On complements of Kazhdan projections in semisimple groups

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Abstract

We prove that for an isometric representation of some groups on certain Banach spaces, the complement of the subspace of invariant vectors is 1-complemented.

1 Introduction

An isometric representation $\pi$ of a locally compact group $G$ on a reflexive Banach space $E$ induces a direct sum decomposition

$$E = E^\pi \oplus E_\pi$$

into the invariant vectors $E^\pi$ and its canonical, $\pi$-invariant complement, $E_\pi$. The corresponding projection $P^\pi$ from $E$ to $E^\pi$ along $E_\pi$ has norm 1 under fairly general conditions. However, typically, the complementary projection $I - P^\pi$ need not be of norm 1. An easy example is that of the regular representation of a finite group $G$ on $\ell_p(G)$, $p > 2$, where $E^\pi$ are the constant functions and $E_\pi$ are the functions on $G$ whose mean is zero (see Example 4 for details).

In this article we give conditions under which, for representations $\pi$ on a Banach space $E$ in a certain class $\mathcal{O}$, consisting of uniformly convex uniformly

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smooth Banach spaces with the Opial property (see Section 2.1 for details), $E^\pi$ is 1-complemented and more precisely that

$$\|I - P^\pi\| = 1.$$ 

**Theorem 1.** Let $G$ be a locally compact group and let $\pi$ be an isometric representation on a Banach space $E \in \mathcal{O}$. Assume that there exists a sequence of elements $\{g_n\} \subset G$, $g_n \to \infty$, such that

$$\text{WOT} - \lim \pi_{g_n} = \text{WOT} - \lim \pi_{g_n^{-1}} = P^\pi.$$ 

Then $E^\pi$ is Birkhoff-James orthogonal to $E^\pi$ and $\|I - P^\pi\| = 1$.

The above theorem in particular applies to class of groups satisfying certain algebraic conditions, introduced as quasi-semisimple groups in [2]. They are defined via the existence of a $KAK$-type decomposition and generation by certain contraction subgroups, see [2, 9] for details. This class of groups includes the classical semisimple Lie groups as well as certain automorphism groups of trees.

Recall that a representation $\pi$ of $G$ on a reflexive Banach space $E$ is a $c_0$-representation if the matrix coefficients $(\pi_{g}v, w)$ vanish at infinity for every $v \in E^\pi$ and $w \in E^*$. As shown in [2], for quasi-semisimple groups this is the case for all isometric representations on reflexive Banach spaces.

**Corollary 2.** Let $\pi$ be a representation of a non-compact locally compact group $G$ on $E \in \mathcal{O}$. If $\pi$ is a $c_0$-representation then $\|1 - P^\pi\| = 1$.

In particular, if $G$ is quasi-semisimple then the above estimate holds for all isometric representations on $E \in \mathcal{O}$.

This is particularly interesting in the context of properties such as property $(T^\ell_p)$, studied by Bekka and Olivier [4]. In that case we obtain a stronger conclusion that holds not only in the algebra of bounded operators on $E \in \mathcal{O}$, but in the the associated maximal group Banach algebra (see Section 3.1).

**Corollary 3.** Let $G$ be quasi-semisimple. Assume that $G$ satisfies uniform property $(T_\mathcal{O})$, namely that there exists a Kazhdan projection $p$ in the Banach algebra $C^\mathcal{O}_{\text{max}}(G)$. Then the element $1 - p$ satisfies $\|1 - p\| = 1$.

We remark that in [5] Bernau and Lacey studied projections $P$ such that both $P$ and $I - P$ are of norm 1, and in particular gave a general description of such projections in $L_p$-spaces.
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2 Banach spaces, duality mappings, and the Opial property

2.1 Duality mappings

Consider a duality mapping \( J : E \to \mathcal{S}(E^*) \), into the collection \( \mathcal{S}(E^*) \) of subsets of \( E^* \), defined by the condition

\[
J(v) = \{ w \in E^* : \langle v, w \rangle = \|v\|^2 \text{ and } \|w\| = \|v\| \}.
\]

A thorough discussion of duality mappings in Banach spaces can be found in e.g. [8].

Denote by \( w^\ast \text{lim} \) the limit in the weak topology on \( E \). A Banach space \( E \) has the Opial property if for every weakly convergent sequence \( \{x_n\} \subseteq E \) with the weak limit \( x_0 \in E \) the inequality

\[
\liminf \|x_n - x_0\| < \liminf \|x_n - y\|
\]

holds for every \( y \neq x_0 \). Every separable Banach space admits an equivalent norm with the Opial property [11].

It is known that a uniformly convex uniformly smooth Banach space has Opial’s property if and only if the following condition is satisfied: for every sequence \( \{x_n\} \) in \( E \), we have

\[
w^\ast \text{lim} x_n = x \iff w^\ast \text{lim} J(x_n - x) = 0.
\]

The second mode of convergence is the so-called \( \Delta \)-convergence. We refer to [18] for details.

By \( \mathcal{O} \) we will denote the class of uniformly convex uniformly smooth Banach spaces with the Opial property. It is known for instance that the spaces \( \ell_p \), \( 1 < p < \infty \), belong to the class \( \mathcal{O} \), as do their infinite direct \( q \)-sums, \( 1 < q < \infty \). However, the spaces \( L_p[0,1] \), \( p \in (1,2) \cup (2,\infty) \) do not belong to the class \( \mathcal{O} \).
2.2 Representations

Let $B(E)$ denote the space of bounded linear operators on $E$. We will consider $B(E)$ with the strong operator topology ($\text{SOT}$) and with the weak operator topology ($\text{WOT}$). Let $G$ be a locally compact group. Let $\pi$ be an isometric representation of $G$ on $E \in \mathcal{O}$ that is continuous in the strong operator topology on $B(E)$. In other words, for every $v \in E$ the orbit map $G \rightarrow E$, $g \mapsto \pi_g v$, is continuous into the norm topology on $E$. The dual space $E^*$ is naturally equipped with an isometric representation $\pi^*$, defined by the formula

$$\pi^*_g = (\pi^*_g)^{-1},$$

for every $g \in G$. If $\pi$ is $\text{SOT}$-continuous then so is $\pi^*$, under the assumption that $E$ is reflexive, see [20, Proposition 4.1.2.3, page 224]. A matrix coefficient of a representation $\pi$ on a Banach space $E$, associated to vectors $v \in E, w \in E^*$, is a function $\psi_{v,w} : G \rightarrow \mathbb{C}$ defined by $\psi_{v,w}(g) = \langle \pi_g v, w \rangle$. When $E$ is reflexive we will say that $\pi$ is a $c_0$-representation if $\psi_{v,w} \in C_0(G)$ for every $v \in E_\pi$ and $w \in E^*$.

2.3 Decompositions and projections

Denote

$$E^\pi = \{ v \in E : \pi_g v = v \text{ for all } g \in G \}.$$

If $E$ is reflexive then there is a direct sum decomposition $E = E^\pi \oplus E_\pi$, where $E_\pi$ is the annihilator of the space $(E^*)^\pi$. Since $E_\pi$ is $\pi$-invariant, the decomposition is in fact a decomposition of $\pi$ into the trivial representation and its complement. See e.g. [1, 3]. Note however, that if $E$ is not reflexive then the projection onto $E^\pi$ might not be equivariant [1, Example 2.29], and in fact there might not be a bounded projection onto $E^\pi$ at all [17, Theorem 1].

Let $P^\pi : E \rightarrow E^\pi$ be the projection along $E_\pi$. The projection $P^\pi$ is known to satisfy $\|P^\pi\| = 1$ in many cases, see [12, 17]. However, for the complementary projection in general we only have

$$\|I - P^\pi\| \leq 2.$$

Example 4. Let $G$ be a finite group. Consider the left regular representation of $G$ on $E = \ell_p(G), 1 \leq p \leq \infty$. The invariant vectors are then the constant functions on $G$ and the complement $E_\pi$ is the subspace of functions satisfying

$$Mf = \frac{1}{\#G} \sum_{h \in G} f(h) = 0.$$
In the case \( p = \infty \) and the group \( \mathbb{Z}_n, n \geq 3 \) it is easy to see that \( \|I - P^\pi\| > 1 \) by considering the vector \( f = (1, 1, \ldots, 1, -1) \in \ell_\infty(\mathbb{Z}_n) \). Then \( \|f\| = 1, Mf = 1 - 2/n \) and \( \|f - Mf\| = 2 - 2/n \). Since \( G \) is finite, clearly, by choosing \( n \in \mathbb{N} \) and \( 2 < p < \infty \), sufficiently large we obtain that the norm of \( \|I - P^\pi\| \) can be arbitrarily close to 2. The same is true for any group with a finite quotient of cardinality at least 3.

Observe that if \( G \) is residually finite with a sequence \( \{N_i\} \) of finite index normal subgroups, \( \cap N_i = \{e\} \), then the previous case shows that for the representation of \( G \) on \( \ell_p(\coprod G/N_i) = (\bigoplus \ell_p(G/N_i))_p \) for \( p = \infty \) the norm \( \|I - P^\pi\| \) will be in fact 2. We can thus choose \( p \) such that the norm of the projection \( I - P^\pi \) will be arbitrarily close to 2.

3 Proofs

Before proving the main theorem we first need a few lemmas regarding duality mappings and their behavior with respect to isometric representations within the class \( \mathcal{O} \) of Banach spaces. For an invertible isometry \( S \) on \( E \) denote \( S = (S^*)^{-1} \).

**Lemma 5.** Let \( E \in \mathcal{O} \) be a Banach space and let \( S \in B(E) \) be an invertible isometry. Then \( S J(v) = J(Sv) \).

**Proof.** First note that since \( S \) is an invertible isometry, so is \( S^* \), and we have that \( \|S^* J(Sv)\| = \|Sv\| = \|Sv\| = \|v\| \). Thus

\[
\langle v, S^* J(Sv) \rangle = \langle Sv, J(Sv) \rangle = \|Sv\|^2 = \|v\|^2 = \langle v, J(v) \rangle.
\]

By uniform convexity and uniform smoothness of \( E \) the duality mapping \( J \) is bijective, which guarantees that there is a unique element \( w \in E^* \) such that \( \|w\| = \|v\| \) and \( \langle v, w \rangle = \|v\|^2 \). This yields

\[
S^* J(Sv) = J(v),
\]

and the claim is proved. \( \Box \)

A vector \( 0 \neq v \in E \) is **Birkhoff-James orthogonal** to \( 0 \neq w \in E \) if

\[
\|v\| \leq \|v + \lambda w\|
\]

for every \( \lambda \in \mathbb{R} \). Given subspaces \( V_1, V_2 \) in \( E \) we will say that \( V_1 \) is Birkhoff-James orthogonal to \( V_2 \) if for every \( v_1 \in V_1 \) and \( v_2 \in V_2 \), \( v_1 \) is Birkhoff-James orthogonal to \( v_2 \).
A useful lemma by Kato [13] states that \( \|v\| \leq \|v + \lambda w\| \) for every \( \lambda > 0 \) if and only if we have
\[
\langle w, J(v) \rangle \geq 0.
\]
In particular this implies that \( v \) is Birkhoff-James orthogonal to \( w \) if we have \( \langle w, J(v) \rangle \geq 0 \) and \( \langle w, J(-v) \rangle \geq 0 \). Of course, this is equivalent to \( \langle w, J(v) \rangle = 0 \).

**Theorem 1.** Let \( G \) be a locally compact group and let \( \pi \) be an isometric representation on a Banach space \( E \in \mathcal{O} \). Assume that there exists a sequence of elements \( \{g_n\} \subset G \), \( g_n \to \infty \), such that
\[
\text{WOT} - \lim \pi g_n = \text{WOT} - \lim \pi^{-1} g_n = P^\pi.
\]
Then \( E_\pi \) is Birkhoff-James orthogonal to \( E^\pi \) and \( \|I - P^\pi\| = 1 \).

**Proof.** Let \( w \in E^\pi \), \( v \in E_\pi \) be arbitrary. We first observe that since \( P^\pi = \text{WOT} - \lim \pi g_n \), then
\[
(P^\pi)^* = \text{WOT} - \lim \pi g_n^* = \text{WOT} - \lim \pi^{-1} g_n^*.
\]
Thus we have
\[
\langle w, J(v) \rangle = \langle P^\pi w, J(v) \rangle \\
= \langle w, (P^\pi)^* J(v) \rangle \\
= \lim \langle w, \pi g_n^{-1} J(v) \rangle \\
= \lim \langle w, J(\pi g_n^{-1} v) \rangle,
\]
where the last equality follows from Lemma 5.

Now, by assumption, the sequence \( \pi g_n^{-1} v \) converges weakly to \( P^\pi v = 0 \) since \( v \in E_\pi \). By the Opial property we have that this is the same as the condition that
\[
J(\pi g_n^{-1} v) \to 0
\]
in the weak topology in \( E^* \). In particular,
\[
\lim \langle w, J(\pi g_n v) \rangle = \langle w, J(v) \rangle = 0.
\]
The norm estimate \( \|I - P^\pi\| = 1 \) is then a direct consequence of Birkhoff-James orthogonality.

The above theorem applies to the class of quasi-semisimple groups, studied in [2]. These groups are defined as satisfying two algebraic conditions: a version of a KAK-decomposition and that the group is generated by certain contraction subgroups, associated to sequences of elements tending off to infinity. We refer to [2] and [9] for definitions and applications.
Proof of Corollary 2. If \( \pi \) is an isometric \( c_0 \)-representation of \( G \) on a reflexive Banach space \( E \) then, for \( v \in E \) and \( w \in E^* \), every matrix coefficient of \( \pi \) of the form \( \psi_{v,w} = \langle \pi_g v, w \rangle \) vanishes as \( g \) goes to infinity. This in particular means, that the wot closure of \( \pi(G) \) in \( B(E) \) is the one-point compactification of \( G \), with the point at infinity being precisely the projection \( P^\pi \). In particular, this ensures that the assumptions of Theorem 1 are satisfied.

The class of quasi-semisimple groups includes the classical semisimple Lie groups and certain automorphism groups of trees. As proved in [2], quasi-semisimple groups satisfy a Veech decomposition of their space \( W(G) \) of weakly almost periodic functions,

\[
W(G) = C_0(G) \oplus \mathbb{C}.
\]

(See also [19] for the classical result of Veech on semisimple Lie groups.) Every matrix coefficient of an isometric representation on a reflexive Banach space \( E \) is an element of \( W(G) \), with the projection (invariant mean on weakly almost periodic functions) \( m : W(G) \to \mathbb{C} \) given by \( m(\psi_{v,w}) = \langle P^\pi v, P^\pi w \rangle \). Thus, the previous argument applies to every isometric representation \( \pi \) of a quasi-semisimple group on \( E \in \mathcal{O} \).

### 3.1 Complements of Kazhdan projections

Let \( \mathcal{O} \) be a class of Banach spaces. The group ring \( CG \) can be completed in the the norm

\[
\|f\| = \sup \|\pi(f)\|,
\]

where the supremum is taken over all isometric representations of \( G \) on Banach spaces \( E \in \mathcal{O} \). This completion is a Banach algebra, which we denote \( C_{\text{max}}^\mathcal{O}(G) \). In the case when \( \mathcal{O} \) consists of the class of Hilbert spaces the algebra is the maximal \( C^\ast \)-algebra of \( G \). A Kazhdan projection \( p \in C_{\text{max}}^\mathcal{O}(G) \) is an idempotent such that \( \pi(p) = P^\pi \) for every isometric representation of \( G \) on \( E \in \mathcal{O} \). \( G \) has uniform property \( (T_\mathcal{O}) \) if a Kazhdan projection exists in \( p \in C_{\text{max}}^\mathcal{O}(G) \). See [10] for a detailed study.

Proof of Corollary 3. Follows from the fact that for every isometric representation of \( G \) on \( E \in \mathcal{O} \) we have \( \|I - P^\pi\| = 1 \). \qed

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