BALANCED RATIONAL CURVES AND RIGID CURVES OF ALL GENERA
ON SOME CALABI-YAU COMPLETE INTERSECTIONS

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ABSTRACT. Let $X$ be either a general hypersurface of degree $n+1$ in $\mathbb{P}^n$ or a general $(2,n)$ complete intersection in $\mathbb{P}^{n+1}$, $n \geq 4$. We construct balanced rational curves on $X$ of all high enough degrees. If $n = 3$ or $g = 1$, we construct rigid curves of genus $g$ on $X$ of all high enough degrees. As an application we construct some rigid bundles on Calabi-Yau threefolds.

A curve $C$ of genus $g$ on a variety $X$ of dimension $m$ is said to be rigid (or sometimes isolated) if its normal bundle $N = N_{C/X}$ has $H^0(N) = 0$, i.e. $C$ does not move on $X$, even infinitesimally. Now in general one has

$$\chi(N) = C.(-K_X) + (m - 3)(1 - g).$$

Therefore, when $X$ is a Calabi-Yau ($K_X$ trivial) 3-fold (or for that matter, a Calabi-Yau $m$-fold and $g = 1$), one always has $\chi(N) = 0$. So in that case rigidity is equivalent to regularity, i.e. vanishing of $H^i(N)$, and also to the pair $(X,C)$ having unobstructed, non-oversize deformations. One ‘expects’ any curve on such $X$ to be rigid: e.g. Clemens has famously conjectured that on a general quintic threefold in $\mathbb{P}^4$, all rational curves are rigid. However the most obvious curves one can construct are often not rigid, so it is a nontrivial problem, partially motivated by Physics [12], [6], to construct rigid, or more generally regular curves (and relatedly, vector bundles) on Calabi-Yau manifolds, in particular those that are complete intersections in projective space, the so-called CICY manifolds, which anyhow contain a lot of non-rigid curves.

Work on this problem goes back to Clemens [1], who constructed infinitely many rigid rational curve on the general quintic in $\mathbb{P}^4$. This was extended to rational curves on all CICY threefolds (CICY3fs) by Ekedahl, Johnsen and Sommervoll [3], and subsequently, based on Clemens’s method, by Kley [4] to any
CICY3f \( X \) and curves of sufficiently low genus \( g \) (depending on \( X \), e.g. \( g < 35 \) for the quintic) and sufficiently high degree (depending on the genus). For the quintic threefold, any \( g \geq 0 \) and large \( d \), Zahariuc [15] constructs a degree- \( d \) map from a smooth curve of genus \( g \) to \( X \) which is set-theoretically isolated (it is not proved the map is an embedding or infinitesimally rigid). Other existence and nonexistence results for curves of small degree (relative to the genus or otherwise) were obtained by Knutsen [5], Clemens and Kley [2] and Yu [13], [14]. I am not aware of results in the literature for curves on higher-dimensional CICYs. Some results on vector bundles are in [6], [11] and references therein.

On the other hand results on balanced rational and irrational curves on Fano hypersurfaces were obtained in [8] and [9].

The purpose of this paper is to enlarge the known collection of ‘good’ curves and bundles on CICY manifolds by constructing on some \( m \)-dimensional CICYs curves \( C \) of all large enough degrees with the following properties:

- \( C \) rigid of any genus, \( m = 3 \); or
- \( C \) rigid of genus 1, \( m \geq 3 \); or
- \( C \) rational and balanced, \( m \geq 3 \).

Precisely, we will prove the following.

**Theorem 1.** Let \( X \) be either a general hypersurface of degree \( n + 1 \) in \( \mathbb{P}^n \) or a general \((2, n)\) complete intersection in \( \mathbb{P}^{n+1} \), \( n \geq 4 \), and let \( e \) be an integer. Then

(i) if \( e \geq 2n - 1 \), there exist smooth rational curves of degree \( e \) on \( X \) with normal bundle \((n - 4)O \oplus 2O(-1)\);

(ii) let \( g \geq 1 \) be an integer and assume \( e \geq 2n(n - 1)(g + 1) + 1 \); if either \( g = 1 \) or \( n = 4 \), then there exists a smooth rigid curve of genus \( g \) and degree \( e \) on \( X \).

As noted above, the assumption \( g = 1 \) or \( n = 4 \) implies \( \chi(N) = 0 \), so in that case rigidity is equivalent to regularity. For \( g = 0 \) we have \( \chi(N) = n - 4 \) so the curve cannot be rigid if \( n > 4 \).

The case \( g = 1, n = 4 \) (already done by Kley [4]) has the following application to rigid vector bundles. See §3 for the proof.

**Corollary 2.** If \( X \) is either a general quintic threefold in \( \mathbb{P}^4 \) or a general \((2, 4)\) complete intersection in \( \mathbb{P}^5 \), then for every integer \( e \geq 48 \) \( X \) carries a rigid, indecomposable, semistable rank-2 vector bundle \( E \) with \( c_1(E) = 0, c_2(E) = e \).

On the \((2, 4)\) complete intersection, two other rigid rank-2 bundles (with odd \( c_1 \)) were constructed by Richard Thomas [11], who also constructs numerous other rigid examples on K3 fibrations. He has also constructed in [10] an example of a curve and a bundle on a CY3f of special moduli that are set-theoretically isolated but not (infinitesimally) rigid.
The idea of the proof of the Theorem is to use a suitable degeneration of $X$ to a reducible normal-crossing variety $X_1 \cup X_2$. In the case where $X$ is a hypersurface we use a so-called quasi-cone degeneration where $X_1$ is ‘resolved quasi-cone’ i.e. the blowup at a point $q$ of a quintic with multiplicity $n$ at $q$, and $X_2$ is a degree-$n$ hypersurface in $\mathbb{P}^n$. In the case where $X$ is a $(2,n)$ complete intersection, $X_0$ is a the complete intersection of a degree-$n$ hypersurface with a reducible quadric, so that $X_1, X_2$ are both degree-$n$ hypersurfaces in hyperplanes $H_1, H_2 \cong \mathbb{P}^{n+1}$ such that $X_1 \cap H_1 \cap H_2 = X_2 \cap H_1 \cap H_2$. We use results from [8] and [9] about existence of curves on a degree-$n$ hypersurface with ‘good’ normal bundle, together with a new notion, introduced in §1, of relative regularity of a curve or bundle, which is analogous to a special case of the notion of ‘ultra-balance’ introduced in [9] but where the ‘test points’ are not general but lie on a divisor in a specified system.

The question of existence of good curves of genus $g > 0$ and high degree on other CICY types of any dimension, or of rigid curves of genus $> 1$ on any CICYs of dimension $> 3$ (where $\chi(N) < 0$) remains open.

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1. RELATIVELY(REGULAR BUNDLES AND CURVES

In this paper we work over $\mathbb{C}$.

The purpose of this section is to study a property of vector bundles which, when applied to normal bundles, is helpful in studying the normal bundle to a union of curves.

Let $C$ be a smooth curve, $0 \neq V \subset H^0(L)$ be a linear system on $C$ and $D \in V$ a reduced member.

**Definition 3.** (i) A vector bundle $E$ on $C$ is said to be regular relative to $D$ if it is regular, i.e. $H^1(E) = 0$ and, for any subset $D_1 \subset D$ and a general quotient $U$ of $E|_{D_1}$ locally of rank 1, the composite map

$$\rho_U : H^0(E) \to E|_{D_1} \to U$$

has maximal rank. $E$ is said to be regular relative to $V$ if it is regular relative to some divisor (or equivalently, a general divisor) $D \in V$.

(ii) A curve $C$ on a variety $X$ endowed with a linear system $V$ is said to be regular relative to $V$ if the corresponding property holds for its normal bundle $N = N_{C/X}$ and the restricted system $V|_C$.

More explicitly, Condition (i) means that for any distinct $p_1, ..., p_k \in D$ and respective general 1-dimensional quotients $U_1, ..., U_k$ of $E|_{p_1}, ..., E|_{p_k}$, the natural
map $H^0(E) \to \bigoplus_{i=1}^k U_i$ has maximal rank. This notion is essentially meaningless in genus 0:

**Lemma 4.** If $D$ is any reduced divisor on $\mathbb{P}^1$ then any regular vector bundle is regular relative to $D$.

**Proof.** We use the above notation. The proof is by induction on $h^0(E)$ which may be assumed $> 0$. We may also assume $D_1 = D$ is nontrivial and write $D = D' + p$. Let $L \subset E$ be a line subbundle of maximal degree which we may assume surjects to $U_p$ which is 1-dimensional. Then letting $E' \subset E$ denote the kernel of $E \to U_p$, we get an exact diagram

$$0 \to H^0(E') \to H^0(E) \to \mathbb{C}_p \to 0$$

As the right vertical arrow is an isomorphism and the left vertical arrow has maximal rank by induction, it is easy to see that the middle vertical arrow likewise has maximal rank. □

In the higher-genus case relative regularity seems related to ultra-balancedness, but there is no implication either way. Relative regularity is stronger in that the support of the quotient is a general divisor in $V$ which may not be a general divisor on $C$; it is weaker in that the quotient must have local rank 1.

The main result of this section is

**Proposition 5.** For $X = \mathbb{P}^n, n \geq 3$ (resp. $X$ a general hypersurface of degree $n$ in $\mathbb{P}^n, n \geq 4$) and $g \geq 0$, there exists a curve of genus $g$ and degree $e \geq 2(g + 1)n$ (resp. $e \geq 2(g + 1)n(n - 1)$) in $X$ that is regular relative to $|-K_X|$.

**Proof.** Case 1: $X = \mathbb{P}^n$.

We follow closely the proof of Theorem 29 in [9], using induction on the genus $g$. As the case $g = 0$ is automatic we begin with the case $g = 1$. Thus we consider a fang of type $\ell$

$$P_0 = P_1 \cup_{\mathbb{P}^n \times \mathbb{P}^{n-1-\ell}} P_2$$

where $P_1$ resp $P_2$ is the blowup of $\mathbb{P}^n$ in $\mathbb{P}^\ell$ resp. $\mathbb{P}^{n-1-\ell}$, with common exceptional divisor $Y = \mathbb{P}^\ell \times \mathbb{P}^{n-1-\ell}$. As $\mathbb{P}^n$ degenerates to $P_0$, hyperplanes have different types of limits depending on the dimension of the limiting intersection with $\mathbb{P}^\ell$. At one extreme, if the limiting intersection is transverse, the limit in $P_0$ will have the form $H_1' \cup H_2'$ where $H_1'$ is the birational transform of a hyperplane in $\mathbb{P}^n$ transverse
to \(\mathbb{P}^\ell\), while \(H'_2\) is the birational transform of a hyperplane containing \(\mathbb{P}^{n-1-\ell}\). This is the type we will use below.

Let 
\[
C_0 = C_1 \cup_{p,q} C_2 \subset P_0
\]
be a genus-1 curve as in loc. cit. with \(C_i\) rational and set \(e_i = \deg(C_i), i = 1, 2\) so \(e = e_1 + e_2 - 2\). Let \(N_0 = N_{C_0/P_0}, N_i = N_{C_i/P_2}\). Then
\[
\chi(N_0) = (e_1 + e_2 - 2)(n + 1),
\]
\[
\chi(N_1) = e_1(n + 1) + (n - 3) - 2(n - \ell - 1), \chi(N_2) = e_2(n + 1) + (n - 3) - 2\ell.
\]
Now let \(\ell = [(n-1)/2]\) which makes \(\chi(N_1) \leq e_1(n + 1)\). Let
\[
H'_{1,1} \cup H'_{2,1}, \ldots, H'_{1,n+1} \cup H'_{2,n+1}
\]
be \(n + 1\) general hyperplane limits as above, and let \(D\) consist of \(s = \chi(N_1)\) many points on \(C_1 \cap (H'_{1,1} \cup \ldots \cup H'_{1,n+1})\) plus \(\ell - s\) many points on \(C_2 \cap (H'_{2,1} \cup \ldots \cup H'_{2,n+1})\), and let \(U\) be a general, locally rank \(\leq 1\) quotient of \(N_D\) and let \(N'\) be the kernel of \(N \to U\). To prove \(H^0(N) \to U\) has maximal rank we may assume \(U\) has length \(\chi(N_1)\), i.e. rank exactly 1 at each point of \(D\). Then we must prove \(H^0(N') = 0\). Now because \(N_1, N_2\) are balanced we have \(H^0(N'|_{C_1}) = 0\) and \(H^0(N'|_{C_2}(-p-q)) = 0\), hence \(H^0(N') = 0\). Thus proves that \(C_0\) is regular relative to the limit of \([(n+1)H]\) hence is smoothing in \(\mathbb{P}^n\) is regular relative to \([(n+1)H]\). This proves our assertion in genus 1.

In the general case we use induction on the genus based on a fan (i.e. fang of type \(\ell = 0\)) degeneration
\[
P_0 = P_1 \cup_E P_2, P_1 = B_q \mathbb{P}^n, P_2 = \mathbb{P}^n
\]
with \(E \subset \mathbb{P}_1\) the exceptional divisor and \(E \subset \mathbb{P}_2\) a hyperplane. We use a limit anticanonical divisor that is a union of limit hyperplanes of the form
\[
D = \bigcup_{i=1}^{n+1} H'_i, H'_i = H'_{i,1} \cup H'_{2,i}
\]
with each \(H'_{1,i} \subset P_1\) the birational transform of a hyperplane through \(q\) and \(H'_{2,i} \subset P_2\) a hyperplane with \(H'_{1,i}.E = H'_{2,i}.E\). This is the opposite extreme of hyperplane limit types from the one used above. We consider a lci curve
\[
C_0 = C_1 \cup C_2
\]
where \(C_2 \subset P_2\) a \(|-K|\)-regular curve of genus \(g-1\) and degree \(e-1\) and
\[
C_1 = C_{1,1} \cup \bigcup_{i=1}^{e-3} L_i
\]
consists of the birational transforms of a plane cubic nodal at \(q\) (so that \(C_{1,1}E = 2\)) plus lines through \(q\). Then an argument similar to the above but simpler with \(H'_i, L_j = 0, \forall i, j\) shows that, for a locally-rank-1 quotient \(U\) of \(N_0|_D\) the map \(H^0(N_0) \to U\) is an isomorphism as required. This concludes the proof in the case \(X = \mathbb{P}^n\).

**Case 2:** \(X \subset \mathbb{P}^n\) of degree \(n\).

We use the usual ‘quasi-cone’ degeneration as in [9], with the same notation:

\[X_0 = X_1 \cup_F X_2 \subset P_1 \cup P_2\]

with \(X_1\) the blow-up at the singular point of a general hypersurface of degree \(n\) in \(\mathbb{P}^a\) with a point \(q\) of multiplicity \(n - 1\), and \(X_2\) a hypersurface of degree \(n - 1\). Projection from \(q\) realizes \(X_1\) as the blow-up of \(\mathbb{P}^{n-1}\) in a \((n-1, n)\) complete intersection \(Y\). As in loc. cit. we consider a curve

\[C_0 = C_1 \cup C_2\]

with \(C_1 \subset X_1\) the birational transform of a curve \(C_1'\) of degree \(e_1\) and genus \(g\) in \(\mathbb{P}^{a-1}\) that is regular with respect to the anticanonical \(|-K_{\mathbb{P}^{a-1}}|\), and \(C_2\) a disjoint union of lines with trivial normal bundle. Here as in loc. cit. \(e = e_1n - a, a \leq n\). Now a limit anticanonical divisor on \(X_0\) has the form \(F'_n\) which is the birational transform of a hypersurface of degree \(n\) in \(\mathbb{P}^{a-1}\) containing \(Y\) while \(N_{C_1/X_1}\) is a general corank-1 modification of \(N_{C_1'/\mathbb{P}^{a-1}}\) at the \(a\) points \(\{p_1, ..., p_a\} = C_1' \cap Y\). From this is is easy to see that \(C_1\) is regular relative to the divisor \(F'_n, C_1\) and hence that \(C_0\) is regular relative to \(F'_n, C_0\). Therefore as \(C_0\) smooths to a curve \(C\) on a general degree-\(n\) hypersurface \(X\), \(C\) is regular relative to \(|-K_X|\).

\[\square\]

## 2. Lines and Conics on Some Hypersurfaces

In this section we will study some rational curves that will serve as ‘tails’ in the construction of good curves as in Theorem 1. First, an easy remark about their normal bundle:

**Lemma 6.** On a general hypersurface of degree \(n\) in \(\mathbb{P}^n\) there exists a line with normal bundle \((n - 3)\mathcal{O} \oplus \mathcal{O}(-1)\) and a conic with normal bundle \((n - 2)\mathcal{O}\).

**Proof.** Line case: Let \(L\) be the line \(V(x_2, ..., x_n)\) in \(\mathbb{P}^n\). A hypersurface \(X\) of degree \(n\) containing \(L\) has equation \(\sum_{i=2}^{n} x_i f_i(x_0, x_1), \deg(f_i) = n - 1\). The normal sequence reads

\[0 \to N_{L/X} \to (n - 1)\mathcal{O}_L(1) \to \mathcal{O}_L(n) \to 0\]

where the right map is \((f_2, ..., f_n)\). Since this map is general, it follows e.g. by [8], Lemma 26 (or by the argument below in the conic case) that the kernel is
balanced, i.e. \( N_{L/X} \simeq (n - 3)O \oplus O(-1) \). Since this bundle is regular, \( L \) moves with \( X \).

In the conic case the normal sequence reads

\[
0 \to N_{C/X} \to O(4) \oplus (n - 2)O(2) \to O(2n) \to 0
\]

where the right map is general. The middle term is not balanced, nevertheless the kernel must be \((n - 2)O\) by openness of balancedness, because there exists a fibrewise injection \((n - 2)O \to O(4) \oplus (n - 2)O(2)\) and, counting degrees, the cokernel must be \(O(2n)\). □

Let \( P \) denote the blow-up of \( \mathbb{P}^n \) at a point \( q \). By a *resolved quasi-cone* in \( P \) we mean the birational transform \( X \subset P \) of a quasi-cone, i.e. a hypersurface \( \tilde{X} \) of degree \( d \) in \( \mathbb{P}^n \) having multiplicity \( d - 1 \) at \( q \), the quasi-vertex. Projection from \( q \) realizes \( X \) as the blow-up of \( \mathbb{P}^3 \) in a \((d, d-1)\) complete intersection curve \( Y \) so that the exceptional divisor corresponds to the unique degree-\((d - 1)\) hypersurface containing \( Y \), while hyperplane sections from \( \mathbb{P}^n \) correspond to hypersurfaces of degree \( d \) containing \( Y \). In particular, taking \( d = n + 1 \), \((n - 1)\)-secant lines to \( Y \) in \( \mathbb{P}^{n-1} \) correspond to conics in \( \tilde{X} \) through the quasi-vertex \( q \).

**Lemma 7.** Let \( X \) be either (a) a general degree-\( n \) hypersurface in \( \mathbb{P}^n \), \( n \geq 4 \) or (b) a general resolved degree-\((n+1)\) quasi-cone in \( P \); let \( L \subset X \) be either (a) a general line or (b) the transform of a general conic through the quasi-vertex; let \( Z \) be either (a) \( Z = X \cap H \) a general hyperplane section from \( \mathbb{P}^n \) or (b) the exceptional divisor; and let \( p = Z.L \) Then:

(i) We have \( N := N_{L/X} = (n - 3)O \oplus O(-1) \).

(ii) Varying \( X \) with fixed \( L, Z \), and identifying \( N|_p \simeq T_pZ \), the upper subspace \((n - 3)O|_p \subset T_pZ\) becomes a general hyperplane.

(iii) There is a deformation \( \{(X_t, L_t) : t \in T\} \), fixing \( Z \), such that the image of the map

\[
T \ni t \mapsto L_t \cap Z
\]

contains a neighborhood of \( p \).

**Remark 8.** From the proof below it will follow that \( X \) actually contains a 1-parameter family of such curves.

**Proof of Lemma.** We begin with Case (a).

(i) This is just Lemma 6 above. Note that it implies that \( L \) moves on \( X \) in a smooth \((n - 3)\)-dimensional family and, because the restriction map \( H^0(N) \to N|_p \) for general \( p \in L \) has \((n - 3)\)-dimensional image, this family fills up an \((n - 2)\)-dimensional scroll that is a divisor on \( X \).
(ii) Let $x_0$ be the equations of $H$ and $x_1, \ldots, x_n$ be general linear forms, and consider the case of a simplex

$$X_0 = H_1 \cup \ldots \cup H_n, H_i = V(x_i)$$

and let $A_1 \cup \ldots \cup A_n = X_0 \cap H$ be its $H$-section. $X_0$ has singular locus

$$S = \bigcup S_{ij}, S_{ij} := H_i \cap H_j, 1 \leq i < j \leq n.$$ 

Let $L \subset H_1$ be a general line and

$$p_j = L.H_j = L.S_{ij}, j = 2, \ldots, n; p = L.H = L.A_1.$$ 

Set $f_0 = x_1 \cdots x_n$, let $g$ be a general degree-$(n - 1)$ form vanishing at $p_2, \ldots, p_n$, and let $X_1 \subset \mathbb{P}^n$ be the hypersurface with equation $f_0 + x_0g$. Thus $X_1$ contains $p_2, \ldots, p_n$ and has $X_1 \cap H = X_0 \cap H = Z$. Set $Q_j = V(g) \cap S_{ij}, j = 2, \ldots, n$, which is a degree-$(n - 1)$ subvariety in $S_{ij}$ containing $p_j$, and $X_1 \cap S_{ij}$ consists of the hyperplane $V(x_0)$ plus $Q_j$. Consider the linear family, depending on $g$

$$\pi = \pi(g) : X(g) = V(f_0 + tx_0g) \subset \mathbb{P}^n \times \mathbb{A}^1 \to \mathbb{A}^1, X_i = \pi^{-1}(t).$$

This is a pencil of hypersurfaces with fixed $H$-section $Z$. Then $X(g)$ is singular at $S \cap X_1$ and, away from $S_0 = S \cap V(x_0, g)$, $X_g$ has singularity of type (3-fold ordinary double point)$\times \mathbb{A}^{n-3}$ and so admits a small resolution $X' \to \mathbb{A}^1$ with fibre $X'_0$. Note that $S_0$ also coincides with the singular locus of a general fibre $X_g$.

The normal bundle $N_{L/X'_0}$ is a corank-1 down modification of $N_{L/H_1}$ at $p_2, \ldots, p_n$. Identifying $N_{L/H_1}|_{p_j} \simeq T_{p_j}S_{1j}$, this is the down modification corresponding to the subspace $T_{p_j}Q_j$, $j = 2, \ldots, n$. Since these subspaces may be chosen generally it follows firstly that the modification is general, so that $N_{L/X'_0} = (n - 3)O \oplus O(-1)$. Moreover clearly, and as one can check by a coordinate computation, as the hyperplanes $T_{p_j}Q_j, j = 2, \ldots, n$ vary, so does the $(n - 3)O$ subsheaf and its fibre at $p$. Explicitly, write $N_{L/H_1} = L_2 \oplus \ldots \oplus L_{n-1}$ where $L_i \simeq O(1)$ and the fibre of $L_i$ at $p_i$ corresponds to $T_{p_i}Q_i, i = 2, \ldots, n - 1$. Then the down modification corresponding to $T_{p_i}Q_i, i = 2, \ldots, n - 1$ replaces each $L_i$ by $O_L$ so it is just $(n - 2)O_L$ with basis $e_2, \ldots, e_{n-1}$. Then if the hyperplane $T_{p_i}Q_i$ is represented by $(\alpha_2, \ldots, \alpha_{n-1})$ in this basis, then the $(n - 3)O$ subsheaf is generated by $\alpha_3e_2 - \alpha_2e_3, \ldots, \alpha_{n-1}e_{n-2} - \alpha_{n-2}e_{n-1}$ and this clearly moves with $T_{p_i}Q_i$.

(iii) What (ii) shows is that the set of limit lines in the family $X(g)$ is a smooth $(n - 3)$-parameter family which traces out on $A_1$ a smooth divisor $D(g)$ whose tangent hyperplane at $p$ corresponds to the aforementioned $(n - 3)O$ subsheaf. This depends on $g$ through the $Q_i$ curves. As $g$ varies, the latter computation shows that $D(g)$ will vary and with it the tangent hyperplane $T_pD(g)$. Therefore the divisors $D(g)$ will sweep out $A_1$, filling up an open set.
3. Proof of Theorem \([1]\) and Corollary \([2]\)

(2, \(n\)) complete intersection case:
Assume first \(e = 2e_1\) even. Let

\[
X_0 = X_1 \cup X_2 \subset \mathbb{P}^{n+1}
\]

where each \(X_i\) is a degree-\(n\) hypersurface in a hyperplane \(P_i \subset \mathbb{P}^{n+1}\) such that

\[
X_1 \cap P_1 \cap P_2 = X_2 \cap P_2 =: Z.
\]

We may assume \(Z\) is a general hyperplane section of \(X_1\) and \(X_2\). A smoothing of \(X_0\) is given by a smoothing of the reducible quadric \(P_1 \cup P_2\), and has total space that is singular along a divisor \(Z_q \subset Z\).

Let \(C_1 \subset X_1\) be a curve of degree \(e_1 = e/2\) and genus \(g\), regular with respect to \(|\mathcal{O}_{X_1}(1)| = | - K_{X_1}| \) (cf. Proposition \([5]\)), and meeting \(Z\) transversely in \(p_1, \ldots, p_{e_1}\) and disjoint from \(Z_q\). Then \(C_1\) moves in a smooth \(e_1 = \chi(N_{C_1/X_1})\)-dimensional family on \(X_1\) and because

\[
e_1 - 2 = \chi(N_{C_1/X_1}(-p_i)) > 0,
\]

each \(p_i\) moves on \(Z\) filling up an analytic open set \(U_i\). By restricting, we may assume the \(U_i\) pairwise disjoint. On the other hand consider a balanced line \(L \subset X_2\) with normal bundle \((n-3)\mathcal{O} \oplus \mathcal{O}(-1)\). As we saw in Lemma \([7]\) as the pair \((X_2, L)\) moves while fixing \(Z\), the point \(L \cap Z\) moves, filling up a Zariski open set \(V\) (NB it is obviously necessary here that the hypersurface move together with the line). As \(V\) must intersect each \(U_i\), we may assume that we have a balanced line \(L_i\) in \(X_2\) through \(p_i\) for \(i = 1, \ldots, e_1\). Moreover by Lemma \([7]\) Case (a), the may assume the upper subspace \(M_i\) of \(N_{L_i/X_1}|_{P_i}\) is general as subspace of \(T_{p_i}Z\). Let \(N_1 \subset N_{C_1/X_1}\) be the down modification corresponding to \(M_1, \ldots, M_{e_1}\).

Now assume \(g \geq 1\), so that either \(g = 1\) or \(n = 4\). Then because \(N_{C_1/X_1}\) is regular relative to \(|\mathcal{O}(1)|\) and the modification is general, we have \(H^0(N_1) = 0\). let

\[
C_2 = \bigcup_{i=1}^{e_1} L_i, C_0 = C_1 \cup C_2.
\]
Now note that $H^0(N_{C_0/X_0}) = H^0(N_1)$ and as we have seen this vanishes. Thanks to our assumption that either $n = 4$ or $g = 1$, we have $\chi(N_{C_0/X_0}) = 0$ so it follow that $H^1(N_{C_0/X_0}) = 0$ as well. Thus, $C_0$ is rigid on $X_0$ and deforms with it to a rigid curve of degree $2e_1 = e$ on a general $(2,n)$ complete intersection in $\mathbb{P}^n$. This completes the proof in case $g \geq 1$.

The case $g = 0$ is similar because $N_{C_1/X_1}$ is balanced and a general modification of a balanced bundle is balanced.

Now consider the case $e = 2e_1 + 1$ odd. The idea is to use the same $C_1$ and to replace one of the lines, say $L_1$, by a suitable conic $M$. Now recall that the smoothing of $X_0$ corresponds to smoothing the reducible quadric with equation $x_1x_2 + tq$. The total space of the family has local equation $x_1x_2 + tq$. As such it is singular in $x_1 = x_2 = t = q = 0$, i.e. the intersection $Z_q$ of $Z$, the double locus of the special fibre, with the quadric $q = 0$. There the total space admits a small resolution $\tilde{X}$ by blowing up $x_1 = q_0$ (this makes sense globally) and the special fibre in $\tilde{X}$ replaces the component $X_2$ by its blowup in $Z_q$. Choosing the quadratic $q$ suitably, we can arrange that $M$ meets $Z_q$ in exactly 1 point and there transversely. Then the birational transform of $M$ meets that of $Z$ in exactly 1 point and has normal bundle $(n-3)O \oplus O(-1)$ by Lemma 6, so we can proceed as before.

Suppose $e$ is even. Here we use a standard quasi-cone degeneration to

$$X_0 = X_1 \cup_Z X_2$$

with $X_1$ a resolved quasi-cone of degree $n + 1$ and $X_2$ a hypersurface of degree $n$ in $\mathbb{P}^n$. The degeneration has smooth total space. As in the $(2,n)$ case, we have a lci curve

$$C_0 = C_1 \cup C_2$$

with $C_1$ a disjoint union of $e_1 = e/2$ many `conics' with normal bundle $(n-3)O \oplus O(-1)$ as in Lemma 7 Case (b), and $C_2$ is a curve of genus $g$ and degree $e_1 = e/2$ on $X_2$ that is regular relative to $| - K_{X_2} | = |O_{X_2}(1)|$, and with

$$C_1 \cap Z = C_2 \cap Z.$$ 

Then an argument as above shows that $H^0(N_{C_0/X_0}) = 0$ so we can conclude as above.

Finally in case $e$ is odd we replace one of the `conics' by a `twisted cubic'. This is obtained by starting with a conic $M \subset \mathbb{P}^{n-1}$ with $2n-1$ points $p_1, \ldots, p_{2n-1}$, choosing general hypersurfaces $F_n, F_{n+1}$ through $p_1, \ldots, p_{2n-1}$ and blowing up $Y = F_n \cap F_{n+1}$. The birational transform $\tilde{M}$ of $M$ meets $Z = \tilde{F}_n$ in 1 point and contributes 2 to the total degree of $C_0$ and its smoothing. Even though the $M$ has normal bundle

$$\begin{aligned}
H^0(N_{C_0/X_0}) &= H^0(N_1) \\
H^1(N_{C_0/X_0}) &= 0 \\
H^2(N_{C_0/X_0}) &= 0
\end{aligned}$$

as we have seen this vanishes. Thanks to our assumption that either $n = 4$ or $g = 1$, we have $\chi(N_{C_0/X_0}) = 0$ so it follow that $H^1(N_{C_0/X_0}) = 0$ as well. Thus, $C_0$ is rigid on $X_0$ and deforms with it to a rigid curve of degree $2e_1 = e$ on a general $(2,n)$ complete intersection in $\mathbb{P}^n$. This completes the proof in case $g \geq 1$.

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$$C_1 \cap Z = C_2 \cap Z.$$ 

Then an argument as above shows that $H^0(N_{C_0/X_0}) = 0$ so we can conclude as above.

Finally in case $e$ is odd we replace one of the `conics' by a `twisted cubic'. This is obtained by starting with a conic $M \subset \mathbb{P}^{n-1}$ with $2n-1$ points $p_1, \ldots, p_{2n-1}$, choosing general hypersurfaces $F_n, F_{n+1}$ through $p_1, \ldots, p_{2n-1}$ and blowing up $Y = F_n \cap F_{n+1}$. The birational transform $\tilde{M}$ of $M$ meets $Z = \tilde{F}_n$ in 1 point and contributes 2 to the total degree of $C_0$ and its smoothing. Even though the $M$ has normal bundle
\( N = \mathcal{O}(4) \oplus (n - 3)\mathcal{O}(2) \) which is unbalanced, the down modification of \( N \) in \( > 0 \) points is balanced and so \( \tilde{M} \) has balanced normal bundle i.e. \( (n - 2)\mathcal{O} \oplus \mathcal{O}(-1) \).

**Remark 9.** Let \( C \) be a rational curve on a CICY \( X \) as in Theorem \(^1\) with normal bundle \( N = (n - 4)O_C \oplus 2O_C(-1) \). Then for \( p \in C \) the image of restriction \( H^0(N) \to N|_p \) is \((n - 4)\)-dimensional. It follows that \( C \) moves in \( X \) filling up an \((n - 3)\)-dimensional ruled subvariety.

**Remark 10.** If \( C \) is as in Remark \(^9\) then clearly \( T_{X|C} = \mathcal{O}(2) \oplus (n - 2)O \oplus O(-1) \) which is not balanced. Thus in the terminology of \(^9\), \( C \) is never ambient-balanced.

**Remark 11.** The above method of constructing curves yields rigid nodal lci curves of any genus on \( X_0 \) for any \( n \). However for \( n > 4, g > 1 \) these curves have \( H^1(N) \neq 0 \), so it’s not clear these curves smooth out with \( X_0 \).

**Proof of Corollary 2.** The proof is based on the Serre construction (see \(^7\), §I.5.1 which, though formulated for projective spaces is actually mostly valid for arbitrary smooth varieties, as already noted in \(^7\), §I.5.3). If \( C \) is an elliptic curve as in the Theorem, Part (ii), then the computations in \(^7\), §I.5.1 show that \( \text{Ext}^1(I_C, \mathcal{O}_X) = \mathcal{O}_C \) and there is an exact sequence

\[
H^1(\mathcal{O}_X) \to \text{Ext}^1(I_C, \mathcal{O}_X) \to H^0(\text{Ext}^1(I_C, \mathcal{O}_X)) \to H^2(\mathcal{O}_X).
\]

Since the extreme groups clearly vanish, the element \( 1 \in H^0(\mathcal{O}_C) \) yields a uniquely determined sheaf \( E \) as an extension of \( I_C \) by \( \mathcal{O}_X \). Then the computations in loc. cit. show that \( E \) is locally free.

Now an easy diagram chase around the exact sequence

\[
0 \to \mathcal{O}_X \to E \to I_C \to 0
\]

shows that \( h^0(E) = 1, h^1(E) = 0 \), so the unique section up to scalars of \( E \) extends to (infinitesimal) deformations, therefore \( C \) deforms with deformations of \( E \). Conversely the functoriality of the Serre construction shows that deformations of \( C \) induce deformations of \( E \). Thus, there is an isomorphism between the deformation functors of \( C \) and \( E \). Therefore since \( C \) is rigid, so is \( E \). Finally since \( \text{Pic}(X) \) is generated by the hyperplane class, indecomposability follows from the Chern classes while (proper) semistability follows from the above exact sequence. \( \square \)

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