Universality of soft theorem from locality of soft vertex operators

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Abstract

The universal behavior of the soft theorem at the tree level is explained by considering the operator product expansion of the soft and hard vertex operators. We find that the world-sheet integral for the soft vertex is determined only by the regions that are close to the hard vertexes after eliminating total derivative terms. This analysis can be applied to massless particles in various theories such as bosonic closed string, closed superstring and heterotic string. When the hard particle is massive, we find that a mixing with different vertex operators occurs.

1 Introduction

In recent years much progress has been made in understanding the origin of the universality of the soft theorems [5]-[33]. For example, the universal behavior of soft graviton is given by

\[ M_{n+1}(q; p_1, \cdots, p_n) = \left[ S^{(0)}(0) + S^{(1)} + S^{(2)}(2) \right] M_n(p_1, \cdots, p_n), \]  

(1.1)

where

\[ S^{(0)} = \sum_{k=1}^{n} \frac{h_{\mu \nu} p_k^\mu p_k^\nu}{p_k \cdot q}, \]
\[ S^{(1)} = -i \sum_{k=1}^{n} \frac{h_{\mu \nu} p_k^\mu q_k^\alpha J_k^{\alpha}}{p_k \cdot q}, \]
\[ S^{(2)} = -\frac{1}{2} \sum_{k=1}^{n} \frac{h_{\mu \nu} q_k^\alpha J_k^{\alpha}}{p_k \cdot q}, \]
\[ J_k^{\alpha} = L_k^{\alpha} + S_k^{\alpha}, \]
\[ L_k^{\alpha} = i \left( p_k^\mu \frac{\partial}{\partial p_k^\alpha} - p_k^\alpha \frac{\partial}{\partial p_k^\mu} \right). \]  

(1.2)

Here \( J_k^{\alpha} \), \( L_k^{\alpha} \) and \( S_k^{\alpha} \) are the total, orbital and spin angular momenta of the k-th particle respectively. From the viewpoint of field theory, the soft theorems are beautifully derived by using the Ward identity [9], although it is not clear why the total angular momentum comes out from the Feynman diagrams when a soft particle is added (Fig.1).

The soft theorems can also be understood in string theory [11]-[28]. The scattering amplitude is expressed by the insertions of the vertex operators on the world-sheet that correspond to in and out states. The soft theorems are obtained by considering the operator product expansion(OPE) of the soft vertex operator with the hard ones. In this paper we give a simple explanation for the universality of the soft theorems in string theory. For a while we focus on the tree amplitudes of bosonic string.

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The soft massless vertex operator in bosonic closed string is given by
\[ V_{\text{soft}}(z, \bar{z}) = h_{\mu\nu} \frac{\partial X^\mu(z)}{p \cdot q} \frac{\partial X^\nu(\bar{z})}{\bar{p} \cdot \bar{q}} \exp(iq \cdot X(z, \bar{z})), \quad (1.3) \]
where \( q \) is the momentum of the soft particle. As we will see in Section 3, by dropping the surface terms at the infinity, eq. (1.3) can be replaced by
\[
\begin{cases}
V_s \equiv \frac{1}{2} h_{\mu\nu} X^\mu(z, \bar{z}) X^\nu(w, \bar{w}) \partial z \partial \bar{z} \exp(iq \cdot X(z, \bar{z})) & \text{for symmetric } h_{\mu\nu} \\
V_a \equiv -h_{\mu\nu} X^\mu(z, \bar{z}) \partial X^\nu(\bar{z}) \partial z \exp(iq \cdot X(z, \bar{z})) & \text{for antisymmetric } h_{\mu\nu} .
\end{cases}
\]
(1.4)

For the soft dilaton or graviton, where \( h_{\mu\nu} \) is symmetric, this operator is superlocal through the linear order in \( q \), while for the B field, where \( h_{\mu\nu} \) is antisymmetric, through the 0-th order. A superlocal operator is highly local in the sense that it takes a nonzero value only when its position coincides with the other operators’ positions (see Section 3). In this paper we will show that the universality of the soft theorems is a direct consequence of this superlocality. We can apply the same analysis for superstring and heterotic string theory.

The structure of this paper is as follows. In Section 2 we review the calculation of the scattering amplitudes in string theory and see how the leading soft theorem arises from the OPE. In Section 3 we introduce the concept of the superlocal operator and see that the soft graviton vertex operator is superlocal through the linear order in \( q \). In section 4 we give a unified explanation for the universality of the soft theorems for graviton, dilaton and B field. In Section 5 we apply this idea for superstring and heterotic string theory. The details of the calculations are given in the Appendixes.

2 Soft graviton/dilaton theorems from OPE

In this section we review the calculation of the scattering amplitudes in string theory and explain how to derive the leading soft graviton or dilaton theorem by using the OPE.

The tree level amplitudes are represented as the insertions of the vertex operators on a complex plane.
\[
M_{N+1}(p_1, \ldots, p_N, q) \sim \int d^2 z \int d^2 w_i (V_{\text{soft}}(q, z) \prod_{i=1}^N V_i(p_i, w_i)).
\]
(2.1)

Here we use the following normalization:
\[
X^\mu(z, \bar{z}) X^\nu(w, \bar{w}) \sim -\frac{\alpha'}{2} \eta^\mu^\nu \ln |z - w|^2.
\]
(2.2)

For convenience we define a disk \( D_i \) of radius \( \epsilon \) around each vertex \( w_i \), and denote the rest bulk region by \( B \), \( B = \C - \bigcup_{i=1}^n D_i \). We evaluate the integration over each of these regions.

First we calculate the contribution from the disk \( D_i \) by using the OPE
\[
V_{\text{soft}}(q, z) : V_i(p, w) := \cdots + |z - w|^{\alpha' p - q - 4} \times \cdots + |z - w|^{\alpha' p - q} \left[ \left( -\frac{i\alpha'}{2} \right)^2 p^\mu p^\nu \right] + O(q) : V_i(p, w) :
\]
(2.3)
We then perform the z integration
\[ \int_{|z|<\epsilon} d^2z |z-w|^{\alpha' p+q+m-2} = \frac{2\pi}{\alpha' p \cdot q + m} e^{\alpha' p+q}. \]
(2.4)

If we pick up the most singular terms \( (m=0) \) for \( q \), the leading soft graviton theorem is reproduced:
\[ \int d^2z : V_{\text{soft}}(q,z) :: V_i(p,w) := -\frac{\pi\alpha' p \cdot q}{2} : V_i(p,w) + O(1). \]
(2.5)

We also evaluate the contributions from the bulk B by some partial integration. The subleading and subsubleading soft graviton or dilaton theorems are reproduced from the disks \( D_i \) and bulk B, if we ignore the higher order corrections in \( \alpha' \) and mixing with different vertex operators. The detail of the calculation for massless hard particles is given in Appendix A. The structure is the same as in field theory. The singular terms in \( q \) arise from diagrams with a soft particle attached to the external lines, while the regular terms arise from the the internal lines (See Figure 2).

In the above calculation there are three problems. First, we cannot control the higher order corrections in \( \alpha' \) because \( \epsilon \) dependence remains in the expansion of \( e^{\alpha' p \cdot q} \). We will calculate these corrections by a better method in the next section. Second, we cannot determine through which order the universality of the soft theorems holds. This also becomes clear in the next section. Third, it seems that the hard particles are mixed to other levels. For example, we assume that the hard particle is massless, \( V_i = h_{\mu\nu} \partial X^\mu \partial X^\nu e^{ip \cdot X} \). Then from the contractions between the exponential functions in the soft and hard part, we would have the following term:
\[ \int d^2z |z-w|^{\alpha' p+q+m} : \partial X^\mu(w) \partial X^\nu(w) \partial X^\sigma(\bar{w}) \partial X^\rho(\bar{w}) \exp(ip \cdot X(w, \bar{w}))(w) : \sim \frac{2\pi}{\alpha' p \cdot q + 2} e^{\alpha' p \cdot q} \partial X^\mu(w) \partial X^\nu(\bar{w}) \partial X^\sigma(w) \partial X^\rho(\bar{w}) \exp(ip \cdot X(w, \bar{w})). \]
(2.6)

We will show that such term does not appear in the next section. It should be canceled if we correctly evaluate the integration over the bulk region B.

### 3 Superlocal operator and universality of soft theorems

We introduce the concept of superlocal operator in this section. A superlocal operator is highly local in the sense that it takes a nonzero value only when its position coincides with the other operators. More precisely, an operator \( O(z) \) is said to be
superlocal when the following equation holds for any operators \( \phi_i(w_i) \):

\[
\int d^2z(\partial \partial^{\nu} \exp(\partial X(z, \bar{z})) \exp(\partial X(z, \bar{z})) = \sum_{a} \sum_{i=1}^{\infty} c_{ai}(O^a_i(w_i) \phi_1(w_1) \cdots \phi_n(w_n)),
\]

(3.1)

where \( c_{ai} \) is a constant, \( O^a_i \) are local operators and \( \phi_i(w_i) \) means that i-th operator is removed. The simplest example of superlocal operator is \( \partial \partial^{\nu} \exp(\partial X(z, \bar{z})) \).

Because there is no contribution from B for any radius \( \epsilon \), we can simply expand \( \exp(\partial X(z, \bar{z})) \) because of the singularity in \( z - w \). We can expand \( \partial \partial^{\nu} \exp(\partial X(z, \bar{z})) \) before expanding in q:

\[
\int \frac{1}{2} : X^\nu(z, \bar{z}) \partial_{\nu} \partial_{\bar{\nu}} \exp(iq \cdot X(z, \bar{z})) : \cdots.
\]

(3.4)

As in the previous section, we divide the complex plane into the disks \( D_1 \) and the bulk region B.

In the bulk B, because there is no singularity in \( z - w \), we can simply expand the soft exponential \( e^{iq \cdot X} \). By expanding in \( q \), we find that the soft vertex operator is 0 through the linear order in q on B:

\[
\partial \partial^{\nu} \exp(\partial X(z)) = \partial \partial^{\nu} (1 + iq \cdot X(z) + O(q^2)) = O(q^2).
\]

(3.5)

Because there is no contribution from B for any radius \( \epsilon \), we can conclude that the soft vertex operator is superlocal through the linear order in q.

In the disk \( D_1 \), however, we cannot simply expand \( e^{iq \cdot X} \) because of the singularity in \( z - w \). We should consider the contraction of the exponential functions \( e^{iq \cdot X} \) and \( e^{ip \cdot X} \) before expanding in q:

\[
\int \frac{1}{2} : X^\nu(z, \bar{z}) \partial_{\nu} \partial_{\bar{\nu}} \exp(iq \cdot X(z, \bar{z})) : \partial \partial^{\nu} \exp(ip \cdot X(w, \bar{w})) \}
\]

(3.6)

where operators at the same position, \( z \) or \( w \), must not be contracted. Then if we expand \( e^{iq \cdot X} \) with respect to \( q \), we can get the leading, subleading, subsubleading soft theorem. The details are given in the following sections.

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**Figure 3:** Contributions from \( \frac{1}{2} : X^\nu(z, \bar{z}) \partial_{\nu} \partial_{\bar{\nu}} \exp(iq \cdot X(z, \bar{z})) \)
4 Soft graviton, dilaton and B field theorem

4.1 Soft graviton/dilaton theorem

4.1.1 General formula

In this section we consider a hard vertex operator of the form, \( V_{\text{hard}} = \partial^{\alpha_1} X \cdots \partial^{\alpha_k} X \cdots \text{exp}(ip \cdot X) \), and we refer to the factors before the exponential function, \( \partial^{\alpha_1} X \cdots \partial^{\alpha_k} X \cdots \), as prefactors. Then we can evaluate the OPE of the soft vertex operator \( V_s \) (eq. (4.1)) and \( V_{\text{hard}} \) as follows.

First in order to simplify the contractions we rewrite the soft and hard vertex operators as

\[
V_s = \frac{\hbar_{\mu \nu}}{2} X^\nu(z, \bar{z}) X^\mu(z, \bar{z}) \partial_\mu \partial_\nu \exp(iq \cdot X(z, \bar{z})) = \frac{\hbar_{\mu \nu}}{2} \lim_{\xi \to 0} \frac{\partial_\nu}{\partial_\xi} \partial_\nu \partial_\xi \exp(iq \cdot X(z, \bar{z}) + i\xi \cdot X(z', \bar{z}')) .
\]

\[
V_{\text{hard}} = A_{\alpha_1 \cdots \alpha_k} \partial^{\alpha_1} X \cdots \partial^{\alpha_k} X \cdots \text{exp}(ip \cdot X(w))
\]

\[
= A_{\alpha_1 \cdots \alpha_k} \left( ip \cdot X(w) + \sum_i i\zeta_i \cdot \partial^{\alpha_i} X + \sum_i i\lambda_i \cdot \bar{\partial}^{\alpha_i} X(\bar{w}) \right)_{\text{multilinear in } i\zeta_i, i\lambda_i}.
\]

Here we keep only the multilinear terms in \( i\zeta_i \) and \( i\lambda_i \), and perform the replacement, \( i\zeta_{\mu_1} \cdots i\lambda_{\sigma_1} \cdots \to A_{\mu_1 \cdots \sigma_1} \), at the end.

The OPE of these operators is given by

\[
:V_s(z) :: V_{\text{hard}}(w) :
\]

\[
= \lim_{\xi \to 0} \frac{\partial_\nu}{\partial_\xi} \partial_\nu \lim_{z' \to z} \frac{1}{2} \hbar_{\mu \nu} \partial_\mu \partial_\nu \exp(iq \cdot X(z) + i\xi \cdot X(z')) \exp \left( \left( ip \cdot X(w) + i \sum_i \zeta_i \cdot \partial^{\alpha_i} X(w) + i \sum_i \lambda_i \cdot \bar{\partial}^{\alpha_i} X(\bar{w}) \right) \right)_{\text{multilinear in } i\zeta_i, i\lambda_i}
\]

\[
= - \lim_{\xi \to 0} \frac{\partial_\mu}{\partial_\xi} \partial_\mu \exp(\alpha' \cdot q + \xi) :|z - w|^{\alpha' \cdot p + q |q + \xi|}
\]

\[
\times \exp \left( - \frac{\alpha'}{2} \sum_i (m_i - 1)! (q_i \cdot \zeta_i) \frac{1}{(z - w)^{m_i}} - \frac{\alpha'}{2} \sum_i (n_i - 1)! (q_i \cdot \lambda_i) \frac{1}{(z - w)^{m_i}} \right)
\]

\[
\times \exp \left( - \frac{\alpha'}{2} \sum_i (m_i - 1)! (q + \xi) \cdot \zeta_i \frac{1}{(z - w)^{m_i}} - \frac{\alpha'}{2} \sum_i (n_i - 1)! (q + \xi) \cdot \lambda_i \frac{1}{(z - w)^{m_i}} \right)
\]

\[
\times \exp \left( i(q + \xi) \cdot X(z) + ip \cdot X(w) + i \sum_i \zeta_i \cdot \partial^{\alpha_i} X(w) + i \sum_i \lambda_i \cdot \bar{\partial}^{\alpha_i} X(\bar{w}) \right)_{\text{multilinear in } i\zeta_i, i\lambda_i}
\]

\[
= \lim_{\xi \to 0} \frac{\partial_\mu}{\partial_\xi} \frac{1}{2} \hbar_{\mu \nu} :|z - w|^{\alpha' \cdot p + q |q + \xi|}
\]

\[
\times \exp \left( - \frac{\alpha'}{2} \sum_i (m_i - 1)! (q + \xi) \cdot \zeta_i \frac{1}{(z - w)^{m_i}} - \frac{\alpha'}{2} \sum_i (n_i - 1)! (q + \xi) \cdot \lambda_i \frac{1}{(z - w)^{m_i}} \right)
\]

\[
\times \exp \left( iq \cdot X(z) + ip \cdot X(w) + i \sum_i \zeta_i \cdot \partial^{\alpha_i} X(w) + i \sum_i \lambda_i \cdot \bar{\partial}^{\alpha_i} X(\bar{w}) \right)_{\text{multilinear in } i\zeta_i, i\lambda_i}
\]

\[
\times \left( \frac{\alpha' \cdot q}{2(z - w)} + \sum_i \frac{\alpha' \cdot m_i q \cdot \zeta_i}{2(z - w)^{m_i + 1}} + i q \cdot \partial X(z) \right)_{\text{multilinear in } i\zeta_i, i\lambda_i}.
\]

The following two facts are crucial.

- First, we focus on the powers of \( z - w \). Because the last line in eq. (4.2) is quadratic in \( q \), the \( z \) integration needs to yield a singular behavior in \( q \) in order to obtain a nonzero result through the linear order in \( q \). Therefore it is sufficient to consider the coefficients of \( |z - w|^{\alpha' \cdot p + q + \xi} \).

- The singular factor that emerges after the \( z \)-integration of \( |z - w|^{\alpha' \cdot p + q + \xi} \) is expanded as

\[
\frac{1}{p \cdot (q + \xi)} = \frac{1}{p \cdot q} \left( 1 - \frac{p \cdot \xi}{p \cdot q} + \left( \frac{p \cdot \xi}{p \cdot q} \right)^2 + \cdots \right).
\]

We are interested in quadratic terms in \( \xi \), and the last line of eq. (4.2) is quadratic in \( q \). Therefore the leading, subleading and subsubleading terms come from the 0-th, first and second order of the expansion of the exponential functions with respect to \( (q + \xi) \), respectively.
In the following we label the degrees of \((q + \xi)\) in each expansion of 
\[
\exp\left( -\frac{\alpha'}{2} \sum_i \frac{(m_i - 1)! (q + \xi) \cdot \zeta_i}{(z - w)^{m_i}} \right) \quad \text{and} \quad \exp(i(q + \xi) \cdot X(z)), \quad \text{as} \quad (a,b,c), \quad \text{for example} \quad (0,0;1).
\]

The leading soft theorem

We keep the 0-th order in \(q + \xi\) in eq\.(4.2). We have only one contribution.

\( (0,0;0) \) terms

\[
- \lim_{\xi \to 0} \frac{1}{2} h_{\mu\nu} \partial_\xi^\mu \partial_{\xi}^\nu : |z - w|^{\alpha' \cdot (q + \xi)} \cdot \exp \left( i p \cdot X(w) + i \sum_i \zeta_i \cdot \partial_{\xi}^{m_i} X(w) + i \sum_i \lambda_i \cdot \partial^{n_i} X(w) \right) 
\times \left( \frac{\alpha' \cdot q}{2(z - w)} + i q \cdot \partial X(z) \right) \bigg|_{\text{multilinear in } i \zeta_i, i \lambda_i}.
\]

In the last line only the product of the first terms in each bracket yields the factor \(|z - w|^{-2}\), and the \(z\) integration becomes

\[
- \lim_{\xi \to 0} \frac{1}{2} h_{\mu\nu} \partial_\xi^\mu \partial_{\xi}^\nu : |z - w|^{\alpha' \cdot (q + \xi)} \cdot \exp \left( i p \cdot X(w) + i \sum_i \zeta_i \cdot \partial_{\xi}^{m_i} X(w) + i \sum_i \lambda_i \cdot \partial^{n_i} X(w) \right) 
\times : \exp \left( i p \cdot X(w) + i \sum_i \zeta_i \cdot \partial_{\xi}^{m_i} X(w) + i \sum_i \lambda_i \cdot \partial^{n_i} X(w) \right) \bigg|_{\text{multilinear in } i \zeta_i, i \lambda_i}.
\]

Here the arrow stands for the integration over \(z\). By using eq\.(4.3) and taking terms that are quadratic in \(\xi\) and multilinear in \(i \zeta_i\) and \(i \lambda_i\), eq\.(4.5) becomes

\[
- \frac{1}{2} h_{\mu\nu} \frac{2 \pi \alpha' \cdot p' \cdot p''}{\alpha' \cdot (p' \cdot q)^3} \left( \frac{\alpha' \cdot q}{2} \right)^2 \cdot \exp(ip \cdot X(w)) \partial^{m_1} X \cdots \partial^{m_i} X \cdots 
\]

Thus the leading soft theorem is obtained for any hard vertex.

The subleading soft theorem

We keep the first order in \((q + \xi)\) in eq\.(4.2). We have the following three contributions.

1. \( (0,0;1) \) terms

\[
- \lim_{\xi \to 0} \frac{1}{2} h_{\mu\nu} \partial_\xi^\mu \partial_{\xi}^\nu : |z - w|^{\alpha' \cdot (q + \xi)} i (q + \xi) \cdot X(z) \cdot \exp \left( x + i p \cdot X(w) + i \sum_i \zeta_i \cdot \partial_{\xi}^{m_i} X(w) + i \sum_i \lambda_i \cdot \partial^{n_i} X(w) \right) 
\times \left( \frac{\alpha' \cdot q}{2(z - w)} + i q \cdot \partial X(z) \right) \bigg|_{\text{multilinear in } i \zeta_i, i \lambda_i}.
\]

We consider the expansion of the last line. If we pick up one of the third terms in each bracket, we cannot have a pole for \(z - w\) or \(\bar{z} - \bar{w}\). Furthermore the product of the second terms in each bracket cannot become \(|z - w|^{-2}\) because the derivatives of \(X\) with respect to both \(z\) and \(\bar{z}\), \(\partial^a \partial^b X(a, b \geq 1)\), are zero.
The remaining terms are

\[- \lim_{\xi \to 0} \frac{1}{2} h_{\mu \nu} \partial^\mu \partial^\nu \left| z - w \right|^{\alpha'^p - q + \xi} \left( \delta_{\mu \nu} \right) \cdot X(z) \cdot \exp \left( \frac{\alpha' p \cdot q}{2(z-w)} \right) + \sum_i \partial^\mu_{\mu} m_i \cdot \partial^\nu_{\nu} X(w) + \sum_i \partial^\mu_{\mu} m_i \cdot \partial^\nu_{\nu} X(\bar{w}_i) \right) :\]

\[
\times \left[ \left( \frac{\alpha' p \cdot q}{2(z-w)} \right) \left( \frac{\alpha' p \cdot q}{2(z-w)} \right) + \left( \frac{\alpha' p \cdot q}{2(z-w)} \right) \right. \left( \sum_i \frac{\alpha' n_i \cdot \lambda_i}{\lambda_i} \right) + \left( \sum_i \frac{\alpha' n_i \cdot \lambda_i}{\lambda_i} \right) \left( \frac{\alpha' p \cdot q}{2(z-w)} \right) \right] \text{ multilinear in } i\xi, i\lambda_i.
\]

\[
\to \lim_{\xi \to 0} - \frac{1}{2} h_{\mu \nu} \partial^\mu \partial^\nu \left( \frac{2\pi}{\alpha' p \cdot (q + \xi)} \right) \left( \frac{\alpha' p \cdot q}{2(z-w)} \right) i(q + \xi) \cdot X(w) + \left( \sum_i \frac{\alpha' q \cdot \lambda_i}{\lambda_i} \right) \left( \frac{\alpha' p \cdot q}{2(z-w)} \right) i(q + \xi) \cdot \partial^\mu_{\mu} X(w)
\]

\[
+ \left( \sum_i \frac{\alpha' q \cdot \lambda_i}{\lambda_i} \right) : i(q + \xi) \cdot \partial^\mu_{\mu} X(w) \right] \exp \left( \frac{ip \cdot X(w) + \sum_i \partial^\mu_{\mu} m_i \cdot \partial^\nu_{\nu} X(w) + i \sum_i \lambda_i \cdot \partial^\nu_{\nu} X(\bar{w}_i)}{2(z-w)} \right) \text{ multilinear in } i\xi, i\lambda_i \quad (4.8)
\]

2. \((0,1;0)\) terms

\[- \lim_{\xi \to 0} \frac{1}{2} h_{\mu \nu} \partial^\mu \partial^\nu \left| z - w \right|^{\alpha'^p - q + \xi} \left( -\frac{\alpha'}{2} \sum_i \frac{(n_i - 1)(q + \xi) \cdot \lambda_i}{z - w} \right) \]

\[
\times \cdot \exp \left( \frac{ip \cdot X(w) + \sum_i \partial^\mu_{\mu} m_i \cdot \partial^\nu_{\nu} X(w) + \sum_i \lambda_i \cdot \partial^\nu_{\nu} X(\bar{w}_i)}{2(z-w)} \right) \quad (4.9)
\]

In the last line only the product of \(\frac{\alpha' p \cdot q}{2(z-w)}\) and \(iq \cdot \partial X(\bar{z})\) can give the form \(|z - w|^{-2}\), after combining with the Taylor expansion of \(X(z)\). Then we have

\[- \lim_{\xi \to 0} - \frac{1}{2} h_{\mu \nu} \partial^\mu \partial^\nu \left| z - w \right|^{\alpha'^p - q + \xi} \left( -\frac{\alpha'}{2} \sum_i \frac{(n_i - 1)(q + \xi) \cdot \lambda_i}{z - w} \right) \]

\[
\times \cdot \exp \left( \frac{ip \cdot X(w) + \sum_i \partial^\mu_{\mu} m_i \cdot \partial^\nu_{\nu} X(w) + \sum_i \lambda_i \cdot \partial^\nu_{\nu} X(\bar{w}_i)}{2(z-w)} \right) \quad \left( \frac{\alpha' p \cdot q}{2(z-w)} \right) \quad (4.10)
\]

3. \((1,0;0)\) terms

We can get the result by replacing \(\lambda_i, n_i,\) and \(X\) with \(\xi_i, m_i,\) and \(X\) in eq. 4.10.

\[- \lim_{\xi \to 0} \partial^\mu \partial^\nu - \frac{1}{2} h_{\mu \nu} \left( -\frac{\alpha'}{2} \sum_i \frac{(q + \xi) \cdot \xi_i}{z} \right) \quad (4.11)
\]

\[
\times q \cdot \partial^\mu_{\mu} X(w) \exp \left( \frac{ip \cdot X(w) + \sum_i \partial^\mu_{\mu} m_i \cdot \partial^\nu_{\nu} X(w) + \sum_i \lambda_i \cdot \partial^\nu_{\nu} X(\bar{w}_i)}{2(z-w)} \right) \quad \text{ multilinear in } i\xi, i\lambda_i.
\]
The subsubleading soft theorem

The second and third terms in the square bracket represent a Lorentz transformation for each index of the prefactors.

Adding eq.(4.8), eq.(4.10) and eq.(4.11), we get

\[
\lim_{\xi \to 0} \frac{1}{2} h_{\mu\nu} \partial^\mu \partial^\nu \frac{2\pi}{\alpha' p \cdot (q + \xi)} \left( \frac{\alpha' p \cdot q}{2} \right) \exp \left( ip \cdot X(w) + i \sum_i \zeta_i \cdot \partial_n^m X(w) + i \sum_i \lambda_i \cdot \bar{\partial}^n X(\bar{w}_i) \right)
\]

\[
\left[ \left( \frac{\alpha' p \cdot q}{2} \right) i(q + \xi) \cdot X(w) + \left( \sum \frac{\alpha' q \cdot \zeta_i}{2} \right) i(q + \xi) \cdot \partial^m X(w) + \left( \sum \frac{\alpha' q \cdot \lambda_i}{2} \right) i(q + \xi) \cdot \bar{\partial}^n X(w) \right]
\]

\[
+ \left( -\frac{\alpha'}{2} \sum_i (q + \xi) \cdot \zeta_i \cdot q \cdot \partial^m X(w) + \left( -\frac{\alpha'}{2} (q + \xi) \cdot \lambda_i \right) q \cdot \bar{\partial}^n X(w) \right)
\]  

(4.12)

\[
= \lim_{\xi \to 0} \frac{1}{2} h_{\mu\nu} \partial^\mu \partial^\nu \frac{2\pi}{\alpha' p \cdot (q + \xi)} \left( \frac{\alpha' p \cdot q}{2} \right) \exp \left( ip \cdot X(w) + i \sum_i \zeta_i \cdot \partial^m X(w) + i \sum_i \lambda_i \cdot \bar{\partial}^n X(\bar{w}_i) \right)
\]

\[
\left[ \left( \frac{\alpha' p \cdot q}{2} \right) i(q + \xi) \cdot X(w) + \left( \sum \frac{\alpha' q \cdot \zeta_i}{2} \right) -p \cdot \xi \frac{1}{(p \cdot q)^2} + \left( \sum \frac{\alpha' q \cdot \lambda_i}{2} \right) -p \cdot \xi \frac{1}{(p \cdot q)^2} \right]
\]

\[
+ \left( \sum \frac{\alpha' q \cdot \zeta_i \cdot \partial_n^m X_d(\bar{w}) (S^{ab})^{cd}}{2} \right) -p^\nu \frac{1}{(p \cdot q)} + \left( \sum \frac{\alpha' q \cdot \lambda_i \cdot \bar{\partial}^n X_d(\bar{w}) (S^{ab})^{cd}}{2} \right) -p^\nu \frac{1}{(p \cdot q)}
\]  

(4.13)

\[
\left[ -ip^\mu q_a L^{a\mu} + \left( \sum q_a \zeta_i \cdot \partial_n^m X_d(\bar{w}) (S^{ab})^{cd} \right) p^\nu + \left( \sum q_a \lambda_i \cdot \bar{\partial}^n X_d(\bar{w}) (S^{ab})^{cd} \right) p^\nu \right]
\]

\[
\times \exp \left( ip \cdot X(w) + i \sum_i \zeta_i \cdot \partial_n^m X(w) + i \sum_i \lambda_i \cdot \bar{\partial}^n X(\bar{w}_i) \right)
\]

multilinear in \(i\zeta, i\lambda_i
\)

where \(L^{a\mu} = p^\mu X^a - p^a X^\mu\), \((S^{ab})^{cd} = \eta^{ac} \eta^{bd} - \eta^{ad} \eta^{bc}\).

The second and third terms in the square bracket represent a Lorentz transformation for each index of the prefactors. Then we can write eq.(4.13) as

\[
\frac{i\alpha' h_{\mu\nu} \pi p^\mu q_a J^{a\nu}}{2p \cdot q} \cdot \partial^m X \cdots \bar{\partial}^n X \cdots \exp \left( ip \cdot X(w) \right),
\]

where \(J^{a\nu} = L^{a\nu} + S^{a\nu} + \bar{S}^{a\nu}\). \(S^{a\nu}\) and \(\bar{S}^{a\nu}\) are the spin angular momentum operators for the holomorphic and antiholomorphic parts, respectively.

The subsubleading soft theorem

We keep the second order terms in \(q + \xi\) in eq.(4.12).
The details are given in Appendix [B] and we write only the last expression here:

\[
\begin{align*}
\frac{\pi \alpha'}{4p \cdot q} & \left[ q_\rho q_\sigma \left( L^{\mu\alpha} L^{\nu\beta} + 2 S^{\mu\alpha} S^{\nu\beta} \right) \right] + \frac{\pi \alpha'}{4p \cdot q} \sum_i (S^{\mu\nu})^{cd} (S^{\rho\sigma})^{ef} q_\rho q_\sigma \bar{X}_d \partial p_c p_f \partial X^e \partial X^f \exp (ip \cdot X(w)) : \partial^{m_1} X^{\nu_1} \cdots \partial^{m_1} X^{\nu_1} \exp (ip \cdot X(w)) : + \\
\frac{\pi \alpha' \alpha}{4p \cdot q} & \sum_i (S^{\mu\nu})^{cd} (S^{\rho\sigma})^{ef} q_\rho q_\sigma \bar{X}_d \partial p_c p_f \partial X^e \partial X^f \exp (ip \cdot X(w)) : \\
\frac{\pi \alpha' \alpha}{4p \cdot q} & \sum_i (S^{\mu\nu})^{cd} (S^{\rho\sigma})^{ef} q_\rho q_\sigma \bar{X}_d \partial p_c p_f \partial X^e \partial X^f \exp (ip \cdot X(w)) : \\
\frac{\pi \alpha' \alpha}{4p \cdot q} & \sum_i (S^{\mu\nu})^{cd} (S^{\rho\sigma})^{ef} q_\rho q_\sigma \bar{X}_d \partial p_c p_f \partial X^e \partial X^f \exp (ip \cdot X(w)) : \\
\frac{\pi \alpha' \alpha}{4p \cdot q} & \sum_i (S^{\mu\nu})^{cd} (S^{\rho\sigma})^{ef} q_\rho q_\sigma \bar{X}_d \partial p_c p_f \partial X^e \partial X^f \exp (ip \cdot X(w)) : \\
\end{align*}
\]

Here the underbrace represents the absence of the factor above it. From the 4-th line to 7-th line there are mixings with different vertex operators. Note that in general we cannot express the subsubleading soft theorem in terms of the total angular momentum.

### 4.1.2 Examples

We consider the following two examples of the above formula.

**Soft graviton/dilaton theorem for hard massless particles**

We assume that the hard vertex is \( V = A_{\rho\sigma} \partial X^\rho (w) \partial X^\sigma (\bar{w}) \exp (ip \cdot X(w, \bar{w})) \), where \( A_{\rho\sigma} \) is a polarization tensor of the hard particle, which can be either symmetric or antisymmetric. Applying the formulas eq.\((4.6)\), eq.\((4.13)\) and eq.\((4.15)\) for this case, we obtain

\[
\begin{align*}
\frac{\pi \alpha' h_{\mu\rho} p^\rho p^\sigma A_{\rho\sigma}}{2p \cdot q} & : \partial X^\rho \partial X^\sigma \exp (ip \cdot X(w)) : + \\
\frac{\alpha' h_{\mu\rho} q_\mu A_{\rho\sigma}}{2p \cdot q} & \left[ j^{\sigma \alpha} : \partial X^\rho \partial X^\sigma \exp (ip \cdot X(w)) : \right] + \\
\frac{\pi \alpha' h_{\mu\rho} A_{\rho\sigma}}{4p \cdot q} & \left[ q_\rho q_\sigma \left( L^{\mu\alpha} L^{\nu\beta} + 2 (S^{\mu\alpha})^\rho_\alpha (S^{\nu\beta})^\sigma_\beta \right) + 2q_\rho L^{\mu\alpha} \left( q_\sigma (S^{\nu\beta})^\rho_\alpha \eta_\beta^\gamma + q_\sigma (S^{\nu\beta})^\rho_\alpha \eta_\gamma^\beta \right) : \partial X^\rho \partial X^\sigma \exp (ip \cdot X(w)) : \right] + \\
C_1(w) & + C_2(w),
\end{align*}
\]

where \( C_1(w) \) and \( C_2(w) \) are the higher order terms in \( \alpha' \):

\[
\begin{align*}
C_1(w) & \equiv - \frac{\pi \alpha' \alpha h_{\mu\rho} A_{\rho\sigma}}{4p \cdot q} : \sum_i (S^{\mu\alpha})^{cd} (S^{\nu\beta})^{ef} q_\rho q_\sigma \partial X d p_c p_f \partial X^e \partial X^f \exp (ip \cdot X(w)) : \quad, \\
C_2(w) & \equiv - \frac{\pi \alpha' \alpha h_{\mu\rho} A_{\rho\sigma}}{4p \cdot q} : \sum_i (S^{\mu\alpha})^{cd} (S^{\nu\beta})^{ef} q_\rho q_\sigma \partial X d p_c p_f \partial X^e \partial X^f \exp (ip \cdot X(w)) :.
\end{align*}
\]

We have only one prefactor for each of the holomorphic and antiholomorphic part. For graviton it is easy to check that the following combination of the spin operators vanishes when it acts on a single vector index:

\[
h_{\mu\rho} q_\rho (S^{\mu\rho})^\sigma_\alpha (S^{\nu\beta})^\alpha_\beta = 0.
\]

Here we have used \( h_{\mu\rho} q^\rho = 0 \) and \( h_\mu^\rho = 0 \). Then the third line of eq.\((4.16)\) becomes
\[
\begin{align*}
\frac{\pi\alpha' h_{\mu
u} h_{\rho\sigma}}{4p \cdot q} \left[ q_\mu q_\nu \left( L^{\mu
u} L^{\rho\sigma} + 2 S^{\mu\alpha} S^{\nu\beta} \right) + 2q_\mu L^{\mu\alpha} \left( q_\nu S^{\beta\nu} + q_\nu S^{\gamma\nu} \right) \right] : \partial X^\rho \partial X^\sigma \exp (ip \cdot X(w)) : \\
= \frac{\pi\alpha' h_{\mu
u} A_{\rho\sigma}}{4p \cdot q} q_\mu q_\nu : \left( p^{\mu\alpha} j^{\rho\nu} \partial X \partial X \exp(ip \cdot X) \right)^{\rho\sigma} :.
\end{align*}
\]

(4.19)

This reproduces the subleading soft graviton theorem of the field theory at the tree level, if we ignore the higher order terms in \(\alpha'\) in eq. (4.16). On the other hand, \(C_1(w)\) and \(C_2(w)\) vanish for soft dilaton as was discussed in [20]. To see this, we take the polarization tensor of dilaton as

\[
h_{\mu
u}(q) = \frac{\eta_{\mu
u} - q_\mu q_\nu}{\sqrt{D - 2}},
\]

(4.20)

where \(\bar{q}^2 = 0, q \cdot \bar{q} = 1\), and \(D\) is the dimension of the spacetime. When we substitute this in \(C_1(w)\) and \(C_2(w)\), the second and third terms in \(h_{\mu\nu}(q), -q_\mu q_\nu - q_\nu q_\mu\), vanish because \(S^{\mu\alpha} q_\alpha q_\beta = 0\). Then we can replace \(h_{\mu\nu}\) with \(\eta_{\mu\nu}\), and \(C_1(w)\) becomes

\[
- \frac{\pi\alpha'^2 \eta_{\mu\nu} A_{\rho\sigma}}{4p \cdot q} : \sum_i (S^{\mu\alpha}) \bar{e}_d (S^{\nu\beta}) \rho_j q_\mu q_\nu \partial X p_i \rho_j \partial X^\sigma \exp (ip \cdot X(w)) : \\
= - \frac{\pi\alpha'^2 \eta_{\mu\nu} A_{\rho\sigma}}{4p \cdot q} : \sum_i (p^\rho q^\sigma - p \cdot q \eta^{\rho\sigma})(q^\rho p^\sigma - p \cdot q \eta^{\rho\sigma}) \partial X \partial X^\sigma \exp (ip \cdot X(w)) : \\
= - \frac{\pi\alpha'^2 A_{\rho\sigma}}{4p \cdot q} : \sum_i (p^\rho q^\sigma - p \cdot q \eta^{\rho\sigma}) \partial X \partial X^\sigma \exp (ip \cdot X(w)) : \\
= - \frac{\pi\alpha'^2 A_{\rho\sigma}}{4p \cdot q} : \sum_i (p^\rho q^\sigma - p \cdot q \eta^{\rho\sigma}) \partial X \partial X^\sigma \exp (ip \cdot X(w)) : \\
= - \frac{\pi\alpha'^2}{4} : \sum_i (A_{\rho\sigma} q^\rho \partial X^\sigma \partial + A_{\rho\sigma} p \cdot q \partial X^\rho \partial X^\sigma \partial \exp (ip \cdot X(w)) :.
\]

(4.21)

Here we have used \(p^2 = 0\) and \(A_{\rho\sigma} p^\rho = 0\). We can drop the first term in eq. (4.21) because it is total derivative. The second term becomes proportional to the original vertex operator, and vanishes by the momentum conservation when we consider all the hard vertex operators:

\[
\begin{align*}
\int d^2 z \langle V_\epsilon (q; z) : V_1 (p_1; w_1) : \cdots : V_n (p_n; w_n) \rangle \\
\rightarrow - \frac{\pi\alpha'^2}{4} \sum_i p_i \cdot q \langle V_1 (p_1; w_1) : \cdots : V_n (p_n; w_n) \rangle \\
= 0,
\end{align*}
\]

(4.22)

where the arrow means that we focus only on the contribution from \(C_1(w)\). The same is true for \(C_2(w)\).

Therefore, the higher order terms in \(\alpha'\) for the soft dilaton and hard massless vertexes become 0. The explicit form of the subsubleading soft dilaton theorem is

\[
A_{\rho\sigma} \left[ -2q_\mu p_\nu \frac{\partial}{\partial p_\nu} \frac{\partial}{\partial p_\mu} + p \cdot q \frac{\partial}{\partial p_\mu} \frac{\partial}{\partial p_\nu} \right] \partial X^\rho \partial X^\sigma \\
+ \frac{2}{p \cdot q} \left( q^\sigma q_\alpha b X^\alpha \partial X^\beta - q_\alpha q^\sigma \partial X^\alpha \partial X^\beta - q_\alpha q_\beta \partial X^\alpha \partial X^\beta + q^\rho q^\sigma \partial X^\rho \partial X^\sigma \right) \\
+ 2 \left( -\frac{\partial}{\partial p_\rho} \partial X^\rho \partial X^\sigma \right) \exp (ip \cdot X(w)).
\]

(4.23)

This is the same result as in [20].

**Soft graviton/dilaton theorem for hard massive particles**

We assume that the hard vertex is \(V(w) = A_{\rho\sigma} \partial X^\rho \partial X^\sigma \exp (ip \cdot X(w))\). Because of the physical conditions, the polarization tensors must satisfy the following equations:

\[
A_{\rho\sigma} = 0, \quad A_{\alpha\sigma} = 0, \quad A_{\rho\sigma} = 0, \quad A_{\alpha\sigma} = 0.
\]
We apply the formula eq. (4.16), eq. (4.13) and eq. (4.15) for this case, and obtain

\[
\begin{align*}
\pi \alpha' h_{\mu \nu} p' \nu' A_{\mu_1 \nu_1 \sigma_1 \sigma_2} : \partial X^\mu \partial X^{\nu'} \partial X^{\sigma_1} \partial X^{\sigma_2} \exp(i p \cdot X(w)) : \\
+ \frac{\pi \alpha' h_{\mu \nu} p' \nu' A_{\mu_1 \nu_1 \sigma_1 \sigma_2}}{2p \cdot q} : J^\mu \partial X^\nu \partial X^{\nu'} \partial X^{\sigma_1} \partial X^{\sigma_2} \exp(i p \cdot X(w)) : \partial p_1 \partial p_2 \partial p_3 \partial p_4 \\
+ \frac{\pi \alpha' h_{\mu \nu} A_{\mu_1 \nu_1 \sigma_1 \sigma_2}}{2p \cdot q} \left[ q_\alpha q_\beta \left( L^{\alpha \beta} \eta^{\alpha \beta} \eta^{\sigma_1 \gamma} \eta^{\sigma_2 \delta} + 2(S^{\alpha \beta} \gamma^{\alpha \beta}) \right) \right. \\
+ 2q_\alpha L^{\alpha \beta} \left( q_\beta (S^\beta) \gamma^{\alpha \beta} \eta^{\sigma_1 \gamma} \eta^{\sigma_2 \delta} + q_\beta (S^\beta) \gamma^{\alpha \beta} \eta^{\sigma_1 \gamma} \eta^{\sigma_2 \delta} \right) \right] \partial X_\alpha \partial X_\beta \partial X_\gamma \partial X_\delta \exp(i p \cdot X(w)) \\
- \frac{\pi \alpha' h_{\mu \nu} A_{\mu_1 \nu_1 \sigma_1 \sigma_2}}{2p \cdot q} : (S^{\alpha \beta}) \partial (S^\beta) \partial X^{\sigma_1} \partial X^{\sigma_2} \exp(i p \cdot X(w)) : \\
- \frac{\pi \alpha' h_{\mu \nu} A_{\mu_1 \nu_1 \sigma_1 \sigma_2}}{2p \cdot q} : (S^{\alpha \beta}) \partial (S^\beta) \partial X^{\sigma_1} \partial X^{\sigma_2} \exp(i p \cdot X(w)) : \\
- \frac{\pi \alpha' h_{\mu \nu} A_{\mu_1 \nu_1 \sigma_1 \sigma_2}}{2p \cdot q} \left[ q_\alpha q_\beta (S^{\alpha \beta} \gamma^{\alpha \beta}) : \partial X^{\sigma_1} \partial X^{\sigma_2} \exp(i p \cdot X(w)) : \right].
\end{align*}
\] (4.24)

When the hard particle is massive, a mixing with different vertex operators appears at subsubleading order as well as the higher order corrections in \(\alpha'\). The subsubleading soft theorem can no longer be written in terms of the total angular momentum.

Furthermore, the resulting vertex operator does not necessarily satisfy the physical condition. However the soft theorem is still obtained by a universal rule, that is, the OPE with a superlocal operator \(V_s\). Here we point out that the third and fourth lines cannot be expressed in terms of the total angular momentum when the hard particle is massive. As we have seen for the massless particles, the contribution vanishes when the spin angular momentum operators act on the holomorphic part twice as in eq. (4.18). On the other hand, for the massive particles, \(S^2(S^2)\) need not be 0 when the each \(S(\tilde{S})\) acts on the different indexes, such as

\[h_{\mu \nu} q_\alpha q_\beta (S^{\alpha \beta}) \partial X^\mu \partial X^\beta \neq 0.\] (4.25)

Therefore we cannot simply write the subsubleading soft theorem by the total angular momentum operator.

### 4.2 Soft B field theorem

The soft B field theorem has been examined in \[22\], and its universal behavior is determined through subleading order. This can also be seen in our formulation. We can rewrite the B field vertex operator as follows:

\[\begin{align*}
: \partial X^\mu (z) \partial X^\nu (\bar{z}) \exp(i \xi \cdot X(z, \bar{z})) : = & \partial \cdot X^\mu (z, \bar{z}) \partial X^\nu (\bar{z}) \exp(i \xi \cdot X(z, \bar{z})) : - : X^\mu (z, \bar{z}) \partial X^\nu (\bar{z}) \partial \cdot X \exp(i \xi \cdot X(z, \bar{z})) : .
\end{align*}\] (4.26)

For the B field we cannot further rewrite it by using partial integration. We find that the contribution from the bulk is \(O(q)\). Therefore the soft theorem holds universally through subleading order, and is obtained in a similar manner to the previous subsection. First, we rewrite the soft vertex as

\[-h_{\mu \nu} X^\mu (z, \bar{z}) \partial X^\nu (\bar{z}) \partial \exp(i \xi \cdot X(z, \bar{z})) = \lim_{\xi \omega \to 0} \lim_{z' \to z} \frac{\partial}{\partial \xi_\mu} \frac{\partial}{\partial \omega} h_{\mu \nu} \partial_\xi \partial_\omega \exp(i \xi \cdot X(z, \bar{z}) + i \xi \cdot X(z', \bar{z}) + i \omega \cdot \partial X(z')).\] (4.27)

The details are given in Appendix \[C\] and the result is

\[- \int_{|z| < \delta} h_{\mu \nu} : X^\mu (z, \bar{z}) \partial X^\nu (z, \bar{z}) \partial \exp(i \xi \cdot X(z, \bar{z})) : = \frac{-i \pi \alpha}{2 \sqrt{2} E} \left( \frac{p' q_\alpha (S^{\alpha \beta} - \tilde{S}^{\alpha \beta}) - (\mu \leftrightarrow \nu) + \tilde{S}^{\alpha \beta}}{p \cdot q} \right) : V(w) : + O(q^2),\] (4.28)

where \(V(w)\) is any hard vertex operator.
5 Soft theorem in other string theories

5.1 superstring

The soft theorems in superstring theory have been examined in various contexts [11] [21] [24]. They can be seen also from our formulation.

The vertex operator of the graviton or dilaton in (0,0) picture can be written as

\[ h_{\mu \nu} \cdot \left( i \partial X^\mu (z) - \frac{\alpha'}{2} q_{\alpha} \psi^\mu (z) \psi^\alpha (z) \right) \left( i \partial X^\nu (\bar{z}) - \frac{\alpha'}{2} q_\beta \bar{\psi}^\nu \bar{\psi}^\beta \right) \exp (i q \cdot X (z, \bar{z})) : \]

Note that \( \psi^\mu (z) \psi^\alpha (z) \) can be regarded as the spin angular momentum operator. We classify the terms in eq. (5.1) as follows:

1. The product of the bosonic parts is the same as in the previous section. It gives the momentum and the angular momentum as well as the other terms in eqs. (4.24).

2. The product of the fermionic parts is quadratic in \( q \) and contribute only to the subsubleading soft theorem. This gives the product of the spin angular momenta of the holomorphic and antiholomorphic parts of the hard vertexes.

3. For the products of the bosonic and fermionic part, we perform partial integration in the bosonic part as in the previous section.

\[ \partial X^\mu (z) \exp (i q \cdot X (z, \bar{z})) = \partial (X^\mu \exp (i q \cdot X (z, \bar{z}))) - X^\mu (z, \bar{z}) \partial \exp (i q \cdot X (z, \bar{z})). \]

Because the fermionic part is multiplied by \( q \), the contribution from the bulk is at least \( O (q^2) \). Therefore, these terms contribute to the subleading and the subsubleading soft theorem.

As an example, let’s consider the contraction of the soft graviton with a hard dilatino or gravitino in NS-R sector. The vertex operator in \(-1, -\frac{1}{2}\) picture in the NS - R sector is given by

\[ V (w, \bar{w}) = u^\alpha_\mu \psi^\nu (w) \exp (-\phi (w)) \tilde{S}_\alpha (\bar{w}) \exp \left( -\frac{\phi (\bar{w})}{2} \right) \exp (i p \cdot X (w, \bar{w})), \]

where \( u^\alpha_\mu \) is a polarization, \( \phi, \bar{\phi} \) are superghosts, \( \tilde{S}_\alpha \) generates the ground state in R sector. In this case the prefactors are purely fermionic, so the spin angular momentum comes from the fermionic part and the orbital angular momentum from the bosonic part separately.

The contraction rules are as follows:

\[ \psi^\mu (z) \psi^\nu (w) \sim \frac{\eta^{\mu \nu}}{z - w}, \]

\[ \psi^\mu (z) S_\alpha (w) \sim \frac{1}{\sqrt{2 (z - w)}} (T^\nu)_\alpha^\beta S_\beta (w), \]

\[ \phi (z) \phi (w) \sim -\ln (z - w). \]

The contributions from the above 1. ~ 3. are as follows:

1. This is essentially the same as in the hard tachyon in bosonic string theory:

\[ -\frac{\pi \alpha'}{4 p \cdot q} \left( 2 p^\mu p^\nu + 2 q_{\alpha} q_{\beta} L^{\alpha \beta} + q_{\alpha} q_{\beta} L^{\alpha \beta} \right) u^\alpha_\mu \psi^\nu (w) \exp (-\phi (w)) \tilde{S}_\alpha (\bar{w}) \exp \left( -\frac{\phi (\bar{w})}{2} \right) \exp (i p \cdot X (w, \bar{w})). \]

2. The holomorphic part gives

\[ -\frac{\alpha'}{2} q_{\alpha} \psi^\mu (z) \psi^\alpha (z) \psi^\nu (w) \sim -\frac{\alpha' (\eta^{\mu \nu} \psi^\alpha - \eta^{\nu \mu} \psi^\alpha)}{2 (z - w)} = -\frac{\alpha' q_{\alpha} (S^{\mu \alpha})^\nu \nu \psi^\nu (w)}{2 (z - w)}, \]

while the antiholomorphic part gives

\[ -\frac{\alpha'}{2} q_{\beta} \bar{\psi}^\nu \bar{\psi}^\beta \bar{S}_\alpha (\bar{w}) \sim -\frac{\alpha' q_{\alpha} (\Gamma^\nu)^\alpha_\beta (\Gamma^\nu)^\beta \bar{S}_\alpha (\bar{w})}{4 (z - w)} = -\frac{\alpha' q_{\alpha} (S^{\nu \beta})^{\delta \beta} \bar{S}_\delta (\bar{w})}{2 (z - w)}. \]

Taking the products of these results and integrating over \( z \), we obtain

\[ \frac{\pi \alpha'}{2 p \cdot q} u^\alpha_\mu \psi^\nu (w) \exp (-\phi (w)) \tilde{S}_\alpha (\bar{w}) \exp \left( -\frac{\phi (\bar{w})}{2} \right) \exp (i p \cdot X (w, \bar{w})). \]
3. For the bosonic part we have

\[ iX^\mu(z, \bar{z}) \partial \exp (i q \cdot X(z, \bar{z})) \quad i \partial X(w, \bar{w}) \]  

\[ = \lim_{\xi \to 0} \lim_{\bar{z} \to z} \frac{\partial}{\partial \xi^\mu} \left[ \exp (i q \cdot X(z, \bar{z}) + i \xi \cdot X(z', \bar{z}')) \right] \quad i \partial X(w, \bar{w}) \]  

\[ = \frac{\alpha' p \cdot q}{2(z - w)} \frac{\partial}{\partial \xi^\mu} \left[ |z - w|^{\alpha' p \cdot q} \exp (i q \cdot X(z, \bar{z})) \right] \exp (i p \cdot X(w, \bar{w})). \]  

(5.11)

After the z integration this gives the momentum and the orbital angular momentum.

For the fermionic part, by the same calculation as eq.(4.8) and eq.(5.9), we get the spin angular momentum for each of the holomorphic and antiholomorphic parts.

Combining these results, we obtain

\[ -\frac{\pi \alpha' h_{\mu\nu}(ip^\mu + i q_\mu L^{\mu a})q_a S^{ab}}{2p \cdot q} u^a_\alpha \psi^\mu(w) \exp (-\phi(w)) S_{\alpha}(\bar{w}) \exp \left( -\frac{\tilde{\phi}(\bar{w})}{2} \right) \exp (i p \cdot X(w, \bar{w})). \]  

(5.12)

Summing up these contributions, we can express the soft theorem as follows:

\[ -\frac{\pi \alpha' h_{\mu\nu}}{4p \cdot q} \left( 2p^\mu p^\nu + 2q_a q_b J^{\mu a} + q_a q_b J^{\nu b} - q_a q_b S^{a b} S^{d b} - q_a q_b \tilde{S}^{a b} \tilde{S}^{d b} \right) \]  

\[ \times u^a_\alpha \psi^\mu(w) \exp (-\phi(w)) S_{\alpha}(\bar{w}) \exp \left( -\frac{\tilde{\phi}(\bar{w})}{2} \right) \exp (i p \cdot X(w, \bar{w})). \]  

(5.13)

As in eq.(4.18), the same combination of the spin operators vanishes when it acts on a single spinor:

\[ h_{\mu\nu} q_a q_b (S^{a b})_j (S^{a b})_k = 0, \]  

(5.14)

where i, j, k are spinor indices. Therefore the fourth and fifth terms are 0 and eq. (5.13) is written in terms of the total angular momentum for soft graviton.

In superstring theory, if there is no bosonic prefactor \( \partial X \) or \( \tilde{\partial} X \), there appears neither the higher order correction in \( \alpha' \) nor a mixing with different vertexes, because the spin angular momentum operator comes only from the fermionic part.

### 5.2 heterotic string

In this section we discuss the soft theorems for gauge bosons in heterotic string theory. The vertex operator of the gauge boson is given by

\[ V_{\text{gauge}}(z) = \zeta_{a b}(z) \partial X^\mu(\bar{z}) \exp (i k \cdot X(z, \bar{z})). \]  

(5.15)

Here \( j^a \) is a holomorphic (1,0) operator that satisfies the current algebra:

\[ j^a(z) j^b(w) \sim \frac{k^{ab}}{(z - w)^2} + \frac{i e^{a b}}{z - w} j^c(w), \]  

(5.16)

where \( k^{a b}, e^{a b} \) are constants.

As in the previous sections, we rewrite \( V_{\text{gauge}} \) as

\[ V_{\text{gauge}}(z) = \tilde{\partial} (\zeta_{a b}(z) \exp (i k \cdot X(z, \bar{z}))) - \zeta_{a b} j^a(z) \tilde{\partial} \exp (i k \cdot X(z, \bar{z})). \]  

(5.17)

We can drop the first term because it is a total derivative. The contribution from the bulk of the second term is \( O(q) \). Therefore the soft theorem holds through the 0-th order in q.

When we evaluate the OPE with a hard vertex, the holomorphic part gives the generator of the gauge symmetry with a pole \( \frac{1}{z - w} \). The antiholomorphic part has exactly the same form as in the case of superstring, and it gives the momentum, the orbital angular momentum and the spin angular momentum.

This is consistent with the results \(^1\) \(^2\) \(^3\) that the soft gauge boson theorem at the tree level is universal through subleading order.

---

\(^1\)Eq.(5.14) holds for dilaton as well as graviton contrary to the case of a vector index in eq.(4.18)
6 Conclusion

We have discussed the soft theorem in string theory in terms of the OPE of the soft and hard vertexes. When the soft vertex is expanded with the soft momentum after some partial integration, it turns out to be a superlocal operator through a certain order in \( q \). As a result, we find that the scattering amplitude is evaluated only by the local integral around the hard vertexes through that order of the soft momentum, which leads to the universal soft behavior.

We have confirmed that the soft behavior of massless particles of bosonic closed string, closed superstring and heterotic string can actually be reproduced by that method. When the hard vertex represents a massive particle, we find that the subsubleading soft theorem can no longer be written in terms of the total angular momentum.

It is known that the universal behavior of the soft theorem breaks down when loop effects are taken into account [24] [31] [32]. It is interesting to see whether loop effects can be evaluated with the superlocal operator. Because the loop effect can be examined by factorizing diagrams in field theory, we expect that the loop effects appear as the pinches of the world-sheet in string theory.

In this paper we have considered scattering amplitudes with one soft particle. It should be able to consider the scattering amplitudes with more than one soft particle in this formulation.

Although we have obtained a simple picture of soft theorems based on the superlocal operator, we still do not have a brief explanation why the total angular momentum operator emerges. Furthermore it is interesting to try to obtain a unified view that describes the higher order correction in \( \alpha' \) and mixing with different vertex operators.

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A Analogy with field theory

In this appendix we show that the soft theorem has the same structure as field theory, if we ignore the higher order corrections of \( \alpha' \) and the mixing with different vertex operators. Note that here we consider the soft vertex before partial integration.

Let’s calculate the OPE between the soft graviton vertex \( V_{\text{soft}}(z, \bar{z}) = h_{\mu \nu} \partial X^\mu(z) \bar{\partial} X^\nu(\bar{z}) \exp(iq \cdot X(z, \bar{z})) \) and a hard vertex. For simplicity we take a graviton as the hard vertex operator: \( V_{\text{hard}}(w, \bar{w}) = A_{\mu \nu} \partial X^\mu(w) \bar{\partial} X^\nu(\bar{w}) \exp(ip \cdot X(w, \bar{w})) \).

In the following we omit polarization tensors.

First, we consider the contraction between the exponential functions in the soft and the hard vertexes.

\[
\int d^2 z \partial X^\mu(z) \bar{\partial} X^\nu(\bar{z}) \exp(iq \cdot X(z, \bar{z})) \partial X^\rho(w) \bar{\partial} X^\sigma(\bar{w}) \exp(ip \cdot X_{\mu \nu}(w, \bar{w})) = \int d^2 z |z - w|^\alpha \beta \partial X^\mu(z) \bar{\partial} X^\nu(\bar{z}) \partial X^\rho(w) \bar{\partial} X^\sigma(\bar{w}) : \exp(iq \cdot X(z, \bar{z}) + ip \cdot X(w, \bar{w})) :. \tag{A.1}
\]

Here we assume that we do not take any contraction among the operators defined on the same points, \( z \) or \( w \), even if the symbol of the normal ordering is not explicitly written. Then we expand the exponential functions with the power of \( q \). In order to indicate that the expanded operators should not be contracted with \( \exp(ip \cdot X(w, \bar{w})) \) anymore, we denote them by the symbol \( X(z, \bar{z}) \). We classify the contributions to eq. (A.1) into the following five cases by the order of \( q \) and the integration regions.

1. The contribution of the 0-th order in \( q \) from the disk around the hard vertex.
   It is given by
   \[
   \int |z| < \varepsilon \int d^2 z \partial X^\mu(z) \bar{\partial} X^\nu(\bar{z}) \partial X^\rho(w) \bar{\partial} X^\sigma(\bar{w}) \exp(ip \cdot X(w, \bar{w})) \\
   \sim -\pi \alpha' \epsilon^\alpha \epsilon^\beta \frac{p^\mu p^\nu}{2p \cdot q} \partial X^\rho(w) \bar{\partial} X^\sigma(\bar{w}) \exp(ip \cdot X(w, \bar{w})) \\
   \sim -\pi \alpha' \frac{p^\mu p^\nu}{2p \cdot q} \partial X^\rho(w) \bar{\partial} X^\sigma(\bar{w}) \exp(ip \cdot X(w, \bar{w})), \tag{A.2}
   \]
where we have ignored the higher order terms in $\alpha'$. This gives the leading soft theorem.

2. The contribution of the $0$-th order in $q$ from the bulk
   Because the singularity in $q$ is not yielded from the bulk, this contribution gives a part of the subleading soft theorem. In fact, the soft vertex operator at this order is a total derivative with respect to $z$, and we can rewrite it to the contour integral around the hard vertex as follows:
   \[
   \int_{\text{bulk}} d^2z \partial X^\nu \partial X^\alpha(z) V(w_1, w_1) \cdots V(w_n, w_n) = -\frac{i}{2} \sum_{i=1}^n \oint_{|z-w_i| = \epsilon} d\bar{z} X^\alpha(z, \bar{z}) \bar{\partial} X^\nu(\bar{z}) V(w_1, w_1) \cdots V(w_n, w_n). \tag{A.3}
   \]
   Then by looking at the pole in the antiholomorphic part of the OPE, we have
   \[
   \sim -\frac{i}{2} \oint_{|z-w| = \epsilon} d\bar{z} X^\alpha(z, \bar{z}) \bar{\partial} X^\nu(\bar{z}) \partial X^\sigma(w) \bar{\partial} X^\sigma(\bar{w}) \exp(ip \cdot X(w, \bar{w}))
   \]
   \[
   \sim \frac{i\pi\alpha'}{2} \frac{\partial}{\partial p_\mu} \left( \partial X^\sigma(w) \bar{\partial} X^\sigma(\bar{w}) \exp(ip \cdot X(w, \bar{w})) \right). \tag{A.4}
   \]
   This is a part of the orbital angular momentum of the subleading soft theorem. The fact that a part of the angular momentum comes out from the bulk is similar to the structure in field theory.

3. The contribution of the first order in $q$ from the disk around the hard vertex
   It is given by
   \[
   iq_\alpha \int_{|z| < \epsilon} d^2z \partial X^\alpha(z) \bar{\partial} X^\nu(\bar{z}) X^{\alpha}(z, \bar{z}) \partial X^\sigma(w) \bar{\partial} X^\sigma(\bar{w}) \exp(ip \cdot X(w, \bar{w})). \tag{A.5}
   \]
   As in the above cases, we take the OPE, and then ignore the higher order terms in $\alpha'$ and the mixing with different vertex operators. We have
   \[
   \sim -\frac{\pi\alpha'}{2q \cdot p} \exp(ip \cdot X(w, \bar{w})) \left[ iq_\alpha \eta^{\nu\sigma} \frac{\partial}{\partial p_\mu} \partial X^\sigma(w) \bar{\partial} X^\sigma(\bar{w}) + q_\alpha \rho_\mu \eta^{\nu\sigma} \frac{\partial}{\partial p_\mu} \partial X^\nu(w) \bar{\partial} X^\sigma(\bar{w}) - p_\mu^{\nu\sigma} \partial X^\sigma(w) \bar{\partial} X^\sigma(\bar{w}) 
   \right.
   \]
   \[
   \left. + q_\alpha \rho_\mu \eta^{\nu\sigma} \partial X^\nu(w) \bar{\partial} X^\sigma(\bar{w}) - p_\mu^{\nu\sigma} \partial X^\nu(w) \bar{\partial} X^\sigma(\bar{w}) \right]. \tag{A.6}
   \]

4. The contribution of the first order in $q$ from the bulk
   By the symmetry of the polarization tensor, we can rewrite the soft vertex operator as follows:
   \[
   iq_\alpha \partial X^\mu(z) \bar{\partial} X^\nu(\bar{z}) X^{\alpha}(z, \bar{z}) = iq_\alpha \partial \left( X^\nu(z, \bar{z}) \bar{\partial} X^\nu(\bar{z}) \bar{\partial} X^\nu(z) \right) - \frac{iq_\alpha}{2} \partial \left( X^\nu(z, \bar{z}) X^{\alpha}(z, \bar{z}) \bar{\partial} X^\nu(\bar{z}) \bar{\partial} X^\nu(z) \right). \tag{A.7}
   \]
   As in [2], the $z$ integration can be replaced by contour integrals around the hard vertexes:
   \[
   \int_{\text{bulk}} d^2z \partial X^\nu(z) \bar{\partial} X^\nu(\bar{z}) iq_\alpha X^{\alpha}(z, \bar{z}) V(w_1, w_1) \cdots V(w_n, w_n) \tag{A.8}
   \]
   \[
   = \frac{q_\alpha}{2} \sum_{i=1}^n \oint_{|z-w_i| = \epsilon} d\bar{z} X^\nu(z, \bar{z}) \bar{\partial} X^\nu(\bar{z}) X^{\alpha}(z, \bar{z}) V(w_1, w_1) \cdots V(w_n, w_n)
   \]
   \[
   + \frac{q_\alpha}{4} \sum_{i=1}^n \oint_{|z-w_i| = \epsilon} d\bar{z} X^\nu(z, \bar{z}) \bar{\partial} X^\nu(\bar{z}) X^{\alpha}(z, \bar{z}) V(w_1, w_1) \cdots V(w_n, w_n) \tag{A.9}
   \]

   When a hard vertex is a graviton, we get
   \[
   \frac{q_\alpha}{2} \oint_{|z-w| = \epsilon} d\bar{z} X^\nu(z, \bar{z}) \bar{\partial} X^\nu(\bar{z}) X^{\alpha}(z, \bar{z}) \partial X^\sigma(w) \bar{\partial} X^\sigma(\bar{w}) \exp(ip \cdot X(w, \bar{w}))
   \]
   \[
   + \frac{q_\alpha}{4} \sum_{i=1}^n \oint_{|z-w_i| = \epsilon} d\bar{z} X^\nu(z, \bar{z}) X^{\alpha}(z, \bar{z}) \partial X^\sigma(w) \bar{\partial} X^\sigma(\bar{w}) \exp(ip \cdot X(w, \bar{w}))
   \]
   \[
   = -\frac{i\pi\alpha'q_\alpha}{2} \left[ -ip_\mu \frac{\partial}{\partial p_\mu} \partial X^\nu(w) \bar{\partial} X^\nu(\bar{w}) - \eta^{\nu\sigma} \frac{\partial}{\partial p_\mu} \partial X^\nu(w) \bar{\partial} X^\sigma(\bar{w}) 
   \right.
   \]
   \[
   \left. + \eta^{\nu\sigma} \frac{\partial}{\partial p_\mu} \partial X^\nu(w) \bar{\partial} X^\sigma(\bar{w}) + \frac{1}{2} ip_\mu \eta^{\nu\rho} \frac{\partial}{\partial p_\rho} \partial X^\nu(w) \bar{\partial} X^\sigma(\bar{w}) \exp(ip \cdot X(w, \bar{w})) \right]. \tag{A.10}
   \]
   This contributes to subsubleading soft theorem.
5. The contribution of the second order in q from the disk around the hard vertex operators

It is given by

\[
\frac{iq_\mu iq_\rho}{2} \int_{|z|<\epsilon} d^2 z \partial X^\mu(z) \bar{\partial} X^\rho(z) \prod_{i} \partial X^\mu(w_i) \bar{\partial} X^\rho(w_j) \exp(ip \cdot X(w, \bar{w}))
\]

\[
\sim - \pi \alpha' q_\mu q_\rho \left[ -p^\mu p^\rho \frac{\partial}{i \partial p_\mu} \partial X^\mu(w) \partial X^\rho(w) + 2ip^\mu \eta^\rho \frac{\partial}{i \partial p_\sigma} \partial X^\rho(w) \partial X^\sigma(w) \right.
\]

\[
- 2ip^\mu \eta^\rho \frac{\partial}{i \partial p_\sigma} \partial X^\rho(w) \partial X^\sigma(w) + 2ip^\mu p^\rho \frac{\partial}{i \partial p_\sigma} \partial X^\rho(w) \partial X^\sigma(w)
\]

\[
+ 2\eta^\rho \eta^\sigma \partial X^\rho(w) \partial X^\sigma(w) - 2\eta^\rho \eta^\sigma \partial X^\rho(w) \partial X^\sigma(w)
\]

\[
- 2\eta^\rho \eta^\sigma \partial X^\rho(w) \partial X^\sigma(w) - 2\eta^\rho \eta^\sigma \partial X^\rho(w) \partial X^\sigma(w)
\]

\[
+ 2\eta^\rho \eta^\sigma \partial X^\rho(w) \partial X^\sigma(w) \right]
\]

where we have ignored the higher order terms in \(\alpha'\) and the mixing with different vertex operators. The sum of the contributions of 4. and 5. gives the subleading soft theorem.

B Subleading soft theorem for graviton and dilaton

We focus on the second order terms in \((q + \xi)\) in eq.(4.2).

For convenience we classify the terms by three types of underlines, \(\underline{\underline{\underline{\alpha}}}, \underline{\underline{\alpha}}\) and \(\underline{\alpha}\). The simple and double lines do not change the number of prefactors, but the wavy lines do. The simple line represents the terms without \(\alpha'\) correction, while the double line stands for the higher order terms in \(\alpha'\).

1. \((0,2;0)\) terms

\[
- \lim_{\xi \to 0} \frac{1}{2} h_{\mu\nu} \partial^\mu \partial^\nu \left[ \frac{z - w}{(z - w)^{n+1}} \right] \exp \left( ip \cdot X(w) + i \sum \xi \cdot \partial_{\mu} X(w) + i \sum \lambda \cdot \bar{\partial}_{\mu} X(w) \right)
\]

\[
\times \left( -\frac{\alpha'}{2} \sum_{i} \left( n_i - 1 \right)! (q + \xi) \cdot \lambda_i \right) \left( -\frac{\alpha'}{2} \sum_{j} \left( n_j - 1 \right)! (q + \xi) \cdot \lambda_j \right)
\]

\[
\times \left( \frac{\alpha'q \cdot q}{2(z - w)} + \sum \frac{\alpha' m_i q \cdot \xi}{2(z - w)^{m_i+1}} + iq \cdot \partial X(z) \right)
\]

\[
\left. \right|_{\text{multilinear in } i\xi, i\lambda} \right)
\]

(B.1)

In the last line only the first term in the first bracket gives the pole \(\frac{1}{z - w}\), and only the third term in the second bracket gives \(\frac{1}{z - w}\).

\[
- \lim_{\xi \to 0} \frac{1}{2} h_{\mu\nu} \partial^\mu \partial^\nu \left[ \frac{z - w}{(z - w)^{n+1}} \right] \exp \left( ip \cdot X(w) + i \sum \xi \cdot \partial_{\mu} X(w) + i \sum \lambda \cdot \bar{\partial}_{\mu} X(w) \right)
\]

\[
\times \left( -\frac{\alpha'}{2} \sum_{i} \left( n_i - 1 \right)! (q + \xi) \cdot \lambda_i \right) \left( -\frac{\alpha'}{2} \sum_{j} \left( n_j - 1 \right)! (q + \xi) \cdot \lambda_j \right)
\]

\[
\times \left( \frac{\alpha'q \cdot q}{2(z - w)} + \sum \frac{\alpha' m_i q \cdot \xi}{2(z - w)^{m_i+1}} + iq \cdot \partial X(z) \right)
\]

\[
\left. \right|_{\text{multilinear in } i\xi, i\lambda} \right)
\]

(B.2)

2. \((2,0;0)\) terms
By replacing \( \lambda_i, n_i, \tilde{X} \) with \( \zeta_i, m_i, X \) in the above expression we get the result for the holomorphic part.

\[
- \lim_{\xi \to 0} \frac{1}{2} h_{\mu\nu} \partial_{\xi}^\mu \partial_{\xi}^\nu \exp \left( -\frac{2\pi}{\alpha' p \cdot (q + \xi)} \sum_i \zeta_i \partial_{w_i}^{m_i} X(w) + \sum_i \lambda_i \partial_{\bar{w}_i}^{\bar{m}_i} X(\bar{w}_i) \right) \\
\times \left( -\frac{\alpha'}{2} \sum_i (m_i - 1)! (q + \xi) \cdot \zeta_i \right) \\
\times \left( -\frac{\alpha'}{2} \sum_j (m_j - 1)! (q + \xi) \cdot \zeta_j \right) \\
\times \left( -\frac{\alpha' p \cdot q \cdot iq \cdot \partial_{\bar{m}_j+m_j} X}{2 (m_j + m_j - 1)!} \right) \multlinear_{\zeta_i, \lambda_i} \tag{B.3}
\]

3. \((1,1,0)\) terms

\[
- \lim_{\xi \to 0} \frac{1}{2} h_{\mu\nu} \partial_{\xi}^\mu \partial_{\xi}^\nu [z - w]^{\alpha' p \cdot (q + \xi)} \exp \left( -\frac{2\pi}{\alpha' p \cdot (q + \xi)} \sum_i \zeta_i \partial_{w_i}^{m_i} X(w) + \sum_i \lambda_i \partial_{\bar{w}_i}^{\bar{m}_i} X(\bar{w}_i) \right) \\
\times \left( -\frac{\alpha'}{2} \sum_i (m_i - 1)! (q + \xi) \cdot \zeta_i \right) \\
\times \left( -\frac{\alpha'}{2} \sum_j (m_j - 1)! (q + \xi) \cdot \lambda_j \right) \\
\times \left( \frac{\alpha' p \cdot q \cdot i q \cdot \partial_{\bar{m}_j+m_j} X}{2(z - w)} \right) \multlinear_{\zeta_i, \lambda_i} \tag{B.4}
\]

In the last line only the product of the third terms in each bracket, \( iq \cdot \partial X \times iq \cdot \partial \bar{X} \) gives the factor \( |z - w|^{-2} \).

\[
- \lim_{\xi \to 0} \frac{1}{2} h_{\mu\nu} \partial_{\xi}^\mu \partial_{\xi}^\nu [z - w]^{\alpha' p \cdot (q + \xi)} \exp \left( -\frac{2\pi}{\alpha' p \cdot (q + \xi)} \sum_i \zeta_i \partial_{w_i}^{m_i} X(w) + \sum_i \lambda_i \partial_{\bar{w}_i}^{\bar{m}_i} X(\bar{w}_i) \right) \\
\times \left( -\frac{\alpha'}{2} \sum_i (m_i - 1)! (q + \xi) \cdot \zeta_i \right) \\
\times \left( -\frac{\alpha'}{2} \sum_j (n_j - 1)! (q + \xi) \cdot \lambda_j \right) \\
\times \left( \frac{\alpha' p \cdot q \cdot i q \cdot \partial_{\bar{m}_j+m_j} X}{2(z - w)} \right) \multlinear_{\zeta_i, \lambda_i} \tag{B.5}
\]

4. \((0,1,1)\) terms

\[
- \lim_{\xi \to 0} \frac{1}{2} h_{\mu\nu} \partial_{\xi}^\mu \partial_{\xi}^\nu [z - w]^{\alpha' p \cdot (q + \xi)} \left( -\frac{\alpha'}{2} \sum_i (n_i - 1)! (q + \xi) \cdot \lambda_i \right) \\
\times (q + \xi) \cdot X(z) : \exp \left( -\frac{2\pi}{\alpha' p \cdot (q + \xi)} \sum_i \zeta_i \partial_{w_i}^{m_i} X(w) + \sum_i \lambda_i \partial_{\bar{w}_i}^{\bar{m}_i} X(\bar{w}_i) \right) \\
\times \left( \frac{\alpha' p \cdot q \cdot i q \cdot \partial_{\bar{m}_j+m_j} X}{2(z - w)} \right) \multlinear_{\zeta_i, \lambda_i} \tag{B.6}
\]

In the last line the third term in the first bracket does not give the pole \( \frac{1}{z - w} \). The product of the second term in the
6. This result is given by replacing $\lambda_i, n_i, X$ with $\zeta_i, m_i X$ in eq. (B.7).

\[ -\lim_{\xi \to 0} \frac{1}{2} h_{\mu\nu} \partial^\mu \xi \partial^\nu \xi \mid z - w \mid^{\alpha' p (q + \xi)} \left( -\frac{\alpha'}{2} \sum_i (n_i - 1)!(q + \xi) \cdot \lambda_i \right) \]

\[ \times i(q + \xi) \cdot X(z) : \exp \left( ip \cdot X(w) + i \sum_i \zeta_i \cdot \partial^m_i X(w) + i \sum_i \lambda_i \cdot \partial^m_i X(w) \right) : \]

\[ \times \left[ \left( \frac{\alpha' p - q}{2(z - w)} \right) \left( \frac{\alpha' p - q}{2(z - w)} + \sum_i \left( \alpha' n_i q \cdot \lambda_i \right) \right) + \left( \sum_i \alpha' m_i q \cdot \lambda_i \right) \right] \]

\[ \times \left( \sum_i \left( \alpha' n_i q \cdot \lambda_i \right) \right) \mid \text{multilinear in } i \zeta_i, i \lambda_i \]

\[ = -\lim_{\xi \to 0} \frac{1}{2} h_{\mu\nu} \partial^\mu \xi \partial^\nu \xi \mid z - w \mid^{\alpha' p (q + \xi)} \exp \left( ip \cdot X(w) + i \sum_i \zeta_i \cdot \partial^m_i X(w) + i \sum_i \lambda_i \cdot \partial^m_i X(w) \right) \left( -\frac{\alpha'}{2} \sum_i (q + \xi) \cdot \lambda_i (n_i - 1)! \right) \]

\[ \times \left[ \left( \frac{\alpha' p - q}{2} \right) \left( \frac{\alpha' p - q}{2} + \frac{n_i - 1}{n_i!} \right) \sum_i \frac{i q \cdot \partial^m_i X(w)}{k!(n_i - k - 1)!} \right] \]

\[ \times \left( \sum_i \alpha' m_i q \cdot \lambda_i \right) \mid \text{multilinear in } i \zeta_i, i \lambda_i \]

\[ \text{(B.7)} \]

5. (1:0:1) terms

This result is given by replacing $\lambda_i, n_i, X$ with $\zeta_i, m_i X$ in eq. (B.7).

\[ -\lim_{\xi \to 0} \frac{1}{2} h_{\mu\nu} \partial^\mu \xi \partial^\nu \xi \mid z - w \mid^{\alpha' p (q + \xi)} \frac{2\pi}{\alpha' p \cdot (q + \xi)} \exp \left( ip \cdot X(w) + i \sum_i \zeta_i \cdot \partial^m_i X(w) + i \sum_i \lambda_i \cdot \partial^m_i X(w) \right) \left( -\frac{\alpha'}{2} \sum_i (q + \xi) \cdot \zeta_i (m_i - 1)! \right) \]

\[ \times \left[ \left( \frac{\alpha' p - q}{2} \right) \left( \frac{\alpha' p - q}{2} + \frac{n_i - 1}{n_i!} \right) \sum_i \frac{i q \cdot \partial^m_i X(w)}{k!(m_i - k - 1)!} \right] \]

\[ \times \left( \sum_i \alpha' m_i q \cdot \lambda_i \right) \mid \text{multilinear in } i \zeta_i, i \lambda_i \]

\[ \text{(B.8)} \]

6. (0:0:2) terms

\[ -\lim_{\xi \to 0} \frac{1}{2} h_{\mu\nu} \partial^\mu \xi \partial^\nu \xi \mid z - w \mid^{\alpha' p (q + \xi)} \frac{i(q + \xi) \cdot X(z)}{2} \]

\[ \times \exp \left( ip \cdot X(w) + i \sum_i \zeta_i \cdot \partial^m_i X(w) + i \sum_i \lambda_i \cdot \partial^m_i X(w) \right) : \]

\[ \times \left( \frac{\alpha' p - q}{2(z - w)} \right) \sum_i \frac{i q \cdot \partial^m_i X(w)}{k!(m_i - k - 1)!} \right] \]

\[ \times \left( \sum_i \alpha' m_i q \cdot \lambda_i \right) \mid \text{multilinear in } i \zeta_i, i \lambda_i \]

\[ \text{(B.9)} \]
In the last line the third term in the first or second brackets does not give the pole of $z - w$.

\[
- \lim_{\xi \to 0} \frac{1}{4} h_{\mu\nu} \partial^\mu \partial^\nu \frac{2\pi}{\alpha' (q + \xi)} \exp \left( i p \cdot X(w) + i \sum_i \zeta_i \cdot \partial^m_{\nu} X(w) + i \sum_i \lambda_i \cdot \partial^m_{\nu} X(\bar{w}) \right)
\]

\[
\times \exp \left( \frac{\alpha' p \cdot q}{2(z - w)} + \sum_i \frac{\alpha' m_i l q \cdot \zeta_i}{2(z - w)^{m_i + 1}} \right) \left( \frac{\alpha' p \cdot q}{2(z - w)} + \sum_i \frac{\alpha' n_i l q \cdot \lambda_i}{2(z - w)^{n_i + 1}} \right) \left( \frac{\alpha' p \cdot q}{2(z - w)} \right)^2 i q \cdot X(w) i q \cdot X(\bar{w})
\]

\[
\times \exp \left( \frac{\alpha' p \cdot q}{2(z - w)} + \sum_i \frac{\alpha' n_i l q \cdot \lambda_i}{2(z - w)^{n_i + 1}} \right) \left( \frac{\alpha' p \cdot q}{2(z - w)} \right)^2 i q \cdot X(w) i q \cdot X(\bar{w})
\]

\[
\times \exp \left( \frac{\alpha' p \cdot q}{2(z - w)} + \sum_i \frac{\alpha' n_i l q \cdot \lambda_i}{2(z - w)^{n_i + 1}} \right) \left( \frac{\alpha' p \cdot q}{2(z - w)} \right)^2 i q \cdot X(w) i q \cdot X(\bar{w})
\]

\[
+ 2 \left( \frac{\sum_{i} \alpha' m_i l q \cdot \zeta_i}{2} \right) \left( \frac{\sum_{i} \alpha' n_i l q \cdot \lambda_i}{2} \right) i q \cdot X(w) i q \cdot X(\bar{w})
\]

\[
\left( \frac{\alpha' p \cdot q}{2} \right)^2 \left( \sum_{i} \alpha' m_i l q \cdot \zeta_i \right) \left( \sum_{i} \alpha' n_i l q \cdot \lambda_i \right) i q \cdot X(w) i q \cdot X(\bar{w})
\]

(B.10)

- First we sum up the simple lines.

\[
- \lim_{\xi \to 0} \frac{1}{4} h_{\mu\nu} \partial^\mu \partial^\nu \frac{2\pi}{\alpha' (q + \xi)} \exp \left( i p \cdot X(w) + i \sum_i \zeta_i \cdot \partial^m_{\nu} X(w) + i \sum_i \lambda_i \cdot \partial^m_{\nu} X(\bar{w}) \right)
\]

\[
\times \exp \left( \frac{\alpha' p \cdot q}{2(z - w)} + \sum_i \frac{\alpha' m_i l q \cdot \zeta_i}{2(z - w)^{m_i + 1}} \right) \left( \frac{\alpha' p \cdot q}{2(z - w)} + \sum_i \frac{\alpha' n_i l q \cdot \lambda_i}{2(z - w)^{n_i + 1}} \right) \left( \frac{\alpha' p \cdot q}{2(z - w)} \right)^2 i q \cdot X(w) i q \cdot X(\bar{w})
\]

\[
\times \exp \left( \frac{\alpha' p \cdot q}{2(z - w)} + \sum_i \frac{\alpha' n_i l q \cdot \lambda_i}{2(z - w)^{n_i + 1}} \right) \left( \frac{\alpha' p \cdot q}{2(z - w)} \right)^2 i q \cdot X(w) i q \cdot X(\bar{w})
\]

\[
\times \exp \left( \frac{\alpha' p \cdot q}{2(z - w)} + \sum_i \frac{\alpha' n_i l q \cdot \lambda_i}{2(z - w)^{n_i + 1}} \right) \left( \frac{\alpha' p \cdot q}{2(z - w)} \right)^2 i q \cdot X(w) i q \cdot X(\bar{w})
\]

\[
+ 2 \left( \frac{\sum_{i} \alpha' m_i l q \cdot \zeta_i}{2} \right) \left( \frac{\sum_{i} \alpha' n_i l q \cdot \lambda_i}{2} \right) i q \cdot X(w) i q \cdot X(\bar{w})
\]

(B.11)
• Secondly we sum up the double lines.

\[-\lim_{\xi \to 0} \frac{1}{2} \frac{\partial^\mu}{\partial x^\mu} \frac{\partial^\nu}{\partial x^\nu} : \frac{2 \pi}{\alpha p \cdot (q + \xi)} \exp \left( ip \cdot X(w) + \sum_i \zeta_i \cdot \partial^{\mu_i} X(\bar{w}_i) + \sum_i \lambda_i \cdot \partial^{\nu_i} X(\bar{w}_i) \right) \left( \frac{\alpha_p \cdot q}{2} \right)^2 \]

\[\left[ \left( \frac{\alpha_p}{2} \sum_l (q + \xi) \cdot \lambda_i (n_i - 1)! \right) \left( \frac{i(q + \xi) \cdot \partial^{\mu_i} X(w)}{n_i !} \right) + \left( - \frac{\alpha_p}{2} \sum_l (q + \xi) \cdot \zeta_i (m_i - 1)! \right) \left( \frac{i(q + \xi) \cdot \partial^{\mu_i} X(w)}{m_i !} \right) \right] \text{multilinear in } \zeta_i, \lambda_i \]

\[= \lim_{\xi \to 0} \frac{1}{2} \frac{\partial^\mu}{\partial x^\mu} \frac{\partial^\nu}{\partial x^\nu} : 2 \pi \exp \left( ip \cdot X(w) + \sum_i \zeta_i \cdot \partial^{\mu_i} X(\bar{w}_i) + \sum_i \lambda_i \cdot \partial^{\nu_i} X(\bar{w}_i) \right) \left( \frac{\alpha_p \cdot q}{2} \right)^2 \]

\[\left[ \sum_l q \cdot \lambda_i q \cdot \partial^{\mu_i} X(w)(p \cdot \xi)^2 - \xi \cdot \lambda_i q \cdot \partial^{\mu_i} X(w)(p \cdot q - q \cdot \lambda_i q \cdot \partial^{\nu_i} X(w)(p \cdot q + \xi - \lambda_i q \cdot \partial^{\nu_i} X(w)(p \cdot q)^2 \right] \]

\[\cdot \left( 2m_i (p \cdot q)^3 \right) \text{multilinear in } \zeta_i, \lambda_i \]

\[= \frac{\pi \alpha^2 h_{ew}}{4 \alpha p \cdot q} \left( \eta^{ae} \eta^{bd} \eta^{ce} \eta^{df} - \eta^{ae} \eta^{bd} \eta^{de} \eta^{cf} - \eta^{ae} \eta^{bd} \eta^{de} \eta^{bf} + \eta^{ae} \eta^{cd} \eta^{de} \eta^{bf} \right) \]

\[\times \left( \sum_i \frac{i q_g q_i \zeta_i \partial^{\mu_i} X a p c P f}{m_i p \cdot q} + \sum_i \frac{i q_g q_i \lambda_i \partial^{\nu_i} X a p c P f}{n_i p \cdot q} \right) \text{multilinear in } \zeta_i, \lambda_i \]

\[= \frac{\pi \alpha^2 h_{ew}}{4 \alpha p \cdot q} \left( \eta^{ae} \eta^{ad} (S^{be}) \eta^{cf} (S^{de}) \right) \left( \frac{\sum_i \frac{i q_g q_i \zeta_i \partial^{\mu_i} X a p c P f}{m_i p \cdot q} + \sum_i \frac{i q_g q_i \lambda_i \partial^{\nu_i} X a p c P f}{n_i p \cdot q} \right) \text{multilinear in } \zeta_i, \lambda_i \]

\[= \frac{\pi \alpha^2 h_{ew}}{4 \alpha p \cdot q} \left( S^{be} \right) \left( S^{de} \right) \left( \frac{\sum_i \frac{i q_g q_i \zeta_i \partial^{\mu_i} X a p c P f}{m_i p \cdot q} + \sum_i \frac{i q_g q_i \lambda_i \partial^{\nu_i} X a p c P f}{n_i p \cdot q} \right) \text{multilinear in } \zeta_i, \lambda_i \]
Finally we sum up the wavy lines.

\[
\begin{align*}
- \lim_{\xi \to 0} \frac{1}{\varepsilon} h_{\mu
u} \partial_{\xi}^\mu \partial_{\xi}^\nu & : \frac{2\pi}{\alpha' p \cdot (q + \xi)} \exp \left( ip \cdot X(w) + i \sum_i \zeta_i \cdot \partial_{\xi}^\mu X(w) + i \sum_i \lambda_i \cdot \partial^\mu X(\bar{w}_i) \right) \\
\left[ \frac{1}{2} \left( -\frac{\alpha'}{2} \sum_i (n_i - 1)! (q + \xi) \cdot \lambda_i \right) \left( -\frac{\alpha'}{2} \sum_j (n_j - 1)! (q + \xi) \cdot \lambda_j \right) \right. & \left. \alpha' p \cdot q \cdot (q + \xi) \cdot \partial^{m_i + n_i} X \right] \\
+ \left[ \frac{1}{2} \left( -\frac{\alpha'}{2} \sum_i (m_i - 1)! (q + \xi) \cdot \zeta_i \right) \left( -\frac{\alpha'}{2} \sum_j (m_j - 1)! (q + \xi) \cdot \zeta_j \right) \right. & \left. \alpha' p \cdot q \cdot (q + \xi) \cdot \partial^{m_i + n_i} X \right] \\
+ \left( \frac{\alpha'}{2} \sum_i (q + \xi) \cdot \zeta_i (m_i - 1)! \right) \left( \frac{\alpha' p \cdot q}{2} \right) & \\
\times \left\{ \left( \frac{m_i - 1}{2} \sum_{k=0}^{m_i - 1} i q \cdot \partial^{m_i + k} X_i (q + \xi) \cdot \partial^k X(w) \right) + \left( \sum_{j \neq i} \alpha' m_i q \cdot \zeta_j \right) \frac{(q + \xi) \cdot \partial^{m_i + n_j} X(w)}{(m_i + m_j)} \right\} \\
+ \left( \frac{\alpha'}{2} \sum_i (q + \xi) \cdot \lambda_i (n_i - 1)! \right) \left( \frac{\alpha' p \cdot q}{2} \right) & \\
\times \left\{ \left( \frac{m_i - 1}{2} \sum_{k=0}^{m_i - 1} i q \cdot \partial^{m_i + k} X_i (q + \xi) \cdot \partial^k X(w) \right) + \left( \sum_{j \neq i} \alpha' n_i q \cdot \lambda_j \right) \frac{(q + \xi) \cdot \partial^{m_i + n_j} X(w)}{(n_i + n_j)} \right\} \\
+ \left( \frac{\alpha' p \cdot q}{2} \right) \left( \sum_i \alpha' m_i q \cdot \zeta_i \right) & \\
\times \left\{ \sum_{k=0}^{m_i - 1} i q \cdot \partial^{m_i + k} X_i (q + \xi) \cdot \partial^k X(w) \right\} \\
+ \left( \frac{\alpha' p \cdot q}{2} \right) \left( \sum_i \alpha' n_i q \cdot \lambda_i \right) & \\
\times \left\{ \sum_{k=0}^{n_i - 1} i q \cdot \partial^{m_i + k} X_i (q + \xi) \cdot \partial^k X(w) \right\} \\
\right]_{\text{multilinear in } \zeta_i, \lambda_i, \bar{w}_i}.
\end{align*}
\]

(B.13)

First we write the terms that include \(X\)’s without any derivative.

\[
\begin{align*}
- \lim_{\xi \to 0} \frac{1}{\varepsilon} h_{\mu\nu} \partial_{\xi}^\mu \partial_{\xi}^\nu & : \frac{2\pi}{\alpha' p \cdot (q + \xi)} \exp \left( ip \cdot X(w) + i \sum_i \zeta_i \cdot \partial_{\xi}^\mu X(w) + i \sum_i \lambda_i \cdot \partial^\mu X(\bar{w}_i) \right) \\
\left[ \left( \frac{1}{2} \left( -\frac{\alpha'}{2} \sum_i (q + \xi) \cdot \zeta_i (m_i - 1)! \right) \left( \frac{\alpha' p \cdot q}{2} \right) \right. & \left. \alpha' p \cdot q \cdot (q + \xi) \cdot \partial^m X(w) \right] \\
+ \left[ \left( \frac{1}{2} \left( -\frac{\alpha'}{2} \sum_i (q + \xi) \cdot \lambda_i (n_i - 1)! \right) \left( \frac{\alpha' p \cdot q}{2} \right) \right. & \left. \alpha' p \cdot q \cdot (q + \xi) \cdot \partial^n X(w) \right] \\
+ \left( \frac{\alpha' p \cdot q}{2} \right) \left( \sum_i \alpha' m_i q \cdot \zeta_i \right) & \\
\times \left\{ \sum_{k=0}^{m_i - 1} i q \cdot \partial^{m_i + k} X_i (q + \xi) \cdot \partial^k X(w) \right\} \\
+ \left( \frac{\alpha' p \cdot q}{2} \right) \left( \sum_i \alpha' n_i q \cdot \lambda_i \right) & \\
\times \left\{ \sum_{k=0}^{n_i - 1} i q \cdot \partial^{m_i + k} X_i (q + \xi) \cdot \partial^k X(w) \right\} \\
\right]_{\text{multilinear in } \zeta_i, \lambda_i, \bar{w}_i}.
\end{align*}
\]

(B.14)
Secondly we write the terms that change one prefactor to two.

\[- \lim_{\xi \to 0} \frac{1}{2} h_{\mu\nu} \partial_{\mu}^{\nu} \frac{2\pi}{\alpha' p \cdot (q + \xi)} \exp \left( i p \cdot X(w) + i \sum_{i} \zeta_i \cdot \partial_{\mu_i}^{\nu_i} X(w) + i \sum_{i} \lambda_i \cdot \partial_{\mu_i}^{\nu_i} X(\bar{w}_i) \right) \]

\[\left[ \left( -\frac{\alpha'}{2} \sum_{i} (q + \xi) \cdot \zeta_i (m_i - 1)! \right) \left( \alpha' p \cdot q \over 2 \right) \left( \sum_{k=1}^{m_i-1} \frac{1}{k!(m_i - k - 1)!} \right) \right]_{\text{multilinear in } \zeta_i, \lambda_i} \]

\[+ \left( -\frac{\alpha'}{2} \sum_{i} (q + \xi) \cdot \lambda_i (n_i - 1)! \right) \left( \alpha' p \cdot q \over 2 \right) \left( \sum_{k=1}^{n_i-1} \frac{1}{k!(n_i - k - 1)!} \right) \frac{1}{(m_i - k)!} \left[ \sum_{k=1}^{m_i-1} \frac{1}{k!(m_i - k - 1)!} \right]_{\text{multilinear in } \zeta_i, \lambda_i} \]

\[= \frac{\alpha' h_{\mu\nu}}{2} \partial_{\mu}^{\nu} \frac{2\pi}{\alpha' p \cdot (q + \xi)} \exp \left( i p \cdot X(w) + i \sum_{i} \zeta_i \cdot \partial_{\mu_i}^{\nu_i} X(w) + i \sum_{i} \lambda_i \cdot \partial_{\mu_i}^{\nu_i} X(\bar{w}_i) \right) \]

\[\times \left[ \left( \sum_{k=1}^{m_i-1} \frac{1}{k!(m_i - k - 1)!} \right) \left( i q \cdot \partial^{m_i-k} X(p \cdot q) \over 2 \right) \left( \sum_{k=1}^{n_i-1} \frac{1}{k!(n_i - k - 1)!} \right) \right]_{\text{multilinear in } \zeta_i, \lambda_i} \]

\[= - \lim_{\xi \to 0} \frac{1}{2} h_{\mu\nu} \partial_{\mu}^{\nu} \frac{2\pi}{\alpha' p \cdot (q + \xi)} \exp \left( i p \cdot X(w) + i \sum_{i} \zeta_i \cdot \partial_{\mu_i}^{\nu_i} X(w) + i \sum_{i} \lambda_i \cdot \partial_{\mu_i}^{\nu_i} X(\bar{w}_i) \right) \]

\[\times \left[ \left( \sum_{k=1}^{m_i-1} \frac{1}{k!(m_i - k - 1)!} \right) \left( i q \cdot \partial^{m_i-k} X(p \cdot q) \over 2 \right) \left( \sum_{k=1}^{n_i-1} \frac{1}{k!(n_i - k - 1)!} \right) \right]_{\text{multilinear in } \zeta_i, \lambda_i} \]

\[= - \lim_{\xi \to 0} \frac{1}{2} h_{\mu\nu} \partial_{\mu}^{\nu} \frac{2\pi}{\alpha' p \cdot (q + \xi)} \exp \left( i p \cdot X(w) + i \sum_{i} \zeta_i \cdot \partial_{\mu_i}^{\nu_i} X(w) + i \sum_{i} \lambda_i \cdot \partial_{\mu_i}^{\nu_i} X(\bar{w}_i) \right) \]

\[\times \left[ \left( \sum_{k=1}^{m_i-1} \frac{1}{k!(m_i - k - 1)!} \right) \left( i q \cdot \partial^{m_i-k} X(p \cdot q) \over 2 \right) \left( \sum_{k=1}^{n_i-1} \frac{1}{k!(n_i - k - 1)!} \right) \right]_{\text{multilinear in } \zeta_i, \lambda_i} \]
Finally we write the terms that change two prefactors to one.

\[
\begin{align*}
&= - \lim_{\xi \to 0} \frac{1}{2} h_{\mu \nu} \partial^\mu \partial^\nu \frac{2 \pi}{\alpha' p \cdot (q + \xi)} \exp \left( i p \cdot X(w) + i \sum_i \zeta_i \cdot \bar{\partial}^{m_i} X(w) + i \sum_i \lambda_i \cdot \bar{\partial}^{n_i} X(w) \right) \\
&\quad \left[ \sum_{i,j,p,q} \alpha'^2 (n_i - 1)! (n_j - 1)! \frac{\alpha' \cdot q}{4(n_i + n_j)} (q + \xi) \cdot \lambda_{ij} \left( \bar{\partial}^{m_i} X(w) \cdot \bar{\partial}^{n_i} X(w) \right) (i \leftrightarrow j) \right] \left[ \sum_{i,j,p,q} \alpha'^2 (m_i - 1)! (m_j - 1)! \frac{\alpha' \cdot q}{4(m_i + m_j)} (q + \xi) \cdot \zeta_{ij} \left( \bar{\partial}^{n_i} X(w) \cdot \bar{\partial}^{m_i} X(w) \right) (i \leftrightarrow j) \right] \\
&\quad = \frac{\pi \alpha'^2}{8 p \cdot q} h_{\mu \nu} \exp \left( i p \cdot X(w) + i \sum_i \zeta_i \cdot \bar{\partial}^{m_i} X(w) + i \sum_i \lambda_i \cdot \bar{\partial}^{n_i} X(w) \right) \\
&\quad \left[ \sum_{i,j,p,q} \alpha'^2 (n_i - 1)! (n_j - 1)! q_a q_b \partial^a X_d p_d p_f \partial^{m_i} X_e \partial^{n_j} X_h \left( S^{\rho e} \right)^{g h} (i \leftrightarrow j) \right] \\
&\quad \left[ \sum_{i,j,p,q} \alpha'^2 (m_i - 1)! (m_j - 1)! q_a q_b \partial^a X_d p_d p_f \partial^{n_j} X_e \partial^{m_i} X_h \left( S^{\rho e} \right)^{g h} (i \leftrightarrow j) \right] \text{multilinear in } \zeta_i, \lambda_i.
\end{align*}
\]

Thus we can summarize all the results as follows:

\[
\begin{align*}
&\frac{\pi \alpha' h_{\mu \nu}}{4 p \cdot q} \left[ q_a q_b \left( L^{\mu a} L^{\nu b} + 2 S^{\mu a} S^{\nu b} \right) + 2 q_a L^{\mu a} \left( q_b S^{\nu b} + q_b \bar{S}^{\nu b} \right) \right] \\
&+ \frac{\pi \alpha'^2 h_{\mu \nu}}{4 p \cdot q} \sum_i \left( S^{\mu a} \right)^{cd} (S^{\nu b})_{\rho j} \frac{q_a q_b \partial^a X_d p_d p_f}{m_i} \partial^{m_i} X \partial^{n_i} X \cdots \exp (i p \cdot X(w)) \\
&+ \frac{\pi \alpha'^2 h_{\mu \nu}}{4 p \cdot q} \sum_i \left( \eta^{\mu a} \eta^{\nu b} S^{\rho j} \right) \frac{q_a q_b \partial^a X_d p_d p_f}{m_i} \partial^{m_i} X \partial^{n_i} X \cdots \exp (i p \cdot X(w)) \\
&+ \frac{\pi \alpha' h_{\mu \nu}}{2 p \cdot q} \sum_{k=1}^{m_i - 1} \left( \frac{m_i - k}{k!(m_i - k - 1)!} \right) q_a q_b \partial^a X_d p_d p_f \partial^{m_i} X \partial^{n_i} X \cdots \exp (i p \cdot X(w)) \\
&+ \frac{\pi \alpha' h_{\mu \nu}}{2 p \cdot q} \sum_{k=1}^{n_i - 1} \left( \frac{n_i - k}{k!(n_i - k - 1)!} \right) q_a q_b \partial^a X_d p_d p_f \partial^{n_i} X \partial^{m_i} X \cdots \exp (i p \cdot X(w)) \\
&+ \frac{\pi \alpha'^2 h_{\mu \nu}}{8 p \cdot q} \sum_{i,j} \left( \frac{(n_i - 1)! (n_j - 1)! q_a q_b \partial^a X_d p_d p_f}{m_i + m_j} \partial^{m_i} X \partial^{n_j} X \cdots \exp (i p \cdot X(w)) \right) \text{multilinear in } \zeta_i, \lambda_i.
\end{align*}
\]
C Soft theorem for B field

As in the previous sections we derive the general formula for the soft B field theorem. The contraction between the B field and the hard vertex operator is as follows:

\[
\lim_{\xi, \omega \to 0} h_{\mu\nu} \partial_{\xi}^{\mu} \partial_{\omega}^{\nu} \partial_{z} \partial_{\bar{z}} : \exp \left( ip \cdot X(z) + i \xi \cdot X(z') + i \omega \cdot \partial X(z') \right) :
\]

\[
\times : \exp \left( \left( ip \cdot X(w) + i \sum \zeta_{i} \cdot \partial_{w}^{\mu} X(w) + i \sum \lambda_{i} \cdot \partial^{\nu i} X(\bar{w}) \right) \right) \text{ multilinear in } \zeta_{i}, \lambda_{i} \nabla_{i}
\]

\[
= \lim_{\xi, \omega \to 0} h_{\mu\nu} \partial_{\xi}^{\mu} \partial_{\omega}^{\nu} \partial_{z} \partial_{\bar{z}} : |z - w|^{\alpha^{\mu} \cdot \eta^{\nu}} |z' - w|^{\alpha'^{\mu} \cdot \xi^{\nu}} \exp \left( -\frac{\alpha' \cdot \eta \cdot \omega}{2(z' - w)} \right)
\]

\[
\times \exp \left( -\frac{\alpha'}{2} \sum \frac{(m_{i} - 1)! q \cdot \zeta_{i}}{(z - w)^{m_{i}}} - \frac{\alpha'}{2} \sum \frac{(m_{i} - 1)! q \cdot \lambda_{i}}{(z - w)^{m_{i}}} \right) \exp \left( -\frac{\alpha'}{2} \sum \frac{(m_{i} - 1)! \xi \cdot \zeta_{i}}{(z' - w)^{m_{i}}} - \frac{\alpha'}{2} \sum \frac{(m_{i} - 1)! \xi \cdot \lambda_{i}}{(z' - w)^{m_{i}}} \right)
\]

\[
\times \exp \left( \frac{\alpha'}{2} \sum \frac{n_{i} \omega \cdot \lambda_{i}}{(z' - w)^{n_{i}}} \right)
\]

\[
\times : \exp \left( i q \cdot X(z) + i \xi \cdot X(z') + i \omega \cdot \partial X(z') + ip \cdot X(w) + i \sum \zeta_{i} \cdot \partial_{w}^{\mu} X(w) + i \sum \lambda_{i} \cdot \partial^{\nu i} X(\bar{w}) \right) \text{ multilinear in } \zeta_{i}, \lambda_{i}
\]

\[
= \lim_{\xi \to 0} h_{\mu\nu} \partial_{\xi}^{\mu} : |z - w|^{\alpha^{\mu} \cdot (q + \xi)}
\]

\[
\times \exp \left( -\frac{\alpha'}{2} \sum \frac{(m_{i} - 1)! (q + \xi) \cdot \zeta_{i}}{(z - w)^{m_{i}}} - \frac{\alpha'}{2} \sum \frac{(m_{i} - 1)! (q + \xi) \cdot \lambda_{i}}{(z - w)^{m_{i}}} \right)
\]

\[
\times \left( i (q + \xi) \cdot X(z) + ip \cdot X(w) + i \sum \zeta_{i} \cdot \partial_{w}^{\mu} X(w) + i \sum \lambda_{i} \cdot \partial^{\nu i} X(\bar{w}) \right)
\]

\[
\times \left( \frac{\alpha' \cdot q}{2(z - w)} + \sum \frac{\alpha' m_{i} q \cdot \zeta_{i}}{2(z - w)^{m_{i} + 1}} + iq \cdot \partial X(z) \right) \left( -\frac{\alpha' \cdot q}{2(z - w)} + \frac{\alpha' n_{i} \lambda_{i}}{2(z - w)^{n_{i} + 1}} + i \partial X^{\nu}(\bar{z}) \right)
\]

\[\text{ multilinear in } \zeta_{i}, \lambda_{i} \]

\[
\text{(C.1)}
\]

We look at the powers of \(z - w\). Because the last line in eq. [C.1] is first-order in \(q\), it does not affect through subleading order unless the \(z\) integration yields the singular behavior in \(q\). Therefore we take only the coefficients of \(|z - w|^{\alpha^{\mu} \cdot (q + \xi) - 2}\).

1. The 0-th order terms in \((q + \xi)\)

\[
\lim_{\xi \to 0} h_{\mu\nu} \partial_{\xi}^{\mu} |z - w|^{\alpha^{\mu} \cdot (q + \xi)} \exp \left( ip \cdot X(w) + i \sum \zeta_{i} \cdot \partial_{w}^{\mu} X(w) + i \sum \lambda_{i} \cdot \partial^{\nu i} X(\bar{w}) \right) :
\]

\[
\times \left( \frac{\alpha' \cdot q}{2(z - w)} + \sum \frac{\alpha' m_{i} q \cdot \zeta_{i}}{2(z - w)^{m_{i} + 1}} + iq \cdot \partial X(z) \right) \left( -\frac{\alpha' \cdot q}{2(z - w)} + \frac{\alpha' n_{i} \lambda_{i}}{2(z - w)^{n_{i} + 1}} + i \partial X^{\nu}(\bar{z}) \right)
\]

\[\text{ multilinear in } \zeta_{i}, \lambda_{i} \]

\[\text{(C.2)}
\]

Only the product of the first terms in each bracket yields the factor \(|z - w|^{-2}\).

\[
h_{\mu\nu} \frac{-2 \pi p^{\nu}}{\alpha'(p, q)^{2}} \exp \left( ip \cdot X(w) + i \sum \zeta_{i} \cdot \partial_{w}^{\mu} X(w) + i \sum \lambda_{i} \cdot \partial^{\nu i} X(\bar{w}) \right) \frac{-\alpha'^{2} p \cdot q^{\nu}}{4} \text{ multilinear in } \zeta_{i}, \lambda_{i} \]

[\text{(C.3)}

We have used the antisymmetry of the tensor \(h_{\mu\nu}\). The leading B field soft theorem does not exist.

2. The first order terms in \((q + \xi)\)

(a) \((0,0;1)\) terms

\[
\lim_{\xi \to 0} h_{\mu\nu} \partial_{\xi}^{\mu} |z - w|^{\alpha^{\mu} \cdot (q + \xi)} \exp \left( ip \cdot X(w) + i \sum \zeta_{i} \cdot \partial_{w}^{\mu} X(w) + i \sum \lambda_{i} \cdot \partial^{\nu i} X(\bar{w}) \right) :
\]

\[
\times i(q + \xi) \cdot X(z) \left( \frac{i \alpha' \cdot q}{2(z - w)} + \sum \frac{i \alpha' m_{i} q \cdot \zeta_{i}}{2(z - w)^{m_{i} + 1}} + iq \cdot \partial X(z) \right) \left( -\frac{i \alpha' \cdot q}{2(z - w)} + \frac{i \alpha' n_{i} \lambda_{i}}{2(z - w)^{n_{i} + 1}} + i \partial X^{\nu}(\bar{z}) \right)
\]

\[\text{ multilinear in } \zeta_{i}, \lambda_{i} \]

[\text{(C.4)}

(23)
We write only the terms that have the factor \(|z - w|^{\alpha' p \cdot (q + \xi) - \frac{2}{m} - 2}\).

\[
\lim_{\xi \to 0} h_{m\nu} e^{2\pi i (q + \xi)} \frac{2\pi}{\alpha' p \cdot (q + \xi)} \exp \left( i p \cdot X(w) + i \sum_{i} \zeta_i \cdot \partial_{w}^{m_i} X(w) + i \sum_{i} \lambda_i \cdot \partial^{m_i} X(\bar{w}_i) \right) \\
\times \left( \frac{-\alpha'}{2} \sum_{i} (m_i - 1)! (q + \xi) \cdot \zeta_i \right) \frac{\alpha' p \cdot q}{2 (z - w)} + \sum_{i} \frac{\alpha' m_i q \cdot \zeta_i}{2 (z - w)^{m_i + 1}} + i q \cdot \partial X(z) \\
\times \left( \frac{-\alpha' p^{\nu}}{2 (z - w)} + \frac{\alpha' n_i ! \lambda_i^{\nu}}{2 (z - w)^{n_i + 1}} + i \partial^{\nu} X(z) \right) \middle|_{\text{multilinear in } \zeta_i, \lambda_i} \tag{C.6}
\]

We write only the terms that have the factor \(|z - w|^{\alpha' p \cdot (q + \xi) - \frac{2}{m} + 2}\).

\[
\lim_{\xi \to 0} h_{m\nu} e^{2\pi i (q + \xi)} \frac{2\pi}{\alpha' p \cdot (q + \xi)} \exp \left( i p \cdot X(w) + i \sum_{i} \zeta_i \cdot \partial_{w}^{m_i} X(w) + i \sum_{i} \lambda_i \cdot \partial^{m_i} X(\bar{w}_i) \right) \\
\times \left( \frac{-\alpha'}{2} \sum_{i} (m_i - 1)! (q + \xi) \cdot \zeta_i \right) \frac{\alpha' p \cdot q}{2 (z - w)} + \sum_{i} \frac{\alpha' m_i q \cdot \zeta_i}{2 (z - w)^{m_i + 1}} + i q \cdot \partial X(z) \\
\times \left( \frac{-\alpha' p^{\nu}}{2 (z - w)} + \frac{\alpha' n_i ! \lambda_i^{\nu}}{2 (z - w)^{n_i + 1}} + i \partial^{\nu} X(z) \right) \middle|_{\text{multilinear in } \zeta_i, \lambda_i} \tag{C.7}
\]

(c) \((0,1;0)\) terms

\[
\lim_{\xi \to 0} h_{m\nu} e^{2\pi i (q + \xi)} \frac{2\pi}{\alpha' p \cdot (q + \xi)} \exp \left( i p \cdot X(w) + i \sum_{i} \zeta_i \cdot \partial_{w}^{m_i} X(w) + i \sum_{i} \lambda_i \cdot \partial^{m_i} X(\bar{w}_i) \right) \\
\times \left( \frac{-\alpha'}{2} \sum_{i} (n_i - 1)! (q + \xi) \cdot \lambda_i \right) \frac{\alpha' p \cdot q}{2 (z - w)} + \sum_{i} \frac{\alpha' m_i q \cdot \zeta_i}{2 (z - w)^{m_i + 1}} + i q \cdot \partial X(z) \\
\times \left( \frac{-\alpha' p^{\nu}}{2 (z - w)} + \frac{\alpha' n_i ! \lambda_i^{\nu}}{2 (z - w)^{n_i + 1}} + i \partial^{\nu} X(z) \right) \middle|_{\text{multilinear in } \zeta_i, \lambda_i} \tag{C.8}
\]
We write only the terms that have the factor \( |z - w|^\alpha p \cdot (q \cdot \bar{\xi})^{-2} \).

\[
\lim_{\xi \to 0} h_{\mu \nu} \frac{2\pi}{\alpha \cdot p \cdot (q \cdot \bar{\xi})} \exp \left( i p \cdot X(w) + i \sum_i \zeta_i \cdot \partial_\mu^{\eta_i} X(w) + i \sum_i \lambda_i \cdot \bar{\partial}_\nu^{\eta_i} X(\bar{w}_i) \right)
\times \left( -\frac{\alpha}{2} \sum_i (q \cdot \bar{\xi}) \cdot \lambda_i \right) \frac{\alpha \cdot p \cdot q \cdot \bar{\partial}_\nu X(\bar{w})}{2} \bigg|_{\text{multilinear in } i\zeta_i, i\lambda_i}
\]

\[
= \frac{\pi \alpha^2 h_{\mu \nu} \bar{\partial}_\nu X(\bar{w})}{2} \sum_i \left( -\lambda_i^2 + q \cdot \bar{\lambda}_i p^\nu \right) \exp \left( i p \cdot X(w) + i \sum_i \zeta_i \cdot \partial_\mu^{\eta_i} X(w) + i \sum_i \lambda_i \cdot \bar{\partial}_\nu^{\eta_i} X(\bar{w}_i) \right) \bigg|_{\text{multilinear in } i\zeta_i, i\lambda_i}.
\]

By summing up these results, we obtain

\[
\frac{\pi \alpha^2 h_{\mu \nu}}{2} \left[ - \frac{i q_\nu \delta_{\mu a} p^\nu}{p \cdot q} \sum_i + \frac{i p^\nu q_\nu \lambda_i c \bar{\partial}_\nu^{\eta_i} X_d(S^\mu \nu)_{cd}}{p \cdot q} \sum_i + \frac{i p^\nu q_\nu \zeta_i c \bar{\partial}_\nu^{\eta_i} X_d(S^\mu \nu)_{cd}}{p \cdot q} \sum_i + i(S^\mu \nu)_{cd} \lambda_i c \bar{\partial}_\nu^{\eta_i} X_d \right]
\times \exp \left( i p \cdot X(w) + i \sum_i \zeta_i \cdot \partial_\mu^{\eta_i} X(w) + i \sum_i \lambda_i \cdot \bar{\partial}_\nu^{\eta_i} X(\bar{w}_i) \right) \bigg|_{\text{multilinear in } i\zeta_i, i\lambda_i}.
\]

On the other hand, we can rewrite the vertex in the alternative form of eq.(4.26)

\[
: \bar{\partial} X^a(z) \partial X^\nu \exp (i q \cdot X(z, \bar{z})) : = \bar{\partial} : \partial X^a(z, \bar{z}) X^\nu(z) \exp (i q \cdot X(z, \bar{z})) : - : X^\nu(z, \bar{z}) \partial X^a(z, \bar{z}) \bar{\partial}_z \exp (i q \cdot X(z, \bar{z})) : .
\]

This contribution is given by changing \( \mu, \lambda_i, \bar{\partial}_\nu^{\eta_i} \) in the square bracket in eq.(C.10)

\[
\frac{\pi \alpha^2 h_{\mu \nu}}{2} \left[ - \frac{i q_\nu \delta_{\mu a} p^\nu}{p \cdot q} \sum_i + \frac{i p^\nu q_\nu \lambda_i c \bar{\partial}_\nu^{\eta_i} X_d(S^\mu \nu)_{cd}}{p \cdot q} \sum_i + \frac{i p^\nu q_\nu \zeta_i c \bar{\partial}_\nu^{\eta_i} X_d(S^\mu \nu)_{cd}}{p \cdot q} \sum_i + i(S^\mu \nu)_{cd} \lambda_i c \bar{\partial}_\nu^{\eta_i} X_d \right]
\times \exp \left( i p \cdot X(w) + i \sum_i \zeta_i \cdot \partial_\mu^{\eta_i} X(w) + i \sum_i \lambda_i \cdot \bar{\partial}_\nu^{\eta_i} X(\bar{w}_i) \right) \bigg|_{\text{multilinear in } i\zeta_i, i\lambda_i}.
\]

Thus averaging these two results, eq.(C.10) and eq.(C.12), we obtain

\[
\frac{\pi \alpha^2 h_{\mu \nu}}{2} \left[ \sum_i + \frac{i p^\nu q_\nu \lambda_i c \bar{\partial}_\nu^{\eta_i} X_d(S^\mu \nu)_{cd} - (\mu \leftrightarrow \nu)}{p \cdot q} \sum_i + \frac{i p^\nu q_\nu \zeta_i c \bar{\partial}_\nu^{\eta_i} X_d(S^\mu \nu)_{cd} - (\mu \leftrightarrow \nu)}{p \cdot q} \sum_i + i(S^\mu \nu)_{cd} \lambda_i c \bar{\partial}_\nu^{\eta_i} X_d \right]
\times \exp \left( i p \cdot X(w) + i \sum_i \zeta_i \cdot \partial_\mu^{\eta_i} X(w) + i \sum_i \lambda_i \cdot \bar{\partial}_\nu^{\eta_i} X(\bar{w}_i) \right) \bigg|_{\text{multilinear in } i\zeta_i, i\lambda_i}
\]

\[
= \frac{i \pi \alpha^2 h_{\mu \nu}}{2} \left( p^\nu q_\nu S^\mu \nu - (\mu \leftrightarrow \nu) \right) \sum_i + \frac{i p^\nu q_\nu S^\mu \nu - (\mu \leftrightarrow \nu)}{p \cdot q} + \frac{1}{2} (S^\mu \nu - S^\nu \mu) \times \partial_\nu^{\eta_i} X^\mu \cdots \partial_\nu^{\eta_i} X^\mu \cdots \exp (i p \cdot X) : .
\]

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