RANK FOUR VECTOR BUNDLES WITHOUT THETA DIVISOR OVER A CURVE OF GENUS TWO

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Abstract. We show that the locus of stable rank four vector bundles without theta divisor over a smooth projective curve of genus two is in canonical bijection with the set of theta-characteristics. We give several descriptions of these bundles and compute the degree of the rational theta map.

1. Introduction

Let $C$ be a complex smooth projective curve of genus 2 and let $\mathcal{M}_r$ denote the coarse moduli space parametrizing semi-stable rank-$r$ vector bundles with trivial determinant over the curve $C$. Let $\mathcal{C} \cong \Theta \subset \text{Pic}^1(C)$ be the Riemann theta divisor in the degree 1 component of the Picard variety of $C$. For any $E \in \mathcal{M}_r$ we consider the locus $\theta(E) = \{ L \in \text{Pic}^1(C) \mid h^0(C, L \otimes E) > 0 \}$, which is either a curve linearly equivalent to $r\Theta$ or $\theta(E) = \text{Pic}^1(C)$, in which case we say that $E$ has no theta divisor. We obtain thus a rational map, the so-called theta map

$$\theta : \mathcal{M}_r \dashrightarrow |r\Theta|,$$

between varieties having the same dimension $r^2 - 1$. We denote by $\mathcal{B}_r$ the closed subvariety of $\mathcal{M}_r$ parametrizing semi-stable bundles without theta divisor. It is known [R] that $\mathcal{B}_2 = \mathcal{B}_3 = \emptyset$ and that $\mathcal{B}_r \neq \emptyset$ for $r \geq 4$.

It was recently shown that $\theta$ is generically finite; see [B1] Theorem A. Moreover the cases of low ranks $r$ have been studied in the past: if $r = 2$ the theta map is an isomorphism $\mathcal{M}_2 \cong \mathbb{P}^3$ [NR] and if $r = 3$ the theta map realizes $\mathcal{M}_3$ as a double covering of $\mathbb{P}^3$ ramified along a sextic hypersurface [O].

In this note we study the next case $r = 4$ and give a complete description of the locus $\mathcal{B}_4$. Our main result is the following

Theorem 1.1. Let $C$ be a curve of genus 2.

1. The locus $\mathcal{B}_4$ is of dimension 0, reduced and of cardinality 16.
2. There exists a canonical bijection between $\mathcal{B}_4$ and the set of theta-characteristics of $C$. Let $E_\kappa \in \mathcal{B}_4$ denote the stable vector bundle associated with the theta-characteristic $\kappa$. Then

$$\Lambda^2 E_\kappa = \bigoplus_{\alpha \in S(\kappa)} \alpha, \quad \text{Sym}^2 E_\kappa = \bigoplus_{\alpha \in J[2] \setminus S(\kappa)} \alpha,$$

where $S(\kappa)$ is the set of 2-torsion line bundles $\alpha \in J[2]$ such that $\kappa \alpha \in \Theta \subset \text{Pic}^1(C)$.

3. If $\kappa$ is odd, then $E_\kappa$ is a symplectic bundle. If $\kappa$ is even, then $E_\kappa$ is an orthogonal bundle with non-trivial Stiefel-Whitney class.

4. The 16 vector bundles $E_\kappa$ are invariant under the tensor product with the group $J[2]$.

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The 16 vector bundles $E_κ$ already appeared in Raynaud’s paper [R] as Fourier-Mukai transforms and were further studied in [Hh] and [He] — see section 2.2. We note that Theorem 1.1 completes the main result of [Hi] which describes the restriction of $B_3$ to symplectic rank-4 bundles. The method of this paper is different and is partially based on [P].

As an application of Theorem 1.1 we obtain the degree of the theta map for $r = 4$. We refer to [BV] for a geometric interpretation of the general fiber of $θ$ in terms of certain irreducible components of a Brill-Noether locus of the curve $θ(E) \subset \text{Pic}^1(C)$.

**Corollary 1.2.** The degree of the rational theta map $θ : M_4 \rightarrow |4θ|$ equals 30.

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**Notations:** If $E$ is a vector bundle over $C$, we will write $H^i(E)$ for $H^i(C, E)$ and $h^i(E)$ for $\dim H^i(C, E)$. We denote the slope of $E$ by $μ(E) := \frac{deg(E)}{rk(E)}$, the canonical bundle over $C$ by $K$ and the degree $d$ component of the Picard variety of $C$ by $\text{Pic}^d(C)$. We denote by $J := \text{Pic}^0(C)$ the Jacobian of $C$ and by $J[n]$ its group of $n$-torsion points. The divisor $Θ_κ \subset J$ is the translate of the Riemann theta divisor $C \cong Θ \subset \text{Pic}^1(C)$ by a theta-characteristic $κ$. The line bundle $O_J(2Θ_κ)$ does not depend on $κ$ and will be denoted by $O_J(2Θ)$.

2. Proof of Theorem 1.1

2.1. The 16 vector bundles $E_κ$. We first show that the set-theoretical support of $B_4$ consists of 16 stable vector bundles $E_κ$, which are canonically labelled by the theta-characteristics of $C$.

We note that $B_4 \neq \emptyset$ by [R], see also [P] Theorem 1.1. We consider a vector bundle $E \in B_4$. Since $B_2 = B_3 = \emptyset$, we deduce that $E$ is stable. We introduce $E' = E^* \otimes K$. Then $μ(E') = 2$ and since $E \in B_4$, we obtain that $h^0(E' \otimes λ^{-1}) = h^1(E \otimes λ) = h^0(E \otimes λ) > 0$ for any $λ \in \text{Pic}^1(C)$. In particular for any $x \in C$ we have $h^0(E' \otimes O_C(-x)) > 0$. On the other hand stability of $E$ implies that $h^0(E) = h^1(E') = 0$. Hence $h^0(E') = 4$ by Riemann-Roch. Thus we obtain that the evaluation map of global sections

$$O_C \otimes H^0(E') \xrightarrow{ev} E'$$

is not of maximal rank. Let us denote by $I := \text{im } ev$ the subsheaf of $E'$ given by the image of $ev$. Then clearly $h^0(I) = 4$. The cases $rk I \leq 2$ are easily ruled out using stability of $E'$. Hence we conclude that $rk I = 3$. We then consider the natural exact sequence

$$0 \rightarrow L^{-1} \rightarrow O_C \otimes H^0(E') \xrightarrow{ev} I \rightarrow 0,$$

where $L$ is the line bundle such that $L^{-1} := \ker ev$.

**Proposition 2.1.** We have $h^0(I^*) = 0$.

**Proof.** Suppose on the contrary that there exists a non-zero map $I \rightarrow O_C$. Its kernel $S \subset I$ is a rank-2 subsheaf of $E'$ and by stability of $E'$ we obtain $μ(S) < μ(E') = 2$, hence $\deg S \leq 3$. Moreover $h^0(S) ≥ h^0(I) − 1 = 3$.

Assume that $\deg S = 3$. Then $S$ is stable and $S$ can be written as an extension

$$0 \rightarrow μ \rightarrow S \rightarrow ν \rightarrow 0,$$

with $\deg μ = 1$ and $\deg ν = 2$. The condition $h^0(S) ≥ 3$ then implies that $μ = O_C(x)$ for some $x \in C$, $ν = K$ and that the extension has to be split, i.e., $S = K \oplus O_C(x)$. This contradicts stability of $S$.

The assumption $\deg S ≤ 2$ similarly leads to a contradiction. We leave the details to the reader. \qed
Now we take the cohomology of the dual of the exact sequence (1) and we obtain — using $h^0(I^*) = 0$ — an inclusion $H^0(E)^* \subset H^0(L)$. Hence $h^0(L) \geq 4$, which implies $\deg L \geq 5$. On the other hand $\deg L = \deg I$ and by stability of $\mathcal{E}'$, we have $\mu(I) < 2$, i.e., $\deg L \leq 5$. So we can conclude that $\deg L = 5$, that $H^0(E)^* = H^0(L)$ and that $I = E_L$, where $E_L$ is the evaluation bundle associated to $L$ defined by the exact sequence

\begin{equation}
0 \longrightarrow E_L^* \longrightarrow H^0(L) \otimes \mathcal{O}_C \xrightarrow{ev} L \longrightarrow 0.
\end{equation}

Moreover the subsheaf $E_L \subset \mathcal{E}'$ is of maximal degree, hence $E_L$ is a subbundle of $\mathcal{E}'$ and we have an exact sequence

\begin{equation}
0 \longrightarrow E_L \longrightarrow \mathcal{E}' \longrightarrow K^4 L^{-1} \longrightarrow 0,
\end{equation}

with extension class $e \in \text{Ext}^1(K^4 L^{-1}, E_L) = H^1(E_L \otimes K^4 L) = H^0(E_L^* \otimes K^5 L^{-1})^*$. Using Riemann-Roch and stability of $E_L$ (see e.g. [Bu]) one shows that

\begin{align*}
&h^0(E_L^* \otimes K^5 L^{-1}) = 7, \quad h^0(E_L^* \otimes K^5 L^{-1}(-x)) = 4, \quad h^0(E_L^* \otimes K^5 L^{-1}(-x - y)) = 1
\end{align*}

for general points $x, y \in C$. In that case we denote by $\mu_{x,y} \in \mathbb{P}H^0(E_L^* \otimes K^5 L^{-1})$ the point determined by the 1-dimensional subspace $H^0(E_L^* \otimes K^5 L^{-1}(-x - y))$. We also denote by

\[ S \subset \mathbb{P}H^0(E_L^* \otimes K^5 L^{-1}) \]

the linear span of the points $\mu_{x,y}$ when $x$ and $y$ vary in $C$ and by $H_e \subset \mathbb{P}H^0(E_L^* \otimes K^5 L^{-1})$ the hyperplane determined by the non-zero class $e$.

Tensoring the sequence (3) with $K^{-4} L(x + y)$ and taking cohomology one shows that $\mu_{x,y} \in H_e$ if and only if $h^0(\mathcal{E}' \otimes K^{-4} L(x + y)) > 0$. Since we assume $\mathcal{E} \in \mathcal{B}_4$, we obtain

\[ S \subset H_e. \]

We consider a general point $x \in C$ such that $h^0(E_L^* \otimes K^5 L^{-1}(-x)) = 4$ and denote for simplicity

\[ A := E_L^* \otimes K^5 L^{-1}(-x). \]

Then $A$ is stable with $\mu(A) = \frac{7}{4}$. We consider the evaluation map of global sections

\[ ev_A : \mathcal{O}_C \otimes H^0(A) \longrightarrow A \]

and consider the set $S_A$ of points $p \in C$ for which $(ev_A)_p$ is not surjective, i.e.

\[ S_A = \{ p \in C \mid h^0(A(-p)) \geq 2 \}. \]

Then we have the following

**Lemma 2.2.** We assume that $x$ is general.

1. If $L^2 \neq K^5$, then the set $S_A$ consists of the 2 distinct points $p_1, p_2$ determined by the relation $\mathcal{O}_C(p_1 + p_2) = K^4 L^{-1}(-x)$.

2. If $L^2 = K^5$, then the set $S_A$ consists of the 2 distinct points $p_1, p_2$ introduced in (1) and the conjugate $\sigma(x)$ of $x$ under the hyperelliptic involution $\sigma$.

**Proof.** Given a point $p \in C$, we tensorize the exact sequence (2) with $K^5 L^{-1}(-x - p)$ and take cohomology:

\[ 0 \longrightarrow H^0(A(-p)) \longrightarrow H^0(L) \otimes H^0(K^5 L^{-1}(-x - p)) \longrightarrow H^0(K^5(-x - p)) \longrightarrow \cdots \]

We note that $h^0(K^5 L^{-1}(-x - p)) = 2$. We distinguish two cases.

(a) The pencil $|K^5 L^{-1}(-x - p)|$ has a base-point, i.e. there exists a point $q \in C$ such that $K^5 L^{-1}(-x - p) = K(q)$, or equivalently $K^4 L^{-1}(-x) = \mathcal{O}_C(p + q)$. Since $x$ is general, we have $h^0(K^4 L^{-1}(-x)) = 1$, which determines $p$ and $q$, i.e., $\{p, q\} = \{p_1, p_2\}$. In this case $|K^5 L^{-1}(-x - p)| = |K(q)| = |K|$ and $h^0(A(-p)) = h^0(K^5(-x)) = 2$. This shows that $p_1, p_2 \in S_A$. 

(b) The pencil \(|K^5L^{-1}(-x-p)|\) is base-point-free. By the base-point-free-pencil-trick, we have \(H^0(A(-p)) \cong H^0(L^2K^{-5}(x+p))\). Since \(\deg L^2K^{-5}(x+p) = 2\), we have \(h^0(L^2K^{-5}(x+p)) = 2\) if and only if \(L^2K^{-5}(x+p) = K\), or equivalently \(\mathcal{O}_C(p) = K^6L^{-2}(-x)\). If \(K^6L^{-2} \neq K\), then for general \(x \in C\) the line bundle \(K^6L^{-2}(-x)\) is not of the form \(\mathcal{O}_C(p)\). If \(K^6L^{-2} = K\), then for any \(x \in C\), \(K^6L^{-2}(-x) = \mathcal{O}_C(\sigma(x))\), which implies that \(\sigma(x) \in S_A\).

This shows the lemma.

\[\Box\]

**Proposition 2.3.** If \(L \neq K^5\), then \(S = \mathbb{P}H^0(E^*_L \otimes K^5L^{-1})\).

**Proof.** We consider a general point \(x \in C\) and the rank-3 bundle \(A\). Let \(B \subset A\) denote the subsheaf given by the image of \(ev_A\). By Lemma 2.2 (1) we have \(\deg B = \deg A - 2 = 5\). Moreover \(H^0(B) = H^0(A)\) and there is an exact sequence

\[0 \longrightarrow M^{-1} \longrightarrow \mathcal{O}_C \otimes H^0(B) \overset{ev_A}{\longrightarrow} B \longrightarrow 0,
\]

with \(M \in \text{Pic}^5(C)\). It follows that the rational map

\[\phi_x : C \dashrightarrow \mathbb{P}H^0(B) = \mathbb{P}H^0(A) = \mathbb{P}^3, \quad y \mapsto \mu_{x,y}
\]

factorizes through

\[C \xrightarrow{\varphi_M} |M|^* \longrightarrow \mathbb{P}H^0(B),
\]

where \(\varphi_M\) is the morphism given by the linear system \(|M|\) and the second map is linear and identifies with the projectivization of the dual of \(\delta\), which is given by the long exact sequence obtained from \([\mathbb{P}]\) by dualizing and taking cohomology:

\[0 \longrightarrow H^0(B^*) \longrightarrow H^0(B)^* \overset{\delta}{\longrightarrow} H^0(M) \longrightarrow H^1(B^*) \longrightarrow \cdots
\]

We obtain that the linear span of \(\text{im } \phi_x\) is non-degenerate if and only if \(h^0(B^*) = 0\).

We now show that \(h^0(B^*) = 0\). Suppose on the contrary that there exists a non-zero map \(B \rightarrow \mathcal{O}_C\). Its kernel \(S \subset B\) is a rank-2 subsheaf of \(A\) with \(\deg S \geq \deg B = 5\), hence \(\mu(S) \geq \frac{5}{2}\), which contradicts stability of \(A\) — recall that \(\mu(A) = \frac{7}{3}\).

This shows that \(\text{im } \phi_x\) spans \(\mathbb{P}H^0(A) \subset \mathbb{P}H^0(E^*_L \otimes K^5L^{-1})\) for general \(x \in C\). We now take 2 general points \(x, x' \in C\) and deduce from \(\dim H^0(A) \cap H^0(A') = \dim H^0(E^*_L \otimes K^5L^{-1}(-x-x')) = 1\) that the linear span of the union \(\mathbb{P}H^0(A) \cap \mathbb{P}H^0(A')\) equals the full space \(\mathbb{P}H^0(E^*_L \otimes K^5L^{-1})\). This shows the proposition.

We deduce from the proposition that the line bundle \(L\) satisfies the relation \(L^2 = K^5\), i.e.

\[L = K^2\kappa
\]

for some theta-characteristic \(\kappa\) of \(C\). In that case we note that \(H^0(E^*_L \otimes K^5L^{-1}) = H^0(E^*_L \otimes L)\) and we can consider the exact sequence

\[0 \longrightarrow H^0(E^*_L \otimes L) \longrightarrow H^0(L) \otimes H^0(L) \overset{\mu}{\longrightarrow} H^0(L^2) \longrightarrow 0,
\]

obtained from \([2]\) by tensoring with \(L\) and taking cohomology. We also note that there is a natural inclusion \(\Lambda^2H^0(L) \subset H^0(E^*_L \otimes L)\), see e.g. \([\mathbb{P}]\) section 2.1. More precisely we can show

**Proposition 2.4.** The linear span \(S\) equals

\[S = \mathbb{P}\Lambda^2H^0(L) \subset \mathbb{P}H^0(E^*_L \otimes L).
\]

**Proof.** Using the standard exact sequences and the base-point-free-pencil-trick, one easily works out that for general points \(x, y \in C\)

\[\mu_{x,y} = \mathbb{P}\Lambda^2H^0(L(-x-y)) \subset \mathbb{P}\Lambda^2H^0(L) \subset \mathbb{P}H^0(E^*_L \otimes L).
\]

This implies that \(S \subset \mathbb{P}\Lambda^2H^0(L)\). In order to show equality one chooses 4 general points \(x_i \in C\) such that their images \(C \to |L|^* = \mathbb{P}^3\) linearly span the \(\mathbb{P}^3\). We denote by \(s_i \in H^0(L)\) the global
section vanishing on the points $x_j$ for $j \neq i$ and not vanishing on $x_i$. Then one checks that for any choice of the indices $i, j, k, l$ such that $\{i, j, k, l\} = \{1, 2, 3, 4\}$ one has $s_i \wedge s_j = \mu_{x_k, x_l}$. Since the 6 tensors $s_i \wedge s_j$ are a basis of $\Lambda^2 H^0(L)$, we obtain equality.

The hyperplane $S = \mathbb{P} \Lambda^2 H^0(L) \subset \mathbb{P} H^0(E^*_L \otimes L)$ determines a unique (up to a scalar) non-zero extension class $e \in H^0(E^*_L \otimes L)^*$ by $S = H_\kappa$, which in turn determines a unique stable vector bundle $\mathcal{E} \in \mathcal{B}_4$, which we will denote by $E_\kappa$.

This shows that $\mathcal{B}_4$ is of dimension 0 and of cardinality 16.

2.2. The Raynaud bundles. In this subsection we recall the construction of the Raynaud bundles introduced in [R] as Fourier-Mukai transforms. We refer to [Hi] section 9.2 for the details and the proofs.

The rank-4 vector bundle $\mathcal{O}_J(2\Theta) \otimes H^0(J, \mathcal{O}_J(2\Theta))^*$ over $J$ admits a canonical $J[2]$-linearization and descends therefore under the duplication map $[2] : J \rightarrow J$, i.e., there exists a rank-4 vector bundle $M$ over $J$ such that

$$[2]^*M \cong \mathcal{O}_J(2\Theta) \otimes H^0(J, \mathcal{O}_J(2\Theta))^*.$$  

**Proposition 2.5.** For any theta-characteristic $\kappa$ of $C$ there exists an isomorphism

$$\xi_\kappa : M \sim \to M^* \otimes \mathcal{O}_J(\Theta_\kappa).$$

Moreover if $\kappa$ is even (resp. odd), then $\xi_\kappa$ is symmetric (resp. skew-symmetric).

Let $\gamma_\kappa : C \rightarrow J$ be the Abel-Jacobi map defined by $\gamma_\kappa(p) = \kappa^{-1}(p)$. We define the Raynaud bundle

$$R_\kappa := \gamma_\kappa^* M \otimes \kappa^{-1}.$$  

Then by [R] the bundle $R_\kappa \in \mathcal{B}_4$. Since $\gamma_\kappa^* \mathcal{O}_J(\Theta_\kappa) = K$ we see that the isomorphism $\xi_\kappa$ induces an orthogonal (resp. symplectic) structure on the bundle $R_\kappa$, if $\kappa$ is even (resp. odd). In particular the bundle $R_\kappa$ is self-dual, i.e., $R_\kappa = R_\kappa^*$. The pull-back $\gamma_\kappa^*(\xi_\kappa')$ for a theta-characteristic $\kappa' = \kappa\alpha$ with $\alpha \in J[2]$ gives an isomorphism

$$R_\kappa \sim \to R_\kappa^* \otimes \alpha,$$

hence a non-zero section in $H^0(\Lambda^2 R_\kappa \otimes \alpha)$ (resp. $H^0(\text{Sym}^2 R_\kappa \otimes \alpha)$) if $h^0(\kappa\alpha) = 1$ (resp. $h^0(\kappa\alpha) = 0$). We deduce that there are isomorphisms

$$\Lambda^2 R_\kappa = \bigoplus_{\alpha \in S(\kappa)} \alpha, \quad \text{Sym}^2 R_\kappa = \bigoplus_{\alpha \in J[2] \setminus S(\kappa)} \alpha.$$  

In particular the 16 bundles $R_\kappa$ are non-isomorphic. Each $R_\kappa$ is invariant under tensor product with $J[2]$. The isomorphisms (5) can be used to prove the relation

$$R_\kappa \otimes \beta = R_{\kappa \beta^2}, \quad \forall \beta \in J[4].$$  

2.3. Symplectic and orthogonal bundles. In this subsection we give a third construction of the bundles in $\mathcal{B}_4$ as symplectic and orthogonal extension bundles. Let $\kappa$ be a theta-characteristic.

If $\kappa$ is odd, then $\kappa = \mathcal{O}_C(w)$ for some Weierstrass point $w \in C$. The construction outlined in [P] section 2.2 gives a unique symplectic bundle $\mathcal{E}_e \in \mathcal{B}_4$ with $e \in H^1(\text{Sym}^2 G)_+$. We denote this bundle by $V_\kappa$.

If $\kappa$ is even, there is an analogue construction, which we briefly outline for the convenience of the reader. The proofs are similar to those given in [Hi]. Using the Atiyah-Bott-fixed-point formula one observes that among all non-trivial extensions

$$0 \longrightarrow \kappa^{-1} \longrightarrow G \longrightarrow \mathcal{O}_C \longrightarrow 0,$$
there are 2 extensions (up to scalar), which are $\sigma$-invariant. We take one of them. Then any non-zero class $e \in H^1(\Lambda^2G) = H^1(\kappa^{-1})$ determines an orthogonal bundle $\mathcal{E}_e$, which fits in the exact sequence

$$0 \to G \to \mathcal{E}_e \to G^* \to 0.$$ \hfill (7)

The composite map

$$\tilde{D}_G : \mathbb{P}H^1(\Lambda^2G) \to \mathcal{M}_4 \overset{\theta}{\to} |4\Theta|, \quad e \mapsto \theta(\mathcal{E}_e)$$

is the projectivization of a linear map

$$\tilde{D}_G : H^1(\Lambda^2G) \to H^0(\text{Pic}^1(C), 4\Theta).$$

Moreover $\text{Im} \tilde{D}_G \subset H^0(\text{Pic}^1(C), 4\Theta)_-$, which can be seen as follows. By [Se] Thm 2 the second Stiefel-Whitney class $w_2(\mathcal{E}_e)$ of an orthogonal bundle $\mathcal{E}_e$ is given by the parity of $h^0(\mathcal{E}_e \otimes \kappa')$ for any theta-characteristic $\kappa'$. This parity can be computed by taking the cohomology of the exact sequence (7) tensorized with $\mathcal{E}_e$ and taking into account that the coboundary map is skew-symmetric. One obtains that $w_2(\mathcal{E}_e) \neq 0$ and one can conclude the above-mentioned inclusion by [B2] Lemma 1.4.

We now observe that by the Atiyah-Bott-fixed-point-formula $h^1(\Lambda^2G)_+ = h^1(\Lambda^2G)_- = 1$. By the argument given in [P] section 2.2 we conclude that one of the two eigenspaces $H^1(\Lambda^2G)_\pm$ is contained in the kernel $\text{ker} \tilde{D}_G$. We denote the corresponding bundle $\mathcal{E}_e$ by $V_\kappa \in \mathcal{B}_4$.

2.4. Three descriptions of the same bundle.

**Proposition 2.6.** For any theta-characteristic $\kappa$ the three bundles $E_\kappa$, $R_\kappa$ and $V_\kappa$ coincide.

**Proof.** If $\kappa$ is odd, this was worked out in detail in [H] section 8 and Theorem 29. If $\kappa$ is even, the proofs are similar. \hfill $\square$

This proposition shows all assertions of Theorem 1.1 except reducedness of $\mathcal{B}_4$.

I am grateful to Olivier Serman for giving me the following fourth description of the bundle $E_\kappa$ for an even theta-characteristic $\kappa$. We recall that an even theta-characteristic $\kappa$ corresponds to a partition of the set of six Weierstrass points of $C$ into two subsets of three points, which we denote by $\{w_1, w_2, w_3\}$ and $\{w_4, w_5, w_6\}$. With this notation we have

**Proposition 2.7.** Let $\kappa$ be an even theta-characteristic. We denote by $A_\kappa$ (resp. $B_\kappa$) the unique stable rank-2 bundle with determinant $\kappa$ and which contains the four 2-torsion line bundles $\mathcal{O}_C$, $\mathcal{O}_C(w_1 - w_2)$, $\mathcal{O}_C(w_1 - w_3)$ and $\mathcal{O}_C(w_2 - w_3)$ (resp. $\mathcal{O}_C$, $\mathcal{O}_C(w_4 - w_5)$, $\mathcal{O}_C(w_4 - w_6)$ and $\mathcal{O}_C(w_5 - w_6)$). Then the orthogonal rank-4 vector bundle $E_\kappa$ is isomorphic to

$${\text{Hom}}(A_\kappa, B_\kappa),$$

equipped with the quadratic form given by the determinant.

We refer to [S] section 5.5 for the proof.

2.5. Reducedness of $\mathcal{B}_4$. We denote by $\mathcal{L}$ the determinant line bundle over the moduli space $\mathcal{M}_4$ and recall that the set $\mathcal{B}_4$ can be identified with the base locus of the linear system $|\mathcal{L}|$. This endows the set $\mathcal{B}_4$ with a natural scheme-structure.

We start with a description of the space of global sections $H^0(\mathcal{M}_4, \mathcal{L})$.

**Proposition 2.8.** For any theta-characteristic $\kappa$ there is a section $s_\kappa \in H^0(\mathcal{M}_4, \mathcal{L})$ with zero divisor

$$\Delta_\kappa := \text{Zero}(s_\kappa) = \{E \in \mathcal{M}_4 \mid h^0(\Lambda^2E \otimes \kappa) > 0\}.$$

The 16 sections $s_\kappa$ form a basis of $H^0(\mathcal{M}_4, \mathcal{L})$. 

Proof. The Dynkin index of the second fundamental representation \( \rho : \mathfrak{sl}_4(\mathbb{C}) \to \text{End}(\Lambda^2 \mathbb{C}^4) \) equals 2 (see e.g. [LS] Proposition 2.6). Moreover the bundle \( \Lambda^2 E \otimes \kappa \) admits a \( K \)-valued non-degenerate quadratic form, which allows to construct the Pfaffian divisor \( s_\kappa \), which is a section of \( \mathcal{L} \) (see [LS]). The space \( H^0(\mathcal{M}_4, \mathcal{L}) \) is a representation of level 2 of the Heisenberg group \( \text{Heis}(2) \), which is a central extension of \( J[2] \) by \( \mathbb{C}^\ast \). One can work out that the sections \( s_\kappa \) generate the 16 one-dimensional character spaces for the \( \text{Heis}(2) \)-action on \( H^0(\mathcal{M}_4, \mathcal{L}) \). This shows that the sections \( s_\kappa \) are linearly independent. \( \square \)

Since \( E_\kappa \in \mathcal{B}_4 \), we have \( E_\kappa \in \Delta_\kappa \) for any theta-characteristic \( \kappa' \). By the deformation theory of determinant and Pfaffian divisors (see e.g. [L], [LS]) the point \( E_\kappa \in \mathcal{M}_4 \) is a smooth point of the divisor \( \Delta_\kappa' \subset \mathcal{M}_4 \) if and only if the following two conditions hold

\begin{align*}
(1) & \quad h^0(\Lambda^2 E_\kappa \otimes \kappa') = 2, \\
(2) & \quad \text{the natural linear form}
\end{align*}

\[ \Phi_{\kappa'} : T_{E_\kappa} \mathcal{M}_4 = H^1(\text{End}_0(E_\kappa)) \longrightarrow \Lambda^2 H^0(\Lambda^2 E_\kappa \otimes \kappa')^* \]

is non-zero.

Moreover if these two conditions holds, then \( T_{E_\kappa} \Delta_\kappa' = \ker \Phi_{\kappa'} \). The map \( \Phi_{\kappa'} \) is built up as follows: the exceptional isomorphism of Lie algebras \( \mathfrak{sl}_4 \cong \mathfrak{so}_6 \) induces a natural vector bundle isomorphism

\[ \text{End}_0(E_\kappa) \xrightarrow{\sim} \Lambda^2(\Lambda^2 E_\kappa). \]

Then \( \Phi_{\kappa'} \) is the dual of the linear map given by the wedge product of global sections

\[ \Lambda^2 H^0(\Lambda^2 E_\kappa \otimes \kappa') \longrightarrow H^0(\Lambda^2(\Lambda^2 E_\kappa) \otimes K) = H^0(\text{End}_0(E_\kappa) \otimes K). \]

Proposition 2.9. The 0-dimensional scheme \( \mathcal{B}_4 \) is reduced.

Proof. Since \( E_\kappa \) is a smooth point of \( \mathcal{M}_4 \) and \( \dim T_{E_\kappa} \mathcal{M}_4 = 15 \), it is sufficient to show that for any theta-characteristic \( \kappa' \neq \kappa \) the divisor \( \Delta_\kappa' \) is smooth at \( E_\kappa \) and that the 15 hyperplanes \( \ker \Phi_{\kappa'} \subset T_{E_\kappa} \mathcal{M}_4 \) are linearly independent: using the isomorphism (3) we obtain that for \( \kappa' \neq \kappa \)

\[ h^0(\Lambda^2 E_\kappa \otimes \kappa') = \# S(\kappa) \cap S(\kappa') = 2 \]

and using the isomorphism (3) we obtain that

\[ \text{End}_0(E_\kappa) = \bigoplus_{\alpha \in J[2] \setminus \{0\}} \alpha. \]

On the other hand one easily sees that if \( \gamma, \delta \in J[2] \) are the two 2-torsion points in the intersection \( S(\kappa) \cap S(\kappa') \), then \( \kappa' = \kappa \gamma \delta \), hence \( \Lambda^2 H^0(\Lambda^2 E_\kappa \otimes \kappa') \cong H^0(K \gamma \delta) \). This implies that the linear form

\[ \Phi_{\kappa'} : \bigoplus_{\alpha \in J[2] \setminus \{0\}} H^1(\alpha) \longrightarrow H^0(K \gamma \delta)^* = H^1(\beta) \]

is projection onto the direct summand \( H^1(\beta) \), where \( \beta = \kappa^{-1} \kappa' \in J[2] \). This description of the linear forms \( \Phi_{\kappa'} \) clearly shows that they are non-zero and linearly independent. \( \square \)

This completes the proof of Theorem 1.1.

3. Proof of Corollary 1.2

Since by Theorem 1.1 \( \mathcal{B}_4 \) is a reduced 0-dimensional scheme of length 16, the degree of the theta map \( \theta \) is given by the formula

\[ \deg \theta + 16 = c_{15}, \]

where \( \frac{c_{15}}{15!} \) is the leading coefficient of the Hilbert polynomial

\[ P(n) = \chi(\mathcal{M}_4, \mathcal{L}^n) = \frac{c_{15}}{15!} n^{15} + \text{lower degree terms}. \]
In order to compute the polynomial $P$ we write

$$P(X) = \sum_{k=0}^{15} \alpha_k Q_k(X), \quad \text{with} \quad Q_k(X) = \frac{1}{k!} (X + 7)(X + 6) \cdots (X + 8 - k)$$

and $Q_0(X) = 1$. Note that $\deg Q_k = k$ and that $c_{15} = \alpha_{15}$. The canonical bundle of $\mathcal{M}_4$ equals $\mathcal{L}^{-8}$. By the Grauert-Riemenschneider vanishing theorem we obtain that $h^i(\mathcal{M}_4, \mathcal{L}^n) = 0$ for any $i \geq 1$ and $n \geq -7$. Hence $P(n) = h^0(\mathcal{M}_4, \mathcal{L}^n)$ for $n \geq -7$. Moreover $P(n) = 0$ for $n = -7, -6, \ldots, -1$ and $P(0) = 1$. The values $P(n)$ for $n = 1, 2, \ldots, 8$ can be computed by the Verlinde formula and with the use of MAPLE. They are given in the following table.

| $n$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $P(n)$ | 16  | 140 | 896 | 4680 | 21024 | 83628 | 300080 | 984539 |

Using the expression (9) of $P$ one straightforwardly deduces the coefficients $\alpha_k$ by increasing induction on $k$: $\alpha_k = 0$ for $k = 0, 1, \ldots, 6$ and the values $\alpha_k$ for $k = 7, \ldots, 15$ are given in the following table.

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 15 |
|-----|---|---|---|---|---|---|---|----|
| $\alpha_k$ | 1 | 8 | 32 | 96 | 214 | 328 | 324 | 184 | 46 |

Hence $\deg \theta = \alpha_{15} - 16 = 30$.

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