Noncommutative Hermitian structures were recently introduced in [66] as an algebraic framework for studying noncommutative complex geometry on quantum homogeneous spaces. In this paper, we introduce the notion of a compact quantum homogeneous Hermitian space which gives a natural set of compatibility conditions between covariant Hermitian structures and Woronowicz’s theory of compact quantum groups. Each such object admits a Hilbert space completion, which possesses a remarkably rich yet tractable structure. The spectral behaviour of the associated Dolbeault–Dirac operators is moulded by the complex geometry of the underlying calculus. In particular, twisting the Dolbeault–Dirac operators by a negative (anti-ample) line bundle is shown to give a Fredholm operator if and only if the top anti-holomorphic cohomology group is finite-dimensional. When this is so, the operator’s index coincides with the holomorphic Euler characteristic of the underlying noncommutative complex structure. Our motivating family of examples, the irreducible quantum flag manifolds $O_q(G/L)$ endowed with their Heckenberger–Kolb calculi, are presented in detail. The noncommutative Bott–Borel–Weil theorem [22] is used to produce a family of Dolbeault–Dirac Fredholm operators for each $O_q(G/L)$. Moreover, following the spectral calculations of [18], the Dolbeault–Dirac operator of quantum projective space is exhibited as a spectral triple in the sense of Connes.

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1. Introduction

Since the emergence of quantum groups in the 1980s, a central role in their presentation and development has been played by $C^*$-algebras. We mention in particular Woronowicz’s seminal notion of a compact quantum group [8]. There exists, however, a stark contrast in the development of the noncommutative topological and the noncommutative differential geometric aspects of the theory. In particular, for the Drinfeld–Jimbo quantum groups, their $C^*$-algebraic $K$-theory has long been known to be the same as for their classical counterparts [61]. By contrast, the unbounded formulation of $K$-homology, which is to say Connes and Moscovici’s theory of spectral triples, remains very poorly understood. Indeed, despite a large number of very important contributions over the last thirty years, there is still no consensus on how to construct a spectral triple for $O_q(SU_2)$, probably the most fundamental example of a quantum group. This can be understood as a consequence of the fact that, at the algebraic level, the construction of
$q$-deformed differential operators is an extremely challenging task. Ultimately this is due to fundamental differences between the quantum and classical cases, most notably the non-trivial braiding on the monoidal category of $U_q(\mathfrak{g})$-modules. The question of how to incorporate this braiding into any $q$-deformed geometry is at the heart of the matter. These difficulties aside, the prospect of reconciling quantum groups and spectral triples still holds great promise for their mutual enrichment. On one hand, it would provide powerful tools from unbounded $KK$-theory with which to study quantum groups. On the other hand, it would provide unbounded $KK$-theory with a large class of examples, of fundamental importance, with which to test and guide the future development of the subject.

There exists a long standing algebraic approach to constructing $q$-deformed differential operators for quantum groups based on the theory of covariant differential calculi. This has its origins in the work of Woronowicz \[87\], with steady advances made in the following decades by many others, most notably Majid \[5\]. As has become increasingly clear in recent years, this approach is particularly suited to the study of those quantum homogeneous spaces where the “worst of the noncommutativity has been quotiented out”. More precisely, differential calculi have seen major successes in the study of the quantum flag manifolds, quantum homogeneous spaces which $q$-deform the coordinate rings of the classical flag manifolds $G/L_S$. These quantum spaces are distinguished by being braided commutative algebra objects in the braided monoidal category of $U_q(\mathfrak{g})$-modules, and have a geometric structure much closer to the classical situation than quantum groups themselves. This is demonstrated by the existence of an essentially unique $q$-deformed de Rham complex for the irreducible quantum flag manifolds, as shown by Heckenberger and Kolb in their seminal series of papers \[34, 35, 36\]. This makes the quantum flag manifolds a far more tractable starting point than quantum groups for investigating $q$-deformed noncommutative geometry.

The classical flag manifolds exhaust the compact connected homogeneous Kähler manifolds \[76, Théorème 1\], providing us with a rich store of geometric structures to exploit. Motivated by this, the notion of a noncommutative Hermitian structure was introduced by the second author in \[66\] to provide a framework in which to study the noncommutative geometry of the quantum flag manifolds. Many of the fundamental results of Hermitian and Kähler geometry follow from the existence of such a structure: Lefschetz decomposition, the Kähler identities, and the proportionality of the Laplace operators. Moreover, in the quantum homogeneous space case, it provides powerful tools with which to study the cohomology of the calculus, tools such as Hodge decomposition, the hard Lefschetz theorem, and the refinement of de Rham cohomology by Dolbeault cohomology. The existence of a Kähler structure was verified for the Heckenberger–Kolb calculus of quantum projective space in \[66\]. This result later extended by Matassa \[53\] to every Heckenberger–Kolb calculus, for all but a finite number of values of $q$. Moreover, further examples are anticipated to arise in due course from more general classes of quantum flag manifolds. Indeed, Kähler structures have recently been discovered in the setting of holomorphic étale groupoids \[7\], promising a much wider domain of application than initially expected.

In this paper we build on this rich algebraic and geometric structure to produce a theory of unbounded differential operators acting on square integrable forms. We do so in the
novel framework of compact quantum homogeneous Hermitian spaces (CQH-Hermitian spaces) which detail a natural set of compatibility conditions between covariant Hermitian structures and Woronowicz’s theory of compact quantum groups. Every CQH-Hermitian space is shown to have a naturally associated Hilbert space completion. Moreover, much of the theory of Hermitian structures carries over to square integrable setting, giving almost-complex and Lefschetz decompositions, as well as bounded representations of $\mathfrak{sl}_2$ and $U_p(\mathfrak{sl}_2)$. The de Rham, holomorphic, and anti-holomorphic differentials also behave extremely well with respect to completion. All three Dirac operators $D_0$, $D_\partial$, and $D_d$ are seen to be essentially self-adjoint, giving access to powerful analytic machinery such as functional calculus.

The spectral and index theoretic properties of these operators are intimately connected with the curvature and cohomology of the underlying calculus. Moreover, they are highly amenable to applications of the concepts and structures of classical complex geometry. As shown in §9, twisting the anti-holomorphic Dolbeault–Dirac operator of a CQH-Kähler space by a negative (anti-ample) line bundle (or the holomorphic Dolbeault–Dirac operator by a positive (ample) line bundle) produces a Fredholm operator if and only if the top anti-holomorphic cohomology group (or the bottom holomorphic cohomology group) is finite-dimensional. Just as in the classical case, Hodge theory then implies that the index of the twisted operator is given by the anti-holomorphic Euler characteristic of the twisted calculus. This invariant can be determined by geometric means. In particular, for positive line bundles, it follows from the Kodaira vanishing theorem for noncommutative Kähler structures that all higher cohomologies vanish, meaning that the index is concentrated in degree zero. (The case of negative line bundles follows similarly through an application of noncommutative Serre duality [67] §6.2.) In practical cases, such as for line bundles over the irreducible quantum flag manifolds $O_q(G/L_S)$, the cohomology groups can be explicitly determined. Indeed, as presented in §11, the irreducible quantum flag manifolds admit a direct noncommutative generalisation of the Borel–Weil–Bott theorem [22, 23], allowing us to construct a countable family of Dolbeault–Dirac Fredholm operators for each $O_q(G/L_S)$.

In order to produce an unbounded $K$-homology class, which is to say a spectral triple, the Dolbeault–Dirac operator $D_\partial$ needs to have compact resolvent, a significant strengthening of the Fredholm condition. Unlike the properties discussed above, this cannot at present be concluded for a general CQH-Hermitian space by geometric means. Hence we must resort to confirming it, in a case by case basis, through explicit calculation of the spectrum of $D_\partial$. In [18], which can be regarded as accompanying the present paper, the authors began the development of a robust framework in which to investigate the compact resolvent condition. This was done under the assumption of restricted multiplicities for the $U_q(\mathfrak{g})$-modules appearing in anti-holomorphic forms of a CQH-Hermitian space, an assumption that allows one to make strong statements about the spectral behaviour of $D_\partial$. The framework was applied to quantum projective space $O_q(\mathbb{C}P^n)$, the simplest family of quantum flag manifolds, allowing us to confirm the compact resolvent condition. Moreover, since each $O_q(\mathbb{C}P^n)$ is a noncommutative Fano space [10, 10], with consequent non-vanishing Euler characteristic, the associated $K$-homology class is necessarily non-trivial. Efforts to extend this result to all the irreducible quantum flag
manifolds are in progress, motivating Conjecture 10.18 below. For a detailed discussion of the next most approachable families of examples, see [18, §7].

To place our efforts in context, we briefly recall the previous constructions in the literature of \( q \)-deformed Dolbeault–Dirac operators for the quantum flag manifolds. (See [18] for a more detailed discussion.) The prototypical example of a spectral triple on a quantum flag manifold is the Dolbeault–Dirac spectral triple on the standard Podleś sphere as introduced by Owczarek [69] and Dąbrowski–Sitarz [20]. This operator was later rediscovered by Majid, at the algebraic level, as the Dolbeault–Dirac operator associated to the noncommutative complex structure of the Podleś sphere [50]. At around the same time as these works, Krähmer introduced an influential algebraic Dirac operator for the irreducible quantum flag manifolds, which gave a commutator realisation of their Heckenberger–Kolb calculi [44]. A series of papers by Dąbrowski, D’Andrea, and Landi, followed, where spectral triples were constructed for the all quantum projective spaces [17, 16]. Matassa would subsequently reconstruct these spectral triples [55] in a more formal manner by connecting with the work of Krähmer and Tucker–Simmons [45]. This approach was then extended to the quantum Lagrangian Grassmannian \( \mathcal{O}_q(L_2) \), a \( C \)-series irreducible quantum flag manifold [51, 55]. The precise relationship between these operators and those presented in §10 is at present unclear. However, given the rigidity of their \( U_q(g) \)-module structures, it is reasonable to expect that they can be understood within the framework of CQH-Hermitian structures. For non-irreducible quantum flags, the only example thus far examined is the full quantum flag manifold \( \mathcal{O}_q(SU_3/T^2) \). In [84] Yuncken and Voigt constructed Fredholm modules for \( \mathcal{O}_q(SU_3/T^2) \) using a quantum version of the BGG complex [73, 36]. As an application, the Baum–Connes conjecture with trivial coefficients was verified for the discrete quantum group dual to \( \mathcal{O}_q(SU_3) \).

Finally, we mention the alternative general approach to noncommutative Hermitian and Kähler geometry due to Fröhlich, Grandjean, and Recknagel [28, 29], as discussed in more detail in §7.4.

The paper is organised as follows: In §2 we recall from [66] the necessary basics of Hermitian and Kähler structures. In §3, we recall the foundations of the theory of compact quantum group algebras, and introduce the notion of a compact quantum homogeneous Hermitian space as a 4-tuple \( \mathcal{H} = (B = A^{co(H)}, \Omega^*, \Omega^{(*)}, \sigma) \), consisting of a quantum homogeneous space \( B = A^{co(H)} \), a differential calculus \( \Omega^* \) over \( B \), and a noncommutative Hermitian structure \( (\Omega^{(*)}, \sigma) \) for \( \Omega^* \).

In §4 we begin our examination of the Hilbert space completion of a CQH-Hermitian space. In particular, we use Takeuchi’s categorical equivalence to show boundedness of morphisms and multiplication operators. This gives us bounded representations of \( \mathfrak{sl}_2 \) and \( U_p(\mathfrak{sl}_2) \) on \( L^2(\Omega^*) \), and allows us to conclude boundedness of the commutators \([D\sigma, b]\), for all \( b \in B = A^{co(H)} \).

In §5 we treat the question of when the Dolbeault–Dirac operator \( D\sigma \) is Fredholm, observing that it is sufficient to prove closure of \( im(D\sigma) \) and finite-dimensional of anti-holomorphic cohomologies. When the operator is Fredholm, we show that its index is given by the anti-holomorphic Euler characteristic of the calculus, and discuss how this can be calculated in the Fano setting.
In §6 we recall the definition of a spectral triple, the basic object in Connes’ theory of noncommutative Riemannian manifolds [13]. We collect all relevant results in the previous sections, and show that a CQH-Hermitian space gives a spectral triple if and only if the point spectrum of $D_\partial$ has finite multiplicity and tends to infinity. We finish by observing non-triviality of the associated $K$-homology class in the Fano setting.

In §7 we discuss the relationship between opposite complex structures and CQH-Hermitian spaces. In particular, we prove that the opposite Dolbeault–Dirac operator $D_\partial$ is unitarily equivalent to $D_\partial$.

In §8, twists by Hermitian holomorphic vector bundles are considered, and we show that the results of §4 carry over directly to this more general setting. Moreover, we see that the Akizuki–Nakano identities can be used to imply a spectral gap for the twisted Dolbeault–Dirac operator $D_{\partial E}$. For the case of positive $E_k$, and negative line bundles $E_{-k}$, this gives us a means of verifying closure of the image of $D_{\partial E_k}$, and respectively $D_{\partial E_{-k}}$, and hence reducing the Fredholm condition to a question about finite dimensionality of bottom holomorphic cohomology groups, or top anti-holomorphic cohomology groups respectively.

In §10 we present our motivating family of examples, the irreducible quantum flag manifolds $O_q(G/L_S)$ endowed with their Heckenberger–Kolb calculi. We recall the covariant noncommutative Kähler structure of each $O_q(G/L_S)$, and show that it always gives a CQH-Kähler space. As an interesting application, we observe non-vanishing of the central Dolbeault cohomology groups, demonstrating that the Heckenberger–Kolb cohomology groups do not suffer from the dimension drop phenomenon occurring in cyclic cohomology.

In §11, we recall the Borel–Weil–Bott theorem for the irreducible quantum flag manifolds, and build upon it to construct a countable family of Dolbeault–Dirac Fredholm operators for each $O_q(G/L_S)$. Finally, the Dolbeault–Dirac operator for quantum projective space is exhibited as a spectral triple.

We finish with an appendix §A detailing the basics of unbounded operators on Hilbert spaces, so as to make the paper more accessible to those coming from an algebraic or geometric background.

Acknowledgements: The authors would like to thank Branimir Čačić, Elmar Wagner, Fredy Díaz García, Marco Matassa, Andrey Krutov, Karen Strung, Simon Brain, Bram Mesland, Adam Rennie, Bob Yuncken, Paolo Saracco, Kenny De Commer, and Matthias Fischmann, Adam–Christiaan van Roosmalen, Jan Šťovíček, and Zhaoting Wei, for many useful discussions during the preparation of this paper. The second author would like to thank IMPAN Wrocław for hosting him in November 2017, and would also like to thank Klaas Landsman and the Institute for Mathematics, Astrophysics and Particle Physics, Radboud University, Nijmegen for hosting him in the winter of 2017 and 2018.

2. Preliminaries on Hermitian Structures

We recall the basic definitions and results for differential calculi, as well as complex, Hermitian, and Kähler structures. For a more detailed introduction see [65], [66], and
2.1. Differential Calculi. A *differential calculus* is a dg-algebra (differential graded algebra) \((\Omega^* \simeq \bigoplus_{k \in \mathbb{N}_0} \Omega^k, d)\) which is generated in degree 0 as a dg-algebra, that is to say, it is generated as an algebra by the elements \(a, db\), for \(a, b \in \Omega^0\). For a given algebra \(B\), a differential calculus over \(B\) is a differential calculus such that \(B = \Omega^0\). A differential calculus is said to be of total degree \(m \in \mathbb{N}\) if \(\Omega^m \neq 0\), and \(\Omega^k = 0\), for all \(k > m\). A differential \(*\)-calculus over a \(*\)-algebra \(B\) is a differential calculus over \(B\) such that the \(*\)-map of \(B\) extends to a (necessarily unique) conjugate linear involutive map \(* : \Omega^* \to \Omega^*\) satisfying \(d(\omega^*) = (d\omega)^*\), and \((\omega \wedge \nu)^* = (-1)^{kl} \nu^* \wedge \omega^*\), for all \(\omega \in \Omega^k\), \(\nu \in \Omega^l\).

2.2. Complex Structures. In this subsection we present the definition of a complex structure, an abstraction of the properties of the de Rham complex of a classical complex manifold.

**Definition 2.1.** An *almost complex structure* \(\Omega^{(\bullet, \bullet)}\), for a differential \(*\)-calculus \((\Omega^*, d)\), is an \(\mathbb{N}_0^2\)-algebra grading \(\bigoplus_{(a,b) \in \mathbb{N}_0^2} \Omega^{(a,b)}\) for \(\Omega^\bullet\) such that, for all \((a, b) \in \mathbb{N}_0^2\):

1. \(\Omega^k = \bigoplus_{a+b=k} \Omega^{(a,b)}\);

2. \(*(\Omega^{(a,b)}) = \Omega^{(b,a)}\).

A complex structure is an almost complex which satisfies

\[
(1) \quad d\Omega^{(a,b)} \subseteq \Omega^{(a+1,b)} \oplus \Omega^{(a,b+1)}, \quad \text{for all} \quad (a, b) \in \mathbb{N}_0.
\]

We call an element of \(\Omega^{(a,b)}\) an \((a, b)\)-form. For \(\text{proj}_{\Omega^{(a+1,b)}}\), and \(\text{proj}_{\Omega^{(a,b+1)}}\), the projections from \(\Omega^{a+b+1}\) onto \(\Omega^{(a+1,b)}\), and \(\Omega^{(a,b+1)}\) respectively, we denote

\[
\overline{\partial}|_{\Omega^{(a,b)}} := \text{proj}_{\Omega^{(a+1,b)}} \circ d, \quad \overline{\partial}|_{\Omega^{(a,b)}} := \text{proj}_{\Omega^{(a,b+1)}} \circ d.
\]

For a complex structure, (1) implies the identities

\[
d = \partial + \overline{\partial}, \quad \overline{\partial} \circ \partial = -\partial \circ \overline{\partial}, \quad \partial^2 = \overline{\partial}^2 = 0.
\]

Thus \((\bigoplus_{(a,b) \in \mathbb{N}_0^2} \Omega^{(a,b)}, \partial, \overline{\partial})\) is a double complex, which we call the Dolbeault double complex of \(\Omega^{(\bullet, \bullet)}\). Moreover, it is easily seen that both \(\partial\) and \(\overline{\partial}\) satisfy the graded Leibniz rule, and that

\[
(2) \quad \partial(\omega^*) = (\overline{\partial}\omega)^*, \quad \overline{\partial}(\omega^*) = (\partial\omega)^*, \quad \text{for all} \quad \omega \in \Omega^*.
\]

These facts can be succinctly expressed by saying that \((\bigoplus_{(a,b) \in \mathbb{N}_0^2} \Omega^{(a,b)}, \partial, \overline{\partial})\) is a bi-graded differential \(*\)-algebra.
2.3. Hermitian and Kähler Structures. We now present the definition of an Hermitian structure, as introduced in [66, §4], which abstracts the properties of the fundamental form of an Hermitian metric.

Definition 2.2. An Hermitian structure \((\Omega^{(a,b)}, \sigma)\) for a differential \(\ast\)-calculus \(\Omega^\bullet\), of even total dimension \(2n\), is a pair consisting of a complex structure \(\Omega^{(a,b)}\), and a central real \((1,1)\)-form \(\sigma\), called the Hermitian form, such that, with respect to the Lefschetz operator

\[ L : \Omega^\bullet \to \Omega^\bullet, \quad \omega \mapsto \sigma \wedge \omega, \]

isomorphisms are given by

\[ L^{n-k} : \Omega^k \to \Omega^{2n-k}, \quad \text{for all } k = 0, \ldots, n-1. \]

For \(L\) the Lefschetz operator of an Hermitian structure, we denote

\[ P(a,b) := \begin{cases} \{ \alpha \in \Omega^{a,b} \mid L^{n-a-b+1}(\alpha) = 0 \}, & \text{if } a + b \leq n, \\ 0 & \text{if } a + b > n. \end{cases} \]

Moreover, we denote \(P^k := \bigoplus_{a+b=k} P(a,b)\), and \(P^\bullet := \bigoplus_{k \in \mathbb{N}_0} P^k\). An element of \(P^\bullet\) is called a primitive form. We now recall Lefschetz decomposition, for a proof see [66, Proposition 4.3].

Proposition 2.3 (Lefschetz decomposition). For \(L\) the Lefschetz operator of an Hermitian structure on a differential calculus \(\Omega^\bullet\), an \(A-A\)-bimodule decomposition of \(\Omega^k\), for all \(k \in \mathbb{N}_0\), is given by

\[ \Omega^k \simeq \bigoplus_{j \geq 0} L^j(P^{k-2j}). \]

We call it the Lefschetz decomposition of \(\Omega^\bullet\).

We finish with the definition of a Kähler structure. This is a simple strengthening of the requirements of an Hermitian structure, but as we will see below, one with profound consequences.

Definition 2.4. A Kähler structure for a differential \(\ast\)-calculus is an Hermitian structure \((\Omega^{(a,b)}, \kappa)\) such that the Hermitian form \(\kappa\) is closed, which is to say \(d\kappa = 0\). We call such a \(\kappa\) a Kähler form.

2.4. The Hodge Map and Metric. In classical Hermitian geometry, the Hodge map of an Hermitian metric is related to the associated Lefschetz decomposition through the well-known Weil formula (see [85, Théorème 1.2] or [39, Proposition 1.2.31]). In the noncommutative setting we take the direct generalisation of the Weil formula for our definition of the Hodge map.

Definition 2.5. The Hodge map associated to an Hermitian structure \((\Omega^{(a,b)}, \sigma)\) is the morphism \(*_\sigma : \Omega^\bullet \to \Omega^\bullet\) uniquely defined by

\[ *_\sigma(L^j(\omega)) = (-1)^{\frac{(n+1)(j+1)}{2}} i^{a-b} \frac{j!}{(n-j-k)!} L^{n-j-k}(\omega), \quad \omega \in P^{(a,b)} \subseteq P^k. \]
Many of the basic properties of the classical Hodge map can now be understood as consequences of the Weil formula. (See [66, §4.3] for a proof.)

Lemma 2.6. Let $\Omega^\bullet$ be a differential $\ast$-calculus, of total dimension $2n$. For $(\Omega^{(\bullet \bullet)}, \sigma)$ a choice of Hermitian structure for $\Omega^\bullet$, and $\ast_\sigma$ the associated Hodge map, it holds that:

1. $\ast_\sigma$ is a $\ast$-map,
2. $\ast_\sigma(\Omega^{(a,b)}) = \Omega^{(n-b,n-a)}$, 
3. $\ast_\sigma^2 (\omega) = (-1)^k \omega$, for all $\omega \in \Omega^k$.

Reversing the classical order of construction, we now define a metric in terms of the Hodge map.

Definition 2.7. The metric associated to the Hermitian structure $(\Omega^{(\bullet \bullet)}, \sigma)$ is the unique map $g_\sigma : \Omega^\bullet \otimes_{\mathbb{R}} \Omega^\bullet \to \mathbb{A}$ for which

$$g_\sigma(\omega \otimes_{\mathbb{R}} \nu) = \ast_\sigma(\ast_\sigma(\omega^\ast) \wedge \nu),$$

for all $\omega, \nu \in \Omega^k$.

The $\mathbb{N}_0^2$-decomposition, and the Lefschetz decomposition, of the de Rham complex of a classical Hermitian manifold are orthogonal with respect to the metric [39, Lemma 1.2.24]. As shown in [66, Lemma 5.2] this carries over to the noncommutative setting. An important consequence of these orthogonalities is that the metric is conjugate symmetric [66, Corollary 5.3].

Lemma 2.8. For a differential calculus $\Omega^\bullet$, endowed with an Hermitian structure $(\Omega^{(\bullet \bullet)}, \sigma)$, it holds that

1. the $\mathbb{N}_0^2$-decomposition of $\Omega^\bullet$ is orthogonal with respect to $g_\sigma$,
2. the Lefschetz decomposition of $\Omega^\bullet$ is orthogonal with respect to $g_\sigma$.

Corollary 2.9. It holds that

$$g_\sigma(\omega \otimes_{\mathbb{R}} \nu) = g_\sigma(\nu \otimes_{\mathbb{R}} \omega)^\ast,$$

for all $\omega, \nu \in \Omega^\bullet$.

2.5. The $\mathfrak{sl}_2$-Representation and the Deformed Hodge Map. As is readily verified [66, Lemma 5.11], the Lefschetz map is adjointable on $\Omega^\bullet$ with respect to $g_\sigma$, with adjoint explicitly given by

$$\Lambda := L^\dagger = \ast_\sigma^{-1} \circ L \circ \ast_\sigma. \quad (4)$$

Taking $L$ and $\Lambda$ together with the counting operator

$$H : \Omega^\bullet \to \Omega^\bullet, \quad H(\omega) := (k - n)\omega, \text{ for } \omega \in \Omega^k,$$

we get the following commutator relations.

Proposition 2.10. We have the relations

$$[H, L] = 2H, \quad [L, \Lambda] = H, \quad [H, \Lambda] = -2\Lambda.$$

Clearly, this gives a representation of $\mathfrak{sl}_2$, which we present formally as such at the level of Hilbert space operators in [41, 12].

We finish with the interesting observation [66, §4.3] that, for any Hermitian structure $(\Omega^{(\bullet \bullet)}, \sigma)$, the associated Hodge map admits a canonical deformation $\ast_{\sigma, p}$, for any...
Proposition 2.11. We can deform the identities in Proposition 2.10. where the twisted commutator bracket is defined by (5)

\[ \sigma_p(L^j(\omega)) = (-1)^{\frac{k(k+1)}{2}} \frac{[j]_p!}{[n-j-k]_p!} L^{n-j-k}(\omega), \quad \omega \in P^{(a,b)} \subseteq P^k, \]

where, the quantum \( p \)-integer, and quantum \( p \)-factorial, are defined by \( [0]_p := 0 \) and \( [0]_p! = 1 \), and for \( m \in \mathbb{N} \),

\[ [m]_p := p^{-(m-1)} + p^{-(m-3)} + \cdots + p^{n-1} \quad [m]_p! := [1]_p [2]_p [3]_p \cdots [m]_p. \]

We call \( p \) the Hodge parameter of the deformation. Lemma 2.6 holds for all values of \( p \), giving a \( p \)-deformed metric \( g_{\sigma,p} \), and hence, a \( p \)-deformed dual Lefschetz map \( \Lambda_p \). As established in [66, §5.3.2], by introducing the operators

\[ H_p, K_p : \Omega^\bullet \to \Omega^\bullet, \quad H_p(\omega) = [k-n]_p \omega, \quad K_p(\omega) = p^{k-n} \omega, \quad \text{for } \omega \in \Omega^k, \]

we can deform the identities in Proposition 2.10.

**Proposition 2.11.** We have the relations

\[ [H_p, L_p] \mid_p - 2 = [2]_p L_p K_p, \quad [L_p, \Lambda_p] = H_p, \quad [H_p, \Lambda_p] \mid_p^2 = -[2]_p^2 K_p \Lambda_p, \]

where the twisted commutator bracket is defined by \([A, B] \mid_p := AB - p^{\pm 2} BA\).

Generalising the undeformed case, these relations imply a representation of \( U_p(\mathfrak{sl}_2) \), which we present formally as such at the level of Hilbert space operators in [44.2].

**Remark 2.12.** It is worth stressing that the Hodge parameter \( p \) need not depend on, or relate to, a deformation parameter of the underlying algebra \( A \). Indeed, the deformed Hodge map is well-defined for algebras which are not deformations and even for the de Rham complex of a classical Hermitian manifold.

### 2.6. Covariant Differential Calculi and Hermitian Structures.

For \( A \) a Hopf algebra, a left \( A \)-comodule algebra \( P \) is an \( A \)-comodule, which is also an algebra, such that the comodule structure map \( \Delta_L : P \to A \otimes P \) is an algebra map. Equivalently, it is a monoid object in \(^A\text{Mod}\), the category of left \( A \)-comodules. A differential calculus \( \Omega^\bullet \) over \( P \) is said to be covariant if the coaction \( \Delta_L : P \to A \otimes P \) extends to a (necessarily unique) \( A \)-comodule algebra structure \( \Delta_L : \Omega^\bullet \to A \otimes \Omega^\bullet \), with respect to which the differential \( d \) is a left \( A \)-comodule map.

For \( \Omega^\bullet \) a covariant differential \( \ast \)-calculus \( \Omega^\bullet \) over \( P \), we say that a complex structure for \( \Omega^\bullet \) is covariant if the \( N^2_0 \)-decomposition is a decomposition in \(^A\text{Mod}\), which is to say, if \( \Omega^{(a,b)} \) is a left \( A \)-sub-comodule of \( \Omega^\bullet \), for each \((a, b) \in N^2_0 \). A direct consequence of covariant is that the maps \( \hat{\partial} \) and \( \overline{\partial} \) are left \( A \)-comodule maps.

A covariant Hermitian structure for \( \Omega^\bullet \) is an Hermitian structure \((\Omega^{(\bullet, \bullet)}, \sigma)\) such that \( \Omega^{(\bullet, \bullet)} \) is a covariant complex structure, and the Hermitian form \( \sigma \) is left \( A \)-coinvariant, which is to say \( \Delta_L(\sigma) = 1 \otimes \sigma \). A covariant Kähler structure is a covariant Hermitian structure which is also a Kähler structure. Note that in the covariant case, in addition to being \( P \)-\( P \)-bimodule maps, \( L, \ast, \sigma \), and \( \Lambda \) are also left \( A \)-comodule maps.
3. CQH-Hermitian Spaces

In this section we introduce the notion of a compact quantum homogeneous Hermitian space, which serves as the formal setting for all our discussions of completed Hermitian structures. It sets out a natural set of compatibility conditions between the theory of compact quantum group algebras and covariant Hermitian structures, motivated by classical geometry, and the irreducible quantum flag manifolds. Throughout this section \( A \) will denote a Hopf algebra defined over \( \mathbb{C} \).

3.1. Compact Quantum Groups Algebras. For \( \Delta_L : V \to A \otimes V \) a left \( A \)-comodule, its space of matrix elements is the sub-coalgebra

\[ \mathcal{C}(V) := \text{span}_{\mathbb{C}} \{(id \otimes f)\Delta_L(v) \mid f \in \text{Lin}_{\mathbb{C}}(V, \mathbb{C}), v \in V\} \subseteq A. \]

A comodule is irreducible if and only if its coalgebra of matrix elements is irreducible, and, for \( W \) another left \( A \)-comodule, \( \mathcal{C}(V) = \mathcal{C}(W) \) if and only if \( V \) is isomorphic to \( W \).

Let us now recall the definition of a cosemisimple Hopf algebra, a natural abstraction of the properties of a reductive algebraic group.

**Definition 3.1.** A Hopf algebra \( A \) is called cosemisimple if it satisfies the following three equivalent conditions:

1. \( A \simeq \bigoplus_{V \in \hat{A}} \mathcal{C}(V) \), where summation is over all equivalence classes of left \( A \)-comodules,
2. the abelian category \( \mathcal{C}^\text{mod}(A) \) of right \( A \)-comodules is semisimple,
3. there exists a unique linear map \( h : A \to \mathbb{C} \), which we call the \textit{Haar functional}, such that \( h(1) = 1 \), and

\[ (id \otimes h) \circ \Delta(a) = h(a)1, \quad (h \otimes id) \circ \Delta(a) = h(a)1. \]

In this paper we will be concerned principally with \textit{Hopf \ast\textendash\textsl{algebras}}, which is to say, Hopf algebras which are also \ast\textendash\textsl{algebras}, and for which the coproduct \( \Delta \) and counit \( \varepsilon \) are \ast\textendash\textsl{algebra maps}. In the cosemisimple setting it is natural to require the following strengthening of the compatibility between the \ast\textendash\textsl{map} and the Hopf algebra.

**Definition 3.2.** A \textit{compact quantum group algebra}, or a \textit{CQGA}, is a cosemisimple Hopf \ast\textendash\textsl{algebra} \( A \) such that \( h(a^*a) > 0 \), for all non-zero \( a \in A \).

3.2. Compact Quantum Groups. Compact quantum group algebras are the algebraic counterpart of Woronowicz’s \textit{\ast\textendash\textsl{algebraic notion of a compact quantum group}. (Note that in the following definition, \( \otimes_{\text{min}} \) denotes the minimal tensor product of two \textit{\textsl{C\textendash\textsl{algebras} \[60, \S6\].})}

**Definition 3.3.** A \textit{compact quantum group}, or simply a \textit{CQG}, is a pair \((A, \Delta)\), where \( A \) is a unital \textit{\textsl{C\textendash\textsl{algebra}}, and \( \Delta \) is a unital \ast\textendash\textsl{homomorphism} \( \Delta : A \to A \otimes_{\text{min}} A \), such that

1. \( (\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta \),
2. the \( \mathbb{C} \)-linear spans of \((A \otimes 1)\Delta(A)\) and \((1 \otimes A)\Delta(A)\) are dense in \( A \otimes_{\text{min}} A \).
Every CQGA can be completed to a CQG, and each such completion admits an extension of \( h \) to a \( C^* \)-algebraic state. Moreover, every CQG arises as the completion of a CQGA. It is important to note that this completion will not, in general, be unique. However, every completion lives between a smallest and a largest completion, analogous to the maximal and minimal tensor product of two \( C^* \)-algebras [80, §5.4.2]. The completion relevant to this paper is the smallest completion, whose construction we now briefly recall. (See [80, §5.4.2] for a more detailed presentation.) For \( h \) the Haar functional of \( A \), an inner product is defined on \( A \) by

\[
\langle \cdot, \cdot \rangle_h : A \times A \to \mathbb{C}, \quad (a, b) \mapsto h(a^* b).
\]

Consider now the faithful \( * \)-representation \( \rho_A : A \to \text{Lin}_\mathbb{C}(A, A) \), defined by \( \rho_A(a)(b) := ab \), where \( \text{Lin}_\mathbb{C}(A, A) \) denotes the \( \mathbb{C} \)-linear operators on \( A \). For all \( a \in A \), the operator \( \rho_A(a) \) is bounded with respect to \( \langle \cdot, \cdot \rangle_h \). Hence, denoting by \( L^2(A) \) the associated Hilbert space completion of \( A \), each operator \( \rho_A(a) \) extends to an element of \( \mathbb{B}(L^2(A)) \). We denote by \( \mathcal{A} \), the corresponding closure of \( \rho(A) \) in \( \mathbb{B}(L^2(A)) \). The coproduct of \( A \) extends to a \( * \)-homomorphism \( \Delta : \mathcal{A} \to \mathcal{A} \otimes_{\text{min}} \mathcal{A} \), and together the pair \((A, \Delta)\) forms a CQG. We call it the reduced CQG associated to \( A \).

### 3.3. CQGA-Homogeneous Spaces.

Let \( \Delta_L : V \to A \otimes V \) be a left \( A \)-comodule. We say that an element \( v \in V \) is coinvariant if \( \Delta_L(v) = 1 \otimes v \). We denote the subspace of all \( A \)-coinvariant elements by \( \text{co}^*(A)V \), and call it the coinvariant subspace of the coaction. We use the analogous conventions for right comodules.

**Definition 3.4.** A homogeneous right \( H \)-coaction on \( A \) is a coaction of the form \((\text{id} \otimes \pi) \circ \Delta\), where \( \pi : A \to H \) is a surjective Hopf algebra map. A quantum homogeneous space \( B := A^{\text{co}(H)} \) is the coinvariant subspace of such a coaction.

As is easily verified, every quantum homogeneous space \( B := A^{\text{co}(H)} \) is a left coideal subalgebra of \( A \). We denote by \( \Delta_L : B \to A \otimes B \) the restriction to \( B \) of the coproduct of \( A \). Moreover, if \( A \) and \( H \) are Hopf \( * \)-algebras, and \( \pi \) is a \( * \)-algebra map, then \( B \) is a \( * \)-subalgebra of \( A \). We finish with a convenient, and natural, definition, which identifies the class of quantum homogeneous spaces we are concerned with in this paper.

**Definition 3.5.** A CQGA-homogeneous space \( \pi : A \to H \) is a quantum homogeneous space such that \( A \) and \( H \) are both CQGAs, and \( \pi \) is a surjective Hopf \( * \)-algebra map.

### 3.4. Takeuchi’s Equivalence.

In this subsection we recall Takeuchi’s equivalence [79], for a quantum homogeneous space \( B := A^{\text{co}(H)} \), in the form most suited to the purpose of this paper. Specifically, we take the simplest extension of the equivalence to a monoidal equivalence [66, §4], while simultaneously restricting to the sub-equivalence between finitely generated \( B \)-modules and finite dimensional \( H \)-comodules [66, Corollary 2.5].

Let \( H_{\text{mod}} \) denote the category whose objects are finite-dimensional left \( H \)-comodules, with morphisms left \( H \)-comodule maps. In what follows, we construct an equivalence between this category and the following, ostensibly more involved, category.

**Definition 3.6.** Let \( A_{\text{mod}}^{\Delta} \) be the category whose objects are left \( A \)-comodules \( \Delta_L : \mathcal{F} \to A \otimes \mathcal{F} \), endowed with a \( B-B \)-bimodule structure, such that
1. $\Delta_L(bf) = \Delta_L(b)\Delta_L(f)$, for all $f \in F, b \in B$.
2. $F$ is finitely-generated as a left $B$-module,
3. $F B^+ = B^+ F$, where $B^+ := B \cap \text{ker}(\varepsilon)$,
and whose morphisms are left $A$-comodule, $B$-$B$-bimodule, maps.

Consider next the functors
\[
\Phi : A \mod \to H \mod, \quad \mathcal{F} \mapsto \mathcal{F}/B^+ \mathcal{F},
\]
\[
\Psi : H \mod \to A \mod, \quad V \mapsto A \boxtimes_H V,
\]
where the left $H$-comodule structure of $\Phi(F)$ is given by $\Delta_L[f] := \pi(f_{\varepsilon(-1)}) \otimes [f_{\varepsilon(0)}]$ (with square brackets denoting the coset of an element in $\Phi(F)$) and the $B$-$B$-module, and left $A$-comodule, structures of $\Psi(V)$ are defined on the first tensor factor.

**Theorem 3.7** (Takeuchi’s Equivalence). An adjoint equivalence of categories between $A \mod$ and $H \mod$ is given by the functors $\Phi$ and $\Psi$ and the natural isomorphisms
\[
C : \Phi \circ \Psi(V) \to V, \quad \sum_i a_i \otimes v_i \mapsto \sum_i \varepsilon(a_i)v_i,
\]
\[
U : \mathcal{F} \to \Psi \circ \Phi(\mathcal{F}), \quad f \mapsto f_{\varepsilon(-1)} \otimes [f_{\varepsilon(0)}].
\]

We define the *dimension* of an object $F \in A \mod$ to be the vector space dimension of $\Phi(F)$.

For $E, F$ two objects in $A \mod$, we denote by $E \otimes_B F$ the usual bimodule tensor product endowed with the standard left $A$-comodule structure. It is easily checked that $E \otimes_B F$ is again an object in $A \mod$, and so, the tensor product $\otimes_B$ gives the category a monoidal structure. With respect to the usual tensor product of comodules in $H \mod$, Takeuchi’s equivalence is given the structure of a monoidal equivalence (see [65 §4] for details) by the morphisms
\[
\mu_{E,F} : \Phi(E) \otimes \Phi(F) \to \Phi(E \otimes_B F), \quad [e] \otimes [f] \mapsto [e \otimes_B f], \quad \text{for } E, F \in A \mod.
\]

In what follows, this monoidal equivalence will be tacitly assumed, along with the implied monoid structure on $\Phi(F)$, for any monoid object $F \in A \mod$. We finish with a useful technical lemma, necessary for our proof of Proposition 4.2.

**Lemma 3.8.** For $V \in H \mod$, with respect to the left $A$-action
\[
A \otimes V \times A \to A \otimes V, \quad (a \otimes v, b) \mapsto ab \otimes v,
\]
it holds that
\[
(A \boxtimes_H V)A = \left\{ \sum_i a_i a \otimes v_i \mid \sum_i a_i \otimes v_i \in A \boxtimes_H V, a \in A \right\} = A \otimes V.
\]

**Proof.** By Takeuchi’s equivalence, $V$ is isomorphic to a comodule of the form $\Phi(F)$, for some $F \in A \mod$. Now for any $1 \otimes [f] \in A \otimes \Phi(F)$, we have
\[
(S^{-1} \otimes \text{id}) \circ (\Delta \otimes \text{id}) \circ U(f) = S^{-1}(f_{\varepsilon(-2)}) \otimes f_{\varepsilon(-1)} \otimes [f_{\varepsilon(0)}] \in A \otimes A \boxtimes H \Phi(F).
\]
From this we see that, for any $a \in A$,
$$a \otimes [f] = f_{(1)} S^{-1}(f_{(2)})a \otimes [f_{(0)}] \in (A \square_H \Phi(F))A.$$
Thus $A \otimes \Phi(F)$ is contained in $(A \square_H \Phi(F))A$. Since the opposite inclusion is obvious, we have established the claimed identity. □

**Remark 3.9.** In Takeuchi’s original formulation [79], the equivalence is stated for a coideal $C$ of a Hopf algebra $A$, such that the functor $A \otimes_C -$, from left $C$-modules to vector spaces, is faithfully flat. As shown in [11, Corollary 3.4.5], for any coideal $*$-subalgebra of a CQGA, faithful flatness is automatic. In particular, it is automatic for any CQGA-homogeneous space.

### 3.5. CQH-Hermitian Spaces

In this subsection we introduce the notion of a CQH-Hermitian space, the formal setting for all of our discussions of Hilbet space completions of Hermitian structures. In essence, the definition organizes the central assumptions of [66] into a compact presentation.

We begin by presenting closed integrals for Hermitian structures, abstracting the situation for a classical manifold without boundary. Note that this is a special case of an orientable differential calculus with closed integral [66, §3.2], where the Hodge map is taken as the orientation. The assumption of a closed integral underpins our discussion of Hodge theory in §3.7. In particular, it is essential for establishing the codifferential formulae in (7).

**Definition 3.10.** Let $(\Omega^{(\bullet, \bullet)}, \sigma)$ be an Hermitian structure of total degree $2n \in \mathbb{N}$. The **integral** is the linear map
$$\int := h \circ *_{\sigma} : \Omega^{2n} \to \mathbb{C}.$$
If $\int d\omega = 0$, for all $\omega \in \Omega^{2n-1}$, then the integral is said to be **closed**, and $(\Omega^{(\bullet, \bullet)}, \sigma)$ is said to be **$\int$-closed**.

As we now see, closure of the integral can be converted into a more manageable representation theoretic condition. The lemma was established in [66, Corollary 3.3] in terms of the penultimate non-zero forms. However, the following version is equivalent, due to the defining Lefschetz isomorphisms of an Hermitian structure.

**Corollary 3.11.** For a CQGA-homogeneous space $\pi : A \to H$, with a given covariant differential $*$-calculus, endowed with a covariant Hermitian structure $(\Omega^{(\bullet, \bullet)}, \sigma)$, the associated integral is closed if the decomposition of $\Phi(\Omega^1)$ into irreducible comodules does not contain the trivial $H$-comodule.

The definition of an Hermitian form $\sigma$ abstracts certain properties of the fundamental form of an Hermitian manifold. Until now, however, we have made no assumption of positive definiteness, which is to say we have not required $\sigma$ to satisfy some noncommutative generalisation of the classical definition of a positive $(1, 1)$-form [39, Definition 4.3.14]. This we now do, as it proves essential for the development of noncommutative Hodge theory, as well as the construction of Hilbert spaces. (See the remark below for some further discussion.)
Definition 3.12. We say that an Hermitian structure $(\Omega^{\bullet\bullet}, \sigma)$ is **positive definite** if the associated metric $g_\sigma$ satisfies

$$g_\sigma(\omega, \omega) \in B_{>0} := \left\{ \sum_{i=1}^l b_i^* b_i \neq 0 \mid b_i \in B, l \in \mathbb{N} \right\}, \quad \text{for all } \omega \in \Omega^\bullet.$$

Moreover, we call $B_{>0}$ the **cone of positive elements** of $B$.

With these definitions introduced, we are now ready to present the definition of a CQH-Hermitian space.

**Definition 3.13.** A compact quantum homogeneous Hermitian space, or simply a CQH-Hermitian space, is a quadruple $H := (B, \Omega^{\bullet\bullet}, \Omega^{\bullet\bullet})$, where

1. $B = A^{co(H)}$ is a CQGA-homogeneous space,
2. $\Omega^{\bullet\bullet}$ is a left $A$-covariant differential $\ast$-calculus over $B$, and an object in $\mathcal{A}_B^{\text{mod}_0}$,
3. $(\Omega^{\bullet\bullet}, \sigma)$ is a covariant, $\int$-closed, positive-definite, Hermitian structure for $\Omega^{\bullet\bullet}$.

We denote by $\dim(H)$ the total dimension of the constituent differential calculus $\Omega^{\bullet\bullet}$.

The assumption of positivity of $g_\sigma$, together with Corollary 2.9 and positivity of the Haar state $h$, immediately imply the following result.

**Corollary 3.14.** For any CQH-Hermitian space, an inner product is given by

$$\langle \cdot, \cdot \rangle_\sigma : \Omega^{\bullet\bullet} \otimes R \Omega^{\bullet\bullet} \rightarrow \mathbb{C}, \quad \omega \otimes R \nu \mapsto h \circ \ast_{\sigma}(\ast_{\sigma}(\omega^\ast) \wedge \nu).$$

We finish with an important consequence of the monoidal structure of Takeuchi’s equivalence. Covariance of the calculus implies that $\Omega^{\bullet\bullet}$ is a left $A$-comodule algebra, or equivalently a monoid object in $\mathcal{A}_B^{\text{mod}_0}$. Since $\Phi$ is a monoidal functor, $\Phi(\Omega^{\bullet\bullet})$ is a monoid object in $\text{mod}_0$. This means that a well-defined algebra structure on $\Phi(\Omega^{\bullet\bullet})$ is given by

$$\wedge : \Phi(\Omega^{\bullet\bullet}) \otimes \Phi(\Omega^{\bullet\bullet}) \rightarrow \Phi(\Omega^{\bullet\bullet}), \quad [\omega] \otimes [\nu] \mapsto [\omega \wedge \nu].$$

**Remark 3.15.** Note that the definition of positivity given in [66, Definition 5.4] is given in local terms, in the category $H^{\text{mod}}$, through Takeuchi’s equivalence. As the presentation of [66, §5.2] makes clear, the two definitions are indeed equivalent. The global presentation, in $\mathcal{A}_B^{\text{mod}_0}$, is adopted in this paper, as it proves to be more natural when considering Hilbert modules in later work.

3.6. **Peter–Weyl Decomposition.** Note that by cosemisimplicity of $A$, the abelian category $H^{\text{mod}}$ is semisimple, and so, $\mathcal{A}_B^{\text{mod}_0}$ is semisimple. For any $\mathcal{F} \in \mathcal{A}_B^{\text{mod}_0}$, we have the decomposition

$$\mathcal{F} \simeq A \square_H \Phi(\mathcal{F}) \simeq \left( \bigoplus_{V \in \hat{A}} \mathcal{C}(V) \right) \square_H \Phi(\mathcal{F}) = \bigoplus_{V \in \hat{A}} \mathcal{C}(V) \square_H \Phi(\mathcal{F}) =: \bigoplus_{V \in \hat{A}} \mathcal{F}_V.$$

We call this the **Peter–Weyl decomposition** of $\mathcal{F}$.

For any $V \in H^{\text{mod}}$, it is easy to see that $\mathcal{C}(V) \simeq V^{\otimes \dim(V)}$, as a left $A$-comodule [42, Proposition 11.8]. Thus, for any left $A$-comodule map $f : \mathcal{F} \rightarrow \mathcal{F}$ it holds that

$$f(\mathcal{F}_V) \subseteq \mathcal{F}_V.$$
More generally, a Peter–Weyl map \( f : \mathcal{F} \rightarrow \mathcal{F} \) is a \( \mathbb{C} \)-linear map satisfying (6). We present some properties of the Peter–Weyl decomposition, and Peter–Weyl maps, in the CQH-Hermitian setting. The proof is completely analogous to the arguments of [66, §5.2], and so, omitted.

**Lemma 3.16.** For a CQH-Hermitian space \( H = \{ B = A^{\text{co}(H), \Omega^*, \Omega^{\bullet \bullet}}, \sigma \} \), the Peter–Weyl decomposition of \( \Omega^* \) is orthogonal with respect to \( \langle \cdot, \cdot \rangle \). Moreover, for any Peter–Weyl map \( f : \Omega^* \rightarrow \Omega^* \), it holds that

1. \( f \) is adjointable on \( \Omega^* \) with respect to \( \langle \cdot, \cdot \rangle \), and its adjoint is a Peter–Weyl map,
2. if \( f \) is self-adjoint with respect to \( \langle \cdot, \cdot \rangle \), then \( f \) is diagonalisable on \( \Omega^* \).

### 3.7. Dirac and Laplace Operators and Hodge Theory

We now recall the noncommutative generalisation of Hodge theory associated to any CQH-Hermitian space. An important application is the identification of the index of the Dolbeault–Dirac operator with the anti-holomorphic Euler characteristic of the calculus, as shown in [43].

For any CQH-Hermitian space, Lemma [3.10] tells us that the exterior derivatives \( \partial, \partial, \overline{\partial} \) are adjointable on \( \Omega^* \). As established in [66, §5.3.3], their adjoints are expressible in terms of the Hodge operator:

(7) \[
\partial^\dagger = -* \circ \partial \circ *\sigma, \quad \overline{\partial} = -* \circ \overline{\partial} \circ *\sigma, \quad \overline{\partial}^\dagger = -* \circ \partial \circ *\sigma.
\]

Just as for the classical case, we define the \( \partial \)-, \( \partial \)-, and \( \overline{\partial} \)-Dirac operators respectively as

\[
\Delta_{\partial} := \partial + \partial^\dagger, \quad \Delta_{\overline{\partial}} := \overline{\partial} + \overline{\partial}^\dagger.
\]

Moreover, we define the \( \partial \)-, \( \overline{\partial} \)-, and \( \overline{\partial} \)-Laplace operators to be

\[
\Delta_{\partial} := (\partial + \partial^\dagger)^2, \quad \Delta_{\overline{\partial}} := (\overline{\partial} + \overline{\partial}^\dagger)^2, \quad \Delta_{\overline{\partial}} := (\overline{\partial} + \overline{\partial}^\dagger)^2.
\]

The \( \partial \)-harmonic, \( \partial \)-harmonic, and \( \overline{\partial} \)-harmonic forms, are defined respectively to be

\[
\mathcal{H}_{\partial} := \ker(\Delta_{\partial}), \quad \mathcal{H}_{\overline{\partial}} := \ker(\Delta_{\overline{\partial}}), \quad \mathcal{H}_{\overline{\partial}} := \ker(\Delta_{\overline{\partial}}).
\]

For any CQH-Hermitian space, Lemma [3.16] tells us that the Dirac and Laplace operators are diagonalisable. Just as in the classical case, it now follows that

(8) \[
\mathcal{H}_{\partial} = \ker(\partial) \cap \ker(\partial^\dagger), \quad \mathcal{H}_{\overline{\partial}} = \ker(\overline{\partial}) \cap \ker(\overline{\partial}^\dagger), \quad \mathcal{H}_{\overline{\partial}} = \ker(\overline{\partial}) \cap \ker(\overline{\partial}^\dagger),
\]

see [66, Lemma 6.1] for details. Moreover, as shown in [66, §6.2], diagonalisability also allows us to conclude the following noncommutative generalisation of Hodge decomposition for Hermitian manifolds.

**Theorem 3.17** (Hodge decomposition). Let \( H = (B, \Omega^*, \Omega^{\bullet \bullet}, \sigma) \) be a CQH-Hermitian space. Direct sum decompositions of \( \Omega^* \), orthogonal with respect to \( \langle \cdot, \cdot \rangle \), are given by

\[
\Omega^* = \mathcal{H}_{\partial} \oplus \partial \Omega^* \oplus \partial^\dagger \Omega^*, \quad \Omega^* = \mathcal{H}_{\overline{\partial}} \oplus \overline{\partial} \Omega^* \oplus \overline{\partial}^\dagger \Omega^*, \quad \Omega^* = \mathcal{H}_{\overline{\partial}} \oplus \overline{\partial} \Omega^* \oplus \overline{\partial}^\dagger \Omega^*.
\]

Moreover, the following projections are isomorphisms

\[
\mathcal{H}_{\partial}^k \rightarrow H_{\partial}^k, \quad \mathcal{H}_{\overline{\partial}}^{(a,b)} \rightarrow H_{\overline{\partial}}^{(a,b)}, \quad \mathcal{H}_{\overline{\partial}}^{(a,b)} \rightarrow H_{\overline{\partial}}^{(a,b)},
\]

where \( H_{\partial}^k, H_{\overline{\partial}}^{(a,b)} \), and \( H_{\overline{\partial}}^{(a,b)} \), denote the cohomology groups of the de Rham, holomorphic, and anti-holomorphic, complexes respectively.
3.8. **CQH-Kähler Spaces.** We finish this section with the obvious specialisation of the CQH-Hermitian space definition to the Kähler setting.

**Definition 3.18.** A compact quantum homogeneous Kähler space, or simply a CQH-Kähler space, is a CQH-Hermitian space $K = (B, \Omega^\bullet, \Omega^{\bullet\bullet}, \kappa)$ such that the Hermitian structure $(\Omega^{\bullet\bullet}, \kappa)$ is a Kähler structure.

As mentioned in [23], this simple strengthening of the requirements of an Hermitian structure has profound consequences. As a first example, we present the following theorem, which gives a direct noncommutative generalisation of the Kähler identities of a classical Kähler manifold. See [66, §7] for a proof.

**Theorem 3.19** (Kähler Identities). For CQH-Kähler space $K$, we have the following relations

\[
[\partial, L] = 0, \quad [\overline{\partial}, L] = 0, \quad [\partial^\dagger, \Lambda] = 0, \quad [\overline{\partial}^\dagger, \Lambda] = 0,
\]

\[
[L, \partial^\dagger] = i\overline{\partial}, \quad [L, \overline{\partial}^\dagger] = -i\partial, \quad [\Lambda, \partial] = i\overline{\partial}^\dagger, \quad [\Lambda, \overline{\partial}] = -i\partial^\dagger.
\]

As direct consequence [66, Corollary 7.6] we have the following important identities.

**Corollary 3.20.** It holds that

\[
\partial \partial^\dagger + \partial^\dagger \partial = 0, \quad \partial^\dagger \overline{\partial} + \overline{\partial} \partial^\dagger = 0, \quad \Delta_\partial = 2\Delta_\partial = 2\Delta_{\overline{\partial}}.
\]

In the classical setting an Hermitian manifold is Kähler if and only if the three Laplacians satisfy this proportionality relation [68, Theorem 3.10]. In both the commutative and noncommutative setting this result has strong cohomological consequences. Corollary 3.20 taken together with Hodge decomposition, implies that de Rham cohomology is refined by Dolbeault cohomology [66, Corollary 7.7], that is

\[
H^k_\partial \simeq \bigoplus_{a+b=k} H^{(a,b)}_\partial \simeq \bigoplus_{a+b=k} H^{(a,b)}_{\overline{\partial}}.
\]

Finally, we recall the hard Lefschetz theorem for a CQH-Kähler space. This result is expressed in terms of primitive cohomology, the definition of which we now recall.

**Definition 3.21.** For a Kähler structure, the $(a, b)$-primitive cohomology group is the vector space

\[
H^{(a,b)}_{\text{prim}} := \ker \left( L^{n-(a+b)+1} : H^{(a,b)} \to H^{(n-b+1, n-a+1)} \right).
\]

Moreover, we denote $H^k_{\text{prim}} := \bigoplus_{a+b=k} H^{(a,b)}_{\text{prim}}$.

As observed in [66, Theorem 7.12], the proof of the classical hard Lefschetz theorem carries over directly from the classical setting, giving us the following theorem.

**Theorem 3.22** (Hard Lefschetz). Let $(\Omega^{\bullet\bullet}, d)$ be a Kähler structure, then it holds that

1. $L^{n-a-b} : H^{(a,b)} \to H^{(n-a-n-b)}$ is an isomorphism, for all $(a, b) \in \mathbb{N}_0$,
2. $H^{(a,b)} \simeq \bigoplus_{i \geq 0} L^i H^{(a-i, b-i)}_{\text{prim}}$.

As a direct consequence of this, we see that for any CQH-Kähler space, the cohomology
4. The Hilbert Space of Square Integrable Forms

In this section we consider the completion of \( \Omega^* \) to a Hilbert space with respect to the inner product of a CQH-Hermitian space. In particular we examine how the various operators associated to Hermitian and Kähler structure behave with respect to this completion.

4.1. Square Integrable Forms. In this subsection we introduce the Hilbert space of square integrable forms of a CQH-Hermitian space. We then observe that the complex and Lefschetz decompositions of the calculus carry over to the completed setting, introduce an alternative description of the Hilbert space in terms of Takeuchi’s equivalence, and finally establish separability of the Hilbert space.

**Definition 4.1.** For a CQH-Hermitian space \( H = (B, \Omega^*, \Omega^{\otimes n}, \sigma) \), we denote by \( L^2(\Omega^*) \) the Hilbert space completion of \( \Omega^* \) with respect to its inner product \( \langle \cdot, \cdot \rangle \), and call it the Hilbert space of square integrable forms of \( H \).

Recall from Lemma 2.8 that the \( N^0 \)-decomposition, and the Lefschetz decomposition, of \( \Omega^* \) are orthogonal with respect to the associated inner product. This implies that we have the following \( L^2 \)-decompositions

\[
L^2(\Omega^*) \simeq \bigoplus_{(a,b) \in N^0} L^2(\Omega^{(a,b)}) , \quad L^2(\Omega^k) \simeq \bigoplus_{j \geq 0} L^2(L^2_j(\mathbb{P}^{(2n-2j)})) .
\]

We now introduce an alternative presentation of \( L^2(\Omega^*) \) coming from Takeuchi’s equivalence. Consider the sesquilinear form on \( \Phi(\Omega^*) \) defined by

\[
\langle \cdot, \cdot \rangle : \Phi(\Omega^*) \otimes \Phi(\Omega^*) \to \mathbb{C} , \quad [\omega] \otimes [\nu] \mapsto [g_{\sigma}(\omega \otimes_B \nu)].
\]

This in turn gives us the sesquilinear form

\[
\langle \cdot, \cdot \rangle_U : A\Box_H \Phi(\Omega^*) \otimes_R A\Box_H \Phi(\Omega^*) \to \mathbb{C} , \quad \sum_{i,j} f_i \otimes v_i \otimes g_j \otimes w_j \mapsto \sum \langle f_i, g_j \rangle_h(v_i, w_j).
\]

**Proposition 4.2.** Let \( H = (B = A^{\text{co}(H)}, \Omega^*, \Omega^{\otimes n}, \sigma) \) be a CQH-Hermitian space.

1. The sesquilinear form \( \langle \cdot, \cdot \rangle_U \) is an inner product.
2. The unit \( U \) of Takeuchi’s equivalence is an isomorphism of the inner product spaces \((\Omega^*, \langle \cdot, \cdot \rangle)\) and \((A\Box_H \Phi(\Omega^*), \langle \cdot, \cdot \rangle_U)\). Hence it extends to an isomorphism between the respective Hilbert space completions \( L^2(\Omega^*) \) and \( L^2(A\Box_H \Phi(\Omega^*)) \).

**Proof.** We begin by showing that \( \langle \cdot, \cdot \rangle \) is well-defined by presenting it as a composition of well-defined maps. Since \( \Omega^* \in \mathbb{A} \text{mod}_0 \), it holds that \( \Omega^* B^+ = B^+ \Omega^* \), implying that

\[
(B^+ \Omega^*)^* = \Omega^* B^+ = B^+ \Omega^* .
\]

Hence, the \( * \)-map of the calculus restricts to a well-defined conjugate linear map \( * : \Phi(\Omega^*) \to \Phi(\Omega^*) \). Moreover, a morphism in \( \mathbb{A} \text{mod}_0 \) is given by

\[
\overline{\gamma} : \Omega^* \otimes_B \Omega^* \to B , \quad \omega \otimes_B \nu \mapsto \sigma(\sigma^*(\omega) \wedge \nu).
\]

Taking the image of \( \overline{\gamma} \) under \( \Phi \), we see that

\[
\langle \cdot, \cdot \rangle = \Phi(\overline{\gamma}) \circ (\otimes_R \text{id}) ,
\]
implying that $\langle \cdot , \cdot \rangle_U$ are well-defined $\mathbb{R}$-linear maps.

We will now show that $\langle \cdot , \cdot \rangle_U$ is the inner product induced on $A \Box_H \Phi(\Omega^\bullet)$ by $\langle \cdot , \cdot \rangle$ and $U$. Note first that

$$\langle \omega, \nu \rangle = h \circ g(\omega, \nu)$$
$$= h(\ast_\sigma(\ast_\sigma(\omega^\ast) \wedge \nu))$$
$$= (h \otimes \text{id}) \circ U(\ast_\sigma(\ast_\sigma(\omega^\ast) \wedge \nu)),$$

where we have used the evident identity $h = (h \otimes \text{id}) \circ U : B \to \mathbb{C}$. Continuing, we see

$$(h \otimes \text{id}) \circ U(\ast_\sigma(\ast_\sigma(\omega^\ast) \wedge \nu)) = (h \otimes \text{id}) \left( \omega_{(1)}^\ast \nu_{(1)} \otimes [\ast_\sigma(\ast_\sigma(\omega_{(0)}^\ast) \wedge \nu_{(0)})] \right)$$
$$= h(\omega_{(1)}^\ast \nu_{(1)})[\ast_\sigma(\ast_\sigma(\omega_{(0)}^\ast) \wedge \nu_{(0)})]$$
$$= h(\omega_{(1)}^\ast \nu_{(1)})[g(\omega_{(0)}, \nu_{(0)})]$$
$$= \langle \omega_{(1)} \otimes \omega_{(0)}, \nu_{(1)} \otimes \nu_{(0)} \rangle_U$$
$$= \langle U(\omega) \otimes \mathbb{R} \ U(\nu) \rangle_U.$$

Thus $U$ is an isomorphism of inner product spaces, which in turn extends to an isomorphism between the Hilbert spaces $L^2(\Omega^\bullet)$ and $L^2(A \Box_H \Phi(\Omega^\bullet))$.

Finally, we come to showing that $\langle \cdot , \cdot \rangle$ is an inner product. By Lemma 3.8, any element $X \in A \otimes \Phi(\Omega^\bullet)$ can be written as a sum of non-zero elements of the form $X = \sum_{k=1}^m Y_k a_k$, where $a_k \in A$, and $(Y_k)_{k \in \mathbb{N}}$ is a choice of orthonormal basis of $A \Box_H \Phi(\Omega^\bullet)$. For convenience, we denote by $((\cdot, \cdot))$ the tensor product of $\langle \cdot , \cdot \rangle_U$ and $\langle \cdot , \cdot \rangle$ acting on $A \otimes \Phi(\Omega^\bullet)$. This means that

$$((X, X)) = \sum_{k=1}^m ((Y_k a_k, Y_k a_k)) = h\left( \sum_{k=1}^m a_k^* g(U^{-1}(Y_k), U^{-1}(Y_k)) a_k \right).$$

Since $H$ is a CQH-Hermitian space, each $g(U^{-1}(Y_k), U^{-1}(Y_k)) \in B_{>0}$. Hence, each summand $a_k^* g(U^{-1}(Y_k), U^{-1}(Y_k)) a_k$ is contained in $B_{>0}$. Positivity of $h$ now implies that $((X, X)) > 0$.

Since $\langle \cdot , \cdot \rangle$ restricts to $\text{id} \otimes \langle \cdot , \cdot \rangle_U$ on the elements of $1 \otimes \Phi(\Omega^\bullet) \otimes 1 \otimes \Phi(\Omega^\bullet)$, we see that $\langle \cdot , \cdot \rangle$ must be positive definite. Conjugate symmetry of $\langle \cdot , \cdot \rangle$ follows directly from conjugate symmetry of $g_\sigma$, as presented in Corollary 2.9, allowing us to conclude that $\langle \cdot , \cdot \rangle$ is an inner product.

We finish by producing a sufficient condition for separability, given in terms of the set $\hat{A}$ of isomorphism classes of irreducible comodules of $A$.

**Lemma 4.3.** The Hilbert space $L^2(\Omega^\bullet)$ is separable if $\hat{A}$ has a countable number of elements.

**Proof.** If $\hat{A}$ is countable, then the Peter–Weyl decomposition of $\Omega^\bullet$ must have a countable number of summands. Moreover, since $\Omega^\bullet_V$ is finite-dimensional, for each $V \in \hat{A}$, it is clear that $\Omega^\bullet_V$ admits a countable Hamel basis. Hence $L^2(\Omega^\bullet)$ is separable. \qed
4.2. Morphisms as Bounded Operators. In this subsection we discuss the extension of endomorphisms of $\Omega^\bullet$ to bounded operators on $L^2(\Omega^\bullet)$. As an application, we produce bounded representations of $\mathfrak{sl}_2$ and $U_p(\mathfrak{sl}_2)$.

**Proposition 4.4.** Every endomorphism $f : \Omega^\bullet \rightarrow \Omega^\bullet$ of the differential calculus is bounded, and hence extends to a bounded operator on $L^2(\Omega^\bullet)$.

**Proof.** Consider the commutative diagram given by Takeuchi’s equivalence

$$
\begin{array}{ccc}
\Omega^\bullet & \xrightarrow{f} & \Omega^\bullet \\
\downarrow{U} & & \downarrow{U^{-1}} \\
A\Box_H \Phi(\Omega^\bullet) & \xrightarrow{\Psi \circ \Phi(f)} & A\Box_H \Phi(\Omega^\bullet).
\end{array}
$$

Since $U$ is an isomorphism of inner product spaces, the morphism $f$ is bounded if and only if $\Psi \circ \Phi(f)$ is bounded. But $\Psi \circ \Phi(f) = \text{id} \otimes \Phi(f)$, and $\Phi(\Omega^\bullet)$ is finite-dimensional by assumption, implying that $\text{id} \otimes \Phi(f)$ is bounded, and hence that $f$ is bounded. □

**Corollary 4.5.** The maps $L, \Lambda,$ and $H$ extend to bounded operators on $L^2(\Omega^\bullet)$. Hence, a representation $\mathfrak{sl}_2 \rightarrow B(L^2(\Omega^\bullet))$ is given by

$$
\rho(E) = L, \quad \rho(K) = K, \quad \rho(F) = \Lambda.
$$

The space of lowest weight vectors of the representation is given by $L^2(P^\bullet)$, the Hilbert space completion of the primitive forms.

**Proof.** Since $L, \Lambda,$ and $H$ are all morphisms, Proposition 4.4 implies that they extend to bounded operators on $L^2(\Omega^\bullet)$. It now follows from Proposition 2.10 that we get a bounded representation of $\mathfrak{sl}_2$. □

**Corollary 4.6.** The Hodge map $\ast_{\sigma}$ extends to a unitary operator on $L^2(\Omega^\bullet)$.

**Proof.** This follows from Proposition 4.4 and unitarity of $\ast_{\sigma}$ as an operator on $\Omega^\bullet$, as established in [66, Lemma 5.10]. □

We recall from §2.5 that the Hodge map can be deformed, resulting in the deformed set of commutation relations presented in Proposition 2.11. This in turn deforms the representation $\rho$ to a representation of $U_q(\mathfrak{sl}_2)$, considered with respect to its presentation in §10. The proof is an immediate consequence of the proof at the level of linear operators on $\Omega^\bullet$, as presented in [66, Corollary 5.14].

**Corollary 4.7.** A representation $\rho_p : U_p(\mathfrak{sl}_2) \rightarrow B(L^2(\Omega^\bullet))$ is given by

$$
\rho_p(E) = L_p, \quad \rho_p(K) = K_p, \quad \rho_p(F) = \Lambda_p.
$$

As we now explain, the representation of $\mathfrak{sl}_2$ given in Corollary 4.5 can be understood as a special case of the representation of $U_p(\mathfrak{sl}_2)$ given above. Let $\tilde{U}_p(\mathfrak{sl}_2)$ be the algebra
generated by the elements $E, F, G,$ and $H,$ subject to the relations

\[
KK^{-1} = K^{-1}K = 1, \quad KE = q^2EK, \quad KF = q^{-2}FK, \quad [G, E] = E(pK + p^{-1}K^{-1}), \quad [G, F] = -(pK + p^{-1}K^{-1})F, \quad [E, F] = G, \quad (p - p^{-1})G = K - K^{-1}.
\]

For $p \neq 1,$ an algebra isomorphism between $U_p(sl_2)$ and $\tilde{U}_p(sl_2)$ is defined by

\[
E \mapsto E, \quad F \mapsto F, \quad K \mapsto K, \quad G \mapsto \frac{K - K^{-1}}{p - p^{-1}}.
\]

For $p = 1,$ the algebra $\tilde{U}_1(sl_2)$ is well-defined, and we have an algebra isomorphism

\[
\tilde{U}_1(sl_2)/\langle K - 1 \rangle \cong U(sl_2).
\]

Since $\langle K - 1 \rangle$ is clearly contained in the kernel of $\rho_p,$ it descends to a representation of $U(sl_2),$ with $\rho$ being its restriction to $sl_2 \subseteq U(sl_2).$

### 4.3. Grading Operators

The calculus admits an obvious $\mathbb{Z}_2$-grading coming from its decomposition into even and odd forms. This gives an operator

\[
\gamma : \Omega^\bullet \to \Omega^\bullet, \quad \gamma(\omega) = (-1)^k\omega, \quad \text{for any } \omega \in \Omega^k.
\]

Associated to the $N_0^2$-decomposition of the complex structure, we have two analogous operators. Denote by $\tau : \Omega^\bullet \to \Omega^\bullet,$ and $\overline{\tau} : \Omega^\bullet \to \Omega^\bullet,$ the unique linear operators for which

\[
\tau(\omega) = a\omega, \quad \overline{\tau}(\omega) = b\omega, \quad \text{for any } \omega \in \Omega^{(a, b)}.
\]

We have yet another operator associated to the Lefschetz decomposition

\[
\lambda : \Omega^\bullet \to \Omega^\bullet, \quad \lambda(\omega) = j\omega, \quad \text{for any } \omega \in L^j(P^\bullet).
\]

Both $\gamma$ and $\lambda$ are $\ast$-maps, while $\overline{\tau} = \ast \circ \tau \circ \ast.$

Since the $N_0, N_0^2,$ and Lefschetz decompositions are all decompositions in the category $A^\bullet_{B, \text{mod}}_0,$ the operators $\gamma, \tau, \overline{\tau},$ and $\lambda,$ are all morphisms. Thus they extend to bounded operators on the Hilbert space $L^2(\Omega^\bullet).$ Orthogonality of the $N_0, N_0^2,$ and Lefschetz decompositions implies that each operator is self-adjoint, while $\gamma$ is moreover unitary. Finally, we note since the $N_0^2$-decomposition is homogeneous with respect to $N_0$-decomposition, and the Lefschetz decomposition is in turn homogeneous with respect to the $N_0^2$-decomposition, all four operators $\gamma, \tau, \overline{\tau},$ and $\lambda,$ pairwise commute. Hence they generate a commutative subalgebra of $\mathbb{B}(L^2(\Omega^\bullet)).$

### 4.4. Bounded Multiplication Maps

In this subsection we prove that every multiplication operator on $\Omega^\bullet$ is bounded with respect to the inner product of the Hermitian structure. As a consequence, we observe that the restriction to $B$ of the bounded representation $\rho$ of $A$ on $L^2(A),$ extends to a bounded representation of $B$ on $L^2(\Omega^\bullet).$

**Proposition 4.8.** For any form $\omega \in \Omega^\bullet,$ a non-zero bounded operator is given by

\[
L_\omega : \Omega^\bullet \to \Omega^\bullet, \quad \nu \mapsto \omega \wedge \nu.
\]
Hence a faithful algebra representation \( \rho : \Omega^* \to \mathcal{B}(L^2(\Omega^*)) \) is uniquely defined by
\[
\rho(\omega)(\nu) = L_\omega(\nu), \quad \text{for all } \omega, \nu \in \Omega^*.
\]

**Proof.** For any \([\omega] \in \Phi(\Omega^*)\), we have a well-defined operator
\[
B_{[\nu]} : \Phi(\Omega^*) \to \Phi(\Omega^*), \quad [\omega] \mapsto [\nu] \land [\omega] = [\nu \land \omega].
\]
This operator is bounded by finite dimensionality of \(\Phi(\Omega^*)\). Moreover, recalling the representation \(\rho : A \to \mathcal{B}(L^2(A))\) introduced at the end of \(\S 2.2\) we see that a bounded operator on \(A \otimes \Phi(\Omega^*)\) is given by \(\rho(a) \otimes B_{[\nu]}\), for all \(a \in A, [\nu] \in \Phi(\Omega^*)\).

We will now show that \(L_\omega\) is bounded by showing that \(U \circ L_\omega \circ U^{-1}\) is bounded. For any \(\nu \in \Omega^*\),
\[
U \circ L_\omega \circ U^{-1}(\nu(-1) \otimes [\nu(0)]) = U \circ L_\omega \circ U^{-1} \circ U(\nu) = U(\omega \land \nu) = \omega(-1) \nu(-1) \otimes [\omega(0) \land \nu(0)] = \rho(\omega(-1))(\nu(-1)) \otimes B_{\omega(0)}[\nu(0)] = \rho(\omega(-1))(\nu(-1)) \otimes B_{\omega(0)}[\nu(0)].
\]
Since every element of \(A \Box_H \Phi(\Omega^*)\) is of the form \(U(\nu) = \nu(-1) \otimes [\nu(0)]\), for some \(\nu \in \Omega^*\), we see that \(U \circ L_\omega \circ U^{-1}\) is bounded.

Since \(\Omega^*\) is dense in \(L^2(\Omega^*)\), and \(L_\omega\) is bounded on \(\Omega^*\) by construction, it is clear that \(L_\omega\) uniquely extends to an element of \(\mathcal{B}(L^2(\Omega^*))\). This gives a well-defined \(\mathbb{C}\)-linear map from \(\Omega^*\) to \(\mathcal{B}(L^2(\Omega^*))\), which moreover, is an algebra map by associativity of the multiplication of \(\Omega^*\). Finally, we note that since \(1 \in B \subseteq \Omega^*\), it is clear that \(\rho\) is faithful. \(\square\)

**Corollary 4.9.** A faithful \(*\)-algebra representation \(\varphi : \mathcal{B} \to \mathcal{B}(L^2(\Omega^*))\) is uniquely determined by \(\varphi(b) = \rho(b)\), for all \(b \in B\).

**Proof.** Proposition \(\S 2.2\) tells us that the unit \(U\) of Takeuchi’s equivalence extends to an isomorphism between \(L^2(\Omega^*)\) and a Hilbert subspace of \(L^2(A) \otimes \Phi(\Omega^*)\). Moreover, since \(U\) is a left \(B\)-module map
\[
U^{-1} \circ \rho(b) \circ U = (\gamma(b) \otimes \text{id}) |_{\Omega^* \Phi(\Omega^*)}.
\]
Now \(\gamma(b) \otimes \text{id}\) acts on the Hilbert space \(L^2(A) \otimes \Phi(\Omega^*)\) as a bounded operator with norm \(|b|_B\), meaning that the norm of \(\rho(b)\) must be less than or equal to \(|b|_B\). Hence \(\rho\) is norm decreasing, meaning that \(\rho\) uniquely extends to an algebra map \(\varphi : \mathcal{B} \to \mathcal{B}(L^2(\Omega^*))\).

To see that \(\rho\) is faithful on \(\mathcal{B}\), consider an element \(b \in \mathcal{B}\), and note that
\[
|b|_\text{op}^2 \geq \|\rho(b)(1)\|_{L^2}^2 = |b|_B^2 = \langle b, b \rangle_h.
\]
Recalling that the extension of \(h\) to a state \(h_r : \mathcal{B} \to \mathbb{C}\) is faithful, and observing that \(\langle b, b \rangle_h = h_r(b^*b)\), for all \(b \in \mathcal{B}\), we see that
\[
|b|_\text{op}^2 \geq \langle b, b \rangle_h = h_r(b^*b) > 0.
\]
Thus \(\rho\) is faithful on \(\mathcal{B}\).
The fact that $\rho : B \to B(L^2(\Omega^*))$ is a $*$-representation follows from
\[
\langle \omega \otimes b\nu \rangle = h \circ \ast_\sigma (\ast_\sigma (\omega^* b) \wedge \nu) \\
= h \circ \ast_\sigma (\ast_\sigma (\omega^* b) \wedge \nu) \\
= h \circ \ast_\sigma (\ast_\sigma ((b^*\omega)^* \wedge \nu) \\
= \langle b^* \omega \otimes \nu \rangle.
\]
Hence by continuity $\rho$ is a $*$-map as claimed. □

We finish with a second consequence of Proposition 4.8, namely boundedness of the various commutator operators associated to a CQH-Hermitian space. This is a direct noncommutative generalisation of an important classical phenomenon [10, §2.4.1], one which is abstracted in the definition of $K$-homology and ultimately spectral triples, as we see in §7.

**Corollary 4.10.** The following operators are all bounded on $\Omega^*$, and hence they extend to bounded operators on $L^2(\Omega^*)$:
\[
[d, \rho(b)], \quad [\partial, \rho(b)], \quad [\partial^\dagger, \rho(b)], \quad [\partial^\dagger, \rho(b)],
\]
for all $b \in B$.

**Proof.** For any $\omega \in \Omega^*$, we have the identity
\[
[d, \rho(b)](\omega) = (d \circ \rho(b) - \rho(b) \circ d)(\omega) \\
= d(b\omega) - bd\omega \\
= db \wedge \omega + bd\omega - bd\omega \\
= db \wedge \omega.
\]
It now follows from Proposition 4.8 that $[d, \rho(b)]$ is a bounded operator on $L^2(\Omega^*)$.

Boundedness of $[\partial, \rho(b)]$ and $[\partial^\dagger, \rho(b)]$ are established similarly.

The operator $[d, \rho(b)]$ is adjointable on $\Omega^*$, in particular
\[
[d, \rho(b)]^\dagger = -(d^\dagger, \rho(b^*)) \\
\]
for all $b \in B$.

Thus we can conclude that $[d^\dagger, \rho(b)]$ is a bounded operator on $\Omega^*$. Boundedness of $[\partial^\dagger, \rho(b)]$ and $[\partial^\dagger, \rho(b)]$ are established similarly. □

**Corollary 4.11.** For all $b \in B$, the operators $[D_d, b]$, $[D_\partial, b]$, and $[D_{\bar{\partial}}, b]$ are bounded.

4.5. Closability and Essential Self-Adjointness. In this subsection we examine closability and essential self-adjointness for unbounded operators on $\Omega^*$. In particular, we show that the unbounded operators $d, \partial$ and $\bar{\partial}$ are closable, and that the Dirac and Laplacian operators are essentially self-adjoint.

**Proposition 4.12.** Every Peter–Weyl map $f : \Omega^* \to \Omega^*$ is closable.

**Proof.** Since $f$ is a Peter–Weyl map by assumption, Lemma 3.16 tells us that it is adjointable on $\Omega^*$. Moreover, since $\Omega^*_V$ is finite dimensional, for every $V \in \hat{A}$, the restriction $f^*_V : \Omega^*_V \to \Omega^*_V$ is bounded. Now for any $\omega \in \Omega^*_V$, consider the linear functional
\[
\Omega^* = \text{dom}(f) \to \mathbb{C}, \quad \nu \mapsto \langle \omega, f(\nu) \rangle.
\]
Boundedness of the functional follows from the inequality
\[ |\langle \omega, f^\dagger(\nu) \rangle| = |\langle f^\dagger(\omega), \nu \rangle| \leq \|f^\dagger(\omega)\| \|\nu\| \leq \|f^\dagger_V\| \|\omega\| \|\nu\|,\]
where \(\|f^\dagger_V\|\) denotes the norm of \(f^\dagger_V\). Hence, \(\omega \in \text{dom}(f^\dagger)\), implying that \(\Omega^* \subseteq \text{dom}(f^\dagger)\), and consequently that \(\text{dom}(f^\dagger)\) is dense in \(L^2(\Omega^*)\). It now follows from the discussion of \(\ref{A.2}\) that \(f\) is closable. \(\square\)

Since every comodule map is automatically a Peter–Weyl map, we have the following immediate consequences of the proposition.

**Corollary 4.13.** Every left \(A\)-comodule map \(f : \Omega^* \to \Omega^*\) is closable.

**Corollary 4.14.** The operators \(d, \partial,\) and \(\bar{\partial}\) are closable.

*Proof.* Since the calculus and complex structure are, by assumption, covariant, the maps \(d, \partial,\) and \(\bar{\partial}\) are comodule maps, and hence closable. \(\square\)

We now come to a corollary which, although not used in what follows, is included as an easy application of Proposition \(\ref{A.12}\). The motivating example is the usual dual pairing between \(U_q(g)\) and \(O_q(G)\) (see \(\ref{10}\)) generalising the classical action of vector fields on forms.

**Corollary 4.15.** Given a dually paired Hopf algebra \((\cdot, \cdot) : W \times A \to \mathbb{C}\), and some \(X \in W\), a closable linear operator is given by
\[ \hat{X} : \Omega^* \to \Omega^*, \quad \omega \mapsto (X, \omega(-1))\omega(0). \]

*Proof.* By construction \(\hat{X}\) is a Peter–Weyl operator, and so, it is closable by Proposition \(\ref{A.12}\). \(\square\)

We finish with a proof of essential self-adjointness for symmetric comodule maps, and the implied essential self-adjointness of the Dirac and Laplacian operators.

**Proposition 4.16.** Every symmetric left \(A\)-comodule map \(f : \Omega^* \to \Omega^*\) is diagonalisable on \(L^2(\Omega^*)\), and moreover, essentially self-adjoint.

*Proof.* Diagonalisability of \(f\) as an operator on \(L^2(\Omega^*)\) follows immediately from Lemma \(\ref{3.16}\) and our assumption that \(f\) is symmetric. Symmetry of \(f\) also implies that its eigenvalues are real. Thus the range of the operators \(f - i \text{id}\) and \(f + i \text{id}\) must be equal to \(\Omega^*\), which is to say, the range of each operator is dense in \(L^2(\Omega^*)\). It now follows from the discussions of \(\ref{A.3}\) that \(f\) is essentially self-adjoint. \(\square\)

**Corollary 4.17.** The Dirac operators \(D_\partial, D_\nabla,\) and \(D_\bar{\partial}\), as well as the Laplace operators \(\Delta_\partial, \Delta_\nabla,\) and \(\Delta_\bar{\partial}\), are essentially self-adjoint.
4.6. Sobolev Spaces and Smooth Sections. In this subsection, which is in effect an extended remark, we make some brief observations about the noncommutative Sobolev spaces associated to any CQH-Hermitian space. Sobolev theory, for a classical compact Hermitian manifold $M$, can be understood as the study of those square integrable forms contained in the domain of the closure of $\Delta^k_\partial$ for $k > 0$. Hence, for any CQH-Hermitian space $H = (B, \Omega^\bullet, \Omega^{\bullet,\bullet}, \sigma)$, we may define its $k$th-Sobolev space to be

$$W^k(\Omega^\bullet) := \text{dom}(\Delta^k_\partial),$$

where we note that $W^0(\Omega^\bullet) = L^2(\Omega^\bullet)$. Moreover, we denote

$$W^\infty(\Omega^\bullet) := \bigcap_{k \in \mathbb{N}_0} W^k(\Omega^\bullet),$$

and call it the space of smooth forms of $H$. The study of these spaces and their connections with noncommutative smoothness, and the theory of operator spaces \cite{57}, presents itself as a promising direction for future research.

In a related observation, we note that $W^\infty(\Omega^\bullet)$ carries an action of the bounded operators $L, \Lambda, \text{ and } H$, as well as the differential operators $d, \partial, \overline{\partial}$, and $\Delta_\overline{\partial}$. In the Kähler setting, it follows from the Kähler identities Theorem \cite{39, §3.B} that the vector space spanned by these operators forms a Lie superalgebra $\mathcal{K}$ with respect to the graded commutator bracket \cite{39, §3.B}. (See \cite{71} or \cite{25} for more details on the structure of $\mathcal{K}$.) Just as for ordinary Lie algebras, $\mathcal{K}$ has an enveloping Hopf superalgebra $U(\mathcal{K})$, which is to say a braided Hopf algebra in the braided category of super vector spaces, see \cite[Example 10.1.3]{49}. Note that by construction $U(\mathcal{K})$ acts on the space of smooth forms $W^\infty(\Omega^\bullet)$. The interaction between the $U(\mathcal{K})$-module structure of $W^\infty(\Omega^\bullet)$, and its analytic construction, presents itself as another interesting topic for investigation.

5. Fredholm Operators and the Holomorphic Euler Characteristic

In this section we use Hodge decomposition to relate the Fredholm property, for the Dolbeault–Dirac operator of a CQH-Hermitian space, to the cohomology of the underlying calculus. We observe that if $D_\overline{\partial}$ is Fredholm, then its index is given by the holomorphic Euler characteristic of the calculus. This is a direct generalisation of the classical relationship between the Dolbeault–Dirac index of an Hermitian manifold and the holomorphic Euler characteristic of the underlying complex manifold. This relationship between index theory and cohomology is one of the major strengths of the paper, allowing us to apply geometric tools to index theory calculations. For example, we observe that for any CQH-Fano space (a special type of CQH-Kähler space) its Dolbeault–Dirac operator will always have non-zero index.

5.1. Fredholm Operators. We begin by recalling the definition of an (unbounded) Fredholm operator, which abstracts the index theoretic properties of elliptic differential operators over a compact manifold.

**Definition 5.1.** For $\mathcal{H}_1$ and $\mathcal{H}_2$ two Hilbert spaces, and $T : \text{dom}(T) \subseteq \mathcal{H}_1 \to \mathcal{H}_2$ a densely defined closed linear operator, we say that $T$ is a Fredholm operator if $\ker(T)$
and \( \mathrm{coker}(T) \) are both finite-dimensional. The \textit{index} of a Fredholm operator \( T \) is then defined to be the integer
\[
\text{index}(T) := \dim(\ker(T)) - \dim(\mathrm{coker}(T)).
\]

As is well-known [75, §2], the image \( \text{im}(T) \) of a Fredholm operator \( T \) is always closed. In practice, however, it often proves easier to first establish closure, and from this establish finite-dimensionality of the cokernel. As we see below, this is the case for the Dolbeault–Dirac operator of a CQH-Hermitian space.

5.2. The Holomorphic Euler Characteristic. We now present the obvious noncommutative generalisation of the holomorphic Euler characteristic of an Hermitian manifold. Note that the definition makes sense for any differential calculus endowed with a complex structure, as it makes no mention of the addition structure of a CQH-Hermitian space.

\textbf{Definition 5.2.} Let \( \Omega^\bullet \) be a differential calculus, of total dimension \( 2n \), endowed with a complex structure \( \Omega^{(\bullet, \bullet)} \). The \textit{holomorphic Euler characteristic} of \( \Omega^{(\bullet, \bullet)} \) is the value
\[
\chi^\partial := \frac{1}{2\dim(H)} \sum_{k=0}^{\dim(H)} (-1)^k \dim(H^{(0,k)}) \in \mathbb{Z} \cup \{\pm \infty\}.
\]

Note that, unlike the case of classical compact complex manifolds, there exist examples of complex structures with infinite holomorphic Euler characteristics, which is to say, \( \chi^\partial \) does not necessarily lie in \( \mathbb{Z} \). (Explicit constructions of such examples will appear in later work.)

5.3. The Fredholm Index. Since \( D^\partial \) is a self-adjoint operator, if it were Fredholm, then its index would necessarily be zero. However, we can alternatively calculate its index with respect to the canonical \( \mathbb{Z}_2 \)-grading of the Hilbert space, a value which is not necessarily zero.

For any CQH-Hermitian space, take the two Hilbert spaces
\[
L^2(\Omega^{(0, \bullet)}_{\text{even}}) := \bigoplus_{k \in \mathbb{N}_0} L^2(\Omega^{(0,k)}), \quad L^2(\Omega^{(0, \bullet)}_{\text{odd}}) := \bigoplus_{k \in \mathbb{N}_0} L^2(\Omega^{(0,2k+1)}).
\]

Consider the restricted operator
\[
D^\partial_{\text{odd}} : \text{dom}(D^\partial) \cap L^2(\Omega^{(0, \bullet)}_{\text{even}}) \to L^2(\Omega^{(0, \bullet)}_{\text{odd}}), \quad x \mapsto D^\partial(x).
\]

We now use Hodge decomposition to relate the index of \( D^\partial_{\text{odd}} \) to the cohomology of the underlying calculus.

\textbf{Lemma 5.3.} The image of \( D^\partial_{\text{odd}} \) is closed, with respect to the Hilbert space norm, if and only if an isomorphism is given by
\[
H^{(0,2k+1)} \to \text{coker}(D^\partial_{\text{odd}}), \quad \alpha \mapsto [\alpha].
\]
Proof. Since $\Delta_{\overline{\partial}}$ commutes with $\overline{\partial}$, and is an operator of degree 0, it is diagonalisable on $\Omega^{(0,\bullet)}_{\text{odd}}$. Let $b$ be a basis element, for some choice of diagonalisation, and denote its non-zero eigenvalue by $\mu$. Now $\overline{\partial}b$ is a non-zero element of $\Omega^{(0,\bullet)}_{\text{even}}$, and

$$b = \Delta_{\overline{\partial}}(\mu^{-1}b) = \overline{\partial} \circ \overline{\partial}(\mu^{-1}b) = D_{\overline{\partial}}(\overline{\partial}(\mu^{-1}b)) \in \operatorname{im}(D_{\overline{\partial}}).$$

Hence $D_{\overline{\partial}}$ must map surjectively onto $\overline{\partial} \Omega^{(0,\bullet)}_{\text{even}}$. A similar argument shows that $D_{\overline{\partial}}$ maps surjectively onto $\overline{\partial} \Omega^{(0,\bullet)}_{\text{even}}$, meaning that $\overline{\partial} \Omega^{(0,\bullet)}_{\text{even}} \oplus \overline{\partial} \Omega^{(0,\bullet)}_{\text{even}} \subseteq \operatorname{im}(D_{\overline{\partial}})$. Thus we see that $\operatorname{im}(D_{\overline{\partial}})$ is closed if and only if it is equal to

$$L^2 \left( \overline{\partial} \Omega^{(0,\bullet)}_{\text{even}} \oplus \overline{\partial} \Omega^{(0,\bullet)}_{\text{even}} \right).$$

Recalling that Hodge decomposition is an orthogonal decomposition [3.17], we see that this is in turn equivalent to the given map in (9) being an isomorphism. $\square$

**Theorem 5.4.** For any CQH-Hermitian space $\mathbf{H}$, the following are equivalent:

1. $D_{\overline{\partial}}$ is an even Fredholm operator,
2. $\operatorname{im}(D_{\overline{\partial}})$ is a closed subspace of $L^2(\Omega^{(0,\bullet)})$ and $\dim(H^{(0,\bullet)}) < \infty$.

Moreover, if $D_{\overline{\partial}}$ is Fredholm, then its index is equal to the holomorphic Euler characteristic of $\Omega^{(\bullet,\bullet)}$, which is to say,

$$\operatorname{index}(D_{\overline{\partial}}) = \chi_{\overline{\partial}}.$$

Proof. Since $D_{\overline{\partial}}$ is diagonalisable on $\Omega^{\bullet,\bullet}$, its closure cannot admit an additional non-trivial zero eigenvector. Hence, the operator and its closure have the same kernel. By the equivalence between cohomology classes and harmonic forms implied by Hodge decomposition, we now have that

$$(10) \quad \dim\left( \ker(D_{\overline{\partial}}) \right) = \dim \left( \bigoplus_{k=0}^{\frac{1}{2}\dim(H)} \mathcal{H}^{(0,k)} \right) = \sum_{k=0}^{\frac{1}{2}\dim(H)} \dim(H^{(0,k)}).$$

By Lemma 5.3 above, we see that the image of $D_{\overline{\partial}}$ is closed if and only if

$$(11) \quad \dim\left( \operatorname{coker}(D_{\overline{\partial}}) \right) = \dim \left( \bigoplus_{k=1}^{\frac{1}{2}\dim(H)} \mathcal{H}^{(0,2k+1)} \right) = \sum_{k=1}^{\frac{1}{2}\dim(H)} \dim(H^{(0,k)}).$$

Now if $D_{\overline{\partial}}$ is Fredholm, then it is closed by definition, and hence by (10) and (11), it must have finite-dimensional anti-holomorphic cohomology groups. Conversely, if $D_{\overline{\partial}}$ is closed and has finite-dimensional anti-holomorphic cohomology groups, then its kernel and cokernel must be finite dimensional. Since it is densely defined by construction, and a closed operator by Corollary 4.14, we see that it must be Fredholm.
Finally, if $D_\nabla^+$ is Fredholm, then its index is given by the holomorphic Euler characteristic, as claimed:

\[
\text{index}(D_\nabla^+) = \dim \left( \ker \left( D_\nabla^+ \right) \right) - \dim \left( \coker \left( D_\nabla^+ \right) \right) = \sum_{k=0}^{\frac{1}{2}\dim(\mathcal{H})} \dim \left( H^{0,k} \right) - \sum_{k=0}^{\frac{1}{2}\dim(\mathcal{H})} \dim \left( H^{0,k+1} \right) - \frac{1}{2} \dim(\mathcal{H}) \dim \left( H^{(0,k)} \right)
\]

\[= \chi_{\nabla}.\]

Remark 5.5. Determining if the operator $D_\nabla$ has closed range is a non-trivial task. By a standard functional analytic argument, $D_\nabla$ will have closed range if and only if its set of non-zero eigenvalues does not have 0 as an accumulation point (see Corollary 9.18). Such bounds can, in general, be quite difficult to produce. However, given the very geometric construction of $D_\nabla$, there are a number of classical geometric techniques to fall back on. In particular, there is the well-studied question of lowest eigenvalue estimates for Dirac operators on spin manifolds in general [27, §5], and hence Hermitian spin manifolds in particular. This is a question intimately connected with Schrödinger–Lichnerowicz [27, §5] and Weitzenböck techniques [59, §14]. A proper treatment of this question, in the noncommutative setting, will appear in later works, while an analogous approach for twists by holomorphic vector bundles appears in [9, §1].

6. CQH Fano Spaces

In this section we recall the basic results of connections, holomorphic structures, Hermitian metrics, and noncommutative Chern connections, as introduced by Beggs and Majid [4]. We then recall the notion of a Fano structure introduced in [67] and the naturally implied notion of a CQH-Fano space. We observe that for such spaces, vanishing results established in [67] imply that the Dolbeault–Dirac operator is Fredholm if and only if its image is closed, whereupon it has non-zero index. Such interactions of geometry and index theory form one of the most important themes of the paper.

6.1. Holomorphic Vector Bundles. We begin by recalling the definition of a noncommutative holomorphic vector bundle, as considered, for example, in [6], [12], and [41]. The definition directly generalises the classical Koszul–Malgrange characterisation of holomorphic bundles [43]. See [67] for a more detailed discussion.

For $\Omega^*$ a differential calculus over an algebra $B$, and $\mathcal{F}$ a left $B$-module, a connection for $\mathcal{F}$ is a $\mathbb{C}$-linear map $\nabla : \mathcal{F} \to \Omega^1 \otimes_B \mathcal{F}$ satisfying

\[\nabla(bf) = db \otimes_B f + b\nabla f, \quad \text{for all } b \in B, f \in \mathcal{F}.\]
Any connection can be extended to a map $\nabla : \Omega^* \otimes_B F \to \Omega^* \otimes_B F$ uniquely defined by
$$\nabla(\omega \otimes_B f) = d\omega \otimes_B f + (-1)^{|\omega|} \omega \wedge \nabla f,$$
for $f \in F$, $\omega \in \Omega^*$,
where $\omega$ is a homogeneous form of degree $|\omega|$. The curvature of a connection is the left $B$-module map $\nabla^2 : F \to \Omega \otimes_B \Omega^2$. A connection is said to be flat if $\nabla^2 = 0$. Since $\nabla^2(\omega \otimes f) = \omega \wedge \nabla^2(f)$, a connection is flat if and only if the pair $(\Omega^* \otimes_B F, \nabla)$ is a complex. With respect to a choice $\Omega^{(\bullet, \bullet)}$ of complex structure on $\Omega^*$, a $(0,1)$-connection on $F$, is a connection with respect to the differential calculus $(\Omega^{(0, \bullet)}, \partial)$. The curvature of a connection is the left $B$-module map $\nabla^2 : F \to \Omega \otimes_B \Omega^2$. A connection is said to be flat if $\nabla^2 = 0$. Since $\nabla^2(\omega \otimes f) = \omega \wedge \nabla^2(f)$, a connection is flat if and only if the pair $(\Omega^* \otimes_B F, \nabla)$ is a complex. With respect to a choice $\Omega^{(\bullet, \bullet)}$ of complex structure on $\Omega^*$, a $(0,1)$-connection on $F$, is a connection with respect to the differential calculus $(\Omega^{(0, \bullet)}, \partial)$.

**Definition 6.1.** A holomorphic vector bundle is a pair $(F, \overline{\partial}_F)$, where $F$ is a finitely generated projective left $B$-module, and $\overline{\partial}_F : F \to F \otimes \Omega^{(0,1)}$ is a flat $(0,1)$-connection.

**6.2. Hermitian Vector Bundles.** Next we recall the generalisation of the classical notion of an Hermitian metric for a vector bundle, which requires us to assume that $B$ is a *-algebra. For a left $B$-module $F$, denote by $F^\vee$ the dual right $B$-module $\text{Hom}_B(F, B)$. Moreover, denote by $F^\vee$ the conjugate right $B$-module of $F$, as defined by
$$F^\vee \otimes_B B \to F, \quad \overline{f} \otimes b \mapsto \overline{b} \overline{f}.$$

**Definition 6.2.** An Hermitian vector bundle is a pair $(F, h_F)$, where $F$ is a finitely generated projective left $B$-module, and $h_F : F^\vee \otimes_B F \to B$ is a left $B$-module map satisfying
1. $h_F(e \otimes_B f) = h_F(f \otimes_B e)^*$, for all $e, f \in F$,
2. $h_F(f \otimes_B e) \in B_{>0}$, for all $f \in F$, such that $f \neq 0$,
where $B_{>0}$ denotes the cone of positive elements of $B$, as defined in Definition 5.12.

We say that a connection $\nabla : F \to \Omega^1 \otimes_B F$ is Hermitian if it satisfies
$$d \circ h(f, g) = h_F(\nabla(f), g) + h_F(f, \nabla(g)).$$

In the classical setting, any Hermitian holomorphic bundle admits a unique extension of its $(0,1)$-connection to an Hermitian connection called the Chern connection, see [39, §4.2] for details. In [4] Beggs and Majid showed that this result carries over to the noncommutative setting. (For an alternative proof, using the conventions of this paper, see [67].)

**Lemma 6.3.** For any Hermitian holomorphic vector bundle $(F, h, \overline{\partial}_F)$, there exists a unique Hermitian connection $\nabla : F \to \Omega^1 \otimes_A F$ satisfying
$$(\text{proj}_{\Omega^{(0,1)}} \otimes_B \text{id}) \circ \nabla = \overline{\partial}_F,$$
where $\text{proj}_{\Omega^{(0,1)}} : \Omega^1 \to \Omega^{(0,1)}$ is the obvious projection. We call $\nabla$ the Chern connection of $(F, h, \overline{\partial}_F)$.

**Remark 6.4.** We consider now the case of quantum homogeneous spaces. From Takeuchi’s equivalence, and the above discussions (see Proposition 2.12), it is clear that an Hermitian structure on $F$ is uniquely determined by a $H$-comodule isomorphism $\Phi(F) \simeq \Phi(F)$. Hence, for irreducible objects $F \in \mathcal{A}/\mathcal{D}$ mod $\Phi$, there is a unique Hermitian structures, up to scalar multiple (see [19, Theorem 11.27]
6.3. Positive and Negative Vector Bundles. We finish with the notion of positivity for a holomorphic Hermitian vector bundle. This directly generalises the classical notion of positivity, a property which is equivalent to ampleness \cite[Proposition 5.3.1]{39}. It was first introduced in \cite[§8.1]{67} and requires a compatibility between Hermitian holomorphic vector bundles and Kähler structures.

**Definition 6.5.** Let $\Omega^\bullet$ be a differential calculus over an algebra $B$, and let $(\Omega^{\bullet,\bullet}, \kappa)$ be a Kähler structure for $\Omega^\bullet$. An Hermitian holomorphic vector bundle $(\mathcal{F}, h_{\mathcal{F}}, \overline{\partial}_{\mathcal{F}})$ is said to be **positive** if there exists a positive definite Kähler form $\kappa$, such that the Chern connection $\nabla$ of $\mathcal{F}$ satisfies

$$\nabla^2(f) = -i\kappa \otimes_B f,$$

for all $f \in \mathcal{F}$. Analogously, $(\mathcal{F}, h_{\mathcal{F}}, \overline{\partial}_{\mathcal{F}})$ is said to be **negative** if there exists a positive definite Kähler form $\kappa$, such that the Chern connection $\nabla$ of $\mathcal{F}$ satisfies

$$\nabla^2(f) = i\kappa \otimes_B f,$$

for all $f \in \mathcal{F}$.

6.4. Noncommutative Fano Structures. In order to produce a holomorphic vector bundle from a complex structure, we recall from \cite[§6.3]{65} a refinement called factorisability. The Dolbeault double complex of every complex manifold is automatically factorisable \cite[§1.2]{39}, as are the Heckenberger–Kolb calculi for the all irreducible flag manifolds, as discussed in §10.

**Definition 6.6.** An almost complex structure for a differential $*$-calculus $\Omega^\bullet$ over a $*$-algebra $B$, is called **factorisable** if we have bimodule isomorphisms

$$\wedge : \Omega^{(a,0)} \otimes_B \Omega^{(0,b)} \simeq \Omega^{(a,b)},$$

and

$$\wedge : \Omega^{(0,b)} \otimes_B \Omega^{(a,0)} \simeq \Omega^{(a,b)},$$

where, as usual, $\wedge$ denotes the multiplication of $\Omega^\bullet$.

An important point to note is that, for any factorisable complex structure $\Omega^{(\bullet,\bullet)}$, the pair $(\Omega^{(n,0)}, \overline{\partial})$ is a holomorphic vector bundle. Moreover, for a **factorisable Hermitian**, or **factorisable Kähler**, structure, which is to say an Hermitian, or Kähler, structure whose constituent complex structure is factorisable, the triple $(\Omega^{(n,0)}, h_{\mathcal{F}}, \overline{\partial}_{\mathcal{F}})$ is an Hermitian holomorphic vector bundle. In particular, $(\Omega^{(n,0)}, h_{\mathcal{F}}, \overline{\partial}_{\mathcal{F}})$ has an associated Chern connection $\nabla$.

**Definition 6.7.** A **Fano structure**, for a differential calculus $\Omega^\bullet$, is a $2n$-dimensional Kähler structure $(\Omega^{\bullet,\bullet}, \kappa)$ for the calculus, such that

1. $\Omega^{(\bullet,\bullet)}$ is a factorisable complex structure,
2. the holomorphic Hermitian vector bundle $(\Omega^{(n,0)}, h_{\mathcal{F}}, \overline{\partial}_{\mathcal{F}})$ is negative.

6.5. CQH-Fano Spaces. We finish this section by introducing the obvious notion of a CQH-Fano space. For sake of clarity, we recall the obvious definitions of covariant holomorphic, and Hermitian, vector bundles.

**Definition 6.8.** An Hermitian vector bundle $(\mathcal{F}, h_{\mathcal{F}})$ is said to be **covariant** if $\mathcal{F}$ is an object in $\mathcal{A}_B\mod_0$, and $h_{\mathcal{F}} : \mathcal{F} \otimes^\vee \mathcal{F} \rightarrow B$ is left $A$-comodule map. Moreover, a holomorphic vector bundle $(\mathcal{F}, \overline{\partial}_{\mathcal{F}})$ is said to be **covariant** if $\mathcal{F}$ is an object in $\mathcal{A}_B\mod_0$, and $\overline{\partial}_{\mathcal{F}} : \mathcal{F} \rightarrow \Omega^{(0,1)} \otimes_B \mathcal{F}$ is a left $A$-comodule map.
It is important to note that, as is established in [4, §4], and again in [67, §7.1], the Chern connection of a covariant Hermitian holomorphic vector bundle is a left $A$-comodule map.

**Definition 6.9.** A compact quantum homogeneous Fano space, or simply a CQH-Fano space, is a CQH-Kähler space $F = (B, \Omega^*, \Omega^{(\bullet\bullet)}, \sigma)$ such that the pair $(\Omega^{(\bullet\bullet)}, \sigma)$ is a Fano structure for $\Omega^*$.

We are interested in CQH-Fano structures because of the following result, established in [67, Corollary 8.9] as a consequence of the noncommutative Kodaira vanishing theorem for Kähler structures [67, Theorem 8.3]. (See §9.5.1 for the statement of the noncommutative Kodaira vanishing theorem, as well as a novel variation on the proof.)

**Theorem 6.10.** For a CQH-Fano space $F = (B, \Omega^*, \Omega^{(\bullet\bullet)}, \kappa)$, it holds that $H^{(0,k)} = 0$, for all $k \neq 0$.

**Corollary 6.11.** For a CQH-Fano space $F$, the operator $D^\pm_\partial$ is Fredholm if and only if its image is closed and $\dim(H^{(0,0)})$ is finite-dimensional. In this case

$$ (12) \quad \chi_\partial = \dim(H^{(0,0)}) \neq 0. $$

**Proof.** The characterisation of $D^\pm_\partial$ as a Fredholm operator follows directly from Theorem 6.10 and Theorem 9.12, as does the identity in (12). Non-triviality of $H^{(0,0)}$ follows from the fact that $D^\pm_\partial(1) = \partial(1) = 0$, where the last identity is a standard consequence of the Leibniz rule, holding for any unital dg-algebra. □

### 7. Dolbeault–Dirac Spectral Triples

In this section we recall the definition of a spectral triple, or unbounded $K$-homology class, the object around which Connes constructed his notion of a noncommutative Riemannian spin manifold [13]. In particular, we discuss when a CQH-Hermitian space gives rise to such a structure. Spectral triples provide a means for calculating the index pairing between the $K$-theory groups of a $C^*$-algebra, and more formally abstract the properties of classical Riemannian spin manifolds. In particular, they abstract the properties of the Dolbeault–Dirac operator on an Hermitian manifold, with a prototypical example being provided by

$$ \left(C^\infty(M), D_\bar{\partial}, L^2(\Omega^{(0,\bullet)}) \right). $$

For a presentation of the classical Dolbeault–Dirac operator of an Hermitian manifold as a commutative spectral triple, see [37] or [27]. For a standard reference on the general theory of spectral triples, see [30] or [10]. A presentation of the relationship between Hermitian and spin manifolds is given in [1, Proposition 3.2].

**7.1. Spectral Triples, $K$-homology, and the Bounded Transform.** We begin by carefully recalling the definition of $K$-homology, which we consider as the main motivation for the introduction of spectral triples.
Definition 7.1. Let $\mathcal{A}$ be a unital separable $C^*$-algebra. A Fredholm module over $\mathcal{A}$ is a triple $(\mathcal{H}, F, \rho)$, where $\mathcal{H}$ is a separable Hilbert space, $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a $*$-representation, and $F : \mathcal{H} \to \mathcal{H}$ a bounded linear operator, such that

$$F^2 - 1, \quad F - F^*, \quad [F, \rho(a)],$$

are all compact operators, for any $a \in \mathcal{A}$. An even Fredholm module is a Fredholm module $(\mathcal{H}, F, \rho)$ together with a $\mathbb{Z}_2$-grading $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ of Hilbert spaces, with respect to which $F$ is a degree 1 operator, and $\rho(a)$ is a degree 0 operator, for each $a \in \mathcal{A}$.

The direct sum of two even Fredholm modules is formed by taking the direct sum of Hilbert spaces, representations, and operators. For $(\mathcal{H}, F, \rho)$ an even Fredholm module, and $u : \mathcal{H} \to \mathcal{H}'$ a degree-0 unitary transformation, the triple $(\mathcal{H}', uFu^*, upu^*)$ is again a Fredholm module. This defines an equivalence relation on Fredholm modules over $\mathcal{A}$, which we call unitary equivalence. Moreover, we say that a norm continuous family of Fredholm modules $(\mathcal{H}, F_t, \rho_t)$, for $t \in [0, 1]$, defines an operator homotopy between the two Fredholm modules $(\mathcal{H}, F_0, \rho)$ and $(\mathcal{H}, F_1, \rho)$.

Definition 7.2. The $K$-homology group $K^0(\mathcal{A})$ of a $C^*$-algebra $\mathcal{A}$ is the abelian group with one generator for each unitary equivalence class of even Fredholm modules, subject to the following relations: For any two even Fredholm modules $\mathcal{M}_0, \mathcal{M}_1$,

1. $[\mathcal{M}_0] = [\mathcal{M}_1]$ if there exists an operator homotopy between $\mathcal{M}_0$ and $\mathcal{M}_1$,
2. $[\mathcal{M}_0 \oplus \mathcal{M}_1] = [\mathcal{M}_0] + [\mathcal{M}_1]$, where $+$ denotes addition in $K^0(\mathcal{A})$.

For any Fredholm module $\mathcal{M} = (\mathcal{H}, F, \rho)$, we see that a Fredholm operator is defined by $F_+ := F|_{\mathcal{H}_+} : \mathcal{H}_+ \to \mathcal{H}_-$. Moreover, a well-defined group homomorphism is given by

$$\text{Index} : K^0(\mathcal{A}) \to \mathbb{Z}, \quad [\mathcal{M}] \mapsto \text{Index}(F_+) = \ker(F_+) - \text{cokernel}(F_+).$$

In practice the calculation of the index of a $K$-homology class, or more generally the pairing with $K$-theory, can be very difficult. However, the work of Baaj and Julg [2], and Connes and Moscovici [15], shows that by considering spectral triples, unbounded representatives of $K$-homology classes, the problem can often become more tractable.

Definition 7.3. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of a unital $*$-algebra $\mathcal{A}$, a separable Hilbert space $\mathcal{H}$, endowed with a faithful $*$-representation $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, and $D : \text{dom}(D) \to \mathcal{H}$ a densely-defined self-adjoint operator, such that

1. $\rho(a)\text{dom}(D) \subseteq \text{dom}(D)$, for all $a \in \mathcal{A}$,
2. $[D, \rho(a)]$ is a bounded operator, for all $a \in \mathcal{A}$,
3. $(D^2 + i)^{-1} \in \mathcal{K}(\mathcal{H})$, where $\mathcal{K}(\mathcal{H})$ denotes the compact operators on $\mathcal{H}$.

An even spectral triple is a quadruple $(\mathcal{A}, \mathcal{H}, D, \gamma)$, consisting of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, and a $\mathbb{Z}_2$-grading $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ of Hilbert spaces $\gamma$, with respect to which $D$ is a degree 1 operator, and $\rho(a)$ is a degree 0 operator, for each $a \in \mathcal{A}$.

Spectral triples are important primarily because they provide unbounded representatives for $K$-homology classes. For a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, its bounded transform is the
operator
\[ b(D) := \frac{D}{\sqrt{1 + D^2}} \in \mathbb{B}(\mathcal{H}), \]
defined via the functional calculus. Denoting by \( \overline{\mathcal{A}} \) the closure of \( \rho(A) \) with respect to the operator topology of \( \mathbb{B}(\mathcal{H}) \), a Fredholm module is given by \( (\mathcal{H}, \rho, b(D)) \). (See [9] for details.) The index of the Fredholm operator \( D_+ \) is clearly equal to the index of the bounded transform. Since the index is an invariant of \( K \)-homology classes, a spectral triple with non-zero index has a non-trivial associated \( K \)-homology class.

7.2. Cores and Domains. In general, proving that \( \text{dom}(D) \) is closed under the action of \( \rho(a) \) can be difficult. The following proposition, proved by Forsyth, Mesland, and Rennie, in [26, Proposition 2.1], gives us the possibility of instead proving the requirement for a core of \( \text{dom}(D) \), something which can in practice be much easier. Recall that a core for a closed operator \( T : \text{dom}(T) \to \mathcal{H} \) is a subset \( X \subseteq \text{dom}(T) \) such that \( T \) is equal to the closure of the restriction of \( T \) to \( X \).

**Proposition 7.4.** Let \( \mathcal{H} \) be a separable Hilbert space, \( D : \text{dom}(D) \subseteq \mathcal{H} \to \mathcal{H} \) a densely-defined closed operator, \( X \subseteq \text{dom}(D) \) a core for \( D \), and \( L \in \mathbb{B}(\mathcal{H}) \) such that
1. \( L(X) \subseteq \text{dom}(D) \),
2. \( [D, L] : X \to \mathcal{H} \) is bounded on \( X \).

Then it holds that \( L(\text{dom}(D)) \subseteq \text{dom}(D) \).

Applying this proposition directly to a general CQH-Hermitian space, we get the following result.

**Corollary 7.5.** For any CQH-Hermitian space \( H = (B, \Omega^*, \Omega^{*\bullet}, \sigma) \), with Dolbeault–Dirac operator \( D_\gamma \), it holds that
\[ \rho(b)\text{dom}(D_\gamma) \subseteq \text{dom}(D_\gamma), \]
for all \( b \in B \).

**Proof.** The subspace \( \Omega^* \subseteq \text{dom}(D_\gamma) \) is a core by construction of the closure of \( D_\gamma \). Moreover, since \( B \) is a subalgebra of \( \Omega^* \), the core is clearly closed under the action of \( \rho(b) \), for all \( b \in B \). Recalling from Proposition 4.10 that \( [D_\gamma, \rho(b)] \) is a bounded operator, for all \( b \in B \), we see Proposition 7.4 implies that \( \rho(b)\text{dom}(D_\gamma) \subseteq \text{dom}(D_\gamma) \) as claimed. \( \square \)

7.3. Spectral Triples and Dolbeault–Dirac Eigenvalues. We would now like a precise criteria for when the Dolbeault–Dirac operator of a CQH-Hermitian space gives a spectral triple. For sake of clarity and convenience, let us recall the relevant properties of the \( D_\gamma \). By Corollary 4.9, we have a faithful \( * \)-representation \( \rho : B \to \mathbb{B}(L^2(\Omega^*)) \). From Corollary 4.17 we know that \( D_\gamma \) is an essentially self-adjoint operator, which is, moreover, densely-defined by construction. By Corollary 4.11 the commutators \( [D_\gamma, \rho(b)] \) are bounded, and by Corollary 7.4 above, \( \rho(b)\text{dom}(D_\gamma) \subseteq \text{dom}(D_\gamma) \), for all \( b \in B \). With respect to the \( \mathbb{Z}_2 \)-grading \( \gamma \) defined in 4.3, the operator \( D_\gamma \) is of degree 1, and \( \rho(b) \) is a degree 0 operator, for all \( b \in B \). Finally, we note that since \( D_\gamma \) is diagonalisable on \( L^2(\Omega^*) \), it has compact resolvent if and only if its eigenvalues tend to infinity and have
finite multiplicity. Collecting these facts together gives the obvious result, which we find convenient to present in the form of a lemma.

**Lemma 7.6.** Let $H = (B = A^\text{co}(H), \Omega^\bullet, \Omega^\bullet(\bullet), \sigma)$ be a CQH-Hermitian space such that $\hat{A}$ (the isomorphism classes of irreducible comodules of $A$) is countable, then an even spectral triple is given by

$$\left( B, L^2(\Omega^{(0,\bullet)}), D_\gamma, \gamma \right),$$

if and only if the eigenvalues of $D_\gamma$ tend to infinity and have finite multiplicity. We call such a spectral triple the Dolbeault–Dirac spectral triple of $H$.

We now come to the $K$-homology classes associated to a Dolbeault–Dirac spectral triple via the bounded transform. Note that since the representation $\overline{\rho} : B \to B(L^2(\Omega^\bullet))$ is an isometric $*$-isomorphism, its image is closed, implying that $\overline{\rho(B)} \simeq B$. Thus the bounded transform takes its image in

$$K^0 \left( \overline{\rho(B)} \right) \simeq K^0(B).$$

The discussions of §5.3 and §6 now give us the following immediate results, which we find convenient to present as corollaries.

**Corollary 7.7.** Let $H = (B, \Omega^\bullet, \Omega^\bullet(\bullet), \sigma)$ be a CQH-Hermitian space with a Dolbeault–Dirac spectral triple. The $K^0(B)$-class of the spectral triple is non-trivial if the holomorphic Euler characteristic of $\Omega^\bullet(\bullet)$ is non-trivial.

**Corollary 7.8.** Let $F = (B, \Omega^\bullet, \Omega^\bullet(\bullet), \sigma)$ be a CQH-Fano space with a Dolbeault–Dirac spectral triple, then the $K^0(B)$-homology class of the spectral triple is non-trivial.

As discussed in the introduction, it is not clear at present how to conclude the compact resolvent condition from the general properties of a CQH-Hermitian space. Hence, in our examples we resort to calculating the spectrum explicitly, and verifying the required eigenvalue growth directly. See, for example, the case of quantum projective space as discussed in §10.1.2.

### 7.4. Fröhlich–Grandjean–Recknagel Sets of Kähler Spectral Data.

At this point we find it interesting to recall an alternative approach to noncommutative Hermitian and Kähler geometry appearing in the literature. In a series of papers [29, 28] Fröhlich, Grandjean, and Recknagel introduced sets of symplectic spectral data, Hermitian spectral data, Kähler spectral data, and hyper-Kähler spectral data. These are essentially spectral triples, modelled on the de Rham–Dirac operator $d + d^1 : \text{dom}(d + d^1) \to L^2(\Omega^\bullet)$, together with additional linear operators on $L^2(\Omega^\bullet)$, generalising the structure of the de Rham complex of symplectic, Hermitian, Kähler, and hyper-Kähler manifolds respectively. The noncommutative 2-torus $T_\alpha$ was taken as the motivating example, while new examples, coming from $C^*$-dynamical systems, have recently been discovered by Guin in [31].

The approach of Fröhlich, Grandjean, and Recknagel shares many commonalities with CQH-Hermitian spaces. Analogues of the Hodge map $*_\sigma$, and the grading operators $\gamma, \tau$ and $\overline{\gamma}$ form part of the definition of an Hermitian spectral data, where they are denoted
8. The Opposite CQH-Hermitian Space

Building on the definition of opposite complex structure, we introduce the notion of opposite CQH-Hermitian space. This serves as a useful formal framework in which to present the holomorphic Dolbeault–Dirac operator $D_\partial$. We then construct a unitary equivalence between $D_\partial$ and $D_{\overline{\partial}}$, showing that, from a spectral point of view, the two operators are essentially the same. We finish, however, by briefly discussing how the two operators might be told apart using equivariance.

8.1. Opposite Complex, Hermitian, and Kähler Structures. We begin by recalling from [65] the notion of an opposite complex structure, which is a direct generalisation of the corresponding classical notion.

Definition 8.1. The opposite almost-complex structure of an almost-complex structure $\Omega^{(\bullet, \bullet)}$ is the $\mathbb{N}_0^2$-algebra grading $\overline{\Omega}^{(a, b)}$, defined by $\overline{\Omega}^{(a, b)} := \Omega^{(b, a)}$, for $(a, b) \in \mathbb{N}_0^2$.

Note that the $*$-map of the calculus sends $\Omega^{(a, b)}$ to $\overline{\Omega}^{(a, b)}$ and vice-versa. Moreover, it is clear that an almost-complex structure is a complex structure if and only if its opposite almost-complex structure is a complex structure. Hence, we can speak of the opposite complex structure of a complex structure.

Lemma 8.2. For any Hermitian structure $(\Omega^{(\bullet, \bullet)}, \sigma)$, it holds that

1. $(\overline{\Omega}^{(\bullet, \bullet)}, -\sigma)$ is an Hermitian structure, which we call the opposite Hermitian structure of $(\Omega^{(\bullet, \bullet)}, \sigma)$,
2. $L_{-\sigma} = -L_\sigma$,
3. $\Lambda_{-\sigma} = -\Lambda_\sigma$,
4. $P^{(a, b)} = \overline{P}^{(b, a)}$, where $\overline{P}^{(b, a)}$ denotes the primitive forms of $(\overline{\Omega}^{(\bullet, \bullet)}, -\sigma)$,
5. $*_{-\sigma} = (-1)^n *_\sigma$,
6. $g_{-\sigma} = g_\sigma$,
7. $(\overline{\Omega}^{(\bullet, \bullet)}, -\sigma)$ is positive definite if and only if $(\Omega^{(\bullet, \bullet)}, \sigma)$ is positive definite.

Proof. Since $\sigma$ is an Hermitian form, $-\sigma$ must be a real central $(1, 1)$-form. For any $\omega \in \Omega^*$, we have $L_{-\sigma}(\omega) = -\sigma \wedge \omega = -(\sigma \wedge \omega) = -L_\sigma(\omega)$, and so, $L_{-\sigma} = -L_\sigma$. From this we see that isomorphisms are given by the maps

$L_{n-k}^{-1} : \overline{\Omega}^k \rightarrow \Omega^{2n-k}$, for all $k = 0, \ldots, n - 1$.

Thus the pair $(\overline{\Omega}^{(\bullet, \bullet)}, -\sigma)$ is an Hermitian structure.

By definition, $\alpha \in P^{(a, b)}$ if and only if $\alpha \in \Omega^{(b, a)}$ and $L_{-\sigma}^{-k+1}(\alpha) = 0$. Explicitly,

$L_{-\sigma}^{-k+1}(\alpha) = (L_\sigma)^{n-k+1}(\alpha) = (-1)^{n-k+1} L_{-\sigma}^{-k+1}(\alpha)$.

Thus, we see that $\alpha \in P^{(a, b)}$ if and only if it is an element of $P^{(b, a)}$. 

*}, $\gamma$, $T$ and $\overline{T}$ respectively (see [29, Definition 2.6] for details). Moreover, analogues of the identities in Corollary 3.20 are taken as part of the definition of a set of Kähler spectral data [29, Definition 2.28].
From the defining formula for the Hodge map, as given in Definition 2.5, we see that, for \( \alpha \in P^{(a,b)} = P^{(b,a)} \subseteq \Omega^k \),

\[
\star_{-\sigma}(L^j_{-\sigma}(\alpha)) = (-1)^{\frac{k(k+1)}{2}} i^{a-b} \frac{j!}{(n-j-k)!} (-L_{\sigma})^{n-j-k}(\alpha)
\]

\[
= i^{2(a-b)} (-1)^{n-j-k} \left( (-1)^{\frac{k(k+1)}{2}} i^{b-a} \frac{j!}{(n-j-k)!} L_{\sigma}^{n-j-k}(\alpha) \right)
\]

\[
= (-1)^{a+b}(-1)^{n-j-k} \star_{\sigma} (L^j_{\sigma}(\alpha))
\]

\[
= (-1)^k(-1)^{n-k} \star_{\sigma} ((-L_{\sigma})^j(\alpha))
\]

\[
= (-1)^n \star_{\sigma} (L^j_{-\sigma}(\alpha)).
\]

Recalling from 4 that \( \Lambda_{\sigma} = \star_{\sigma}^{-1} \circ L_{\sigma} \circ \star_{\sigma} \), we now see that \( \Lambda_{-\sigma} = -\Lambda_{\sigma} \).

The fact that \( g_{\sigma} = g_{-\sigma} \) now follows from the proportionality of Hodge maps, and the definition of the metric, as we see from

\[
g_{-\sigma}(\omega, \nu) = \star_{-\sigma} \left( \star_{-\sigma} (\omega^*) \wedge \nu \right) = (-1)^{2n} \star_{\sigma} \left( \star_{\sigma} (\omega^*) \wedge \nu \right) = g_{\sigma}(\omega, \nu).
\]

From this it follows immediately that \((\Omega^*(\bullet), -\sigma)\) is positive definite if and only if \((\Omega^*(\bullet), \sigma)\) is positive definite.

The following lemma, presenting the opposite representation of \( \mathfrak{sl}_2 \), follows immediately from Lemma 8.2 above.

**Corollary 8.3.** The representation \( \overline{\rho} : \mathfrak{sl}_2 \to B(L^2(\Omega^*)) \) associated to the opposite Hermitian structure \((\Omega^*(\bullet), -\sigma)\) is given explicitly by

\[
\overline{\rho}(E) = -L_{\sigma}, \quad \rho(K) = K, \quad \overline{\rho}(F) = -\Lambda_{\sigma}.
\]

We note that for a Kähler structure \((\Omega^*(\bullet), \sigma)\), its opposite Hermitian structure is clearly again a Kähler structure. Hence we can speak of the opposite Kähler structure of a Kähler structure. Moreover, recalling Definition 6.5, we see that a positive vector bundle is negative with respect to the opposite Kähler structure, and conversely a negative vector bundle is positive with respect to the opposite Kähler structure.

### 8.2. Opposite CQH-Hermitian Spaces.

In this subsection we consider opposite Hermitian structures in the context of CQH-Hermitian spaces, introducing the notion of an opposite CQH-Hermitian space. We begin with the following lemma, which is an immediate consequence of the definition of a CQH-Hermitian space, and Lemma 8.2.

**Lemma 8.4.** For any CQH-Hermitian space \( \mathbf{H} = (B, \Omega^*, \Omega^*(\bullet), \sigma) \), a CQH-Hermitian space is given by

\[
\overline{\mathbf{H}} = (B, \Omega^*, \overline{\Omega^*(\bullet)}, -\sigma).
\]

We call \( \overline{\mathbf{H}} \) the opposite CQH-Hermitian space of \( \mathbf{H} \).
Proposition 8.6. There exists a unitary equivalence two operators

The above proposition implies that the triple $(\Omega^0, D_\partial)$ satisfies the requirements of a spectral triple if and only if $(B, L^2(\Omega^{(0,•)}), D_\partial)$ does. Moreover, $u$ induces a unitary equivalence between the bounded transform of both operators, meaning that they have the same $K$-homology class.

We should note, however, that the restriction of $u$ to $\Omega^•$ is not necessarily a left $A$-comodule map. Thus taking the $A$-comodule structure into account, it can prove possible to distinguish between the two operators $D_\partial$ and $D_\overline{\partial}$. While we will not do so here, this
fact can be formally presented in the context of equivariant spectral triples. See [78] for a presentation of equivariant spectral triples.

9. Twisted Dolbeault–Dirac Fredholm Operators

In this section we treat twists of the Dolbeault complex by Hermitian holomorphic vector bundles, observing that the constructions of §4 and §5 naturally extend to this more general setting. A significant difference between the twisted and untwisted cases is that, even in the Kähler setting, the Laplacian operators $\Delta_{\partial F}$ and $\Delta_{\partial F}$ are no longer guaranteed to coincide. Just as in the classical case, they differ by a possibly non-trivial curvature operator $[i\nabla^2, L_F]$. Exploiting this difference, we show that when $[i\nabla^2, L_F]$ is positive, finite-dimensionality of the anti-holomorphic cohomology groups is enough to guarantee that $D_{\overline{\partial} F}$ is Fredholm. This highlights the intimate relationship between the algebraic and analytic properties of a CQH-Hermitian space, and in particular, how the spectral properties of Dolbeault–Dirac operators are moulded by the geometry of the underlying calculus.

9.1. Hermitian Vector Bundle Hilbert Spaces. For any Hermitian vector bundle $(F, h)$, we recall that an inner product is given by

\[ \langle \cdot, \cdot \rangle : F \times F \rightarrow \mathbb{C}, \quad (e, f) \mapsto h(h(e)(f)). \]

**Definition 9.1.** We denote by $L^2(F)$ the completion of $F$ with respect to $\langle \cdot, \cdot \rangle$, and call it the Hilbert space of square integrable sections of $F$.

When dealing with the Hilbert space of square integrable forms $L^2(\Omega^*)$, we found it useful to consider an alternative presentation in Proposition 4.2, given in terms of Takeuchi’s equivalence. This result generalises directly to the setting of Hermitian vector bundles, as we present in this subsection.

First we generalise to the untwisted case the sesquilinear form introduced in §4.1. Let $B = A^{co(H)}$ be a CQGA homogeneous space, and $(F, h)$ a covariant Hermitian vector bundle over $B$, we have an associated sesquilinear form defined by

\[ (\cdot, \cdot) : \Phi(F) \otimes_R \Phi(F) \rightarrow \mathbb{C}, \quad [f] \otimes_R [h] \mapsto [h(f)(g)]. \]

This in turn gives us the sesquilinear form

\[ \langle \cdot, \cdot \rangle_U : A^\square_H \Phi(F) \otimes_R A^\square_H \Phi(F) \rightarrow \mathbb{C}, \quad \sum_{i,j} f_i \otimes v_i \otimes_R g_j \otimes w_j \mapsto \sum_{i,j} (f_i, g_j) h(v_i, w_j). \]

The proof of the following lemma is a direct generalisation of the proof given for the untwisted case in Proposition 4.2 and hence is omitted.

**Lemma 9.2.** Let $B = A^{co(H)}$ be a CQGA homogeneous space, and $(F, h)$ a covariant Hermitian vector bundle over $B$.

1. The sesquilinear form $(\cdot, \cdot)$ is an inner product, implying that $\langle \cdot, \cdot \rangle_U$ is an inner product.
2. The unit $U$ of Takeuchi’s equivalence is an isomorphism of the inner product spaces $(F, \langle \cdot, \cdot \rangle)$ and $(A^\square_H \Phi(F), \langle \cdot, \cdot \rangle_U)$. Hence it extends to an isomorphism between the respective Hilbert space completions $L^2(F)$ and $L^2(A^\square_H \Phi(F))$. 
Just as established in Proposition 4.4 for the special case of $L^2(\Omega^*)$, this lemma now implies that morphisms extend to bounded operators on the Hilbert space $L^2(F)$.

**Corollary 9.3.** Every morphism $f : F \to F$ in $A_B$ mod is bounded, and hence extends to a bounded operator on $L^2(F)$.

Next we observe that Proposition 4.8 which established boundedness of multiplication operators on $\Omega^*$, also extends to the setting of Hermitian vector bundles. We express this in terms of module objects in $A_B$ mod, defined over a general monoid object. This generalises the fact that $B$ is a monoid object in $A_B$ mod, and that any other object is a module object over $B$. Moreover, it generalises the fact that $\Omega^*$ is a monoid object in $A_B$ mod, and hence a module object over itself.

**Lemma 9.4.** Let $(\mathcal{R}, \mu_\mathcal{R})$ be a monoid object in $A_B$ mod, and let $(\mathcal{N}, h)$ be a covariant Hermitian vector bundle such that $\mathcal{N}$ is additionally a $\mathcal{R}$-module object. Then, for all $r \in \mathcal{R}$, a bounded operator is given by

$$L_r : \mathcal{N} \to \mathcal{N}, \quad n \mapsto \mu_\mathcal{R}(r \otimes n).$$

9.2. **Tensor Products of Hermitian Vector Bundles.** In this subsection we consider tensor products of Hermitian vector bundles. This general procedure will be used in 9.3 to produce an Hermitian structure for twisted differential forms $\Omega^* B \otimes F$.

**Lemma 9.5.** Let $(F, h_F)$ and $(K, h_K)$ be two Hermitian vector bundles, defined over a $*$-algebra $B$. An Hermitian structure for the tensor product $F \otimes B K$ is given by

$$h_{F \otimes K}(f \otimes_B k, f' \otimes_B k') = h_K(k \otimes_B h_F(f \otimes_B k)k').$$

**Proof.** The defining properties of a bimodule Hermitian connection easily imply that $h_F \otimes h_G$ is a well-defined morphism in the category $A_B$ mod. Conjugate symmetry follows from the calculation

$$(h_F \otimes h_G(e \otimes_B f \otimes_B e' \otimes_B f'))^* = h_F(T \otimes_B h_G(\overline{e} \otimes_B e')f') = h_F(h_G(\overline{e} \otimes_B e')f) = h_F(T \otimes_B h_G(\overline{e}, e)f) = h_F \otimes h_G(e \otimes_B f').$$

By positive definiteness of $h_G$, we know that $h_F(\overline{e} \otimes_B e) \in B_{>0}$, and so, it is expressible in the form $\sum b_i^* b_i$, for $b_i \in B$. This implies that

$$h_F \otimes h_G(e \otimes_B f \otimes_B e' \otimes_B f') = h_F(T \otimes_B \left( \sum_{i=1}^k b_i^* b_i \right) f') = \sum_{i=1}^k h_F(b_i^* e \otimes_B b_i f') \geq 0,$$

where the last inequality follows from positivity of $h_F$. Thus we see that $h_F \otimes h_G$ is a metric as claimed. □

**Remark 9.6.** It is instructive to observe that the associated Hilbert space completion $L^2(F \otimes_B K)$ is not equal to the usual tensor product of the two Hilbert spaces $L^2(F)$.

**Remark 9.7.** In the context of Section 9.3, the space $L^2(F \otimes_B K)$ is a natural completion of $L^2(F \otimes B K)$. The proof in 9.3 shows that $h_F \otimes h_G$ is a bounded operator on $L^2(F \otimes B K)$.
and $L^2(K)$. In fact, it proves useful to think of $L^2(F \otimes_B K)$ as a type of Hilbert space analogue of the interior product of Hilbert modules [44 §4].

9.3. The Twisted Dolbeault Forms. Let $H = (B, \Omega^*, \Omega^{\bullet, \bullet}, \sigma)$ be a CQH-Hermitian space, and $(F, h)$ an Hermitian vector bundle over $B$. By the discussions of the previous subsection, we can take the tensor product of their two Hermitian structures to produce a new Hermitian structure $g_F$ on $\Omega^* \otimes_B F$, with associated inner product

\[ \langle \cdot, \cdot \rangle_F := h \circ g_F. \]

The associated Hilbert space $L^2(\Omega^* \otimes_B F)$ now has a bounded linear operator for every morphism on $\Omega^* \otimes_B F$. In particular, a representation of $\rho: U_p(sl_2) \to B(L^2(\Omega^* \otimes_B F))$ is given by

\[ \rho(E) = L_p \otimes_B id_F, \quad \rho(K) = K \otimes_B id_F, \quad \rho(F) = \Lambda_p \otimes_B id_F. \]

Just as in the untwisted case considered in §4.2, this reduces to a representation of $sl_2$ when $p = 1$. Moreover, we again get grading operators $\gamma \otimes_B id_F$, $\lambda \otimes_B id_F$, $T \otimes_B id_F$, $\overline{T} \otimes_B id_F$.

As before, all four operators are self-adjoint, and in particular, $\gamma \otimes_B id_F$ is a $C^*$-algebraic projection. Moreover, the operators generate a commutative subalgebra of $B(L^2(\Omega^* \otimes_B F))$.

9.4. Twisted Dolbeault–Dirac Operators. We begin by presenting the natural generalisation of Lemma 3.16 to the setting of Hermitian vector bundles. The proof is completely analogous to the arguments of [66 §5.2], and so, omitted. (For the reader’s convenience, we refer to §3.6 for the definition of the Peter–Weyl decomposition of an arbitrary object $F \in \mathfrak{h} \mathfrak{m} \mathfrak{o} \mathfrak{d} \mathfrak{e}$, and the notion of a Peter–Weyl map.)

Lemma 9.7. For a covariant Hermitian vector bundle $(F, h)$ over a CQGA homogeneous space, the Peter–Weyl decomposition of $F$ is orthogonal with respect to the associated inner product $\langle \cdot, \cdot \rangle$. Moreover, for any Peter–Weyl map $f: F \to F$, it holds that

1. $f$ is adjointable on $F$ with respect to $\langle \cdot, \cdot \rangle$, and its adjoint is a Peter–Weyl map,
2. if $f$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle$, then is is diagonalisable on $F$.

It follows that for any covariant holomorphic vector bundle $(F, \overline{\partial} F)$, the map $\overline{\partial} F$ is adjointable. Just as for the untwisted case, the adjoint $\overline{\partial}^t F$ admits an analogous, if slightly more involved, presentation in terms of the Hodge map, see [67 §5.3] for further details.

Definition 9.8. For a covariant Hermitian holomorphic vector bundle $(F, \overline{\partial} F)$, over a CQH-Hermitian space $(B, \Omega^*, \Omega^{\bullet, \bullet}, \sigma)$, its Dirac and Laplace operators are respectively defined by

\[ D_{\overline{\partial} F} := \overline{\partial} F + \overline{\partial}^t F, \quad \Delta_{\overline{\partial} F} := D_{\overline{\partial} F}^2 = \overline{\partial} F \overline{\partial} F + \overline{\partial} F \overline{\partial}^t F. \]

Moreover, we denote $H^0_{\overline{\partial} F} := \ker(\Delta_{\overline{\partial} F})$, and call it the space of harmonic elements.

In terms of the twisted Laplacians and twisted harmonic forms, we have the direct generalisation of Theorem 3.17 as established in [67 Theorem 6.4].
Theorem 9.9 (Hodge Decomposition). Let $(\mathcal{F}, h, \overline{\partial}_F)$ be an Hermitian holomorphic vector bundle over a CQH-Hermitian space $(B, \Omega^\bullet, \Omega^{(\bullet, \bullet)}, \sigma)$, then an orthogonal decomposition of $A$-comodules is given by
\[
\Omega^{(0, \bullet)}_F = H^{(0, \bullet)}_{\overline{\partial}_F} \oplus \overline{\partial}_F \left( \Omega^{0, \bullet}_F \right) \oplus \overline{\partial}_F \left( \Omega^{(0, \bullet)}_F \right).
\]
Furthermore, the projection $H^{(0, \bullet)}_{\overline{\partial}_F} \to H^{(0, \bullet)}_{\overline{\partial}_F}$ defined by $\alpha \mapsto [\alpha]$ is an isomorphism.

Next, we observe that the proofs of Proposition 4.12 and Proposition 4.10 carry over to the setting of covariant Hermitian vector bundles, giving us the following lemma.

Lemma 9.10. For any covariant Hermitian vector bundle $(\mathcal{F}, h)$, every Peter–Weyl map $f : \mathcal{F} \to \mathcal{F}$ is closable. Moreover, if $f$ is symmetric, then it is essentially self-adjoint, and diagonalisable.

As a direct consequence, twisted Dirac operators have the same analytic properties as in the untwisted case presented in [41] and [45].

Corollary 9.11. For any CQH-Hermitian space $(B, \Omega^\bullet, \Omega^{(\bullet, \bullet)}, \sigma)$, with an Hermitian holomorphic vector bundle $(\mathcal{F}, h, \overline{\partial}_F)$, the twisted Dirac operator $D_{\overline{\partial}_F}$ is diagonalisable, essentially self-adjoint, and the commutators $[D_{\overline{\partial}_F}, \rho(b)]$ are bounded, for all $b \in B$.

Proof. By Lemma 9.10, we need only prove boundedness of commutators. To this end, observe that, for $\omega \otimes_B f \in \Omega^k \otimes_B \mathcal{F}$, we have
\[
[D_{\overline{\partial}_F}, \rho(b)](\omega \otimes_B f) = D_{\overline{\partial}_F}(b \omega \otimes_B f) - b D_{\overline{\partial}_F}(\omega \otimes_B f)
\]
\[
= \overline{\partial}(b \omega) \otimes_B f + (-1)^k b \omega \otimes_B \overline{\partial}_F(f) - b \overline{\partial}_F(\omega \otimes_B f)
\]
\[
= \overline{\partial} b \wedge \omega \otimes_B f + b \overline{\partial}_F \omega \otimes_B f + (-1)^k b \omega \otimes_B \overline{\partial}_F(f) - b \overline{\partial}_F(\omega \otimes_B f)
\]
\[
= \overline{\partial} b \wedge \omega \otimes_B f + b \overline{\partial}_F(\omega \otimes_B f) - b \overline{\partial}_F(\omega \otimes_B f)
\]
\[
= \overline{\partial} b \wedge \omega \otimes_B f.
\]
It now follows from Lemma 9.4 that $[D_{\overline{\partial}_F}, \rho(b)]$ is a bounded operator. Moreover, Lemma 9.7 tells us that $[D_{\overline{\partial}_F}, \rho(b)]$ is adjointable. The adjoint operator is given by
\[
[D_{\overline{\partial}_F}, \rho(b)]^* = -[D_{\overline{\partial}_F}, \rho(b^*)].
\]
In particular, $[D_{\overline{\partial}_F}, \rho(b)]$ must be a bounded operator, for all $b \in B$. Thus we see that $[D_{\overline{\partial}_F}, \rho(b)]$ is a bounded operator as claimed.

We now come to the Fredholm property for twisted Dolbeault–Dirac operators. Just as for the untwisted case, we introduce the restricted operator
\[
D_{\overline{\partial}_F}^+: \text{dom}(D_{\overline{\partial}_F}) \cap L^2(\Omega^{(0, \bullet)} \otimes_B \mathcal{F}) \to L^2(\Omega^{0, \bullet} \otimes_B \mathcal{F}), \quad x \mapsto D_{\overline{\partial}_F}(x).
\]
The following lemma is established exactly as for 9.12, hence we state it without proof.

Theorem 9.12. For any CQH-Hermitian space $H = (B = A_{\text{coH}}, \Omega^\bullet, \Omega^{(\bullet, \bullet)}, \sigma)$, and any Hermitian holomorphic vector bundle $(\mathcal{F}, h, \overline{\partial}_F)$, the following are equivalent:

1. $D_{\overline{\partial}_F}^+$ is an even Fredholm operator,
2. \( \text{im}(D^+_\overline{\partial}_\mathcal{F}) \) is a closed subspace of \( L^2(\Omega^{0,\bullet} \otimes_B \mathcal{F}) \) and \( \dim \left( H^{(0,\bullet)}_{\overline{\partial}_\mathcal{F}} \right) < \infty. \)

Moreover, if \( D^+_{\overline{\partial}} \) is Fredholm, then its index is equal to \( \chi_{\overline{\partial}_\mathcal{F}} \) the \( \mathcal{F} \)-twisted holomorphic Euler characteristic of \( \Omega^{(\bullet,\bullet)} \), that is

\[
\text{index}(D^+_\overline{\partial}_\mathcal{F}) = \chi_{\overline{\partial}_\mathcal{F}} = \frac{1}{2} \dim(\mathcal{H}) \sum_{i=1} \dim \left( H^{(0,i)}_{\overline{\partial}_\mathcal{F}} \right) \in \mathbb{Z} \cup \{\pm \infty\}.
\]

Taking these properties, together with the observation that Proposition 7.4 carries over directly to the twisted setting, we get the following direct generalisation of Lemma 7.6.

**Lemma 9.13.** For any CQH-Hermitian spaces \( \mathcal{H} \), and an Hermitian holomorphic vector bundle \( (\mathcal{F},h,\partial_\mathcal{F}) \), an even spectral triple is given by

\[
\left( B, L^2(\Omega^{0,\bullet} \otimes_B \mathcal{F}), D_{\overline{\partial}_\mathcal{F}}, \gamma \right),
\]

if and only if the eigenvalues of \( D_{\overline{\partial}_\mathcal{F}} \) tend to infinity and have finite multiplicity.

9.5. **Spectral Gaps and Dolbeault–Dirac Fredholm Operators.** In this subsection we recall the Nakano and Akizuki–Nakano identities for a Hermitian holomorphic vector bundle \( \mathcal{F} \) over a CQH-Kähler space. We observe as a consequence that when the Chern–Lefschetz operator \([i\nabla^2, \Lambda_\mathcal{F}]\) acts positively, the point spectrum of the Laplacian \( \Delta_{\overline{\partial}} \) has a non-zero lower bound. In other words we are able to conclude a spectral gap purely from knowledge of the curvature of the underlying differential calculus. For the special case of a positive line bundle \( \mathcal{E} \), this allows to conclude that \( D_{\overline{\partial}_\mathcal{E}} \) is a Fredholm operator from purely cohomological data, one of the strongest results of the paper.

9.5.1. **The Akizuki–Nakano Identity and the Kodaira Vanishing Theorem.** For the twisted Dolbeault complex of a CQH-Kähler space, the following direct generalisation of the Kähler identities was established in [67, Theorem 7.6]. (For a discussion of the classical situation, of which this is a direct generalisation, see [39, §5.3] or [21, §VII.1].)

**Theorem 9.14** (Nakano identities). Let \( K = (B, \Omega^*, \Omega^{(\bullet,\bullet)}, \kappa) \) be a CQH-Kähler space, and \( (\mathcal{F},h,\overline{\partial}_\mathcal{F}) \) an Hermitian holomorphic vector bundle. Denoting the Chern connection of \( \mathcal{F} \) by \( \nabla_\mathcal{F} = \overline{\partial}_\mathcal{F} + \partial_\mathcal{F} \), it holds that

\[
\begin{align*}
[L_\mathcal{F}, \partial_\mathcal{F}] &= 0, & [L_\mathcal{F}, \overline{\partial}_\mathcal{F}] &= 0, & [\Lambda_\mathcal{F}, \partial_\mathcal{F}] &= 0, & [\Lambda_\mathcal{F}, \overline{\partial}_\mathcal{F}] &= 0, \\
[L_\mathcal{F}, \overline{\partial}_\mathcal{F}] &= i \overline{\partial}_\mathcal{F}, & [L_\mathcal{F}, \overline{\partial}_\mathcal{F}] &= -i \partial_\mathcal{F}, & [\Lambda_\mathcal{F}, \partial_\mathcal{F}] &= i \overline{\partial}_\mathcal{F}, & [\Lambda_\mathcal{F}, \overline{\partial}_\mathcal{F}] &= -i \overline{\partial}_\mathcal{F}.
\end{align*}
\]

As observed in [67, Corollary 7.8], these identities imply that the classical relationship, between the Laplacians \( \Delta_{\overline{\partial}} \) and \( \Delta_{\partial_\mathcal{F}} \), carries over to the noncommutative setting. Note that, unlike the untwisted case presented in Corollary 3.20, the operators differ by a not necessarily trivial curvature operator.

**Corollary 9.15** (Akizuki–Nakano identity). It holds that

\[
\Delta_{\overline{\partial}_\mathcal{F}} = \Delta_{\partial_\mathcal{F}} + [i\nabla^2, \Lambda_\mathcal{F}].
\]
We now observe that the noncommutative Kodaira vanishing theorem, originally established in [67, Theorem 8.3], admits an alternative proof using the Akizuki–Nakano identity. The proof uses the following identity, which, since it is also used in establishing Theorem 9.21 below, we present as a separate lemma.

**Lemma 9.16.** Let $\mathcal{E}_+$, and $\mathcal{E}_-$, be positive, and respectively negative, line bundles over a $2n$-dimensional CQH-Kähler space. It holds that

$$[i\nabla^2, \Lambda_{\mathcal{E}_\pm}](\omega \otimes e) = \pm (k - n)(\omega \otimes e), \quad \text{for all } \omega \otimes e \in \Omega^k \otimes \mathcal{E}_\pm.$$

**Proof.** For $\mathcal{E}_+$, the claimed identity follows from

$$[i\nabla^2, \Lambda_{\mathcal{E}_+}](\omega \otimes e) = i\nabla^2 \circ \Lambda_{\mathcal{E}_+}(\omega \otimes e) - i\Lambda_{\mathcal{E}_+} \circ \nabla^2(\omega \otimes e)$$

$$= i\Lambda(\omega) \wedge \nabla^2(e) - i\Lambda_{\mathcal{E}_+}(\omega \wedge \nabla^2(e))$$

$$= (\Lambda(\omega) \wedge \kappa) \otimes e - \Lambda_{\mathcal{E}_+}(\omega \wedge \kappa \otimes e)$$

$$= (L \circ \Lambda(\omega)) \otimes e - (\Lambda \circ L(\omega)) \otimes e$$

$$= ([L, \Lambda](\omega)) \otimes e$$

$$= (k - n)\omega \otimes e.$$  

The case of $\mathcal{E}_-$ is completely analogous, amounting to a change of sign. \(\square\)

With the above lemma in hand, we now re-establish the Kodaira vanishing theorem for CQH-Kähler spaces. (We note that while the original proof was presented in a more general setting, as careful examination will confirm, the argument below can be extended accordingly.)

**Theorem 9.17 (Kodaira Vanishing).** Let $\mathcal{E}_+$, and $\mathcal{E}_-$, be positive, and respectively negative, line bundles over a $2n$-dimensional CQH-Kähler space $K = (B, \Omega^*, \Omega^{\bullet, \bullet}, \kappa)$. Then it holds that

1. $H^{(a,b)}_{\partial\mathcal{E}_+} = 0$, for all $a + b > n$,  
2. $H^{(a,b)}_{\partial\mathcal{E}_-} = 0$, for all $a + b < n$.

**Proof.** Since $\mathcal{E}_+$ is positive, it follows from the above lemma that $[i\nabla^2, \Lambda_{\mathcal{E}_+}]$ is a positive operator, for all $a + b > n$. Since $\Delta_B$ is a positive operator, it follows from the Akizuki–Nakano identity that we can have no $\partial\mathcal{E}_+$-harmonic forms in $\Omega^{a,b}_B$, whenever $a + b > n$. It now follows from the identification of harmonic forms and cohomology classes that $H^{(a,b)}_{\partial\mathcal{E}_+} = 0$, for all $a + b > n$. The proof for the negative bundle $\mathcal{E}_-$ is completely analogous. \(\square\)

**9.5.2. A Spectral Gap.** With these general results in hand, we are now ready to conclude some spectral properties of twisted Dolbeault–Dirac operators from the behaviour of the curvature of their Chern connection.

**Lemma 9.18.** The operator $[i\nabla^2, \Lambda_{\mathcal{F}}]$ is a self-adjoint morphism in the category $\mathcal{A}^1_{/\mathbb{Z}}\mod_0$. Hence it is diagonalisable with a necessarily finite number of eigenvalues. Moreover, it holds that

$$\sigma_P(\Delta_{\mathcal{F}}) \subseteq [c_{\mathcal{F}}, \infty), \quad \text{where } c_{\mathcal{F}} := \min\left(\sigma_P[i\nabla^2, \Lambda_{\mathcal{F}}]\right).$$ (14)
Proof. Since $\nabla$ is a connection, $\nabla^2$ is necessarily a $B$-module map. Moreover, since $h$ is covariant, $\nabla^2$ must also be a left-$A$-comodule map, and hence it is a morphism in $H^\text{mod}_0$. Since it is the difference of two self-adjoint operators, it must also be self-adjoint on $\Omega^* \otimes_B F$, and hence diagonalisable by Lemma 3.16. Finally, since $\Phi(\Omega^* \otimes_B F)$ is finite-dimensional (as it is an object in $H^\text{mod}_0$) the operator $\Phi([\Lambda F, i\nabla^2])$ must have a finite number of eigenvalues, and hence $[i\nabla^2, \Lambda F]$ has a finite number of eigenvalues. Finally, by the Akizuki–Nakano identity, and positivity of $\Delta_{\partial F}$, it follows that the eigenvalues of $\Delta_{\partial F}$ are always greater than the eigenvalues of $[\Lambda F, i\nabla^2]$, giving us the inclusion in (14). □

9.5.3. Dolbeault–Dirac Fredholm Operators. The argument of this corollary can now be adapted to provide an effective means of verifying the Fredholm condition for $D_{\partial F}^+$. To do so, we will need the following generalisation of [18, Proposition 3.3] to the twisted setting. The proof, which is completely analogous to the untwisted case, is omitted.

**Lemma 9.19.** For a CQH-Hermitian space $(B = A^\text{coH}(\Omega^*, \Omega^{(\bullet, \bullet)}, \sigma), \partial F)$, a left $A$-covariant Hermitian holomorphic vector bundle $(F, h, \overline{\partial F})$, left $A$-comodule isomorphisms are given by

1. $\overline{\partial F}_F : \overline{\partial F}(\Omega^* \otimes_B F) \to \overline{\partial F}(\Omega^* \otimes_B F)$,
2. $\partial_F^+ : \partial_F(\Omega^* \otimes_B F) \to \partial_F^+(\Omega^* \otimes_B F)$.

Using this lemma, we can now provide sufficient conditions for $D_{\partial F}^+$ to be a Fredholm operator. This is done in terms of certain positivity conditions for either the odd or the even twisted anti-holomorphic forms.

**Corollary 9.20.** If the complex $\Omega^{(0, \bullet)} \otimes_B F$ has finite-dimensional cohomologies, and if

$$-i\Lambda_F \circ \nabla^2 : \Omega^{(0, \bullet)}_{\text{odd}} \otimes_B F \to \Omega^{(0, \bullet)}_{\text{odd}} \otimes_B F,$$

is a positive operator, or if

$$-i\Lambda_F \circ \nabla^2 : \Omega^{(0, \bullet)}_{\text{even}} \otimes_B F \to \Omega^{(0, \bullet)}_{\text{even}} \otimes_B F,$$

is a positive operator, then $\partial_{\partial F}^+$ is a Fredholm operator.

**Proof.** By Lemma 9.19 above, the non-zero point spectrum of the Laplacian operator $\Delta_{\partial F} : \Omega^{(0, \bullet)}_F \to \Omega^{(0, \bullet)}_F$ is equal to the non-zero point spectrum of the restricted operator $\Delta_{\partial F}^+ : \Omega^{(0, \bullet)}_{\text{even}} \to \Omega^{(0, \bullet)}_{\text{even}}$. Positivity of $\Delta_{\partial F}$ implies that the eigenvalues of $\Delta_{\partial F}$ are always greater than the eigenvalues of $[\Lambda F, i\nabla^2]$, which reduces to $-i\Lambda_F \circ \nabla^2$ on $\Omega^{(0, \bullet)} \otimes_B F$. Thus, if $-i\nabla^2 \circ \Lambda F$ acts as a positive operator on $\Omega^{(0, \bullet)}_{\text{odd}} \otimes_B F$, then the non-zero point spectrum of $\Delta_{\partial F}^+ : \Omega^{(0, \bullet)}_{\text{even}} \otimes_B F \to \Omega^{(0, \bullet)}_{\text{even}} \otimes_B F$ is bounded below by a non-zero positive scalar. This in turn implies that the absolute value of the non-zero eigenvalues of $D_{\partial F}$ are bounded below. Let us now identify $L^2(\Omega^* \otimes_B F)$ with the $\ell^2$-sequences for some choice of diagonalisation $\{e_n\}_{n \in \mathbb{N}_0}$ of $D_{\partial F}$, where $D_{\partial F}^+(e_n) =: \mu_n e_n$. For any such $\ell^2$-sequence
\[ \sum_{n=0}^{\infty} a_n e_n, \] we see that
\[ \left\| \sum_{n=0}^{\infty} \mu_n^{-1} a_n e_n \right\| \leq \sup_{n \in \mathbb{N}_0} |\mu_n|^{-1} \left\| \sum_{n=0}^{\infty} a_n e_n \right\| < \infty. \]

Hence \( \sum_{n=0}^{\infty} \mu_n^{-1} a_n e_n \) is a well-defined element of \( L^2(\Omega^\bullet \otimes \mathcal{F}) \). Moreover, since it is clear
\[ D_{\partial \mathcal{F}} \left( \sum_{n=0}^{\infty} \mu_n^{-1} a_n e_n \right) = \sum_{n=0}^{\infty} a_n e_n, \]
we now see that the image of \( D_{\partial \mathcal{F}} \) is equal to
\[ L^2 \left( \partial \mathcal{F} (\Omega^\bullet \otimes_B \mathcal{F}) \oplus \mathcal{F}^j (\Omega^\bullet \otimes_B \mathcal{F}) \right). \]

In particular, the image of \( D_{\partial \mathcal{F}} \) is closed. Finally, we note that
\[ \text{im} \left( D_{\partial \mathcal{F}} \right) = \text{im} \left( D_{\partial \mathcal{F}} \right) \cap L^2(\Omega^{0,\bullet}), \]
and hence, as the intersection of two closed sets, \( \text{im}(D_{\partial \mathcal{F}}) \) must be closed. The corollary now follows from Theorem 9.12.

The assumption that \( -i \nabla^2 \circ \Lambda_{\mathcal{F}} \) acts as a positive operator on \( \Omega^{0,\bullet} \otimes_B \mathcal{F} \) implies, in a completely analogous manner, that \( \text{im}(D_{\partial \mathcal{F}}^+) \) is closed. Hence, in this case, \( \text{im}(D_{\partial \mathcal{F}}^+) \) will again be a Fredholm operator.

As we now see, upon restricting to the case of a positive line bundle \( \mathcal{E} \), this result simplifies, allowing us to conclude that \( D_{\partial \mathcal{E}} \) is a Fredholm operator from purely cohomological data. This result will be used in §11 to construct Dolbeault–Dirac Fredholm operators for all the irreducible quantum flag manifolds.

**Theorem 9.21.** If \( \mathcal{E}_- \) is a negative line bundle over an \( 2n \)-dimensional CQH-Kähler space, then the twisted Dirac operator
\[ D_{\partial \mathcal{E}_-}^\pm : \text{dom}(D_{\partial \mathcal{E}_-}) \cap L^2(\Omega^{0,\bullet} \otimes_B \mathcal{E}_-) \rightarrow L^2(\Omega^{0,\bullet} \otimes_B \mathcal{E}_-), \]
is a Fredholm operator if and only if \( H^{(0,n)}_{\partial \mathcal{E}_-} \) is finite dimensional. Moreover, in this case
\[ \text{Index} \left( D_{\partial \mathcal{E}_-}^\pm \right) = \dim \left( H^{(0,n)}_{\partial \mathcal{E}_-} \right). \]

**Proof.** Since \( \mathcal{E}_- \) is by assumption a negative line bundle, for any \( \omega \otimes e \in \Omega^{(0,k)} \otimes_B \mathcal{E}_- \), it follows from Lemma 9.10 that
\[ -i \Lambda_{\mathcal{E}_-} \circ \nabla^2 (\omega \otimes e) = [i \nabla^2, \Lambda_{\mathcal{E}_-}] (\omega \otimes e) = (n-k) \omega \otimes e. \]

Thus we see that \( -i \Lambda_{\mathcal{E}_-} \circ \nabla^2 \) is a positive operator on \( \Omega^{(0,k)} \), for all \( k < n \). Moreover, by the Kodaira vanishing theorem it holds that \( H^{(0,k)}_{\partial \mathcal{E}_-} = 0 \), for all \( k = 0, \ldots, n-1 \). Corollary 9.20 now implies that \( D_{\partial \mathcal{E}_-}^\pm \) is a Fredholm operator if and only if \( H^{(0,n)}_{\partial \mathcal{E}_-} \neq 0. \)
Recall from §8.1 that, with respect to the opposite choice of Kähler structure ($\Omega^{•,•}$, $-\kappa$), positive line bundles are negative, and negative line bundles are positive. Thus by considering the opposite CQH-Kähler space, we easily arrive at the following corollary.

**Corollary 9.22.** If $E_+$ is a positive line bundle over an $n$-dimensional CQH-Kähler space, then the twisted Dirac operator $D^+_\partial E_+$ is a Fredholm operator if and only if $H^0(\partial E_+)$ is finite dimensional. Moreover, in this case,

$$\text{Index}(D^+_\partial E_-) = \dim(H^0(\partial E_+)).$$

### 9.6. The Chern–Dirac and Chern–Laplace Operators.

Let $(F, h, \partial F)$ be an Hermitian holomorphic vector bundle over a CQH-Hermitian space. We observe that $\nabla$, the associated Chern connection of $F$, is an adjointable operator, with adjoint given explicitly by $\nabla^\dagger := (\partial F + \overline{\partial} F)^\dagger = \overline{\partial}_F + \overline{\partial} F$. In direct analogy with the untwisted case, we introduce the twisted de Rham-Dirac and twisted Laplace operators

$$D_{\nabla} := \nabla + \nabla^\dagger,$$

$$\Delta_{\nabla} := \nabla \circ \nabla^\dagger + \nabla^\dagger \circ \nabla,$$

We can now follow the arguments given above for twisted and untwisted Dirac operators and conclude analogous analytic properties about $D_{\nabla}$ and $\Delta_{\nabla}$. So as to avoid tedious repetition, we will not do so, but contend ourselves with the observation that the triple

$$(B, L^2(\Omega^{•,•} \otimes_B F), D_{\nabla})$$

is a spectral triple if and only if the point spectrum of $D_{\nabla}$ (which is automatically countable) tends to infinity and all eigenspaces finite-dimensional.

It is natural to ask if, in the Kähler case, the equality of the untwisted Laplacians given in (3.20) carries over to the twisted setting. To do so we will need the following important corollary of the Nakano identities established in [67, Corollary 7.7].

**Corollary 9.23.** It holds that

1. $\partial_F \overline{\partial} F + \overline{\partial} F \partial F = 0$,
2. $\overline{\partial}_F \overline{\partial} F + \overline{\partial} F \overline{\partial}_F = 0$.

Using this corollary, we will now show that the operators do not coincide. Instead, just as in the Akizuki–Nakano identity, differ by the curvature operator $[\Lambda_F, i\nabla^2]$.

**Proposition 9.24.** For any CQH-Kähler space $K = (B, \Omega^{•,•}, \Omega^{•,•}, \kappa)$, and any Hermitian holomorphic vector bundle $(F, h, \overline{\partial} F)$, the following identities holds on $\overline{\Omega}^{•,•}$:

$$\Delta_{\nabla} = \Delta_{\partial_F} + \Delta_{\overline{\partial} F} = 2\Delta_{\partial_F} - [\Lambda_F, i\nabla^2] = 2\Delta_{\overline{\partial} F} + [\Lambda_F, i\nabla^2].$$

(17)

**Proof.** We begin by expanding the expression for $\nabla^\dagger \circ \nabla$ as follows

$$\nabla \circ \nabla^\dagger = (\partial_F + \overline{\partial} F) \circ (\partial^\dagger_F + \overline{\partial}^\dagger F) = \partial_F \circ \partial^\dagger_F + \partial_F \circ \overline{\partial}^\dagger F + \overline{\partial} F \circ \partial^\dagger_F + \overline{\partial} F \circ \overline{\partial}^\dagger F.$$

Recalling now the Nakano identities from Theorem 9.14 we see that this expression is equal to

$$\partial_F \circ [\Lambda_F, \overline{\partial} F] - \partial_F \circ i[\Lambda_F, \partial_F] + \overline{\partial} F \circ i[\Lambda_F, \partial F] - \overline{\partial} F \circ i[\Lambda_F, \partial F].$$
Expanding the commutator brackets and regrouping gives us the expression
\[-\partial_F \circ \overline{\partial}_F + \overline{\partial}_F \circ \partial_F \circ i \Lambda_F + (\partial_F + \overline{\partial}_F) \circ i \Lambda_F \circ (\overline{\partial}_F - \partial_F).\]

Another application of the Nakano identities yields
\[-\partial_F \circ \overline{\partial}_F + \overline{\partial}_F \circ \partial_F \circ i \Lambda_F + (\partial_F + \overline{\partial}_F) \circ (i \overline{\partial}_F \circ \Lambda_F + \overline{\partial}_F^\dagger - i \partial_F \circ \Lambda_F + \overline{\partial}_F^\dagger).\]

Removing the obvious cancelling terms, we finally arrive at the expression
\[\nabla \circ \nabla^\dagger = \partial_F \circ \partial_F^\dagger + \partial_F \circ \overline{\partial}_F \circ \partial_F^\dagger + \overline{\partial}_F \circ \partial_F^\dagger + \overline{\partial}_F \circ \overline{\partial}_F^\dagger.\]

An analogous calculation for \(\nabla^\dagger \circ \nabla\) yields the identity
\[\nabla^\dagger \circ \nabla = \partial_F^\dagger \circ \partial_F + \partial_F^\dagger \circ \overline{\partial}_F + \overline{\partial}_F \circ \partial_F^\dagger + \overline{\partial}_F \circ \overline{\partial}_F^\dagger.\]

Corollary 9.23 above now implies that
\[\Delta \circ \nabla = \partial_F \circ \partial_F^\dagger + \partial_F \circ \overline{\partial}_F \circ \partial_F^\dagger + \overline{\partial}_F \circ \partial_F^\dagger + \overline{\partial}_F \circ \overline{\partial}_F^\dagger = \Delta_\theta + \Delta_{\overline{\partial}}.\]

Finally, the other identities in (17) can now be concluded from the Akizuki–Nakano identity in Corollary 9.15.

10. THE IRREDUCIBLE QUANTUM FLAG MANIFOLDS AS CQH-FANO SPACES

In this section we present the motivating set of examples for the general theory of CQH-Hermitian spaces: the irreducible quantum flag manifolds \(O_{q}(G/L_S)\) endowed with their Hekenberger–Kolb differential calculi. We recall the covariant Kähler structure for each \(O_{q}(G/L_S)\), which is unique up to real scalar multiple, and present the associated CQH-Kähler space
\[K_S := \left( O_{q}(G/L_S), \Omega^*, \Omega^{\bullet, \bullet}, \kappa \right).\]

We finish with the special case of quantum projective space, discussing the spectral properties of its Dolbeault–Dirac operator and the associated spectral triple.

10.1. Drinfeld–Jimbo Quantum Groups. Let \(\mathfrak{g}\) be a finite-dimensional complex semisimple Lie algebra of rank \(r\). We fix a Cartan subalgebra \(\mathfrak{h}\) with corresponding root system \(\Delta \subseteq \mathfrak{h}^*\), where \(\mathfrak{h}^*\) denotes the linear dual of \(\mathfrak{h}\). With respect to a choice of simple roots \(\Pi = \{\alpha_1, \ldots, \alpha_r\}\), denote by \((\cdot, \cdot)\) the symmetric bilinear form, induced on \(\mathfrak{h}^*\) by the Killing form of \(\mathfrak{g}\), normalised so that any shortest simple root \(\alpha_i\) satisfies \((\alpha_i, \alpha_i) = 2\). The coroot \(\alpha_i^\vee\) of a simple root \(\alpha_i\) is defined by
\[\alpha_i^\vee := d_i \alpha_i = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}\]
where \(d_i := \frac{2}{(\alpha_i, \alpha_i)}\).

The Cartan matrix \((a_{ij})\) of \(\mathfrak{g}\) is defined by
\[a_{ij} := \langle \alpha_i^\vee, \alpha_j \rangle.\]
Let $q \in \mathbb{R}$ such that $q \neq -1, 0, 1$, and denote $q_i := q^{d_i}$. The quantised enveloping algebra $U_q(\mathfrak{g})$ is the noncommutative associative algebra generated by the elements $E_i, F_i$, and $K_i, K_i^{-1}$, for $i = 1, \ldots, r$, subject to the relations

\[ K_i E_j = q_i^{a_{ij}} E_j K_i, \quad K_i F_j = q_i^{-a_{ij}} F_j K_i, \quad K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \]

along with the quantum Serre relations

\[
\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \frac{1-a_{ij}}{r} \right]_{q_i} E_i^{1-a_{ij}-r} E_j E_i^r = 0, \quad \text{for } i \neq j,
\]

\[
\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \frac{1-a_{ij}}{r} \right]_{q_i} F_i^{1-a_{ij}-r} F_j F_i^r = 0, \quad \text{for } i \neq j,
\]

where we have used the $q$-binomial coefficient

\[
\left[ \frac{n}{r} \right]_q := \frac{[n]_q!}{[r]_q! [n-r]_q!}.
\]

A Hopf algebra structure is defined on $U_q(\mathfrak{g})$ by

\[ \Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i, \]
\[ S(E_i) = -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i, \quad S(K_i) = K_i^{-1}, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1. \]

A Hopf $\ast$-algebra structure, called the compact real form, is defined by

\[ K_i^* := K_i, \quad E_i^* := K_i F_i, \quad F_i^* := E_i K_i^{-1}. \]

10.2. **Type 1 Representations.** The set of fundamental weights $\{\varpi_1, \ldots, \varpi_r\}$ of $\mathfrak{g}$ is the dual basis of simple coroots $\{\alpha_1^\vee, \ldots, \alpha_r^\vee\}$, that is

\[ (\alpha_i^\vee, \varpi_j) = \delta_{ij}, \quad \text{for all } i, j = 1, \ldots, r. \]

We denote by $\mathcal{P}$ the integral weight lattice of $\mathfrak{g}$, which is to say the $\mathbb{Z}$-span of the fundamental weights. Moreover, $\mathcal{P}^+$ denotes the cone of dominant integral weights, which is to say the $\mathbb{N}_0$-span of the fundamental weights.

The elements $K_i$ are simultaneously diagonalisable on any finite-dimensional $U_q(\mathfrak{g})$-module $V$. We call the corresponding eigenspaces weight spaces, and call an element of a weight space a weight vector. A vector $v \in V$ is a highest weight vector if it is a weight vector and $E_i \triangleright v = 0$, for all $i = 1, \ldots, r$. For any dominant integral weight $\mu \in \mathcal{P}^+$, there exists a finite-dimensional irreducible $U_q(\mathfrak{g})$-module $V_\mu$, unique up to isomorphism, with a highest weight vector $v$ satisfying

\[ K_i \triangleright v = q^{\alpha_i \cdot \mu} v = q^{(\alpha_i^\vee, \mu)} v, \quad \text{for all } i = 1, \ldots, r. \]

We call any such $V_\mu$, or a finite direct sum of such modules, a type-1 representation. For any highest weight vector $v$, of weight $\mu$, we find it convenient to denote

\[ \text{wt}(v) := \mu, \quad \text{wt}_i(v) := \mu_i, \quad \text{where } \mu = \sum_{i=1}^r \mu_i \varpi_i. \]
The category of type 1 representations consists of type 1 representation as objects, and $U_q(\mathfrak{g})$-module maps as morphisms. It admits the structure of a braided monoidal category (coming from the $h$-adic quasi-triangular structure of the Drinfeld–Jimbo algebras). Explicitly, for $V$ and $W$ two finite-dimensional irreducible representations, the braiding is completely determined by the formula

$$\widehat{R}_{V,W}(v_{hw} \otimes w_{lw}) = \sum_{k,l,j} (\widehat{R}_{V,W})^{kl}_{ij} f_k \otimes e_j,$$

where $v_{hw}$ and $w_{lw}$ are a choice of highest weight vector for $V$, and a lowest weight vector for $W$, respectively. Given a choice of bases $\{e_i\}_{i=1}^{\dim(V)}$, and $\{f_i\}_{i=1}^{\dim(W)}$, for two finite dimensional $U_q(\mathfrak{g})$-modules $V$, and $W$, its associated $R$-matrix $R_{ij}^{kl}$ is defined by

As has been long known, it follows from Lusztig and Kashiwara’s theory of crystal bases [11, 12] that one can choose a weight basis for any $U_q(\mathfrak{g})$-module such that the associated $R$-matrix coefficients are Laurent polynomials in $q$. (See [12], and reference therein, for a more detailed discussion.) For sake of clarity, and subsequent referral, we present this result as a formal lemma.

**Lemma 10.1.** For $V$ an object in the category of type 1 representations of $U_q(\mathfrak{g})$, one can choose a basis of $V$, composed of weight vectors, such that the $R$-matrix coefficients are Laurent polynomials in $q$. We call such a basis a Laurent basis of $V$.

10.3. Quantum Coordinate Algebras and the Quantum Flag Manifolds. Let $V$ be a finite-dimensional $U_q(\mathfrak{g})$-module, $v \in V$, and $f \in V^*$, the linear dual of $V$. Consider the function $c_{v,f}^V : U_q(\mathfrak{g}) \to \mathbb{C}$ defined by $c_{v,f}^V(X) := f(X(v))$. The coordinate ring of $V$ is the subspace

$$C(V) := \text{span}_\mathbb{C} \{ c_{v,f}^V \mid v \in V, f \in V^* \} \subseteq U_q(\mathfrak{g})^*.$$

It is easily checked that $C(V) \subseteq U_q(\mathfrak{g})^*$, and moreover that a Hopf subalgebra of $U_q(\mathfrak{g})^*$ is given by

$$\mathcal{O}_q(G) := \bigoplus_{\mu \in \mathcal{P}^+} C(V_\mu).$$

We call $\mathcal{O}_q(G)$ the quantum coordinate algebra of $G$, where $G$ is the compact, connected, simply-connected, simple Lie group having $\mathfrak{g}$ as its complexified Lie algebra.

For $S$ a subset of simple roots, consider the Hopf $*$-subalgebra

$$U_q(I_S) := \langle K_i, E_j, F_j \mid i = 1, \ldots, r; j \in S \rangle.$$

From the Hopf $*$-algebra embedding $\iota : U_q(I_S) \hookrightarrow U_q(\mathfrak{g})$, we get the dual Hopf $*$-algebra map $\iota^* : U_q(\mathfrak{g})^* \to U_q(I_S)^*$. By construction $\mathcal{O}_q(G) \subseteq U_q(\mathfrak{g})^*$, so we can consider the restriction map

$$\pi_S := \iota|_{\mathcal{O}_q(G)} : \mathcal{O}_q(G) \to U_q(I_S)^*,$$
and the Hopf $*$-subalgebra $\mathcal{O}_q(L_S) := \pi_S(\mathcal{O}_q(G)) \subseteq U_q(L_S)^\circ$. The CQGA-homogeneous space associated to the surjective Hopf $*$-algebra map $\pi_S : \mathcal{O}_q(G) \to \mathcal{O}_q(L_S)$, is called the quantum flag manifold associated to $S$ and denoted by

$$\mathcal{O}_q(G/L_S) := \mathcal{O}_q(G)^{\text{co}(\mathcal{O}_q(L_S))}.$$  

Denoting $\mu_S := \sum_{s \in S} \omega_s$, we choose for $V_{\mu_S}$ a weight basis $\{v_i\}$, with corresponding dual basis $\{f_i\}_i$ for the dual module $V_{-w_0(\mu_S)} \simeq V_{\mu_S}$, where $w_0$ denotes the longest element in the Weyl group of $\mathfrak{g}$. As shown in [34, Proposition 3.2], a set of generators for $\mathcal{O}_q(G/L_S)$ is given by

$$z_{ij} := c^\mu_S_{f_i, v_j} c^{-w_0(\mu_S)}_{v_i, f_j}$$

for $i, j = 1, \ldots, N := \dim(V_{\mu_S})$, where $v_N$, and $f_N$, are the highest weight basis elements of $V_{\mu_S}$, and $V_{-w_0(\mu_S)}$, respectively, and for ease of notation we have written

$$c^\mu_S_{f_i, v_j} := c^\mu_{f_i, v_j}, \quad c^{-w_0(\mu_S)}_{v_i, f_j} := c^{-w_0(\mu_S)}_{v_i, f_j}.$$  

### 10.4. First-Order Calculi and Maximal Prolongations.

In this subsection, we quickly recall some details about first-order differential calculi necessary for our discussion of the Heckenberger–Kolb calculi below. A first-order differential calculus over an algebra $B$ is a pair $(\Omega^1, d)$, where $\Omega^1$ is an $B$-$B$-bimodule and $d : B \to \Omega^1$ is a linear map satisfying the Leibniz rule, $d(ab) = adb + (da)b$, for $a, b \in B$, and for which $\Omega^1 = \text{span}_C\{adb \mid a, b \in B\}$. The notions of differential map, and left-covariance (when the calculus is defined over a principal left $A$-comodule algebra $B$), have obvious first-order analogues, for details see [65, §2.4]. The direct sum of two first-order differential calculi $(\Omega^1, d_1)$ and $(\Gamma^1, d_2)$ is the first-order calculus $(\Omega^1 \oplus \Gamma^1, d_1 + d_2)$. Finally, we say that a left-covariant first-order differential calculus over $B$ is irreducible if it does not possess any non-trivial quotients by a left-covariant $B$-$B$-bimodule.

We say that a differential calculus $(\Gamma^\bullet, d_\Gamma)$ prolongs a first-order calculus $(\Omega^1, d_\Omega)$ if there exists a bimodule isomorphism $\varphi : \Omega^1 \to \Gamma^1$ such that $d_\Gamma = \varphi \circ d_\Omega$. It can be shown [65, §2.5] that any first-order calculus admits an extension $\Omega^\bullet$ which is maximal in the sense that there exists a unique differential map from $\Omega^\bullet$ onto any other extension of $\Omega^1$. We call this extension the maximal prolongation of the first-order calculus. It is important to note that the maximal prolongation of a left-covariant calculus is automatically left-covariant.

### 10.5. The Heckenberger–Kolb Calculi.

If $S = \{\alpha_1, \ldots, \alpha_r\} \setminus \{\alpha_i\}$, where $\alpha_i$ has coefficient 1 in the expansion of the highest root of $\mathfrak{g}$, then we say that the associated quantum flag manifold is irreducible. In the classical limit of $q = 1$, these homogeneous spaces reduce to the family of compact Hermitian symmetric spaces $\mathfrak{g}$. These algebras are also referred to as the cominiscule quantum flag manifolds, reflecting terminology in the classical setting. Presented below is a useful diagrammatic presentation of the set of simple roots defining the irreducible quantum flag manifolds, along with the names associated to the various series.

The irreducible quantum flag manifolds are distinguished by the existence of an essentially unique $q$-deformation of their classical de Rham complex. The existence of such
a canonical deformation is one of the most important results in the study the noncommutative geometry of quantum groups, serving as a solid base from which to investigate more general classes of quantum spaces. We present the calculus in two steps. First we give Heckenberger and Kolb’s classification of first-order calculi over $\mathcal{O}_q(G/L_S)$ as established in [34, Theorem 7.2].

| $A_n$ | $\mathcal{O}_q(\text{Gr}_k,n+1)$ | quantum Grassmannian | $k(n-k+1)$ |
|-------|---------------------------------|----------------------|-------------|
| $B_n$ | $\mathcal{O}_q(\mathbb{Q}_{2n+1})$ | odd quantum quadric | $2n-1$ |
| $C_n$ | $\mathcal{O}_q(\mathbb{L}_n)$ | quantum Lagrangian Grassmannian | $\frac{n(n+1)}{2}$ |
| $D_n$ | $\mathcal{O}_q(\mathbb{Q}_{2n})$ | even quantum quadric | $2(n-1)$ |
| $D_n$ | $\mathcal{O}_q(\mathbb{S}_n)$ | quantum spinor variety | $\frac{2(n-1)}{2}$ |
| $E_6$ | $\mathcal{O}_q(\mathbb{P}^2)$ | quantum Cayley plane | $16$ |
| $E_7$ | $\mathcal{O}_q(F)$ | quantum Freudenthal variety | $27$ |

**Table 1.** Irreducible Quantum Flag Manifolds: organised by series, with defining crossed node numbered according to [38 §11.4], CQGA-homogeneous space symbol and name, as well as the complex dimension $M$ of the corresponding classical complex manifold.

**Theorem 10.2.** There exist exactly two non-isomorphic irreducible left-covariant first-order differential calculi $\Omega^{(1,0)}$ and $\Omega^{(0,1)}$ of finite dimension over $\mathcal{O}_q(G/L_S)$. We call the direct sum of these two calculi the first-order Heckenberger–Kolb calculus of $\mathcal{O}_q(G/L_S)$, and denote it by $\Omega_1^q(G/L_S)$.

We next recall Heckenberger and Kolb’s verification that the maximal prolongation of $\Omega_1^q(G/L_S)$ has classical dimension [35 Proposition 3.11].

**Proposition 10.3.** For any irreducible quantum flag manifold, we denote by $\Omega^\bullet_q(G/L_S)$ the maximal prolongation of the first-order differential calculus $\Omega_1^q(G/L_S)$. The covariant
differential calculus $\Omega_q^*(G/L_S)$ is of classical dimension, which is to say
\[
\dim(\Phi(\Omega^k)) = \binom{2M}{k}, \quad \text{for all } k = 0, \ldots, 2M,
\]
where $M$ is the complex dimension of the corresponding classical manifold, as presented in Table 1.

**Example 10.4.** Since it is discussed in some detail below, we consider here the special case of quantum projective space $\mathcal{O}_q(\mathbb{CP}^n)$, the simplest type of quantum Grassmannian.Explicitly, it is the $A_n$-type irreducible quantum flag manifold corresponding to the first, or the last, crossed node of the Dynkin diagram, which is to say, the nodes

\[
\bullet\cdots\circ\cdots\circ\cdots\circ
\]

or

\[
\circ\cdots\bullet\cdots\circ\cdots\circ.
\]

For historical reasons, the quantum projective line $\mathcal{O}_q(\mathbb{CP}^1)$, the simplest example of a quantum flag manifold, is usually called the Podleś sphere and is denoted by $\mathcal{O}_q(S^2)$.

For this special case, the Heckenberger–Kolb calculus of $\mathcal{O}_q(S^2)$ is usually known as the Podleś calculus and is denoted by $\Omega_q^*(S^2)$.

10.6. **Generators and Relations for the Differential Calculus $\Omega_q^*(G/L_S)$**. For each irreducible quantum flag manifold, the defining relations of the maximal prolongation $\Omega_q^*(G/L_S)$ are a subtle and intricate $q$-deformation of the classical Grassmann anti-commutation relations. (For example, see [70] for an explicit presentation of the relations of the Podleś calculus $\Omega_q^*(S^2)$.)

In an impressive technical achievement, a complete $R$-matrix description of the general $\Omega_q^*(G/L_S)$ relations was established in [35, §3.3]. We recall here this presentation, following the original conventions of Heckenberger and Kolb. In particular, we use the following $R$-matrix notations, defined with respect to the index set $J := \{1, \ldots, \dim(V_{\omega_s})\}$:

\[
\hat{R}_{V_{\omega_s},V_{\omega_s}}(v_i \otimes v_j) := \sum_{k,l \in J} \hat{R}_{ij}^{kl} v_k \otimes v_l,
\]

\[
\hat{R}_{V_{-\omega_0(\omega_s)},V_{\omega_s}}(f_i \otimes v_j) := \sum_{k,l \in J} \hat{R}_{ij}^{kl} v_k \otimes f_l,
\]

\[
\hat{R}_{V_{\omega_s},V_{-\omega_0(\omega_s)}}(v_i \otimes f_j) := \sum_{k,l \in J} \hat{R}_{ij}^{kl} f_k \otimes v_l,
\]

\[
\hat{R}_{V_{-\omega_0(\omega_s)},V_{-\omega_0(\omega_s)}}(f_i \otimes f_j) := \sum_{k,l \in J} \hat{R}_{ij}^{kl} f_k \otimes f_l.
\]

In addition, we denote by $\hat{R}^-, \hat{R}, \hat{R}$, and $\hat{R}^-$, the inverse matrices of $\hat{R}, \hat{R}^-, \hat{R}$, and $\hat{R}$ respectively. The calculus $\Omega_q^*(G/L_S)$ can be described as the tensor algebra of the $\mathcal{O}_q(G/L_S)$-bimodule $\Omega_q^1(G/L_S)$, subject to three sets of matrix relations, given in terms of the coordinate matrix $z := (z_{ij})_{(ij)}$: First are the holomorphic relations

\[
\hat{Q}_{12} \hat{R}_{23} \partial z \wedge \partial z = 0,
\]

\[
\hat{P}_{34} \hat{R}_{23} \partial z \wedge \partial z = 0,
\]

where we have used leg notation, and have denoted

\[
\hat{Q} := \hat{R} + q(\omega_s,\omega_s) - (\alpha_s,\alpha_s) \text{id}, \quad \hat{P} := \hat{R} - q(\omega_s,\omega_s) \text{id}.
\]
Second are the anti-holomorphic relations
\[ \hat{P}_{12} \hat{R}_{23} \partial z \wedge \overline{\partial z} = 0, \quad \overline{Q}_{34} \hat{R}_{23} \partial z \wedge \overline{\partial z} = 0, \]
where we have again used leg notation, and have denoted
\[ \hat{P} := \hat{R} - q^{\{\alpha_s, \alpha_s\}} \text{id}, \quad \overline{Q} := \overline{R} + q^{\{\alpha_s, \alpha_s\} - \{\alpha_s, \alpha_s\}} \text{id}. \]
Finally, we have the cross-relations
\[ \overline{\partial z} \wedge \partial z = -q^{-\{\alpha_s, \alpha_s\}} T_{1234} \overline{\partial z} \wedge \partial z + q^{\{\alpha_s, \alpha_s\} - \{\alpha_s, \alpha_s\}} \overline{\partial z} \wedge \partial z, \]
where we have again used leg notation, and have denoted
\[ T_{1234} := \hat{R}_{23} \hat{R}_{12} \overline{Q}_{34} \hat{R}_{23}, \quad C_{kl} := \sum_{i \in I} \hat{R}^{-ii}_{kl}. \]

Because of the complicated nature of the relations, we find it helpful to highlight exactly which of its properties are used below. First, we note that when \( q = 1 \), the relations reduce to the usual anti-commutating Grassmann relations. The second relevant property is that the commutation relations for the \( q \neq 1 \) case are generated in degree two (just as for any maximal prolongation), with generators given by certain linear combinations of 2-forms of type
\[ \partial z_{ab} \wedge \partial z_{ab}, \quad \overline{\partial z}_{ab} \wedge \partial z_{ab}, \quad \partial z_{ab} \wedge \overline{\partial z}_{ab}, \quad \overline{\partial z}_{ab} \wedge \overline{\partial z}_{ab}, \quad \text{for } a, b \in J. \]
In particular, if the chosen basis of \( V_{\overline{\alpha}_s} \) is a Laurent basis, then these coefficients are Laurent polynomials in \( q \).

10.7. Noncommutative Complex Structures. The first example of a Heckenberger–Kolb calculus to be discovered was the Podleś calculus for the Podleś sphere [70]. As part of its construction it was demonstrated to be a \( \ast \)-calculus. The extension of this result to the general setting of irreducible quantum flag manifolds was not considered in [34, 35]. It was subsequently observed in [65, Proposition 3.4] that \( \Omega^\bullet_q(\mathbb{C}P^n) \) is a \( \ast \)-calculus. The general result, for all irreducible quantum flag manifolds, was later established by Matassa in [53, Theorem 4.2].

**Theorem 10.5.** For each irreducible quantum flag manifold \( O_q(G/L_S) \), its Heckenberger–Kolb calculus \( \Omega^\bullet_q(G/L_S) \) is a \( \ast \)-differential calculus.

In [53] it was also observed that each \( \ast \)-calculus \( \Omega^\bullet_q(G/L_S) \) carries a natural complex structure. We present this result, along with some additional observations which can easily be concluded from the presentation of \( \Omega^\bullet_q(G/L_S) \) given in [110]. A careful proof of these results, established within the formal framework of [63], will appear in [63].

**Lemma 10.6.** Let \( O_q(G/L_S) \) be an irreducible quantum flag manifold, and \( \Omega^\bullet_q(G/L_S) \) its Heckenberger–Kolb calculus.

1. The decomposition \( \Omega^1_q(G/L_S) = \Omega^{(1,0)} \oplus \Omega^{(0,1)} \) extends to a (necessarily unique) almost-complex structure \( \Omega^{(\bullet,\bullet)} \) on \( \Omega^\bullet_q(G/L_S) \).
2. The almost-complex structure \( \Omega^{(\bullet,\bullet)} \) is covariant, and it is the unique such structure on \( \Omega^\bullet_q(G/L_S) \).
3. The almost-complex structure $\Omega^{(\bullet \bullet)}$ is integrable, which is to say, it is a complex structure.

4. The complex structure $\Omega^{(\bullet \bullet)}$ is factorisable.

As we now recall, $\Phi(\Omega^1)$ admits a concrete description in terms of the complex structure $\Omega^{(\bullet \bullet)}$. Consider the subset of $J := \{1, \ldots, \dim(V_\omega)\}$ given by

$$J_{(1)} := \{i \in I \mid (\omega_s, \omega_s - \alpha_s - \mathrm{wt}(v_i))\}.$$

From [35] Proposition 3.6, bases of $\Phi(\Omega^{(1,0)})$, and $\Phi(\Omega^{(0,1)})$, are given respectively by

$$\{e_i^+ := [\partial z_{iM}] \mid \text{for } i \in J_{(1)}\}, \quad \{e_i^- := [\overline{\partial} z_{iM}] \mid \text{for } i \in J_{(1)}\}.$$

Moreover, as shown in [35] §3.3, a basis of $\Phi(\Omega^{(n,b)})$ is given by

$$\Theta := \{e_K^+ \wedge e_K^- \mid K \in O(a), L \in O(b)\},$$

where $O(l)$ is the set of all ordered subsets of $\{1, \ldots, M\}$ with $l$ elements.

**Example 10.7.** Let us now focus on quantum projective space $O_q(\mathbb{C}P^n)$, the quantum flag manifold corresponding to the first crossed node of the $A_n$ Dynkin diagram. The basis of $\Phi(\Omega_q^{(\bullet \bullet)})$ reduces to

$$e_i^+ = [\partial z_{i+1,n}], \quad e_i^- = [\overline{\partial} z_{n+1,i}], \quad \text{for } i = 1, \ldots, n.$$

Moreover, the relations of the algebras $\Phi(\Omega^{(\bullet \bullet)})$, and $\Phi(\Omega^{(0,\bullet)})$, reduce to the standard quantum affine space, and its dual, respectively:

$$e_j^+ \wedge e_i^+ = -q e_i^+ \wedge e_j^+, \quad e_j^- \wedge e_i^- = -q^{-1} e_i^- \wedge e_j^-,$$

for $i \leq j$.

### 10.8. Noncommutative Kähler Structures

We begin by recalling from Lemma [10.6] that the covariant complex structure $\Omega^{(\bullet \bullet)}$ of any Heckenberger–Kolb calculus is factorisable. In particular, we have that

$$\Phi(\Omega^{(1,1)}) \simeq \Phi(\Omega^{(1,0)} \otimes_{O_q(G/L_S)} \Omega^{(0,1)}) \simeq \Phi(\Omega^{(1,0)}) \otimes \Phi(\Omega^{(0,1)}).$$

Since the $*$-map of the calculus restricts to a real linear isomorphism between $\Omega^{(1,0)}$ and $\Omega^{(0,1)}$, it must hold that $\Phi(\Omega^{(1,0)})$ and $\Phi(\Omega^{(0,1)})$ are conjugate comodules. For any comodule $V$ of a CQGA, its conjugate comodule and its dual comodule are isomorphic [12, Theorem 11.27]. Hence, recalling that $\Phi(\Omega^{(0,1)})$ is an irreducible $O_q(L_S)$-comodule, we see that the left $O_q(L_S)$-coinvariant elements in $\Phi(\Omega^{(1,1)})$ form a 1-dimensional complex vector space. This in turn implies that

$$U\left(\text{co}(O_q(G))\Omega^{(1,1)}\right) = \text{co}(O_q(G))\left(G \square_H \Phi(\Omega^{(1,1)})\right)$$

$$= 1 \square_H \Phi(\Omega^{(1,1)})$$

$$= 1 \otimes \left(O_q(L_S)\Phi(\Omega^{(1,1)})\right)$$

$$\simeq 1 \otimes \mathbb{C}$$

Moreover, since the $*$-map sends left $O_q(G)$-coinvariant forms to left $O_q(G)$-coinvariant forms, we see that each $\Omega^{(1,1)}$ contains a coinvariant form $\kappa$, which is unique up to complex scalar multiple. If in addition we require $\kappa$ to be real, which is to say $\kappa^* = \kappa$, then the form is uniquely determined up to real scalar multiple.
For the special case of $\mathcal{O}_q(\mathbb{C}P^n)$, the pair $(\Omega^{\bullet\bullet}, \kappa)$ was shown to be a Kähler structure in [66, §4.4], for all $q \neq -1, 0$. Moreover, $(\Omega^{\bullet\bullet}, \kappa)$ was shown to be positive definite for all $q$ sufficiently close to 1. This motivates the following conjecture, originally proposed in [66, Conjecture 4.25].

**Conjecture 10.8.** For $\Omega^{\bullet\bullet}_q(G/L)$ the Heckenberger–Kolb calculus of the irreducible quantum flag manifold $\mathcal{O}_q(G/L)$, the pair $(\Omega^{\bullet\bullet}, \kappa)$ is a positive definite covariant Kähler structure, for all $q \in \mathbb{R}\setminus\{0, -1\}$.

By extending the representation theoretic argument given in [66, §4.4] for the case of $\mathcal{O}_q(\mathbb{C}P^n)$, the form $\kappa$ is readily seen to be a closed central element of $\Omega^\bullet$. In more detail, a direct examination confirms that the $\mathcal{O}_q(L_S)$-comodules $V^{(2,1)} \simeq V^{(2,0)} \otimes V^{(0,1)}$ and $V^{(1,2)} \simeq V^{(1,0)} \otimes V^{(0,2)}$ do not contain a copy of the trivial comodule. Hence, there can be no non-trivial map from $\mathbb{C}\kappa = \text{co}(\mathcal{O}_q(G))(\Omega^{(1,1)})$ to either $\Omega^{(2,1)}$ or $\Omega^{(1,2)}$, implying that $d\kappa = 0$. As shown by Matassa in [53], the form $\kappa$ can be explicitly presented as

$$\kappa = i \sum_{a,b,k \in J} q^{(2\rho, \text{wt}(v_a))} z_{ab} d z_{bk} \wedge d z_{ka},$$

where $\rho$ denotes the Weyl element of $h^*$, which is to say $\rho := \sum_{i=1}^r \omega_i$. Using this formulation it proves possible to express $\kappa$ as an exact form, whence one can conclude that $d\kappa = 0$. Moreover, it was shown that

$$[\kappa] = i \sum_{a \in J(1)} q^{(2\rho, \text{wt}(v_a))} e^+_a \otimes e^-_a.$$  

Using this description the following result was established in [53, Theorem 5.10].

**Theorem 10.9.** Let $\Omega^{\bullet\bullet}_q(G/L_S)$ be the Heckenberger–Kolb calculus of the irreducible quantum flag manifold $\mathcal{O}_q(G/L_S)$. The pair $(\Omega^{\bullet\bullet}, \kappa)$ is a covariant Kähler structure for all $q \in \mathbb{R}_{>0}\setminus F$, where $F$ is a, not necessarily non-empty, finite subset of $\mathbb{R}_{>0}$. Moreover, any elements of $F$ are necessarily non-transcendental.

10.9. CQH-Kähler Spaces. In this subsection we build on Theorem 10.9 above and produce a CQH-Kähler space structure for all the irreducible quantum flag manifolds, providing us with a rich family of examples to which to apply the framework of this paper.

**Lemma 10.10.** For every irreducible quantum flag manifold $\mathcal{O}_q(G/L_S)$, there exists an open interval $I$ around 1, such that $(\Omega^{\bullet\bullet}, \kappa)$ is a positive definite Kähler structure for the Heckenberger–Kolb calculus of $\mathcal{O}_q(G/L_S)$, for all $q \in I$.

**Proof.** It follows from [66, Lemma 5.7] that positive definiteness of the bilinear map

$$(\cdot, \cdot) : \Phi(\Omega^\bullet) \otimes_{\mathbb{R}} \Phi(\Omega^\bullet) \to \mathbb{C}$$

would imply positive definiteness of he Kähler structure. We start by looking at the $q = 1$ case, recalling that when $q = 1$, the $R$-matrix of any $\mathcal{U}_q(g)$-module reduces to
the identity matrix. This implies that the relations in Theorem 10.3 reduce to the usual Grassmann relations, giving us the isomorphism
\[ \Lambda^\bullet(\Phi(\Omega^1)) \simeq \Phi(\Omega^\bullet(G/L_S)), \]
where \( \Lambda^\bullet(\Phi(\Omega^1)) \) denotes the usual Grassmann exterior algebra of \( \Phi(\Omega^1) \). Moreover
\[ (e_k^-, e_l^-) = *_\kappa( (e_k^-)^\ast \wedge e_l^-) = \frac{i}{(M-1)!} *_\kappa( e_k^- \wedge [k]^{M-1} \wedge e_l^+ ). \]
In the commutative setting, it follows from (22) that
\[ [k]^{M-1} := i^{M^2-1} (M-1)! \sum_{K \in O(n-1)} e_K^+ \wedge e_{\bar{K}}^- . \]
Inserting this identity into (23), we see that
\[ (e_k^+, e_l^+) = i^{M^2} \sum_{K \in O(n-1)} *_\kappa( e_k^+ \wedge e_K^+ \wedge e_{\bar{K}}^- \wedge e_l^- ), \]
which gives a non-zero answer if and only if \( k = 1 \). Analogous calculations show that \( (e_k^-, e_l^-) \) is non-zero if and only if \( k = l \). Since \( \Phi(\Omega^{1(0)}) \) and \( \Phi(\Omega^{0,1}) \) are orthogonal by Lemma 2.8 we see that the degree 1 elements of \( \Theta \) form an orthogonal basis of \( \Phi(\Omega^\bullet) \). In particular, in the sense of [39, Definition 1.2.11], the scalar product \((\cdot, \cdot)\) is compatible with the decomposition
\[ \Phi(\Omega^1) = \Phi(\Omega^{1,0}) \oplus \Phi(\Omega^{0,1}) . \]
It now follows from the classical Weil formula [39, Theorem 1.2.31] that the \((\cdot, \cdot)\) coincides with the standard extension of \((\cdot, \cdot)\) to a sesquilinear form on the exterior algebra \( \Lambda^\bullet(\Phi(\Omega^1(G/L_S))) \). In particular, \( \Theta \) forms an orthogonal basis of \( \Phi(\Omega^\bullet) \).

Returning to the case where \( q \) is not necessarily equal to 1, we observe that since the commutation relations of \( \Omega^\bullet \) (as presented in 10.6) have \( R \)-matrix entry coefficients, the commutation relations of \( \Phi(\Omega^\bullet) \) must be linear combinations, with \( R \)-matrix coefficients, of the degree 2 basis elements \( e_k^+ \wedge e_l^-, e_k^- \wedge e_l^+ \), and \( e_k^- \wedge e_l^- \), for \( k, l \in J \). In particular, taking the basis of the fundamental representation \( V^\omega \) to be a Laurent basis (as defined in Lemma 10.1) the coefficients of the relations will be Laurent polynomials in \( q \). Hence there exists a Laurent polynomial \( p(q) \) such that
\[ *_\kappa(1) = \frac{p(q)}{M!} e_j^+ \wedge e_{\bar{j}}^- \]
Denoting by \( F \) the finite set of real numbers for which \( (\Omega^{1,\bullet}, \kappa) \) is not a Kähler structure. We write \( I_0 \) for the largest open interval around 1 which does not contain an element of \( F \). For any pair of basis elements \( e_K^+ \wedge e_L^- \) and \( e_A^+ \wedge e_B^- \), consider the function
\[ f_{KLAB} : I_0 \to \mathbb{C}, \quad q \mapsto (e_K^+ \wedge e_L^-, e_A^+ \wedge e_B^-) , \]
where we have identified \( \Phi(\Omega_q(G/L_S)) \) with \( \mathbb{C} \). Since the commutation relations have Laurent polynomial coefficients, there exists a Laurent polynomial \( p'(q) \) such that
\[ f_{KLAB}(q) = p'(q) e_j^+ \wedge e_{\bar{j}}^- \]
Taken together with (24), this means that each \( f_{KLAB} \) is a Laurent polynomial in \( q \), and hence a continuous function taking values in the real numbers. Since \( f_{KLAB}(1) > 0 \)
(following from orthogonality of the basis at \( q = 1 \)) there must exist an interval \( I_{KLAB} \) around 1 such that \( f_{KLAB} \) is strictly positive on \( I_{KLAB} \). The required interval \( I \) is now given by the intersection

\[
I := I_0 \cap I_{KLAB},
\]

where summation takes place over all pairs of basis elements \( e^+_K \wedge e^-_L, e^+_A \wedge e^-_B \in \Theta \).

With positive definiteness in hand, we are now ready to show that, for each irreducible quantum flag manifold, its Kähler structure gives a CQH-Kähler space. This is one of the principal results of the paper, and provides us with a rich family of examples to which to apply the general theory of CQH-Kähler spaces.

**Theorem 10.11.** For each irreducible quantum flag manifold \( O_q(G/L_S) \), there exists an open interval \( I \subseteq \mathbb{R} \) around 1, such that a CQH-Kähler space is given by the quadruple

\[
H_S := (O_q(G/L_S), \Omega^{*}, \Omega^{(\bullet \bullet)}), \quad \text{for all } q \in I.
\]

**Proof.** By construction, each quantum flag manifold \( O_q(G/L_S) \) is a CQGA-homogeneous space, and \( \Omega^{*}(G/L_S) \) is a left \( O_q(G) \)-covariant differential \( \ast \)-calculus over \( O_q(G/L_S) \). It follows from [35, Corollary 3.5] that

\[
\Omega^*(G/L_S) \in \Theta_{O_q(G/L_S)} \mod 0.
\]

By Lemma 10.6 the complex structure \( \Omega^{(\bullet \bullet)} \) is covariant. Moreover, for \( I \) the interval identified in Lemma 10.10, the pair \( (\Omega^{(\bullet \bullet)}, \kappa) \) is a positive definite covariant Kähler structure, for all \( q \in I \).

It remains to establish closure of the integral with respect to the Kähler structure. This will be done by showing \( \Phi(\Omega^{(0,1)}) \) does not contain a copy of the trivial \( O_q(L_S) \)-comodule, and then appealing to Lemma 3.11. Note that since the case of \( O_q(\mathbb{C}P^n) \) has been dealt with in [66, Lemma 3.4.4], it follows from Table II that we can restrict our attention to those irreducible quantum flag manifolds for which \( \Phi(\Omega^{(0,1)}) \) has dimension strictly greater than 1. It follows from Theorem 10.2 that \( \Phi(\Omega^{(0,1)}) \) is irreducible as a \( O_q(L_S) \)-comodule. Hence \( \Phi(\Omega^{(0,1)}) \) cannot contain a copy of the trivial comodule, implying that the integral is closed. We can now conclude that the quadruple given in (25) is indeed a CQH-Kähler space. \( \square \)

10.10. **CQH-Fano Spaces.** Based on the arguments and results around the noncommutative Bott–Borel–Weil theorem presented in §11.2, the following result were established in [23].

**Theorem 10.12.** Let \( O_q(G/L_S) \) be an irreducible quantum flag manifold, with \( q \in \mathbb{R}_{>0} \) such that \( (\Omega^{(\bullet \bullet)}, \kappa) \) is a Kähler structure. Then the Kähler structure is a Fano structure.

As a direct consequence we get that each of the CQH-Kähler spaces presented in Theorem 10.13 is a CQH-Fano, allowing us to calculate the Euler characteristic of each constituent complex structure.
Corollary 10.13. For each irreducible quantum flag manifold $O_q(G/L_S)$, there exists an open interval $I \subseteq \mathbb{R}$ around 1, such that a CQH-Fano space is given by the quadruple

$$H_S := \left( O_q(G/L_S), \Omega^\bullet, \Omega^{\bullet\bullet}, \kappa \right),$$

for all $q \in I$.

Moreover, the holomorphic Euler characteristic of $\Omega^{\bullet\bullet}$ is given by

$$\chi_{\mathbb{C}} = \dim(H^{0,0}) = 1.$$  

Proof. It follows directly from Theorem 10.13 and Theorem 10.12 that we get a CQH-Fano space for every irreducible quantum flag manifold. The first equality in (27) now follows from Corollary 6.11. The second equality, giving the dimension of $H^{(0,0)}$, was established in [22] as the trivial line bundle case of the quantum Borel–Weil theorem presented in Theorem 11.3 below. \qed

We finish by observing that since we have non-trivial Euler characteristic for each irreducible quantum flag manifold, any example for which the associated Dolbeault–Dirac operator gives a spectral triple will necessarily have a non-trivial associated $K$-homology class.

Cyclic Cohomology and Central Column Dolbeault Cohomology. Cyclic cohomology $HC^k$, as independently introduced by Connes [13] and Tysgan [81], is the standard replacement for de Rham cohomology in noncommutative geometry. However, when applied to quantum group examples, it fails to preserve classical dimension. This phenomenon is informally known as dimension drop, and is regarded by many as an unpleasant feature of the theory. For example, it was shown by Masuda, Nakagami, and Watanabe [51] that the cyclic cohomology of $O_q(SU_2)$ satisfies $HC^3(O_q(SU_2)) = 0$. This work was extended by Feng and Tsygan [24], who computed the cyclic cohomology of each Drinfeld–Jimbo coordinate algebra $O_q(G)$. They showed that $HC^k(O_q(G)) = 0$, for all $k$ greater than the rank of $G$. Vanishing of cohomology occurs even at the level of the quantum flag manifolds. For the simplest case, which is to say the Podleś sphere, its cyclic cohomology satisfies $HC^2(O_q(S^2)) = 0$ [52].

As we now show, the dimension drop phenomenon does not occur for the de Rham cohomology of the Heckenberger–Kolb calculi, a fact which proposes it as a more natural cohomology theory. Just as for any classical compact Kähler manifold, it follows directly from the hard Lefschetz theorem 3.22 that the central column Dolbeault cohomology groups $H^{(k,k)}$ are non-zero for any positive definite Kähler structure. In particular, for any such structure, its even de Rham cohomology groups $H^{2k}$ are non-zero. As an important application of Lemma 10.10, we can now conclude non-vanishing of the even cohomology groups of each irreducible quantum flag manifold.

Theorem 10.14. For any irreducible quantum flag manifold $O_q(G/L_S)$, such that $q \in I$, it holds that the Dolbeault cohomology $H^{\bullet\bullet}$ of its Heckenberger–Kolb calculus $\Omega^{\bullet\bullet}(G/L_S)$, endowed with the complex structure $\Omega^{\bullet\bullet}$, satisfies

$$H^{(k,k)} \neq 0,$$

for all $k = 1, \ldots, M$. 

In particular, the de Rham cohomology $H^\bullet$ of $\Omega^\bullet_\ast(G/L_S)$ satisfies

$$H^{2k} \neq 0, \quad \text{for all } k = 1, \ldots, M.$$ 

**Remark 10.15.** Twisted cyclic cohomology was introduced in [46] as an attempt to address this unpleasant aspect of cyclic cohomology. It generalises cyclic cohomology through the introduction of an algebra automorphism $\sigma$, which when $\sigma = \text{id}$ reduces to ordinary cyclic cohomology. For $\mathcal{O}_q(SU_n)$, with $\sigma$ chosen to be the modular automorphism of the Haar state [42, §11.3.4], the dimension of the twisted cyclic cohomology coincides with the classical dimension [33]. Analogous results were obtained for the Podleś sphere in [32]. The relationship between twisted cyclic cohomology and the cohomology of the Heckenberger–Kolb calculi is at present unclear.

10.12. The Dolbeault–Dirac Spectral Triple of Quantum Projective Space. Finally we come to the question of which irreducible quantum flag manifolds have associated Dolbeault–Dirac spectral triples. As discussed earlier, we do not at present have an effective means of verifying the compact resolvent condition. Instead, we resort to explicitly calculating the point spectrum of the Dolbeault–Dirac operator. In general, this is a very challenging technical task. However, as shown in [18], under the assumption of restricted multiplicities for the $U_q(\mathfrak{g})$-modules occurring in $\Omega^{(0,\bullet)}$, the task becomes significantly more tractable. In particular, for the special case of quantum projective space $\mathcal{O}_q(\mathbb{CP}^n)$, the spectrum of $D_\partial$ was completely determined in [18, §6].

**Theorem 10.16.** The point spectrum of the Dolbeault–Dirac operator $D_\partial$, associated to the CQH-Kähler space

$$\left( \mathcal{O}_q(\mathbb{CP}^n), \Omega^\bullet, \Omega^{(\bullet,\bullet)}, \kappa \right)$$

has finite multiplicity and tends to infinity. Hence, a pair of unitarily equivalent even spectral triples is given by

$$\left( \mathcal{O}_q(\mathbb{CP}^n), L^2(\Omega^{(0,\bullet)}), D_\partial, \gamma \right), \quad \left( \mathcal{O}_q(\mathbb{CP}^n), L^2(\Omega^{(\bullet,\bullet)}), D_\partial, \gamma \right).$$

Recall from Theorem 10.12 that the Kähler structure of the Heckenberger–Kolb calculus of each irreducible quantum flag manifold, and in particular each quantum projective space, is of Fano type. The following result is now implied from Corollary 7.8.

**Corollary 10.17.** The $K$-homology class associated to the pair of spectral triples in (28) is non-trivial. In particular

$$\text{index}(D_\partial) = \text{index}(D_\partial) = 1.$$ 

Efforts to extend this result to all the irreducible quantum flag manifolds are in progress. See [18, §7] for a detailed discussion of the next most approachable families of examples. Here we satisfy ourselves with presenting this goal as a formal conjecture.

**Conjecture 10.18.** For any irreducible quantum flag manifold $\mathcal{O}_q(G/L_S)$, the point spectrum of the Dolbeault–Dirac operator $D_\partial$, associated to the CQH-Kähler space

$$\left( \mathcal{O}_q(G/L_S), \Omega^\bullet, \Omega^{(\bullet,\bullet)}, \kappa \right)$$

is non-trivial. In particular

$$\text{index}(D_\partial) = \text{index}(D_\partial) = 1.$$
has finite multiplicity and tends to infinity. Hence, a pair of unitarily equivalent even spectral triples, with non-trivial associated $K$-homology class, is given by
\[
\left( O_q(G/L_S), L^2(\Omega^{(0,\bullet)}), D_\partial, \gamma \right), \quad \left( O_q(G/L_S), L^2(\Omega^{(0,\bullet)}), D_\partial, \gamma \right).
\]

Example 10.19. For the special case of the Podleś sphere, we now give an explicit presentation of the Dolbeault–Dirac spectrum. For details on the derivation of these values, as well as a presentation of the general quantum projective space case, see [18].

The point spectrum of the Dolbeault–Dirac operator $D_\partial : \Omega^{(0,\bullet)} \to \Omega^{(0,\bullet)}$ is given by
\[
\mu_k := [k]_q^2 [k + 1]_q^2, \quad \text{for } k \in \mathbb{N}_0.
\]
Each eigenspace $V_{\mu_k}$ is a $U_q(\mathfrak{sl}_2)$-module, with a $U_q(\mathfrak{sl}_2)$-module isomorphism given by
\[
V_{\mu_k} \simeq V_{2k\omega_1} \oplus V_{2k\omega_1}.
\]

In particular, the multiplicity of the eigenvalue $\mu_k$ is given by $\dim(V_{\mu_k}) = 4k + 2$.

It is important to note that $D_\partial$ is not an isospectral deformation of the classical Dolbeault–Dirac operator, which is to say, the spectrum is not invariant under deformation. This phenomenon extends to general quantum projective space, and conjecturally to all the irreducible quantum flag manifolds.

11. Twisted Dolbeault–Dirac Operators for $O_q(G/L_S)$

In this section we apply the general theory of [9] to the irreducible quantum flag manifolds and prove that upon twisting by appropriate line bundles, their Dolbeault–Dirac operators can be shown to be Fredholm operators. This is one of the most important results of the paper, showing that by applying the powerful tools of classical complex geometry to quantum spaces, one can prove general results about the spectral behaviour of their $q$-deformed differential operators. This is in contrast to the isospectral approach advanced by Connes and Landi [14]. Here one sets out a classical spectrum in advance, and then builds a noncommutative geometry around it (see [19, 82, 62] for examples).

As shown in Example 10.19 above, the differential calculus approach will not in general leave the spectrum unchanged, and we consider this as a fundamental property of the noncommutative geometry of quantum groups.

11.1. Line Bundles over the Irreducible Quantum Flag manifolds. In this subsection, we recall the necessary definitions and results about noncommutative line bundles over the irreducible quantum flag manifolds. For a more detailed discussion see [22] or [23].

Classically, the algebra $\mathfrak{l}_S$ is reductive, and hence decomposes into a direct sum $\mathfrak{l}_S^\circ \oplus \mathfrak{u}_1$, comprised of a semisimple part and a commutative part, respectively. In the quantum setting, we are thus motivated to consider the subalgebra
\[
U_q(\mathfrak{l}_S^\circ) := \langle K_i, E_i, F_i \mid i \in S \rangle \subseteq U_q(\mathfrak{l}_S).
\]

From the Hopf $*$-algebra embedding $\iota : U_q(\mathfrak{l}_S^\circ) \hookrightarrow U_q(\mathfrak{g})$, we get the dual Hopf $*$-algebra map $\iota^* : U_q(\mathfrak{g})^\circ \to U_q(\mathfrak{l}_S^\circ)^\circ$. By construction $O_q(G) \subseteq U_q(\mathfrak{g})^\circ$, so we can consider the
AND MÖRDELL THAT EACH $\Phi(E)$ IS A $q$-BUNDLE OVER $U_1$.

Moreover, the CQGA-homogeneous space associated to the Hopf $*$-algebra map $\pi^\natural$.

It follows directly from the defining relations of the Drinfeld–Jimbo quantum groups that, for any set of integers $a_1, \ldots, a_r$, the element

$$K_1^{a_1} \cdots K_s^{a_{s-1}} K_s K_{s+1}^{a_s} \cdots K_r^{a_r}$$

commutes with any other element of $U_q(g)$ up to a power of $q$. Finding an element which is genuinely commutative reduces to solving a system of linear equations in the variables $a_i$, with Cartan matrix coefficients. By invertibility of the Cartan matrix, this system admits a unique solution. Motivated by conventions of parabolic geometry [83], we denote this central element by $K_E$. Now with respect to the usual tensor product of Hopf algebras, a Hopf algebra isomorphism

$$\phi : U_q(I_S) \to U_q(I_S^c) \otimes U(U_1)$$

is uniquely defined by setting

$$\phi(E_i) := E_i \otimes 1, \quad \phi(K_i) := K_i \otimes 1, \quad \phi(F_i) := F_i \otimes 1, \quad \text{for } i \notin S,$$

and, denoting by $X$ the generator of $U_1$,

$$\phi(K_E) := 1 \otimes X.$$

The fact that $\phi(U_q(I_S^c)) = U_q(I_S^c) \otimes 1$, together with centrality of $K_E$, implies that $O_q(G/L^S)$ is closed under the action of $K_E$. Thus we have a well-defined $U(U_1)$-action on $O_q(G/L^S)$, or equivalently a $O(U_1)$-coaction. This implies an associated $\mathbb{Z}$-grading

$$O_q(G/L^S) \simeq \bigoplus_{k \in \mathbb{Z}} \mathcal{E}_k.$$

Each $\mathcal{E}_k$ is clearly a bimodule over $\mathcal{E}_0 = O_q(G/L_S)$. Moreover, since the action of $U(U_1)$ clearly commutes with the left $O_q(G)$-coaction on $O_q(G/L^S)$, each $\mathcal{E}_k$ is an $O_q(G)$-sub-bimodule of $O_q(G/L^S)$. It was shown in [22, Lemma 4.1] that

$$\mathcal{E}_k \in O_q(G/L_S) \mod_0,$$

and moreover that each $\Phi(E_k)$ is a 1-dimensional space. Using the general theory of principal comodule algebras, it was shown in [22] that each $E_k$ is projective as a left $O_q(G/L_S)$-module. Thus, when $q = 1$, each $\mathcal{E}_k$ reduces to the space of sections of a line bundle over $G/L_S$.

**Example 11.1.** For the special case of quantum projective space $O_q(\mathbb{CP}^n)$, the quantum homogeneous space $O_q(G/L^S) = O_q(S^{2n-1})$ is given by the odd dimensional quantum sphere $O_q(S^{2n-1})$, where the decomposition into line bundles is well known [58, 63].

For the case of the quantum quadrics $O_q(Q_n)$, the quantum homogeneous space $O_q(G/L^S)$ is a $q$-deformation of the coordinate ring of $V^2(\mathbb{R}^n)$, the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^n$. 

**NONCOMMUTATIVE DOLBEAULT–DIRAC FREDHOLM OPERATORS**
11.2. The Bott–Borel–Weil Theorem. The Borel–Weil theorem [77] is an elegant geometric procedure for constructing all unitary irreducible representations of a compact Lie group. In this section we recall its noncommutative generalisation, as introduced in [22], and its role in establishing positivity and negativity for noncommutative line bundles, as presented in [23].

Classically, the line bundles over the irreducible flag manifolds admit a unique holomorphic structure. If we additionally assume left $O_q(G)$-covariance, then uniqueness can be extended to the noncommutative setting (see [22] for details).

**Proposition 11.2.** Each $E_k$ possesses a unique covariant $(0,1)$-connection, which we denote by $\bar{\partial}E_k$. Moreover, each $\bar{\partial}E_k$ is flat and hence forms a covariant holomorphic structure for $E_k$.

We note that since $\bar{\partial}E_k$ is covariant, its kernel, which is to say $H^0(E_k)$, is a $U_q(g)$-module. The following theorem, established in [22], directly generalises the classical Borel–Weil theorem. In particular, it demonstrates that $H^0(E_k)$ is finite dimensional, explicitly identifying it as a fundamental representation of $U_q(g)$.

**Theorem 11.3 (Borel–Weil).** For each irreducible quantum flag manifold $O_q(G/L_S)$, we have $U_q(g)$-module isomorphisms

1. $H^0(E_k) \cong V_{k\varpi_s}$, for all $k \in \mathbb{N}_0$,
2. $H^0(E_{-k}) = 0$, for all $k \in \mathbb{N}$.

As observed in [23], every line bundle over $O_q(G/L_S)$ must be positive, flat, or negative. Combining this observation with the noncommutative Kodaira vanishing theorem, it proves relatively easy to observe the following corollary of the Borel–Weil theorem (see [23] for details).

**Corollary 11.4.** For all $k \in \mathbb{N}$, it holds that $E_k > 0$, and $E_{-k} < 0$.

Through another application of the noncommutative Kodaira vanishing theorem, combined with noncommutative Serre duality [67 §6.2], the following noncommutative generalisation of the Bott–Borel–Weil theorem [8] was established in [23].

**Corollary 11.5 (Bott–Borel–Weil).** For each irreducible quantum flag manifold $O_q(G/L_S)$, we have $U_q(g)$-module isomorphisms

1. $H^{(0,i)}(E_l) = 0$, for all $l \in \mathbb{Z}$, and $i = 1, \ldots, M - 1$,
2. $H^{(0,M)}(E_k) = 0$, for all $k \in \mathbb{N}_0$,
3. $H^{(0,M)}(E_{-k}) \cong V_{-w_0(k\varpi_s)}$, for all $k \in \mathbb{N}$.

11.3. Dolbeault–Dirac Fredholm Operators. With the appropriate cohomological and positivity results recalled, we are now ready to construct twisted Dolbeault–Dirac Fredholm operators for the irreducible quantum flag manifolds. This forms one of the most important results of the paper, producing explicit evidence of the geometry of the underlying calculus moulding the spectral behaviour of its noncommutative differential operators.

**Theorem 11.6.** For each irreducible quantum flag manifold $O_q(G/L_S)$, and any $k \in \mathbb{N}$, it holds that
1. the $\mathcal{E}_{-k}$-twisted Dolbeault–Dirac operator $D^+_{\overline{\partial}_{\mathcal{E}_{-k}}}$ is a Fredholm operator,
2. the $\mathcal{E}_k$-twisted Dolbeault–Dirac operator $D^+_{\overline{\partial}_{\mathcal{E}_k}}$ is a Fredholm operator,
3. $\text{Index}(D^+_{\overline{\partial}_{\mathcal{E}_{-k}}}) = \text{Index}(D^+_{\overline{\partial}_{\mathcal{E}_k}}) = \dim(V_{k\pi\sigma}),$

Proof. Recall from Corollary 11.4 that $\mathcal{E}_k > 0$, and $\mathcal{E}_{-k} < 0$, for all $k \in \mathbb{N}$. Moreover, by Bott–Borel–Weil, we know that $H^{(0,0)}(\mathcal{E}_{\pm k})$ and $H^{(0,\mathcal{M})}(\mathcal{E}_{\pm k})$ are finite dimensional, for all $k \in \mathbb{N}$. The theorem now follows from Theorem 9.21. □

Example 11.7. Returning to the instructive example of the quantum projective space, we recall the explicit curvature calculations presented in [50] for the Podleś sphere, and for all quantum projective spaces in [23]. For $l \in \mathbb{Z}$, the curvature of the Chern connection of the Hermitian holomorphic line bundles $(\mathcal{E}_l, h, \overline{\partial}_{\mathcal{E}_l})$ is given by

$$\nabla^2(e) = -\text{sign}(l) i[l]_q \kappa \otimes e, \quad \text{for all } e \in \mathcal{E}_l. \quad (30)$$

Hence we can see explicitly that $\mathcal{E}_l > 0$, if $l$ is positive, while $\mathcal{E}_l < 0$, if $l$ is negative. Moreover, we see from Corollary 14 that, for each $k \in \mathbb{N},$

$$\sigma_P(D_{\overline{\partial}_{\mathcal{E}_{-k}}}, \sigma_P(D_{\overline{\partial}_{\mathcal{E}_k}}) \subseteq (-\infty, [-k]_q] \cup ([k]_q, \infty),$$

giving us an explicit non-zero lower bound for the spectra of these operators.

Equation (30) shows the classical integer curvatures of the line bundles over $\mathbb{C}P^n$ being $q$-deformed to $q$-integers. The case of the quantum 2-plane Grassmannians $O_q(Gr_{2,n+1})$ is also discussed in [23], where it is again seen to have $q$-integer curvature. In general, the deformation of geometric integer quantities to $q$-integers is a ubiquitous phenomenon in the noncommutative geometry of quantum groups.

Appendix A. Elementary Results on Unbounded Operators

In this appendix, we present the rudiments of the theory of unbounded operators on Hilbert spaces, with a view to making the paper more accessible to those coming from an algebraic or geometric background. Where proofs are omitted, we provide references to the standard texts [74] and [37].

A.1. Closed and Closable Operators. Let $T : \text{dom}(T) \to \mathcal{H}$ be a not necessarily bounded operator on a Hilbert space $\mathcal{H}$, with $\text{dom}(T)$ denoting its domain of definition. The graph of $T$ is the subset

$$\mathcal{G}(T) := \{(x, T(x)) \mid x \in \text{dom}(T)\} \subseteq \mathcal{H} \oplus \mathcal{H}.$$

We say that an operator $T : \text{dom}(T) \to \mathcal{H}$ is closed if its graph $\mathcal{G}(T)$ is closed in the direct sum $\mathcal{H} \oplus \mathcal{H}$. Equivalently, $T$ is closed if for any sequence $\{x_n\}_{n}$ in $\text{dom}(T)$ converging to $x \in \mathcal{H}$, such that $\{T(x_n)\}_{n}$ converges to $y \in \mathcal{H}$, we necessarily have $x \in \text{dom}(T)$ and $T(x) = y$. Finally, we say that an operator is closable if the closure of its graph in $\mathcal{H} \oplus \mathcal{H}$ is the graph of a (necessarily closed) operator.
A.2. Adjoints of Unbounded Operators. For \( T : \text{dom}(T) \to \mathcal{H} \) a densely defined operator, there is an associated operator \( T^\dagger \), called its adjoint, generalising the adjoint of a bounded operator: The domain of \( T^\dagger \) consists of those elements \( x \in \mathcal{H} \) such that
\[
\psi_x : \text{dom}(T) \to \mathbb{C}, \quad y \mapsto \langle x | T(y) \rangle
\]
is a continuous linear functional. By the Riesz representation theorem, there exists a unique \( z \in \mathcal{H} \), such that
\[
\langle z, y \rangle = \langle x, T(y) \rangle, \quad \text{for all } y \in \text{dom}(T).
\]
The operator \( T^\dagger \) is then defined as
\[
T^\dagger : \text{dom}(T^\dagger) \to \mathcal{H}, \quad x \mapsto z.
\]
As established in [74, Theorem 13.8], for a densely defined operator \( T : \text{dom}(T) \to \mathcal{H} \), it holds that
\[
\mathcal{G}(T^\dagger) = \{( -y, x ) \mid (x, y) \in \mathcal{H} \oplus \mathcal{H} \}^\perp.
\]
Consequently, the adjoint of a densely-defined operator is always closed. From this it is easy to conclude that, if \( T^\dagger \) is densely defined on \( \mathcal{H} \), then
\[
\mathcal{G}( (T^\dagger)^\dagger ) = \overline{\mathcal{G}(T)}.
\]
Thus any operator \( T : \text{dom}(T) \to \mathcal{H} \) whose adjoint is densely defined is necessarily closable.

A.3. Essentially Self-Adjoint Operators. A densely defined operator \( T \) is said to be symmetric if it holds that
\[
\langle T(x), y \rangle = \langle x, T(y) \rangle, \quad \text{for all } x, y \in \text{dom}(T).
\]
For any symmetric operator \( T \) it is easy to see that \( \text{dom}(T) \subseteq \text{dom}(T^\dagger) \). Thus, from the discussion of the previous subsection, every densely defined symmetric operator is automatically closable. An operator \( T \) is said to be self-adjoint if it is symmetric and \( \text{dom}(T) = \text{dom}(T^\dagger) \), and is said to be essentially self-adjoint if it is closable and its closure is self-adjoint. As explained in [74, §13.20], a densely symmetric operator is essentially self-adjoint if the operators \( T + i \text{id}_\mathcal{H} \) and \( T - i \text{id}_\mathcal{H} \) have dense range.

A.4. Operator Spectra and Functional Calculus. A complex number \( \lambda \) is said to be in the resolvent set \( \rho(D) \) of an unbounded operator \( D : \text{dom}(D) \to \mathcal{H} \), if the operator
\[
D - \lambda \text{id}_\mathcal{H} : \text{dom}(D) \to \mathcal{H},
\]
has a bounded inverse, that is, if there exists a bounded operator \( S : \mathcal{H} \to \text{dom}(D) \) such that \( S \circ (T - \lambda \text{id}_\mathcal{H}) = \text{id}_{\text{dom}(D)} \) and \( (T - \lambda \text{id}_\mathcal{H}) \circ S = \text{id}_\mathcal{H} \). The spectrum of \( D \), which we denote by \( \sigma(D) \), is the complement of \( \rho(D) \) in \( \mathbb{C} \). Just as in the unbounded case, self-adjoint operators have real spectrum. In particular, \( D + i \text{id} \) is always invertible, giving sense to the compact resolvent condition of a spectral triple.

We now recall functional calculus for unbounded self-adjoint operators: For any self-adjoint \( D \), and a bounded Borel function \( f : \sigma(T) \to \mathbb{C} \), one can associate a bounded function \( f(T) : \mathcal{H} \to \mathcal{H} \). This extends the usual functional calculus for bounded operators (see [37, §1.8] for details). For the special case when \( D \) is diagonalisable, the case
of interest in this paper, $f(D)$ admits a simple explicit description: Let $\{e_n\}_{n \in \mathbb{N}_0}$ be any diagonalisation of $D$, where $D(e_n) = \lambda_n e_n$, then $f(D)$ is the unique bounded linear operator defined by

$$f(D)(e_n) = f(\lambda_n) e_n,$$

for all $n \in \mathbb{N}_0$.

This gives sense to the definition of the bounded transform of a spectral triple in [13].

References

[1] M. F. Atiyah, *Riemann surfaces and spin structures*, Ann. Sci. École Norm. Sup. (4), 4 (1971), pp. 47–62.

[2] S. Baaj and P. Julg, *Théorie bivariante de Kasparov et opérateurs non bornés dans les $C^*$-modules hilbertiens*, C. R. Acad. Sci. Paris Sér. I Math., 296 (1983), pp. 875–878.

[3] R. J. Baston and M. G. Eastwood, *The Penrose transform. Its interaction with representation theory*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1989. Oxford Science Publications.

[4] E. Beggs and S. Majid, *Spectral triples from bimodule connections and Chern connections*, J. Noncommut. Geom., 11 (2017), pp. 669–701.

[5] Quantum Riemannian Geometry, vol. 355 of Grundlehren der mathematischen Wissenschaften, Springer International Publishing, 1 ed., 2019.

[6] E. Beggs and P. S. Smith, *Noncommutative complex differential geometry*, J. Geom. Phys., 72 (2013), pp. 7–33.

[7] S. Bhattacharjee, I. Biswas, and D. Goswami, *Generalized symmetry in noncommutative complex geometry*, arXiv preprint math.QA/1907.04673.

[8] R. Bott, *Homogeneous vector bundles*, Ann. of Math. (2), 66 (1957), pp. 203–248.

[9] A. Carey and J. Phillips, *Unbounded Fredholm modules and spectral flow*, Canad. J. Math., 50 (1998), pp. 673–718.

[10] A. L. Carey, J. Phillips, and A. Rennie, *Spectral triples: examples and index theory*, in Noncommutative geometry and physics: renormalisation, motives, index theory, ESI Lect. Math. Phys., Eur. Math. Soc., Zürich, 2011, pp. 175–265.

[11] A. Chirvasitu, *Relative Fourier transforms and expectations on coideal subalgebras*, J. Algebra, 516 (2018), pp. 271–297.

[12] A. Chirvasitu and M. Tucker-Simmons, *Remarks on quantum symmetric algebras*, J. Algebra, 397 (2014), pp. 589–608.

[13] A. Connes, *Noncommutative geometry*, Academic Press, Inc., San Diego, CA, 1994.

[14] A. Connes and G. Landi, *Noncommutative manifolds, the instanton algebra and isospectral deformations*, Comm. Math. Phys., 221 (2001), pp. 141–159.

[15] A. Connes and H. Moscovici, *The local index formula in noncommutative geometry*, Geom. Funct. Anal., 5 (1995), pp. 174–243.

[16] F. D’Andrea and L. Dąbrowski, *Dirac operators on quantum projective spaces*, Comm. Math. Phys., 295 (2010), pp. 731–790.

[17] F. D’Andrea, L. Dąbrowski, and G. Landi, *The noncommutative geometry of the quantum projective plane*, Rev. Math. Phys., 20 (2008), pp. 979–1006.

[18] B. Das, R. Ő Buachalla, and P. Somberg, *Dolbeault–Dirac spectral triples on quantum homogeneous spaces*, arXiv preprint math.QA/1903.07599.

[19] L. Dąbrowski, G. Landi, A. Sitarz, W. Van Suijlekom, and J. C. Várilly, *The Dirac operator on $SU_q(2)$*, Comm. Math. Phys., 259 (2005), pp. 729–759.

[20] L. Dąbrowski and A. Sitarz, *Dirac operator on the standard Podleś quantum sphere*, in Noncommutative geometry and quantum groups (Warsaw, 2001), vol. 61 of Banach Center Publ., Polish Acad. Sci. Inst. Math., Warsaw, 2003, pp. 49–58.

[21] J.-P. Demailly, *Complex analytic and differential geometry*. Available at https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf.
[22] F. Díaz García and R. Ó Buachalla, A Borel–Weil theorem for the irreducible quantum flag manifolds. In preparation.
[23] F. Díaz García, R. Ó Buachalla, K. R. Strung, A. Krutov, and P. Somberg, Positive and negative line bundles over the irreducible quantum flag manifolds. In preparation.
[24] P. Feng and B. Tsygan, Hochschild and cyclic homology of quantum groups, Comm. Math. Phys., 140 (1991), pp. 481–521.
[25] J. M. Figueroa-O’Farrill, C. Köhl, and B. Spence, Supersymmetry and the cohomology of (hyper)Kähler manifolds, Nuclear Phys. B, 503 (1997), pp. 614–626.
[26] I. Forsyth, B. Mesland, and A. Rennie, Dense domains, symmetric operators and spectral triples, New York J. Math., 20 (2014), pp. 1001–1020.
[27] T. Friedrich, Dirac operators in Riemannian geometry, vol. 25 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2000. Translated from the 1997 German original by Andreas Nestke.
[28] J. Fröhlich, O. Grandjean, and A. Recknagel, Supersymmetric quantum theory, noncommutative geometry, and gravitation, in Symétries quantiques (Les Houches, 1995), North-Holland, Amsterdam, 1998, pp. 221–385.
[29] J. Fröhlich, O. Grandjean, and A. Recknagel, Supersymmetric quantum theory and noncommutative geometry, Comm. Math. Phys., 203 (1999), pp. 119–184.
[30] J. M. Gracia-Bondía, J. C. Várilly, and H. Figueroa, Elements of noncommutative geometry, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Boston, Inc., Boston, MA, 2001.
[31] S. Guin, Noncommutative Kähler structure on $C^*$-dynamical systems, J. Geom. Phys., 146 (2019), p. 103492.
[32] T. Hadfield, Twisted cyclic homology of all Podleś quantum spheres, J. Geom. Phys., 57 (2007), pp. 339–351.
[33] T. Hadfield and U. Krähmer, On the Hochschild homology of quantum $SL(N)$, C. R. Math. Acad. Sci. Paris, 343 (2006), pp. 9–13.
[34] I. Heckenberger and S. Kolb, The locally finite part of the dual coalgebra of quantized irreducible flag manifolds, Proc. London Math. Soc. (3), 89 (2004), pp. 457–484.
[35] I. Heckenberger and S. Kolb, De Rham complex for quantized irreducible flag manifolds, J. Algebra, 305 (2006), pp. 704–741.
[36] N. Higson and J. Roe, Analytic $K$-homology, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000. Oxford Science Publications.
[37] J. E. Humphreys, Introduction to Lie algebras and representation theory, vol. 9 of Graduate Texts in Mathematics, Springer-Verlag, New York-Berlin, 1978. Second printing, revised.
[38] D. Huybrechts, Complex geometry: an introduction, universitext, Springer–Verlag, 2005.
[39] M. Kashiwara, On crystal bases of the $Q$-analogue of universal enveloping algebras, Duke Math. J., 63 (1991), pp. 465–516.
[40] M. Khalkhali, G. Landi, and W. D. van Suijlekom, Holomorphic structures on the quantum projective line, Int. Math. Res. Not. IMRN, (2011), pp. 851–884.
[41] A. Klimyk and K. Schmudgen, Quantum Groups and Their Representations, Texts and Monographs in Physics, Springer-Verlag, 1997.
[42] J.-L. Koszul and B. Malgrange, Sur certaines structures fibrées complexes, Arch. Math. (Basel), 9 (1958), pp. 102–109.
[43] U. Krähmer, Dirac operators on quantum flag manifolds, Lett. Math. Phys., 67 (2004), pp. 49–59.
[44] U. Krähmer and M. Tucker-Simmons, On the Dolbeault-Dirac operator of quantized symmetric spaces, Trans. London Math. Soc., 2 (2015), pp. 33–56.
[45] J. Kustermans, G. J. Murphy, and L. Tuset, Quantum groups, differential calculi and the eigenvalues of the laplacian, Trans. Amer. Math. Soc., 357 (2005), pp. 4681–4717.
[46] E. C. Lance, Hilbert $C^*$-modules, vol. 210 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 1995. A toolkit for operator algebraists.
[48] G. Lusztig, *Introduction to quantum groups*, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2010. Reprint of the 1994 edition.

[49] S. Majid, *Foundations of quantum group theory*, Cambridge University Press, Cambridge, 1995.

[50] ———, *Noncommutative Riemannian and spin geometry of the standard q-sphere*, Comm. Math. Phys., 256 (2005), pp. 255–285.

[51] T. Masuda, Y. Nakagami, and J. Watanabe, *Noncommutative differential geometry on the quantum SU(2). I. An algebraic viewpoint*, K-Theory, 4 (1990), pp. 157–180.

[52] ———, *Noncommutative differential geometry on the quantum two sphere of Podleś. I. An algebraic viewpoint*, K-Theory, 5 (1991), pp. 151–175.

[53] ———, *Kähler structures on quantum irreducible flag manifolds*, J. Geom. Phys., 145 (2019), pp. 103477, 16.

[54] ———, *Dolbeault-Dirac operators, quantum Clifford algebras and the Parthasarathy formula*, Adv. Appl. Clifford Algebr., 27 (2017), pp. 1581–1609.

[55] ———, *On the Dolbeault-Dirac operators on quantum projective spaces*, J. Lie Theory, 28 (2018), pp. 211–244.

[56] ———, *The parthasarathy formula and a spectral triple for the quantum lagrangian Grassmannian of rank two*, arXiv preprint math.QA/1810.06456, 2018.

[57] B. Mesland, *Unbounded bivariant K-theory and correspondences in noncommutative geometry*, J. Reine Angew. Math., 691 (2014), pp. 101–172.

[58] U. Meyer, *Projective quantum spaces*, Lett. Math. Phys., 35 (1995), pp. 91–97.

[59] A. Moroianu, *Lectures on Kähler geometry*, vol. 69 of London Mathematical Society Student Texts, Cambridge University Press, Cambridge, 2007.

[60] G. J. Murphy, *C∗-algebras and operator theory*, Academic Press, Inc., Boston, MA, 1990.

[61] G. Nagy, *Deformation quantization and K-theory*, in Perspectives on quantization (South Hadley, MA, 1996), vol. 214 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1998, pp. 111–134.

[62] S. Neshveyev and L. Tuset, *The Dirac operator on compact quantum groups*, J. Reine Angew. Math., 641 (2010), pp. 1–20.

[63] R. O Buachalla, *Conjugate pairs and first-order complex structures*. In preparation.

[64] R. O Buachalla, *Quantum bundle description of quantum projective spaces*, Comm. Math. Phys., 316 (2012), pp. 345–373.

[65] ———, *Noncommutative complex structures on quantum homogeneous spaces*, J. Geom. Phys., 99 (2016), pp. 154–173.

[66] ———, *Noncommutative Kähler structures on quantum homogeneous spaces*, Adv. Math., 322 (2017), pp. 892–939.

[67] R. O Buachalla, J. Šťovíček, and A.-C. van Roosmalen, *A Kodaira vanishing theorem for noncommutative Kähler structures*. arXiv preprint math.QA/1801.08125, 2018.

[68] Y. Ogawa, *Operators on almost Hermitian manifolds*, J. Differential Geometry, 4 (1970), pp. 105–119.

[69] R. Owczarek, *Dirac operator on the Podleś sphere*, vol. 40, 2001, pp. 163–170. Clifford algebras and their applications (Ixtapa, 1999).

[70] P. Podleś, *Differential calculus on quantum spheres*, Lett. Math. Phys., 18 (1989), pp. 107–119.

[71] E. Poletaeva, *Superconformal algebras and Lie superalgebras of the Hodge theory*, J. Nonlinear Math. Phys., 10 (2003), pp. 141–147.

[72] A. Polishchuk and A. Schwarz, *Categories of holomorphic vector bundles on noncommutative two-tori*, Comm. Math. Phys., 236 (2003), pp. 135–159.

[73] M. Rosso, *An analogue of B.G.G. resolution for the quantum SL(N)-group*, in Symplectic geometry and mathematical physics (Aix-en-Provence, 1990), vol. 99 of Progr. Math., Birkhäuser Boston, Boston, MA, 1991, pp. 422–432.

[74] W. Rudin, *Functional analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, second ed., 1991.

[75] M. Schechter, *Basic theory of Fredholm operators*, Ann. Scuola Norm. Sup. Pisa (3), 21 (1967), pp. 261–280.
[76] J.-P. Serre, Repr´esentations lin´eaires et espaces homog`enes k¨ahl´eriens des groupes de Lie compacts (d’apr`es Armand Borel et Andr´e Weil), in S´eminaire Bourbaki, Vol. 2, Soc. Math. France, Paris, 1995, pp. Exp. No. 100, 447–454.

[77] ———, Repr´esentations lin´eaires et espaces homog`enes k¨ahl´eriens des groupes de Lie compacts (d’apr`es Armand Borel et Andr´e Weil), in S´eminaire Bourbaki, Vol. 2, Soc. Math. France, Paris, 1995, pp. Exp. No. 100, 447–454.

[78] A. Sitarz, Equivariant spectral triples, in Noncommutative geometry and quantum groups (Warsaw, 2001), vol. 61 of Banach Center Publ., Polish Acad. Sci. Inst. Math., Warsaw, 2003, pp. 231–263.

[79] M. Takeuchi, Relative Hopf modules—equivalences and freeness criteria, J. Algebra, 60 (1979), pp. 452–471.

[80] T. Timmermann, An invitation to quantum groups and duality, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Z¨urich, 2008. From Hopf algebras to multiplicative unitaries and beyond.

[81] B. L. Tsygan, Homology of matrix Lie algebras over rings and the Hochschild homology, Uspekhi Mat. Nauk, 38 (1983), pp. 217–218.

[82] W. van Suijlekom, L. D˙abrowski, G. Landi, A. Sitarz, and J. C. V´arilly, The local index formula for SU_q(2), K-Theory, 35 (2005), pp. 375–394 (2006).

[83] A. Cap and J. Slovak, Parabolic geometries. I, vol. 154 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2009. Background and general theory.

[84] C. Voigt and R. Yuncken, Equivariant Fredholm modules for the full quantum flag manifold of SU_q(3), Doc. Math., 20 (2015), pp. 433–490.

[85] A. Weil, Introduction `a l’´etude des vari´et´es k¨ahl´eriennes, no. 1267 in Publications de l’Institut de Math´ematique de l’Universit´e de Nancago, VI., Hermann, Paris, 1958.

[86] S. L. Woronowicz, Compact matrix pseudogroups, Comm. Math. Phys., 111 (1987), pp. 613–665.

[87] ———, Twisted SU(2) group. An example of a noncommutative differential calculus, Publ. Res. Inst. Math. Sci., 23 (1987), pp. 117–181.

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