Modeling the social media relationships of Irish politicians using a generalized latent space stochastic blockmodel

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Abstract

Dáil Éireann is the principal chamber of the Irish parliament. The 31st Dáil was in session from March 11th, 2011 to February 6th, 2016. Many of the members of the Dáil were active on social media and many were Twitter users who followed other members of the Dáil. The pattern of following amongst these politicians provides insights into political alignment within the Dáil. We propose a new model, called the generalized latent space stochastic blockmodel, which extends and generalizes both the latent space model and the stochastic blockmodel to study social media connections between members of the Dáil. The probability of an edge between two nodes in a network depends on their respective class labels as well as latent positions in an unobserved latent space. The proposed model is capable of representing transitivity, clustering, as well as disassortative mixing. A Bayesian method with Markov chain Monte Carlo sampling is proposed for estimation of model parameters. Model selection is performed using the WAIC criterion and models of different number of classes or dimensions of latent space are compared. We use the model to study Twitter following relationships of members of the Dáil and interpret structure found in these relationships. We find that the following relationships amongst politicians is mainly driven by past and present political party membership. We also find that the modeling outputs are informative when studying voting within the Dáil.
1 Introduction

Networks are widely used to represent relational data. In social network data, vertices typically represent individual nodes while edges represent ties or relationships between nodes. For example, on the Twitter social medium platform, the nodes are users and the following of users can be represented by edges. Twitter is widely used by many public figures, including politicians. In this work, we study the following connections of Irish politicians on Twitter to see if the links reveal insights about alignment of the politicians in the parliament.

A number of network models have been proposed in recent years, including latent space models (Hoff et al., 2002; Handcock et al., 2007), stochastic blockmodels (Holland et al., 1983; Snijders and Nowicki, 1997; Nowicki and Snijders, 2001), overlapping stochastic blockmodels (Latouche et al., 2011, 2014) and mixed membership stochastic blockmodels (Airoldi, 2006; Airoldi et al., 2008); Salter-Townshend et al. (2012) contains a detailed review of network models.

The stochastic blockmodel (Snijders and Nowicki, 1997; Nowicki and Snijders, 2001) assumes that each node belongs to one latent class. Conditional on latent class assignments, the probability of an edge between two nodes depends only on the latent classes to which the nodes belong. These models are able to represent both assortative mixing, indicated by within-block probabilities being larger than between-block probabilities, and disassortative mixing, indicated by the reverse. However, they are unable to capture transitivity within classes, which is observed in many real-world networks.

The latent space model (Hoff et al., 2002) assigns each node to a latent position in a Euclidean space. The probability of an edge between two nodes in the latent space is the logistic transformation of a linear function of the Euclidean distance between their respective latent positions and their covariates. This model takes advantage of the properties of Euclidean space to naturally represent transitivity and reciprocity. The latent position cluster model (Handcock et al., 2007) extends the latent space model by accounting for clustering. In this model, the latent positions of nodes are drawn from a finite mixture of multivariate normal distributions. Krivitsky et al. (2009) further extended the latent position cluster model by allowing node-specific random effects.

While the latent space model and its extensions are able to capture transitivity, reciprocity, and
homophily, they are unable to represent disassortative mixing – the phenomena of nodes being more likely to have ties with nodes which are different from them. Social networks tend to be assortative when patterns of friendships are strongly affected by race, age, and language; however, disassortative mixing is more common in biological networks.

We propose the generalized latent space stochastic blockmodel, which addresses the issues of stochastic blockmodels as well as latent space models. The resulting model can be viewed as a generalization of stochastic blockmodels as well as latent space models. The model generalizes the latent space stochastic blockmodel recently developed in Fosdick et al. (2018) to permit directed edges and to relax the assortative mixing assumption. Both of these extensions are essential for effectively modeling the Irish politician Twitter following data.

In Section 2 we introduce a data set that contains the social media connections between Irish politicians on the Twitter social media platform. In Section 3 we describe the generalized latent space stochastic blockmodel (GLSSBM). In Section 4 we study the proposed model in more detail. In Section 5 we describe model estimation based on Monte Carlo Markov Chain (MCMC) sampling. In Section 6 we propose a method for model comparison so that the number of classes and latent space dimension can be selected. In Section 7 we use the generalized latent space stochastic blockmodel to investigate the Twitter following relationships between the Irish politicians. We discuss the structure that is revealed by the model and its connection to the politicians’ political alignment within the parliament. In Section 8 we discuss the model and possible extensions.

2 Irish Politicians Twitter Data

The Irish legislature is called the Oireachtas, and it consists of the President of Ireland and a parliament with two chambers: Seanad Éireann and Dáil Éireann. Dáil Éireann is the principal chamber of the Irish parliament, and it is directly elected from constituencies that elect either three, four or five seats (members). Each seat represents between 20,000 and 30,000 Irish citizens. Further details of the Irish political system can be found in Coakley and Gallagher (2018).

The 31st Irish Dáil (March 11th, 2011 - February 6th, 2016) had 166 members who were affiliated
with nine political parties or were independent. Of these, 147 members had accounts on the Twitter social media platform (Table 1). The government consisted of a coalition between Fine Gael and Labour and the opposition had a number of parties including Fianna Fáil, Sinn Féin, as well as a number of small parties and independent politicians.

Table 1: Political Parties in the 31st Irish Dáil on January 1st, 2016. The number of members of each party and the number Twitter users are also shown. On this date, Fine Gael and Labour were in a coalition government.

| Party                                | Members | Twitter |
|--------------------------------------|---------|---------|
| Fianna Fáil                          | 21      | 17      |
| Fine Gael                            | 66      | 61      |
| Independent                          | 18      | 17      |
| Labour                               | 34      | 29      |
| People Before Profit Alliance        | 1       | 1       |
| Renua Ireland                        | 3       | 2       |
| Sinn Féin                            | 14      | 12      |
| Social Democrats                     | 3       | 3       |
| Socialist Party                      | 3       | 3       |
| United Left                          | 2       | 2       |
| All                                  | 166     | 147     |

A directed network consisting of the following relationship of the 147 accounts was constructed, where there is an edge from member $i$ to member $j$ if member $i$ follows member $j$. A plot of the network is given in Figure 1, where the nodes are colored according to their political party.

The plot in Figure 1 shows that nodes within the same party are clustered together. The Fruchterman Reingold force-directed layout used in the plot puts nodes closer together if they are connected. Thus, there is an indication that political party affiliation is related to Twitter following.

Data were collected on the voting within the Dáil from January 1st, 2016 to February 6th, 2016. In this time period, a total of 23 votes took place. Each member of the Dáil was recorded as voting Tá (Yes), Níl (No) or Absent/Abstain for each vote. The period was characterized by having a large number of Absent/Abstain, with 30.7–71.7% abstaining in the votes, which can be explained by a few factors. Firstly, the political parties have an informal agreement that on some votes both the government and opposition will have the same number of members absent so that the outcome of the vote isn’t altered. Secondly, the term of the Dáil is five years, so the politicians knew that the election would be called before March 11th, 2016; the date of the election is chosen...
Figure 1: A force directed layout of the Irish politicians Twitter follower network. The nodes are colored by political affiliation of each member of the parliament with a Twitter account.

by the Taoiseach (Head of Government). Thus, many politicians were campaigning locally in their constituencies in anticipation of the upcoming election.

We are interested in exploring the structure of the Twitter following network for the members of the Irish parliament. In particular, we want to determine if there are subgroups in the network with similar connectivity patterns, to determine if political party is related to the structure in the network, and to determine whether the voting in the Dáil is related to the structure in the network. These questions are particularly interesting in a parliament with such a large number of independent members, because there is less prior information available on how these members interact with each other and with the members who are affiliated with political parties.

It is worth noting that Fianna Fáil and Fine Gael are traditionally the two main parties in Ireland, but Fianna Fáil had their worst election performance ever in the election of the 31st Dáil. Members
of these parties tend to take opposite positions on many matters. Fine Gael and Labour have been in coalition on a number of occasions, with Labour being the minor party of the coalition, but Fianna Fáil and Labour were in coalition from December 14th, 1992 to December 15th, 1994. Fine Gael and Labour are closely aligned on many policies but take different stances on other policies. Hansen (2009) used a Bayesian ideal point model to estimate the location of the parties relative to each other and the results showed a government-opposition divide that explained the location of parties relative to each other.

Hansen (2010) found that Fianna Fáil and Fine Gael tend to vote in opposite ways in the parliament, as do Fianna Fáil and Labour, whereas Fine Gael and Labour tend to vote together. Hansen (2010) also found that minor party members tend to vote against the government unless they are part of the government coalition, whereas independent members tend to vote en bloc either in support of or in opposition to the government, depending on the context.

Bolleyer and Weeks (2009) provide a detailed description of the role of independents in the Irish context from the electoral stage through to their role in parliament and in government. Weeks (2009) studied the role of independent candidates in Irish politics and provided a typology of independent politicians (not just those elected). His typology had six categories of independents, and those most relevant in this data are national single-issue independents, who are concerned with some ideological national issue, community independents, who are entirely focused on representing their local community with little concern in national issues, and apostate independents, who flit between party and independent status (some of these either resigned or were expelled from a party or failed to secure a nomination to be a candidate for the party). Many of the independent members of the 31st Dáil can be categorized into these categories.

Farrell et al. (2015) discuss the role of political party on voting within the Dáil, arguing that Irish politicians have historically shown strong party discipline in voting. One reason for this is that the political parties impose high penalties for those who do not follow the party line. However, the Irish electoral system is very candidate-focused, with candidates from the same party competing for election in a constituency (eg. Gormley and Murphy, 2008a, b). However, historically, very few candidates have rebelled against the party when voting in the Dáil.
3 Model

The network being modeled is a binary, directed network, though the model developed herein can easily be adjusted to undirected networks. We let \( N \) denote the number of nodes (politicians) in the network and denote the \( N \times N \) adjacency matrix by \( Y_N \). We let \( y_{ij} = 1 \) if node \( i \) follows node \( j \) and \( y_{ij} = 0 \) otherwise.

We partition nodes into \( K \) latent classes and denote the class assignment by \( \gamma = (\gamma_1, \gamma_2, ..., \gamma_N) \). We assume that the class assignments of the nodes are independent and identically distributed. That is,

\[
\gamma_i|\pi \sim \text{multinomial}(1; \pi_1, \ldots, \pi_K),
\]

where \( \pi_k \) is the probability of a node being assigned to class \( k \).

We also assign each node a latent position in the Euclidean space \( \mathbb{R}^d \), the collection of which is denoted by \( Z = (Z_1, Z_2, ..., Z_N) \). We further assume that, conditional on class label \( \gamma_i = k \), the latent position \( Z_i \) is drawn from a multivariate normal distribution \( Z_i \sim \text{MVN}(0, \sigma_k^2 I_d) \), where \( \sigma = (\sigma_1, \ldots, \sigma_K) \) are the standard deviations of the latent positions within each class.

If two nodes belong to the same class \( k \), we assume that the probability there is an edge between them depends only on their latent positions, and model the connection probability as

\[
\text{logit } P(Y_{ij} = 1 \mid \gamma_i = \gamma_j = k, Z_i, Z_j) = \beta_k - \|Z_i - Z_j\|,
\]

where \( \| \cdot \| \) is the Euclidean norm and \( \beta = (\beta_1, \ldots, \beta_K) \) are block-specific intercepts. Note that conditional on same class membership, the edge probability is independent of all other edges.

If two nodes have different class labels, we assume that the presence of an edge depends only on their respective class labels. That is,

\[
P(Y_{ij} = 1 \mid \gamma_i = k, \gamma_j = l, k \neq l) = \tau_{kl},
\]

where \( \{\tau_{kl}\}_{1 \leq k, l \leq K} \) are between-block probabilities.
Hence, conditional on class labels and latent positions, the likelihood function is:

\[
P(Y_N | \gamma, Z, \tau, \beta) = \prod_{1 \leq k < l \leq K} \tau_{kl} e_{kl} (1 - \tau_{kl})^{n_{kl} - e_{kl}} \\
\prod_{k=1}^{K} \prod_{i,j : \gamma_i = \gamma_j = k} \left( e_{\beta_k - \|Z_i - Z_j\|} \frac{1}{1 + e^{\beta_k - \|Z_i - Z_j\|}} \right)^{Y_{ij}} \left( \frac{1}{1 + e^{\beta_k - \|Z_i - Z_j\|}} \right)^{1 - Y_{ij}},
\]

where

\[
e_{kl} = \sum_{1 \leq i \neq j \leq N} Y_{ij} I(\gamma_i = k) I(\gamma_j = l), \quad n_k = \sum_{i=1}^{N} I(\gamma_i = k), \quad n_{kl} = n_k n_l
\]

and \(I(\cdot)\) is the indicator function.

The model can be viewed as a generalization of the latent space model (Hoff et al., 2002) and the stochastic blockmodel (Nowicki and Snijders, 2001). If we reduce the number of classes to one, the model reduces to the latent space model. On the other hand, if all nodes in the same class have the same latent positions, the model reduces to the stochastic blockmodel.

## 4 Motivation for the model

### 4.1 Comparison with latent position cluster model

The generalized latent space stochastic blockmodel is closely related to the latent position cluster model (Handcock et al., 2007). The latent position cluster model puts all the nodes in a single geometric space. The positions of the points in this space are determined using a finite mixture model, giving rise to “clustering” in the latent space. Both models take into account heterogeneity within groups, transitivity, reciprocity, and clustering. However, there are a few key differences.

Firstly, in the latent position cluster model, within-cluster edges are more likely than between-cluster edges. On the other hand, the generalized latent space stochastic blockmodel can capture scenarios with disassortative mixing (Newman, 2002), where between-group edges are more likely than within-group edges. This is a crucial piece of flexibility, as it may be desirable to fit a model that does not require connectivity of nodes within the same group to be greater than that between
groups.

Secondly, by having a different intercept parameter $\beta_k$ for each class, the model is more flexible because it can allow for different within-group connectivity. In particular, the maximum within-block edge probability for block $k$ is given by $\exp(\beta_k)/(1 + \exp(\beta_k))$. Thus, different intercept parameters for each block facilitates different maximum edge probabilities in each block.

Furthermore, the model also allows nodes with low connectivity between each other to be grouped together. In this case, the group is characterized by the common connection probabilities with other groups.

4.2 Connection with latent position cluster model

4.2.1 Approximation of between cluster connection probability

We now demonstrate that when the clusters in the latent position cluster model are sufficiently far apart from each other, an approximation can be derived for the connection probabilities between nodes from different clusters. This approximation shows a connection between the latent position cluster model and the latest space stochastic blockmodel.

Consider the latent position cluster model with $N$ nodes $\{V_1, \ldots, V_N\}$ and $K$ clusters with a $d$ dimensional latent space, as in Handcock et al. (2007).

Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_N)$ and $Z = (Z_1, Z_2, \ldots, Z_N)$ denote the cluster memberships and latent positions of the nodes, respectively.

For $1 \leq k \neq l \leq K$, define

\[
\overline{d}_k := \max_{\gamma_i = \gamma_j = k} \|Z_i - Z_j\|
\]

\[
\overline{d}_{kl} := \max_{\gamma_i = k, \gamma_j = l} \|Z_i - Z_j\|
\]

\[
d_{kl} := \min_{\gamma_i = k, \gamma_j = l} \|Z_i - Z_j\|.
\]

That is, $\overline{d}_k$ denotes the maximum Euclidean distance between any two nodes in cluster $k$, $\overline{d}_{kl}$ and $d_{kl}$ denote the maximum and minimum Euclidean distance between nodes in cluster $k$ and $l$, respectively.
respectively.

Now, for any two clusters \( k \) and \( l \) with \( k \neq l \), let \( V_s \) and \( V_t \) be two nodes such that \( \gamma_s = k \), \( \gamma_t = l \) and \( \|Z_s - Z_t\| = d_{kl} \). Consider two arbitrary nodes such that \( \gamma_i = k \), \( \gamma_j = l \), we have

\[
\|Z_i - Z_j\| \leq \|Z_i - Z_s\| + \|Z_s - Z_t\| + \|Z_t - Z_j\|
\]

Taking the maximum over all \( i, j \) such that \( \gamma_i = k \) and \( \gamma_j = l \) on the left hand side, we have

\[
d_{kl} \leq \overline{d}_k + \overline{d}_l + \overline{d}_{kl}
\]

Moreover, for any two nodes \( V_i \) and \( V_j \) in clusters \( k \) and \( l \), respectively, the connection probability \( p_{ij} \) can be bounded by

\[
\frac{e^{\beta - \overline{d}_{kl}}}{1 + e^{\beta - \overline{d}_{kl}}} \leq p_{ij} \leq \frac{e^{\beta - \overline{d}_{kl}}}{1 + e^{\beta - \overline{d}_{kl}}}
\]

for some constant \( \beta \).

In the cases where clusters are sufficiently far apart from each other, i.e. \( d_{kl} \gg \overline{d}_k + \overline{d}_l \), we have

\[
\frac{e^{\beta - \overline{d}_{kl}}}{1 + e^{\beta - \overline{d}_{kl}}} - \frac{e^{\beta - d_{kl}}}{1 + e^{\beta - d_{kl}}} \approx 0
\]

Hence, for all \( V_i, V_j \) such that \( \gamma_i = k \) and \( \gamma_j = l \) and \( k \neq l \),

\[
p_{ij} \approx \tau_{kl} \approx \frac{e^{\beta - \overline{d}_{kl}}}{1 + e^{\beta - \overline{d}_{kl}}} \approx \frac{e^{\beta - d_{kl}}}{1 + e^{\beta - d_{kl}}}
\]

We have shown that the connectivity between two clusters which are far apart in the latent position cluster model is similar to the between-cluster connectivity in the generalized latent space stochastic blockmodel. As a result, in many cases, not much is lost by modelling the between-cluster ties with a single probability, as we do here, rather than through the latent space.
4.2.2 Improving within-cluster flexibility

Under the latent position cluster model (Handcock, 2007), the connection probability between any two nodes is bounded above by \( e^\beta / (1 + e^\beta) \). A single upper bound can be too restrictive in cases where some clusters are “significantly more connected” than others. To see this, for \( k = 1, \ldots, K \), we first define

\[
\bar{p}_k = \min_{\gamma_i = \gamma_j = k} p_{ij} \\
\underline{p}_k = \max_{\gamma_i = \gamma_j = k} p_{ij}
\]

That is, \( \bar{p}_k \) and \( \underline{p}_k \) are the maximum and minimum connection probability between nodes in cluster \( k \). Suppose \( \underline{p}_k = 1 - \delta \), \( \bar{p}_l = \xi \), and \( \underline{p}_k \gg \bar{p}_l \) for some \( k \neq l \). Then we require

\[
1 - \delta \leq \frac{e^\beta}{1 + e^\beta}.
\]

If \( \delta \to 0 \), then we must have \( \beta \to \infty \). On the other hand, we have

\[
\xi = \frac{e^{\beta - d_l}}{1 + e^{\beta - d_l}}.
\]

As \( \beta \to \infty \), we must have \( d_l \to \infty \). That is, \( \|Z_i - Z_j\| \to \infty \) for all \( V_i, V_j \) such that \( \gamma_i = \gamma_j = l \). That is, all nodes in cluster \( l \) have to be far apart from each other in the latent space.

To avoid the situation above, it is desirable to allow \( \beta \) to take different values on different clusters. As a result, the connection probability between two nodes in the same cluster (i.e. \( \gamma_i = \gamma_j = k \)) can be represented as

\[
p_{ij} = \frac{e^{\beta_k - \|Z_i - Z_j\|}}{1 + e^{\beta_k - \|Z_i - Z_j\|}} \leq \frac{e^{\beta_k}}{1 + e^{\beta_k}}.
\]

4.3 Comparison with stochastic blockmodel

The model specified in Section 3 can be viewed as a generalization of the stochastic blockmodel (Nowicki and Snijders, 2001) with heterogeneity within classes. That is, we extend the stochastic blockmodel by fitting a latent space model for each class while retaining between class connection structure. By assigning each node to a latent space, properties such as clustering and transitivity
that are typically observed in the real world can be better captured.

5 Bayesian Estimation

5.1 Prior Specification

We fit the model specified in Section 3 in a Bayesian paradigm using MCMC sampling. We assign prior distributions for the parameters $\pi$, $\tau$, and $\beta$ as follows:

\[
\pi \sim \text{Dirichlet}(T, \ldots, T), \\
\tau_{kl} \sim \text{beta}(E_0, E_1) \\
\beta_k \sim \mathcal{N}(\mu_{\beta_k}, (\sigma_{\beta_k}^2)),
\]

where in the second line $1 \leq k < l \leq K$, and in the last line $1 \leq k \leq K$.

As discussed in Nowicki and Snijders (2001), small values of $T$ tend to favor unequal class sizes. We set $T = 10$ to discourage very small classes that are almost empty. We apply the uniform prior distribution to $\tau_{kl}$ by setting $E_0 = E_1 = 1$. We set $\mu_{\beta_k} = 0$ and $\sigma_{\beta_k}^2 = 5$. We set the prior variance for the latent positions $\sigma_k^2 = 5$.

5.2 Full conditional distributions

With the prior assumptions given above, the full conditional posterior distributions can be expressed as:

\[
[\pi \mid \text{others}] \sim \text{Dirichlet}(T + n_1, \ldots, T + n_c) \\
[\tau_{kl} \mid \text{others}] \sim \text{beta}(E_0 + n_{kl} - e_{kl}, E_1 + e_{kl}) \\
[\beta_k \mid \text{others}] \propto P(Y \mid \gamma, Z, \tau, \beta) \mathcal{N}(\mu_{\beta_k}, (\sigma_{\beta_k}^2))
\]
\[ [\gamma_i = k \mid \text{others}] \propto \pi_k \prod_{l \neq k} \prod_{i \neq j} \left( \tau_{kl}^\gamma (1 - \tau_{kl})^{1 - Y_{ij}} \right)^{I(\gamma_j = l)} \prod_{i \neq j} \left( \frac{e^{\beta_k - \|Z_i - Z_j\|}}{1 + e^{\beta_k - \|Z_i - Z_j\|}} \right)^{I(\gamma_j = k)} \]  

(5)

\[ [Z_i \mid \gamma_i = k, \text{others}] \propto P(Y_N \mid X, Z, \tau, \beta) \text{MVN}(Z_i; 0, \sigma_k^2 I_d). \]  

(6)

5.3 MCMC algorithm

In designing the algorithm to sample from the posterior distribution, we first note that if two nodes have different class labels, their respective latent positions are irrelevant. Hence, the class label \( \gamma_i \) and latent position \( Z_i \) are updated simultaneously. On iteration \( t + 1 \) of the algorithm, we use the following proposal to propose a new pair \((\gamma^*_i, Z^*_i)\) from the existing pair \((\gamma^t_i, Z^t_i)\).

We let

\[ q((\gamma^t_i, Z^t_i), (\gamma^*_i, Z^*_i)) = q(\gamma^t_i, \gamma^*_i)q(Z^t_i, Z^*_i \mid \gamma^*_i) \]  

(7)

be the proposal distribution for the class label and latent position, where \( q(\gamma^t_i, \gamma^*_i) \) is the probability of proposing the class label \( \gamma^*_i \) given current class label \( \gamma^t_i \). The proposal distribution is specified in equation (5).

If \( \gamma^*_i = \gamma^t_i \), we let

\[ q(Z^t_i, Z^*_i \mid \gamma^*_i) = \text{MVN}(Z^*_i; Z^t_i, \delta I_d) \]  

(8)

If \( \gamma^*_i \neq \gamma^t_i \) and \( \sum_{j \neq i} I(\gamma_j = \gamma^*_i) > 0 \), that is, if the proposed class is non-empty, we let

\[ q(Z^t_i, Z^*_i \mid \gamma^*_i) = \text{MVN}(Z^*_i; \overline{Z}_{\gamma^*_i}, \delta I_d) \]  

(9)

where \( \overline{Z}_{\gamma^*_i} \) is the average latent position of nodes in class \( \gamma^*_i \), and we set \( \delta = 1 \).

If \( \gamma^*_i \neq \gamma^t_i \) and \( \sum_{j \neq i} I(\gamma_j = \gamma^*_i) = 0 \), that is, if the proposed class is empty, we let

\[ q(Z^t_i, Z^*_i \mid \gamma^*_i) = \text{MVN}(Z^*_i; 0, \sigma_k^2 I_d). \]  

(10)

That is, we simulate from the prior distribution for the latent position.

Our Metropolis-within-Gibbs algorithm (Metropolis et al., 1953) to sample from the posterior is
then as follows:

**Step 1:**

For vertices $i = 1, 2, \ldots, n$, we use Metropolis-Hastings to update $(\gamma_i, Z_i)$.

1. Propose $(\gamma_i^*, Z_i^*)$ according to equations (7) - (10).

2. With probability equal to

   $$
   \frac{P(Y_N | \gamma^*, Z^*, \tau, \beta)q((\gamma_i^*, Z_i^*), (\gamma_t^*, Z_t^*))p(\gamma_t^*, Z_t^*)}{P(Y_N | \gamma^t, Z^t, \tau, \beta)q((\gamma_i^t, Z_i^t), (\gamma_t^t, Z_t^t))p(\gamma_t^t, Z_t^t)}
   $$

   set $(\gamma_i^{t+1}, Z_i^{t+1}) = (\gamma_i^*, Z_i^*)$. Otherwise, set $(\gamma_i^{t+1}, Z_i^{t+1}) = (\gamma_t^t, Z_t^t)$.

**Step 2:**

For $k = 1, 2, \ldots, c$, we use Metropolis-Hasting to update $\beta_k$.

1. Propose $\beta_k^* \sim N(\beta_k^t, \delta_\beta)$.

2. With probability equal to

   $$
   \frac{P(Y_N | \gamma, Z, \tau, \beta^*)p(\beta_k^*; \mu_k^\beta, (\sigma_k^\beta)^2)}{P(Y_N | \gamma, Z, \tau, \beta^t)p(\beta_k^t; \mu_k^\beta, (\sigma_k^\beta)^2)}
   $$

   set $\beta_k^{t+1} = \beta_k^*$. Otherwise, set $\beta_k^{t+1} = \beta_k^t$.

**Step 3:**

Update $\pi$ and $\tau$ according to the conditional distributions given in (2) and (3).

**5.4 Model identifiability**

The likelihood function is invariant to reflections, rotations, and translations of the latent spaces and permutation of class labels. As in the case of latent position cluster model (Handcock et al., 2007), the non-identifiability issues can be resolved by Procrustes transformation (Sibson, 1979) and post-processing the MCMC output. We first fix a reference set of latent positions for the nodes, which can be obtained by a standard method of multidimensional scaling to one minus the
adjacency matrix of the network. At each iteration of MCMC sampling, we perform a Procrustes transformation with translation and rotation to minimize the distance between simulated and reference latent positions. We then post-process the MCMC output by applying the relabelling algorithm (Celeux et al., 2000) to solve the label switching problem.

6 Model selection

A number of model selection criteria and algorithms have been proposed for selecting the latent space dimension and number of clusters in the latent position cluster model. The majority of methods are concerned with choosing the appropriate number of clusters. Handcock et al. (2007) used a BIC approximation to the conditional posterior model probabilities of the latent position cluster model, where the conditioning is on an estimate of latent positions. For each value of the number of clusters, they computed the conditional posterior model probabilities and chose the value of the number of clusters which returned the highest probability. Ryan et al. (2017) proposed a Bayesian model selection method for the latent position cluster model by collapsing the model to integrate out the model parameters which allows posterior inference over the number of components.

Likewise, a few model selection criterion have been proposed for selecting the number of groups for a stochastic blockmodel. Côme and Latouche (2015) proposed the integrated classification likelihood criterion. Their focus was on choosing the number of classes and maximum a posteriori estimates of the class labels for each observation with other parameters integrated out. Wang and Bickel (2017) proposed a penalized likelihood criterion for selecting the optimal number of blocks.

In this paper, we adopt the Watanabe-Akaike information criterion (WAIC) (Watanabe, 2010) to choose the number of blocks for the generalized latent space stochastic blockmodel. WAIC is a method for estimating point-wise out-of-sample prediction accuracy from a fitted Bayesian model and can be interpreted as a computationally convenient approximation to cross-validation by using posterior simulations (Gelman et al., 2014).
Let \( \{(\gamma^t, Z^t, \tau^t, \beta^t)\}_{t=1}^{S} \) denote the posterior samples, and let \( V_t^{S} \log P(Y_N | \gamma^t, Z^t, \tau^t, \beta^t) \) denote the sample variance of the log-likelihood evaluated at the posterior samples. Then the WAIC for the generalized latent space stochastic blockmodel is given by

\[
\text{WAIC} = \log \left( \frac{1}{S} \sum_{t=1}^{S} P(Y_N | \gamma^t, Z^t, \tau^t, \beta^t) \right) - V_t^{S} \log P(Y_N | \gamma^t, Z^t, \tau^t, \beta^t) \tag{11}
\]

\[
= \text{(prediction accuracy)} - \text{(penalty)} .
\]

The first term on the right-hand side of equation (11) measures the model fit, while the second term penalizes model complexity; models with higher WAIC value are preferred.

### 7 Irish Politician Twitter Data Structure

We fit the generalized latent space stochastic blockmodel with varying number of blocks and dimensions to the Irish politicians Twitter follower data. We perform model selection using WAIC. The results are given in Table 2.

We note that a zero-dimensional latent space corresponds to a stochastic blockmodel, and a 1-block model corresponds to the latent space model. Based on the results in Table 2, a generalized latent space stochastic blockmodel with nine blocks and two-dimensional latent spaces is selected.

The posterior mean of the block-to-block connection probabilities between the nine blocks calculated from the MCMC samples are provided in Table 3. The off-diagonal entry \((k, l)\), \(k \neq l\) represents the probability that a node from block \(k\) follows a node from block \(l\). The diagonal elements give the range of within block connection probabilities for nodes within the blocks.

It can be seen that the block-to-block connection probabilities exhibit considerable asymmetry. For example, the probability that a member of block 1 connects with block 2 is \(\tau_{12} = 0.14\), whereas the probability that a member of block 2 connects with block 1 is \(\tau_{21} = 0.51\).

Furthermore, some between-block connection probabilities give evidence of disassortative mixing. For example, some politicians within block 1 have probability 0.12 of connecting with each other, whereas the probability of connecting with a politician from block 8 is 0.33. Likewise, some
Table 2: Model selection for the generalized latent space stochastic blockmodel using WAIC. The model with the highest WAIC is the most preferred (marked in bold).

| Dimension | No. of Blocks | Pred | Penalty | WAIC  |
|-----------|---------------|------|---------|-------|
| 0         | 1             | -13196 | 1      | -13197 |
| 0         | 2             | -10992 | 5      | -10997 |
| 0         | 3             | -10128 | 5      | -10123 |
| 0         | 4             | -9468  | 9      | -9477  |
| 0         | 5             | -9089  | 14     | -9103  |
| 0         | 6             | -8850  | 20     | -8870  |
| 0         | 7             | -8526  | 25     | -8551  |
| 0         | 8             | -8396  | 37     | -8434  |
| 0         | 9             | -8145  | 46     | -8191  |
| 0         | 10            | -8044  | 55     | -8191  |
| 1         | 1             | -10251 | 106    | -10357 |
| 1         | 2             | -9199  | 89     | -9288  |
| 1         | 3             | -8809  | 91     | -8900  |
| 1         | 4             | -8637  | 90     | -8727  |
| 1         | 5             | -8524  | 84     | -8609  |
| 1         | 6             | -8489  | 83     | -8573  |
| 1         | 7             | -8514  | 91     | -8605  |
| 1         | 8             | -8267  | 85     | -8352  |
| 1         | 9             | -7905  | 96     | -8001  |
| 1         | 10            | -8137  | 103    | -8240  |
| 2         | 1             | -8562  | 179    | -8811  |
| 2         | 2             | -8466  | 172    | -8638  |
| 2         | 3             | -8385  | 163    | -8548  |
| 2         | 4             | -8166  | 174    | -8340  |
| 2         | 5             | -8155  | 170    | -8325  |
| 2         | 6             | -8121  | 166    | -8287  |
| 2         | 7             | -8112  | 168    | -8280  |
| 2         | 8             | -7985  | 170    | -8155  |
| 2         | 9             | -7586  | 161    | -7748  |
| 2         | 10            | -7718  | 169    | -7887  |
Table 3: Posterior mean block-to-block connection probabilities. For the within block connections, the range of estimated connection probabilities is shown. Probabilities greater than .5 are highlighted.

| From/To | 1        | 2        | 3        | 4        | 5        | 6        | 7        | 8        | 9        |
|---------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 1       | 0.12–1.00| 0.14     | 0.19     | 0.84     | 0.70     | 0.03     | 0.07     | 0.33     | 0.68     |
| 2       | 0.51     | 0.21–1.00| 0.30     | 0.06     | 0.06     | 0.09     | 0.14     | 0.30     | 0.86     |
| 3       | 0.49     | 0.20     | 0.01–1.00| 0.21     | 0.05     | 0.02     | 0.38     | 0.15     | 0.86     |
| 4       | 0.31     | 0.07     | 0.06     | 0.00–1.00| 0.08     | 0.03     | 0.01     | 0.29     | 0.34     |
| 5       | 0.84     | 0.10     | 0.08     | 0.04     | 0.00–1.00| 0.02     | 0.02     | 0.35     | 0.45     |
| 6       | 0.38     | 0.41     | 0.20     | 0.16     | 0.09     | 0.00–1.00| 0.06     | 0.28     | 0.65     |
| 7       | 0.60     | 0.25     | 0.88     | 0.02     | 0.07     | 0.01     | 0.00–0.96| 0.26     | 0.92     |
| 8       | 0.87     | 0.48     | 0.27     | 0.43     | 0.41     | 0.11     | 0.15     | 0.77–1.00| 0.95     |
| 9       | 0.72     | 0.52     | 0.82     | 0.05     | 0.12     | 0.05     | 0.42     | 0.45     | 0.54–1.00|

politicians in block 8 have probability 0.77 of connecting with each others, whereas the probability of them connecting with a politician from block 1 is 0.88.

The posterior block allocation probabilities are presented in Table 4, where we can see that blocks 2, 3, 4 and 7 contain more members than the other blocks.

Table 4: Posterior mean of the block allocation probabilities.

| Block | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|       | 0.07| 0.15| 0.19| 0.15| 0.09| 0.08| 0.14| 0.06| 0.08|

Table 5 presents the relationships between block memberships and political party memberships.

Table 5: Political party and maximum a posteriori block membership.

|          | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|----------|---|---|---|---|---|---|---|---|---|
| Fianna Fáil | 5 | - | - | 2 | 9 | - | - | 1 | - |
| Fine Gael  | - | - | 30| 1 | - | 2 | 21| - | 7 |
| Independent| 2 | 1 | 2 | 5 | 2 | 2 | 2 | 1 | - |
| Labour     | - | 24| - | - | - | 3 | - | - | 2 |
| People Before Profit Alliance | - | - | - | 1 | - | - | - | - | - |
| Remua Ireland | - | - | 2 | - | - | - | - | - | - |
| Sinn Fén   | - | - | 12| - | - | - | - | - | - |
| Social Democrats | - | - | - | 1 | - | 2 | - | - | - |
| Socialist Party | - | - | 3 | - | - | - | - | - | - |
| United Left | - | - | 1 | - | 1 | - | - | - | - |

The two parties in government (Fine Gael and Labour) account for most of the members of blocks 2, 3 and 7 whereas block 4 contains all of the members of the opposition party, Sinn
Féin. Interestingly, most Fine Gael politicians fall in to block 3 or block 7. The block-to-block interaction probabilities of these blocks are $\tau_{37} = 0.38$ and $\tau_{73} = 0.88$, so the two blocks are capturing asymmetry in the Twitter following within the party.

It is worth noting that when we compare the block membership of the 9-block GLSSBM model to the 8-block GLSSBM model, in the 8-block model most of the Fine Gael politicians fall into a single block; adding the extra block finds extra structure in this group of politicians.

By examining the maximum $a$ posteriori block membership of each party member, we developed text descriptors for each block; these are given in Table 6.

| Block | Descriptor |
|-------|------------|
| 1     | Fianna Fáil and Fianna Fáil leaning independents |
| 2     | Labour and former Labour |
| 3     | Fine Gael and former Fine Gael |
| 4     | Sinn Féin and Left |
| 5     | Fianna Fáil (mainly Munster) |
| 6     | Fine Gael and Labour (Government + former) |
| 7     | Fine Gael and former (Rural) |
| 8     | High profile independents |
| 9     | Fine Gael and Labour (Government, Suburban) |

We can see that the block membership is largely dictated by party membership, but with some aspects of geography too.

The posterior mean latent positions for each node within its maximum $a$ posteriori block membership is shown in Figure 2. The locations of nodes within the block allow us to see additional structure among politicians, beyond what is possible with a stochastic blockmodel.

There are two Renua members and two independent members located within block 3, which is dominated by Fine Gael. These members were former members of the party and either left or were expelled from the party during the sitting of the 31st Dáil because of their stance taken on the issue of abortion; thus these independents and Renua members closely correspond to the apostate independent category of Weeks (2009) because they are following the party that they were affiliated with in the past. Similarly, in block 2, the independent member (Broughan) is a former member of the Labour party. Broughan left the party in 2014 because of disagreements
Figure 2: Posterior mean latent positions within each block. The politicians are colored by political party. A scale has been added to each plot to show how the link probabilities depend on distance; the left of the scale is the highest probability and the probability decreases as the distance from the left increases on the scale.
over government economic policy.

Whilst block 5 is dominated by Fianna Fáil members, there are two independent members in the block who have a historical track record of voting with Fianna Fáil. Interestingly, Healy-Rae is the son of an apostate independent who failed to get a nomination to run for Fianna Fáil in 1997; however, Healy-Rae would be categorized as a community independent using the typology of Weeks (2009). Lowry, the other independent in this block, was expelled from the Fine Gael party in 1994, but would now be considered to be a community independent under Weeks’s typology.

The members of block 8 are high profile independent or small party politicians. Ross is a high profile journalist who ran on a change platform, and it could be argued that he is a national single-interest independent in the typography of Weeks (2009). Donnelly and Murphy were originally independent candidates who formed the Social Democrat party with Shortall (in block 6). Keaveney was a member of the Labour party when elected but left the party in 2013 and joined Fianna Fáil.

As a next step in the analysis, we considered each vote in the Dáil and grouped all pairs of politicians into three groups on the basis of their voting agreement: agree, disagree or at least one was absent. Kernel density estimates of the edge probabilities were formed for each group and each vote. The resulting plot for one vote is shown in Figure 3; the equivalent plot for the remaining 22 votes has similar structure (see Appendix A).

We can see from the plot that the politicians who agree in the Dáil votes tend to have higher edge probabilities than those who disagree, with the absent group having intermediate probabilities. Thus, the model edge probabilities have some predictive importance in terms of voting in the Dáil. This result can be partially explained by the effect of political party on Twitter following and on the effect of political party on voting in the Dáil (Farrell et al., 2015). However, the result helps us understand the role that independent politicians play in the Dáil.

Finally, we completed a 10-fold cross-validation study to assess the model performance for edge prediction. The results are shown in a box plot in Figure 4, where it can be seen that the edge probabilities from the GLSSBM model are better for edge prediction than the SBM model, but the LSM model is very competitive with the GLSSBM model.
8 Discussion

A new model has been proposed in this paper, namely, the generalized latent space stochastic blockmodel. The model generalizes both the latent space model as well as the stochastic blockmodel and simultaneously captures transitivity, clustering and disassortative mixing of networks. A fully Bayesian approach using MCMC sampling was used to estimate the class memberships, latent positions, and model parameters.

The model has enabled discovering interesting structure in the Irish politician Twitter following network. The resulting estimated model parameters, estimated block memberships and estimated latent positions reveal the relationships between political parties, within political parties and the relationship between independent politicians and small party politicians and the main parties of the Dáil. Most interestingly, many apostate independents remained a member of the block dominated by the party for which they are former members.
The generalized latent space stochastic blockmodel can be extended in a number of ways. It could be extended to model integer-valued networks by assuming the number of edges between each pair follows a Poisson or negative binomial distribution. Furthermore, covariates could be incorporated into the model through the use of node-specific covariates and edge covariates.

Additionally, although Bayesian estimation with MCMC sampling is highly flexible, it is computationally prohibitive and unsuitable for larger networks. If the main interest is in the posterior mode, variational methods (eg. Smidl and Quinn, 2006; Bishop, 2006) may be more appropriate; this approach to inference is well established some other network models (eg, Airoldi, 2006; Salter-Townshend and Murphy, 2013).

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A Kernel Density Estimates for Edge Probabilities

The kernel density estimates of the edge probabilities for each of the votes from January 1st, 2016 to February 6th, 2016.
Vote 19

Vote 20

Vote 21

Vote 22
B Traceplots from MCMC chain

B.1 Traceplots for $\gamma$
B.2 Traceplots for $\beta$
B.3 Traceplots for $\tau$

- Traceplot for $\tau[1,2]$
- Traceplot for $\tau[1,3]$
- Traceplot for $\tau[1,4]$
- Traceplot for $\tau[1,5]$
- Traceplot for $\tau[1,6]$
- Traceplot for $\tau[1,7]$
- Traceplot for $\tau[1,8]$
- Traceplot for $\tau[1,9]$