Exact formulas of the transition probabilities of the multi-species asymmetric simple exclusion process

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Abstract
We provide the exact formulas for the transition probabilities of the $N$-particle multi-species asymmetric simple exclusion process (ASEP). Our formulas improve the ones obtained by Tracy and Widom in 2013 and we provide an alternate way to prove the initial condition, which is different from the method Tracy and Widom used. Also, we prove the integrability in the sense that the $R$ matrix satisfies the Yang-Baxter equation.

1 Introduction

The exact formulas of the transition probabilities of the $exactly solvable$ models may be a good starting point to study interesting distributions and their asymptotic behaviours [5,9,10,13-15,18].

In the multi-species versions of these models, particles in the system may belong to different classes and there is a hierarchy of these classes. In this paper, we consider the multi-species version of the asymmetric simple exclusion processes. We consider $N$-particle systems which consist of $k = 1, \cdots, N$ species, labelled $1, \cdots, k$. The rules for the multi-species ASEP are as follows: a particle at $x \in \mathbb{Z}$ waits an exponential random time with rate 1 and then chooses $x + 1$ with probability $p$ or $x - 1$ with probability $q = 1 - p$ to jump. If the particle at $x$ belongs to species $l$ and the chosen site to jump is already occupied by another particle belonging to species $l' \geq l$, then the jump is prohibited, but if $l' < l$, the particle belonging to $l$ jumps to the chosen site by interchanging sites with the particle belonging to $l'$. Here, we may view an empty site as a “particle” labelled $0$.

A state of the process is denoted by a pair $(X, \pi)$ where $X = (x_1, \cdots, x_N) \in \mathbb{Z}^N$ with $x_1 < \cdots < x_N$ and $\pi = \pi_1 \pi_2 \cdots \pi_N$ is a permutation of a multi-set $\mathcal{M} = [i_1, \cdots, i_N]$ with elements taken from $\{1, \cdots, k\}$. Each $x_i$ represents the position of the $i$th leftmost particle and $\pi_i$ represents the species the $i$th leftmost particle belongs to.

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The purpose of this paper is to provide a method to find an exact and explicit formula of the probability \( P_{(Y,\nu)}(X,\pi;t) \) that the system is in \((X,\pi)\) at time \(t\), given an initial state \((Y,\nu)\) for each \(M\). If \(M = [i,i,\ldots,i]\), then a permutation of \([i,i,\ldots,i]\) is uniquely \(ii\ldotsi\), so the system is the ASEP, and if \(M = [1,2,\ldots,N]\), then all \(N\) particles in the system belong to all different species. The exact formulas of the transition probabilities of the ASEP, written \( P_{(Y)}(X;t) \), were obtained by Tracy and Widom \([15]\), generalizing Schütz’s formulas \([14]\) for the totally asymmetric simple exclusion process (TASEP). Tracy and Widom’s formulas of the transition probabilities of the ASEP are in the form of the \(N\)-fold contour integral

\[
P_{(Y)}(X;t) = \int_c \cdots \int_c \sum_{\sigma \in S_N} A_{\sigma} \prod_{i=1}^{N} \left( \xi_{\sigma(i)}^{x_{\sigma(i)}-y_{\sigma(i)}-1} e^{\varepsilon(\xi_{i})t} \right) d\xi_1 \cdots d\xi_N,
\]

where \(c\) implies \(\frac{1}{2\pi i} \int_c\) and the contour \(c\) is a circle centered at the origin with sufficiently small radius. The sum in (1) is over all permutations \(\sigma\) in the symmetric group \(S_N\) and

\[
A_{\sigma} = \prod_{(\beta,\alpha)} S_{\beta\alpha}
\]

where

\[
S_{\beta\alpha} = \frac{p + q\xi_\alpha \xi_\beta - \xi_\beta}{p + q\xi_\alpha \xi_\beta - \xi_\alpha}
\]

and the product is over all inversions \((\beta,\alpha)\) with \(\beta > \alpha\) in \(\sigma\). If \(\sigma\) is the identity permutation, written \(Id\), then we define \(A_{\sigma} = 1\), and

\[
\varepsilon(\xi_i) = \frac{p}{\xi_i} + q\xi_i - 1.
\]

Tracy and Widom have shown that there is a formula analogous to (1) for the transition probabilities of the multi-species ASEP but their formula was not given fully explicitly as mentioned by the authors in \([17, p. 458]\). To be more specific, Tracy and Widom’s formulas for the multi-species ASEP are in the form (using our notations) of

\[
P_{(Y,\nu)}(X,\pi;t) = \int_c \cdots \int_c \sum_{\sigma \in S_N} A_{\nu,\pi}^{\sigma} \prod_{i=1}^{N} \left( \xi_{\sigma(i)}^{x_{\sigma(i)}-y_{\sigma(i)}-1} e^{\varepsilon(\xi_{i})t} \right) d\xi_1 \cdots d\xi_N,
\]

where \(A_{\nu,\pi}^{\sigma}\) was not given explicitly except a special case in \([17]\). Recently, for the two-species TASEP, the author gave a formula of \(A_{\nu,\pi}^{\sigma}\) when \(\nu = \pi\) \([11,12]\).

In finding the formula (3) fully explicitly, we obtain two by-products. First, we prove the integrability of the multi-species ASEP in the sense that the “R matrix” satisfies the Yang-Baxter equation by using the coordinate Bethe Ansatz. The R matrix is defined in Definition 2.1. In fact, Alcaraz and Bariev studied the integrability but they claimed the integrability without proof. We quote their statement, “we can verify by a long and straightforward calculation that for arbitrary number of classes \(N\) and ... \([1, p. 660]\). We prove the integrability in more systematic way. The second by-product is a new method of the proof for the initial condition of (3). Since a special case
(ν = π = 11⋯1) of (3) is the formula for the ASEP, our proof will be an alternative to Tracy and Widom’s proof in [16].

Recently, Borodin and Wheeler posted a work on the colored stochastic vertex models a bit earlier before the first version of this paper was posted [3]. In their work, the multi-species ASEP is a degeneration of the colored stochastic vertex models. Also, we introduce the following references for other multi-species models, where their integrability or some probability distributions are studied. [4–8].

The paper is organized as follows. In Section 2 we introduce some notations and consider the two-particle systems which will play a role of building-blocks for \( N \)-particle systems. In Section 3 we consider the three-particle systems. The integrability is proved in Proposition 3.1. In Section 4 we provide our main results for \( N \)-particle systems.

### 2 Preliminary

The approach we take in this paper is similar to that in [4,11,12] because the model in this paper is a generalization of the two-species ASEP. We first introduce some notations and generalize the section 2.1 in [12] to the asymmetric case.

#### 2.1 Notations

The state space of the multi-species ASEP with \( N \) particles is countable, so we may view \( P_{(Y,\nu)}(X,\pi; t) \) as a matrix element of an infinite matrix, denoted \( \mathbf{P}(t) \), which is a member of a probability semi-group \( \{ \mathbf{P}(t) : t \geq 0 \} \). The rows are labelled by \( (X,\pi) \) and the columns are labelled by \( (Y,\nu) \). We assume that if \( \nu < \nu' \) and \( \pi < \pi' \) where \( < \) is the lexicographical order, the row \( (X,\pi') \) is below the row \( (X,\pi) \) and the column \( (Y,\nu') \) is to the right of the column \( (Y,\nu) \) as below, and we also assume that the order of the labels of the rows from the top to the bottom is the same as the order of the labels of the columns from the left to the right.

\[
\vdots \\
(X,\pi) \\
\vdots \\
(X,\pi') \\
\vdots
\]

Let \( \mathbf{P}_Y(X; t) \) be a sub-matrix of \( \mathbf{P}(t) \), obtained by taking the rows labelled \( (X, \cdot) \) and the columns labelled \( (Y, \cdot) \). In this case, we simply omit \( X \) and \( Y \) in the labels so that \( \mathbf{P}_Y(X; t) \) is written
That is, we label the rows and the columns of the \( N^N \times N^N \) matrix, \( \mathbf{P}_Y(X; t) \), with the permutations of the multi-sets of cardinality \( N \) with elements taken from \( \{1, \ldots, N\} \) in the lexicographical order from the top to the bottom and from the left to the right, respectively. Similarly, we label the rows and the columns of any \( m^n \times m^n \) matrices with the permutations of the multi-sets of cardinality \( n \) with elements taken from \( \{1, \ldots, m\} \) in the lexicographical order from the top to the bottom and from the left to the right, respectively. Sometimes, we will omit the labels of the rows and the columns if there is no confusion in the context.

The semigroup \( \{ \mathbf{P}(t) : t \geq 0 \} \) is uniform, and \( \mathbf{P}(t) \) is the unique solution to the forward equation which is an element-wise matrix differential equation

\[
\frac{d}{dt} \mathbf{P}(t) = \mathbf{G} \mathbf{P}(t)
\]

subject to the initial condition \( \mathbf{P}(0) = \mathbf{I}_\infty \) where \( \mathbf{I}_\infty \) is the infinite identity matrix and \( \mathbf{G} \) is the generator. (We use the notation \( \mathbf{I}_n \) for \( n \times n \) identity matrix and \( \mathbf{0}_n \) for \( n \times n \) zero matrix.) The initial condition \( \mathbf{P}(0) = \mathbf{I}_\infty \) implies that the sub-matrices \( \mathbf{P}_Y(X; t) \) satisfy

\[
\mathbf{P}_Y(X; 0) = \begin{cases} 
\mathbf{I}_{N^N} & \text{if } X = Y \\
\mathbf{0}_{N^N} & \text{otherwise.}
\end{cases}
\]

### 2.2 Two-particle systems

In a two-particle system with species 1 or 2, either both particles belong to the same species or particles belong to two different species. Now, we find the formula for

\[
\mathbf{P}_Y(X; t) = \begin{bmatrix}
\begin{array}{c}
\begin{bmatrix}
P_{Y,11}(X, 11; t) & P_{Y,12}(X, 11; t) & P_{Y,21}(X, 11; t) & P_{Y,22}(X, 11; t)
\end{array}
\end{bmatrix} \\
\begin{array}{c}
\begin{bmatrix}
P_{Y,11}(X, 12; t) & P_{Y,12}(X, 12; t) & P_{Y,21}(X, 12; t) & P_{Y,22}(X, 12; t)
\end{array}
\end{bmatrix} \\
\begin{array}{c}
\begin{bmatrix}
P_{Y,11}(X, 21; t) & P_{Y,12}(X, 21; t) & P_{Y,21}(X, 21; t) & P_{Y,22}(X, 21; t)
\end{array}
\end{bmatrix} \\
\begin{array}{c}
\begin{bmatrix}
P_{Y,11}(X, 22; t) & P_{Y,12}(X, 22; t) & P_{Y,21}(X, 22; t) & P_{Y,22}(X, 22; t)
\end{array}
\end{bmatrix}
\end{array}
\end{bmatrix}
\]

The element-wise derivative of \( \mathbf{P}_Y(X; t) = \mathbf{P}_Y(x_1, x_2; t) \), which is a sub-matrix of the left-hand
Suppose that for each \( d \) or side in (4), is written as either

\[
\frac{d}{dt} P_Y(x_1, x_2; t) = p P_Y(x_1 - 1, x_2; t) + p P_Y(x_1, x_2 - 1; t) + q P_Y(x_1 + 1, x_2; t) + q P_Y(x_1, x_2 + 1; t) - 2 P_Y(x_1, x_2; t) \quad \text{for } x_1 < x_2 - 1
\]

or

\[
\frac{d}{dt} P_Y(x_1, x_2; t) = p P_Y(x_1 - 1, x_2; t) + q P_Y(x_1, x_2 + 1; t)
\]

subject to the initial condition (5) with \( N = 2 \). Let \( U(x_1, x_2; t) \) be a 4 × 4 matrix whose entries are functions on \( \{(x_1, x_2) \in \mathbb{Z}^2 : x_1 \leq x_2\} \times [0, \infty) \) which satisfies

\[
p U(x, x; t) + q U(x + 1, x + 1; t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & p & 0 \\ 0 & q & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} U(x, x + 1; t) \quad \text{for all } x \in \mathbb{Z}.
\]

Suppose that for each \( (x_1, x_2) \in \mathbb{Z}^2 \) with \( x_1 < x_2 \), \( U(x_1, x_2; t) \) satisfies

\[
\frac{d}{dt} U(x_1, x_2; t) = p U(x_1 - 1, x_2; t) + p U(x_1, x_2 - 1; t) + q U(x_1 + 1, x_2; t) + q U(x_1, x_2 + 1; t) - 2 U(x_1, x_2; t).
\]

Then, (7) and (10) imply that \( U(x_1, x_2; t) \) satisfies (7) when \( x_1 < x_2 - 1 \) and (8) when \( x_1 = x_2 - 1 \). Moreover, if \( U(x_1, x_2; t) \) satisfies

\[
U(x_1, x_2; 0) = \begin{cases} I_4 & \text{if } (x_1, x_2) = (y_1, y_2); \\ 0_4 & \text{if } (x_1, x_2) \neq (y_1, y_2) \text{ and } x_1 < x_2 \end{cases}
\]

for given \( (y_1, y_2) \) with \( y_1 < y_2 \), then \( U(x_1, x_2; t)|_{(x_1, x_2) : x_1 < x_2} = P_Y(X; t) \) by uniqueness. A solution of (10) by the Bethe Ansatz is, for each \( \xi_1, \xi_2 \in \mathbb{C} \setminus \{0\} \),

\[
(A_{12} \xi_1^{x_1-y_1-1} \xi_2^{x_2-y_2-1} + A_{21} \xi_1^{x_1-y_2-1} \xi_2^{x_2-y_1-1}) e^{\epsilon(\xi_1, \xi_2)t}
\]

where

\[
\epsilon(\xi_1, \xi_2) = \frac{p}{\xi_1} + \frac{p}{\xi_2} + q \xi_1 + q \xi_2 - 2
\]

and, \( A_{12} \) and \( A_{21} \) are 4 × 4 matrices whose entries are independent of \( x_1, x_2 \) and \( t \). If we apply (9) to (12), we obtain

\[
[(p + q \xi_1 \xi_2)I_4 - \xi_2 \tilde{B}] A_{12} = -[(p + q \xi_1 \xi_2)I_4 - \xi_1 \tilde{B}] A_{21}
\]
where
\[
\mathbf{\tilde{B}} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & p & p & 0 \\
0 & q & q & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]
and
\[
A_{21} = -[(p + q\xi_2)\mathbf{I}_4 - \xi_1 \mathbf{\tilde{B}}]^{-1} [(p + q\xi_2)\mathbf{I}_4 - \xi_2 \mathbf{\tilde{B}}] A_{12}
\]
\[
\begin{bmatrix}
-p + q\xi_2 - \xi_2 & 0 & 0 & 0 \\
0 & (p - q\xi_2)(\xi_2 - 1) & p(\xi_2 - \xi_1) & 0 \\
0 & 0 & (p - q\xi_2)(\xi_1 - 1) & 0 \\
0 & 0 & 0 & -p + q\xi_2 - \xi_2
\end{bmatrix}
\]
\[
A_{12}.
\]
For later use, we let
\[
S_{\beta\alpha} = -\frac{p + q\xi_\alpha\xi_\beta - \xi_\beta}{p + q\xi_\alpha\xi_\beta - \xi_\alpha}, \quad P_{\beta\alpha} = \frac{(p - q\xi_\alpha)(\xi_\beta - 1)}{p + q\xi_\alpha\xi_\beta - \xi_\alpha}
\]
\[
T_{\beta\alpha} = \frac{\xi_\beta - \xi_\alpha}{p + q\xi_\alpha\xi_\beta - \xi_\alpha}, \quad Q_{\beta\alpha} = \frac{(p - q\xi_\beta)(\xi_\alpha - 1)}{p + q\xi_\alpha\xi_\beta - \xi_\alpha}
\]
and
\[
\mathbf{\tilde{R}}_{\beta\alpha} = \begin{bmatrix}
S_{\beta\alpha} & 0 & 0 & 0 \\
0 & P_{\beta\alpha} & pT_{\beta\alpha} & 0 \\
0 & qT_{\beta\alpha} & Q_{\beta\alpha} & 0 \\
0 & 0 & 0 & S_{\beta\alpha}
\end{bmatrix}.
\]
A contour integral of (12) with respect to \(\xi_1, \xi_2\) over some contours with (14) still satisfies (9) and (10). We claim that the formula of \(\mathbf{P}_Y(X; t)\) is given by
\[
\mathbf{P}_Y(X; t) = \int_c \int_c \left( \mathbf{I}_4 \xi_1^{x_1 - y_1 - 1} \xi_2^{x_2 - y_2 - 1} + \mathbf{\tilde{R}}_{21} \mathbf{\tilde{R}}_{21} \xi_1^{x_1 - y_1 - 1} \xi_2^{x_2 - y_2 - 1} \right) e^{c(\xi_1, \xi_2)} d\xi_1 d\xi_2
\]
where \(c\) is a circle centered at the origin with some radius (which will be given later). We will omit the proof because we provide a general statement and its proof in Theorem 4.2.

The formula (17) provides the formulas for \(\mathbf{P}_{Y,ij}(X, kl; t)\) for \(i, j, k, l = 1\) or \(2\) but these formulas essentially provide the formulas for \(\mathbf{P}_{Y,ij}(X, kl; t)\) for all \(i, j, k, l = 1, \ldots, N - 1\) or \(N\), which are for the two-particle system with species \(1, \ldots, N - 1\), or \(N\). These are the elements of \(N^2 \times N^2\) matrix, \(\mathbf{P}_Y(X; t)\), and we have an analogous formula to (17) for this \(\mathbf{P}_Y(X; t)\). In the formula of the \(N^2 \times N^2\) matrix \(\mathbf{P}_Y(X; t)\), we call the matrix corresponding to \(\mathbf{\tilde{R}}_{21}\) in (17) the \(\mathbf{R}\) matrix. Each element of \(\mathbf{R}\) describes the interaction of two particles. In fact, the elements of \(\mathbf{R}\) may be immediately read off from the elements of \(\mathbf{\tilde{R}}_{21}\) but we give the following definition for later uses.
**Definition 2.1.** $R_{\beta \alpha}$ is an $N^2 \times N^2$ matrix whose rows and columns are labelled $ij$ and $kl$, respectively, for $i, j, k, l = 1, \ldots, N$ and whose elements are given by

$$
[R_{\beta \alpha}]_{ij,kl} =
\begin{cases}
S_{\beta \alpha} & \text{if } ij = kl \text{ with } i = j; \\
P_{\beta \alpha} & \text{if } ij = kl \text{ with } i < j; \\
Q_{\beta \alpha} & \text{if } ij = kl \text{ with } i > j; \\
pT_{\beta \alpha} & \text{if } ij = lk \text{ with } i < j; \\
qT_{\beta \alpha} & \text{if } ij = lk \text{ with } i > j; \\
0 & \text{for all other cases.}
\end{cases}
$$

(18)

We will often omit the subscripts $\beta, \alpha$ in $R_{\beta \alpha}, S_{\beta \alpha}, P_{\beta \alpha}, Q_{\beta \alpha}, T_{\beta \alpha}$ for notational simplicity if there is no confusion.

It is also possible to obtain $R_{\beta \alpha}$ by the direct method as in (13). In this case,

$$
R_{\beta \alpha} = -[(p + q\xi_\alpha \xi_\beta)I_{N^2} - \xi_\alpha B]^{-1}[(p + q\xi_\alpha \xi_\beta)I_{N^2} - \xi_\beta B]
$$

(19)

where $B$ is an $N^2 \times N^2$ matrix defined by

$$
[B]_{ij,kl} =
\begin{cases}
1 & \text{if } ij = kl \text{ with } i = j; \\
p & \text{if either } ij = kl \text{ or } ij = lk \text{ with } i < j; \\
q & \text{if either } ij = kl \text{ or } ij = lk \text{ with } i > j; \\
0 & \text{for all other cases.}
\end{cases}
$$

(20)

### 3 Integrability

Now, we find the formula of the $N^3 \times N^3$ matrix $P_Y(X; t)$ whose entries are $P_{(Y_{1,1}, Y_{1,2}, Y_{2,2})}(X, \pi_1 \pi_2 \pi_3; t)$ where $\nu_1, \nu_2, \nu_3, \pi_1, \pi_2, \pi_3 = 1, \ldots, N - 1$ or $N$. Let $U(x_1, x_2, x_3; t)$ be an $N^3 \times N^3$ matrix whose entries are functions on $\{(x_1, x_2, x_3) \in \mathbb{Z}^3 : x_1 \leq x_2 \leq x_3\} \times [0, \infty)$ which satisfies

$$
pU(x, x, y; t) + qU(x + 1, x + 1, y; t) = (B \otimes I_N)U(x, x + 1, y; t),
$$

(21)

$$
pU(z, x; t) + qU(z, x + 1, x + 1; t) = (I_N \otimes B)U(z, x, x + 1; t)
$$

(22)

for all possible $x, y, z$. Suppose that for each $(x_1, x_2, x_3) \in \mathbb{Z}^3$ with $x_1 < x_2 < x_3$, $U(x_1, x_2, x_3; t)$ satisfies

$$
\frac{d}{dt}U(x_1, x_2, x_3; t) = pU(x_1 - 1, x_2, x_3; t) + pU(x_1, x_2 - 1, x_3; t) + pU(x_1, x_2, x_3 - 1; t)
$$

$$
+ qU(x_1 + 1, x_2, x_3; t) + qU(x_1, x_2 + 1, x_3; t) + qU(x_1, x_2, x_3 + 1; t)
$$

$$
- 3U(x_1, x_2, x_3; t).
$$

(23)
Moreover, if $U(x_1, x_2, x_3; t)$ satisfies

$$U(x_1, x_2, x_3; 0) = \begin{cases} I_{N^3} & \text{if } (x_1, x_2, x_3) = (y_1, y_2, y_3); \\ 0_{N^3} & \text{if } (x_1, x_2, x_3) \neq (y_1, y_2, y_3) \text{ and } x_1 < x_2 < x_3 \end{cases} \quad (24)$$

for given $Y = (y_1, y_2, y_3)$ with $y_1 < y_2 < y_3$, then the restriction of $U(x_1, x_2, x_3; t)$ on $\{(x_1, x_2, x_3) \in \mathbb{Z}^3 : x_1 < x_2 < x_3\}$ is $P_Y(X; t)$ by uniqueness. The solution of (23) by the Bethe Ansatz is

$$\sum_{\sigma \in S_3} A_{\sigma} e^{x_1 - y_{\sigma(1)} - 1} e^{x_2 - y_{\sigma(2)} - 1} e^{x_3 - y_{\sigma(3)} - 1} = e^{(\xi_1, \xi_2, \xi_3)t} \quad (25)$$

where

$$e(\xi_1, \xi_2, \xi_3) = \frac{p}{\xi_1} + \frac{p}{\xi_2} + \frac{p}{\xi_3} + q\xi_1 + q\xi_2 + q\xi_3 - 3$$

for each $\xi_1, \xi_2, \xi_3 \in \mathbb{C} \setminus \{0\}$. Here, $A_{\sigma}$ is an $N^3 \times N^3$ matrix whose entries are independent of $x_1, x_2, x_3$ and $t$. If we apply (21) to (25), then we obtain

$$A_{213} = -[(p + q\xi_1\xi_2)I_{N^3} - B\xi_1 \otimes I_N]^{-1}[(p + q\xi_1\xi_2)I_{N^3} - B\xi_2 \otimes I_N]A_{123},$$

$$A_{312} = -[(p + q\xi_1\xi_3)I_{N^3} - B\xi_1 \otimes I_N]^{-1}[(p + q\xi_1\xi_3)I_{N^3} - B\xi_3 \otimes I_N]A_{132},$$

$$A_{321} = -[(p + q\xi_2\xi_3)I_{N^3} - B\xi_2 \otimes I_N]^{-1}[(p + q\xi_2\xi_3)I_{N^3} - B\xi_3 \otimes I_N]A_{231},$$

and if we apply (22) to (25), then we obtain

$$A_{132} = -[(p + q\xi_2\xi_3)I_{N^3} - I_N \otimes B\xi_2]^{-1}[(p + q\xi_2\xi_3)I_{N^3} - I_N \otimes B\xi_3]A_{123},$$

$$A_{231} = -[(p + q\xi_1\xi_3)I_{N^3} - I_N \otimes B\xi_1]^{-1}[(p + q\xi_1\xi_3)I_{N^3} - I_N \otimes B\xi_3]A_{213},$$

$$A_{321} = -[(p + q\xi_1\xi_2)I_{N^3} - I_N \otimes B\xi_1]^{-1}[(p + q\xi_1\xi_2)I_{N^3} - I_N \otimes B\xi_2]A_{312}.$$ (26)

Also, we obtain

$$I_N \otimes R_{\beta\alpha} = -[(p + q\xi_\alpha\xi_\beta)I_{N^3} - I_N \otimes B\xi_\alpha]^{-1}[(p + q\xi_\alpha\xi_\beta)I_{N^3} - I_N \otimes B\xi_\beta],$$

$$R_{\beta\alpha} \otimes I_N = -[(p + q\xi_\alpha\xi_\beta)I_{N^3} - B\xi_\alpha \otimes I_N]^{-1}[(p + q\xi_\alpha\xi_\beta)I_{N^3} - B\xi_\beta \otimes I_N]$$

from (19) in the same way as in [12] p.8. Let $T_i$ be a simple transposition in $S_N$, that is, a permutation such that $(T_i\sigma)(i) = \sigma(i+1)$ and $(T_i\sigma)(i+1) = \sigma(i)$, and $(T_i\sigma)(k) = \sigma(k)$ for $k \neq i, i+1$ for any given $\sigma \in S_N$. Then, (26) and (27) are written as

$$A_{T_1\sigma} = (R_{\sigma(2)\sigma(1)} \otimes I_N)A_{\sigma} \quad \text{and}$$

$$A_{T_2\sigma} = (I_N \otimes R_{\sigma(3)\sigma(2)})A_{\sigma},$$

respectively. Hence, we have the following expressions for $A_{\sigma}$:

$$A_{213} = (R_{21} \otimes I_N)A_{123},$$

$$A_{132} = (I_N \otimes R_{32})A_{123},$$

$$A_{312} = (R_{31} \otimes I_N)(I_N \otimes R_{32})A_{123},$$

$$A_{231} = (I_N \otimes R_{31})(R_{21} \otimes I_N)A_{123},$$

$$A_{321} = (R_{32} \otimes I_N)(I_N \otimes R_{31})(R_{21} \otimes I_N)A_{123} \quad \text{or} \quad (I_N \otimes R_{21})(R_{31} \otimes I_N)(I_N \otimes R_{32})A_{123}$$

(30)
Since $A_{321}$ can be expressed in two ways, it is required that these two expressions are consistent. Let us recall that $R$ is an $N^2 \times N^2$ matrix.

**Proposition 3.1.** For arbitrary $N = 3, 4, \cdots$, the matrix $R$ satisfies the following relations.

\[
\begin{align*}
(a) \quad & (R_{\beta \alpha} \otimes I_N)(R_{\alpha \beta} \otimes I_N) = (I_N \otimes R_{\beta \alpha})(I_N \otimes R_{\alpha \beta}) = I_{N^2}, \\
(b) \quad & (R_{\gamma \beta} \otimes I_N)(I_N \otimes R_{\gamma \alpha})(R_{\beta \alpha} \otimes I_N) = (I_N \otimes R_{\gamma \alpha})(R_{\gamma \alpha} \otimes I_N)(I_N \otimes R_{\gamma \beta}).
\end{align*}
\]

**Remark 3.1.** The equation (32) is called the Yang-Baxter equation, so the multi-species ASEP is **integrable** in the sense that the matrix $R$ satisfies the Yang-Baxter equation. The equation (32) was claimed in [1] without proof. In [4], the Yang-Baxter equation for the two-species TASEP with $N = 3$ was verified.

The proof of Proposition 3.1 relies on some results on the sub-matrices of $(R \otimes I_N)$ and $(I_N \otimes R)$. Let $(R \otimes I_N)_{[i, j, k]}$ be a sub-matrix of $(R \otimes I_N)$ which is obtained by taking the rows and columns of $(R \otimes I_N)$ whose labels are the permutations of the multi-set $[i, j, k]$. For example,

\[
(R \otimes I_N)_{[1, 2, 3]} = \begin{bmatrix}
123 & 132 & 213 & 231 & 312 & 321 \\
123 & P & 0 & pT & 0 & 0 & 0 \\
132 & 0 & P & 0 & 0 & pT & 0 \\
213 & qT & 0 & Q & 0 & 0 & 0 \\
231 & 0 & 0 & 0 & P & 0 & pT \\
312 & 0 & qT & 0 & 0 & Q & 0 \\
321 & 0 & 0 & 0 & qT & 0 & Q \\
\end{bmatrix}, \quad (R \otimes I_N)_{[1, 1, 2]} = \begin{bmatrix}
112 & 121 & 211 \\
112 & S & 0 & 0 \\
121 & 0 & P & pT \\
211 & 0 & qT & Q \\
\end{bmatrix}.
\]

Similarly, let $(I_N \otimes R)_{[i, j, k]}$ be a sub-matrix of $(I_N \otimes R)$ which is obtained by taking the rows and columns whose labels are the permutations of the multi-set $[i, j, k]$. For example,

\[
(I_N \otimes R)_{[1, 2, 3]} = \begin{bmatrix}
123 & 132 & 213 & 231 & 312 & 321 \\
123 & P & pT & 0 & 0 & 0 & 0 \\
132 & qT & Q & 0 & 0 & 0 & 0 \\
213 & 0 & 0 & P & pT & 0 & 0 \\
231 & 0 & 0 & qT & Q & 0 & 0 \\
312 & 0 & 0 & 0 & P & pT & 0 \\
321 & 0 & 0 & 0 & 0 & qT & Q \\
\end{bmatrix}, \quad (I_N \otimes R)_{[1, 1, 2]} = \begin{bmatrix}
112 & 121 & 211 \\
112 & P & pT & 0 \\
121 & qT & Q & 0 \\
211 & 0 & 0 & S \\
\end{bmatrix}.
\]

**Lemma 3.2.** For each multi-set $[i, j, k]$ where $i, j, k = 1, \cdots, N$, the following relations hold.
(a) Let $n$ be the number of permutations of $[i, j, k]$.

$$
(R_{\beta\alpha} \otimes I_N)[i,j,k] (R_{\alpha\beta} \otimes I_N)[i,j,k] = (I_N \otimes R_{\beta\alpha})[i,j,k] (I_N \otimes R_{\alpha\beta})[i,j,k] = I_n.
$$

(b) (Yang-Baxter equation for sub-matrices)

$$
(R_{\gamma\beta} \otimes I_N)[i,j,k] (I_N \otimes R_{\gamma\alpha})[i,j,k] (R_{\beta\alpha} \otimes I_N)[i,j,k] = (I_N \otimes R_{\beta\alpha})[i,j,k] (R_{\gamma\alpha} \otimes I_N)[i,j,k] (I_N \otimes R_{\gamma\beta})[i,j,k].
$$

Proof. It suffices to show the statements for $[i, j, k] = [1, 1, 2], [2, 2, 1], \text{ and } [1, 2, 3]$. The statements for these cases can be shown by direct computations of the matrices.

The main idea of the proof of Proposition 3.1 is that if two permutation $ijk$ and $lmn$ are not from the same multi-set, then the $(ijk, lmn)$ elements of the matrices $(R_{\gamma\beta} \otimes I_N)$ and $(I_N \otimes R_{\gamma\alpha})$ are zero, and $(R_{\gamma\beta} \otimes I_N)$ and $(I_N \otimes R_{\gamma\alpha})$ can be made block-diagonal by re-ordering rows and columns. To be more specific, we interchange rows and columns of $(R_{\gamma\beta} \otimes I_N)$ and $(I_N \otimes R_{\gamma\alpha})$ so that all the permutations of a given multi-set are grouped and the order of the labels of the rows is the same as the order of the labels of the columns. Then, the matrix obtained in this manner is block-diagonal. For example, the rows and the columns may be labelled as follows.

Proof of Proposition 3.1. Let $A$ be either $R \otimes I_N$ or $I_N \otimes R$ and let $A'$ be the matrix obtained by re-ordering the rows and columns of $A$ as above. There is a finite sequence of permutation
matrices, $P_1, \ldots, P_n$ such that

$$A' = (P_n \cdots P_1)A(P_1' \cdots P_n')$$

for some $n$.

Then, the order of the labels of the rows of $A'$ is the same as the order of the labels of the columns of $A'$, and $A'$ is block-diagonal. Also, it is obvious that

$$A = (P_1' \cdots P_n')A'(P_n \cdots P_1)$$

by the fact that $P_i^{-1} = P_i^t$.

(a) If $(R_{\beta \alpha} \otimes I_N)(R_{\alpha \beta} \otimes I_N)$ is pre-multiplied by $P_n \cdots P_1$ and post-multiplied by $P_1' \cdots P_n'$, then

$$\begin{align*}
(P_n \cdots P_1)(R_{\beta \alpha} \otimes I_N)(R_{\alpha \beta} \otimes I_N)(P_1' \cdots P_n') \\
= (P_n \cdots P_1)(R_{\beta \alpha} \otimes I_N)(P_1' \cdots P_n')(P_n \cdots P_1)(R_{\alpha \beta} \otimes I_N)(P_1' \cdots P_n') \\
= (R_{\beta \alpha} \otimes I_N)'(R_{\alpha \beta} \otimes I_N)' = I_{N^3}
\end{align*}$$

by Lemma 3.2 (a). Finally,

$$(R_{\beta \alpha} \otimes I_N)(R_{\alpha \beta} \otimes I_N) = (P_1^{-1} \cdots P_n^{-1})I_{N^3}(P_n \cdots P_1) = I_{N^3}.$$ 

Similarly, $(I_N \otimes R_{\beta \alpha})(I_N \otimes R_{\alpha \beta}) = I_{N^3}$.

(b) If $(R_{\gamma \beta} \otimes I_N)(I_N \otimes R_{\gamma \alpha})(R_{\beta \alpha} \otimes I_N)$ is pre-multiplied by $P_n \cdots P_1$ and post-multiplied by $P_1' \cdots P_n'$, then

$$\begin{align*}
(P_n \cdots P_1)(R_{\gamma \beta} \otimes I_N)(I_N \otimes R_{\gamma \alpha})(R_{\beta \alpha} \otimes I_N)(P_1' \cdots P_n') \\
= (R_{\gamma \beta} \otimes I_N)'(I_N \otimes R_{\gamma \alpha})'(R_{\beta \alpha} \otimes I_N)' = (I_N \otimes R_{\gamma \beta})'(I_N \otimes R_{\gamma \alpha})'(I_N \otimes R_{\gamma \beta})'
\end{align*}$$

by Lemma 3.2 (b). Finally,

$$\begin{align*}
(R_{\gamma \beta} \otimes I_N)(I_N \otimes R_{\gamma \alpha})(R_{\beta \alpha} \otimes I_N) \\
= (P_1' \cdots P_n')(I_N \otimes R_{\beta \alpha})'(I_N \otimes R_{\gamma \beta})'(I_N \otimes R_{\gamma \beta})'(P_n \cdots P_1) \\
= (I_N \otimes R_{\beta \alpha})(R_{\gamma \alpha} \otimes I_N)(I_N \otimes R_{\gamma \beta}).
\end{align*}$$

As in $N = 2$ case, the formula for $P_Y(X; t)$ is obtained by the contour integral of $[25]$ with respect to $\xi_1, \xi_2, \xi_3$ over some circles centered at the origin. The proof is included in the proof for the $N$-particle system in the next section.
4 $N$-particle systems

We extend the result for the 3-particle system to the $N$-particle system. We will find the formula for $N^N \times N^N$ matrix $P_Y(X;t)$. Let $U(X;t)$ be an $N^N \times N^N$ matrix whose entries are functions on $\{X \in \mathbb{Z}^N : X = (x_1, \ldots, x_N) \text{ with } x_1 \leq \cdots \leq x_N \} \times [0, \infty)$ which satisfies

$$
p U(x_1, \ldots, x_{i-1}, x_i, x_{i+2}, \ldots, x_N; t) + q U(x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_N; t) = (I_N^{(i-1)} \otimes B \otimes I_N^{(N-i-1)}) U(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_N; t)
$$

for all $i = 1, \ldots, N - 1$. Suppose that for each $(x_1, \ldots, x_N) \in \mathbb{Z}^N$ with $x_1 < \cdots < x_N$, $U(x_1, \ldots, x_N; t)$ satisfies

$$
d U(x_1, \ldots, x_N; t) = \sum_{i=1}^N [p U(x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_N; t) + q U(x_1, \ldots, x_{i-2}, x_i + 1, x_{i+1}, \ldots, x_N; t)]
$$

and $U(x_1, \ldots, x_N; t)$ satisfies

$$
U(x_1, \ldots, x_N; 0) = \begin{cases} I_{N^N} & \text{if } (x_1, \ldots, x_N) = (y_1, \ldots, y_N) \\ 0_{N^N} & \text{if } (x_1, \ldots, x_N) \neq (y_1, \ldots, y_N) \text{ and } x_1 < \cdots < x_N \\ \end{cases}
$$

for given $Y = (y_1, \ldots, y_N)$ with $y_1 < \cdots < y_N$. Then the restriction of $U(x_1, \ldots, x_N; t)$ on $\{(x_1, \ldots, x_N) \in \mathbb{Z}^N : x_1 < \cdots < x_N \}$ is $P_Y(X;t)$ by the same argument as the $N = 3$ case. The solution of (36) by the Bethe Ansatz is

$$
\sum_{\sigma \in S_N} A_{\sigma} \prod_{i=1}^N \left( \xi_{\sigma(i)}^{x_i-y_{\sigma(i)-1}} e^{\epsilon(\xi_i)t} \right)
$$

where

$$
\epsilon(\xi_i) = \frac{p}{\xi_i} + q \xi_i - 1
$$

for $\xi_i \in \mathbb{C} \setminus \{0\}$. Here, $A_{\sigma}$ is an $N^N \times N^N$ matrix whose entries are independent of $x_1, \ldots, x_N$ and $t$, but it needs to satisfy some conditions in order that (37) satisfies (35). Basically, the idea for the formula of $A_{\sigma}$ is the same as in [12, Section 2.3.2]. Here, we slightly modify the case in [12, Section 2.3.2] for the multi-species ASEP. Let us define $N^N \times N^N$ matrices

$$
T_i(\beta, \alpha) := I_N^{(l-1)} \otimes R_{\beta \alpha} \otimes I_N^{(N-l-1)}
$$

for $l = 1, \ldots, N - 1$. Then, we immediately obtain the following results,

$$
T_i(\beta, \alpha)T_j(\delta, \gamma) = T_j(\delta, \gamma)T_i(\beta, \alpha) \quad \text{if } |i-j| \geq 2,
$$

$$
T_i(\gamma, \beta)T_j(\gamma, \alpha)T_i(\beta, \alpha) = T_j(\beta, \alpha)T_i(\gamma, \alpha)T_i(\gamma, \beta) \quad \text{if } |i-j| = 1,
$$

$$
T_i(\beta, \alpha)T_i(\alpha, \beta) = I_{N^N}.
$$
where (40) and (41) are obtained by Proposition 3.1, and (39) is obtained by a property of tensor product, \((A \otimes B)(C \otimes D) = (AC \otimes BD)\). Let \(T_i\) be a simple transposition which interchanges the \(i\)th entry and the \((i+1)st\) entry and leaves the other entries fixed, that is,

\[
T_i \sigma = \sigma'
\]

where \(\sigma = \sigma(1)\ldots\sigma(i)\sigma(i+1)\ldots\sigma(N)\) and \(\sigma' = \sigma(1)\ldots\sigma(i+1)\sigma(i)\ldots\sigma(N)\). Since \(T_1, \ldots, T_{N-1}\) generate \(S_N\), for any given \(\sigma \in S_N\), there is a finite sequence \(a_1, \ldots, a_n\) for some \(n\) where \(a_i \in \{1, 2, \ldots, N-1\}\) such that

\[
\sigma = T_{a_n} \ldots T_{a_1}.
\]

(42)

For this finite sequence \(a_1, \ldots, a_n\), let \(T_{a_k} \ldots T_{a_1} = \sigma(k), (k = 1, \ldots, n)\) and the identity permutation \((12\ldots N) = \sigma(0)\). Let us define

\[
A_{\sigma} := T_{a_n} \ldots T_{a_1} = T_{a_n}(\sigma^{(n-1)}(a_n+1), \sigma^{(n-1)}(a_n)) \ldots T_{a_1}(\sigma^{(0)}(a_1+1), \sigma^{(0)}(a_1)).
\]

(43)

Although the expression (43) is not unique because the expression (42) is not unique, (43) is well defined by (39), (40) and (41). Also, we note that the pairs of numbers

\[
(\sigma^{(0)}(a_1+1), \sigma^{(0)}(a_1)), (\sigma^{(1)}(a_2+1), \sigma^{(1)}(a_2)), \ldots, (\sigma^{(n-1)}(a_n+1), \sigma^{(n-1)}(a_n))
\]

are exactly the inversions of \(\sigma\).

**Lemma 4.1.** Let \(A_{\sigma}\) in (37) be given as (43). Then, (37) satisfies (35) for each \(i\).

**Proof.** Substituting (37) into (35) for \(i\),

\[
\sum_{\sigma \in S_N} \left( p A_{\sigma} \xi_1^{x_1} \ldots \xi_{i-1}^{x_{i-1}} \xi_i^{x_i} \xi_{i+1}^{x_{i+1}} \ldots \xi_N^{x_N} + q A_{\sigma} \xi_1^{x_1} \ldots \xi_{i-1}^{x_{i-1}} \xi_i^{x_i} \xi_{i+1}^{x_{i+1}} \ldots \xi_N^{x_N} \right) = 0_{NN}.
\]

(44)

If \(\sigma'\) is an even permutation, then \(T_i \sigma'\) is an odd permutation, so if we express (44) as a sum over the alternating group \(A_N\),

\[
\sum_{\sigma' \in A_N} \left( \left[ I_{NN} (p + q \xi_{\sigma'(i)}) \xi_{\sigma'(i+1)} \right] - \left[ I_{NN}^{(i-1)} \otimes B \otimes I_{NN}^{(N-i-1)} \right] \xi_{\sigma'(i+1)} \right] A_{\sigma'}
\]

\[
+ \left[ I_{NN} (p + q \xi_{\sigma'(i+1)} \xi_{\sigma'(i)}) - \left( I_{NN}^{(i-1)} \otimes B \otimes I_{NN}^{(N-i-1)} \right) \xi_{\sigma'(i)} \right] A_{T_i \sigma'} \right] \xi_1^{x_1} \ldots \xi_{i-1}^{x_{i-1}} \xi_i^{x_i} \xi_{i+1}^{x_{i+1}} \ldots \xi_N^{x_N} = 0_{NN},
\]

(45)
and a sufficient condition for (45) is that for each \( \sigma \in \mathcal{S}_N \),

\[
\mathbf{A}_{T_i \sigma} = -\left[ \mathbf{I}_{NN} (p + q \xi_{\sigma(i+1)} \xi_{\sigma(i)}) - \left( \mathbf{I}_N^{(i-1)} \otimes \mathbf{B} \otimes \mathbf{I}_N^{(N-i-1)} \right) \xi_{\sigma(i)} \right]^{-1} 
\times \left[ \mathbf{I}_{NN} (p + q \xi_{\sigma(i+1)} \xi_{\sigma(i)}) - \left( \mathbf{I}_N^{(i-1)} \otimes \mathbf{B} \otimes \mathbf{I}_N^{(N-i-1)} \right) \xi_{\sigma(i+1)} \right] \mathbf{A}_\sigma 
\times \left( \mathbf{I}_N^{(i-1)} \otimes \mathbf{R}_{\sigma(i+1) \sigma(i)} \otimes \mathbf{I}_N^{(N-i-1)} \right) \mathbf{A}_\sigma.
\] (46)

Now, it remains to show that if \( \mathbf{A}_\sigma \) is given by (43), then (46) is satisfied. This is immediately obtained because

\[
\mathbf{A}_{T_i \sigma} = \mathbf{T}_i \mathbf{A}_{\sigma} \cdots \mathbf{T}_\alpha \mathbf{A} = \mathbf{T}_i \left( \sigma(i+1), \sigma(i) \right) \mathbf{A}_\sigma 
= \left( \mathbf{I}_N^{(i-1)} \otimes \mathbf{R}_{\sigma(i+1) \sigma(i)} \otimes \mathbf{I}_N^{(N-i-1)} \right) \mathbf{A}_\sigma
\]

by (38).

Finally, if we integrate (37) with respect to \( \xi_i \)'s, it still satisfies (35) and (36), hence we obtain the following result for \( p, q \neq 0 \).

**Theorem 4.2.** Let \( c \) be a circle centered at the origin with radius \( r \) such that

\[
0 < r < \frac{-1 + \sqrt{1 + 4pq}}{2q}
\]

and \( \mathbf{A}_\sigma \) be given as (38). Then,

\[
\mathbf{P}_Y(X; t) = \int_c \cdots \int_c \sum_{\sigma \in \mathcal{S}_N} \mathbf{A}_\sigma \prod_{i=1}^{N} \left( \xi_{\sigma(i)}^{x_i - y_{\sigma(i)}} - 1 \right) e^{\xi_i t} d\xi_1 \cdots d\xi_N.
\] (48)

It remains to prove the initial condition of (48). The proof relies on the following lemmas.

**Lemma 4.3.** For a permutation \( \sigma = \sigma(1) \sigma(2) \cdots \sigma(N) \neq \text{Id} \), let \( k \) be the smallest \( \alpha \) such that \( (\beta, \alpha) \) is an inversion of \( \sigma \), and let \( k = \sigma(n) \). Then, \( n > k \).

**Proof.** Since \( k \) is the smallest \( \alpha \) such that \( (\beta, \alpha) \) is an inversion of \( \sigma \), all the \( (k-1) \) numbers less than \( k \) (if any) are located to the left of \( k \). Also, there is at least one number larger than \( k \) to the left of \( k \). Hence, there are at least \( k \) numbers to the left of \( k \), which implies that \( n > k \). \qed

**Lemma 4.4.** Let \( (\beta, \alpha) \) be an inversion in \( \sigma \in \mathcal{S}_N \) \( (\beta, \alpha) \neq \text{Id} \). For each \( (\beta, \alpha) \), choose one of \( S_{\beta \alpha} \), \( P_{\beta \alpha} \), \( Q_{\beta \alpha} \), \( pT_{\beta \alpha} \), \( qT_{\beta \alpha} \) and 0, and call it \( E_{\beta \alpha} \). If \( c \) is a circle centered at the origin with radius \( r \) in (47), then

\[
\int_c \cdots \int_c \prod_{(\beta, \alpha) \in \Sigma} E_{\beta \alpha} \prod_{i=1}^{N} \xi_{\sigma(i)}^{x_i - y_{\sigma(i)}} \ d\xi_1 \cdots d\xi_N = 0
\]
for any choice of $E_{\beta\alpha}$. (The product is over all inversions $(\beta, \alpha)$ of $\sigma$.)

**Proof.** If there is a zero in the product, the statement is trivial. Suppose that there is no zero in the product. Let $k$ be the smallest $\alpha$ such that $(\beta, \alpha)$ is an inversion of $\sigma$ and $k = \sigma(n)$. Recalling the forms of $E_{\beta\alpha}$ from (15), we see that the variable $\xi_k$ in $E_{\beta k}$ factors in $\prod_{(\beta, \alpha)} E_{\beta\alpha}$ has poles $\frac{p}{1-q_{\xi_k}}$. But if $r$ is given by (47),

$$\left| \frac{p}{1-q_{\xi_k}} \right| \geq \frac{p}{1+qr} > r$$

so the pole is outside the contour $c$. For the variable $\xi_k$ in $\prod_{i} \xi_i^{x_i-y_{\sigma(i)}^{-1}}$, since $x_n > x_k \geq y_k$ by Lemma 4.3, $\xi_k^{x_n-y_k^{-1}}$ has no pole. Hence, the integral with respect to $\xi_k$ is zero, and we obtain the required result. \hfill \square

**Proof of Theorem 4.2.** It suffices to show that for $X = (x_1, \ldots, x_N)$ and $Y = (y_1, \ldots, y_N)$ such that $x_1 < \cdots < x_N$, $y_1 < \cdots < y_N$ and $x_i \geq y_i$ for all $i$

$$\int_c \cdots \int_c \sum_{\sigma \in S_N} A_{\sigma} \prod_{i=1}^{N} \left( \frac{x_i-y_{\sigma(i)}}{x_i-y_{\sigma(i)}^{(i)}} \right) d\xi_1 \cdots d\xi_N = \begin{cases} I_{N} & \text{if } X = Y \\ 0 & \text{if } X \neq Y. \end{cases}$$

Since

$$\int_c \cdots \int_c I_{N} \prod_{i=1}^{N} \left( \frac{x_i-y_{\sigma(i)}}{x_i-y_{\sigma(i)}^{(i)}} \right) d\xi_1 \cdots d\xi_N = \begin{cases} I_{N} & \text{if } X = Y \\ 0 & \text{if } X \neq Y, \end{cases}$$

we need to show

$$\int_c \cdots \int_c \sum_{\sigma \in S_N} A_{\sigma} \prod_{i=1}^{N} \left( \frac{x_i-y_{\sigma(i)}}{x_i-y_{\sigma(i)}^{(i)}} \right) d\xi_1 \cdots d\xi_N = 0_{N}$$

for any $(x_1, \ldots, x_N)$ and $(y_1, \ldots, y_N)$ such that $x_1 < \cdots < x_N$, $y_1 < \cdots < y_N$ and $x_i \geq y_i$. Actually, we will show that

$$\int_c \cdots \int_c A_{\sigma} \prod_{i=1}^{N} \left( \frac{x_i-y_{\sigma(i)}}{x_i-y_{\sigma(i)}^{(i)}} \right) d\xi_1 \cdots d\xi_N = 0_{N}$$

for each $\sigma \neq Id$. Since $A_{\sigma}$ is expressed as in (43), an arbitrary $(\pi, \nu)$ element of $A_{\sigma}$ is expressed as

$$[A_{\sigma}]_{\pi,\nu} = \sum_{\nu_2, \ldots, \nu_n} \left[ T_{a_1} \left( \sigma^{-1}(a_n + 1), \sigma^{(n-1)}(a_n) \right) \right]_{\pi,\nu_n} \times \cdots \times \left[ T_{a_2} \left( \sigma^{(1)}(a_2 + 1), \sigma^{(1)}(a_2) \right) \right]_{\nu_3,\nu_2} \left[ T_{a_1} \left( \sigma^{(0)}(a_1 + 1), \sigma^{(0)}(a_1) \right) \right]_{\nu_2,\nu_1}. \tag{49}$$

Since each element of the matrix

$$T_{a_1} \left( \sigma^{(i-1)}(a_i + 1), \sigma^{(i-1)}(a_i) \right)$$

is one of $S_{\beta\alpha}$, $P_{\beta\alpha}$, $Q_{\beta\alpha}$, $pT_{\beta\alpha}$, $qT_{\beta\alpha}$ and 0 where $\beta = \sigma^{(i-1)}(a_i + 1)$ and $\alpha = \sigma^{(i-1)}(a_i)$, the sum (49) is expressed as the sum of the terms in the form of $\prod_{(\beta, \alpha)} E_{\beta\alpha}$ where $E_{\beta\alpha}$ is one of $S_{\beta\alpha}$, $P_{\beta\alpha}$, $Q_{\beta\alpha}$,
\[ pT_{\beta\alpha}, \ qT_{\beta\alpha} \text{ and 0, and the product is over all inversions } (\beta, \alpha) \text{ of } \sigma. \text{ Hence,} \]

\[
\int_{c} \cdots \int_{c} [A_{x}]_{\pi,\nu} \prod_{i=1}^{N} (\xi_{\sigma(i)}^{x_{i}-y_{\sigma(i)}-1}) \ d\xi_{1} \cdots d\xi_{N} = 0.
\]

by Lemma[14] This completes the proof. \[ \square \]

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