Magnetized topological black holes of dimensionally continued gravity

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ABSTRACT: In this paper, a large family of topological black hole solutions of dimensionally continued gravity are derived. The action of Lovelock gravity is coupled to the exponential electrodynamics and the equations of motion are solved in the presence of a pure magnetic source. We work out the metric functions in terms of parameter $\beta$ of exponential electrodynamics, and magnetic charge. Further, we couple Lovelock gravity to power-Yang-Mills theory and construct black holes, in diverse dimensions, having Yang-Mills magnetic charge. The thermodynamics of resulting magnetized black hole solutions in the framework of two different models is also studied. The thermodynamical quantities like Hawking temperature, entropy and specific heat capacity at constant charge are found. We also calculate the free energy density and the total mass of the black hole and show that the resulting thermodynamical quantities satisfy the first law of black hole thermodynamics. We also study the magnetized hairy black holes of dimensionally continued gravity.
1 Introduction

Since the theory of general relativity is non-renormalizable, therefore, higher derivative gravitational theories have been investigated, because the higher derivative corrections to the familiar Einstein’s theory produce a power-counting renormalizable theory [1]. Further, modifying gravity gives an alternative way to understand the acceleration and expansion of the universe without the introduction of dark energy in the model. Thus, this is another reason to study higher derivative theories. Among the different higher derivative modified theories, Lovelock gravity [2], which contains dimensionally continued Euler characteristics, has a unique property that, in four dimensions, it reduces to general relativity. The field equations corresponding to the Lovelock gravity contain only the metric and its first two derivatives, thus the linearized form of this theory is free of ghosts. The second order Lovelock gravity, i.e., the Gauss-Bonnet gravity emerges in string theory in the low energy limit [3, 4]. However, due to the presence of a lot of Lovelock coefficients in Lovelock gravity, it is very difficult to interpret the physical meaning of the solution, therefore Banados, Teitelboim and Zanelli [5] proposed a suitable choice of these coefficients which allows us to write the solution in an explicit form. This theory of gravity that is obtained from this particular choice of Lovelock coefficients is known as dimensionally continued gravity (DCG). In the literature [5–7] neutral and charged black hole solutions of DCG have been studied. The hairy black holes of DCG have also been constructed recently [8, 9].

It has been shown that the standard Maxwell’s theory is not always workable for studying electromagnetic fields. In 1934 nonlinear electrodynamics was proposed by Born and Infeld which has the property of cancelling the divergences of electron’s self-energy [10]. Later in 1936, Heisenberg and Euler also put forward a nonlinear electromagnetic theory to explain the quantum electrodynamics phenomena [11]. The Born-Infeld electrodynamics could also be reproduced in the framework of string theory [12]. The action of Born-Infeld theory also governs the dynamics of D3-branes [13]. Other theories, that have recently been found from a particular form of Born-Infeld theory, include Dirac-Born-Infeld inflation theory and Eddington-inspired Born-Infeld theory [14, 15]. These theories have also been used in the study of dark energy, holographic entanglement entropy and holographic superconductors [16–18]. The first black hole solution of Einstein’s theory with Born-Infeld electrodynamical source was given by Hoffmann [19]. Subsequently, many spherical black hole solutions which are asymptotic to the Reissner-Nordström solution were derived with other nonlinear electrodynamical sources [20, 21]. The exponential electrodynamics model [22] is also used to construct the asymptotic Reissner-Nordström black hole solution having magnetic charge.

Black hole solutions of modified gravities have also been studied in the framework of nonlinear electromagnetic theory, for example, the electrically charged black hole solutions of DCG have been constructed [23] where Born-Infeld electromagnetic field is taken as a source of gravity. In this work we derive magnetized black hole solutions of DCG coupled with exponential electrodynamics and then generalize our solution to black holes which contain scalar hair.

More recently black hole solution has been found where the nonlinear electromagnetic
source is expressed in powers of Maxwell’s invariant \((F_{\mu\nu}F^{\mu\nu})^q\), where \(q\) is an arbitrary real number \([24]\). The nonlinearity involved in this power-Maxwell theory is radically different from the familiar Born-Infeld electrodynamics. One property of power-Maxwell formalism is that for the special case of \(q = d/4\), where \(d\) represents the dimension of spacetime, it yields the traceless matter tensor which indicates the satisfaction of conformal invariance. Instead of considering the power-Maxwell theory, we investigate black hole solutions with power-Yang-Mills source \([25]\) in this paper, that is, we can choose the source of gravity as \((F^{(a)}_{\mu\nu}F^{(a)\mu\nu})^q\), where \(F^{(a)}_{\mu\nu}\) denotes the Yang-Mills field with \(1 \leq a \leq \frac{1}{2}(d-1)(d-2)\) and \(q\) is a real number. In Ref. \([25]\) Lovelock black holes with a power-Yang-Mills source have been studied and magnetically charged solutions are obtained. In this paper we construct black hole solutions and hairy black hole solutions of DCG in the framework of power-Yang-Mills field.

Thermodynamics of black holes is an interesting aspect of the subject that attracts much attention not only in linear Maxwell’s theory but also in the nonlinear theories \([26, 27]\). For instance, thermodynamics of spherically symmetric black holes with exponential electromagnetic source has been discussed \([22]\). Similarly, thermodynamics of Lovelock black holes in the framework of power-Yang-Mills theory has been studied \([25]\). In our work we also study thermodynamics of the resulting black hole solutions in DCG within the two different models.

The plan of the paper is as follows. In Section 2, the exponential electrodynamics model is coupled with Lovelock gravity and we find a family of magnetized black hole solutions of DCG depending on the parameter \(\beta\) of exponential electrodynamics. In this section the metric functions are calculated and we study thermodynamics of black holes with exponential electrodynamical source. We also work out the hairy black hole solution of DCG within this model. In Section 3, we couple Lovelock gravity to power-Yang-Mills theory and obtain the magnetized black hole solutions with and without scalar hair of DCG. We also study thermodynamics of these black hole solutions. In Section 4, the Yang-Mills hierarchies are discussed. Finally we conclude our results in Section 5.

2 Topological black holes of DCG coupled to exponential electrodynamics

2.1 Magnetized black hole solution

The action function for Lovelock gravity with exponential electromagnetic source \([22]\) in diverse dimensions is written in the form

\[
I = \frac{1}{16\pi G} \int d^d x \sqrt{-g} \left[ \sum_{p=0}^{n-1} \frac{\alpha_p}{2^p} \delta^{[\mu_1...\mu_{2p}]} R_{\mu_1\mu_2}...R_{\nu_2\nu_{2p}}^{\nu_1...\nu_{2p}} + L(F) \right],
\]

where

\[
L(F) = -F \exp (-\beta F),
\]

and \(F = F_{\mu\nu}F^{\mu\nu} = (B^2 - E^2)/2\), where \(\beta\) is the parameter of our model, \(B\) is the magnetic field and \(E\) is the electric field. Here \(G\) is the Newtonian constant, \(\delta^{[\mu_1...\mu_{2p}]}\) is the generalized
Kronecker delta of order \(2p\) and \(d = n + 2\) where \(n\) is a natural number. The coefficients \(\alpha_p\) in (2.1) represent arbitrary constants, however, in the particular case of DCG, they are chosen in the form

\[
\alpha_p = C_p^{n-1} \frac{(d - 2p - 1)!}{(d - 2)!l^{2(n-p-1)}},
\]

where \(l\) is related to cosmological constant. It is worth noting that DCG becomes Born-Infeld theory in even dimensions and Chern-Simons theory in odd dimensions [5, 6]. Varying (2.1) with respect to the electromagnetic potential \(A_\mu\) yields the equations of motion of nonlinear electromagnetic field

\[
\partial_\mu \left[ \sqrt{-g} F^{\mu\nu} \left( 1 - \beta F \right) \exp \left( -\beta F \right) \right] = 0.
\]

After taking the variation of the action (2.1) with respect to the metric tensor, \(g_{\mu\nu}\), one can arrive at the equations of gravitational field

\[
\sum_{p=0}^{n-1} \frac{\alpha_p}{2p+1} \delta_{\mu\rho_1...\rho_p} R^{\rho_1\rho_2}...R^{\rho_{2p-1}\rho_{2p}} = T^\nu_\mu,
\]

where \(T^\nu_\mu\) represents the matter tensor given by

\[
T^\nu_\mu = \exp \left( -\beta F \right) \left[ (\beta F - 1) F^{\mu\lambda} \phi^\nu_\lambda + F g^{\mu\nu} \right].
\]

One can easily find the trace of the matter tensor in the form

\[
T = -4\beta F^2 \exp \left( -\beta F \right),
\]

which implies that for this model, the scale invariance is completely broken down, however, in the limit \(\beta \to 0\) one comes to the classical Maxwell’s electrodynamics because the trace of the matter tensor becomes zero.

Since we want to determine the magnetically charged static black hole solution, so we assume a pure magnetic field such that \(E = 0\), which yields from Maxwell’s invariant [22], the form \(F = (B(r))^2 / 2 = Q^2 / 2r^4\), where \(Q\) represents the magnetic charge. Now, our static and spherically symmetric line element is in the form

\[
ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 (h_{ij} dx^i dx^j),
\]

where \(h_{ij} dx^i dx^j\) is the metric of \(n\) dimensional hypersurface having constant curvature. Now using the above line element and substituting (2.6) in Eq. (2.5), the equation of motion becomes

\[
\frac{d}{dr} \left[ \sum_{p=0}^{n-1} \frac{\alpha_p(d - 2)!}{2(d - 2p - 1)!} r^{d-1} \left( \frac{k - f(r)}{r^2} \right)^p \right] = 32\pi G \exp \left( -\beta Q^2 \right) \left( \frac{\beta Q^4}{2r^{2d-4}} - \frac{Q^2}{2r^{2d-4}} \right),
\]
where $k = 0, 1, -1$ associated to the co-dimensions-2 hypersurface with planar, spherical and hyperbolic topology respectively. If we chose $\alpha$ from Eq. (2.3), that is, for the case of DCG, the above equation becomes

$$\frac{d}{dr} \left[ r^{d-1} \left( \frac{1}{l^2} + \frac{k - f(r)}{r^2} \right)^{n-1} \right] = 64\pi G \exp \left( -\frac{\beta Q^2}{2r^{2d-4}} \right) \left( \frac{\beta Q^4}{2r^{4d-8}} - \frac{Q^2}{2r^{2d-4}} \right).$$

Integration of the above equation with respect to $r$ yields

$$f(r) = \frac{r^2}{l^2} + k - r^2 \left[ \frac{16\pi G m}{r^{d-1}\Sigma_{d-2}} + \frac{\delta_d}{r^{d-1}} + \frac{\pi GQ}{(d-2)r^d} \left( \Gamma \left( \frac{2d - 5}{2d - 4}, \frac{\beta Q^2}{2r^{2d-4}} \right) - 2\Gamma \left( \frac{4d - 9}{2d - 4}, \frac{\beta Q^2}{2r^{2d-4}} \right) \right) \right]^{\frac{1}{d-1}},$$

where $m$ is a constant of integration which is associated to the ADM mass of black hole, $\Sigma_{d-2}$ represents the volume of $n$-dimensional hypersurface. The reason for the appearance of additional constant $\delta_d$ in Eq. (2.11) is that one can expect the horizon of the black hole to shrink to a single point when $m \to 0$ and the function $\Gamma(s, x)$ is the incomplete Gamma function.

Now let’s discuss the asymptotic behavior of metric function at $r = 0$. We take $k = 1$ here (the cases $k = 0, -1$ can be studied in a similar manner). Thus

$$f(r) = 1 + \frac{r^2}{l^2} - m^{\frac{1}{d-1}} \left[ r^{d-5} - \frac{\delta_d}{m(d-3)} r^{d-5} + \frac{\pi G}{m(d-3)(d-2)} \exp \left( -\frac{\beta Q^2}{2r^{2d-4}} \right) \right]
\times \left( \frac{32r^{d-4}(3 - d)}{\beta(d - 2)} - \frac{Q^2r^{2d-15}}{2d - 4} - \frac{Q^2r^{d-15}}{r^d} + O(r^{5d-12}) \right).$$

The above series expansion shows that metric function becomes infinite for $d = 2, 3$, however, for any value of $d \geq 4$ the metric functions are finite and regular for all values of $r$. Furthermore, the metric function (2.11) indicates that the resulting black hole solution is non-asymptotically flat, since in the limit $r \to \infty$, the metric function diverges.

### 2.2 Thermodynamics of black holes of DCG with exponential magnetic source

The event horizons can be found by solving the equation $f(r_h) = 0$. Thus from Eq. (2.11), we can write the ADM mass of the black hole in the form as

$$m = \frac{r_h^{d-1}\Sigma_{d-2}}{16\pi G} \left( \frac{1}{l^2} + \frac{k}{r_h^2} \right)^{d-3} - \frac{\delta_d}{16\pi G} - \frac{Q^{\frac{1}{d-2}}\Sigma_{d-2}^{\frac{5-2d}{d-2}}}{(d-2)2^{\frac{5-2d}{d-2}}}$$

$$\times \left[ \Gamma \left( \frac{2d - 5}{2d - 4}, \frac{\beta Q^2}{2r_h^{2d-4}} \right) - 2\Gamma \left( \frac{4d - 9}{2d - 4}, \frac{\beta Q^2}{2r_h^{2d-4}} \right) \right].$$

The Hawking temperature of the black hole is given by

$$T_H(r_h) = \frac{r_h}{2\pi l^2} - \frac{r_h}{2\pi} [G(r_h)]^{\frac{1}{d-1}} - \frac{r_h^2}{4\pi} [G(r_h)]^{\frac{2-d}{d-2}} W(r_h),$$
Differentiating (2.14) we get

\[ G(r_h) = \frac{16\pi Gm}{\Sigma_{d-2} r_h^{d-2}} + \frac{\delta_d}{r_h^{d-1}} + \frac{\pi GQ^{\frac{d-2}{2}}}{(d-2)} \frac{\beta^{\frac{2-d}{2}}}{2^2 \Sigma_{d-1} r_h^{d-1}} H(r_h), \tag{2.15} \]

\[ W(r_h) = \frac{16\pi Gm(1 - d)}{r_h^{d-2}} + \frac{(1 - d)\delta_d}{r_h^{d-1}} + \frac{\pi GQ^{\frac{d-2}{2}}(1 - d)\beta^{\frac{2-d}{2}}}{(d-2)2^2 \Sigma_{d-1} r_h^{d-1}} H(r_h) \]

\[ + \frac{16\pi GQ^2}{(d-2)r_h^{d-2}} \exp \left( -\frac{\beta Q^2}{2r_h^{2d-4}} \right) \left( (2d-4)r_h^{2d-4} + (4 - 2d)\beta^{\frac{3d-7}{2}} 2^2 \Sigma_{d-1} \right) \tag{2.16} \]

and

\[ H(r_h) = \Gamma \left( \frac{2d-5}{2d-4}, \frac{\beta Q^2}{2r_h^{2d-4}} \right) - 2\Gamma \left( \frac{4d-9}{2d-4}, \frac{\beta Q^2}{2r_h^{2d-4}} \right). \tag{2.17} \]

The Wald entropy is defined by

\[ S = -2\pi \int d^d x \sqrt{h} \frac{\partial L}{\partial R_{\mu\rho\lambda}} \epsilon_{\mu\nu}\epsilon_{\rho\lambda} \]

\[ = \frac{(d-3)\Sigma_{d-2} r_h^{d-2}}{4kG(6 - d)} \left( \frac{k}{r_h} + \frac{1}{l^2} \right) d^{-3} F_1 \left[ 1, \frac{d}{2} \frac{8 - d}{2}, \frac{-r_h^2}{k^2} \right], \tag{2.18} \]

where \( \epsilon_{\mu\nu} \) is the normal bivector of the \( t = \text{const} \) and \( r = r_h \) hypersurface such that \( \epsilon_{\rho\lambda}\epsilon^{\rho\lambda} = -2 \), \( F_1 \) is the Gaussian hypergeometric function. The magnetic potential conjugate to the magnetic charge \( Q \) is given by

\[ \Phi_m = \frac{\partial m}{\partial Q} = \frac{\Sigma_{d-2}}{(d-2)} \exp \left( -\frac{\beta Q^2}{2r_h^{2d-4}} \right) \left[ \frac{2Q}{r_h^{2d-5}} - \frac{2\beta Q^3}{r_h^{4d-9}} \right] - \frac{\Sigma_{d-2} Q^{\frac{d-2}{2}} \beta^{\frac{2-d}{2}}}{(d-2)2^2 \Sigma_{d-1}} H(r_h). \tag{2.19} \]

Using the thermodynamical quantities obtained above, we can now easily verify that the first law of thermodynamics \[28\]

\[ dm = T_H dS + \Phi_m dQ, \tag{2.20} \]

is satisfied. Now the specific heat capacity \[29–31\] at a constant charge \( Q \) is given by

\[ C_Q = T_H \frac{\partial S}{\partial T_H} \bigg|_Q. \tag{2.21} \]

Differentiating (2.14) we get

\[ \frac{\partial T_H}{\partial r_h} = \frac{1}{2\pi l^2} - \frac{G(r_h) \frac{1}{\pi}}{2\pi} - \frac{r_h}{2\pi(d-3)G(r_h) \frac{\beta^{\frac{2-d}{2}}}{2^2 \Sigma_{d-1}}} \frac{dG(r_h)}{dr_h} \]

\[ - \left( \frac{r_h}{2\pi} G(r_h) \frac{\beta^{\frac{2-d}{2}}}{2^2 \Sigma_{d-1}} + \frac{(d-2)r_h^2}{4\pi(3-d)G(r_h) \frac{\beta^{\frac{2-d}{2}}}{2^2 \Sigma_{d-1}}} \frac{dG(r_h)}{dr_h} \right) W(r_h) \]

\[ - \frac{r_h^2 G(r_h) \frac{\beta^{\frac{2-d}{2}}}{2^2 \Sigma_{d-1}}}{4\pi} \frac{dW(r_h)}{dr_h}, \tag{2.22} \]
where
\[
\frac{dW(r_h)}{dr_h} = \frac{16\pi G m (d-1)}{\Sigma_{d-2} r_h^{d+1}} + \frac{d(d-1)\delta_d}{r_h^{d+1}} + \frac{\pi G d (d-1) Q^{\frac{1-d}{2}} H(r_h)}{(d-2)^2 \frac{21 - 10d}{2d - 4} \beta^2 r_h^{d+1}} + \frac{\pi G (1 - d) Q^\frac{1}{d-2}}{(d-2)^2 \frac{21 - 10d}{2d - 4} r_h^{d+1}} \frac{dH(r_h)}{dr_h} + \frac{16\pi G Q^2 (2d - 4)^2}{(d-2)^3 r_h^{2d-3}} \exp \left( -\frac{\beta Q^2}{2 r_h^{2d-4}} \right) + \frac{8\pi G Q^4 \beta (2d - 4)^2}{r_h^{2d-11} (d-2)} - \frac{32\pi G Q^2 (2d - 4)^2}{(d-2)^4 r_h^{4d-7}} \right] \times \exp \left( -\frac{\beta Q^2}{2 r_h^{2d-4}} \right) \left( r_h^{2d-4} - \beta \frac{3d-7}{2d-2} Q^2 \frac{2d-14}{2d-2} \frac{9-4d}{2d-2} \right),
\]
(2.23)

\[
\frac{dG(r_h)}{dr_h} = \frac{16\pi G m (1 - d)}{\Sigma_{d-2} r_h^{d+1}} + \frac{(1 - d)\delta_d}{r_h^{d+1}} + \frac{\pi G Q^\frac{1}{d-2} \beta^2 r_h^{d+1}}{(d-2)^2 \frac{21 - 10d}{2d - 4} r_h^{d+1}} \left( (d-1)H(r_h) + r_h \frac{dH(r_h)}{dr_h} \right),
\]
(2.24)

and
\[
\frac{dH(r_h)}{dr_h} = \left( \frac{4 - 2d}{2d - 2} \frac{3d - 9}{2d - 4} \frac{Q^2}{r_h^{2d-8}} \right) \exp \left( -\frac{\beta Q^2}{2 r_h^{2d-4}} \right) \left[ 1 - \frac{\beta \frac{3d-7}{2d-2}}{Q^2} \right].
\]
(2.25)

Putting Eqs. (2.14), (2.18) and (2.22) in the general expression of heat capacity (2.21) we obtain
\[
C_Q = \left( \frac{r_h}{2\pi l^2} - \frac{r_h}{2\pi} [G(r_h)]^\frac{1}{d-2} - \frac{r_h^2}{4\pi} \frac{Z(r_h)}{G(r_h)} \right) \frac{(d-3) \Sigma_{d-2} r_h d Z(r_h)}{4G(6-d)\Sigma_{d-2} r_h d W(r_h)} \left( \frac{1}{r_h^2} + \frac{1}{l^2} \right)^{d-3},
\]
(2.26)

where
\[
Z(r_h) = F_1 \left[ 1, \frac{d}{2}, \frac{8 - d}{2}, \frac{r_h^2}{l^2} \right] \left( \frac{d}{r_h} + \frac{(d-3)r_h^2}{r_h^2 + l^2} \right) - \frac{2r_h l^2}{r_h^2 + l^2} F_1 \left[ 2, \frac{d + 2}{2}, \frac{10 - d}{2}, -\frac{r_h^2}{l^2} \right],
\]
(2.27)

and
\[
X(r_h) = \frac{1}{2\pi l^2} - \frac{G(r_h)}{2\pi} \frac{1}{d-3} - \frac{r_h}{2\pi (d-3) G(r_h)} \frac{dG(r_h)}{dr_h} - \left( \frac{r_h}{2\pi} G(r_h) \right)^\frac{d-2}{d-3}
\]
\[+ \frac{(d-2)r_h^2}{4\pi (3-d) G(r_h)} \frac{dG(r_h)}{dr_h} \] \left( \frac{r_h^2}{4\pi} \right) \frac{dW(r_h)}{dr_h}. \nn
(2.28)

The above Eq. (2.26) represents the general expression for black hole’s heat capacity for any value of nonlinear electrodynamical parameter $\beta$. The black hole is stable if the Hawking temperature and heat capacity are positive. The black hole becomes unstable in the region where Hawking temperature or heat capacity become negative. The point at which the sign of Hawking temperature changes corresponds to the first order phase transition of black hole. The maximum of Hawking temperature corresponds to the second order phase transition since the heat capacity at that point is singular.

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2.3 Hairy black holes of DCG with exponential magnetic source

The action describing Lovelock-Scalar gravity with nonlinear exponential electrodynamics source [32] is defined by

\[
I = \frac{1}{16\pi G} \int d^d x \sqrt{-g} \left[ \sum_{p=0}^{n-1} \frac{a_p}{2^p} \delta^\mu_{\nu_1} \ldots \delta^\mu_{\nu_{2p}} \left( a_p R^\mu_{\nu_1 \nu_2} \ldots R^{\mu_{2p-1} \nu_{2p}} \right) + 16\pi G b_p \phi^d \delta^\mu_{\nu_1} \ldots \delta^\mu_{\nu_{2p}} S_{\mu_1 \nu_2} \ldots S^{\mu_{2p-1} \nu_{2p}} \right] + 4\pi G L(F)
\]

where \(L(F)\) corresponds to the Lagrangian density describing the exponential electromagnetic field [22]. The second term in the integrand which contains \(\phi^d\) denotes the Lagrangian density of the scalar field, where

\[
S_{\mu \nu}^\sigma = \phi^2 R_{\mu \nu}^\sigma - 2 \delta^\sigma_{[\mu} \delta^\rho_{\nu]} \partial_\rho \phi \partial_\sigma \phi - 4 \phi \delta^\rho_{[\mu} \partial_\nu] \partial^\sigma \phi + 48 \delta^\rho_{[\mu} \partial_\nu \phi \partial^\sigma \phi.
\]

Note that, by putting the scalar function \(\phi\) equal to zero we will get the action function defined in (2.1). The matter tensor corresponding to the scalar field is given by

\[
[T_{\mu}^\nu(s)] = \sum_{p=0}^{d-3} \frac{b_p}{2^p+1} \phi^d \delta^\nu_{\lambda_1 \lambda_2 \ldots \lambda_{2p}} S_{\lambda_1 \lambda_2} \ldots S_{\lambda_{2p-1} \lambda_{2p}}.
\]

Thus the equation of motion for the scalar field becomes

\[
\sum_{p=0}^{d-3} \frac{b_p(d-2p)}{2^p} \phi^d \delta^\nu_{\lambda_1 \lambda_2 \ldots \lambda_{2p}} S_{\lambda_1 \lambda_2} \ldots S_{\lambda_{2p-1} \lambda_{2p}} = 0.
\]

Taking variation of (2.29) with respect to metric tensor, we get the field equations as

\[
\sum_{p=0}^{n-1} \frac{a_p}{2^p+1} \delta^\mu_{\nu_1} \ldots \delta^\mu_{\nu_{2p}} R_{\nu_1 \nu_2} \ldots R^{\mu_{2p-1} \nu_{2p}} = 16\pi G [T_{\mu}^\nu(M)] + 16\pi G [T_{\mu}^\nu(s)],
\]

where \([T_{\mu}^\nu(M)]\) corresponds to the matter tensor of exponential electromagnetic field and \(a_p\) corresponds to the value defined as in Eq. (2.3) so that the Lovelock gravity becomes DCG. If we take the scalar configuration as [32]

\[
\phi = \frac{N}{r},
\]

the equation describing the scalar field becomes

\[
\sum_{p=0}^{d-3} \frac{b_p[d(d-1)](d^2 - d + 4p^2)}{(d-2p-1)!} N^{-2p} = 0.
\]

Using the assumption of pure magnetic field and substituting Eqs. (2.31) and (2.32) in Eq. (2.33) we get

\[
f(r) = k + \frac{r^2}{l^2} - \frac{r^2}{l^2} \left[ \frac{16\pi G M}{\Sigma_{d-2} r^d - 1} + \frac{32\pi Y}{r^d} + \frac{\delta_d}{r^d - 1} \right]
+ \frac{\pi Q \beta^\frac{\pi}{4} \beta^5 \Gamma \left( \frac{2d-5}{2d-4}, \frac{\beta Q^2}{2d-4} \right)}{(d-2)^2} \left( \Gamma \left( \frac{4d-9}{2d-4}, \frac{\beta Q^2}{2d-4} \right) - \frac{1}{2} \right) \left( \frac{1}{2} \right),
\]

(2.36)
where we have
\[ Y = \sum_{p=0}^{d-3} b_p \frac{(d-2)!}{(d-2p-2)!} N^{d-2p}. \]  

(2.37)

Thus the line element (2.8) with metric function given by (2.36) describes the hairy black hole solution of DCG sourced by exponential electrodynamics. For the four dimensional case, it is seen from Eq. (2.35) that all the coefficients \( b_p \) vanish, thus in the case of \( d = 4 \), hairy black holes do not exist. It is easy to see that, by taking \( Y = 0 \), this black hole solution reduces to the black hole with no scalar hair which we derived earlier in Section 2.2.

3 Topological black holes of DCG coupled to power-Yang-Mills theory

3.1 Black hole solution with power-Yang-Mills source

The action function describing the Lovelock-power-Yang-Mills theory is given by
\[ I = \frac{1}{16\pi G} \int d^d x \sqrt{-g} \sum_{p=0}^{n-1} \frac{\alpha_p}{2^p} \delta^{\mu_1 \ldots \mu_{2p}}_{\nu_1 \ldots \nu_{2p}} R_{\mu_1 \mu_2 \ldots \nu_{2p-1} \nu_{2p}} + (\Upsilon)^q, \]  

(3.1)

where \( \Upsilon \) is the Yang-Mills invariant defined as
\[ \Upsilon = \sum_{a=1}^{n(n+1)/2} \left( F^{a}_a F^{(a)\lambda \sigma} \right), \]  

(3.2)

\( q \) is a positive real parameter and the Yang-Mills field is defined by
\[ F^{(a)} = dA^{(a)} + \frac{1}{2\eta} C^{(a)}_{(b)(c)} A^{(b)} \wedge A^{(c)}, \]  

(3.3)

where, \( C^{(a)}_{(b)(c)} \) represents the structure constants of \( \frac{n(n+1)}{2} \)-parameter Lie group \( G \), \( n = d - 2 \), \( \eta \) denotes the coupling constant and \( A^{(a)} \) are the \( So(n+1) \) gauge group Yang-Mills potentials. The structure constants have been determined in Ref. [33]. We should keep in mind, that the internal indices \([a, b, ...]\) make no difference whether we write them in contravariant or covariant form. Taking variation of action defined in (3.1) with respect to the metric tensor yields the Lovelock field equations (2.5) with matter tensor of Yang-Mills field given by
\[ T^{(a)\nu}_\mu = -\frac{1}{2} \left[ \delta^{\nu}_\mu \Upsilon^q - 4q Tr \left( F^{(a)\mu \lambda} \right) \Upsilon^{q-1} \right]. \]  

(4.4)

The equations of Yang-Mills field can be obtained if we vary the action with respect to gauge potentials \( A^{(a)} \)
\[ d(*F^{(a)}\Upsilon^{q-1}) + \frac{1}{\eta} C^{(a)}_{(b)(c)} \Upsilon^{q-1} A^{(b)} \wedge *F^{(c)} = 0, \]  

(3.5)

where \( * \) denotes the duality of a field. By taking the static metric in the form (2.8), it can easily be checked that power-Yang-Mills equations are satisfied for the choice of Yang-Mills potentials.
gauge potential one-forms [34, 35] defined as
\[ A^{(a)} = \frac{Q}{r^2} C_{(a)}^{(i)} x^i dx^j, \]
where \( Q \) represents the Yang-Mills magnetic charge and \( 2 \leq j + 1 \leq i \leq d - 1 \). Thus the symmetric matter tensor (3.4), with
\[ \Upsilon = \frac{n(n - 1)Q^2}{r^4} = \frac{(d - 2)(d - 3)Q^2}{r^4}, \]
becomes
\[ T^{(a)\nu} = -\frac{1}{2} \Upsilon^{ij} \text{diag}[1, 1, \zeta, \zeta, \ldots, \zeta], \zeta = \left( 1 - \frac{4q}{(d - 2)} \right). \] (3.8)

Thus using Eqs. (2.8) and (3.8) in the field equations (2.5), we get
\[ \frac{d}{dr} \left[ r^{d-1} \left( \frac{1}{r^2} + \frac{k - f(r)}{r^2} \right)^{d-3} \right] = -\frac{32\pi G Q^2 [d - 2]^{q} (d - 3)^q}{r^{4q}}. \] (3.9)

Direct integration with respect to \( r \) yields
\[ f(r) = k + r^2 \left( \frac{1}{r^2} - \frac{1}{r^2} \right)^{d-3} \left[ \frac{32\pi G Q^2 [d - 2]^{q} (d - 3)^q}{r^{4q}} \right] + \frac{16\pi G m}{\Sigma_{d-2} r^{d-1}} + \frac{\delta}{r^{d-1}} \] (3.10)

The above value of \( f(r) \) in line element (2.8) shows the higher dimensional magnetized black hole solution of DCG for any value of the parameter \( q \). However, it indicates that \( q = 1 \) will give the solution of DCG coupled to the standard Yang-Mills theory and for \( q = 1/4 \) the solution does not exist. By substituting the Yang-Mills magnetic charge \( Q = 0 \), it gives the neutral black hole solution in DCG.

### 3.2 Thermodynamics of black holes of DCG with power-Yang-Mills magnetic source

It is observed from the metric function (3.10) that the physical properties of such type of black holes depend on the parameter \( q \). The horizons of the black hole are given by \( f(r_h) = 0 \), where \( r_h \) represents the location of horizon. Thus
\[ m = \frac{\Sigma_{d-2} r^{d-1}}{16\pi G} \left( \frac{1}{r^2} + \frac{k}{r^2} \right)^{d-3} \left[ \frac{32\pi G Q^2 [d - 2]^{q} (d - 3)^q}{r^{4q}} \right] + \frac{16\pi G m}{\Sigma_{d-2} r^{d-1}} + \frac{\delta}{r^{d-1}}. \] (3.11)

Eq. (3.11) gives the ADM mass of the black hole in terms of horizon radius and Yang-Mills magnetic charge \( Q \). The black hole’s Hawking temperature [29] is given by
\[ T_H = \frac{1}{4\pi} \frac{df}{dr} |_{r=r_h} \]
\[ = \frac{r_h}{2\pi l^2} \left[ \frac{32\pi G Q^2 [d - 2]^{q} (d - 3)^q}{r^{4q+2}} + \frac{16\pi G m}{\Sigma_{d-2} r^{d-1}} + \frac{\delta}{r^{d-1}} \right]^{\frac{1}{4q}} - \frac{r_h^2}{4\pi (d - 3)} \left[ \frac{32\pi G Q^2 [d - 2]^{q} (d - 3)^q}{r^{4q+2}} + \frac{16\pi G m}{\Sigma_{d-2} r^{d-1}} + \frac{\delta}{r^{d-1}} \right]^{\frac{1}{4q}} \]
\[ \times \left[ \frac{32\pi G (2 - d - 4q) Q^2 [d - 2]^{q} (d - 3)^q}{(r^{4q+2})} + \frac{(1 - d) 16\pi G m}{\Sigma_{d-2} r^{d-1}} + \frac{(1 - d) \delta}{r^{d-1}} \right]. \] (3.12)
where charge is given by

\[ H = \frac{k}{r_h^d} + \frac{1}{4l^2} \left( \frac{d}{r_h^d} \right)^{d-3} F_1 \left[ 1, \frac{d}{2}, \frac{8-d}{2}, \frac{-r_h^2}{kl^2} \right]. \tag{3.13} \]

The Yang-Mills magnetic potential conjugate to Yang-Mills magnetic charge \( Q \) is given by

\[ \Phi_m = -64\pi Gq(d-2)^q(d-3)^qQ^{2q-1}. \tag{3.14} \]

Thus, using the above thermodynamical quantities, it can be seen that the first law of black hole thermodynamics given by \( (2.20) \) is satisfied. The free energy density of the resulting black hole can also be calculated as \( [28] \)

\[ \Xi = m - T_H S, \tag{3.15} \]

where \( m, T_H \) and \( S \) are the ADM mass, Hawking temperature and entropy of the black hole given by \( (3.11), (3.12) \) and \( (3.13) \), respectively. Similarly, the heat capacity at constant charge is given by

\[ C_Q = \frac{\left[ H_1(r_h) \right]^{d+1} - H_1^{d+2}}{2\pi^2} - \frac{H_1^{d+2}}{4\pi(d-3)H_1^{d-1}} \left[ (d-3)\Sigma_{d-2} H_2(r_h) \right] \left( \frac{k}{r_h^d} + \frac{1}{4l^2} \right)^{d-3} \]

\[ 4kG \left[ \frac{1}{2\pi^2} - \frac{H_1^{d+1}}{2\pi} - \frac{r_h H_1^{d+3}}{4\pi(d-3)} \frac{dH_1}{dr_h} \right] - \frac{r_h H_1^{d+3}}{4\pi(d-3)} \frac{d^2 H_1}{dr_h^2} + \frac{r_h^2 H_1^{d+2}}{4\pi(d-3)^2} \left( \frac{dH_1}{dr_h} \right)^2 \]. \tag{3.16}

where

\[ H_1(r_h) = \frac{32\pi GQ^{2q}[(d-2)(d-3)]^q}{(4q-1)r_h^{4q+d-2}} + \frac{16\pi Gm}{\Sigma_{d-2} r_h^{d-1}} + \frac{\delta}{r_h^{d-1}}, \tag{3.17} \]

\[ \frac{dH_1(r_h)}{dr_h} = \frac{32\pi GQ^{2q}(2-4q-d)[(d-2)(d-3)]^q}{(4q-1)r_h^{4q+d-1}} + \frac{16\pi G(1-d)m}{\Sigma_{d-2} r_h^{d}} + \frac{(1-d)\delta}{r_h^d}, \tag{3.18} \]

\[ \frac{d^2 H_1(r_h)}{dr_h^2} = \frac{32\pi GQ^{2q}(2-4q-d)(1-4q-d)[(d-2)(d-3)]^q}{(4q-1)r_h^{4q+d}} \]

\[ + \frac{16\pi Gd(1-d)m}{\Sigma_{d-2} r_h^{d+1}} + \frac{d(d-1)\delta}{r_h^{d+1}}, \tag{3.19} \]

and

\[ H_2(r_h) = \left[ \frac{d}{r_h} + \frac{r_h^2}{kl^2 + r_h^2} \right] F_1 \left( 1, \frac{d}{2}, \frac{8-d}{2}, \frac{-r_h^2}{kl^2} \right) + \frac{2r_h d}{kl^2(d-2)} F_1 \left( 2, \frac{d+2}{2}, \frac{10-d}{2}, \frac{-r_h^2}{kl^2} \right). \tag{3.20} \]
3.3 Hairy black holes of DCG with power-Yang-Mills magnetic source

The action function for Lovelock-scalar gravity with power-Yang-Mills source [32] is defined by

\[
I = \frac{1}{16\pi G} \int d^dx \sqrt{-g} \left[ \sum_{p=0}^{n-1} \frac{a_p}{2^p} \delta^{\mu_1 \cdots \mu_{2p}}_{\nu_1 \cdots \nu_{2p}} \left( a_p R^\mu_1 \nu_2 \cdots R^{\nu_{2p-1}} \nu_{2p} + 16\pi G b_p \phi^{d-4p} S^\mu_1 \nu_2 \cdots S^{\nu_{2p-1}} \nu_{2p} + 4\pi G(\Upsilon)^q \right) \right],
\]

where \((\Upsilon)\) corresponds to the power-Yang-Mills invariant defined by (3.2) and (3.3) and we have used the Lagrangian density of the scalar field given by

\[
L_s = \sum_{p=0}^{n-1} \frac{b_p}{2^p} (d-4p)^{\mu_1 \cdots \mu_{2p}} \rho^\nu_1 \nu_2 \cdots S^\mu_1 \nu_2 \cdots S^{\nu_{2p-1}} \nu_{2p},
\]

where \(S^\mu_\nu\) is given by Eq. (2.30). Note that, by putting scalar function \(\phi\) equal to zero, the above action function reduces to the case of DCG coupled to power-Yang-Mills field, i.e., (3.1).

Taking variation of (3.21) with respect to the metric tensor, we get the field equations as

\[
\sum_{p=0}^{n-1} \frac{a_p}{2^p} \delta^{\mu_1 \cdots \mu_{2p}}_{\nu_1 \cdots \nu_{2p}} \rho^\nu_1 \nu_2 \cdots R^\mu_1 \nu_2 \cdots R^{\nu_{2p-1}} \nu_{2p} = 16\pi G [T^\mu_\nu(M)] + 16\pi G [T^\mu_\nu(s)],
\]

where \([T^\mu_\nu(M)]\) corresponds to the matter tensor of power-Yang-Mills field while \(a_p\) corresponds to the value defined as in Eq. (2.3) so that the Lovelock gravity becomes DCG. The stress-energy tensor corresponding to the scalar field is given by (2.31) and the equation of motion of scalar field is given by (2.32). Choosing the scalar configuration (2.34) and using Eqs. (2.31) and (2.32) in Eq. (3.23), we get the metric function in the form

\[
f(r) = k + \frac{r^2}{l^2} - r^2 \left[ 16\pi G m \frac{32\pi Y}{rd} + \frac{32\pi G Q(2q(d-2)^q(d-3)^q)}{(4q-1)r^{4q+d-2}} \right]^{\frac{1}{d-4}},
\]

where \(Y\) is given by (2.37). Thus the line element (2.8) with metric function given by (3.24) gives the hairy black hole solution of DCG in the background of power-Yang-Mills field. Hence, we derive a large family of black hole solutions for any value of real parameter \(q\) except for the case \(q = 1/4\). It is easy to see that, by taking \(Y = 0\), this black hole solution reduces to the black hole with no scalar hair which we derived in Section 3.1.

4 Black holes of DCG and Yang-Mills hierarchies

In this section we study the possible black holes of DCG whose gravitational field is sourced by the superposition of different power-Yang-Mills field. The Yang-Mills hierarchies in diverse dimensions have been discussed in the literature [36]. Here we begin with an action defined as

\[
I = \frac{1}{16\pi G} \int d^dx \sqrt{-g} \left[ \sum_{p=0}^{n-1} \frac{a_p}{2^p} \delta^{\mu_1 \cdots \mu_{2p}}_{\nu_1 \cdots \nu_{2p}} R^\mu_1 \nu_2 \cdots R^{\nu_{2p-1}} \nu_{2p} + \sum_{j=0}^{q} c_j(\Upsilon)^j \right],
\]
where $\Upsilon$ represents the Yang-Mills invariant defined in (3.2), $c_j, j \geq 1$ is a coupling constant. The variation of the above action with respect to the metric tensor gives (2.5) with the matter tensor given by

$$T_{\mu\nu} = -\frac{1}{2} \sum_{j=0}^{q} c_j \left( \delta^\nu_{\mu} \Upsilon^j - 4j Tr(F^{(a)\mu\sigma}) \right). \quad (4.2)$$

The Yang-Mills equations are determined by varying (4.1) with respect to the gauge potential $A^{(a)}$

$$\sum_{j=0}^{q} c_j \left[ d(*F^{(a)} \Upsilon^{j-1}) + \frac{1}{\eta} C^{(a)}_{(b)(c)} \Upsilon^{j-1} A^{(b)} \wedge^* F^{(c)} \right] = 0. \quad (4.3)$$

Our $d = n + 2$-dimensional line element ansatz is given by (2.8) and the power-Yang-Mills field ansatz would be chosen as before such that the matter tensor takes the form

$$T_{\nu\mu} = -\frac{1}{2} \sum_{j=0}^{q} c_j \Upsilon^j \text{diag}[1, 1, \xi, \xi, ..., \xi], \quad (4.4)$$

where $\xi = 1 - \frac{4j}{d-2}$. Thus using Eqs. (2.3), (2.8) and (4.4) in (2.5) we get the metric function

$$f(r) = k + \frac{r^2}{l^2} - r^2 \left[ 32\pi G \sum_{j=0}^{q} \frac{(d-2)j(d-3)jQ^{2j}}{(4j-1)r^{4j+d-2}} + \frac{16\pi mG}{\Sigma_{d-2} \delta} + \frac{\delta}{r^{d-1}} \right]^{\frac{1}{d-3}}. \quad (4.5)$$

The above equation indicates that for $j = 0$ the neutral black hole solution is obtained, for $j = 4$ the solution is undefined and for taking a unique value of $j = q$ we get the solution obtained in Section 3.1. The ADM mass in terms of the outer horizon radius $r_h$ is

$$m = \frac{\Sigma_{d-2} r_h^{d-1}}{16\pi G} \left( \frac{1}{r^2} + \frac{k}{r_h^2} \right)^{d-3} - \frac{\Sigma_{d-2} \delta}{16\pi G} - \sum_{j=0}^{q} \frac{2Q^{2j} \Sigma_{d-2} [(d-2)(d-3)]^j}{(4j-1) r_h^{4j-1}}. \quad (4.6)$$

When superposition of different power-Yang-Mills sources are taken into account, the metric function corresponding to hairy black hole solution of DCG takes the form

$$f(r) = k + \frac{r^2}{l^2} - r^2 \left[ 32\pi G \sum_{j=0}^{q} \frac{(d-2)j(d-3)jQ^{2j}}{(4j-1)r^{4j+d-2}} + \frac{16\pi mG}{\Sigma_{d-2} \delta} + \frac{\delta}{r^{d-1}} + \frac{32\pi Y}{r^{d}} \right]^{\frac{1}{d-3}}, \quad (4.7)$$

where $Y$ is given by (2.37).

5 Summary and conclusion

In this paper, new magnetized black hole solutions of DCG are constructed. In order to do this we studied black holes of DCG in the framework of both exponential electrodynamics and power-Yang-Mills theory. Firstly, we derived both topological black hole solutions and
hairy black hole solutions of DCG with pure exponential magnetic source. These solutions depend on the parameter $\beta$ of exponential electrodynamics. For the model of exponential electrodynamics, there is no need to impose the condition on the matter tensor for making it traceless, so we conclude that the scale and dual invariances are completely violated here. Further, the components of the matter tensor obtained from the Lagrangian density of this model satisfy all the energy conditions along with causality and unitarity principles. For any value of parameter $\beta$, it is possible to obtain a solution which is regular at the origin. Moreover, these solutions are non-asymptotically flat for nonzero value of constant $l$ while in the limit $l$ approaches to infinity one can get the asymptotically flat solutions. It is shown that the metric functions are finite at the origin, this finiteness property of metric functions is due to the nonlinear behavior of electromagnetic field characterized by Lagrangian density (2.2). Secondly, we use a model of power-Yang-Mills theory and derive a large family of topological black hole solutions in DCG. These solutions depend on the parameter $q$ and are also non-asymptotically flat for any nonzero value of $l$. The case $q = 1$ gives the solutions of black holes with standard Yang-Mills source and for $q = 1/4$ the solution does not exist. The hairy black hole solutions are also derived in the framework of power-Yang-Mills theory which are reducible to black holes with no scalar hair for $Y = 0$ and to neutral black holes for $Q = 0$.

The thermodynamics of topological black holes is also studied within both the exponential electrodynamics and power-Yang-Mills theory. Thermodynamical quantities such as entropy, Hawking temperature and specific heat capacity at constant charge of resulting magnetized higher dimensional black holes of DCG have been worked out. The first law of black hole thermodynamics has also been shown to hold for these black hole solutions.

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