On the Russo-Dye Theorem for positive linear maps

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Abstract. We revisit a classical result, the Russo-Dye Theorem, stating that every positive linear map attains its norm at the identity.

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1 Introduction

Let $\mathbb{M}_n$ denote the space of complex $n \times n$ matrices and let $\mathbb{M}_n^+$ stand for the positive (semi-definite) cone. A linear map $\Phi : \mathbb{M}_n \to \mathbb{M}_m$ is called positive if $\Phi(\mathbb{M}_n^+) \subset \mathbb{M}_m^+$. A fundamental fact, the Russo-Dye Theorem [7], asserts that every positive linear map attains its norm at the identity. This is discussed in two nice books, [2, pp. 41–44] and, in the operator algebras setting, [6, Corollary 2.9]. If $I$ denotes the identity and $\| \cdot \|_\infty$ the operator norm, the theorem can be stated as:

Theorem 1.1. If $\Phi : \mathbb{M}_n \to \mathbb{M}_m$ is a positive linear map then, for all contractions $A \in \mathbb{M}_n$,

$$\|\Phi(A)\|_\infty \leq \|\Phi(I)\|_\infty.$$ 

This short note presents several refinements of Theorem 1.1. The contractive property of $A$ can be written as an operator inequality, $|A| \leq I$ where $|A| = (A^*A)^{1/2}$ is the right modulus. This suggests that for a general matrix $Z$ the data of its two modulus $|Z|$ and $|Z^*|$ should be useful to obtain more general forms of the Russo-Dye Theorem. For instance, we have the following consequence of our main result.

Proposition 1.2. Let $Z \in \mathbb{M}_n$, $J \in \mathbb{M}_n^+$, and let $\Phi : \mathbb{M}_n \to \mathbb{M}_m$ be a positive linear map. If $|Z| \leq J$ and $|Z^*| \leq J$, then there exists a unitary $V \in \mathbb{M}_m$ such that

$$|\Phi(Z)| \leq \frac{\Phi(J) + V\Phi(J)V^*}{2}.$$ 

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If $Z = A$ is a contraction and $J = I$, Proposition 1.2 claims that

$$|\Phi(A)| \leq \frac{\Phi(I) + V\Phi(I)V^*}{2}$$  \hfill (1.1)

and this clearly improves Theorem 1.1. Let us illustrate that with some eigenvalue inequalities for the Schur product with $S \in \mathbb{M}_n^+$. Let $s_i^j$ be the diagonal entries of $S$ rearranged in decreasing order, and similarly let $\lambda_i^j(H)$ denote the eigenvalues of the Hermitian matrix $H$ arranged in decreasing order. Theorem 1.1 for the Schur multiplier $\Phi(T) = S \circ T$ says that

$$\lambda_1^j(|S \circ A|) \leq s_1^j$$

for all contractions $A \in \mathbb{M}_n$. From (1.1) we infer several other inequalities, for example,

$$\lambda_3^1(|S \circ A|) \leq s_2^1.$$  \hfill (1.2)

We shall prove a considerable improvement of Proposition 1.2. This yields several eigenvalue inequalities which extend and complete the Russo-Dye Theorem. Of course the Russo-Dye Theorem holds for positive linear maps acting on unital $C^*$-algebras and, at the end of the paper, we state our main result in this setting.

## 2 Positive linear maps and geometric mean

Let $Z \in \mathbb{M}_n, J \in \mathbb{M}_n^+$. We wish to state a version of Proposition 1.2 under a more general assumption than the domination $|Z|, |Z^*| \leq J$. Given two nonnegative functions $f(t)$ and $g(t)$, we use the notation

$$Z \leq_{f,g} J \iff f(|Z|) \leq J \text{ and } g(|Z^*|) \leq J.$$

So, the assumption of Proposition 1.2 can be written $Z \leq_{f,g} J$ with $f(t) = g(t) = t$.

We also sharpen Proposition 1.2 by using the geometric mean instead of the arithmetic mean. We refer to [1] or [2] for a background on the geometric mean $A \# B$ of two matrices $A, B \in \mathbb{M}_n^+$. Since the arithmetic-geometric mean inequality

$$A \# B \leq \frac{A + B}{2}$$

holds, the following theorem extends and improves Proposition 1.2.

**Theorem 2.1.** Let $Z \in \mathbb{M}_n, J \in \mathbb{M}_n^+$, and let $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_m$ be a positive linear map. Suppose that $Z \leq_{f,g} J$ for some functions satisfying $f(t)g(t) = t^2$. Then, there exists a unitary $V \in \mathbb{M}_m$ such that

$$|\Phi(Z)| \leq \Phi(J) \# V\Phi(J)V^*.$$
Proof. We first prove the case when $Z = N$ is normal, $f(t) = g(t) = t$, and $J = |N|$. The theorem then reads as

$$|\Phi(N)| \leq \Phi(|N|) \# V \Phi(|N|) V^*.$$  \hfill (2.1)

Observe that

$$\begin{pmatrix} |N| & N^* \\ N & |N| \end{pmatrix} \succeq 0.$$

We may restrict $\Phi$ to the unital $C^*$-algebra spanned by $N$, and thus we may assume that $\Phi$ is completely positive thanks to Stinespring’s lemma \[8\]. Hence,

$$\begin{pmatrix} \Phi(|N|) & \Phi(N^*) \\ \Phi(N) & \Phi(|N|) \end{pmatrix} \succeq 0.$$

Next, applying a unitary congruence

$$\begin{pmatrix} I & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} \Phi(|N|) & \Phi(N^*) \\ \Phi(N) & \Phi(|N|) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & V^* \end{pmatrix},$$

where $V^*$ is the unitary factor in the polar decomposition of $\Phi(N)$, we obtain

$$\begin{pmatrix} \Phi(|N|) & |\Phi(N)| \\ |\Phi(N)| & V \Phi(|N|) V^* \end{pmatrix} \succeq 0.$$

It then follows from the maximal property of $\#$ that (2.1) holds.

Now, suppose that $Z = A$ is a contraction. Since

$$|A| = \frac{W + W^*}{2}$$

for some unitary $W$, we infer from the polar decomposition that

$$A = \frac{U_0 + U_1}{2}$$

some unitaries $U_0$ and $U_1$. Applying (2.1) to the normal (unitary) matrix

$$N = \begin{pmatrix} U_0 & 0 \\ 0 & U_1 \end{pmatrix}$$

and to the positive linear map from $\mathbb{M}_2(\mathbb{M}_n)$ to $\mathbb{M}_n$,

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \mapsto \Phi \left( \frac{P + S}{2} \right),$$

we obtain

$$|\Phi(A)| \leq \Phi(I) \# V \Phi(I) V^*.$$  \hfill (2.2)

This establishes the theorem when $Z$ is a contraction, $f(t) = g(t) = t$, and $J = I$.  

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We turn to the general case, \( f(|Z|) \leq J \) and \( g(|Z^*|) \leq J \) with \( f(t)g(t) = t^2 \). Define a map \( \Psi : \mathbb{M}_n \to \mathbb{M}_m \) by
\[
\Psi(X) = \Phi(J^{1/2}XJ^{1/2}).
\]
Observe that
\[
\Psi(I) = \Phi(J), \quad \text{and} \quad \Psi(Y) = \Phi(Z)^{(2.3)}
\]
where \( Y = J^{-1/2}ZJ^{-1/2} \) and \( J^{-1} \) is the generalized inverse. Thanks to the polar decomposition \( Z = |Z^*|U = U|Z| \) and the condition \( \sqrt{f(t)g(t)} = t \),
\[
Y = J^{-1/2}g^{1/2}(|Z^*|)Uf^{1/2}(|Z|)J^{-1/2}.
\]
From the assumption \( Z \leq J \) we infer that \( J^{-1/2}g^{1/2}(|Z^*|) \) and \( f^{1/2}(|Z|)J^{-1/2} \) are contractions, and so \( Y \) is a contraction too. Applying (2.2) to \( Y \) and \( \Psi \) yields
\[
|\Psi(Y)| \leq \Psi(I)\#V\Psi(I)V^*
\]
and coming back to (2.3) completes the proof.

In the course of the proof, we have noted two special cases.

**Corollary 2.2.** Let \( \Phi : \mathbb{M}_n \to \mathbb{M}_m \) be a positive linear map and let \( N \in \mathbb{M}_n \) be normal. Then, there exists a unitary \( V \in \mathbb{M}_m \) such that
\[
|\Phi(N)| \leq \Phi(|N|)\#V\Phi(|N|)V^*.
\]

**Corollary 2.3.** Let \( A \in \mathbb{M}_n \) be a contraction and let \( \Phi : \mathbb{M}_n \to \mathbb{M}_m \) be a positive linear map. Then, for some unitary \( V \in \mathbb{M}_m \),
\[
|\Phi(A)| \leq \Phi(I)\#V\Phi(I)V^*.
\]

Another extension of the Russo-Dye Theorem is given in the next corollary.

**Corollary 2.4.** Let \( Z \in \mathbb{M}_n \), \( J \in \mathbb{M}_n^+ \), and let \( \Phi : \mathbb{M}_n \to \mathbb{M}_m \) be a positive linear map. If \( J \geq E \), the range projection of \( Z \), and \( J \geq Z^*Z \), then, for some unitary \( V \in \mathbb{M}_m \),
\[
|\Phi(Z)| \leq \Phi(J)\#V\Phi(J)V^*.
\]

**Proof.** Apply Theorem 2.1 with \( f(t) = t^2 \) and \( g(t) \) such that \( g(0) = 0 \) and \( g(s) = 1 \) for \( s > 0 \).

The next statement extends Corollary 2.2.

**Corollary 2.5.** Let \( \Phi : \mathbb{M}_n \to \mathbb{M}_m \) be a positive linear map, let \( Z \in \mathbb{M}_n \) be invertible, and \( \rho \) the spectral radius of \( |Z^*||Z|^{-1} \). Then, there exists a unitary \( V \in \mathbb{M}_m \) such that
\[
|\Phi(Z)| \leq \sqrt{\rho} \Phi(|Z|)\#V\Phi(|Z|)V^*.
\]
Proof. Note that $|Z^*| \leq \rho |Z|$ and apply Theorem 2.1 for the matrices $Z$ and $J = \sqrt{\rho} |Z|$ with the functions $f(t) = \sqrt{\rho} t$ and $g(t) = (1/\sqrt{\rho}) t$.

In general the inequality of Corollary 2.5 is sharp as shown with the transpose map $\Phi(X) = X^T$ on $\mathbb{M}_2$ and

$$Z = \begin{pmatrix} 0 & 1 \\ k & 0 \end{pmatrix}.$$ 

Indeed, the spectral radius of $|Z^*||Z|^{-1}$ then equals $k$, and if $e_2$ denotes the second vector of the canonical basis of $\mathbb{C}^2$, the inequality

$$k = \langle e_2, Z^T e_2 \rangle \leq c \langle e_2, |Z|^T V |Z|^T V^* e_2 \rangle$$

entails

$$k \leq c \langle e_2, |Z|^T e_2 \rangle \# \langle e_2, V |Z|^T V^* e_2 \rangle \leq c \sqrt{k}$$

and so $c \geq \sqrt{k}$.

For 2-positive maps, Theorem 2.1 can be improved.

Theorem 2.6. Let $Z \in \mathbb{M}_n$ and let $\Phi : \mathbb{M}_n \to \mathbb{M}_m$ be a 2-positive linear map. Then, for any pair of nonnegative functions $f(t)$ and $g(t)$ satisfying $f(t)g(t) = t^2$, there exists a unitary $V \in \mathbb{M}_m$ such that

$$|\Phi(Z)| \leq \Phi(f(|Z|)) \# V \Phi(g(|Z^*|)) V^*.$$ 

Proof. From the polar decomposition $Z = U|Z|$ we have $Z = \sqrt{g(|Z^*|)} U \sqrt{f(|Z|)}$, hence

$$\begin{pmatrix} f(|Z|) & Z^* \\ Z & g(|Z^*|) \end{pmatrix} \geq 0.$$ 

Since $\Phi$ is 2-positive, we then have

$$\begin{pmatrix} \Phi(f(|Z|)) & \Phi(Z^*) \\ \Phi(Z) & \Phi(g(|Z^*|)) \end{pmatrix} \geq 0.$$ 

Arguing as in the proof of Theorem 2.1 we then infer that

$$|\Phi(Z)| \leq \Phi(f(|Z|)) \# V \Phi(g(|Z^*|)) V^*$$

for some unitary $V \in \mathbb{M}_m$. 

Corollary 2.7. Let $\Phi : \mathbb{M}_n \to \mathbb{M}_m$ be a 2-positive linear map, let $Z \in \mathbb{M}_n$ and let $p \in (\infty, \infty)$. Then, there exists a unitary $V \in \mathbb{M}_m$ such that

$$|\Phi(Z)| \leq \Phi(|Z|^{1+p}) \# V \Phi(|Z^*|^{1-p}) V^*.$$ 


Here, if $Z$ is not invertible, the nonpositive powers of $|Z|$ and $|Z^*|$ are understood as generalized inverses, in particular $|Z|^0$ is the support projection of $Z$ and $|Z^*|^0$ the range projection. The case $p = 0$ reads as

$$|\Phi(Z)| \leq \Phi(|Z|)\#V\Phi(|Z^*|)V^*.$$  \hfill (2.4)

However, in general (2.4) does not hold if the 2-positivity assumption is dropped, as shown by the next example.

**Example 2.8.** Let $\Phi : M_2 \to M_2$, $\Phi(X) = X + X^T$, and let

$$Z = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$$

Then, for any pair of unitary matrices $U, V \in M_2$,

$$25 = \det |\Phi(Z)| > \det \{U\Phi(|Z|)U^*\#V\Phi(|Z^*|)V^*\} = 16.$$  

3 Some consequences

From these results follow some eigenvalue inequalities. Given $S, T \in M_n^+$, recall that the (weak) log-majorisation $S \prec_{\text{wlog}} T$ means that a series of $n$ eigenvalue inequalities holds: For $k = 1, \ldots, n$,

$$\prod_{j=1}^k \lambda_j^+(S) \leq \prod_{j=1}^k \lambda_j^+(T),$$

where $\lambda_j^+(\cdot)$ still stand for the eigenvalues arranged in nonincreasing order.

Theorem 2.1 and Corollary 2.3 are equivalent, though the most useful statement is Theorem 2.1. For instance, if $Z \in M_n$ is a shift (a weighted permutation matrix), then $|Z^*|$ and $|Z|$ are diagonal matrices, and comparing with a diagonal $J$ seems natural. A similar remark holds if $Z$ is paranormal. For a general matrix $Z$, the Kato supremum, $[5]$

$$|Z| \vee |Z^*| = \lim_{p \to \infty} \left( \frac{|Z|^p + |Z^*|^p}{2} \right)^{1/p}$$

may be used as $J$ in Theorem 2.1 with $f(t) = g(t) = t$. Some other natural choices for $J$ are $(|Z|^q + |Z^*|^q)^{1/q}$, $1 \leq q < \infty$, still with $f(t) = g(t) = t$.

We list some consequences of Theorem 2.1.

**Corollary 3.1.** Let $Z \in M_n$, $J \in M_n^+$, and let $\Phi : M_n \to M_m$ be a positive linear map. If $Z \leq_{f,g} J$ for some functions satisfying $f(t)g(t) = t^2$, then,

$$|\Phi(Z)| \prec_{\text{wlog}} \Phi(J)$$

and

$$\lambda_{j+k+1}^+(|\Phi(Z)|) \leq \sqrt{\lambda_{j+1}^+(\Phi(J))\lambda_{k+1}^+(\Phi(J))}$$

for all $j, k = 0, 1, \ldots, n$.  

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Proof. By a well-known property of the geometric mean, Theorem 2.1 says that
\[ |\Phi(Z)| \leq \sqrt{\Phi(J)U\sqrt{\Phi(J)V^*}} \]
for some unitary matrices $U$ and $V$. From Horn’s inequalities we then get the log-majorisation, while the second series of inequalities follows from a standard min-max principle. \[ \square \]

The Schur product of a matrix with the identity yields the diagonal part of the matrix. From the second statement of Corollary 3.1 we obtain:

**Corollary 3.2.** Let $S \in \mathbb{M}_n^+$ and let $A \in \mathbb{M}_n$ be a contraction. If $s_1^t \geq \ldots \geq s_n^t$ denote the diagonal entries of $S$ arranged in decreasing order, then, for all $j, k = 0, 1, \ldots$,
\[ \lambda_{j+k+1}^\downarrow(\langle S \circ A \rangle) \leq \sqrt{s_{j+1}^\downarrow s_{k+1}^\downarrow}. \]

**Remark 3.3.** For $j = k = 1$, we recapture (1.2). If $S \in \mathbb{M}_n^+$ is expansive, Corollary 3.2 shows that
\[ \lambda_{2j+1}^\downarrow(S \circ S^{-1}) \leq s_{j+1}^\downarrow. \]
If $S \in \mathbb{M}_n^+$ is a contraction, then
\[ \lambda_{2j+1}^\downarrow(S \circ S) \leq s_{j+1}^\downarrow. \]

**Corollary 3.4.** Let $Z \in \mathbb{M}_n$, $J \in \mathbb{M}_n^+$, and let $\Phi : \mathbb{M}_n \to \mathbb{M}_m$ be a positive linear map. If $Z \leq_{f,g} J$ for some functions satisfying $f(t)g(t) = t^2$, then
\[ \left\{ \prod_{j=1}^k \lambda_j^\uparrow(|\Phi(Z)|) \right\}^2 \leq \prod_{j=1}^k \lambda_j^\uparrow(\Phi(J))\lambda_j^\uparrow(\Phi(J)) \]
for all $k = 1, \ldots, n$.

**Proof.** Let $S$ be any subspace of $\mathbb{C}^m$, and denote by $A_S$ the compression of $A \in \mathbb{M}_m$ onto $S$. From Theorem 2.1 and Ando’s inequality [1 Theorem 3] we infer
\[ |\Phi(Z)|_S \leq (\Phi(J))_S \# (V\Phi(J)V^*)_S. \]
So,
\[ \det |\Phi(Z)|_S \leq \left\{ \det (\Phi(J))_S \det (\Phi(J))_{V^*(S)} \right\}^{1/2}. \]
Letting $S$ be the spectral subspace corresponding to the $k$ smallest eigenvalues of $\Phi(J)$ and using standard min-max principles completes the proof. \[ \square \]
Every $n$-by-$n$ complex matrix $Z$ has a decomposition in real and imaginary parts, its so-called Cartesian decomposition, $Z = X + iY$ where $X$ and $Y$ are self-adjoints. In general, the operator inequality
\[ |Z| \leq |X| + |Y| \] (3.1)
does not hold. However, a well-known substitute is the (weak) log-majorisation
\[ |Z| \prec_{\log} |X| + |Y|, \] (3.2)
see Corollary 2.11 for a stronger statement. We may use Corollary 2.2 to obtain further extensions of the triangle inequality (3.2).

**Corollary 3.5.** Let $\Phi : \mathbb{M}_n \to \mathbb{M}_m$ be a positive linear map and let $Z \in \mathbb{M}_n$ with Cartesian decomposition $Z = X + iY$. Then, there exists a unitary $V \in \mathbb{M}_m$ such that
\[ |\Phi(Z)| \leq \Phi(|X| + |Y|) \# V \Phi(|X| + |Y|) V^* . \]

**Proof.** Consider the normal matrix in $\mathbb{M}_{2n}$
\[ N = \begin{pmatrix} X & 0 \\ 0 & iY \end{pmatrix} \]
and apply Corollary 2.2 to this matrix and the positive linear map $\Psi : \mathbb{M}_{2n} \to \mathbb{M}_m$ defined as $\Psi = \Phi \circ \Lambda$ where $\Lambda$ is the partial trace on $\mathbb{M}_{2n} = \mathbb{M}_2(\mathbb{M}_n)$,
\[ \Lambda \left( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = A + D. \]
Since $\Psi(N) = \Phi(Z)$ and $\Psi(|N|) = \Phi(|X| + |Y|)$, the corollary is established. \qed

**Corollary 3.6.** If $\Phi : \mathbb{M}_n \to \mathbb{M}_m$ is a positive linear map and $Z \in \mathbb{M}_n$ has Cartesian decomposition $Z = X + iY$, then,
\[ |\Phi(Z)| \prec_{\log} \Phi(|X| + |Y|). \]

**Corollary 3.7.** If $Z \in \mathbb{M}_n$ has Cartesian decomposition $Z = X + iY$, then,
\[ \|(|X| + |Y|)^{-1/2}Z(|X| + |Y|)^{-1/2}\|_{\infty} \leq 1. \]

**Proof.** This is a direct consequence of Corollary 3.6 applied to the positive linear map $T \mapsto (|X| + |Y|)^{-1/2}T(|X| + |Y|)^{-1/2}$. \qed

**Remark 3.8.** Corollary 3.7 is surprising. We could think that this follows from the polar decomposition $Z = |Z^*|^{1/2}U|Z|^{1/2}$ and the plausible fact that $|Z|^{1/2}(|X| + |Y|)^{-1/2}$ is contractive. But in general this matrix is not contractive, as it would be equivalent to the untrue inequality (3.1).
Corollary 3.9. If $Z \in \mathbb{M}_n$ has Cartesian decomposition $Z = X + iY$, then,

$$\rho(Z(|X| + |Y|^{-1})) \leq 1.$$ 

Proof. This is a straightforward consequence of Corollary 3.7. □

Remark 3.10. Corollary 3.9 seems surprising too: It may happen that

$$\|Z(|X| + |Y|^{-1})\|_\infty > 1.$$ 

Otherwise we would have $|Z|^2 \leq (|X| + |Y|)^2$, entailing the untrue inequality (3.1).

4 Comments

This article is a continuation of [4]. Indeed, the idea of revisiting the Russo-Dye Theorem comes from that paper, where Corollary 2.2 is proved with several related sharp inequalities. We can state a version of Theorem 2.1 for operator algebras. Let $\mathcal{A}$ be a unital $C^*$-algebra, and let $\mathcal{B}$ be a von Neumann algebra.

Theorem 4.1. Let $Z \in \mathcal{A}$, $J \in \mathcal{A}^+$, and let $\Phi : \mathcal{A} \to \mathcal{B}$ be a positive linear map. Suppose that $Z \leq_{f,g} J$ for some continuous functions satisfying $f(t)g(t) = t^2$. Then, there exists a partial isometry $V \in \mathcal{B}$ such that

$$|\Phi(Z)| \leq \Phi(J)^\#V\Phi(J)V^*.$$ 

The proof follows exactly the same steps as in the matrix case. Note that $V^*$ is the partial isometry occurring in the polar decomposition $\Phi(Z) = V^*|\Phi(Z)|$, in general it is not a unitary operator, and it belongs to the von Neumann algebra generated by $\Phi(Z)$. If $\mathcal{A}$ is also a von Neumann algebra, then $f(t)$ and $g(t)$ can be bounded Borel functions.

The assumption of Corollary 2.5 can be replaced by the condition that $|Z^*| \leq \rho |Z|$ for some $\rho > 0$, which makes sense also for noninvertible operators. With $\rho = 1$, Corollary 2.5 then takes the following form for infinite dimensional operators.

Corollary 4.2. Let $Z \in \mathcal{A}$ be semi-hyponormal and let $\Phi : \mathcal{A} \to \mathcal{B}$ be a positive linear map. Then, there exists a partial isometry $V \in \mathcal{B}$ such that

$$|\Phi(Z)| \leq \Phi(|Z|)^\#V\Phi(|Z|)V^*.$$ 

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