HESSIAN GEOMETRY AND PHASE CHANGES OF MULTI-TAUB-NUT METRICS

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ABSTRACT. We study the Hessian geometry of toric multi-Taub-NUT metrics and their phase change phenomena via the images of their moment maps. This generalizes an earlier paper on toric Gibbons-Hawking metrics.

1. Introduction

This is a sequel to [17] and [18] in which the Kepler metrics and the toric Gibbons-Hawking metrics are studied from the point of view of Hessian geometry [13], respectively. These are techniques developed in the setting of toric canonical Kähler metrics on compact toric manifolds in Kähler geometry [8, 1, 2], and also in the study of toric Sasaki-Einstein metrics [7, 11, 12] that arise in AdS/CFT correspondence in string theory [10, 16]. Our motivation is to test the applicability of some techniques developed in string theory to problems in classical gravity and general relativity, and to gain some new insights to these problems in doing this. For the examples in our earlier papers mentioned above, by examining the images of the moment maps, one can easily visualize the degenerations of Kähler metrics, more importantly, one can study their phase changes [18, 15], a concept borrowed from statistical physics and introduced in [3, 4] in the context of Kähler geometry. In this paper we will show that the results in [18] can be generalized to toric multi-Taub-NUT spaces with mild modifications, and more importantly, we will report a new type of phase transition not known from our earlier papers [3, 4, 18, 15].

Both the Gibbons-Hawking metrics and the multi-Taub-NUT metrics are gravitational instantons in Euclidean gravity in dimension four of type $A_{n-1}$. They share the same construction that shows that they are both hyperkähler with a triholomorphic circle action. They live on the same space: The crepant resolution of $\mathbb{C}^2/\mathbb{Z}_n$, but with different asymptotic behavior near the infinity: The Gibbons-Hawking metrics are asymptotically locally Euclidean (ALE) while the multi-Taub-NUT metrics are asymptotically locally flat (ALF). This means they have the following asymptotic form respectively:

\begin{align}
(1) & \quad g_{n-\text{GH}} \sim dr^2 + r^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2), \\
(2) & \quad g_{n-\text{TB}} \sim dr^2 + \sigma_1^2 + r^2(\sigma_2^2 + \sigma_3^2),
\end{align}

where $\sigma_1, \sigma_2, \sigma_3$ are left-invariant one-forms on $S^3/\mathbb{Z}_n$.

As in the case of Gibbons-Hawking metrics, we will focus on the toric multi-Taub-NUT spaces and consider the convex bodies that arise as images of natural torus actions on them. Our main result is that the metrics and the complex structures can be reconstructed from some functions on the convex bodies. Considerations of the degenerations of these convex bodies lead to the consideration of phase transition phenomena of the multi-Taub-NUT metrics. We will also consider another
kind of phase transition. Recall the Gibbons-Hawking metrics and the multi-Taub-NUT metrics are given by the same construction, the only difference is a positive parameter in the latter. When this parameter becomes zero, the multi-Taub-NUT metrics become the Gibbons-Hawking metrics. We will let the parameter to further become negative. Then the total space is divided into two regions, in one of which the metrics remain positive definite, but on the other, the metrics become negative definite.

The rest of the paper is arranged as follows. In §2 after recalling the Gibbons-Hawking construction of the multi-Taub-NUT metrics, we study the moment maps of the toric cases and the phase transition of their images. Hessian geometry is used in §3 to explicitly construct local complex coordinates on multi-Taub-NUT spaces. We describe the phase change for 1-Taub-NUT metrics in §4. We end the paper by some concluding remarks in §5.

2. Phase Transitions of Moment Map Images of Toric Multi-Taub-NUT Metrics

In this Section we first recall the Gibbons-Hawking construction [6] of multi-Taub-NUT metrics, then we focus on the toric cases and consider their moment maps. We explain how the moment map images undergo a phase transition.

2.1. Gibbons-Hawking construction of multi-Taub-NUT metrics. Given \( n \) distinct points \( \vec{p}_1, \ldots, \vec{p}_n \) in \( \mathbb{R}^3 \), consider a function \( V \) defined by

\[
V_\epsilon(r) = \frac{\epsilon}{2} + \frac{1}{2} \sum_{j=1}^{n} \frac{1}{|r - \vec{p}_j|},
\]

for some fixed constant \( \epsilon \in \mathbb{R} \). Since \( V_\epsilon \) is a harmonic function on a simply connected domain \( U \subset \mathbb{R}^3 - \{\vec{p}_1, \ldots, \vec{p}_n\} \) be a domain one has

\[
* dV_\epsilon = -d\alpha
\]

for some smooth one-form \( \alpha \), where * is the Hodge star-operator. As in the case of toric Gibbons-Hawking metrics [18, §2.2], we will let the points \( \vec{p}_1, \ldots, \vec{p}_n \) lie in a line, say, \( \vec{p}_j = (0, 0, c_j) \), \( j = 1, \ldots, n \), \( c_1 < c_2 < \cdots < c_n \), we take \( U := \mathbb{R}^3 - \{(0, 0, z) \mid z \geq c_1\} \) we take:

\[
\alpha = -\frac{1}{2} \sum_{j=1}^{n} \frac{xdy - ydx}{r_j(r_j - z + c_j)},
\]

where \( r_j = \sqrt{x^2 + y^2 + (z - c_j)^2} \). On the principal bundle \( U \times S^1 \to U \) with connection 1-form \( d\varphi + \alpha \), where \( \varphi \) is the natural coordinate on \( S^1 \), i.e., \( e^{i\varphi} \in S^1 \), consider

\[
g_\epsilon = \frac{1}{V_\epsilon} (d\varphi + \alpha)^2 + V_\epsilon \cdot (dx^2 + dy^2 + dz^2).
\]

For \( \epsilon > 0 \), this is the multi-Taub-NUT metric in local coordinates \( \{\varphi, x, y, z\} \); for \( \epsilon = 0 \), this is the Gibbons-Hawking metric; for \( \epsilon < 0 \), \( g_\epsilon \) defines a Riemannian metric in the region \( \Omega_+ \) where \( V_\epsilon > 0 \), and \( -g_\epsilon \) defines a Riemannian metric in the region \( \Omega_- \) where \( V_\epsilon < 0 \).
2.2. The regions $\Omega_+$ and $\Omega_-$ for $\epsilon < 0$. Let $U_{\pm}$ be the regions in $\mathbb{R}^3 - \{\vec{p}_1, \ldots, \vec{p}_n\}$ on which $V_\epsilon$ takes the $\pm$-sign, so that $\Omega_{\pm}$ are circle bundles over them.

Write $\epsilon = -a$ for some $a > 0$. Let us first consider the case of $n = 1$ and take $c_1 = 0$. Then we have

$$V_\epsilon = -\frac{a}{2} + \frac{1}{2\sqrt{\rho^2 + z^2}},$$

where $\rho = \sqrt{x^2 + y^2}$. When $V_\epsilon > 0$, $0 < \rho^2 < \frac{1}{\epsilon^2} - z^2$. When $V_\epsilon < 0$, $\rho^2 > \frac{1}{\epsilon^2} - z^2$. This imposes no bounds on $z$.

Now we move on to the general case. Fix $z$, and regard $V_{-a}$ as function of $\rho = \rho^2$,

$$V_{-a} = -\frac{a}{2} + \sum_{j=1}^{n} \frac{1}{\sqrt{\rho + (z - c_j)^2}},$$

then we have

$$\frac{\partial V_{-a}}{\partial \rho} = -\frac{1}{4} \sum_{j=1}^{n} \frac{1}{(\rho + (z - c_j)^2)^{3/2}}.$$

Therefore, there exists a smooth function $p_a(z)$ such that when $\rho = \sqrt{p_a(z)}$, $V_{-a} = 0$, when $\rho < \sqrt{p_a(z)}$, $V_{-a} > 0$, and when $\rho > \sqrt{p_a(z)}$, $V_{-a} < 0$. In other words, $U_+$ is given by $\rho^2 < p_a(z)$ and $U_-$ is given by $\rho^2 > p_a(z)$.

2.3. Complex structures. Consider the almost complex structure given by:

$$J_\epsilon^*(V^{1/2}_\epsilon(d\varphi + \alpha)) = -V^{1/2}_\epsilon dz, \quad J_\epsilon^*(V^{1/2}_\epsilon dz) = V^{1/2}_\epsilon (d\varphi + \alpha),$$

$$J_\epsilon^*(V^{1/2}_\epsilon dx) = -V^{1/2}_\epsilon dy, \quad J_\epsilon^*(V^{1/2}_\epsilon dy) = V^{1/2}_\epsilon dx.$$

As in [18], one can check that it is integrable, with the space of type $(1, 0)$-forms generated by:

$$dx + \sqrt{-1}dy, \quad (d\varphi + \alpha) + \sqrt{-1}V_\epsilon dz.$$

When $\epsilon < 0$, $J_\epsilon$ is undefined when $V_\epsilon = 0$, and one has to consider $J_\epsilon$ in $\Omega_+$ and $\Omega_-$ separately.

2.4. Symplectic structures. The complex structure $J_\epsilon$ is compatible with the Riemannian metric $g_\epsilon$, with the symplectic form given by:

$$\omega_\epsilon = (d\varphi + \alpha) \wedge dz + V_\epsilon dx \wedge dy.$$

Of course when $\epsilon < 0$, $\omega_\epsilon$ is degenerate along $V_\epsilon = 0$. As in the case of Gibbons-Hawking metric (i.e., $g_0$), the multi-Taub-NUT metric (i.e. $g_\epsilon$ for $\epsilon > 0$) is hyperkähler. For $\epsilon < 0$, $g_\epsilon$ is hyperkähler in the region $\Omega_+$ where $V_\epsilon > 0$, and $-g_\epsilon$ is hyperkähler in the region $\Omega_-$ where $V_\epsilon < 0$.

2.5. Torus action and moment map on a toric multi-Taub-NUT space. Using the explicit choice [5] of $\alpha$ in $U$ we have

$$\omega_\epsilon = \left(d\varphi - \frac{n}{2} \sum_{j=1}^{n} \frac{xdy - ydx}{r_j(r_j - z + c_j)}\right) \wedge dz + \frac{1}{2} \left(\epsilon + \frac{n}{2} \sum_{j=1}^{n} \frac{1}{r_j}\right) dx \wedge dy.$$
Let \( x + y\sqrt{-1}y = \rho e^{\sqrt{-1}\theta} \), then
\[
\omega_\epsilon = \left( d\phi - \frac{1}{2} \frac{\rho^2 d\theta}{r_j(r_j - z + c_j)} \right) \wedge dz + \frac{1}{2} \left( \epsilon + \sum_{j=1}^{n} \frac{1}{r_j} \right) \rho d\rho \wedge d\theta
\]
\[
= -dz \wedge d\phi + \frac{1}{2} \left( \frac{r_j + z - c_j}{r_j} \right) dz + \left( \epsilon + \sum_{j=1}^{n} \frac{1}{r_j} \right) \rho d\rho \wedge d\theta
\]
\[
= d\mu_1 \wedge d\theta_1 + d\mu_2 \wedge d\theta_2,
\]
where
\[
\theta_1 = \phi, \quad \mu_1 = -z,
\]
\[
\theta_2 = \theta, \quad \mu_2 = \frac{1}{4} \epsilon (x^2 + y^2) + \frac{1}{2} \sum_{j=1}^{n} (r_j + z - c_j).
\]
Here we have used:
\[
d\mu_2 = \frac{1}{2} \left( \epsilon (xdx + ydy) + \sum_{j=1}^{n} \frac{xdx + ydy + (z - c_j)dz}{r_j} + dz \right).
\]

The Hamiltonian functions \( \mu_1 \) and \( \mu_2 \) generate a 2-torus action given in local coordinates \((\phi, x, y, z)\) by:
\[
(e^{i\theta_1}, e^{i\theta_2}) \cdot (\phi, x, y, z) = (\phi + \theta_1, x \cos \theta_2 - y \sin \theta_2, x \sin \theta_2 + y \cos \theta_2, z).
\]

Note for \( \epsilon < 0 \), even though \( \omega_\epsilon \) is not defined everywhere, the group action and the moment map are defined everywhere. The group action is smooth everywhere, but the moment map is not.

**2.6. Phase transition of images of the moment maps.** In this subsection, we explain how the images of the moment maps undergo a phase change when the parameter \( \epsilon \) changes from a positive number to a negative number.

When \( \epsilon \geq 0 \), note
\[
\mu_2 = \frac{\epsilon}{4} (x^2 + y^2) + \frac{1}{2} \sum_{j=1}^{n} \sqrt{x^2 + y^2 + (z - c_j)^2 + z - c_j}
\]
\[
\geq \frac{1}{2} \sum_{j=1}^{n} |z - c_j| + z - c_j.
\]
Since
\[
|z - c_j| + z - c_j = \begin{cases} 
0, & \text{if } z \leq c_j, \\
2(z - c_j), & \text{if } z \geq c_j,
\end{cases}
\]
it is easy to see that the image of the moment map is the convex region given by the following inequalities:
\[
l_0 := \mu_2 \geq 0,
\]
\[
l_1 := \mu_2 + (\mu_1 + c_1) \geq 0,
\]
\[
l_2 := \mu_2 + (\mu_1 + c_1) + (\mu_1 + c_2) \geq 0,
\]
\[
\vdots
\]
\[
l_n := \mu_2 + \sum_{j=1}^{n} (\mu_1 + c_j) \geq 0.
\]
This is the same as in the case of toric Gibbons-Hawking metrics.

Now we consider the case of $\epsilon < 0$. Let us first consider the case of $n = 1$ and take $c_1 = 0$. Write $\epsilon = -a$ for some $a > 0$. Then we have

$$\mu_2 = -\frac{a}{4} \rho^2 + \frac{1}{2}(\sqrt{\rho^2 + z^2} + z).$$

Consider the function

$$f_z(p) = -\frac{a}{4} p + \frac{1}{2}(\sqrt{p + z^2} + z),$$

as a function of $p$. We have

$$f'_z(p) = -\frac{a}{4} + \frac{1}{2} \frac{1}{\sqrt{p + z^2}} = \frac{1}{2} V_a.$$

So we have $f'_z(x) > 0$ for $0 \leq x < \frac{a}{\sqrt{a^2 - z^2}}$, and $f_z(0) = \frac{1}{2} (|z| + z)$, $f_z(\frac{a}{a^2 - z^2}) = \frac{1}{4a} (1 + az)^2$, it follows that in $\Omega^+$ we have:

$$(18) \quad \frac{1}{2} (|z| + z) \leq \mu_2 < \frac{1}{4a} (1 + az)^2.$$

Hence moment map image of $\Omega^+$ is given by the following inequalities:

$$(19) \quad \frac{1}{2} (|\mu_1| - \mu_1) \leq \mu_2 < \frac{1}{4a} (1 - a\mu_1)^2.$$

This is no longer a convex set. Its boundary has three pieces: an interval on the line $\mu_2 = 0$, an interval on the line $\mu_2 = -\mu_1$, and they are both tangent to a portion of the curve $\mu_2 = \frac{1}{4a} (1 - a\mu_1)^2$ for $-\frac{1}{a} < \mu_1 < \frac{1}{a}$.

We have $f'_z(x) < 0$ for $x \geq \frac{1}{\sqrt{a^2 - z^2}}$, and $f_z(\frac{a}{a^2 - z^2}) = \frac{1}{4a} (1 + az)^2$, so we have:

$$(20) \quad \mu_2 < \frac{1}{4a} (1 + az)^2.$$

The moment image of the region $\Omega^-$ is given by:

$$(21) \quad \mu_2 < \frac{1}{4a} (1 - a\mu_1)^2.$$

In this case we obtain an unexpected convexity result as follows: The union of the moment image of $\Omega^+$ with the complement of the moment image of $\Omega^-$ is convex.

Now we move on to the general case:

$$\mu_2 = -\frac{a}{4} \rho^2 + \frac{1}{2} \sum_{j=1}^{n} (\sqrt{\rho^2 + (z - c_j)^2} + z - c_j).$$

Consider the function

$$f_z(p) = -\frac{a}{4} p + \frac{1}{2} \sum_{j=1}^{n} (\sqrt{p + (z - c_j)^2} + z - c_j),$$

as a function of $p$. We have

$$f'_z(p) = -\frac{a}{4} + \frac{1}{2} \sum_{j=1}^{n} \frac{1}{\sqrt{p + (z - c_j)^2}} = \frac{1}{2} V_a,$$

$$f''_z(p) = -\frac{1}{8} \sum_{j=1}^{n} \frac{1}{\sqrt{(p + (z - c_j)^2)^2}} < 0.$$
Recall that, when \( 0 \leq \rho = \sqrt{p_a(z)} \), \( V_{-a} = 0 \), when \( \rho < \sqrt{p_a(z)} \), \( V_{-a} > 0 \), and so \( \mu_2 \) is bounded between \( f_2(0) = \frac{1}{2} \sum_{j=1}^{n} (|z - c_j| + z - c_j) \) and \( f_2(p_a(z)) = -\frac{a}{4} p_a(z) + \frac{1}{2} \sum_{j=1}^{n} (\sqrt{p_a(z)} + (z - c_j)^2 + z - c_j) \). Since \( z = -\mu_1 \), the moment image of \( \Omega_+ \) is given by:

\[
\frac{1}{2} \sum_{j=1}^{n} (|\mu_1 + c_j| - \mu_1 - c_j) \leq \mu_2
\]

When \( \rho > \sqrt{p_a(z)} \), \( V_{-a} < 0 \), \( \mu_2 < f_2(p_a(z)) \), and the moment image of the region \( \Omega_- \) is

\[
\mu_2 < -\frac{a}{4} p_a(-\mu_1) + \frac{1}{2} \sum_{j=1}^{n} (\sqrt{p_a(-\mu_1)} + (-\mu_1 - c_j)^2 - \mu_1 - c_j).
\]

We conjecture that the boundary of this region is a convex curve which is concave up and tangent to the boundary of the moment image in the \( \epsilon > 0 \) case.

3. Hessian Geometry of Toric Multi-Taub-NUT Spaces

In this Section we find the complex potential functions of the toric multi-Taub-NUT metrics and use them to define local complex coordinates.

3.1. Symplectic coordinates and Hessian geometry for toric multi-Taub-NUT spaces. The following result can be obtained by a simple modification of the corresponding result in [13]:

**Proposition 3.1.** In the above symplectic coordinates, the multi-Taub-NUT metric takes the following form:

\[
g_\epsilon = \sum_{i,j=1}^{2} \left( \frac{1}{2} G_{ij} \, d\mu_i \, d\mu_j + 2G^{ij} \, d\theta_i \, d\theta_j \right),
\]

where the coefficient matrices \((G_{ij})_{i,j=1,2}\) and \((G^{ij})_{i,j=1,2}\) are given by:

\[
(G_{ij})_{i,j=1,2} = \left( 2V_\epsilon + \frac{\rho^2}{2V_\epsilon} \sum_{j=1}^{n} \frac{1}{\tau_j(r_j - (z - c_j))} \right)^2 \sum_{j=1}^{n} \frac{1}{\tau_j(r_j - (z - c_j))},
\]

\[
(G^{ij})_{i,j=1,2} = \left( -\frac{1}{4V_\epsilon} \sum_{j=1}^{n} \frac{\rho^2}{\tau_j(r_j - (z - c_j))} - \frac{1}{8V_\epsilon} \sum_{j=1}^{n} \frac{\rho^2}{\tau_j(r_j - (z - c_j))} \right)^2.
\]

These matrices are inverse to each other. The complex potential and the Kähler potential are given by the following formulas respectively:

\[
\psi_\epsilon = \frac{1}{2} \sum_{j=1}^{n} \left( (r_j + (z - c_j)) \log(r_j + (z - c_j)) + (r_j - (z - c_j)) \log(r_j - (z - c_j)) \right) + \frac{\epsilon}{2} z^2 + C_1 \mu_1 + C_2 \mu_2.
\]
And the Kähler potential is given by:

\[
\psi^{\nu} = - \sum_{j=1}^{n} c_j \log(r_j - (z - c_j)) + \frac{\epsilon}{2} z^2 + C_1 \mu_1 + C_2 \mu_2.
\]

for some constants \( C_1 \) and \( C_2 \).

### 3.2. Hessian local complex coordinates.

The function \( \psi \) is called the complex potential because one can find local complex coordinates \( z_1 \) and \( z_2 \) so that

\[
\frac{dz_i}{z_i} = \frac{1}{2} \sum_{j=1}^{2} \frac{\partial^2 \psi}{\partial \mu_i \partial \mu_j} d\mu_j + \sqrt{-1}d\theta_i = \frac{1}{2} \sum_{i,j=1}^{2} G_{ij} d\mu_j + \sqrt{-1}d\theta_i
\]

is of type \((1,0)\). We have

\[
\frac{dz_1}{z_1} = -\frac{\epsilon}{2} dz + \frac{1}{2} d \sum_{j=1}^{n} \log(r_j - (z - c_j)) + \sqrt{-1}d\theta_1,
\]

\[
\frac{dz_2}{z_2} = d \log \rho + \sqrt{-1}d\theta_2 = d \log(x + y \sqrt{-1}).
\]

Therefore, we take

\[
z_1 = \prod_{j=1}^{n} (r_j - (z - c_j))^{1/2} \cdot e^{-\frac{\epsilon}{2} z + \sqrt{-1} \theta_1},
\]

\[
z_2 = x + \sqrt{-1}y.
\]

A simple modification by changing \( V \) to \( V_\epsilon \) in the proof of Theorem 4.1 in [18] then proves the following:

**Theorem 3.1.** The metrics \( g_\epsilon \) and Kähler forms \( \omega_\epsilon \) are given in local complex coordinates \( z_1, z_2 \) as follows:

\[
g_\epsilon = \frac{1}{V_\epsilon} \frac{dz_1}{z_1} \frac{d\bar{z}_1}{\bar{z}_1} - \frac{1}{2V_\epsilon} \sum_{j=1}^{n} \frac{r_j + z - c_j}{r_j} \left( \frac{dz_1}{z_1} \frac{d\bar{z}_2}{\bar{z}_2} + \frac{dz_2}{z_2} \frac{d\bar{z}_1}{\bar{z}_1} \right)
\]

\[
+ \left[ V_\epsilon \rho^2 + \frac{1}{4V_\epsilon} \left( \sum_{j=1}^{n} \frac{r_j + z - c_j}{r_j} \right)^2 \right] \frac{dz_2}{z_2} \frac{d\bar{z}_2}{\bar{z}_2},
\]

\[
\omega_\epsilon = \frac{1}{2\sqrt{-1}} \left( \frac{1}{V_\epsilon} \frac{dz_1}{z_1} \wedge \frac{d\bar{z}_1}{\bar{z}_1} - \frac{1}{2V_\epsilon} \sum_{j=1}^{n} \frac{r_j + z - c_j}{r_j} \left( \frac{dz_1}{z_1} \wedge \frac{d\bar{z}_2}{\bar{z}_2} + \frac{dz_2}{z_2} \wedge \frac{d\bar{z}_1}{\bar{z}_1} \right)
\]

\[
+ \left[ V_\epsilon \rho^2 + \frac{1}{4V_\epsilon} \left( \sum_{j=1}^{n} \frac{r_j + z - c_j}{r_j} \right)^2 \right] \frac{dz_2}{z_2} \wedge \frac{d\bar{z}_2}{\bar{z}_2} \right).
\]

### 3.3. The \((\alpha, \beta)\)-coordinates.

As in [18], we make the following change of variables:

\[
z_1 = \beta_1, \quad z_2 = \alpha_1 \beta_1.
\]

By a simple calculation we have:
Theorem 3.2. The metrics $g_\epsilon$ are given in local complex coordinates $\alpha_1, \beta_1$ as follows:

$$g_\epsilon = \left[ V_\epsilon \rho^2 + \frac{1}{4V_\epsilon} \left( \sum_{j=1}^{n} \frac{r_j + z - c_j}{r_j} \right)^2 \right] \frac{d\alpha_1}{\alpha_1} \cdot \frac{d\bar{\alpha}_1}{\bar{\alpha}_1}$$

$$+ \left[ V_\epsilon \rho^2 - \frac{1}{2V_\epsilon} \sum_{j=1}^{n} \frac{r_j + z - c_j}{r_j} + \frac{1}{4V_\epsilon} \left( \sum_{j=1}^{n} \frac{r_j + z - c_j}{r_j} \right)^2 \right]$$

$$(35)$$

$$\cdot \left( \frac{d\alpha_1}{\alpha_1} \cdot \frac{d\bar{\beta}_1}{\beta_1} + \frac{d\beta_1}{\beta_1} \cdot \frac{d\bar{\alpha}_1}{\bar{\alpha}_1} \right)$$

$$+ \left[ \frac{1}{V_\epsilon} + V_\epsilon \rho^2 - \frac{1}{V_\epsilon} \sum_{j=1}^{n} \frac{r_j + z - c_j}{r_j} + \frac{1}{4V_\epsilon} \left( \sum_{j=1}^{n} \frac{r_j + z - c_j}{r_j} \right)^2 \right] \frac{d\beta_1}{\beta_1} \cdot \frac{d\bar{\beta}_1}{\bar{\beta}_1}. $$

As in [18 §4.5], one can introduce $(\alpha_i, \beta_i)$ for $i = 1, \ldots, n$, such that

$$\alpha_{i+1} = \alpha_i^2 \beta_i, \qquad \beta_{i+1} = \alpha_i^{-1}, $$

for $i = 1, \ldots, n - 1$. So the underlying space for the family of metrics $g_\epsilon$ is the crepant resolution of $\mathbb{C}^2/\mathbb{Z}_n$.

4. Phase Transition of The 1-Taub-NUT Metrics with Respect to $\epsilon$

In this Section we discuss the phase transition of the 1-Taub-NUT metrics with respect to the parameter $\epsilon$.

4.1. The complex potential in the $n = 1$ case. One can take $c_1 = 0$. From the equation

$$\frac{\epsilon}{4} \rho^2 + \sqrt{\rho^2 + z^2} + z = 2\mu_2$$

we can solve for $\rho^2$:

$$\rho^2 = -4 - \frac{2\epsilon \mu_2 - (2 - \epsilon z) + \sqrt{8\epsilon \mu_2 + (2 - \epsilon z)^2}}{\epsilon^2} = 4\mu_2(\mu_2 - z) + \cdots$$

and

$$r_1 = \frac{-2 + \sqrt{8\epsilon \mu_2 + (2 - \epsilon z)^2}}{\epsilon} = 2\mu_2 - z + \cdots, $$

where $\cdots$ are higher order terms in $\epsilon$.

The complex potential is

$$\psi = \frac{\epsilon z - 2 + \sqrt{8\epsilon \mu_2 + (2 - \epsilon z)^2}}{2\epsilon} \log \frac{\epsilon z - 2 + \sqrt{8\epsilon \mu_2 + (2 - \epsilon z)^2}}{\epsilon}$$

$$+ \frac{-\epsilon z - 2 + \sqrt{8\epsilon \mu_2 + (2 - \epsilon z)^2}}{2\epsilon} \log \frac{-\epsilon z - 2 + \sqrt{8\epsilon \mu_2 + (2 - \epsilon z)^2}}{\epsilon}$$

$$+ C_1 \mu_1 + C_2 \mu_2 + \frac{\epsilon}{2} \mu_1^2,$$

for some constants $C_1, C_2$. 
4.2. The 1-Taub-NUT metrics in \((\alpha, \beta)\)-coordinates. In this case write \(\alpha_1 = \alpha\) and \(\beta_1 = \beta\). The 1-Taub-NUT metric becomes:

\[
g_e = \frac{e^{2r^2} + 2er - e^{2r}z + 2 - 2ez}{2(1 + er)} \cdot e^{-\epsilon z} d\alpha d\bar{\alpha}
+ \frac{\epsilon(2 + er)}{2(1 + er)} \cdot \left(\bar{\alpha} \delta d\bar{\beta} + \bar{\beta} d\alpha d\bar{\alpha}\right)
+ \frac{e^{2r^2} + 2er + e^{2r}z + 2 + 2ez}{2(1 + er)} \cdot e^{\epsilon z} d\beta d\bar{\beta}.
\]

Let us now analyze the phase transition of the family of metrics (41). From (45) and (41), we have

\[
\rho^2 = |z_2|^2,
\]

\[
((|z_2|^2 + z^2)^{1/2} - z)e^{-\epsilon z} = |z_1|^2.
\]

Or in the \((\alpha, \beta)\)-coordinates

\[
\rho^2 = |\alpha|^2|\beta|^2,
\]

\[
(||\alpha|^2|\beta|^2 + z^2)^{1/2} - z)e^{-\epsilon z} = |\beta|^2.
\]

In order to consider the solution of the second equation, let

\[
f(z) = ((|\alpha|^2|\beta|^2 + z^2)^{1/2} - z)e^{-\epsilon z}
\]

be regarded as a function in \(z\) with other variables as parameters, and consider its derivative:

\[
f'(z) = e^{-\epsilon z}(-\epsilon((|\alpha|^2|\beta|^2 + z^2)^{1/2} - z) + \frac{z}{(|\alpha|^2|\beta|^2 + z^2)^{1/2}} - 1)
= -e^{-\epsilon z}((|\alpha|^2|\beta|^2 + z^2)^{1/2} - z)(\epsilon + \frac{1}{(|\alpha|^2|\beta|^2 + z^2)^{1/2}})
= -2Ve^{-\epsilon z}((|\alpha|^2|\beta|^2 + z^2)^{1/2} - z).
\]

And so when \(\epsilon \geq 0, \alpha \beta \neq 0\), we have \(f'(z) < 0\), and therefore, there is only one \(z\) such that (41) holds, and by inverse function theorem, \(z\) is a smooth function in \(|\alpha|^2\) and \(|\beta|^2\). Near \(|\alpha|^2 = |\beta|^2 = \epsilon = 0\), \(z\) is analytic in these variables, and one can see that

\[
z = (|\alpha|^2 - |\beta|^2)\left(\frac{1}{2} - \frac{|\alpha|^2 + |\beta|^2}{4} \epsilon + \frac{3|\alpha|^4 + 2|\alpha|^2|\beta|^2 + 3|\beta|^4}{16} \epsilon^2 + \cdots\right),
\]

\[
r = \frac{|\alpha|^2 + |\beta|^2}{2} + (|\alpha|^2 - |\beta|^2)\left(-\frac{\epsilon}{4} + \frac{3|\alpha|^4 + 3|\beta|^2 + 3|\beta|^2}{16} \epsilon^2 + \cdots\right).
\]

When \(\epsilon < 0\), write \(\epsilon = -a\). To have \(V_e > 0\), we need \(|z| < \frac{1}{\alpha} - |\alpha|^2|\beta|^2\), in particular, \(|\alpha||\beta| < a\). So this is the region on the \((\alpha, \beta)\)-plane where (41) defines a hyperkähler metric by the same argument as above. When \(V_e < 0\), we have \(|z| > \frac{1}{\alpha} - |\alpha|^2|\beta|^2\) and \(r \geq \frac{1}{\epsilon}, \) in particular, \(|\alpha||\beta| > a\). Because \(r\) and \(z\) are smooth functions in \(|\alpha|^2\) and \(|\beta|^2\) in this region, (41) defines a negative of a hyperkähler metric there.

To summarize, we have proved the following:
Theorem 4.1. Suppose that $z$ and $r$ are determined by

\[(|\alpha|^2 |\beta|^2 + z^2)^{1/2} - z) e^{-\epsilon z} = |\beta|^2, \tag{47}\]
\[r = \sqrt{|\alpha|^2 |\beta|^2 + z^2}. \tag{48}\]

Then for $\epsilon \geq 0$, (41) determines a hyperkähler metric on $\mathbb{C}^2$. For $\epsilon < 0$, (41) determines a hyperkähler metric in the region defined by $|\alpha \beta| < a$ in $\mathbb{C}^2$, and negative of a hyperkähler metric in the region defined by $|\alpha \beta| > a$ in $\mathbb{C}^2$.

5. Concluding Remarks

Based on the computations in an early paper [18], we consider the Hessian geometry of toric multi-Taub-NUT metrics. This means to find the moment maps together with their images, and complex potential functions on the moment map images, and use them to introduce some local complex coordinates to express the metrics and the Kähler forms. Then in the same fashion as in [18], one can discuss the phase change of the multi-Taub-NUT metrics by allowing the parameters $c_j$'s to be complex numbers. Since there is little difference in the treatments, we have omitted a discussion on this, but instead focus on the phase transition associated with the parameter $\epsilon$. Explicit expressions when $\epsilon$ is a nonzero real number are no longer possible so for explicit examples we have focused on the case of $n = 1$. In this example we present a formula for a family of hyperkähler metrics on $\mathbb{C}^2$ in (41). The discovery of this formula is a demonstration of the power of Hessian geometry.

We can also take the parameter $\epsilon$ to be a complex number. This will lead us again to complexified Kähler forms as in [18]. We do not go into this because we plan to write a separate paper to treat complexifications and phase transitions of Kähler forms more generally.

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