ON SOME PROBLEMS INVOLVING HARDY’S FUNCTION

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ABSTRACT. Some problems involving the classical Hardy function

\[ Z(t) := \zeta(\frac{1}{2} + it)(\chi(\frac{1}{2} + it))^{-1/2}, \quad \zeta(s) = \chi(s)\zeta(1 - s) \]

are discussed. In particular we discuss the odd moments of \( Z(t) \) and the distribution of its positive and negative values.

1. Definition of Hardy’s function

Let \( Z(t) \) be the classical Hardy function (see e.g., [7]), defined as

\[ Z(t) := \zeta(\frac{1}{2} + it)(\chi(\frac{1}{2} + it))^{-1/2}, \]

where \( \chi(s) \) comes from the familiar functional equation for \( \zeta(s) \) (see e.g., [7, Chapter 1]), namely \( \zeta(s) = \chi(s)\zeta(1 - s) \), so that

\[ \chi(s) = 2^s\pi^{s-1}\sin(\frac{1}{2}\pi s)\Gamma(1 - s), \quad \chi(s)\chi(1 - s) = 1. \]

It follows that

\[ \chi(\frac{1}{2} + it) = \chi(\frac{1}{2} - it) = \chi^{-1}(\frac{1}{2} + it), \]

so that \( Z(t) \in \mathbb{R} \) when \( t \in \mathbb{R} \) and \( |Z(t)| = |\zeta(\frac{1}{2} + it)| \). Thus the zeros of \( \zeta(s) \) on the “critical line” \( \Re s = 1/2 \) correspond to the real zeros of \( Z(t) \), which makes \( Z(t) \) an invaluable tool in the study of the zeros of the zeta-function on the critical line. Alternatively, if we use the symmetric form of the functional equation for \( \zeta(s) \), namely

\[ \pi^{-s/2}\zeta(s)\Gamma(\frac{1}{2}s) = \pi^{-(1-s)/2}\zeta(1-s)\Gamma(\frac{1}{2}(1-s)), \]

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then for \( t \in \mathbb{R} \) we obtain
\[
Z(t) = e^{i\theta(t)} \zeta\left(\frac{1}{2} + it\right), \quad e^{i\theta(t)} := \pi^{-it/2} \frac{\Gamma\left(\frac{1}{4} + \frac{1}{2}it\right)}{|\Gamma\left(\frac{1}{4} + \frac{1}{2}it\right)|}, \quad \theta(t) \in \mathbb{R}.
\]

Hardy’s original application of \( Z(t) \) was to show that \( \zeta(s) \) has infinitely many zeros on the critical line \( \Re s = 1/2 \) (see e.g., E.C. Titchmarsh [23]). The argument is briefly as follows. Suppose on the contrary that, for \( T \geq T_0 \), the function \( Z(t) \) does not change sign. Then
\[
\int_T^{2T} |Z(t)| \, dt = \left| \int_T^{2T} Z(t) \, dt \right|. \tag{1.1}
\]
But it is not difficult to show (see [7, Chapter 9] and (2.4) for an even slightly sharper bound) that
\[
\int_T^{2T} |Z(t)| \, dt = \int_T^{2T} |\zeta\left(\frac{1}{2} + it\right)| \, dt \gg T. \tag{1.2}
\]
On the other hand, to bound the integral on the right-hand side of (1.1) we can use the approximate functional equation (this is weakened form of the so-called Riemann–Siegel formula; for a proof see [7] or [23])
\[
Z(t) = 2 \sum_{n \leq \sqrt{t/(2\pi)}} n^{-1/2} \cos \left( t \log \frac{\sqrt{t/(2\pi)}}{n} - \frac{t}{2} - \frac{\pi}{8} \right) + O\left( \frac{1}{t^{1/4}} \right).
\]
If this expression is integrated and the second derivative test is applied (see [7] or [23]) it follows that
\[
\int_T^{2T} Z(t) \, dt = O(T^{3/4}). \tag{1.3}
\]
Thus from (1.1)–(1.3) we obtain
\[
T \ll \int_T^{2T} |Z(t)| \, dt \ll T^{3/4},
\]
which is a contradiction. This proves that \( \zeta(s) \) has infinitely many zeros on the critical line. Later Hardy refined his argument to show that \( \zeta(s) \) has \( \gg T \) zeros \( \beta + i\gamma \) satisfying \( 0 < \gamma \leq T \). A. Selberg (see [22] or [23] for a proof) improved this bound to \( \gg T \log T \), which is one of the most important results of analytic number theory. His method was later used and refined by many mathematicians. The latest result is by S. Feng [1], who proved that at least 41.73% of the zeros of \( \zeta(s) \) are on the critical line and at least 40.75% of the zeros of \( \zeta(s) \) are simple and on the critical line.
2. The distribution of values of Hardy’s function

One of the aims of this paper is to study the distribution of positive and negative values of \( Z(t) \). To this end let

\[
I_+(T) := \int_{T, Z(t) > 0}^T Z(t) \, dt, \quad I_-(T) := \int_{T, Z(t) < 0}^T Z(t) \, dt
\]

and

\[
J_+(T) := \mu\{T < t \leq 2T : Z(t) > 0\},
J_-(T) := \mu\{T < t \leq 2T : Z(t) < 0\},
\]

where \( \mu(\cdot) \) denotes measure. Note that by the author’s result [10] we have

\[
I_+(T) + I_-(T) = \int_T^{2T} Z(t) \, dt = O_\varepsilon(T^{1/4+\varepsilon}),
\]

which significantly improves on Hardy’s bound \( O(T^{3/4}) \) (cf. (1.3)). Here and later \( \varepsilon \) denotes arbitrarily small constants, not necessarily the same ones at each occurrence, while \( F = O_\varepsilon(G) \) means that the \( O \)-constant depends on \( \varepsilon \). Later M. Jutila [13] and M.A. Korolev [17] independently removed the \( \varepsilon \) in (2.2) and proved that the integral is actually \( \Omega_\pm(T^{1/4}) \). As usual, \( f = \Omega_+(g) \) means that \( \limsup f/g > 0, f = \Omega_-(g) \) means that \( \limsup f/g < 0, \) and \( f = \Omega_\pm(g) \) that both \( f = \Omega_+(g) \) and \( f = \Omega_-(g) \) hold. Thus the problem of the true order of the integral of Hardy’s function is settled, up to the numerical values of the constants that are involved. The primitive of Hardy’s function will be discussed in more detail in the next section.

Returning to the discussion on \( I_\pm(T) \), note that we have

\[
I_+(T) - I_-(T) = \int_T^{2T} Z(t) \, dt - \int_T^{2T} Z(t) \, dt
\]

\[
= \int_T^{2T} |Z(t)| \, dt = \int_T^{2T} |\zeta(\frac{1}{2} + it)| \, dt,
\]

since \( |Z(t)| = |\zeta(\frac{1}{2} + it)| \). But from K. Ramachandra [21] we know that

\[
T(\log T)^{1/4} \ll \int_T^{2T} |\zeta(\frac{1}{2} + it)| \, dt \ll T(\log T)^{1/4},
\]

and from (2.2)–(2.3) it follows that

\[
I_+(T) = \frac{1}{2} \int_T^{2T} |\zeta(\frac{1}{2} + it)| \, dt + O_\varepsilon(T^{1/4+\varepsilon}),
\]
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\[-I_-(T) = \frac{1}{2} \int_T^{2T} |\zeta(\frac{1}{2} + it)| \, dt + O(\varepsilon(T^{1/4 + \varepsilon})],\]

and as mentioned, by [13] or [17] we can dispense with “\(\varepsilon\)” in the above two formulas. In view of (2.4) we obtain then

THEOREM 1. We have

\[
T (\log T)^{1/4} \ll I_+ (T) \ll T(\log T)^{1/4},
\]

\[
T (\log T)^{1/4} \ll -I_- (T) \ll T(\log T)^{1/4}.\]

If one could sharpen (2.4) to an asymptotic formula, then we could sharpen (2.5) and solve the following

Problem 1. Prove that there is a constant \(A > 0\) such that

\[
I_+ (T) = (A + o(1))T(\log T)^{1/4}, \quad -I_- (T) = (A + o(1))T(\log T)^{1/4} \quad (T \to \infty).
\]

3. The primitive of Hardy’s function

Let us define

\[
F(T) := \int_0^T Z(t) \, dt,
\]

so that \(F(t)\) is a primitive function of Hardy’s function \(Z(t)\). In the aforementioned work [13] M. Jutila actually proved the following result.

THEOREM A. Let \(T\) be a large positive number and write \(\sqrt{T/(2\pi)} = L + \theta\) with \(L \in \mathbb{N}\) and \(0 \leq \theta < 1\). Define

\[
\theta_0 = \min \left( |\theta - 1/4|, |\theta - 3/4| \right).
\]

Then for \(\theta_0 \neq 0\) we have

\[
F(T) = \left( \frac{T}{2\pi} \right)^{1/4} (-1)^L K(\theta) + O(T^{1/6} \log T) + O \left( \min(T^{1/4}, T^{1/8}\theta_0^{-3/4}) \right),
\]

where \(K(x) = 0\) for \(0 \leq x < 1/4\) and \(3/4 < x \leq 1\), \(K(x) = 2\pi\) for \(1/4 < x < 3/4\), and further

\[
F(T) = \left( \frac{T}{2\pi} \right)^{1/4} (-1)^L \frac{4\pi}{3} + O(T^{1/6} \log T) \quad \text{for } \theta = 1/4,
\]

\[
F(T) = \left( \frac{T}{2\pi} \right)^{1/4} (-1)^L \frac{2\pi}{3} + O(T^{1/6} \log T) \quad \text{for } \theta = 3/4.
\]
The main contribution to $F(T)$ comes from the expression

$$F_1(T) := \left( \frac{T}{2\pi} \right)^{1/4} (-1)^L K(\theta) = 2\pi \left( \frac{T}{2\pi} \right)^{1/4} (-1)^{\lfloor T/(2\pi) \rfloor}$$

when $1/4 < \theta = \{ \sqrt{T/(2\pi)} \} < 3/4$, and $F_1(T) = 0$ otherwise. In [14] Jutila derived another expression for $F(T)$, different from (3.1), where $F_1(T)$ is replaced by a smoothed expression involving the integral of $K(\theta + t) - K(t)$, plus an error term which is only $O(T^{1/6} \log T)$. On the average $F_1(T)$ is small, as shown by the following

**THEOREM 2.** We have

$$\int_0^T F_1(t) \, dt = O(T^{3/4}),$$

$$\int_0^T F_1(t) \, dt = \Omega_\pm(T^{3/4}).$$

**Proof.** Setting $u(x) = (-1)^{\lfloor x \rfloor} K(\{x\})$ ($\lfloor x \rfloor$ is the integer part of $x$ and $\{x\}$ is its fractional part) we see that this is an odd function satisfying $u(x + 2) = u(x)$, and it follows that we have the Fourier series (for $x - \frac{1}{4} \notin \mathbb{Z}$, $x - \frac{3}{4} \notin \mathbb{Z}$)

$$u(x) = 4 \sum_{n=1}^{\infty} \cos\left( \frac{\pi n}{4} \right) - \cos\left( \frac{3\pi n}{4} \right) \frac{\sin(\pi n x)}{n}.$$

Setting $a(n) := \cos\left( \frac{\pi n}{4} \right) - \cos\left( \frac{3\pi n}{4} \right)$ we see that $a(2k) = 0$ and that $a(n) = \sqrt{2}$ for $n \equiv 1, 7 \pmod{8}$ and $a(n) = -\sqrt{2}$ for $n \equiv 3, 5 \pmod{8}$. This gives

$$u(x) = 4 \sum_{k=1}^{\infty} \frac{a(2k - 1)}{2k - 1} \sin\left( \pi(2k - 1)x \right) \quad (x - \frac{1}{4} \notin \mathbb{Z}, \; x - \frac{3}{4} \notin \mathbb{Z}).$$

Hence with the change of variable $t = 2\pi x^2$ and integration by parts we obtain, since the above series is boundedly convergent and can be integrated termwise,

$$\int_0^T F_1(t) \, dt = \int_0^T \left( \frac{t}{2\pi} \right)^{1/4} u \left( \left( \frac{t}{2\pi} \right)^{1/2} \right) \, dt$$

$$= 16\pi \int_0^{\sqrt{T/2\pi}} x^{3/2} \sum_{k=1}^{\infty} \frac{a(2k - 1)}{2k - 1} \sin(\pi(2k - 1)x) \, dx$$

$$= 16\pi \sum_{k=1}^{\infty} \frac{a(2k - 1)}{2k - 1} \int_0^{\sqrt{T/2\pi}} x^{3/2} \sin(\pi(2k - 1)x) \, dx$$

$$= -16 \left( \frac{T}{2\pi} \right)^{3/4} \sum_{k=1}^{\infty} \frac{a(2k - 1)}{(2k - 1)^2} \cos\left( \pi(2k - 1) \sqrt{\frac{T}{2\pi}} \right) + O(T^{1/4}).$$
Since the last series is absolutely convergent we obtain the $O$-estimate of (3.3). The Omega-results follow if we take $T = 2\pi(\frac{3}{4} + 2m)^2$ and $T = 2\pi(\frac{1}{4} + 2m)^2$ with $m \in \mathbb{N}$. In fact the last $O$-term stands for a function that is $O(T^{1/4})$ and also $\Omega_\pm(T^{1/4})$. This proves (3.3). However the true order of the primitive of $F(T)$ remains elusive, since it is not obvious how small will be, when integrated, the expression standing for the $O$-term in Jutila’s expression for $F(T)$.

**Problem 2.** What is the true order of

$$\int_0^T F(t) \, dt$$

Is it perhaps true that

$$\int_0^T F(t) \, dt = O(T^{3/4}), \quad \int_0^T F(t) \, dt = \Omega_\pm(T^{3/4})?$$

**4. The cubic moment of Hardy’s function**

The analogous problem when $Z(t)$ in $I_\pm(T)$ is replaced by $Z^3(t)$ is much harder. Namely there is an old problem of mine (posed in Oberwolfach, 2003), which is stated here as

**Problem 3.** Does there exist a constant $0 < c < 1$ such that

$$\int_0^T Z^3(t) \, dt = O(T^c)? \quad (4.1)$$

One may naturally ask for bounds for higher moments of $Z(t)$. However, only odd moments are interesting in this context, because of their oscillating nature. Unfortunately when $k > 4$ not much, in general, is known on the moments of $|\zeta(\frac{1}{2} + it)|^k$ (see e.g., [7]). I have not been able to prove (4.1) yet, although I feel that there must be a lot of cancelation between the positive and negative values of $Z(t)$ and I am certain that it must be true. What can be proved is (see [12])

$$\int_0^{2T} Z^3(t) \, dt = 2\pi \sqrt{\frac{2}{3}} \sum_{(\frac{T}{\pi n})^{3/2} \leq n \leq (\frac{T}{\pi})^{3/2}} d_3(n) n^{-\frac{1}{6}} \cos\left(3\pi n^{\frac{2}{3}} + \frac{1}{8}\pi\right) + O(\sqrt{T}^{1/4+\epsilon}), \quad (4.2)$$

where as usual the divisor function $d_3(n)$ (generated by $\zeta^3(s)$) denotes the number of ways $n$ can be written as a product of three factors. The difficulty is in the estimation of the exponential sum on the right-hand side of (4.2). I can show that

$$\sum_{N \leq n \leq 2N} d_3(n) n^{-1/6} \cos\left(3\pi n^{\frac{2}{3}} + \frac{1}{8}\pi\right) \ll_{\epsilon} N^{2/3+\epsilon}. \quad (4.3)$$
However, this just gives the bound $O_\varepsilon(T^{1+\varepsilon})$ for the integral on the left-hand side of (4.2). This is unfortunately weak, since by the Cauchy-Schwarz inequality for integrals we easily obtain a better result, namely

$$\left| \int T Z^3(t) \, dt \right| \leq \left( \int T \left| \zeta\left(\frac{1}{2} + it\right)\right|^2 \, dt \int T \left| \zeta\left(\frac{1}{2} + it\right)\right|^4 \, dt \right)^{1/2} \ll T (\log T)^{5/2} \quad (4.4)$$

on using the well-known elementary bounds (see e.g., [7] or [23])

$$\int T \left| \zeta\left(\frac{1}{2} + it\right)\right|^2 \, dt \ll T \log T, \quad \int T \left| \zeta\left(\frac{1}{2} + it\right)\right|^4 \, dt \ll T \log^4 T.$$

For the cubic moment of $Z(t)$ I know of no better bound than (4.4), and any improvement thereof would be interesting. A nice feature of the exponential sum in (4.3) is that it is “pure” in the sense that the argument of the cosine depends only on $n$, and not on any other quantity. A reasonable conjecture is that

$$\int_0^T Z^3(t) \, dt = O_\varepsilon(T^{3/4+\varepsilon}), \quad \int_0^T Z^3(t) \, dt = \Omega_\pm(T^{3/4}).$$

Actually (4.3) holds if $d_3(n)$ is replaced by $d_k(n)$ (the number of ways $n$ can be written as a product of $k$ factors, so that $d_1(n) \equiv 1$), provided that we have

$$\int_0^T \left| \zeta\left(\frac{1}{2} + it\right)\right|^k \, dt \ll_{\varepsilon,k} T^{1+\varepsilon} \quad (k \in \mathbb{N}). \quad (4.5)$$

In (4.5) it is assumed that $k$ is fixed. If (4.5) holds for any $k \geq k_0$, then this is equivalent to the Lindelöf hypothesis that $\left| \zeta\left(\frac{1}{2} + it\right)\right| \ll_{\varepsilon} |t|^\varepsilon$ (see e.g., E.C. Titchmarsh [23]).

The assertion related to (4.3) is contained in

**THEOREM 3.** If (4.5) holds for some fixed $k \in \mathbb{N}$, then

$$\sum_{N \leq n \leq 2N} d_k(n) n^{-1/6} \cos(3\pi n^2 + \frac{1}{8} \pi) \ll_\varepsilon N^{2/3+\varepsilon}. \quad (4.6)$$

The bound (4.5) is at present known to hold only when $k \leq 4$, in which case we have a non-trivial result. It is unclear whether (4.6), in the general case, has an arithmetic meaning as in the case $k = 3$ (cf. (4.2)).

**Proof of Theorem 3.** From the Perron inversion formula (see e.g., [7, Appendix]) we have

$$\sum_{n \leq x} d_k(n) = \frac{1}{2\pi i} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} \frac{\zeta^k(w)x^w}{w} \, dw + O_\varepsilon(N^\varepsilon) \quad (T \asymp x, x \asymp N). \quad (4.7)$$
We replace the segment of integration by \([\frac{1}{2} - iT, \frac{1}{2} + iT]\), passing over the pole \(w = 1\) of order \(k\) of \(\zeta^k(w)\). The residue is \(xP_{k-1}(\log x)\), with \(P_{k-1}(\log x)\) a polynomial of degree \(k - 1\) in \(\log x\). It follows by the residue theorem that

\[
\Delta_k(x) := \sum_{n \leq x} d_k(n) - xP_{k-1}(\log x) = \frac{1}{2\pi i} \int_{\frac{1}{2} - iT}^{\frac{1}{2} + iT} \frac{\zeta^k(w)x^w}{w}\,dw + R_k(x, T),
\]

say. Here \(\Delta_k(x)\) is the error term in the asymptotic formula for the sumatory function of \(d_k(n)\), and \(R_k(x, T)\) stands for the error term in (4.7) plus the contribution over the segments \([\frac{1}{2} \pm iT, 1 \pm \varepsilon \pm iT]\), so that

\[
R_k(x, T) \ll_{\varepsilon} \int_{\frac{1}{2}}^{1 + \varepsilon} |\zeta(\sigma + iT)|^k x^{\sigma T^{-1}}\,d\sigma + N^\varepsilon.
\]

If we integrate (4.9) over \(T\) from \(T_1\) to \(2T_1\), similarly as was done in [11], we shall find a value of \(T (\geq N)\) such that \(R_k(x, T) \ll_{\varepsilon, k} N^\varepsilon\). It is actually this value of \(T\) that is taken in (4.7). In this process we need the bound, which follows by (4.5) and the convexity of mean values (see [7, Chapter 8]). This is

\[
\int_0^T |\zeta(\sigma + it)|^k\,dt \ll_{\varepsilon, k} T^{1 + \varepsilon} \quad \left(\frac{1}{2} \leq \sigma \leq 1\right).
\]

We write now the sum in (4.6) as

\[
\int_{N}^{2N} x^{-1/6} \cos \left(3\pi x^{2/3} + \frac{1}{8}\pi \right) \,d\left(\sum_{n \leq x} d_k(n)\right)
\]

and use (4.8). The contribution of \(xP_{k-1}(\log x)\), by the first derivative test ([7, Lemma 2.1]), will be \(\ll N^{1/6 + \varepsilon}\). In the portion pertaining to \(\Delta_k(x)\) we integrate by parts the term \(R_k(x, T)\). Since \(R_k(x, T) \ll_{\varepsilon, k} N^\varepsilon\), trivial estimation will yield a contribution which is \(\ll_{\varepsilon} N^{1/2 + \varepsilon}\), and this is probably optimal. After differentiating the remaining expression over \(x\) and writing \(w = \frac{1}{2} + iv\) and the cosine as a sum of exponentials, we are left with a multiple of

\[
\int_{-T}^{T} \zeta^k(\frac{1}{2} + iv) \left\{ \int_{N}^{2N} x^{-2/3} \exp \left(iv \log x \pm 3\pi i x^{2/3}\right) \,dx \right\} \,dv.
\]

The function in the exponential is \(iF_\pm(x)\) with

\[
F_\pm(x) = F_\pm(x, v) := v \log x \pm 3\pi x^{2/3},
\]
so that
\[ F'_\pm(x) = \frac{v}{x} \pm 2\pi x^{-1/3}, \quad F''_\pm(x) = -\frac{v}{x^2} \pm \frac{2}{3} \pi x^{-1/3}. \]

Suppose \( v > 0 \), the other case being analogous. The saddle point \( x_0 \) (root of \( F'_\pm(x) = 0 \) in this case) is \( x_0 = (v/(2\pi))^{3/2} \), and \( x_0 \in [N, 2N] \) for \( v \asymp N^{2/3} \). When this is not satisfied, the contribution is, by the first derivative test, \( \ll \varepsilon N^{1/3+\varepsilon} \).

For \( v \asymp N^{2/3} \) we have
\[ |F''_\pm(x_0)|^{-1/2} \asymp v \asymp N^{2/3}. \]

Hence by the second derivative test and (4.5) we have that this contribution is, with suitable constants \( 0 < C_1 < C_2 \),
\[ \ll \int_{C_1 N^{2/3}}^{C_2 N^{2/3}} |\zeta(1/2 + iv)|^k \, dv \cdot N^{-2/3} N^{2/3} \ll \varepsilon N^{2/3+\varepsilon}. \]

Therefore (4.6) follows, and Theorem 3 is proved. If one wants to have explicitly the main term produced by this method, one can use the saddle point method (see e.g., [7, Chapter 2]).

5. Further problems on the distribution of values

If (4.1) holds with some \( 0 < c < 1 \) then, similarly as in the case of \( I_\pm(T) \), we obtain
\[ \int_{T, Z(t) > 0}^{2T} Z^3(t) \, dt = \frac{1}{2} \int_T^{2T} |\zeta(1/2 + it)|^3 \, dt + O(T^c), \]
\[ \int_{T, Z(t) < 0}^{2T} Z^3(t) \, dt = \frac{1}{2} \int_T^{2T} |\zeta(1/2 + it)|^3 \, dt + O(T^c). \quad (5.1) \]

Note that no asymptotic formula exists yet for the integral on the right-hand side of (5.1). In general it is conjectured that (known to be true only when \( k = 2, 4 \))
\[ \int_0^T |\zeta(1/2 + it)|^k \, dt = (c_k + o(1)) T (\log T)^{k^2/4} \quad (k \in \mathbb{N}, \ T \to \infty), \]

where (see J. P. Keating and N.C. Snaith [16]), with \( d_k(n) \) generated by \( \zeta^k(s) \), the value of \( c_k \) for even \( k \) is
\[ c_k = \frac{a_k g_k}{\Gamma(1 + k^2)}, \quad a_k = \prod_p \left( 1 - \frac{1}{p} \right)^{k^2} \sum_{p} d_k^2(p^j), \quad g_k = (k^2)! \prod_{j=0}^{k-1} \frac{j!}{(j + k)!}, \]

and the product is taken over all primes \( p \). In the case \( k = 3 \) all that is currently known is
\[ T (\log T)^{3/4} \ll \int_0^T |\zeta(1/2 + it)|^3 \, dt \ll T (\log T)^{5/2}. \]
The upper bound follows as in (4.4), and the lower bound is a special case of the
general lower bound (see e.g., K. Ramachandra [21])
\[
\int_0^T |\zeta(\frac{1}{2} + it)|^k \, dt \gg_k T (\log T)^{k^2/4} \quad (k \in \mathbb{N}).
\]

The problems of the asymptotic evaluation of \( I_\pm(T) \) are clearly connected to
the distribution of the values of \( \zeta(\frac{1}{2} + it) \) and \( |\zeta(\frac{1}{2} + it)| \). In [9] I proved, using
results of A. Selberg (see D.A. Hejhal [6]) on the distribution of values of \( \zeta(s) \),
that
\[
\mu(A_c(T)) = \frac{T}{2} + O \left( T (\log \log \log T)^2 \sqrt{\log \log T} \right), \quad (5.2)
\]
where \( c > 0 \) is any constant and
\[
A_c(T) := \left\{ 0 < t \leq T : |\zeta(\frac{1}{2} + it)| \leq c \right\}.
\]

Unfortunately one does not see how to put (5.2) to use in connection with the
distribution of positive values of \( Z(t) \). It is hard to determine asymptotically the
order of \( J_+(T) \), \( J_-(T) \), defined by (2.1). We formulate the following

**Problem 4.** Is it true that there exist constants \( A_+ > 0, A_- > 0 \) such that
\[
J_+(T) := \mu\left\{ T < t \leq 2T : Z(t) > 0 \right\} = (A_+ + o(1))T \quad (T \to \infty), \quad (5.3)
\]
\[
J_-(T) := \mu\left\{ T < t \leq 2T : Z(t) < 0 \right\} = (A_- + o(1))T \quad (T \to \infty)? \quad (5.4)
\]

Obviously \( A_+ + A_- = 1 \) (if \( A_+, A_- \) exist). The asymptotic formula (5.2) gives
rise to the thought that in that case maybe \( A_+ = A_- = 1/2 \). On the other hand,
things may not be that simple. If one assumes the famous Riemann Hypothesis
(that all complex zeros of \( \zeta(s) \) lie on \( \Re s = 1/2 \)) and the simplicity of zeta zeros
(these very strong conjectures seem to be independent in the sense that it is not
known whether either of them implies the other one) then (since \( Z(0) = -1/2 \))
\[
\mu\left\{ T < t \leq 2T : Z(t) > 0 \right\} = \sum_{T < \gamma_{2n} \leq 2T} (\gamma_{2n} - \gamma_{2n-1}) + O(1), \quad (5.5)
\]
where \( 0 < \gamma_1 < \gamma_2 < \ldots \) are the ordinates of complex zeros of \( \zeta(s) \). The sum in
(5.5) is connected to the sum \( (\alpha \geq 0 \text{ is fixed}) \)
\[
\sum_{\alpha} (T) := \sum_{\gamma_n \leq T} (\gamma_n - \gamma_{n-1})^\alpha,
\]
which was investigated in [8]. The sum $\sum_\alpha(T)$ in turn can be connected to the Gaussian Unitary Ensemble hypothesis (see A.M. Odlyzko [19], [20]) and the pair correlation conjecture of H.L. Montgomery [18]. Both of these conjectures assume the Riemann Hypothesis and e.g., the former states that, for

$$0 \leq \alpha < \beta < \infty, \quad \delta_n = \frac{1}{2\pi} (\gamma_{n+1} - \gamma_n) \log \left( \frac{\gamma_n}{2\pi} \right),$$

we have

$$\sum_{\gamma_n \leq T, \delta_n \in [\alpha, \beta]} 1 = \left( \int_\alpha^\beta p(0, u) \, du + o(1) \right) \frac{T}{2\pi} \log \left( \frac{T}{2\pi} \right) \quad (T \to \infty),$$

where $p(0, u)$ is a certain probabilistic density, given by complicated functions defined in terms of prolate spheroidal functions. In fact, in [8] I have proved that, if the RH and the Gaussian Unitary Ensemble hypothesis hold, then for $\alpha \geq 0$ fixed and $T \to \infty$,

$$\sum_\alpha(T) = \left( \int_0^\infty p(0, u) u^\alpha \, du + o(1) \right) \left( \frac{2\pi}{\log(\frac{T}{2\pi})} - 1 \right)^{\alpha-1} T.$$

Also note that, since $\Re \log \zeta(\frac{1}{2} + it) = \log |Z(t)|$, a classical result of A. Selberg (see [22]) gives, for any real $\alpha < \beta$,

$$\lim_{T \to \infty} \frac{1}{T} \mu \left\{ t : t \in [T, 2T], \alpha < \frac{\log |Z(t)|}{\sqrt{\frac{1}{2} \log \log T}} < \beta \right\} = \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta e^{-\frac{1}{2}x^2} \, dx,$$

but here we are interested in the distribution of values of $Z(t)$ and not $|Z(t)|$.

Recently J. Kalpokas and J. Steuding [15] proved, among other things, that for $\phi \in [0, \pi)$,

$$\sum_{0 < t \leq T, \zeta(\frac{1}{2} + it) \in e^{i\phi} \mathbb{R}} \zeta(\frac{1}{2} + it) = \left( 2e^{i\phi} \cos \phi \right) \frac{T}{2\pi} \log \frac{T}{2\pi e} + O_\varepsilon(T^{1/2+\varepsilon}), \quad (5.6)$$

and an analogous result holds for the sums of $|\zeta(\frac{1}{2} + it)|^2$. It is unclear whether (5.6) and the other approaches mentioned above can be put to use in connection with our problems.

Although (5.3) and (5.4) seem difficult to prove, one can at least show that

$$\mathcal{J}_+(T) \gg T(\log T)^{-1/2}, \quad (5.7)$$
and a similar bound for $J_-(T)$. Namely from (2.5) we have, by the Cauchy-Schwarz inequality,

$$T(\log T)^{1/4} \ll I_+(T) \leq \left( \int_{T, Z(t) > 0} 1 \, dt \int_T^{2T} |\zeta(\frac{1}{2} + it)|^2 \, dt \right)^{1/2},$$

which easily gives (5.7), since

$$\int_{T, Z(t) > 0} 1 \, dt = \mu \left\{ T < t \leq 2T : Z(t) > 0 \right\} = J_+(T).$$

6. The distribution of values of $E(T)$

The problem analogous to the evaluation of $I_\pm(T)$ can be considered for $E(T)$ (see [7, Chapter 15]), the error term in the mean square formula for $|\zeta(\frac{1}{2} + it)|^2$. Namely one has (here $C_0 = -\Gamma'(1)$ is Euler’s constant) the defining relation

$$E(T) = \int_0^T |\zeta(\frac{1}{2} + it)|^2 \, dt - T \log \frac{T}{2\pi} - (2C_0 - 1)T.$$

Thus the function $E(T)$ is continuous for $T > 0$, and it is known that $E(T) = O(T^\alpha)$ with $1/4 \leq \alpha < 1/3$ (see [7, Chapter 15]). Moreover its mean value is $\pi$, as J.L. Hafner and I showed in [2]-[3] that if we define

$$G(T) := \int_0^T (E(t) - \pi) \, dt,$$

one can then obtain precise expressions for $G(T)$ which, in particular, yield

$$G(T) = O(T^{3/4}), \quad G(T) = \Omega_\pm(T^{3/4}).$$

Let us also define

$$J_+(T) := \int_{T, E(t) > \pi} (E(t) - \pi) \, dt, \quad J_-(T) := \int_{T, E(t) < \pi} (E(t) - \pi) \, dt.$$

Then, similarly to the discussion on $I_\pm(T)$, we obtain

$$J_+(T) = \frac{1}{2} \int_T^{2T} |E(t) - \pi| \, dt + O(T^{3/4}), \quad -J_-(T) = \frac{1}{2} \int_T^{2T} |E(t) - \pi| \, dt + O(T^{3/4}).$$

(6.1)
Note that D.R. Heath-Brown [4] proved that the moments

\[ D_k := \lim_{X \to \infty} X^{-1-k/4} \int_0^X |E(t)|^k \, dt \] (6.2)

exist for any real \( k \in [0, 9] \), and so do the odd moments of \( E(t) \) for \( k = 1, 3, 5, 7 \) or 9. If \( D_k \) exists for some \( k \geq 1 \), then \( D_k > 0 \). Namely from a result of D.R. Heath-Brown and K.-M. Tsang [5] it follows that

\[ \int_0^X |E(t)| \, dt \gg X^{5/4}, \] (6.3)

so that \( D_1 > 0 \). If \( k > 1 \), then by Hölder’s inequality

\[ \int_0^X |E(t)| \, dt \ll_k \left( \int_0^X |E(t)|^k \, dt \right)^{1/k} X^{1-1/k}, \]

so that (6.3) yields

\[ \int_0^X |E(t)|^k \, dt \gg_k X^{1+k/4}, \]

implying that \( D_k > 0 \) if \( D_k \) exists. Since

\[ \int_T^{2T} |E(t)| \, dt - \pi T \leq \int_T^{2T} |E(t) - \pi| \, dt \leq \int_T^{2T} |E(t)| \, dt + \pi T, \] (6.4)

then using (6.2) (for \( k = 1 \)) once with \( X = T \) and once with \( X = 2T \), we obtain from (6.1) and (6.4)

THEOREM 4. As \( T \to \infty \) we have, for some constant \( C > 0 \),

\[ J_+(T) = (C + o(1))T^{5/4}, \]
\[ -J_-(T) = (C + o(1))T^{5/4}. \]

We conclude by stating two related problems.

**Problem 5.** Does there exist a constant \( B > 0 \) such that

\[ \int_0^T |E(t)| \, dt = BT^{5/4} + O(T)? \]

**Problem 6.** Do there exist a constants \( B_+, B_- > 0 \) such that, for \( T \to \infty \),

\[ \mu \left\{ 0 \leq t \leq T : E(t) > \pi \right\} = (B_+ + o(1))T, \]
\[ \mu \left\{ 0 \leq t \leq T : E(t) < \pi \right\} = (B_- + o(1))T? \]
Obviously $B_+ + B_- = 1$ (if $B_+, B_-$ exist), and maybe $B_+ = B_- = 1/2$. This is analogous to (5.3)–(5.4). Generalizations to the distribution of positive and negative values of other functions of arithmetic interest may be clearly considered as well.

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