Starting from the well-known quantum Miura-like transformation for the non simply-laced Lie algebras $B_N$, we give an explicit construction of the Casimir $WB_3$ algebras. We reserve the notation $WB_N$ for the Casimir $\mathcal{W}$ algebras of type $\mathcal{W}(2, 4, 6, \cdots, 2N, N + \frac{1}{2})$ which contains one fermionic field. It is seen that $WB_3$ algebra is closed and associative for all values of the central charge $c$. 

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† This article is dedicated to the 225th anniversary of the Istanbul Technical University.

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1. Introduction

Extended (super) conformal algebras which consist of the Virasoro algebra and with additional higher spin fields play an important role in our understanding the structure of two-dimensional (super) conformal field theories (see e.g. Refs. 1-4). The idea to extend the Virasoro algebra with the introduction of higher integer and a half integer conformal spin generators is also seem to be relevant in two-dimensional conformal field theories. It is known that a Casimir algebra corresponding to a simple Lie algebra $\mathcal{L}_N$ can be denoted by $\mathcal{WL}_N$. Therefore Casimir $\mathcal{WB}_N$ algebras based on non-simply laced Lie algebra $\mathcal{B}_N$. With the above identification, one says that $\mathcal{WB}_3$ algebra is the simplest superconformal algebra. A construction of this type algebras was presented first by Fateev and Luk’yakov. This algebra coincides with Neveu–Schwarz algebra ($N=1$ super Virasoro algebra). Then Figueroa-O’Farrill et al. have proven that $\mathcal{WB}_3$ algebra are associative for all values of central charge $c$. And also Ahn shown an explicit construction of the Casimir $\mathcal{WB}_2$ algebra from a fermion proposed by Watts in Ref.8.

In this paper we examine a Casimir algebra for a non simply-laced Lie algebras $\mathcal{L}_N = \mathcal{B}_N$, which are called Casimir $\mathcal{WB}_N$ algebras, for $N = 3$, and also show that $\mathcal{WB}_3$ algebra is closed and associative for all values of $c$. We must emphasize here that we reserve the notation $\mathcal{WB}_N$ for Casimir $\mathcal{W}$ algebras of type $\mathcal{W}(2, 4, 6, \cdots, 2N + \frac{1}{2})$ which contains one fermionic field. Since there were three different $\mathcal{W}$ algebras apart from our notation $\mathcal{WB}_N$, algebra type consistent for generic central charge $c$.

This paper is organized as follows. In section 2, we give a basis for Casimir $\mathcal{WB}_N$ algebras by using the well-known Miura-like transformation with free massless bosonic field and a free fermion field. In section 3, we constructed a free field realization of the fermionic Casimir $\mathcal{WB}_N$ algebra by calculating explicitly all nontrivial OPEs among primary fields. All results were given by the shorthand notation. In section 4, we constructed a vertex operator extension of the Casimir $\mathcal{WB}_N$ algebras by using the help of Mathematica Package OPEDefs.m of Thielemans under the MathematicaTM in Ref.15.

2. The Casimir $\mathcal{WB}_N$ Algebra Basis

In this section we give a primary basis for the Casimir $\mathcal{WB}_N$ algebras from the free massless bosonic fields and a free fermion field realization point of view.

The Casimir $\mathcal{WB}_N$ algebra generated by a set of chiral currents $U_{2k}(z)$, of conformal dimension $2k$ ($k = 1, \cdots, N$), together with a fermionic field $U_{N + \frac{1}{2}}(z)$, of conformal dimension $N + \frac{1}{2}$. The following Miura-like transformation

$$R_N(z) = \prod_{j=1}^{N}(\alpha_j \partial_z - h_j(z)),$$

and a free fermion field $b(z)$, of conformal dimension $\frac{1}{2}$, give the fermionic field

$$U_{N + \frac{1}{2}}(z) = R_N(z) \cdot b(z)$$

of the Casimir $\mathcal{WB}_N$ Algebra. By taking OPE of $d_{N + \frac{1}{2}}(z)$ with itself one generates a set of fields $U_{2k}(z)$ ($k = 1, \cdots, N$) above mentioned. In the Miura-like transformation, the symbol : $\star$ : shows the normal ordering, and $\alpha_0$ is a free parameter. $h_j(z) = i\mu_j \partial \varphi(z)$ has $N$ component which are Feigin Fuchs-type of free massless scalar fields, here, $\mu_i$’s, $(i = 1, \cdots, N)$ are the weights of the fundamental (vector) representation of $\mathcal{B}_N$, satisfying $\mu_i, \mu_j = \delta_{ij}$. The simple roots of $\mathcal{B}_N$ are given by $\alpha_i = \mu_i - \mu_{i+1}$, $(i = 1, \cdots, N - 1)$ and
\[ \alpha_N = \mu_N. \] A free scalar bosonic field \( \varphi(z) \) and a free fermionic field \( b(z) \) are single-valued function on the complex plane and its mode expansion are given by

\[
i \partial \varphi(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}, \quad b(z) = \sum_{n \in \mathbb{Z}} b_n z^{-n-\frac{1}{2}}. \tag{2.3}\]

respectively. Canonical quantization give the commutator and anti-commutator relations

\[
[a_m, a_n] = m \delta_{m+n,0}, \quad \{b_m, b_n\} = \delta_{mn}, \tag{2.4}\]

and these relations are equivalent to the contractions, as \( z_{12} = z_1 - z_2 \)

\[
h_i(z_1) h_j(z_2) = \frac{\delta_{ij}}{z_{12}}, \quad b(z_1) b(z_2) = \frac{1}{z_{12}} \tag{2.5}\]

respectively.

First of all, in the example of \( WB_1 \), fermionic field \( U_\frac{z}{2} \) can be obtained by expanding \( U_N \) for \( N = 3 \), which is following

\[
U_\frac{z}{2} = (h_1 h_2 b)(z) - a_o (h_1 h')(z) - a_o (h_2 h')(z) - a_o (h_2 h')(z) + a_o^2 h''(z) \tag{2.6}\]

Then, by taking OPE of \( U_\frac{z}{2} \) with itself, under the following normalization,

\[
Q(z) = \sqrt{\frac{3675}{2 (49 + c) (21 + 4c)}} U_\frac{z}{2} \tag{2.7}\]

gives

\[
Q(z_1) Q(z_2) = \frac{2c/7}{z_{12}^7} + \frac{2W_2}{z_{12}^5} + \frac{W_4}{z_{12}^4} + \frac{S_4}{z_{12}^3} + \frac{S_6}{z_{12}} \tag{2.8}\]

where

\[
W_2(z) = \frac{1}{2} \sum_{i=1}^{N} \left( (h_i h_i)(z) + a_o (i - N - \frac{1}{2}) h'_i(z) \right) + (b'b)(z) \tag{2.9}\]

We observe that \( W_2(z) \equiv U_\frac{z}{2} \equiv T(z) \) has spin-2, which is called the stress-energy tensor. The standard OPE of \( T(z) \) with itself is

\[
T(z_1) T(z_2) = \frac{c/2}{z_{12}^4} + \frac{2 T}{z_{12}^2} + \frac{\partial T}{z_{12}} \tag{2.10}\]

where the central charge, for \( B_N \), is given by

\[
c = (N + \frac{1}{2}) (1 - 2N(2N - 1) \alpha_0^2) \tag{2.11}\]

We can also write one of the two statements in (8), i.e. \( S_4(z) \), since \( S_6(z) \) is too large, which is given by

\[
S_4(z) = (1 - 2 \alpha_0^2) \left( \sum_{i<j}^3 (h_i h_j h_i)(z) + \sum_{i=1}^3 (h_i h_i h')(z) \right.
\]

\[+ \alpha_o \sum_{i<j}^3 (2j - 7) (h_i h_j h')(z) - 2 \alpha_o \sum_{i<j}^3 (h_i h_j h')(z) - \alpha_o \sum_{i=1}^3 (h_i b'' b)(z) + \]
We can decompose $S$ and respectively. Thus, above normalized fields leads to new decompositions for $S$ and

\[ +\alpha_0^2 \sum_{i<j}^3 (7-2j)(h_i h'_j)(z) + \alpha_0 \sum_{i<j}^3 (2i-5)(h_i h_j)(z) + \alpha_0 \sum_{i<j}^3 (2i-5)(h_i b b)(z) \]  

\[ +\alpha_0^2 \sum_{i<j}^3 (2i-5)(2j-7)(h_i h'_j)(z) + \alpha_0^2 \sum_{i=1}^3 (i-1)(i-4)(h_i h'_j)(z) \]  

\[ +\frac{1}{2} \sum_{i<j}^3 (1-2(i^2-6i+11)\alpha_0^2)(h''_i h_i)(z) - \frac{1}{6} \sum_{i<j}^3 (7-2i)(1-(i^2-7i+18)\alpha_0^2)\alpha_0 h'''(z) \]

\[ +\frac{1}{6} (1-6\alpha_0^2)(b'''b)(z) \]

We can decompose $S_4(z)$ and $S_6(z)$ to the fields $W_{2k}(z)$'s, as follows

\[ S_4(z) = \alpha_4 W_4(z) + \alpha_2 W_2 W_2(z) + \alpha_3 W_2''(z) \]

and

\[ S_6(z) = \alpha_6 W_6''(z) + \alpha_4 (W_2' W_4')(z) + \alpha_5 W_2''(z) + \alpha_3 W_2''(z) + \alpha_2 (W_2 W_2)(z) + \alpha_1 (W_2 W_4)(z) \]

Notice that $W_4(z)$ and $W_6(z)$ are not a primary field under the stress-energy tensor $T(z)$. Therefore we can give the primary fields $U_{2k}(z)$ in the following normalizations

\[ U_4(z) = \sqrt{\frac{(21+4c)(22+5c)}{2(19+6c)(161+8c)}} W_4(z) \equiv U(z) \]

\[ U_6(z) = \sqrt{\frac{3(24+c)(-1+2c)(21+4c)(68+7c)}{5(-1+c)(19+6c)(161+8c)(403+22c)}} W_6(z) \equiv R(z) \]

respectively. Thus, above normalized fields leads to new decompositions for $S_4(z)$ and $S_6(z)$. One obtains

\[ S_4(z) = a_4 U(z) + a_2 (TT)(z) + a_3 T''(z) \]

and

\[ S_6(z) = a_6 U''(z) + a_4 (T''T)(z) + a_5 (T'T')(z) + a_3 (TT')(z) + a_2 (TTT)(z) + a_1 (TTU)(z) \]

where all $a_i$'s are given by

\[ a_4^2 = \frac{2 E_4 E_2}{E_4 E_4}, a_2 = \frac{37}{E_4}, a_3 = \frac{3 E_3}{2 E_4}, a_4 = \frac{E_4}{12 E_4}, a_5 = \frac{25 E_4 E_4 E_4^2}{64 E_4 E_4 E_4}, a_6 = \frac{5}{2 E_{10} E_{11} E_4}, a_7 = \frac{5}{2 E_{10} E_{11} E_4} \]

\[ a_8 = \frac{5}{12} E_{10} E_{11} E_4, a_9 = \frac{5 E_{14}}{E_{10} E_{11} E_4}, a_{10} = \frac{5 E_{14}}{3 E_4 E_{10} E_{11} E_4}, a_{11}^2 = \frac{1250}{9} E_3 E_4 E_4 \]

where we set

\[ E_1 = 19 + 6c, E_2 = 161 + 8c, E_3 = 21 + 4c, E_4 = 22 + 5c, E_5 = -3 + c, E_6 = -49 + 4c, E_7 = 14 + c \]

\[ E_8 = 24 + c, E_9 = -793 + 103c + 62c^2, E_{10} = -1 + 2c, E_{11} = 68 + 7c, E_{12} = 270 + 457c + 52c^2 \]

\[ E_{13} = -27 - 246c + 5c^2 + 2c^3, E_{14} = 41 + 270c, E_{15} = -1 + c \]
3. \(WB_3\) OPEs With Shorthand Notation

In this section we present the results of our explicit calculations of \(WB_3\) OPEs. In all these OPEs, the composite fields are included only up to conformal spin 10, the higher order composite fields neglected due to their formal complexity. In view of this shorthand notation, we verify the OPEs of the complete Casimir \(WB_3\) algebra

\[
\begin{align*}
Q \times Q &= \frac{2c}{7} I + C_{QQ}^u U + C_{QQ}^r R \\
U \times Q &= C_{uQ}^u Q \\
U \times U &= \frac{c}{4} I + C_{uu}^u U + C_{uu}^r R \\
R \times Q &= C_{RQ}^u Q + \tilde{C}_{RQ}^{(UQ)} (U Q) \\
U \times R &= C_{uR}^u U + C_{uR}^r R + \tilde{C}_{(U)R}^{(UU)} (U U) + \tilde{C}_{RR}^{(UR)} (U R)
\end{align*}
\]

Here we write down the expressions for the coefficients in the above OPE’s for \(WB_3\) algebras.

\[
\begin{align*}
(C_{QQ}^u)^2 &= a_1^2, \quad (C_{QQ}^r)^2 = a_2^2, \quad (C_{uu}^u)^2 = \frac{49 E_1 E_2}{32 E_3 E_4}, \quad (C_{uu}^r)^2 = \frac{2 E_{17}^2}{E_3 E_2 E_3 E_4} \\
(C_{uR}^r)^2 &= \frac{80 E_1^2 E_{12}^2 E_15 E_{16}}{3 E_1 E_2 E_3 E_8 E_{10} E_{11}}, \quad (C_{uR}^u)^2 = \frac{245 E_1 E_2 E_{12} E_{16}}{432 E_3 E_8 E_{10} E_{11}}, \quad (\tilde{C}_{RQ}^{(UQ)})^2 = \frac{245 E_{19} E_{21} E_{16}}{6 E_{18} E_4 E_8 E_{15}} \\
(C_{uR}^r)^2 &= \frac{320 E_{10}^2 E_{13}^2 E_{15}^2}{27 E_1 E_2 E_3 E_8 E_{10} E_{11}}, \quad (C_{uR}^u)^2 = \frac{E_{10}^2 E_{13}^2 E_{16}^2}{18 E_1 E_2 E_3 E_4 E_8 E_{15}^2}, \quad (\tilde{C}_{RR}^{(UR)})^2 = \frac{400 E_{13}^2 E_{16}}{3 E_4 E_8 E_{15} E_{16}} \\
(C_{R}^r)^2 &= \frac{2 E_{10}^2 E_{13}^2 E_{15}^2}{81 E_1 E_2 E_3 E_4 E_8^2}, \quad (C_{RR}^r)^2 = \frac{2 E_{20}^2}{120 E_2 E_3 E_{10} E_{11} E_8 E_{15} E_{16}^2}, \quad (\tilde{C}_{RR}^{(UR)})^2 = \frac{120 E_2 E_3 E_{10} E_{11} E_8 E_{15} E_{16}^2}{E_4 E_8 E_{15} E_{16}}
\end{align*}
\]

where we set

\[
\begin{align*}
E_{16} &= 403 + 22 c, \quad E_{17} = 4214 + 627 c + 34 c^2, \quad E_{18} = 49 + c, \quad E_{19} = 812 + 13 c, \quad E_{20} = -259341632 \\
&- 36692548 c + 37936207 c^2 + 3693737 c^3 + 97882 c^4 + 104 c^5, \quad E_{21} = 522592 + 25091 c + 442 c^2 \\
E_{22} &= 14 + 11 c
\end{align*}
\]

4. (OPEs) for Chiral Vertex Operators

We define two kinds of vertex operators which corresponds to the long and short roots of \(B_N\)

\[
V^L_\beta(z) = e^{i \beta \cdot \phi(z)} \quad : (4.1)
\]

and

\[
V^S_\beta(z) = b(z) e^{i \beta \cdot \phi(z)} \quad : (4.2)
\]

Here a non-simple root \(\beta\) including the long and short roots of \(B_N\), is given by

\[
\beta = \sum_{i=1}^{N} m_i \alpha_i \quad : (4.3)
\]
and The Fubini-Veneziano field \( \varphi(z) \), which has conformal spin-0
\[
\varphi(z) = q - ip \ln z + i \sum_{n \neq 0} \frac{1}{n} a_n z^{-n}
\] (4.4)

By using conformal spin-0 contraction \( \varphi(z) \varphi(w) = -\ln |z - w| \), The standard OPEs are of two forms:
\[
\mathcal{V}_\beta^l(z) \mathcal{V}_\beta^l(w) = (z - w)^{\beta \dot{\beta}} : \mathcal{V}_\beta^l(z) \mathcal{V}_\beta^l(w) : 
\] (4.5)
and
\[
\mathcal{V}_\beta^s(z) \mathcal{V}_\beta^s(w) = \frac{1}{(z - w)^2} \mathcal{V}_\beta^l(z) \mathcal{V}_\beta^l(w) 
\] (4.6)

We can tell that the operators \( \mathcal{V}_\beta^l(z) \) and \( \mathcal{V}_\beta^s(z) \) carrie a root \( \beta \). From
\[
h_j(z) \mathcal{V}_\beta^l(w) = \frac{\theta_j}{z - w} \mathcal{V}_\beta^l(w) + \cdots
\] (4.5)
and
\[
h_j(z) \mathcal{V}_\beta^s(w) = \frac{\theta_j}{z - w} \mathcal{V}_\beta^s(w) + \cdots
\] (4.6)
where
\[
\theta_j = \theta_j(\beta) \equiv (\beta, \mu_j)
\] (4.6)

The OPEs with the stress-energy tensor \( T(z) \) is
\[
T(z) \mathcal{V}_\beta^l(w) = \frac{h^l(\beta)}{(z - w)^2} \mathcal{V}_\beta^l(w) + \frac{(\eta^l \mathcal{V}_\beta^l)(w)}{z - w} + \cdots 
\] (4.7)
where \((\eta^l \mathcal{V}_\beta^l)(z) \) *, is given by
\[
(\eta^l \mathcal{V}_\beta^l)(z) = \sum_i \theta_i (h_i \mathcal{V}_\beta^l)(z)
\] (4.8)
and similarly, for the short root
\[
T(z) \mathcal{V}_\beta^s(w) = \frac{h^s(\beta)}{(z - w)^2} \mathcal{V}_\beta^s(w) + \frac{(\eta^s \mathcal{V}_\beta^s)(w)}{z - w} + \cdots 
\] (4.9)
where \((\eta^s \mathcal{V}_\beta^s)(z) \) *, is given by
\[
(\eta^s \mathcal{V}_\beta^s)(z) = \sum_i \theta_i (h_i \mathcal{V}_\beta^s)(z) + (\partial h^l \mathcal{V}_\beta^l)(z)
\] (4.10)

Thus the vertex operators \( \mathcal{V}_\beta^l(z) \) and \( \mathcal{V}_\beta^s(z) \) are a conformal field of spin \( h^l(\beta) \) and \( h^s(\beta) \) which are algebraic in \( \beta \), are given by respectively
\[
h^l(\beta) = \frac{1}{2} \sum_i \{ \theta_i^2 + 2 \alpha_0 (N - i + \frac{1}{2}) \theta_i \}
\] (4.11)
and
\[
h^s(\beta) = h^l(\beta) + \frac{1}{2}
\] (4.12)
* if we take \( \beta = \alpha_i \), a simple root, then \((\eta^l \mathcal{V}_\beta^l)(z) = \partial \mathcal{V}_\beta^l(z) \) and \((\eta^s \mathcal{V}_\beta^s)(z) = \partial \mathcal{V}_\beta^s(z) \).
Another important OPEs between the Fermionic field \( U_{N+\frac{1}{2}}(z) \) and the Vertex Operators have been calculated explicitly, the lower order singler terms are neglected due to their formal complexity. We have

\[
U_{N+\frac{1}{2}}(z) V_\beta^l(w) = \frac{U_{N+\frac{1}{2}}^l(\beta)}{(z-w)^N} V_\beta^s(w) + \cdots \tag{4.13}
\]

and also

\[
U_{N+\frac{1}{2}}(z) V_\beta^s(w) = \frac{U_{N+\frac{1}{2}}^s(\beta)}{(z-w)^{N+1}} V_\beta^l(w) + \cdots \tag{4.14}
\]

where

\[
U_{N+\frac{1}{2}}^l(\beta) = (-1)^N \prod_{i=1}^{N} \left( \theta_i + (N-i+1)\alpha_0 \right) \tag{4.15}
\]

and

\[
U_{N+\frac{1}{2}}^s(\beta) = (-1)^N \prod_{i=1}^{N} \left( \theta_i + (N-i)\alpha_0 \right) \tag{4.16}
\]

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