NOTE ON IMPROVED BOHR INEQUALITY FOR HARMONIC MAPPINGS

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Abstract. In this paper, we give a new generalization of the Bohr inequality in refined form both for bounded analytic functions, and for sense-preserving harmonic functions with analytic part being bounded.

1. Introduction

Let \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) denote the open unit disk and \( H_\infty \) denote the class of all bounded analytic functions \( f \) on the unit disk \( \mathbb{D} \) with the supremum norm \( \| f \|_\infty := \sup_{z \in \mathbb{D}} |f(z)| \). Also, let

\[
\mathcal{B} = \{ f \in H_\infty : \| f \|_\infty \leq 1 \} \quad \text{and} \quad \mathcal{B}_0 = \{ \omega \in \mathcal{B} : \omega(0) = 0 \}.
\]

Note that if \( |f(z)| = 1 \) for some \( z \in \partial \mathbb{D} \), then, by the maximum principle, it follows that \( f \) should be unimodular constant functions. So, one can conveniently exclude constant functions. In 1914, the following theorem was proved by Harald Bohr \[4\] for \( 0 \leq r \leq 1/6 \) and then, it was improved to \( 0 \leq r \leq 1/3 \) independently by M. Riesz, I. Schur and F. W. Wiener.

**Theorem A.** (Bohr, 1914) If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B} \), then

\[
\sum_{n=0}^{\infty} |a_n| r^n \leq 1
\]

holds for \( 0 \leq r \leq 1/3 \). The number \( 1/3 \) is optimal: for \( 1/3 < r < 1 \), there exists a function \( f_0(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{B} \) such that \( \sum_{n=0}^{\infty} |b_n| r^n > 1 \), namely, \( f_0(z) = \frac{a - z}{1 - az} \) with \( a \in (0, 1) \) is close to 1 from left.

The number \( 1/3 \) is usually referred as Bohr radius for the family \( \mathcal{B} \). Few other proofs of Bohr’s theorem are known in the literature. See for instance, Paulsen, et al. \[19\], Sidon \[21\] and Tomic \[22\]. Tomic used analogue of the argument of Sidon. Another elementary proof of this result, which uses a result of L. Fejér \[7\], can also be found in the paper of Landau \[16\]. For further details, we refer to the survey articles \[1, 8\] and the references therein. In the case of functions in \( \mathcal{B}_0 \), the optimal value of Bohr radius is \( 1/\sqrt{2} \) which was obtained by Bombieri \[5\], where one can obtain several deep results in...
this article (see also [6]). For some refinements and investigations on Bohr, we refer to
the recent articles (cf. [11, 13]).

Let \( \mathcal{F} \) consist of sequences \( \varphi = \{ \varphi_n(r) \}_{n=0}^{\infty} \) of nonnegative continuous functions in \([0, 1)\) such that the series \( \sum_{n=0}^{\infty} \varphi_n(r) \) converges locally uniformly with respect to \( r \in [0, 1) \). Also, for \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B} \) and \( f_0(z) := f(z) - f(0) \), we let for convenience

\[
\Phi_N(r) := \sum_{n=N}^{\infty} \varphi_n(r) \quad \text{and} \quad B_N(f, \varphi, r) := \sum_{n=N}^{\infty} |a_n| \varphi_n(r) \quad \text{for} \quad N \geq 0,
\]

so that \( B_0(f, \varphi, r) = |a_0| \varphi_0(r) + B_1(f, \varphi, r) \). In addition, we also let

\[
A(f_0, \varphi, r) := \sum_{n=1}^{\infty} |a_n|^2 \left( \frac{\varphi_{2n}(r)}{1 + |a_0|} + \Phi_{2n+1}(r) \right) \quad \text{and} \quad \|f_0\|^2 = \sum_{n=1}^{\infty} |a_n|^2 r^{2n}.
\]

In particular, when \( \varphi_n(r) = r^n \), we let \( B_N(f, \varphi, r) = B_N(f, r) \) and observe that the formula for \( A(f_0, \varphi, r) \) takes the following simple form

\[
A(f_0, r) := \left( \frac{1}{1 + |a_0|} + \frac{r}{1 - r} \right) \|f_0\|^2,
\]

since \( \Phi_{2n+1}(r) = r^{2n+1}/(1 - r) \).

Based on the recent investigation on this topic [9], the following two results were proved in [20].

**Theorem B.** ([20] Proof of Theorem 1]) Suppose \( f \in \mathcal{B} \), \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( \{ \varphi_n(r) \}_{n=1}^{\infty} \) belongs to \( \mathcal{F} \). Then we have the following inequality:

\[
B_1(f, \varphi, r) + A(f_0, \varphi, r) \leq (1 - |a_0|^2) \Phi_1(r) \quad \text{for} \quad r \in [0, 1).
\]

**Lemma C.** Let \( \{ \psi_n(r) \}_{n=1}^{\infty} \) be a decreasing sequence of nonnegative functions in \([0, r_\psi)\), and \( g, h \) be analytic in \( \mathbb{D} \) such that \( |g'(z)| \leq k|h'(z)| \) in \( \mathbb{D} \) and for some \( k \in [0, 1] \), where \( h(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \). Then

\[
\sum_{n=1}^{\infty} |b_n|^2 \psi_n(r) \leq k^2 \sum_{n=1}^{\infty} |a_n|^2 \psi_n(r) \quad \text{for} \quad r \in [0, r_\psi).
\]

**Proof.** Note that the condition \( |g'(z)| \leq k|h'(z)| \) for \( z \in \mathbb{D} \) may be rewritten in terms of subordination as \( g'(z) \prec kh'(z) \) for \( z \in \mathbb{D} \). Now apply [20] Theorem 4] (see also [20] Proof of Theorem 5)) to obtain the desired conclusion. \( \square \)

In this paper, we prove two general results which in particular yield a number of recently known results as special choices of \( \varphi_n(r) \)'s.
2. Main Results

Theorem 1. Suppose that $f(z) = h(z) + g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n$ is harmonic mapping of the disk $\mathbb{D}$ such that $|g'(z)| \leq k|h'(z)|$ in $\mathbb{D}$ and for some $k \in [0, 1]$. Assume that $\{\varphi_n(r)\}_{n=1}^{\infty}$ is a decreasing sequence of functions from $\mathcal{F}$, $\Phi_1(r) = \sum_{n=1}^{\infty} \varphi_n(r)$ and $p \in (0, 2]$. If

$$1 > \frac{2}{p} \left( \frac{1 + r^m}{1 - r^m} \right) (1 + k)\Phi_1(r),$$

then the following sharp inequality holds:

$$|h(z^m)|^p + B_1(h, \varphi, r) + B_1(g, \varphi, r) + A(h_0, \varphi, r) \leq 1 \quad \text{for} \quad r \leq R_{m,p}^k,$$

where $R_{m,p}^k$ is the minimal positive root of the equation

$$1 = \frac{2}{p} \left( \frac{1 + x^m}{1 - x^m} \right) (1 + k)\Phi_1(x).$$

In the case when $1 < \frac{2}{p} \left( \frac{1 + x^m}{1 - x^m} \right) (1 + k)\Phi_1(x)$ in some interval $(R_{m,p}^k, R_{m,p}^k + \epsilon)$, the number $R_{m,p}^k$ cannot be improved.

Proof. From the classical Schwarz-Pick lemma and Theorem B, it follows easily that

$$|h(z^m)|^p + B_1(h, \varphi, r) + A(h_0, \varphi, r) \leq \left( \frac{r^m + a}{1 + r^m a} \right)^p + (1 - a^2)\Phi_1(r),$$

where $a = |h(0)| \in [0, 1)$. For $h \in \mathcal{B}$, as an application of Schwarz-Pick lemma, we have the inequality $|a_n| \leq 1 - a^2$ for $n \geq 1$. By assumption $|g'(z)| \leq k|h'(z)|$ in $\mathbb{D}$, where $k \in [0, 1]$ and so, by Lemma C, it follows that

$$\sum_{n=1}^{\infty} |b_n|^2 \varphi_n(r) \leq k^2 \sum_{n=1}^{\infty} |a_n|^2 \varphi_n(r) \leq k^2 (1 - a^2)^2 \Phi_1(r).$$

Consequently, it follows from the classical Schwarz inequality that

$$\sum_{n=1}^{\infty} |b_n| \varphi_n(r) \leq \sqrt{\sum_{n=1}^{\infty} |b_n|^2 \varphi_n(r) \sqrt{\Phi_1(r)}} \leq k(1 - a^2)\Phi_1(r).$$

Thus, we have

$$|h(z^m)|^p + B_1(h, \varphi, r) + B_1(g, \varphi, r) + A(h_0, \varphi, r) \leq \left( \frac{r^m + a}{1 + r^m a} \right)^p + (1 - a^2)(1 + k)\Phi_1(r)$$

$$= 1 - \Psi_p(a)$$

(6)

where

$$\Psi_p(a) = 1 - (1 - a^2)(1 + k)\Phi_1(r) - \left( \frac{r^m + a}{1 + r^m a} \right)^p, \quad a \in [0, 1].$$

Now, as in the proof of Lemma 3.1 in [17], we need to determine conditions such that $\Psi_p(a) \geq 0$ for all $a \in [0, 1]$. Note that $\Psi_p(1) = 0$. We claim that $\Psi_p$ is a decreasing function.
of \( a \), under the conditions of the theorem. A direct computation shows that

\[
\Psi'_p(a) = 2a(1 + k)\Phi_1(r) - p(1 - r^{2m})\frac{(r^m + a)^{p-1}}{(1 + r^m)^{p+1}}
\]

and

\[
\Psi''_p(a) = 2(1 + k)\Phi_1(r) - p(1 - r^{2m})\frac{(r^m + a)^{p-2}}{(1 + r^m)^{p+2}}[p - 1 - 2ar^m - (p + 1)r^{2m}].
\]

Evidently, \( \Psi''_p(a) \geq 0 \) for all \( a \in [0, 1] \), whenever \( 0 < p \leq 1 \). Hence for \( r \leq R_{m,p}^k \),

\[
\Psi'_p(a) \leq \Psi'_p(1) = 2(1 + k)\Phi_1(r) - p\left(\frac{1 - r^m}{1 + r^m}\right) \leq 0,
\]

by the assumption that (3) holds. Thus, for each \( r \leq R_{m,p}^k \) and \( 0 < p \leq 1 \), \( \Psi_p(a) \) is a decreasing function of \( a \in [0, 1] \), which in turn implies that \( \Psi_p(a) \geq \Psi_p(1) = 0 \) for all \( a \in [0, 1] \) and the desired inequality follows from (6). Next, we show that condition \( \Psi'_p(1) \leq 0 \) is also sufficient for the function \( \Psi_p(a) \) to be decreasing on \( [0, 1] \) in the case when \( 1 < p \leq 2 \). From the proof of Lemma 3.1 in [17] it was known that \( \Phi(\sqrt{r}) \geq a^{p-1} \) for all \( r \in [0, 1) \).

In view of the above discussion, for \( r \leq R_{m,p}^k \), we find that

\[
\Psi'_p(a) = 2a(1 + k)\Phi_1(r) - p\left(\frac{1 - r^m}{1 + r^m}\right)\Phi(r)
\]

\[
\leq a^{p-1}\left[2a^{2-p}(1 + k)\Phi_1(r) - p\left(\frac{1 - r^m}{1 + r^m}\right)\right]
\]

\[
\leq a^{p-1}\left[2(1 + k)\Phi_1(r) - p\left(\frac{1 - r^m}{1 + r^m}\right)\right] = a^{p-1}\Psi'_p(1) \leq 0,
\]

since \( 0 \leq a^{2-p} \leq 1 \) for \( 1 < p \leq 2 \). Again, \( \Psi_p(a) \) is a decreasing on \( [0, 1] \), whenever \( 1 < p \leq 2 \) which implies that \( \Psi_p(a) \geq \Psi_p(1) = 0 \) for all \( a \in [0, 1] \) and thus, the desired inequality (4) holds.

Now let us prove that \( R_{m,p}^k \) is an optimal number, we consider the function

\[
h(z) = \frac{a - z}{1 - az} = a - (1 - a^2)\sum_{n=1}^{\infty}a^{n-1}z^n, \quad a \in [0, 1).
\]

and \( g(z) = \lambda kh(z) \), where \( |\lambda| = 1 \). Also, let \( h_0(z) = h(z) - h(0) \). Then it is a simple exercise to see that
\[
|h(-r^m)|^p + B_1(h, \varphi, r) + B_1(g, \varphi, r) + A(h_0, \varphi, r)
= \left( \frac{r^m + a}{1 + r^m a} \right)^p + (1 - a^2)(1 + k) \sum_{n=1}^{\infty} a^{n-1} \varphi_n(r) + (1 - a^2) \sum_{n=1}^{\infty} a^{2n-2} \left[ \frac{\varphi_{2n}(r)}{1 + a} + \Phi_{2n+1}(r) \right] 
= 1 + (1 - a) Q_p(a, r) + O((1 - a)^2),
\]
where
\[
Q_p(a, r) = (1 + a)(1 + k) \sum_{n=1}^{\infty} a^{n-1} \varphi_n(r) - \frac{1}{1 - a} \left( 1 - \left( \frac{r^m + a}{1 + r^m a} \right)^p \right),
\]
and it is easy to see that the last expression on the left is bigger than or equal to 1 if \(Q_p(a, r) \geq 0\). In fact, for \(r > R_{m,p}^k\) and \(a\) close to 1, we see that
\[
\lim_{a \to 1^-} Q_p(a, r) = \left[ 2(1 + k) \Phi_1(r) - p \left( \frac{1 - r^m}{1 + r^m} \right) \right] > 0,
\]
showing that the number \(R_{m,p}^k\) in (5) is best possible.

The most fundamental special case is perhaps \(\varphi_n(r) = r^n (n \geq 1)\) and \(k = 0\). In this case, Theorem 1 gives the following.

Corollary 1. (\cite{17 Lemma 3.3}) Suppose that \(f \in B\) and \(f(z) = \sum_{n=0}^{\infty} a_n z^n\) with \(a = |f(0)|\) and \(f_0(z) = f(z) - f(0)\). Then for \(p \in (0, 2]\), we have the sharp inequality:
\[
|h(0)|^p + B_1(f, r) + A(f_0, r) \leq 1 \quad \text{for } r \leq R_p = \sqrt{4p + 1 + p + 1},
\]
where \(A(f_0, r)\) is defined by (2). The radius \(R_p\) is best possible.

Remark 1. We mention now few other simple special cases.

1. For \(\varphi_n(r) = r^n (n \geq 1)\), Theorem 1 gives a refinement of \cite[Theorem 5]{3}. In this special case, if we allow \(m \to \infty\), then the resulting inequality (4) takes the form
\[
|h(0)|^p + \sum_{n=1}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n + A(h_0, r) \leq 1
\]
for \(r \leq R_p^k = \frac{p}{2(1 + k + p)}\). The constant \(R_p^k\) cannot be improved. In particular, \(p = 1, 2\) in (7) with the substitution \(k = \frac{K - 1}{K + 1}\), the inequality yields the following refined forms of two well-known results, namely, \cite[Theorem 1.1]{15} and \cite[Theorem 1.2]{15}, respectively:

(a) \(|h(0)| + \sum_{n=1}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n + A(h_0, r) \leq 1 \quad \text{for } r \leq \frac{K + 1}{5K + 1}\). The constant \(\frac{K + 1}{5K + 1}\) is sharp.
(b) $|h(0)|^2 + \sum_{n=1}^{\infty} |a_n|r^n + \sum_{n=1}^{\infty} |b_n|r^n + A(h_0,r) \leq 1$ for $r \leq \frac{K+1}{3K+1}$. The constant $\frac{K+1}{3K+1}$ is sharp.

(2) The case $k = 0$ of (7) gives the refined form of the classical Bohr’s inequality and the case $k = 1$, contains the refined Bohr inequality for sense-preserving harmonic mappings $f(z) = h(z) + g(z)$ of the disk $\mathbb{D}$ (see [15, Corollary 1.4]).

In the case of analytic functions, one can easily get the following result and, since the proof of it follows on the similar lines of the proof of Theorem 1, we omit the details.

**Theorem 2.** Let $f \in \mathcal{B}$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $0 < p \leq 2$ and $\{\varphi_n(r)\}_{n=0}^{\infty}$ belongs to $\mathcal{F}$. If

$$1 > \frac{2}{p} \frac{(1 + r^m)}{(1 - r^m)} \Phi_N(r),$$

for some $N \geq 1$ then the following sharp inequality holds:

$$|f(z^m)|^p + B_N(f, \varphi, r) \leq 1 \text{ for all } r \leq \rho^N_{m,p},$$

where $\rho^N_{m,p}$ is the minimal positive root of the equation

$$1 = \frac{2}{p} \frac{(1 + x^m)}{(1 - x^m)} \Phi_N(x).$$

In the case when $1 < \frac{2}{p} \frac{(1 + x^m)}{(1 - x^m)} \Phi_N(x)$ in some interval $(\rho^N_{m,p}, \rho^N_{m,p} + \epsilon)$, the number $\rho^N_{m,p}$ cannot be improved.

Note that for $N = 1$, the number $\rho^N_{m,p}$ coincides with the number $R^0_{m,p}$ of Theorem 1.

**Remark 2.** We mention now several useful remarks concerning some special cases of Theorem 2 for $f \in \mathcal{B}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

(1) For $\varphi_n(r) = r^n (n \geq 1)$, we easily have the following sharp inequality

$$|f(z^m)|^p + \sum_{n=N}^{\infty} |a_n|r^n \leq 1 \text{ for all } r \leq \rho^N_{m,p},$$

where $0 < p \leq 2$ and $\rho^N_{m,p}$ is the minimal positive root of the equation

$$2(1 + r^m)r^N - p(1 - r)(1 - r^m) = 0.$$

In particular, the case $m = 1$ and $p = 1, 2$ in (8) yields the well-known result (see [10, Theorem 1]).

(2) For $p = 1$ and $\varphi_n(r) = r^n (n \geq 1)$, we obtain the result [10, Theorem 2].

(3) For $N = 1$, and $\varphi_n(r) = r^n (n \geq 1)$, if we allow $m \to \infty$ in Theorem 2 we obtain that $\rho^1_{m,p} \to \frac{p}{2 + p}$. Thus, we easily have the following inequality which contains the classical Bohr inequality (i.e., the case $p = 1$):

$$|a_0|^p + \sum_{n=1}^{\infty} |a_n|r^n \leq 1 \text{ for } r \leq R(p) = \frac{p}{2 + p}$$

and the constant $\frac{p}{2 + p}$ cannot be improved.
(4) For the case $m = 1 = p$, $\varphi_{2n}(r) = r^{2n} \ (n \geq 1)$ and $\varphi_{2n-1} = 0 \ (n \geq 1)$, we easily have

$$|f(z)| + \sum_{n=1}^{\infty} |a_{2n}| r^{2n} \leq 1 \text{ for } r \leq R = \sqrt{2} - 1$$

and the radius $R = \sqrt{2} - 1$ is the best possible. This was obtained in [18, Lemma 2.8].

(5) If we allow $m \to \infty$, with $\varphi_{kn}(r) = r^{kn} \ (n \geq 1)$ for each fixed $k \geq 1$ and $\varphi_m = 0$ for $m \neq kn$, we obtain the following the sharp inequality

$$|a_0|^p + \sum_{n=1}^{\infty} |a_{kn}| r^{kn} \leq 1 \text{ for } r \leq k^{p \over 2 + p}.$$ 

For $p = 1$, this gives [2, Lemma 2.1]. For $p = 2$ and $k = 1$, this is a well-known result (cf. [19, Corollary 2.7]).

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