The fractional quantum Hall effect at filling factor 5/2: numerically searching for non-abelian anyons

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Abstract. We review some theoretical progress towards understanding the origin of the fractional quantum Hall effect at filling factor 5/2 using numerical techniques. The leading candidate thought to describe the physics is the Moore-Read Pfaffian state which supports non-abelian anyonic excitations. Interestingly, these excitations have possible applications in topologically protected quantum information processing. Specifically, we discuss how the finite-thickness of realistic quasi-two-dimensional systems stabilizes the Moore-Read Pfaffian in the fractional quantum Hall effect at 5/2 and the use of bilayer systems in the second Landau level to investigate the possible quantum phase transition between abelian and non-abelian fractional quantum Hall states.

1. Introduction
The prediction and discovery of new particles is usually reserved for particle physics, and yet, condensed matter physics has a very real possibility of observing never-before-observed non-abelian anyonic particles in the fractional quantum Hall effect [1] (FQHE) at filling factor \( \nu = 5/2 \). Beyond being of fundamental interest, non-abelian anyons, should they exist, could provide a platform for fault-tolerant topological quantum information processing [2, 3].

The search for non-abelian anyons was initiated with the Moore-Read Pfaffian [4] wavefunction in 1991 as a possible many-body description for the FQHE at \( \nu = 5/2 \), first observed in 1987 [5]. The Moore-Read (MR) Pfaffian wavefunction is written, for an even number of electrons \( N_e \), as

\[
\Psi_{\text{MR}} = \text{Pf} \left\{ \frac{1}{z_i - z_j} \right\} \prod_{i<j}^{N_e} (z_i - z_j)^2 e^{-\sum_{i=1}^{N_e} z_i^2/4l^2} \tag{1}\]

where \( z = x - iy \) is the electron position in the complex plane (\( l \) is defined below).

\[
\text{Pf} \left\{ \frac{1}{z_i - z_j} \right\} = \mathcal{A} \left\{ \frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \ldots \frac{1}{z_{N_e-1} - z_N} \right\} \tag{2}
\]

defines the Pfaffian operator where \( \mathcal{A} \) is the antisymmetrization operator. A remarkable property of \( \Psi_{\text{MR}} \) is that it is a representative of a universality class with low-energy excitations that obey non-abelian braiding statistics [4, 6, 7]. The exotic braiding statistics of \( \Psi_{\text{MR}} \) and the FQHE
at $\nu = 5/2$ were tied to the field of topologically protected quantum computation [2, 3] by Das Sarma, Freedman, and Nayak in 2005 by applying theoretical ideas of Kitaev [8] and Freedman [9] to a real experimental system, i.e., the FQHE, and the search for non-abelian anyons took on renewed urgency. Specifically, reference [2] proposed that one could braid the world lines of the non-abelian anyonic-particle excitations of the 5/2 FQHE and process quantum information in a topologically protected way.

The FQHE occurs when electrons are confined to a quasi-two-dimensional plane, such as at the interface between the two semiconductors (e.g., GaAs/AlGaAs heterostructures), in the presence of a strong perpendicular magnetic field of strength $B$. The Hamiltonian describing the physics is

$$H_{\text{Realistic}} = \sum_{i=1}^{N_e} \left( \frac{p + \frac{\epsilon}{\epsilon} A}{2m} \right)^2 + \sum_{i<j} \frac{\epsilon^2}{\epsilon|\mathbf{r}_i - \mathbf{r}_j|} + g\mu\mathbf{B} \cdot \mathbf{S} + \text{(other terms)}$$

(3)

where $\epsilon$ is the dielectric constant of the host semiconductor (in GaAs/AlGaAs $\epsilon \approx 12$), $m$ is the effective electron mass (in GaAs/AlGaAs, $m = 0.07m_0$, where $m_0$ is the bare electron mass), $l = \sqrt{e\hbar/cB}$ is the magnetic length (hence $r_i$ above is dimensionless), $A$ is the vector potential such that $\nabla \times A = B\hat{z}$, and $\mathbf{S}$ is the total spin of the $N_e$ system with $g = -0.44$ (again for GaAs/AlGaAs). The kinetic energy is quantized into Landau levels (LLs) with energy $\hbar\omega(n + 1/2)$, where $\omega = eB/mc$ is the cyclotron frequency and $n = 0, 1, 2, \ldots$, each with a macroscopically large degeneracy of $1/(2\pi l^2) = eB/hc = B/\phi_0$, where $\phi_0 = hc/e$ is the magnetic flux quantum. Thus, the degeneracy is equal to the number of flux quanta $N_\Phi$ per area piercing the quasi-two-dimensional surface. The filling factor $\nu = \rho 2\pi l^2 = N_e/N_\Phi$ ($\rho$ is the electron density) defines how many LLs are filled (completely or fractionally/partially). The lowest Landau level (LLL) has $n = 0$ and, due to the electron spin, corresponds to $0 < \nu < 2$. The second LL (SLL) obtains for $2 < \nu < 4$ since the LLL for spin-up and spin-down electrons is completely filled, hence, $\nu = 5/2 = 2 + 1/2$ concerns the half-filled second LL.

When the filling factor in the highest unoccupied Landau is less than unity all the electrons have the same kinetic energy and the first term in equation 3 is an irrelevant constant. The last term in equation 3 labeled “other terms” denote single-particle interactions between the electrons and the ions of the semiconductor as well as disorder. All FQHE experiments are done at very high magnetic fields ($B = 3 - 40$ T), at very low temperatures ($T < 500$ mK), and in very clean samples with mobilities mostly greater than $10^7$ cm$^2$/Vs. Hence, it is usually a good approximation to take the large field limit ($\hbar\omega = \hbar eB/mc \to \infty$) and neglect the “other terms”. Furthermore, the large magnetic field likely polarizes the electron spin making $g\mu\mathbf{B} \cdot \mathbf{S}$ another irrelevant constant. This approximation leaves only the electron-electron Coulomb interaction

$$H_0 = \sum_{i<j} \frac{\epsilon^2}{\epsilon|\mathbf{r}_i - \mathbf{r}_j|}$$

(4)

as the Hamiltonian describing $N_e$ electrons fractionally filling the highest unoccupied LL. The FQHE occurs when the strongly interacting electrons with filling factor $p/q$ (in the highest unoccupied LL), with $p$ and $q$ integers, condense into a new type of emergent incompressible quantum fluid with fractionally charged anyonic excitations [12, 13, 14]. This quantum fluid, being incompressible, is a uniform-density unique ground state with a finite energy gap to excitations. Sample disorder converts this energy gap into a mobility gap and the phenomena of

1 Full spin polarization at $\nu = 5/2$ is not experimentally established definitively, however, theoretically the ground state of $H_0$ has been shown to be spin polarized [10, 11]. Thus we take this approximation throughout this work.
the FQHE can be understood [15, 16]. While there have been approximately 70 different odd-denominator FQHE states (q odd) observed in the lowest LL (i.e., \( n = 0 \) and \( \nu < 1 \)) the FQHE in the second LL is relatively rare, and more fragile (small energy gaps), with approximately 8 FQHE states so far observed, including the even-denominator state at \( \nu = 5/2 \) which is the only even-denominator FQHE experimentally observed in single-layer systems.

In terms of strongly or weakly correlated problems, \( H_0 \) defines an essentially infinitely strongly interacting problem; there is only the interaction term in the Hamiltonian, no small parameter, and no normal state from which to construct a perturbation theory. Thus, only non-perturbative methods are applicable. One method is to propose variational wavefunctions—the approach of Laughlin [12], the Composite Fermion theory of Jain [14, 16], and the MR Pfaffian theory.

Another method to solve equation 4 is exact diagonalization for finite-sized systems. This can be done for systems up to approximately 20 electrons. Exact (Lanczos) diagonalization can be readily achieved for Hamiltonian matrices of dimension up to \( 10^6 \times 10^6 \). For a finite sized system the filling factor is \( \nu = \lim_{N_\Phi \to \infty} N_e/N_\Phi \) where \( N_\Phi \) is the number of magnetic flux quanta piercing the surface of the sample, so the Hamiltonian matrix dimension depends on filling factor. In this review, we will report on results where both the wavefunction and exact diagonalization methods are used in combination to elucidate the physics and, needless to say, both methods rely heavily on numerics.

Although suggested earlier [17], the MR Pfaffian description of the FQHE at \( \nu = 5/2 \) gained serious traction after the work of Morf [10] in 1998. Reference [10] found that the Coulomb Hamiltonian \( H_0 \), for electrons in the second LL (SLL), is very likely spin-polarized. This is a necessary condition for the MR Pfaffian to be a viable ansatz. (However, Read and Green [18] find that full spin-polarization might not be totally necessary for the FQHE at \( \nu = 5/2 \) to be in the same non-abelian universality class of the MR Pfaffian, see their figure 1.) Further numerical work by Feiguin et al. [11] have more definitively shown that \( H_0 \) for the SLL is spin-polarized and, theoretically at least, the full spin-polarization of the ground state for an ideal, strictly two-dimensional, Coulomb Hamiltonian in the SLL is very well established. The second thing coming out of Morf’s work [10] was that the wavefunction overlap (which is, of course, a quantitative measure of how similar one wavefunction is to another, with a value of zero being completely dissimilar, or orthogonal, and a value of unity being completely similar) between the exact ground state of the Coulomb Hamiltonian in the SLL and the MR Pfaffian was impressively large at \( \sim 0.8 – 0.9 \) and could be made equal to essentially unity by perturbing the Hamiltonian very slightly—namely, increasing the first Haldane pseudopotential \( V_1 \) by approximately 10%.

While Morf’s result considered the spherical geometry, this result was soon corroborated using the torus geometry [19]. Furthermore, recently it has been demonstrated that the spin-polarized ground state of the Coulomb Hamiltonian in the SLL is adiabatically connected to the MR Pfaffian state [20] and that the ground state displays even-odd oscillations as a function of particle number [21] indicative of quasiparticle pairing.

Nagging questions still remained. While an overlap of \( \sim 0.8 – 0.9 \) is certainly impressive since a randomly chosen wavefunction would have an overlap of the order of \( 1/(\text{number of basis states}) \sim 0 \), FQHE theoreticians had gotten used to overlaps with variational ansatz being \( \sim 0.99 \), for example, the Laughlin wavefunction for \( \nu =1/(\text{odd}) \) and Jain composite fermion wavefunctions describing \( \nu = n/(2p \pm 1) \) (n and p integers). Furthermore, while hardening the short-range interaction, increasing the Haldane pseudopotential \( V_1 \), makes the overlap between the ground state and the MR Pfaffian \( \sim 0.99 \) there is no physical mechanism for which one can increase \( V_1 \), i.e., experimentalists cannot simply increase \( V_1 \) by some Haldane pseudopotential “knob”. But, certain realistic effects of quasi-two-dimensional systems that are ignored in equation 4 such as finite-thickness (which we discuss at length here), Landau level mixing, etc., can change the values of the \( V_m \)’s with respect to their values for the strictly two-dimensional system. However, these effects change all the \( V_m \)’s and it is impossible to change the relative
strength of only $V_1$ with respect to the other $V_m$'s. Hence, while the MR Pfaffian is seemingly a good description of a slightly perturbed Coulomb Hamiltonian in the SLL, perhaps $H_0$ is not the correct Hamiltonian to describe the physics in the SLL.

The plan of this review is as follows: in section 2 we report on recent results [22, 23, 24] finding that the finite-thickness of realistic quasi-two-dimensional systems stabilizes the MR Pfaffian description of the FQHE at $\nu = 5/2$. That is, the MR Pfaffian turns out to be an exceptionally good description of the physics for a more realistic Hamiltonian. We also briefly touch on the role of particle-hole symmetry breaking [25]. In section 3, we report on recent investigations [26, 27, 28] concerning the competition between abelian and non-abelian FQH states in bilayer systems as an alternative scenario to probe the possible non-abelian nature of the 5/2 FQHE.

2. Finite-thickness effects in single-layer systems
A realistic effect of quasi-two-dimensional systems that we review in this work is the finite-thickness effect. The quasi-two-dimensional plane in which the electrons reside (densities are $\sim 1 \times 10^{11}$ cm$^{-2}$) in GaAs/AlGaAs quantum wells have a typical thickness of $\sim 30$ nm. The electrons in the direction perpendicular to the two-dimensional surface have only some probability of being within some length $w$ of the interface, thus, finite-thickness softens the Coulomb interaction between electrons at short distances, leading to an effective electron-electron interaction $V(|r_i - r_j|)$. Starting with equation 4 we replace the pure Coulomb potential $(e^2/\varepsilon |r|)$ with a general potential $V(r)$. This general Hamiltonian $H$ can be parameterized by Haldane pseudopotentials (alluded to above)

$$H = \sum_{i<j}^N V(|r_i - r_j|) = \sum_{k=0}^{\infty} \sum_{i<j}^N V_{2k+1}(m_{ij})$$

(5)

where $\hat{P}_m(m_{ij})$ projects onto states of relative angular momentum $m_{ij} = m$.

$$V_m^{(n)} = \int_0^{\infty} dq \int dq' L_n\left(\frac{q^2}{2}\right)^2 L_m(q'^2) e^{-q^2} \left\{ \frac{1}{2\pi} \int dr e^{iqr} V(r) \right\}$$

(6)

are the Haldane pseudopotentials defined as the energy of a pair of electrons with relative angular momentum $m$ confined to the $n$-th LL interacting via a potential $V(r)$.

In order to model the effect of finite-thickness a variety of models are possible. For this review, we focus on the model where the electrons are confined to the quasi-two-dimensional plane in an infinite-square-well potential [29]. Encouragingly, we find that our results do not qualitatively depend on the finite-thickness model (in reference [23] we considered the Fang-Howard heterostructure model [30, 31] and the Zhang-Das Sarma model [32]). Furthermore, if a more direct experimental comparison is desired, such as comparing the calculated energy gaps to the experimentally measured activation gaps, one can use the local density approximation to self-consistently find the single-particle wavefunction in a quantum-well of finite height as has been done previously [33]. However, as we are only interested in qualitative features and trends, the infinite-square-well serves our purposes.

To consider finite-thickness effects one needs the effective interaction depending on the finite-thickness model to calculate the Haldane pseudopotentials using equation 6. For the infinite-square-well, the Fourier transform of the effective interaction is

$$\tilde{V}(q) = \frac{e^2}{\varepsilon} \left\{ \frac{3qw + \frac{8\pi^2}{\varepsilon} - 32\pi^4(1-\exp(-qw))}{(qw)^2 + 4\pi^2} \right\}$$

(7)
which is found by using the single-particle states of the infinite square-well \( \eta(z) = \sqrt{2/w} \cos(\pi z/w) \) and calculating

\[
\tilde{V}(q) = \frac{e^2 I}{\varepsilon q} \int_0^d dz \int_0^d dz' |\eta(z)|^2 |\eta(z')|^2 e^{-|z-z'|}.
\]

We briefly mention the spherical and torus geometries in which we work. In the spherical geometry, the electrons are confined to the surface of a sphere with radius \( R = \sqrt{N_\Phi}/2 \) where a magnetic monopole of strength \( N_\Phi/2 \) (where \( N_\Phi \) is an integer according to Dirac) is placed at the center of the sphere producing a radial magnetic field everywhere perpendicular to the surface with total magnetic flux of \( N_\Phi \phi_0 \). Total angular momentum \( L \) is a good quantum number and totally rotationally symmetric uniform states have \( L = 0 \). In the spherical geometry, the filling fraction is defined as \( \nu = \lim_{N_e \to \infty} N_e/N_\Phi \) and \( N_\Phi = \nu^{-1}N_e + S \) where \( S \) is known as the “shift” – a topological quantum number that arises due to the curved surface of the sphere and allows one to distinguish potential FQH states of different topological order \[34\]. In the torus geometry, the electrons are confined to a period rectangular plane with unequal sides \( a \) and \( b \) (we define the aspect ration as \( a/b \)). The energy is a function of pseudomomenta \( (K_x = 2\pi \hbar n_x/a, K_y = 2\pi \hbar n_y/b) \) in a two-dimensional Brillouin zone with \( N_0^2 \) points \( (n_x, n_y = 0, \ldots, N_0) \) with \( N_0 \) being the greatest common factor of \( n_x \) and \( n_y \). The filling factor on the torus is simply \( \nu = N_e/N_\Phi \) and there is no “shift”. Instead, states have a ground state degeneracy that depends on their topological origin \[35\]. We utilize both geometries and use the spherical “shift” and ground state degeneracy as complimentary diagnostics of possible FQH states.

2.1. Overlaps

In this section we discuss how the finite-thickness of realistic quasi-two-dimensional systems, in which the FQHE at \( \nu = 5/2 \) is observed, stabilizes the non-abelian MR Pfaffian state.

Figure 1a shows the overlap between the exact ground state of \( H \) (equation 5) and the MR Pfaffian wavefunction as a function of finite-thickness \( w \) for \( N_e = 8, 10, \) and 12 electrons, respectively. As mentioned above the overlap for the strictly two-dimensional system \( (w = 0) \) is encouragingly large. However, inclusion of finite-thickness produces a ground state with a larger overlap with the MR Pfaffian than zero thickness with a maximum near \( w = 4l \) that is largely insensitive to the confinement model and particle number \[22, 23\]. Hence, this effect is very likely a real effect that will survive the thermodynamic limit. It should be pointed out that other “standard” FQH states, i.e., the Laughlin and Composite Fermion states, are uniformly supressed with finite-thickness. In contrast, softening the Coulomb interaction via the finite-thickness effect, an effect that is realized in nature, produces just the right interaction to stabilize and favor the MR Pfaffian description of the FQHE at \( \nu = 5/2 \).

In figure 1a, on the overlap curve, we also indicate examples of the quantum-well widths of experimental samples where the FQHE at \( \nu = 5/2 \) has been observed. Interestingly, the experimental sample width is uniformly sub-optimal. Perhaps it is possible to experimentally “engineer” samples of width \( w \sim 4l \) for which the FQHE at \( \nu = 5/2 \) would be more likely to be described by the MR Pfaffian wavefunction.

Before discussing topological degeneracy of the ground state in the torus geometry we point out that the first two pseudopotentials, \( V_1^{(n)} \) and \( V_3^{(n)} \), do not by themselves determine the physics. Needless to say, finite-thickness effects change all the pseudopotentials and the first two \( V_m^{(n)} \)'s do not determine the physics and manipulating the ratio of \( V_1^{(n)}/V_3^{(n)} \) is not akin to changing the thickness of the two-dimensional layer. To illustrate this, in figure 1b we reproduce an example shown in reference \[23\] where we arbitrarily varied \( V_3^{(1)} \) and \( V_5^{(1)} \) away from their strictly two-dimensional Coulomb values. It was found that there is a region of decreasing \( V_3^{(1)} \).
and $V^{(1)}_5$ where the overlap is nearly unity, hence, one can leave $V^{(1)}_1$ completely alone and simply vary the $m = 3$ and $m = 5$ pseudopotentials and yet still drive the system into the MR Pfaffian phase. On the same figure, we show the overlap for the square-well potential at different values of thickness from $w = 0$ to $10l$. Since finite-thickness changes all the $V^{(1)}_m$'s we only show the path traced out in the $\delta V_3-\delta V_5$ plane. Thus, it is clear that one cannot unambiguously parameterize the finite-thickness effect by changing only the first one or two pseudopotentials.

2.2. Topological Degeneracy

Another measure of stability of the MR Pfaffian state is found when one considers the torus geometry. On the torus, quantum states with topological order display ground state degeneracies. For the MR Pfaffian wavefunction the degeneracy is three-fold [18]. Hence, an approximately three-fold degeneracy of the ground state of our Hamiltonian (equation 5) allows us to distinguish it from other candidate states with different topological order. For example, the Composite Fermion fermi liquid is expected to show shell effects that are very sensitive to the aspect ratio and no particular ground state degeneracy. This is what is theoretically found [39]. Another candidate wavefunction is Halperin’s 331 state [40] (which is only applicable for two-component systems, either bilayer systems or spin) which has a completely different ground state degeneracy signature compared to the MR Pfaffian. The three-fold ground state degeneracy of the MR Pfaffian occur at pseudompmenta $K = (N_0/2, 0), (0, N_0/2), (N_0/2, N_0/2)$.

For $w = 0$, we find that the spectra is very sensitive to aspect ratio for all system sizes studied, indicative of a Composite Fermion fermi liquid ground state. Remarkably, upon increasing the thickness $w$ to a value where we see the largest overlap, i.e., $w \sim 4l$, we discover the appearance

\[ V = \frac{e^2}{\varepsilon l} \]
of a three-fold degeneracy at the quantum numbers corresponding to the Moore-Read Pfaffian. This is shown in figure 2a-f. To the best of our knowledge, this is the first time this effect has been seen in a topological system generated from a Hamiltonian consisting of only two-body interactions.

Another interesting aspect of the MR Pfaffian wavefunction is that it is not particle-hole symmetric. In the spherical geometry, the MR Pfaffian exists when the particle number versus magnetic flux relationship is $N_{\Phi} = 2N_e - 3$ and $-3$. Since the shift is an order one effect it vanishes in the thermodynamic limit. A half-filled LL is particle-hole symmetric but under particle-hole symmetry the system at $N_{\Phi} = 2N_e - 3$ transforms to $N_{\Phi} = 2N_e + 1$ since $N_e + N_h = N_{\Phi} + 1$ (where $N_h$ is the number of holes). The particle-hole conjugate has a shift of $+1$, different from $-3$, hence, the state with $N_{\Phi} = 2N_e + 1$ has been termed the anti-Pfaffian \cite{41, 42} and is topologically distinct from the MR Pfaffian.

On the torus, the shift is absent and the existence of the anti-Pfaffian would lead to a two-fold degeneracy in the energy spectrum. However, in a finite size system this exact degeneracy is slightly lifted. For a nearly square unit cell, i.e., $a/b \approx 0.99$, if the MR Pfaffian is a good description of the physics, one would expect to see an approximately two-fold degeneracy for the ground state energy at the $K$ values corresponding to the MR Pfaffian/anti-Pfaffian. For $w \approx 4l$ we see this approximate two-fold degeneracy (see figure 2g) while for $w = 0$ it is absent.

The observation of the finite-thickness effect providing stabilizing effects for the MR Pfaffian
state for the FQHE at $\nu = 5/2$ in both the torus and spherical geometries gives us great confidence that this effect is robust and real and not a finite-sized artifact. We also encourage experimentalists to take seriously our suggestion of “engineering” systems with $w \sim 4d$ to produce systems more likely to have a MR Pfaffian ground state.

2.3. Particle-hole symmetry breaking

The explicit particle-hole symmetry breaking of the MR Pfaffian wavefunction was briefly mentioned above. This quality lay relatively dormant [43, 6, 44] until recently when it was realized that the consequences of this rather simple fact, due to the MR Pfaffian being the exact ground state of a three-body potential, are interestingly far reaching. The Hamiltonian $H_3$

$$H_3 = \sum_{i<j<k} \hat{P}_{ijk}(3N_\Phi - 3)$$

produces the Moore-Read Pfaffian as an exact zero-energy ground state [17] (here $H_3$ is written in the spherical geometry). $\hat{P}_{ijk}(L)$ projects onto electron triplets with total angular momentum $L$. The particle-hole transform of $H_3$, denoted $\overline{H}_3$, produces the anti-Pfaffian as its zero energy ground state and consists of a term exactly the minus of $H_3$ as well as a two-body term, a chemical potential, and a constant. In contrast, the particle-hole conjugate of the two-body Coulomb Hamiltonian $\overline{H} = H$ and, hence, it respects particle-hole symmetry. Thus, if the MR Pfaffian is to be a good description of the ground state of $H$ then perhaps $H$ spontaneously breaks particle-hole symmetry.

If we consider the total ground state energy to be similar to a Landau free energy and the electron number $N_e$ as the order parameter (strictly speaking we should take the electron number away from half-filling) then if the total ground state energy exhibits a “mexican hat”-like structure as a function of $N_e$ this would indicate spontaneously broken particle-hole symmetry. Specifically, we would expect that whenever $N_e$ is even and corresponds to the MR Pfaffian ($N_\Phi = 2N_e - 3$) or anti-Pfaffian ($N_\Phi = 2N_e + 1$) the total ground state energy will be lowest. We can now attempt to answer whether the two-body Coulomb interaction, with or without finite-thickness effects, spontaneously breaks particle-hole symmetry.

In the left panels of figure 3 we show the total ground state energy as a function of $N_e$ for the Coulomb Hamiltonian $H$ in the SLL for total values of magnetic flux $N_\Phi = 13$ to 17. The MR Pfaffian (anti-Pfaffian) has $N_\Phi = 2N_e - 3$ ($N_\Phi = 2N_e + 1$) and $N_e$ must be an even integer. Hence, only $N_\Phi = 13, 15, 17$ contain even integer $N_e$ solutions to $N_\Phi = 2N_e - 3$ and $N_\Phi = 2N_e + 1$, namely, $N_e = 8$ and 10 for the MR Pfaffian and $N_e = 6$ and 8 for the anti-Pfaffian. It is found that the overall structure of the total ground state energy versus $N_e$ is nearly a parabola around half-filling reflecting the charging energy of a finite-sized system and no sign of spontaneously broken particle-hole symmetry is observed. This result obtains even when we include finite-thickness effects. Thus, the two-body electronic Coulomb Hamiltonian does not spontaneously break particle-hole symmetry with or without finite-thickness. Experimentally, this could mean that the true ground state responsible for the FQHE at $\nu = 5/2$ is not strictly in the same universality class of the Pfaffian (Moore-Read or anti-Pfaffian). Or, the true ground state is a superposition of the MR Pfaffian and anti-Pfaffian forming a new particle-hole symmetric state with necessarily different topological order, and most likely abelian excitations, and edge physics. Or, something external explicitly breaks particle-hole symmetry such as disorder or Landau level mixing. Note that Landau level mixing is known to produce three-body terms perturbatively that explicitly break particle-hole symmetry [45]. Recently, theoretical investigations of LL mixing in the FQHE at 5/2 have been done with no consistent conclusion so far [46, 47].

Interestingly, however, adding $H_3$ and $\overline{H}_3$ cancels the particle-hole symmetry breaking three-body terms and produces a particle-hole symmetric Hamiltonian which we label $H_2$. It turns
Figure 3. Total ground state energy of the Coulomb Hamiltonian in the SLL (with zero-thickness) and $H_2$ as functions of $N_e$ near half-filling for different values of magnetic flux $N_\Phi$. It is found that the ground state of the two-body Coulomb Hamiltonian for $\nu = 5/2$ does not spontaneously break particle-hole symmetry with or without finite-thickness while the ground state of $H_2$ does spontaneously break particle-hole symmetry. (From reference [25]).

out that $H_2$ spontaneously breaks particle-hole symmetry, as can be seen in the right panels of figure 3. Here a clear “mexican-hat”-like structure is observed in the total ground state energy as a function of $N_e$. Perhaps the ground state of $H_2$ might provide an interesting ansatz for the FQHE at $\nu = 5/2$ since it spontaneously breaks particle-hole symmetry and yet arises as a ground state solution to a particle-hole symmetric two-body Hamiltonian. The topological order of the ground state of $H_2$ has yet to be determined, so the nature of quasiparticle excitations are not known.

We briefly mention that Biddle et al. [24] have corroborated the finite-thickness stabilization of the MR Pfaffian state by investigating entanglement measures (both entanglement entropy and spectra) recently applied to the FQHE [48, 49].

3. Competition between abelian and non-abelian phases in bilayer FQH systems

The current experimental status of the FQHE at $\nu = 5/2$ is as follows. Besides being observed many times by different groups over the years, recent experimental work has measured the fractionally charged excitations of $e/4$ expected from the MR Pfaffian theory through both direct shot noise measurements [51, 52, 50] and local measurements with comparison to other FQH states [53]. Observations of interference oscillations of quasiparticles [54] is also consistent with fractional charge $e/4$ and counter propagating neutral modes expected from theories in the same universality class as the MR Pfaffian have been recently observed [55]. The FQHE at $\nu = 5/2$ needs to be nearly spin-polarized in order for the MR Pfaffian theory to be a viable explanation and, yet, recent experimental work [56, 57] points to the possibility of an unpolarized ground state. Completely definitive experimental investigations of spin-polarization are currently lacking.

We point out, however, that the interpretation of the shot noise measurements is evidently more complicated than originally thought [50].
The experimental situation, along with recent experimental developments [58, 59, 60] of the observation of the FQHE at \( \nu = 1/2 \) and 1/4 in very wide quantum wells at high magnetic fields, have led us to consider alternative scenarios where one could probe the non-abelian nature of the FQHE at \( \nu = 5/2 \). In particular, we have recently investigated bilayer systems which may provide an avenue for the proposed non-abelian physics to show itself in new and unexplored ways. We investigate two questions: (1) whether the analog of the non-abelian state at \( \nu = 5/2 \) might also exist in the LLL at \( \nu = 1/2 \) for single-layer systems, and (2) what does the bilayer FQHE at total filling factor \( \nu = 5/2 \) teach us about abelian to non-abelian quantum phase transitions.

Bilayer, or two-component, systems come in two experimental varieties. One is an actual bilayer consisting of two parallel quasi-two-dimensional systems separated from one another by a separation \( d \) with an experimentally adjustable tunneling barrier [61]. The other variety is a single wide-quantum-well [59, 60, 62, 63, 64]. In a wide-quantum-well, with a typical width of \( \sim 70 \) nm, the electrostatic electron interaction creates an effective bilayer by depleting the electron density in the center of the wide-quantum-well. See reference [27] for an in-depth discussion.

Our simple model for bilayer FQHE most closely resembles that of a true bilayer [61]:

\[
H_{\text{Bilayer}} = \sum_{i<j} N_e \left[ V_{\text{intra}}(|r_i - r_j|) + V_{\text{intra}}(|\tilde{r}_i - \tilde{r}_j|) + V_{\text{inter}}(|r_i - \tilde{r}_j|) \right] + t H_{\text{Tunneling}}.
\] (10)

\( V_{\text{intra}}(r) \) is the electron-electron interaction between electrons in a single-layer with position \( r_i \) and \( \tilde{r}_j \), for, say, the left and right layer, respectively. \( V_{\text{inter}}(r) = e^2/\varepsilon(\sqrt{r^2 + D^2}) \) is the interaction between electrons in individual layers separated by a distance \( d \), \( d > w \) by definition. The last term \( H_{\text{Tunneling}} \) controls the tunneling of electrons between layers, i.e., it is a pseudo-spin operator, with large \( t \) denoting strong tunneling.

In order to discuss the physics of the bilayer problem let us concentrate first on the well known \( \nu_{\text{total}} = \nu_{\text{left}} + \nu_{\text{right}} = 1/4 + 1/4 = 1/2 \) bilayer problem. For very small \( d \) and large \( t \) we have a single-layer problem with filling factor \( \nu_{\text{total}} = 1/2 \) which is known to be a Composite Fermion fermi liquid state [65, 66] that, while exotic in it’s own right, does not display the FQHE. For very large separation \( d \) we have two independent layers each with \( \nu_{\text{right/left}} = 1/4 \) and each layer is again known to form Composite Fermion fermi liquid states that do not display the FQHE. However, at intermediate values of layer separation \( d \sim 2l \) and very weak tunneling the FQHE is observed [67, 68, 69, 62, 61]. The 331 state

\[
\Psi_{331} = \prod_{i<j} \frac{N_e}{2} \left( z_i - z_j \right)^3 \prod_{i<j} \left( \tilde{z}_i \right)^3 \frac{N_e}{2} \prod_{i} \prod_{j} \left( z_i \right)^2 \left( \tilde{z}_j \right)^2 e^{-\sum \frac{N_e}{2} / (4l^2 + 4l^2)}
\] (11)

due to Halperin [40] is known to be a good description of the FQHE for bilayer systems with total filling \( \nu_{\text{total}} = 1/2 \) at intermediate \( d \) and weak tunneling. For intermediate \( d \) and strong tunneling the system is basically a wide-quantum-well of width \( w + d \), albeit single-component, with total filling factor 1/2 and again a non-FQHE Composite Fermion fermi liquid obtains. We also point out that the excitations of the Halperin 331 state are abelian and would not support topologically protected quantum information processing.

Instead of a bilayer at \( \nu_{\text{total}} = 1/2 \), we start with a system with vanishing separation \( d \) and strong tunneling \( d \) with total filling factor \( \nu_{\text{total}} = 5/2 \). This is a single-component system that is known to display the FQHE thought to be described by the non-abelian MR Pfaffian state that can be stabilized by finite-thickness \( w \). If we now increase the separation and decrease the tunneling strength we eventually are left with two independent layers with
\[ \nu_{\text{right/left}} = \frac{5}{4} = 1 + \frac{1}{4} \text{ which would presumably form a Composite Fermion fermi liquid in each layer and not display the FQHE (see section VD in reference [26] for a discussion on the bilayer FQHE in higher Landau levels and the additional complications that are present compared to bilayer FQHE in the LLL).} \]

In order to study this problem we exactly diagonalize \( H_{\text{Bilayer}} \), in the spherical geometry, for different values of layer thickness \( w \), layer separation \( d \), and tunneling strength \( t \). Due to the complexity of the problem and large size of the Hilbert space we only consider one system size where the total number of electrons is \( N_e = 8 \) and the total magnetic flux is \( N\Phi = 13 \). This, as noted above, corresponds to a shift of “−3” and since both the MR Pfaffian and the 331 state have a shift of “−3” we can compare both ansatz to the exact ground state of \( H_{\text{Bilayer}} \) on an equal footing. After obtaining the exact ground state we ask ourselves two questions: (1) what is the physics?, and (2) will the system display the FQHE? To answer the first we numerically calculate the wave function overlap between the exact ground state and the MR Pfaffian or 331 state, then we operationally define the system to be in the MR Pfaffian or 331 “phase” depending upon which overlap is larger. To answer the second question, we numerically obtain the excitation gap (provided the ground state is a uniform state with total angular momentum \( L = 0 \)) and recall that in order for a system to display the FQHE it must have a non-zero energy gap.

Figure 4 shows the FQHE energy gap as a function of layer separation \( d \) and tunneling strength \( t \) for single-layer widths of \( w = 0 \) for the lowest and second LLs \(^3\). Superimposed on these plots is a dashed line that represents the “phase” boundary between the 331 or MR Pfaffian phases (the MR Pfaffian phase is labeled Pf on these figures) where this boundary is defined above. Regions of parameter space with a sizable FQHE energy gap are expected to display the FQHE whereas regions with small or vanishing gaps will not.

What is immediately clear from these results is that the FQHE energy gap as a function of \( d \) and \( t \) is qualitatively different in the SLL compared to the LLL. Specifically, in the LLL there is a ridge of large energy gap that is greatest for weak tunneling and intermediate layer separation, i.e., when the system is largely two-component. This region of large FQHE energy gap is described well by Halperin’s abelian 331 state. Furthermore the ridge of positive gap monotonically increases curving upward in a parabolic fashion and, notably, is always on the 331 side of the “phase” boundary line. The symbols superimposed on the LLL FQHE energy gap figures correspond to different bilayer experiments, see caption of figure 4. The groups of symbols with circles around them correspond to experiments where the FQHE at \( \nu_{\text{total}} = 1/2 \) was observed whereas in the experiments corresponding to the non-circled symbols no FQHE was observed. While there is some ambiguity in assigning an experiment to a single point in the \( d-t \) phase space it is qualitatively clear that the FQHE observed in bilayer systems at \( \nu_{\text{total}} = 1/2 \) are described by the abelian 331 state as opposed to the non-abelian MR Pfaffian state.

In the SLL the result is qualitatively different from the LLL. Here we see two ridges of sizable FQHE energy gap as a function of \( d \) and \( t \). One ridge is qualitatively similar to the ridge observed in the LLL (where the system is largely two-component) and on the 331 side of the “phase” boundary, while the other ridge is in the small layer separation and increasing tunneling region, i.e., where the system is largely one-component, and on the MR Pfaffian side of the “phase” boundary. Hence, a bilayer system with \( \nu_{\text{total}} = 5/2 \) has a much richer quantum phase diagram than the LLL bilayer with \( \nu_{\text{total}} = 1/2 \). In particular, it appears possible to experimentally prepare a system with sizable tunneling \( t \) and small layer separation \( d \) that displays the FQHE at \( \nu_{\text{total}} = 5/2 \) described by the non-abelian MR Pfaffian state and tune experimental parameters in such a way to drive a quantum phase transition into an abelian FQHE described by Halperin’s 331 state. While it is difficult to definitively determine, via finite

\(^3\) Three other single-layer widths \( w \) are shown in reference [26] but, qualitatively, they are similar so we do not show them here.
size exact diagonalization, whether there would be a quantum phase transition or crossover going from the non-abelian phase to the abelian phase, our results are consistent with those of Read and Green [18] who obtained a (topologically) similar quantum phase diagram. Theoretically, the FQHE energy gap necessarily collapses across the quantum phase boundary since the two states, 331 and MR Pfaffian, are topologically distinct. Thus, we suspect a true quantum phase transition to occur.

In reference [27], we also investigated the possible quantum phase transition between the abelian 331 and non-abelian MR Pfaffian state in the torus geometry. By looking for the topological ground state degeneracy signature of the two states we corroborated the results presented above for the spherical geometry.

Before concluding, we mention that in reference [28] Scarola et al. looked at the bilayer problem under an electrostatic quantum-well tilt [60]. In these tilted samples, there is substantial subband mixing and it was found [28] that a generalization of the Halperin 331 state due to Scarola and Jain [70] provide a number of partial subband polarized abelian FQHE states for even-denominator filling factors \( \nu = 1/2 \) and 1/4.

Hence, bilayer FQHE systems provide a wide variety of possibilities for observing new FQHE states as well as shedding light on the nature of the FQHE at \( \nu = 5/2 \).

4. Conclusions
We have provided a brief review of some theoretical searches for non-abelian anyons in the FQHE using numerical techniques. Specifically, we reported on results that showed that the finite-thickness of realistic quasi-two-dimensional systems stabilizes the MR Pfaffian non-abelian description of the FQHE at \( \nu = 5/2 \) and suggested the possibility of engineering wavefunctions that would be more likely to experimentally display a non-abelian FQHE. This result was
bolstered by the fact that it was achieved using the complimentary spherical and torus geometries and was largely independent of the finite-thickness model implemented. Lastly, we reported on the use of bilayer systems in the second Landau level as alternative experimental platforms to investigate the possible quantum phase transition between abelian and non-abelian FQHE states. Needless to say, definitive experimental observation of non-abelian anyons in the FQHE would be an extremely exciting and important discovery.

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