ON THE POSSIBLE EXCEPTIONS FOR THE TRANSCENDENCE OF THE LOG-GAMMA FUNCTION AT RATIONAL ENTRIES

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Abstract. In a very recent work [JNT 129, 2154 (2009)], Gun and co-workers have claimed that the number \( \log \Gamma(x) + \log \Gamma(1 - x) \), \( x \) being a rational number between 0 and 1, is transcendental with at most one possible exception, but the proof presented there in that work is incorrect. Here in this paper, I point out the mistake they committed and I present a theorem that establishes the transcendence of those numbers with at most two possible exceptions. As a consequence, I make use of the reflection property of this function to establish a criteria for the transcendance of \( \log \pi \), a number whose irrationality is not proved yet. I also show that each pair \( \{ \log \left( \pi / \sin(\pi x) \right), \log \left( \pi / \sin(\pi y) \right) \} \), \( x \) and \( y \) being rational numbers between 0 and 1, contains at least one transcendental number. This has an interesting consequence for the transcendence of the product \( \pi \cdot e \), another number whose irrationality is not proved.

1. Introduction

The gamma function, defined as \( \Gamma(x) := \int_0^\infty t^{x-1} e^{-t} \, dt \), \( x > 0 \), has attracted much interest since its introduction by Euler, appearing frequently in both mathematics and natural sciences problems. The transcendental nature of this function at rational values of \( x \) in the open interval \((0, 1)\), to which we shall restrict our attention hereafter, is enigmatic, just a few special values having their transcendence established. Such special values are: \( \Gamma(\frac{1}{2}) = \sqrt{\pi} \), whose transcendence follows from the Lindemann’s proof that \( \pi \) is transcendental (1882) \[1\], \( \Gamma(\frac{1}{4}) \), as shown by Chudnovsky (1976) \[2\], \( \Gamma(\frac{1}{3}) \), as proved by Le Lionnais (1983) \[3\], and \( \Gamma(\frac{1}{5}) \), as can be deduced from a theorem of Schneider (1941) on the transcendence of the beta function at rational entries \[4\]. The most recent result in this line was obtained by Grinspan (2002), who showed that at least two of the numbers \( \Gamma(\frac{1}{5}), \Gamma(\frac{2}{5}) \) and \( \pi \) are algebraically independent \[5\]. For other rational values of \( x \) in the interval \((0, 1)\), not even irrationality was established for \( \Gamma(x) \).
The function \( \log \Gamma(x) \), known as the log-gamma function, on the other hand, received less attention with respect to the transcendence at rational points. In a recent work, however, Gun, Murty and Rath (GMR) have presented a “theorem” asserting that \[6\]:

**Conjecture 1.** The number \( \log \Gamma(x) + \log \Gamma(1-x) \) is transcendental for any rational value of \( x \), \( 0 < x < 1 \), with at most one possible exception.

This has some interesting consequences. For a better discussion of these consequences, let us define a function \( f : (0, 1) \rightarrow \mathbb{R}_+ \) as follows:

\[
(1.1) \quad f(x) := \log \Gamma(x) + \log \Gamma(1-x).
\]

Note that \( f(1-x) = f(x) \), which implies that \( f(x) \) is symmetric with respect to \( x = \frac{1}{2} \). By taking into account the well-known reflection property of the gamma function

\[
(1.2) \quad \Gamma(x) \cdot \Gamma(1-x) = \frac{\pi}{\sin(\pi x)},
\]

valid for all \( x \notin \mathbb{Z} \), and being \( \log [\Gamma(x) \cdot \Gamma(1-x)] = \log \Gamma(x) + \log \Gamma(1-x) \), one easily deduces that

\[
(1.3) \quad f(x) = \log \left[ \frac{\pi}{\sin(\pi x)} \right] = \log \pi - \log \sin(\pi x).
\]

From this logarithmic expression, one promptly deduces that \( f(x) \) is differentiable (hence continuous) in the interval \( (0, 1) \), its derivative being \( f'(x) = -\pi \cot(\pi x) \). The symmetry of \( f(x) \) around \( x = \frac{1}{2} \) can be taken into account for proofing that, being Conjec. 1 true, the only exception (if there is one) has to take place for \( x = \frac{1}{2} \) (see the Appendix). From Eq. (1.3), we promptly deduce that \( \log \pi - \log \sin(\pi x) \) is transcendental for all rational \( x \) in \( (0, 1) \), the only possible exception being \( f(\frac{1}{2}) = \log \pi = 1.1447298858\ldots \)

All these consequences would be impressive, but the proof presented in Ref. [6] for Conjec. 1 is incorrect. This is because those authors implicitly assume that \( f(x_1) \neq f(x_2) \) for every pair of distinct rational numbers \( x_1, x_2 \) in \( (0, 1) \), which is not true, as may be seen in Fig. 1 where the symmetry of \( f(x) \) around \( x = \frac{1}{2} \) can be appreciated. To be explicit, let me exhibit a simple counterexample: for the pair \( x_1 = \frac{1}{4} \) and \( x_2 = \frac{3}{4} \), Eq. (1.3) yields \( f(x_1) = f(x_2) = \log \pi + \log \sqrt{2} \) and then \( f(x_1) - f(x_2) = 0 \). This null result clearly makes it invalid their conclusion that \( f(x_1) - f(x_2) \) is a non-null Baker period.

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1 This is an interesting number whose irrationality is not yet established.
2 In fact, null results are found for every pair of rational numbers \( x_1, x_2 \in (0, 1) \) with \( x_1 + x_2 = 1 \) (i.e., symmetric with respect to \( x = \frac{1}{2} \)).
Here in this short paper, I take Conjec. on the transcendence of \( f(x) = \log \Gamma(x) + \log \Gamma(1 - x) \) into account for setting up a theorem establishing that there are at most two possible exceptions for the transcendence of \( f(x) \), \( x \) being a rational in \((0, 1)\). This theorem is proved here based upon a careful analysis of the monotonicity of \( f(x) \), taking also into account its obvious symmetry with respect to \( x = \frac{1}{2} \). Interestingly, this yields a criteria for the transcendence of \( \log \pi \), an important number in the study of the algebraic nature of special values of a general class of \( L \)-functions \([7]\). This reformulation of the GMR “theorem” allows us to exhibit an infinity of pairs of logarithms of certain algebraic multiples of \( \pi \) whose elements are not both algebraic.

2. Transcendence of \( \log \Gamma(x) + \log \Gamma(1 - x) \) and Exceptions

For simplicity, let us define \( \mathbb{Q}_{(0, 1)} \) as \( \mathbb{Q} \cap (0, 1) \), i.e. the set of all rational numbers in the real open interval \((0, 1)\), which is a countable infinite set. My theorem on the transcendence of \( \log \Gamma(x) + \log \Gamma(1 - x) \) depends upon the fundamental theorem of Baker (1966) on the transcendence of linear forms in logarithms. We record this as:

**Lemma 2.1** (Baker). Let \( \alpha_1, \ldots, \alpha_n \) be nonzero algebraic numbers and \( \beta_1, \ldots, \beta_n \) be algebraic numbers. Then the number

\[
\beta_1 \log \alpha_1 + \ldots + \beta_n \log \alpha_n
\]

is either zero or transcendental. The latter case arises if \( \log \alpha_1, \ldots, \log \alpha_n \) are linearly independent over \( \mathbb{Q} \) and \( \beta_1, \ldots, \beta_n \) are not all zero.

**Proof.** See theorems 2.1 and 2.2 of Ref. \[8\]. \( \square \)

Now, let us define a Baker period according to Refs. \[9, 10\].

**Definition 2.2** (Baker period). A Baker period is any linear combination in the form \( \beta_1 \log \alpha_1 + \ldots + \beta_n \log \alpha_n \), with \( \alpha_1, \ldots, \alpha_n \) nonzero algebraic numbers and \( \beta_1, \ldots, \beta_n \) algebraic numbers.

From Baker’s theorem, it follows that

**Corollary 2.3.** Any non-null Baker period is necessarily a transcendental number.

Now, let us demonstrate the following theorem, which comprises the main result of this paper.

**Theorem 2.4** (Main result). The number \( \log \Gamma(x) + \log \Gamma(1 - x) \) is transcendental for all \( x \in \mathbb{Q}_{(0, 1)} \), with at most two possible exceptions.
Proof. Let \( f(x) \) be the function defined in Eq. (1.1). From Eq. (1.3), \( f(x) = \log \pi - \log \sin (\pi x) \) for all real \( x \in (0, 1) \). Let us divide the open interval \((0, 1)\) into two adjacent subintervals by doing \((0, 1) \equiv (0, \frac{1}{2}] \cup [\frac{1}{2}, 1)\). Note that \( \sin (\pi x) \) — and thus \( f(x) \) — is either a monotonically increasing or decreasing function in each subinterval. Now, suppose that \( f(x_1) \) and \( f(x_2) \) are both algebraic numbers, for some pair of distinct real numbers \( x_1 \) and \( x_2 \) in \( (0, \frac{1}{2}] \). Then, the difference

\[
(2.1) \quad f(x_2) - f(x_1) = \log \sin (\pi x_1) - \log \sin (\pi x_2)
\]

will, itself, be an algebraic number. However, as the sine of any rational multiple of \( \pi \) is an algebraic number \([11, 12]\), then Lemma 2.1 guarantees that, being \( x_1, x_2 \in \mathbb{Q} \), then \( \log \sin (\pi x_1) - \log \sin (\pi x_2) \) is either null or transcendental. Since \( \sin (\pi x) \) is a continuous, monotonically increasing function in \((0, \frac{1}{2}]\), then \( \sin \pi x_1 \neq \sin \pi x_2 \) for all \( x_1 \neq x_2 \) in \((0, \frac{1}{2}]\). Therefore, \( \log \sin (\pi x_1) \neq \log \sin (\pi x_2) \) and then \( \log \sin (\pi x_1) - \log \sin (\pi x_2) \) is a non-null Baker period. From Corol. 2.3 we know that non-null Baker periods are transcendental numbers, which contradicts our initial assumption. Then, there is at most one exception for the transcendence of \( f(x) \), \( x \in \mathbb{Q} \cap (0, \frac{1}{2}] \).

Clearly, as \( \sin (\pi x) \) is a continuous and monotonically decreasing function for \( x \in [\frac{1}{2}, 1) \), an analogue assertion applies to this complementary subinterval, which yields another possible exception for the transcendence of \( f(x) \), \( x \in \mathbb{Q} \cap [\frac{1}{2}, 1) \). \( \square \)

It is most likely that not even one exception takes place for the transcendence of \( \log \Gamma(x) + \log \Gamma(1-x) \) with \( x \in \mathbb{Q}(0,1) \). If this is true, it can be deduced that \( \log \pi \) is transcendental. If there are exceptions, however, then their number — either one or two, according to Theorem 2.4 — will determine the transcendence of \( \log \pi \). The next theorem summarizes these connections between the existence of exceptions to the transcendence of \( f(x) \), \( x \in \mathbb{Q}(0,1) \), and the transcendence of \( \log \pi \).

**Theorem 2.5 (Exceptions).** With respect to the possible exceptions to the transcendence of \( \log \Gamma(x) + \log \Gamma(1-x) \), \( x \in \mathbb{Q}(0,1) \), exactly one of the following statements is true:

(i) There are no exceptions, hence \( \log \pi \) is a transcendental number;

(ii) There is only one exception and it has to be for \( x = \frac{1}{2} \), hence \( \log \pi \) is an algebraic number;

(iii) There are exactly two exceptions for some \( x \neq \frac{1}{2} \), hence \( \log \pi \) is a transcendental number.
Proof. If \( f(x) = \log \Gamma(x) + \log \Gamma(1 - x) \) is a transcendental number for every \( x \in \mathbb{Q}_{(0,1)} \), item(i), it suffices to put \( x = \frac{1}{2} \) in Eq. (1.3) for finding that \( f(\frac{1}{2}) = \log \pi \) is transcendental. If there is exactly one exception, item (ii), then it has to take place for \( x = \frac{1}{2} \), otherwise (i.e., for \( x \neq \frac{1}{2} \)) the symmetry property \( f(1 - x) = f(x) \) would yield algebraic values for two distinct arguments, namely \( x \) and \( 1 - x \). Therefore, \( f(\frac{1}{2}) = \log \pi \) is the only exception, thus it is an algebraic number. If there are two exceptions, item (iii), both for \( x \neq \frac{1}{2} \), then they have to be symmetric with respect to \( x = \frac{1}{2} \), otherwise, by the property \( f(1 - x) = f(x) \), we would find more than two exceptions, which is prohibited by Theorem 2.4. Indeed, if one of the two exceptions is for \( x = \frac{1}{2} \), then the other, for \( x \neq \frac{1}{2} \), would yield a third exception, corresponding to \( 1 - x \neq \frac{1}{2} \), which is again prohibited by Theorem 2.4. Then the two exceptions are for values of the argument distinct from \( \frac{1}{2} \) and then \( f(\frac{1}{2}) = \log \pi \) is a transcendental number. \( \square \)

From this theorem, it is straightforward to conclude that

Criteria 1 (Algebraicity of \( \log \pi \)). The number \( \log \pi \) is algebraic if and only if \( \log \Gamma(x) + \log \Gamma(1 - x) \) is a transcendental number for every \( x \in \mathbb{Q}_{(0,1)} \), except \( x = \frac{1}{2} \).

The symmetry of the possible exceptions for the transcendence of \( \log \Gamma(x) + \log \Gamma(1 - x) \) around \( x = \frac{1}{2} \) yields the following conclusion.

Corollary 2.6 (Pairs of logarithms). Every pair \( \{ \log [\pi / \sin(\pi x)], \log [\pi / \sin(\pi y)] \} \), with both \( x \) and \( y \) rational numbers in the interval \((0,1)\), \( y \neq 1 - x \), contains at least one transcendental number.

By fixing \( x = \frac{1}{2} \) in this corollary, one has

Corollary 2.7 (Pairs containing \( \log \pi \)). Every pair \( \{ \log \pi, \log [\pi / \sin(\pi y)] \} \), \( y \) being a rational in \((0,1)\), \( y \neq \frac{1}{2} \), contains at least one transcendental number.

3. The log-gamma function and the transcendence of \( \pi \cdot e \)

An interesting consequence of Corol. 2.7, together the famous Hermite-Lindemann (HL) theorem, is that the algebraicity of \( \log \Gamma(y) + \log \Gamma(1 - y) \) for some \( y \in \mathbb{Q}_{(0,1)} \) implies the transcendence of \( \pi \cdot e = 8.5397342226 \ldots \), another number whose irrationality is not established yet. Let me proof this assertion based upon a logarithmic version of the HL theorem.

Lemma 3.1 (HL). For any non-zero complex number \( w \), one at least of the two numbers \( w \) and \( \exp(w) \) is transcendental.
Proof. See Ref. [13] and references therein. □

Lemma 3.2 (HL, logarithmic version). For any positive real number \( z \), \( z \neq 1 \), one at least of the real numbers \( z \) and \( \log z \) is transcendental.

Proof. It is enough to put \( w = \log z \), \( z \) being a non-negative real number, in Lemma 3.1 and to exclude the singularity of \( \log z \) at \( z = 0 \). □

Theorem 3.3 (Transcendence of \( \pi e \)). If the number \( \log \Gamma(y) + \log \Gamma(1 - y) \) is algebraic for some \( y \in \mathbb{Q}_{(0,1)} \), then the number \( \pi \cdot e \) is transcendental.

Proof. Let us denote by \( \overline{\mathbb{Q}} \) the set of all algebraic numbers and by \( \overline{\mathbb{Q}}^* \) the set of all non-null algebraic numbers. First, note that \( k(y) := 1 / \sin(\pi y) \in \overline{\mathbb{Q}}^* \) for every \( y \in \mathbb{Q}_{(0,1)} \) and that, from Eq. (1.3), \( \log \Gamma(y) + \log \Gamma(1 - y) = \log [k(y) \pi] \). Now, note that if \( \log [k(y) \pi] \in \overline{\mathbb{Q}} \) for some \( y = \tilde{y} \), then \( 1 + \log [k(\tilde{y}) \pi] \) is also an algebraic number. Therefore, \( \log e + \log [k(\tilde{y}) \pi] = \log [k(\tilde{y}) \pi e] \in \overline{\mathbb{Q}} \) and, by Lemma 3.2, the number \( k(\tilde{y}) \pi e \) has to be either transcendental or 1. However, it cannot be equal to 1 because this would imply that \( k(y) = 1 / (\pi e) < 1 \), which is not possible because \( 0 < \sin(\pi y) \leq 1 \) implies that \( k(y) \geq 1 \). Therefore, the product \( k(\tilde{y}) \pi e \) is a transcendental number. Since \( k(\tilde{y}) \in \overline{\mathbb{Q}}^* \), then \( \pi \cdot e \) has to be transcendental. □

4. Summary

In this work, the transcendental nature of \( \log \Gamma(x) + \log \Gamma(1 - x) \) for rational values of \( x \) in the interval \((0,1)\) has been investigated. I have first shown that the proof presented in Ref. [6] for the assertion that \( \log \Gamma(x) + \log \Gamma(1 - x) \) is transcendental for any rational value of \( x \), \( 0 < x < 1 \), with at most one possible exception is incorrect. I then reformulate their conjecture, presenting and proofing a theorem that establishes the transcendence of \( \log \Gamma(x) + \log \Gamma(1 - x) \), \( x \) being a rational in \((0,1)\), with at most two possible exceptions. The careful analysis of the number of possible exceptions has yielded a criteria for the number \( \log \pi \) to be algebraic. I have also shown that each pair \( \{ \log [\pi / \sin(\pi x)], \log [\pi / \sin(\pi y)] \} \), \( x, y \in \mathbb{Q} \), \( y \neq 1 - x \), contains at least one transcendental number. This occurs, in particular, with the pair \( \{ \log \pi, \log [\pi / \sin(\pi y)] \} \), \( y \neq \frac{1}{2} \). At last, I have shown that if \( \log [\pi / \sin(\pi y)] \) is algebraic for some \( y \neq \frac{1}{2} \), then the product \( \pi \cdot e \) has to be transcendental.

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Appendix

Let us show that the assumption that Conjec. 1 is true — i.e., that $f(x) = \log \Gamma(x) + \log \Gamma(1-x)$ is transcendental with at most one possible exception, $x$ being a rational in $(0,1)$ — implies that if one exception exists then it has to be just $f(\frac{1}{2}) = \log \pi$.

The fact that $f(1-x) = f(x)$ for all $x \in (0,1)$ implies that, if the only exception would take place for some rational $x$ distinct from $\frac{1}{2}$, then automatically there would be another rational $1-x$, distinct from $x$, at which the function would also assume an algebraic value (in fact, the same value obtained for $x$). However, Conjec. 1 restricts the number of exceptions to at most one. Then, we have to conclude that if an exception exists, it has to be for $x = \frac{1}{2}$, where $f(x)$ evaluates to $\log \pi$. \hfill \Box

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Figure 1. The graph of the function \( f(x) = \log \Gamma(x) + \log \Gamma(1 - x) = \log \pi - \log [\sin (\pi x)] \) in the interval \((0, 1)\). Since \( f(1 - x) = f(x) \), the graph is symmetric with respect to \( x = \frac{1}{2} \). Note that, as \( 0 < \sin (\pi x) \leq 1 \) for all \( x \in (0, 1) \), then \( \log \sin (\pi x) \leq 0 \), and then \( f(x) \geq \log \pi \) and the minimum of \( f(x) \), \( x \) being in the interval \((0, 1)\), is attained just for \( x = \frac{1}{2} \), where \( f(x) \) evaluates to \( \log \pi \). The dashed lines highlight the coordinates of this point.