Replica symmetry breaking in the minority game

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Abstract - We extend and complete recent work concerning the analytic solution of the minority game. Nash equilibria (NE) of the game have been found to be related to the ground states of a disordered hamiltonian with replica symmetry breaking (RSB), signalling the presence of a large number of them. Here we study the number of NE both analytically and numerically. We then analyze the stability of the recently-obtained replica-symmetric solution and, in the region where it becomes unstable, derive the solution within one-step RSB approximation. We are finally able to draw a detailed phase diagram of the model.

1 Introduction

The minority game has drawn much attention recently as a toy model of a market [1, 2, 3]. In the simplest possible case, when no public information [4] is present, its definition is fairly simple. At each time step, N players have to choose between two actions, such as buying a certain stock or selling it. Those who end up in the minority side win. This mechanism can be obtained by abstracting the well known law of supply-and-demand. When the majority of traders is buying a certain asset it is convenient to be a seller, for prices are likely to be high, and viceversa. The minority side has an advantage.

The full complexity of the model arises in the presence of public information, which is modeled by the occurrence of one of $P$ events representing, e.g., some political news or a price change. In the minority game agents resort to choice rules or information-processing devices – called strategies henceforth – which suggest them whether to “buy” or “sell” given the information they have received. Then each player acts according to the suggestion of his best performing strategy. This mechanism allows to tackle the central problem one faces when trying to understand the collective behaviour of systems of heterogeneous agents interacting under strategic interdependence (as in markets), that is, how agents react to public information (e.g., political news or price changes) and the feedback effects that these reactions have on public information.

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Early numerical simulations \cite{2, 5, 6, 7} have shown a remarkably rich behaviour where both cooperativity and crowd effects \cite{6, 7} arise. Much emphasis was initially put on the emergence of a cooperative phase in the stationary state as compared to the reference situation of random agents – i.e., agents who toss a coin to decide which action to take. Agents in the minority game are able to coordinate their actions and reduce the global waste of resources below the level corresponding to random agents.

Later work \cite{9, 10, 11, 12} has revealed that the agents’ adaptive dynamics minimizes a global function, related to market predictability, and that the system undergoes a phase transition between an asymmetric and a symmetric, unpredictable market, as the ratio $\alpha = P/N$ decreases. A full characterization of the model’s behaviour for $N \to \infty$ was derived studying the minima of the global function by the replica method \cite{10, 11, 12}. It was also realized that agents can greatly improve their performance and global efficiency if they account for their own impact on the market \cite{4, 10, 11}. In this case the steady state is a Nash equilibrium, that is a configuration where no agent can improve his performance by changing his behavior if others stick to theirs. Also in this case the dynamics is related asymptotically to the minima of a global function, and hence statistical mechanics again allows one to describe in detail the stationary state. However, for Nash equilibria replica symmetry breaking (RSB) occurs. This makes the replica symmetric calculation of Refs. \cite{10, 11} only an approximation.

In this paper we move the first steps towards a complete characterisation of the set of NE of the minority game. Our analysis will be slightly more general, for we shall be able to embody the cooperative state as well. First, we shall briefly outline the replica approach, showing that the replica-symmetric solution has a limited validity due to entropy arguments. Then we compute the number of Nash equilibria, following Ref. \cite{13}, and show that there are exponentially many (in $N$). By analogy with the de Almeida-Thouless stability analysis of the SK spin glass model \cite{14, 15}, we find the phase transition line (AT line) separating the replica symmetric from the RSB phase. Finally we shall break the replica symmetry and study the solution in the one-step RSB approximation (1RSB). Probably the exact solution requires infinitely many steps of RSB, but 1RSB provides already an extremely close agreement with numerical results. Finally, we will be able to draw complete phase diagram of the model within 1RSB.

Our discussion will focus on the statistical mechanical properties of the model. The economic and game-theoretic aspects of the model, which are discussed in detail elsewhere \cite{3, 4, 11, 12}, will only be described briefly.

2 The model

2.1 Basic definitions

The essential ingredients of the minority game are:

• $N$ players, labeled by the index $i$;

\footnote{It should be mentioned at this point that the replica approach fails to describe the system’s behaviour in a certain range of the parameters, as discussed in Refs \cite{4, 18}. This point will be made more precise later in the text.}
• for each player \( i \) a strategy variable \( s_i \in \{ \pm 1 \} \), saying which strategy (+1 or −1) player \( i \) is adopting (we restrict ourselves to the case where players have two strategies each);

• \( P \) different information patterns, labeled by the index \( \mu \).

At each time step \( t \) all players receive the same information \( \mu \), drawn at random with equal probability in \( \{1, \ldots, P\} \) \[16\]. Strategies \( s \) are the label of information processing devices, that suggest an action as a value of a binary variable (like “buy” or “sell”) upon receiving information \( \mu \):

\[
s : \{1, \ldots, P\} \ni \mu \mapsto a_{i,s}^\mu \in \{\pm 1\}.
\]

Two such strategies are assigned to each agent and they are drawn at random and independently for each agent, from the set of all \( 2^P \) such functions. In practice, the \( a_{i,s}^\mu \) play the role of quenched disorder, analogous to the random couplings \( \{J_{ij}\} \) in spin glass models.

It is convenient to make the dependence of \( a_{i,s}^\mu \) on \( s \) explicit by introducing auxiliary random variables \( \omega_{i}^\mu \) and \( \xi_{i}^\mu \) such that

\[
a_{i,s}^\mu = \omega_{i}^\mu + s \xi_{i}^\mu.
\]

Clearly, both \( \xi_{i}^\mu \) and \( \omega_{i}^\mu \) take on values in \( \{0, \pm 1\} \) but they are not independent.

The payoff to player \( i \) under information \( \mu \) is defined as

\[
u_{i}^\mu(s_i, s_{-i}) = -a_{i,s}^\mu A^\mu, \quad A^\mu = \sum_{j=1}^{N} a_{j,s}^\mu,
\]

where \( s_{-i} = \{s_j\}_{j \neq i} \). It is positive whenever \( i \) is in the minority group, whence the name of the game. Moreover, players interact with each other only through a global quantity (namely, \( A^\mu \)). This feature clarifies the mean-field character of the model. The total loss experienced by players under information \( \mu \) simply reads

\[
- \sum_{i=1}^{N} u_{i}^\mu(s_i, s_{-i}) = (A^\mu)^2,
\]

which is always positive.

### 2.2 Dynamics

A snapshot configuration of the system corresponds to a point \( \{s_i\}_{i=1}^{N} \) in the (pure) strategy space \( \{\pm 1\}^N \). The game is repeated and at each time step players face the problem of choosing the strategy to follow. By assumption, each player keeps a “score” \( U_{i,s}(t) \) for each strategy \( s = \pm 1 \) and updates it as the game proceeds. In the beginning, players set \( U_{i,\pm 1}(0) = 0 \). Then for \( t \geq 0 \) scores are updated according to the map

\[
U_{i,s}(t+1) = U_{i,s}(t) - \frac{1}{P} a_{i,s}^\mu(t) \left[ A^\mu(t) - \eta \left( a_{i,s,(t)}^\mu - a_{i,s}^\mu(t) \right) \right],
\]

where \( \eta \in \mathbb{R} \) and \( s_{i}(t) \) denotes the strategy that player \( i \) actually uses at time \( t \). The term proportional to \( \eta \) is introduced to model agents who account for
their market impact. We refer the reader to Ref. [11] for a detailed discussion of this term.

Let it suffice to say that with \( \eta = 0 \) Eq. (8) reduces to the standard minority game dynamics: In this case agents reward (penalize) strategies which would have prescribed a sign opposite (equal) to that of \( A^\mu \). In doing so, agents ignore the fact that if they actually had played those strategies, their contribution to \( A^\mu \), and hence \( A^\mu \) itself, could have changed. With \( \eta = 1 \) instead, agents correctly accounting for their contribution to \( A^\mu \). Hence the reward to strategy \( s \) is really the payoff that agent \( i \) would have received had he played that strategy. The parameter \( \eta \) tunes the extent to which agents account for their market impact.

Following [7], we assume that the probability with which player \( i \) chooses the strategy to adopt at time \( t \) depends on the strategy’s score as follows:

\[
\text{Prob}(s_i(t) = \pm 1) = C \exp[\Gamma U_{i,\pm 1}(t)],
\]

where \( C \) is a normalization constant and \( \Gamma > 0 \) is the learning rate. With this rule, the most successful strategy is more likely to be chosen.

Note that \( A^\mu \) is the contribution of \( N \) terms whereas the term proportional to \( \eta \) in Eq. (8) is of order one. One may naively argue that the \( \eta \) term is irrelevant, as \( N \to \infty \). This is not so for exactly the same reason for which the Onsager reaction term – or cavity field – is relevant in mean-field spin glass theory [14]. Indeed Refs. [10, 11] have shown that for \( \eta = 1 \) the dynamics converges to a NE. This means that, in a sense, players have become fully sophisticated. For \( \eta = 0 \), instead, agents converge to a sub-optimal state.

We introduce the continuous variables (“soft spins”) \( \phi_i(t) \in [-1, 1] \) as

\[
\phi_i = \langle s_i \rangle
\]

where \( \langle \cdots \rangle \) stands for the average over the distribution of \( s_i \) in the stationary state. The system is then described by a point \( \{\phi_i\}_{i=1}^N \) in the hypercube \([-1, 1]^N\). Hence the \( \phi_i \)'s are the relevant dynamical variables and the phase space is \([-1, 1]^N\).

The analytic study of the dynamics Eq. (8) has been carried out in Ref. [1]. We shall therefore omit the details and limit ourselves to a brief outline of the results. One can show that the stationary states of the dynamics correspond to the minima of the function

\[
H_\eta = \sum_{i \neq j}^{1,N} \xi^i_{s_i} \xi^j_{s_j} \phi_i \phi_j + 2 \sum_{i = 1}^{N} \Omega^i \xi^i_{s_i} \phi_i + \eta \sum_{i = 1}^{N} \left( \xi^i_{s_i} \right)^2 \left( 1 - \phi_i^2 \right) + \left( \Omega^i \right)^2, \tag{8}
\]

\[\text{In Refs. [4], [11], [12] the term proportional to } \eta \text{ is } \eta \delta(s_i s_i(t))/P. \text{ This leads, however, to the same results of the last term in Eq. (7) in the statistical mechanics approach. The reason is that the approach of Refs. [4], [11] is based on the average of the evolution equation in the stationary state. Observing that } \delta(s_i s_i(t)) \phi_i(t) \text{ is 1 when } s_i(t) = s \text{ and a random sign, with zero average, otherwise, we find that the time average of the last term of Eq. (7) is the same as the average of } \eta(\delta(s_i s_i(t)) - 1)/P. \text{ Hence the two equations are equivalent (apart from an irrelevant constant } -\eta/P).\]

\[\text{For } \eta = 0, \text{ our approach gives correct results only for sufficiently large } \Gamma, \text{ namely for } \Gamma > \Gamma_c(\alpha) \text{ of Ref. [2].}\]

\[\text{While the contribution of other spins to the effective field acting on spin } i \text{ have fluctuating signs, the contribution of spin } i \text{ has always the sign of spin } i. \text{ Mean-field theory needs to be corrected with the subtraction of the self-interaction from the effective-field, which becomes a cavity field. Likewise the contribution of agent } j \neq i \text{ to } A^\mu \text{ is uncorrelated with the action of agent } i, \text{ whereas the contribution of agent } i \text{ itself is totally correlated with his action.}\]
where $\cdots = (1/P) \sum_{\mu=1,P} \cdots$ and $\Omega^\mu = \sum_{i=1,N} w_i^\mu$.

Analyzing the stationary states of the dynamics is then equivalent to minimizing $H_\eta$. The limiting cases $\eta = 0$ and $\eta = 1$ correspond to $H_0 = \langle A^\mu \rangle^2$ and $H_1 = \langle (A^\mu)^2 \rangle$ respectively. $H_0$ (whose minima describe the standard minority game) is related to the market’s predictability or available information, as explained at length in Refs. [9, 10, 11]. In fact, if $\langle A^\mu \rangle \neq 0$, then one can predict that the action $a^\mu = -\text{sign} \langle A^\mu \rangle$ is more likely to be successful than the other whenever pattern $\mu$ arises. $H_1$ is the long time total loss of players averaged over $\mu$, as is clear from Eq. (4). In previous works this quantity is usually denoted as $\sigma^2$. We stress the fact that the minima of $H_1$ are the game’s NE.

It is easy to understand that [10, 11]:

1. $H_0$ is a positive definite quadratic form. Hence for $\eta = 0$ there is a unique stationary state, corresponding to the cooperative state observed in early numerical simulations;

2. for any $\eta > 0$ both the global efficiency and the individual payoffs are sensibly improved with respect to the $\eta = 0$ case;

3. for $\eta = 1$ there is a large number of stationary states, i.e., of NE. These states have $\phi^2_i = 1$, i.e., agents play pure strategies.

Point 1. has been treated extensively in previous works. Here we shall focus on points 2. and 3., namely on the $\eta > 0$ case.

3 Replica approach

3.1 Replica-symmetric theory

In order to minimize the function in Eq. (8) we can resort to statistical mechanics methods, for

$$\min_{\{\phi_i\}_{i=1}^N \in [-1,+1]^N} H_\eta = -\lim_{\beta \to \infty} \frac{1}{\beta} \log Z(\beta) \equiv \lim_{\beta \to \infty} F_\eta(\beta),$$

where $Z(\beta)$ is the canonical partition function associated to $H_\eta$. Further, since $H_\eta$ contains quenched disorder we need to apply the replica formalism [14] to analyze its ground states. If we let $J = \{a_{ij}^\mu\}$ denote collectively the disorder variables and $E_J(\cdots)$ denote statistical average over $J$, the “typical free energy” $F_\eta(\beta)$ can be obtained from the identity

$$E_J[\log Z(\beta)] = \lim_{n \to 0} \frac{E_J[Z^n(\beta)] - 1}{n}.$$

A long but standard computation (see appendices in Refs [11, 12]) leads to the following expression:

$$F_\eta(\beta) = \frac{\alpha}{2\beta n} \text{Tr} \left\{ \log \left[ \left(1 + \frac{\beta}{\alpha}\right) I_n + \frac{\beta}{\alpha} q \right] \right\} + \frac{\alpha \beta}{2n} \sum_{a \neq b} r_{ab} q_{ab} +$$

$$- \frac{1}{\beta n} \log \left\{ \text{Tr}_s \left[ \exp \left( \frac{\alpha \beta^2}{2} \sum_{a \neq b} r_{ab} q_{ab} \right) \right] \right\} + \frac{\eta}{2} (1 - Q_a).$$

5
where $I_n$ is the $n$ dimensional unit matrix, $\hat{q}$ is the overlap matrix with elements $q_{ab} = \langle \phi_a \phi_b \rangle$ ($a, b = 1, \ldots, n; a \neq b$), and $Q_a = (1/N) \sum_{i=1,N} (\phi_i^a)^2$ ($a = 1, \ldots, n$). The quantities $r_{ab}$ and $R_a$ appear as Lagrange multipliers associated to $q_{ab}$ and $Q_a$, respectively.

Imposing the replica-symmetric (RS) Ansatz ($q_{ab} = q$ for all $a \neq b$, $Q_a = Q$ for all $a$, and similarly for $r_{ab}$ and $R_a$) one obtains

$$F^{(RS)}_\eta(\beta) = \frac{\alpha}{2} \frac{1+q}{\alpha+\beta(Q-q)} + \frac{\alpha}{2\beta} \log \left[ 1 + \frac{\beta(Q-q)}{\alpha} \right] + \frac{\beta}{2} (RQ - rq) +$$

$$- \frac{1}{\beta} \left\langle \log \int_1^{-1} \exp[-\beta V(s;z)] ds \right \rangle_z + \frac{\eta}{2} (1 - Q).$$

with

$$V(s;z) = -\sqrt{\frac{\alpha r}{2} z s} - \frac{\alpha \beta}{4} (R - r_1) s^2. \quad (14)$$

The ground state properties of $H_\eta$ in the $N \to \infty$ limit can now be studied by solving the the saddle point equations (obtained by setting equal to zero the derivatives of $F^{(RS)}_\eta(\beta)$ with respect to $Q$, $q$, $R$ and $r$) in the $\beta \to \infty$ limit, since

$$\lim_{N \to \infty} \min_{\{\phi_i\}_{i \in [-1,1]^N}} \frac{H_\eta}{N} = \lim_{\beta \to \infty} \frac{F^{(RS)}_\eta(\beta)|_{s.p.}}{N}, \quad (15)$$

the subscript s.p. indicating the function $F^{(RS)}_\eta(\beta)$ computed at the saddle point values of $Q$, $q$, $R$ and $r$. This procedure yields the so-called RS solution.

For $\eta = 0$ this solution is characterised by a phase transition at $\alpha_c \approx 0.3374 \ldots$ separating a symmetric phase ($\alpha < \alpha_c$) with $H_0 = 0$ from an asymmetric one ($\alpha > \alpha_c$) with $H_0 > 0$. The “spin susceptibility” $\chi = \beta(Q-q)/\alpha$ diverges as $\alpha \to \alpha_c^+$. Also, it is known that this solution is stable against replica symmetry breaking for all values of the control parameter $\alpha$. A detailed account of this case can be found in Ref. [10].

We expect replica symmetry to breakdown for all $\eta > 0$ at certain critical values of $\alpha$, denoted by $\alpha_{RT}(\eta)$. In order to test this prediction, we study the stability of the RS solution for generic $\eta$.

We shall see that, in the $\beta \to \infty$ limit a RSB phase arises. It is natural then to ask whether there is a critical temperature $\beta_c$ separating a high temperature behavior from a low temperature one. This question, even if not directly related to the Minority Game, may be of interest in its own and can be answered at the RS level. Without going into details, let it suffice to mention that setting $q = 0$ in the RS saddle point equations – which is correct for the high temperature phase – one finds $\beta_c = 0$ for all values of $\alpha$ and $\eta$.

### 3.2 Entropy of the RS solution for $\eta = 1$

In random Ising spin systems useful indications about the stability of the RS solution are provided by the zero temperature entropy, namely

$$S^{(RS)}_\eta(\beta \to \infty) = \lim_{\beta \to \infty} \beta^2 \frac{\partial F^{(RS)}_\eta(\beta)}{\partial \beta^2}. \quad (16)$$
This quantity is non-negative due to the discreteness of the configuration space of the model (i.e., \{\pm 1\}^N). If its zero temperature limit turns out to be negative, the corresponding solution is unstable and further steps in the approximation are needed.

In our case, this point is more subtle than it seems. In fact, \(H_\eta\) is defined for continuous variables \(\phi_i \in [-1, +1]\), and not for Ising (i.e., discrete) variables. The corresponding configuration space is not a discrete set of points, but rather a continuum. Therefore, in principle, the zero temperature entropy need not be non-negative.

For \(\eta = 1\), however, \(H_\eta\) attains its minima on the corners of the phase space \([-1, 1]^N\), i.e., \(\phi_i = \pm 1\) for all \(i\). Hence the zero temperature entropy calculation is revealing. Leaving details of the computation aside, the final result is that for \(\alpha > 1/\pi\)

\[
S_1^{(\text{RS})}(\beta \to \infty) = \frac{\alpha}{2} \left[ \frac{C}{\alpha + C} - \log \left( 1 + \frac{C}{\alpha} \right) \right], \quad C = \frac{\alpha}{\sqrt{\pi \alpha - 1}}.
\]

This is negative for all \(\alpha > 1/\pi\) and it diverges at \(\alpha = 1/\pi\). For \(\alpha < 1/\pi\) one finds \(S_1 = -\infty\). This means that RSB occurs for all values of \(\alpha\) at \(\eta = 1\), or, in other words, that the study of NE requires RSB.

### 3.3 The number of Nash equilibria

A crucial feature of the occurrence of replica symmetry breaking is gained from the study of the number of minima of \(H_1\). We show here that an exponentially (in \(N\)) large number of such minima occurs, i.e., that the game possesses an exponentially large number of NE. Our analytic results will be supported by numerical investigations.

In order to compute the number of Nash equilibria (NE) we use the fact that NE are in pure strategies\(^5\), or, that at any NE \(\phi_i = \pm 1\) for all \(i\). Keeping this in mind, we start by considering that NE satisfy the condition

\[
\phi_i \left[ u_i^\eta (+1, s_i) - u_i^\eta (-1, s_i) \right] = 2 \left[ (\xi_i^\eta)^2 - A^\eta \xi_i^\eta \phi_i \right] \geq 0, \quad \forall i.
\]

Hence an indicator function for NE (using \(2(\xi_i^\eta)^2 \approx 1\)) is

\[
I_{\text{NE}}(\{\phi_i\}) = \prod_{i=1}^{N} \theta \left( 1 - 2A^\eta \xi_i^\eta \phi_i \right)
\]

\((I_{\text{NE}} = 1\) if \(\{\phi_i\}\) is a NE and = 0 otherwise) and the number of NE is just obtained summing over all configuration \(\{\phi_i\} \in \{\pm 1\}^N\) (an operation which we denote by \(\text{Tr}_\phi\)). Hence \(N_{\text{NE}} = \text{Tr}_\phi I_{\text{NE}}(\{\phi_i\})\). Following \([13]\) we take the average over the disorder and introduce the integral representation of the \(\theta\) function.

We arrive at

\[
E_J(N_{\text{NE}}) = \frac{N^2 \alpha^3}{(2\pi)^2} \int_{-\infty}^{\infty} d\gamma \ d\Gamma \ d\omega \ d\Omega \exp \left[ N \Sigma(\gamma, \Gamma, \omega, \Omega) \right]
\]

\(^5\)This is a consequence of the fact that \(H_1\) is an harmonic function of \(\phi_i\), i.e. \(\nabla^2 \phi \equiv 0\). This implies that extrema occurs on the corners.

\(^6\) The reader is warned that the \(\Omega\) appearing here is in no relation with the \(\Omega^\mu\) introduced in Section 2.
with
\[
\Sigma(\alpha, \gamma, \Gamma, \omega, \Omega) = \alpha \omega \gamma + \alpha^2 \Omega \Gamma - \frac{\alpha}{2} \log \left[ (1 + \gamma)^2 + 2 \Gamma \right] + \log \left[ 1 + \text{erf} \left( \frac{1 - \omega}{2 \sqrt{\Omega}} \right) \right] .
\] (21)

Eq. (20) is dominated by the saddle point of \( \phi \), which is attained at \( \omega = 1 - \gamma \), \( \Omega = \frac{\gamma}{\alpha (1 + \gamma)} \) and \( \Gamma = \frac{\gamma^2 (1 + \gamma)}{2 (1 - \gamma)} \), where \( \gamma \) is the root of the equation
\[
\frac{\gamma^2 \alpha (1 + \gamma)}{4 (1 - \gamma)} = \log \left( \gamma \sqrt{\frac{\alpha (1 + \gamma)}{1 - \gamma}} \right) - \log \left\{ \alpha \gamma^2 \left[ 1 + \text{erf} \left( \frac{\gamma}{2 \sqrt{\frac{\alpha (1 + \gamma)}{1 - \gamma}}} \right) \right] \right\} .
\] (22)

In terms of the solution \( \gamma^* \) of this equation we have
\[
\Sigma(\alpha) = \frac{\alpha \gamma^*}{2} (2 - \gamma^*) - \frac{\alpha}{2} \log \left( 1 + \frac{\gamma^*}{1 - \gamma^*} \right) + \log \left[ 1 + \text{erf} \left( \frac{\gamma^*}{2 \sqrt{\frac{\alpha (1 + \gamma^*)}{1 - \gamma^*}}} \right) \right] .
\] (23)

The behavior of \( \Sigma \) as a function of \( \alpha \) is shown in Fig. 1. As expected (see [4]), as \( \alpha \to 0 \) the number of NE grows as \( 2^N \). Numerical results from exact enumeration for \( N \leq 20 \) are in very good agreement, which shows that the so-called annealed approximation used here (i.e., taking the average of \( \mathcal{N}_{NE} \)) is sufficient and one does not need to introduce replicas (to compute the average of \( \log \mathcal{N}_{NE} \)) at this level.

### 3.4 de Almeida-Thouless line

A more thorough analysis can be obtained using the de Almeida-Thouless (AT) protocol [15]. In order to investigate the stability of the RS ground states of \( H_\eta \) against RSB we compute the matrix of the second derivatives of the general expression for the free energy, Eq. (12), with respect to \( q_{ab} \) and \( r_{ab} \). The conditions for RSB are then obtained by studying the effect of fluctuations in the direction of RSB. This analysis results in an instability line – i.e., a family of points where the RS solution becomes unstable – in the parameter space \((\alpha, \eta)\), called the AT line and denoted by \( \alpha_{AT}(\eta) \). The resulting equation has to be solved (numerically) together with the RS saddle point equations.

An outline of the calculation is reported in the appendix. We have studied particularly the so called replicon mode, namely those eigenvectors of the stability matrix which are symmetric under interchange of all but two of the indices. The replicon mode is typically responsible for the onset of the RSB instability. Points on the AT line are found to satisfy the following stability condition:
\[
\alpha \left[ 1 - \eta \left( 1 + \frac{\beta (Q - q)}{\alpha} \right) \right]^2 = 1 .
\] (24)

The “susceptibility” \( \chi \equiv \beta (Q - q) / \alpha \) remains finite as \( \beta \to \infty \), so that the zero temperature behaviour can be safely detected. Results are reported in Fig. 2. For \( \eta = 0 \) replica symmetry is preserved for all \( \alpha \). The point \( \alpha_c = 0.3374 \ldots \) where the second order phase transition occurs in the standard MG, separates a line of first order phase transitions, for \( \alpha < \alpha_c \), from a second order line. For \( \eta = 0^+ \) one finds RSB for \( \alpha_{AT} = 1^+ \). Finally, for \( \eta = 1 \), RS is broken for all \( \alpha \).
3.5 Replica symmetry breaking

The one step breaking of replica permutation symmetry is expressed by the Parisi Ansatz for the $q_{ab}$'s and the $Q_n$'s, where an additional parameter denoted by $m$ is introduced: $Q_n = Q$ (all $a$), $q_{ab} = q_1$ (all $a \neq b$ such that $|a-b| \leq m$) and $q_{ab} = q_0$ (otherwise). The “free energy” is the same as in Eq. (12), but this time the overlap matrix has to be parameterised as

$$\hat{q} = q_0 \epsilon_n \epsilon_n^T + (q_1 - q_0) \mathbb{I}_m \otimes \epsilon_m \epsilon_m^T + (Q - q_1) \mathbb{I}_n,$$

where $\epsilon_n$ is the $n$-dimensional column vector with all components equal to one (so that $\epsilon_n \epsilon_n^T$ is the $n$ dimensional matrix with all elements equal to one) and we have used the standard tensor product.

We need to consider the matrix

$$\hat{T} = \left(1 + \frac{\beta}{\alpha}\right) \mathbb{I}_n + \frac{\beta}{\alpha} \hat{q},$$

that is

$$\hat{T} = \left[1 + \frac{\beta}{\alpha}(Q - q_1)\right] \mathbb{I}_n + \frac{\beta}{\alpha}(1 + q_0) \epsilon_n \epsilon_n^T + \frac{\beta}{\alpha}(q_1 - q_0) \mathbb{I}_m \otimes \epsilon_m \epsilon_m^T.$$

Using the identities

$$\mathbb{I}_n = \mathbb{I}_m \otimes \mathbb{I}_n, \quad \epsilon_n \epsilon_n^T = \mathbb{I}_m \otimes \epsilon_m \epsilon_m^T,$$

we can decompose $\hat{T}$ into tensor products and write its determinant $|\hat{T}|$ in a straightforward way (we need $|\hat{T}|$ because $\text{Tr}(\log \hat{T}) = \log |\hat{T}|$). We get

$$|\hat{T}| = \left[1 + \frac{\beta}{\alpha}(Q - q_1) + n \frac{\beta}{\alpha}(1 + q_0) + m \frac{\beta}{\alpha}(q_1 - q_0)\right] \times\left[1 + \frac{\beta}{\alpha}(Q - q_1) + m \frac{\beta}{\alpha}(q_1 - q_0)\right]^{-\frac{1}{\beta}} \left[1 + \frac{\beta}{\alpha}(Q - q_1) \right]^{n - \frac{m}{\beta}}.$$

This means that the first factor on the r.h.s. is an eigenvalue of $\hat{T}$ with multiplicity one, the second one is an eigenvalue with multiplicity $\frac{n}{m} - 1$, and the third one is an eigenvalue with multiplicity $n - \frac{m}{\beta}$.

Putting the latter formula into Eq. (12) and taking the $n \to 0$ limit as requested by the replica trick, we obtain the one step replica symmetry broken free energy $F_n^{(1RSB)}(\beta)$, whose final expression is

$$F_n^{(1RSB)}(\beta) = \frac{\alpha}{2} \left[1 + q_0 \right] + \frac{\eta}{2} \left(1 - Q\right) + \frac{\alpha}{2} \log \left[1 + m \frac{\beta}{\alpha} \right] + \frac{\alpha}{2} \log \left[1 + \frac{\beta}{\alpha} \right] + \frac{\alpha}{2} \left[RQ + (m - 1) r_1 q_1 - m^2 r_0 q_0 \right] + \frac{\alpha \beta}{4} \left[RQ + (m - 1) r_1 q_1 - m^2 r_0 q_0 \right] - \frac{1}{m \beta} \left(\log \left(\int_{-1}^{1} ds \exp[-\beta V(s; y, z)]\right)^m\right)_y z,$$
where $\langle \cdots \rangle_x$ denotes again the average over the unit gaussian variables $x$ and

$$V(s; y, z) = -\sqrt{\frac{\alpha r_0}{2}} z s - \sqrt{\frac{\alpha (r_1 - r_0)}{2}} y s - \frac{\alpha \beta}{4} (R - r_1) s^2.$$  (30)

$F^{(1\text{RSB})}_\eta(\beta)$ depends on seven parameters: the three overlap matrix elements $Q, q_0, q_1$, their related Lagrange multipliers $R, r_0$ and $r_1$, and $m$. Their values have to be determined self-consistently from the seven saddle point equations obtained by setting to zero the derivatives of the free energy with respect to the above parameters. These equations can be solved numerically.

One finds three different regimes in the $(\alpha, \eta)$ plane when $\beta \to \infty$ (Figure 2):

1. For $\alpha < \alpha_0 \approx 0.09012 \ldots$ (all $\eta > 0$) one has $H_\eta = 0$. The solution does not depend on $\eta$ as long as $\eta > 0$. The self-overlap is $Q = 1$ signalling that agents play pure strategies ($\phi_i = \pm 1$) but off diagonal overlaps $q_1 > q_0$ are both less than 1. This suggests that NE are organized in a complex geometric structure. The parameter $m$ attains a finite value.

2. For $\alpha_0 < \alpha < \alpha_1(\eta)$ (all $\eta > 0$) the solution has $H_\eta > 0$, it is independent of $\eta$ (for $\eta > 0$) and $1 = Q = q_1 > q_0$. The spin susceptibility $\chi = \beta(Q - q_1)/\alpha$ attains a finite value in the limit $\beta \to \infty$, which diverges as $\alpha \to \alpha_0$. Again agents play pure strategies and $q_0 < 1$ is the typical overlap between two NE. The parameter $m$ vanishes as $1/\beta$ (indeed the $\beta m$ is finite as $\beta \to \infty$). The line $\alpha_1(\eta)$ is determined by the solution of

$$\eta = \frac{1}{\alpha + \beta(Q - q_1)}.\quad (31)$$

3. In between the line $\alpha_1(\eta)$ and the stability line $\alpha_{\text{AT}}(\eta)$ the solution has $H_\eta > 0$ and $1 > Q = q_1 > q_0$. Hence agents do not play pure strategies. The solution in this region depends on $\eta$.

We stress again that NE are in pure strategies, since at $\eta = 1$ for all values of $\alpha$ one finds $Q = 1$.

Figure 3 shows that the one-step calculation for $H_1/N$ agrees very well with numerical simulations and it represents a considerable improvement over the replica symmetric result\footnote{Note that numerical results refer to a typical NE which need not be the ground state of $H_1$.}. Further steps of RSB, most probably infinitely many, are likely to be needed to recover exact results. However, already the one step calculation provides a rather good approximation.

4 Conclusion

Summarizing, we have analyzed the solution of the minority game by means of statistical mechanics methods. Our starting point has been the study of Refs \cite{10, 11}, where the NE of the game have been mapped onto the ground states of a disordered hamiltonian with RSB, suggesting the existence of a very large number of them. First, we have computed (both analytically and numerically)
the number of NE, showing that actually they are exponentially many in $N$ (number of players). Then, we probed the stability of the replica symmetric theory developed in Ref. [10]. After showing the necessity of RSB by simple entropy considerations, we have calculated the instability line (AT line) using the de Almeida-Thouless method. Finally, we have derived the broken-replica-symmetry solution, drawing the complete phase diagram of the model. All our results are in excellent agreement with computer experiments.

To our knowledge, the minority game is the first example of a market game that requires the full use of spin glass theory in order to uncover its behaviour. Remarkably, many features actually observed in real markets can be recovered within the simple setup of the minority game [3, 4, 5]. Understanding real markets is among the most challenging theoretical problems ahead of us. This work suggests that statistical mechanics of disordered systems may be a valuable tool in this endeavour.

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A Calculation of the AT line

The stability matrix has dimension $n(n-1) \times n(n-1)$ and is given by

$$ C = \begin{pmatrix} A^{(ab,cd)} & D^{(ab,cd)} \\ D^{(ab,cd)} & B^{(ab,cd)} \end{pmatrix} $$ (32)

where

$$ A^{(ab,cd)} = \frac{\partial^2 (nF)}{\partial q_{ab} \partial q_{cd}}, \quad B^{(ab,cd)} = \frac{\partial^2 (nF)}{\partial r_{ab} \partial r_{cd}}, \quad \text{and} \quad D^{(ab,cd)} = \frac{\partial^2 (nF)}{\partial q_{ab} \partial r_{cd}} $$ (33)

($F$ denotes shortly the replica-symmetric free energy.)

Introducing the “perturbation” of the RS solution in the form

$$ \delta q_{ab} = \zeta_{ab} \quad \text{and} \quad \delta r_{ab} = x\zeta_{ab} $$ (34)

with the condition $\sum_b \zeta_{ab} = 0$ for all $a$, it is possible to show that this condition is satisfied for all $n$ by

$$ \zeta_{ab} = \zeta \quad (a, b) \neq (1, 2) $$

$$ \zeta_{1b} = \zeta_{2b} = \frac{1}{2}(3-n)\zeta \quad b \neq 1, 2 $$

$$ \zeta_{12} = \frac{1}{2}(2-n)(3-n)\zeta \quad (a, b) = (1, 2) $$

$$ \zeta_{aa} = 0 \quad \forall a. $$ (35)

The relevant eigenvalue equations (the so-called replicon mode) are given by

$$ \sum_{cd} \left( A^{(ab,cd)} + xD^{(ab,cd)} \right) \zeta_{cd} = \lambda \zeta_{ab} $$

$$ \sum_{cd} \left( xA^{(ab,cd)} + D^{(ab,cd)} \right) \zeta_{cd} = \lambda x \zeta_{ab} $$ (36)
One needs to find an expression for the matrix elements $A$, $B$ and $D$.

There are three different types of matrix elements in the replica symmetric state, corresponding to the cases $(a, b) = (c, d)$, $a = c$ and $(a, b) \neq (c, d)$ respectively. For the $A^{(ab,cd)}$ they are $A^{(ab,ab)}$, $A^{(ab,ad)}$ and $A^{(ab,cd)}$. It is simple to show that in the replica symmetric state

$$
A^{(ab,ab)} = -\alpha\beta (E_{ab} E_{aa} + E_{ab}^2)
$$

$$
A^{(ab,ad)} = -\alpha\beta E_{ab} (E_{ab} + E_{aa})
$$

$$
A^{(ab,cd)} = -2\alpha\beta E_{ab}^2
$$

where

$$
E_{ab} = \beta q[\alpha + \beta(Q - q)]^{-2} \quad \text{and} \quad E_{aa} = E_{ab} + [\alpha + \beta(Q - q)]^{-1}.
$$

For the $B^{(ab,cd)}$ we find

$$
B^{(ab,ab)} = -\alpha^2\beta^3 \left( (\langle s^2 \rangle^2)_z - \langle \langle s^2 \rangle \rangle^2_z \right)
$$

$$
B^{(ab,ad)} = -\alpha^2\beta^3 \left( (\langle s^2 \rangle \langle s \rangle)_z - \langle \langle s^2 \rangle \rangle^2_z \right)
$$

$$
B^{(ab,cd)} = -\alpha^2\beta^3 \left( \langle \langle s \rangle^4 \rangle_z - \langle \langle s^2 \rangle \rangle^2_z \right)
$$

As for $D^{(ab,cd)}$, one finds the general result

$$
D^{(ab,cd)} = \alpha\beta (\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}).
$$

It is important to notice that the relevant combinations of matrix elements appearing in the eigenvalue equations are of the form $A^{(ab,cd)} - 2A^{(ab,ad)} + A^{(ab,cd)}$, and that the eigenvalues can be shown to depend only on

$$
a = A^{(ab,cd)} - 2A^{(ab,ad)} + A^{(ab,cd)} \quad \text{and} \quad b = B^{(ab,cd)} - 2B^{(ab,ad)} + B^{(ab,cd)}
$$

via the simple formula

$$
\lambda_{\pm} = \frac{1}{2} \left( a + b \pm \sqrt{(a - b)^2 + 4} \right).
$$

Putting things together and solving the eigenvalue equations, one finds that one of the eigenvalue (namely $\lambda_-$) is constant in sign (at least at low temperatures). The second one, instead, changes sign and signals the onset of RSB instability. The equation corresponding to $\lambda_+ = 0$ in the end reads

$$
\frac{\alpha\beta^2}{\alpha^2[1 + \beta(Q - q)/\alpha]^2} \langle (\langle s^2 \rangle - \langle s \rangle^2)^2 \rangle_z = 1.
$$

Calculating explicitly the averages appearing in the above formula one arrives at the AT line reported in the text:

$$
\alpha[1 - \eta(1 + \beta(Q - q)/\alpha)]^2 = 1.
$$
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Figure 1: Logarithm of the average number of NE divided by $N$ ($\Sigma$) as a function of $\alpha$. 
Figure 2: Phase diagram of the Minority Game, where the shaded region corresponds to the replica-symmetric phase and the light region to the RSB phase. The AT line marking the boundary between the RS and RSB phases is denoted by $\alpha_{\text{AT}}(\eta)$. The point $\alpha_c \simeq 0.3374\ldots$ separates a line of second order transitions (full heavy line for $\alpha > \alpha_c$) from a first order line of critical points (heavy dashed line for $\alpha < \alpha_c$). Light dashed lines refer to the different regimes found in the solution of the one-step approximation in the RSB phase (see text for details).
Figure 3: Behaviour of $\sigma^2/N \equiv H_1/N$, corresponding to NE. The one-step solution is compared to numerical data and to the RS solution (dashed line).