MULTILABELLED VERSIONS OF SPERNER’S AND FAN’S LEMMAS AND APPLICATIONS

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Abstract. We propose a general technique related to the polytopal Sperner lemma for proving old and new multilabeled versions of Sperner’s lemma. A notable application of this technique yields a cake-cutting theorem where the number of players and the number of pieces can be independently chosen. We also prove multilabeled versions of Fan’s lemma, a combinatorial analogue of the Borsuk-Ulam theorem, and exhibit applications to fair division and graph coloring.

1. Introduction

Sperner’s lemma and Fan’s lemma are classical results of combinatorial topology. Notably, they provide elementary constructive proofs of the topological theorems of Brouwer and Borsuk-Ulam and their applications. The typical object involved with these lemmas is a pseudomanifold with an integer labeling of the vertices. Then, a constraint on the labeling imposes the existence of a simplex with a certain pattern. For Sperner’s lemma, such a simplex is the discrete analogue of a Brouwer fixed point, and for Fan’s lemma, such a simplex is the discrete analogue of the antipodal point whose existence is asserted by the Borsuk-Ulam theorem.

These lemmas can be extended to results involving many labelings, in which one seeks the existence of a simplex on which the labelings together exhibit a special pattern. The oldest results of this type are probably Bapat’s generalization of Sperner’s lemma [7] and Lee-Shih’s generalization of Fan’s formula [14].

One of the contributions of this paper is a new technique for proving multilabeled versions of Sperner’s lemma. Interestingly, it relies on another generalization of the Sperner lemma, namely the polytopal Sperner lemma [8]. We are not only able to recover several known results using this method, but also to find new ones that seem to be of interest. One of these new results is a theorem about envy-free cake-cutting. The original envy-free cake-cutting theorem, due to Stromquist [22], ensures under mild conditions on player preferences that there always exists a division of a cake into \( k \) connected pieces and an envy-free assignment of these pieces to \( k \) players, i.e., an assignment such that each player prefers the piece she is assigned. Recently, Asada et al. [4] showed that such a division exists without knowing the preferences of a “secretive player”: it is possible to divide the cake into \( k \) connected pieces so that whatever choice is made by the “secretive player”, there is an envy-free assignment of the remaining pieces to the \( k - 1 \) other players. A special case of our theorem is the following dual version, which is new in the cake-cutting literature.

Corollary 1.1. For any instance of the cake-cutting problem with \( k \) players, there exists a division of cake into \( k - 1 \) connected pieces so that no matter which player leaves, there is an envy-free assignment of the pieces to the remaining \( k - 1 \) players.

We show that Fan’s lemma admits multilabeled generalizations too. This is another contribution of this paper. These generalizations extend known applications of Fan’s lemma to graph coloring and “continuous necklace-splitting” problems. An illustration of these applications is the following multicoloring version of a theorem by Simonyi and Tardos [21, Theorem 1]. The quantity
Corollary 1.2. In any collection of $m$ proper colorings of a graph $G$, there is a vertex adjacent to at least $\left\lfloor \frac{1}{2m} \text{ind}(\text{Hom}(K_2,G)) \right\rfloor + 1$ distinct colors in each of the colorings.

2. Multilabeled versions of Sperner’s lemma

2.1. Results on multiple labelings and cake division. Given a triangulation $T$ of the standard $(n-1)$-dimensional simplex $\Delta^{n-1} = \langle v_1, \ldots, v_n \rangle$, a labeling of its vertices $V(T) \to [n]$ is a Sperner labeling if each vertex $v$ of $T$ is labeled by an integer $j$ such that $v_j$ is a vertex of the minimal face of $\Delta^{n-1}$ containing $v$. Sperner’s celebrated lemma is the following statement.

Sperner’s lemma. Any triangulation of $\Delta^{n-1}$ with a Sperner labeling has an $(n-1)$-dimensional simplex whose vertices get distinct labels.

Our main results regarding multilabeled versions of Sperner’s lemma are the following two theorems.

Theorem 2.1 (Multilabeled Sperner lemma). Let $T$ be a triangulation of $\Delta^{n-1}$ and let $\lambda_1, \ldots, \lambda_m$ be Sperner labelings on $T$.

(1) For any choice of positive integers $k_1, \ldots, k_m$ such that $k_1 + \cdots + k_m = m + n - 1$, there exists a simplex $\sigma \in T$ on which, for each $i$, the labeling $\lambda_i$ uses at least $k_i$ distinct labels.

(2) For any choice of positive integers $\ell_1, \ldots, \ell_n$ such that $\ell_1 + \cdots + \ell_n = m + n - 1$, there exists a simplex $\tau \in T$ on which, for each $j$, the label $j$ is used in at least $\ell_j$ labelings.

An illustration of the possible patterns asserted by the theorem is given by Figure 1.

The proof of Theorem 2.1 will actually show that part (1) holds with an additional property: each label is used by at least one of the $\lambda_i$ on the simplex $\sigma$. When $m = 1$, both parts (1) and (2) reduce to the usual Sperner’s lemma. Other choices of parameters (e.g., $\lambda_1 = \cdots = \lambda_m$ with $\ell_1, \ldots, \ell_n \geq 1$) also yield the usual Sperner’s lemma.

![Figure 1](image_url)

Figure 1. Let $n = 3$ (with labels 1, 2, 3) and $m = 2$ (with labelings $\lambda_1, \lambda_2$). For $k_1 + k_2 = 2 + 2 = 4$, Theorem 2.1 part (1) asserts the existence of a simplex $\sigma \in T$ like this one, in which the first labeling uses $k_1 = 2$ labels (1 and 3) and the second labeling uses $k_2 = 2$ labels (2 and 3). This simplex $\sigma$ also exhibits an instance asserted by Theorem 2.1 part (2) for $\ell_1 + \ell_2 + \ell_3 = 1 + 1 + 2 = 4$, since the label 1 appears in 1 labeling, label 2 appears in 1 labeling, and label 3 appears in 2 labelings.

Originally conjectured by the first author, part (1) has been proved by Babson [5] and an elementary and constructive proof has recently been found by Frick et al. [10]. The proof we propose is new but shares some common points with the second proof. In the next section, we will provide yet another proof of this result, with an approach that clearly departs from the other ones. A kind of “dual” statement of part (1) is present in Section 6 of the cited paper by Frick et al. However, part (2) seems to be the natural “dual” version, and we get it with almost the same proof as for part (1).
The second theorem is formulated as a “cake-cutting” result. Even if we actually prove Sperner-type results implying it in a standard way, we felt that the cake-cutting formulation makes the statement of the theorem more appealing.

The traditional setting of the cake-cutting problem is the following. We are given a cake to divide among players. Since the cake will be cut with parallel knives, we can identify the cake with the segment [0, 1] so that knife cuts are just points of this segment, and a division is just a partition of cake into intervals (the fact that a boundary point belongs or not to a given interval does not matter). The players have preferences satisfying the following two assumptions. In any division of cake, each player prefers at least one piece of positive length—the “hungry” assumption—and may prefer several pieces. And, if a player prefers a particular piece in each division of a converging sequence of divisions, then she also prefers this piece for the limit division—the “closed preferences” assumption. An assignment of pieces to players is envy-free if each player prefers the piece she is assigned.

**Theorem 2.2.** Consider an instance of the cake-cutting problem with k players. The following holds for all integers 1 ≤ p, q ≤ k.

(1) For any subset of p players, there is a division of the cake into k pieces such that, no matter which \( \left\lceil \frac{k-p}{p} \right\rceil \) pieces we select, there is an envy-free assignment of some p of the other pieces to the p players.

(2) There is a division of the cake into q pieces such that, no matter which \( \left\lceil \frac{k-q}{q} \right\rceil \) players leave, there is an envy-free assignment of the pieces to some q of the remaining players.

When p = q = k, we get the usual envy-free result due to Stromquist [22]. When p = k − 1, we get the “secretive player” extension of Asada et al. [4] mentioned in the introduction. When p = 1, we get that it is always possible to divide a cake into an arbitrary number of pieces a player is indifferent between. The case q = 1 says something obvious. And Corollary 1.1 is the special case q = k − 1, which appears to be a new result in the cake-cutting literature.

### 2.2. Proofs

All multilabeled versions of Sperner’s lemma considered in this paper involve a triangulation \( T \) of the standard \((n-1)\)-dimensional simplex \( \Delta^{n-1} = \langle v_1, \ldots, v_n \rangle \) with \( m \) Sperner labelings. Our proofs are based on the following construction. Consider the standard \((m-1)\)-dimensional simplex \( \Delta^{m-1} = \langle u_1, \ldots, u_m \rangle \) and the polytope

\[
P = \Delta^{m-1} \times \Delta^{n-1}
\]

whose vertices are \( \{(u_i, v_j) : i \in [m], j \in [n]\} \). Choose any triangulation \( \overline{T} \) of \( P \) that refines the product decomposition \( \Delta^{m-1} \times T \) without adding new vertices; this will ensure that any simplex in \( \overline{T} \) projects naturally to a simplex in \( T \).

We are given \( m \) Sperner labelings \( \lambda_1, \ldots, \lambda_m \) on \( T \), as in Theorem 2.1. For the proof of Theorem 2.2, these labelings are given by the players’ preferences. Then define the map \( \lambda : V(T) \to V(P) \) by

\[
\lambda(u_i, v) = (u_i, v_{\lambda_i(v)}).
\]

**Lemma 2.3.** The map \( \lambda \) maps each vertex of \( \overline{T} \) to a vertex of the minimal face of \( P \) containing it.

**Proof.** Consider a vertex \((u_i, v)\) of \( \overline{T} \), and let \( \sigma \times \tau \) be the minimal face of \( P \) containing it. The simplex \( \sigma \) must be the single vertex \( u_i \) itself and \( v \) must be contained in the interior of \( \tau \). Since \( \lambda_i \) is a Sperner labeling, \( \lambda_i(v) \) must be an integer \( j \) such that \( v_j \) is a vertex of \( \tau \). Thus \( \lambda(u_i, v) \) is \((u_i, v_j)\), which is a vertex of \( \sigma \times \tau \), as desired. \( \square \)

The lemma above actually says that \( \lambda \) is a kind of ‘polytopal’ Sperner labeling on \( \overline{T} \), using vertices of \( P \) as labels. There is a polytopal Sperner lemma [8], which asserts the existence of a number of simplices of \( \overline{T} \) whose vertices get different labels. The result crucially depends on the
following lemma, which appears in [8] Proposition 3] and whose underlying topological intuition is well-known: when a ball is mapped to itself so that the restriction to the boundary has degree 1 (as a map of spheres), then the interior of the ball must be covered as well.

**Lemma 2.4.** Any continuous map \( f \) from a polytope to itself, which satisfies \( f(F) \subseteq F \) for all faces \( F \) of the polytope, is surjective.

The map \( \lambda \), which has been defined for each vertex of \( \overline{T} \), can be affinely extended on each simplex of \( T \) to produce a piecewise affine map \( \lambda : P \to P \). It is a continuous map and Lemma 2.3 implies that it satisfies the conditions of Lemma 2.4.

Each of our results is then obtained by choosing a special point \( p \) in \( P \) and by analyzing the nature of a simplex \( \sigma \) in \( \overline{T} \) such that \( p \in \lambda(\sigma) \). Choose \( \sigma \) of minimal dimension; then each vertex of \( \sigma \) is mapped to a distinct vertex of \( P \). Let \( G(\sigma) \) denote the bipartite graph with vertex bipartition \( \{u_1, \ldots, u_m\}, \{v_1, \ldots, v_n\} \), and with edges \( u_iv_j \) such that \( (u_i, v_j) \) is a vertex of \( \lambda(\sigma) \). By construction, \( G(\sigma) \) is a simple bipartite graph with at most \( m + n - 1 \) edges.

The point \( p \), as a weighted convex combination of the vertices \( (u_i, v_j) \), corresponds in a natural way to a set of weights \( w_e \geq 0 \) on the edges \( e = u_iv_j \) of \( G(\sigma) \). Writing \( p = (a_i) \in \Delta^{m-1} \times \Delta^{n-1} \) and expressing \( a = (a_i) \) and \( b = (b_j) \) in barycentric coordinates, we can see that

\[
a_i = \sum_{e \in \delta_G(\sigma)(u_i)} w_e \quad \text{and} \quad b_j = \sum_{e \in \delta_G(\sigma)(v_j)} w_e,
\]

where the notation \( \delta_G(x) \) denotes the edges of a graph \( G \) incident to a vertex \( x \).

With the following lemma, the proof of Theorem 2.1 is almost immediate.

**Lemma 2.5.** Consider a bipartite graph \( G = (X, Y, E) \) with non-negative weights \( \omega_e \) on its edges \( e \in E \) and with positive integer weights \( s_x \) on its vertices \( x \in X \) such that \( \sum_{x \in X} s_x = |E| \). If for all \( x \in X \) and \( y \in Y \) we have

\[
\sum_{e \in \delta_G(X)(x)} \omega_e > \frac{s_x - 1}{|Y|} \quad \text{and} \quad \sum_{e \in \delta_G(Y)(y)} \omega_e > \frac{1}{|Y|},
\]

then the degree of each vertex in \( X \) is exactly \( s_x \).

**Proof.** The non-negativity of \( \omega_e \) implies that \( \omega_e \leq 1/|Y| \) for all edges \( e \) of \( G \). Then the first inequality above implies that \( \frac{\deg(x)}{|Y|} > \frac{s_x - 1}{|Y|} \) for all \( x \in X \). Since the \( s_x \) are integers, we see \( \deg(x) \geq s_x \) for all \( x \in X \). But since the sum of the degrees \( \deg(x) \) is equal to the sum of the weights \( s_x \), we have \( \deg(x) = s_x \) for all \( x \in X \). \( \square \)

**Proof of Theorem 2.1 part [1]** We choose \( p = (a_i) \) such that \( a_i = \frac{k_i - 1}{m} + \frac{1}{mn} \) and \( b_j = \frac{1}{n} \). According to Lemma 2.5 with \( G = G(\sigma) \), \( X = \{u_1, \ldots, u_m\}, Y = \{v_1, \ldots, v_n\} \), \( \omega_e = w_e \), and \( s_{u_i} = k_i \), the degree in \( G(\sigma) \) of each \( u_i \) is \( k_i \). This means that \( \lambda_i \) assigns at least \( k_i \) distinct labels on the vertices of \( \sigma \), the projection of \( \sigma \in T \) to a simplex \( \sigma \in T \).

**Proof of Theorem 2.1 part [2]** We choose \( p = (a_i) \) such that \( a_i = \frac{1}{m} \) and \( b_j = \frac{\ell_j - 1}{m} + \frac{1}{mn} \). According to Lemma 2.5 with \( G = G(\sigma) \), \( X = \{v_1, \ldots, v_n\}, Y = \{u_1, \ldots, u_m\} \), \( \omega_e = w_e \), and \( s_{v_j} = \ell_j \), the degree in \( G(\sigma) \) of each \( v_j \) is \( \ell_j \). This means that \( j \) is used by at least \( \ell_j \) labelings on the vertices of \( \sigma \), the projection of \( \sigma \in T \) to a simplex \( \sigma \in T \).

We remark that if one desires an algorithmic way to find \( \sigma \), the path-following proof of the polytopal Sperner lemma [8] can be adapted for this purpose.

To prove Theorem 2.2, we proceed with a technique introduced by the second author [23] and identify each point \( y = (y_1, \ldots, y_n) \) in \( \Delta^{n-1} \) with a division of the cake into \( n \) pieces: numbering the pieces from left to right, \( y_j \) is the length of the \( j \)-th piece. For each player \( i \) and each vertex \( v \)
of $T$, we define $\lambda_i(v)$ to be the number (in $[n]$) of a preferred piece in the division encoded by $v$. The “hungry” assumption implies that the $\lambda_i$ are Sperner labelings of $T$.

Instead of Lemma 2.5, we use the following lemma.

Lemma 2.6. Consider a bipartite graph $G = (X, Y, E)$ with non-negative weights $\omega_e$ on its edges $e \in E$. If for all $x \in X$ and $y \in Y$ we have

$$\sum_{e \in \delta_G(x)} \omega_e = \frac{1}{|X|} \text{ and } \sum_{e \in \delta_G(y)} \omega_e = \frac{1}{|Y|},$$

then for any subset $Y' \subseteq Y$ of size $\left\lceil \frac{|Y| - |X|}{|X|} \right\rceil$, there exists a matching covering $X$ and missing $Y'$.

Proof. We apply Hall’s marriage theorem. Let $X'$ be any non-empty subset of $X$. The weight on the edges incident to $X'$ is not larger than the weight of the edges incident to $N(X')$. This, with the hypotheses on the weights, implies $\frac{|X'|}{|X|} \leq \frac{|N(X')|}{|Y|}$. Then $\left(1 + \frac{|Y| - |X|}{|X|}\right) |X'| \leq |N(X')|$ and since $X'$ is non-empty, we obtain

$$|X'| + \left\lceil \frac{|Y| - |X|}{|X|} \right\rceil \leq |N(X')|.$$

Thus in a graph $H$ obtained from $G$ by removing any subset $Y' \subseteq Y$ with $\left\lceil \frac{|Y| - |X|}{|X|} \right\rceil$ vertices, we have the inequality $|X'| \leq |N_H(X')|$ that holds for all non-empty subset $X'$ of $X$. \hfill \Box

To prove parts (1) and (2), we choose $p = (a, b)$ such that $a_i = \frac{1}{m}$ and $b_j = \frac{1}{n}$, but with different meanings of $m$ and $n$ in each instance.

Proof of Theorem 2.2, part (1) Set $m = p$ and $n = k$. We apply Lemma 2.6 with $G = G(\bar{\sigma})$, $X = \{u_1, \ldots, u_m\}$, $Y = \{v_1, \ldots, v_n\}$, and $\omega_e = w_e$. No matter what $\left\lfloor \frac{n-m}{m} \right\rfloor$ vertices we remove from $\{v_1, \ldots, v_n\}$, the graph $G(\bar{\sigma})$ has a matching covering the vertices in $\{u_1, \ldots, u_m\}$. This matching assigns the $m$ players to distinct pieces in an envy-free way; the triangulation $T$ being arbitrary, compactness of the space of divisions (and the closed preferences assumption) ensures the existence of the desired division. \hfill \Box

Proof of Theorem 2.2, part (2) Set $m = k$ and $n = q$. We apply Lemma 2.6 with $G = G(\bar{\sigma})$, $X = \{v_1, \ldots, v_n\}$, $Y = \{u_1, \ldots, u_m\}$, and $\omega_e = w_e$. No matter what $\left\lfloor \frac{m-n}{m} \right\rfloor$ vertices we remove from $\{u_1, \ldots, u_m\}$, the graph $G(\bar{\sigma})$ has a matching covering the vertices in $\{v_1, \ldots, v_n\}$. This matching assigns the $n$ pieces to distinct players in an envy-free way. We conclude as in part (1). \hfill \Box

Remark 2.1. While Theorems 2.1 and 2.2 can also be obtained with the averaging technique introduced by Gale [11] for proving his “permutation” generalization of the KKM lemma—the technique used in the papers Asada et al. [4] and Frick et al. [10]—our approach makes clear the symmetry between the labelings and the labels.

3. Multilabeled versions of Fan’s lemma

We now establish multilabeled versions of Fan’s lemma in direct analogy to the two parts of the multilabeled Sperner lemma.

3.1. Fan’s lemma and $\mathbb{Z}_2$-index. A free simplicial $\mathbb{Z}_2$-complex is a simplicial complex on which there is a free $\mathbb{Z}_2$-action. A Fan labeling of a free simplicial $\mathbb{Z}_2$-complex is a labeling of its vertices with non-zero integers such that (i) no adjacent vertices have labels that sum to zero (adjacency condition), and (ii) the two vertices of any orbit have labels that sum to zero (antisymmetry condition).
In such a labeling, a simplex is alternating (with respect to the labeling) if the signs alternate when the vertices are ordered according to the absolute value of their labels. The sign of an alternating simplex (either positive or negative) is the sign of its first (lowest) label by absolute value.

The following lemma is due to Fan [9]. It deals with a centrally symmetric triangulation of a $d$-dimensional sphere $S^d$, which is a free simplicial $\mathbb{Z}_2$-complex via the antipodal map. The antisymmetry condition for a Fan labeling requires then that labels at antipodal vertices sum to zero.

**Fan’s lemma.** In any centrally symmetric triangulation of $S^d$ with a Fan labeling, there is an alternating $d$-dimensional simplex.

Tucker’s lemma [24] is equivalent to the special case: In a centrally symmetric triangulation of $S^d$ with a Fan labeling, there is at least one vertex with a label whose absolute value is larger than $d$.

It has recently been realized [2, Proposition 1] that Fan’s lemma remains true when the $d$-sphere $S^d$ is replaced by any free simplicial $\mathbb{Z}_2$-complex with $\mathbb{Z}_2$-index equal to $d$. We remind the reader that the $\mathbb{Z}_2$-index of a free simplicial $\mathbb{Z}_2$-complex $K$, denoted ind($K$), is the minimal dimension of the sphere to which there is a continuous map from $K$ commuting with the $\mathbb{Z}_2$-action. If $K$ is a triangulated sphere, the $\mathbb{Z}_2$-action is the antipodal map, and the Borsuk-Ulam theorem is the equality ind($S^d$) = $d$.

**Fan’s lemma for a simplicial $\mathbb{Z}_2$-complex.** In any free simplicial $\mathbb{Z}_2$-complex $K$ with a Fan labeling, there is an alternating $\text{ind}(K)$-dimensional simplex.

In fact, by symmetry, there is a positive alternating simplex as well as a negative alternating simplex.

### 3.2. Coincidences of alternating simplices.

The next theorem can be seen as the Fan-type generalization of Theorem 2.1 part (1). Using the derivation of Sperner’s lemma from Fan’s lemma (see [16, 25]), this theorem, when $K$ is a centrally symmetric triangulation of the $(n-1)$-dimensional sphere, actually provides yet another proof of Theorem 2.1 part (1).

**Theorem 3.1.** Let $\lambda_1, \ldots, \lambda_m$ be $m$ Fan labelings of a free simplicial $\mathbb{Z}_2$-complex $K$. For any choice of positive integers $d_1, \ldots, d_m$ summing to $\text{ind}(K)$, there exists a simplex $\sigma$ in $K$ that for each $i$ has a $d_i$-dimensional alternating face with respect to $\lambda_i$.

An illustration of the possible patterns asserted by the theorem is given by Figure 2.

Fan’s lemma is the special case $m = 1$ on a triangulated $d$-sphere (whose index is $d$).

![Figure 2](image-url)
Alishahi [11, Lemma 17] introduced a new technique for proving Fan’s lemma and its generalizations for other group actions. This technique consists in defining a labeling on the first barycentric subdivision of the centrally symmetric triangulation $T$ – the labels recording the length of the alternation on the simplices of $T$ – and then in using Tucker’s lemma to show the existence of alternating simplices of certain dimensions. Our approach for proving Theorem 3.1 relies on this idea, and will again use this technique in the proof of Theorem 3.4 below.

We shall use the following notation. Let $\text{alt}_i(\sigma)$ be the number of vertices of a maximal alternating face of $\sigma$ with respect to the labeling $\lambda_i$. Let $s_{\lambda_i}(\sigma)$ be the sign of the alternation of such a face, i.e., the sign of the label in the alternation with the smallest absolute value. It is straightforward to check that $s_{\lambda_i}(\sigma)$ is well-defined. Given a simplicial complex $K$, let $\text{sd}(K)$ denote the barycentric subdivision of $K$: vertices of $\text{sd}(K)$ are the simplices of $K$, and the simplices of $\text{sd}(K)$ are chains of faces $K$ ordered by inclusion.

Proof of Theorem 3.1. For $\sigma \in K$, let $i^*(\sigma)$ be the smallest labeling index $i$ such that $\sigma$ does not have an alternating face of dimension $d_i$, according to $\lambda_i$ (hence $\text{alt}_{\lambda_i}(\sigma) \leq d_i$) and if there no such index $i$, call $\sigma$ desirable and let $i^*(\sigma) = m$. Our goal is to show the existence of desirable simplices. Define

$$\mu(\sigma) = \pm [d_1 + \cdots + d_{i^*(\sigma)} - 1 + \text{alt}_{i^*(\sigma)}(\sigma)]$$

where the sign is chosen to be the sign of the alternation in $\lambda_{i^*(\sigma)}$. Since $d_1 + \cdots + d_{i^*(\sigma)} - 1$ is at most $\text{ind}(K) - d_m$ and $\text{alt}_{i^*(\sigma)}(\cdot)$ is at most $\text{dim} K + 1$, we see $\mu$ is a labeling of the vertices of $\text{sd}(T)$ by non-zero integers being at most $\text{ind}(K) - d_m + \text{dim}(K) + 1$ in absolute value. Note that a simplex $\sigma$ is desirable if and only if $\mu(\sigma) > \text{ind}(K)$.

We claim $\mu$ is a Fan labeling of $\text{sd}(K)$. Antisymmetry of $\mu$ follows easily from its definition. For the adjacency condition of $\mu$ on $\text{sd}(K)$, we consider $\tau$ a face of $\sigma$, and check that $\mu(\tau)$ and $\mu(\sigma)$ cannot sum to 0. We have $i^*(\tau) \leq i^*(\sigma)$. If $i^*(\tau) < i^*(\sigma)$, then $|\mu(\tau)| \leq d_1 + \cdots + d_{i^*(\sigma)} - 1 < |\mu(\sigma)|$ (even when $\sigma$ is desirable), hence $\mu(\tau)$ and $\mu(\sigma)$ cannot sum to 0. Suppose now that $i^*(\tau) = i^*(\sigma)$. Then denoting this value by $i^*$, we see that $|\mu(\tau)| = |\mu(\sigma)|$ can only happen when $\text{alt}_{i^*}(\tau) = \text{alt}_{i^*}(\sigma)$ (even when $\tau$ or $\sigma$ is desirable). But then the maximal alternating face of $\tau$ is a maximal alternating face of $\sigma$, and the label of lowest absolute values in $\tau$ and $\sigma$ must have the same sign, hence $\mu(\tau) = \mu(\sigma)$ so their sum is not 0, either.

Then by Fan’s lemma for simplicial $\mathbb{Z}_2$-complex, there is a simplex of dimension $\text{ind}(K)$ in $\text{sd}(K)$ specified by a maximal chain of simplices in $K$:

$$\sigma_0 \subseteq \cdots \subseteq \sigma_{\text{ind}(K)}$$

with alternation

$$1 \leq \mu(\sigma_0) < \cdots < (-1)^{\text{ind}(K)} \mu(\sigma_{\text{ind}(K)}).$$

This latter chain of inequalities implies immediately that $\mu(\sigma_{\text{ind}(K)}) > \text{ind}(K)$, which implies that $\sigma_{\text{ind}(K)}$ is desirable.

The proof above relies on Fan’s lemma which is known to have constructive proof when $K$ is a triangulated $d$-sphere containing a flag of hemispheres [17]. In that latter case, we have thus also a constructive method for finding a desirable simplex in our multilabeled extension of the Fan lemma.

We show now two applications of Theorem 3.1. A typical consequence of Fan’s lemma in combinatorics is the existence of large colorful bipartite subgraphs in proper colorings of graphs with large topological lower bounds. The first application is a strengthening of this result. The second one is in the context of the Hobby-Rice theorem, also called the “continuous necklace-splitting theorem”, and consensus-halving.
3.2.1. **Graph application.** Since the foundational paper of Lovász solving the Kneser conjecture [15], there is a whole machinery for designing topological lower bounds on the chromatic number of graphs. The **Hom complex** of a graph \( G = (V, E) \), denoted \( \text{Hom}(K_2, G) \), is the poset whose elements are the pairs \((A, B)\) of non-empty disjoint subsets of \( V \) inducing a complete bipartite graph and whose order \( \preceq \) is given by: \((A, B) \preceq (A', B')\) if \( A \subseteq A' \) and \( B \subseteq B' \).

We shall associate \( \text{Hom}(K_2, G) \) with its order complex: a simplicial complex whose vertices are poset elements and simplices are poset chains. There is a natural free \( \mathbb{Z}_2 \)-action on this complex given by exchanging \( A \) and \( B \).

One of the largest topological lower bounds is provided by the \( \mathbb{Z}_2 \)-index of the Hom complex:

\[
\chi(G) \geq \text{ind}(\text{Hom}(K_2, G)) + 2.
\]

In a properly colored graph, a **colorful subgraph** is a subgraph whose vertices get distinct colors. The above inequality was strengthened by Simonyi, Tardif, and Zsbán [20], who showed that in a properly colored graph \( G \), there is in fact a colorful complete bipartite subgraph \( K_{[d/2]+1,[d/2]+1} \) where \( d = \text{ind}(\text{Hom}(K_2, G)) \). We can use our Theorem 3.1 to extend their result to multiple proper colorings.

**Theorem 3.2.** Consider a graph \( G \) colored with \( m \) proper colorings \( c_1, \ldots, c_m \). For any choice of positive integers \( d_1, \ldots, d_m \), summing to \( \text{ind}(\text{Hom}(K_2, G)) \), there exists a complete bipartite subgraph that for each \( i \) contains a colorful \( K_{[d_i/2]+1,[d_i/2]+1} \) with respect to \( c_i \).

**Proof.** Let \( G = (V, E) \). Think of each coloring as a map \( c_i : V \to \{1, 2, \ldots\} \). Recall that each vertex of \( \text{Hom}(K_2, G) \) is a pair \((A, B)\) of non-empty disjoint subsets of \( V \) that induce a complete bipartite subgraph of \( G \). Define

\[
\lambda_i(A, B) = \pm \max_{v \in A \cup B} c_i(v)
\]

with sign \(-\) (resp. \(+\)) if the maximum is attained in \( A \) (resp. \( B \)). Then \( \lambda_i \) is a Fan labeling, since antisymmetry is obvious from the definition, and the adjacency condition follows from \( c_i \) being a proper coloring: if \((A, B) \preceq (A', B')\) and \( \max_{v \in A \cup B} c_i(v) \) is achieved in \( A \) while \( \max_{v \in A' \cup B'} c_i(v) \) is achieved in \( B' \), then the two maxima must be different \( c_i \) colors because each vertex in \( A \subseteq A' \) is adjacent to each vertex in \( B' \).

Since \( \sum_i d_i = \text{ind}(\text{Hom}(K_2, G)) \), Theorem 3.1 implies there exists a simplex of \( \text{Hom}(K_2, G) \) with alternating \( d_i \)-dimensional faces, for \( i = 1, \ldots, m \). Each such face is a chain

\[
(A_0, B_0) \preceq (A_1, B_1) \preceq \cdots \preceq (A_{d_i}, B_{d_i})
\]

on which \( \lambda_i \) alternates; by the definition of \( \lambda_i \), there are \( v_j \in A_j \cup B_j \) for \( j = 0, \ldots, d_i \) such that \( \lambda_i(A_j, B_j) = c_i(v_j) \) and \( c_i(v_0) < \cdots < c_i(v_{d_i}) \).

Suppose without loss of generality \( v_0 \in B_0 \). Then \( a = \max_{v \in A_0} c_i(v) \) is achieved by some \( v = \tilde{v} \) and since \( c_i \) is proper, the color \( a \) is strictly less than \( c_i(v_0) \). This means that \( \tilde{v} \) in the bipartite subgraph \((A_{d_i}, B_{d_i})\) but must be distinct from the \( d_i + 1 \) vertices \( v_j, j = 0, \ldots, d_i \) because they all have larger colors than \( \tilde{v} \). The bipartite subgraph induced by these \( d_i + 2 \) vertices must be colorful according to \( c_i \) and of type \( K_{[d_i/2]+1,[d_i/2]+1} \) as desired. \( \square \)

Choosing \( d_i \simeq \frac{1}{m} \text{ind}(\text{Hom}(K_2, G)) \) for all \( i \) produces Corollary 1.2 stated in the introduction.

3.2.2. **Consensus-halving.** Given a family \( \mathcal{M} \) of absolutely continuous measures on \([0, 1]\), we define a \( t \)-**splitting** to be a partition of \([0, 1]\) into \( t + 1 \) intervals such that

\[
\mu\left( \bigcup_{\text{odd } j} I_j \right) = \mu\left( \bigcup_{\text{even } j} I_j \right)
\]

where \( \mu \) is the total mass of the measures in \( \mathcal{M} \).
for all $\mu \in \mathcal{M}$. The Hobby-Rice theorem [13] asserts that if $\mathcal{M}$ has size $t$, then there is a $t$-splitting of $\mathcal{M}$. We can prove a generalization of this result.

**Theorem 3.3.** Consider finite collections $\mathcal{M}_1, \ldots, \mathcal{M}_m$ of absolutely continuous measures on $[0, 1]$, a positive integer $n$, and positive integers $k_1, \ldots, k_m$ summing to $m + n - 1$. Then there exists a partition of $[0, 1]$ into $n$ intervals $I_1, \ldots, I_n$ such that, for each $i \in [m]$, one of the properties holds:

- These intervals provide an $(n - 1)$-splitting of $\mathcal{M}_i$.
- The equality

$$\mu \left( \bigcup_{\text{odd } j} I_j \right) - \mu \left( \bigcup_{\text{even } j} I_j \right) = \pm \max_{\mu' \in \mathcal{M}_i} \left| \mu' \left( \bigcup_{\text{odd } j} I_j \right) - \mu' \left( \bigcup_{\text{even } j} I_j \right) \right|$$

is attained for at least $k_i$ measures $\mu$ in $\mathcal{M}_i$, each sign, $+$ and $-$, being attained at least $\lfloor k_i/2 \rfloor$ times.

Theorem 3.3 generalizes the Hobby-Rice theorem already for $m = 1$: if $\mathcal{M}_1$ has $n - 1$ measures, then the second possibility cannot hold since $k_1 = n$, and thus there must be an $(n - 1)$-splitting. The case $m = 1$ is actually close to a result by Simonyi [19], which ensures that one can somehow “control” the imbalance in an $(n - 1)$-splitting when the number of measures is $n$.

If we remove the condition on the finiteness of the $\mathcal{M}_i$, the theorem does not hold anymore, as shown with density functions of arbitrarily small support.

The Hobby-Rice theorem also yields a social science interpretation in terms of consensus-halving [13]. If $[0, 1]$ represents a cake to be cut and each measure in $\mathcal{M}$ represents a hungry person, then the theorem says there is a consensus-halving among $t$ people with just $t$ cuts: for such a cut, there is consensus among all people in $\mathcal{M}$ that the odd-index pieces are the same size as the even-index pieces.

An interpretation of Theorem 3.3 in that spirit goes as follows: Consider $m$ finite groups of people and positive integers $d_1, \ldots, d_m$ summing to $n - 1$. Then given any cake, there exists a division of that cake into $n$ pieces, each assigned to one of two portions $A$ and $B$, such that for each group, either: all people in that group believe $A$ and $B$ are exactly the same size; or: there exists $\gamma_i > 0$ such that at least $\lfloor (d_i + 1)/2 \rfloor$ believe $A$ is larger by exactly $\gamma_i$ and at least $\lfloor (d_i + 1)/2 \rfloor$ believe that $B$ is larger by exactly $\gamma_i$ and everyone else is somewhere in between.

**Proof of Theorem 3.3.** Let us first assume that no $\mathcal{M}_i$ admits an $(n - 1)$-splitting.

We arbitrarily index the measures in each $\mathcal{M}_i$ with positive integers: $\mu_1^i, \mu_2^i, \ldots$ Let $T$ be a centrally symmetric triangulation of the $(n - 1)$-dimensional unit $L_1$-sphere

$$\partial \diamond^n = \left\{ (y_1, \ldots, y_n) \in \mathbb{R}^n : \sum_{j=1}^n |y_j| = 1 \right\}.$$ 

Let $Y_0 = 0$ and $Y_j = \sum_{j'=1}^j |y_j|$. For each vertex $v$ of $T$ with coordinates $(y_1, \ldots, y_n)$, we define $\lambda_i(v) = \pm a^*$, where $a^*$ is the smallest $a$ for which $\left| \sum_{j=0}^{n-1} \text{sign}(y_j) \mu_a^i([Y_j, Y_{j+1}]) \right|$ is maximal, and the sign of $\lambda_i(v)$ is the sign of the expression in the absolute value.

We check that $\lambda_i$ is a Fan labeling. Antisymmetry is clearly satisfied. By compactness of $\partial \diamond^n$, finiteness of the $\mathcal{M}_i$’s, and continuity of the measures, there is some $\delta > 0$ so that

$$\max_a \left| \sum_{j=0}^{n-1} \text{sign}(y_j) \mu_a^i([Y_j, Y_{j+1}]) \right| > \delta.$$
for all $i$. We can choose $T$ with a sufficiently small mesh size so that the left-hand side of the above inequality varies by less than $\delta$ on any simplex. This implies the labeling satisfies the adjacency condition.

Then, since the dimension of $\partial{\Box^n}$ is $n - 1$, according to Theorem 3.1 there is a simplex with an alternating $(k_i - 1)$-face with respect to each $\lambda_i$. Making the mesh size of $T$ go to 0 and using the compactness of $\partial{\Box^n}$, we get the existence of intervals $J_1, \ldots, J_n$ and a partition $A, B$ of $[n]$ such that, for each $i$, the equality

$$
\mu \left( \bigcup_{j \in A} J_j \right) - \mu \left( \bigcup_{j \in B} J_j \right) = \pm \max_{\mu' \in \mathcal{M}_i} \left| \mu' \left( \bigcup_{j \in A} J_j \right) - \mu' \left( \bigcup_{j \in B} J_j \right) \right|
$$

is attained with the ‘$+$’ sign for at least $\lfloor k_i/2 \rfloor$ measures $\mu$ in $\mathcal{M}_i$ and is attained with the ‘$-$’ sign for at least $\lceil k_i/2 \rceil$ measures $\mu$ in $\mathcal{M}_i$. Now, by merging consecutive intervals whose indices belong to the same subset $A$ or $B$, we get the desired statement, except that we may have less than $n$ intervals; in that case, just add empty intervals to complete the collection.

Now, let us consider the general case. We arbitrarily choose a collection $P$ of $n$ absolutely continuous measures $\rho_1, \ldots, \rho_n$ on $[0, 1]$ with disjoint supports and with total weight equal to some $\varepsilon > 0$. We define $\mathcal{M}_i = \mathcal{M}_i \cup P$. No $\mathcal{M}_i$ has an $(n - 1)$-splitting. Applying what we already proved on $\mathcal{M}_1, \ldots, \mathcal{M}_m$, we get that there are $n$ intervals $I_1, \ldots, I_n$ (depending on $\varepsilon$) such that, for each $i$, one of the two properties holds:

a) $\left| \mu \left( \bigcup_{j \text{ odd}} I_j \right) - \mu \left( \bigcup_{j \text{ even}} I_j \right) \right| \leq \varepsilon$ for all $\mu \in \mathcal{M}_i$.

b) $\mu \left( \bigcup_{j \text{ odd}} I_j \right) - \mu \left( \bigcup_{j \text{ even}} I_j \right) = \pm \max_{\mu' \in \mathcal{M}_i} \left| \mu' \left( \bigcup_{j \text{ odd}} I_j \right) - \mu' \left( \bigcup_{j \text{ even}} I_j \right) \right|$ is attained for at least $k_i$ measures in $\mathcal{M}_i$, each sign, $+$ and $-$, being attained at least $\lfloor k_i/2 \rfloor$ times.

Indeed, this follows from noting that for each $i$, either we are in case [a] or the maximum in not attained for a measure in $P$.

Consider a sequence of values for $\varepsilon$ converging to 0. Up to taking a subsequence, we can assume that each $i$ falls under the same case [a] or the same case [b] for all these values of $\varepsilon$, and, by finiteness of the $\mathcal{M}_i$’s, we can even assume that when it falls under case [b], the equality is always attained with exactly the same measures and the same signs. By compactness and absolute continuity of the measures, we get the sought conclusion. \hfill $\square$

The necklace-splitting theorem ensures that there is a certain rounding property for the Hobby-Rice theorem when one works with “beads” instead of measures; see [3, 12]. We do not know whether the rounding property still holds for Theorem 3.3.

### 3.3. Further multilabeled generalizations of Fan’s lemma

Similarly, the next theorem can be seen as the Fan-type generalization of Theorem 2.1 part (2). Corollary 3.5 provides an alternative proof of Theorem 2.1 part (2) in a same way Theorem 3.1 provides an alternative proof of Theorem 2.1 part (1).

**Theorem 3.4.** Consider a free simplicial $\mathbb{Z}_2$-complex $K$ with $\text{ind}(K) = n - 1$. Let $\lambda_1, \ldots, \lambda_m$ be $m$ Fan labelings of $K$ with labels $\{\pm 1, \ldots, \pm N\}$. For any positive integers $\ell_1, \ell_2, \ldots, \ell_N$ summing to $m + N - 1$, there exists a simplex $\sigma = \langle v_1, \ldots, v_n \rangle$ in $K$ and $n$ label numbers $1 \leq j_1 \leq \cdots \leq j_n \leq N$ and $n$ indices $i_1, \ldots, i_n$ in $[m]$ such that

(a) $\lambda_{i_k}(v_k) = (-1)^k j_k$,

(b) if $j_k = j_{k+1}$, then $i_k < i_{k+1}$, and

(c) for each $k$, labels $+j_k$ or $-j_k$ are present on $\sigma$ in at least $\ell_{j_k}$ of the labelings.
Note here the theorem asserts that there is an “alternating” sequence of labels, though the label absolute values $j_k$ are not necessarily distinct. Property (a) guarantees the asserted alternating labels appear on different vertices of $\sigma$. Property (b) shows that if a sequence of asserted alternating labels have identical absolute values, then those labels successively appear in a sequence of labelings of increasing index.

**Proof of Theorem 3.4.** For $\sigma \in K$, let $r_j(\sigma)$ be the number of labelings in which $-j$ or $+j$ is present on $\sigma$. We define $\mu(\sigma)$ as follows. Let $j^*(\sigma)$ be the largest $j$ such that $r_j(\sigma) \geq \ell_j$. Such a $j^*(\sigma)$ exists because $\sum_{j=1}^{N}(\ell_j - 1) = m - 1$ and $\sum_{j=1}^{N}r_j(\sigma) \geq m$ (each labeling contributes at least one unit to this sum). Let $i^*(\sigma)$ be the largest $i$ such that $\lambda_i(v) = \pm j^*(\sigma)$ for a vertex $v \in \sigma$.

Now set

$$
\mu(\sigma) = \pm \lfloor m \cdot (j^*(\sigma) - 1) + i^*(\sigma) \rfloor
$$

where the $\pm$ sign is determined by the sign of the label $\pm j^*(\sigma)$ that appears in $\sigma$ (both labels cannot appear in $\sigma$ since $\lambda_i^*(\sigma)$ is a Fan labeling, hence adjacent vertices in $\sigma$ cannot have labels that sum to zero).

We claim $\mu$ is a Fan labeling on $\text{sd}(K)$ with labels from $\{-1, \ldots, \pm mN\}$. Antisymmetry of $\mu$ follows from the symmetry of $j^*$ and $i^*$ and the antisymmetry of $\lambda_i$ on $K$. Also, $\mu$ satisfies the adjacency condition, because if $\tau \subset \sigma$ are adjacent simplices in $\text{sd}(K)$, then either $j^*(\tau) < j^*(\sigma)$ in which case $\mu(\tau) + \mu(\sigma) \neq 0$, or $j^*(\tau) > j^*(\sigma)$ in which case $\mu(\tau) + \mu(\sigma) \neq 0$, or $j^*(\tau) = j^*(\sigma)$ and $i^*(\tau) < i^*(\sigma)$ in which case $\mu(\tau) + \mu(\sigma) \neq 0$.

So $\mu$ is a Fan labeling and according to Fan’s lemma, there is an $(n-1)$-dimensional simplex in $\text{sd}(K)$, which corresponds to a chain of simplices in $K$ of successive dimension (here $\sigma_1$ has dimension 0):

$$
\sigma_1 \subseteq \cdots \subseteq \sigma_n
$$

with

$$
1 \leq -\mu(\sigma_1) < \cdots < (-1)^n\mu(\sigma_n) \leq mN.
$$

The fact that $|\mu|$ increases going from $\sigma_{k-1}$ to $\sigma_k$ means that $j^*$ and/or $i^*$ also increases by including a vertex $v_k$ in the simplex $\sigma_k$; hence this $v_k$ must be labeled by $\pm j^*(\sigma_k)$ in one of the labelings $\lambda_i$. Then setting $\sigma$ to be $\sigma_n$, and $j_k$ to be $\lfloor |\mu(\sigma_k)|/m \rfloor$ gives the desired result. □

We say that $m$ Fan labelings $\lambda_1, \ldots, \lambda_m$ are compatible if $\lambda_i(u) + \lambda_i(u') \neq 0$ for any adjacent vertices $u, u'$ and any pair $i, i'$ (where $u$ is not considered as being adjacent to itself).

**Corollary 3.5.** Consider a free simplicial $\mathbb{Z}_2$-complex $K$ with $\text{ind}(K) = n - 1$. Let $\lambda_1, \ldots, \lambda_m$ be $m$ compatible Fan labelings of $K$ with labels $\{\pm 1, \ldots, \pm N\}$. For any positive integers $\ell_1, \ell_2, \ldots, \ell_N$ summing to $m + N - 1$, there exists a simplex $\sigma$ and $n$ label absolute values $1 \leq j_1 < \cdots < j_n \leq N$ such that for each $k$, the label $(-1)^k j_k$ is present on $\sigma$ in at least $\ell_{j_k}$ of the labelings.

**Proof.** We apply Theorem 3.4. Because of the additional condition, property (a) implies that $j_k < j_{k+1}$ for all indices $k$. □

**Remark 3.1.** In the proofs of Theorems 3.1 and 3.4, the labeling $\mu$ is increasing. It implies that the same proofs actually provide slightly stronger versions of these results, where the $\mathbb{Z}_2$-index $\text{ind}(\cdot)$ is replaced by the always non-smaller cross-index $\text{Xind}(\cdot)$; see [20] for definition of this latter index and discussion about it. In turn, it implies that the slightly stronger versions of Theorem 3.2 Corollary 1.2 and Corollary 3.5 with the cross-index in place of the index are also true. However, while there are simplicial complexes with distinct $\mathbb{Z}_2$-index and cross-index, it is not known whether this can occur for Hom complexes of graphs; see [20].

Note that in contrast to Theorem 3.1 Corollary 3.5 requires the Fan labelings to be compatible. The following example below shows that without this hypothesis, the corollary is not true.
Example 3.1. Let $T$ be a triangulation of the unit 2-sphere $S^2$ whose mesh size is small, e.g., the simplices have diameter not larger than 0.01. Consider regions of $S^2$ defined as follows:

$$A_{xyz}^1 = \{(x, y, z) : x^2 \geq 0.6\}$$
$$A_{xyz}^2 = \{(x, y, z) : x^2 \leq 0.6, y^2 \geq 0.1\}$$
$$A_{xyz}^3 = \{(x, y, z) : x^2 \leq 0.6, y^2 \leq 0.1\}.$$

Observe that these sets cover $S^2$. The set $A_{xyz}^1$ is the union of two disjoint closed components: $C_{+1}^{xyz}$ (on which $x > 0$) and $C_{-1}^{xyz}$ (on which $x < 0$). Similarly, $A_{xyz}^2$ is the union of disjoint closed components: $C_{+2}^{xyz}$ (on which $y > 0$) and $C_{-2}^{xyz}$ (on which $y < 0$). And $A_{xyz}^3$ is the union of disjoint closed components: $C_{+3}^{xyz}$ (on which $z > 0$) and $C_{-3}^{xyz}$ (on which $z < 0$). Hence the collection

$$C^{xyz} = \{C_{+1}^{xyz}, C_{-1}^{xyz}, C_{+2}^{xyz}, C_{-2}^{xyz}, C_{+3}^{xyz}, C_{-3}^{xyz}\}$$

is a cover of $S^2$. By permuting the order of $x, y, z$ we obtain other covers $C^{yxz}$ and $C^{zxy}$.

Now define the labeling $\lambda_1$ by setting $\lambda_1(v) = k$ for the smallest $|k|$ such that $v \in C_k^{xyz}$. The disjointness of the sets $C_{+1}^{xyz}, C_{-1}^{xyz}$ ensures that labeling is well defined, and that the labels at neighboring vertices do not sum to zero since the mesh size of $T$ is small. The symmetry of $C_{+1}^{xyz}, C_{-1}^{xyz}$ ensures that vertex labels at antipodal points sum to zero. Thus $\lambda_1$ is a Fan labeling.

Similarly, define $\lambda_2(v) = k$ for the smallest $|k|$ such that $v \in C_k^{yxz}$, and $\lambda_3(v) = k$ for the smallest $|k|$ such that $v \in C_k^{zxy}$. They, too, are Fan labelings.

Note that the sets in $\{A_{1}^{xyz}, A_{2}^{yxz}, A_{3}^{zxy}\}$ are pairwise disjoint, as are the sets in $\{A_{3}^{xyz}, A_{3}^{yxz}, A_{3}^{zxy}\}$. However, $A_{2}^{xyz} \cap A_{3}^{yxz} \cap A_{2}^{zxy} \neq \emptyset$ since it contains the point $p = \left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$.

Then the labelings $\lambda_i : V(K) \to \{\pm 1, \pm 2, \pm 3\}$ satisfy the hypotheses of Corollary 3.5 with $n = N = m = 3$, except that $\lambda_i(u) + \lambda_i(v')$ may be 0 for adjacent vertices $u, u'$ in $T$ and a pair $i, i'$. In particular, $\lambda_1(u) = -2$ and $\lambda_2(u') = +2$ for adjacent vertices $u, u'$ within 0.01 of $p$. Note that the conclusion of Corollary 3.5 now fails for $\ell_1 = 2, \ell_2 = 1, \ell_3 = 2$, since we must have $j_1 = 1, j_2 = 2, j_3 = 3$, but the pairwise disjointness of the closed sets $A_1^{xyz}, A_1^{yxz}, A_1^{zxy}$ and the small mesh size of $T$ imply that at most one labeling can have label $\pm 1$ on the same simplex. So the label $j_1$ does not appear in $\ell_1$ of the labelings.

Example 3.1 satisfies the conditions of Theorem 3.4 for $n + 1 = N = m = 3$. We can see how it applies when $\ell_1 = 2, \ell_2 = 1, \ell_3 = 2$. Since $A_1^{xyz}, A_1^{yxz}, A_1^{zxy}$ are pairwise disjoint, no two labelings from $\lambda_1, \lambda_2, \lambda_3$ can simultaneously be $\pm 1$ for vertices on the same simplex $\sigma$. Similarly, since $A_2^{xyz}, A_3^{yxz}, A_3^{zxy}$ are pairwise disjoint, no two labelings from $\lambda_1, \lambda_2, \lambda_3$ can simultaneously be $\pm 3$ on the same simplex $\sigma$. Thus Theorem 3.4 must produce a simplex $\sigma$ and an alternating sequence of labels $-2, +2, -2$ on $\sigma$ each of which makes the redundant claim that the total number of $\pm 2$ labels is at least $\ell_2 = 1$. (For instance, this will occur for a simplex near $p$.) From the proof of the theorem, we learn that each of the labels $-2, +2, -2$ asserted by the theorem must be achieved by different labelings.

The next result is a multilabeled version of Bacon’s lemma [6], which states that in a Fan labeling of $S^{n-1}$, with labels taken in $\{\pm 1, \ldots, \pm n\}$, all feasible subsets of $n$ distinct labels occur as labels of a simplex. Its proof illustrates the fact that Gale’s averaging trick (mentioned in Remark 2.1) can also be used for proving Fan-type results.

Proposition 3.6. Consider a free simplicial $\mathbb{Z}_2$-complex $K$ with $\text{ind}(K) = n - 1$. Let $\lambda_1, \ldots, \lambda_n$ be $n$ compatible Fan labelings of $K$ with labels in $\{\pm 1, \ldots, \pm n\}$ with the additional condition that $\lambda_i(u) + \lambda_{i'}(u) \neq 0$ for any pair of indices $i, i'$ and any vertex $u$. Then, for every $(\alpha_1, \ldots, \alpha_n) \in \{-1, +1\}^n$, there exists a simplex $\sigma$ of $K$ and a permutation $\pi$ of $[n]$ such that for each $j \in [n]$, the integer $\alpha_j \cdot j$ is a value taken by $\lambda_{\pi(j)}$ on $\sigma$. 


Note that the condition of the proposition is stricter than of Corollary \[3.5\] since it requires that among the labels found at a single vertex, no pair of labels sums to zero.

**Proof of Proposition \[3.6\].** Use each \(\lambda_i\) to construct a piecewise affine map \(L_i\) from the underlying space \(\|K\|\) to \(\partial^n\) in the following way: for each vertex \(v\) set \(L_i(v) = \pm e_k\lambda_i(v)\) (where \(e_k\) is the \(k\)-th basis vector) choosing the sign to agree with \(\lambda_i(v)\), then extend \(L_i\) linearly across each simplex. Using Gale’s averaging trick mentioned in Remark \[2.1\], we define \(L \in \|x\|\) for every \(x \in \|K\|\). Because of the compatibility conditions and \(\lambda_i(u) + \lambda_j(u') \neq 0\), the map \(L\) has still its image in \(\partial^n\). Since \(L\) is a \(\mathbb{Z}_2\)-map and since \(K\) has \(\mathbb{Z}_2\)-index equal to \(n-1\), the map \(L\) is surjective. (Otherwise, a \(\mathbb{Z}_2\)-map from \(K\) into \(S^{n-2}\) would exist by a standard topological argument.)

Consider \(p = (p_1, \ldots, p_n)\) defined by \(p_j = \frac{1}{n} \alpha_j\). Since \(L\) is surjective, there exists \(y \in K\) such that \(L(y) = p\). Define \(z_{ij}\) to be the \(j\)-th component of \(L_i(y)\); note that \(z_{ij}\) is positive if and only if \(+j\) appears as a \(\lambda_i\) label in a minimal simplex containing \(y\), because \(\lambda_i\) is a Fan labeling. We have \(\sum_{i=1}^n |z_{ij}| = 1\) for all \(j\) (since \(L(y) = p\)) and \(\sum_{j=1}^n |z_{ij}| = 1\) for all \(i\) (since \(L(y) \in \partial^n\)).

Consider the bipartite graph \(G\), with on one side the vertices \(i = 1, \ldots, n\) and on the other side the vertices \(j = 1, \ldots, n\), and with edges the pairs \((ij)\) such that \(z_{ij} \neq 0\). For every subset \(X\) of \(j\)-vertices, we have

\[
|X| = \sum_{i=1}^n \sum_{j \in X} |z_{ij}| \leq \sum_{i \in N(X)} \sum_{j=1}^n |z_{ij}| = |N(X)|,
\]

the inequality following from noting the right double sum is a sum over more edges. Then Hall’s marriage theorem ensures that \(G\) has a matching covering the \(j\)-vertices. For any such \(j\), we define \(\pi(j)\) to be the integer \(i\) with which \(j\) is matched.

The pair we are looking for is \((\sigma, \pi)\), where \(\sigma\) is a simplex of \(K\) containing \(y\). Indeed, we have \(z_{\pi(j)j} \neq 0\), which implies that \(\lambda_{\pi(j)}(v) = \alpha_j \cdot j\) for at least one vertex of \(\sigma\).

\[\square\]

**Remark 3.2.** We probably could get other generalizations of Proposition \[3.6\] in the spirit of the “Sperner”-versions of Theorem \[2.2\] with a similar approach, by playing with the point \(p\) used in the proof. But we were not interested in going further in that direction.

4. **Bapat’s theorem and Lee-Shih’s formula.**

As mentioned in the introduction, Bapat’s theorem \[7\] is the first multilabeled version of Sperner’s lemma. It implies Gale’s permutation generalization of the KKM lemma, but it is more general, since it has a quantitative conclusion.

**Bapat’s theorem.** Let \(T\) be a triangulation of \(\Delta^{n-1}\) with \(n\) Sperner labelings \(\lambda_1, \ldots, \lambda_n\). Consider the pairs \((\sigma, \pi)\), where \(\sigma\) is an \((n-1)\)-dimensional simplex of \(T\) and \(\pi\) is a bijection \(V(\sigma) \to [n]\), such that the \(\lambda_{\pi(v)}(v)\) for \(v \in V(\sigma)\) are all different. For all such pairs, order the vertices of \(\sigma\) so that \(\lambda_{\pi(v)}(v)\) is increasing along them. Then the difference between the number of such pairs with \(\sigma\) positively oriented by this order and the number of such pairs with \(\sigma\) negatively oriented is equal to \(n!\) in absolute value.

While writing this paper, the natural question of whether a similar generalization of Fan’s lemma holds arose. We were not able to settle this question. Note that however for some special triangulations, it is easy to get a statement in that spirit.

Any centrally symmetric triangulation \(T\) of \(S^{n-1}\) provides a triangulation \(T/\mathbb{Z}_2\) of the \((n-1)\)-dimensional projective space, obtained by identifying antipodal simplices.

**Theorem 4.1.** Let \(T\) be a centrally symmetric triangulation of \(S^{n-1}\) with \(n\) compatible Fan labelings \(\lambda_1, \ldots, \lambda_n\). If the simplicial complex \(T/\mathbb{Z}_2\) is balanced, then there are at least \(n!\) pairs \((\sigma, \pi)\) with \(\sigma \in T\) and \(\pi\) a bijection \(V(\sigma) \to [n]\), such that

\[
0 < -\lambda_{\pi(v_1)}(v_1) < \lambda_{\pi(v_2)}(v_2) < \cdots < (-1)^n \lambda_{\pi(v_n)}(v_n)
\]
where \( \langle v_1, \ldots, v_n \rangle = \sigma \).

A \( d \)-dimensional simplicial complex is balanced if there is a coloring of its vertices with \( d + 1 \) colors such that every \( d \)-dimensional simplex is colorful, i.e., has its vertices of distinct colors.

**Proof of Theorem 4.1.** Denote by \( \lambda_1, \ldots, \lambda_n \) the \( n \) compatible Fan labelings. Fix an arbitrary coloring \( c: V(T) \to [n] \) such that each \((n-1)\)-dimensional simplex is colorful. For each permutation \( \pi' \) of \([n]\), the labeling \( \lambda^{\pi'} \) defined by \( \lambda^{\pi'}(v) = \lambda^{\pi'(c(v))}(v) \) is a Fan labeling and there is a negative alternating simplex according to this labeling. It is then easy to see that for any pair of distinct permutations \( \pi' \), each \((d-1)\)-dimensional simplex \( \sigma \) has at least one vertex where the two values of \( \pi'(c(v)) \) are different. Each choice of \( \pi' \) provides thus a different pair \((\sigma, \pi)\) with the desired property (and with \( \pi = \pi' \circ c \)). \( \square \)

We do not know whether the statement still holds if we remove the balancedness condition. Fan proved his lemma by induction on the dimension on the sphere, with the help of a formula relating the number of alternating simplices of a pseudomanifold to the number of alternating simplices with positive sign on its boundary \[9\]. A similar formula exists for multilabelings: this is precisely the Lee-Shih formula mentioned in the introduction. Unfortunately, because of issues related to the orientation of the sphere, mimicking Fan’s proof with Lee-Shih’s formula does not seem to lead to any non-trivial result.

**Acknowledgments.** This material is based upon work supported by the National Science Foundation under Grant No. DMS-1440140 while the authors were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 2017 semester.

**References**

[1] Meysam Alishahi. Colorful subhypergraphs in uniform hypergraphs. *The Electronic Journal of Combinatorics*, 24:#P1.23, 2017.
[2] Meysam Alishahi, Hossein Hajiabolhassan, and Frédéric Meunier. Strengthening topological colorful results for graphs. *European Journal of Combinatorics*, 64:27–44, 2017.
[3] Noga Alon and Douglas B West. The Borsuk-Ulam theorem and bisection of necklaces. *Proceedings of the American Mathematical Society*, 98(4):623–628, 1986.
[4] Megumi Asada, Florian Frick, Vivek Pisharody, Maxwell Polevy, David Stoner, Ling Hei Tsang, and Zoe Wellner. Fair division and generalizations of Sperner- and KKM-type results. *SIAM Journal of Discrete Mathematics*, to appear.
[5] Eric Babson. Meunier conjecture. *arXiv preprint arXiv:1209.0102*, 2012.
[6] Philip Bacon. Equivalent formulations of the Borsuk-Ulam theorem. *Canad. J. Math.*, 18:492–502, 1966.
[7] Ravindra B Bapat. A constructive proof of a permutation-based generalization of Sperner’s lemma. *Mathematical Programming*, 44(1):113–120, 1989.
[8] Jesus A De Loera, Elisha Peterson, and Francis Edward Su. A polytopal generalization of Sperner’s lemma. *Journal of Combinatorial Theory, Series A*, 100(1):1–26, 2002.
[9] Ky Fan. A generalization of Tucker’s combinatorial lemma with topological applications. *Annals of Mathematics*, 56:431–437, 1952.
[10] Florian Frick, Kelsey Houston-Edwards, and Frédéric Meunier. Achieving rental harmony with a secretive roommate. *arXiv preprint arXiv:1702.07325v2*, 2017.
[11] David Gale. Equilibrium in a discrete exchange economy with money. *International Journal of Game Theory*, 13(1):61–64, 1984.
[12] Charles H Goldberg and Douglas B West. Bisection of circle colorings. *SIAM Journal on Algebraic Discrete Methods*, 6(1):93–106, 1985.
[13] Charles R Hobby and John R Rice. A moment problem in \( L_1 \) approximation. *Proceedings of the American Mathematical Society*, 16(4):665–670, 1965.
[14] Shyh-Nan Lee and Man-Hsiang Shih. A counting lemma and multiple combinatorial Stokes’ theorem. *European Journal of Combinatorics*, 19(8):969–979, 1998.
[15] László Lovász. Kneser’s conjecture, chromatic number and homotopy. *Journal of Combinatorial Theory, Series A*, 25:319–324, 1978.
[16] Kathryn L Nyman and Francis Edward Su. A Borsuk-Ulam equivalent that directly implies Sperner’s lemma. *American Mathematical Monthly*, 120(4):346–354, 2013.

[17] Timothy Prescott and Francis Edward Su. A constructive proof of Ky Fan’s generalization of Tucker’s lemma. *J. Combin. Theory Ser. A*, 111(2):257–265, 2005.

[18] Forest W Simmons and Francis Edward Su. Consensus-halving via theorems of Borsuk-Ulam and Tucker. *Math. Social Sci.*, 45(1):15–25, 2003.

[19] Gábor Simonyi. Necklace bisection with one cut less than needed. *The Electronic Journal of Combinatorics*, 15:#N16, 2008.

[20] Gábor Simonyi, Claude Tardif, and Ambrus Zsbán. Colourful theorems and indices of homomorphism complexes. *The Electronic Journal of Combinatorics*, 20:#P10, 2013.

[21] Gábor Simonyi and Gábor Tardos. Local chromatic number, Ky Fan’s theorem, and circular colorings. *Combinatorica*, 26:587–626, 2006.

[22] Walter Stromquist. How to cut a cake fairly. *The American Mathematical Monthly*, 87(8):640–644, 1980.

[23] Francis Edward Su. Rental harmony: Sperner’s lemma in fair division. *The American Mathematical Monthly*, 106(10):930–942, 1999.

[24] Albert William Tucker. Some topological properties of disk and sphere. *Proc. First Canad. Math. Congr. Montreal*, pages 285–309, 1945.

[25] Rade T Živaljević. Oriented matroids and Ky Fan’s theorem. *Combinatorica*, 30(4):471–484, 2010.

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