From Hamiltonians to complex symplectic transformations

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Abstract. Gaussian unitaries are specified by a second order polynomial in the bosonic operators, that is, by a quadratic polynomial and a linear term. From the Hamiltonian other equivalent representations of the Gaussian unitaries are obtained, such as Bogoliubov and real symplectic transformations. The paper develops an alternative representation, called complex symplectic transformation, which is more compact and is comprehensive of both Bogoliubov and real symplectic transformations. Moreover, it has other advantages. One of the main results of the theory, not available in the literature, is that the final displacement is not simply given by the linear part of the Hamiltonian, but depends also on the quadratic part. In particular, it is shown that by combining squeezing and rotation, it is possible to achieve a final displacement with an arbitrary amount.

Symbols and terminology

:= equal by definition

\( I_n \) identity operator of \( \mathcal{H} \)

\( I_n \) identity matrix of size \( n \)

\( A^\dagger \) adjoint of operator \( A \)

\( \overline{x} \) complex conjugate of the scalar \( x \)

\( A^T \) transpose of matrix \( A \)

\( \bar{A} \) conjugate of the matrix \( A \)

\( \mathcal{H} = \mathcal{H}^{\otimes n} \) bosonic Hilbert space

\( q_i \) and \( p_i \) quadrature operators of the \( i \)th mode

\( a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad a_* = \begin{bmatrix} a_1^* \\ \vdots \\ a_n^* \end{bmatrix} \) vectors of creation and annihilation operators

\( \xi = \begin{bmatrix} a \\ a_* \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \\ a_1^* \\ \vdots \\ a_n^* \end{bmatrix} \) vector of bosonic operators

\( \Omega = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix} \) fundamental symplectic matrix

\( H = \frac{1}{2} \xi^\dagger \mathcal{H} \xi + \xi^\dagger h \) quadratic Hamiltonian

\( (H, h) \) matrix representation of \( H \)

\( S = \begin{bmatrix} E & F \\ F^T & T \end{bmatrix} \)

\( S_0 \) real symplectic matrix

\( D(\alpha), \alpha \in \mathbb{C}^n \) \( n \)-mode displacement operator

\( R(\phi), \phi \) Hermitian matrix \( n \)-mode rotation operator

\( Z(z), z \) symmetric matrix \( n \)-mode squeeze operator

The special symbol \( a_* \) is introduced to denote the column vector of the \( n \) creation operators. The reason is that, in our conventions, \( a^\dagger \) is the conjugate transpose of the column vector \( a \) and therefore it denotes a row vector. The overline denotes the complex conjugate. Boldface is used only for the matrices \( H \) and \( S \) and for the column vectors \( h \) and \( s \).

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I. INTRODUCTION

In the past three decades the attention to multimode Gaussian states and transformations has obtained an increasing interest from both the theoretical and the application point of view. Concepts and protocols, such as entanglement and teleportation, initially intended only for discrete quantum systems, have been extended to continuous variable systems, allowing more efficient implementation and measurements, in particular in the applications of linear optics. As a consequence, the characterization of Gaussian states and transformations plays a fundamental role, as witnessed by the large number of review and tutorial papers devoted to this topic (see f.i. [1–7]).

Gaussian states are traditionally characterized by waveforms or by Wigner functions, expressed as Gaussian functions of the canonical position and momentum coordinates. In this approach, Gaussian transformations are defined as unitary transformations that preserve the Gaussian nature of the states. In a different approach, adopted in this note, suitable definitions are given for Gaussian transformations, and Gaussian states are defined as obtained by Gaussian transformations applied to the vacuum state (pure Gaussian states) or, more generally, to thermal states (mixed Gaussian states).

We consider an $n$–mode bosonic system characterized by a Hilbert space $\mathcal{H} = \mathcal{H}_0^n$ with annihilation operators $a_1, \ldots, a_n$ and creation operators $a_1^*, \ldots, a_n^*$, satisfying the usual bosonic commutation relations $[a_i, a_j] = [a_i^*, a_j^*] = 0$ and $[a_i, a_j^*] = \delta_{ij}$. In this context the Gaussian unitaries may have several specifications, as shown in Fig. 1:

1) **Hamiltonian specification**, given by a second–order polynomial in the bosonic operators;
2) **Bogoliubov specification**, based on Bogoliubov transformations;
3) **real symplectic specification** in the phase space,
4) **complex symplectic specification** in the phase space.

These specifications are equivalent in representing the whole class of Gaussian unitaries in the sense that it is possible to obtain any specification from the others [12].

The Hamiltonian specification is supported by a fundamental theorem [8], which states that a unitary operator $U = e^{-iH}$, where the Hamiltonian $H$ is a second–order polynomial in the bosonic operators $a_i^*$ and $a_i$, is a Gaussian unitary. Then $H$ can be handled using a matrix representation $(H, h)$, having the structure

$$H = \begin{bmatrix} A & B \\ B^T & A \\ \end{bmatrix}, \quad h = \begin{bmatrix} h \\ 0 \\ \end{bmatrix}$$ (1)

where in the $n$–mode $H$ is a $2n \times 2n$ complex matrix and $h$ is a $2n$ complex vector. From Hamiltonians one can derive Bogoliubov transformations, which usually are formulated in terms of the vectors $a, a^*$ containing the bosonic operators $a_i, a_i^*$ and have the form

$$a \rightarrow Ea + Fa^* + s$$ (2)

where $E, F$ are $n \times n$ complex matrices and $s$ is an $n$ complex vector. In this paper we consider a more efficient approach where the two bosonic vectors $a$ and $a^*$ are stored in a single vector $\xi$ of size $2n$, thus obtaining the compact form

$$\xi \rightarrow S \xi + s, \quad \xi = \begin{bmatrix} a \\ a^* \end{bmatrix}$$ (3)

where $(S, s)$ have the same structure and dimension as $(H, h)$ in (1).

$$S = \begin{bmatrix} E & F \\ F & E \end{bmatrix}, \quad s = \begin{bmatrix} s \\ 0 \end{bmatrix}$$ (4)
The form (3) is developed in [7] and, although achieved with a trivial recast of symbols, has several advantages with respect to the traditional Bogoliubov form (2), namely: 1) $S$ is directly given by an exponential of $H$ as $S = e^{-iH}$, 2) $S$ generates directly the Bogoliubov matrices $E$ and $F$, as indicated in (4), and 3) $S$ gives directly the traditional real symplectic matrix $S_0$ of the phase space. In other words, the compact form (3) provides both the Bogoliubov transformation and the passage to the phase space. For these reasons we call $S$ complex symplectic matrix and the compact form (3) complex symplectic transformation.

In this paper a particular attention is also devoted to the linear terms $h$ appearing in the Hamiltonian and $s$ in the complex symplectic transformation. For convenience we call $s$ displacement amount or simply displacement. In the review papers the linear term is often ignored by saying that it may be compensated by local operations. In other cases the Hamiltonians and the Bogoliubov transformations are presented as alternative representations. This may inspire the erroneous idea that the displacements are exclusively due to the linear part $h$ of the Hamiltonian representation: in symbols $s = h$. On the contrary, in the deduction of the complex symplectic transformation we have discovered that the displacement $s$ depends not only on the linear term but also, in a relevant manner, on the quadratic term.

The paper is organized as follows. In Section II we introduce the matrix representation $(H, h)$ of a quadratic Hamiltonian and then we prove the fundamental result on the complex symplectic transformation, giving the symplectic pair $(S, s)$. The proof is based on a sophisticated application of the Hadamard Lemma. In particular we will find that the complex symplectic matrix $S$ depends only on the quadratic part $H$ of the Hamiltonian, while the displacement amount $s$ depends both on $H$ and $h$. We also derive from the complex symplectic matrix $S$ the standard real symplectic matrix $S_0$. In Section III we apply the theory of complex symplectic transformations to the fundamental Gaussian unitaries (displacement, rotation, and squeezing) with the main target of illustrating the dependence of the displacement amount $s$ on both $H$ and $h$. In Section IV we will discuss the possibility of amplification of the displacement amount by combining rotation and squeezing. The appendix collects a few theoretical topics related to the complex symplectic transformations.

II. GENERATION OF COMPLEX SYMPLECTIC TRANSFORMATIONS

II.1. Specification of a quadratic Hamiltonian

A Hamiltonian $H$ given by a second–order polynomial in the bosonic operators can be written in the form

$$H = \frac{1}{2} \sum_{r,s=1}^{n} [A_{rs}a_{r}^*a_{s} + \overline{A}_{rs}a_{r}a_{s}^*] + \frac{1}{2} \sum_{r,s=1}^{n} [B_{rs}a_{r}^*a_{s} + \overline{B}_{rs}a_{r}a_{s}^*] + \sum_{r=1}^{n} (h_{r}a_{r}^* + \overline{h}_{r}a_{r}) .$$

Collecting the coefficients $A_{rs}$, $B_{rs}$ and $h_{r}$ in the matrices $A$, $B$, and in the column vector $h$, the Hamiltonian takes the compact form

$$H = \frac{1}{2} \xi^{*} H \xi + \xi^{*} h$$

where

$$\xi := \begin{bmatrix} a \\ a_{*} \end{bmatrix} \quad \text{with} \quad a = \begin{bmatrix} a_{1} \\ \vdots \\ a_{n} \end{bmatrix}, \quad a_{*} = \begin{bmatrix} a_{1}^{*} \\ \vdots \\ a_{n}^{*} \end{bmatrix}$$

and

$$H = \begin{bmatrix} A & B \\ B & A \end{bmatrix}, \quad h = \begin{bmatrix} h \\ \overline{h} \end{bmatrix} .$$

$H$ is a $2n \times 2n$ Hermitian matrix and $h$ is a $2n$ complex column vector. The pair $(H, h)$ gives the matrix representation of $H$. The Hermitian nature of the Hamiltonian implies the conditions

$$A = A^{*}, \quad B = B^{T},$$

that is, $A$ must be Hermitian and $B$ must be symmetric.

For later use it is convenient to decompose the Hamiltonian (6) into the quadratic and the linear parts, namely

$$H = H_{q} + H_{\ell} \quad \text{with} \quad H_{q} = \frac{1}{2} \xi^{*} H \xi, \quad H_{\ell} = \xi^{*} h .$$
II.2. The fundamental result

**Theorem 1.** A Gaussian unitary $U = e^{-iH}$ with $H = \frac{1}{2} \xi^* \xi + h^* \xi$ applied to the vector $\xi$ of bosonic operators gives the affine transformation

$$e^{iH} \xi e^{-iH} = S \xi + s ,$$

where

$$S = e^{-i\Omega H} , \quad s = \Psi h$$

with

$$\Omega = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}$$

($I_n$ is the identity matrix of order $n$) and

$$\Psi = \sum_{n=1}^{\infty} \frac{1}{n!} (-i\Omega H)^{n-1} (-i\Omega) = (S - I_{2n})H^{-1} .$$

Note that (13) holds when the matrix $H$ is invertible. However, the series can be summed in a closed form also when $H$ is singular, as shown in Appendix A.

The theorem states the passage from the bosonic representation $(H, h)$ to the complex symplectic representation $(S, s)$, where $S$ is a $2n \times 2n$ complex symplectic matrix and $s$ is a $2n$ complex vector. For convenience we call $s$ “displacement” or “displacement amount”.

The structure of the matrix $H$ in (8) is invariant with respect to matrix addition and multiplication. Since this is also the structure of the matrix $-i\Omega$, it follows that the matrices $S$ and $\Psi$ have the same structure as $H$ and $s$ has the same structure as $h$. Namely,

$$S = \begin{bmatrix} E & F \\ F & E \end{bmatrix} , \quad \Psi = \begin{bmatrix} P & Q \\ Q & P \end{bmatrix} , \quad s = \begin{bmatrix} s \\ \bar{s} \end{bmatrix} .$$

In terms of blocks (11) reads

$$e^{iH} \begin{bmatrix} a \\ a^* \end{bmatrix} e^{-iH} = \begin{bmatrix} E & F \\ F & E \end{bmatrix} \begin{bmatrix} a \\ a^* \end{bmatrix} + \begin{bmatrix} s \\ \bar{s} \end{bmatrix}$$

that is,

$$e^{iH} a e^{-iH} = E a + F a^* + s \quad e^{iH} a^* e^{-iH} = E a^* + F a + \bar{s}$$

with

$$s = P h + Q \bar{h} .$$

The complex symplectic matrix $S$ satisfies the condition

$$S \Omega S^* = \Omega .$$

This equation is related to the fact that the Gaussian transformation preserves the above cited bosonic correlation relations. As a matter of fact, these relations applied to the vector bosonic operator $\xi$ may be expressed in the compact form $[\xi, \xi^*] = \Omega$, from which for the output bosonic operator $S \xi + s$ one gets

$$[S \xi + s, \xi^* S^* + s^*] = S [\xi, \xi^*] S^* = S \Omega S = \Omega ,$$

so that the bosonic correlation relations are preserved.

Two particular cases are of interest:
1) \( h = 0 \): the Hamiltonian reduces to the quadratic part \((H = H_q)\) and one gets the linear transformation
\[
e^{iH_0} \xi e^{-iH_0} = S \xi .
\]

2) \( H = 0 \): the Hamiltonian reduces to the linear part \((H = H_Q)\) and one gets the simple displacement
\[
e^{iH_0} \xi e^{-iH_0} = \xi - i\Omega h .
\]

**Remark.** In this paper we do not make any reference to Lie groups and algebras [10], but it would be easy to verify that the matrix \(-\Omega H\), with \(H\) as in (8) satisfying conditions (9) form a Lie algebra, which through \(S = e^{-i\Omega H}\) generates the complex symplectic Lie group of the matrices \(S\) satisfying the condition (19).

II.3. Proof of Theorem 1

With the Hamiltonian \(H\) decomposed as in (10) one gets the commutation relations
\[
[i H_\ell , a_k] = i[a^*_\ell h, a_k] = -i h_k e_k , \quad [i H_\ell , a^*_k] = i[a^*_\ell \Omega h, a_k^*] = i h_k^* e_k
\]
with \(e_k\) a \(n\)-vector with zero entries except a 1 entry in the \(k\)-th place. In a compact form
\[
[i H_\ell , \xi] = -i \left[ \begin{array}{c} h \\ -h \end{array} \right] = -i \Omega h .
\]
Moreover one gets
\[
[i H_q , a_k] = \frac{i}{2} \left[ \xi H_\xi , a_k \right] = \frac{i}{2} [a^* Aa + a^* B a_* + a^* \Omega B a_* + a^* \Omega A a_* , a_k]
\]
\[
= \frac{i}{2} \left\{ [a^* , a_k] Aa + [a^* , a_k] B a_* + a^* B [a_* , a_k] + a^* \Omega A [a_* , a_k] \right\}
\]
\[
= -\frac{i}{2} (a_k A a + e_k B a_* + e_k B^T a_* + e_k A^* a) = -i (e_k A a + e_k B a_*)
\]
and in compact form
\[
[i H_q , a] = -i (A a + B a_*) .
\]
A similar computation gives
\[
[i H_q , a_*] = i (\Omega A a_* + \Omega B a)
\]
and combining (24) and (25) yields
\[
[i H_q , \xi] = i \left[ \begin{array}{cc} -A & -B \\ B^T & \Omega \end{array} \right] \xi = -i \Omega H \xi .
\]
The above relations are used in the Hadamard identity allowing one to write
\[
e^{iH} \xi e^{-iH^*} = \sum_{n=0}^{\infty} \frac{1}{n!} D_n ,
\]
where the operator vectors \(D_n\) are evaluated by setting \(D_0 = \xi\) and recursively \(D_{n+1} = [iH , D_n]\). Recursion gives
\[
D_n = (-i\Omega H)^n \xi + (-i\Omega H)^{n-1} (-i\Omega h) .
\]
Indeed
\[
D_1 = [iH , \xi] = [i H_q , \xi] + [i H_\ell , \xi] = -i \Omega H \xi - i\Omega h
\]
and (28) holds true for \(n = 1\). Moreover, provided that (28) holds true for \(n\),
\[
D_{n+1} = [iH , D_n] = [i H_q , (-i\Omega H)^n \xi] + (-i\Omega H)^n [iH , \xi] = (-i\Omega H)^n D_1 .
\]
It follows
\[
e^{iH} \xi e^{-iH^*} = \sum_{n=0}^{\infty} \frac{1}{n!} (-i\Omega H)^n \xi + \sum_{n=1}^{\infty} \frac{1}{n!} (-i\Omega H)^{n-1} (-i\Omega h)
\]
\[
= e^{-i\Omega H} \xi + \sum_{n=1}^{\infty} \frac{i^n}{n!} (\Omega H)^{n-1} (-i\Omega h) = e^{-i\Omega H} \xi + s .
\]
II.4. The fundamental Gaussian unitaries

To proceed it is convenient to introduce the fundamental Gaussian unitaries (FUs). These unitaries were formulated for multimode systems by Ma and Rhode [8], through the following definitions:

1) **Displacement operator**

\[ D(\alpha) := e^{\alpha^T a^* - a^* a}, \quad \alpha = [\alpha_1, \ldots, \alpha_n]^T \in \mathbb{C}^n \]  

(32)

which is the same as the Weyl operator.

2) **Rotation operator**

\[ R(\phi) := e^{i \alpha a^*}, \quad \phi \quad n \times n \text{ Hermitian matrix} \]  

(33)

3) **Squeeze operator** [15]

\[ Z(z) := e^{\frac{1}{2}(a^* z a^* - a^T z^* a)}, \quad z \quad n \times n \text{ symmetric matrix} \]  

(34)

The importance of these operators, illustrated in Fig. 2, is established by the following:

**Theorem 2** [8]. The most general Gaussian unitary is given by the combination of the three fundamental Gaussian unitaries \( D(\alpha), Z(z), \) and \( R(\phi), \) cascaded in any arbitrary order, that is,

\[ Z(z) D(\alpha) R(\phi), \quad R(\phi) D(\alpha) Z(z), \quad \text{etc.} \]  

(35)

It is interesting to remark that the above definitions can be obtained from the matrix representation of the Hamiltonian

\[ H = \begin{bmatrix} A & B \\ B & A \end{bmatrix}, \quad h = \begin{bmatrix} h \\ h \end{bmatrix} \rightarrow H = \frac{1}{2} \xi^* H \xi + \xi^* h. \]

by setting to zero two of the submatrices \( A, B, h. \) Specifically (see Appendix B):

1) **Displacement** with \( A = B = 0 \) and \( h \in \mathbb{C}^n \)

\[ H = \xi^* h = h^T a^* + h^* a \quad \rightarrow \quad D(\alpha) = e^{-iH} = e^{\alpha^T a^* - a^* a}, \quad \alpha = -ih \]  

(36)

2) **Rotation** with \( B = 0, \) \( h = 0 \) and \( A \) \( n \times n \) Hermitian

\[ H = \frac{1}{2}(a^* A a + a^T A^* a^*) \quad \rightarrow \quad R(\phi) = e^{-iH} = e^{i \alpha a^*}, \quad \phi = -A \]  

(37)

3) **Squeezing** with \( A = 0, \) \( h = 0 \) and \( B \) \( n \times n \) symmetric

\[ H = \frac{1}{2}(a^* B a + a^T B^* a^*) \quad \rightarrow \quad Z(z) := e^{\frac{1}{2}(a^* z a^* - a^T z^* a)}, \quad z = -iB. \]  

(38)

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**Gaussian unitaries**

- **\( D(\alpha) \)**: displacement
- **\( R(\phi) \)**: rotation
- **\( Z(z) \)**: squeezing

**Fig. 2** – The class of Gaussian unitaries with the subclasses of displacement, rotation, and squeezing.
We consider in particular the displacement. In the above definition $D(\alpha)$ is formulated in terms of the bosonic vectors $a, a^\star$. But for the interpretation of Theorem 1 is is convenient to give a formulation in terms of the single vector $\xi$, as

$$D_Y(\xi) := e^{\xi^\star \Omega}Y \text{ with } Y = \begin{bmatrix} y \\ \bar{y} \end{bmatrix}, \ y \in \mathbb{C}^n.$$  

(39)

This definition ensures that $D_Y(\xi)$ provides the shift of $Y$ indicated in the symbol

$$D_Y(\xi) D_Y(\xi) = \xi + Y.$$  

(40)

On the other hand, the standard form provides the transformations

$$D^\star(\alpha) a D(\alpha) = \alpha + a, \quad D^\star(\alpha) a^\star D(\alpha) = \alpha^\star + \bar{\alpha}$$

which are equivalent to

$$D^\star(\alpha) \xi D(\alpha) = \xi + \begin{bmatrix} \alpha \\ \bar{\alpha} \end{bmatrix}.$$  

(41)

Comparison of (40) and (41) gives

$$Y = \begin{bmatrix} y \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \alpha \\ \bar{\alpha} \end{bmatrix}.$$  

(42)

II.5. Interpretation of the fundamental result

The complex symplectic transformation $S(\xi + s)$ can be expressed in two different ways in terms of the two transformations:

- the linear transformation $S_H(\xi) = e^{-i\Omega H} \xi$, with input $\xi$ and output $S(\xi)$,
- a displacement $D_Y(\xi)$ with input $\xi$ and output $\xi + Y$, where $Y$ is a $2n$ complex vector as in (42).

Then the transformation $S(\xi + s)$ can be expressed by

1) the linear transformation $S_H(\cdot)$ followed by the displacement $D_s(\cdot)$, giving

$$D_s(S_H(\xi)) = D_Y(S(\xi)) = S(\xi + s)$$

or

2) the displacement $D_Y(\cdot)$ with $Y = S^{-1}s$ followed by the linear transformation $S_H(\cdot)$, giving

$$S_H(D_Y(\xi)) = S_H(\xi + Y) = e^{-i\Omega H} \xi + e^{-i\Omega H} Y = e^{-i\Omega H} (\xi + s).$$

The two equivalent interpretations are illustrated in Fig. 3. Note that the linear transformation is common to both cascades, while the displacement is different.

II.6. The complex symplectic matrix of a general Gaussian unitary

For a general Gaussian unitary (see Theorem 2) it is possible to calculate the complex symplectic matrix $S$ in a closed form.

**Theorem 3.** For the general Gaussian unitary $R(\phi)Z(z)D(\alpha)$ the complex symplectic matrix is given by

$$S = \begin{bmatrix} \cosh(r) e^{i\phi} & \sinh(r) e^{i\theta} e^{-i\phi} \\ \sinh(r) e^{-i\theta} e^{i\phi} & \cosh(r) e^{-i\phi} \end{bmatrix}.$$  

(43)

where the squeeze matrix is decomposed in the polar form $z = r e^{i\theta}$. 


**Proof** Consider two \( n \)-mode Hamiltonians \( H_{sq} \) and \( H_{rot} \) given by the matrix representations

\[
H_{sq} = \begin{bmatrix} 0 & iz \\ -iz & 0 \end{bmatrix}, \quad H_{rot} = -\begin{bmatrix} \phi & 0 \\ 0 & \phi^T \end{bmatrix}
\]

where \( z = r \, e^{i\theta} \) is the squeeze matrix and \( \phi \) the rotation matrix. Considering that \( H_{disp} = 0 \), the global symplectic matrix is obtained as

\[
S = S_{sq} S_{rot} = e^{-i \Omega H_{sq}} e^{-i \Omega H_{rot}}. \tag{44}
\]

Now the evaluation of \( S_{rot} \) is immediate

\[
S = e^{-i \Omega H_{rot}} = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi^T} \end{bmatrix}. \tag{45}
\]

For the evaluation of \( S_{sq} \) we use the general formula

\[
S_{sq} = \begin{bmatrix} E_{sq} & F_{sq} \\ -F_{sq}^T & E_{sq}^T \end{bmatrix} \tag{46}
\]

where \( E_{sq} \) and \( F_{sq} \) are evaluated in [8] and read

\[
E_{sq} = \cosh(r), \quad F = \sinh(r) \, e^{i\theta}. \tag{47}
\]

Then combination of the above results gives (43).

**II.7. Relation with real symplectic transformations**

Real symplectic transformations refer to the quadrature operators arranged in the form \( \xi_0 = [q,p]^T \) and has the same affine structure seen for complex symplectic transformation (3), namely

\[
\xi_0 \rightarrow S_0 \xi_0 + s_0. \tag{48}
\]

The relation between the two affine transformations is easily obtained considering that the quadrature operators are related to the bosonic operator as \( q_i = (a_i + a_i^\dagger)/\sqrt{2}, \ p_i = -i \, (a_i - a_i^\dagger)/\sqrt{2}, \) and in compact form

\[
\xi_0 = L \xi \quad \text{with} \quad L = \frac{1}{\sqrt{2}} \begin{bmatrix} I_n & I_n \\ -iI_n & iI_n \end{bmatrix}. \tag{49}
\]

Then, relation \( \xi \rightarrow S \xi + s \) gives (48) with

\[
S_0 = L \, S \, L^*, \quad s_0 = L \, s. \tag{50}
\]
Now it easy to see that the matrix $S_0$ and the vector $s_0$ are real. In fact, (50) gives explicitly
\[
S_0 = \begin{bmatrix} \Re(E + F) & -\Im(E - F) \\ \Im(E + F) & \Re(E - F) \end{bmatrix}, \quad s_0 = \sqrt{2} \begin{bmatrix} \Re s \\ \Im s \end{bmatrix}
\]  
(51)

From (50) we find the symplectic condition for the matrix $S_0$
\[
S_0 \Omega_0 S_0^T = \Omega_0 \quad \text{with} \quad \Omega_0 = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.
\]  
(52)

**Remark.** In this paper, in order to avoid a proliferation of notations, we adopt the symbols $S$ and $\Omega$ which in the literature are usually reserved to the real symplectic matrices appearing in the Gaussian transformations of canonical operators, strictly related to the complex symplectic transformations [9]. As a consequence of the unitary similarity (50) the sets of the real and complex symplectic groups are isomorphic.

**III. APPLICATIONS TO FUNDAMENTAL GAUSSIAN UNITARIES**

We apply the previous theory on complex symplectic transformations to combinations of fundamental Gaussian unitaries. More specifically, we develop the cases:

1) rotation+displacement,
2) squeezing+displacement,
3) squeezing+rotation+displacement (which represents the most general Gaussian unitaries [8][14]).

The results will be expressed in the general $n$ mode and illustrated in the single and in the two mode. We assume that the matrix $H$ is not singular, so that we can apply the closed–form formula (13). The detail of the deduction is given in Appendix C.

**III.1. Rotation+displacement**

The Hamiltonian $H$ is given by the matrix representation
\[
H = -\begin{bmatrix} \phi & 0 \\ 0 & \phi^T \end{bmatrix}, \quad h = \begin{bmatrix} h \\ \bar{h} \end{bmatrix}
\]  
(53)

where $\phi$ is an $n \times n$ Hermitian matrix and $h$ is an arbitrary complex vector of size $n$.

**Proposition 1** The complex symplectic matrix is given by
\[
S = e^{-i\Omega H} = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi^T} \end{bmatrix}.
\]  
(54)

The matrix giving the displacement results in
\[
\Psi = -\begin{bmatrix} (e^{i\phi} - I_n)\phi^{-1} & 0 \\ 0 & (e^{-i\phi^T} - I_n)\phi^{-T} \end{bmatrix}, \quad \phi \in [-\pi, \pi)^n
\]  
(55)

where $\phi^{-T} := (\phi^{-1})^T$.

Expression (55) is a special case of the second of (14) with $P = -(e^{i\phi} - I_n)\phi^{-1}$ and $Q = 0$, so that
\[
s = -(e^{i\phi} - I_n)\phi^{-1} h, \quad \phi \in [-\pi, \pi)^n
\]  
(56)

which depends on the companion Gaussian unitary (in this case the rotation).

**Remark:** periodicity of $S$ and $\Psi$

The matrix $S$ is periodic with respect to the phase matrix $\phi$. In fact, $e^{i(\phi + 2k\pi I_n)} = e^{i\phi} e^{i2\pi k I_n} = e^{i\phi}$, $k \in \mathbb{Z}$. Then, the specification of $\Psi$ can be confined in the $n$–dimensional period $P = [-\pi, \pi)^n$. Also the matrix $\Psi$ should be periodic with the same periodicity, $\Psi(\phi + k I_n) = \Psi(\phi)$. But the expression given by (55) is aperiodic and is correct only if $\phi$ is confined in the period $\phi \in [-\pi, \pi)^n$. On the other hand, it is possible to get an unconstrained expression using the identity
\[
H = -\log \left[ \exp(iH) \right]
\]  
(57)
where with log the principal value of the logarithm should be intended [16]. As we will see in the single mode, the use of identity (57) leads to cumbersome expressions, so that we prefer the aperiodic expressions like (55) with the indication of the validity in a period.

**Example 1.** We discuss (56) in the single mode, where we get

\[
\begin{align*}
H &= \begin{bmatrix} -\phi & 0 \\ 0 & -\phi \end{bmatrix}, \quad -i\Omega H = \begin{bmatrix} i\phi & 0 \\ 0 & -i\phi \end{bmatrix}, \\
S &= \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}, \quad H^{-1} = \begin{bmatrix} -\frac{1}{\phi} & 0 \\ 0 & -\frac{1}{\phi} \end{bmatrix}, \\
\Psi &= \begin{bmatrix} -\frac{1+e^{i\phi}}{\phi} & 0 \\ 0 & -\frac{1+e^{-i\phi}}{\phi} \end{bmatrix}, \\
P(\phi) &= -\frac{1+e^{i\phi}}{\phi} = -ie^{i\phi/2} \frac{\sin(\phi/2)}{\phi/2}, \quad \phi \in [-\pi, \pi).
\end{align*}
\]

This is the “aperiodic” solution which is correct only in the interval \(\phi \in [-\pi, \pi)\). To get an unconstrained periodic expression we can use the identity (57), which gives

\[
\begin{align*}
H &= \begin{bmatrix} i \log(e^{-i\phi}) & 0 \\ 0 & i \log(e^{i\phi}) \end{bmatrix}, \quad H^{-1} = \begin{bmatrix} \frac{i}{\log(e^{-i\phi})} & 0 \\ 0 & \frac{i}{\log(e^{i\phi})} \end{bmatrix}, \\
\Psi_0 &= \begin{bmatrix} i \frac{1+e^{i\phi}}{\log(e^{-i\phi})} & 0 \\ 0 & i \frac{1+e^{i\phi}}{\log(e^{i\phi})} \end{bmatrix}, \quad P_0(\phi) = -i \frac{1+e^{i\phi}}{\log(e^{i\phi})},
\end{align*}
\]

Note that \(P_0(\phi)\) is periodic as shown in Fig. 4.

**Fig. 4 — Plot of \(P(\phi)\) (aperiodic) and \(P_0(\phi)\) (periodic)**

**Example 2 (Beam splitter).** A beam splitter is modeled as a two–mode rotation operator with rotation matrix

\[
\phi = \begin{bmatrix} 0 & -i\beta \\ i\beta & 0 \end{bmatrix}, \quad \beta \in [-\pi, \pi)
\]

(59)

The corresponding matrices \(H\) and \(S\) result in

\[
\begin{align*}
H &= \begin{bmatrix} 0 & i\beta & 0 & 0 \\ -i\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\beta \\ 0 & 0 & i\beta & 0 \end{bmatrix}, \quad S = \begin{bmatrix} \cos \beta & \sin \beta & 0 & 0 \\ -\sin \beta & \cos \beta & 0 & 0 \\ 0 & 0 & \cos \beta & \sin \beta \\ 0 & 0 & -\sin \beta & \cos \beta \end{bmatrix}.
\end{align*}
\]
The inverse of $H$ is

$$H^{-1} = \begin{bmatrix}
0 & i & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{i}{\beta} \\
0 & 0 & \frac{i}{\beta} & 0
\end{bmatrix}$$

Hence

$$\Psi = \begin{bmatrix}
-\sin \beta & \frac{i(\cos \beta - 1)}{\beta} & 0 & 0 \\
\frac{i(\cos \beta - 1)}{\beta} & -\sin \beta & 0 & 0 \\
0 & 0 & \frac{i \sin \beta}{\beta} & -\frac{i(\cos \beta - 1)}{\beta} \\
0 & 0 & -\frac{i(\cos \beta - 1)}{\beta} & \frac{i \sin \beta}{\beta}
\end{bmatrix}, \quad \beta \in [0, 2\pi)$$

$$P = \begin{bmatrix}
-\frac{i \sin \beta}{\beta} & \frac{i(\cos \beta - 1)}{\beta} \\
\frac{i(\cos \beta - 1)}{\beta} & -\frac{i \sin \beta}{\beta}
\end{bmatrix}, \quad \beta \in [0, 2\pi), \quad Q = 0.$$  

Again, $\Psi$ and $P$ are aperiodic, while they should be periodic in $\beta$. The remedy is the limitation of $\beta$ as in (59) or the application of identity (57). But the latter solution gives a cumbersome expression.

### III.2. Squeezing+displacement

The Hamiltonian $H$ is given by the matrix representation (see (34))

$$H = \begin{bmatrix}
0 & i z \\
-i z & 0
\end{bmatrix}, \quad h = \begin{bmatrix}
h \\
h
\end{bmatrix}$$  

(60)

where $z$ is an $n \times n$ symmetric matrix and $h$ is an arbitrary complex vector of size $n$. The squeeze matrix must be decomposed in polar form as $z = r e^{i \theta}$, where $r$ is positive semidefinite and $\theta$ is Hermitian.

**Proposition 2** The complex symplectic matrix is given by (see (43))

$$S = e^{-i \Omega H} = \begin{bmatrix}
\cosh(r) & \sinh(r) e^{i \theta} \\
\sinh(r) e^{-i \theta} & \cosh(r)
\end{bmatrix}$$  

(61)

The matrix giving the displacement results in

$$\Psi = \begin{bmatrix}
\sinh(r) e^{i \theta} (-i z^{-1}) \\
\cosh(r) - I_n
\end{bmatrix} \begin{bmatrix}
\sinh(r) e^{i \theta} (-i z^{-1}) \\
\cosh(r) - I_n
\end{bmatrix}^{-1}$$  

(62)

Expression (62) is a special case of the second of (14) with

$$P = \sinh(r) e^{i \theta} (-i z^{-1}) \quad , \quad Q = (\cosh(r) - I_n) i z^{-1}.$$  

(63)

Hence

$$s = \sinh(r) e^{i \theta} (-i z^{-1}) h + (\cosh(r) - I_n) i z^{-1} h.$$  

(64)

**Example 3.** In the single mode, (64) gives

$$s = e^{i \theta} \sinh r (-i z^{-1}) h + (\cosh r - 1) i z^{-1} h$$

$$= -i \frac{\sinh(r)}{r} h + i \frac{\cosh(r) - 1}{r e^{-i \theta}} h.$$  

(65)

where $h = i \alpha$ (see (36)).

The plot of $|s|$ as a function of $r$ and as a function of $\theta$ and $h = 1$ is shown in Fig. 5.
Fig. 5 – The displacement amount \( |s| \) (left) as a function of \( r \) for 6 values of \( \theta \) and \( h = 1 \) and (right) as a function of \( \theta \) for 6 values of \( r \) and \( h = 1 \) in the squeezing +displacement

### III.3. Squeezing+rotation +displacement

Consider two \( n \)-mode Hamiltonians \( H_{sq} \) and \( H_{rot} \) given by the matrix representations

\[
H_{sq} = \begin{bmatrix}
0 & i \tau e^{i\theta} \\
-i e^{-i\theta} \tau^T & 0
\end{bmatrix}, \quad H_{rot} = -\begin{bmatrix}
\phi & 0 \\
0 & \phi^T
\end{bmatrix}
\]

and a general linear term

\[
h = \begin{bmatrix}
h \\
h
\end{bmatrix}.
\]

This specifies the most general Gaussian unitary [8]. The global symplectic matrix is obtained as

\[
S = S_{sq} S_{rot} = e^{-i \Omega H_{sq}} e^{-i \Omega H_{rot}}
\]

and reads on (see (43))

\[
S = \begin{bmatrix}
\cosh(r) e^{i\phi} & \sinh(r) e^{i\theta} e^{-i\phi^T} \\
\sinh(r) e^{-i\theta} e^{i\phi} & \cosh(r) e^{-i\phi^T}
\end{bmatrix}.
\]

But we want to obtain this expression starting from a single Hamiltonian. Note that in general

\[
e^{-i \Omega H_{sq}} e^{-i \Omega H_{rot}} \neq e^{-i \Omega H_{sq}} e^{-i \Omega H_{rot}}.
\]

The single Hamiltonian is given by (see [8])

\[
H = i\Omega \log(S).
\]

For the evaluation of \( \log(S) \) we can use the standard methods of calculation of a function of a matrix [16]. Then we apply (13) to evaluate the matrix \( \Psi \) and (62) to evaluate the displacement \( s = \Psi h \).

**Example 4.** We develop the above procedure in the single mode. For the evaluation of the \( 2 \times 2 \) matrix \( H \), according to (68), we can use the Sylvester interpolation method [12], which gives

\[
H = d_0 I_2 + d_1 i \Omega S := \begin{bmatrix}
A & B \\
\overline{B} & \overline{A}
\end{bmatrix}
\]

The coefficient \( d_0 \) and \( d_1 \) are given by

\[
d_0 = \frac{\lambda_+^2 + 1}{\lambda_+^2 - 1} \log(\lambda_+) \quad , \quad d_1 = \frac{2\lambda_+}{\lambda_+^2 - 1} \log(\lambda_+) \quad |\Re(E)| \neq 1
\]

where \( \lambda_\pm \) are the eigenvalues of \( S \), which result in

\[
\lambda_- = \Re(E) - \sqrt{\Re(E)^2 - 1} \quad , \quad \lambda_+ = \Re(E) + \sqrt{\Re(E)^2 - 1}
\]
and 

\[ E = \cosh(r)e^{i\phi}, \quad F = \sinh(r)e^{i(\theta - \phi)}, \quad L = \sqrt{\Re(E)^2 - 1}. \]

We find

\[ A = \frac{\Im(E)}{2L} \log \left( \frac{\Re(E) - L}{\Re(E) + L} \right), \quad B = -i \frac{F}{2L} \log \left( \frac{\Re(E) - L}{\Re(E) + L} \right) \]

Hence

\[ \Psi = (S - I_2)H^{-1} = \begin{bmatrix} 2\text{Re}(E+1)|E|^2L(\Re(E) - 1) & 2iFL^*[-(E-1)|E|^2 + E + |F|^2(\Re(E) - 1)] \\ 2iFL^*[|E|^2 - (E+1)E^* + |F|^2(\Re(E) - 1)] & 2\text{Re}(E+1)|E|^2L(\Re(E) - 1) \end{bmatrix} \]

\[ X = \log(\Re(E) - L), \quad Y = \log(\Re(E) + L). \]

In particular

\[ P = \frac{2i(E+1)|E|^2L(\Re(E) - 1)}{(X - Y)[|E|^2 - 1]^2 - |F|^2\Im(E)^2]} \quad \text{(71)} \]

\[ Q = \frac{2iFL^*[-(E-1)|E|^2 + E + |F|^2(\Re(E) - 1)]}{[X^* - Y^*][|E|^2 - 1]^2 - |F|^2\Im(E)^2]} \quad \text{(72)} \]

The displacement amount \( s \) is related to the \( h \) amount as in (62), that is, \( s = P + Q \sqrt{h} \), where the coefficients \( P \) and \( Q \) can be expressed in terms of rotation and squeezing parameters \( \phi \), \( r \), and \( \theta \) as

\[ P = -\frac{2i\Delta}{2A(\cos(\phi) \cosh(r) + 1)}, \quad Q = \frac{2iA^*e^{i\theta - i\phi}}{\Delta \cos(\phi) \cosh(r) + 1} \]

with

\[ \Delta = \sqrt{\cos^2(\phi) \cosh^2(r) - 1}, \quad A = \log \left[ \frac{-\Delta + \cos(\phi) \cosh(r)}{\Delta + \cos(\phi) \cosh(r)} \right]. \quad \text{(73)} \]

Fig. 6 shows the displacement amount \( |s| \) as a function of \( \theta \) for four values of \( r \) for \( \phi = 0.3 \). Note that the curves increase with \( r \) and, for a given \( r \), have a maximum for \( \theta = \pi \). Fig. 7 shows the behavior of the displacement amount as a function of the phase \( \phi \). This behavior exhibits divergences at two points of the period \( [0, 2\pi] \), as shown in Fig. 6. In fact the denominators of \( P \) and \( Q \) vanish at such points.

![Fig. 6 - The displacement amount |s| as a function of \( \theta \) for \( h = 1 \), \( \phi = 0.5 \), and for four values of \( r \) in the general case of single mode](image-url)
Fig. 7 – The displacement amount \(|s|\) as a function of \(\phi\) for \(h = 1\) for different ranges of \(\phi\).

Fig. 8 and Fig. 9 show other representations of the displacement amount.

Fig. 8 – The displacement amount \(|s|\) as a function of \(r\) and \(\theta\) for \(h = 1\).

Fig. 9 – The displacement amount \(|s|\) as a function of \(\theta\) and \(\phi\) for \(h = 1\) and \(r = 0.1\).
Alternative evaluation for the single mode

In Appendix D we consider a specific evaluation for the single mode leading to simpler results. We find

\[
P = -\frac{i \sinh(T) (-e^{i\phi} \cosh(r) (\cosh(T) + 1) + e^{2i\phi} \cosh^2(r) + \sinh^2(r) + \cosh(T))}{T \left( \sinh^2(r) + (\cosh(T) - e^{i\phi} \cosh(r))^2 \right)}
\]

\[
Q = \frac{ie^{-i(\phi-\theta)} \sinh(r) \sinh(T) (\cosh(T) - 1)}{T \left( \sinh^2(r) + (\cosh(T) - e^{i\phi} \cosh(r))^2 \right)}
\]

where

\[
T = \cosh^{-1}(\cos(\phi) \cosh(r)) .
\]

Note that if \( \cos(\phi) \cosh(r) < 1 \), \( T \) turns out to be imaginary, namely \( T = i \cos^{-1}(\cos(\phi) \cosh(r)) \).

**Example 5.** We consider a two-mode Gaussian unitary given by a beam splitter followed by a Caves–Schumaker unitary, followed by a two-mode displacement. The beam splitter is specified by the rotation matrix

\[
\phi = \begin{bmatrix} 0 & -i\beta \\ i\beta & 0 \end{bmatrix}, \quad \beta \in \mathbb{R}
\]

and has the following Hamiltonian representation and symplectic matrix

\[
H_b = \begin{bmatrix} 0 & i\beta & 0 & 0 \\ -i\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\beta \\ 0 & 0 & i\beta & 0 \end{bmatrix}, \quad S_b = \begin{bmatrix} \cos \beta & \sin \beta & 0 & 0 \\ -\sin \beta & \cos \beta & 0 & 0 \\ 0 & 0 & \cos \beta & \sin \beta \\ 0 & 0 & -\sin \beta & \cos \beta \end{bmatrix}
\]

The Caves–Schumacher unitary is a squeezer specified by the squeeze matrix

\[
z = \begin{bmatrix} 0 & e^{i\theta}r \\ e^{i\theta}r & 0 \end{bmatrix}, \quad r, \theta \in \mathbb{R}
\]

and has the following Hamiltonian representation and symplectic matrix

\[
H_c = \begin{bmatrix} 0 & 0 & 0 & 0 & ie^{i\theta}r \\ 0 & 0 & ie^{i\theta}r & 0 & 0 \\ 0 & -ie^{-i\theta}r & 0 & 0 & 0 \\ -ie^{-i\theta}r & 0 & 0 & 0 & 0 \end{bmatrix}, \quad S_c = \begin{bmatrix} \cosh r & 0 & 0 & e^{i\theta} \sinh r \\ 0 & \cosh r & e^{i\theta} \sinh r & 0 \\ 0 & e^{-i\theta} \sinh r & \cosh r & 0 \\ e^{-i\theta} \sinh r & 0 & 0 & \cosh r \end{bmatrix}
\]

The global symplectic matrix is given by

\[
S = S_c S_b = \begin{bmatrix} \cosh \beta \cosh r & \cosh r \sin \beta & e^{i\theta} \sin \beta \sinh r & e^{i\theta} \cosh \beta \sinh r \\ -\cosh r \sin \beta & \cosh \beta \cosh r & e^{i\theta} \cosh \beta \sinh r & e^{i\theta} \sin \beta \sinh r \\ e^{-i\theta} \sin \beta \sinh r & e^{-i\theta} \cosh \beta \sinh r & \cosh r \cosh \beta & \cosh r \sin \beta \\ e^{-i\theta} \cosh \beta \sinh r & -e^{-i\theta} \sin \beta \sinh r & -\cosh r \sin \beta & \cosh r \cosh \beta \end{bmatrix}
\]

The problem is the evaluation of the matrix log given by (68). Since the \( 4 \times 4 \) matrix \( S \) has not distinct eigenvalues we cannot use Sylvester’s formula, but the matrix log can be evaluated via Jordan canonical form [16]. Considering that \( S \) has two distinct eigenvalues with multiplicity 2, specifically

\[
\mu_{1,2} = \cos(\beta) \cosh(r) \mp \frac{1}{2} \sqrt{2 \cosh^2(\beta) \cosh(2r) + \cos(2\beta) - 3}
\]

the related Jordan form is

\[
H = i\Omega \log S = i\Omega \sum_{m=0}^{3} d_m S^m
\]

where

\[
d_0 = -\frac{(\mu_1 - \mu_2) [\mu_1^2 + \mu_2^2] + (\mu_1 - 3\mu_2) \mu_1^3 \log(\mu_2) + (3\mu_1 - \mu_2) \mu_2^3 \log(\mu_1)}{(\mu_1 - \mu_2)^3}
\]
\[
\begin{align*}
d_1 &= \frac{(\mu_1 - \mu_2)(\mu_1 + \mu_2) [\mu_1^2 + \mu_2 \mu_1 + \mu_2^2] + 6\mu_1^2\mu_2^2(\log(\mu_2) - \log(\mu_1))}{\mu_1(\mu_1 - \mu_2)^3\mu_2} \\
\end{align*}
\]
\[
\begin{align*}
d_2 &= \frac{-2\mu_1^3 + 2\mu_3^3 + 3\mu_2(\mu_1 + \mu_2)\mu_1(\log(\mu_1) - \log(\mu_2))}{\mu_1(\mu_1 - \mu_2)^3\mu_2} \\
\end{align*}
\]
\[
\begin{align*}
d_3 &= \frac{\mu_1^2 - \mu_2^2 + 2\mu_2\mu_1(\log(\mu_2) - \log(\mu_1))}{\mu_1(\mu_1 - \mu_2)^3\mu_2}
\end{align*}
\]

Finally, we evaluate the matrix \( \Psi = (S - I_4)H^{-1} \), but we find a very complicated expression. However, we are interested in the evaluation of the coefficients \( P \) and \( Q \) for which we have obtained a readable expression. We let

\[
M = \sqrt{[e^{2r} + 1]^2 \cos^2(\beta) - 4e^{2r}} = e^r \sqrt{2 \cos(2\beta) \cosh^2(r) + \cosh(2r) - 3}
\]

\[
a = 2 \log \left[ \cos(\beta) \cosh(r) - \frac{1}{2} Me^{-r} \right]
\]
\[
b = 2 \log \left[ \cos(\beta) \cosh(r) + \frac{1}{2} Me^{-r} \right]
\]
\[
c = \log \left[ 2 \cos(\beta) \cosh(r) - Me^{-r} \right]
\]
\[
d = \log \left[ 2 \cos(\beta) \cosh(r) + Me^{-r} \right]
\]
\[
h = e^{i\theta} \sqrt{[e^{2r} + 1]^2 \cos(2\beta) - 6e^{2r} + e^{4r} + 1}
\]
\[
k = \log 4 + 2r \log \left[ [e^{2r} + 1] \cos(\beta) + M \right]
\]
\[
m = \log 4 + 2r \log \left[ [e^{2r} + 1] \cos(\beta) - M \right]
\]

Then we get

\[
P = i \frac{[a + b] (\cos \beta - 1) + 2i \sqrt{2} [e^{2r} + 1] e^{i\theta} \sin^2 \beta (c - d)}{hk m}
\]
\[
Q = i \frac{[a + b] \sin \beta - 2i \sqrt{2} [e^{2r} + 1] e^{i\theta} \sin \beta (\cos \beta - 1)(c - d)}{hk m}
\]

**IV. AMPLIFICATION OF THE DISPLACEMENT**

Starting from a linear Hamiltonian \( H = H_\ell = \xi^* \hbar \) we can generate a displacement whose amount is just given by \( \hbar \). But, introducing a Gaussian unitary containing rotation and squeezing we can modify the displacement amount as

\[
s = \Psi \hbar .
\]

In particular, acting on the rotation and squeezing parameters we can obtain an amplification of the displacement, as seen for the single and the two–mode. In the following consideration we first consider the case squeezing+displacement and then we consider the general case where also the rotation is present.
IV.1. The displacement amount with only squeezing

We have seen in Fig. 3 and in Fig. 4 that in the presence of only squeezing we get both attenuation and amplification, where a fundamental role is played by the squeeze phase $\theta$. In fact, the displacement amount is given by (65), that is,

$$s = -i \sinh \frac{r}{r} \cosh \left( \frac{r - 1}{r} \right) e^{i \theta}.$$

(76)

For convenience we consider the case $h = 1$, so that the displacement amount is given by

$$|s(r, \theta)| = \left| \sinh \frac{r}{r} e^{\frac{i \theta}{2}} (\cosh(r - 1)) \right| = \frac{2 \sinh(r/2) \sqrt{\cosh(r) - \sinh(r) \cos(\theta)}}{r}.$$

The function $|s(r, \theta)|$, illustrated in Fig. 10, has a maximum at $\theta = \pi$ given by $(e^r - 1)/r$ and a minimum at $\theta = 0$, given by $e^{-r}(e^r - 1)/r$. The maximum and the minimum are as illustrated in Fig. 11 (left) as a function of $r$.

In the $(r, \theta)$–plane the separation between the amplification and attenuation regions is determined by the condition

$$|s(r, \theta)| = \frac{2 \sinh(r/2) \sqrt{\cosh(r) - \sinh(r) \cos(\theta)}}{r} = 1$$

as illustrated in Fig. 11 (right).

IV.2. The displacement amount in the general case

The presence of only rotation does not provide amplification but only attenuation. In fact, with $h = 1$ we have the amount (see (58))

$$|s| = \frac{\left| \sin(\frac{\phi}{2}) \right|}{\phi/2} \leq 1 \quad \phi \in [-\pi, +\pi].$$

(77)
However, the rotation when combined with squeezing, provides huge amplifications, as already illustrated in Fig. 6, where the curves of $|s|$ versus $\phi$ exhibit divergences. We reconsider the plots around the divergences

$$\phi_r = \arccos(-1/\cosh(r)), \quad 2\pi - \phi_r.$$ 

For instance, with $r = 0.5$ we find $\phi_r = 2.66121$.

The conclusion seems that, combining squeezing and rotation, one can achieve arbitrarily huge displacement amounts!

V. CONCLUSIONS

Starting form the matrix representation $(H, h)$ of the Hamiltonian we have derived the matrix representation $(S, s)$ of the complex symplectic transformation, with the direct passage from the bosonic Hilbert space to the phase space. This approach has several advantages with respect to the traditional one, based of Bogoliubov transformation and real symplectic transformations, as illustrated in the introduction. The derivation was carried out in the general $n$-mode arriving in any case at explicit closed-form results.

In particular, we have focused our attention on the relation between the linear terms $h$ and $s$, not developed in the literature of Gaussian unitaries. This relation becomes $s = h$ in the presence of displacement only, that is, with $H = 0$. In all the other cases $s$ turns out to be strongly dependent on the quadratic part $H$ of the Hamiltonian. As illustrated with the application to combination of fundamental Gaussian unitaries, from a given $h \neq 0$, it is possible to achieve an $s$ arbitrarily large. In the authors’ opinion this topic deserves a further development with the help of an experimental verification.

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APPENDIX

Appendix A: Sum of the series $\Psi$ when $H$ is singular

The series $\Psi$ defined by (13) can be summed in a closed form using the identity

$$\sum_{n=1}^{\infty} \frac{1}{n!} C^{n-1} = (e^C - I_p)C^{-1}$$

where $C$ is a nonsingular $p \times p$ matrix. If the matrix $H$ is not singular, the application of (78) to the series $\Psi$ gives the result written in (13).

The series can be summed in closed form also when $H$ is singular, using the Jordan decomposition of the matrix $-i\Omega H$, say

$$-i\Omega H = V \Lambda V^{-1}.$$ 

If $r < 2n$ is the rank of $H$ we decompose the diagonal matrix $\Lambda$ in the form

$$\Lambda = \begin{bmatrix} \Lambda_r & 0 \\ 0 & \Lambda_0 \end{bmatrix}$$

where $\Lambda_r$ contains the Jordan blocks corresponding to the non vanishing eigenvalues of $-i\Omega H$ and $\Lambda_0$ contains the Jordan blocks corresponding to the zero eigenvalues. Then the series gives

$$\Psi = \sum_{n=1}^{\infty} \frac{1}{n!} (-i\Omega H)^{n-1} (-i\Omega) = V \sum_{n=1}^{\infty} \frac{1}{n!} \begin{bmatrix} \Lambda_r^{n-1} & 0 \\ 0 & \Lambda_0^{n-1} \end{bmatrix} V^{-1}(-i\Omega)$$

Since $\Lambda_r$ is non singular (78) gives

$$W_r = \sum_{n=1}^{\infty} \frac{1}{n!} \Lambda_r^{n-1} = (e^{\Lambda_r} - 1)\Lambda_r^{-1}.$$
Moreover, $\Lambda_0$ is nilpotent so that the series
\[ W_0 = \sum_{n=1}^{\infty} \frac{1}{n!} \Lambda_0^{n-1} \]
reduces to a finite sum. In conclusion:

**Proposition 3.** If $r < 2n$ is the rank of $H$, in the Jordan decomposition $-i\Omega H = VAV^{-1}$, the matrix $\Lambda$ is decomposed as in (79), where $\Lambda_r$ is regular and $\Lambda_0$ is nilpotent. Then
\[ \Psi = -i V \begin{bmatrix} W_r & 0 \\ 0 & W_0 \end{bmatrix} V^{-1}(-i\Omega) \]
where $W_r$ is given by (80) and $W_0$ by (81).

**Example 6 (Single mode).** In the single mode the matrix $H$ has the structure
\[ H = \begin{bmatrix} \alpha & \beta \\ \beta^* & \alpha \end{bmatrix}. \]
Its singularity implies $|\beta| = |\alpha|$ and $\beta = \alpha e^{i\phi}$, so that one gets
\[ -i\Omega H = -i\alpha \begin{bmatrix} 1 & -i e^{i\phi} \\ -e^{-i\phi} & -1 \end{bmatrix}. \]
The matrix $-i\Omega H$ has a double eigenvalue $\lambda = 0$. On the other hand, it cannot be diagonalizable because in this case it should vanish. On the contrary, it is nilpotent since $(-i\Omega H)^2 = 0$. As a consequence, it follows
\[ S = e^{-i\Omega H} = I - i\Omega H = \begin{bmatrix} 1 - i\alpha & -i\alpha e^{i\phi} \\ i\alpha e^{-i\phi} & 1 + i\alpha \end{bmatrix} \]
\[ \Psi = \sum_{n=1}^{\infty} \frac{1}{n!} (-i\Omega H)^{n-1} (-i\Omega) = \left( I - \frac{1}{2} i\Omega H \right) (-i\Omega) = \begin{bmatrix} -\frac{\alpha}{2} - i & \frac{1}{2} i\alpha e^{i\phi} \\ \frac{1}{2} i\alpha e^{-i\phi} & -\frac{\alpha}{2} + i \end{bmatrix}. \]

**Example 7 (A two mode).** We consider a degenerate two–mode rotation specified by the Hermitian matrix
\[ \phi = \begin{bmatrix} \phi & 0 \\ 0 & \phi^* \end{bmatrix} \quad \text{with} \quad \phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{12}^* & \phi_{22} \end{bmatrix}. \]
The matrix representation of the Hamiltonian is
\[ H = -\begin{bmatrix} \phi_{11} & \phi_{12} & 0 & 0 \\ \phi_{12}^* & \phi_{22} & 0 & 0 \\ 0 & 0 & \phi_{11} & \phi_{12}^* \\ 0 & 0 & \phi_{12} & \phi_{22} \end{bmatrix} \]
We have $\det H = (|\phi_{12}|^2 - \phi_{11}\phi_{22})^2$, so that a degenerate case is obtained with
\[ |\phi_{12}| = \sqrt{\phi_{11}\phi_{22}} \quad \rightarrow \quad \text{rank } H = 2. \]
In this case we have
\[ -i\Omega H = -i \begin{bmatrix} \phi_{11} & \sqrt{\phi_{11}\phi_{22}} & 0 & 0 \\ \sqrt{\phi_{11}\phi_{22}} & \phi_{22} & 0 & 0 \\ 0 & 0 & -\phi_{11} & -\sqrt{\phi_{11}\phi_{22}} \\ 0 & 0 & -\sqrt{\phi_{11}\phi_{22}} & -\phi_{22} \end{bmatrix} \]
and $r = \text{rank}(H) = 2$. The matrices of the Jordan decomposition are
\[ V = \begin{bmatrix} 0 & -\frac{\phi_{22}}{\sqrt{\phi_{11}\phi_{22}}} & 0 & \frac{\phi_{11}}{\sqrt{\phi_{11}\phi_{22}}} \\ 0 & 1 & 0 & 0 \\ -\frac{\phi_{22}}{\sqrt{\phi_{11}\phi_{22}}} & 0 & 0 & \frac{\phi_{11}}{\sqrt{\phi_{11}\phi_{22}}} \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i(\phi_{11} + \phi_{22}) & 0 \\ 0 & 0 & 0 & i(\phi_{11} + \phi_{22}) \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]
\[
V^{-1} = \begin{bmatrix}
0 & 0 & -\sqrt{\phi_{11}\phi_{22}} & \phi_{11} \\
\sqrt{\phi_{11}\phi_{22}} & \phi_{11} + \phi_{22} & 0 & 0 \\
0 & 0 & \phi_{11} + \phi_{22} & 0 \\
\sqrt{\phi_{11}\phi_{22}} & \phi_{22} & 0 & 0
\end{bmatrix}
\]

\[
A_r = \begin{bmatrix}
-i(\phi_{11} + \phi_{22}) & 0 & 0 \\
0 & i(\phi_{11} + \phi_{22}) & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \Lambda_0 = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}.
\]

Hence

\[
\sum_{n=1}^{\infty} \frac{1}{n!} \Lambda_n^n = (e^{A_r} - I_r) \Lambda^{-1} = \begin{bmatrix}
\frac{i(-1 + e^{-i(\phi_{11} + \phi_{22})})}{\phi_{11} + \phi_{22}} & 0 \\
0 & -\frac{i(-1 + e^{i(\phi_{11} + \phi_{22})})}{\phi_{11} + \phi_{22}}
\end{bmatrix}, \quad W_0 = 0.
\]

**Appendix B: Hamiltonian representation of the fundamental unitaries**

In the literature the multimode fundamental Gaussian unitaries (displacement, rotation, and squeezing) are usually expressed in terms of the bosonic vectors \(a\) and \(a^\dagger\) (see f.i. [8]). Then, it may be useful to find the relations between these expressions and the corresponding quantities \(H\) and \(h\).

The usual representation of a \(n\)-modal displacement operator is given by

\[
D(\alpha) = e^{a^\dagger \alpha a - \alpha^\dagger a}, \quad \alpha = [\alpha_1, \ldots, \alpha_n]^T \in \mathbb{C}^n,
\]

while its representation in terms of the Hamiltonian is (see (83))

\[
e^{-iHt} = e^{-ih^*\xi} = e^{-ih^*a - ih^*a^\dagger}
\]

coinciding with (83), provided that \(H = 0\) and \(h = \begin{bmatrix} i\alpha \\ -i\alpha^T \end{bmatrix}\).

The rotation operator is usually given by (see (33))

\[
R(\phi) = e^{ia^\dagger \phi a},
\]

where \(\phi\) is a Hermitian \(n \times n\) matrix. The present representation of the rotation is given by the Hamiltonian

\[
H_r = \frac{1}{2} \sum_{r,s=1}^{n} \left[ A_{rs}a_r^\dagger a_s + A_{rs}a_r a_s^\dagger \right] = \frac{1}{2} \sum_{r,s=1}^{n} \left[ A_{rs}a_r^\dagger a_s + A_{sr}(a_r^\dagger a_s - \delta_{rs}) \right] = a^\dagger Aa - \frac{1}{2} \text{Tr}(A)
\]

so that, apart an irrelevant phasor,

\[
e^{-iH_r} = e^{-ia^\dagger Aa}
\]

coinciding with (84) provided that \(h = 0\) and \(A = -\phi\), i.e., \(H = \begin{bmatrix} -\phi & 0 \\ 0 & -\phi \end{bmatrix}\).

Finally, the squeezing operator is given by (see (34))

\[
Z(z) = e^{\frac{1}{2}(a^\dagger z a + a z^\dagger)}
\]

with \(z\) a symmetric \(n \times n\) matrix. The Hamiltonian is given by

\[
H_s = \frac{1}{2} \sum_{r,s=1}^{n} \left[ B_{rs}a_r^\dagger a_s^\dagger + B_{rs}a_r a_s \right] = \frac{1}{2}(a^\dagger Ba + a^\dagger B^\dagger a)
\]

so that

\[
e^{-iH_s} = e^{-\frac{1}{2}(a^\dagger Ba + a^\dagger B^\dagger a)}
\]

coincides with (84), provided that \(h = 0\) and \(B = iz\), i.e., \(H = \begin{bmatrix} 0 & iz \\ -iz & 0 \end{bmatrix}\).
The expressions of the FUs in terms of the vectors \( a, a^* \) (traditional form) and in terms of the single vector \( \xi \) are summarized in the following table:

| Displacement | \( D(\alpha) := e^{a^T \alpha - a^* a} \) | \( \alpha \in \mathbb{C}^n \) | \( D_Y(\xi) = e^{\xi^* \Omega Y} \) | \( Y = \begin{bmatrix} \alpha \\ \alpha^* \end{bmatrix} \) |
| --- | --- | --- | --- | --- |
| Rotation | \( R(\phi) := e^{i a^* \phi a} \) | \( \phi \in \mathbb{R} \times \mathbb{R} \) Hermitian | \( R(\xi) = e^{i \xi^* \Phi \xi} \) | \( \Phi = \begin{bmatrix} \phi \\ 0 \\ 0 \end{bmatrix} \) |
| Squeezing | \( Z(z) := e^{z \frac{1}{2} (a^* z a_s - a^T z^* a_s)} \) | \( z \in \mathbb{R} \times \mathbb{R} \) symmetric | \( Z(\xi) = e^{i \xi^* \xi} \) | \( \xi = \begin{bmatrix} 0 \\ z \\ z^* \end{bmatrix} \) |

**Appendix C: Proof of Propositions 1 and 2**

For Proposition 1, starting from

\[-i \Omega H = \begin{bmatrix} i \phi & 0 \\ 0 & -i \phi^T \end{bmatrix}\]

one finds immediately

\[S = \begin{bmatrix} e^{i \phi} & 0 \\ 0 & e^{-i \phi^T} \end{bmatrix}.\]

Then, from (13) it follows

\[
\Psi = (S - I_{2n})H^{-1} = -\begin{bmatrix} (e^{i \phi} - I_n) \phi^{-1} & 0 \\ 0 & (e^{-i \phi^T} - I_n)(\phi^{-1})^T \end{bmatrix}.
\]

For Proposition 2, starting from (60) and using the polar decomposition \( z = re^{i \theta} \) gives

\[-i \Omega H = \begin{bmatrix} 0 & re^{i \theta} \\ re^{-i \theta^T} & 0 \end{bmatrix}\]

As shown in [8], the corresponding Bogoliubov transformation gives

\[e^{i \Omega H} \begin{bmatrix} a \\ a^* \end{bmatrix} e^{-i \Omega H} = \begin{bmatrix} \cosh(r)a + \sinh(r)e^{i \theta}a_s \\ \cosh(r^T)a_s + \sinh(r^T)e^{-i \theta}a \end{bmatrix},\]

so that the complex symplectic matrix is

\[S = \begin{bmatrix} \cosh(r) & \sinh(r)e^{i \theta} \\ \sinh(r^T)e^{-i \theta} & \cosh(r^T) \end{bmatrix}.\]

In conclusion, from (13) one gets

\[
\Psi = \begin{bmatrix} \cosh(r) - I_n & \sinh(r)e^{i \theta} \\ \sinh(r^T)e^{-i \theta} & \cosh(r^T) - I_n \end{bmatrix} \begin{bmatrix} 0 & i \zeta^{-1} \\ -i \zeta^{-1} & 0 \end{bmatrix},
\]

from which (62) follows.
Appendix D: Alternative evaluation of $\Psi$ in the single mode

We consider the matrix $\mathbf{H}$ in the single mode

$$
\mathbf{H} = \begin{bmatrix} a & b \\ b^* & a \end{bmatrix}, \quad a > 0, \ b \in \mathbb{C}
$$

and we evaluate the corresponding symplectic matrix $\mathbf{S}$. We find

$$
\mathbf{S} = e^{-i\Omega\mathbf{H}} = \begin{bmatrix} \cosh(T) - \frac{i a \sinh(T)}{T} & -\frac{i b \sinh(T)}{T} \\ \frac{i b^* \sinh(T)}{T} & \cosh(T) + \frac{i a \sinh(T)}{T} \end{bmatrix}
$$

(85)

where

$$
T = \sqrt{|b|^2 - a^2}, \quad T^2 = |b|^2 - a^2.
$$

On the other hand we know that the symplectic matrix matrix has the form

$$
\mathbf{S} = \begin{bmatrix} \cosh(r)e^{i\phi} & \sinh(r)e^{i(\theta-\phi)} \\ \sinh(r)e^{-i(\theta-\phi)} & \cosh(r)e^{-i\phi} \end{bmatrix}
$$

(86)

Now we assume to know the parameters $r, \theta, \phi$ and we want to evaluate the parameters $a, b$. To this end we equate the first rows of (85) and (86)

$$
cosh(T) - \frac{i a \sinh(T)}{T} = \cosh(r)e^{i\phi}, \quad -\frac{i b \sinh(T)}{T} = \sinh(r)e^{i(\theta-\phi)}
$$

(87)

Assuming as known $T$, the solution is

$$
a = -iT \frac{\cosh(T) - e^{i\phi} \cosh(r)}{\sinh(T)}, \quad b = iT \frac{\sinh(r)e^{i(\theta-\phi)}}{\sinh(T)}
$$

(88)

To calculate $T$ we take the real part of the first of (87)

$$
cosh(T) = \cosh(r) \cos(\phi) \quad \rightarrow \quad T = \cosh^{-1}[\cosh(r) \cos(\phi)]
$$

Hence

$$
T = \cosh^{-1}[\cosh(r) \cos(\phi)]
$$

and

$$
\sinh(T) = \sqrt{\cosh(T)^2 - 1} = \sqrt{\cosh^2(r) \cos^2(\phi) - 1}
$$

This complete the evaluation of $\mathbf{H}$ from $\mathbf{S}$.

Once evaluated the Hamiltonian matrix in terms of the parameters $r, \theta, \phi$, we can calculated the matrix $\Psi = (\mathbf{S} - I_2)\mathbf{H}^{-1}$ as a function of the same parameters. We find

$$
\Psi = \begin{bmatrix}
-\frac{i \tanh(T)(-e^{i\phi} \cosh(r)(\cosh(T) + 1) + e^{2i\phi} \cosh^2(r) + \sinh^2(r) + \cosh(T))}{T(\sinh^2(r) + (\cosh(T) - e^{i\phi} \cosh(r))^2)} & \frac{i e^{-i(\theta-\phi)} \sinh(r)(\cosh(T) - 1) \tanh(T)}{T(\sinh^2(r) + (\cosh(T) - e^{i\phi} \cosh(r))^2)} \\
\frac{e^{-i\phi} \sinh(r) \tanh(T)(\sin(\phi) - i \cos(\phi))(2 \cos(\phi) \cosh(r) - \cosh(T) - 1)}{T(\sinh^2(r) + (\cosh(T) - e^{i\phi} \cosh(r))^2)} & \frac{i e^{-i\phi} \tanh(T)(\cosh(r) + e^{2i\phi}) - e^{i\phi} (\cosh(T) + 1)}{T(\sinh^2(r) + (\cosh(T) - e^{i\phi} \cosh(r))^2)}
\end{bmatrix}
$$

In particular

$$
P = -\frac{i \sinh(T)(-e^{i\phi} \cosh(r)(\cosh(T) + 1) + e^{2i\phi} \cosh^2(r) + \sinh^2(r) + \cosh(T))}{T(\sinh^2(r) + (\cosh(T) - e^{i\phi} \cosh(r))^2)}
$$

$$
Q = \frac{i e^{-i(\phi-\theta)} \sinh(r) \cosh(T)(\cos(T) - 1)}{T(\sinh^2(r) + (\cosh(T) - e^{i\phi} \cosh(r))^2)}
$$

where

$$
cosh(T) = \cos(\phi) \cosh(r) \\
\sinh(T) = \sqrt{\cos^2(\phi) \cosh^2(r) - 1}.
$$
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