An orbital entanglement in two-electron quantum dots in a magnetic field

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Abstract. We consider a simple model for two-electron quantum dots (QDs) in the perpendicular magnetic field. With the aid of this model we study the degree of the orbital entanglement as a function of the magnetic field strength and the QD shape. We found that the circular QD exhibits no entanglement in the magnetic field. Although the entanglement increases with the increase of the QD deformation at a fixed value of the magnetic field, it approaches the constant value at the fixed deformation with the increase of the magnetic field.

1. Introduction
Nowadays there is an enormous experimental and theoretical activity focused on the study of properties of few-electron quantum dots (QDs) [1, 2, 3]. This interest is motivated by: i) possible applications of QDs in nanoelectronics and ii) due to a rapid development of the field of quantum computing. It is expected that QDs could lead to novel device applications in fields such as quantum cryptography and information storage. It is also widely believed that the entangled states of the electrons confined in a QD may give a natural realization of a quantum bit or "qubit" [4]. However, an entanglement being one of the most subtle and intriguing phenomena in nature is not yet well understood [5]. The questions how to efficiently produce and control it, for example, in QDs are among fundamental as well as technological problems.

In general, in finite fermionic systems the entanglement is associated with a correlated dynamics of electrons. The correlated motion of electrons in QDs and, therefore, the entanglement could be controlled by externally applied electromagnetic fields or by varying the parameters of QDs. The simplest QD with the essential features of more complex systems contains two electrons. When two electron move in the external field created by the confining potential and by a homogeneous constant magnetic field, the spin part of the total wave function evolves from the singlet to the triplet state (see [3] and references therein). At the same time the spatial part changes in accordance with the antisymmetric nature of the total wave function. The spin singlet wave function is always maximally entangled. In this case one can study how the external field and parameters of the system would influence the spatial function and, therefore, the spatial entanglement which may have important effects in various applications.

Recently we developed an analytical approach to study the degree of entanglement in a model of two-coupled harmonic oscillators as a function of time [6]. Note, that in condensed matter physics this model is used as a starting point for analysis of electronic properties of QDs in a
perpendicular magnetic field [3]. In this communication we report the results of a detailed study of the effect of the magnetic field and shape parameters of a two-electron QD on the degree of the orbital entanglement.

2. The Model
In theoretical studies of QDs the underlying lattice of the semiconductor material is taken into account by using an effective mass for the conduction electrons. For vertical QDs, the resulting confining potential is, to a good approximation, harmonic [1]. In particular, for the typical voltage ~ 1 V applied to the gate, the confining potential is some eV deep which is large compared to the few meV of the confining frequency [7, 8]. Hence, the electron wave function is localized close to the minimum of the well which always can be approximated by a parabolic potential. Indeed, this approximation enables one to reproduce quite well experimental ground state transitions under the perpendicular magnetic field, observed in recent experiments with few-electron QDs (see details in [3, 9, 10]).

We consider a system of two electrons whose motion is restricted to the xy plane by a parabolic potential. The system is also subject to an external magnetic field applied in the vertical direction (z). The full Hamiltonian thus reads

\[ H = \sum_{i=1}^{2} \left( -i c \hat{\psi}_{i} \right) + V(r_1, r_2) + H_{\text{spin}} = \sum_{i=1}^{2} \left[ \frac{1}{2m^*} \left( p - \frac{e}{c} A \right)^2 + \frac{1}{2} m^* \left( \omega_{x}^2 x^2 + \omega_{y}^2 y^2 \right) \right] + V(r_1, r_2) + H_{\text{spin}}, \]

where \( H_{\text{spin}} = g^* \mu_B (s_1 + s_2) \cdot B \) describes the Zeeman energy and \( \mu_B = |e| \hbar / 2m_e c \) is the Bohr magneton. Here \( m^* \) and \( g^* \) are the effective electron mass and \( g \)-factor, respectively, and we use the planar polar coordinates \((r^2 = x^2 + y^2)\). The confining potential is approximated with a 2D harmonic oscillator with two different confining frequencies \( \omega_x \) and \( \omega_y \), which are not equal in a general case. Although the three-dimensional nature of QDs is important for analysis of experimental data [3], one is able to reproduce the experimental results with the effective two-dimensional electron-electron interaction [11, 9].

For the perpendicular magnetic field \( B \parallel z \), we choose a gauge described by the vector \( A = (B \times r)/2 = (1/2)B(\hat{y}, \hat{x}, 0) \). In the following we assume \( g^* = 0 \) (the parabolic quantum well with \( g^* = 0 \) can be grown [12]) and neglect the contribution of the Zeeman term. In this communication we consider the evolution of the entanglement in the ground state of the QD due to the applied magnetic field and shape parameters. To proceed further, we neglect the interaction between electrons \( V(r_1, r_2) = 0 \) in order to shed light on the spatial entanglement of the single-electron dynamics due to the magnetic field. In this case the spatial part of the total wave function is factorized. In particular, for the singlet state \( \Psi(r_1, r_2, s_1, s_2) = \psi(r_1) \psi(r_2) \xi(s_1, s_2) \) (\( \xi(s_1, s_2) \) is the spin wave function), there is a spin entanglement, while the spatial motion of each electron is independent. The spatial dynamics for each electron is described by the Hamiltonian

\[ \hbar = \frac{p_x^2 + p_y^2}{2m^*} + \frac{m^*}{2} \left( \omega_x^2 x^2 + \omega_y^2 y^2 \right) - \omega_LLz \]

(2)

In the chosen gauge the magnetic field contributions are taken into account by dint of an effective parabolic confinement with two effective frequencies \( \omega_1^2 = \omega_x^2 + \omega_L^2 \), \( \omega_2^2 = \omega_y^2 + \omega_L^2 \), and with a magnetic rotation \( \omega_LLz \). Here \( \omega_L = |e|B/2m^* c \) is the Larmor frequency and \( L_z = xp_y - yp_x \) is an orbital momentum. The Hamiltonian (2) can be recast in the form of two coupled oscillators

\[ \hbar = \hbar \omega_1 \hat{c}_1^\dagger \hat{c}_1 + \hbar \omega_2 \hat{c}_2^\dagger \hat{c}_2 + i \hbar g_1 \left( \hat{c}_1^\dagger \hat{c}_2 - \hat{c}_2^\dagger \hat{c}_1 \right) - i \hbar g_2 \left( \hat{c}_1^\dagger \hat{c}_2^\dagger - \hat{c}_2 \hat{c}_1 \right), \]

(3)
where \( \hat{c}_{1,2} (\hat{c}_{1,2}^\dagger) \) are the annihilation (creation) boson operators of the fields with the energies \( \hbar \omega_{1,2} \). The interplay between these fields is governed by dint of the coupling constants

\[
g_1 = \frac{\omega_L \omega_1 + \omega_2}{2 \sqrt{\omega_1 \omega_2}}, \quad g_2 = \frac{\omega_L \omega_1 - \omega_2}{2 \sqrt{\omega_1 \omega_2}},
\]

(4)

By means of the Bogoliubov transformation \( \hat{a}_k = \sum_{m=1}^{2} \left( A^k_m \hat{c}_m + B^k_m \hat{c}_m^\dagger \right) \), the Hamiltonian (3) is reduced to the diagonal form

\[
h = \sum_{k=1}^{2} \hbar \Omega_k \left( \hat{a}_k^\dagger \hat{a}_k + 1/2 \right)
\]

(5)

with the two eigenmodes

\[
\Omega_k^2 = |\eta_+ + 2(g_1^2 - g_2^2) + (-1)^{k+1} \Delta|/2, \quad \Delta = |\eta_- + 4\eta_+(g_1^2 - g_2^2) + 8\omega_1 \omega_2 (g_1^2 + g_2^2)|^{1/2},
\]

(6)

where \( \eta_{\pm} = \omega_1^2 \pm \omega_2^2 \). The coefficients \( A^k_m \) and \( B^k_m \) are defined in [13].

3. Entanglement of the ground state

Here, we focus upon the eigenstate of the Hamiltonian (5) with \( \hat{n}_1 = \langle \hat{a}_1^\dagger \hat{a}_1 \rangle = 0 \), \( \hat{n}_2 = \langle \hat{a}_2^\dagger \hat{a}_2 \rangle = 0 \), which is the ground state of the QD. In this case, the quantum state of the fields \( \hat{c}_1 \) and \( \hat{c}_2 \) governed by the Hamiltonian (3) can be described by a Gaussian Wigner function of the form (see details in [6])

\[
W^E(\xi) = (4\pi^2 \sqrt{\det V})^{-1} \exp \left\{ -\frac{1}{2} \xi^T V^{-1} \xi \right\}
\]

(7)

Here, \( \xi^T = (q_1, p_1, q_2, p_2) \) is a transposed four-vector whose elements are the quadrature-component variables defined as \( \xi_i = (q_i + ip_i)/\sqrt{2} \). The variance matrix

\[
V = \begin{pmatrix} X_1 & Y \\ Y^T & X_2 \end{pmatrix}
\]

(8)

is determined by the matrices \( X_1 \) and \( Y \) which forms are given in [6]. The Gaussian form of the Wigner function enables one to calculate analytically the measure of entanglement between the fields in the form of the logarithmic negativity via the symplectic spectrum of the partial transpose of the variance matrix. As a result, the logarithmic negativity

\[
E = -\frac{1}{2} \log_2 (4A),
\]

(9)

where

\[
A = B - \sqrt{B^2 - \det V}, \quad B = \frac{\det X_1 + \det X_2}{2} - \det Y,
\]

(10)

determines the degree of entanglement for \( E > 0 \). For \( E \leq 0 \) the composite state is separable according to the Peres-Horodecki criterion [14].

In Fig.1 one observes that a circular QD \( (\omega_x = \omega_y) \) does not manifest the orbital entanglement. It means that there appears no correlations between coordinate and momentum distributions of an electron, corresponding to the mutually orthogonal projections, i.e., the distributions of \( x \) \( (y) \)-values and \( p_y \) \( (p_x) \)-values are independent. The deviation from the circular shape \( \omega_y/\omega_x \neq 1 \)
(the onset of deformation) at nonzero value of the magnetic field $\omega_L/\omega_x$ yields the correlations between the coordinate and the momentum of the mutually orthogonal projections. Moreover, as can be seen in Fig.1, the entanglement increases with the increase of the deformation. The increase of the magnetic field leads to the saturation effect in the entanglement for a fixed QD deformation. For example, at $\omega_y/\omega_x = 2$ the entanglement reaches the value $E \sim 0.4$ at $\omega_L/\omega_x \approx 2.5$ which remains almost unchanged at higher values of the magnetic field.

4. Summary
We consider a simple model for two-electron two-dimensional QDs with the anisotropic parabolic confinement. The considered model demonstrates that the orbital entanglement can be controlled in time by appropriate choice of the magnetic field strength and the choice of the shape parameters of the systems. The obtained results show that even the simple system exhibits rich dynamics from the point of view of the information theory and possible physical applications. Further analysis is required for better understanding of the orbital entanglement in two-electron QDs in the magnetic field.

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