Asymptotics of Reinforcement Learning with Neural Networks

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Abstract

We prove that a single-layer neural network trained with the Q-learning algorithm converges in distribution to a random ordinary differential equation as the size of the model and the number of training steps become large. Analysis of the limit differential equation shows that it has a unique stationary solution which is the solution of the Bellman equation, thus giving the optimal control for the problem. In addition, we study the convergence of the limit differential equation to the stationary solution. As a by-product of our analysis, we obtain the limiting behavior of single-layer neural networks when trained on i.i.d. data with stochastic gradient descent under the widely-used Xavier initialization.

1 Introduction

Reinforcement learning with neural networks (frequently called “deep reinforcement learning”) has had a number of recent successes, including learning to play video games [22, 23], mastering the game Go [31], and robotics [16]. In deep reinforcement learning, a neural network is trained to learn the optimal action given the current state.

Despite many advances in applications, a number of mathematical questions remain open regarding reinforcement learning with neural networks. Our paper studies the Q-learning algorithm with neural networks (typically called “deep Q-learning”), which is a popular reinforcement learning method for training a neural network to learn the optimal control for a stochastic optimal control problem. The deep Q-learning algorithm uses a neural network to approximate the value of an action \( a \) in a state \( x \) [22]. This neural network approximator is called the “Q-network”. The Q-learning algorithm estimates the Q-network by taking stochastic steps which attempt to train the Q-network to satisfy the Bellman equation.

The literature on (deep) reinforcement learning and Q-learning is substantial. Instead of providing a complete literature review here we refer interested readers to classical texts [2, 19, 35], to the more recent book [12], and to the extensive survey on recent developments in [1]. The majority of reinforcement learning algorithms are based on some variation of the Q-learning or policy gradient methods [36]. Q-learning originated in [40] and proofs of convergence can be found in [41, 37]. The neural network approach to reinforcement learning (i.e., using Q-networks) was proposed in [22]. More recent developments include deep recurrent Q-networks [13], dueling architectures for deep reinforcement learning [39], double Q-learning [28], bootstrapped deep Q-networks [27], and asynchronous methods for deep reinforcement learning [24]. Although the performance of Q-networks has been extensively studied in numerical experiments, there has been relatively little theoretical investigation.

We study the behavior of a single-layer Q-network in the asymptotic regime of large numbers of hidden units and large numbers of training steps. We prove that the Q-network (which models the value function for the related optimal control problem) converges to the solution of a random ordinary differential equation (ODE). We characterize the limiting random ODE in both the infinite and finite time horizon discounted reward cases. Then, we study the behavior of the solution to the limiting random ODE as time \( t \to \infty \).

In the infinite time horizon case, we show that the limit ODE has a unique stationary solution which equals the solution of the associated Bellman equation. Thus, the unique stationary solution of the limit

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Q-network gives the optimal control for the problem. In the infinite time horizon case, we also show that the limit ODE converges to the unique stationary solution for small values of the discount factor. Convergence of ODEs to stationary solutions has been studied in related problems in the classical papers [3,37]. The difference in our work is that, in contrast to [3,37], we study the effect of a neural network as a function approximator in the Q-learning algorithm.

The presence of a neural network in the Q-learning algorithm introduces additional technical challenges, which lead us to be able to prove, in the infinite time horizon case, convergence of the limiting ODE to the stationary solution only for small values of the discount factor. We elaborate more on this issue in Remark 3.6. The situation is somewhat different in the finite time horizon case, where we can prove that the limit ODE converges to a global minimum, which is the solution of the associated Bellman equation, for all values of the discount factor in (0,1].

As a by-product of our analysis, we also prove that a single-layer neural network trained on i.i.d. data with stochastic gradient descent under the Xavier initialization [10] converges to a limit ODE. In addition to characterizing the limiting behavior of the neural network as the number of hidden units and stochastic gradient descent steps grow to infinity, we also obtain that the neural network in the limit converges to a global minimum with zero training loss (see Section 4). Convergence to a global minimum for a neural network in regression or classification on i.i.d. data (not in the reinforcement learning setting) has been recently proven in [6], [7], and [38]. Our result shows that convergence to a global minimum can also be viewed as a simple consequence of the limit ODE for neural networks.

The rest of the paper is organized as follows. The Q-learning algorithm is introduced in Section 2. Section 3 presents our main theorems. Section 4 discusses the limiting behavior of single-layer neural networks when trained on i.i.d. data with stochastic gradient descent under the Xavier initialization. Section 5 contains the proof for the infinite time horizon reinforcement learning case. The proof for the finite time horizon reinforcement learning case is in Section 6. Section 7 contains a proof that a certain matrix in the limit ODE is positive definite, which is useful for establishing convergence properties of the limiting ODEs. Appendix A collects the proofs of intermediate results.

2 Q-learning Algorithm

We consider a Markov decision problem defined on the finite state space $\mathcal{X} \subset \mathbb{R}^d$. For every state $x \in \mathcal{X}$ there is a finite set $A \subset \mathbb{R}^d$ of actions that can be taken. The homogeneous Markov chain $x_k \in \mathcal{X}$ has a probability transition function $P[x_{k+1} = z|x_k = x, a_j = a] = p(z|x, a)$ which governs the probability that $x_{k+1} = z$ given that $x_k = x$ and $a_j = a$. For every state $x$ and action $a$ there is a reward function $r(x,a)$. Let $\lambda$ denote an admissible control policy (i.e., it is chosen based on a probability law such that it depends only on the history up to the present).

For a given initial state $x \in \mathcal{X}$ and admissible control policy $\lambda$, the infinite time horizon reward is defined to be

$$W_\lambda(x) = \mathbb{E}_\lambda \left[ \sum_{j=0}^{\infty} \gamma^j r(x_j, a_j) | x_0 = x \right],$$

where the actions $a_j$ for $j \geq 0$ are chosen according to the policy $\lambda$ and $\gamma \in (0,1]$ is the discount factor.

Let $V(x,a)$ be the reward given that we start at state $x \in \mathcal{X}$, action $a \in A$ is taken, and the optimal policy is subsequently used. As is well known (see for example [19]), $\max_{a \in A} V(x,a) = \sup_{\lambda} W_\lambda(x)$ and the maximum expected reward $V$ satisfies the Bellman equation

$$0 = r(x,a) + \gamma \max_{z \in \mathcal{X}} \max_{a' \in A} V(z, a') p(z|x, a) - V(x,a), \quad (2.1)$$

where $a^*(x) = \arg\max_{a \in A} V(x,a)$ is an optimal policy. The Bellman equation (2.1) can be derived using the principle of optimality for dynamic programming.

In the finite time horizon case, for a given initial state $x \in \mathcal{X}$ and admissible control policy $\lambda$, the finite time horizon reward is defined to be

$$W_\lambda(J,x) = \mathbb{E}_\lambda \left[ \sum_{j=0}^{J} \gamma^j r_j | x_0 = x \right], \quad (2.2)$$
where \( r_j = r(j, x_j, a_j) \) for \( j = 0, 1, \ldots, J - 1 \) and \( r_J = r(J, x_J) \).

Similar to the infinite time horizon discount case, the optimal control \( a^*(j, x) \) is given by the solution to the Bellman equation

\[
V(j, x, a) = r(j, x, a) + \gamma \max_{a'} V(j + 1, z, a') p(z|x, a), \quad j = 0, 1, \ldots, J - 1,
\]

\[
V(J, x, a) = r(J, x),
\]

(2.3)

with the optimal control given by \( a^*(j, x) = \arg \max_a V(j, x, a) \).

In principle, the Bellman equations (2.1) and (2.3) can be solved to find the optimal control. However, there are two obstacles. First, the transition probability function \( p(z|x, a) \) (i.e., the state dynamics) may not be known. Secondly, even if it is known, the state space may be too high-dimensional for standard numerical methods to solve (2.1) and (2.3) due to the curse of dimensionality. For these reasons, reinforcement learning methods can be used to learn the solution to the Bellman equations (2.1) and (2.3).

Reinforcement learning approximates the solution to the Bellman equation with a function approximator, which typically is a neural network model. The parameters \( \theta \) (i.e., the weights) of the neural network are estimated using the Q-learning algorithm. The neural network \( Q(x, a; \theta) \) in Q-learning is referred to as a “Q-network”.

The Q-learning algorithm attempts to minimize the objective function

\[
L(\theta) = \sum_{(x, a) \in X \times A} \left[ (Y(x, a) - Q(x, a; \theta))^2 \right] \pi(x, a),
\]

(2.4)

where \( \pi(x, a) \) is a probability mass function (to be specified later on) which is strictly positive for every \( (x, a) \in X \times A \) and the “target” \( Y \) is

\[
Y(x, a) = r(x, a) + \gamma \max_{a'} Q(x', a'; \theta) p(x'|x, a).
\]

In the case of the infinite time horizon problem (and analogously for the finite time horizon problem), if \( L(\theta) = 0 \), then \( Q(x, a; \theta) \) is a solution to the Bellman equation (2.1). In practice, the hope is that the Q-learning algorithm will learn a model \( Q \) such that \( L(\theta) \) is small and therefore \( Q(x, a; \theta) \) is a good approximation for the Bellman solution \( V(x, a) \).

The Q-learning updates for the parameters \( \theta \) are:

\[
\theta_{k+1} = \theta_k - \alpha G_k,
\]

\[
G_k = \left( r(x_k, a_k) + \gamma \max_{a'} Q(x_{k+1}, a'; \theta_k) - Q(x_k, a_k; \theta_k) \right) \nabla \theta Q(x_k, a_k; \theta_k),
\]

(2.5)

where \((x_k, a_k)\) is an ergodic Markov chain with \( \pi(x, a) \) as its limiting distribution.

The Q-network, which models the value of a state \( x \) and action \( a \), is the neural network

\[
Q^N(x, a; \theta) = \frac{1}{\sqrt{N}} \sum_{i=1}^N C^i \sigma(W^i \cdot (x, a)),
\]

(2.6)

where \( C^i \in \mathbb{R}, W^i \in \mathbb{R}^d, x \in X \subseteq \mathbb{R}^{dX}, a \in A \subset \mathbb{R}^{dA}, d = d_X + d_A, \) and \( \sigma(\cdot) : \mathbb{R} \to \mathbb{R} \). The parametric model (2.6) receives an input vector containing both the state and action in the enlarged Euclidean space \( \mathbb{R}^d \). This formulation is a common choice in practice; see for example [8]. Other variations of the parametric model (2.6), such as an input vector of the state and an output vector which is the length of the number of possible actions, are of course possible and can also be studied using this paper’s techniques.

The number of hidden units is \( N \) and the output is scaled by a factor \( \frac{1}{\sqrt{N}} \), which is commonly used in practice and is called the “Xavier initialization” [10]. The set of parameters that must be estimated is

\[
\theta = (C^1, \ldots, C^N, W^1, \ldots, W^N) \in \mathbb{R}^{(1+d)N}.
\]
In the infinite-time horizon case, the Q-learning algorithm for training the parameters $\theta$ is

$$C_{k+1} = C_k + \frac{\alpha^N}{\sqrt{N}} \left( r_k + \gamma \max_{a' \in A} Q^N(x_{k+1}, a'; \theta_k) - Q^N(x_k, a_k; \theta_k) \right) \sigma(W_k^i \cdot (x_k, a_k)),$$

$$W_{k+1} = W_k^i + \frac{\alpha^N}{\sqrt{N}} \left( r_k + \gamma \max_{a' \in A} Q^N(x_{k+1}, a'; \theta_k) - Q^N(x_k, a_k; \theta_k) \right) C_k^i \sigma'(W_k^i(x_k, a_k))(x_k, a_k),$$

$$Q^N(x, a; \theta_k) = \frac{1}{\sqrt{N}} \sum_{i=1}^N C_k^i \sigma(W_k^i \cdot (x, a)),$$ (2.7)

for $k = 0, 1, \ldots$. We assume that the action $a_k$ is sampled uniformly at random from all possible actions $A$ (i.e., “pure exploration”).

In this paper, we study the asymptotic behavior of the Q-network $Q^N(x, a; \theta_k)$ as the number of hidden units $N$ and number of stochastic gradient descent iterates $k$ go to infinity. As we will see, after appropriate scalings, the Q-network converges to the solution of a limiting ODE.

It is worthwhile noting that the Q-learning algorithm is similar to the stochastic gradient descent algorithm (which we also discuss in Section 4), the Q-learning update directions $G_k$ are not necessarily unbiased estimates of a descent direction for the objective function $L(\theta)$. The Q-learning algorithm calculates its update by taking the derivative of $L(\theta)$ while treating the target $Y$ as a constant. Since $Y$ actually depends upon $\theta$,

$$\mathbb{E}[G_k \theta_k, x_k, a_k] \neq \frac{1}{2} \nabla_{\theta} \left( Y(x_k, a_k) - Q(x_k, a_k; \theta_k) \right)^2.$$ (2.8)

This fact together with the presence of the neural network function approximator leads to certain difficulties in the proofs. We will return to this issue in Remark 3.6.

Let us next present the main results of the paper in Section 3.

3 Main results

In this section we present our main results. We start with the infinite time horizon setting. We consider the Q-network (2.6) which models the value of a state and action. The parameters $\theta$ for the Q-network are trained using the Q-learning algorithm (2.7). We prove that, as the number of hidden units and training steps become large, the Q-network converges in distribution to a random ordinary differential equation.

Assumption 3.1. Our results are proven under the following assumptions:

- The activation function $\sigma \in C^2_b(\mathbb{R})$, i.e. $\sigma$ is twice continuously differentiable and bounded.
- The randomly initialized parameters $(C_0^i, W_0^i)$ are i.i.d., mean-zero random variables with a distribution $\mu_0(dw, dw)$, We assume that $\mu_0$ is absolutely continuous with respect to Lebesgue measure.
- The random variable $C_0^i$ is bounded and $\langle \|w\|, \mu_0 \rangle < \infty$.
- The reward function $r$ is uniformly bounded in its arguments.
- The Markov chain $x_k$ has a limiting distribution $\pi$, namely

$$\pi(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N 1_{\{x_k = x\} \mid x_0 = z},$$

exists, is independent of the initial state $z$, $\sum_{x \in \mathcal{X}} \pi(x) = 1$ and $\pi(x) > 0$ for all $x \in \mathcal{X}$.

- $\mathcal{X}$ and $\mathcal{A}$ are finite, discrete spaces.
We shall also assume that the action $a \in \mathcal{A}$ is sampled uniformly at random from all possible actions (referred to as “pure exploration”). The uniform distribution of the actions combined with the fact that $x_k$ is assumed to have a limiting distribution $\pi(x)$ imply that the Markov chain $\zeta_k = (x_k, a_k)$ will have limiting distribution $\pi(x, a) = \frac{1}{K} \pi(x)$ where $K = |\mathcal{A}|$. In addition, the Markov chain $(x_{k+1}, x_k, a_k)$ will have $\pi(x', x, a) = p(x'|x, a)\pi(x, a)$ as its limiting distribution.\footnote{There is a slight abuse of notation due to denoting all of these distributions with $\pi$. In our calculations, the specific limit distribution being used is clear via its argument $x, (x, a)$, or $(x', x, a)$.}

**Assumption 3.2.** Certain properties of the limit ODE also require the following assumptions:

- The activation function $\sigma$ is non-polynomial (e.g., a tanh or sigmoid function).
- If $C_0 \in [-B, B]$, $\mu_0(\Gamma) > 0$ for any set $\Gamma \subset [-B, B] \times \mathbb{R}^1 d$ with positive Lebesgue measure.

Define the empirical measure

$$\nu^N_k = \frac{1}{N} \sum_{i=1}^N \delta_{C_i}, W_k.$$  \hspace{1cm} (3.1)

In addition, let us set $Q^N_k(x, a) = Q^N(x, a; \theta_k)$ and define the scaled processes

$$h^N_t(x, a) = Q^N_{[N]}(x, a),$$

$$\mu^N_t = \nu^N_{[N]}.$$ 

Using Assumption\ref{assumption:3.1} we know that $\mu^N_0 \xrightarrow{d} \mu_0$ and $h^N_0 \xrightarrow{d} \mathcal{G}$ as $N \rightarrow \infty$ where $\mathcal{G}$ is a mean-zero Gaussian random variable.

The variable $h^N_t$ is the output of the neural network after $\frac{1}{\epsilon} \times 100\%$ of the training has been completed.

We will study convergence in distribution of the random process $(\mu^N_t, h^N_t)$ as $N \rightarrow \infty$ in the space $D_E([0, T])$ where $E = M(\mathbb{R}^1 d) \times \mathbb{R}^{[X \times \mathcal{A}]}$. $D_E([0, T])$ is the Skorokhod space and $M(S)$ is the space of probability measures on $S$.

Before presenting the first main convergence result, Theorem\ref{theorem:3.4} we present a lemma stating that a certain matrix $A$ which appears in the limit ODE is positive definite.

**Lemma 3.3.** Let Assumption\ref{assumption:3.1} hold. Consider the matrix $A$ with elements

$$A_{\zeta, \zeta'} = \alpha \langle \sigma(w \cdot \zeta)\sigma(w \cdot \zeta'), \mu_0 \rangle + \langle \epsilon^2 \sigma'(w \cdot \zeta)\sigma'(w \cdot \zeta')\zeta \top \zeta', \mu_0 \rangle,$$

for $\zeta, \zeta' \in \{\zeta^{(1)}, \ldots, \zeta^{(M)}\}$ where $\zeta^{(i)} \in \mathcal{S} \subset \mathbb{R}^d$ are distinct (i.e., $\zeta^{(i)} \neq \zeta^{(j)}$ for $i \neq j$).

Then, the matrix $A$ is positive definite.

**Proof.** The proof is deferred to Section\ref{section:7} \hfill \qed

We then have the following theorem.

**Theorem 3.4.** Let Assumption\ref{assumption:3.1} hold and let the learning rate be $\alpha^N = \frac{\alpha}{N}$. The process $(\mu^N_t, h^N_t)$ converges in distribution in the space $D_E([0, T])$ as $N \rightarrow \infty$ to $(\mu_t, h_t)$, for $t \in [0, T]$ which, for $(x, a) \in \mathcal{X} \times \mathcal{A}$, satisfies, for every $f \in C^b_2(\mathbb{R}^{1+1})$, the random ODE

$$h_t(x, a) = h_0(x, a) + \int_0^t \sum_{(x', a') \in \mathcal{X} \times \mathcal{A}} \pi(x', a')A_{x, a, x', a'}\left(r(x', a') + \gamma \max_{z \in \mathcal{X}} h_s(z, a'')p(z|x', a') - h_s(x', a')\right)ds,$$

$$h_0(x, a) = \mathcal{G}(x, a),$$

$$< f, \mu_t > = < f, \mu_0 >.$$ \hspace{1cm} (3.2)

The tensor $A$ is

$$A_{x, a, x', a'} = \alpha \langle \sigma(w \cdot (x', a'))\sigma(w \cdot (x, a)), \mu_0 \rangle + \langle \epsilon^2 \sigma'(w \cdot (x', a'))\sigma'(w \cdot (x, a)) (x', a') \top (x, a), \mu_0 \rangle.$$

Furthermore, if Assumption\ref{assumption:3.2} holds, (3.2) has a unique stationary point which equals the solution $V$ of the Bellman equation (2.1).
Lemma 3.5. Let $\gamma < \frac{2}{1+R}$ where $K$ is the number of possible actions in the set $\mathcal{A}$. Then, we have

$$\lim_{t \to \infty} \sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} |h_t(x,a) - V(x,a)| = 0.$$  

Remark 3.6. Convergence of ODEs of the type (3.2) to solutions of the corresponding stationary equation has been studied in the literature in [3, 37]. The difference between our case and these earlier works is the nature of the matrix $A$ which appears in the ODE (e.g., see equation (3.2) with the matrix $A$). In previous papers such as [3, 37], the matrix $A$ is either an identity matrix or a diagonal matrix with diagonal elements uniformly bounded away from zero and with an upper bound of one. In our case, the Q-learning algorithm with a neural network produces an ODE with a matrix $A$ that is not a diagonal matrix. The arguments of [3, 37] do not establish convergence in the case where $A$ is non-diagonal. Lemma 3.5 proves convergence for a non-diagonal matrix $A$ for small $\gamma$.

Despite our best efforts we did not succeed in proving Lemma 3.5 for all $0 < \gamma < 1$ in our general case with the non-diagonal matrix $A$, which is produced by the neural network approximator in the Q-learning algorithm. As we discussed in Section 2 the difficulties that arise here are also related to the fact that the Q-learning algorithm calculates its update by taking the derivative of $L(\theta)$ while treating the target $Y$ as a constant. Hence, the asymptotic dynamics of the Q-network as $N$ and $k$ grow to infinity, which is the solution to the ODE (3.2), may not necessarily move in the descent direction of the limiting objective function (this is in contrast to the standard regression problem with i.i.d. data that we study in Section 4).

However, as shown in Theorem 3.8 below, one can prove convergence for all values of the discount factor $\gamma \in (0, 1)$ in the finite time horizon case. We are able to prove convergence for all $\gamma \in (0, 1]$ because in the finite time horizon case one can study the large time limit of the limiting ODE recursively.

We now consider the finite time horizon problem. The Q-network, which models the value of state $x$ and action $a$ at time $j$, is

$$Q(j, x, a) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} C^i \sigma(W^i \cdot (x, a)), $$

where $C^i \in \mathbb{R}^J$, $W^i \in \mathbb{R}^d$, $d = d_x + d_a$, and $\sigma : \mathbb{R} \to \mathbb{R}$. Note that the parameter $W^i$ is shared across all times $j$.

The model parameters $\theta$ are trained using the Q-learning algorithm. The parameter updates are given by, for training iterations $k = 0, 1, \ldots$ and times $j = 0, \ldots, J-1$, the following equations:

$$C_{k+1}^{i,j} = C_k^{i,j} + \frac{\alpha}{\sqrt{N}} \left( r_{k,j} + \gamma \max_{a' \in \mathcal{A}} Q^N(j+1, x_{k,j+1}, a'; \theta_k) - Q^N(j, x_{k,j}, a_{k,j}; \theta_k) \right) \sigma(W^i_k \cdot (x_{k,j}, a_{k,j})), $$

$$W_{k+1}^{i,j} = W_k^{i,j} + \frac{\alpha}{\sqrt{N}} \sum_{j=0}^{J-1} \left( r_{k,j} + \gamma \max_{a' \in \mathcal{A}} Q^N(j+1, x_{k,j+1}, a'; \theta_k) - Q^N(j, x_{k,j}, a_{k,j}; \theta_k) \right)$$

$$\times C_k^{i,j} \sigma(W^i_k(x_{k,j}, a_{k,j}))(x_{k,j}, a_{k,j}), $$

$$Q^N(j, x, a; \theta_k) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} C_k^{i,j} \sigma(W^i_k \cdot (x, a)), $$

where $(x_{k,j}, a_{k,j})_{k=1}^{N}$ are independent random variables and $r_{k,j} = r(j, x_{k,j}, a_{k,j})$. For notational convenience, define $Q^N_k(j, x, a) = Q^N(j, x, a; \theta_k)$ and $Q^N(J, x, a; \theta_k) = r(J, x)$.
Assumption 3.7. Certain properties of the limit ODE also require the following assumptions:

- The activation function $\sigma$ is non-polynomial (e.g., tanh or sigmoid functions).
- If $C_0^\top \in [-B, B]^j$, $\mu_0(\Gamma) > 0$ for any set $\Gamma \subset [-B, B]^j \times \mathbb{R}^d$ with positive Lebesgue measure.

Similar to the infinite time horizon case, we define the processes $h_k^{N,j} = \frac{1}{N} \sum_{i=1}^{N} \delta_{C_k^i,j,W_i}$, $\mu_k^{N,j} = \gamma_{|Nt|}$, and $h_0^{N,j} = Q_{|Nt|}(j, x, a)$. We will study convergence in distribution of the random process $(\mu_k^{N,j}, h_0^{N,j})$ as $N \to \infty$ in the space $D_E([0, T])$ where $E = \mathcal{M}(\mathbb{R}^{j+d}) \times \mathbb{R}^{|X | \times |A |}$. Denote the probability distribution of $(x_k,j,a_k,j)$ denoted as $\pi_j(x_k,a_k)$. We then have the following theorem.

**Theorem 3.8.** Let Assumption 3.1 hold and let the learning rate be $\alpha^N = \frac{\alpha}{N}$. The process $(\mu_k^{N,j}, h_k^{N,j})$ converges in distribution in the space $\mathcal{M}(\mathbb{R}^{j+d})$ to $(\mu_t, h_t)$, which, for $t \in [0, T]$, $j = 1, 2, \ldots, J$ and $(x, a) \in X \times A$, satisfies, for every $f \in C_2^b(\mathbb{R}^{1+d})$, the random ODE

$$
h_t(j, x, a) = h_0(j, x, a) + \mathbb{E}_0 \left[ \sum_{(x', a') \in X \times A} \pi_j(x', a') A_{x,a,x',a'} \left( r(j, x', a') - h_t(j, x', a') \right) \right] ds,
$$

where $j = 0, 1, \ldots, J - 1$ and $h_t(J, x, a) = r(J, x)$. The tensor $A$ is

$$
A_{x,a,x',a'} = \alpha (\sigma(w \cdot (x', a'))\sigma(w \cdot (x, a)), \mu_0) + \left( \alpha^2 \sigma'(w \cdot (x', a'')\sigma'(w \cdot (x, a))(x', a')^\top(x, a), \mu_0) \right).
$$

Furthermore, if Assumption 3.7 holds, the neural network converges to the solution of the Bellman equation (2.3):

$$
\lim_{t \to \infty} \sup_{j, x, a} |h_t(j, x, a) - V(j, x, a)| = 0.
$$

**Proof.** The proof of this result is in Section 6.

4 A special case: neural networks and regression

The asymptotic approach developed in this paper can be used to study other popular cases in machine learning. For example, consider the case of the objective function (2.1) but with $y_k$ now being independent samples from a fixed distribution. Then, (2.4) is simply the mean-squared error objective function for regression. Using the same techniques as we employ on (the more difficult) Q-learning problem discussed in the previous section, we can establish the asymptotic behavior of neural network models used in regression. Let the neural network be

$$
g^N(x; \theta) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} C_i \sigma(W_i \cdot x),
$$

where $C_i \in \mathbb{R}$, $W_i \in \mathbb{R}^d$, $x \in \mathbb{R}^d$, and $\sigma(\cdot) : \mathbb{R} \to \mathbb{R}$. The objective function is

$$
\mathcal{L}^N(\theta) = \mathbb{E} \left[ (Y - g^N(X; \theta))^2 \right],
$$

where the data $(X, Y) \sim \pi(dx, dy)$, $Y \in \mathbb{R}$, and the parameters $\theta = (C^1, \ldots, C^N, W^1, \ldots, W^N) \in \mathbb{R}^{N \times (1+d)}.$
The model parameters \( \theta \) are trained using stochastic gradient descent. The parameter updates are given by:

\[
C^{i}_{k+1} = C^{i}_{k} + \frac{\alpha^{N}}{\sqrt{N}} (y^{k} - g^{N}_{k}(x^{k})) \sigma(W^{i}_{k} \cdot x^{k}),
\]

\[
W^{i}_{k+1} = W^{i}_{k} + \frac{\alpha^{N}}{\sqrt{N}} (y^{k} - g^{N}_{k}(x^{k})) C^{i}_{k} \sigma'(W^{i}_{k} \cdot x^{k})x^{k},
\]

\[
g^{N}_{k}(x) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} C^{i}_{k} \sigma(W^{i}_{k} \cdot x),
\]

for \( k = 0, 1, \ldots \). \( \alpha^{N} \) is the learning rate (which may depend upon \( N \)). The data samples are \( (x^{k}, y^{k}) \) are i.i.d. samples from a distribution \( \pi(dx, dy) \).

In Theorem 4.2 we prove that a neural network with the Xavier initialization (i.e., with the \( \frac{1}{\sqrt{N}} \) normalization in the formula for \( g^{N}(x; \theta) \)) and trained with stochastic gradient descent converges in distribution to a random ODE as the number of units and training steps become large. Although the pre-limit problem of optimizing a neural network is non-convex (and therefore the neural network may converge to a local minimum), the limit equation minimizes a quadratic objective function. In Theorem 4.3, we also show that the neural network (in the limit) will converge to a global minimum with zero loss on the training set. Convergence to a global minimum for a neural network has been recently proven in \([6, 7, 38]\). Our result shows that convergence to a global minimum can also be viewed as a simple consequence of the mean-field limit for neural networks.

For completeness, we also mention here that other scaling regimes have also been studied in the literature. In particular, \([1, 25, 29, 32, 33]\) study the asymptotics of single-layer neural networks with a \( \frac{1}{\sqrt{N}} \) normalization; that is, \( g^{N}(x; \theta) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} C^{i} \sigma(W^{i} \cdot x) \). \([34]\) studies the asymptotics of deep (i.e., multi-layer) neural networks with a \( \frac{1}{\sqrt{N}} \) normalization in each hidden layer. The \( \frac{1}{\sqrt{N}} \) normalization is convenient since the single-layer neural network is then in a traditional mean-field framework where it can be described via an empirical measure of the parameters. In the single layer case, the limit for the neural network satisfies a partial differential equation. As discussed in \([25]\), it is not necessarily true that the limiting equation (a PDE in this case) will converge to the global minimum of an objective function with zero training error. However, the \( \frac{1}{\sqrt{N}} \) normalization that we study in this paper is more widely-used in practice (see \([10]\) and, importantly, as we demonstrate in Theorem 4.3, the limit equation converges to a global minimum with zero training error.

Lastly, we mention here that \([15]\) proved, using different methods, a limit for neural networks with a \( \frac{1}{\sqrt{N}} \) Xavier initialization when they are trained with continuous-time gradient descent. Our result in Theorem 4.2 proves a limit for neural networks trained with the (standard) discrete-time stochastic gradient descent algorithm which is used in practice, and rigorously passes from discrete time (where the stochastic gradient descent updates evolve) to continuous time.

**Assumption 4.1.** We impose the following assumptions:

- The activation function \( \sigma \in C^{2}_{0}(\mathbb{R}) \), i.e., \( \sigma \) is twice continuously differentiable and bounded.
- The randomly initialized parameters \( (C^{i}_{0}, W^{i}_{0}) \) are i.i.d., mean-zero random variables with a distribution \( \mu_{0}(dc, dw) \).
- The random variable \( C^{i}_{0} \) has compact support and \( \langle \|w\|, \mu_{0} \rangle < \infty \).
- The sequence of data samples \( (x^{k}, y^{k}) \) is i.i.d. from the probability distribution \( \pi(dx, dy) \). In particular, there is a fixed dataset of \( M \) data samples \( (x^{(i)}, y^{(i)})_{i=1}^{M} \) and therefore \( \pi(dx, dy) = \frac{1}{M} \sum_{i=1}^{M} \delta_{(x^{(i)}, y^{(i)})}(dx, dy) \).

Note that the last assumption also implies that \( \pi(dx, dy) \) has compact support.

Following the asymptotic procedure developed in this paper, we can study the limiting behavior of the network output \( g^{N}_{k}(x) = g^{N}(x; \theta^{k}) \) for \( x \in \mathcal{D} = \{x^{(1)}, \ldots, x^{(M)}\} \) as the number of hidden units \( N \) and stochastic gradient descent steps \( k \) simultaneously become large, after appropriately relating \( k \) and \( N \). The network output converges in distribution to the solution of a random ODE as \( N \to \infty \).
For this purpose, let us recall the empirical measure defined in (3.1). Note that the neural network output can be written as the inner-product

\[ g_k^N(x) = \left\langle c\sigma(w \cdot x), \sqrt{N} \nu_k^N \right\rangle. \]

Due to Assumption 4.1 as \( N \to \infty \) and for \( x \in \mathcal{D} \),

\[ g_0^N(x) \overset{\mathcal{D}}{\to} \mathcal{G}(x), \]

where \( \mathcal{G} \in \mathbb{R}^M \) is a Gaussian random variable. We also of course have that

\[ \nu_0^N \overset{\mathcal{D}}{\to} \nu_0 \equiv \mu_0. \]

Define the scaled processes

\[ h_t^N = g_t^N(x^{(1)}), \quad \mu_t^N = \nu_t^N \]

where \( g_k^N = \left( g_k^N(x^{(1)}), \ldots, g_k^N(x^{(M)}) \right) \), \( h_t^N(x) = g_t^N(x^{(1)})(x) \), and \( h_t^N = \left( h_t^N(x^{(1)}), \ldots, h_t^N(x^{(M)}) \right) \).

Now, we are ready to state the main result of this section, Theorem 4.2.

**Theorem 4.2.** Let Assumption 4.1 hold, set \( \alpha^N = \frac{1}{N} \) and define \( E = \mathcal{M}(\mathbb{R}^{1+d}) \times \mathbb{R}^M \). The process \((\mu_t^N, h_t^N)\) converges in distribution in the space \( D_E([0, T]) \) as \( N \to \infty \) to \((\mu_t, h_t)\) which satisfies, for every \( f \in C^2_b(\mathbb{R}^{1+d}) \), the random ODE

\[
\begin{align*}
\dot{h}_t(x) &= h_0(x) + \alpha \int_{X \times Y} (y - h_t(x')) \langle \sigma(w \cdot x)\sigma(w \cdot x'), \mu_t \rangle \pi(dx', dy) dt \\
&+ \alpha \int_{X \times Y} (y - h_t(x')) \langle c^2 \sigma'(w \cdot x)\sigma'(w \cdot x)x^\top x', \mu_t \rangle \pi(dx', dy) dt,
\end{align*}
\]

\[ h_0(x) = \mathcal{G}(x), \]

\[ (f, \mu_t) = (f, \mu_0). \] \( \square \)

**Proof.** The proof of this theorem is omitted because it is exactly parallel to the proof of Theorem 3.4.

Recall that \( \mathcal{G} \in \mathbb{R}^M \) is a Gaussian random variable; see equation (4.1). In addition, note that the limit equation (4.2) is a constant, i.e. \( \mu_t = \mu_0 \) for \( t \in [0, T] \). Therefore, (4.2) reduces to

\[
\begin{align*}
\dot{h}_t(x) &= h_0(x) + \alpha \int_{X \times Y} (y - h_t(x')) \langle \sigma(w \cdot x)\sigma(w \cdot x'), \mu_0 \rangle \pi(dx', dy) dt \\
&+ \alpha \int_{X \times Y} (y - h_t(x')) \langle c^2 \sigma'(w \cdot x)\sigma'(w \cdot x)x^\top x', \mu_0 \rangle \pi(dx', dy) dt,
\end{align*}
\]

\[ h_0(x) = \mathcal{G}(x). \] \( \square \)

Since (4.3) is a linear equation in \( C_{\mathbb{R}^M}([0, T]) \), the solution \( h_t \) is unique.

To better understand (4.3), define the matrix \( A = \mathbb{R}^{M \times M} \) where

\[
A_{x,x'} = \frac{1}{M} \left\langle \sigma(w \cdot x)\sigma(w \cdot x'), \mu_0 \right\rangle + \frac{1}{M} \left\langle c^2 \sigma'(w \cdot x)\sigma'(w \cdot x)x^\top x', \mu_0 \right\rangle,
\]

where \( x, x' \in \mathcal{D} \). \( A \) is finite-dimensional since we fixed a training set of size \( M \) in the beginning. Then, (4.3) becomes

\[
\begin{align*}
\dot{h}_t &= A \left( \dot{Y} - h_t \right) dt, \\
h_0 &= \mathcal{G},
\end{align*}
\]

9
where $\hat{Y} = (y^{(1)}, \ldots, y^{(M)})$.

Therefore, $h_t$ is the solution to a continuous-time gradient descent algorithm which minimizes a quadratic objective function.

$$\frac{dh_t}{dt} = -\frac{1}{2} \nabla_h J(\hat{Y}, h_t),$$

$$J(y, h) = (y - h)^\top A (y - h),$$

$$h_0 = G.$$

Therefore, even though the pre-limit optimization problem is non-convex, the neural network’s limit will minimize a quadratic objective function.

An interesting question is whether $h_t \to \hat{Y}$ as $t \to \infty$. That is, in the limit of large numbers of hidden units and many training steps, does the neural network model converge to a global minimum with zero training error? Theorem 4.3 shows that $h_t \to \hat{Y}$ as $t \to \infty$ if $A$ is positive definite. Lemma 3.3 proves that, under reasonable hyperparameter choices and if the data samples are distinct, $A$ will be positive definite.

**Theorem 4.3.** If Assumption 3.2 holds and the data samples are distinct, then

$$h_t \to \hat{Y} \quad \text{as} \quad t \to \infty.$$

**Proof.** Consider the transformation $\tilde{h}_t = h_t - \hat{Y}$. Then,

$$d\tilde{h}_t = -A\tilde{h}_tdt,$$

$$\tilde{h}_0 = G - \hat{Y}.$$

Then, $\tilde{h}_t \to 0$ (and consequently $h_t \to \hat{Y}$) as $t \to \infty$ if $A$ is positive definite. Lemma 3.3 proves that $A$ is positive definite under the Assumption 3.2 and if the data samples are distinct.

In connection to Theorem 4.3 we mention that the data samples in the dataset will be distinct with probability 1 if the random variable $X$ has a probability density function.

5 Proof of Convergence in Infinite time Horizon Case

In this section we prove Theorem 5.3. The proof is divided into three parts. Let $\rho^N$ be the probability measure of a convergent subsequence of $(\mu^N, h^N)_{0 \leq t \leq T}$. In Section 5.1 we write the prelimit in a form that is convenient in order to establish the desired limiting behavior. In Section 5.2 we prove that any limit point of $\rho^N$ is a probability measure of the random ODE (3.2). In Section 5.3 we prove that the sequence $\rho^N$ is relatively compact (which implies that there is a subsequence $\rho^{N_k}$ which weakly converges). In Section 5.4 we prove that the limit point is unique. These three results are collected together in Section 5.5 to prove that $(\mu^N, h^N)$ converges in distribution to $(\mu, h)$.
5.1 Evolution of the Pre-limit Process

For notational convenience, let \( Q^N(x, a; \theta_k) = Q^N_k(x, a) \), \( \zeta = (x, a) \), and \( \zeta_k = (x_k, a_k) \). We study the evolution of \( Q^N_k(x, a) \) during training.

\[
Q^N_{k+1}(\zeta) = Q^N_k(\zeta) + \frac{1}{\sqrt{N}} \sum_{i=1}^{N} C_{k+1}^i \sigma(W^i_{k+1} \cdot \zeta) - \frac{1}{\sqrt{N}} \sum_{i=1}^{N} C_{k}^i \sigma(W^i_{k} \cdot \zeta)
\]

\[
= Q^N_k(\zeta) + \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left( C_{k+1}^i \sigma(W^i_{k+1} \cdot \zeta) - C_{k}^i \sigma(W^i_{k} \cdot \zeta) \right)
\]

\[
= Q^N_k(\zeta) + \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left( (C_{k+1}^i - C_{k}^i) \sigma(W^i_{k+1} \cdot \zeta) + (\sigma(W^i_{k+1} \cdot \zeta) - \sigma(W^i_{k} \cdot \zeta)) C_{k}^i \right)
\]

\[
= Q^N_k(\zeta) + \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left( (C_{k+1}^i - C_{k}^i) \right) \sigma(W^i_{k+1} \cdot \zeta) + \frac{1}{2} \sigma''(W^i_{k+1} \cdot \zeta) (W^i_{k+1} - W^i_{k})^2 C_{k}^i
\]

for points \( W^i_{k+1} \) and \( W^i_{k+1} \) in the line segment connecting the points \( W^i_{k} \) and \( W^i_{k+1} \). Let \( \alpha^N = \frac{\alpha}{N} \). Substituting (2.7) into (5.1) yields

\[
Q^N_{k+1}(\zeta) = Q^N_k(\zeta) + \frac{\alpha}{N^2} \left( r_k + \gamma \max_{a' \in A} Q^N_k(x_{k+1}, a') - Q^N_k(\zeta_k) \right) \sum_{i=1}^{N} \sigma(W^i_{k} \cdot \zeta_k) \sigma(W^i_{k} \cdot \zeta)
\]

\[
+ \frac{\alpha}{N^2} \left( r_k + \gamma \max_{a' \in A} Q^N_k(x_{k+1}, a') - Q^N_k(\zeta_k) \right) \sum_{i=1}^{N} \sigma'(W^i_{k} \cdot \zeta) \sigma'(W^i_{k} \cdot \zeta_k) \zeta_k^T \zeta (C_{k}^i)^2
\]

\[
+ O_p(N^{-3/2}).
\]

We can re-write the evolution of \( Q^N_k(\zeta) \) in terms of the empirical measure \( \nu^N_k \),

\[
Q^N_{k+1}(\zeta) = Q^N_k(\zeta) + \frac{\alpha}{N} \left( r_k + \gamma \max_{a' \in A} Q^N_k(x_{k+1}, a') - Q^N_k(\zeta_k) \right) \left\langle \sigma(w \cdot \zeta_k) \sigma(w \cdot \zeta), \nu^N_k \right\rangle
\]

\[
+ \frac{\alpha}{N} \left( r_k + \gamma \max_{a' \in A} Q^N_k(x_{k+1}, a') - Q^N_k(\zeta_k) \right) \left\langle \sigma'(w \cdot \zeta) \sigma'(w \cdot \zeta_k) \zeta_k^T \zeta, \nu^N_k \right\rangle
\]

\[
+ O_p(N^{-3/2}).
\]

Using (5.3), we can write the evolution of \( h^N_t(\zeta) \) for \( t \in [0, T] \) as

\[
h^N_t(\zeta) = h^N_0(\zeta) + \sum_{k=0}^{[Nt]-1} (Q^N_{k+1}(\zeta) - Q^N_k(\zeta))
\]

\[
= h^N_0(\zeta) + \frac{\alpha}{N} \sum_{k=0}^{[Nt]-1} \left( r_k + \gamma \max_{a' \in A} Q^N_k(x_{k+1}, a') - Q^N_k(\zeta_k) \right) \left\langle \sigma(w \cdot \zeta_k) \sigma(w \cdot \zeta), r^N_k \right\rangle
\]

\[
+ \frac{\alpha}{N} \sum_{k=0}^{[Nt]-1} \left( r_k + \gamma \max_{a' \in A} Q^N_k(x_{k+1}, a') - Q^N_k(\zeta_k) \right) \left\langle \sigma'(w \cdot \zeta) \sigma'(w \cdot \zeta_k) \zeta_k^T \zeta, \nu^N_k \right\rangle
\]

\[
+ O_p(N^{-1/2})
\]
This can then be rewritten as follows

\[
h_i^N(\zeta) = h_i^0(\zeta) + \sum_{k=0}^{N-1} (Q_k^N(\zeta) - Q_k^0(\zeta))
\]

\[
= h_i^0(\zeta) + \alpha \int_0^t \sum_{(\zeta', x') \in X \times x} \left( r(\zeta') + \gamma \max_{x''} h_i^N(x'', a'') - h_i^N(\zeta') \right)
\times \langle \sigma(w \cdot \zeta') \sigma(w \cdot \zeta), \mu_i^N \rangle \pi(x'', \zeta')ds
\]

\[
+ \alpha \int_0^t \sum_{(\zeta', x') \in X \times x} \left( r(\zeta') + \gamma \max_{x''} h_i^N(x'', a'') - h_i^N(\zeta') \right)
\times \langle \epsilon^2 \sigma'(w \cdot \zeta) \sigma'(w \cdot \zeta) \zeta^T \zeta, \mu_i^N \rangle \pi(x'', \zeta')ds
\]

\[+ M_1^{1,N} + M_1^{2,N} + M_1^{3,N} + o_p(N^{-1/2}), \tag{5.4}\]

where \(\pi(x'', \zeta') = p(x''|\zeta') \pi(\zeta')\). The fluctuation terms are

\[
M_1^{1,N}(\zeta) = -\frac{1}{N} \sum_{k=0}^{N-1} Q_k^N(\zeta)B_{\zeta',\zeta,k} + \frac{1}{N} \sum_{k=0}^{N-1} \sum_{\zeta' \in X \times A} Q_k^N(\zeta')B_{\zeta',\zeta,k}^N \pi(\zeta'),
\]

\[
M_1^{2,N}(\zeta) = \frac{1}{N} \sum_{k=0}^{N-1} r_k B_{\zeta',\zeta,k} - \frac{1}{N} \sum_{k=0}^{N-1} \sum_{\zeta' \in X \times A} r(\zeta')B_{\zeta',\zeta,k}^N \pi(\zeta'),
\]

\[
M_1^{3,N}(\zeta) = \frac{1}{N} \sum_{k=0}^{N-1} \gamma \max_{a''} Q_k^N(x_{k+1}, a'')B_{\zeta',\zeta,k}^N - \frac{1}{N} \sum_{k=0}^{N-1} \sum_{(\zeta', x') \in X \times x} \gamma \max_{a''} Q_k^N(x'', a'')B_{\zeta',\zeta,k}^N \pi(x'', \zeta'),
\]

where

\[
B_{\zeta',\zeta,k}^N = \alpha \left( \langle \sigma(w \cdot \zeta') \sigma(w \cdot \zeta), \nu_k^N \rangle + \langle \sigma'(w \cdot \zeta) \sigma'(w \cdot \zeta) \zeta^T \zeta, \nu_k^N \rangle \right). \tag{5.5}\]

Later on, in Lemma 5.3 we prove that that the fluctuation terms \(M_1^{1,N}(\zeta)\) go to zero in \(L^1\) as \(N \to \infty\).

The evolution of the empirical measure \(\nu_k^N\) can be characterized in terms of their projection onto test functions \(f \in C_0^\infty(\mathbb{R}^{1+d})\). A Taylor expansion yields

\[
\langle f, \nu_{k+1}^N \rangle = f(C_k^{i+k}, W_{k+1}^i) - f(C_k^i, W_k^i)
\]

\[
= \frac{1}{N} \sum_{i=1}^N \left( f(C_k^{i+k}, W_{k+1}^i) - f(C_k^i, W_k^i) \right)
\]

\[
= \frac{1}{N} \sum_{i=1}^N \partial_x f(C_k^i, W_k^i)(C_k^{i+k} - C_k^i) + \frac{1}{N} \sum_{i=1}^N \nabla_w f(C_k^i, W_k^i)(W_{k+1}^i - W_k^i)
\]

\[
+ \frac{1}{N} \sum_{i=1}^N \partial_{w}^2 f(C_k^i, W_k^i)(C_k^{i+k} - C_k^i)^2 + \frac{1}{N} \sum_{i=1}^N (C_k^{i+k} - C_k^i) \nabla_{cw} f(C_k^i, W_k^i)(W_{k+1}^i - W_k^i)
\]

\[+ \frac{1}{N} \sum_{i=1}^N (W_{k+1}^i - W_k^i) \nabla_{w}^2 f(C_k^i, W_k^i)(W_{k+1}^i - W_k^i), \tag{5.6}\]

for points \(\tilde{C}_k^i, \tilde{W}_k^i\) in the segments connecting \(C_k^{i+k}\) with \(C_k^i\) and \(W_{k+1}^i\) with \(W_k^i\), respectively.
Substituting (2.7) into (5.6) yields

\[
\langle f, \nu_{k+1}^N \rangle - \langle f, \nu_k^N \rangle = \sum_{i=1}^{N} \partial_x f(C_k^i, W_k^i) \alpha \left( r_k + \gamma \max_{a' \in A} Q_k^N(x_{k+1}, a') - Q_k^N(\zeta_k) \right) \sigma(W_k^i \cdot \zeta_k)
\]

\[
+ N^{-5/2} \sum_{i=1}^{N} \alpha \left( r_k + \gamma \max_{a' \in A} Q_k^N(x_{k+1}, a') - Q_k^N(\zeta_k) \right) C_k^i \sigma'(W_k^i \cdot \zeta_k) (\zeta_k)
\]

\[
= N^{-3/2} \alpha \left( r_k + \gamma \max_{a' \in A} Q_k^N(x_{k+1}, a') - Q_k^N(\zeta_k) \right) \langle \partial_x f(c, w) \sigma(w \cdot \zeta_k), \nu_k^N \rangle
\]

\[
+ N^{-3/2} \alpha \left( r_k + \gamma \max_{a' \in A} Q_k^N(x_{k+1}, a') - Q_k^N(\zeta_k) \right) \langle \sigma'(w \cdot \zeta_k) \nabla_w f(c, w) \cdot \zeta_k, \nu_k^N \rangle + O_p \left( N^{-2} \right).
\]

(5.7)

Similarly, we can also obtain that

\[
\langle f, \mu_t^N \rangle = \langle f, \mu_0^N \rangle + \sum_{k=0}^{N(t)-1} \left( \langle f, \nu_{k+1}^N \rangle - \langle f, \nu_k^N \rangle \right)
\]

\[
= \langle f, \mu_0^N \rangle + N^{-3/2} \sum_{k=0}^{N(t)-1} \alpha \left( r_k + \gamma \max_{a' \in A} Q_k^N(x_{k+1}, a') - Q_k^N(\zeta_k) \right) \langle \partial_x f(c, w) \sigma(w \cdot \zeta_k), \nu_k^N \rangle
\]

\[
+ N^{-3/2} \sum_{k=0}^{N(t)-1} \alpha \left( r_k + \gamma \max_{a' \in A} Q_k^N(x_{k+1}, a') - Q_k^N(\zeta_k) \right) \langle \sigma'(w \cdot \zeta_k) \nabla_w f(c, w) \cdot \zeta_k, \nu_k^N \rangle + O_p \left( N^{-1} \right).
\]

(5.8)

5.2 Identification of the Limit

We must first establish that \( M_t^{1,N}, M_t^{2,N}, M_t^{3,N} \) \( p \to 0 \) as \( N \to \infty \). For this purpose, we first prove two lemmas.

**Lemma 5.1.** Consider a Markov chain \( z_k \) on a finite, discrete space \( S \) with a unique limiting distribution \( q(z) \) and a random function \( f^N : S \to \mathbb{R} \). Suppose \( f^N \) is uniformly bounded in \( L^2 \) with respect to \( N \). Then,

\[
\lim_{N \to \infty} \mathbb{E} \left[ \frac{1}{N} \sum_{k=0}^{N-1} f^N(z_k) - \sum_{z \in S} f^N(z) q(z) \right] = 0.
\]

The proof of Lemma 5.1 should be known. However, given that we could not locate an exact reference, we provide its short proof in the Appendix A.

**Lemma 5.2.** Consider the notation and assumptions of Lemma 5.1. Define the quantity

\[
M_t^N = \frac{1}{N} \sum_{k=0}^{N(t)-1} f_k^N(z_k) - \frac{1}{N} \sum_{k=0}^{N(t)-1} \sum_{z \in S} f_k^N(z) q(z),
\]

where the function \( f_k^N \) satisfies

\[
\sup_{z \in S} \mathbb{E} \left[ |f_k^N(z) - f_{k-1}^N(z)| \right] \leq \frac{C}{N},
\]

\[
\sup_{0 \leq k \leq N} \sup_{z \in S} \mathbb{E} \left[ |f_k^N(z)|^2 \right] < C.
\]

(5.9)

Then we have that,

\[
\lim_{N \to \infty} \sup_{t \in (0, T]} \mathbb{E}|M_t^N| = 0.
\]
Proof. For any $K \in \mathbb{N}$ and $\Delta = \frac{1}{K}$, we have

\[
M_i^N = \sum_{j=0}^{K-1} \Delta \frac{1}{[\Delta N]} \sum_{k=j[\Delta N]}^{(j+1)[\Delta N] - 1} \left( f_k^N(z_k) - \sum_{z \in S} f_k^N(z)q(z) \right) + o(1)
\]

\[
= \sum_{j=0}^{K-1} \Delta \frac{1}{[\Delta N]} \sum_{k=j[\Delta N]}^{(j+1)[\Delta N] - 1} \left( f_j^N(z_k) - \sum_{z \in S} f_j^N(z)q(z) \right)
\]

\[
+ \sum_{j=0}^{K-1} \Delta \frac{1}{[\Delta N]} \sum_{k=j[\Delta N]}^{(j+1)[\Delta N] - 1} \left[ \left( f_k^N(z_k) - \sum_{z \in S} f_k^N(z)q(z) \right) - \left( f_j^N(z_k) - \sum_{z \in S} f_j^N(z)q(z) \right) \right]
\]

\[
+ o(1),
\]

(5.10)

where the term $o(1)$ goes to zero, at least, in $L^1$ as $N \to \infty$. We will need to show that, for each $j = 0, 1, \ldots, K - 1$,

\[
\frac{1}{[\Delta N]} \sum_{k=j[\Delta N]}^{(j+1)[\Delta N] - 1} 1_{z_k = s} \overset{p}{\to} q(s) \quad \text{as} \quad N \to \infty.
\]

(5.11)

This can be proven in the following way. We already know that $\frac{1}{[\Delta N]} \sum_{k=0}^{j[\Delta N] - 1} 1_{z_k = s} \overset{p}{\to} (j + 1)q(s)$ as $N \to \infty$. Of course, we also have that $\frac{1}{[\Delta N]} \sum_{k=0}^{j[\Delta N] - 1} 1_{z_k = s} \overset{p}{\to} jq(s)$ as $N \to \infty$. Then, it must hold that

\[
\frac{1}{[\Delta N]} \sum_{k=j[\Delta N]}^{(j+1)[\Delta N] - 1} 1_{z_k = s} \overset{p}{\to} q(s).
\]

Combining (5.11) and Lemma 5.1, we can show that for each $j = 0, 1, \ldots, K - 1$,

\[
\lim_{N \to \infty} \mathbb{E} \left[ \frac{1}{[\Delta N]} \sum_{k=j[\Delta N]}^{(j+1)[\Delta N] - 1} \left( f_j^N(z_k) - \sum_{z \in S} f_j^N(z)q(z) \right) \right] = 0.
\]
Lemma 5.3. \( M \)

**Proof.** The process concluding the proof of the lemma.

Indeed, recalling the notation we have

\[
E \left[ \sum_{j=0}^{K-1} \Delta \frac{1}{[\Delta N]} \sum_{k=j[\Delta N]}^{(j+1)[\Delta N]-1} \left[ \left( f_k^N(z_k) - \sum_{z \in S} f_k^N(z)q(z) \right) - \left( f_{j[\Delta N]}^N(z_k) - \sum_{z \in S} f_{j[\Delta N]}^N(z)q(z) \right) \right] \right] \\
\leq C \sum_{j=0}^{K-1} \Delta \frac{1}{[\Delta N]} \sum_{k=j[\Delta N]}^{(j+1)[\Delta N]-1} \frac{k-j[\Delta N]}{N} \\
\leq C \sum_{j=0}^{K-1} \Delta \frac{1}{[\Delta N]} \sum_{k=j[\Delta N]}^{(j+1)[\Delta N]-1} \frac{k[j\Delta N]}{N} \\
\leq C \sum_{j=0}^{K-1} \Delta \frac{1}{[\Delta N]} |\Delta N|^2 \\
\leq C \sum_{j=0}^{K-1} \Delta \frac{1}{N} \\
\leq C \Delta.
\]

Collecting our results, we have shown that

\[
\lim_{N \to \infty} \sup_{t \in [0,T]} \mathbb{E}|M_t^N| \leq C \frac{T}{K}.
\]

Note that \( K \) was arbitrary. Consequently, we obtain

\[
\lim_{N \to \infty} \sup_{t \in [0,T]} \mathbb{E}|M_t^N| = 0,
\]

concluding the proof of the lemma.

This now allows us to prove the following lemma.

**Lemma 5.3.** \( M_t^{1,N}, M_t^{2,N}, M_t^{3,N} \to 0 \) as \( N \to \infty \).

**Proof.** The process \( Q_k^N(x,a) \) satisfies the uniform \( L^2 \) bound in equation (5.9) due to Lemma 5.6. It also satisfies the regularity condition in equation (5.9). Indeed, recalling the notation \( \zeta = (x,a) \) and \( \zeta_k = (x_k,a_k) \), we have

\[
\mathbb{E} \left[ |Q_{k+1}^N(\zeta) - Q_k^N(\zeta)| \right] \leq \frac{1}{N^\gamma} \sum_{i=1}^{N} \mathbb{E} \left[ |(C_{k+1} - C_k)\sigma(W_{k+1}^i \cdot \zeta) + \sigma'(W_{k+1}^i \cdot \zeta)\zeta^\top(W_{k+1}^i - W_k^i)C_{k+1}| \right] \\
\leq \frac{1}{N^2} \sum_{i=1}^{N} \mathbb{E} \left[ |\alpha(r_k + \gamma \max_{a' \in \mathcal{A}} Q_k^N(x_k+1,a') - Q_k^N(\zeta_k))\sigma(W_{k+1}^i \cdot \zeta_k)\sigma(W_{k+1}^i \cdot s)| \right] \\
\leq \frac{1}{N^2} \sum_{i=1}^{N} \mathbb{E} \left[ |\alpha(r_k + \gamma \max_{a' \in \mathcal{A}} Q_k^N(x_k+1,a') - Q_k^N(\zeta_k))\sigma'(W_{k+1}^i \cdot \zeta_k)\sigma(W_{k+1}^i \cdot \zeta_k)\zeta^\top(C_{k+1}^i)| \right] \leq \frac{C}{N}.
\]

(5.12)
where we have used the bounds from Lemmas 5.3 and 5.6, the boundedness of $\sigma(\cdot)$ and $\sigma'(\cdot)$, and the Cauchy-Schwartz inequality.

In addition,
\[
\mathbb{E} \left[ \max_{a \in A} Q_{k+1}^N(x, a) - \max_{a \in A} Q_k^N(x, a) \right] \leq \mathbb{E} \left[ \max_{a \in A} |Q_{k+1}^N(x, a) - Q_k^N(x, a)| \right] \\
\leq \sum_{a \in A} \mathbb{E} \left[ |Q_{k+1}^N(x, a) - Q_k^N(x, a)| \right] \\
\leq \frac{C}{N},
\]  
(5.13)

where we have used the bound 5.12.

The term $B_{0,a,x',a',k}^N$ that appears in the formula for $M_t^{1,N}$ can be treated analogously using (5.7) and Lemma 5.6. The result for $M_t^{1,N}$ then immediately follows from Lemmas 5.1 and 5.2 and the triangle inequality. Using the same approach, one can obtain the claim for $M_t^{2,N}$ and $M_t^{3,N}$, and the proof is omitted due to the similarity of the argument.

Let $\rho^N$ be the probability measure of a convergent subsequence of $(\mu^N, h^N)_{0 \leq t \leq T}$. Each $\rho^N$ takes values in the set of probability measures $\mathcal{M}(D_E([0, T]))$. Relative compactness, proven in Section 5.3, implies that there is a subsequence $\rho^{N_k}$ which weakly converges. We must prove that any limit point $\rho$ of a convergent subsequence $\rho^{N_k}$ will satisfy the evolution equation (3.2).

**Lemma 5.4.** Let $\rho^{N_k}$ be a convergent subsequence with a limit point $\rho$. Then, $\rho$ is a Dirac measure concentrated on $(\mu, h) \in D_E([0, T])$ and $(\mu, h)$ satisfies equation (3.2).

**Proof.** We define a map $F(\mu, h) : D_E([0, T]) \to \mathbb{R}_+$ for each $t \in [0, T]$, $f \in C^2_b(\mathbb{R}^{1+d})$, $g_1, \cdots, g_p \in C^1_b(\mathbb{R}^{1+d})$, $q_1, \cdots, q_p \in C_b(X \times A)$, and $0 \leq s_1 < \cdots < s_p \leq t$.

\[
F(\mu, h) = \left| \langle f, \mu \rangle - \langle f, \mu_0 \rangle \right| + \langle g_1, \mu_{s_1} \rangle + \cdots + \langle g_p, \mu_{s_p} \rangle \\
+ \sum_{(x, a) \in X \times A} h_t(x, a) - h_0(x, a) - \alpha \int_0^t \sum_{(x', a') \in X \times A} \left( r(x', a') + \gamma \int_{a'' \in A} \max_i h_s^N(x'', a'')p(x''|x', a') - h_s^N(x', a') \right) \\
\times \left( \langle \sigma(w \cdot (x', a')) \sigma(w \cdot (x, a)), \mu_s^N \rangle + \langle c^2 \sigma'(w)\sigma'(w \cdot (x', a')), (x', a') \rangle \right) \pi(x', a') ds \\
\times q_1(h_{s_1}) \times \cdots \times q_p(h_{s_p}).
\]

Then, using equations 5.4 and 5.8, we obtain
\[
\mathbb{E}_{\rho^N}[F(\mu, h)] = \mathbb{E}[F(\mu^N, h^N)] \\
\leq \mathbb{E} \left[ \mathcal{O}(N^{-1/2}) + \prod_{i=1}^P g_i(h_{s_i}^N) \right] \\
+ \mathbb{E} \left[ (M_t^{1,N} + M_t^{2,N} + M_t^{3,N} + \mathcal{O}(N^{-1/2})) \prod_{i=1}^P g_i(h_{s_i}^N) \right] \\
\leq C \left( \mathbb{E}\left[M_{1,N}(t)\right] + \mathbb{E}\left[M_{2,N}(t)\right] + \mathbb{E}\left[M_{3,N}(t)\right] \right) + O(N^{-1/2}).
\]

Therefore, using Lemma 5.3
\[
\lim_{N \to \infty} \mathbb{E}_{\rho^N}[F(\mu, h)] = 0.
\]
Since $F(\cdot)$ is continuous and $F(\mu^N)$ is uniformly bounded (due to the uniform boundedness results of Section 5.3),
\[ E_p[F(\mu, h)] = 0. \]
Since this holds for each $t \in [0, T], f \in C_b^2(\mathbb{R}^{1+d})$ and $g_1, \ldots, g_p, q_1, \ldots, q_p \in C_b(\mathbb{R}^{1+d})$, $(\mu, h)$ satisfies the evolution equation (3.2). \qed

5.3 Relative Compactness

In this section we prove that the family of processes $\{\mu^N, h^N\}_N$ is relatively compact. Section 5.3.1 proves compact containment. Section 5.3.2 proves regularity. Section 5.3.3 combines these results to prove relative compactness.

5.3.1 Compact Containment

We first establish a priori bounds for the parameters $(C_i^j, W_i^j)$.

**Lemma 5.5.** For all $i \in \mathbb{N}$ and all $k$ such that $k/N \leq T$,
\[ |C_i^k| < C < \infty \]
\[ \mathbb{E} \| W_i^k \| < C < \infty. \]

**Proof.** The unimportant finite constant $C < \infty$ may change from line to line. We first observe that
\[ |C_i^j| \leq |C_i^j| + |C_i^{j+1}| \leq |C_i^j| + \alpha N^{-3/2} \left| r_k + \gamma \max_{a' \in \mathcal{A}} Q_k^N(x_{k+1}, a') - Q_k^N(x_k, a_k) \right| |\sigma(W_i^j \cdot x_k)| \]
\[ \leq |C_i^j| + \frac{C |r_k|}{N^{3/2}} + \frac{C}{N^2} \sum_{i=1}^N |C_i^j|, \]
where the last inequality follows from the definition of $Q_k^N(x, a)$ and the uniform boundedness assumption on $\sigma(\cdot)$.

Then, we subsequently obtain that
\[ |C_i^j| = |C_i^0| + \sum_{j=1}^k \left( |C_i^j| - |C_i^{j-1}| \right) \]
\[ \leq |C_i^0| + \sum_{j=1}^k \frac{C}{N^{3/2}} + \frac{C}{N^2} \sum_{j=1}^k \sum_{i=1}^N |C_i^j| \]
\[ \leq |C_i^0| + \frac{C}{\sqrt{N}} + \frac{C}{N^2} \sum_{j=1}^k \sum_{i=1}^N |C_i^{j-1}|. \]
\[ (5.14) \]

This implies that
\[ \frac{1}{N} \sum_{i=1}^N |C_i^j| \leq \frac{1}{N} \sum_{i=1}^N |C_i^0| + \frac{C}{\sqrt{N}} + \frac{C}{N^2} \sum_{j=1}^k \sum_{i=1}^N |C_i^{j-1}|, \]
\[ \frac{1}{N} \sum_{i=1}^N |C_i^j| \leq \frac{1}{N} \sum_{i=1}^N |C_i^0| + \frac{C}{\sqrt{N}} + \frac{C}{N^2} \sum_{j=1}^k \sum_{i=1}^N |C_i^{j-1}|, \]

Let us now define $m_k^N = \frac{1}{N} \sum_{i=1}^N |C_i^j|$. Since the random variables $C_i^j$ take values in a compact set, we have that $\frac{1}{N} \sum_{i=1}^N |C_i^j| + \frac{C}{\sqrt{N}} < C < \infty$. Then,
\[ m_k^N \leq C + \frac{C}{N} \sum_{j=1}^k m_{j-1}^N. \]
By the discrete Gronwall lemma and using $k/N \leq T$,
\[
m_k^N \leq C \exp\left(\frac{Ck}{N}\right) \leq C. \tag{5.15}
\]

Note that the constants may depend on $T$. We can now combine the bounds (5.15) and (5.14) to yield, for any $0 \leq k \leq \lceil TN \rceil$,
\[
|C_k^i| \leq |C_0^i| + \frac{C}{\sqrt{N}} + \frac{C}{N^2} \sum_{j=1}^{k} m_{j-1}^N
\leq |C_0^i| + \frac{C}{\sqrt{N}} + \frac{C}{N^2} \sum_{j=1}^{k} C_2
\leq |C_0^i| + \frac{C}{\sqrt{N}} + \frac{C}{N}
\leq C, \tag{5.16}
\]
where the last inequality follows from the random variables $C_0^i$ taking values in a compact set.

Now, we turn to the bound for $\| W_k^i \|$. We start with the bound (using Young’s inequality)
\[
\| W_{k+1}^i \| \leq \| W_k^i \| + C \frac{1}{N^{3/2}} \left( r_k + \gamma \max_{a' \in A} Q_k^N(x_{k+1}, a') - Q_k^N(x_k, a_k) \right) \| C_k^i \| \| \sigma'(W_k^i \cdot x_k) \| \| x_k \|
\leq \| W_k^i \| + C \left( \frac{1}{N^{3/2}} + \frac{1}{N^2} \sum_{j=1}^{N} |C_j^i|^2 \right)
\leq \| W_k^i \| + \frac{C}{N},
\]
for a constant $C < \infty$ that may change from line to line. Taking an expectation, using Assumption 3.1, the bound (5.16), and using the fact that $k/N \leq T$, we obtain
\[
\mathbb{E} \| W_k^i \| \leq C < \infty,
\]
for all $i \in \mathbb{N}$ and all $k$ such that $k/N \leq T$, concluding the proof of the lemma.

Using the bounds from Lemma 5.5 we can now establish a bound for $Q_k^N(x, a)$ for $(x, a) \in \mathcal{X} \times \mathcal{A}$.

**Lemma 5.6.** For all $i \in \mathbb{N}$, all $k$ such that $k/N \leq T$,
\[
\mathbb{E} \sup_{(x, a) \in \mathcal{X} \times \mathcal{A}} \left| Q_k^N(x, a) \right|^2 < C < \infty.
\]

**Proof.** Recall equation (5.2), which describes the evolution of $Q_k^N(x, a)$. Recall the notation $\zeta = (x, a)$ and $\zeta_k = (x_k, a_k)$.
\[
Q_{k+1}^N(\zeta) = Q_k^N(\zeta) + \frac{\alpha}{N^2} \left( r_k + \gamma \max_{a' \in A} Q_k^N(x_{k+1}, a') - Q_k^N(\zeta_k) \right) \sum_{i=1}^{N} (W_k^i \cdot \zeta_k) \sigma(W_k^i \cdot \zeta)
+ \frac{\alpha}{N^2} \left( r_k + \gamma \max_{a' \in A} Q_k^N(x_{k+1}, a') - Q_k^N(\zeta_k) \right) \sum_{i=1}^{N} \sigma(W_k^i \cdot \zeta) \sigma'(W_k^i \cdot \zeta) \zeta_i (C_i^2) + \tilde{C}(\omega) \frac{1}{N^{3/2}},
\]
where $\tilde{C}(\omega)$ is a random variable (independent of $N$) that is bounded in mean square sense. This leads to the bound
\[
\sup_{\zeta \in \mathcal{X} \times \mathcal{A}} |Q_{k+1}^N(\zeta)| \leq \sup_{\zeta \in \mathcal{X} \times \mathcal{A}} |Q_k^N(\zeta)| + \frac{C}{N} \sup_{\zeta \in \mathcal{X} \times \mathcal{A}} |Q_k^N(\zeta)| + \frac{\tilde{C}(\omega)}{N}.
\]
We now square both sides of the above inequality.

\[
\sup_{\zeta \in \mathcal{X} \times \mathcal{A}} |Q^N_{k+1}(\zeta)|^2 \leq \left( \sup_{\zeta \in \mathcal{X} \times \mathcal{A}} |Q^N_k(\zeta)| + \frac{C}{N} \sup_{\zeta \in \mathcal{X} \times \mathcal{A}} |Q^N_k(\zeta)| + \frac{C}{N} \right)^2
\]

\[
\leq \sup_{\zeta \in \mathcal{X} \times \mathcal{A}} |Q^N_k(\zeta)|^2 + \frac{C}{N} \sup_{\zeta \in \mathcal{X} \times \mathcal{A}} |Q^N_k(\zeta)|^2 + \frac{C^2(\omega)}{N},
\]

where the last line uses Young's inequality. Therefore, we obtain

\[
| \sup_{\zeta \in \mathcal{X} \times \mathcal{A}} Q^N_{k+1}(\zeta)|^2 - \sup_{\zeta \in \mathcal{X} \times \mathcal{A}} |Q^N_k(\zeta)|^2 \leq \frac{C}{N} \sup_{\zeta \in \mathcal{X} \times \mathcal{A}} |Q^N_k(\zeta)|^2 + \frac{C^2(\omega)}{N}.
\]

Then, using a telescoping series, we have

\[
\sup_{\zeta \in \mathcal{X} \times \mathcal{A}} |Q^N_k(\zeta)|^2 = \sup_{\zeta \in \mathcal{X} \times \mathcal{A}} |Q^N_0(\zeta)|^2 + \sum_{j=1}^{k} \left( \sup_{\zeta \in \mathcal{X} \times \mathcal{A}} |Q^N_j(\zeta)|^2 - \sup_{\zeta \in \mathcal{X} \times \mathcal{A}} |Q^N_{j-1}(\zeta)|^2 \right)
\]

\[
\leq \sup_{\zeta \in \mathcal{X} \times \mathcal{A}} |Q^N_0(\zeta)|^2 + \sum_{j=1}^{k} \left( \frac{C}{N} \sup_{\zeta \in \mathcal{X} \times \mathcal{A}} |Q^N_{j-1}(\zeta)|^2 + \frac{C^2(\omega)}{N} \right)
\]

\[
\leq \sup_{\zeta \in \mathcal{X} \times \mathcal{A}} |Q^N_0(\zeta)|^2 + \frac{C}{N} \sum_{j=1}^{k} \sup_{\zeta \in \mathcal{X} \times \mathcal{A}} |Q^N_{j-1}(\zeta)|^2 + C^2(\omega).
\]

Taking expectations, we subsequently obtain

\[
\mathbb{E}\left[ \sup_{\zeta \in \mathcal{X} \times \mathcal{A}} |Q^N_k(\zeta)|^2 \right] \leq \mathbb{E}\left[ \sup_{\zeta \in \mathcal{X} \times \mathcal{A}} |Q^N_0(\zeta)|^2 \right] + \frac{C}{N} \sum_{j=1}^{k} \mathbb{E}\left[ \sup_{\zeta \in \mathcal{X} \times \mathcal{A}} |Q^N_{j-1}(\zeta)|^2 \right] + C. \quad (5.17)
\]

Recall that

\[
Q^N_0(\zeta) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} C^i_0 \sigma(W^i_0 \cdot (\zeta)),
\]

where \((C^i_0, W^i_0)\) are i.i.d., mean-zero random variables. Then,

\[
\mathbb{E}\left[ \sup_{\zeta} |Q^N_0(\zeta)|^2 \right] \leq \mathbb{E}\left[ \sum_{\zeta \in \mathcal{X} \times \mathcal{A}} |Q^N_0(\zeta)|^2 \right]
\]

\[
\leq \sum_{\zeta \in \mathcal{X} \times \mathcal{A}} \mathbb{E}\left[ \left( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} C^i_0 \sigma(W^i_0 \cdot (\zeta)) \right)^2 \right]
\]

\[
\leq \frac{C}{N} \sum_{i=1}^{N} \mathbb{E}\left[ (C^i_0)^2 \right]
\]

\[
\leq C.
\]

Substituting this bound into equation (5.17) produces the desired bound

\[
\mathbb{E}\left[ \sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} |Q^N_k(x,a)|^2 \right] \leq C,
\]

for any \(0 \leq k \leq \lfloor NT \rfloor\).

We now prove compact containment for the process \(\{(\mu^N_t, h^N_t), t \in [0,T]\}_{N \in \mathbb{N}}\). Recall that \((\mu^N_t, h^N_t) \in D_E([0,T])\) where \(E = \mathcal{M}(\mathbb{R}^{1+d}) \times \mathbb{R}^M\) and \(M = |\mathcal{X} \times \mathcal{A}|\).
**Lemma 5.7.** For each $\eta > 0$, there is a compact subset $\mathcal{K}$ of $E$ such that

$$\sup_{N \in \mathbb{N}, 0 \leq t \leq T} \mathbb{P}[(\mu^N_t, h^N_t) \notin \mathcal{K}] < \eta.$$ 

**Proof.** For each $L > 0$, define $K_L = [-L, L]^{1+d}$. Then, we have that $K_L$ is a compact subset of $\mathbb{R}^{1+d}$, and for each $t \geq 0$ and $N \in \mathbb{N}$,

$$\mathbb{E} [\mu^N_t(\mathbb{R}^{1+d} \setminus K_L)] = \frac{1}{N} \sum_{i=1}^{N} \mathbb{P} \left[ \left| C^i_{N,t} \right| + \| W^i_{N,t} \| \geq L \right] \leq \frac{C}{L}.$$ 

where we have used Markov’s inequality and the bounds from Lemma 5.6. We define the compact subsets of $\mathcal{M}(\mathbb{R}^{1+d})$

$$\hat{K}_L = \left\{ \nu : \nu(\mathbb{R}^{1+d} \setminus K_{(L+j)^2}) < \frac{1}{\sqrt{L+j}} \text{ for all } j \in \mathbb{N} \right\}$$

and we observe that

$$\mathbb{P} \left[ \mu^N_t \notin \hat{K}_L \right] \leq \sum_{j=1}^{\infty} \mathbb{P} \left[ \mu^N_t(\mathbb{R}^{1+d} \setminus K_{(L+j)^2}) > \frac{1}{\sqrt{L+j}} \right] \leq \sum_{j=1}^{\infty} \mathbb{E} [\mu^N_t(\mathbb{R}^{1+d} \setminus K_{(L+j)^2})] \leq \sum_{j=1}^{\infty} \frac{C}{(L+j)^{3/2}}.$$ 

Given that $\lim_{L \to \infty} \sum_{j=1}^{\infty} \frac{C}{(L+j)^{3/2}} = 0$, we have that, for each $\eta > 0$, there exists a compact set $\hat{K}_L$ such that

$$\sup_{N \in \mathbb{N}, 0 \leq t \leq T} \mathbb{P}[(\mu^N_t, h^N_t) \notin \hat{K}_L] < \frac{\eta}{2}.$$ 

Due to Lemma 5.6 and Markov’s inequality, we also know that, for each $\eta > 0$, there exists a compact set $U = [-B, B]^M$ such that

$$\sup_{N \in \mathbb{N}, 0 \leq t \leq T} \mathbb{P}[h^N_t \notin U] < \frac{\eta}{2}.$$ 

Therefore, for each $\eta > 0$, there exists a compact set $\hat{K}_L \times [-B, B]^M \subset E$ such that

$$\sup_{N \in \mathbb{N}, 0 \leq t \leq T} \mathbb{P}[(\mu^N_t, h^N_t) \notin \hat{K}_L \times [-B, B]^M] < \eta.$$ 

\[\square\]

### 5.3.2 Regularity

We now establish regularity of the process $\mu^N$ in $D_M(\mathbb{R}^{1+d})([0, T])$. Define the function $q(z_1, z_2) = \min \{|z_1 - z_2|, 1\}$ where $z_1, z_2 \in \mathbb{R}$.

**Lemma 5.8.** Let $f \in C^2_b(\mathbb{R}^{1+d})$. For any $\delta \in (0, 1)$, there is a constant $C < \infty$ such that for $0 \leq u \leq \delta$, $0 \leq v \leq \delta \wedge t$, $t \in [0, T]$,

$$\mathbb{E} \left[ q(\langle f, \mu^N_{t+u} \rangle, \langle f, \mu^N_t \rangle) \right] \mathbb{E} \left[ q(\langle f, \mu^N_t \rangle, \langle f, \mu^N_{t-v} \rangle) \right] \leq C\delta + \frac{C}{N^{3/2}}.$$ 

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Proof. We start by noticing that a Taylor expansion gives for \(0 \leq s \leq t \leq T\)
\[
|\langle f, \mu_s^N \rangle - \langle f, \mu_t^N \rangle| = |\langle f, \nu_{\langle N \rangle}^N \rangle - \langle f, \nu_{\langle N \rangle}^N \rangle|
\]
\[
\leq \frac{1}{N} \sum_{i=1}^{N} |f(C_{\langle N \rangle}^i, W_{\langle N \rangle}^i) - f(C_{\langle N \rangle}^i, W_{\langle N \rangle}^i)|
\]
\[
\leq \frac{1}{N} \sum_{i=1}^{N} |\partial_k f(C_{\langle N \rangle}^i, W_{\langle N \rangle}^i)||C_{\langle N \rangle}^i - C_{\langle N \rangle}^i|
\]
\[
+ \frac{1}{N} \sum_{i=1}^{N} \| \nabla f(C_{\langle N \rangle}^i, W_{\langle N \rangle}^i) \| ||W_{\langle N \rangle}^i - W_{\langle N \rangle}^i||,
\]
(5.18)
for points \(C^i, W^i\) in the segments connecting \(C_{\langle N \rangle}^i\) with \(C_{\langle N \rangle}^i\) and \(W_{\langle N \rangle}^i\) with \(W_{\langle N \rangle}^i\), respectively.

Let’s now establish a bound on \(|C_{\langle N \rangle}^i - C_{\langle N \rangle}^i|\) for \(s < t \leq T\) with \(0 < t - s \leq \delta < 1\).

\[
E\left[|C_{\langle N \rangle}^i - C_{\langle N \rangle}^i| \big| F_s^N \right] = E\left[ \left| \sum_{k=1}^{\lceil N/3 \rceil} (C_{k+1}^i - C_k^i) \right| \big| F_s^N \right]
\]
\[
\leq E\left[ \sum_{k=1}^{\lceil N/3 \rceil} |\alpha(r_k + \gamma \max_{a' \in A} Q_k^N(x_{k+1}, a') - Q_k^N(x_k, a_k))| \frac{1}{N^{3/2}} \sigma(W_k^i \cdot x_k) \right] \big| F_s^N \right]
\]
\[
\leq \frac{1}{N^{3/2}} \sum_{k=1}^{\lceil N/3 \rceil} C \leq \frac{C}{\sqrt{N}} (t-s) + \frac{C}{N^{3/2}}
\]
\[
\leq \frac{C}{\sqrt{N}} \delta + \frac{C}{N^{3/2}},
\]
(5.19)
where Assumption 3.2 was used as well as the bounds from Lemmas 5.5 and 5.6.

Let’s now establish a bound on \(\|W_{\langle N \rangle}^i - W_{\langle N \rangle}^i\|\) for \(s < t \leq T\) with \(0 < t - s \leq \delta < 1\). We obtain

\[
E\left[ \|W_{\langle N \rangle}^i - W_{\langle N \rangle}^i\| \big| F_s^N \right] = E\left[ \left| \sum_{k=1}^{\lceil N/3 \rceil} (W_{k+1}^i - W_k^i) \right| \big| F_s^N \right]
\]
\[
\leq E\left[ \sum_{k=1}^{\lceil N/3 \rceil} |\alpha(r_k + \gamma \max_{a' \in A} Q_k^N(x_{k+1}, a') - Q_k^N(x_k, a_k))| \frac{1}{N^{3/2}} C_k \sigma'(W_k^i \cdot x_k) x_k \right] \big| F_s^N \right]
\]
\[
\leq \frac{1}{N^{3/2}} \sum_{k=1}^{\lceil N/3 \rceil} C
\]
\[
\leq \frac{C}{\sqrt{N}} (t-s) + \frac{C}{N} \leq \frac{C}{\sqrt{N}} \delta + \frac{C}{N^{3/2}}.
\]
(5.20)
where we have again used the bounds from Lemmas 5.5 and 5.6.

Now, we return to equation (5.18). Due to Lemma 5.5 the quantities \((\bar{C}_{\langle N \rangle}^i, \bar{W}_{\langle N \rangle}^i)\) are bounded in expectation for \(0 < s < t \leq T\). Therefore, for \(0 < s < t \leq T\) with \(0 < t - s \leq \delta < 1\)
\[
E \left[ |\langle f, \mu_s^N \rangle - \langle f, \mu_t^N \rangle| \big| F_s^N \right] \leq C\delta + \frac{C}{N^{3/2}}
\]
where \(C < \infty\) is some unimportant constant. Then, the statement of the Lemma follows.

We next establish regularity of the process \(h_t^N\) in \(D_{RM}([0, T])\). For the purposes of the following lemma, let the function \(q(z_1, z_2) = \min\{|z_1 - z_2|, 1\}\) where \(z_1, z_2 \in \mathbb{R}^M\) and \(\|z\| = |z_1| + \cdots + |z_M|\).
Lemma 5.9. For any $\delta \in (0, 1)$, there is a constant $C < \infty$ such that for $0 \leq u \leq \delta < 1$, $0 \leq v \leq \delta \land t$, $t \in [0, T]$,

$$E \left[ q(h^N_{t+u}, h^N_{t})q(h^N_{t}, h^N_{t-v})|\mathcal{F}^N_t \right] \leq C\delta + \frac{C}{N}.$$ 

Proof. Recall that

$$Q_{k+1}^N(\zeta) = Q_k^N(\zeta) + \frac{1}{\sqrt{N}} \sum_{i=1}^N \left( C_{k+1}^i - C_k^i \right) \sigma(W_{k+1}^i \cdot \zeta) + \sigma'(W_{k+1}^i \cdot \zeta) \zeta^\top (W_{k+1}^i - W_k^i) C_k^i \right).$$

Therefore,

$$h^N_{t}(\zeta) - h^N_{s}(\zeta) = Q_{[Nt]}^N(\zeta) - Q_{[Ns]}^N(\zeta) = \sum_{k=\lceil [Ns] \rceil}^{\lceil [Nt] \rceil} (Q_{k+1}^N(\zeta) - Q_k^N(\zeta)) = \sum_{k=\lceil [Ns] \rceil}^{\lceil [Nt] \rceil} \frac{1}{\sqrt{N}} \sum_{i=1}^N \left( C_{k+1}^i - C_k^i \right) \sigma(W_{k+1}^i \cdot \zeta) + \sigma'(W_{k+1}^i \cdot \zeta) \zeta^\top (W_{k+1}^i - W_k^i) C_k^i \right).$$

This yields the bound

$$|h^N_{t}(\zeta) - h^N_{s}(\zeta)| \leq \sum_{k=\lceil [Ns] \rceil}^{\lceil [Nt] \rceil} |Q_{k+1}^N(\zeta) - Q_k^N(\zeta)| \leq \sum_{k=\lceil [Ns] \rceil}^{\lceil [Nt] \rceil} \frac{1}{\sqrt{N}} \sum_{i=1}^N \left( |C_{k+1}^i - C_k^i| + \|W_{k+1}^i - W_k^i\| \right),$$

where we have used the boundedness of $\sigma'$ (from Assumption 3.1) and the bounds from Lemma 5.5.

Taking expectations,

$$E \left[ \sup_\zeta |h^N_{t}(\zeta) - h^N_{s}(\zeta)| \bigg| \mathcal{F}^N_s \right] \leq \frac{1}{\sqrt{N}} \sum_{i=1}^N \sum_{k=\lceil [Ns] \rceil}^{\lceil [Nt] \rceil} E \left[ |C_{k+1}^i - C_k^i| + \|W_{k+1}^i - W_k^i\| \bigg| \mathcal{F}^N_s \right].$$

Using the bounds (5.19) and (5.20),

$$E \left[ \sup_\zeta |h^N_{t}(\zeta) - h^N_{s}(\zeta)| \bigg| \mathcal{F}^N_s \right] \leq \frac{1}{\sqrt{N}} \sum_{i=1}^N \left( C \sqrt{N} (t-s) + \frac{C}{N^{3/2}} \right) = C(t-s) + \frac{C}{N}.$$  

(5.21)

Therefore, we have obtained that

$$E \left[ \|h^N_{t} - h^N_{s}\| \bigg| \mathcal{F}^N_s \right] \leq C(t-s) + \frac{C}{N}.$$ 

The statement of the Lemma then follows.
5.3.3 Combining our results to prove relative compactness

**Lemma 5.10.** The family of processes \( \{ \mu^N, h^N \}_{N \in \mathbb{N}} \) is relatively compact in \( D_E([0, T]) \).

**Proof.** Combining Lemmas 5.7 and 5.9 and Theorem 8.6 of Chapter 3 of [3] proves that \( \{ \mu^N \}_{N \in \mathbb{N}} \) is relatively compact in \( D_M(\mathbb{R}^{1+d})([0, T]) \). (See also Remark 8.7 B of Chapter 3 of [3] regarding replacing \( \sup_N \) with \( \lim_N \) in the regularity condition B of Theorem 8.6.) Similarly, combining Lemmas 5.7 and 5.9 proves that \( \{ h^N \}_{N \in \mathbb{N}} \) is relatively compact in \( D_M([0, T]) \).

From these, we finally obtain that \( \{ \mu^N, h^N \}_{N \in \mathbb{N}} \) is relatively compact as a \( D_E([0, T]) \)–valued random variable where \( E = M(\mathbb{R}^{1+d}) \times \mathbb{R}^M \).

\( \square \)

5.4 Uniqueness

We prove uniqueness of the limit equation (3.2) for \( h_t \). Suppose there are two solutions \( h_t^1 \) and \( h_t^2 \). Let us define their difference to be \( \phi_t = h_t^1 - h_t^2 \).

Recall that \( A \) is the tensor

\[
A_{x,y} = \alpha \langle \sigma(w \cdot (x', a'))\sigma(w \cdot (x, a)), \mu_0 \rangle + \langle c^2 \sigma'(w \cdot (x', a'))\sigma'(w \cdot (x, a))(x', a')^\top (x, a), \mu_0 \rangle.
\]

For notational convenience, define \( \zeta = (x, a), \zeta' = (x', a') \), and

\[
A_{\zeta,\zeta'} = \alpha \langle \sigma(w \cdot \zeta')\sigma(w \cdot \zeta), \mu_0 \rangle + \alpha \langle c^2 \sigma'(w \cdot \zeta')\sigma'(w \cdot \zeta)\zeta', \mu_0 \rangle.
\]

The matrix \( A \) is positive definite; see Section 7 for the proof. We also define

\[
G_s(\zeta) = \gamma \sum_{x'' \in \mathcal{X}} \left[ \max_{\alpha'' \in \mathcal{A}} h^1_s(x'', a'') - \max_{\alpha'' \in \mathcal{A}} h^2_s(x'', a'') \right] p(x''|\zeta).
\]

Note that

\[
|G_s(\zeta)| \leq \gamma \sum_{x'' \in \mathcal{X}} \max_{\alpha'' \in \mathcal{A}} |\phi_s(x'', a'')| p(x''|\zeta)
\]

\[
\leq C \sum_{(x,a) \in \mathcal{X} \times \mathcal{A}} |\phi_s(\zeta)|,
\]

where we have used the inequality \( |\max_y f(y) - \max_y g(y)| \leq \max_y |f(y) - g(y)| \).

Then, \( \phi_t \), at the point \( \zeta \), i.e. \( \phi_t(\zeta) \) satisfies the following equation

\[
\phi_t(\zeta) = \int_0^t \sum_{\zeta'' \in \mathcal{X} \times \mathcal{A}} \left( G_s(\zeta') - \phi_s(\zeta') \right) A_{\zeta,\zeta'} \pi(\zeta') ds
\]

\[
\phi_0(\zeta) = 0,
\]

The latter, using (5.22) and the boundedness of the elements \( A_{\zeta,\zeta'} \), implies,

\[
|\phi_t(\zeta)|^2 = 2 \int_0^t \phi_s(\zeta) \phi_s(\zeta) ds
\]

\[
= 2 \int_0^t \phi_s(\zeta) \int_{\mathcal{X} \times \mathcal{A}} \left( G_s(\zeta') - \phi_s(\zeta') \right) A_{\zeta,\zeta'} \pi(\zeta') ds
\]

\[
\leq C \int_0^t \phi_s(\zeta) \sum_{\zeta'' \in \mathcal{X} \times \mathcal{A}} |\phi_s(\zeta')| ds.
\]

Then, summing over all possible \( \zeta \in \mathcal{X} \times \mathcal{A} \) gives, due to the finiteness of the state space

\[
\sum_{\zeta \in \mathcal{X} \times \mathcal{A}} |\phi_t(\zeta)|^2 \leq C \int_0^t \left( \sum_{\zeta \in \mathcal{X} \times \mathcal{A}} |\phi_s(\zeta)| \right)^2 ds.
\]

\[
\leq C \int_0^t \sum_{\zeta \in \mathcal{X} \times \mathcal{A}} |\phi_s(\zeta)|^2 ds.
\]
An application of Gronwall’s inequality proves that \( \phi_t(\zeta) = 0 \) for all \( 0 \leq t \leq T \) and for all \( \zeta \in \mathcal{X} \times \mathcal{A} \). Therefore, the solution \( h_t \) is indeed unique.

5.5 Proof of Convergence

We now combine the previous results of Sections 5.3 and 5.2 to prove Theorem 3.8. Let \( \rho^N \) be the probability measure corresponding to \((\mu^N, h^N)\). Each \( \rho^N \) takes values in the set of probability measures \( \mathcal{M}(D\mathcal{E}([0, T])) \). Relative compactness, proven in Section 5.3, implies that every subsequence \( \rho^{N_k} \) has a further sub-sequence \( \rho^{N_{k_m}} \) which weakly converges. Section 5.2 proves that any limit point \( \rho \) of \( \rho^{N_{k_m}} \) will satisfy the evolution equation (3.2). Equation (3.2) has a unique solution (proven in Section 5.4). Therefore, by Prokhorov’s Theorem, \( \rho^N \) weakly converges to \( \rho \), where \( \rho \) is the distribution of \((\mu, h)\), the unique solution of (3.2). That is, \((\mu^N, h^N)\) converges in distribution to \((\mu, h)\).

5.6 Analysis of the Limit Equation

It is easy to show that there is a unique stationary point of the limit equation (3.2) where \( h = V \), the solution of the Bellman equation (2.1). We define \( \zeta, \zeta', \) and \( A_{\zeta, \zeta'} \) as in Section 5.4. Any stationary point \( h \) of (3.2) must satisfy

\[
0 = \sum_{\zeta' \in \mathcal{X} \times \mathcal{A}} A_{\zeta, \zeta'} \pi(\zeta') \left( r(\zeta') + \gamma \sum_{z \in \mathcal{X}} \max_{a''} h(z, a'') p(z|\zeta') - h(\zeta') \right). \tag{5.23}
\]

Let \( B \) be a matrix where \( B_{\zeta, \zeta'} = A_{\zeta, \zeta'} \pi(\zeta') \). Since, by Lemma 3.3, \( A \) is positive definite and \( \pi(\zeta') > 0 \), \( B \) is also positive definite. Therefore, we can re-write (5.23) as

\[
0 = B(r + \gamma U - h), \tag{5.24}
\]

(5.24) is exactly the Bellman equation (2.1), which has the unique solution \( V \). Therefore, \( h_t \) has a unique stationary point which equals the solution \( V \) of the Bellman equation.

We now prove convergence of \( h_t \) to \( V \) for small \( \gamma \). Define \( \phi_t = h_t - V \) where \( V \) is the unique solution to the Bellman equation (2.1). We also define the matrix \( G_{\zeta, t} \) where

\[
G_{\zeta, t} = \sum_{x'' \in \mathcal{X}} \left[ \max_{a'' \in \mathcal{A}} h_s(x'', a'') - \max_{a'' \in \mathcal{A}} V(x'', a'') \right] p(x''|\zeta) \cdot \frac{1}{2} \phi_t^T A^{-1} \phi_t.
\]

Note that

\[
|G_{\zeta, t}| \leq \sum_{x'' \in \mathcal{X}} \max_{a'' \in \mathcal{A}} |\phi_s(x'', a'')| p(x''|\zeta).
\]

Then, \( \phi_t(\zeta) \) satisfies

\[
d\phi_t = -A \left( \pi \odot (\phi_t - \gamma G_t) \right) dt,
\]

where \( \odot \) is the element-wise product. The matrix \( A \) is positive definite. Thus, \( A^{-1} \) exists and is also positive definite. Define the process

\[
Y_t = \frac{1}{2} \phi_t^T A^{-1} \phi_t.
\]
Then,
\[
dY_t = \phi_t^\top A^{-1} d\phi_t
\]
\[
= -\phi_t^\top A^{-1} A \left( \pi \odot (\phi_t - \gamma G_t) \right) dt
\]
\[
= -\phi_t^\top \left( \pi \odot (\phi_t - \gamma G_t) \right) dt
\]
\[
= -\pi \cdot \phi_t^2 dt + \gamma \phi_t^\top (\pi \odot G_t) dt,
\]
where \( \phi_t^2 \) denotes the element-wise square \( \phi_t^2 = \phi_t \odot \phi_t \).

Let us now study the second term in equation (5.25). Let \( \Gamma_t := \gamma \phi_t^\top (\pi \odot G_t) \). Then,
\[
|\Gamma_t| \leq \gamma \sum_{\zeta} |\pi(\zeta)\phi_t(\zeta)G_{\zeta,t}|
\]
\[
\leq \frac{\gamma}{2} \sum_{\zeta} \pi(\zeta)\phi_t(\zeta)^2 + \frac{\gamma}{2} \sum_{x,a} \pi(\zeta)G_{\zeta,t}^2
\]
\[
= \frac{\gamma}{2} \pi \cdot \phi_t^2 + \frac{\gamma}{2} \sum_{\zeta} \pi(\zeta)G_{\zeta,t}^2,
\]

We can bound the second term as
\[
\sum_{\zeta} \pi(\zeta)G_{\zeta,t}^2 \leq \sum_{\zeta} \pi(\zeta) \sum_{x''} \max_{a'' \in A} |\phi_s(x'', a'')|^2 p(x''|\zeta)
\]
\[
\leq \sum_{x,a} \pi(\zeta) \sum_{x''} |\phi_s(x'', a'')|^2 p(x''|\zeta)
\]
\[
\leq K \sum_{x,a} \pi(\zeta) \sum_{x''} |\phi_s(x'', a'')|^2 \frac{1}{K} p(x''|\zeta)
\]
\[
= K \sum_{x,a} \pi(\zeta) \phi_s(\zeta)^2
\]
\[
= K \pi \cdot \phi_t^2.
\]

Consequently,
\[
|\Gamma_t| \leq \frac{\gamma}{2} \pi \cdot \phi_t^2 + \frac{K \gamma}{2} \pi \cdot \phi_t^2.
\]

Suppose \( \gamma < \frac{2}{1+K} \). Then, there exists an \( \epsilon > 0 \) such that
\[
\frac{dY_t}{dt} \leq -\epsilon \pi \cdot \phi_t^2.
\]

\( Y_t \) is clearly decreasing in time \( t \) and, since \( A \) is positive definite, has a lower bound of zero. We also have the following upper bound using Young’s inequality and the finite number of states in \( X \times A \):
\[
Y_t = \sum_{\zeta, \zeta'} \phi_t(\zeta)A_{\zeta,\zeta'}^{-1} \phi_t(\zeta')
\]
\[
\leq C \phi_t^\top \phi_t,
\]
where \( C > 0 \). This leads to the lower bound \( \phi_t^\top \phi_t \geq \frac{Y_t}{\pi} \) and the bound
\[
\frac{dY_t}{dt} \leq -\min_{x,a} \pi(x,a) \times \phi_t^\top \phi_t
\]
\[
\leq -C_0 Y_t,
\]
where \( C_0 > 0 \). By Gronwall’s inequality,

\[
Y_t \leq Y_0 e^{-C_0 t}.
\]

Consequently,

\[
\lim_{t \to \infty} Y_t = 0,
\]

concluding the proof of Lemma \( \ref{lemma:convergence} \) due to the positive-definiteness of the matrix \( A \).

6 Proof of Convergence in Finite Time Horizon Case

In this section we address the proof of Theorem \( \ref{thm:finite_convergence} \). The proof for the finite time horizon case is essentially exactly the same as the proof for the infinite time-horizon case. The main difference is that we can prove for any \( 0 < \gamma < 1 \) that the limit equation \( h_t \) converges to the Bellman equation solution \( V \) as \( t \to \infty \).

Let us begin by calculating the pre-limit evolution of the neural network output \( Q^{N}_k(j, x, a) \). For convenience, let \( \zeta = (x, a) \).

Substituting (3.3) into (6.1) yields

\[
Q^{N}_{k+1}(j, \zeta) = Q^{N}_k(j, \zeta) + \frac{1}{\sqrt{N}} \sum_{i=1}^{N} C^{i,j}_{k+1} \sigma(W^{i}_{k+1} \cdot \zeta) - \frac{1}{\sqrt{N}} \sum_{i=1}^{N} C^{i,j}_{k} \sigma(W^{i}_{k} \cdot \zeta)
\]

\[
= Q^{N}_k(j, \zeta) + \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left( C^{i,j}_{k+1} \sigma(W^{i}_{k+1} \cdot \zeta) - C^{i,j}_{k} \sigma(W^{i}_{k} \cdot \zeta) \right)
\]

\[
= Q^{N}_k(j, \zeta) + \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left( (C^{i,j}_{k+1} - C^{i,j}_{k}) \sigma(W^{i}_{k+1} \cdot \zeta) + (\sigma(W^{i}_{k+1} \cdot \zeta) - \sigma(W^{i}_{k} \cdot \zeta)) C^{i,j}_{k} \right)
\]

\[
= Q^{N}_k(j, \zeta) + \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left( (C^{i,j}_{k+1} - C^{i,j}_{k}) \sigma(W^{i}_{k+1} \cdot \zeta) + \sigma'(W^{i,*}_{k} \cdot \zeta) C^{i,j}_{k} \right)
\]

\[
+ \left[ \sigma'(W^{i}_{k} \cdot \zeta) \zeta^\top (W^{i}_{k+1} - W^{i}_{k}) + \frac{1}{2} \sigma''(W^{i,*}_{k} \cdot \zeta) \left( (W^{i}_{k+1} - W^{i}_{k}) \zeta^\top \right)^2 \right] C^{i,j}_{k},
\]

(6.1)

for points \( W^{i,*}_{k} \) and \( W^{i,**}_{k} \) in the line segment connecting the points \( W^{i}_{k} \) and \( W^{i}_{k+1} \). Let \( \alpha^N = \frac{\alpha}{N} \). Substituting (3.3) into (6.1) yields

\[
Q^{N}_{k+1}(j, \zeta) = Q^{N}_k(j, \zeta) + \frac{\alpha}{N^2} \left( r_j + \gamma \max_{a_i \in \mathcal{A}} Q^{N}_k(j + 1, x_{j+1}, a') - Q^{N}_k(j, \zeta) \right) \sum_{i=1}^{N} \sigma(W^{i}_{k} \cdot \zeta) \sigma(W^{i}_{k} \cdot \zeta)
\]

\[
+ \frac{\alpha}{N^2} \sum_{m=0}^{J-1} \left( r_m + \gamma \max_{a_i \in \mathcal{A}} Q^{N}_k(m + 1, x_{m+1}, a') - Q^{N}_k(m, \zeta_m) \right) \sum_{i=1}^{N} \sigma(W^{i}_{k} \cdot \zeta) \sigma(W^{i}_{k} \cdot \zeta) \zeta_m^\top \mathcal{C}^{i,j}_k \zeta_m + O_p(N^{-3/2}).
\]

(6.2)

We can then re-write the evolution of \( Q^{N}_k(j, x, a) \) in terms of the empirical measure \( \nu^N_k \).

\[
Q^{N}_{k+1}(j, \zeta) = Q^{N}_k(j, \zeta) + \frac{\alpha}{N} \left( r_j + \gamma \max_{a_i \in \mathcal{A}} Q^{N}_k(j + 1, x_{j+1}, a') - Q^{N}_k(j, \zeta) \right) \left( \sigma(w \cdot \zeta) \sigma(w \cdot \zeta), \nu^N_k \right)
\]

\[
+ \frac{\alpha}{N} \sum_{m=0}^{J-1} \left( r_m + \gamma \max_{a_i \in \mathcal{A}} Q^{N}_k(m + 1, x_{m+1}, a') - Q^{N}_k(m, \zeta_m) \right) \left( \sigma'(w \cdot \zeta) \sigma'(w \cdot \zeta) \zeta_m^\top \mathcal{C}^{i,j}_k \zeta_m, \nu^N_k \right)
\]

\[
+ O_p(N^{-3/2}).
\]

(6.3)
Lemma 6.1. The sequence in the set of probability measures

6.1 Identification of the Limit, Relative Compactness, and Uniqueness

The result is obtained by following the exact same steps as in the proofs of Lemmas 5.7, 5.8, 5.9, and Proof.

The proof follows the same steps as in Section 5.4, and we do not repeat it here.

Lemma 6.2. Let \( N \) be the probability measure of a convergent subsequence of \((\mu^N, h^N)\) due to the fact that the random variables \( C_{0j}, W_0 \) are assumed to be mean zero, independent random variables (see Assumption 3.1), the terms \( A_{j,m}^{l,m} \) with \( m \neq j \) will become zero in the limit as \( N \to \infty \) in the expression for (6.4).

Using the same analysis as in the infinite time horizon case (see Lemma 5.3), we can show that \( M_t^N \to 0 \) as \( N \to \infty \).

6.1 Identification of the Limit, Relative Compactness, and Uniqueness

Let \( \rho^N \) be the probability measure of a convergent subsequence of \((\mu^N, h^N)_{0 \leq t \leq T}\). Each \( \rho^N \) takes values in the set of probability measures \( \mathcal{M}(D_E([0, T])) \). We can prove the following results.

Lemma 6.1. The sequence \( \rho^N \) is relatively compact in \( \mathcal{M}(D_E([0, T])) \).

Proof. The result is obtained by following the exact same steps as in the proofs of Lemmas 5.7, 5.8, 5.9 and 5.10 Therefore, its proof will not be repeated here.

Lemma 6.2. Let \( \rho^{N_k} \) be a convergent subsequence with a limit point \( \rho \). Then, \( \rho \) is a Dirac measure concentrated on \((\mu, h) \in D_E([0, T])\) and \( (\mu, h) \) satisfies equation (3.4).

Proof. The proof is exactly the same as in Lemma 5.4 and we do not repeat it here. We only note here for completeness that due to the fact that the random variables \( C_{0j}, W_0 \) are assumed to be mean zero, independent random variables (see Assumption 3.1), the terms \( A_{j,m}^{l,m}(s) \) with \( m \neq j \) will become zero in the limit as \( N \to \infty \) in the expression for (6.4).

Lemma 6.3. The solution \((\mu, h)\) to the equation (3.4) is unique.

Proof. The proof follows the same steps as in Section 5.3 and we do not repeat it here.

Combining Lemmas 6.1 6.2 and 6.3 proves that \((\mu^N, h^N) \xrightarrow{d} (\mu, h)\) as \( N \to \infty \).
6.2 Analysis of Limit Equation

Let \( \phi_t(j, x, a) = h_t(j, x, a) - V(j, x, a) \) where \( V(j, x, a) \) is the solution to the Bellman equation (2.3). Note that \( h_t(J, x, a) = V(J, x, a) = r(J, x) \) and thus \( \phi_t(J, x, a) = 0 \). Then,

\[
d\phi_t(J - 1, x, a) = - \sum_{(x', a') \in X \times A} \pi_{J-1}(x', a') A_{x,a,x',a'} \phi_t(J - 1, x', a') dt. \tag{6.5}
\]

Let \( \zeta = (x, a) \) and \( \zeta' = (x', a') \). By Lemma 3.3 the matrix \( A_{\zeta, \zeta'} \) is positive definite and recall that \( \pi_{j-1}(\zeta') > 0 \) for every \( \zeta' \in X \times A \). Therefore, using the same analysis as in Section 5.6, we can show that, for \( t > T \) in Section 5.6, we can show that, for \( t > T \),

\[
\lim_{t \to \infty} \phi_t(j+1, y) = 0. \tag{6.6}
\]

In fact, using induction, we can prove that \( \lim_{t \to \infty} \phi_t(j, x, a) = 0 \) for \( j = 0, 1, \ldots, J \). Indeed, let us assume that \( \lim_{t \to \infty} \phi_t(j+1, y) = 0 \) for each \( \zeta \in X \times A \). Let \( Y_t = \frac{1}{2} \phi_{t,j+1}^T A^{-1} \phi_{t,j} \) where \( \phi_{t,j} = \phi_t(j, \cdot) \). The process \( Y_t \) satisfies the differential equation

\[
dY_t = \phi_{t,j+1}^T A^{-1} d\phi_{t,j} = - \phi_{t,j+1}^T A^{-1} A \left( \pi_j \circ (\phi_{t,j} - \gamma G_t) \right) dt = - \phi_{t,j}^T \left( \pi_j \circ (\phi_{t,j} - \gamma G_t) \right) dt = - \pi_j \cdot \phi_{t,j+1}^2 dt + \gamma \phi_{t,j+1} \left( \pi_j \circ G_{t,j+1} \right) dt,
\]

where the vector \( G_{t,j+1} \) is given by

\[
G_{t,j+1}(\zeta) = \sum_{x'' \in X} \left[ \max_{a'' \in A} h_t(j+1, x'', a'') - \max_{a'' \in A} V(j+1, x'', a'') \right] p(x''|\zeta). \tag{6.7}
\]

Let \( \Gamma_t := \gamma \phi_{t,j}^T (\pi_j \circ G_{t,j+1}) \). Then,

\[
|\Gamma_t| \leq \gamma \sum_{\zeta \in X \times A} |\pi_j(\zeta)\phi_{t,j}(\zeta)G_{t,j+1}(\zeta)| \\
\leq \frac{\gamma}{2} \sum_{\zeta \in X \times A} \pi_j(\zeta) \phi_{t,j}(\zeta)^2 + \frac{\gamma}{2} \sum_{\zeta \in X \times A} \pi_j(\zeta) G_{t,j+1}(\zeta)^2 \\
= \frac{\gamma}{2} \pi_j \cdot \phi_{t,j}^2 + \frac{\gamma}{2} \sum_{\zeta} \pi_j(\zeta) G_{t,j+1}(\zeta)^2 \tag{6.8}
\]

We can bound the second term \( \Gamma_t^2 := \frac{\gamma}{2} \sum_{\zeta} \pi_j(\zeta) G_{t,j+1}(\zeta)^2 \) as

\[
|\Gamma_t|^2 = \frac{\gamma}{2} \sum_{\zeta \in X \times A} \pi_j(\zeta) G_t(\zeta)^2 \\
\leq \frac{\gamma}{2} \sum_{\zeta \in X \times A} \pi_j(\zeta) \sum_{x'' \in X} \max_{a'' \in A} |\phi_t(j+1, x'', a'')|^2 p(x''|\zeta). \tag{6.9}
\]

Consequently, \( \lim_{t \to \infty} \Gamma_t^2 = 0 \). We are now in a position to prove the convergence of \( Y_t \). Similar to the analysis in Section 5.6, we can show that, for \( t > s \),

\[
Y_t \leq Y_s - C_0 \int_s^t Y_s ds + \int_s^t |\Gamma_t^2| ds, \tag{6.10}
\]

where \( C_0 > 0 \). We now choose an (arbitrary) \( \epsilon > 0 \). Since \( \lim_{t \to \infty} \Gamma_t^2 = 0 \), there exists a \( T_0 \) such that \( |\Gamma_t^2| < \frac{\epsilon C_0}{3} \) for \( t > T_0 \). Suppose that there exists a \( T_1 > T_0 \) such that \( Y_t > \epsilon \) for \( t > T_1 \). Then, for \( t \geq T_1 \),
\[
Y_t \leq Y_{T_1} - C_0 \int_{T_1}^t Y_s ds + \int_{T_1}^t |\Gamma_s^2| ds \\
\leq Y_{T_1} - C_0 \int_{T_1}^t \epsilon ds + \int_{T_1}^t \epsilon C_0 \frac{\epsilon}{3} ds \\
\leq Y_{T_1} - \frac{2C_0\epsilon}{3} (t - T_1).
\]

(6.11)

This upper bound implies that \( Y_t < 0 \) for some \( t > T_1 \). However, \( Y_t \geq 0 \) for all \( t \geq 0 \) and thus this is a contradiction. Consequently, there exists a \( T_2 > T_0 \) such that \( Y_{T_2} = \epsilon \).

Suppose that there exists a \( T_4 > T_2 \) such that \( Y_{T_4} > \epsilon \). Define the time \( T_3 = \max\{t : T_2 \leq t \leq T_4, Y_t = \epsilon\} \). Then, we obtain

\[
Y_{T_4} \leq Y_{T_3} - C_0 \int_{T_3}^{T_4} Y_s ds + \int_{T_3}^{T_4} |\Gamma_s^2| ds \\
\leq Y_{T_3} - \frac{2C_0\epsilon}{3} (T_4 - T_3) \\
\leq \epsilon,
\]

(6.12)

which is a contradiction.

Therefore, for any \( \epsilon > 0 \), there exists a \( T_2 > 0 \) such that \( Y_t \leq \epsilon \) for all \( t \geq T_2 \). Since \( \epsilon \) is arbitrary, we have proven that

\[
\lim_{t \to \infty} Y_t = 0.
\]

(6.13)

Therefore, if \( \lim_{t \to \infty} \phi_{i,j+1} = 0 \), we have shown that \( \lim_{t \to \infty} \phi_{i,j} = 0 \). By induction, \( \lim_{t \to \infty} \phi_{i,j} = 0 \) for \( j = 0, 1, \ldots, J - 1 \). This concludes the convergence proof for Theorem 3.8.

7 Proof that \( A \) is positive definite-Lemma 3.3

We now prove Lemma 3.3. Recall the matrix \( A \) with elements \( A_{\zeta, \zeta'} \) for \( \zeta, \zeta' \in \{\zeta^{(1)}, \ldots, \zeta^{(M)}\} \) where \( \zeta^{(i)} \in \mathcal{S} \subset \mathbb{R}^d \). Furthermore, each \( \zeta^{(i)} \) is distinct (i.e., \( \zeta^{(i)} \neq \zeta^{(j)} \) for \( i \neq j \)). We prove that \( A \) is positive definite under Assumption 3.2 (or equivalently under Assumption 3.7).

Let \( U = \left(U(\zeta^{(1)}), \ldots, U(\zeta^{(M)})\right) \), where \( U(\zeta) \) is defined as

\[
U(\zeta) = \sqrt{\alpha} \sigma(W \cdot \zeta) + \sqrt{C} \sigma'(W \cdot \zeta) \zeta,
\]

(7.1)

where \( (W, C) \sim \mu_0 \). Since \( C \) is a mean-zero random variable which is independent of \( W \),

\[
\mathbb{E}\left[U(\zeta)U(\zeta')\right] = \mathbb{E}\left[\alpha \sigma(W \cdot \zeta)\sigma(W \cdot \zeta') + \alpha C^2 \sigma'(W \cdot \zeta)\sigma'(W \cdot \zeta')\zeta\zeta'\right] = A_{\zeta, \zeta'}.
\]

(7.2)

Note that if \( \sigma(\cdot) \) is an odd function (e.g., the tanh function) and the distribution of \( W \) is even, then \( A \) is a covariance matrix.

To prove that \( A \) is positive definite, we need to show that \( z^T A z > 0 \) for every non-zero \( z \in \mathbb{R}^M \).

\[
z^T A z = z^T \mathbb{E}\left[U U^T\right] z \\
= \mathbb{E}\left[(z^T U)^2\right] \\
= \alpha \mathbb{E}\left[\sum_{i=1}^M z_i (\sigma(\zeta^{(i)} \cdot W) + C \sigma'(W \cdot \zeta^{(i)}) \zeta^{(i)})^2\right].
\]

(7.3)
The functions $\sigma(\zeta(i) \cdot W)$ are linearly independent since the $\zeta(i)$ are distinct (by Corollary 4.3 of [14]). Therefore, for each non-zero $z$, there exists a point $w^*$ such that

$$\sum_{i=1}^{M} z_i \sigma(\zeta(i) \cdot w^*) \neq 0.$$ 

Consequently, there exists an $\epsilon > 0$ such that

$$\left( \sum_{i=1}^{M} z_i \sigma(\zeta(i) \cdot w^*) \right)^2 > \epsilon.$$

Since $\sigma(w \cdot \zeta) + c\sigma'(w \cdot \zeta)\zeta$ is a continuous function, there exists a set $B = \{c, w : \|w - w^*\| + \|c\| < \eta\}$ for some $\eta > 0$ such that for $(c, w) \in B$

$$\left( \sum_{i=1}^{M} z_i (\sigma(\zeta(i) \cdot W) + C\sigma'(W \cdot \zeta(i))\zeta(i)) \right)^2 > \frac{\epsilon}{2}.$$

Then, we obtain that

$$\mathbb{E}\left[ \left( \sum_{i=1}^{M} z_i (\sigma(\zeta(i) \cdot W) + C\sigma'(W \cdot \zeta(i))\zeta(i)) \right)^2 \right] \geq \mathbb{E}\left[ \left( \sum_{i=1}^{M} z_i (\sigma(\zeta(i) \cdot W) + C\sigma'(W \cdot \zeta(i))\zeta(i)) \right)^2 1_{C, W \in B} \right]$$

$$\geq \mathbb{E}\left[ \frac{\epsilon}{2} 1_{C, W \in B} \right]$$

$$= \frac{K \epsilon}{2},$$

where $K > 0$. Therefore, for every non-zero $z \in \mathbb{R}^M$, we do have

$$z^T A z > 0,$$

and $A$ is positive definite, concluding the proof of the Lemma.

**A Proofs of Lemma [5.1]**

**Proof of Lemma [5.1]** We begin by recognizing that

$$\frac{1}{N} \sum_{k=0}^{N-1} f^N(z_k) - \sum_{z \in S} f^N(z) q(z) = \sum_{s \in S} \left( \frac{1}{N} \sum_{k=0}^{N-1} 1_{z_k = s} - f^N(s)q(s) \right)$$

$$= \sum_{s \in S} f^N(s) \left( \frac{1}{N} \sum_{k=0}^{N-1} 1_{z_k = s} - q(s) \right).$$

Of course, we have that $\frac{1}{N} \sum_{k=0}^{N-1} 1_{z_k = s} \xrightarrow{p} q(s)$ and, since $\frac{1}{N} \sum_{k=0}^{N-1} 1_{z_k = s}$ is uniformly bounded, a special case of Vitali’s theorem gives

$$\lim_{N \to \infty} \mathbb{E}\left[ \left( \frac{1}{N} \sum_{k=0}^{N-1} 1_{z_k = s} - q(s) \right)^2 \right] = 0.$$

Using the Cauchy-Schwartz inequality, we have

$$\mathbb{E}\left[ \frac{1}{N} \sum_{k=0}^{N-1} f^N(z_k) - \sum_{z \in S} f^N(z) q(z) \right] \leq \sum_{s \in S} \mathbb{E}\left[ f^N(s) \left( \frac{1}{N} \sum_{k=0}^{N-1} 1_{z_k = s} - q(s) \right) \right]$$

$$\leq \sum_{s \in S} \mathbb{E}\left[ (f^N(s))^2 \right]^\frac{1}{2} \mathbb{E}\left[ \left( \frac{1}{N} \sum_{k=0}^{N-1} 1_{z_k = s} - q(s) \right)^2 \right]^\frac{1}{2}$$

$$\leq C \sum_{s \in S} \mathbb{E}\left[ \left( \frac{1}{N} \sum_{k=0}^{N-1} 1_{z_k = s} - q(s) \right)^2 \right]^\frac{1}{2}.$$
Therefore, we obtain that

\[
\lim_{N \to \infty} E \left[ \left| \frac{1}{N} \sum_{k=0}^{N-1} f^N(z_k) - \sum_{z \in S} f^N(z) q(z) \right| \right] = 0,
\]
concluding the proof of the lemma.

References

[1] K. Arulkumaran, M.P. Deisenroth, M. Brundage, and A.A. Bharath. A brief survey of deep reinforcement learning [arXiv:1708.05866] 2017.

[2] D.P. Bertsekas and J. Tsitsiklis. Neuro-Dynamic Programming. Athena Scientific, 106, 1996.

[3] V. S. Borkar. Asynchronous Stochastic Approximations. SIAM J. Control Optim., 36(3), p. 840–851, 1998.

[4] L. Chizat, and F. Bach. On the global convergence of gradient descent for over-parameterized models using optimal transport. Advances in Neural Information Processing Systems (NeurIPS). p. 3040-3050, 2018.

[5] G. E. Dahl, D. Yu, L. Deng, and A. Acero. Context-dependent pre-trained deep neural networks for large-vocabulary speech recognition. Audio, Speech, and Language Processing, IEEE Transactions on, 20(1), p. 3042, 2012.

[6] S. Du, J. Lee, H. Li, L. Wang, and X. Zhai. Gradient Descent Finds Global Minima of Deep Neural Networks. Proceedings of the 36th International Conference on Machine Learning, Long Beach, California, PMLR 97, 2019.

[7] S. Du, X. Zhai, B. Poczos, and A. Singh. Gradient Descent Provably Optimizes Over-Parameterized Neural Networks. ICLR, 2019.

[8] G. Dulac-Arnold, R. Evans, H. van Hasselt, P. Sunehag, J. Lillicrap, J. Hunt, T. Mann, T. Weber, T. Degris and Ben Coppin. Deep Reinforcement Learning in Large Discrete Action Spaces. arXiv: 1512.07679v2, 2015.

[9] S. Ethier and T. Kurtz. Markov Processes: Characterization and Convergence. 1986, Wiley, New York, MR0838085.

[10] X. Glorot and Y. Bengio. Understanding the difficulty of training deep feedforward neural networks. Proceedings of the thirteenth international conference on artificial intelligence and statistics, p. 249–256, 2010.

[11] A. Graves, A.-r. Mohamed, and G. E. Hinton. Speech recognition with deep recurrent neural networks. In Proc. ICASSP, 2013.

[12] I. Goodfellow, Y. Bengio, and A. Courville. Deep Learning. Cambridge: MIT Press, 2016.

[13] M. Hausknecht, and P. Stone. Deep recurrent Q-learning for partially observable mdps. AAAI 2015 Fall Symposium, 2015.

[14] Y. Ito. Nonlinearity creates linear independence. Advances in Computational Mathematics, Vol. 5, pp. 189–203, 1996.

[15] A. Jacot, F. Gabriel, and C. Hongler. Neural Tangent Kernel: Convergence and Generalization in Neural Networks. 32nd Conference on Neural Information Processing Systems (NeurIPS), Montreal, Canada 2018.

[16] J. Kober and J. Peters. Reinforcement learning in robotics: A survey. In Reinforcement Learning. Springer, p. 579-610, 2012.
[17] A. Krizhevsky, I. Sutskever, and G. Hinton. Imagenet classification with deep convolutional neural networks. In Advances in Neural Information Processing Systems 25, p. 1106-1114, 2012.

[18] C. Kuan and K. Hornik. Convergence of learning algorithms with constant learning rates. IEEE Transactions on Neural Networks, Vol. 2, No. (5), p. 484–489, 1991.

[19] H.J. Kushner and G.G. Yin. Stochastic approximation and recursive algorithms and applications. Springer-Verlag, New York, 2003.

[20] K. Hornik, M. Stinchcombe, and H. White. Multilayer feedforward networks are universal approximators. Neural Networks, Vol. 2, No. 5, p. 359–366, 1989.

[21] K. Hornik. Approximation capabilities of multilayer feedforward networks. Neural Networks, Vol. 4, No. 2, (1991), pp. 251–257.

[22] V. Mnih, K. Kavukcuoglu, D. Silver, A. A. Rusu, J. Veness, M. G. Bellemare, A. Graves, M. Riedmiller, A. K. Fidjeland, G. Ostrovski, et al., Human-level control through deep reinforcement learning, Nature, vol. 518, no. 7540, p. 529-533, 2015.

[23] V. Mnih, K. Kavukcuoglu, D. Silver, A. Graves, I. Antonoglou, D. Wierstra and M. Riedmiller, Playing Atari with Deep Reinforcement Learning, arXiv: 1312.5602, 2013.

[24] V. Mnih, A.P. Badia, M. Mirza, A. Graves, T. Lillicrap, T. Harley, D. Silver and K. Kavukcuoglu. Asynchronous methods for deep reinforcement learning. International Conference on Machine Learning, 2016.

[25] S. Mei, A. Montanari, and P. Nguyen. A mean field view of the landscape of two-layer neural networks Proceedings of the National Academy of Sciences, Vol. 115, Index 33, E7665-E767, 2018.

[26] B. Perthame. Perturbed dynamical systems with an attracting singularity and weak viscosity limits in Hamilton-Jacobi equations. Transactions of the American Mathematical Society, Vol. 317, No. 2, p. 723–748, 1990.

[27] I. Osband, C. Blundell, A. Pritzel, and B. Van Roy. Deep exploration via bootstrapped DQN. In Advances in neural information processing systems, p. 4026-4034, 2016.

[28] H. Van Hasselt, A. Guez, and D. Silver. Deep reinforcement learning with double Q-learning. In Thirtieth AAAI conference on artificial intelligence, 2016.

[29] G. M. Rotskoff and E. Vanden-Eijnden. Neural Networks as Interacting Particle Systems: Asymptotic Convexity of the Loss Landscape and Universal Scaling of the Approximation Error. arXiv:1805.00915, 2018.

[30] P. Sermanet, K. Kavukcuoglu, S. Chintala, and Y. LeCun. Pedestrian detection with unsupervised multi-stage feature learning. In Proc. International Conference on Computer Vision and Pattern Recognition (CVPR 2013). IEEE, 2013.

[31] D. Silver, J. Schrittwieser, K. Simonyan, I. Antonoglou, A. Huang, A. Guez, T. Hubert, L. Baker, M. Lai, A. Bolton, et al., Mastering the game of Go without human knowledge, Nature, vol. 550, no. 7676, p. 354, 2017.

[32] J. Sirignano and K. Spiliopoulos. Mean Field Analysis of Neural Networks: A Law of Large Numbers. SIAM Journal on Applied Mathematics, 2019.

[33] J. Sirignano and K. Spiliopoulos. Mean Field Analysis of Neural Networks: A Central Limit Theorem. Stochastic Processes and their Applications, 2019.

[34] J. Sirignano and K. Spiliopoulos. Mean Field Analysis of Deep Neural Networks. arXiv:1903.04440, 2019.

[35] R.S. Sutton and A. Barto. Reinforcement Learning: An Introduction. MIT Press, 106, 1998.
[36] R.S. Sutton, D.A. McAllester, S.P. Singh, S. P. and Y. Mansour. Policy gradient methods for reinforcement learning with function approximation. In Advances in Neural Information Processing Systems, 2000.

[37] J.N. Tsitsiklis. Asynchronous stochastic approximation and Q-learning. Machine Learning, Vol. 16, pp. 185–202, 1994.

[38] D. Zou, Y. Cao, D. Zhou, and Q. Gu. Stochastic Gradient Descent Optimizes Over-parameterized Deep ReLU Networks. arXiv: 1811.08888, 2018.

[39] Z. Wang, T. Schaul, M. Hessel, H. Hasselt, M. Lanctot and N. Freitas. Dueling network architectures for deep reinforcement learning. ICML’16 Proceedings of the 33rd International Conference on International Conference on Machine Learning - Vol. 48, p. 1995-2003, 2016.

[40] C.I.C.H. Watkins. Learning from delayed rewards. PhD thesis, University of Cambridge, Cambridge, UK, 1989.

[41] C.I.C.H. Watkins and P. Dayan. Q-learning. Machine learning, Vol. 8, p. 279-292, 1992.