A CLASSIFICATION OF REAL INDECOMPOSABLE SOLVABLE LIE ALGEBRAS
OF SMALL DIMENSION WITH CODIMENSION ONE NILRADICALS

by

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A Classification of Real Indecomposable Solvable Lie Algebras
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This thesis was concerned with classifying the real indecomposable solvable Lie algebras with
codimension one nilradicals of dimensions two through seven. This thesis was organized into three
chapters.

In the first, we described the necessary concepts and definitions about Lie algebras as well as a
few helpful theorems that are necessary to understand the project. We also reviewed many concepts
from linear algebra that are essential to the research.

The second chapter was occupied with a description of how we went about classifying the Lie
algebras. In particular, it outlined the basic premise of the classification: that we can use the auto-
morphisms of the nilradical of the Lie algebra to find a basis with the simplest structure equations
possible. In addition, it outlined a few other methods that also helped find this basis. Finally, this
chapter included a discussion of the canonical forms of certain types of matrices that arose in the
project.

The third chapter presented a sample of the classification of the seven dimensional Lie algebras.
In it, we proceeded step-by-step through the classification of the Lie algebras whose nilradical was
one of four specifically chosen because they were representative of the different types that arose
during the project.
In the appendices, we presented our results in a list of the multiplication tables of the isomorphism classes found.
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Chapter 1

A REVIEW OF FREQUENTLY USED CONCEPTS

This chapter will deal with reviewing those concepts and definitions that are most helpful in discussing this classification of solvable Lie algebras. We will begin our discussion with a few basic definitions and concepts of Lie theory. This will be followed by the major linear algebra concepts needed for our classification.

1.1 Lie Algebras: Definitions and Concepts

We first define a Lie algebra and the various types of Lie algebra homomorphisms.

**Definition 1.1.1.** A **Lie algebra** is a vector space, $\mathfrak{g}$, over a field, $\mathbb{F}$, coupled with a mapping

$\mathbf{[\cdot, \cdot]} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$

that is

1. **bilinear** ($\mathbf{[cX + Y, Z]} = c\mathbf{[X, Z]} + \mathbf{[Y, Z]}$; $\mathbf{[X, cY + Z]} = c\mathbf{[X, Y]} + \mathbf{[X, Z]}$),

2. **skew-symmetric** ($\mathbf{[X, Y]} = -\mathbf{[Y, X]}$),

3. and satisfies the **Jacobi property**

$$\mathbf{[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]} = 0$$

for all $c \in \mathbb{F}$ and $X, Y, Z \in \mathfrak{g}$. 

The map $[,]$ is called the Lie bracket.

**Definition 1.1.2.** If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, then a linear transformation $T : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism if for all $X, Y \in \mathfrak{g}$ we have that

$$T([X, Y]) = [T(X), T(Y)].$$

If $\mathfrak{g} = \mathfrak{h}$, then $T$ is a Lie algebra endomorphism. If $T$ is bijective, then $T$ is a Lie algebra isomorphism. If $T$ is both a Lie algebra endomorphism and an isomorphism, we call $T$ a Lie algebra automorphism.

There are a number of different properties of Lie algebras. In fact, the classification of Lie algebras into isomorphism classes is done by finding canonical forms for algebras with certain properties that are preserved by isomorphism. As such, it is necessary to discuss a few of these properties here.

**Definition 1.1.3.** The derived algebra, denoted $D\mathfrak{g}$, is the set

$$\{ X \in \mathfrak{g} \mid \text{for some } Y, Z \in \mathfrak{g}, X = [Y, Z] \}.$$  

An alternative way to describe the derived algebra is

$$D\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}],$$

where $[\mathfrak{g}, \mathfrak{g}]$ denotes all possible Lie brackets between vectors in $\mathfrak{g}$. This gives rise to another concept called the *derived series*, given by a series of $D^i\mathfrak{g}$, for all $i \in \mathbb{N}$, where each $D^i\mathfrak{g}$ is defined inductively by

$$D^i\mathfrak{g} = [D^{i-1}\mathfrak{g}, D^{i-1}\mathfrak{g}] \text{ with } D^1\mathfrak{g} = D\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}].$$

Another useful series is the *lower central series*, given by a series of $D_i\mathfrak{g}$, for all $i \in \mathbb{N}$, where each $D_i\mathfrak{g}$ is defined by

$$D_i\mathfrak{g} = [D_{i-1}\mathfrak{g}, \mathfrak{g}] \text{ with } D_1\mathfrak{g} = D\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}].$$

Note that for all $i \in \mathbb{N}$, $D^i\mathfrak{g} \subseteq D_i\mathfrak{g}$.

The dimensions of each $D^i\mathfrak{g}$ and $D_i\mathfrak{g}$ is invariant under a Lie algebra isomorphism.

These facts give us enough information to discuss the difference between two special types of Lie algebras. So we give two definitions. When in context, let 0 denote the zero vector.
Definition 1.1.4. A Lie algebra is solvable if for some \( i \in \mathbb{N} \), \( D^i \mathfrak{g} = \{0\} \).

Definition 1.1.5. A Lie algebra is nilpotent if for some \( i \in \mathbb{N} \), \( D_i \mathfrak{g} = \{0\} \).

It is clear then that because \( D^i \mathfrak{g} \subseteq D_i \mathfrak{g} \) for all \( i \in \mathbb{N} \), all nilpotent Lie algebras are also solvable. In addition, as the dimensions of \( D^i \mathfrak{g} \) and \( D_i \mathfrak{g} \) are invariant under a Lie algebra isomorphism, it follows that whether an algebra is solvable or nilpotent is invariant as well.

There is another property that is important to note as it plays a large role in the type of algebras we classify in this paper. So we offer the following definition.

Definition 1.1.6. A Lie algebra, \( \mathfrak{g} \), is said to be decomposable if there exist lower dimensional algebras \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) such that

\[
\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2
\]

where \( \oplus \) denotes a Lie algebra direct sum. A Lie algebra is said to be indecomposable if it is not decomposable.

Another useful tool is the concept of the nilradical of a solvable Lie algebra.

Definition 1.1.7. The nilradical of a Lie algebra, \( \mathfrak{g} \), is its maximal nilpotent ideal. We’ll denote it \( NR(\mathfrak{g}) \). The codimension of the nilradical of a Lie algebra is the difference between the dimension of the entire Lie algebra and the dimension of the nilradical, that is \( \dim \mathfrak{g} - \dim NR(\mathfrak{g}) \).

The nilradical is similar to the radical of a Lie algebra, which is defined to be the maximal solvable ideal. The radical is useful in classifying non-solvable Lie algebras, as the Levi decomposition states that every Lie algebra can be written as the semi-direct product of its radical and a semisimple Lie algebra. However, in a solvable Lie algebra, the radical is obviously the entire algebra. So we use the nilradical instead. The nilradical is unique in a Lie algebra and if two algebras are isomorphic, then their nilradicals are isomorphic as well. Thus it becomes the perfect object by which to classify the algebra.

Moreover, there is a useful theorem that states that the derived algebra, \( D\mathfrak{g} \), of a solvable Lie algebra, \( \mathfrak{g} \), is contained in the nilradical of \( \mathfrak{g} \). That is, \( D\mathfrak{g} \subseteq NR(\mathfrak{g}) \).

In this classification, we classify the indecomposable solvable Lie algebras over \( \mathbb{R} \) of dimensions two through seven with codimension one nilradicals. So we have most of the necessary definitions
and concepts. However, there are a few more useful tools that we’ll use in the classification that we’ll describe here.

The first is a derivation.

**Definition 1.1.8.** A linear transformation \( T : \mathfrak{g} \to \mathfrak{g} \) is a derivation if it satisfies the Leibniz rule. That is for all \( X, Y \in \mathfrak{g} \), we have

\[
T([X,Y]) = [T(X), Y] + [X, T(Y)].
\]

The space of derivations form a Lie algebra with the Lie bracket given by the commutator \([T, U] = T \circ U - U \circ T\). The exponential of a derivation, \( e^T \) is an automorphism of \( \mathfrak{g} \).

The next tool is a representation of a Lie algebra.

**Definition 1.1.9.** A Lie algebra representation is a Lie algebra homomorphism \( \rho : \mathfrak{g} \to \mathfrak{gl}(V) \), where \( \mathfrak{gl}(V) \) denotes the Lie algebra of all linear transformations of a vector space \( V \) whose Lie bracket is given by the commutator.

A special Lie algebra representation is the adjoint or \( \text{ad} \) representation of a Lie algebra \( \mathfrak{g} \).

**Definition 1.1.10.** The \( \text{ad} \) representation of a Lie algebra, \( \mathfrak{g} \), is a Lie algebra homomorphism \( \text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \) such that for all \( X, Y \in \mathfrak{g} \),

\[
\text{ad}(X)(Y) = [X,Y].
\]

We move on now to review a few Linear algebra concepts.

### 1.2 Linear Algebra: Change of Basis Matrices and Real Jordan Canonical Form

Here we give a discussion about invertible Linear transformations on a Vector space and how this idea extends to Lie Algebra isomorphisms. Most importantly though, we’ll talk about how these invertible operators can be used to “move” a matrix into Jordan Canonical Form. We begin by defining the matrix representation of a transformation.
Definition 1.2.1. Let $V$ and $W$ be finite dimensional vector spaces over a field $\mathbb{F}$ with bases $\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{w_1, \ldots, w_m\}$ respectively. Let $T : V \rightarrow W$ be a linear transformation, then we could express $T$ evaluated on every basis vector $v_i$ in the following way

$$T(v_i) = \sum_{j=1}^{m} c^j_i w_j,$$

for $1 \leq i \leq n$, where $c^j_i \in \mathbb{F}$. Then the matrix representation of $T$ denoted $[T]_\beta^\gamma$ is the $m \times n$ matrix given by

$$([T]_\beta^\gamma)_{i,j} = c^j_i.$$

We also note that if $v \in V$, then $v = \sum_{i=1}^{n} a^i v_i$ and so we could define the $m \times 1$ matrix $[v]_\beta$ by $([v]_\beta)_i = a^i$. This will give us the property that

$$[T]_\beta^\gamma [v]_\beta = [T(v)]_\gamma.$$

This is how the matrix representation is useful. It simplifies evaluation of a linear transformation on a vector to matrix multiplication.

Now we can define an invertible linear transformation and see how this relates to its matrix representation.

Definition 1.2.2. Let $T : V \rightarrow W$ and $U : W \rightarrow V$ be linear transformations. Let $I_V : V \rightarrow V$ and $I_W : W \rightarrow W$ be the identity transformations on $V$ and $W$ respectively. If $T \circ U = I_V$ and $U \circ T = I_W$, then $T$ is invertible and $T^{-1} = U$.

As we’ve seen that evaluation of a transformation $T$ can be reduced to left multiplication of $[T]_\beta^\gamma$, then we can see that $[T \circ U]_\beta^\gamma = [T]_\beta^\gamma [U]_\beta^\gamma$. Because of this, we have that $T$ is an invertible linear transformation if and only if $[T]_\beta^\gamma$ is an invertible matrix. Which reduces to $\det[T]_\beta^\gamma \neq 0$. This shows quite easily then that if $T : V \rightarrow W$ invertible, then $\dim V = \dim W$. Of course a linear transformation is invertible if and only if it is bijective. Hence any Lie algebra isomorphism is an invertible linear transformation. It should be noted that an invertible $n \times n$ matrix is an element of the group $GL(n, \mathbb{F})$ where $\mathbb{F}$ is the field of scalars. We will usually denote such a matrix in this manner.

Next we define a change of basis matrix. These are important because the matrix representation of any Lie algebra automorphism can be interpreted as a change of basis matrix.
Definition 1.2.3. Let $V$ be a finite-dimensional vector space and let $\beta$ and $\beta'$ be two bases for $V$. Let $Q = [I_V]_{\beta'}^\beta$, then we call $Q$ the change of basis matrix that changes $\beta'$-coordinates into $\beta$-coordinates.

Let $T : V \to V$ be a linear transformation. Then if $Q$ is the change of basis matrix that changes $\beta'$-coordinates into $\beta$-coordinates, we have that

$$[T]_{\beta'}^{\beta} = Q^{-1}[T]_{\beta'}^{\beta}Q.$$  

This is because

$$Q[T]_{\beta'}^{\beta'} = [I_V]_{\beta}^{\beta'}[T]_{\beta'}^{\beta} = [I_{TV}]_{\beta}^{\beta'}[T]_{\beta}^{\beta'} = [I_{TV}]_{\beta}^{\beta'}[I_V]_{\beta}^{\beta'} = [I_{TV}]_{\beta}^{\beta'}Q.$$  

Hence, we can see how changing the basis on a vector space affects the matrix representation of a linear transformation. Analogously, we also can see how changing the basis of a Lie algebra affects the matrix representation of a Lie algebra endomorphism, which is a concept we use almost constantly throughout the computations in this paper. It also plays a key role in the discussion of Jordan canonical form, which we'll review next.

The reader is cited to any standard Linear Algebra text for a more in depth study of Jordan canonical form including the proofs of the following statements. The real Jordan canonical form is explained in detail in the book by Hirsch and Smale [2].

Let $T : V \to V$ be a linear transformation on a vector space $V$ over a field $F$.

Definition 1.2.4. An eigenvector of $T$ is a vector, $v \in V$, such that $(T - \lambda I)(v) = 0$ for some $\lambda \in F$. The value $\lambda$ is called the eigenvalue corresponding to the eigenvector $v$. A generalized eigenvector, corresponding to the eigenvalue $\lambda \in F$, of $T$ is a vector, $v' \in V$, such that $(T - \lambda I)^p(v) = 0$ for some positive integer $p$.

Definition 1.2.5. The subspace of $V$

$$\{v \in V \mid (T - \lambda I)^p(v) = 0 \text{ for some positive integer } p\}$$

is called the generalized eigenspace of $T$ corresponding to $\lambda$ and is denoted $K_\lambda$. If $K_\lambda$ consists only of eigenvectors of $T$, then $K_\lambda$ is simply called the eigenspace of $T$ corresponding to $\lambda$. 
Definition 1.2.6. The polynomial \( q(\lambda) = \det(T - \lambda I) \) is called the characteristic polynomial of \( T \) and has degree equal to the dimension of \( V \).

The roots of the characteristic polynomial are the eigenvalues of \( T \). In addition, the multiplicity of any root of the characteristic polynomial is equal to the dimension of its corresponding generalized eigenspace.

Definition 1.2.7. Let \( v \) be a generalized eigenvector of \( T \). If \( p \) is the smallest integer such that \( (T - \lambda I)^p(v) = 0 \), then the set \( \{(T - \lambda I)^{p-1}(v), (T - \lambda I)^{p-2}(v), \ldots, (T - \lambda I)(v), v\} \) is called a cycle of generalized eigenvectors corresponding to \( \lambda \) of length \( p \). The vectors \( (T - \lambda I)^{p-1}(v) \) and \( v \) are called the initial vector and end vector of the cycle respectively.

Clearly the initial vector of any cycle is an eigenvector of \( T \).

Theorem 1.2.1. If \( \beta = \{v_1, \ldots, v_p\} \) is a cycle of generalized eigenvectors corresponding to \( \lambda \in \mathbb{F} \) where \( v_i = (T - \lambda I)^{p-i}(v_p) \). Then \( T(v_1) = \lambda v_1 \) and for all \( i \) such that \( 2 \leq i \leq p \),

\[
T(v_i) = \lambda v_i + v_{i-1}.
\]

Proof. We know that \( v_1 \) is an eigenvector and so \( (T - \lambda I)(v_1) = 0 \) or equivalently \( T(v_1) = \lambda v_1 \).

Also

\[
(T - \lambda I)(v_i) = (T - \lambda I)^{p-(i-1)}(v_p) = v_{i-1}.
\]

The result follows immediately. 

Every cycle of generalized eigenvectors is a linearly independent set. Moreover, if the initial vectors of the cycles are linearly independent, then the union of any number of cycles is linearly independent. Furthermore, for any \( K_\lambda \), there exists a basis for \( K_\lambda \) consisting of disjoint cycles of generalized eigenvectors.

Thus if the characteristic polynomial splits over \( \mathbb{F} \), then \( V \) is the direct sum of generalized eigenspaces and has a basis, \( \beta \), consisting of disjoint cycles of generalized eigenvectors. In this basis, \( T \) is in what is called Jordan canonical form. The Jordan canonical form of a transformation \( T \) is unique up to the ordering of its eigenvalues. If \( A \) is the matrix representation of \( T \) in a given basis,
\[ Q^{-1}AQ = [T]^\beta_{\beta}. \]

Thus the Jordan canonical form of the matrix representation, \( A \), of a linear transformation, \( T \), if it exists, is a matrix of the form

\[
\begin{pmatrix}
\lambda_1 & a_{1,1} \\
\lambda_1 & a_{1,2} \\
\vdots & \ddots \\
\lambda_1 & a_{2,1} \\
\vdots & \\
\vdots & \\
\lambda_n \\
\end{pmatrix}
\]

where the \( a_{i,j} \in \{0, 1\} \). In addition, the \( \lambda_i \) are the eigenvalues of \( A \). The square block belonging to each \( \lambda_i \) corresponds to a generalized eigenspace of \( A \) and the block is called a Jordan block.

For example, let \( A \) be a 6 \times 6 matrix whose characteristic polynomial splits over \( \mathbb{R} \). If, for some invertible \( Q \),

\[
Q^{-1}AQ = \begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & 1 & 0 \\
0 & 0 & 0 & 0 & 5 & 1 \\
0 & 0 & 0 & 0 & 0 & 5 \\
\end{pmatrix},
\]

then this is a Jordan canonical form of \( A \). The eigenvalues of \( A \) are 2, 3, and 5. The Jordan blocks are

\[
\begin{pmatrix}
2 & 0 \\
0 & 2 \\
\end{pmatrix},
\begin{pmatrix}
5 & 1 & 0 \\
0 & 5 & 1 \\
0 & 0 & 5 \\
\end{pmatrix},
\]

and each of these blocks correspond to a generalized eigenspace of \( A \).

We are considering real Lie algebras which are vector spaces over \( \mathbb{R} \). However, not all polynomials split over \( \mathbb{R} \). It is true, though, that as \( \mathbb{C} \) is algebraically closed, every polynomial with real coefficients splits over \( \mathbb{C} \). Thus the complexification of \( V, V_c \), can always be written as the direct sum of generalized eigenspaces and the complex Jordan canonical form always exists for such a matrix. But it is important to note that as any real linear transformation \( T \) will have a characteristic polynomial with real coefficients, any complex root will come in a conjugate pair. Hence if \( z \) is a complex eigenvalue of a linear transformation \( T \), then \( \overline{z} \) is as well. In fact, if \( v \) is an eigenvector of \( A \) corresponding to \( z \), then \( \overline{v} \) is an eigenvector corresponding to \( \overline{z} \).
Let $z = a + bi$ be an eigenvalue of $T$ corresponding to an eigenvector $v_1$. Then $v_2 = v_1$ is an eigenvector corresponding to $a - bi$ and a submatrix of the Jordan canonical form of the matrix representation of $T$ is

$$\begin{pmatrix} a + bi & 0 \\ 0 & a - bi \end{pmatrix}.$$ 

Let $v'_1 = v_1 + v_2$ and $v'_2 = -i(v_1 - v_2)$. Then $v'_1$ and $v'_2$ are real vectors that form a basis for the space spanned by the eigenspaces of $v_1$ and $v_2$. In this basis the submatrix block above becomes

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$ 

If a cycle of generalized eigenvectors of length two corresponds to a complex eigenvalue, then we have a Jordan block of the form

$$\begin{pmatrix} a + bi & 1 & 0 & 0 \\ 0 & a + bi & 0 & 0 \\ 0 & 0 & a - bi & 1 \\ 0 & 0 & 0 & a - bi \end{pmatrix}.$$ 

Then using a similar trick as described above but on both the eigenvectors and the generalized eigenvectors, this block will become

$$\begin{pmatrix} a & b & 1 & 0 \\ -b & a & 0 & 1 \\ 0 & 0 & a & b \\ 0 & 0 & -b & a \end{pmatrix}.$$ 

This can be easily extended to a generalized eigenspace of any dimension.

If we use this to deal with a characteristic polynomial with complex roots, then the resulting form of $T$ is called the real Jordan canonical form and the basis for $V$, that puts $T$ into real Jordan canonical form is also a basis for $V$. Thus we can still consider $V$ as a vector space over $\mathbb{R}$ and every linear transformation $T : V \to V$ has a real Jordan canonical form.

This completes our review of frequently used concepts. The reader should have the necessary information to understand the mechanics behind the paper. In the next chapter, we’ll discuss the methods we use in classifying Lie algebras and also the canonical forms of a few special types of matrices.
This chapter is devoted to discussing in detail methods that will be utilized frequently throughout the classification. As they are discussed here, when they are used in the text the reader will be referred back to this chapter for a more detailed explanation. In this chapter, we will also be building off the topics reviewed in the previous chapter.

2.1 Classifying Solvable Lie Algebras with Codimension One Abelian Nilradicals

The first of these methods that we will discuss is that of how we’ll classify a solvable Lie algebra with a codimension one abelian nilradical.

To begin, let $\mathfrak{g}$ be an $n$-dimensional solvable Lie algebra with such a nilradical, $NR(\mathfrak{g})$. Now choose a basis for $\mathfrak{g}$, $\beta = \{e_1, e_2, \ldots, e_n\}$, such that the set $\gamma = \{e_1, e_2, \ldots, e_{n-1}\}$ forms a basis for the nilradical. As $\mathfrak{g}$ is solvable, we know that $D\mathfrak{g} \subseteq NR(\mathfrak{g})$, and this yields

$$[e_i, e_j] = \sum_{k=1}^{n-1} A_{i,j}^k e_k$$

for all $1 \leq i, j \leq n$. Also as $NR(\mathfrak{g})$ is abelian we have that $[e_i, e_j] = 0$ for all $1 \leq i, j \leq n - 1$. Combining these two facts, we have that the structure equations of $\mathfrak{g}$ are simply

$$[e_i, e_n] = \sum_{k=1}^{n-1} A_{i,n}^k e_k$$
for all $1 \leq i \leq n - 1$. And so it would only make sense to direct our discussion towards $\text{ad}(e_n)$.

From the structure equations we have that

$$\text{ad}(e_n)_{\beta} = \begin{pmatrix} -A_{1,n}^1 & -A_{1,n}^2 & \cdots & -A_{1,n}^{n-1} & 0 \\ -A_{2,n}^1 & -A_{2,n}^2 & \cdots & -A_{2,n}^{n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -A_{1,n}^{n-1} & -A_{2,n}^{n-1} & \cdots & -A_{n-1,n}^{n-1} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$  

It is clear that we lose no information of the structure constants if we restrict this transformation to act only on the nilradical. Thus we will consider $\text{ad}(e_n)$ in the following way

$$\text{ad}(e_n)_{|NR(g)} = \begin{pmatrix} -A_{1,n}^1 & -A_{2,n}^1 & \cdots & -A_{n-1,n}^{n-1} \\ -A_{1,n}^2 & -A_{2,n}^2 & \cdots & -A_{n-1,n}^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{1,n}^{n-1} & -A_{2,n}^{n-1} & \cdots & -A_{n-1,n}^{n-1} \end{pmatrix}$$

but we’ll refer to it as $\text{ad}(e_n)$.

Now we note that as $NR(g)$ is abelian ($[a,b] = 0$ for all $a,b \in NR(g)$), any change of basis strictly on the vectors in the nilradical will not change the structure constants of the nilradical. To compute the matrix representation of $\text{ad}(e_n)$ after we apply such a change of basis, we first construct a change of basis matrix $M$ and conjugate $\text{ad}(e_n)$ by it in the following way

$$M^{-1} \text{ad}(e_n) M.$$  

We can pick $M \in GL(NR(g))$ arbitrarily so this expression can move $\text{ad}(e_n)$ into real Jordan canonical form.

This can always be done when the nilradical is abelian and codimension one, and so this method is used quite often in our classification. It would also be helpful in our classification discussion to be familiar with the possible real Jordan canonical forms for each dimension through dimension six. We will discuss this a little later.

We should also note that we can make one more change that won’t affect the structure equations of the nilradical. This basis change is simply exchanging $f_6$ by a constant multiple of itself. This change won’t affect the nilradical basis at all, but can change values in the $\text{ad}(f_6)$ matrix. So this is usually the type of basis change we’ll make at the end of each classification. However, for some real Jordan canonical forms, we add into the basis change a scaling of one or more of the nilradical basis
vectors. This is done to keep certain values the same while scaling $f_6$. For instance, when we have a Jordan block that has any number of ones on the super diagonal. These added scalings are simply reapplying one or more of the automorphisms of the nilradical. This doesn’t mean much in this case where any invertible linear transformation is an automorphism of the nilradical, but it will become more relevant as we move further through our classification. So, to keep from being confusing in our use of the automorphisms, we’ll simply make the change when making our final change of basis.

### 2.2 Classifying Solvable Lie Algebras with Codimension One Non-abelian Nilradicals

This section is a natural extension of the preceding one. We will see many of the same ideas used, as well as extensions to them.

We are given a Lie algebra $\mathfrak{g}$ with a codimension one non-abelian nilradical, where the first $n - 1$ vectors in a basis for $\mathfrak{g}$ constitute a basis for $NR(\mathfrak{g})$. We will always apply the necessary isomorphism to $NR(\mathfrak{g})$ to move it into a canonical form. We utilize known texts for a classification of nilpotent Lie algebras of the required dimensions [7, 1].

As $D\mathfrak{g} \subseteq NR(\mathfrak{g})$, we can still view $\text{ad} (e_n)$ restricted to the nilradical and not lose any information on the structure equations involving $e_n$. Arbitrarily, this is given by

$$\text{ad} (e_n) = \begin{pmatrix} A_1^1 & A_1^2 & \cdots & A_1^{n-1} \\ A_2^1 & A_2^2 & \cdots & A_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n-1}^1 & A_{n-1}^2 & \cdots & A_{n-1}^{n-1} \end{pmatrix}.$$  

This is the matrix that we would like to simplify. And we have quite a few tools to do it.

#### 2.2.1 The Jacobi Property

The next step is to require the algebra to satisfy the Jacobi property. This will eliminate several of the $A_i^j$ in the $\text{ad} (e_n)$ matrix simultaneously. In fact, as $D\mathfrak{g} \subseteq NR(\mathfrak{g})$, $\text{ad} (e_n)$ maps $NR(\mathfrak{g})$ to $NR(\mathfrak{g})$, and by the Jacobi property, we have, for $i, j < n$, that

$$\text{ad} (e_n) ([e_i, e_j]) = [e_n, [e_i, e_j]] = -[e_i, [e_j, e_n]] - [e_j, [e_n, e_i]]$$

$$= [e_i, [e_n, e_j]] + [[e_n, e_i], e_j] = [e_i, \text{ad} (e_n) (e_j)] + [\text{ad} (e_n) (e_i), e_j].$$
Then \( \text{ad} (e_n) \) is a derivation of the nilradical. This implies that in general, \( \text{ad} (e_n) \) is an arbitrary derivation of the nilradical. This significantly simplifies the possible general form of \( \text{ad} (e_n) \).

Note that we didn’t check the Jacobi property when the codimension one nilradical was abelian. If the nilradical is abelian then \([u, v] = 0\) for all \(u, v \in NR(g)\). Then, as \( Dg \subseteq NR(g) \), we see that \([e_n, e_i] \in NR(g)\) for all \(i < n\). This yields that for all \(i, j < n\), we have that

\[
\begin{align*}
[e_n, [e_i, e_j]] + [e_i, [e_j, e_n]] + [e_j, [e_n, e_i]] &= 0.
\end{align*}
\]

Hence the Jacobi property was already satisfied.

### 2.2.2 Perturbing \( e_n \)

We begin our discussion here by making an observation. Consider an \( n \)-dimensional Lie algebra with a codimension one abelian nilradical and basis \( \beta = \{e_1, e_2, \ldots, e_n\} \), where the first \( n - 1 \) vectors constitute a basis for the nilradical.

Pick a new basis

\[
e_1 = e_1
\]

\[
e_2 = e_2
\]

\[
\vdots
\]

\[
e_{n-1} = e_{n-1}
\]

\[
e_n = e_n + \sum_{k=1}^{n-1} \lambda_k e_k
\]

and consider the structure equations. We have that, for \(i, j \leq n - 1\), \([e_i, e_j] = 0\) simply because the first \( n - 1 \) vectors still form a basis for the abelian nilradical. Then the only brackets left to consider are those of the form

\[
[e_n, e_i].
\]

Thus we have that for \(i \leq n - 1\) (as of course \([\hat{e}_n, \hat{e}_n] = 0\))

\[
\begin{align*}
[\hat{e}_n, e_i] &= \left[ e_n + \sum_{k=1}^{n-1} \lambda_k e_k, e_i \right] \\
&= [e_n, e_i] + \sum_{k=1}^{n-1} \lambda_k [e_k, e_i] \\
&= [e_n, e_i]
\end{align*}
\]
Thus we can see that the structure equations do not change for this kind of change of basis. That is, for an algebra with an abelian nilradical, perturbing a vector outside of the nilradical by a linear combination of vectors inside the nilradical will not change the structure equations of the algebra. However, a scaling on $e_n$ would change the structure equations, something we used quite often in classifying algebras of this type.

The important implication of this observation is that if the nilradical is not abelian, then perturbing a vector outside the nilradical by a linear combination of vectors inside the nilradical will change the structure equations of the nilradical. But it will only change those structure equations that have a nonzero bracket involving the vector we perturbed by.

We will use this idea to our advantage and perturb $e_n$ by a particular linear combination of vectors in the nilradical in order to simplify the structure equations (or rather to zero out a few of the $A^i_j$ terms in the $\text{ad} (e_n)$ matrix). We will also remember that we can still change the structure equations by scaling $e_n$.

One more simple way to see how exactly we can change the structure equations involving $e_n$ is that the idea explained above reduces to perturbing $\text{ad} (e_n)$ by linear combinations of the ad matrices of the nilradical basis vectors. This is because

$$\text{ad} \left( e_n + \sum_{k=1}^{n-1} \lambda_k e_k \right) = \text{ad} (e_n) + \sum_{k=1}^{n-1} \lambda_k \text{ad} (e_k).$$

This gives us a better viewpoint to see exactly what $A^i_j$’s we can change or eliminate.

### 2.2.3 The Automorphisms of $NR(\mathfrak{g})$

Recall that when $NR(\mathfrak{g})$ was abelian, we could apply any change of basis matrix to $\text{ad} (e_n)$, which is what allowed us to put $\text{ad} (e_n)$ into real Jordan canonical form. We could do this because we knew that it wouldn’t change the structure equations of the nilradical.

We can then permute the basis of the nilradical in any way that doesn’t change its structure equations. In other words, we can conjugate $\text{ad} (e_n)$ by any change of basis matrix representing a transformation that doesn’t change the structure equations of the nilradical. The obvious and only choice of transformations are the automorphisms of the nilradical. This is consistent with our previous work with abelian nilradicals because the automorphisms of an abelian algebra consist of
all invertible linear transformations on that algebra.

These automorphisms will not affect the structure equations of the nilradical by definition, but they do affect the makeup of \( \text{ad}(e_n) \). We can then use these to our advantage to simplify \( \text{ad}(e_n) \).

We stated last chapter that if \( T \) is a derivation of a Lie algebra, then \( e^T \) is an automorphism of that Lie algebra. As such, the automorphisms of \( NR(g) \) can be found by first computing the derivations of \( NR(g) \) and then exponentiating them.

If the Lie algebra of derivations has a semisimple part via its Levi decomposition, then the group corresponding to that semisimple subalgebra is a subgroup of the automorphism group. As \( \text{ad}(e_n) \) is an arbitrary derivation, we can then use this subgroup of the automorphism group to move the semisimple part of \( \text{ad}(e_n) \) into a canonical form. In the next section, we discuss the semisimple algebras that arose as we classified from one nilradical to the next. Specifically, we find the canonical forms of the matrices representing these semisimple subalgebras when only conjugation by its corresponding group is allowed.

## 2.3 Canonical Forms

In this section, we will discuss the possible canonical forms of different sets of matrices. In each case, we only allow conjugation by a particular group of matrices. We will first consider the set of all \( n \times n \) matrices, which is the Lie algebra \( \mathfrak{gl}(n, \mathbb{R}) \), and allow conjugation by elements of the group \( \text{GL}(n, \mathbb{R}) \); this, of course, results in the real Jordan canonical form. The second set of matrices considered are those in the symplectic Lie algebra \( \mathfrak{sp}(4, \mathbb{R}) \) and we allow conjugation by the group \( \text{SP}(4, \mathbb{R}) \). The final set of matrices considered form a representation of the Lie algebra \( \mathfrak{so}(3, 1, \mathbb{R}) \), but an non-equivalent representation to the usual one. In that case, we will allow conjugation by the group of matrices corresponding to that representation.

### 2.3.1 Real Jordan Canonical Forms of Matrices of Small Dimension

Let \( a \in \mathfrak{gl}(n, \mathbb{R}) \) and \( A \in \text{GL}(n, \mathbb{R}) \). The canonical forms of \( a \), of course, is the real Jordan canonical form, which has already been described in detail in Chapter 1. In this section, we simply enumerate explicitly the real Jordan canonical forms of matrices of dimension six or less. We can describe these
possible forms by taking a look at how the generalized eigenspaces of a transformation split the vector space, \( V \), into a direct sum of smaller vector spaces over \( \mathbb{R} \).

Dimension one is trivial as the one dimensional \( \mathbb{R} \) cannot be split. Thus the only real Jordan canonical form for a one dimensional linear transformation is

\[
(\lambda)
\]

For dimension two, \( \mathbb{R}^2 \) can be split in the following ways

\[
\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R} \\
\mathbb{R}^2 = \mathbb{R}^2
\]

The case where \( \mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R} \) is easy as we can build it from all the combinations of one dimensional real Jordan canonical forms. This will give us the case where \( a \) is diagonalizable over \( \mathbb{R} \). When \( \mathbb{R}^2 = \mathbb{R}^2 \), we have two cases: either the transformation is diagonalizable over \( \mathbb{C} \) but not \( \mathbb{R} \), or it can be put into general real Jordan canonical form over \( \mathbb{R} \). Thus we have three possibilities for the two-dimensional case.

\[
\begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix}, \\
\begin{pmatrix}
\lambda_1 & \lambda_2 \\
-\lambda_2 & \lambda_1
\end{pmatrix}, \\
\begin{pmatrix}
\lambda & 1 \\
0 & \lambda
\end{pmatrix}
\]

We will order the following lists in a similar manner to the pattern above.

Next we consider linear transformations on three-dimensional vector spaces. We can split \( \mathbb{R}^3 \) in the following ways

\[
\mathbb{R}^3 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \\
\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R} \\
\mathbb{R}^3 = \mathbb{R}^3
\]

As will always be the case, the splitting of \( \mathbb{R}^n \) into one-dimensional eigenspaces, here that is \( \mathbb{R}^3 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \), yields the situation that \( a \) is diagonalizable over the reals. The case where \( \mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R} \) does something of note. The \( \mathbb{R}^2 \) will follow the two cases for \( \mathbb{R}^2 \) given above, while the \( \mathbb{R} \) piece simply yields a one-dimensional eigenspace. Actually it will follow that however \( \mathbb{R}^n \) breaks up into eigenspaces, the dimension of each eigenspace will determine what real Jordan canonical forms that
block can have. As such, we will refer quite often to information found previously when determining the general real Jordan canonical forms for a particular dimension. Finally, \( \mathbb{R}^3 \) only yields one case of a three-dimensional generalized eigenspace. The complex eigenspaces do not come into play here as they must appear in conjugate pairs (as we’re working with characteristic polynomials with real coefficients) and we have an odd-dimensional space. Thus dimension three only yields these four cases, up to an ordering of eigenvalues of course.

\[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix},
\begin{pmatrix}
\lambda_1 & \lambda_2 & 0 \\-
\lambda_2 & \lambda_1 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix},
\begin{pmatrix}
\lambda_1 & 1 & 0 \\
0 & \lambda_1 & 0 \\
0 & 0 & \lambda_2
\end{pmatrix},
\begin{pmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 1 \\
0 & 0 & \lambda
\end{pmatrix}.
\]

We now consider dimension four. We have the splitting of \( \mathbb{R}^4 \), thus

\[
\mathbb{R}^4 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}
\]

\[
\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R} \oplus \mathbb{R}
\]

\[
\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2
\]

\[
\mathbb{R}^4 = \mathbb{R}^3 \oplus \mathbb{R}
\]

\[
\mathbb{R}^4 = \mathbb{R}^4
\]

Using the method described above, we can look at the previous dimensions for the possible forms of each eigenspace of that dimension and combining them together according to the direct sum listed. This will cover the first four cases. As for \( \mathbb{R}^4 = \mathbb{R}^4 \), we have a few possibilities. We could simply have a four-dimensional real generalized eigenspace, or we could have two two-dimensional complex generalized eigenspaces, in which case, we would move them to the real case using \( 2 \times 2 \) blocks with a two-dimensional identity block in the strictly upper triangular piece. This is described in more detail in the Chapter. For dimension four then, we have the following nine cases (always up to an ordering of the eigenvalues).

\[
\begin{pmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 \\
0 & 0 & 0 & \lambda_4
\end{pmatrix},
\begin{pmatrix}
\lambda_1 & \lambda_2 & 0 & 0 \\-
\lambda_2 & \lambda_1 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 \\
0 & 0 & 0 & \lambda_4
\end{pmatrix},
\begin{pmatrix}
\lambda_1 & 1 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 \\
0 & 0 & \lambda_2 & 0 \\
0 & 0 & 0 & \lambda_3
\end{pmatrix}.
\]
For dimension five, we split $\mathbb{R}^5$ in the following ways

$$\mathbb{R}^5 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$$

$$\mathbb{R}^5 = \mathbb{R}^2 \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$$

$$\mathbb{R}^5 = \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}$$

$$\mathbb{R}^5 = \mathbb{R}^3 \oplus \mathbb{R} \oplus \mathbb{R}$$

$$\mathbb{R}^5 = \mathbb{R}^3 \oplus \mathbb{R}^2$$

$$\mathbb{R}^5 = \mathbb{R}^4 \oplus \mathbb{R}$$

$$\mathbb{R}^5 = \mathbb{R}^5$$

We again use the same technique already described and also note that the only case that $\mathbb{R}^5 = \mathbb{R}^5$ generates is a five-dimensional generalized eigenspace. Thus we have the following 12 cases for dimension five.
Finally, for dimension six, we can split $\mathbb{R}^6$ up into eigenspaces thus:

\[
\begin{align*}
\mathbb{R}^6 &= \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \\
\mathbb{R}^6 &= \mathbb{R}^2 \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \\
\mathbb{R}^6 &= \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R} \\
\mathbb{R}^6 &= \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R} \\
\mathbb{R}^6 &= \mathbb{R}^3 \oplus \mathbb{R} \oplus \mathbb{R} \\
\mathbb{R}^6 &= \mathbb{R}^3 \oplus \mathbb{R}^2 \oplus \mathbb{R} \\
\mathbb{R}^6 &= \mathbb{R}^3 \oplus \mathbb{R}^3 \\
\mathbb{R}^6 &= \mathbb{R}^4 \oplus \mathbb{R} \oplus \mathbb{R} \\
\mathbb{R}^6 &= \mathbb{R}^4 \oplus \mathbb{R}^2 \\
\mathbb{R}^6 &= \mathbb{R}^5 \oplus \mathbb{R} \\
\mathbb{R}^6 &= \mathbb{R}^6
\end{align*}
\]

Following the same method we can find all the forms up to the last case. For $\mathbb{R}^6 = \mathbb{R}^6$, we can have either a six-dimensional real generalized eigenspace, or a six-dimensional complex generalized eigenspace, again with identity matrices on the strictly upper triangular part. Thus for dimension
six, we have the following 23 possible real Jordan canonical forms

\[
\begin{pmatrix}
\lambda_1 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 & 0 \\
0 & 0 & 0 & \lambda_4 & 0 \\
0 & 0 & 0 & 0 & \lambda_5
\end{pmatrix}, \quad \begin{pmatrix}
\lambda_1 & \lambda_2 & 0 & 0 & 0 \\
-\lambda_2 & \lambda_1 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 & 0 \\
0 & 0 & 0 & \lambda_4 & 0 \\
0 & 0 & 0 & 0 & \lambda_5
\end{pmatrix}, \\
\begin{pmatrix}
\lambda_1 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 & 0 \\
0 & 0 & 0 & \lambda_4 & 0 \\
0 & 0 & 0 & 0 & \lambda_5
\end{pmatrix}, \quad \begin{pmatrix}
\lambda_1 & \lambda_2 & 0 & 0 & 0 \\
-\lambda_2 & \lambda_1 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & \lambda_4 & 0 \\
0 & 0 & -\lambda_4 & \lambda_3 & 0 \\
0 & 0 & 0 & \lambda_5 & \lambda_6
\end{pmatrix}, \\
\begin{pmatrix}
\lambda_1 & \lambda_2 & 0 & 0 & 0 \\
-\lambda_2 & \lambda_1 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & \lambda_4 & 0 \\
0 & 0 & -\lambda_4 & \lambda_3 & 0 \\
0 & 0 & 0 & \lambda_5 & \lambda_6
\end{pmatrix}, \quad \begin{pmatrix}
\lambda_1 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & 1 & 0 & 0 \\
0 & 0 & \lambda_3 & \lambda_4 & 0 \\
0 & 0 & -\lambda_4 & \lambda_3 & 0 \\
0 & 0 & 0 & \lambda_5 & \lambda_6
\end{pmatrix}.
\]
Lemma 2.3.1. Let $A = a^t J + J a = 0$. Then $a^t \in \mathfrak{sp}(2n, \mathbb{R})$. 

2.3.2 Canonical Forms of Matrices in $\mathfrak{sp}(4, \mathbb{R})$

Let $V = \mathbb{R}^{2n}$. Let $J \in GL(V)$ be given as

$$J = \begin{pmatrix} 0 & A_n \\ -A_n & 0 \end{pmatrix}$$

where $A_n$ is the $n \times n$ identity. Note that $J$ is skew-symmetric and has the property that $J^2 = I_{2n}$.

The symplectic Lie algebra, $\mathfrak{sp}(2n, \mathbb{R})$, is defined as

$$\mathfrak{sp}(2n, \mathbb{R}) = \{ a \in \text{Hom}(V, V) \mid a^t J + J a = 0 \}.$$
Proof. If $a \in \mathfrak{sp}(2n, \mathbb{R})$, then $a^t J + Ja = 0$. Multiply on the left by $J$ to obtain $Ja^t J - a = 0$. Then multiply on the right: $-Ja^t - aJ = 0$ or equivalently $(a^t)^t J + Ja^t = 0$. ■

Define $\omega : V \times V \to \mathbb{R}$ by $\omega(x, y) = x^t Jy$ for all $x, y \in V$. This is called a symplectic form. It can also be viewed as a map $V_c \times V_c \to \mathbb{C}$ with the same rule of assignment. We say that $\omega$ is non-degenerate if whenever $z \in V$ is such that $\omega(z, y) = 0$ for all $y \in V$, then $z = 0$.

**Proposition 2.3.2.** The symplectic form, $\omega : V \times V \to \mathbb{R}$ ($\omega : V_c \times V_c \to \mathbb{C}$), is a non-degenerate skew-symmetric bilinear form on $V$ ($V_c$).

Proof. From its construction, it is clearly bilinear over $\mathbb{R}$ or $\mathbb{C}$. To prove skew-symmetry, we note that as $\omega(x, y)$ can be viewed as a $1 \times 1$ matrix, we have that $(\omega(x, y))^t = \omega(x, y)$. Then, as $J^t = -J$, this yields that

$$\omega(x, y) = (\omega(x, y))^t = (x^t Jy)^t = y^t J^t x = -y^t Jx = -\omega(y, x).$$

Finally to prove non-degeneracy, let $z \in V$ be such that $\omega(z, y) = z^t Jy = 0$ for all $y \in V$. As $J$ is invertible, it has zero kernel, thus $z = 0$. ■

We will always denote the symplectic form by $\omega$ as it will be clear from the context whether we mean the complex or real form.

Now we define the symplectic group. The symplectic group, $Sp(2n, \mathbb{R})$ is defined as

$$Sp(2n, \mathbb{R}) = \{ A \in GL(V) \mid \omega(Ax, Ay) = \omega(x, y) \}.$$ 

That is, it is the group that preserves the symplectic form.

**Lemma 2.3.3.** $a \in \mathfrak{sp}(2n, \mathbb{R})$ if and only if $\omega(ax, y) = -\omega(x, ay)$ for all $x, y \in V$. $A \in Sp(2n, \mathbb{R})$ if and only if $A^t J A = J$.

Proof. As $a \in \mathfrak{sp}(2n, \mathbb{R})$, we have that $a^t J + Ja = 0$. Let $x, y \in V$. This gives us

$$\omega(ax, y) = (ax)^t Jy = x^t a^t Jy = -x^t Jay = -\omega(x, ay).$$
Next assume that $\omega(ax, y) = -\omega(x, ay)$ for all $x, y \in V$. Then

$$\omega(ax, y) = -\omega(x, ay)$$
$$\begin{aligned}(ax)^tJy &= -x^tJay \\
x^ta^tJy &= -x^tJay \end{aligned}$$

As this is true for all $x, y \in V$, this implies that $a^tJ = -Ja$ or equivalently that $a^tJ + Ja = 0$. Hence $a \in \mathfrak{sp}(2n, \mathbb{R})$.

To prove the statement about $\mathfrak{sp}(2n, \mathbb{R})$, first assume that $A \in \mathfrak{sp}(2n, \mathbb{R})$. Then

$$\omega(Ax, Ay) = \omega(x, y)$$
$$\begin{aligned}(Ax)^tJAy &= x^tJy \\
x^tA^tJAy &= x^tJy \end{aligned}$$

As this is true for all $x, y \in V$, we see that $A^tJA = J$.

If we assume first that $A^tJA = J$, then $A$ clearly preserves the symplectic form and hence $A \in \mathfrak{sp}(2n, \mathbb{R})$. ■

We now wish to conjugate $a \in \mathfrak{sp}(2n, \mathbb{R})$ by an arbitrary element $A \in \mathfrak{sp}(2n, \mathbb{R})$, that is, $A^{-1}aA$.

**Lemma 2.3.4.** If $a \in \mathfrak{sp}(2n, \mathbb{R})$ and $A \in \mathfrak{sp}(2n, \mathbb{R})$, then $A^{-1}aA \in \mathfrak{sp}(2n, \mathbb{R})$.

**Proof.** If $A \in \mathfrak{sp}(2n, \mathbb{R})$, then $\omega(Ax, y) = \omega(A^{-1}Ax, A^{-1}y) = \omega(x, A^{-1}y)$. This yields

$$\omega(A^{-1}aAx, y) = \omega(aAx, Ay) = -\omega(Ax, aAy) = -\omega(x, A^{-1}aAy).$$

Then by Lemma 2.3.3 $A^{-1}aA \in \mathfrak{sp}(2n, \mathbb{R})$. ■

If $a_1, a_2 \in \mathfrak{sp}(2n, \mathbb{R})$ are such that $A^{-1}a_1A = a_2$ for some $A \in \mathfrak{sp}(2n, \mathbb{R})$, we say that $a_1$ and $a_2$ are *symplectically similar*.

This naturally brings up the question: what kind of canonical forms could $a \in \mathfrak{sp}(2n, \mathbb{R})$ have if this were the only kind of change of basis allowed? In four dimensions, the result is as follows
Theorem 2.3.5. Let \( a \in \mathfrak{sp}(4, \mathbb{R}) \), then \( a \) is symplectically similar to one of the following ten matrices. We call this the real symplectic canonical form of the matrix.

\[
\begin{align*}
(1) & \quad \begin{pmatrix}
\lambda & 0 & 0 & 0 \\
0 & \mu & 0 & 0 \\
0 & 0 & -\lambda & 0 \\
0 & 0 & 0 & -\mu
\end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}, \\
(2) & \quad \begin{pmatrix}
\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & \varepsilon \\
0 & 0 & -\lambda & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \lambda \in \mathbb{R}, \quad \varepsilon^2 = 1, \\
(3) & \quad \begin{pmatrix}
\lambda & 1 & 0 & 0 \\
0 & \lambda & 0 & 0 \\
0 & 0 & -\lambda & 0 \\
0 & 0 & 0 & -\lambda
\end{pmatrix}, \quad \lambda \in \mathbb{R}, \\
(4) & \quad \begin{pmatrix}
0 & 0 & 0 & \varepsilon \\
0 & 0 & 0 & \varepsilon \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \varepsilon^2 = 1,
\end{align*}
\]

Lemma 2.3.6. Let \( a \in \mathfrak{sp}(2n, \mathbb{R}) \) and let \( \lambda \) be an eigenvalue of \( a \). Then \( -\lambda \) is also an eigenvalue of \( a \).

Proof. As \( a \in \mathfrak{sp}(2n, \mathbb{R}) \), we have that \( a^tJ + Ja = 0 \) or equivalently \( JaJ^{-1} = -a^t \). If \( \lambda \) is an eigenvalue of \( a \), then \( \det(a - \lambda I_{2n}) = 0 \). This yields the following

\[
0 = \det(a - \lambda I_{2n}) = \det(J(a - \lambda I_{2n})J^{-1}) = \det(JaJ^{-1} - \lambda J I_{2n}J^{-1})
\]

\[
= \det(-a^t - \lambda I_{2n}) = \det(-(a + \lambda I_{2n})^t) = (-1)^{2n} \det((a + \lambda I_{2n})^t) = \det(a + \lambda I_{2n})
\]

which, of course, implies that \( -\lambda \) is also an eigenvalue of \( a \). 

We know that the characteristic polynomial of \( a \) always splits over \( \mathbb{C} \), thus we have that the complexification of \( V \), \( V_c \), is the direct sum of the generalized eigenspaces of \( a \). This allows us to write \( V_c \) in the following manner

\[
V_c = K_0 \oplus K_{\lambda_1} \oplus K_{-\lambda_1} \oplus K_{\lambda_2} \oplus K_{-\lambda_2} \oplus \cdots \oplus K_{\lambda_k} \oplus K_{-\lambda_k}
\]
where $1 \leq k \leq n$, $K_\mu$ is the generalized eigenspace corresponding to the eigenvalue $\mu$, $\lambda_i \neq 0$ for all $i$, and the $\lambda_i$ are distinct and such that $\lambda_i \neq -\lambda_j$ for any pair $i, j$. Note that if all the eigenvalues of $a$ are real, then its characteristic polynomial splits over $\mathbb{R}$ and $V$ decomposes in the manner above where all the summands are subspaces over $\mathbb{R}$.

**Proposition 2.3.7.** If $\mu \neq -\eta$, then $K_\eta$ and $K_\mu$ are symplectically orthogonal to each other. That is, $\omega(K_\eta, K_\mu) = 0$.

*Proof.* Let $v \in K_\eta$ be an eigenvector of $a$ (i.e. $av = \eta v$) and let $w \in K_\mu$ be an eigenvector of $a$.

Then we have that
\[
\eta \omega(v, w) = \omega(\eta v, w) = \omega(av, w) = -\omega(v, aw) = -\omega(v, \mu w) = -\mu \omega(v, w).
\]

This implies that $(\eta + \mu) \omega(v, w) = 0$. As $\mu \neq -\eta$, this implies that $\omega(v, w) = 0$.

Now let $\{w, w'\} \subseteq K_\mu$ be a cycle of generalized eigenvectors. Then we have that
\[
\eta \omega(v, w') = \omega(\eta v, w') = \omega(av, w') = -\omega(v, aw')
\]
\[
= -\omega(v, \mu w' + w) = -\mu \omega(v, w') - \omega(v, w) = -\mu \omega(v, w').
\]

This implies that $(\eta + \mu) \omega(v, w') = 0$. And as $\mu \neq -\eta$, this gives us that $\omega(v, w') = 0$.

Now we have that any eigenvector in $K_\eta$ is symplectically orthogonal to any eigenvector and “one-step” generalized eigenvector in $K_\mu$. Proceed inductively on the step of the generalized eigenvector in $K_\mu$ and this will imply that any eigenvector in $K_\eta$ is symplectically orthogonal to any vector in $K_\mu$.

Next let $\{v, v'\} \subseteq K_\eta$ be a cycle of generalized eigenvectors. Then for any eigenvector $w \in K_\mu$, we have that $\omega(v', w) = 0$ by the above argument and the skew-symmetry of $\omega$. Let $\{w, w'\}$ be a cycle of generalized eigenvectors and recall that $\omega(v, w') = 0$. This will give us
\[
\eta \omega(v', w') = \omega(v, w') + \eta \omega(v', w') = \omega(\eta v' + v, w') = \omega(av', w')
\]
\[
= -\omega(v', aw') = -\omega(v', \mu w' + w) = -\mu \omega(v', w') - \omega(v', w) = -\mu \omega(v', w')
\]
which implies that $(\eta + \mu) \omega(v', w') = 0$. And as $\mu \neq -\eta$, this yields that $\omega(v', w') = 0$. Therefore any one-step generalized eigenvector in $K_\eta$ is symplectically orthogonal to any eigenvector and any
one-step generalized eigenvector in $K_\mu$. Again proceed inductively on the step of the generalized eigenvector in $K_\mu$ and we’ll obtain that any one-step generalized eigenvector in $K_\eta$ is symplectically orthogonal to any vector in $K_\mu$.

Finally, continue this argument inductively on the step of the generalized eigenvector in $K_\eta$ and this will yield that every vector in $K_\eta$ is symplectically orthogonal to every vector in $K_\mu$. Therefore whenever $\mu \neq -\eta$, $K_\eta$ is symplectically orthogonal to $K_\mu$ as proposed.

Note that in the above proposition, we make no assumption as to the value of $\eta$. It is, in fact, true if $\eta = 0$. However, if $\eta \neq 0$, then this leads immediately to a useful corollary.

**Corollary 2.3.8.** If $\eta \neq 0$, then $K_\eta$ is symplectically orthogonal to itself.

**Proposition 2.3.9.** $\omega$ is non-degenerate on $K_\mu \oplus K_{-\mu}$ for all $\mu \neq 0$. In addition, $\omega$ is non-degenerate on $K_0$.

**Proof.** First, consider the case when $\mu \neq 0$. Let $z \in K_\mu \oplus K_{-\mu}$ be such that $\omega(z, y) = 0$ for all $y \in K_\mu \oplus K_{-\mu}$. Then as $K_\mu$ and $K_{-\mu}$ are both symplectically orthogonal to the rest of the space, we have that $\omega(z, y) = 0$ for all $y \in V$, which implies that $z = 0$ because $\omega$ is non-degenerate on $V$.

For an analogous reason, $\omega$ is non-degenerate on $K_0$. ■

**Corollary 2.3.10.** If $v \in K_\mu$ is nonzero, then there exists a nonzero $w \in K_{-\mu}$ such that $\omega(v, w) \neq 0$.

**Proof:** If $\mu = 0$, then this is a direct consequence of the proposition. If $\mu \neq 0$, then by Corollary 2.3.8, $K_\mu$ is symplectically orthogonal to itself. Thus $\omega$ is totally degenerate on $K_\mu$. However, as $\omega$ is non-degenerate on $K_\mu \oplus K_{-\mu}$, we have that for any nonzero $v \in K_\mu$, there exists a $z \in K_\mu \oplus K_{-\mu}$ such that $\omega(v, z) \neq 0$. But we can write $z = u + w$, where $u \in K_\mu$ and $w \in K_{-\mu}$. Then as $\omega(v, u) = 0$, it must be that $\omega(v, w) \neq 0$. ■

We can say a little more when $v \in K_\mu$ is an eigenvector.

**Proposition 2.3.11.** If $v \in K_\mu$ is an eigenvector and $w \in K_{-\mu}$ such that $\omega(v, w) \neq 0$, then $w$ is the end vector in any cycle of generalized eigenvectors to which it belongs.

**Proof.** Assume to the contrary that there is a cycle of generalized eigenvectors such that $w$ is not the end vector. This implies that there is a generalized eigenvector $w'$ in the cycle such that...
\[ aw' = -\mu w' + w. \] Then we have that

\[ \mu \omega(v, w') = \omega(\mu v, w') = \omega(\omega v, w') = -\omega(v, aw') = -\omega(v, -\mu w' + w) = \mu \omega(v, w') - \omega(v, w) \]

But this implies that \(-\omega(v, w) = 0\), which is a contradiction to the fact that \(\omega(v, w) \neq 0\). Thus \(w\) is the end vector in any cycle of generalized eigenvectors to which it belongs. ■

We now investigate what happens when two eigenvectors \(v \in K_\mu\) and \(w \in K_{-\mu}\) are such that \(\omega(v, w) \neq 0\). We already know by the previous proposition that any cycle of generalized eigenvectors to which they belong must necessarily have length 1. But there is considerably more that we can say as well. For convenience in the following proposition, let \((\cdot)\) denote the span of the vectors inside and \((\cdot)\perp\) denote the symplectically orthogonal complement.

**Proposition 2.3.12.** If \(v \in K_\mu\) and \(w \in K_{-\mu}\) are eigenvectors such that \(\omega(v, w) \neq 0\). Then

1. \((v, w)\perp \oplus (v, v) = V.\)
2. \((v, w)\perp\) is \(a\)-invariant.
3. \(\omega\) is non-degenerate on \((v, w)\perp\).

**Proof.** 1. First, we show that \((v, w)\perp \cap (v, v) = 0.\) Let \(z \in (v, w)\perp \cap (v, v)\), then \(z = c_1 v + c_2 w\) for some \(c_1, c_2 \in \mathbb{R}\.\) This yields that \(z - c_1 v - c_2 w = 0\), which implies that \(\omega(z - c_1 v - c_2 w, y) = 0\) for all \(y \in V\.\) In particular, as \(\omega(z, w) = \omega(z, v) = 0\), we have

\[ 0 = \omega(z - c_1 v - c_2 w, w) = \omega(z, w) - c_1 \omega(v, w) - c_2 \omega(w, w) = -c_1 \omega(v, w), \]

\[ 0 = \omega(z - c_1 v - c_2 w, v) = \omega(z, v) - c_1 \omega(v, v) - c_2 \omega(w, v) = -c_2 \omega(w, v). \]

As \(\omega(v, w) \neq 0\), this implies that \(c_1 = c_2 = 0\) and hence \(z = 0\).

Next, we show that \((v, w)\perp \oplus (v, v) = V.\) Let \(z \in V\) be arbitrary. As \(\omega(v, w) \neq 0\), we can let

\[ \tilde{z} = z - \frac{\omega(z, w)}{\omega(v, w)} v - \frac{\omega(z, v)}{\omega(w, v)} w. \]

Then

\[ \omega(\tilde{z}, v) = \omega \left( z - \frac{\omega(z, w)}{\omega(v, w)} v - \frac{\omega(z, v)}{\omega(w, v)} w, v \right) = \omega(z, v) - \frac{\omega(z, w)}{\omega(v, w)} \omega(v, v) + \frac{\omega(z, v)}{\omega(w, v)} \omega(w, v) = 0, \]

and

\[ \omega(\tilde{z}, w) = \omega \left( z - \frac{\omega(z, w)}{\omega(v, w)} v - \frac{\omega(z, v)}{\omega(w, v)} w, w \right) = \omega(z, w) - \frac{\omega(z, w)}{\omega(v, w)} \omega(v, v) + \frac{\omega(z, v)}{\omega(w, v)} \omega(w, v) = 0. \]
Thus \( \tilde{z} \in (v, w)^\perp \) and we can write \( z \) as
\[
z = \tilde{z} + \frac{\omega(z, w)}{\omega(v, w)} v + \frac{\omega(z, v)}{\omega(w, v)} w.
\]
Therefore \( (v, w)^\perp + (v, w) = V \).

2. Let \( z \in (v, w)^\perp \). Then
\[
\omega(az, v) = -\omega(z, av) = -\omega(z, \mu v) = -\mu \omega(z, v) = 0,
\]
\[
\omega(az, w) = -\omega(z, aw) = -\omega(z, -\mu w) = \mu \omega(z, w) = 0.
\]
Thus \( az \in (v, w)^\perp \).

3. Let \( z \in (v, w)^\perp \) be such that \( \omega(z, y) = 0 \) for all \( y \in (v, w)^\perp \). Then as \( \omega(z, v) = \omega(z, w) = 0 \), we have that \( \omega(z, y) = 0 \) for all \( y \in (v, w) \). But \( V = (v, w)^\perp \oplus (v, w) \), so \( \omega(z, y) = 0 \) for all \( y \in V \). Thus \( z = 0 \) by the non-degeneracy of \( \omega \) on \( V \).

This leads to a useful corollary.

**Corollary 2.3.13.** If \( v, w \) is a cycle of generalized eigenvectors in \( K_0 \) such that \( \omega(v, w) \neq 0 \), then the conclusion of Proposition 2.3.12 holds.

**Proof.** Parts 1. and 3. hold for exactly the same reasons. It suffices to prove that 2. holds in this case. Note that if \( v, w \) is a cycle of generalized eigenvectors such that \( \omega(v, w) \neq 0 \), then \( v \) and \( w \) belong to a generalized eigenspace corresponding to the eigenvalue 0.

Let \( z \in (v, w)^\perp \). Then
\[
\omega(az, v) = -\omega(z, av) = -\omega(z, 0) = 0,
\]
and
\[
\omega(az, w) = -\omega(z, aw) = -\omega(z, v) = 0.
\]
Thus \( az \in (v, w)^\perp \) and we see that 2. holds.

Note that Parts 1. and 3. of Proposition 2.3.12 actually hold for any pair of vectors, \( v, w \in V \) such that \( \omega(v, w) \neq 0 \). We can then prove quite easily the following proposition.

**Proposition 2.3.14.** The dimension of \( K_0 \) is even and \( \dim K_\mu = \dim K_{-\mu} \) for all \( \mu \neq 0 \).
Proof. Let $V = K_0$. If $\dim K_0 = 0$, then the result is trivial. Assume then that $\dim K_0 > 0$. We also have that $\dim K_0 \neq 1$, otherwise $\omega$ is degenerate on $K_0$. Thus $\dim K_0 \geq 2$.

Let $v_1 \in K_0$ be nonzero, then as $\omega$ is non-degenerate on $K_0$, there exists a nonzero $w_1 \in K_0$ such that $\omega(v_1, w_1) \neq 0$. Then as parts 1. and 3. of Proposition 2.3.12 hold for $v_1$ and $w_1$, we have that

$$K_0 = (v_1, w_1) \oplus (v_1, w_1)^\perp$$

and that $\omega$ is non-degenerate on $(v_1, w_1)^\perp$. Also note that $\dim(v_1, w_1)^\perp = \dim K_0 - 2$. If $\dim(v_1, w_1)^\perp = 0$ then we’re done. If $\dim(v_1, w_1)^\perp \neq 0$, then it must be that $\dim(v_1, w_1)^\perp \geq 2$, otherwise $\omega$ is degenerate on $K_0$.

Let $v_2 \in (v_1, w_1)^\perp$ be nonzero, then as $\omega$ is non-degenerate on $(v_1, w_1)^\perp$, there exists a nonzero $w_2 \in (v_1, w_1)^\perp$ such that $\omega(v_2, w_2) \neq 0$. Then, because of parts 1. and 3. of Proposition 2.3.12 we can write

$$K_0 = (v_1, w_1) \oplus (v_2, w_2) \oplus (v_1, w_1, v_2, w_2)^\perp.$$ 

We continue this process until it terminates. It is guaranteed to terminate because $\dim K_0 < \infty$. Thus $K_0$ is even dimensional.

The proof that $\dim K_\mu = \dim K_{-\mu}$ for all $\mu \neq 0$ is almost identical except that the $v_i$ vectors come from $K_\mu$ and the $w_i$ vectors come from $K_{-\mu}$. Then $K_\mu \oplus K_{-\mu}$ decomposes into a direct sum of subspaces each spanned by two vectors, one from $K_\mu$ and one from $K_{-\mu}$. This puts a basis from $K_\mu$ into one-to-one correspondence with a basis for $K_{-\mu}$ and yields the result. ■

A final lemma that will prove useful, particularly in finding the canonical forms of $a \in \mathfrak{sp}(4, \mathbb{R})$, is the following.

Lemma 2.3.15. Any matrix in the following sets of matrices in $\mathfrak{sp}(2, \mathbb{R})$ or $\mathfrak{sp}(4, \mathbb{R})$ is not symplectically similar to the other matrix in the set to which it belongs.

\[
(1) \left\{ a_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \right\}
\]

\[
(2) \left\{ a_3 = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix}, \quad a_4 = \begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix} \right\}
\]
The two matrices are arbitrary just as in the previous part. Again, the equation condition would yield the equation
\[
\text{det } A = 0.
\]

Solving this equation, we find that
\[
A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}
\]

Thus in order to satisfy the equation
\[
A a = 0,
\]

we require \( \text{det } A \neq 0 \) if we require \( A a \) to be an arbitrary element of \( \text{sl}(2, \mathbb{R}) \). Then forcing \( A a \) to satisfy the defining symplectic condition would yield the equation

\[
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & c_1 + c_4 \\ -c_1 - c_4 & 0 \end{pmatrix}
\]

Solving this equation, we find that \( c_4 = -c_1 \), that is, \( b \) must be a trace free matrix. Hence \( \text{sp}(2, \mathbb{R}) = \text{sl}(2, \mathbb{R}) \) and consequently, \( \text{Sp}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R}) \).

To prove that the two matrices in set (1) are not symplectically similar to each other first we’ll call the two matrices \( a_1 \) and \( a_2 \) respectively. Then let \( A \in \text{GL}(2, \mathbb{R}) \) be arbitrary. Solving the equation \( A^{-1} a_1 A = a_2 \) is equivalent to solving \( a_1 A = A a_2 \) as long as we make the restriction that \( \det A \neq 0 \). If \( A = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \), then this equation becomes

\[
\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}
\]

Thus in order to satisfy the equation \( A^{-1} a_1 A = a_2 \), \( A \) would have to have the form \( \begin{pmatrix} d_1 & d_2 \\ 0 & -d_1 \end{pmatrix} \) with \( d_1 \neq 0 \), which always has negative determinant. As every element in \( \text{Sp}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R}) \) has determinant 1, no symplectic group element can make the required move. Thus \( a_1 \) and \( a_2 \) are not symplectically similar.

Next we prove that the matrices in set (2) are symplectically dissimilar. Let \( A \in \text{GL}(2, \mathbb{R}) \) be arbitrary just as in the previous part. Again, the equation \( A^{-1} a_3 A = a_4 \) is equivalent to solving \( a_3 A = A a_4 \) if we require \( \det A \neq 0 \). Solving this equation, we obtain that \( A \) must have the form \( \begin{pmatrix} d_1 & d_2 \\ d_2 & -d_1 \end{pmatrix} \), which has the property that \( \det A = -(d_1^2 + d_2^2) \). This is a nonpositive value,
which implies that \( A \not\in Sp(2, \mathbb{R}) \) because \( \det A = 1 \) whenever \( A \in Sp(2, \mathbb{R}) \). Thus there is no symplectic change of basis that will, by conjugation, move \( a_3 \) to \( a_4 \).

In order to prove, that the matrices in set (3) are symplectically dissimilar, we let \( A \in GL(4, \mathbb{R}) \) be given by the matrix

\[
A = \begin{pmatrix}
    d_1 & d_2 & d_3 & d_4 \\
    d_5 & d_6 & d_7 & d_8 \\
    d_9 & d_{10} & d_{11} & d_{12} \\
    d_{13} & d_{14} & d_{15} & d_{16}
\end{pmatrix}
\]

with \( \det A \neq 0 \). We consider again the equation \( a_5A = Aa_6 \). Solving this equation, we obtain the following

\[
\begin{pmatrix}
    0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 1 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
    d_1 & d_2 & d_3 & d_4 \\
    d_5 & d_6 & d_7 & d_8 \\
    d_9 & d_{10} & d_{11} & d_{12} \\
    d_{13} & d_{14} & d_{15} & d_{16}
\end{pmatrix}
= 
\begin{pmatrix}
    0 & 1 & 0 & 0 \\
    0 & 0 & 0 & -1 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
    d_1 & d_2 & d_3 & d_4 \\
    d_5 & d_6 & d_7 & d_8 \\
    d_9 & d_{10} & d_{11} & d_{12} \\
    d_{13} & d_{14} & d_{15} & d_{16}
\end{pmatrix}
= 
\begin{pmatrix}
    0 & 1 & 0 & 0 \\
    0 & 0 & 0 & -1 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
    0 & 1 & -d_4 & -d_2 \\
    0 & d_5 & -d_8 & -d_6 \\
    0 & d_9 & -d_{12} & -d_{10} \\
    0 & d_{13} & -d_{16} & -d_{14}
\end{pmatrix}
\]

This implies that any invertible linear transformation \( A \) that by conjugation will move \( a_5 \) to \( a_6 \) is of the form

\[
\begin{pmatrix}
    d_1 & d_2 & d_3 & d_4 \\
    0 & d_1 & -d_4 & -d_2 \\
    0 & 0 & -d_1 & 0 \\
    0 & 0 & d_2 & -d_1
\end{pmatrix}
\]

If we require this matrix to satisfy \( A^tJA = J \) so that it is also in the symplectic group, then we get, aside from other things, that \((d_1)^2 = -1\) and hence there is no real symplectic matrix that will, by conjugation, move \( a_5 \) to \( a_6 \).

Finally, to prove that the matrices in set (4) are symplectically dissimilar, we let \( A \in GL(4, \mathbb{R}) \) be arbitrary as above. We consider again the equation \( a_7A = Aa_8 \). Solving this equation, we have that any invertible matrix \( A \) that by conjugation will move \( a_7 \) to \( a_8 \) is of the form

\[
\begin{pmatrix}
    d_1 & d_2 & d_3 & d_4 \\
    -d_2 & d_1 & -d_4 & d_3 \\
    0 & 0 & -d_1 & -d_2 \\
    0 & 0 & d_2 & -d_1
\end{pmatrix}
\]

Requiring this to satisfy \( A^tJA = J \), will require, amongst other things, that \((d_1)^2 + (d_2)^2 = -1\), which is impossible as \( d_1, d_2 \in \mathbb{R} \). Therefore the two matrices are symplectically dissimilar. ■
We now have the tools necessary to compute the canonical forms of $a \in \mathfrak{sp}(4, \mathbb{R})$. We will consider first those $a$ that have real eigenvalues and then those where one or more of the eigenvalues are complex. We’ll classify according to the decomposition of $V$ into generalized eigenspaces of $a$ when the eigenvalues are real and $V_c$ when some or all of them are complex. The cases will follow these nine decompositions of $V$ or $V_c$.

1. $V = K_\lambda \oplus K_{-\lambda} \oplus K_\mu \oplus K_{-\mu}$ where $\lambda, \mu \in \mathbb{R}$ are both nonzero and $\mu \neq \pm \lambda$.

2. $V = K_\lambda \oplus K_{-\lambda}$ where $\lambda \in \mathbb{R}$ is nonzero.

3. $V = K_\lambda \oplus K_{-\lambda} \oplus K_0$ where $\lambda \in \mathbb{R}$ is nonzero.

4. $V = K_0$.

5. $V_c = K_\lambda \oplus K_{-\lambda} \oplus K_{\mu i} \oplus K_{-\mu i}$ where $\lambda, \mu \in \mathbb{R}$ are both nonzero.

6. $V_c = K_0 \oplus K_{\mu i} \oplus K_{-\mu i}$ where $\mu \in \mathbb{R}$ is nonzero.

7. $V_c = K_z \oplus K_{-z} \oplus K_{\mu i} \oplus K_{-\mu i}$ where $z = \lambda + \mu i$ for some nonzero $\lambda, \mu \in \mathbb{R}$.

8. $V_c = K_{\mu i} \oplus K_{-\mu i} \oplus K_{\eta i} \oplus K_{-\eta i}$ where $\mu, \eta \in \mathbb{R}$ are both nonzero and $\eta \neq \pm \mu$.

9. $V_c = K_{\mu i} \oplus K_{-\mu i}$ where $\mu \in \mathbb{R}$ is nonzero.

This enumerates every possible decomposition of $V$ or $V_c$ into generalized eigenspaces of $a \in \mathfrak{sp}(4, \mathbb{R})$.

**Case 1:** $V = K_\lambda \oplus K_{-\lambda} \oplus K_\mu \oplus K_{-\mu}$

We also assume in this case that $\lambda, \mu \in \mathbb{R}$ are both nonzero and $\mu \neq \pm \lambda$. As $\dim V = 4$ it must be that each of these generalized eigenspaces are one dimensional. This implies, of course, that $a$ is diagonalizable.

Then pick eigenvectors $v_1 \in K_\lambda$ and $w_1 \in K_{-\lambda}$ such that $\omega(v_1, w_1) \neq 0$ and decompose them out as in Proposition 2.3.12. Let $v_2 \in K_\mu$ and $w_2 \in K_{-\mu}$. Then $\{v_2, w_2\}$ is a basis for $(v_1, w_1)^\perp$ and $\omega(v_2, w_2) \neq 0$ by the non-degeneracy of $\omega$ on $(v_1, w_1)^\perp$.

Make the additional change that if $\omega(v_1, w_1) < 0$, replace $w_1$ with $-w_1$ and relabel it as $w_1$. In this way, we can ensure that $\omega(v_1, w_1) > 0$. Do the same for $\omega(v_2, w_2)$. Then we pick a basis, $\beta$, for
$V$ consisting of eigenvectors in the following way,

$$\beta = \left\{ \frac{1}{\sqrt{\omega(v_1, w_1)}} v_1, \frac{1}{\sqrt{\omega(v_2, w_2)}} v_2, \frac{1}{\sqrt{\omega(v_1, w_1)}} w_1, \frac{1}{\sqrt{\omega(v_2, w_2)}} w_2 \right\}$$

and relabel the vectors $v_1', v_2', w_1'$, and $w_2'$. Then $\beta$ has the properties that $\omega(v_i', v_j') = \omega(w_i', w_j') = 0$ and $\omega(v_i', w_j') = \delta_{ij}$. Thus the matrix

$$W = (v_1' \quad v_2' \quad u_1' \quad u_2')$$

is in $Sp(4, \mathbb{R})$ as $W^t J W = J$. In this basis, $a$ is of form (1) presented in Theorem 2.3.5 with $\lambda \neq \pm \mu$

**Case 2:** $V = K_\lambda \oplus K_{-\lambda}$

In this case, we also assume that $\lambda \in \mathbb{R}$ is nonzero. Note that as $\dim K_\lambda = \dim K_{-\lambda}$, both of these generalized eigenspaces are of dimension two.

If $a$ is diagonalizable, then we proceed exactly as in Case 1 with $\mu = \pm \lambda$ depending on how the eigenvectors are to be ordered. Then the real symplectic canonical form of $a$ is of form (1) presented in Theorem 2.3.5 with $\mu = \pm \lambda$.

On the other hand, if $a$ is not diagonalizable, then either $K_\lambda$ contains a cycle of generalized eigenvectors of length two or $K_{-\lambda}$ does.

**Lemma 2.3.16.** $K_\lambda$ contains a cycle of generalized eigenvectors of length two if and only if $K_{-\lambda}$ contains one.

**Proof:** First, let $v, v' \in K_\lambda$ be a cycle of length two. Let $w \in K_{-\lambda}$ be an eigenvector, then by Proposition 2.3.11 $\omega(w, v) = 0$. Thus if $K_{-\lambda}$ contains only eigenvectors, then $\omega$ is degenerate as $v$ is symplectically orthogonal to the entire space. As $\dim K_{-\lambda} = 2$, it must then contain a cycle of generalized eigenvectors of length two. The converse is analogous. ■

As $K_\lambda$ and $K_{-\lambda}$ both contain a cycle of length two, let $\{v_1, v_2\}$ be a cycle of generalized eigenvectors in $K_\lambda$ and $\{w_1, w_2\}$ be one in $K_{-\lambda}$. Then each these sets forms a basis for its respective space. In addition, as $v_1$ is an eigenvector in $K_\lambda$ and $w_1$ is not the end vector in its cycle, we have by Proposition 2.3.11 that $\omega(v_1, w_1) = 0$, and by Corollary 2.3.8 $\omega(v_1, v_2) = \omega(w_1, v_2) = 0$. This implies, as $\omega$ is non-degenerate on $K_\lambda \oplus K_{-\lambda}$, that $\omega(v_1, w_2) \neq 0$ and $\omega(v_2, w_1) \neq 0$. It is, however, possible that
\(\omega(v_2, w_2) \neq 0\). In that case, consider the cycle of generalized eigenvectors \(\{w_1, w_2 - \frac{\omega(v_2, w_2)}{\omega(v_2, w_1)}w_1\}\). If we relabel the elements of this cycle as \(v_1'\) and \(w_2'\), then it has the properties that \(\omega(v_2, w_2') = 0\) and \(\omega(v_1, w_2') = \omega(v_1, w_2)\). In addition, we have the following

\[
\lambda \omega(v_2, w_2') + \omega(v_1, w_2') = \omega(\lambda v_2 + v_1, w_2') = \omega(\lambda v_2 + v_1, w_2') = -\omega(v_2, aw_2')
\]

\[
= -\omega(v_2, -\lambda w_2' + w_1') = \lambda \omega(v_2, w_2') - \omega(v_2, w_1')
\]

which implies that \(\omega(v_1, w_2') = -\omega(v_2, w_1')\). Thus if \(\omega(v_1, w_2') < 0\), then \(\omega(v_2, w_1') > 0\). If this is the case, then replace \(w_2'\) with \(-w_2'\) and relabel to guarantee that \(\omega(v_1, w_2') > 0\) as well. Note now that \(aw_2' = -\lambda w_2' - w_1'\). If \(\omega(v_1, w_2') > 0\) then \(\omega(v_2, w_1') < 0\). In this case, replace \(w_1'\) with \(-w_1'\) and relabel to guarantee that \(\omega(v_2, w_1') > 0\). Again, this will make \(aw_2' = -\lambda w_2' - w_1'\). After making either of these changes, we pick a new basis, \(\beta\), for \(V\) in the following way

\[
\beta = \left\{ \frac{1}{\sqrt{\omega(v_1, w_2')}} v_1, \frac{1}{\sqrt{\omega(v_2, w_1')}} v_2, \frac{1}{\sqrt{\omega(v_1, w_2')}} w_2', \frac{1}{\sqrt{\omega(v_2, w_1')}} w_1' \right\}
\]

and relabel the vectors \(v_1'', v_2'', w_1'',\) and \(w_2''\) respectively. Then the matrix

\[
W = \begin{pmatrix} v_1'' & v_2'' & w_1'' & w_2'' \end{pmatrix}
\]

is such that \(W^tJW = J\) and is in \(Sp(4, \mathbb{R})\). In addition, in this basis, \(a\) is of form (3) presented in Theorem 2.3.3 with \(\lambda \neq 0\).

**Case 3:** \(V = K_\lambda \oplus K_{-\lambda} \oplus K_0\)

We also assume that \(\lambda \in \mathbb{R}\) is nonzero. By Proposition 2.3.14 \(\dim K_0\) is even. Then we have here that \(\dim K_0 = 2\) and \(\dim K_{\lambda} = \dim K_{-\lambda} = 1\).

If \(a\) is diagonalizable, then these generalized eigenspaces consist only of eigenvectors and we proceed as in Case 1. In this case, the real symplectic canonical form of \(a\) is form (1) from Theorem 2.3.3 with \(\mu = 0\) and \(\lambda \neq 0\).

If \(a\) is not diagonalizable, then \(K_0\) contains a cycle of generalized eigenvectors of length two. Let \(v_1\) and \(w_1\) be eigenvectors in \(K_\lambda\) and \(K_{-\lambda}\) respectively. Then \(\omega(v_1, w_1) \neq 0\) and we can split them off as in Proposition 2.3.12. Pick a basis for \((v_1, w_1)^\perp = K_0, \{v_2', w_2'\}\), such that the two vectors form a cycle of generalized eigenvectors of length two. We know that \(\omega(v_2', w_2') \neq 0\) because \(\omega\) is
non-degenerate. As with the previous case if \( \omega(v_1, w_1) < 0 \), then replace \( w_1 \) with \(-w_1\) and relabel. However, if \( \omega(v'_2, w'_2) < 0 \), we can’t replace \( w'_2 \) so trivially. This is because if we change the sign of \( w'_2 \), then \( \omega(v'_2, w'_2) > 0 \) but \( aw'_2 = -v'_2 \) which by Lemma 2.3.15 is symplectically dissimilar to the case when \( \omega(v'_2, w'_2) > 0 \) in the first place. However, we still make the change so that \( \omega(v'_2, w'_2) > 0 \).

Then we can pick a basis \( \beta \) given by

\[
\beta = \left\{ \frac{1}{\sqrt{\omega(v_1, w_1)}} v_1, \frac{1}{\sqrt{\omega(v'_2, w'_2)}} v'_2, \frac{1}{\sqrt{\omega(v_1, w_1)}} w_1, \frac{1}{\sqrt{\omega(v'_2, w'_2)}} w'_2 \right\}
\]

and relabel the vectors \( v'_1, v''_1, w''_1, \) and \( w''_2 \) respectively. Then \( \beta \) has the properties that \( \omega(v''_1, v''_j) = \omega(w''_1, w''_j) = 0 \) and \( \omega(v''_i, w''_j) = \delta_{ij} \). Then the matrix

\[
W = \begin{pmatrix}
v''_1 & v'_2 & w''_1 & w''_2
\end{pmatrix}
\]

is in \( Sp(4, \mathbb{R}) \) as \( W^t J W = J \). In this basis, \( a \) is of form (2) presented in Theorem 2.3.5 with \( \lambda \neq 0 \).

**Case 4:** \( V = K_0 \)

In this case, \( K_0 \) is obviously of dimension four.

If \( a \) is diagonalizable, then, as its only eigenvalue is 0, \( a \) is the zero transformation and is already in real symplectic canonical form, which is form (1) in Theorem 2.3.5 with \( \lambda = \mu = 0 \).

If \( a \) is not diagonalizable, then one of the following statements is true.

1. There is a basis for \( K_0 \) consisting of three cycles of generalized eigenvectors, two of length one and one of length two.
2. There is a basis for \( K_0 \) consisting of two cycles of generalized eigenvectors of length two.
3. There is a basis for \( K_0 \) consisting of one cycle of generalized eigenvectors of length four.

A basis for \( K_0 \) consisting of a cycle of length three and a cycle of length one is impossible. To show this, let \( \{v, v', v''\} \) be a cycle of length three and let \( \{w\} \) be a cycle of length one such that \( \{v, v', v'', w\} \) is a basis for \( K_0 \). Then, by Proposition 2.3.11 and the non-degeneracy of \( \omega \), \( \omega(v'', w) \neq 0 \), and \( \omega(v, v'') \neq 0 \). Moreover, \( w \) and \( v \) are symplectically orthogonal to everything else in the space and the nonzero vector \( v - \frac{\omega(v, v'')}{\omega(w, v'')} w \) is symplectically orthogonal to the entire space, which is impossible.
If the first possibility is true, let a basis for $V$ be given by \{v_1, w_1, v_2, w_2\} where $v_1$ and $w_1$ are cycles of generalized eigenvectors of length one and $v_2, w_2$ is a cycle of length two. Then $\omega(v_1, v_2) = 0$ and $\omega(w_1, v_2) = 0$ because $v_2$ is not the end vector in its cycle. This implies that if $\omega(v_1, w_1) = 0$, then $\omega(v_1, w_2) \neq 0$ because $\omega$ is non-degenerate. But this tells us that the nonzero vector $u = v_2 - \frac{\omega(v_2, w_2)}{\omega(w_2, w_2)} v_1$ is symplectically orthogonal to the entire space. This, of course, can’t happen and hence $\omega(v_1, w_1) \neq 0$. Then we follow the second argument in Case 3 to obtain that the real symplectic canonical form of $a$ is form (2) in Theorem 2.3.5 with $\lambda = 0$.

For the second possibility above, let \{v_1, v_2, w_1, w_2\} be a basis for $K_0$ such that $v_1, v_2$ and $w_1, w_2$ are cycles of generalized eigenvectors. We know that $\omega(v_1, w_1) = 0$ by Proposition 2.3.11. If $\omega(v_1, v_2) = \omega(w_1, w_2) = 0$, then the second argument presented in Case 2 holds here as well. This allows $\lambda = 0$ in form (3) in Theorem 2.3.5.

If one of these values is nonzero, then order the basis so that $\omega(v_1, v_2) \neq 0$. Then $v_1$ and $v_2$ decompose off as in Corollary 2.3.13. Let \{w'_1, w'_2\} be a cycle of generalized eigenvectors in $(v_1, v_2)^\perp$. This implies that $\omega(w'_1, w'_2) \neq 0$ because of the non-degeneracy of $\omega$ on $(v_1, v_2)^\perp$ and the fact that \{w'_1, w'_2\} form a basis for $(v_1, v_2)^\perp$. This gives us three possibilities,

- $\omega(v_1, v_2) > 0$ and $\omega(w'_1, w'_2) > 0$.

- $\omega(v_1, v_2) > 0$ and $\omega(w'_1, w'_2) < 0$. In which case, we replace $w'_2$ with $-w'_2$ and relabel it so that $\omega(w'_1, w'_2) > 0$. Making this change will, however, result in $aw'_2 = -w'_1$. This possibility includes the instance where $\omega(v_1, v_2) < 0$ and $\omega(w'_1, w'_2) > 0$ because we can just relabel the vectors appropriately to get the desired condition.

- $\omega(v_1, v_2) < 0$ and $\omega(w'_1, w'_2) < 0$. In which case, we replace $v_2$ and $w'_2$ with $-v_2$ and $-w'_2$ respectively and relabel them so that $\omega(v_1, v_2) > 0$ and $\omega(w'_1, w'_2) > 0$. However, this will result in $av_2 = -v_1$ and $aw'_2 = -w'_1$.

By Lemma 2.3.15 none of these three possibilities are symplectically similar to any other one. However, the second situation is symplectically similar to the canonical form presented in the last argument of Case 3 with $\lambda = 0$. This is because $a$, in the basis the second case would produce, is of
the form
\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
And if we can conjugate by the matrix
\[
\begin{pmatrix}
\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 1 & -1 \\
-1 & -1 & 0 & 1
\end{pmatrix},
\]
which is in $Sp(2n, \mathbb{R})$ because $A^tJA = J$, we obtain
\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix}.
\]
However, as the first and third possibilities are not symplectically similar to the second one, they are not similar to the canonical form presented in Case 3 either.

Thus in each remaining case, after making its outlined correction, we pick a basis, $\beta$ for $K_0$ in the following way
\[
\beta = \left\{ \frac{1}{\sqrt{\omega(v_1, v_2)}} v_1, \frac{1}{\sqrt{\omega(w_1', w_2')}} w_1', \frac{1}{\sqrt{\omega(v_1, v_2)}} v_2, \frac{1}{\sqrt{\omega(w_1', w_2')}} w_2' \right\}
\]
and relabel the vectors $v_1'', w_1'', v_2'', w_2''$. Then the matrix
\[
W = \begin{pmatrix} v_1'' & w_1'' & v_2'' & w_2'' \end{pmatrix}
\]
is in $Sp(4, \mathbb{R})$ because $W^tJW = J$. In this basis, $a$ is of form (4) in Theorem 2.3.3.

Finally, in the third instance, where $K_0$ has a basis consisting of a cycle of generalized eigenvector of length four, we let $\{v_1, v_2, v_3, v_4\}$ be a cycle of generalized eigenvectors. Then this set also defines a basis for $K_0$. We have, by Proposition 2.3.11 that $\omega(v_1, v_2) = \omega(v_1, v_3) = 0$ and that $\omega(v_1, v_4) \neq 0$. We also note that
\[
\omega(v_1, v_4) = \omega(av_2, v_4) = -\omega(v_2, av_4) = -\omega(v_2, v_3)
\]
implying as well that $\omega(v_2, v_3) \neq 0$ and
\[
\omega(v_2, v_4) = \omega(av_3, v_4) = -\omega(v_3, av_4) = -\omega(v_3, v_3) = 0.
\]
Next, we pick a new basis in the following manner

\[
\{ v_1, v_2, v_3 - \frac{\omega(v_3, v_4)}{2\omega(v_1, v_4)}v_1, v_4 - \frac{\omega(v_3, v_4)}{2\omega(v_1, v_4)}v_2 \}
\]

and relabel the vectors \( v'_1, v'_2, v'_3, \) and \( v'_4. \) This new basis is still a cycle of generalized eigenvectors with all the properties above, except that

\[
\omega(v'_3, v'_4) = \omega \left( v_3 - \frac{\omega(v_3, v_4)}{2\omega(v_1, v_4)}v_1, v_4 - \frac{\omega(v_3, v_4)}{2\omega(v_1, v_4)}v_2 \right)
= \omega(v_3, v_4) - \frac{\omega(v_3, v_4)}{2\omega(v_1, v_4)}\omega(v_3, v_2) - \frac{\omega(v_3, v_4)}{2\omega(v_1, v_4)}\omega(v_1, v_4) + \left( \frac{\omega(v_3, v_4)}{2\omega(v_1, v_4)} \right)^2 \omega(v_1, v_2)
= \omega(v_3, v_4) - \frac{\omega(v_3, v_4)}{2\omega(v_3, v_2)}\omega(v_3, v_2) - \frac{1}{2}\omega(v_3, v_4)
= 0
\]

If \( \omega(v'_1, v'_4) > 0, \) then \( \omega(v'_2, v'_4) < 0 \) and we replace \( v'_4 \) with \(-v'_3\) and relabel. This will yield that \( \omega(v'_1, v'_4) = \omega(v'_2, v'_4) > 0, \) but that \( \alpha v'_4 = -v'_3 \) and \( \alpha v'_3 = -v'_2. \) On the other hand, if \( \omega(v'_1, v'_4) < 0, \) then \( \omega(v'_2, v'_4) > 0 \) and we replace \( v'_4 \) with \(-v'_3\) and relabel. This gives us that \( \omega(v'_1, v'_4) = \omega(v'_2, v'_4) > 0, \) but that \( \alpha v'_4 = -v'_3. \) We know by Lemma 2.3.15 that these two cases are not symplectically similar. However, we make whatever change is necessary to guarantee that \( \omega(v'_1, v'_4) = \omega(v'_2, v'_3) > 0. \)

In either case, we pick a basis, \( \beta, \) for \( V \) in the following way

\[
\beta = \left\{ \frac{1}{\sqrt{\omega(v'_1, v'_4)}} v'_1, \frac{1}{\sqrt{\omega(v'_2, v'_4)}} v'_2, \frac{1}{\sqrt{\omega(v'_1, v'_4)}} v'_3, \frac{1}{\sqrt{\omega(v'_1, v'_4)}} v'_4 \right\}
\]

and relabel the vectors \( v''_1, v''_2, v''_3, \) and \( v''_4. \) Then the matrix

\[
W = (v''_1 v''_2 v''_3 v''_4)
\]

is such that \( W^t JW = J \) and hence is in \( Sp(4, \mathbb{R}). \) In this basis, \( a \) is of form (5) in Theorem 2.3.5

**Case 5:** \( V_c = K_\lambda \oplus K_{-\lambda} \oplus K_\mu \oplus K_{-\mu} \)

We also assume here that \( \lambda, \mu \in \mathbb{R} \) are both nonzero. In this case, we note that \( a \) is diagonalizable over \( \mathbb{C} \) and that each of the generalized eigenspaces in the decomposition of \( V_c \) is one dimensional.

Let \( v_1 \in K_\lambda \) and \( w_1 \in K_{-\lambda} \). Then \( \omega(v_1, w_1) \neq 0, \) because \( \omega \) is non-degenerate on \( K_\lambda \oplus K_{-\lambda} \), and \( v_1 \) and \( w_1 \) split off as in Proposition 2.3.12. Pick \( v_2 \in K_\mu \) and \( w_2 \in K_{-\mu} \) such that \( w_2 = \overline{v_2}. \) Then \( \omega(v_2, w_2) \neq 0 \) again because of the non-degeneracy of \( \omega \) on \( (v_1, w_1)^{\perp} = K_\mu \oplus K_{-\mu}. \) We pick
two new vectors $v'_2 = v_2 + w_2$ and $w'_2 = -i(v_2 - w_2)$. The set \{$v'_2, w'_2$\} is still a basis for $K_{\mu i} \oplus K_{-\mu i}$ but the vectors are real. Moreover the set \{$v_1, w_1, v'_2, w'_2$\} forms a basis for $V$ when considered over $\mathbb{R}$. In addition,

$$\omega(v'_2, w'_2) = \omega(v_2 + w_2, -i(v_2 - w_2))$$

$$= -i\omega(v_2, v_2) + i\omega(v_2, w_2) - i\omega(w_2, v_2) + i\omega(w_2, w_2) = 2i\omega(v_2, w_2) \neq 0$$

and $\omega(v'_2, w'_2) \in \mathbb{R}$ because $v'_2$ and $w'_2$ are real vectors. We also have that

$$av'_2 = a(v_2 + w_2) = av_2 + aw_2 = \mu iv_2 - \mu iw_2 = \mu(v_2 - w_2) = -\mu w'_2$$

and

$$aw'_2 = -i(aw_2 - aw_2) = -i(\mu iv_2 + \mu iw_2) = -\mu^2(v_2 + w_2) = \mu(v_2 + w_2) = \mu v'_2.$$ 

If $\omega(v_1, w_1) < 0$, then replace $w_1$ with $-w_1$ and relabel just as we’ve done before. Also if $\omega(v'_2, w'_2) < 0$, then we’ll replace $w'_2$ with $-w'_2$ and relabel. While this will make $\omega(v'_2, w'_2) > 0$, it will also have the effect that $av'_2 = \mu v'_2$ and $aw'_2 = -\mu v'_2$. We know by Lemma 2.3.15 that these two cases are symplectically dissimilar, but we nonetheless make the change so that $\omega(v'_2, w'_2) > 0$.

In either instance, after ensuring that $\omega(v_1, w_1)$ and $\omega(v'_2, w'_2)$ are both positive, we pick a new basis, $\beta$, for $V$ in the following way

$$\beta = \left\{ \frac{1}{\sqrt{\omega(v_1, w_1)}} v_1, \frac{1}{\sqrt{\omega(v'_2, w'_2)}} v'_2, \frac{1}{\sqrt{\omega(v_1, w_1)}} w_1, \frac{1}{\sqrt{\omega(v'_2, w'_2)}} w'_2 \right\}$$

and relabel the vectors $v''_1, v''_2, w''_1$, and $w''_2$. Then the matrix

$$W = \begin{pmatrix} v''_1 & v''_2 & v''_3 & v''_4 \end{pmatrix}$$

is such that $W^t J W = J$ and hence $W \in Sp(4, \mathbb{R})$. In this basis, $a$ is of form (6) in Theorem 2.3.5 with $\lambda \neq 0$. We also assume $\mu > 0$ to make a distinction between the dissimilar cases.

**Case 6:** $V_c = K_0 \oplus K_{\mu i} \oplus K_{-\mu i}$

Also assume in this section that $\mu \in \mathbb{R}$ is nonzero. Note that dim $K_0 = 2$ and dim $K_{\mu i} = \text{dim } K_{-\mu i} = 1$ as well.

If $a$ is diagonalizable over $\mathbb{C}$, then we proceed as in Case 5, except that $v_1$ and $w_1$ are eigenvectors in $K_0$ such that $\omega(v_1, w_1) \neq 0$. This will lead to form (6) in Theorem 2.3.5 with $\lambda = 0$. 
On the other hand, if $a$ is not diagonalizable over $\mathbb{C}$, then there exists a basis for $K_0$ consisting of a cycle of generalized eigenvectors of length two. Let $\{v_1, w_1\}$ be a cycle of generalized eigenvectors in $K_0$. Then this set also forms a basis for $K_0$ and it must be that $\omega(v_1, w_1) \neq 0$ or $\omega$ is degenerate on $K_0$. Thus the span of these two vectors, $K_0$, decomposes out as in Corollary 2.3.13 and note that $(v_1, w_1) = K_{\mu i} \oplus K_{-\mu i}$. Next let $v_2 \in K_{\mu i}$ and let $w_2 \in K_{-\mu i}$ be such that $w_2 = \overline{v_2}$. Then $\omega(v_2, w_2) \neq 0$. Let $v'_2 = v_2 + w_2$ and $w'_2 = -i(v_2 + w_2)$. Clearly $v'_2$ and $w'_2$ span $K_{\mu i} \oplus K_{-\mu i}$ and have the property that $\omega(v'_2, w'_2) \neq 0$. Note that $av'_2 = -\mu w'_2$, $aw'_2 = \mu v'_2$.

The vectors $\{v_1, w_1, v'_2, w'_2\}$ now form a basis for $V$ when considered as a vector space over $\mathbb{R}$.

If $\omega(v_1, w_1) < 0$, then we replace $w_1$ with $-w_1$ and relabel so that this value is guaranteed to be positive. However, this gives $w_1$ the property that $aw_1 = -v_1$. By Lemma 2.3.15 we know that this is not symplectically similar to the case where $\omega(v_1, w_1) > 0$ in the first place. Similarly, if $\omega(v'_2, w'_2) < 0$, then we replace $w'_2$ with $-w'_2$ and relabel so that this values is positive. Making this change will yield the equations $av'_2 = \mu w'_2$ and $aw'_2 = -\mu v'_2$. And again by Lemma 2.3.15 this is not symplectically similar to the case where $\omega(v'_2, w'_2) > 0$ in the first place. After making the necessary changes, we pick a new basis, $\beta$, for $V$, as follows

$$\beta = \left\{ \frac{1}{\sqrt{\omega(v_1, w_1)}}v_1, \frac{1}{\sqrt{\omega(v'_2, w'_2)}}v'_2, \frac{1}{\sqrt{\omega(v_1, w_1)}}w_1, \frac{1}{\sqrt{\omega(v'_2, w'_2)}}w'_2 \right\}$$

and relabel the vectors $v''_1$, $v''_2$, $w''_1$, and $w''_2$. Then the matrix

$$W = \begin{pmatrix} v''_1 & v''_2 & v''_3 & v''_4 \end{pmatrix}$$

is such that $W^t JW = J$ and hence $W \in Sp(4, \mathbb{R})$. In this basis, $a$ is of form (7) in Theorem 2.3.5. We also assume that $\mu > 0$ to make a distinction between the dissimilar cases.

**Case 7: $V_c = K_z \oplus K_{-z} \oplus K_{\pi} \oplus K_{-\pi}$**

In this section, we also assume that $z = \lambda + \mu i$ for some $\lambda, \mu \in \mathbb{R}$ such that $\lambda, \mu \neq 0$. Also it is clear that $a$ is diagonalizable and that every generalized eigenspace of $a$ in the decomposition of $V_c$ is one-dimensional. Before we proceed further, we present a lemma that will be helpful.
**Lemma 2.3.17.** \( \omega(v, w) = \omega(\overline{v}, \overline{w}) \) for all \( v, w \in V \).

**Proof.** Let \( v, w \in V \). Then \( \omega(v, w) = \overline{v^tJw} = \overline{v^tJw} = \omega(\overline{v}, \overline{w}) \). \( \blacksquare \)

Let \( v_1 \in K_z, v_2 \in K_\overline{z}, w_1 \in K_{-z}, \) and \( w_2 \in K_{-\overline{z}} \) be eigenvectors such that \( v_2 = \overline{v_1} \) and \( w_2 = \overline{w_1} \). Then \( \{v_1, v_2, w_1, w_2\} \) is a basis for \( V_c \). In addition, \( \omega(v_1, w_1) \neq 0 \) and \( \omega(v_2, w_2) \neq 0 \) while any other pair in the basis is symplectically orthogonal. From this basis for \( V_c \), we construct a basis for \( V \) in the following way:

\[
\begin{align*}
    v'_1 &= v_1 + v_2, & v'_2 &= -i(v_1 - v_2), & w'_1 &= w_1 + w_2, & w'_2 &= -i(w_1 - w_2).
\end{align*}
\]

Note that

\[
\omega(v_1, w_1) = \omega(\overline{v_1}, \overline{w_1}) = \omega(v_2, w_2).
\]

Let \( \Re(x) \) and \( \Im(x) \) denote the real and imaginary parts of \( x \) respectively. It follows then that

\[
\omega(v'_1, w'_1) = \omega(v_1 + v_2, w_1 + w_2) = \omega(v_1, w_1) + \omega(v_1, w_2) + \omega(v_2, w_1) + \omega(v_2, w_2) = \omega(v_1, w_1) + \omega(v_2, w_2) = 2\Re(\omega(v_1, w_1))
\]

and

\[
\begin{align*}
    \omega(v'_2, w'_2) &= \omega(-i(v_1 - v_2), -i(w_1 - w_2)) = -\omega(v_1 - v_2, w_1 - w_2) \\
    &= -\omega(v_1, w_1) + \omega(v_1, w_2) + \omega(v_2, w_1) - \omega(v_2, w_2) \\
    &= -(\omega(v_1, w_1) + \omega(v_2, w_2)) = -\omega(v'_1, w'_1)
\end{align*}
\]

Also, we have that

\[
\begin{align*}
    \omega(v'_1, w'_2) &= \omega(v_1 + v_2, -i(w_1 - w_2)) = -i\omega(v_1, w_1) + i\omega(v_1, w_2) - i\omega(v_2, w_1) + i\omega(v_2, w_2) \\
    &= -i(\omega(v_1, w_1) - \omega(v_2, w_2)) = 2\Im(\omega(v_1, w_1))
\end{align*}
\]

\[
\begin{align*}
    \omega(v'_2, w'_1) &= \omega(-i(v_1 - v_2), w_1 + w_2) = -i\omega(v_1, w_1) - i\omega(v_1, w_2) + i\omega(v_2, w_1) + i\omega(v_2, w_2) \\
    &= -i(\omega(v_1, w_1) - \omega(v_2, w_2)) = \omega(v'_1, w'_2)
\end{align*}
\]
and

\[ \omega(v'_1, v'_2) = \omega(v_1 + v_2, -i(v_1 - v_2)) = 0 \]
\[ \omega(w'_1, w'_2) = \omega(w_1 + w_2, -i(w_1 - w_2)) = 0 \]

Furthermore, we have that

\[ av'_1 = a(v_1 + v_2) = (\lambda + \mu i)v_1 + (\lambda - \mu i)v_2 = \lambda (v_1 + v_2) + \mu i (v_1 - v_2) = \lambda v'_1 - \mu v'_2, \]
\[ av'_2 = a(-i(v_1 - v_2)) = -i(\lambda + \mu i)v_1 + i(\lambda - \mu i)v_2 = \mu (v_1 + v_2) - \lambda i (v_1 - v_2) = \mu v'_1 + \lambda v'_2, \]

and similarly

\[ aw'_1 = -\lambda w'_1 + \mu w'_2, \quad aw'_2 = -\mu w'_1 - \lambda w'_2. \]

As \( \omega(v_1, w_1) \neq 0 \), we know that either \( \mathbb{R}(\omega(v_1, w_1)) \neq 0 \), or \( \mathbb{R}(\omega(v_1, w_1)) = 0 \) and \( \mathbb{S}(\omega(v_1, w_1)) \neq 0 \).

If \( \mathbb{R}(\omega(v_1, w_1)) \neq 0 \), then let

\[ v''_1 = v'_1 + \frac{\omega(v'_1, w'_2)}{\omega(v'_1, w'_1)} v'_2, \quad v''_2 = v_2 - \frac{\omega(v'_1, w'_2)}{\omega(v'_1, w'_1)} v'_1, \quad w''_1 = w'_1, \quad w''_2 = w'_2. \]

The set \( \{ v''_1, v''_2, w''_1, w''_2 \} \) is still a basis for \( V \) and has the following properties

\[ \omega(v''_1, w''_2) = \omega(v'_1 + \frac{\omega(v'_1, w'_2)}{\omega(v'_1, w'_1)} v'_2, w'_2) = \omega(v'_1, w'_2) + \frac{\omega(v'_1, w'_2)}{\omega(v'_1, w'_1)} \omega(v'_2, w'_2) \]
\[ = \omega(v'_1, w'_2) - \frac{\omega(v'_1, w'_2)}{\omega(v'_1, v'_1)} \omega(v'_1, w'_1) = 0 \]
\[ \omega(v''_2, w''_1) = \omega(v'_2 - \frac{\omega(v'_1, w'_2)}{\omega(v'_1, v'_1)} v'_1, w'_1) = \omega(v'_2, w'_1) - \frac{\omega(v'_1, w'_2)}{\omega(v'_1, v'_1)} \omega(v'_1, w'_1) \]
\[ = \omega(v'_2, w'_1) - \frac{\omega(v'_1, w'_1)}{\omega(v'_1, v'_1)} \omega(v'_1, w'_1) = 0 \]
\[ \omega(v''_1, w''_1) = \omega(v'_1 + \frac{\omega(v'_1, w'_2)}{\omega(v'_1, w'_1)} v'_2, w'_1) = \omega(v'_1, w'_1) + \frac{\omega(v'_1, w'_2)}{\omega(v'_1, w'_1)} \omega(v'_2, w'_1) \]
\[ \omega(v''_2, w''_2) = \omega(v'_2 - \frac{\omega(v'_1, w'_2)}{\omega(v'_1, v'_1)} v'_1, w'_2) = \omega(v'_2, w'_2) - \frac{\omega(v'_1, w'_2)}{\omega(v'_1, v'_1)} \omega(v'_1, w'_2) \]
\[ = -\omega(v'_1, w'_2) - \frac{\omega(v'_1, w'_2)}{\omega(v'_1, v'_1)} \omega(v'_1, w'_2) = 0 \] and \( \omega(v''_1, v''_2) = \omega(v''_1, w''_2) = 0 \). Clearly, as \( \omega \) is non-degenerate, \( \omega(v''_1, w''_1) = -\omega(v''_2, w''_2) \neq 0 \). In
and as $w_1'' = w_1'$ and $w_2'' = w_1'$, their relationship remains unchanged.

If $\Re(\omega(v_1, w_1)) = 0$, then $\Im(\omega(v_1, w_1)) \neq 0$. This implies that $\omega(v_1, w_1') = \omega(v_2, w_2') = 0$ and $\omega(v_1', w_2') = \omega(v_2', w_1') \neq 0$. Let

$$v_1'' = v_1' + v_2', \quad v_2'' = v_2' - v_1', \quad w_1'' = w_1' + w_2', \quad w_2'' = w_2' - w_1'.$$

The set $\{v_1'', v_2'', w_1'', w_2''\}$ is still a basis for $V$ and has the properties that

$$\omega(v_1'', w_2'') = \omega(v_1' + v_2', w_2' - w_1') = \omega(v_1', w_2') - \omega(v_1', w_1') + \omega(v_2', w_1') - \omega(v_2', w_1') = 0$$

$$\omega(v_2'', w_1'') = \omega(v_2' - v_1', w_1' + w_2') = \omega(v_2', w_1') + \omega(v_2', w_1') - \omega(v_1', w_1') - \omega(v_1', w_1') = 0$$

$$\omega(v_1'', w_1'') = \omega(v_1' + v_2', w_1' + w_2') = \omega(v_1', w_1') + \omega(v_1', w_2') + \omega(v_2', w_1') + \omega(v_2', w_2') = 2\omega(v_1', w_2')$$

$$\omega(v_2'', w_2'') = \omega(v_2' - v_1', w_2' - w_1') = \omega(v_2', w_2') - \omega(v_2', w_1') - \omega(v_1', w_2') + \omega(v_1', w_1') = -\omega(v_1', w_1')$$

and $\omega(v_1'', v_2'') = \omega(w_1'', w_2'') = 0$. Then $\omega(v_1'', w_1'') = -\omega(v_2'', w_2'') \neq 0$. In addition,

$$av_1'' = a(v_1' + v_2') = \lambda v_1' - \mu v_2' + \mu v_1' + \lambda v_2' = \lambda v_1'' - \mu v_2'',$$

$$av_2'' = a(v_2' - v_1') = \mu v_1' + \lambda v_2' - \lambda v_1' + \mu v_2' = \mu v_1'' + \lambda v_2'',$$

and similarly

$$aw_1'' = -\lambda w_1'' + \mu w_2'', \quad aw_2'' = -\mu w_1'' - \lambda w_2''.$$

In either case, we know that $\omega(v_1'', w_1'') = -\omega(v_2'', w_2'') \neq 0$. If $\omega(v_1'', w_1'') < 0$, then $\omega(v_2'', w_2'') > 0$ and we replace $w_1''$ with $-w_1''$ and relabel. This will make both values positive, but will yield that $aw_1'' = -\lambda w_1'' - \mu w_2''$ and $aw_2'' = \mu w_1'' - \lambda w_2''$. On the other hand, if $\omega(v_1'', w_1'') > 0$, then $\omega(v_2'', w_2'') < 0$ and we replace $w_2''$ with $-w_2''$ and relabel. This will again make both values positive but will yield that $aw_1'' = -\lambda w_1'' - \mu w_2''$ and $aw_2'' = \mu w_1'' - \lambda w_2''$.

After this final change, we pick a new basis, $\beta$, for $V$ as follows

$$\beta = \left\{ \frac{1}{\sqrt{\omega(v_1'', w_1'')}} v_1'', \frac{1}{\sqrt{\omega(v_2'', w_2'')}} v_2'', \frac{1}{\sqrt{\omega(v_1', w_1'')}} w_1'', \frac{1}{\sqrt{\omega(v_2', w_2'')}} w_2'' \right\}.$$
and relabel the vectors \( v_1'', v_2'' \), \( w_1'' \), and \( w_2'' \) respectively. This implies that the matrix

\[
W = \begin{pmatrix} v_1'' & v_2'' & v_3'' & v_4'' \end{pmatrix}
\]

is in \( Sp(4, \mathbb{R}) \) as \( W^t JW = J \). In this basis, \( a \) is of form (8) in Theorem 2.3.5 with \( \lambda \neq 0 \).

**Case 8:** \( V_c = K_\mu \oplus K_- \mu \oplus K_\eta \oplus K_- \eta \)

Also we assume that \( \mu, \eta \in \mathbb{R} \) are both nonzero and \( \eta \neq \pm \mu \). Let \( v_1 \in K_\mu \), then \( v_2 = \eta_1 \in K_- \mu \). We are guaranteed that \( \omega(v_1, v_2) \neq 0 \). This pair then decomposes out according to Proposition 2.3.12. However, note that \((v_1, v_2)^\perp \) is \( K_\eta \oplus K_- \eta \). Let \( w_1 \in K_\eta \), then \( w_2 = \eta_1 \in K_- \eta \) and we are again guaranteed that \( \omega(w_1, w_2) \neq 0 \). In addition, every pair of vectors from the set \( \{v_1, v_2, w_1, w_2\} \) other than \( v_1, v_2 \) and \( w_1, w_2 \) are symplectically orthogonal. Moreover, this set forms a basis for \( V_c \).

Let

\[
v_1' = v_1 + v_2, \quad v_2' = -i(v_1 - v_2), \quad w_1' = w_1 + w_2, \quad w_2' = -i(w_1 - w_2),
\]

then the set \( \{v_1', v_2', w_1', w_2'\} \) forms a basis for \( V \) over \( \mathbb{R} \) such that

\[
aw_1' = -\mu w_2', \quad aw_2' = \mu w_1', \quad aw_1' = -\eta w_2', \quad aw_2' = \eta w_1'
\]

and \( \omega(v_1', v_2') \neq 0 \) and \( \omega(w_1', w_2') \neq 0 \).

If \( \omega(v_1', v_2') < 0 \), then replace \( v_2' \) with \(-v_2'\) and relabel so that \( \omega(v_1', v_2') > 0 \). Doing this will, however, yield that \( aw_1' = \mu v_2' \) and \( aw_2' = -\mu v_1' \). By Lemma 2.3.15, this is not symplectically similar to the case where \( \omega(v_1', v_2') > 0 \) in the first place. The same can be said for \( w_1' \) and \( w_2' \).

After making the necessary changes, we pick a new basis, \( \beta \), for \( V \) as follows

\[
\beta = \left\{ \begin{array}{c}
\frac{1}{\sqrt{\omega(v_1', v_2')}} v_1', \\
\frac{1}{\sqrt{\omega(w_1', w_2')}} w_1', \\
\frac{1}{\sqrt{\omega(v_1', v_2')}} v_2', \\
\frac{1}{\sqrt{\omega(w_1', w_2')}} w_2'
\end{array} \right\}
\]

and relabel the vectors \( \{v_1'', v_1', v_2'', v_2'\} \). Then the matrix

\[
W = \begin{pmatrix} v_1'' & v_1' & v_2'' & v_2' \end{pmatrix}
\]

is in \( Sp(4, \mathbb{R}) \) as \( W^t JW = J \). In this base \( a \) is of form (9) in Theorem 2.3.5 with \( \eta \neq \pm \mu, \eta, \mu \neq 0 \).
Case 9: $V_c = K_\mu \oplus K_{-\mu}$

Also assume that $\mu \in \mathbb{R}$ is nonzero. Note that $\dim K_\mu = \dim K_{-\mu}$.

If $a$ is diagonalizable, then let $v_1 \in K_\mu$, then $v_2 = \overline{v_1} \in K_{-\mu}$. If $\omega(v_1, v_2) = 0$, then we let $w_1 \in K_{-\mu}$ and pick $w_2 = \overline{w_1} \in K_\mu$ such that $w_1 \neq v_2$, and we’re guaranteed that $\omega(v_1, w_1) \neq 0$ and $\omega(v_2, w_2) \neq 0$. We also know that $\omega(v_1, w_2) = \omega(w_1, v_2) = 0$ because $v_1, w_2 \in K_\mu$ and $v_2, w_1 \in K_{-\mu}$. Moreover,

$$\omega(v_1, w_1) = \omega(\overline{v_1}, w_1) = \omega(v_2, w_2),$$
$$\omega(w_1, w_2) = \omega(w_1, \overline{w_2}) = \omega(w_2, w_1) = -\omega(w_1, w_2).$$

Let

$$w'_1 = w_1 - \frac{\omega(w_1, w_2)}{2\omega(v_2, w_2)} v_2, \quad w'_2 = w_2 + \frac{\omega(w_1, w_2)}{2\omega(v_2, w_2)} v_1.$$

Then we have that

$$\omega(w'_1, w'_2) = \omega\left( w_1 - \frac{\omega(w_1, w_2)}{2\omega(v_2, w_2)} v_2, w_2 + \frac{\omega(w_1, w_2)}{2\omega(v_2, w_2)} v_1 \right)$$
$$= \omega(w_1, w_2) + \frac{\omega(w_1, w_2)}{2\omega(v_1, w_2)} \omega(w_1, v_1) - \frac{\omega(w_1, w_2)}{2\omega(v_2, w_2)} \omega(v_2, w_2) = 0.$$ 

Furthermore,

$$\overline{w'_1} = w_1 - \frac{\omega(w_1, w_2)}{2\omega(v_2, w_2)} v_2, \quad \overline{w'_2} = w_2 + \frac{\omega(w_1, w_2)}{2\omega(v_1, w_1)} v_1 = w'_2$$

Relabel $w'_1$ and $w'_2$ as $w_1$ and $w_2$ and then proceed precisely as in Case 7. This allows $\lambda = 0$ in form (8) of Theorem 2.3.3.

Now if $\omega(v_1, v_2) \neq 0$, then these two decompose out according to Proposition 2.3.12 and we pick $w_1 \in K_\mu$ and $w_2 = \overline{w_1} \in K_{-\mu}$ such that $w_1 \neq v_1$. Then $\omega(w_1, w_2) \neq 0$ and we proceed precisely as in Case 8. This allows $\eta = \mu$ in form (9) of Theorem 2.3.3. However, we must still require that $\eta \neq -\mu$ because the following three matrices in $\mathfrak{sp}(4, \mathbb{R})$,

$$a_1 = \begin{pmatrix} 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & \mu \\ \mu & 0 & 0 & 0 \\ 0 & -\mu & 0 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & -\mu \\ -\mu & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 0 & \mu & 0 & 0 \\ -\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & -\mu & 0 \end{pmatrix}.$$
are symplectically similar conjugating by the matrices in $Sp(4, \mathbb{R})$

\[
A_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
1 & 1 & \frac{1}{2} & 0 \\
1 & 1 & 0 & \frac{1}{2} \\
-1 & 1 & 0 & \frac{1}{2} \\
1 & -1 & \frac{1}{2} & 0
\end{pmatrix}.
\]

That is, $(A_1)^{-1}a_1A_1 = a_2$ and $(A_2)^{-1}a_2A_2 = a_3$.

If $a$ is not diagonalizable, then either $K_{\mu i}$ contains a cycle of generalized eigenvectors of length two or $K_{-\mu i}$ does. However, Lemma 2.3.16 shows, in fact, that they both do.

Let $\{v_1, v_2\}$ be a cycle of generalized eigenvectors in $K_{\mu i}$, then $\{w_1 = v_1, w_2 = v_2\}$ is a cycle of generalized eigenvectors in $K_{-\mu i}$.

By Proposition 1.10 and the non-degeneracy of $\omega$, we know that $\omega(v_1, w_2) \neq 0$, $\omega(v_2, w_1) \neq 0$, and $\omega(v_1, w_1) = 0$. In fact, we have

\[
\mu i\omega(v_2, w_2) + \omega(v_1, w_2) = \omega(\mu iv_2 + v_1, w_2) = \omega(aw_2, w_2) = -\omega(v_2, aw_2)
\]

\[
= -\omega(v_2, -\mu iv_2 + w_1) = \mu i\omega(v_2, w_2) - \omega(v_2, w_1),
\]

which implies that $\omega(v_1, w_2) = -\omega(v_2, w_1)$. With this information, we pick a new basis for $V_c$ as follows

\[
v'_1 = v_1 + w_1, \quad v'_2 = v_2 + w_2, \quad w'_1 = -i(v_1 - w_1), \quad w'_2 = -i(v_2 - w_2).
\]

As these four vectors are real, they actually form a basis for $V$. In addition, we have that

\[
av'_1 = -\mu w'_1, \quad aw'_1 = \mu v'_1, \quad av'_2 = -\mu w'_2 + v'_1, \quad aw'_2 = \mu v'_2 + w'_1.
\]

We also have the following

\[
\omega(v'_1, w'_2) = \omega(v_1 + w_1, -i(v_2 - w_2)) = -i\omega(v_1, v_2) + i\omega(v_1, w_2) + i\omega(w_1, v_2) + i\omega(w_1, w_2) = 0,
\]

\[
\omega(v'_2, w'_1) = \omega(v_2 + w_2, -i(v_1 - w_1)) = -i\omega(v_2, v_1) + i\omega(v_2, w_1) - i\omega(w_2, v_1) + i\omega(w_2, w_1) = 0,
\]

\[
\omega(v'_1, w'_1) = \omega(v_1 + w_1, -i(v_1 - w_1)) = 0,
\]

\[
\omega(v'_1, v'_2) = \omega(v_1 + w_1, v_2 + w_2) = \omega(v_1, v_2) + \omega(v_1, w_2) + \omega(w_1, v_2) + \omega(w_1, w_2) = 2\omega(v_1, w_2),
\]

\[
\omega(w'_1, w'_2) = \omega(-i(v_1 - w_1), -i(v_2 - w_2)) = -\omega(v_1 - w_1, v_2 - w_2)
\]

\[
= -\omega(v_1, v_2) + \omega(v_1, w_2) + \omega(w_1, v_2) - \omega(w_1, w_2) = \omega(v'_1, v'_2).
\]
However, it is possible that $\omega(v_2', w_2')$ is nonzero. To fix this potential problem, we pick another new basis for $V$ in the following way

$$v_1' = v_1', \quad v_2' = v_2' + \frac{\omega(v_2', w_2')}{2\omega(v_2', v_1')} w_1', \quad w_1'' = w_1', \quad w_2'' = w_2' - \frac{\omega(v_2', w_2')}{2\omega(v_2', v_1')} v_1'. $$

The set $\{v_1'', v_2'', w_1'', w_2''\}$ still forms a basis for $V$, but now

$$\omega(v_2'', w_2'') = \omega\left(v_2' + \frac{\omega(v_2', w_2')}{2\omega(v_2', v_1')} w_1', w_2' - \frac{\omega(v_2', w_2')}{2\omega(v_2', v_1')} v_1'\right)$$

$$= \omega(v_2', w_2') - \frac{\omega(v_2', w_2')}{2\omega(v_2', v_1')} \omega(v_2', v_1') + \frac{\omega(v_2', w_2')}{2\omega(v_2', v_1')} \omega(w_1', w_2')$$

$$= \omega(v_2', w_2') - \frac{\omega(v_2', w_2')}{2\omega(v_2', v_1')} \omega(v_2', v_1') = \frac{\omega(v_2', w_2')}{2} = 0.$$

In addition,

$$a v_2'' = a \left(v_2' + \frac{\omega(v_2', w_2')}{2\omega(v_2', v_1')} v_1'\right) = a v_2' + \frac{\omega(v_2', w_2')}{2\omega(v_2', v_1')} a v_1' = -\mu w_2' + v_1' + \mu \frac{\omega(v_2', w_2')}{2\omega(v_2', v_1')} v_1' = -\mu w_2' + v_1''$$

and

$$a w_2'' = a \left(w_2' - \frac{\omega(v_2', w_2')}{2\omega(v_2', v_1')} v_1'\right) = a w_2' - \frac{\omega(v_2', w_2')}{2\omega(v_2', v_1')} a v_1' = \mu v_2' + w_1' + \mu \frac{\omega(v_2', w_2')}{2\omega(v_2', v_1')} v_1' = \mu v_2' + w_1''.$$

Finally, $\omega(v_1'', w_1'') = \omega(v_1', w_1'') = 0$ and $\omega(v_2'', w_2'') = \omega(v_2', w_2'') = \omega(w_1', w_2') = \omega(v_1', w_2')$.

If $\omega(v_1'', w_1'') = \omega(w_1'', w_2'') < 0$, then replace the vectors $v_2''$ and $w_2''$ with $-v_2''$, and $-w_2''$ respectively and relabel. This will make the values in question positive, but will result in

$$a v_2'' = -\mu w_2'' - v_1'', \quad a w_2'' = \mu v_2'' - w_1''.$$

Lemma 2.3.15 shows, however, that this case is not symplectically similar to the case where $\omega(v_1', v_1'') = \omega(w_1', w_1'') > 0$ in the first place. But we still make the change to guarantee that $\omega(v_1', v_1'') = \omega(w_1', w_1'') > 0$.

After making the appropriate changes, we can pick a final basis for $V, \beta$, as follows

$$\beta = \left\{ \frac{1}{\sqrt{\omega(v_1'', v_1'')}} v_1'', \frac{1}{\sqrt{\omega(v_1', v_1'')}} v_1', \frac{1}{\sqrt{\omega(v_1'', v_2'')}} v_2'', \frac{1}{\sqrt{\omega(v_1', v_2'')}} v_2', \frac{1}{\sqrt{\omega(v_2'', v_1'')}} w_1'', \frac{1}{\sqrt{\omega(v_2', v_1'')}} w_1', \frac{1}{\sqrt{\omega(v_2'', v_2'')}} w_2'', \frac{1}{\sqrt{\omega(v_2', v_2'')}} w_2' \right\}$$
and relabel the vectors $v''_1$, $w''_1$, $v''_2$, and $w''_2$. Then the matrix

$$W = (v''_1 \quad w''_1 \quad v''_2 \quad w''_2)$$

is in $Sp(4, \mathbb{R})$ because $W^t J W = J$. In this basis, $a$ is of form (10) in Theorem 2.3.5.

This covers all of the possible cases of $a \in \mathfrak{sp}(4, \mathbb{R})$ and completes the proof of Theorem 2.3.5.

### 2.3.3 Canonical Forms of Matrices in a Nonstandard Representation of $\mathfrak{so}(3, 1, \mathbb{R})$

Let $V = \mathbb{R}^4$. Let $J_1, J_2 \in GL(V)$ be given as

$$J_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $J_1$ and $J_2$ are skew-symmetric and have the property that $J_i^2 = -I_4$, where $I_4$ denotes the $4 \times 4$ identity. Let $\mathfrak{h}(J_2)$ be defined by

$$\mathfrak{h}(J_2) = \{ a \in \text{Hom}(V, V) \mid a^t J_i + J_i a = 0 \text{ for all } i \in \{1, 2\} \}.$$ 

Clearly $\mathfrak{h}(J_2)$ is a subalgebra of $\mathfrak{sp}(4, \mathbb{R})$ as $J_1$ is the $J$ used in the section on $\mathfrak{sp}(4, \mathbb{R})$. This is why we use the notation $\mathfrak{h}(J_2)$ to denote it; that is, it is the subalgebra, $\mathfrak{h}$, of $\mathfrak{sp}(4, \mathbb{R})$ whose elements are also skew-symmetric about the matrix $J_2$. As such, many of our proofs will refer to those in that section.

However, before we get started, we should note a few interesting properties of $\mathfrak{h}(J_2)$. Let $D_1$ be an arbitrary $4 \times 4$ matrix. Then we can use Maple to solve the equation $D_1^t J_i + J_i D_1 = 0$ for all $i \in \{1, 2\}$. Doing so will force $D_1$ to be of the form

$$D_1 = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix},$$

with $A, B, C$ all $2 \times 2$ matrices such that $A = \begin{pmatrix} y & -p \\ p & y \end{pmatrix}$ and $B$ and $C$ are trace-free symmetric. This
implies that a basis for $\mathfrak{h}(J_2)$ is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.
\]

Label these elements $X_1, \ldots, X_6$. As the Lie bracket in a matrix Lie algebra is given by the commutator, then using this basis, the Lie algebra $\mathfrak{h}(J_2)$ has the multiplication table

\[
\begin{array}{cccccc}
X_1 & X_2 & X_3 & X_4 & X_5 & X_6 \\
0 & 0 & -2X_3 & -2X_4 & 2X_5 & 2X_6 \\
0 & 0 & 2X_4 & -2X_3 & 2X_6 & -2X_5 \\
2X_3 & -2X_4 & 0 & 0 & -X_1 & -X_2 \\
2X_4 & 2X_3 & 0 & 0 & X_2 & -X_1 \\
-2X_5 & -2X_6 & X_1 & -X_2 & 0 & 0 \\
-2X_6 & 2X_5 & X_2 & X_1 & 0 & 0 \\
\end{array}
\]

We make the change of basis

\[
Y_1 = \frac{1}{2}X_1, \quad Y_2 = \frac{\sqrt{2}}{4}(X_3 + X_4 + X_5 + X_6), \quad Y_3 = \frac{\sqrt{2}}{4}(X_3 - X_4 + X_5 - X_6),
\]

\[
Y_4 = \frac{\sqrt{2}}{4}(X_3 + X_4 - X_5 - X_6), \quad Y_5 = \frac{\sqrt{2}}{4}(X_3 - X_4 - X_5 + X_6), \quad Y_6 = -\frac{1}{2}X_2.
\]

In this basis, the multiplication table becomes

\[
\begin{array}{cccccc}
Y_1 & Y_2 & Y_3 & Y_4 & Y_5 & Y_6 \\
0 & -Y_4 & -Y_5 & -Y_2 & -Y_3 & 0 \\
Y_4 & 0 & -Y_6 & Y_1 & 0 & -Y_3 \\
Y_5 & Y_6 & 0 & 0 & Y_1 & Y_2 \\
Y_2 & -Y_1 & 0 & 0 & Y_6 & -Y_5 \\
Y_3 & 0 & -Y_1 & -Y_6 & 0 & Y_4 \\
Y_6 & 0 & Y_3 & -Y_2 & Y_5 & -Y_4 & 0 \\
\end{array}
\]

We will show that this is the multiplication table for $\mathfrak{so}(3,1,\mathbb{R})$ and hence $\mathfrak{h}(J_2)$ is a representation of $\mathfrak{so}(3,1,\mathbb{R})$. We’ll call this representation $\rho_1$. We’ll also show that $\rho_1$ is not equivalent to the standard representation of $\mathfrak{so}(3,1,\mathbb{R})$.

Let $\rho_2$ denote the standard representation of $\mathfrak{so}(3,1,\mathbb{R})$, which consists of those matrices that are skew-symmetric about the matrix

\[
M = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
that is, it consists of those matrices, \( a \), such that \( a^t M + Ma = 0 \). Let \( D_2 \) be an arbitrary \( 4 \times 4 \) matrix. Then we use Maple to solve the equation \( D_2^t M + MD_2 = 0 \). We find then that to satisfy the equation, \( D_2 \) must be of the form

\[
D_2 = \begin{pmatrix} 0 & d_1 & d_2 & d_3 \\ d_1 & 0 & -d_4 & -d_5 \\ d_2 & d_4 & 0 & -d_6 \\ d_3 & d_5 & d_6 & 0 \end{pmatrix}.
\]

This implies that a basis for the standard representation of \( \mathfrak{so}(3, 1, \mathbb{R}) \) is given by the following six matrices

\[
\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
\]

\[
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Call these elements \( \gamma_1, \ldots, \gamma_6 \) respectively. If we compute the multiplication table of this Lie algebra in this basis, we’ll obtain the second multiplication table given in this section. Thus, as abstract Lie algebras these two representations are isomorphic. This shows that \( \rho_1 \) is a representation of \( \mathfrak{so}(3, 1, \mathbb{R}) \). But \( \rho_1 \) and \( \rho_2 \) are not equivalent as there exists no \( T \in GL(\mathbb{R}^4) \) such that \( T \circ \rho_1 = \rho_2 \circ T \).

We show this by proving the equivalent statement that there exists no invertible \( 4 \times 4 \) matrix \( T \) such that \( TY_i = \gamma_i T \) for \( 1 \leq i \leq 6 \). Let \( T \) be an arbitrary \( 4 \times 4 \) matrix. Then we use Maple to solve the system \( TY_i = \gamma_i T \) for \( 1 \leq i \leq 6 \), which yields that \( T = 0 \). Hence \( \rho_1 \) and \( \rho_2 \) are not equivalent.

As the two representations are not equivalent, we cannot rely on the classification of the canonical forms of matrices in \( \rho_2 \) to discover those canonical forms of the matrices in the representation of \( \rho_1 \), that is, the canonical forms of matrices in \( \mathfrak{h}(J_2) \). Instead we find these canonical forms directly. We begin by proving a few facts about \( \mathfrak{h}(J_2) \).

**Lemma 2.3.18.** Let \( a \in \mathfrak{h}(J_2) \). Then \( a^t \in \mathfrak{h}(J_2) \).

*Proof.* If \( a \in \mathfrak{h}(J_2) \), then \( a^t J_1 + J_1 a = a^t J_2 + J_2 a = 0 \). For either case, multiply on the left by \( J_i \) to obtain \( J_i a^t J_i - a = 0 \). Then multiply on the right by \( J_i \): \( -J_i a^t - a J_i = 0 \) or equivalently \( (a^t)^t J_1 + J_1 a^t = (a^t)^t J_2 + J_2 a^t = 0 \). □
Define \( \omega_i : V \times V \to \mathbb{R} \) by \( \omega_i(x, y) = x^t J_i y \) for all \( x, y \in V \) and \( i \in \{1, 2\} \). Then the \( \omega_i \) are a pair of symplectic forms. They can also be viewed as maps \( V_c \times V_c \to \mathbb{C} \) with the same rules of assignment.

**Proposition 2.3.19.** These forms, \( \omega_i : V \times V \to \mathbb{R} \) (\( \omega_i : V_c \times V_c \to \mathbb{C} \) for \( i \in \{1, 2\} \), are non-degenerate skew-symmetric bilinear forms on \( V \) (\( V_c \)).

**Proof.** From their construction, they are clearly bilinear over \( \mathbb{R} \) or \( \mathbb{C} \). To prove skew-symmetry, we note that as \( \omega_i(x, y) \) can be viewed as a \( 1 \times 1 \) matrix, we have that \( (\omega_i(x, y))^t = \omega_i(x, y) \). Then, as \( J_i^t = -J_i \), this yields that for \( i \in \{1, 2\} \)

\[
\omega_i(x, y) = (\omega_i(x, y))^t = (x^t J_i y)^t = y^t J_i^t x = -y^t J_i x = -\omega_i(y, x).
\]

Finally, to prove non-degeneracy, let \( z \in V \) be such that \( \omega_i(z, y) = z^t J_i y = 0 \) for all \( y \in V \) for some \( i \in \{1, 2\} \). As both \( J_i \) are invertible, each has zero kernel, thus \( z = 0 \). ■

We will always denote these symplectic forms by simply \( \omega_1 \) or \( \omega_2 \) as it will be clear from the context whether we mean the complex or real form.

**Proposition 2.3.20.** The forms \( \omega_1 \) and \( \omega_2 \) have the following properties for all \( x, y \in V \).

1. \( \omega_1(J_2 x, y) = \omega_1(x, J_2 y) = -\omega_2(J_1 x, y) = -\omega_2(x, J_1 y) \).
2. \( \omega_i(J_i x, y) = -\omega_i(x, J_i y) \) for all \( i \in \{1, 2\} \).
3. \( \omega_i(J_1 J_2 x, y) = \omega_i(x, J_1 J_2 y) = -\omega_i(J_2 J_1 x, y) = -\omega_i(x, J_2 J_1 y) \) for all \( i \in \{1, 2\} \).

**Proof.** First note that \( J_1 J_2 = -J_2 J_1 \). Let \( x, y \in V \). Then we have that

\[
\omega_1(x, J_2 y) = x^t J_1 J_2 y = -x^t J_2 J_1 y = (J_2 x)^t J_1 y = \omega_1(J_2 x, y),
\]

\[
\omega_2(J_1 x, y) = (J_1 x)^t J_2 y = -x^t J_1 J_2 y = x^t J_2 J_1 y = \omega_2(x, J_1 y),
\]

\[
\omega_1(x, J_2 y) = x^t J_1 J_2 y = -(J_1 x)^t J_2 y = -\omega_2(J_1 x, y).
\]

Also we have, for all \( i \in \{1, 2\} \), that

\[
\omega_i(J_i x, y) = (J_i x)^t J_i y = -x^t J_i J_i y = -\omega_i(x, J_i y).
\]
The third statement is a direct consequence of the previous two and the fact that $J_1J_2 = -J_2J_1$. ■

Now we define the group preserving $J_1J_2$. This group, which we'll call $H(J_2)$ is defined as

$$H(J_2) = \{ A \in GL(V) \mid \omega_i(Ax, Ay) = \omega_i(x, y) \text{ for all } i \in \{1, 2\} \}.$$ 

That is, it is the group that preserves the symplectic forms.

**Lemma 2.3.21.** $a \in h(J_2)$ if and only if $\omega_i(ax, y) = -\omega_i(x, ay)$ for all $x, y \in V$ and $i \in \{1, 2\}$. $A \in H(J_2)$ if an only if $A^tJ_iA = J_i$ for all $i \in \{1, 2\}$.

**Proof.** As $a \in h(J_2)$, we have that $a^tJ_i + J_ia = 0$ for all $i \in \{1, 2\}$. Let $x, y \in V$. This gives us that for all $i \in \{1, 2\}$

$$\omega_i(ax, y) = (ax)^tJ_iy = x^ta^tJ_iy = -x^tJ_iay = -\omega_i(x, ay).$$

Next assume that $\omega_i(ax, y) = -\omega_i(x, ay)$ for all $x, y \in V$ and $i \in \{1, 2\}$. Then

$$\omega_i(ax, y) = -\omega_i(x, ay)$$

$$(ax)^tJ_iy = -x^tJ_iay$$

$$x^t a^tJ_iy = -x^tJ_iay$$

for all $i \in \{1, 2\}$. As this is true for all $x, y \in V$, this implies that $a^tJ_i = -J_ia$ or equivalently that $a^tJ_i + J_ia = 0$. Hence $a \in h(J_2)$.

To prove the statement about $H(J_2)$, first assume that $A \in H(J_2)$. Then for all $i \in \{1, 2\}$

$$\omega_i(Ax, Ay) = \omega_i(x, y)$$

$$(Ax)^tJ_iAy = x^tJ_iy$$

$$x^tA^tJ_iAy = x^tJ_iy$$

As this is true for all $x, y \in V$, we see that $A^tJ_iA = J_i$.

If we assume first that $A^tJ_iA = J_i$, then $A$ clearly preserves the symplectic forms and hence $A \in H(J_2)$. ■

We now wish to conjugate $a \in sp(2n, \mathbb{R})$ by an arbitrary element $A \in Sp(2n, \mathbb{R})$, that is, $A^{-1}aA$. 

Lemma 2.3.22. If \( a \in \mathfrak{h}(J_2) \) and \( A \in H(J_2) \), then \( A^{-1}aA \in \mathfrak{h}(J_2) \).

Proof. If \( A \in H(J_2) \), then \( \omega_i(Ax, y) = \omega_i(A^{-1}Ax, A^{-1}y) = \omega_i(x, A^{-1}y) \). This yields

\[
\omega_i(A^{-1}aAx, y) = \omega_i(aAx, Ay) = -\omega_i(Ax, aAy) = -\omega_i(x, A^{-1}aAy).
\]

Then by Lemma 2.3.21 \( A^{-1}aA \in \mathfrak{h}(J_2) \). ■

If \( a_1, a_2 \in \mathfrak{h}(J_2) \) are such that \( A^{-1}a_1A = a_2 \) for some \( A \in H(J_2) \), we say that \( a_1 \) and \( a_2 \) are \( \mathfrak{h} \)-symplectically similar.

This naturally brings up the question: what kind of canonical forms could \( a \in \mathfrak{h}(J_2) \) have if this were the only kind of change of basis allowed? The result is as follows

Theorem 2.3.23. Let \( a \in \mathfrak{h}(J_2) \), then \( a \) is \( \mathfrak{h} \)-symplectically similar to one of the following three matrices. We call this the real \( \mathfrak{h} \)-symplectic canonical form of the matrix.

\[
\begin{align*}
(1) & \quad \begin{pmatrix}
\lambda & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 \\
0 & 0 & -\lambda & 0 \\
0 & 0 & 0 & -\lambda
\end{pmatrix}, \quad \lambda \in \mathbb{R}, \\
(2) & \quad \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
(3) & \quad \begin{pmatrix}
\lambda & \varepsilon \mu & 0 & 0 \\
-\varepsilon \mu & \lambda & 0 & 0 \\
0 & 0 & -\lambda & \varepsilon \mu \\
0 & 0 & -\varepsilon \mu & -\lambda
\end{pmatrix}, \quad \lambda \geq 0, \varepsilon \mu > 0, \varepsilon^2 = 1.
\end{align*}
\]

The remainder of this section is the proof of this theorem. We present first some preliminary facts that will allow us to do so.

Lemma 2.3.24. Let \( a \in \mathfrak{h}(J_2) \) and let \( \lambda \) be an eigenvalue of \( a \). Then \( -\lambda \) is also an eigenvalue of \( a \).

Proof. As \( \mathfrak{h}(J_2) \subseteq \mathfrak{sp}(4, \mathbb{R}) \), this result follows by Lemma 2.3.16. ■

Lemma 2.3.25. If \( v \in V \) is an eigenvector of \( a \in \mathfrak{h}(J_2) \) corresponding to the eigenvalue \( \lambda \), then \( J_1J_2v \) is as well. Furthermore, if \( v \) and \( J_1J_2v \) are linearly dependent, then \( v = \pm iJ_1J_2v \). This implies that if \( v \) is real, then \( v \) and \( J_1J_2v \) are linearly independent. Finally, if \( v = \pm iJ_1J_2v \) and we write \( v = u + iw \), then \( w = \pm J_1J_2u \).

Proof. If \( v \in V \) is an eigenvector of \( a \in \mathfrak{h}(J_2) \) corresponding to the eigenvalue \( \lambda \in \mathbb{C} \), then \( av = \lambda v \). Also by Lemma 2.3.18 \( a^t \in \mathfrak{h}(J_2) \). This yields that

\[
a(J_1J_2v) = -J_1a^tJ_2v = J_1J_2av = \lambda(J_1J_2v).
\]
Before we go on, note that \((J_1J_2)^2 = -J_2J_1J_2 = -I_4\). Assume that \(v\) and \(J_1J_2v\) are linearly dependent. Then consider the equation

\[ c_1v + c_2J_1J_2v = 0 \]

Multiply on the left by \(J_1J_2\).

\[ c_1J_1J_2v - c_2v = 0 \]

Then we see that \(c_1J_1J_2v = c_2v\). Now multiply the original equation by \(c_1\) and make this substitution.

\[ (c_1)^2v + c_2c_1J_1J_2v = 0 \]
\[ (c_1)^2v + (c_2)^2v = 0 \]
\[ ((c_1)^2 + (c_2)^2) v = 0 \]

As \(v \neq 0\), this implies that \((c_1)^2 + (c_2)^2 = 0\) or equivalently that \(c_2 = \pm c_1i\). Thus if \(v\) and \(J_1J_2v\) are linearly dependent, then \(v = \pm iJ_1J_2v\) (simply let \(c_1 = 1\)). If \(v\) is real, then \(c_1, c_2 \in \mathbb{R}\) and we see that \(c_1 = c_2 = 0\).

If \(v\) is complex, then we can write \(v = u + iw\) where \(u, w \in V\). If \(c_2 = \pm c_1i\), then let \(c_1 = 1\) and we obtain that \(\pm iJ_1J_2v = v\). This yields the following

\[ \pm iJ_1J_2v = v \]
\[ \pm i(J_1J_2u + iJ_1J_2w) = u + iw \]
\[ \mp J_1J_2w \pm iJ_1J_2u = u + iw \]

This implies that \(w = \pm J_1J_2u\). Then \(w = \pm J_1J_2u\) and we can write \(v = u \pm iJ_1J_2u\).

**Lemma 2.3.26.** If \(\{v_1, \ldots, v_k\}\) is a cycle of generalized eigenvectors corresponding to \(\lambda \in \mathbb{C}\), then so is \(\{J_1J_2v_1, \ldots, J_1J_2v_k\}\).

**Proof.** First, note that if \(\{v_1, \ldots, v_k\}\) is a cycle of generalized eigenvectors corresponding to \(\lambda \in \mathbb{C}\), then, by definition, \(v_i = (a - \lambda I_n)^{k-i}v_k\) for all \(1 \leq i \leq k\).
We now proceed by induction if $k = 1$, then by Lemma 2.25, if $v_1$ is an eigenvector then so is $J_1J_2v_1$ and the statement holds.

Assume that the statement is true for $k = m$.

Finally let $k = m + 1$, that is $\{v_1, \ldots, v_{m+1}\}$ is a cycle of generalized eigenvectors. Then $\{v_1, \ldots, v_m\}$ is a cycle of generalized eigenvectors of length $m$. Thus, by assumption, $\{J_1J_2v_1, \ldots, J_1J_2v_m\}$ is a cycle of generalized eigenvectors of length $m$ as well. This implies that $J_1J_2v_i = (a - \lambda I_n)^{m-i}J_1J_2v_m$.

As $\{v_1, \ldots, v_{m+1}\}$ is a cycle, then we have that $v_m = (a - \lambda I)v_{m+1}$. This implies

$$(a - \lambda I_n)J_1J_2v_{m+1} = (aJ_1J_2 - \lambda I_nJ_1J_2)v_{m+1} = (-J_1a^tJ_2 - J_1J_2(\lambda I_n))v_{m+1}$$

$$= (J_1J_2a - J_1J_2(\lambda I_n))v_{m+1} = J_1J_2((a - \lambda I_n)v_{m+1}) = J_1J_2v_m.$$  

This yields that

$$J_1J_2v_i = (a - \lambda I_n)^{m-i}J_1J_2v_m = (a - \lambda I_n)^{m-i}(a - \lambda I_n)J_1J_2v_{m+1}$$

$$= (a - \lambda I_n)^{m+1-i}J_1J_2v_{m+1}. \quad (2.1)$$

Hence $\{J_1J_2v_1, \ldots, J_1J_2v_{m+1}\}$ is a cycle of generalized eigenvectors. Therefore, by induction, the statement holds for all $k \in \mathbb{N}$. Note that if $v_1$ and $J_1J_2v_1$ are linearly independent, then these cycles are linearly independent of each other. ■

We know that the characteristic polynomial of $a$ always splits over $\mathbb{C}$, thus we have that the complexification of $V$, $V_c$, is the direct sum of the generalized eigenspaces of $a$. This allows us to write $V_c$ in the following manner

$$V_c = K_0 \oplus K_{\lambda_1} \oplus K_{-\lambda_1} \oplus K_{\lambda_2} \oplus K_{-\lambda_2} \oplus \cdots \oplus K_{\lambda_k} \oplus K_{-\lambda_k}$$

where $1 \leq k \leq n$, $K_\mu$ is the generalized eigenspace corresponding to the eigenvalue $\mu$, $\lambda_i \neq 0$ for all $i$, and the $\lambda_i$ are distinct and such that $\lambda_i \neq -\lambda_j$ for any pair $i, j$. Note that if all the eigenvalues of $a$ are real, then its characteristic polynomial splits over $\mathbb{R}$ and $V$ decomposes in the manner above where all the summands are subspaces over $\mathbb{R}$. 
Proposition 2.3.27. If $\mu \neq -\eta$, then $K_\eta$ and $K_\mu$ are $\mathfrak{h}$-symplectically orthogonal to each other. That is, $\omega_i(K_\eta, K_\mu) = 0$.

Proof. $K_\eta$ and $K_\mu$ are orthogonal with respect to $\omega_1$ by Lemma 2.3.7. The proof that they are orthogonal with respect to $\omega_2$ is analogous to the proof of that lemma. ■

Note that the above proposition is true if $\eta = 0$. However, if $\eta \neq 0$, then this leads immediately to a useful corollary.

Corollary 2.3.28. If $\eta \neq 0$, then $K_\eta$ is $\mathfrak{h}$-symplectically orthogonal to itself.

Proposition 2.3.29. For $i \in \{1, 2\}$, $\omega_i$ is non-degenerate on $K_\mu \oplus K_{-\mu}$ for all $\mu \neq 0$. In addition, $\omega_i$ is non-degenerate on $K_0$.

Proof. As $\mathfrak{h}(J_2) \subseteq \mathfrak{sp}(4, \mathbb{R})$, then by Proposition 2.3.9, this result is clear for $\omega_1$. The proof for $\omega_2$ is analogous to the proof of that proposition. ■

Corollary 2.3.30. If $v \in K_\mu$ is nonzero, then there exists a nonzero $w_1 \in K_{-\mu}$ such that $\omega_1(v, w_1) \neq 0$ and a nonzero $w_2 \in K_{-\mu}$ such that $\omega_2(v, w_2) \neq 0$.

Proof: The existence of $w_1$ is guaranteed by Corollary 2.3.10. The proof of the existence of $w_2$ is analogous to the proof of that corollary. Note that it is possible that $w_2 = w_1$. ■

We can say a little more when $v \in K_\mu$ is an eigenvector.

Proposition 2.3.31. If $v \in K_\mu$ is an eigenvector and $w \in K_{-\mu}$ such that $\omega_1(v, w) \neq 0$ or $\omega_2(v, w) \neq 0$, then $w$ is the end vector in any cycle of generalized eigenvectors to which it belongs.

Proof. As $\mathfrak{h}(J_2) \subseteq \mathfrak{sp}(4, \mathbb{R})$, this is true for $\omega_1$ by Proposition 2.3.11. The proof for $\omega_2$ is analogous to the proof of that proposition. ■

Proposition 2.3.32. The dimension of $K_0$ is even and $\dim K_\mu = \dim K_{-\mu}$ for all nonzero $\mu \in \mathbb{C}$. Moreover, if $\mu \in \mathbb{R}$, then $\dim K_\mu$ is even.

Proof. As $\mathfrak{h}(J_2) \subseteq \mathfrak{sp}(4, \mathbb{R})$, the first two properties follow immediately.
Let $\mu \in \mathbb{R}$. If $\dim K_\mu = 0$, then we’re done. Assume then that $\dim K_\mu \neq 0$. Let $\{v_1, \ldots, v_k\} \subseteq K_\mu$ be a cycle of generalized eigenvectors. Then by Lemma 2.3.26, $\{J_1J_2v_1, \ldots, J_1J_2v_k\}$ is also a cycle of generalized eigenvectors. Furthermore, As the $v_i$ are real vectors, then by Lemma 2.3.25, $v_1$ and $J_1J_2v_1$ are linearly independent and hence their entire cycles are linearly independent of each other. Therefore every cycle of generalized eigenvectors is paired with another cycle of generalized eigenvectors of the same length to which it is linearly independent.

As there exists a basis for $K_\mu$ consisting of disjoint cycles of generalized eigenvectors of $a$, we have that $\dim K_\mu$ must be even. $$\blacksquare$$

Lemma 2.3.33. Let $\mu$ be an eigenvalue of $a \in \mathfrak{h}(J_2)$. Then $\dim K_{\mu}$ is 2 and $a$ is diagonalizable.

Proof. As $\dim K_{-\mu} = \dim K_{\mu}$ and $\dim V = 4$, then clearly $\dim K_{\mu} \leq 2$.

Now assume to the contrary the $\dim K_{\mu} = 1$. Let $v_1 \in K_{\mu}$ be nonzero. By Lemma 2.3.25, and as $\dim K_{\mu} = 1$, we have that $v_1$ and $J_1J_2v_1$ must be linearly dependent and consequently $v_1 = \pm iJ_1J_2v_1$. Assume first that $v_1 = iJ_1J_2v_1$. Then by the same lemma, we can write $v_1 = u + iJ_1J_2u$ for some real vector $u$. This implies that $v_2 = u - iJ_1J_2u \in K_{-\mu}$. In addition, as $\dim K_{-\mu} = \dim K_{\mu}$, we have that $\{v_1, v_2\}$ is a basis for $K_{\mu} \oplus K_{-\mu}$. However, for $j \in \{1, 2\}$, we have by Lemma 2.3.26 that

$$\omega_j(v_1, v_2) = \omega_j(u + iJ_1J_2u, u - iJ_1J_2u)$$

$$= \omega_j(u, u) - i\omega_j(u, J_1J_2u) + i\omega_j(J_1J_2u, u) + \omega_j(J_1J_2u, J_1J_2u)$$

$$= -i\omega_j(u, J_1J_2u) + \omega_j(u, J_1J_2u) = 0.$$  

This implies that $\omega_j$ is degenerate on $K_{\mu} \oplus K_{-\mu}$ contradicting Proposition 2.3.29. By a similar argument the same result can be shown if $v_1 = -iJ_1J_2v_1$. Thus $\dim K_{\mu} \geq 2$ and consequently $\dim K_{\mu} = 2$.

Next we show that $a$ is diagonalizable. As $\dim K_{\mu} = \dim K_{-\mu} = 2$, we have that $V = K_{\mu} \oplus K_{-\mu}$. Thus if $a$ is not diagonalizable, then there exists a basis for $K_{\mu}$ consisting of a cycle of generalized eigenvectors of length two. Let $\{v_1, v_2\}$ be such a cycle. Then, by Lemma 2.3.26, $\{J_1J_2v_1, J_1J_2v_2\}$ is also a cycle of generalized eigenvectors in $K_{\mu}$. However, as $\dim K_{\mu} = 2$, this implies that $v_1$ and $J_1J_2v_1$ are linearly dependent and hence $v_2$ and $J_1J_2v_2$ are as well. By Lemma 2.3.25, we have then that $v_1 = \pm iJ_1J_2v_1$. 


Assume first that \( v_1 = iJ_1J_2v_1 \). Then \( v_2 = iJ_1J_2v_2 \) as well or \( \{J_1J_2v_1, J_1J_2v_2\} \) is not a cycle of generalized eigenvectors. This is because if \( v_2 = -iJ_1J_2v_2 \), then

\[-ia(J_1J_2v_2) = a(-iJ_1J_2v_2) = a\omega v_2 = \mu iv_2 + v_1 = \mu(-iJ_1J_2v_2) + iJ_1J_2v_1 = -i(\mu iJ_1J_2v_2 - J_1J_2v_1)\]

which implies that

\[a(J_1J_2v_2) = \mu iJ_1J_2v_2 - J_1J_2v_1\]

Now as \( v_1 = iJ_1J_2v_1 \) and \( v_2 = iJ_1J_2v_2 \), then, by Lemma 2.3.20, we have that \( v_1 = u_1 + iJ_1J_2u_1 \) and \( v_2 = u_2 + iJ_1J_2u_2 \) for real vectors \( u_1, u_2 \).

Let \( w_1 = \bar{v}_1 = u_1 - iJ_1J_2u_1 \) and \( w_2 = \bar{v}_2 = u_2 - iJ_1J_2u_2 \), then \( \{w_1, w_2\} \) is a cycle of generalized eigenvectors in \( K_{-\mu i} \). For \( j \in \{1, 2\} \), we have by Corollary 2.3.20 that \( \omega_j(v_1, v_2) = 0 \) and by Proposition 2.3.31 that \( \omega_j(v_1, w_1) = 0 \). However, we also have by Lemma 2.3.20 that

\[
\omega_j(v_1, w_2) = \omega_j(u_1 + iJ_1J_2u_1, u_2 - iJ_1J_2u_2)
= \omega_j(u_1, u_2) - i\omega_j(u_1, J_1J_2u_2) + i\omega_j(J_1J_2u_1, u_2) + \omega_j(J_1J_2u_1, J_1J_2u_2)
= \omega_j(u_1, u_2) - i\omega_j(u_1, J_1J_2u_2) + i\omega_j(u_1, J_1J_2u_2) - \omega_j(u_1, u_2) = 0
\]

As \( v_1 \neq 0 \), this implies that \( \omega_j \) is degenerate on \( V \), which is impossible. By a similar argument, we get the same result if \( v_1 = -iJ_1J_2v_1 \). Therefore \( a \) is diagonalizable. ■

**Lemma 2.3.34.** The following two matrices in \( \mathfrak{h}(J_2) \) are not \( \mathfrak{h} \)-symplectically similar for all \( \lambda, \mu \neq 0 \).

\[
a_1 = \begin{pmatrix}
\lambda & \mu & 0 & 0 \\
-\mu & \lambda & 0 & 0 \\
0 & 0 & -\lambda & \mu \\
0 & 0 & -\mu & -\lambda \\
\end{pmatrix},
a_2 = \begin{pmatrix}
\lambda & -\mu & 0 & 0 \\
\mu & \lambda & 0 & 0 \\
0 & 0 & -\lambda & -\mu \\
0 & 0 & \mu & -\lambda \\
\end{pmatrix}.
\]

**Proof:** We prove this by showing that the equation \( a_1A = Aa_2 \) has no solution in \( H(J_2) \). Let \( A \in GL(V) \) be given by

\[
A = \begin{pmatrix}
d_1 & d_2 & d_3 & d_4 \\
d_5 & d_6 & d_7 & d_8 \\
d_9 & d_{10} & d_{11} & d_{12} \\
d_{13} & d_{14} & d_{15} & d_{16} \\
\end{pmatrix}.
\]

By requiring \( A \) to satisfy the equation above, we find that \( A \) must have the form

\[
\begin{pmatrix}
d_1 & d_2 & 0 & 0 \\
d_2 & -d_1 & 0 & 0 \\
0 & 0 & d_{11} & d_{12} \\
0 & 0 & d_{12} & -d_{11} \\
\end{pmatrix}.
\]
However, if we also require $A$ to be an element in $H(J_2)$, that is, require it to satisfy the additional equations $A^i J_i A = J_i$ for all $i \in \{1, 2\}$, then we get, amongst other things, that $d_1 d_{11} + d_2 d_{12}$ is equal to both 1 and -1, which is, of course, impossible. Thus there is no solution to $a_1 A = A a_2$ in $H(J_2)$ and consequently $a_1$ and $a_2$ are not $\mathfrak{h}$-symplectically similar. ■

We now have the tools necessary to compute the canonical forms of $a \in \mathfrak{h}(J_2)$. We will consider first those $a$ that have real eigenvalues and then those where one or more of the eigenvalues are complex. We’ll classify according to the decomposition of $V$ into generalized eigenspaces of $a$ when the eigenvalues are real and $V_c$ when some or all of them are complex. The cases will follow these four decompositions of $V$ or $V_c$.

1. $V = K_\lambda \oplus K_{-\lambda}$ where $\lambda \in \mathbb{R}$ is nonzero.
2. $V = K_0$.
3. $V_c = K_z \oplus K_{-z} \oplus K_\pi \oplus K_{-\pi}$ where $z = \lambda + \mu i$ for some $\lambda, \mu \in \mathbb{R}$ such that $\lambda, \mu > 0$.
4. $V_c = K_{\mu i} \oplus K_{-\mu i}$ where $\mu \in \mathbb{R}$ is positive.

By Proposition 2.3.32 and Lemma 2.3.33 this enumerates every possible decomposition of $V$ or $V_c$ into generalized eigenspaces of $a \in \mathfrak{h}(J_2)$.

**Case 1:** $V = K_\lambda \oplus K_{-\lambda}$

Also assume that $\lambda \in \mathbb{R}$ is nonzero. We know, by Proposition 2.3.32 that $\dim K_\lambda = \dim K_{-\lambda} = 2$ and, by Lemma 2.3.25 $K_\lambda$ and $K_{-\lambda}$ must contain at least two dimensions of eigenvectors. Thus $K_\lambda$ and $K_{-\lambda}$ contain only eigenvectors and hence $a$ is diagonalizable.

Let $v_1 \in K_\lambda$, then by Corollary 2.3.30 there exists a $w_1 \in K_{-\lambda}$ such that $\omega_1(v_1, w_1) \neq 0$. Let $v_2 = J_1 J_2 v_1$ and $w_2 = J_1 J_2 w_1$. Then by Lemma 2.3.25 $v_2$ and $w_2$ are eigenvectors of $a$ corresponding to $\lambda$ and $-\lambda$ respectively and are linearly independent of $v_1$ and $w_1$. Then the set $\{v_1, v_2, w_1, w_2\}$ is a basis for $V$ and has the following properties. Because $v_1, v_2 \in K_\lambda$ and $w_1, w_2 \in K_{-\lambda}$, we have that $\omega_1(v_1, v_2) = \omega_1(w_1, w_2) = 0$. In addition, though, we have that

\[
\begin{align*}
\omega_1(v_1, w_2) &= \omega_1(v_1, J_1 J_2 w_1) = -\omega_1(J_1 v_1, J_2 w_1) = \omega_2((J_1)^2 v_1, w_1) = -\omega_2(v_1, w_1), \\
\omega_2(v_1, w_2) &= \omega_2(v_1, J_1 J_2 w_1) = -\omega_1(v_1, (J_2)^2 w_1) = \omega_1(v_1, w_1).
\end{align*}
\]
Also

\[ \omega_1(v_2, w_1) = \omega_1(J_1J_2v_1, w_1) = -\omega_1(w_1, J_1J_2v_1) = \omega_2(w_1, v_1) = -\omega_2(v_1, w_1), \]

\[ \omega_2(v_2, w_1) = \omega_2(J_1J_2v_1, w_1) = -\omega_2(w_1, J_1J_2v_1) = -\omega_1(v_1, w_1) = \omega_1(v_1, w_1), \]

and

\[ \omega_1(v_2, w_2) = -\omega_1(J_1J_2v_1, J_1J_2w_1) = -\omega_1((J_1)^2J_2v_1, J_2w_1) = \omega_1(v_2, J_2w_1) = \omega_1(v_1, (J_2)^2w_1) = -\omega_1(v_1, w_1), \]

\[ \omega_2(v_2, w_2) = \omega_2(J_1J_2v_1, J_1J_2w_1) = \omega_2((J_1)^2J_2v_1, J_2w_1) = -\omega_2(v_2, J_2w_1) = \omega_2((J_2)^2v_1, w_1) = -\omega_2(v_1, w_1). \]

At this point, we pick a new basis in the following way

\[ v'_1 = v_1, \quad v'_2 = v_2 \]

\[ w'_1 = w_1 - \frac{\omega_1(v_2, w_1)}{\omega_1(v_2, w_2)} w_2, \quad w'_2 = J_1J_2w'_1 \]

Note that as \( \omega_1(v_2, w_2) = -\omega_1(v_1, w_1) \), we have that \( \omega_1(v_2, w_2) \neq 0 \) and so this basis change makes sense. Note that \( w'_1, w'_2 \in K_{-\lambda} \) and so are still eigenvectors. This implies that \( \omega_i(v'_1, v'_2) = \omega_i(w'_1, w'_2) = 0 \). Furthermore as \( w'_2 = J_1J_2w'_1 \), the properties computed above still hold. In addition,

\[ \omega_1(v'_2, w'_1) = \omega_1(v_2, w_1 - \frac{\omega_1(v_2, w_1)}{\omega_1(v_2, w_2)} w_2) = \omega_1(v_2, w_1) - \frac{\omega_1(v_2, w_1)}{\omega_1(v_2, w_2)} \omega_1(v_2, w_2) = 0, \]

which implies that

\[ \omega_2(v'_2, w'_2) = \omega_1(v'_1, w'_1) = -\omega_2(v'_1, w'_1) = \omega_1(v'_2, w'_1) = 0. \]

Moreover,

\[ \omega_1(v'_1, w'_1) = \omega_1(v_1, w_1 - \frac{\omega_1(v_2, w_1)}{\omega_1(v_2, w_2)} w_2) = \omega_1(v_1, w_1) - \frac{\omega_1(v_2, w_1)}{\omega_1(v_2, w_2)} \omega_1(v_1, w_2) \]

\[ = \omega_1(v_1, w_1) + \frac{\omega_1(v_1, w_1)}{\omega_1(v_2, w_2)} \omega_1(v_1, w_2) = 2\omega_1(v_1, w_1) \]

and hence

\[ \omega_2(v'_1, w'_2) = \omega_2(v'_2, w'_1) = -\omega_1(v'_2, w'_2) = \omega_1(v'_1, w'_1) = 2\omega_1(v_1, w_1) \neq 0. \]
Finally, if $\omega_1(v_1', w_1') > 0$, then replace $v_1'$, $w_1'$, and $w_2'$ with $-v_1'$, $-w_1'$, and $-w_2'$ and relabel them $v_1'$, $w_1'$, and $w_2'$ respectively. If $\omega_1(v_1', w_1') < 0$, then replace $v_2'$, $w_1'$, and $w_2'$ with $-v_2'$, $-w_1'$, and $-w_2'$ and relabel them $v_2'$, $w_1'$, and $w_2'$ respectively. This will ensure the following relationship

$$\omega_1(v_1', w_1') = \omega_1(v_2', w_2') = \omega_2(v_1', w_2') = -\omega_2(v_2', w_1') > 0.$$ 

Then we pick a new basis for $V$, call it $\beta$, as follows

$$\beta = \left\{ \frac{1}{\sqrt{\omega_1(v_1', w_1')}} v_1', \frac{1}{\sqrt{\omega_1(v_1', w_1')}} v_2', \frac{1}{\sqrt{\omega_1(v_1', w_1')}} w_1', \frac{1}{\sqrt{\omega_1(v_1', w_1')}} w_2' \right\},$$

and relabel the vectors $v_1''$, $v_2''$, $w_1''$, and $w_2''$ respectively. Then the matrix

$$W = \begin{pmatrix} v_1'' & v_2'' & w_1'' & w_2'' \end{pmatrix}$$

has the properties that $W^t J_1 W = J_1$ and $W^t J_2 W = J_2$ and is thus in $H(J_2)$. Finally, in this basis, $a$ is of form (1) in Theorem 2.3.23 with $\lambda \neq 0$.

**Case 2: $V = K_0$**

If $a$ is diagonalizable, then, as its only eigenvalue is 0, $a$ is the zero transformation and is already in form (1) in Theorem 2.3.23 with $\lambda = 0$.

If $a$ is not diagonalizable, then there exists a cycle of generalized eigenvectors of length two in $K_0$. Let $\{v_1, v_2\}$ be such a cycle. Let $w_1 = J_1 J_2 v_1$ and $w_2 = J_1 J_2 v_2$. Then by Lemma 2.3.20 $\{w_1, w_2\}$ is also a cycle of generalized eigenvectors and linearly independent to $\{v_1, v_2\}$. Then $\{v_1, v_2, w_1, w_2\}$ is a basis for $K_0$. By Proposition 2.3.31 we see that $\omega_i(v_1, w_1) = 0$. Furthermore

$$\omega_1(v_2, w_2) = \omega_1(v_2, J_1 J_2 v_2) = -\omega_1(J_1 v_2, J_2 v_2) = \omega_2(J_1 v_2, J_1 v_2) = 0$$

$$\omega_2(v_2, w_2) = \omega_2(v_2, J_1 J_2 v_2) = -\omega_1(J_2 v_2, J_2 v_2) = 0$$

In addition, we have that

$$\omega_1(v_1, w_2) = \omega_1(v_1, J_1 J_2 v_2) = -\omega_1(J_1 v_1, J_2 v_2) = \omega_2((J_1)^2 v_1, v_2) = -\omega_2(v_1, v_2),$$

$$\omega_2(v_1, w_2) = \omega_2(v_1, J_1 J_2 v_2) = -\omega_1(v_1, (J_2)^2 v_2) = \omega_1(v_1, v_2).$$
Also

$$\omega_1(w_1, v_2) = \omega_1(J_1J_2v_1, v_2) = -\omega_1(v_2, J_1J_2v_1) = \omega_2(v_2, v_1) = -\omega_2(v_1, v_2),$$

$$\omega_2(w_1, v_2) = \omega_2(J_1J_2v_1, v_2) = -\omega_2(v_2, J_1J_2v_1) = -\omega_1(v_2, v_1) = \omega_1(v_1, v_2),$$

and

$$\omega_1(w_1, w_2) = -\omega_1(J_1J_2v_1, J_1J_2v_2) = -\omega_1((J_1)^2J_2v_1, J_2v_2)$$

$$= \omega_1(J_2v_1, J_2v_2) = \omega_1(v_1, (J_2)^2v_2) = -\omega_1(v_1, v_2),$$

$$\omega_2(w_1, w_2) = \omega_2(J_1J_2v_1, J_1J_2v_2) = \omega_2((J_1)^2J_2v_1, J_2v_2)$$

$$= -\omega_2(J_2v_1, J_2v_2) = \omega_2((J_2)^2v_1, v_2) = -\omega_2(v_1, v_2).$$

As \(\omega\) is non-degenerate on \(V\), we have that \(\omega_1(v_1, v_2)\) and \(\omega_2(v_1, v_2)\) are not both zero.

First assume that \(\omega_2(v_1, v_2) \neq 0\). Then we solve the equation

$$c_1x^2 + 2c_2x - c_1 = 0$$

where \(c_1 = \omega_2(v_1, v_2)\) and \(c_2 = \omega_1(v_1, v_2)\). The discriminant of this equation is \(4(c_2)^2 + 4(c_1)^2\). As \(c_1, c_2 \in \mathbb{R}\) and \(c_1 \neq 0\), we see that the discriminant is strictly positive and this quadratic equation has two real solutions. Let \(x_0 \in \mathbb{R}\) be a solution to the equation. Then pick a new basis as follows

$$v_1' = v_1 - x_0w_1,$$

$$v_2' = v_2 - x_0w_2,$$

$$w_1' = J_1J_2v_1',$$

$$w_2' = J_1J_2v_2'.$$

Note that \(\{v_1', v_2'\}\) is still a cycle of generalized eigenvectors. Then by Lemma 2.3.26 \(\{w_1', w_2'\}\) is also one as well. Also note that

$$\omega_i(v_1', w_1') = \omega_i(v_2', w_2') = 0 \text{ for all } i \in \{1, 2\},$$

and by a similar argument to that above, we have that

$$\omega_1(v_1', v_2') = \omega_2(v_1', w_2') = \omega_2(w_1', v_2') = -\omega_1(w_1', w_2'),$$

$$\omega_2(v_1', v_2') = -\omega_1(v_1', w_2') = -\omega_1(w_1', v_2') = -\omega_2(w_1', w_2').$$
However, in this basis we also have that

\[ \omega_2(v_1', v_2') = \omega_2(v_1 - x_0w_1, v_2 - x_0w_2) = \omega_2(v_1, v_2) - x_0\omega_2(v_1, w_2) - x_0\omega_2(w_1, v_2) + (x_0)^2\omega_2(w_1, w_2) \]

\[ = \omega_2(v_1, v_2) - 2x_0\omega_1(v_1, v_2) - (x_0)^2\omega_2(v_1, v_2) = -(c_1(x_0)^2 + 2cx_0 - c_1) = 0. \]

Then as \( \omega \) is non-degenerate, we must have that

\[ \omega_1(v_1', v_2') = \omega_2(v_1', w_1') = \omega_2(w_1', v_2') = -\omega_1(w_1', w_2') \neq 0. \]

Now assume that \( \omega_2(v_1, v_2) = 0 \) in the first place, then \( \omega_1(v_1, v_2) \neq 0 \). In this case, simply relabel \( v_1, v_2, w_1 \), and \( w_2 \) as \( v_1', v_2', w_1', \) and \( w_2' \) respectively and we have the exact situation as described above.

If \( \omega_1(v_1', v_2') > 0 \), then let

\[ v_1'' = v_1' \quad v_2'' = v_2' \]
\[ w_1'' = -w_1' \quad w_2'' = w_2' \]

On the other hand, if \( \omega_1(v_1', v_2') < 0 \), then let

\[ v_1'' = -w_1' \quad v_2'' = -w_2' \]
\[ w_1'' = -v_1' \quad w_2'' = w_2' \]

Either case will yield that

\[ \omega_1(v_1'', v_2'') = \omega_1(w_1'', w_2'') = \omega_2(v_1'', w_2'') = -\omega_2(w_1'', v_2'') \]

and any other pair on either \( \omega_i \) products to 0. However, it also yields that

\[ a_{v_2''} = v_1'' \quad \text{but} \quad a_{w_2''} = -w_1''. \]

Then we pick a final basis for \( V \),

\[ \left\{ \frac{1}{\sqrt{\omega_1(v_1', v_2')}} v_1', \frac{1}{\sqrt{\omega_1(v_1', v_2')}} v_2', \frac{1}{\sqrt{\omega_1(v_1', v_2')}} v_1'', \frac{1}{\sqrt{\omega_1(v_1', v_2')}} v_2'' \right\} \]

and relabel the vectors \( v_1'', w_1'', v_2'' \), and \( w_2'' \) respectively. Then the matrix

\[ W = \begin{pmatrix} v_1'' & v_1'' & v_2'' & w_2'' \end{pmatrix} \]

is in \( H(J_2) \) because \( W^T J_i W = J_i \) for all \( i \in \{1, 2\} \). Finally, in this basis, \( a \) is of form (2) in Theorem
Case 3: $V_c = K_z \oplus K_{-z} \oplus K_{\tau} \oplus K_{-\tau}$

We also assume that $z = \lambda + \mu i$ such that $\lambda, \mu > 0$. Before we begin, we extend a lemma proved in the section on the symplectic Lie algebra.

Lemma 2.3.35. $\omega_i(v, w) = \omega_i(\bar{v}, \overline{w})$ for all $v, w \in V$ and $i \in \{1, 2\}$.

Proof. Let $v, w \in V$ and $i \in \{1, 2\}$. Then $\omega_i(v, w) = v^i J_i w = \overline{v^i J_i w} = \omega_i(\bar{v}, \overline{w})$. ■

Clearly in this case, each eigenspace must be of dimension one and consequently contains only eigenvectors. Hence $a$ is diagonalizable. Let $v_1 \in K_z$ and $w_1 \in K_{-z}$, and let $v_2 = \overline{v_1}$ and $w_2 = \overline{w_1}$.

Then $v_2 \in K_{\tau}$ and $w_2 \in K_{-\tau}$. By the non-degeneracy of $\omega_i$, we know that $\omega_i(v_1, w_1) \neq 0$ and $\omega_i(v_2, w_2) \neq 0$ for all $i \in \{1, 2\}$ and any other pair products to 0 using either form. In fact,

$$\omega_i(v_1, w_1) = \omega_i(\overline{v_1}, \overline{w_1}) = \omega_i(v_2, w_2).$$

Now we make the change of basis

$$v'_1 = v_1 + v_2 \quad v'_2 = J_1 J_2 v'_1$$
$$w'_1 = w_1 + w_2 \quad w'_2 = J_1 J_2 w'_1$$

As $J_1 J_2 v_1 \in K_z$ and $K_z$ is one dimensional, we see that $v_1$ and $J_1 J_2 v_1$ are linearly dependent. Then by Lemma 2.3.23 we have that $\Im(v_1) = \pm J_1 J_2 \Re(v_1)$. The same can be said of $w_1$ and $J_1 J_2 w_1$. This implies that we have

$$v'_1 = 2 \Re(v_1) \quad v'_2 = \pm 2 \Im(v_1) = \mp i(v_1 - v_2)$$
$$w'_1 = 2 \Re(w_1) \quad w'_2 = \pm 2 \Im(w_2) = \mp i(w_1 - w_2)$$

Clearly $\omega_i(v'_1, v'_2) = \omega_i(w'_1, w'_2) = 0$, but in addition, this implies, as before, that

$$\omega_1(v'_1, w'_1) = \omega_2(v'_1, w'_2) = \omega_2(v'_2, w'_1) = -\omega_1(v'_2, w'_2)$$
$$\omega_2(v'_1, w'_1) = -\omega_1(v'_1, w'_2) = -\omega_2(v'_2, w'_1) = -\omega_2(v'_2, w'_2)$$

Furthermore, we have that

$$\omega_1(v'_1, w'_1) = \omega_1(v_1 + v_2, w_1 + w_2) = \omega_1(v_1, w_1) + \omega_1(v_1, w_2) + \omega_1(v_2, w_1) + \omega_1(v_2, w_2)$$
$$= \omega_1(v_1, w_1) + \omega_1(v_2, w_2) = 2 \Re(\omega_1(v_1, w_1))$$
and

\[ \omega_1(v'_1, w'_2) = \omega_1(v_1 + v_2, \mp i(w_1 - w_2)) \]

\[ = \mp i\omega_1(v_1, w_1) \pm i\omega_1(v_1, w_2) \pm i\omega_1(v_2, w_1) \pm i\omega_1(v_2, w_2) \]

\[ = \mp i(\omega_1(v_1, w_1) - \omega_1(v_2, w_2)) = \pm 2\Im(\omega_1(v_1, w_1)) \]

Together, these imply that

\[ 2\Re(\omega_1(v_1, w_1)) = \omega_1(v'_1, w'_2) = \omega_2(v'_2, w'_2) = \omega_2(v'_2, w'_1) = -\omega_1(v'_2, w'_2) \]

\[ \pm 2\Im(\omega_1(v_1, w_1)) = -\omega_2(v'_1, w'_1) = \omega_1(v'_1, w'_2) = \omega_1(v'_2, w'_1) = \omega_2(v'_2, w'_2). \]

Finally, we also have that

\[ av'_1 = \lambda v'_1 \mp \mu v'_2, \quad av'_2 = \pm \mu v'_1 + \lambda v'_2, \]

\[ aw'_1 = -\lambda w'_1 \pm \mu w'_2, \quad aw'_2 = \mp \mu v'_1 - \lambda v'_1. \]

As \( \omega_1(v_1, w_1) \neq 0 \), we have that either \( \Re(\omega_1(v_1, w_1)) \neq 0 \), or \( \Re(\omega_1(v_1, w_1)) = 0 \) and \( \Im(\omega_1(v_1, w_1)) \neq 0 \).

If \( \Re(\omega_1(v_1, w_1)) \neq 0 \), then let

\[ v''_1 = v'_1 + \frac{\omega_1(v'_1, w'_2)}{\omega_1(v'_1, w'_1)} v'_2, \quad v''_2 = J_1J_2v''_2, \]

\[ w''_1 = w'_1, \quad w''_2 = w'_2. \]

The set \( \{v''_1, v''_2, w''_1, w''_2\} \) is still a basis for \( V \). We know that

\[ v''_2 = J_1J_2v''_1 = J_1J_2 \left( v'_1 + \frac{\omega_1(v'_1, w'_2)}{\omega_1(v'_1, w'_1)} v'_2 \right) \]

\[ = v'_2 + \frac{\omega_1(v'_1, w'_2)}{\omega_1(v'_1, w'_1)} (J_1J_2)^2 v'_1 = v'_2 - \frac{\omega_1(v'_1, w'_2)}{\omega_1(v'_1, w'_1)} v'_1. \]

Then we have the following properties

\[ \omega_1(v''_1, w''_2) = \omega_1 \left( v'_1 + \frac{\omega_1(v'_1, w'_2)}{\omega_1(v'_1, w'_1)} v'_2, w'_2 \right) = \omega_1(v'_1, w'_2) + \frac{\omega_1(v'_1, w'_2)}{\omega_1(v'_1, w'_1)} \omega_1(v'_2, w'_2) \]

\[ = \omega_1(v'_1, w'_2) - \frac{\omega_1(v'_1, w'_2)}{\omega_1(v'_1, w'_1)} \omega_1(v'_1, w'_1) = 0, \]
The set \( \{ \omega \} \) then we have the following properties
\[
\omega_1(v''_1, w''_1) = \omega_1 \left( v'_1 + \frac{\omega_1(v'_1, w'_2)}{\omega_1(v'_1, w'_1)} v'_2, w'_1 \right) = \omega_1(v'_1, w'_1) + \frac{\omega_1(v'_1, w'_2)}{\omega_1(v'_1, w'_1)} \omega_1(v'_2, w'_1),
\]
and \( \omega_1(v''_1, w''_1) = \omega_1(v''_1, w''_1) = 0 \). Clearly, as \( \omega_1 \) and \( \omega_2 \) are non-degenerate, we have that \( \omega_1(v''_1, w''_1) \neq 0 \). Thus we have as well that
\[
\omega_1(v''_1, w''_1) = \omega_2(v_1, w_2) = \omega_2(v_2, w_1) = -\omega_1(v_2, w_2) \neq 0
\]
and
\[
\omega_2(v''_1, w''_1) = -\omega_1(v''_1, w''_1) = -\omega_1(v''_1, w''_1) = -\omega_2(v_2, w_2) = 0.
\]
In addition,
\[
av''_1 = a \left( v'_1 + \frac{\omega_1(v'_1, w'_2)}{\omega_1(v'_1, w'_1)} v'_2 \right) = \lambda v'_1 + \mu v'_2 + \frac{\omega_1(v'_1, w'_2)}{\omega_1(v'_1, w'_1)} (\pm \mu v'_1 + \lambda v'_2) = \lambda v''_1 + \mu v''_2,
\]
\[
av''_2 = a \left( v'_2 - \frac{\omega_1(v'_1, w'_2)}{\omega_1(v'_1, w'_1)} v'_1 \right) = \pm \mu v'_1 + \lambda v'_2 - \frac{\omega_1(v'_1, w'_2)}{\omega_1(v'_1, w'_1)} (\lambda v'_1 + \mu v'_2) = \pm \mu v''_1 + \lambda v''_2.
\]
and as \( w''_1 = w'_1 \) and \( w''_2 = w'_2 \), their relationship remains unchanged.

Now we consider the other possibility. If \( \Re(\omega_1(v_1, w_1)) = 0 \), then \( \Im(\omega_1(v_1, w_1)) \neq 0 \). Then let
\[
v''_1 = v'_1 + v'_2, \quad v''_2 = J_1 J_2(v''_1),
\]
\[
w''_1 = w'_1 + w'_2, \quad w''_2 = J_1 J_2(w''_1).
\]
The set \( \{ v''_1, v''_2, w''_1, w''_2 \} \) is still a basis for \( V \). We also have that
\[
v''_2 = J_1 J_2 v''_1 = J_1 J_2(v'_1 + v'_2) = v'_2 + (J_1 J_2)^2 v'_1 = v'_2 - v'_1,
\]
\[
w''_2 = J_1 J_2 w''_1 = J_1 J_2(w'_1 + w'_2) = w'_2 + (J_1 J_2)^2 w'_1 = w'_2 - w'_1.
\]
Then we have the following properties
\[
\omega_1(v''_1, w''_2) = \omega_1(v'_1 + v'_2, w'_2 - w'_1) = \omega_1(v'_1, w'_2) - \omega_1(v'_1, w'_1) + \omega_1(v'_2, w'_2) - \omega_1(v'_2, w'_1) = 0
\]
\[
\omega_1(v''_1, w''_1) = \omega_1(v'_1 + v'_2, w'_1 + w'_2) = \omega_1(v'_1, w'_1) + \omega_1(v'_1, w'_2) + \omega_1(v'_2, w'_1) + \omega_1(v'_2, w'_2)
\]
\[
= 2 \omega_1(v'_1, w'_2)
\]
and \( \omega_1(v''_1, v''_2) = \omega_1(w''_1, w''_2) = 0 \). Then \( \omega_1(v''_1, w''_1) \neq 0 \) and we have that
\[
\omega_1(v''_1, w''_1) = \omega_2(v_1, w_2) = \omega_2(v_2, w_1) = -\omega_1(v_2, w_2) \neq 0
\]
\[
\omega_2(v''_1, w''_1) = -\omega_1(v''_1, w''_1) = -\omega_1(v''_1, w''_1) = -\omega_2(v_2, w_2) = 0.
\]
In addition, 
\[ av''_1 = a(v'_1 + v'_2) = \lambda v'_1 \mp \mu v'_2 \pm \mu v'_1 + \lambda v'_2 = \lambda v''_1 \mp \mu v''_2, \]
\[ av''_2 = a(v'_2 - v'_1) = \pm \mu v'_2 + \lambda v'_1 \mp \mu v'_2 = \pm \mu v''_1 + \lambda v''_2, \]
and similarly
\[ aw''_1 = -\lambda w''_1 \pm \mu w''_2, \quad aw''_2 = \mp \mu w''_1 - \lambda w''_2. \]

In either case, we know that
\[ \omega_1(v''_1, w''_1) = \omega_2(v_2, w_2) = -\omega_1(v''_2, w''_2) \neq 0. \]

If \( \omega_1(v''_1, w''_1) < 0 \), then replace \( v''_1 \) with \( -v''_1 \) and relabel. This will make
\[ \omega_1(v''_1, w''_1) = \omega_2(v_2, w_2) = -\omega_2(v_1, w_1) = -\omega_1(v''_2, w''_2) > 0, \]
but will yield that \( av''_1 = \lambda v''_1 \pm \mu v''_2 \) and \( av''_2 = \mp \mu v''_1 + \lambda v''_2 \). On the other hand, if \( \omega_1(v''_1, w''_1) > 0 \),
then replace \( v''_2 \) with \( -v''_2 \) and relabel. This will again give us the same situation as above.

Finally, we pick a basis, \( \beta \), for \( V \) in the following way
\[ \beta = \left\{ \frac{1}{\sqrt{\omega_1(v''_1, w''_1)}} v''_1, \frac{1}{\sqrt{\omega_1(v''_2, w''_1)}} v''_2, \frac{1}{\sqrt{\omega_1(v''_1, w''_2)}} v''_1, \frac{1}{\sqrt{\omega_1(v''_2, w''_2)}} w''_2 \right\} \]
and relabel the vectors \( v''_1, v''_2, w''_1, \) and \( w''_2 \) respectively. Then the matrix
\[ W = \begin{pmatrix} v''_1 & v''_2 & w''_1 & w''_2 \end{pmatrix} \]
is in \( H(J_2) \) because \( W^t J_i W = J_i \) for all \( i \in \{1, 2\} \). In this basis, \( a \) is of form (3) in Theorem 2.3.23 with \( \lambda \neq 0 \). The \( \varepsilon \) is present because by Lemma 2.3.34 the two cases are not \( \Theta \)-symplectically similar.

**Case 4:** \( V_c = K_{\mu i} \oplus K_{-\mu i} \)

We also assume that \( \mu \in \mathbb{R} \) such that \( \mu > 0 \). In addition, by Lemma 2.3.33 we have that \( a \) is diagonalizable.

Now let \( v_1 \in K_{\mu i} \) be an eigenvector. As \( v_1 \) is complex, it is not clear that \( J_1 J_2 v_1 \) is linearly independent of \( v_1 \). If \( v_1 \) and \( J_1 J_2 v_1 \) are linearly independent, then let \( v'_1 = v_1 + i J_1 J_2 v_1 \) and
Let \( v_2 = v_1 - iJ_1J_2v_1 \). Then \( v_1' \) and \( v_2' \) are linearly independent and such that

\[
iJ_1J_2v_1' = v_1', \quad \text{and} \quad -iJ_1J_2v_2' = v_2'.
\]

This implies by Proposition 2.3.20 that for all \( v_1 \) and \( v_2 \) as are \( \omega \) and \( \omega \), then we have that \( \omega(v_1', v_2') = \omega(v_1', v_2') = 0 \). Furthermore

\[
w_1' = \overline{v_1'} = v_1 + iJ_1J_2v_1 = v_1 + iJ_1J_2v_1 = \overline{v_1} - iJ_1J_2\overline{v_1}
\]

\[
w_2' = \overline{v_2'} = v_1 - iJ_1J_2v_1 = \overline{v_1} - iJ_1J_2\overline{v_1} + iJ_1J_2\overline{v_1}
\]

Now consider the situation where \( v_1 \) and \( J_1J_2v_1 \) are linearly dependent, then \( \pm iJ_1J_2v_1 = v_1 \). Let \( v_2 \in K_{\mu} \) be another eigenvector linearly independent of \( v_1 \). If \( J_1J_2v_2 \) and \( v_2 \) are linearly independent then we proceed as above using \( v_2 \) as \( v_1 \). If \( J_1J_2v_2 \) and \( v_2 \) are linearly dependent, then we know that \( \pm iJ_1J_2v_2 = v_2 \).
If the sign on \( i \) is different between the two above equations, then the two vectors

\[ v_1 + v_2 \quad \text{and} \quad J_1 J_2 (v_1 + v_2) \]

are linearly independent. To see this, write \( \pm i J_1 J_2 v_1 = v_1 \), then \( \mp i J_1 J_2 v_2 = v_2 \). Then

\[ J_1 J_2 (v_1 + v_2) = J_1 J_2 v_1 + J_1 J_2 v_2 = \mp iv_1 \pm iv_2 = \mp i(v_1 - v_2) \]

which is clearly independent of \( v_1 + v_2 \). If this is the case, we relabel \( v_1 + v_2 \) as simply \( v_1 \) and proceed as in the situation where \( v_1 \) and \( J_1 J_2 v_1 \) were linearly independent to begin with.

If the sign on \( i \) is the same in the two equations, then let \( v'_1 = v_1 + iv_2 \) and \( v'_2 = v_1 - iv_2 \). Then \( v'_1 \) and \( v'_2 \) are linearly independent and \( \pm i J_1 J_2 v'_1 = v'_1 \) and \( \mp i J_1 J_2 v'_2 = v'_2 \). If \( -i J_1 J_2 v'_1 = v'_1 \), then relabel \( v'_1 \) and \( v'_2 \) as \( v'_2 \) and \( v'_1 \) respectively. After making this change if necessary, this ensures that

\[ i J_1 J_2 v'_1 = v'_1, \quad \text{and} \quad -i J_1 J_2 v'_2 = v'_2. \]

Then we proceed as in the argument above after it reached this relation.

This covers every possible decomposition of \( V \) or \( V_c \) into generalized eigenspaces of \( a \) and so completes the proof of Theorem 2.3.23.
Chapter 3

A SAMPLE OF THE CLASSIFICATION OF DIMENSION SEVEN

As has been stated before, the method we use to find all the possible solvable indecomposable Lie algebras with codimension one nilradicals focuses on the nilradical of the algebra. The possible nilradicals of a Lie algebra of dimension \( n \) are all the nilpotent algebras of dimension \( n - 1 \). We reference the classifications of Winternitz and Gong for a list of possible nilpotent algebras of dimensions one through six and give the list in Appendix A [7, 1]. Since much of the classification of these algebras is the same from one nilradical to the next, we offer in this section the step by step classification of the seven dimensional algebras that stem from four nilradicals of dimension six. The derivations of the first nilradical form a solvable Lie algebra. The semi-simple part of the Lie algebra formed by the derivations of the second nilradical is isomorphic to \( \mathfrak{sl}(2, \mathbb{R}) \). The semi-simple part of the Lie algebra formed by the derivations of the third nilradical is isomorphic to \( \mathfrak{sp}(4, \mathbb{R}) \). And the semi-simple part of the Lie algebra formed by the derivations of the fourth nilradical is isomorphic to the representation of \( \mathfrak{so}(3, 1, \mathbb{R}) \) discussed in Chapter 2. These four nilradicals are representative of the different situations we have to deal with when classifying these algebras through dimension seven, and as such, the classification of the algebras from the other nilradicals are similar. In Appendix B we give a complete table of all solvable indecomposable Lie algebras with codimension one nilradicals from dimension two through dimension seven.
3.1 A Solvable Derivation Algebra

In this section, we classify those seven-dimensional algebras, \( g \), whose nilradical, \( NR(g) \), is isomorphic to one with structure equations

\[
[f_2, f_5] = f_1, \quad [f_3, f_4] = -f_1, \quad [f_3, f_6] = f_2, \quad [f_4, f_6] = f_3, \quad [f_5, f_6] = f_4.
\]

This is Nilradical 16 listed under the six dimensional nilradicals in Appendix \( A \). Let \( \{f_1, \ldots, f_6\} \) be a basis for the nilradical and pick the vectors so that they have the above structure equations. Complete this to a basis for \( g \) by including a vector \( f_7 \not\in NR(g) \). Then as \( Dg \in NR(g) \), we can view \( ad(f_7) \) as a transformation on the nilradical. After requiring the algebra to satisfy the Jacobi property, we have that \( ad(f_7) \) is of the form

\[
\begin{pmatrix}
2x_5 + 3y_6 & b_1 & c_1 & d_1 & x_1 & y_1 \\
0 & x_5 + 3y_6 & c_2 & 0 & x_2 & -d_1 \\
0 & 0 & x_5 + 2y_6 & c_2 & 0 & c_1 \\
0 & 0 & 0 & x_5 + y_6 & c_2 & -b_1 \\
0 & 0 & 0 & 0 & x_5 & 0 \\
0 & 0 & 0 & 0 & 0 & y_6
\end{pmatrix}.
\]

The other nonzero \( ad \) matrices are

\[
\begin{align*}
ad(f_2) &= \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \\
ad(f_3) &= \begin{pmatrix}
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \\
ad(f_4) &= \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \\
ad(f_5) &= \begin{pmatrix}
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \\
ad(f_6) &= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

This will allow us to zero out the \( x_1, d_1, c_1, b_1, \) and \( c_2 \) positions in one simple perturbing of \( f_7 \).

Then the only entries in \( ad(f_7) \) that we don’t have under control are \( x_5, y_6, x_2 \) and \( y_1 \). We’ll find that if \( x_5 \) and \( y_6 \) were simultaneously 0, then algebra becomes nilpotent, so clearly that cannot
happen. However, to deal with the other two entries, we now turn our attention to computing the automorphisms of the nilradical.

By using Maple to compute the derivations and exponentiate them, we find that the automorphism group of the nilradical is generated by the following nine one-parameter groups of transformations.

\[
\begin{align*}
A_1 &= \begin{pmatrix} (s_1)^3 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & s_1 & 0 & 0 & 0 \\
0 & 0 & 0 & (s_1)^2 & 0 & 0 \\
0 & 0 & 0 & 0 & (s_1)^3 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{s_1} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & s_2 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -s_2 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
A_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\
0 & (s_3)^3 & 0 & 0 & 0 & 0 \\
0 & 0 & s_3 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{s_3} & 0 & 0 \\
0 & 0 & 0 & 0 & (\frac{1}{s_3})^3 & 0 \\
0 & 0 & 0 & 0 & 0 & (s_3)^2 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 0 & s_4 & 0 & 0 & \frac{(s_3)^2}{2} \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & s_4 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
A_5 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & s_5 & (s_3)^3 & 0 & 0 \\
0 & 0 & 1 & s_5 & \frac{(s_3)^3}{6} & 0 \\
0 & 0 & 0 & 1 & s_5 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_6 = \begin{pmatrix} 1 & 0 & 0 & s_6 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -s_6 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
A_7 &= \begin{pmatrix} 1 & 0 & 0 & 0 & s_7 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & s_8 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
A_9 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & s_9 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
\end{align*}
\]

However, conjugation by \(A_2, A_4, A_5, A_6, \) and \(A_7\) will only affect the positions that we’ll zero out by a basis change. As such, they become less useful in simplifying the \(\text{ad}(f_7)\) matrix.

As there is no semisimple part of the derivation algebra, we use only a single parent case.
3.1.1 Parent Case 1:

We start with the ad \( f_7 \) matrix of the form

\[
\operatorname{ad} (f_7) = \begin{pmatrix}
2x_5 + 3y_6 & b_1 & c_1 & d_1 & x_1 & y_1 \\
0 & x_5 + 3y_6 & c_2 & 0 & x_2 & -d_1 \\
0 & 0 & x_5 + 2y_6 & c_2 & x_3 & 0 \\
0 & 0 & 0 & x_5 + y_6 & c_2 & -b_1 \\
0 & 0 & 0 & 0 & x_5 & 0 \\
0 & 0 & 0 & 0 & 0 & y_6
\end{pmatrix}.
\]

Conjugate \( \operatorname{ad} (f_7) \) by \( A_8 \) and note that if \( y_6 \neq 0 \), then we could let \( s_8 = -\frac{x_5}{3y_6} \) which would zero out the \( x_2 \) position. That is, if we apply the automorphism \( A_8 \) with \( s_8 = -\frac{x_5}{3y_6} \), the resulting \( \operatorname{ad} (f_7) \) matrix would have a 0 in the \( x_2 \) position. It also happens to change the \( x_1 \) position, but we will simply relabel what ends up in that position as \( x_1 \). We’ll usually use this same general method when conjugating by an automorphism. At any rate, this yields two possible cases.

1. In this first case, either \( y_6 \neq 0 \) and we moved the \( x_2 \) position to 0, or \( y_6 = 0 \) and \( x_2 = 0 \) already.

2. In this second case, \( y_6 = 0 \), but \( x_5 \neq 0 \).

Subcase 1:

The resulting \( \operatorname{ad} (f_7) \) matrix is as follows

\[
\operatorname{ad} (f_7) = \begin{pmatrix}
2x_5 + 3y_6 & b_1 & c_1 & d_1 & x_1 & y_1 \\
0 & x_5 + 3y_6 & c_2 & 0 & 0 & -d_1 \\
0 & 0 & x_5 + 2y_6 & c_2 & x_3 & 0 \\
0 & 0 & 0 & x_5 + y_6 & c_2 & -b_1 \\
0 & 0 & 0 & 0 & x_5 & 0 \\
0 & 0 & 0 & 0 & 0 & y_6
\end{pmatrix}.
\]

Next conjugate by \( A_9 \). In similar manner as above, if \( x_5 \neq -y_6 \), then we can pick \( s_9 = -\frac{y_1}{2(y_6 + x_5)} \) and this conjugation will result in moving \( y_1 \) to 0. Thus we have another two cases.

1. \( x_5 \neq -y_6 \) and we can move \( y_1 \) to 0, or \( x_5 = -y_6 \) and \( y_1 = 0 \) already.

2. \( x_5 = -y_6 \), but \( y_1 \neq 0 \).
Subcase 1.1:

We have \( \text{ad}(f_7) \) as follows

\[
\text{ad}(f_7) = \begin{pmatrix}
2x_5 + 3y_6 & b_1 & c_1 & d_1 & x_1 & 0 \\
0 & x_5 + 3y_6 & c_2 & 0 & 0 & -d_1 \\
0 & 0 & x_5 + 2y_6 & c_2 & 0 & c_1 \\
0 & 0 & 0 & x_5 + y_6 & c_2 & -b_1 \\
0 & 0 & 0 & 0 & x_5 & 0 \\
0 & 0 & 0 & 0 & 0 & y_6
\end{pmatrix}.
\]

We already know that \( x_5 \) and \( y_6 \) are not simultaneously 0. This allows us to bifurcate on this as well.

1. \( y_6 \neq 0 \).

2. \( y_6 = 0 \), which implies that \( x_5 \neq 0 \) or the algebra is nilpotent.

Subcase 1.1.1:

In this section, simply make the basis change

\[
e_i = f_i \quad \text{for} \quad 1 \leq i \leq 6, \quad \text{and} \quad e_7 = -\frac{1}{y_6}(f_7 - x_1f_2 + d_1f_3 - c_1f_4 + b_1f_5 + c_2f_6)
\]

and let \( a = \frac{x_5}{y_6} \). Notice how we perturbed \( f_7 \). This makes use of the other \( \text{ad} \) matrices and zeros out the \( x_1, d_1, c_1, b_1, \) and \( c_2 \) positions. This yields the structure equations

\[
\begin{align*}
[e_2, e_5] &= e_1, & [e_4, e_6] &= e_2, & [e_4, e_6] &= e_3, & [e_4, e_6] &= e_3, \\
[e_1, e_7] &= (2a + 3)e_1, & [e_2, e_7] &= (a + 3)e_2, & [e_3, e_7] &= (a + 2)e_3, & [e_4, e_7] &= (a + 1)e_4, \\
[e_5, e_7] &= ae_5, & [e_6, e_7] &= e_6,
\end{align*}
\]

with \( a \in \mathbb{R} \). In the table in Appendix B this is \([7,[6,16],1,1]\).

Subcase 1.1.2:

Here we assumed that \( y_6 = 0 \) and so we have the \( \text{ad}(f_7) \) matrix

\[
\text{ad}(f_7) = \begin{pmatrix}
2x_5 & b_1 & c_1 & d_1 & x_1 & 0 \\
0 & x_5 & c_2 & 0 & 0 & -d_1 \\
0 & 0 & x_5 & c_2 & 0 & c_1 \\
0 & 0 & 0 & x_5 & c_2 & -b_1 \\
0 & 0 & 0 & 0 & x_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
Then we make the basis change

\[ e_i = f_i \text{ for } 1 \leq i \leq 6, \text{ and } e_7 = -\frac{1}{x_5}(f_7 - x_1f_2 + d_1f_3 - c_1f_4 + b_1f_5 + c_2f_6), \]

which yields the structure equations

\[ [e_2,e_5] = e_1, \quad [e_3,e_6] = e_2, \quad [e_4,e_6] = e_3, \quad [e_4,e_6] = e_3, \quad [e_1,e_6] = 2e_1, \]
\[ [e_2,e_7] = e_2, \quad [e_3,e_7] = e_3, \quad [e_4,e_7] = e_4, \quad [e_5,e_7] = e_5. \]

This is \([7,[6,16],1,2]\) in the table.

**Subcase 1.2:**

In this section, we assumed that \(x_5 = -y_6\) and \(y_1 \neq 0\). This will yield the \(\text{ad}(f_7)\) matrix

\[
\text{ad}(f_7) = \begin{pmatrix}
y_6 & b_1 & c_1 & d_1 & x_1 & y_1 \\
0 & 2y_6 & c_2 & 0 & 0 & -d_1 \\
0 & 0 & y_6 & c_2 & 0 & c_1 \\
0 & 0 & 0 & 0 & c_2 & -b_1 \\
0 & 0 & 0 & 0 & -y_6 & 0 \\
0 & 0 & 0 & 0 & 0 & y_6
\end{pmatrix},
\]

and implies that \(y_6 \neq 0\) or the algebra is nilpotent. At this point, we conjugate by \(A_3\). As \(y_6\) and \(y_1\) are both nonzero, we can pick \(s_3 = \frac{\sqrt{|y_1y_6|}}{y_1}\), which will scale the \(y_1\) position to \(\pm y_6\). This also requires us to relabel some of the entries of \(\text{ad}(f_7)\) as we did previously. This is how we'll usually deal with an automorphism that scales the entries of \(\text{ad}(f_7)\). Then we make the basis change

\[ e_i = f_i \text{ for } 1 \leq i \leq 6, \text{ and } e_7 = -\frac{1}{y_6}(f_7 - x_1f_2 + d_1f_3 - c_1f_4 + b_1f_5 + c_2f_6) \]

and let \(\varepsilon = \pm 1\). This will yield the structure equations

\[ [e_2,e_5] = e_1, \quad [e_3,e_6] = e_2, \quad [e_4,e_6] = e_3, \quad [e_4,e_6] = e_3, \quad [e_1,e_6] = e_1, \]
\[ [e_2,e_7] = 2e_2, \quad [e_3,e_7] = e_3, \quad [e_5,e_7] = -e_5, \quad [e_6,e_7] = \varepsilon e_1 + e_6. \]

with \(\varepsilon^2 = 1\). In the table, this is \([7,[6,16],1,3]\).
Subcase 2:

In this section, we assumed that $y_6 = 0$ and $x_2 \neq 0$. This gives us the ad ($f_7$) matrix

\[
ad (f_7) = \begin{pmatrix}
2x_5 & b_1 & c_1 & d_1 & x_1 & y_1 \\
0 & x_5 & c_2 & 0 & x_2 & -d_1 \\
0 & 0 & x_5 & c_2 & 0 & c_1 \\
0 & 0 & 0 & x_5 & c_2 & -b_1 \\
0 & 0 & 0 & 0 & x_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

and implies that $x_5 \neq 0$ or the algebra is nilpotent. We now conjugate by $A_9$. As $x_5 \neq 0$, we pick $s_9 = -\frac{y_1}{x_5}$, which will zero out the $y_1$ position. Then we conjugate by $A_1$, and as $x_5$ and $x_2$ are both nonzero, we can let $s_1 = \frac{\sqrt{x_5(x_2)^2}}{x_2}$, which will scale the $x_2$ position to $x_5$. Finally, we make the basis change

\[e_i = f_i \quad \text{for} \quad 1 \leq i \leq 6, \quad \text{and} \quad e_7 = -\frac{1}{x_5}(f_7 - x_1f_2 + d_1f_3 - c_1f_4 + b_1f_5 + c_2f_6).
\]

This gives us the structure equations

\[\begin{align*}
[e_2, e_5] &= e_1, & [e_3, e_6] &= e_2, & [e_4, e_6] &= e_3, & [e_4, e_5] &= e_3, & [e_1, e_6] &= 2e_1, \\
[e_2, e_7] &= e_2, & [e_3, e_7] &= e_3, & [e_4, e_7] &= e_4, & [e_5, e_7] &= e_2 + e_5.
\end{align*}\]

This is $[7,[6,16],1,4]$ in the table and completes the classification of seven dimensional algebras with this nilradical.

Now that the reader has the idea of how we use the automorphisms to simplify the ad ($f_7$) matrix, we will, to conserve space, refrain from giving the explicit value of the automorphism parameter that will make the desired change, but simply state its existence and the parameter it pivots on. We now move on to the classification of an algebra with a nilradical whose derivation algebra has a semisimple part isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

### 3.2 A Derivation Algebra with Semisimple Part Isomorphic to $\mathfrak{sl}(2, \mathbb{R})$

In this section, we classify those algebras, $\mathfrak{g}$, with nilradical, $\mathcal{N}R(\mathfrak{g})$, isomorphic to the six dimensional nilpotent algebra with structure equations

\[\begin{align*}
[f_4, f_5] &= f_2, & [f_4, f_6] &= f_3, & [f_5, f_6] &= f_4.
\end{align*}\]
This is Nilradical 4 from the list of six dimensional nilradicals in Appendix A. We will also assume that we have a basis for \( g \) such that \( \{ f_1, \ldots, f_6 \} \) forms a basis for \( NR(g) \) and has the structure equations given above. Let the last vector in \( g \), \( f_7 \), be an arbitrary vector not contained in \( NR(g) \).

By the Jacobi property, the ad matrix of \( f_7 \) must be of the form

\[
\text{ad} (f_7) = \begin{pmatrix}
a_1 & 0 & 0 & 0 & x_1 & y_1 \\
a_2 & 2x_5 + y_6 & y_5 & -y_4 & x_2 & y_2 \\
a_3 & x_6 & x_5 + 2y_6 & x_4 & x_3 & y_3 \\
0 & 0 & 0 & 0 & x_5 + y_6 & y_4 \\
0 & 0 & 0 & 0 & 0 & x_6 \\
0 & 0 & 0 & 0 & 0 & y_6 \\
\end{pmatrix}.
\]

First, we look at the other nonzero ad matrices. They are

\[
\text{ad} (f_4) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
\text{ad} (f_5) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
\text{ad} (f_6) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

This will allow us, by perturbing \( f_7 \), to annihilate, \( y_4, x_4 \), and, if we can move the \( x_2 \) value to the \( y_3 \) value, both of these as well, by perturbing \( f_7 \).

We now take a look at the automorphisms. Again we use Maple to compute a basis for the derivation algebra and to compute the Levi decomposition of it. Then we find that a basis for the semisimple part is

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

Label these as \( D_1, D_2, D_3 \) respectively. These three vectors form a basis for a subalgebra isomorphic to \( \mathfrak{sl}(2, \mathbb{R}) \). This is clear by matching these basis vectors with the corresponding standard basis vectors of \( \mathfrak{sl}(2, \mathbb{R}) \),

\[
D_1 \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad D_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D_3 \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]
However, we can use another basis vector, this time from the radical of the derivation algebra, namely,

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix},
\]

which we’ll call \(D_4\), to do the following. Note that

\[
\frac{1}{2}(D_2 + D_4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad -\frac{1}{2}(D_2 - D_4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.
\]

Then the subalgebra with basis \(\{\frac{1}{2}(D_2 + D_4), D_1, D_3, -\frac{1}{2}(D_2 - D_4)\}\) is isomorphic to \(\text{gl}(2, \mathbb{R})\) via the map

\[
\frac{1}{2}(D_2 + D_4) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D_1 \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad D_3 \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad -\frac{1}{2}(D_2 - D_4) \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]

which is easily checked, as the structure equations of the two algebras are the same.

Note that the lower right hand \(2 \times 2\) blocks of the derivation vectors are identical to the basis vectors of \(\text{gl}(2, \mathbb{R})\). This leads us to suspect that if we exponentiate the derivation vectors, then we can find a nilradical automorphism that has an arbitrary \(GL(2, \mathbb{R})\) matrix in the lower right hand \(2 \times 2\) block. And sure enough, we use Maple to check that the matrix

\[
A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a(ad - bc) & b(ad - bc) & 0 & 0 & 0 \\ 0 & c(ad - bc) & d(ad - bc) & 0 & 0 & 0 \\ 0 & 0 & 0 & ad - bc & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & c & d \end{pmatrix}
\]

is an automorphism of the nilradical.

We next use Maple to compute a complete basis for the derivation algebra and exponentiate it. Then we find that the following one-parameter groups of transformations generate the automorphism group of the nilradical.

\[
A_1 = \begin{pmatrix} s_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ s_2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},
\]
Note that $A_8$ and $A_9$ could be used to zero out the $y_4$ and $x_4$ positions. As we can do this with a simple perturbing of $f_7$, these automorphisms will be less useful than the others in simplifying $ad(f_7)$.

Now, we start classifying the possible forms of $ad(f_7)$. First, we note that by block multiplication,
we can conjugate ad \((f_7)\) by \(A_0\) and pick \(a, b, c, d\) to put the lower right hand \(2 \times 2\) block of ad \((f_7)\) into real Jordan form. This yields three parent cases depending on the real Jordan form of this \(2 \times 2\) block. They are

1. \[
\begin{pmatrix}
0 & 0 & 0 & 0 & x_1 & y_1 \\
0 & 2\lambda_1 + \lambda_2 & 0 & -y_4 & x_2 & y_2 \\
0 & 0 & \lambda_1 + 2\lambda_2 & x_4 & x_3 & y_3 \\
0 & 0 & 0 & \lambda_1 + \lambda_2 & x_4 & y_4 \\
0 & 0 & 0 & 0 & 0 & \lambda_2
\end{pmatrix},
\]

2. \[
\begin{pmatrix}
0 & 0 & 0 & 0 & x_1 & y_1 \\
0 & 3\lambda_1 & \lambda_2 & -y_4 & x_2 & y_2 \\
0 & -\lambda_2 & 3\lambda_1 & x_4 & x_3 & y_3 \\
0 & 0 & 0 & 2\lambda_1 & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda_1 & \lambda_2 \\
0 & 0 & 0 & 0 & -\lambda_2 & \lambda_1
\end{pmatrix},
\] with \(\lambda_2 \neq 0\) and the eigenvalues ordered so that \(\lambda_1 \geq \lambda_2 \geq 0\).

3. \[
\begin{pmatrix}
0 & 0 & 0 & 0 & x_1 & y_1 \\
0 & 3\lambda & 1 & -y_4 & x_2 & y_2 \\
0 & 0 & 3\lambda & x_4 & x_3 & y_3 \\
0 & 0 & 0 & 2\lambda & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda & 1 \\
0 & 0 & 0 & 0 & 0 & \lambda
\end{pmatrix}.
\]

### 3.2.1 Parent Case 1:

We consider here, the ad \((f_7)\) matrix

\[
\text{ad} (f_7) = \begin{pmatrix}
0 & 0 & 0 & 0 & x_1 & y_1 \\
0 & 2\lambda_1 + \lambda_2 & 0 & -y_4 & x_2 & y_2 \\
0 & 0 & \lambda_1 + 2\lambda_2 & x_4 & x_3 & y_3 \\
0 & 0 & 0 & \lambda_1 + \lambda_2 & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2
\end{pmatrix}.
\]

We note that if \(a_1, \lambda_1, \lambda_2\) are all simultaneously 0, then the algebra is nilpotent.

We start by conjugating by \(A_2\), which will allow us to move the \(a_2\) position to 0 if \(a_1 \neq 2\lambda_1 + \lambda_2\).

We have two cases then.

1. \(a_1 \neq 2\lambda_1 + \lambda_2\) and we move \(a_2\) to 0, or \(a_1 = 2\lambda_1 + \lambda_2\) and \(a_2 = 0\) already.

2. \(a_1 = 2\lambda_1 + \lambda_2\), but \(a_2 \neq 0\).
Subcase 1.1:

In this section, the $a_2$ position is 0, which makes the ad ($f_7$) matrix

$$\text{ad} (f_7) = \begin{pmatrix}
a_1 & 0 & 0 & 0 & x_1 & y_1 \\
0 & 2\lambda_1 + \lambda_2 & 0 & -y_4 & x_2 & y_2 \\
a_3 & 0 & \lambda_1 + 2\lambda_2 & x_4 & x_3 & y_3 \\
0 & 0 & 0 & \lambda_1 + \lambda_2 & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2
\end{pmatrix}.$$  

Next we conjugate by $A_3$ and find that we can move $a_3$ to 0 if $a_1 \neq \lambda_1 + 2\lambda_2$. This gives us another two cases.

1. $a_1 \neq \lambda_1 + 2\lambda_2$ and we move $a_3$ to 0, or $a_1 = \lambda_1 + 2\lambda_2$ and $a_3 = 0$ already.

2. $a_1 = \lambda_1 + 2\lambda_2$, but $a_3 \neq 0$.

Subcase 1.1.1:

Here we have

$$\text{ad} (f_7) = \begin{pmatrix}
a_1 & 0 & 0 & 0 & x_1 & y_1 \\
0 & 2\lambda_1 + \lambda_2 & 0 & -y_4 & x_2 & y_2 \\
0 & 0 & \lambda_1 + 2\lambda_2 & x_4 & x_3 & y_3 \\
0 & 0 & 0 & \lambda_1 + \lambda_2 & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2
\end{pmatrix}.$$  

We conjugate by $A_{10}$ at this point and find that if $a_1 \neq \lambda_1$, then we can move the $x_1$ position to 0.

This yields two cases.

1. $a_1 \neq \lambda_1$ and we move $x_1$ to 0, or $a_1 = \lambda_1$ and $x_1 = 0$ already.

2. $a_1 = \lambda_1$, but $x_1 \neq 0$.

Subcase 1.1.1.1:

In this subcase, we have the ad ($f_7$) matrix as

$$\text{ad} (f_7) = \begin{pmatrix}
a_1 & 0 & 0 & 0 & x_1 & y_1 \\
0 & 2\lambda_1 + \lambda_2 & 0 & -y_4 & x_2 & y_2 \\
0 & 0 & \lambda_1 + 2\lambda_2 & x_4 & x_3 & y_3 \\
0 & 0 & 0 & \lambda_1 + \lambda_2 & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2
\end{pmatrix}.$$  

We next conjugate by $A_{13}$. This allows us to move the $y_1$ position to 0, if $a_1 \neq \lambda_2$ and hence yields two cases.
1. $a_1 \neq \lambda_2$ and we move $y_1$ to 0, or $a_1 = \lambda_2$ and $y_1 = 0$ already.

2. $a_1 = \lambda_1$, but $y_1 \neq 0$.

**Subcase 1.1.1.1.1:**

Here we have the $\text{ad}(f_7)$ matrix

$$\text{ad}(f_7) = \begin{pmatrix}
a_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2\lambda_1 + \lambda_2 & 0 & -y_4 & x_2 & y_2 \\
0 & 0 & \lambda_1 + 2\lambda_2 & x_4 & x_3 & y_3 \\
0 & 0 & 0 & \lambda_1 + \lambda_2 & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2
\end{pmatrix}.$$  

We conjugate by $A_{14}$ and see that if $\lambda_1 \neq 0$, we can move the $y_2$ position to 0. We again have two cases.

1. $\lambda_1 \neq 0$ and we move $y_2$ to 0, or $\lambda_1 = 0$ and $y_2 = 0$ already.

2. $\lambda_1 = 0$, but $y_2 \neq 0$.

**Subcase 1.1.1.1.1.1:**

In this section, we have the $\text{ad}(f_7)$ matrix

$$\text{ad}(f_7) = \begin{pmatrix}
a_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2\lambda_1 + \lambda_2 & 0 & -y_4 & x_2 & 0 \\
0 & 0 & \lambda_1 + 2\lambda_2 & x_4 & x_3 & y_3 \\
0 & 0 & 0 & \lambda_1 + \lambda_2 & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2
\end{pmatrix}.$$  

In this section, we conjugate by $A_{12}$ and find that we can move the $x_3$ position to 0, if $\lambda_2 \neq 0$. This yields yet another two cases.

1. $\lambda_2 \neq 0$ and we move $x_3$ to 0, or $\lambda_2 = 0$ and $x_3 = 0$ already.

2. $\lambda_2 = 0$, but $x_3 \neq 0$.

**Subcase 1.1.1.1.1.1.1:**

In this section, we have the $\text{ad}(f_7)$ matrix

$$\text{ad}(f_7) = \begin{pmatrix}
a_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2\lambda_1 + \lambda_2 & 0 & -y_4 & x_2 & 0 \\
0 & 0 & \lambda_1 + 2\lambda_2 & x_4 & 0 & y_3 \\
0 & 0 & 0 & \lambda_1 + \lambda_2 & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2
\end{pmatrix}.$$
We conjugate here by $A_{11}$. This allows us to move the $x_2$ value to the value of $y_3$, if $\lambda_2 \neq -\lambda_1$ and thus yields two cases.

1. $\lambda_2 \neq -\lambda_1$ and we move the $x_2$ position to the $y_3$ value, or $\lambda_2 = -\lambda_1$ and $x_2 = y_3$ already.

2. $\lambda_2 = -\lambda_1$, but $x_2 \neq y_3$.

**Subcase 1.1.1.1.1.1:**

Here the ad $(f_7)$ matrix is of the form

$$\text{ad} (f_7) = \begin{pmatrix}
    a_1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 2\lambda_1 + \lambda_2 & 0 & -y_4 & y_3 & 0 \\
    0 & 0 & \lambda_1 + 2\lambda_2 & x_4 & 0 & y_3 \\
    0 & 0 & 0 & \lambda_1 + \lambda_2 & x_4 & y_3 \\
    0 & 0 & 0 & 0 & \lambda_1 & 0 \\
    0 & 0 & 0 & 0 & 0 & \lambda_2
\end{pmatrix}.$$

At this point, we make the basis change

$$f'_i = f_i \text{ for } 1 \leq i \leq 6, \text{ and } f'_7 = f_7 - y_4f_4 - y_4f_5 + x_4f_6.$$

This will yield that

$$\text{ad} (f'_7) = \begin{pmatrix}
    a_1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 2\lambda_1 + \lambda_2 & 0 & 0 & 0 & 0 \\
    0 & 0 & \lambda_1 + 2\lambda_2 & 0 & 0 & 0 \\
    0 & 0 & 0 & \lambda_1 + \lambda_2 & 0 & 0 \\
    0 & 0 & 0 & 0 & \lambda_1 & 0 \\
    0 & 0 & 0 & 0 & 0 & \lambda_2
\end{pmatrix}.$$

We see here that if $a_1 = 0$, or if $\lambda_1 = \lambda_2 = 0$, then the algebra decomposes. In addition, we can conjugate by $A_{00}$ with $a = d = 0$ and $b = c = 1$ to swap the order of $\lambda_1$ and $\lambda_2$. We order these two so that $\frac{\lambda_1}{a_1} \geq \frac{\lambda_2}{a_1}$. Finally, as $a_1 \neq 0$, we make the change of basis

$$e_i = f'_i \text{ for } 1 \leq i \leq 6, \text{ and } e_7 = -\frac{1}{a_1}f'_7,$$

and let $a = \frac{\lambda_1}{a_1}$ and $b = \frac{\lambda_2}{a_1}$. This yields the structure equations

\begin{align*}
[e_4, e_5] &= e_2, & [e_4, e_6] &= e_3, & [e_5, e_6] &= e_4, & [e_1, e_7] &= e_1, & [e_2, e_7] &= (2a + b)e_2, \\
[e_3, e_7] &= (a + 2b)e_3, & [e_4, e_7] &= (a + b)e_4, & [e_5, e_7] &= ae_5, & [e_6, e_7] &= be_6,
\end{align*}

with $a \geq b$ and $a^2 + b^2 \neq 0$. In the table in Appendix B this is $[7,[6,4],[1,1]]$. 

Subcase 1.1.1.1.1.1.2:

In this section, we assume that $x_2 \neq y_3$ and $\lambda_2 = -\lambda_1$, yielding the \( \text{ad} (f_7) \) matrix
\[
\text{ad} (f_7) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
\lambda_1 & 0 & -y_4 & x_2 & 0 \\
0 & -\lambda_1 & x_4 & 0 & y_3 \\
0 & 0 & 0 & x_4 & y_4 \\
0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & -\lambda_1
\end{pmatrix}.
\]

We make the change of basis
\[
f_i' = f_i \quad \text{for} \quad 1 \leq i \leq 6, \quad \text{and} \quad f'_7 = f_7 - y_3 f_4 - y_4 f_5 + x_4 f_6.
\]

This will yield the \( \text{ad} (f'_7) \) matrix
\[
\text{ad} (f'_7) = \begin{pmatrix}
a_1 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1 & 0 & x_2 - y_3 & 0 \\
0 & 0 & -\lambda_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\lambda_1
\end{pmatrix}.
\]

Now note that if $a_1 = 0$, then the algebra decomposes. Hence $a_1 \neq 0$. Then we conjugate by $A_4$ and as $x_2 \neq y_3$ and $a_1 \neq 0$, we can scale $x_2 - y_3$ to $a_1$. Finally we make the change of basis
\[
e_i = f'_i \quad \text{for} \quad 1 \leq i \leq 6, \quad e_7 = -\frac{1}{a_1} f'_7,
\]

and let $a = \frac{\lambda_1}{a_1}$. This will yield the structure equations
\[
[e_4, e_5] = e_2, \quad [e_4, e_6] = e_3, \quad [e_5, e_6] = e_4, \quad [e_1, e_7] = e_1, \\
[e_2, e_7] = a e_2, \quad [e_3, e_7] = -a e_3, \quad [e_5, e_7] = e_2 + a e_5, \quad [e_6, e_7] = -a e_6,
\]

with $a \in \mathbb{R}$. This is [7,6,4,1,2].

Subcase 1.1.1.1.1.1.2:

Here, we assumed that $\lambda_2 = 0$, but $x_3 \neq 0$ giving us the \( \text{ad} (f_7) \) matrix
\[
\text{ad} (f_7) = \begin{pmatrix}
a_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2\lambda_1 & 0 & -y_4 & x_2 & 0 \\
0 & 0 & \lambda_1 & x_4 & x_3 & y_3 \\
0 & 0 & 0 & \lambda_1 & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
We conjugate by $A_{11}$ and find that if $\lambda_1 \neq 0$, then we can move the $x_2$ position to equal $y_3$. This yields two possibilities.

1. $\lambda_1 \neq 0$ and we make the $x_2$ position equal $y_3$, or $\lambda_1 = 0$ and $x_2 = y_3$ already.

2. $\lambda_1 = 0$, but $x_2 \neq y_3$.

**Subcase 1.1.1.1.1.2.1:**

We have in this case, that

$$\text{ad}(f_7) = \begin{pmatrix}
a_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2\lambda_1 & 0 & -y_4 & y_3 & 0 \\
0 & 0 & \lambda_1 & x_4 & x_3 & y_3 \\
0 & 0 & 0 & \lambda_1 & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$  

We note now that $a_1 \neq 0$ or the algebra is decomposable. This allows us to conjugate by $A_4$ and, as $x_3$ is nonzero as well, scale $x_3$ to $\pm a_1$. Then we make the basis change

$$e_i = f_i, \quad \text{for } 1 \leq i \leq 6, \quad \text{and } e_7 = -\frac{1}{a_1}(f_7 - y_3 f_4 - y_4 f_5 + x_4 f_6),$$

and let $a = \frac{\lambda_1}{a_1}$. This will yield the structure equations

$$[e_4, e_5] = e_2, \quad [e_4, e_6] = e_3, \quad [e_5, e_6] = e_4, \quad [e_1, e_7] = e_1,$$

$$[e_2, e_7] = 2ae_2, \quad [e_3, e_7] = ae_3, \quad [e_4, e_7] = ae_4, \quad [e_5, e_7] = \epsilon e_3 + ae_5,$$

with $a \in \mathbb{R}$ and $\epsilon^2 = 1$. This is $[7,[6,4],1,3]$ in the table.

**Subcase 1.1.1.1.1.1.2.2:**

We assumed here that $\lambda_1 = 0$ and $x_2 \neq y_3$. This yields the $\text{ad}(f_7)$ matrix

$$\text{ad}(f_7) = \begin{pmatrix}
a_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -y_4 & x_2 & 0 \\
0 & 0 & x_4 & x_3 & y_3 & 0 \\
0 & 0 & 0 & x_4 & y_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$  

This implies that $a_1 \neq 0$ or the algebra is nilpotent.

We make the change of basis

$$f_i' = f_i \quad \text{for } 1 \leq i \leq 6, \quad \text{and } f_7' = f_7 - y_3 f_4 - y_4 f_5 + x_4 f_6.$$
This will yield the ad \((f_7)\) matrix

\[
ad (f_7) = \begin{pmatrix}
a_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x_2 - y_3 & 0 \\
0 & 0 & 0 & 0 & x_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

As \(a_1, x_3, \) and \(x_2 - y_3\) are all nonzero values, we can conjugate by \(A_4A_7\) and pick \(s_4\) and \(s_7\) to simultaneously scale \(x_2 - y_3\) to \(a_1\) and \(x_3\) to \(\pm a_1\). Finally, we make the change of basis

\[
e_i = f'_i \quad \text{for} \quad 1 \leq i \leq 6, \quad e_7 = -\frac{1}{a_1}f'_7.
\]

The resulting structure equations are

\[
[e_4, e_5] = e_2, \quad [e_4, e_6] = e_3, \quad [e_5, e_6] = e_4, \quad [e_1, e_7] = e_1, \quad [e_5, e_7] = e_2 + \varepsilon e_3,
\]

with \(\varepsilon^2 = 1\). This is \([7, [6, 4], 1, 4]\).

**Subcase 1.1.1.1.1.2:**

In this section, we assumed that \(\lambda_1 = 0\) and \(y_2 \neq 0\). This yields the ad \((f_7)\) matrix

\[
ad (f_7) = \begin{pmatrix}
a_1 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & -y_4 & x_2 & y_2 \\
0 & 0 & 2\lambda_2 & x_4 & x_3 & y_3 \\
0 & 0 & 0 & \lambda_2 & x_4 & y_4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2 \\
\end{pmatrix}.
\]

Next we conjugate by \(A_{11}A_{12}\) and find that if \(\lambda_2 \neq 0\), then we can pick \(s_{11}\) and \(s_{12}\) to make the \(x_2\) position equal \(y_3\) and \(x_3\) equal 0. Also, if \(\lambda_2 = 0\), then we can conjugate by \(A_5\), without affecting any previous changes, and as \(y_2 \neq 0\), we can still make the \(x_2\) position equal the \(y_3\) position. We have two cases then.

1. \(\lambda_2 \neq 0\) and we make \(x_2\) equal to \(y_3\) and \(x_3\) equal to 0, or \(\lambda_2 = 0\) and we still make the \(x_2\) position equal to the \(y_3\) position and the \(x_3\) position is zero afterwards.

2. \(\lambda_2 = 0\) and we still make the \(x_2\) position equal the \(y_3\) position, but the \(x_3\) position is nonzero afterwards.
Subcase 1.1.1.1.2.1:

Here we have the \( \text{ad} (f_7) \) matrix
\[
\text{ad} (f_7) = \begin{pmatrix}
a_1 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & -y_4 & y_3 & y_2 \\
0 & 0 & 2\lambda_2 & x_4 & 0 & y_3 \\
0 & 0 & 0 & \lambda_2 & x_4 & y_4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2 \\
\end{pmatrix}.
\]

We conjugate by \( A_0 \) with \( a = d = 0 \) and \( b = c = 1 \), and we’ll end up with the matrix
\[
\text{ad} (f_7) = \begin{pmatrix}
a_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2\lambda_2 & 0 & x_4 & -y_3 & 0 \\
0 & 0 & \lambda_2 & -y_4 & -y_2 & -y_3 \\
0 & 0 & 0 & \lambda_2 & -y_4 & -x_4 \\
0 & 0 & 0 & 0 & 0 & \lambda_2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

If we relabel, \( \lambda_2, -y_3, -x_4, -y_4 \), and \( -y_2 \) as \( \lambda_1, y_3, y_4, x_4 \), and \( x_3 \) respectively, then this is the same as the \( \text{ad} (f_7) \) matrix in Subcase 1.1.1.1.1.1.2.1. Thus we don’t obtain anything new from this subcase.

Subcase 1.1.1.1.2.2:

Here we have that \( \lambda_2 = 0 \) and \( x_3 \neq 0 \), which yields the \( \text{ad} (f_7) \) matrix
\[
\text{ad} (f_7) = \begin{pmatrix}
a_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -y_4 & x_2 & y_2 \\
0 & 0 & 0 & x_4 & x_3 & y_3 \\
0 & 0 & 0 & 0 & x_4 & y_4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

This implies that \( a_1 \neq 0 \) or the algebra is nilpotent. We then conjugate by \( A_4 A_7 \) and, as \( x_3 \) and \( y_2 \) are both nonzero as well, scale both \( x_3 \) and \( y_2 \) to \( \pm a_1 \). Finally, we make the change of basis
\[
e_i = f_i, \text{ for } 1 \leq i \leq 6, \text{ and } e_7 = -\frac{1}{a_1}(f_7 - y_3 f_4 - y_4 f_5 + x_4 f_6).
\]

Then the structure equations are
\[
[e_4, e_5] = e_2, \quad [e_4, e_6] = e_3, \quad [e_5, e_6] = e_4, \quad [e_1, e_7] = e_1, \quad [e_5, e_7] = \delta e_3, \quad [e_6, e_7] = \varepsilon e_2,
\]
with \( \varepsilon^2 = \delta^2 = 1 \). In Appendix B this is \([7,[6,4],[1,5]]\).
Subcase 1.1.1.1.2:

In this section, we have the ad \( f_7 \) matrix

\[
\text{ad} (f_7) = \\
\begin{pmatrix}
\lambda_2 & 0 & 0 & 0 & 0 & y_1 \\
0 & 2\lambda_1 + \lambda_2 & 0 & -y_4 & x_2 & y_2 \\
0 & 0 & \lambda_1 + 2\lambda_2 & x_4 & x_3 & y_3 \\
0 & 0 & 0 & \lambda_1 + \lambda_2 & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2
\end{pmatrix},
\]

with \( y_1 \neq 0 \). Next, we conjugate by \( A_{14} \) and find that if \( \lambda_1 \neq 0 \), we can move the \( y_2 \) position to 0. However, if \( \lambda_1 = 0 \), then we can conjugate by \( A_2 \) without affecting any previous changes and, as \( y_1 \neq 0 \), we can still make the \( y_2 \) position equal 0.

We can do something similar for the \( y_3 \) position. If we conjugate by \( A_{15} \), then we can change the \( y_3 \) position to equal the \( x_2 \) position whenever \( \lambda_2 \neq -\lambda_1 \). However, if \( \lambda_2 = -\lambda_1 \), then we can apply \( A_3 \) without affecting any previous changes and, as \( y_1 \neq 0 \), we can still change the \( y_3 \) position to equal \( x_2 \).

Now we conjugate by \( A_{12} \) and we see that we can only make the \( x_3 \) position equal to 0, if \( \lambda_2 \neq 0 \). This gives us two cases.

1. \( \lambda_2 \neq 0 \) and we make the \( x_3 \) position equal 0, or \( \lambda_2 = 0 \) and \( x_3 = 0 \) already.

2. \( \lambda_2 = 0 \), but \( x_3 \neq 0 \).

Subcase 1.1.1.1.2.1:

Here we have the ad \( f_7 \) matrix

\[
\text{ad} (f_7) = \\
\begin{pmatrix}
\lambda_2 & 0 & 0 & 0 & 0 & y_1 \\
0 & 2\lambda_1 + \lambda_2 & 0 & -y_4 & x_2 & 0 \\
0 & 0 & \lambda_1 + 2\lambda_2 & x_4 & 0 & x_2 \\
0 & 0 & 0 & \lambda_1 + \lambda_2 & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2
\end{pmatrix},
\]

We have that \( \lambda_1 \) and \( \lambda_2 \) are not simultaneously 0. We will bifurcate on this here and consider both of the following cases individually.

1. \( \lambda_1 \neq 0 \).

2. \( \lambda_1 = 0 \), which implies that \( \lambda_2 \neq 0 \) or the algebra is nilpotent.
Subcase 1.1.1.1.2.1.1:

Here, we conjugate by $A_1$ and, as $y_1$ and $\lambda_1$ are both nonzero, scale $y_1$ to $\lambda_1$. Then we make the change of basis

$$e_i = f_i, \text{ for } 1 \leq i \leq 6, \text{ and } e_7 = -\frac{1}{\lambda_1}(f_7 - x_2f_4 - y_4f_5 + x_4f_6),$$

and let $a = \frac{\lambda_2}{\lambda_1}$. We end up with the structure equations

$$[e_4, e_5] = e_2, \quad [e_4, e_6] = e_3, \quad [e_5, e_6] = e_4, \quad [e_1, e_7] = ae_1, \quad [e_2, e_7] = (a + 2)e_2,$$

$$[e_3, e_7] = (2a + 1)e_3, \quad [e_4, e_7] = (a + 1)e_4, \quad [e_5, e_7] = e_5, \quad [e_6, e_7] = e_1 + ae_6,$$

with $a \in \mathbb{R}$. This is $[7, [6, 4], 1, 6]$.

Subcase 1.1.1.1.2.1.2:

In this section, we assumed that $\lambda_1 = 0$, which yields the $\text{ad}(f_7)$ matrix

$$\text{ad}(f_7) = \begin{pmatrix}
\lambda_2 & 0 & 0 & 0 & 0 & y_1 \\
0 & \lambda_2 & 0 & -y_4 & x_2 & 0 \\
0 & 0 & 2\lambda_2 & x_4 & 0 & x_2 \\
0 & 0 & 0 & \lambda_2 & x_4 & y_4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2
\end{pmatrix}.$$ 

Conjugate by $A_1$, and this time, as $y_1$ and $\lambda_2$ are both nonzero, scale $y_1$ to $\lambda_2$. Then make the change of basis

$$e_i = f_i, \text{ for } 1 \leq i \leq 6, \text{ and } e_7 = -\frac{1}{\lambda_2}(f_7 - x_2f_4 - y_4f_5 + x_4f_6),$$

which will yield the structure equations

$$[e_4, e_5] = e_2, \quad [e_4, e_6] = e_3, \quad [e_5, e_6] = e_4, \quad [e_1, e_7] = e_1,$$

$$[e_2, e_7] = e_2, \quad [e_3, e_7] = 2e_3, \quad [e_4, e_7] = e_4, \quad [e_6, e_7] = e_1 + e_6.$$

This is $[7, [6, 4], 1, 7]$. 

Subcase 1.1.1.1.2.2:

In this section, we assume that $\lambda_2 = 0$ and $x_3 \neq 0$. This yields that the $\text{ad}(f_7)$ matrix is

$$\text{ad}(f_7) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & y_1 \\
0 & 2\lambda_1 & 0 & -y_4 & x_2 & 0 \\
0 & 0 & \lambda_1 & x_4 & x_3 & x_2 \\
0 & 0 & 0 & \lambda_1 & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$  

We next conjugate by $A_1A_4$ and, as $y_1$, $x_3$, and $\lambda_1$ are all nonzero, scale $y_1$ to $\lambda_1$ and $x_3$ to $\pm\lambda_1$.

Finally, we make the basis change

$$e_i = f_i, \text{ for } 1 \leq i \leq 6, \text{ and } e_7 = \frac{1}{\lambda_1}(f_7 - x_2f_4 - y_4f_5 + x_4f_6),$$

and the structure equations become

$$[e_4, e_5] = e_2, \quad [e_4, e_6] = e_3, \quad [e_5, e_6] = e_4, \quad [e_2, e_7] = 2e_2,$$

$$[e_3, e_7] = e_3, \quad [e_4, e_7] = e_4, \quad [e_5, e_7] = \varepsilon e_3 + e_5, \quad [e_6, e_7] = e_1,$$

with $\varepsilon^2 = 1$. In the table, this is $[7,[6,4],[1,8]]$.

Subcase 1.1.1.2:

We assumed in this section that $a_1 = \lambda_1$ but $x_1 \neq 0$. This gives us the $\text{ad}(f_7)$ matrix

$$\text{ad}(f_7) = \begin{bmatrix}
\lambda_1 & 0 & 0 & 0 & 0 & x_1 \\
0 & 2\lambda_1 + \lambda_2 & 0 & -y_4 & x_2 & y_2 \\
0 & 0 & \lambda_1 + 2\lambda_2 & x_4 & x_3 & y_3 \\
0 & 0 & 0 & \lambda_1 & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2
\end{bmatrix}.$$  

Here we conjugate by $A_{13}$ and, if $\lambda_2 \neq \lambda_1$, move $y_1$ to 0. However, if $\lambda_2 = \lambda_1$, then we conjugate by $A_6$ instead and, as $x_1 \neq 0$, still move $y_1$ to 0. At this point, if we conjugate by $A_0$ with $a = d = 0$ and $b = c = 1$, we'll get that

$$\text{ad}(f_7) = \begin{bmatrix}
\lambda_1 & 0 & 0 & 0 & 0 & x_1 \\
0 & 2\lambda_2 + \lambda_1 & 0 & x_4 & -y_3 & -x_3 \\
0 & 0 & \lambda_2 + 2\lambda_1 & -y_4 & -y_2 & -x_2 \\
0 & 0 & 0 & \lambda_2 + \lambda_1 & -y_4 & -x_4 \\
0 & 0 & 0 & 0 & \lambda_2 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_1
\end{bmatrix}.$$
If we relabel \( \lambda_1, \lambda_2, x_1, -x_4, -y_3, -y_4, -y_2, \) and \(-x_2\) as \( \lambda_2, \lambda_1, y_1, y_4, x_2, y_2, x_4, x_3, \) and \( y_3 \) respectively, we’ll have the same \( \text{ad}(f_7) \) matrix as in Subcase 1.1.1.1.2. Thus there are no new isomorphism classes to be found here.

**Subcase 1.1.2:**

In this section, we assumed that \( a_1 = \lambda_1 + 2\lambda_2 \) and that \( a_3 \neq 0 \). This will give us the \( \text{ad}(f_7) \) matrix

\[
\text{ad}(f_7) = \begin{pmatrix}
\lambda_1 + 2\lambda_2 & 0 & 0 & 0 & x_1 & y_1 \\
0 & 2\lambda_1 + \lambda_2 & 0 & -y_4 & x_2 & y_2 \\
a_3 & 0 & \lambda_1 + 2\lambda_2 & x_4 & x_3 & y_3 \\
0 & 0 & 0 & \lambda_1 + \lambda_2 & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2
\end{pmatrix}.
\]

We conjugate next by \( A_{10} \). If \( \lambda_2 \neq 0 \), this allows us to move the \( x_1 \) position to 0 and hence gives us two cases.

1. \( \lambda_2 \neq 0 \) and we move \( x_1 \) to 0, or \( \lambda_2 = 0 \) and \( x_1 = 0 \) already.

2. \( \lambda_2 = 0 \), but \( x_1 \neq 0 \).

**Subcase 1.1.2.1:**

Here we have the \( \text{ad}(f_7) \) matrix

\[
\text{ad}(f_7) = \begin{pmatrix}
\lambda_1 + 2\lambda_2 & 0 & 0 & 0 & 0 & y_1 \\
0 & 2\lambda_1 + \lambda_2 & 0 & -y_4 & x_2 & y_2 \\
a_3 & 0 & \lambda_1 + 2\lambda_2 & x_4 & x_3 & y_3 \\
0 & 0 & 0 & \lambda_1 + \lambda_2 & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2
\end{pmatrix}.
\]

We next conjugate by \( A_{13} \). If \( \lambda_2 \neq -\lambda_1 \), we move \( y_1 \) to 0. This yields another two cases

1. \( \lambda_2 \neq -\lambda_1 \) and we make the \( y_1 \) position equal to 0, or \( \lambda_2 = -\lambda_1 \) and \( y_1 = 0 \) already.

2. \( \lambda_2 = -\lambda_1 \), but \( y_1 \neq 0 \).

**Subcase 1.1.2.1.1:**

In this section, we have

\[
\text{ad}(f_7) = \begin{pmatrix}
\lambda_1 + 2\lambda_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 2\lambda_1 + \lambda_2 & 0 & -y_4 & x_2 & y_2 \\
a_3 & 0 & \lambda_1 + 2\lambda_2 & x_4 & x_3 & y_3 \\
0 & 0 & 0 & \lambda_1 + \lambda_2 & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2
\end{pmatrix}.
\]
We conjugate by $A_{12}$ and, if $\lambda_2 \neq 0$, move $x_3$ to 0. If $\lambda_2 = 0$, then we conjugate by $A_{10}$ instead and, as $a_3 \neq 0$, we still move $x_3$ to 0.

Similarly, we next conjugate by $A_{15}$ and, if $\lambda_2 \neq -\lambda_1$, then we move $y_3$ to $x_2$. If $\lambda_2 = -\lambda_1$, then we conjugate by $A_{13}$ instead and, as $a_3 \neq 0$, we still move $y_3$ to $x_2$.

Then we conjugate by $A_{14}$. If $\lambda_1 \neq 0$, then we can move $y_2$ to 0. This yields two cases.

1. $\lambda_1 \neq 0$ and we move $y_2$ to 0, or $\lambda_1 = 0$ and $y_2 = 0$ already.

2. $\lambda_1 = 0$, but $y_2 \neq 0$.

**Subcase 1.1.2.1.1.1:**

We have the ad $(f_7)$ matrix

$$
ad(f_7) = \begin{pmatrix}
\lambda_1 + 2\lambda_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 2\lambda_1 + \lambda_2 & 0 & -y_4 & x_2 & 0 \\
a_3 & 0 & \lambda_1 + 2\lambda_2 & x_4 & 0 & x_2 \\
0 & 0 & 0 & \lambda_1 + \lambda_2 & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2
\end{pmatrix}.
$$

We know that $\lambda_1$ and $\lambda_2$ are not simultaneously 0, otherwise the algebra is nilpotent. This gives us another two cases.

1. $\lambda_1 \neq 0$.

2. $\lambda_1 = 0$, which implies that $\lambda_2 \neq 0$.

**Subcase 1.1.2.1.1.1.1:**

Here we conjugate by $A_1$ and, as $a_3$ and $\lambda_1$ are both nonzero, we scale $a_3$ to $\lambda_1$. Then we make the change of basis

$$
e_i = f_i, \text{ for } 1 \leq i \leq 6, \text{ and } e_7 = -\frac{1}{\lambda_1}(f_7 - x_2f_4 - y_4f_5 + x_4f_6),$$

and let $a = \frac{\lambda_1}{\lambda_1}$. This yields the structure equations

$$
[e_4, e_5] = e_2, \quad [e_4, e_6] = e_3, \quad [e_5, e_6] = e_4, \quad [e_1, e_7] = (2a + 1)e_1 + e_3,
$$

$$
[e_2, e_7] = (a + 2)e_2, \quad [e_3, e_7] = (2a + 1)e_3, \quad [e_4, e_7] = (a + 1)e_4, \quad [e_5, e_7] = e_5,
$$

$$
[e_6, e_7] = ae_6,
$$

with $a \in \mathbb{R}$. This is $[7, [6,4],[1,9]]$ in the table.
Subcase 1.1.2.1.1.2:

In this section, we have that

\[
\text{ad} \left( f_7 \right) = \begin{pmatrix}
2\lambda_2 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & -y_4 & x_2 & 0 \\
a_3 & 0 & 2\lambda_2 & x_4 & 0 & x_2 \\
0 & 0 & 0 & \lambda_2 & x_4 & y_4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2 \\
\end{pmatrix}.
\]

Then we conjugate by \( A_1 \) and, as \( a_3 \) and \( \lambda_2 \) are both nonzero, we scale \( a_3 \) to \( \lambda_2 \). Finally, we make the change of basis

\[
e_i = f_i, \quad \text{for } 1 \leq i \leq 6, \quad \text{and } e_7 = -\frac{1}{\lambda_2} (f_7 - x_2 f_4 - y_4 f_5 + x_4 f_6).
\]

Then we obtain the structure equations

\[
\begin{align*}
\{e_4, e_5\} &= e_2, & \{e_4, e_6\} &= e_3, & \{e_5, e_6\} &= e_4, & \{e_1, e_7\} &= 2e_1 + e_3, \\
\{e_2, e_7\} &= e_2, & \{e_3, e_7\} &= 2e_3, & \{e_4, e_7\} &= e_4, & \{e_6, e_7\} &= e_6.
\end{align*}
\]

This is \([7,[6,4],1,10]\).

Subcase 1.1.2.1.1.2:

In this section, we assumed that \( \lambda_1 = 0 \) and \( y_2 \neq 0 \). This yields that

\[
\text{ad} \left( f_7 \right) = \begin{pmatrix}
2\lambda_2 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & -y_4 & x_2 & y_2 \\
a_3 & 0 & 2\lambda_2 & x_4 & 0 & x_2 \\
0 & 0 & 0 & \lambda_2 & x_4 & y_4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2 \\
\end{pmatrix}.
\]

Note that if \( \lambda_2 = 0 \), then the algebra is nilpotent. Then we conjugate by \( A_1 A_7 \) and, as \( a_3 \), \( y_2 \), and \( \lambda_2 \) are all nonzero, scale \( a_3 \) to \( \lambda_2 \) and \( y_2 \) to \( \pm \lambda_2 \). Finally, we make the change of basis

\[
e_i = f_i, \quad \text{for } 1 \leq i \leq 6, \quad \text{and } e_7 = -\frac{1}{\lambda_2} (f_7 - x_2 f_4 - y_4 f_5 + x_4 f_6).
\]

This yields the structure equations

\[
\begin{align*}
\{e_4, e_5\} &= e_2, & \{e_4, e_6\} &= e_3, & \{e_5, e_6\} &= e_4, & \{e_1, e_7\} &= 2e_1 + e_3, \\
\{e_2, e_7\} &= e_2, & \{e_3, e_7\} &= 2e_3, & \{e_4, e_7\} &= e_4, & \{e_6, e_7\} &= \epsilon e_2 + e_6,
\end{align*}
\]

with \( \epsilon^2 = 1 \). In Appendix B, this is \([7,[6,4],1,11]\).
Subcase 1.1.2.1.2:

Here we had that $\lambda_2 = -\lambda_1$ and $y_1 \neq 0$. This yields the $\text{ad}(f_7)$ matrix

$$\text{ad}(f_7) = \begin{pmatrix}
-\lambda_1 & 0 & 0 & 0 & 0 & y_1 \\
0 & \lambda_1 & 0 & -y_4 & x_2 & y_2 \\
a_3 & 0 & -\lambda_1 & x_4 & x_3 & y_3 \\
0 & 0 & 0 & 0 & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & -\lambda_1
\end{pmatrix}.$$ 

Also we see that $\lambda_1 \neq 0$ or the algebra is nilpotent. Then we conjugate by $A_{12}A_{13}A_{14}$ and, as $\lambda_1$ and $a_3$ are both nonzero, move $y_2$ and $x_3$ to 0 and $y_3$ to $x_2$. Next we conjugate by $A_1A_4$ and, as $a_3$, $y_1$, and $\lambda_1$ are all nonzero, scale $a_3$ and $y_1$ both to $-\lambda_1$. Finally, we make the basis change

$$e_i = f_i, \text{ for } 1 \leq i \leq 6, \text{ and } e_7 = \frac{1}{\lambda_1}(f_7 - x_2f_4 - y_4f_5 + x_4f_6),$$

and obtain the structure equations

$$[e_4, e_5] = e_2, \quad [e_4, e_6] = e_3, \quad [e_5, e_6] = e_4, \quad [e_1, e_7] = e_1 + e_3,$$

$$[e_2, e_7] = -e_2, \quad [e_3, e_7] = e_3, \quad [e_5, e_7] = -e_5, \quad [e_6, e_7] = e_1 + e_6.$$

This is $[7, [6,4], 1,12]$.

Subcase 1.1.2.2:

In this section, we have assumed that $\lambda_2 = 0$ and $x_1 \neq 0$. Then

$$\text{ad}(f_7) = \frac{1}{\lambda_1}(f_7 - x_2f_4 - y_4f_5 + x_4f_6),$$

and we see that $\lambda_1 \neq 0$ or the algebra is nilpotent. We now conjugate by $A_{13}A_{14}A_{15}$ and, as $\lambda_1$ and $x_1$ are both nonzero, move $y_1$, $y_2$, and $x_3$ to 0 and $y_3$ to $x_2$. Then we conjugate by $A_1A_4$ and, as $x_1$, $a_3$, and $\lambda_1$ are all nonzero, scale $a_3$ to $\lambda_1$ and $x_1$ to $\pm \lambda_1$. Finally, we make the change of basis

$$e_i = f_i, \text{ for } 1 \leq i \leq 6, \text{ and } e_7 = \frac{1}{\lambda_1}(f_7 - x_2f_4 - y_4f_5 + x_4f_6),$$

and arrive at the structure equations

$$[e_4, e_5] = e_2, \quad [e_4, e_6] = e_3, \quad [e_5, e_6] = e_4, \quad [e_1, e_7] = e_1 + e_3,$$

$$[e_2, e_7] = 2e_2, \quad [e_3, e_7] = e_3, \quad [e_4, e_7] = e_4, \quad [e_5, e_7] = \epsilon e_1 + e_5.$$
with \( \varepsilon^2 = 1 \). This is \([7,6,4,1,13]\) in the table.

**Subcase 1.2:**

We assumed here that \( a_1 = 2\lambda_1 + \lambda_2 \) and \( a_2 \neq 0 \). This will yield that

\[
\text{ad}(f_7) = \begin{pmatrix}
2\lambda_1 + \lambda_2 & 0 & 0 & 0 & x_1 & y_1 \\
a_2 & 2\lambda_1 + \lambda_2 & 0 & -y_4 & x_2 & y_2 \\
a_3 & 0 & \lambda_1 + 2\lambda_2 & x_4 & x_3 & y_3 \\
0 & 0 & 0 & \lambda_1 + \lambda_2 & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2
\end{pmatrix}.
\]

If \( \lambda_2 \neq \lambda_1 \), then we conjugate by \( A_3 \) and move \( a_3 \) to 0. If, on the other hand, \( \lambda_2 = \lambda_1 \), then we conjugate by \( A_5 \) and, as \( a_2 \neq 0 \), we still move \( a_3 \) to 0. Finally, we conjugate by \( A_0 \) with \( a = d = 0 \) and \( b = c = 1 \), and arrive at

\[
\text{ad}(f_7) = \begin{pmatrix}
2\lambda_1 + \lambda_2 & 0 & 0 & 0 & y_1 & x_1 \\
0 & \lambda_1 + 2\lambda_2 & 0 & x_4 & -y_3 & -x_3 \\
-a_2 & 0 & 2\lambda_1 + \lambda_2 & -y_4 & -y_2 & -x_2 \\
0 & 0 & 0 & \lambda_1 + \lambda_2 & -y_4 & -x_4 \\
0 & 0 & 0 & 0 & \lambda_2 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_1
\end{pmatrix}.
\]

If we relabel \( \lambda_1, \lambda_2, -a_2, y_1, x_1, -x_4, -y_3, -x_3, -y_4, -y_2, \) and \(-x_2\) as \( \lambda_2, \lambda_1, a_3, x_1, y_1, y_4, x_2, y_2, x_4, x_3, \) and \( y_3 \) respectively, then this is the same \( \text{ad}(f_7) \) matrix as the one in Subcase 1.1.2.

### 3.2.2 Parent Case 2:

In the second parent case, we classify the algebras with an \( \text{ad}(f_7) \) matrix of the form

\[
\text{ad}(f_7) = \begin{pmatrix}
a_1 & 0 & 0 & 0 & x_1 & y_1 \\
a_2 & 3\lambda_1 & \lambda_2 & -y_4 & x_2 & y_2 \\
a_3 & -\lambda_2 & 3\lambda_1 & x_4 & x_3 & y_3 \\
0 & 0 & 0 & 2\lambda_1 & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda_1 & \lambda_2 \\
0 & 0 & 0 & 0 & -\lambda_2 & \lambda_1
\end{pmatrix},
\]

with \( \lambda_2 \neq 0 \) and the eigenvalues ordered and labeled so that \( \frac{\lambda_1}{\lambda_2} \geq 0 \).

We first conjugate by \( A_2A_3 \) and pick \( s_2 \) and \( s_3 \) to be as follows

\[
s_2 = \frac{\lambda_2a_3 - 3\lambda_1a_2 + a_1a_2}{(\lambda_2)^2 + (a_1 - 3\lambda_1)^2}, \quad s_3 = \frac{a_1a_3 - 3\lambda_1a_3 - \lambda_2a_2}{(\lambda_2)^2 + (a_1 - 3\lambda_1)^2}.
\]

As \( \lambda_2 \neq 0 \), these denominators are nonzero. Picking \( s_2 \) and \( s_3 \) in this manner will move both the \( a_2 \) and \( a_3 \) positions to 0 simultaneously. This is the common type of change that we make when we have a real Jordan block of this type in the \( \text{ad} \) matrix we’re trying to simplify.
Next conjugate by \( A_{10}A_{13} \) and, again as \( \lambda_2 \neq 0 \), we move the \( x_1 \) and \( y_1 \) positions to 0. If \( \lambda_1 \neq 0 \), then we conjugate by \( A_{11}A_{12}A_{14} \), and, as \( \lambda_2 \neq 0 \), we pick \( s_{11}, s_{12}, \) and \( s_{14} \) to simultaneously move the \( y_2 \) and \( x_3 \) positions to 0 and make the \( x_2 \) position equal to the \( y_3 \) position (call the common value \( y_3 \)). If \( \lambda_1 = 0 \), then we conjugate by \( A_{11}A_{12} \) and, as \( \lambda_2 \neq 0 \), move \( x_3 \) to 0 and make the \( x_2 \) position equal to the \( y_3 \) position. This yields two cases.

1. \( \lambda_1 \neq 0 \) and we make all three changes, or \( \lambda_1 = 0 \) and we move \( x_3 \) to 0 and \( x_2 \) to \( y_3 \) while \( y_2 = 0 \) already.

2. \( \lambda_1 = 0 \) and we move \( x_3 \) to 0 and \( x_2 \) to \( y_3 \), but \( y_2 \neq 0 \).

**Subcase 2.1:**

We have the \( \text{ad} (f_7) \) matrix here as

\[
\begin{pmatrix}
  a_1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 3\lambda_1 & \lambda_2 & -y_4 & y_3 & 0 \\
  0 & -\lambda_2 & 3\lambda_1 & x_4 & 0 & y_3 \\
  0 & 0 & 0 & 2\lambda_1 & x_4 & y_4 \\
  0 & 0 & 0 & \lambda_1 & \lambda_2 & 0 \\
  0 & 0 & 0 & 0 & -\lambda_2 & \lambda_1 
\end{pmatrix}
\]

Note that \( a_1 \neq 0 \) or the algebra decomposes. We then make the change of basis

\[
e_i = f_i, \quad \text{for } 1 \leq i \leq 6, \quad \text{and } e_7 = -\frac{1}{\lambda_2} (f_7 - y_3 f_4 - y_4 f_5 + x_4 f_6),
\]

and let \( a = \frac{a_1}{\lambda_2} \) and \( b = \frac{\lambda_1}{\lambda_2} \). This yields the structure equations

\[
[e_4, e_5] = e_2, \quad [e_4, e_6] = e_3, \quad [e_5, e_6] = e_4, \quad [e_1, e_7] = a e_1, \quad [e_2, e_7] = 3 b e_2 - e_3,
\]

\[
[e_3, e_7] = e_2 + 3 b e_3, \quad [e_4, e_7] = 2 b e_4, \quad [e_5, e_7] = b e_5 - e_6, \quad [e_6, e_7] = e_5 + b e_6,
\]

with \( a \neq 0 \) and \( b \geq 0 \). In the table in Appendix B, this is \([7,[6,4],[2,1]]\).

**Subcase 2.2:**

Here we assumed that \( \lambda_1 = 0 \) and, while we moved \( x_3 \) to 0 and \( x_2 \) to \( y_3 \), that \( y_2 \neq 0 \). This gives us the \( \text{ad} (f_7) \) matrix

\[
\begin{pmatrix}
  a_1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & \lambda_2 & -y_4 & y_3 & y_2 \\
  0 & -\lambda_2 & 0 & x_4 & 0 & y_3 \\
  0 & 0 & 0 & 2\lambda_1 & x_4 & y_4 \\
  0 & 0 & 0 & 0 & \lambda_2 & 0 \\
  0 & 0 & 0 & 0 & -\lambda_2 & 0 
\end{pmatrix}
\]
Again, note that if \( a_1 = 0 \), then the algebra is decomposable. Then conjugate by \( A_4A_7 \) and let \( s_7 = s_4 \). As \( \lambda_2 \) and \( y_2 \) are both nonzero, this will allow us to pick \( s_4 \) to scale \( y_2 \) to \( \pm \lambda_2 \). Then make the change of basis

\[
e_i = f_i, \quad \text{for } 1 \leq i \leq 6, \quad \text{and } e_7 = -\frac{1}{\lambda_2}(f_7 - y_3f_4 - y_4f_5 + x_4f_6),
\]

and let \( a = \frac{a_4}{\lambda_2} \). This yields the structure equations

\[
[e_4, e_5] = e_2, \quad [e_4, e_6] = e_3, \quad [e_5, e_6] = e_4, \quad [e_1, e_7] = a e_1,
\]

\[
[e_2, e_7] = -e_3, \quad [e_3, e_7] = e_2, \quad [e_5, e_7] = -e_6, \quad [e_6, e_7] = \varepsilon e_2 + e_5,
\]

with \( a \neq 0 \) and \( \varepsilon^2 = 1 \). This is \([7,[6,4],2,2]\).

### 3.2.3 Parent Case 3:

In this final section, we classify those algebras whose \( \text{ad} (f_7) \) matrix is of the form

\[
\text{ad} (f_7) = \begin{pmatrix}
a_1 & 0 & 0 & 0 & x_1 & y_1 \\
a_2 & 3\lambda & 1 & -y_4 & x_2 & y_2 \\
a_3 & 0 & 3\lambda & x_4 & x_3 & y_3 \\
0 & 0 & 0 & 2\lambda & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda & 1 \\
0 & 0 & 0 & 0 & 0 & \lambda
\end{pmatrix}.
\]

First, if \( a_1 \neq 3\lambda \), we conjugate by \( A_2A_3 \) and pick \( s_2 \) and \( s_3 \) as follows

\[
s_2 = \frac{a_3}{a_1 - 3\lambda}, \quad s_3 = \frac{a_1a_2 - 3\lambda a_2 + a_3}{(a_1 - 3\lambda)^2}.
\]

This will move both the \( a_2 \) and \( a_3 \) positions to 0 simultaneously. If, on the other hand, \( a_1 = 3\lambda \), then we conjugate by \( A_3 \) and let \( s_3 = -a_2 \). This will move the \( a_2 \) position to 0. This is usually how we'll proceed if a real Jordan block of this type is in the \( \text{ad} \) matrix we're trying to simplify. We have two cases.

1. \( a_1 \neq 3\lambda \) and we move \( a_2 \) and \( a_3 \) to 0, or \( a_1 = 3\lambda \) and we still move \( a_2 \) to 0, while \( a_3 = 0 \) already.

2. \( a_1 = 3\lambda \) and we still move \( a_2 \) to 0, but \( a_3 \neq 0 \).
Subcase 3.1:

Here we have the ad \((f_7)\) matrix

\[
\text{ad}(f_7) = \begin{pmatrix}
0 & 0 & 0 & 0 & x_1 & y_1 \\
0 & 3\lambda & 0 & -y_4 & x_2 & y_2 \\
0 & 0 & 3\lambda & x_4 & x_3 & y_3 \\
0 & 0 & 0 & 2\lambda & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda & 1 \\
0 & 0 & 0 & 0 & 0 & \lambda
\end{pmatrix}.
\]

Now, if \(a_1 \neq \lambda\), then we conjugate by \(A_{10}A_{13}\) and move \(x_1\) and \(y_1\) to 0. If \(a_1 = \lambda\), then we conjugate by just \(A_{10}\) instead and pick \(s_{10}\) to move \(y_1\) to 0. This gives us two cases.

1. \(a_1 \neq \lambda\) and we move \(x_1\) and \(y_1\) to 0, or \(a_1 = \lambda\) and we move \(y_1\) to 0 and \(x_1 = 0\) already.

2. \(a_1 = \lambda\) and we still move \(y_1\) to 0, but \(x_1 \neq 0\).

Subcase 3.1.1:

In this section, we consider the case when the ad \((f_7)\) matrix is of the form

\[
\text{ad}(f_7) = \begin{pmatrix}
a_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 3\lambda & 1 & -y_4 & x_2 & y_2 \\
0 & 0 & 3\lambda & x_4 & x_3 & y_3 \\
0 & 0 & 0 & 2\lambda & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda & 1 \\
0 & 0 & 0 & 0 & 0 & \lambda
\end{pmatrix}.
\]

If \(\lambda \neq 0\), then we conjugate by \(A_{11}A_{12}A_{14}\) and pick \(s_{11}, s_{12},\) and \(s_{14}\) to move the \(y_2\) and \(x_3\) positions to 0 and make the \(x_2\) position equal to the \(y_3\) position (label the common value \(y_3\)). If \(\lambda = 0\), then we conjugate by \(A_{11}A_{12}\) and pick \(s_{11}\) and \(s_{12}\) to move \(y_2\) to 0 and \(x_2\) to \(y_3\). This gives us two cases.

1. \(\lambda \neq 0\) and we move \(x_3\) and \(y_2\) to 0 and \(x_2\) to \(y_3\), or \(\lambda = 0\) and we still move \(y_2\) to 0 and \(x_2\) to \(y_3\), while \(x_3 = 0\) already.

2. \(\lambda = 0\) and while we still move \(y_2\) to 0 and \(x_2\) to \(y_3\), \(x_3 \neq 0\).

Subcase 3.1.1.1:

Here we end up with

\[
\text{ad}(f_7) = \begin{pmatrix}
a_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 3\lambda & 1 & -y_4 & x_3 & y_3 \\
0 & 0 & 3\lambda & x_4 & 0 & y_3 \\
0 & 0 & 0 & 2\lambda & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda & 1 \\
0 & 0 & 0 & 0 & 0 & \lambda
\end{pmatrix}.
\]
Note that $a_1 \neq 0$ or the algebra decomposes. This allows us to make the change of basis
\[ e_1 = f_1, \quad e_3 = (a_1)^2 f_3, \quad e_5 = f_5, \quad e_i = a_1 f_i, \quad \text{for } i \in \{2, 4, 6\}, \quad \text{and } e_7 = -\frac{1}{a_1} (f_7 - y_3 f_4 - y_4 f_5 + x_4 f_6), \]
and let $a = \frac{A}{a_1}$. Then we have the structure equations
\[
\begin{align*}
[e_4, e_3] &= e_2, & [e_4, e_6] &= e_3, & [e_5, e_6] &= e_4, & [e_1, e_7] &= e_1, & [e_2, e_7] &= 3ae_2, \\
[e_3, e_7] &= e_2 + 3ae_3, & [e_4, e_7] &= 2ae_4, & [e_5, e_7] &= ae_5, & [e_6, e_7] &= e_5 + ae_6,
\end{align*}
\]
with $a \in \mathbb{R}$. In the table in Appendix B this is $[7,[6,4],3,1]$.

**Subcase 3.1.1.2:**

In this section, we have that
\[
\text{ad} (f_7) = \begin{pmatrix}
  a_1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & -y_4 & y_3 & 0 \\
  0 & 0 & x_4 & x_3 & y_3 \\
  0 & 0 & 0 & x_4 & y_4 \\
  0 & 0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
with $x_3 \neq 0$. Note also that $a_1 \neq 0$ or the algebra is nilpotent and decomposable. Now we conjugate by $A_4A_7$ and let $s_7 = s_4$. As $a_1$ and $x_3$ are both nonzero, this allows us to scale $x_3$ to $\pm (a_1)^3$. Then we make the change of basis
\[ e_1 = f_1, \quad e_3 = (a_1)^2 f_3, \quad e_5 = f_5, \quad e_i = a_1 f_i, \quad \text{for } i \in \{2, 4, 6\}, \quad \text{and } e_7 = -\frac{1}{a_1} (f_7 - y_3 f_4 - y_4 f_5 + x_4 f_6), \]
which will yield the structure equations
\[
\begin{align*}
[e_4, e_3] &= e_2, & [e_4, e_6] &= e_3, & [e_5, e_6] &= e_4, & [e_1, e_7] &= e_1, \\
[e_3, e_7] &= e_2, & [e_5, e_7] &= \varepsilon e_3, & [e_6, e_7] &= e_5,
\end{align*}
\]
with $\varepsilon^2 = 1$. This is $[7,[6,4],3,2]$.

**Subcase 3.1.2:**

We assumed here that $a_1 = \lambda$ and that while we moved $y_1$ to 0, $x_1 \neq 0$.
\[
\text{ad} (f_7) = \begin{pmatrix}
  \lambda & 0 & 0 & 0 & x_1 & 0 \\
  0 & 3\lambda & 1 & -y_4 & x_2 & y_2 \\
  0 & 0 & 3\lambda & x_4 & x_3 & y_3 \\
  0 & 0 & 0 & 2\lambda & x_4 & y_4 \\
  0 & 0 & 0 & 0 & \lambda & 1 \\
  0 & 0 & 0 & 0 & 0 & \lambda
\end{pmatrix}.
\]
This implies that $\lambda \neq 0$ or the algebra is nilpotent. Then we conjugate by $A_{11}A_{12}A_{14}$ and, as $\lambda \neq 0$, move $y_2$ and $x_3$ to 0 and make the $x_2$ position equal the $y_3$ position. Next we conjugate by $A_1$ and, as $\lambda$ and $x_1$ are both nonzero, scale $x_1$ to $\lambda$. Finally, we make the basis change

$$e_1 = f_1, \quad e_3 = \lambda^2 f_3, \quad e_5 = f_5, \quad e_i = \lambda f_i, \quad \text{for } i \in \{2, 4, 6\}, \quad \text{and } e_7 = -\frac{1}{\lambda}(f_7 - y_3 f_4 - y_4 f_5 + x_4 f_6),$$

and we arrive at the structure equations

$$[e_4, e_5] = e_2, \quad [e_4, e_6] = e_3, \quad [e_5, e_6] = e_4, \quad [e_1, e_7] = e_1, \quad [e_2, e_7] = 3e_2,$$

$$[e_3, e_7] = e_2 + 3e_3, \quad [e_4, e_7] = 2e_4, \quad [e_5, e_7] = e_5 + e_5, \quad [e_6, e_7] = e_5 + e_6.$$

In the table, this is $[7,6,4],3,3$.

**Subcase 3.2:**

In this section, we assumed that $a_1 = 3\lambda$ and while we moved $a_2$ to 0, $a_3 \neq 0$. This yields that $\text{ad}(f_7)$ matrix

$$\text{ad}(f_7) = \begin{pmatrix}
3\lambda & 0 & 0 & 0 & x_1 & y_1 \\
0 & 3\lambda & 1 & -y_4 & x_2 & y_2 \\
a_3 & 0 & 3\lambda & x_4 & x_3 & y_3 \\
0 & 0 & 0 & 2\lambda & x_4 & y_4 \\
0 & 0 & 0 & 0 & \lambda & 1 \\
0 & 0 & 0 & 0 & 0 & \lambda
\end{pmatrix}.$$

This implies that $\lambda \neq 0$ or the algebra is nilpotent. This allows us to conjugate by $A_{10}A_{11}A_{12}A_{13}A_{14}$ and pick $s_{10}, \ldots, s_{14}$ to move $x_1, y_1, y_2,$ and $x_3$ to 0 and make the $x_2$ position equal to the $y_3$ position. Then we conjugate by $A_1$ and, as $a_3$ and $\lambda$ are both nonzero, scale $a_3$ to $\lambda^3$. Finally, we make the basis change

$$e_1 = f_1, \quad e_3 = \lambda^2 f_3, \quad e_5 = f_5, \quad e_i = \lambda f_i, \quad \text{for } i \in \{2, 4, 6\}, \quad \text{and } e_7 = -\frac{1}{\lambda}(f_7 - y_3 f_4 - y_4 f_5 + x_4 f_6),$$

and arrive at the structure equations

$$[e_4, e_5] = e_2, \quad [e_4, e_6] = e_3, \quad [e_5, e_6] = e_4, \quad [e_1, e_7] = 3e_1 + e_3, \quad [e_2, e_7] = 3e_2,$$

$$[e_3, e_7] = e_2 + 3e_3, \quad [e_4, e_7] = 2e_4, \quad [e_5, e_7] = e_5, \quad [e_6, e_7] = e_5 + e_6.$$

This is $[7,6,4],3,4$.

This completes the classification of seven dimensional algebras with this nilradical.
3.3 A Derivation Algebra with Semisimple Part Isomorphic to $\mathfrak{sp}(4, \mathbb{R})$

In this section, we classify those seven-dimensional Lie algebras, $\mathfrak{g}$, whose nilradical, $NR(\mathfrak{g})$, is isomorphic to the six-dimensional nilpotent algebra with structure equations

$$[f_3, f_5] = f_2, \quad [f_4, f_6] = f_2.$$ 

This is Nilradical 5 from the six-dimensional nilradicals listed in Appendix A. We again assume that we have a basis for $\mathfrak{g}$ such that the first six vectors, $f_1, \ldots, f_6$, form a basis for $NR(\mathfrak{g})$ and have the structure equations given above. Let $f_7$ be any vector not in $NR(\mathfrak{g})$. Then by the Jacobi property, $\text{ad} (f_7)$ must be of the form

$$\text{ad} (f_7) = \begin{pmatrix}
a_1 & 0 & c_1 & d_1 & x_1 & y_1 \\
b_2 & c_2 & d_2 & x_2 & y_2 \\
0 & 0 & c_3 & d_3 & x_3 & x_4 \\
0 & 0 & c_4 & d_4 & x_4 & y_4 \\
0 & 0 & c_5 & c_6 & b_2 - c_3 & -c_4 \\
0 & 0 & c_6 & d_6 & -d_3 & b_2 - d_4
\end{pmatrix}.$$ 

Let $m = \frac{b_2}{2}$, $n = c_3 - \frac{b_2}{2}$, and $p = d_4 - \frac{b_2}{2}$. Then we can rewrite $\text{ad} (f_7)$ as

$$\text{ad} (f_7) = \begin{pmatrix}
a_1 & 0 & c_1 & d_1 & x_1 & y_1 \\
2m & c_2 & d_2 & x_2 & y_2 \\
0 & 0 & m + n & d_3 & x_3 & x_4 \\
0 & 0 & c_4 & m + p & x_4 & y_4 \\
0 & 0 & c_5 & c_6 & m - n & -c_4 \\
0 & 0 & c_6 & d_6 & -d_3 & m - p
\end{pmatrix}.$$ 

Then the lower right hand $4 \times 4$ block is of the form $mI_4 + a$ where $I_4$ denotes the $4 \times 4$ identity and $a$ is of the form

$$a = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix},$$

where $A, B, C$ are all $2 \times 2$ matrices, $A$ is arbitrary, and $B$ and $C$ are symmetric. Then $a$ is an arbitrary element in the symplectic Lie algebra $\mathfrak{sp}(4, \mathbb{R})$ [5]. This will become especially important in a moment.

Next we look at the other nonzero $\text{ad}$ matrices. They are

$$\text{ad} (f_3) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \text{ad} (f_4) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

and
we find that a basis for the semisimple part is formed by the following matrices.

Using Maple to compute a basis for the derivation algebra and computing its Levi decomposition, this will effectively allow us to move the $e_2$, $d_2$, $x_2$, and $y_2$ positions to 0 by perturbing $f_7$.

To deal with the rest of the parameters in $\text{ad}(f_7)$, we must consider the automorphisms of $NR(g)$.

Using Maple to compute a basis for the derivation algebra and computing its Levi decomposition, we find that a basis for the semisimple part is formed by the following matrices.

\[
\begin{align*}
\text{ad}(f_5) &= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \\
\text{ad}(f_6) &= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\end{align*}
\]

This will effectively allow us to move the $c_2$, $d_2$, $x_2$, and $y_2$ positions to 0 by perturbing $f_7$.

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

which we’ll call $D_1, \ldots, D_{10}$ respectively. The lower right hand $4 \times 4$ submatrices of $D_1, \ldots, D_{10}$ are all linearly independent and in $\mathfrak{sp}(4, \mathbb{R})$ as they are of the form discussed above. As $\dim \mathfrak{sp}(4, \mathbb{R}) = 10$, then this set of submatrices form a basis for $\mathfrak{sp}(4, \mathbb{R})$, and clearly the semisimple part of our derivation algebra is isomorphic to $\mathfrak{sp}(4, \mathbb{R})$. Then there exists an automorphism of $NR(g)$ of the form

\[
\begin{pmatrix}
I_2 & 0 \\
0 & S
\end{pmatrix},
\]

where $I_2$ is the $2 \times 2$ identity and $S$ is an arbitrary element in the symplectic group $Sp(4, \mathbb{R})$. Call this automorphism $A_0$. 

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If we use Maple to compute a full basis of derivations and exponentiate them, we find that the automorphism group of $NR(\mathfrak{g})$ is generated by the following one parameter groups of transformations

$$
\begin{align*}
A_1 &= \begin{pmatrix}
s_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, &
A_2 &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & s_2 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \\
A_3 &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & s_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, &
A_4 &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \\
A_5 &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, &
A_6 &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \\
A_7 &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, &
A_8 &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \\
A_9 &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, &
A_{10} &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \\
A_{11} &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, &
A_{12} &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \\
A_{13} &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, &
A_{14} &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & s_{14} & 0
\end{pmatrix}.
\end{align*}

\[
A_{15} = \begin{pmatrix}
1 & 0 & 0 & 0 & s_{15} & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad A_{16} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & s_{16} & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad A_{17} = \begin{pmatrix}
1 & 0 & 0 & 0 & s_{17} & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad A_{18} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & s_{18} \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad A_{19} = \begin{pmatrix}
1 & 0 & 0 & 0 & s_{19} & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad A_{20} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & s_{20} \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad A_{21} = \begin{pmatrix}
1 & 0 & 0 & 0 & s_{21} & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

Note that conjugating \( \text{ad}(f_7) \) by \( A_5, A_{11}, A_{16}, \) or \( A_{20} \) will affect most significantly those entries in \( \text{ad}(f_7) \) that we’re going to move to zero by perturbing \( f_7 \); consequently, these will be less useful in simplifying \( \text{ad}(f_7) \) that the others.

We consider now conjugating \( \text{ad}(f_7) \) by \( A_0 \). By block multiplication, the symplectic submatrix of \( A_0, S \), will only affect the lower right hand \( 4 \times 4 \) piece of \( \text{ad}(f_7) \). And as that part of \( \text{ad}(f_7) \) could be written as \( mI_4 + a \), we have that

\[
S^{-1}(mI_4 + a)S = S^{-1}(mI_4)S + S^{-1}aS = mI_4 + S^{-1}aS,
\]

As \( S \in Sp(4, \mathbb{R}) \) and \( a \in \mathfrak{sp}(4, \mathbb{R}) \) arbitrarily. Then, by the section on the real symplectic canonical form in Chapter 2, conjugation by \( A_0 \) can be used to put \( a \) into real symplectic canonical form. This yields ten parent cases.

1. \( \text{ad}(f_7) = \begin{pmatrix}
a_1 & 0 & c_1 & d_1 & x_1 & y_1 \\
a_2 & 2m & c_2 & d_2 & x_2 & y_2 \\
0 & 0 & m + \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & m + \mu & 0 & 0 \\
0 & 0 & 0 & 0 & m - \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & m - \mu
\end{pmatrix} \).
2. \( \text{ad} (f_7) = \begin{pmatrix}
  a_1 & 0 & c_1 & d_1 & x_1 & y_1 \\
  a_2 & 2m & c_2 & d_2 & x_2 & y_2 \\
  0 & 0 & m + \lambda & 0 & 0 & 0 \\
  0 & 0 & m & 0 & \varepsilon & 0 \\
  0 & 0 & 0 & m - \lambda & 0 & 0 \\
  0 & 0 & 0 & 0 & m & 0 \\
\end{pmatrix} \), with \( \varepsilon^2 = 1 \).

3. \( \text{ad} (f_7) = \begin{pmatrix}
  a_1 & 0 & c_1 & d_1 & x_1 & y_1 \\
  a_2 & 2m & c_2 & d_2 & x_2 & y_2 \\
  0 & 0 & m + \lambda & 1 & 0 & 0 \\
  0 & 0 & 0 & m + \lambda & 0 & 0 \\
  0 & 0 & 0 & 0 & m - \lambda & 0 \\
  0 & 0 & 0 & 0 & -1 & m - \lambda \\
\end{pmatrix} \).

4. \( \text{ad} (f_7) = \begin{pmatrix}
  a_1 & 0 & c_1 & d_1 & x_1 & y_1 \\
  a_2 & 2m & c_2 & d_2 & x_2 & y_2 \\
  0 & 0 & m & 0 & \varepsilon & 0 \\
  0 & 0 & 0 & m & 0 & \varepsilon \\
  0 & 0 & 0 & 0 & m & 0 \\
  0 & 0 & 0 & 0 & -1 & m \\
\end{pmatrix} \), with \( \varepsilon^2 = 1 \).

5. \( \text{ad} (f_7) = \begin{pmatrix}
  a_1 & 0 & c_1 & d_1 & x_1 & y_1 \\
  a_2 & 2m & c_2 & d_2 & x_2 & y_2 \\
  0 & 0 & m & 1 & 0 & 0 \\
  0 & 0 & 0 & m & 0 & \varepsilon \\
  0 & 0 & 0 & 0 & m & 0 \\
  0 & 0 & 0 & 0 & -1 & m \\
\end{pmatrix} \), with \( \varepsilon^2 = 1 \).

6. \( \text{ad} (f_7) = \begin{pmatrix}
  a_1 & 0 & c_1 & d_1 & x_1 & y_1 \\
  a_2 & 2m & c_2 & d_2 & x_2 & y_2 \\
  0 & 0 & m + \lambda & 0 & 0 & 0 \\
  0 & 0 & 0 & m & 0 & \varepsilon \mu \\
  0 & 0 & 0 & 0 & m - \lambda & 0 \\
  0 & 0 & 0 & -\varepsilon \mu & 0 & m \\
\end{pmatrix} \), with \( \mu \neq 0 \) and \( \varepsilon^2 = 1 \).

7. \( \text{ad} (f_7) = \begin{pmatrix}
  a_1 & 0 & c_1 & d_1 & x_1 & y_1 \\
  a_2 & 2m & c_2 & d_2 & x_2 & y_2 \\
  0 & 0 & m & 0 & \varepsilon & 0 \\
  0 & 0 & 0 & m & 0 & \delta \mu \\
  0 & 0 & 0 & 0 & m & 0 \\
  0 & 0 & 0 & -\delta \mu & 0 & m \\
\end{pmatrix} \), with \( \mu \neq 0 \) and \( \varepsilon^2 = \delta^2 = 1 \).

8. \( \text{ad} (f_7) = \begin{pmatrix}
  a_1 & 0 & c_1 & d_1 & x_1 & y_1 \\
  a_2 & 2m & c_2 & d_2 & x_2 & y_2 \\
  0 & 0 & m + \lambda & \mu & 0 & 0 \\
  0 & 0 & -\mu & m + \lambda & 0 & 0 \\
  0 & 0 & 0 & 0 & m - \lambda & \mu \\
  0 & 0 & 0 & 0 & -\mu & m - \lambda \\
\end{pmatrix} \), with \( \mu \neq 0 \) and the eigenvalues ordered so that \( \frac{\delta}{\mu} \geq 0 \).

9. \( \text{ad} (f_7) = \begin{pmatrix}
  a_1 & 0 & c_1 & d_1 & x_1 & y_1 \\
  a_2 & 2m & c_2 & d_2 & x_2 & y_2 \\
  0 & 0 & m & 0 & \varepsilon \mu & 0 \\
  0 & 0 & 0 & m & \varepsilon \eta & 0 \\
  0 & 0 & -\varepsilon \mu & 0 & m & 0 \\
  0 & 0 & 0 & -\varepsilon \eta & 0 & m \\
\end{pmatrix} \), with \( \eta \neq -\mu, \eta, \mu \neq 0 \), and \( \varepsilon^2 = 1 \).
10. \( \text{ad} (f_7) = \begin{pmatrix} a_1 & 0 & c_1 & d_1 & x_1 & y_1 \\ a_2 & 2m & c_2 & d_2 & x_2 & y_2 \\ 0 & 0 & m & \mu & 1 & 0 \\ 0 & 0 & -\mu & m & 0 & 1 \\ 0 & 0 & 0 & 0 & m & \mu \\ 0 & 0 & 0 & 0 & -\mu & m \end{pmatrix} \), with \( \mu \neq 0 \) and the eigenvalues ordered so that \( \frac{\mu}{\mu} \geq 0 \).

From here, the classification follows in a similar manner as the previous sections and so we omit the remainder of the classification proof. The resultant algebras, however, are in the table in Appendix B.

3.4 A Derivation Algebra with Semisimple Part Isomorphic to \( \text{so}(3,1,\mathbb{R}) \).

In this section, we classify those seven dimensional Lie algebras, \( g \), whose nilradical, \( NR(g) \), is isomorphic to the six dimensional nilpotent algebra with structure equations

\[
[f_3, f_5] = f_2, \quad [f_3, f_6] = f_1, \quad [f_4, f_3] = -f_1, \quad [f_4, f_6] = f_2.
\]

This is Nilradical 9 from the six dimensional nilradicals listed in Appendix A. We again assume that we have a basis for \( g \) such that the first six vectors, \( f_1, \ldots, f_6 \), form a basis for \( NR(g) \) and have the structure equations given above. Let \( f_7 \) be any vector not in \( NR(g) \). Then by the Jacobi property, \( \text{ad} (f_7) \) must be of the form

\[
\text{ad} (f_7) = \begin{pmatrix} c_3 + x_5 & -c_4 + x_6 & c_1 & d_1 & x_1 & y_1 \\ c_4 - x_6 & c_3 + x_5 & c_2 & d_2 & x_2 & y_2 \\ 0 & 0 & c_3 & -c_4 & x_3 & x_4 \\ 0 & 0 & c_4 & c_3 & x_4 & -x_3 \\ 0 & 0 & c_5 & c_6 & x_5 & -x_6 \\ 0 & 0 & c_6 & -c_5 & x_6 & -x_5 \end{pmatrix}.
\]

Let \( m = \frac{1}{2}(c_4 + x_5) \), \( n = \frac{1}{2}(-c_4 + x_6) \), \( p = \frac{1}{2}(c_4 - x_5) \), and \( q = \frac{1}{2}(c_4 + x_6) \). Then we can rewrite \( \text{ad} (f_7) \) as

\[
\text{ad} (f_7) = \begin{pmatrix} 2m & 2n & c_1 & d_1 & x_1 & y_1 \\ -2n & 2m & c_2 & d_2 & x_2 & y_2 \\ 0 & 0 & m + p & n - q & x_3 & x_4 \\ 0 & 0 & -n + q & m + p & x_4 & -x_3 \\ 0 & 0 & c_5 & c_6 & m - p & -n - q \\ 0 & 0 & c_6 & -c_5 & n + q & m - p \end{pmatrix}.
\]

Let \( I_4 \) denote the \( 4 \times 4 \) identity and let

\[
K = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\]
Then the lower right hand $4 \times 4$ submatrix of $\text{ad}(f_7)$ is of the form $mI_4 + nK + a$ where $a$ is of the form

$$a = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix},$$

with $A, B, C$ all $2 \times 2$ matrices such that $A = \begin{pmatrix} \frac{p}{q} & -q \\ q & p \end{pmatrix}$ and $B$ and $C$ are trace-free symmetric. Then $a$ is an arbitrary element of $h(J_2)$, the nonstandard representation of $\mathfrak{so}(3,1,\mathbb{R})$ that we studied in Chapter 2. Again, this will be important later.

Next, we look at the other nonzero ad matrices. They are

$$\text{ad}(f_3) = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$\text{ad}(f_4) = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$\text{ad}(f_5) = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$\text{ad}(f_6) = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{pmatrix}. $$

If we can move $y_1$ to $-x_1$, $d_1$ to $-c_2$, and $d_2$ to $c_1$ in $\text{ad}(f_7)$, then we can perturb $f_7$ to annihilate the entire upper left hand $2 \times 4$ block.

In order to do this and to take care of the remaining parameters, we need to compute the derivation algebra. We use Maple to compute a basis for the Lie algebra of derivations of $NR(g)$ and its Levi decomposition. After doing this, we’ll find that a basis for the semisimple part of the derivation algebra is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$
which we’ll call \( D_1, \ldots, D_6 \) respectively. Note that the lower right hand \(4 \times 4\) submatrices of \( D_1, \ldots, D_6 \) form a basis \( \mathfrak{h}(J_2) \). This implies that the semisimple part of the derivation algebra is naturally isomorphic to \( \mathfrak{h}(J_2) \). In addition, this implies that the matrix

\[
A_0 = \begin{pmatrix} I_2 & 0 \\ 0 & S \end{pmatrix},
\]

with \( S \) an arbitrary member of \( H(J_2) \) is an automorphism of \( NR(\mathfrak{g}) \).

At this point, we use Maple to compute a complete basis of the Lie algebra of derivations of \( NR(\mathfrak{g}) \). We then exponentiate them to find that the automorphism group of \( NR(\mathfrak{g}) \) is generated by the following one parameter groups of transformations:

\[
\begin{align*}
A_1 &= \begin{pmatrix} s_1 & 0 & 0 & 0 & 0 \\ 0 & s_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & s_1 & 0 \\ 0 & 0 & 0 & 0 & s_1 \end{pmatrix}, & A_2 &= \begin{pmatrix} \cos(s_2) & -\sin(s_2) & 0 & 0 & 0 & 0 \\ \sin(s_2) & \cos(s_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sin(s_2) & \cos(s_2) \end{pmatrix}, \\
A_3 &= \begin{pmatrix} 1 & 0 & s_3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, & A_4 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & s_4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
A_5 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & s_5 & 0 & 0 \\ 0 & 0 & 0 & s_6 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{s_6} \end{pmatrix}, & A_6 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cos(s_6) & -\sin(s_6) & 0 \\ 0 & 0 & \sin(s_6) & \cos(s_6) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
A_7 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & s_7 & 0 \\ 0 & 0 & 0 & 0 & -s_7 \end{pmatrix}, & A_8 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & s_8 & 1 \\ 0 & 0 & 0 & 0 & s_8 \end{pmatrix}.
\]
Now note that if $S \in H(J_2)$, then $S^t J_i S = J_i$ for all $i \in \{1, 2\}$. This implies that $J_i S = (S^t)^{-1} J_i$ and $S^t J_i = J_i S^{-1}$. Then, as $(J_i)^{-1} = -J_i$, we have

$$S^{-1} J_i = (-J_i S)^{-1} = (-(S^t)^{-1} J_i)^{-1} = J_i S^t$$

for all $i \in \{1, 2\}$. Hence we have that

$$S^{-1} J_1 J_2 S = J_1 S^t J_2 S = J_1 J_2 S^{-1} S = J_1 J_2.$$ 

Moreover, note that $J_1 J_2 = K$ and hence $S^{-1} K S = K$.

By block multiplication, we have that the lower right hand $4 \times 4$ block of $(A_0)^{-1} \text{ad}(f_2) A_0$ will be $S^{-1}(mI_4 + nK + a)S$. This yields

$$S^{-1}(mI_4 + nK + a)S = m(S^{-1} I_4 S) + n(S^{-1} KS) + S^{-1} a S = mI_4 + nK + S^{-1} a S.$$ 

As $S$ is an arbitrary member of the group $H(J_2)$ and $a$ is an arbitrary member of $h(J_2)$, then we
pick $S$ to put $a$ into real $\mathfrak{h}$-symplectic canonical form. This yields three parent cases

1. $\text{ad} (f_7) = \begin{pmatrix}
2m & 2n & c_1 & d_1 & x_1 & y_1 \\
-2n & 2m & c_2 & d_2 & x_2 & y_2 \\
0 & 0 & m + \lambda & n & 0 & 0 \\
0 & 0 & -n & m + \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & m - \lambda & -n \\
0 & 0 & 0 & 0 & n & m - \lambda \\
\end{pmatrix}$.

2. $\text{ad} (f_7) = \begin{pmatrix}
2m & 2n & c_1 & d_1 & x_1 & y_1 \\
-2n & 2m & c_2 & d_2 & x_2 & y_2 \\
0 & 0 & m & n & \varepsilon & 0 \\
0 & 0 & -n & m & 0 & -\varepsilon \\
0 & 0 & 0 & 0 & m & -n \\
0 & 0 & 0 & 0 & n & m \\
\end{pmatrix}$, with $\varepsilon^2 = 1$.

3. $\text{ad} (f_7) = \begin{pmatrix}
2m & 2n & c_1 & d_1 & x_1 & y_1 \\
-2n & 2m & c_2 & d_2 & x_2 & y_2 \\
0 & 0 & m + \lambda & n - \varepsilon \mu & 0 & 0 \\
0 & 0 & -n + \varepsilon \mu & m + \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & m - \lambda & -n - \varepsilon \mu \\
0 & 0 & 0 & 0 & n + \varepsilon \mu & m - \lambda \\
\end{pmatrix}$, with $\mu > 0$ and $\varepsilon^2 = 1$.

From here, the classification runs similar to the previous section and so we omit the remainder of the classification proof. However, the algebras are listed in Appendix B.

This completes our sample of the classification of the seven dimensional algebras and the text of this paper. The multiplication tables of all the isomorphism classes of indecomposable solvable Lie algebras of dimension two through dimension seven with codimension one nilradicals can be found in Appendix B.
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Appendix A

A LIST OF NILPOTENT ALGEBRAS USED AS NILRADICALS

In this appendix, we simply list and number the multiplication tables of the nilpotent algebras from dimension three through dimension six (for dimension one and two the only nilpotent algebra is the abelian algebra). These are what we used as nilradicals for our classification.

The lists were compiled by other authors. Dimensions three through five are from Winternitz’s list; wherever possible, we’ve indicated the appropriate reference in Mubarakzyanov’s as well [7, 8]. In the classification, Winternitz and Mubarakzyanov’s names are abbreviated to Win and Mubar respectively. Dimension six is from Gong’s classification [1]. In using Gong’s classification, we have made a change of basis for every algebra. If \([e_1, \ldots, e_6]\) is the basis that Gong used, then we used the basis \([-e_6, \ldots, -e_1]\) for every six dimensional nilpotent algebra except number five; for that algebra, we instead used the basis \([-e_6, -e_5, -e_4, -e_2, -e_3, -e_1]\). This was done for consistency in our classification.

In addition, we have imposed the numbering and will use it in the numbering scheme of Appendix [3]. Also we don’t list the abelian algebra of each dimension, \(m\), but we will refer to it as \([m, 0]\).
A.1 Dimension Three

[3, 1]

| e1 | e2 | e3 | Win: [3, 1] |
|----|----|----|-----------|
| e1 | .  | .  | e1        |
| e2 | .  | e1 | e2        |
| e3 | .  | .  | e3        |

A.2 Dimension Four

[4, 1]

| e1 | e2 | e3 | e4 | Win: [4, 1] |
|----|----|----|----|-----------|
| e1 | .  | .  | .  | e1        |
| e2 | .  | e1 | e2 | e2        |
| e3 | .  | .  | .  | e3        |
| e4 | .  | .  | .  | e4        |

[4, 2]

| e1 | e2 | e3 | e4 | Win: [4, 1] |
|----|----|----|----|-----------|
| e1 | .  | .  | .  | e1        |
| e2 | .  | e1 | e2 | e2        |
| e3 | .  | .  | .  | e3        |
| e4 | .  | .  | .  | e4        |

A.3 Dimension Five

[5, 1]

| e1 | e2 | e3 | e4 | e5 | Win: [5, 1] |
|----|----|----|----|----|-----------|
| e1 | .  | .  | .  | .  | e1        |
| e2 | .  | e1 | e2 | e2 | e2        |
| e3 | .  | .  | .  | .  | e3        |
| e4 | .  | .  | .  | .  | e4        |
| e5 | .  | .  | .  | .  | e5        |

[5, 2]

| e1 | e2 | e3 | e4 | e5 | Win: [5, 2] |
|----|----|----|----|----|-----------|
| e1 | .  | .  | .  | .  | e1        |
| e2 | .  | e1 | e2 | e2 | e2        |
| e3 | .  | .  | .  | .  | e3        |
| e4 | .  | .  | .  | .  | e4        |
| e5 | .  | .  | .  | .  | e5        |

[5, 3]

| e1 | e2 | e3 | e4 | e5 | Win: [5, 1] |
|----|----|----|----|----|-----------|
| e1 | .  | .  | .  | .  | e1        |
| e2 | .  | .  | e1 | e1 | e1        |
| e3 | .  | .  | e2 | e2 | e2        |
| e4 | .  | .  | e3 | e3 | e3        |
| e5 | .  | .  | e4 | e4 | e4        |

[5, 4]

| e1 | e2 | e3 | e4 | e5 | Win: [5, 2] |
|----|----|----|----|----|-----------|
| e1 | .  | .  | .  | .  | e1        |
| e2 | .  | .  | e1 | e1 | e1        |
| e3 | .  | .  | e2 | e2 | e2        |
| e4 | .  | .  | e3 | e3 | e3        |
| e5 | .  | .  | e4 | e4 | e4        |

[5, 5]

| e1 | e2 | e3 | e4 | e5 | Win: [5, 3] |
|----|----|----|----|----|-----------|
| e1 | .  | .  | .  | .  | e1        |
| e2 | .  | .  | e1 | e1 | e1        |
| e3 | .  | .  | e2 | e2 | e2        |
| e4 | .  | .  | e3 | e3 | e3        |
| e5 | .  | .  | e4 | e4 | e4        |

[5, 6]

| e1 | e2 | e3 | e4 | e5 | Win: [5, 4] |
|----|----|----|----|----|-----------|
| e1 | .  | .  | .  | .  | e1        |
| e2 | .  | .  | e1 | e1 | e1        |
| e3 | .  | .  | e2 | e2 | e2        |
| e4 | .  | .  | e3 | e3 | e3        |
| e5 | .  | .  | e4 | e4 | e4        |

[5, 7]

| e1 | e2 | e3 | e4 | e5 | Win: [5, 5] |
|----|----|----|----|----|-----------|
| e1 | .  | .  | .  | .  | e1        |
| e2 | .  | .  | e1 | e1 | e1        |
| e3 | .  | .  | e2 | e2 | e2        |
| e4 | .  | .  | e3 | e3 | e3        |
| e5 | .  | .  | e4 | e4 | e4        |

Mubar: [5, 1]

Mubar: [5, 2]

Mubar: [5, 3]

Mubar: [5, 4]

Mubar: [5, 5]

Mubar: [5, 6]

Mubar: [5, 7]
A.4 Dimension Six

| 5, 8 | 6, 5 |
|---|---|
| e₁ e₂ e₃ e₄ e₅ | e₁ e₂ e₃ e₄ e₅ e₆ |
| e₁ | . . . . . | e₁ | . . . . . |
| e₂ | . . e₁ | e₂ | . . . . . |
| e₃ | . e₁ e₂ | e₃ | . . e₂ |
| e₄ | . e₃ | e₄ | . . e₂ |
| e₅ | . e₄ | e₅ | . . |
| e₆ | . | e₆ | . |

| 6, 1 | 6, 6 |
|---|---|
| e₁ e₂ e₃ e₄ e₅ e₆ | e₁ e₂ e₃ e₄ e₅ e₆ |
| e₁ | . . . . . | e₁ | . . . . . |
| e₂ | . . . . . | e₂ | . . . . . |
| e₃ | . e₂ | e₃ | . . e₂ |
| e₄ | . e₂ e₃ | e₄ | . . e₂ |
| e₅ | . e₄ | e₅ | . e₄ |
| e₆ | . | e₆ | . |

| 6, 2 | 6, 7 |
|---|---|
| e₁ e₂ e₃ e₄ e₅ e₆ | e₁ e₂ e₃ e₄ e₅ e₆ |
| e₁ | . . . . . | e₁ | . . . . . |
| e₂ | . . . . . | e₂ | . . . . . |
| e₃ | . e₂ | e₃ | . e₂ |
| e₄ | . e₃ | e₄ | . e₂ |
| e₅ | . e₄ | e₅ | . |
| e₆ | . | e₆ | . |

| 6, 3 | 6, 8 |
|---|---|
| e₁ e₂ e₃ e₄ e₅ e₆ | e₁ e₂ e₃ e₄ e₅ e₆ |
| e₁ | . . . . . | e₁ | . . . . . |
| e₂ | . . . . . | e₂ | . . . . . |
| e₃ | . e₂ | e₃ | . e₁ |
| e₄ | . e₃ | e₄ | . |
| e₅ | . e₄ | e₅ | . e₁ |
| e₆ | . | e₆ | . |

| 6, 4 | 6, 9 |
|---|---|
| e₁ e₂ e₃ e₄ e₅ e₆ | e₁ e₂ e₃ e₅ e₆ |
| e₁ | . . . . . | e₁ | . . . . . |
| e₂ | . . . . . | e₂ | . . . . . |
| e₃ | . e₂ e₃ | e₃ | . e₂ e₁ |
| e₄ | . e₃ | e₄ | . e₁ e₂ |
| e₅ | . e₄ | e₅ | . |
| e₆ | . | e₆ | . |

| 6, 5 | 6, 6 |
|---|---|
| e₁ e₂ e₃ e₄ e₅ e₆ | e₁ e₂ e₃ e₄ e₅ e₆ |
| e₁ | . . . . . | e₁ | . . . . . |
| e₂ | . . . . . | e₂ | . . . . . |
| e₃ | . e₃ | e₃ | . |
| e₄ | . e₄ | e₄ | . |
| e₅ | . | e₅ | . |
| e₆ | . | e₆ | . |
|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | .  |
| e₂ | .  | .  | .  | .  | .  | .  |
| e₃ | .  | .  | e₁ | .  | .  | .  |
| e₄ | .  | e₂ | e₃ | .  | .  | .  |
| e₅ | .  | e₄ | .  | .  | .  | .  |
| e₆ | .  | .  | .  | .  | .  | .  |

Gong: [6, N65]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | .  |
| e₂ | .  | .  | e₁ | .  | .  | .  |
| e₃ | .  | .  | e₂ | .  | .  | .  |
| e₄ | .  | e₃ | .  | .  | .  | .  |
| e₅ | .  | e₄ | .  | .  | .  | .  |
| e₆ | .  | .  | .  | .  | .  | .  |

Gong: [6, N611]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | .  |
| e₂ | .  | .  | e₁ | .  | .  | .  |
| e₃ | .  | e₁ | .  | .  | .  | .  |
| e₄ | .  | e₂ | e₃ | .  | .  | .  |
| e₅ | .  | e₄ | .  | .  | .  | .  |
| e₆ | .  | .  | .  | .  | .  | .  |

Gong: [6, N612]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | .  |
| e₂ | .  | .  | e₁ | .  | .  | .  |
| e₃ | .  | e₁ | .  | .  | .  | .  |
| e₄ | .  | e₂ | e₃ | .  | .  | .  |
| e₅ | .  | e₄ | .  | .  | .  | .  |
| e₆ | .  | .  | .  | .  | .  | .  |

Gong: [6, N613]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | .  |
| e₂ | .  | .  | e₁ | .  | .  | .  |
| e₃ | .  | e₁ | .  | .  | .  | .  |
| e₄ | .  | e₂ | e₃ | .  | .  | .  |
| e₅ | .  | e₄ | .  | .  | .  | .  |
| e₆ | .  | .  | .  | .  | .  | .  |

Gong: [6, N614]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | .  |
| e₂ | .  | e₁ | .  | .  | .  | .  |
| e₃ | .  | .  | e₁ | .  | .  | .  |
| e₄ | .  | e₂ | e₃ | .  | .  | .  |
| e₅ | .  | e₄ | .  | .  | .  | .  |
| e₆ | .  | .  | .  | .  | .  | .  |

Gong: [6, N615]
| 6, 20 | 6, 25 |
|-------|-------|
| e₁  e₂  e₃  e₄  e₅  e₆ | e₁  e₂  e₃  e₄  e₅  e₆ |
| Gong: [6, N625a] | Gong: [6, N629a] |
| e₁  .  .  .  .  . | e₁  .  .  .  .  . |
| e₂  .  .  – e₁  . | e₂  .  .  .  .  . |
| e₃  .  .  – e₁  . | e₃  .  .  e₂  e₁ |
| e₄  .  e₂  e₃ | e₄  .  – e₁  e₂ |
| e₅  .  e₄ | e₅  .  e₄ |
| e₆  . | e₆  . |

| 6, 21 | 6, 26 |
|-------|-------|
| e₁  e₂  e₃  e₄  e₅  e₆ | e₁  e₂  e₃  e₄  e₅  e₆ |
| Gong: [6, N626] | Gong: [6, N6210] |
| e₁  .  .  .  .  . | e₁  .  .  .  .  . |
| e₂  .  .  e₁  .  . | e₂  .  .  .  .  . |
| e₃  .  .  e₂ | e₃  .  .  e₂ |
| e₄  .  e₁  e₃ | e₄  .  e₁  e₂ |
| e₅  .  e₄ | e₅  .  e₃ |
| e₆  . | e₆  . |

| 6, 22 | 6, 27 |
|-------|-------|
| e₁  e₂  e₃  e₄  e₅  e₆ | e₁  e₂  e₃  e₄  e₅  e₆ |
| Gong: [6, N627] | Gong: [6, N631] |
| e₁  .  .  .  .  . | e₁  .  .  .  .  . |
| e₂  .  .  .  .  . | e₂  .  .  e₁  . |
| e₃  .  .  e₂ | e₃  .  .  e₁  . |
| e₄  .  e₁  e₃ | e₄  .  .  e₂ |
| e₅  .  e₄ | e₅  .  e₃ |
| e₆  . | e₆  . |

| 6, 23 | 6, 28 |
|-------|-------|
| e₁  e₂  e₃  e₄  e₅  e₆ | e₁  e₂  e₃  e₄  e₅  e₆ |
| Gong: [6, N628] | Gong: [6, N631a] |
| e₁  .  .  .  .  . | e₁  .  .  .  .  . |
| e₂  .  .  .  .  . | e₂  .  .  e₁  . |
| e₃  .  .  e₂  e₁ | e₃  .  .  e₁  . |
| e₄  .  .  e₂ | e₄  .  .  e₂ |
| e₅  .  e₄ | e₅  .  e₃ |
| e₆  . | e₆  . |

| 6, 24 | 6, 29 |
|-------|-------|
| e₁  e₂  e₃  e₄  e₅  e₆ | e₁  e₂  e₃  e₄  e₅  e₆ |
| Gong: [6, N629] | Gong: [6, N632] |
| e₁  .  .  .  .  . | e₁  .  .  .  .  . |
| e₂  .  .  .  .  . | e₂  .  e₁  . |
| e₃  .  .  e₁ | e₃  .  .  .  . |
| e₄  .  e₁  e₂ | e₄  .  e₁  . |
| e₅  .  e₄ | e₅  .  e₄ |
| e₆  . | e₆  . |
\begin{tabular}{c|cccccc}
| 6, 30 | e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
|-------|-----|-----|-----|-----|-----|-----|
| e_1  | .   | .   | .   | .   | .   | .   |
| e_2  | .   | .   | .   | .   | .   | .   |
| e_3  | .   | .   | .   | e_1 | .   | .   |
| e_4  | .   | .   | e_2 | .   | .   | .   |
| e_5  | .   | .   | e_4 | .   | .   | .   |
| e_6  | .   | .   | .   | .   | .   | .   |
\end{tabular}

Gong: [6, N633]

\begin{tabular}{c|cccccc}
| 6, 31 | e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
|-------|-----|-----|-----|-----|-----|-----|
| e_1  | .   | .   | .   | .   | .   | .   |
| e_2  | .   | .   | .   | .   | .   | .   |
| e_3  | .   | .   | .   | e_1 | .   | .   |
| e_4  | .   | .   | e_2 | .   | .   | .   |
| e_5  | .   | .   | e_4 | .   | .   | .   |
| e_6  | .   | .   | .   | .   | .   | .   |
\end{tabular}

Gong: [6, N634]

\begin{tabular}{c|cccccc}
| 6, 32 | e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
|-------|-----|-----|-----|-----|-----|-----|
| e_1  | .   | .   | .   | .   | .   | .   |
| e_2  | .   | .   | .   | .   | .   | .   |
| e_3  | .   | .   | .   | e_1 | .   | .   |
| e_4  | .   | .   | e_1 | .   | .   | .   |
| e_5  | .   | .   | e_2 | .   | .   | .   |
| e_6  | .   | .   | .   | .   | .   | .   |
\end{tabular}

Gong: [6, N635]

\begin{tabular}{c|cccccc}
| 6, 33 | e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
|-------|-----|-----|-----|-----|-----|-----|
| e_1  | .   | .   | .   | .   | .   | .   |
| e_2  | .   | .   | .   | .   | .   | .   |
| e_3  | .   | .   | .   | .   | .   | .   |
| e_4  | .   | e_1 | e_2 | .   | .   | .   |
| e_5  | .   | e_3 | .   | .   | .   | .   |
| e_6  | .   | .   | .   | .   | .   | .   |
\end{tabular}

Gong: [6, N636it]
Appendix B

MULTIPLICATION TABLES

In this appendix, we will list all of the isomorphism classes of two through seven dimensional real solvable indecomposable Lie algebras with codimension one nilradicals. Before we list them, however, we will describe the numbering system to the reader. Any given Lie algebra on the list will be given a number sequence. For example,

\[ [5, [4, 1], 3, 2]. \]

The first number in the list corresponds to the dimension of the Lie algebra; our example is a five dimensional algebra. The second list corresponds to the algebra’s nilradical. This first number is the dimension of the nilradical and the second number corresponds to the numbering of the nilradicals given in Appendix A with the convention that as the abelian nilradical was given no number, we will number it with a 0. Our example has the first four dimensional non-abelian nilradical. The third number corresponds to the number of the parent case. For instance, if there was a $2 \times 2$ block that was moved into real Jordan canonical form, it would create three “parent” cases, one for each possible real Jordan canonical form. Our example is in the third parent case. Finally, the fourth number is the number of the algebra produced in that parent case. To put it all together, our example is the second algebra that came from the third parent case of the first non-abelian nilradical of a five dimensional Lie algebra.

Wherever possible, we have indicated the appropriate reference to the classification lists of both Winternitz and Mubarakzyanov. As in Appendix A, Winternitz and Mubarakzyanov’s names are abbreviated to Win and Mubar respectively.
B.1 Dimension Two

$[2, [1, 0], 1, 1]$

| $e_1$ | $e_2$ |
|-------|-------|
| $e_1$ | $e_1$ |

B.2 Dimension Three

$[3, [2, 0], 1, 1]$

| $e_1$ | $e_2$ | $e_3$ |
|-------|-------|-------|
| $e_1$ | $a e_1 - e_2$ |
| $e_2$ | $e_1 + a e_2$ |
| $e_3$ | $b e_3$ |

parameters: $[a, b]$



\[0 \leq a, b \neq 0\]

B.3 Dimension Four

$[4, [3, 0], 2, 1]$

| $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|-------|-------|-------|-------|
| $e_1$ | $e_2$ | $a$ |
| $e_2$ | $e_1 + a e_2$ |
| $e_3$ | $b e_3$ |
| $e_4$ | |

Win: $[4, 6]$

parameters: $[a, b]$

\[0 \leq a, b \neq 0\]

$[4, [3, 0], 3, 1]$

| $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|-------|-------|-------|-------|
| $e_1$ | $e_3$ |
| $e_2$ | $e_1 + a e_2$ |
| $e_3$ | $e_3$ |
| $e_4$ | |

Win: $[4, 2]$

\[0 \leq a\]

$[4, [3, 0], 4, 1]$

| $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|-------|-------|-------|-------|
| $e_1$ | $e_2$ | $e_3$ |
| $e_2$ | $e_1 + a e_2$ |
| $e_3$ | $e_2 + e_3$ |
| $e_4$ | |

Win: $[4, 4]$

\[0 \leq a\]

$[4, [3, 0], 5, 1]$

| $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|-------|-------|-------|-------|
| $e_1$ | $e_2$ | $e_3$ |
| $e_2$ | $e_1 + a e_2$ |
| $e_3$ | $e_2 + e_3$ |
| $e_4$ | |

Win: $[4, 8]$

\[0 \leq a\]

$[4, [3, 0], 6, 1]$

| $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|-------|-------|-------|-------|
| $e_1$ | $e_2$ | $e_3$ |
| $e_2$ | $e_1 + a e_2$ |
| $e_3$ | $e_2 + e_3$ |
| $e_4$ | |

Win: $[4, 10]$

\[0 \leq a\]

$[4, [3, 0], 7, 1]$

| $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|-------|-------|-------|-------|
| $e_1$ | $e_2$ | $e_3$ |
| $e_2$ | $e_1 + a e_2$ |
| $e_3$ | $e_2 + e_3$ |
| $e_4$ | |

Win: $[4, 11]$

\[0 \leq a\]

$[4, [3, 1], 1, 1]$

| $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|-------|-------|-------|-------|
| $e_1$ | $e_2$ | $e_3$ |
| $e_2$ | $e_1 + a e_2$ |
| $e_3$ | $e_2 + e_3$ |
| $e_4$ | |

Win: $[4, 9]$

\[0 \leq a\]

$[4, [3, 1], 2, 1]$

| $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|-------|-------|-------|-------|
| $e_1$ | $e_2$ | $e_3$ |
| $e_2$ | $e_1 + a e_2$ |
| $e_3$ | $e_2 + e_3$ |
| $e_4$ | |

Win: $[4, 11]$

\[0 \leq a\]
B.4 Dimension Five

[5, [4, 0], 1, 1]

\[
\begin{array}{cccc}
E_1 & E_2 & E_3 & E_4 & E_5 \\
\cdots & \cdots & \cdots & \cdots & E_1 \\
E_2 & \cdots & a E_2 & \cdots & \cdots \\
E_3 & \cdots & b E_3 & \cdots & \cdots \\
E_4 & \cdots & c E_4 & \cdots & \cdots \\
E_5 & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

Win: [5, 7]

Win: [5, 8]

Mubar: [5, 7]

Mubar: [5, 8]

parameters: [a, b, c]

\([-1 \leq c, c \leq b, b \leq a, a \leq 1, a \neq 0, b \neq 0, c \neq 0]\)

[5, [4, 0], 2, 1]

\[
\begin{array}{cccc}
E_1 & E_2 & E_3 & E_4 & E_5 \\
\cdots & \cdots & \cdots & \cdots & E_1 \\
E_2 & \cdots & E_1 + a E_2 & \cdots & \cdots \\
E_3 & \cdots & b E_3 & \cdots & \cdots \\
E_4 & \cdots & c E_4 & \cdots & \cdots \\
E_5 & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

Win: [5, 13]

Win: [5, 14]

Mubar: [5, 13]

Mubar: [5, 14]

parameters: [a, b, c]

\([-1 \leq a, 0 \leq b, b \neq 0, c \neq 0]\)

[5, [4, 0], 3, 1]

\[
\begin{array}{cccc}
E_1 & E_2 & E_3 & E_4 & E_5 \\
\cdots & \cdots & \cdots & \cdots & E_1 \\
E_2 & \cdots & E_1 + a E_2 & \cdots & \cdots \\
E_3 & \cdots & E_3 & \cdots & \cdots \\
E_4 & \cdots & b E_4 & \cdots & \cdots \\
E_5 & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

Win: [5, 8]

Win: [5, 9]

Mubar: [5, 8]

Mubar: [5, 9]

parameters: [a, b]

\([-1 \leq b, b \leq 1, b \neq 0]\)

[5, [4, 0], 4, 1]

\[
\begin{array}{cccc}
E_1 & E_2 & E_3 & E_4 & E_5 \\
\cdots & \cdots & \cdots & \cdots & a E_1 - E_2 \\
E_2 & \cdots & E_1 + a E_2 & \cdots & \cdots \\
E_3 & \cdots & b E_3 - c E_4 & \cdots & \cdots \\
E_4 & \cdots & c E_3 + b E_4 & \cdots & \cdots \\
E_5 & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

Win: [5, 17]

Win: [5, 16]

Mubar: [5, 17]

Mubar: [5, 16]

parameters: [a, b, c]

\([-1 \leq a, 0 < c, c \leq 1]\)

[5, [4, 0], 5, 1]

\[
\begin{array}{cccc}
E_1 & E_2 & E_3 & E_4 & E_5 \\
\cdots & \cdots & \cdots & \cdots & a E_1 \\
E_2 & \cdots & E_1 + E_2 & \cdots & \cdots \\
E_3 & \cdots & a E_3 & \cdots & \cdots \\
E_4 & \cdots & E_3 + a E_4 & \cdots & \cdots \\
E_5 & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

Win: [5, 14]

Win: [5, 15]

Mubar: [5, 14]

Mubar: [5, 15]

parameters: [a]

\([-1 \leq a, a \leq 1]\)

[5, [4, 0], 6, 1]

\[
\begin{array}{cccc}
E_1 & E_2 & E_3 & E_4 & E_5 \\
\cdots & \cdots & \cdots & \cdots & E_1 \\
E_2 & \cdots & E_1 + E_2 & \cdots & \cdots \\
E_3 & \cdots & a E_3 & \cdots & \cdots \\
E_4 & \cdots & E_3 + a E_4 & \cdots & \cdots \\
E_5 & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

Win: [5, 15]

Win: [5, 15]

Mubar: [5, 15]

Mubar: [5, 15]

parameters: [a]

\([-1 \leq a, a \leq 1]\)

[5, [4, 0], 7, 1]

\[
\begin{array}{cccc}
E_1 & E_2 & E_3 & E_4 & E_5 \\
\cdots & \cdots & \cdots & \cdots & a E_1 \\
E_2 & \cdots & E_1 + a E_2 & \cdots & \cdots \\
E_3 & \cdots & E_2 + a E_3 & \cdots & \cdots \\
E_4 & \cdots & E_4 & \cdots & \cdots \\
E_5 & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

Win: [5, 10]

Win: [5, 11]

Mubar: [5, 10]

Mubar: [5, 11]

parameters: [a]

\([-1 \leq a, a \neq 0]\)
| [5, [4, 0], 8, 1] | [5, [4, 1], 1, 3] |
|---|---|
| \(e_1\) | \(e_1\) |
| \(e_2\) | \(e_1\) |
| \(e_3\) | \(e_2\) |
| \(e_4\) | \(e_3\) |
| \(e_5\) | \(e_4\) |
| \(e_1 \ldots \) \(a e_1 - e_2\) \(e_1\) \(e_2\) | \(e_1 \ldots \) \(e_1\) \(e_2\) |
| \(e_2 \ldots \) \(e_1 + a e_2\) | \(e_2 \ldots \) \(e_2 + e_3\) |
| \(e_3 \ldots \) \(e_1 + e_3 - e_4\) | \(e_3 \ldots \) \(e_1 + (a + 1) e_4\) |
| \(e_4 \ldots \) \(e_2 + e_3 + a e_4\) | \(e_4 \ldots \) \(e_3 + e_4\) |
| \(e_5 \ldots \) \(e_3 + e_4\) | \(e_5 \ldots \) |

parameters: \([a]\) \n\([0 \leq a]\]

| [5, [4, 0], 9, 1] | [5, [4, 1], 1, 4] |
|---|---|
| \(e_1\) | \(e_1\) |
| \(e_2\) | \(e_2\) |
| \(e_3\) | \(e_3\) |
| \(e_4\) | \(e_4\) |
| \(e_5\) | \(e_5\) |
| \(e_1 \ldots \) \(e_1\) \(e_1 \ldots \) \((a+1) e_1\) \(e_2\) \(e_2\) \(e_1 + (a+1) e_4\) | \(e_1 \ldots \) \(e_1\) \(e_2\) \(e_3\) \(e_4\) \(e_5\) |
| \(e_2 \ldots \) \(e_1 + e_2\) | \(e_2 \ldots \) \(a e_3\) |
| \(e_3 \ldots \) \(e_2 + e_3\) | \(e_3 \ldots \) \(e_1 + e_3\) |
| \(e_4 \ldots \) \(b e_4\) | \(e_4 \ldots \) \(e_1 + e_4\) |
| \(e_5 \ldots \) \(e_1 + e_4\) | \(e_5 \ldots \) |

parameters: \([a, b]\) \n\([[-1 \leq a, a \leq 1, b \neq 0]]\]

| [5, [4, 1], 1, 2] | [5, [4, 1], 2, 1] |
|---|---|
| \(e_1\) | \(e_1\) |
| \(e_2\) | \(e_2\) |
| \(e_3\) | \(e_3\) |
| \(e_4\) | \(e_4\) |
| \(e_5\) | \(e_5\) |
| \(e_1 \ldots \) \(a e_1\) \(e_1 \ldots \) \(2 a e_1\) \(e_2\) | \(e_1 \ldots \) \(2 a e_1\) |
| \(e_2 \ldots \) \(e_1\) \(e_2 + e_4\) | \(e_2 \ldots \) \(e_2 + e_3\) |
| \(e_3 \ldots \) \(a e_3\) | \(e_3 \ldots \) \(e_2 + a e_3\) |
| \(e_4 \ldots \) \(e_4\) | \(e_4 \ldots \) \(b e_4\) |
| \(e_5 \ldots \) | \(e_5 \ldots \) |

parameters: \([a, b]\) \n\([0 \leq a, b \neq 0]\]
### Table 1: 

|      | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ |
|------|-------|-------|-------|-------|-------|
| **5, [4, 1], 2, 2** |       |       |       |       |       |
| $e_1$ | $\ldots$ |       |       | $2a e_1$ |       |
| $e_2$ |       | $e_1$ |       | $a e_3 - e_3$ |       |
| $e_3$ |       |       | $e_2 + a e_3$ |       |       |
| $e_4$ |       |       |       | $e_1 + 2a e_4$ |       |
| $e_5$ |       |       |       |       |       |

Parameters: $a$ 

Conditions: $0 \leq a$

- **Win:** [5, 26]  
- **Mubar:** [5, 26]

### Table 2: 

|      | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ |
|------|-------|-------|-------|-------|-------|
| **5, [4, 2], 1, 1** |       |       |       |       |       |
| $e_1$ | $\ldots$ |       |       | $e_1$ |       |
| $e_2$ |       | $e_1$ |       | $e_2$ |       |
| $e_3$ |       | $e_2$ |       | $e_3$ |       |
| $e_4$ |       |       | $e_4$ |       |       |
| $e_5$ |       |       |       |       |       |

Parameters: $a$ 

Conditions: 

### Table 3: 

|      | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ |
|------|-------|-------|-------|-------|-------|
| **5, [4, 1], 3, 1** |       |       |       |       |       |
| $e_1$ | $\ldots$ |       |       | $2a e_1$ |       |
| $e_2$ |       | $e_1$ |       | $e_2$ |       |
| $e_3$ |       | $e_2 + e_3$ |       |       |       |
| $e_4$ |       |       | $e_1 + 2e_4$ |       |       |
| $e_5$ |       |       |       |       |       |

Parameters: $a$ 

Conditions: 

### Table 4: 

|      | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ |
|------|-------|-------|-------|-------|-------|
| **5, [4, 2], 1, 2** |       |       |       |       |       |
| $e_1$ | $\ldots$ |       |       | $e_1$ |       |
| $e_2$ |       | $e_1$ |       | $e_2$ |       |
| $e_3$ |       | $e_2$ |       | $e_3$ |       |
| $e_4$ |       |       | $e_4$ |       |       |
| $e_5$ |       |       |       |       |       |

### Table 5: 

|      | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ |
|------|-------|-------|-------|-------|-------|
| **5, [4, 2], 1, 3** |       |       |       |       |       |
| $e_1$ | $\ldots$ |       |       | $3e_1$ |       |
| $e_2$ |       | $e_1$ |       | $2e_2$ |       |
| $e_3$ |       | $e_2$ |       | $e_3$ |       |
| $e_4$ |       |       | $e_3 + e_4$ |       |       |
| $e_5$ |       |       |       |       |       |

### Table 6: 

|      | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ |
|------|-------|-------|-------|-------|-------|
| **5, [4, 2], 1, 4** |       |       |       |       |       |
| $e_1$ | $\ldots$ |       |       | $e_1$ |       |
| $e_2$ |       | $e_1$ |       | $e_2$ |       |
| $e_3$ |       | $e_2$ |       | $e_1 + e_3$ |       |
| $e_4$ |       |       | $e_4$ |       |       |
| $e_5$ |       |       |       |       |       |

Parameters: $c$ 

Conditions: $|e|^2 = 1$
### B.5 Dimension Six

#### [6, [5, 0], 1, 1]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | e₁ |
| e₂ | .  | .  | a  | e₂ | .  | e₂ |
| e₃ | .  | .  | b  | e₃ | .  | e₃ |
| e₄ | .  | .  | c  | e₄ | .  | e₂ |
| e₅ | .  | .  | d  | e₅ | .  | e₅ |
| e₆ | .  | .  | .  | .  | .  | .  |

Parameters: \([a, b, c, d]\)

\([-1 \leq d, d \leq c, c \leq b, b \leq a, a \neq 0, b \neq 0, c \neq 0, d \neq 0]\)

#### [6, [5, 0], 2, 1]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | .  | .  | .  | a  | e₁ | .  |
| e₂ | .  | .  | .  | e₁+a | e₂ | e₂ |
| e₃ | .  | .  | b  | e₃ | .  | e₃ |
| e₄ | .  | .  | c  | e₄ | .  | e₃ |
| e₅ | .  | .  | d  | e₅ | .  | e₅ |
| e₆ | .  | .  | .  | .  | .  | .  |

Parameters: \([a, b, c, d]\)

\([0 \leq c, c \leq b, b \leq a, a \neq 0, b \neq 0, c \neq 0, d \neq 0]\)

#### [6, [5, 0], 3, 1]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | .  | .  | .  | a  | e₁ | .  |
| e₂ | .  | .  | .  | e₁+a | e₂ | e₂ |
| e₃ | .  | .  | e  | e₃ | .  | e₃ |
| e₄ | .  | .  | b  | e₄ | .  | e₃ |
| e₅ | .  | .  | c  | e₅ | .  | e₅ |
| e₆ | .  | .  | .  | .  | .  | .  |

Parameters: \([a, b, c]\)

\([-1 \leq c, c \leq b, b \leq 1, b \neq 0, c \neq 0]\)

#### [6, [5, 0], 4, 1]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | (a−1) | e₆ |
| e₂ | .  | .  | .  | e₁+a | e₂ | e₂ |
| e₃ | .  | .  | b  | e₃-c | e₃ | e₃ |
| e₄ | .  | .  | c  | e₄+b | e₄ | e₄ |
| e₅ | .  | .  | d  | e₅ | .  | e₅ |
| e₆ | .  | .  | .  | .  | .  | .  |

Parameters: \([a, b, c, d]\)

\([0 \leq a, 0 < c, c \leq 1, d \neq 0]\)

#### [6, [5, 0], 5, 1]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | .  | .  | a  | e₁ | .  | e₁ |
| e₂ | .  | .  | e₁+a | e₂ | e₂ | e₂ |
| e₃ | .  | .  | b  | e₃ | .  | e₃ |
| e₄ | .  | .  | c  | e₄ | .  | e₄ |
| e₅ | .  | .  | d  | e₅ | .  | e₅ |
| e₆ | .  | .  | .  | .  | .  | .  |

Parameters: \([a, b, c]\)

\([c \neq 0, 0 \leq b]\)

#### [6, [5, 0], 6, 1]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | .  | .  | a  | e₁ | .  | e₁ |
| e₂ | .  | .  | e₁+a | e₂ | e₂ | e₂ |
| e₃ | .  | .  | b  | e₃ | .  | e₃ |
| e₄ | .  | .  | e₃+b | e₄ | e₄ | e₄ |
| e₅ | .  | .  | e  | e₅ | .  | e₅ |
| e₆ | .  | .  | .  | .  | .  | .  |

Parameters: \([a, b]\)

\([b \leq a]\)

#### [6, [5, 0], 7, 1]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | .  | .  | a  | e₁ | .  | e₁ |
| e₂ | .  | .  | e₁+a | e₂ | e₂ | e₂ |
| e₃ | .  | .  | e₂+a | e₃ | e₃ | e₃ |
| e₄ | .  | .  | e  | e₄ | e₄ | e₄ |
| e₅ | .  | .  | b  | e₅ | e₅ | e₅ |
| e₆ | .  | .  | .  | .  | .  | .  |

Parameters: \([a, b]\)

\([-1 \leq b, b \leq 1, b \neq 0]\)
### [6, [5, 0], 8, 1]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | a₁e₁ |
| e₂ | .  | .  | .  | .  | e₁+a₂e₂ |
| e₃ | .  | .  | .  | e₂+a₃e₃ |
| e₄ | .  | .  | b₄e₄−e₅ |
| e₅ | .  | .  | e₄+b₅e₅ |
| e₆ | .  | .  |

parameters: [a, b]

[[0 ≤ b]]

### [6, [5, 0], 9, 2]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | .  |
| e₂ | .  | .  | .  | .  | e₁+a₂e₂ |
| e₃ | .  | .  | .  | e₂+a₃e₃ |
| e₄ | .  | .  | e₄ |
| e₅ | .  | .  | e₄+e₅ |
| e₆ | .  | .  |

parameters: [a, b]

[[0 ≤ a], [b ≤ a, a² + b² ≠ 0, −1 ≤ c, c ≤ 1, c ≠ 0]]

### [6, [5, 0], 9, 1]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | e₁ |
| e₂ | .  | .  | .  | .  | e₁+a₂e₂ |
| e₃ | .  | .  | .  | e₂+a₃e₃ |
| e₄ | .  | .  | a₄e₄ |
| e₅ | .  | .  | e₄+a₅e₅ |
| e₆ | .  | .  |

parameters: [a]

[[[]]]

### [6, [5, 0], 10, 1]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | a₁e₁−e₂ |
| e₂ | .  | .  | .  | .  | e₁+a₂e₂ |
| e₃ | .  | .  | e₁+a₃e₃−e₄ |
| e₄ | .  | .  | e₂+e₃+a₄e₄ |
| e₅ | .  | .  | b₅e₅ |
| e₆ | .  | .  |

parameters: [a, b]

[[0 ≤ a, b ≠ 0]]

### [6, [5, 0], 11, 1]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | e₁ |
| e₂ | .  | .  | .  | .  | e₁+a₂e₂ |
| e₃ | .  | .  | .  | e₂+a₃e₃ |
| e₄ | .  | .  | .  | e₃+a₄e₄ |
| e₅ | .  | .  | e₄ |
| e₆ | .  | .  |

parameters: [a, b]

[[[]]]

### [6, [5, 0], 12, 1]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | e₁ |
| e₂ | .  | .  | .  | .  | e₁+a₂e₂ |
| e₃ | .  | .  | .  | e₂+a₃e₃ |
| e₄ | .  | .  | .  | e₃+a₄e₄ |
| e₅ | .  | .  | e₄+a₅e₅ |
| e₆ | .  | .  |

parameters: [a]

[[[]]]

### [6, [5, 0], 13, 2]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | (a+b₁)e₁ |
| e₂ | .  | .  | .  | .  | a₂e₂ |
| e₃ | .  | .  | .  | b₃e₃ |
| e₄ | .  | .  | e₄ |
| e₅ | .  | .  | e₅ |
| e₆ | .  | .  |

parameters: [a, b, c]

[[b ≤ a, a² + b² ≠ 0, −1 ≤ c, c ≤ 1, c ≠ 0]]

### [6, [5, 0], 14, 1]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | (a+b₁)e₁ |
| e₂ | .  | .  | .  | .  | a₂e₂ |
| e₃ | .  | .  | .  | b₃e₃ |
| e₄ | .  | .  | e₄ |
| e₅ | .  | .  | b₅e₅ |
| e₆ | .  | .  |

parameters: [a, b]

[[b ≤ a]]
| $[6, [5, 1], 1, 3]$ | $[6, [5, 1], 1, 4]$ | $[6, [5, 1], 1, 5]$ | $[6, [5, 1], 1, 6]$ | $[6, [5, 1], 1, 7]$ | $[6, [5, 1], 1, 8]$ | $[6, [5, 1], 2, 1]$ | $[6, [5, 1], 3, 1]$ |
|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| $e_1$ $e_2$ $e_3$ $e_4$ $e_5$ $e_6$ | $e_1$ $e_2$ $e_3$ $e_4$ $e_5$ $e_6$ | $e_1$ $e_2$ $e_3$ $e_4$ $e_5$ $e_6$ | $e_1$ $e_2$ $e_3$ $e_4$ $e_5$ $e_6$ | $e_1$ $e_2$ $e_3$ $e_5$ $e_6$ | $e_1$ $e_2$ $e_3$ $e_5$ $e_6$ | $e_1$ $e_2$ $e_3$ $e_4$ $e_5$ $e_6$ | $e_1$ $e_2$ $e_3$ $e_5$ $e_6$ |
| $e_1$ $e_2$ $e_3$ $e_4$ $e_5$ $e_6$ | $e_1$ $e_2$ $e_3$ $e_4$ $e_5$ $e_6$ | $e_1$ $e_2$ $e_3$ $e_4$ $e_5$ $e_6$ | $e_1$ $e_2$ $e_3$ $e_4$ $e_5$ $e_6$ | $e_1$ $e_2$ $e_3$ $e_5$ $e_6$ | $e_1$ $e_2$ $e_3$ $e_5$ $e_6$ | $e_1$ $e_2$ $e_3$ $e_4$ $e_5$ $e_6$ | $e_1$ $e_2$ $e_3$ $e_5$ $e_6$ |
| parameters: $[a, b]$ | parameters: $[a]$ | parameters: $[a, b, c]$ | parameters: $[a]$ | | | | |
| $[6, [5, 1], 1, 3]$ | $[6, [5, 1], 1, 4]$ | $[6, [5, 1], 1, 5]$ | $[6, [5, 1], 1, 6]$ | $[6, [5, 1], 1, 7]$ | $[6, [5, 1], 1, 8]$ | $[6, [5, 1], 2, 1]$ | $[6, [5, 1], 3, 1]$ |

Mubar: $[6, 17]$  
Mubar: $[6, 18]$  
Mubar: $[6, 19]$  
Mubar: $[6, 20]$  
Mubar: $[6, 15]$  
Mubar: $[6, 16]$  
Mubar: $[6, 35]$  
Mubar: $[6, 25]$  

$([-1 \leq a, b \leq 1])$  
$([-1 \leq a, b \leq 1])$  
$([-a^2 + b^2 \neq 0, b \leq a, 0 \leq c])$  

$([-1 \leq a, b \leq 1])$  
$([-1 \leq a, b \leq 1])$
### $[6, [5, 1], 3, 2]$  

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
|-------|-------|-------|-------|-------|-------|
| $\ldots$ | $\ldots$ | $(a+1)e_1$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $e_2$ | $e_1$ | $\ldots$ | $e_2$ | $\ldots$ | $\ldots$ |
| $e_3$ | $\ldots$ | $\ldots$ | $ae_3$ | $\ldots$ | $\ldots$ |
| $e_4$ | $\ldots$ | $e_1+(a+1)e_4$ | $\ldots$ | $e_4+(a+1)e_5$ | $\ldots$ |
| $e_5$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $e_6$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

Parameters: $[a]$  
\begin{align*}
&[-1 \leq a, a \leq 1]\end{align*}

### $[6, [5, 1], 3, 3]$  

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
|-------|-------|-------|-------|-------|-------|
| $\ldots$ | $\ldots$ | $(a+1)e_1$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $e_2$ | $\ldots$ | $e_1$ | $ae_2+e_5$ | $\ldots$ | $\ldots$ |
| $e_3$ | $\ldots$ | $\ldots$ | $e_3$ | $\ldots$ | $\ldots$ |
| $e_4$ | $\ldots$ | $\ldots$ | $ae_4$ | $\ldots$ | $\ldots$ |
| $e_5$ | $\ldots$ | $e_1+ae_5$ | $\ldots$ | $e_4+ae_5$ | $\ldots$ |
| $e_6$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

Parameters: $[a]$  
\begin{align*}
&[[\text{parameters: } a = 0]]

### $[6, [5, 1], 3, 4]$  

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
|-------|-------|-------|-------|-------|-------|
| $\ldots$ | $\ldots$ | $\ldots$ | $e_1$ | $\ldots$ | $\ldots$ |
| $e_2$ | $\ldots$ | $e_1$ | $e_2+e_5$ | $\ldots$ | $\ldots$ |
| $e_3$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $e_4$ | $\ldots$ | $\ldots$ | $e_1+e_4$ | $\ldots$ | $\ldots$ |
| $e_5$ | $\ldots$ | $\ldots$ | $e_4+e_5$ | $\ldots$ | $\ldots$ |
| $e_6$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

Parameters: $[a, b, c]$  
\begin{align*}
&[[e^2 = c]]

### $[6, [5, 1], 4, 1]$  

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
|-------|-------|-------|-------|-------|-------|
| $\ldots$ | $\ldots$ | $2ae_1$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $e_2$ | $\ldots$ | $e_1$ | $ae_2-e_3$ | $\ldots$ | $\ldots$ |
| $e_3$ | $\ldots$ | $\ldots$ | $e_2+a_e_3$ | $\ldots$ | $\ldots$ |
| $e_4$ | $\ldots$ | $\ldots$ | $be_4$ | $\ldots$ | $\ldots$ |
| $e_5$ | $\ldots$ | $\ldots$ | $ce_5$ | $\ldots$ | $\ldots$ |
| $e_6$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

Parameters: $[a, b, c]$  
\begin{align*}
&[[0 \leq a, c \leq b, b \neq 0, c \neq 0]]

### $[6, [5, 1], 4, 2]$  

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
|-------|-------|-------|-------|-------|-------|
| $\ldots$ | $\ldots$ | $2ae_1$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $e_2$ | $\ldots$ | $e_1$ | $ae_2-e_3$ | $\ldots$ | $\ldots$ |
| $e_3$ | $\ldots$ | $\ldots$ | $e_2+a_e_3$ | $\ldots$ | $\ldots$ |
| $e_4$ | $\ldots$ | $\ldots$ | $be_4$ | $\ldots$ | $\ldots$ |
| $e_5$ | $\ldots$ | $\ldots$ | $ce_5$ | $\ldots$ | $\ldots$ |
| $e_6$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

Parameters: $[a, b]$  
\begin{align*}
&[[0 \leq a, b \neq 0]]

### $[6, [5, 1], 5, 1]$  

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
|-------|-------|-------|-------|-------|-------|
| $\ldots$ | $\ldots$ | $2ae_1$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $e_2$ | $\ldots$ | $e_1$ | $ae_2-e_3$ | $\ldots$ | $\ldots$ |
| $e_3$ | $\ldots$ | $\ldots$ | $e_2+a_e_3$ | $\ldots$ | $\ldots$ |
| $e_4$ | $\ldots$ | $\ldots$ | $be_4$ | $\ldots$ | $\ldots$ |
| $e_5$ | $\ldots$ | $\ldots$ | $ce_5$ | $\ldots$ | $\ldots$ |
| $e_6$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

Parameters: $[a, b, c]$  
\begin{align*}
&[[0 \leq a, 0 < c]]

### $[6, [5, 1], 5, 2]$  

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
|-------|-------|-------|-------|-------|-------|
| $\ldots$ | $\ldots$ | $2ae_1$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $e_2$ | $\ldots$ | $e_1$ | $ae_2-e_3+e_4$ | $\ldots$ | $\ldots$ |
| $e_3$ | $\ldots$ | $\ldots$ | $e_2+a_e_3$ | $\ldots$ | $\ldots$ |
| $e_4$ | $\ldots$ | $\ldots$ | $ae_4-e_5$ | $\ldots$ | $\ldots$ |
| $e_5$ | $\ldots$ | $\ldots$ | $e_4+ae_5$ | $\ldots$ | $\ldots$ |
| $e_6$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

Parameters: $[a]$  
\begin{align*}
&[[0 \leq a]]

### $[6, [5, 1], 6, 1]$  

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
|-------|-------|-------|-------|-------|-------|
| $\ldots$ | $\ldots$ | $2ae_1$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $e_2$ | $\ldots$ | $\ldots$ | $ae_2-e_3$ | $\ldots$ | $\ldots$ |
| $e_3$ | $\ldots$ | $\ldots$ | $e_2+a_e_3$ | $\ldots$ | $\ldots$ |
| $e_4$ | $\ldots$ | $\ldots$ | $be_4$ | $\ldots$ | $\ldots$ |
| $e_5$ | $\ldots$ | $\ldots$ | $e_4+be_5$ | $\ldots$ | $\ldots$ |
| $e_6$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

Parameters: $[a, b]$  
\begin{align*}
&[[0 \leq a]]
| [6, [5, 1], 6, 2] | [6, [5, 1], 6, 2] | [6, [5, 1], 7, 4] | [6, [5, 1], 7, 4] |
|----------------|----------------|----------------|----------------|
| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
| 2 $a$ $e_1$ | . | . | . | . | . | 2 $a$ $e_1$ | . | . | . | . | . | . | . | . | . | . | . |
| $a$ $e_2$ | . | $e_1$ | . | . | $a$ $e_2$ | $e_1$ | . | . | . | . | $a$ $e_2$ | . | . | . | . | . | . |
| $e_2$ | . | . | $e_2$ | $a$ $e_3$ | . | $e_2$ | $a$ $e_3$ | . | . | $e_2$ | $a$ $e_3$ | . | . | . | . | . | . |
| $e_4$ | . | . | . | $e_1$ | $e_3$ | . | $e_1$ | $e_3$ | . | . | $e_1$ | $e_3$ | . | . | . | . | . | . |
| $e_5$ | . | . | . | $e_4$ | $a$ $e_5$ | . | $e_4$ | $a$ $e_5$ | . | . | $e_4$ | $a$ $e_5$ | . | . | . | . | . | . |
| $e_6$ | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |

parameters: $[a]$

parameters: $[0 \leq a]$}

| [6, [5, 1], 7, 1] | [6, [5, 1], 7, 1] | [6, [5, 1], 7, 5] | [6, [5, 1], 7, 5] |
|----------------|----------------|----------------|----------------|
| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
| 2 $a$ $e_1$ | . | . | . | . | . | 2 $a$ $e_1$ | . | . | . | . | . | 2 $a$ $e_1$ | . | . | . | . | . | . |
| $a$ $e_2$ | . | $e_1$ | . | . | $a$ $e_2$ | $e_1$ | . | . | . | . | $a$ $e_2$ | . | . | . | . | . | . |
| $e_2$ | . | . | $e_2$ | $a$ $e_3$ | . | $e_2$ | $a$ $e_3$ | . | . | $e_2$ | $a$ $e_3$ | . | . | . | . | . | . |
| $e_3$ | . | . | . | $e_1$ | $e_3$ | . | $e_1$ | $e_3$ | . | . | $e_1$ | $e_3$ | . | . | . | . | . | . |
| $e_4$ | . | . | . | $e_4$ | $a$ $e_5$ | . | $e_4$ | $a$ $e_5$ | . | . | $e_4$ | $a$ $e_5$ | . | . | . | . | . | . |
| $e_5$ | . | . | . | . | $b$ $e_5$ | . | . | . | . | $b$ $e_5$ | . | . | . | . | . | . |
| $e_6$ | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |

parameters: $[a, b]$

parameters: $[a, b]$

parameters: $[0 \leq a, b]$}

| [6, [5, 1], 7, 2] | [6, [5, 1], 7, 2] | [6, [5, 1], 8, 1] | [6, [5, 1], 8, 1] |
|----------------|----------------|----------------|----------------|
| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
| 2 $a$ $e_1$ | . | . | . | . | . | 2 $a$ $e_1$ | . | . | . | . | . | 2 $a$ $e_1$ | . | . | . | . | . | . |
| $a$ $e_2$ | . | $e_1$ | . | . | $a$ $e_2$ | $e_1$ | . | . | . | . | $a$ $e_2$ | . | . | . | . | . | . |
| $e_2$ | . | . | $e_2$ | $a$ $e_3$ | . | $e_2$ | $a$ $e_3$ | . | . | $e_2$ | $a$ $e_3$ | . | . | . | . | . | . |
| $e_3$ | . | . | . | $e_1$ | $e_3$ | . | $e_1$ | $e_3$ | . | . | $e_1$ | $e_3$ | . | . | . | . | . | . |
| $e_4$ | . | . | . | $e_4$ | $a$ $e_5$ | . | $e_4$ | $a$ $e_5$ | . | . | $e_4$ | $a$ $e_5$ | . | . | . | . | . | . |
| $e_5$ | . | . | . | . | $e_1+2a e_5$ | . | $e_1+2a e_5$ | . | . | $e_1+2a e_5$ | . | . | . | . | . | . |
| $e_6$ | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |

parameters: $[a]$

parameters: $[0 \leq a]$}

| [6, [5, 1], 7, 3] | [6, [5, 1], 7, 3] | [6, [5, 1], 9, 1] | [6, [5, 1], 9, 1] |
|----------------|----------------|----------------|----------------|
| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
| 2 $a$ $e_1$ | . | . | . | . | . | 2 $a$ $e_1$ | . | . | . | . | . | 2 $a$ $e_1$ | . | . | . | . | . | . |
| $a$ $e_2$ | . | $e_1$ | . | . | $a$ $e_2$ | $e_1$ | . | . | . | . | $a$ $e_2$ | . | . | . | . | . | . |
| $e_2$ | . | . | $e_2$ | $a$ $e_3$ | . | $e_2$ | $a$ $e_3$ | . | . | $e_2$ | $a$ $e_3$ | . | . | . | . | . | . |
| $e_3$ | . | . | . | $e_2$ | $a$ $e_3$ | . | $e_2$ | $a$ $e_3$ | . | . | $e_2$ | $a$ $e_3$ | . | . | . | . | . | . |
| $e_4$ | . | . | . | $e_4$ | $a$ $e_5$ | . | $e_4$ | $a$ $e_5$ | . | . | $e_4$ | $a$ $e_5$ | . | . | . | . | . | . |
| $e_5$ | . | . | . | . | $e_5$ | . | $e_5$ | . | . | $e_5$ | . | . | . | . | . | . |
| $e_6$ | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |

parameters: $[a]$

parameters: $[0 \leq a]$}
### [6, [5, 1], 9, 2]

| e₁, e₂, e₃, e₄, e₅, e₆ | Mubar: [6, 28] |
|-------------------------|----------------|
| e₁ | . . . . . | 2e₁ |
| e₂ | . e₁ . . | e₂ |
| e₃ | . . . e₂+e₃ | |
| e₄ | . . e₄ | |
| e₅ | . e₄+2e₅ | |
| e₆ | . | |

### [6, [5, 2], 1, 1]

| e₁, e₂, e₃, e₄, e₅, e₆ | Mubar: [6, 39] |
|-------------------------|----------------|
| e₁ | . . . . . | (a+2)e₁ |
| e₂ | . e₁ . (a+1)e₂ | |
| e₃ | . e₂ . a e₃ | |
| e₄ | . . e₄ | |
| e₅ | . . b e₅ | |
| e₆ | . | |

parameters: [a, b]

[[b ≠ 0]]

### [6, [5, 1], 9, 3]

| e₁, e₂, e₃, e₄, e₅, e₆ | Mubar: [6, 29] |
|-------------------------|----------------|
| e₁ | . . . . . | 2e₁ |
| e₂ | . e₁ . . | e₂ |
| e₃ | . . . e₂+e₃ | |
| e₄ | . . e₁+2e₄ | |
| e₅ | . . e₄+2e₅ | |
| e₆ | . | |

### [6, [5, 2], 1, 2]

| e₁, e₂, e₃, e₄, e₅, e₆ | Mubar: [6, 42] |
|-------------------------|----------------|
| e₁ | . . . . . | (a+2)e₁ |
| e₂ | . e₁ . (a+1)e₂ | |
| e₃ | . e₂ . a e₃ | |
| e₄ | . . e₄+e₅ | |
| e₅ | . . e₅ | |
| e₆ | . | |

parameters: [a]

[[]]

### [6, [5, 1], 9, 4]

| e₁, e₂, e₃, e₄, e₅, e₆ | Mubar: [6, 31] |
|-------------------------|----------------|
| e₁ | . . . . . | 2e₁ |
| e₂ | . e₁ . . | e₂ |
| e₃ | . . . e₂+e₃ | |
| e₄ | . . e₄ | |
| e₅ | . e₄+e₅ | |
| e₆ | . | |

### [6, [5, 2], 1, 3]

| e₁, e₂, e₃, e₄, e₅, e₆ | Mubar: [6, 44] |
|-------------------------|----------------|
| e₁ | . . . . . | 3e₁ |
| e₂ | . e₁ . 2e₂ | |
| e₃ | . e₂ . e₃ | |
| e₄ | . . e₃+e₄ | |
| e₅ | . . a e₅ | |
| e₆ | . | |

parameters: [a]

[[a ≠ 0]]

### [6, [5, 2], 1, 4]

| e₁, e₂, e₃, e₄, e₅, e₆ | Mubar: [6, 40] |
|-------------------------|----------------|
| e₁ | . . . . . | (a+2)e₁ |
| e₂ | . e₁ . (a+1)e₂ | |
| e₃ | . e₂ . a e₃ | |
| e₄ | . . e₄ | |
| e₅ | . . e₁+(a+2)e₅ | |
| e₆ | . | |

parameters: [a]

[[]]
\[
\begin{array}{cccccc}
[6, [5, 2], 1, 5] & Mubar: [6, 43] & [6, [5, 2], 1, 9] & Mubar: [6, 47] \\
\begin{array}{cccccc}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
e_1 & \ldots & \ldots & e_1 & \ldots & \ldots \\
e_2 & \ldots & e_1 & \ldots & \ldots & e_2 \\
e_3 & \ldots & e_2 & \ldots & e_3 & \ldots \\
e_4 & \ldots & e_4 + e_5 & \ldots & e_4 + e_5 \\
e_5 & \ldots & e_1 + e_5 & \ldots & e_1 + e_5 \\
e_6 & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}
\end{array}
\]

parameters: \([a, \varepsilon]\)

\[[a \neq 0, \varepsilon^3 = \varepsilon] \]

\[
\begin{array}{cccccc}
[6, [5, 2], 1, 6] & Mubar: [6, 45] & [6, [5, 2], 1, 10] & Mubar: [6, 49] \\
\begin{array}{cccccc}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
e_1 & \ldots & \ldots & 3e_1 & \ldots & \ldots \\
e_2 & \ldots & e_1 & \ldots & 2e_2 & \ldots \\
e_3 & \ldots & e_2 & \ldots & e_3 & \ldots \\
e_4 & \ldots & e_3 + e_4 & \ldots & e_3 + e_4 \\
e_5 & \ldots & e_1 + 3e_5 & \ldots & e_1 + 3e_5 \\
e_6 & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\end{array}
\]

parameters: \([a]\)

\[[a] \]

\[
\begin{array}{cccccc}
[6, [5, 2], 1, 7] & Mubar: [6, 41] & [6, [5, 2], 1, 11] & Mubar: [6, 48] \\
\begin{array}{cccccc}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
e_1 & \ldots & \ldots & (a+2)e_1 & \ldots & \ldots \\
e_2 & \ldots & e_1 & \ldots & (a+1)e_2 & \ldots \\
e_3 & \ldots & e_2 & \ldots & e_3 + e_5 & \ldots \\
e_4 & \ldots & e_4 & \ldots & e_4 \\
e_5 & \ldots & a + e_5 & \ldots & a + e_5 \\
e_6 & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\end{array}
\]

parameters: \([a, \varepsilon]\)

\[[a] \]

\[
\begin{array}{cccccc}
[6, [5, 2], 1, 8] & Mubar: [6, 46] & [6, [5, 2], 1, 12] & Mubar: [6, 50] \\
\begin{array}{cccccc}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
e_1 & \ldots & \ldots & 3e_1 & \ldots & \ldots \\
e_2 & \ldots & e_1 & \ldots & 2e_2 & \ldots \\
e_3 & \ldots & e_2 & \ldots & e_3 + e_5 & \ldots \\
e_4 & \ldots & e_3 + e_4 & \ldots & e_3 + e_4 \\
e_5 & \ldots & e_5 & \ldots & e_5 \\
e_6 & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\end{array}
\]

parameters: \([\varepsilon]\)

\[[\varepsilon^3 = \varepsilon] \]

\[[\varepsilon^3 = \varepsilon] \]
### Parameters: $[a, b]$

#### $[6, [5, 3], 1, 1]$

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
|-------|-------|-------|-------|-------|-------|
| $e_1$ |       |       |       | $e_1$ |       |
| $e_2$ |       |       |       | $a e_2$ |       |
| $e_3$ |       | $e_1$ | $(1-b) e_3$ |       |       |
| $e_4$ |       | $e_2$ | $(a-b) e_4$ |       |       |
| $e_5$ |       |       |       | $b e_5$ |       |
| $e_6$ |       |       |       |       |       |

Parameters: $[-1 \leq a, a \leq 1]$

### Parameters: $[a]$

#### $[6, [5, 3], 1, 2]$

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
|-------|-------|-------|-------|-------|-------|
| $e_1$ |       |       |       | $e_1$ |       |
| $e_2$ |       |       |       | $2 a e_2$ |       |
| $e_3$ |       | $e_1$ | $(1-a) e_3$ |       |       |
| $e_4$ |       | $e_2$ | $a e_4$ |       |       |
| $e_5$ |       |       |       | $e_4+a e_5$ |       |
| $e_6$ |       |       |       |       |       |

Parameters: $[-1/2 \leq a, a \leq 1/2]$

### Parameters: $[a]$

#### $[6, [5, 3], 1, 3]$

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
|-------|-------|-------|-------|-------|-------|
| $e_1$ |       |       |       | $2 e_1$ |       |
| $e_2$ |       |       |       | $a e_2$ |       |
| $e_3$ |       |       |       | $e_1$ | $e_3$ |
| $e_4$ |       | $e_2$ | $(a-1) e_4$ |       |       |
| $e_5$ |       |       |       | $e_3+e_5$ |       |
| $e_6$ |       |       |       |       |       |

Parameters: $[a, a]$

#### $[6, [5, 3], 1, 4]$

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
|-------|-------|-------|-------|-------|-------|
| $e_1$ |       |       |       | $e_1$ |       |
| $e_2$ |       |       |       | $(a+1) e_2$ |       |
| $e_3$ |       |       |       | $(1-a) e_3$ |       |
| $e_4$ |       | $e_2$ | $e_1+e_4$ |       |       |
| $e_5$ |       |       |       | $a e_5$ |       |
| $e_6$ |       |       |       |       |       |

Parameters: $[-2 \leq a, a \leq 2]$

### Parameters: $[a]$

#### $[6, [5, 3], 1, 5]$

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
|-------|-------|-------|-------|-------|-------|
| $e_1$ |       |       |       | $e_1$ |       |
| $e_2$ |       |       |       | $(1-a) e_2$ |       |
| $e_3$ |       |       |       | $e_1 e_2+(1-a) e_3$ |       |
| $e_4$ |       |       |       | $(1-2 a) e_4$ |       |
| $e_5$ |       |       |       | $a e_5$ |       |
| $e_6$ |       |       |       |       |       |

Parameters: $[a]$

#### $[6, [5, 3], 1, 6]$

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
|-------|-------|-------|-------|-------|-------|
| $e_1$ |       |       |       | $3 e_1$ |       |
| $e_2$ |       |       |       | $2 e_2$ |       |
| $e_3$ |       |       |       | $e_1 e_2+2 e_3$ |       |
| $e_4$ |       |       |       | $e_2$ | $e_4$ |
| $e_5$ |       |       |       | $e_4+e_5$ |       |
| $e_6$ |       |       |       |       |       |

Parameters: $[0 \leq a, a \leq 2]$

### Parameters: $[e]$

#### $[6, [5, 2], 1, 13]$

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
|-------|-------|-------|-------|-------|-------|
| $e_1$ |       |       |       |       |       |
| $e_2$ |       |       |       |       |       |
| $e_3$ |       | $e_2$ | $e_3$ |       |       |
| $e_4$ |       |       |       | $e_4$ |       |
| $e_5$ |       |       |       |       | $e_5$ |
| $e_6$ |       |       |       |       |       |

Parameters: $[e^3 = e]$
### [6, [5, 3], 1, 7]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | . . . . . | 2e₁ |
| e₂ | . . . . | e₂ |
| e₃ | . . | e₁ e₂ + e₃ |
| e₄ | . | e₂ |
| e₅ | . | e₃ + e₅ |
| e₆ | . |

Mubar: [6, 62]

### [6, [5, 3], 1, 8]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | . . . . | e₁ |
| e₂ | . . . | e₂ |
| e₃ | . . | e₁ e₂ + e₃ |
| e₄ | . | e₂ |
| e₅ | . |
| e₆ | . |

parameters: [ε]

[[ε² = 1]]

### [6, [5, 3], 1, 9]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | . . . . | e₁ |
| e₂ | . . | a e₂ |
| e₃ | . . | e₁ e₁ + e₃ |
| e₄ | . | e₂ a e₄ |
| e₅ | . |
| e₆ | . |

parameters: [α]

[[-1 ≤ α, α ≤ 1]]

### [6, [5, 3], 1, 10]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | . . . . | e₁ |
| e₂ | . . . | e₂ |
| e₃ | . . | e₁ e₁ + e₃ |
| e₄ | . | e₂ |
| e₅ | . | e₄ |
| e₆ | . |

Mubar: [6, 59]

### [6, [5, 3], 1, 11]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | . . . . . | e₄ |
| e₂ | . . . . | e₂ |
| e₃ | . . | e₁ e₁ + e₃ |
| e₄ | . | e₂ e₁ + e₄ |
| e₅ | . |
| e₆ | . |

parameters: [ε]

[[ε² = 1]]

### [6, [5, 3], 1, 12]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | . . . . . |
| e₂ | . . . . |
| e₃ | . . | e₁ e₃ |
| e₄ | . | e₂ |
| e₅ | . | - e₅ |
| e₆ | . |

Mubar: [6, 53]

### [6, [5, 3], 2, 1]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | . . . . | a e₁ - e₂ |
| e₂ | . . . . | e₁ + a e₂ |
| e₃ | . . | e₁ (a-b) e₃ - e₄ |
| e₄ | . | e₂ e₃ + (a-b) e₄ |
| e₅ | . | b e₅ |
| e₆ | . |

parameters: [α, b]

[[0 ≤ a, b]]

### [6, [5, 3], 2, 2]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ |
|---|----|----|----|----|----|----|
| e₁ | . . . . | a e₁ - e₂ |
| e₂ | . . . . | e₁ + a e₂ |
| e₃ | . . | e₁ a e₃ - e₄ |
| e₄ | . | e₂ e₁ + e₃ + a e₄ |
| e₅ | . |
| e₆ | . |

parameters: [α]

[[0 ≤ a]]
\[ [6, [5, 3], 3, 1] \]

\[
\begin{array}{cccccc}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
 \cdot & \cdot & \cdot & \cdot & \cdot & a \cdot e_1 \\
 \cdot & \cdot & \cdot & e_1 + a \cdot e_2 \\
 \cdot & \cdot & e_1 & (a-1) \cdot e_3 \\
 \cdot & e_2 & e_3 + (a-1) \cdot e_4 \\
 \cdot & \cdot & \cdot & \cdot & . \\
 \cdot & \cdot & \cdot & \cdot & . \\
 \end{array}
\]

parameters: \([a]\]

\[ [6, [5, 4], 1, 2] \]

\[
\begin{array}{cccccc}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
 \cdot & \cdot & \cdot & \cdot & \cdot & 4 \cdot e_1 \\
 \cdot & \cdot & \cdot & e_1 & 3 \cdot e_2 \\
 \cdot & \cdot & e_2 & 2 \cdot e_3 \\
 \cdot & e_3 & e_4 & e_4 \\
 \cdot & e_4 & e_4 & e_5 \\
 \cdot & \cdot & \cdot & . \\
 \end{array}
\]

Mubar: [6, 65]

\[ [6, [5, 3], 3, 2] \]

\[
\begin{array}{cccccc}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
 \cdot & \cdot & \cdot & \cdot & 2 \cdot e_1 \\
 \cdot & \cdot & \cdot & e_1 + 2 \cdot e_2 \\
 \cdot & \cdot & e_1 & e_3 \\
 \cdot & e_2 & e_3 + e_4 \\
 \cdot & \cdot & e_4 + e_5 \\
 \cdot & \cdot & \cdot & . \\
 \end{array}
\]

Mubar: [6, 66]

\[ [6, [5, 4], 1, 3] \]

\[
\begin{array}{cccccc}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
 \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & e_1 & e_2 \\
 \cdot & \cdot & e_2 & e_3 \\
 \cdot & e_3 & e_4 \\
 \cdot & \cdot & \cdot & . \\
 \cdot & \cdot & \cdot & . \\
 \end{array}
\]

Mubar: [6, 74]

\[ [6, [5, 3], 3, 3] \]

\[
\begin{array}{cccccc}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
 \cdot & \cdot & \cdot & \cdot & e_1 \\
 \cdot & \cdot & \cdot & e_1 + e_2 \\
 \cdot & \cdot & e_1 & e_2 + e_3 \\
 \cdot & e_2 & e_3 + e_4 \\
 \cdot & \cdot & e_4 + e_5 \\
 \cdot & \cdot & \cdot & . \\
 \end{array}
\]

parameters: \([a]\]

\[ [6, [5, 4], 1, 4] \]

\[
\begin{array}{cccccc}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
 \cdot & \cdot & \cdot & \cdot & e_1 \\
 \cdot & \cdot & \cdot & e_1 & e_2 \\
 \cdot & \cdot & e_2 & e_3 + a \cdot e_1 + e_3 \\
 \cdot & e_3 & e_4 + e_5 \\
 \cdot & \cdot & \cdot & . \\
 \cdot & \cdot & \cdot & . \\
 \end{array}
\]

parameters: \([a]\]

\[ [6, [5, 4], 1, 1] \]

\[
\begin{array}{cccccc}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
 \cdot & \cdot & \cdot & (a+3) \cdot e_1 \\
 \cdot & \cdot & e_1 & (a+2) \cdot e_2 \\
 \cdot & \cdot & e_2 & (a+1) \cdot e_3 \\
 \cdot & e_3 & e_4 \\
 \cdot & \cdot & e_5 \\
 \cdot & \cdot & \cdot & . \\
 \end{array}
\]

parameters: \([a]\]

\[ [6, [5, 4], 1, 5] \]

\[
\begin{array}{cccccc}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
 \cdot & \cdot & \cdot & \cdot & e_1 \\
 \cdot & \cdot & \cdot & e_1 & e_2 \\
 \cdot & \cdot & e_2 & e_3 + e_4 \\
 \cdot & e_3 & e_4 + e_5 \\
 \cdot & \cdot & \cdot & . \\
 \cdot & \cdot & \cdot & . \\
 \end{array}
\]

parameters: \([a]\]

\[ [6, [5, 4], 1, 5] \]

\[
\begin{array}{cccccc}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
 \cdot & \cdot & \cdot & \cdot & e_1 \\
 \cdot & \cdot & \cdot & e_1 & e_2 \\
 \cdot & \cdot & e_2 & e_3 + e_4 \\
 \cdot & e_3 & e_4 + e_5 \\
 \cdot & \cdot & \cdot & . \\
 \cdot & \cdot & \cdot & . \\
 \end{array}
\]

parameters: \([a]\]

\[ [6, [5, 4], 1, 1] \]

\[
\begin{array}{cccccc}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
 \cdot & \cdot & \cdot & \cdot & \cdot & (a+3) \cdot e_1 \\
 \cdot & \cdot & e_1 & (a+2) \cdot e_2 \\
 \cdot & \cdot & e_2 & (a+1) \cdot e_3 \\
 \cdot & e_3 & e_4 \\
 \cdot & \cdot & e_5 \\
 \cdot & \cdot & \cdot & . \\
 \end{array}
\]

parameters: \([a]\]

\[ [6, [5, 4], 1, 5] \]

\[
\begin{array}{cccccc}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
 \cdot & \cdot & \cdot & \cdot & \cdot & e_1 \\
 \cdot & \cdot & \cdot & e_1 & e_2 \\
 \cdot & \cdot & e_2 & e_3 + e_4 \\
 \cdot & e_3 & e_4 + e_5 \\
 \cdot & \cdot & \cdot & . \\
 \cdot & \cdot & \cdot & . \\
 \end{array}
\]

parameters: \([a]\]

\[ [6, [5, 4], 1, 1] \]

\[
\begin{array}{cccccc}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
 \cdot & \cdot & \cdot & \cdot & \cdot & e_1 \\
 \cdot & \cdot & \cdot & e_1 & e_2 \\
 \cdot & \cdot & e_2 & e_3 + e_4 \\
 \cdot & e_3 & e_4 + e_5 \\
 \cdot & \cdot & \cdot & . \\
 \cdot & \cdot & \cdot & . \\
 \end{array}
\]

parameters: \([a]\]

\[ [6, [5, 4], 1, 5] \]

\[
\begin{array}{cccccc}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
 \cdot & \cdot & \cdot & \cdot & \cdot & e_1 \\
 \cdot & \cdot & \cdot & e_1 & e_2 \\
 \cdot & \cdot & e_2 & e_3 + e_4 \\
 \cdot & e_3 & e_4 + e_5 \\
 \cdot & \cdot & \cdot & . \\
 \cdot & \cdot & \cdot & . \\
 \end{array}
\]

parameters: \([a]\]

\[ [6, [5, 4], 1, 1] \]

\[
\begin{array}{cccccc}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
 \cdot & \cdot & \cdot & \cdot & \cdot & e_1 \\
 \cdot & \cdot & \cdot & e_1 & e_2 \\
 \cdot & \cdot & e_2 & e_3 + e_4 \\
 \cdot & e_3 & e_4 + e_5 \\
 \cdot & \cdot & \cdot & . \\
 \cdot & \cdot & \cdot & . \\
 \end{array}
\]

parameters: \([a]\]

\[ [6, [5, 4], 1, 5] \]
\[6, [5, 6], 2, 2\]

\[
\begin{array}{cccccc}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
 e_1 & . & . & . & . & . \\
 e_2 & . & e_1 & . & e_2 & . \\
 e_3 & . & e_1 & . & . & . \\
 e_4 & . & . & e_4 & . & . \\
 e_5 & . & . & . & e_5 & . \\
 e_6 & . & . & . & . & . \\
\end{array}
\]

Mubar: [6, 84]

\[6, [5, 6], 3, 1\]

\[
\begin{array}{cccccc}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
 e_1 & . & . & . & . & 2 e_1 \\
 e_2 & . & e_1 & . & (a+1) e_2 & . \\
 e_3 & . & e_1 & e_2+(a+1) e_3 & . \\
 e_4 & . & . & (1-a) e_4-e_5 & . \\
 e_5 & . & . & (1-a) e_5 & . \\
 e_6 & . & . & . & . \\
\end{array}
\]

parameters: \([a]\)

\([0 \leq a]\)

\[6, [5, 6], 6, 1\]

\[
\begin{array}{cccccc}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
 e_1 & . & . & . & . & 2 a e_1 \\
 e_2 & . & e_1 & . & (a+b) e_2 & . \\
 e_3 & . & e_1 & a e_3-e_5 & . \\
 e_4 & . & . & (a-b) e_4 & . \\
 e_5 & . & . & e_3+a e_5 & . \\
 e_6 & . & . & . & . \\
\end{array}
\]

parameters: \([a, b]\)

\([0 \leq a, 0 \leq b]\)

\[6, [5, 6], 7, 1\]

\[
\begin{array}{cccccc}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
 e_1 & . & . & . & . & 2 a e_1 \\
 e_2 & . & e_1 & . & a e_2 & . \\
 e_3 & . & e_1 & a e_3-e_5 & . \\
 e_4 & . & . & e e_4+a e_4 & . \\
 e_5 & . & . & e_3+a e_5 & . \\
 e_6 & . & . & . & . \\
\end{array}
\]

parameters: \([a, e]\)

\([e^2 = 1]\)

\[6, [5, 6], 8, 1\]

\[
\begin{array}{cccccc}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
 e_1 & . & . & . & . & 2 a e_1 \\
 e_2 & . & e_1 & . & (a+b) e_2-e_3 & . \\
 e_3 & . & e_1 & e_2+(a+b) e_3 & . \\
 e_4 & . & . & (a-b) e_4-e_5 & . \\
 e_5 & . & . & e_4+(a-b) e_5 & . \\
 e_6 & . & . & . & . \\
\end{array}
\]

parameters: \([a, b]\)

\([0 \leq a, 0 \leq b]\)
| $e_1$, $[5, 6]$, 9, 1 | $e_1$, $[5, 7]$, 1, 3 |
|----------------------|----------------------|
| $e_1$ . . . . 2 $e_1$ | $e_1$ . . . . 3 $e_1$ |
| $e_2$ . . $e_1$ . $e_2 - e_4$ | $e_2$ . . $e_1$ 2 $e_2$ |
| $e_3$ . . $e_1$ $a e_2 + b e_3$ | $e_3$ . $e_1$ $e_2$ $e_3$ |
| $e_4$ . . $e_2 + a e_4$ | $e_4$ . . $e_2 + 2 e_4$ |
| $e_5$ . . $b e_3 + a e_5$ | $e_5$ . $e_3 + 2 e_5$ |
| $e_6$ . | $e_6$ . |

parameters: $[a, b]$

$[0 \leq a, -1 < b, b \leq 1, b \neq 0]$

| $e_1$, $[5, 6]$, 10, 1 |
|----------------------|
| $e_1$ . . . . 2 $e_1$ |
| $e_2$ . . $e_1$ . $a e_2 - e_3$ |
| $e_3$ . . $e_1$ $e_2 + a e_3$ |
| $e_4$ . . $b e_3 + a e_5$ |
| $e_5$ . | $e_6$ . |

parameters: $[a]$

$[0 \leq a]$

| $e_1$, $[5, 7]$, 1, 1 |
|----------------------|
| $e_1$ . . . . $(a+2) e_1$ |
| $e_2$ . . $e_1$ $(a+1) e_2$ |
| $e_3$ . . $e_1$ $2 e_4$ |
| $e_4$ . . $e_5$ |
| $e_5$ . | $e_6$ . |

parameters: $[a]$

$[0 \leq a]$

| $e_1$, $[5, 7]$, 1, 2 |
|----------------------|
| $e_1$ . . . . 2 $e_1$ |
| $e_2$ . . $e_1$ $e_2$ |
| $e_3$ . . $e_1$ $e_2$ |
| $e_4$ . . $e_1 + 2 e_4$ |
| $e_5$ . . $e_5$ |
| $e_6$ . | $e_6$ . |

parameters: $[a]$

$[0 \leq a]$

| $e_1$, $[5, 7]$, 1, 3 |
|----------------------|
| $e_1$ . . . . 3 $e_1$ |
| $e_2$ . . $e_1$ 2 $e_2$ |
| $e_3$ . $e_1$ $e_2$ $e_3$ |
| $e_4$ . . $e_2 + 2 e_4$ |
| $e_5$ . $e_3 + 2 e_5$ |
| $e_6$ . | $e_6$ . |

parameters: $[a, b]$

$[0 \leq a, -1 < b, b \leq 1, b \neq 0]$

| $e_1$, $[5, 7]$, 1, 4 |
|----------------------|
| $e_1$ . . . . 4 $e_1$ |
| $e_2$ . . $e_1$ 3 $e_2$ |
| $e_3$ . $e_1$ $e_2$ $e_3$ |
| $e_4$ . . $2 e_4$ |
| $e_5$ . $e_5$ |
| $e_6$ . | $e_6$ . |

parameters: $[a]$

$[0 \leq a]$

| $e_1$, $[5, 7]$, 1, 5 |
|----------------------|
| $e_1$ . . . . $e_1$ |
| $e_2$ . . $e_1$ $e_1 + 2 e_2$ |
| $e_3$ . $e_1$ $e_2$ $e_3$ |
| $e_4$ . . $e_4$ |
| $e_5$ . $e_5$ |
| $e_6$ . | $e_6$ . |

parameters: $[e]$

$[0 \leq e, e]$

| $e_1$, $[5, 7]$, 1, 6 |
|----------------------|
| $e_1$ . . . . 5 $e_1$ |
| $e_2$ . . $e_1$ 4 $e_2$ |
| $e_3$ . $e_1$ $e_2$ 3 $e_3$ |
| $e_4$ . $e_3$ 2 $e_4$ |
| $e_5$ . $e_5$ |
| $e_6$ . | $e_6$ . |

parameters: $[e]$

$[0 \leq e, e]$

| $e_1$, $[5, 8]$, 1, 1 |
|----------------------|
| $e_1$ . . . . 5 $e_1$ |
| $e_2$ . . $e_1$ 4 $e_2$ |
| $e_3$ . $e_1$ $e_2$ 3 $e_3$ |
| $e_4$ . $e_3$ 2 $e_4$ |
| $e_5$ . $e_5$ |
| $e_6$ . | $e_6$ . |
### B.6 Dimension Seven

#### [7, [6, 0], 1, 1]

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|------|-------|-------|-------|-------|-------|
| $e_1$ | . . . . . | $a e_1 - e_2$ |
| $e_2$ | . . . . . | $e_1 + a e_2$ |
| $e_3$ | . . . . . | $b e_3$ |
| $e_4$ | . . . . . | $c e_4$ |
| $e_5$ | . . . . . | $d e_5$ |
| $e_6$ | . . . . . | $t e_6$ |
| $e_7$ | . . . . . | . |

Parameters: $[a, b, c, d, t]$

\[ -1 \leq t, t \leq d, d \leq c, c \leq b, b \leq a, a \leq 1, a \neq 0, b \neq 0, c \neq 0, d \neq 0, t \neq 0 \]

#### [7, [6, 0], 2, 1]

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|------|-------|-------|-------|-------|-------|
| $e_1$ | . . . . . | $a e_1 - e_2$ |
| $e_2$ | . . . . . | $e_1 + a e_2$ |
| $e_3$ | . . . . . | $b e_3$ |
| $e_4$ | . . . . . | $c e_4$ |
| $e_5$ | . . . . . | $d e_5$ |
| $e_6$ | . . . . . | $t e_6$ |
| $e_7$ | . . . . . | . |

Parameters: $[a, b, c, d, t]$

\[ t \leq d, d \leq c, c \leq b, 0 \leq a, b \neq 0, c \neq 0, d \neq 0, t \neq 0 \]

#### [7, [6, 0], 3, 1]

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|------|-------|-------|-------|-------|-------|
| $e_1$ | . . . . . | $a e_1$ |
| $e_2$ | . . . . . | $e_1 + a e_2$ |
| $e_3$ | . . . . . | $e_3$ |
| $e_4$ | . . . . . | $b e_4$ |
| $e_5$ | . . . . . | $c e_5$ |
| $e_6$ | . . . . . | $d e_6$ |
| $e_7$ | . . . . . | . |

Parameters: $[a, b, c, d]$

\[ -1 \leq d, d \leq c, c \leq b, b \neq 0, c \neq 0, d \neq 0 \]

#### [7, [6, 0], 4, 1]

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|------|-------|-------|-------|-------|-------|
| $e_1$ | . . . . . | $a e_1 - e_2$ |
| $e_2$ | . . . . . | $e_1 + a e_2$ |
| $e_3$ | . . . . . | $b e_3$ |
| $e_4$ | . . . . . | $c e_4 + b e_4$ |
| $e_5$ | . . . . . | $d e_5$ |
| $e_6$ | . . . . . | $t e_6$ |
| $e_7$ | . . . . . | . |

Parameters: $[a, b, c, d, t]$

\[ 0 \leq a, 0 < c, c \leq 1, t \leq d, d \neq 0, t \neq 0 \]

#### [7, [6, 0], 5, 1]

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|------|-------|-------|-------|-------|-------|
| $e_1$ | . . . . . | $a e_1$ |
| $e_2$ | . . . . . | $e_1 + a e_2$ |
| $e_3$ | . . . . . | $b e_3$ |
| $e_4$ | . . . . . | $e_3 + b e_4$ |
| $e_5$ | . . . . . | $c e_5$ |
| $e_6$ | . . . . . | $d e_6$ |
| $e_7$ | . . . . . | . |

Parameters: $[a, b, c, d]$

\[ 0 \leq b, d \leq c, c \neq 0, d \neq 0 \]

#### [7, [6, 0], 6, 1]

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|------|-------|-------|-------|-------|-------|
| $e_1$ | . . . . . | $a e_1$ |
| $e_2$ | . . . . . | $e_1 + a e_2$ |
| $e_3$ | . . . . . | $b e_3$ |
| $e_4$ | . . . . . | $e_3 + b e_4$ |
| $e_5$ | . . . . . | $c e_5$ |
| $e_6$ | . . . . . | $c e_6$ |
| $e_7$ | . . . . . | . |

Parameters: $[a, b, c]$

\[ b \leq a, -1 \leq c, c \leq 1, c \neq 0 \]
\[ \begin{align*}
\text{parameters: } & [a, b, c, d, t] \\
& [[0 \leq a, 0 < t, t \leq c, c \leq 1]] \\
\end{align*} \]

\[ \begin{align*}
\text{parameters: } & [a, b] \\
& [[-1 \leq b, b \leq a, a \leq 1]] \\
\end{align*} \]

\[ \begin{align*}
\text{parameters: } & [a, b, c, d] \\
& [[0 \leq b, 0 < d, d \leq 1]] \\
\end{align*} \]

\[ \begin{align*}
\text{parameters: } & [a, b, c] \\
& [[-1 \leq c, c \leq b, b \leq 1, b \neq 0, c \neq 0]] \\
\end{align*} \]

\[ \begin{align*}
\text{parameters: } & [a, b, c] \\
& [[b \leq a, 0 \leq c]] \\
\end{align*} \]

\[ \begin{align*}
\text{parameters: } & [a, b, c] \\
& [[0 \leq b, c \neq 0]] \\
\end{align*} \]
### Parameters

| Parameters | Constraints |
|------------|-------------|
| $[7, [6, 0], 13, 1]$ | $[a, b]$ | $[-1 \leq b, b \leq 1, b \neq 0]$ |
| $[7, [6, 0], 14, 1]$ | $[a]$ | $[-1 \leq a, a \leq 1]$ |
| $[7, [6, 0], 15, 1]$ | $[a, b, c]$ | $[0 \leq a, 0 < c]$ |
| $[7, [6, 0], 16, 1]$ | $[a, b]$ | $[0 \leq a]$ |
### [7, [6, 0], 19, 1]

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| $a e_4$ | $e_1 + a e_2$ | $e_2 + a e_3$ | $e_3 + a e_4$ | $e_5 + b e_6$ |       |       |

parameters: $[a, b]$

$[0 \leq b]$

### [7, [6, 0], 20, 1]

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|
|       | $e_1 + e_2$ | $e_2 + e_3$ | $e_3 + e_4$ | $a e_5$ | $e_5 + a e_6$ |       |

parameters: $[a]$

$[a]$

### [7, [6, 0], 20, 2]

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|
|       |       | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5 + e_6$ |

$[a = 0]$

### [7, [6, 0], 21, 1]

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| $a e_4$ | $e_1 + a e_2$ | $e_2 + a e_3$ | $e_3 + a e_4$ | $e_4 + a e_5$ | $e_6$ |       |

parameters: $[a]$

$[a \neq 0]$

### [7, [6, 1], 1, 1]

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| $a e_1$ | $5 e_2$ | $4 e_3$ | $3 e_4$ | $2 e_5$ | $e_6$ |       |

parameters: $[a]$

$[a \neq 0]$

### [7, [6, 1], 1, 2]

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| $5 e_1 + e_2$ | $5 e_2$ | $4 e_3$ | $3 e_4$ | $2 e_5$ | $e_6$ |       |

parameters: $[a]$

$[a \neq 0]$
### [7, [6, 1], 1, 3]

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
| --- | --- | --- | --- | --- | --- | --- |
| \(2e_1\) | \(5e_2\) | \(4e_3\) | \(3e_4\) | \(2e_5\) | \(e_6\) | \(e_7\) |

### [7, [6, 1], 1, 4]

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
| --- | --- | --- | --- | --- | --- | --- |
| \(2e_1\) | \(5e_2\) | \(4e_3\) | \(3e_4\) | \(2e_5\) | \(e_6\) | \(e_7\) |

### [7, [6, 1], 1, 5]

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
| --- | --- | --- | --- | --- | --- | --- |
| \(2e_1\) | \(5e_2\) | \(4e_3\) | \(3e_4\) | \(2e_5\) | \(e_6\) | \(e_7\) |

### [7, [6, 1], 1, 6]

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
| --- | --- | --- | --- | --- | --- | --- |
| \(2e_1\) | \(5e_2\) | \(4e_3\) | \(3e_4\) | \(2e_5\) | \(e_6\) | \(e_7\) |

### [7, [6, 1], 1, 7]

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
| --- | --- | --- | --- | --- | --- | --- |
| \(2e_1\) | \(5e_2\) | \(4e_3\) | \(3e_4\) | \(2e_5\) | \(e_6\) | \(e_7\) |

### [7, [6, 2], 1, 1]

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
| --- | --- | --- | --- | --- | --- | --- |
| \((a+3b)e_1\) | \((a+2b)e_2\) | \((a+b)e_3\) | \(ae_4\) | \(be_5\) | \(ae_6\) | \(be_7\) |

#### Parameters: \([a, b]\) \(\left[\left[a^2 + b^2 \neq 0\right]\right]\)

### [7, [6, 2], 1, 2]

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
| --- | --- | --- | --- | --- | --- | --- |
| \((a+3)e_1 + e_2\) | \((a+3)e_2\) | \((a+3)e_3\) | \((a+2)e_4\) | \((a+1)e_5\) | \(ae_6\) | \(be_7\) |

#### Parameters: \([a]\) \(\left[\left[a\neq 0\right]\right]\)

### [7, [6, 2], 1, 3]

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
| --- | --- | --- | --- | --- | --- | --- |
| \((a\epsilon + e_1 + e_2\epsilon)\) | \((a\epsilon + e_1 + e_2)\) | \((a\epsilon + e_1 + e_2)\) | \((a\epsilon + e_1 + e_2)\) | \((a\epsilon + e_1 + e_2)\) | \((a\epsilon + e_1 + e_2)\) | \((a\epsilon + e_1 + e_2)\) |

#### Parameters: \([a, \epsilon]\) \(\left[\left[\epsilon^2 = 1\right]\right]\)
### [7, [6, 2], 1, 4]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
|    |    |    |    |    |    | e1 |
|    |    |    |    | e2 | e2 | e2 |
| e3 |    |    |    | e2 | e3 | e3 |
| e4 |    |    |    | e3 | e4 | e4 |
| e5 |    | e4 | e5 | e6 | e7 | e7 |

Parameters: \([a]\\)


### [7, [6, 2], 1, 8]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
|    |    |    |    |    |    | e1 |
|    |    |    |    |    | e2 | e2 |
| e3 |    |    |    | e2 | e3 | e3 |
| e4 |    |    |    | e3 | e4 | e4 |
| e5 |    | e4 | e5 | e6 | e7 | e7 |

Parameters: \([a]\\)


### [7, [6, 2], 1, 5]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
|    |    |    |    |    |    | e1 |
|    |    |    |    |    | e2 | e2 |
| e3 |    |    |    | e2 | e3 | e3 |
| e4 |    |    |    | e3 | e4 | e4 |
| e5 |    | e4 | e5 | e6 | e7 | e7 |

Parameters: \([a]\\)


### [7, [6, 2], 1, 9]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
|    |    |    |    |    |    | e1 |
|    |    |    |    |    | e2 | e2 |
| e3 |    |    |    | e2 | e3 | e3 |
| e4 |    |    |    | e3 | e4 | e4 |
| e5 |    | e4 | e5 | e6 | e7 | e7 |

Parameters: \([a]\\)


### [7, [6, 2], 1, 6]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
|    |    |    |    |    |    | e1 |
|    |    |    |    |    | e2 | e2 |
| e3 |    |    |    | e2 | e3 | e3 |
| e4 |    |    |    | e3 | e4 | e4 |
| e5 |    | e4 | e5 | e6 | e7 | e7 |

Parameters: \([a]\\)


### [7, [6, 2], 1, 10]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
|    |    |    |    |    |    | e1 |
|    |    |    |    |    | e2 | e2 |
| e3 |    |    |    | e2 | e3 | e3 |
| e4 |    |    |    | e3 | e4 | e4 |
| e5 |    | e4 | e5 | e6 | e7 | e7 |


### [7, [6, 2], 1, 7]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
|    |    |    |    |    |    | e1 |
|    |    |    |    |    | e2 | e2 |
| e3 |    |    |    | e2 | e3 | e3 |
| e4 |    |    |    | e3 | e4 | e4 |
| e5 |    | e4 | e5 | e6 | e7 | e7 |

Parameters: \([a]\\)


### [7, [6, 2], 1, 11]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
|    |    |    |    |    |    | e1 |
|    |    |    |    |    | e2 | e2 |
| e3 |    |    |    | e2 | e3 | e3 |
| e4 |    |    |    | e3 | e4 | e4 |
| e5 |    | e4 | e5 | e6 | e7 | e7 |

Parameters: \([a]\\)
### [7, [6, 2], 1, 12]

|   | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ |  |  |  |  |  |  | $e_1$ |
| $e_2$ |  |  |  |  |  |  | $e_2$ |
| $e_3$ |  |  |  |  |  |  | $e_2$ |
| $e_4$ |  |  |  |  |  |  | $e_3$ |
| $e_5$ |  |  |  |  |  |  | $e_3$ |
| $e_6$ |  |  |  |  |  |  | $e_6$ |
| $e_7$ |  |  |  |  |  |  |  |

### [7, [6, 2], 1, 13]

|   | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ |  |  |  |  |  |  | $e_1$ |
| $e_2$ |  |  |  |  |  |  | $a e_2$ |
| $e_3$ |  |  |  |  |  |  | $a e_3$ |
| $e_4$ |  |  |  |  |  |  | $a e_4$ |
| $e_5$ |  |  |  |  |  |  | $a e_5$ |
| $e_6$ |  |  |  |  |  |  | $e_6$ |
| $e_7$ |  |  |  |  |  |  |  |

parameters: $[a, b, c]$

[[{$a^2 = 1$}]]

### [7, [6, 2], 1, 14]

|   | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ |  |  |  |  |  |  | $e_1$ |
| $e_2$ |  |  |  |  |  |  | $a e_2$ |
| $e_3$ |  |  |  |  |  |  | $a e_3$ |
| $e_4$ |  |  |  |  |  |  | $a e_4$ |
| $e_5$ |  |  |  |  |  |  | $a e_5$ |
| $e_6$ |  |  |  |  |  |  | $e_6$ |
| $e_7$ |  |  |  |  |  |  |  |

parameters: $[a, b, c]$

[[{$a^2 = 1$}]]

### [7, [6, 2], 1, 15]

|   | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ |  |  |  |  |  |  | $e_1 + e_2$ |
| $e_2$ |  |  |  |  |  |  | $e_2$ |
| $e_3$ |  |  |  |  |  |  | $e_3$ |
| $e_4$ |  |  |  |  |  |  | $e_4$ |
| $e_5$ |  |  |  |  |  |  | $e_5$ |
| $e_6$ |  |  |  |  |  |  | $e_6$ |
| $e_7$ |  |  |  |  |  |  |  |

parameters: $[a, c]$

[[{$a^2 = 1$}]]

### [7, [6, 2], 1, 16]

|   | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ |  |  |  |  |  |  | $e_1 + a e_2$ |
| $e_2$ |  |  |  |  |  |  | $e_2$ |
| $e_3$ |  |  |  |  |  |  | $e_3$ |
| $e_4$ |  |  |  |  |  |  | $e_4$ |
| $e_5$ |  |  |  |  |  |  | $e_5$ |
| $e_6$ |  |  |  |  |  |  | $e_6$ |
| $e_7$ |  |  |  |  |  |  |  |

### [7, [6, 2], 1, 17]

|   | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ |  |  |  |  |  |  | $e_1$ |
| $e_2$ |  |  |  |  |  |  | $a e_2$ |
| $e_3$ |  |  |  |  |  |  | $a e_3$ |
| $e_4$ |  |  |  |  |  |  | $a e_4$ |
| $e_5$ |  |  |  |  |  |  | $a e_5$ |
| $e_6$ |  |  |  |  |  |  | $e_6$ |
| $e_7$ |  |  |  |  |  |  |  |

parameters: $[a, b, c]$

[[{$a^2 = 1$}]]

### [7, [6, 2], 1, 18]

|   | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ |  |  |  |  |  |  | $e_1$ |
| $e_2$ |  |  |  |  |  |  | $4 a e_2$ |
| $e_3$ |  |  |  |  |  |  | $3 a e_3$ |
| $e_4$ |  |  |  |  |  |  | $2 a e_4$ |
| $e_5$ |  |  |  |  |  |  | $a e_5$ |
| $e_6$ |  |  |  |  |  |  | $e_6$ |
| $e_7$ |  |  |  |  |  |  |  |

### [7, [6, 2], 1, 19]

|   | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ |  |  |  |  |  |  | $4 e_1 + e_2$ |
| $e_2$ |  |  |  |  |  |  | $4 e_2$ |
| $e_3$ |  |  |  |  |  |  | $3 e_3$ |
| $e_4$ |  |  |  |  |  |  | $2 e_4$ |
| $e_5$ |  |  |  |  |  |  | $e_5$ |
| $e_6$ |  |  |  |  |  |  | $e_6$ |
| $e_7$ |  |  |  |  |  |  |  |
| $7, [6, 2], 1, 20$ | $7, [6, 3], 1, 1$ |
|-------------------|-------------------|
| $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ | $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ |
| $e_1$ | $e_1$ |
| $e_2$ | $e_2$ |
| $e_3$ | $(a+2b)e_2$ |
| $e_4$ | $(a+b)e_3$ |
| $e_5$ | $2be_4$ |
| $e_6$ | $ae_5$ |
| $e_7$ | $be_6$ |

parameters: $[a, b]$  
$[a^2 + b^2 \neq 0]$  

| $7, [6, 2], 1, 21$ |
|-------------------|
| $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ |
| $e_1$ | $e_1$ |
| $e_2$ | $4e_2$ |
| $e_3$ | $3e_3$ |
| $e_4$ | $2e_4$ |
| $e_5$ | $e_5$ |
| $e_6$ | $e_1 + e_5 + e_6$ |
| $e_7$ | $e_7$ |

parameters: $[a]$  
$[\square]$  

| $7, [6, 2], 1, 22$ |
|-------------------|
| $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ |
| $e_1$ | $e_1$ |
| $e_2$ | $4e_2$ |
| $e_3$ | $3e_3$ |
| $e_4$ | $2e_4$ |
| $e_5$ | $e_4 + e_5$ |
| $e_6$ | $a(e_1 + e_5 + e_6)$ |
| $e_7$ | $e_7$ |

parameters: $[a]$  
$[\square]$  

| $7, [6, 2], 1, 23$ |
|-------------------|
| $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ |
| $e_1$ | $e_1$ |
| $e_2$ | $e_2$ |
| $e_3$ | $e_2 + e_3$ |
| $e_4$ | $e_4 + e_2 + e_3$ |
| $e_5$ | $e_5$ |
| $e_7$ | $e_7$ |

parameters: $[a, \varepsilon]$  
$[\varepsilon^2 = 1]$  

| $7, [6, 3], 1, 2$ |
|-------------------|
| $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ |
| $e_1$ | $e_1$ |
| $e_2$ | $2ae_2$ |
| $e_3$ | $ae_3$ |
| $e_4$ | $e_2 + 2ae_4$ |
| $e_5$ | $e_3$ |
| $e_6$ | $ae_6$ |
| $e_7$ | $e_7$ |

parameters: $[a]$  
$[\square]$  

| $7, [6, 3], 1, 3$ |
|-------------------|
| $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ |
| $e_1$ | $e_1$ |
| $e_2$ | $(a+2)e_2$ |
| $e_3$ | $(a+1)e_3$ |
| $e_4$ | $e_2 + 2e_4$ |
| $e_5$ | $e_3$ |
| $e_6$ | $e_1 + e_6$ |
| $e_7$ | $e_7$ |

parameters: $[a]$  
$[\square]$  

| $7, [6, 3], 1, 4$ |
|-------------------|
| $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ |
| $e_1$ | $e_1$ |
| $e_2$ | $2e_2$ |
| $e_3$ | $e_3$ |
| $e_4$ | $e_2$ |
| $e_5$ | $e_3$ |
| $e_6$ | $e_1$ |
| $e_7$ | $e_7$ |
### [7, [6, 3], 1, 5]

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
|--------|--------|--------|--------|--------|--------|--------|
| \(e_8\) | \(e_9\) | \(e_{10}\) | \(e_{11}\) | \(e_{12}\) | \(e_{13}\) | \(e_{14}\) |

### [7, [6, 3], 1, 6]

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
|--------|--------|--------|--------|--------|--------|--------|
| \(a e_2\) | \(2 a e_3\) | \(2 a e_4\) | \(e_5\) | \(- e_4\) | \(e_6\) | \(e_7\) |

parameters: \([a]\)

### [7, [6, 3], 1, 7]

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
|--------|--------|--------|--------|--------|--------|--------|
| \(a e_2\) | \(3 a e_3\) | \(3 a e_4\) | \(e_5\) | \(- e_4\) | \(e_6\) | \(e_7\) |

parameters: \([a]\)

### [7, [6, 3], 1, 8]

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
|--------|--------|--------|--------|--------|--------|--------|
| \(a e_2\) | \((a+2) e_2\) | \((a+1) e_3\) | \(2 e_4\) | \(e_5\) | \(- e_4\) | \(e_7\) |

parameters: \([a]\)

### [7, [6, 3], 1, 9]

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
|--------|--------|--------|--------|--------|--------|--------|
| \(e_8\) | \(e_9\) | \(e_{10}\) | \(e_{11}\) | \(e_{12}\) | \(e_{13}\) | \(e_{14}\) |

### [7, [6, 3], 1, 10]

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
|--------|--------|--------|--------|--------|--------|--------|
| \(e_8\) | \(2 e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(- e_4\) | \(e_7\) |

### [7, [6, 3], 1, 11]

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
|--------|--------|--------|--------|--------|--------|--------|
| \(e_8\) | \(3 e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(- e_4\) | \(e_7\) |

### [7, [6, 3], 1, 12]

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
|--------|--------|--------|--------|--------|--------|--------|
| \(e_8\) | \(3 e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(- e_4\) | \(e_7\) |
### [7, [6, 3], 1, 13]

| \( \mathbf{e}_1 \) | \( \mathbf{e}_2 \) | \( \mathbf{e}_3 \) | \( \mathbf{e}_4 \) | \( \mathbf{e}_5 \) | \( \mathbf{e}_6 \) | \( \mathbf{e}_7 \) |
|---|---|---|---|---|---|---|
| \( e_1 \) | . | . | . | . | . | \( e_1 \) |
| \( e_2 \) | . | . | . | . | \( 4ae_2 \) | |
| \( e_3 \) | . | . | \( e_2 \) | \( 3ae_3 \) | | |
| \( e_4 \) | . | \( e_2 \) | . | \( 2ae_4 \) | | |
| \( e_5 \) | . | \( e_3 \) | \( e_4+2ae_5 \) | | | |
| \( e_6 \) | . | . | \( ae_6 \) | | | |
| \( e_7 \) | | | | | | |

**Parameters:** \( \{a\} \)

### [7, [6, 3], 1, 17]

| \( \mathbf{e}_1 \) | \( \mathbf{e}_2 \) | \( \mathbf{e}_3 \) | \( \mathbf{e}_4 \) | \( \mathbf{e}_5 \) | \( \mathbf{e}_6 \) | \( \mathbf{e}_7 \) |
|---|---|---|---|---|---|---|
| \( e_1 \) | . | . | . | . | . | \( 2e_1 \) |
| \( e_2 \) | . | . | . | . | \( (a+2)e_2 \) | |
| \( e_3 \) | . | . | \( e_2 \) | \( (a+1)e_3 \) | | |
| \( e_4 \) | . | \( e_2 \) | . | \( e_1+2e_4 \) | | |
| \( e_5 \) | . | \( e_3 \) | . | \( ae_5 \) | | |
| \( e_6 \) | . | . | \( e_6 \) | | | |
| \( e_7 \) | | | | | | |

**Parameters:** \( \{a\} \)

### [7, [6, 3], 1, 18]

| \( \mathbf{e}_1 \) | \( \mathbf{e}_2 \) | \( \mathbf{e}_3 \) | \( \mathbf{e}_4 \) | \( \mathbf{e}_5 \) | \( \mathbf{e}_6 \) | \( \mathbf{e}_7 \) |
|---|---|---|---|---|---|---|
| \( e_1 \) | . | . | . | . | . | . |
| \( e_2 \) | . | . | . | . | \( e_2 \) | |
| \( e_3 \) | . | . | \( e_2 \) | \( e_2+e_3 \) | | |
| \( e_4 \) | . | \( e_2 \) | . | \( e_1 \) | | |
| \( e_5 \) | . | \( e_3 \) | \( e_5 \) | | | |
| \( e_6 \) | . | . | . | \( -e_4 \) | | |
| \( e_7 \) | | | | | | |

### [7, [6, 3], 1, 19]

| \( \mathbf{e}_1 \) | \( \mathbf{e}_2 \) | \( \mathbf{e}_3 \) | \( \mathbf{e}_4 \) | \( \mathbf{e}_5 \) | \( \mathbf{e}_6 \) | \( \mathbf{e}_7 \) |
|---|---|---|---|---|---|---|
| \( e_1 \) | . | . | . | . | . | . |
| \( e_2 \) | . | . | . | . | \( e_2 \) | |
| \( e_3 \) | . | . | \( e_2+e_3 \) | | | |
| \( e_4 \) | . | \( e_2 \) | . | \( e_1 \) | | |
| \( e_5 \) | . | \( e_3 \) | \( e_5 \) | | | |
| \( e_6 \) | . | . | . | \( -e_4 \) | | |
| \( e_7 \) | | | | | | |

### [7, [6, 3], 1, 20]

| \( \mathbf{e}_1 \) | \( \mathbf{e}_2 \) | \( \mathbf{e}_3 \) | \( \mathbf{e}_4 \) | \( \mathbf{e}_5 \) | \( \mathbf{e}_6 \) | \( \mathbf{e}_7 \) |
|---|---|---|---|---|---|---|
| \( e_1 \) | . | . | . | . | . | \( 2e_1 \) |
| \( e_2 \) | . | . | . | . | \( 3e_2 \) | |
| \( e_3 \) | . | . | \( e_2 \) | \( 2e_3 \) | | |
| \( e_4 \) | . | \( e_2 \) | . | \( e_1+2e_4 \) | | |
| \( e_5 \) | . | \( e_3 \) | \( e_5 \) | | | |
| \( e_6 \) | . | . | . | \( -e_5+e_6 \) | | |
| \( e_7 \) | | | | | | |
| 7, [6, 3], 1, 21 | 7, [6, 3], 1, 22 | 7, [6, 3], 1, 23 | 7, [6, 3], 1, 24 |
|------------------|------------------|------------------|------------------|
| \( e_1 \)       | \( e_1 \)       | \( e_1 \)       | \( e_1 \)       |
| \( e_2 \)       | \( e_2 \)       | \( e_2 \)       | \( e_2 \)       |
| \( e_3 \)       | \( e_2 \)       | \( e_2 \)       | \( e_2 \)       |
| \( e_4 \)       | \( e_2 \)       | \( e_2 \)       | \( e_2 \)       |
| \( e_5 \)       | \( e_3 \)       | \( e_3 \)       | \( e_3 \)       |
| \( e_6 \)       | \( e_3 \)       | \( e_3 \)       | \( e_3 \)       |
| \( e_7 \)       | \( e_3 \)       | \( e_3 \)       | \( e_3 \)       |

parameters: \([a]\)

| 7, [6, 3], 1, 25 | 7, [6, 3], 1, 26 | 7, [6, 3], 1, 27 | 7, [6, 3], 1, 28 |
|------------------|------------------|------------------|------------------|
| \( e_1 \)       | \( e_1 \)       | \( e_1 \)       | \( e_1 \)       |
| \( e_2 \)       | \( e_2 \)       | \( e_2 \)       | \( e_2 \)       |
| \( e_3 \)       | \( e_2 \)       | \( e_2 \)       | \( e_2 \)       |
| \( e_4 \)       | \( e_2 \)       | \( e_2 \)       | \( e_2 \)       |
| \( e_5 \)       | \( e_3 \)       | \( e_3 \)       | \( e_3 \)       |
| \( e_6 \)       | \( e_3 \)       | \( e_3 \)       | \( e_3 \)       |
| \( e_7 \)       | \( e_3 \)       | \( e_3 \)       | \( e_3 \)       |
\[
\begin{array}{cccccccc}
7, [6, 3], 1, 29 & & & & & & & \\
\hline
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
e_1 & \cdot & \cdot & \cdot & \cdot & \cdot & 2e_1 + e_2 \\
e_2 & \cdot & \cdot & \cdot & \cdot & \cdot & 2e_2 \\
e_3 & \cdot & \cdot & e_2 & e_3 \\
e_4 & \cdot & e_2 & e_3 & e_4 \\
e_5 & \cdot & e_3 & \cdot & \cdot \\
e_6 & \cdot & e_5 & \cdot & \cdot \\
e_7 & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

\[
\begin{array}{cccccccc}
7, [6, 4], 1, 1 & & & & & & & \\
\hline
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
e_1 & \cdot & \cdot & \cdot & \cdot & \cdot & e_1 \\
e_2 & \cdot & \cdot & \cdot & \cdot & a & e_2 \\
e_3 & \cdot & \cdot & \cdot & -a & e_3 \\
e_4 & \cdot & e_2 & e_3 & \cdot \\
e_5 & \cdot & e_4 & e_2 + e_5 & \cdot \\
e_6 & \cdot & \cdot & -a & e_6 & \cdot \\
e_7 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

\text{parameters: } [a, b] 
\text{\quad } \{b \leq a, a^2 + b^2 \neq 0\}

\[
\begin{array}{cccccccc}
7, [6, 4], 1, 2 & & & & & & & \\
\hline
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
e_1 & \cdot & \cdot & \cdot & \cdot & \cdot & e_1 \\
e_2 & \cdot & \cdot & \cdot & \cdot & a & e_2 \\
e_3 & \cdot & \cdot & \cdot & -a & e_3 \\
e_4 & \cdot & e_2 & e_3 & \cdot \\
e_5 & \cdot & e_4 & e_2 + e_5 & \cdot \\
e_6 & \cdot & \cdot & -a & e_6 & \cdot \\
e_7 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

\text{parameters: } [a] 
\text{\quad } \{\}

\[
\begin{array}{cccccccc}
7, [6, 4], 1, 3 & & & & & & & \\
\hline
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
e_1 & \cdot & \cdot & \cdot & \cdot & \cdot & e_1 \\
e_2 & \cdot & \cdot & \cdot & \cdot & 2a & e_2 \\
e_3 & \cdot & \cdot & \cdot & a & e_3 \\
e_4 & \cdot & e_2 & e_3 & a & e_4 \\
e_5 & \cdot & e_4 & e_3 + a & e_5 \\
e_6 & \cdot & \cdot & \cdot & \cdot & \cdot \\
e_7 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

\text{parameters: } [a, \varepsilon] 
\text{\quad } \{\varepsilon^2 = 1\}

\[
\begin{array}{cccccccc}
7, [6, 4], 1, 4 & & & & & & & \\
\hline
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
e_1 & \cdot & \cdot & \cdot & \cdot & \cdot & e_1 \\
e_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
e_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
e_4 & \cdot & e_2 & e_3 & \cdot & e_3 \\
e_5 & \cdot & e_4 & e_2 + \varepsilon e_3 & \cdot \\
e_6 & \cdot & \cdot & \cdot & \cdot & \cdot \\
e_7 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

\text{parameters: } [e] 
\text{\quad } \{e^2 = 1\}

\[
\begin{array}{cccccccc}
7, [6, 4], 1, 5 & & & & & & & \\
\hline
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
e_1 & \cdot & \cdot & \cdot & \cdot & \cdot & e_1 \\
e_2 & \cdot & \cdot & \cdot & \cdot & a & e_2 \\
e_3 & \cdot & \cdot & \cdot & (a+2) & e_3 \\
e_4 & \cdot & e_2 & e_3 & (a+1) & e_4 \\
e_5 & \cdot & e_4 & e_5 & \cdot & e_7 \\
e_6 & \cdot & e_5 & \cdot & \cdot & \cdot \\
e_7 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

\text{parameters: } [e, \delta] 
\text{\quad } \{\delta^2 = 1\}

\[
\begin{array}{cccccccc}
7, [6, 4], 1, 6 & & & & & & & \\
\hline
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
e_1 & \cdot & \cdot & \cdot & \cdot & \cdot & ae_1 \\
e_2 & \cdot & \cdot & \cdot & \cdot & (a+2) & e_2 \\
e_3 & \cdot & \cdot & \cdot & (2a+1) & e_3 \\
e_4 & \cdot & e_2 & e_3 & (a+1) & e_4 \\
e_5 & \cdot & e_4 & e_5 & \cdot & e_7 \\
e_6 & \cdot & e_6 & \cdot & \cdot & \cdot \\
e_7 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

\text{parameters: } [a] 
\text{\quad } \{\}

\[
\begin{array}{cccccccc}
7, [6, 4], 1, 7 & & & & & & & \\
\hline
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
e_1 & \cdot & \cdot & \cdot & \cdot & \cdot & e_1 \\
e_2 & \cdot & \cdot & \cdot & \cdot & 2e_3 & e_2 \\
e_3 & \cdot & \cdot & \cdot & 2e_3 & \cdot & \cdot \\
e_4 & \cdot & e_2 & e_3 & e_4 & \cdot \\
e_5 & \cdot & e_4 & \cdot & \cdot & \cdot \\
e_6 & \cdot & \cdot & \cdot & e_1 + e_6 & \cdot \\
e_7 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

\text{parameters: } [e] 
\text{\quad } \{e^2 = 1\}
| Parameters: $[a]$ | Parameters: $[c]$ |
|------------------|------------------|
| $[[c^2 = 1]]$    | $[[c^2 = 1]]$    |

- **[7, [6, 4], 1, 8]**
  
  
  \[
  \begin{array}{cccccc}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_7 \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  e_2 & \cdots & \cdots & 2e_2 & \cdots & \cdots \\
  e_3 & \cdots & \cdots & \cdots & e_3 & \cdots \\
  e_4 & e_2 & e_3 & e_4 & \cdots & \cdots \\
  e_5 & e_4 & e_3 + e_5 & \cdots & \cdots & \cdots \\
  e_6 & \cdots & e_1 & \cdots & \cdots & \cdots \\
  e_7 & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \end{array}
  \]

- **[7, [6, 4], 1, 9]**
  
  \[
  \begin{array}{cccccc}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_7 \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  e_2 & \cdots & \cdots & (2a+1)e_1 + e_3 & \cdots & \cdots \\
  e_3 & \cdots & \cdots & (a+2)e_2 & \cdots & \cdots \\
  e_4 & \cdots & e_2 & e_3 & (a+1)e_4 & \cdots \\
  e_5 & e_4 & e_5 & \cdots & \cdots & \cdots \\
  e_6 & \cdots & a_0 & \cdots & \cdots & \cdots \\
  e_7 & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \end{array}
  \]

- **[7, [6, 4], 1, 10]**
  
  \[
  \begin{array}{cccccc}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_7 \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  e_2 & \cdots & \cdots & 2e_1 + e_3 & \cdots & \cdots \\
  e_3 & \cdots & \cdots & e_2 & \cdots & \cdots \\
  e_4 & \cdots & e_2 & e_3 & e_4 & \cdots \\
  e_5 & \cdots & e_4 & \cdots & \cdots & \cdots \\
  e_6 & \cdots & e_6 & \cdots & \cdots & \cdots \\
  e_7 & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \end{array}
  \]

- **[7, [6, 4], 1, 11]**
  
  \[
  \begin{array}{cccccc}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_7 \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  e_2 & \cdots & \cdots & 2e_1 + e_3 & \cdots & \cdots \\
  e_3 & \cdots & \cdots & e_2 & \cdots & \cdots \\
  e_4 & \cdots & e_2 & e_3 & e_4 & \cdots \\
  e_5 & \cdots & e_4 & \cdots & \cdots & \cdots \\
  e_6 & \cdots & e_2 + e_6 & \cdots & \cdots & \cdots \\
  e_7 & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \end{array}
  \]

- **[7, [6, 4], 1, 12]**
  
  \[
  \begin{array}{cccccc}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_7 \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  e_2 & \cdots & \cdots & e_1 + e_3 & \cdots & \cdots \\
  e_3 & \cdots & \cdots & -e_2 & \cdots & \cdots \\
  e_4 & \cdots & e_2 & e_3 & \cdots & \cdots \\
  e_5 & \cdots & e_4 & e_5 & \cdots & \cdots \\
  e_6 & \cdots & e_1 + e_6 & \cdots & \cdots & \cdots \\
  e_7 & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \end{array}
  \]

- **[7, [6, 4], 1, 13]**
  
  \[
  \begin{array}{cccccc}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_7 \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  e_2 & \cdots & \cdots & 2e_2 & \cdots & \cdots \\
  e_3 & \cdots & \cdots & e_3 & \cdots & \cdots \\
  e_4 & \cdots & e_2 & e_3 & e_4 & \cdots \\
  e_5 & \cdots & e_4 & e_5 & \cdots & \cdots \\
  e_6 & \cdots & \cdots & e_5 & \cdots & \cdots \\
  e_7 & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \end{array}
  \]

- **[7, [6, 4], 2, 1]**
  
  \[
  \begin{array}{cccccc}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_7 \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  e_2 & \cdots & \cdots & a_e_1 & \cdots & \cdots \\
  e_3 & \cdots & \cdots & 3b + e_2 - e_3 & \cdots & \cdots \\
  e_4 & \cdots & e_2 & 3b e_3 & \cdots & \cdots \\
  e_5 & \cdots & e_4 & b e_3 - e_6 & \cdots & \cdots \\
  e_6 & \cdots & e_3 + b e_6 & \cdots & \cdots & \cdots \\
  e_7 & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \end{array}
  \]

- **[7, [6, 4], 2, 2]**
  
  \[
  \begin{array}{cccccc}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_7 \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  e_2 & \cdots & \cdots & a e_1 & \cdots & \cdots \\
  e_3 & \cdots & \cdots & -e_3 & \cdots & \cdots \\
  e_4 & \cdots & e_2 & e_3 & \cdots & \cdots \\
  e_5 & \cdots & e_4 & -e_6 & \cdots & \cdots \\
  e_6 & \cdots & e_2 + e_6 & \cdots & \cdots & \cdots \\
  e_7 & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \end{array}
  \]

- **[7, [6, 4], 2, 3]**
  
  \[
  \begin{array}{cccccc}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_7 \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  e_2 & \cdots & \cdots & a e_1 & \cdots & \cdots \\
  e_3 & \cdots & \cdots & -e_3 & \cdots & \cdots \\
  e_4 & \cdots & e_2 & e_3 & \cdots & \cdots \\
  e_5 & \cdots & e_4 & -e_6 & \cdots & \cdots \\
  e_6 & \cdots & e_2 + e_6 & \cdots & \cdots & \cdots \\
  e_7 & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \end{array}
  \]

- **[7, [6, 4], 2, 4]**
  
  \[
  \begin{array}{cccccc}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_7 \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  e_2 & \cdots & \cdots & a e_1 & \cdots & \cdots \\
  e_3 & \cdots & \cdots & -e_3 & \cdots & \cdots \\
  e_4 & \cdots & e_2 & e_3 & \cdots & \cdots \\
  e_5 & \cdots & e_4 & -e_6 & \cdots & \cdots \\
  e_6 & \cdots & e_2 + e_6 & \cdots & \cdots & \cdots \\
  e_7 & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \end{array}
  \]
\[7, [6, 4], 3, 1\]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | .   | .   | e₁ |
| e₂ | .   | .   | .   | .   | 3a e₂ | .   |
| e₃ | .   | .   | .   | e₂+3a e₃ | .   | .   |
| e₄ | .   | e₂ | e₃ | 2a e₄ | .   | .   |
| e₅ | .   | e₄ | a e₅ | .   | e₅+a e₅ | .   |
| e₆ | .   | .   | .   | .   | .   | .   |
| e₇ | .   | .   | .   | .   | .   | .   |

parameters: \([a]\)

\( (e^2 = 1) \)

\[7, [6, 4], 3, 2\]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | .   | .   | e₁ |
| e₂ | .   | .   | .   | .   | e₁ e₂ | .   |
| e₃ | .   | .   | .   | e₂ | .   | .   |
| e₄ | .   | e₂ | e₃ | .   | .   | e₁ e₄ |
| e₅ | .   | e₄ | e₅ | .   | e₅ e₆ | .   |
| e₆ | .   | .   | .   | .   | .   | .   |
| e₇ | .   | .   | .   | .   | .   | .   |

parameters: \([e]\)

\( (e^2 = 1) \)

\[7, [6, 4], 3, 3\]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | .   | .   | e₁ |
| e₂ | .   | .   | .   | .   | 3e₂ | .   |
| e₃ | .   | .   | .   | e₂+3e₃ | .   | .   |
| e₄ | .   | e₂ | e₃ | 2e₄ | .   | .   |
| e₅ | .   | e₄ | e₅ | e₁+e₅ | .   | .   |
| e₆ | .   | .   | .   | e₅+e₆ | .   | .   |
| e₇ | .   | .   | .   | .   | .   | .   |

parameters: \([a]\)

\( (a^2 = 1) \)

\[7, [6, 4], 3, 4\]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | .   | 3e₁+e₃ | .   |
| e₂ | .   | .   | .   | .   | 3e₂ | .   |
| e₃ | .   | .   | .   | e₂+3e₃ | .   | .   |
| e₄ | .   | e₂ | e₃ | 2e₄ | .   | .   |
| e₅ | .   | e₄ | e₅ | .   | e₅+e₆ | .   |
| e₆ | .   | .   | .   | .   | .   | .   |
| e₇ | .   | .   | .   | .   | .   | .   |

\[7, [6, 5], 1, 1\]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | .   | .   | e₁ |
| e₂ | .   | .   | .   | 2a e₂ | .   | .   |
| e₃ | .   | e₂ | (a+b) e₃ | .   | .   |
| e₄ | .   | e₂ | (a+c) e₄ | .   | .   |
| e₅ | .   | (a-b) e₅ | .   | .   |
| e₆ | .   | (a-c) e₆ | .   | .   |
| e₇ | .   | .   | .   | .   | .   | .   |

parameters: \([a, b, c]\)

\( (a^2+b^2 \neq 0, 0 \leq c, c \leq b) \)

\[7, [6, 5], 1, 2\]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | (1-b) e₁ | .   | .   |
| e₂ | .   | .   | .   | 2e₂ | .   | .   |
| e₃ | .   | e₂ | (a+1) e₃ | .   | .   |
| e₄ | .   | e₂ | (b+1) e₄ | .   | .   |
| e₅ | .   | (1-a) e₅ | .   | .   |
| e₆ | .   | e₁+(1-b) e₆ | .   | .   |
| e₇ | .   | .   | .   | .   | .   | .   |

parameters: \([a, b]\)

\( (0 \leq a) \)

\[7, [6, 5], 1, 3\]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | -a e₁ | .   | .   |
| e₂ | .   | .   | .   | .   | .   | .   |
| e₃ | .   | e₂ | e₃ | .   | .   | e₃ |
| e₄ | .   | e₂ | a e₄ | .   | .   | e₄ |
| e₅ | .   | .   | -e₅ | .   | .   | e₅ |
| e₆ | .   | e₁-a e₆ | .   | .   | e₆ |
| e₇ | .   | .   | .   | .   | .   | .   |

parameters: \([a]\)

\( (a^2 = 1) \)

\[7, [6, 5], 1, 4\]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | -e₁ | .   | .   |
| e₂ | .   | .   | .   | .   | .   | .   |
| e₃ | .   | e₂ | .   | .   | .   | e₂ |
| e₄ | .   | e₂ | e₄ | .   | .   | e₄ |
| e₅ | .   | .   | .   | .   | .   | .   |
| e₆ | .   | e₁-e₆ | .   | .   | e₆ |
| e₇ | .   | .   | .   | .   | .   | .   |

\[7, [6, 5], 1, 4\]
### [7, [6, 5], 1, 5]

| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
|-----------------------|
| e₁ . . . . . . (1-a) e₁ |
| e₂ . . . . . 2 e₂ |
| e₃ . . e₂ . (a+1) e₃ |
| e₄ . . e₂ (a+1) e₄ |
| e₅ . . e₁+(1-a) e₅ |
| e₆ . . e₁+(1-a) e₆ |
| e₇ . . . . . . |

parameters: [a]

[[[]]]

### [7, [6, 5], 1, 6]

| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
|-----------------------|
| e₁ . . . . . . − e₁ |
| e₂ . . . . . 2 e₂ |
| e₃ . . e₂ . e₃ |
| e₄ . . e₂ e₄ |
| e₅ . . e₁−e₅ |
| e₆ . . e₁−e₆ |
| e₇ . . . . . . |

parameters: [a, b]

[[0 ≤ b, b ≤ a]]

### [7, [6, 5], 1, 7]

| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
|-----------------------|
| e₁ . . . . . . 2 e₁+e₂ |
| e₂ . . . . . 2 e₂ |
| e₃ . . e₂ . (a+1) e₃ |
| e₄ . . e₂ (b+1) e₄ |
| e₅ . . (1−a) e₅ |
| e₆ . . (1−b) e₆ |
| e₇ . . . . . . |

parameters: [a, b]

[[0 ≤ a]]

### [7, [6, 5], 1, 8]

| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
|-----------------------|
| e₁ . . . . . . e₂ |
| e₂ . . . . . . |
| e₃ . . e₂ . e₃ |
| e₄ . . e₂ a e₄ |
| e₅ . . − e₅ |
| e₆ . . − a e₆ |
| e₇ . . . . . . |

parameters: [a]

[[0 ≤ a]]

### [7, [6, 5], 1, 9]

| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
|-----------------------|
| e₁ . . . . . . 2 e₁+e₂ |
| e₂ . . . . . 2 e₂ |
| e₃ . . e₂ . (a+1) e₃ |
| e₄ . . e₂ . |
| e₅ . . (1−a) e₅ |
| e₆ . . e₁+2 e₆ |
| e₇ . . . . . . |

parameters: [a]

[[0 ≤ a]]

### [7, [6, 5], 1, 10]

| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
|-----------------------|
| e₁ . . . . . . e₀ |
| e₂ . . . . . . |
| e₃ . . e₂ . e₃ |
| e₄ . . e₂ . |
| e₅ . . e₁+2 e₅ |
| e₆ . . e₁+2 e₆ |
| e₇ . . . . . . |

parameters: [a, b]

[[0 ≤ b, b ≤ a]]

### [7, [6, 5], 1, 11]

| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
|-----------------------|
| e₁ . . . . . . 2 e₁+e₂ |
| e₂ . . . . . 2 e₂ |
| e₃ . . e₂ . |
| e₄ . . e₂ . |
| e₅ . . e₁+2 e₅ |
| e₆ . . e₁+2 e₆ |
| e₇ . . . . . . |

parameters: [a, b]

[[0 ≤ a]]

### [7, [6, 5], 2, 1]

| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
|-----------------------|
| e₁ . . . . . . e₁ |
| e₂ . . . . . . |
| e₃ . . e₂ . (a+b) e₃ |
| e₄ . . e₂ . a e₄ |
| e₅ . . (a−b) e₅ |
| e₆ . . a₄+a e₆ |
| e₇ . . . . . . |

parameters: [a, b]

[[0 ≤ b]]
### [7, [6, 5], 2, 2]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|-----|----|----|----|----|----|----|
| e₁  | e₂  | e₃  | (1−a)e₁ | e₄  | e₅  | e₆  |
| e₂  | e₃  | e₄  | 2e₂  | e₅  | e₆  | e₇  |
| e₃  | e₄  | e₅  | e₆  | e₇  | e₈  |
| e₄  | e₅  | e₆  | e₇  | e₈  |

**Parameters:** \([a]\)

### [7, [6, 5], 2, 6]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|-----|----|----|----|----|----|----|
| e₁  | e₂  | e₃  | e₄  | e₅  | e₆  | e₇  |
| e₂  | e₃  | e₄  | 2e₂  | e₅  | e₆  | e₇  |
| e₃  | e₄  | e₅  | e₆  | e₇  | e₈  |
| e₄  | e₅  | e₆  | e₇  | e₈  |

**Parameters:** \([a]\)

### [7, [6, 5], 2, 7]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|-----|----|----|----|----|----|----|
| e₁  | e₂  | e₃  | e₄  | e₅  | e₆  | e₇  |
| e₂  | e₃  | e₄  | e₅  | e₆  | e₇  | e₈  |
| e₃  | e₄  | e₅  | e₆  | e₇  | e₈  |
| e₄  | e₅  | e₆  | e₇  | e₈  |

**Parameters:** \([a]\)

### [7, [6, 5], 2, 8]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|-----|----|----|----|----|----|----|
| e₁  | e₂  | e₃  | e₄  | e₅  | e₆  | e₇  |
| e₂  | e₃  | e₄  | e₅  | e₆  | e₇  | e₈  |
| e₃  | e₄  | e₅  | e₆  | e₇  | e₈  |
| e₄  | e₅  | e₆  | e₇  | e₈  |

**Parameters:** \([0 ≤ a]\)

### [7, [6, 5], 2, 9]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|-----|----|----|----|----|----|----|
| e₁  | e₂  | e₃  | e₄  | e₅  | e₆  | e₇  |
| e₂  | e₃  | e₄  | 2e₂  | e₅  | e₆  | e₇  |
| e₃  | e₄  | e₅  | e₆  | e₇  | e₈  |
| e₄  | e₅  | e₆  | e₇  | e₈  |

**Parameters:** \([0 ≤ a]\)
### 7, [6, 5], 2, 10

| e₁  | e₂  | e₃  | e₄  | e₅  | e₆  | e₇  |
|-----|-----|-----|-----|-----|-----|-----|
|     |     |     |     |     |     | e₂  |
| e₁  |     |     |     |     | e₃  |     |
| e₁  |     |     |     |     |     |     |
| e₅  |     |     |     |     |     |     |
| e₆  |     |     |     |     |     |     |
|     |     |     |     |     |     |     |

### 7, [6, 5], 3, 1

| e₁  | e₂  | e₃  | e₄  | e₅  | e₆  | e₇  |
|-----|-----|-----|-----|-----|-----|-----|
|     |     |     |     |     |     | e₂  |
| e₁  |     |     |     |     | e₃  |     |
| e₁  |     |     |     |     |     |     |
| e₅  |     |     |     |     |     |     |
| e₆  |     |     |     |     |     |     |
|     |     |     |     |     |     |     |

parameters: [a, b]

[[0 ≤ b]]

### 7, [6, 5], 3, 2

| e₁  | e₂  | e₃  | e₄  | e₅  | e₆  | e₇  |
|-----|-----|-----|-----|-----|-----|-----|
|     |     |     |     |     |     | e₂  |
| e₁  |     |     |     |     | e₃  |     |
| e₁  |     |     |     |     |     |     |
| e₅  |     |     |     |     |     |     |
| e₆  |     |     |     |     |     |     |
|     |     |     |     |     |     |     |

parameters: [a]

[[0 ≤ a]]

### 7, [6, 5], 3, 5

| e₁  | e₂  | e₃  | e₄  | e₅  | e₆  | e₇  |
|-----|-----|-----|-----|-----|-----|-----|
|     |     |     |     |     |     | e₁  |
| e₁  |     |     |     |     | e₃  |     |
| e₁  |     |     |     |     |     |     |
| e₅  |     |     |     |     |     |     |
| e₆  |     |     |     |     |     |     |
|     |     |     |     |     |     |     |

### 7, [6, 5], 3, 6

| e₁  | e₂  | e₃  | e₄  | e₅  | e₆  | e₇  |
|-----|-----|-----|-----|-----|-----|-----|
|     |     |     |     |     |     | e₂  |
| e₁  |     |     |     |     | e₃  |     |
| e₁  |     |     |     |     |     |     |
| e₅  |     |     |     |     |     |     |
| e₆  |     |     |     |     |     |     |
|     |     |     |     |     |     |     |

### 7, [6, 5], 3, 7

| e₁  | e₂  | e₃  | e₄  | e₅  | e₆  | e₇  |
|-----|-----|-----|-----|-----|-----|-----|
|     |     |     |     |     |     | e₁  |
| e₁  |     |     |     |     | e₃  |     |
| e₁  |     |     |     |     |     |     |
| e₅  |     |     |     |     |     |     |
| e₆  |     |     |     |     |     |     |
|     |     |     |     |     |     |     |
\[ \begin{array}{cccccccc}
& e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
[7, [6, 5], 4, 1] & e_1 & \ldots & \ldots & \ldots & e_1 \\
e_2 & \ldots & \ldots & \ldots & 2a e_2 \\
e_3 & \ldots & e_2 & a e_3 \\
e_4 & \ldots & e_2 & a e_4 \\
e_5 & \ldots & e_3 + a e_5 \\
e_6 & \ldots & e_4 + a e_6 \\
e_7 & . \\
\end{array} \]

parameters: \([a]\)
| $[7, [6, 5], 6, 3]$ | $[7, [6, 5], 7, 2]$ |
|----------------------|----------------------|
| $e_1$ . . . . . . . 2 $a e_3 + e_2$ | $e_1$ . . . . . . . 2 $a e_4 + e_2$ |
| $e_2$ . . . . . . . 2 $a e_2$ | $e_2$ . . . . . . . 2 $a e_2$ |
| $e_3$ . . . . . . . $(a+b) e_3$ | $e_3$ . . . . . . . $a e_3$ |
| $e_4$ . . . . . . . $a e_4 - e_6$ | $e_4$ . . . . . . . $a e_4 - e_6$ |
| $e_5$ . . . . . . . $(a-b) e_5$ | $e_5$ . . . . . . . $c e_3 + a e_5$ |
| $e_6$ . . . . . . . $e_4 + a e_6$ | $e_6$ . . . . . . . $e_4 + a e_6$ |
| $e_7$ . . . . . . . . | $e_7$ . . . . . . . . |

parameters: $[a, b]$

$[[0 \leq a, 0 \leq b]]$

| $[7, [6, 5], 6, 4]$ | $[7, [6, 5], 7, 3]$ |
|----------------------|----------------------|
| $e_1$ . . . . . . . 2 $a e_3 + e_2$ | $e_1$ . . . . . . . $e_2$ |
| $e_2$ . . . . . . . 2 $a e_2$ | $e_2$ . . . . . . . . |
| $e_3$ . . . . . . . $e_1 + e_2$ | $e_3$ . . . . . . . $e_1$ |
| $e_4$ . . . . . . . $a e_4 - e_6$ | $e_4$ . . . . . . . $e_2 - e_6$ |
| $e_5$ . . . . . . . $e_1 + 2 a e_5$ | $e_5$ . . . . . . . $c e_3$ |
| $e_6$ . . . . . . . $e_4 + a e_6$ | $e_6$ . . . . . . . $e_4$ |
| $e_7$ . . . . . . . . | $e_7$ . . . . . . . . |

parameters: $[a]$

$[[0 \leq a]]$

| $[7, [6, 5], 7, 1]$ | $[7, [6, 5], 8, 1]$ |
|----------------------|----------------------|
| $e_1$ . . . . . . . $a e_4$ | $e_1$ . . . . . . . $a e_4$ |
| $e_2$ . . . . . . . 2 $b e_2$ | $e_2$ . . . . . . . 2 $b e_2$ |
| $e_3$ . . . . . . . $b e_3$ | $e_3$ . . . . . . . $(b+c) e_3 - e_4$ |
| $e_4$ . . . . . . . $b e_4 - e_6$ | $e_4$ . . . . . . . $e_2 e_3 + (b+c) e_4$ |
| $e_5$ . . . . . . . $e_3 + b e_5$ | $e_5$ . . . . . . . $(b-c) e_5 - e_6$ |
| $e_6$ . . . . . . . $e_4 + b e_6$ | $e_6$ . . . . . . . $e_5 + (b-c) e_6$ |
| $e_7$ . . . . . . . . | $e_7$ . . . . . . . . |

parameters: $[a, b, c]$

$[[a \neq 0, c^2 = 1]]$

| $[7, [6, 5], 8, 1]$ |
|----------------------|
| $e_1$ . . . . . . . $a e_4$ |
| $e_2$ . . . . . . . 2 $b e_2$ |
| $e_3$ . . . . . . . $(b+c) e_3 - e_4$ |
| $e_4$ . . . . . . . $e_2 e_3 + (b+c) e_4$ |
| $e_5$ . . . . . . . $(b-c) e_5 - e_6$ |
| $e_6$ . . . . . . . $e_5 + (b-c) e_6$ |
| $e_7$ . . . . . . . . |

parameters: $[a, b, c]$

$[[a \neq 0, 0 \leq c]]$
| $[7, [6, 5], 8, 2]$ | $[7, [6, 5], 10, 1]$ |
|---------------------|---------------------|
| $e_1 : \ldots : \ldots : 2ae_1 + e_2$ | $e_1 : \ldots : \ldots : a e_1$ |
| $e_2 : \ldots : \ldots : 2ae_2$ | $e_2 : \ldots : \ldots : 2be_2$ |
| $e_3 : \ldots : e_2 : (a+b)e_3 - e_4$ | $e_3 : \ldots : e_2 : be_3 - e_4$ |
| $e_4 : \ldots : e_2 : e_3 + (a+b)e_4$ | $e_4 : \ldots : e_2 : e_3 + be_4$ |
| $e_5 : \ldots : (a-b)e_5 - e_6$ | $e_5 : \ldots : e_3 + be_5 - e_6$ |
| $e_6 : \ldots : e_5 + (a-b)e_6$ | $e_6 : \ldots : e_4 + e_5 + be_6$ |
| $e_7$ | $e_7$ |

Parameters: $[a, b]$  
$[0 \leq b]$  

| $[7, [6, 5], 9, 1]$ | $[7, [6, 5], 10, 2]$ |
|---------------------|---------------------|
| $e_1 : \ldots : \ldots : a e_1$ | $e_1 : \ldots : \ldots : a e_1$ |
| $e_2 : \ldots : \ldots : 2ae_2$ | $e_2 : \ldots : \ldots : 2be_2$ |
| $e_3 : \ldots : e_2 : be_3 - e_5$ | $e_3 : \ldots : e_2 : a e_3 - e_4$ |
| $e_4 : \ldots : e_2 : e_3 - ce_5$ | $e_4 : \ldots : e_2 : a e_4$ |
| $e_5 : \ldots : e_3 + be_5$ | $e_5 : \ldots : e_3 + e_5 - e_6$ |
| $e_6 : \ldots : e_4 + be_6$ | $e_6 : \ldots : e_4 + e_5 + a e_6$ |
| $e_7$ | $e_7$ |

Parameters: $[a, b, c]$  
$[0 \neq a, 0 \leq b, -1 < c, c \leq 1, c \neq 0]$  

| $[7, [6, 5], 9, 2]$ | $[7, [6, 6], 1, 1]$ |
|---------------------|---------------------|
| $e_1 : \ldots : \ldots : 2ae_1 + e_2$ | $e_1 : \ldots : \ldots : e_1$ |
| $e_2 : \ldots : \ldots : 2ae_2$ | $e_2 : \ldots : \ldots : (a+c)e_2$ |
| $e_3 : \ldots : e_2 : a e_3 - e_5$ | $e_3 : \ldots : (b+c)e_3$ |
| $e_4 : \ldots : e_2 : e_4 - be_6$ | $e_4 : \ldots : e_2 : a e_4$ |
| $e_5 : \ldots : e_3 + a e_5$ | $e_5 : \ldots : e_3 : b e_5$ |
| $e_6 : \ldots : e_4 + a e_6$ | $e_6 : \ldots : e_5 : c e_6$ |
| $e_7$ | $e_7$ |

Parameters: $[a, b]$  
$[0 \leq a, -1 < b, b \leq 1, b \neq 0]$  

Parameters: $[a, b, c]$  
$[b \leq a]$
### 7, [6, 6], 1, 8

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | . | . | . | . | (1-α)e₁ | . |
| e₂ | . | . | . | (2-α)e₂ | . | . |
| e₃ | . | . | . | e₃ | . | . |
| e₄ | . | . | . | e₂ | e₃ + e₄ | . |
| e₅ | . | e₃ | . | a e₅ | . | . |
| e₆ | . | e₁ + (1-α)e₀ | . | . | . |
| e₇ | . | . | . | . | . | . |

Parameters: [α]

### 7, [6, 6], 1, 9

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | . | . | . | . | e₁ | . |
| e₂ | . | . | . | . | e₂ | . |
| e₃ | . | . | . | . | e₃ | . |
| e₄ | . | . | . | e₂ | e₃ + e₄ | . |
| e₅ | . | e₃ | . | e₂ + e₅ | . |
| e₆ | . | e₁ | . | . | . |
| e₇ | . | . | . | . | . | . |

Parameters: [ε]

### 7, [6, 6], 1, 10

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | . | . | . | . | . | . |
| e₂ | . | . | . | . | e₂ | . |
| e₃ | . | . | . | . | e₃ | . |
| e₄ | . | . | . | e₂ | e₃ + e₄ | . |
| e₅ | . | e₃ | . | e₂ + e₅ | . |
| e₆ | . | e₁ | . | . | . |
| e₇ | . | . | . | . | . | . |

Parameters: [c]

### 7, [6, 6], 1, 11

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | . | . | . | . | e₁ | . |
| e₂ | . | . | . | (a+b)e₂ | . | . |
| e₃ | . | . | . | 2be₃ | . | . |
| e₄ | . | e₂ | . | ae₄ | . | . |
| e₅ | . | e₃ | . | be₅ | . | . |
| e₆ | . | e₅ + b e₆ | . | . | . |
| e₇ | . | . | . | . | . | . |

Parameters: [a, b]

### 7, [6, 6], 1, 12

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | . | . | . | . | e₁ | . |
| e₂ | . | . | . | . | ae₂ | . |
| e₃ | . | . | . | . | . | . |
| e₄ | . | . | . | e₂ + ae₄ | . | . |
| e₅ | . | e₃ | . | . | . |
| e₆ | . | e₅ | . | . | . |
| e₇ | . | . | . | . | . | . |

Parameters: [α]

### 7, [6, 6], 1, 13

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | . | . | . | . | e₁ | . |
| e₂ | . | . | . | . | 3ae₂ | . |
| e₃ | . | . | . | 2ae₃ | . | . |
| e₄ | . | . | . | e₂ + ae₄ | . | . |
| e₅ | . | e₃ | . | ae₅ | . | . |
| e₆ | . | e₅ + ae₆ | . | . | . |
| e₇ | . | . | . | . | . | . |

Parameters: [α]

### 7, [6, 6], 1, 14

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | . | . | . | . | e₁ | . |
| e₂ | . | . | . | . | ae₂ | . |
| e₃ | . | . | . | 2ae₃ | . | . |
| e₄ | . | . | . | e₂ + ae₄ | . | . |
| e₅ | . | e₃ | . | ae₅ | . | . |
| e₆ | . | e₅ + ae₆ | . | . | . |
| e₇ | . | . | . | . | . | . |

Parameters: [α]

### 7, [6, 6], 1, 15

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | . | . | . | . | e₁ | . |
| e₂ | . | . | . | . | . | . |
| e₃ | . | . | . | . | . | . |
| e₄ | . | . | . | e₂ + e₄ | . | . |
| e₅ | . | e₃ | . | . | . |
| e₆ | . | e₅ | . | . | . |
| e₇ | . | . | . | . | . | . |
### 7, [6, 6], 1, 16

| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
|------------------------|
| e₁ . . . . . . . e₁    |
| e₂ . . . . . . .      |
| e₃ . . . . . . .      |
| e₄ . . e₂ e₃         |
| e₅ . . e₃ e₂ e₃     |
| e₆ . . e₂ e₃ e₄   |
| e₇ . . . . . . .     |

Parameters: [ε]

\[ \varepsilon^2 = 1 \]

### 7, [6, 6], 1, 17

| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
|------------------------|
| e₁ . . . . . . . a e₁  |
| e₂ . . . . . . . (b+1) e₂ |
| e₃ . . . . . . . (a+b) e₃ |
| e₄ . . . e₂ e₃  |
| e₅ . . e₃ e₁+α e₅ |
| e₆ . . . b e₆   |
| e₇ . . . . . . . |

Parameters: [a, b]

### 7, [6, 6], 1, 18

| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
|------------------------|
| e₁ . . . . . . . e₁    |
| e₂ . . . . . . . . a e₂ |
| e₃ . . . . . . . (a+1) e₃ |
| e₄ . . . e₂ . .      |
| e₅ . . e₃ e₁+ε e₅ |
| e₆ . . . . a e₆    |
| e₇ . . . . . . .     |

Parameters: [a]

### 7, [6, 6], 1, 19

| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
|------------------------|
| e₁ . . . . . . .      |
| e₂ . . . . . . .      |
| e₃ . . . . . . .      |
| e₄ . . . . . . .      |
| e₅ . . e₃ e₁ .      |
| e₆ . . . e₂ e₃     |
| e₇ . . . . . . .     |

### 7, [6, 6], 1, 20

| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
|------------------------|
| e₁ . . . . . . . a e₁  |
| e₂ . . . . . . . (2−a) e₂ |
| e₃ . . . . . . . e₃     |
| e₄ . . . e₂ e₃+e₄ |
| e₅ . . . e₃ e₁+α e₅ |
| e₆ . . . (1−a) e₆ |
| e₇ . . . . . . .     |

Parameters: [a]

### 7, [6, 6], 1, 21

| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
|------------------------|
| e₁ . . . . . . . e₁    |
| e₂ . . . . . . . −e₂   |
| e₃ . . . . . . .       |
| e₄ . . . e₂ e₃      |
| e₅ . . . e₃ e₁+e₅   |
| e₆ . . . . e₆       |
| e₇ . . . . . .       |

### 7, [6, 6], 1, 22

| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
|------------------------|
| e₁ . . . . . . . a e₁  |
| e₂ . . . . . . . (a+1) e₂ |
| e₃ . . . . . . . 2 a e₃  |
| e₄ . . . . e₂ e₄     |
| e₅ . . . e₃ e₁+α e₅ |
| e₆ . . . . e₅+α e₆ |
| e₇ . . . . . .       |

Parameters: [a]

### 7, [6, 6], 1, 23

| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
|------------------------|
| e₁ . . . . . . . e₁    |
| e₂ . . . . . . . e₂    |
| e₃ . . . . . . . 2 e₃  |
| e₄ . . . . . e₂       |
| e₅ . . . e₃ e₁+α e₅ |
| e₆ . . . . e₅+α e₆ |
| e₇ . . . . . .       |

### 7, [6, 6], 1, 24

| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
|------------------------|
| e₁ . . . . . . . e₁    |
| e₂ . . . . . . . e₂    |
| e₃ . . . . . . . 2 e₃  |
| e₄ . . . . . e₂       |
| e₅ . . . e₃ e₁+α e₅ |
| e₆ . . . . e₅+α e₆ |
| e₇ . . . . . .       |
### [7, [6, 6], 1, 24]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | e₁ | .  |
| e₂ | .  | .  | .  | .  | 3e₂| .  |
| e₃ | .  | .  | .  | 2e₃| .  | .  |
| e₄ | .  | e₂| e₃+2e₄| .  | .  | .  |
| e₅ | .  | e₃| e₁+e₅| .  | .  | e₅+e₆|
| e₆ | .  | .  | e₄+e₆| .  | .  | .  |

### [7, [6, 6], 1, 25]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | a | e₁| .  |
| e₂ | .  | .  | .  | .  | 2e₂| .  |
| e₃ | .  | .  | .  | (a+1) | e₃| .  |
| e₄ | .  | e₂| e₄| .  | .  | .  |
| e₅ | .  | e₃| e₁+a | e₅| .  | e₄+e₆|
| e₆ | .  | .  | .  | .  | e₄ | .  |
| e₇ | .  | .  | .  | .  | .  | .  |

### [7, [6, 6], 1, 26]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | e₁| .  |
| e₂ | .  | .  | .  | .  | .  | .  |
| e₃ | .  | .  | .  | e₃| .  | .  |
| e₄ | .  | e₂| .  | .  | .  | .  |
| e₅ | .  | e₃| e₁+e₅| .  | e₄| .  |
| e₆ | .  | .  | .  | .  | .  | e₆|
| e₇ | .  | .  | .  | .  | .  | .  |

### [7, [6, 6], 1, 27]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | .  |
| e₂ | .  | .  | .  | .  | .  | .  |
| e₃ | .  | .  | .  | .  | e₃| .  |
| e₄ | .  | e₂| e₃+e₄| .  | .  | .  |
| e₅ | .  | e₃| e₁| .  | e₄| .  |
| e₆ | .  | .  | .  | e₄+e₆| .  | .  |
| e₇ | .  | .  | .  | .  | .  | .  |

### [7, [6, 6], 1, 28]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | (a+b) | e₁+e₃|
| e₂ | .  | .  | .  | .  | (b+1) | e₂| .  |
| e₃ | .  | .  | .  | (a+b) | e₃| .  | .  |
| e₄ | .  | .  | e₂| e₄| .  | .  | .  |
| e₅ | .  | e₃| a | e₅| .  | b | e₆|
| e₆ | .  | .  | .  | .  | .  | .  | .  |
| e₇ | .  | .  | .  | .  | .  | .  | .  |

Parameters: [a, b]

### [7, [6, 6], 1, 29]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | (a+1) | e₁+e₃| .  |
| e₂ | .  | .  | .  | .  | a | e₂| .  |
| e₃ | .  | .  | .  | (a+1) | e₃| .  | .  |
| e₄ | .  | e₂| .  | .  | .  | .  | .  |
| e₅ | .  | e₃| e₅| .  | a | e₆| .  |
| e₆ | .  | .  | e₆| .  | .  | .  | .  |
| e₇ | .  | .  | .  | .  | .  | .  | .  |

Parameters: [a]

### [7, [6, 6], 1, 30]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | e₁+e₃| .  | .  |
| e₂ | .  | .  | .  | e₂| .  | .  |
| e₃ | .  | .  | .  | e₃| .  | .  |
| e₄ | .  | e₂| .  | .  | .  | .  |
| e₅ | .  | e₃| .  | .  | .  | e₆|
| e₆ | .  | .  | .  | .  | e₆| .  |
| e₇ | .  | .  | .  | .  | .  | .  |

### [7, [6, 6], 1, 31]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | (2a−1)| e₁+e₃| .  |
| e₂ | .  | .  | .  | a | e₂| .  |
| e₃ | .  | .  | (2a−1) | e₃| .  | .  |
| e₄ | .  | e₂| e₄| .  | .  | .  |
| e₅ | .  | e₃| e₂+a | e₅| .  | e₆+a | e₆|
| e₆ | .  | .  | (a−1) | e₆| .  | .  |
| e₇ | .  | .  | .  | .  | .  | .  |

Parameters: [a]
| 7, [6, 6], 1, 32 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 . . . . . . 2e1 + e3 |
| e2 . . . . . e2 |
| e3 . . . . . 2e3 |
| e4 . . . . . e2 . |
| e5 . . . . . e2 + e5 |
| e6 . . . . . e6 |
| e7 . . . . . . |

| 7, [6, 6], 1, 33 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 . . . . . . ae1 + e3 |
| e2 . . . . . (a+1)e2 |
| e3 . . . . . e3 |
| e4 . . . . . e2 e4 |
| e5 . . . . . e3 |
| e6 . . . . . e1+a e6 |
| e7 . . . . . . |

parameters: [a]

| 7, [6, 6], 1, 34 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 . . . . . . e1 + e3 |
| e2 . . . . . e2 |
| e3 . . . . . e3 |
| e4 . . . . . e2 . |
| e5 . . . . . e3 |
| e6 . . . . . e1 + e6 |
| e7 . . . . . . |

| 7, [6, 6], 1, 35 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 . . . . . . e1 + e3 |
| e2 . . . . . e2 |
| e3 . . . . . e3 |
| e4 . . . . . e2 - e4 |
| e5 . . . . . e3 e2 |
| e6 . . . . . e1 + e6 |
| e7 . . . . . . |

| 7, [6, 6], 1, 36 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 . . . . . . 2ae1 + e3 |
| e2 . . . . . e2 |
| e3 . . . . . 2ae3 |
| e4 . . . . . e2 e4 |
| e5 . . . . . e3 e5 |
| e6 . . . . . e5 + a e6 |
| e7 . . . . . . |

parameters: [a]

| 7, [6, 6], 1, 37 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 . . . . . . 2e1 + e3 |
| e2 . . . . . e2 |
| e3 . . . . . 2e3 |
| e4 . . . . . e2 |
| e5 . . . . . e3 e5 |
| e6 . . . . . e5 + e6 |
| e7 . . . . . . |

| 7, [6, 6], 1, 38 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 . . . . . . 2e1 + e3 |
| e2 . . . . . e2 |
| e3 . . . . . 2e3 |
| e4 . . . . . e2 |
| e5 . . . . . e3 e5 |
| e6 . . . . . e5 + e6 |
| e7 . . . . . . |

| 7, [6, 6], 1, 39 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 . . . . . . e3 |
| e2 . . . . . e2 |
| e3 . . . . . . |
| e4 . . . . . e2 e4 |
| e5 . . . . . e3 |
| e6 . . . . . e5 + e6 |
| e7 . . . . . . |
### Parameters:

#### [7, [6, 6], 1, 40]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| .  | .  | .  | .  | (a+1)e1+e3 | .  | .  |
| e2 | .  | .  | .  | .  | e2 | .  |
| e3 | .  | .  | .  | (a+1)e3 | .  | .  |
| e4 | .  | e2 | .  | e4 | .  | .  |
| e5 | e3 | e5 | .  | .  | .  | .  |
| e6 | .  | e4 | .  | .  | .  | .  |
| e7 | .  | .  | .  | .  | .  | .  |

#### [7, [6, 6], 1, 41]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| .  | .  | .  | .  | .  | e1+e3 | .  |
| e2 | .  | .  | .  | .  | .  | e2 |
| e3 | .  | .  | .  | .  | .  | e3 |
| e4 | .  | e2 | .  | e4 | .  | .  |
| e5 | e3 | e5 | .  | .  | .  | .  |
| e6 | .  | e4 | .  | .  | .  | .  |
| e7 | .  | .  | .  | .  | .  | .  |

#### [7, [6, 6], 1, 42]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| .  | .  | .  | .  | .  | e1+e3 | .  |
| e2 | .  | .  | .  | .  | .  | e2 |
| e3 | .  | .  | .  | .  | .  | e3 |
| e4 | .  | e2 | .  | e4 | .  | .  |
| e5 | e3 | .  | .  | .  | .  | .  |
| e6 | .  | e4 | .  | .  | .  | .  |
| e7 | .  | .  | .  | .  | .  | .  |

#### [7, [6, 6], 1, 43]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| .  | .  | .  | .  | .  | e1+e3 | .  |
| e2 | .  | .  | .  | .  | .  | e2 |
| e3 | .  | .  | .  | .  | .  | e3 |
| e4 | .  | e2 | .  | e4 | .  | .  |
| e5 | .  | e3 | .  | .  | .  | .  |
| e6 | .  | .  | .  | .  | .  | .  |
| e7 | .  | .  | .  | .  | .  | .  |

#### [7, [6, 6], 1, 44]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| .  | .  | .  | .  | .  | e1+e3 | .  |
| e2 | .  | .  | .  | .  | .  | e2 |
| e3 | .  | .  | .  | .  | .  | e3 |
| e4 | .  | e2 | .  | e4 | .  | .  |
| e5 | e3 | e5 | .  | .  | .  | .  |
| e6 | .  | .  | .  | .  | .  | .  |
| e7 | .  | .  | .  | .  | .  | .  |

#### [7, [6, 6], 1, 45]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| .  | .  | .  | .  | .  | e1+e3 | .  |
| e2 | .  | .  | .  | .  | .  | e2 |
| e3 | .  | .  | .  | .  | .  | e3 |
| e4 | .  | e2 | .  | e4 | .  | .  |
| e5 | e3 | .  | .  | .  | .  | .  |
| e6 | .  | .  | .  | .  | .  | .  |
| e7 | .  | .  | .  | .  | .  | .  |

#### [7, [6, 6], 1, 46]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| .  | .  | .  | .  | .  | e1+e3 | .  |
| e2 | .  | .  | .  | .  | .  | e2 |
| e3 | .  | .  | .  | .  | .  | e3 |
| e4 | .  | e2 | .  | e4 | .  | .  |
| e5 | .  | e3 | e1+e5 | .  | .  | .  |
| e6 | .  | .  | .  | .  | .  | .  |
| e7 | .  | .  | .  | .  | .  | .  |

#### [7, [6, 6], 1, 47]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| .  | .  | .  | .  | .  | e1+e3 | .  |
| e2 | .  | .  | .  | .  | .  | e2 |
| e3 | .  | .  | .  | .  | .  | e3 |
| e4 | .  | e2 | .  | e4 | .  | .  |
| e5 | .  | e3 | e2 | .  | .  | .  |
| e6 | .  | .  | .  | .  | .  | .  |
| e7 | .  | .  | .  | .  | .  | .  |

#### Parameters:

- [a]  

### Parameters:

- [ε, δ]  

#### Parameters:

- [ε^2 = 1, δ^2 = 1]
### [7, [6, 6], 1, 48]

|   | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | ... | ... | $e_3$ | ... | ... | $e_5$ | ... |
| $e_2$ | ... | ... | ... | $e_2+e_4$ | ... | ... | ... |
| $e_3$ | ... | ... | $e_3$ | ... | ... | ... | ... |
| $e_4$ | ... | ... | ... | ... | ... | ... | ... |
| $e_5$ | ... | ... | ... | ... | ... | ... | ... |
| $e_6$ | ... | ... | ... | ... | ... | ... | ... |
| $e_7$ | ... | ... | ... | ... | ... | ... | ... |

Parameters: $[a]$

[[]]

### [7, [6, 6], 1, 49]

|   | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | ... | ... | $e_3$ | ... | ... | $e_5$ | ... |
| $e_2$ | ... | ... | ... | $e_2+e_4$ | ... | ... | ... |
| $e_3$ | ... | ... | $e_3$ | ... | ... | ... | ... |
| $e_4$ | ... | ... | ... | ... | ... | ... | ... |
| $e_5$ | ... | ... | ... | ... | ... | ... | ... |
| $e_6$ | ... | ... | ... | ... | ... | ... | ... |
| $e_7$ | ... | ... | ... | ... | ... | ... | ... |

Parameters: $[a, b, c]$

$[[a \neq 0, 0 \leq b]]$

### [7, [6, 6], 1, 50]

|   | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | ... | ... | $e_3$ | ... | ... | $e_5$ | ... |
| $e_2$ | ... | ... | ... | $e_2+e_4$ | ... | ... | ... |
| $e_3$ | ... | ... | $e_3$ | ... | ... | ... | ... |
| $e_4$ | ... | ... | ... | ... | ... | ... | ... |
| $e_5$ | ... | ... | ... | ... | ... | ... | ... |
| $e_6$ | ... | ... | ... | ... | ... | ... | ... |
| $e_7$ | ... | ... | ... | ... | ... | ... | ... |

Parameters: $[c]$

$[[x^2 = 1]]$

### [7, [6, 6], 1, 51]

|   | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | ... | ... | $e_3$ | ... | ... | $e_5$ | ... |
| $e_2$ | ... | ... | ... | $3e_2$ | ... | ... | ... |
| $e_3$ | ... | ... | $e_3$ | ... | ... | ... | ... |
| $e_4$ | ... | ... | $e_2+2e_4$ | ... | ... | ... | ... |
| $e_5$ | ... | ... | $e_3$ | $e_5$ | ... | ... | ... |
| $e_6$ | ... | ... | ... | ... | $e_5+e_6$ | ... | ... |
| $e_7$ | ... | ... | ... | ... | ... | ... | ... |

Parameters: $[a, b]$

$[[a \neq 0, 0 \leq b]]$

### [7, [6, 6], 1, 52]

|   | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | ... | ... | $e_3$ | ... | ... | $e_5$ | ... |
| $e_2$ | ... | ... | ... | $2e_2$ | ... | ... | ... |
| $e_3$ | ... | ... | $e_3$ | ... | ... | ... | ... |
| $e_4$ | ... | ... | $e_2+e_4$ | ... | ... | ... | ... |
| $e_5$ | ... | ... | $e_3$ | ... | ... | ... | ... |
| $e_6$ | ... | ... | ... | $e_4+e_6$ | ... | ... | ... |
| $e_7$ | ... | ... | ... | ... | ... | ... | ... |

Parameters: $[a]$

[[[]]]
### [7, [6, 6], 2, 4]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | .   | .   | .   | e₁ |
| e₂ | .   | .   | .   | a₂ e₂ | .   | .   | .   |
| e₃ | .   | .   | e₂ + e₃ | .   | .   | .   | .   |
| e₄ | .   | e₂ | a e₄ | .   | .   | .   | .   |
| e₅ | .   | e₃ | e₄ + e₅ | .   | .   | .   | .   |
| e₆ | .   | e₁ | e₆ | .   | .   | .   | .   |
| e₇ | .   | .   | .   | .   | .   | .   | .   |

Parameters: [a]

[[0 ≤ a]]

### [7, [6, 6], 3, 1]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | .   | .   | .   | e₁ |
| e₂ | .   | .   | .   | a₂ e₂ | .   | .   | .   |
| e₃ | .   | .   | e₂ + (a + b) e₃ | .   | .   | .   | .   |
| e₄ | .   | e₂ | a e₄ | .   | .   | .   | .   |
| e₅ | .   | e₃ | e₄ + e₅ | .   | .   | .   | .   |
| e₆ | .   | e₁ | e₆ | .   | .   | .   | .   |
| e₇ | .   | .   | .   | .   | .   | .   | .   |

Parameters: [a, b]

[[[]]]

### [7, [6, 6], 3, 2]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | a e₁ | .   | .   | .   |
| e₂ | .   | .   | .   | a₁ e₂ | .   | .   | .   |
| e₃ | .   | .   | e₂ + (a + b) e₃ | .   | .   | .   | .   |
| e₄ | .   | e₂ | e₄ | .   | .   | .   | .   |
| e₅ | .   | e₃ | e₄ + e₅ | .   | .   | .   | .   |
| e₆ | .   | e₁ | e₆ | .   | .   | .   | .   |
| e₇ | .   | .   | .   | .   | .   | .   | .   |

Parameters: [a]

[[[]]]

### [7, [6, 6], 3, 3]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | e₁ | .   | .   | .   |
| e₂ | .   | .   | .   | e₂ | .   | .   | .   |
| e₃ | .   | .   | e₂ + e₃ | .   | .   | .   | .   |
| e₄ | .   | e₂ | .   | .   | .   | .   | .   |
| e₅ | .   | e₃ | e₄ | .   | .   | .   | .   |
| e₆ | .   | e₁ + e₆ | .   | .   | .   | .   | .   |
| e₇ | .   | .   | .   | .   | .   | .   | .   |

### [7, [6, 6], 3, 4]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | .   | .   | .   | e₁ |
| e₂ | .   | .   | .   | a e₂ | .   | .   | .   |
| e₃ | .   | .   | e₂ + a e₃ | .   | .   | .   | .   |
| e₄ | .   | e₂ | a e₄ | .   | .   | .   | .   |
| e₅ | .   | e₃ | e₄ + a e₅ | .   | .   | .   | .   |
| e₆ | .   | e₁ | a e₆ | .   | .   | .   | .   |
| e₇ | .   | .   | .   | .   | .   | .   | .   |

Parameters: [a]

[[[]]]

### [7, [6, 6], 3, 5]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | .   | .   | .   | e₁ |
| e₂ | .   | .   | .   | 2 a e₂ | .   | .   | .   |
| e₃ | .   | .   | e₂ + 2 a e₃ | .   | .   | .   | .   |
| e₄ | .   | e₂ | a e₄ | .   | .   | .   | .   |
| e₅ | .   | e₃ | e₄ + a e₅ | .   | .   | .   | .   |
| e₆ | .   | e₅ | e₆ | .   | .   | .   | .   |
| e₇ | .   | .   | .   | .   | .   | .   | .   |

Parameters: [a]

[[[]]]

### [7, [6, 6], 3, 6]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | .   | .   | .   | e₁ |
| e₂ | .   | .   | .   | 2 a e₂ | .   | .   | .   |
| e₃ | .   | .   | e₂ + 2 a e₃ | .   | .   | .   | .   |
| e₄ | .   | e₂ | a e₄ | .   | .   | .   | .   |
| e₅ | .   | e₃ | e₄ + a e₅ | .   | .   | .   | .   |
| e₆ | .   | e₅ | e₆ | .   | .   | .   | .   |
| e₇ | .   | .   | .   | .   | .   | .   | .   |

Parameters: [a]

[[[]]]
| 7, [6, 6], 3, 8 | 7, [6, 6], 3, 12 |
|-----------------|-----------------|
| e₁  e₂  e₃  e₄  e₅  e₆  e₇ |
| e₁  . . . . . .  e₁ |
| e₂  . . . . . . (a+1) e₂ |
| e₃  . . . . . e₂+(a+1) e₃ |
| e₄  .  e₂  e₁+e₄ |
| e₅  . e₃ e₄+e₅ |
| e₆  .  a e₆ |
| e₇  . |

parameters: [a]

| 7, [6, 6], 3, 9 | 7, [6, 6], 3, 13 |
|-----------------|-----------------|
| e₁  e₂  e₃  e₄  e₅  e₆  e₇ |
| e₁  . . . . . .  . |
| e₂  . . . . . .  e₂ |
| e₃  . . . . . e₂+e₃ |
| e₄  .  e₂ e₁+e₄ |
| e₅  . e₃ e₄+e₅ |
| e₆  .  . e₆ |
| e₇  . |

parameters: [a]

| 7, [6, 6], 3, 10 | 7, [6, 6], 3, 14 |
|-----------------|-----------------|
| e₁  e₂  e₃  e₄  e₅  e₆  e₇ |
| e₁  . . . . . .  e₁ |
| e₂  . . . . . .  e₂ |
| e₃  . . . . . e₂+e₃ |
| e₄  .  e₂ e₁+e₄ |
| e₅  . e₃ e₄+e₅ |
| e₆  .  . e₆ |
| e₇  . |

| 7, [6, 6], 3, 11 | 7, [6, 6], 3, 15 |
|-----------------|-----------------|
| e₁  e₂  e₃  e₄  e₅  e₆  e₇ |
| e₁  . . . . . .  e₁ |
| e₂  . . . . . .  2e₂ |
| e₃  . . . . . e₂+2e₃ |
| e₄  .  e₂ e₁+e₄ |
| e₅  . e₃ e₄+e₅ |
| e₆  .  e₅+e₆ |
| e₇  . |

| 7, [6, 6], 3, 12 | 7, [6, 6], 3, 13 |
|-----------------|-----------------|
| e₁  e₂  e₃  e₄  e₅  e₆  e₇ |
| e₁  . . . . . .  e₁+e₃ |
| e₂  . . . . . .  e₂ |
| e₃  . . . . . e₂+e₃ |
| e₄  .  e₂ e₁+e₄ |
| e₅  . e₃ e₄+e₅ |
| e₆  .  . e₆ |
| e₇  . |

parameters: [a]

| 7, [6, 6], 3, 13 | 7, [6, 6], 3, 14 |
|-----------------|-----------------|
| e₁  e₂  e₃  e₄  e₅  e₆  e₇ |
| e₁  . . . . . .  e₁+e₃ |
| e₂  . . . . . .  e₂ |
| e₃  . . . . . e₂+e₃ |
| e₄  .  e₂ e₁+e₄ |
| e₅  . e₃ e₄+e₅ |
| e₆  .  . e₆ |
| e₇  . |

parameters: [a]

| 7, [6, 6], 3, 14 | 7, [6, 6], 3, 15 |
|-----------------|-----------------|
| e₁  e₂  e₃  e₄  e₅  e₆  e₇ |
| e₁  . . . . . .  e₁+e₃ |
| e₂  . . . . . .  e₂ |
| e₃  . . . . . e₂+e₃ |
| e₄  .  e₂ e₁+e₄ |
| e₅  . e₃ e₄+e₅ |
| e₆  .  . e₆ |
| e₇  . |
### $[7, [6, 6], 3, 16]$  

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| $\ldots$ | $\ldots$ | $2e_1 + e_3$ | $\ldots$ | $\ldots$ | $2e_2$ | $\ldots$ |
| $e_3$ | $\ldots$ | $\ldots$ | $e_2 + 2e_3$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $e_4$ | $\ldots$ | $e_2$ | $e_4$ | $\ldots$ | $e_3$ | $e_4+e_5$ |
| $e_5$ | $\ldots$ | $e_3$ | $e_4+e_5$ | $\ldots$ | $e_5+e_6$ | $\ldots$ |
| $e_7$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

Parameters: $[a, b, c]$  

### $[7, [6, 6], 3, 17]$  

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| $\ldots$ | $\ldots$ | $\ldots$ | $e_1 + e_3$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $e_2$ | $\ldots$ | $\ldots$ | $a e_2$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $e_3$ | $\ldots$ | $\ldots$ | $(b+2c) e_3$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $e_4$ | $\ldots$ | $e_3$ | $(b+c) e_4$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $e_5$ | $\ldots$ | $e_4$ | $b e_5$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $e_6$ | $\ldots$ | $e_4$ | $c e_6$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $e_7$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

Parameters: $[a, b, c]$  

### $[7, [6, 7], 1, 1]$  

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $e_1$ | $\ldots$ | $\ldots$ |
| $e_2$ | $\ldots$ | $\ldots$ | $\ldots$ | $a e_2$ | $\ldots$ | $\ldots$ |
| $e_3$ | $\ldots$ | $\ldots$ | $\ldots$ | $(b+2c) e_3$ | $\ldots$ | $\ldots$ |
| $e_4$ | $\ldots$ | $\ldots$ | $\ldots$ | $(b+c) e_4$ | $\ldots$ | $\ldots$ |
| $e_5$ | $\ldots$ | $\ldots$ | $\ldots$ | $b e_5$ | $\ldots$ | $\ldots$ |
| $e_6$ | $\ldots$ | $\ldots$ | $\ldots$ | $c e_6$ | $\ldots$ | $\ldots$ |
| $e_7$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

Parameters: $[a, b, c]$  

### $[7, [6, 7], 1, 2]$  

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $e_1$ | $\ldots$ | $\ldots$ |
| $e_2$ | $\ldots$ | $\ldots$ | $\ldots$ | $a e_2$ | $\ldots$ | $\ldots$ |
| $e_3$ | $\ldots$ | $\ldots$ | $\ldots$ | $b e_3$ | $\ldots$ | $\ldots$ |
| $e_4$ | $\ldots$ | $\ldots$ | $e_3$ | $e_4$ | $\ldots$ | $\ldots$ |
| $e_5$ | $\ldots$ | $\ldots$ | $\ldots$ | $e_4$ | $e_5+b e_5$ | $\ldots$ |
| $e_6$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $e_7$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

Parameters: $[a, b, c]$  

### $[7, [6, 7], 1, 3]$  

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $e_1$ | $\ldots$ | $\ldots$ |
| $e_2$ | $\ldots$ | $\ldots$ | $\ldots$ | $a e_2$ | $\ldots$ | $\ldots$ |
| $e_3$ | $\ldots$ | $\ldots$ | $\ldots$ | $(2a+b) e_3$ | $\ldots$ | $\ldots$ |
| $e_4$ | $\ldots$ | $\ldots$ | $\ldots$ | $(a+b) e_4$ | $\ldots$ | $\ldots$ |
| $e_5$ | $\ldots$ | $\ldots$ | $\ldots$ | $b e_5$ | $\ldots$ | $\ldots$ |
| $e_6$ | $\ldots$ | $\ldots$ | $\ldots$ | $e_2 + a e_6$ | $\ldots$ | $\ldots$ |
| $e_7$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

Parameters: $[a, b]$  

### $[7, [6, 7], 1, 4]$  

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $e_1$ | $\ldots$ | $\ldots$ |
| $e_2$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $e_3$ | $\ldots$ | $\ldots$ | $\ldots$ | $a e_3$ | $\ldots$ | $\ldots$ |
| $e_4$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $e_3$ | $a e_4$ |
| $e_5$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $e_4$ | $e_3 + a e_5$ |
| $e_6$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $e_2$ |
| $e_7$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

Parameters: $[a, c]$  

### $[7, [6, 7], 1, 5]$  

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $e_1$ | $\ldots$ | $\ldots$ |
| $e_2$ | $\ldots$ | $\ldots$ | $\ldots$ | $a e_2$ | $\ldots$ | $\ldots$ |
| $e_3$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $3b e_3$ | $\ldots$ |
| $e_4$ | $\ldots$ | $\ldots$ | $\ldots$ | $e_3$ | $2 b e_4$ | $\ldots$ |
| $e_5$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $b e_5$ |
| $e_6$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $e_5 + b e_6$ | $\ldots$ |
| $e_7$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

Parameters: $[a, b]$  

### $[7, [6, 7], 1, 4]$  

- $[1, a] = 1$  
- $[2, a] = 1$  
- $[3, a] = 1$  
- $[4, a] = 1$  
- $[5, a] = 1$  
- $[6, a] = 1$  
- $[7, a] = 1$
\[ [7, [6, 7], 1, 6] \]

\[
\begin{array}{cccccc}
& e_1 & e_2 & e_3 & e_4 & e_5 & e_7 \\
e_1 & . & . & . & . & . & e_1 \\
e_2 & . & . & . & . & a & e_2 \\
e_3 & . & . & . & . & . & . \\
e_4 & . & . & . & e_3 & . & . \\
e_5 & . & . & e_4 & e_5 & . & . \\
e_6 & . & . & . & . & e_5 & . \\
e_7 & . & . & . & . & . & . \\
\end{array}
\]

parameters: \([a, \varepsilon]\)

\([-1 \leq a, a \leq 1, a \neq 0, \varepsilon^2 = 1]\)

\[ [7, [6, 7], 1, 7] \]

\[
\begin{array}{cccccc}
& e_1 & e_2 & e_3 & e_4 & e_5 & e_7 \\
e_1 & . & . & . & . & . & e_1 \\
e_2 & . & . & . & . & a & e_2 \\
e_3 & . & . & . & . & (a+2) & e_3 \\
e_4 & . & . & . & e_3 & (a+1) & e_4 \\
e_5 & . & . & e_4 & e_2+a & e_5 \\
e_6 & . & . & . & . & e_1+e_6 & . \\
e_7 & . & . & . & . & . & . \\
\end{array}
\]

parameters: \([a, b]\)

\[ [7, [6, 7], 1, 8] \]

\[
\begin{array}{cccccc}
& e_1 & e_2 & e_3 & e_4 & e_5 & e_7 \\
e_1 & . & . & . & . & . & e_1 \\
e_2 & . & . & . & . & a & e_2 \\
e_3 & . & . & . & . & (a+2) & e_3 \\
e_4 & . & . & . & e_3 & (a+1) & e_4 \\
e_5 & . & . & e_4 & e_2+a & e_5 \\
e_6 & . & . & . & . & e_1+e_6 & . \\
e_7 & . & . & . & . & . & . \\
\end{array}
\]

parameters: \([a]\)

\[ [7, [6, 7], 1, 9] \]

\[
\begin{array}{cccccc}
& e_1 & e_2 & e_3 & e_4 & e_5 & e_7 \\
e_1 & . & . & . & . & . & . \\
e_2 & . & . & . & e_2 & . & . \\
e_3 & . & . & . & e_3 & . & . \\
e_4 & . & . & . & e_3 & e_4 & . \\
e_5 & . & . & e_4 & e_2+e_5 & . & . \\
e_6 & . & . & . & . & e_1 & . \\
e_7 & . & . & . & . & . & . \\
\end{array}
\]

parameters: \([a]\)

\[ [7, [6, 7], 1, 10] \]

\[
\begin{array}{cccccc}
& e_1 & e_2 & e_3 & e_4 & e_5 & e_7 \\
e_1 & . & . & . & . & . & e_1 \\
e_2 & . & . & . & . & a & e_2 \\
e_3 & . & . & . & . & 3 & a e_3 \\
e_4 & . & . & . & e_3 & 2 & a e_4 \\
e_5 & . & . & e_4 & e_2+a & e_5 \\
e_6 & . & . & . & . & e_5+e & e_6 \\
e_7 & . & . & . & . & . & . \\
\end{array}
\]

parameters: \([a]\)

\[ [7, [6, 7], 1, 11] \]

\[
\begin{array}{cccccc}
& e_1 & e_2 & e_3 & e_4 & e_5 & e_7 \\
e_1 & . & . & . & . & . & e_1 \\
e_2 & . & . & . & a & e_2+e_3 \\
e_3 & . & . & . & a & e_3 \\
e_4 & . & . & . & e_3 & (a-b) & e_4 \\
e_5 & . & . & e_4 & (a-2b) & e_5 \\
e_6 & . & . & . & b & e_6 & . \\
e_7 & . & . & . & . & . & . \\
\end{array}
\]

parameters: \([a, b]\)

\[ [7, [6, 7], 1, 12] \]

\[
\begin{array}{cccccc}
& e_1 & e_2 & e_3 & e_4 & e_5 & e_7 \\
e_1 & . & . & . & . & . & e_1 \\
e_2 & . & . & . & a & e_2+e_3 \\
e_3 & . & . & . & a & e_3 \\
e_4 & . & . & . & e_3 & . & . \\
e_5 & . & . & e_4 & -a & e_5 \\
e_6 & . & . & . & e_2+a & e_6 \\
e_7 & . & . & . & . & . & . \\
\end{array}
\]

parameters: \([a]\)
| 7, [6, 7], 1, 13 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 . . . . . . . |
| e2 . . . . . . a e3+e3 |
| e3 . . . . . . a e3 |
| e4 . . . . . . e3 (a-1)e4 |
| e5 . . . . . . e4 (a-2)e5 |
| e6 . . . . . . e1+e6 |
| e7 . . . . . . . |

parameters: [a]  
[[[]]]

| 7, [6, 7], 1, 14 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 . . . . . . . |
| e2 . . . . . . . e2+e3 |
| e3 . . . . . . . e3 |
| e4 . . . . . . . e3 e4 |
| e5 . . . . . . . e4 e5 |
| e6 . . . . . . . e1 |
| e7 . . . . . . . . |

| 7, [6, 7], 1, 15 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 . . . . . . . |
| e2 . . . . . . . 3 a e2+e3 |
| e3 . . . . . . . 3 a e3 |
| e4 . . . . . . . 2 a e4 |
| e5 . . . . . . . a e5 |
| e6 . . . . . . . e5+a e6 |
| e7 . . . . . . . . |

parameters: [a]  
[[[]]]

| 7, [6, 7], 1, 16 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 . . . . . . . |
| e2 . . . . . . . e1 |
| e3 . . . . . . . e3 |
| e4 . . . . . . . e3 |
| e5 . . . . . . . e3 |
| e6 . . . . . . . e1 |
| e7 . . . . . . . . |

parameters: [c]  
[[c^2 = 1]]

| 7, [6, 7], 1, 17 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 . . . . . . . |
| e2 . . . . . . . a e2+e3 |
| e3 . . . . . . . a e3 |
| e4 . . . . . . . e3 e4 |
| e5 . . . . . . . e4 e2+e3 |
| e6 . . . . . . . . |
| e7 . . . . . . . . |

parameters: [a, c]  
[[c^2 = 1]]

| 7, [6, 7], 1, 18 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 . . . . . . . |
| e2 . . . . . . . e2+e3 |
| e3 . . . . . . . e3 |
| e4 . . . . . . . e3 e4 |
| e5 . . . . . . . e4 e2+e5 |
| e6 . . . . . . . e1 |
| e7 . . . . . . . . |

parameters: [c]  
[[c^2 = 1]]

| 7, [6, 7], 1, 19 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 . . . . . . . |
| e2 . . . . . . . e3 |
| e3 . . . . . . . . |
| e4 . . . . . . . e3 |
| e5 . . . . . . . e3 e2 |
| e6 . . . . . . . e5 |
| e7 . . . . . . . . |

parameters: [c]  
[[c^2 = 1]]
### [7, [6, 7], 1, 20]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
|    |    |    | 2  |    |    |    |
|    |    |    | 2  | e2|    |    |
|    |    |    |    | e3|    |    |
|    |    |    | (a+1)|   | e4|    |
|    |    |    | (a-1)|   |    |    |
|    |    |    |    |    |    |    |

Parameters: [a]  
[[]]

### [7, [6, 7], 2, 1]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
|    |    |    | 2  |    |    |    |
|    |    |    | 2  | e2|    |    |
|    |    |    |    | e3|    |    |
|    |    |    | (b+2)|   | e4|    |
|    |    |    | (b+c)|   |    |    |
|    |    |    |    |    |    |    |

Parameters: [a, b, c]  
[0 ≤ a]

### [7, [6, 7], 1, 21]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
|    |    |    |    |    |    |    |
|    |    |    | 2  |    |    |    |
|    |    |    |    | e3|    |    |
|    |    |    |    | e3|    |    |
|    |    |    |    |    |    |    |

### [7, [6, 7], 2, 2]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
|    |    |    |    |    |    |    |
|    |    |    | 2  |    |    |    |
|    |    |    |    | e3|    |    |
|    |    |    |    | e3|    |    |
|    |    |    |    |    |    |    |

Parameters: [a, b]  
[0 ≤ a]

### [7, [6, 7], 2, 3]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
|    |    |    |    |    |    |    |
|    |    |    | 3  |    |    |    |
|    |    |    |    | e3|    |    |
|    |    |    |    |    |    |    |

Parameters: [a, b]  
[0 ≤ a]
### Table 1: Parameters in Different Settings

| Setting | Parameters |
|---------|------------|
| 7, [6, 7], 2, 4 | \(a, \varepsilon\) \([0 \leq a, \varepsilon^2 = 1]\) |
| 7, [6, 7], 3, 1 | \(a, b\) \([a, b] = 1\) |
| 7, [6, 7], 3, 2 | \(a\) \([\varepsilon] = 1\) |
| 7, [6, 7], 3, 3 | \(a\) \([\varepsilon] = 1\) |
| 7, [6, 7], 3, 4 | \(a, \varepsilon\) \([\varepsilon^2 = 1]\) |
| 7, [6, 7], 3, 5 | \(\varepsilon\) \([\varepsilon^2 = 1]\) |
| 7, [6, 7], 3, 6 | \(a\) \([\varepsilon] = 1\) |
| 7, [6, 7], 3, 7 | \(a\) \([\varepsilon] = 1\) |
| 7, [6, 7], 3, 8 | 7, [6, 7], 3, 12 |
|-----------------|-----------------|
| $e_1$ | $e_1$ |
| $e_2$ | $e_2$ |
| $e_3$ | $e_3 + e_2$ |
| $e_4$ | $3 e_3$ |
| $e_5$ | $2 e_4$ |
| $e_6$ | $e_5 + e_6$ |
| $e_7$ | . |

parameters: $[\epsilon]$

$[[\epsilon^2 = 1]]$

| 7, [6, 7], 3, 9 | 7, [6, 7], 3, 13 |
|-----------------|-----------------|
| $e_1$ | $e_1$ |
| $e_2$ | $e_1 + e_2$ |
| $e_3$ | $3 a e_3$ |
| $e_4$ | $2 a e_4$ |
| $e_5$ | $a e_5$ |
| $e_6$ | $a e_6$ |
| $e_7$ | . |

parameters: $[a]$

$[[]]$
### 7, [6, 7], 3, 16

|   | e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|---|----|----|----|----|----|----|----|
| e1 | .   | .   | .   | .   | .   | e3 | e1 + e3 |
| e2 | .   | .   | .   | .   | e1 | e1 | e1 + e2 |
| e3 | .   | .   | e3  | .   | .   | .   | e3   |
| e4 | .   | .   | e3  | (1−a)e4 | .   | .   | e4 (1−2a)e5 |
| e5 | .   | e4 | (1−2a)e5 | .   | a | e6 |   |
| e6 | .   | .   | .   | .   | .   | .   | e7   |
| e7 | .   | .   | .   | .   | .   | .   | .    |

Parameters: [a]  
[[3, 18]]

### 7, [6, 7], 3, 17

|   | e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|---|----|----|----|----|----|----|----|
| e1 | .   | .   | .   | .   | e3 | .   | .   |
| e2 | .   | .   | .   | .   | e1 | .   | e1   |
| e3 | .   | .   | .   | .   | .   | .   | e3   |
| e4 | .   | .   | e3  | .   | e4−e4 | .   | .   |
| e5 | .   | e4−2e5 | .   | .   | e6 | e7 |   |
| e6 | .   | .   | .   | .   | .   | .   | e7   |

Parameters: [a, b, c]  
[[−1 ≤ a, a ≤ 1, −1 ≤ b, c ≤ 1, b + c ≠ 1]]

### 7, [6, 7], 3, 18

|   | e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|---|----|----|----|----|----|----|----|
| e1 | .   | .   | .   | .   | e3 | .   | e1 + e3 |
| e2 | .   | .   | .   | .   | e1 | e1 | e1 + e2 |
| e3 | .   | .   | e3  | .   | .   | .   | e3   |
| e4 | .   | .   | e3  | .   | .   | .   | e3   |
| e5 | .   | e4 | .   | .   | .   | .   | e7   |
| e6 | .   | .   | .   | .   | .   | .   | e7   |

Parameters: [a, b]  
[[−1 ≤ a, a ≤ 1]]

### 7, [6, 7], 3, 19

|   | e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|---|----|----|----|----|----|----|----|
| e1 | .   | .   | .   | .   | 3e1 + e3 | .   | .   |
| e2 | .   | .   | .   | e1+3e2 | .   | .   | .   |
| e3 | .   | .   | 3e3  | .   | .   | .   | .   |
| e4 | .   | .   | e3  | 2e4 | .   | .   | .   |
| e5 | .   | e4 | e5  | .   | e5 | e5 | .   |
| e6 | .   | .   | e5 + e6 | .   | .   | .   | e7   |

Parameters: [a]  
[[6, 7], 3, 16]

### 7, [6, 7], 3, 20

|   | e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|---|----|----|----|----|----|----|----|
| e1 | .   | .   | .   | .   | .   | e3 | e1 + e3 |
| e2 | .   | .   | .   | .   | e1 | e1 | e1 + e2 |
| e3 | .   | .   | e3  | .   | .   | .   | e3   |
| e4 | .   | .   | e3  | e4 | .   | .   | e4 e2 + e5 |
| e5 | .   | e4 | e5  | .   | e5 | e5 | .   |
| e6 | .   | .   | .   | .   | .   | .   | e7   |

Parameters: [c]  
[[r^2 = 1]]

### 7, [6, 8], 1, 1

|   | e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|---|----|----|----|----|----|----|----|
| e1 | .   | .   | .   | .   | .   | (a+1)e1 | e1   |
| e2 | .   | .   | .   | (b+c)e2 | .   | .   | e2   |
| e3 | .   | e1 | .   | .   | .   | a | e3   |
| e4 | .   | .   | .   | a | e4 | .   | e4   |
| e5 | .   | e2 | b | e5 | .   | c | e6   |
| e6 | .   | .   | .   | .   | .   | .   | e7   |

Parameters: [a, b, c]  
[[−1 ≤ a, a ≤ 1, −1 ≤ c, b ≤ 1, b^2 + c^2 ≠ 0]]

### 7, [6, 8], 1, 2

|   | e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|---|----|----|----|----|----|----|----|
| e1 | .   | .   | .   | .   | .   | (a+1)e1 | e1   |
| e2 | .   | .   | .   | (a+b+1)e2 | .   | .   | e2   |
| e3 | .   | e1 | .   | .   | .   | e3 | e3   |
| e4 | .   | .   | .   | a | e4 | .   | e4   |
| e5 | .   | e2 | b | e5 | .   | e1+(a+1)e6 | e5   |
| e6 | .   | .   | .   | .   | .   | .   | e7   |

Parameters: [a, b]  
[[−1 ≤ a, a ≤ 1]]

### 7, [6, 8], 1, 3

|   | e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|---|----|----|----|----|----|----|----|
| e1 | .   | .   | .   | .   | .   | .   | e1   |
| e2 | .   | .   | .   | .   | .   | .   | e2   |
| e3 | .   | e1 | .   | .   | .   | e3 | e3   |
| e4 | .   | .   | .   | .   | .   | e4 | e4   |
| e5 | .   | e2 | e5 | .   | e6 | .   | e6   |
| e6 | .   | .   | .   | .   | .   | .   | e7   |

Parameters: [a]  
[[6, 7], 3, 20]
### [7, [6, 8], 1, 4]

|   | e₁  | e₂  | e₃  | e₄  | e₅  | e₆  | e₇  |
|---|-----|-----|-----|-----|-----|-----|-----|
| e₁ | .    | .    | .    | a   | e₁  |     |     |
| e₂ | .    | .    | .    | (a-1) | e₂  |     |     |
| e₃ | .    | e₁  | .    | e₃  |     |     |     |
| e₄ | .    | .    | e₂+(a-1) | e₄  |     |     |     |
| e₅ | .    | e₂  | - e₅ |     | e₁+a e₆ |     |     |
| e₆ | .    | e₁+a e₆ |     |     |     |     |     |
| e₇ |     |     |     |     |     |     |     |

parameters: [a]

### [7, [6, 8], 3, 1]

|   | e₁  | e₂  | e₃  | e₄  | e₅  | e₆  | e₇  |
|---|-----|-----|-----|-----|-----|-----|-----|
| e₁ | .    | .    | .    | (a+1) | e₁  |     |     |
| e₂ | .    | .    | .    | 2 b e₂  |     |     |     |
| e₃ | .    | e₁  | .    | e₃  |     |     |     |
| e₄ | .    | .    | e₂+2 b e₄ |     |     |     |     |
| e₅ | .    | b e₅  | - e₆  |     | e₁+a e₄ |     |     |
| e₆ | .    | e₅+b e₆ |     |     |     |     |     |
| e₇ |     |     |     |     |     |     |     |

parameters: [a, b]

### [7, [6, 8], 1, 5]

|   | e₁  | e₂  | e₃  | e₄  | e₅  | e₆  | e₇  |
|---|-----|-----|-----|-----|-----|-----|-----|
| e₁ | .    | .    | .    | e₁  |     |     |     |
| e₂ | .    | .    | .    | e₂  |     |     |     |
| e₃ | .    | e₁  | .    | e₃  |     |     |     |
| e₄ | .    | .    | e₂+e₄ |     | e₁+e₆ |     |     |
| e₅ | .    | e₂  | .    |     |     |     |     |
| e₆ | .    | e₁+e₆ |     |     |     |     |     |
| e₇ |     |     |     |     |     |     |     |

parameters: [a, b, c]

### [7, [6, 8], 2, 1]

|   | e₁  | e₂  | e₃  | e₄  | e₅  | e₆  | e₇  |
|---|-----|-----|-----|-----|-----|-----|-----|
| e₁ | .    | .    | .    | (a+b) e₁ |     |     |     |
| e₂ | .    | .    | .    | 2 c e₂  |     |     |     |
| e₃ | .    | e₁  | .    | a e₃  |     |     |     |
| e₄ | .    | .    | b e₄  |     |     |     |     |
| e₅ | .    | e₂  | c e₅  | - e₆  |     |     |     |
| e₆ | .    | e₅+c e₆ |     |     |     |     |     |
| e₇ |     |     |     |     |     |     |     |

parameters: [a, b, c]

### [7, [6, 8], 2, 2]

|   | e₁  | e₂  | e₃  | e₄  | e₅  | e₆  | e₇  |
|---|-----|-----|-----|-----|-----|-----|-----|
| e₁ | .    | .    | .    | (a+2 b) e₁ |     |     |     |
| e₂ | .    | .    | .    | 2 b e₂  |     |     |     |
| e₃ | .    | e₁  | .    | a e₃  |     |     |     |
| e₄ | .    | .    | e₂+2 b e₄ |     |     |     |     |
| e₅ | .    | e₂  | b e₅  | - e₆  |     |     |     |
| e₆ | .    | e₅+b e₆ |     |     |     |     |     |
| e₇ |     |     |     |     |     |     |     |

parameters: [a, b]

### [7, [6, 8], 3, 3]

|   | e₁  | e₂  | e₃  | e₄  | e₅  | e₆  | e₇  |
|---|-----|-----|-----|-----|-----|-----|-----|
| e₁ | .    | .    | .    | 2 e₁  |     |     |     |
| e₂ | .    | .    | .    | 2 e₂  |     |     |     |
| e₃ | .    | e₁  | .    |     |     |     |     |
| e₄ | .    | .    | e₂+2 e₄ |     |     |     |     |
| e₅ | .    | e₂  | e₅  |     | e₅+e₆ |     |     |
| e₆ | .    | e₅+e₆ |     |     |     |     |     |
| e₇ |     |     |     |     |     |     |     |

### [7, [6, 8], 3, 4]

|   | e₁  | e₂  | e₃  | e₄  | e₅  | e₆  | e₇  |
|---|-----|-----|-----|-----|-----|-----|-----|
| e₁ | .    | .    | .    | (a+1) | e₁  |     |     |
| e₂ | .    | .    | .    | (2 a+2) | e₂  |     |     |
| e₃ | .    | e₁  | .    | e₃  |     |     |     |
| e₄ | .    | .    | e₂+e₄ |     | e₁+(a+1) e₅ |     |     |
| e₅ | .    | e₂  | e₁+(a+1) | e₅  |     |     |     |
| e₆ | .    | e₅+(a+1) | e₆  |     |     |     |     |
| e₇ |     |     |     |     |     |     |     |

parameters: [a]

### [7, [6, 8], 3, 4]
### [7, [6, 9], 1, 3]

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|
| $-2e_2$ | $2e_1$ | $-e_1$ | $e_3$ | $e_3$ | $-e_6$ | $e_1$ |

### [7, [6, 9], 1, 4]

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|
| $-2e_2$ | $2e_1$ | $e_3$ | $-e_1$ | $e_3$ | $e_1+2e_6$ | $e_1$ |

### [7, [6, 9], 2, 1]

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|
| $-2e_2$ | $2e_1$ | $-e_1$ | $e_3$ | $e_3+e_5+a_6$ | $-e_4-a_5+e_6$ | $e_1$ |

parameters: $[a, b, c, e]$

[[c^2 = 1]]

### [7, [6, 9], 3, 1]

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|
| $2a_e_1-2b_e_2$ | $2b_e_1+2a_e_2$ | $(a+c)e_3+(-b-c)e_4$ | $-e_1$ | $(b+c)e_3+(a+c)e_4$ | $-e_1$ | $(-b+c)e_5+(a-c)e_6$ |

parameters: $[a, b, c, e]$

[[0 ≤ c^2 = 1]]

### [7, [6, 9], 3, 2]

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|
| $2a_e_1+2b_e_2$ | $-2b_e_1+2a_e_2$ | $2e_1$ | $-e_1$ | $-2e_5+2a_e_6$ | $e_1$ | $e_5$ |

parameters: $[a]$

([],)

### [7, [6, 10], 1, 1]

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|
| $e_1$ | $a_e_2$ | $b_e_3$ | $(c+d)e_4$ | $e_4$ | $c_e_5$ | $d_e_6$ |

parameters: $[a, b, c, d]$

[[−1 ≤ b, b ≤ a, a ≤ 1, a ≠ 0, b ≠ 0, d ≤ c, c^2 + d^2 ≠ 0]]
### [7, [6, 10], 1, 2]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| .. | .. | .. | .. | .. | .. | e₁ |
| .. | .. | .. | a | e₂ | .. | .. |
| .. | .. | (b+c) | e₃ | e₄ | .. | .. |
| .. | .. | .. | .. | e₅ | .. | .. |
| .. | e₄ | .. | .. | .. | .. | .. |
| .. | .. | .. | .. | .. | .. | e₇ |

parameters: [a, b, c]

\([-1 \leq a, a \leq 1, a \neq 0, c \leq b]\)

### [7, [6, 10], 1, 3]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| .. | .. | .. | .. | .. | .. | e₁ |
| .. | .. | .. | a | e₂ | .. | .. |
| .. | .. | (b+c) | e₃ | e₄ | .. | .. |
| .. | .. | .. | .. | e₅ | .. | .. |
| .. | e₄ | .. | .. | .. | .. | .. |
| .. | .. | .. | .. | .. | .. | e₇ |

parameters: [a, b, c]

\([-1 \leq a, a \leq 1, a \neq 0]\)

### [7, [6, 10], 1, 4]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| .. | .. | .. | .. | .. | .. | e₁ |
| .. | .. | .. | a | e₂ | .. | .. |
| .. | .. | b | e₃ | e₄ | .. | .. |
| .. | .. | .. | .. | e₅ | .. | .. |
| .. | e₄ | .. | .. | .. | .. | .. |
| .. | .. | .. | .. | .. | .. | e₇ |

parameters: [a, b]

\([-1 \leq a, a \leq 1, a \neq 0]\)

### [7, [6, 10], 1, 5]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| .. | .. | .. | .. | .. | .. | e₁ |
| .. | .. | .. | a | e₂ | .. | .. |
| .. | .. | .. | .. | e₃ | .. | .. |
| .. | .. | .. | .. | e₄ | .. | .. |
| .. | .. | e₅ | .. | .. | .. | .. |
| .. | .. | .. | .. | .. | .. | e₇ |

parameters: [a, b]

\([-1 \leq a, a \leq 1, a \neq 0]\)

### [7, [6, 10], 1, 6]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| .. | .. | .. | .. | .. | .. | e₁ |
| .. | .. | .. | a | e₂ | .. | .. |
| .. | .. | .. | .. | e₃ | .. | .. |
| .. | .. | .. | .. | e₄ | .. | .. |
| .. | .. | e₅ | .. | .. | .. | .. |
| .. | .. | .. | .. | .. | .. | e₇ |

parameters: [a, b]

\([-1 \leq a, a \leq 1, a \neq 0]\)

### [7, [6, 10], 1, 7]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| .. | .. | .. | .. | .. | .. | e₁ |
| .. | .. | .. | a | e₂ | .. | .. |
| .. | .. | .. | .. | e₃ | .. | .. |
| .. | .. | .. | .. | e₄ | .. | .. |
| .. | .. | e₅ | .. | .. | .. | .. |
| .. | .. | .. | .. | .. | .. | e₇ |

parameters: [a]

\([-1 \leq a, a \leq 1, a \neq 0]\)
| $[7, [6, 10], 1, 8] $ | $[7, [6, 10], 3, 1] $ |
|---|---|
| $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ | $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ |
| $e_1$ | $e_1$ |
| $e_2$ | $e_2$ |
| $e_3$ | $e_3$ |
| $e_4$ | $e_4$ |
| $e_5$ | $e_5$ |
| $e_6$ | $e_6$ |
| $e_7$ | $e_7$ |
| parameters: $[a]$ | parameters: $[a,b,c]$ |
| $[-1 \leq a, a \leq 1]$ | $[-1 \leq b, b \leq a, a \neq 0, b \neq 0]$ |

| $[7, [6, 10], 2, 1] $ | $[7, [6, 10], 3, 2] $ |
|---|---|
| $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ | $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ |
| $e_1$ | $e_1$ |
| $e_2$ | $e_2$ |
| $e_3$ | $e_3$ |
| $e_4$ | $e_4$ |
| $e_5$ | $e_5$ |
| $e_6$ | $e_6$ |
| $e_7$ | $e_7$ |
| parameters: $[a,b,c,d]$ | parameters: $[a,b]$ |
| $[c \leq b, b \leq a, a \neq 0, b \neq 0, c \neq 0, 0 \leq d]$ | $[-1 \leq a, a \leq 1, a \neq 0]$ |

| $[7, [6, 10], 2, 2] $ | $[7, [6, 10], 3, 3] $ |
|---|---|
| $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ | $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ |
| $e_1$ | $e_1$ |
| $e_2$ | $e_2$ |
| $e_3$ | $e_3$ |
| $e_4$ | $e_4$ |
| $e_5$ | $e_5$ |
| $e_6$ | $e_6$ |
| $e_7$ | $e_7$ |
| parameters: $[a,b,c]$ | parameters: $[a,b]$ |
| $[b \leq a, a \neq 0, b \neq 0, 0 \leq c]$ | $[-1 \leq a, a \leq 1, a \neq 0]$ |
\[ 7, \{6, 10\}, 3, 4 \]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | .   | e₁ |    |
| e₂ | .   | .   | .   | .   | a e₂ |    |
| e₃ | .   | .   | .   | .   | e₄ |    |
| e₄ | .   | .   | .   | .   | a e₃ |    |
| e₅ | .   | .   | .   | .   | 2 a e₄ |    |
| e₆ | .   | .   | e₄ | e₅+a e₅ | e₆ |    |
| e₇ | .   | .   | .   | .   | .   |    |

parameters: \([a]\)  
\([-1 \leq a, a \leq 1, a \neq 0]\)

\[ 7, \{6, 10\}, 4, 2 \]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | .   | .   | a e₁-e₂ |
| e₂ | .   | .   | .   | .   | .   | e₁+a e₂ |
| e₃ | .   | .   | .   | .   | e₃ | b e₃ |
| e₄ | .   | .   | .   | .   | (b+c) e₄ |    |
| e₅ | .   | .   | e₄ | e₃+b e₅ | e₆ |    |
| e₆ | .   | .   | c e₆ |    |    |    |
| e₇ | .   | .   | .   | .   | .   |    |

parameters: \([a, b, c]\)  
\([0 \leq a, c \leq b]\)

\[ 7, \{6, 10\}, 3, 5 \]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | .   | .   | e₁ |
| e₂ | .   | .   | .   | .   | 2 a e₂+e₄ |    |
| e₃ | .   | .   | .   | .   | a e₃ |    |
| e₄ | .   | .   | .   | .   | 2 a e₄ |    |
| e₅ | .   | .   | e₄ | e₃+a e₅ | e₆ |    |
| e₆ | .   | e₅+a e₆ | .   | .   |    |    |
| e₇ | .   | .   | .   | .   | .   |    |

parameters: \([a]\)  
\([[]]\)

\[ 7, \{6, 10\}, 4, 3 \]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | .   | .   | a e₁-e₂ |
| e₂ | .   | .   | .   | .   | .   | a e₂ |
| e₃ | .   | .   | .   | .   | e₃ | b e₃ |
| e₄ | .   | .   | .   | .   | (b+c) e₄ |    |
| e₅ | .   | .   | e₄ | e₃+b e₅ | e₆ |    |
| e₆ | .   | .   | c e₆ |    |    |    |
| e₇ | .   | .   | .   | .   | .   |    |

parameters: \([a, b, c]\)  
\([0 \leq a]\)

\[ 7, \{6, 10\}, 4, 1 \]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | .   | a e₁-e₂ |    |
| e₂ | .   | .   | .   | .   | a e₂ |    |
| e₃ | .   | .   | .   | .   | b e₃ |    |
| e₄ | .   | .   | .   | .   | (c+d) e₄ |    |
| e₅ | .   | .   | e₄ | e₅ | c e₅ |    |
| e₆ | .   | .   | d e₆ | .   | .   |    |
| e₇ | .   | .   | .   | .   | .   |    |

parameters: \([a, b, c, d]\)  
\([0 \leq a, b \neq 0, d \leq c, c^2 + d^2 \neq 0]\)

\[ 7, \{6, 10\}, 4, 4 \]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | .   | a e₁-e₂ |    |
| e₂ | .   | .   | .   | .   | a e₂ |    |
| e₃ | .   | .   | .   | .   | b e₃+e₄ |    |
| e₄ | .   | .   | .   | .   | b e₄ |    |
| e₅ | .   | .   | e₄ | e₃+b e₅ | e₆ |    |
| e₆ | .   | .   | c e₆ |    | .   |    |
| e₇ | .   | .   | .   | .   | .   |    |

parameters: \([a, b]\)  
\([0 \leq a]\)
\[\begin{array}{cccccc}
\text{[7, [6, 10], 5, 1]} & \text{[7, [6, 10], 5, 4]} \\
\hline
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
e_1 & \ldots & \ldots & a e_1 - e_2 & \ldots & \ldots & \ldots \\
e_2 & \ldots & \ldots & e_1 + a e_2 & \ldots & \ldots & \ldots \\
e_3 & \ldots & \ldots & b e_3 & \ldots & \ldots & \ldots \\
e_4 & \ldots & \ldots & 2 e_4 & \ldots & \ldots & \ldots \\
e_5 & \ldots & \ldots & e_4 c e_5 + a e_6 & \ldots & \ldots & \ldots \\
e_6 & \ldots & \ldots & c e_5 + a e_6 & \ldots & \ldots & \ldots \\
e_7 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}\]

\text{parameters: \([a, b, c, d]\) for \([0 \leq a, b, c, d < 0]\)]

\[\begin{array}{cccccc}
\text{[7, [6, 10], 5, 2]} & \text{[7, [6, 10], 5, 5]} \\
\hline
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
e_1 & \ldots & \ldots & \ldots & a e_1 - e_2 & \ldots & \ldots \\
e_2 & \ldots & \ldots & \ldots & e_1 + a e_2 & \ldots & \ldots \\
e_3 & \ldots & \ldots & \ldots & b e_3 & \ldots & \ldots \\
e_4 & \ldots & \ldots & \ldots & 2 a e_4 & \ldots & \ldots \\
e_5 & \ldots & \ldots & \ldots & e_4 e_1 + c e_2 + a e_5 - e_6 & \ldots & \ldots \\
e_6 & \ldots & \ldots & \ldots & e_5 + a e_6 & \ldots & \ldots \\
e_7 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}\]

\text{parameters: \([a, b, c]\) for \([0 \leq a, b, c < 0]\)]

\[\begin{array}{cccccc}
\text{[7, [6, 10], 5, 3]} & \text{[7, [6, 10], 5, 6]} \\
\hline
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
e_1 & \ldots & \ldots & \ldots & a e_1 - e_2 & \ldots & \ldots \\
e_2 & \ldots & \ldots & \ldots & e_1 + a e_2 & \ldots & \ldots \\
e_3 & \ldots & \ldots & \ldots & b e_3 & \ldots & \ldots \\
e_4 & \ldots & \ldots & \ldots & 2 a e_4 & \ldots & \ldots \\
e_5 & \ldots & \ldots & \ldots & e_4 e_1 + c e_2 + a e_5 - e_6 & \ldots & \ldots \\
e_6 & \ldots & \ldots & \ldots & e_5 + a e_6 & \ldots & \ldots \\
e_7 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}\]

\text{parameters: \([a, b]\) for \([0 \leq a, b < 0]\)]

\[\begin{array}{cccccc}
\text{[7, [6, 10], 5, 6]} \\
\hline
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
e_1 & \ldots & \ldots & \ldots & a e_1 - e_2 & \ldots & \ldots \\
e_2 & \ldots & \ldots & \ldots & e_1 + a e_2 & \ldots & \ldots \\
e_3 & \ldots & \ldots & \ldots & 2 a e_4 & \ldots & \ldots \\
e_4 & \ldots & \ldots & \ldots & 2 a e_4 & \ldots & \ldots \\
e_5 & \ldots & \ldots & \ldots & e_4 e_1 + a e_5 - e_6 & \ldots & \ldots \\
e_6 & \ldots & \ldots & \ldots & e_5 + a e_6 & \ldots & \ldots \\
e_7 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}\]

\text{parameters: \([a]\) for \([0 \leq a]\)]
\[
\begin{align*}
&[7, [6, 10], 6, 1] \quad & [7, [6, 10], 6, 4] \\
\begin{array}{cccccc}
\text{\textbf{e}}_1 & \text{\textbf{e}}_2 & \text{\textbf{e}}_3 & \text{\textbf{e}}_4 & \text{\textbf{e}}_5 & \text{\textbf{e}}_6 & \text{\textbf{e}}_7 \\
\text{e}_1 & \ldots & \ldots & \ldots & \ldots & \ldots & \alpha \text{e}_1 - \text{e}_2 \\
\text{e}_2 & \ldots & \ldots & \ldots & \ldots & \ldots & \text{e}_1 + \alpha \text{e}_2 \\
\text{e}_3 & \ldots & \ldots & \ldots & \beta \text{e}_3 & & \\
\text{e}_4 & \ldots & \ldots & \ldots & \ldots & \text{e}_2 \text{e}_4 & \\
\text{e}_5 & \ldots & \ldots & \ldots & \text{e}_4 & \text{c} \text{e}_5 \\
\text{e}_6 & \ldots & \ldots & \text{e}_5 + b \text{e}_6 \\
\text{e}_7 & \ldots & \\
\end{array}
\end{align*}
\]
\begin{align*}
\text{parameters: } [\alpha, \beta, \gamma] \\
&[[0 \leq \alpha, \beta \neq 0]]
\end{align*}

\[
\begin{align*}
&[7, [6, 10], 6, 2] \quad & [7, [6, 10], 7, 1] \\
\begin{array}{cccccc}
\text{\textbf{e}}_1 & \text{\textbf{e}}_2 & \text{\textbf{e}}_3 & \text{\textbf{e}}_4 & \text{\textbf{e}}_5 & \text{\textbf{e}}_6 & \text{\textbf{e}}_7 \\
\text{e}_1 & \ldots & \ldots & \ldots & \ldots & \ldots & \alpha \text{e}_1 \\
\text{e}_2 & \ldots & \ldots & \ldots & \ldots & \ldots & \text{e}_1 + \alpha \text{e}_2 \\
\text{e}_3 & \ldots & \ldots & \ldots & \beta \text{e}_3 & & \\
\text{e}_4 & \ldots & \ldots & \ldots & 2 \beta \text{e}_4 & \\
\text{e}_5 & \ldots & \ldots & \ldots & \text{e}_4 & \text{c} \text{e}_5 \\
\text{e}_6 & \ldots & \ldots & \text{e}_5 + b \text{e}_6 \\
\text{e}_7 & \ldots & \\
\end{array}
\end{align*}
\]
\begin{align*}
\text{parameters: } [\alpha, \beta] \\
&[[0 \leq \alpha]]
\end{align*}

\[
\begin{align*}
&[7, [6, 10], 6, 3] \quad & [7, [6, 10], 7, 2] \\
\begin{array}{cccccc}
\text{\textbf{e}}_1 & \text{\textbf{e}}_2 & \text{\textbf{e}}_3 & \text{\textbf{e}}_4 & \text{\textbf{e}}_5 & \text{\textbf{e}}_6 & \text{\textbf{e}}_7 \\
\text{e}_1 & \ldots & \ldots & \ldots & \ldots & \ldots & \alpha \text{e}_1 - \text{e}_2 \\
\text{e}_2 & \ldots & \ldots & \ldots & \ldots & \ldots & \text{e}_1 + \alpha \text{e}_2 \\
\text{e}_3 & \ldots & \ldots & \ldots & \beta \text{e}_3 & & \\
\text{e}_4 & \ldots & \ldots & \ldots & \text{e}_2 \text{e}_4 & \\
\text{e}_5 & \ldots & \ldots & \ldots & \text{e}_4 & \text{c} \text{e}_5 \\
\text{e}_6 & \ldots & \ldots & \text{e}_5 + b \text{e}_6 \\
\text{e}_7 & \ldots & \\
\end{array}
\end{align*}
\]
\begin{align*}
\text{parameters: } [\alpha, \beta] \\
&[[0 \leq \alpha]]
\end{align*}
### [7, [6, 10], 7, 3]

|     | e_1 | e_2 | e_3 | e_4 | e_5 | e_6 | e_7 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| e_1 |     |     |     |     |     | .   | .   |
| e_2 |     |     |     |     | .   | .   | e_1 |
| e_3 |     |     |     |     |     |     | (a+1) e_3 + e_4 |
| e_4 |     |     |     |     |     |     | (a+1) e_4 |
| e_5 |     |     | e_4 | e_5 |     |     |     |
| e_6 |     |     |     | a e_6 |     |     |     |
| e_7 |     |     |     |     |     |     | .   |

Parameters: [a]
([-1 ≤ a, a ≤ 1])

### [7, [6, 10], 7, 4]

|     | e_1 | e_2 | e_3 | e_4 | e_5 | e_6 | e_7 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| e_1 |     |     |     |     |     | .   | .   |
| e_2 |     |     |     |     | .   | .   | e_1 |
| e_3 |     |     |     |     |     |     |     |
| e_4 |     |     |     |     |     | .   | e_4 |
| e_5 |     |     | e_4 | a e_5 |     |     |     |
| e_6 |     |     |     | b e_6 |     |     |     |
| e_7 |     |     |     |     |     |     | .   |

Parameters: [a, b]
([-b ≤ a])

### [7, [6, 10], 7, 5]

|     | e_1 | e_2 | e_3 | e_4 | e_5 | e_6 | e_7 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| e_1 |     |     |     |     |     | .   | e_1 |
| e_2 |     |     |     |     | .   | .   | e_2 |
| e_3 |     |     |     |     | .   | a e_3 | e_2 |
| e_4 |     |     |     |     |     | (a+b) e_4 | e_3 + e_4 |
| e_5 |     | e_4 | e_3+a e_5 |     |     |     |     |
| e_6 |     |     |     | b e_6 |     |     |     |
| e_7 |     |     |     |     |     |     | .   |

Parameters: [a, b]
([-b ≤ a])

### [7, [6, 10], 7, 6]

|     | e_1 | e_2 | e_3 | e_4 | e_5 | e_6 | e_7 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| e_1 |     |     |     |     |     | .   | .   |
| e_2 |     |     |     |     | .   | .   | e_1 |
| e_3 |     |     |     |     |     |     |     |
| e_4 |     |     |     |     |     | .   | (a+1) e_4 |
| e_5 |     | .   | e_4 | e_3 + e_5 |     |     |     |
| e_6 |     | .   | a e_6 |     |     |     |     |
| e_7 |     |     |     |     |     |     | .   |

Parameters: [a]
([-])

### [7, [6, 10], 7, 7]

|     | e_1 | e_2 | e_3 | e_4 | e_5 | e_6 | e_7 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| e_1 |     |     |     |     |     | .   | .   |
| e_2 |     |     |     |     | .   | .   | e_1 |
| e_3 |     |     |     |     |     |     |     |
| e_4 |     |     |     |     | .   | e_4 | e_3 + e_4 |
| e_5 |     | .   | e_4 | e_3+a e_5 |     |     |     |
| e_6 |     | .   | a e_6 |     |     |     |     |
| e_7 |     |     |     |     |     |     | .   |

Parameters: [a, b]
([-])

### [7, [6, 10], 7, 8]

|     | e_1 | e_2 | e_3 | e_4 | e_5 | e_6 | e_7 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| e_1 |     |     |     |     |     | .   | e_1 |
| e_2 |     |     |     |     | .   | .   | e_1 |
| e_3 |     |     |     |     | .   | a e_3 + e_4 | e_2 |
| e_4 |     |     |     |     | .   | a e_4 | e_3 + e_4 |
| e_5 |     | e_4 | e_3+a e_5 |     |     |     |     |
| e_6 |     |     |     | b e_6 |     |     |     |
| e_7 |     |     |     |     |     |     | .   |

Parameters: [a]
([-])

### [7, [6, 10], 7, 9]

|     | e_1 | e_2 | e_3 | e_4 | e_5 | e_6 | e_7 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| e_1 |     |     |     |     |     | .   | .   |
| e_2 |     |     |     |     | .   | .   | e_1 |
| e_3 |     |     |     |     | .   | e_3 + e_4 | e_2 |
| e_4 |     |     |     |     | .   | e_4 | e_3 + e_4 |
| e_5 |     | .   | e_4 | e_3 + e_5 |     |     |     |
| e_6 |     | .   |     |     |     |     |     |
| e_7 |     |     |     |     |     |     | .   |

Parameters: [a, b]
([-])
### [7, [6, 10], 7, 10]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| ...| ...| ...| ...| (a+1)e1| ...| ...|
| ...| ...| e1| ...| (a+1)e2| ...| ...|
| ...| ...| ...| ...| e3| ...| ...|
| ...| ...| ...| e4| ...| ...| ...|
| ...| e4| ...| e3| e5| ...| ...|
| ...| ...| e6| ...| ...| ...| ...|
| ...| ...| ...| ...| ...| ...| ...|

Parameters: [a]

### [7, [6, 10], 7, 11]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| ...| ...| ...| ...| ...| ...| ...|
| ...| ...| ...| ...| e1| e4| ...|
| ...| ...| ...| ...| e2| ...| ...|
| ...| ...| ...| ...| ...| e3| ...|
| ...| ...| ...| e4| ...| ...| ...|
| ...| e4| ...| e3| ...| ...| ...|
| ...| ...| e6| ...| ...| ...| ...|
| ...| ...| ...| ...| ...| ...| ...|

### [7, [6, 10], 7, 12]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| ...| ...| ...| ...| ...| ...| ...|
| ...| ...| ...| ...| e1| e4| ...|
| ...| ...| ...| ...| e2| ...| ...|
| ...| ...| ...| ...| ...| e3| ...|
| ...| ...| ...| e4| ...| ...| ...|
| ...| e4| ...| e3| e5| ...| ...|
| ...| ...| e6| ...| ...| ...| ...|
| ...| ...| ...| ...| ...| ...| ...|

### [7, [6, 10], 7, 13]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| ...| ...| ...| ...| ...| ...| ...|
| ...| ...| ...| ...| e1| e4| ...|
| ...| ...| ...| ...| e2| ...| ...|
| ...| ...| ...| ...| ...| e3| ...|
| ...| ...| ...| e4| ...| ...| ...|
| ...| e4| ...| e3| ...| ...| ...|
| ...| ...| e6| ...| ...| ...| ...|
| ...| ...| ...| ...| ...| ...| ...|

Parameters: [a, b]

### [7, [6, 10], 7, 14]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| ...| ...| ...| ...| ...| ...| ...|
| ...| ...| ...| ...| ...| ...| e1|
| ...| ...| ...| ...| ...| e2| ...|
| ...| ...| ...| ...| ...| ...| e3|
| ...| ...| ...| ...| ...| ...| e4|
| ...| ...| ...| ...| ...| ...| e5|
| ...| ...| ...| ...| ...| ...| e6|
| ...| ...| ...| ...| ...| ...| ...|

Parameters: [a]

### [7, [6, 10], 7, 15]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| ...| ...| ...| ...| ...| ...| ...|
| ...| ...| ...| ...| ...| ...| ...|
| ...| ...| ...| ...| e1| e4| ...|
| ...| ...| ...| ...| e2| ...| ...|
| ...| ...| ...| ...| ...| e3| ...|
| ...| ...| ...| e4| ...| ...| ...|
| ...| e4| ...| e3| e5| ...| ...|
| ...| ...| e6| ...| ...| ...| ...|
| ...| ...| ...| ...| ...| ...| ...|

Parameters: [a]

### [7, [6, 10], 7, 16]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| ...| ...| ...| ...| ...| ...| ...|
| ...| ...| ...| ...| ...| ...| e1|
| ...| ...| ...| ...| ...| e2| ...|
| ...| ...| ...| ...| ...| ...| e3|
| ...| ...| ...| ...| ...| ...| e4|
| ...| ...| ...| ...| ...| ...| e5|
| ...| ...| ...| ...| ...| ...| e6|
| ...| ...| ...| ...| ...| ...| ...|

Parameters: [a]

### [7, [6, 10], 7, 17]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| ...| ...| ...| ...| ...| ...| ...|
| ...| ...| ...| ...| ...| ...| e1|
| ...| ...| ...| ...| ...| e2| ...|
| ...| ...| ...| ...| ...| ...| e3|
| ...| ...| ...| ...| ...| ...| e4|
| ...| ...| ...| ...| ...| ...| e5|
| ...| ...| ...| ...| ...| ...| e6|
| ...| ...| ...| ...| ...| ...| ...|

Parameters: [a]
### $[7, [6, 10], 7, 18]$ 

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-----------------------------|
| e₁ | . | . | . | . | . | . | . |
| e₂ | . | . | . | . | . | . | e₁ |
| e₃ | . | . | . | . | . | . | e₃ + e₄ |
| e₄ | . | . | . | . | . | . | e₄ + e₅ |
| e₅ | . | e₄ | e₂ | . | . | . | e₆ + e₇ |
| e₆ | . | . | e₃ + e₆ | . | . | . | . |
| e₇ | . | . | . | . | . | . | . |

### $[7, [6, 10], 7, 19]$ 

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-----------------------------|
| e₁ | . | . | . | . | . | . | . |
| e₂ | . | . | . | . | . | . | e₁ |
| e₃ | . | . | . | e₃ + e₄ | . | . | e₄ |
| e₄ | . | . | . | . | . | . | e₅ + e₆ |
| e₅ | . | e₄ | e₂ | . | . | . | e₇ |
| e₆ | . | . | e₃ + e₆ | . | . | . | . |
| e₇ | . | . | . | . | . | . | . |

### $[7, [6, 10], 7, 20]$ 

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-----------------------------|
| e₁ | . | . | . | . | . | . | . |
| e₂ | . | . | . | . | . | . | e₁ + e₄ |
| e₃ | . | . | . | . | . | . | e₂ + e₅ |
| e₄ | . | . | . | . | . | . | e₃ |
| e₅ | . | e₄ | e₂ + e₅ | . | . | . | e₇ |
| e₆ | . | . | e₃ + e₆ | . | . | . | . |
| e₇ | . | . | . | . | . | . | . |

### $[7, [6, 10], 7, 21]$ 

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-----------------------------|
| e₁ | . | . | . | . | . | . | . |
| e₂ | . | . | . | . | . | . | e₁ + e₄ |
| e₃ | . | . | . | . | . | . | e₂ + e₅ |
| e₄ | . | . | . | . | . | . | e₃ |
| e₅ | . | e₄ | e₂ + e₅ | . | . | . | e₇ |
| e₆ | . | . | e₃ + e₆ | . | . | . | . |
| e₇ | . | . | . | . | . | . | . |

### $[7, [6, 10], 8, 1]$ 

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-----------------------------|
| e₁ | . | . | . | . | . | . | . |
| e₂ | . | . | . | . | . | . | a e₁ |
| e₃ | . | . | . | . | . | . | e₁ + a e₂ |
| e₄ | . | . | . | . | . | . | b e₃ |
| e₅ | . | . | . | . | . | . | 2 c e₄ |
| e₆ | . | . | . | . | . | . | e₄ c e₅ + e₆ |
| e₇ | . | . | . | . | . | . | e₅ + c e₆ |

parameters: $[a, b, c]$

$[[b \neq 0, 0 \leq c]]$

### $[7, [6, 10], 8, 2]$ 

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-----------------------------|
| e₁ | . | . | . | . | . | . | . |
| e₂ | . | . | . | . | . | . | a e₁ |
| e₃ | . | . | . | . | . | . | e₁ + a e₂ |
| e₄ | . | . | . | . | . | . | 2 b e₃ |
| e₅ | . | . | . | . | . | . | b e₄ |
| e₆ | . | . | . | . | . | . | e₄ b e₅ + e₆ |
| e₇ | . | . | . | . | . | . | e₅ + b e₆ |

parameters: $[a, b]$

$[[0 \leq b]]$

### $[7, [6, 10], 8, 3]$ 

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-----------------------------|
| e₁ | . | . | . | . | . | . | . |
| e₂ | . | . | . | . | . | . | 2 a e₁ + e₄ |
| e₃ | . | . | . | . | . | . | e₁ + 2 a e₂ |
| e₄ | . | . | . | . | . | . | b e₃ |
| e₅ | . | . | . | . | . | . | 2 a e₄ |
| e₆ | . | . | . | . | . | . | e₄ a e₅ + e₆ |
| e₇ | . | . | . | . | . | . | e₅ + a e₆ |

parameters: $[a, b]$

$[[0 \leq a, b \neq 0]]$
| 7, [6, 10], 9, 1 | 7, [6, 10], 9, 5 |
|------------------|------------------|
| e1 e2 e3 e4 e5 e6 e7 | e1 e2 e3 e4 e5 e6 e7 |
| e1 e2 e3 e4 e5 e6 e7 | e1 e2 e3 e4 e5 e6 e7 |
| parameters: [a, b] | parameters: [a] |

| 7, [6, 10], 9, 2 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 e2 e3 e4 e5 e6 e7 |
| parameters: [a] |

| 7, [6, 10], 9, 3 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 e2 e3 e4 e5 e6 e7 |
| parameters: [a] |

| 7, [6, 10], 9, 4 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 e2 e3 e4 e5 e6 e7 |
| parameters: [a] |

| 7, [6, 10], 9, 6 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 e2 e3 e4 e5 e6 e7 |

| 7, [6, 10], 9, 7 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 e2 e3 e4 e5 e6 e7 |

| 7, [6, 10], 9, 8 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 e2 e3 e4 e5 e6 e7 |

| 7, [6, 10], 9, 5 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 e2 e3 e4 e5 e6 e7 |
| parameters: [a] |

| 7, [6, 10], 9, 6 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 e2 e3 e4 e5 e6 e7 |

| 7, [6, 10], 9, 7 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 e2 e3 e4 e5 e6 e7 |

| 7, [6, 10], 9, 8 |
|------------------|
| e1 e2 e3 e4 e5 e6 e7 |
| e1 e2 e3 e4 e5 e6 e7 |
### [7, [6, 10], 9, 9]

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-------------------------------|
| e₁    . . . . . . . . a e₁    |
| e₂    . . . . . e₁+a e₂      |
| e₃    . . . . . e₃            |
| e₄    . . . . 2 a e₄         |
| e₅    . . e₄ e₂+a e₅        |
| e₆    . . e₅+a e₆            |
| e₇    . . . . . . . . .      |

parameters: [a]

### [7, [6, 10], 9, 13]

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-------------------------------|
| e₁    . . . . . . . . . e₁    |
| e₂    . . . . . e₁+e₂         |
| e₃    . . . . . e₃            |
| e₄    . . . . 2 e₃+e₄         |
| e₅    . . e₄ e₁+e₅           |
| e₆    . . e₅+e₆              |
| e₇    . . . . . . . . .      |

### [7, [6, 10], 9, 10]

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-------------------------------|
| e₁    . . . . . . . . e₁      |
| e₂    . . . . . e₁+e₂         |
| e₃    . . . . . e₃            |
| e₄    . . . . 2 e₄            |
| e₅    . . e₄ e₂+e₅           |
| e₆    . . e₅+e₆              |
| e₇    . . . . . . . . .      |

### [7, [6, 10], 9, 11]

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-------------------------------|
| e₁    . . . . . . . . e₄      |
| e₂    . . . . . e₃            |
| e₃    . . . . . e₃            |
| e₄    . . . . .              |
| e₅    . . e₄ e₂              |
| e₆    . . e₅                 |
| e₇    . . . . . . . . .      |

### [7, [6, 10], 9, 14]

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-------------------------------|
| e₁    . . . . . . . . e₁      |
| e₂    . . . . . e₁+e₂         |
| e₃    . . . . . e₃            |
| e₄    . . . . 2 e₄            |
| e₅    . . e₄ e₁+e₅           |
| e₆    . . e₅+e₆              |
| e₇    . . . . . . . . .      |

### [7, [6, 10], 9, 15]

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-------------------------------|
| e₁    . . . . . . . . e₁      |
| e₂    . . . . . e₁+e₂         |
| e₃    . . . . . e₃            |
| e₄    . . . . (b+1) e₄       |
| e₅    . . e₄ e₁+e₅           |
| e₆    . . e₅+e₆              |
| e₇    . . . . . . . . .      |

### [7, [6, 10], 10, 1]

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-------------------------------|
| e₁    . . . . . . . . a e₁    |
| e₂    . . . . . e₁+e₂         |
| e₃    . . . . . e₃            |
| e₄    . . . . 2 a e₄         |
| e₅    . . e₄ e₁+a e₅        |
| e₆    . . e₅+a e₆            |
| e₇    . . . . . . . . .      |

parameters: [a]

### [7, [6, 10], 10, 1]

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-------------------------------|
| e₁    . . . . . . . . a e₁    |
| e₂    . . . . . e₁+e₂         |
| e₃    . . . . . e₂+e₃         |
| e₄    . . . . (b+1) e₄       |
| e₅    . . e₄ e₅              |
| e₆    . . b e₆              |
| e₇    . . . . . . . . .      |

parameters: [a, b]

([-1 ≤ b, b ≤ 1])
### [7, [6, 10], 10, 2]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | .   | .   | (a+1) | e₁+e₄ |
| e₂ | .   | .   | .   | e₁+(a+1) | e₂ |
| e₃ | .   | .   | e₂+(a+1) | e₃ |
| e₄ | .   | .   | (a+1) | e₄ |
| e₅ | e₄ | e₅ | a  | e₆ |
| e₆ | .   | e₆ | .   | .   |
| e₇ | .   | .   | .   | .   |

Parameters: [a]

$([-1 \leq a, a \leq 1])$

### [7, [6, 10], 11, 1]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | .   | .   | a  | e₁ |
| e₂ | .   | .   | .   | e₁+a  | e₂ |
| e₃ | .   | .   | e₂+a | e₃ |
| e₄ | .   | .   | 2a  | e₄ |
| e₅ | e₄ | b e₅−e₆ | e₅ |
| e₆ | .   | e₅+b | e₆ |
| e₇ | .   | .   | .   | .   |

Parameters: [a, b]

$([0 \leq b])$

### [7, [6, 10], 10, 3]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | .   | .   | .   | e₁ |
| e₂ | .   | .   | .   | e₁+e₂ | e₂ |
| e₃ | .   | .   | e₂+e₃ | e₃ |
| e₄ | .   | .   | (a+1) | e₄ |
| e₅ | e₄ | e₃+e₅ | a  | e₆ |
| e₆ | .   | .   | a  | e₆ |
| e₇ | .   | .   | .   | .   |

Parameters: [a]

$([0 \leq a])$

### [7, [6, 10], 11, 2]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | 2a e₁+e₄ | e₁ |
| e₂ | .   | .   | e₁+2a | e₂ |
| e₃ | .   | .   | e₂+2a | e₃ |
| e₄ | .   | .   | 2a  | e₄ |
| e₅ | e₄ | a e₅−e₆ | e₅ |
| e₆ | .   | e₅+a | e₆ |
| e₇ | .   | .   | .   | .   |

Parameters: [a]

$([-1 \leq a, a \leq 1])$

### [7, [6, 10], 12, 1]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | .   | e₁ |
| e₂ | .   | .   | e₁+e₂ | e₂ |
| e₃ | .   | .   | e₂+e₃ | e₃ |
| e₄ | .   | .   | 2a  | e₄ |
| e₅ | e₄ | a e₅ | e₅ |
| e₆ | .   | e₅+a | e₆ |
| e₇ | .   | .   | .   | .   |

Parameters: [a]

$([0 \leq a])$

### [7, [6, 10], 12, 2]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .   | .   | .   | .   | .   | .   | .   |
| e₂ | .   | .   | .   | e₁ |
| e₃ | .   | .   | .   | e₂ |
| e₄ | .   | .   | 2e₁ | e₄ |
| e₅ | e₄ | e₅ | e₅ |
| e₆ | .   | e₅+e₆ | e₆ |
| e₇ | .   | .   | .   | .   |
| \[7, [6, 10], 12, 3\] |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| \(e_1\) | \(\ldots\) | \(\ldots\) | \(2e_1 + e_4\) |
| \(e_2\) | \(\ldots\) | \(\ldots\) | \(e_1 + 2e_2\) |
| \(e_3\) | \(\ldots\) | \(\ldots\) | \(e_2 + 2e_3\) |
| \(e_4\) | \(\ldots\) | \(\ldots\) | \(2e_4\) |
| \(e_5\) | \(\ldots\) | \(\ldots\) | \(e_4 + e_5\) |
| \(e_6\) | \(\ldots\) | \(\ldots\) | \(e_5 + e_6\) |
| \(e_7\) |   |   |   |

| \[7, [6, 12], 1, 1\] |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| \(e_1\) | \(\ldots\) | \(\ldots\) | \(7e_1\) |
| \(e_2\) | \(\ldots\) | \(\ldots\) | \(e_1 + 5e_2\) |
| \(e_3\) | \(\ldots\) | \(\ldots\) | \(e_2 + 4e_3\) |
| \(e_4\) | \(\ldots\) | \(\ldots\) | \(e_2 + 3e_4\) |
| \(e_5\) | \(\ldots\) | \(\ldots\) | \(e_4 + 2e_5\) |
| \(e_6\) | \(\ldots\) | \(\ldots\) | \(e_6\) |
| \(e_7\) |   |   |   |

| \[7, [6, 13], 1, 1\] |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| \(e_1\) | \(\ldots\) | \(\ldots\) | \(7e_1\) |
| \(e_2\) | \(\ldots\) | \(\ldots\) | \(e_1 + 2e_2\) |
| \(e_3\) | \(\ldots\) | \(\ldots\) | \(e_2 + 3e_3\) |
| \(e_4\) | \(\ldots\) | \(\ldots\) | \(e_2 + 3e_4\) |
| \(e_5\) | \(\ldots\) | \(\ldots\) | \(e_4 + 2e_5\) |
| \(e_6\) | \(\ldots\) | \(\ldots\) | \(e_6\) |
| \(e_7\) |   |   |   |

| \[7, [6, 14], 1, 1\] |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| \(e_1\) | \(\ldots\) | \(\ldots\) | \(5e_1\) |
| \(e_2\) | \(\ldots\) | \(\ldots\) | \(e_1 + 3e_2\) |
| \(e_3\) | \(\ldots\) | \(\ldots\) | \(e_1 + 4e_3\) |
| \(e_4\) | \(\ldots\) | \(\ldots\) | \(e_1 + 3e_4\) |
| \(e_5\) | \(\ldots\) | \(\ldots\) | \(e_4 + 2e_5\) |
| \(e_6\) | \(\ldots\) | \(\ldots\) | \(e_6\) |
| \(e_7\) |   |   |   |

| \[7, [6, 15], 1, 1\] |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| \(e_1\) | \(\ldots\) | \(\ldots\) | \((a + 4)e_1\) |
| \(e_2\) | \(\ldots\) | \(\ldots\) | \(e_1 + (a + 3)e_2\) |
| \(e_3\) | \(\ldots\) | \(\ldots\) | \(e_2 + (a + 2)e_3\) |
| \(e_4\) | \(\ldots\) | \(\ldots\) | \(e_3 + (a + 1)e_4\) |
| \(e_5\) | \(\ldots\) | \(\ldots\) | \(e_4 + ae_5\) |
| \(e_6\) | \(\ldots\) | \(\ldots\) | \(e_6\) |
| \(e_7\) |   |   |   |

parameters: \([a]\)
### [7, [6, 15], 1, 2]

|    | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | .  | .  |
| e₂ | .  | .  | .  | e₁ | e₂ | .  | .  |
| e₃ | .  | .  | e₂ | e₃ | e₄ | e₅ | .  |
| e₄ | .  | e₃ | e₄ | .  | .  | .  | .  |
| e₅ | .  | e₄ | e₅ | .  | .  | .  | .  |
| e₆ | .  | .  | .  | .  | .  | .  | .  |
| e₇ | .  | .  | .  | .  | .  | .  | .  |

**Parameters:** $[ε]$

$[[ε² = 1]]$

### [7, [6, 15], 1, 6]

|    | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | .  | .  |
| e₂ | .  | .  | e₁ | e₂ | e₃ | e₄ | e₅ |
| e₃ | .  | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
| e₄ | e₅ | e₆ | e₇ | .  | .  | .  | .  |
| e₅ | e₆ | e₇ | .  | .  | .  | .  | .  |
| e₆ | e₇ | .  | .  | .  | .  | .  | .  |
| e₇ | .  | .  | .  | .  | .  | .  | .  |

**Parameters:** $[ε]$

$[[ε² = 1]]$

### [7, [6, 15], 1, 3]

|    | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | .  | .  |
| e₂ | .  | .  | e₁ | e₂ | e₃ | e₄ | .  |
| e₃ | .  | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
| e₄ | .  | e₃ | e₄ | e₅ | e₆ | e₇ | .  |
| e₅ | .  | e₄ | e₅ | e₆ | e₇ | .  | .  |
| e₆ | .  | .  | e₇ | .  | .  | .  | .  |
| e₇ | .  | .  | .  | .  | .  | .  | .  |

**Parameters:** $[ε]$

$[[ε² = 1]]$

### [7, [6, 16], 1, 1]

|    | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | .  | .  |
| e₂ | .  | .  | e₁ | e₂ | e₃ | e₄ | e₅ |
| e₃ | .  | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
| e₄ | .  | e₃ | e₄ | e₅ | e₆ | e₇ | .  |
| e₅ | .  | e₄ | e₅ | e₆ | e₇ | .  | .  |
| e₆ | .  | .  | e₇ | .  | .  | .  | .  |
| e₇ | .  | .  | .  | .  | .  | .  | .  |

**Parameters:** $[a]$

$[[]]$

### [7, [6, 15], 1, 4]

|    | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | .  | .  |
| e₂ | .  | .  | e₁ | e₂ | e₃ | e₄ | .  |
| e₃ | .  | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
| e₄ | .  | e₃ | e₄ | e₅ | e₆ | e₇ | .  |
| e₅ | .  | e₄ | e₅ | e₆ | e₇ | .  | .  |
| e₆ | .  | .  | e₇ | .  | .  | .  | .  |
| e₇ | .  | .  | .  | .  | .  | .  | .  |

**Parameters:** $[a]$

$[[]]$

### [7, [6, 16], 1, 2]

|    | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | .  | .  |
| e₂ | .  | .  | e₁ | e₂ | e₃ | e₄ | e₅ |
| e₃ | .  | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
| e₄ | .  | e₃ | e₄ | e₅ | e₆ | e₇ | .  |
| e₅ | .  | e₄ | e₅ | e₆ | e₇ | .  | .  |
| e₆ | .  | .  | e₇ | .  | .  | .  | .  |
| e₇ | .  | .  | .  | .  | .  | .  | .  |

**Parameters:** $[a]$  

$[[]]$

### [7, [6, 16], 1, 3]

|    | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | .  | .  |
| e₂ | .  | .  | e₁ | e₂ | e₃ | e₄ | e₅ |
| e₃ | .  | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
| e₄ | .  | e₃ | e₄ | e₅ | e₆ | e₇ | .  |
| e₅ | .  | e₄ | e₅ | e₆ | e₇ | .  | .  |
| e₆ | .  | .  | e₇ | .  | .  | .  | .  |
| e₇ | .  | .  | .  | .  | .  | .  | .  |

**Parameters:** $[ε]$

$[[ε² = 1]]$
### [7, [6, 16], 1, 4]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| .  | .  | .  | .  | 2e1| .  | .  |
| .  | .  | e1 | e2 | .  | .  | .  |
| .  | e1 | e2 | e3 | .  | e4 | e5 |
| .  | .  | e4 | e2+e5| .  | .  | .  |
| .  | .  | .  | .  | .  | .  | .  |
| .  | .  | .  | .  | .  | .  | .  |

parameters: [a]

### [7, [6, 17], 1, 1]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| .  | .  | .  | .  | .  | (a+3)e1| .  |
| .  | .  | e1 | e2 | .  | (a+2)e2| e3 |
| .  | e1 | e2 | e3 | .  | 2e4| .  |
| .  | .  | e3 | e5 | .  | e6| .  |
| .  | .  | e5 | .  | .  | .  | .  |

### [7, [6, 17], 1, 2]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| .  | .  | .  | .  | .  | .  | (a+3)e1|
| .  | .  | e1 | e2 | .  | 3e2| e3 |
| .  | e1 | e2 | e3 | .  | e4| .  |
| .  | .  | e3 | e5 | .  | e6| .  |
| .  | .  | e5 | .  | .  | .  | .  |

parameters: [a]

### [7, [6, 17], 1, 3]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| .  | .  | .  | .  | e1 | .  | .  |
| .  | .  | e1 | e2 | .  | 3e2| e3 |
| .  | e1 | e2 | e3 | .  | e4| .  |
| .  | .  | e3 | e5 | .  | e6| .  |
| .  | .  | e5 | .  | .  | .  | .  |

### [7, [6, 17], 1, 5]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| .  | .  | .  | .  | .  | 4e1| .  |
| .  | .  | e1 | e2 | .  | 3e2| e3 |
| .  | e1 | e2 | e3 | .  | e4| .  |
| .  | .  | e3 | e5 | .  | e6| .  |
| .  | .  | e5 | .  | .  | .  | .  |

### [7, [6, 18], 1, 1]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| .  | .  | .  | .  | .  | (a+3)e1| .  |
| .  | .  | e1 | e2 | .  | 3e2| e3 |
| .  | e1 | e2 | e3 | .  | e4| .  |
| .  | .  | e3 | e5 | .  | e6| .  |
| .  | .  | e5 | .  | .  | .  | .  |

parameters: [a]

### [7, [6, 18], 1, 2]

| e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|----|----|----|----|----|----|----|
| .  | .  | .  | .  | .  | e1| .  |
| .  | .  | e1 | e2 | .  | e3| e4 |
| .  | e1 | e2 | e3 | .  | e4| .  |
| .  | .  | e3 | e5 | .  | e4| .  |
| .  | .  | e5 | .  | .  | e5| e6 |
| .  | .  | e6 | .  | .  | .  | .  |

| e7 |
### [7, [6, 18], 1, 3]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | e₁ |    |
| e₂ | .  | .  | e₁ | .  | .  | .  |    |
| e₃ | .  | .  | e₁ | e₃ | .  | .  |    |
| e₄ | .  | .  | e₃ | e₄ | .  | .  |    |
| e₅ | .  | e₄ | e₃⁺e₅| . | .  | e₂ | e₃⁺e₅|
| e₆ | .  | e₂ | e₃⁺e₅| . | .  | .  |    |
| e₇ |    |    |    |    |    |    |    |

parameters: [ε]

[[ε² = 1]]

### [7, [6, 18], 1, 4]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | e₁ |    |
| e₂ | .  | .  | e₁ | .  | .  | .  |    |
| e₃ | .  | .  | e₁ | e₃ | .  | .  |    |
| e₄ | .  | .  | e₃ | e₄ | .  | .  |    |
| e₅ | .  | e₃⁺e₄ | .  | .  | e₂ | e₃⁺e₄ |     |
| e₆ | .  | e₄ | e₃⁺e₄ | .  | .  | .  |    |
| e₇ |    |    |    |    |    |    |    |

parameters: [a, ε]

[[ε² = 1]]

### [7, [6, 18], 1, 5]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | 3e₁ | .  | .  |    |
| e₂ | .  | .  | e₁ | e₁+3e₂ | .  | .  |    |
| e₃ | .  | .  | e₁ | 2e₃ | .  | .  |    |
| e₄ | .  | .  | e₃ | e₄ | .  | .  |    |
| e₅ | .  | e₄ | e₃ | e₄ | .  | .  |    |
| e₆ | .  | e₄ | .  | .  | .  | .  |    |
| e₇ |    |    |    |    |    |    |    |

### [7, [6, 18], 1, 6]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | 6e₁ | .  | .  |    |
| e₂ | .  | .  | e₁ | 3e₂ | .  | .  |    |
| e₃ | .  | .  | e₁ | 5e₃ | .  | .  |    |
| e₄ | .  | .  | e₃ | 4e₄ | .  | .  |    |
| e₅ | .  | e₄ | e₃⁺3e₅ | .  | .  | .  |    |
| e₆ | .  | .  | .  | .  | .  | .  |    |
| e₇ |    |    |    |    |    |    |    |

parameters: [ε]

[[ε² = 1]]
### $[7, [6, 19], 1, 4]$  
\[
\begin{array}{cccccccc}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
  \cdot & \cdot & \cdot & \cdot & 2e_1 & \cdot & \cdot \\
  e_2 & \cdot & \cdot & e_1 & 2e_2 & \cdot & \cdot \\
  e_3 & \cdot & \cdot & e_1 & e_3 & \cdot & \cdot \\
  e_4 & \cdot & e_2 & e_3 & e_4 & \cdot & \cdot \\
  e_5 & \cdot & e_4 & e_5 & e_3+e_5 & \cdot & \cdot \\
  e_6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  e_7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

Parameters: $[\varepsilon]$

$[\varepsilon^2 = 1]$  

### $[7, [6, 21], 1, 1]$  
\[
\begin{array}{cccccccc}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
  \cdot & \cdot & \cdot & \cdot & (2a+1)e_1 & \cdot & \cdot \\
  e_2 & \cdot & \cdot & e_1 & (a+1)e_2 & \cdot & \cdot \\
  e_3 & \cdot & \cdot & e_1 & 2ae_3 & \cdot & \cdot \\
  e_4 & \cdot & \cdot & e_2 & ae_4 & \cdot & \cdot \\
  e_5 & \cdot & e_3 & (2a-1)e_5 & e_6 & \cdot & \cdot \\
  e_6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  e_7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

Parameters: $[a]$

$[]$  

### $[7, [6, 19], 1, 5]$  
\[
\begin{array}{cccccccc}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
  \cdot & \cdot & \cdot & \cdot & 4e_1 & \cdot & \cdot \\
  e_2 & \cdot & \cdot & e_1 & 3e_2 & \cdot & \cdot \\
  e_3 & \cdot & \cdot & e_1 & 3e_3 & \cdot & \cdot \\
  e_4 & \cdot & e_2 & e_3 & e_4+2e_4 & \cdot & \cdot \\
  e_5 & \cdot & e_4 & e_5 & \cdot & \cdot & \cdot \\
  e_6 & \cdot & \cdot & -e_3+e_6 & \cdot & \cdot & \cdot \\
  e_7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

Parameters: $[a]$

$[]$  

### $[7, [6, 20], 1, 1]$  
\[
\begin{array}{cccccccc}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
  \cdot & \cdot & \cdot & \cdot & 4e_1 & \cdot & \cdot \\
  e_2 & \cdot & \cdot & -e_1 & 3e_2-ae_3 & \cdot & \cdot \\
  e_3 & \cdot & \cdot & -e_1 & ae_2+3e_3 & \cdot & \cdot \\
  e_4 & \cdot & e_2 & e_3 & 2e_4 & \cdot & \cdot \\
  e_5 & \cdot & e_4 & e_5 & -ae_6 & \cdot & \cdot \\
  e_6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  e_7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

Parameters: $[\varepsilon]$

$[\varepsilon^2 = 1]$  

### $[7, [6, 20], 1, 2]$  
\[
\begin{array}{cccccccc}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
  \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  e_2 & \cdot & \cdot & -e_1 & \cdot & -e_3 & \cdot \\
  e_3 & \cdot & \cdot & -e_1 & \cdot & e_2 & \cdot \\
  e_4 & \cdot & e_2 & e_3 & \cdot & e_1 & \cdot \\
  e_5 & \cdot & e_4 & \cdot & e_6 & \cdot & \cdot \\
  e_6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  e_7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

Parameters: $[\varepsilon]$

$[\varepsilon^2 = 1]$  

### $[7, [6, 21], 1, 2]$  
\[
\begin{array}{cccccccc}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
  \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  e_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  e_2 & \cdot & \cdot & e_1 & \cdot & e_2 & \cdot \\
  e_3 & \cdot & \cdot & e_1 & \cdot & 2e_3 & \cdot \\
  e_4 & \cdot & \cdot & e_2 & \cdot & e_4 & \cdot \\
  e_5 & \cdot & e_3 & \cdot & e_5 & \cdot & \cdot \\
  e_6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  e_7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

Parameters: $[\varepsilon]$

$[\varepsilon^2 = 1]$  

### $[7, [6, 21], 1, 3]$  
\[
\begin{array}{cccccccc}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
  \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  e_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  e_2 & \cdot & \cdot & e_1 & \cdot & e_2 & \cdot \\
  e_3 & \cdot & \cdot & e_1 & \cdot & 2e_3 & \cdot \\
  e_4 & \cdot & \cdot & e_2 & \cdot & e_4 & \cdot \\
  e_5 & \cdot & e_3 & \cdot & e_5 & \cdot & \cdot \\
  e_6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  e_7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

Parameters: $[\varepsilon]$

$[\varepsilon^2 = 1]$  

### $[7, [6, 21], 1, 4]$  
\[
\begin{array}{cccccccc}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
  \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  e_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  e_2 & \cdot & \cdot & e_1 & \cdot & e_2 & \cdot \\
  e_3 & \cdot & \cdot & e_1 & \cdot & 2e_3 & \cdot \\
  e_4 & \cdot & \cdot & e_2 & \cdot & e_4 & \cdot \\
  e_5 & \cdot & e_3 & \cdot & e_5+2e_5 & \cdot & \cdot \\
  e_6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  e_7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

Parameters: $[\varepsilon]$

$[\varepsilon^2 = 1]$
| 7, [6, 21], 1, 5 |
|------------------|
| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
| e₁ . . . . . . e₁ |
| e₂ . . e₁ . . e₂ |
| e₃ . . . . e₁ . |
| e₄ . . e₂ e₃ |
| e₅ . e₃ e₁ e₅ |
| e₆ . e₂+e₆ |
| e₇ . |

parameters: [a]

| 7, [6, 22], 1, 2 |
|------------------|
| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
| e₁ . . . . . . 2e₁ |
| e₂ . . . . e₂ |
| e₃ . . . e₂ e₃ |
| e₄ . . e₁ e₃ e₄ |
| e₅ . e₄ e₅ |
| e₆ . . |
| e₇ . |

| 7, [6, 21], 1, 6 |
|------------------|
| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
| e₁ . . . . . . 2e₁ |
| e₂ . . e₁ . . e₂ |
| e₃ . . . . e₁ 2e₃ |
| e₄ . . e₂ e₂+e₄ |
| e₅ . e₃ a e₁+2e₅ |
| e₆ . . |
| e₇ . |

| 7, [6, 22], 1, 3 |
|------------------|
| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
| e₁ . . . . . . e₁ |
| e₂ . . . . 3e₂ |
| e₃ . . . e₂ 2e₃ |
| e₄ . . e₁ e₃ e₄ |
| e₅ . e₄ . |
| e₆ . . e₁+e₆ |
| e₇ . |

parameters: [ε]

| 7, [6, 22], 1, 4 |
|------------------|
| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
| e₁ . . . . . . 2e₁ |
| e₂ . . . . e₂ |
| e₃ . . . e₂ e₃ |
| e₄ . . e₁ e₃ e₄ |
| e₅ . e₄ e₅ |
| e₆ . e₂+e₅ |
| e₇ . |

| 7, [6, 22], 1, 5 |
|------------------|
| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
| e₁ . . . . . . e₁ |
| e₂ . . . . 2e₂ |
| e₃ . . . e₂ e₃ |
| e₄ . . e₁ e₃ e₄ |
| e₅ . e₄ . |
| e₆ . e₁ e₅ |
| e₇ . |

parameters: [ε]

| 7, [6, 22], 1, 7 |
|------------------|
| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
| e₁ . . . . . . 3e₁ |
| e₂ . . e₁ . . 2e₂+e₃ |
| e₃ . . . . e₁ 2e₃ |
| e₄ . . e₂ e₄+e₅ |
| e₅ . e₃ e₅ |
| e₆ . . e₄+e₅+e₆ |
| e₇ . |

parameters: [a]

| 7, [6, 22], 1, 1 |
|------------------|
| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
| e₁ . . . . . . (2a+1)e₁ |
| e₂ . . . . (a+3)e₂ |
| e₃ . . . e₂ (a+2)e₃ |
| e₄ . e₁ e₃ (a+1)e₄ |
| e₅ . e₄ a e₅ |
| e₆ . e₆ |
| e₇ . |

parameters: [α]

| 7, [6, 22], 1, 5 |
|------------------|
| e₁ e₂ e₃ e₄ e₅ e₆ e₇ |
| e₁ . . . . . . −e₁ |
| e₂ . . . . 2e₂ |
| e₃ . . . e₂ e₃ |
| e₄ . . e₁ e₃ . |
| e₅ . e₄ e₅ |
| e₆ . e₁ e₅ |
| e₇ . |

parameters: [ε]
\[\begin{align*}
&[7, [6, 22], 1, 6] \\
&\begin{array}{ccccccc}
& e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
&e_1 & . & . & . & . & . & 2e_1 \\
e_2 & . & . & . & . & . & e_2 \\
e_3 & . & . & e_2 & e_3 \\
e_4 & . & e_1 & e_3 & e_2 + e_4 \\
e_5 & . & e_4 & a e_2 + e_3 + e_5 \\
e_6 & . & . & . \\
e_7 & . & & & \\
\end{array}
\end{align*}\]

parameters: \([a, \epsilon]\)

\([x^2 = 1]\)

\[\begin{align*}
&[7, [6, 23], 1, 3] \\
&\begin{array}{ccccccc}
& e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
&e_1 & . & . & . & . & . & . \\
e_2 & . & . & . & . & . & e_2 \\
e_3 & . & . & e_2 & e_1 \\
e_4 & . & . & e_2 & e_4 \\
e_5 & . & e_4 & e_2 + e_5 \\
e_6 & . & . & . \\
e_7 & . & . & & \\
\end{array}
\end{align*}\]

parameters: \([\epsilon]\)

\([x^2 = 1]\)

\[\begin{align*}
&[7, [6, 22], 1, 7] \\
&\begin{array}{ccccccc}
& e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
&e_1 & . & . & . & . & . & 3e_1 \\
e_2 & . & . & . & . & . & 4e_2 \\
e_3 & . & . & e_2 & e_1 + 3e_3 \\
e_4 & . & e_1 & e_3 & 2e_4 \\
e_5 & . & e_4 & e_5 \\
e_6 & . & . & e_5 + e_6 \\
e_7 & . & & & \\
\end{array}
\end{align*}\]

\[\begin{align*}
&[7, [6, 23], 1, 4] \\
&\begin{array}{ccccccc}
& e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
&e_1 & . & . & . & . & . & 3e_1 \\
e_2 & . & . & . & . & . & 5e_2 \\
e_3 & . & . & e_2 & e_1 & 2e_3 \\
e_4 & . & . & e_2 & 4e_4 \\
e_5 & . & e_4 & e_1 + 3e_5 \\
e_6 & . & . & e_6 \\
e_7 & . & & & \\
\end{array}
\end{align*}\]

\[\begin{align*}
&[7, [6, 23], 1, 1] \\
&\begin{array}{ccccccc}
& e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
&e_1 & . & . & . & . & . & 3e_1 \\
e_2 & . & . & . & . & . & (a+2)e_2 \\
e_3 & . & . & e_2 & e_1 & 2e_3 \\
e_4 & . & . & e_2 & (a+1)e_4 \\
e_5 & . & . & a e_3 \\
e_6 & . & . & e_6 \\
e_7 & . & & & \\
\end{array}
\end{align*}\]

parameters: \([a]\)

\([x^2 = 1]\)

\[\begin{align*}
&[7, [6, 23], 1, 5] \\
&\begin{array}{ccccccc}
& e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
&e_1 & . & . & . & . & . & 3e_1 \\
e_2 & . & . & . & . & . & 2e_2 \\
e_3 & . & . & e_2 & e_1 & 2e_3 \\
e_4 & . & . & e_2 & e_4 \\
e_5 & . & . & e_4 & . \\
e_6 & . & . & e_6 \\
e_7 & . & & & \\
\end{array}
\end{align*}\]

\[\begin{align*}
&[7, [6, 23], 1, 2] \\
&\begin{array}{ccccccc}
& e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
&e_1 & . & . & . & . & . & . \\
e_2 & . & . & . & . & . & e_2 \\
e_3 & . & . & . & e_2 & e_1 \\
e_4 & . & . & . & e_2 & e_4 \\
e_5 & . & . & . & e_4 & e_5 \\
e_6 & . & . & . & . \\
e_7 & . & & & \\
\end{array}
\end{align*}\]

\[\begin{align*}
&[7, [6, 23], 1, 6] \\
&\begin{array}{ccccccc}
& e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
&e_1 & . & . & . & . & . & . \\
e_2 & . & . & . & . & . & 2e_2 \\
e_3 & . & . & e_2 & e_1 & . \\
e_4 & . & . & e_2 & e_4 \\
e_5 & . & . & e_4 & e_5 \\
e_6 & . & . & . & - e_3 \\
e_7 & . & & & \\
\end{array}
\end{align*}\]
\[
\begin{array}{c|ccccccc}
[7, [6, 23], 1, 7] & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
 e_1 & . & . & . & . & . & . & . \\
e_2 & . & . & . & 2 & e_2 & . & . \\
e_3 & . & e_2 & e_1 & e_1 & . & . & . \\
e_4 & . & e_2 & a & e_2 + e_4 & . & . & . \\
e_5 & . & . & e_4 & e_5 & . & . & . \\
e_6 & . & . & -a & e_3 & . & . & . \\
e_7 & . & . & . & . & . & . & . \\
\end{array}
\]

Parameters: \([a]\)

\[
\begin{array}{c|ccccccc}
[7, [6, 24], 1, 1] & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
 e_1 & . & . & . & . & . & . & . \\
e_2 & . & . & . & (2a+1) & e_1 & . & . \\
e_3 & . & e_1 & (a+1) & e_3 & . & . & . \\
e_4 & . & e_1 & e_2 & (a+1) & e_4 & . & . \\
e_5 & . & e_4 & a & e_5 & . & . & . \\
e_6 & . & . & . & . & . & . & . \\
e_7 & . & . & . & . & . & . & . \\
\end{array}
\]

Parameters: \([a]\)

\[
\begin{array}{c|ccccccc}
[7, [6, 24], 1, 2] & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
 e_1 & . & . & . & . & 2 & e_1 & . \\
e_2 & . & . & . & . & e_2 & . \\
e_3 & . & . & e_1 & . & e_3 & . \\
e_4 & . & e_1 & e_2 & e_4 & . & . \\
e_5 & . & e_4 & e_5 & . & . & . \\
e_6 & . & . & . & . & . & . \\
e_7 & . & . & . & . & . & . & . \\
\end{array}
\]

\[
\begin{array}{c|ccccccc}
[7, [6, 24], 1, 3] & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
 e_1 & . & . & . & . & . & e_1 & . \\
e_2 & . & . & . & . & 2 & e_2 & . \\
e_3 & . & . & e_1 & . & e_3 & . \\
e_4 & . & . & e_1 & e_2 & e_4 & . \\
e_5 & . & . & e_4 & . & . & . \\
e_6 & . & . & . & . & . & . \\
e_7 & . & . & . & . & . & . & . \\
\end{array}
\]

Parameters: \([e]\)

\([e^2 = 1]\)

\[
\begin{array}{c|ccccccc}
[7, [6, 24], 1, 4] & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
 e_1 & . & . & . & . & 2 & e_1 & . \\
e_2 & . & . & . & . & e_2 & . \\
e_3 & . & . & e_1 & . & e_3 & . \\
e_4 & . & . & e_1 & e_2 & e_4 & . \\
e_5 & . & . & e_4 & e_5 & . & . \\
e_6 & . & . & . & . & . & . \\
e_7 & . & . & . & . & . & . & . \\
\end{array}
\]

Parameters: \([e]\)

\([e^2 = 1]\)
### [7, [6, 24], 1, 5]

| $e_1$  | $e_2$  | $e_3$  | $e_4$  | $e_5$  | $e_6$  | $e_7$  |
|--------|--------|--------|--------|--------|--------|--------|
|        |        |        |        | 2$e_1$ |        |        |
|        |        |        |        | $e_2$  |        |        |
|        |        | $e_1$  | $e_3$  |        |        |        |
| $e_4$  | $e_1$  | $e_2$  | $e_3$  |        |        |        |
| $e_5$  |        |        |        | $e_3$  | $e_4$  |        |
| $e_6$  |        |        |        |        |        |        |
| $e_7$  |        |        |        |        |        |        |

### [7, [6, 24], 1, 6]

| $e_1$  | $e_2$  | $e_3$  | $e_4$  | $e_5$  | $e_6$  | $e_7$  |
|--------|--------|--------|--------|--------|--------|--------|
|        |        |        |        |        |        |        |
|        |        |        |        |        |        |        |
| $e_4$  | $e_1$  | $e_2$  | $e_3$  |        |        |        |
| $e_5$  |        |        |        |        |        |        |
| $e_6$  |        |        |        |        |        |        |
| $e_7$  |        |        |        |        |        |        |

### [7, [6, 24], 1, 7]

| $e_1$  | $e_2$  | $e_3$  | $e_4$  | $e_5$  | $e_6$  | $e_7$  |
|--------|--------|--------|--------|--------|--------|--------|
|        |        |        |        |        |        |        |
|        |        |        |        |        |        |        |
|        |        |        |        |        |        |        |
|        |        |        |        |        |        |        |
|        |        |        |        |        |        |        |
|         |         |         |         |         |         | $e_3$  |
| $e_7$  |        |        |        |        |        |        |

### [7, [6, 25], 1, 1]

| $e_1$  | $e_2$  | $e_3$  | $e_4$  | $e_5$  | $e_6$  | $e_7$  |
|--------|--------|--------|--------|--------|--------|--------|
|        |        |        |        |        |        |        |
|        |        |        |        |        |        |        |
|        |        |        |        |        |        |        |
| $e_3$  | $e_2$  |        |        |        |        |        |
| $e_4$  | $e_1$  | $e_2$  |        |        |        |        |
| $e_5$  |        |        |        |        |        |        |
| $e_6$  |        |        |        |        |        |        |
| $e_7$  |        |        |        |        |        |        |

### [7, [6, 25], 1, 2]

| $e_1$  | $e_2$  | $e_3$  | $e_4$  | $e_5$  | $e_6$  | $e_7$  |
|--------|--------|--------|--------|--------|--------|--------|
|        |        |        |        |        |        |        |
|        |        |        |        |        |        |        |
|        | $e_3$  | $e_2$  |        |        |        |        |
| $e_4$  | $e_1$  | $e_2$  |        |        |        |        |
| $e_5$  |        |        |        |        |        |        |
| $e_6$  |        |        |        |        |        |        |
| $e_7$  |        |        |        |        |        |        |

### [7, [6, 26], 1, 1]

| $e_1$  | $e_2$  | $e_3$  | $e_4$  | $e_5$  | $e_6$  | $e_7$  |
|--------|--------|--------|--------|--------|--------|--------|
|        |        |        |        |        |        |        |
|        |        |        |        |        |        |        |
|        |        |        |        |        |        |        |
| $e_3$  | $e_2$  |        |        |        |        |        |
| $e_4$  | $e_1$  | $e_2$  |        |        |        |        |
| $e_5$  |        |        |        |        |        |        |
| $e_6$  |        |        |        |        |        |        |
| $e_7$  |        |        |        |        |        |        |

### [7, [6, 26], 1, 2]

| $e_1$  | $e_2$  | $e_3$  | $e_4$  | $e_5$  | $e_6$  | $e_7$  |
|--------|--------|--------|--------|--------|--------|--------|
|        |        |        |        |        |        |        |
|        |        |        |        |        |        |        |
|        |        |        |        |        |        |        |
|        |        |        |        |        |        |        |
|        |        |        |        |        |        |        |
|        |        |        |        |        |        |        |
|        |        |        |        |        |        |        |

### Parameters

- $[7, [6, 24], 1, 5]$: $a$ parameters
- $[7, [6, 24], 1, 6]$: $a$ parameters
- $[7, [6, 24], 1, 7]$: $a$ parameters
- $[7, [6, 25], 1, 1]$: $a$ parameters
- $[7, [6, 25], 1, 2]$: $a$ parameters
- $[7, [6, 26], 1, 1]$: $a$ parameters
- $[7, [6, 26], 1, 2]$: $a$ parameters
### [7, [6, 26], 1, 3]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
|    |    | -e₁|    |    |    |    |
| e₂ |    |    |    | e₂ |    |    |
| e₃ |    | e₂ |    | 2e₃|    |    |
| e₄ |    | e₁ | e₂ |    |    |    |
| e₅ |    | e₄ |    | -e₅|    |    |
| e₆ |    |    | e₂+e₆|    |    |    |
| e₇ |    |    |    |    |    |    |

Parameters: [e]

\[ \langle e^2 = 1 \rangle \]

### [7, [6, 26], 1, 4]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
|    |    |    |    |    |    | e₁ |
| e₂ |    |    |    |    |    | 2e₂ |
| e₃ |    | e₂ |    | 2e₃|    |    |
| e₄ |    | e₁ | e₂ | e₄ |    |    |
| e₅ |    |    | e₄ |    |    |    |
| e₆ |    |    | e₁+e₆|    |    |    |
| e₇ |    |    |    |    |    |    |

### [7, [6, 26], 1, 5]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
|    |    |    |    | 5e₁|    |    |
| e₂ |    |    |    | 4e₂|    |    |
| e₃ |    | e₂ |    | 2e₃|    |    |
| e₄ |    | e₁ | e₂ | 3e₄|    |    |
| e₅ |    | e₄ | e₃+2e₅|    |    |    |
| e₆ |    |    | e₆ |    |    |    |
| e₇ |    |    |    |    |    |    |

### [7, [6, 26], 1, 6]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
|    |    |    |    |    |    | e₁ |
| e₂ |    |    |    |    |    | 2e₂ |
| e₃ |    | e₂ |    | 2e₃|    |    |
| e₄ |    | e₁ | e₂ | e₄ |    |    |
| e₅ |    |    | e₄ |    |    |    |
| e₆ |    |    | e₁+e₆|    |    |    |
| e₇ |    |    |    |    |    |    |

Parameters: [a]

\[ \langle \rangle \]

### [7, [6, 26], 1, 7]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
|    |    |    |    | 2e₁|    |    |
| e₂ |    |    |    | e₂ |    |    |
| e₃ |    | e₂ |    |    |    |    |
| e₄ |    | e₁ | e₂ | e₄ |    |    |
| e₅ |    | e₄ | e₃ | e₅ |    |    |
| e₆ |    |    | e₃+e₅|    |    |    |
| e₇ |    |    |    |    |    |    |

Parameters: [e, b]

\[ \langle \rangle \]

### [7, [6, 26], 1, 8]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
|    |    |    |    |    | 4e₁|    |
| e₂ |    |    |    |    | 5e₂|    |
| e₃ |    | e₂ |    | e₃+4e₃|    |    |
| e₄ |    | e₁ | e₂ | 3e₄|    |    |
| e₅ |    | e₄ | e₅ |    |    |    |
| e₆ |    |    | e₅+e₆|    |    |    |
| e₇ |    |    |    |    |    |    |

### [7, [6, 26], 1, 9]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
|    |    |    |    |    |    |    |
| e₂ |    |    |    |    |    |    |
| e₃ |    | e₂ |    | 2e₃|    |    |
| e₄ |    | e₁ | e₂ | 2e₄|    |    |
| e₅ |    | e₄ | e₅ | e₆ |    |    |
| e₆ |    |    |    |    |    |    |
| e₇ |    |    |    |    |    |    |

Parameters: [a, b]

\[ \langle \rangle \]

### [7, [6, 27], 1, 1]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
|    |    |    |    |    |    |    |
| e₂ |    |    |    |    |    |    |
| e₃ |    | e₁ |    | (a+1)e₂|    |    |
| e₄ |    | e₂ |    | ae₄ |    |    |
| e₅ |    | e₃ | e₅ | b₅ |    |    |
| e₆ |    |    | e₆ |    |    |    |
| e₇ |    |    |    |    |    |    |

Parameters: [a, b]

\[ \langle \rangle \]
\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
\textbf{[7, [6, 27], 1, 10]} & \textbf{[7, [6, 28], 1, 2]} & \textbf{[7, [6, 27], 1, 11]} & \textbf{[7, [6, 28], 1, 3]} & \textbf{[7, [6, 27], 1, 12]} & \textbf{[7, [6, 28], 1, 4]} & \textbf{[7, [6, 28], 1, 5]} \\
\hline
\textbf{\(e_1\)} & \textbf{\(e_1\)} & \textbf{\(e_1\)} & \textbf{\(e_1\)} & \textbf{\(e_1\)} & \textbf{\(e_1\)} & \textbf{\(e_1\)} \\
\textbf{\(e_2\)} & \textbf{\(e_2\)} & \textbf{\(e_2\)} & \textbf{\(e_2\)} & \textbf{\(e_2\)} & \textbf{\(e_2\)} & \textbf{\(e_2\)} \\
\textbf{\(e_3\)} & \textbf{\(e_3\)} & \textbf{\(e_3\)} & \textbf{\(e_3\)} & \textbf{\(e_3\)} & \textbf{\(e_3\)} & \textbf{\(e_3\)} \\
\textbf{\(e_4\)} & \textbf{\(e_4\)} & \textbf{\(e_4\)} & \textbf{\(e_4\)} & \textbf{\(e_4\)} & \textbf{\(e_4\)} & \textbf{\(e_4\)} \\
\textbf{\(e_5\)} & \textbf{\(e_5\)} & \textbf{\(e_5\)} & \textbf{\(e_5\)} & \textbf{\(e_5\)} & \textbf{\(e_5\)} & \textbf{\(e_5\)} \\
\textbf{\(e_6\)} & \textbf{\(e_6\)} & \textbf{\(e_6\)} & \textbf{\(e_6\)} & \textbf{\(e_6\)} & \textbf{\(e_6\)} & \textbf{\(e_6\)} \\
\textbf{\(e_7\)} & \textbf{\(e_7\)} & \textbf{\(e_7\)} & \textbf{\(e_7\)} & \textbf{\(e_7\)} & \textbf{\(e_7\)} & \textbf{\(e_7\)} \\
\hline
\end{tabular}
\end{table}

parameters: \([a, b]\)
| 7, [6, 28], 1, 6 | 7, [6, 29], 1, 2 |
|---|---|
| \( e_1 \) | \( e_1 \) |
| \( e_2 \) | \( e_2 \) |
| \( e_3 \) | \( e_3 \) |
| \( e_4 \) | \( e_4 \) |
| \( e_5 \) | \( e_5 \) |
| \( e_6 \) | \( e_6 \) |
| \( e_7 \) | \( e_7 \) |

parameters: \([a, \varepsilon]\)

\([\varepsilon^2 = 1]\)

| 7, [6, 28], 1, 7 | 7, [6, 29], 1, 3 |
|---|---|
| \( e_1 \) | \( e_1 \) |
| \( e_2 \) | \( e_2 \) |
| \( e_3 \) | \( e_3 \) |
| \( e_4 \) | \( e_4 \) |
| \( e_5 \) | \( e_5 \) |
| \( e_6 \) | \( e_6 \) |
| \( e_7 \) | \( e_7 \) |

parameters: \([\varepsilon]\)

\([\varepsilon^2 = 1]\)

| 7, [6, 29], 1, 1 | 7, [6, 29], 1, 4 |
|---|---|
| \( e_1 \) | \( e_1 \) |
| \( e_2 \) | \( e_2 \) |
| \( e_3 \) | \( e_3 \) |
| \( e_4 \) | \( e_4 \) |
| \( e_5 \) | \( e_5 \) |
| \( e_6 \) | \( e_6 \) |
| \( e_7 \) | \( e_7 \) |

parameters: \([a, b]\)

\([b \leq a]\)

| 7, [6, 29], 1, 4 | 7, [6, 29], 1, 2 |
|---|---|
| \( e_1 \) | \( e_1 \) |
| \( e_2 \) | \( e_2 \) |
| \( e_3 \) | \( e_3 \) |
| \( e_4 \) | \( e_4 \) |
| \( e_5 \) | \( e_5 \) |
| \( e_6 \) | \( e_6 \) |
| \( e_7 \) | \( e_7 \) |

parameters: \([a, \varepsilon]\)

\([\varepsilon^2 = 1]\)
### Parameters $\epsilon$

\[
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3 \\
\epsilon_4 \\
\epsilon_5 \\
\epsilon_6 \\
\epsilon_7
\end{bmatrix}
\]

### Parameters $a$

\[
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5 \\
a_6 \\
a_7
\end{bmatrix}
\]

### Parameters $a, b$

\[
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5 \\
a_6 \\
a_7
\end{bmatrix}
\]
### [7, [6, 29], 2, 3]

| | e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|---|---|---|---|---|---|---|
| e₁ | . . . . . . . | 2 a e₁ |
| e₂ | . e₁ . . . | a e₂ − e₃ |
| e₃ | . . . . | e₂ + a e₃ |
| e₄ | . . e₁ | 2 a e₄ |
| e₅ | . . e₄ e₁ + 2 a e₅ |
| e₆ | . . . . . . |
| e₇ | . . . . |

Parameters: [a, ε]

[[0 ≤ a, ε² = 1]]

### [7, [6, 29], 2, 4]

| | e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|---|---|---|---|---|---|---|
| e₁ | . . . . . . . |
| e₂ | . e₁ . . . | − e₃ |
| e₃ | . . . . . . e₂ |
| e₄ | . . e₁ . . . |
| e₅ | . . . . e₁ e₃ |
| e₆ | . . . . e₅ |
| e₇ | . . . . |

Parameters: [ε]

[[ε² = 1]]

### [7, [6, 29], 3, 1]

| | e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|---|---|---|---|---|---|---|
| e₁ | . . . . . . . | 2 e₁ |
| e₂ | . e₁ . . . | e₂ |
| e₃ | . . . . | e₂ + e₃ |
| e₄ | . . e₁ (2−a) e₄ |
| e₅ | . . e₄ (2−2a) e₅ |
| e₆ | . e₆ |
| e₇ | . . . . |

Parameters: [a]

[[ ]]

### [7, [6, 29], 3, 2]

| | e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|---|---|---|---|---|---|---|
| e₁ | . . . . . . |
| e₂ | . e₁ . . . |
| e₃ | . . . . e₂ |
| e₄ | . . e₁ − e₄ |
| e₅ | . . . . e₂ e₅ |
| e₆ | . . . . e₆ |
| e₇ | . . . . |

Parameters: [a, b]

[[ ]]
| $[7, [6, 30], 1, 10]$ | $[7, [6, 30], 1, 13]$ |
|-------------------|-------------------|
| $e_1$ | $e_1$ |
| $e_2$ | $e_2$ |
| $e_3$ | $e_3$ |
| $e_4$ | $e_4$ |
| $e_5$ | $e_5$ |
| $e_6$ | $e_6$ |
| $e_7$ | $e_7$ |

parameters: $[\varepsilon]$  
$[[\varepsilon^2 = 1]]$

| $[7, [6, 30], 1, 11]$ | $[7, [6, 30], 1, 14]$ |
|-------------------|-------------------|
| $e_1$ | $e_1$ |
| $e_2$ | $e_2$ |
| $e_3$ | $e_3$ |
| $e_4$ | $e_4$ |
| $e_5$ | $e_5$ |
| $e_6$ | $e_6$ |
| $e_7$ | $e_7$ |

parameters: $[\varepsilon]$  
$[[\varepsilon^2 = 1]]$

| $[7, [6, 30], 1, 12]$ | $[7, [6, 30], 1, 15]$ |
|-------------------|-------------------|
| $e_1$ | $e_1$ |
| $e_2$ | $e_2$ |
| $e_3$ | $e_3$ |
| $e_4$ | $e_4$ |
| $e_5$ | $e_5$ |
| $e_6$ | $e_6$ |
| $e_7$ | $e_7$ |

parameters: $[\alpha, \varepsilon]$  
$[[\varepsilon^2 = 1]]$
\[ \gamma, \delta \]

| Parameters: \( a \) |\( \beta \) |
|---|---|
| \( \varepsilon, \delta \) | \( \eta \) |

\[ \gamma, \delta \]
### [7, [6, 30], 1, 24]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| .  | .  | .  | .  | .  | 2e₁ | .  |
| .  | .  | .  | .  | 3e₂ | .  | .  |
| .  | .  | e₁ | e₂ | e₃ | .  | e₄ |
| .  | e₂ | e₃ | .  | .  | e₄ | .  |
| .  | e₄ | e₅ | e₆ | .  | e₇ | .  |
| .  | .  | .  | .  | .  | .  | .  |

Parameters: \([e] \)  
\(\{\varepsilon^2 = 1\}\)

### [7, [6, 30], 1, 25]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| .  | .  | .  | .  | .  | 2e₁ | .  |
| .  | .  | .  | .  | .  | e₂ | .  |
| .  | e₁ | e₂ | e₃ | .  | .  | e₄ |
| .  | e₂ | e₃ | .  | .  | .  | e₅ |
| .  | e₄ | e₅ | e₆ | .  | .  | e₇ |
| .  | .  | .  | .  | .  | .  | .  |

Parameters: \([e, \delta] \)  
\(\{\varepsilon^2 = 1, \delta^2 = 1\}\)

### [7, [6, 30], 1, 26]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| .  | .  | .  | .  | .  | 2e₁ | .  |
| .  | .  | .  | .  | .  | 2e₂ | .  |
| .  | e₁ | e₂ | e₃ | .  | .  | e₄ |
| .  | e₂ | e₃ | .  | .  | .  | e₅ |
| .  | e₃ | e₄ | e₅ | .  | .  | e₆ |
| .  | .  | .  | .  | .  | .  | e₇ |

Parameters: \([e, \delta] \)  
\(\{\varepsilon^2 = 1, \delta^2 = 1\}\)

### [7, [6, 30], 1, 27]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| .  | .  | .  | .  | .  | .  | .  |
| .  | .  | .  | .  | .  | e₁ | e₂ |
| .  | e₁ | e₂ | e₃ | .  | e₄ | e₅ |
| .  | e₂ | e₃ | .  | .  | e₅ | e₆ |
| .  | e₃ | e₄ | e₅ | .  | e₆ | e₇ |
| .  | .  | .  | .  | .  | .  | .  |

Parameters: \([e] \)  
\(\{\varepsilon^2 = 1\}\)

### [7, [6, 31], 1, 1]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| .  | .  | .  | .  | .  | (a+b)e₁ | .  |
| .  | .  | .  | .  | .  | (2a+1)e₂ | .  |
| .  | e₁ | .  | .  | e₃ | b e₄ | e₅ |
| .  | e₂ | .  | .  | e₄ | (a+1)e₅ | e₆ |
| .  | e₃ | e₄ | a e₅ | .  | .  | e₇ |
| .  | .  | .  | .  | .  | .  | .  |

Parameters: \([a, b]\)  
\(\{\}\)  
\(\{\varepsilon^2 = 1\}\)

### [7, [6, 31], 1, 2]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| .  | .  | .  | .  | .  | (a+1)e₁ | .  |
| .  | .  | .  | .  | .  | 2e₂ | .  |
| .  | e₁ | .  | .  | e₃ | a e₄ | e₅ |
| .  | e₂ | .  | .  | e₄ | e₅ | e₆ |
| .  | e₃ | e₄ | e₅ | e₆ | .  | e₇ |
| .  | .  | .  | .  | .  | .  | .  |

Parameters: \([a] \)  
\(\{\}\)  
\(\{\varepsilon^2 = 1\}\)

### [7, [6, 31], 1, 3]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|
| .  | .  | .  | .  | .  | .  | e₁ |
| .  | .  | .  | .  | .  | e₂ | .  |
| .  | e₁ | .  | .  | e₃ | e₄ | e₅ |
| .  | e₂ | .  | .  | e₄ | e₅ | e₆ |
| .  | e₃ | e₄ | e₅ | e₆ | .  | e₇ |
| .  | .  | .  | .  | .  | .  | .  |

Parameters: \([e] \)  
\(\{\varepsilon^2 = 1\}\)
\begin{align*}
\text{parameters: } & [a, \epsilon] \\
\text{([}\epsilon^2 = 1]\text{)} \\
\text{[7, [6, 31], 1, 4]} \\
\begin{array}{cccccccc}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
e_1 & . & . & . & . & . & . & a e_1 \\
e_2 & . & . & . & . & . & . & e_2 \\
e_3 & . & . & . & . & . & . & a e_3 \\
e_4 & . & e_2 & e_4 & e_4 & e_4 & e_4 & e_4 \\
e_5 & . & . & . & e_4 & e_4 & e_4 & e_4 \\
e_6 & . & . & . & . & . & e_2 & e_2 \\
e_7 & . & . & . & . & . & . & .
\end{array}
\end{align*}

\begin{align*}
\text{parameters: } & [\epsilon] \\
\text{([}\epsilon^2 = 1]\text{)} \\
\text{[7, [6, 31], 1, 5]} \\
\begin{array}{cccccccc}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
e_1 & . & . & . & . & . & . & e_1 \\
e_2 & . & . & . & . & . & . & e_2 \\
e_3 & . & e_1 & . & e_3 & e_3 & e_3 & e_3 \\
e_4 & . & e_2 & . & . & . & . & e_4 \\
e_5 & . & . & . & e_4 & e_4 & e_4 & e_4 \\
e_6 & . & . & . & . & . & e_2 & e_2 \\
e_7 & . & . & . & . & . & . & .
\end{array}
\end{align*}

\begin{align*}
\text{parameters: } & [a] \\
\text{[7, [6, 31], 1, 6]} \\
\begin{array}{cccccccc}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
e_1 & . & . & . & . & . & . & 2 a e_1 \\
e_2 & . & . & . & (a+1) e_2 & . & . & e_2 \\
e_3 & . & . & . & a e_3 & . & . & e_3 \\
e_4 & . & . & . & (a+1) e_4 & . & . & e_4 \\
e_5 & . & . & e_4 & e_4 & e_4 & e_4 & e_4 \\
e_6 & . & . & . & e_6 & e_6 & e_6 & e_6 \\
e_7 & . & . & . & . & . & . & .
\end{array}
\end{align*}

\begin{align*}
\text{parameters: } & [\epsilon] \\
\text{([}\epsilon^2 = 1]\text{)} \\
\text{[7, [6, 31], 1, 7]} \\
\begin{array}{cccccccc}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
e_1 & . & . & . & . & . & . & 2 e_1 \\
e_2 & . & . & . & . & . & . & 2 e_2 \\
e_3 & . & e_1 & . & e_3 & e_3 & e_3 & e_3 \\
e_4 & . & e_2 & e_4 & e_4 & e_4 & e_4 & e_4 \\
e_5 & . & e_4 & e_4 & e_4 & e_4 & e_4 & e_4 \\
e_6 & . & . & . & . & . & e_3 & e_3 \\
e_7 & . & . & . & . & . & . & .
\end{array}
\end{align*}

\begin{align*}
\text{parameters: } & [\epsilon] \\
\text{([}\epsilon^2 = 1]\text{)} \\
\text{[7, [6, 31], 1, 8]} \\
\begin{array}{cccccccc}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
e_1 & . & . & . & . & . & . & e_1 \\
e_2 & . & . & . & . & . & . & e_2 \\
e_3 & . & . & . & . & . & . & e_3 \\
e_4 & . & . & . & . & . & . & e_4 \\
e_5 & . & . & . & . & . & . & e_5 \\
e_6 & . & . & . & . & . & . & e_6 \\
e_7 & . & . & . & . & . & . & .
\end{array}
\end{align*}

\begin{align*}
\text{parameters: } & [a] \\
\text{[7, [6, 31], 1, 9]} \\
\begin{array}{cccccccc}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
e_1 & . & . & . & . & . & . & e_1 \\
e_2 & . & . & . & . & . & . & 2 a e_2 \\
e_3 & . & . & . & (1-a) e_3 & . & . & e_3 \\
e_4 & . & . & . & . & . & . & e_4 \\
e_5 & . & . & . & a e_5 & . & . & e_5 \\
e_6 & . & . & . & . & . & . & e_6 \\
e_7 & . & . & . & . & . & . & .
\end{array}
\end{align*}

\begin{align*}
\text{parameters: } & [\epsilon] \\
\text{([}\epsilon^2 = 1]\text{)} \\
\text{[7, [6, 31], 1, 10]} \\
\begin{array}{cccccccc}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
e_1 & . & . & . & . & . & . & e_1 \\
e_2 & . & . & . & . & . & . & e_2 \\
e_3 & . & . & . & . & . & . & e_3 \\
e_4 & . & . & . & . & . & . & e_4 \\
e_5 & . & . & . & . & . & . & e_5 \\
e_6 & . & . & . & . & . & . & e_6 \\
e_7 & . & . & . & . & . & . & .
\end{array}
\end{align*}

\begin{align*}
\text{parameters: } & [a] \\
\text{[7, [6, 31], 1, 11]} \\
\begin{array}{cccccccc}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
e_1 & . & . & . & . & . & . & 2 a e_3 \\
e_2 & . & . & . & . & . & . & 4 e_2 \\
e_3 & . & . & . & . & . & . & e_3 \\
e_4 & . & . & . & . & . & . & e_4 \\
e_5 & . & . & . & . & . & . & e_5 \\
e_6 & . & . & . & . & . & . & e_6 \\
e_7 & . & . & . & . & . & . & .
\end{array}
\end{align*}
|  | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | . . . . . . | $3 e_2$ | $e_3$ | . . | $e_4$ | . | $e_5 + e_6$ |
| $e_2$ | . . . . . . | . | $e_5$ | . | $e_6$ | . | . |
| $e_3$ | $a e_3$ | $e_2 e_4$ | . | . | . | . | . |
| $e_4$ | $a e_4$ | $e_5$ | . | . | . | . | . |
| $e_5$ | . | $e_6$ | . | . | . | . | . |
| $e_6$ | . | . | . | . | . | . | . |
| $e_7$ | . | . | . | . | . | . | . |

parameters: $[a]$

[[ ]]

|  | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | . . . . . . | $3 a e_1$ | $e_2$ | . | $e_3$ | . | $e_4 + 2 e_3$ |
| $e_2$ | . . . . . . | $a e_2$ | $e_5$ | . | $e_6$ | . | . |
| $e_3$ | $e_4 e_5$ | . | $e_6$ | . | $e_7$ | . | . |
| $e_4$ | . | $e_7$ | . | . | . | . | . |
| $e_5$ | . | . | . | . | . | . | . |
| $e_6$ | . | . | . | . | . | . | . |
| $e_7$ | . | . | . | . | . | . | . |

parameters: $[\epsilon]$

[[ ]]
### [7, [6, 31], 1, 20]

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-----------------------------|
| e₁ | . . . . . . . . 4e₁ |
| e₂ | . . . . . . 3e₂ |
| e₃ | . . e₁ . e₂+3e₃ |
| e₄ | . e₂ . 2e₄ |
| e₅ | . e₄ e₅+e₆ |
| e₆ | . e₆ |
| e₇ | . |

Parameters: [a, b]

### [7, [6, 31], 1, 21]

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-----------------------------|
| e₁ | . . . . . . . . ae₁ |
| e₂ | . . . . . . e₂ |
| e₃ | . . e₁ . e₁+ae₃ |
| e₄ | . e₂ . e₄ |
| e₅ | . e₄ . |
| e₆ | . be₂+e₆ |
| e₇ | . |

Parameters: [a, b]

### [7, [6, 31], 1, 22]

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-----------------------------|
| e₁ | . . . . . . . . e₁ |
| e₂ | . . . . . . e₂ |
| e₃ | . . e₁ . e₁+e₃ |
| e₄ | . e₂ . e₄ |
| e₅ | . e₄ . |
| e₆ | . ae₂ |
| e₇ | . |

Parameters: [a]

### [7, [6, 31], 1, 23]

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-----------------------------|
| e₁ | . . . . . . . . . |
| e₂ | . . . . . . e₂ |
| e₃ | . . e₁ . e₁ |
| e₄ | . e₂ . e₄ |
| e₅ | . e₄ e₅ |
| e₆ | . ae₂+e₆ |
| e₇ | . |

Parameters: [a]

### [7, [6, 31], 1, 24]

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-----------------------------|
| e₁ | . . . . . . . . e₁ |
| e₂ | . . . . . . e₂ |
| e₃ | . . e₁ . e₁+e₃ |
| e₄ | . e₂ . e₄ |
| e₅ | . e₄ e₅ |
| e₆ | . aae₂ |
| e₇ | . |

Parameters: [a]

### [7, [6, 31], 1, 25]

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-----------------------------|
| e₁ | . . . . . . . . (a+1)e₁ |
| e₂ | . . . . . . (2a+1)e₂ |
| e₃ | . . e₁ . e₃ |
| e₄ | . e₂ . e₄ |
| e₅ | . e₄ aae₅ |
| e₆ | . e₃+e₆ |
| e₇ | . |

Parameters: [a]

### [7, [6, 31], 1, 26]

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-----------------------------|
| e₁ | . . . . . . . . e₁ |
| e₂ | . . . . . . 2e₂ |
| e₃ | . . e₁ . e₃ |
| e₄ | . e₂ . e₄ |
| e₅ | . e₄ e₅ |
| e₆ | . e₃ |
| e₇ | . |

Parameters: [a]

### [7, [6, 31], 1, 27]

| e₁, e₂, e₃, e₄, e₅, e₆, e₇ |
|-----------------------------|
| e₁ | . . . . . . . . e₁ |
| e₂ | . . . . . . e₂ |
| e₃ | . . e₁ . e₁+e₃ |
| e₄ | . e₂ . e₄ |
| e₅ | . e₄ e₅ |
| e₆ | . e₃+e₆ |
| e₇ | . |

Parameters: [c]

[c² = 1]
### Parameters:

- \([a, b] \quad ([r^2 = 1])\)

### Equations

#### 1. \([7, [6, 31], 1, 28]\)

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
|---|---|---|---|---|---|---|
| 2\(e_1\) | \(3e_2\) | \(e_1\) | \(-e_1 + 2e_4\) | \(e_4\) | \(e_5 + e_6\) | \(e_3 + e_6\) |

#### 2. \([7, [6, 31], 1, 29]\)

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
|---|---|---|---|---|---|---|
| \((2a + 1)e_1 + e_2\) | \((2a + 1)e_2\) | \(e_1\) | \((a + 1)e_3 + e_4\) | \(e_2\) | \(ae_5\) | \(e_6\) |

#### 3. \([7, [6, 31], 1, 30]\)

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
|---|---|---|---|---|---|---|
| \(2e_1 + e_2\) | \(2e_2\) | \(e_1\) | \(e_3 + e_4\) | \(e_2\) | \(e_4\) | \(e_6\) |

#### 4. \([7, [6, 31], 1, 31]\)

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
|---|---|---|---|---|---|---|
| \(2e_1 + e_2\) | \(2e_2\) | \(e_1\) | \(e_3 + e_4\) | \(e_2\) | \(e_4\) | \(e_6\) |

#### 5. \([7, [6, 31], 1, 32]\)

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
|---|---|---|---|---|---|---|
| \(e_1 + e_2\) | \(e_2\) | \(e_1\) | \(e_3 + e_4\) | \(e_2\) | \(e_4\) | \(e_6\) |

#### 6. \([7, [6, 31], 1, 33]\)

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
|---|---|---|---|---|---|---|
| \(3e_1 + e_2\) | \(5e_2\) | \(e_1\) | \(2e_3 + e_4\) | \(e_2\) | \(e_6\) | \(ae_1 + e_3 + e_6\) |

#### 7. \([7, [6, 31], 1, 34]\)

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
|---|---|---|---|---|---|---|
| \(e_1 + e_2\) | \(e_2\) | \(e_1\) | \(e_3 + e_4\) | \(e_2\) | \(e_4\) | \(ae_1 + e_3 + e_6\) |

#### 8. \([7, [6, 32], 1, 1]\)

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
|---|---|---|---|---|---|---|
| \(ae_1\) | \((b + 1)e_2\) | \(e_1\) | \((a - 1)e_4\) | \(e_2\) | \(be_6\) | |

#### Parameters:

- \([a, b] \quad ([r^2 = 1])\)

- \([-1 \leq b, b \leq 1]\)
### [7, [6, 32], 1, 2]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|-----|-----|-----|-----|-----|-----|-----|
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |

Parameters: [a]

### [7, [6, 32], 1, 3]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|-----|-----|-----|-----|-----|-----|-----|
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |

Parameters: [a] (with additional parameters)

### [7, [6, 32], 1, 4]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|-----|-----|-----|-----|-----|-----|-----|
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |

Parameters: [a] (with additional parameters)

### [7, [6, 32], 1, 5]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|-----|-----|-----|-----|-----|-----|-----|
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |

Parameters: [a] (with additional parameters)

### [7, [6, 32], 1, 6]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|-----|-----|-----|-----|-----|-----|-----|
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |

Parameters: [a] (with additional parameters)

### [7, [6, 32], 1, 7]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|-----|-----|-----|-----|-----|-----|-----|
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |

Parameters: [a] (with additional parameters)

### [7, [6, 32], 1, 8]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|-----|-----|-----|-----|-----|-----|-----|
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |

Parameters: [a] (with additional parameters)

### [7, [6, 32], 1, 9]

| e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|-----|-----|-----|-----|-----|-----|-----|
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |
|  1  |  1  |  1  |  1  |  1  |  1  |  1  |

Parameters: [a] (with additional parameters)
### [7, [6, 32], 1, 10]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇  |
|---|----|----|----|----|----|----|-----|
| e₁ | .   | .   | 2e₁|    |     |     |     |
| e₂ | .   | .   | .   | e₂ |     |     |     |
| e₃ | .   | .   | e₁ | 2e₃|     |     |     |
| e₄ | .   | e₁  | e₂ + e₄|    |     |     |     |
| e₅ | .   | e₂  | e₄ + e₅|    |     |     |     |
| e₆ | .   |     |     |     |     |     |     |
| e₇ | .   |     |     |     |     |     |     |

### [7, [6, 32], 1, 11]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇  |
|---|----|----|----|----|----|----|-----|
| e₁ | .   | .   | 4e₁|    |     |     |     |
| e₂ | .   | .   | 3e₂|    |     |     |     |
| e₃ | .   | e₁ | 2e₃|    |     |     |     |
| e₄ | .   | e₁ | e₂ + 3e₄|    |     |     |     |
| e₅ | .   | e₂  | e₅|     |     |     |     |
| e₆ | .   | e₃ + 2e₆|    |     |     |     |     |
| e₇ | .   |     |     |     |     |     |     |

### [7, [6, 32], 1, 12]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇  |
|---|----|----|----|----|----|----|-----|
| e₁ | .   | .   | (a+1)e₁|    |     |     |     |
| e₂ | .   | .   | (a+1)e₂|    |     |     |     |
| e₃ | .   | .   | e₃|     |     |     |     |
| e₄ | .   | e₁ | a₄|     |     |     |     |
| e₅ | .   | e₂  | e₃ + e₅|    |     |     |     |
| e₆ | .   | e₄ + a₆|    |     |     |     |     |
| e₇ | .   |     |     |     |     |     |     |

Parameters: [a]  
\([-1 \leq a, a \leq 1]\)

### [7, [6, 32], 1, 13]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇  |
|---|----|----|----|----|----|----|-----|
| e₁ | .   | .   | e₁|    |     |     |     |
| e₂ | .   | .   | .  | e₂ |     |     |     |
| e₃ | .   | .   | e₁| .   |     |     |     |
| e₄ | .   | e₁  | e₂ + e₄|    |     |     |     |
| e₅ | .   | e₂  | e₃|     |     |     |     |
| e₆ | .   | e₄ + e₆|    |     |     |     |     |
| e₇ | .   |     |     |     |     |     |     |

### [7, [6, 32], 1, 14]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇  |
|---|----|----|----|----|----|----|-----|
| e₁ | .   | .   | (a+1)e₁|    |     |     |     |
| e₂ | .   | .   | e₁ + (a+1)e₂|    |     |     |     |
| e₃ | .   | .   | e₁|     |     |     |     |
| e₄ | .   | e₁  | a₄|     |     |     |     |
| e₅ | .   | e₂  | b₃ + e₈|    |     |     |     |
| e₆ | .   | .   | (b-1)e₄ + a₆|    |     |     |     |
| e₇ | .   |     |     |     |     |     |     |

Parameters: [a, b]  
\([-1 \leq a, a \leq 1]\)

### [7, [6, 32], 1, 15]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇  |
|---|----|----|----|----|----|----|-----|
| e₁ | .   | .   | e₁|    |     |     |     |
| e₂ | .   | .   | e₁ + e₂|    |     |     |     |
| e₃ | .   | .   | e₁|     |     |     |     |
| e₄ | .   | e₁  | e₄|     |     |     |     |
| e₅ | .   | e₂  | a₃|     |     |     |     |
| e₆ | .   | e₂ + (a-1)e₄ + e₆|    |     |     |     |     |
| e₇ | .   |     |     |     |     |     |     |

Parameters: [a]  
\([\text{empty}]\)

### [7, [6, 32], 1, 16]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇  |
|---|----|----|----|----|----|----|-----|
| e₁ | .   | .   | 2e₁|    |     |     |     |
| e₂ | .   | .   | e₁ + 2e₂|    |     |     |     |
| e₃ | .   | .   | e₁|     |     |     |     |
| e₄ | .   | e₁  | e₄|     |     |     |     |
| e₅ | .   | e₂  | a₃ + e₄ + e₅|    |     |     |     |
| e₆ | .   | (a-1)e₄ + e₆|    |     |     |     |     |
| e₇ | .   |     |     |     |     |     |     |

Parameters: [a]  
\([\text{empty}]\)
### [7, [6, 32], 1, 17]

$$\begin{array}{cccccc}
\text{e}_1 & \text{e}_2 & \text{e}_3 & \text{e}_4 & \text{e}_5 & \text{e}_7 \\
\text{e}_1 & \ldots & \ldots & \ldots & 2\text{e}_1 \\
\text{e}_2 & \ldots & \ldots & \ldots & \text{e}_1+2\text{e}_2 \\
\text{e}_3 & \ldots & \ldots & \text{e}_1 & \text{e}_3 \\
\text{e}_4 & \ldots & \text{e}_1 & \ldots & \text{e}_4 \\
\text{e}_5 & \ldots & \text{e}_2 & \text{a}\text{e}_3+\text{b}\text{e}_4+\text{e}_5 \\
\text{e}_6 & \ldots & \text{e}_3+(a-1)\text{e}_4+\text{e}_6 \\
\text{e}_7 & \ldots & \ldots & \ldots & \ldots & \\
\end{array}$$

Parameters: $[a, b]$

### [7, [6, 32], 2, 1]

$$\begin{array}{cccccc}
\text{e}_1 & \text{e}_2 & \text{e}_3 & \text{e}_4 & \text{e}_5 & \text{e}_7 \\
\text{e}_1 & \ldots & \ldots & \ldots & \ldots & \text{b}\text{e}_1 \\
\text{e}_2 & \ldots & \ldots & \ldots & 2\text{a}\text{e}_2 \\
\text{e}_3 & \ldots & \ldots & \text{e}_1 & \text{e}_3+(b-a)\text{e}_4 \\
\text{e}_4 & \ldots & \text{e}_1 & \ldots & \text{b}\text{e}_4+\text{e}_5 \\
\text{e}_5 & \ldots & \text{e}_2 & \text{a}\text{e}_5+\text{e}_6 \\
\text{e}_6 & \ldots & \text{e}_3+\text{a}\text{e}_6 \\
\text{e}_7 & \ldots & \ldots & \ldots & \ldots & \\
\end{array}$$

Parameters: $[a, b]$

### [7, [6, 32], 1, 18]

$$\begin{array}{cccccc}
\text{e}_1 & \text{e}_2 & \text{e}_3 & \text{e}_4 & \text{e}_5 & \text{e}_7 \\
\text{e}_1 & \ldots & \ldots & \ldots & \ldots & \text{e}_1 \\
\text{e}_2 & \ldots & \ldots & \ldots & \text{e}_1+\text{e}_2 \\
\text{e}_3 & \ldots & \ldots & \ldots & \text{e}_1+\text{e}_3 \\
\text{e}_4 & \ldots & \ldots & \ldots & \ldots & \text{e}_4 \\
\text{e}_5 & \ldots & \text{e}_2 & \text{a}\text{e}_3+\text{e}_5 \\
\text{e}_6 & \ldots & \ldots & \ldots & \ldots & \ldots \\
\text{e}_7 & \ldots & \ldots & \ldots & \ldots & \\
\end{array}$$

Parameters: $[a]$
### $[7, [6, 33], 1, 5]$  

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| $e_1$ | .     | .     | .     | .     | .     | .     |
| $e_2$ | .     | .     | .     | .     | .     | .     |
| $e_3$ | .     | .     | .     | $2e_3$| .     | .     |
| $e_4$ | .     | $e_1$| $e_2$| .     | .     | .     |
| $e_5$ | $e_3$ | $e_5$ | .     | .     | .     | .     |
| $e_6$ | .     | .     | $e_1+e_6$| .     | .     | .     |
| $e_7$ | .     | .     | .     | .     | .     | .     |

### $[7, [6, 33], 1, 6]$  

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| $e_1$ | .     | .     | .     | .     | .     | .     |
| $e_2$ | .     | .     | .     | .     | .     | .     |
| $e_3$ | .     | .     | .     | $-e_3$| .     | .     |
| $e_4$ | .     | $e_1$| $e_2$| $e_4$ | .     | .     |
| $e_5$ | $e_3$ | $e_5$ | .     | .     | .     | .     |
| $e_6$ | .     | .     | $e_1+e_6$| .     | .     | .     |
| $e_7$ | .     | .     | .     | .     | .     | .     |

**Parameters:** $[e]$  

$[(e^2 = 1)]$

### $[7, [6, 33], 1, 7]$  

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| $e_1$ | .     | .     | .     | .     | .     | .     |
| $e_2$ | .     | .     | .     | .     | .     | .     |
| $e_3$ | .     | .     | .     | $2e_2$| .     | .     |
| $e_4$ | .     | $e_1$| $e_2$| $e_4$ | .     | .     |
| $e_5$ | $e_3$ | .     | .     | .     | .     | .     |
| $e_6$ | .     | .     | $e_1+e_3+e_6$| .     | .     | .     |
| $e_7$ | .     | .     | .     | .     | .     | .     |

**Parameters:** $[e]$  

$[(e^2 = 1)]$

### $[7, [6, 33], 1, 8]$  

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| $e_1$ | .     | .     | .     | .     | .     | .     |
| $e_2$ | .     | .     | .     | .     | .     | .     |
| $e_3$ | .     | .     | .     | $2e_3$| .     | .     |
| $e_4$ | .     | $e_1$| $e_2$| .     | .     | .     |
| $e_5$ | $e_3$ | $e_5$ | .     | .     | .     | .     |
| $e_6$ | .     | .     | $e_1+e_6$| .     | .     | .     |
| $e_7$ | .     | .     | .     | .     | .     | .     |

**Parameters:** $[a]$  

$([-2 \leq a, a \leq 2])$

### $[7, [6, 33], 2, 1]$  

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| $e_1$ | .     | .     | .     | .     | .     | .     |
| $e_2$ | .     | .     | .     | $(a+b)e_2-e_3$| .     | .     |
| $e_3$ | .     | .     | .     | $e_2+(a+b)e_3$| .     | .     |
| $e_4$ | .     | $e_1$| $e_2$| $ae_4-e_5$| .     | .     |
| $e_5$ | $e_3$ | $e_5$ | .     | .     | .     | .     |
| $e_6$ | .     | .     | $b e_6$| .     | .     | .     |
| $e_7$ | .     | .     | .     | .     | .     | .     |

**Parameters:** $[a, b]$  

$[0 \leq a]$

### $[7, [6, 33], 2, 2]$  

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| $e_1$ | .     | .     | .     | .     | .     | .     |
| $e_2$ | .     | .     | .     | $3ae_2-e_3$| .     | .     |
| $e_3$ | .     | .     | .     | $e_2+3ae_3$| .     | .     |
| $e_4$ | .     | $e_1$| $e_2$| $ae_4-e_5$| .     | .     |
| $e_5$ | $e_3$ | $e_5$ | .     | .     | .     | .     |
| $e_6$ | .     | .     | $e_1+2ae_6$| .     | .     | .     |
| $e_7$ | .     | .     | .     | .     | .     | .     |

**Parameters:** $[a]$  

$[0 \leq a]$

### $[7, [6, 33], 2, 3]$  

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| $e_1$ | .     | .     | .     | .     | .     | .     |
| $e_2$ | .     | .     | .     | $ae_2-e_3$| .     | .     |
| $e_3$ | .     | .     | .     | $e_2+ae_3$| .     | .     |
| $e_4$ | .     | $e_1$| $e_2$| $e_3+ae_4-e_5$| .     | .     |
| $e_5$ | $e_3$ | $e_5$ | .     | .     | .     | .     |
| $e_6$ | .     | .     | $e_4+ae_5$| .     | .     | .     |
| $e_7$ | .     | .     | .     | .     | .     | .     |

**Parameters:** $[a]$  

$[0 \leq a]$
### [7, [6, 33], 2, 4]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | .  | 2e₁ |
| e₂ | .  | .  | .  | .  | .  | -e₃ |    |
| e₃ | .  | .  | .  | e₂ |    |    |    |
| e₄ | .  | e₁ | e₂ | e₃ | -e₅|    |    |
| e₅ | .  | e₃ | e₄ |    |    |    |    |
| e₆ | .  | a  | e₅ |    |    |    |    |
| e₇ |    |    |    |    |    |    |    |

Parameters: [a]

### [7, [6, 33], 3, 1]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | .  | 2e₁ |
| e₂ | .  | .  | .  | .  | .  | (a+1)e₂|    |
| e₃ | .  | .  | .  | e₂+(a+1)e₃|    |    |    |
| e₄ | .  | e₁ | e₂ | e₄ |    |    |    |
| e₅ | .  | e₃ | e₄| +e₅|    |    |    |
| e₆ | .  | a  | e₅|    |    |    |    |
| e₇ |    |    |    |    |    |    |    |

Parameters: [a]

### [7, [6, 33], 3, 2]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | .  |    |
| e₂ | .  | .  | .  | .  | .  | e₂ |    |
| e₃ | .  | .  | .  | e₂+e₃|    |    |    |
| e₄ | .  | e₁ | e₂ | .  |    |    |    |
| e₅ | .  | e₃ | e₄ |    |    |    |    |
| e₆ | .  | e₆ |    |    |    |    |    |
| e₇ |    |    |    |    |    |    |    |

Parameters: [a]

### [7, [6, 33], 3, 3]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | 2e₁|    |
| e₂ | .  | .  | .  | .  | .  | e₂ |    |
| e₃ | .  | .  | .  | e₂+e₃|    |    |    |
| e₄ | .  | e₁ | e₂ | e₄ |    |    |    |
| e₅ | .  | e₃ | e₄| +e₅|    |    |    |
| e₆ | .  | .  |    |    |    |    |    |
| e₇ |    |    |    |    |    |    |    |

### [7, [6, 33], 3, 4]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | .  |    |
| e₂ | .  | .  | .  | .  | .  | e₂ |    |
| e₃ | .  | .  | .  | e₂+e₃|    |    |    |
| e₄ | .  | e₁ | e₂ | .  |    |    |    |
| e₅ | .  | e₃ | e₄|    |    |    |    |
| e₆ | .  | e₃| +e₆|    |    |    |    |
| e₇ |    |    |    |    |    |    |    |

### [7, [6, 33], 3, 5]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | 2e₁|    |    |
| e₂ | .  | .  | .  | .  | 3e₂|    |    |
| e₃ | .  | .  | .  | e₂+3e₃|    |    |    |
| e₄ | .  | e₁ | e₂ | e₄ |    |    |    |
| e₅ | .  | e₃ | e₄| +e₅|    |    |    |
| e₆ | .  | e₁+2e₆|    |    |    |    |    |
| e₇ |    |    |    |    |    |    |    |

### [7, [6, 33], 3, 6]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | 2e₁|    |
| e₂ | .  | .  | .  | .  | .  | e₂ |    |
| e₃ | .  | .  | .  | e₂+e₃|    |    |    |
| e₄ | .  | e₁ | e₂ | e₃+e₄|    |    |    |
| e₅ | .  | e₃ | a  | e₂+e₄+e₅|    |    |    |
| e₆ | .  | .  |    |    |    |    |    |
| e₇ |    |    |    |    |    |    |    |

Parameters: [a]

### [7, [6, 33], 4, 1]

|   | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|---|----|----|----|----|----|----|----|
| e₁ | .  | .  | .  | .  | .  | 2e₁|    |
| e₂ | .  | .  | .  | .  | .  | e₁+2e₂|    |
| e₃ | .  | .  | .  | e₂+2e₃|    |    |    |
| e₄ | .  | e₁ | e₂ | e₄ |    |    |    |
| e₅ | .  | e₃ | e₄| +e₅|    |    |    |
| e₆ | .  | e₅| +e₆|    |    |    |    |
| e₇ |    |    |    |    |    |    |    |