LARGE SCALE INDEX OF MULTI-PARTITIONED MANIFOLDS

THOMAS SCHICK AND MOSTAFA E. ZADEH

Abstract. Let $M$ be a complete Riemannian spin manifold, partitioned by $q$ two-sided hypersurfaces $M_1, \ldots, M_q$ which have a compact transverse intersection $N$ and which are also coarsely transversal. Let $E \to M$ be a bundle of finitely generated projective Hilbert $A$-modules, where $A$ is an auxiliary $C^*$-algebra. An example is $A = C^*\pi_1(M)$ and $E$ the Mishchenko line bundle.

In this situation, we define a coarse multi-partitioned index $\text{ind}_p(D_M) \in K_{n-q}(A)$ of the spin Dirac operator twisted by $E$.

Our main result is the computation of this multi-partitioned index as $\text{ind}(D_N) \in K_{n-q}(A)$, the index of the Dirac operator on the compact manifold $N$, twisted by the restriction of $E$ to $N$.

We establish the following main application: if the scalar curvature of $M$ is uniformly positive (or even if it is uniformly positive only on one of the quadrants defined by the partitioning hypersurfaces) then the multi-partitioned index vanishes: $\text{ind}_p(D_M) = 0$. Consequently, $\text{ind}(D_N)$ is an obstruction to uniformly positive scalar curvature on $M$. For example, $\text{ind}(D_N) \neq 0$ if $N$ is enlargeable and if there is a map $f: M \to N$ such that the composition with the inclusion $N \to M$ has non-zero degree.

The proof of the multi-partitioned index theorem proceeds in two steps: first we establish a strong new localization property of the multi-partitioned index. This allows to reduce to the case $M = N \times \mathbb{R}^q$. For this special case, standard methods for the explicit calculation of the index in this product situation can be adapted to obtain the result.

1. Introduction

Consider a complete Riemannian manifold spin manifold $(M, g)$. If $M$ is non-compact, which is the situation we are interested in, the classical index theory of elliptic operators (like the Dirac operator) usually cannot be applied because of lack of the Fredholm property.

In this situation, using non-commutative geometry and operator algebras, John Roe has initiated an adapted index theory which we call “large scale index theory” or sometimes “coarse index theory” (compare e.g. [4]). The main player is the Roe algebra $C^*(X)$, associated to a complete proper metric space $X$. This is an algebra of operators (acting on function spaces on $X$), mildly depending on choices. Its K-theory $K_*(C^*(X))$ is canonically and functorially associated to $X$, with functorially for proper and uniformly expansive maps. Given an auxiliary $C^*$-algebra $A$, there is a variant $C^*(X; A)$ “with coefficients in $A$” which contains refined information.

If $M$ is a complete Riemannian spin manifold of dimension $n$ and $E \to M$ is a bundle of finitely generated projective Hilbert $A$-modules with connection, we have the twisted Dirac operator $D_E$. One of the main virtues of large scale index theory is the construction of its coarse index $\text{ind}_c(D_E) \in K_n(C^*(M; A))$. This index contains information about the geometry: on the one hand it depends on the metric only up to bilipschitz equivalence. On the other hand, it vanishes if the metric has uniformly positive scalar curvature (actually, it suffices to have this property on sufficiently large subsets of $M$). It is therefore important to get information about this coarse index. In the present paper, we will do this for multi-partitioned manifolds.

Definition 1.1. A complete Riemannian manifold $n$-dimensional $M$ is multi-partitioned by codimension 1 hypersurfaces $M_1, \ldots, M_q$ if

- each of the $M_k$ is a two-sided, separating $M = M_1^+ \cup M_1^- \cup \cdots \cup M_q^+ \cup M_q^-$ with $M_k^+ \cap M_k^- = M_k$;
- $N := \bigcap_{k=1}^q M_k$ is compact, and in a neighborhood of $N$ the $M_i$ intersect mutually transversally. In particular, $N$ is itself a submanifold of dimension $n - q$ with trivial normal bundle;
Lemma 1.2. If $M$ is a multi-partitioned manifold, partitioned by $M_1, \ldots, M_q$, the signed distance to the $M_k$ defines a proper and uniformly expansive map $f : M \to \mathbb{R}^q$ which is smooth near $N := f^{-1}(0)$ and such that 0 is a regular value.

Definition 1.3. If $M$ is a multi-partitioned spin manifold and $E \to M$ is a Hilbert $A$-module bundle with connection, we define the multi-partitioned index of the Dirac operator twisted by $E$ as

$$\text{ind}_p(D_E) := \kappa(f_*(\text{ind}_c(D_E))) \in K_{n-q}(A).$$

Here, $f_* : K_*(C^*(M; A)) \to K_*(C^*(\mathbb{R}^q; A))$ is obtained by functoriality of the K-theory of the twisted Roe algebra and

$$\kappa : K_*(C^*(\mathbb{R}^q; A)) \to K_{n-q}(A)$$

is a canonical isomorphism we establish in Proposition 2.12.

Our main result is the calculation of this multi-partitioned manifold index:

Theorem 1.4. Let $M$ be a complete spin manifold of dimension $n$ with proper continuously expansive map $f : M \to \mathbb{R}^q$. Assume that $f$ is smooth near $N := f^{-1}(0)$ and that 0 is a regular value. This implies that $N$ is a compact submanifold with trivial normal bundle of dimension $n - q$. In particular, $N$ inherits a spin structure from $M$. For example, $M$ could be multi-partitioned by $M_1, \ldots, M_q$. Let $E \to M$ be a Hilbert $A$-module bundle with connection. Then

$$\kappa(f_*(\text{ind}_c(D_{M,E}))) = \text{ind}(D_{N,E|N}) \in K_{n-q}(A).$$

Here, $\text{ind}(D_{N,E|N}) \in K_{n-q}(A)$ is the Mishchenko-Fomenko index of the Dirac operator on the compact spin manifold $N$, twisted by the Hilbert $A$-module bundle $E|_N$.

The first version of this theorem is the partitioned manifold index theorem of Roe and Higson, for the case $q = 1$ and $A = \mathbb{C}$ with several proofs, e.g. in [3] or [4]. Note that here only the case $n$ odd is interesting (and treated), as otherwise the target group $K_1(\mathbb{C}) = 0$. In [17] [18], the approach of [4] is generalized, still for $q = 1$, to arbitrary coefficient algebras $A$, as long as $n$ is odd. Finally, Siegel treats the case of multi-partitioned manifolds, but with $A = \mathbb{C}$ and with additional geometric restrictions in [6].

All these proofs consist of two steps. The first is a reduction to the product case $M = N \times \mathbb{R}^q$, and the second is a more or less explicit calculation in this product case.

We develop a new and particularly strong method for the reduction step. Indeed, we prove in particular the following:

Proposition 1.5. Assume that $f : M \to \mathbb{R}^q$ with Hilbert $A$-module bundle $E \to M$ and $f' : M' \to \mathbb{R}^q$ with Hilbert $A$-module bundle $E' \to M'$ are two complete Riemannian spin manifolds as in Theorem 1.4. Assume there are open neighborhoods $U$ of $f^{-1}(0)$ in $M$ and $U'$ of $f^{-1}(0)$ in $M'$ and a spin-structure preserving isometry $\psi : U \to U'$ which is covered by an isometry $\Psi : E|_U \to E'|_{U'}$.

Then $f_*(\text{ind}_c(D_E)) = f'_*(\text{ind}_c(D_{E'})) \in K_n(C^*(\mathbb{R}^q; A))$.

Indeed, Proposition 1.5 is a corollary of a localization theorem for classes in $K_*(C^*(\mathbb{R}^q; A))$: if two such are obtained as indices of operators which coincide on an arbitrary non-empty open subset of $\mathbb{R}^q$, then they are already equal.

In Section 4 we apply the Theorem 1.4 to obtain obstructions to metrics with uniformly positive scalar curvature at infinity:

Theorem 1.6. Let $M$ be a multi-partitioned $n$-dimensional spin manifold, with partitioning hypersurfaces $M_1, \ldots, M_q$ and $N = \bigcap_{k=1}^q M_k$. The inclusion of $N$ into $M$ induces maps $\iota_1 : \pi_1 N \to \pi_1 M$ and therefore $\iota_* : C^*_\text{max} \pi_1(N) \to C^*_\text{max} \pi_1(M)$. Assume that the image of the Rosenberg index $\text{ind}(D_N)$ of the Dirac operator twisted with the Mishchenko line bundle under $\iota_* : K_*(C^*_\text{max} \pi_1(N)) \to K_*(C^*_\text{max} \pi_1(M))$ is non-zero.
Then there is no Riemannian metric on $M$ which is bilipschitz to the given metric and which is uniformly positive on the quadrant $\bigcap_{k=1}^{\infty} M_k^+$. In particular there is no such metric which is uniformly positive outside a compact subset of $M$.

The assumptions on the index are satisfied e.g. if

1. $\text{ind}(D_N) \neq 0 \in K_s(C_*^{\max} \pi_1 N)$ and $e: \pi_1(N) \to \pi_1(M)$ is split injective

2. $N$ is enlargeable and there is a map $r: M \to N$ such that $N \hookrightarrow M \xrightarrow{r} N$ has non-zero degree.

Note that for each compact manifold $N$ and for $n \geq 3$ the space $N \times \mathbb{R}^n$ admits a metric with uniformly positive scalar curvature (see M. Gromov and B. Lawson [1, page 298]). As a consequence of Theorem 1.4 we conclude that no such metric can be bilipschitz equivalent to the product metric $g_N + dx_1^2 + \cdots + dx_2^2$.

2. BASICS OF LARGE SCALE INDEX THEORY

We start with the definition of the Roe algebra $C^*(M; A)$ and a companion, the structure algebra $D^*(M; A)$, inside which $C^*(M; A)$ is an ideal. For simplicity, we assume that $M$ is a Riemannian manifold throughout (possibly with boundary). We also fix an auxiliary $C^*$-algebra $A$. We assume familiarity with the concepts of Hilbert $A$-modules, the algebra of bounded adjointable operators on them, compact Hilbert $A$-module operators and (unbounded) regular self-adjoint operators.

**Definition 2.1.** Given a Hermitian vector bundle $S \to X$, consider the Hilbert $A$-module $L^2(S \otimes l^2(A))$ of square integrable sections of $S \otimes l^2(A)$ with $A$-valued inner product given by integration of the point-wise $A$-valued inner product.

We define the structure algebra $D^*(X; A)$ as $C^*$-closure of the algebra of bounded adjointable operators $T$ on $L^2(S \otimes l^2(A))$ which satisfy

1. $T$ has finite propagation, i.e. there is $R > 0$ such that $\text{supp}(T s) \subset U_R(\text{supp}(s))$ for each $s \in L^2(S \otimes l^2(A))$.

2. $T$ is pseudolocal: for any compactly supported continuous functions $\phi, \psi$ with $\phi \psi = 0$ the operator $\phi T \psi$ is a compact Hilbert $A$-module operator.

We define the Roe algebra $C^*(X; A)$ as the norm closure of operators $T$ as above which satisfy

1. $T$ has finite propagation

2. $T$ is locally compact, i.e. for every compactly supported continuous function $\phi$, $T \phi$ and $\phi T$ are compact Hilbert $A$-module operators, where we let $\phi$ act on $L^2(S \otimes l^2(A))$ as multiplication operator.

One checks immediately that $C^*(X; A)$ is an ideal in $D^*(X; A)$.

**Definition 2.2.** Let $f: X \to Y$ be a continuous map between complete Riemannian manifolds.

- $f$ is called proper if the inverse image of every compact subset of $Y$ is compact
- $f$ is uniformly expansive if for every $r > 0$ there is $s > 0$ such that $d(f(x), f(y)) \leq s$ whenever $d(x, y) \leq r$.

Given, in addition, Hermitian bundles $E \to X$ and $F \to Y$, an isometric embedding $V: L^2(E \otimes l^2(A)) \to L^2(F \otimes l^2(A))$ is said to cover $f$ in the $D^*$-sense if $V$ is a norm-limit of operators $V$ such that

- $V$ has finite propagation, i.e. there is $R > 0$ such that $\text{supp}(V s) \subset U_R(f(\text{supp}(s)))$

- whenever $\phi$ is a compactly supported continuous function on $X$ and $\psi$ is a compactly supported continuous function on $Y$ such that $\phi \cdot \psi \circ f = 0$ then $\psi V \phi$ is compact.

Indeed, it would suffice if $V$ is a norm limit of such operators.

The proof of [5, Lemma 7.7] carries over to the case with coefficients and shows that one can always find such an isometry $V$ covering $f$ in the $D^*$-sense, even with fixed $R > 0$.

Given such an isometry $V$, $\text{Ad}_V(T) := VTV^*$ then defines a map $D^*(X; A) \to D^*(Y; A)$ which restricts to $C^*(X; A) \to C^*(Y; A)$.

As in [5, Lemma 3], the induced map on K-theory does not depend on the choice of $V$, but only on $f$. This implies that
• $C^*(X; A)$ and $D^*(X; A)$ and therefore also the quotient algebra $D^*(X; A)/C^*(X; A)$ are well defined up to non-canonical isomorphism

• $K_* (C^*(X; A))$, $K_* (D^*(X; A))$, and $K_* (D^*(X; A)/C^*(X; A))$ are well defined up to canonical isomorphism and are functorial for proper continuous uniformly expansive maps.

We define $K_* (X; A) := K_{*+1} (D^*(X; A)/C^*(X; A))$, the locally finite $K$-homology of $X$ with coefficients in the $C^*$-algebra $A$.

In the original papers of Roe the compactness condition was forgotten to be mentioned. It is introduced (with this terminology) in [12, Definition 1.7] or (under the name “covers topologically”) in [16, Definition 2.4].

**Proposition 2.3.** Our large scale $K$-theory groups are homotopy invariant in the following sense:

If $H : X \times [0, 1] \to Y$ is a continuous proper uniformly expansive homotopy between $H_0, H_1 : X \to Y$, then

- $(H_0)_* = (H_1)_* : K_* (C^*(X; A)) \to K_* (C^*(Y; A))$
- $(H_0)_* = (H_1)_* : K_* (D^*(X; A)) \to K_* (D^*(Y; A))$
- $(H_0)_* = (H_1)_* : K_* (X; A) \to K_* (Y; A)$

**Proof.** The proof of this rather non-trivial result given in [8, Lemma 7.8] carries over to the case with coefficients.

We will have to establish vanishing for these $K$-groups in a number of situations. The most powerful and useful concept in this context is flasqueness.

**Definition 2.4.** A complete Riemannian manifold $M$ (possibly with boundary) is called flasque if there is a continuous, proper and uniformly expansive $e$ map $f : M \to M$ with the following properties:

1. there is a continuous uniformly expansive proper homotopy between $f$ and the identity
2. for every compact subset $K \subset M$ there is an $N \in \mathbb{N}$ such that $f^N (M) \cap K = \emptyset$.

**Example 2.5.** For an arbitrary manifold $X$, the product $X \times [0, \infty)$ is flasque.

**Proof.** The map $f : X \times [0, \infty) \to X \times [0, \infty); (x, t) \mapsto (x, t + 1)$ satisfies the conditions required in the definition of flasqueness.

**Proposition 2.6.** If $M$ is flasque, then

$$K_* (C^*(M; A)) = 0; \quad K_* (D^*(M; A)) = 0; \quad K_* (M; A) = 0.$$ $

**Proof.** The proof of [14, Proposition 9.4], based on an Eilenberg swindle, can be adapted to the situation with $D^*$ and with coefficients, using the homotopy invariance of $D^*$ of Proposition 2.3.

**Remark 2.7.** We have set up the theory such that everything works uniformly for $C^*$, $D^*$ and $D^*/C^*$. If one is only interested in working with $C^*$, one can weaken quite a few of the assumptions. For example, in the case of $C^*$ we have functoriality for maps which are not necessarily continuous and the homotopy invariance for coarse homotopies which are not necessarily continuous. We will not need this flexibility in this article, therefore decided not to develop these details.

The final ingredients we will need from the basics of large scale index theory is a Mayer-Vietoris principle.

**Definition 2.8.** Let $M$ be a complete Riemannian manifold and $Y \subset M$ a closed subset. We define $C^*(Y \subset M; A) \subset C^*(M; A)$ as the closure of the set of all operators $T \in C^*(M; A)$ which have support near $Y$, i.e. such that there is $R > 0$ such that $T \phi = 0$ and $\phi T = 0$ whenever supp$(\phi) \cap U_R(Y) = \emptyset$.

We define $D^*(Y \subset M; A)$ as the closure of the set of all operators $T \in D^*(M; A)$ such that $T$ has support near $Y$ and in addition $T \phi$ and $\phi T$ are compact Hilbert $A$-module operators whenever supp$(\phi) \cap Y = \emptyset$. Then $C^*(Y \subset M; A)$ and $D^*(Y \subset M; A)$ are both ideals in $D^*(M; A)$.
Proposition 2.9. In the situation of Definition 2.8 the canonical maps
\[ C^*(Y; A) \hookrightarrow C^*(Y \subset M; A); \quad D^*(Y; A) \hookrightarrow D^*(Y \subset M; A); \]
\[ D^*(Y; A) \cap C^*(Y; A) \rightarrow D^*(Y \subset M; A) \cap C^*(Y \subset M; A) \]
induce isomorphisms in K-theory.

Proof. The proof for \( A = \mathbb{C} \) given in [16] carries over to the case with coefficients. \( \square \)

Definition 2.10. Assume that \( M = M_1 \cup M_2 \) with intersection \( M_0 := M_1 \cap M_2 \) for closed subsets \( M_1, M_2 \). This decomposition is called coarsely excisive if for each \( r > 0 \) there is \( s > 0 \) such that \( U_r(M_1) \cap U_r(M_2) \subset U_s(M_0) \).

Theorem 2.11. Assume that \( M \) is a complete Riemannian manifold with a coarsely excisive decomposition \( M = M_1 \cup M_2 \) into closed subsets. Then we have long exact Mayer-Vietoris sequences
\[ K_j(C^*(M_1; A)) \oplus K_j(C^*(M_2; A)) \rightarrow K_j(C^*(M; A)) \xrightarrow{\delta_{MV}} K_{j-1}(C^*(M_0; A)) \rightarrow \]
\[ K_j(D^*(M_1; A)) \oplus K_j(D^*(M_2; A)) \rightarrow K_j(D^*(M; A)) \xrightarrow{\delta_{MV}} K_{j-1}(D^*(M_0; A)) \rightarrow \]
\[ K_j(M_1; A) \oplus K_j(M_2; A) \rightarrow K_j(M; A) \xrightarrow{\delta_{MV}} K_{j-1}(M_0; A) \rightarrow \]
They are compatible with the long exact sequences in K-theory of the extensions \( 0 \rightarrow C^* \rightarrow D^* \rightarrow D^*/C^* \rightarrow 0 \).

Proof. The main results of [14, Section 3] establish this for the case \( A = \mathbb{C} \), and the method covers without change the general case. \( \square \)

Proposition 2.12. For an arbitrary complete Riemannian manifold \( M \) and coefficient \( C^* \)-algebra \( A \) there is a commuting diagram with horizontal isomorphisms
\[ K_*(D^*(M \times \mathbb{R}; A)) \xrightarrow{\delta_{MV}} K_{*-1}(D^*(M; A)) \]
\[ K_*(D^*(M \times \mathbb{R}; A)/C^*(M \times \mathbb{R}; A)) \xrightarrow{\delta_{MV}} K_{*-1}(D^*(M; A)/C^*(M; A)) \]
\[ K_{*-1}(C^*(M \times \mathbb{R}; A)) \xrightarrow{\delta_{MV}} K_{*-2}(C^*(M; A)). \]

In particular, for arbitrary \( q \geq 0 \) we obtain a canonical isomorphism
\[ \kappa: K_*(C^*(\mathbb{R}^q; A)) \rightarrow K_{*-q}(A). \]

Proof. The decomposition \( M \times \mathbb{R} = M \times (-\infty, 0] \cup M \times [0, \infty) \) is coarsely excisive, and the half spaces \( M \times (-\infty, 0] \), \( M \times [0, \infty) \) are flasque. Combining Proposition 2.6 and Theorem 2.11, the Mayer-Vietoris boundary map gives an isomorphism
\[ \delta_{MV}: K_j(D^*(M \times \mathbb{R}; A)) \xrightarrow{\approx} K_{j-1}(D^*(M; A)), \]
and similarly for the other algebras.

The \( q \)-fold iteration of this then gives an isomorphism \( K_j(C^*(\mathbb{R}^q; A)) \rightarrow K_{j-q}(C^*(\mathbb{R}^0; A)) \). Finally, \( K_j(C^*(\mathbb{R}^0; A)) \xrightarrow{\approx} K_j(A) \) with a canonical isomorphism, as \( C^*(pt; A) = K(f^2(A)) \).

We define \( \kappa: K_j(C^*(\mathbb{R}^q; A)) \rightarrow K_{j-q}(A) \) as the composition of these isomorphisms. \( \square \)

Finally, we describe how to define the coarse index of a twisted Dirac operator and we state its main properties.

Assume therefore that \( M \) is a complete even dimensional spin manifold (without boundary) and \( E \rightarrow M \) a Hilbert \( A \)-module bundle with connection. The twisted Dirac operator \( D_E \) with its natural domain is then a regular self-adjoint unbounded operator on the Hilbert \( A \)-module \( L^2(S \otimes E) \) where \( S \)...
is the spinor bundle of \( M \). Because we assume that \( M \) is even dimensional, \( S = S^+ \oplus S^- \) is \( \mathbb{Z}/2 \)-graded and \( D_E \) is an odd operator with respect to this grading.

Using that the fibers of \( E \) are finitely generated projective, choose an isometric embedding of \( E \) into \( M \times \mathbb{L}^2(\mathcal{A}) \) as Hilbert \( \mathcal{A} \)-module bundle. By the generalization of Kuipers theorem the unitary group of \( \mathbb{L}^2(\mathcal{A}) \) is contractible. It follows that any two such embeddings are homotopic. Let \( V : L^2(S \otimes E) \to L^2(S \otimes \mathbb{L}^2(\mathcal{A})) \) be the induced isometric embedding, unique up to homotopy of such embeddings. Any operator \( T \) on \( L^2(S \otimes E) \) can then, via \( VTV^* \) be considered as an operator on \( L^2(E \otimes \mathbb{L}^2(\mathcal{A})) \) in a way which is unique up to homotopy, and we will do so without further mentioning it in the sequel.

For such an operator \( D_E \) a functional calculus for continuous bounded function on \( \text{spec}(\mathcal{D}) \subset \mathbb{R} \) is developed, compare \[.\] Let \( \chi : \mathbb{R} \to [-1,1] \) an odd continuous function such that \( \chi(t) \xrightarrow{t \to \pm \infty} \pm 1 \). Then \( \chi(D_E) \) is still an odd operator. We also can define the wave operators \( t \mapsto e^{itD_E} \), a one-parameter operator group which has the unit propagation property (i.e. \( e^{itD_E} \) has propagation \( t \)). Local elliptic regularity, the Fourier inversion formula and unit propagation of the wave operator imply, as in \[.\] that \( \chi(D_E) \in D^*(M;A) \) and that \( \chi(D_E)^2 - 1 \in C^*(M;A) \).

**Definition 2.13.** Choose a measurable fiberwise isometry \( S^+ \to S^- \) with induced isometry of propagation zero \( V : L^2(S^+ \otimes E) \to L^2(S^- \otimes E) \).

Because \( \chi(D_E)^2 - 1 \in C^*(M;A) \), the operator \( V^*\chi(D_E)_+ : L^2(S_+ \otimes E) \to L^2(S_+ \otimes E) \) (extended by the identity on the complement of \( \text{Im}(V) \)) is a unitary operator in \( D^*(M;A)/C^*(M;A) \) and therefore represents a class \( [D_E] \in K_1(D^*(M;A)/C^*(M;A)) = K_0(M;A) \), the fundamental \( K \)-homology class.

We have the long exact sequence in \( K \)-theory associated to the extension

\[
0 \to C^*(M;A) \to D^*(M;A) \to D^*(M;A)/C^*(M;A) \to 0
\]

with boundary map \( \partial : K_1(D^*(M;A)/C^*(M;A)) \to K_0(C^*(M;A)) \).

We define the large scale index \( \text{ind}_c(D_E) := \partial([D_E]) \in K_0(C^*(M;A)) \).

Because of homotopy invariance, \([D_E]\) as well as \( \text{ind}_c(D_E) \) do not depend on the choices made.

If the dimension of \( M \) is odd, \( ([\chi(D_E) - 1]/2) \in D^*(M;A)/C^*(M;A) \) is a projector and therefore represents a fundamental class \([D_E] \in K_0(M;A)\).

In this case, we define the large scale index \( \text{ind}_c(D_E) := \partial([D_E]) \in K_1(C^*(M;A)) \).

**Remark 2.14.** A more uniform way to construct the fundamental class and the large scale index is the use of the \( n \)-multigraded Dirac operator on an \( n \)-dimensional spin manifold, as in \[.\] Chapter 10, which generalizes immediately to Dirac operators twisted by a Hilbert \( \mathcal{A} \)-module bundle. This is more convenient in particular when dealing with product situations. We chose here the more pedestrian approach of \[.\]

Of course, \( ([\chi(D_E) - 1]/2) \) is a projector in \( D^*(M;A)/C^*(M;A) \) also for even dimensional manifolds. Because of the extra symmetries, however, it represents \( 0 \in K_0(M;A) \) and therefore is of no use.

**Remark 2.15.** If \( M \) is compact, then \( C^*(M;A) \cong K(\mathbb{L}^2(\mathcal{A})) \) which together with Morita invariance of \( K \)-theory induces a canonical isomorphism \( K_*(C^*(M;A)) \cong K_* (\mathcal{A}) \).

Under this identification, it is well known but non-trivial that \( \text{ind}_c(D_E) \in K_{\dim N}(C^*(M;A)) \) is mapped to the Mishchenko-Fomenko index of \( D_E \) as defined in \[.\] under the isomorphism \( K_*(C^*(M;A)) \to K_*(\mathcal{A}) \). For the purposes of this paper, for a compact manifold \( N \) we define \( \text{ind}(D_E) \in K_{\dim N}(\mathcal{A}) \) as the image of \( \text{ind}_c(D_E) \) under the isomorphism.

**Proposition 2.16.** Let \( M \) be a complete Riemannian spin manifold and \( E \) a Hilbert \( \mathcal{A} \)-module bundle on \( M \), as above.

If \( 0 \) is not in the spectrum of \( D_E \), by the Schrödinger-Lichnerowicz formula in particular if \( M \) has uniformly positive scalar curvature and if the Hilbert \( \mathcal{A} \)-module bundle \( E \) is flat, then

\[
\text{ind}_c(D_E) = 0 \in K_{\dim M}(C^*(M;E)).
\]

**Proof.** In this case, we can use for \( \chi \) a function such that \( \chi^2 = 1 \) on the spectrum of \( D_E \). Consequently, the formula defining \([D_E]\) shows that this class has a lift to \( K_{\dim M+1}(D^*(M;A)) \). Because of the exactness of the \( K \)-theory sequence

\[
K_* (D^*(M;A)) \to K_{*-1}(M;A) \xrightarrow{\partial} K_{*-1}(C^*(M;A))
\]
this implies that \( \text{ind}_c(D_E) = \partial([D_E]) = 0 \).

\textbf{Proposition 2.17.} Given a complete Riemannian spin manifold \( M \) of dimension \( n \) with Hilbert \( A \)-module bundle \( E \to M \), write \( E \) also for the pullback to \( M \times \mathbb{R} \). The Mayer-Vietoris isomorphism \( \delta_{MV} \) of Proposition 2.12 sends \([D_{M \times \mathbb{R}, E}]\) to \([D_{M, E}]\) and consequently also \( \text{ind}_c(D_{M \times \mathbb{R}, E}) \) to \( \text{ind}_c(D_{M, E}) \):

\[
\begin{align*}
K_{n+1}(M \times \mathbb{R}; E) &\xrightarrow{\delta_{MV}} K_n(M; E); \\
\frac{\partial}{\partial} &\rightarrow \frac{\partial}{\partial}
\end{align*}
\]

\[
\begin{align*}
K_{n+1}(C^*(M \times \mathbb{R}; E^n)) &\xrightarrow{\delta_{MV}} K_n(C^*(M; E^n)); \\
\text{ind}_c(D_{M \times \mathbb{R}, E^n}) &\rightarrow \text{ind}_c(D, E^n)
\end{align*}
\]

\textbf{Proof.} This crucial property of the Dirac operator is based on the principle that “boundary of Dirac is Dirac”. The proof of the corresponding statement for \( A = \mathbb{C} \) given in [16, Lemma 4.6] and based on the precise meaning of “boundary of Dirac is Dirac” as treated in [1, Chapter 11] carries over to the case with coefficients.

\textbf{Remark 2.18.} More generally, given complete Riemannian manifolds \( M_1 \) with Hilbert \( A \)-module bundle \( E_1 \to M_1 \) and \( M_2 \) with Hilbert \( A \)-module bundle \( E_2 \to M_2 \), the external tensor product of operators on \( L^2(M_1, S_1 \otimes I^2(A_1)) \otimes L^2(M_2, S_2 \otimes I^2(A_2)) \) defines an algebra homomorphism \( C^*(M_1; A_1) \otimes C^*(M_2; A_2) \) (which in general is not an isomorphism) and an induced map in K-theory

\[
\alpha: K_i(C^*(M_1; A_1)) \otimes K_j(C^*(M_2; A_2)) \to K_{i+j}(M_1 \times M_2; A_1 \otimes A_2).
\]

In this situation, we have a product formula for the index of the Dirac operator:

If \( M_1, M_2 \) (and therefore also \( M_1 \times M_2 \)) are spin manifolds then for the coarse indices we obtain

\[
\alpha(\text{ind}_c(D_{M_1, E_1}) \otimes \text{ind}_c(D_{M_2, E_2})) = \text{ind}_c(D_{M_1 \times M_2, E_1 \otimes E_2}) \in K_{\dim(M_1) + \dim(M_2)}(C^*(M_1 \times M_2; A_1 \otimes A_2)).
\]

Even for \( A_1 = A_2 = \mathbb{C} \), this well-known result is non-trivial to prove. The best route is prehaps to reformulate the coarse index via KK-theory and use appropriate formulas for Kasparov products. This proof carries over without change to general \( C^* \)-coefficients.

\section{Multi-partitioned manifolds and their large scale index}

Throughout this section, assume that \( M \) is a complete Riemannian manifold which is multi-partitioned by the separating hypersurfaces \( M_1, \ldots, M_q \). Recall that this means in particular that the latter are coarse transversal in the sense of Definition 1.1 and near their common intersection \( N := \bigcap_{k=1}^q M_k \) the mutual intersections is transversal in the usual sense, and such that finally \( N \) is compact.

We now prove Lemma 1.2.

\textbf{Definition 3.1.} We write \( M = M_k^+ \cup \overline{M}_k^+ \) for the decomposition of \( M \) induced by the hypersurface \( M_k \). Define \( h_k: M \to \mathbb{R} \) as the signed distance to \( M_k \), i.e. \( h_k(x) = d(x, M_k) \) if \( x \in M_k^+ \) and \( h_k(x) = -d(x, M_k) \) if \( x \in \overline{M}_k^- \). Set \( f: M \to \mathbb{R}; x \mapsto (h_1(x), \ldots, h_q(x)) \).

By the triangle inequality, for an arbitrary subset \( X \subset M \) we have \( |d(x, H) - d(y, H)| \leq d(x, y) \). Moreover, as \( M \) has a length metric and \( M_k \) is separating, \( x \in M_k^+ \) and \( y \in \overline{M}_k^- \) satisfy \( d(x, M_k) + d(y, M_k) \leq d(x, y) \). It follows that \( h_k: M \to \mathbb{R} \) is a 1-Lipschitz map, therefore \( f \) is a \( \sqrt{q} \)-Lipschitz map. The condition that \( N \) is compact and the \( M_k \) are coarse transversal implies that the inverse image of every bounded subset of \( \mathbb{R}^k \) under \( f \) is bounded.

This finishes the proof of Lemma 1.2.

We have now explained all the ingredients for the statement of Theorem 1.4. Indeed, in Section 3 we essentially already proved the model case of this theorem, which reads as follows:

\textbf{Lemma 3.2.} Let \( N \) be a compact \( n \)-dimensional spin manifold, \( E \to N \) a Hilbert \( A \)-modul bundle. Write \( E \) also for the pullback to \( N \times \mathbb{R}^q \). Let \( f: N \times \mathbb{R}^q \to \mathbb{R}^q \) be the projection. For this special case, the assertion of Theorem 1.4 holds.
Proof. By naturality of the Mayer-Vietoris sequence, the following diagram is commutative:

\[
\begin{array}{cccc}
K_{n+q}(C^*(N \times \mathbb{R}^q; A)) & \xrightarrow{f_*} & K_{n+q}(C^*(\mathbb{R}^q; A)) \\
\cong \left\downarrow \delta_M^* \right. & \cong \left\downarrow \delta_M^* \\
K_n(C^*(N; A)) & \xrightarrow{pr_*} & K_n(C^*(\mathbb{R}^q; A)) \\
\downarrow \cong & & \downarrow \cong \\
K_n(A) & \xrightarrow{=} & \cong & K_n(A).
\end{array}
\]

By definition, the right vertical composition is \(\kappa\) so that \(\text{ind}_p(D_{N \times \mathbb{R}^q,E}) \in K_n(A)\) is the image of \(\text{ind}_c(D_{N \times \mathbb{R}^q,E})\) under the map to the right lower corner.

However, by Proposition 2.17, \(\delta_M^* \left(\text{ind}_c(D_{N \times \mathbb{R}^q,E})\right) = \text{ind}_c(D_{N,E}) \in K_n(C^*(N; A))\), and the latter is mapped to \(\text{ind}(D_{N,E}) \in K_n(A)\) under the isomorphism \(K_n(C^*(N; A)) \to K_n(A)\) by Remark 2.12.

The main novelty of this note is the localization result for the partitioned manifold index. It follows from the following localization result for the K-theory of \(C^*(\mathbb{R}^q; A)\).

Definition 3.3. Two operators \(T_1, T_2 \in D^*(\mathbb{R}^q; A)\) are said to coincide on the open set \(U \subset \mathbb{R}^q\) if and only if \(T_1 s = T_2 s\) for all \(s \in \text{supp}(s) \subset U\).

Proposition 3.4. Let \(T_1, T_2 \in D^*(\mathbb{R}^q; A)\) be two operators which coincide on the non-empty open set \(U \subset \mathbb{R}^q\). Assume that \([T_1]\) and \([T_2]\) represent elements in \(K_j(D^*(\mathbb{R}^q; A)/C^*(\mathbb{R}^q; A))\), i.e. are either idempotents (for \(j\) even) or invertible (for \(j\) odd) modulo \(C^*(\mathbb{R}^q; A)\).

Just from the fact that \(T_1\) and \(T_2\) coincide on \(U\), it then follows that

\([T_1] = [T_2] \in K_j(D^*(\mathbb{R}^q; A)/C^*(\mathbb{R}^q; A))\).

Proof. By translation invariance of \(\mathbb{R}^q\) and making \(U\) smaller, if necessary, we can assume that \(U = B_r(0)\) for some \(r > 0\).

We use the auxiliary space \(\mathbb{R}^q \setminus U\). We apply the Mayer-Vietoris principle to the decomposition

\[
\mathbb{R}^q \setminus B_r(0) = (\mathbb{R}^q \setminus B_r(0)) \cup (\mathbb{R}^q \setminus B_r(0)).
\]

The intersection is \(\mathbb{R}^{q-1} \setminus B_r(0)\) and the half spaces are flasque. For \(q > 1\) the decomposition is coarsely excisive and, using Proposition 2.6 and Theorem 2.11 we get a Mayer-Vietoris isomorphism

\[
K_*(\mathbb{R}^q \setminus B_r(0); A) \to K_{q-1}(\mathbb{R}^{q-1} \setminus B_r(0); A).
\]

Finally, in the case \(q = 1\), write \(\mathbb{R}' := (-\infty, -r] \cup [r, \infty) \subset \mathbb{R}\). Then \(D^*(\mathbb{R}'; A)/C^*(\mathbb{R}'; A)\) decomposes as a direct sum

\[
(D^*((-\infty, -r] \subset \mathbb{R}'; A) + C^*(\mathbb{R}'; A))/C^*(\mathbb{R}'; A) \oplus (D^*([r, \infty); A) + C^*(\mathbb{R}'; A))/C^*(\mathbb{R}'; A)
\]

By the isomorphism theorems for the two summands we have

\[
(D^*((-\infty, -r] \subset \mathbb{R}'; A) + C^*(\mathbb{R}'; A))/C^*(\mathbb{R}'; A) \cong D^*((-\infty, -r]; A)/C^*((-\infty, -r]; A),
\]

\[
(D^*([r, \infty); A) + C^*(\mathbb{R}'; A))/C^*(\mathbb{R}'; A) \cong D^*([r, \infty); A)/C^*([r, \infty); A).
\]

Therefore

\[
K_*(\mathbb{R}'; A) = K_*(-\infty, -r]; A) \oplus K_*(r, \infty); A) = 0
\]

again using that the half line is flasque.

By assumption, \(T_1, T_2\) coincide on \(U\). We claim that this implies that \(T_1 - T_2 \in D^*(Y \subset \mathbb{R}; A)\) with \(Y = \mathbb{R}^q \setminus U\) and with subspace algebra defined in Definition 2.8. The support condition is automatic, as \(U_r(Y) = \mathbb{R}^q\). If \(\phi: \mathbb{R} \to \mathbb{C}\) has support on \(U\) then \((T_1 - T_2)\phi = 0\) by assumption, therefore \((T_1 - T_2)\phi\) is compact. Because \(T_1, T_2\) are in \(D^*(\mathbb{R}^q; A)\), the commutator \([\phi, T_1 - T_2]\) is a compact Hilbert \(A\)-module operator. This gives then the required remaining compactness of \(\phi(T_1 - T_2)\).

From this, we conclude that the images of \(T_1\) and \(T_2\) in \(D^*(\mathbb{R}^q; A)/(D^*(Y \subset \mathbb{R}; A) + C^*(\mathbb{R}^q; A))\) coincide.
By Proposition 2.3, \[ K_*(D^*(Y \subset \mathbb{R}^q; A)) \cong K_{*-1}(Y; A) = 0 \]
Therefore, the long exact K-theory sequence of the extension \[ 0 \to D^*(Y \subset \mathbb{R}^q; A)/C^*(Y \subset \mathbb{R}^q; A) \to D^*(\mathbb{R}^q; A)/C^*(\mathbb{R}^q; A) \]
\[ \to D^*(\mathbb{R}^q; A)/(D^*(Y \subset \mathbb{R}^q; A) + C^*(\mathbb{R}^q; A)) \to 0 \]
gives the isomorphism, induced by the projection \[ K_*(\mathbb{R}^q; A) \to K_{*+1}(D^*(\mathbb{R}^q; A)/(D^*(Y \subset \mathbb{R}^q; A) + C^*(\mathbb{R}^q; A))). \]
We observed above that the images of \( T_1 \) and \( T_2 \) in the right hand algebra coincide. Because of the isomorphism, \([T_1] = [T_2] \in K_*(\mathbb{R}^q; A), \) as we had to prove. \( \square \)

The localization theorem, Proposition 1.3, now is a rather direct corollary, as we want to prove next. Assume therefore the situation of Proposition 1.3, with two manifolds \( f: M \to \mathbb{R}^q, f': M' \to \mathbb{R}^q \) which are locally isomorphic on open neighborhoods \( U, U' \) of the inverse images \( N, N' \) of 0 via isomorphisms \( \psi, \Psi \). As \( f \) is proper and continuous and \( U \) is an open neighborhood of \( N \) (and the corresponding situation for \( M' \)), if we choose \( t > 0 \) sufficiently small then \( f^{-1}(B_t(0)) \subset U \) and \( (f')^{-1}(B_t(0)) \subset U' \). Choose \( r > 0 \) such that \( U_r(f^{-1}(B_t(0))) \subset f^{-1}(B_t(0)). \) Because \( U, U' \) are isometric, the same is then also true for \( M' \). Next, choose a smooth chopping function \( \chi \) as for the definition of \( [D_E] \) such that its Fourier transform \( \hat{\chi} \) (which is a distribution which is smooth outside 0) has support in \( (-r/4, r/4) \). By the Fourier inversion formula and unit propagation speed of the wave operator (which implies that \( \chi(D_E), \chi(D_E) \) have propagation \( r/4 \), \( \chi(D_E)s = \Psi\chi(D_E)\Psi^{-1}s \) for each \( s \) with support in \( f^{-1}(B_t(0)) \)). Next, for the construction of \( f: D^*(M; A) \to D^*(\mathbb{R}^q; A) \) and \( f': D^*(M'; A) \to D^*(\mathbb{R}^q; A) \) choose isometries \( V, V' \) as in Definition 2.2 with propagation smaller than \( r/4 \). These isometries can be constructed locally and patched together. We can therefore in addition arrange that
\[
(3.1) \quad V_s = V'\Psi_s \quad \text{if} \quad \text{supp}(s) \subset U_{3r/4}(f^{-1}(B_{t/2}(0))) \subset U.
\]
As \( \langle V^*u, s \rangle_{L^2(S^{1}\otimes E(A))} = \langle u, Vs \rangle_{L^2(\mathbb{R}^{q}\times\mathbb{T}(A))} \), the fact that \( V \) has propagation \( r/4 \) implies that if \( \text{supp}(u) \subset B_{t/2}(0) \) then the support of \( V^*u \) is contained in \( U_{r/4}(f^{-1}(B_{t/2}(0))) \). Then Equation (3.1) implies that \( \Psi V^*u = (V')^*u \) for these \( u \).

Taken together, we get immediately that \( V\chi(D_E)V^*u = V'\chi(D_E)(V')^*s \) if \( u \) is supported on \( B_{t/2}(0) \). This implies that \( f_*([\chi(D_E)] + 1/2) \) and \( f'_*([\chi(D_E')] + 1/2) \) coincide on \( B_{t/2}(0) \subset \mathbb{R}^q \). If \( M \) has even dimension, in addition we have to choose the measurable bundle isomorphisms \( V_\delta, V'_\delta \) between the positive and negative spinor bundles. Again, this construction is local and we can therefore arrange that for sections supported on \( U \) the isometry \( \Psi \) intertwines these bundle isomorphisms, i.e. \( \Psi V_\delta s = V'_\delta \Psi s \) if \( \text{supp}(s) \subset U \).

It then follows that, if \( u \) is supported on \( B_{t/2}(0) \subset \mathbb{R}^q \) then \[
f_*([\chi(D_E)] + 1/2) = V_\delta \chi(D_E)V^*u = V'_\delta \chi(D_E')(V')^*u = f'_*(V_\delta \chi(D_E')) + (u).
\]
To summarize: in all dimensions the classes \( f_*([D_E]) \) and \( f'_*([D_E']) \) coincide on the non-empty open subset \( B_{t/2}(0) \subset \mathbb{R}^q \). By Proposition 3.4, \( f'_*([D_E]) = f'_*([D_E']) \in K_{\dim M}(\mathbb{R}^q; A). \) By naturality of the boundary map of the K-theory long exact then also \[
f_*([\text{ind}_c(D_E)]) = f'_*([\text{ind}_c(D_{E'})]) \in K_{\dim M}(C^*(\mathbb{R}^q; A)),
\]
as we have to prove.

Finally, we are in the position to give the proof of the multi-partitioned manifold index theorem 1.4. Given \( f: M \to \mathbb{R}^q \) as in Theorem 1.4, by homotopy invariance of the index we can deform the metric on \( M \) and connection on \( E \) in a neighborhood \( U \) of \( N \) such that it is isometric to a neighborhood of \( N \times \{0\} \) in \( N \times \mathbb{R}^q \) (with product structure) without changing \( \text{ind}(D_{M,E}). \)
By Proposition 4.4, which we just proved, \( f_*(\text{ind}_c(D_{M,E})) = f_*(\text{ind}_c(D_{N \times \mathbb{R} \equiv E|N})) \). But for the latter we already proved in Lemma 4.2 that \( f_*(\text{ind}_c(D_{N \times \mathbb{R} \equiv E|N})) = \text{ind}(D_{N,E|N}) \subseteq K_*(A) \). Therefore Theorem 4.4 is established.

4. Application to metrics with positive scalar curvature

In this section we apply the multi-partitioned manifold index theorem to prove non-existence theorems for metrics with positive scalar curvature.

The following lemma is stated (rather its first part) in [14, page 22] without proof for the case \( A = C \). In [18], a proof of this special case is provided using the Friedrich extension of symmetric operators which are bounded below. A sketch of a proof using similar ideas for general \( A \) is given in [15]. Simultaneously, a detailed proof using Fourier inversion techniques was given in the Göttingen thesis of Daniel Pape [14] and will appear in [18].

Lemma 4.1. Let \( M \) be a spin manifold with spin bundle \( S \) and let \( E \) be a flat Hilbert \( A \)-module bundle on \( M \). Let \( D_E \) be a Dirac operator twisted by \( E \).

(1) If there is a constant \( C > 0 \) such that the scalar curvature of \( g \) is greater than \( C \) outside \( Y \), then \( \text{ind}_c(D_E) \) is in the image of \( K_*(C^{\infty}(Y \subset M; A)) \rightarrow K_*(C^{\infty}(M; A)) \).

(2) Let \((M',g')\) be another complete manifold. If \( f: M \rightarrow M' \) is a proper and uniformly expanding map with \( f(Y) \subset Y' \subset M' \) then \( f_* (\text{ind} D) \subseteq K_*(C^{\infty}(M', S \otimes E')) \) takes its value in the image of \( K_*(C^{\infty}(Y' \subset M'; A)) \rightarrow K_*(C^{\infty}(M'; A)) \).

Proof. The proof of the first part can be found in the references listed above. The second part is a direct consequence of the first part and of naturality. \( \square \)

With this Lemma, we are in the position to prove Theorem 4.5. Assume therefore that \( f: M \rightarrow \mathbb{R}^q \) is a proper and uniformly expansive map defined on a complete spin manifold \( M \) and assume that \( f \) is smooth near \( N := f^{-1}(0) \) such that 0 is a regular value. Let \( L := M \times_{\pi} C^{\infty}_{\text{max}} \pi \) be the Mishchenko line bundle, where \( \pi := \pi_1 M \) and \( \tilde{M} \) is the universal covering of \( M \), a flat Hilbert \( C^{\infty}_{\text{max}} \pi \)-bundle. If the scalar curvature of \( M \) is uniformly positive on \( \bigcap_{k=1}^n M^+_k \), then by Lemma 4.4 \( f_*(\text{ind}_c(D_{M,L})) \) lies in the image of \( K_*(C^{\infty}(\mathbb{R}^q \setminus \{0,\infty\}^q \subset \mathbb{R}^q; C^{\infty}_{\text{max}} \pi)) \rightarrow K_*(C^{\infty}(\mathbb{R}^q; C^{\infty}_{\text{max}} \pi)) \).

Now, \( \mathbb{R}^q \setminus \{0,\infty\}^q \) is a flasque space, therefore by Proposition 4.5 \( K_*(C^{\infty}(\mathbb{R}^q \setminus \{0,\infty\}^q; C^{\infty}_{\text{max}} \pi)) = K_*(C^{\infty}(\mathbb{R}^q \setminus \{0,\infty\}^q \subset \mathbb{R}^q; C^{\infty}_{\text{max}} \pi)) \) vanishes.

It follows, under the positivity assumption on the scalar curvature, that \( f_*(\text{ind}_c(D_{M,L})) = 0 \).

On the other hand, by Theorem 4.4 \( k_f \cdot (\text{ind}_c(D_{M,L})) = \text{ind}(D_{N,L|N}) \subseteq K_{\text{max}}(C^{\infty}_{\text{max}} \pi_1 N) \). Finally, if \( L_N \) is the Mishchenko bundle of \( N \), a Hilbert \( C^{\infty}_{\text{max}} \pi_1 N \)-module bundle, then by Lemma 3.1 \( i_* : C^{\infty}_{\text{max}} \pi_1 N \rightarrow C^{\infty}_{\text{max}} \pi \) sends \( \text{ind}(D_{N,L|N}) \subseteq K_*(C^{\infty}_{\text{max}} \pi_1 N) \) to \( \text{ind}(D_{N,L|N}) \subseteq K_*(C^{\infty}_{\text{max}} \pi) \).

We get the desired contradiction: if the image of \( \text{ind}(D_{N,L|N}) \subseteq K_*(C^{\infty}_{\text{max}} \pi) \) is non-zero, then no metric with uniformly positive scalar curvature on \( \bigcap_{k=1}^n M^+_k \) can exist.

If \( \iota : \pi_1 N \rightarrow \pi \) is split injective with split \( s : \pi \rightarrow \pi_1 N \), by functoriality of the maximal group \( C^{\infty}\)-algebra and of \( C^{\infty}\)-algebra K-theory the composition \( s \circ i_* : K_*(C^{\infty}_{\text{max}} \pi_1 N) \rightarrow K_*(C^{\infty}_{\text{max}} \pi) \rightarrow K_*(C^{\infty}_{\text{max}} \pi_1 N) \) is the identity, therefore \( i_* \) is injective.

Definition 4.2. Let \( N \) be a closed spin manifold of dimension \( n \). The manifold \( N \) is enlargeable if (for one and then for all fixed Riemannian metric on \( N \)) for each real number \( \epsilon > 0 \) there is a covering projection \( N_{\epsilon} \rightarrow N \), with lifted metric, and a smooth map \( f_{\epsilon} : N_{\epsilon} \rightarrow S^n \) such that the function \( f \) is constant outside a compact subset \( K \) of \( N \); the degree of \( f_{\epsilon} \) is non-zero; and the map \( f_{\epsilon} : N_{\epsilon} \rightarrow (S^n, g_0) \) is \( \epsilon \)-contracting. Being \( \epsilon \)-contracting means that \( \|T_x f_{\epsilon} \| \leq \epsilon \) for each \( x \in N_{\epsilon} \), where \( T_x f_{\epsilon} : T_x N_{\epsilon} \rightarrow T_{f_{\epsilon}(x)} S^n \).

Enlargeability is an obstruction to the existence of a metric with positive scalar curvature. Actually, the main result of [2] states that enlargeability allows to construct a homomorphism

\[
e : K_{\dim N}(C^{\infty}_{\text{max}} \pi_1 N) \rightarrow \left( \prod_{k \in \mathbb{N}} \mathbb{Z} \right) / \left( \bigoplus_{k \in \mathbb{N}} \mathbb{Z} \right)
\]
which sends \( \text{ind}(D_{N,L,N}) \) to \([\deg(f_1), \deg(f_{1/2}), \deg(f_{1/3}), \ldots] \neq 0\), where \( f_{1/k} \) are the maps from the definition of enlargeability.

Let \( r: M \to N \) be a map such that the composition \( N \hookrightarrow M \xrightarrow{r} N \) has degree \( d \neq 0 \). It follows that the coverings \( N_\epsilon \) of the definition of enlargeability can (by pulling the original ones back through \( r \) and the inclusion) be replaced by coverings which are restrictions of coverings \( M_\epsilon \to M \). In this situation, the construction of \( \mathcal{J} \) gives a factorisation of the homomorphism \( e \) of (4.1) as

\[
K_{\dim N}(C^{*}_{\text{max}}\pi_1 N) \to K_{\dim N}(C^{*}_{\text{max}}\pi) \to \left( \prod_{k \in \mathbb{N}} \mathbb{Z} \right) / \left( \bigoplus_{k \in \mathbb{N}} \mathbb{Z} \right).
\]

As \( \text{ind}(D_{N,L_N}) \) has non-zero image under this composition, its image \( \text{ind}(D_{N,L_{N}}) \in K_{\dim (N)}(C^{*}_{\text{max}}\pi) \) is non-zero.

This shows that indeed the two conditions listed at the end of Theorem 1.6 imply that the general assumption of that theorem are satisfied.

5. Summary

In this article, we introduced the concept of multi-partitioned manifold \( M \) and defined a partitioned index using large scale index theory. Throughout we allow for coefficients in a \( C^* \)-algebra \( A \).

We prove a multi-partitioned manifold index theorem, which equates the multi-partitioned index of the Dirac operator with the usual higher index of the compact hypersurface \( N \) which is the intersection of all the parts.

The proof of this theorem is based on a new and strong localization theorem for such a multi-partitioned index: it depends only on the geometry near \( N \), which, by homotopy invariance of the index we can assume to be of standard product form.

We applied this theorem to obtain obstructions to the existence of metrics of positive scalar curvature on \( M \) in terms of the compact submanifold \( N \).

It would be important and interesting to

1. find explicit example situations where our theorem, in particular the obstruction to positive scalar curvature, can be applied
2. to supply detailed proofs for our rather sketchy arguments about the large scale index theory of Dirac operators twisted with Hilbert \( A \)-module bundles in Section 2, like Propositions 2.3, 2.6, 2.9, 2.17 or Remarks 2.13, 2.18.

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