MASS FORMULAS AND THE BASIC LOCUS OF UNITARY SHIMURA VARIETIES

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Abstract. In this article we compute the mass associated to any unimodular lattice in a Hermitian space over an arbitrary CM field under a condition at 2. We study the geometry and arithmetic of the basic locus of the GU(r, s)-Shimura variety associated to an imaginary quadratic field modulo a good prime \( p > 2 \). We give explicit formulas for the numbers of irreducible and connected components of the basic locus, and of points of the zero-dimensional Ekedahl-Oort (EO) stratum, as well as of the irreducible components of basic EO strata when the signature is either \((1, n-1)\) or \((2, 2)\).

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1. Introduction

Let \( p \) be a prime number and \( k = \overline{\mathbb{F}}_p \) an algebraic closure of \( \mathbb{F}_p \). The classical Eichler-Deuring mass formula gives a weighted number of the isomorphism classes of the supersingular elliptic curves over \( k \):

\[
\sum_{[E] \text{ supersingular}} \frac{1}{|\text{Aut}(E)|} = \frac{p - 1}{24}.
\]

A higher dimensional generalization was obtained by Ekedahl [13] and Katsura-Oort [36] (based on the mass formula for its arithmetic counterpart by Hashimoto-Ibukiyama [29]) where supersingular elliptic curves are replaced by principally polarized superspecial abelian varieties. Recall that an abelian variety \( A \) over \( k \) is said be to superspecial (resp. supersingular) if it is isomorphic (resp. isogenous) to a product of supersingular elliptic curves. Let \( A_g \) denote the (coarse) moduli space over \( k \) of principally polarized abelian varieties of dimension \( g \). Extending earlier works of Ibukiyama, Katsura, and Oort [34, 36] in lower dimensions, Li and Oort [49] investigated the geometry of the supersingular locus \( S_g \) of \( A_g \), the closed subvariety parameterizing supersingular points. They determined the dimension and expressed the number of the irreducible components as a class number of the quaternion Hermitian lattices in question. For the Siegel moduli spaces \( A_{g,1,N} \) with a prime-to-\( p \) level-\( N \) structure, the second author [75] obtained an explicit formula for the number of irreducible components of the supersingular locus.

For Shimura varieties other than Siegel modular varieties, Bachmat and Goren [2] studied the supersingular locus of Hilbert-Blumenthal modular surfaces. The second author [73, 74] studied the supersingular locus of lower-dimensional Hilbert-Blumenthal moduli spaces. He also computed the number of superspecial points in an arbitrary PEL-type Shimura variety of type \( C \) modulo a good prime \( p \) [76], using Shimura’s mass formula for quaternionic unitary groups [69].

The goal of this paper is to extend the above results to the reduction modulo \( p \) of a unitary Shimura variety associated with a unimodular Hermitian lattice. Let \( E \) be an imaginary quadratic extension over \( \mathbb{Q} \). Let \( (V, \varphi) \) be a Hermitian space of dimension \( n \) over \( E \) with signature \( (r, s) \). Let \( \Lambda \subset V \) an \( O_E \)-lattice on which \( \varphi \) is unimodular. Let \( G = GU(\Lambda, \varphi) \) be the unitary similitude group of \( (\Lambda, \varphi) \). Let \( N \geq 3 \) be an integer with \( p \nmid N \). We consider the open compact subgroup

\[
K := G(\mathbb{Z}_p)K^p(N) \subset G(A_f)
\]

where \( G(\mathbb{Z}_p) \) is a hyperspecial subgroup of \( G(\mathbb{Q}_p) \) and \( K^p(N) \) is the kernel of the reduction homomorphism \( G(\overline{\mathbb{Z}}^p) \to G(\overline{\mathbb{Z}}^p/N\overline{\mathbb{Z}}^p) \). We write \( M_K \) for the associated moduli space, with good reduction at \( p \), classifying principally polarized abelian schemes \( A \) of relative dimension \( n \) with an \( O_E \)-action and a level \( N \) structure (Section 1.1). Let \( M_K := M_K \otimes \mathbb{K} \) denote the base extension of \( M_K \) to \( k \). To each \( k \)-point in \( M_K \) we can attach a \( p \)-divisible group \( A[p^\infty] \) over \( k \) equipped with additional structures. Its isogeny type defines the \textit{Newton stratification} on \( M_K \). There is a unique closed Newton stratum, called the basic locus and denoted by \( M_K^{\text{bas}} \). Furthermore, the isomorphism type of the \( p \)-torsion of \( A[p^\infty] \) with additional structures defines the \textit{Ekedahl-Oort (EO) stratification} [33]. There is a unique 0-dimensional EO stratum, denoted by \( M_K^{0} \). This is always non-empty and \( M_K^{0} \subset M_K^{\text{bas}} \). If \( p \) is inert in \( E \), or \( r = s \) and \( p \) is split in \( E \), the basic locus \( M_K^{\text{bas}} \) (resp. \( M_K^{0} \)) coincides with the supersingular locus (resp. superspecial locus) of \( M_K \).

The geometry of the basic locus \( M_K^{\text{bas}} \) has been studied by many people: Vollaard [82] and Vollaard-Wedhorn [83] for \((r, s) = (1, n - 1)\) with \( p \) inert in \( E \); Howard-Pappas [33] for \((r, s) = (2, 2)\) again with \( p \) inert; Fox [14] for \((r, s) = (2, 2)\) with \( p \) split; Imai and Fox [15] for \((r, s) = (2, n - 2)\) with \( p \) inert. The geometry of the EO stratification for arbitrary signature \((r, s)\) has been studied by Wooding in her thesis [84]. Further, Görtz-He [21] and Görtz-He-Nie [22, 23] developed more group-theoretic approaches to giving a concrete description of the basic locus of Shimura varieties.
Note that such a description of the basic locus has been used to compute intersection numbers of special cycles by Kudla and Rapoport [44, 45].

In [12], De Shalit and Goren have studied the basic locus $\mathcal{M}^\text{bas}_K$ intensively when $(r, s) = (1, 2)$ and $p$ is inert in $E$. They derived a formula relating the number of irreducible components of $\mathcal{M}^\text{bas}_K$ (and the cardinality of $\mathcal{M}^e_K$) to the second Chern class of the complex algebraic surface $\mathbf{M}_K(\mathbb{C})$. An explicit formula for the second Chern class of a connected component of $\mathbf{M}_K(\mathbb{C})$ was given by Holzapfel [32]; see Example 6.3.11.

In this paper, we study the basic locus $\mathcal{M}^\text{bas}_K$ for an arbitrary signature $(r, s)$ (including $rs = 0$) and any unramified prime $p > 2$. We give explicit formulas (Theorem 6.3.1) for

(i) the number of irreducible components of the basic locus $\mathcal{M}^\text{bas}_K$, and

(ii) the cardinality of the 0-dimensional stratum $\mathcal{M}^0_K$.

We also treat irreducible components of basic EO strata of possibly positive dimension when $(r, s) = (1, n - 1)$ and $p$ is inert in $E$. In this case, Vollaard and Wedhorn [83, Section 6.3] proved that for each odd integer $1 \leq t \leq n$ there exists a unique EO stratum $\mathcal{M}^{(t)}_K$ of dimension $\frac{1}{2}(t - 1)$ which are contained in $\mathcal{M}^\text{bas}_K$. Let $\overline{\mathcal{M}}^{(t)}_K$ denote the Zariski closure of $\mathcal{M}^{(t)}_K$ in $\mathcal{M}_K$. We give an explicit formula (Theorem 6.3.3) for

(iii) the number of irreducible components of $\overline{\mathcal{M}}^{(t)}_K$.

We remark that if $(r, s) = (1, n - 1)$ and $p$ is split in $E$, then $\mathcal{M}^\text{bas}_K = \mathcal{M}^e_K$.

Further we study geometrically connected components of the moduli space $\mathbf{M}_K$. We give an explicit formula for the number of connected components of the complex Shimura variety $\mathbf{M}_K \otimes \mathbb{C}$ (Theorem 3.3.2, Remark 4.1.3, and Example 6.3.9). There exists a smooth compactification of $\mathcal{M}_K$ to the second Chern class of the complex algebraic surface $\mathbf{M}_K(\mathbb{C})$. Let $\mathbf{M}^{(t)}$ denote the Zariski closure of $\mathcal{M}^{(t)}_K$ in $\mathcal{M}_K$. We give an explicit formula (Theorem 6.3.7) for

We give a summary of each section. In Section 2 we study the mass of a unimodular Hermitian lattice over a CM field $E$. We first compute the local densities of unimodular Hermitian lattices over local fields using the results of Gan and J.-K. Yu [17] and Cho [6, 7]. Then we derive an exact mass formula for a unimodular Hermitian lattice over a CM field $E$ under the assumption that 2 is unramified in the maximal totally real subfield of $E$ (Theorem 2.3.4). We note that an exact formula for the unimodular lattices defined by identity matrices was obtained earlier by Hashimoto and Koseki [30, Theorem 5.7] by different techniques.

From Section 3 we restrict ourselves to the case where $E$ is an imaginary quadratic extension over $\mathbb{Q}$. In Section 3.1 we describe the similitude factor of $\mathbf{G}(\mathbb{Z}_\ell)$ for each prime $\ell$. This result will be used in Section 4. Let $(r, s)$ be a pair of non-negative integers, and assume that $\mathbf{G}_\mathbb{R}$ is isomorphic to the real Lie group $\mathbf{GU}(r, s)$. When $rs > 0$, the number of connected components of the complex Shimura variety associated to $\mathbf{G}_\mathbb{Q}$ of level $\mathbf{G}(\mathbb{Z})$ can be expressed by a class number of the quotient torus $D := \mathbf{G}_\mathbb{Q}/\mathbf{G}^\text{der}_\mathbb{Q}$, where $\mathbf{G}^\text{der}_\mathbb{Q}$ denotes the derived subgroup of $\mathbf{G}_\mathbb{Q}$. The natural projection $\nu : \mathbf{G}_\mathbb{Q} \to D$ is identified with the product of the similitude and determinant characters. We compute the class number of $D$, using our description of the similitude factors and Kirschmer’s description of the determinants [38]. We thus obtain an explicit formula for the number of connected components of the complex Shimura variety in question.

From Section 4 we fix a prime $p > 2$ which is unramified in $E$. In this section we study the basic locus $\mathcal{M}^\text{bas}_K$ and the 0-dimensional EO stratum $\mathcal{M}^0_K$. We describe the Dieudonné module $M$ of the $p$-divisible group with additional structures attached to a point of $\mathcal{M}^e_K$. Further we compute the
group $J_b$ of automorphisms of the isocrystal $M[1/p]$ with additional structures, and also compute the stabilizer of $M$ in $J_b(\mathbb{Q}_p)$ (Proposition 4.4.2).

In Section 5 we study the affine Deligne-Lusztig variety $X_\mu(b)$ associated with a basic element $b$ of $G(L)$ and the minuscule coweight $\mu$ given by the Shimura datum. The set of its irreducible components $\text{Irr}(X_\mu(b))$ admits an action of $J_b(\mathbb{Q}_p)$. By the work of Xiao-X. Zhu [85], Hamacher-Viehmann [27], Nie [55], and Zhou-Y. Zhu [86]. The set of orbits $J_b(\mathbb{Q}_p)/\text{Irr}(X_\mu(b))$ is in natural bijection with the “Mirkovic-Vilonen basis” of a certain weight space of a representation of the dual group of $G_{\mathbb{Q}_p}$. We compute the dimension of this weight space explicitly, and give a formula for the cardinality of the set $J_b(\mathbb{Q}_p)/\text{Irr}(X_\mu(b))$. We also study an action of $J_b(\mathbb{Q}_p)$ on the set of connected components of $X_\mu(b)$.

In Section 6 we first study a mass formula for the inner from associated to the basic locus, and then we state and prove the main theorems. For each $k$-point $(A, \mu, \lambda, \eta)$ of $\mathcal{M}_K^{\text{bas}}$, we define a similitude group $I$ of auto-quasi-isogenies of the tuple $(A, \mu, \lambda)$. This group is an inner form of $G_\mathbb{Q}$ satisfying $I_{\mathbb{Q}_p} \simeq J_b$ and $I_{A, \mu} \simeq G_{\mathbb{Q}_p}$. The mass of $I$ with respect to an open compact subgroup $U$ of $I(A_f)$ is then defined as a weighted cardinality of the double coset space $I(\mathbb{Q})\backslash I(A_f)/U$, and denoted by $\text{Mass}(I, U)$ (Definition 6.2.3). Let $I_p$ be a maximal parahoric subgroup of $J_b(\mathbb{Q}_p)$ and regard $I_p G(\hat{\mathbb{P}})$ as a subgroup of $I(A_f)$. We give an explicit formula for $\text{Mass}(I, I_pG(\hat{\mathbb{P}}))$ (Theorem 6.2.8). The proof of this formula consists of two steps. First we derive an equality relating the mass of $I$ to a mass of its subgroup $I^1$ consisting of elements with trivial similitude factor. Here we use the description of the similitude factor of $G(\hat{\mathbb{P}})$ given in Section 3.1. Then we show that the mass of $I^1$ equals the mass of the unimodular lattice $\Lambda$ multiplied by the reciprocal of the volume of $I^1(\mathbb{Q}_p) \cap I_p$.

For each irreducible component $Z$ of $X_\mu(b)$, its stabilizer $I^Z_p$ in $J_b(\mathbb{Q}_p)$ is a parahoric subgroup with maximum volume by [31]. Moreover, the $p$-adic uniformization theorem of Rapoport-Zink [63] implies that

$$|\text{Irr}(\mathcal{M}_K^{\text{bas}})| = |J_b(\mathbb{Q}_p)\backslash \text{Irr}(X_\mu(b))| \cdot \text{Mass}(I, I_p G(\hat{\mathbb{P}})) \cdot |G(\hat{\mathbb{P}}) : K^p(N)|,$$

and thus we obtain an explicit formula. We also derive formulas for the numbers of points of $\mathcal{M}_K^{\text{bas}}$ and connected components of $\mathcal{M}_K^{\text{bas}}$. Further we illustrate these results in low-dimensional examples. In particular, we compute the number of irreducible components of $\text{EO}$ strata in $\mathcal{M}_K^{\text{bas}}$ when $(r, s) = (1, n - 1)$ or $(r, s) = (2, 2)$ (Example 6.3.13).

In Section 7 we give an application of the main theorems to the arithmetic of mod $p$ automorphic forms. In a letter to Tate [67], Serre proved that the systems of Hecke eigenvalues appearing in the space of mod $p$ modular forms are the same as those appearing in the space of algebraic modular forms on a quaternion algebra over $\mathbb{Q}$. This result can be regarded as a mod $p$ analogue of the Jacquet-Langlands correspondence. Serre’s result was generalized to the Siegel case by Ghitza [19], to the case of $\text{GU}(r, s)$ Shimura varieties with $p$ inert by Reduzzi [64], and to the Hodge type case by the authors [71]. This correspondence, combined with the formula for the cardinality of $\mathcal{M}_K^{\text{bas}}$, gives an explicit upper bound for the number of the systems of Hecke eigenvalues appearing in automorphic forms on $\mathcal{M}_K$ (Theorem 7.4.1).

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2. Mass formula for unimodular Hermitian lattices

In this section, we give an exact mass formula for a unimodular Hermitian lattice.

2.1. Unimodular Hermitian lattices over local fields.
2.1.1. Let $F$ be a non-Archimedean local field of characteristic zero, and let $O_F$ be its ring of integers. Let $\mathbb{F}_q$ be the residue field of $O_F$, and $q$ its cardinality. If $2 \mid q$, we assume that $F$ is an unramified finite extension of $\mathbb{Q}_2$. Let $(E, \cdot)$ be one of the following $F$-algebras with involution:

- $E$ is a quadratic field extension of $F$, $a \mapsto \bar{a}$ is the non-trivial automorphism of $E/F$;
- $E = F \oplus F$, $(a, b) = (b, a)$.

Denote by $O_E$ the maximal order in $E$, that is, $O_E$ is the ring of integers of $E$ and $O_E = O_F \oplus O_F$ in each case, respectively. We write $N = N_{E/F}$ for the norm map of $E/F$ given by $N(b) = b \cdot b$.

2.1.2. A Hermitian space over $E$ is a free $E$-module $V$ of finite rank equipped with a Hermitian form $\varphi : V \times V \to E$. By definition, the form $\varphi$ satisfies that

$$\varphi(x + y, z) = \varphi(x, z) + \varphi(y, z), \quad \varphi(ax, by) = a \bar{b} \cdot \varphi(x, y),$$

for $x, y, z \in V$ and $a, b \in E$. A Hermitian lattice over $O_E$ is an $O_E$-lattice $\Lambda$ (that is, a free $O_E$-module of finite rank) equipped with a form $\varphi : \Lambda \times \Lambda \to O_E$ satisfying relations (2.1) for $x, y, z \in \Lambda$ and $a, b \in O_E$. In this paper, we assume that $\varphi$ is non-degenerate on $V := \Lambda \otimes_{O_F} F$, namely, for any $x \in V$, the condition $\varphi(x, V) = 0$ implies that $x = 0$. A Hermitian lattice $\Lambda$ is called unimodular if $\Lambda$ coincides with its dual lattice $\Lambda^\vee = \Lambda^{\varphi} := \{x \in V \mid \varphi(x, \Lambda) \subset O_E\}$.

2.1.3. Let $E/F$ be a quadratic field extension. The scale $s(\Lambda)$ and the norm $n(\Lambda)$ of a Hermitian lattice $\Lambda$ are the $O_E$-ideals defined as

$$s(\Lambda) := \{\varphi(x, y) \mid x, y \in \Lambda\}, \quad n(\Lambda) := \sum_{x \in \Lambda} \varphi(x, x) \cdot O_E.$$

If $s(\Lambda) = n(\Lambda)$, the lattice $\Lambda$ is called normal. Otherwise the lattice $\Lambda$ is called subnormal. For two lattices $\Lambda$ and $\Lambda'$, one has

$$s(\Lambda \oplus \Lambda') = s(\Lambda) + s(\Lambda'), \quad n(\Lambda \oplus \Lambda') = n(\Lambda) + n(\Lambda').$$

If $\Lambda$ is unimodular, then $s(\Lambda) = O_E$.

We call a Hermitian space or lattice isotropic if it contains an element $x$ with $\varphi(x, x) = 0$. Otherwise we call it anisotropic.

Let $\{e_i\}$ be a basis of a Hermitian space $V$ over $E$. Let $d(V)$ denote the image of $\det(\varphi(e_i, e_j))$ by the natural projection $F^x \to F^x/N(E^x)$. Then $d(V)$ is independent of the choice of $\{e_i\}$. We call $d(V)$ the determinant of $V$. By [35, Theorem 3.1], the dimension and determinant are complete isomorphism invariants of Hermitian spaces. For a Hermitian lattice $\Lambda$ with basis $\{e_i\}$ over $O_E$, we call the matrix $(\varphi(e_i, e_j))$ the Gram matrix of $\{e_i\}$. We often write $\Lambda = (\varphi(e_i, e_j))$. If $\Lambda$ admits an orthogonal basis $\{e_i\}$ such that $\varphi(e_i, e_i) = a_i$, we also write $\Lambda = (a_1) \oplus \cdots \oplus (a_n)$. Let $d(\Lambda)$ denote the image of $\det(\varphi(e_i, e_j))$ by the natural projection $F^x \to F^x/N(O_F^x)$. Then $d(\Lambda)$ is independent of the choice of $\{e_i\}$. We call $d(\Lambda)$ the determinant of $\Lambda$. If $\Lambda$ is unimodular, then $d(\Lambda) \in O_F^x/N(O_F^x)$. If $E/F$ is unramified, any unimodular lattice may be written $\Lambda = (1) \oplus \cdots \oplus (1)$ by [35, Section 7]. If $E/F$ is ramified and $2 \mid q$, the rank and determinant are the complete isomorphism invariants of unimodular lattices by [35, Section 8]. If $E/F$ is ramified and $2 \mid q$, then the rank, norm, and determinant are complete isomorphism invariants of unimodular lattices by [35, Proposition 10.4].

2.1.4. Assume now that $E/F$ is a ramified field extension. Let $\mathcal{D} = \mathcal{D}_{E/F}$ denote the relative different of $E/F$, and let $d_{E/K} = N(\mathcal{D})$ denote the discriminant ideal of $E/F$. When $E/F$ is a ramified extension, we choose a uniformizer $\pi$ of $O_E$ as follows. If $2 \mid q$, one can choose a uniformizer $\varpi$ of $O_F$ such that $E = F(\sqrt{\varpi})$, and then the element $\pi := \sqrt{\varpi}$ is a uniformizer of $O_E$. We have $\mathcal{D} = (2\pi) = (\pi)$ and $d_{E/K} = (\varpi)$. 


Suppose that $E/F$ is a ramified extension with $2 \mid q$. Recall we assume $F/\mathbb{Q}_2$ is an unramified extension. Then we have the following two cases:

(RU) $E = F(\sqrt{1 + 2u})$ for some unit $u$ in $O_F$. In this case, the element $\pi := 1 + \sqrt{1 + 2u}$ is a uniformizer of $O_E$. We have $O_E = O_F[\pi] \cong O_F[X]/(X^2 - 2X - 2u)$ and hence $\mathcal{D} = (2\sqrt{1 + 2u}) = (2)$. It follows that $d_{E/K} = (4)$.

(RP) $E = F(\sqrt{2\delta})$ for some element $\delta \in O_F$ with $\delta \equiv 1 \pmod{2}$. The element $\pi := \sqrt{2\delta}$ is a uniformizer of $O_E$. We have $\mathcal{D} = (2\sqrt{2\delta})$ and $d_{E/F} = (8)$.

**Lemma 2.1.5.** Assume that $E/F$ is a ramified quadratic field extension. Let $\Lambda$ be a unimodular lattice of rank $n$ over $O_E$ and $H$ be the rank-two lattice defined by $H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

1. If $2 \mid q$, then
   \[
   \Lambda \simeq \begin{cases} 
   H^{(n-1)/2} \oplus ((-1)^{(n-1)/2} \cdot d(\Lambda)) & \text{if } n \text{ is odd}; \\
   H^{(n-2)/2} \oplus (1) \oplus ((-1)^{(n-2)/2} \cdot d(\Lambda)) & \text{if } n \text{ is even}. 
   \end{cases}
   \]

2. Assume $2 \mid q$ and $F$ is unramified over $\mathbb{Q}_2$. If $n$ is odd, then
   \[
   \Lambda \simeq H^{(n-1)/2} \oplus ((-1)^{(n-1)/2} \cdot d(\Lambda)).
   \]

If $n$ is even, then
\[
\Lambda \simeq \begin{cases} 
H^{(n-2)/2} \oplus (1) \oplus ((-1)^{(n-2)/2} \cdot d(\Lambda)) & \text{if } \Lambda \text{ is normal;} \\
H^{n/2} & \text{if } \Lambda \text{ is subnormal in RU;} \\
H^{(n-2)/2} \oplus \begin{pmatrix} 2\delta & 1 \\ 1 & 2b \end{pmatrix}, & b \in O_F \text{ if } \Lambda \text{ is subnormal in RP.}
\end{cases}
\]

**Proof.** If $2 \mid q$, then the assertion follows from [35, Section 8]. If $2 \mid q$, the assertion follows from [6, Theorem 2.10], or [35, Propositions 10.2 and 10.3].

**Lemma 2.1.6.** Suppose that $2 \mid q$ and $E/F$ is ramified in RP. Then there are two distinct isomorphism classes of unimodular subnormal lattices $L$ of rank two over $O_E$. Moreover, the following conditions are equivalent:

1. $L \simeq \begin{pmatrix} 2\delta & 1 \\ 1 & 2b \end{pmatrix}$ for an element $b \in O_F$ such that the equation $z^2 + z \equiv b \pmod{2}$ has a solution $z$ in $\mathbb{F}_q = O_F/2O_F$.
2. The quadratic form $\psi : L/\pi L \to \mathbb{F}_q$ induced by the reduction $\varphi(x, x) \pmod{2}$ is isotropic.
3. $d(L) = -1$ as elements of the quotient group $O_E^\times/\mathcal{N}(O_E^\times)$.
4. $L \simeq H$.

**Proof.** Let $L$ be a unimodular subnormal lattice of rank two. By [35, (9.1) and Proposition 9.1.a], there are inclusions $O_E \supseteq n(L) \supseteq n(H) = 2O_E$. This implies that $n(L) = 2O_E$ since $n(L)$ cannot be $\pi O_E$ by definition. Hence the isomorphism types of $L$ are classified only by the determinant, which takes value in $O_E^\times/\mathcal{N}(O_E^\times) \cong \mathbb{Z}/2\mathbb{Z}$. This implies the equivalence of (3) and (4).

As in Lemma 2.1.5, there is an isomorphism $L \simeq \begin{pmatrix} 2\delta & 1 \\ 1 & 2b \end{pmatrix}$ for some $b \in O_F$. The equivalence of (1) and (2) follows from [7, Remark 4.6]. We show the equivalence of (1) and (3). Suppose that there exist $x, y \in O_F$ such that $\mathcal{N}(x + \sqrt{2\delta}y) = -d(L)$, and equivalently $x^2 - 2\delta y^2 = 1 - 4\delta b$. Then $x \equiv 1 \pmod{2}$, and we can write $x = 1 + 2w$ for some $w \in O_F$. It follows that $4(w + w^2) - 2\delta y^2 = -4\delta b$. Since $\delta \equiv 1 \pmod{2}$, we have $y \equiv 0$ and $w + w^2 \equiv -b \equiv 0 \pmod{2}$. Conversely, suppose that

\*RU and RP are the abbreviations for ramified unit and ramified prime respectively.
$z + z^2 \equiv b \equiv -\delta b \pmod{2}$ for some $z \in \mathbb{F}_q$. By Hensel’s lemma, there is an element $w \in O_F$ such that $w + w^2 = -\delta b$. It follows that 

$$N(1 + 2w) = (1 + 2w)^2 = 1 + 4(w + w^2) = 1 - 4\delta b = -d(L).$$

\[ \boxdot \]

**Example 2.1.7.** Let $E/F$ be as in Lemma 2.1.6. The lattice $L = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ is a unimodular subnormal lattice. The above proof shows that $L$ satisfies those equivalent conditions if and only if the equation $z + z^2 = 1$ has a solution in $\mathbb{F}_q$, that is, $\mathbb{F}_q$ contains the quadratic extension of $\mathbb{F}_2$.

### 2.2. Unitary groups and local densities.

#### 2.2.1. Let $(V, \varphi)$ be a Hermitian space over $E$ as in Section 2.1 and let $\Lambda$ be a unimodular $O_E$-lattice in $V$. Let $G^1 = U(V, \varphi)$ be the unitary group associated to $(V, \varphi)$. By definition $G^1$ is the reductive group over $F$ whose group of $R$-values for any commutative $F$-algebra $R$ is given by

$$G^1(R) = \{ g \in (\text{End}_R(V \otimes_F R))^\times \mid \varphi(gx, gy) = \varphi(x, y), \ x, y \in V \otimes_F R \}. \tag{2.2}$$

As well-known, the group $G^1$ is connected. We define a naive integral model $G'$ over $O_F$ of $G^1$ by

$$G'(R) = \{ g \in (\text{End}_R(\Lambda \otimes_{O_F} R))^\times \mid \varphi(gx, gy) = \varphi(x, y), \ x, y \in \Lambda \otimes_{O_F} R \}$$

for any commutative $O_F$-algebra $R$. By the work of Gan and J.-K. Yu [17, Proposition 3.7], there exists a unique smooth affine group scheme $G'$ over $O_F$ such that $G^1 \otimes_{O_F} F = G^1$ and $G^1(R) = G'(R)$ for any étale $O_F$-algebra $R$.

#### 2.2.2. Let $\overline{G}$ denote the maximal reductive quotient of the special fiber $G^1 \otimes_{O_F} \mathbb{F}_q$. By Gan-Yu [17, Proposition 6.2.3 and Section 9], there is an isomorphism of groups over $\mathbb{F}_q$:

$$\overline{G} \simeq \begin{cases} U_n & \text{if } E/F \text{ is an unramified quadratic field extension;} \\ GL_n & \text{if } E = F \oplus F. \end{cases} \tag{2.3}$$

Here, $U_n$ denotes a unitary group in $n$ variables over $\mathbb{F}_q$, which is unique up to isomorphism.

Assume that $E/F$ is a ramified quadratic field extension. Then

$$\overline{G} \simeq \begin{cases} O_n & \text{if } 2 \nmid q, n \text{ is odd;} \\ O_n & \text{if } 2 \mid q, n \text{ is even}, \ d(\Lambda) = (-1)^{n/2}; \\ 2O_n & \text{if } 2 \nmid q, n \text{ is even}, \ d(\Lambda) \neq (-1)^{n/2}; \\ Sp_{n-1} \times \mathbb{Z}/2\mathbb{Z} & \text{if } 2 \mid q, n \text{ is odd, in RU;} \\ SO_n \times \mathbb{Z}/2\mathbb{Z} & \text{if } 2 \mid q, n \text{ is odd, in RP;} \\ Sp_{n-2} \times \mathbb{Z}/2\mathbb{Z} & \text{if } 2 \mid q, n \text{ is even}, \Lambda \text{ is normal in RU;} \\ SO_{n-1} \times \mathbb{Z}/2\mathbb{Z} & \text{if } 2 \mid q, n \text{ is even}, \Lambda \text{ is normal in RP;} \\ Sp_n & \text{if } 2 \mid q, n \text{ is even, } \Lambda \text{ is subnormal in RU;} \\ O_n & \text{if } 2 \mid q, n \text{ is even, } \Lambda \text{ is subnormal in RP, } d(\Lambda) = (-1)^{n/2}; \\ 2O_n & \text{if } 2 \mid q, n \text{ is even, } \Lambda \text{ is subnormal in RP, } d(\Lambda) \neq (-1)^{n/2}. \end{cases} \tag{2.4}$$

Here, $O_n$ denotes the split orthogonal group in $n$ variables, $2O_n$ denotes the quasi-split but non-split orthogonal group, $Sp_n$ denotes the symplectic group, $SO_n$ denotes the split special orthogonal group, and $\mathbb{Z}/2\mathbb{Z}$ denotes the constant group scheme of order 2. The determinant $d(\Lambda)$ takes value in the quotient group $O_E^\times / N(O_E^\times)$.

These isomorphisms were given by Gan-Yu [17, Proposition 6.2.3] when $2 \nmid q$, and by Cho [6, 17] when $2 \mid q$. Here we recall Cho’s construction. The function $x \mapsto \varphi(x, x) \pmod{2}$ induces a
quadratic form $\psi : \Lambda/\pi\Lambda \rightarrow \mathbb{F}_q$, which we regard as an additive polynomial. We define a lattice $B$ as the sublattice of $\Lambda$ such that $B/\pi\Lambda$ is the kernel of $\psi$. By [6, Remark 2.11], one has

$$B = \begin{cases} 
H^{(n-1)/2} \oplus (\pi)e_1 & \text{if } n \text{ is odd;} \\
H^{(n-2)/2} \oplus (\pi)e_1 \oplus O_Ee_2 & \text{if } n \text{ is even and } \Lambda \text{ is normal;} \\
H^{n/2} & \text{if } n \text{ is even and } \Lambda \text{ is subnormal.}
\end{cases}$$

In the RU case, the reduction $\varphi \pmod{\pi}$ induces an alternating and bilinear form on $B/\pi\Lambda$, and we define $Y$ as the sublattice of $\Lambda$ such that $Y/\pi\Lambda$ is the radical of this form. In the RP case, the reduction $2^{-1}\psi \pmod{2}$ induces a quadratic form on $B/\pi\Lambda$, and we define $Z$ as the sublattice of $\Lambda$ such that $Z/\pi\Lambda$ is the radical of this form. Then, for any étale local $O_F$-algebra $R$ and any element $\tilde{m} \in G_1(R)$ with reduction $m \in G_1(R \otimes_{O_F} \mathbb{F}_q)$, the action of $\tilde{m}$ on $\Lambda \otimes_{O_F} R$ preserves $B$, $Y$, and $Z$. Furthermore, there exists a unique morphism of algebraic groups over $\mathbb{F}_q$ $f : G_1 \otimes \mathbb{F}_q \rightarrow \{\text{Sp}(B/Y, \varphi \pmod{\pi}) \text{ in the RU case; } O(B/Z, 2^{-1}\psi \pmod{2})_{\text{red}} \text{ in the RP case,}\}$

such that the image $f(m) \in \text{GL}(B \otimes_{O_F} R/Y \otimes_{O_F} R)$ (resp. $\text{GL}(B \otimes_{O_F} R/Z \otimes_{O_F} R)$) is induced by the action of $\tilde{m}$ on $\Lambda \otimes_{O_F} R$. Here, $O(B/Z, 2^{-1}\psi \pmod{2})_{\text{red}}$ denotes the reduced subgroup scheme of $O(B/Z, 2^{-1}\psi \pmod{2})$. There is also a surjective morphism $g : G_1 \otimes \mathbb{F}_q \rightarrow (\mathbb{Z}/2\mathbb{Z})^\epsilon$, where $\epsilon = 0$ if $n$ is even and $\Lambda$ is subnormal, or $\epsilon = 1$ otherwise. The product $f \times g$ gives the projection from $G_1 \otimes \mathbb{F}_q$ to its maximal reductive quotient, by [6, Theorem 4.12] and [7, Theorem 4.11]. The isomorphism types of $\text{Sp}(B/Y, \varphi \pmod{\pi})$ and $O(B/Z, 2^{-1}\psi \pmod{2})_{\text{red}}$ are given in [6, Remark 4.7] and [7, Remark 4.6] respectively.

**Definition 2.2.3.** The local density of $\Lambda$ is the quantity

$$\beta_\Lambda := \lim_{N \to \infty} q^{-N \dim G_1} \cdot |G_1'(O_F/\varpi^N O_F)|,$$

where $\varpi$ is a uniformizer of $O_F$.

The limit stabilizes for $N$ sufficiently large. By the results of Gan-Yu [17, Theorem 7.3] and Cho ([6, Theorem 5.2], [7, Theorem 5.2], and [6, Remark 5.3]), the local density of a unimodular lattice $\Lambda$ can be computed via the formula

$$\beta_\Lambda = q^{-N \dim G_1} \cdot |G_1'(\mathbb{F}_q)|, \quad N = \begin{cases} 
n & \text{if } 2 \mid q \text{ and } \Lambda \text{ is subnormal;} \\
0 & \text{otherwise.}
\end{cases}$$

In Table 1 we see the dimensions and orders of finite classical groups appearing in $G_1'$. 

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Table 1. The dimensions and orders of finite classical groups

| $U$   | dim $U$ | $|U(F_q)|$                 |
|-------|---------|---------------------------|
| $GL_n$ | $n^2$   | $q^{\frac{n(n-1)}{2}} \cdot \prod_{i=1}^{n}(q^i - 1)$ |
| $U_n$  | $n^2$   | $q^{\frac{n(n-1)}{2}} \cdot \prod_{i=1}^{n}(q^i - (-1)^i)$ |
| $O_{2m+1}$ | $m(2m+1)$ | $q^{m^2} \cdot \prod_{i=1}^{m}(q^{2i} - 1)$ |
| $O_{2m}$ | $m(2m-1)$ | $2q^{m(m-1)}(q^m - 1) \cdot \prod_{i=1}^{m-1}(q^{2i} - 1)$ |
| $\text{Sp}_{2m}$ | $m(2m+1)$ | $q^{m^2} \cdot \prod_{i=1}^{m}(q^{2i} - 1)$ |
| $\text{SO}_{2m+1}$ | $m(2m+1)$ | $q^{m^2} \cdot \prod_{i=1}^{m}(q^{2i} - 1)$ |

A computation shows the following:

Lemma 2.2.4. The local density $\beta_\Lambda$ of a unimodular lattice $\Lambda$ is given by

$$\beta_\Lambda = \begin{cases} \prod_{i=1}^{\nu} (1 - (-1)^i \cdot q^{-i}) & \text{if } E/F \text{ is an unramified quadratic field extension;} \\ \prod_{i=1}^{\nu} (1 - q^{-i}) & \text{if } E = F \oplus E, \end{cases}$$

and if $E/F$ is a ramified quadratic field extension, then

$$\beta_\Lambda = \begin{cases} 2 \cdot \prod_{i=1}^{\nu} (1 - q^{-2i}) & \text{if } \nu \text{ is odd;} \\ 2(1 + q^{-n/2})^{-1} \cdot \prod_{i=1}^{\nu/2} (1 - q^{-2i}) & \text{if } \nu \text{ is even, } 2 \nmid q, \ d(\Lambda) = (-1)^{\nu/2}; \\ 2(1 - q^{-n/2})^{-1} \cdot \prod_{i=1}^{\nu/2} (1 - q^{-2i}) & \text{if } \nu \text{ is even, } 2 \nmid q, \ d(\Lambda) \neq (-1)^{\nu/2}; \\ 2q^n \cdot (1 + q^{-n/2})^{-1} \cdot \prod_{i=1}^{\nu/2} (1 - q^{-2i}) & \text{if } \nu \text{ is even, } 2 \mid q, \ \Lambda \text{ normal;} \\ 2q^n \cdot (1 - q^{-n/2})^{-1} \cdot \prod_{i=1}^{\nu/2} (1 - q^{-2i}) & \text{if } \nu \text{ is even, } 2 \mid q, \ \Lambda \text{ subnormal in } RU; \\ 2q^n \cdot (1 + q^{-n/2})^{-1} \cdot \prod_{i=1}^{\nu/2} (1 - q^{-2i}) & \text{if } \nu \text{ is even, } 2 \mid q, \ \Lambda \text{ subnormal in } RP, \ d(\Lambda) = (-1)^{\nu/2}; \\ 2q^n \cdot (1 - q^{-n/2})^{-1} \cdot \prod_{i=1}^{\nu/2} (1 - q^{-2i}) & \text{if } \nu \text{ is even, } 2 \mid q, \ \Lambda \text{ subnormal in } RP, \ d(\Lambda) \neq (-1)^{\nu/2}. \end{cases}$$

2.3. Exact mass formula for unimodular lattices over CM-fields.

2.3.1. Let $F$ be a totally real number field of degree $d$ over $\mathbb{Q}$, with ring of integers $O_F$ and ring of adeles $\mathbb{A}_F$. We assume that $2$ is unramified in $F$. Let $E$ be a totally imaginary quadratic extension of $F$, with ring of integers $O_E$. The non-trivial automorphism of $E$ over $F$ is denoted by $a \mapsto \bar{a}$.

For a finite place $v$ of $F$, $F_v$ denotes the corresponding completion of $F$, $O_{F_v}$ denotes the ring of integers of $F_v$, $\mathbb{F}_v$ denotes the residue field of $O_{F_v}$, and $q_v$ denotes the cardinality of $\mathbb{F}_v$. We write $E_v = E \otimes_F F_v$ and $O_{E_v} = O_E \otimes_{O_F} O_{F_v}$. If $E_v/F_v$ is a ramified quadratic field extension, let $\pi_v$ denote a uniformizer of $O_{E_v}$.

Let $(V, \varphi)$ be a Hermitian space over $E$, by which we mean a finite dimensional vector space $V$ over $E$ equipped with a non-degenerate Hermitian form $\varphi : V \times V \rightarrow E$, in the sense that $\varphi$ satisfies relations (2.1) for $x, y, z \in V$ and $a, b \in E$. Let $G^1 = U(V, \varphi)$ be the unitary group associated with $(V, \varphi)$, which is a connected reductive group over $F$.

2.3.2. Let $\Lambda$ be a lattice in $V$, by which we mean a finitely generated $O_E$-submodule of $V$ such that $\Lambda \otimes_{O_F} F \simeq V$ and $\varphi(\Lambda, \Lambda) \subset O_F$. For a finite place $v$ of $F$, we write $\Lambda_v = \Lambda \otimes_{O_{F_v}} O_{E_v}$. Let $K_v$ be the stabilizer of $\Lambda_v$ in $G^1(F_v)$, and let

$$K = G^1(F \otimes \mathbb{R}) \times \prod_{v \text{ finite}} K_v \subset G^1(\mathbb{A}_F).$$
The set of isomorphism classes of the genus of Λ is indexed by \( \Sigma := G^1(F) \backslash G^1(\Lambda_F)/K \). For each class \([g]\) ∈ \( \Sigma \), represented by \( g \in G^1(\Lambda_F) \), let \( \Gamma_g := G^1(F \otimes \mathbb{R}) \cap gKg^{-1} \). Then the quotient \( \Gamma_g \backslash G^1(F \otimes \mathbb{R}) \) is of finite volume with respect to any Haar measure on \( G^1(F \otimes \mathbb{R}) \) [17, Section 10.4].

Let \( \mu_{c,R} \) be the Haar measure on the compact form of the real Lie group \( G^1(F \otimes \mathbb{R}) \) which gives the group volume 1. We then transfer \( \mu_{c,R} \) to \( G^1(F \otimes \mathbb{R}) \) and obtain a Haar measure on \( G^1(F \otimes \mathbb{R}) \), which is denoted again by \( \mu_{c,R} \).

**Definition 2.3.3.** The mass of \( \Lambda \) is defined by

\[
\text{Mass}(\Lambda) := \sum_{[g] \in \Sigma} \int_{\Gamma_g \backslash G^1(F \otimes \mathbb{R})} \mu_{c,R}.
\]

Let \( \chi = \chi_E \) be the Dirichlet character corresponding to \( E/F \). It satisfies that \( \chi(v) = 1, -1, \) or 0 according as \( v \) is split, inert, or ramified in \( E \). We use the convention that \( \chi^j = 1 \) if \( j \) is even, and \( \chi^j = \chi \) if \( j \) is odd. Let \( L_F(s, \chi^j) \) be the \( L \)-series over \( F \).

**Theorem 2.3.4.** Let \( \Lambda \) be a unimodular lattice in \( V \). Then we have that

\[
\text{Mass}(\Lambda) = \frac{(-1)^s \cdot 2^{nd+w}}{2^{nd+w}} \cdot \prod_{j=1}^{n} L_F(1 - j, \chi_E^j) \cdot \prod_{v \mid d_{E/F}} \kappa_v
\]

where \( n := \dim_E V \), \( d := [F : \mathbb{Q}] \), \( d_{E/F} \) denotes the discriminant of \( E/F \), \( w \) denotes the number of places \( v \) with \( v \mid d_{E/F} \), and the quantities \( s \) and \( \kappa_v \) are given by

\[
s = \begin{cases} 
0 & \text{if } n \text{ is odd;} \\
\frac{n}{2} & \text{if } n \text{ is even,}
\end{cases}
\]

\[
\kappa_v = \begin{cases} 
1 & \text{if } n \text{ is odd;} \\
q_v^{n/2} + 1 & \text{if } n \text{ is even, } v \nmid 2, \ d(\Lambda_v) = (-1)^{n/2}; \\
q_v^{n/2} - 1 & \text{if } n \text{ is even, } v \mid 2, \ \Lambda_v \text{ normal in } RU; \\
q_v^{n/2} \cdot (q_v^{n/2} - 1) & \text{if } n \text{ is even, } v \mid 2, \ \Lambda_v \text{ normal in } RP; \\
2 & \text{if } n \text{ is even, } v \mid 2, \ \Lambda_v \text{ subnormal in } RU; \\
q_v^{n/2} + 1 & \text{if } n \text{ is even, } v \mid 2, \ \Lambda_v \text{ subnormal in } RP, \ d(\Lambda_v) = (-1)^{n/2}; \\
q_v^{n/2} - 1 & \text{if } n \text{ is even, } v \mid 2, \ \Lambda_v \text{ subnormal in } RP, \ d(\Lambda_v) = (1)^{n/2}.
\end{cases}
\]

Here, the determinant \( d(\Lambda_v) \) takes value in the quotient group \( O_{E_v}^\times / \mathcal{N}_{E_v/F_v}(O_{E_v}^\times) \).

**Proof.** The mass formula of Gan-Yu [17, Theorem 10.20] shows that

\[
\text{Mass}(\Lambda) = \left( \prod_{i=1}^{n} \frac{(2\pi)^i}{(i-1)!} \right)^{-d} \cdot |O_F/d_{E/F}|^{\frac{n(n+1)}{4}} \cdot \tau(G^1) \cdot d_F^2 \cdot \prod_{v: \text{finite}} \beta_v
\]

where \( \beta_v = \beta_{\Lambda_v} \) denotes the local density of \( \Lambda_v \), \( \tau(G^1) \) denotes the Tamagawa number of \( G^1 \), and \( d_F \in \mathbb{Z} \) denotes the absolute discriminant of \( F \). By Kottwitz [40] and Ono [56, p. 128], we have \( \tau(G^1) = 2 \).

We describe the product \( \prod_{v: \text{finite}} \beta_v \) by the \( L \)-series. Let \( L_{F_v}(j, \chi^j) \) be the local factor of the \( L \)-series at a finite place \( v \) of \( F \):

\[
L_{F_v}(j, \chi^j) = \left( 1 - \frac{\chi^j(v)}{q_v^j} \right)^{-1}.
\]
If \( v \) is unramified in \( E \), then Lemma 2.2.4 implies that \( \beta_v^{-1} = \prod_{j=1}^{n} L_{F_v}(j, \chi^j) \). For a place \( v \) which is ramified in \( E \), we define a rational number \( \lambda_v \) by the relation

\[
(2.7) \quad \beta_v^{-1} = \frac{1}{2} \prod_{j=1}^{n} L_{F_v}(j, \chi^j) \cdot \lambda_v.
\]

Then, Lemma 2.2.4 implies that

\[
(2.8) \quad \lambda_v = \begin{cases} 
1 & \text{if } n \text{ is odd;} \\
1 + q_v^{-n/2} & \text{if } n \text{ is even, } v \nmid 2, d(\Lambda_v) = (-1)^{n/2}; \\
1 - q_v^{-n/2} & \text{if } n \text{ is even, } v \mid 2, d(\Lambda_v) \neq (-1)^{n/2}; \\
2 \cdot q_v^{-n} & \text{if } n \text{ is even, } v \mid 2, \Lambda_v \text{ subnormal in RU}; \\
q_v^{-n} \cdot (1 + q_v^{-n/2}) & \text{if } n \text{ is even, } v \mid 2, \Lambda_v \text{ subnormal in RP, } d(\Lambda_v) = (-1)^{n/2}; \\
q_v^{-n} \cdot (1 - q_v^{-n/2}) & \text{if } n \text{ is even, } v \mid 2, \Lambda_v \text{ subnormal in RP, } d(\Lambda_v) \neq (-1)^{n/2}.
\end{cases}
\]

The product of the local densities \( \beta_v \) of \( \Lambda_v \) over all finite places \( v \) of \( F \) can be written as

\[
\prod_{v: \text{finite}} \beta_v^{-1} = \frac{1}{2^w} \prod_{j=1}^{n} L_{F}(j, \chi^j) \cdot \prod_{v|d_{E/F}} \lambda_v.
\]

Let \( \zeta_F(s) \) and \( \zeta_E(s) \) be the Dedekind zeta functions of \( F \) and \( E \), respectively. Then

\[
\zeta_E(s) = \zeta_F(s) \cdot L_F(s, \chi).
\]

The functional equations for \( \zeta_F(s) \) and \( \zeta_E(s) \) imply that

\[
L_F(j, \chi^j) = \zeta_F(j) = \left( \frac{(-1)^i}{2} \cdot \frac{(2\pi)^j}{(j - 1)!} \right)^d \cdot \zeta_F(1 - j) \cdot d_{F}^{1 - j} \quad \text{if } j \text{ is even};
\]

\[
L_F(j, \chi^j) = \left( \frac{(-1)^i}{2} \cdot \frac{(2\pi)^j}{(j - 1)!} \right)^d \cdot L_F(1 - j, \chi) \cdot d_{F}^{1 - j} \cdot |O_F/d_{E/F}|^{1/2 - j} \quad \text{if } j \text{ is odd}.
\]

These equalities and (2.6) imply that

\[
(2.9) \quad \text{Mass}(\Lambda) = \frac{(-1)^s \cdot 2 \cdot \gamma}{2^{nd+w}} \prod_{j=1}^{n} L_F(1 - j, \chi^j) \cdot \prod_{v|d_{E/F}} \lambda_v
\]

where

\[
\gamma = \begin{cases} 
1 & \text{if } n \text{ is odd;} \\
|O_F/d_{E/F}|^{n/2} & \text{if } n \text{ is even}.
\end{cases}
\]

For a finite place \( v \) which is ramified in \( E \), we put

\[
(2.10) \quad \kappa_v = \begin{cases} 
\lambda_v & \text{if } n \text{ is odd;} \\
O_{E_v/F_v}/d_{E_v/F_v}^{n/2} \cdot \lambda_v & \text{if } n \text{ is even},
\end{cases}
\]

where \( d_{E_v/F_v} \) denotes the discriminant of \( E_v/F_v \). Then \( \prod_{v|d_{E/F}} \kappa_v = \gamma \cdot \prod_{v|d_{E/F}} \lambda_v \).
As in Section 2.1.4 if \( v \) is ramified in \( E \), then

\[
|O_{F_v}/d_{E_v/F_v}| = \begin{cases} 
q_v & \text{if } v \nmid 2; \\
q_v^2 & \text{if } v | 2 \text{ and } E_v/F_v \text{ is RU}; \\
q_v^3 & \text{if } v | 2 \text{ and } E_v/F_v \text{ is RP}.
\end{cases}
\]

Equalities (2.8), (2.9), (2.10), and (2.11) imply the assertion. \( \Box \)

3. Connected components of complex unitary Shimura varieties

3.1. Similitude factors of Hermitian lattices

3.1.1. Let \( \ell \) be a prime. Let \((E,\bar{\imath})\) be an étale quadratic algebra over \( F = \mathbb{Q}_\ell \) with involution as in Section 2.1.1 that is, \( E/\mathbb{Q}_\ell \) is a quadratic field extension, or \( E \simeq \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell \). Denote by \( O_E \) the maximal order in \( E \). Let \( N = N_{E/\mathbb{Q}_\ell} \) denote the norm map of \( E/\mathbb{Q}_\ell \).

Let \((V,\varphi)\) be a Hermitian space over \( E \), and let \( n = \dim_E V \). We write \( G = GU(V,\varphi) \) for the unitary similitude group associated to \((V,\varphi)\), which is a reductive group over \( \mathbb{Q}_\ell \) defined by

\[
G(R) = \{(g,c) \in (\text{End}_{E\otimes \mathbb{Q}_\ell} R(V_R))^\times \times R^\times \mid \varphi(gx,gy) = c \cdot \varphi(x,y) \text{ for } x,y \in V_R\}
\]

for any \( \mathbb{Q}_\ell \)-algebra \( R \). Here we write \( V_R = V \otimes \mathbb{Q}_\ell R \). The similitude character is defined by the second projection:

\[
sim : G \to \mathbb{Q}_m, \quad (g,c) \mapsto c.
\]

Its kernel \( G^1 \) is the unitary group \( U(V,\varphi) \) (Section 2.2.1). Note that the similitude factor \( c \) is uniquely determined by \( g \). We write \( \text{sim}(g) := \text{sim}((g,c)) = c \) by abuse of notation.

Let \( E = \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell \). Let \( \{e_i\} \) be a basis of \( V \) over \( E \) and write \( \varphi(e_i,e_j) = (a_{ij},b_{ij}) \in E \). Then \( a_{ij} = b_{ji} \). The matrix \( \Phi_1 := (a_{ij}) \) is invertible since \( \varphi \) is non-degenerate. We have that

\[
G(Q_\ell) \simeq \{(c, A, B) \in \mathbb{Q}_\ell^\times \times \text{GL}_n(\mathbb{Q}_\ell)^2 \mid B^t \Phi_1 A = c \cdot \Phi_1 \} \simeq \mathbb{Q}_\ell^\times \times \text{GL}_n(\mathbb{Q}_\ell)
\]

where the second morphism is given by \((c, A, B) \mapsto (c, A)\). Then the similitude character is identified with the first projection and hence it maps \( G(Q_\ell) \) onto \( \mathbb{Q}_\ell^\times \). If \( E/\mathbb{Q}_\ell \) is a quadratic field extension, then the similitude character maps \( G(Q_\ell) \) onto \( N(E^\times) \) or \( \mathbb{Q}_\ell^\times \) according as \( n \) is odd or even (see (3.13)).

3.1.2. Here we assume \( n = 2 \). We recall the quaternion algebra associated to \((V,\varphi)\) which was constructed by Shimura [68, Section 2]. We fix a basis of \( V \) and write \( \Phi \) for its Gram matrix. We identify \( \text{End}_E(V) \) with \( \text{Mat}_2(E) \). Let \( \iota \) denote its canonical involution, given by \( \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\iota = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \). We define a subring \( B \) of \( \text{End}_E(V) \) by

\[
B := \{g \in \text{End}_E(V) \mid g^t \Phi \cdot g = \Phi \cdot g^t \cdot g = \det(g) \cdot \Phi \}.
\]

We observe that each element \( g \) of \( B \) satisfies \( \tilde{g}^t \Phi \cdot g = \Phi \cdot g^t \cdot g = \det(g) \cdot \Phi \). By [68, Propositions 2.6 and 2.8], the subring \( B \) satisfies the following properties:

- \( B \) is a quaternion algebra over \( \mathbb{Q}_\ell \);
- \( B^\times = \{g \in G(Q_\ell) \mid \det(g) = \text{sim}(g)\} \) where we regard \( G(Q_\ell) \) as a subset of \( \text{End}_E(V) \);
- \( B \) is a division algebra if and only if \((V,\varphi)\) is anisotropic.
3.1.3. Let \( \Lambda \subset V \) be a Hermitian \( O_E \)-lattice, not necessarily unimodular for a moment. Let \( K = K_{\Lambda} := \text{Stab}_{G(\mathbb{Q})}(\Lambda) \) denote the stabilizer of \( \Lambda \) in \( G(\mathbb{Q}) \). Then the similitude character induces a homomorphism from \( K \) to \( \mathbb{Z}^\times \). We compute its image, denoted by \( \text{sim}(K) \).

We fix a basis \( \{e_i\} \) of the lattice \( \Lambda \) over \( O_E \), inducing an isomorphism \( \text{End}_{O_E}(\Lambda) \simeq \text{Mat}_n(O_E) \). Let \( \Phi \) be the Gram matrix of \( \{e_i\} \). Then we have an identification

\[
K \simeq \{(g, c) \in \text{GL}_n(O_E) \times \mathbb{Z}_\ell^\times \mid \tilde{g}^t \cdot \Phi \cdot g = c \cdot \Phi\}. \tag{3.4}
\]

Let \( c \in \mathbb{N}(O_E^\times) \) and let \( u \) be an element in \( O_E^\times \) such that \( \mathbb{N}(u) = c \). Then \((uI_n, c) \in K\) where \( I_n \) denotes the identity matrix. Hence we always have inclusions

\[
\mathbb{N}(O_E^\times) \subset \text{sim}(K) \subset \mathbb{Z}_\ell^\times. \tag{3.5}
\]

**Lemma 3.1.4.** If \( E/\mathbb{Q}_\ell \) is an unramified quadratic field extension or \( E = \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell \), then \( \text{sim}(K) = \mathbb{Z}_\ell^\times \).

**Proof.** If \( E/\mathbb{Q}_\ell \) is an unramified field extension, then \( \mathbb{Z}_\ell^\times = \mathbb{N}(O_E^\times) \). Hence \( \text{sim}(K) = \mathbb{Z}_\ell^\times \) by \([3.5]\). If \( E = \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell \), then \( K \simeq \mathbb{Z}_\ell^\times \times \text{GL}_n(\mathbb{Z}_\ell) \), similarly to \([3.2]\). The similitude character is identified with the first projection, and hence \( \text{sim}(K) = \mathbb{Z}_\ell^\times \).

**Lemma 3.1.5.** Assume that \( E/\mathbb{Q}_\ell \) is a ramified quadratic field extension.

1. If \( n \) is odd, then \( \text{sim}(K) = \mathbb{N}(O_E^\times) \).
2. If \( n \) is even and \( \Lambda \) is unimodular, then \( \text{sim}(K) = \mathbb{Z}_\ell^\times \).

**Proof.** (1) As in \([3.4]\), any element \( g \in K \) satisfies that \( \mathbb{N}(\det(g)) = \det(g)^n \), and equivalently \( \mathbb{N}(\det(g) \cdot \det(g)^{1/n}) = \det(g) \). Thus we have \( \text{sim}(K) \subset \mathbb{N}(O_E^\times) \) and hence \( \text{sim}(K) = \mathbb{N}(O_E^\times) \) by \([3.5]\).

(2) Let \( \Lambda \) be a unimodular lattice and let \( \Phi \) be the Gram matrix of a basis of \( \Lambda \) over \( O_E \). We show that for any element \( c \in \mathbb{Z}_\ell^\times \) there exists \( g \in \text{GL}_n(O_E) \) such that \( \tilde{g}^t \cdot \Phi \cdot g = c \cdot \Phi \). We may assume that \( \Lambda \) is of rank two: In fact, the lattice \( \Lambda \) has a splitting \( \Lambda = \bigoplus_{1 \leq i \leq n/2} \Lambda_i \) for some lattices \( \Lambda_i \) of rank two as in Lemma 2.1.3. Let \( \{e_1^i, e_2^i\} \) be a basis of \( \Lambda_i \) over \( O_E \) with Gram matrix \( \Phi_i \) for \( 1 \leq i \leq n/2 \). Then \( \{(e_1^i, e_2^i)_{1 \leq i \leq n/2}\} \) is a basis of \( \Lambda \) and has Gram matrix \( \Phi = \text{diag}(\Phi_1, \ldots, \Phi_{n/2}) \).

Suppose that for an element \( c \in \mathbb{Z}_\ell^\times \) there exist matrices \( g_i \in \text{GL}_2(O_E) \) such that \( \tilde{g}^t_i \cdot \Phi \cdot g_i = c \cdot \Phi_i \) for all \( i \). Then the matrix \( g = \text{diag}(g_1, \ldots, g_{n/2}) \) satisfies \( \tilde{g}^t \cdot \Phi \cdot g = c \cdot \Phi \), as desired. By Lemma 2.1.5, we only need to consider three cases: (i) The case \( \Lambda = H \); (ii) The case \( \ell = 2 \), \( \Lambda \) is subnormal in \( \text{RP} \), and \( \Lambda \not\cong H \); (iii) The case \( \Lambda = (1) \oplus (-\alpha) \) for an \( \alpha \in O_E^\times \).

(i) The case \( \Lambda = H = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \). In this case, for any \( c \in \mathbb{Z}_\ell^\times \), the matrix \( g = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \) satisfies the desired property.

(ii) The case \( \ell = 2 \), \( \Lambda \) is subnormal in \( \text{RP} \), and \( \Lambda \not\cong H \). Before discussing the case \( \ell = 2 \), we recall some facts on a ramified quadratic field extension \( E/\mathbb{Q}_2 \). We have that \( E = \mathbb{Q}_2(\sqrt{\theta}) \) where \( \theta = 3, 7, 2, 6, 10, \) or \( 14 \). We consider the inclusions

\[
1 + 8\mathbb{Z}_2 = (\mathbb{Z}_2^\times)^2 \subset \mathbb{N}(O_E^\times) \subset \text{sim}(K) \subset \mathbb{Z}_2^\times.
\]

The quotient group \( \mathbb{Z}_2^\times/(\mathbb{Z}/8\mathbb{Z})^\times = \mathbb{Z}/2\mathbb{Z} \). Further, the subgroup \( \mathbb{N}(O_E^\times)/(\mathbb{Z}_2^\times)^2 \subset \mathbb{Z}_2^\times/(\mathbb{Z}_2^\times)^2 \) is of order two. In the second column in Table 3, we see an example of an element \( a \in O_E^\times \) such that \( \mathbb{N}(a) \mod 8 \) generates this subgroup. To prove \( \text{sim}(K) = \mathbb{Z}_2^\times \), it suffices to find an element \( g \in K \) such that \( \det(g) \not\in \mathbb{N}(a) \mod 8 \).

Now let \( \Lambda \) be as above. By Lemma 2.1.6 and Example 2.1.7, we may choose a basis of \( \Lambda \) so that \( \Phi = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \). Table 2 shows explicit examples of elements \( g \in \text{GL}_2(O_E) \) such that \( \tilde{g}^t \cdot \Phi \cdot g = c \cdot \Phi \) with \( c = \text{sim}(g) \not\in \mathbb{N}(a) \mod 8 \).
(iii) The case \( \Lambda = (1) \oplus (-\alpha) \) for an element \( \alpha \in \mathbb{Z}_\ell^\times \). In this case, \( \Lambda \) is normal. If \( \alpha \in \mathbb{N}(O_E^\times) \), then \( d(\Lambda) = -1 \) as elements of \( \mathbb{Z}_\ell^\times / \mathbb{N}(O_E^\times) \) and hence \( \Lambda \simeq H \). Therefore we may assume that \( \alpha \notin \mathbb{N}(O_E^\times) \). Then \( V \) is anisotropic by [35, (3.1)]. Let \( B \) be the quaternion division algebra over \( \mathbb{Q}_\ell \) associated with the space \( V = \Lambda \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \), as in (3.3). A computation shows

\[
\begin{align*}
B = \left\{ \left( \begin{array}{cc} a & \alpha c \\ c & \bar{\alpha} \end{array} \right) \in \text{Mat}_2(E) \mid a, c \in E \right\}.
\end{align*}
\]

We put \( O_\Lambda := \{ g \in B \mid g \cdot \Lambda \subset \Lambda \} = B \cap \text{Mat}_2(O_E) \). Then \( O_\Lambda \) is a \( \mathbb{Z}_\ell \)-order of \( B \). Further, the second property of \( B \) after (3.3) implies that \( \det(O_\Lambda^\times) \subset \text{sim}(\mathbb{K})(\subset \mathbb{Z}_\ell^\times) \). Hence it suffices to show that \( \mathbb{Z}_\ell^\times \subset \det(O_\Lambda^\times) \).

First assume that \( \ell \neq 2 \). Let \( E \oplus E_z \) be a two-dimensional \( E \)-vector space, equipped with a structure of a quaternion algebra over \( \mathbb{Q}_\ell \) defined by the relations

\[
z^2 = \alpha, \ za = \bar{a}z, \ a \in E.
\]

Then there is an isomorphism of quaternion algebras

\[
E \oplus E_z \sim B, \quad a + bz \mapsto \left( \begin{array}{cc} a & \alpha b \\ \bar{b} & \bar{\alpha} \end{array} \right),
\]

which identifies the \( \mathbb{Z}_\ell \)-order \( O_E \oplus O_E z \) with \( O_\Lambda \). Let \( \pi \) be a uniformizer of \( O_E \) such that \( \bar{\pi} = -\pi \). Then \( \pi O_\Lambda \) is a two-sided ideal of \( O_\Lambda \). Further, we have an isomorphism of \( \mathbb{F}_\ell \)-algebras

\[
O_\Lambda / \pi O_\Lambda \simeq \mathbb{F}_\ell[T]/(T^2 - \bar{\alpha}),
\]

where we write \( \bar{\alpha} = \alpha \mod \ell \in \mathbb{F}_\ell \). This shows that \( O_\Lambda / \pi O_\Lambda \) is a separable \( \mathbb{F}_\ell \)-algebra since \( \ell \neq 2 \). Note that \( (\pi O_\Lambda)^2 \subset \ell O_\Lambda \), and hence \( \pi O_\Lambda \) is contained in the Jacobson radical of \( O_\Lambda \), as in [65, Exercise 39.1]. Suppose \( O_\Lambda / \pi O_\Lambda \simeq \mathbb{F}_\ell \times \mathbb{F}_\ell \). Then its idempotents have lifts in \( O_\Lambda \) by [9, Theorem 6.7], and this contradicts to the fact that \( B \) is a division algebra. It follows that \( O_\Lambda / \pi O_\Lambda \) is a quadratic field extension of \( \mathbb{F}_\ell \), generated by \( z \mod \pi \). Therefore \( \mathbb{Q}_\ell(z) \) is an unramified quadratic field extension of \( \mathbb{Q}_\ell \), with the ring of integers \( \mathbb{Z}_\ell[z] \). Hence we have \( \mathbb{Z}_\ell^\times = \mathbb{N}_{\mathbb{Q}_\ell(z)/\mathbb{Q}_\ell}(\mathbb{Z}_\ell[z]^\times) \subset \det(O_\Lambda^\times) \), as desired.

Next we assume that \( \ell = 2 \) and \( \alpha \notin \mathbb{N}(O_E^\times) \). By equality (3.6), the group \( \det(O_\Lambda^\times) \) consists of elements of the form \( \mathbb{N}(a) - \alpha \mathbb{N}(c) \) for some \( a, c \in O_E \). Table 3 shows some examples of \( a \in O_E^\times \) and \( c \in O_E \) such that \( \mathbb{N}(a) \) and \( \mathbb{N}(a) - \alpha \mathbb{N}(c) \mod 8 \) generate the multiplicative group \( (\mathbb{Z}/8\mathbb{Z})^\times \). The argument in case (ii) thus implies \( \det(O_\Lambda^\times) = \mathbb{Z}_2^\times \).

\[
\square
\]
3.2. Unitary Shimura varieties.

Definition 3.2.1. A unitary PEL datum is a 5-tuple \( \mathcal{D}_Q = (E, \bar{\tau}, V, \langle , \rangle, h_0) \) where

- \( E \) is an imaginary quadratic extension of \( \mathbb{Q} \);
- \( b \mapsto b \) for \( b \in E \) is the non-trivial automorphism of \( E/\mathbb{Q} \);
- \( V \) is an \( E \)-vector space of dimension \( n > 0 \) with a non-degenerate alternating \( \mathbb{Q} \)-bilinear form \( \langle , \rangle : V \times V \to \mathbb{Q} \) such that
  \[
  \langle bx, y \rangle = \langle x, by \rangle
  \]
  for all \( x, y \in V \) and \( b \in E \);
- \( h_0 : C \to \text{End}_{E \otimes \mathbb{R}}(V_R) \) is an \( \mathbb{R} \)-linear algebra homomorphism, such that
  \[
  \langle h_0(z)x, h_0(z)y \rangle = \langle x, y \rangle \quad \text{for all } z \in C, \ x, y \in V_R = V \otimes \mathbb{R},
  \]
  and that the pairing \( \langle x, y \rangle \mapsto \langle x, h_0(\sqrt{-1})y \rangle \) is symmetric and definite (positive or negative) on \( V_R \).

Let \( G = \text{GU}(V, \langle , \rangle) \) be the \( \mathbb{Q} \)-group of \( E \)-linear \( \langle , \rangle \)-similitudes on \( V \): For every commutative \( \mathbb{Q} \)-algebra \( R \), the group of its \( R \)-values is given by
\[
G(R) = \{ (g, c) \in \text{End}_E(V_R)^\times \times R^\times \mid \langle gx, gy \rangle = c \cdot \langle x, y \rangle \text{ for all } x, y \in V_R \}
\]
where \( V_R = V \otimes \mathbb{R} \).

We may write \( E = \mathbb{Q}(\sqrt{d_E}) \), where \( d_E \) is the discriminant of \( E \). Then there exists (see [51, Lemma A.7]) a unique non-degenerate Hermitian form \( \varphi : V \times V \to E \) such that \( \langle , \rangle \) can be written as
\[
\langle x, y \rangle = \text{Tr}_{E/\mathbb{Q}} \left( (\sqrt{d_E})^{-1} \cdot \varphi(x, y) \right).
\]
Further, the algebraic \( \mathbb{Q} \)-group \( G \) is isomorphic to the unitary similitude group \( \text{GU}(V, \varphi) \) as defined in (3.3). In particular, one has \( G_\mathbb{R} \simeq \text{GU}(r, s) \) over \( \mathbb{R} \), where \( (r, s) \) is the signature of \( \varphi_\mathbb{R} \). If we replace \( \sqrt{d_E} \) by \( -\sqrt{d_E} \) in (3.3), then \( G_\mathbb{R} \simeq \text{GU}(s, r) \); however, we have \( \text{GU}(r, s) = \text{GU}(s, r) \).

We define a homomorphism \( h : \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m, \mathbb{C}} \to G_\mathbb{R} \) by restricting \( h_0 \) to \( \mathbb{C}^\times \). Composing \( h(\mathbb{C}) \) with the map \( \mathbb{C}^\times \to \mathbb{C}^\times \times \mathbb{C}^\times \) where \( z \mapsto (z, 1) \) then gives \( h_\mathbb{R} : \mathbb{C}^\times \to G(\mathbb{R}) \simeq \text{GL}_n(\mathbb{C}) \times \mathbb{C}^\times \). Up to conjugation, we have \( h_\mathbb{R} = (\text{diag}(z^r, 1^s), z) \) or \( (\text{diag}(z^r, 1^s), z) \). Let \( E \) be the reflex field associated with \( \mathcal{D}_Q \), that is, the field of definition of the conjugacy class of \( h_\mathbb{R} \). Then either \( E = \mathbb{Q} \) when \( r = s \) or \( E = E \) when \( r \neq s \).

Let \( X \) be the \( \mathbb{G}(\mathbb{R}) \)-conjugacy class of \( h \). Then \( (G, X) \) is a Shimura datum. For any open compact subgroup \( K \subset G(\mathbb{A}_f) \), we write \( \text{Sh}_K(G, X) \) for the Shimura variety of level \( K \) associated to \( (G, X) \). Then \( \text{Sh}_K(G, X) \) is a smooth quasi-projective variety of dimension \( rs \) over \( E \). The set of complex points of \( \text{Sh}_K(G, X) \) is identified, as a complex manifold, with
\[
\text{Sh}_K(G, X)(\mathbb{C}) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K.
\]

\footnote{If \( rs = 0 \), it does not satisfy the axiom \( \text{II} \) (2.1.1.3), however, we still can consider the Shimura variety associated to it using (3.9).}
Definition 3.2.2. An integral unitary PEL datum is a tuple $\mathcal{D} = (E, \gamma, O_E, V, \langle, \rangle, \Lambda, h_0)$, where
- $(E, \gamma, V, \langle, \rangle, h_0)$ is a unitary PEL datum;
- $O_E$ is the ring of integers of $E$;
- $\Lambda$ is a full $O_E$-lattice in $V$ on which $\langle, \rangle$ takes value in $\mathbb{Z}$.

Equality (3.5) implies the following.

Lemma 3.2.3. Let $\Lambda^{\nu,\langle,\rangle} = \{x \in V \mid \langle x, \Lambda \rangle \subset \mathbb{Z}\}$ be the dual lattice of $\Lambda$ with respect to $\langle, \rangle$, and let $\Lambda^{\nu,\varphi}$ be the dual lattice with respect to $\varphi$ as defined in Section 2.1.2. Then $\Lambda^{\nu,\langle,\rangle} = \Lambda^{\nu,\varphi}$ as $O_E$-submodules of $V$. In particular, $(\Lambda, \langle, \rangle)$ is self-dual if and only if $(\Lambda, \varphi)$ is self-dual (unimodular).

Let us fix an integral unitary PEL datum $\mathcal{D} = (E, \gamma, O_E, V, \langle, \rangle, \Lambda, h_0)$. This gives a model $\mathrm{GU}(\Lambda^{\langle,\rangle}) = \mathrm{GU}(\Lambda, \varphi)$ of $G$ over $\mathbb{Z}$, which is again denoted by $G$. Its group of $R$-values for a commutative ring $R$ is given by

$$G(R) = \{ (g, c) \in \mathrm{End}_{O_E\otimes R}(\Lambda_R)^{\times} \times R^{\times} \mid \varphi(gx, gy) = c \cdot \varphi(x, y) \text{ for all } x, y \in \Lambda_R \}$$

where $\Lambda_R = \Lambda \otimes R$.

3.2.4. Let $G_{\text{der}} = \mathrm{SU}(V, \varphi)$ be the derived subgroup of $G_Q$. Then $G_{\text{der}}$ is a simply connected semisimple algebraic group over $\mathbb{Q}$. Kneser’s theorem states that the group $H^1(Q_\ell, G_{\text{der}}^{\text{der}})$ is trivial for every finite prime $\ell$ [60, Theorem 6.4, p. 284]. Further, the Hasse principle holds true, that is, the map $H^1(Q_\ell, G_{\text{der}}^{\text{der}}) \to \prod_{\ell \in S} H^1(Q_\ell, G^{\text{der}})$ is injective [60, Theorem 6.6, p. 286].

Let $D$ be the quotient torus $D := G_Q/G_{\text{der}}^{\text{der}}$ over $\mathbb{Q}$, and let $\nu : G_Q \to D$ be the natural projection. Kottwitz [11, Section 7] described $D$ and $\nu$ as follows. Let $T^E$ denote the Weil restriction $\mathrm{Res}_{E/Q} G_{m,E}$, and let $T^{E,1}$ denote the kernel of the norm homomorphism $N = N_{E/Q} : T^E \to G_{m,Q}$. Then, for any $\mathbb{Q}$-algebra $R$ we have an isomorphism

$$D(R) \simeq \{ (x, c) \in T^E(R) \times R^{\times} \mid N(x) = c^n \},$$

under which $\nu$ is identified with the product $\det \times \mathrm{sim}, (g, c) \mapsto (\det g, c)$. Furthermore we have isomorphisms

$$f : D \to \begin{cases} T^E & \text{if } n \text{ is odd;} \\ T^{E,1} \times G_{m,Q} & \text{if } n \text{ is even.} \end{cases}$$

The similitude character $\mathrm{sim} : G_Q \to G_{m,Q}$ is equal to $N \circ f \circ \nu$ if $n$ is odd, and $\mathrm{pr}_2 \circ f \circ \nu$ if $n$ is even.

Let $D(\mathbb{R})^0$ be the identity component of $D(\mathbb{R})$. From [31,11] it follows that $D(\mathbb{R})^0 \simeq C^\times$ if $n$ is odd and $D(\mathbb{R})^0 \simeq S^1 \times \mathbb{R}_{>0}$ if $n$ is even, where $S^1$ denotes the unit circle. We write $E^1 := T^{E,1}(Q) = \{ x \in E^X \mid N(x) = 1 \}$. Then

$$D(\mathbb{Q})_\infty := D(\mathbb{Q}) \cap D(\mathbb{R})^0 = \begin{cases} E^\times & \text{if } n \text{ is odd;} \\ E^1 \times \mathbb{Q}_{>0} & \text{if } n \text{ is even.} \end{cases}$$

For a prime $\ell$, we write $E_\ell = E \otimes Q_\ell$. By Kneser’s theorem, we have that

$$\nu(G(Q_\ell)) = D(Q_\ell) = \begin{cases} E^\times_\ell & \text{if } n \text{ is odd;} \\ E^1_\ell \times Q^\times_{\ell} & \text{if } n \text{ is even.} \end{cases}$$

The similitude character $\mathrm{sim} : G(Q_\ell) \to G(Q_\ell)^\times$ is given by

$$\mathrm{sim}(G(Q_\ell)) = \begin{cases} N(E^\times_\ell) & \text{if } n \text{ is odd;} \\ Q^\times_\ell & \text{if } n \text{ is even.} \end{cases}$$

For any algebraic torus $T$ over $\mathbb{Q}$, let $T(\mathbb{Z}_\ell)$ denote the unique maximal open compact subgroup of $T(\mathbb{Q}_\ell)$. We write $O^1_{E_\ell} := T^{E,1}(\mathbb{Z}_\ell) = \{ x \in O^X_{E_\ell} \mid N(x) = 1 \}$. For $\ell = 2$, we also define a subgroup $O^0_{E_2}$ of $O^X_{E_2}$ by $O^0_{E_2} := \{ u \bar{u}^{-1} \mid u \in O^X_{E_2} \}$. Then $O^0_{E_2} \subset O^1_{E_2}$. When $E_2/Q_2$ is ramified, Hilbert’s Theorem 90 implies that $[O^0_{E_2} : O^0_{E_2}] = 2$, as in [38, Remark 3.4].
We write \( \Lambda : \Lambda \otimes \mathbb{Z}_\ell \). By Kirschner \cite{38}, Theorem 3.7, the subgroup \( \det(G^1(\mathbb{Z}_\ell)) \subset O_{E_\ell}^1 \) is described as

\[
\det(G^1(\mathbb{Z}_\ell)) = \begin{cases} 
O_{E_\ell}^1 & \text{if } \ell = 2, \ E_2/\mathbb{Q}_2 \text{ is RU, } \Lambda_2 \simeq H^{n/2}; \\
0 & \text{otherwise,}
\end{cases}
\]

where \( H = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) denotes a rank-two \( O_{E_\ell}^1 \)-lattice.

\textbf{Lemma 3.2.5.} Let \( \ell \) be a prime. If \( n \) is odd, or \( n \) is even and \((\Lambda, \varphi)\) is unimodular, then

\[ D(\mathbb{Z}_\ell) = \nu(G(\mathbb{Z}_\ell)), \]

unless \( n \) is even, \( E_2/\mathbb{Q}_2 \) is RU, and \( \Lambda_2 \simeq H^{n/2} \), in which case

\[ [D(\mathbb{Z}_\ell) : \nu(G(\mathbb{Z}_\ell))] = 2. \]

\textbf{Proof.} There is a commutative diagram

\[
\begin{array}{cccc}
1 & \longrightarrow & G^1(\mathbb{Z}_\ell) & \longrightarrow & G(\mathbb{Z}_\ell) & \sim & \sim & \sim & \sim & \longrightarrow & 1 \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
1 & \longrightarrow & O_{E_\ell}^1 & \longrightarrow & D(\mathbb{Z}_\ell) & \longrightarrow & D(\mathbb{Z}_\ell) & \longrightarrow & 1
\end{array}
\]

where the horizontal sequences are exact. By the description of \( D \) in \((3.11)\), the group \( \text{pr}_2(D(\mathbb{Z}_\ell)) \)

equals \( N(O_{E_\ell}^2) \) or \( \mathbb{Z}_\ell^2 \) according as \( n \) is odd or even. It follows from Lemmas \( 3.1.4 \) and \( 3.1.5 \) that

\[
[D(\mathbb{Z}_\ell) : \nu(G(\mathbb{Z}_\ell))] = [O_{E_\ell}^1 : \det(G^1(\mathbb{Z}_\ell))].
\]

This and \((3.14)\) imply the assertion. \( \square \)

\subsection{3.3. The number of connected components.}

\textbf{3.3.1.} Let \( K \) be any open compact subgroup of \( G(\mathbb{A}_f) \). Let \( \text{Sh}_K(G, X)_\mathbb{C} \) be the Shimura variety of level \( K \), and \( \pi_0(\text{Sh}_K(G, X)_\mathbb{C}) \) be the set of its connected components. Let \( X^+ \) be the connected component of the Hermitian symmetric domain \( X \) containing the base point \( h_0 \). We write \( G(\mathbb{R})_+ \)

for the stabilizer of \( X^+ \) in \( G(\mathbb{R}) \), and write \( G(\mathbb{Q})_+ := G(\mathbb{Q}) \cap G(\mathbb{R})_+ \). As in \[11\] 2.1.3, we have

\[ \pi_0(\text{Sh}_K(G, X)_\mathbb{C}) \simeq G(\mathbb{Q}) \backslash G(\mathbb{A})/G(\mathbb{R})_+, K \simeq G(\mathbb{Q}) \backslash G(\mathbb{A}_f)/K. \]

Kneser's theorem and the Hasse principle for \( G^{\text{der}} \) imply that \( \nu(G(\mathbb{Q})_+) = D(\mathbb{Q})_\infty \), and hence \( \nu \)

induces a surjective map

\[
\nu : G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K \rightarrow D(\mathbb{Q})_\infty \backslash D(\mathbb{A}_f)/\nu(K).
\]

Now we assume that \( rs > 0 \). Then the group \( G^{\text{der}}(\mathbb{R}) \) is not compact. The strong approximation theorem \((37, 60 \text{ Theorem 7.12, p. 427})\) therefore implies that \( G^{\text{der}}(\mathbb{Q}) \) is dense in \( G^{\text{der}}(\mathbb{A}_f) \). It follows that the map in \((3.17)\) is a bijection. For more details, see \[51\] Section 5.

\textbf{Theorem 3.3.2.} Assume that \( rs > 0 \). We write \( D(\mathbb{Z}) := \prod_{\ell \text{prime}} D(\mathbb{Z}_\ell) \subset D(\mathbb{A}_f) \).

\textbf{(1)} If \( n \) is odd, or \( n \) is even and \( \Lambda \) is unimodular, then

\[
|\pi_0(\text{Sh}_G(\mathbb{Z})(G, X)_\mathbb{C})| = \begin{cases} 
2^{-w} \cdot h(E) & \text{if } n \text{ is even, } \Lambda_2 \simeq H^{n/2}, d_E \equiv 4 \pmod{8}, \ E \neq \mathbb{Q}(\sqrt{-1}); \\
2^{1-w} \cdot h(E) & \text{otherwise},
\end{cases}
\]

where \( h(E) \) denotes the class number of the field \( E \), \( w \) denotes the number of primes which are ramified in \( E \).
(2) For an integer \( N \geq 3 \), let \( K(N) \) be the kernel of the reduction map \( G(\hat{Z}) \to G(\hat{Z}/N\hat{Z}) \). Then \[
|\pi_0(Sh_{K(N)}(G, X)_{c})| = \begin{cases} |D(\hat{Z}) : \nu(K(N))| \cdot |\mu_E|^{-1} \cdot h(E) & \text{if } n \text{ is odd;} \\ |D(\hat{Z}) : \nu(K(N))| \cdot |\mu_E|^{-1} \cdot 2^{1-w} \cdot h(E) & \text{if } n \text{ is even,} \end{cases}
\]
where \( \mu_E \) denotes the group of roots of unity in \( E \).

The case \( rs = 0 \) will be treated in Example 2.1.1.

Proof. We regard the group \( \mu_E \) as a subgroup of \( T^{E,1}(\mathbb{A}_f) \) via the natural embedding. Note that \( T^{E,1} \) can be embedded in \( D \) by (3.10). Let \( \mu_E \cap \nu(K) \) be the intersection taken in \( D(\mathbb{A}_f) \). Further, let \( h(D) = |D(\hat{Q}) \setminus D(\mathbb{A}_f)/D(\hat{Z})| \) denote the class number of \( D \). Then, by [26] Lemma 6.1 and Theorem 6.3], we have an equality

\[
|\pi_0(Sh_{K(N)}(G, X)_{c})| = |D(\hat{Q}) \setminus D(\mathbb{A}_f)/\nu(K)| = \frac{|D(\hat{Z}) : \nu(K)|}{|\mu_E : \mu_E \cap \nu(K)|} \cdot h(D).
\]

(1) We apply formula (3.18) to \( K = G(\hat{Z}) = \text{Stab}_{\hat{G}(\mathbb{A}_f)}^{\hat{x}} \). First recall that \( E_2/\mathbb{Q}_2 \) is RU if and only if \( d_{E} \equiv 4 \mod 8 \) as in Section 2.1.1. It follows from Lemma 3.1.5 that

\[
|D(\hat{Z}) : \nu(G(\hat{Z}))| = \prod_{\ell \equiv \text{prime}} |D(\mathbb{Z}_\ell) : \nu(G(\mathbb{Z}_\ell))| = \begin{cases} 2 & \text{if } d_{E} \equiv 4 \mod 8 \text{ and } \Lambda_2 \simeq H^{n/2}; \\ 1 & \text{otherwise.} \end{cases}
\]

Next we compute the index \( |\mu_E : \mu_E \cap \nu(G(\hat{Z}))| \). If \( d_{E} \not\equiv 4 \mod 8 \) or \( \Lambda_2 \not\simeq H^{n/2} \), then formula (3.19) implies that \( D(\hat{Z}) = \nu(G(\hat{Z})) \) and hence \( \mu_E = \mu_E \cap \nu(G(\hat{Z})) \). Now we assume that \( d_{E} \equiv 4 \mod 8 \) and \( \Lambda_2 \simeq H^{n/2} \). Note that we have \( \det(G(1)(\hat{Z})) = O_1^1(\hat{Z}) \cdot \det(G(1)(\hat{Z})) \) by diagram chasing in (3.15), and hence \( \mu_E \cap \nu(G(\hat{Z})) = \mu_E \cap \nu(\hat{Z}) \). This and (3.14) imply that \( |\mu_E : \mu_E \cap \nu(G(\hat{Z}))| = |\mu_E : \mu_E \cap O_{E_2}^1| \) where we regard \( \mu_E \) as a subgroup of \( O_{E_2}^1 \). Recall that \( E_2 = \mathbb{Q}_2(\sqrt{\theta}) \) for a \( \theta \in \mathbb{Z}_2^\times \). If we put \( a = \sqrt{\theta} \), then \( a \in O_{E_2}^1 \) and \( a = -a \). Hence \( -1 = a^2 \in O_{E_2}^1 \). If we further assume \( E \not\equiv \mathbb{Q}(\sqrt{-1}) \), then \( \{ \pm 1 \} = \mu_E = \mu_E \cap O_{E_2}^1 \). Suppose that \( E = \mathbb{Q}(\sqrt{-1}) \). Then the elements \( \pm \sqrt{-1} \) are uniformizers of \( O_{E_2} \). On the other hand, we have \( O_{E_2}^1 = \{ x \in O_{E_2} \mid x^2 = 1 \in D_{E_2/\mathbb{Q}_2} \} \) by [38] Lemma 3.5 where \( D_{E_2/\mathbb{Q}_2} \) denotes the different of \( E_2/\mathbb{Q}_2 \), and further \( D_{E_2/\mathbb{Q}_2} = (2) \) as in Section 2.1.1. It follows that \( \pm \sqrt{-1} \not\in O_{E_2} \) and hence \( |\mu_E : \mu_E \cap O_{E_2}^1| = \{ \pm 1 \} = 2 \).

As a result,

\[
|\mu_E : \mu_E \cap \nu(G(\hat{Z}))| = \begin{cases} 2 & \text{if } E = \mathbb{Q}(\sqrt{-1}) (d_{E} \equiv 4 \mod 8) \text{ and } \Lambda_2 \simeq H^{n/2}; \\ 1 & \text{otherwise.} \end{cases}
\]

Equalities (3.19) and (3.20) imply that

\[
|\mu_E : \mu_E \cap \nu(G(\hat{Z}))| = \begin{cases} 2 & \text{if } d_{E} \equiv 4 \mod 8, \Lambda_2 \simeq H^{n/2} \text{ and } E \not\equiv \mathbb{Q}(\sqrt{-1}); \\ 1 & \text{if } d_{E} \equiv 4 \mod 8, \Lambda_2 \simeq H^{n/2} \text{ and } E = \mathbb{Q}(\sqrt{-1}); \end{cases}
\]

Finally, the formulas of Shyr [70] Formula (16)] and Guo-Shee-Yu [16] Section 5.1] show that

\[
h(D) = \begin{cases} h(T^E) & \text{if } n \text{ is odd;}; \\ h(T^{E,1}) = 2^{1-w} \cdot h(E) & \text{if } n \text{ is even.} \end{cases}
\]

Equalities (3.18), (3.21), and (3.22) imply assertion (1).

(2) We apply formula (3.18) to \( K = K(N) \) for \( N \geq 3 \). By definition we have \( \nu(K(N)) \subset 1 + N\hat{Z} \).

Furthermore \( \mu_E \cap (1 + N\hat{Z}) = 1 \) since \( N \geq 3 \) (cf. [34] Lemma, p. 207]). It follows that

\[
|\mu_E : \mu_E \cap \nu(K(N))| = |\mu_E|.
\]
Equalities (3.18), (3.23), and (3.22) imply assertion (2).

4. The basic locus of unitary Shimura varieties

4.1. We recall the moduli space of abelian varieties with good reduction at $p$ following Kottwitz [41] and Lan [46]. Let $\mathcal{D} = (E, \tau, O_E, V, \langle , \rangle, A, h_0)$ be an integral unitary PEL datum as in Definition 3.2.2. Let $\psi$ be the Hermitian form on $V$ associated to the pairing $\langle , \rangle$, and $(r,s)$ its signature. We assume that $\Lambda$ is self-dual with respect to $\langle , \rangle$ (or equivalently, $(\Lambda, \psi)$ is unimodular; see Lemma 3.2.3). We fix a prime $p > 2$ which is unramified in $E$. Let $\bar{Q}$ be the algebraic closure of $Q$ in $C$ and fix an embedding $\iota_p: Q \hookrightarrow \bar{Q}$. Denote by $\Sigma_E = \text{Hom}(E, C) = \text{Hom}(E, \bar{Q})$ the set of embeddings of $E$ into $C$. Let $d_E$ be the discriminant of $E/Q$. We write $\Sigma_E = \{\tau, \bar{\tau}\}$, where we choose $\tau: E \to C$ such that $\text{Im}(\tau(\sqrt{d_E})) < 0$.

Let $G := \text{GU}(\Lambda, \langle , \rangle) \simeq \text{GU}(\Lambda, \psi)$. Then $G(Z_p)$ is a hyperspecial subgroup of $G(Q_p)$. For an integer $N \geq 3$ with $p \nmid N$, we define an open compact subgroup $K^p(N)$ of $G(A_f)$ by

$$K^p(N) := \ker(G(\bar{Z}^p) \to G(\bar{Z}^p/N\bar{Z}^p)).$$

Further we put $K := G(Z_p) \cdot K^p(N) \subset G(A_f)$.

Let $O_E$ be the ring of integers of the reflex field $E$ of $\mathcal{D}$ and set $O_{E,(p)} := O_E \otimes \mathbb{Z}(p)$. Let $M_K = M_K(\mathcal{D})$ be the contravariant functor from the category of locally Noetherian schemes over $O_{E,(p)}$ to the category of sets which takes a connected scheme $S$ over $O_{E,(p)}$ to the set of isomorphism classes of tuples $(A, \iota, \lambda, \bar{\eta})$ where

- $A$ is an abelian scheme over $S$;
- $\iota$ is a homomorphism $O_E \to \text{End}(A)$ which satisfies the Kottwitz determinant condition of signature $(r,s)$, that is, we have an equality of polynomials

$$\det(T - \iota(b); \text{Lie}(A)) = (T - \tau(b))^s(T - \bar{\tau}(b))^s \in O_S[T]$$

for all $b \in O_E$, where $\{\tau, \bar{\tau}\}$ are embeddings of $E$ into $C$;

- $\lambda: (A, \iota) \sim (A', \iota')$ is an $O_E$-linear principal polarization, that is, $(A', \iota')$ is the dual abelian scheme of $A$ with a homomorphism $\iota': O_E \to \text{End} A'$ given by $b \mapsto (\iota(b))'$, and $\lambda: A \sim A'$ is a principal polarization preserving $\iota$ and $\iota'$;

- $\bar{\eta}$ is a $\pi_1(S, \bar{s})$-invariant $K^p(N)$-orbit of $O_E \otimes \bar{Z}^p$-linear isomorphisms $\eta: \Lambda \otimes \bar{Z}^p \sim \tilde{T}^p(A_s)$ which preserve the pairings

$$\langle , \rangle : \Lambda \otimes \bar{Z}^p \times \Lambda \otimes \bar{Z}^p \to \bar{Z}^p,$$

$$\lambda : \tilde{T}^p(A_s) \times \tilde{T}^p(A_s) \to \bar{Z}^p(1)$$

up to a scalar in $\bar{Z}^p$. Here, $\tilde{T}^p(A_s)$ is the prime-to-$p$ Tate module and $s$ is a geometric point of $S$.

Two tuples $(A, \iota, \lambda, \bar{\eta})$ and $(A', \iota', \lambda', \bar{\eta}')$ are said to be isomorphic if there exists an $O_E$-linear isomorphism of abelian schemes $f: A \sim A'$ such that $\lambda = f^\vee \circ \lambda' \circ f$ and $\bar{\eta}' = \bar{f} \circ \bar{\eta}$. See [46] 1.4.1 for more details.

**Theorem 4.1.2** ([41], [46] Ch.2]). The contravariant functor $M_K$ is represented by a smooth quasi-projective scheme (denoted again by) $M_K$ over $O_{E,(p)}$.

**Remark 4.1.3.** (1) In [41] Kottwitz defined a moduli problem using prime-to-$p$ isogeny classes of abelian schemes with a $\mathbb{Z}(p)$-polarization. By [46] Proposition 1.4.3.4, this moduli problem is isomorphic to $M_K$ defined above, under the assumption that $\Lambda$ is self-dual.
(2) By \([11]\) Section 8 the complex algebraic variety \(\mathbf{M}_K \otimes \mathbb{C}\) is a finite disjoint union of complex Shimura varieties indexed by the finite set \(\ker^1(\mathbb{Q}, G) := \ker \left( H^1(\mathbb{Q}, G) \to \prod_{\ell \leq \infty} H^1(\mathbb{Q}_\ell, G) \right)\). By \([79]\) Lemma 8.8, the group \(G = \text{GU}(V, \langle \cdot, \cdot \rangle)\) (for an imaginary quadratic field \(E\)) satisfies the Hasse principle and hence that \(\mathbf{M}_K \otimes \mathbb{C} \simeq \text{Sh}_K(G, X)_\mathbb{C}\). The number of connected components of \(\mathbf{M}_K \otimes \mathbb{C}\) then is computed by Theorem 3.3.2 (2).

Let \(p\) be a prime ideal of \(O_E\) lying on \(p\) that corresponds to the embedding \(\iota_p : \mathcal{O} \hookrightarrow \mathcal{O}_p\). Let \(O_{E,p}\) be the localization of \(O_E\) at \(p\), and let \(k\) be an algebraic closure of the residue field of \(O_{E,p}\). Let \(\mathcal{M}_K := \mathbf{M}_K \otimes_{O_{E,p}} k\) denote the base change of \(\mathbf{M}_K\) to \(k\).

4.2. Newton and Ekedahl-Oort strata.

4.2.1. Let \(\mathbb{F}\) be a perfect field of characteristic \(p\). Let \(W(\mathbb{F})\) be the ring of Witt-vectors over \(\mathbb{F}\) with the Frobenius morphism \(\sigma : W(\mathbb{F}) \to W(\mathbb{F})\). Let \(W(\mathbb{F})[F, V]\) be the quotient ring of the associative free \(W(\mathbb{F})\)-algebra generated by the indeterminates \(F\) and \(V\) with respect to the relations

\[
FV = VF = p, \quad Fa = a\sigma F, \quad V a^\sigma = aV \quad \text{for all } a \in W(\mathbb{F}).
\]

A Dieudonné module \(M\) over \(W(\mathbb{F})\) is a left \(W(\mathbb{F})[F, V]\)-module which is finitely generated as a \(W(\mathbb{F})\)-module. A polarization on \(M\) is an alternating form \(\langle \cdot, \cdot \rangle : M \times M \to W(\mathbb{F})\) satisfying

\[
\langle Fx, y \rangle = \langle x, Vy \rangle^\sigma \quad \text{for all } x, y \in M.
\]

A polarization is called a principal polarization if it is a perfect pairing. By Dieudonné theory, there is an equivalence of categories between the category of \(p\)-divisible groups over \(\mathbb{F}\) and the category of Dieudonné modules that are free \(W(\mathbb{F})\)-modules.

Let \(\mathcal{D} = (E_\ell, \iota, O_E, V, \langle \cdot, \cdot \rangle, \Lambda, h_0)\) and \(p > 2\) be as in Section 4.1.1. We set \(E_p := E \otimes_{\mathbb{Q}} \mathbb{Q}_p\) and \(O_{E_p} := O_E \otimes \mathbb{Z}_p\). A \(p\)-divisible group with \(\mathcal{D}\)-structure over \(k\) is a triple \((X, \iota, \lambda)\) consisting of a \(p\)-divisible group \(X\) over \(k\) of height \(2n\), an action \(\iota : O_{E_p} \to \text{End}(X)\), and an \(O_{E_p}\)-linear isomorphism \(\lambda : X \to X^t\) such that \(\lambda^t = -\lambda\). Furthermore, the induced \(O_{E_p}\)-action on \(\text{Lie}(X)\) is assumed to satisfy the determinant condition governed by \(\mathcal{D}\).

For a \(p\)-divisible group with \(\mathcal{D}\)-structure \((X, \iota, \lambda)\), let \(M\) be the covariant Dieudonné module of the \(p\)-divisible group \(X\). Then \(\iota\) induces an action of \(O_{E_p}\) on \(M\) and \(\lambda\) induces a principal polarization \(\langle \cdot, \cdot \rangle : M \times M \to W(k)\), satisfying that

- the action of \(O_{E_p}\) commutes with the operators \(F\) and \(V\), and
- \(\langle ax, y \rangle = \langle x, a^*y \rangle\) for all \(x, y \in M, a \in O_{E_p}\).

4.2.2. In the rest of this section, we write \(G := G_{\mathbb{Z}_p}\). This is a connected reductive group scheme over \(\mathbb{Z}_p\) and its generic fiber \(G_{\mathbb{Q}_p}\) is unramified. Let \(L\) denote the fraction field of \(W(k)\). Let \(C(G)\) denote the set of \(G(W(k))\)-\(\sigma\)-conjugacy classes \([b] := \{g^{-1}b\sigma(g) \mid g \in G(W(k))\}\) of elements \(b \in G(L)\). By \([63]\) Theorem 3.16, there exists an isomorphism \(M \xrightarrow{\sim} \Lambda \otimes W(k)\) preserving \(O_{E_p}\)-actions and the pairings \(\langle \cdot, \cdot \rangle\), under which \(F\) corresponds to \(b(\text{id} \otimes \sigma)\) for some \(b \in G(L)\). The class \([b] \in C(G)\) does not depend on the choice of a fixed isomorphism, and we therefore obtain an injective map

\[
\left\{ \text{isomorphism classes of } p\text{-divisible groups with } \mathcal{D}\text{-structure over } k \right\} \hookrightarrow C(G).
\]

Let \(B(G)\) be the set of \(G(L)\)-\(\sigma\)-conjugacy classes \([b] := \{g^{-1}b\sigma(g) \mid g \in G(L)\}\) of elements \(b \in G(L)\). The above map with the natural projection \(C(G) \to B(G)\) then induces an injective map

\[
\left\{ \text{isogeny classes of } p\text{-divisible groups with } \mathcal{D}\text{-structure over } k \right\} \hookrightarrow B(G).
\]

We recall a description of \(B(G)\). We fix a maximal torus \(T\) of \(G\) and a Borel subgroup \(B\) containing \(T\) both defined over \(\mathbb{Q}_p\). Let \((X^*(T), \Psi, X_*(T), \Psi^\vee, \Delta)\) be the corresponding based
root datum. Let $W := N_G(T)/T$ be the associated Weyl group. By work of Kottwitz \[39\], a class $[b] \in B(G)$ is uniquely determined by two invariants: the Kottwitz point $\kappa_G(b) \in \pi(G)_\sigma$ is $\pi_1(G)/(1-\sigma)(\pi_1(G))$ and the Newton point $\nu_G(b) \in (X_s(T)_\mathbb{Q})^\theta$. Here $\pi_1(G)$ denotes the Borovoi’s fundamental group, and $X_s(T)_\mathbb{Q}^+$ denotes the set of dominant elements in $X_s(T)_\mathbb{Q} := X_s(T) \otimes \mathbb{Q}$. For $\lambda, \lambda' \in X_s(T)_\mathbb{Q}$, we write $\lambda \leq \lambda'$ if the difference $\lambda' - \lambda$ is a non-negative rational linear combination of positive coroots. The set $B(G)$ naturally forms a partially ordered set with $[b] \leq [b']$ if $\kappa_G(b) = \kappa_G(b')$ and $\nu_G(b) \leq \nu_G(b')$. We call a class $[b] \in B(G)$ basic if $\nu_G(b) \in X_s(Z_G)_\mathbb{Q}$, where $Z_G \subset T$ is the center of $G$. From \[39\ Section 5\] it follows that $[b] \in B(G)$ is basic if and only if it is minimal with respect to the above partial order.

We now describe the image of the map (4.5). Let $[\mu]$ be the conjugacy class of the cocharacter $\mu_b$ defined by the PEL datum $\mathcal{D}$ as in Section 4.2. Under the fixed embedding $t_p : \mathbb{Q} \to \mathbb{Q}_p$, we can regard $[\mu]$ as a $W$-orbit in $X_s(T)$. The dominant representative of $[\mu]$ in $X_s(T)$ is denoted by $\mu$. We set $\mu^\circ := \frac{1}{m} \sum_{i=0}^{m-1} \sigma^i(\mu) \in X_s(T)_\mathbb{Q}$ where $m$ is the degree of a splitting field of $G$ over $\mathbb{Q}_p$. Further $\mu^\circ$ denotes the image of $\mu$ under the projection $X_s(T) \to \pi_1(G)_\sigma$. As in \[12\ Section 6\] and \[61\ Section 4\], we define a set $B(G, \mu)$ by

\begin{equation}
B(G, \mu) = \{ [b] \in B(G) \mid \nu_G(b) \leq \mu^\circ, \kappa_G(b) = \mu^\circ \}.
\end{equation}

Then by \[52\ 13\ \[50\ 11\], the image of the map (4.5) is equal to the subset $B(G, \mu)$. There is a unique basic class in $B(G, \mu)$ by \[39\ Section 5\].

4.2.3. Let $(A, \iota, \lambda, \bar{\eta})$ be a $k$-point of $M_k$. Then $\iota$ and $\lambda$ induce a $\mathcal{D}$-structure on the $p$-divisible group $A[p^\infty]$ of $A$, and we can attach to it a class $[b] \in C(G)$. We thus obtain a map

$$M_k(k) \to C(G).$$

Each fiber of this map is called a central leaf on $M_k$.

Further, there exists a map

$$N_t : M_k \to B(G, \mu)$$

such that it takes a $k$-point $(A, \iota, \lambda, \bar{\eta})$ to the class $[b]$ attached to $(A[p^\infty], \iota, \lambda)$. Each fiber is a locally closed subset of $M_k$ \[52\]. By definition, the Newton stratum attached to $[b] \in B(G, \mu)$ is the fiber $N_t^{-1}([b])$ endowed with the reduced scheme structure. It is non-empty for every $[b] \in B(G, \mu)$ \[81\ Theorem 1.6 (1)\]. The Newton stratum attached to the unique basic class $[b] \in B(G, \mu)$ is called the basic locus and denoted by $M^\text{bas}_k$. This is closed in $M_k$.

Let $S \subset W$ denote the set of simple reflections corresponding to $B$. To the cocharacter $\mu$ associated to $\mathcal{D}$, one can attach a non-empty subset $J \subset S$ \[81\ A.5\]. Let $W_J$ denote the subgroup of $W$ generated by $J$, and let $J'W$ denote the set of representatives of minimal length of $W_J \backslash W$:

$$J'W := \{ w \in W \mid \ell(sw) > \ell(w) \text{ for all } s \in J \}.$$ 

For a $k$-point $(A, \iota, \lambda, \bar{\eta})$ of $M_k$, let $(A, \iota, \lambda)[p]$ be the $p$-torsion $A[p]$ of $A$ endowed with the $O_E$-action induced by $\iota$ and the isomorphism $A[p] \cong \hat{A}[p]^\theta$ induced by $\lambda$. By \[53\ Theorem 6.7\], we can associate to $(A, \iota, \lambda)[p]$ an element $\zeta(x) \in J'W$, and the element $\zeta(x)$ determines the isomorphism class of $(A, \iota, \lambda)[p]$. The Ekedahl-Oort (EO) stratum attached to $w \in J'W$ is the locally closed reduced subscheme $M^\text{EO}_k$ of $M_k$ whose $k$-points consist of $x \in M_k(k)$ with $\zeta(x) = w$. The EO stratum is non-empty for all $w \in J'W$ \[81\ Theorem 1.2 (1)\]. The EO stratum $M^\text{EO}_k$ attached to the identity element $e \in J'W$ is the unique EO stratum of dimension zero.

4.2.4. Let $(X, \iota, \lambda)$ be a $p$-divisible group with $\mathcal{D}$-structure over $k$ and let $b \in G(L)$ such that the isomorphism class of $(X, \iota, \lambda)$ corresponds to $[b] \in C(G)$ via (4.4). Further we put $H := \ker(G(W(k)) \to G(k))$. By \[81\ Remark 8.1\], the following conditions are equivalent.

- For all $b' \in HbH$ there exists an element $g \in G(W(k))$ such that $gb'\sigma(g) = b$. 

Any $p$-divisible group with $\mathcal{D}$-structure $(X', \iota', \lambda)$ with $(X', \iota', \lambda)[p] \simeq (X, \iota, \lambda)[p]$ is isomorphic to $(X, \iota, \lambda)$.

An element $b \in G(L)$ and also $(X, \iota, \lambda)$ are called minimal if they satisfy these equivalent conditions.

**Proposition 4.2.5** ([84 Proposition 9.17]). (1) For each $k$-point $(A, \iota, \lambda, \tilde{\eta})$ on the $0$-dimensional EO stratum $\mathcal{M}_K^e$, the associated $p$-divisible group with $\mathcal{D}$-structure $(A[p^\infty], \iota, \lambda)$ is minimal. Furthermore the set $\mathcal{M}_K^e(k)$ is a central leaf.

(2) There is an inclusion $\mathcal{M}_K^e \subset \mathcal{M}_K^\text{bas}$.

4.2.6. Let $\Sigma$ be a Dieudonné module over $W$. Let $D$ gives rise to a decomposition $\sigma$. Here $\tilde{\eta}$ that $\tilde{\eta}$.

For each $i \in \mathbb{Z}/2\mathbb{Z}$, we have that $\tilde{\eta}$.

Let $(A, \iota, \lambda, \tilde{\eta})$ be a $k$-point of $\mathcal{M}_K$, and $(A[p^\infty], \iota, \lambda)$ be the associated $p$-divisible group with $\mathcal{D}$-structure. Let $M$ be the covariant Dieudonné module of $A[p^\infty]$. The induced $O_{E_p}$-action on $M$ gives rise to a decomposition $M = M_1 \oplus M_2 = \oplus_i \mathbb{Z}/2\mathbb{Z} M_i$, where $M_i$ is the $\sigma_i$-component of $M$.

For each $i \in \mathbb{Z}/2\mathbb{Z}$, we have that $\langle M_i, M_i \rangle = 0$, and

$$F(M_i) \subset M_{i+1}, \quad V(M_i) \subset M_{i+1}, \quad \dim_k(M_1/V(M_2)) = r \quad \text{if } p \text{ is inert;}$$

$$F(M_i) \subset M_i, \quad V(M_i) \subset M_i, \quad \dim_k(M_1/V(M_1)) = r \quad \text{if } p \text{ is split.}$$

Here, the right equalities follows from the determinant condition ([44]).

4.2.7. Now we fix a $k$-point $(A, \iota, \lambda, \tilde{\eta})$ of the $0$-dimensional EO stratum $\mathcal{M}_K^e$. We assume first that $p$ is inert in $E$. In this case, the abelian variety $A$ over $k$ is superspecial ([84 Section 3.5.1]).

As in [19 Proposition 6], $A$ has a canonical model $\tilde{A}$ over $\mathbb{F}_{p^2}$, in which the geometric Frobenius is $[-p]$. Moreover, the covariant Dieudonné module $\tilde{M}$ of its $p$-divisible group $\tilde{A}[p^\infty]$ is isomorphic as a Dieudonné module over $W(\mathbb{F}_{p^2})$ to the direct sum of $n$-copies of the Dieudonné module $\mathcal{D}$ which is defined by

$$(\mathcal{D}, F, V) = \left( W(\mathbb{F}_{p^2})^{\oplus n}, \begin{pmatrix} -p & 1 \\ p & -1 \end{pmatrix}^{\sigma'}, \begin{pmatrix} pI_r & 0 \\ 0 & I_s \end{pmatrix} \right).$$

Here $\sigma'$ is the Frobenius morphism of $W(\mathbb{F}_{p^2})$. In particular one has $F = -V$ on $\tilde{M}$.

Since the correspondence $A \mapsto \tilde{A}$ is functorial, $\tilde{A}$ is equipped with additional structure corresponding to $\iota$ and $\lambda$. Thus $\tilde{M}$ has a principal polarization $\langle , \rangle : \tilde{M} \times \tilde{M} \to W(\mathbb{F}_{p^2})$, and an $O_{E_p}$-action, inducing a decomposition $\tilde{M} = \tilde{M}_1 \oplus \tilde{M}_2$.

**Proposition 4.2.8.** Assume that $p$ is inert in $E$. Then there is a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of $\tilde{M}$ over $W(\mathbb{F}_{p^2})$ such that

(i) it is a standard symplectic basis with respect to $\langle , \rangle$, that is, it satisfies that

$$\langle e_1, e_2 \rangle = 0 = \langle f_1, f_2 \rangle, \quad \langle e_1, f_2 \rangle = \delta_{ij} = -\langle f_1, e_i \rangle;$$

(ii) the elements $e_1, \ldots, e_n$ span $\tilde{M}_1$ and $f_1, \ldots, f_n$ span $\tilde{M}_2$;

(iii) the $\sigma'$-linear (resp. $\sigma'^{-1}$-linear) operator $F$ (resp. $V$) is given by

$$F = \begin{pmatrix} -pI_r & I_s \\ -pI_s & I_r \end{pmatrix} \sigma', \quad V = \begin{pmatrix} pI_r & 0 \\ 0 & pI_s \end{pmatrix} \sigma'^{-1}. $$

Proposition 4.2.10. Let $M \simeq \mathbb{D}^{\oplus n}$, there is an element $x \in M$ such that $(x, F(x))$ is a unit of $W(\mathbb{F}_p)$. We may assume that $x$ belongs to $\tilde{M}_1$ or $\tilde{M}_2$ since $\langle \tilde{M}_1, F(\tilde{M}_{i+1}) \rangle = 0$. We have that $\langle x, F(x) \rangle = \langle x, -V(x) \rangle = -\langle F(x), x \rangle^\sigma = \langle x, F(x) \rangle^\sigma$ and hence $\langle x, F(x) \rangle$ belongs to $W(\mathbb{F}_p)^\times$. Multiplying $x$ by an element of $W(\mathbb{F}_p)^\times$, we may further assume that $\langle x, F(x) \rangle = 1$ or $-1$ according as $x$ belongs to $\tilde{M}_1$ or $\tilde{M}_2$, since the norm map $W(\mathbb{F}_p)^\times \to W(\mathbb{F}_p)^\times, a \mapsto a^\sigma$ is surjective. Let $\tilde{N} := W(\mathbb{F}_p)x \oplus W(\mathbb{F}_p^\times)F(x)$. Then $\tilde{M} = \tilde{N} \oplus \tilde{N}^\perp$ where $\tilde{N}^\perp$ denotes the dual of $\tilde{N}$ with respect to $\langle , \rangle$. Furthermore one has $F = -V$ on $\tilde{N}^\perp$, and it follows that $\tilde{N}^\perp \simeq \mathbb{D}^{\oplus n-1}$. Moreover we have $\dim_k ((\tilde{N} \cap \tilde{M}_{i+1})/\tilde{N} \cap \tilde{M}_1) = 1$ if $x \in \tilde{M}_j$.

By induction, there exist elements $e_1, \ldots, e_r$ of $\tilde{M}_1$ and $f_{n-r}, \ldots, f_n$ of $\tilde{M}_2$ such that $\tilde{M}_1$ is spanned by $e_1, \ldots, e_r, F(f_{n-r}), \ldots, F(f_n)$, $\tilde{M}_2$ is spanned by $F(e_1), \ldots, F(e_r), f_{n-r}, \ldots, f_n$, and further $\langle e_i, F(e_i) \rangle = \langle F(f_j), f_j \rangle = 1$. □

4.2.9. Next we consider the case $p$ is split in $E$. We start by recalling the definition of minimal $p$-divisible groups (without additional structure) introduced by Oort [57]. Let $a, b$ be coprime non-negative integers. Let $M$ be the Dieudonné module with basis $e_1, e_2, \ldots, e_{a+b}$ over $W(k)$ on which the actions of $F$ and $V$ are given by $F(e_i) = e_{i+b} + V(e_i) = e_{i+a}$. Here we use the notation $e_{i+a+b} = pe_i$. We write $H_{a, b}$ for the the $p$-divisible group over $k$ corresponding to $M$. Then $H_{a, b}$ has height $a + b$ and is isoclinic of slope $b/(a + b)$. We call a $p$-divisible group $X$ over $k$ Oort-minimal if $X$ is isomorphic to a product $\prod_i H_{a_j, b_j}$ for some $a_j, b_j$.

Let $(A, \iota, \lambda, \eta) \in M^R(k)$ and let $M$ be the Dieudonné module associated with $(A[p\infty], \iota, \lambda)$. We set $m = \gcd(r, s)$, $r' = r/m$, $s' = s/m$, and $n' = n/m$.

Proposition 4.2.10. Assume that $p$ is split in $E$. Then there is a decomposition $M = (M')^{\oplus m}$ of $M$ into mutually orthogonal copies of a Dieudonné module $M'$ over $W(k)$ with an $O_{E_p}$-action, which has a basis $e_1, \ldots, e_{n'}, f_1, \ldots, f_{n'}$ such that

(i) it is a standard symplectic basis with respect to $\langle , \rangle$, that is, $\langle e_1, e_2 \rangle = 0 = \langle f_1, f_2 \rangle$, $\langle e_1, f_2 \rangle = \delta_{11} = -\langle f_1, e_1 \rangle$;

(ii) the elements $e_1, \ldots, e_{n'}$ span $M'_1$ and $f_1, \ldots, f_{n'}$ span $M'_2$ where $M' = M'_1 \oplus M'_2$ is the decomposition induced by the action of $O_{E_p}$;

(iii) the $\sigma$-linear (resp. $\sigma^{-1}$-linear) operator $F$ (resp. $V$) is given by

$$
\begin{pmatrix}
I_{r'} & pI_{s'} \\
pI_{s'} & I_{r'}
\end{pmatrix}
\sigma,
\begin{pmatrix}
I_{r'} & pI_{s'} \\
pI_{s'} & I_{r'}
\end{pmatrix}
\sigma^{-1}.
$$

Proof. Let $A_{W(k)} = A_1 \oplus A_2$ be the decomposition induced by an action of $O_E \otimes W(k) = W(k) \oplus W(k)$. Then we have $G(L) \simeq \mathbb{G}_m(L) \times \GL(A_1[1/p])$. Furthermore there exists an isomorphism $M \simeq A_{W(k)}$ identifying $M_1$ with $A_1$, under which $F$ corresponds to an element $b$ of $\mathbb{G}_m(L) \times \GL(A_1[1/p])$. Note that its $G(L)$-conjugacy class $[b] \in B(G, \mu)$ is basic by Proposition 4.2.5 (2). On the other hand, one has $F(M_1), V(M_1) \subset M_1$ by (4.7), and hence $M_1$ can be regarded as a Dieudonné module without additional structure. We write $b_1 \in \GL(A_1[1/p])$ for the element corresponding to the operator $F|_{M_1}$ under the identification $M_1 \simeq A_2$. Then $b_1 = p\gamma_2(b)$ by the construction.

Let $X$ be the $p$-divisible group over $k$ corresponding to the Dieudonné module $M_1$. By Proposition 4.2.5 (1), the element $b \in G(L)$ is minimal, and therefore $b_1$ is minimal as an element of $\GL(A_1[1/p]) \simeq \GL_n(L)$. It follows from 57 (see also 54 Corollary 9.13) that $X$ is Oort-minimal. Further, the above argument shows that the $\GL_n(L)$-conjugacy class of $b_1$ is basic, and hence $X$ is isoclinic.
In addition we have that $\dim_k M_1/V(M_1) = r$ and $\dim_k M_1/F(M_1) = s$ by (4.7). These conditions imply that $X \simeq H_{\mathfrak{p}, \mathfrak{s}'}$, i.e., there is a decomposition $M_1 = (M_1')^{\oplus m}$ of $M_1$ into a Dieudonné module $M_1'$ with a basis $e_i, \ldots, e_{n'}$ such that $F(e_i) = e_{i+1}$ and $V(e_i) = pe_{i+1}$.

There exist elements $f_1, \ldots, f_{n'}$ of $M_2$ satisfying condition (i) since $\langle \cdot, \cdot \rangle$ is a perfect pairing and $M_1^+ = M_1$. Let $M'_2$ be the submodule of $M_2$ spanned by $f_1, \ldots, f_{n'}$ and let $M' = M_1' \oplus M'_2$, so that condition (ii) is satisfied. Using relation (4.3) we determine the actions of $F$ and $V$ on $M'_2$ and obtain (4.9).

By (4.9), the abelian variety $A$ over $k$ is superspecial (i.e., $FM = VM$) if and only if $r = s$. See also [84] Section 3.5.1.

4.3. Maximal parahoric subgroups of unitary groups.

4.3.1. We briefly recall some facts about maximal parahoric subgroups of unitary (similitude) groups and of inner forms of $GL_n$. Our reference is Tits [72].

First let $E_p/Q_p$ be the unramified quadratic field extension. Let $N = N_{E_p/Q_p}$ denote the norm map of $E_p/Q_p$. The quotient group $Q_p^\times /N E_p^\times$ is of order two, having $\{1, p\}$ as a system of representatives. The determinant $d(V)$ of a Hermitian space $(V, \phi)$ takes value in $Q_p^\times /N E_p^\times$, as in Section 2.1.2.

Let $H$ be the Hermitian lattice of rank two over $OE_p$ having a basis $\{e_1, e_2\}$ with Gram matrix

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\]

and let $\mathbb{H} := H \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then $d(\mathbb{H}) = -1 \in Q_p^\times /N E_p^\times$. Let $(1)$ and $(p)$ denote the lattices of rank one with Gram matrix $(1)$ and $(p)$, respectively. Further, let $V_1 := (1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $V_p := (p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Let $(V, \phi)$ be a non-degenerate Hermitian space of dimension $n$ over $E_p$. Then there is an isomorphism

\[
V \simeq \begin{cases}
\mathbb{H}^{(n-1)/2} \oplus V_1 & \text{if } n \text{ is odd, } d(V) = 1; \\
\mathbb{H}^{(n-1)/2} \oplus V_p & \text{if } n \text{ is odd, } d(V) = p; \\
\mathbb{H}^{n/2} & \text{if } n \text{ is even, } d(V) = 1; \\
\mathbb{H}^{(n/2)-1} \oplus V_1 \oplus V_p & \text{if } n \text{ is even, } d(V) = p.
\end{cases}
\]

Let $G = GU(V, \phi)$ (resp. $G^1 = U(V, \phi)$) denote the unitary similitude group (resp. the unitary group) of $(V, \phi)$ over $\mathbb{Q}_p$. When $n$ is odd, the groups $G$ of the above two spaces with $d(V) = 1$ and $p$ are isomorphic and unramified. When $n$ is even, the group $G$ is unramified or not according as $d(V) = 1$ or $p$. The same statements hold true for the groups $G^1$.

Let $H(1)$ be the $OE_p$-lattice in $\mathbb{H}$ spanned by the basis $\{e_1, pe_2\}$, whose Gram matrix is

\[
\begin{pmatrix}
1 & \quad p
\end{pmatrix}
\]

Let $t$ be an integer such that $0 \leq t \leq n$ and $d(V) = p^t \in Q_p^\times /N (E_p^\times)$. Under the identification in (4.10), we define an $OE_p$-lattice $L_t$ in $V$ by

\[
L_t := \begin{cases}
H^{(n-t-1)/2} \oplus H(1)^{t}/2 & \text{if } n \text{ is odd, } d(V) = 1 \ (t \text{ is even}); \\
H^{(n-t)/2} \oplus H(1)^{t-1}/2 & \text{if } n \text{ is odd, } d(V) = p \ (t \text{ is odd}); \\
H^{(n-t)/2} \oplus H(1)^{t-1}/2 & \text{if } n \text{ is odd, } d(V) = 1 \ (t \text{ is even}); \\
H^{(n-t-1)/2} \oplus H(1)^{t-1}/2 & \text{if } n \text{ is odd, } d(V) = p \ (t \text{ is odd}).
\end{cases}
\]

Note that $L_t$ is unimodular if and only if $d(V) = 1$ and $t = 0$. Conversely, any unimodular lattice $\Lambda$ of rank $n$ over $OE_p$ satisfies $d(\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = 1$ and $\Lambda \simeq L_0$.

We define subgroups $P_t$ of $G(Q_p)$ and $P^1_t$ of $G^1(Q_p)$ by

\[
P_t := \text{Stab}_{G(Q_p)} L_t, \quad P^1_t := \{ g \in P_t \mid \text{sim}(g) = 1 \}.
\]
Then $\mathcal{P}_t$ (resp. $\mathcal{P}_t^1$) is a maximal parahoric subgroup of $G(\mathbb{Q}_p)$ (resp. $G^1(\mathbb{Q}_p)$). Further, any maximal parahoric subgroup of $G(\mathbb{Q}_p)$ (resp. $G^1(\mathbb{Q}_p)$) is conjugate to a subgroup $\mathcal{P}_t$ (resp. $\mathcal{P}_t^1$) for some $t$. In Table 4, we give special and hyperspecial parahoric subgroups among $(\mathcal{P}_t)_t$.

**Table 4. Special and hyperspecial parahoric subgroups among $(\mathcal{P}_t)_t$**

| $n$ | $d(V)$ | special | hyperspecial |
|-----|--------|---------|--------------|
| odd | 1      | $\mathcal{P}_0, \mathcal{P}_{n-1}$ | $\mathcal{P}_0$ |
| odd | $p$    | $\mathcal{P}_1, \mathcal{P}_n$ | $\mathcal{P}_n$ |
| even | 1      | $\mathcal{P}_0, \mathcal{P}_n$ | $\mathcal{P}_0, \mathcal{P}_n$ |
| even | $p$    | $\mathcal{P}_1, \mathcal{P}_{n-1}$ | none |

Let $\mathcal{P}_t$ (resp. $\mathcal{P}_t^1$) denote the smooth model of $\mathcal{P}_t$ (resp. $\mathcal{P}_t^1$) over $\mathbb{Z}_p$, and let $\mathcal{P}_t$ (resp. $\mathcal{P}_t^1$) denote the maximal reductive quotient of the special fiber $\mathcal{P}_t \otimes \mathbb{F}_p$ (resp. $\mathcal{P}_t^1 \otimes \mathbb{F}_p$). Then there are isomorphisms of algebraic groups over $\mathbb{F}_p$,

$$\mathcal{P}_t \simeq G(U_{n-t} \times U_t), \quad \mathcal{P}_t^1 \simeq U_{n-t} \times U_t.$$  

Here, by $G(U_{n-t} \times U_t)$ we mean the group of pairs of matrices $(g_1, g_2) \in GU_{n-t} \times GU_t$ having the same similitude factor.

4.3.2. Next let $m$ be a divisor of $n$ and $n' := n/m$. Let $B$ be a division algebra over $\mathbb{Q}_p$ of degree $n'^2$ and let $O_B$ be the unique maximal order of $B$. Then the algebraic group $G^1 := \text{Res}_{B/\mathbb{Q}_p} \text{GL}_{m,B}$ is an inner form of the unramified group $GL_n, \mathbb{Q}_p$. The group $G^1(\mathbb{Q}_p) = \text{GL}_m(B)$ contains a unique conjugacy class of maximal parahoric subgroup, represented by $\text{GL}_m(O_B)$. In particular, any maximal parahoric subgroup of $G^1(\mathbb{Q}_p)$ is special. The maximal reductive quotient of the special fiber over $\mathbb{F}_p$ of its smooth model is isomorphic to $\text{Res}_{\mathbb{F}_q/\mathbb{F}_p} \text{GL}_{m,\mathbb{F}_q}$, where $\mathbb{F}_q$ denotes the finite field of order $q := p^{n'}$.

4.4. The group of automorphisms of a basic isocrystal.

4.4.1. For each $b \in G(L)$ we define an algebraic group $J_b$ over $\mathbb{Q}_p$ with functor of points

$$J_b(R) = \{ g \in G(L \otimes_{\mathbb{Q}_p} R) \mid g^{-1}b\sigma(g) = b \}$$

for any $\mathbb{Q}_p$-algebra $R$. Up to isomorphism, $J_b$ depends only on the $G(L)$-$\sigma$-conjugacy class $[b]$. By [39], the class $[b]$ is basic if and only if $J_b$ is an inner form of $G_{\mathbb{Q}_p}$.

Let $(A, \iota, \lambda, \bar{\eta})$ be a $k$-point of $\mathcal{M}_K$, and let $(A[p^\infty], \iota, \lambda)$ be the associated $p$-divisible group with $\mathcal{D}$-structure. Let $(M, F, V)$ be the Dieudonné module of $(A[p^\infty], \iota, \lambda)$, and let $M[1/p] = M \otimes_{W(k)} L$ be the associated isocrystal. Let $b$ be the element of $G(L)$ corresponding to the operator $F$ via an identification $M \simeq A_{W(k)}$ as in Section 4.2.2. Then there is an isomorphism

$$J_b(R) \simeq \{ (g, c) \in \text{GL}_{(E \otimes \mathbb{Q}_L) \otimes \mathbb{Q}_p} R(M[1/p]_R) \times R^\times \mid (F \otimes \text{id}_R)g = g(F \otimes \text{id}_R), \langle gx, gy \rangle = c(x, y) \}$$

for any $\mathbb{Q}_p$-algebra $R$, where $M[1/p]_R := M[1/p] \otimes_{\mathbb{Q}_p} R$.

Assume now that $(A, \iota, \lambda, \bar{\eta})$ is lying on the 0-dimensional EO stratum $\mathcal{M}_K^c$. Then the class $[b]$ is the unique basic element of $B(G, \mu)$ by Proposition 4.2.5 (2). Let

$$I_p^c := \text{Stab}_{J_b(\mathbb{Q}_p)} M.$$  

**Proposition 4.4.2.** (1) Assume that $p$ is inert in $E$. Let $(V, \phi)$ be a Hermitian space over $E_p$ with $d(V) = p^s$. Then

$$J_b \simeq \text{GU}(V, \phi).$$

In particular, $J_b$ is unramified if and only if $rs$ is even.
The subgroup $I_p^e$ of $J_b(Q_p)$ is a maximal parahoric subgroup, which is conjugate to the subgroup $P_s$ defined in Section 4.3.

(2) Assume that $p$ is split in $E$. Let $m = \gcd(r, s)$ and $n' = n/m$. Let $B$ be a division algebra over $Q_p$ of degree $n'^2$ with $\mathrm{inv}(B) = r/n$. Then

$$J_b \simeq \mathrm{Res}_{B/Q_p} \mathrm{GL}_{m,B} \times \mathbb{G}_{m,Q_p}.$$}

Let $O_B$ be the unique maximal order of $B$. Then the subgroup $I_p^e$ is a maximal parahoric subgroup of $J_b(Q_p)$ conjugate to $\mathrm{GL}_m(O_B) \times \mathbb{Z}_p^\times$, under the identification $J_b(Q_p) \simeq \mathrm{GL}_m(B) \times \mathbb{Z}_p^\times$.

Proof. (1) As in Section 4.2.7, $M$ has a model $\tilde{M}$ over $W(F_{p^2})$. By functoriality, we have

$$J_b(Q_p) \simeq \{(g, c) \in \mathrm{GL}_{E\otimes Q(W(F_{p^2})[1/p])} \tilde{M}[1/p] \times \mathbb{Q}_p^\times \mid Fg = gF, \langle gx, gy \rangle = c(x, y)\}.$$ Let $(g, c)$ be an element of $I_p^e \subset J_b(Q_p)$. Then $c \in \mathbb{Z}_p^\times$. As in Proposition 4.2.8, we choose bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ of $\tilde{M}_1$ and $\tilde{M}_2$, respectively. Since $g$ preserves $\tilde{M}_1$ and $\tilde{M}_2$, one can write $g = \mathrm{diag}(g_1, g_2)$ where we regard $g_1, g_2$ as elements of $\mathrm{Mat}_n(W(F_{p^2}))$ via the fixed bases. Then we have

$$\begin{pmatrix} g_1' & g_2' \\ \cdot & I_n \end{pmatrix} \begin{pmatrix} I_n \\ \cdot \end{pmatrix} = \begin{pmatrix} g_1 & g_2 \\ \cdot & I_n \end{pmatrix},$$

and hence $g_2' \cdot g_1 = c \cdot I_n$. Further, the relation $Fg = gF$ and equality (4.8) show that

$$\Psi \cdot g_2'' = g_2 \Psi, \quad \Psi := \mathrm{diag}(1^r, (-p)^s),$$

where $\sigma'$ denotes the Frobenius morphism of $W(F_{p^2})$.

Now we identify $O_{E_p}$ with $W(F_{p^2})$. Then $\det(\Psi) = p^s$ as elements of $\mathbb{Q}_p^\times / \mathbb{N}_{E_p/Q_p}(E_p^\times)$. Further, the Hermitian lattice of rank $n$ over $O_{E_p}$ defined by $\Psi$ is isomorphic to the lattice $(\mathcal{L}_s, \phi)$ as in (4.11). It follows that

$$I_p^e \simeq \{(g_1, c) \in \mathrm{GL}_n(O_{E_p}) \times \mathbb{Z}_p^\times \mid (g_1')^t \cdot \Psi \cdot g_1 = c \cdot \Psi \} \simeq \mathrm{GU}(\mathcal{L}_s, \phi)(\mathbb{Z}_p).$$

A similar argument can be applied to the module $\tilde{M}[1/p] \otimes_{\mathbb{Q}_p} R$ for any $\mathbb{Q}_p$-algebra $R$, and hence

$$J_b \simeq \mathrm{GU}(\mathcal{L}_s \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \phi) = \mathrm{GU}(\mathcal{V}, \phi).$$

(2) Assume first that $r$ and $s$ are relatively prime. We choose a basis of $M = M' = M_1' \oplus M_2'$ over $W(k)$ as in Proposition 4.2.10. Then for each element $(g, c)$ of $I_p^e$, we can write $g = \mathrm{diag}(g_1, g_2)$ with $g_1, g_2 \in \mathrm{Mat}_n(W(k))$. Similarly to the previous case, we have $g_2' \cdot g_1 = c \cdot I_n$. Further, equality (4.9) shows that

$$g_i \begin{pmatrix} pI_s \\ I_r \end{pmatrix} = \begin{pmatrix} pI_s \\ I_r \end{pmatrix} g_i^\sigma, \quad i = 1, 2.$$ It follows that $I_p^e \simeq I_p^{e,1} \times \mathbb{Z}_p^\times$ where

$$I_p^{e,1} = \left\{g_1 \in \mathrm{GL}_n(W(k)) \mid g_1 \begin{pmatrix} pI_s \\ I_r \end{pmatrix} = \begin{pmatrix} pI_s \\ I_r \end{pmatrix} g_1^\sigma \right\}.$$ The group $I_p^{e,1}$ is therefore isomorphic to the automorphism group of the Dieudonné module $M_1'$ without additional structure given in Proposition 4.2.10. Recall that $M_1'$ corresponds to an Oort-minimal $p$-divisible group over $k$. Let $B$ be a division algebra of degree $n^2$ with $\mathrm{inv}(B) = n$. From Lemmas 3.1 and 3.4 it follows that $I_p^{e,1} \simeq O_B^\times$ as groups and $J \simeq \mathrm{Res}_{B/Q_p} \mathrm{G}_{m,B} \times \mathbb{G}_{m,Q_p}$ as algebraic groups over $Q_p$.

For general $r$ and $s$, there is a decomposition $M = (M')^\oplus m$ and thus we see the assertion. \hfill \qed

Let $[b] \in B(G, \mu)$ be as above and let $J_b^{\text{der}}$ be the derived subgroup of $J_b$. The following lemma will be used to describe the connected components of $\mathcal{M}_K^{\text{bas}}$ in Section 6.
Lemma 4.4.3. Assume \( r, s > 0 \). Then \( J^\text{der}_b(\mathbb{Q}_p) \) is compact if and only if \( p \) is inert in \( E \) and \( (r, s) = (1, 1) \), or \( p \) is inert in \( E \) and \( \gcd(r, s) = 1 \).

Proof. Assume first that \( p \) is inert in \( E \). Then we have that \( J^\text{der}_b \simeq \text{SU}(\mathcal{V}, \phi) \) for a Hermitian space \( (\mathcal{V}, \phi) \) of dimension \( n \geq 2 \). When \( n \geq 3 \), the space \( \mathcal{V} \) is always isotropic and \( J^\text{der}_b(\mathbb{Q}_p) \) is not compact. When \( n = 2 \), namely \( (r, s) = (1, 1) \), we have \( d(\mathcal{V}) = p \). Hence \( \mathcal{V} \) is anisotropic and \( J^\text{der}_b(\mathbb{Q}_p) \) is compact. Next assume that \( p \) is split in \( E \). Then \( J^\text{der}_b(\mathbb{Q}_p) \) is isomorphic to the group \( \{ g \in \text{GL}_m(B) \mid \text{nr}(g) = 1 \} \), where \( \text{nr} \) denotes the reduced norm on \( \text{GL}_m(B) \). This group is compact if and only if \( m = 1 \). \( \square \)

5. Irreducible and connected components of affine Deligne-Lusztig varieties

5.1. Irreducible components of ADLVs for unramified groups.

5.1.1. Let \( k \) be an algebraic closure of \( \mathbb{F}_p \). Let \( W(k) \) be the ring of Witt vectors over \( k \) with Frobenius automorphism \( \sigma \), and let \( L = W(k)[1/p] \) be the fraction field of \( W(k) \). Let \( G \) be a connected reductive group scheme over \( \mathbb{Z}_p \). Then \( G_\mathbb{Q}_p \) is automatically unramified, and \( G(\mathbb{Z}_p) \) is a hyperspecial subgroup of \( G(\mathbb{Q}_p) \). We fix a maximal torus \( T \) of \( G \) and a Borel subgroup \( B \) containing \( T \). We use the same notations as those in Section 4.2.2. For a dominant coweight \( \mu \in X_*(T)^+ \) and an element \( b \in G(L) \), we set

\[
X_\mu(b)(k) = \{ g \in G(L)/G(W(k)) \mid g^{-1}b\sigma(g) \in G(W(k))\mu(p)G(W(k)) \}.
\]

We can identify \( X_\mu(b)(k) \) with the set of \( k \)-points of a locally closed subscheme \( X_\mu(b) \) of the Witt vector partial affine flag variety \( \text{Gr}_G \) constructed by Zhu [87] and Bhatt-Scholze [3]. The scheme \( X_\mu(b) \) is locally of perfectly finite type over \( k \) [27, Lemma 1.1]. We call \( X_\mu(b) \) the affine Deligne-Lusztig variety associated to \( (G, \mu, b) \). Note that affine Deligne-Lusztig varieties \( X_\mu(b) \) depends only on the \( \sigma \)-conjugacy class \( [b] \in B(G) \) up to isomorphism. Further \( X_\mu(b) \) is non-empty if and only if \( [b] \in B(G, \mu) [18] \).

We write \( \text{Irr}(X_\mu(b)) \) for the set of irreducible components of the affine Deligne-Lusztig variety \( X_\mu(b) \), and \( \text{Irr}^{\text{top}}(X_\mu(b)) \) for the set of those components which are top-dimensional. It is conjectured that \( X_\mu(b) \) is equi-dimensional (see [27, Theorem 3.4]). However, it always holds true if \( \mu \) is minuscule.

For each \( b \in G(L) \), let \( J_b \) be an algebraic group over \( \mathbb{Q}_p \) as defined in (1.12). Then \( J_b(\mathbb{Q}_p) \) acts on the set \( X_\mu(b) \) by left multiplication. This action induces an action of \( J_b(\mathbb{Q}_p) \) on \( \text{Irr}^{\text{top}}(X_\mu(b)) \).

5.1.2. Let \( X_*(T)^\sigma \) and \( X_*(T)_\sigma \) denote the \( \sigma \)-invariants and coinvariants of \( X_*(T) \), respectively. For each \( \lambda \in X_*(T) \), let \( \lambda^\sigma \) denote its image in \( \pi_1(G)_\sigma \), \( \lambda \) denote its image in \( X_*(T)_\sigma \), and \( \lambda^\natural \) denote its \( \sigma \)-average in \( X_*(T)_\mathbb{Q} \). Then we have a canonical isomorphism \( X_*(T)_\sigma,\mathbb{Q} \cong X_*(T')_\mathbb{Q} \) where \( \lambda \mapsto \lambda^\natural \).

Now we fix \( \mu \in X_*(T)^+ \) and \( [b] \in B(G, \mu) \). Let \( \kappa_G(b) \in \pi_1(G)_\sigma \) denote the Kottwitz point and \( \nu_G(b) \in X_*(T')_\mathbb{Q}^{\nu,\sigma} \) denote the Newton point. Hamacher and Viehmann [27, Lemma/Definition 2.1] proved that the set

\[
\{ \lambda \in X_*(T)_\sigma \mid \lambda^\natural = \kappa_G(b), \lambda^\sigma \leq \nu_G(b) \}
\]

has a unique maximum element \( \Delta_G(b) \) characterized by the property that \( \lambda_G(b)^\natural = \kappa_G(b) \) and that \( \nu_G(b) - \lambda_G(b)^\sigma \) is equal to a linear combination of simple coroots with coefficients in \( [0, 1) \cap \mathbb{Q} \). This element can be regarded as “the best integral approximation” of the newton point \( \nu_G(b) \).

Let \( \hat{G} \) be the Langlands dual of \( G \) defined over \( \overline{\mathbb{Q}}_\ell \) with \( \ell \neq p \). Let \( \hat{B} \) be a Borel subgroup of \( \hat{G} \) with maximal torus \( \hat{T} \) such that there exists a bijection \( X_*(T)^+ \simeq X^*(\hat{T})^+ \). We write \( V_\mu \) for the irreducible \( \hat{G} \)-module of highest weight \( \mu \). Let \( V_\mu(\Delta_G(b)) \) be the sum of \( \lambda \)-weight spaces \( V_\mu(\lambda) \) for \( \lambda \in X_*(T) = X^*(\hat{T}) \) satisfying \( \lambda = \Delta_G(b) \).
The following theorem was conjectured by Chen and X. Zhu, and proved by Zhou-Y. Zhu and Nie.

**Theorem 5.1.3** ([86] Theorem A, [55] Theorem 4.10). There exists a canonical bijection between the set $J_b(\mathbb{Q}_p)\backslash \text{Irr}^{\top}(X_\mu(b))$ and the Mirković-Vilonen basis of $V_\mu(\Delta_G(b))$ constructed in [52]. In particular,

$$|J_b(\mathbb{Q}_p)\backslash \text{Irr}^{\top}(X_\mu(b))| = \dim V_\mu(\Delta_G(b)).$$

Note that this theorem has first been shown by Xiao-X. Zhu [85] for $[b]$ such that the $\mathbb{Q}_p$-ranks of $J_b$ and $G$ coincide, and by Hamacher-Viehmann [27] if $\mu$ is minuscule and either $G$ is split or $b$ is superbasic.

Let $W$ denote the absolute Weyl group of $G$. If $\mu$ is minuscule, then the dimension of $V_\mu(\lambda)$ for a $\lambda \in X_*(T)$ equals 1 or 0 according as $\lambda$ belongs to the $W$-orbit of $\mu$ or not. Hence we have

$$(5.1) \quad \dim V_\mu(\Delta_G(b)) = \#(W \cdot \mu \cap \{\lambda \in X_*(T) \mid \lambda \leq \Delta_G(b)\}).$$

### 5.2. Irreducible components of ADLVs for unitary similitude groups.

#### 5.2.1. Let $\mathcal{D}$, $p$, $G$ be as in Section 4.14

We write $E_p = E \otimes \mathbb{Q}_p$, $V_p = V \otimes \mathbb{Q}_p$, and $\Lambda_p = \Lambda \otimes \mathbb{Z}_p$. Then $V_p$ is of rank $n$ over $E_p$, and equipped with a Hermitian form $\varphi$. Further, the $O_{E_p}$-lattice $\Lambda_p$ is unimodular. In the rest of this section, let $G := G_{\mathbb{Q}_p} \simeq GU(V_p, \varphi)$.

Assume first that $p$ is inert in $E$. For an $i \in \{1, \ldots, n\}$, write $i^\vee = n - i + 1$. Let $\Phi$ denote the anti-diagonal matrix whose $(i, i^\vee)$-entry is 1. By the classification of Hermitian spaces and lattices in [4.10] and [4.11], the space $V_p$ has a basis $\{e_i, \ldots, e_n\}$ over the field $E_p$ with Gram matrix $\Phi$. In other words, the Hermitian form $\varphi$ on $V_p$ is defined by $\varphi(e_i, e_{j^\vee}) = \delta_{ij}$ for $i, j \in I$.

Fix an isomorphism $\text{End}(V_p \otimes \mathbb{Q}_p) = \text{End}(V_p) \otimes \mathbb{Q}_p \iso \text{Mat}_n(\mathbb{Q}_p)^2$ by $A \otimes \alpha \mapsto (A\alpha, \bar{\alpha})$. The matrix $\Phi \in \text{End}(V_p)$ corresponds to $(\Phi, \bar{\Phi})$ and we have

$$G(\mathbb{Q}_p) = \{(t_0, A, B) \in \mathbb{G}_m(\mathbb{Q}_p) \times \text{GL}_n(\mathbb{Q}_p)^2 \mid (B^t, A^t)(\Phi, \bar{\Phi})(A, B) = (t_0 \Phi, t_0 \bar{\Phi})\}$$

$$= \{(t_0, A, B) \in \mathbb{G}_m(\mathbb{Q}_p) \times \text{GL}_n(\mathbb{Q}_p)^2 \mid B = t_0 \Phi^{-1}(A^t)^{-1} \bar{\Phi}\}$$

$$\simeq \mathbb{G}_m(\mathbb{Q}_p) \times \text{GL}_n(\mathbb{Q}_p) \quad \text{(via the first and second projections)}.$$

Let $\mathbb{G}_m(\mathbb{Q}_p) \subset \text{GL}_n$ be the diagonal maximal torus, and let $T_{\mathbb{Q}_p} = \mathbb{G}_m(\mathbb{Q}_p) \times \mathbb{G}_n(\mathbb{Q}_p)$ be a maximal torus of $\mathbb{G}_m(\mathbb{Q}_p) \times \text{GL}_n(\mathbb{Q}_p)$. Further, let $B$ be the Borel subgroup of $G(\mathbb{Q}_p)$ given by the product of $\mathbb{G}_m(\mathbb{Q}_p)$ and the group of upper triangular matrices in $\text{GL}_n(\mathbb{Q}_p)$. Let $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n$ be the $\mathbb{Z}$-basis of $X^*(T)$ defined by $\varepsilon_i(t) = t_i$ for any $t = (t_0, t_1, \ldots, t_n) \in T_{\mathbb{Q}_p}$. Let $\varepsilon^*_i \in X_*(T)$ denote the dual of $\varepsilon_i$. Then the Frobenius $\sigma$ acts on $X_*(T)$ by

$$(5.2) \quad \sigma(\varepsilon^*_i) = \varepsilon^*_0 + \cdots + \varepsilon^*_i.$$

The $\sigma$-average of a cocharacter $\lambda = \sum_{i=0}^n \lambda_i \varepsilon^*_i \in X_*(T)$ is

$$\lambda^\vee = \frac{1}{2}(\lambda + \sigma \lambda) = \lambda_0 \varepsilon^*_0 + \frac{1}{2} \{(\lambda_0 + \lambda_1 - \lambda_n) \varepsilon^*_1 + \cdots + (\lambda_0 + \lambda_n - \lambda_1) \varepsilon^*_n\}.$$ 

The map $X_*(T) \rightarrow \mathbb{Z}^2$ where $\sum_{i=0}^n a_i \varepsilon^*_i \mapsto (a_0, \sum_{i=1}^n a_i)$ induces an identification

$$(5.3) \quad \pi_1(G) = X_*(T) / \sum_{\alpha \in \Phi^+} \mathbb{Z} \alpha^* \simeq \mathbb{Z}^2.$$

Here, $\Phi^+$ denotes the set of positive roots corresponding to $B$, and $\alpha^*$ denotes the coroot corresponding to $\alpha$. Under this identification, we have that $\sigma(1, 0) = (1, n)$ and $\sigma(0, 1) = (0, -1)$. Hence
(0, n) and (0, 2) generate the submodule \((1 - \sigma)\pi_1(G)\). Further, \(\pi_1(G)\)' is generated by either \((2, n)\) when \(n\) is odd, or \((1, n/2)\) when \(n\) is even. It follows that

\[
\pi_1(G)_\sigma \simeq \begin{cases} 
\mathbb{Z} & \text{if } n \text{ is odd;} \\
\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n \text{ is even,}
\end{cases}
\text{ and } \pi_1(G)^{\sigma} \simeq \mathbb{Z}.
\]

The simple roots are \(\Delta = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n\}\) and the simple coroots are \(\Delta^\vee = \{\varepsilon_1^* - \varepsilon_2^*, \ldots, \varepsilon_{n-1}^* - \varepsilon_n^*\}\). Hence, a cocharacter \(\lambda = \sum_{i=0}^n \lambda_i \varepsilon_i^* \in X_*(T)\) is dominant if and only if \(\lambda_i \geq \lambda_{i+1}\) for all \(i \geq 1\).

Let \([\mu]\) be the conjugacy class of the cocharacter \(\mu_h\) defined by \(\mathcal{D}\), regarded as a \(W\)-orbit in \(X_*(T)\) as in Section 4.2.2. Let \(\mu\) be the dominant representative of \([\mu]\). Then \(\mu = \sum_{i=0}^n \varepsilon_i^*\) or \(\sum_{i=0}^n \varepsilon_i^*\). Now we assume that \(\mu = \sum_{i=0}^n \varepsilon_i^*\). This assumption will be justified by the fact that replacing \(r\) with \(s = n-r\) does not change the final result, see Proposition 5.2.3. Let \([b]\) be the basic \(\sigma\)-conjugacy class in \(B(G, \mu)\). The Kottwitz point \(\kappa_G(b)\) is equal to the image of \(\mu\) in \(\pi_1(G)_\sigma\), and hence

\[
\kappa_G(b) = \begin{cases} 
1 & \text{if } n \text{ is odd;} \\
(1, r \text{ mod } 2) & \text{if } n \text{ is even.}
\end{cases}
\]

The Newton point \(\nu_G(b) \in X_*(T)_\mathbb{Q}\) satisfies that

\[
\nu_G(b) \leq \mu^\circ = \varepsilon_0^* + \varepsilon_1^* + \cdots + \varepsilon_{\min(r,n-r)}^* + \frac{1}{2}(\varepsilon_{\min(r,n-r)+1}^* + \cdots + \varepsilon_{\max(r,n-r)}^*).\]

Further, \(\nu_G(b)\) factors through the \(\sigma\)-invariant part of the center \(Z_G\) of \(G\) since \([b]\) is basic. The subgroup \(X_*(Z_G)\) of \(X_*(T)\) is generated by \(\varepsilon_0^*, \varepsilon_1^* + \cdots + \varepsilon_n^*\). It follows that

\[
\nu_G(b) = \varepsilon_0^* + \frac{1}{2}(\varepsilon_1^* + \cdots + \varepsilon_n^*) = (\varepsilon_0^*)^\circ \in X_*(T)_\mathbb{Q}^{\sigma}.
\]

Lemma 5.2.2. We define an element \(\tilde{\lambda}\) of \(X_*(T)\) by

\[
\tilde{\lambda} = \begin{cases} 
\varepsilon_0^* + \varepsilon_{n/2+1}^* & \text{if } n \text{ is even and } r \text{ is odd;} \\
\varepsilon_0^* & \text{otherwise.}
\end{cases}
\]

Then we have an equality as elements of \(X_*(T)_\sigma\):

\[
\Delta_G(b) = \tilde{\lambda}.
\]

Proof. The image of \(\tilde{\lambda}\) in \(\pi_1(G)_\sigma\) is equal to the Kottwitz point \(\kappa_G(b)\) by (5.5). Further,

\[
\nu_G(b) - \tilde{\lambda}^\circ = \begin{cases} 
-\varepsilon_{n/2+1}^* & \text{if } n \text{ is even and } r \text{ is odd;} \\
0 & \text{otherwise,}
\end{cases}
\]

and \(\varepsilon_{n/2}^* - \varepsilon_{n/2-1}^*\) is a simple coroot. The characterization of \(\Delta_G(b)\) implies the assertion. \(\square\)

Proposition 5.2.3. We have that

\[
|J_b(\mathbb{Q}_p)\backslash \text{Irr}(X_*(b))| = \begin{cases} 
\binom{n/2-1}{(r-1)/2} & \text{if } n \text{ is even and } r \text{ is odd;} \\
\binom{n/2}{[r/2]} & \text{if } n \text{ is odd or } r \text{ is even.}
\end{cases}
\]

Proof. We consider the following collections of indices:

\[
\mathcal{I} \subset \begin{cases} 
\{1, \ldots, n/2 - 1\} & \text{if } n \text{ is even and } r \text{ is odd;} \\
\{1, \ldots, [n/2]\} & \text{otherwise,}
\end{cases}
\text{ and } |\mathcal{I}| = [r/2].
\]
The number of such collections \( \mathcal{I} \) is equal to \( \left( \frac{n/2}{r} - 1 \right)/2 \) if \( n \) is even and \( r \) is odd, and to \( \left( \frac{n/2}{r} \right) \) otherwise.

The \( W \)-orbits of \( \mu \) are \( \{ \varepsilon_0 + \sum_{i \in J} \varepsilon_i^* \in X_\ast(T) \mid J \subset \{1, \ldots, n\}, |J| = r \} \). It follows from Lemma \ref{lem:2.2} that the assignment \( \mathcal{I} \mapsto \lambda + \sum_{i \in \mathcal{I}} (\varepsilon_i^* + \varepsilon_i^*) \in X_\ast(T) \) induces a bijection from the set of collections \( \mathcal{I} \) satisfying \ref{eq:5.6} to the intersection \( W \cdot \mu \cap \{ \lambda \in X_\ast(T) \mid \lambda = \Delta_G(b) \} \). The assertion thus follows from Theorem \ref{thm:1.3} and equality \ref{eq:5.1}.

5.2.4. Next we assume that \( p \) is split in \( E \). Then \( E_p = \mathbb{Q}_p \oplus \mathbb{Q}_p \). There is an isomorphism \( G \simeq \mathbb{G}_m \times GL_n \mathbb{Q}_p \) over \( \mathbb{Q}_p \), and in particular \( G \) is split. We fix a maximal torus \( T \) and a Borel subgroup \( B \) of \( G \) as in the previous case. Let \( \mu \in X_\ast(T)^+ \) be the coweight defined by \( \mathcal{D} \), and let \( [b] \) be the basic \( \sigma \)-conjugacy class in \( B(G, \mu) \).

**Proposition 5.2.5.** We have that

\[ |J_b(\mathbb{Q}_p) \backslash \text{Irr}(X_\mu(b))| = 1. \]

**Proof.** Since the Frobenius \( \sigma \) acts on \( X_\ast(T) \) trivially, the set \( \{ \lambda \in X_\ast(T) \mid \lambda = \Delta_G(b) \} \) is a singleton \( \{ \Delta_G(b) \} \). Hence the assertion follows from Theorem \ref{thm:1.3} and equality \ref{eq:5.1}.

For a description of \( \Delta_G(b) \) in this case, see \cite[Example 2.3]{27}.

5.3. Connected components of ADLVs for unitary similitude groups.

5.3.1. Let \( w : X_\ast(T) \to \pi_1(G) \) denote the canonical projection. As in \cite{39}, there exists a map \( w_G : G(L) \to \pi_1(G) \) sending an element \( b \in G(\mathbb{Z}_p) \lambda(p)G(\mathbb{Z}_p) \subset G(L) \) for \( \lambda \in X_\ast(T) \) to \( w(\lambda) \). For each \( b \in G(L) \), the projection of \( w_G(b) \) to \( \pi_1(G)_\sigma \) coincides with \( \kappa_G(b) \).

Let \( \mu \in X_\ast(T)^+ \) be the coweight defined by \( \mathcal{D} \), and \( [b] \in B(G, \mu) \) be the basic \( \sigma \)-conjugacy class, as in Section 5.2. Then \( \kappa_G(b) = \mu^s = w(\mu) \text{ mod } (1 - \sigma) \in \pi_1(G)_\sigma \). Hence \( w_G(b) - w(\mu) = (1 - \sigma)(c_{b, \mu}) \) for an element \( c_{b, \mu} \in \pi_1(G) \), whose \( \pi_1(G)_\sigma \)-coset is uniquely determined by this relation.

We note that the adjoint group of \( G \) is simple. Further \( \mu \) is minuscule and the pair \( (\mu, b) \) is Hodge-Newton indecomposable in the sense of \cite[Section 2.2.5]{5}. By the work of Viehmann \cite{80} and Chen-Kisin-Viehmann \cite[Theorem 1.1]{5}, the set \( \pi_0(X_\mu(b)) \) of connected components of \( X_\mu(b) \) is described as follows. If \( rs > 0 \), then \( w_G \) induces a bijection

\[ \pi_0(X_\mu(b)) \simeq c_{b, \mu} \pi_1(G)^\sigma. \]

If \( rs = 0 \), then \( X_\mu(b) \simeq G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \) is discrete.

Moreover, the group \( J_b(\mathbb{Q}_p) \) acts transitively on \( \pi_0(X_\mu(b)) \) by \cite[Theorem 1.2]{5}.

**Proposition 5.3.2.** Assume that \( rs > 0 \).

1. There is a bijection

\[ \pi_0(X_\mu(b)) \simeq \begin{cases} \mathbb{Z} & \text{if } p \text{ is inert in } E; \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } p \text{ is split in } E. \end{cases} \]

2. As in Proposition \ref{prop:4.4.2}, we fix an identification

\[ J_b(\mathbb{Q}_p) \simeq \begin{cases} \text{GU}(V, \phi)(\mathbb{Q}_p) & \text{if } p \text{ is inert in } E; \\ \text{GL}_m(B) \times \mathbb{Q}_p^\times & \text{if } p \text{ is split in } E. \end{cases} \]

Let \( J^0_b \) denote the stabilizer in \( J_b(\mathbb{Q}_p) \) of a fixed connected component of \( X_\mu(b) \). Then

\[ J^0_b \simeq \begin{cases} \{ g \in \text{GU}(V, \phi)(\mathbb{Q}_p) \mid v_p(\text{sim } g) = 0 \} & \text{if } p \text{ is inert in } E; \\ \{ (g, c) \in \text{GL}_m(B) \times \mathbb{Q}_p^\times \mid v_p(\text{nrd}(g)) = v_p(c) = 0 \} & \text{if } p \text{ is split in } E, \end{cases} \]

where \( v_p \) denotes the \( p \)-adic valuation on \( \mathbb{Q}_p \), and \( \text{nrd} \) denotes the reduced norm on \( \text{GL}_m(B) \).
Proof. (1) When $p$ is inert in $E$, the assertion follows from (5.7) and (5.3). A similar geometric argument works for the case $p$ is split, and hence we have $\pi_1(G)^p = \pi_1(G) \simeq \mathbb{Z}^2$ as in (5.3).

(2) Assume that $p$ is inert in $E$. We write $V_p = V \otimes \mathbb{Q}_p$. We fix an isomorphism $E_p \otimes \mathbb{Q}_p L \simeq L \oplus L$, which induces a decomposition $V_p \otimes \mathbb{Q}_p L \simeq V_1 \oplus V_2$. Then $G(L) = \text{GU}(V_p, \varphi)(L) \simeq \text{GL}_L(V_1) \times L^\times$.

Under this identification, the map $w_G$ is given by

$$w_G : G(L) \to \pi_1(G) \simeq \mathbb{Z} \oplus \mathbb{Z},$$

$$(h, c) \mapsto (v_p(\det h), v_p(c))$$

where $v_p$ denotes the $p$-adic valuation on $L$.

On the other hand, as in the proof of Proposition 4.4.2 (1), we have an inclusion $\text{End}_{E_p} V \hookrightarrow \text{End}_L V_1$, under which the inclusion $J_0(\mathbb{Q}_p) \hookrightarrow G(L) \simeq \text{GL}_L(V_1) \times L^\times$ can be regarded as the one sending $g \in J_0(\mathbb{Q}_p) = \text{GU}(V, \varphi)(\mathbb{Q}_p) \subset \text{GL}_{E_p}(V)$ to $(g, \text{sim}(g))$. It follows that, for each element of $c_{b, \mu} \pi_1(G)^p \subset \pi_1(G)$, its stabilizer in $J_0(\mathbb{Q}_p)$ consists of elements $g$ with $v_p(\det(g)) = 0$ and $v_p(\text{sim}(g)) = 0$. However, the latter condition is sufficient since $\mathbb{N}_{E_p/\mathbb{Q}_p}(\det(g)) = \text{sim}(g)^n$.

The case $p$ is split in $E$ can be proved similarly. \qed

6. Mass formula for the inner form associated to the basic locus

6.1. The group of self quasi-isogenies of an abelian variety.

6.1.1. Let $\mathcal{D}$ and $p$ be as in Section 4.1.1 and $G = \text{GU}(A, \langle , \rangle)$. We assume that $A$ is self-dual with respect to $\langle , \rangle$. We write $\mu$ for the cocharacter defined by $\mathcal{D}$. Let $K := G(\mathbb{Z}_p) \mathbb{K}^\theta(N)$ with $N \geq 3$ and $p \nmid N$.

Let $(A, \iota, \lambda, \bar{v})$ be a $k$-point of $\mathcal{M}_K$. Let $\text{End}^0(A)$ denote the $\mathbb{Q}$-algebra of self quasi-isogenies of the abelian variety $A$. It admits an injective homomorphism $\iota : E \to \text{End}^0(A)$. We regard $E$ as a subalgebra of $\text{End}^0(A)$ via $\iota$, and write $\text{End}^0_E(A)$ for its centralizer.

Lemma 6.1.2. If $(A, \iota, \lambda, \bar{v})$ is lying on the basic locus $\mathcal{M}_K^{\text{bas}}$, then $\text{End}^0_E(A)$ is a central simple algebra over $E$.

Proof. There exist a finite field $\mathbb{F}_q \subset k$, and a model $(A', \iota')$ over $\mathbb{F}_q$ of $(A, \iota)$ such that $\text{End}^0(A') \simeq \text{End}^0(A)$. Let $A' \sim \prod_{i=1}^n (A'_i)^{b_i}$ be the decomposition into components up to isogeny, where each abelian variety $A'_i$ is simple and $A'_i \not\sim A'_j$ for any $i \neq j$. Then we have an isomorphism $\text{End}^0(A') \simeq \prod_{i=1}^n \text{Mat}_{b_i}(D_i)$, where $D_i := \text{End}^0(A'_i)$ is a division algebra over $\mathbb{Q}$. Let $\pi_i$ be the Frobenius endomorphism of $A'_i$. Then the center $Z(\text{End}^0(A'))$ of $\text{End}^0(A')$ is equal to $\prod_{i=1}^n Q(\pi_i)$. Moreover, the assumption and [78] Proposition 4.2 imply that $Z(\text{End}^0(A'))$ is contained in $E$. Hence the abelian variety $A'$ has only one component, say $(A'_1)^{b_1}$, and the algebra $\text{End}^0(A') = \text{Mat}_{b_1}(D_1)$ is a central simple algebra over $Z(\text{End}^0(A')) = Q(\pi_1)$. It follows from the double centralizer theorem that the ring $\text{End}^0_E(A') \simeq \text{End}^0_E(A)$ is a central simple algebra over $E$. \qed

By Lemma 6.1.2 there exist a finite extension $E'/E$ and an isomorphism $f : \text{End}^0_E(A) \otimes_E E' \sim \text{Mat}_{b_1}(E')$. The reduced norm of an element $g \in \text{End}^0_E(A)$ is defined by $\text{nr}(g) := \det(f(g \otimes 1))$, which takes value in $E$.

6.1.3. For a $k$-point $(A, \iota, \lambda, \bar{v})$ of $\mathcal{M}_K$, we define an algebraic group $I$ over $\mathbb{Q}$ by

$$I(R) := \{(g, c) \in (\text{End}^0_E(A) \otimes R) \times R_\times \mid g^* \cdot g = c \cdot \text{id}\},$$

where $R$ is a $\mathbb{Q}$-algebra, and $g \mapsto g^*$ is the Rosati involution of $\text{End}^0_E(A)$ induced by the principal polarization $\lambda$. The similitude character $\text{sim} : I \to \mathbb{G}_{m, \mathbb{Q}}$ is defined by $(g, c) \mapsto c$. 
Theorem 6.1.4. Assume that \((A, \iota, \lambda, \bar{\eta})\) is lying on the basic locus \(\mathcal{M}_K^{\text{bas}}\).

(1) The group \(I\) is an inner form of \(G_Q\), and is such that \(I(\mathbb{R})\) is compact modulo center. Further, there are isomorphisms
\[
I_{Q_\ell} \simeq \begin{cases} G_{Q_\ell} & \text{if } \ell \neq p; \\ J_b & \text{if } \ell = p,
\end{cases}
\]
where \(b \in G(L)\) is a representative of the unique basic class \([b] \in B(G_{Q_\ell}, \mu)\).

(2) For any point \((A', \iota', \lambda', \bar{\eta}') \in \mathcal{M}_K^{\text{bas}}(k)\), the associated group \(I'\) over \(Q\) is isomorphic to \(I\) as inner forms of \(G_Q\).

(3) There is an isomorphism of perfect schemes
\[
\Theta : I(\mathbb{Q}) \backslash X_\mu(b) \times G(\mathbb{A}_f^p)/K^p(N) \xrightarrow{\sim} \mathcal{M}_K^{\text{bas}, \text{pf}},
\]
where \(\mathcal{M}_K^{\text{bas}, \text{pf}}\) denotes the perfection of \(\mathcal{M}_K^{\text{bas}}\).

(4) Assume that \((A, \iota, \lambda, \bar{\eta}) \in \mathcal{M}_K^{\text{bas}}(k)\). Let \(I_p^e\) be the stabilizer in \(J_b(\mathbb{Q}_p)\) of the associated Dieudonné module, as in \([4.13]\). Then \(\Theta\) induces a bijection
\[
I(\mathbb{Q}) \backslash I(\mathbb{A}_f)/I_p^e \cdot K^p(N) \xrightarrow{\sim} \mathcal{M}_K^{e}(k).
\]

Proof. Assertions (1), (2), and (3) are proved by Rapoport and Zink in \([93]\). Note that they constructed an isomorphism from a quotient of what is now called a Rapoport-Zink formal scheme to the completion of the integral model along the basic locus (see also Remark \([4.1.3]\) (2)). An isomorphism using an affine Deligne-Lusztig variety was proved in \([85, \text{Corollary 7.2.16}]\) and \([31, \text{Proposition 5.2.2}]\) for a Hodge-type Shimura variety.

By \([71, \text{Corollary 3.4}]\), the morphism \(\Theta\) induces a bijection from the double coset space in (4) to the central leaf passing through \((A, \iota, \lambda, \bar{\eta})\) in \(\mathcal{M}_K(k)\). This leaf coincides with \(\mathcal{M}_K^{e}(k)\) by Proposition \([4.2.5]\) (1), and hence assertion (4) follows. \(\square\)

6.1.5. Assume that \((A, \iota, \lambda, \bar{\eta}) \in \mathcal{M}_K^{\text{bas}}(k)\). Since \(I\) is an inner form of \(G_Q\), its derived subgroup \(I^{\text{der}}\) is again simply connected, and the quotient torus \(I/I^{\text{der}}\) is isomorphic to \(D = G_Q/G_Q^{\text{der}}\) (Section \([3.2.4]\)). Let \(\nu : I \rightarrow D\) denote the projection. The reduced norms of elements of \(\text{End}_K^{\text{der}}(A_x)\) induce a homomorphism \(\text{nr}_D : I \rightarrow T_E\) of algebraic groups over \(Q\). We write \(N = N_{E/\mathbb{Q}}\) for the norm map of \(E/\mathbb{Q}\). Then each element \((g, c) \in I\) satisfies that \(N(\text{nr}_D(g)) = c^n\). The projection \(\nu\) is thus equal to the product \(\text{nr}_D \times \text{sim}\) under the identification in \([3.10]\).

Let \(\mathbb{Q}_{>0}\) (resp. \(\mathbb{R}_{>0}\)) denote the group of positive rational (resp. real) numbers. The norm group \(N(E^\times)\) is contained in \(\mathbb{Q}_{>0}\) since \(E\) is an imaginary quadratic field. We write \(E^1 := \{x \in E \mid N(x) = 1\}\).

Lemma 6.1.6. We have that
\[
\nu(I(\mathbb{Q})) = \begin{cases} E^\times & \text{if } n \text{ is odd}; \\ E^1 \times \mathbb{Q}_{>0} & \text{if } n \text{ is even}.
\end{cases}
\]

Proof. The projection \(\nu : I \rightarrow D\) induces a commutative diagram
\[
\begin{array}{ccc}
I(\mathbb{Q}) & \xrightarrow{\nu} & D(\mathbb{Q}) \\
\downarrow & & \downarrow \\
I(\mathbb{R}) & \xrightarrow{\nu} & D(\mathbb{R})
\end{array}
\quad \quad \begin{array}{ccc}
& & \downarrow \\
H^1(\mathbb{Q}, I^{\text{der}}) & & H^1(\mathbb{R}, I^{\text{der}})
\end{array}
\]
where the horizontal sequences are exact, and the left and middle vertical arrows are injective. Furthermore, Kneser's theorem and Hasse principle imply that the right vertical arrow is also injective. A diagram chasing therefore shows \(\nu(I(\mathbb{Q})) = D(\mathbb{Q}) \cap \nu(I(\mathbb{R}))\).
Since \( I(\mathbb{R}) \) is connected, its image \( \nu(I(\mathbb{R})) \) is the identity component \( D(\mathbb{R})^0 \) of \( D(\mathbb{R}) \). As in Section 3.2.4, it follows that the group \( \nu(I(\mathbb{Q})) = D(\mathbb{Q})_{\infty} \) is isomorphic to \( E^x \) if \( n \) is odd and to \( E^1 \times \mathbb{Q}_{>0} \) if \( n \) is even. Their similitude factors are \( N(E^x) \) and \( \mathbb{Q}_{>0} \), respectively. \( \square \)

For each prime \( \ell \) we write \( E_\ell = E \otimes \mathbb{Q}_\ell \). Similarly to the case of \( G_\mathbb{Q} \) (see (3.13)), we have

\[
\nu(I(\mathbb{Q}_\ell)) = D(\mathbb{Q}_\ell) = \begin{cases} E^x_\ell, & \text{if } n \text{ is odd;} \\ E_\ell \times \mathbb{Q}_\ell^x, & \text{if } n \text{ is even.} \end{cases}
\]

**Lemma 6.1.7.** The group \( I(\mathbb{Q}) \) is dense in \( I(\mathbb{Q}_p) \).

**Proof.** Since the norm one subgroup \( T^{E,1} \) of \( T^E = \text{Res}_{E/\mathbb{Q}} G_{m,E} \) is a unitary group, it is a \( \mathbb{Q} \)-rational algebraic variety and hence satisfies the weak approximation property [60, Propositions 7.3 and 7.4, p. 402-403]. Thus, \( E^1 \) is dense in \( E_\mathbb{Q}^1 \). From Lemma 6.1.6 and equality (6.1) it follows that \( \nu(I(\mathbb{Q})) \) is dense in \( \nu(I(\mathbb{Q}_p)) \). On the other hand, since \( I^{\text{der}} \) is a \( \mathbb{Q} \)-simple group, it has weak approximation [60, Proposition 7.11, p. 422]. Therefore, \( I^{\text{der}}(\mathbb{Q}) \) is dense in \( I^{\text{der}}(\mathbb{Q}_p) \). Since \( \nu(I(\mathbb{Q})) = \nu(I(\mathbb{Q}_p)) \) and \( I(\mathbb{Q}) \supset I^{\text{der}}(\mathbb{Q}_p) \), we have \( I(\mathbb{Q}) = I(\mathbb{Q}_p) \). This proves the lemma. \( \square \)

### 6.2. Mass formula for maximal parahoric subgroups at \( p \).

**6.2.1.** Let \( (A, \iota, \lambda, \bar{\eta}) \in \mathcal{M}_K^{\text{bas}}(k) \). We fix identifications

\[
I(\mathbb{Q}_p) = J_b(\mathbb{Q}_p) = \begin{cases} \text{GU}(V, \phi)(\mathbb{Q}_p) & \text{if } p \text{ is inert in } E; \\ \text{GL}_m(B) \times \mathbb{Q}_p^x & \text{if } p \text{ is split in } E. \end{cases}
\]

Here \( (V, \phi) \) is a Hermitian space over \( E_p \) with \( d(V) = p^s \), and \( B \) is a division algebra over \( \mathbb{Q}_p \) of degree \( (n/m)^2 \) with \( \text{inv}(B) = \tau/n \), see Proposition 4.4.2

Let \( I_p \) be a maximal parahoric subgroup of \( I(\mathbb{Q}_p) \). As in Section 4.3 we have that

\[
I_p \sim_{\text{conj}} \begin{cases} \mathcal{P}_t & \text{for some } 0 \leq t \leq n \text{ with } t \equiv s \pmod{2} \text{ if } p \text{ is inert in } E; \\ \text{GL}_m(O_B) \times \mathbb{Z}_p^x & \text{if } p \text{ is split in } E. \end{cases}
\]

Similarly, we fix an identification \( I(\mathbb{A}_f^p) = G(\mathbb{A}_f^p) \) and regard \( G(\mathbb{Z}_p) \) as a subgroup of \( I(\mathbb{A}_f^p) \).

**Lemma 6.2.2.** The similitude character induces a surjective map

\[
sim : I(\mathbb{Q}) \backslash I(\mathbb{A}_f)/I_p G(\mathbb{Z}_p) \to \begin{cases} N(\mathbb{A}_{E,f}^x)/N(E^x) \cdot N(\mathbb{O}_E^\times) & \text{if } n \text{ is odd,} \\ \mathbb{A}_f^x/\mathbb{Q}_{>0} \cdot \mathbb{Z}_p^x & \text{if } n \text{ is even.} \end{cases}
\]

Moreover, the cardinality \( \tau \) of the quotient group in the RHS is given by

\[
\tau = \begin{cases} 2^{w-1} & \text{where } w = \#\{\ell : \text{prime, } \ell \mid d_E\} \text{ if } n \text{ is odd;} \\ 1 & \text{if } n \text{ is even.} \end{cases}
\]

**Proof.** By (6.1), the group \( \text{sim}(I(\mathbb{A}_f)) \) equals to either \( N(\mathbb{A}_{E,f}^x) \) when \( n \) is odd, or \( \mathbb{A}_f^x \) when \( n \) is even. For \( \ell \neq p \), the subgroup \( G(\mathbb{Z}_\ell) \) is the stabilizer of the unimodular lattice \( \Lambda_\ell \) in \( \text{GU}(V_\ell, \varphi) \), and therefore Lemma 3.1.3 shows that \( \text{sim}(G(\mathbb{Z}_\ell)) \) equals either \( N(O_{\hat{E}}^\times) \) when \( n \) is odd, or \( \mathbb{Z}_p^x \) when \( n \) is even. Further, when \( p \) is inert in \( E \), then \( \mathcal{P}_t \) is by definition the stabilizer of an \( O_{E_p} \)-lattice \( \mathcal{L}_t \), see (4.11). Hence Lemma 3.1.4 implies that \( \text{sim}(I_p) = \text{sim}(\mathcal{P}_t) = N(O_{E_p}^\times) = \mathbb{Z}_p^x \). When \( p \) is split in \( E \), then \( \text{sim}(I_p) = \text{sim}(\text{GL}_m(O_B) \times \mathbb{Z}_p^x) = \mathbb{Z}_p^x \). These and Lemma 6.1.6 imply the first assertion.

If \( n \) is even, the quotient group is trivial. Suppose that \( n \) is odd. We consider the inclusions

\[
N(\mathbb{O}_E^\times) \cdot \mathbb{Q}_{>0} \subset N(\mathbb{A}_{E,f}^x) \cdot \mathbb{Q}_{>0} \subset \mathbb{A}_f^x.
\]
As in [26, Formula (4.7)], we have
\[ N(A_{E,f}^\times) / N(E^\times) \cdot N(\hat{O}_E^\times) \simeq N(A_{E,f}^\times) / Q_{>0} \cdot N(\hat{O}_E^\times) / Q_{>0}. \]

Further, a direct computation shows that
\[ [A_{E,f}^\times : N(\hat{O}_E^\times) \cdot Q_{>0}] = \prod_{\ell} [\hat{Z}_{\ell}^\times : N(\hat{O}_{\ell}^\times)] = 2^w. \]

Moreover, an equality \( N(C^\times) = \mathbb{R}_{>0} \) and the norm index theorem imply that
\[ [A_{E,f}^\times : N(A_{E,f}^\times) \cdot Q_{>0}] = [A^\times : N(A_{E}^\times) : Q^\times] = 2. \]

Equalities (6.3), (6.4), and (6.5) show that
\[ [N(A_{E,f}^\times) : N(E^\times) \cdot N(\hat{O}_E^\times)] = [N(A_{E,f}^\times) : Q_{>0} : N(\hat{O}_E^\times) : Q_{>0}] = 2^{w-1}. \]

\[ \square \]

We write \( I^1 \) for the kernel of the similitude character of \( I \). Let \( \Gamma \) be an arithmetic subgroup of \( I(\mathbb{Q}) \). Then \( \text{sim}(\Gamma) \subset Z^\times \cap Q_{>0} = \{1\} \) by Lemma 6.1.6 and hence \( \Gamma \subset I^1(\mathbb{Q}) \). Since \( I^1(\mathbb{R}) \) is compact, \( \Gamma \) is finite.

Let \( U \) be an open compact subgroup of \( I(\mathbb{A}_f) \). Let \([g]\) be a double coset in \( I(\mathbb{Q}) \backslash I(\mathbb{A}_f) / U \), represented by \( g \in I(\mathbb{A}_f) \). Then \( \Gamma_g := I(\mathbb{Q}) \cap gUg^{-1} \) is finite. A similar statement holds true also for an open compact subgroup \( U^1 \) of \( I^1(\mathbb{A}_f) \) and a coset \([g] \in I^1(\mathbb{Q}) \backslash I^1(\mathbb{A}_f) / U^1 \).

**Definition 6.2.3.** The *mass of \( I \) with respect to \( U \)" is defined by
\[ \text{Mass}(I, U) := \sum_{[g] \in I(\mathbb{Q}) \backslash I(\mathbb{A}_f) / U} \frac{1}{|\Gamma_g|}. \]

The *mass of \( I^1 \) with respect to \( U^1 \)" is defined similarly and denoted by Mass\((I^1, U^1)\).

We write \( I^1_p := I_p \cap I^1(\mathbb{Q}_p) \).

**Proposition 6.2.4.** Let \( \tau \) be as in Lemma 6.2.2. Then
\[ \text{Mass}(I, I^1_p G(\hat{\mathbb{Z}}^p)) = \tau \cdot \text{Mass}(I^1, I^1_p G(\hat{\mathbb{Z}}^p)). \]

**Proof.** We put \( U := I^1_p G(\hat{\mathbb{Z}}^p) \) and \( U^1 := I^1_p G(\hat{\mathbb{Z}}^p) \). The map in Lemma 6.2.2 induces a decomposition
\[ I(\mathbb{Q}) \backslash I(\mathbb{A}_f) / U = \bigcup_{i=1}^\tau I(\mathbb{Q}) \backslash I(\mathbb{Q}) I^1(\mathbb{A}_f) f_i U / U, \]
where \( f_1, \ldots, f_\tau \) are some elements of \( I(\mathbb{A}_f) \). The right multiplication by \( f_i^{-1} \) induces a bijection
\[ I(\mathbb{Q}) \backslash I(\mathbb{Q}) I^1(\mathbb{A}_f) f_i U / U \overset{\sim}{\rightarrow} I(\mathbb{Q}) \backslash I(\mathbb{Q}) I^1(\mathbb{A}_f) f_i U f_i^{-1} / f_i U f_i^{-1}. \]

We put \( U^1_i := f_i U f_i^{-1} \cap I^1(\mathbb{A}_f) \). Then a computation of similitude factors shows \( U^1_i = f_i U^1 f_i^{-1} \). Furthermore, there is a bijection
\[ I^1(\mathbb{Q}) \backslash I^1(\mathbb{A}_f) / U^1_i \overset{\sim}{\rightarrow} I(\mathbb{Q}) \backslash I(\mathbb{Q}) I^1(\mathbb{A}_f) f_i U f_i^{-1} / f_i U f_i^{-1}. \]

In fact, the natural projection induces a well-defined map. We show this map is injective. Suppose that two elements \( x_1, x_2 \in I^1(\mathbb{A}_f) \) satisfy \( \gamma x_1 u = x_2 \) for some \( \gamma \in I(\mathbb{Q}) \) and \( u \in f_i U f_i^{-1} \). Then sim\((\gamma) \cdot \text{sim}(u) = 1 \). We have sim\((\gamma) \in Q_{>0} \) as in Lemma 6.1.6 and sim\((u) \in \hat{Z}^\times \). It follows that sim\((\gamma) = \text{sim}(u) = 1 \), and hence \( \gamma \in I^1(\mathbb{Q}) \) and \( u \in U^1_i \), as desired.

For each \( 1 \leq i \leq \tau \), let \( g_{i1}, \ldots, g_{im_i} \) be elements of \( I^1(\mathbb{A}_f) \) such that \( g_{ij} f_i, \ldots, g_{im_i} f_i \) are representatives of double cosets in \( I(\mathbb{Q}) \backslash I(\mathbb{Q}) I^1(\mathbb{A}_f) f_i U / U \). Since \( \Gamma_{g_{ij} f_i} \subset I^1(\mathbb{Q}) \), we have
\[ \Gamma_{g_{ij} f_i} = I^1(\mathbb{Q}) \cap g_{ij} U^1_i g_{ij}^{-1}. \]

\[ \square \]
It follows that
\[
\text{Mass}(I, U) = \sum_{i=1}^{\tau} \sum_{j=1}^{m_i} \frac{1}{|\Gamma_{g_{ij}f_i}|} = \sum_{i=1}^{\tau} \sum_{j=1}^{m_i} \frac{1}{|I_1(Q) \cap g_{ij}U_1g_{ij}^{-1}|} = \sum_{i=1}^{\tau} \text{Mass}(I_1, U_1) = \tau \cdot \text{Mass}(I_1, U_1). \tag{6.6}
\]

6.2.5. We write \( G^1 = U(\Lambda, \varphi) \) for the kernel of the similitude character of \( G \). Recall that \( I_1^1 \) is a compact inner form of \( G^1_1 \). Let \( \mu_{I^1_1(\mathbb{R})} \) be the Haar measure on \( I^1_1(\mathbb{R}) \) which gives this group volume one. The pull-back of \( \mu_{I^1_1(\mathbb{R})} \) via an inner twist gives a Haar measure on \( G^1_1(\mathbb{R}) \), denoted by \( \mu_{G^1_1(\mathbb{R})} \). Further, for each prime \( \ell \), let \( \mu_{G^1_1(\mathbb{Q}_\ell)} \) be the Haar measure on \( G^1_1(\mathbb{Q}_\ell) \) which gives \( G^1_1(\mathbb{Z}_\ell) \) volume one. Finally we put \( \mu_{G^1_1(A)} := \mu_{G^1_1(\mathbb{R})} \times \prod_{\ell} \mu_{G^1_1(\mathbb{Q}_\ell)} \). By Definition 2.3.3 we have that
\[
\text{Mass}(\Lambda) = \int_{G^1(\mathbb{Q}) \backslash G^1_1(\mathbb{A})} \mu_{G^1_1(\mathbb{A})}. \tag{6.7}
\]

Let \( \mu_{I^1_1(\mathbb{Q}_p)} \) be the Haar measure on \( I^1_1(\mathbb{Q}_p) \) which gives \( I^1_1 \) volume one. For \( \ell \neq p \), we put \( \mu_{I^1_1(\mathbb{Q}_\ell)} = \mu_{G^1_1(\mathbb{Q}_\ell)} \) under the fixed identification \( I^1_1(\mathbb{Q}_\ell) \simeq G^1_1(\mathbb{Q}_\ell) \). Further we put \( \mu_{I^1_1(\mathbb{A})} := \mu_{I^1_1(\mathbb{R})} \times \prod_{\ell} \mu_{I^1_1(\mathbb{Q}_\ell)} \). Then
\[
\text{Mass}(I^1_1, I^1_1 \mathbb{G}^1_1(\mathbb{\mathbb{A}})) = \int_{I^1_1(\mathbb{Q}) \backslash I^1_1(\mathbb{A})} \mu_{I^1_1(\mathbb{A})}. \tag{6.8}
\]

Fix an inner twisting \( f : I^1_1_{\mathbb{Q}_p} \rightarrow G^1_1_{\mathbb{Q}_p} \) over an extension of \( \mathbb{Q}_p \), and let \( \mu_{I^1_1(\mathbb{Q}_p)}^* \) be the Haar measure on \( I^1_1(\mathbb{Q}_p) \) defined by the pull-back \( \mu_{I^1_1(\mathbb{Q}_p)} := f^*(\mu_{G^1_1(\mathbb{Q}_p)}) \). Let
\[
\lambda_p(I_p) := \left( \int_{I^1_1(\mathbb{Q}_p)} \mu_{I^1_1(\mathbb{Q}_p)}^* \right)^{-1}. \tag{6.9}
\]

It follows from (6.10) and (6.11) that
\[
\frac{\text{Mass}(I^1_1, I^1_1 \mathbb{G}^1_1(\mathbb{\mathbb{A}}))}{\text{Mass}(\Lambda)} = \lambda_p(I_p). \tag{6.12}
\]

**Proposition 6.2.6.** We have that
\[
\lambda_p(I_p) = \begin{cases} 
\left( \prod_{i=1}^{n} (p^i - (-1)^i) \right) \cdot \left( \prod_{j=1}^{n-t} (p^j - (-1)^j) \cdot \prod_{k=1}^{t} (p^k - (-1)^k) \right)^{-1} & \text{if } p \text{ is inert and } I_p \sim_{\text{conj}} \mathcal{P}_i; \\
\left( \prod_{i=1}^{n} (p^i - 1) \right) \cdot \left( \prod_{j=1}^{m} (p^{m-j} - 1) \right)^{-1} & \text{if } p \text{ is split.}
\end{cases}
\]

**Proof.** We compute \( \lambda_p(I_p) \) using Gan-Hanke-Yu’s argument \[16\]. Note that Prasad \[59\] Prop. 2.3 gave a similar formula for a semi-simple and simply connected algebraic group.

We recall the canonical Haar measures on \( G^1_1(\mathbb{Q}_p) \) and \( I^1_1(\mathbb{Q}_p) \) constructed in \[21\] Section 4. Let \( \omega_{G^1_1_{\mathbb{Q}_p}} \) be an invariant differential of top degree on \( G^1_1_{\mathbb{Q}_p} \) with nonzero reduction on \( G^1_1_{\mathbb{Q}_p} \) and let \( |\omega_{G^1_1_{\mathbb{Q}_p}}| \) be the associated Haar measure on \( G^1_1(\mathbb{Q}_p) \). Let \( |\omega_{I^1_1_{\mathbb{Q}_p}}| \) on \( I^1_1_{\mathbb{Q}_p} \) be the Haar measure associated with
the pull-back $f^*(\omega)$. Since $G^1_{\mathbb{Z}_p}$ is unramified with reductive model $G^1_{\mathbb{Q}_p}$, we have $\int_{G^1_{\mathbb{Z}_p}} |\omega_{G^1_{\mathbb{Z}_p}}| = p^{-\dim G^1_{\mathbb{Z}_p} \cdot |G^1_{\mathbb{Z}_p}|}$ as in [24, p. 294]. It follows that $\mu_{G^1_{\mathbb{Q}_p}} = p^{-\dim G^1_{\mathbb{Z}_p} \cdot |G^1_{\mathbb{Z}_p}| \cdot |\omega^*_{G^1_{\mathbb{Z}_p}}|}$.

Now let $I_p^1$ be the smooth model over $\mathbb{Z}_p$ of $I_p^1$, and let $\overline{I}_p$ be the maximal reductive quotient of the special fiber $I_p^1 \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. Further let $N(G^1_{\mathbb{Z}_p})$ (resp. $N(\overline{I}_p)$) denote the number of positive roots of $G^1_{\mathbb{Z}_p}$ (resp. $\overline{I}_p$). By a computation of the volume of an Iwahori subgroup of $I^1(\mathbb{Q}_p)$ [16, (2.6), (2.11) and (2.12)], we have that

\begin{equation}
\lambda_p(I_p) = \left( p^{-\dim G^1_{\mathbb{Z}_p} \cdot |G^1_{\mathbb{Z}_p}|} \cdot \int_{I_p^1} |\omega^*_{I_p^1}| \right)^{-1} = \frac{p^{-N(G^1_{\mathbb{Z}_p}) \cdot |G^1_{\mathbb{Z}_p}|}}{p^{-N(\overline{I}_p)} \cdot |\overline{I}_p|}.
\end{equation}

Let $U_t$ denote the unitary group in $n$ variables over $\mathbb{F}_p$, and $\mathbb{F}_q$ denote the field of order $q := p^{n/m}$. Then we have isomorphisms of algebraic groups over $\mathbb{F}_p$ (Section 4.3):

$$G^1_{\mathbb{F}_p} \simeq \begin{cases} U_n & \text{if } p \text{ is inert in } E \text{ and } I_p^1 \sim_{\text{cong}} P^1_t; \\ GL_n, & \text{if } p \text{ is split in } E. \\
\overline{I}_p \simeq \begin{cases} U_{n-t} \times U_t & \text{if } p \text{ is inert in } E \text{ and } I_p^1 \sim_{\text{cong}} P^1_t; \\ \text{Res}_{\mathbb{F}_q/\mathbb{F}_p} GL_{m,\mathbb{F}_q} & \text{if } p \text{ is split in } E. 
\end{cases}
$$

Moreover, we have that

\begin{equation}
N(U_n) = N(GL_n) = \frac{n(n-1)}{2}, \quad N(\text{Res}_{\mathbb{F}_q/\mathbb{F}_p} GL_{m,\mathbb{F}_q}) = \frac{n}{m} \cdot \frac{m(m-1)}{2}.
\end{equation}

Equalities (6.14) and (6.15), and Table 1 imply the assertion. $\square$

**Remark 6.2.7.** The rational function $\lambda_p(I_p)$ is in fact a polynomial with integer coefficients. Indeed, if $p$ is split in $E$, we can write $\lambda_p(I_p) = \prod_{1 \leq i \leq n, m|t} (p^i - 1)$.

Assume that $p$ is inert in $E$. Formula (6.14) shows that $\lambda_p(I_p)$ equals the prime-to-$p$ factor of the fraction $|U_n(\mathbb{F}_p)|/|U_{n-t}(\mathbb{F}_p) \times U_t(\mathbb{F}_p)|$. Since $U_{n-t} \times U_t$ can be embedded into $U_n$, the fraction is an integer for any prime $p > 2$. From Gauss' lemma [11, Ch.9, Ex.2] it follows that the coefficients of the polynomial $\lambda_p(I_p)$ are integers, as desired. We can also show this fact by induction, using the following relation: If we write $\lambda_p(n, t)$ for the expression of $\lambda_p(I_p)$ (with $p$ inert) in Proposition 6.2.6, then

$$\lambda_p(n, t) = p^t \cdot \lambda_p(n - 1, t) + (-1)^{n-t} \cdot \lambda_p(n - 1, t - 1).$$

**Theorem 6.2.8.** Let $\chi$ be the Dirichlet character of $E/\mathbb{Q}$. We use the convention that $\chi^i = 1$ or $\chi$ according as $j$ is even or odd. Let $L(-, \chi^i)$ be the Dirichlet $L$-function associated to $\chi^i$. Let $I_p^1$ be a maximal parahoric subgroup of $I(\mathbb{Q}_p)$ and let $\lambda_p(I_p)$ be the associated number defined in (6.12) whose formula is given in Proposition 6.2.6.

(1) The mass of $I$ with respect to $I_p G(\widehat{\mathbb{Z}}^p)$ is

$$\text{Mass} \left( I, I_p G(\widehat{\mathbb{Z}}^p) \right) = \varepsilon \cdot \prod_{j=1}^n L(1-j, \chi^j) \cdot \prod_{\ell \mid d_E} \kappa_\ell \cdot \lambda_p(I_p)$$
where \( d_E \) denotes the discriminant of \( E \), and quantities \( \varepsilon \) and \( \kappa_\ell \) for primes \( \ell \mid d_E \) are given by

\[
\varepsilon = \begin{cases} 
\frac{1}{2n} & \text{if } n \text{ is odd;} \\
\frac{(-1)^{n/2}}{2^{n+w-1}} & \text{if } n \text{ is even, where } w = \#\{\ell : \text{prime, } \ell \mid d_E\}, 
\end{cases}
\]

\[
\kappa_\ell = \begin{cases} 
1 & \text{if } n \text{ is odd;} \\
\ell^{n/2} + 1 & \text{if } n \text{ is even, } \ell \neq 2, \ d(A_\ell) = (-1)^{n/2}; \\
\ell^{n/2} - 1 & \text{if } n \text{ is even, } \ell \neq 2, \ d(A_\ell) \neq (-1)^{n/2}; \\
2^n - 1 & \text{if } n \text{ is even, } \ell = 2, \ d_E \equiv 4 \pmod{8}, \ A_2 \text{ normal;} \\
2^n \cdot (2^n - 1) & \text{if } n \text{ is even, } \ell = 2, \ d_E \equiv 4 \pmod{8}, \ A_2 \text{ subnormal;} \\
2^n + 1 & \text{if } n \text{ is even, } \ell = 2, \ d_E \equiv 0 \pmod{8}, \ A_2 \text{ subnormal, } d(A_2) = (-1)^{n/2}; \\
2^n - 1 & \text{if } n \text{ is even, } \ell = 2, \ d_E \equiv 0 \pmod{8}, \ A_2 \text{ subnormal, } d(A_2) \neq (-1)^{n/2}. 
\end{cases}
\]

Here we write \( \Lambda_\ell = \Lambda \otimes \mathbb{Q}_\ell \), and its determinant \( d(A_\ell) \) takes value in \( \mathbb{Z}_\ell / \mathcal{N}_{E_\ell / \mathbb{Q}_\ell}(O_{E_\ell}^\times) \).

(2) Let \( \varepsilon, \kappa_\ell, \) and \( \lambda_p(I_p) \) be as above. For an integer \( N \geq 3 \) with \( p \nmid N \), we have that

\[
|I(\mathbb{Q}) \setminus I(A_f)/I_pK^p(N)| = [G(\mathbb{Q}): K^p(N)] \cdot \varepsilon \cdot \prod_{j=1}^{n} L(1 - j, \chi^j) \cdot \prod_{\ell \mid d_E} \kappa_\ell \cdot \lambda_p(I_p).
\]

Proof. Theorem 2.3.4, Proposition 6.2.4, and equality (6.13) imply assertion (1). For any representative \( g \) of a double coset in \( I(\mathbb{Q}) \setminus I(A_f)/I_pK^p(N) \), the intersection \( \Gamma_g = I(\mathbb{Q}) \cap gI_pK^p(N)g^{-1} \) is trivial by Serre’s lemma. Hence assertion (2) follows from assertion (1). \( \Box \)

6.3. Main theorems and examples. For a scheme \( S \) over \( k \), we write \( \text{Irr}(S) \) for the set of irreducible components of \( S \). Recall that \( \mathcal{M}_K^{\text{bas}} \) and \( \mathcal{M}_K^e \) denote the basic locus and the 0-dimensional EO stratum of \( \mathcal{M}_K \), respectively, where \( K = G(\mathbb{Z}_p)K^p(N) \) with \( N \geq 3 \) and \( p \nmid N \).

Theorem 6.3.1. We have that

\[
|\text{Irr}(\mathcal{M}_K^{\text{bas}})| = |G(\mathbb{Q}): K^p(N)] \cdot \varepsilon \cdot \prod_{j=1}^{n} L(1 - j, \chi^j) \cdot \prod_{\ell \mid d_E} \kappa_\ell \cdot \lambda_p^{\text{bas}} \cdot \rho^{\text{bas}},
\]

\[
|\mathcal{M}_K^e(k)| = |G(\mathbb{Q}): K^p(N)] \cdot \varepsilon \cdot \prod_{j=1}^{n} L(1 - j, \chi^j) \cdot \prod_{\ell \mid d_E} \kappa_\ell \cdot \lambda_p^e,
\]

where \( d_E \) denotes the discriminant of \( E/\mathbb{Q} \), \( \chi \) denotes the Dirichlet character associated to \( E/\mathbb{Q} \), the quantities \( \varepsilon \) and \( \kappa_\ell \) for primes \( \ell \mid d_E \) are as defined in (6.16) and (6.17), and the quantities
\( \lambda_p^{\text{bas}}, \lambda_p^{e}, \text{and } \rho^{\text{bas}} \) are given by

\[
\lambda_p^{\text{bas}} = \begin{cases} 
1 & \text{if } p \text{ is inert, } rs \text{ is even; } \\
p^n - 1 & \text{if } p \text{ is inert, } rs \text{ is odd; } \\
\left( \prod_{h=1}^{n} (p^h - 1) \right) \cdot \left( \prod_{i=1}^{m} (p^{m-i} - 1) \right)^{-1}, \quad m := \gcd(r, s) & \text{if } p \text{ is split, }
\end{cases}
\]

(6.20)

\[
\lambda_p^{e} = \begin{cases} 
\left\lfloor \frac{n}{2} \right\rfloor & \text{if } p \text{ is inert, } rs \text{ is even; } \\
\left\lfloor \frac{r}{2} \right\rfloor & \text{if } p \text{ is split, } \\
\left\lceil \frac{n}{2} - 1 \right\rceil & \text{if } p \text{ is inert, } rs \text{ is odd; } \\
1 & \text{if } p \text{ is split. }
\end{cases}
\]

(6.21)

\[
\rho^{\text{bas}} = \begin{cases} 
\left( \prod_{h=1}^{n} (p^h - (-1)^h) \right) \cdot \left( \prod_{i=1}^{r} (p^i - (-1)^i) \right) \cdot \left( \prod_{j=1}^{s} (p^j - (-1)^j) \right)^{-1} & \text{if } p \text{ is inert; } \\
1 & \text{if } p \text{ is split. }
\end{cases}
\]

(6.22)

In particular, if (i) \( p \) is split in \( E \), (ii) \( rs = 0 \), or (iii) \( n \) is even and \( (r, s) = (1, n - 1) \) or \( (n - 1, 1) \), then

\[
|\text{Irr}(\mathcal{M}_K^{\text{bas}})| = |\mathcal{M}_K^{e}(k)|.
\]

Proof. For each \( Z \in \text{Irr}(X_\mu(b)) \), let \( I_p^Z \) denote the stabilizer of \( Z \) in \( J_0(\mathbb{Q}_p) \). Then we have a bijection

\[
\prod_{[Z] \in J_0(\mathbb{Q}_p)\setminus\text{Irr}(X_\mu(b))} J_0(\mathbb{Q}_p)/I_p^Z \sim \text{Irr}(X_\mu(b)).
\]

We fix identifications \( I(\mathbb{Q}_p) = J_0(\mathbb{Q}_p) \) and \( I(\mathbb{A}_f^p) = \mathbf{G}(\mathbb{A}_f^p) \). By Theorem 6.1.4 (2), the Rapoport-Zink uniformization map induces a bijection (see [31 Theorem A])

\[
\prod_{[Z] \in J_0(\mathbb{Q}_p)\setminus\text{Irr}(X_\mu(b))} I(\mathbb{Q})\backslash I(\mathbb{A}_f)/I_p^Z \text{KP}(N) \sim \text{Irr}(\mathcal{M}_K^{\text{bas}}).
\]

(6.23)

By He-Zhou-Zhu [31 Theorem 4.1.2 and Proposition 2.2.5] and Nie [55], the stabilizer \( I_p^Z \) is a parahoric subgroup which has the maximal volume among all the parahoric subgroups of \( J_0(\mathbb{Q}_p) \). In particular, the cardinality \( |I(\mathbb{Q})\backslash I(\mathbb{A}_f)/I_p^Z \text{KP}(N)| \) does not depend on \( Z \in \text{Irr}(X_\mu(b)) \). If \( p \) is split in \( E \), then \( I_p^Z \) is conjugate to \( \text{GL}_m(O_B) \times \mathbb{Z}_p^\times \), as in (6.2). Suppose that \( p \) is inert in \( E \). By (6.2) and (6.12), the maximal parahoric subgroup \( I_p^Z \) is conjugate to some \( \mathcal{P}_t \) such that \( \lambda_p(\mathcal{P}_t) \) is minimal among \( (\mathcal{P}_t)_t \) with \( t \equiv s \pmod{2} \). It follows from Proposition 6.2.6 that \( I_p^Z \) is conjugate to \( \mathcal{P}_0 \) or \( \mathcal{P}_n \) if \( rs \) is even, and to \( \mathcal{P}_1 \) or \( \mathcal{P}_{n-1} \) if \( rs \) is odd. Hence Proposition 5.2.3 and Theorem 6.2.8 (2) imply formula (6.18).

Theorem 6.1.4 (3) and Theorem 6.2.8 (2) imply formula (6.19). \( \square \)

Corollary 6.3.2. If \( rs = 0 \), or \( p \) is inert in \( E \) and \( rs \) is even, then the number \( |\text{Irr}(\mathcal{M}_K^{\text{bas}})| \) of irreducible components of the basic locus is independent of \( p \) and is a constant number depending only on the input PEL datum \( \mathcal{D} \) and \( N \). For other cases, the number \( |\text{Irr}(\mathcal{M}_K^{\text{bas}})| \) grows to infinity with \( p \).

Remark 6.3.3. The former condition is equivalent to that the basic element \( [b] \in B(G, \mu) \) is unramified in the sense of Xiao-Zhu [55 Section 4.2], and is further equivalent to that \( \dim_k \mathcal{M}_K = 2 \cdot \dim_k \mathcal{M}_K^{\text{bas}} \).
Now we assume that \((r, s) = (1, n - 1)\). The case where \(p\) is split in \(E\) has been studied by Harris and Taylor [28] for their proof of the local Langlands conjecture for \(GL_n\). In this case the EO and Newton stratifications coincide, and in particular \(\mathcal{M}_K^{\text{bas}} = \mathcal{M}_K\). If \(p\) is inert in \(E\), Vollaard and Wedhorn proved that for each odd integer \(1 \leq t \leq n\) there exists a unique \(EO\) stratum of dimension \(\frac{1}{2}(t - 1)\) in \(\mathcal{M}_K^{\text{bas}}\), denoted by \(\mathcal{M}_K(t)\) [33, Section 6.3]. Let \(\overline{\mathcal{M}_K(t)}\) be the Zariski closure of \(\mathcal{M}_K(t)\) in \(\mathcal{M}_K\). Note that \(\mathcal{M}_K^{\text{bas}} = \mathcal{M}_K^{(n)}\) or \(\mathcal{M}_K^{(n-1)}\) according as \(n\) is odd or even.

**Theorem 6.3.4.** Assume that \((r, s) = (1, n - 1)\) and \(p\) is inert in \(E\). Let \(t\) be an odd integer such that \(1 \leq t \leq n\). Then

\[
|\text{Irr}(\overline{\mathcal{M}_K(t)})| = |\text{G}(\mathbb{Z}_p) : K^p(N)| \cdot \varepsilon \cdot \prod_{j=1}^{n} L(1-j, \chi^j) \cdot \prod_{\ell | d_E} \kappa_{\ell} \cdot \lambda_p^{(t)}
\]

where \(\varepsilon\) and \(\kappa_{\ell}\) are given as in (6.16) and (6.17), and \(\lambda_p^{(t)}\) is given by

\[
\lambda_p^{(t)} = \left( \prod_{h=1}^{n} (p^h - (-1)^h) \right) \cdot \left( \prod_{i=1}^{t} (p^i - (-1)^i) \cdot \prod_{j=1}^{n-t} (p^j - (-1)^j) \right)^{-1}.
\]

**Proof.** The group \(J_b\) is isomorphic to \(GU(\mathcal{V}, \phi)\) with \(d(\mathcal{V}) = p^{n-1}\). By (4.10), we have

\[
\mathcal{V} \simeq \begin{cases} \mathbb{H}^{(n-1)/2} \oplus V_1 & \text{if } n \text{ is odd;} \\ \mathbb{H}^{n/2-1} \oplus V_1 \oplus V_p & \text{if } n \text{ is even.} \end{cases}
\]

Further, the lattice \(\mathcal{L}_{n-t}\) is of orbit type \(t\) in the sense of [33]. We regard its stabilizer \(\mathcal{P}_{n-t}\) as a subgroup of \(J_b(\mathbb{Q}_p)\). By [33, Proposition 6.3], there is a bijection

\[
I(\mathbb{Q}) \backslash J_b(\mathbb{Q}_p) \cdot \text{G}(\mathbb{A}_f^p) / \mathcal{P}_{n-t} \cdot K^p(N) \xrightarrow{\sim} \text{Irr}(\overline{\mathcal{M}_K(t)}).
\]

Hence the assertion follows from Theorem 6.2.8 (2). \(\square\)

6.3.5. By [46, Theorem 6.4.1.1], there is a toroidal compactification \(\mathcal{M}_K^{\text{tor}}\) of the integral model \(\mathcal{M}_K\), which is proper and smooth over \(O_{E, (\rho)}\). This implies that (see [46, Corollary 6.4.1.2])

\[
\pi_0(\mathcal{M}_K) \simeq \pi_0(\text{Sh}_K(\mathcal{G}, X)_C).
\]

When \(rs = 0\), the set \(\text{Sh}_K(\mathcal{G}, X)_C\) is discrete; see Example 6.3.9. When \(rs > 0\), the map in (3.17) induces a bijection

\[
\pi_0(\text{Sh}_K(\mathcal{G}, X)_C) \simeq D(\mathbb{Q}_\infty) \backslash D(A_f) / \nu(K) = D(\mathbb{Q}_\infty) \backslash D(A_f) / D(Z_p) \nu(K^p(N)).
\]

Here \(D(\mathbb{Q}_\infty)\) is the intersection of \(D(\mathbb{Q})\) with the connected component \(D(\mathbb{R})^0\) of \(D(\mathbb{R})\). Further, \(D(Z_p)\) is the unique maximal open compact subgroup of \(D(\mathbb{Q}_p)\), which satisfies that \(D(Z_p) = \nu(G(\mathbb{Z}_p))\) by Lemma 3.2.3. The cardinality of \(\pi_0(\text{Sh}_K(\mathcal{G}, X)_C)\) is given in Theorem 3.3.2 (2).

Let \(J_b^0\) be the stabilizer in \(J_b(\mathbb{Q}_p)\) of a connected component of \(X_{\mu}(b)\). Then \(J_b(\mathbb{Q}_p)\) acts transitively on \(\pi_0(X_{\mu}(b))\), and hence \(\pi_0(X_{\mu}(b)) \simeq J_b(\mathbb{Q}_p) / J_b^0\). The Rapoport-Zink uniformization therefore induces a bijection

\[
I(\mathbb{Q}) \backslash I(A_f) / J_b^0 K^p(N) \simeq \pi_0(\mathcal{M}_K^{\text{bas}}).
\]

Note that this bijection together with the description of \(J_b^0\) in Proposition 5.3.2 (2) gives a generalization of the result of Vollaard and Wedhorn in [33, Proposition 6.4], where they considered the case \(p\) is inert in \(E\) and \((r, s) = (1, n - 1)\).

Let \(\pi_0(\text{Sh}_G(z_p)(\mathcal{G}, X)_C) := \lim_{N \to \infty} \pi_0(\text{Sh}_G(z_p)K^p(N)(\mathcal{G}, X)_C)\) where \(N\) runs through all prime-to-\(p\) positive integers. We define similarly \(\pi_0(\mathcal{M}_G^{\text{bas}}(z_p))\) and \(\mathcal{M}_G^{(k)}(z_p)\).
Lemma 6.3.6. Assume that $rs > 0$. Then the group $G(\mathbb{A}_f^p)$ acts transitively on $\pi_0(\text{Sh}_{G(\mathbb{Z}_p)}(G, X)_{\mathbb{C}})$, $\pi_0(M\mathbb{G}_{b,\text{bas}}(\mathbb{Z}_p))$, and $M\mathbb{G}_{b,\text{bas}}(\mathbb{Z}_p)(k)$. 

Proof. The group $D(\mathbb{Q})_\infty$ is dense in $D(\mathbb{Q}_p)$ by (6.12) and (6.13). Hence the set $D(\mathbb{Q})_\infty \setminus D(\mathbb{Q}_p)/D(\mathbb{Z}_p)$ is a singleton. It follows from (6.24) that $G(\mathbb{A}_f^p)$ acts transitively on $\pi_0(\text{Sh}_{G(\mathbb{Z}_p)}(G, X)_{\mathbb{C}})$. Similarly, the set $I(\mathbb{Q})\setminus I(\mathbb{Q}_p)/U_p$ is a singleton for any open subgroup $U_p$ of $I(\mathbb{Q}_p)$, since $I(\mathbb{Q})$ is dense in $I(\mathbb{Q}_p)$. It follows from (6.25) and Theorem 6.3.4 (4) that the group $G(\mathbb{A}_f^p) = I(\mathbb{A}_f^p)$ acts transitively on $\pi_0(M\mathbb{G}_{\text{bas}}(\mathbb{Z}_p))$ and $M\mathbb{G}_{b,\text{bas}}(\mathbb{Z}_p)(k)$. \hfill \Box

The natural $G(\mathbb{A}_f^p)$-equivariant map $\pi_0(M\mathbb{G}_{\text{bas}}) \rightarrow \pi_0(M_{\mathbb{K}})$ is therefore surjective.

Theorem 6.3.7. Assume that $rs > 0$. Then there is a bijection 

$$\pi_0(M\mathbb{G}_{\text{bas}}) \simeq \pi_0(M_{\mathbb{K}}),$$

unless $p$ is inert in $E$ and $(r, s) = (1, 1)$ or $p$ is split in $E$ and $\gcd(r, s) = 1$, in which cases we have

$$|\pi_0(M\mathbb{G}_{\text{bas}})| = |\text{Irr}(M\mathbb{G}_{\text{bas}})|/|\rho\mathbb{G}_{\text{bas}}|.$$

Proof. The projection $\nu$ induces a surjective map

$$\nu : \pi_0(M\mathbb{G}_{\text{bas}}) \simeq I(\mathbb{Q})\setminus I(\mathbb{A}_f)/J_0^b K^p(N) \rightarrow \nu(I(\mathbb{Q}))\setminus D(\mathbb{A}_f)/\nu(J_0^b) \cdot \nu(K^p(N)).$$

By equality (3.12) and Lemma 6.1.6 we have $\nu(I(\mathbb{Q})) = D(\mathbb{Q})_\infty$. Further, Proposition 5.3.2 implies that $\nu(J_0^b) = D(\mathbb{Z}_p)$. It follows from (6.24) that the double coset space in the RHS has the same cardinality as the set $\pi_0(\text{Sh}_{G(\mathbb{Z}_p)}(G, X)_{\mathbb{C}}) \simeq \pi_0(M_{\mathbb{K}})$.

On the other hand, for each $g \in I(\mathbb{A}_f)$, there is a surjective map

$$(6.26) \quad I^\text{der}(\mathbb{Q})\setminus I^\text{der}(\mathbb{A}_f)/I^\text{der}(\mathbb{A}_f) \cap gJ_0^b K^p(N)g^{-1} \rightarrow I(\mathbb{Q})\setminus D(\mathbb{A}_f)/gJ_0^b K^p(N)/J_0^b K^p(N) = \nu^{-1}(\nu(g)).$$

The subgroup $J_0^b$ is normal in $I(\mathbb{Q}_p)$ and contains $I^\text{der}(\mathbb{Q}_p) = \{g \in I(\mathbb{Q}_p) \mid \text{nr}(g) = 1, \text{sim}(g) = 1\}$. Hence $I^\text{der}(\mathbb{A}_f) \cap gJ_0^b K^p(N)g^{-1} = I^\text{der}(\mathbb{Q}_p) \cdot I^\text{der}(\mathbb{A}_f) \cap gK^p(N)g^{-1}$.

Suppose that $I^\text{der}(\mathbb{Q}_p)$ is not compact. The strong approximation theorem [37] can then be applied to the simply connected group $I^\text{der}(\mathbb{R}) \times I^\text{der}(\mathbb{Q}_p)$. Hence the LHS of (6.26) consists of a single element. It follows that the map $\nu$ is bijective, and in particular $\pi_0(M\mathbb{G}_{\text{bas}})$ and $\pi_0(M_{\mathbb{K}})$ have the same cardinality. Thus the surjective map $\pi_0(M\mathbb{G}_{\text{bas}}) \rightarrow \pi_0(M_{\mathbb{K}})$ in this case is bijective.

Suppose that $I^\text{der}(\mathbb{Q}_p)$ is compact. Then $J_0^b$ is also compact since we have an exact sequence

$$1 \rightarrow I^\text{der}(\mathbb{Q}_p) \rightarrow J_0^b \rightarrow D(\mathbb{Z}_p) \rightarrow 1.$$ Further, $J_0^b$ contains any parahoric subgroup of $J_0(\mathbb{Q}_p)$, and hence it is the unique maximal parahoric subgroup. It follows from (6.23) that

$$|\pi_0(M\mathbb{G}_{\text{bas}})| = |I(\mathbb{Q})\setminus I(\mathbb{A}_f)/J_0^b K^p(N)| = |\text{Irr}(M\mathbb{G}_{\text{bas}})|/|\rho\mathbb{G}_{\text{bas}}|.$$

The assertion thus follows from Lemma 4.4.3. \hfill \Box

Remark 6.3.8. In an alternative definition, $I^\text{der}(\mathbb{Q}_p)$ is compact if and only if the corresponding basic element $b \in G(\mathbb{L})$ is superbasic, see [27] Section 4. Lemma 6.3.6 implies that irreducible components of $M\mathbb{G}_{\text{bas}}$ and points of $M_{\mathbb{K}}(k)$ are equally distributed to each connected component $Y$ of $M\mathbb{G}_{\text{bas}}$. Therefore, in the superbasic case we have that

$$|\text{Irr}(Y)| = \rho\mathbb{G}_{\text{bas}}$$

and

$$|(M_{\mathbb{K}} \cap Y)(k)| = \lambda\mathbb{G}_{b,\text{bas}}.$$
same cardinality as the set $\text{Sh}_K(G, X)_C$. Further we have $\mathcal{M}_K = \mathcal{M}^{bas}_K = \mathcal{M}^e_K$, whose cardinality is given by Theorem 6.3.1 with $\lambda^{bas}_p = \lambda^e_p = \rho^{bas} = 1$.

When $(r, s) = (0, 1)$ or $(1, 0)$, by the relation $L(0, \chi) = 2h(E)/|\mu_E|$ we have that

\[(6.27) \quad |\mathcal{M}_K(k)| = \left|G(\mathbb{Z}^p) : K^0(N)\right| \cdot h(E)/|\mu_E|.
\]

We use the following lemma in the examples below.

**Lemma 6.3.10.** Let $q$ be a power of $p$ and $H \subset \mathbb{P}^n (n \geq 1)$ be the Fermat hypersurface defined by $X_0^{q+1} + \cdots + X_n^{q+1} = 0$. Then

\[(6.28) \quad |H(\mathbb{F}_{q^2})| = \frac{(q^{n+1} + (-1)^n)(q^n + (-1)^{n+1})}{q^2 - 1}.
\]

**Proof.** Let

\[S_n \coloneqq \#\{[x_0 : \cdots : x_n] \in \mathbb{P}^n(\mathbb{F}_{q^2}) \mid x_0^{q+1} + \cdots + x_n^{q+1} = 0 \} = \#H(\mathbb{F}_{q^2}),
\]

\[A_n \coloneqq \#\{([x_0, \ldots, x_{n-1}] \in \mathbb{A}^n(\mathbb{F}_{q^2}) \mid x_0^{q+1} + \cdots + x_{n-1}^{q+1} + 1 = 0 \}.
\]

We separate the cases where $x_1 = 0$ or $x_n = 1$; this gives $S_n = S_{n-1} + A_n$ and $S_n = A_1 + \cdots + A_n$.

By solving the equation $-X_0^{q+1} = X_0^{q+1} + \cdots + X_{n-1}^{q+1} + 1$ in $\mathbb{A}^n(\mathbb{F}_{q^2})$, we obtain

\[A_n = \#\{x_0^{q+1} + \cdots + x_{n-1}^{q+1} + 1 = 0 \} + \#\{x_0^{q+1} + \cdots + x_{n-1}^{q+1} \neq 1 \} \cdot (q + 1)
\]

\[= A_{n-1} + (q + 1)(q^{2n-2} - A_{n-1}) = q^{2n-1} + q^{2n-2} - qA_{n-1}.
\]

Put $B_n := A_n - q^{2n-1}$, then $B_1 = A_1 - q = 1$ and $B_n = (-q) \cdot B_{n-1}$. Therefore, $B_n = (-q)^{n-1}$ and $A_n = q^{2n-1} - (-q)^{n-1}$. We compute

\[S_n = A_1 + \cdots + A_n = \frac{q(q^{2n} - 1)}{q^2 - 1} + \frac{(-q)^{n-1} - 1}{q - 1} = \frac{(q^{n+1} + (-1)^n)(q^n + (-1)^{n+1})}{q^2 - 1}.
\]

\[\Box
\]

**Example 6.3.11** (Picard modular surfaces). Assume that $(r, s) = (1, 2)$. In this case, the complex Shimura variety $M_{K, C} = M_K \otimes \mathbb{C}$ is of dimension two, and called a Picard modular surface. When $p$ is inert in $E$, $M^{bas}_K$ coincides the supersingular locus, which is of dimension one, and $M^e_K$ coincides with the superspecial locus. Now we fix a connected component $M_{K, C}^0$ of $M_{K, C}$, and let $\mathcal{M}^0_K$ be the corresponding connected component of $\mathcal{M}_K$ through a smooth compactification $\overline{M}_K$ of $M_K$. Write $N$ for the number of irreducible components of $M^{bas}_K$ which are contained in $M^0_K$. When $p$ is inert in $E$, De Shalit and Goren [12] proved that

\[(6.29) \quad 3N = c_2(M_{K, C}^0),
\]

where $c_2(M_{K, C}^0)$ is the top Chern class of the smooth model $\overline{M}_{K, C}^0$ of $M_{K, C}^0$, which depends only on $M_{K, C}^0$. They also showed that under the condition that $N$ is sufficiently large depending on $p$ (see [12] Theorem 2.1(iii)), the number of superspecial points in $\mathcal{M}^0_K$ is equal to

\[(6.30) \quad \frac{c_2(M_{K, C}^0)}{3} \cdot \frac{p^3 + 1}{p + 1}.
\]

The class $c_2(M_{K, C}^0)$ had been computed explicitly by Holzapfel [32], Main Theorem 5A.4.7.

\[(6.31) \quad c_2(M_{K, C}^0) = \left[G^{\text{der}}(\mathbb{Z}) : K^{\text{der}}(N)\right] \cdot \left|d_E\right|^{5/2} \cdot L(3, \chi),
\]

\[\text{Formula } [32] \text{ (5A.4.3), p. 325} \text{ (as well as [12] (1.12) and (1.14)) contains an unnecessary factor ‘3’ in its RHS, although it is computed correctly in the proof [32] line 16, p. 326].
where we write $G^\text{der}(\hat{Z}) := G(\hat{Z}) \cap G^\text{der}(A_f)$ and $K^\text{der}(N) := \ker (G^\text{der}(\hat{Z}) \to G(\hat{Z}/N\hat{Z}))$. Formulas (6.39), (6.30), (6.31), and the functional equation imply that

\begin{equation}
N = -[G^\text{der}(\hat{Z}) : K^\text{der}(N)] \cdot \frac{1}{48} \cdot L(-2, \chi), \quad \text{and} \quad |(M_0^\text{bas} \cap M^e_\text{K})(k)| = N \cdot (p^2 - p + 1).
\end{equation}

On the other hand, Theorem 6.3.1 shows that if $p$ is inert in $E$ then

\begin{equation}
|\text{Irr}(M^\text{bas}_K)| = -[G(\hat{Z}^p) : K^p(N)] \cdot \frac{1}{48} \cdot \frac{h(E)}{[\mu_E]} \cdot L(-2, \chi),
\end{equation}

\begin{equation}
|M^e_\text{K}(k)| = -[G(\hat{Z}^p) : K^p(N)] \cdot \frac{1}{48} \cdot \frac{h(E)}{[\mu_E]} \cdot L(-2, \chi) \cdot (p^2 - p + 1).
\end{equation}

Here we have an equality

\[
[G(\hat{Z}^p) : K^p(N)] = [G(\hat{Z}) : K(N)] = [G^\text{der}(\hat{Z}) : K^\text{der}(N)] \cdot [D(\hat{Z}) : \nu(K(N))].
\]

Further, Theorem 3.3.2 shows that $M^\text{bas}_K$ has $[D(\hat{Z}) : \nu(K(N))] \cdot [\mu_E]^{-1} \cdot h(E)$ connected components. From these results one deduces (6.32).

In this case, every irreducible component of $M^\text{bas}_K$ is isomorphic to the Fermat curve $C : X_{p+1}^{p+1} + X_1^{p+1} + X_2^{p+1} = 0$ of degree $p + 1$ and it has $|C(\mathbb{F}_{p^2})| = p^3 + 1$ superspecial points. Every superspecial point is contained in $p + 1$ irreducible components; see [32] Theorem 4.

When $p$ is split in $E$, the basic locus $M^\text{bas}_K$ is zero-dimensional and we have

\begin{equation}
|\text{Irr}(M^\text{bas}_K)| = |M^\text{bas}_K(k)| = -[G(\hat{Z}^p) : K^p(N)] \cdot \frac{1}{48} \cdot \frac{h(E)}{[\mu_E]} \cdot L(-2, \chi) \cdot (p - 1)(p^2 - 1).
\end{equation}

**Example 6.3.12** ($n = 2$ or $n = 4$). For simplicity we assume that 2 is unramified in $E$; this implies that $\kappa_2 = 1$ in (6.17). Set $S^q_E := \{ \ell|d_E : G^1_{\mathbb{Q}_l} \text{ is quasi-split} \}$ and $S^q_{q_E} := \{ \ell|d_E : G^1_{\mathbb{Q}_l} \text{ is not quasi-split} \}$, and set

\[
L(2, E, N) := [G(\hat{Z}) : K(N)] \cdot \frac{1}{2^w \cdot 12} \cdot \frac{h(E)}{[\mu_E]} \cdot \prod_{\ell \in S^q_E} (\ell + 1) \prod_{\ell \in S^q_{q_E}} (\ell - 1), \quad \text{and}
\]

\[
L(4, E, N) := -[G(\hat{Z}) : K(N)] \cdot \frac{1}{2^w \cdot 5760} \cdot \frac{h(E)}{[\mu_E]} \cdot L(-2, \chi) \cdot \prod_{\ell \in S^q_E} (\ell^2 + 1) \prod_{\ell \in S^q_{q_E}} (\ell^2 - 1).
\]

For $(r, s) = (1, 1)$, the basic locus $M^\text{bas}_K$ is zero-dimensional and we have (for both $p$ inert and split)

\begin{equation}
|M^\text{bas}_K(k)| = |\text{Irr}(M^\text{bas}_K)| = L(2, E, N) \cdot (p - 1).
\end{equation}

For $(r, s) = (1, 3)$, if $p$ is inert in $E$, then $M^\text{bas}_K$ is one-dimensional and we have

\begin{equation}
|M^\text{bas}_K(k)| = |\text{Irr}(M^\text{bas}_K)| = L(4, E, N) \cdot (p - 1)(p^2 + 1).
\end{equation}

In this case, every irreducible component of $M^\text{bas}_K$ is isomorphic to the Fermat curve $C : X_{p+1}^{p+1} + X_1^{p+1} + X_2^{p+1} = 0$ of degree $p + 1$ and it has $|C(\mathbb{F}_{p^2})| = p^3 + 1$ superspecial points. Every superspecial point is contained in $p^3 + 1$ irreducible components; see [32] Example G, p. 595).

If $p$ is split in $E$, then $M^\text{bas}_K$ is zero-dimensional and we have

\begin{equation}
|M^e_\text{K}(k)| = |\text{Irr}(M^\text{bas}_K)| = L(4, E, N) \cdot (p - 1)(p^2 - 1)(p^3 - 1).
\end{equation}

For $(r, s) = (2, 2)$, if $p$ is inert in $E$, then $M^\text{bas}_K$ is two-dimensional and we have

\begin{equation}
|M^e_\text{K}(k)| = L(4, E, N) \cdot (p^2 - p + 1)(p^2 + 1), \quad |\text{Irr}(M^\text{bas}_K)| = 2L(4, E, N).
\end{equation}
In this case, every irreducible component of $\mathcal{M}_K^{\text{bas}}$ is isomorphic to the Fermat surface $S : X_0^{p+1} + X_1^{p+1} + X_2^{p+1} + X_3^{p+1} = 0$ of degree $p + 1$ and it has $|S(\mathbb{F}_p^2)| = (p^2 + 1)(p^3 + 1)$ superspecial points; see [33].

If $p$ is split in $E$, then $\mathcal{M}_K^{\text{bas}}$ is 1-dimensional and we have

\begin{equation}
|\mathcal{M}_K^{\text{bas}}(k)| = |\text{Irr}(\mathcal{M}_K^{\text{bas}})| = L(4, E, N) \cdot (p - 1)(p^3 - 1).
\end{equation}

In this case, every irreducible component of $\mathcal{M}_K^{\text{bas}}$ is isomorphic to $\mathbb{P}^1$ and it has $p^2 + 1$ superspecial points. Every superspecial point is contained in $p^2 + 1$ irreducible components [14].

**Example 6.3.13** (Basic EO strata for GU(2,2)). Assume that $p$ is inert in $E$. By [33], the EO stratification agrees with the Bruhat-Tits stratification on the basic locus. There are two 2-dimensional EO strata, one 1-dimensional EO stratum, and the unique 0-dimensional EO stratum.

The union of all basic EO strata is the basic locus $\mathcal{M}_K^{\text{bas}}$.

Let $\mathcal{M}_K^{\text{cr}}$ be one of the two 2-dimensional basic EO strata. Then the closure of every irreducible component of $\mathcal{M}_K^{\text{cr}}$ is an irreducible component of $\mathcal{M}_K^{\text{bas}}$. Hence the stabilizer in $J_b(\mathbb{Q}_p)$ of an irreducible component of its preimage in the Rapoport-Zink space is an open compact subgroup with maximal volume. This implies that $\mathcal{M}_K^{\text{cr}}$ has a multiple of $L(4, E, N)$ irreducible components. From (6.35) it follows that $\mathcal{M}_K^{\text{cr}}$ has $L(4, E, N)$ irreducible components.

We know the number of the points in $\mathcal{M}_K^{\text{cr}}(k)$ from (6.35). Every irreducible component of the closure $X$ of the 1-dimension EO stratum is isomorphic to $\mathbb{P}^1$ and has $p^2 + 1$ superspecial points. On the other hand, every superspecial point is contained in $p + 1$ irreducible components of $X$. Using the incidence relation, the 1-dimensional basic EO stratum has $L(4, E, N) \cdot (p^3 + 1)$ irreducible components.

Assume that $p$ is split in $E$. By [14], the basic locus is the union of the 1-dimensional basic EO stratum and the unique 0-dimensional EO stratum. Thus, by (6.39) both the 1-dimensional stratum and the 0-dimensional stratum have $L(4, E, N) \cdot (p - 1)(p^3 - 1)$ irreducible components.

### 7. Upper bound on the number of Hecke eigensystems of mod $p$ automorphic forms

Let $\mathcal{D}$ and $p$ be as in Section [1.1.1] and let $K := G(\mathbb{Z}_p) K^p(N)$ where $N \geq 3$ and $p \nmid N$. Let $[\mu]$ be the conjugacy class of the cocharacter $\mu_h$ defined by $\mathcal{D}$. We can find a representative $\mu$ of $[\mu]$ that extends to a cocharacter $\mu : \mathbb{G}_{m, W(k)} \to \mathbb{G}_{W(k)}$. Let $L(\mu)$ be the centralizer of $\mu$ in $\mathbb{G}_{W(k)}$. Then $L(\mu)$ is a reductive group over $W(k)$ [81, A.6].

We fix a maximal torus $T$ of the $k$-group $L(\mu) \otimes_{W(k)} k$, and a Borel subgroup $B$ containing $T$. Let $X^+(T)^+$ denote the set of dominant weights with respect to $B$. To each $\xi \in X^+(T)^+$, one can associate a vector bundle $\mathcal{V}(\xi)$ on $\mathcal{M}_K$, called the automorphic bundle of weight $\xi$; see [47, Definition 6.7] and [71, Section 4.1].

Let $\mathcal{M}_K^{\text{tor}}$ be a toroidal compactification of $\mathcal{M}_K$, and let $\mathcal{M}_K^{\text{tor}} := \mathcal{M}_K^{\text{tor}} \otimes_k k$. By [47, Section 6B], there exits a canonical extension of $\mathcal{V}(\xi)$ to $\mathcal{M}_K^{\text{tor}}$, denoted by $\mathcal{V}^{\text{can}}(\xi)$. The space of mod $p$ automorphic forms is then defined as

$$
\mathcal{A}(\mathcal{D}, N) := \bigoplus_{\xi \in X^+(T)^+} H^0(\mathcal{M}_K^{\text{tor}}, \mathcal{V}^{\text{can}}(\xi)).
$$

We remark that $\mathcal{M}_K$ is compact or has the boundary with codimension larger than one, and hence Koecher’s principle holds for $\mathcal{M}_K$ by [48, Theorem 2.3]:

$$H^0(\mathcal{M}_K^{\text{tor}}, \mathcal{V}^{\text{can}}(\xi)) \simeq H^0(\mathcal{M}_K, \mathcal{V}(\xi)).$$

The space $\mathcal{A}(\mathcal{D}, N)$ admits an action of the unramified Hecke algebra

$$
\mathcal{H} := \bigotimes_{\ell \neq p} \mathcal{H}_\ell(G_{\mathbb{Q}_\ell}, K_\ell; \mathbb{Z}_p).
$$
We say that a system of Hecke eigenvalues \((b_T)_{T \in \mathcal{H}} \in k^\mathcal{H}\) appears in \(A(\mathcal{D}, N)\) if there exists an element \(f \in A(\mathcal{D}, N)\) such that \(Tf = b_T f\) for all \(T \in \mathcal{H}\).

**Theorem 7.0.1.** Let \(\mathcal{N}(\mathcal{D}, N)\) denote the number of the systems of prime-to-\(p\) Hecke eigenvalues appearing in \(A(\mathcal{D}, N)\). Then

\[
\mathcal{N}(\mathcal{D}, N) \leq |G(\hat{\mathbb{Z}}^p) : K^p(N)| \cdot \varepsilon \cdot \prod_{j=1}^{n} L(1, \chi^2) \cdot \prod_{\ell \mid d_\ell} \kappa_{\ell} \cdot \lambda_{p}^p \cdot \nu_p
\]

where \(\varepsilon, \kappa_{\ell}, \lambda_{p}^p\) are as defined in \((6.16), (6.17), (5.21)\), and \(\nu_p\) is given by

\[
\nu_p = \begin{cases} 
p^{(r(r-1)+s(s-1))/2} \cdot p^{n-2}(p-1)(p+1)^2 & \text{if } p \text{ is inert and } rs \neq 0; 
p^{n(n-1)/2} \cdot p^{n-1}(p-1)(p+1) & \text{if } p \text{ is inert and } rs = 0; 
p^{n(m-1)/2} \cdot p^{n-m}(p^{n/m} - 1) & \text{if } p \text{ is split.}
\end{cases}
\]

**Proof.** We fix a point \((A, \varepsilon, \lambda, \eta)\) of the 0-dimensional EO stratum \(\mathcal{M}'_k(k)\). Let \(I\) be the associated algebraic group over \(\mathbb{Q}\). Then \(I(\mathbb{Q}_p) \simeq J_{\mathbf{b}}(\mathbb{Q}_p)\) for an element \(b\) representing the basic class \([b] \in B(G_{\mathbb{Q}_p}, \mu)\). Let \((A[p^\infty], \varepsilon, \lambda)\) be the associated \(p\)-divisible group with \(\mathcal{D}\)-structure, and let \(M\) be its Dieudonné module. By Proposition 4.4.2, the stabilizer \(I_p'^{e}\) of \(M\) in \(I(\mathbb{Q}_p)\) is a maximal parahoric subgroup. Let \(I_p'^{e}\) be its smooth model over \(\mathbb{Z}_p\) and \(\Gamma_p^e\) be the maximal reductive quotient of the special fiber \(I_p^e \otimes_{\mathbb{Z}_p} \mathbb{F}_p\). Then

\[
\Gamma_p^e \simeq \begin{cases} 
G(U_r \times U_s) & \text{if } p \text{ is inert in } E; 
(\text{Res}_{\mathbb{F}_q/\mathbb{F}_p} \text{GL}_m) \times \mathbb{G}_{m, \mathbb{F}_p} & \text{if } p \text{ is split in } E,
\end{cases}
\]

where \(G(U_r \times U_s)\) denotes the group of pairs of matrices \((g_1, g_2) \in GU_r \times GU_s\) having the same similitude factor, and \(q := p^{n/m}\).

On the other hand, the group \(\Gamma_p^e(\mathbb{F}_p)\) acts on the \(k\)-vector space \(M/pM\), preserving the subspace \(VM/pM\). We write \(I(p)\) for the image of the induced homomorphism \(\Gamma_p^e(\mathbb{F}_p) \rightarrow \text{GL}_k(VM/pM) \times \text{GL}_k(M/VM)\). Then a computation in the proof of Proposition 4.4.2 shows that \(I(p) \simeq \Gamma_p^e(\mathbb{F}_p^e)\).

Let \(\text{Irr}_k(I(p))\) denote the set of isomorphism classes of simple \(k[I(p)]\)-modules over \(k\). By [71, Theorem 0.1], the systems of prime-to-\(p\) Hecke eigenvalues appearing in the space \(A(\mathcal{D}, N)\) are the same as those appearing in the space of algebraic modular forms on \(I\) with varying weights \(V_\tau \in \text{Irr}_k(I(p))\). As a corollary, we have an inequality ([71, Corollary 5.7])

\[(7.1) \quad \mathcal{N}(\mathcal{D}, N) \leq |\mathcal{M}'_k(k)| \cdot \sum_{V_\tau \in \text{Irr}_k(I(p))} \dim_k V_\tau.
\]

When \(p\) is inert in \(E\), Reduzzi showed that ([64, Section 5.2 (5)])

\[(7.2) \quad \sum_{V_\tau \in \text{Irr}_k(I(p))} \dim_k V_\tau \leq \begin{cases} 
p^{(r(r-1)+s(s-1))/2} \cdot p^{n-2}(p-1)(p+1)^2 & \text{if } rs \neq 0; 
p^{n(n-1)/2} \cdot p^{n-1}(p-1)(p+1) & \text{if } rs = 0.
\end{cases}
\]

Now we assume that \(p\) is split in \(E\). We first compute the cardinality of \(\text{Irr}_k(I(p))\). By [66, Corollary 3 of Theorem 42], this cardinality equals the number of \(p\)-regular conjugacy classes of \(I(p)\), where an element of \(I(p)\) is said to be \(p\)-regular if its order is prime to \(p\). Further, this number is equal to \(p^{l} \cdot |Z(F_p)|\) where \(Z\) be the center of \(\Gamma_p\) and \(l\) be the semisimple rank of \(\Gamma_p^e\) ([4, Theorem 3.7.6 (ii)]). We have \(Z(F_p) \simeq \mathbb{F}_q^\times \times \mathbb{F}_p^\times\), and the derived subgroup of \(\Gamma_p^e\) is isomorphic to \(\text{Res}_{\mathbb{F}_q/\mathbb{F}_p} \text{SL}_m, \mathbb{F}_q\), whose rank is \(l = (n/m) \times (m-1) = n - n/m\). It follows that

\[(7.3) \quad |\text{Irr}_k(I(p))| = p^{n-n/m} \cdot (q-1)(p-1).
\]
Next we give an upper bound of \( \dim_k V_\tau \). The group \( I(p) \) is a finite group of Lie type, and hence has a structure of a split \( BN \)-pair of characteristic \( p \) by [14, Section 1.18]. It follows from [8, Corollaries 3.5 and 5.11] that the dimension of a simple \( k[I(p)] \)-module \( V_\tau \) is no larger than the order of a \( p \)-Sylow subgroup of \( I(p) = GL_m(F_q) \times \mathbb{F}_p^\times \), which is equal to \( q^{m(m-1)/2} \). Hence we have

\[
\dim_k V_\tau \leq q^{m(m-1)/2} = p^{n(m-1)/2}.
\]

Formulas (7.1), (7.2), (7.3), (7.4), and (6.19) imply the assertion. \( \square \)

**Corollary 7.0.2** (Asymptotics). If we fix \( N \geq 3 \) and \( n = \dim_E V \), then

\[
N(\mathcal{Z}, N) = O(p^{n(n+1)/2+1}) \quad \text{as} \quad p \to \infty.
\]

**Remark 7.0.3.** Ghitza [20] gave an explicit upper bound for the number of the systems of Hecke eigenvalues of mod \( p \) Siegel modular forms using Hashimoto-Ibukiyama-Ekedahl’s mass formula [20, 13]. For automorphic forms on \( \mathcal{M}_K \) with inert \( p \), Reduzzi [64] has proved that \( N(\mathcal{Z}, N) = O(p^{n^2+n-rs+1}) \) as \( p \to \infty \). He observed that the superspecial locus of \( \mathcal{M}_K \) (which is \( \mathcal{M}_K^{ss} \)) is embedded into the one of a Siegel modular variety, and used Hashimoto-Ibukiyama-Ekedahl’s mass formula. One sees that his bound is improved by Corollary 7.0.2.

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