Global well-posedness in the Energy space for the
Benjamin-Ono equation on the circle

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Abstract. We prove that the Benjamin-Ono equation is locally well-posed in $H^{1/2}(\mathbb{T})$. This leads to a global well-posedness result in $H^{1/2}(\mathbb{T})$ thanks to the energy conservation.

1 Introduction, main results and notations

1.1 Introduction

This paper is devoted to the study of the Cauchy problem for the Benjamin-Ono equation on the circle

\begin{equation}
\begin{aligned}
\partial_t u + \mathcal{H} \partial_x^2 u - u \partial_x u &= 0, \quad (t, x) \in \mathbb{R} \times \mathbb{T}, \\
u(0, x) &= u_0(x),
\end{aligned}
\end{equation}

where $\mathbb{T} = \mathbb{R}/2\pi \mathbb{Z}$, $u$ is real-valued and $\mathcal{H}$ is the Hilbert transform defined for $2\pi$-periodic functions with mean value zero by

$$\mathcal{H}(f)(0) = 0 \quad \text{and} \quad \mathcal{H}(f)(\xi) = -i \text{sgn}(\xi) \hat{f}(\xi), \quad \xi \in \mathbb{Z}^* .$$

The Benjamin-Ono equation arises as a model for long internal gravity waves in deep stratified fluids, see [3]. This equation is formally completely integrable (cf. [2], [8]) and thus possesses an infinite number of conservation laws. These conservation laws permit to control the $H^{n/2}$-norms, $n \in \mathbb{N}$, and thus to derive global well-posedness results in Sobolev spaces. The Cauchy problem on the real line has been extensively studied these last years (cf. [21], [1], [11], [20], [19], [15], [12]). Recently, T. Tao [22] has pushed the well-posedness theory to $H^1(\mathbb{R})$ by using an appropriate gauge transform. In the periodic setting, the local well-posedness of (BO) is known in $H^s(\mathbb{T})$ for $s > 3/2$ (cf. [11], [14]), by standard compactness methods which do
not take advantage of the dispersive effects of the equation. Thanks to the
conservation laws mentioned above and an interpolation argument, this leads
to global well-posedness in $H^s(T)$ for $s > 3/2$ (cf. [11]). Very recently, F.
Ribaud and the author [18] have improved the global well-posedness result
to $H^1(T)$ by using the gauge transform introduced by T. Tao [22] combining
with Strichartz estimates derived in [3] for the Schrödinger group on the
one-dimensional torus.

The aim of this paper is to improve the local and global well-posedness to
$H^{1/2}(T)$ which is the energy space for (BO). Recall that the momentum
and the energy for the Benjamin-Ono equation are given by

$$M(u) = \int_T u^2 \quad \text{and} \quad E(u) = \frac{1}{2} \int_T |D_x^{1/2}u|^2 + \frac{1}{6} \int_T u^3 . \quad (1)$$

Let us underline that in order to prove qualitative properties as stability of
travelling waves, a well-posedness result in the energy space is often very
useful. Our strategy is to combine the gauge transform of T. Tao with
estimates in Bourgain spaces.

1.2 Notations

For $x, y \in \mathbb{R}$, $x \sim y$ means that there exists $C_1, C_2 > 0$ such that
$C_1|x| \leq |y| \leq C_2|x|$ and $x \preceq y$ means that there exists $C_2 > 0$ such that
$|x| \leq C_2|y|$. For a Banach space $X$, we denote by $\| \cdot \|_X$ the norm in $X$.
We will use the same notations as in [6] and [7] to deal with Fourier trans-
form of space periodic functions with a large period $\lambda$. $(d\xi)_\lambda$ will be the
renormalized counting measure on $\lambda^{-1}\mathbb{Z}$ :

$$\int a(\xi) \cdot d\xi = \frac{1}{\lambda} \sum_{\xi \in \lambda^{-1}\mathbb{Z}} a(\xi) .$$

As written in [7], $(d\xi)_\lambda$ is the counting measure on the integers when $\lambda = 1$
and converges weakly to the Lebesgue measure when $\lambda \to \infty$. In all the
text, all the Lebesgue norms in $\xi$ will be with respect to the measure $(d\xi)_\lambda$.
For a $(2\pi\lambda)$-periodic function $\varphi$, we define its space Fourier transform on
$\lambda^{-1}\mathbb{Z}$ by

$$\hat{\varphi}(\xi) = \int_{\mathbb{R}/(2\pi\lambda)} e^{-i\xi x} f(x) \, dx, \quad \forall \xi \in \lambda^{-1}\mathbb{Z} .$$

We denote by $V(\cdot)$ the free group associated with the linearized Benjamin-
Ono equation,

$$\hat{V}(t) \hat{\varphi}(\xi) = e^{-i\xi|\xi|t} \hat{\varphi}(\xi), \quad \xi \in \lambda^{-1}\mathbb{Z} .$$
We define the Sobolev spaces $H^s_{\lambda}$ for $(2\pi\lambda)$-periodic functions by
\[
\|\varphi\|_{H^s_{\lambda}} = \|\langle \xi \rangle^s \hat{\varphi}(\xi)\|_{L^2_{\xi}} = \|J_s^\lambda \varphi\|_{L^2_{\lambda}},
\]
where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$ and $\hat{J^\lambda_s} \varphi(\xi) = \langle \xi \rangle^s \hat{\varphi}(\xi)$.
For $s \geq 0$, the closed subspace of zero mean value functions of $H^s_{\lambda}$ will be denoted by $\dot{H}^s_{\lambda}$.
The Lebesgue spaces $L^q_{\lambda}$, $1 \leq q \leq \infty$, will be defined as usually by
\[
\|\varphi\|_{L^q_{\lambda}} = \left( \int_{\mathbb{R}/(2\pi\lambda)\mathbb{Z}} |\varphi(x)|^q \, dx \right)^{1/q}
\]
with the obvious modification for $q = \infty$.
In the same way, for a function $u(t,x)$ on $\mathbb{R} \times \mathbb{R}/(2\pi\lambda)\mathbb{Z}$, we define its space-time Fourier transform by
\[
\hat{u}(\tau,\xi) = \mathcal{F}_{t,x}(u)(\tau,\xi) = \int_{\mathbb{R}} \int_{\mathbb{R}/(2\pi\lambda)\mathbb{Z}} e^{-i(\tau t + \xi x)} u(t,x) \, dx \, dt, \quad \forall (\tau,\xi) \in \mathbb{R} \times \lambda^{-1}\mathbb{Z}.
\]
We define the Bourgain spaces $X^{b,s}_{\lambda}$, $Z^{b,s}_{\lambda}$ and $Y^s_{\lambda}$ of $(2\pi\lambda)$-periodic (in $x$) functions respectively endowed with the norm
\[
\|u\|_{X^{b,s}_{\lambda}} = \|\langle \tau + \xi |\xi| \rangle^b \langle \xi \rangle^s \hat{u}\|_{L^2_{\tau,\xi}} = \|\langle \tau \rangle^b \langle \xi \rangle^s \mathcal{F}_{t,x}(V(t)u)\|_{L^2_{\tau,\xi}},
\]
\[
\|u\|_{Z^{b,s}_{\lambda}} = \|\langle \tau + \xi |\xi| \rangle^b \langle \xi \rangle^s \hat{u}\|_{L^2_{\xi}L^1_{\tau}} = \|\langle \tau \rangle^b \langle \xi \rangle^s \mathcal{F}_{t,x}(V(t)u)\|_{L^2_{\xi}L^1_{\tau}},
\]
and
\[
\|u\|_{Y^s_{\lambda}} = \|u\|_{X^{1/2,s}_{\lambda}} + \|u\|_{Z^{0,s}_{\lambda}}.
\]
For $T > 0$ and a function space $B_{\lambda} = X^{b,s}_{\lambda}$, $Z^{b,s}_{\lambda}$ or $Y^s_{\lambda}$, we denote by $B_{T,\lambda}$ the corresponding restriction in time space endowed with the norm
\[
\|u\|_{B_{T,\lambda}} = \inf_{w \in B_{\lambda}} \{ \|w\|_{B_{\lambda}}, \, w(\cdot) \equiv u(\cdot) \text{ on } [0,T] \}.
\]
Recall that $Y^s_{T,\lambda} \hookrightarrow Z^{0,s}_{T,\lambda} \hookrightarrow C([0,T];H^s_{\lambda})$.
$L^p_{\tau}L^q_{\lambda}$ and $L^p_{\tau}L^q_{\lambda}$ will denote the Lebesgue spaces
\[
\|u\|_{L^p_{\tau}L^q_{\lambda}} = \left( \int_{\mathbb{R}} \|u(t,\cdot)\|_{L^q_{\lambda}}^p \, dt \right)^{1/p} \quad \text{and} \quad \|u\|_{L^p_{\tau}L^q_{\lambda}} = \left( \int_{0}^{T} \|u(t,\cdot)\|_{L^q_{\lambda}}^p \, dt \right)^{1/p}
\]
with the obvious modification for $p = \infty$.
Finally, for all function spaces of $(2\pi\lambda)$-periodic functions, we will drop the
We will denote by $P_+$ and $P_-$ the projection on respectively the positive and the negative spatial Fourier modes. Moreover, for $a \geq 0$, we will denote by $P_a$ and $P_{>a}$ the projection on respectively the spatial Fourier modes of absolute value equal or less than $a$ and the spatial Fourier modes strictly larger than $a$.

### 1.3 Main result

**Theorem 1.1** For all $u_0 \in H^s(\mathbb{T})$ with $s \geq 1/2$ and all $T > 0$, there exists a unique solution

$$u \in C([0,T];H^s(\mathbb{T})) \cap X_{T,\lambda}^{1/2,0}$$

of the Benjamin-Ono equation (BO).

Moreover $u \in C_b(\mathbb{R},H^{1/2}(\mathbb{T}))$ and the map $u_0 \mapsto u$ is continuous from $H^s(\mathbb{T})$ into $C([0,T],H^s(\mathbb{T}))$ and Lipschitz on every bounded set from $H^s(\mathbb{T})$ into $C([0,T],H^s(\mathbb{T}))$.

**Remark 1.1** Actually, we prove that the flow-map is Lipschitz on every bounded set of any hyperplan of $H^s(\mathbb{T})$ with a prescribed mean value.

**Remark 1.2** For KdV-like equation,

$$u_t + \partial_x D^\alpha_x u = uu_x, \quad \alpha \geq 0, \quad (5)$$

one can easily prove (see Proposition 7.1 in the appendix) that the map $u_0 \mapsto u$ is not uniformly continuous in $H^s(\mathbb{T})$ for $s > 0$. This has no relation with the order of the dispersion and is only related to the nonlinear transport equation $u_t = uu_x$. More precisely, the key point is that if $u(t,x)$ is a solution of (5) then $u(t,x + \omega t) + \omega$, where $\omega$ is any constant, is also a solution (see [10] or [23]). Therefore, we think that a good notion of stability for this type of equations is the uniform continuity of the flow-map on bounded sets of hyperplans of functions with fixed mean value. Indeed, the restriction on such hyperplans prevent to perform the above transformation. In this point of view, (BO) shares the same behavior as KdV since by Theorem 1.1 and [5], the flow-map for these both equations is Lipschitz on any bounded balls of such hyperplans. This is in sharp contrast with the real line case where the flow-map of KdV is uniformly continuous on every bounded set of $H^s(\mathbb{R})$, $s \geq 0$ ([10]), whereas the flow-map of (BO) is not uniformly continuous on the unit ball of $H^s(\mathbb{R})$ for $s > 0$ ([16]).
The main tools to prove Theorem 1.1 are the gauge transformation of T. Tao and the Fourier restriction spaces introduced by Bourgain. Recall that in order to solve (BO), T. Tao \[22\] performed a kind of complex Cole-Hopf transformation

\[ W = P_\text{+} (e^{-iF/2}) \]

where \( F \) is a primitive of \( u \). In the periodic setting, requiring that \( u \) has mean value zero, we can take \( F = \partial_x^{-1}u \) the unique zero mean value primitive of \( u \). By the mean value theorem, it is then easy to check that the above gauge transformation is Lipschitz from \( L^2_\lambda \) to \( L^\infty_\lambda \). This property, which is not true on the real line, is crucial to derive the regularity of the flow-map. On the other hand, when one expresses \( u \) in terms of \( w = \partial_x W \) one gets something like

\[ u = e^{iF}w + R(u) \]

which is not so good since multiplication by gauge function as \( e^{iF} \) behaves not so well in Bourgain spaces \[2\]. For this reason we will take \( u \) and \( w \) with different space regularities in the scale of Bourgain spaces. Indeed, whereas \( w \) will belong to \( Y^{1/2}_{T,\lambda} \), \( u \) will belong to \( X^{1/2,0}_{T,\lambda} \cap C([0,T];H^{1/2}(T)) \). Note that this bad behavior is compensated by the fact that Bourgain spaces permit to gain one derivative when estimating the nonlinear term of the equation \[13\] satisfied by \( w \).

This paper is organized as follows: In the next section we recall some linear estimates in Bourgain type spaces. In Section 3 we introduce the gauge transform and establish the key nonlinear estimates. Section 4 is devoted to the proof of the local existence result for small data in \( \dot{H}^s(T) \) and Section 5 to the uniqueness and the regularity of the flow-map. Finally, in Section 6 we use dilation arguments to extend the result for arbitrary large data and thus prove Theorem 1.1. The appendix is divided in 3 parts. In the first part we present a proof communicated to us by N. Tzvetkov of the crucial linear estimate \[6\] first proved by Bourgain \[4\]. In the second part we present a brief proof of the lack of uniform continuity of the flow-map for KdV type equations on the circle (see Remark \[12\]). Finally the third part is devoted to the proof of technical lemmas needed in Section 3.

\[1\] Note that projecting (BO) on the non negative frequencies, one gets the following equation:

\[ \partial_t (P_+ u) - i\partial_x^2 P_+ u = -P_+ (uu_x) \]

\[2\] Let us note that Bourgain spaces do not enjoy an algebra property.
2 Linear Estimates

One of the main ingredients is the following linear estimate due to Bourgain [4]. We present in the appendix a shorter proof of this result.

\[ \| v \|_{L^4_{\lambda^2, \lambda}} \lesssim \| v \|_{X^{3/8,0}_{\lambda^2, \lambda}}. \]  

(6)

This estimate is proved in [4] for Bourgain spaces associated with the Schrödinger group and for a period equal to 1. The corresponding estimate for the Benjamin-Ono group follows by writing \( v \) as the sum of its positive and negative frequency parts. Also the estimate for any period \( \lambda \geq 1 \) follows directly from dilation arguments. The following classical lemma (cf. [10]) enables to deduce a localized version in time of the above estimate with a gain of a small factor of \( T \).

**Lemma 2.1** For any \( T > 0 \) and \( 0 \leq b \leq 1/2 \), it holds

\[ \| v \|_{X^{b,0}_{T, \lambda}} \lesssim T^{1/2-b} \| v \|_{X^{1/2,0}_{T, \lambda}}. \]  

(7)

Combining (6) and (7) we deduce that for \( 3/8 \leq b \leq 1/2 \) and \( 0 < T \leq \lambda^2 \), it holds

\[ \| v \|_{L^4_{T, \lambda}} \lesssim T^{b-3/8} \| v \|_{X^{b,0}_{T, \lambda}}. \]  

(8)

Let us now state some estimates for the free group and the Duhamel operator. Let \( \psi \in C^\infty_0([-2,2]) \) be a time function such that \( 0 \leq \psi \leq 1 \) and \( \psi \equiv 1 \) on \([1,1]\). The following linear estimates are well-known (cf. [4], [9]).

**Lemma 2.2** For all \( \varphi \in H^s_{\lambda} \), it holds:

\[ \| \psi(t)V(t)\varphi \|_{Y^s_{\lambda}} \lesssim \| \varphi \|_{H^s_{\lambda}}. \]  

(9)

**Lemma 2.3** For all \( G \in X^{1/2,s}_{-1, \lambda} \cap Z^{1,s}_{-1, \lambda} \), it holds

\[ \| \psi(t) \int_0^t V(t-t')G(t') \, dt' \|_{X^s_{\lambda}} \lesssim \| G \|_{X^{1/2,s}_{-1, \lambda}} + \| G \|_{Z^{1,s}_{-1, \lambda}}. \]  

(10)

Let us recall that (10) is a direct consequence of the following one dimensional (in time) inequalities (cf. [9]): for any function \( f \in S(\mathbb{R}) \), it holds

\[ \| \psi(t) \int_0^t f(t') \, dt' \|_{H^{1/2}_{-1, \lambda}} \lesssim \| f \|_{H^{-1/2}_{-1, \lambda}} + \| F_{\lambda} (f) \|_{L^1_{\lambda}} \]

and

\[ \| F_{\lambda} (\psi(t) \int_0^t f(t') \, dt') \|_{L^1_{\lambda}} \lesssim \| F_{\lambda} (f) \|_{L^1_{\lambda}}. \]
3 Gauge transform and nonlinear estimates

3.1 Gauge transform

Let $\lambda \geq 1$ and $u$ be a smooth $(2\pi \lambda)$-periodic solution of (BO) with initial data $u_0$. In the sequel, we assume that $u_0$ has mean value zero. Otherwise we do the change of unknown:

$$v(t, x) = u(t, x - t \int u) - \int u$$  \hspace{1cm} (11)

where $\int u = P_0(u) = \frac{1}{2\pi \lambda} \int_{\mathbb{R}/(2\pi \lambda) \mathbb{Z}} u$ is the mean value of $u$. Since $\int u$ is preserved by the flow, it is easy to see that $v$ satisfies (BO) with $u_0 - \int u_0$ as initial data. We are thus reduced to the case of zero mean value initial data. We define $F = \partial_x^{-1} u$ which is the periodic, zero mean value, primitive of $u$,

$$\hat{F}(0) = 0 \quad \text{and} \quad \hat{F}(\xi) = \frac{1}{i \xi} \hat{u}(\xi), \quad \xi \in \lambda^{-1} \mathbb{Z}^*.$$  

Following T. Tao [22], we introduce the gauge transform

$$W = P_+(e^{-iF/2})$$  \hspace{1cm} (12)

Since $F$ satisfies

$$F_t + \mathcal{H} F_{xx} = \frac{F_x^2}{2} - \frac{1}{2} \int F_x^2 = \frac{F_x^2}{2} - \frac{1}{2} P_0(F_x^2),$$

we can then check that $w = W_x = -i \frac{1}{2} P_+(e^{-iF/2} F_x) = -\frac{i}{2} P_+(e^{-iF/2} u)$ satisfies

$$w_t - iw_{xx} = -\partial_x P_+ [ e^{-iF/2} (P_-(F_{xx}) - \frac{i}{4} P_0(F_x^2)) ]$$

$$= -\partial_x P_+ (WP_-(u_x)) + \frac{i}{4} P_0(F_x^2) w.$$  \hspace{1cm} (13)

On the other hand, one can write $u$ as

$$u = e^{iF/2} e^{-iF/2} F_x = 2i e^{iF/2} \partial_x (e^{-iF/2}) = 2ie^{iF/2}w + 2ie^{iF/2} \partial_x P_-(e^{-iF/2})$$  \hspace{1cm} (14)

thus

$$P_{>1} u = 2i P_{>1} (e^{iF/2}w) + 2i P_{>1} (e^{iF/2} \partial_x P_-(e^{-iF/2}))$$

$$= 2i P_{>1} (e^{iF/2}w) + 2i P_{>1} (P_{>1}(e^{iF/2}) \partial_x P_-(e^{-iF/2})).$$  \hspace{1cm} (15)

The remaining of this section is devoted to the proof of the following crucial nonlinear estimates on $u$ and $w$.  

Proposition 3.1 Let \( u \in X_{T,\lambda}^{1/2,0} \cap L_T^\infty \dot{H}_x^s \) be a solution of (BO) and \( w \in Y_{T,\lambda}^s \) satisfying (13)-(14). Then for \( 0 < T \leq 1 \) and \( 0 \leq s \leq 1 \), it holds

\[
\|u\|_{Y_{T,\lambda}^s} \lesssim (1 + \|u_0\|_{L_x^2}) \|u_0\|_{H_x^s} + T^{1/8} \|w\|_{X_{T,\lambda}^{1/2,s}} \left( \|u\|_{X_{T,\lambda}^{1/2,0}} + \|u\|_{X_{T,\lambda}^{1/2,0}}^2 \right).
\]

(16)

Moreover for \( 1/2 \leq s \leq 1 \) and \((p,q) = (\infty,2)\) or \((4,4)\),

\[
\|J_x^s u\|_{L_T^p L_x^q} \lesssim \|u_0\|_{L_x^2} + (1 + \|u\|_{L_T^\infty H_x^{1/2}}) \left( \|w\|_{Y_{T,\lambda}^s} + \|u\|_{L_T^\infty H_x^{1/2}}^2 \right).
\]

(17)

Remark 3.1 It is worth noting that (18) can also be rewritten in a convenient way for \( s \geq 0 \). We choose the above expression involving the \( L_T^\infty H_x^{1/2} \)-norm only for simplicity. The restriction \( s \geq 1/2 \) in our well-posedness result is due to the loss of one half derivative in (17) which can be explained by the bad behavior of Bourgain spaces with respect to multiplication (see [14]).

3.2 Proof of Proposition 3.1

Let us first prove (18). In this purpose, we need the two following lemmas proven in the appendix (see also [17] and [18]). The first one treat the multiplication with the gauge function \( e^{-iF/2} \) in Sobolev spaces. The second one shows that, due to the frequency projections, we can share derivatives when taking the \( H^s \)-norm of the second term of the right-hand side to (14).

Lemma 3.1 Let \( 2 \leq q < \infty \). Let \( f_1 \) and \( f_2 \) be two real-valued functions of \( L_x^q \) with mean value zero and let \( g \in L_x^q \) such that \( J_x^\alpha g \in L_x^q \) with \( 0 \leq \alpha \leq 1 \). Then

\[
\left\| J_x^\alpha \left( e^{i\partial_x^{-1} f_1} g \right) \right\|_{L_x^q} \lesssim \|J_x^\alpha g\|_{L_x^q} (1 + \|f_1\|_{L_x^q}),
\]

(19)

and

\[
\left\| J_x^\alpha \left( (e^{-i\partial_x^{-1} f_1} - e^{-i\partial_x^{-1} f_2}) g \right) \right\|_{L_x^q} \lesssim \|J_x^\alpha g\|_{L_x^q} \left( \|f_1 - f_2\|_{L_x^q} + \|e^{-i\partial_x^{-1} f_1} - e^{-i\partial_x^{-1} f_2}\|_{L_x^\infty} (1 + \|f_1\|_{L_x^q}) \right).
\]

(20)
Lemma 3.2 Let $\alpha \geq 0$ and $1 < q < \infty$ then

$$
\left\| D_x^\alpha P_+ \left( f P_ - \partial_x g \right) \right\|_{L^q_x} \lesssim \left\| D_x^{\gamma_1} f \right\|_{L^\infty_x} \left\| D_x^{\gamma_2} g \right\|_{L^q_x},
$$

with $1 < q_i < \infty$, $1/q_1 + 1/q_2 = 1/q$ and \( \gamma_1 \geq \alpha, \gamma_2 \geq 0 \) and \( \gamma_1 + \gamma_2 = \alpha + 1 \).

We first prove Lemma 3.1. Since $u$ is real-valued, it holds

$$
\left\| J_x^s u \right\|_{L^p_x L^q_x} \lesssim \left\| P_1 u \right\|_{L^p_x L^q_x} + \left\| D_x^s P_{>1} u \right\|_{L^p_x L^q_x}.
$$

From Lemma 3.1, Lemmas 3.1-3.2 Sobolev inequalities and (8), we infer that for $0 < T \leq 1$, $1/2 \leq s \leq 1$ and $(p, q) = (\infty, 2)$ or $(4, 4)$,

$$
\left\| D_x^s P_{>1} u \right\|_{L^p_x L^q_x} \lesssim (1 + \left\| u \right\|_{L^\infty_T L_x^1}) \left\| J_x^s w \right\|_{L^p_x L^q_x} + \left\| D_x^s P_{>1} e^{iF/2} \right\|_{L^\infty_T L_x^q} \left\| u \right\|_{L^p_x L^q_x} \\
\quad \lesssim (1 + \left\| u \right\|_{L^\infty_T H^{1/2}_x}) \left\| w \right\|_{L^p_x L^q_x} + \left\| T^{1/p} D_x^{s+1/2-1/(2q)} P_{>1} e^{iF/2} \right\|_{L^\infty_T L_x^1} \left\| u \right\|_{L^p_x H^{1/2}_x},
$$

with

$$
\left\| D_x^{s+1/2-1/(2q)} P_{>1} e^{iF/2} \right\|_{L^\infty_T L_x^2} \lesssim \left\| u \right\|_{L^\infty_T H^{1/2}_x} (1 + \left\| u \right\|_{L^\infty_T L_x^2}).
$$

On the other hand, by the Duhamel formulation of the equation, the unitarity of $V(t)$ in $L^2_x$, the continuity of $\partial_x P_1$ in $L^2_x$ and Sobolev inequalities, we get for $(p, q) = (\infty, 2)$ or $(4, 4)$,

$$
\left\| P_1 u \right\|_{L^p_x L^q_x} \lesssim \left\| u_0 \right\|_{L^p_x} + \left\| u^2 \right\|_{L^1_x L^q_x} \lesssim \left\| u_0 \right\|_{L^p_x} + T \left\| u \right\|_{L^p_x H^{1/2}_x}^2.
$$

This completes the proof of Lemma 3.1.

To prove (17) we start by noticing that

$$
\left\| u \right\|_{X^{1,s-1}_{p,q}} \lesssim \left\| J_x^{s-1} u \right\|_{L^2_x} + \left\| J_x^{s-1} (u_1 + H u_{xx}) \right\|_{L^2_T},
$$

and thus by the equation and Leibniz rule for fractional derivatives (cf. (13)) we deduce that

$$
\left\| u \right\|_{X^{1,s-1}_{p,q}} \lesssim T^{1/2} \left\| u \right\|_{L^\infty_T L^2_x} + \left\| u \right\|_{L^4_T L^\infty_x} \left\| J_x^s u \right\|_{L^4_T}.
$$
Interpolating between (23) and the obvious estimate
\[ \|u\|_{X^{0,s}_{T,\lambda}} \lesssim T^{1/2} \|u\|_{L^\infty_t H^s_x}, \]
(17) follows.

To prove (16), we will first need to establish two non-linear estimates. These estimates enlighten the good behavior in Bourgain spaces of the non-linear term of (13). The frequency projections lead to the smoothing relation (28) which enables somehow to gain one derivative. In the following lemmas we will assume that the functions are supported in time in \([-2T, 2T]\).

Moreover, since all the norms appearing in the right-hand side of the inequalities only see the size of the module of the Fourier transform, we can always assume that all the functions have non-negative Fourier transforms.

**Lemma 3.3** For any \(s \geq 0\) and \(0 < T \leq 1\),
\[ \left\| \partial_x P_+ (W P_- (u_x)) \right\|_{X^{0,-1/2,s}_{\lambda}} \lesssim T^{1/4} \left\| W_x \right\|_{X^{1/2,s}_{\lambda}} \|u\|_{X^{1/2,0}_{\lambda}}. \]
(24)

**Proof.** As we wrote above, we assume that the functions have time support in \([-2T, 2T]\) and non-negative Fourier transforms. By duality it thus suffices to show that
\[ I = \left| \int_A \langle \xi \rangle^{s} \hat{h}(\tau, \xi) \langle \xi_1 \rangle^{s} \hat{f}(\tau_1, \xi_1) \hat{g}(\tau_2, \xi_2) \right| \]
\[ \lesssim T^{1/4} \left\| h \right\|_{L^2_{t,\lambda}} \left\| f \right\|_{L^2_{t,\lambda}} \left\| g \right\|_{L^2_{t,\lambda}} \]
(25)
where \((\tau_2, \xi_2) = (\tau - \tau_1, \xi - \xi_1), \sigma = \sigma(\tau, \xi) = \tau + \xi |\xi|, \sigma_i = \sigma(\tau_i, \xi_i), i = 1, 2,\)

and, due to the frequency projections, the domain of integration \(A \subset \mathbb{R}^2 \times (\lambda^{-1} \mathbb{Z})^2\) is given by
\[ A = \{ (\tau, \tau_1, \xi, \xi_1) \in \mathbb{R}^2 \times (\lambda^{-1} \mathbb{Z})^2, \xi \geq 1/\lambda, \xi_1 \geq 1/\lambda, \xi - \xi_1 \leq -1/\lambda \} . \]

Note that in the domain of integration above,
\[ \xi_1 \geq |\xi_2| \quad \text{and} \quad \xi_1 \geq \xi . \]
(26)

We thus get
\[ I \lesssim \int_A \frac{\xi_1^{1/2} \hat{h}(\tau, \xi) \hat{f}(\tau_1, \xi_1) |\xi_2|^{1/2} \hat{g}(\tau_2, \xi_2)}{\langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} \]
(27)
Moreover, in $A$, we have

$$\sigma_1 + \sigma_2 - \sigma = \xi_1^2 - \xi_2^2 - \xi^2 = -2\xi_2 \xi \ .$$

(28)

Therefore, if $\sigma$ is dominant then by Plancherel and (8),

$$I \leq \int_A \hat{h}(\tau, \xi) \hat{f}(\tau_1, \xi_1) \hat{g}(\tau_2, \xi_2)$$

$$\lesssim \|h\|_{L^2_{t,\lambda}} \|F^{-1}(\hat{f})\|_{L^4_{t,\lambda}} \|F^{-1}(\hat{g})\|_{L^4_{t,\lambda}}$$

$$\lesssim T^{1/4} \|h\|_{L^1_{t,\lambda}} \|f\|_{L^1_{t,\lambda}} \|g\|_{L^2_{t,\lambda}} \ .$$

(29)

Finally, it is clear that the cases $\sigma_1$ and $\sigma_2$ dominant can be treated in exactly the same way.

**Lemma 3.4** For any $s \geq 0$ and $0 < T \leq 1$,

$$\left\| \partial_x P_+(W P_-(u_x)) \right\|_{X^{-1/2+\varepsilon,s}_\lambda} \lesssim T^{1/8} \|W_x\|_{X^{1/2,s}_\lambda} \|u\|_{X^{1/2,0}_\lambda} \ .$$

(30)

**Proof.** First note that by Cauchy-Schwarz in $\tau$,

$$\left\| \frac{(\xi)^s}{\langle \sigma \rangle} F(\partial_x P_+(W P_-(u_x))) \right\|_{L^2_{t,1}} \lesssim \left\| \partial_x P_+(W P_-(u_x)) \right\|_{X^{-1/2+\varepsilon,s}_\lambda}, \ \varepsilon > 0,$$

which can be estimated in the regions $\{\langle \sigma_1 \rangle \geq \langle \sigma \rangle / 10\}$ and $\{\langle \sigma_2 \rangle \geq \langle \sigma \rangle / 10\}$ as in the proof of the preceding lemma (with $T^{1/8}$ instead of $T^{1/4}$), since we only need the weight $\langle \sigma \rangle^{3/8}$ in those regions.

Moreover, in the region $\{\xi_1 \leq 1\}$, using (26) and then (8) it is easy to show that

$$\left\| \partial_x P_+(W P_-(u_x)) \right\|_{X^{-1/2+\varepsilon,s}_\lambda} \lesssim \left\| W_x u \right\|_{L^2_{t,\lambda}}$$

$$\lesssim \left\| W_x \right\|_{L^4_{t,\lambda}} \left\| u \right\|_{L^4_{t,\lambda}}$$

$$\lesssim T^{1/4} \left\| W_x \right\|_{X^{1/2,0}_\lambda} \left\| u \right\|_{X^{1/2,0}_\lambda} .$$

Now in the region $\{\langle \sigma \rangle \geq 10\langle \sigma_1 \rangle, \ \langle \sigma \rangle \geq 10\langle \sigma_2 \rangle, \ \xi_1 > 1\}$, we proceed as in [7]. By (26), in this region we have :

$$\langle \sigma \rangle \sim \langle \xi \xi_2 \rangle \ .$$

(31)
By symmetry we can moreover assume that $|\sigma_1| \geq |\sigma_2|$. We note that proving (30) is equivalent to proving

$$I \lesssim \|f\|_{L^2_{\tau,\xi}} \|g\|_{L^2_{\tau,\xi}} ,$$

where

$$I = \left\| \chi_{(\xi \geq 1/\lambda)} \int_{B(\tau,\xi)} \frac{(\xi)^s \xi (\xi_1)^{-s} (\xi_1)^{-1} \hat{f}(\tau_1,\xi_1)|\xi_2| \hat{g}(\tau_2,\xi_2)}{(\sigma)^{1/2}(\sigma_1)^{1/2}} \right\|_{L^2_{\tau}L^1_{\xi}}$$

with $B(\tau,\xi) \subset \mathbb{R} \times \lambda^{-1} \mathbb{Z}$ given by

$$B(\tau,\xi) = \{(\tau_1,\xi_1) \in \mathbb{R} \times \mathbb{Z}/\lambda, \xi_1 \geq 1, \xi - \xi_1 \leq -1/\lambda, \langle \sigma \rangle \geq 10\langle \sigma_1 \rangle, |\sigma_1| \geq |\sigma_2|\} .$$

Recall that (26) holds in the domain of integration of (33). We divide this domain into 2 subregions.

- The subregion $\max(|\sigma_1|,|\sigma_2|) \geq (\xi|\xi_2|)^{1/16}$. We will assume that $\max(|\sigma_1|,|\sigma_2|) = |\sigma_1|$ since the other case can be treated in exactly the same way. Then, by (26) and (31), we get

$$I \lesssim \left\| \int_{B_1(\tau,\xi)} \frac{\hat{f}(\tau_1,\xi_1)\hat{g}(\tau_2,\xi_2)}{\langle \sigma \rangle^{1/2}\langle \sigma_1 \rangle^{3/8}\langle \sigma_2 \rangle^{1/2}} \right\|_{L^2_{\tau}L^1_{\xi}}$$

where

$$B_1(\tau,\xi) = \{(\tau_1,\xi_1) \in B(\tau,\xi), \sigma_1 \geq (\xi|\xi_2|)^{1/16}\}$$

and by applying Cauchy-Schwarz in $\tau$ we obtain thanks to (3),

$$I \lesssim \left\| \int_{B_1(\tau,\xi)} \frac{\hat{f}(\tau_1,\xi_1)\hat{g}(\tau_2,\xi_2)}{\langle \sigma_1 \rangle^{3/8}\langle \sigma_2 \rangle^{1/2}} \right\|_{L^2_{\tau}L^2_{\xi},\lambda} \lesssim \left\| \mathcal{F}^{-1}\left(\frac{\hat{f}}{\langle \sigma \rangle^{3/8}}\right)\right\|_{L^4_{\lambda}} \left\| \mathcal{F}^{-1}\left(\frac{\hat{g}}{\langle \sigma \rangle^{1/2}}\right)\right\|_{L^4_{\lambda}} \lesssim T^{1/8}\|f\|_{L^2_{\tau,\lambda}}\|g\|_{L^2_{\tau,\lambda}} .$$

- The subregion $\max(|\sigma_1|,|\sigma_2|) \leq (\xi|\xi_2|)^{1/16}$. Changing the $\tau,\tau_1$ integrals in $\tau_1,\tau_2$ integrals in (33) and using (31), we infer that

$$I \lesssim \left\| \int_{|\xi|\geq 1/\lambda} \int_{\tau_1 = -\xi_1^2 + O(|\xi|\xi_2|/16)} \frac{\hat{f}(\tau_1,\xi_1)}{\langle \tau_1 + \xi_1^2 \rangle^{1/2}} \int_{\tau_2 = \xi_2^2 + O(|\xi|\xi_2|/16)} \frac{\hat{g}(\tau_2,\xi_2)}{\langle \tau_2 - \xi_2^2 \rangle^{1/2}} \right\|_{L^2_{\xi}}$$
with \( C(\xi) = \{ \xi_1 \in \lambda^{-1} \mathbb{Z}, \xi_1 \geq 1, \xi - \xi_1 \leq -1/\lambda \} \). Applying Cauchy-Schwarz inequality in \( \tau_1 \) and \( \tau_2 \) and recalling that \( \xi_1 \geq 1 \) we get

\[
I \lesssim \left\| \chi_{\{\xi \geq 1/\lambda\}} \int_{C(\xi)} \langle \xi_1 \rangle^{-1} \langle \xi_2 \rangle \hat{K}_1(\xi_1) \hat{K}_2(\xi_2) \right\|_{L^2_\xi}
\]

where

\[
K_1(\xi) = \left( \int_{\tau} \frac{\hat{f}(\tau, \xi)^2}{(\tau + \xi^2)} \right)^{1/2} \quad \text{and} \quad K_2(\xi) = \left( \int_{\tau} \frac{\hat{g}(\tau, \xi)^2}{(\tau - \xi^2)} \right)^{1/2}.
\]

Therefore, by using \( \| \| \), Hölder and then Cauchy-Schwarz inequalities,

\[
I \lesssim \left\| \langle \xi \rangle^{-\frac{3}{2}} \int_{\xi_1 \in \lambda^{-1} \mathbb{Z}} K_1(\xi_1) K_2(\xi_2) \right\|_{L^2_\xi}
\]

\[
\lesssim \left\| \int_{\xi_1 \in \lambda^{-1} \mathbb{Z}} K_1(\xi_1) K_2(\xi_2) \right\|_{L^\infty_\xi}
\]

\[
\lesssim \left( \int_{\xi \in \lambda^{-1} \mathbb{Z}} K_1(\xi)^2 \right)^{1/2} \left( \int_{\xi \in \lambda^{-1} \mathbb{Z}} K_2(\xi)^2 \right)^{1/2}
\]

\[
\lesssim \| f \|_{X^{-1/2,0}_\lambda} \| g \|_{X^{-1/2,0}_\lambda}
\]

\[
\lesssim T \| f \|_{L^2_{t,\lambda}} \| g \|_{L^2_{t,\lambda}},
\]

(36)

where we used the dual estimate of \( \| \| \) in the last step.

### 3.2.1 Proof of (16).

To complete the proof of \( \| \| \) it remains to treat the second term of the right-hand side of \( \| \| \). This term is mainly harmless. Indeed by Cauchy-Schwarz inequality in \( \tau \), Sobolev inequalities in time and Minkowski inequality,

\[
\| P_0(u^2)w \|_{Z^{-1,s}_\lambda} + \| P_0(u^2)w \|_{X^{-1/2,s}_\lambda} \lesssim \| P_0(u^2)w \|_{X^{-1/2+\varepsilon,\varepsilon'}_\lambda} \lesssim \| P_0(u^2)w \|_{L^{1+\varepsilon}H^{s}_\lambda},
\]

for some \( 0 < \varepsilon, \varepsilon' \ll 1 \). Assuming that \( u \) and \( w \) are supported in time in \([-2T, 2T]\), by Hölder inequality in time and \( \| \| \) we get

\[
\| P_0(u^2)w \|_{L^{1+\varepsilon}H^{s}_\lambda} \lesssim T^{1/8} \| J_x^s w \|_{L^4_x L^4_\lambda} \| P_0(u^2) \|_{L^4_x L^4_\lambda} \lesssim T^{1/8} \| w \|_{X^{1/2,s}_\lambda} \| P_0(u^2) \|_{L^2_{t,\lambda}},
\]

where we used that \( \| \|_{L^4_\lambda} \leq \| \|_{L^2_\lambda} \) since \( \lambda \geq 1 \). Hence, the following estimate holds:

\[
\| P_0(u^2)w \|_{Z^{-1,s}_\lambda} + \| P_0(u^2)w \|_{X^{-1/2,s}_\lambda} \lesssim T^{1/8} \| w \|_{X^{1/2,s}_\lambda} \| u \|_{X^{1/2,s}_\lambda}^2.
\]

(37)
Now, by the Duhamel formulation of (13), for \( 0 < T \leq 1 \) and \(-T \leq t \leq T\), we have
\[
w(t) = \psi(t) \left[ V(t)w(0) - \int_0^t V(t-t')\partial_x P_+ \left( P_-(\psi_T \tilde{u}_x)\psi_T \tilde{W} \right)(t') \, dt' \right] + \frac{i}{4} \int_0^t V(t-t') \left( P_0(\psi_T^2 \tilde{u}^2)\psi_T \tilde{W}_x \right)(t') \, dt',
\]
where \( \psi_T(\cdot) = \psi(\cdot/T) \), \( \tilde{u} \) is an extension of \( u \) satisfying \( \| \tilde{u} \|_{X^{1/2,0}_\lambda} \leq 2 \| u \|_{X^{1/2,0}_T} \) and \( \tilde{W} \) is an extension of \( W \) satisfying \( \| \tilde{W}_x \|_{X^{1/2,s}_\lambda} \leq 2 \| W_x \|_{X^{1/2,s}_T} \). At this stage, it is worth noticing that the multiplication by \( \psi_T \) is continuous in \( X^{1/2}_s \lambda \) and \( Y^s \lambda \) with a norm which does not depend on \( T > 0 \) or \( s \in \mathbb{R} \). Therefore, combining Lemmas 2.2-2.3, 3.3-3.4 and (37), we infer that for \( s \geq 0 \),
\[
\| w \|_{Y^s_{T,\lambda}} \lesssim \| w(0) \|_{H^s} + T^{1/8} \| w \|_{X^{1/2,s}_{T,\lambda}} \left( \| u \|_{X^{1/2,0}_T} + \| u \|_{X^{1/2,0}_{T,\lambda}}^2 \right).
\]
This proves (16) since by Lemma 3.1 for \( 0 \leq s \leq 1 \),
\[
\| w(0) \|_{H^s} = \| \partial_x P_+ e^{-i\partial_x^{-1}u_0/2} \|_{H^s} \lesssim \| u_0 e^{-i\partial_x^{-1}u_0/2} \|_{H^s} \lesssim (1 + \| u_0 \|_{L^2_\lambda}) \| u_0 \|_{H^s}.
\]

4 Local existence for small data

We will now prove the local well-posedness result for small data, the result for arbitrary large data will then follow from scaling arguments. More precisely, for some small \( 0 < \varepsilon \ll 1 \) depending only on the implicit constant contained in the above estimates\(^3\), we will prove a local well-posedness result for initial data belonging to the closed ball \( B_{\varepsilon,\lambda} \) of \( \dot{H}^{1/2}_\lambda \) defined by
\[
B_{\varepsilon,\lambda} = \left\{ \varphi \in \dot{H}^{1/2}_\lambda, \| \varphi \|_{H^{1/2}_\lambda} \lesssim \varepsilon \right\},
\]
with \( \lambda \geq 1 \).

4.1 Uniform estimate

Let \( u_0 \) belonging to \( \dot{H}^{\infty}_\lambda \cap B_{\varepsilon,\lambda} \). We want first to show that the emanating solution \( u \in C(\mathbb{R}; \dot{H}^{\infty}_\lambda) \), given by the classical well-posedness results (cf.\(^3\))
\(^3\)In this stage, it worth recalling that these implicit constants do not depend on the period \( \lambda \).
satisfies
\[ \|u\|_{X^{1/2,0}_{1,\lambda}} + \|u\|_{L^\infty_t H^{1/2}_\lambda} \lesssim \varepsilon^2 \quad \text{and} \quad \|w\|_{Y^{1/2}_{1,\lambda}} \lesssim \varepsilon^2 . \] (39)

Clearly, since \( u \) satisfies the equation, \( u \) belongs in fact to \( C^\infty(\mathbb{R}; H^\infty_\lambda) \). Thus, for any \( 0 < T \leq 1 \), \( u \) and \( w \) belong to \( Y^\infty_{T,\lambda} \) and from the linear estimates we easily deduce that
\[ \|u\|_{X^{1/2,0}_{T,\lambda}} \lesssim \|u_0\|_{L^2} + \|\partial_x (u^2)\|_{L^2_{T,\lambda}} \lesssim \|u_0\|_{H^{1/2}} + T^{1/2}\|u\|_{L^\infty_T H^1_\lambda} \]
Recalling also (16), by a continuity argument we can thus assume that
\[ \|u\|_{X^{1/2,0}_{T,\lambda}} + \|u\|_{L^\infty_t H^{1/2}_\lambda} \lesssim \varepsilon \quad \text{and} \quad \|w\|_{Y^{1/2}_{T,\lambda}} \lesssim \varepsilon \] (40)
for some \( 0 < T < 1 \). But (16) then clearly ensures that \( \|w\|_{Y^{1/2}_{T,\lambda}} \lesssim \varepsilon^2 \) and (18) ensures that
\[ \|J_{1/2} u\|_{L^1_{T,\lambda}} + \|u\|_{L^\infty_t H^{1/2}_\lambda} \lesssim \varepsilon^2 . \]
We thus deduce from (17) that \( \|u\|_{X^{1/2,0}_{T,\lambda}} \lesssim \varepsilon^2 \) and (39) is proven. It then follows from (16) and (18) that for \( 1/2 \leq s \leq 1 \),
\[ \|u\|_{L^\infty_t H^s_\lambda} + \|w\|_{Y^s_{T,\lambda}} \lesssim (1 + \|u_0\|_{L^2_\lambda})\|u_0\|_{H^s_\lambda} . \] (41)

4.1.1 Local existence

Let \( u_0 \in B_{\varepsilon,\lambda} \cap H^s_\lambda \) with \( 1/2 \leq s \leq 1 \) and let \( \{u_0^n\} \subset \dot{H}^\infty(T) \cap B_{\varepsilon,\lambda} \) converging to \( u_0 \) in \( H^s(T) \). We denote by \( u_n \) the solution of (BO) emanating from \( u_0^n \). From standard existence theorems (see for instance [1], [11]), \( u_n \in C(\mathbb{R}; \dot{H}^\infty_\lambda) \). According to (39),
\[ \|u_n\|_{X^{1/2,0}_{1,\lambda}} + \|u_n\|_{L^\infty_t H^{1/2}_\lambda} \lesssim \varepsilon^2 \]
and (11) ensures that
\[ \|u_n\|_{L^\infty H^s_\lambda} \lesssim (1 + \|u_0\|_{L^2_\lambda})\|u_0\|_{H^s_\lambda} \]
uniformly in \( n \). We can thus pass to the limit up to a subsequence. We then obtain the existence of a solution \( u \in X^{1/2,0}_{1,\lambda} \cap L^\infty_t \dot{H}^s_\lambda \) to the Benjamin-Ono equation with \( u_0 \) as initial data (there is no problem to pass to the limit on the nonlinear term here).
5 Continuity, uniqueness and regularity of the flow map for small data solutions

We are going to prove that the flow-map is Lipschitz from $B_{\varepsilon,\lambda} \cap H^s_\lambda$ to $X^{1/2,0}_{1,\lambda} \cap L_1^- H^s_\lambda$. The continuity of $t \mapsto u(t)$ in $H^s_\lambda$ will follow directly. So, let $u_1$ and $u_2$ be two solutions of (BO) in $X^{1/2,0}_{1,\lambda} \cap C([0,T];\dot{H}^s_\lambda)$ associated with initial data $\varphi_1$ and $\varphi_2$ in $B_{\varepsilon,\lambda} \cap H^s_\lambda$. We assume that they satisfy

\[
\|u_i\|_{L^\infty_T \dot{H}^{s/2}} + \|u_i\|_{X^{1/2,0}_{1,\lambda}} \lesssim \varepsilon^2, \quad i = 1, 2.
\] (42)

for some $0 < T \leq 1$ and where $0 < \varepsilon \ll 1$ is taken as above.

We set $W_i = P_+(e^{-iF_i/2})$ with $F_i = \partial_x^{-1} u_i$.

5.1 Regularity and estimate on $w_i = \partial_x P_+(e^{-i\partial_x^{-1} u_i/2})$

The first step consists in showing that $w_i = \partial_x P_+(e^{-iF_i/2})$, $i = 1, 2$, belongs to $Y_{T,\lambda}^s$ and satisfies (13) with $u$ and $u_0$ replaced by $u_i$ and $\varphi_i$. To simplify the notations, we drop the index $i$ for a while. Since $u \in C([0,T];H^{1/2}_\lambda) \cap X^{1/2,0}_{1,\lambda}$ and satisfies (BO), $u_t \in C([0,T];H^{-3/2}_\lambda)$. Therefore $F \in C([0,T];H^{3/2}_\lambda) \cap C^1([0,T];H^{-1/2}_\lambda) \cap X^{1/2,1}_{1,\lambda}$. The following calculations are thus justified:

\[
\partial_t W = \partial_t P_+(e^{-iF/2}) = -\frac{i}{2} P_+ (F_t e^{-iF/2}) = -\frac{i}{2} P_+ \left( e^{-iF/2}(-\mathcal{H} F_{xx} + F_x^2/2 - P_0(F_x^2)/2) \right)
\]

and

\[
\partial_x W = \partial_x P_+(e^{-iF/2}) = P_+ \left( e^{-iF/2}(-F_x^2/4 - iF_{xx}/2) \right)
\] .

It follows that $W$ satisfies at least in a distributional sens,

\[
W_t - iW_{xx} = -P_+ (e^{-iF/2}(P_- F_{xx} - iP_0(F_x^2))/4) = -P_+ (W P_- F_{xx}) + \frac{i}{4} P_0(F_x^2) W .
\] (43)

Therefore $w = \partial_x W$ satisfies and $W = \partial_x^{-1} w = W - \int W$ satisfies:

\[
\begin{cases}
\hat{W}_t - i\hat{W}_{xx} = -P_{>0}(\hat{W} P_- F_{xx}) + \frac{i}{4} P_0(F_x^2) \hat{W} \\
W(0) = W(0) - \int W(0)
\end{cases}
\] (44)

Since $F \in C([0,T];H^{3/2}_\lambda)$, one has $\hat{W} \in C([0,T];H^{3/2}_\lambda) \hookrightarrow X^{0,3/2}_{T,\lambda}$. Moreover, using Lemma 3.2 one can easily check that the right-hand side member
of \( L_1 \) belongs to \( C([0, T]; L^2_X) \). Therefore, by \( \ref{14} \), \( \hat{W} \in X_{T, \lambda}^{1, 0} \) and by interpolation we deduce that \( \hat{W} \in Y_{T, \lambda}^{1/2} \). On the other hand, on account of estimate \( \ref{16} \) it is easy to construct by a Picard iterative scheme a zero mean value solution of \( \ref{44} \) (for \( F \in X_{T, \lambda}^{1/2, 1} \) given) which belongs to \( Y_{T, \lambda}^{s+1} \) and such that \( w = \hat{W}_x \) satisfies \( \ref{16} \). Therefore, if we prove the uniqueness of the solution \( \hat{W} \) of \( \ref{44} \) in \( Y_T^{1/2} \cap C([0, T]; H_T^{1/2}) \), for any fixed \( F \in X_{T, \lambda}^{1/2, 1} \), we are done. To prove this uniqueness result, we need the following two lemmas, the proof of which are slight modifications of those of Lemmas \( \ref{3.3} \) and \( \ref{3.4} \). As in Section 3, we assume that \( F \) and \( \hat{W} \) are supported in time in \([-2T, 2T] \).

**Lemma 5.1** For any \( 0 < T < 1 \) and \( \lambda \geq 1 \),

\[
\left\| P_{>0}(\hat{W}P_-(F_{xx})) \right\|_{X_{\lambda}^{-1/2,1/2}} \lesssim T^{1/4} \lambda^{1/2} \| \hat{W} \|_{X_{\lambda}^{1/2,1/2}} \| F_x \|_{X_{\lambda}^{1/2,0}} \tag{45}
\]

**Proof.** As mentioned above, the proof is essentially the same as for Lemma \( \ref{3.3} \). By duality it thus suffices to show that

\[
I = \left| \int_A \frac{\langle \xi \rangle^{1/2} \hat{h}(\tau, \xi) \langle \xi_1 \rangle^{-1/2} \hat{f}(\tau_1, \xi_1) \xi_2 \hat{g}(\tau_2, \xi_2)}{\langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} \right| \lesssim T^{1/4} \lambda^{1/2} \| h \|_{L_{t, \lambda}^2} \| f \|_{L_{t, \lambda}^2} \| g \|_{L_{t, \lambda}^2} \tag{46}
\]

with the same set \( A \) as in \( \ref{25} \). We divide \( A \) into two regions.

- **\( \xi \geq |\xi_2| \).** Then by \( \ref{26} \),

\[
I \lesssim \int_A \frac{\hat{h}(\tau, \xi) \hat{f}(\tau_1, \xi_1) \xi_2 \hat{g}(\tau_2, \xi_2)}{\langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}
\]

and the result follows, since by \( \ref{28} \), \( |\sigma_1 + \sigma_2 - \sigma| \geq 2|\xi_2|^2 \).

- **\( \xi \leq |\xi_2| \).** Since \( \xi \geq 1/\lambda \) in \( A \), we have

\[
I \lesssim \int_A \frac{\langle \xi \rangle^{1/2} \hat{h}(\tau, \xi) \hat{f}(\tau_1, \xi_1) \xi_2^{1/2} \hat{g}(\tau_2, \xi_2)}{\langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} \lesssim \lambda^{1/2} \int_A \frac{\langle \xi \rangle^{1/2} \hat{h}(\tau, \xi) \hat{f}(\tau_1, \xi_1) \xi_2^{1/2} \hat{g}(\tau_2, \xi_2)}{\langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}
\]

and the result follows thanks to \( \ref{28} \).

**Lemma 5.2** For any \( 0 < T < 1 \) and \( \lambda \geq 1 \),

\[
\left\| P_{>0}(\hat{W}P_-(F_{xx})) \right\|_{Z_{\lambda}^{-1,1/2}} \lesssim T^{1/8} \lambda^{1/2} \| \hat{W} \|_{X_{\lambda}^{1/2,1/2}} \| F_x \|_{X_{\lambda}^{1/2,0}} \tag{47}
\]

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Proof. The proof is the same as for Lemma 3.4 up to some straightforward modifications similar to the ones of the preceding lemma. It will thus be omitted.

On the other hand, proceeding exactly as for the obtention of (37), it is easy to see that
\[
\|P_0(F_x^2)\tilde{W}\|_{Z^{-1,s}_\lambda} + \|P_0(F_x^2)\tilde{W}\|_{X_{\lambda}^{-1/2,s}} \lesssim T^{1/8}\|\tilde{W}\|_{X_{\lambda}^{1/2,s}}\|F_x\|_{X_{\lambda}^{1/2,0}}^2. \tag{48}
\]
Combining (44), (48), Lemmas 5.1-5.2 and Lemma 2.2-2.3 and proceeding as in the proof of (16), we deduce that for \(F \in X_{\lambda,T}^{1/2}\), fixed there exists a unique solution \(\tilde{W} \in Y_{1/2}^{s} \cap C([0,T];\dot{H}_\lambda^{1/2})\) of (44) with initial data \(\tilde{W}_0 \in \dot{H}_\lambda^{1/2}\). Therefore, by the arguments given above, we can conclude that \(w_i = \partial_x P_+\left(e^{-iF_i/2}\right)\) belongs to \(Y_{1/2,T}^{s}\) and satisfies (16). In particular, by (42), we infer that for \(0 < T \leq 1\) and \(i = 1, 2\),
\[
\|w_i\|_{Y_{1/2,T}^{s}} \lesssim (1 + \|\varphi_i\|_{L^2})\|\varphi_i\|_{H^s} \tag{49}
\]
and thus thanks to (18),
\[
\|w_i\|_{Y_{1/2,T}^{s}} + \|J_{1/2}^i u\|_{L^4_{1/2,T}^{s}} \lesssim \varepsilon^2. \tag{50}
\]

5.2 Lipschitz bound in \(X_{\lambda,T}^{1/2,0} \cap L_T^{\infty} H_\lambda^s\)

We set
\[
z = w_1 - w_2 = -iP_+\left(e^{-iF_1/2}u_1\right) + iP_+\left(e^{-iF_2/2}u_2\right)
\]
with \(F_i = \partial_x^{-1}u_i\). Obviously, \(z\) satisfies
\[
z_t - iz_{xx} = -\partial_x P_+\left[P_-(\partial_x u_1 - \partial_x u_2)W_1\right] - \partial_x P_+\left[P_-(\partial_x u_2)(W_1 - W_2)\right] + \frac{i}{4}\left(P_0(u_1^2)z + P_0(u_1^2 - u_2^2)w_2\right). \tag{51}
\]
On account of Lemmas 2.2-2.3, 3.3-3.4 and (37), we thus infer that, for \(0 \leq s \leq 1\),
\[
\|z\|_{Y_{1/2,T}^{s}} \lesssim \|z(0)\|_{H_\lambda^s} + T^{1/8}\left[\|z\|_{X_{\lambda,T}^{1/2,s}}\left(\|u_1\|_{X_{\lambda,T}^{1/2,0}} + \|u_1\|_{X_{\lambda,T}^{1/2,0}}^2\right) + \|u_1 - u_2\|_{X_{\lambda,T}^{1/2,0}}\|u_2\|_{X_{\lambda,T}^{1/2,0}}(1 + \|u_1\|_{X_{\lambda,T}^{1/2,0}} + \|u_2\|_{X_{\lambda,T}^{1/2,0}})\right].
\]
Therefore, thanks to (42) and (50) for $0 < T < 1$,

$$
\|z\|_{Y_{T,\lambda}^\infty} \lesssim \left( 1 + \|\varphi_1\|_{\dot{H}_x^1} (1 + \lambda^{1/2}) \right) \|\varphi_1 - \varphi_2\|_{\dot{H}_x^1} + T^{1/8} \|w_2\|_{X_{T,\lambda}^{1/2,\infty}} \|u_1 - u_2\|_{X_{T,\lambda}^{1/2,0}} ,
$$

(52)

since, by Lemma 3.1, it can be easily seen that

$$
\|z(0)\|_{\dot{H}_x^1} \lesssim \|\varphi_1 - \varphi_2\|_{\dot{H}_x^1} \left( 1 + \|\varphi_1\|_{\dot{H}_x^1} + \|\varphi_2\|_{L_x^2} \right) + \|e^{-iF_1(0)} - e^{-iF_2(0)}\|_{L_x^\infty} \|\varphi_1\|_{\dot{H}_x^1} (1 + \|\varphi_1\|_{L_x^2})
$$

with

$$
\|e^{-iF_1(0)} - e^{-iF_2(0)}\|_{L_x^\infty} \lesssim \|\partial_x^{-1} (\varphi_1 - \varphi_2)\|_{L_x^\infty} \lesssim \lambda^{1/2} \|\varphi_1 - \varphi_2\|_{L_x^2} .
$$

On the other hand, writing the equation satisfied by $v = u_1 - u_2$, proceeding as in (23) and using (50) we get, for $0 < T < 1$,

$$
\|v\|_{X_{T,\lambda}^{1/2}} \lesssim \|e^{-iF_1(0)} - e^{-iF_2(0)}\|_{L_x^\infty} \|\varphi_1 - \varphi_2\|_{L_x^1} + \|\varphi_1\|_{\dot{H}_x^1} \left( 1 + \|\varphi_1\|_{L_x^2} \right)
$$

Interpolating this last inequality with the obvious inequality

$$
\|v\|_{X_{T,\lambda}^{0,1/2}} \lesssim T^{1/2} \|v\|_{L_x^\infty \dot{H}_x^{1/2}} ,
$$

it follows that, for $0 < T < 1$,

$$
\|v\|_{X_{T,\lambda}^{1/2,0}} \lesssim \|\varphi_1 - \varphi_2\|_{L_x^1} + \|\varphi_1\|_{\dot{H}_x^1} \left( 1 + \|\varphi_1\|_{L_x^2} \right) .
$$

Now, proceeding as in (44), we infer that

$$
v = \partial_x F_1 - \partial_x F_2 = 2ie^{iF_1/2} \left[ z + \partial_z P_-(e^{-iF_1/2} - e^{-iF_2/2}) \right] + 2i(e^{iF_1/2} - e^{iF_2/2}) \left( w_2 + \partial_z P_-(e^{-iF_2/2}) \right)
$$

and thus

$$
P_{>1} v = 2iP_{>1}(e^{iF_1/2} z) + 2iP_{>1} \left[ P_{>1}(e^{iF_1/2}) \partial_z P_-(e^{-iF_1/2} - e^{-iF_2/2}) \right] + 2iP_{>1}(e^{iF_1/2} - e^{iF_2/2}) w_2 + 2iP_{>1} \left[ P_{>1}(e^{iF_1/2} - e^{iF_2/2}) \partial_z P_-(e^{-iF_2/2}) \right].
$$

(55)
Therefore, by Lemmas 3.1-3.2 and (50), for $1/2 \leq s \leq 1$, $0 < T < 1$ and $(p,q) = (\infty,2)$ or $(4,4)$ we get as in (18)

$$
\|J_x^s v\|_{L_T^p L_\lambda^q} \lesssim \|P_1 v\|_{L_T^p L_\lambda^q} + (1 + \|u_1\|_{L_T^{\infty} L_\lambda^{1/2}}) \|z\|_{Y_T^p,\lambda} + \|u_1\|_{L_T^{\infty} H_\lambda^{1/2}} \|u_1 e^{-iF_1/2} - u_2 e^{-iF_2/2}\|_{L_T^{\infty} H_\lambda^{1/2}} + \|J_x^s (e^{iF_1/2} - e^{iF_2/2})w_2\|_{L_T^p L_\lambda^q} + \|u_1 e^{iF_1/2} - u_2 e^{iF_2/2}\|_{L_T^{\infty} H_\lambda^{1/2}} \|u_2\|_{L_T^{\infty} H_\lambda^{1/2}}
$$

$$
\lesssim \|P_1 v\|_{L_T^p L_\lambda^q} + \|z\|_{Y_T^p,\lambda} + \left(\|v\|_{L_T^{\infty} H_\lambda^{1/2}} + \|e^{iF_1/2} - e^{iF_2/2}\|_{L_T^p L_\lambda^q}\right) \left(\|w_2\|_{Y_T^p,\lambda} + \varepsilon^2\right). \tag{56}
$$

Moreover, since the functions $u_i$, $i = 1, 2$, are real-valued, by the mean value theorem and Sobolev inequalities,

$$
\|e^{iF_1/2} - e^{iF_2/2}\|_{L_T^{\infty} H_\lambda^2} \lesssim \|\partial_x^{-1}u_1 - \partial_x^{-1}u_2\|_{L_T^{\infty} L_\lambda^2} \lesssim \|\partial_x^{-1}v\|_{L_T^{\infty} L_\lambda^2} + \|v\|_{L_T^{\infty} L_\lambda^2} \tag{57}
$$

and writing the equation satisfied by $v$, using the unitarity of $V(\cdot)$ in $L_\lambda^2$ and (5), it is easily seen that

$$
\|\partial_x^{-1}v\|_{L_T^{\infty} L_\lambda^2} \lesssim \|\partial_x^{-1}\varphi_1 - \partial_x^{-1}\varphi_2\|_{L_\lambda^2} + T^{1/4}\|v\|_{X_{T,\lambda}^{1/2,0}}(\|u_1\|_{X_{T,\lambda}^{1/2,0}} + \|u_2\|_{X_{T,\lambda}^{1/2,0}}) \lesssim \lambda \|\varphi_1 - \varphi_2\|_{L_\lambda^{1/2}} + T^{1/4}\varepsilon^2\|v\|_{X_{T,\lambda}^{1/2,0}}. \tag{58}
$$

Gathering (52), (56), (58) and the obvious estimate (see (22))

$$
\|P_1 v\|_{L_T^p L_\lambda^q} \lesssim \|\varphi_1 - \varphi_2\|_{L_\lambda^2} + T\|v\|_{L_T^{\infty} H_\lambda^{1/2}}(\|u_1\|_{L_T^{\infty} H_\lambda^{1/2}} + \|u_2\|_{L_T^{\infty} H_\lambda^{1/2}}), \tag{59}
$$

we finally deduce that for $0 < T < 1$ and $(p,q) = (\infty,2)$ or $(4,4)$,

$$
\|J_x^s v\|_{L_T^p L_\lambda^q} \lesssim \left(1 + \|\varphi_1\|_{H_\lambda^1} + \varepsilon^2\lambda\right) \|\varphi_1 - \varphi_2\|_{H_\lambda^1} + \left(\|v\|_{L_T^{\infty} H_\lambda^{1/2}} + \|v\|_{X_{T,\lambda}^{1/2,0}} + \lambda \|\varphi_1 - \varphi_2\|_{L_\lambda^2}\right)(\|w_2\|_{Y_T^p,\lambda} + \varepsilon^2). \tag{60}
$$

In particular, taking $s = 1/2$, we deduce from (12), (50), (51) and (60) that

$$
\|u_1 - u_2\|_{L_T^{\infty} H_\lambda^{1/2}} + \|J_x^{1/2}(u_1 - u_2)\|_{L_T^4 L_\lambda^4} \lesssim (1 + \varepsilon^2\lambda) \|\varphi_1 - \varphi_2\|_{H_\lambda^{1/2}}. \tag{61}
$$

It then follows from (54) that

$$
\|u_1 - u_2\|_{X_{T,\lambda}^{1/2,0}} \lesssim (1 + \varepsilon^2\lambda) \|\varphi_1 - \varphi_2\|_{H_\lambda^{1/2}} \tag{62}
$$

and from (49) and (60) that

$$
\|u_1 - u_2\|_{L_T^{\infty} H_\lambda^s} \lesssim \left(1 + (\|\varphi_1\|_{H_\lambda^1} + \|\varphi_2\|_{H_\lambda^1})(1 + \lambda)\right) \|\varphi_1 - \varphi_2\|_{H_\lambda^s}, \quad 1/2 \leq s \leq 1. \tag{63}
$$
5.3 Continuity of the trajectory uniqueness and regularity of the flow-map

Let \( u_0 \in B_{\varepsilon, \lambda} \cap H_{\lambda}^s \), \( 1/2 \leq s \leq 1 \). With \([\text{12}],\text{[13]}\) in hand, we observe that the approximative sequence \( u^n \) constructed for the local existence result is a Cauchy sequence in \( C([0,1]; \dot{H}_{\lambda}^s) \cap X_{1,\lambda}^{1/2,0} \) since \( \|u_n\|_{X_{1,\lambda}^{1/2,0}} + \|u_n\|_{L_t^\infty H_{\lambda}^{1/2}} \lesssim \varepsilon^2 \) and \( u_{0,n} \) converges to \( u_0 \) in \( \dot{H}_{\lambda}^s \). Hence, \( u \) belongs to \( C([0,1]; \dot{H}_{\lambda}^s) \cap X_{1,\lambda}^{1/2,0} \).

Now let \( v \) be another solution emanating from \( u_0 \) belonging to the same class of regularity as \( u \). By Lebesgue monotone convergence theorem, there exists \( k > 0 \) such that \( \|P_{c-k} + P_{c,k}v\|_{X_{1,\lambda}^{1/2,0}} \lesssim \varepsilon^2 \). On the other hand, using Lemma \([\text{2,2}],\text{[2,3]}\) it is easy to check that

\[
\|P_k v\|_{X_{1,\lambda}^{1/2,0}} \lesssim \|u_0\|_{L_{x,\lambda}^2} + k\|v\|_{L_{x,\lambda}^2} \lesssim \|u_0\|_{L_{x,\lambda}^2} + T^{1/4}k\|v\|_{X_{1,\lambda}^{1/2,0}}^2.
\]

Therefore, for \( T > 0 \) small enough we can require that \( v \) satisfies the smallness condition \([\text{12}]\) and thus by \([\text{13}]\), \( u_2 \equiv u \) on \([0,T]\). This proves the uniqueness result for initial data belonging to \( B_{\varepsilon, \lambda} \). Moreover, \([\text{13}]\) clearly ensures that the flow-map is Lipschitz from \( B_{\varepsilon, \lambda} \cap H_{\lambda}^s \) into \( C([0,1]; \dot{H}_{\lambda}^s) \).

6 Proof of Theorem 1.1

We used the dilation symmetry argument to extend the result for arbitrary large data. First note that if \( u(t,x) \) is a \( 2\pi \)-periodic solution of \( \text{BO} \) on \([0,T]\) with initial data \( u_0 \) then \( u_\lambda(t,x) = \lambda^{-1}u(\lambda^{-2}t, \lambda^{-1}x) \) is a \( (2\pi\lambda) \)-periodic solution of \( \text{BO} \) on \([0,\lambda^2T]\) emanating from \( u_{0,\lambda} = \lambda^{-1}u_0(\lambda^{-1}x) \).

Let \( u_0 \in \dot{H}^s(\mathbb{T}) \) with \( 1/2 \leq s \leq 1 \). If \( u_0 \) belongs to \( B_{\varepsilon,1} \), we are done. Otherwise, we set

\[
\lambda = \varepsilon^{-4}\|u_0\|_{H_{\lambda}^{1/2}}^2 \geq 1 \quad \text{so that} \quad \|u_{0,\lambda}\|_{H_{\lambda}^{1/2}} \leq \varepsilon^2.
\]

Therefore, \( u_{0,\lambda} \) belongs to \( B_{\varepsilon,\lambda} \) and we are reduce to the case of small initial data. Hence, there exists a unique local solution \( u_\lambda \in C([0,1]; \dot{H}_{\lambda}^s) \cap X_{1,\lambda}^{1/2,0} \) of \( \text{BO} \) emanating from \( u_{0,\lambda} \). This proves the existence and uniqueness in \( C([0,T]; \dot{H}^s(\mathbb{T})) \cap X_{1,\lambda}^{1/2,0} \) of the solution \( u \) emanating from \( u_0 \) with \( T = T(||u_0||_{H_{\lambda}^{1/2}}) \) and \( T \to +\infty \) as \( ||u_0||_{H_{\lambda}^{1/2}} \to 0 \). The global well-posedness result follows thus directly since, by combining the conservation laws \([\text{1}]\) with Sobolev inequalities, it can be easily checked that \( ||u(t)||_{H_{\lambda}^{1/2}} \) remains uniformly bounded for all \( t > 0 \). The fact that the flow-map is Lipschitz on
every bounded set of $\dot{H}^s(\mathbb{T})$ follows as well since $\lambda$ only depends on $\|u_0\|_{\dot{H}^{1/2}}$.

Note that the change of unknown preserves the continuity of the solution and the continuity of the flow-map in $H^s(\mathbb{T})$. Moreover, the Lipschitz property (on bounded sets) of the flow-map is also preserved on the hyperplans of $H^s(\mathbb{T})$ with fixed mean value.

**Remark 6.1** We prove Theorem 1.1 for $1/2 \leq s \leq 1$. Of course, the case $s > 1$ can be treated in the same way up to obvious modifications. On the other hand, we think that by the present approach completed with some new ingredients, one can go down to $H^s(\mathbb{T})$, $s > 0$, or even perhaps to $L^2(\mathbb{T})$. It would be then very interesting to know if $L^2(\mathbb{T})$ is the limit space for the uniform continuity of the flow-map.

7 Appendix

7.1 A shorter proof of Estimate (6)

We present here a very nice shorter proof communicated to us by Nikolay Tzvetkov of the crucial inequality (6) (with $\lambda = 1$ and the Schrödinger group $U(\cdot)$) proven by Bourgain in [4]. The estimate for the Benjamin-Ono group can be deduced simply by writing $v$ as the sum of its positive and negative frequency parts.

Note that if $v(t,x)$ is $2\pi$-periodic in $t$ and $x$ then $U(t)v$ is also $2\pi$-periodic in $t$ and $x$. By density of the $2\pi$-periodic functions in $L^4([0,2\pi] \times \mathbb{T})$, it thus suffices to prove the result for such functions. We will thus work in the function space $X^{b,0}(\mathbb{T}^2)$ endowed with the norm

$$
\|v\|_{X^{b,0}} = \left( \sum_{\tau \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \langle \tau - n^2 \rangle^{2b} |\hat{v}(\tau,n)|^2 \right)^{1/2}.
$$

Let $v \in X^{b,0}(\mathbb{T}^2)$, $b \geq 3/8$. We introduce a Littlewood-Paley decomposition of $v$:

$$
v = \sum_{M-dyadic} v_M
$$

where $\text{supp} \hat{v}_M \subset \{(\tau,n) \in \mathbb{Z}^2, \langle \tau - n^2 \rangle \sim M\}$. Note that

$$
\|v\|_{X^{b,0}}^2 \sim \sum_M M^{2b} \|v_M\|_{L^4_{\tau,n}}^2.
$$
By the triangle inequality, we have
\[
\|v\|_{L^2_{t,x}}^4 = \|v^2\|_{L^2_{t,x}} = \left\| \sum_{M_1, M_2} v_{M_1} v_{M_2} \right\|_{L^2_{t,x}} \lesssim \sum_{M_1, M_2} \|v_{M_1} v_{M_2}\|_{L^2_{t,x}} \lesssim \sum_{M_1 \geq M_2} \|v_{M_1} v_{M_2}\|_{L^2_{t,x}}.
\]

The proof of (6) is based on the following lemma:

**Lemma 7.1**
\[
\left\| v_{M_1} v_{M_2} \right\|_{L^2_{t,x}} \lesssim \left( M_1 \wedge M_2 \right)^{1/2} \left( M_1 \vee M_2 \right)^{1/4} \left\| v_{M_1} \right\|_{L^2_{t,x}} \left\| v_{M_2} \right\|_{L^2_{t,x}} . \quad (64)
\]

Indeed, with this lemma in hand, rewriting \( M_1 = 2^l M_2 \) with \( l \in \mathbb{N} \), we get the following chain of inequalities:
\[
\sum_{M_1 \geq M_2} \left\| v_{M_1} v_{M_2} \right\|_{L^2_{t,x}} \lesssim \sum_{l \geq 0} \sum_{M_2} M_2^{1/2} (2^l M_2)^{1/4} \left\| v_{M_2} \right\|_{L^2_{t,x}} \left\| v_{2^l M_2} \right\|_{L^2_{t,x}}
\]
\[
\lesssim \sum_{l \geq 0} \sum_{M_2} M_2^{3/8} \left\| v_{M_2} \right\|_{L^2_{t,x}} (2^l M_2)^{3/8} 2^{-l/8} \left\| v_{2^l M_2} \right\|_{L^2_{t,x}}
\]
\[
\lesssim \sum_{l \geq 0} 2^{-l/8} \left( \sum_{M_2} M_2^{3/4} \left\| v_{M_2} \right\|_{L^2_{t,x}}^2 \right)^{1/2} \left( \sum_{M_2} (2^l M_2)^{3/4} \left\| v_{2^l M_2} \right\|_{L^2_{t,x}}^2 \right)^{1/2}
\]
\[
\lesssim \|u\|_{X^{3/8,0}}^2 .
\]

It thus remains to prove Lemma [7.1]. By Cauchy-Schwarz in \((\tau_1, n_1)\) we infer that
\[
\left\| v_{M_1} v_{M_2} \right\|_{L^2_{t,x}}^2 = \sum_{\tau, n} \left| \sum_{\tau_1, n_1} \hat{v}_{M_1}(\tau_1, n_1) \hat{v}_{M_2}(\tau - \tau_1, n - n_1) \right|^2
\]
\[
\lesssim \sum_{\tau, n} \alpha(\tau, n) \sum_{\tau_1, n_1} \left| \hat{v}_{M_1}(\tau_1, n_1) \hat{v}_{M_2}(\tau - \tau_1, n - n_1) \right|^2
\]
\[
\lesssim \sup_{\tau, n} \alpha(\tau, n) \|v_{M_1}\|_{L^2_{t,x}}^2 \left\| v_{M_2} \right\|_{L^2_{t,x}}^2 ,
\]
where
\[
\alpha(\tau, n) = \# \{(\tau_1, n_1), (\tau_1, n_1) \in \text{supp} \hat{v}_{M_1} \text{ and } (\tau_1, n_1) \in \text{supp} \hat{v}_{M_2} \}
\]
\[
\lesssim \# \{(\tau_1, n_1), (\tau_1 - n_1^2) \sim M_1 \text{ and } (\tau - \tau_1 - (n - n_1)^2) \sim M_2 \}
\]
\[
\lesssim (M_1 \wedge M_2) \# \{n_1, (\tau - n_1^2 - (n - n_1)^2) \lesssim M_1 + M_2 \}
\]
\[
\lesssim (M_1 \wedge M_2)(M_1 + M_2)^{1/2}
\]
\[
\lesssim (M_1 \wedge M_2)(M_1 \vee M_2)^{1/2} ,
\]

since \( \partial^2_{n_1} (\tau - n_1^2 - (n - n_1)^2) = -4 \).
7.2 On the lack of uniform continuity for KdV type equations on the circle

We prove the following proposition:

**Proposition 7.1** Let \((P_\alpha), \alpha \geq 0,\) denote the Cauchy problem associated with
\[ u_t + D_x^{2\alpha} \partial_x u = uu_x, \quad (t,x) \in \mathbb{R} \times \mathbb{T} \] (65)

The following assertions hold:

1. Let \(\alpha \geq 0.\) For any \(s > 3/2\) and any \(t > 0,\) the flow-map \(u_0 \mapsto u(t)\) is not uniformly continuous from any ball of \(H^s(\mathbb{T})\) centered at the origin to \(H^s(\mathbb{T}).\)

2. If \(\alpha \geq 1/2\) then for any \(s > 0\) and any \(T > 0,\) the map \(u_0 \mapsto u\) (if it exists) is not uniformly continuous from any ball of \(H^s(\mathbb{T})\) centered at the origin to \(C([0,T]; H^s(\mathbb{T})).\)

3. If \(0 \leq \alpha < 1/2\) then the same result holds for any \(s > 1/2.\)

**Remark 7.1** By compactness methods using energy estimates it is known that \((P_\alpha), \alpha \geq 0,\) is locally well-posed in \(H^s(\mathbb{T})\) for \(s > 3/2\) (cf. [7]). Of course this can be improved as soon as \(\alpha\) is large enough. For instance, for \(\alpha = 1/2\) (Benjamin-Ono equation) we proved well-posedness for \(s \geq 1/2\) whereas for \(\alpha = 1\) (KdV) well-posedness is known for \(s \geq -1/2\) ([14], [6]).

**Proof.** The proof is a replay of the proof of Koch-Tzvetkov [16] for the Benjamin-Ono equation on the real-line (see also [20] for other dispersive equations). Actually, it is even much simpler. The key reason is that, in the periodic setting, if \(u(t,\cdot)\) is a solution of (7.1) emanating from \(\varphi\) then \(u(t,\cdot + \omega t) + \omega\) is exactly a solution of (7.1) emanating from \(\varphi(\cdot) + \omega.\)

For \(\lambda = 2^n,\) we set
\[ \varphi_\lambda = \lambda^{-s} \sin(\lambda x), \]
so that \(\varphi_\lambda\) is a \(2\pi\)-periodic function with norm \(\|\varphi_\lambda\|_{H^s} \sim 1.\) We denote by \(U_\alpha(\cdot)\) the free group associated with \((P_\alpha)\),
\[ \hat{U}_\alpha(t)\varphi(n) = e^{-i|n|^{2\alpha}nt} \hat{\varphi}(n), \quad n \in \mathbb{Z}. \]

The proposition will be a direct consequence of the fact that the free evolution of \(\varphi_\lambda\) given by
\[ (U_\alpha(t)\varphi_\lambda)(x) = \lambda^{-s} \sin(-\lambda^{2\alpha+1}t + \lambda x) \]
is a good first approximation on some time interval of the solution $u_\lambda$ emanating from $\varphi_\lambda$. More precisely, we have the following key lemma.

**Lemma 7.2** Let $\alpha \geq 0$ and $s > 1/2$, then there exists $0 < \mu < 1$ such that for $0 < t \lesssim \lambda^{s-1/2}$ the following equality holds in $H^s(\mathbb{T})$:

$$u_\lambda(t, \cdot) = (U_\alpha(t) \varphi_\lambda)(\cdot) + O(\lambda^{-\mu}) \quad . \tag{66}$$

Moreover, if $\alpha \geq 1/2$ and $0 < s \leq 1/2$, then there exist $0 < \mu < 1$ such that for $0 < t \lesssim \lambda^{s-1}$ the following equality holds in $H^s(\mathbb{T})$:

$$u_\lambda(t, \cdot) = (U_\alpha(t) \varphi_\lambda)(\cdot) + O(\lambda^{-\mu}) \quad . \tag{67}$$

Let us assume this lemma for a while. For $\lambda \geq 1$, we choose a time $t_\lambda \in [\lambda^{-1+0^+}, \lambda^{-s-\frac{1}{2}+0^+}]$ in the case $s > 1/2$ and a time $t_\lambda \in [\lambda^{-1+0^+}, \lambda^{s-1}]$ in the case $\alpha \geq 1/2$ and $0 < s < 1/2$. We then set

$$\varphi_{\lambda, \omega_i}(x) = \varphi_\lambda(x) + \omega_i \quad \text{and} \quad u_{\lambda, \omega_i}(t, x) = u_\lambda(t, x + \omega_i t) + \omega_i, \quad i = 1, 2,$

with $\omega_1 = (\lambda t_\lambda)^{-1} \pi/2$ and $\omega_2 = -(\lambda t_\lambda)^{-1} \pi/2$.

Obviously $\|\varphi_{\lambda, \omega_1} - \varphi_{\lambda, \omega_2}\|_{H^s} = (\lambda t_\lambda)^{-1} \pi = O(\lambda^{-0^+})$. On the other hand, from the triangular inequality and Lemma 7.2 we deduce that

$$\|u_{\lambda, \omega_1}(t_\lambda, \cdot) - u_{\lambda, \omega_2}(t_\lambda, \cdot)\|_{H^s} \gtrsim \|(U_\alpha(t_\lambda) \varphi_\lambda)(\cdot) + \omega_1 t_\lambda) - (U_\alpha(t_\lambda) \varphi_\lambda)(\cdot) + \omega_2 t_\lambda)\|_{H^s} - |\omega_1 - \omega_2|$$

$$\gtrsim \|(U_\alpha(t_\lambda) \varphi_\lambda)(\cdot) + \omega_1 t_\lambda) - (U_\alpha(t_\lambda) \varphi_\lambda)(\cdot) + \omega_2 t_\lambda)\|_{H^s} - |\omega_1 - \omega_2|$$

$$\gtrsim \|2u_\lambda(t_\lambda, \cdot) - (U_\alpha(t_\lambda) \varphi_\lambda)(\cdot)\|_{H^s} - |\omega_1 - \omega_2|$$

$$\gtrsim 1 - O(\lambda^{-0^+}).$$

Where in the last step we computed

$$(U_\alpha(t_\lambda) \varphi_\lambda)(\cdot) + \omega_1 t_\lambda) - (U_\alpha(t_\lambda) \varphi_\lambda)(\cdot) + \omega_2 t_\lambda)$$

$$= \lambda^{-s} \left[ \sin \left( -\lambda^{2s+1} t_\lambda + \lambda x + \pi/2 \right) - \sin \left( -\lambda^{2s+1} t_\lambda + \lambda x - \pi/2 \right) \right]$$

$$= 2\lambda^{-s} \cos \left( -\lambda^{2s+1} t_\lambda + \lambda x \right).$$

Note that, for $s > 3/2$, $t_\lambda$ can be taken arbitrary large when $\lambda$ goes to infinity. This proves the proposition since $u_{\lambda, \omega_i}$ is the exact solution of $(P_\alpha)$ emanating from $\varphi_{\lambda, \omega_i}$.
Proof of Lemma 7.2. Obviously, we have
\[
\partial_t(U_\alpha(t)\varphi_\lambda) + D_x^{2\alpha}\partial_x(U_\alpha(t)\varphi_\lambda) - (U_\alpha(t)\varphi_\lambda)\partial_x(U_\alpha(t)\varphi_\lambda) = -(U_\alpha(t)\varphi_\lambda)\partial_x(U_\alpha(t)\varphi_\lambda) = \frac{1}{2} \lambda^{1-2s}\sin\left(-2\lambda^{2\alpha+1}t + 2\lambda x\right).
\]
Calling \(F_\lambda\) the term in right-hand side of the above equality, we thus get
\[
\|F_\lambda(t)\|_{L^2} \lesssim \lambda^{1-2s}, \quad \forall t \in \mathbb{R}.
\]
Setting now \(v_\lambda = u_\alpha - U_\alpha(t)\varphi_\lambda\), it is easily checked that \(v_\lambda\) verifies
\[
\partial_t v_\lambda + D_x^{2\alpha}\partial_x v_\lambda = v_\lambda \partial_x v_\lambda + \partial_x(v_\lambda(U_\alpha(t)\varphi_\lambda)) + F_\lambda.
\]
Taking the \(L^2\)-scalar product with \(v_\lambda\), we infer that
\[
\frac{1}{2} \frac{d}{dt}\|v_\lambda(t)\|_{L^2} \lesssim \|\partial_x(U_\alpha(t)\varphi_\lambda)\|_{L^\infty}\|v_\lambda(t)\|_{L^2} + \|F_\lambda(t)\|_{L^2}.
\]
Since \(\|\partial_x(U_\alpha(t)\varphi_\lambda)\|_{L^\infty} \lesssim \lambda^{1-s}\) we thus deduce from Gronwall lemma that for \(0 < T \leq \lambda^{s-1}\),
\[
\|v_\lambda\|_{L^\infty_T L^2} \leq T\|F_\lambda\|_{L^\infty_T L^2} \lesssim \lambda^{1-2s}T. \quad (68)
\]
On the other hand, classical energy estimates on solutions to (70) lead to
\[
\frac{d}{dt}\|u_\lambda(t)\|_{H^{\frac{3}{2}+}} \lesssim \|u_\lambda(t)\|_{H^{\frac{3}{2}+}}^2 \Rightarrow \|u_\lambda(t)\|_{H^{\frac{3}{2}+}} \leq \frac{\|\varphi_\alpha\|_{H^{\frac{3}{2}+}}}{1 - c\, t\|\varphi_\alpha\|_{H^{\frac{3}{2}+}}}. \quad (69)
\]
Thus for \(0 < T \leq \|\varphi_\alpha\|_{H^{\frac{3}{2}+}}^{-1}/2 \sim \lambda^{s-(\frac{3}{2}+)}\), one has
\[
\|u_\lambda(t)\|_{H^{\frac{3}{2}+}} \lesssim \|\varphi_\alpha\|_{H^{\frac{3}{2}+}} \lesssim \lambda^{(\frac{3}{2}+)-s}
\]
Combining this estimate with the classical energy estimate
\[
\|u_\lambda(t)\|_{H^r} \lesssim \exp(C\, t\|u_\lambda\|_{H^{\frac{3}{2}+}})\|\varphi_\alpha\|_{H^r}, \quad r \geq 0,
\]
we obtain that for \(0 < t \leq \lambda^{s-(\frac{3}{2}+)}\),
\[
\|u_\lambda(t)\|_{H^r} \lesssim \|\varphi_\alpha\|_{H^r} \lesssim \lambda^{r-s}. \quad (69)
\]
Since it is easy to check that for any \(r \in \mathbb{R}\), \(\|U_\alpha(t)\varphi_\lambda\|_{H^r} \lesssim \lambda^{r-s}\), we thus infer that for \(r > 3/2\) and \(0 < T \leq \lambda^{s-(\frac{3}{2}+)}\),
\[
\|v_\lambda\|_{L^\infty_T H^r} \lesssim \lambda^{r-s}. \quad (69)
\]
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For $s > 1/2$, interpolating between (68) and (69), with $r > s$, yields (66).

For $\alpha > 1/2$, we use the conservation of the $L^2$-norm and of the energy

$$E(u) = \frac{1}{2} \int_{\mathbb{R}/2\pi\mathbb{Z}} |D_x^\alpha u|^2 + \frac{1}{6} \int_{\mathbb{R}/2\pi\mathbb{Z}} u^3$$

to get after some calculations,

$$\|u(t)\|_{H^\alpha} \lesssim \|\varphi\|_{H^\alpha} \lesssim \lambda^{\alpha-s}, \quad \forall t \in \mathbb{R}.$$ 

The result for $0 < s \leq 1/2$ follows by interpolating this inequality with (68).

Finally, for $\alpha = 1/2$, i.e. the Benjamin-Ono equation, we interpolate (68) with the estimate given by the next conservation law which controls the $H^1$-norm of the solution.

### 7.3 Proof of Lemmas 3.1 and 3.2

The proofs of Lemmas 3.1 and 3.2 can be found in the appendix of [17] in the context of the $L^p_x L^q_t$ spaces. We present here short proofs for sake of completeness.

#### 7.3.1 Proof of Lemma 3.1

We set $F = \partial_x^{-1} f_1$ which is allowed since $f_1$ has zero mean value. We first notice that

$$\left\| D_x \left( e^{\pm i F} g \right) \right\|_{L_x^\lambda} \lesssim \left\| D_x e^{\mp i F} \right\|_{L_x^\lambda} \left\| g \right\|_{L_x^\infty} + \left\| e^{\mp i F} \right\|_{L_x^\infty} \| D_x g \|_{L_x^\lambda} \lesssim \left\| f_1 \right\|_{L_x^\lambda} \left\| J_x g \right\|_{L_x^\lambda} + \| D_x g \|_{L_x^\lambda} \lesssim (1 + \left\| f_1 \right\|_{L_x^\lambda}) \left\| J_x g \right\|_{L_x^\lambda},$$

(70)

where we used that $f_1$ is real-valued. Interpolating between (70) and the obvious inequality $\left\| (e^{\mp i F}) g \right\|_{L_x^\lambda} \lesssim \| g \|_{L_x^\lambda}$, (19) follows.

Finally, (20) can be obtained exactly in the same way, using that

$$\| D_x (e^{-i \partial_x^{-1} f_1} - e^{-i \partial_x^{-1} f_2}) \|_{L_x^\lambda} \lesssim \| e^{-i \partial_x^{-1} f_1} - e^{-i \partial_x^{-1} f_2} \|_{L_x^\infty} \| f_1 \|_{L_x^\lambda} + \| f_1 - f_2 \|_{L_x^\lambda}. $$

#### 7.3.2 Proof of Lemma 3.2

We need to introduce a few notations to deal with the Littlewood-Paley decomposition (cf. [13] or [17]). We consider $\Delta_k$ and $S_k$ the two operators respectively defined for $(2\pi \lambda)$-periodic functions by

$$\Delta_k(f) = \mathcal{F}^{-1} \left( \eta(2^{-k} \xi) \hat{f}(\xi) \right) \quad \text{and} \quad S_k(f) = \mathcal{F}^{-1} \left( p(2^{-k} \xi) \hat{f}(\xi) \right), \xi \in \lambda^{-1} \mathbb{Z},$$

where $\eta \in C_c^\infty$ is a smooth cut-off function.

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where \( \eta \) is a smooth non-negative function supported in \( \{ \xi , 1/2 \leq |\xi| \leq 2 \} \), such that \( \sum_{-\infty}^{+\infty} \eta(2^{-k}|\xi|) = 1 \) for \( \xi \neq 0 \), and where

\[
p(\xi) = \sum_{j=-3}^{\infty} \eta(2^{-j}|\xi|) = 1 - \sum_{j=2}^{\infty} \eta(2^{-j}|\xi|)
\]

Note that \( p(0) = 1 \) and \( \text{Supp } p \subset (-1/4, 1/4) \). We also consider the operators \( \Delta_k \) and \( \tilde{S}_k \) respectively defined by

\[
\Delta_k(f) = \mathcal{F}^{-1}\left(\tilde{\eta}(2^{-k}|\xi|)\hat{f}(\xi)\right) \quad \text{and} \quad \tilde{S}_k(f) = \mathcal{F}^{-1}\left(\tilde{\eta}(2^{-k}|\xi|)\hat{f}(\xi)\right), \quad \xi \in \lambda^{-1}\mathbb{Z},
\]

where \( \tilde{\eta} \) has the same properties as \( \eta \) except that \( \tilde{\eta} \) is supported in \( \{ \xi , 1/8 \leq |\xi| \leq 8 \} \) and that \( \tilde{\eta} = 1 \) on \( \{ \xi , 1/4 \leq |\xi| \leq 4 \} \), and where \( \tilde{p} \in C_0^\infty(\mathbb{R}) \) with \( p(\xi) = 1 \) for \( \xi \in [-100, 100] \). Clearly this implies that

\[
\forall j \in \mathbb{Z}, \quad \tilde{\Delta}_j \circ \Delta_j = \Delta_j.
\]

We then define the operators

\[
\Delta_k^{j}(f) = \mathcal{F}^{-1}\left(\eta^{j}(2^{-k}|\xi|)\hat{f}(\xi)\right), \quad \tilde{\Delta}_k^{j}(f) = \mathcal{F}^{-1}\left(\tilde{\eta}^{j}(2^{-k}|\xi|)\hat{f}(\xi)\right),
\]

\[
S_k^{j}(f) = \mathcal{F}^{-1}\left(p^{j}(2^{-k}|\xi|)\hat{f}(\xi)\right), \quad \tilde{S}_k^{j}(f) = \mathcal{F}^{-1}\left(\tilde{p}^{j}(2^{-k}|\xi|)\hat{f}(\xi)\right),
\]

where \( \eta^{j}(\xi) = |\xi|^j \eta(\xi) \), \( \tilde{\eta}^{j}(\xi) = |\xi|^j \tilde{\eta}(\xi) \), \( p^{j}(\xi) = |\xi|^j p(\xi) \) and \( \tilde{p}^{j}(\xi) = |\xi|^j \tilde{p}(\xi) \). Finally we denote by \( M \) the maximal operator.

First, note that the zero-modes of \( f \) and \( g \) are not involved in the expression \( P_+(f \partial_z g) \). We are thus allowed to use an homogeneous Littlewood-Paley decomposition of \( f \) and \( g \). Now, since clearly \( P_+(\Delta_l f \Delta_k (P_- \partial_z g)) = 0 \) as soon as \( l \leq k - 3 \), one has

\[
P_+\left(\sum_k S_k(f)\Delta_k(P_- \partial_z g)\right) = 0.
\]

Therefore, the usual homogeneous Littlewood-paley decomposition leads to

\[
D_x^\alpha \left( P_+ f P_- \partial_z g \right) = D_x^\alpha P_+ \left[ \sum_k \Delta_k(f)S_k(P_- \partial_z g) + \sum_{|j| \leq 2} \sum_k \Delta_{k-j}(f)\Delta_k(P_- \partial_z g) \right]
\]

\[
= P_+ \left[ \sum_k D_x^\alpha \tilde{\Delta}_k \left( \Delta_k(f)S_k(P_- \partial_z g) \right) + \sum_{|j| \leq 2} \sum_k D_x^\alpha \tilde{S}_k \left( \Delta_{k-j}(f)\Delta_k(P_- \partial_z g) \right) \right]
\]

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\[
\begin{align*}
\sum_k & \hat{\Delta}_k^{\alpha} \left( \Delta_{-}^{\gamma_1} (D_x^{\gamma_1} f) \hat{S}_k^{1-\gamma_2} (P_- D_x^{\gamma_2} g) \right) \\
+ & \sum_{|j| \leq 2} 2^{j \gamma_1} \sum_k \hat{S}_k^{\alpha} \left( \Delta_{k-j}^{\gamma_1} (D_x^{\gamma_1} f) \Delta_k^{1-\gamma_2} (P_- D_x^{\gamma_2} g) \right) \\
= & P_+ \left[ \sum_k \hat{\Delta}_k^{\alpha} \left( \Delta_{-}^{\gamma_1} (D_x^{\gamma_1} f) \hat{S}_k^{1-\gamma_2} (P_- D_x^{\gamma_2} g) \right) \right] \\
+ P_+ \left[ \sum_{|j| \leq 2} 2^{j \gamma_1} \sum_k \hat{S}_k^{\alpha} \left( \Delta_{k-j}^{\gamma_1} (D_x^{\gamma_1} f) \Delta_k^{1-\gamma_2} (P_- D_x^{\gamma_2} g) \right) \right] \\
= & I + II .
\end{align*}
\]

By the continuity of \( P_+ \) and \( \hat{S}_k^{\alpha} \) in \( L^q_{\lambda} \), Cauchy-Schwarz inequality and Littlewood-Paley square function theorem, for \( 1 < q < \infty \),

\[
\|II\|_{L^q_{\lambda}} \lesssim \|D_x^{\gamma_1} f\|_{L^{q_1}_{\lambda}} \|D_x^{\gamma_2} g\|_{L^{q_2}_{\lambda}} \quad \text{with} \quad 1/q_1 + 1/q_2 = 1/q, \quad 1 < q < \infty .
\]

On the other hand, by the Littlewood-Paley square function theorem,

\[
\|I\|_{L^q_{\lambda}} \lesssim \left\| M(P_- D_x^{\gamma_2} g) \left( \sum_k |\Delta_{-}^{\gamma_1} (D_x^{\gamma_1} f)|^2 \right)^{1/2} \right\|_{L^q_{\lambda}}
\]

and Hölder inequality and the continuity of the maximal operator on \( L^p_{\lambda} \), \( 1 < p \leq \infty \), yields the result.

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