EQUIVALENCE BETWEEN THE MORITA CATEGORIES OF ÉTALE LIE GROUPOIDS AND OF LOCALLY GROUPLIKE HOPF ALGEBROIDS

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Abstract. Any étale Lie groupoid $G$ is completely determined by its associated convolution algebra $\mathcal{C}^\infty_c(G)$ equipped with the natural Hopf algebroid structure. We extend this result to the generalized morphisms between étale Lie groupoids: we show that any principal $H$-bundle $P$ over $G$ is uniquely determined by the associated $\mathcal{C}^\infty_c(G)$-$\mathcal{C}^\infty_c(H)$-bimodule $\mathcal{C}^\infty_c(P)$ equipped with the natural coalgebra structure. Furthermore, we prove that the functor $\mathcal{C}^\infty_c$ gives an equivalence between the Morita category of étale Lie groupoids and the Morita category of locally grouplike Hopf algebroids.

1. Introduction

The ideas and tools of noncommutative geometry have given us an insight into a large new class of spaces, which seemed unattainable from the point of view of the classical topology and geometry. Lie groupoids and their convolution algebras provide models for many such singular spaces, for example orbifolds, spaces of orbits of Lie group actions and leaf spaces of foliations [4, 5, 9, 16, 17]. A singular space may, however, be represented by different Lie groupoids which are weakly equivalent to each other. For example, the foliation groupoids (and in particular the holonomy groupoids of foliations) may be represented by étale Lie groupoids [7, 17]. It turns out that two Lie groupoids are weakly equivalent if and only if they are isomorphic in the Morita category of Lie groupoids, the category in which morphisms are isomorphism classes of principal bundles [8, 10, 15, 18, 19].

We are therefore primarily interested in those algebraic invariants of Lie groupoids which are functorially defined on the Morita category, thus respecting the weak equivalence.

The Connes convolution algebra $\mathcal{C}^\infty_c(G)$ of smooth functions with compact support [5, 24] on an étale Lie groupoid $G$ is an example of such an invariant. Indeed, the map $\mathcal{C}^\infty_c$ can be extended to a functor from the Morita category of étale Lie groupoids to the Morita category of algebras [19]. More precisely, if $G$ and $H$ are étale Lie groupoids and if $P$ is a principal $H$-bundle over $G$, then the space $\mathcal{C}^\infty_c(P)$ of smooth functions with compact support on $P$ has a natural structure of a $\mathcal{C}^\infty_c(G)$-$\mathcal{C}^\infty_c(H)$-bimodule. Furthermore, the composition of principal bundles is reflected as the tensor product of the corresponding bimodules.

The convolution algebra $\mathcal{C}^\infty_c(G)$ admits an additional structure of a coalgebra over the commutative ring $\mathcal{C}^\infty_c(M)$ of smooth functions with compact support on the base manifold $M$ of objects of $G$, which turns $\mathcal{C}^\infty_c(G)$ into a Hopf algebroid over $\mathcal{C}^\infty_c(M)$ [20, 22]. Moreover, the $\mathcal{C}^\infty_c(G)$-$\mathcal{C}^\infty_c(H)$-bimodule $\mathcal{C}^\infty_c(P)$ has a natural coalgebra structure over $\mathcal{C}^\infty_c(M)$ as well, compatible with the coalgebra structures on $\mathcal{C}^\infty_c(G)$ and $\mathcal{C}^\infty_c(H)$ in a natural way [20]. The Hopf algebroid structure on $\mathcal{C}^\infty_c(G)$ determines the étale Lie groupoid $G$ uniquely [22]. In fact, one can reconstruct

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$G$ out of $\mathcal{C}_c^\infty(G)$ as the spectral étale groupoid associated to the Hopf algebroid $\mathcal{C}_c^\infty(G)$. The Hopf algebroids isomorphic to those associated to étale Lie groupoids can be characterized as those which are locally grouplike (see [22] and Definition 6.4 below).

In this paper we show how to reconstruct a principal $H$-bundle $P$ over $G$ out of the associated $\mathcal{C}_c^\infty(G)\mathcal{C}_c^\infty(H)$-bimodule $\mathcal{C}_c^\infty(P)$ equipped with the natural coalgebra structure. Moreover, we show that a $\mathcal{C}_c^\infty(G)\mathcal{C}_c^\infty(H)$-bimodule $\mathcal{M}$ with a $\mathcal{C}_c^\infty(M)$-coalgebra structure is isomorphic to the bimodule $\mathcal{C}_c^\infty(P)$ of a principal $H$-bundle $P$ over $G$ if and only if $\mathcal{M}$ is principal and locally grouplike (see Definition 5.2), and that the principal bundle $P$ is uniquely determined by $\mathcal{M}$ up to an isomorphism. Furthermore, we show that locally grouplike Hopf algebroids and locally grouplike principal bimodules form a category $\text{LgHoALGD}$, which is equivalent to the Morita category $\text{EtGPD}$ of étale Lie groupoids (Theorem 5.1). The equivalence is given by the functor $\mathcal{C}_c^\infty: \text{EtGPD} \to \text{LgHoALGD}$.

2. Preliminaries

For the convenience of the reader, and to fix the notations, we begin by summarizing some basic definitions and results that will be used in the rest of this paper. We shall write $\mathbb{F}$ for our base field, which can be $\mathbb{R}$ or $\mathbb{C}$. Throughout the paper, all manifolds and maps between them are assumed to be smooth. This is not essential: the results hold true if one replaces this by any class of differentiability $\mathcal{C}^k$, $k = 0, 1, 2, \ldots$. The manifolds are not assumed to be Hausdorff.

2.1. Lie groupoids and principal bundles. First, we recall the notion of a Lie groupoid and the definition of the Morita category of Lie groupoids. For detailed exposition and many examples, we refer the reader to one of the books [13, 17, 18] and references cited there.

A Lie groupoid over a Hausdorff manifold $M$ is given by a manifold $G$ and a structure of a category on $G$ with objects $G_0 = M$, in which all arrows are invertible and all the structure maps

$$G \times_{G_0} G \xrightarrow{\text{mult}} G \xrightarrow{\text{inv}} G \xrightarrow{s} G_0 \xrightarrow{\text{uni}} G$$

are smooth, with the source map $s$ a submersion. We allow manifold $G$ to be non-Hausdorff, but we assume that the fibers of the source map are Hausdorff. If $g \in G$ is any arrow with source $s(g) = x$ and target $t(g) = y$, and $g' \in G$ is another arrow with $s(g') = y$ and $t(g') = y'$, then the product $g'g = \text{mult}(g',g)$ is an arrow from $x$ to $y'$. The map $s$ assigns to each $x \in G_0$ the identity arrow $1_x = \text{uni}(x)$ in $G$, and we often identify $G_0$ with $\text{uni}(G_0)$. The map $\text{inv}$ maps each $g \in G$ to its inverse $g^{-1}$. We write $G(x,y) = s^{-1}(x) \cap t^{-1}(y)$.

A left action of a Lie groupoid $G$ on a manifold $P$ along a map $\pi: P \to G_0$ is a map $\mu: G \times_{G_0}^\pi P \to P$, $(g,p) \mapsto g \cdot p$, which satisfies $\pi(g \cdot p) = t(g)$, $1_{\pi(p)} \cdot p = p$ and $g' \cdot (g \cdot p) = (g'g) \cdot p$, for all $g', g \in G$ and $p \in P$ with $s(g') = t(g)$ and $s(g) = \pi(p)$. We define right actions of Lie groupoids on manifolds in a similar way.

Let $G$ and $H$ be Lie groupoids. A principal $H$-bundle over $G$ is a manifold $P$, equipped with a left action $\mu$ of $G$ along a surjective submersion $\pi: P \to G_0$ and a right action $\eta$ of $H$ along $\phi: P \to H_0$, such that $\phi(g \cdot p) = \phi(p)$, $\pi(p \cdot h) = \pi(p)$ and $g \cdot (p \cdot h) = (g \cdot p) \cdot h$ for every $g \in G$, $p \in P$ and $h \in H$ with $s(g) = \pi(p)$ and $\phi(p) = t(h)$, and such that $(pr_1, \eta): P \times_H^\phi P \to P \times_{G_0}^\pi P$ is a diffeomorphism.

A map $f: P \to P'$ between principal $H$-bundles $P$ and $P'$ over $G$ is equivariant if it satisfies $\pi'(f(p)) = \pi(p)$, $\phi'(f(p)) = \phi(p)$ and $f(g \cdot p \cdot h) = g \cdot f(p) \cdot h$, for every $g \in G$, $p \in P$ and $h \in H$ with $s(g) = \pi(p)$ and $\phi(p) = t(h)$. Any such
map is automatically a diffeomorphism. Principal $H$-bundles $P$ and $P'$ over $G$ are isomorphic if there exists an equivariant diffeomorphism between them.

If $P$ is a principal $H$-bundle over $G$ and $P'$ is a principal $K$-bundle over $H$, for another Lie groupoid $K$, one can construct the principal $K$-bundle $P \otimes_H P'$ over $G$ [18, 19]. It is the space of orbits $(P \times_H P')/H$ with respect to the natural action of $H$ along $\phi \circ \rho_0$, given by $(p, p') \cdot h = (p \cdot h, h^{-1} \cdot p')$. The actions of $G$ and $K$ on $P \otimes_H P'$ along $\pi \otimes \pi': p \otimes p' \mapsto \pi(p)$ respectively $\phi \otimes \phi': p \otimes p' \mapsto \phi(p')$ are given by $g \cdot (p \otimes p') = (g \cdot p) \otimes p'$ and $(p \otimes p') \cdot k = p \otimes (p' \cdot k)$, where $p \otimes p'$ denotes the orbit of $(p, p')$ in $P \otimes_H P'$.

As an example, any Lie groupoid $G$ can be seen as a principal $G$-bundle over $G$ with the actions given by the groupoid multiplication along the maps $t$ respectively $s$. This bundle behaves as the identity for the tensor product, up to an isomorphism.

The Morita category $\text{GPD}$ of Lie groupoids consists of Lie groupoids as objects and isomorphism classes of principal bundles as morphisms between them: a principal $H$-bundle over $G$ represents a morphism from $G$ to $H$, while the composition of morphisms is induced by the tensor product. The morphisms in $\text{GPD}$ are sometimes referred to as Hilsen-Raschitz maps or generalised morphisms between Lie groupoids. Two Lie groupoids are Morita equivalent if they are isomorphic in the category $\text{GPD}$.

A Lie groupoid is étale if all its structure maps are local diffeomorphisms. A bisection of an étale Lie groupoid $G$ is an open subset $U$ of $G$ such that both $s|_U$ and $t|_U$ are injective. Any such bisection $U$ gives a diffeomorphism $\tau_U: s(U) \to t(U)$ by $\tau_U = t|_U \circ (s|_U)^{-1}$.

The Morita category $\text{EtGPD}$ of étale Lie groupoids is the full subcategory of $\text{GPD}$ with étale Lie groupoids as objects. If $G$ and $H$ are étale Lie groupoids and $P$ a principal $H$-bundle over $G$, then the corresponding map $\pi: P \to G_0$ is automatically a local diffeomorphism.

2.2. The bimodule associated to a principal bundle. In this subsection we review the construction of the principal bimodule assigned to a principal bundle. Our exposition closely follows [19], where all the work is done in the Hausdorff setting. It turns out that essentially the same ideas also work in the non-Hausdorff case if we use the proper notion of a smooth function with compact support.

We first recall the definition of a smooth function with compact support on a non-Hausdorff manifold as given in [6]. Let $P$ be a manifold and let $C_c^\infty(P)$ denote the sheaf of germs of smooth $\mathbb{R}$-valued functions on $P$. The stalk of this sheaf at a point $p \in P$ is a commutative algebra with identity. If $P$ is Hausdorff, we can identify (compactly supported) smooth functions on $P$ with (compactly supported) continuous sections of $C_c^\infty(P)$, and we denote the commutative algebra of compactly supported smooth $\mathbb{R}$-valued functions on $P$ by $C_c^\infty(P)$. For a general $P$, we consider the space $\Gamma_4(P, C_c^\infty)$ of not-necessarily continuous sections of the sheaf $C_c^\infty(P)$. For any Hausdorff open subset $U \subset P$ there is a monomorphism $C_c^\infty(U) \to \Gamma_4(P, C_c^\infty)$, which maps $f \in C_c^\infty(U)$ to the extension of $f$ to $P$ by zero. The vector space $C_c^\infty(P)$ of smooth functions with compact support on $P$ is by definition the image of the map $\bigoplus_U C_c^\infty(U) \to \Gamma_4(P, C_c^\infty)$, where $U$ runs over all (or a cover of) Hausdorff open subsets of $P$. This definition agrees with the classical one if $P$ is Hausdorff. We will denote the extension of $f \in C_c^\infty(U)$ to $P$ again by $f \in C_c^\infty(P)$, and identify the space $C_c^\infty(U)$ with its image in $C_c^\infty(P)$.

For any $f \in C_c^\infty(P)$ we define the support of $f$ by supp$(f) = \{p \in P \mid f_p \neq 0\}$, where $f_p$ is the value of the section $f$ at the point $p$. So defined support agrees with the classical one in the Hausdorff case, and is a compact set, although not closed in general. Every $f \in C_c^\infty(P)$ with support in some Hausdorff open subset $U \subset P$ can be identified with $C_c^\infty(U)$. We will often work with smooth functions on
the total space $P$ of a local diffeomorphism $\pi : P \to M$ into a Hausdorff manifold $M$. In this case, we say that $U \subset P$ is $\pi$-elementary if $\pi|_U$ is injective. The open $\pi$-elementary subsets of $P$ are Hausdorff and together they cover $P$. We have a natural identification $C^\infty_c(U) \cong C^\infty_c(\pi(U))$, for every $\pi$-elementary open subset $U \subset P$, identifying $f_0 \in C^\infty_c(\pi(U))$ with $f = f_0 \circ \pi|_U \in C^\infty_c(U) \subset C^\infty_c(P)$.

Let $\psi : P \to P'$ be a smooth map. For any $p \in P$ we have a homomorphism of algebras $\psi_p^\# : (C^\infty_c(P))_{\psi(p)} \to (C^\infty_c(P'))_p$ given by the composition with $\psi$. If $\psi$ is a local diffeomorphism, this homomorphism has the inverse $\psi_*^\# = (\psi_*^\#)^{-1}$, and we can define a linear map $\psi_* : C^\infty_c(P) \to C^\infty_c(P')$ by

$$
\psi_*(f) = \sum_{p \in \psi^{-1}(p')} \psi_*^\#(f_p).
$$

In this way $C^\infty_c$ becomes a functor from the category of smooth manifolds and local diffeomorphisms between them to the category of vector spaces.

We can use this definition of smooth functions with compact support to construct the bimodule associated to a principal bundle over étale Lie groupoids [19]. Let $P$ be a principal $H$-bundle over $G$ and let $P'$ be a principal $K$-bundle over $H$, where $G$, $H$, and $K$ are étale Lie groupoids. Define a bilinear map

$$
\rho = \rho_{P,P'} : C^\infty_c(P) \times C^\infty_c(P') \to C^\infty_c(P \times H P')
$$

by

$$
\rho(f,f')(p,p') = (\pr_1^\#)(f,p')(\pr_2^\#)(f',p'),
$$

where $\pr_1$ and $\pr_2$ are the projections from $P \times H P'$ to $P$ respectively $P'$. To show that $\rho(f,f')$ is indeed a smooth function with compact support on $P \times H P'$, we can assume that $f \in C^\infty_c(U)$ and $f' \in C^\infty_c(U')$, where $U$ is a $\pi$-elementary open subset of $P$ and $U'$ a $\pi'$-elementary open subset of $P'$. The support of $\rho(f,f')$ is then contained in the Hausdorff $\pr_1$-elementary open subset $U \times H_0 P'$ of $P \times H_0 P'$, and

$$
(1) \quad \rho(f,f')|_{U \times H_0 U'} = (f(f_0^\# \circ \phi)) \circ \pr_1|_{U \times H_0 U'},
$$

where $f_0^\# \in C^\infty_c(\pi'(U'))$ is such that $f' = f_0^\# \circ \pi'|_{U'}$. The support $S = \text{supp}(f) \cap \phi^{-1}(\text{supp}(f_0^\#))$ of the function $f(f_0^\# \circ \phi) \in C^\infty_c(U)$ is compact and lies in the set $\pr_1(U \times H_0 U') = U \cap \phi^{-1}(\pi'(U'))$. Indeed, since $\text{supp}(f_0^\#)$ is compact in the Hausdorff manifold $H_0$, it is closed, so $\phi^{-1}(\text{supp}(f_0^\#))$ is closed as well. The set $S$ is then a closed subspace of the compact space $\text{supp}(f)$ and therefore compact. This shows that $\rho(f,f')$ is a smooth function with compact support inside $U \times H_0 U'$.

Define the map

$$
\varphi = \varphi_{P,P'} = q_+ \circ \rho : C^\infty_c(P) \times C^\infty_c(P') \to C^\infty_c(P \times H P'),
$$

where $q$ is the quotient projection $P \times H P' \to P \times H P'$, which is in fact a local diffeomorphism. If we choose $f \in C^\infty_c(U)$ and $f' \in C^\infty_c(U')$ as in the equation (1), the function $\varphi(f,f')$ has the support in $U \times H U' = q(U \times H U')$ and is given by

$$
\varphi(f,f') = (f(f_0^\# \circ \phi)) \circ ((\pi|_U)^{-1} \circ (\pi \circ \pi'))|_{U \times H U'}.
$$

Now choose another étale Lie groupoid $L$ and a principal $L$-bundle $P''$ over $K$, and observe that there is a natural diffeomorphism from $(P \times H P') \circ K P''$ to $P \times H (P' \circ_K P'')$ which maps $(\varphi \circ p) \circ p'$ to $p \circ (p' \circ p'')$. Straight from the definition one can see that for any $f \in C^\infty_c(P)$, $f' \in C^\infty_c(P')$ and $f'' \in C^\infty_c(P'')$ we have the associativity law

$$
\varphi(\varphi(f,f'),f'') = \varphi(f,\varphi(f',f'')) ,
$$

using the natural identification $C^\infty_c((P \times H P') \circ_K P'') \cong C^\infty_c(P \times H (P' \circ_K P''))$.
The equivariant diffeomorphisms \( \theta_{G,G} : G \otimes G \to G, \theta_{G,P} : G \otimes G \to P \) and \( \theta_{P,H} : P \otimes H \to P \), given by \( \theta_{G,G}(g' \otimes g) = g'g \), \( \theta_{G,P}(g \otimes p) = g \cdot p \) and \( \theta_{P,H}(p, h) = p \cdot h \), induce the action maps

\[
\lambda_{G,G} = (\theta_{G,G})_+ \circ \phi : C_c^\infty(G) \times C_c^\infty(G) \to C_c^\infty(G), \\
\lambda_{G,P} = (\theta_{G,P})_+ \circ \phi : C_c^\infty(G) \times C_c^\infty(P) \to C_c^\infty(P)
\]

and

\[
\lambda_{P,H} = (\theta_{P,H})_+ \circ \phi : C_c^\infty(P) \times C_c^\infty(H) \to C_c^\infty(P).
\]

The map \( \lambda_{G,G} \) is precisely the convolution product on the algebra \( C_c^\infty(G) \) [5] [6] [19] [22]. Furthermore, the maps \( \lambda_{G,P} \) and \( \lambda_{P,H} \) turn \( C_c^\infty(P) \) into a \( C_c^\infty(G) \)-\( C_c^\infty(H) \)-bimodule [19]. Explicitly, choose a function \( f \in C_c^\infty(P) \) with support in a \( \pi \)-elementary open set \( U \subset P \), and functions \( u \in C_c^\infty(G) \) and \( v \in C_c^\infty(H) \) with supports in bissections \( U_a \subset G \) respectively \( U_v \subset H \). Write \( f = f_0 \circ \pi|_U, u = u_0 \circ t|_{U_a} \) and \( v = v_0 \circ t|_{U_v} \) for \( f_0 \in C_c^\infty(\pi(U)), u_0 \in C_c^\infty(t(U_a)) \) and \( v_0 \in C_c^\infty(t(U_v)) \). Then

\[
u f = (u_0(f_0 \circ \tau_{U_v}^{-1})) \circ \pi|_{U \times s_0 U_a},
\]

and

\[
fv = (f_0(u_0 \circ \phi \circ (\pi|_U)^{-1})) \circ \pi|_{U \times s_0 U_a}.
\]

Now choose a principal \( H \)-bundle \( P \) over \( G \) and a principal \( K \)-bundle \( P' \) over \( H \), and interpret \( H \) as a principal \( H \)-bundle over \( H \). For \( f \in C_c^\infty(P), f' \in C_c^\infty(P') \) and \( v \in C_c^\infty(H) \) we have

\[
\phi(fv, f') = \phi(\phi(f, v), f') = \phi(f, \phi(v, f')) = \phi(f, vf') \in C_c^\infty(P \otimes_H P'),
\]

where we have identified \( fv \in C_c^\infty(P) \) with \( \phi(f, v) \in C_c^\infty(P \otimes H) \) and \( vf' \in C_c^\infty(P') \) with \( \phi(v, f') \in C_c^\infty(H \otimes H P') \). The map \( \phi \) thus induces a homomorphism

\[
\Omega = \Omega_{P,P'} : C_c^\infty(P) \otimes C_c^\infty(H) C_c^\infty(P') \to C_c^\infty(P \otimes_H P')
\]

of \( C_c^\infty(G) \)-\( C_c^\infty(K) \)-bimodules, which is in fact an isomorphism. Indeed, this has been proven in [19] in the Hausdorff case, but literally the same proof applies in the general case as well.

### 2.3. Principal bimodules over Hopf algebroids

Next, we review the notions of a Hopf algebroid and of a principal bimodule over Hopf algebroids, following [20] [22]. Throughout this paper, we will assume that all our algebras are over the field \( \mathbb{F} \) and that they are associative, but not necessarily commutative. Recall that an algebra \( A \) has local identities in a commutative subalgebra \( A_0 \subset A \) if for any \( a_1, \ldots, a_k \in A \) there exists \( a_0 \in A_0 \) such that \( a_0 a_i = a_i a_0 = a_i \) for all \( i = 1, \ldots, k \). A commutative algebra has local identities if it has local identities in itself. A left module \( M \) over a commutative algebra \( A_0 \) with local identities is locally \( A_0 \)-unitary if for any \( m_1, \ldots, m_k \in M \) there exists \( a_0 \in A_0 \) such that \( a_0 m_i = m_i \) for all \( i = 1, \ldots, k \). Analogously one defines the notions of a right locally \( B_0 \)-unitary \( B_0 \)-module and of a locally \( A_0 \)-\( B_0 \)-unitary \( A_0 \)-\( B_0 \)-bimodule, for any commutative algebras \( A_0 \) and \( B_0 \) with local identities. In particular, if \( A \) is an algebra with local identities in a commutative subalgebra \( A_0 \subset A \), then \( A \) is an \( A_0 \)-\( A_0 \)-unitary \( A \)-\( A \)-bimodule. In this case we shall write \( A \otimes_{A_0} A \) for the tensor product of two copies of \( A \) with respect to the right action of \( A_0 \) on the first factor and the left action of \( A_0 \) on the second factor, while the notation \( A \otimes_{A_0} A \) will stand for the tensor product taken with respect to the left action of \( A_0 \) on both factors.

Suppose that \( A_0 \) is a commutative algebra with local identities. Recall that a left \( A_0 \)-coalgebra is a left \( A_0 \)-unitary module \( C \), together with \( A_0 \)-linear maps \( \Delta : C \to C \otimes_{A_0} C \) (comultiplication) and \( \epsilon : C \to A_0 \) (counit) such that \( (\epsilon \otimes \id) \circ \Delta = \id, (\id \otimes \epsilon) \circ \Delta = \id \) and \( (\Delta \otimes \id) \circ \Delta = (\id \circ \Delta) \circ \Delta \) (coassociativity). We do not assume here that our coalgebras are necessarily cocommutative, although our
examples will be such. A homomorphism \( \theta : C \to C' \) of left \( A_0 \)-coalgebras is a homomorphism of left \( A_0 \)-modules that respects the coalgebra structures, i.e. \( \epsilon = \epsilon' \circ \theta \) and \( (\theta \otimes \theta) \circ \Delta = \Delta' \circ \theta \).

An \( A_0 \)-bialgebroid is an algebra \( A \) such that \( A_0 \) is a commutative subalgebra of \( A \) in which \( A \) has local identities, together with a structure of a left \( A_0 \)-coalgebra on \( A \) such that

(i) \( \epsilon|_{A_0} = \text{id} \), \( \Delta|_{A_0} \) is the canonical embedding \( A_0 \subset A \otimes_{A_0}^h A \) and the two right actions of \( A_0 \) on \( A \otimes_{A_0}^h A \) coincide on \( \Delta(A) \),

(ii) \( \epsilon(ab) = \epsilon(ac(b)) \) and

(iii) \( \Delta(ab) = \Delta(a)\Delta(b) \)

for any \( a, b \in A \). Note that by (i), the product of \( A \) induces the componentwise product \( \Delta(A) \otimes (A \otimes_{A_0}^h A) \to A \otimes_{A_0}^h A \), which is used in (iii). The comultiplication in an \( A_0 \)-bialgebroid \( A \) is also a homomorphism of right \( A_0 \)-modules with respect to any of the right \( A_0 \)-actions on \( A \otimes_{A_0}^h A \), and it induces a homomorphism of left \( A_0 \)-modules \( \overline{\Delta} : A \otimes_{A_0}^h A \to A \otimes_{A_0}^h A \) by \( \overline{\Delta}(a \otimes b) = \Delta(a)(a_0 \otimes b) \), where \( a_0 \) is any element of \( A_0 \) with \( a_0b = b \). An \( A_0 \)-bialgebroid \( A \) is principal if \( \overline{\Delta} \) is an isomorphism. A homomorphism between \( A_0 \)-bialgebroids is a homomorphism of algebras which is also a homomorphism of left \( A_0 \)-coalgebras.

A Hopf \( A_0 \)-algebroid is an \( A_0 \)-bialgebroid \( A \), together with an \( F \)-linear involution \( S : A \to A \) (antipode) such that \( S|_{A_0} = \text{id} \), \( S(ab) = S(b)S(a) \) for any \( a, b \in A \), and \( \mu_A \circ (S \otimes \text{id}) \circ \Delta = \epsilon \circ S \), where \( \mu_A : A \otimes_{A_0}^h A \to A \) denotes the multiplication. (This definition is slightly stronger than the definition of a Hopf algebra over \( A_0 \) given in [22], while the notion of a principal Hopf algebroid is slightly weaker than that of an étale Hopf algebroid given in [20].)

Similar notions have been studied in [12] [11] [25] [26] and more recently in [11] [2] [3] [11].) A homomorphism between Hopf \( A_0 \)-algebroids is a homomorphism of \( A_0 \)-bialgebroids which intertwines the antipodes.

**Example 2.1.** (1) For any sheaf \( \pi : P \to M \) over a Hausdorff manifold \( M \), the space \( C^\infty_c(P) \) has a natural structure of a left \( C^\infty_c(M) \)-coalgebra [21] [23]. The algebra \( C^\infty_c(M) \) acts on the space \( C^\infty_c(P) \) by \( u_0f_p = \pi^*_f(\pi_0(u_0))f_p \), for any \( u_0 \in C^\infty_c(M) \) and \( f \in C^\infty_c(P) \). The comultiplication on \( C^\infty_c(P) \) is given by \( \Delta = \Omega^{-1}_{\pi,\pi} \circ \text{diag} \), where \( \text{diag} : P \to P \times_M P \) is the diagonal map and \( \Omega_{\pi,\pi} : C^\infty_c(P) \otimes C^\infty_c(M) \to C^\infty_c(P \times_M M) \) is the natural isomorphism given by \( \Omega_{\pi,\pi}(f \otimes f')_{(p,p')} = \langle p_1 \rangle_{(p,p')}f_p \langle p_2 \rangle_{(p,p')}f'_{p'} \) [21] [23]. The counit is \( \epsilon = \pi_+ \). Explicitly, if \( U \subset P \) is a \( \pi \)-elementary open subset, \( f_0 \in C^\infty_c(\pi(U)) \) and \( u_0 \in C^\infty_c(M) \), then

\[
u_0(f_0 \circ \pi|_U) = (u_0f_0) \circ \pi|_U,\]

\[
\Delta(f_0 \circ \pi|_U) = (f_0 \circ \pi|_U) \otimes (u_0' \circ \pi|_U) = (u_0' \circ \pi|_U) \otimes (f_0 \circ \pi|_U)
\]

and

\[
\epsilon(f_0 \circ \pi|_U) = f_0,
\]

where \( u_0' \in C^\infty_c(\pi(U)) \) is any function which satisfies \( u_0'f_0 = f_0 \).

In particular, the convolution algebra \( C^\infty_c(G) \) of an étale Lie groupoid \( G \) has a natural structure of a left \( C^\infty_c(G_0) \)-coalgebra, induced by the target map. The antipode \( S = \text{inv}_+ : C^\infty_c(G) \to C^\infty_c(G) \) turns \( C^\infty_c(G) \) into a principal Hopf \( C^\infty_c(G_0) \)-algebroid [22].

(2) Any commutative algebra \( A_0 \) with local identities is a principal Hopf \( A_0 \)-algebroid in the trivial way.

Suppose that \( A \) is an \( A_0 \)-bialgebroid and that \( B \) is a \( B_0 \)-bialgebroid. A preprincipal \( A-B \)-bimodule is a locally \( A_0-B_0 \)-unitary \( A-B \)-bimodule \( \mathcal{M} \) such that

(i) the two right \( B_0 \)-module structures on \( \mathcal{M} \otimes_{A_0} \mathcal{M} \) coincide on \( \Delta(\mathcal{M}) \),
(ii) $\epsilon(mb) = \epsilon(me(b))$ and $\epsilon(am) = \epsilon(ac(m))$,

(iii) $\Delta(am) = \Delta(a)\Delta(m)$ and $\Delta(mb) = \Delta(m)\Delta(b)$

for any $a \in A$, $b \in B$ and $m \in \mathcal{M}$. The bimodule structure of a preprincipal $A$-$B$-bimodule $\mathcal{M}$ induces the componentwise products $\Delta(A) \otimes (\mathcal{M} \otimes_{A_0} \mathcal{M}) \to \mathcal{M} \otimes_{A_0} \mathcal{M}$ and $\Delta(\mathcal{M}) \otimes (B \otimes_{B_0} B) \to \mathcal{M} \otimes_{A_0} \mathcal{M}$. The existence of these two partially defined products gives the meaning to the condition (iii) in the definition, which also implies that $\Delta$ is right $B_0$-linear. The comultiplication in $\mathcal{M}$ induces a homomorphism of $A_0$-$B$-bimodules $\Delta: \mathcal{M} \otimes_{B_0} B \to \mathcal{M} \otimes_{A_0} \mathcal{M}$ by $\Delta(m \otimes b) = \Delta(m)(b_0 \otimes b)$, where $b_0$ is any element of $B_0$ such that $b_0 b = b$. A principal $A$-$B$-bimodule is a preprincipal $A$-$B$-bimodule $\mathcal{M}$ such that $\epsilon$ is surjective and $\Delta$ is an isomorphism. A homomorphism of preprincipal $A$-$B$-bimodules is a homomorphism of $A$-$B$-bimodules which is also a homomorphism of left $A_0$-coalgebras.

Note that a preprincipal $A$-$B$-bimodule is in particular a preprincipal $A_0$-$B$-bimodule as well as a preprincipal $A$-$B_0$-bimodule. If $\mathcal{M}$ is a principal $A$-$B$-bimodule, then it is also a principal $A_0$-$B$-bimodule. Furthermore, any $A_0$-bialgebroid $A$ is also a preprincipal $A$-$A$-bimodule, which is principal if and only if $A$ is principal as an $A_0$-bialgebroid.

Let $\mathcal{M}$ be a preprincipal $A$-$B$-bimodule and let $\mathcal{N}$ be a principal $B$-$C$-bimodule, for a $C_0$-bialgebroid $C$. There is a natural structure of a preprincipal $A$-$C$-bimodule on the tensor product $\mathcal{M} \otimes_B \mathcal{N}$ given by

$$\Delta(m \otimes n) = \sum_{i,j} (m_i' \otimes n_j') \otimes (m_i'' \otimes n_j'')$$

and

$$\epsilon(m \otimes n) = \epsilon(me(n)),$$

for any $m \otimes n \in \mathcal{M} \otimes_B \mathcal{N}$ with $\Delta(m) = \sum_i m_i' \otimes m_i''$ and $\Delta(n) = \sum_j n_j' \otimes n_j''$ [20].

If $B$, $\mathcal{M}$ and $\mathcal{N}$ are all principal, then $\mathcal{M} \otimes_B \mathcal{N}$ is principal as well; this was proved in [20] in the cocommutative case, however the cocommutativity assumption was not used in the proof.

We shall denote by $\text{BiALGD}$ the Morita category of bialgebroids: objects of $\text{BiALGD}$ are pairs $(A, A_0)$, where $A$ is an $A_0$-bialgebroid, a morphism from $(A, A_0)$ to $(B, B_0)$ in the category $\text{BiALGD}$ is an isomorphism class of preprincipal $A$-$B$-bimodules, while the composition is induced by the tensor product. The principal bialgebroids and the principal bimodules form a subcategory $\text{PrBiALGD}$ of $\text{BiALGD}$. Similarly, we have the Morita category $\text{HoALGD}$ of Hopf algebroids and isomorphism classes of preprincipal bimodules as morphisms between them, as well as its subcategory $\text{PrHoALGD}$ of principal Hopf algebroids and isomorphism classes of principal bimodules.

A smooth bialgebroid is a pair $(A, M)$, where $M$ is a smooth Hausdorff manifold and $A$ is a $C^\infty(M)$-bialgebroid. We have the Morita category $\text{BiALGD}^\infty$ of smooth bialgebroids, in which a morphism from $(A, M)$ to $(B, N)$ is an isomorphism class of preprincipal $A$-$B$-bimodules, and the composition is induced by the tensor product. The principal smooth bialgebroids and the principal bimodules form a subcategory $\text{PrBiALGD}^\infty$ of $\text{BiALGD}^\infty$. The natural functor $\text{BiALGD}^\infty \to \text{BiALGD}$, which maps $(A, M)$ to $(A, C^\infty_c(M))$, is therefore fully-faithful, and the same is true for its restriction $\text{PrBiALGD}^\infty \to \text{PrBiALGD}$. Note that if $A$ is an $A_0$-bialgebroid and if $A_0 \cong A_0^\infty \cong C^\infty_c(M)$ for a Hausdorff manifold $M$, then $M$ is in fact determined uniquely up to a canonical diffeomorphism: any isomorphism $C^\infty_c(M) \cong C^\infty_c(N)$ is induced by a unique diffeomorphism between $M$ and $N$. Moreover, the set $M$ can be in this case identified with the set $A_0$ of surjective multiplicative functionals on $A_0$, so there is a natural structure of a smooth manifold on $A_0$ such that $A_0$ and $C^\infty_c(A_0)$ are canonically isomorphic.
Analogously, we have the Morita category $\text{HoALGD}^\infty$ of smooth Hopf algebroids and isomorphism classes of preprincipal bimodules as morphisms between them, and also its subcategory $\text{PrHoALGD}^\infty$ of principal smooth Hopf algebroids and isomorphism classes of principal bimodules. The natural functors $\text{HoALGD}^\infty \to \text{HoALGD}$ and $\text{PrHoALGD}^\infty \to \text{PrHoALGD}$ are fully-faithful.

**Example 2.2.** (1) Let $G$ and $H$ be étale Lie groupoids and let $P$ be a principal $H$-bundle over $G$. The $C_c^\infty(G)$-$C_c^\infty(H)$-bimodule $C_c^\infty(P)$ (Subsection 2.2) carries a natural structure of a principal $C_c^\infty(G)$-$C_c^\infty(H)$-bimodule \([20]\). Indeed, the coalgebra structure of $P$ is given by the sheaf $\pi : P \to G_0$ (Example 2.1 (1)), while the principalness follows because $\Delta$ is induced by the diffeomorphism $(pr_1, \eta) : P \times_{H_0} H \to P \times_P P$. Indeed, we have $C_c^\infty(P) \otimes_{C_c^\infty(H_0)} C_c^\infty(P) \cong C_c^\infty(P \times_{H_0} P)$ (Example 2.1 (1)). Furthermore, with the methods of the proof of \([19\), Theorem 2.4], one can easily show that we also have an isomorphism $C_c^\infty(P) \otimes_{C_c^\infty(H_0)} C_c^\infty(H) \cong C_c^\infty(P \times_{H_0} H)$ induced by the map $\rho_{P,H}$ given in Subsection 2.2.

The isomorphism $\Omega$, described in Subsection 2.2, respects the coalgebra structure and is therefore an isomorphism of principal bimodules. To sum up, we have a functor $C_c^\infty : \text{EtGPD} \to \text{PrHoALGD}^\infty$, which maps an étale Lie groupoid $G$ to the associated principal smooth Hopf algebroid $(C_c^\infty(G), G_0)$ and an isomorphism class of a principal $H$-bundle $P$ over $G$ to the isomorphism class of the principal $C_c^\infty(G)$-$C_c^\infty(H)$-bimodule $C_c^\infty(P)$.

(2) Let $x$ be a point of a Hausdorff manifold $M$. We have the quotient epimorphism of commutative algebras $C_c^\infty(M) \to (C_c^\infty(M)_x)^{\times}$, mapping a function $f \in C_c^\infty(M)_x$ to its germ at $x$. With respect to this epimorphism, the space $(C_c^\infty)_x$ has a natural structure of a principal $(C_c^\infty)_x$-$C_c^\infty$-bialgebroid. Thus, if $A$ is a $C_c^\infty(M)$-bialgebroid, if $B$ is a $B_0$-bialgebroid and if $M$ a (pre)principal $A$-$B$-bialgebroid, then $(C_c^\infty)_x \otimes_{C_c^\infty(M)} M$ is a (pre)principal $(C_c^\infty)_x$-$B$-bialgebroid.

3. The Morita category of locally grouplike Hopf algebroids

3.1. Locally grouplike Hopf algebroids. Suppose that $C = (C, \Delta, \varepsilon)$ is a left $C_c^\infty(M)$-coalgebra, for a Hausdorff manifold $M$. Choose a point $x \in M$ and write $C_c^\infty(M)_x = (C_c^\infty)_x$. There is the associated local left $C_c^\infty(M)_x$-coalgebra $(C_x, \Delta_x, \varepsilon_x)$ at $x$ \([21\), \([23\), given as the quotient $C_x = C/\ker(C)$ with respect to the left $C_c^\infty(M)$-submodule $N_x(C) = \{c \in C \mid f \varepsilon_c = 0 \text{ for some } f \in C_c^\infty(M) \text{ with } (f)_x = 1\}$ of $C$, and with the induced coalgebra structure. The equivalence class of an element $c \in C$ in $C_x$ will be denoted by $c_x$. Note that we have a natural isomorphism of left $C_c^\infty(M)_x$-modules $C_x \cong C_c^\infty(M)_x \otimes_{C_c^\infty(M)} C$.

An element $c \in C$ is weakly grouplike \([21\), \([23\) if there exists $c' \in C$ such that $\Delta(c) = c \otimes c'$. We denote by $G_w(C)$ the set of weakly grouplike elements of $C$. We may also consider the set $G(C_x) = \{c \in C_x \mid \Delta_x(c) = c \otimes c, \varepsilon_x(c) = 1\}$ of grouplike elements of the coalgebra $C_x$. The connection between the weakly grouplike elements of $C$ and the grouplike elements of $C_x$ is as follows: every $c \in G_w(C)$, which is normalised at $x$ (i.e. $c(c)_x = 1$), projects to a grouplike element $c_x \in G(C_x)$. Conversely, any $c \in G(C_x)$ can be written as $c = c_x$ for some $c \in G_w(C)$ normalised at $x$ \([21\), \([23\).

A weakly grouplike element $a$ of a Hopf $C_c^\infty(M)$-algebroid $A$ is $S$-invariant if there exists $a' \in A$ such that $\Delta(a) = a \otimes a'$ and $\Delta(S(a)) = S(a') \otimes S(a)$. Write $G^w_0(A)$ for the set of $S$-invariant weakly grouplike elements of $A$. The set of arrows of $A$ with target $y \in M$ is given by $G^3(A_y) = \{a_y \mid a \in G_w^0(A), \varepsilon(a)_y = 1\} \subset G(A_y)$. 


For example, the weakly grouplike elements of the left $C^\infty_c(M)$-coalgebra $C^\infty_c(P)$ of a sheaf $\pi : P \to M$ are precisely the functions on $P$ with compact support in a $\pi$-elementary open subset of $P$ [21, 23]. Similarly, the $S$-invariant weakly grouplike elements of the Hopf algebroid $C^\infty_c(G)$, associated to an étale Lie groupoid $G$, are the functions on $G$ with compact support in a bisection of $G$ [22].

Definition 3.1. A locally grouplike Hopf algebroid is a smooth Hopf algebroid $(A, M)$ such that the $C^\infty_c(M)_y$-module $A_y$ is freely generated by the set of arrows $G^S(A_y)$ with target $y$, for every $y \in M$.

The smooth Hopf algebroid $(C^\infty_c(G), G_0)$ of an étale Lie groupoid $G$ is an example of a locally grouplike Hopf algebroid. In fact, the converse is true as well: For any locally grouplike Hopf algebroid $(A, M)$ there exists a spectral étale Lie groupoid $G_{sp}(A)$ over $M$ such that $C^\infty_c(G_{sp}(A)) \cong A$ [22]. Indeed, a smooth Hopf algebroid $(A, M)$ is locally grouplike if and only if the $S$-invariant weakly grouplike elements normally generate $A$ and are normally linearly independent. In particular, any locally grouplike Hopf algebroid $(A, M)$ is cocommutative and principal; furthermore, it satisfies $(S \otimes \text{id}) \circ \Delta \circ (S \otimes \text{id}) = \text{id}$ and $G(A_y) = G^S(A_y)$, for any $y \in M$.

Definition 3.2. Let $(A, M)$ and $(B, N)$ be locally grouplike Hopf algebroids. A principal $A$-$B$-bimodule $M$ is locally grouplike if the set of grouplike elements $G(M_x)$ freely generates the $C^\infty_c(M)_x$-module $M_x$, for every $x \in M$.

For locally grouplike Hopf algebroids $(A, M)$ and $(B, N)$, a principal $A$-$B$-bimodule $M$ is locally grouplike if and only if the weakly grouplike elements of $M$ normally generate $M$ and are normally linearly independent [21, 23]. This implies that locally grouplike principal bimodules are cocommutative.

The principal bimodules associated to principal bundles are locally grouplike [21, 23], and are in fact the only examples of locally grouplike principal bimodules, up to an isomorphism. We will show (see Theorem 5.1) that locally grouplike Hopf algebroids and locally grouplike principal bimodules form a subcategory $\text{LgHoALGD}$ of $\text{PrHoALGD}^{\infty}$. Moreover, we will prove that the category $\text{LgHoALGD}$ is equivalent to the Morita category $\text{EtGPD}$ of étale Lie groupoids; an explicit equivalence is given by the functor $C^\infty_c : \text{EtGPD} \to \text{LgHoALGD}$.

3.2. The moment map. Let $M$ and $N$ be Hausdorff manifolds, let $A$ be a $C^\infty_c(M)$-bialgebroid and let $B$ be a $C^\infty_c(N)$-bialgebroid. Suppose that $\mathcal{M}$ is a preprincipal $A$-$B$-bimodule. The bimodule $\mathcal{M}$ is in particular a left $C^\infty_c(M)$-coalgebra, hence there is the associated spectral sheaf

$$\pi = \pi_{sp} : \mathcal{E}_{sp}(\mathcal{M}) \to M$$

(see [21, 23]). Its stalk $\mathcal{E}_{sp}(\mathcal{M})_x$ over a point $x$ is by definition the set $G(\mathcal{M}_x)$, while the topology on $\mathcal{E}_{sp}(\mathcal{M})$ is given by the basis of $\pi$-elementary open subsets $m_W = \{ m_x \in G(\mathcal{M}_x) | x \in W \}$, where $W$ is any open subset of $M$ and $m \in G_w(\mathcal{M})$ any weakly grouplike element normalised on $W$ (i.e. $\epsilon(m)|_W = 1$).

Suppose that $m \in G_w(\mathcal{M})$ is normalised on an open subset $W$ of $M$. Define a linear map $T_{W,m} : C^\infty_c(N) \to C^\infty (W)$ by

$$T_{W,m}(v_0) = \epsilon(mv_0)|_{\bar{W}}.$$ 

This map is a homomorphism of algebras. To see this, choose $m' \in M$ with $\Delta(m) = m \otimes m'$. Note that $m = \epsilon(m)m'$. Since $\Delta$ is a homomorphism of right $C^\infty_c(N)$-modules and the two right $C^\infty_c(N)$-actions on $\Delta(\mathcal{M})$ coincide, we have $\Delta(mv_0) = mv_0 \otimes m'$ for any $v_0 \in C^\infty_c(N)$, thus in particular $mv_0 \in G_w(\mathcal{M})$ and
\( m v_0 = \epsilon(m' v_0)m. \) Now the statement follows from the equalities

\[
T_{W,m}(v_0 v'_0) = \epsilon(m v_0 v'_0)|_W
= \epsilon(\epsilon(m' v_0)m v'_0)|_W
= \epsilon(m' v_0)|_W \epsilon(m v'_0)|_W
= \epsilon(\epsilon(m) m' v_0)|_W \epsilon(m v'_0)|_W
= \epsilon(\epsilon(m) m m' v_0)|_W
= \epsilon(m v_0)|_W \epsilon(m v'_0)|_W

(\epsilon is \( C^\infty(M) \)-linear)
\]

\( m = \epsilon(m) m' \)

\[
= T_{W,m}(v_0) T_{W,m}(v'_0),
\]

for any \( v_0, v'_0 \in C^\infty(N). \)

Now pick \( x \in W \) and define a map \( T_{W,m} : C^\infty_c(N) \to F \) by

\[
T_{W,m}(v_0) = T_{W,m}(v_0)(x).
\]

This map is a nontrivial (because \( M \) is locally unitary) multiplicative linear functional on the algebra \( C^\infty_c(N) \), and therefore given by the evaluation at a unique point \( z = \phi_{W,m}(x) \in N \). Since this is true for any \( x \in W \), we have the map

\[
\phi_{W,m} : W \to N,
\]

uniquely determined by the property that

\[
(4) \quad \epsilon(m v_0)(x) = v_0(\phi_{W,m}(x))
\]

for any \( v_0 \in C^\infty_c(N). \)

Recall that \( M_x = C^\infty_c(M)_x \otimes C^\infty_c(M)_x \) \( B \)-bimodule, for any \( x \in M \). Combining the equation (4) with the equalities \( m v_0 = \epsilon(m' v_0)m \) and \( \epsilon(m' v_0)|_W = \epsilon(m v'_0)|_W \) (the last follows from \( m = \epsilon(m) m' \)), we get the equality

\[
(5) \quad (v_0 \circ \phi_{W,m}) x m_x = m_x v_0,
\]

which holds for any \( x \in W \) and any \( v_0 \in C^\infty_c(N). \)

Choose \( x_0 \in W \) and real functions \( \psi_1, \ldots, \psi_k \in C^\infty_c(N) \) such that \( (\psi_1, \ldots, \psi_k)|_U : U \to \mathbb{R}^k \) is a local chart on an open neighbourhood \( U \) of the point \( z_0 = \phi_{W,m}(x_0) \in N \). For each \( i = 1, \ldots, k \) and any \( x \in W \) we have \( (\psi_i \circ \phi_{W,m})(x) = \epsilon(m v_i)(x) \), which shows that \( \psi_i \circ \phi_{W,m} \) is a smooth function on \( W \). From this we conclude that \( \phi_{W,m} \) is smooth on a neighbourhood of any point \( x_0 \in W \). Using the diffeomorphism \( \pi|_{m_W} : m_W \to W \) we get a smooth map

\[
\phi_{n_{mW,m}} = \phi_{W,m} \circ \pi|_{m_W} : m_W \to N.
\]

Suppose that \( n \) is another element of \( G_{w}(M) \), normalised on an open subset \( V \) of \( M \), such that \( m_W \cap n_V \neq \emptyset \). Choose any point \( m_x = n_x \in m_W \cap n_V \), for any \( x \in V \cap W \). By definition, this means that there exists \( v_0 \in C^\infty_c(M) \) such that \( (v_0)|_x = 1 \) and \( u_0 m = u_0 n \). We can find an open neighbourhood \( U \subset V \cap W \) of \( x \) such that \( u_0|_U = 1 \). We have \( m_{x'} = (u_0 m)_{x'} = (u_0 n)_{x'} = n_{x'} \) for all \( x' \in U \) and consequently \( m_{U} = n_{U} \). We will show that the maps \( \phi_{m_{W,m}} \) and \( \phi_{n_{m,n}} \) agree on the set \( m_{U} = n_{U} \). Indeed, for any \( x' \in U \) and any \( v_0 \in C^\infty_c(N) \) we have

\[
\epsilon(m v_0)|_{x'} = \epsilon_{x'}(m_{x'} v_0) = \epsilon_{x'}(n_{x'} v_0) = \epsilon(n v_0)|_{x'},
\]

which implies \( v_0(\phi_{W,m}(x')) = v_0(\phi_{n,m}(x')) \). Since \( v_0 \in C^\infty_c(N) \) was arbitrary and since the algebra \( C^\infty_c(N) \) separates the points of \( N \), we conclude that the maps \( \phi_{W,m} \) and \( \phi_{n,m} \) agree on \( U \). The same is then true for the maps \( \phi_{m_{W,m}} \) and \( \phi_{n_{m,n}} \) on the set \( m_{U} = n_{U} \). Gluing together these locally defined maps we get a globally defined smooth map

\[
\phi : E_{sp}(M) \to N,
\]

the moment map of the preprincipal \( A-B \)-bimodule \( M \).
For any \( p \in G(M_x) \) we define the effect germ \( \phi^e_p = \phi_{p \circ (\pi_p)^{-1}} \) of \( p \), which is the germ at \( x \) of a map \( (M, x) \to (N, \phi(p)) \). The equation (56) yields
\[
((v_0)_{\phi(p)} \circ \phi^e_p) p = pv_0
\]
for any \( v_0 \in C^\infty_c(N) \). In particular, this equation shows that the left \( C^\infty_c(M)_x \)-submodule
\[
\mathcal{M}_x^B = C^\infty_c(M)_x p \subset M_x
\]
is also a right \( C^\infty_c(N) \)-submodule of \( M_x \). Moreover, the right \( C^\infty_c(N) \)-action on \( \mathcal{M}_x^B \) induces a right \( C^\infty_c(N)_{\phi(p)} \)-action on \( \mathcal{M}_x^B \), and so \( \mathcal{M}_x^B \) is in fact a \( C^\infty_c(M)_x \)-\( C^\infty_c(N)_{\phi(p)} \)-bimodule.

### 3.3. Tensor product of locally grouplike principal bimodules.

Let \((A, M), (B, N)\) and \((C, N')\) be locally grouplike Hopf algebroids, let \( M \) be a locally grouplike principal \( A \)-\( B \)-bimodule and let \( N \) be a locally grouplike principal \( B \)-\( C \)-bimodule. Write \( A_0 = C^\infty_c(M) \) and \( B_0 = C^\infty_c(N) \). We know that \( M \otimes B_0 N \) is a principal \( A \)-\( C \)-bimodule. Furthermore, since \( M \) is also a preprincipal \( A \)-\( B_0 \)-bimodule and \( N \) is a principal \( B_0 \)-\( C \)-bimodule, it follows that \( M \otimes B_0 N \) is a preprincipal \( A \)-\( C \)-bimodule.

Take any \( x \in M \). Since \((A_0)_x\) is a principal Hopf \((A_0)_x\)-algebroid and also a principal \((A_0)_x \)-\( A_0 \)-\( B_0 \)-bimodule (Example (2.2) (2)), we have an isomorphism of preprincipal \((A_0)_x \)-\( B_0 \)-\( C \)-bimodules
\[
(M \otimes B_0 N)_x \cong (A_0)_x \otimes_{A_0} M \otimes B_0 N \cong M_x \otimes_{B_0} N
\]
and an isomorphism of principal \((A_0)_x \)-\( C \)-bimodules
\[
(M \otimes B N)_x \cong (A_0)_x \otimes_{A_0} M \otimes_{B_0} N \cong M_x \otimes B N
\].

Furthermore, the local coalgebra \((M \otimes A_0)_x\) of the product coalgebra \( M \otimes A_0 M \) simply is the product coalgebra \( M \otimes (A_0)_x \), \( M_x = M \otimes (A_0)_x M_x \) over \((A_0)_x\) (see also Proposition 2.1) and 24).

**Lemma 3.3.** The map \( \Delta : M \otimes B_0 N \to M \otimes A_0 M \) is an isomorphism of left \( A_0 \)-coalgebras, and induces an isomorphism \( \overline{\Delta} : M_x \otimes B_0 N \to M_x \otimes A_0 M_x \) of left \((A_0)_x \)-coalgebras, for every \( x \in M \).

**Proof.** Since the left \( A_0 \)-module \( M \) is generated by \( G(M) \), a straightforward calculation shows that \( \overline{\Delta} \) is an isomorphism of coalgebras over \( A_0 \). Thus we have the induced isomorphism of \((A_0)_x \)-coalgebras \( (M \otimes B_0) \otimes (A_0)_x \sim (M \otimes A_0 M)_x \), which we combine with the isomorphisms \((M \otimes A_0 M)_x \cong M_x \otimes A_0 M_x \) and \((M \otimes B_0)_x \cong (A_0)_x \otimes A_0 M \otimes B_0 N \) to obtain \( \overline{\Delta} \). Alternatively, one can also describe \( \overline{\Delta} \) as the isomorphism \( \overline{\Delta} \) associated to the principal \((A_0)_x \)-\( B \)-bimodule \( M_x \cong (A_0)_x \otimes A_0 M \).

We will next describe the grouplike elements of the coalgebra \( M_x \otimes B_0 N \). Since \( M \) is a locally grouplike principal \( A \)-\( B \)-bimodule, the set \( G(M_x) \) freely generates the left \((A_0)_x \)-module \( M_x \), for every \( x \in M \). Choose any \( p \in G(M_x) \). Observe that we have the natural isomorphisms of left \((A_0)_x \)-modules
\[
M_x^p \otimes B_0 N \cong M_x^p \otimes (B_0)_{\phi(p)} \otimes B_0 N \cong M_x^p \otimes (B_0)_{\phi(p)} N_{\phi(p)} \cdot N_{\phi(p)}
\]
Since \( N \) is locally grouplike as well, it follows that \( N_{\phi(p)} \) is a free left \((B_0)_{\phi(p)} \)-module with the basis \( G(N_{\phi(p)}) \). For any \( q \in G(N_{\phi(p)}) \) we have the left \((A_0)_x \)-module \( M_x^p \otimes (B_0)_{\phi(p)} N_{\phi(p)} = (A_0)_x(p \otimes q) \). Therefore the left \((A_0)_x \)-modules
\[
M_x^p \otimes B_0 N \cong M_x^p \otimes (B_0)_{\phi(p)} N_{\phi(p)} \cong M_x^p \otimes (B_0)_{\phi(p)} \bigoplus_{q \in G(N_{\phi(p)})} N_{\phi(p)}^q
\]
\[
\cong \bigoplus_{q \in G(N_{\phi(p)})} (M_x^p \otimes (B_0)_{\phi(p)} N_{\phi(p)}^q)
\]
are free. The explicit base of the free \((A_0)_x\)-module \(M^p \otimes B_0 N\) is the set \(L_p = \{[p, q] | q \in G(N_{\phi(p)})\}\), where \([p, q]\) denotes the element of \(M^p \otimes B_0 N\) which corresponds to the element \(p \otimes q \in M^p \otimes (B_0)_x N_{\phi(p)}\) by the above isomorphism.

**Lemma 3.4.** (i) The coalgebra \(M_x \otimes B_0 N\) over \((A_0)_x\) is freely generated by its grouplike elements \(G(M_x \otimes B_0 N) = \{[p, q] | p \in G(M_x), q \in G(N_{\phi(p)})\}\).

(ii) The coalgebra \(M_x \otimes B_0 N\) is a direct sum \(\bigoplus_{p \in G(M_x)} M^p \otimes B_0 N\) of sub-coalgebras over \((A_0)_x\), and each subcoalgebra \(M^p \otimes B_0 N\) is freely generated by its grouplike elements \(G(M^p \otimes B_0 N) = \{[p, q] | q \in G(N_{\phi(p)})\}\).

(iii) The coalgebra \(M_x \otimes A_0 \cdot M_x\) is a free left \((A_0)_x\)-module, generated by its grouplike elements \(G(M_x \otimes A_0 \cdot M_x) = \{p \otimes p' | p, p' \in G(M_x)\} \cong G(M_x) \times G(M_x)\).

**Proof.** (i) It is clear that \(M_x \otimes B_0 N\) is a free \((A_0)_x\)-module with the basis \(L = \cup_p L_p\). We need to show that \(G(M_x \otimes B_0 N) = L\). Straight from the definition of the structure maps of the coalgebra \(M_x \otimes B_0 N\) it follows \(L \subset G(M_x \otimes B_0 N)\). To prove the converse inclusion, choose any \(u \in G(M_x \otimes B_0 N)\) and write it in the form \(u = \sum_{p, q} a_{pq} [p, q]\) (where \(p \in G(M_x)\) and \(q \in G(N_{\phi(p)})\)) for uniquely determined \(a_{pq} \in (A_0)_x\). Then we have

\[
 u \otimes u = \sum_{p, q, p', q'} a_{pq} a_{p'q'} [p, q] \otimes [p', q']
\]

and

\[
 \Delta(u) = \sum_{p, q} a_{pq} [p, q] \otimes [p, q].
\]

Since the element \(u\) is grouplike, we have \(\Delta(u) = u \otimes u\) and \(\epsilon(u) = 1\). Therefore, by checking the components of \(\Delta(u)\) and \(u \otimes u\), we see that

\[
 a_{pq}^2 = a_{pq} \quad \text{for all } p, q,
\]

(6)

\[
 a_{pq} a_{p'q'} = 0 \quad \text{if } p \neq p' \text{ or } q \neq q',
\]

and

(7)

\[
 \sum_{p, q} a_{pq} = 1.
\]

The equation (7) implies that there exist \(p_0\) and \(q_0\) such that \(a_{pq_0}\) is invertible in \((A_0)_x\). Combining this with the equation (6) we see that \(a_{pq} = 0\) if \(p \neq p_0\) or \(q \neq q_0\), and that \(a_{pq_0}=1\). We conclude that \(u = [p_0, q_0] \in L\). Parts (ii) and (iii) follow analogously.

**Proposition 3.5.** The locally grouplike Hopf algebroids and the locally grouplike principal bimodules form a subcategory \(\text{LgHoALGD}\) of the category \(\text{PrHoALGD}^{\infty}\).

**Proof.** Let \((A, M), (B, N)\) and \((C, N')\) be locally grouplike Hopf algebroids, let \(\mathcal{M}\) be a locally grouplike principal \(A-B\)-bimodule and let \(\mathcal{N}\) be a locally grouplike principal \(B-C\)-bimodule. We have to show that the tensor product \(\mathcal{M} \otimes B \mathcal{N}\) is locally grouplike. If we restrict the isomorphism \(\Delta_x\) (Lemma 3.3) to the submodule \(M_x \otimes B_x B\), for some \(x \in M\) and \(p \in G(M_x)\), we obtain an isomorphism \(M_x^p \otimes B_x B \cong M_x^p \otimes A_x M_x \cong M_x \) of coalgebras over \((A_0)_x\). Indeed, this follows from Lemma 3.4 by considering the explicit basis of both \((A_0)_x\)-modules. It follows that we have the isomorphisms of coalgebras over \((A_0)_x\)

\[
 (\mathcal{M} \otimes B \mathcal{N})_x \cong M_x \otimes B \mathcal{N} \cong (M_x^p \otimes B_x B) \otimes B \mathcal{N} \cong M_x^p \otimes B_x \mathcal{N}.
\]

Since \(M_x^p \otimes B_x \mathcal{N}\) is freely generated by its grouplike elements by Lemma 3.4 so is \((\mathcal{M} \otimes B \mathcal{N})_x\).
4. The principal bundle associated to a locally grouplike principal bimodule

Let \((A,M)\) and \((B,N)\) be locally grouplike Hopf algebroids and let \(\mathcal{M}\) be a locally grouplike principal \(A\)-\(B\)-bimodule. Denote by \(G = G_{sp}(A)\) and \(H = G_{sp}(B)\) the spectral étale Lie groupoids associated to \((A,M)\) respectively \((B,N)\) \([22]\). In particular, we have \(G_0 = M\) and \(H_0 = N\). Recall that there are natural isomorphisms of Hopf algebroids \(C_c^\infty(G) \cong A\) and \(C_c^\infty(H) \cong B\), so we may regard \(\mathcal{M}\) as a principal \(C_c^\infty(G)\)\(-\)\(C_c^\infty(H)\)-bimodule. In this section we will construct an associated principal \(H\)-bundle \(P = \mathcal{M}_x\) over \(G\) such that the principal \(C_c^\infty(G)\)\(-\)\(C_c^\infty(H)\)-bimodules \(C_c^\infty(P)\) and \(\mathcal{M}\) are isomorphic.

For the manifold \(P\) we take the total space of the spectral sheaf \(\pi_{sp} : \mathcal{E}_{sp}(\mathcal{M}) \to G_0\) associated to the \(C_c^\infty(G_0)\)-coalgebra \(\mathcal{M}\) (see Subsection 3.2),

\[P = \mathcal{E}_{sp}(\mathcal{M}).\]

Put \(\pi = \pi_{sp}\). We also have the associated moment map \(\phi : P \to H_0\) constructed in Subsection 3.2.

The locally grouplike Hopf algebroid \((A,G_0)\) is in particular a coalgebra over \(C_c^\infty(G_0)\). Thus we have the associated spectral sheaf \(\pi_{sp}(A) : \mathcal{E}_{sp}(A) \to G_0\), which is in fact equal to the target map of the spectral étale Lie groupoid \(t : G_{sp}(A) \to G_0\) because \((A,G_0)\) is locally grouplike. In particular, we have \(\mathcal{E}_{sp}(A) = G_{sp}(A) = G\), while \(G^G(A_y) = G(A_y) = y^{-1}(y)\) for any \(y \in G_0\). Recall that an arrow \(g \in G(x,y)\) can be represented as \(g = a_y\) by an element \(a \in G_w^G(A)\) normalised on an open neighbourhood \(W_a \subset G_0\) of \(y\). Such a pair \((W_a,a)\) induces a diffeomorphism \(\tau_{W_a,a} : V_{W_a,a} \to W_a\), defined on an open subset \(V_{W_a,a} \subset G_0\), which is determined by the property that

\[(u_0a = a(u_0 \circ \tau_{W_a,a})\text{ for any } u_0 \in C_c^\infty(W_a)\text{ [22]. We have } x = s(a_y) = \tau_{W_a,a}^{-1}(y).\text{ The effect germ } \tau_x = (\tau_{W_a,a})_x \text{ of } g, \text{ which is a germ at } x \text{ of a diffeomorphism } (G_0,x) \to (G_0,y), \text{ depends only on } g \text{ and not on the choice of } a \text{ and } W_a.\text{ The equation (5) may be rewritten as}

\[((u_0)_x \circ \tau_{y}^{-1})g = gu_0^t,\text{ for any } u_0 \in C_c^\infty(G_0)\text{. For another arrow } g' \in G(y,y'), \text{ represented as } g' = a_y' \text{ by } a' \in G_w^G(A) \text{ normalised on an open neighbourhood } W_{a'} \subset G_0 \text{ of } y', \text{ the product of } g' \text{ and } g \text{ is given by } g'g = a_y'a_y = (a'a)y'. \text{ Any function } a_0 \in C_c^\infty(G_0) \text{ with } (a_0)_x = 1 \text{ represents the identity arrow } 1_x = (a_0)_x \text{ at } x, \text{ while } a_y^{-1} = (S(a))_x.\]

4.1. Construction of the actions of \(G\) and \(H\) on \(P\). Let \(g \in G(x,y), p \in P\) and \(h \in H(z',z)\) be such that \(\pi(p) = x\) and \(\phi(p) = z\). Choose \(a \in G_w^G(A)\), \(m \in G_w(M)\) and \(b \in G_w^B(B)\) such that \(a\) is normalised on an open neighbourhood \(W_a \subset G_0\) of \(y\), \(m\) is normalised on an open neighbourhood \(W_m \subset G_0\) of \(x\) and \(b\) is normalised on an open neighbourhood \(W_b \subset H_0\) of \(z\) with \(g = a_y, p = m_x, h = b_z\).

**Lemma 4.1.** The elements \(am\) and \(mb\) of \(\mathcal{M}\) are weakly grouplike and normalised on \(W_{am} = \tau_{W_a,a}(V_{W_a,a} \cap W_m)\) respectively \(W_{mb} = \tau(m_{W_m} \cap \phi^{-1}(W_b))\).

**Proof.** From \(\Delta(a) = a \otimes a'\), \(\Delta(b) = b \otimes b'\) and \(\Delta(m) = m \otimes m'\) we get \(\Delta(am) = \Delta(a) \Delta(m) = (a \otimes a') (m \otimes m') = am \otimes a'm'\) and \(\Delta(mb) = (m \otimes m')(b \otimes b') = mb \otimes m'b'\). Next, for any \(y' \in W_{am} \subset W_a\) and any function \(u_0 \in C_c^\infty(W_{am})\) with
which shows that

Similarly, the right $B$-action on $M$ is a group-like element. A direct computation shows that this image is exactly $g, p$, so $W_{am}$ is normalised on $W_{mb}$. Finally, choose any $x' \in W_{mb} \subset W_m$. Then $\phi(m_{x'}) \in W_b$ and therefore $\epsilon(b)(\phi(m_{x'})) = 1$. The equation \eqref{eq:epsilon} yields

so $mb$ is normalised on $W_{mb}$. \hfill \square

Therefore we may define

and

We have to show that this two definitions are independent of the choice of $a, m$ and $b$. To this end, observe that the left $A$-action on $M$, as a map $A \otimes_{A_0} M \rightarrow M$, is a homomorphism of left $A_0$-coalgebras, and induces a homomorphism of left $(A_0)_y$-coalgebras

By Lemma \ref{lem:mu} we know that $[g, p] \in G(A_y \otimes_{A_0} M)$, so its image in $M_y$ is a group-like element. A direct computation shows that this image is exactly $\mu(g, p)$, which shows that $\mu$ is well defined as a map

Similarly, the right $B$-action $M \otimes_{B_0} B \rightarrow M$ on $M$ is a homomorphism of left $A_0$-coalgebras, and gives a homomorphism of left $(A_0)_x$-coalgebras

Again by Lemma \ref{lem:mu} we know that $[p, h] \in G(M_x \otimes_{B_0} B)$, so its image in $M_x$ is a group-like element, equal to $\eta(p, h)$. This shows that $\eta$ is well defined as a map

Proposition 4.2. The map $\mu$ is a left action of $G$ on $P$ along $\pi$.

Proof. Suppose that $m \in G_u(M)$ is normalised on an open neighbourhood $W_m \subset G_0$ of a point $x \in M$. Let $a, a' \in G^\circ_0(A)$ be normalised on open subsets $W_a$ respectively $W_{a'}$ of $G_0$, and let $y \in W_a$ and $y' \in W_{a'}$ be such that $a_y$ is an arrow from $x$ to $y$ and $a'_{y'}$ is an arrow from $y$ to $y'$.

Straight from the definition of $\mu$ it follows $\pi(a_y \cdot m_x) = y$, thus $\mu$ acts along the map $\pi$. Furthermore, we have

The unit arrow $1_x \in G$ can be represented by a smooth function $u_0 \in C_c^\infty(G_0)$ satisfying $(u_0)_x = 1$, thus

To prove that $\mu$ is smooth, first observe that the map $w: G \times_{G_0} P \rightarrow G_0$, given by $w = s \circ \text{pr}_1 = \pi \circ \text{pr}_2$, is a local diffeomorphism. The neighbourhood $W_1 = \{(u_0)_{x'} \in G_0 \mid \text{pr}_2(u_0)_{x'} = 1\}$ is open.
Proof. Let \( b, b' \in G_w^0(B) \) be normalised on open subsets \( W_b \) respectively \( W_{b'} \) of \( H_0 \), and suppose that \( z \in W_b \) and \( z' \in W_{b'} \) are such that \( b_z \) is an arrow from \( z' \) to \( z \) and \( b'_{z'} \) is an arrow from \( z'' \) to \( z' \). Furthermore, assume that \( m \in G_w(M) \) is normalised on an open neighbourhood \( W_m \) of a point \( x \in G_0 \) such that \( \phi(m_x) = z \). Choose \( m' \in M \) with \( \Delta(m) = m \otimes m' \). Since \( \epsilon(m'v'_0)|_W = \epsilon(m'v'_0)|_{W_0} \) and \( mv'_0 = \epsilon(m'v'_0)m \) for any \( v'_0 \in C^\infty(H_0) \), it follows that

\[
v_0(\phi(m_x \cdot b_z)) = \epsilon(mbv_0)(x) = \epsilon(v_0 \circ \tau_{W_{b_z}, b})(x) = \epsilon(v_0 \circ \tau_{W_{b_z}, b})(x) = \epsilon(mv_0 \circ \tau_{W_{b_z}, b})(x) = \epsilon(mv_0 \circ \tau_{W_{b_z}, b})(x) = \epsilon(v_0 \circ \tau_{W_{b_z}, b})(x) = v_0(z')
\]

for arbitrary \( v_0 \in C^\infty(W_{b_z}, b) \). If \( \phi(m_x \cdot b_z) \) and \( z' \in W_{b_z, b} \) were different points of \( H_0 \), we could choose \( v_0 \in C^\infty(W_{b_z, b}) \) such that \( v_0(\phi(m_x \cdot b_z)) \neq v_0(z') \). The above calculation thus shows that \( \phi(m_x \cdot b_z) = z' \).

Next we have

\[
(m_x \cdot b_z) \cdot b'_{z'} = (mb)_x \cdot b'_{z'} = (mb')_x = m_x \cdot (bb')_x = m_x \cdot (b_z b'_{z'}).
\]

If we represent the identity arrow \( 1_b \) by \( v_0 \in C^\infty(H_0) \) with \( (v_0)_x = 1 \), we get

\[
m_x \cdot 1_b = m_x(v_0)_x = m_x v_0 = (v_0 \circ \phi_{W_m, m})_x m_x = m_x.
\]

Finally, we show that \( \eta \) is smooth. Note that the projection \( \text{pr}_1 : P \times H, H \to P \) is a local diffeomorphism. The neighbourhood \( W_1 = m_{W_m} \times H_0 \) of the point \( (m_x, b_z) \in P \times H \) is mapped by \( \pi \circ \text{pr}_1 \) diffeomorphically onto \( W_{mb} = \pi(m_{W_m} \cap \phi^{-1}(W_3)) \). Similarly, the neighbourhood \( W_2 = (mb)_{W_m} \) of the point \( (m_x, b_z) \in P \) is mapped by \( \pi|_{W_2} \) diffeomorphically onto \( W_{mb} \). We can locally express \( \eta \) as \( \eta|_{W_1} = (\pi|_{W_2})^{-1} \circ (\pi \circ \text{pr}_1)|_{W_1} \) and conclude that \( \eta \) is smooth. \( \square \)

4.2. Principalness of \( P \). Finally, we need to show that \( P \) constructed above is indeed a principal \( H \)-bundle over \( G \).

Proposition 4.4. The manifold \( P \), with the actions \( \mu \) and \( \eta \), is a principal \( H \)-bundle over \( G \).

Proof. Let \( g \in G(x, y), p \in P \) and \( h \in H(z') \) be such that \( \pi(p) = x \) and \( \phi(p) = z \). Choose \( a \in G_w^0(A) \), \( m \in G_w(M) \), and \( b \in G_w^0(B) \) such that \( a \) is normalised on an open neighbourhood \( W_a \subset G_0 \) of \( y \), \( m \) is normalised on an open neighbourhood \( W_m \subset G_0 \) of \( x \) and \( b \) is normalised on an open neighbourhood \( W_b \subset H_0 \) of \( z \) with \( g = a_y, p = m_x \) and \( h = b_z \).

(i) Straight from the definition of the action of \( H \) on \( P \) it follows that \( \pi(m_x \cdot b_z) = \pi(m_x) \), which proves that \( H \) acts along the fibers of the map \( \pi \). Next, choose
arbitrary \( v_0 \in C^\infty_c(H_0) \) and \( u_0 \in C^\infty_c(W_a) \) with \((u_0)_y = 1\). Then
\[
v_0(\phi(a_y \cdot m_x)) = \epsilon(amv_0)(y) = u_0(y)\epsilon(amv_0)(y)
\]
\[
= \epsilon(\mu a \epsilon(mv_0))(y) = \epsilon(a_0 \circ \tau_{W,a})\epsilon(mv_0)(y)
\]
\[
= \epsilon(u_0(\epsilon(mv_0) \circ \tau_{W,a}^{-1})a)(y)
\]
\[
= u_0(y)(\epsilon(mv_0) \circ \tau_{W,a}^{-1})(y)e(a)(y)
\]
\[
= \epsilon(mv_0)(x)e(a)(y) = \epsilon(mv_0)(x)
\]
\[
= v_0(\phi(m_x)).
\]

The function \( u_0 \in C^\infty_c(W_a) \) was used to ensure that \((u_0 \circ \tau_{W,a})\epsilon(mv_0) \in C^\infty_c(V_{W,a})\).

Since \( v_0 \in C^\infty_c(H_0) \) was arbitrary, we conclude that \( \phi(a_y \cdot m_x) = \phi(m_x) \), which shows that \( G \) acts along the fibers of the map \( \phi \).

(ii) Both actions commute as a result of
\[
(a_y \cdot m_x) \cdot b_x = (am)_y \cdot b_x = (amb)_y = a_y \cdot (mb)_x = a_y \cdot (m_x \cdot b_x).
\]

(iii) The map \( \pi \) is surjective because \( \epsilon \) is surjective. Finally, we have to show that \((pr_1, \eta) : P \times H_0 \to P \times G_0 \) is a diffeomorphism. To see that it is a bijection, it is sufficient to show that it restricts to a bijection between the corresponding fibers over any \( x \in G_0 \), that is from \( \bigcup_{p \in \pi^{-1}(x)} \{ p \} \times t^{-1}(\phi(p)) \subset P \times H_0 \) to \( \pi^{-1}(x) \times \pi^{-1}(x) \subset P \times G_0 \). By Lemma 3.3 we know that \( \Sigma_x : M_x \otimes_{B_0} B \to M_x \otimes_{A_0} M_x \) is an isomorphism of coalgebras over \((A_0)_x\), so it restricts to a bijection between \( G(M_x \otimes_{B_0} B) \) and \( G(M_x \otimes_{A_0} M_x) \). By Lemma 5.4 applied to \( N = B \), we may identify \( G(M_x \otimes_{B_0} B) \) with \( \bigcup_{p \in \Sigma^{-1}(x)} \{ p \} \times t^{-1}(\phi(p)) \subset P \times H_0 \) and \( G(M_x \otimes_{A_0} M_x) \) with \( G(M_x) \times G(M_x) = \pi^{-1}(x) \times \pi^{-1}(x) \). Observe that the bijection
\[
\bigcup_{p \in \Sigma^{-1}(x)} \{ p \} \times t^{-1}(\phi(p)) \to \pi^{-1}(x) \times \pi^{-1}(x)
\]
given by \( \Sigma_x \) is in fact of the form \( (p, h) \mapsto (p, p \cdot h) \), thus equal to the restriction of \((pr_1, \eta) : P \times H_0 \to P \times G_0 \) to the fiber over \( x \). This proves that \((pr_1, \eta) : P \times H_0 \to P \times G_0 \) is a bijection. Furthermore, the map \((pr_1, \eta) \) is a local diffeomorphism because \( H \) is étale, and therefore it is a diffeomorphism.

5. Equivalence of the Morita categories

In this section we state and prove the main result of this paper:

**Theorem 5.1.** The functor \( C^\infty_c : EtGPD \to LgHoALGD \) is an equivalence between the Morita category of étale Lie groupoids and the Morita category of locally grouplike Hopf algebroids.

Before we give the proof, let us start with two lemmas. Let \((A, G_0) \) and \((B, H_0) \) be locally grouplike Hopf algebroids with the associated spectral étale groupoids \( G \) respectively \( H \). We denote by \( M_* \) the principal \( H \)-bundle \( P \to G \) associated to a locally grouplike principal \( A \)-\( B \)-bimodule \( M \), as in Section 4. Suppose that \( \theta : M \to M' \) is a homomorphism of principal \( A \)-\( B \)-bimodules. In particular, \( \theta : M \to M' \) is a homomorphism of coalgebras over \( A_0 = C^\infty_c(G_0) \), so we have the associated map \( E_{sp}(\theta) : E_{sp}(M) \to E_{sp}(M') \) of spectral sheaves over \( G_0 \). 21 22 24. We can describe the map \( E_{sp}(\theta) \) as follows: If a point \( p \in G(M_x) = E_{sp}(M)_x \) is represented by an element \( m \in G_w(M) \) normalised at \( x \in G_0 \) (i.e. \( p = m_x \)), then \( \theta(m) \in G_w(M') \) is normalised at \( x \) as well and
\[
E_{sp}(\theta)(p) = (\theta(m))_x \in G(M'_x) = E_{sp}(M'_x).
\]
Since \( M_* = E_{sp}(M) \) and \( M'_* = E_{sp}(M') \) as sheaves over \( M \), we shall write \( \theta_* = E_{sp}(\theta) : M_* \to M'_* \).
Lemma 5.2. The map $\theta_* : \mathcal{M}_* \to \mathcal{M}'_*$ is an equivariant map of principal $H$-bundles over $G$.

Proof. Suppose that $m \in G_w(M)$ is normalised at $x \in G_0$, $a \in G^w_0(A)$ is normalised at $y \in G_0$ and $b \in G^w_0(B)$ is normalised at $\phi(m_x) = z$, such that $a_y \in G(x,y)$ and $b_z \in H(z', z)$. Straight from the definition of $\theta_*$ it follows that $\pi' \circ \theta_* = \pi$.

Furthermore, for any $v_0 \in C^\infty(H_0)$ we have

$$v_0(\phi'(\theta_* (m_x))) = \epsilon'(\theta(m) v_0)(x) = \epsilon'(\theta(m v_0))(x) = \epsilon(m v_0)(x) = v_0(\phi(m_x)),$$

which shows that $\phi' \circ \theta_* = \phi$. Next, the equalities

$$\theta_*(a_y \cdot m_x) = \theta_*(a(y)m)_x = (a(y)m)_x = a_y \cdot \theta_*(m_x)$$

and

$$\theta_*(m_x \cdot b_z) = \theta_*(m(b)_x) = \theta(m(b)_x) = \theta(m)_x \cdot b_z = \theta_*(m_x) \cdot b_z$$

show that $\theta_*$ is equivariant. \hfill $\Box$

For any principal $H$-bundle $P$ over $G$ there is a natural isomorphism $\Phi: P \to \mathcal{E}_\mathfrak{sp}(C^\infty_c(P)) = C^\infty_c(P)_*$ of sheaves over $G$, \cite[Lemma 5.3.]{21,23}, given by

$$\Phi(p) = f_{\pi(p)},$$

where $f \in G_w(C^\infty_c(P))$ with $f_p = 1$. With respect to the natural isomorphisms $G \cong G_{sp}(C^\infty_c(G))$ and $H \cong G_{sp}(C^\infty_c(H))$, given by the same formula as $\Phi$, we may regard $C^\infty_c(P)_*$ as a principal $H$-bundle over $G$.

Lemma 5.3. The map $\Phi: P \to C^\infty_c(P)_*$ is an isomorphism of principal $H$-bundles over $G$.

Proof. Let $g \in G(x,y)$, $h \in H(z', z)$ and $p \in P$ with $\pi(p) = x$ and $\phi(p) = z$.

Suppose that $u \in G^w_0(C^\infty_c(G))$, $v \in G^w_0(C^\infty_c(H))$ and $f \in G_w(C^\infty_c(P))$ satisfy $u_g = 1$, $v_h = 1$ and $f_p = 1$. For any $v_0 \in C^\infty_c(H_0)$ we have $(fv_0)(p) = f(p) v_0(\phi(p)) = v_0(\phi(p))$, by the equation (3). Since $fv_0 \in G_w(C^\infty_c(P))$ we have $(fv_0)(p) = \epsilon(fv_0)(\pi(p))$ and hence

$$v_0(\phi(p)) = \epsilon(fv_0)(\pi(p)) = v_0(\phi(f_{\pi(p)})) = v_0(\phi(\Phi(p))),$$

which shows that $\phi \circ \Phi = \phi$.

By Lemma 1.1, we have $uf \in G_w(C^\infty_c(P))$, $vf \in G_w(C^\infty_c(P))$, $\epsilon(uf)_p = 1$ and $\epsilon(fv)_p = 1$. Furthermore, we have the equalities $(uf)_y = 1$ respectively $(fv)_y = 1$ by the equations (2) and (3), so

$$\Phi(g \cdot p) = (uf)_p = u_y \cdot f_x = g \cdot \Phi(p)$$

and

$$\Phi(p \cdot h) = (fv)_p = f_x \cdot v_z = \Phi(p \cdot h),$$

which shows that $\Phi$ is equivariant. \hfill $\Box$

Proof of Theorem 5.7. (i) Any locally grouplike Hopf algebroid $(A, M)$ is isomorphic to the locally grouplike Hopf algebroid $(\mathcal{C}^\infty_c(Gsp(A)), M)$ \cite[22,]{24}, so $\mathcal{C}^\infty_c$ is essentially surjective.

(ii) Let $G$ and $H$ be étale Lie groupoids and let $\mathcal{M}$ be a locally grouplike principal $\mathcal{C}^\infty_c(G)$-$\mathcal{C}^\infty_c(H)$-bimodule. The map $\Psi: \mathcal{C}^\infty_c(\mathcal{M}_*) \to \mathcal{M}_*$, given by

$$\Psi(\sum_{i=1}^k f_i \circ \pi|_{(m_i)_{W_{m_i}}}) = \sum_{i=1}^k f_i m_i,$$

is an isomorphism of coalgebras over $\mathcal{C}^\infty_c(G_0)$ \cite[Theorem 2.5]{21}, see also \cite[23]{24}, where $m_i \in G_w(M)$, $W_{m_i}$ is an open subset of $G_0$ such that $m_i$ is normalised on $W_{m_i}$ and $f_i \in C^\infty_c(W_{m_i})$, for any $i = 1, \ldots, k$. 

We will show that \( \Psi(u f) = u \Psi(f) \) and \( \Psi(f v) = \Psi(f) v \) for any \( u \in C^\infty_c(G) \), \( f \in C^\infty_c(M) \) and \( v \in C^\infty_c(H) \). We can assume that \( f = f_0 \circ \pi|_U \), where \( U = m_{W_m} \) for some \( m \in G_u(M) \) normalized on an open subset \( W_m \subset G_0 \) and \( f_0 \in C^\infty_c(W_m) \). Furthermore, we may assume that \( u = u_0 \circ \ell|_U \) and \( v = v_0 \circ \ell|_{U'_v} \), where \( U'_v \subset G \) respectively \( U'_v \subset H \) are bisections, \( u_0 \in C^\infty_c(\ell(U'_v)) \) and \( v_0 \in C^\infty_c(\ell(U'_v)) \). Choose elements \( a \in C^\infty_c(U'_v) \subset G_u^S(C^\infty_c(G)) \), normalised on an open neighbourhood \( W_a \subset \ell(U'_v) \) of \( \text{supp}(u_0) \), and \( b \in C^\infty_c(U'_v) \subset G_u^S(C^\infty_c(H)) \), normalised on an open neighbourhood \( W_b \subset \ell(U'_v) \) of \( \text{supp}(v_0) \). Write \( U_u = t^{-1}(W_a) \cap U'_v \) and \( U_v = t^{-1}(W_b) \cap U'_v \). By the equations \((2)\) and \((3)\) we get
\[
uf = (u_0(f_0 \circ \tau^{-1}_{u_0,a})) \circ \pi|_{\mu(U_u \times_G U_a)}
\]
and
\[
fv = (f_0(v_0 \circ \varphi_{W_m,m})) \circ \pi|_{\eta(U \times_H U_v)}.
\]
Denote \( W_{am} = \tau_{W_{am}}(W_{am} \cap W_m) \) and \( W_{mb} = \pi(m_{W_m} \cap \phi^{-1}(W_b)) \). The elements \( am, mb \in G_u(M) \) are, by Lemma \( \[ \ref{lem:isomorphism} \] \) normalised on the supports of the functions \( u_0(f_0 \circ \tau^{-1}_{W_{am},a}) \in C^\infty_c(W_{am}) \) respectively \( f_0(v_0 \circ \varphi_{W_m,m}) \in C^\infty_c(W_{mb}) \). Moreover, \( \mu(U_u \times_G U_0) = (am)_{W_{am}} \) and \( \eta(U \times_H U_v) = (mb)_{W_{mb}} \), which implies
\[
\Psi(uf) = u_0(f_0 \circ \tau^{-1}_{W_{am},a})am
\]
and
\[
\Psi(fv) = f_0(v_0 \circ \varphi_{W_m,m})mb.
\]
On the other hand we have \( u \Psi(f) = ufam = uamf = u_0(f_0 \circ \tau^{-1}_{W_{am},a})am \) respectively \( \Psi(f)v = fmav = formb = f_0(v_0 \circ \varphi_{W_m,m})mb \), which shows that \( \Psi \) is an isomorphism of locally grouplike principal bimodules. This proves that the functor \( C^\infty_c \) is full.

(iii) Finally, we show that \( C^\infty_c \) is faithful. Choose principal \( H \)-bundles \( P \) and \( P' \) over \( G \), and suppose that there exists an isomorphism \( \theta : C^\infty_c(P) \to C^\infty_c(P') \) of locally grouplike principal \( C^\infty_c(G) \)-\( C^\infty_c(H) \)-bimodules. The principal bundles \( C^\infty_c(P)_a \) and \( C^\infty_c(P')_a \) are isomorphic by Lemma \( \[ \ref{lem:isomorphism} \] \), so \( P \) and \( P' \) are isomorphic as well by Lemma \( \[ \ref{lem:isomorphism} \] \) and therefore represent the same morphism from \( G \) to \( H \) in the category \( \text{EtGPD} \).

\[ \square \]

References

[1] C. Blohmann, A. Weinstein, Group-like objects in Poisson geometry and algebra. Preprint arXiv: math.SG/0701499 (2007).
[2] C. Blohmann, X. Tang, A. Weinstein, Hopfian structure and modules over irrational rotation algebras. Preprint arXiv: math.OA/0604105 (2006).
[3] G. Böhm, K. Szlachányi, Hopf algebroids with bijective antipodes: axioms, integrals, and duals. J. Algebra 274 (2004) 708–750.
[4] A. Cannas da Silva, A. Weinstein, Geometric Models for Noncommutative Algebras. Berkeley Mathematics Lecture Notes 10, American Mathematical Society, Providence, Rhode Island (1999).
[5] A. Connes, Noncommutative Geometry. Academic Press, San Diego (1994).
[6] M. Crainic, I. Moerdijk, A homotopy theory for étale groupoids. J. Reine Angew. Math. 521 (2000) 25–46.
[7] M. Crainic, I. Moerdijk, Foliation groupoids and their cyclic homology. Adv. Math. 157 (2001) 177–197.
[8] A. Haefliger, Structures feuilletées et cohomologie à valeur dans un faisceau de groupoïdes. Comment. Math. Helv. 32 (1958) 248–329.
[9] A. Haefliger, Groupoids and foliations. Groupoids in analysis, geometry, and physics, Contemp. Math. 282 (2001) 83–100.
[10] M. Hilsum, G. Skandalis, Morphismes K-orientés d’espaces de feuilles et fonctorialité en théorie de Kasparov (d’après une conjecture d’A. Connes). Ann. Sci. École Norm. Sup. 20 (1987) 325–390.
[11] M. Kapranov, Free Lie algebroids and the space of paths. Preprint arxiv: math.AG/0702584 (2007).
EQUIVALENCE BETWEEN THE MORITA CATEGORIES

[12] J.-H. Lu, Hopf algebroids and quantum groupoids. International J. Math. 7 (1996) 47–70.
[13] K. C. H. Mackenzie, General Theory of Lie Groupoids and Lie Algebroids. London Math.
Soc. Lecture Note Ser. 213, Cambridge University Press, Cambridge (2005)
[14] G. Maltsiniotis, Groupoïdes quantiques de base non commutative. Comm. Algebra 28 (2000)
3441–3501.
[15] I. Moerdijk, Classifying toposes and foliations. Ann. Inst. Fourier (Grenoble) 41 (1991) 189–
209.
[16] I. Moerdijk, Orbifolds as groupoids: an introduction. Contemp. Math. 310 (2002) 205-222.
[17] I. Moerdijk, J. Mrčun, Introduction to Foliations and Lie Groupoids. Cambridge Studies in
Advanced Mathematics 91, Cambridge University Press, Cambridge (2003).
[18] I. Moerdijk, J. Mrčun, Lie groupoids, sheaves and cohomology. Poisson Geometry, Deform-
ation Quantisation and Group Representations, London Math. Soc. Lecture Note Ser. 323,
Cambridge University Press, Cambridge, (2005) 145–272.
[19] J. Mrčun, Functoriality of the bimodule associated to a Hilsum-Skandalis map. K-Theory 18
(1999) 235–253.
[20] J. Mrčun, The Hopf algebroids of functions on étale groupoids and their principal Morita
equivalence. J. Pure Appl. Algebra 160 (2001) 249–262.
[21] J. Mrčun, On spectral representation of coalgebras and Hopf algebroids. Preprint arXiv:
math.QA/0208199 (2002).
[22] J. Mrčun, On duality between étale groupoids and Hopf algebroids. J. Pure Appl. Algebra
210 (2007) 267–282.
[23] J. Mrčun, Sheaf coalgebras and duality. Preprint.
[24] J. Renault, A Groupoid Approach to C*-algebras. Lecture Notes in Math. 793, Springer, New
York (1980).
[25] M. Takeuchi, Groups of algebras over $A \otimes \mathbb{T}$. J. Math. Soc. Japan 29 (1977) 459–492.
[26] P. Xu, Quantum groupoids and deformation quantization. C. R. Acad. Sci. Paris 326 (1998)
289–294.

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