A Laguerre Polynomial Orthogonality and the Hydrogen Atom

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Abstract

The radial part of the wave function of an electron in a Coulomb potential is the product of a Laguerre polynomial and an exponential with the variable scaled by a factor depending on the degree. This note presents an elementary proof of the orthogonality of wave functions with differing energy levels. It is also shown that this is the only other natural orthogonality for Laguerre polynomials. By expanding in terms of the usual Laguerre polynomial basis an analogous strange orthogonality is obtained for Meixner polynomials.

1 Introduction

The time-independent Schrödinger equation for one electron subject to a Coulomb potential is

$$-\frac{\hbar^2}{2m} \Delta \psi (x) - \frac{c}{||x||} \psi (x) = E \psi (x),$$

for certain constants $m, c$ (see Robinett [7], Ch. 18). Rewrite the equation as

$$\Delta \psi (x) + \frac{k}{||x||} \psi (x) = -\lambda \psi (x),$$

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where \( k \) is a positive parameter and \( \lambda \) denotes the rescaled energy level; let \( x \in \mathbb{R}^N \) (usually \( N = 3 \), of course) and \( ||x|| = \left( \sum_{j=1}^{N} x_j^2 \right)^{1/2} \). We separate variables by considering solutions of the form \( \psi (x) = \phi (||x||) p_l (x) \) where \( p_l \) is a harmonic homogeneous polynomial of degree \( l = 0, 1, 2, \ldots \) (the angular momentum quantum number). Then \( \phi \) must satisfy the equation

\[
\frac{d^2}{dr^2} \phi (r) + \frac{N + 2l - 1}{r} \frac{d}{dr} \phi (r) + \left( \frac{k}{r} + \lambda \right) \phi (r) = 0.
\] (1)

Using spherical polar coordinates for \( \mathbb{R}^N \) we see that \( \phi \) must be square-integrable for the measure \( r^2 l + N - 1 \, dr \) on \( \{ r : r > 0 \} \) (we restrict our attention to the point spectrum). Furthermore solutions for the same value of \( l \) but different energy levels must be orthogonal for this measure, by the self-adjointness of the equation. The topic of this note is a direct elementary proof of this orthogonality. It will also be shown that these functions and the standard set form the only possible (in a sense to be specified) orthogonal sets of scaled Laguerre polynomials. Then by expanding the functions in terms of the usual Laguerre basis an analogous orthogonality is obtained for Meixner polynomials. This orthogonal set incorporates a parameter dependent on the degree and is a set of eigenfunctions of a self-adjoint second-order difference operator resembling equation (1).

In equation (1) set \( \lambda = -\mu^2 \) (the bound energy levels are negative) and substitute \( \phi (r) = e^{-\mu r} g (2\mu r) \), then \( g (s) \) satisfies

\[
\frac{d^2}{ds^2} g (s) + (N + 2l - 1 - s) \frac{d}{ds} g (s) + \frac{1}{2} \left( 1 + \frac{k}{\mu} - N - 2l \right) g (s) = 0.
\]

The integrable solutions are obtained by setting \( \frac{1}{2} \left( 1 + \frac{k}{\mu} - N - 2l \right) = n \) for some \( n = 0, 1, 2, 3, \ldots \) then \( \phi (r) = L_n^\alpha (2\mu r) \exp (-\mu r) \) where \( \mu_n = \frac{k}{2(\alpha + 2n + 1)} \) and \( \alpha = N + 2l - 2 \); see (3) below (\( \phi \) is not normalized). The energy levels are \( -\mu_n^2 \) (specifically \( -\left( \frac{k}{4(n+l+1)} \right)^2 \) for \( N = 3 \)).

These radial Laguerre-type functions continue to be useful in the research literature: Beckers and Debergh \( [1] \) used them for supersymmetric properties of the parastatistical hydrogen atom, Blaive and Cadilhac \( [2] \) derived an expansion in powers of \( \mu_n \) (for fixed \( l \)), Dehesa et al \( [3] \) found approximations for entropies which involve integrals of the logarithms of the Laguerre polynomials, Xu and Kong \( [9] \) constructed shift operators for the radial Hamiltonian.
The Pochhammer symbol is defined by

\[(a)_n = \prod_{i=1}^{n} (a + i - 1), \ a \in \mathbb{R},\]

and the hypergeometric \((\text{2F1})\) series is

\[\text{2F1}(a, b; c; x) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j j!} x^j.\]

The basic facts about the Laguerre polynomial of index \(\alpha\) and degree \(n\) that will be needed are:

\[L_\alpha^n(x) = \frac{(\alpha + 1)_n}{n!} \sum_{j=0}^{n} \frac{(-n)_j}{(\alpha + 1)_j j!} x^j,\]

\[
\int_0^\infty L^\alpha_m(x) L^\alpha_n(x) x^\alpha e^{-x} \, dx = \delta_{mn} \Gamma (\alpha + 1) \frac{(\alpha + 1)_n}{n!}, \tag{2}
\]

\[
\left( x \frac{d^2}{dx^2} + (\alpha + 1 - x) \frac{d}{dx} + n \right) L^\alpha_n(x) = 0, \tag{3}
\]

\[(n + 1) L^\alpha_{n+1}(x) - (\alpha + 1 + 2n - x) L^\alpha_n(x) + (n + \alpha) L^\alpha_{n-1}(x) = 0. \tag{4}\]

Their proofs and further information are in Szegö [8], Ch.5. The author gratefully acknowledges a useful discussion with R. Yáñez.

## 2 Orthogonality relation

Fix a parameter \(\alpha > -1\). The scaling factor \(k\) appearing in equation (1) is set equal to 1.

**Definition 1** For \(n = 0, 1, 2, \ldots\) and \(x \in \mathbb{R}\) let

\[\phi_n(x) = L^\alpha_n \left( \frac{x}{\alpha + 2n + 1} \right) \exp \left( -\frac{x}{2(\alpha + 2n + 1)} \right).\]

**Theorem 2** For \(m, n = 0, 1, 2, \ldots\) the following orthogonality relation holds:

\[
\int_0^\infty \phi_m(x) \phi_n(x) x^{\alpha+1} \, dx = \frac{\delta_{mn}}{n!} (\alpha + 2n + 1)^{\alpha+3} \Gamma (\alpha + 1 + n).\]
The proof is composed of two parts. First, the case \( m = n \) follows from the orthogonality relation (2) and the three-term recurrence (4) in the form

\[
x L_n^\alpha(x) = -(n+1)L_n^\alpha(x) + (\alpha+2n+1)L_n^\alpha(x) + (\alpha+n)L_{n-1}^\alpha(x),
\]

and thus \( \int_0^\infty x L_n^\alpha(x)^2 x^\alpha e^{-x} \, dx = (\alpha+2n+1) \Gamma(\alpha+1+n)/n! \). In the integral \( \int_0^\infty \phi_n(x)^2 x^{\alpha+1} \, dx \) change the variable to \( s = \frac{x}{\alpha+2n+1} \).

Second, fix \( m \neq n \) and in the integral \( \int_0^\infty \phi_m(x) \phi_n(x) x^{\alpha+1} \, dx \) change the variable to

\[
s = \frac{x}{2} \left( \frac{1}{\alpha+2m+1} + \frac{1}{\alpha+2n+1} \right),
\]

and let

\[
\beta = \frac{\alpha+2n+1}{\alpha+m+n+1};
\]

then the desired integral equals \( \left( \frac{(\alpha+2m+1)(\alpha+2n+1)}{\alpha+m+n+1} \right)^{\alpha+2} \) times

\[
\int_0^\infty L_m^\beta(s) L_n^\alpha((2-\beta)s) s^{\alpha+1} e^{-s} \, ds.
\]

We will show this equals zero by expressing the two polynomials in terms of \( L_j^{\alpha+1}(s) \), which simplifies evaluation of the integral. Recall the identities

\[
L_n^\alpha(x) = L_n^{\alpha+1}(x) - L_{n-1}^{\alpha+1}(x),
\]

\[
L_\delta^\beta(bx) = \sum_{j=0}^{n} \frac{(\delta+1+j)(\alpha+n-j)}{(n-j)!} b^j (1-b)^{n-j} L_j^{\alpha}(x),
\]

which hold for arbitrary \( x, b \in \mathbb{R}, \delta > -1 \) and \( n = 0, 1, 2, \ldots \). Then (with \( \delta = \alpha + 1 \) in (5))

\[
L_m^\alpha(\beta s) = L_m^{\alpha+1}(\beta s) - L_{m-1}^{\alpha+1}(\beta s)
\]

\[
= \sum_{j=0}^{m} \frac{(\alpha+2+j)(m-1-j)}{(m-j)!} \beta^j (1-\beta)^{m-1-j} L_j^{\alpha+1}(s)
\]

\[
\times ((\alpha + 1 + m) - (m-j))
\]

\[
= \sum_{j=0}^{m} \frac{(\alpha+2+j)(m-1-j)}{(m-j)!} \beta^j (1-\beta)^{m-1-j} (j-n(2-\beta)) L_j^{\alpha+1}(s)
\]
\[ L_n^\alpha ((2 - \beta) s) = L_{n-1}^{\alpha+1} ((2 - \beta) s) - L_{n-1}^{\alpha+1} ((2 - \beta) s) \]
\[ = \sum_{j=0}^{n} \frac{(\alpha + 2 + j)_{n-1-j}}{(n-j)!} (2 - \beta)^j (\beta - 1)^{n-1-j} L_j^{\alpha+1} (s) \]
\times ((\beta - 1) (\alpha + 1 + n) - (n-j))
\[ = \sum_{j=0}^{n} \frac{(\alpha + 2 + j)_{n-1-j}}{(n-j)!} (2 - \beta)^j (\beta - 1)^{n-1-j} (j - m\beta) L_j^{\alpha+1} (s). \]

Now multiply the two expressions together and integrate with respect to the measure \( s^{\alpha+1} e^{-s} ds \) on \( \{ s : s > 0 \} \). By the orthogonality of the integral (5) equals \( \Gamma (\alpha + 2) \) times
\[ \frac{(\alpha + 2)_{m-1} (\alpha + 2)_{n-1}}{m! n!} (1 - \beta)^{m-1} (\beta - 1)^{n-1} \]
\times \[ \sum_{j=0}^{\min(m,n)} \frac{(-m-j) (-n-j)!}{(\alpha + 2)_{j} j!} \left( \frac{\beta (\beta - 2)}{(1-\beta)^2} \right)^j (j - n (2 - \beta)) (j - m\beta). \]

The formula holds when one of \( m, n = 0 \) by the convention \( (\alpha + 2)_{-1} = \frac{1}{\alpha+1} \).

Let
\[ \gamma = \frac{\beta (\beta - 2)}{(1-\beta)^2} = 1 - \frac{1}{(1-\beta)^2} \]
then
\[ (j - n (2 - \beta)) (j - m\beta) = j (\alpha + 1 + j) + \frac{\gamma}{\gamma - 1} (mn - j (\alpha + m + n + 1)), \]
since \( \beta (2 - \beta) = \frac{\gamma}{\gamma - 1} \). The second line of the sum (8) can be written as
\[ mn\gamma \, _2\!F_1 (1 - m, 1 - n; \alpha + 2; \gamma) + \frac{\gamma mn}{\gamma - 1} \, _2\!F_1 (-m, -n; \alpha + 2; \gamma) \]
\[ - \frac{\gamma^2 mn (\alpha + m + n + 1)}{(\gamma - 1)(\alpha + 2)} \, _2\!F_1 (1 - m, 1 - n; \alpha + 3; \gamma); \]
which equals \( mn\frac{\gamma}{\gamma - 1} \) times
\[ (\gamma - 1) \, _2\!F_1 (1 - m, 1 - n; \alpha + 2; \gamma) + \, _2\!F_1 (-m, -n; \alpha + 2; \gamma) \]
\[ - \frac{\gamma (\alpha + m + n + 1)}{\alpha + 2} \, _2\!F_1 (1 - m, 1 - n; \alpha + 3; \gamma); \]
and this expression is zero by the contiguity relations for the \(_2\!F_1\)-series (by the straightforward calculation of the coefficient of \( \gamma^j \) for each \( j \)). This completes the proof of the theorem.
3.1 The case \( \kappa \neq 0, 1 \)

We show that the equations \( \langle \phi_0, \phi_1 \rangle = \langle \phi_0, \phi_2 \rangle = \langle \phi_1, \phi_2 \rangle = 0 \) are incompatible unless \( \kappa = 0 \) or 1. The equation \( \langle \phi_0, \phi_1 \rangle = 0 \) implies \( \gamma_1 = \frac{\alpha + 1}{2 + \kappa} \). The condition \( \langle \phi_0, \phi_2 \rangle = 0 \) is equivalent to a certain quadratic equation \( q_2 (\gamma_2) = 0 \) (not displayed here). The discriminant of \( q_2 \) equals \( 16 \kappa (\alpha + \kappa + 1) (\alpha + 2) \). Integrability requires \( \alpha + \kappa + 1 > 0 \) hence there are real solutions for \( \gamma_2 \) only if \( \kappa \geq 0 \). The conditions \( \langle \phi_1, \phi_2 \rangle = 0 \) and \( \gamma_1 = \frac{\alpha + 1}{2 + \kappa} \) are equivalent to a certain cubic equation \( q_3 (\gamma_2) = 0 \). The resultant of \( q_2 \) and \( q_3 \) (a polynomial in the coefficients which is zero if and only if there exists a simultaneous root) is found to be the product of \( \kappa (\kappa - 1) \) by a polynomial in \( (\alpha + 1) \) and \( \kappa \) with all coefficients nonnegative. This shows that \( q_2 (\gamma_2) = 0 = q_3 (\gamma_2) \) is impossible for \( \kappa \geq 0, \alpha > -1 \) unless \( \kappa = 0 \) or 1.

3.2 The case \( \kappa = 0 \) or 1

For the case \( \kappa = 0 \) the condition \( \langle \phi_0, \phi_n \rangle = 0 \) (for \( n > 0 \)) and the formula (8) with \( \delta = \alpha \) show that \( (\beta - 1)^n = 0 \) where \( \beta = \frac{2}{\lambda + \gamma} \), hence \( \gamma_n = 1 = \gamma_0 \) for all \( n \) (the variable \( \beta \) is not identical to the one used in Section 2, because the scaling parameters \( \gamma_n \) are initially undetermined).
For the case $\kappa = 1$, $\langle \phi_0, \phi_1 \rangle = 0$ implies $\gamma_1 = \frac{\alpha + 1}{\alpha + 3}$; further for $n \geq 2$ the condition $\langle \phi_0, \phi_n \rangle = 0$ and formulas (3), (4) imply

$$(\beta - 1)^{n-1} ((\alpha + 1 + n)(\beta - 1) - n) = 0,$$

(where $\beta = \frac{2}{1 + \gamma_n}$) with the solutions $\beta = 1$ and $\frac{\alpha + 2n + 1}{\alpha + n + 1}$. Thus $\gamma_n = \frac{\alpha + 1}{\alpha + 2n + 1}$ or 1. But $\gamma_1 = \frac{\alpha + 1}{\alpha + 3}$, and the computation of $\langle \phi_1, \phi_n \rangle$ with $\gamma_n = 1$ yields

$$\langle \phi_1, \phi_n \rangle = (-1)^{n-1} n ((\alpha + 3)n - 1) \frac{(\alpha + 3)(\alpha + 3)_{n-2}}{(\alpha + 1)^{n-2}(\alpha + 2)},$$

which is nonzero for $n \geq 2$. This rules out the possibility that $\gamma_n = 1$ and shows that $\gamma_n = \frac{\alpha + 1}{\alpha + 2n + 1}$ for all $n$ is necessary. The sufficiency was proved already.

### 4 Change of Basis and Meixner Polynomials

Since there is another obvious orthogonal basis for $L^2 \left(x^{\alpha+1}dx\right)$, the standard Laguerre functions $\{L^{\alpha+1}_m(cx)e^{-cx/2} : m = 0, 1, 2, \ldots\}$, with some scaling parameter $c$, the computation of the transformation coefficients will give rise to an infinite orthogonal matrix. The entries will be expressed as Meixner polynomials: for parameters $\gamma > 0$ and $0 < c < 1$ the Meixner polynomial of degree $n$ is given by

$$M_n(x; \gamma, c) = \binom{x}_n \binom{n}{\gamma} c^n (1 - c)^{-\gamma},$$

with orthogonality relations

$$\sum_{x=0}^{\infty} \frac{(\gamma)_x e^x M_n(x; \gamma, c) M_m(x; \gamma, c)}{x!} = \frac{\delta_{mn} n!}{(\gamma)_n c^n (1 - c)^\gamma},$$

satisfying the second order difference equation

$$c(x + \gamma)f(x + 1) - (x + (x + \gamma)c)f(x) + x f(x - 1) = n(c - 1)f(x),$$

and the three-term recurrence

$$xM_n(x; \gamma, c) = \frac{c(\gamma + n)}{c - 1} M_{n+1}(x; \gamma, c) + \frac{n + (n + \gamma)c}{1 - c} M_n(x; \gamma, c) + \frac{n}{c - 1} M_{n-1}(x; \gamma, c)$$

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(see the handbook of Koekoek and Swarttouw \[1\], p.45).

Now consider the two orthogonal bases \( \{ q_{0,n} : n \geq 0 \} \) and \( \{ q_{1,m} : m \geq 0 \} \) for \( L^2 ((0, \infty), x^{\alpha+1} dx) \) where \( q_{0,n} (x) = L_n^{\alpha+1} (\beta x) e^{-\beta x/2} \) and \( q_{1,m} (x) = L_m^{\alpha+1} ((2 - \beta) x) e^{-(2 - \beta) x/2} \) (for \( 0 < \beta < 2 \), again not identical to previous usage of this symbol). By use of the expansion (7) for \( \beta \neq 1 \) it follows easily that

\[
\Gamma (\alpha + 2)^{-1} \int_0^\infty q_{0,n} (x) q_{1,m} (x) x^{\alpha+1} dx
\]

\[
= \Gamma (\alpha + 2)^{-1} \int_0^\infty L_n^{\alpha+1} (\beta x) L_m^{\alpha+1} ((2 - \beta) x) e^{-x} x^{\alpha+1} dx
\]

\[
= (1 - \beta)^{n+m} (-1)^m \frac{\Gamma (\alpha + 2) \Gamma (\alpha + 2)}{n! m!} _2F_1 \left( -n, -m; \alpha + 2; \frac{\beta (\beta - 2)}{(1 - \beta)^2} \right);
\]

and the \( _2F_1 \)-sum equals \( M_n \left( m; \alpha + 2, (1 - \beta)^2 \right) \). Of course this calculation is only for illustration and is not novel; the orthogonality of Meixner polynomials is a simple consequence of that of the Laguerre polynomials and the values \( \Gamma (\alpha + 2)^{-1} \int_0^\infty (q_{0,n} (x))^2 x^{\alpha+1} dx = \beta^{-\alpha-2 \frac{(\alpha+2)}{n}} \) and the similar result for \( q_{1,m} \).

### 4.1 The transformation matrix

We consider the analogous computation for the two bases

\[
\left\{ L_m^{\alpha+1} \left( \frac{x}{\alpha + 1} \right) e^{-x/(2(\alpha+1))} : m \geq 0 \right\} \text{ and } \{ \phi_n : n \geq 0 \text{ \} (from definition[2]. The scaling for the former is chosen to force equality at } m = n = 0.

\[\text{Proposition 3 } \text{For } m, n \geq 0 \text{ let } A_n = \frac{1}{2} \left( \frac{1}{\alpha + 2n + 1} + \frac{1}{\alpha + 1} \right), u_n = \frac{n}{\alpha + n + 1} \text{ and } \]

\[I(n, m) = \Gamma (\alpha + 2)^{-1} \int_0^\infty L_n^\alpha \left( \frac{x}{\alpha + 2n + 1} \right) L_m^{\alpha+1} \left( \frac{x}{\alpha + 1} \right) e^{-A_n x} x^{\alpha+1} dx, \]

then \( I(0, 0) = (\alpha + 1)^{\alpha+2}; I(0, n) = 0 = I(n, 0) \text{ for } n > 0; A_n = \frac{n}{(\alpha + 1)(1 + u_n)} \) and

\[I(n, m) = ((\alpha + 1) (1 + u_n))^{\alpha+2} \frac{(\alpha + 3)_{m-1} (\alpha + 2)_{n-1}}{(m-1)! (n-1)!} (-1)^{m-1} (1 - u_n^2)
\]

\[\times u_n^{m+n-3} \cdot \frac{\Gamma (1 - n, 1 - m; \alpha + 3; 1 - u_n^{-2})}{2F_1 (1 - n, 1 - m; \alpha + 3; 1 - u_n^{-2})}.
\]

\[\text{Proof. In } I(n, m) \text{ change the integration variable to } s = A_n x \text{ and let } \]

\[\beta = (A_n (\alpha + 2n + 1))^{-1} = \frac{\alpha + 1}{\alpha + n + 1}, \text{ then } \]

\[I(n, m) = \Gamma (\alpha + 2)^{-1} A_n^{-\alpha-2} \int_0^\infty L_n^\alpha (\beta s) L_m^{\alpha+1} ((2 - \beta) s) e^{-s} s^{\alpha+1} ds. \]
By formulae (7) and (6)

\[
L_\alpha(n)(\beta s) = \sum_{j=0}^{n} \frac{(\alpha + 2 + j)_{n-1-j}}{(n-j)!} \beta^j (1 - \beta)^{n-1-j} L_{j+1}^{\alpha+1}(s) \\
\times ((1 - \beta) (\alpha + n + 1) - (n - j)) \\
= \sum_{j=0}^{n} \frac{j (\alpha + 2 + j)_{n-1-j}}{(n-j)!} \beta^j (1 - \beta)^{n-1-j} L_{j+1}^{\alpha+1}(s),
\]

for \(n > 0\) while \(L^{\alpha}_{0}(\beta s) = 1\). By formula (7)

\[
L_{\alpha+1}(m)((2 - \beta) s) = \sum_{j=0}^{m} \frac{(\alpha + 2 + j)_{m-j}}{(m-j)!} (2 - \beta)^j (\beta - 1)^{m-j} L_j^{\alpha+1}(s).
\]

Multiply the two sums and integrate with respect to \(\Gamma (\alpha + 2)^{-1} e^{-s} s^{\alpha+1} ds\) and by use of the orthogonality relations obtain

\[
I(n,m) = A_n^{-\alpha-2}(\alpha + 2)_{m} (\alpha + 2)_{n-1} (-1)^m \\
\times \sum_{j=1}^{\min(m,n)} \frac{j}{(n-j)! (m-j)! j! (\alpha + 2)_j} \left(\frac{\beta (\beta - 2)}{(1 - \beta)^2}\right)^j,
\]

for \(n > 0\); and \(I(0,0) = A_0^{-\alpha-2} = (\alpha + 1)^{\alpha+2} I(0,m) = 0 \) for \(m > 0\). When \(n > 0\) let \(u_n = \frac{n}{\alpha+n+1}\), then \(A_n = ((1 + u_n) (\alpha + 1))^{-1}\), \(\beta = 1 - u_n, 2 - \beta = 1 + u_n\). In the sum replace \(j\) by \(i+1\) (and \((n-j)!\) by \((n-1)! / (1-n)_i (-1)^i\) and similarly for \((m-1)!\)) to obtain the claimed result. ■

Certainly the integral could be computed for any scaling of the \(\{L_{\alpha+1}^{\alpha+1}\}\) basis, but our particular choice gives the neatest formulae. Motivated by the value of \(I(n,m)\) we write the following:

**Definition 4** For \(\alpha \geq 0\) and \(n = 0, 1, 2, 3 \ldots\) (and \(u_n = \frac{n}{\alpha+n+1}\)) define the function

\[
h_n^{\alpha}(x) = u_n^{\alpha+1} (x + 1) M_n(x; \alpha + 3, u_n^{2}) \text{ for } x \geq -1.
\]

**Corollary 5** For \(n > 0\),

\[
I(n,m) = ((\alpha + 1) (1 + u_n))^{\alpha+2} (\alpha + 3)_{m-1} (\alpha + 2)_{n-1} \\
\times (-1)^{m-1} (1 - u_n^{2}) u_n^{n-2} h_n^{\alpha}(m-1).
\]
Let
\[ N_m = \Gamma (\alpha + 2)^{-1} \int_0^\infty L_{m+1}^\alpha \left( \frac{x}{\alpha + 1} \right)^2 e^{-x/(\alpha+1)} x^{\alpha+1} \, dx \]
\[ = (\alpha + 1)^{\alpha+2} \frac{(\alpha + 2)_m}{m!}, \]
and recall from Theorem 2 that \( \|\phi_n\|^2 = (\alpha + 2n + 1)^{\alpha+3} \frac{(\alpha+2)_{n+1}}{n!} \) (dividing by the normalizing \( \Gamma (\alpha + 2) \)). By the orthogonality of the two bases the matrix \( \{ I(n,m) \| \phi_n \|^N_{1/2} \}_{n,m=0}^\infty \) is orthogonal. The entries in row 0 and column 0 are zero except \( I(0,0) = (\alpha + 1)^{\alpha+2} = \|\phi_0\|^N_{1/2} \). The row orthogonality demonstrates the orthogonality of the functions \( \{ h_\alpha^n \} \); indeed \( \sum_{m=0}^\infty N_{m-1}^{-1} I(n,m) I(l,m) = \delta_{nl} \|\phi_n\|^2 \). After some manipulation this implies for \( n,l \geq 1 \) that
\[ \sum_{m=1}^\infty \frac{(\alpha + 2)_m}{m!} h_\alpha^{n-1} (m-1) h_\alpha^{l-1} (m-1) \]
\[ = \delta_{nl} (n-1)! \frac{(\alpha + 2)^{\alpha+2} (\alpha + n + 1)^{2\alpha+2n+4}}{(\alpha + 1)^{\alpha+4} (\alpha + 2n + 1)^{\alpha+3} n^{2n-3}}. \]
For \( n = l \) this can be verified from the Meixner polynomial relations: the sum equals
\[ (\alpha + 2) \sum_{x=0}^\infty \frac{(\alpha + 3)_x}{x!} (x + 1) u_n^{2x} M_{n-1} (x; \alpha + 3, u_n^2)^2; \]
then \( (x + 1) M_{n-1} (x; \alpha + 3, u_n^2) \) can be expanded in terms of \( \{ M_j (x; \alpha + 3, u_n^2) : j = n - 2, n - 1, n \} \) by use of formula (11). The coefficient of \( M_{n-1} \) is \( \frac{n(\alpha + n + 1)}{\alpha+1} \) and together with (11) this produces the desired result. The orthogonality of \( \{ h_\alpha^n \}_{n=0}^\infty \) can also be established by exhibiting these functions as eigenfunctions of a self-adjoint operator.

4.2 A self-adjoint operator

First let \( M_{n-1} (x; \alpha + 3, u_n^2) = u_n^{-1} g(x) \) and substitute in the second-order difference equation (10) to obtain:
\[ (x + \alpha + 3) g(x + 1) + (\alpha + 1) g(x) + x g(x - 1) = (u_n + u_n^{-1}) (x + 1) g(x); \]
replace \( g(x) \) by \( \frac{f(x)}{x+1} \) then \( f \) satisfies

\[
\frac{x + \alpha + 3}{x + 2} f(x + 1) + \frac{\alpha + 1}{x + 1} f(x) + f(x - 1) = (u_n + u_n^{-1}) f(x).
\]

It is easy to see that the operator

\[
D f(x) = \frac{x + \alpha + 3}{x + 2} f(x + 1) + \left( \frac{\alpha + 1}{x + 1} - 2 \right) f(x) + f(x - 1)
\]

defined for functions on \( x = -1, 0, 1, 2, \ldots \) such that \( f(-1) = 0 \) is symmetric for the inner product \( \langle f_1, f_2 \rangle = \sum_{x=0}^{\infty} \frac{(\alpha + 2)x + 1}{(x+1)!} f_1(x) f_2(x) \). Further

\[
D h_n^\alpha = \frac{(u_n - 1)^2}{u_n} h_{n-1}^\alpha \quad \text{(for } n \geq 1) \text{, and } \frac{(u_n - 1)^2}{u_n} = \frac{(\alpha + 1)^2}{n(\alpha + n + 1)},
\]

so here is another proof of the orthogonality of the set \( \{h_n^\alpha : n \geq 0\} \).

Note that \( D \) can be written as

\[
D f(x) = (f(x + 1) - 2f(x) + f(x - 1)) + \frac{\alpha + 1}{x + 2} (f(x + 1) - f(x)) + (\alpha + 1) \left( \frac{1}{x + 1} + \frac{1}{x + 2} \right) f(x),
\]

which is a discrete analogue of the radial Coulomb equation (1). It is clear that the difference equation

\[
D f(x) = -\lambda f(x) \tag{12}
\]

with initial conditions \( f(0) = 1 \) and \( f(-1) = 0 \) has a unique solution for any \( \lambda \). To find a formal expression for the solution set

\[
f(x) = u^x (x + 1) \sum_{j=0}^{\infty} a_j (-x)_j
\]

with \( u, a_j \) to be determined, and substitute in the equation (the series terminates for each \( x = 0, 1, 2, \ldots \)). After the usual manipulations we obtain (for \( j \geq 0 \)):

\[
-u^2 (j + 1) (j + \alpha + 3) a_{j+1} + u (u (\alpha + 2j + 3) + \alpha + 1 - (2 - \lambda) (j + 1)) a_j - \left( (u - 1)^2 + \lambda u \right) a_{j-1} = 0.
\]
To make this a two-term recurrence, solve $u^2 - (2 - \lambda) u + 1 = 0$ for $u$; when \( \lambda < 0 \) select the solution with $0 < u < 1$. Substituting $\lambda u = -(u - 1)^2$ in the recurrence results in

$$-u^2 (j + 1) (j + \alpha + 3) a_{j+1} + (1 + u) \{(j (u - 1) + u (\alpha + 2) - 1) a_j = 0.$$ (13)

First we handle the special case $\lambda = 0, u = 1$ with the solution $a_{j+1} = \frac{2(\alpha+1)}{(j+1)(j+\alpha+3)} a_j$ and denote the corresponding solution $f(x)$ of (12) by $h_\alpha^\infty(x)$ where

$$h_\alpha^\infty(x) = (x + 1) \sum_{j=0}^{\infty} \frac{(-x)_j}{(\alpha + 3)_j} \frac{(2 (\alpha + 1))^j}{j!},$$

a hypergeometric series of type $1F_1$ (and resembling a Laguerre polynomial).

Second, for $u \neq 1$ let $\gamma = \frac{1 - u (\alpha + 2)}{u - 1}$ then equation (13) becomes

$$-u^2 (j + 1) (j + \alpha + 3) a_{j+1} + (u^2 - 1) (j - \gamma) a_j = 0,$$

with the corresponding solution of (12)

$$f(x) = u^x (x + 1) \left( -x, -\gamma; \alpha + 3; 1 - u^{-2} \right),$$

generally only meaningful for $x = -1, 0, 1, 2, \ldots$. In this parametrization,

$$u = \frac{1 + \gamma}{\alpha + 2 + \gamma} \quad \text{and} \quad \lambda = \frac{(\alpha + 1)^2}{(\gamma + 1) (\alpha + \gamma + 2)}.$$ (The terminating solutions of course were found already, $\gamma = n$ for $h_n^\alpha$.)

The asymptotic behavior of Meixner polynomials and their zeros has been investigated by Jin and Wong [4], [5], a typical situation being the properties of $M_n(nx; \gamma, c)$ as $n \to \infty$ for fixed $x, \gamma, c$. But here $c$ depends on $n$, and a simple pointwise result is valid:

**Proposition 6** \( \lim_{n \to \infty} h_n^\alpha(x) = h_\infty^\alpha(x) \) for each $x \in \mathbb{R}$.

**Proof.** In $h_n^\alpha$ the argument of the $\left( -x, -\gamma; \alpha + 3; 1 - u^{-2} \right)$ is $1 - u_{n+1}^{-2} = -\frac{(\alpha+1)(\alpha+2n+3)}{(n+1)^2}$. Then $u_{n+1} \to 1$ and

$$\frac{(-x)_j (-n)_j}{(\alpha + 3)_j j!} \left( -\frac{(\alpha + 1)(\alpha + 2n + 3)}{(n+1)^2} \right)^j \to \frac{(-x)_j}{(\alpha + 3)_j j!} (2 (\alpha + 1))^j,$$

for each $j$ as $n \to \infty$. Since $\left| (-n)_j / n^j \right| \leq 1$ the dominated convergence theorem applies to prove the claim. \( \blacksquare \)
For fixed $k = 1, 2, 3, \ldots$ the first (in increasing order) $k$ zeros of $h_n^\alpha$ tend to the first $k$ zeros of $h_\infty^\alpha$.

In this paper we gave an elementary proof of the strange orthogonality of Laguerre functions associated with the wave functions of the Coulomb potential and found an analogous orthogonality for Meixner polynomials.

References

[1] Beckers, J. and Debergh, N.: On a parastatistical hydrogen atom and its supersymmetric properties, *Phys. Lett. A* 178 (1993), 43-46.

[2] Blaive, B. and Cadilhac, M.: Upper limit of the discrete hydrogen-like wave functions: expansion in the inverse principal quantum number $n^{-1}$, *J. Math. Phys.* 38 (1997), 6061-6071.

[3] Dehesa, J., Yáñez, R., Aptekarev, A. and Buyarov, V.: Strong asymptotics of Laguerre polynomials and information entropies of two-dimensional harmonic oscillator and one-dimensional Coulomb potentials, *J. Math. Phys.* 39 (1998), 3050-3060.

[4] Jin, X-S. and Wong, R.: Uniform asymptotic expansions for Meixner polynomials, *Constr. Approx.* 14 (1998), 113-150.

[5] Jin, X-S. and Wong, R.: Asymptotic formulas for the zeros of the Meixner polynomials, *J. Approx. Theory* 96 (1999), 281-300.

[6] Koekoek, R. and Swarttouw, R.: *The Askey Scheme of Hypergeometric Orthogonal Polynomials and its q-Analogue*, Report 98-17, Delft University of Technology, 1998.

[7] Robinett, R.: *Quantum Mechanics*, Oxford University Press, New York, Oxford, 1997.

[8] Szegö, G.: *Orthogonal Polynomials* 3rd ed., Colloquium Publications vol. 23, Amer. Math. Soc., Providence, 1967.

[9] Xu, B-W. and Kong, F-M.: Factorization of the radial Schrödinger equation of the hydrogen atom, *Phys. Lett. A* 259 (1999), 212-214.