Long Time Behavior of 2D Water Waves with Point Vortices

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Abstract: In this paper, we study the motion of the two dimensional inviscid incompressible, infinite depth water waves with point vortices in the fluid. We show that the Taylor sign condition $-\partial_{\vec{n}} P \geq 0$ can fail if the point vortices are sufficiently close to the free boundary, so the water waves could be subject to Taylor instability. Assuming the Taylor sign condition, we prove that the water wave system is locally wellposed in Sobolev spaces. Moreover, we show that if the water waves is symmetric with a symmetric vortex pair traveling downward initially, then the free interface remains smooth for a long time, and for initial data of size $\epsilon \ll 1$, the lifespan is at least $O(\epsilon^{-2})$.

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1. Introduction

We consider the motion of an inviscid and incompressible ideal fluid with a free surface in two space dimensions (that is, the interface separating the fluid and the vacuum is one dimensional), such as the surface waves in the ocean. We refer such fluid as water waves. We denote the fluid region by $\Omega(t)$, with a free interface $\Sigma(t)$. The equations of motion are Euler’s equations, coupled to the motion of the boundary, and with vanishing boundary condition for the pressure. It is assumed that the fluid region is below the air region. Assume that the density of the fluid is 1, the gravitational field is $-\vec{k}$, where $\vec{k}$
is the unit vector pointing in the upward vertical direction. When the surface tension is zero, the motion of the fluid is described by

\[
\begin{align*}
\begin{cases}
\frac{\partial}{\partial t} v + v \cdot \nabla v &= -\nabla P - (0, 1) \\
\text{div} v &= 0, \\
P &= 0
\end{cases}
\end{align*}
\tag{1.1}
\]

where \( v \) is the fluid velocity, \( P \) is the fluid pressure.

This system, and many variants and generalizations, has been extensively studied in the literature. One distinguishing feature of the paper under consideration is the presence of point vortices, that is, the vorticity distribution \( \omega := \text{curl} v \) is a linear combination of dirac masses, i.e.,

\[
\omega(\cdot, t) = \sum_{j=1}^{N} \lambda_j \delta_{z_j(t)}(\cdot),
\tag{1.2}
\]

where \( \{z_j(t)\} \subset \Omega(t) \), and \( \lambda_j \in \mathbb{R} \). Since the 2d Euler is transported following the fluid, if the initial vorticity of (1.1) is a dirac delta mass, then \( \omega \) might remain as such a point vortex for all \( t \). The question is, as the singular vorticity \( \lambda_j \delta_{z_j(t)} \) generates a singularly rotational part \( \frac{\lambda_j}{\pi} \nabla^\perp \log |z - z_j(t)| \) of \( v \) (Here, \( z = (x, y) \) and \( \nabla^\perp = (-\partial_y, \partial_x) \)), following what vector field should \( z_j(t) \) move? Since the above singular vector field is purely rotational about \( z_j(t) \) and does not move that particle \( z_j(t) \), it is reasonable to expect that the dynamics of \( z_j(t) \) is governed only by the remaining smooth part of \( v \), that is,

\[
\dot{z}_j(t) = \left. \left( v(z, t) - \sum_{j=1}^{N} \frac{\lambda_j}{\pi} \nabla^\perp \log |z - z_j(t)| \right) \right|_{z=z_j(t)}.
\tag{1.3}
\]

This well-known result is rigorously established by considering a family of vortex patch solutions whose initial vorticity limiting (weakly) to a dirac delta mass (see [34] Theorem 4.1, 4.2 for more details).

The system (1.1)–(1.2)–(1.3) is a model for the motion of submerged bodies (see e.g. [10,19]) and it is believed to give some insight into the problem of turbulence ([34], chap 4, §4.6). To the best of our knowledge, rigorous mathematical analysis for this system has not been done previously, while there have been many numerical studies on this system (see for example, [18,22,25,35,43,47]) and studies on the local in time well-posedness of the Cauchy problem in the irrotational case (that is, no vorticity), and for regular vortex distributions.

We obtain three main results for this system: (1) a detailed study of the Taylor sign condition, (2) a local well-posedness result in Sobolev spaces for general distributions of point vortices satisfying the Taylor sign condition, and (3) an extended lifespan results in the case of two symmetric point vortices.

The so-called Taylor-sign condition on the pressure is an important stability condition for the water waves problem. If the Taylor sign condition fails, the system is, in general, unstable, see, for example, [4,6,21,41]. In the irrotational case and without a bottom the validity of the Taylor sign condition was shown by Wu [51,52], and was the key to obtaining the first local-in-time existence results for large data in Sobolev
spaces. In the case of non-trivial vorticity or with a bottom the Taylor sign condition can fail and the sign condition has to be part of the assumptions for the initial data. In the case of point vortices analyzed in this paper, we can obtain an explicit formula for the Taylor sign coefficient, by adapting the analysis of Wu. This formula shows how the Taylor sign condition can fail (and explicit examples are provided where the point vortices are too close to the interface) and gives sufficient criteria for it to hold.

Assuming the Taylor sign condition, we prove the local-wellposedness of the problem. In the irrotational case, Nalimov [36], Yosihara [57] and Craig [17] proved local well-posedness for 2d water waves equation for small initial data. In S. Wu's breakthrough works [51,52] she proved local-in-time well-posedness without smallness assumption. Since then, a lot of interesting local well-posedness results were obtained, see for example [2,3,11,14,30,32,33,37,40,58], and the references therein. See also [39,48–50] for water waves with non-smooth interface. For the formation of splash singularities, see for example [8,9,15,16]. Regarding the local-in-time wellposedness with regular vorticity, see [11,11,29,33,37,38,58]. To our best knowledge, rigorous mathematical analysis for water waves with point vortices has not been done previously. In this paper, working in Riemann mapping variables, we prove the local well-posedness by an iteration scheme. The formulation is close to that of Wu (See [55]), but some additional care is needed to deal with the point vortices.

Our third result is an extended lifespan result in the case of two symmetric point vortices. In the irrotational case, almost global and global well-posedness for water waves were proved in [1,23,31,53,54], and see also [26,27,46,59]. In the rotational case, assuming constant vorticity, Iffrim and Tataru [28] proved extended lifespan for 2d inviscid incompressible infinite depth water waves. In [5], Bieri, Miao, Shahshahani, and Wu prove cubic lifespan for the motion of a self-gravitating incompressible fluid in a bounded domain with a free boundary for small initial data for the irrotational case and the case of constant vorticity. In [24], Ginsberg considered the 3 dimensional gravity water waves with small initial data. Assuming small initial data and let the vorticity goes to zero, Ginsberg was able to obtain an almost global lifespan for the solution. However, if the size of the vorticity and the velocity and of the same order, Ginsberg’s method fails to give an extended lifespan. In this paper, we consider a simplified scenario consisting of two vortices that are symmetric relative to the vertical axis and have the same strength and opposite rotation. Under the assumption that the interface and the velocity also have appropriate symmetries, and that all the data are of sufficiently small size $\epsilon$, we construct solutions for times of the order $\epsilon^{-2}$. This is done by noticing that a cancellation in the leading order total contribution of the vortices, so that this can be treated as an integrable-in-time perturbation of the irrotational problem.

1.1. Outline of the paper. In §2, we provide some analytical tools that will be used in later sections, as well as the lagrangian formulation and the Riemann variable formulation for the water waves with point vortices. In §3, we state the main results of this paper, and discuss the difficulty and the main strategy. In §4, we give a systematic investigation of the Taylor sign condition. We give examples that Taylor sign condition fails. We also give a sufficient condition which implies the strong Taylor sign condition. In §5, we prove Theorem 2. In §6, we prove Theorem 3.
2. Notations and Preliminaries

2.1. Convention and notations. We identify $\mathbb{R}^2$ with $\mathbb{C}$, a point $(x, y)$ is identified with the complex number $x + iy$. Let $z \in \mathbb{C}$, we use $\Re\{z\}, \Im\{z\}$ to represent the real and imaginary part of $z$, respectively.

We use $[A, B]$ to represent the commutator $[A, B] := AB - BA$. Given a function $g(\cdot, t) : \mathbb{R} \to \mathbb{R}$, the composition $f(\cdot, t) \circ g := f(g(\cdot, t), t)$. For a function $h(\cdot, t) : \mathbb{R} \to \mathbb{C}$, we say $h$ is holomorphic in $\Omega(t)$ if there is some holomorphic function $G : \Omega(t) \to \mathbb{C}$ such that $h(\alpha, t) = G(z(\alpha, t), t)$. Here, $z(\alpha, t)$ is a parametrization of $\partial\Omega(t)$.

2.2. Function spaces and tools from harmonic analysis. Let $s \geq 0$. $H^s$ represents the Sobolev space $H^s(\mathbb{R})$, which is defined as:

$$H^s(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}) : \|f\|_{H^s}^2 := \int_{-\infty}^{\infty} (1 + |\xi|^{2s}) |\hat{f}(\xi)|^2 d\xi < \infty \right\}.$$

We use $L^p$ to denote $L^p(\mathbb{R})$. Define $\Lambda$ by

$$\Lambda f(\xi) = |\xi| \hat{f}(\xi). \quad (2.1)$$

2.2.1. The singular integrals and commutators

**Definition 2.1 (Hilbert transform).** Assume that $z(\alpha)$ satisfies

$$\beta_0 |\alpha - \beta| \leq |z(\alpha) - z(\beta)| \leq \beta_1 |\alpha - \beta|, \quad \forall \ \alpha, \beta \in \mathbb{R}, \quad (2.2)$$

where $0 < \beta_0 < \beta_1 < \infty$ are constants. The Hilbert transform associates to the curve $z(\alpha)$ is defined as

$$\mathcal{H} f(\alpha) := \frac{1}{\pi i} p.v. \int_{-\infty}^{\infty} \frac{z_{\beta}(\beta)}{z(\alpha) - z(\beta)} f(\beta) d\beta. \quad (2.3)$$

The standard Hilbert transform is the Hilbert transform associated to $z(\alpha) = \alpha$, that is,

$$\mathcal{H} f(\alpha) := \frac{1}{\pi i} p.v. \int_{-\infty}^{\infty} \frac{1}{\alpha - \beta} f(\beta) d\beta. \quad (2.4)$$

It is well-known (see, for example the celebrated paper by Guy David [20], Theorem 6) that if $z(\alpha)$ satisfies (2.2), then $\mathcal{H}$ is bounded on $L^2$.

**Lemma 2.1.** Assume that $z(\alpha)$ satisfies (2.2), then

$$\|\mathcal{H} f\|_{L^2} \leq C \|f\|_{L^2}, \quad (2.5)$$

for some constant depends on $\beta_0$ and $\beta_1$ only.

From now on, we assume $z$ satisfies (2.2) (possibly with different constants $\beta_0$ and $\beta_1$).

We can use the Hilbert transform to characterize the boundary value of holomorphic functions. Such characterization is classical, the reader can see for example Proposition 2.1 in [56].
Lemma 2.2. Let \( \Omega \subset \mathbb{C} \) be a domain with \( C^1 \) boundary \( \Sigma : z = z(\alpha), \alpha \in \mathbb{R} \), oriented clockwise. Let \( \delta \) be the Hilbert transform associated to \( \Omega \).

(a.) Let \( g \in L^p \) for some \( 1 < p < \infty \). Then \( g \) is the boundary value of a holomorphic function \( G \) on \( \Omega \) with \( G(z) \to 0 \) at infinity if and only if

\[
(I - \delta)g = 0.
\] (2.6)

(b.) Let \( f \in L^p \) for some \( 1 < p < \infty \). Then \( \frac{1}{2}(I + \delta)f \) is the boundary value of a holomorphic function \( \mathcal{G} \) on \( \Omega \), with \( \mathcal{G}(z) \to 0 \) as \( |z| \to \infty \).

(c.) \( \delta 1 = 0 \).

2.2.2. Singular integrals and commutator estimates

Let \( m \geq 0 \) be an integer. Denote

\[
S_1(A, f) = \text{p.v.} \int \prod_{j=1}^{m} \frac{A_j(\alpha) - A_j(\beta)}{\gamma_j(\alpha) - \gamma_j(\beta)} \frac{f(\beta)}{\gamma_0(\alpha) - \gamma_0(\beta)} \, d\beta.
\] (2.7)

\[
S_2(A, f) = \int \prod_{j=1}^{m} \frac{A_j(\alpha) - A_j(\beta)}{\gamma_j(\alpha) - \gamma_j(\beta)} f(\beta) \, d\beta.
\] (2.8)

The two lemmas below are due, in their original form, to Calderon [7], Coifman, McIntosh, Meyer [13], Coifman, David, and Meyer [12]. See also Wu [53] for the proof of the second part of these lemmas using these results and the \( Tb \) Theorem.

Lemma 2.3. Assume each \( \gamma_j(j = 0, \ldots, m) \) satisfies

\[
C_{0, j} |\alpha - \beta| \leq |\gamma_j(\alpha) - \gamma_j(\beta)| \leq C_{1, j} |\alpha - \beta|,
\] (2.9)

for some constants \( C_{0, j}, C_{1, j}, j = 0, \ldots, m \). Then there is a constant \( C = C(C_{0,0}, \ldots, C_{0, m}) \) such that the following statements hold.

(1) For any \( f \in L^2 \), \( A'_i \in L^\infty \), \( 1 \leq i \leq m \),

\[
\| S_1(A, f) \|_{L^2} \leq C \| A'_1 \|_{L^\infty} \cdots \| A'_m \|_{L^\infty} \| f \|_{L^2}.
\]

(2) For any \( f \in L^\infty \), \( A'_i \in L^\infty \), \( 2 \leq i \leq m \), \( A'_1 \in L^2 \),

\[
\| S_1(A, f) \|_{L^2} \leq C \| A'_1 \|_{L^2} \| A'_2 \|_{L^\infty} \cdots \| A'_m \|_{L^\infty} \| f \|_{L^\infty}.
\]

Proof. This is Proposition 3.2 in Wu [53]. \( \square \)

Lemma 2.4. Assume each \( \gamma_j(j = 0, \ldots, m) \) satisfies

\[
C_{0, j} |\alpha - \beta| \leq |\gamma_j(\alpha) - \gamma_j(\beta)| \leq C_{1, j} |\alpha - \beta|,
\] (2.10)

for some constants \( C_{0, j}, C_{1, j}, j = 1, \ldots, m \). Then there is a constant \( C = C(C_{0,0}, \ldots, C_{0, m}) \) such that the following statements hold.

(1) For any \( f \in L^2 \), \( A'_i \in L^\infty \), \( 1 \leq i \leq m \),

\[
\| S_2(A, f) \|_{L^2} \leq C \| A'_1 \|_{L^\infty} \cdots \| A'_m \|_{L^\infty} \| f \|_{L^2}.
\]

(2) For any \( f \in L^\infty \), \( A'_i \in L^\infty \), \( 2 \leq i \leq m \), \( A'_1 \in L^2 \),

\[
\| S_2(A, f) \|_{L^2} \leq C \| A'_1 \|_{L^2} \| A'_2 \|_{L^\infty} \cdots \| A'_m \|_{L^\infty} \| f \|_{L^\infty}.
\]
Proof. This is Proposition 3.3 in Wu [53]. □

The following lemma is an important corollary of Lemmas 2.3 and 2.4, whose proof can be found in proposition 2.3 in Totz and Wu [44].

**Lemma 2.5.** Assume \( s \geq 4 \). Assume each \( \gamma_j (j = 1, \ldots, m) \) satisfies

\[
C_{0,j} |\alpha - \beta| \leq |\gamma_j(\alpha) - \gamma_j(\beta)| \leq C_{1,j} |\alpha - \beta|,
\]

for some constants \( C_{0,j}, C_{1,j}, j = 1, \ldots, m \).

**Lemma 2.6.** Suppose \( s \geq 4 \). Assume that \( z(\alpha) \) satisfies

\[
C_0 |\alpha - \beta| \leq |z(\alpha) - z(\beta)| \leq C_1 |\alpha - \beta|,
\]

and \( z_\alpha - 1 \in H^{s-1} \). Let \( \mathcal{S} \) be the Hilbert transform associated with \( z(\alpha) \). The following statements hold:

a. For any \( f \in H^s, A'_i \in W^{s-2,\infty}, 1 \leq i \leq m, \)

\[
\| S_2(A, f) \|_{H^s} \leq C \| A'_1 \|_{W^{s-2,\infty}} \cdots \| A'_m \|_{W^{s-2,\infty}} \| f \|_{H^s}.
\]

b. For any \( f \in W^{s-1,\infty}, A'_i \in W^{s-2,\infty}, 2 \leq i \leq m, A'_1 \in H^s, \)

\[
\| S_2(A, f) \|_{H^s} \leq C \| A'_1 \|_{H^{s-1}} \| A'_2 \|_{W^{s-2,\infty}} \cdots \| A'_m \|_{W^{s-2,\infty}} \| f \|_{W^{s-1,\infty}},
\]

where \( C \) is a constant depends on \( C_{0,j}, \| \partial_\alpha \gamma_j - 1 \|_{H^{s-1}}, j = 1, \ldots, m \).

An immediate consequence of Lemma 2.5 is the following commutator estimates.

**Lemma 2.6.** Suppose \( s \geq 4 \). Assume that \( z(\alpha) \) satisfies

\[
C_0 |\alpha - \beta| \leq |z(\alpha) - z(\beta)| \leq C_1 |\alpha - \beta|,
\]

and \( z_\alpha - 1 \in H^{s-1} \). Let \( \mathcal{S} \) be the Hilbert transform associated with \( z(\alpha) \). The following statements hold:

a. For any \( f \in H^{s-1}, g \in H^s, \)

\[
\|[f, \mathcal{S}] \frac{g_\alpha}{z_\alpha} \|_{H^s} \leq C \| f \|_{H^{s-1}} \| g \|_{H^s}.
\]

b. For any \( f \in H^s, g \in H^{s-1}, \)

\[
\|[f, \mathcal{S}] \frac{g_\alpha}{z_\alpha} \|_{H^s} \leq C \| f \|_{H^s} \| g \|_{H^{s-1}},
\]

for some constant \( C \) depends on \( C_0 \) and \( \| z_\alpha - 1 \|_{H^{s-1}} \).

**Proof.** By definition,

\[
[f, \mathcal{S}] \frac{g_\alpha}{z_\alpha} = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{z_\beta(\beta)(f(\alpha) - f(\beta)) g_\beta(\beta)}{z(\alpha) - z(\beta)} \frac{g_\beta(\beta)}{z_\beta} d\beta = \pi i S_2(f, g).
\]

So the lemma follows from Lemma 2.5. □
2.2.3. Double layer potentials and its adjoint  We define the double layer potential operator as follows.

**Definition 2.2 (Double layer potential).** Suppose \( z(\alpha) \) satisfies (2.2). Let \( \Sigma \) be the curve parametrized by \( z(\alpha) \), and \( \Omega \) the region in \( \mathbb{C} \) bounded above by \( \Sigma \). Let \( \vec{n} \) be the outward normal of \( \Omega \). The so-called double layer potential operator \( \kappa \) is defined by, for \( f \in L^2 \),

\[
\kappa f(\alpha) := p.v. \int_{-\infty}^{\infty} \Re \left\{ \frac{1}{\pi i} \frac{z_\beta}{z(\alpha, t) - z(\beta, t)} \right\} f(\beta) d\beta. \tag{2.13}
\]

The adjoint of the double layer potential \( \kappa^* \) is defined as

\[
\kappa^* f(\alpha) := p.v. \int_{-\infty}^{\infty} \Re \left\{ -\frac{1}{\pi i} \frac{z_\alpha}{|z_\alpha|} \frac{|z_\beta|}{z(\alpha) - z(\beta)} \right\} f(\beta) d\beta. \tag{2.14}
\]

By Lemma 2.1, \( \| \kappa f \|_{L^2} \leq C \| f \|_{L^2} \), so \( \kappa f \) is well-defined as an \( L^2 \) function. Similarly, for \( f \in L^2 \), \( \| \kappa^* f \|_{L^2} \leq C \| f \|_{L^2} \), so \( \kappa^* f \) is also well-defined as an \( L^2 \) function. Moreover, we have the following celebrated results due to Verchota [45]. See also [13, 42].

**Lemma 2.7.** Let \( \Sigma \) be a Jordan curve parametrized by \( z(\alpha) \) such that \( z(\alpha) \) satisfies (2.2) and \( z(\alpha) \) approaches \( \alpha \) at infinity. Then \( I \pm \kappa : L^2 \to L^2 \) and their adjoints \( I \pm \kappa^* : L^2 \to L^2 \) are invertible, with

\[
\|(I \pm \kappa)^{-1} f\|_{L^2} \leq C \| f \|_{L^2}, \tag{2.15}
\]

and

\[
\|(I \pm \kappa^*)^{-1} f\|_{L^2} \leq C \| f \|_{L^2}, \tag{2.16}
\]

for some constant \( C > 0 \) depends only on \( \beta_0 \) and \( \beta_1 \).

2.3. The point vortices. Suppose we have, in a domain \( D \), an initial profile of vorticity given by \( \omega(z) = \sum_{j=1}^{N} \lambda_j \delta_{z_j}(z) \), where \( N \) is a positive integer, \( \{z_j : j = 1, \cdots, N\} \subset D \) are distinct points, \( \lambda_j \in \mathbb{R} \), and \( \delta_{z_j} \) is the Dirac measure concentrated on the point \( z_j \). Then the single component \( \lambda_j z_j \) is called a point vortex at \( z_j \) with strength \( \lambda_j \).

Since the fundamental solution of Laplacian in \( \mathbb{R}^2 \) is \( -\frac{1}{2\pi} \log \sqrt{x^2 + y^2} \), a point vortex \( \lambda_j \delta_{z_j} \) generates a velocity field

\[
v_j(z) := \frac{\lambda_j i}{2\pi} \frac{1}{z - z_j},
\]

which rotates about \( z_j \) at a constant angular velocity.
2.4. Some estimates involving point vortices.

**Lemma 2.8.** Assume that \(d_I(t) = \inf_{\alpha \in \mathbb{R}} \min_{1 \leq j \leq N} |z(\alpha, t) - z_j(t)| > 0\) and \(z(\alpha, t)\) satisfies

\[C_0|\alpha - \beta| \leq |z(\alpha) - z(\beta)| \leq C_1|\alpha - \beta|.
\]

Let \(k > 1\). Then

\[
\int_{-\infty}^{\infty} \frac{1}{|z_j(t) - z(\beta, t)|^k} d\beta \leq C d_I(t)^{-k+1},
\]

(2.17)

where \(C = 4C_0^{-1} + 4C_1^{k-1} (k-1)C_0^{k-1}\).

**Proof.** We may assume that \(d_I(t) = d(z_j(t), z(0, t))\).

\[
\int_{-\infty}^{\infty} \frac{1}{|z_j(t) - z(\beta, t)|^k} d\beta = \int_{|z(0, t) - z(\beta, t)| \leq 2d_I(t)} \frac{1}{|z_j(t) - z(\beta, t)|^k} d\beta + \int_{|z(0, t) - z(\beta, t)| \geq 2d_I(t)} \frac{1}{|z_j(t) - z(\beta, t)|^k} d\beta
\]

:= I + II.

Denote

\[E := \{\beta : |z(0, t) - z(\beta, t)| \leq 2d_I(t)\}.
\]

Since \(C_0|\beta - 0| \leq |z(\beta, t) - z(0, t)|\), we have for \(\beta \in E\),

\[|\beta - 0| \leq \frac{2}{C_0} d_I(t).
\]

Therefore

\[I \leq 4C_0^{-1} d_I(t)^{-k+1}.
\]

For \(\beta \in E^c\), we have

\[|z(\beta, t) - z(0, t) - d_I(t)| \geq |z(\beta, t) - z(0, t)| - d_I(t) \geq \frac{1}{2} |z(\beta, t) - z(0, t)|
\]

\[\geq \frac{C_0}{2} |\beta - 0|.
\]

(2.18)

Also, for \(\beta \in E^c\), use chord-arc condition (2.12), we have

\[C_1|\beta - 0| \geq |z(\beta, t) - z(0, t)| \geq 2d_I(t).
\]

(2.19)

So

\[|\beta| \geq \frac{2}{C_1} d_I(t).
\]

(2.20)
Therefore, for $II$, we have
\[
II \leq \frac{2^k}{C_0^k} \int_{|\beta| \geq \frac{2}{k+1}C_0^k} |\beta|^{-k} d\beta = 2^k \frac{C_0^{k-1}}{(k-1)C_0^k} \int_{|\beta| \geq \frac{2}{k+1}C_0^k} |\beta|^{-k} dI(t)^{-k+1} = 4 \frac{C_0^{k-1}}{(k-1)C_0^k} \int_{|\beta| \geq \frac{2}{k+1}C_0^k} |\beta|^{-k} dI(t)^{-k+1}.
\]

\[\square\]

**Corollary 2.1.** Under the same assumptions as in Lemma 2.8, given $m \geq 2$, there exist $C = (k+1)!C(C_0, C_1, \|z_\alpha - 1\|_{H^{m-1}})$ such that
\[
\| \frac{1}{(z(\alpha, t) - z_j(t))^k} \|_{H^m} \leq C (d_I(t)^{-k+1/2} + d_I(t)^{-k-m+1/2}).
\]

In particular, if $d_I(t) \geq 1$, then we have
\[
\| \frac{1}{(z(\alpha, t) - z_j(t))^k} \|_{H^m} \leq C d_I(t)^{-k+1/2}.
\]

2.5. Governing equation for the free boundary. Throughout this paper, we denote the fluid region at time $t$ by $\Omega(t)$, with a nonself-intersect free surface $\Sigma(t)$. For a point $x + iy \in \Sigma(t)$, we assume $|y| \to 0$ as $|x| \to \infty$. That is, $\Sigma(t)$ approaches the real axis at $\pm \infty$.

The system (1.1)–(1.2) is completely determined by the free surface $\Sigma(t)$, the trace of the velocity $v$ along the free surface, and the position of the point vortices. On one hand, it’s natural to study this system in lagrangian coordinates, so we present a lagrangian formulation of (1.1)–(1.2), which will be used in studying the long time existence. On the other hand, because of the moving boundary, it’s convenient to study the Taylor sign and construct solutions in Riemann mapping variables. So we present the Riemann mapping formulation as well.

2.5.1. Lagrangian formulation We parametrize the free surface by Lagrangian coordinates, i.e., let $\alpha$ be such that
\[
z_t(\alpha, t) = v(z(\alpha, t), t).
\]
\[
P \Big|_{\Sigma(t)} \equiv 0 \text{ implies that } \nabla P \Big|_{\Sigma(t)} \text{ is along the normal direction, so we can write } \nabla P \text{ as } -ia z_\alpha, \text{ where } a = -\frac{\partial P}{\partial n} \frac{1}{|z_\alpha|} \text{ is real valued. Here, } \vec{n} = i \frac{z_\alpha}{|z_\alpha|} \text{ is the unit outward normal to } \Omega(t). \text{ So the trace of the momentum equation } v_t + v \cdot \nabla v = -\nabla P - (0, 1) \text{ can be written as }
\]
\[
z_{tt} - i a z_\alpha = -i.
\]
We decompose $z_t$ as $\tilde{z}_t = f + p$, where $p = -\sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \frac{1}{z(\alpha, t) - z_j(t)}$. Note that $f$ is holomorphic in $\Omega(t)$ with the value at the boundary $\Sigma(t)$ given by $\tilde{z}_t + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi(z(\alpha, t) - z_j(t))}$. Lemma 2.2 implies that $f$ is the boundary value of a holomorphic function in $\Omega(t)$ if and only if
\[
(I - \delta f) = 0.
\]
where $\hat{f}$ is the Hilbert transform associated with the curve $z(\alpha, t)$, i.e.,

$$
\hat{f}(\alpha) := \frac{1}{\pi i} \text{p.v.} \int_{-\infty}^{\infty} \frac{z_\beta}{z(\alpha, t) - z(\beta, t)} f(\beta) d\beta.
$$

(2.26)

Because of the singularity of the velocity at the point vortices, we don’t have $(I - \hat{\delta}) \bar{z}_t = 0$. However, the following lemma asserts that $\bar{z}_t$ is almost holomorphic, in the sense that $(I - \hat{\delta}) \bar{z}_t$ consists of lower order terms.

**Lemma 2.9 (Almost holomorphicity).** Assume that $z(\cdot, t) \in L^2(\mathbb{R})$ satisfies (2.2) and $\bar{z}_t$ is the boundary value of a velocity field $\bar{v}$ in $\Omega(t)$ such that $\bar{v} + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi(z - z_j(t))}$ is holomorphic in $\Omega(t)$. Then we have

$$
(I - \hat{\delta}) \bar{z}_t = -\frac{i}{\pi} \sum_{j=1}^{N} \frac{\lambda_j}{z(\alpha, t) - z_j(t)}.
$$

(2.27)

**Proof.** Since $\bar{z}_t + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi(z(\alpha, t) - z_j(t))}$ is the boundary value of a holomorphic function in $\Omega(t)$, by lemma 2.2,

$$(I - \hat{\delta})(\bar{z}_t + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi(z(\alpha, t) - z_j(t))}) = 0,
$$

we have

$$
(I - \hat{\delta}) \bar{z}_t = -\sum_{j=1}^{N} (I - \hat{\delta}) \frac{\lambda_j i}{2\pi(z(\alpha, t) - z_j(t))}.
$$

(2.28)

Since $\frac{1}{z(\alpha, t) - z_j(t)}$ is boundary value of the holomorphic function $\frac{1}{\bar{z} - z_j(t)}$ in $\Omega(t)$, by lemma 2.2 again, we have

$$
(I - \hat{\delta}) \frac{1}{z(\alpha, t) - z_j(t)} = \frac{2}{z(\alpha, t) - z_j(t)}.
$$

(2.29)

(2.28) together with (2.29) complete the proof of the lemma. □

So the system (1.1) is reduced to a system of equations for the free boundary coupled with the dynamic equation for the motion of the point vortices:

$$
\begin{align*}
\frac{\partial z}{\partial t} - i a z = & -i \\
\frac{\partial^2 z}{\partial t^2} - i a \frac{\partial z}{\partial t} = & (v - \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi(z - z_j(t))})|_{z = z_j} \\
(I - \hat{\delta}) f = & 0.
\end{align*}
$$

(2.30)

Note that $v$ can be recovered from (2.30). Indeed, we have

$$
\bar{v}(z, t) + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi(z - z_j(t))} = \frac{1}{2\pi i} \int \frac{z_\beta}{z - z(\beta)} \left( \bar{z}_t(\beta, t) + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi(z(\beta, t) - z_j(t))} \right) d\beta.
$$

(2.31)
So the system (1.1) and the system (2.30) are equivalent. The quantity $a|z_\alpha|$ plays an important role in the study of water waves.¹

**Definition 2.3** (The Taylor sign condition and the strong Taylor sign condition).

1. If $a|z_\alpha| \geq 0$ pointwisely, then we say the Taylor sign condition holds.
2. If there is some positive constant $c_0$ such that $a|z_\alpha| \geq c_0 > 0$ pointwisely, then we say the strong Taylor sign condition holds.

It is well known that when surface tension is neglected and the Taylor sign condition fails, the motion of the water waves can be subject to the Taylor instability [4,6,21,41,56]. For irrotational incompressible infinite depth water waves without surface tension, S. Wu [51,52] showed that the strong Taylor sign condition always holds provided that the interface is non-self-intersecting and smooth. For water waves with nonzero vorticity, the Taylor sign has to be assumed. To derive a useful formula for the Taylor sign coefficient and use the iteration method to construct solutions to (2.30), we use the Riemann mapping formulation.

### 2.5.2. The Riemann mapping formulation.

Let $\mathbb{P}_- = \{z \in \mathbb{C} : \Im\{z\} < 0\}$. Let $\Phi(\cdot, t) : \Omega(t) \to \mathbb{P}_-$ be the Riemann mapping such that $\Phi \to 1$ as $z \to \infty$. Denote

\[
\begin{align*}
  h(\alpha, t) &:= \Phi(z(\alpha), t), \\
  Z(\alpha, t) &:= z \circ h^{-1}(\alpha, t), \\
  B &= h_t \circ h^{-1}, \\
  D_t &= \partial_t + B\partial_\alpha, \\
  A &= (ah_\alpha) \circ h^{-1}, \\
  F &= f \circ h^{-1}.
\end{align*}
\]

(2.32)

In Riemann mapping variables, the system (2.30) becomes

\[
\begin{align*}
  (D_t^2 - iA\partial_\alpha) Z &= -i \\
  \frac{d}{dt}z_j(t) &= (v - \frac{\lambda_j i}{2\pi(z - z_j)}) \bigg|_{z=z_j} \\
  (I - \mathbb{H}) F &= 0.
\end{align*}
\]

(2.33)

Here, $\mathbb{H}$ is the standard Hilbert transform which is defined by

\[
\mathbb{H}f(\alpha) := \frac{1}{\pi i} p.v. \int_{-\infty}^{\infty} \frac{1}{\alpha - \beta} f(\beta) d\beta.
\]

(2.34)

Denote

\[
A_1 := A|Z_\alpha|^2.
\]

(2.35)

Since $(a|z_\alpha|) \circ h^{-1} = A|Z_\alpha| = \frac{A_1}{|Z_\alpha|}$, it’s clear that the Taylor sign condition holds if and only if

\[
\inf_{\alpha \in \mathbb{R}} \frac{A_1}{|Z_\alpha|} \geq 0,
\]

(2.36)

¹ Indeed, $a|z_\alpha| = -\frac{\partial P}{\partial n}\bigg|_{\Sigma(t)}$. 

and the strong Taylor sign condition holds if and only if
\[
\inf_{\alpha \in \mathbb{R}} \frac{A_1}{|Z_\alpha|} > 0.
\] (2.37)

**Convention:** Let \( \Sigma_0 \) be the initial free interface, and \( z_0(\alpha) \) be the parametrization of \( \Sigma_0 \) by the Riemann mapping.

### 2.6. Basic identities.

**Lemma 2.10.** Let \( s \geq 4 \) and \( 0 < T < \infty \). Assume that \( z \) is a solution to (2.30) such that \( z \in C([0, T]; H^{s+1/2}) \cap C^1([0, T]; H^s) \), \( f \in C^1(\mathbb{R} \times [0, T]) \) and \( f_\alpha(\alpha, t) \to 0 \) as \( |\alpha| \to \infty \). We have

\[
[\partial_t, \mathcal{S}] f = \left[ z_t, \mathcal{S} \right] \frac{f_\alpha}{z_\alpha}
\] (2.38)

\[
[\partial^2_t, \mathcal{S}] f = \left[ z_{tt}, \mathcal{S} \right] \frac{f_\alpha}{z_\alpha} + 2\left[ z_t, \mathcal{S} \right] \frac{f_{t\alpha}}{z_\alpha} - \frac{1}{\pi i} \int \left( \frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)} \right)^2 f_\beta(\beta, t) d\beta
\] (2.39)

\[
[a\partial_\alpha, \mathcal{S}] f = \left[ az_\alpha, \mathcal{S} \right] \frac{f_\alpha}{z_\alpha}, \quad \partial_\alpha \mathcal{S} f = z_\alpha \mathcal{S} \frac{f_\alpha}{z_\alpha}
\] (2.40)

\[
[\partial^2_t - ia\partial_\alpha, \mathcal{S}] f = 2\left[ z_t, \mathcal{S} \right] \frac{f_{t\alpha}}{z_\alpha} - \frac{1}{\pi i} \int \left( \frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)} \right)^2 f_\beta(\beta, t) d\beta.
\] (2.41)

**Lemma 2.10** is an easy consequence of integrating by parts. For details, see lemma 2.1 of [53].

**Lemma 2.11.** Let \( D_t = \partial_t + b\partial_\alpha \), then

\[
[D_t^2, \partial_\alpha] = -D_t(b_\alpha)\partial_\alpha - b_\alpha D_t \partial_\alpha - b_\alpha \partial_\alpha D_t
\] (2.42)

\[
[D_t^2, \partial^k_\alpha] = \sum_{m=0}^{k-1} \left[ \partial^m_\alpha(D_t b_\alpha) \partial^{k-m}_\alpha + \partial^m_\alpha(b_\alpha \partial^{k-m}_\alpha D_t) + \partial^m_\alpha(b_\alpha [b\partial_\alpha, \partial^{k-m}_\alpha]) + \partial^m_\alpha b_\alpha \partial^{k-m}_\alpha D_t \right.
\]

\[
+ \partial^m_\alpha b_\alpha [b\partial_\alpha, b] \partial^{k-m-1}_\alpha \right]
\] (2.43)

**Proof.** By direct calculation,

\[
[D_t^2, \partial^k_\alpha] = -\sum_{m=0}^{k-1} \left[ \partial^m_\alpha(D_t b_\alpha) \partial^{k-m}_\alpha + \partial^m_\alpha(b_\alpha D_t \partial^{k-m}_\alpha) + \partial^m_\alpha b_\alpha \partial_\alpha D_t \partial^{k-m-1}_\alpha \right]
\]

\[
= -\sum_{m=0}^{k-1} \left[ \partial^m_\alpha(D_t b_\alpha) \partial^{k-m}_\alpha + \partial^m_\alpha(b_\alpha \partial^{k-m}_\alpha D_t) + \partial^m_\alpha(b_\alpha [b\partial_\alpha, \partial^{k-m}_\alpha]) + \partial^m_\alpha b_\alpha \partial^{k-m}_\alpha D_t \right.
\]

\[
+ \partial^m_\alpha b_\alpha [b\partial_\alpha, b] \partial^{k-m-1}_\alpha \right]
\]

\( \square \)
3. Main Results

3.1. Main result one: the Taylor sign condition. Our first result is a formula for the Taylor sign coefficient $A_1$. We also use this formula to construct counter-examples for which the Taylor sign condition fails and find a criterion for strong Taylor sign condition to hold.

3.1.1. Statement of the theorem  Recall that $\tilde{z}_t = f + p$, where $p = -\sum_{j=1}^{N} \frac{\lambda_j}{2\pi} z(\alpha,t) - z_j(t)$. Let $F = f \circ h^{-1}$. Recall that $\Phi(\cdot, t) : \Omega(t) \to \mathbb{P}_-$ is the Riemann mapping.

Theorem 1. Denote

$$c_j^\beta := (\Phi^{-1})(\omega_j^\beta, t), \quad \omega_j^\beta := \Phi(z_j(t), t).$$

$$\beta_0(t) := \inf_{\alpha \in \mathbb{R}} |Z_\alpha(\alpha, t)|, \quad M_0(t) := \|F(\cdot, t)\|_{\infty}$$

$$\tilde{\lambda} := \sum_{j=1}^{N} |\lambda_j| \frac{1}{2\pi}, \quad \tilde{d}_I(t) := \min_{1 \leq j \leq N} \inf_{\alpha \in \mathbb{R}} |\alpha - \Phi(z_j(t), t)|,$$

$$\tilde{d}_P(t) = \min_{j \neq k} |z_j(t) - z_k(t)|.$$  \hspace{1cm} (3.3)

Assume that the system (2.33) admits a solution $Z(\alpha, t)$ on $[0, T_0]$ such that

$$(Z_\alpha - 1, D_t Z) \in C([0, T_0]; H^3 \times H^3),$$

and

$$0 < \tilde{d}_I(t) < \infty, \quad 0 < \tilde{d}_P(t) < \infty, \quad \forall t \in [0, T_0].$$

(1) (Formula for the Taylor sign condition) We have

$$A_1(\alpha, t) = 1 + \frac{1}{2\pi} \int \frac{|D_t Z(\alpha, t) - D_t Z(\beta, t)|^2}{(\alpha - \beta)^2} d\beta - \sum_{j=1}^{N} \frac{\lambda_j}{\pi} \Re \left\{ \frac{D_t Z - \dot{z}_j(t)}{c_j^\beta(\alpha - w_j^\beta)^2} \right\},$$

$$\hspace{1cm} (3.4)$$

(2) (Failure of the Taylor sign condition) ³

**One point vortex** Assume that at time $t \in [0, T_0]$, the interface is $\Sigma(t) = \mathbb{R}$, and the fluid velocity is $v(z, t) = \frac{\lambda_i}{2\pi} \frac{1}{z - z_1(t)}$, i.e., it is generated by a single point vortex $z_1(t) := x(t) + iy(t)$. Then

a. If $\frac{\lambda_i^2}{|y|^3} < \frac{8\pi^2}{3}$, then the strong Taylor sign condition holds. We have

$$\inf_{\alpha \in \mathbb{R}} A_1(\alpha, t) \geq 1 - \frac{3}{8\pi^2} \frac{\lambda_i^2}{|y|^3} > 0.$$  \hspace{1cm} (3.5)

b. If $\frac{\lambda_i^2}{|y|^3} > \frac{8\pi^2}{3}$, then the Taylor sign condition fails, i.e., there exists $\alpha \in \mathbb{R}$ such that $A_1(\alpha, t) < 0$.

³ The assumption that $(Z_\alpha - 1, D_t Z) \in H^3 \times H^3$ is of course not optimal.
³ This is a conditional result, we assume a priori the existence of the solution.
c. If \( \frac{\lambda^2}{|y|^2} = \frac{8\pi^2}{3} \), then the 'Degenerate Taylor sign condition' holds, i.e.,

\[
A_1(\alpha, t) > 0, \quad \forall \alpha \neq x(t); \quad \text{and} \quad A_1(x(t), t) = 0.
\] (3.6)

**Two point vortices** Assume that at time \( t \in [0, T_0] \), the interface is \( \Sigma(t) = \mathbb{R} \), and the fluid velocity is \( v(z, t) = \sum_{j=1}^{2} \frac{\lambda_j}{2\pi} \frac{1}{z - z_j(t)} \), where \( \lambda_1 = -\lambda_2 := \lambda \). Assume \( z_1(t) = -x + iy \), \( z_2(t) = x + iy \), where \( x > 0, \ y < 0 \).

If

\[
1 + \frac{\lambda^2}{4\pi^2} \frac{x^4 + 4y^4 - 7x^2y^2}{|y|(x^2 + y^2)^3} < 0,
\] (3.7)

then Taylor sign condition fails.

(3) (A criterion for strong Taylor sign condition) If

\[
\frac{\tilde{\lambda}^2}{d_I(t)^2} + \frac{\tilde{\lambda}^2}{d_I(t)^2 d_P(t)} + \frac{2M_0 \tilde{\lambda}}{d_I(t)^2} < \beta_0,
\] (3.8)

then the strong Taylor sign condition holds.

3.1.2. Difficulty and the strategy It’s difficult to calculate the Taylor sign coefficient in lagrangian coordinates. To overcome this difficulty, we follow S. Wu’s work (section 3 in [51]) to calculate the Taylor sign condition. To construct examples for which the Taylor sign condition fails, we consider initially still water waves with its motion purely generated by the point vortices. We can derive a formula for \( A_1 \) in terms of the intensity and location of these point vortices, from which we can see that Taylor sign condition could fail if the point vortices are close to the interface. Part (1) is proved in Corollary 4.4, part (2) is proved by examples from §4.3, §4.4, and part (3) is proved in Proposition 3.

3.2. Main result two: local wellposeness. Our second result confirms that (2.30) is locally wellposed in Sobolev spaces, provided that the strong Taylor sign condition holds initially.

3.2.1. Statement of the theorem Denote

\[
z_0(\alpha) = z(\alpha, 0), \quad \xi_0(\alpha) := z_0(\alpha) - \alpha, \quad v_0(\alpha) := z_t(\alpha, 0), \quad (3.9)
\]

\[
d_I(t) := \inf_{\alpha \in \mathbb{R}} \min_{1 \leq j \leq N} |z(\alpha, t) - z_j(t)|, \quad d_P(t) := \min_{1 \leq i, j \leq N, i \neq j} |z_i(t) - z_j(t)|.
\] (3.10)

Here, \( d_I(t) \) represents the distance of the point vortices to the free boundary, the 'I' means the interface. \( d_P(t) \) represents the distance among the point vortices, 'P' means the point vortices. Let \( \Sigma_0 \) be the curve parametrized by \( z_0 \), and \( \Omega_0 \) the initial fluid region bounded above by \( \Sigma_0 \).
Theorem 2 (The local wellposedness). Assume \( s \geq 4 \). Assume the initial vorticity \( \omega_0 = \sum_{j=1}^{N} \lambda_j \delta_{z_j,0} \), where \( N \geq 1 \) is an integer, \( \lambda_j \in \mathbb{R} \), and \( \{z_j,0\}_{j=1}^{N} \subset \Omega_0 \) are distinct points in \( \Omega(0) \). Assume \( (\xi_0, v_0) \in H^{s+1/2} \times H^{s+1/2} \), satisfying

(H1) Strong Taylor sign assumption. There is some \( \alpha_0 > 0 \) such that

\[
\inf_{\alpha \in \mathbb{R}} a(\alpha, 0) |\partial_\alpha z_0(\alpha)| \geq 2\alpha_0 > 0.
\] (3.11)

(H2) Chord-arc assumption. There are constants \( C_0, C_1 > 0 \) such that

\[
C_0|\alpha - \beta| \leq |z(\alpha, 0) - z(\beta, 0)| \leq C_1|\alpha - \beta|.
\] (3.12)

Then there exists \( T_0 > 0 \) such that (2.30) admits a unique solution

\[
(z_\alpha - 1, z_t, z_{tt}) \in C([0, T_0]; H^{s-1/2} \times H^{s+1/2} \times H^s),
\]

\[
z_j \in C^2([0, T_0]; \Omega(t)), \quad j = 1, \ldots, N,
\]

with \( T_0 \) depends on \( s, \sum_{j=1}^{N} |\lambda_j|, \| (\partial_\alpha \xi_0, v_0) \|_{H^{s-1/2} \times H^{s+1/2}}, d_f(0)^{-1}, d_p(0)^{-1}, C_0, C_1, \alpha_0 \), and

\[
\inf_{t \in [0, T_0]} \inf_{\alpha \in \mathbb{R}} a(\alpha, t)|z_\alpha| \geq \alpha_0/2.
\] (3.13)

\[
|z(\alpha, t) - z(\beta, t)| \geq \frac{1}{2} C_0|\alpha - \beta|, \quad \forall \alpha, \beta \in \mathbb{R}, \quad t \in [0, T_0].
\] (3.14)

Moreover, if \( T_0^* \) is the maximal lifespan, then either \( T_0^* = \infty \), or \( T_0^* < \infty \), but

\[
\lim_{T \to T_0^* -} \| (z_t, z_{tt}) \|_{C([0, T]; H^{s+1/2} \times H^s)} + \sup_{0 \leq t < T_0^*} \frac{|\alpha - \beta|}{|z(\alpha, t) - z(\beta, t)|} + \sup_{t \to T_0^*} (d_f(t)^{-1} + d_p(t)^{-1}) = \infty
\] (3.15)

or

\[
\lim_{t \to T_0^* -} \inf_{\alpha \in \mathbb{R}} a(\alpha, t)|z_\alpha(\alpha, t)| \leq 0,
\] (3.16)

or

\[
\sup_{\alpha \neq \beta, 0 \leq t < T_0^*} \frac{|z(\alpha, t) - z(\beta, t)|}{|\alpha - \beta|} = \infty.
\] (3.17)
3.2.2. The main difficulty and the strategy of proof of Theorem 2  The momentum equation \( z_{tt} - iaz_\alpha = -i \) is fully nonlinear. So the main difficulty is to obtain its quasilinearization. If there is no point vortices in the water waves, S. Wu observed that one can obtain quasilinearization of the system (2.30) by taking one time derivative to the momentum equation. This is still true for water waves with point vortices: take \( \partial_t \) on both sides of \( (\partial_t^2 + i a \partial_\alpha)^2 \bar{z} = i \), we obtain
\[
(\partial_t^2 + i a \partial_\alpha)^2 \bar{z}_t = -i a_t \bar{z}_\alpha.  
\] (3.18)

We decompose \( \bar{z}_t \) as \( \bar{z}_t = f + p \), where \( p = -\sum_{j=1}^N \frac{\lambda_j}{2\pi} \frac{1}{z(\alpha,t) - z_j(t)} \). A key observation is that \( (\partial_t^2 + i a \partial_\alpha) p \) consists of lower order terms. Apply \( I - \mathcal{S} \) on both sides of equation (3.18), we obtain
\[
-i(I - \mathcal{S}) a_t \bar{z}_\alpha = g_1 + g_2,  
\] (3.19)
where
\[
g_1 := 2[z_{tt}, \mathcal{S}] \frac{\bar{z}_{t\alpha}}{z_\alpha} + 2[z_t, \mathcal{S}] \frac{\bar{z}_{t\alpha}}{z_\alpha} - \frac{1}{\pi} \int \left( \frac{z_t(\alpha, t) - z_\alpha(t)}{z(\alpha, t) - z(\beta, t)} \right)^2 \bar{z}_t \beta d\beta.  
\] (3.20)
\[
g_2 := \frac{i}{\pi} \sum_{j=1}^N \lambda_j \left( \frac{2z_{tt} + i - \partial_t^2 z_j}{(z(\alpha, t) - z_j(t))^2} - 2 \frac{(z_t - \bar{z}_j(t))^2}{(z(\alpha, t) - z_j(t))^3} \right).  
\] (3.21)

So \( a_t \bar{z}_\alpha \) is of lower order. The quasilinear system
\[
\begin{aligned}
(\partial_t^2 + i a \partial_\alpha) \bar{z}_t &= -i a_t \bar{z}_\alpha \\
\bar{z}_j(t) &= (v - \frac{\lambda_j}{2\pi} \frac{1}{z - z_j(t)})|_{z = z_j(t)} \\
(I - \mathcal{S}) f &= 0
\end{aligned}  
\] (3.22)
is of hyperbolic type as long as the Taylor sign condition \( a|z_\alpha| \geq a_0 > 0 \) holds. The the local wellposedness of (3.22) can be proved by standard iteration and fixed point method. A proof of the local wellposedness of the 3d counterpart of (3.22) without point vortices was carried out in [52]. While the local wellposedness can be proved directly in Lagrangian coordinates (as in the 3d counterpart by Wu [52]), to avoid getting involved in the complicated Hilbert transform and layer potentials associated to time dependent free interfaces, we prove the existence and uniqueness in Riemann mapping coordinates. The process is as follows:

(a) Obtain a quasilinear system in Riemann mapping variables, which is the version of (3.19) in Riemann mapping variables.
(b) Prove the local wellposedness of the linear system:
\[
\begin{aligned}
(D_t^2 + AA) F &= G, \\
F(\cdot, 0) &= F_0, \quad D_t F(\cdot, 0) = F_1 \\
D_t &= \partial_t + B \partial_\alpha, \\
B \in C([0, T]; H^{s+1/2}), \quad G \in C([0, T]; H^s), \quad F_0 \in H^{s+1/2} \quad \text{and} \quad F_1 \in H^s 
\end{aligned}  
\] (3.23)
where \( B \in C([0, T]; H^{s+1/2}), G \in C([0, T]; H^s), F_0 \in H^{s+1/2} \) and \( F_1 \in H^s \) are given. This system is equivalent to the linear system (5.14) studied by Wu in [51].
(c) Using the existence and uniqueness of (3.23), we construct a sequence of approximate solutions to the water waves with point vortices in Riemann mapping variables. We need to pay some attention to the quantities due to the point vortices. Other than this, the argument is almost the same as that in [51].
3.3. The third main result: the long time behavior. Our third result is concerned with the extended lifespan of the water waves with point vortices. Assuming small data, we show that if the water wave is symmetric with a symmetric vortex pair traveling downward initially, then the free interface remains smooth for a long time, and for initial data satisfying \( \| (\Lambda^{1/2} \xi_0, f_0, f_1) \|_{H^s \times H^{s+1/2}} \leq \epsilon \ll 1, \)
then the lifespan is of size \( O(\epsilon^{-2}) \). Here, \( f_0, f_1 \) are the initial value of \( f, f_1 \), respectively.

We make the following assumptions:

(H3) Vortex pair assumption. Assume \( N = 2 \), i.e., there are two point vortices, with positions \( z_1(t) = x_1(t) + i y_1(t) \), \( z_2(t) = x_2(t) + i y_2(t) \), strength \( \lambda_1, \lambda_2 \), respectively. Assume further that \( z_1(t) \) and \( z_2(t) \) are symmetric about the \( y \)-axis, i.e.,
\[
x_1(t) = -x_2(t) =: -x(t) < 0, \quad y_1(t) = y_2(t) := y(t) < 0,
\]
and assume \( \lambda_1 = -\lambda_2 := \lambda < 0 \).

(H4) Symmetry assumption. Assume that velocity field \( v = v_1 + i v_2 \) satisfies: \( v_1 \) odd in \( x \), \( v_2 \) even in \( x \), and the free boundary \( \Sigma(t) \) is symmetric about the \( y \)-axis.

(H5) Smallness assumption. Assume that at \( t = 0 \),
\[
\| (\Lambda^{1/2} \xi_0, f_0, f_1) \|_{H^s \times H^{s+1/2}} \leq \epsilon, \quad \lambda^2 + |\lambda x(0)| \leq c_0 \epsilon,
\]
for some constant \( c_0 = c_0(s) \). We can take \( c_0 = \frac{1}{(s+1)^2!} \).

(H6) Vortex-vortex interaction. Assume \( \frac{|\lambda|}{\chi(0)} \geq M \epsilon \) for some constant \( M \gg 1 \) (say, \( M = 100 \pi \)).

(H7) Vortex-interface interaction. Assume \( d_I(0) \geq 1 \).

**Theorem 3** (Long time behavior). Let \( s \geq 4 \). Assume (H3)–(H7). There exists \( \epsilon_0 > 0 \) and \( \delta > 0 \) such that for all \( 0 < \epsilon \leq \epsilon_0 \), the lifespan \( T_0 \geq \delta \epsilon^{-2} \). Moreover,
\[
\| \Lambda^{1/2} (z(\alpha, t) - \alpha) \|_{H^s} \leq 6 \epsilon, \quad \| f \|_{H^{s+1/2}} \leq 6 \epsilon, \quad \| f_1 \|_{H^s} \leq 6 \epsilon, \quad \forall \ t \in [0, \delta \epsilon^{-2}].
\]

Here, \( \delta \) is an absolute constant independent of \( \epsilon \) and \( s \).

**Remark 3.1.** The assumption that \( d_I(0) \geq 1 \) can be relaxed. Moreover,

1. The assumption \( \frac{|\lambda|}{\chi(0)} \geq M \epsilon \) ensures that the point vortices travel downward at \( t = 0 \). In the proof of Theorem 3, we will show that when \( \frac{|\lambda|}{\chi(0)} = M \epsilon \), the velocity of the point vortices is comparable to \( \epsilon \), which is slow in some sense. Theorem 3 demonstrates that even if the point vortices moves at an initial velocity as slow as \( M \epsilon \), the water waves still remain smooth and small for a long time.
2. Assumption (H7) implies that the strong Taylor sign condition holds initially.
3. The assumptions (H5), (H6), (H7) do allow \( x(0) \) to be as small as we want. So \( \frac{|\lambda|}{\chi(0)} \) can be very large.

---

\(^4\) \( f_1 \) is not arbitrary, it must satisfy some compatible condition. See (5.23).
3.3.1. Difficulty and the strategy of the proof of Theorem 3 To illustrate the idea of studying long time behavior, we begin with the following toy model.

**Toy model:** Consider

\[ u_{tt} + \Lambda u = u_t^p + \frac{C}{(\alpha + it)^m}, \quad p \geq 2, m \geq 2 \]  

(3.25)

for some constant C such that \(|C| \lesssim \epsilon\). Define an energy

\[ E_s(t) = \sum_{k \leq s} \int |\partial^k \alpha u_t(\alpha, t)|^2 + |\Lambda^{1/2} \partial^k \alpha u|^2 d\alpha. \]  

(3.26)

Then we have

\[ \frac{d}{dt} E_s(t) \lesssim E_s(t)^{(p+1)/2} + \epsilon(1 + |t|)^{(m-1/2)} E_s(t)^{1/2}. \]  

(3.27)

Assume \( E_s(0) \lesssim \epsilon^2 \). By the bootstrap argument, we can prove

\[ E_s(t) \lesssim \epsilon^2, \quad \forall \ t \lesssim \epsilon^{1-p}. \]  

(3.28)

If the nonlinearity is at least cubic, i.e., \( p \geq 3 \), then the lifespan is at least \( \epsilon^{-2} \).

**The water waves:** If we can find \( \theta, \theta \approx z_t \), such that

\[ (\partial^2_t - i A \partial_\alpha) \theta = -2[z_t, \delta_\alpha 1 z_\alpha + \delta_\alpha^{-1} z_{\alpha t}] + \frac{1}{\pi i} \int_{-\infty}^{\infty} \left( \frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)} \right)^2 (z - \bar{z})_\beta d\beta := g. \]  

(3.29)

g is cubic, while \( a - 1 \) contains first order terms, so \( (\partial^2_t + \Lambda) \theta \) contains quadratic terms, which does not imply cubic lifespan. To resolve the problem, S. Wu [53] found that the fully nonlinear transform \( \theta := (I - \delta)(z - \bar{z}) \) satisfies

\[ \left\{ \begin{array}{l} (D^2_t - i A \partial_\alpha) \zeta = -i \\ (I - \mathcal{H}) D_t \zeta = 0 \end{array} \right. \]  

(3.30)

and (3.29) becomes

\[ (D^2_t - i A \partial_\alpha) \theta \circ \kappa^{-1} = -2[D_t \zeta, \mathcal{H} \frac{1}{\zeta_\alpha} + \mathcal{H} \frac{1}{\zeta_\alpha}] \partial_\alpha D_t \zeta \]  

\[ + \frac{1}{\pi i} \int_{-\infty}^{\infty} \left( \frac{D_t \zeta(\alpha, t) - D_t \zeta(\beta, t)}{\zeta(\alpha, t) - \zeta(\beta, t)} \right)^2 (\zeta - \bar{\zeta})_\beta d\beta, \]  

(3.31)

where

\[ \mathcal{H} f(\alpha) := \frac{1}{\pi i} p.v. \int_{-\infty}^{\infty} \frac{\zeta_\beta}{\zeta(\alpha, t) - \zeta(\beta, t)} f(\beta) d\beta. \]  

(3.32)
Long Time Behavior of 2D Water Waves

She realized that there exists a change of variables $\kappa$ such that

\[
(I - \mathcal{H})b = -[D_t \zeta, \mathcal{H}] \frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha},
\]

\[
(I - \mathcal{H})(A - 1) = i[D_t \zeta, \mathcal{H}] \frac{\partial_\alpha D_t \bar{\zeta}}{\zeta_\alpha} + i[D_t^2 \zeta, \mathcal{H}] \frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha}.
\]

(3.33) \hspace{1cm} (3.34)

So $b, A - 1$ are quadratic. Using this, S. Wu was able to prove the almost global existence for the irrotational water waves with small localized initial data. The method implies lifespan of order $O(\epsilon^{-2})$ for nonlocalized data of size $O(\epsilon)$ for irrotational water waves.

Assume there are point vortices in the fluid. We use S. Wu’s change of variables, by taking $\kappa : \mathbb{R} \rightarrow \mathbb{R}$, satisfying that for $\zeta = z \circ \kappa^{-1}$,

\[
(I - \mathcal{H})(\bar{\zeta} - \alpha) = 0.
\]

(3.35)

In new variables, by direct calculation, we have

\[
(D_t^2 - i A \partial_\alpha) \bar{\theta} = G_c + G_d,
\]

where $\bar{\theta} = (I - \mathcal{H})(\zeta - \bar{\zeta}), \quad A = (a \kappa_\alpha) \circ \kappa^{-1}, \quad b = \kappa_t \circ \kappa^{-1}, \quad D_t = \partial_t + b \partial_\alpha$. Let $\bar{\mathcal{F}} = f \circ \kappa^{-1}, \quad q = p \circ \kappa^{-1}$. We have

\[
G_c := -2[\bar{\mathcal{F}}, \mathcal{H}] \frac{\partial_\alpha \bar{\mathcal{F}}}{\zeta_\alpha} + \bar{\mathcal{F}} \cdot \frac{\partial_\alpha}{\zeta_\alpha} \frac{1}{\pi i} \int \left( \frac{D_t \zeta(\alpha, t) - D_t \zeta(\beta, t)}{\zeta(\alpha, t) - \zeta(\beta, t)} \right)^2 (\zeta - \bar{\zeta}) d\beta.
\]

(3.37)

\[
G_d := -2[q, \mathcal{H}] \frac{\partial_\alpha q}{\zeta_\alpha} - 2[\bar{\mathcal{F}}, \mathcal{H}] \frac{\partial_\alpha \bar{q}}{\zeta_\alpha} - 2[q, \mathcal{H}] \frac{\partial_\alpha q}{\zeta_\alpha} - 4D_t q.
\]

(3.38)

\[
(I - \mathcal{H})b = -[D_t \zeta, \mathcal{H}] \frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} - \frac{i}{\pi} \sum_{j=1}^{2} \frac{\lambda_j (D_t \zeta(\alpha, t) - \bar{\zeta}_j(t))}{\zeta(\alpha, t) - \bar{\zeta}_j(t)^2}.
\]

(3.39)

\[
(I - \mathcal{H})A = 1 + i[D_t \zeta, \mathcal{H}] \frac{\partial_\alpha \bar{\mathcal{F}}}{\zeta_\alpha} + i[D_t^2 \zeta, \mathcal{H}] \frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} - (I - \mathcal{H}) \frac{1}{2\pi} \sum_{j=1}^{2} \frac{\lambda_j (D_t \zeta(\alpha, t) - \bar{\zeta}_j(t))}{\zeta(\alpha, t) - \bar{\zeta}_j(t)^2}.
\]

(3.40)

To control the acceleration $D_t^2 \zeta$, we consider $\bar{\sigma} = (I - \mathcal{H}) D_t \bar{\mathcal{F}}$. We have

\[
(D_t^2 - i A \partial_\alpha) \bar{\sigma} = \tilde{G},
\]

(3.41)

where

\[
\tilde{G} = (I - \mathcal{H})(D_t G + i \frac{\partial_\alpha}{a} \circ \kappa^{-1} A((I - \mathcal{H})(\zeta - \bar{\zeta}(\alpha)))_\alpha) - 2[D_t \zeta, \mathcal{H}] \frac{\partial_\alpha D_t^2 (I - \mathcal{H})(\zeta - \bar{\zeta})}{\zeta_\alpha} + \frac{1}{\pi i} \int \left( \frac{D_t \zeta(\alpha, t) - D_t \zeta(\beta, t)}{\zeta(\alpha, t) - \zeta(\beta, t)} \right)^2 \partial_\beta D_t (I - \mathcal{H})(\zeta - \bar{\zeta}) d\beta.
\]

(3.42)
The difficulty:

(1) $G_c$ is cubic, while $G_d$ consists of quadratic and first order terms. $G_d$ is the contribution from the point vortices. Similarly, $b$, $A - 1$ contains first order terms due to the presence of point vortices.

(2) Each term of $G_d$ contains factors of the form $\sum_{j=1}^{N} \frac{\lambda_j}{(\zeta(\alpha, t) - z_j(t))^k}$ for some $k \geq 1$. It’s possible that the point vortices travel upward and get closer and closer to the free interface. In that case, $G_d$ becomes very large.

(3) The strong interaction between the point vortices could excite the water waves and make it significantly larger in a short time. Assume two point vortices $z_1(t) = -x + iy$, $z_2(t) = x + iy$, with strength $\lambda_1$ and $\lambda_2$, respectively. Then the velocity of $z_1$ is

$$\dot{z}_1 = -\frac{\lambda_2 i}{2\pi(z_2 - z_1)} + \tilde{U}(z_1(t), t), \quad \dot{z}_2 = \frac{\lambda_1 i}{2\pi(z_2 - z_1)} + \tilde{U}(z_2(t), t).$$

Here, $\tilde{U}$ is the holomorphic extension of $f$ in $\Omega(t)$. Roughly speaking, if $\frac{|\lambda|}{2\pi|z_2 - z_1|}$ is large, then $|\dot{z}_j(t)|$ is large. Since $p_t = \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{z_j - \dot{z}_j(t)}{(z(\alpha, t) - \dot{z}_j(t))^2}$, in general, $\|p_t\|_{H^s}$ could be large as well. Therefore, small data theory does not directly apply in such a situation. Moreover, $\tilde{G}$ contains the term $\sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} (\dot{z}_j)^2 (\zeta(\alpha, t) - z_j(t))^2$, which is even worse than $p_t$ if $\dot{z}_j(t)$ is large. In theorem 3, we do allow $\|\frac{\lambda}{z_1 - z_2}\|$ large. See Remark 3.1.

(4) If the point vortices collide, after the collision, we cannot use the same system to describe the motion of the fluid anymore, for the reason that the vorticity after the collision is not the same as that before it, which violates the conservation of vorticity.

The idea: Intuitively, if each point vortices $z_j(t)$ moves away from the free boundary rapidly, with the factor $\frac{1}{(z(\alpha, t) - z_j(t))}$ decaying in time at least at a linear rate, then we could overcome the difficulties (1) and (2). We will show that this indeed is true if (H3)–(H6) holds initially. To overcome the difficulty (3), we use $\lambda_1 = -\lambda_2$ from assumption (H3), and by direct calculation, $\|p_t\|_{H^s}$ does not depend on $\dot{z}_1, \dot{z}_2$, resolving difficulty (3).

Also, although the term $\left\| \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{(\dot{z}_j)^2}{(z(\alpha, t) - z_j(t))^3} \right\|_{H^s}$ could be large at time $t = 0$, yet its long time effect remains small, i.e.,

$$\int_{0}^{T} \left\| \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{(\dot{z}_j)^2}{(z(\alpha, t) - z_j(t))^3} \right\|_{H^s} dt \leq C' \epsilon,$$

for some constant $C'$ which is independent of $\frac{|\lambda|}{\pi x(0)}$.

So we are able to overcome the difficulty of large vortex-vortex interaction, provided that the point vortices keeps traveling away from the free interface.

However, the motion of point vortices interferes with the motion of the water waves, it’s not obvious at all that why the point vortices should escape to the deep water (toward...
\( y = -\infty \). Indeed, in some cases, they can travel upwards toward the interface. Recall that the velocity of \( z_j \) is given by
\[
\dot{z}_j(t) = \left. \left( v - \frac{\lambda_j i}{2\pi} \frac{1}{z - z_j(t)} \right) \right|_{z=z_j(t)}.
\] (3.43)

The point vortices could interact with each other and with the water waves. In general, it’s not always true that \( \text{Im}\{\dot{z}_j(t)\} < 0 \) (i.e., travels downward).

In the following, we discuss how the number of point vortices, the sign of \( \lambda_j \), and the strength of \( |\lambda_j| \) affect the motion of point vortices.

(1) For \( N = 1 \), the motion of the point vortex is hard to predict except for some special cases.

(2) For \( N = 2 \), \( \lambda_1 = \lambda_2 \). The two point vortices are more likely to rotate about each other and excite the fluid.

(3) \( N = 2 \), \( \lambda_1 = -\lambda_2 = \lambda > 0 \). In this case, the point vortices are more likely to move upward and getting closer and closer to the interface and hence cause the Taylor sign condition to fail.

(4) \( N = 2 \), \( \lambda_1 = -\lambda_2 = \lambda < 0 \) and \( \frac{\lambda}{|z_1 - z_2|} \) is relatively large (say, \( \frac{\lambda}{|z_1 - z_2|} \gg \epsilon \), where \( \epsilon \) is the size of the initial data), then the point vortices moving straight downward rapidly and hence the term \( \frac{1}{z(\alpha,t) - z_j(t)} \) decays like \( t^{-1} \).

(5) \( N \geq 3 \). This is far beyond understood. Indeed, even for 2d Euler equations (fixed boundary) with point vortices, the problem is still not fully understood. When \( N = 3 \), the problem resembles the three-body problem.

(6) Vortex pairs. With \( 2N \) point vortices, denoted by \( \{(z_{i1}, z_{i2})|_{i=1}^N \}. \) Let the corresponding strength be \( \lambda_{i1}, \lambda_{i2} \), respectively. Assume \( \lambda_{i1} = -\lambda_{i2} < 0 \). Assume \( |z_{i1} - z_{i2}| \) is sufficiently small, while different pairs are sufficiently far away from each other. Then the point vortices travel downward, at least for a short time. It’s likely that the factor \( \frac{1}{z(\alpha,t) - z_ij} \) decays linearly in time. So long time existence will not be a surprise. For brevity, we consider only one vortex pair.

Therefore, from the above discussion, if we assume \( N = 2 \) and \( \lambda_1 = -\lambda_2 < 0 \), we expect that the point vortices keep traveling downward at a speed comparable to its initial speed, hence
\[
|\zeta(\alpha,t) - z_j(t)|^{-1} = O\left(\frac{1}{\alpha + i \frac{|\lambda|}{\bar{x}(0)} t}\right),
\] (3.44)
and we can expect to have
\[
(D_t^2 - i \Lambda \partial_\alpha)\tilde{\theta} = G_c + O\left(\frac{\epsilon}{(\alpha + i \frac{|\lambda|}{\bar{x}(0)} t)^2}\right),
\] and
\[
(D_t^2 - i \Lambda \partial_\alpha)\tilde{\theta} = (\partial_t^2 + \Lambda)\tilde{\theta} + cubic + O\left(\frac{\epsilon}{(\alpha + i \frac{|\lambda|}{\bar{x}(0)} t)^2}\right),
\] hence
\[
(\partial_t^2 + \Lambda)\tilde{\theta} = cubic + O\left(\frac{\epsilon}{(\alpha + i \frac{|\lambda|}{\bar{x}(0)} t)^2}\right).
\]
Similarly, at least formally, we have
\[(\partial_t^2 + \Lambda)\tilde{\sigma} = \text{cubic} + O\left(\frac{\epsilon}{(\alpha + i |\lambda| x(0) t)^2}\right)\.

From the discussion on the Toy model, we expect to prove lifespan of order $O(\epsilon^{-2})$ for small nonlocalized data of size $O(\epsilon)$.

Let’s summarize our previous discussion in a more precisely way as the following.

Step 1. Change of variables. Let $\kappa$ be the change of variables such that $(I - H)(\zeta - \alpha) = 0$, where $\zeta = z \circ \kappa^{-1}$. Then we derive the formula (3.33) for the quantity $b$ and the formula (3.34) for the quantity $A - 1$.

Step 2. Nonlinear transform. Let $\tilde{\theta} = \theta \circ \kappa^{-1}$, $\tilde{\sigma} = D_t \tilde{\theta}$, where $\tilde{\theta} = (I - H)(\zeta - \bar{\xi})$. Then we derive water wave equations (3.36)–(3.38) for $\tilde{\theta}$ and water wave equations (3.41)–(3.42) for $\tilde{\sigma}$.

Step 3. Bootstrap assumption. Assume that on $[0, T]$, we have
\[\|\zeta - \alpha\|_{H^s} \leq 5\epsilon, \quad \|\bar{\zeta}\|_{H^{s+1/2}} \leq 5\epsilon, \quad \|D_t \bar{\zeta}\|_{H^s} \leq 5\epsilon, \quad \forall \, t \in [0, T],\]
(3.45)
where $\bar{\zeta} = f \circ \kappa^{-1}$.

Step 4. Control the motion of $z_j(t)$. Under the bootstrap assumption (3.45), we show that for any $t \in [0, T]$,
\[\frac{1}{2} \leq \frac{x(t)}{x(0)} \leq 2.\]
(3.46)

In another words, the trajectory of the point vortices are almost parallel to each other.

Therefore, we obtain decay estimate
\[d_I(t)^{-1} = \left(\min_{j=1,2} \inf_{\alpha \in \mathbb{R}} |\zeta(\alpha, t) - z_j(t)|\right)^{-1} \leq (1 + \frac{|\lambda|}{20\pi x(0)})^{-1}.\]
(3.47)

Step 5. Energy estimates. Denote $\theta_k = (I - \mathcal{H}) \partial_\alpha^k \tilde{\theta}$, $\sigma_k = (I - \mathcal{H}) \partial_\alpha^k \tilde{\sigma}$. Define energy
\[E(t) = \sum_{0 \leq k \leq s} \left\{ \int \frac{1}{A} |D_t \theta_k|^2 + \int \frac{1}{A} |D_t \sigma_k|^2 + i \int \theta_k \bar{\sigma}_k + i \int \sigma_k \bar{\theta}_k \right\}.\]
(3.48)

By energy estimates, use the time decay of $\frac{1}{|\zeta(\alpha, t) - z_j(t)|}$, we obtain control of $D_t \tilde{\theta}$ and $D_t \tilde{\sigma}$. As a consequence, by bootstrap argument, we show that $T^* \geq \delta \epsilon^{-2}$.

Then we can change of variables back to lagrangian coordinates to conclude the proof of Theorem 3.

4. Proof of Theorem 1

In this section we prove Theorem 1. We investigate systematically the Taylor sign condition of water waves with point vortices in Riemann mapping variables.
4.1. The Taylor sign condition in Riemann variables. Recall that \( \tilde{z}_t + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi(z(\alpha, t) - z_j(t))} \) is holomorphic, i.e., there is a holomorphic function \( \mathcal{U}(z, t) \) in \( \Omega(t) \) such that

\[
\mathcal{U}(z(\alpha, t), t) = \tilde{z}_t(\alpha, t) + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi(z(\alpha, t) - z_j(t))}. 
\]

So we have

\[
\tilde{z}_{tt} = \partial_t \tilde{z}_t = \partial_t \left( \mathcal{U}(z(\alpha, t), t) - \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi(z(\alpha, t) - z_j(t))} \right) 
= \mathcal{U}_z(z(\alpha, t), t)z_t + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi(z(\alpha, t) - z_j(t))}^2.
\]

Note that

\[
\dot{z}_j(t) = (v - \frac{\lambda_j i}{2\pi(z - z_j(t))}) \bigg|_{z=z_j(t)} = U_z(z_j(t), t) - \sum_{k:k \neq j} \frac{\lambda_k i}{2\pi(z_k(t) - z_j(t))}.
\]

Let \( \Phi(\cdot, t) : \Omega(t) \to \mathbb{P}_- \) be the Riemann mapping such that \( \Phi_z \to 1 \) as \( z \to \infty \). Let \( h(\alpha, t) := \Phi(z(\alpha, t), t) \). Denote

\[
Z(\alpha, t) := z \circ h^{-1}(\alpha, t), \quad B = h_t \circ h^{-1}, \quad D_t := \partial_t + B\partial_{\alpha}, \quad A := (ah_{\alpha}) \circ h^{-1}.
\]

Use (4.1), apply \( h^{-1} \) on both sides of \( \tilde{z}_{tt} + i a \tilde{z}_\alpha = i \), we obtain

\[
\mathcal{U}_z \circ Z(\alpha, t)D_t Z + \mathcal{U}_t \circ Z(\alpha, t) + \sum_{j=1}^{N} \frac{\lambda_j i(D_t Z(\alpha, t) - \dot{z}_j(t))}{2\pi(Z(\alpha, t) - z_j(t))^2} + iA\tilde{Z}_\alpha = i.
\]

Multiply by \( Z_\alpha \) on both sides of (4.5), and denote

\[
A_1 := A|Z_\alpha|^2,
\]

we obtain

\[
\mathcal{U}_z \circ ZZ_\alpha D_t Z + \mathcal{U}_t \circ ZZ_\alpha + \sum_{j=1}^{N} \frac{\lambda_j i(D_t ZZ_\alpha - \lambda_j i \dot{z}_j(t)Z_\alpha)}{2\pi(Z(\alpha, t) - z_j(t))^2} + iA_1 = iZ_\alpha.
\]

Apply \( I - \mathbb{H} \) on both sides of the above equation, then take imaginary parts, we obtain

\[
A_1 = 1 - Im \left\{ (I - \mathbb{H})(\mathcal{U}_z \circ ZZ_\alpha D_t Z) + (I - \mathbb{H}) \sum_{j=1}^{N} \frac{\lambda_j i(D_t ZZ_\alpha - \dot{z}_j(t)Z_\alpha)}{2\pi(Z(\alpha, t) - z_j(t))^2} \right\}.
\]

(4.7)
Note that
\[ U_z \circ ZZ_\alpha = \partial_\alpha (D_\iota \tilde{Z} + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi(Z(\alpha, t) - z_j(t))}) = \partial_\alpha D_\iota \tilde{Z} - \sum_{j=1}^{N} \frac{\lambda_j i Z_\alpha}{2\pi(Z(\alpha, t) - z_j(t))^2} \]
is holomorphic. So we have
\[
(I - \mathbb{H})U_z \circ ZZ_\alpha D_\iota Z = [D_\iota Z, \mathbb{H}](\partial_\alpha D_\iota \tilde{Z} - \sum_{j=1}^{N} \frac{\lambda_j i Z_\alpha}{2\pi(Z(\alpha, t) - z_j(t))^2})
\]  \hspace{1cm} (4.8)

We know that
\[
-I m[D_\iota Z, \mathbb{H}] \partial_\alpha D_\iota \tilde{Z} = \frac{1}{2\pi} \int \left| \frac{D_\iota Z(\alpha, t) - D_\iota Z(\beta, t)}{(\alpha - \beta)^2} \right|^2 d\beta \geq 0. \hspace{1cm} (4.9)
\]
Also,
\[
- \sum_{j=1}^{N}[D_\iota Z, \mathbb{H}] \frac{\lambda_j i Z_\alpha}{2\pi(Z(\alpha, t) - z_j(t))^2} + (I - \mathbb{H}) \sum_{j=1}^{N} \frac{\lambda_j i (D_\iota ZZ_\alpha - \hat{z}_j(t) Z_\alpha)}{2\pi(Z(\alpha, t) - z_j(t))^2}
\]
\[
= \sum_{j=1}^{N}[D_\iota Z, I - \mathbb{H}] \frac{\lambda_j i Z_\alpha}{2\pi(Z(\alpha, t) - z_j(t))^2} + (I - \mathbb{H}) \sum_{j=1}^{N} \frac{\lambda_j i (D_\iota ZZ_\alpha - \hat{z}_j(t) Z_\alpha)}{2\pi(Z(\alpha, t) - z_j(t))^2}
\]
\[
= \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} D_\iota Z(\alpha, t) \frac{Z_\alpha}{(Z(\alpha, t) - z_j(t))^2} - \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} (I - \mathbb{H}) \frac{\hat{z}_j(t) Z_\alpha}{(Z(\alpha, t) - z_j(t))^2}
\]
\[
= \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} (I - \mathbb{H}) \frac{Z_\alpha}{(Z(\alpha, t) - z_j(t))^2} (D_\iota Z - \hat{z}_j(t)). \hspace{1cm} (4.10)
\]
So we have
\[
A_1 = 1 + \frac{1}{2\pi} \int \frac{|D_\iota Z(\alpha, t) - D_\iota Z(\beta, t)|^2}{(\alpha - \beta)^2} d\beta
\]
\[
- I m \left\{ \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \left( (I - \mathbb{H}) \frac{Z_\alpha}{(Z(\alpha, t) - z_j(t))^2} \right)(D_\iota Z - \hat{z}_j(t)) \right\}
\]
\[
= 1 + \frac{1}{2\pi} \int \frac{|D_\iota Z(\alpha, t) - D_\iota Z(\beta, t)|^2}{(\alpha - \beta)^2} d\beta
\]
\[
- \sum_{j=1}^{N} \frac{\lambda_j}{2\pi} Re \left\{ \left( (I - \mathbb{H}) \frac{Z_\alpha}{(Z(\alpha, t) - z_j(t))^2} \right)(D_\iota Z - \hat{z}_j(t)) \right\}. \hspace{1cm} (4.11)
\]
To get an estimate as sharp as possible for the Taylor sign condition, we’d like to get rid of the Hilbert transform \( \mathbb{H} \) in the formula above.
It's easy to see that, if $Z = \alpha$, then
$$\frac{Z_\beta}{(Z_\beta(t) - z_j(t))^2}(D_t Z - \dot{z}_j(t))$$
is boundary value of a holomorphic function in $\mathbb{P}_+$, so
$$\left((I - \mathbb{H})\frac{Z_\alpha}{(Z(\alpha, t) - z_j(t))^2}\right)(D_t Z - \dot{z}_j(t)) = 2\frac{Z_\alpha}{(Z(\alpha, t) - z_j(t))^2}(D_t Z - \dot{z}_j(t)).$$

For the general case, we use the following lemma.

**Lemma 4.1.** Let $z_0 \in \Omega(t)$. Under the assumptions of Theorem 1, we have

$$\left((I - \mathbb{H})\frac{1}{Z(\alpha, t) - z_0}\right) = \frac{2}{c_1(\alpha - w_0)}, \quad c_1 = (\Phi^{-1})_{\alpha}(w_0), \quad w_0 = \Phi(z_0, t).$$

(4.12)

**Proof.** Note that $Z(\alpha, t) = \Phi^{-1}(\alpha, t)$. So $Z(\alpha, t) - z_0$ is the boundary value of $\Phi^{-1}(z, t) - z_0$ in the lower half plane. Since $\Phi^{-1}$ is 1-1 and onto, $\Phi^{-1}(z, t) - z_0$ has a unique zero $w_0 := \Phi(z_0, t)$, so $\frac{1}{Z(\alpha, t) - z_0}$ has exactly one pole of multiplicity one. For $z$ near $w_0$, we have

$$\Phi^{-1}(z, t) - z_0 = c_1(z - w_0) + \sum_{n=2}^{\infty} c_n(z - w_0)^n, \quad \text{where} \quad c_1 = (\Phi^{-1})_{\alpha}(w_0) \neq 0.$$

(4.13)

Therefore, we have

$$\frac{1}{Z(\alpha, t) - z_0} - \frac{1}{c_1(\alpha - w_0)} \text{ is holomorphic in } \mathbb{P}_-, \text{ and hence by Lemma 2.2,}$$

$$\left((I - \mathbb{H})\frac{1}{Z(\alpha, t) - z_0} - \frac{1}{c_1(\alpha - w_0)}\right) = 0.$$

(4.14)

Since $\frac{1}{c_1(\alpha - w_0)}$ is holomorphic in $\mathbb{P}_+$, by Lemma 2.2,

$$\left((I - \mathbb{H})\frac{1}{Z(\alpha, t) - z_0}\right) = \left((I - \mathbb{H})\frac{1}{c_1(\alpha - w_0)}\right) = \frac{2}{c_1(\alpha - w_0)}.$$

(4.15)

\[\Box\]

**Corollary 4.1.** Under the assumptions of Theorem 1, we have

$$\frac{Z_\alpha}{(Z(\alpha, t) - z_j(t))^2} = \frac{2}{(\Phi^{-1})_{\alpha}(\Phi(z_j(t)))(\alpha - \Phi(z_j(t)))^2}.$$

(4.16)

**Proof.** We have

$$\frac{Z_\alpha}{(Z(\alpha, t) - z_j(t))^2} = -\partial_\alpha(I - \mathbb{H})\frac{1}{Z(\alpha, t) - z_j(t)}$$

$$= -\partial_\alpha\frac{2}{(\Phi^{-1})_{\alpha}(\Phi(z_j(t)))(\alpha - \Phi(z_j(t)))}$$

$$= \frac{2}{(\Phi^{-1})_{\alpha}(\Phi(z_j(t)))(\alpha - \Phi(z_j(t)))^2}.$$

\[\Box\]
Corollary 4.2. Under the assumptions of Theorem 1, we have

\[ A_1 = 1 + \frac{1}{2\pi} \int \frac{|D_t Z(\alpha, t) - D_t Z(\beta, t)|^2}{(\alpha - \beta)^2} d\beta - \sum_{j=1}^{N} \frac{\lambda_j}{\pi} \text{Re} \left\{ \frac{D_t Z - \hat{z}_j}{c_0^{(\alpha - w_0^j)^2}} \right\}, \]

(4.17)

where

\[ c_0^j = (\Phi^{-1})_z(w_0^j), \quad \omega_0^j = \Phi(z_j). \]  

(4.18)

4.2. A formula for \( A_1 \) when \( Z(\alpha, t) = \alpha, D_t Z = \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \frac{1}{\alpha - z_j(t)} \). If the point vortices are very close to the interface, then Taylor sign condition can fail. To see this, we study the special case when \( Z(\alpha, t) = \alpha \) and \( D_t Z(\alpha, t) = \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \frac{1}{\alpha - z_j(t)} \).

Since the integral term of the formula (4.17) is nonlocal, in order to obtain a more convenient form of (4.17), we use residue theorem to calculate this integral. We’ll use the following formula.

Lemma 4.2. Let \( w_1, w_2 \in \mathbb{P}_+ \). Then

\[ \int_{-\infty}^{\infty} \frac{1}{(\beta - w_1)(\beta - w_2)} d\beta = \frac{2\pi i}{w_2 - w_1}. \]  

(4.19)

Proof. \( w_2 \) is the only residue of \( \frac{1}{(\beta - w_1)(\beta - w_2)} \) in \( \mathbb{P}_+ \). By residue Theorem,

\[ \int_{-\infty}^{\infty} \frac{1}{(\beta - w_1)(\beta - w_2)} d\beta = \frac{2\pi i}{w_2 - w_1}. \]

\( \square \)

As a consequence, we have

Corollary 4.3. Assume that the solution \( Z(\alpha, t) \) to (2.33) exists at time \( t \). Assume further that \( Z(\alpha, t) = \alpha, D_t Z = \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \frac{1}{\alpha - z_j(t)} \). Then we have

\[ \frac{1}{2\pi} \int \frac{|D_t Z(\alpha, t) - D_t Z(\beta, t)|^2}{(\alpha - \beta)^2} d\beta = \sum_{1 \leq j, k \leq N} \frac{\lambda_j \lambda_k}{(2\pi)^2} \frac{1}{(\alpha - z_j)(\alpha - z_k)} \frac{i}{z_k - z_j}. \]  

(4.20)

Proof. We have

\[ D_t Z(\alpha, t) - D_t Z(\beta, t) = \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \frac{\beta - \alpha}{(\alpha - z_j)(\beta - z_j)}. \]  

(4.21)
So we have

\[
\left| \frac{\mathcal{D}_t Z(\alpha, t) - \mathcal{D}_t Z(\beta, t)}{\alpha - \beta} \right|^2 = \sum_{j=1}^{N} \frac{\lambda_j}{2\pi} \frac{1}{(\alpha - z_j)(\beta - z_j)} \left( \alpha - \frac{1}{\beta - z_j} \right)^2
\]

\[(4.22)\]

Apply lemma 4.2, we have

\[
\int_{-\infty}^{\infty} \frac{1}{(\alpha - z_j)(\beta - z_j)(\alpha - z_k)(\beta - z_k)} d\beta = \frac{1}{(\alpha - z_j)(\alpha - z_k)} \frac{2\pi i}{z_k - z_j}
\]

So we have

\[
\frac{1}{2\pi} \int \frac{\left| \mathcal{D}_t Z(\alpha, t) - \mathcal{D}_t Z(\beta, t) \right|^2}{(\alpha - \beta)^2} d\beta = \sum_{1 \leq j, k \leq N} \frac{\lambda_j \lambda_k}{(2\pi)^2} \frac{1}{(\alpha - z_j)(\alpha - z_k)} \frac{2\pi i}{z_k - z_j}
\]

\[
= \sum_{1 \leq j, k \leq N} \frac{\lambda_j \lambda_k}{(2\pi)^2} \frac{1}{(\alpha - z_j)(\alpha - z_k)} \frac{i}{z_k - z_j}
\]

\[(4.23)\]

**Corollary 4.4.** Assume that the solution \( Z(\alpha, t) \) to (2.33) exists at time \( t \). Assume further that \( Z(\alpha, t) = \alpha \), \( \mathcal{D}_t Z = \sum_{j=1}^{N} \frac{\lambda_j}{2\pi} \frac{1}{\alpha - z_j(t)} \). Then

\[
A_1 = 1 + \sum_{1 \leq j, k \leq N} \frac{\lambda_j \lambda_k}{(2\pi)^2} \frac{1}{(\alpha - z_j)(\alpha - z_k)} \frac{i}{z_k - z_j} - \sum_{j=1}^{N} \frac{\lambda_j}{\pi} \text{Re} \left\{ \frac{\mathcal{D}_t Z - z_j}{(\alpha - z_j)^2} \right\}
\]

\[(4.24)\]

In the following two subsections, we use Corollary 4.4 to construct counter-examples for which the Taylor sign condition fails.

### 4.3. One point vortex: an example that the Taylor condition fails

We have the following characterization. Recall that \( \left| \frac{A_1}{\left| Z_{\alpha} \right|} = -\frac{\partial P}{\partial n} \right. \)

**Proposition 1.** Assume that the solution \( Z(\alpha, t) \) to (2.33) exists at time \( t \). Assume further that \( Z(\alpha, t) = \alpha \), the interface is \( \Sigma(t) = \mathbb{R} \), the fluid velocity is \( v(z, t) = \frac{\lambda_i}{2\pi} \frac{1}{z - z_1(t)} \), i.e., it is generated by a single point vortex \( z_1(t) := x(t) + iy(t) \). Then

1. If \( \frac{\lambda^2}{|y|^3} < \frac{8\pi^2}{3} \), the strong Taylor sign condition holds. We have

\[
\inf_{\alpha \in \mathbb{R}} A_1(\alpha, t) \geq 1 - \frac{3}{8\pi^2} \frac{\lambda^2}{|y|^3} > 0.
\]

2. If \( \frac{\lambda^2}{|y|^3} > \frac{8\pi^2}{3} \), the Taylor sign condition fails, i.e., there exists \( \alpha \in \mathbb{R} \) such that \( A_1(\alpha, t) < 0 \).
(3) If \( \frac{\lambda^2}{|y|^3} = \frac{8\pi^2}{3} \), the ’Degenerate Taylor sign condition’ holds, i.e.,

\[
A_1(\alpha, t) > 0, \quad \forall \alpha \neq x(t); \quad \text{and} \quad A_1(x(t), t) = 0. \tag{4.25}
\]

**Remark 4.1.** The quantity \( \frac{\lambda^2}{|y|^3} \) is a measurement of the interface-vortex interaction. \( \lambda \) is the intensity of the point vortex, and \( |y| \) is the distance from the point vortex to the interface.

**Proof.** Note that

\[
\dot{z}_1(t) = 0.
\]

So we have

\[
- \sum_{j=1}^{N} \frac{\lambda_j}{\pi} \text{Re} \left\{ \frac{D_j Z - \dot{z}_j}{(\alpha - z_j)^2} \right\} = -\frac{\lambda}{\pi} \text{Re} \left\{ \frac{D_z Z(\alpha, t)}{(\alpha - z_1(t))^2} \right\} = -\frac{\lambda}{\pi} \text{Re} \left\{ \frac{1}{(\alpha - z_1(t))^2} \frac{\lambda i}{2\pi \bar{\alpha} - \bar{z}_1(t)} \right\} = \frac{\lambda^2}{2\pi^2 |\alpha - z_1(t)|^4}. \tag{4.26}
\]

Therefore, by Corollary 4.4, we have

\[
A_1(\alpha, t) = 1 + \frac{\lambda^2}{4\pi^2} \frac{1}{|\alpha - z_1|^2} \frac{i}{-2i y^3} + \frac{\lambda^2}{2\pi^2} \frac{y}{|\alpha - z_1(t)|^4}.
\]

Without loss of generality, we can assume \( x = 0 \). Setting \( \partial_\alpha A_1(\alpha, t) = 0 \), it’s easy to see that \( A_1(\alpha, t) \) admits a unique local minimum at \( \alpha = x = 0 \). Moreover, it’s easy to see that

\[
\lim_{\alpha \to \pm \infty} A_1(\alpha, t) = 1. \tag{4.27}
\]

Therefore, \( \inf_{\alpha \in \mathbb{R}} A(\alpha, t) \leq 0 \) if and only if \( A_1(0, t) \leq 0 \).

We have

\[
A_1(0, t) = 1 - \frac{\lambda^2}{8\pi^2} \frac{1}{y^3} + \frac{\lambda^2}{2\pi^2} \frac{1}{y^3} = 1 - \frac{3\lambda^2}{8\pi^2} \frac{1}{|y|^3}.
\]

If \( \frac{\lambda^2}{|y|^3} < \frac{8\pi^2}{3} \), then \( A(\alpha, t) \geq A(0, t) > 0 \). If \( \frac{\lambda^2}{|y|^3} > \frac{8\pi^2}{3} \), then \( A_1(0, t) < 0 \). If \( \frac{\lambda^2}{|y|^3} = \frac{8\pi^2}{3} \), then \( A_1(0, t) = 0 \), and for \( \alpha \neq 0 \), \( A_1(\alpha, t) > A_1(0, t) = 0 \). \( \square \)
4.4. Two point vortices: another example that Taylor sign condition fails. Assume that \( Z(\alpha, t) = \alpha \). Assume \( D_t Z = \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{1}{\alpha - z_j(t)} \). Assume \( z_1(t) = -x(t) + iy(t), z_2(t) = x(t) + iy(t) \), with \( x(t) > 0, y(t) < 0 \), and \( \lambda_1 = -\lambda_2 := \lambda \). Let’s calculate \( A_1(0, t) \).

We have

\[
D_t Z(\alpha, t) = \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{1}{\alpha - z_j(t)} = \frac{\lambda i}{2\pi} \frac{z_1(t) - z_2(t)}{(\alpha - z_1(t)) (\alpha - z_2(t))}.
\]

Since \( z_1 - z_2 = -2x \),

\[
D_t Z(0, t) = \frac{\lambda i}{\pi} \frac{x}{x^2 + y^2}.
\]

At \( \alpha = 0 \), we have

\[
\sum_{j=1}^{2} \frac{\lambda_j}{\pi} \frac{1}{(0 - z_j(t))^2} D_t Z(0) = -\frac{\lambda i}{\pi} \frac{x}{x^2 + y^2} \frac{\lambda}{\pi} \frac{4xyi}{(x^2 + y^2)^2} = \frac{4\lambda^2 x^2 y}{\pi^2 (x^2 + y^2)^3}.
\]

(4.28)

We have

\[
\dot{z}_1 = -\frac{\dot{\lambda}_i}{2\pi} \frac{1}{z_1 - z_2} = \frac{\dot{\lambda}_i}{4\pi x}.
\]

\[
\dot{z}_2 = \frac{\dot{\lambda}_i}{2\pi} \frac{1}{z_2 - z_1} = \frac{\dot{\lambda}_i}{4\pi x}.
\]

So we have

\[
\sum_{j=1}^{2} \frac{\lambda_j}{\pi} \frac{1}{(0 - z_j(t))^2} \dot{z}_j(t) = \frac{\dot{\lambda}_i}{4\pi x} \sum_{j=1}^{2} \frac{\lambda_j}{\pi} \frac{1}{(0 - z_j(t))^2} = \frac{\dot{\lambda}_i}{4\pi x} \frac{4xyi}{(x^2 + y^2)^2} = -\frac{\lambda^2 y}{\pi^2 (x^2 + y^2)^2}.
\]

So we have

\[
-\sum_{j=1}^{N} \frac{\lambda_j}{\pi} \text{Re} \left\{ \frac{1}{(0 - z_j(t))^2} (D_t Z(0, t) - \dot{z}_j(t)) \right\} = \frac{\lambda^2 y (3x^2 - y^2)}{\pi^2 (x^2 + y^2)^3}.
\]

On the other hand, we have

\[
\sum_{1 \leq j, k \leq 2} \frac{\lambda_j \lambda_k}{(2\pi)^2} \frac{1}{(0 - z_j)(0 - z_k)} \frac{i}{z_k - z_j} = \frac{\lambda^2}{4\pi^2} \frac{1}{|z_1|^2} \frac{i}{\bar{z}_1 - z_1} + \frac{\lambda^2}{4\pi^2} \frac{1}{|z_2|^2} \frac{i}{\bar{z}_2 - z_2} - \frac{\lambda^2}{4\pi^2} \frac{1}{z_1 \bar{z}_2 - z_1 - \bar{z}_2} - \frac{\lambda^2}{4\pi^2} \frac{1}{\bar{z}_1 \bar{z}_2 - \bar{z}_1 - z_2}.
\]
\[
\begin{align*}
&= \frac{\lambda^2}{4\pi^2} \frac{1}{\|z\|^2} \left( \frac{1}{\|y\|} + 2 Re \left\{ \frac{\lambda^2}{8\pi^2} \frac{i}{(x - iy)^3} \right\} \right) \\
&= \frac{\lambda^2}{4\pi^2} \frac{1}{x^2 + y^2} \frac{1}{\|y\|} + \frac{\lambda^2}{4\pi^2} \frac{y^3 - 3x^2 y}{(x^2 + y^2)^3} \\
&= \frac{\lambda^2}{4\pi^2} \frac{x^4 + 5x^2 y^2}{\|y\|(x^2 + y^2)^3}.
\end{align*}
\]

Therefore, by Corollary 4.4, we have

\[
A_1(0, t) = 1 + \frac{\lambda^2}{4\pi^2} \frac{x^4 + 5x^2 y^2}{\|y\|(x^2 + y^2)^3} + \frac{\lambda^2}{\pi^2} \frac{y(3x^2 - y^2)}{(x^2 + y^2)^3} \\
= 1 + \frac{\lambda^2}{4\pi^2} \frac{x^4 + 5x^2 y^2 - 12x^2 y^2 + 4y^4}{\|y\|(x^2 + y^2)^3} \\
= 1 + \frac{\lambda^2}{4\pi^2} \frac{x^4 + 4y^4 - 7x^2 y^2}{\|y\|(x^2 + y^2)^3}.
\]

So we have

**Proposition 2.** Assume that the solution \(Z(\alpha, t)\) to (2.33) exists at time \(t\). Assume further that \(Z(\alpha, t) = \alpha\), the fluid velocity is \(v(z, t) = \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{1}{z - z_j(t)}\), where \(\lambda_1 = -\lambda_2 := \lambda\).

Assume

\[
z_1(t) = -x + iy, \quad z_2(t) = x + iy, \quad \text{where} \quad x > 0, y < 0.
\]

If

\[
1 + \frac{\lambda^2}{4\pi^2} \frac{x^4 + 4y^4 - 7x^2 y^2}{\|y\|(x^2 + y^2)^3} < 0,
\]

then Taylor sign condition fails.

**Corollary 4.5.** Under the assumption of Proposition 2, if \(|x| = |y|\) and \(\frac{\lambda^2}{|y|^3} > 16\pi^2\), then the strong Taylor sign condition fails.

### 4.5. A criterion that implies the strong Taylor sign condition.

If the vortex-vortex, vortex-interface interaction is weak, then the Taylor sign condition holds. Let’s recall that we denote \(\mathcal{U}\) by

\[
\tilde{U}(z, t) := v(z, t) - \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \frac{1}{z - z_j(t)}.
\]

Let \(F\) be the boundary value of \(\mathcal{U}\), i.e.,

\[
F(\alpha, t) := \mathcal{U}(Z(\alpha, t), t).
\]

We have the following.
**Proposition 3.** Assume \( \inf_{\alpha \in \mathbb{R}} |Z_\alpha| = \beta_0, \|F\|_\infty = M_0. \) Denote

\[
\tilde{\lambda} =: \frac{\sum_{j=1}^{N} |\lambda_j|}{\pi}, \quad \tilde{d}_f(t) := \min_{1 \leq j \leq N} \frac{\inf_{\alpha \in \mathbb{R}} |\alpha - \Phi(z_j(t))|}{\inf_{1 \leq j \leq N} \alpha} \quad \text{and} \quad \tilde{d}_p(t) = \min_{j \neq k} |z_j(t) - z_k(t)|.
\]

If

\[
\frac{\tilde{\lambda}^2}{2\tilde{d}_f(t)^3\beta_0} + \frac{\tilde{\lambda}^2}{2\tilde{d}_f(t)^2\tilde{d}_p(t)} + \frac{2M_0\tilde{\lambda}}{\tilde{d}_f(t)^2} < \beta_0, \quad (4.33)
\]

then the strong Taylor sign condition holds.

**Proof.** Use the formula

\[
A_1 = 1 + \frac{1}{2\pi} \int \frac{|D_\alpha Z(\alpha, t) - D_\alpha Z(\beta, t)|^2}{(\alpha - \beta)^2} d\beta - \sum_{j=1}^{N} \frac{\lambda_j}{\pi} Re \left\{ \frac{D_\alpha Z - \dot{z}_j}{c_j^0(\alpha - \omega_0^j)^2} \right\}. \quad (4.34)
\]

For \( \sum_{j=1}^{N} \frac{\lambda_j}{\pi} Re \left\{ \frac{D_\alpha Z - \dot{z}_j}{c_j^0(\alpha - \omega_0^j)^2} \right\} \), we have

\[
\left| \sum_{j=1}^{N} \frac{\lambda_j}{\pi} Re \left\{ \frac{D_\alpha Z - \dot{z}_j}{c_j^0(\alpha - \omega_0^j)^2} \right\} \right| \leq \sum_{j=1}^{N} \frac{|\lambda_j|}{\pi} \max_{1 \leq j \leq N} |c_j^0| -1 \left( \|D_\alpha Z\|_\infty + |\dot{z}_j| \right) \left( \inf_{\alpha \in \mathbb{R}} |\alpha - \Phi(z_j)| \right)^{-2}.
\]

Since \( Z(\alpha) = \Phi^{-1}(\alpha) \), we have \( \partial_\alpha \Phi^{-1}(\alpha) = Z_\alpha \) and \( \partial_\alpha \Phi^{-1}(\alpha) \) is the boundary value of \( (\Phi^{-1})_z \). Note that \( (\Phi^{-1})_z \) never vanishes. By maximum module principle of holomorphic functions (apply to \( \frac{1}{(\Phi^{-1})_z} \)),

\[
|c_j^0| = |(\Phi^{-1})_z(\Phi(z_j))| \geq \inf_{\alpha \in \mathbb{R}} |Z_\alpha| \geq \beta_0. \quad (4.35)
\]

Since

\[
Z(\alpha, t) - z_j(t) = \Phi^{-1}(\alpha, t) - \Phi^{-1}(\omega_0^j) = \Phi^{-1}(\tau')(\alpha - \omega_0^j) \quad (4.36)
\]

for some \( \tau' \in \mathbb{P}_- \), so we have

\[
|Z(\alpha, t) - z_j(t)| \geq \beta_0 |\alpha - \omega_0^j|. \quad (4.37)
\]

Therefore,

\[
|D_\alpha Z| \leq |F| + \sum_{j=1}^{N} \frac{|\lambda_j|}{2\pi \beta_0 |\alpha - \omega_0^j|} \leq M_0 + \frac{\tilde{\lambda}}{2\tilde{d}_f(t)\beta_0}
\]

similarly,

\[
|\dot{z}_j(t)| = |\dot{F} + \sum_{k \neq j} \frac{\lambda_k i}{2\pi \beta_0 (z_j(t) - z_k(t))}| \leq M_0 + \sum_{j=1}^{N} \frac{|\lambda_j|}{2\pi \tilde{d}_p(t)} = M_0 + \frac{\tilde{\lambda}}{2\tilde{d}_p(t)}.
\]
So we obtain
\[ \left| \sum_{j=1}^{N} \frac{\lambda_j}{\pi} \text{Re} \left\{ \frac{\mathcal{D}_t Z - \dot{z}_j}{c_i(\alpha - \omega_0^j)^2} \right\} \right| \leq \beta_0^{-1} \frac{\tilde{\lambda}}{d_l(t)^2}(M_0 + \frac{\tilde{\lambda}}{2d_l(t)\beta_0} + M_0 + \frac{\tilde{\lambda}}{2d_l(t)^2}) \]
\[ \leq \beta_0^{-1} \left( \frac{\tilde{\lambda}^2}{2d_l(t)^3\beta_0} + \frac{\tilde{\lambda}^2}{2d_l(t)^2d_p(t)} + \frac{2M_0\tilde{\lambda}}{d_l(t)^2} \right). \]

If (4.33) holds, then
\[ \left| \sum_{j=1}^{N} \frac{\lambda_j}{\pi} \text{Re} \left\{ \frac{\mathcal{D}_t Z - \dot{z}_j}{c_i(\alpha - \omega_0^j)^2} \right\} \right| < 1. \]

Then \( A_1 > 0 \), so strong Taylor sign condition holds. \( \Box \)

In particular, if \( Z_\alpha \sim 1, d_l(t) \gtrsim 1, M_0 \ll 1, \) and \( |\lambda| \ll 1 \), then the strong Taylor sign condition holds.

5. Local Wellposedness: Proof of Theorem 2

In this section we prove Theorem 2. Denote
\[ \tilde{\lambda} = N \max_{1 \leq j \leq N} |\lambda_j|. \] (5.1)

**Plan of this section:** In §5.1, we derive the system (5.18), which is the quasilinearization for (2.33). In §5.2, we state an existence and uniqueness theorem for a linear system. In §5.3, we obtain the estimates that are necessary for proving the wellposedness of (5.18). In §5.5, we use iteration and fixed point method to prove the wellposedness of (5.18). In §5.6, we change of variables back to lagrangian coordinates and conclude the proof of Theorem 2. In §5.7, we show that the water wave with point vortices preserve certain symmetries.

5.1. Reduction to a quasilinear system. Let \( u = \mathcal{D}_t \tilde{Z} = F + Q \), recall that \( F \) is the boundary value of a holomorphic function in \( \mathbb{P}_- \), and \( Q = -\sum_{j=1}^{N} \frac{\lambda_j}{2\pi} \frac{1}{Z(\alpha, t) - \dot{z}_j(t)} \). We have
\[ \begin{cases} 
(\mathcal{D}_t^2 + iA\partial_\alpha)u = -i\mathcal{D}_t A\tilde{Z}_\alpha, \\
(I - \mathbb{H})F = 0.
\end{cases} \] (5.2)

Rewrite (5.2) as
\[ \begin{cases} 
(\mathcal{D}_t^2 + iA\partial_\alpha)F = -i\mathcal{D}_t A\tilde{Z}_\alpha - (\mathcal{D}_t^2 + iA\partial_\alpha)Q, \\
(I - \mathbb{H})F = 0.
\end{cases} \] (5.3)

It’s easy to see that \((\mathcal{D}_t^2 + iA\partial_\alpha)Q\) is of lower order (will be clear when we estimate this term). We need to show that \( \mathcal{D}_t A\tilde{Z}_\alpha \) is also of lower order.
5.1.1. Formula for $D_tA\bar{Z}_\alpha$ Multiply by $Z_\alpha$ on both sides of (5.3), we obtain

$$Z_\alpha(D_t^2 + iA\partial_\alpha)F = -Z_\alpha(D_t^2 + iA\partial_\alpha)Q - iD_t\mathcal{A}|Z_\alpha|^2.$$  (5.4)

We manipulate $(D_t^2 + iA\partial_\alpha)F$ as follows. Assume $F = \mathcal{U}(Z(\alpha, t), t)$, where $\mathcal{U}$ is a holomorphic function in $\Omega(t)$. Then we have

$$D_tF = D_tZ\mathcal{U}_Z + \mathcal{U}_t, \quad \partial_\alpha F = \mathcal{U}_Z Z_\alpha$$  (5.5)

$$D_t^2F = D_t^2Z\mathcal{U}_Z + (D_tZ)^2\mathcal{U}_{ZZ} + 2D_tZ\mathcal{U}_{Zt} + \mathcal{U}_{tt}.$$  (5.6)

We have

$$\mathcal{U}_Z = \frac{F_\alpha}{Z_\alpha}, \quad \mathcal{U}_{ZZ} = \frac{\partial_\alpha}{Z_\alpha} \left( \frac{\partial_\alpha}{Z_\alpha} F \right), \quad \mathcal{U}_t = \frac{\partial_\alpha}{Z_\alpha} (D_tF - \frac{F_\alpha}{Z_\alpha} D_tZ).$$  (5.7)

So we have

$$-iD_t\mathcal{A}|Z_\alpha|^2$$

$$= Z_\alpha \left( D_t^2 Z \frac{F_\alpha}{Z_\alpha} + iA F_\alpha + (D_t Z)^2 \frac{\partial_\alpha}{Z_\alpha} \left( \frac{\partial_\alpha}{Z_\alpha} F \right) + 2D_t Z \frac{\partial_\alpha}{Z_\alpha} (D_t F - \frac{F_\alpha}{Z_\alpha} D_t Z) + \mathcal{U}_{tt} \right)$$

$$+ Z_\alpha(D_t^2 + iA\partial_\alpha)Q.$$  (5.8)

Applying $I - \mathbb{H}$ on (5.8), using the fact that $\mathcal{U}_{tt}$, $F$, $Z_\alpha$ are holomorphic, we have

$$(I - \mathbb{H})\mathcal{U}_{tt} = 0, \quad (I - \mathbb{H})F = 0, \quad (I - \mathbb{H})(Z_\alpha - 1) = 0.$$  

So we obtain

$$-i(I - \mathbb{H})D_t\mathcal{A}|Z_\alpha|^2$$

$$= (I - \mathbb{H})Z_\alpha(D_t^2 + iA\partial_\alpha)Q + (I - \mathbb{H}) \left( D_t^2 Z F_\alpha + iA Z_\alpha F_\alpha + (D_t Z)^2 \frac{\partial_\alpha}{Z_\alpha} \left( \frac{\partial_\alpha}{Z_\alpha} F \right) 

+ 2D_t Z \frac{\partial_\alpha}{Z_\alpha} (D_t F - \frac{F_\alpha}{Z_\alpha} D_t Z) \right)$$

$$= (I - \mathbb{H})Z_\alpha(D_t^2 + iA\partial_\alpha)Q + 2[D_t^2 Z, \mathbb{H}] F_\alpha + \{((D_t Z)^2, \mathbb{H})\partial_\alpha \left( \frac{\partial_\alpha}{Z_\alpha} F \right) + 2[D_t Z, \mathbb{H}] \partial_\alpha (D_t F - \frac{F_\alpha}{Z_\alpha} D_t Z).$$

Using integration by parts on each term, after cancelations we have

$$\{((D_t Z)^2, \mathbb{H})\partial_\alpha \frac{F_\alpha}{Z_\alpha} = 2[D_t Z, \mathbb{H}] \partial_\alpha \left( \frac{F_\alpha}{Z_\alpha} D_t Z \right)$$

$$\frac{1}{\pi i} \int \left( \frac{D_t Z(\alpha, t) - D_t Z(\beta, t)}{\alpha - \beta} \right)^2 \frac{F_\beta}{Z_\beta} d\beta.$$  

So

$$D_t\mathcal{A}|Z_\alpha|^2 = -Im \left\{ (I - \mathbb{H})Z_\alpha(D_t^2 + iA\partial_\alpha)Q + 2[D_t^2 Z, \mathbb{H}] F_\alpha + 2[D_t Z, \mathbb{H}] \partial_\alpha D_t F + \frac{1}{\pi i} \int \left( \frac{D_t Z(\alpha, t) - D_t Z(\beta, t)}{\alpha - \beta} \right)^2 \frac{F_\beta}{Z_\beta} d\beta \right\}.$$  (5.9)
We write \( \mathcal{D}_t A\bar{Z}_\alpha \) as
\[
\mathcal{D}_t A\bar{Z}_\alpha = \mathcal{D}_t A|Z_\alpha|^2 \frac{1}{Z_\alpha},
\]
where \( \mathcal{D}_t A|Z_\alpha|^2 \) is given by (5.9), and \( \frac{1}{Z_\alpha} \) is given by
\[
\frac{1}{Z_\alpha} = i \frac{\mathcal{D}_t^2 \bar{Z} - i}{A_1}.
\]

5.1.2. Formula for \( B \) Recall that \( h(\alpha, t) = \Phi(z(\alpha, t), t) \), where \( \Phi \) is the Riemann mapping. So we have
\[
h_t = \frac{h_\alpha}{z_\alpha}, \quad \Phi_z = \frac{h_\alpha}{z_\alpha}.
\]

Precomposite with \( h^{-1} \) on both sides of the above,
\[
\mathcal{B} = h_t \circ h^{-1} = \Phi_t \circ Z + \frac{\mathcal{D}_t Z}{Z_\alpha}.
\]

Apply \( I - \mathbb{H} \) then take real part, we obtain
\[
\mathcal{B} = \Re\{[\mathcal{D}_t Z, \mathbb{H}](\frac{1}{Z_\alpha} - 1)\} + 2\Re\{\mathcal{D}_t Z\}.
\]

5.1.3. The quasilinearization Since \( F \) is holomorphic, we have \( i \partial_\alpha F = \Lambda F \). Using (5.3), we have
\[
(\mathcal{D}_t^2 + A\Lambda) F = -i \mathcal{D}_t A\bar{Z}_\alpha - (\mathcal{D}_t^2 + i A\partial_\alpha) Q.
\]

Using \( \frac{1}{2}(I + \mathbb{H}) F = F \), applying \( \frac{1}{2}(I + \mathbb{H}) \) on both sides of (5.15), we obtain\(^6\)
\[
(\mathcal{D}_t^2 + A\Lambda) F = [\mathcal{D}_t^2 + A\Lambda, \mathbb{H}] F - \frac{i}{2}(I + \mathbb{H}) \mathcal{D}_t A\bar{Z}_\alpha - \frac{1}{2}(I + \mathbb{H})(\mathcal{D}_t^2 + i A\partial_\alpha) Q := G.
\]

In lagrangian coordinates, the velocity of the point vortex \( z_j \) is given by
\[
\dot{z}_j(t) = \check{U}(z_j(t), t) + \sum_{1 \leq k \leq N, k \neq j} \frac{\lambda_i i}{2\pi} \frac{1}{z_k(t) - z_j(t)},
\]
where
\[
\check{U}(z, t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{z_\beta(\beta, t)}{z - z(\beta, t)} f(\beta, t) d\beta.
\]

\(^6\) The reason that we apply \( \frac{1}{2}(I + \mathbb{H}) \) on both sides of (5.15) is that it is easier to prove the equivalence of (5.16) and (5.3). To show that (5.15) is equivalent to (5.3), we need to prove that a solution \( F \) to (5.15) satisfies \( (I - \mathbb{H}) F = 0 \), which means that \( F = (I + \mathbb{H})g \) for some function \( g \).
Change of variables, we have
\[ U(z, t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{Z_{2}(\beta, t)}{z - Z(\beta, t)} F(\beta, t) d\beta. \] (5.17)

So we obtain a quasilinear system
\[
\begin{aligned}
(D_{t}^{2} + \mathcal{A} \Lambda) F &= G, \\
\mathcal{A} &= \frac{A_{1}}{|Z_{\alpha}|^{2}} = \frac{|D_{t}^{2} \bar{Z} - i|}{A_{1}}, \\
\frac{1}{Z_{\alpha}} &= i \frac{D_{t}^{2} \bar{Z} - i}{A_{1}}, \\
G &= [D_{t}^{2} + \mathcal{A} \Lambda, \mathbb{H}] F - \frac{i}{2} (I + \mathbb{H}) D_{t} \mathcal{A} \bar{Z}_{\alpha} - \frac{1}{2} (I + \mathbb{H}) (D_{t}^{2} + i A \partial_{\alpha}) Q, \\
D_{t} \bar{Z} &= F + Q, \\
Q &= -\sum_{j=1}^{N} \frac{\lambda_{j} i}{2\pi} z_{j}(t), \\
\dot{z}_{j}(t) &= \mathcal{U}(z_{j}(t), t) + \sum_{1 \leq k \leq N, k \neq j} \frac{\lambda_{k}}{2\pi} \frac{1}{z_{j}(t) - z_{j}(t)},
\end{aligned}
\] (5.18)

where
\[
\begin{aligned}
A_{1} &= 1 + \frac{1}{2\pi} \int \frac{|D_{t} Z(\alpha, t) - D_{t} Z(\beta, t)|^{2}}{(\alpha - \beta)^{2}} \frac{d\beta}{d\alpha}, \\
&= -\sum_{j=1}^{N} \frac{\lambda_{j}}{2\pi} Re \left\{ \left( (I - \mathbb{H}) \frac{Z_{\alpha}}{(Z(\alpha, t) - z_{j}(t))^{2}} \right) (D_{t} Z - \dot{z}_{j}(t)) \right\}. 
\end{aligned}
\] (5.19)

\(D_{t}, \mathcal{A}|_{Z_{\alpha}}^{2}\) is given by (5.9). Note that \(G\) is determined by \(Z, F, D_{t} F\) and \(\{z_{j}\}\). So we can write \(G\) as \(G(Z, F, D_{t} F, \{z_{j}(t)\})\). Here, \(\{z_{n}^{0}\}\) means \(\{z_{1}(t), \cdots, z_{N}(t)\}\). For the same reason, \(\mathcal{B}, \mathcal{A}\) are determined by \(Z, F, D_{t} F\) and \(\{z_{j}\}\), so we write
\[
\mathcal{B} = \mathcal{B}(Z, F, D_{t} F, \{z_{j}\}), \quad \mathcal{A} = \mathcal{A}(Z, F, D_{t} F, \{z_{j}\}).
\]

Remark 5.1. Unlike the irrotational case considered in [51] by Wu, it is quite complicated to derive an equivalent system of the water waves using only \(F, D_{t} F, \) and \(\{z_{j}\}\), because \(Q = -\sum_{j=1}^{N} \frac{\lambda_{j} i}{2\pi} z_{j}(t)\) depends explicitly on \(Z(\alpha, t)\).

5.1.4. The initial data  For the reader’s convenience, we list some notations as follows

\[
\begin{aligned}
Z_{0}(\alpha) &= \text{Initial free interface in Riemann variables, that is, } Z_{0}(\alpha) := Z(\alpha, 0), \\
F_{0}(\alpha) &= \text{Initial value of } F(\alpha, t), \text{ that is, } F_{0}(\alpha) := F(\alpha, 0), \\
F_{1}(\alpha) &= \text{Initial value of } D_{t} F, \text{ that is, } F_{1}(\alpha) := D_{t} F(\alpha, 0), \\
Q_{0}(\alpha) &= \text{Initial value of } Q(\alpha, t), \text{ that is, } Q_{0}(\alpha) := Q(\alpha, 0), \\
z_{j,0} &= \text{Initial value of } z_{j}(t), \text{ that is, } z_{j,0} := z_{j}(0), \\
\dot{z}_{j,0} &= \text{Initial value of } \dot{z}_{j}(t), \text{ that is, } \dot{z}_{j,0} := \dot{z}_{j}(0), \\
Q_{1}(\alpha) &= \text{Initial value of } D_{t} Q, \text{ that is, } Q_{1}(\alpha) := D_{t} Q(\alpha, 0), \\
A_{1,0}(\alpha) &= \text{Initial value of } A_{1}(\alpha, t), \text{ that is, } A_{1,0} := A_{1}(\alpha, 0),
\end{aligned}
\]
We parametrize $F$ such that $U$ by Riemann mapping, so $z_0(\alpha) = Z_0(\alpha)$. Recall that

$$Q = -\sum_{j=1}^{N} \frac{\lambda_j}{2\pi} \frac{1}{Z(\alpha, t) - z_j(t)}.$$  

$$\mathcal{D}_t Q = \sum_{j=1}^{N} \frac{\lambda_j}{2\pi} \frac{D_z Z - \dot{z}_j(t)}{(Z(\alpha, t) - z_j(t))^{3/2}}.$$  

$$\dot{z}_j(t) = \sum_{1 \leq k \leq N \atop k \neq j} \frac{\lambda_j}{2\pi} \frac{1}{z_k(t) - z_j(t)} + \mathcal{U}(z_j(t), t),$$

where $\mathcal{U}(z_j(t), t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{Z(\beta, t)}{z_j(t) - Z(\beta, t)} F(\beta, t) d\beta$ is determined by $Z$ and $F$. So

$$\dot{z}_{j,0} = \sum_{1 \leq k \leq N \atop k \neq j} \frac{\lambda_j}{2\pi} \frac{1}{z_k,0 - z_j,0} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\partial_{\beta} Z_0(\beta)}{z_j,0 - Z_0(\beta)} F_0(\beta) d\beta.$$  

$$Q_1 = \sum_{j=1}^{N} \frac{\lambda_j}{2\pi} \left( \bar{u}_0 - z_j,0 \right).$$  

$$A_{1,0} = 1 + \frac{1}{2\pi} \int |\bar{u}_0(\alpha) - \bar{u}_0(\beta)|^2 \frac{1}{(\alpha - \beta)^2} d\beta - \sum_{j=1}^{N} \frac{\lambda_j}{2\pi} \Im \left( (I - \mathbb{H}) \frac{\partial_{\alpha} Z_0(\alpha)}{(Z_0(\alpha) - z_j(t))^{3/2}} (\bar{u}_0(\alpha) - \dot{z}_{j,0}) \right).$$

By the equation, we have $\mathcal{D}_t F = -\mathcal{D}_t Q + i - iA\bar{Z}_\alpha = -\mathcal{D}_t Q + i - i A \frac{1}{Z_0}$. So we require $F_1$ to satisfy

$$F_1 = -Q_1 + i + iA_{1,0} \frac{1}{\partial_{\alpha} Z_0}.$$  

The initial data for the system (5.18) is

$$Z(\alpha, 0) = Z_0(\alpha), \quad F(\alpha, 0) = F_0(\alpha), \quad \mathcal{D}_t F(\alpha, 0) = F_1, \quad \{z_j(0)\} = \{z_{j,0}\}.$$  

such that $F_1$ satisfies the compatibility condition (5.23), and $A_0$, $Z_0$ satisfy

$$\inf_{\alpha \in \mathbb{R}} A_0(\alpha) \geq 2\alpha_0.$$  

$$|Z_0(\alpha) - Z_0(\beta)| \geq C_0 |\alpha - \beta|, \quad \forall \alpha, \beta \in \mathbb{R}.$$  

Denote

$$\frac{1}{4} M_0^2 := \|F_0\|_{H^{s+1/2}}^2 + \|F_1\|_{H^s}^2 + \|\partial_{\alpha} Z_0 - 1\|_{H^{s-1/2}}^2 + \lambda^2.$$  

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Without loss of generality, we assume $\tilde{\lambda} \leq \frac{1}{4} M_0$.

**The constants:** We fix positive constants $s \geq 4, M_0, C_0, \alpha_0, d_{L,0}, d_{P,0}$. Throughout the rest of this section, if not specified, we use $C$ to denote a constant that depends continuous on $\tilde{\lambda}, s, M_0, C_0, \alpha_0, d_{L,0}^{-1}, d_{P,0}^{-1}$.

5.2. The linear system. Fix $T \in (0, \infty)$. Consider the linear system

\begin{equation}
\begin{cases}
(D_t^2 + A\Lambda) F = G, \\
F(\cdot, 0) = F_0, \quad D_t F(\cdot, 0) = F_1
\end{cases}
\end{equation}

where $B \in C([0, T]; H^{s+1/2})$, $G \in C([0, T]; H^s)$, $F_0 \in H^{s+1/2}$ and $F_1 \in H^s$ are given. If we let $W := D_t F$, then the system (5.28) is equivalent to

\begin{equation}
\begin{cases}
W_t + BW_\alpha + A\Lambda F = G, \\
F_t + BF_\alpha - W = 0, \\
(F(\cdot, 0), W(\cdot, 0)) = (F_0, F_1).
\end{cases}
\end{equation}

The system (5.29) has been studied by Wu in [51]. The following proposition is Theorem 5.10 and Lemma 5.8 in [51].

**Proposition 4.** Let $0 < T < \infty, s \geq 4, A-1 \in C([0, T]; H^s), B \in C([0, T]; H^{s+1/2})$, and $A_t \in C([0, T]; H^2)$. Let $G \in C([0, T]; H^s)$. Assume $A \geq \alpha_0$, for some constant $\alpha_0 > 0$. If $(F_0, F_1) \in H^{s+1/2} \times H^s$, then there is a unique solution $F$ of (5.28) such that $(F, D_t F) \in C^j ([0, T]; H^{s+1/2-j/2} \times H^{s-j/2}), j = 0, 1$. Moreover,

\begin{equation}
\| (F, D_t F) \|_{H^{s+1/2} \times H^s} \leq \delta_0 e^{\delta_1 t} \| (F_0, F_1) \|_{H^{s+1/2} \times H^s}^2 + \int_0^t e^{\delta_1 (t-\tau)} \| G(\tau) \|_{H^s}^2 d\tau,
\end{equation}

where

\[
\delta_0 = \frac{\| A \|_{L^\infty}}{\alpha_0} + 2(1 + k_0) \sup_{0 \leq t \leq T} \| A - 1 \|_{H^2},
\]

and

\[
\delta_1 = k(1 + \frac{1}{\alpha_0})(1 + \sup_{0 \leq t \leq T} \| A - 1 \|_{H^s})^2 + (1 + \sup_{0 \leq t \leq T} \| B \|_{H^{s+1/2}}^2) + \sup_{0 \leq t \leq T} \| A_t \|_{H^2},
\]

$k_0, k$ are constants depend only on $s$.

5.3. Estimates. Most of the estimates in this subsection can be found in [51] (Lemma 5.1-5.6). The main difference in our situation is the presence of the point vortices. The proof of those estimates that do not involve the point vortices can be proved in the same way as in [51] (indeed, follows almost immediately from Lemma 5.1), so we omit the proof. We give the proof of those estimates involve explicitly the point vortices.
5.3.1. Additional convention Throughout this subsection, we use the following convention: Given \((Z, F, \mathcal{D}_t F, \{z_j\})\), then we always define \(\mathcal{D}_t, \mathcal{A}, G, \Sigma(t), \Omega(t), d_I(t), d_P(t), \mathcal{U}, Q, \mathcal{D}_t Q, \hat{z}_j, \hat{\lambda}\) by

\[
\begin{align*}
\Sigma(t) & : \text{ the curve parametrized by } Z(\alpha, t), \\
\Omega(t) & : \text{ the region bounded above by } \Sigma(t), \\
\mathcal{D}_t & = \partial_t + B\partial_\alpha, \\
\mathcal{B} & = \mathcal{B}(Z, F, \mathcal{D}_t F, \{z_j\}), \\
\mathcal{A} & = \mathcal{A}(Z, F, \mathcal{D}_t F, \{z_j\}), \\
G & = G(Z, F, \mathcal{D}_t F, \{z_j\}), \\
d_I(t) & = \inf_{\alpha \in \mathbb{R}} \min_{1 \leq j \leq N} |Z(\alpha, t) - z_j(t)|, \\
d_P(t) & = \min_{1 \leq j, k \leq N} |z_j(t) - z_k(t)|, \\
U(z, t) & = \frac{1}{2\pi} \int_0^\infty \frac{Z_p(\beta, t)}{\bar{z} - Z(\beta, t)} F(\beta, t) d\beta, \\
Q & = -\sum_{j=1}^N \frac{\lambda_j}{2\pi} Z(\alpha, t) - z_j(t), \\
\mathcal{D}_t Q & = \sum_{j=1}^N \frac{\lambda_j}{2\pi} \mathcal{D}_t Z - z_j(t), \\
\hat{z}_j(t) & = \sum_{1 \leq k \leq N, k \neq j} \frac{\lambda_j}{2\pi} \frac{z_k(t) - z_j(t)}{\bar{z}_k(t) - z_j(t)} + \tilde{U}(z_j(t), t), \\
\hat{\lambda} & = N \max_{1 \leq j \leq n} |\lambda_j|.
\end{align*}
\]

If \((Z, F, \mathcal{D}_t F, \{z_j\})\) is replaced by \((Z^0, F^0, \mathcal{D}_t^0 F^0, \{z_j^0\})\), then \(\mathcal{D}_t^0, \mathcal{A}^0, G^0, \Sigma^0(t), \Omega^0(t), d_I^0(t), d_P^0(t), U^0, Q^0, \mathcal{D}_t^0 Q^0, \hat{z}_j\) are defined similarly.

5.3.2. A priori assumption We’ll derive estimates under the following a priori assumption. These assumptions are verified during the process of iteration in §5.5.

(AS1) \(Z_\alpha - 1 \in C([0, T]; H^{s-1/2}), \mathcal{D}_t F \in C([0, T]; H^{s+1/2}), \mathcal{D}_t^2 F \in C([0, T]; H^s),\)
\(z_j(t) \in C^2([0, T]; \Omega(t)).\)

(AS2) \(\sup_{0 \leq t \leq T} \left( \|F(\cdot, t)\|_{H^{s+1/2}}^2 + \|\mathcal{D}_t F(\cdot, t)\|_{H^s}^2 + \|Z_{\alpha}(\cdot, t) - 1\|_{H^{s-1/2}}^2 + \hat{\lambda}^2 \right) \leq 4\delta_0^2 M_0^2.\)

(AS3) \(\mathcal{A}(\alpha, t)|Z_\alpha| \geq a_0, \quad \forall \ t \in [0, T].\)

(AS4) \(|Z(\alpha, t) - Z(\beta, t)| \geq \frac{2}{3} C_0 |\alpha - \beta|, \quad \forall \ t \in [0, T].\)

(AS5) \(d_I(t) \geq \frac{1}{2} d_{I,0}, \quad d_P(t) \geq \frac{1}{2} d_{P,0}, \quad t \in [0, T].\)

5.4. A priori estimates.

**Lemma 5.1.** Let \(r \geq 0, q > 1/2, \text{ and } s \geq 1.\) For \(a, u \in \mathcal{S}, \text{ where } \mathcal{S} \text{ represents the Schwartz functions on } \mathbb{R}. \) we have

(a) \(\|a, \mathcal{H}\|_{H^r} \leq k_0 \|a\|_{H^{r+p}} \|u\|_{H^{q-p}}, \quad p \geq 0.\)

(b) \(\|a, \mathcal{A}^s\|_{L^2} \leq k_0(\|a\|_{H^s} \|u\|_{H^q} + \|u\|_{H^q} \|a\|_{H^{q-s}}).\)

(c) \(\|a(1 + \mathcal{A})^s u\|_{L^2} \leq k_0(\|a\|_{H^s} \|u\|_{H^q} + \|u\|_{H^q} \|a\|_{H^{s+1}}).\)

(d) \(\|a, \mathcal{A}^{1/2}(1 + \mathcal{A})^{1/2} u\|_{L^2} \leq k_0(\|a\|_{H^{1/2} + 1/2} \|u\|_{H^q} + \|u\|_{H^q} \|a\|_{H^{1/2} + 1/2}).\)

(e) \(\|a, \mathcal{A}^p u\|_{L^2} \leq k_0 \|a\|_{H^{1+p}} \|u\|_{H^{q-p}}, \quad 0 < p \leq 1,\)

where \(k_0\) is a constant independent of \(a\) and \(u.\)
See [57] for the proof.

It’s easy to obtain the following lemma.

**Lemma 5.2.** Let $0 < T < \infty$. Assume (AS1)–(AS5), then $\frac{1}{Z_\alpha} - 1 \in C([0, T]; H^{s-1/2})$, and
\[
\left\| \frac{1}{Z_\alpha} - 1 \right\|_{H^{s-1/2}} \leq CM_0. \tag{5.32}
\]

**Lemma 5.3.** Let $0 < T < \infty$. Assume (AS1)–(AS5), then $Q \in C([0, T]; H^{s+1/2})$, and
\[
\|Q(\cdot, t)\|_{H^{s+1/2}} \leq CM_0. \tag{5.33}
\]

**Proof.** Without loss of generality, we assume that $s$ is an integer. Recall that $Q = -\sum_{j=1}^{N} \lambda_j \frac{1}{2\pi} \frac{1}{Z(\alpha, t) - z_j(t)}$. By definition,
\[
\|Q\|^2_{H^{s+1/2}} = \|Q\|_{L^2}^2 + \|\dot{Q}\|_{H^{s+1/2}}^2. \tag{5.34}
\]

Here $\dot{Q}$ represents the homogeneous Sobolev space. Denote $Q_j = \frac{1}{Z(\alpha, t) - z_j(t)}$. By Triangle inequality,
\[
\|Q\|_{H^{s+1/2}} \leq \bar{\lambda} \max_{1 \leq j \leq N} \|Q_j\|_{H^{s+1/2}} = \bar{\lambda} \max_{1 \leq j \leq N} \|\partial^s Q_j\|_{H^{1/2}}. \tag{5.35}
\]

Let $h_j = \frac{1}{\alpha - z_j(t)}$. Then $Q_j = h_j \circ z$. Using the Faa di Bruno formula,
\[
\partial^s Q_j(\alpha, t) = \sum_{k=1}^{s} \sum_{\Lambda_{s,k}} \frac{s!}{k_1!k_2! \cdots k_s!} h_j^{(1)}(Z(\alpha, t), t) \prod_{j=1}^{s} \left( \frac{\partial^j Z(\alpha, t)}{j!} \right)^{k_j}
= -\frac{\partial^s Z(\alpha - 1)}{(Z(\alpha, t) - z_j(t))^2}
+ \sum_{k=2}^{s} \sum_{\Lambda_{s,k}} \frac{s!}{k_1!k_2! \cdots k_s!} h_j^{(1)}(Z(\alpha, t), t) \prod_{j=1}^{s} \left( \frac{\partial^j Z(\alpha, t)}{j!} \right)^{k_j}
:= I_1 + I_2,
\]

where
\[
\Lambda_{s,k} := \{(k_1, \ldots, k_s) \in (\mathbb{N} \cup \{0\})^s : \sum_{j=1}^{s} k_j = k, \sum_{j=1}^{n} jk_j = s\}.
\]

Note that $h_j^{(k)}(Z(\alpha, t), t) \in H^{s-1/2}$ and applying lemma 2.8, it’s easy to obtain that for fixed $1 \leq k \leq s$,
\[
\|h_j^{(k)}(Z(\alpha, t), t)\|_{H^{s-1/2}} \leq C. \tag{5.37}
\]

Clearly,
\[
\|I_1\|_{H^{1/2}} \leq C\|Z_\alpha - 1\|_{H^{s-1/2}}, \tag{5.38}
\]

and since for the terms in $I_2$, the derivative taken on $Z(\alpha, t)$ is at most $s - 1$, so we have
\[
\|I_2\|_{H^{1/2}} \leq \|I_2\|_{H^1} \leq C\|Z_\alpha - 1\|_{H^{s-1}}. \tag{5.39}
\]

So we complete the proof of the lemma. \qed
As a corollary, we have

**Corollary 5.1.** Let $0 < T < \infty$. Assume (AS1)–(AS5), then $D_t Z \in C([0, T]; H^{s+1/2})$. Moreover,

$$\|D_t Z\|_{H^{s+1/2}} \leq CM_0.$$  

**Lemma 5.4.** Let $0 < T < \infty$. Assume (AS1)–(AS5), then

$$\sup_{t \in [0,T]} (|\ddot{z}_j(t)| + |\dot{z}_j(t)|) \leq CM_0, \quad j = 1, \ldots, N. \quad (5.40)$$

**Proof:** The main tool is the maximum principle for holomorphic functions. Recall that

$$\dot{z}_j(t) = \sum_{1 \leq k \leq N, k \neq j} \frac{\lambda_k i}{2\pi(z_j(t) - z_k(t))} + \tilde{U}(z_j(t), t),$$

where $U(Z, t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{Z(\beta, t)}{Z - Z(\beta, t)} F(\beta, t) d\beta$ is holomorphic in $\Omega(t)$ with boundary value $F$ on $\Sigma(t)$.

**Estimate** $\dot{z}_j$: Clearly,

$$\sum_{1 \leq k \leq N, k \neq j} \frac{\lambda_k i}{2\pi(z_j(t) - z_k(t))} \leq \sum_{j=1}^{N} \frac{\lambda_j}{2\pi} |z_j(t) - z_k(t)|^{-1} \leq \frac{\lambda}{2\pi} d^{-1}.$$

By maximum principle,

$$|\tilde{U}(z_j(t), t)| \leq \|U(\cdot, t)\|_{L^\infty(\Sigma(t))} = \|F\|_{\infty} \leq \|F\|_{H^1} \leq CM_0.$$  

So we obtain

$$|\dot{z}_j(t)| \leq CM_0 + \frac{\lambda}{\pi} d^{-1}. \quad (5.41)$$

**Estimate** $\ddot{z}_j(t)$: Take time derivative on both sides of $\dot{z}_j(t) = \sum_{k: k \neq j} \frac{\lambda_k i}{2\pi(z_j(t) - z_k(t))} + \tilde{U}(z_j(t), t)$, we obtain

$$\ddot{z}_j(t) = -\sum_{k: k \neq j} \frac{\lambda_k i\dot{z}_j(t) - \dot{z}_k(t)}{2\pi(z_j(t) - z_k(t))^2} + \tilde{U}_z(z_j(t), t)\dot{z}_j(t) + \tilde{U}_t(z_j(t), t) \quad (5.42)$$

$U_z$ is a holomorphic function in $\Omega(t)$ with boundary value $\frac{F_{\alpha}}{Z_{\alpha}}$. By maximum principle,

$$|U_z(z_j(t), t)| \leq \|U_z\|_{L^\infty(\Sigma(t))} = \left\| \frac{F_{\alpha}}{Z_{\alpha}} \right\|_{L^\infty} \leq \left\| F_{\alpha} \right\|_{L^\infty} \left\| \frac{1}{Z_{\alpha}} \right\|_{\infty} \leq CM_0/C_0.$$  

$U_t$ is holomorphic in $\Omega(t)$ with boundary value

$$U_t(Z(\alpha, t), t) = D_t U(Z(\alpha, t), t) - \frac{\partial_{\alpha} U(Z(\alpha, t), t)}{Z_{\alpha}} D_t Z = D_t F - \frac{F_{\alpha}}{Z_{\alpha}} D_t Z. \quad (5.43)$$
By maximum principle, Sobolev embedding,

\[ |\mathcal{U}_t(z_j(t), t)| \leq \|\mathcal{U}_t\|_{L^\infty(\Sigma(t))} \leq \bigg\| \mathcal{D}_t \mathcal{U}(Z(\alpha, t), t) - \frac{\partial_q \mathcal{U}(Z(\alpha, t), t)}{Z_\alpha} \mathcal{D}_t Z \bigg\|_{L^\infty} \leq \|\mathcal{D}_t F\|_{H^1} + \frac{1}{Z_\alpha} \|\mathcal{F}_\alpha\|_{H^1} \|\mathcal{D}_t Z\|_{H^1} \leq CM_0.

Clearly,

\[ \sum_{k:k \neq j} \frac{\lambda_k i \ddot{z}_j(t) - \ddot{z}_k(t)}{2\pi (z_j(t) - z_k(t))^2} \leq \tilde{\lambda} d_p(t)^{-2} \max_{1 \leq j \leq N} |\dot{z}_j(t)| \leq CM_0.

So the proof of the lemma is concluded. \(\square\)

**Lemma 5.5.** Let \(0 < T < \infty\). Assume \((A1)\)–\((A5)\), then \(B \in C([0, T]; H^{s+1/2})\), and

\[ \|B\|_{H^{s+1/2}} \leq CM_0. \tag{5.44} \]

**Proof.** Recall that

\[ B = \Re\{[\mathcal{D}_t Z, \mathbb{H}](\frac{1}{Z_\alpha} - 1)\} + 2\Re \mathcal{D}_t Z. \]

Then (5.45) follows from Lemma 5.1, Corollary 5.1, and Lemma 5.2. \(\square\)

**Lemma 5.6.** Let \(0 < T < \infty\). Assume \((A1)\)–\((A5)\), then \(\mathcal{D}_t Q \in C([0, T]; H^{s+1/2})\), \(\mathcal{D}_t^2 Q \in C([0, T]; H^3)\), and

\[ \|\mathcal{D}_t Q(\cdot, t)\|_{H^{s+1/2}} \leq CM_0, \tag{5.45} \]
\[ \|\mathcal{D}_t^2 Q(\cdot, t)\|_{H^s} \leq CM_0, \tag{5.46} \]
\[ \|\mathcal{A}Q_\alpha\|_{H^s} \leq CM_0, \tag{5.47} \]
\[ \|\mathcal{D}_t^2 Q + i\mathcal{A}Q_\alpha\|_{H^s} \leq CM_0. \tag{5.48} \]

**Proof.** Recall that \(Q = -\sum_{j=1}^N \frac{\lambda_j i}{2\pi} \frac{1}{Z(\alpha, t) - z_j(t)}\). So we have

\[ \mathcal{D}_t Q = \sum_{j=1}^N \frac{\lambda_j i}{2\pi} \frac{\mathcal{D}_t Z - \dot{z}_j(t)}{(Z(\alpha, t) - z_j(t))^2}, \]
\[ \mathcal{D}_t^2 Q = -\sum_{j=1}^N \frac{\lambda_j i}{\pi} \frac{(\mathcal{D}_t Z - \dot{z}_j(t))^2}{(Z(\alpha, t) - z_j(t))^3} + \sum_{j=1}^N \frac{\lambda_j i}{2\pi} \frac{\mathcal{D}_t^2 Z - \ddot{z}_j(t)}{(Z(\alpha, t) - z_j(t))^2}, \]
\[ \mathcal{A}Q_\alpha = \sum_{j=1}^N \frac{\lambda_j i}{2\pi} \frac{i\mathcal{A}Z_\alpha}{(Z(\alpha, t) - z_j(t))^2} = -\sum_{j=1}^N \frac{\lambda_j i}{2\pi} \frac{\mathcal{D}_t^2 Z - i}{(Z(\alpha, t) - z_j(t))^2}. \]

Here, for the last equality we’ve used \(\mathcal{D}_t^2 Z - i\mathcal{A}Z_\alpha = -i\). Then (5.45)–(5.46)–(5.47)–(5.48) can be proved the same way as the same proof as in Lemma 5.3. \(\square\)

**Lemma 5.7.** Let \(0 < T < \infty\). Assume \((A1)\)–\((A5)\), then
(a) \( \inf_{\alpha \in \mathbb{R}} A_1 \geq \alpha_0 C_0/2 \).

(b) \( \frac{1}{A_1} - 1, A_1 - 1 \in C([0, T]; H^s) \). Moreover,

\[
\left\| \frac{1}{A_1} - 1 \right\|_{H^s} + \left\| A_1 - 1 \right\|_{H^s} \leq CM_0.
\]

(c) \( \frac{1}{Z_\alpha} - 1 \in C([0, T]; H^s) \), and

\[
\left\| \frac{1}{Z_\alpha} - 1 \right\|_{H^s} \leq CM_0.
\]

Proof. (a) follows from the definition of \( A_1 \). We have \( A_1 = A|Z_\alpha|^2 \). (b) follows from (a), (5.19), Lemma 5.1, Corollary 5.1, Lemma 5.4, and Lemma 5.6.

For (c), we have

\[
\frac{1}{Z_\alpha} - 1 = i \frac{D_t^2 Z - i}{A_1} - 1 = \frac{iD_t F + iD_t Q + 1 - A_1}{A_1}.
\]

Then (c) follows from Lemma 5.6 and part (b) of this Lemma. \( \Box \)

Lemma 5.8. Let \( 0 < T < \infty \). Assume (AS1)–(AS5), then \( D_t Q \in C([0, T]; H^{s+1/2}) \), \( D_t^2 Q \in C([0, T]; H^s) \), and

\[
\left\| [D_t^2 + i A\partial_\alpha, \mathbb{H}] F \right\|_{H^s} \leq CM_0. \tag{5.49}
\]

Proof. Integration by parts, we obtain

\[
[D_t^2, \mathbb{H}] F = [D_t, \mathbb{H}] D_t F + D_t[D_t, \mathbb{H}] F = [B, \mathbb{H}] \partial_\alpha D_t F + [D_t B, \mathbb{H}] F_\alpha = 2[B, \mathbb{H}] \partial_\alpha D_t F + [D_t B, \mathbb{H}] F_\alpha - \frac{1}{\pi i} \int \left( \frac{B(\alpha, t) - B(\beta, t)}{\alpha - \beta} \right)^2 F_\beta(\beta, t)d\beta,
\]

\([A\partial_\alpha, \mathbb{H}] F = [A, \mathbb{H}] F_\alpha.\]

Then applying Lemma 5.1 and Lemma 5.5, we obtain

\[
\left\| [D_t^2, \mathbb{H}] F \right\|_{H^s} \leq CM_0. \tag{5.50}
\]

Since \( A \) as \( A|Z_\alpha| \frac{1}{Z_\alpha} = |D_t^2 Z + i| \frac{1}{Z_\alpha} \), applying Lemma 5.1 and Lemma 5.7, we obtain

\[
\left\| [A\partial_\alpha, \mathbb{H}] F \right\|_{H^s} \leq CM_0. \tag{5.51}
\]

(5.50) and (5.51) give (5.49). \( \Box \)

The following two lemmas are easy consequence of the previous estimates.

Lemma 5.9. Let \( 0 < T < \infty \). Assume (AS1)–(AS5), then

(a) \( G = G(Z, F, D_t F, \{z_j\}) \in C([0, T]; H^s) \), and

\[
\left\| G(Z, F, D_t F, \{z_j\}) \right\|_{H^s} \leq CM_0. \tag{5.52}
\]
(b) \( A - 1 = A(Z, F, \mathcal{D}_t F, \{z_j\}) - 1 \in C([0, T]; H^s) \), \( B = B(Z, F, \mathcal{D}_t F, \{z_j\}) \in C([0, T]; H^{s+1/2}) \), and
\[
\|A(Z, F, \mathcal{D}_t F, \{z_j\}) - 1\|_{H^s} \leq CM_0, \quad (5.53)
\]
\[
\|B(Z, F, \mathcal{D}_t F, \{z_j\})\|_{H^{s+1/2}} \leq CM_0. \quad (5.54)
\]

(c) Assume \((Z^0, F^0, \mathcal{D}^{(0)}_t F^0, \{z^0_j\})\) which follows the convention in (5.31). Assume
\[
(Z^0_a - 1, F^0, \mathcal{D}^{(0)}_t F^0) \in C([0, T]; H^{s-1/2} \times H^{s+1/2} \times H^s), \quad \text{and} \quad \{z^0_j\} \in C^2([0, T]; (\Omega^0)^N).
\]
Assume that
\[
\sup_{0 \leq t \leq T} \left( \|F^0(\cdot, t)\|^2_{H^{s+1/2}} + \|\mathcal{D}^{(0)}_t F^0(\cdot, t)\|^2_{H^s} + \|Z^0_a(\cdot, t) - 1\|_{H^{s-1/2}}^2 + \lambda^2 \right) \leq 4 \delta^2 M_0^2.
\]
\[
A(0) \partial_t Z^0 | \geq \alpha_0, \quad \forall \ \alpha, \beta \in \mathbb{R}, \quad t \in [0, T],
\]
\[
|Z^0(\alpha, t) - Z^0(\beta, t)| \geq \frac{1}{2} C_0 |\alpha - \beta|, \quad \forall \ \alpha, \beta \in \mathbb{R},
\]
and
\[
d^0_1(t) \geq \frac{1}{2} d_{I,0}, \quad d^0_p(t) \geq \frac{1}{2} d_{P,0} \quad t \in [0, T].
\]

Then
\[
\|G(Z, F, \mathcal{D}_t F, \{z_j\}) - G(Z^0, F^0, \mathcal{D}^{(0)}_t F^0, \{z^0_j\})\|_{H^s} \leq CM_0(\|F - F_0\|_{H^{s+1/2}} + \|\mathcal{D}_t F - \mathcal{D}^{(0)}_t F^0\|_{H^s} + \|\{z_j\} - \{z^0_j\}\| + \|Z_a - Z^0_a\|_{H^{s-1/2}}), \quad (5.55)
\]
\[
\|A(Z, F, \mathcal{D}_t F, \{z_j\}) - A(0) Z^0(\cdot, t)\|_{H^s} \leq CM_0(\|F - F_0\|_{H^{s+1/2}} + \|\mathcal{D}_t F - \mathcal{D}^{(0)}_t F^0\|_{H^s} + \|\{z_j\} - \{z^0_j\}\| + \|Z_a - Z^0_a\|_{H^{s-1/2}}), \quad (5.56)
\]
\[
\|B(Z, F, \mathcal{D}_t F, \{z_j\}) - B(0) Z^0(\cdot, t)\|_{H^{s+1/2}} \leq CM_0(\|F - F_0\|_{H^{s+1/2}} + \|\mathcal{D}_t F - \mathcal{D}^{(0)}_t F^0\|_{H^s} + \|\{z_j\} - \{z^0_j\}\| + \|Z_a - Z^0_a\|_{H^{s-1/2}}). \quad (5.57)
\]

Lemma 5.10. Let \(0 < T < \infty\). Assume (AS1)-(AS5). Then
(a) \( \mathcal{A}_t \in C([0, T]; H^2) \), and
\[
\|\mathcal{A}_t\|_{H^2} \leq CM_0, \quad \|\mathcal{A}_t\|_{\infty} \leq CM_0, \quad (5.58)
\]
(b) \( \|\mathcal{A}(\cdot, t) - A_0\|_{H^2} \leq CM_0 t. \)

Proof. Using \( \mathcal{D}^2_t F = G - \mathcal{A} \Lambda F \), by Lemma 5.9, we have
\[
\|\mathcal{D}^2_t F\|_{H^{s-1/2}} \leq \|G - \mathcal{A} \Lambda F\|_{H^{s-1/2}} \leq \|G\|_{H^{s-1/2}} + (1 + \|\mathcal{A} - 1\|_{H^{s-1/2}}) \|\Lambda F\|_{H^{s-1/2}} \leq CM_0.
\]
(a) is the direct consequence of the following facts:
(1) \[ A = \frac{A_1}{|Z_\alpha|^2}, \]

(2) \[ |Z_\alpha| \geq C_0 \text{ and } Z_\alpha - 1 \in H^{s-1/2}, \]

(3) \[
A_1 = 1 + \frac{1}{2\pi} \int \frac{[D_t Z(\alpha, t) - D_t Z(\beta, t)]^2}{(\alpha - \beta)^2} d\beta
- \sum_{j=1}^{N} \frac{\lambda_j}{2\pi} \text{Re}\left\{ (I - \mathbb{H})(\frac{Z_\alpha}{(Z(\alpha, t) - z_j(t))^2})(D_t Z - \dot{z}_j(t)) \right\}.
\]

Using Facts (1) and (2), it’s a standard estimate to obtain that
\[ \partial_t |Z_\alpha|^2 \in C([0, T]; H^2), \]
and
\[ \left\| \partial_t |Z_\alpha|^2 \right\|_{H^2} \leq CM_0. \]

Using Fact (3), it’s also standard to obtain
\[ \left\| \partial_t A_1 \right\|_{H^2} \leq CM_0. \]

So we have
\[ \left\| \mathcal{A}_t \right\|_{H^2} \leq \left\| (A_1)_{t} \frac{1}{|Z_\alpha|^2} \right\|_{H^2} + \left\| A_1 \left( \frac{1}{|Z_\alpha|^2} \right) \right\|_{H^2} \leq CM_0, \]
which gives (a).

(b) is the direct consequence of \( \mathcal{A}(\cdot, t) - \mathcal{A}_0 = \int_0^t \mathcal{A}_t(\cdot, \tau)d\tau \) and the estimate in (a).

\[ \square \]

5.5. The wellposedness of the quasilinear system (5.18). Let’s recall that we chose the Riemann mapping as the initial parametrization, the initial data \((Z_0, F_0, F_1, \{z_j\})\) coming from the water wave system has the property that \(A_0|\partial_\alpha Z_0| \geq 2\alpha_0\), for some \(\alpha_0 > 0\). We use the wellposedness of the linear system (5.28), the energy estimates, the iteration method, and the fixed point theorem to prove the wellposedness of the quasilinear system (5.18).

Let \(\Sigma(t)\) be the curve parametrized by \(Z(\alpha, t)\), and \(\Omega(t)\) the region in \(\mathbb{C}\) bounded above by \(\Sigma(t)\). Let \(Z_0(\alpha) := Z(\alpha, 0) = z_0(\alpha)\). Recall that
\[
d_1(t) = \inf_{\alpha \in \mathbb{C}} \min_{1 \leq j \leq N} |Z(\alpha, t) - z_j(t)|, \quad d_\mathcal{P}(t) = \min_{1 \leq j, k \leq N} |z_j(t) - z_k(t)|,
\]
and \(d_{1,0} := d_1(0), d_{\mathcal{P},0} := d_\mathcal{P}(0)\). Recall that
\[ \delta_0 := \alpha_0^{-1} \|\mathcal{A}_0\|_{L^\infty} + 2(1 + k_0\|\mathcal{A}_0 - 1\|_{H^2}). \]

Since \(\mathcal{A}_0 \geq 2\alpha_0\), we have \(\delta_0 \geq 3\).

**Theorem 4.** Let \(s \geq 4\). Assume that \(F_0 \in H^{s+1/2}, F_1 \in H^{s}, \{z_j\} \subset \Omega(0), A_0|Z_\alpha| \geq 2\alpha_0, d_{1,0} > 0, d_{\mathcal{P},0} > 0\). Then there exists \(T > 0\), such that the system (5.18)–(5.24) has a unique solution \((Z, F, D_tF, \{z_j\})\) satisfying
(a) \( Z - \alpha, F, D_t F \in C^j([0, T]; H^{s+1/2-j} \times H^{s+1/2-j}) \times C^j([0, T]; H^s), j = 0, 1, \text{ and } z_j \in C^2([0, T]; \Omega(t)), \; j = 1, \cdots, N. \) Moreover,

\[
\sup_{0 \leq t \leq T} \left( \| F(\cdot, t) \|_{H^{s+1/2}}^2 + \| D_t F(\cdot, t) \|_{H^s}^2 + \| Z_\alpha(\cdot, t) - 1 \|_{H^{s-1/2}}^2 + \bar{\lambda}^2 \right) \leq 4\delta_0^2 M_0^2.
\]

(b) \( |Z(\alpha, t) - Z(\beta, t)| \geq \frac{1}{2} C_0 |\alpha - \beta|, \; \alpha, \beta \in \mathbb{R}, \; t \in [0, T]. \)

(c) \( \frac{1}{2} d_{l,0} \leq d_l(t), \; \frac{1}{2} d_{p,0} \leq d_p(t), \; t \in [0, T]. \)

Here, \( T \) depends on \( s, \bar{\lambda}, \alpha_0, C_0, d_{l,0}^{-1}, d_{p,0}^{-1}, M_0. \)

**Proof.** Recall that

\[
\frac{1}{4} M_0^2 := \| F_0 \|_{H^{s+1/2}}^2 + \| F_1 \|_{H^s}^2 + \| \partial_\alpha Z_0 - 1 \|_{H^{s-1/2}}^2 + \bar{\lambda}^2.
\]  

Let \( T > 0 \) to be determined, define

\[
S_T = \left\{ \left( \xi, F, D_t F, \{z_j\} \right) \Big| Z = \alpha + \xi, \; \xi \in H^{s+1/2}, (\text{AS1})-(\text{AS5}) \text{ hold} \right\}.
\]

So

\[
S_T \subset C([0, T]; H^{s+1/2}) \times C([0, T]; H^{s+1/2}) \times C([0, T]; H^s) \times \{C^2([0, T]; \mathbb{C})\}.
\]

We denote \( D_t^{(n)} \) by \( \partial_t + B^n \partial_\alpha \), where \( B^n \) is the \( n \)-th approximation of \( B \), which will be constructed shortly.

**The zero-th approximation.** We take \( F^0 = F_0, D_t^{(-1)} F^0 := F_1, \{z_j^0\} = \{z_{j,0}\} \). We take \( Z^0 = Z_0, \xi^0 = Z^0 - \alpha \). Take \( G^0 = G(Z^0, F_0, F_1, \{z_{j,0}\}), A^0 = A_0. \) Clearly, for arbitrary \( T > 0, \)

\[
(\xi^0, F^0, D_t^0 F^0, \{z_j^0\}) \in S_T.
\]

**The \( n \)-th approximation.** Assume we have constructed \( (\xi^n, F^n, D_t^{(n)} F^n, \{z_j^n\}) \) such that

\[
(\xi^n, F^n, D_t^{(n)} F^n, \{z_j^n\}) \in S_T.
\]

**The \( (n + 1) \)-th approximation.** Let’s construct \( (\xi^{n+1}, F^{n+1}, D_t^{(n)} F^{n+1}, \{z_j^{n+1}\}) \) as follows.

**Step 1.** Define

\[
\begin{aligned}
G^{(n)} &:= G(Z^n, F^n, D_t^{(n-1)} F^n, \{z_j^n(t)\}) \\
B^n &:= B(Z^n, F^n, D_t^{(n-1)} F^n, \{z_j^n(t)\}) \\
A^n &:= A(Z^n, F^n, D_t^{(n)} F^n, \{z_j^n(t)\}), \\
Q^n &:= - \sum_{j=1}^N \frac{\lambda_{ji}}{2\pi} \frac{1}{Z^n(\alpha, t) - z_j^n(t)},
\end{aligned}
\]

and define \( U^n \) by

\[
U^n(Z, t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\partial_\beta Z^n(\beta, t)}{Z - Z^n(\beta, t)} F^n(\beta, t) d\beta.
\]
Let $\Sigma^n(t)$ be the curve parametrized by $Z^n(\alpha, t)$, and $\Omega(t)^n$ the region bounded above by $Z^n$.

**Step 2.** $F^{n+1}$ is defined as the solution of

\[
\begin{cases}
(D^{(n)}_t)^2 F^{n+1} + A^n F^{n+1} = G^n, \\
F^{n+1}(-, 0) = F_0, \quad D^{(n)} F^{n+1}(-, 0) = F_1.
\end{cases}
\]  

(5.64)

Define $(z_j^{n+1})$ by

\[
\begin{cases}
\frac{d}{dt} z_{j}^{n+1}(t) = \Phi^n(z_j^n(t), t) + \sum_{1 \leq k \leq N} \frac{\lambda_k t}{2\pi} \frac{1}{z_k(t) - z_j^n(t)}, \\
z_j^{n+1}(0) = z_j(0).
\end{cases}
\]  

(5.65)

By Lemma 5.9 and Lemma 5.10, we have $A^n - 1 \in C([0, T]; H^s)$, $B^n \in C([0, T]; H^{s+1/2})$, $G^n \in C([0, T]; H^s)$, and $\partial_t A^n \in C([0, T]; H^2)$. Moreover,

\[
\sup_{0 \leq t \leq T} \left( \|G^n(\cdot, t)\|_{H^s} + \|Q^n(\cdot, t)\|_{H^{s+1/2}} + \|B^n(\cdot, t)\|_{H^{s+1/2}} + \|A^n(\cdot, t) - 1\|_{H^s} \right) \leq C,
\]  

(5.66)

and

\[
\sup_{0 \leq t \leq T} (|\frac{d}{dt} z_j^n(t)| + |\frac{d^2}{dt^2} z_j^n(t)|) \leq C, \quad j = 1, \cdots, N.
\]  

(5.67)

Then by Proposition 4, there is a unique solution $(F^{n+1}, D^{(n)} F^{n+1}) \in C^l([0, T]; H^{s+1/2-l/2}) \times C([0, T]; H^{s-l/2}), l = 0, 1$, to the system (5.64). By standard ODE existence and uniqueness theorem, there is a unique solution $z_j^{n+1} \in C^2([0, T]; \Omega(t)^n)$, $j = 1, \cdots, N$ to the system (5.65).

Define $Z^{n+1}$ by solving

\[
\begin{cases}
D^{(n)}_t Z^{n+1} = \tilde{F}^n + \tilde{Q}^n, \\
Z^{n+1}(\alpha, 0) = Z_0(\alpha).
\end{cases}
\]  

(5.68)

Define $\xi^{n+1} = Z^{n+1} - \alpha$.

Define

\[
\begin{align*}
G^{(n+1)} & := G(Z^{n+1}, F^{n+1}, D^{(n)}_t F^{n+1}, \{z_j^n(t)\}) \\
B^{n+1} & := B(Z^{n+1}, F^{n+1}, D^{(n)}_t F^{n+1}, \{z_j^n(t)\}) \\
D^{(n+1)}_t & := \partial_t + B^{n+1} \partial_\alpha, \\
A^{n+1} & := A(Z^{n+1}, F^{n+1}, D^{(n)}_t F^{n+1}, \{z_j^n(t)\}), \\
Q^{n+1} & := Q(Z^{n+1}, z^{n+1}), \\
D^{(n)}_t Q^{n+1} & := D_t Q(Z^{n+1}, z^{n+1}, F^{n+1}).
\end{align*}
\]  

(5.69)

Let $\Sigma^{n+1}(t)$ be the curve parametrized by $Z^{n+1}(\alpha, t)$, and let $\Omega^{n+1}(t)$ be the region bounded above by $\Sigma^{n+1}(t)$. Define $U^{n+1}$ by

\[
U^{n+1}(Z, t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{Z^{n+1}(\beta, t)}{Z - Z^{n+1}(\beta, t)} F^{n+1}(\beta, t) d\beta.
\]
By Lemma 5.9, we have $G^{n+1}, A^{n+1} - 1 \in C([0, T]; H^s), B^{n+1} \in C([0, T]; H^{s+1/2})$. We show that $(\xi^{n+1}, F^{n+1}, D_t^{(n)} F^{n+1}, \{z^{n+1}_j\}) \in S_T$.

**Estimate** $\|\xi^{n+1}\|_{H^{s+1/2}}^2$:
Using (5.68), we have

$$\partial_t Z^{n+1} = \tilde{Q}^n + \tilde{F}^n - B^n \partial_\alpha Z^{n+1}. \tag{5.70}$$

So we obtain

$$\partial_t (Z^{n+1} - \alpha) = \tilde{Q}^n + \tilde{F}^n - B^n - B^n \partial_\alpha (Z^{n+1} - \alpha). \tag{5.71}$$

Since

$$\|B^n\|_{H^{s+1/2}} + \|\tilde{Q}^n\|_{H^{s+1/2}} + \|\tilde{F}^n\|_{H^{s+1/2}} \leq C,$$

by standard energy estimates, we have

$$\frac{d}{dt} \|Z^{n+1} - \alpha\|_{H^{s+1/2}}^2 \leq CM_0^2 + C \|Z^{n+1} - \alpha\|_{H^{s+1/2}}^2. \tag{5.72}$$

Indeed, in the above energy estimates, the worst term is

$$\Re \int \Lambda^{s+1/2}(Z^{n+1} - \alpha) \Lambda^{s+1/2}(B^n \partial_\alpha (Z^{n+1} - \alpha)),$$

which can be estimated as

$$\Re \int \Lambda^{s+1/2}(Z^{n+1} - \alpha) \Lambda^{s+1/2}(B^n \partial_\alpha (Z^{n+1} - \alpha))
= \Re \int \Lambda^{s+1/2}(Z^{n+1} - \alpha) B^n \partial_\alpha \Lambda^{s+1/2}(Z^{n+1} - \alpha)
+ \Re \int \Lambda^{s+1/2}(Z^{n+1} - \alpha) [\Lambda^{s+1/2}, B^n] \partial_\alpha (Z^{n+1} - \alpha)
:= I + II.$$

Clearly, integration by parts,

$$I = -\frac{1}{2} \int \partial_\alpha B^n |\Lambda^{s+1/2}(Z^{n+1} - \alpha)|^2 \leq C \|Z^{n+1} - \alpha\|_{H^{s+1/2}}^2.$$

By Lemma 5.1, we obtain

$$II \leq C \|Z^{n+1} - \alpha\|_{H^{s+1/2}}^2.$$

So we obtain (5.72). By Grönwall’s inequality, we obtain

$$\|\xi^{n+1}\|_{H^{s+1/2}} = \|Z^{n+1} - \alpha\|_{H^{s+1/2}} \leq (\|Z_0 - \alpha\|_{H^{s+1/2}}^2 + 2CM_0^2) e^{2Ct}. \tag{5.73}$$

Then we can take $T_1$ such that

$$e^{2Ct_1} \leq \frac{1}{2} \delta_0, \quad 2CT_1 e^{2Ct_1} \leq \frac{1}{2} \delta_0. \tag{5.74}$$

Note that $\delta_0 \geq 3$. 

So we obtain
\[
\sup_{0 \leq t \leq T_1} \| \xi^{n+1}_t \|_{H^{s+1/2}}^2 \leq \delta_0^2 M_0^2. \tag{5.75}
\]

**Estimate** \( \| (F^{n+1}, D^n_t F^{n+1}) \|_{H^{s+1/2} \times H^s}^2 \):

By Proposition 4,
\[
\| (F^{n+1}, D^n_t F^{n+1}) \|_{H^{s+1/2} \times H^s}^2 \leq \delta_{0,n} e^{\delta_1 t} \| (F_0, F_1) \|_{H^{s+1/2} \times H^s}^2 + \int_0^t e^{\delta_{1,n}(t-\tau)} \| G^n(\tau) \|_{H^s}^2 d\tau, \tag{5.76}
\]

where
\[
\delta_{0,n} = \frac{\| A^n \|_{L^\infty_t (\mathbb{R} \times [0,T])}}{\alpha_0} + 2(1 + k_0 \sup_{0 \leq t \leq T} \| A^n - 1 \|_{H^2}), \tag{5.77}
\]
\[
\delta_{1,n} = k(1 + \frac{1}{\alpha_0}) \{ (1 + \sup_{0 \leq t \leq T} \| A^n - 1 \|_{H^s})^2 + (1 + \sup_{0 \leq t \leq T} \| B^n \|_{H^{s+1/2}})^2 + \sup_{0 \leq t \leq T} \| A^n_t \|_{H^2} \}, \tag{5.78}
\]
is as in Proposition 4. By Lemma 5.10, we have
\[
\| A^n - 1 \|_{H^2} \leq \| A_0 - 1 \|_{H^2} + \| A(\cdot, t) - A_0 \|_{H^2}
\leq \| A_0 - 1 \|_{H^2} + \int_0^t \| A_t(\cdot, \tau) \|_{H^2} d\tau
\leq \| A_0 - 1 \|_{H^2} + t \sup_{0 \leq t \leq T} \| A_t(\cdot, t) \|_{H^2}
\leq \| A_0 - 1 \|_{H^2} + C M_0 t,
\]
and similarly,
\[
\| A \|_{L^\infty_t} \leq \| A_0 \|_{L^\infty} + C M_0 t.
\]

So we obtain \( \delta_{0,n} \leq \delta_0 + C M_0 t \); similarly, \( \delta_{1,n} \leq \delta_1 + C M_0 t \). So we have
\[
\delta_{0,n} \leq 2 \delta_0, \quad \delta_{1,n} \leq 2 \delta_1, \quad 0 \leq t \leq \min\{ \frac{\delta_0}{C M_0}, \frac{\delta_1}{C M_0} \}. \tag{5.79}
\]

By Lemma 5.9, there is some constant \( C \) such that
\[
\sup_{0 \leq \tau \leq T} \| G^n(\tau) \|_{H^s} \leq C. \tag{5.80}
\]

Combing (5.76), (5.79), (5.80), we have for \( 0 \leq t \leq 1 \),
\[
\| (F^{n+1}, D^n_t F^{n+1}) \|_{H^{s+1/2} \times H^s}^2 \leq 2 \delta_0 e^{2 \delta_1 t} \frac{1}{4} M_0^2 + t e^{2 \delta_1 t} C M_0^2.
\]
So there is $T_2 \leq \min\{T, T_1\}$ depends continuous on $s, \tilde{\lambda}, \alpha_0, M_0, C_0, d_{f,0}^{-1}, d_{P,0}^{-1}$ such that

$$\sup_{0 \leq t \leq T_2} \| (F^{n+1}, D_0^{(n)} F^{n+1}) \|_{H_1^{1,2} \times H^s} \leq \delta_0^2 M_0^2. \quad (5.81)$$

**Estimate** $A^{n+1}_s | Z^{n+1}_\alpha$ :

By Lemma 5.10, we have

$$\| A^{n+1}_s \|_{L^{\infty}} \leq CM_0.$$ 

So we have

$$\left| A^{n+1}_s(\alpha, t) - A^{n+1}_s(\alpha, 0) \right| \leq \int_0^t |\partial_t A^{n+1}_s(\alpha, \tau)| d\tau \leq CM_0 t.$$ 

Similarly, we have

$$| Z^{n+1}_\alpha(\alpha, t) - Z^{n+1}_\alpha(\alpha, 0) | \leq CM_0 t.$$ 

So we obtain

$$A^{n+1}_s | Z^{n+1}_\alpha(\alpha, t) | = A^{n+1}_s(\alpha, 0) | Z^{n+1}_\alpha(\alpha, 0) | + \left( A^{n+1}_s | Z^{n+1}_\alpha(\alpha, t) | - A^{n+1}_s(\alpha, 0) | Z^{n+1}_\alpha(\alpha, 0) | \right)$$

$$= A_0(\alpha) |\partial_\alpha Z(\alpha, 0) | + \left( A^{n+1}_s | Z^{n+1}_\alpha(\alpha, t) | - A^{n+1}_s(\alpha, 0) | Z^{n+1}_\alpha(\alpha, 0) | \right)$$

$$\geq 2\alpha_0 - CM_0 t.$$ 

Choose $T_3 = \min\{T_2, \frac{a_0}{CM_0}\}$, we get if $t \leq T_2$,

$$A^{n+1}_s(\alpha, t) = A^{n+1}_s(\alpha, t) - A^{n+1}_s(\alpha, 0) + A^{n+1}_s(\alpha, 0) \geq 2\alpha_0 - \alpha_0 = \alpha_0.$$ 

**Estimate** $| Z^{n+1}_\alpha(\alpha, t) - Z^{n+1}(\beta, t) |$ : By (5.88), we have

$$\partial_t (Z^{n+1}_\alpha(\alpha, t) - Z^{n+1}_\beta(\beta, t)) = \left[ (\tilde{F}^{n}_\alpha(\alpha, t) - \tilde{F}^{n}_\beta(\beta, t)) + (\tilde{Q}^{n}_\alpha(\alpha, t) - \tilde{Q}^{n}_\beta(\beta, t)) \right]$$

$$- (B^n_\alpha(\alpha, t) Z_\alpha(\alpha, t) - B^n_\beta(\beta, t) Z_\beta(\beta, t)).$$

It’s easy to see that

$$\left| \left[ (\tilde{F}^{n}_\alpha(\alpha, t) - \tilde{F}_\beta(\beta, t)) + (\tilde{Q}^{n}_\alpha(\alpha, t) - \tilde{Q}_\beta(\beta, t)) \right] - (B^n_\alpha(\alpha, t) Z_\alpha(\alpha, t)$$

$$- B^n_\beta(\beta, t) Z_\beta(\beta, t)) \right| \leq CM_0 |\alpha - \beta|.$$ 

So we obtain

$$| Z(\alpha) - Z(\beta) | - CM_0 |\alpha - \beta| \leq | Z^{n+1}_\alpha(\alpha, t) - Z^{n+1}_\beta(\beta, t) | \leq | Z(\alpha)$$

$$- Z(\beta) | + CM_0 |\alpha - \beta|.$$ 

Choose $T_4 = \min\{T_3, \frac{C_0}{2CM_0}\}$, then we have

$$\frac{1}{2} C_0 |\alpha - \beta| \leq | Z^{n+1}_\alpha(\alpha, t) - Z^{n+1}_\beta(\beta, t) | \quad \forall \alpha, \beta \in R, \ t \in [0, T_4]. \quad (5.84)$$
Estimate $d_I^{n+1}(t)$ : To estimate $d_I^{n+1}(t)$, recall that $d_I^{n+1}(t) = \min_{1 \leq j, k \leq N} |Z^{n+1}(\alpha, t) - z_j^{n+1}(t)|$. Using $Z^{n+1}(\alpha, 0) = Z_0(\alpha, 0), z_j^{n+1}(0) = z_j(0)$, we have

$$|Z^{n+1}(\alpha, t) - z_j^{n+1}(t)| \geq |Z_0(\alpha) - z_j(0)| - |Z^{n+1}(\alpha, t) - Z_0(\alpha)| - |z_j^{n+1}(t) - z_j(0)|$$

$$\geq |Z_0(\alpha) - z_j(0)| - |Z^{n+1}(\alpha, t) - Z^{n+1}(\alpha, 0)| - |z_j^{n+1}(t) - z_j^{n+1}(0)|$$

$$\geq d_{I,0} - \int_0^t |\partial_t Z^{n+1}(\alpha, \tau)| d\tau - \int_0^t |z_j^{n+1}(\tau)| d\tau$$

$$\geq d_{I,0} - CM_0 t.$$

So we obtain $d_I^{n+1}(t) \geq d_{I,0} - CM_0 t$. Choose $T_5 = \min\{T_4, \frac{d_{I,0}}{CM_0}\}$, we obtain

$$d_I^{n+1}(t) \geq \frac{1}{2} d_{I,0}, \quad \forall \ t \in [0, T_5]. \quad (5.85)$$

Estimate $d_p^{n+1}(t)$ : To estimate $d_p^{n+1}(t)$, recall that $d_p^{n+1}(t) = \min_{1 \leq j, k \leq N} |z_j^{n+1}(t) - z_k^{n+1}(t)|$. For $j \neq k$, using $z_j^{n+1}(0) = z_j(0), z_k^{n+1}(0) = z_k(0)$,

$$|z_j^{n+1}(t) - z_k^{n+1}(t)| \geq |z_j^{n+1}(0) - z_k^{n+1}(0)| - |z_j^{n+1}(t) - z_j^{n+1}(0)| - |z_k^{n+1}(t) - z_k^{n+1}(0)|$$

$$\geq d_{P,0} - CM_0 t.$$

So we have $d_p^{n+1}(t) \geq d_{P,0} - CM_0 t$. Choosing $T_6 = \min\{T_5, \frac{d_{P,0}}{2CM_0}\}$, then we have

$$d_p^{n+1}(t) \geq \frac{1}{2} d_{P,0}, \quad \forall \ t \in [0, T_6]. \quad (5.86)$$

Therefore, if we choose $T \leq T_6$, then we have

$$(\partial_\alpha Z^{n+1} - 1, F^{n+1}, D_t^{n+1} F^{n+1}, \{z_j^{n+1}\}) \in S_T.$$

Error estimates and convergence of the approximate solutions. We show that $\{(Z^k - \alpha, F^k, D_t^{(k-1)} F^k, \{z_j^k\})\}$ is a Cauchy sequence in some Banach space. Let

$$\begin{align*}
\hat{F} &= F^{k+1} - F^k, \\
\hat{G} &= G_k - G^{k-1} - ((D_t^{(k)})^2 - (D_t^{(k-1)})^2) F^k - (A^k - A^{k-1}) F^k, \\
\hat{Z} &= Z^{k+1} - Z^k, \\
\hat{z}_j &= z_j^{k+1} - z_j^k.
\end{align*}$$

(5.87)

Then $\hat{F}$ satisfies

$$\begin{align*}
(\partial_t D_t^{(n)}) \hat{F} &+ A^n \hat{F} = \hat{G}, \\
\hat{F}(\cdot, 0) &= 0, \quad D_t^{(n)} \hat{F}(\cdot, 0) = 0.
\end{align*}$$

(5.88)

And $\hat{z}_j$ satisfies

$$\begin{align*}
\frac{d}{dt} \hat{z}_j &= U^k_{\text{error}} + S T^k_{\text{error}}, \\
\hat{z}_j(0) &= 0.
\end{align*}$$

(5.89)
Here,
\[ \mathcal{U}^k_{error} = \mathcal{U}^k(z_j(t), t) - \mathcal{U}^{k-1}(z_j(t), t) , \]
and
\[ S\mathcal{P}^k_{error} = \sum_{1 \leq l \leq N} \frac{\lambda_i l}{2\pi} \left( \frac{1}{z_l(t) - z_j(t)} - \frac{1}{z_{k-1}^{l}(t) - z_{j}^{l-1}(t)} \right) . \]

Denote
\[ \mathcal{E}^k(t) := \| (\hat{\mathcal{F}}, \mathcal{D}_t^{(k)} \hat{F}) \|_{H^{s-1/2} \times H^{s-1}}^2 + \| \hat{Z} \|_{H^{s-1/2}}^2 + |\hat{z}_j|^2 . \]  

(5.90)

It’s elementary to check that
\[ |\hat{z}_j(t)|^2 + \left| \frac{d}{dt} \hat{z}_j(t) \right|^2 \leq CM_0^2 t \mathcal{E}^{k-1}(t) . \]  

(5.91)

Using Lemma 5.9, we have
\[ \| \hat{G} \|_{H^{s-1/2}} \leq CM_0 \mathcal{E}^{k-1} . \]  

(5.92)

Using energy estimates, it’s easy to obtain
\[ \| (\hat{\mathcal{F}}, \mathcal{D}_t^{(k)} \hat{F}) \|_{H^{s-1/2} \times H^{s-1}} \leq CM_0^2 t \mathcal{E}^{k-1} . \]  

(5.93)

To estimate \( \hat{Z} \), using (5.70), we have
\[
\partial_t \hat{Z} = \left( \tilde{Q}^k + \tilde{F}^k - B^k \partial_\alpha Z^{k+1} \right) - \left( \tilde{Q}^{k-1} + \tilde{F}^{k-1} - B^{k-1} \partial_\alpha Z^k \right) \\
= (\tilde{Q}^k - \tilde{Q}^{k-1}) + (\tilde{F}^k - \tilde{F}^{k-1}) - (B^k - B^{k-1}) \partial_\alpha Z^k - B^k \partial_\alpha \hat{Z} .
\]

Using energy estimates similar to that for (5.68), we obtain
\[ \| \hat{Z} \|_{H^{s-1/2}} \leq CM_0^2 t \mathcal{E}^{k-1} . \]  

(5.94)

So we obtain
\[ \mathcal{E}^k(t) \leq CM_0^2 t \mathcal{E}^{k-1} . \]  

(5.95)

Choosing \( T \) smaller if necessary, we obtain
\[ \sup_{0 \leq t \leq T} \mathcal{E}^k(t) \leq c \sup_{0 \leq t \leq T} \mathcal{E}^{k-1}(t) , \]  

(5.96)

for some constant \( 0 < c < 1 \). So \((\xi^n, F^n, \mathcal{D}_t^{n-1} F^n, \{z_j^n\})\) is a Cauchy sequence in \( C([0, T_0]; H^{s-1/2} \times H^{s-1/2} \times H^{s-1} \times \mathbb{C}^N) \). So
\[ (\xi^n, F^n, \mathcal{D}_t^{n-1} F^n, \{z_j^n\}) \to (\xi, F, \mathcal{D}_t F, \{z_j\}) \]  

(5.97)

in \( C([0, T]; H^{s-1/2} \times H^{s-1/2} \times H^{s-1} \times \mathbb{C}^N) \). Since \((\xi^n, F^n, \mathcal{D}_t^{n-1} F^n, \{z_j^n\})\) is bounded in \( C([0, T]; H^{s+1/2} \times H^{s+1/2} \times H^s \times \mathbb{C}^N) \), we have
\[ (\xi, F, \mathcal{D}_t F, \{z_j\}) \in C([0, T]; H^{s+1/2} \times H^{s+1/2} \times H^s \times \mathbb{C}^N) . \]  

(5.98)
Define $Z(\alpha, t) = \xi(\alpha, t) + \alpha$. So $Z_\alpha - 1 \in C([0, T]; H^{s-1/2})$. So we have

$$
\begin{align*}
\begin{cases}
\mathcal{D}_t^2 F + iAF_\alpha = G, \\
F(\cdot, 0) = F_0, \quad \mathcal{D}_t F(\cdot, 0) = F_1.
\end{cases}
\end{align*}
$$

(5.98)

It’s easy to check that

$$
Q^n \to - \sum_{j=1}^N \frac{\lambda_j i}{2\pi} \frac{1}{Z(\alpha, t) - z_j(t)} \text{ in } C([0, T]; H^{s+1/2}).
$$

(5.99)

Let $\Sigma(t)$ be the curve parametrized by $Z(\alpha, t)$ and $\Omega(t)$ the region bounded above by $\Sigma(t)$. Then we have

$$
\mathcal{U}^n(z, t) \to \frac{1}{2\pi i} \int \frac{Z_\beta(\beta, t)}{z - Z(\beta, t)} F(\beta, t) d\beta \quad \text{uniformly in } \Omega(t).
$$

(5.100)

So we can verify that

$$
\dot{z}_j(t) = \mathcal{U}(z_j(t), t) + \sum_{1 \leq k \leq N, k \neq j} \frac{\lambda_k i}{2\pi} \frac{1}{z_k(t) - z_j(t)}.
$$

(5.101)

Then it’s easy to check that $d_1(t) \geq \frac{1}{2}d_{1,0}, d_P(t) \geq \frac{1}{2}d_{P,0}$, and $\{z_j(t)\} \in C^2([0, T]; \Omega(t))$.

To show that $(Z, F, \mathcal{D}_t F, \{z_j\})$ gives rise to a solution to the water wave system (2.33), we need to show that $(I - \mathbb{H}) F = 0$. We have $(I - \mathbb{H}) F_0 = 0$. Note that

$$
(I - \mathbb{H}) \left\{ \mathcal{D}_t^2 F + iAF_\alpha = G \right\} \iff (\mathcal{D}_t^2 + iA\partial_\alpha)(I - \mathbb{H}) F = 0.
$$

(5.102)

$(I - \mathbb{H}) F_0 = 0$ implies $(I - \mathbb{H}) F \equiv 0$. So $(Z, F, \mathcal{D}_t F, \{z_j\})$ is a solution to the water wave system (2.33). The uniqueness is obtained by standard energy estimates, which we omit.

\[\square\]

5.6. The proof of the theorem. To prove Theorem 2, we need to change of variables back to lagrangian coordinates. Solve

$$
\begin{align*}
\begin{cases}
\frac{dh}{dt} = \mathcal{B}(h, t), \\
h(\alpha, 0) = \alpha.
\end{cases}
\end{align*}
$$

(5.103)

By standard ODE existence and unique theorem, (5.103) admits a unique classical solution on $[0, T]$, and since $\mathcal{B} \in C([0, T]; H^{s+1/2})$, we have $h \in C([0, T]; H^{s+1/2})$. Moreover, there exists a $0 < T_0 \leq T$, depending on $\|B(t)\|_{H^2}$, such that

$$
h(\alpha, t) - h(\alpha', t) \geq \frac{1}{2}(\alpha - \alpha').
$$
for $0 \leq t \leq T_0$, $\alpha < \alpha'$, and $h(\alpha, t) - \alpha \in C([0, T_0]; H^{s+1/2})$. Let $z(\alpha, t) = Z(h(\alpha, t), t)$, $a(\alpha, t) = A(h(\alpha, t), t)\omega_0(\alpha, t)$, $f(\alpha, t) = F(h(\alpha, t), t)$. Then it’s easy to see that

$$
\begin{aligned}
&z_{tt} - i\alpha z_{\alpha} = -i, \\
&\hat{z}_j = \overline{U(z_j(t), t)} + \sum_{1 \leq k \leq N \atop k \neq j} \frac{\lambda_j i}{2\pi} \frac{1}{z_k(t) - z_j(t)}, \\
&(I - \delta_j) f = 0.
\end{aligned}
$$

(5.104)

Then the proof of Theorem 2 is concluded.

5.7. Preservation of symmetries. In this subsection we show that symmetric water waves with symmetric point vortices preserve the symmetry. Such symmetry is well-known if there is no point vortex.

Define the $H^k_e, H^k_\alpha$ by

$$
H^k_e = \left\{ f \in H^k(\mathbb{R}) : \Re\{f\} \text{ is odd, } \Im\{f\} \text{ is even} \right\},
$$

$$
H^k_\alpha = \left\{ f \in H^k(\mathbb{R}) : \Re\{f\} \text{ is even, } \Im\{f\} \text{ is odd} \right\}.
$$

**Theorem 5.** Under the assumptions of Theorem 2, if we assume further that

(a) $z_0 - \alpha \in H^{r+1/2}_e$, $v_0 \in H^{r+1/2}_e$.

(b) $N = 2N_0$, where $N_0 \geq 1$ is an integer. The initial vorticity $\omega_0$ is

$$
\omega_0 = \sum_{j=1}^{N_0} (\lambda_j \delta_{z_j,1}(0) - \lambda_j \delta_{z_j,2}(0)),
$$

where $\lambda_j \in \mathbb{R}$ and $z_{j,1}(0), z_{j,2}(0) \in \Omega(0)$; $j = 1, \cdots, N_0$. Moreover, $z_{j,1}(0)$ and $z_{j,2}(0)$ are symmetric about the y-axis, that is,

$$
\Re\{z_{j,1}(0)\} = -\Re\{z_{j,2}(0)\}, \quad \Im\{z_{j,1}(0)\} = \Im\{z_{j,2}(0)\}.
$$

Then the water wave system (2.30) admits a unique solution $(z(\alpha, t), z_t(\alpha, t), \{(z_{j,1}(t), z_{j,2}(t))\})$ on $[0, T]$ such that

$$(z_\alpha - 1, z_t, z_{tt}) \in C([0, T]; H^{s-1/2}_e \times H^{s+1/2}_e \times H^s_e).$$

**Theorem 5** can be proved by repeating the proof of Theorem 4 for initial data $(Z_0 - \alpha, F_0, F_1) \in H^{s+1/2}_e \times H^{s+1/2}_e \times H^s_e$, and each pair $z_{j,1}(0), z_{j,2}(0)$ is symmetric about the vertical axis. We give only a sketch of proof here.

**Proof.** It’s easy to verify that if $Z(\alpha, t) - \alpha \in H^{s+1/2}_e$ and $(z_{j,1}, z_{j,2})$ satisfies (b), then

$$
Q = -\sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \frac{1}{Z(\alpha, t) - z_{j,1}(t)} - \frac{\lambda_j i}{2\pi} \frac{1}{Z(\alpha, t) - z_{j,2}(t)} \in H^{s+1/2}_e.
$$

Repeat the proof of Theorem 4, it’s straightforward to verify that , if

(1) $Z^n - \alpha, \in C([0, T]; H^{r+1/2}_e)$,
For $D_t^{(n-1)} F^n \in C([0, T]; H_{\epsilon}^{s+1/2})$, $(D_t^{(n-1)})^2 F^n \in C([0, T]; H_{\epsilon}^s)$.

\[ \{ z_{n,1}(t) \} = -\Re\{ z_{n,2}(t) \}, \quad \Im\{ z_{n,1}(t) \} = \Im\{ z_{n,2}(t) \}, \]

then

(A) For $G^n = G^n (Z^n, D_t^{(n-1)} F^n, (D_t^{(n-1)})^2 F^n, \{(z_{n,1}, z_{n,2})\})$, we have $G^n \in C([0, T]; H_{\epsilon}^s)$. 

(B) For

\[ B^n = B^n (Z^n, D_t^{(n-1)} F^n, (D_t^{(n-1)})^2 F^n, \{(z_{n,1}, z_{n,2})\}) \]

and

\[ A^n = A^n (Z^n, D_t^{(n-1)} F^n, (D_t^{(n-1)})^2 F^n, \{(z_{n,1}, z_{n,2})\}), \]

we have $B^n \in C([0, T]; H_{\epsilon}^{s+1/2})$ and $A^n \in C([0, T]; H_{\epsilon}^s)$.

Then use exactly the proof as in the previous subsections, for each $n$, we can construct approximate solutions $(Z^n, D_t^{(n-1)} F^n, (D_t^{(n-1)})^2 F^n, \{(z_{n,1}, z_{n,2})\})$, such that $Z^n - \alpha \in H_{\epsilon}^{s+1/2}$, $D_t^{(n-1)} F^n \in H_{\epsilon}^{s+1/2}$, $(D_t^{(n-1)})^2 F^n \in H_{\epsilon}^s$, and $\{ z_{n,1}(t) \} = -\Re\{ z_{n,2}(t) \}$, $\Im\{ z_{n,1}(t) \} = \Im\{ z_{n,2}(t) \}$. And we can show that there exists $(Z, F, D_t F, \{(z_{1,1}, z_{1,2})\})$ such that

\[ Z^n_0 - 1 \to Z_\alpha - 1 \quad \text{in} \quad C([0, T]; H_{\epsilon}^{s-1/2}), \]
\[ F^n \to F \quad \text{in} \quad C([0, T]; H_{\epsilon}^{s-1/2}), \]
\[ D_t^{(n-1)} F^n \to D_t F \quad \text{in} \quad C([0, T]; H_{\epsilon}^{s-1}), \]
\[ \{(z_{n,1}, z_{n,2})\} \to \{(z_{1,1}, z_{1,2})\} \quad \text{in} \quad C^2 ([0, T]). \]

Then we must have $Z - \alpha \in H_{\epsilon}^{s+1/2}$, $F \in H_{\epsilon}^{s+1/2}$, and $\{ z_{1,1}(t) \} = -\Re\{ z_{1,2}(t) \}$, $\Im\{ z_{1,1}(t) \} = \Im\{ z_{1,2}(t) \}$. So the water waves preserves the symmetry (a), (b).  

6. Long Time Behavior for Small Data and the Proof of Theorem 3

In this section we prove Theorem 3.

6.1. Derivation of the cubic structure. As was explained in the introduction, the main difficulty of studying long time behavior of the system (2.30) is to find a cubic structure for this system. In [53], S. Wu used $\theta := (I - \mathcal{S})(z - \bar{z})$ and showed that $\partial_t^2 - ia \partial_{\alpha} \theta$ is cubic for the irrotational case. We use the same $\theta$ here. Using lemma 2.10,

\[
(\partial_t^2 - ia \partial_{\alpha}) \theta = (I - \mathcal{S}) (\partial_t^2 - ia \partial_{\alpha})(z - \bar{z}) - (\partial_t^2 - ia \partial_{\alpha}, \mathcal{S})(z - \bar{z})
\]
\[
= -2 (I - \mathcal{S}) \partial_t \bar{z}_t - 2 [z_t, \mathcal{S}] \frac{\partial_{\alpha} (z_t - \bar{z}_t)}{z_{\alpha}^2} + \frac{1}{\pi i} \int \left( \frac{z_t (\alpha, t) - z_t (\beta, t)}{z (\alpha, t) - z (\beta, t)} \right)^2 (z - \bar{z}) \beta d\beta
\]
\[
= -2 \partial_t (I - \mathcal{S}) \bar{z}_t - 2 [z_t, \mathcal{S}] \frac{\partial_{\alpha} \bar{z}_t}{z_{\alpha}} + \frac{1}{\pi i} \int \left( \frac{z_t (\alpha, t) - z_t (\beta, t)}{z (\alpha, t) - z (\beta, t)} \right)^2 (z - \bar{z}) \beta d\beta.
\]
Long Time Behavior of 2D Water Waves

Decompose $\tilde{z}_t = f + p$ as before, with $p = -\sum_{j=1}^{2} \frac{\lambda_j^i}{2\pi z(\alpha, t) - z_j(t)}$. Since $(I - \mathcal{S})p = 2p$, we have

$$-2\partial_t (I - \mathcal{S})\tilde{z}_t = -2\partial_t (I - \mathcal{S})p = -4p_t,$$

and

$$-2[z_t, \mathcal{S}]\frac{\partial_\alpha z_t}{z_\alpha} = -2[\tilde{f}, 2\mathcal{S}]\frac{\partial_\alpha f}{z_\alpha} - 2[\tilde{p}, \mathcal{S}]\frac{\partial_\alpha p}{z_\alpha} - 2[\tilde{f}, \mathcal{S}]\frac{\partial_\alpha \tilde{p}}{z_\alpha} - 2[\tilde{p}, \mathcal{S}]\frac{\partial_\alpha \tilde{p}}{z_\alpha}.$$

Since $f$ is holomorphic, we have $[f, \mathcal{S}]\frac{\partial_\alpha f}{z_\alpha} = 0$, and hence $[\tilde{f}, \mathcal{S}]\frac{\partial_\alpha \tilde{f}}{z_\alpha} = 0$, so

$$-2[z_t, \mathcal{S}]\frac{\partial_\alpha z_t}{z_\alpha} = -2[\tilde{f}, \mathcal{S}]\frac{\partial_\alpha f}{z_\alpha} + \mathcal{S} \frac{\partial_\alpha \tilde{f}}{z_\alpha}.$$

which is cubic. So we obtain

$$(\partial_t^2 - i\alpha \partial_\alpha)\theta = -2[\tilde{f}, \mathcal{S}]\frac{\partial_\alpha f}{z_\alpha} + \frac{1}{\pi i} \int \left( \frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)} \right)^2 (z - \bar{z})_\beta d\beta$$

$$-2[\tilde{p}, \mathcal{S}]\frac{\partial_\alpha \tilde{p}}{z_\alpha} - 2[\tilde{f}, \mathcal{S}]\frac{\partial_\alpha \tilde{p}}{z_\alpha} + 2[\tilde{p}, \mathcal{S}]\frac{\partial_\alpha \tilde{p}}{z_\alpha} - 4p_t.$$

Denote

$$g_c := -2[\tilde{f}, \mathcal{S}]\frac{\partial_\alpha f}{z_\alpha} + \frac{1}{\pi i} \int \left( \frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)} \right)^2 (z - \bar{z})_\beta d\beta. \quad (6.3)$$

$$g_d := -2[\tilde{p}, \mathcal{S}]\frac{\partial_\alpha \tilde{p}}{z_\alpha} - 2[\tilde{f}, \mathcal{S}]\frac{\partial_\alpha \tilde{p}}{z_\alpha} - 2[\tilde{p}, \mathcal{S}]\frac{\partial_\alpha \tilde{p}}{z_\alpha} - 4p_t. \quad (6.4)$$

To control $z_{tt}$, we consider the quantity

$$\sigma := (I - \mathcal{S})\partial_t \theta = (I - \mathcal{S})\partial_t (I - \mathcal{S})(z - \bar{z}).$$

We have

$$(\partial_t^2 - i\alpha \partial_\alpha)\sigma = (I - \mathcal{S})(\partial_t^2 - i\alpha \partial_\alpha)\partial_t (I - \mathcal{S})(z - \bar{z}) = [\partial_t^2 - i\alpha \partial_\alpha, \mathcal{S}]\partial_t (I - \mathcal{S})(z - \bar{z})$$

$$+ (I - \mathcal{S})(\partial_t g + i\alpha \partial_t ((I - \mathcal{S})(z - \bar{z})))_\alpha \partial_t (I - \mathcal{S})(z - \bar{z}). \quad (6.5)$$

Here, $g = g_c + g_d$. Use lemma 2.10,

$$(\partial_t^2 - i\alpha \partial_\alpha)\sigma = (I - \mathcal{S})(\partial_t^2 - i\alpha \partial_\alpha)\partial_t (I - \mathcal{S})(z - \bar{z}) - [\partial_t^2 - i\alpha \partial_\alpha, \mathcal{S}]\partial_t (I - \mathcal{S})(z - \bar{z})$$

$$= (I - \mathcal{S})(\partial_t g + i\alpha \partial_t ((I - \mathcal{S})(z - \bar{z})))_\alpha \partial_t (I - \mathcal{S})(z - \bar{z})$$

$$+ \frac{1}{\pi i} \int \left( \frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)} \right)^2 ((I - \mathcal{S})(z - \bar{z}))_{\beta} d\beta$$

$$:= \tilde{g}_1 + \tilde{g}_2 + \tilde{g}_3.$$
Remark 6.1. We have
\[ \tilde{g}_1 = (I - \hat{\zeta}) \partial_t g_c + (I - \hat{\zeta}) \partial_t g_d + (I - \hat{\zeta}) i a_t (I - \hat{\zeta})(z - \bar{z})_\alpha \]
\[ := \tilde{g}_{11} + \tilde{g}_{12} + \tilde{g}_{13}. \tag{6.7} \]
Note that \( \tilde{g}_{11} \) and \( \tilde{g}_{13} \) are obvious cubic or enjoy nice time decay. As one can see later, \( \tilde{g}_2 \) is cubic as well. Since \( a_t \bar{z}_\alpha \) consists of quadratic nonlinearities and terms with sufficiently fast time decay, as long as the point vortices move away from the interface at a speed which has a positive lower bound, so \( (\partial^2_t - ia_t \partial_\alpha)(I - \hat{\zeta}) \partial_t (I - \hat{\zeta})(z - \bar{z}) \) consists of cubic or higher order nonlinearities, or nonlinearities with rapid time decay, as long as the point vortices move away from the interface rapidly.

6.2. Change of coordinates. Note that \((a - 1)\theta_\alpha\) involves quadratic nonlinearities, which does not directly lead to cubic lifespan. To resolve the problem, we use the diffeomorphism \( \kappa : \mathbb{R} \to \mathbb{R} \) such that \( \bar{\zeta} - \alpha \) is holomorphic, where \( \bar{\zeta} = z \circ \kappa^{-1} \). This \( \kappa \) was used in [53] [44] for the irrotational case. Here we need to derive the formulae for \( b \) and \( A \) for the case with point vortices. Let \( \Psi_1 \) be the holomorphic function on \( \Omega(t) \) such that \( \bar{\zeta} - \alpha = \Psi_1 \circ \zeta \).

We denote
\[ D_t \zeta = z_t \circ \kappa^{-1}, \quad A := (a \kappa_\alpha) \circ \kappa^{-1}, \quad b = \kappa_t \circ \kappa^{-1}. \]
Then
\[ \kappa_t = b \circ \kappa. \tag{6.8} \]
Suppose we know \( b \), then we can recover \( \kappa \) by solving the ODE (6.8).

Denote
\[ \tilde{f} = f \circ \kappa^{-1}, \quad q = p \circ \kappa^{-1}. \tag{6.9} \]
Since \( f \) is the boundary value of the holomorphic function \( \mathcal{U} \) on \( \Omega(t) \), we have
\[ \tilde{f}(\alpha, t) = \mathcal{U}(\zeta(\alpha, t), t). \tag{6.10} \]
In new variables, the water wave system (2.30) can be written as
\[
\begin{cases}
D^2_t \zeta - i A \zeta_\alpha = -i d \zeta (t) = (v - \frac{2\pi (\bar{z} - z_j)}{2\pi (\zeta(\alpha, t) - z_j)}) |_{z = z_j} \\
(I - \mathcal{H})(D_t \zeta + \sum_{j=1}^N \frac{\lambda_{ji}}{2\pi (\zeta(\alpha, t) - z_j(t))}) = 0 \\
(I - \mathcal{H})(\bar{\zeta} - \alpha) = 0.
\end{cases} \tag{6.11}
\]
Here, \( \mathcal{H} \) is the Hilbert transform associates with \( \zeta \), i.e.
\[ \mathcal{H} f (\alpha) := \frac{1}{\pi i} \text{p.v.} \int_{-\infty}^{\infty} \frac{\zeta_\beta (\beta, t)}{\zeta(\alpha, t) - \zeta(\beta, t)} f(\beta) d\beta. \tag{6.12} \]
We choose the initial change of variables to be the identity, that is, \( \kappa(\alpha, 0) = \alpha \). By the assumption of Theorem 3,
\[ \| \Lambda^{1/2}(\zeta(\alpha, 0) - \alpha) \|_{H^s} + \| \tilde{f}(\cdot, 0) \|_{H^{s+1/2}} + \| D_t \tilde{f}(\cdot, 0) \|_{H^s} \leq \epsilon. \tag{6.13} \]
We need to derive formula for \( b, D_t b, \) and \( A \).
6.2.1. Formula for the quantities \( b \) and \( D_t b \)  

Note that

\[
D_t \tilde{\zeta} = \mathcal{U} \circ \zeta - \frac{i}{2\pi} \sum_{j=1}^{2} \frac{\lambda_j}{\zeta(\alpha, t) - z_j(t)}, \quad \lambda_1 = -\lambda_2 = \lambda. \tag{6.14}
\]

Also, \( D_t \tilde{\zeta} \) can be written as

\[
D_t \tilde{\zeta} = D_t (\tilde{\zeta} - \alpha) + b = D_t \zeta \Psi_\zeta \circ \zeta + \Psi_t \circ \zeta + b. \tag{6.15}
\]

By (6.14) and (6.15), we have

\[
\mathcal{U} \circ \zeta - \frac{i}{2\pi} \sum_{j=1}^{2} \frac{\lambda_j}{\zeta(\alpha, t) - z_j(t)} = D_t \zeta \Psi_\zeta \circ \zeta + \Psi_t \circ \zeta + b. \tag{6.16}
\]

Apply \( I - \mathcal{H} \) on both sides of the above equation, use the fact that

\[
(I - \mathcal{H}) \Psi_t \circ \zeta = 0, \quad (I - \mathcal{H}) \mathcal{U} \circ \zeta = 0, \quad \Psi_\zeta \circ \zeta = \frac{\tilde{\zeta}_\alpha - 1}{\zeta_\alpha},
\]

we obtain

\[
(I - \mathcal{H}) b = -(I - \mathcal{H}) D_t \tilde{\zeta} \frac{\tilde{\zeta}_\alpha - 1}{\zeta_\alpha} - (I - \mathcal{H}) \frac{i}{2\pi} \sum_{j=1}^{2} \frac{\lambda_j}{\zeta(\alpha, t) - z_j(t)}
\]

\[
= -[D_t \zeta, \mathcal{H}] \frac{\tilde{\zeta}_\alpha - 1}{\zeta_\alpha} - 2 \frac{i}{2\pi} \sum_{j=1}^{2} \frac{\lambda_j}{\zeta(\alpha, t) - z_j(t)} \tag{6.17}
\]

\[
= -[D_t \zeta, \mathcal{H}] \frac{\tilde{\zeta}_\alpha - 1}{\zeta_\alpha} - \frac{i}{\pi} \sum_{j=1}^{2} \frac{\lambda_j}{\zeta(\alpha, t) - z_j(t)},
\]

where we’ve used the fact that \( \frac{1}{\zeta(\alpha, t) - z_j(t)} \) is boundary value of a holomorphic function in \( \Omega(t)^c \), so

\[
(I - \mathcal{H}) \frac{1}{\zeta(\alpha, t) - z_j(t)} = \frac{2}{\zeta(\alpha, t) - z_j(t)}. \tag{6.18}
\]

So \( b \) is quadratic plus terms with sufficient rapid time decay, as long as \( z_j(t) \) moves away from the interface rapidly.

We need a formula for \( D_t b \) as well. Use \( (I - \mathcal{H}) b = -[D_t \zeta, \mathcal{H}] \frac{\tilde{\zeta}_\alpha - 1}{\zeta_\alpha} - \frac{i}{\pi} \sum_{j=1}^{2} \frac{\lambda_j}{\zeta(\alpha, t) - z_j(t)} \), change of variables, we get

\[
(I - \mathcal{H}) b = -[\zeta_t, \mathcal{H}] \frac{\tilde{\zeta}_\alpha - 1}{\zeta_\alpha} - \frac{i}{\pi} \sum_{j=1}^{2} \frac{\lambda_j}{\zeta(\alpha, t) - z_j(t)}. \tag{6.19}
\]
So we have

\[
(I - \bar{\gamma}) \partial_t b \circ \kappa = \left[ z_t, \bar{\gamma} \right] \frac{\partial_\alpha b \circ \kappa}{z_\alpha} - \left[ z_{tt}, \bar{\gamma} \right] \frac{\bar{z}_t}{z_\alpha} - \left[ z_t, \bar{\gamma} \right] \frac{\bar{z}_{t\alpha}}{z_\alpha} + \frac{1}{\pi i} \int \left( \frac{z_t(\alpha, t) - z_t(\beta, t)}{\bar{z}(\alpha, t) - \bar{z}(\beta, t)} \right)^2 (\bar{z}_\beta(\beta, t) - 1) d\beta
\]

\[
+ \frac{i}{\pi} \sum_{j=1}^2 \frac{\lambda_j (z_t - \bar{z}_j(t))}{(\bar{z}(\alpha, t) - z_j(t))^2}.
\]

(6.20)

Changing coordinates by precomposing with \(\kappa^{-1}\), we obtain

\[
(I - \bar{\gamma}) D_t b = \left[ D_t \xi, \bar{\gamma} \right] \frac{\partial_\alpha b}{\xi_\alpha} - \left[ D_t^2 \xi, \bar{\gamma} \right] \frac{\bar{\xi}_\alpha}{\xi_\alpha} - \left[ D_t \xi, \bar{\gamma} \right] \frac{\partial_\alpha D_t \bar{\xi}}{\xi_\alpha}
\]

\[
+ \frac{1}{\pi i} \int \left( \frac{D_t \xi(\alpha, t) - D_t \xi(\beta, t)}{\xi(\alpha, t) - \xi(\beta, t)} \right)^2 (\bar{\xi}_\beta(\beta, t) - 1) d\beta
\]

\[
+ \frac{i}{\pi} \sum_{j=1}^2 \frac{\lambda_j (D_t \xi - \bar{z}_j(t))}{(\xi(\alpha, t) - z_j(t))^2}.
\]

(6.21)

So \(D_t b\) is quadratic plus terms with sufficient rapid time decay, as long as \(z_j(t)\) moves away from the interface rapidly.

6.3. The quantity \(A\). Since \(\partial_\alpha \tilde{\xi} = \partial_\alpha U(\xi(\alpha, t), t) = U(t) \circ \xi \circ \xi_\alpha\), we have

\[
U(t) \circ \xi = \frac{\partial_\alpha \tilde{\xi}}{\xi_\alpha}.
\]

(6.22)

Use \(D_t^2 \tilde{\xi} + i A \xi_\alpha = i\). We have

\[
D_t^2 \tilde{\xi} = D_t (D_t \tilde{\xi}) = D_t U \circ \xi - \frac{i}{2 \pi} D_t \sum_{j=1}^2 \frac{\lambda_j}{\xi(\alpha, t) - z_j(t)}
\]

\[
= D_t U \circ \xi + \frac{i}{2 \pi} \sum_{j=1}^2 \frac{\lambda_j (D_t \xi(\alpha, t) - \bar{z}_j(t))}{(\xi(\alpha, t) - z_j(t))^2}
\]

\[
= D_t \xi \frac{\partial_\alpha \tilde{\xi}}{\xi_\alpha} + \frac{i}{2 \pi} \sum_{j=1}^2 \frac{\lambda_j (D_t \xi(\alpha, t) - \bar{z}_j(t))}{(\xi(\alpha, t) - z_j(t))^2}.
\]

(6.23)

Also,

\[
i A \xi_\alpha = i A + i A \partial_\alpha (\tilde{\xi} - \alpha) = i A + i A \xi_\alpha \Psi(t) \circ \xi = i A + (D_t^2 \tilde{\xi} + i) \Psi(t) \circ \xi.
\]

(6.24)

So we have

\[
i A = i - D_t \xi \frac{\partial_\alpha \tilde{\xi}}{\xi_\alpha} - \frac{i}{2 \pi} \sum_{j=1}^2 \frac{\lambda_j (D_t \xi(\alpha, t) - \bar{z}_j(t))}{(\xi(\alpha, t) - z_j(t))^2} - (D_t^2 \tilde{\xi} + i) \Psi(t) \circ \xi.
\]

(6.25)
Apply \( I - \mathcal{H} \) on both sides of (6.25), use the fact that \((I - \mathcal{H}) \frac{\delta}{\zeta} = 0\), \((I - \mathcal{H}) \Psi_{\zeta} \circ \xi = 0\), we obtain

\[
i(I - \mathcal{H})A = i - [D_t \xi, \mathcal{H}] \frac{\partial_{\alpha} \delta}{\zeta} - [D_t^2 \xi, \mathcal{H}] \frac{\zeta_{\alpha} - 1}{\zeta} \\
- (I - \mathcal{H}) \sum_{j=1}^{2} \frac{\lambda_j (D_t \xi(\alpha, t) - \dot{z}_j(t))}{(\xi(\alpha, t) - z_j(t))^2}.
\]

(6.26)

So we obtain

\[
(I - \mathcal{H})A = 1 + i[D_t \xi, \mathcal{H}] \frac{\partial_{\alpha} \delta}{\zeta} + i[D_t^2 \xi, \mathcal{H}] \frac{\zeta_{\alpha} - 1}{\zeta} \\
- (I - \mathcal{H}) \sum_{j=1}^{2} \frac{\lambda_j (D_t \xi(\alpha, t) - \dot{z}_j(t))}{(\xi(\alpha, t) - z_j(t))^2}.
\]

(6.27)

So \( A - 1 \) is quadratic plus terms with rapid time decay, as long as the point vortices move away from the interface with a speed that has a positive lower bound.

6.4. The quantity \( \frac{a_v}{a} \circ \kappa^{-1} \). We need a formula for \( \frac{a_v}{a} \circ \kappa^{-1} \) as well. Use (3.19) and (3.20), (3.21), then change of variables, we obtain

\[
(I - \mathcal{H}) \frac{a_t}{a} \circ \kappa^{-1} A_{\zeta_{\alpha}} \\
= 2i[D_t^2 \xi, \mathcal{H}] \frac{\partial_{\alpha} \delta}{\zeta} + 2i[D_t \xi, \mathcal{H}] \frac{\partial_{\alpha} D_t^2 \zeta}{\zeta} \\
- \frac{1}{\pi} \int \left( \frac{D_t \xi(\alpha, t) - D_t \xi(\beta, t)}{\zeta(\alpha, t) - \zeta(\beta, t)} \right)^2 (D_t^2 \zeta) d\beta \\
- \frac{1}{\pi} \sum_{j=1}^{2} \lambda_j \left( \frac{2D_t^2 \xi + i - D_t \xi^2}{(\zeta(\alpha, t) - z_j(t))^2} - 2 \frac{(D_t \xi - \dot{z}_j(t))^2}{(\zeta(\alpha, t) - z_j(t))^3} \right).
\]

(6.28)

So \( \frac{a_v}{a} \circ \kappa^{-1} \) is quadratic plus terms with rapid time decay, as long as the point vortices move away from the interface with a speed that has a positive lower bound.

6.5. Cubic structure in new variables. Denote

\[
\tilde{\theta} := (I - \mathcal{H})(\xi - \tilde{\xi}), \quad \tilde{\sigma} := (I - \mathcal{H})D_t \tilde{\theta}.
\]

(6.29)

We sum up the calculations above, which show that \((D_t^2 - i A \partial_{\alpha}) \tilde{\theta}\) and \((D_t^2 - i A \partial_{\alpha}) \tilde{\sigma}\) consist of cubic terms and terms with rapid time decay, as long as the point vortices move away from the interface rapidly. Recall that \( q = p \circ \kappa^{-1} \), so \( q \) is given by

\[
q = - \sum_{j=1}^{2} \frac{\lambda_j i}{\pi} \frac{1}{\zeta(\alpha, t) - z_j(t)}.
\]

(6.30)
We have

\[
\begin{cases}
(D_t^2 - i A \partial_\alpha) \theta = G \\
(D_t^2 - i A \partial_\alpha) \sigma = \tilde{G}
\end{cases}
\]  
(6.31)

where \( G = G_c + G_d \), with

\[
G_c := -2[\bar{\delta}, \mathcal{H}] \frac{\partial_\alpha \bar{\delta}}{\zeta_\alpha} + \frac{1}{\pi i} \int \left( \frac{D_t \xi(\alpha, t) - D_t \xi(\beta, t)}{\xi(\alpha, t) - \xi(\beta, t)} \right)^2 (\xi - \bar{\xi}) \beta d\beta.
\]  
(6.32)

\[
G_d := -2[\bar{q}, \mathcal{H}] \frac{\partial_\alpha \bar{q}}{\zeta_\alpha} - 2[\bar{\delta}, \mathcal{H}] \frac{\partial_\alpha \bar{q}}{\zeta_\alpha} - 2[\bar{q}, \mathcal{H}] \frac{\partial_\alpha \bar{q}}{\zeta_\alpha} - 4D_t q,
\]  
(6.33)

and

\[
\tilde{G} = (I - \mathcal{H})(D_t G + i \frac{\alpha_t}{\alpha} \circ \kappa^{-1} A((I - \mathcal{H})(\xi - \bar{\xi})_\alpha))
\]

\[
- 2[D_t \xi, \mathcal{H}] \frac{\partial_\alpha D^2_t (I - \mathcal{H})(\xi - \bar{\xi})}{\xi_\alpha}
\]

\[
+ \frac{1}{\pi i} \int \left( \frac{D_t \xi(\alpha, t) - D_t \xi(\beta, t)}{\xi(\alpha, t) - \xi(\beta, t)} \right)^2 \partial_\beta D_t (I - \mathcal{H})(\xi - \bar{\xi}) d\beta.
\]  
(6.34)

6.5.1. Evolution equation for higher order derivatives Apply \( \partial_\alpha^k \) on both sides of (6.195), we have

\[
\begin{cases}
(D_t^2 - i A \partial_0) \theta_k = G^\theta_k \\
(D_t^2 - i A \partial_0) \sigma_k = G^\sigma_k
\end{cases}
\]  
(6.35)

where for \( 0 \leq k \leq s \),

\[
\theta_k = (I - \mathcal{H}) \partial_\alpha^k \tilde{\theta}, \quad \sigma_k = (I - \mathcal{H}) \partial_\alpha^k \tilde{\sigma}.
\]  
(6.36)

\[
G^\theta_k = (I - \mathcal{H})(\partial_\alpha^k G + [D_t^2 - i A \partial_\alpha, \partial_\alpha^k \tilde{\theta}]) - [D_t^2 - i A \partial_\alpha, \mathcal{H}] \partial_\alpha^k \tilde{\theta},
\]  
(6.37)

and

\[
G^\sigma_k = (I - \mathcal{H})(\partial_\alpha^k \tilde{G} + [D_t^2 - i A \partial_\alpha, \partial_\alpha^k \tilde{\theta}]) - [D_t^2 - i A \partial_\alpha, \mathcal{H}] \partial_\alpha^k \tilde{\sigma}.
\]  
(6.38)

6.6. Energy functional. Define

\[
E_k^\theta := \int \frac{1}{A} |D_t \theta_k|^2 + i \theta_k \bar{\partial_\alpha \theta_k} d\alpha.
\]  
(6.39)

By Wu’s basic energy lemma (lemma 4.1, [53]), we have

\[
\frac{d}{dt} E_k^\theta = \int \frac{2}{A} Re D_t \theta_k \tilde{G}_k - \int \frac{1}{A} \frac{\alpha_t}{\alpha} \circ \kappa^{-1} |D_t \theta_k|^2.
\]  
(6.40)
Define
\[ E^\sigma_k := \int_1^A |D_t \sigma_k|^2 + i \sigma_k \bar{\sigma}_k \sigma_k d\alpha. \] (6.41)

Then we have
\[ \frac{d}{dt} E^\sigma_k = \int_2^A \Re D_t \sigma_k \bar{G}_k - \int_1^A \frac{a_t}{A} \kappa^{-1} |D_t \sigma_k|^2. \] (6.42)

Define
\[ E_s := \sum_{k=0}^s (E^\theta_k + E^\sigma_k). \] (6.43)

6.7. The bootstrap assumption and some preliminary estimates. To obtain a priori energy estimates, we make the following bootstrap assumption: Let \( T_0 \geq 0 \), we assume
\[ \|\zeta_\alpha - 1\|_{H^s} \leq 5\epsilon, \quad \|\vec{\Phi}\|_{H^{s+1/2}} \leq 5\epsilon, \quad \|D_t \vec{\Phi}\|_{H^s} \leq 5\epsilon, \quad \forall \ t \in [0, T_0]. \] (6.44)

Remark 6.2. The assumptions of Theorem 3 imply that the bootstrap assumption holds at \( T_0 = 0 \).

As a consequence of (6.44), we have

Lemma 6.1 (Chord-arc condition). Assume the assumptions of Theorem 3 holds. Assume also the bootstrap assumption (6.44), we have
\[ (1 - 5\epsilon)|\alpha - \beta| \leq |\zeta(\alpha, t) - \zeta(\beta, t)| \leq (1 + 5\epsilon)|\alpha - \beta|, \quad \forall \ t \in [0, T_0]. \] (6.45)

Proof.
\[ |\zeta(\alpha, t) - \zeta(\beta, t)| = |\alpha - \beta + (\zeta(\alpha, t) - \alpha) - (\zeta(\beta, t) - \beta)|. \] (6.46)

Note that
\[ |\zeta(\alpha, t) - \alpha - (\zeta(\beta, t) - \beta)| \leq \|\zeta_\alpha - 1\|_{\infty} |\alpha - \beta| \leq 5\epsilon |\alpha - \beta|. \] (6.47)

So the conclusion follows by Triangle inequality. \( \square \)

Lemma 6.2. Assume the assumptions of Theorem 3 hold. Assume also the bootstrap assumption (6.44), we have for \( \epsilon \) sufficiently small,
\[ \|U(\cdot, t)\|_{L^\infty(\Omega(t))} \leq 5\epsilon, \] (6.48)
\[ \|U_\xi(\cdot, t)\|_{L^\infty(\Omega(t))} \leq 6\epsilon, \] (6.49)
\[ |\Re\{U(x + iy, t)\}| \leq 6\epsilon |x|, \] (6.50)
\[ \|U_t\|_{L^\infty(\Omega(t))} \leq 6\epsilon, \] (6.51)
\[ \|U_{t\xi}\|_{L^\infty(\Omega(t))} \leq 10\epsilon, \] (6.52)
\[ \|U_{t\xi}\|_{L^\infty(\Omega(t))} \leq 10\epsilon. \] (6.53)
Proof. For (6.48), by maximum principle, we have
\[ \| U \|_{L^\infty(\Omega(t))} = \| U \|_{L^\infty(\Sigma(t))} = \| \mathcal{F}(t) \|_\infty \leq 5\varepsilon. \] (6.54)
By bootstrap assumption (6.44), for \( \varepsilon \) sufficiently small, we have
\[ \| U_\xi (\xi(\alpha, t), t) \|_\infty = \left\| \frac{\partial_\alpha \mathcal{F}(\alpha, t)}{\xi_\alpha} \right\|_\infty \leq \| \mathcal{F} \|_{H^k} \cdot \| \mathcal{F} \|_{H^1} \leq \frac{5\varepsilon}{1 - 5\varepsilon} \leq 6\varepsilon. \] (6.55)
By maximum principle, we have
\[ \| U_\xi (\xi(\alpha, t), t) \|_{L^\infty(\Omega(t))} \leq \| U_\xi (\xi(\alpha, t), t) \|_\infty \leq 6\varepsilon. \] (6.56)
Note that
\[ D_t \mathcal{F}(\alpha, t) = D_t U(\xi(\alpha, t), t) = U_t \circ \xi + D_t \xi U_\xi \circ \xi. \] (6.57)
So we have for \( \varepsilon \) sufficiently small (say, \( \varepsilon < 1/36 \)),
\[ \| U (\xi(\alpha, t), t) \|_{L^\infty(\Omega(t))} \leq \| D_t \mathcal{F} \|_{L^\infty} + \| D_t \xi U_\xi \circ \xi \|_{L^\infty} \leq 6\varepsilon. \] (6.58)
Use the fact that \( \mathfrak{M} \{ U \} \) is odd, and the estimate \( \| U_\xi \|_{L^\infty} \leq 6\varepsilon \), we have
\[ \| \mathfrak{M} \{ U(x + iy, t) \} \| = \| \mathfrak{M} \{ U(x + iy, t) \} - \mathfrak{M} \{ U(0 + iy, t) \} \| \leq \| U_\xi \|_{L^\infty} |x| \leq 6\varepsilon |x|. \]
Note that
\[ U_\xi \xi (\xi(\alpha, t), t) = \left( \frac{\partial_\alpha}{\xi_\alpha} \right)^2 U(\xi(\alpha, t), t) = \frac{1}{\xi_\alpha^2} \mathcal{F} \alpha^2 - \frac{\xi_\alpha}{\xi_\alpha} \mathcal{F} \alpha. \] (6.59)
For \( \varepsilon \) sufficiently small, by maximum principle, we have
\[ \| U_\xi \xi (\xi(\alpha, t), t) \|_{L^\infty(\Omega(t))} \leq \frac{1}{\inf_{\alpha \in \mathbb{R}^k} \xi_\alpha^2} \| \mathcal{F} \|_{H^3} + \frac{\| \xi_\alpha \|_{L^\infty}}{\inf_{\alpha \in \mathbb{R}^k} \xi_\alpha} \| \mathcal{F} \|_{H^1} \leq \frac{1}{(1 - 5\varepsilon)^3} 5\varepsilon + \frac{5\varepsilon}{(1 - 5\varepsilon)^2} 5\varepsilon \leq 10\varepsilon. \] (6.60)
Maximum principle implies \( \| U_\xi \xi \|_{L^\infty(\Omega(t))} \leq \| U_\xi \|_{L^\infty(\partial \Omega(t))} \). Since \( U_\xi \xi (\xi(\alpha, t), t) = \frac{\partial_\alpha \xi U(\xi(\alpha, t), t)}{\xi_\alpha} \), by (6.57), we have
\[ \| U_\xi \circ \xi \|_{L^\infty} = \left\| \frac{\partial_\alpha U(\xi(\alpha, t), t)}{\xi_\alpha} \right\|_{L^\infty} \leq \| \partial_\alpha U(\xi(\cdot, t), t) \|_{L^\infty} \left\| \frac{1}{\xi_\alpha} \right\|_{L^\infty} \leq 10\varepsilon. \]
Assume the bootstrap assumption (6.44), we can obtain control of various characteristics of the point vortices.

Conjecture. We use \( K_s \) to denote a constant that depends on \( s \). We’ll use \( K_s \sim \frac{(s + 12)^2}{(s + 7)^2} \). \( K_s \) can be different at different places, up to an absolute multiplicity constant. We also use \( C \) to represent an absolute constant.

We’ll need the following lemma. Similar versions of this lemma have been appeared in [53].
Lemma 6.3. Assume the bootstrap assumption (6.44), let \( f, h \) be real functions. Assume
\[
(I - \mathcal{H}) h \tilde{\xi}_\alpha = g \quad \text{or} \quad (I - \mathcal{H}) h = g.
\]
Then we have for any \( t \in [0, T_0] \),
\[
\|h\|_{H^s} \leq 2\|g\|_{H^s}.
\]
(6.61)

We’ll use the following estimate a lot.

Lemma 6.4. Assume the assumptions of Theorem 3 hold. Assume also the bootstrap assumption (6.44), and assume a priori that \( d_1(t) \geq 1, \frac{1}{2} \leq \frac{x(t)}{x(0)} \leq 2, \forall \ t \in [0, T_0] \). Then we have \( \forall \ t \in [0, T_0] \),
\[
\|q\|_{H^s} \leq K_s^{-1} \varepsilon d_1(t)^{-3/2}.
\]
(6.62)

Proof. We prove (6.62). The proof of (6.63) is similar. Let \( s \) be a positive integer, we have
\[
\|q\|_{H^s}^2 \leq \sum_{n=0}^{s} \int_{-\infty}^{\infty} \left| \sum_{j=1}^{2} \frac{\lambda_{ji}}{2\pi} \frac{1}{\zeta(\alpha, t) - z_j(t)} \right|^2 \, d\alpha.
\]
(6.65)

Denote \( f_j(\alpha, t) := \frac{\lambda_{ji}}{2\pi} \frac{1}{\alpha - z_j(t)} \), \( g := \zeta(\alpha, t) \). Then \( \frac{\lambda_{ji}}{2\pi} \frac{1}{\zeta(\alpha, t) - z_j(t)} = f_j(g(\alpha, t), t) \). By chain rule for composite functions, we have
\[
\partial \alpha^nf_j(g) = \sum_{k=1}^{n} \sum_{(k_1)!(...)(k_n)!} \frac{n!}{(k_1)!(...)(k_n)!} \partial \alpha^kf_j(\cdot, t) \circ g \prod_{l=1}^{n} \left( \frac{\partial \alpha^l g}{l!} \right)^{k_l},
\]
(6.64)

where the second summation is over all non-negative integers \((k_1,...,k_n)\) such that
\[
\begin{align*}
\sum_{l=1}^{n} k_l &= k \\
\sum_{l=1}^{n} l k_l &= n.
\end{align*}
\]
(6.65)

So we have
\[
\partial \alpha^n q = \sum_{k=1}^{n} \sum_{(k_1)!(...)(k_n)!} \frac{n!}{(k_1)!(...)(k_n)!} \left( \sum_{j=1}^{2} \partial \alpha^kf_j(\cdot, t) \circ g \prod_{l=1}^{n} \left( \frac{\partial \alpha^l g}{l!} \right)^{k_l} \right).
\]
(6.66)

Note that
\[
\begin{align*}
\sum_{j=1}^{2} \partial \alpha^k f_j(\cdot, t) \circ g &= \sum_{j=1}^{2} \frac{\lambda_{ji}}{2\pi} \frac{(-1)^k k!}{(\zeta(\alpha, t) - z_j(t))^{k+1}} \\
&= \frac{\lambda i (-1)^k k!}{2\pi} \sum_{m=0}^{k} \frac{z_1 - z_2}{(\zeta(\alpha, t) - z_1(t))^{k+1-m}(\zeta(\alpha, t) - z_2(t))^{m+1}}
\end{align*}
\]
(6.67)
use $z_1 - z_2 = 2x(t)$, similar to the proof of lemma 2.8, we have

$$\left\| \sum_{j=1}^{2} \partial^k_{\alpha} f_j(\cdot, t) \circ g \right\|_{L^2} \leq 100(k + 1)!|\lambda x(t)|d_I(t)^{-3/2}. \tag{6.69}$$

Therefore,

$$\| \partial^n_{\alpha} q \|_{L^2} = \left\| \sum_{k=1}^{n} \sum_{(k_1)! \ldots (k_n)!} \frac{n!}{(k_1)! \ldots (k_n)!} \left( \sum_{j=1}^{2} \partial^k_{\alpha} f_j(\cdot, t) \circ g \right) \prod_{l=1}^{n} \left( \frac{\partial^l g}{l!} \right)^{k_l} \right\|_{L^2} \leq \sum_{k=1}^{n} \sum_{(k_1)! \ldots (k_n)!} \frac{n!}{(k_1)! \ldots (k_n)!} \prod_{l=1}^{n} \| \partial^{k_l}_{\alpha} g \|_{L^2} \left\| \sum_{j=1}^{2} \partial^k_{\alpha} f_j(\cdot, t) \circ g \right\|_{L^2}. \tag{6.70}$$

For $l = 1$, we bound $\partial^l g$ by $1 + 5\epsilon$. For $l \geq 2$, we bound $\| \partial^l_{\alpha} g \|_{L^2}$ by $5\epsilon$. We choose $\epsilon$ small so that $(1 + 5\epsilon)^s \leq 2$. We bound $(k + 1)!$ by $(n + 1)!$. Use the assumption $x(t) \leq 2x(0)$. Use

$$\prod_{l=1}^{n} \| \partial^{k_l}_{\alpha} g \|_{L^2} \leq \prod_{j=1}^{n} (1 + 5\epsilon)^{k_j} \leq (1 + 5\epsilon)^s, \tag{6.71}$$

we obtain

$$\| \partial^n_{\alpha} q \|_{L^2} \leq \sum_{k=1}^{n} \sum_{(k_1)! \ldots (k_n)!} \frac{n!}{(k_1)! \ldots (k_n)!} \prod_{l=1}^{n} (1 + 5\epsilon)^{k_l} \left( \frac{1}{(l!)^{k_l}} \right) \times (100(k + 1)!|\lambda x(t)|d_I(t)^{-3/2})$$

$$\leq 400S(n)|\lambda| x(0) (n + 1)!d_I(t)^{-3/2}, \tag{6.72}$$

where

$$S(n) = \sum_{k=1}^{n} \sum_{(k_1)! \ldots (k_n)!} \frac{n!}{(k_1)! \ldots (k_n)!} \prod_{l=1}^{n} \frac{1}{(l!)^{k_l}} \tag{6.73}$$

is called the bell number. We can bound $S(n)$ by

$$S(n) \leq n!. \tag{6.74}$$

So we have

$$\| \partial^n_{\alpha} q \|_{L^2} \leq 400|\lambda x(0)| n! (n + 1)!d_I(t)^{-3/2}. \tag{6.75}$$

Therefore,

$$\| q \|_{H^s} \leq \left( \sum_{n=0}^{s} \| \partial^n_{\alpha} q \|_{L^2}^2 \right)^{1/2} \leq \left( \sum_{n=0}^{s} (400|\lambda x(0)| n! (n + 1)!d_I(t)^{-3/2})^2 \right)^{1/2}$$

$$\leq 400((s + 2)!)^2 |\lambda x(0)|d_I(t)^{-3/2} \leq K_s^{-1} \epsilon d_I(t)^{-3/2}. \tag{6.79}$$
Corollary 6.1. Assume the assumptions of Theorem 3 hold and assume the bootstrap assumption (6.44), and assume a priori that $d_1(t) \geq 1$, $\frac{1}{2} \leq \frac{x(t)}{x(0)} \leq 2$, $\forall \ t \in [0, T_0]$. Then we have

$$\sup_{t \in [0, T]} \| D_t \xi \|_{H^s} \leq 6\epsilon, \quad \forall \ t \in [0, T_0].$$

Corollary 6.2. Assume the assumptions of Theorem 3 hold and assume the bootstrap assumption (6.44), and assume a priori that $d_1(t) \geq 1$, $\frac{1}{2} \leq \frac{x(t)}{x(0)} \leq 2$, $\forall \ t \in [0, T_0]$. Then we have

$$\| b \|_{H^s} \leq C\epsilon^2 + K^{-1}_s \epsilon d_1(t)^{-3/2}, \quad \forall \ t \in [0, T_0]$$

for some absolute constant $C > 0$.

Proof.

$$(I - \mathcal{H})b = -[D_t \xi, \mathcal{H}] \overline{\xi} - \frac{1}{\xi} \sum_{j=1}^2 \frac{\lambda_j}{\pi} \frac{\lambda_j}{\xi(\alpha, t) - z_j(t)}.$$

By lemma 2.6 and lemma 6.4, we have

$$\| (I - \mathcal{H})b \|_{H^s} \leq \left\| [D_t \xi, \mathcal{H}] \overline{\xi} - \frac{1}{\xi} \sum_{j=1}^2 \frac{\lambda_j}{\pi} \frac{\lambda_j}{\xi(\alpha, t) - z_j(t)} \right\|_{H^s} \leq C\epsilon^2 + K^{-1}_s \epsilon d_1(t)^{-3/2}.$$

So we have

$$\| b \|_{H^s} \leq C\epsilon^2 + K^{-1}_s \epsilon d_1(t)^{-3/2}. \quad (6.82)$$

$\Box$

Remark 6.3. Again, the a priori assumption $d_1(t) \geq 1$, $\frac{1}{2} \leq \frac{x(t)}{x(0)} \leq 2$, $\forall \ t \in [0, T_0]$ will be justified by a bootstrap argument.

We need to estimate $\dot{z}_j$ and $\ddot{z}_j$ in a more precise way rather than using the rough estimates in lemma 5.4. Let’s first derive the estimate for $\dot{z}_j$, then use this estimate to control $x(t)$ over time. We use the control of $x(t)$ to estimate $\ddot{z}_j$.

Lemma 6.5. Assume the assumptions of Theorem 3 and the bootstrap assumption (6.44), and assume a priori that $d_1(t) \geq 1$, $\frac{1}{2} \leq \frac{x(t)}{x(0)} \leq 2$, $\forall \ t \in [0, T_0]$. Then we have

$$\sup_{t \in [0, T_0]} |\dot{z}_1(t) - \dot{z}_2(t)| \leq 10\epsilon. \quad (6.83)$$

$$\sup_{t \in [0, T_0]} |\dot{z}_j(t) - \frac{\lambda i}{4\pi x(t)}| \leq 5\epsilon. \quad (6.84)$$

$$\sup_{t \in [0, T_0]} |\dot{z}_1^2 - \dot{z}_2^2| \leq 6|\lambda| \epsilon + 120\epsilon^2 x(t). \quad (6.85)$$
Proof. Note that
\[ \dot{z}_1 = \frac{\lambda_2 i}{2\pi(z_1 - z_2)} + \tilde{U}(z_1, t) = \frac{\lambda_2 i}{2\pi x(t)} + \tilde{U}(z_1, t). \] (6.86)

Similarly,
\[ \dot{z}_2 = \frac{\lambda_1 i}{4\pi x(t)} + \tilde{U}(z_2, t). \]

By Lemma 6.2, we have
\[ |\dot{z}_1(t) - \dot{z}_2(t)| = |\dot{U}(z_1, t) - \dot{U}(z_2, t)| \leq 2\|\dot{U}\|_{\infty(\partial\Omega_1(t))} \leq 10\epsilon, \]
and
\[ |\dot{z}_j(t) - \frac{\lambda_1 i}{4\pi x(t)}| = |\tilde{U}(z_j(t), t)| \leq 5\epsilon. \] (6.87)

We have
\[ \dot{z}_j^2 = (\frac{\lambda_1 i}{4\pi x(t)})^2 + 2\frac{\lambda_1 i}{4\pi x(t)} \tilde{U}(z_j(t), t) + (\tilde{U}(z_j(t), t))^2. \]

By mean value theorem, bootstrap assumption (6.44), lemma 6.2, we have
\[ |\dot{U}(z_1(t), t)^2 - \dot{U}(z_2(t), t)^2| = |(\dot{U}(z_1(t), t) + \dot{U}(z_2(t), t))\dot{U}_c(\tilde{x} + i\tilde{y}(t), t)(z_1(t)
- z_2(t))| \leq 120\epsilon^2 x(t). \] (6.88)

Here, \( \tilde{x} \in (0, x(t)) \).

By (6.50) of lemma 6.2, we have
\[ |\dot{z}_1^2 - \dot{z}_2^2| = \left| \frac{2\lambda_1 i}{4\pi x(t)} (\dot{U}(z_1(t), t) - \dot{U}(z_2(t), t)) + \dot{U}(z_1(t), t)^2 - \dot{U}(z_2(t), t)^2 \right| \]
\[ = \left| \frac{\lambda_1 i}{\pi x(t)} \mathcal{M}[\dot{U}(z_1(t), t)] + \dot{U}(z_1(t), t)^2 - \dot{U}(z_2(t), t)^2 \right| \]
\[ \leq \frac{|\lambda|}{\pi x(t)} \|\dot{U}_c\|_{\infty} x(t) + \|\dot{U}(z_1(t), t)^2 - \dot{U}(z_2(t), t)^2\| \]
\[ \leq 6|\lambda|\epsilon + 120\epsilon^2 x(t). \]

\[ \square \]

Another consequence of the bootstrap assumption (6.44) is the following description of the motion of the point vortices, which is the key control of this paper.

**Proposition 5** (key control). Assume the assumptions of Theorem 3 and assume the bootstrap assumption (6.44), we have
\[ \frac{1}{2} \leq \frac{x(t)}{x(0)} \leq 2, \quad 0 \leq t \leq T_0. \] (6.89)
Proof. It suffices to prove the case that \( x(t) \) is increasing on \( 0 \leq t \leq T_0 \). The case that \( x(t) \) is decreasing follows in a similar way, and other cases are controlled by these two cases.

Denote

\[
T := \left\{ T \in [0, T_0] \mid \dot{y}(t) \leq -\frac{|\lambda|}{20\pi x(0)}, \quad 1/2 \leq \frac{x(t)}{x(0)} \leq 2, \quad \hat{d}_I(t) \right\} 
\geq 1 + \frac{|\lambda|}{20\pi x(0)} t, \quad \forall t \in [0, T].
\] (6.90)

Let’s assume \( \mathcal{U} = \mathcal{U}_1 + i\mathcal{U}_2 \), where \( \mathcal{U}_1, \mathcal{U}_2 \) are real (we remind the readers that \( \mathcal{U} \) is the holomorphic extension of \( f \). Recall also the notations that \( z_1(t) = -x(t) + iy(t), \ z_2(t) = x(t) + iy(t), \ x(t) > 0, \ y(t) < 0 \). From the proof of lemma 6.5, we have

\[
\begin{aligned}
\dot{z}_1(t) &= \mathcal{U}(z_1(t), t) + \frac{\lambda z_1 i}{2\pi} \frac{1}{z_1(t) - z_2(t)} = \mathcal{U}(z_1(t)) - \frac{|\lambda| i}{4\pi x(t)}, \\
\dot{z}_2(t) &= \mathcal{U}(z_2(t), t) + \frac{\lambda z_2 i}{2\pi} \frac{1}{z_2(t) - z_1(t)} = \mathcal{U}(z_2(t)) - \frac{|\lambda| i}{4\pi x(t)}.
\end{aligned}
\] (6.91)

So we have

\[
\dot{y}(t) = -\mathcal{U}_2(z_2(t), t) - \frac{|\lambda|}{4\pi x(t)}. \tag{6.92}
\]

By maximum principle and the bootstrap assumption (6.44), we have

\[
|\mathcal{U}_2(z_2(t), t)| \leq |\mathcal{U}(z_2(t), t)| \leq \|\mathcal{U}(\cdot, t)\|_{L^\infty(\Omega(t))} = \|\mathfrak{F}(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq 5\varepsilon. \tag{6.93}
\]

For \( M \) relatively large (we take \( M = 200\pi \)), at \( t = 0 \), we have \( \frac{|\lambda|}{4\pi x(0)} \geq \frac{200\pi \varepsilon}{4\pi} = 50\varepsilon \). So we have

\[
\dot{y}(0) = -\mathcal{U}_2(z_2(0), 0) - \frac{|\lambda|}{4\pi x(0)} \leq -\frac{9|\lambda|}{40\pi x(0)}. \tag{6.94}
\]

So \( 0 \in \mathcal{T} \) and therefore \( \mathcal{T} \neq \emptyset \). Clearly, by the definition of \( \mathcal{T} \), since \( \dot{y}(t), x(t) \) and \( \hat{d}_I(t) \) are continuous, so \( \mathcal{T} \) is closed in \( [0, T_0] \). To prove \( \mathcal{T} = [0, T_0] \), it suffices to prove that if (6.90) holds on \( [0, T] \) with \( T < T_0 \), then there exists \( \delta > 0 \) such that (6.90) holds on \( [T, T + \delta] \).

Let \( T \in \mathcal{T} \).

By (6.91), we have \( \dot{x}(t) = \Re\{\mathcal{U}(z_2(t), t)\} \). Use (6.50) of lemma 6.2, use the fact that \( \Re\{\mathcal{U}\} \) is odd, by mean value theorem, we have

\[
\dot{x}(t) = \Re\{\mathcal{U}(z_2(t), t)\} - \Re\{\mathcal{U}(0 + iy(t), t)\} = \Re\{\mathcal{U}_1(\tilde{x} + iy(t), t)x(t)\}, \tag{6.95}
\]

for some \( \tilde{x} \in (0, x(t)) \).

Since

\[
\mathcal{U}(z, t) = \frac{1}{2\pi i} \int \frac{\zeta_\beta}{z - \zeta(\beta, t)} \mathfrak{F}(\beta, t) d\beta. \tag{6.96}
\]

So we have \( \forall \ t \in [0, T_0] \),

\[
\partial_x \mathcal{U}(z, t) = -\frac{1}{2\pi i} \int \frac{\zeta_\beta}{(z - \zeta(\beta, t))^2} \mathfrak{F}(\beta, t) d\beta. \tag{6.97}
\]
By Cauchy-Schwartz inequality and lemma 2.8, we have

\[
|\partial_x U(\tilde{x} + iy(t), t)| \leq \frac{1}{2\pi} \left( \int |\tilde{x} + iy(t) - \zeta(\beta, t)|^4 d\beta \right)^{1/2} \|\zeta\|_{L^\infty} \|F\|_{L^2} \leq C \epsilon \hat{d}_I(t)^{-3/2}, \quad \forall \ t \in [0, T_0],
\]

for some absolute constant \( C > 0 \). By direct calculation, we can see that \( C \leq 2 \). So we obtain

\[
\dot{x}(t) \leq 2\epsilon \hat{d}_I(t)^{-3/2} x(t).
\]  

(6.99)

So we have

\[
\frac{d}{dt} \ln \frac{x(t)}{x(0)} \leq 2\epsilon \int_0^t \left( 1 + \frac{|\lambda|}{20\pi x(0)} \right)^{-3/2} d\tau.
\]

(6.100)

Then we have for all \( t \in [0, T] \),

\[
x(t) \leq x(0) \exp \left\{ 2\epsilon \int_0^t \left( 1 + \frac{|\lambda|}{20\pi x(0)} \right)^{-3/2} d\tau \right\}
\]

\[
\leq x(0) \exp \left\{ 4\epsilon \frac{20\pi x(0)}{|\lambda|} \right\}
\]

\[
\leq x(0) e^{2\epsilon \frac{40\pi}{|\lambda|}} = e^{\frac{2}{3} x(0)} \leq \frac{3}{2} x(0).
\]

(6.101)

By the continuity of \( x(t) \), there exists \( \delta > 0 \) such that

\[
\sup_{t \in [0, T + \delta)} \frac{x(t)}{x(0)} \leq 2.
\]

(6.102)

Next we show that by choosing \( \delta > 0 \) smaller if necessary, we have

\[
\dot{y}(t) \leq -\frac{|\lambda|}{20\pi x(0)}, \quad \hat{d}_I(t) \geq 1 + \frac{|\lambda|}{20\pi x(0)} t, \quad \forall \ t \in [0, T + \delta). \]

(6.103)

Proof of (6.103): By the definition of \( T \), the bootstrap assumption (6.44), and the fact that \( \frac{|\lambda|}{x(0)} \geq 200\pi \epsilon \), we have

\[
\dot{y}(t) \leq 5\epsilon - \frac{|\lambda|}{8\pi x(0)} \leq -\frac{|\lambda|}{10\pi x(0)}, \quad \forall \ t \in [0, T].
\]

(6.104)

By Fundamental theorem of calculus, we have

\[
y(t) = y(0) + \int_0^t \dot{y}(\tau) d\tau.
\]

(6.105)

We have

\[
\zeta(\alpha, t) - z_j(t) = \zeta(\alpha, 0) - z_j(0) + \int_0^t \partial_t (\zeta(\alpha, \tau) - z_j(\tau)) d\tau
\]

\[
= \zeta(\alpha, 0) - z_j(0) + \int_0^t D_t \zeta(\alpha, \tau) d\tau
\]

\[
- \int_0^t b(\alpha, \tau) \partial_t \zeta(\alpha, \tau) d\tau - \int_0^t \dot{z}_j(\tau) d\tau.
\]

(6.106)
So we have
\[
\text{Im}\{\zeta(\alpha, t) - z_j(t)\} = \text{Im}\left\{\zeta(\alpha, 0) - z_j(0) - \int_0^t \dot{z}_j(\tau) d\tau\right\} + \text{Im}\int_0^t D_\tau \zeta(\alpha, \tau) d\tau \\
- \int_0^t b(\alpha, \tau) \text{Im}\{\partial_\alpha \zeta(\alpha, \tau)\} d\tau.
\] (6.107)

By Sobolev embedding and the bootstrap assumption, we have
\[
\left|\int_0^t D_\tau \zeta(\alpha, \tau) d\tau\right| \leq 6\epsilon t, \quad \forall\ t \in [0, T].
\] (6.108)

By Corollary 6.2 and Sobolev embedding, we have
\[
\left|\int_0^t b(\alpha, \tau) \partial_\alpha \zeta(\alpha, \tau) d\tau\right| \leq (C\epsilon^2 + K_s^{-1}\epsilon d_I(0)) (1 + 5\epsilon) t \leq (C\epsilon^2 + K_s^{-1}\epsilon) t.
\] (6.109)

Note that
\[
\text{Im}\{\zeta(\alpha, 0) - z_j(0)\} \geq \hat{d}_I(0) \geq 1, \quad -\text{Im}\{\dot{z}_j(\tau)\} \geq |\lambda|\frac{10\pi x(0)}{18\pi x(0)} > 0,
\]
so we have
\[
\text{Im}\{\zeta(\alpha, t) - z_j(t)\} \geq \inf_{\alpha \in \mathbb{R}} \text{Im}\{\zeta(\alpha, 0) - z_j(0)\} + (\frac{|\lambda|}{10\pi x(0)} - 6\epsilon - (C\epsilon^2 + K_s^{-1}\epsilon)) t \\
\geq 1 + \frac{|\lambda|}{18\pi x(0)} t.
\] (6.110)

So we have
\[
\hat{d}_I(t) = \min_{j=1,2} \inf_{\alpha \in \mathbb{R}} \text{Im}\{\zeta(\alpha, t) - z_j(t)\} \geq 1 + \frac{|\lambda|}{18\pi x(0)} t, \quad \forall\ t \in [0, T].
\] (6.111)

By (6.104), (6.111), the continuity of \(\hat{d}_I(t)\), and the continuity of \(y'(t)\), choosing \(\delta > 0\) smaller if necessary, we have (6.103).

Since \(\mathcal{T}\) is both closed and open as a subspace of \([0, T_0]\), so we must have
\[
\mathcal{T} = [0, T_0],
\] (6.112)

which concludes the proof of the lemma. \(\square\)

Because
\[
d_I(t) = \min_{j=1,2} \inf_{\alpha \in \mathbb{R}} |\zeta(\alpha, t) - z_j(t)| \geq \hat{d}_I(t),
\]
we have the following estimate.

**Corollary 6.3** [Decay estimate] Assume the assumptions of Theorem 3 and assume the bootstrap assumption (6.44), we have \(\forall\ t \in [0, T_0]\),
\[
d_I(t)^{-1} \leq (1 + \frac{|\lambda|}{20\pi x(0)})^{-1}.
\] (6.113)
We need to estimate \( \ddot{z}_j \) and \( \ddot{z}_1 - \ddot{z}_2 \) as well.

**Convention:** From now on, if the domain of \( t \) is not specified, we assume \( t \in [0, T_0] \) by default.

**Lemma 6.6.** Assume the assumptions of Theorem 3 and assume the bootstrap assumption (6.44), we have \( \forall t \in [0, T_0] \),

\[
|\ddot{z}_j(t)| \leq 10\varepsilon + \frac{6|\lambda|}{x(t)}\varepsilon. \tag{6.114}
\]

\[
|\ddot{z}_1(t) - \ddot{z}_2(t)| \leq 220\varepsilon^2 x(t) + \varepsilon(20 x(t) + \frac{5|\lambda|}{\pi}) \tag{6.115}
\]

**Proof.** Take time derivative of (6.86), we have

\[
\ddot{z}_j(t) = -\frac{\lambda i x'(t)}{4\pi x(t)^2} + \ddot{U}_\zeta(z_j(t), t) \dot{z}_j(t) + \ddot{U}_t(z_j, t). \tag{6.116}
\]

We have \( x'(t) = \Re\{U(z_2(t), t) \}. \) By lemma 6.2, we have

\[
|\ddot{U}_\zeta(z_j(t), t)| \leq ||\ddot{U}_\zeta(\cdot, t)||_{L^\infty(\Omega(t))} \leq ||\ddot{U}_\zeta(\xi(\alpha, t), t)||_\infty \leq 6\varepsilon. \tag{6.117}
\]

By lemma 6.2, we have

\[
||\ddot{U}_t(\cdot, t)||_{L^\infty(\Omega(t))} \leq ||\ddot{U}_t(\xi(\alpha, t), t)||_{L^\infty} \leq 6\varepsilon. \tag{6.118}
\]

Apply lemma 6.2 again, we have

\[
||\Re\{U(z_j(t), t)\}|| \leq 6\varepsilon x(t).
\]

So we obtain

\[
|\ddot{z}_j(t)| \leq \frac{[\lambda]||\Re\{U(z_j, t)\}|}{4\pi x(t)^2} + |\ddot{U}_\zeta(z_j(t), t)| |\dot{z}_j(t)| + |\ddot{U}_t(z_j, t)|
\]
\[
\leq \frac{6|\lambda|\varepsilon}{4\pi x(t)} + 6\varepsilon (\frac{|\lambda|}{4\pi x(t)} + 6\varepsilon) + 6\varepsilon
\]
\[
\leq \frac{6|\lambda|\varepsilon}{\pi x(t)} + 10\varepsilon.
\]

Here, we assume \( \varepsilon^2 \) sufficiently small such that \( 36\varepsilon^2 \leq 4\varepsilon \). We have

\[
|\ddot{z}_1(t) - \ddot{z}_2(t)|
\]
\[
= |\ddot{U}_\zeta(z_1(t), t)\ddot{z}_1(t) + \ddot{U}_t(z_1(t), t) - \ddot{U}_\zeta(z_2(t), t)\ddot{z}_2(t) - \ddot{U}_t(z_2(t), t)|
\]
\[
\leq |\ddot{U}_\zeta(z_1(t), t) - \ddot{U}_\zeta(z_2(t), t)| |\ddot{z}_1(t)| + |\ddot{U}_\zeta(z_2(t), t)| |\ddot{z}_1(t) - \ddot{z}_2(t)| + |\ddot{U}_t(z_1(t), t) - \ddot{U}_t(z_2(t), t)|
\]
\[
\leq ||\ddot{U}_\zeta||_\infty |z_1 - z_2| |\ddot{z}_1(t)| + ||\ddot{U}_\zeta||_\infty ||\ddot{U}_t||_\infty |z_1(t) - \ddot{U}(z_2(t), t)|
\]
\[
+ ||\Re\{U(t)\}||_{L^\infty(\Omega(t))} |z_1 - z_2|.
\]

By lemma 6.2,

\[
||\ddot{U}_\zeta||_\infty \leq 10\varepsilon, \tag{6.119}
\]

and

\[
||\ddot{U}_t||_{L^\infty(\Omega(t))} \leq 10\varepsilon. \tag{6.120}
\]
Since \( \Re \{ \mathcal{U}_t \} \) is odd in \( x \) and \( \Im \{ \mathcal{U}_t \} \) is even in \( x \), by mean value theorem, we have

\[
|\mathcal{U}_t(z_1, t) - \mathcal{U}_t(z_2, t)| = 2|\Re \mathcal{U}_t(z_2, t) - \Re \mathcal{U}_t(0, y, t)| x = 2|\Re \mathcal{U}_x(\bar{x}, y, t)| x(t)
\]

(6.121)

\[
\leq 2\|\mathcal{U}_t x\|_{L^\infty(\Omega(t)), x(t)} \leq 20\epsilon x(t)
\]

(6.122)

for some \( \bar{x} \in (0, x(t)) \).

So we obtain

\[
|\bar{z}_1(t) - \bar{z}_2(t)|
\]

\[
\leq 10\epsilon (2x(t)) (\frac{|\lambda|}{4\pi x(t)} + 5\epsilon) + (6\epsilon) 20\epsilon x(t) + (10\epsilon) 2x(t)
\]

(6.123)

\[
= 220\epsilon^2 x(t) + \epsilon (20x(t) + \frac{5|\lambda|}{\pi}).
\]

\(\square\)

Next, we estimate the quantity \( \left\| \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \left( \frac{\bar{z}_j}{(\zeta(\alpha, t) - z_j(t))^2} \right) \right\|_{H^s} \), the quantity \( \left\| \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} (\zeta(\alpha, t) - z_j(t))^2 \right\|_{H^s} \), and the quantity \( \left\| \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} |(\zeta(\alpha, t) - z_j(t))^2| \right\|_{H^s} \). These quantities arise from the energy estimates.

**Lemma 6.7.** Assume the assumptions of Theorem 3 and assume the bootstrap assumption (6.44). Then we have

\[
\left\| \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \left( \frac{\bar{z}_j}{(\zeta(\alpha, t) - z_j(t))^2} \right) \right\|_{H^s} \leq K_s^{-1} \epsilon \frac{|\lambda|}{x(0)} d_1(t)^{-5/2} + C\epsilon^2.
\]

(6.124)

**Proof.** Replace \( \bar{z}_j \) by

\[
\bar{z}_j(t) = \frac{\lambda_i}{4\pi x(t)} + \bar{U}(z_j(t), t).
\]

We have

\[
\sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \left( \frac{\bar{z}_j}{(\zeta(\alpha, t) - z_j(t))^2} \right)
\]

\[
= \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \left( \frac{\lambda_i}{4\pi x(t)} + \bar{U}(z_j(t), t) \right) \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \left( \frac{1}{(\zeta(\alpha, t) - z_j(t))^2} \right)
\]

\[
+ \frac{\lambda_i (\bar{U}(z_1(t), t) - \bar{U}(z_2(t), t))}{2\pi} \frac{1}{(\zeta(\alpha, t) - z_2(t))^2}.
\]

By lemma 6.4 (and use the proof of lemma 6.4 to estimate the term \( \left\| \frac{1}{(\zeta(\alpha, t) - z_2(t))^2} \right\|_{H^s} \)), we have

\[
\left\| \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \left( \frac{\bar{z}_j}{(\zeta(\alpha, t) - z_j(t))^2} \right) \right\|_{H^s}
\]
\[
\frac{|\lambda|}{4\pi x(t)} \left| \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{\lambda_j i}{(\lambda(t) - z_j(t))^2} \right| + \frac{|\lambda|}{2\pi} \left| \sum_{j=1}^{2} \frac{1}{(\lambda(t) - z_j(t))^2} \right| H^s \\
+ \frac{|\lambda|}{2\pi} \frac{(\tilde{U}(z_1(t), t) - \tilde{U}(z_2(t), t))}{(\lambda(t) - z_2(t))^2} \right| H^s \\
\leq \frac{|\lambda|}{4\pi x(t)} K_s^{-1} \epsilon d_1(t)^{-5/2} + K_s^{-1} \epsilon d_1(t)^{-5/2} + K_s^{-1} \epsilon^2 d_1(t)^{-5/2} + K_s^{-1} \epsilon^2 d_1(t)^{-3/2}.
\]

Here, we use lemma 6.2 to estimate
\[
\left| \frac{U(z_1(t), t) - U(z_2(t), t)}{x(t)} \right| \leq 12\epsilon, \quad (6.125)
\]
and we use the proof of lemma 6.4 to estimate
\[
\left| \frac{\lambda x(t)}{(\lambda(t) - z_2(t))^2} \right| H^s \leq K_s^{-1} \epsilon d_1(t)^{-3/2}. \quad (6.126)
\]

Since \(d_1(t) \geq 1\), we simply estimate \(\left| \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{\lambda_j i}{(\lambda(t) - z_j(t))^2} \right| H^s\) by \(6.124\). □

**Lemma 6.8.** Assume the assumptions of Theorem 3 and assume the bootstrap assumption (6.44). Then we have
\[
\left| \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{(\dot{\lambda}_j)^2}{(\lambda(t) - z_j(t))^3} \right| H^s \leq K_s^{-1} \epsilon^2 + K_s^{-1} \epsilon \frac{|\lambda|}{x(0)} d_1(t)^{-5/2}. \quad (6.127)
\]

**Proof.**
\[
\sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{(\dot{\lambda}_j)^2}{(\lambda(t) - z_j(t))^3} = \frac{\lambda i}{2\pi} \frac{(\dot{\lambda}_1)^2}{(\lambda(t) - z_1(t))^3} - \frac{1}{(\lambda(t) - z_2(t))^3} \\
+ \frac{\lambda i}{2\pi} \frac{(\dot{\lambda}_1)^2 - (\dot{\lambda}_2)^2}{(\lambda(t) - z_2(t))^3} \equiv I + II.
\]
We have
\[
|I| = \left| \frac{\lambda i}{2\pi} \frac{(\dot{\lambda}_1)^2}{2x(t)} \left\{ \frac{1}{(\lambda(t) - z_1(t))^3(\lambda(t) - z_2(t))} + \frac{1}{(\lambda(t) - z_1(t))^2(\lambda(t) - z_2(t)^2)} + \frac{1}{(\lambda(t) - z_1(t))(\lambda(t) - z_2(t))^3} \right\} \right|,
\]
and
\[
\dot{\lambda}_1^2 = -\frac{\lambda^2}{16\pi^2 x(t)^2} + \frac{\lambda i}{2\pi x(t)} \tilde{U}(z_1(t), t) + (\tilde{U}(z_1(t), t))^2.
\]
Use the proof of lemma 6.4, it’s easy to see that
\[
\left\| \frac{1}{(\zeta(\alpha, t) - z_1(t))^3(\zeta(\alpha, t) - z_2(t))} \right\|_{H^s} \leq \frac{((s + 6)!)^2 d_l(t)^{-7/2}}{2} \quad (6.128)
\]
\[
\left\| \frac{1}{(\zeta(\alpha, t) - z_1(t))^2(\zeta(\alpha, t) - z_2(t)^2)} \right\|_{H^s} \leq \frac{((s + 6)!)^2 d_l(t)^{-7/2}}{2} \quad (6.129)
\]
\[
\left\| \frac{1}{(\zeta(\alpha, t) - z_1(t))^{\zeta(\alpha, t) - z_2(t)^3}} \right\|_{H^s} \leq \frac{((s + 6)!)^2 d_l(t)^{-7/2}}{2}. \quad (6.130)
\]

Use the assumption that \( \lambda^2 + |\lambda x(0)| \leq \frac{1}{((s + 12))^2} \epsilon \) and the fact that \( \frac{1}{2} x(0) \leq x(t) \leq 2x(0) \), we have
\[
\| I \|_{H^s} \leq \frac{|\lambda| x(t)}{\pi} \left( \frac{\lambda^2}{16\pi^2 x(t)^2} + \frac{|\lambda|}{2\pi x(t)} \right) \times 5\epsilon + 25\epsilon^2) \frac{((s + 6)!)^2 d_l(t)^{-7/2}}{2} \\
\leq K_s^{-1} \epsilon^2 d_l(t)^{-5/2} + K_s^{-1} \epsilon \frac{|\lambda|}{x(0)} d_l(t)^{-5/2}.
\]

By lemma 6.5, we have
\[
\| II \|_{H^s} \leq \frac{|\lambda| z_1^2 - z_2^2}{2\pi} \left\| \frac{1}{(\zeta(\alpha, t) - z_2(t)^3)} \right\|_{H^s} \\
\leq \frac{|\lambda| (6|\lambda| \epsilon + 120\epsilon^2 x(t))}{2\pi} \frac{((s + 6)!)^2 d_l(t)^{-5/2}}{2} \\
\leq K_s^{-1} \epsilon^2 d_l(t)^{-5/2}.
\]

Here, we use the assumption
\[
\lambda^2 + |\lambda x(0)| \leq c_0 \epsilon, \quad c_0 = \frac{1}{((s + 12))^2}. \quad (6.131)
\]

So we have
\[
\left\| \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{\tilde{z}_j^2}{(\zeta(\alpha, t) - z_j(t))^3} \right\|_{H^s} \leq K_s^{-1} \epsilon^2 + K_s^{-1} \epsilon \frac{|\lambda|}{x(0)} d_l(t)^{-5/2}.
\]

\[\square\]

**Lemma 6.9.** Assume the assumptions of Theorem 3 and assume the bootstrap assumption (6.44). Then we have
\[
\left\| \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{\tilde{z}_j}{(\zeta(\alpha, t) - z_j(t))^2} \right\|_{H^s} \leq K_s^{-1} \epsilon^2 d_l(t)^{-5/2}. \quad (6.132)
\]

**Proof.** We have
\[
\left\| \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{\tilde{z}_j}{(\zeta(\alpha, t) - z_j(t))^2} \right\|_{H^s} \leq \left\| \sum_{j=1}^{2} \frac{\lambda_j i \tilde{z}_1(t)}{2\pi} \frac{1}{(\zeta(\alpha, t) - z_j(t))^2} \right\|_{H^s}
\]
\[ + \left\| \frac{\lambda i (\dot{z}_1(t) - \dot{z}_2(t))}{2\pi} \frac{1}{(\zeta(\alpha, t) - z_2(t))^2} \right\|_{H^s} := I + II. \]

By the proof of lemma 6.4 and by lemma 6.6, we have

\[ I \leq |\dot{z}_1(t)| \left\| \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi (\zeta(\alpha, t) - z_j(t))^2} \right\|_{H^s} \]

\[ \leq (10\epsilon + \frac{6|x(0)|\epsilon}{x(t)}) |\lambda x(0)| ((s + 6)! d_I(t)^{-5/2}) \]

\[ \leq K_s^{-1} \epsilon^2 d_I(t)^{-5/2}. \]

By lemma 6.4 and lemma 6.6, we have

\[ II \leq |\dot{z}_1(t) - \dot{z}_2(t)| \left\| \frac{\lambda i}{2\pi (\zeta(\alpha, t) - z_2(t))^2} \right\|_{H^s} \]

\[ \leq \left( \frac{220\epsilon^2 x(t) + \epsilon(20x(t) + \frac{5|\lambda|}{\pi})}{x(t)^2} \right) |\lambda| ((s + 6)! d_I(t)^{-5/2}) \]

\[ \leq K_s^{-1} \epsilon^2 d_I(t)^{-5/2}. \]

Here, we’ve used the fact that \( \lambda^2 + |\lambda x(0)| \leq \frac{1}{(s+12)! \epsilon}. \) So we obtain

\[ \left\| \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi (\zeta(\alpha, t) - z_j(t))^2} \right\|_{H^s} \leq K_s^{-1} \epsilon^2 d_I(t)^{-5/2}. \] (6.133)

\[ \square \]

6.8. Estimates for quantities involved in the energy estimates. In this subsection, we derive estimates for various quantities that show up in energy estimates.

6.8.1. Control \( \| \partial_{\alpha} \tilde{\theta} \|_{H^s} \) by \( \| \xi_{\alpha} - 1 \|_{H^s} \).

**Lemma 6.10.** Assume the assumptions of Theorem 3 and assume the assumption (6.44), for \( 1 \leq k \leq s + 1 \), we have

\[ \| \partial_{\alpha}^k \tilde{\theta} - 2\partial_{\alpha}^{k-1}(\xi_{\alpha} - 1) \|_{L^2} \leq C \epsilon^2. \] (6.134)

**Proof.** Since \( (I - H)(\tilde{\zeta} - \alpha) = 0 \), we have \( (I + \tilde{H})(\zeta - \alpha) = 2(\zeta - \alpha) \). Therefore,

\[ \partial_{\alpha}^k \tilde{\theta} = \partial_{\alpha}^k (I - H)(\zeta - \tilde{\zeta}) = \partial_{\alpha}^k (I - H)(\zeta - \alpha) \]

\[ = \partial_{\alpha}^k (I + \tilde{H} - (\tilde{H} + H))(\zeta - \alpha) \]

\[ = 2\partial_{\alpha}^{k-1}(\xi_{\alpha} - 1) - \partial_{\alpha}^k (\tilde{H} + H))(\zeta - \alpha). \]

It’s easy to obtain that for \( 1 \leq k \leq s + 1 \),

\[ \| \partial_{\alpha}^k (\tilde{H} + H)(\zeta - \alpha) \|_{L^2} \leq C \| \xi_{\alpha} - 1 \|_{H^s}^2 \leq C \epsilon^2. \] (6.135)

So we have

\[ \| \partial_{\alpha}^k \tilde{\theta} - 2\partial_{\alpha}^{k-1}(\xi_{\alpha} - 1) \|_{L^2} \leq C \epsilon^2. \] (6.136)

So we obtain (6.134). \( \square \)
Corollary 6.4. Assume the assumptions of Theorem 3 and the bootstrap assumption (6.44), we have

\[ \| \partial_\alpha \tilde{\theta} \|_{H^s} \leq 11 \epsilon. \] (6.137)

6.8.2. Compare \( \| D_t \tilde{\theta} \|_{H^s} \) with \( \| D_t \xi \|_{H^s} \) and \( \| D_t \tilde{\sigma} \|_{H^s} \) with \( \| D_t^2 \xi \|_{H^s} \). We need to show that \( D_t \tilde{\theta} \) and \( D_t \xi \) are equivalent in certain sense. We have the following:

Lemma 6.11. Assume the assumptions of Theorem 3 and a priori assumption (6.44), we have

\[ \| D_t \tilde{\theta} - 2(\tilde{\xi} - q) \|_{H^{s+1}/2} \leq C \epsilon^2. \] (6.138)

\[ \| D_t \tilde{\sigma} - 4(D_t \tilde{\xi} - D_t q) \|_{H^s} \leq C \epsilon^2. \] (6.139)

Proof. Recall that \( D_t \xi = \tilde{\xi} + \tilde{q} \), where \((I - \mathcal{H})\tilde{\xi} = 0, (I + \mathcal{H})q = 0\). So we have

\( (I + \mathcal{H})\tilde{\xi} = 2\tilde{\xi}, \quad (I + \mathcal{H})\tilde{q} = 0 \).

We have

\[
D_t \tilde{\theta} = D_t (I - \mathcal{H})(\xi - \tilde{\xi}) = (I - \mathcal{H})(D_t \xi - D_t \tilde{\xi}) - [D_t \xi, \mathcal{H}] \frac{\partial_\alpha (\xi - \tilde{\xi})}{\xi_\alpha} \\
= (I - \mathcal{H})(\tilde{\xi} + \tilde{q} - \tilde{\xi} - q) - [D_t \xi, \mathcal{H}] \frac{\partial_\alpha (\xi - \tilde{\xi})}{\xi_\alpha} \\
= (I + \mathcal{H})\tilde{\xi} + (I + \mathcal{H})\tilde{q} - (\mathcal{H} + \mathcal{H})D_t \xi - 2q - [D_t \xi, \mathcal{H}] \frac{\partial_\alpha (\xi - \tilde{\xi})}{\xi_\alpha} \\
= 2\tilde{\xi} - 2q - (\mathcal{H} + \mathcal{H})D_t \xi - [D_t \xi, \mathcal{H}] \frac{\partial_\alpha (\xi - \tilde{\xi})}{\xi_\alpha}. \] (6.140)

It’s easy to obtain that under a priori assumption (6.44),

\[ \| -(\mathcal{H} + \mathcal{H})D_t \xi - [D_t \xi, \mathcal{H}] \frac{\partial_\alpha (\xi - \tilde{\xi})}{\xi_\alpha} \|_{H^{s+1}/2} \leq C \epsilon \| D_t \xi \|_{H^{s+1}/2} \leq C \epsilon^2, \] (6.141)

for some absolute constant \( C > 0 \).

By triangle inequality,

\[ \| D_t \tilde{\theta} - 2(\tilde{\xi} - q) \|_{H^{s+1}/2} \leq \| -(\mathcal{H} + \mathcal{H})D_t \xi - [D_t \xi, \mathcal{H}] \frac{\partial_\alpha (\xi - \tilde{\xi})}{\xi_\alpha} \|_{H^{s+1}/2} \leq C \epsilon^2. \] (6.142)

So we obtain (6.138).

By (6.140), use

\( (I - \mathcal{H})\tilde{\xi} = 2\tilde{\xi} - (\mathcal{H} + \mathcal{H})\tilde{\xi}, \quad (I - \mathcal{H})q = 2q \), (6.143)
we have

\[
D_t \tilde{\sigma} = D_t(I - \mathcal{H}) D_t \tilde{\beta} = D_t(I - \mathcal{H}) \left\{ 2\tilde{\xi} - 2q - (\mathcal{H} + \tilde{\mathcal{H}}) D_t \xi - [D_t \xi, \mathcal{H}] \frac{\partial \alpha (\xi - \tilde{\xi})}{\xi \alpha} \right\} 
\]

\[
= 4D_t \tilde{\xi} - 4D_t q + D_t(I - \mathcal{H}) \left\{ - (\mathcal{H} + \tilde{\mathcal{H}}) D_t \xi - [D_t \xi, \mathcal{H}] \frac{\partial \alpha (\xi - \tilde{\xi})}{\xi \alpha} \right\} - D_t(\tilde{\mathcal{H}} + \mathcal{H}) \tilde{\xi}. 
\]

Therefore,

\[
\| D_t \tilde{\sigma} - 4(D_t \tilde{\xi} - D_t q) \|_{H^s} \leq C \epsilon^2. 
\]

(6.144)

**Corollary 6.5.** Assume the bootstrap assumption (6.44), we have

\[
\left\| D_t \tilde{\beta} \right\|_{H^{s+1/2}} \leq 11 \epsilon, \quad \left\| D_t \tilde{\sigma} \right\|_{H^s} \leq 21 \epsilon. 
\]

(6.148)

### 6.8.3. Estimate the quantity \( \frac{\alpha_t}{a} \circ \kappa^{-1} \)

Recall that

\[
(I - \mathcal{H}) \frac{\alpha_t}{a} \circ \kappa^{-1} \lambda \zeta = \frac{2i[D_t^2 \xi, \mathcal{H}(t)]}{\zeta \alpha} \frac{\partial \alpha (D_t \xi)}{\zeta \alpha} - \frac{1}{\pi} \int \left( \frac{D_t \xi(\alpha, t) - D_t \xi(\beta, t)}{\zeta(\alpha, t) - \zeta(\beta, t)} \right)^2 (D_t \xi)_{\beta} \, d\beta 
\]

\[
- \frac{1}{\pi} \sum_{j=1}^{2} \lambda_j \left( \frac{2D_t^2 \xi + i - \partial_t^2 z_j}{\zeta(\alpha, t) - z_j(t)} \right)^2 - \frac{2}{\pi} \left( \frac{D_t \xi - \tilde{z}_j(t)}{\zeta(\alpha, t) - z_j(t)} \right)^2. 
\]

(6.149)

By lemma 2.6, the a priori assumption (6.44), we have

\[
\left\| 2i[D_t^2 \xi, \mathcal{H}] \frac{\partial \alpha (D_t \xi)}{\zeta \alpha} \right\|_{H^s} \leq C \| D_t^2 \xi \|_{H^s} \| D_t \xi \|_{H^s} \leq C \epsilon^2. 
\]

(6.150)

\[
\left\| 2i[D_t \xi, \mathcal{H}] \frac{\partial \alpha (D_t \xi)}{\zeta \alpha} \right\|_{H^s} \leq C \| D_t \xi \|_{H^s} \| D_t^2 \xi \|_{H^s} \leq C \epsilon^2. 
\]

(6.151)

\[
\left\| \frac{1}{\pi} \int \left( \frac{D_t \xi(\alpha, t) - D_t \xi(\beta, t)}{\zeta(\alpha, t) - \zeta(\beta, t)} \right)^2 (D_t \xi)_{\beta} \, d\beta \right\|_{H^s} \leq \| D_t \xi \|_{H^s}^3 \leq C(5 \epsilon)^3 \leq C \epsilon^2. 
\]

(6.152)

\[
\left\| \frac{1}{\pi} \sum_{j=1}^{2} \frac{2\lambda_j D_t^2 \xi}{\zeta(\alpha, t) - z_j(t)} \right\|_{H^s} \leq \| D_t^2 \xi \|_{H^s} \left\| \sum_{j=1}^{2} \frac{\lambda_j}{\zeta(\alpha, t) - z_j(t)} \right\|_{H^s} \leq K^2 \epsilon \alpha \| d_1(t) \|^{-5/2}. 
\]

(6.153)
So we have
\[
\left\| \frac{1}{\pi} \sum_{j=1}^{2} \frac{2\lambda_j D_t^2 \xi}{(\xi(\alpha, t) - z_j(t))^2} \right\|_{H^s} \leq \| D_t^2 \xi \|_{H^s} \leq K_s^{-1} \epsilon^2.
\] (6.155)

By lemma 6.7, we have
\[
\left\| \frac{2}{\pi} \sum_{j=1}^{2} \frac{\lambda_j i D_t \xi}{(\xi(\alpha, t) - z_j(t))^3} \right\|_{H^s} \leq 2 \| D_t \xi \|_{H^s} \left\| \frac{2}{\pi} \sum_{j=1}^{2} \frac{\lambda_j i \zeta_j(t)}{(\xi(\alpha, t) - z_j(t))^3} \right\|_{H^s} \leq 12\epsilon K_s^{-1} \epsilon d_1(t)^{-5/2} \leq K_s^{-1} \epsilon^2 d_1(t)^{-5/2}.
\] (6.156)

By lemma 6.8 and lemma 6.9, we have
\[
\left\| \frac{1}{\pi} \sum_{j=1}^{2} \frac{\lambda_j \zeta_j \dot{z}_j(t)}{(\xi(\alpha, t) - z_j(t))^2} \right\|_{H^s} + \left\| \frac{1}{\pi} \sum_{j=1}^{2} \frac{\lambda_j 2 \dot{z}_j(t)^2}{(\xi(\alpha, t) - z_j(t))^3} \right\|_{H^s} \leq K_s^{-1} \epsilon \left\| \frac{\lambda_j}{x(0)} \right\|_{H^s} \leq (6.157)
\]

So we obtain
\[
\left\| (I - \mathcal{H}) \frac{a_t}{a} \circ \kappa^{-1} A \xi \right\|_{H^s} \leq C \epsilon^2 + K_s^{-1} \epsilon \left\| \frac{\lambda_j}{x(0)} \right\|_{H^s} d_1(t)^{-5/2}.
\] (6.158)

By lemma 6.3 and Sobolev embedding, we have
\[
\left\| \frac{a_t}{a} \circ \kappa^{-1} \right\|_{\infty} \leq C \epsilon^2 + K_s^{-1} \epsilon \left\| \frac{\lambda_j}{x(0)} \right\|_{H^s} d_1(t)^{-5/2}.
\] (6.159)

### 6.8.4. Estimate the quantity $A$. Recall that
\[
(I - \mathcal{H})A = 1 + i[D_t \xi, \mathcal{H}] \frac{\partial a}{\xi} + i[D_t^2 \xi, \mathcal{H}] \zeta - \frac{1}{\xi} \zeta + (I - \mathcal{H}) \frac{1}{2\pi} \sum_{j=1}^{2} \frac{\lambda_j (D_t \zeta (\alpha, t) - \dot{z}_j(t))}{(\xi(\alpha, t) - z_j(t))^2}.
\] (6.160)

By lemma 2.6, lemma 6.4, lemma 6.7, we have
\[
\left\| (I - \mathcal{H})(A - 1) \right\|_{H^s} \leq \| D_t \zeta \|_{H^s} \| \xi \|_{H^s} + \| D_t^2 \xi \|_{H^s} \| \zeta \|_{H^s}
\]
\[ \leq C \epsilon^2 + K_s^{-1} \epsilon d_I(t)^{-5/2}. \]

So we have
\[ \| A - 1 \|_{H^s} \leq C \epsilon^2 + K_s^{-1} \epsilon d_I(t)^{-5/2}. \] (6.161)

**Corollary 6.6.** Assume the assumptions of Theorem 3 and assume the bootstrap assumption 6.44. For \( \epsilon \) sufficiently small, we have
\[ \inf_{\alpha \in \mathbb{R}} A(\alpha, t) \geq \frac{9}{10}, \quad \forall t \in [0, T_0]. \] (6.162)
\[ \sup_{\alpha \in \mathbb{R}} A(\alpha, t) \leq \frac{10}{9}, \quad \forall t \in [0, T_0]. \] (6.163)

6.8.5. Estimate the quantity \( D_I b \). Recall that
\[
(I - \mathcal{H}) D_I b = [D_t \xi, \mathcal{H}] \frac{\partial D_I b}{\zeta_\alpha} - [D_t^2 \xi, \mathcal{H}] \frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} - [D_t \xi, \mathcal{H}] \frac{\partial_{\alpha} D_I \bar{\zeta}}{\bar{\zeta}_\alpha} + \frac{1}{\pi} \int \left( \frac{D_t \xi(\alpha, t) - D_t \xi(\beta, t)}{\zeta(\alpha, t) - \zeta(\beta, t)} \right)^2 \bar{\zeta}_\beta(\beta, t) - 1) d\beta
\]
\[ + \frac{i}{\pi} \sum_{j=1}^{2} \frac{\lambda_j (D_I \xi - \bar{z}_j(t))}{(\zeta(\alpha, t) - \bar{z}_j(t))^2} \right) \right] \]
\[ \leq 2 \| D_t \xi \|_{H^{s/2}} \| b \|_{H^s} + C \| D_t^2 \xi \|_{H^{s/2}} \| \zeta_\alpha - 1 \|_{H^s} + C \| D_t \xi \|_{H^s}^2
\]
\[ + \sum_{j=1}^{2} \| \frac{\lambda_j (D_I \xi - \bar{z}_j(t))}{\zeta(\alpha, t) - \bar{z}_j(t))^2} \|_{H^s} \]
\[ \leq C \epsilon (C \epsilon^2 + K_s^{-1} \epsilon d_I(t)^{-5/2}) + C \epsilon^2 + K_s^{-1} \epsilon^2 d_I(t)^{-5/2} + K_s^{-1} \epsilon d_I(t)^{-5/2} + C \epsilon^3
\]
\[ \leq C \epsilon^2 + K_s^{-1} \epsilon d_I(t)^{-5/2}. \]

By lemma 6.3, we have
\[ \| D_I b \|_{H^s} \leq C \epsilon^2 + K_s^{-1} \epsilon d_I(t)^{-5/2}. \] (6.165)
6.8.6. **Estimate** $\|G\|_{H^s}$. Recall that $G = G_c + G_d$, with

$$G_c := -2[\tilde{h}, \mathcal{H}] \frac{1}{\zeta} \frac{1}{\alpha} + \frac{1}{\pi i} \int \left( \frac{D_t \tilde{\zeta}(\alpha, t) - D_t \tilde{\zeta}(\beta, t)}{\zeta(\alpha, t) - \zeta(\beta, t)} \right)^2 (\tilde{\zeta} - \bar{\tilde{\zeta}}) d\beta := G_{c1} + G_{c2}. \quad (6.166)$$

$$G_d := -2[\tilde{q}, \mathcal{H}] \frac{\partial_q \tilde{\zeta}}{\zeta} - 2[\tilde{h}, \mathcal{H}] \frac{\partial_q \bar{q}}{\zeta} - 2[\tilde{q}, \mathcal{H}] \frac{\partial_q \bar{q}}{\zeta} - 4D_t q := G_{d1} + G_{d2} + G_{d3} + G_{d4}. \quad (6.167)$$

We rewrite $G_{c1}$ as

$$G_{c1} = -\frac{4}{\pi} \int \left( \frac{D_t \tilde{\zeta}(\alpha, t) - D_t \tilde{\zeta}(\beta, t)}{\zeta(\alpha, t) - \zeta(\beta, t)} \right)^2 \partial_\beta \tilde{\zeta}(\beta, t) d\beta. \quad (6.168)$$

By lemma 2.6, we have

$$\|G_{c1}\|_{H^s} \leq C \|\tilde{\zeta}\|_{H^s} \|\zeta\| - 1 \|H^s\| \|\tilde{\zeta}\|_{H^s} \leq C \epsilon^3, \quad (6.169)$$

for some constant $C$ depends on $s$ only. Similarly,

$$\|G_{c2}\|_{H^s} \leq C \|D_t \zeta\|_{H^s} \|\zeta\| - 1 \|H^s\| \leq C \epsilon^3. \quad (6.170)$$

By lemma 2.6, we have

$$\|G_{d1}\|_{H^s} + \|G_{d2}\|_{H^s} \leq C \|q\|_{H^s} \|\tilde{\zeta}\|_{H^s} \leq K_s^{-1} \epsilon^2 d_I(t)^{-3/2}. \quad (6.171)$$

Similarly,

$$\|G_{d3}\|_{H^s} \leq C \|q\|_{H^s}^2 \leq K_s^{-1} \epsilon^2 d_I(t)^{-3/2}. \quad (6.172)$$

Use

$$D_t \tilde{q} = \sum_{j=1}^{2} \frac{\lambda_j t}{2\pi} \frac{D_t \zeta - \bar{\zeta}_j}{(\zeta(\alpha, t) - \bar{\zeta}_j(t))^2}, \quad (6.173)$$

by lemma 6.7, lemma 6.4, we have

$$\|G_{d4}\|_{H^s} \leq 4 \|D_t \zeta\|_{H^s} \left\| \sum_{j=1}^{2} \frac{\lambda_j}{2\pi} \frac{1}{(\zeta(\alpha, t) - \bar{\zeta}_j(t))^2} \right\|_{H^s} + 4 \left\| \sum_{j=1}^{2} \frac{\lambda_j \bar{\zeta}_j}{2\pi} \frac{1}{(\zeta(\alpha, t) - \bar{\zeta}_j(t))^2} \right\|_{H^s} \leq C \epsilon^2 d_I(t)^{-5/2} + K_s^{-1} \epsilon \frac{|\lambda|}{x(0)} d_I(t)^{-5/2}. \quad (6.174)$$

So we obtain

$$\|G\|_{H^s} \leq C \epsilon^3 + K_s^{-1} \epsilon^2 d_I(t)^{-3/2} + K_s^{-1} \epsilon \frac{|\lambda|}{x(0)} d_I(t)^{-5/2}. \quad (6.175)$$

As a consequence,

$$\|(I - \mathcal{H})G\|_{H^s} \leq 3 \|G\|_{H^s} \leq C \epsilon^3 + K_s^{-1} \epsilon^2 d_I(t)^{-3/2} + K_s^{-1} \epsilon \frac{|\lambda|}{x(0)} d_I(t)^{-5/2}. \quad (6.176)$$
6.8.7. Estimate $\|(I - \mathcal{H})[D_t^2 - i A \partial_\alpha, \partial_\alpha^k]\tilde{\theta}\|_{L^2}$. By lemma 2.11, we have

$$[D_t^2, \partial_\alpha^k]\tilde{\theta} = -\sum_{m=0}^{k-1} \left[ \partial_\alpha^m(D_t b_\alpha)\partial_\alpha^{k-m} \tilde{\theta} + \partial_\alpha^m(b_\alpha \partial_\alpha^{k-m} D_t \tilde{\theta}) + \partial_\alpha^m(b_\alpha[b_\alpha, \partial_\alpha^{k-m}] \tilde{\theta}) + \partial_\alpha b_\alpha \partial_\alpha^{k-m} D_t \tilde{\theta} + \partial_\alpha b_\alpha[b_\alpha, \partial_\alpha^{k-m}] \tilde{\theta} \right].$$

The quantity $\|\partial_\alpha^m(D_t b_\alpha)\partial_\alpha^{k-m} \tilde{\theta}\|_{L^2}$. For $0 \leq m \leq k - 1, k \leq s$, we have

$$\|\partial_\alpha^m(D_t b_\alpha)\partial_\alpha^{k-m} \tilde{\theta}\|_{L^2} \leq \|D_t b_\alpha\|_{H^m} \|\partial_\alpha^{k-m} \tilde{\theta}\|_{H^m}.$$  \hspace{1cm} (6.175)

Since $D_t b_\alpha = \partial_\alpha D_t b + b b_\alpha$, we have

$$\|D_t b_\alpha\|_{H^m} \leq \|\partial_\alpha D_t b\|_{H^m} + \|b b_\alpha\|_{H^m} \leq \|D_t b_\alpha\|_{H^s} + \|b\|^2_{H^s} \leq C \epsilon^2 + K_s^{-1} \epsilon d_1(t) t^{-5/2} + (C \epsilon^2 + K_s^{-1} \epsilon d_1(t) t^{-3/2})^2 \leq C \epsilon^2 + K_s^{-1} \epsilon d_1(t) t^{-5/2}$$

and since $k - m \geq 1$, by Corollary 6.4, we have

$$\|\partial_\alpha^{k-m}\tilde{\theta}\|_{H^s} \leq 11 \epsilon,$$  \hspace{1cm} (6.176)

we obtain

$$\|\partial_\alpha^m(D_t b_\alpha)\partial_\alpha^{k-m} \tilde{\theta}\|_{L^2} \leq \|D_t b_\alpha\|_{H^s} \|\partial_\alpha^{k-m} \tilde{\theta}\|_{H^s} \leq C \epsilon^3 + K_s^{-1} \epsilon^2 d_1(t) t^{-3/2}.$$  \hspace{1cm} (6.177)

The quantity $\partial_\alpha^m(b_\alpha \partial_\alpha^{k-m} D_t \tilde{\theta})$. Similar to the previous case, we have for $0 \leq m \leq k - 1, k \leq s$, and assume bootstrap assumption (6.44),

$$\|\partial_\alpha^m(b_\alpha \partial_\alpha^{k-m} D_t \tilde{\theta})\|_{L^2} \leq \|b_\alpha\|_{H^{k-1}} \|D_t \tilde{\theta}\|_{H^k} \leq C \epsilon^3 + K_s^{-1} \epsilon^2 d_1(t) t^{-3/2}.$$  \hspace{1cm} (6.178)

The quantity $\partial_\alpha^m(b_\alpha[b_\alpha, \partial_\alpha^{k-m}] \tilde{\theta})$. We have for $0 \leq m \leq k - 1, k \leq s$, and assume bootstrap assumption (6.44),

$$\|\partial_\alpha^m(b_\alpha[b_\alpha, \partial_\alpha^{k-m}] \tilde{\theta})\|_{L^2} \leq C \|b_\alpha\|_{H^{k-1}} \|\partial_\alpha^{k-m}\tilde{\theta}\|_{H^1} \leq C \epsilon^3.$$  \hspace{1cm} (6.179)

The quantity $\partial_\alpha^m b_\alpha \partial_\alpha^{k-m} D_t \tilde{\theta}$. We have for $0 \leq m \leq k - 1, k \leq s$,

$$\|\partial_\alpha^m b_\alpha \partial_\alpha^{k-m} D_t \tilde{\theta}\|_{L^2} \leq C \|b_\alpha\|_{H^{k-1}} \|D_t \tilde{\theta}\|_{H^s} \leq C \epsilon^3 + K_s^{-1} \epsilon^2 d_1(t) t^{-3/2}.$$  \hspace{1cm} (6.180)

The quantity $\partial_\alpha^m b_\alpha[b_\alpha, \partial_\alpha] \partial_\alpha^{k-m-1} \tilde{\theta}$. We have for $0 \leq m \leq k - 1, k \leq s$,

$$\|\partial_\alpha^m b_\alpha[b_\alpha, \partial_\alpha] \partial_\alpha^{k-m-1} \tilde{\theta}\|_{L^2} \leq C \|b\|^2_{H^s} \|D_t \tilde{\theta}\|_{H^s} \leq C \epsilon^3.$$  \hspace{1cm} (6.181)

So we obtain

$$\|[D_t^2, \partial_\alpha^k]\tilde{\theta}\|_{L^2} \leq C \epsilon^3 + K_s^{-1} C \epsilon^2 d_1(t) t^{-3/2}.$$  \hspace{1cm} (6.182)

The quantity $\|[i A \partial_\alpha, \partial_\alpha^k]\tilde{\theta}\|_{L^2}$ Use similar argument, we obtain

$$\|[i A \partial_\alpha, \partial_\alpha^k]\tilde{\theta}\|_{L^2} \leq C \epsilon^3 + K_s^{-1} C \epsilon^2 d_1(t) t^{-3/2}.$$  \hspace{1cm} (6.183)

So we obtain

$$\|((I - \mathcal{H})[D_t^2 - i A \partial_\alpha, \partial_\alpha^k]\tilde{\theta})\|_{L^2} \leq C \epsilon^3 + K_s^{-1} C \epsilon^2 d_1(t) t^{-3/2}.$$  \hspace{1cm} (6.184)
Estimate $\|D_t^2 - iA\partial_\alpha, \mathcal{H}\partial^k_\alpha\hat{\theta}\|_{L^2}$ Note that by identity (2.41),

$$
[D_t^2 - iA\partial_\alpha, \mathcal{H}]\partial^k_\alpha\hat{\theta} = 2[D_t, \mathcal{H}]\partial_\alpha\frac{\partial^k_\alpha}{\xi_\alpha} - \frac{1}{\pi i} \int \left( \frac{\xi(\alpha, t) - \xi(\beta, t)}{\xi(\alpha, t) - \xi(\beta, t)} \right)^2 \partial_\beta\partial^k_\beta\hat{\theta} d\beta
$$

(6.185)

Clearly, for $k \leq s$, and assume (6.44), we have

$$
\|\left[ D_t, \mathcal{H} \right] \frac{\partial_\alpha\partial^k_\alpha}{\xi_\alpha} \|_{L^2} \leq C \epsilon^3.
$$

(6.186)

Estimate $\left\| D_t, \mathcal{H} \right\|_{L^2}$

$[D_t, \mathcal{H}]\frac{\partial_\alpha\partial^k_\alpha}{\xi_\alpha}$ is not obvious cubic. However, since $\partial^k_\alpha\hat{\theta}$ is almost anti-holomorphic, and $D_t \xi = \tilde{\zeta} + \tilde{q}$, with $\tilde{\zeta}$ anti-holomorphic and $\tilde{q}$ decays rapidly in time as long as the point vortices move away from the free interface rapidly, we expect this quantity consists of cubic terms and quadratic terms which decay rapidly. To see this, decompose

$$
\partial^k_\alpha\hat{\theta} := \frac{1}{2} (I - \mathcal{H})\partial^k_\alpha\hat{\theta} + \frac{1}{2} (I + \mathcal{H})\partial^k_\alpha\hat{\theta}.
$$

Note that for $k \geq 1$,

$$
(I + \mathcal{H})\partial^k_\alpha\hat{\theta} = (I + \mathcal{H})\partial^k_\alpha(I - \mathcal{H})(\zeta - \tilde{\zeta}) = -[\partial^k_\alpha, \mathcal{H}]\tilde{\theta}
$$

$$
= -\sum_{m=0}^{k-1} \partial^m_\alpha[\xi_\alpha - 1, \mathcal{H}]\frac{\partial_\alpha\partial^{k-m-1}_\alpha\hat{\theta}}{\xi_\alpha}.
$$

By lemma 2.6 and lemma 6.10,

$$
\| (I + \mathcal{H})\partial^k_\alpha\hat{\theta} \|_{L^2} \leq C \| \xi_\alpha - 1 \|_{H^k} \| \partial_\alpha\hat{\theta} \|_{H^{k-1}} \leq C \epsilon^2.
$$

(6.187)

Therefore, by lemma 2.6, we have

$$
\left\| [D_t, \mathcal{H}]\frac{\partial_\alpha\frac{1}{2}(I + \mathcal{H})\partial^k_\alpha\hat{\theta}}{\xi_\alpha} \right\|_{L^2} \leq C \epsilon^3.
$$

(6.188)

We rewrite $[D_t, \mathcal{H}]\frac{\partial_\alpha\frac{1}{2}(I - \mathcal{H})\partial^k_\alpha\hat{\theta}}{\xi_\alpha}$ as

$$
[D_t\xi, \mathcal{H}]\frac{\partial_\alpha\frac{1}{2}(I - \mathcal{H})\partial^k_\alpha\hat{\theta}}{\xi_\alpha} = \frac{1}{2} (I + \mathcal{H})D_t\xi, \mathcal{H}]\frac{\partial_\alpha\frac{1}{2}(I - \mathcal{H})\partial^k_\alpha\hat{\theta}}{\xi_\alpha} + \frac{1}{2} (I - \mathcal{H})D_t\xi, \mathcal{H}]\frac{\partial_\alpha\frac{1}{2}(I - \mathcal{H})\partial^k_\alpha\hat{\theta}}{\xi_\alpha} := I + II.
$$

Clearly, $II = 0$. Since

$$
\frac{1}{2} (I + \mathcal{H})D_t\xi = \frac{1}{2} (I + \mathcal{H})\tilde{q} + \frac{1}{2} (\mathcal{H} + \tilde{\mathcal{H}})\tilde{\zeta},
$$

(6.189)
Use lemma 2.6, lemma 6.4, and similar to the estimate of $\|G_{d1}\|_{H^s}$ in §6.8.6, we have

$$\left\| \frac{1}{2}(I + \mathcal{H})\bar{q}, \mathcal{H} \right\|_{L^2} \xrightarrow{\xi} C \left\| q \right\|_{H^k} \left\| \partial_{\alpha} \theta \right\|_{H^{k-1}} \leq K_s^{-1} \epsilon^2 d_I(t)^{-3/2}. \tag{6.190}$$

It’s easy to obtain

$$\left\| (\mathcal{H} + \bar{\mathcal{H}})\bar{\mathcal{G}} \right\|_{H^s} \leq C \epsilon^2. \tag{6.191}$$

So we obtain

$$\left\| [(\mathcal{H} + \bar{\mathcal{H}})\bar{\mathcal{G}}, \mathcal{H}] \frac{\partial_{\alpha} \frac{1}{2}(I - \mathcal{H})\bar{\theta}}{\xi} \right\|_{H^s} \leq C \epsilon^3. \tag{6.192}$$

Therefore,

$$\left\| (I - \mathcal{H})[D_t^2 - iA\partial_{\alpha}, \mathcal{H}]\frac{\partial_{\alpha} \frac{1}{2}(I - \mathcal{H})\bar{\theta}}{\xi} \right\|_{L^2} \leq 3 \left\| D_t^2 - iA\partial_{\alpha}, \mathcal{H} \right\|_{L^2} \leq C \epsilon^3 + K_s^{-1} \epsilon^2 d_I(t)^{-3/2}. \tag{6.193}$$

6.8.9. Estimate for $\|G_k^\theta\|_{L^2}$. Collect the estimates from (6.174), (6.184), (6.193), we obtain

$$\|G_k^\theta\|_{L^2} \leq C \epsilon^3 + K_s^{-1} \epsilon^2 d_I(t)^{-5/2} + K_s^{-1} \epsilon \frac{\|\lambda\|}{x(0)} d_I(t)^{-5/2}. \tag{6.194}$$

6.9. Estimate $\left\| (I - \mathcal{H})\frac{\partial_{\alpha} \bar{G}}{\xi} \right\|_{L^2}$. Recall that

$$\bar{G} = (I - \mathcal{H})(D_t G + \frac{a}{A} \circ \kappa^{-1} A((I - \mathcal{H})(\xi - \bar{\xi}))_{\alpha}) - 2[D_t \xi, \mathcal{H}] \frac{\partial_{\alpha} D_t^2 (I - \mathcal{H})(\xi - \bar{\xi})}{\xi} + \frac{1}{\pi i} \int \frac{D_t \xi(\alpha, t) - D_t \xi(\beta, t)}{\xi(\alpha, t) - \xi(\beta, t)}^2 (D_t (I - \mathcal{H})(\xi - \bar{\xi}))_{\beta} d\beta. \tag{6.195}$$

6.9.1. Estimate $\|D_t G\|_{H^k}$ $D_t G$ is given by

$$D_t G = (\partial_t g) \circ \kappa^{-1}.$$

$g = g_c + g_d$, and

$$\partial_t g_c = \partial_t \left\{ -2[f_{\beta}, \bar{\delta}] \frac{1}{\bar{z}_a} + \bar{\delta} \partial_{\alpha} \frac{1}{\bar{z}_a} \right\} + \frac{1}{\pi i} \int \left( \frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)} \right)^2 (z - \bar{z})_{\beta} d\beta$$

$$= -2[f_{\beta}, \bar{\delta}] \frac{1}{\bar{z}_a} + \bar{\delta} \partial_{\alpha} \frac{1}{\bar{z}_a} - 2[f_{\beta}, \bar{\delta}] \frac{1}{\bar{z}_a} + \bar{\delta} \partial_{\alpha} \frac{1}{\bar{z}_a}.$$
So we have

\[ D_t G_c = -2[D_t \tilde{\mathcal{S}}, \mathcal{H}] + \left( \frac{1}{\zeta} \right) \partial \frac{\partial \tilde{\mathcal{S}}}{\zeta} - 2[D_t \tilde{\mathcal{S}}, \mathcal{H}] + \left( \frac{1}{\zeta} \right) \partial \partial \tilde{\mathcal{S}} \]

Recall that

\[ g_d := -2[\tilde{p}, \tilde{\mathcal{S}}] \frac{\partial}{\zeta} \tilde{f} - 2[\tilde{f}, \tilde{\mathcal{S}}] \frac{\partial}{\zeta} \tilde{p} - 2[\tilde{p}, \tilde{\mathcal{S}}] \frac{\partial}{\zeta} \tilde{p} - 4p_t. \] (6.196)

So

\[ \partial_t g_d = -2[\tilde{p}, \tilde{\mathcal{S}}] \frac{\partial}{\zeta} \tilde{f} - 2[\tilde{f}, \tilde{\mathcal{S}}] \frac{\partial}{\zeta} \tilde{p} - 2[\tilde{p}, \tilde{\mathcal{S}}] \frac{\partial}{\zeta} \tilde{p} - 4p_t. \]

So we have

\[ D_t G_d = -2[\tilde{q}, \mathcal{H}] \frac{\partial}{\zeta} \tilde{\mathcal{S}} - \frac{2}{\zeta} \int \left( \tilde{q}(\alpha, t) - \tilde{q}(\beta, t) \right) \partial \tilde{f}(\beta, t) d\beta \]

\[ - 2[\tilde{q}, \mathcal{H}] \frac{\partial}{\zeta} \tilde{\mathcal{S}} \]

\[ - 2[\tilde{q}, \mathcal{H}] \frac{\partial}{\zeta} \tilde{\mathcal{S}} \]

\[ - 2[\tilde{q}, \mathcal{H}] \frac{\partial}{\zeta} \tilde{\mathcal{S}} \]

\[ - 4D_t^2 q. \] (6.197)

\[ \|D_t G_c\|_{H^k} \leq C_s(\|D_t \tilde{\mathcal{S}}\|_{H^k}\|\zeta\|_{H^k} - 1\|\zeta\|_{H^k} + \|\tilde{\mathcal{S}}\|_{H^k}^2 \|D_t \zeta\|_{H^k}. \]
\[
\sum_{j=1}^{2} \frac{\lambda_{j}^{i}}{2\pi} \frac{D_{t}^{2} \zeta - \ddot{z}_{j}(t)}{\zeta(\alpha, t) - z_{j}(t)}^{2} - \sum_{j=1}^{2} \frac{\lambda_{j}^{i} (D_{t} \zeta)^{2} - 2D_{t} \zeta \dot{z}_{j}}{\zeta(\alpha, t) - z_{j}(t)}^{2} - \sum_{j=1}^{2} \frac{\lambda_{j}^{i} \ddot{z}_{j}^{2}}{\zeta(\alpha, t) - z_{j}(t)}^{2}.
\]

Use lemma 6.7, lemma 6.8, lemma 6.9, we have

\[
\left\|4D_{t}^{2}q\right\|_{H^{k}} \leq C \varepsilon^{3} + K_{s}^{-1} \varepsilon^{2} d_{I}(t)^{-3/2} + K_{s}^{-1} \varepsilon \frac{|\lambda|}{x(0)} d_{I}(t)^{-3/2}.
\]

(6.201)

Then we have

\[
\left\|D_{t}G\right\|_{H^{k}} \leq C \varepsilon^{3} + K_{s}^{-1} \varepsilon^{2} d_{I}(t)^{-3/2} + K_{s}^{-1} \varepsilon \frac{|\lambda|}{x(0)} d_{I}(t)^{-3/2}.
\]

(6.202)

Therefore,

\[
\left\|(I - \mathcal{H})D_{t}G\right\|_{H^{k}} \leq C \varepsilon^{3} + K_{s}^{-1} \varepsilon^{2} d_{I}(t)^{-3/2} + K_{s}^{-1} \varepsilon \frac{|\lambda|}{x(0)} d_{I}(t)^{-3/2}.
\]

(6.203)

6.9.2. Estimate \[
\left\|2[D_{t} \zeta, \mathcal{H}] \frac{\partial_{\alpha} D_{t}^{2}(I - \mathcal{H})(\zeta - \tilde{\zeta})}{\zeta_{\alpha}}\right\|_{H^{k}}
\]

The way that we estimate for this quantity is the same as that for \([D_{t} \zeta, \mathcal{H}] \frac{\partial_{\alpha} \partial_{k} \partial_{\bar{\alpha}}}{\zeta_{\alpha}}\). We obtain

\[
\left\|2[D_{t} \zeta, \mathcal{H}] \frac{\partial_{\alpha} D_{t}^{2}(I - \mathcal{H})(\zeta - \tilde{\zeta})}{\zeta_{\alpha}}\right\|_{H^{k}} \leq C \varepsilon^{3} + K_{s}^{-1} \varepsilon^{2} d_{I}(t)^{-3/2}.
\]

(6.204)

So we obtain

\[
\left\|(I - \mathcal{H}) \partial_{\alpha} \frac{\partial_{k} \tilde{G}}{\zeta_{\alpha}}\right\|_{L^{2}} \leq C \varepsilon^{3} + K_{s}^{-1} \varepsilon^{2} d_{I}(t)^{-3/2} + K_{s}^{-1} \varepsilon \frac{|\lambda|}{x(0)} d_{I}(t)^{-3/2}.
\]

(6.205)
6.9.3. Estimate $[D_t^2 - iA \partial_\alpha, \mathcal{H}] \partial^k_\alpha \tilde{\sigma}$

Use

$$[D_t^2 - iA \partial_\alpha, \mathcal{H}] \partial^k_\alpha \tilde{\sigma} = 2[D_t \xi, \mathcal{H}] \frac{\partial_\alpha D_t \partial^k_\alpha \tilde{\sigma}}{\xi_\alpha} - \frac{1}{\pi i} \int \left( \frac{D_t \xi(\alpha, t) - D_t \xi(\beta, t)}{\xi(\alpha, t) - \xi(\beta, t)} \right)^2 \partial^{k+1}_\beta \tilde{\sigma}(\beta, t) d\beta$$

$$:= I_1 + I_2.$$

Clearly,

$$\|I_2\|_{L^2} \leq C \|D_t \xi\|_{H^k} \|\tilde{\sigma}\|_{H^k} \leq C \epsilon^3. \quad (6.206)$$

Note that

$$[D_t \xi, \mathcal{H}] \frac{\partial_\alpha D_t \partial^k_\alpha \tilde{\sigma}}{\xi_\alpha} = [D_t \xi, \mathcal{H}] \frac{\partial_\alpha \partial^k_\alpha D_t \tilde{\sigma}}{\xi_\alpha} + [D_t \xi, \mathcal{H}] \frac{\partial_\alpha [D_t, \partial^k_\alpha \tilde{\sigma}]}{\xi_\alpha}.$$

The second term $[D_t \xi, \mathcal{H}] \frac{\partial_\alpha [D_t, \partial^k_\alpha \tilde{\sigma}]}{\xi_\alpha}$ is cubic, it’s easy to obtain

$$\left\| [D_t \xi, \mathcal{H}] \frac{\partial_\alpha [D_t, \partial^k_\alpha \tilde{\sigma}]}{\xi_\alpha} \right\|_{L^2} \leq C \epsilon^3. \quad (6.207)$$

The way that we estimate for this quantity is the same as that for $[D_t \xi, \mathcal{H}] \frac{\partial_\alpha \partial^k_\alpha \tilde{\sigma}}{\xi_\alpha}$. We obtain

$$\left\| [D_t \xi, \mathcal{H}] \frac{\partial_\alpha \partial^k_\alpha D_t \tilde{\sigma}}{\xi_\alpha} \right\|_{H^k} \leq C \epsilon^3 + K_s^{-1} \epsilon^2 d_I(t)^{-3/2}. \quad (6.208)$$

So we obtain

$$\left\| [D_t^2 - iA \partial_\alpha, \mathcal{H}] \partial^k_\alpha \tilde{\sigma} \right\|_{L^2} \leq C \epsilon^3 + K_s^{-1} \epsilon^2 d_I(t)^{-3/2}. \quad (6.209)$$

6.9.4. Estimate $\left\| (I - \mathcal{H})[D_t^2 - iA \partial_\alpha, \partial^k_\alpha \tilde{\sigma}] \right\|_{L^2}$

The way that we estimate this quantity is the same as that for $\left\| (I - \mathcal{H})[D_t^2 - iA \partial_\alpha, \partial^k_\alpha \tilde{\theta}] \right\|_{L^2}$. We obtain

$$\left\| (I - \mathcal{H})[D_t^2 - iA \partial_\alpha, \partial^k_\alpha \tilde{\sigma}] \right\|_{L^2} \leq C \epsilon^3 + K_s^{-1} \epsilon^2 d_I(t)^{-3/2}. \quad (6.210)$$

6.9.5. Estimate for $\left\| G_k^\sigma \right\|_{L^2}$

Collect the estimates from (6.203), (6.205), (6.209), (6.210), we obtain

$$\left\| G_k^\sigma \right\|_{L^2} \leq C \epsilon^3 + K_s^{-1} \epsilon^2 d_I(t)^{-3/2} + K_s^{-1} \epsilon \frac{|\lambda|}{\chi(0)} d_I(t)^{-3/2}. \quad (6.211)$$
6.10. Energy estimates. We derive energy estimates in this subsection. We’ll prove the following.

**Proposition 6.** Assume the assumptions of Theorem 3, assume the bootstrap assumption (6.44), we have for all $t \in [0, T_0]$,

$$
\frac{d}{dt} \mathcal{E}_s(t) \leq C \epsilon^4 + K_s^{-1} \epsilon^3 d_I(t)^{-3/2} + K_s^{-1} \epsilon^2 \frac{t}{x(0)} d_I(t)^{-5/2}.
$$

(6.212)

**Proof.** From (6.40) and (6.42), we have

$$
\frac{d}{dt} \mathcal{E}_s(t) = \frac{d}{dt} \sum_{k=0}^{s} \left( E_k^\theta + E_k^\sigma \right) = \sum_{k=0}^{s} \left( \int \frac{2}{A} \text{Re} D_t \theta_k \overline{G_k^\theta} - \int \frac{1}{A} \frac{a_t}{a} \kappa^{-1} |D_t \theta_k|^2 \right.
$$

$$
+ \int \frac{2}{A} \text{Re} D_t \sigma_k \overline{G_k^\sigma} - \int \frac{1}{A} \frac{a_t}{a} \kappa^{-1} |D_t \sigma_k|^2 \right).
$$

By Corollary 6.5, we have

$$
\|D_t \bar{\theta}\|_{H^s} \leq 11 \epsilon, \quad \|D_t \bar{\sigma}\|_{H^s} \leq 21 \epsilon.
$$

(6.214)

By Corollary 6.6, (6.159), (6.194), (6.211), we have

$$
\frac{d}{dt} \mathcal{E}_s(t) \leq \sum_{k=0}^{s} \left( 2 \left\| \frac{1}{A} \right\|_{L^\infty} \|D_t \theta_k\|_{L^2} \|G_k^\theta\|_{L^2} + \left\| \frac{1}{A} \right\|_{L^\infty} \frac{a_t}{a} \kappa^{-1} \right\| D_t \theta_k \right\|_{L^2}^2
$$

$$
+ 2 \left\| \frac{1}{A} \right\|_{L^\infty} \|D_t \sigma_k\|_{L^2} \|G_k^\sigma\|_{L^2} + \left\| \frac{1}{A} \right\|_{L^\infty} \frac{a_t}{a} \kappa^{-1} \right\| D_t \sigma_k \right\|_{L^2}^2 \right)
$$

$$
\leq \sum_{k=0}^{s} \left( 4 \times 11 \epsilon \left( C \epsilon^3 + K_s^{-1} \epsilon^2 d_I(t)^{-3/2} + K_s^{-1} \epsilon \frac{t}{x(0)} d_I(t)^{-5/2} \right) + 4 \times (C \epsilon^2 + K_s^{-1} \epsilon \frac{t}{x(0)} d_I(t)^{-5/2}) \times (11 \epsilon)^2
$$

$$
+ 4 \times 21 \epsilon \left( C \epsilon^3 + K_s^{-1} \epsilon^2 d_I(t)^{-3/2} + K_s^{-1} \epsilon \frac{t}{x(0)} d_I(t)^{-5/2} \right) + 4 \times (C \epsilon^2 + K_s^{-1} \epsilon \frac{t}{x(0)} d_I(t)^{-5/2}) \times (21 \epsilon)^2 \right)
$$

$$
\leq C \epsilon^4 + K_s^{-1} \epsilon^3 d_I(t)^{-3/2} + K_s^{-1} \epsilon^2 \frac{t}{x(0)} d_I(t)^{-5/2}.
$$

Here, we simply bound $\frac{1}{A}$ by $\frac{1}{2}$. □

Before we use the bootstrap argument to complete the proof of Theorem 3, we need to show that the energy $\mathcal{E}_s$ is equivalent to

$$
4 \left( \|D_t \bar{\theta}\|_{H^s}^2 + \|D_t \bar{\sigma}\|_{H^s}^2 + \|\Lambda^{1/2} \bar{\theta}\|_{H^s}^2 + \|\Lambda^{1/2} \bar{\sigma}\|_{H^s}^2 \right) \leq C \epsilon^4.
$$

(6.215)

**Lemma 6.12.** Assume the assumptions of Theorem 3, and assume the bootstrap assumption (6.44). Then we have

$$
\mathcal{E}_s - 4 \left( \|D_t \bar{\theta}\|_{H^s}^2 + \|D_t \bar{\sigma}\|_{H^s}^2 + \|\Lambda^{1/2} \bar{\theta}\|_{H^s}^2 + \|\Lambda^{1/2} \bar{\sigma}\|_{H^s}^2 \right) \leq C \epsilon^3.
$$
Proof. Recall that

\[ \mathcal{E}_s = \sum_{k=0}^s \left\{ \int \frac{1}{A} |D_t \theta_k|^2 + i \theta_k \bar{\theta}_k d\alpha + \int \frac{1}{A} |D_t \sigma_k|^2 + i \sigma_k \bar{\sigma}_k d\alpha \right\}, \]

where

\[ \theta_k = (I - \mathcal{H}) \partial^k \tilde{\theta}, \quad \sigma_k = (I - \mathcal{H}) \partial^k \tilde{\sigma}, \quad \tilde{\theta} := (I - \mathcal{H})(\zeta - \bar{\zeta}), \quad \tilde{\sigma} := (I - \mathcal{H})D_t \tilde{\theta}. \]

(6.216)

It’s easy to obtain that

\[ \| A - 1 \|_{H^s} \leq C \epsilon. \]

(6.217)

So we have

\[ \mathcal{E}_s = \sum_{k=0}^s \left\{ \int |D_t \theta_k|^2 + i \theta_k \bar{\theta}_k d\alpha + \int |D_t \sigma_k|^2 + i \sigma_k \bar{\sigma}_k d\alpha \right\} + O(\epsilon^3). \]

(6.218)

We have

\[ \theta_k = \partial^k (I - \mathcal{H})\tilde{\theta} + [\partial^k, \mathcal{H}]\tilde{\theta} = 2\partial^k \tilde{\theta} + [\partial^k, \mathcal{H}]\tilde{\theta}. \]

(6.219)

Similarly, we have

\[ \| D_t \sigma_k - 2\partial^k D_t \tilde{\theta} \|_{L^2} \leq C \epsilon^2. \]

(6.220)

Therefore,

\[ \| D_t \sigma_k - 2\partial^k D_t \tilde{\sigma} \|_{L^2} \leq C \epsilon^2. \]

(6.221)

Decompose \( \tilde{\theta} \) as

\[ \tilde{\theta} = \frac{1}{2}(I + \mathbb{H})\tilde{\theta} + \frac{1}{2}(I - \mathbb{H})\tilde{\theta}. \]

(6.222)

Note that since \( \tilde{\theta} = (I - \mathcal{H})(\zeta - \bar{\zeta}) \), it’s easy to obtain

\[ \| \Lambda^{1/2} \frac{1}{2}(I + \mathbb{H})\tilde{\theta} \|_{H^s} \leq C \epsilon^2. \]

(6.223)

Then we have

\[ \| \Lambda^{1/2} \tilde{\theta} \|_{H^s}^2 - \| \Lambda^{1/2} \frac{1}{2}(I - \mathbb{H})\tilde{\theta} \|_{H^s}^2 \leq C \epsilon^3. \]

(6.224)
Note that
\[ \left\| \Lambda^{1/2} (I - \mathcal{H}) \tilde{\theta} \right\|_{H^s}^2 = i \sum_{k=0}^{s} \int \tilde{\theta} D_t^k (I - \mathcal{H}) \tilde{\theta} D_t^{k+1} (I - \mathcal{H}) \tilde{\theta} d\alpha. \]  
(6.227)

Use the fact that
\[ (I - \mathcal{H}) \tilde{\theta} = 2 \tilde{\theta} + (\mathcal{H} - \mathcal{H}) \tilde{\theta}, \]  
(6.228)
and use
\[ \left\| \partial^k \Lambda^{1/2} (\mathcal{H} - \mathcal{H}) \tilde{\theta} \right\|_{L^2} \leq C \epsilon^2, \]  
(6.229)
we obtain
\[ \left| \int i \theta \partial^k \tilde{\theta} d\alpha - 4 \left\| \partial^k \Lambda^{1/2} \tilde{\theta} \right\|_{L^2} \right| \leq C \epsilon^3. \]  
(6.230)

Similarly,
\[ \left| \int i \sigma \partial^k \sigma d\alpha - 4 \left\| \partial^k \Lambda^{1/2} \tilde{\sigma} \right\|_{L^2} \right| \leq C \epsilon^3. \]  
(6.231)

By (6.221), (6.222), (6.230), and (6.231), we obtain
\[ \left| \mathcal{E}_s - 4 \sum_{k=0}^{s} \left\{ \left| \tilde{\theta} D_t^k \tilde{\theta} \right|_{L^2}^2 + \left| \partial^k D_t \tilde{\sigma} \right|_{L^2}^2 + \left| \partial^k \Lambda^{1/2} \tilde{\theta} \right|_{L^2}^2 + \left| \partial^k \Lambda^{1/2} \tilde{\sigma} \right|_{L^2}^2 \right\} \right| \leq C \epsilon^3. \]  
(6.232)

\[ \boxdot \]

\textbf{Corollary 6.7.} Assume the assumptions of Theorem 3, then
\[ \mathcal{E}_s (0) \leq 17 \epsilon^2. \]  
(6.233)

\textbf{Proposition 7.} Assume the assumptions of Theorem 3, there exists \( \delta > 0 \) such that
\[ \| \xi - 1 \|_{H^s} \leq 5 \epsilon, \quad \| \tilde{\sigma} \|_{H^{s+1/2}} \leq 5 \epsilon, \quad \| D_t \tilde{\sigma} \|_{H^s} \leq 5 \epsilon \quad t \in [0, \delta \epsilon^{-2}]. \]  
(6.234)

Indeed, we can choose \( \delta \) to be an absolute constant.

\textit{Proof.} Let \( \delta > 0 \) to be determined. Let
\[ T := \left\{ T \in [0, \delta \epsilon^{-2}] : \| \xi - 1 \|_{H^s} \leq 5 \epsilon, \quad \| \tilde{\sigma} \|_{H^{s+1/2}} \leq 5 \epsilon, \quad \| D_t \tilde{\sigma} \|_{H^s} \leq 5 \epsilon, \quad \forall t \in [0, T] \right\}. \]  
(6.235)

At \( t = 0 \), we have
\[ \| \tilde{\sigma} \|_{H^{s+1/2}} + \| D_t \tilde{\sigma} \|_{H^s} \leq \frac{3}{2} \epsilon. \]  
(6.236)
To obtain estimate of \( \| \varepsilon - 1 \|_{H^s} \), use \( D_t^2 \varepsilon - i \varepsilon = -i \), we have
\[
\varepsilon - 1 = \frac{D_t^2 \varepsilon - i(A - 1)}{iA}.
\] (6.237)

We have \( D_t^2 \varepsilon = D_t \tilde{\varepsilon} + D_t \tilde{q} \), and
\[
D_t q = \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{D_t \varepsilon - \dot{z}_j}{(\varepsilon(\alpha, t) - z_j(t))^2}.
\]

We have
\[
\| D_t q \|_{H^s} \leq C \epsilon^2 + K_s^{-1} \epsilon. \tag{6.238}
\]

Use (6.161), we obtain
\[
\| \varepsilon - 1 \|_{H^s} \leq \| D_t \tilde{\varepsilon} \|_{H^s} + C \epsilon^2 + K_s^{-1} \epsilon \leq 2 \epsilon. \tag{6.239}
\]

Therefore, \( 0 \in T \), so \( T \neq \emptyset \). Since \( \| \varepsilon - 1 \|_{H^s}, \| \tilde{\varepsilon} \|_{H^{s+1/2}}, \| D_t \tilde{\varepsilon} \|_{H^s} \) are continuous in \( t \), we have \( T \) is closed. To prove \( T = [0, \delta \epsilon^{-2}] \), it suffices to prove that if \( T_0 = \delta \epsilon^{-2} \), then there exists \( c > 0 \) such that \([0, T_0 + c] \subset T \).

Assume \( T_0 \in T \) and assume \( T_0 < \delta \epsilon^{-2} \). By Proposition 6, we have for any \( t_0 \leq T_0 \),
\[
\mathcal{E}_s(t_0) = \mathcal{E}_s(0) + \int_0^{t_0} \frac{d}{dt} \mathcal{E}_s(t) dt
\leq 17 \epsilon^2 + \int_0^{t_0} (C \epsilon^4 + K_s^{-1} \epsilon^3 d_1(t) -3/2 + K_s^{-1} \epsilon^2 \frac{|\lambda|}{x(0)} d_1(t)^{-5/2}) dt
\leq 17 \epsilon^2 + C \epsilon^4 T_0 + K_s^{-1} \epsilon^3 \int_0^{t_0} ((1 + \frac{|\lambda|}{20 \pi x(0)} t)^{-1})^{3/2} dt
+ K_s^{-1} \epsilon^2 \frac{|\lambda|}{x(0)} \int_0^{t_0} ((1 + \frac{|\lambda|}{20 \pi x(0)} t)^{-1})^{3/2} dt
\leq 17 \epsilon^2 + C \epsilon^4 T_0 + K_s^{-1} \epsilon^3 \frac{x(0)}{|\lambda|} + K_s^{-1} \epsilon^2.
\]

Since \( \frac{|\lambda|}{x(0)} \geq M \epsilon \), we have
\[
K_s^{-1} \epsilon^3 \frac{x(0)}{|\lambda|} \leq K_s^{-1} \epsilon^3 (M)^{-1} \epsilon^{-1} = K_s^{-1} M^{-1} \epsilon^2 \leq \frac{1}{2} \epsilon^2.
\]

Since \( T_0 \leq \delta \epsilon^{-2} \), if we choose \( \delta \leq C^{-1} \), then
\[
C \epsilon^4 T_0 \leq \epsilon^2.
\]

Therefore we have
\[
\sup_{t \in [0, T_0]} \mathcal{E}_s(t) \leq 19 \epsilon^2.
\]

By lemma 6.12, we obtain
\[ 4 \sum_{k=0}^{s} \left\{ \| \partial^{k} \alpha \|_{L^{2}}^{2} + \| \partial^{k} \alpha \Lambda^{1/2} \|_{L^{2}}^{2} + \| \partial^{k} \Lambda^{1/2} \|_{L^{2}}^{2} \right\} \leq \mathcal{E}_{s} + C \epsilon^{3} \leq 20 \epsilon^{2}. \quad (6.240) \]

So we have
\[ \| D_{t} \tilde{\theta} \|_{H^{s+1/2}} + \| D_{t} \tilde{\sigma} \|_{H^{s}} + \| \Lambda^{1/2} \tilde{\theta} \|_{H^{s}} \leq 5 \epsilon. \quad (6.241) \]

By lemma 6.11, we obtain
\[ \| \mathcal{F} \|_{H^{s+1/2}} \leq K_{s}^{-1} \epsilon + \frac{1}{2} \| D_{t} \tilde{\theta} \|_{H^{s+1/2}} \leq 3 \epsilon. \quad (6.242) \]
\[ \| D_{t} \mathcal{F} \|_{H^{s}} \leq \frac{1}{4} \| D_{t} \tilde{\sigma} \|_{H^{s}} + K_{s}^{-1} \epsilon \leq 2 \epsilon. \quad (6.243) \]

Since \( \tilde{\zeta} - \alpha \) is holomorphic, we have
\[ \tilde{\theta} = (I - \mathcal{H})(\tilde{\zeta} - \zeta) = (I - \mathcal{H})(\zeta - \alpha) = 2(\zeta - \alpha) - (\mathcal{H} + \tilde{\mathcal{H}})(\zeta - \alpha). \quad (6.244) \]

It’s easy to obtain
\[ \| \Lambda^{1/2}(\tilde{\theta} - 2(\zeta - \alpha)) \|_{H^{s}} = \| \Lambda^{1/2}(\mathcal{H} + \tilde{\mathcal{H}})(\zeta - \alpha) \|_{H^{s}} \leq C \epsilon^{2}. \quad (6.245) \]

By (6.241), we obtain
\[ 2 \| \Lambda^{1/2}(\zeta - \alpha) \|_{H^{s}} \leq \| \Lambda^{1/2} \tilde{\theta} \|_{H^{s}} + C \epsilon^{2} \leq 6 \epsilon. \quad (6.246) \]

So we have
\[ \| \Lambda^{1/2}(\zeta - \alpha) \|_{H^{s}} \leq 3 \epsilon. \quad (6.247) \]

To obtain control of \( \| \zeta_{\alpha} - 1 \|_{H^{s}}, \) again we use
\[ \zeta_{\alpha} - 1 = \frac{D_{t}^{2} \zeta - i(A - 1)}{i A}. \quad (6.248) \]

It’s easy to obtain
\[ \| \zeta_{\alpha} - 1 \|_{H^{s}} \leq \| D_{t} \mathcal{F} \|_{H^{s}} + C \epsilon^{2} + K_{s}^{-1} \epsilon \leq 3 \epsilon. \quad (6.249) \]

By continuity, we can choose \( c > 0 \) sufficiently small such that
\[ \| \zeta_{\alpha} - 1 \|_{H^{s}} \leq 5 \epsilon, \quad \| \mathcal{F} \|_{H^{s+1/2}} \leq 5 \epsilon, \quad \| D_{t} \mathcal{F} \|_{H^{s}} \leq 5 \epsilon, \quad \forall t \in [0, T_{0} + c). \quad (6.250) \]

So we must have \( T = [0, \delta \epsilon^{-2}], \) for some absolute constant \( \delta > 0. \) \( \square \)
6.11. Change of variables back to lagrangian coordinates. Next, we need to change of variables back to system (2.30). So we need to control $\kappa$ on time interval $[0, \delta \epsilon^{-2}]$. We have

$$\kappa_t = b \circ \kappa. \quad (6.251)$$

So we have

$$\partial_t \kappa = b_\alpha \circ \kappa \kappa_\alpha. \quad (6.252)$$

Recall that

$$(I - \mathcal{H}) b = - [D_t \xi, \mathcal{H}] \frac{\tilde{\xi}_\alpha - 1}{\zeta_\alpha} - \frac{i}{\pi} \sum_{j = 1}^{2} \frac{\lambda_j}{\zeta(\alpha, t) - z_j(t)}. \quad (6.253)$$

So we have

$$(I - \mathcal{H}) b_\alpha = [\zeta_\alpha, \mathcal{H}] b - \partial_\alpha [D_t \xi, \mathcal{H}] \frac{\tilde{\xi}_\alpha - 1}{\zeta_\alpha} + \frac{i}{\pi} \sum_{j = 1}^{2} \frac{\lambda_j \xi_\alpha}{(\zeta(\alpha, t) - z_j(t))^2}. \quad (6.254)$$

Clearly,

$$\left\| [\zeta_\alpha, \mathcal{H}] b - \partial_\alpha [D_t \xi, \mathcal{H}] \frac{\tilde{\xi}_\alpha - 1}{\zeta_\alpha} \right\| \leq C \epsilon^2, \quad (6.255)$$

and

$$\left\| \frac{i}{\pi} \sum_{j = 1}^{2} \frac{\lambda_j \xi_\alpha}{(\zeta(\alpha, t) - z_j(t))^2} \right\|_{H^1} \leq K_s^{-1} d_I(t)^{-5/2} \epsilon. \quad (6.256)$$

for some absolute constant $C > 0$. By lemma 6.3, we have

$$\| b_\alpha \|_{H^1} \leq C \epsilon^2 + K_s^{-1} d_I(t)^{-5/2} \epsilon. \quad (6.257)$$

By Sobolev embedding, we have

$$\| b_\alpha \circ \kappa \|_{\infty} = \| b_\alpha \|_{\infty} \leq \| b_\alpha \|_{H^1} \leq C \epsilon^2 + K_s^{-1} d_I(t)^{-5/2} \epsilon. \quad (6.258)$$

It’s easy to obtain that

$$\| \kappa_\alpha(\cdot, 0) - 1 \|_{\infty} \leq C \epsilon. \quad (6.259)$$

So we obtain

$$\kappa_\alpha(\alpha, t) - \kappa_\alpha(\alpha, 0) = \int_0^t \kappa_\alpha(\alpha, \tau) d\tau \quad (6.260)$$

$$= \int_0^t b_\alpha \circ \kappa(\alpha, \tau) \kappa_\alpha(\alpha, \tau) d\tau. \quad (6.261)$$
Let $\delta_1 > 0$ be a constant to be determined.

$$
T_1 := \left\{ T \in [0, \delta_1 \epsilon^{-2}] : \sup_{t \in [0, T]} \|\kappa_\alpha(\cdot, t) - \kappa_\alpha(\cdot, 0)\|_\infty \leq \frac{1}{10} \right\}. \quad (6.262)
$$

In particular, if $t \in T_1$, then for $\epsilon$ sufficiently small, we have $\frac{4}{5} \leq \kappa_\alpha \leq \frac{6}{5}$. Also, $T_1$ is closed. For $T \in T_1$, we have for any $t \in [0, T]$,\[
\left| \kappa_\alpha(\alpha, t) - \kappa_\alpha(\alpha, 0) \right| \leq \int_0^t \left( C \epsilon^2 + K_s^{-1} d_I(\tau)^{-5/2} \epsilon \right) d\tau \quad (6.263)
\]
\[
\leq \int_0^t \left( C \epsilon^2 + K_s^{-1} (1 + \frac{|\lambda|}{20 \pi x(0)})^{-5/2} \epsilon \right) d\tau \quad (6.264)
\]
\[
\leq C \epsilon^2 t + K_s^{-1} \frac{20 \pi x(0)}{|\lambda|} \frac{2}{3} \epsilon \quad (6.265)
\]
\[
\leq C \epsilon^2 t + \frac{1}{15K_s}. \quad (6.266)
\]

Here we’ve used the assumption $\frac{|\lambda|}{x(0)} \geq 200 \pi \epsilon$. Choose $\delta_1 = \frac{1}{30C}$. Then we have

$$
\sup_{t \in [0, T]} \|\kappa_\alpha(\cdot, t) - \kappa_\alpha(\cdot, 0)\|_\infty \leq \frac{1}{20}. \quad (6.267)
$$

Therefore, $T_1$ is open in $[0, \delta_1 \epsilon^{-2}]$, we must have $T_1 = [0, \delta_1 \epsilon^{-2}]$.

Let $\delta_0 := \min\{\delta, \delta_1\}$. Since $\kappa_\alpha \geq \frac{3}{5}$ on $[0, \delta_0 \epsilon^{-2}]$, we can change of variables back to lagrangian coordinates and conclude the proof of Theorem 3.

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