Phase Transition in the Maximal Influence Problem: When Do We Need Optimization?

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Abstract. Considerable efforts were made in recent years in devising optimization algorithms for influence maximization in networks. Here we ask: “When do we need optimization?” and apply insights from statistical mechanics, and direct simulations, to characterize the parameter-space region where optimization is required. We find that this region is due to a well known physical phase transition of the network, and that it vanishes as a power-law with the network size. We show that also from a utility-maximization perspective (when considering the optimization costs), for large networks standard optimization is profitable only in a vanishing parameter region near the phase transition. Finally, we introduce a constant-time optimization approach, and demonstrate it through a simple algorithm that manages to give similar results to standard optimization methods in terms of the influenced-set size, while improving the results in terms of the net utility.

1 Introduction

Assume you are an advertiser trying to maximize the spread of your message in some large social network. Relevant examples abound from the spreading of news, opinions and political ideas, to adoption of new products or trends. The process starts from an initial seed-set of influencers, and then spreads in the network from peer-to-peer until reaching a final influence, i.e., a final number of influenced nodes. If the contagion rule is stochastic then the influence of every seed-set has some distribution, and different seed-sets may have different distributions. The question of finding the seed-set for which the expected influence is maximal has been formalized in [15] as the maximal influence problem, which is NP-hard for a wide family of influence spreading models, and a large body of work has since discussed the development of approximation algorithms for its solution, e.g., [15][26][22][11][13][27][14][20][18].

The literature however does not explore several important aspects of this problem: Is this problem actually hard? How do the parameters of the system affect the empirical hardness? How much worse will we do if we simply take a random choice? And when does one gain from investing in optimization?

Based on the answers we find to these questions, we also look at their implications from a utility-maximization point of view. Each influenced node gives the advertiser some value, the optimization procedure has some cost, and the
total utility needs to be maximized. It turns out that in many cases optimization is similar to trying to find “a needle in a stack of needles”: the choice is among similar outcomes, and the cost of optimization exceeds its marginal profit.

**Evaluation metrics and models.** We compare the empirical performance of the greedy hill-climbing algorithm [15], which is known to have the best performance guarantee, to the case of choosing the seed-set at random. The empirical performance of a seed-set is taken as the median influence gained by this set. Denote the empirical performance of the optimized set (using hill-climbing) as $opt$, and the empirical performance gained by a random choice as $rand$. A main metric that we consider is the *marginal gain* from optimization, defined as the difference: $opt - rand$, which is the typical number of influenced nodes added by the optimization process above the random choice benchmark.

Before going into our results, let us describe shortly the models we use. The spreading model we consider is the standard independent cascade model, as described in [15], where the interaction between nodes is characterized by a parameter $p$, which is the probability of an influenced node to affect each of its neighbors (see Section 2.2 for more details). Under this model, we evaluate the marginal gain over random networks using direct simulations. The results we present are for Erdős - Rényi (ER) networks, with a mean degree of 4 and sizes of 1,000 to 10,000 nodes. The results were robust for other types of random networks; we tried power-law, random regular, and small-world networks, as well as ER networks with different mean degrees. Standard statistical mechanics considerations show that each of these networks has a percolation phase transition, meaning that there is a *critical point* in the parameter space, $p_c$, at which the influence shifts from local to network scale influence.

**Better than random – only in a measure-zero parameter range.** We start by characterizing the cases in which influence maximization requires optimization, i.e., where optimization gives significantly better results than a random choice. We show that these cases are centered at a narrow region of the parameter space, near the critical point. Figure 1 shows the localization near the critical point of the marginal gain for ER networks with different sizes. As can be seen, the parameter-space region where the marginal gain is substantial becomes narrow for larger networks.

A simple way to measure the width of the parameter region where optimization is needed, is to test for which values of $p$ the marginal gain from optimization is at least 1% of the network size $N$. We find that the width of this region decays to zero as a power-law of the network size. Figure 2 shows that the fraction of the parameter space where optimization gives a marginal gain of at least 1% of the nodes decreases as $N^{-\alpha}$, with $\alpha = 0.4$. This was robust to other measures of the width. For further details see Section 3.1.

The practical meaning of these findings is that for large networks optimization is likely to give substantial gains only if the system is near its critical point, in a region that reduces to a zero measure of the parameter space. To the best

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1 For each network size the results are averaged over an ensemble of 10 networks. The differences between individual networks of each size were minor (see Section 3.1).
Fig. 1: Marginal gain from optimization in respect to the random benchmark, presented in percents of the network size, in Erdős - Rényi networks of different sizes with a mean degree of 4.

Fig. 2: Width of the parameter region where the marginal gain from optimization is at least 1% of the nodes, as a function of $N$. The data points are simulation results and the line is the fitted power-law curve: $A \cdot N^{-a}$, $A = 2.75 \pm 0.4$, $a = 2/5 \pm 0.02$. The inset shows the logarithmic scale fit.

of our knowledge, we are the first to identify the percolation phase transition as the source for the differences between seed-sets, and for any significant sub-optimality of a random choice – and thus the need for optimization.

**Economic implications.** Next, we turn to look at the gained utility, i.e., considering also the optimization costs, where the cost is proportional to the optimization running time. We find that away from the critical point, the marginal gain from running optimization algorithms in respect to the random benchmark is small, and does not increase with the network size, while the cost of optimization does increase with the network size; as a result, for every given cost of computation-per-time-unit, in large enough networks the price of optimization exceeds its marginal profit and the marginal utility from optimization in respect to the random benchmark becomes negative. This leaves only a narrow parameter region near the phase transition in which the marginal gain does scale with
the network size, and “it pays” to run optimization procedures. Specifically, considering the nearly-optimal running time obtained by state-of-the-art algorithms (e.g., [4,26]), the width of the region where optimization gives a positive marginal utility diminishes as a power-law of $N$. I.e., for almost all of the parameter space optimization procedures on large networks result in negative marginal utility. See Section 3.2 for the full analysis and results.

Constant-time optimization. To address this problem, that for most of the possible parameter values standard optimization may be wasteful, we introduce a constant-time optimization approach that achieves high performance while maintaining constant computation costs, which are independent of the network size.

We demonstrate this approach through a simple algorithm we call random-local-optimization, which relies on local optimization over small random sub-networks. Figure 3 shows the median influence achieved by the random-local-optimization and by the hill-climbing algorithms, as a function of the parameter $p$, for ER networks of size $N = 10,000$. We see that in terms of the influenced set size, the simple and fast random-local-optimization algorithm achieves similar performance to hill-climbing in most of the parameter space, except near the critical point, where hill-climbing has a modest advantage. Moreover, in Section 4 we show that in terms of the gained utility, the random-local-optimization algorithm outperforms the standard optimization results, for the case of large networks and unknown parameters, even when considering the running time costs of state-of-the-art optimization methods. See Section 4 for more details of our algorithm and the evaluation results.

This constant time approach may be applicable to large networks where global optimization is not practical, or to partial information scenarios, where the system parameters are unknown. The short computation time obtained this way may also be relevant for time-varying networks, in which global computation which leads to long solution times may render the solutions irrelevant.

![Fig. 3: Median influence of the seed-sets obtained by hill-climbing and random-local-optimization in Erdös-Rényi networks of size $N = 10,000$ and a mean degree of 4, in percents of the network size. The advantage of hill-climbing is modest and centered near the phase transition at $p_c$.](image-url)
2 Preliminaries

2.1 The Maximal Influence Problem

The maximal influence problem was first discussed in the context of marketing \cite{8}, and later formalized in the seminal work of Kempe, Kleinberg and Tardos (2003) as a general algorithmic problem, with many papers to follow, including works from algorithmic game theory and mechanism design perspectives \cite{21,22}, targeted immunization \cite{11,10}, competing influences \cite{9,5,2}, and more (for reviews, see \cite{19,16}). The formal algorithmic problem is simple: \textit{for a parameter }$k$, \textit{find the }$k$\textit{-node seed-set that gives the maximal average influence.} Simple as it may seem, the maximal influence problem is NP-hard for a family of cascade and threshold influence models, and the vast majority of the related literature discusses approximation algorithms for its solution.

The practical hardness of the problem is different from its formal one. Since there are only \(\binom{N}{k} = \mathcal{O}(N^k)\) sets of size \(k\), for a \textit{fixed} constant \(k = K\), the task of finding the maximal influence set is polynomial in the network size \(N\), and although finding the exact average influence of a seed-set is known to be \#P-hard \cite{26}, approximating it well is easy by a direct Monte-Carlo simulation (see the proof regarding sampling time in \cite{16}). However, for real-world purposes where networks may consist of billions of nodes, an exhaustive search solution is not practical even for small \(k\) in spite of being polynomial. The practical hardness of the problem stems from the sheer size of the networks one may need to work with, and even linear algorithms may become expensive – depending on the actual networks in discussion. This provides motivation to seek for sub-linear algorithms for this problem, as we suggest in Section 4.

2.2 The Influence Model and the Phase Transition

We consider the independent cascade model of network spreading, as described in \cite{15} (see also \cite{23,11}). The idea of the model is that each link between two nodes is characterized by some probability, and each influenced node tries only once to affect each of its neighbors, and succeeds with the probability associated with the link between them. We will consider a uniform probability \(p\) for all links. The process progresses in discrete time steps, and after an influenced node tries to affect all its neighbors, it becomes inactive in the following time steps (but is still counted as an influenced node). The model is equivalent to the discrete Susceptible-Infected-Removed model (SIR) \cite{17}, studied in the context of epidemic spreading, with a removal probability of 1.

A useful way to look at the diffusion of influence in the network, is the \textit{static picture}, as opposed to the dynamic picture described above. In the static picture, instead of tossing a coin at each step of the process to see whether a link is conductive or blocked, all coins are tossed at the beginning, and the network is broken into clusters. The two pictures are equivalent, and this equivalence has been noted and used for proving the sub-modularity of the influence function in \cite{15}. In addition, the static picture also points out to the similarity of our problem
to the well studied field of bond percolation (see, e.g., \cite{24,12}). Relevant results from percolation theory are the existence of a critical threshold $p_c$, at which a giant cluster appears (i.e., a cluster of the order of the network diameter), with a power-law distribution of the cluster sizes near this critical point, and the scaling of the average cluster size, which scales as $|p - p_c|^{-\gamma}$ near the critical point, with $\gamma > 0$. The influence function discussed in the maximal influence problem is the average over the cluster size distribution around each specific set of nodes. The questions of whether, and to what extent is there a difference between various initial seed-sets, depend on the extent to which the cluster sizes are correlated to specific network sites, and on the cluster size distribution. The distribution depends on the parameter $p$, which plays a central role in determining who are the important nodes, and how much are they different than the average nodes.

2.3 Examples: The Importance of the Influence Parameters

Let us start with two motivating examples that demonstrate the importance of the influence parameters for the result of the optimization process, and show the interplay between these parameters and the network structure. We look at small toy-networks where the influence can be easily solved exactly, but their structure still demonstrates basic phenomena that play an important role in influence maximization in large and complex networks as well.

![Fig. 4: Simple network examples](image)

**Fig. 4:** Simple network examples: network (a) illustrates that the maximal-influence node depends on the parameters; the best node is node 1 for $p > 4/5$, and nodes 2 or 13 for $p < 4/5$. In the star network (b), the central node has the highest influence in every realization. In network (c), the best node is not the best in every realization: node 1 has the maximal influence in this network only in a $\left[1 - p(1 - p)^3\right]$ fraction of realizations.

Our first example demonstrates that the best set of a given network depends on the influence parameters. Assume we try to find a single seed node in the network in Figure 4(a). Every link has a probability $p$ to be conductive, and the average influence of every node can be calculated. The result is that if $p > 4/5$, then the best node is node 1, who has only two neighbors, while for $p < 4/5$, the optimal nodes are nodes 2 or 13, who are equivalent with 6 neighbors each. This illustrates how in real networks, large hubs can be important when the influence probability is below some threshold, but if the probability is higher, low degree nodes who are close to several large hubs may become more important than

\[^2\text{This kind of calculation is manageable in tree networks; if there are cycles such calculation will demand counting all non-backtracking trajectories between all seeds and targets, which becomes unfeasible in large networks.}\]
the hubs themselves. Therefore, the question of “who are the most influential nodes?” should in fact be answered by: “What are the influence parameters?”.

Our second example demonstrates that the desired best seed-set is not always the best one in each specific scenario, i.e., it does not give the maximal influence in every realization of the random process. The star network in Figure 4(b) is a non-typical case where the most influential node does have the maximal influence in every realization. In contrast, in the network in Figure 4(c), node 1 has the best average influence, but there is a probability of $p(1 - p)^2$ in each realization that nodes 2 and 3 will have higher influence. In more complex networks the differences between many of the seed-sets may be small, and this gives hope that some partial and fast optimization may suffice to yield good results, as there may be many “nearly best” seed-sets in the network.

3 When Do We Need Optimization?

Our goal in this section is to characterize the cases in which optimization procedures are required. The answer depends both on the networks in discussion and on the influence parameters. Special networks can be constructed such that the optimization can be dramatically important, as in the case of a star network, or completely useless, as in the case of a uniform lattice where all nodes are equivalent, and the solution is every set with uniform distances between its nodes. However, such special cases are far from being typical representations of real-world networks, and in order to get a more general picture we look at random networks, and focus our question on the effect of the system’s parameters on the necessity of optimization.

As explained in the previous section, the strength of peer-to-peer interaction is characterized by the parameter $p$, which is the probability of an influenced node to affect each of its neighbors. We measure the effects of optimization for different values of this parameter, and ask: For which range of parameters “it pays” to run optimization algorithms?

3.1 Doing Better Than Random

A natural way to test whether and when do we need optimization algorithms is to compare to the random choice benchmark. Optimization takes time and effort and it will be useless if it gives similar results to an uninformed shot in the dark. We test for which parameters does optimization using the hill-climbing algorithm proposed in [15] give higher influence than a random choice, and measure the marginal gain from optimization.

We denote the median influence gained using the hill-climbing optimization procedure as $opt$, and the median influence of the random choice as $rand$. Figure 1 shows the marginal gain, $opt - rand$, as a function of the parameter $p$, on a sample of Erdős - Rényi networks of different sizes, with a mean degree of 4 and an initial seed-set of size $k = 3$. As mentioned in Section 1 the results were
robust across other random networks; we have tested power-law degree distributions, random regular networks and small-world networks (using the Watts and Strogatz model [28]), as well as other choices of mean degree and seed-set sizes.

The practical meaning of the results presented in Figure 1 is that optimization is substantially better than a random choice only in a specific region of the parameter space, near the critical point. In all networks, the marginal gain has a peak centered near the critical point, as calculated from bond percolation theory ($p_c = 1/3$ for these networks). The marginal gain is close to zero both above and below this peak, which can be intuitively understood in the limits of low and high probabilities: far below the critical point, spreading is local and is related mainly to first neighbor influence, and the possibilities for optimization are limited. At high probabilities, at the end of each realization of the random process almost all nodes are connected as one giant influenced cluster; the influence is then high, however a random choice also has a high probability to hit this same cluster and gain the same influence – and so again optimization is not much better than a random choice. The narrow localization of the curve is also consistent with finite size scaling results from statistical mechanics, which suggest that the transition region below $p_c$ in which the power-law scaling of the clusters is valid, diminishes as a power law of $N^{-1/3}$.

In order to measure the width of the parameter range where optimization is needed (the “optimization region”), we test for which values of $p$ the marginal gain from optimization is at least 1% of the nodes. We find that the width of this region decays to zero as a power-law of the network size. Figure 2 shows the region width as a function of the network size for our network sample, and the inset shows a power-law fit on logarithmic scale with an exponent of $2/5 \pm 0.02$. Each data point is the width in parameter space in which the marginal gain is at least 1% of the network, averaged over 10 networks of each size. The differences between the width results calculated on each network separately for each size were small, and were mainly near the phase transition, with maximal standard deviations of 0.009 to 0.015 for $N = 10,000$ to $N = 1,000$, respectively. For most of the parameter range (away from $p_c$) the differences between networks were an order of magnitude lower. We also tested other methods to estimate the width – the standard deviation and the Full-Width-at-Half-Maximum (FWHM) of the marginal-gain curves – and both gave the same results for the exponent.

### 3.2 Considering Optimization Costs

In order to get a more concrete picture of whether and when optimization procedures are profitable, we should also take into account the costs of optimization, and to see when does one have a positive marginal net utility from running optimization procedures. The following results are a corollary of the above results, that optimization has a significant marginal gain only in a vanishing region, together with the fact that computation costs increase with the network size.

Optimization has direct costs resulting from computation time (see, e.g., [25]), as well as possibly other indirect costs incurred by the time delay. We model
these costs as a constant cost-per-time-unit $C$, and the total cost of optimization is proportional to the running time: cost = $C \cdot T$. The optimization time $T$ scales with the network size in the standard methods, with state-of-the-art algorithms achieving impressive developments with nearly linear running times; but still, for large networks the optimization cost $C \cdot T$ may be substantial, and if the marginal gain from optimization is small, this cost may exceed the marginal gains. Each influenced node $i$ gives the user some value $v_i$, and we consider the simple case of a uniform value for all nodes $v_i = v$. The condition under which optimization has positive marginal utility is $U_{opt} > U_{rand}$. Substituting $U_{opt} = v \cdot opt - C \cdot T$ and $U_{rand} = v \cdot rand - C$ we have the condition:

$$\frac{1}{T - 1} (opt - rand) > \frac{C}{v}$$

(1)

We can see the competition between low computation costs on the one hand, and large data sets which demand long running times on the other hand. More importantly, in light of the sharp localization of the marginal gain term $(opt - rand)$ in the parameter space discussed above (see Figure 1), we expect that eq.1 will be satisfied only in a narrow parameter region; as we show next, this is indeed the case for standard optimization algorithms.

In order to estimate the width $w$ of the region in which standard optimization gives positive marginal utility (i.e., when we now also consider the costs), we use simulation results of hill-climbing [15], which is proved to have the optimal approximation guarantee, and calculate the optimization costs incurred by the running time of $NlogN$ achieved by state-of-the-art algorithms.

Substituting this running time and the empirical results for $(opt - rand)$ into eq.1 we can test for which values of $p$ does the inequality hold; this defines a width $w$ in the parameter space, in which the marginal utility is positive. We then look at the functional dependance of $w$ on the network size $N$, and extrapolate the results for larger networks. Figure 5 shows the results of $w$ as a function of the network size for example values of the relative cost $C/v$. We find that $w$ has a clear power-law decay with $N$, in consistency with our previous results. Notice that using different values of the relative cost $C/v$ only amounts to a multiplicative factor, and does not change the power-law decrease with $N$.

In the bottom line, the range of parameters in which it pays to use optimization algorithms is narrowing with a power-law of the network size. If the system happens to be close to its phase transition, it may be useful to choose the initial seed-set using optimization algorithms; but in real-world scenarios the parameters may not be known, and cannot be easily tuned, and in these cases, for large networks there is a small probability to gain anything from running global optimization algorithms. A practical approach to deal with this unfortunate situation is to bound the cost of the optimization process such that it will not scale with the network size, and then see how well can we do. We demonstrate this approach in the next section.

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3 We ignore here any cost incurred by the first $k$ seed nodes starting the process. The question of optimizing the deployment of a budget between costs of the initial seeds is discussed in [22]; here we focus on the costs of the optimization process itself.
4 Constant Time Optimization

We have seen that standard optimization may be wasteful for most of the possible parameters, yet it may be profitable if the system happens to be close to its critical point. Therefore, we suggest an approach that on the one hand, avoids high costs – for the case that the system is away from its phase transition – and on the other hand, if the system is close to its critical point, will capture a significant part of the optimization gains, and will not do much worse than standard optimization. The key idea is to maintain constant optimization costs, that are independent of the network size. We demonstrate this approach by a simple algorithm we call random-local-optimization.

**Random-local-optimization:** In order to maintain a constant running time it is needed to avoid iterating through the whole network data. The basic idea is that in order to find a seed-set of \( k \) nodes, the algorithm chooses \( k \) random nodes and builds bounded sized sub-networks around them. The optimization is then performed only over these sub-networks, and for each sub network independently. Specifically, a user chooses some constant mass \( M \), which is the number of nodes in each sub-network, and the algorithm gets this constant as input. The algorithm then takes \( k \) random nodes to be the initial \( k \) sub-networks. Then, while each sub-network mass is less than \( M \), the algorithm incrementally adds the neighbors of each current sub-network. When this process terminates, there are \( k \) sub-networks of size \( M \), and for each network we need to find one target node. The best node in each sub-network can be found with an arbitrarily small error by iterating through the sub-network nodes and using Monte-Carlo sampling in each sub-network separately. The total running time is \( O(k \cdot M^3) \), which is independent of the network size.

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4 This simple algorithm does not guarantee there will be no overlap between sub-networks, however, the constant time idea is relevant for large networks, where the network size comes to our advantage. An overlap is possible, but the probability decreases with the network size.

5 See [16] for the proof of the Monte-Carlo sampling time.
This is different from the algorithms suggested by [4,26,1] in two main aspects: first, the local sub-networks are not restricted to local trees, but may include cycles, as long as they are within the predetermined sub-network size. This may allow us to capture effects of highly connected local cliques. Second, and more importantly, the running time of random-local-optimization is independent of the network size; local sub-networks are not constructed around all (or a fraction of) the nodes in the network, but only around \( k \) nodes. The algorithm does not need to know the whole network, nor its size.

As we show next, even our simple implementation of the constant-time approach gives similar results to standard optimization in terms of the influenced set size, and considering the optimization costs and uncertainty of the system parameters, our random-local-optimization gives higher expected utility. First, we look at the influenced set size: Figure 3 shows the median influence gained by random-local-optimization and by standard optimization using hill-climbing, in percents of the network size. It can be seen that random-local-optimization gives similar results to standard optimization in most of the parameter space, with a modest advantage to hill-climbing only near the critical point.

Next we turn to look at the utility measure. Figure 6 shows the median utility gained by using random-local-optimization and by standard optimization (using hill climbing) on ER networks of size \( N = 10,000 \), after deducting the optimization time costs with a relative cost of \( C/v = 10^{-3} \) per time unit. The results are presented in percents of the total value in the network, \( v \cdot N \). For the costs of standard optimization we considered the running time of \( N \log N \) of state-of-the-art methods, which are faster than hill-climbing, but have lower or similar performance guarantees. As discussed in Section 3.2 since the total cost is the product \( C \cdot T \), choosing a smaller value for the constant \( C/v \) is equivalent to taking a longer running time. In the standard optimization methods this will be equivalent to measuring the performance on larger networks.
As can be seen, the utility gained by random-local-optimization is similar to standard optimization for most values of $p$. A closer look shows (see the zoom-in images in the insets) that our algorithm seems to achieve the goals of the constant time optimization approach: (i) it gives higher utility for most of the parameter range; (ii) near the critical point it is not much worse than hill-climbing. We also see that at the low phase, i.e., at low probabilities, standard optimization may result in negative utility. This is because the optimization costs remain high (of the order of the network size), even when the added value is small.

Finally, we compare the performance of random-local-optimization and of standard optimization in a situation of uncertainty of the influence parameters. Since the true value of $p$ is unknown, we have some prior distribution $f(p)$ regarding its value, and specifically in the following we consider a uniform prior. Figure 7 depicts the expected marginal utilities (again, in respect to the random benchmark) over a uniform prior of the values of $p$, $f(p) \sim U(0, 1)$, as a function of the network size, presented in percents of the total value. We can see that for this natural situation where $p$ is unknown, our local optimization manages to give higher utility than standard optimization, with a gap that increases with the network size.

5 Conclusion

The maximal influence problem and the related algorithms focus on the structural aspect of network influence, and search for seed-sets that have some structural advantage in the network. In this study we have found that such structural differences between seed-sets become significant only near a phase transition of the network, in a parameter region that shrinks as a power-law of the network size.
size, while away from the phase transition most seed-sets give similar results. One implication we have shown is that when there are optimization costs, optimizing over the whole network is profitable only in a narrow parameter region, while usually a user needs to decide whether or not to invest the optimization costs without knowing the actual parameters. This implies that for the structural aspect of optimization, sub-linear algorithms are more attractive and profitable, and we have demonstrated one example in this vein by the random-local-optimization algorithm, which maintains constant optimization time. We see it as a challenge for future work to further refine this constant-time approach, and to find improved constant-time algorithms.

In light of the results we have seen, we suggest that the study of influence maximization should extend to questions that are not purely network-structural: Which nodes are more susceptible? (Or which links are more conductive?) What are the efficient methods to measure, map and predict the spreading parameters? How do the parameters change with time? Is there a coupling between the (local) spreading probability and the (global) state of network influence?

Moreover, the relation we have found of a physical phase transition of the network to the possible gains from optimization may be more general than the case of influence maximization. In fact, the results of the present study raise a generic conjecture that there is a class of computationally difficult optimization problems that are actually relevant only in a very limited (zero-measure) range of parameters, which is related to statistical-mechanics phase transitions of the underlying systems. The establishment and characterization of this class of problems poses a fascinating direction for future research.

Acknowledgments

We would like to thank Noam Nisan for fruitful discussions and advice throughout this research. We also thank Yaron Singer and Gali Noti for helpful conversations, and Dietrich Stauffer, Roi Reichart and Effi Levi for comments on an earlier draft.

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