8 equations when inverting the (11). Another feature of decomposition (11) is existence of the poles \(1/p^2\) in different terms. So to avoid this unphysical singularity, we should impose some constraints on the coefficients at zero point.

3. Let us construct the basis which is the most convenient at multiplication of spin-tensors.

This basis is built from the operators (7) and the projection operators \(\Lambda^\pm\)
\[\Lambda^\pm = \frac{\sqrt{p^2} \pm \hat{p}}{2\sqrt{p^2}},\]  

where we assume \(p^2 > 0\). Ten elements of this basis look as
\[P_1 = \Lambda^+ P^{3/2}, \quad P_3 = \Lambda^+ P_{11}^{1/2}, \quad P_5 = \Lambda^+ P_{22}^{1/2}, \quad P_7 = \Lambda^+ P_{21}^{1/2}, \quad P_9 = \Lambda^+ P_{12}^{1/2},\]
\[P_2 = \Lambda^- P^{3/2}, \quad P_4 = \Lambda^- P_{11}^{1/2}, \quad P_6 = \Lambda^- P_{22}^{1/2}, \quad P_8 = \Lambda^- P_{21}^{1/2}, \quad P_{10} = \Lambda^- P_{12}^{1/2},\]  

(12)

where tensor indices are omitted. We will call (12) as the \(\Lambda\)-basis.
Decomposition of a spin-tensor in this basis has the following form:

\[ S_{\mu\nu}(p) = \sum_{i=1}^{10} \mathcal{P}_i^{\mu\nu} \overline{S}_i(p^2). \]  

(13)

The coefficients \( \overline{S}_i \) are calculated in analogy with \( \gamma \)-matrix decomposition. Besides, we found (with computer analytical computation) the matrix relating the \( \Lambda \)-basis with the \( \gamma \)-matrix basis and convinced ourselves that this matrix is not singular. So the elements of this basis (12) are independent. It is easy to connect the expansion coefficients (11) and (13) between themselves.

\[
\begin{align*}
\overline{S}_1 &= S_1 + \sqrt{p^2} S_2, \\
\overline{S}_2 &= S_1 - \sqrt{p^2} S_2, \quad \text{etc.}
\end{align*}
\]  

(14)

The \( \Lambda \)-basis has very simple multiplicative properties which are represented in the Table I. The first six basis elements are projection operators, while the remaining four elements are nilpotent. We are convinced by direct calculations that there are no other projection operators besides indicated.

Now we can return to the Dyson-Schwinger equation (5). Let us denote the inverse dressed propagator \((G^{-1})^{\mu\nu}\) and free one \((G_0^{-1})^{\mu\nu}\) by \(S^{\mu\nu}\) and \(S_0^{\mu\nu}\) respectively. Decomposing the \(S^{\mu\nu}\), \(S_0^{\mu\nu}\) and \(J^{\mu\nu}\) in \(\Lambda\)-basis according to (13) we reduce the equation (5) to set of equations for the
scalar coefficients

\[ \overline{S}_i(p^2) = \overline{S}_0(p^2) + \overline{J}_i(p^2) \]

So the values \( \overline{S}_i \) are defined by the bare propagator and the self-energy and may be considered as known.

The dressed propagator also can be found in this form

\[ G^{\mu\nu} = \sum_{i=1}^{10} \mathcal{P}^{\mu\nu}_i \cdot \overline{G}_i(p^2) \]  

(15)

The existing 6 projection operators take part in the decomposition of \( g^{\mu\nu} \):

\[ g^{\mu\nu} = \sum_{i=1}^{6} \mathcal{P}^{\mu\nu}_i. \]  

(16)

Now solving the equation

\[ G^{\mu\nu} S^{\nu\lambda} = g^{\mu\lambda} \]

in \( \Lambda \)-basis, we obtain a set of equations for the scalar coefficients \( \overline{G}_i \). The equations are easy to solve due to simple properties of the \( \Lambda \)-basis — see Table I.

The solution is:

\[ \overline{G}_1 = \frac{1}{\overline{S}_1}, \quad \overline{G}_2 = \frac{1}{\overline{S}_2}, \]

\[ \overline{G}_3 = \overline{S}_6 \overline{\Delta}_1, \quad \overline{G}_4 = \overline{S}_5 \overline{\Delta}_2, \quad \overline{G}_5 = \overline{S}_4 \overline{\Delta}_2, \quad \overline{G}_6 = \overline{S}_3 \overline{\Delta}_1, \]

\[ \overline{G}_7 = -\overline{S}_7 \overline{\Delta}_1, \quad \overline{G}_8 = -\overline{S}_8 \overline{\Delta}_2, \quad \overline{G}_9 = -\overline{S}_9 \overline{\Delta}_2, \quad \overline{G}_{10} = -\overline{S}_{10} \overline{\Delta}_1, \]  

(17)

where

\[ \overline{\Delta}_1 = \overline{S}_3 \overline{S}_6 - \overline{S}_7 \overline{S}_{10}, \quad \overline{\Delta}_2 = \overline{S}_4 \overline{S}_5 - \overline{S}_8 \overline{S}_9. \]  

(18)

The \( \overline{G}_1, \overline{G}_2 \) terms which describe the spin 3/2 have the usual resonance form and could be obtained from (11) as well. As for \( \overline{G}_3-\overline{G}_{10} \) coefficients which describe the spin 1/2 contributions, they have a more complicated structure. Let us consider the denominators of (17) in more details.

\[ \overline{\Delta}_1 = \overline{S}_3 \overline{S}_6 - \overline{S}_7 \overline{S}_{10} = (S_3 + \sqrt{p^2} S_4)(S_5 - \sqrt{p^2} S_6) - (S_7 + \sqrt{p^2} S_8)(S_9 - \sqrt{p^2} S_{10}), \]

\[ \overline{\Delta}_2 = \overline{\Delta}_1(\sqrt{p^2} \rightarrow -\sqrt{p^2}). \]  

(19)
The appearance of $\sqrt{p^2}$ factor is typical for fermions. To see it one can find the dressed propagator of the spin 1/2 Dirac particle with a help of the projection operators $\Lambda^\pm$.

$$\frac{1}{\hat{p} - m} \Rightarrow \frac{1}{\hat{p} - m - \Sigma(p)} = \Lambda^+ G^+ + \Lambda^- G^- \tag{20}$$

The dressed unrenormalized propagator is:

$$(G^+)^{-1} = -m - A(p^2) + \sqrt{p^2}(1 - B(p^2)),$$

$$(G^-)^{-1} = -m - A(p^2) - \sqrt{p^2}(1 - B(p^2)), \tag{21}$$

where $A, B$ are the self-energy components:

$$\Sigma(p) = A(p^2) + \hat{p}B(p^2).$$

Note that use of the projection operators $\Lambda^\pm$ in the case of Dirac particle is useful but not necessary technical step. After renormalization thus obtained propagator will coincide with expression given in any textbook. As for the apparent branch point $\sqrt{p^2}$, it is canceled in total expression (20). The same is true for the dressed Rarita-Schwinger propagator (17).

Note that the propagator’s denominators (21) are not similar by their structure to the Dirac case (21). The nearest analogy for the Rarita-Schwinger field propagator is the joint dressing of two Dirac fermions with account for their mutual transition. Probably this analogy will be useful in the renormalization.

**Conclusion**

Thus we obtained the simple analytical expression (17) for the interacting Rarita-Schwinger field propagator which accounts for all spin components. To derive it we introduced the spin-tensor basis (12) with very simple multiplicative properties. This basis is singular and it seems unavoidable (recall the vector field case). Nevertheless the singularity of a basis is not so big obstacle in its use. We did not suppose here any symmetry properties of the self-energy $J^{\mu\nu}$ restricting ourselves by general case. Of course the concrete form of interaction will lead to some symmetry of $J^{\mu\nu}$ and it will be important at renormalization.

Note some features of the obtained answer (17). First, if there is only one spin 1/2 pole it can not appear in $\Delta_1$ and $\Delta_2$ simultaneously. Second, it is well known that $\Psi^\mu$ field contains two spin 1/2 components (two irreducible representations) associated with $\mathcal{P}_{11}^{1/2}$ and $\mathcal{P}_{22}^{1/2}$ operators. However it turns out that these two representations are not completely independent since they have common denominators $\Delta_1, \Delta_2$. 

The obtained dressed propagator \([17]\) solves an algebraic part of the problem, the following step is renormalization. Note that the investigation of dressed propagator is the alternative for more conventional method based on equations of motion (see, i.e Ref. \([17]\) and references therein). The natural requirement for the renormalization is that the spin 1/2 components should remain unphysical after dressing. In other words the denominators \(\Delta_1, \Delta_2\) should not acquire of zero in the complex energy plane. However a problem of the renormalization needs a more careful consideration.

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