Development of analytical calculation method for axisymmetric oscillations of circular and annular plates on variable Winkler elastic foundation

Yu Krutii1,*, M Surianinov1, V Osadchiy1, V Kolomiichuk1

1Odesa State Academy of Civil Engineering and Architecture, 4, Didrihsona str.,
Odesa 65029, Ukraine

*E-mail: yurii.krutii@gmail.com

Abstract. An analytical method for calculating axisymmetric vibrations of circular and annular plates on a variable elastic Winkler base is presented. The proposed calculation method is based on the exact solution of the oscillation equation with the subsequent development of a method for its numerical implementation. To construct an exact solution, the direct integration method is used, which was previously developed by one of the authors for differential equations with continuous variable coefficients. The proposed method does not require the discretization of the considered structure and it is a real alternative to the application of approximate methods during solving this types of problems. Based on the exact solution of the differential equation, this method will allow to obtain better picture of oscillations in comparison with approximate methods. In fact, the exact solution has qualitative information and forms the most complete picture of the physical phenomenon that is researched.

1. Introduction

Circular and annular plates are widely used model in the calculation of structural elements of construction objects, machine building, shipbuilding, aircraft building, instrument-making and other fields of technology. The theory of calculating such plates is discussed in detail in the works [1-3] and many others.

One of the most actual problems is the calculation of circular and annular plates lying on a solid elastic foundation. In particular, the basement of circular structures, which rest on the ground, are calculated exactly as a circular plate on an elastic foundation. Winkler model is often used among the models of ground foundations. In this case, in integral form there is a number of modifications of this model, which allow to take into account the heterogeneous properties of the foundation, both in plan and depth. The most common modification is the model of the variable modulus of subgrade reaction. Such a model, for example, is used in calculating the stress-strain state of the structure foundations lying on the loess soils, in which subsidence is characteristic. In this case, the modulus of subgrade reaction is a variable value, depending on the coordinates in which the draft of the foundation is determined.

Calculations of the plates lying on an elastic foundation with a variable modulus of subgrade reaction are extremely rare in the scientific periodicals. For rectangular plates, these calculations are known only for static bending problems. In the article [4] the calculation of such plates is carried out by the method of finite elements, and in [5] – by Galerkin method. In our opinion, the work [6] deserves special attention, in which free oscillations of the circular plate, lying on a variable elastic foundation, are investigated. The modulus of subgrade reaction changes according to the power law in the direction of radius.

This article is devoted to the development of an analytical method for calculating on the axisymmetric oscillations of circular and annular plates with constant cylindrical stiffness $D$ lying on the solid variable Winkler elastic foundation with a variable modulus of subgrade reaction. In figure 1 the annular plate is shown. Here are $a$ and $b$ – radiiuses of the outer and inner contour circles of the
plate, \( r \) – radial coordinate \((0 \leq r \leq a)\). In particular, with the value \( b = 0 \) we obtain a solid circular plate (figure 2).

Free axisymmetric oscillations of the plates occur when the elastic \( R(r,t) \) and the conditions for fixing the edges are not dependent on the polar angle \( \theta \). In such oscillations, only three dynamic internal forces act in the plate, namely, radial and circular bending moments \( M_r(r,t) \) and \( M_\theta(r,t) \), as well as radial shear force \( Q_r(r,t) \) (figure 3). The torque moment \( M_{\theta r}(r,t) \) and the circular shear force \( Q_\theta(r,t) \) are zero because of the axial symmetry of the oscillations.

According to Winkler hypothesis, the reaction of the foundation \( R(r,t) \) to the plate and the dynamic deflection of the plate \( W(r,t) \) are associated with equality

\[
R(r,t) = -k(r)W(r,t),
\]

where \( k(r) \) – the continuous variable modulus of subgrade reaction. Relatively \( k(r) \) we take \( k(r) = k_0A(r) \), where \( k_0 \) – the value of the modulus of subgrade reaction at some characteristic point of the plate; \( A(r) \) – a dimensionless continuous function that expresses the law of changing the modulus of subgrade reaction from the radial coordinate.

The differential equation of oscillations in our case has the form [7]

\[
D \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial W}{\partial r} \right) \right] + k_0A(r)W + \rho h \frac{\partial^2 W}{\partial r^2} = 0. \tag{\#}
\]

The purpose of the work is to develop an analytical method of calculation on free oscillations of circular and annular plates under the conditions described above. To achieve the goal, the following tasks are formulated and solved:
- to find the exact solution of the equation \(#\);
- to obtain the formulas for dynamic internal force movements in the plate;
- to obtain an analytical representation for the frequency of oscillations;
- to specify analytical method of numerical realization of exact predicted solutions.

2. Materials and methods
The offered method of calculation is based on the exact solution of the oscillation equation \(#\), followed by the development of the method for its numerical implementation. To construct an exact solution, this publication uses direct integration method, developed in the case of differential equations...
with continuous variable coefficients in the work [8]. The principle of the direct integration method can also be realized, for example, according to the publications [9-13], where it solves some problems of the mechanics of the deformed solid body. It is characteristic that differential equations with variable coefficients are in the role of mathematical models of these problems.

3. The exact solution of the oscillation equation and the formulas for internal forces

When the deflection function \( W(r,t) \) is found from the equation (\( * \)), the dynamic angle of rotation, as well as the dynamic forces in the plate, are determined with the known formulas [7]:

\[
\varphi(r,t) = \frac{\partial W}{\partial r}; \\
M_r(r,t) = -D \left( \frac{\partial^2 W}{\partial r^2} + \frac{\mu}{r} \frac{\partial W}{\partial r} \right); \\
M_\theta(r,t) = -D \left( \frac{\partial^2 W}{\partial r^2} + \frac{1}{r} \frac{\partial W}{\partial r} \right); \\
Q_r(r,t) = -D \left( \frac{\partial^2 W}{\partial r^2} + \frac{1}{r^2} \frac{\partial W}{\partial r} \right),
\]

The solution (1) is searched in the form

\[
W(r,t) = w(r)T(t),
\]

where \( w(r) \) is the amplitude function of deflection, depending only on \( r \) coordinate, \( T(t) \) – time function. Using (5) in the equation (\( * \)) and dividing the variables there, we obtain two ordinary differential equations:

\[
\ddot{T}(t) + \omega^2 T(t) = 0; \\
D \frac{1}{r} \frac{d}{dr} \left[ r \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} \left( r \frac{1}{r} \frac{d}{dr} \right) \right) \right] + (k_0 A(r) - \rho \omega^2)w = 0.
\]

It is easy to write out the solution of equation (6)

\[
T(t) = T(0)\cos \omega t + \frac{\dot{T}(0)}{\omega}\sin \omega t,
\]

where \( T(0), \dot{T}(0) \) are the parameters of the initial conditions of movement, \( \omega \) – unknown plate oscillation frequency. The main mode of oscillations is defined as the solution of the equation (7), which we rewrite in the form to shorten the record

\[
D \Delta \Delta w + (k_0 A(r) - \rho \omega^2)w = 0,
\]

where \( \Delta = d^2/dr^2 + 1/r d/dr \) – Laplace operator.

We denote with \( X_n(r), Y_n(r) \ (n = 1, 2) \) the fundamental solutions of the equation (8) and for solutions \( Y_n(r) \) we accept representations

\[
Y_n(r) = X_n(r)\ln \frac{r}{a} + Z_n(r) \quad (n = 1, 2),
\]

where \( Z_n(r) \) – unknown functions. Substituting (9) into the equation (8), after transformations we get

\[
\left[ D \Delta \Delta X_n(r) + (k_0 A(r) - \rho \omega^2)X_n(r) \right] \ln \frac{r}{a} + D \Delta \Delta Z_n(r) + ....
\]
\[(k_o A(r) - \rho \omega^2)Z_n(r) + 4D \frac{1}{r} \frac{d^3 X_n(r)}{dr^3} = 0 . \]  

Since \( X_n(r) (n = 1, 2) \) are fundamental solutions of the equation (8), the left part in logarithm (10) is obliged to be identically equal to zero. Therefore, instead of the equality (10), we can write:

\[ D \Delta \Delta X_n(r) + (k_o A(r) - \rho \omega^2)X_n(r) = 0 \quad (n = 1, 2) ; \]  
\[ D \Delta \Delta Z_n(r) + (k_o A(r) - \rho \omega^2)Z_n(r) = -4D \frac{1}{r} \frac{d^3 X_n(r)}{dr^3} \quad (n = 1, 2) . \]

The solutions of the equation (11) will be found in the form

\[ X_n(r) = \sum_{k=0}^{\infty} \sum_{m=0}^{k} (-K_o)^{k-m} K^{2m} \alpha_{n, m, k-m}(r) , \]

where \( \alpha_{n, m, k-m}(r) (n = 1, 2) \) are unknown functions, which we consider to be continuous with derivatives from the first to the fourth order; \( K_o = a^4 k_o / D \) is a known dimensionless parameter; \( K \) is an unknown dimensionless parameter, which is connected with the oscillation frequency by the formula

\[ K^2 = a^4 \rho \omega^2 / D . \]

So we assume that the series (13), as well as similar series, made up from the first four derivative functions \( \alpha_{n, m, k-m}(r) (n = 1, 2) \), converge uniformly. In this case, a series differential operation will be possible.

Substituting (13) into the equation (11), we have

\[ \sum_{k=0}^{\infty} \sum_{m=0}^{k} (-K_o)^{k-m} K^{2m} \Delta \Delta \alpha_{n, m, k-m}(r) - \frac{1}{a^2} A(r) \sum_{k=0}^{\infty} \sum_{m=0}^{k} (-K_o)^{k-m} K^{2m} \alpha_{n, m, k-m}(r) = \]

\[ -\frac{1}{a^2} \sum_{m=0}^{\infty} \sum_{k=m}^{k} (-K_o)^{k-m} K^{2(m+1)} \alpha_{n, m, k-m}(r) = 0 . \]

In the second and third sums, we shift the index \( k \) by one, that is, we replace \( k \) by \( k - 1 \). The same operation is carried out in the third sum with index \( m \). As a result, we get

\[ \sum_{k=0}^{\infty} \sum_{m=0}^{k} (-K_o)^{k-m} K^{2m} \Delta \Delta \alpha_{n, m, k-m}(r) - \frac{1}{a^2} A(r) \sum_{k=0}^{\infty} \sum_{m=0}^{k} (-K_o)^{k-m} K^{2m} \alpha_{n, m, k-m}(r) = \]

\[ -\frac{1}{a^2} \sum_{k=1}^{\infty} \sum_{m=1}^{k} (-K_o)^{k-m} K^{2m} \alpha_{n, m-1, k-m}(r) = 0 . \]

Then, after the transformation, it is necessary to implement the equality

\[ \Delta \Delta \alpha_{n, 0, 0}(r) + \sum_{k=1}^{\infty} K^{2k} \left( \Delta \Delta \alpha_{n, k, 0}(r) - \frac{1}{a^2} \alpha_{n, k-1, 0}(r) \right) + \sum_{k=1}^{\infty} (-K_o)^{k} \left( \Delta \Delta \alpha_{n, 0, k}(r) - \frac{1}{a^2} A(r) \alpha_{n, 0, k-1}(r) \right) + \]

\[ + \sum_{k=2}^{\infty} \sum_{m=1}^{k-1} (-K_o)^{k-m} K^{2m} \left( \Delta \Delta \alpha_{n, m, k-m}(r) - \frac{1}{a^2} \alpha_{n, m-1, k-m}(r) - \frac{1}{a^2} A(r) \alpha_{n, m, k-m-1}(r) \right) = 0 . \]
To satisfy it, we equate to zero all the variable coefficients at the powers of $(-K_0)^{k-m}K^{2m}$ $(k = 0, 1, 2, ...)(m = 0, 1, 2, ..., k)$. In this case, in the corresponding equations, we move from the operational form of record to the usual one:

$$\frac{1}{r} \frac{d}{dr} \left[ r \frac{d}{dr} \left( r \frac{d\alpha_{0,0}(r)}{dr} \right) \right] = 0 \quad (n = 1, 2); \quad (15)$$

$$\frac{1}{r} \frac{d}{dr} \left[ r \frac{d}{dr} \left( r \frac{d\alpha_{n,k,0}(r)}{dr} \right) \right] = \frac{1}{a^4} \alpha_{n,k-1,0}(r) \quad (k = 1, 2, 3, ...); \quad (16)$$

$$\frac{1}{r} \frac{d}{dr} \left[ r \frac{d}{dr} \left( r \frac{d\alpha_{0,k}(r)}{dr} \right) \right] = \frac{1}{a^4} A(r)\alpha_{n,0,k-1}(r) \quad (k = 1, 2, 3, ...); \quad (17)$$

$$\frac{1}{r} \frac{d}{dr} \left[ r \frac{d}{dr} \left( r \frac{d\alpha_{n,m,k-m}(r)}{dr} \right) \right] = \frac{1}{a^4} \alpha_{n,m-1,k-m}(r) + \frac{1}{a^4} A(r)\alpha_{n,m,k-m-1}(r) \quad (k = 2, 3, 4, ...), (m = 1, 2, 3, ..., k - 1). \quad (18)$$

As $\alpha_{n,0,0}(r)(n = 1, 2)$, we will select the following functions

$$\alpha_{n,0,0}(r) = \left( \frac{r}{a} \right)^{2n-2} \quad (n = 1, 2). \quad (19)$$

It is easy to verify that each of them satisfies the equation (15). Further, integrating the equations (16)-(18) and assuming integration constants are zero, we obtain:

$$\alpha_{n,k,0}(r) = \frac{1}{a^4} \int_0^r \int_0^r \int_0^r r \alpha_{n,k-1,0}(r) dr dr dr \quad (k = 1, 2, 3, ...); \quad (20)$$

$$\alpha_{n,0,k}(r) = \frac{1}{a^4} \int_0^r \int_0^r \int_0^r r A(r)\alpha_{n,0,k-1}(r) dr dr dr \quad (k = 1, 2, 3, ...); \quad (21)$$

$$\alpha_{n,m,k-m}(r) = \frac{1}{a^4} \int_0^r \int_0^r \int_0^r r (\alpha_{n,m-1,k-m}(r) + A(r)\alpha_{n,m,k-m-1}(r)) dr dr dr \quad (k = 2, 3, 4, ...), (m = 1, 2, 3, ..., k - 1). \quad (22)$$

Formulas (20)-(22) are recurrent. Functions $\alpha_{n,k,0}(r), \alpha_{n,0,k}(r), \alpha_{n,m,k-m}(r)$, which we call the generating ones, are sequentially defined with the help of these formulas according to the known initial function $\alpha_{n,0,0}(r)$. For such functions, the equality (11) is satisfied identically by the formation.

As we can see, this equation is similar to the equation (11) intrinsically. The difference is only that the equation (12) contains a non-zero right part. Therefore, acting like the previous one, we can obtain the solutions of the equation (12). For brevity sake, we will write only the final formulas, which determine the solutions that are being searched for:

$$Z_n(r) = \sum_{k=0}^{n} \sum_{m=0}^{k} (-K_0)^{k-m}K^{2m}\beta_{n,m,k-m}(r); \quad (23)$$

$$\beta_{n,0,0}(r) = \alpha_{n,0,0}(r) = \left( \frac{r}{a} \right)^{2n-2} \quad (n = 1, 2); \quad (24)$$
\[ \beta_{n,k,0}(r) = \frac{1}{a^2} \int_0^r \int_0^r \int_0^r \left( \beta_{n,k-1,0}(r) - \frac{4a^4}{r} \frac{d^3\alpha_{n,k,0}(r)}{dr^3} \right) dr \, dr \, dr \quad (k=1,2,3,...); \]  

\[ \beta_{n,0,k}(r) = \frac{1}{a^2} \int_0^r \int_0^r \int_0^r \left( A(r)\beta_{n,0,k-1}(r) - \frac{4a^4}{r} \frac{d^3\alpha_{n,0,k}(r)}{dr^3} \right) dr \, dr \, dr \quad (k=1,2,3,...); \]  

\[ \beta_{n,m,k-m}(r) = \frac{1}{a^2} \int_0^r \int_0^r \int_0^r \left( \beta_{n,m-1,k-m}(r) + A(r)\beta_{n,m,k-m-1}(r) - \frac{4a^4}{r} \frac{d^3\alpha_{n,m,k-m}(r)}{dr^3} \right) dr \, dr \, dr; \]  

\[ (k=2,3,4,...) \quad (m=1,2,3,...,k-1). \]

Series (13), (23) converge uniformly. Here are the demonstration of convergence, for example, for series (13).

Firstly, note that the generating functions \( \alpha_{n,k,0}(r) \) \( (k=1,2,3,...) \) can be found by formulas (19), (20) in an explicit form:

\[ \alpha_{n,k,0}(r) = c_{n,k,0} \left( \frac{r}{a} \right)^{2n+4k-2} \quad (k=1,2,3,...); \]

\[ c_{n,k,0} = \frac{1}{(2^n(n+2k-1))}. \]

For other generating functions, based on the properties of certain integrals, we obtain estimates:

\[ \alpha_{n,0,k}(r) \leq \frac{\gamma}{a^2} \int_0^r \int_0^r \int_0^r r \alpha_{n,0,k-1}(r) dr \, dr \, dr \quad (k=1,2,3,...); \]

\[ \alpha_{n,m,k-m}(r) \leq \frac{\gamma}{a^2} \int_0^r \int_0^r \int_0^r r(\alpha_{n,m-1,k-m}(r) + \gamma\alpha_{n,m,k-m-1}(r)) dr \, dr \, dr \quad (k=2,3,4,...) \quad (m=1,2,3,...,k-1). \]

where \( \gamma = \max_{n\geq 0} A(r) \). Performing sequential operations, prescribed by recurrent formulas (30), (31) for the specified values of indices \( k, m \) and taking into account the equality (28), (29), concerning the generating functions (20)-(22), we have general result

\[ \alpha_{n,m,k-m}(r) \leq \frac{\gamma^{k-m}C^m_n}{(2^n(n+2k-1)!)^2} \left( \frac{r}{a} \right)^{2n+4k-2} \quad (k=1,2,3,...) \quad (m=1,2,3,...,k), \]

where \( C^m_n \) is the number of combinations from \( k \) to \( m \).

Taking into account (32), for the series (13) we have

\[ |X_n(r)| \leq \left( \frac{r}{a} \right)^{2n-2} \sum_{k=0}^{\infty} \frac{1}{(2^n(n+2k-1)!)^2} \left( \frac{r}{a} \right)^{4k} C^m_n (\gamma K_0)^k K^m = \]

\[ = \left( \frac{r}{a} \right)^{2n-2} \sum_{k=0}^{\infty} (K^2 + \gamma K_0)^k \frac{1}{(2^n(n+2k-1)!)^2} \left( \frac{r}{a} \right)^{4k}. \]

Using the D’Alembert principle to calculate the convergence radius of the last power series, we get

\[ R = \frac{2^4}{K^2 + \gamma K_0} \lim_{k \to \infty} ((n+2k)(n+2k+1))^2 = \infty. \]
Thus it is proved that the series (13) converge uniformly. Similarly, it is possible to prove uniform convergence for the series (23), as well as for similar series, where the functions of variable coefficients will be derived from the initial and generating functions $\alpha_{n,k}(r)$, $\beta_{n,k}(r)$ ($k = 0, 1, 2, \ldots$) from the first to the fourth order.

Now let’s prove that $X_n(r)$, $Y_n(r)$ ($n = 1, 2$) is linearly independent. Let us first assume that the identity has been fulfilled

$$C_1X_1(r) + C_2X_2(r) + C_3Y_1(r) + C_4Y_2(r) = 0,$$

and among the constants $C_1, C_2, C_3, C_4$ are not all zero. Taking into account (9) and separately equating to zero the expression with logarithm, we get

$$C_3X_1'(r) + C_4X_2'(r) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (-K_0)^{k+m} K^{2m}\left[C_3\alpha_{1,m,k-m}(r) + C_4\alpha_{2,m,k-m}(r)\right] = 0.$$  \hspace{1cm} (34)

This identity can only be performed under the following conditions:

$$C_3\alpha_{1,m,k-m}(r) + C_4\alpha_{2,m,k-m}(r) = 0 \quad (k = 0, 1, 2, \ldots) (m = 0, 1, 2, \ldots),$$

In particular, when $k = 0$, $m = 0$, should be performed: $C_3 + C_4 (r/a)^2 = 0$. Hence

$$C_3 = C_4 = 0.$$ \hspace{1cm} (35)

Then from the formula (34) we conclude that the solutions $X_n(r)$ ($n = 1, 2$) are linearly independent. Considering this fact, from the identity (33) taking into account (35) we get: $C_1 = C_2 = C_3 = C_4 = 0$.

Consequently, the solutions $X_n(r)$, $Y_n(r)$ ($n = 1, 2$) are also linearly independent.

Thus, we can state that the formulas (9), (13), (19)-(27) define four fundamental solutions $X_n(r)$, $Y_n(r)$ ($n = 1, 2$) of the equation (8). In addition, based on the analysis of the listed formulas and the results of the work [8], we conclude that $X_n(r)$, $Z_n(r)$ ($n = 1, 2$) functions are dimensionless.

Consequently, the functions $Y_n(r)$ ($n = 1, 2$) will also be dimensionless.

As a result, we will have for the general solution of the equation (8)

$$w(r) = C_1X_1(r) + C_2X_2(r) + C_3Y_1(r) + C_4Y_2(r),$$ \hspace{1cm} (36)

where $C_1, C_2, C_3, C_4$ are arbitrary constants having a bend dimension. In order to highlight a dimensionless multiplier in the right part of the formula (36), let’s put $C_n = \lambda_n (n = 1, 2, 3, 4)$, where $\lambda_n$ is arbitrary dimensionless constants. Then the formula for amplitude bends (36) can be written as:

$$w(r) = \lambda_1 w_0(r);$$ \hspace{1cm} (37)

$$w_0(r) = \lambda_1 X_1(r) + \lambda_2 X_2(r) + \lambda_3 Y_1(r) + \lambda_4 Y_2(r),$$ \hspace{1cm} (38)

where $w_0(r)$ is dimensionless function.

In a similar form, we also present formulas for the first three derivatives of the $w(r)$ function:

$$\frac{dw}{dr} = \tilde{w}_0(r),$$ \hspace{1cm} (39)

$$\tilde{w}_0(r) = \lambda_1 \tilde{X}_1(r) + \lambda_2 \tilde{X}_2(r) + \lambda_3 \tilde{Y}_1(r) + \lambda_4 \tilde{Y}_2(r);$$ \hspace{1cm} (40)

$$\frac{d^2w}{dr^2} = \frac{1}{a} \tilde{w}_0(r),$$ \hspace{1cm} (41)
\[ \hat{w}_0(r) = \lambda_1 \hat{X}_1(r) + \lambda_2 \hat{X}_2(r) + \lambda_3 \hat{Y}_1(r) + \lambda_4 \hat{Y}_2(r); \quad (42) \]

\[ \frac{d^3 w}{dr^3} = \frac{1}{a^2} \hat{w}_0(r), \quad (43) \]

\[ \hat{w}_0(r) = \lambda_1 \hat{X}_1(r) + \lambda_2 \hat{X}_2(r) + \lambda_3 \hat{Y}_1(r) + \lambda_4 \hat{Y}_2(r), \quad (44) \]

where

\[ \hat{X}_n(r) = a \frac{dX_n(r)}{dr}, \quad \hat{X}_n(r) = a^2 \frac{d^2 X_n(r)}{dr^2}, \quad \hat{X}_n(r) = a^3 \frac{d^3 X_n(r)}{dr^3} \quad (n = 1, 2), \quad (45) \]

\[ \hat{Y}_n(r) = a \frac{dY_n(r)}{dr}, \quad \hat{Y}_n(r) = a^2 \frac{d^2 Y_n(r)}{dr^2}, \quad \hat{Y}_n(r) = a^3 \frac{d^3 Y_n(r)}{dr^3} \quad (n = 1, 2). \quad (46) \]

Unlike the derivative functions \( X_n(r), Y_n(r) \), the functions \( \hat{X}_n(r), \hat{Y}_n(r) \) are dimensionless [8]. Then we will call them dimensionless derivatives. However, taking into account (9), for dimensionless derivatives (46) we get:

\[ \hat{Y}_n(r) = \hat{X}_n(r) \ln \frac{r}{a} + a \frac{X_n(r)}{r} + \hat{Z}_n(r); \]

\[ \hat{Y}_n(r) = \hat{X}_n(r) \ln \frac{r}{a} + 2a \frac{X_n(r)}{r} - \left( \frac{a}{r} \right)^2 X_n(r) + \hat{Z}_n(r) \]

\[ \hat{Y}_n(r) = \hat{X}_n(r) \ln \frac{r}{a} + 3a \frac{X_n(r)}{r} - 3 \left( \frac{a}{r} \right)^2 \hat{X}_n(r) + 2 \left( \frac{a}{r} \right)^3 X_n(r) + \hat{Z}_n(r), \]

where

\[ \hat{Z}_n(r) = a \frac{dZ_n(r)}{dr}, \quad \hat{Z}_n(r) = a^2 \frac{d^2 Z_n(r)}{dr^2}, \quad \hat{Z}_n(r) = a^3 \frac{d^3 Z_n(r)}{dr^3} \]

are dimensionless derivative functions \( Z_n(r) \).

Thus, the amplitude function of bends and its first three derivatives are expressed with the dimensionless functions \( w_0(r), \hat{w}_0(r), \hat{w}_0(r), \hat{w}_0(r) \). In this case, the formulas for the dynamic movements (5), (1) and the dynamic forces (2)-(4) taking into account (37), (39), (41), (43) will appear as:

\[ W(r,t) = a w_0(r)T(t); \quad (47) \]

\[ \varphi(r,t) = \hat{w}_0(r)T(t); \quad (48) \]

\[ M_s(r,t) = -\frac{D}{a} \left( \hat{w}_0(r) + \frac{a}{r} \hat{w}_0(r) \right)T(t); \quad (49) \]

\[ M_s(r,t) = -\frac{D}{a} \left( \mu \hat{w}_0(r) + \frac{a}{r} \hat{w}_0(r) \right)T(t); \quad (50) \]

\[ Q_s(r,t) = -\frac{D}{a^2} \left( \hat{w}_0(r) + \frac{a}{r} \hat{w}_0(r) - \left( \frac{a}{r} \right)^2 \hat{w}_0(r) \right)T(t). \quad (51) \]

As a result, we can state that the dynamic movements of \( W(r,t), \varphi(r,t) \) and internal forces \( M_s(r,t), M_s(r,t), Q_s(r,t) \) are expressed with dimensionless fundamental functions of the \( X_n(r), Y_n(r) \) \( (n = 1, 2) \) equation (8) and their dimensionless derivatives. This makes it possible to operate only dimensionless values when calculating plates for oscillations.
4. Analytical representation for oscillation frequency

Directly from the equality (14) we find

$$\omega = \frac{K}{a^2} \sqrt{\frac{D}{\rho h}}. \quad (52)$$

The formula (52) establishes the analytical dependence of the oscillation frequency on other parameters of the mechanical system, which is considered. In fact, the frequency definition is reduced to finding a dimensionless parameter $K$, which we will call the coefficient of oscillations. Since the fundamental functions $X_n(r), Y_n(r) \ (n = 1, 2)$ depend exactly on the coefficient of oscillations, the frequency equations will be used to find it, which will be obtained after the implementation of the desired boundary conditions.

5. Analytical method of numerical realization of exact solutions

Formally, the exact formulas (37)-(44), (47)-(51) are obtained, which are necessary for the research of free oscillations of circular and annular plates under different boundary conditions. However, the implementation of these formulas in practice is connected with necessity of a repeated integral evaluation (20)-(22), (25)-(27), which define the generating functions. These evaluations can be difficult to do. Since the variable modulus of subgrade reaction is often set by polynomials [6], we suggest to specify the analytical method for evaluating the aforementioned integrals for this particular case. In fact, this means the representation of generating functions by polynomials.

As it is clear from the formulas (28), (29), the generating functions $\alpha_{n,k,i}(r) \ (k = 1, 2, 3, \ldots)$ are already represented by polynomials. When considering other generating functions, we assume that the dimensionless function $a$ is defined by a polynomial

$$A(r) = A_0 + A_1 \left( \frac{r}{a} \right) + A_2 \left( \frac{r}{a} \right)^2 + \ldots + A_s \left( \frac{r}{a} \right)^s. \quad (53)$$

In this case, it is directly from formula (21) taking into account (19), (53), that the generating functions $\alpha_{n,0,k}(r) \ (k = 1, 2, 3, \ldots)$ a will also be polynomials. The smallest and largest degrees of these polynomials will be respectively equal to $2n + 4k - 2$ and $ks + 2n + 4k - 2$. Consequently, the functions $\alpha_{n,0,k}(r)$ can be represented in the form

$$\alpha_{n,0,k}(r) = \left( \frac{r}{a} \right)^{2n+4k-2} \sum_{j=0}^{k} c_{n,0,k,j} \left( \frac{r}{a} \right)^j \quad (k = 1, 2, 3, \ldots), \quad (54)$$

where $c_{n,0,k,j} -$ coefficients, which have to be determined. Then

$$\alpha_{n,0,k-1}(r) = \left( \frac{r}{a} \right)^{2n+4k-6} \sum_{j=0}^{(k-1)} c_{n,0,k-1,j} \left( \frac{r}{a} \right)^j. \quad (55)$$

In this case, for setting the polynomials (53) and (55) we will have

$$A(r)\alpha_{n,0,k-1}(r) = \left( \frac{r}{a} \right)^{2n+4k-6} \sum_{j=0}^{k} c_{n,0,k-1,j} \left( \frac{r}{a} \right)^j, \quad (56)$$

where

$$e_{n,0,k-1,j} = \sum_{i=0}^{j} A_{j-i} c_{n,0,k-1,i}. \quad (57)$$
moreover

\[ A_{j-i} = 0, \text{ if } j - i > s; \quad c_{n,0,k-1,j} = 0, \text{ if } i > (k-1)s. \]  

(58)

Put its value (56) into the formula (21) instead of the \( A(r)\alpha_{n,0,k-1,j}(r) \) its value and integrate it.

As a result, we get

\[
\alpha_{n,0,k}(r) = \left(\frac{r}{a}\right)^{2n+4k-2} \sum_{j=0}^{s} e_{n,0,k-1,j} \left(\frac{r}{a}\right)^j,
\]

(59)

where \( p_{n,k,j} = (2n + 4k + j - 4)(2n + 4k + j - 2). \) Comparing the equality (54), (59) and taking into account (57), we will have the following recurrence formula for required coefficients

\[
c_{n,0,k-1,j} = \frac{\sum_{j=0}^{s} A_{j-i} c_{n,0,k-1,j}}{p_{n,k,j}} (k = 1, 2, 3, \ldots) (j = 0, 1, \ldots, ks).
\]

(60)

At the initial stage of calculations, when \( k = 1, \) the formula (60) requires the initial values \( c_{n,0,0,i} \) \((i = 0, 1, 2, \ldots, j)\). Putting \( k = 1 \) into the formula (55) and comparing the result with the equality (19), we find

\[ c_{n,0,0,0} = 1. \]

(61)

The remaining initial values, as it is clear from the formula (58), when \( k = 1, \) will be zero

\[ c_{n,0,0,i} = 0 \quad (i = 1, 2, 3, \ldots, j). \]

(62)

Consequently, the coefficients of the polynomials (54) are fully determined by the formulas (60)-(62), taking into account the conditions (58).

Analyzing the formula (22) taking into account (28), (29), (53), (54), we conclude that the generating functions \( \alpha_{n,m,k-m}(r) \) are also polynomials, their smallest and largest degrees will respectively be equal to \( 2n + 4k - 2 \) and \((k - m)s + 2n + 4k - 2. \) So we can write down

\[
\alpha_{n,m,k-m}(r) = \left(\frac{r}{a}\right)^{2n+4k-2} \sum_{j=0}^{s} c_{n,m,k-m,j} \left(\frac{r}{a}\right)^j \quad (k = 2, 3, 4, \ldots) (m = 1, 2, 3, \ldots, k-1),
\]

(63)

where \( c_{n,m,k-m,j} \) - required coefficients. Moving in the formula (63) the indices \( k \) and \( m \) by one, we get

\[
\alpha_{n,m-1,k-m}(r) = \left(\frac{r}{a}\right)^{2n+4k-6} \sum_{j=0}^{s} c_{n,m-1,k-m,j} \left(\frac{r}{a}\right)^j,
\]

(64)

and moving by one only index \( k \), we have

\[
\alpha_{n,m,k-m-1}(r) = \left(\frac{r}{a}\right)^{2n+4k-6} \sum_{j=0}^{s} c_{n,m,k-m-1,j} \left(\frac{r}{a}\right)^j.
\]

(65)

Moreover, for setting the polynomials (53) and (65) we find

\[
A(r)\alpha_{n,m,k-m-1}(r) = \left(\frac{r}{a}\right)^{2n+4k-6} \sum_{j=0}^{s} e_{n,m,k-m-1,j} \left(\frac{r}{a}\right)^j,
\]

(66)
where

\[ e_{n,m,k-m-1,j} = \sum_{i=0}^{j} A_{j-i} c_{n,m,k-m-1,i}, \]

(67)

moreover

\[ A_{j-i} = 0, \text{ if } j-i > s; \quad c_{n,m,k-m-1,i} = 0, \text{ if } i > (k-m-1)s. \]

(68)

In the formula (22), replace \( \alpha_{n,m-1,k-m}(r) \) and \( A(r)\alpha_{n,m,k-m-1}(r) \) with their values (64) and (66), and then integrate. As a result, we get

\[ \alpha_{n,m,k-m}(r) = \left( \frac{r}{a} \right)^{2n+4k-2} \sum_{j=0}^{(k-m)s} \frac{c_{n,m-1,k-m,j} + c_{n,m,k-m-1,j} \left( \frac{r}{a} \right)^j}{p_{n,k,j}^2} \]

(69)

\[(k = 2,3,4,...) \quad (m = 1,2,3,...,k-1). \]

Comparing the equality (63) and (69) with each other, and also taking into account account (67), we come to the next recurrence formula for required coefficients

\[ c_{n,m,k-m,j} = \frac{c_{n,m-1,k-m,j} + \sum_{i=0}^{j} A_{j-i} c_{n,m,k-m-1,i}}{p_{n,k,j}^2} \]

(70)

\[(k = 2,3,4,...) \quad (m = 1,2,3,...,k-1) \quad (j = 0,1,2,...,(k-m)s). \]

At the first stage of calculations according to the formula (70), when \( k = 2, \quad m = 1 \), the initial values \( c_{n,0,1,j} (j = 0,1,2,...,s) \) and \( c_{n,1,0,j} (i = 0,1,2,...,j) \) will be used. The first of them are found out according to the formula (60). For the second ones, when \( i = 0 \), we find by formula (29)

\[ c_{n,1,0,0} = \frac{1}{(2^2(n+1))^2}. \]

(71)

The remaining initial values will be zero

\[ c_{n,1,0,j} = 0 \quad (i = 1,2,3,...,j), \]

(72)

that follows directly from the condition (68) at the values \( k = 2, \quad m = 1 \).

Thus, using formulas (70)-(72), taking into account conditions (68), the coefficients of the polynomials (63) are completely determined.

In a similar way, we can obtain the representations in the form of polynomials for the generating functions (25)-(27). So we will write the final formulas:

\[ \beta_{n,k,0}(r) = d_{n,k,0,0} \left( \frac{r}{a} \right)^{2n+4k-2} \quad (k = 1,2,3,...); \]

\[ d_{n,0,0,0} = 1, \quad d_{n,k,0,0} = \frac{d_{n,k-1,0,0}}{p_{n,k,0}^2} - 4(2n + 4k - 3) \frac{c_{n,k,0,0}}{p_{n,k,0}} \quad (k = 1,2,3,...); \]

\[ \beta_{n,0,k}(r) = \left( \frac{r}{a} \right)^{2n+4k-2} \sum_{j=0}^{k} d_{n,0,k-j,0} \left( \frac{r}{a} \right)^j \quad (k = 1,2,3,...); \]

\[ d_{n,0,0,0} = 1, \quad d_{n,0,k,0} = \frac{\sum_{i=0}^{j} A_{j-i} d_{n,0,k-i,0}}{p_{n,k,0}^2} - 4(2n + 4k + j - 3) \frac{c_{n,k,0,j}}{p_{n,k,0}} \quad (k = 1,2,3,...) \quad (j = 0,1,...,ks). \]
where

\[ A_{j-1} = 0, \text{ if } j-i > s; \quad d_{n,0,k-1,i} = 0, \text{ if } i > (k-1)s; \]

\[
\beta_{n,m,k,m}(r) = \left(\frac{r}{a}\right)^{2n+4k-2} \sum_{j=0}^{(k-m)s} d_{n,m,k-m,j} \left(\frac{r}{a}\right)^j \quad (k = 2,3,4,...) \quad (m = 1,2,3,...,k-1);
\]

\[ d_{n,1,0,0} = \frac{1}{(2^n+1)^2} - \frac{4(2n+1)}{(2^n+1)^3}; \]

\[ d_{n,m,k,m,j} = \frac{\sum_{i=0}^{j} A_{j-i} d_{n,m,k-m,i}}{p_{n,k,j}} - 4(2n+4k+j-3) \frac{c_{n,m,k-m,j}}{p_{n,k,j}} \quad (k = 2,3,4,...) \quad (m = 1,2,3,...,k-1) \quad (j = 0,1,2,...,(k-m)s), \]

where

\[ A_{j-i} = 0, \text{ if } j-i > s; \quad d_{n,m,k-m-1,i} = 0, \text{ if } i > (k-m-1)s. \]

As a result, all the generating functions are represented by the polynomials. Thus, in the same way, the analytical method for the numerical implementation of exact solutions is proposed. The format of this publication does not allow to consider numerical examples here. The authors plan to devote the following article to this question.

### 6. Results and discussion

Whenever the research of any physical phenomenon is reduced to a differential equation, the key question is constructing its exact (analytical) solution. However, the researchers often deal with known mathematical problem, there is no universal method for integrating differential equations with variable coefficients. This may explain the predominant use of approximate methods. This includes the problem of free axisymmetric oscillations of circular and annular plates on a variable elastic foundation. In this work, for the case where the foundation is described by Winkler model with continuous variable modulus of subgrade reaction, these difficulties were overcome. As a result, there are the following results:

1. The exact solution of the differential equation of oscillations has been set;
2. Formulas for dynamic movements and internal forces in the plate have been obtained;
3. The analytical representation for the oscillation frequency has been obtained;
4. The analytical method of numerical realization of predicted exact solutions has been proposed.

As a matter of fact, the solution of the initial problem is only reduced to the realization of desired boundary conditions and finding appropriate dimensionless oscillation coefficients from the obtained frequency equation.

### 7. Conclusions

A new analytical method of calculation for free axisymmetric oscillations of circular and annular plates on a continuous variable elastic foundation has been developed. The proposed method does not require the discretization of the considered structure and it is a real alternative to the application of approximate methods during solving this types of problems. Based on the exact solution of the differential equation, this method will allow to obtain better picture of oscillations in comparison with approximate methods. In fact, the exact solution has qualitative information and forms the most complete picture of the physical phenomenon that is researched.

### References

[1] Timoshenko S and Woinowsky-Krieger S 1959 Theory of Plates and Shells McGraw-Hill
[2] Ponomarev S, Biderman V, Likharev K, Makushin V, Malinin N and Feodos’ev V 1956 *Raschety na prochnost’ v mashinostroyeni* Mashgiz

[3] Vaynberg D, Vaynberg Ye 1970 *Raschet plastin* Budivel'nik

[4] Witt M 1974 Rozwiązanie płyty spoczywającej na podłożu sprężystym o zmiennym współczynniku podatności metoda elementów skończonych *Pr. nauk. Inst. inz. Lad. Pwr* 13 pp 143-149

[5] Mofid M and Noroozi M 2009 A plate on Winkler foundation with variable coefficient *Transaction A: Civil Engineering* 16 pp 249-255

[6] Doronin A and Soboleva V 2014 Sobstvennyye kolebaniya krugloy plastinki, lezhashchej na peremennom uprugom osnovanii tipa Vinklera *Vestnik Nizhegorodskogo universiteta im. Lobachevskogo* 4 pp 254-258

[7] Filippov A 1970 *Kolebaniya deformiruemyh system* Mashinostroenie

[8] Krutii Y 2016 *Rozrobka metodu roz’yazannya zadach stiykosti i kolyvan’ deformivnykh system zi zminnyny neperervnymy parametramy* Sc. D. diss., Lutsk National Technical University

[9] Krutii Y 2018 Construction of a solution of the problem of stability of a bar with arbitrary continuous parameters *J. Math. Sci* 231 pp 665-677

[10] Shvab’yuk V, Krutii Y and Sur’yaninov M 2016 Investigation of the free vibrations of bar elements with variable parameters using the direct integration method *Strength of Materials* 48 pp 384-393

[11] Krutii Y 2015 Analysis of longitudinal oscillations for systems with continuous variable parameters using force integration method *Technical journal* 9 pp 420-425

[12] Krutii Y, Surianinov M and Vandynskyi V 2018 Development of the method for calculation of cantilever construction's oscillations taking into account own weight *Eastern-European Journal of Enterprise Technologies* 93 pp 13-19

[13] Krutii Y, Surianinov M, Vandynskyi V 2019 Analytic formulas for the cantilever structures’ natural frequencies with taking into account the own weight *Materials Science Forum* 968 pp 450–459