AN OPERATOR THEORETIC APPROACH TO THE PRIME NUMBER THEOREM

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Abstract. In this short note, we establish an operator theoretic version of the Wiener-Ikehara tauberian theorem, and point out how this leads to a new proof of the Prime number theorem that should be accessible to anyone with a basic knowledge of operator theory.

1. Introduction

We begin by stating a version of the Wiener-Ikehara tauberian theorem, due to Korevaar [2]. To this end, we remark that the distributional Fourier transform of \( v \in L^\infty(\mathbb{R}) \) such that \( \lim_{|x| \to \infty} v(x) = 0 \) is called a pseudo-function. Also, we denote the complex variable by \( s = \sigma + it \).

**Theorem 1.1.** Let \( S(t) \) be a non-decreasing function with support in \([0, \infty)\), and suppose that the Laplace transform

\[
\mathcal{L}S(s) = \int_0^\infty S(u)e^{-su}du
\]

exists for \( \sigma > 1 \), and, for some constant \( A \), let

\[
g(s) = \mathcal{L}S(s) - \frac{A}{s-1}.
\]

If \( g(s) \) coincides with a pseudo-function on every bounded interval on the abscissa \( \sigma = 1 \) then

\[
\lim_{u \to \infty} \frac{S(u)}{e^u} = A.
\]

Conversely, if this limit holds, then \( g \) extends to a pseudo-function on \( \sigma = 1 \).

We point out that Ikehara, a student of Wiener, originally established his tauberian theorem in order to find a simple analytic proof for the Prime number theorem.

Below, we state and prove an operator theoretic generalisation of the Wiener-Ikehara-Korevaar theorem. Since the machinery of operator theory allows us to avoid the delicate manipulations of limits required in Korevaar’s proof, our hope is that this will provide a more accessible route to the Prime number theorem for anyone with a basic familiarity of operator theory.

We also mention that J.-P. Kahane has a functional analytic proof of the Prime number theorem, which is rather ingenious [1].
To motivate our approach, we recall some ideas from [4]. Specifically, we define, for intervals $I \subset \mathbb{R}$ symmetric with respect to the origin, the following operator on $L^2(I)$ (which we consider as a subspace of $L^2(\mathbb{R})$):

$$W_I : f \mapsto \lim_{\epsilon \to 0^+} \frac{1}{\pi} \int_I f(\tau) \text{Re} \frac{\zeta(1 + \epsilon + i(t - \tau))}{1 + \epsilon + i(t - \tau)} \, d\tau,$$

where $\zeta(s) = \sum_{n \in \mathbb{N}} n^{-s}$ is the Riemann zeta function. It is well-known that $\frac{\zeta(s)}{s} = \frac{1}{s - 1} + \psi(s)$,

(1)

where $\psi$ is an entire function. Plugging (1) into the formula for $W_I$, and noting that the term $1/(s - 1)$ leads to the appearance of the Poisson kernel, we obtain

$$W_I f(t) = f(t) + \Psi_I f,$$

(2)

where $\Psi_I$ is readily seen to be a compact operator on $L^2(I)$. Since $\zeta(s)/s$ is the Laplace transform of $\pi_N(e^u)$, where $\pi_N$ denotes the counting function of the integers, it follows from Plancherel’s theorem that

$$W_I f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\pi_N(e^{iu})}{e^{iu}} \hat{f}(u)e^{int} \, du.$$

(3)

In particular, by the Fourier inversion formula, this means that

$$\Psi_I f(t) = W_I f(t) - f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \left( \frac{\pi_N(e^{iu})}{e^{iu}} - 1 \right) \hat{f}(u)e^{int} \, du.$$

(4)

This relation allows us to connect the operator theoretic properties of $W_I$ and $\Psi_I$ to the arithmetic properties of the counting function $\pi_N$. More generally, in [4], properties of the operators corresponding to counting functions $\pi_K$ for subsets $K \subset \mathbb{N}$ were studied. In particular, it was observed that the resulting operator is closely related to the frame operator for the set of vectors $\{n^{\pm i\tau - 1/2}\}_{n \in K}$ which appears in the study of Hilbert spaces of Dirichlet series (see also [4]).

We now state our main result, where $\text{Id}$ denotes the identity operator on $L^2(I)$.

**Theorem 1.2.** Let $S(x)$ be a non-decreasing function on $[0, \infty)$ and $I \subset \mathbb{R}$ an interval that is symmetric with respect to the origin. Suppose that the Laplace transform $G(s) := \mathcal{L}\{S(e^u)\}(s)$ exists for $\sigma > 1$, and for $\epsilon > 0$ consider the operators

$$W_{S,I,\epsilon} : f \in L^2(I) \longmapsto \frac{1}{\pi} \int f(\tau) \text{Re} G\left(1 + \epsilon + i(t - \tau)\right) \, d\tau.$$

Then, for every $I$ sufficiently large, we have that as $\epsilon \to 0^+$, the operators $W_{S,I,\epsilon}$ converge in the weak operator norm to an operator $W_{S,I}$ satisfying

$$W_{S,I} = A\text{Id} + \Psi_{S,I},$$

for some constant $A$ and compact operator $\Psi_{S,I}$ on $L^2(I)$, if and only if

$$\lim_{u \to \infty} \frac{S(e^u)}{e^u} = A.$$
Before discussing the proof of the above result, we point out to the reader how the Prime number theorem follows from Theorem 1.2. This argument should be clear to anyone familiar with Ikehara’s theorem, but we include it to keep the note self-contained.

**Corollary 1.3. (The Prime number theorem)** Let \( \pi_P(x) \) be the counting function for the prime numbers. Then

\[
\lim_{x \to \infty} \pi_P(x) \cdot \frac{\ln x}{x} = 1.
\]

**Proof.** Let \( \zeta_P(s) = \sum_{p \text{ prime}} p^{-s} \). Then

\[
\frac{\zeta_P(s)}{s} = \mathcal{L}\{\pi_P(e^u)\}(s),
\]

By taking the logarithm of the Euler product formula, and using a first order Taylor approximation on the terms \( \log(1 - p^{-s}) \), we find that

\[
\log \zeta(s) = \zeta_P(s) + \sum_{p \text{ prime}} O(p^{-2s}).
\]

In combination with (1), this yields the well-known formula

\[
\frac{\zeta_P(s)}{s} = \log \frac{1}{s - 1} + \psi_P(s),
\]

where \( \psi_P(s) \) is analytic in a neighbourhood of \( \{\Re s > 1\} \).

We now combine formulas (5) and (6), and differentiate, to obtain the relation

\[
\mathcal{L}\{u\pi_P(e^u)\} = \frac{1}{s - 1} - \psi_P(e^u).
\]

Finally, by the same reasoning used to deduce (2) from (1), this implies that for the choice \( S(x) = \pi_P(x) \ln x \), we have

\[
W_{S,I}f(t) = \Id + \Psi_{S,I}f,
\]

where \( \Psi_{S,I} \) is a compact operator on \( L^2(I) \) for all bounded and symmetric intervals \( I \). By Theorem 1.2, the Prime number theorem now follows. \( \Box \)

2. **Proof of theorem 1.2**

We first suppose that \( S(e^u)/e^u \to A \) as \( u \to \infty \). In the same way that we arrived at (4), we have

\[
\Psi_{S,I}f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \left( \frac{S(e^{iu})}{e^{iu}} - A \right) \hat{f}(u)e^{iut}du.
\]

Since \( \Psi_{S,I} \) is an operator on \( L^2(I) \), where \( I \) is a bounded interval, and \( h(u) \) decays, it readily follows that \( \Psi_{S,I} \) is compact (see, e.g., Lemma 2 in [4]).
We now consider the converse implication (which is the one needed to prove the Prime number theorem). To this end, let $S$ be a fixed non-decreasing function, and suppose that $W_{S,I} = A\text{Id} + \Psi_{S,I}$ for a compact operator $\Psi_{S,I}$ for $I$ large enough.

First, we show that $\Psi_{S,I}$ bounded implies that $g(u) := S(e^u)/e^u$ is bounded. Letting $\{e_n\}_{n \in \mathbb{Z}}$ denote the standard orthonormal exponential basis for $L^2(I)$, we compute

$$\langle W_{S,I}\epsilon_n, \epsilon_n \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} g(u)|I| \left( \frac{\sin((u|I|)/2)}{|u|/2 - \pi n} \right)^2 e^{-|u|} du.$$  

To obtain a contradiction, suppose that $g(u)$ is not bounded. Then there exists a sequence $u_k$ of positive numbers tending to infinity, such that $g(u_k) \geq k$. But since $S$ is non-decreasing, we get, for $\Delta u > 0$, that

$$g(u + \Delta u) = \frac{S(e^{|u_k| + \Delta u})}{e^{u_k + \Delta u}} \geq \frac{S(e^{u_k})}{e^{u_k}} e^{-\Delta u} \geq \frac{k}{e^{\Delta u}}.$$ 

It now follows that, for sufficiently large $I$, the sequence $\langle W_{S,I}\epsilon_n, \epsilon_n \rangle$ is unbounded, which is absurd.

Next, we recall that if $\Psi_{S,I}$ is compact then

$$\langle \Psi_{S,I}\epsilon_n, \epsilon_n \rangle \to 0 \quad \text{as} \quad n \to \infty.$$ 

Using the fact that $g(u)$, and therefore $h(u)$, defined as in the first part of the proof, is bounded, a straight-forward computation gives

$$\langle \Psi_{S,I}\epsilon_n, \epsilon_n \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} h(u)|I| \left( \frac{\sin((u|I|)/2)}{|u|/2 - \pi n} \right)^2 du. \quad (8)$$ 

To arrive at a contradiction, suppose that the limit $h(u) \to 0$ does not hold as $|u| \to \infty$. There are now two cases. In the first, we suppose that there exists an $\epsilon > 0$ and an unbounded sequence $u_k$ of real numbers so that

$$h(u_k) = \frac{S(e^{|u_k|})}{e^{u_k}} - A \geq \epsilon.$$ 

As above, it follows that there exists a fixed $\Delta u > 0$ so that for all $k \in \mathbb{N}$ and $u \in [u_k, u_k + \Delta u]$ we have

$$h(u) \geq \frac{\epsilon}{2}.$$ 

It is now straight-forward to apply this estimate to the integral expression in (8) to see that for all $I$ large enough, there exists a constant $c = c(I) > 0$ so that for infinitely many $n$, we have

$$| \langle \Psi_{S,I}\epsilon_n, \epsilon_n \rangle | \geq c.$$ 

In the remaining case, we suppose that there exist an unbounded sequence of real numbers $u_k$ so that $h(u) \leq -\epsilon$. This case is settled as above with the adjustment that we consider intervals of the type $[u_k - \Delta u, u_k]$. $\square$
3. Remarks

While working on a manuscript containing these results, the author was made aware of similar efforts by Franz Luef and Eirik Skrettingland, who, in a rather technical paper, obtain a generalised version of Wiener’s tauberian theorem for operators (see [3]). Curiously enough, the results of Luef and Skrettingland only apply when the underlying space is $L^2(\mathbb{R}^n)$ for even $n$, and therefore do not seem to imply the results in this note. We plan to investigate the relation between these Tauberian theorems in a forthcoming publication.

References

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