Density Functional Theory studies of cluster states in nuclei

J.-P. Ebran, 1 E. Khan, 2 T. Nikšić, 3 and D. Vretenar 3

1CEA,DAM,DIF, F-91297 Arpajon, France
2Institut de Physique Nucléaire, Université Paris-Sud, IN2P3-CNRS, F-91406 Orsay Cedex, France
3Physics Department, Faculty of Science, University of Zagreb, 10000 Zagreb, Croatia

(Dated: June 11, 2014)

The framework of nuclear energy density functionals is applied to a study of the formation and evolution of cluster states in nuclei. The relativistic functional DD-ME2 is used in triaxial and reflection-asymmetric relativistic Hartree-Bogoliubov calculations of relatively light $N = Z$ and neutron-rich nuclei. The role of deformation and degeneracy of single-nucleon states in the formation of clusters is analysed, and interesting cluster structures are predicted in excited configurations of Be, C, O, Ne, Mg, Si, S, Ar and Ca $N = Z$ nuclei. Cluster phenomena in neutron-rich nuclei are discussed, and it is shown that in neutron-rich Be and C nuclei cluster states occur as a result of molecular bonding of $\alpha$-particles by the excess neutrons, and also that proton covalent bonding can occur in $^{10}$C.

PACS numbers: 21.60.Jz, 21.10.Gv, 27.20.+n, 27.30.+t

I. INTRODUCTION

Nuclear Energy Density Functional (EDF) provides a comprehensive and accurate description of ground-state properties and collective excitations over the whole nuclide chart. In the last decade energy density functionals (EDFs) have also been successfully applied to studies of clustering phenomena, and this framework enables a consistent microscopic analysis of the formation and evolution of cluster structures, that is not limited to lightest nuclei. To describe the phenomenon of nuclear clustering already in the most basic EDF implementation, the self-consistent mean-field level, it is necessary to break as many spatial symmetries of the nuclear system as possible, and this implies a considerable computational cost. This explains the rather recent application of EDF-based methods to detailed quantitative studies of nuclear clustering. Consequently this approach allows for an understanding of the coexistence of cluster states and mean-field-type states. Cluster structures can, in fact, be considered as a transitional phase between the quantum liquid (nucleonic matter) phase and a crystal phase that does not occur in finite nuclei. Similar phase transition between the liquid and crystal phases are found in studies of mesoscopic systems such as quantum dots, or bosons in a rotating trap.

The solid (crystal) vs. quantum liquid nature of nuclear matter was analyzed using the quantity parameter, defined as the ratio of the zero-point kinetic energy of the confined nucleon to its potential energy. The typical value obtained for nuclear matter is characteristic for a quantum liquid phase and reflects the well-known fact, recently confirmed by microscopic self-consistent Green’s function calculation, that a nucleon in nuclear matter has a large mean-free path of 4 to 5 fm. The quantity parameter, however, is defined for infinite homogeneous systems and its applicability to finite nuclei is limited by the fact that it does not include any nuclear mass or size dependence. Cluster states in finite nuclei introduce an additional phase of nucleonic matter, and to analyze localization and the phenomenon of clustering a quantity must be considered that is sensitive to the nucleon number and size of the nucleus. This is the localization parameter introduced in Refs. [4–6]. Its value increases with mass and describes the gradual transition from a hybrid phase in light nuclei, characterized by the spatial localization of individual nucleon states that leads to the formation of cluster structures, toward the Fermi liquid phase in heavier nuclei. The relationship between the quantality and the localization parameters is detailed in Appendix A.

In this work we apply nuclear EDF to a study of the formation and evolution of cluster states in nuclei. The framework of nuclear EDFs and the role of spatial localization of the individual single-nucleon states is reviewed in section II. Section III presents an analysis of the role of deformation and pronounced level degeneracy on the formation of clusters, and includes a number of examples of cluster structures in excited states. Cluster phenomena and molecular bonding in neutron-rich nuclei are discussed in section IV, and section V contains a short summary and conclusion of the present study.

II. NUCLEAR ENERGY DENSITY FUNCTIONALS

The framework of EDFs provides a global approach to nuclear structure and enables an accurate description of ground-state properties and collective excitations over the whole chart of nuclides. At a moderate computational cost modern non-relativistic and relativistic EDFs can describe the evolution of structure phenomena from relatively light systems to superheavy nuclei, and from the valley of $\beta$-stability to the particle drip-lines.

The nuclear EDF is built from powers and gradients of ground-state nucleon densities and currents, representing distributions of matter, spins, momentum and kinetic...
In principle a nuclear EDF can incorporate all short-range correlations related to the repulsive core of the inter-nucleon interaction, and long-range correlations mediated by nuclear resonance modes. An additional functional of the pairing density is included to account for effects of superfluidity in open-shell nuclei.

The ground-state energy and density of a given system can be determined by minimizing an EDF with respect to the 3-dimensional density. The self-consistent scheme introduces a local effective single-particle potential, such that the exact ground-state density of the interacting system of particles equals the ground-state density of the auxiliary non-interacting system, expressed in terms of the lowest occupied single-particle orbitals. The many-body dynamics is represented by independent nucleons moving in local self-consistent mean-field (SCMF) potentials that correspond to the actual density and current distributions of a given nucleus.

A broad range of nuclear structure phenomena have been analyzed using Skyrme, Gogny and relativistic EDFs [14–20]. These global functionals present different realizations of a universal nuclear EDF governed by the underlying theory of strong interactions. With relatively small sets of global parameters determined by empirical properties of nucleonic matter and data on finite nuclei, structure models based on Skyrme, Gogny or relativistic functionals provide a consistent description of a vast quantity of nuclear data. Even though results for ground-state observables (e.g., binding energies, charge radii) obtained with different functionals are rather similar and of comparable agreement with data, calculated quantities that are not directly observable can show marked differences. One such quantity is the auxiliary local SCMF potential. In Fig. 1 we plot the neutron single-particle energy. In Fig. 2 we display the binding energies of 36Ar calculated with the Skyrme functional SLy4 [21], the Gogny effective interaction D1S [22, 23], and the relativistic density functional DD-ME2 [24]. The levels are labelled by the Nilsson quantum numbers, and dotted lines denote the position of the Fermi level. None of these functionals predict very similar ground-state properties (cf. also Fig. 3) and, therefore, similar ordering and density of levels close to the Fermi surface, the depths of the corresponding SCMF potentials are markedly different. The deepest potential corresponds to the relativistic functional DD-ME2 (−82.4 MeV), whereas the potential of the Skyrme functional SLy4 is fairly shallow (−72.4 MeV). The position of the 1s state shows that the effective depth of the D1S potential lies between the ones of DD-ME2 and SLy4. One finds the same picture for the proton states except, of course, for the effect of Coulomb repulsion.

In Ref. [4] we found qualitatively the same difference for the SCMF potentials of 20Ne calculated with SLy4 and DD-ME2. Even though the SCMF potential is not an observable, a deeper confining potential leads to a more pronounced localization of the single nucleon wave functions and enhances the probability of formation cluster structures in excited states close to the energy threshold for α-particle emission. The formation of nuclear clusters is similar to a transition from a superfluid to a Mott insulator phase in a gas of ultracold atoms held in a three-dimensional optical lattice potential [25–27]. As the potential depth of the lattice is increased, a transition is observed from a phase in which each atom is spread out over the entire lattice, to the insulating phase in which atoms are localized with no phase coherence across the lattice. In the nuclear case one cannot, of course, vary the depth of the single-nucleon potential because the nucleus is a self-bound system. However, the same effect can be obtained by performing self-consistent calculations using different EDFs or effective interactions, as illustrated in Fig. 1 for SLy4, D1S and DD-ME2.

To investigate the role of deformation in the formation of clusters, we perform deformation-constrained SCMF calculations by imposing constraints on the mass multipole moments of a nucleus. The corresponding equations (Schrödinger-like for non-relativistic functionals, or Dirac-like for relativistic EDFs, with the Hamiltonian defined as the functional derivative of the corresponding EDF with respect to density) are solved in the intrinsic frame of reference attached to the nucleus, in which the shape of the nucleus can be arbitrarily deformed. In the present study we employ SCMF models that allow breaking both the axial and reflection symmetries. As an illustration in Fig. 4 we display the binding energies of the self-conjugate nucleus 36Ar as functions of the axial quadrupole deformation parameter β2, calculated with SLy4 and D1S using the Hartree-Fock-Bogoliubov model, and with the functional DD-ME2 employing the relativistic Hartree-Bogoliubov approach. Pairing correlations...
FIG. 2: Self-consistent binding energy curves of \( ^{36}\text{Ar} \) as functions of the quadrupole deformation parameter \( \beta_2 \), calculated with the functionals SLy4, D1S and DD-ME2. The insets display the corresponding intrinsic nucleon density distributions in the reference frame defined by the principal axes of the nucleus.

are taken into account by a delta-pairing force for calculations with the Skyrme functional, whereas for the RHB calculations with DD-ME2 the pairing interaction is separable in momentum space, and determined by two parameters adjusted to reproduce the Gogny pairing gap in symmetric nuclear matter. The curves of the total energy as a function of quadrupole deformation are obtained in a SCMF approach by imposing a constraint on the axial quadrupole moment. The parameter \( \beta_2 \) is directly proportional to the intrinsic mass quadrupole moment. For all three functionals the calculated equilibrium shape of \( ^{36}\text{Ar} \) is a slightly oblate, axially symmetric quadrupole ellipsoid with \( \beta_2 \approx -0.2 \). For the equilibrium deformation and few additional values of \( \beta_2 \), in the insets of Fig. 2 we also include the corresponding intrinsic nucleon density distributions in the reference frame defined by the principal axes of the nucleus. Here one already observes an interesting effect that was previously noted in our studies of Refs. [4–6], namely that deeper potentials lead to a more pronounced spatial localization of nucleonic densities. In general, we find that relativistic functionals, when compared to Skyrme and Gogny functionals, are characterized by deeper SCMF potentials. As noted in Ref. [4], the depth of a relativistic potential is determined by the difference between two large fields: an attractive (negative) Lorentz scalar potential of magnitude around 400 MeV, and a repulsive Lorentz vector potential of roughly 320 MeV (plus the repulsive Coulomb potential for protons). The sum of these potentials (about 700 MeV) determines the effective single-nucleon spin-orbit force in a unique way, whereas in a non-relativistic EDF framework the spin-orbit potential is included in a purely phenomenological way, with a strength parameter adjusted to empirical energy spacings between spin-orbit partner states. In the relativistic case the scalar and vector fields determine both the effective spin-orbit force and the SCMF potential, and the latter is generally found to be deeper than the non-relativistic mean-field potentials. In the following sections we, therefore, perform SCMF calculations based on the relativistic functional DD-ME2, which predicts equilibrium density distributions that are more localized, often with pronounced cluster structures.

III. DEFORMATIONS AND EXCITED CONFIGURATIONS

A unique feature of light nuclei is the coexistence of the nuclear mean-field and cluster structures, as expressed by the well-known Ikeda diagram [27–31]. A certain degree of localization of nucleonic densities is already present in mean-field ground-state configurations [4, 52, 53], and this facilitates the formation of cluster structures in excited states. Close to the particle emission threshold continuum effects become important for a quantitative description of nuclear clustering [34]. Deformation in light nuclei plays, of course, an important role in the formation of clusters [11, 24, 51, 55]. The relationship between \( \alpha- \)
clusters and single-particle states in deformed nuclei has been extensively studied \cite{28,30,36}. For instance, the Bayman-Bohr theorem \cite{37} states that the SU(3) shell model wave function of a ground state is in most cases equivalent to the cluster Brink wave function in the limit when the inter-alpha distance vanishes. However, this important link only relates a cluster wave function to a mean-field type one in this specific limit. The present EDF-based approach allows one to go a step further and establish a link between cluster states and the single-particle spectrum.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig3.png}
\caption{Mean value of the energy gap between consecutive occupied neutron levels as a function of the axial quadrupole deformation parameter $\beta_2$ for $^{12}$C (a) and $^{20}$Ne (b). The insets display the total nucleonic density at the corresponding deformation. To limit the vertical scale the maximum mean value of the energy gap in the plot does not exceed 5 MeV.}
\end{figure}

As stated by Rae \cite{38}, the degeneracy of single-nucleon states at a given deformation could generate clusters because of levels crossing. Here we analyze how degeneracy affects the formation of $\alpha$ clusters in self-conjugate nuclei. As noted by Aberg \cite{36}, an isolated level of the single-particle energy spectrum in a deformed self-conjugate $N = Z$ nucleus can correspond to an alpha-cluster, because of both time-invariance symmetry and isospin symmetry: two protons and two neutrons have similar wave functions, and the localization of these functions facilitates the formation of $\alpha$-clusters. Hence, pronounced level degeneracy (or isolated levels in the case of alpha-clustering) allows to explain: i) why $N = Z$ and deformed nuclei favor cluster formation, ii) the link between the depth of the confining potential and cluster formation and, iii) why cluster structures mainly occur in light nuclei. The second point is related to the fact that pronounced degeneracy is driven by the depth of the potential \cite{39}, and this issue has already been analyzed in our previous studies \cite{4–6}. The answer to the third question comes from the fact that level density is generally smaller in lighter nuclei and this favors the occurrence of isolated single-particle levels or degeneracy at certain deformations.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig4.png}
\caption{(Color online) Total positive-parity projected density plots obtained for a number of excited states in $N = Z$ nuclei. For each nucleus the first density on the bottom corresponds to the equilibrium configuration. Other selected densities are displayed in order of increasing excitation energy.}
\end{figure}

Let us illustrate this concept using the microscopic EDF framework with the examples of axially deformed quadrupole shapes of $^{12}$C and $^{20}$Ne. By increasing the prolate quadrupole deformation in a self-consistent calculation with the constraint on the axial quadrupole moment, these nuclei display a series of cluster configurations until eventually reaching the linear $\alpha$-chain structure (Fig 3). To show the role of level degeneracy, Fig 3
FIG. 5: Self-consistent energy surfaces of \(^{8}\)Be, calculated with DD-ME2 by imposing constraints on both the axial quadrupole and octupole deformation parameters \(\beta_2\) and \(\beta_3\) (left), and the corresponding positive parity-projected energy surfaces (right).

FIG. 6: Same as in the caption to Fig. 5 but for the isotopes \(^{12}\)C.
displays the mean value of the energy gap between consecutive occupied neutron single-particle levels as a function of the deformation parameter $\beta$. At deformations for which the maximum mean value of the energy gap exceeds 5 MeV, we only plot this value so that the scale of the vertical axis does not become too large to display.

A pronounced correlation between the enhancement of energy gaps between the single-particle levels and alpha-cluster formation can clearly be identified. Both for $^{12}$C and $^{20}$Ne the density profiles show pronounced localization of $\alpha$ clusters at deformations at which the mean value of the energy gap between consecutive levels exhibits a sharp increase.

Recent cranking SCMF calculations of high-spin rotating nuclei produced interesting exotic cluster configurations such as, for instance, in $^{16}$O and $^{40}$Ca [41–43]. In the present study cluster shapes occur as local minima at large deformations in the $(\beta_2, \gamma, \beta_3, \beta_2)$ hypersurface. As an illustration, figure displays a sample of various cluster shapes in self-conjugate nuclei, obtained in the triaxial and reflection-asymmetric RHB calculation using the functional DD-ME2. For instance, the axial quadrupole and octupole nucleonic distributions in $^8$Be and $^{12}$C correspond to local minima on the energy surfaces function of axial quadrupole and octupole deformations displayed in Fig. 5 and Fig. 6. The self-consistent reflection-asymmetric axial energy surfaces are calculated with DD-ME2 by imposing constraints on both the axial quadrupole and octupole deformation parameters $\beta_2$ and $\beta_3$, respectively. The 3D energy maps and their projections on the $\beta_2 - \beta_3$ plane in the left column are obtained in SCMF calculations. The corresponding positive parity-projected energy surfaces are shown in the right column. For the equilibrium deformations and few additional local minima the intrinsic nucleon density distributions in the reference frame defined by the principal axes of the nucleus are shown in the insets. The projected energy surface of $^8$Be displays a deep minimum at very large quadrupole deformation that corresponds to a two-$\alpha$ configuration in agreement with a large number of previous studies [11, 12, 14, 43]. $^{12}$C offers the possibility to investigate properties of three-center clusters. Linear chains of $\alpha$-particles are predicted at very large prolate quadrupole deformations. A further possibility for three-center systems involves the formation of triangular shapes characterized by a discrete symmetry, and such structures are found in the region of oblate deformations.

IV. CLUSTER STRUCTURES IN NEUTRON-RICH NUCLEI

Low-lying states in some light neutron-rich nuclei can be described by molecular bonding ($\pi$ or $\sigma$) of $\alpha$-particles by the excess neutrons [27, 28, 31]. Figure 7 displays the total, proton and neutron positive-parity projected densities with axially symmetric quadrupole and octupole deformations of Be isotopes in their equilibrium configurations, calculated using the RHB model with the DD-ME2 functional. One clearly notices the two-$\alpha$ structure, except in $^{11,12}$Be which display nearly spherical shapes because of the N=8 shell closure (the calculation for the odd-N isotopes is performed using the equal filling approximation).

To analyze the cluster content of Be isotopes, we investigate the partial densities built from the occupied single-particle states. Fig. 8 displays the total neutron distribution of $^8$Be at equilibrium deformation, and details its decomposition into partial densities of each of the two occupied Nilsson states. A very similar picture is found for the proton density distributions. The partial densities provide a very clear picture of the formation of the two $\alpha$ clusters in the total density distribution.

In the case of the neutron-rich Be isotopes, decomposing the total density into the $\alpha + \alpha$ core and the valence density of the additional neutrons, supports the interpretation of these systems as genuine nuclear molecules. The valence neutrons stabilize the two-center cluster structure of the $Z = 4$ and $N = 4$ core with $\pi$-like (Fig. 9 and 11) and $\sigma$-like (Fig. 10 and 12) molecular bonding.

Interesting configurations are predicted for the even-A carbon isotopes in Fig. 13 where we plot the mean-field and parity-projected energy surfaces of $^{10,12,14,16}$C as functions of the axial quadrupole and octupole deformation parameters $\beta_2$ and $\beta_3$. Figs. 14 and 15 display the excess neutron molecular orbits of the carbon isotopes in excited configurations of $^{14}$C and $^{16}$C. The decomposition of the density of an excited configuration of
FIG. 8: (Color online) Contour plot of the $^8\text{Be}$ neutron density (a), and surface plots of the partial densities of each of the two occupied Nilsson states in the (Oxz) plane (b) and (c).

FIG. 9: (Color online) Left panel: total (a), proton (b) and neutron (c) intrinsic densities of $^{10}\text{Be}$ at equilibrium deformation. Right panel, from bottom to top: 3D density of the $\alpha + \alpha$ core; contour plots of the core density and the density of the two valence neutrons in the (Oxz) plane; 3D density of the two valence neutrons.

$^{14}\text{C}$ (cf. Fig. 6) in terms of the $3-\alpha$ core and two valence neutrons is shown in Fig. 14. We note that in this case pairing correlations between the two valence neutrons tend to favor a reflection asymmetric chain configuration. Accordingly, the intrinsic reflection-asymmetric chain configuration $\alpha - 2n - \alpha - \alpha$, with the two valence neutrons forming a $\pi$-bond between two $\alpha$, is found at lower energy with respect to the reflection symmetric chain $\alpha - n - \alpha - n - \alpha$. A reflection-symmetric configuration with four valence neutrons is favored in $^{16}\text{C}$, as shown Fig. 15.

Molecular structures with proton covalent bonding are also predicted. The relatively weak Coulomb effect in light nuclei allows one to study isospin charge symmetry. As the mirror nucleus of $^{10}\text{Be}$, $^{10}\text{C}$ is expected to display a covalent 2-center chain configuration with a pair of protons as the covalent bond. Moreover, $^{10}\text{C}$ presents a unique super-Borromean or Brunnian nuclear system [44, 45]. Results of our calculation displayed in Fig. 19.
FIG. 13: Same as in the caption to Fig. 5 but for the isotopes $^{10,12,14,16}\text{C}$. 
support the picture of a proton-bonding molecular structure of this nucleus. On the oblate side, $^{10}$C displays a triangular shape with the cluster structure $\alpha - \alpha - 2p$

V. CONCLUSION

The formation of cluster states in nuclei has been analyzed using a theoretical framework based on nuclear EDF. By performing Hartree-Fock-Bogoliubov (HFB) calculations with Skyrme and Gogny functionals, and relativistic Hartree-Bogoliubov (RHB) calculations with the functional DD-ME2, the axial and reflection symmetries being broken, we have shown that a deeper self-consistent mean-field confining potential leads to a more pronounced localization of the single nucleon wave functions and enhances the probability of formation cluster structures in excited states. In particular, since the relativistic functional DD-ME2 produces the deepest potential among the considered functionals, we have used DD-ME2 in a series of axially-symmetric quadrupole and octupole constrained RHB calculations of relatively light $N = Z$, as well as neutron-rich nuclei. The role of deformation and degeneracy of single-nucleon states in the formation of clusters has been analyzed in detail, and a number of interesting cluster structures have been predicted in excited configurations that correspond to local minima on the parity-projected energy maps as functions of the quadrupole and octupole deformation parameters.

A particularly interesting topic is the occurrence of cluster configurations in neutron-rich nuclei. We have shown that in neutron-rich Be and C nuclei cluster states occur as a result of molecular bonding ($\pi$ or $\sigma$) of $\alpha$-particles by the excess neutrons, and also that proton covalent bonding can occur in $^{10}$C.

Results obtained in this study demonstrate the feasibility of using nuclear EDF to explore the occurrence and evolution of $\alpha$-cluster structures in relatively light $N = Z$ and neutron-rich nuclei. When compared to dedicated cluster models, this framework allows for a microscopic description of the coexistence of cluster states and mean-field-type states at low energies. The SCMF approach does not assume a priori the existence of nucleon cluster structures, rather energy density functionals implicitly include many-body correlations that enable the formation of cluster states starting from microscopic single-nucleon degrees of freedom. For a quantitative description of cluster states, however, EDF-based structure models have to be developed that go beyond the static mean-field approximation, and include collective correlations related to the restoration of symmetries broken at the mean-field level, and to fluctuations of collective vary.
ables. These models can then be employed in analyses of cluster phenomena related to shell evolution and shape transitions, including detailed predictions of excitation spectra and electromagnetic transition rates. Work along these lines is under progress.

Acknowledgement

This work was supported by the Institut Universitaire de France. The authors thank Peter Schuck for reading the manuscript and many valuable discussions.

Appendix A: The localization and quantality parameters

We shall provide here an analytical relationship between the quantality and the localisation parameter in the case of nuclei. The localisation parameter is used to describe clustering as hybrid states between crystal and quantum liquid [4–6] whereas the quantality is used to investigate the quantum liquid vs. crystal behavior of the interaction [12]. The quantality is defined as

\[ \Lambda = \frac{\hbar^2}{m r_0^2 V'_0} \]  

where \( r_0 \) is the typical inter-nucleon distance and \( V'_0 \) the characteristic magnitude of the inter-particle interaction (\( V'_0 \approx 100 \) MeV in the case of the nucleon-nucleon interaction). As discussed by Mottelson the quantum liquid phase is obtained for \( \Lambda > 0.1 \), whereas the crystal phase is characterized by values of \( \Lambda < 0.1 \). Nuclei, of course, are in the quantum liquid phase. However, the quantality parameter Eq. [A1] depends on the nucleon-nucleon interaction only, and does not take into account the finite size effects at work in nuclei. Hence the localization parameter is defined as [4–6]

\[ \alpha = \frac{b}{r_0} = \frac{\sqrt{2} A^{1/6}}{(2mV'_0 r_0^2)^{1/4}} \]  

where \( b \) is the typical dispersion of the single-nucleon wave function, and \( V'_0 \) is the depth of the confining potential (\( V'_0 \approx 75 \) MeV in the case of the nucleon mean-field [4]). One can therefore use \( \alpha \) to analyze the evolution of nuclear configurations with respect to the number of constituents \( A \) and, in particular, systems where finite-size effects are relevant (\( A < 10^9 \)). The crystal, cluster and liquid phases then correspond to \( \alpha < 1, \alpha \sim 1, \) and \( \alpha > 1 \), respectively.

In order to relate quantality and the localisation parameter, one needs to relate the depth of the mean-field potential \( V'_0 \) to the magnitude of the nucleon-nucleon interaction \( V'_0 \). Considering a short range n-n interaction \( V'_2(r, r') \), that can qualitatively be described by a hard core for \( r < r_0 \) and an attractive part of magnitude \(-V'_0 \) in the region between \( r_0 \) and \( r_0+a \) (Fig. [17]):

\[ V'_2(r, r') = V'_2(R) = -V'_0 \]  

for \( R \) between \( r_0 \) and \( r_0+a \), with \( R \equiv |r - r'| \).

The n-n interaction can also be approximated by

\[ V'_2(r, r') = -V'_0 \delta(r - r) \]  

This can be justified by the short range approximation of the nucleon-nucleon interaction, and such a zero-range approximation is successfully used, for instance, in Skyrme functionals. To be compatible with Eq. [A3], \( a < r_0 \).

The mean-field potential \( V(r) \) is, to a good approximation, the mean value of the n-n interaction over the nucleonic density [39]:

\[ V(r) \simeq \int V'_2(r, r') \rho(r')d\bar{r} = -V'_0 \rho(r - r_0) \]  

Eq. [A5] expresses the fact that in a saturated system characterized by a short-range interaction, the mean-field potential displays the spatial dependence that corresponds to the shape of the density. From Eq. [A5] the depth of the mean-field potential is:

\[ V'_0 \approx \rho''_0 \rho_0 \]  

where \( \rho_0 = 3/(4\pi r_0^3) \) corresponds to the saturation density. Moreover, Eq. [A3] and [A4] yield

\[ \int V'_2(R)d\bar{R} = -V'_0 \rho''_0 \]  

and thus

\[ V'_0 = \frac{4}{3} \pi V'_0 \left[(r_0 + a)^3 - r_0^3\right] . \]  

FIG. 17: Simple approximation of the n-n potential used to derive the relation between the localization and quantality parameters Eq. [A11].
Inserting Eq. (A8) in (A6), one finally obtains
\[ V_0 = \gamma V'_0 \]  \quad (A9)
with
\[ \gamma \equiv \left(1 + \frac{a}{r_0}\right)^3 - 1. \]  \quad (A10)

Therefore, the relation between the depth of the mean-field potential \( V_0 \) and the magnitude of the n-n interaction \( V'_0 \) is linear and only depends, in this simple approximation, on the ratio \( a/r_0 \), that is, the width of the attractive part of the n-n interaction over the equilibrium distance between the nucleons. In finite nuclei, for typical values of \( r_0 \) and \( a \) one gets \( \gamma \approx 3/4 \). This is in agreement with the empirical values \( V_0 = 75 \text{ MeV} \) and \( V'_0 = 100 \text{ MeV} \) \[\text{[12 13]}\] Inserting now Eq. (A1) into Eq. (A2) with \( \gamma = 3/4 \), one finds the relation between the localization and quantity parameters
\[ \alpha \approx A^{1/6} \Lambda^{1/4}. \]  \quad (A11)

[1] T. Ichikawa, J.A. Maruhn, N. Itagaki and S. Ohkubo, Phys. Rev. Lett. 107, 112501 (2011).
[2] T. Ichikawa, J.A. Maruhn, N. Itagaki, K. Matsuyanagi, P.-G. Reinhard, and S. Ohkubo, Phys. Rev. Lett. 109, 232503 (2012).
[3] M. Girod and P. Schuck, Phys. Rev. Lett. 111, 132503 (2013).
[4] J.P. Ebran, E. Khan, T. Nikšić and D. Vretenar, Phys. Rev. C 87, 044307 (2013).
[5] J.P. Ebran, E. Khan, T. Nikšić and D. Vretenar, Phys. Rev. C 89, 031303(R) (2014).
[6] J.A. Maruhn, M. Kimura, S. Schramm, P.-G. Reinhard, H. Horiuchi and A. Tohsaki, Phys. Rev. C 74, 044311 (2006).
[7] P. Arumugam, B.K. Sharma, S. K. Patra and Raj K. Gupta, Phys. Rev. C 71, 064308 (2005).
[8] J.M. Yao, N. Itagaki, and J. Meng, Arxiv1403:7940.
[9] T. Ichikawa, J.A. Maruhn, N. Itagaki and S. Ohkubo, Phys. Rev. Lett. 109, 232503 (2012).
[10] J.A. Maruhn, M. Kimura, S. Schramm, P.-G. Reinhard, H. Horiuchi and A. Tohsaki, Phys. Rev. C 74, 044311 (2006).
[11] C. Yannouleas and U. Landman, Rep. Prog. Phys. 70 (2007) 2067
[12] B. Mottelson, Nuclear Structure, Les Houches, Session LXVI, 25 (1996).
[13] A. Rios and V. Somá, Phys. Rev. Lett. 108, 012501 (2012).
[14] M. Bender, P.H. Heenen, P.G. Reinhard, Rev. Mod. Phys. 75, 121 (2003).
[15] G.A. Lalazissis, P. Ring, D. Vretenar, Extended Density Functionals in Nuclear Structure Physics (Lecture Notes in Physics 641, Springer, 2004).
[16] J. Stone, P.G. Reinhard, Prog. Part. Nucl. Phys. 58, 587 (2007).
[17] J. Erler, P. Klüpfel, P.G. Reinhard, J. Phys. G 38, 033101 (2011).
[18] D. Vretenar, A.V. Afanasjev, G. Lalazissis, P. Ring, Phys. Rep. 409, 101 (2005).
[19] J. Meng, H. Toki, S. Zhou, S. Zhang, W. Long, Prog. Part. Nucl. Phys. 57, 470 (2006).
[20] T. Nikšić, D. Vretenar, P. Ring, Prog. Part. Nucl. Phys. 66, 519 (2011).
[21] E. Chabanat, P. Bonche, P. Haensel, J. Meyer, R. Schaeffer, Nucl. Phys. A 635, 231-256 (1998).
[22] J. F. Berger, M. Girod, and D. Gogny, Nucl. Phys. A 428, 23c (1984).
[23] J. F. Berger, M. Girod, and D. Gogny, Comp. Phys. Comm. 63, 365 (1991).
[24] G.A. Lalazissis, T. Nikšić, D. Vretenar, P. Ring, Phys. Rev. C 71, 024312 (2005).
[25] M. Greiner, O. Mandel, T. Esslinger, T.W. Hänsch and I. Bloch, Nature 415, 39 (2002).
[26] D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, and P. Zoller, Phys. Rev. Lett. 81, 3108 (1998).
[27] W.v Oertzen, M. Freer, Y. Kanada-En’yo, Phys. Rep. 432, 43 (2006).
[28] Y. Kanada-En’yo, H. Horiuchi, Prog. Theor. Phys. Suppl. 142, 205 (2001).
[29] K. Ikeda, N. Takigawa and H. Horiuchi, Prog. Theor. Phys. Suppl. 464 (1968).
[30] H. Horiuchi, Lecture Notes in Physics 818, 57 (2011).
[31] H. Horiuchi, K. Ikeda, and K. Kato, Prog. Theor. Phys. Suppl. 192, 1 (2012).
[32] P.-G. Reinhard, J.A. Maruhn, A.S. Umar, V.E. Oberacker, Phys. Rev. C 83, 034312 (2011).
[33] Bo Zhou, Zhongzhou Ren, Chang Xu, Y. Funaki, T. Yamada, A. Tohsaki, H. Horiuchi, P. Schuck, and G. Röpke, Phys. Rev. C86, 014301 (2012).
[34] J. Okołowicz, M. Płoszajczak, W. Nazarewicz, Prog. Theor. Phys. Suppl. 196, 230 (2012).
[35] L.M. Robledo, G.F. Bertsch, Phys. Rev. C 84, 054302 (2011).
[36] S. Aberg, L.-O. Jönsson, Z. Phys. A 349, 205 (1994).
[37] B.F. Bayman, A. Bohr, Nucl. Phys. A 9, 956 (1958).
[38] W.D.M. Rae, Proc of the 5th Int. Conf. on clustering aspects in nuclear and subnuclear systems, Kyoto, Japan, Phys. Society of Japan, 80 (1989).
[39] P. Ring, P. Schuck, The Nuclear Many-Body Problem, Springer-Verlag, Heidelberg, (1980).
[40] Y. Funaki, M. Girod, H. Horiuchi, G. Röpke, P. Schuck, A. Tohsaki and T. Yamada, J. Phys. G 37, 046407 (2010).
[41] M.D. Brink, Proceedings of the International School of Physics Enrico Fermi, Academic Press, New York, 36, 247 (1966).
[42] A. Tohsaki, H. Horiuchi, P. Schuck, and G. Ropke, Phys. Rev. Lett., 87, 192501, (2001).
[43] R.B. Wiringa, S.C. Pieper, J. Carlson and V.R. Pandharipande, Phys. Rev. C62, 014001 (2000).
[44] R.J. Charity, K. Mercurio, L.G. Sobotka, J.M. Elson, M. Famiño, A. Banu, C. Fu, L. Trache, and R. E. Tribble,
Phys. Rev. C 75, 051304(R).

[45] N. Curtis, N.L. Achouri, N.I. Ashwood, H.G. Bohlen, W.N. Catford, N.M. Clarke, M. Freer, P.J. Haigh, B. Laurent, N.A. Orr, N.P. Patterson, N. Soiè, J.S. Thomas and V. Ziman, Phys. Rev. C 77, 021301 (2008).

[46] A.N. James, P.T. Andrews, P. Kirkby, B.G. Lowe, Nucl. Phys. A138 (1969) 145.