PERTURBATIONS OF ISOMETRIES BETWEEN BANACH SPACES

RAFAŁ GÓRAK

Abstract. We prove a very general theorem concerning the estimation of the expression \( \|T(a+b) - \frac{Ta+Tb}{2}\| \) for different kinds of maps \( T \) satisfying some general perturbed isometry condition. It can be seen as a quantitative generalization of the classical Mazur-Ulam theorem. The estimates improve the existing ones for bi-Lipschitz maps. As a consequence we also obtain a very simple proof of the result of Gevitz which answers the Hyers-Ulam problem and we prove a non-linear generalization of the Banach-Stone theorem which improves the results of Jarosz and more recent results of Dutrieux and Kalton.

1. Introduction

The aim of this paper is to prove a very general theorem (Theorem 2.1) that will allow us to obtain several facts concerning approximate preservation of midpoints by different kinds of maps with perturbed isometry condition. Let us define the main notion of this paper:

Definition 1.1. Let \( T : E \to F \) be a function between two metric spaces \((E, d_E)\) and \((F, d_F)\). Assume that there is a function \( \mu : \mathbb{R}_+ \to \mathbb{R}_+ \) (where \( \mathbb{R}_+ = \{x \in \mathbb{R}; x \geq 0\} \)) which is non-decreasing and such that the following conditions hold:

(i) \( T \) is a bijection.
(ii) \( d_F(Tx, Ty) \leq \mu(d_E(x, y)) \) and \( d_E(T^{-1}f, T^{-1}g) \leq \mu(d_F(f, g)) \) for all \( x, y \in E \) and \( f, g \in F \).

Then \( T \) is called a \( \mu \)-isometry.

In our article we consider (except Corollary 3.4) \( \mu \)-isometries between Banach spaces only. It should be noticed that following [7] for a given map \( T : E \to F \) we can easily find the optimal \( \mu \) which is \( \mu(t) = t + \varepsilon_T(t) \) where

\[
\varepsilon_T(t) = \sup\{\|Tx - Ty\| - \|x - y\| : \|x - y\| \leq t \text{ or } \|Tx - Ty\| \leq t\}.
\]

Lindenstrauss and Szankowski consider maps \( T \) that are surjective but not necessarily injective as \( \mu \)-isometries. However they observed that one can easily reduce the considerations to the bijective case when \( t \to \infty \):

Date: January 20, 2013.
2010 Mathematics Subject Classification. 46E40, 46B20.
Key words and phrases. Mazur-Ulam theorem, Banach-Stone theorem, function space, isometry.
Fact 1.2. Let $T : E \mapsto F$ be a surjective map between Banach spaces $E$ and $F$, respectively. If $\varepsilon_T : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is well defined ($\forall t \in \mathbb{R}_+ \varepsilon_T(t) < \infty$) and $\exists \delta_0 > 0 \frac{\varepsilon_T(\delta_0)}{\delta_0} < 1$ then there exists a bijection $\tilde{T} : E \mapsto F$ such that:

$$\forall x \in E \parallel Tx - \tilde{T}x\parallel \leq 2\delta_0 + 2\varepsilon_T(\delta_0).$$

Hence $\tilde{T}$ is a $\mu$-isometry for $\mu(t) = t + \varepsilon_T(t) + 4\delta_0 + 4\varepsilon_T(\delta_0)$. In particular $\varepsilon_{\tilde{T}}(t) \sim \varepsilon_T(t)$ as $t \to \infty$ (if only $\varepsilon_T(t) \to \infty$).

Proof. For the sake of completeness we sketch the proof. Let us consider the maximal set $A \subset E$ such that all the points are in the distance at least $\delta_0$ from each other. Then for every $a \neq b, a, b \in A$ we have $\delta_0 - \varepsilon_T(\delta_0) \leq \|Ta - Tb\|$, hence $T|A$ is injective. Moreover $T(A)$ is a $\delta_0 + \varepsilon_T(\delta_0)$ dense in $F$ (that is the distance of every element of $F$ from $T(A)$ is not greater than $\delta_0 + \varepsilon_T(\delta_0)$)). This shows that the density character of $E$ and $F$ are equal. Now it is easy to construct a decomposition of $E = \bigcup_{a \in A} E_a$ and $F = \bigcup_{a \in A} F_a$ such that for all $a \in A$:

1. $a \in E_a, Ta \in F_a$;
2. $|E_a| = |F_a|$;
3. $\text{diam} E_a \leq \delta_0$ and $\text{diam}F_a \leq \delta_0 + \varepsilon_T(\delta_0)$.

By the standard set theoretical reasoning we can extend $T|A$ to the required $\mu$-isometry $\tilde{T} : E \mapsto F$. 

Hence in further considerations we stick to the notion of $\mu$-isometry as it provides sufficient generality and by considering bijective maps we avoid some easy but rather technical problems.

When considering the $\mu$-isometry $T$ one should rather think that $T$ is not necessarily the perturbated isometry (since it may easily happen that there is no isometry to be perturbated) but $T$ satisfies the perturbated isometry condition. Hence the following natural question arises: "How can you perturbate the definition of an isometry between Banach spaces so that its existence implies the existence of an isometry?". If the answer to the above question is positive then another one can be asked: "How far is the perturbated isometry from an isometry?". It appears that Lindstrauus and Szankowski in [7] answered these questions for the class of all Banach spaces and for all $\mu$-isometries. However one can investigate the above problems for some subclasses of Banach spaces (such as function spaces which leads to generalizations of the Banach-Stone theorem).

Let us discuss now, in more details, some examples of $\mu$-isometries for different functions $\mu$ and the results related to both questions asked above. Let $T$ be a $\mu$-isometry between Banach spaces $E$ and $F$. If $\mu(t) = t$ then $T$ is just an isometry. Let us consider now $\mu(t) = t + L$ for some constant $L \geq 0$. Such maps are called $L$-isometries. More generally $L$-isometry $T$ is a surjective map between Banach spaces for which $\varepsilon_T(t) \leq L$. But as we have already noticed, Fact 1.2 allows us to reduce considerations to the bijective case (see Corollary 3.1 where we show how it is done). Hyers and Ulam asked whether $L$-isometries are close to isometries. The question was
answered positively for all pairs of Banach spaces $E$ and $F$ by Gevitz in [4] (let us say that $L$ can be as large as we please).

Szankowski and Lindenstrauss gave a complete characterization of such $\mu$-isometries whose existence implies the existence of an isometry. More precisely:

**Theorem 1.3.** Let $T : E \mapsto F$ be a $\mu$-isometry between Banach spaces $E$ and $F$ where $\mu(t) = t + \varepsilon_T(t)$, $T(0) = 0$ such that the condition $\int_1^\infty \frac{\varepsilon_T(t)}{t} dt < \infty$ is satisfied. Then there exists an isometry $I : E \mapsto F$ such that

$$\|Tx - Ix\| = o(\|x\|) \text{ as } \|x\| \to \infty.$$  

Moreover the result is sharp (see [7] for more details) in the case when $E$ and $F$ are general Banach spaces.

Let us consider now $\mu(t) = Mt$. In this case $T$ is a bi-Lipschitz map (or Lipschitz equivalence). It means that distances between points are perturbed according to the inequalities

$$\frac{1}{M} \|x - y\| \leq \|Tx - Ty\| \leq M \|x - y\| \text{ for all } x, y \in E.$$  

Obviously if $M = 1$ then $T$ is just an isometry. Let us look at the case when $M > 1$. Unfortunately, no matter how close to one $M$ is, we cannot guarantee the existence of an isometry between general Banach spaces $E$ and $F$. Clearly $\int_1^\infty \frac{(M-1)t}{t^2} dt = \infty$ ($M > 1$) hence you can find in [7] a construction of Banach spaces $E$ and $F$ that are $\mu$-isometric for $\mu(t) = Mt$ but they are not isometric. However, for some particular class of Banach spaces $E$ and $F$ one can obtain some interesting positive results even for more general case that is when $\mu(t) = Mt + L$ (maps that are bi-Lipschitz for large distances). Indeed let us consider $E = C_0(X)$ and $F = C_0(Y)$, the spaces of continuous real valued functions vanishing at $\infty$ on locally compact spaces $X$ and $Y$, respectively. Spaces $C_0(X)$ and $C_0(Y)$ are endowed with the sup norms. It appears that in this case one can obtain more than Theorem 1.

**Theorem 1.4.** Let $T : C_0(X) \mapsto C_0(Y)$ be a $\mu$-isometry, where $X$ and $Y$ are locally compact spaces, $\mu(t) = Mt + L$ ($M \geq 1$, $L \geq 0$) and $T(0) = 0$. Then there exists an absolute constant $M_0 > 1$ and functions $\delta : [1, \infty) \mapsto \mathbb{R}_+$, $\Delta : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ such that whenever $M < M_0$ then there exists an isometry $I : C_0(X) \mapsto C_0(Y)$ such that

$$\|Tf - If\| \leq \delta(M) \|f\| + \Delta(M, L) \text{ for all } f \in C_0(X).$$  

Moreover, $\Delta(M, 0) = 0$ and $\lim_{M \to 1^+} \delta(M) = 0$. In particular, from the Banach-Stone theorem, the spaces $X$ and $Y$ are homeomorphic. It is known that $M_0 \leq \sqrt{2}$ and the equality holds if we assume additionally that $T$ is linear (see [3] and [5] for the discussion).

The first of such results was obtained by Jarosz in [9] but for $L = 0$ only. However the value of $M_0$ which he obtains is very close to 1 as well as the function $\delta$ is far from being optimal ($\delta(M) = O((M - 1)^{0.1})$ as $M \searrow 1$ and $\Delta(M, 0) = 0$ in his result). Later Dutriex and Kalton in [2] obtained the
value of \( M_0 = \sqrt{\frac{17}{16}} \) (in their language the condition \( M < M_0 \) can be seen as the inequality \( d_N(C_0(X), C_0(Y)) < M_0^2 \)) but they do not provide any estimation like (1.1) (this time \( L \) can be positive). Finally the author in [5] improved the constant to \( M_0 = \sqrt{\frac{5}{3}} \) and showed that \( \delta(M) = 26(M - 1) \). Moreover \( \Delta(M,0) = 0 \) hence the result improved both, the constant \( M_0 \) obtained in [2] and the function \( \delta \) obtained in [9] as well as showed the existence of \( \delta \) and \( \Delta \) if \( L > 0 \). However, the proof works only for \( X \) and \( Y \) compact and it is not that easy to extend it to the locally compact case. We will do this in the last section of this paper by applying the main result of Section 2.

It appears that in the proofs of most of the above results the estimation of \( \|T(a + b) - \frac{T a + T b}{2}\| \) is crucial and far from being obvious. Moreover the results estimating this expression can be regarded as generalizations of the Banach-Mazur theorem so in some sense they are of independent interest. We deal with this problem in the next section.

2. APPROXIMATE PRESERVATION OF MIDPOINTS BY \( \mu \)-ISOMETRIES

We present here a very general method of estimating \( \|T(\frac{a+b}{2}) - \frac{T a + T b}{2}\| \) for \( \mu \)-isometries \( T \). It should be mentioned that some results of this kind are already obtained in [7] (in fact this is the most demanding part of the article). However the method presented here has several important advantages. First of all it has astonishingly simple proof and it covers the result of Gevitz (Corollary 3.1) which answers the famous Hyers-Ulam problem (the proofs in the original paper [4] or in the survey paper of Rassias [8] are clearly more complicated). Secondly, applying our result for \( \mu \)-isometries where \( \mu(t) = Mt + L \), we obtain new and elegant estimates (they are interesting even in the Lipschitz case that is when \( L = 0 \)). This will allow us to prove new results concerning the nonlinear version of the Banach-Stone theorem. Finally, although our theorem does not cover the result of Lindenstrauss and Szankowski in full generality, it gives their result for particular functions \( \mu(t) = t + \varepsilon(t) \) such as \( \mu(t) = t + t^\alpha \) where \( \alpha \in [0,1) \) (see Section 4). It is very tempting (due to the simplicity of the proof below) to investigate whether Theorem 2.1 gives us the result from [7] in full generality.

Before we formulate and prove the main result let us say that the idea of it comes from a very beautiful proof of the classical Mazur-Ulam theorem due to Väisälä (see [10]).

**Theorem 2.1.** Let \( T : E \mapsto F \) be a \( \mu \)-isometry between two normed spaces \( (E, \| \cdot \|_E) \) and \( (F, \| \cdot \|_F) \). Assume that \( \mu : \mathbb{R}_+ \mapsto \mathbb{R}_+ \) is such that \( \mu(t)/2 \leq \mu(t/2) \). Then for all \( a, b \in E \) and \( n \in \mathbb{Z}_+ \):

\[
\|T(\frac{a+b}{2}) - \frac{T a + T b}{2}\|_F \leq \mu^{n+1}(\frac{\|a - b\|_E}{2^n})
\]

where \( \mu^n = \mu \circ \mu \circ \ldots \mu \) (\( \mu \) composed \( n \) times).
Proof. Let us consider the set $W_E(\mu)$ consisting of all maps $T$ that are $\mu$-isometries on $E$ and moreover, let $\text{Im} T$ be a normed space. Fix $a,b$ in the space $E$ and set $z = \frac{a+b}{2}$. Denote:

$$
\lambda(\mu) = \sup\{\|Tz - \frac{Ta + Tb}{2}\|_F \mid T \in W_E(\mu), F = \text{Im} T\}.
$$

Let us observe that for $T \in W_E(\mu)$ we have:

$$
\|Tz - \frac{Ta + Tb}{2}\|_F \leq \frac{1}{2}(\|Tz - Ta\|_F + \|Tz - Tb\|_F) \leq \frac{1}{2}(2\mu(\frac{\|a-b\|_E}{2})) = \mu(\frac{\|a-b\|_E}{2}).
$$

Hence

(2.1)

$$
\lambda(\mu) \leq \mu(\frac{\|a-b\|_E}{2})
$$

and one can see that $\lambda(\mu)$ is finite. For some $T \in W_E(\mu)$ let us define $\Psi$ and $\Psi'$ to be the reflections with respect to $z$ and $\frac{Ta+Tb}{2}$, respectively. Consider a new bijection on $E$ defined as a composition $S = \Psi T^{-1} \Psi' T$. It is easy to check that $S \in W_E(\mu \circ \mu)$, $Sa = a$ and $Sb = b$. We have:

$$
2\|Tz - \frac{Ta + Tb}{2}\|_F = \|\Psi' Tz - Tz\|_F \leq \mu(\|T^{-1} \Psi' Tz - T^{-1} Tz\|_E)
$$

$$
= \mu(\|Sz - z\|_E) = \mu(\|Sz - \frac{Sa + Sb}{2}\|_E).
$$

Concluding

$$
\lambda(\mu) \leq \frac{1}{2} \mu(\lambda(\mu \circ \mu)) \leq \mu(\frac{\lambda(\mu^{o2})}{2}).
$$

Hence:

$$
\lambda(\mu^{o2n}) \leq \mu^{o2n}(\frac{\lambda(\mu^{o2n+1})}{2}).
$$

Applying the above formula recursively we obtain:

$$
\lambda(\mu) = \lambda(\mu^{o1}) \leq \mu^{o1}(\frac{\lambda(\mu^{o2})}{2}) \leq \mu^{o1}(\frac{1}{2} \mu^{o2}(\frac{\lambda(\mu^{o4})}{2})) \leq \mu^{o1 \circ o2}(\frac{\lambda(\mu^{o4})}{4}) \leq \ldots
$$

Finally:

$$
\lambda(\mu) \leq \mu^{o1 \circ o2 \ldots o2^{n-1}}(\frac{\lambda(\mu^{o2n})}{2^n}) = \mu^{o(2^{n-1})}(\frac{\lambda(\mu^{o2n})}{2^n}).
$$

From the estimation (2.1) we have

$$
\lambda(\mu) \leq \mu^{o(2^{n+1}-1)}(\frac{\|a-b\|_E}{2^{n+1}}).
$$

□
3. Applications

The result from the previous section gives us a very simple proof of the main result from [4] as a consequence, which answers the question of Hyers and Ulam. More precisely:

**Corollary 3.1.** Let $T$ be an $L$-isometry between Banach spaces $E$ and $F$ such that $T(0) = 0$. Then there exist constants $A$ and $B$, depending on $L$ only, such that

$$\|T\left(\frac{a + b}{2}\right) - \frac{T a + T b}{2}\| \leq A\sqrt{\|a - b\|} + B \text{ for all } a, b \in E.$$

As a corollary from that estimation, Gevirtz easily obtains (relying on the result of Gruber) that the map $I: E \mapsto F$ defined as $I x = \lim_{n \to \infty} \frac{T(2^n x)}{2^n}$ is an isometry such that $\|T x - I x\| \leq 5L$ (later the constant was improved to $2L$ which appears to be optimal).

**Proof.** Let us first assume that $T$ is a $\mu$-isometry for $\mu(t) = t + L$. Applying Theorem 2.1 for $\mu(t) = t + L$, we obtain

$$\|T\left(\frac{a + b}{2}\right) - \frac{T a + T b}{2}\| \leq \|a - b\| + 2^{n+1}L.$$

Taking $n = \lfloor \log_2 \sqrt{\|a - b\|} \rfloor - 1$ we have

$$\|T\left(\frac{a + b}{2}\right) - \frac{T a + T b}{2}\| = O(\sqrt{\|a - b\|})$$

as $\|a - b\| \to \infty$. By applying Fact 1.2, we easily get the estimation for all $L$-isometries, not only the bijective ones. □

For further applications of Theorem 2.1 we need the following simple observation:

**Lemma 3.2.** Let $\mu(t) = t + \varepsilon(t)$ where $\varepsilon : \mathbb{R}_+ \mapsto \mathbb{R}_+ \setminus \{0\}$ is a non-decreasing function. Then

$$\int_{t}^{\mu^{\infty}(t)} \frac{1}{\varepsilon(x)} dx \leq n.$$

**Proof.** Let us notice that $\frac{1}{\varepsilon}$ is a non-increasing function, hence

$$\int_{t}^{\mu^{\infty}(t)} \frac{1}{\varepsilon(x)} dx \leq \sum_{k=0}^{n-1} \frac{1}{\varepsilon(\mu^k(t))}(\mu^{k+1}(t) - \mu^k(t)) = n.$$

We obtain the following:

**Corollary 3.3.** Let $T : E \mapsto F$ be a $\mu$-isometry for $\mu(t) = (1 + \varepsilon)t + L$ where $0 < \varepsilon < 0.2$. Then:

$$\|T\left(\frac{a + b}{2}\right) - \frac{T a + T b}{2}\| \leq 3\varepsilon\|a - b\| + \frac{4}{\varepsilon}L$$

for all $a, b \in E$. 

Let us explain that for \( \varepsilon \geq 0.2 \), we easily obtain

\[
\| T\left(\frac{a+b}{2}\right) - \frac{T(a+Tb)}{2} \| \leq \frac{1+\varepsilon}{2} \|a-b\| + \frac{L}{2}
\]

which is a better estimate than the one from the above corollary when \( \|a-b\| \to \infty \). The above result is the most interesting when \( \varepsilon \) is close to 0 and \( \|a-b\| \to \infty \).

**Proof.** From Theorem 2.1 we obtain that:

\[
\| T\left(\frac{a+b}{2}\right) - \frac{T(a+Tb)}{2} \| \leq \mu^{\circ(k-1)}(\frac{d}{k}) \leq \mu^{\circ k}(\frac{d}{k})
\]

where \( k = 2^{n+1} \) and \( d = \|a-b\| \). From Lemma 3.2 we get that

\[
\int_{\frac{d}{k}}^{\frac{1}{\varepsilon x+L}} \frac{1}{\varepsilon x+L} dx \leq k.
\]

Hence \( \mu^{\circ k}(\frac{d}{k}) \leq \frac{e^k}{k} d + \frac{1}{\varepsilon}(e^k-1)L \). Function \( k \mapsto \frac{e^k}{k} \) has its minimum at \( k = \frac{1}{\varepsilon} \) which is \( \varepsilon e \). Since in our application \( k = 2^{n+1} \) we have to find \( n \) so that \( 2^{n+1} \) is as close to \( \frac{1}{\varepsilon} \) as possible. For \( \varepsilon < 0.2 < \frac{1}{\sqrt{2}} \) there exists \( n \in [\log_2 \frac{1}{\varepsilon} - 1.5; \log_2 \frac{1}{\varepsilon} - 0.5] \cap \mathbb{Z}_+ \). Hence \( 2^{n+1} = k \in \left[ \frac{1}{\sqrt{2}}; \frac{1}{\varepsilon} \right] \) and this interval contains \( 1 \). Checking the values of \( \frac{e^k}{k} \) at the endpoints we obtain that \( \mu^{\circ k}(\frac{d}{k}) \leq 3\varepsilon d + \frac{4}{\varepsilon} L \). \( \square \)

For the Lipschitz case \( (L = 0) \) similar estimations can be found in [11]. Vestfrid obtains the inequality \( \| T\left(\frac{a+b}{2}\right) - \frac{T(a+Tb)}{2} \| \leq 6\varepsilon \|a-b\| \). So one can see that the above result improves the existing estimate as well as extends it onto maps that are not necessarily continuous \( (L > 0) \). By applying the above Corollary we can also obtain some interesting estimates for bi-Lipschitz maps between \( \xi \)-dense subspaces of Banach spaces (nets in particular):

**Corollary 3.4.** Let us consider a \( \mu \)-isometry \( T : A \mapsto B \) from a \( \xi_E \) dense set in Banach space \( E \) onto a \( \xi_F \) dense set in \( F \), where \( \mu(t) = (1+\varepsilon)t \) and \( 0 < \varepsilon < 0.2 \). Then for every \( a, b \in A \) and every \( z \in A \) such that \( \|\frac{a+b}{2} - z\| \leq \xi_E \) we have:

\[
\| Tz - \frac{T(a+Tb)}{2} \| \leq 3\varepsilon \|a-b\| + \frac{34(\xi_E + \xi_F)}{\varepsilon}.
\]

**Proof.** Using a simple Fact 1.5 from [5] (or reasoning similarly as in the proof of Fact 1.2) we obtain a map \( \tilde{T} : E \mapsto F \) which is a \( \mu \)-isometry for \( \mu(t) = (1+\varepsilon)t + 4\xi_F + 3\xi_E \) and \( \|\tilde{T}x - Tx\| \leq 2\xi_F + 2\xi_E \) for all \( x \in A \). Let us take any \( z \in A \) such that \( \|\frac{a+b}{2} - z\| \leq \xi_E \). Applying Corollary 3.3 to the map \( \tilde{T} \), we obtain the desired estimation. \( \square \)

We will show now how Corollary 3.3 allows us to obtain improvements on the constant \( M_0 \) and the function \( \delta \) in Theorem 1.4 for all **locally compact** spaces.
Theorem 3.5. Let $X$ and $Y$ be locally compact spaces. Consider a $\mu$-isometry $T : C_0(X) \mapsto C_0(Y)$ where $\mu(t) = Mt + L$ ($M \geq 1, L \geq 0$). If $M < M_0 = \sqrt{\frac{16}{15}}$ then there exists a homeomorphism $\varphi : X \mapsto Y$ and a continuous map

$\lambda : X \mapsto \{-1, 1\}$ such that for every $f \in C_0(X)$

\[(3.1) \quad \|Tf - If\| \leq 76(M - 1)\|f\| + \Delta\]

where $I$ is the isometry defined as $If(y) = \lambda(\varphi^{-1}(y))f(\varphi^{-1}(y))$. The constant $\Delta$ depends on $M$ and $L$ only. Moreover, for $L = 0$ we have $\Delta = 0$.

As we can see the constant $M_0$ improves the result obtained by Dutrieux and Kalton. However, more important is the estimation $\delta(M) \leq 76(M - 1)$ that is far better then the previously known, obtained by Jarosz in [9].

Proof. Let us assume that indeed $1 < M < \sqrt{\frac{16}{15}}$. If $M = 1$ then the above theorem easily follows from the mentioned solution of the Hyers-Ulam problem (Corollary 3.1) and from the Banach-Stone theorem. Let us first recall the construction of the homeomorphism $\varphi$ and the function $\lambda$ from [5].

In the construction, when dealing with topology of general topological spaces, we use the notion of Moore-Smith convergence. $\Sigma$ will always denote a directed set and whenever we write $a \rightarrow a$ we always mean $\lim_{\sigma \in \Sigma} a_\sigma = a$.

Definition 3.6. $(f^m_\sigma)_{\sigma \in \Sigma} \subset C(X)$ is the $m$-peak sequence at $x \in X$, for some directed set $\Sigma$ if

- $\|f^m_\sigma\| = |f^m_\sigma(x)| = m$ for all $\sigma \in \Sigma$,
- $\lim_{\sigma \in \Sigma} f^m_\sigma(x \setminus U) \equiv 0$ uniformly for all open neighborhoods $U$ of $x$.

The set of $m$-peak sequences at $x$ we denote by $P^X_m(x)$.

Definition 3.7. Let $D > 0$ and $m > 0$. We define the following: $S^D_m(x) = \{y \in Y; \exists (f^m_\sigma)_{\sigma \in \Sigma} \in P^X_m(x) \exists y_\sigma \rightarrow y \forall \sigma \in \Sigma T f^m_\sigma(y^m_\sigma) \geq Dm \text{ and } T(-f^m_\sigma)(y^m_\sigma) \leq -Dm\}$.

In [5] author proves that for suitably chosen $D$ and $m$ we can define $\varphi(x) = S^D_m(x)$ that appears to be a homeomorphism between $X$ and $Y$. In all the steps in [2] where we prove that $\varphi$ is a homeomorphism the only place were compactness is crucial is Fact 2.4. We will modify its proof using Corollary 3.3 so that it works for the locally compact case.

Fact 3.8. Let us consider $D$ such that $D = 14 - 13M$. There exists $m_0$ (depending on $M$ and $L$) such that for all $m > m_0$ we have $S^D_m(x) \neq \emptyset$ for all $x \in X$. Moreover if $L = 0$ then $m_0 = 0$.

Proof. Let us take any $(\tilde{f}^m_\sigma)_{\sigma \in \Sigma} \in P^X_m(x)$ such that for all $\sigma \in \Sigma \tilde{f}^m_\sigma(x) = m$ and pick one $\sigma_0 \in \Sigma$. Let us define $\tilde{g}^m_\sigma = \tilde{f}^m_\sigma + \tilde{f}$ where $\tilde{f} = \tilde{f}^m_{\sigma_0}$. We have

\[\forall \sigma \in \Sigma \|T\tilde{g}^m_\sigma - T(-\tilde{g}^m_\sigma)\| \geq 2Mm - L.\]
Hence $\forall \sigma \in \Sigma$ there exists $y^m_{\sigma} \in Y$ such that $|T\tilde{g}^m_{\sigma}(y^m_{\sigma}) - T(-\tilde{g}^m_{\sigma})(y^m_{\sigma})| \geq \frac{2}{M}m - L$. Let us observe that numbers $T\tilde{g}^m_{\sigma}(y^m_{\sigma})$ and $T(-\tilde{g}^m_{\sigma})(y^m_{\sigma})$ must be of different signs. Assume the contrary. Since $\|T(\pm\tilde{g}^m_{\sigma})\| \leq Mm + L$ we have $Mm + L \geq \frac{2}{M}m - L$ which is impossible for $m$ large enough, say $m > m'_0$ (or for all $m > 0$ if $L = 0$), provided $\frac{2}{M} > M$ (that is if $M < \sqrt{2}$). We can and we do assume that $\forall \sigma \in \Sigma T\tilde{g}^m_{\sigma}(y^m_{\sigma}) \geq 0$ or $\forall \sigma \in \Sigma T\tilde{g}^m_{\sigma}(y^m_{\sigma}) \leq 0$. Let us define:

- If $\forall \sigma \in \Sigma T\tilde{g}^m_{\sigma}(y^m_{\sigma}) \geq 0$ then $f^m_{\sigma} = \tilde{f}^m_{\sigma}$, $f = \tilde{f}$ and $g^m_{\sigma} = \frac{f^m_{\sigma} + f}{2}$.
- If $\forall \sigma \in \Sigma T\tilde{g}^m_{\sigma}(y^m_{\sigma}) \leq 0$ then $f^m_{\sigma} = -\tilde{f}^m_{\sigma}$, $f = -\tilde{f}$ and $g^m_{\sigma} = \frac{f^m_{\sigma} + f}{2}$.

Hence $Tg^m_{\sigma}(y^m_{\sigma}) - T(-g^m_{\sigma})(y^m_{\sigma}) \geq \frac{2}{M}m - L$. Because $\|T(\pm g^m_{\sigma})\| \leq Mm + L$ then

$$Tg^m_{\sigma}(y^m_{\sigma}) \geq \left(\frac{2}{M} - M\right)m - 2L$$

$$T(-g^m_{\sigma})(y^m_{\sigma}) \leq -\left(\frac{2}{M} - M\right)m + 2L.$$ 

Since $g^m_{\sigma} = \frac{f^m_{\sigma} + f}{2}$ and by Corollary 3.3 we obtain

$$\|T(\pm g^m_{\sigma}) - \frac{T(\pm f^m_{\sigma}) + T(\pm f)}{2}\| \leq 3(M - 1)m + \frac{4}{M - 1}L.$$ 

Hence

$$Tf^m_{\sigma}(y^m_{\sigma}) \geq \left(\frac{4}{M} - 9M + 6\right)m - (5 + \frac{8}{M - 1})L,$$

$$T(-f^m_{\sigma})(y^m_{\sigma}) \leq -\left(\frac{4}{M} - 9M + 6\right)m + (5 + \frac{8}{M - 1})L,$$

$$Tf(y^m_{\sigma}) \geq \left(\frac{4}{M} - 9M + 6\right)m - (5 + \frac{8}{M - 1})L.$$ 

Let us consider $m_0 \geq m'_0$ such that

$$\left(\frac{4}{M} - 9M + 6\right)m_0 - (5 + \frac{8}{M - 1})L \geq (14 - 13M)m$$

for all $m > m_0$ (we can do so since $\frac{4}{M} - 9M + 6 > 14 - 13M > 0$ for all positive $M \neq 1$). By the compactness of the set

$$\{y \in Y : Tf(y) \geq (14 - 13M)m\}$$

for $m > m_0$ we can assume that $y^m_{\sigma} \rightarrow y \in S^D_m(x)$. Let us notice that for $L = 0$ we have $m_0 = 0$. \hfill \square

Now the proof of Theorem 3.3 is exactly the same as the proof of Theorems 2.1 and Corollary 3.4 from [5]. Firstly, it is proven in [5] Section 2 that $\varphi(x) = S^D_m(x)$ is a homeomorphism for suitably chosen $m > m_2$ (where $m_2 = 0$ if $L = 0$) if

(i) $D$ is so that $S^D_m(x) \neq \emptyset$ for all $x \in X$;
(ii) $1 - \varepsilon(M)M - \varepsilon(M) > 0$ where $\varepsilon(M) = 2M - 1 - D$. 


In the compact case the condition (i) means that it is enough to take
\[ D = 4 - 3M < \frac{2}{M} - M \] (see Fact 2.4 in [5]). This, together with condition (ii), leads to a conclusion that indeed \( M < \sqrt{\frac{2}{5}} \). In the locally compact case we have already shown (Fact 3.8) that we can take \( D = 14 - 13M \). Now the condition (ii) leads us to the inequality \( M < \sqrt{\frac{16}{15}} \).

For every \( x \in X \) and \( m > m_0 \) let us define (following [5] Section 3) \( \lambda_m(x) = \frac{f_m^m(x)}{\|f_m^m(x)\|} \) where the family \( (f_m^m)_{\sigma \in \Sigma} \in P_m^X(x) \) is such that:

- \( \forall \sigma_0, \sigma_1 \in \Sigma \) \( \frac{f_m^m(x)}{\|f_m^m(x)\|} = \frac{f_m^m(x)}{\|f_m^m(x)\|} \) (\( \lambda_m(x) \) does not depend on \( \sigma \)).
- \( \exists y_\sigma \to y \in S_m^D(x) \) such that for every \( \sigma \in \Sigma \) we have \( Tf_m^{m\sigma}(y_m^{m\sigma}) \geq Dm \) and \( T(-f_m^{m\sigma})(y_m^{m\sigma}) \leq -Dm \).

The existence of the above family for every \( x \in X \) is exactly what was shown in the proof of Fact 3.8. Let us say that the function \( \lambda \) from the formulation of Theorem 3.5 is defined as \( \lambda_m \) for \( m \) sufficiently large.

In order to prove (3.1) it is enough to notice that Fact 2.7 in [5] works also for \( X \) locally compact and hence gives us the estimation
\[
\|Tf(\varphi(x)) - |f(x)|\| \leq \varepsilon(M)M\|f\| + \Delta = 15(M^2 - M)\|f\| + \Delta
\]
for all \( f \in C_0(X), x \in X \) and some constant \( \Delta \) depending on \( M, L \) and such that \( \Delta = 0 \) if \( L = 0 \).

Repeating the reasoning of Section 3 from [5] for \( D = 14 - 13M \) we obtain a slightly modified Fact 3.1 (only one constant is changed):

**Fact 3.9.** Assume that \( |f(x)| > 30(M - 1)\|f\| \) and let \( \|f\| = m \). Then for \( m > m_3 \) (\( m_3 \geq 0 \) depends on \( M \) and \( L \) only), the sign of \( Tf(\varphi(x)) \) is the same as the sign of \( \lambda_m(x)f(x) \). If \( L = 0 \) then \( m_3 = 0 \).

As a consequence, reasoning in exactly the same way as in the proof of Corollary 3.4 in [5] we get (3.1) where \( \lambda \equiv \lambda_m \) for \( m > m_3 \). Summarizing the proof let us just mention that having at hand Fact 3.8 it is very easy to modify the reasoning from [5]. One should only keep in mind that this time \( D = 14 - 13M \).

\[ \square \]

4. Final remarks

Natural directions of further investigations and some open problems arise from both of the above sections. First of all, as we have already mentioned, it would be very interesting to see how the result of Szankowski and Lindenstrauss follows from Theorem 2.1. For instance, if we consider \( \varepsilon_T(t) = t + t^\alpha \) for \( \alpha \in [0, 1) \), by using Fact 3.2 one can obtain that
\[
\|T\left(\frac{a + b}{2}\right) - Ta + Tb\| = O(\|a - b\|\frac{1}{T^\alpha}) \quad \text{as} \quad \|a - b\| \to \infty
\]
which is sufficient to show that \( Ix = \lim_{n \to \infty} T(T^{0.2}x) \) is the required isometry in Theorem 1.3.

Another interesting question concerns the expression \( \|T\left(\frac{a + b}{2}\right) - Ta + Tb\| \) and its optimal estimation when \( T \) is a \( \mu \)-isometry for \( \mu(t) = (1 + \varepsilon)t \) and
\( \varepsilon \to 0 \). We have already seen that 
\[ \| T \left( \frac{a+b}{2} \right) - \frac{T(a)+T(b)}{2} \| = O(\varepsilon \| a - b \|) \] as \( \varepsilon \to 0 \).

It is very easy to show that this is everything one can obtain in the general case. Indeed, as Vestfrid noticed in [11], consider 
\[ T : \mathbb{R} \to \mathbb{R} \text{ defined as:} \]
\[
T(x) = \begin{cases} 
(1 + \varepsilon)x & \text{if } x \geq 0 \\
\frac{1}{1+\varepsilon}x & \text{if } x < 0.
\end{cases}
\]

However the exact value of the constant below remains unknown:

\[
K = \limsup_{\varepsilon \to 0} K_\varepsilon
\]

where

\[
K_\varepsilon = \sup \frac{\| T \left( \frac{a+b}{2} \right) - \frac{T(a)+T(b)}{2} \|}{\varepsilon \| a - b \|}.
\]

Supremum is taken over all \( T \)-\( \mu \)-isometries between Banach spaces, where \( \mu(t) = (1+\varepsilon)t \), and over all pairs of points \( a \neq b \) from the domain of \( T \).

The above example shows that \( K \geq 0.5 \) and Corollary 3.3 shows that \( K \leq 3 \). It is worth to notice that a simple analysis of the proof of Corollary 3.3 gives us that \( \liminf_{\varepsilon \to 0} K_\varepsilon \leq e \).

Finally it is of a great interest to find the optimal constant \( M_0 \) and the optimal estimation of \( \delta \) in Theorem 1.4. In particular it is still unknown whether the constant \( M_0 = \sqrt{2} \) is the optimal one or not.

We skip the detailed discussion on this problem and we direct the reader to the final section in [5].

**REFERENCES**

[1] Y. Benyamini; J. Lindenstrauss : Geometric Nonlinear Functional Analysis, Colloquium Publications, vol. 48, AMS (1993)

[2] Y. Dutrieux; N. Kalton : Perturbations of isometries between C(K)-spaces, Studia Math. 166 (2005), no. 2, 181–197

[3] M. Cambern : On isomorphisms with small bound, Proc. Amer. Math. Soc. 18 (1967), 1062–1066

[4] J. Gevitz : Stability of Isometries on Banach Spaces, Proc. Amer. Math. Soc. 89 (1983), no. 4, 633–636

[5] R. Górák : Coarse version of the Banach-Stone theorem, Journal of Mathematical Analysis and Applications 377 (2011), 406–413

[6] N. Kalton: The nonlinear geometry of Banach spaces, Rev. Mat. Complut. 21 (2008), no. 1, 7-60

[7] J. Lindenstrauss and A. Szankowski : Non-linear perturbations of isometries, Astérisque 131 (1985), 357-371

[8] T. M. Rassias : Isometries and Approximate Isometries, Int. J. Math. Math. Sci. 25 (2001), no. 2, 73–91

[9] K. Jarosz : Nonlinera generalization of Banach-Stone theorem, Studia Math. 93 (1989), no. 2, 97–107

[10] J. Väisälä : A proof of the Mazur-Ulam theorem, Amer. Math. Monthly 110 (2003), no. 7, 633-635

[11] I. Vestfrid : Affine properties and injectivity of quasi isometries, Isr. J. Math. 141 (2004), 185–210

**Rafał Górák**, Technical University of Warsaw, Pl. Politechniki 1, 00-661 Warszawa, Poland