CANONICAL STABILITY IN TERMS OF SINGULARITY INDEX FOR ALGEBRAIC THREEFOLDS

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Introduction

Throughout the ground field is always supposed to be algebraically closed of characteristic zero. Let \(X\) be a smooth projective threefold of general type, denote by \(\phi_m\) the m-canonical map of \(X\) which is nothing but the rational map naturally associated with the complete linear system \(|mK_X|\). Since, once given such a 3-fold \(X\), \(\phi_m\) is birational whenever \(m \gg 0\), thus a quite interesting thing is to find the optimal bound for such an \(m\). This bound is important because it is not only crucial to the classification theory, but also strongly related to other problems. For example, it can be applied to determine the order of the birational automorphism group of \(X\) ([21], Remark in §1). To fix the terminology, we say that \(\phi_m\) is stably birational if \(\phi_t\) is birational onto its image for all \(t \geq m\). It is well-known that the parallel problem in surface case was solved by Bombieri ([1]) and others. In the 3-dimensional case, many authors have studied the problem, in quite different ways. Because, in this paper, we are interested in the results obtained by M. Hanamura ([7]), we don’t plan to mention more references here. According to 3-dimensional MMP, \(X\) has a minimal model which is a normal projective 3-fold with only \(\mathbb{Q}\)-factorial terminal singularities. Though \(X\) may have many minimal models, the singularity index (namely the canonical index) of any of its minimal models is uniquely determined by \(X\). Denote by \(r\) the canonical index of minimal models of \(X\). When \(r = 1\), we know that \(\phi_6\) is stably birational by virtue of [3], [6], [13] and [14]. When \(r \geq 2\), M. Hanamura proved the following theorem.

**Theorem 0. (Theorem (3.4) of [7])** Let \(X\) be a smooth projective threefold of general type with a minimal model of the canonical index \(r\). Then \(\phi_{n_0(r)}\) is stably birational onto its image, where \(n_0(r)\) is a function defined as

\[
\begin{align*}
* & \quad r = 2 \quad 3 \leq r \leq 5 \quad r \geq 6 \\
n_0(r) & \quad 13 \quad 4r + 4 \quad 4r + 3.
\end{align*}
\]

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Noting that the output $n_0(1)$ of Hanamura’s method is actually 7 (rather than 6), it is reasonable to believe that the bound in Theorem 0 is not optimal. On the other hand, we don’t know whether the canonical index $r$ is bounded or not, actually $r$ can be strangely large for some $X$. This suggests that to find the optimal bounds for $n_0(r)$ should still be very interesting. As far as our method can tell in this paper, the results are as the following

**Main Theorem.** Let $X$ be a smooth projective threefold of general type with a minimal model of the canonical index $r$. Then

(i) $\phi_m$ is generically finite whenever $m \geq l_0(r)$, where $l_0(r)$ is a function defined as

* $r = 2$ 3 $r \leq 5$ 6 $r \geq 6$

$l_0(r)$ 10 2$r + 5$ 2$r + 4$.

(ii) $\phi_{m_0(r)}$ is stably birational onto its image, where $m_0(r)$ is a function defined as

* $r = 2$ $r = 3$ $r = 4$ $r = 5$ $r = 6$ $r \geq 7$

$m_0(r)$ 11 15 17 18 19 2$r + 6$.

As an application of our method, we shall present the following

**Corollary.** Let $X$ be a smooth projective 3-fold of general type. Then $\phi_9$ is birational if $p_g(X) \geq 2$.

**Remark.** The above corollary is an improvement to Kollár’s result (Corollary 4.8 of [11]) that $\phi_{16}$ is birational if $p_g(X) \geq 2$. Actually, Kollár proved there that $\phi_{11k+5}$ is birational if $P_k := h^0(X, kK_X) \geq 2$, where $k$ is a positive integer. Recently, [4] improved this result to the level that either $\phi_{7k+3}$ or $\phi_{7k+5}$ is birational under the same condition.

For readers’ convenience, we briefly explain the whole technique of this paper. According to Hanamura’s result that $|(r + 2)K_X|$ is not composed of a pencil, we can take a general member $S_2$ of the movable part of this system. Actually we can suppose that $S_2$ is smooth. Then we use the Matsuki-Tankeev principle to reduce the birationality problem to a parallel one for the adjoint system $|K + L|$ on the surface $S_2$ which is a smooth projective surface of general type. We shall inevitably treat a very delicate case in which $L$ is the round-up of certain nef and big $\mathbb{Q}$-divisor $A$, i.e. $L = \lceil A \rceil$. Instead of applying Reider’s result, we go on reducing to the problem on a curve. The technical point is to estimate the degree of the divisor in question on the curve. The Kawamata-Ramamijam-Viehweg vanishing theorem played an important role in the whole context.

1. Preliminaries

Let $X$ be a normal projective variety of dimension $d$. We denote by $\text{Div}(X)$ the group of Weil divisors on $X$. An element $D \in \text{Div}(X) \otimes \mathbb{Q}$ is called a $\mathbb{Q}$-divisor. A $\mathbb{Q}$-divisor $D$ is said to be $\mathbb{Q}$-Cartier if $mD$ is a Cartier divisor for some positive integer $m$. For a $\mathbb{Q}$-Cartier divisor $D$ and an irreducible curve $C \subset X$, we can define the intersection number $D \cdot C$ in a natural way. A $\mathbb{Q}$-Cartier divisor $D$ is called
nef (or numerically effective) if $D \cdot C \geq 0$ for any effective curve $C \subset X$. A nef divisor $D$ is called big if $D^d > 0$. We say that $X$ is $\mathbb{Q}$-factorial if every Weil divisor on $X$ is $\mathbb{Q}$-Cartier. For a Weil divisor $D$ on $X$, write $\mathcal{O}_X(D)$ as the corresponding reflexive sheaf. Denote by $K_X$ a canonical divisor of $X$, which is a Weil divisor. $X$ is called minimal if $K_X$ is a nef $\mathbb{Q}$-Cartier divisor. $X$ is said to be of general type if the Kodaira dimension $\text{Kod}(X) = \dim(X)$. For a positive integer $m$, we set $\omega_X^{[m]} := \mathcal{O}_X(mK_X)$ and call $P_m(X) := \dim \mathcal{H}^0(X, \omega_X^{[m]})$ the $m$-th plurigenus of $X$.

We remark that numerical equivalence type if the Kodaira dimension $\text{Kod}(X) = \dim(X)$. For a positive integer $m$, we set $\omega_X^{[m]} := \mathcal{O}_X(mK_X)$ and call $P_m(X) := \dim \mathcal{H}^0(X, \omega_X^{[m]})$ the $m$-th plurigenus of $X$. We remark that $P_m(X)$ is an important birational invariant.

$X$ is said to have only canonical singularities (resp. terminal singularities) according to Reid ([15]) if the following two conditions hold:

(i) for some positive integer $r$, $rK_X$ is Cartier;

(ii) for some resolution $f : Y \rightarrow X$, $K_Y = f^*(K_X) + \sum a_iE_i$ for $0 \leq a_i \in \mathbb{Q}$ (resp. $0 < a_i$) $\forall i$, where the $E_i$ vary amongst all the exceptional divisors on $Y$. The minimal $r$ that satisfies (i) is called the canonical index of $X$ and can be also written as $r(X)$.

According to Mori’s MMP ([10], [12]), when $V$ is a smooth projective threefold, there exists a birational map $\sigma : V \dasharrow X$ where $X$ can be a minimal 3-fold with only $\mathbb{Q}$-factorial terminal singularities. Usually, $X$ is not uniquely determined by $V$, but the canonical index $r(X)$ is.

Let $D = \sum a_iD_i$ be a $\mathbb{Q}$-divisor on $X$ where the $D_i$ are distinct prime divisors and $a_i \in \mathbb{Q}$. We define

the round-down $\lfloor D \rfloor := \sum \lfloor a_i \rfloor D_i$, where $\lfloor a_i \rfloor$ is the integral part of $a_i$,

the round-up $\lceil D \rceil := -\lfloor -D \rfloor$,

the fractional part $\{D\} := D - \lfloor D \rfloor$.

Remark 1.1. Suppose that $X$ has only canonical singularities and that $f : V \rightarrow X$ is a resolution. We have

$$P_m(X) = \mathcal{H}^0\left(V, \mathcal{O}_V(\lceil f^*(mK_X) \rceil)\right) = \mathcal{H}^0\left(V, \mathcal{O}_V(\lceil f^*(mK_X) \rceil)\right) = P_m(V)$$

for any positive integer $m$.

We always use the Kawamata-Ramanujam-Viehweg vanishing theorem in the following form.

Vanishing Theorem. ([9] or [18]) Let $X$ be a smooth complete variety, $D \in \text{Div}(X) \otimes \mathbb{Q}$. Assume the following two conditions:

(i) $D$ is nef and big;

(ii) the fractional part of $D$ has supports with only normal crossings. Then $\mathcal{H}^i(X, \mathcal{O}_X(K_X + \lceil D \rceil)) = 0$ for all $i > 0$.

Most of our notations are standard within algebraic geometry except the following which we are in favor of: $\sim_{\text{lin}}$ means linear equivalence while $\sim_{\text{num}}$ means numerical equivalence.

2. Some lemmas

2.1 The Matsuki-Tankeev principle. This principle is tacitly used throughout our argument. Suppose $X$ is a smooth variety, $|M|$ is a base point free system on $X$ and $D$ is a divisor with $|D| \neq \emptyset$. We want to know when $\Phi_{|D+M|}$ is birational. The following principles are due to Tankeev and Matsuki respectively.
(P1). (Lemma 2 of [17]) Suppose $|M|$ is not composed of a pencil, i.e.

$$\dim \Phi|_M(X) \geq 2$$

and take a general member $Y \in |M|$. If the restriction of $\Phi|_{D+M}$ to $Y$ is birational, then $\Phi|_{D+M}$ is birational.

(P2). (see the proof of the main theorem in [14]) Suppose $|M|$ is composed of a pencil and take the Stein factorization of

$$\Phi|_M : X \xrightarrow{f} C \xrightarrow{\nu} W \subset \mathbb{P}^N,$$

where $W$ is the image of $X$ through $\Phi|_M$ and $f$ is a fibration onto a smooth curve $C$. Let $F$ be a general fiber of $f$. If we know (say by the vanishing theorem) that $\Phi|_{D+M}$ can distinguish general fibers of $f$ (i.e. separates any two points of the respective fibers) and its restriction to $F$ is birational, then $\Phi|_{D+M}$ is also birational.

Lemma 2.2. Let $X$ be a smooth projective variety of dimension $d$, $D \in \text{Div}(X) \otimes \mathbb{Q}$ be a $\mathbb{Q}$-divisor on $X$. Then the following assertions are true:

(i) if $S$ is a smooth reduced irreducible divisor on $X$ and $S$ is not a fractional component of $D$, then $\lceil D \rceil|_S \geq \lceil D \rceil|_S$;

(ii) if $\pi : X' \longrightarrow X$ is a birational morphism, then $\pi^*(\lceil D \rceil) \geq \lceil \pi^*(D) \rceil$.

Proof. This lemma is very easy to check. □

Lemma 2.3. Let $S$ be a smooth projective surface of general type and $L$ be a nef and big divisor on $S$. Then $\Phi|_{K_S+mL}$ is birational in the following cases:

(i) $m \geq 4$;

(ii) $m = 3$ and $L^2 \geq 2$.

Proof. This is a direct result of Corollary 2 of [16]. □

Lemma 2.4. (Lemma (3.2) of [7]) Let $X$ be a minimal threefold of general type with canonical index $r \geq 2$. Then $\dim\Phi_{mr+s}(X) \geq 2$ in the following cases, where $m$ is a positive integer and $0 \leq s < r$:

(i) $r = 2$ and $m \geq 3$;

(ii) $r = 3$ and $m \geq 2$;

(iii) $r = 4, 5, 0 \leq s \leq 2$ and $m \geq 2$; $r = 4, 5, s \geq 3$ and $m \geq 1$;

(iv) $r \geq 6, 0 \leq s \leq 1$ and $m \geq 2$; $r \geq 6, s \geq 2$ and $m \geq 1$.

Lemma 2.5. Under the same assumption as in Lemma 2.4, the plurigenus $P_{mr+s}(X) \geq 3$ in one of the following cases:

(i) $r = 2$ and $m \geq 2$;

(ii) $r \geq 3, 0 \leq s \leq 1$ and $m \geq 2$; $r \geq 3, s \geq 2$ and $m \geq 1$.

Proof. This is obvious from the proof of Lemma (3.2) in [7]. In order to be precise and to cite it many times in this paper, let us recall the estimation there.

When $r \geq 3$, $r$ is even and $s \geq 2$, we have

$$P_{mr+s}(X) \geq \frac{1}{12} \left\{ 2r^2m^3 + (6s - 3)r m^2 + \left( 6s^2 - 6s - \frac{1}{2}r^2 \right) m \right\} (rK_X^3) \geq \frac{1}{12}(r^2 + 6r + 9).$$ (2.1)
When $r \geq 3$, $r$ is odd and $s \geq 2$, we have

$$P_{mr+s}(X) \geq \frac{1}{12} \left\{ (mr+s)(mr+s-1)(2mr+2s-1) + m \left( -\frac{1}{2} r^3 + \frac{1}{2} r \right) ight. \\
- s(s-1)(2s-1) \right\} (K_X^3) \\
\geq \frac{1}{8} (r^2 + 6r + 8)$$

(2.2)

When $s = 1$, we have

$$P_{mr+1}(X) \geq \frac{1}{12} r(m^2 - 1)(2rm + 3)(rK_X^3)$$

(2.3)

When $s = 0$, we have

$$P_{mr}(X) \geq \frac{1}{12} r(m^2 - 1)(2rm - 3)(rK_X^3)$$

(2.4)

The above four formulae give the result. □

The following lemma is sufficient to derive our result, though it seems that one might exploit its potential.

**Lemma 2.6.** Let $S$ be a smooth projective surface of general type and $L$ be a nef divisor on $S$ such that $|L|$ gives a generically finite map. Then

(i) $L^2 \geq h^0(S, L) - 2$; if $\Phi_L$ is not birational, then $L^2 \geq 2h^0(S, L) - 4$.

(ii) if $p_g(S) > 0$, then $L^2 \geq 2h^0(S, L) - 4$.

(iii) $K_S + L$ is always effective.

**Proof.** The first part is trivial. One should note that a non-degenerate surface in $\mathbb{P}^N$ has degree $\geq N - 1$. In order to prove the second part, we may suppose that $|L|$ is base point free. Let $C$ be a general member of $|L|$, then

$$h^0(C, L|_C) \geq h^0(S, L) - 1.$$  

Noting that $C$ is moving and that we have the following exact sequence

$$0 \longrightarrow \mathcal{O}_S(K_S - C) \longrightarrow \mathcal{O}_S(K_S) \longrightarrow \mathcal{O}_C(K_S|_C) \longrightarrow 0,$$

the inclusion

$$H^0(S, K_S - C) \hookrightarrow H^0(S, K_S)$$

is proper. So $h^0(C, K_S|_C) > 0$, which means $h^1(C, L|_C) > 0$ and $L|_C$ is special. Thus Clifford’s theorem implies that

$$L^2 = \text{deg}(L|_C) \geq 2h^0(C, L|_C) - 2 \geq 2h^0(S, L) - 4.$$  

Finally, (iii) is an easy exercise by Riemann-Roch. □
Lemma 2.7. Let $X$ be a smooth projective variety of dimension $\geq 2$. Let $D$ be a divisor on $X$, $h^0(X, \mathcal{O}_X(D)) \geq 2$ and $S$ be a smooth irreducible divisor on $X$ such that $S$ is not a fixed component of $|D|$. Denote by $M$ the movable part of $|D|$ and by $N$ the movable part of $|D|_S$ on $S$. Suppose the natural restriction map

$$H^0(X, \mathcal{O}_X(D)) \xrightarrow{\theta} H^0(S, \mathcal{O}_S(D|_S))$$

is surjective. Then $M|_S \geq N$ and thus

$$h^0(S, \mathcal{O}_S(M|_S)) = h^0(S, \mathcal{O}_S(N)) = h^0(S, \mathcal{O}_S(D|_S)).$$

Proof. Denote by $Z$ the fixed part of $|D|$. Because $S$ is not a fixed component of $|Z|$, we see that $Z|_S \geq 0$. Thus $D|_S \geq M|_S$. Considering the natural map

$$H^0(X, \mathcal{O}_X(M)) \xrightarrow{\theta_0} H^0(S, \mathcal{O}_S(M|_S)),$$

we have

$$h^0(S, \mathcal{O}_S(M|_S)) \geq \dim \text{C} \text{im}(\theta_0) = \dim \text{C} \text{im}(\theta) = h^0(S, \mathcal{O}_S(D|_S)).$$

This means $h^0(S, \mathcal{O}_S(M|_S)) = h^0(S, \mathcal{O}_S(D|_S))$ and so $M|_S \geq N$ on $S$. □

3. The generic finiteness

This section is devoted to study the generic finiteness of $\phi_m$. Whenever we mention a minimal 3-fold, we mean one with only $\mathbb{Q}$-factorial terminal singularities. The following theorem is the easy part of the Main Theorem.

Theorem 3.1. Let $X$ be a projective minimal 3-fold of general type with $2 \leq r(X) \leq 5$. Then $\phi_{4r+3}$ is stably birational.

Proof. According to Lemma 2.5, $P_{m_1}(X) \geq 3$ for $m_1 \geq r + 2$. Take necessary blowing-ups $\pi : X' \to X$ along nonsingular centers, according to Hironaka, such that $X'$ is smooth and $|m_1 K_{X'}|$ defines a morphism (of course, $|m_1 K_{X'}|$ may have fixed components). Denote by $M$ the movable part of $|m_1 K_{X'}|$. We have

$$|K_{X'} + 3\pi^*(rK_X) + M| \subset |(m_1 + 3r + 1)K_{X'}|.$$ 

First we note that $K_{X'} + 3\pi^*(rK_X)$ is effective according to Lemma 2.5. If $|M|$ is not composed of a pencil, then a general member of it is an irreducible smooth projective surface $S$ of general type. Set $L := \pi^*(rK_X)|_S$, which is a nef and big Cartier divisor on $S$ with $L^2 \geq 2$. Using the vanishing theorem, we get

$$|K_{X'} + 3\pi^*(rK_X) + S|_S = |K_S + 3L|.$$ 

The right system gives a birational map by Lemma 2.3. So (P1) implies what we want in this case. If $|M|$ is composed of a pencil, we take the Stein-factorization of

$$\Phi_{m_1} : X' \to C \to W.$$
where $W$ is the image of $X'$ through $\Phi|_{M'|}$ and $f$ is a fibration onto the smooth curve $C$. Generically, $M$ can be written as a disjoint union of fibers of $f$, i.e.
$M \sim_{\text{lin}} \sum_{i=1}^{a} F_{i}$. The $F_{i}$’s are irreducible smooth surfaces of general type. The effectiveness of $K_{X'} + 3\pi^{*}(rK_{X})$ implies that $|K_{X'} + 3\pi^{*}(rK_{X}) + M|$ can distinguish general fibers of the morphism $\Phi|_{M'}$. On the other hand, we have the following exact sequence

$$H^{0}\left(X', K_{X'} + 3\pi^{*}(rK_{X}) + M\right) \longrightarrow \bigoplus_{i=1}^{a} H^{0}(F_{i}, K_{F_{i}} + 3L_{i}) \longrightarrow 0$$

where $L_{i} := \pi^{*}(rK_{X})|_{F_{i}}$ is a nef and big divisor with $L_{i}^{2} \geq 2$. This shows that the system $|K_{X'} + 3\pi^{*}(rK_{X}) + M|$ can also distinguish different components in a general fiber of $\Phi|_{M'}$ and the restriction to each $F_{i}$ gives a birational map. Thus, by (P2), we have completed the proof. □

Remark 3.2. Throughout this paper, we shall deal with the same situation as in the proof of Theorem 3.1. In order to avoid unnecessary redundancy, we give the definition of so-called generic irreducible element of a moving system $|M|$ on a variety $V$. Using our notations in the proof of Theorem 3.1, we shall call $S$ (respectively, $F_{i}$) a generic irreducible element of $|M|$ ignoring whether it is composed of a pencil or not. In our case, we always use both (P1) and (P2).

Theorem 3.3. Let $X$ be a projective minimal threefold of general type with the canonical index $r$. Then $\phi_{m}$ is generically finite whenever $m \geq l_{0}(r)$, where $l_{0}(r)$ is a function defined as the following

$$l_{0}(r) = \begin{cases} 
10 & \text{if } r = 2, \\
2r + 5 & \text{if } 3 \leq r \leq 5, \\
2r + 4 & \text{if } r \geq 6.
\end{cases}$$

Proof. The idea of the proof is quite simple. We formulate our proof through steps.

Step 1. Set up for the proof.

First, we define

$$m_{2} = \begin{cases} 
6 & \text{if } r = 2, \\
r + 3 & \text{if } 3 \leq r \leq 5, \\
r + 2 & \text{if } r \geq 6.
\end{cases}$$

Take a birational modification $\pi : X' \rightarrow X$, according to Hironaka, such that

(1) $X'$ is smooth;

(2) $|m_{2}K_{X'}|$ defines a morphism;

(3) the fractional part of $\pi^{*}(K_{X})$ has supports with only normal crossings.

Denote by $M_{2}'$ the movable part of $|m_{2}K_{X'}|$ and by $S_{2}'$ a general member of $|M_{2}'|$. From Lemma 2.4, we know that $\dim \phi_{m_{2}}(X) \geq 2$. Given any integer $t \geq r + 2$, we know that $|tK_{X'}|$ is always effective according to Lemma 2.5. If $|M_{2}'|$ has already given a generically finite map, then $\phi_{t+m_{2}}$ is generically finite and thus the theorem is true in this situation. So from now on, we suppose $\dim \phi_{m_{2}}(X) = 2$. From (2.1), (2.2), (2.3) and (2.4), we have

$$h^{0}(X', S_{2}') = P_{m_{2}}(X) \geq \begin{cases} 
12 & \text{if } r = 2, \\
7 & \text{if } r = 3, \\
\frac{1}{2}(r^{2} + 10r + 24) & \text{if } 4 \leq r \leq 5, \\
\frac{1}{2}(r^{2} + 6r + 2) & \text{if } r \geq 6.
\end{cases}$$
On the surface $S'_2$, we set $L'_2 := M'_2|_{S'_2}$. Then $|L'_2|$ is composed of a pencil. We can write

$$L'_2 \sim_{\text{lin}} \sum_{i=1}^{a'_2} C_i \sim_{\num} a'_2 C,$$

where $a'_2 \geq h^0(S'_2, L'_2) - 1$ and $C$ is a generic irreducible element of $|L'_2|$. Since $h^0(S'_2, L'_2) \geq h^0(X', S'_2) - 1$, we get

$$a'_2 \geq \begin{cases} 10 & \text{if } r = 2, \\ 5 & \text{if } r = 3, \\ \frac{1}{8}(r^2 + 10r + 8) & \text{if } 4 \leq r \leq 5, \\ \frac{1}{8}(r^2 + 6r - 8) & \text{if } r \geq 6. \end{cases}$$

Step 2. Reduce to the problem on a curve. Because

$$|K_{X'} + \gamma(t - 1)\pi^*(K_X) \cap + S'_2| \subset |(t + m_2)K_{X'}|,$$

it is sufficient to prove the generic finiteness of $\Phi|_{K_{X'} + \gamma(t - 1)\pi^*(K_X) \cap + S'_2}|$. Noting that $K_{X'} + \gamma(t - 1)\pi^*(K_X) \cap$ is effective, we only have to verify that

$$|K_{X'} + \gamma(t - 1)\pi^*(K_X) \cap + S'_2| \mid_{S'_2}$$

gives a generically finite map by virtue of (P1).

The vanishing theorem gives

$$|K_{X'} + \gamma(t - 1)\pi^*(K_X) \cap + S'_2| \mid_{S'_2} = |K_{S'_2} + \gamma(t - 1)\pi^*(K_X) \cap| \mid_{S'_2},$$

so we are reduced to verify the same property for

$$|K_{S'_2} + \gamma(t - 1)\pi^*(K_X) \cap| \mid_{S'_2}.$$

Since

$$K_{S'_2} + \gamma(t - 1)\pi^*(K_X) \cap \mid_{S'_2} = \left( K_{X'} + \gamma(t - 1)\pi^*(K_X) \cap \right) \mid_{S'_2} + L'_2$$

and $K_{X'} + \gamma(t - 1)\pi^*(K_X) \cap$ is effective by Lemma 2.5, the system

$$|K_{S'_2} + \gamma(t - 1)\pi^*(K_X) \cap| \mid_{S'_2}$$

can distinguish general fibers of $\Phi|_{L'_2}|$. So it is sufficient to show that

$$\Phi|_{K_{S'_2} + \gamma(t - 1)\pi^*(K_X) \cap} \mid_{C}$$

is a finite map for a generic irreducible element $C$ of $|L'_2|$.

Step 3. Verifying the finiteness on $C$.

Since $m_2\pi^*(K_X) \geq S'_2$, we can write

$$m_2\pi^*(K_X) \mid_{C} = L'_2 + E + \ldots + a'C + E.$$
where $E_Q$ is an effective $\mathbb{Q}$-divisor on $S'_2$. So we have

$$(t - 1)\pi^*(K_X)|_{S'_2} \sim_{\text{num}} \frac{(t - 1)a'_2}{m_2}C + \frac{t - 1}{m_2}E_Q$$

and

$$(t - 1)\pi^*(K_X)|_{S'_2} - C - \frac{1}{a'_2}E_Q \sim_{\text{num}} (t - 1)\left(1 - \frac{m_2}{(t - 1)a'_2}\right)\pi^*(K_X)|_{S'_2}.$$

Set $\alpha := 1 - \frac{m_2}{(t - 1)a'_2}$, it is easy to verify that $\alpha > 0$. This shows that

$$H^1(S'_2, K_{S'_2} + \gamma(t - 1)\pi^*(K_X)|_{S'_2} - \frac{1}{a'_2}E_Q \gamma - C) = 0$$

according to the vanishing theorem. Thus we have the exact sequence

$$H^0\left(S'_2, K_{S'_2} + \gamma(t - 1)\pi^*(K_X)|_{S'_2} - \frac{1}{a'_2}E_Q \gamma\right) \longrightarrow H^0(C, K_C + D) \longrightarrow 0,$$

where $D := \gamma(t - 1)\pi^*(K_X)|_{S'_2} - \frac{1}{a'_2}E_Q \gamma|_C$ is a divisor on $C$ with positive degree. In fact,

$$\deg(D) \geq (t - 1)\alpha\pi^*(K_X)|_{S'_2} \cdot C > 0.$$  

Noting that $C$ is a smooth curve of genus $\geq 2$, we see that $|K_C + D|$ gives a finite map. Therefore

$$\Phi|_{K_{S'_2} + \gamma(t - 1)\pi^*(K_X)|_{S'_2} - \frac{1}{a'_2}E_Q \gamma}|_C$$

is generically finite. So $\Phi|_{K_{S'_2} + \gamma(t - 1)\pi^*(K_X)|_{S'_2}}|_C$ is also generically finite. This derives the generic finiteness of $\phi_{t + m_2}$. \qed

4. The birationality: $r \geq 4$

Suppose $X$ is a projective minimal 3-fold of general type with the canonical index $r$. First we take a birational modification $\pi : X' \longrightarrow X$, according to Hironaka, such that

(i) $X'$ is smooth;
(ii) the system $|(r + 2)K_X'|$ defines a morphism;
(iii) the fractional part of $\pi^*(K_X)$ has supports with only normal crossings.

Denote by $M_2$ the movable part of $|(r + 2)K_X'|$. We know from Lemma 2.4 that $|M_2|$ is not composed of a pencil when $r \geq 6$.

From now on, we assume that $r \geq 4$ and that $|(r + 2)K_X'|$ is not composed of a pencil. Take a general member $S_2 \in |M_2|$. Then $S_2$ is a smooth projective surface of general type. Set $L_2 := M_2|_{S_2}$. Then $L_2$ is a nef divisor on the surface $S_2$. We have already known from the proof of Lemma 2.5 that, for $r \geq 4$,

$$h^0(X', S_2) = P_{r+2}(X) \geq \frac{1}{8}(r^2 + 6r + 8).$$
Theorem 4.1. Let $X$ be a projective minimal 3-fold of general type with the canonical index $r \geq 4$. If $\dim \phi_{r+2}(X) = 3$, then $\phi_{m_1(r)}$ is stably birational where

$$m_1(r) = \begin{cases} 
16, & \text{if } r = 4 \\
2r + 7, & \text{if } 5 \leq r \leq 6 \\
2r + 6, & \text{if } r \geq 7.
\end{cases}$$

Proof. Given an integer $t_1 > 0$, first we note that

$$|K_{X'} + (t_1 + r + 2)\pi^*(K_X)^\nu + S_2| \subset |(t_1 + 2r + 5)K_{X'}|.$$

In order to prove the birationality of $\phi_{t_1 + 2r + 5}$, it is sufficient to prove the same thing for $\Phi_{|K_{X'} + (t_1 + r + 2)\pi^*(K_X)^\nu + S_2|}$.

Step 1. Reduce to the problem on a surface.

Since $K_{X'} + (t_1 + r + 2)\pi^*(K_X)^\nu$ is effective by Lemma 2.5, it is enough to verify the same thing for its restriction to $S_2$ by virtue of (P1). The vanishing theorem gives the exact sequence

$$H^0(X', K_{X'} + (t_1 + r + 2)\pi^*(K_X)^\nu + S_2) \rightarrow H^0(S_2, K_{S_2} + (t_1 + r + 2)\pi^*(K_X)^\nu|_{S_2}) \rightarrow 0.$$

This means

$$|K_{X'} + (t_1 + r + 2)\pi^*(K_X)^\nu + S_2|_{S_2} = |K_{S_2} + (t_1 + r + 2)\pi^*(K_X)^\nu|_{S_2} |.$$

And from Lemma 2.2, we have

$$|K_{S_2} + (t_1 + r + 2)\pi^*(K_X)^\nu|_{S_2} \subset |K_{S_2} + (t_1 + r + 2)\pi^*(K_X)^\nu|_{S_2} |.$$

So, sometimes, it is enough to show that $|K_{S_2} + (t_1 + r + 2)\pi^*(K_X)^\nu|_{S_2} |$ gives a birational map. Under the assumption of this theorem, it is obvious that $|L_2|$ gives a generically finite map.

Step 2. Reduce to the problem on a curve.

We suppose $\overline{C}$ is the general member of the movable part of $|L_2|$. Since

$$K_{S_2} + (t_1 + r + 2)\pi^*(K_X)^\nu|_{S_2} = (K_{X'} + (t_1 + r + 2)\pi^*(K_X)^\nu)|_{S_2} + L_2 \geq L_2,$$

by (P1), we only have to verify the birationality of $\Phi_{|K_{S_2} + (t_1 + r + 2)\pi^*(K_X)^\nu|_{S_2} |}$

for a general member $\overline{C}$. It is obvious that

$$K_{S_2} + (t_1 + r + 2)\pi^*(K_X)^\nu|_{S_2} \geq K_{S_2} + (t_1 + r + 2)\pi^*(K_X)^\nu|_{S_2} + L_2 \geq K_{S_2} + (t_1 + r + 2)\pi^*(K_X)^\nu|_{S_2} + \overline{C}.$$

It’s sufficient to show that $\Phi_{|K_{S_2} + (t_1 + r + 2)\pi^*(K_X)^\nu|_{S_2} + \overline{C}} |$ is birational.

Step 3. Verifying the embedding on $\overline{C}$.

The vanishing theorem gives

$$|K_{S_2} + (t_1 + r + 2)\pi^*(K_X)^\nu|_{S_2} + \overline{C} | = |K_{\overline{C}} + D_0 |.$$
where $D_0 := \lceil t_1 \pi^*(K_X) \rceil_{S_2}$ is a divisor on $\mathcal{C}$ with

$$\deg(D_0) \geq t_1 \pi^*(K_X) \cdot C.$$ 

It’s clear that the theorem follows whenever $\deg(D_0) \geq 3$. Although, in general, $\pi^*(K_X) \cdot C$ is a positive rational number, we can estimate it in this situation.

Note that, if $|C|$ has already given a birational map, then so does $|K_{S_2} + \lceil (t_1 + r + 2) \pi^*(K_X) \rceil_{S_2}|$ because

$$K_{S_2} + \lceil (t_1 + r + 2) \pi^*(K_X) \rceil_{S_2} \geq L_2.$$ 

So we may suppose that $|C|$ gives a generically finite, non-birational map on the surface $S_2$. According to Lemma 2.6(i), we get

$$C^2 \geq 2h^0(S_2, C) - 4 \geq 1/4(r^2 + 6r) - 4.$$ 

Thus we get

$$(r + 2) \pi^*(K_X) \cdot C \geq C^2 \geq 1/4(r^2 + 6r) - 4.$$ 

One can easily obtain

$$\pi^*(K_X) \cdot C \begin{cases} \geq 1, & \text{if } r = 4 \\ \geq 10/7, & \text{if } r = 5 \\ \geq 7/4, & \text{if } r = 6 \\ > 2, & \text{if } r \geq 7. \end{cases}$$ 

Thus we can see that, whenever $t_1 \geq m_1(r) - 2r - 5$,

$$\deg(D_0) \geq \lceil t_1 \pi^*(K_X) \rceil_{S_2} \cdot C \geq 3.$$ 

We have proved that $\phi_{m_1(r)}$ is stably birational under assumption of the theorem. □

**Theorem 4.2.** Let $X$ be a projective minimal 3-fold of general type with the canonical index $r \geq 4$. If $\dim \phi_{r+2}(X) = 2$, then $\phi_{2r+6}$ is stably birational.

**Proof.** In this case, we note that $|L_2|$ is a base point free pencil on the surface $S_2$. We can write

$$L_2 \sim_{\text{lin}} \sum_{i=1}^{a_2} C_i \sim_{\text{num}} a_2 C,$$ 

where

$$a_2 \geq h^0(S_2, L_2) - 1 \geq 1/8(r^2 + 6r - 8)$$ 

and $C$ denotes a generic irreducible element of $|L_2|$. Given an integer $t_2 > 0$, we want to prove the birationality of $\phi_{t_2+2r+5}$. For the same reason as in the proof of Theorem 4.1, it is sufficient to verify the birationality of $\Phi_{|K_{S_2} + \lceil (t_2 + r + 2) \pi^*(K_X) \rceil_{S_2}|}$.

Step 1. Reduce to the problem on a curve.
Since $K_{X'} + \gamma(t_2 + r + 2)\pi^*(K_X) \sim$ is effective by Lemma 2.5, we can see that
\[ K_{S_2} + \gamma(t_2 + r + 2)\pi^*(K_X) \mid_{S_2} \geq L_2. \]
This shows that $\Phi|_{K_{S_2} + \gamma(t_2 + r + 2)\pi^*(K_X) \mid_{S_2}}$ can distinguish different fibers of $\Phi|_{L_2}$.

On the other hand, we have
\[ (t_2 + r + 2)\pi^*(K_X) \geq t_2\pi^*(K_X) + S_2. \]
So we get
\[ K_{S_2} + \gamma(t_2 + r + 2)\pi^*(K_X) \mid_{S_2} \geq K_{S_2} + \gamma t_2\pi^*(K_X) \mid_{S_2} + L_2. \]
The vanishing theorem gives the following exact sequence
\[ H^0 \left( S_2, K_{S_2} + \gamma t_2\pi^*(K_X) \mid_{S_2} + L_2 \right) \rightarrow \bigoplus_{i=1}^{a_2} H^0 (C_i, K_{C_i} + D_i) \rightarrow 0 \]
where the $D_i$ s are divisors on the curve $C_i$ with positive degree. This means that
the map $\Phi|_{K_{S_2} + \gamma t_2\pi^*(K_X) \mid_{S_2}}$ can distinguish disjoint irreducible components
in a general fiber of $\Phi|_{L_2}$. Thus the map $\Phi|_{K_{S_2} + \gamma(t_2 + r + 2)\pi^*(K_X) \mid_{S_2}}$ also has this
property. In order to apply (P2), we are reduced to verify that
\[ \Phi|_{K_{S_2} + \gamma(t_2 + r + 2)\pi^*(K_X) \mid_{S_2}} \mid_C \]
is an embedding for a generic irreducible element $C$ of $|L_2|$. Actually, it is sufficient
to verify this property for $\Phi|_{K_{S_2} + \gamma(t_2 + r + 2)\pi^*(K_X) \mid_{S_2}} \mid_C$.

Step 2. Calculation on $C$.

We can write
\[ (r + 2)\pi^*(K_X) \sim_{\text{lin}} S_2 + \bar{E}_Q, \]
where $\bar{E}_Q$ is an effective $\mathbb{Q}$-divisor. So one has
\[ (r + 2)\pi^*(K_X) \mid_{S_2} \sim_{\text{lin}} S_2 \mid_{S_2} + E_Q \sim_{\text{num}} a_2 C + E_Q \]
where $E_Q = \bar{E}_Q \mid_{S_2}$ is an effective $\mathbb{Q}$-divisor on $S_2$. Considering the system
\[ K_{S_2} + \gamma(t_2 + r + 2)\pi^*(K_X) \mid_{S_2} - \frac{1}{a_2} E_Q \]
we have
\[ K_{S_2} + \gamma(t_2 + r + 2)\pi^*(K_X) \mid_{S_2} \geq K_{S_2} + \gamma(t_2 + r + 2)\pi^*(K_X) \mid_{S_2} - \frac{1}{a_2} E_Q. \]
Note that
\[ (t_2 + r + 2)\pi^*(K_X) \mid_{S_2} - \frac{1}{a_2} E_Q - C \sim_{\text{num}} (t_2 + r + 2) \left( 1 - \frac{r + 2}{a_2(t_2 + r + 2)} \right) \pi^*(K_X) \mid_{S_2}, \]
which is a nef and big $\mathbb{Q}$-divisor on $S_2$ since $\beta := 1 - \frac{r + 2}{a_2(t_2 + r + 2)} > 0$. Thus, by the
vanishing theorem,
\[ H^1 \left( S_2, K_{S_2} + \gamma(t_2 + r + 2)\pi^*(K_X) \mid_{S_2} - \frac{1}{a_2} E_Q \right) = 0. \]
This would give the following exact sequence

\[ H^0\left(S_2, K_{S_2} + \gamma(t_2 + r + 2)\pi^*(K_X)|_{S_2} - \frac{1}{a_2}E_{Q}\right) \rightarrow H^0(C, K_C + D) \rightarrow 0, \]

where \( D := \gamma(t_2 + r + 2)\pi^*(K_X)|_{S_2} - \frac{1}{a_2}E_{Q}\). Now the main task is to show that \( \text{deg}(D) \geq 3 \), which implies that \( K_C + D \) is very ample since

\[ \text{deg}(K_C + D) \geq 2g(C) + 1. \]

In fact, we note that \( r\pi^*(K_X)|_{S_2} \) is a nef and big Cartier divisor on \( S_2 \), so \( r\pi^*(K_X)|_{S_2} \cdot C \) is a positive integer. And we have

\[
\text{deg}(D) \geq \left( (t_2 + r + 2)\pi^*(K_X)|_{S_2} - \frac{1}{a_2}E_{Q} \right) \cdot C = (t_2 + r + 2)\beta\pi^*(K_X)|_{S_2} \cdot C
\]

\[
= \left( t_2 + r + 2 - \frac{r + 2}{a_2} \right)\pi^*(K_X)|_{S_2} \cdot C
\]

\[
\geq r\pi^*(K_X)|_{S_2} \cdot C + (3 - \frac{r + 2}{a_2})\pi^*(K_X)|_{S_2} \cdot C.
\]

It is easy to see \( 3 - \frac{r + 2}{a_2} > 0 \). So \( \text{deg}(D) \geq 3 \) follows whenever \( r\pi^*(K_X)|_{S_2} \cdot C \geq 2 \), which will be proved in the next step.

Step 3. Estimating \( r\pi^*(K_X)|_{S_2} \cdot C \) by studying \( \phi_{3r+5} \).

We claim that \( r\pi^*(K_X)|_{S_2} \cdot C \geq 2 \). This can be derived from our studying \( \phi_{3r+5} \).

We have to use a lot of notations to perform the calculation.

We know that

\[ K_{X'} + \gamma(2r + 2)\pi^*(K_X) + S_2 \leq (3r + 5)K_{X'}. \]

The vanishing theorem gives

\[ |K_{X'} + \gamma(2r + 2)\pi^*(K_X) + S_2||_{S_2} = |K_{S_2} + \gamma(2r + 2)\pi^*(K_X)||_{S_2}| \]

\[
\supset |K_{S_2} + \gamma(2r + 2)\pi^*(K_X)|_{S_2} \supset |K_{S_2} + \gamma(2r + 2)\pi^*(K_X)|_{S_2} - \frac{1}{a_2}E_{Q}||. \]

(4.1)

Because

\[ (2r + 2)\pi^*(K_X)|_{S_2} - \frac{1}{a_2}E_{Q} - C \sim_{\text{num}} (2r + 2 - \frac{r + 2}{a_2})\pi^*(K_X)|_{S_2} \]

is a nef and big \( \mathbb{Q} \)-divisor, the vanishing theorem gives

\[ |K_{S_2} + \gamma(2r + 2)\pi^*(K_X)|_{S_2} - \frac{1}{a_2}E_{Q}||_{C} = |K_{C} + D_{3r+5}|, \]

(4.2)

where \( D_{3r+5} := \gamma(2r + 2)\pi^*(K_X)|_{S_2} - \frac{1}{a_2}E_{Q}||_{C} \) is a divisor on \( C \) with

\[ \text{deg}(D_{3r+5}) \geq (2r + 2 - \frac{r + 2}{a_2})\pi^*(K_X)|_{S_2} \cdot C \]

\[ = 2r\pi^*(K_X)|_{S_2} \cdot C + (2 - \frac{r + 2}{a_2})\pi^*(K_X)|_{S_2} \cdot C > 2. \]
Now let $M_{3r+5}$ be the movable part of $|(3r + 5)K_{X'}|$ and $M'_{3r+5}$ be the movable part of $|K_{X'} + r(2r + 2)\pi^*(K_X) + S_2|$. Then it's clear that

$$(3r + 5)\pi^*(K_X) \geq M_{3r+5} \geq M'_{3r+5}.$$ 

Let $L_{3r+5}$ be the movable part of $|K_{S_2} + r(2r + 2)\pi^*(K_X)|_{S_2}$ and $L'_{3r+5}$ be the movable part of $|K_{S_2} + r(2r + 2)\pi^*(K_X)|_{S_2} - \frac{1}{a_2}E\mathbb{Q}^\parallel$.

Then $L_{3r+5} \geq L'_{3r+5}$. From (4.1) and Lemma 2.7, we have $M_{3r+5}|_{S_2} \geq L_{3r+5}$. From (4.2) and Lemma 2.7, we have

$$h^0(C, L'_{3r+5}|_{C}) = h^0(C, K_C + D_{3r+5}) = g(C) - 1 + \deg(D_{3r+5}) \geq g(C) + 2.$$ 

Using R-R and the Clifford's theorem and noting that $g(C) \geq 2$, one can easily see that $L'_{3r+5} \cdot C \geq 2g(C) + 1 \geq 5$. So we get

$$(3r + 5)\pi^*(K_X)|_{S_2} \cdot C \geq M_{3r+5}|_{S_2} \cdot C \geq L_{3r+5} \cdot C \geq L'_{3r+5} \cdot C \geq 5$$

i.e. $r\pi^*(K_X)|_{S_2} \cdot C \geq \frac{5r}{3r+5} > 1$. Noting that $r\pi^*(K_X)|_{S_2} \cdot C$ is an integer, we see $r\pi^*(K_X)|_{S_2} \cdot C \geq 2$. The proof is completed. \(\Box\)

From Theorem 4.1 and Theorem 4.2, we instantly have the following

**COROLLARY 4.3.** Let $X$ be a projective minimal 3-fold of general type with the canonical index $r \geq 6$. Then $\phi_{\text{mo}(r)}$ is stably birational.

**Proof.** The main point is $\dim\phi_{r+2}(X) \geq 2$ for $r \geq 6$ according to Lemma 2.4. \(\Box\)

For $4 \leq r \leq 5$, we have to treat the case with $\dim\phi_{r+2}(X) = 1$. We shall use a similar method as above by studying the system $|(r+3)K_{X'}|$ because $\dim\phi_{r+3}(X) \geq 2$. First we take a birational modification $\pi : X' \rightarrow X$, according to Hironaka, such that

(i) $X'$ is smooth;

(ii) the system $|(r+3)K_{X'}|$ defines a morphism;

(iii) the fractional part of $\pi^*(K_X)$ has supports with only normal crossings.

Denote by $M_3$ the movable part of $|(r+3)K_{X'}|$. Take a general member $S_3 \in |M_3|$. Then $S_3$ is a smooth irreducible projective surface of general type. Set $L_3 := M_3|_{S_3}$. Then $L_3$ is a nef divisor on the surface $S_3$. Taking $s = 3$ and using (2.1) and (2.2), we have

$$h^0(X', S_3) = P_{r+3}(X) \geq \frac{1}{8}(r^2 + 10r + 24).$$

Thus $h^0(S_3, L_3) \geq h^0(X', S_3) + 1 \geq \frac{1}{8}(r^2 + 10r + 16)$.
Theorem 4.4. Let $X$ be a projective minimal 3-fold of general type with the canonical index $r = 4$, 5. Suppose $\dim \phi_{r+3}(X) = 3$. Then

(i) if $r = 4$, $\phi_{17}$ is stably birational;
(ii) if $r = 5$, $\phi_{18}$ is stably birational.

Proof. Given an integer $t_3 > 0$, we note that

$$|K_{X'} + (t_3 + r + 3)\pi^*(K_X) + S_3| \subset |(t_3 + 2r + 7)K_{X'}|.$$ 

In order to prove the birationality of $\phi_{t_3 + 2r + 7}$, it is sufficient to prove the same thing for $\Phi|K_{X'} + (t_3 + r + 3)\pi^*(K_X) + S_3|$. 

Step 1. Reduce to the problem on a surface.

Since $K_{X'} + (t_3 + r + 3)\pi^*(K_X)$ is effective by Lemma 2.5, it is enough to verify the same thing for its restriction to $S_3$ by virtue of (P1). The vanishing theorem gives

$$|K_{X'} + (t_3 + r + 3)\pi^*(K_X) + S_3|_{S_3} = |K_{S_3} + (t_3 + r + 3)\pi^*(K_X)|_{S_3}.$$ 

And from Lemma 2.2, we have

$$|K_{S_3} + (t_3 + r + 3)\pi^*(K_X)|_{S_3} \subset |K_{S_3} + (t_3 + r + 3)\pi^*(K_X) + S_3|_{S_3}.$$ 

So, sometimes, it is enough to show that $|K_{S_3} + (t_3 + r + 3)\pi^*(K_X)|_{S_3}$ gives a birational map. Under the assumption of this theorem, it is obvious that $|L_3|$ gives a generically finite map.

Step 2. Reduce to the problem on a curve.

We suppose $C'$ is the general member of the movable part of $|L_3|$. Since

$$K_{S_3} + (t_3 + r + 3)\pi^*(K_X)|_{S_3} = (K_{X'} + (t_3 + r + 3)\pi^*(K_X)|_{S_3} + L_3 \geq L_3,$$

by (P1), we only have to verify the birationality of $\Phi|K_{S_3} + (t_3 + r + 3)\pi^*(K_X)|_{S_3} |_{C'}$ for a general member $C'$. It is obvious that

$$K_{S_3} + (t_3 + r + 3)\pi^*(K_X)|_{S_3} \geq K_{S_3} + t_3\pi^*(K_X)|_{S_3} + L_3$$

$$\geq K_{S_3} + t_3\pi^*(K_X)|_{S_3} + C'.$$

It’s sufficient to show that $\Phi|K_{S_3} + (t_3 + r + 3)\pi^*(K_X)|_{S_3} + C'|_{C'}$ is birational.

Step 3. Verifying the embedding on $C'$.

The vanishing theorem gives

$$|K_{S_3} + t_3\pi^*(K_X)|_{S_3} + C'|_{C'} = |K_{C'} + D_1|,$$

where $D_1 := t_3\pi^*(K_X)|_{S_3} + C'$ is a divisor on $C'$ with

$$\deg(D_1) \geq t_3\pi^*(K_X)|_{S_3} \cdot C'.$$

It’s clear that the theorem follows whenever $\deg(D_1) \geq 3$. Although, in general, $\pi^*(K_X)|_{C'}$ is only a rational number, we can still estimate it in this situation.
Note that, if $|C'|$ has already given a birational map, then so does $|K_{S_3} + \gamma(t_3 + r + 3)\pi^*(K_X)|_{S_3}|$ because

$$K_{S_3} + \gamma(t_3 + r + 3)\pi^*(K_X)_{S_3} \geq L_3.$$ 

So we may suppose that $|C'|$ gives a generically finite, non-birational map on the surface $S_3$. According to Lemma 2.6(i), we get

$$C'^2 \geq 2h^0(S_2, C') - 4 \geq \frac{1}{4}(r^2 + 10r).$$

Thus we get

$$(r + 3)\pi^*(K_X)|_{S_3} \cdot C' \geq C'^2 \geq \frac{1}{4}(r^2 + 10r).$$

One can easily obtain

$$\pi^*(K_X)|_{S_3} \cdot C' \begin{cases} \geq 2, & \text{if } r = 4 \\ \geq \frac{19}{8}, & \text{if } r = 5. \end{cases}$$

Thus we can see that, whenever $t_3 \geq 2$ if $r = 4$ or $t_3 \geq 1$ if $r = 5$,

$$\deg(D_1) \geq \gamma t_3 \pi^*(K_X)|_{S_3} \cdot C' \geq 3.$$ 

We have proved the theorem. □

**Theorem 4.5.** Let $X$ be a projective minimal 3-fold of general type with the canonical index $r = 4, 5$. Suppose $\dim \phi_{r+3}(X) = 2$. Then

(i) if $r = 4$, $\phi_{16}$ is stably birational;

(ii) if $r = 5$, $\phi_{18}$ is stably birational.

**Proof.** In this case, we note that $|L_3|$ is a base point free pencil on the surface $S_3$. We can write

$$L_3 \sim_{\text{lin}} \sum_{i=1}^{a_3} C_i \sim_{\text{num}} a_3 C,$$

where

$$a_3 \geq h^0(S_3, L_3) - 1 \geq \frac{1}{8}(r^2 + 10r + 8)$$

and $C$ denotes a generic irreducible element of $|L_3|$. Given an integer $t_4 > 0$, we want to prove the birationality of $\phi_{t_4+2r+7}$. For the same reason as in the proof of Theorem 4.4, it is sufficient to verify the birationality of $\Phi|_{K_{S_3} + \gamma(t_4+r+3)\pi^*(K_X)}|_{S_3}|$.

Step 1. Reduce to the problem on a curve.

Since $K_X + \gamma(t_4 + r + 3)\pi^*(K_X)$ is effective by Lemma 2.5, we can see that

$$K_{S_3} + \gamma(t_4 + r + 3)\pi^*(K_X)_{S_3} \geq L_3.$$ 

This shows that $\Phi|_{K_{S_3} + \gamma(t_4+r+3)\pi^*(K_X)}|_{S_3}|$ can distinguish different fibers of $\Phi|_{L_3}|$. On the other hand, we have

$$(t_4 + r + 2)\pi^*(K_X) \geq t_4 \pi^*(K_X) + S.$$
So we get

\[ K_{S_3} + \gamma(t_4 + r + 3)\pi^*(K_X)|_{S_3} \geq K_{S_3} + \gamma t_4 \pi^*(K_X)|_{S_3} + L_3. \]

The vanishing theorem gives the following exact sequence

\[ H^0\left( S_3, K_{S_3} + \gamma t_4 \pi^*(K_X)|_{S_3} + L_3 \right) \rightarrow \bigoplus_{i=1}^{a_3} H^0(C_i, K_{C_i} + D_i) \rightarrow 0 \]

where the \( D_i \)'s are divisors on the curve \( C_i \) with positive degree. This means that the map \( \Phi|_{K_{S_3} + \gamma t_4 \pi^*(K_X)|_{S_3} + L_3} \) can distinguish disjoint irreducible components in a general fiber of \( \Phi|_{L_3} \). Thus the map \( \Phi|_{K_{S_3} + \gamma(t_4 + r + 3)\pi^*(K_X)|_{S_3}} \) also has this property. In order to apply (P2), we are reduced to verify that

\[ \Phi|_{K_{S_3} + \gamma(t_4 + r + 3)\pi^*(K_X)|_{S_3}}|_C \]

is an embedding for a generic irreducible element \( C \) of \( |L_3| \). Actually, it is sufficient to verify this property for \( \Phi|_{K_{S_3} + \gamma(t_4 + r + 3)\pi^*(K_X)|_{S_3}}|_C \).

Step 2. Calculation on \( C \).

We can write

\[ (r + 3)\pi^*(K_X) \sim_{\text{lin}} S_3 + \overline{E_Q}, \]

where \( \overline{E_Q} \) is an effective \( \mathbb{Q} \)-divisor. So one has

\[ (r + 3)\pi^*(K_X)|_{S_3} \sim_{\text{lin}} S_3|_{S_3} + E_Q \sim_{\text{num}} a_3 C + E_Q \]

where \( E_Q = \overline{E_Q}|_{S_3} \) is an effective \( \mathbb{Q} \)-divisor on \( S_3 \). Considering the system

\[ |K_{S_3} + \gamma(t_4 + r + 3)\pi^*(K_X)|_{S_3} - \frac{1}{a_3} E_Q| \]

we have

\[ K_{S_3} + \gamma(t_4 + r + 3)\pi^*(K_X)|_{S_3} \geq K_{S_3} + \gamma(t_4 + r + 3)\pi^*(K_X)|_{S_3} - \frac{1}{a_3} E_Q| \]

Note that

\[ (t_4 + r + 3)\pi^*(K_X)|_{S_3} - \frac{1}{a_3} E_Q - C \sim_{\text{num}} (t_4 + r + 3)\left( 1 - \frac{r + 3}{a_3(t_4 + r + 3)} \right)\pi^*(K_X)|_{S_3}, \]

which is a nef and big \( \mathbb{Q} \)-divisor on \( S_3 \) since \( \gamma := 1 - \frac{r + 3}{a_3(t_4 + r + 3)} > 0 \). Thus, by the vanishing theorem,

\[ H^1\left( S_3, K_{S_3} + \gamma(t_4 + r + 3)\pi^*(K_X)|_{S_3} - \frac{1}{a_3} E_Q - C \right) = 0. \]

This would give the following exact sequence

\[ H^0\left( S_3, K_{S_3} + \gamma(t_4 + r + 3)\pi^*(K_X)|_{S_3} - \frac{1}{a_3} E_Q \right) \rightarrow H^0(C, K_{C} + D) \rightarrow 0. \]
where \( D := \gamma(t_4 + r + 3)\pi^*(K_X)|_{S_3} - \frac{1}{a_3}E_{|S_3|} \). Now the main task is to show that \( \deg(D) \geq 3 \), which implies that \( K_C + D \) is very ample since

\[
\deg(K_C + D) \geq 2g(C) + 1.
\]

In fact, we note that \( r\pi^*(K_X)|_{S_3} \) is a nef and big Cartier divisor on \( S_3 \), so \( r\pi^*(K_X)|_{S_3} \cdot C \) is a positive integer. And we have

\[
\deg(D) \geq \left( t_4 + r + 3 - \frac{r + 3}{a_3} \right) \pi^*(K_X)|_{S_3} \cdot C = \left( t_4 + r + 3 \right) \gamma\pi^*(K_X)|_{S_3} \cdot C \]

\[
\geq r\pi^*(K_X)|_{S_3} \cdot C + (4 - \frac{r + 3}{a_3}) \pi^*(K_X)|_{S_3} \cdot C.
\]

It is easy to see \( 4 - \frac{r + 3}{a_3} > 0 \). So \( \deg(D) \geq 3 \) follows whenever \( r\pi^*(K_X)|_{S_3} \cdot C \geq 2 \), which will be proved in the next step.

Step 3. Estimating \( r\pi^*(K_X)|_{S_3} \cdot C \) by studying \( \phi_{3r+5} \).

We claim that \( r\pi^*(K_X)|_{S_3} \cdot C \geq 2 \). This can be derived from our studying \( \phi_{3r+5} \).

We have to use a lot of notations to perform the calculation.

We know that

\[
K_{X'} + \gamma(2r + 1)\pi^*(K_X) \geq S_3 \leq (3r + 5)K_{X'}.
\]

The vanishing theorem gives

\[
|K_{X'} + \gamma(2r + 1)\pi^*(K_X) \geq S_3|_{S_3} = |K_{S_3} + \gamma(2r + 1)\pi^*(K_X)|_{S_3} \]

\[
\supset |K_{S_3} + \gamma(2r + 1)\pi^*(K_X)|_{S_3} \geq |K_{S_3} + \gamma(2r + 1)\pi^*(K_X)|_{S_3} - \frac{1}{a_3}E_{|S_3|} |_{S_3}, \quad (4.3)
\]

Because

\[
(2r + 1)\pi^*(K_X)|_{S_3} - \frac{1}{a_3}E_{|S_3|} - C \sim_{\text{num}} (2r + 1 - \frac{r + 3}{a_3})\pi^*(K_X)|_{S_3}
\]

is a nef and big \( \mathbb{Q} \)-divisor, the vanishing theorem gives

\[
|K_{S_3} + \gamma(2r + 1)\pi^*(K_X)|_{S_3} - \frac{1}{a_3}E_{|S_3|}|_{C} = |K_C + D_{3r+5}|, \quad (4.4)
\]

where \( D_{3r+5} := \gamma(2r + 1)\pi^*(K_X)|_{S_3} - \frac{1}{a_3}E_{|S_3|}|_{C} \) is a divisor on \( C \) with

\[
\deg(D_{3r+5}) \geq (2r + 1 - \frac{r + 3}{a_3})\pi^*(K_X)|_{S_3} \cdot C
\]

\[
= 2r\pi^*(K_X)|_{S_3} \cdot C + (1 - \frac{r + 3}{a_3})\pi^*(K_X)|_{S_3} \cdot C > 2.
\]

Now let \( M_{3r+5} \) be the movable part of \(|(3r + 5)K_{X'}|\) and \( M'_{3r+5} \) be the movable part of \(|K_{X'} + \gamma(2r + 1)\pi^*(K_X)|_{S_3}|. \) Then it’s clear that

\[
(3r + 5)\pi^*(K_X) \geq M_{3r+5} \geq M'_{3r+5}.
\]
Let $L_{3r+5}$ be the movable part of $|K_{S_3} + (2r + 1)\pi^*(K_X)|_{S_3}$ and $L'_{3r+5}$ be the movable part of $|K_{S_3} + (2r + 1)\pi^*(K_X)|_{S_3} - \frac{1}{a_3}E_{\mathbb{Q}}|_{\ast}$. Then $L_{3r+5} \geq L'_{3r+5}$. From (4.3) and Lemma 2.7, we have $M_{3r+5}|_{S_3} \geq L_{3r+5}$. From (4.4) and Lemma 2.7, we have $h_0(C, L'_{3r+5}|_{C}) = h_0(C, K_C + D_{3r+5}) = g(C) - 1 + \deg(D_{3r+5}) \geq g(C) + 2$.

Using R-R and the Clifford's theorem and noting that $g(C) \geq 2$, one can easily see that $L'_{3r+5} \cdot C \geq 2g(C) + 1 \geq 5$. So we get

$$(3r+5)\pi^*(K_X)|_{S_3} \cdot C \geq M_{3r+5}|_{S_3} \cdot C \geq L_{3r+5} \cdot C \geq L'_{3r+5} \cdot C \geq 5$$

i.e. $r\pi^*(K_X)|_{S_3} \cdot C \geq \frac{5r}{3r+5} > 1$. Noting that $r\pi^*(K_X)|_{S_3} \cdot C$ is an integer, we see $r\pi^*(K_X)|_{S_3} \cdot C \geq 2$. The proof is completed.

From Theorem 3.1, we can take $m_0(2) = 11$ and $m_0(3) = 15$. Theorems 4.4 and 4.5 imply $m_0(4) = 17$ and $m_0(5) = 18$. Therefore the main theorem follows.

5. Threefolds with positive geometric genus

Throughout this section, we still suppose $X$ is a projective minimal 3-fold of general type. Our aim is to study 3-folds with big geometric genus using the method of the Main Theorem. Kollár (Corollary 4.8 of [11]) proved that $\phi_{16}$ is birational if $p_g(X) \geq 2$. Reviewing the parallel results on surfaces and Gorenstein 3-folds, one should expect a better bound for the birationality of $\phi_m$.

To begin the argument, we first take a birational modification $\pi : X' \rightarrow X$ according to Hironaka such that

(i) $X'$ is smooth;

(ii) $|K_{X'}|$ gives a morphism;

(iii) the fractional part of $\pi^*(K_X)$ has supports with only normal crossings.

Set $g := \phi_{1} \circ \pi$ and take the Stein-factorization of

$$g : X' \xrightarrow{f} W \rightarrow W' \subset \mathbb{P}^N$$

where $W'$ is the image of $X'$ through $g$ and $f$ is a fibration. Let $M$ be the movable part of $|K_{X'}|$. We can write

$$K_{X'} \sim_{\text{lin}} M + E' \quad \text{and} \quad \pi^*(K_X) =_{\mathbb{Q}} M + E_{\mathbb{Q}}$$

where $E'$ is an effective divisor and $E_{\mathbb{Q}}$ is an effective $\mathbb{Q}$-divisor.
Proposition 5.1. Let $X$ be a minimal projective threefold of general type. If $\dim \phi_1(X) \geq 2$, then

(i) $\phi_4$ is generically finite;

(ii) $\phi_3$ is generically finite provided $p_g(X) \geq 4$.

Proof. Because $p_g(X) > 0$, it is sufficient to prove for the case when $\dim \phi_1(X) = 2$. Let $S \in |M|$ be the general member. Then $S$ is a smooth projective surface of general type. We have $S|_S \sim_{\text{num}} aC$, where $C$ is a smooth curve and $a = p_g(X) - 2$. Considering the system $|K_X' + \pi^*(K_X) + 2S|$, we have

$$|K_X' + \pi^*(K_X)^\gamma + 2S|_S = |K_S + \pi^*(K_X)^\gamma|_S + S|_S.$$ 

Besides, we have

$$|K_S + \pi^*(K_X)|_S + S|_S|_C = |K_C + D|,$$

where $D := \pi^*(K_X)|_S^\gamma|_C$ is a divisor on $C$ of positive degree. Thus $|K_C + D|$ gives a finite map and so does $\phi_4$. This derives (i). If $p_g(X) \geq 4$, then $a \geq 2$. By the vanishing theorem, we get

$$|K_X' + \pi^*(K_X)^\gamma + 2S|_S = |K_S + \pi^*(K_X)^\gamma|_S$$

$$\supset |K_S + \pi^*(K_X)|_S^\gamma|.$$ 

We can write $\pi^*(K_X)|_S = S|_S + E_\mathbb{Q}$, where $E_\mathbb{Q}$ is an effective $\mathbb{Q}$-divisor. It is obvious that

$$\pi^*(K_X)|_S - \frac{1}{a}E_\mathbb{Q} - C \sim_{\text{num}} \left(1 - \frac{1}{a}\right)\pi^*(K_X)|_S.$$ 

So

$$H^1\left(S, K_S + \pi^*(K_X)|_S - \frac{1}{a}E_\mathbb{Q} - C\right) = 0,$$

which gives

$$|K_S + \pi^*(K_X)|_S - \frac{1}{a}E_\mathbb{Q} - C|_C = |K_C + D|,$$

where $D := \pi^*(K_X) - \frac{1}{a}E_\mathbb{Q}|_C$ is a divisor on $C$ of positive degree. Thus $|K_C + D|$ gives a generically finite map. By (P2), $\phi_3$ is also generically finite. \square

Proposition 5.2. Let $X$ be a minimal projective threefold of general type. If $\dim \phi_1(X) \geq 2$, then

(i) $\phi_8$ is birational;

(ii) $\phi_6$ is birational provided $p_g(X) \geq 4$.

Proof. If $\dim \phi_1(X) = 3$, then it is very easy to prove the birationality of $\phi_6$ by standard argument. We mainly discuss the case when $\dim \phi_1(X) = 2$. To prove (i), let $M_4$ be the movable part of $|4K_X'|$. We can modify $\pi$, if necessary, such that $|M_4|$ is also base point free. We have

$$|K_X' + \pi^*(K_X)^\gamma + M_4 + 2S|_S = |K_S + \pi^*(K_X)^\gamma|_S + M_4 + S|_S,$$

where $L_4 := M_4|_S$ is nef and $\Phi|_{L_4}$ is generically finite. It is not difficult to see that the right system gives a birational map using the method which has been applied frequently in this paper. So $\phi_6$ is birational according to (P1).
If \( p_g(X) \geq 4 \), denote by \( M_3 \) the moving part of \(|3K_X|\). For the same reason, we can suppose \(|M_3|\) is also base point free. We have

\[
|K_X + 4\pi^*(K_X) + S|_S = |K_S + 4\pi^*(K_X)|_s \supset |K_S + \pi^*(K_X)|_s \supset |K_S + \pi^*(K_X)|_s \oplus L_3|,
\]

where \( L_3 := M_3|s \), which is a nef and big divisor on \( S \). And \(|L_3|\) gives a generically finite map. Because

\[
\pi^*(K_X)|_S - C - \frac{1}{a}E_Q
\]

is nef and big, the vanishing theorem will imply that

\[
|K_S + \pi^*(K_X)|_S - \frac{1}{a}E_Q \oplus L_3|_C = |K_C + L_3|_C + D|,
\]

where \( D = \pi^*(K_X)|_S - \frac{1}{a}E_Q \oplus L_3 \) is a divisor of positive degree. Thus \( \deg(L_3|_C + D) \geq 3 \) and \( K_C + L_3|_C + D \) is very ample. This shows that \( \phi_6 \) is birational. □

**Proposition 5.3.** Let \( X \) be a minimal projective threefold of general type. If \( \dim \phi_1(X) = 1 \), then

(i) \( \phi_6 \) is birational;

(ii) \( \phi_6 \) is birational provided \( p_g(X) \geq 12 \).

**Proof.** (i). In this case, \( W \) is a nonsingular curve. We set \( b := g(W) \), the genus of \( W \).

If \( b > 0 \), then \( \phi_1 \) is actually a morphism. In this case, there is no need to make the modification \( \pi \), i.e. \( X' = X \). Though \( K_X \) is not Cartier, it is a Weil divisor. We can still define the system \(|K_X|\) in a natural way. We have \( M \sim_{\text{lin}} \sum S_i \), where the \( S_i \) are fibers of \( f \). Noting that the singularities on \( X \) are all isolated, a general \( S_i \) is a smooth projective surface of general type. Using Kawamata’s vanishing theorem ([10]) for \( \mathbb{Q} \)-Cartier Weil divisors, we have \( H^1(X, \omega^k_X) = 0 \) whenever \( k > 1 \). Thus \( \phi_6|_{S_i} = \Phi|_{5K_{S_i}} \) is birational. According to (P2), \( \phi_6 \) is birational.

If \( b = 0 \), \( W = \mathbb{P}^1 \) and we have the fibration \( f : X' \rightarrow \mathbb{P}^1 \). Let \( S \) be a general fiber of \( f \). Then \( S \) is a smooth projective surface of general type. We divide \( S \) into two categories:

(I) \((K_{S_0}^2, p_g(S)) \neq (1, 2)\) and \((2, 3);\)

(II) the rest.

where \( S_0 \) denotes the minimal model of \( S \).

Suppose \( S \) is of type (I). There is a common property for these surfaces that the 3-canonical maps are birational. To deal with this situation, We can use Kollár’s technique (the proof of Corollary 4.8 in [11]). Because \( p_g(X) > 0 \), we have \( p_g(S) > 0 \). Let \( \sigma : S \rightarrow S_0 \) be the contraction onto the minimal model. According to Theorem 3.1 in [5], we see that \(|2K_{S_0}|\) is base point free. So the movable part of \(|2K_S|\) is \( \sigma^*(2K_{S_0}) \). We have \( H^0(\omega^2_X) = H^0(\mathbb{P}^1, f^*\omega^2_X) \) and an injection \( \mathcal{O}(1) \hookrightarrow f^*\omega^2_X \), and hence an injection \( \mathcal{O}(5) \hookrightarrow f^*\omega^2_X \). This gives an injection

\[
\mathcal{O}(5) \otimes f^*\omega^2_X \hookrightarrow f^*\omega^2_X,
\]

where

\[
\mathcal{O}(5) \otimes f^*\omega^2_X = \mathcal{O}(1) \otimes f^*\omega^2_X.
\]
It is well-known that $f_\ast \omega_X^2/\mathbb{P}^1$ is a sum of line bundles of non-negative degree on $\mathbb{P}^1$. The local sections of $f_\ast \omega_X^2$, give the bicanonical map for $S$, and all these extend to global sections of $\mathcal{O}(5) \otimes f_\ast \omega_X^2$. Moreover the sections of $\mathcal{O}(1) \otimes f_\ast \omega_X^2$ separate different fibers. Suppose $M_7$ is the movable part of $|7K_X'|$. Because $\phi_7 = \Phi_{|M_7|}$, we can see from the above argument that $M_7|_S \geq \sigma^*(2K_{S_0})$. Now considering the system $|K_{X'} + 7\pi^*(K_X)\cap + S|$, we have

$$|K_{X'} + 7\pi^*(K_X)\cap + S|_S = |K_S + 7\pi^*(K_X)\cap|_S$$

$$\supset |K_S + M_7|_S \supset |K_S + \sigma^*(2K_{S_0})|. $$

Because $|K_S + \sigma^*(2K_{S_0})|$ gives a birational map, we see that $\phi_9$ is birational. Suppose $S$ is of type (II) and $(K_{S_0}^2, p_g(S)) = (2, 3)$. We want to show that $\phi_S$ is birational. We know that the movable part of $|K_{X'}|$ is linearly equivalent to a disjoint union of irreducible copies of $S$. We have

$$|K_{X'} + 3\pi^*(K_X)\cap + S|_S = |K_S + 3\pi^*(K_X)\cap|_S.$$

Because the movable part of $|K_S|$ gives a finite map onto $\mathbb{P}^2$, we can see that $\phi_3$ is generically finite. Denote by $M_3$ the movable part of $|3K_{X'}|$ and by $M_5$ the movable part of $|5K_{X'}|$. In order to prove the birationality of $\phi_S$, we should study $|M_5|_S$. We have $K_{X'} + 3\pi^*(K_X)\cap + S \leq 5K_{X'}$. The vanishing theorem gives

$$|K_{X'} + 3\pi^*(K_X)\cap + S|_S = |K_S + 3\pi^*(K_X)\cap|_S.$$

We suppose that $K_0$ is the movable part of $|K_S|$. Denote by $M_+$ the movable part of $|K_{X'} + 3\pi^*(K_X)\cap + S|$. Then $M_+ \leq M_5$. By Lemma 2.7, we can see that $M_+|_S$ contains the movable part of $|K_S + 3\pi^*(K_X)\cap|_S$. Let $L$ be the movable part of $|M_5|_S$. Then $K_S + 3\pi^*(K_X)\cap|_S \geq K_0 + L$ and $K_0 + L$ is movable. So $M_+|_S \geq K_0 + L$. On the other hand, because $M_5 \geq M_+$, so $M_5|_S \geq K_0 + L$. Now it is time to study the $\phi_S$. For an obvious reason, we can suppose that $|M_5|$ is base point free. This assumption means that $M_5$ is nef and big. The vanishing theorem gives the exact sequence

$$H^0 \left(X', K_{X'} + \pi^*(K_X)\cap + M_5 + S \right) \longrightarrow H^0 \left(S, K_S + \pi^*(K_X)\cap|_S + M_5|_S \right) \longrightarrow 0.$$

We note that

$$K_S + \pi^*(K_X)\cap|_S + M_5|_S \geq K_S + \pi^*(K_X)|_S \cap + L + K_0.$$ 

Since $\pi^*(K_X)|_S$ is nef and big, $\pi^*(K_X)|_S \cap$ is effective, $\dim \Phi_{|K_0|}(S) = 2$ and $\dim \Phi_{|L|}(S) = 2$, using our method again, it is easy to see that $|K_S + \pi^*(K_X)|_S \cap + L + K_0|$ gives a birational map. Which shows that

$$|K_{X'} + \pi^*(K_X)\cap | + M_5 + S|$$

gives a birational map and so does $\phi_S$.

Suppose $S$ is of type (II) and $(K_{S_0}^2, p_g(S)) = (1, 2)$. We want to show that $\phi_9$ is birational. This is the most frustrating case because $\Phi_{|K_{S_0}|}$ is not birational. We recall that $|K_{S_0}|$ has no fixed component, that it has exactly one base point.
and that a general member of this system is a smooth irreducible curve of genus 2. Thus the movable part $C$ of $|K_S|$ is also a smooth curve of genus two. Furthermore $C \leq \sigma^*(K_{S_0})$. Because

$$|K_{X'} + \pi^* (K_X)^\gamma + S|_S = |K_S + \pi^* (K_X)^\gamma|_S \supset |K_S| \supset |C|,$$

we see that $\dim \phi_3(X) \geq 2$. We still denote by $M_3$ the movable part of $|3K_{X'}|$ and by $M'_3$ the movable part of $|K_{X'} + \pi^* (K_X)^\gamma + S|$. According to Lemma 2.7, $M_3|_S \geq M'_3|_S \geq C$. Now we consider the system

$$|K_{X'} + 4\pi^* (K_X)^\gamma + M_3 + S|.$$

Actually we can take further modification to $\pi$ such that $|M_3|$ is also base point free. This means we can suppose $M_3$ is nef. By the Kawamata-Viehweg vanishing theorem, we have

$$|K_{X'} + 4\pi^* (K_X)^\gamma + M_3 + S|_S = |K_S + 4\pi^* (K_X)^\gamma|_S + M_3|_S \supset |K_S + 4\pi^* (K_X)^\gamma|_S + C.$$

We can use the vanishing theorem once more so that we get

$$|K_S + 4\pi^* (K_X)^\gamma|_S + C|_C = |K_C + D|,$$

where $D := 4\pi^* (K_X)^\gamma|_C$. So if we can prove $4\pi^* (K_X)|_S \cdot C > 2$, then $\deg(D) \geq 3$ which induces the birationality of $\phi_3$, because

$$|K_{X'} + 4\pi^* (K_X)^\gamma + M_3 + S| \subset |9K_{X'}|.$$

Thus we only have to prove the following claim.

Claim. $\xi := \pi^* (K_X)|_S \cdot C \geq \frac{3}{5}$.

The idea is to find an initial estimation to $\xi$ by first proving that $\phi_{10}$ is birational. Then we can optimize this estimation by an infinite programme. We will find that the limit estimation is $\frac{3}{5}$. Actually our second estimation is enough for us to show the birationality of $\phi_3$. We present a better estimation here hoping that it might be useful to prove the birationality of $\phi_8$ in future.

From now on, we prove the claim. Let $M_5$, $M_7$ and $M_{10}$ be the movable part of $|5K_{X'}|$, $|7K_{X'}|$ and $|10K_{X'}|$ respectively. For the same reason, we can suppose they are all nef. We can see that

$$|K_{X'} + 3\pi^* (K_X)^\gamma + S|_S = |K_S + 3\pi^* (K_X)^\gamma|_S \supset |K_S + M_3|_S \supset |K_S + C|.$$

Because $g(S) = 0$, $|K_S + C|$ gives a generically finite map. So $\phi_5$ is generically finite. Suppose $L_5$ is the movable part of $|M_5|_S$. Then $\dim \Phi_{|L_5|}(S) = 2$. Therefore we can see that $L_5 \cdot C \geq 2$ for a general element $C$. We also have

$$|K_{X'} + 5\pi^* (K_X)^\gamma + S|_S \supset |K_{X'} + 5\pi^* (K_X)^\gamma|_S \supset |C + L_5|.$$

(5.1)
Noting that $C + L_5$ is movable and by Lemma 2.7, we see that $M'_7|_S \geq C + L_5$ where $M'_7$ is the movable part of $|K_X' + \gamma 5\pi^*(K_X)| + S|$. Because $M_7 \geq M'_7$, $M_7|_S \geq C + L_5$. We have

$$|K_X' + \gamma \pi^*(K_X)| + M_7 + S|_S = |K_S + \gamma \pi^*(K_X)| + M_7 + S|_S$$

$$\supset |K_S + \gamma \pi^*(K_X)| + S + L_5 + C|.$$

(5.2)

Now it is obvious that

$$|K_S + \gamma \pi^*(K_X)| + S + L_5 + C| = |K_C + G|,$$

where $G := (\gamma \pi^*(K_X)| + L_5)|_C$ is a divisor of degree $\geq 3$ and so $h^0(C, K_C + G) \geq g(C) + 2 = 4$. Suppose $L_{10}$ is the movable part of $|M_{10}|$ and $M'_{10}$ is the movable part of $|K_X' + \gamma \pi^*(K_X)| + M_7 + S|$. Let $L'_{10}$ be the movable part of

$$|K_S + \gamma \pi^*(K_X)| + S + L_5 + C|.$$

By Lemma 2.7, we have $M'_{10}|_S \geq L'_{10}$ and

$$h^0(C, L'_{10}|_C) = h^0(K_C + G) \geq 4.$$

Noting that $M_{10} \geq M'_{10}$, we have $L_{10} \geq L'_{10}$ and $h^0(C, L_{10}|_C) \geq 4$. Because $C$ is a curve of genus 2, by R-R, we see that $L_{10} \cdot C \geq 5$. This means that

$$10\pi^*(K_X)| + S \cdot C \geq M_{10}|S \cdot C \geq L_{10} \cdot C \geq 5.$$

So we get $\xi \geq \frac{1}{2}$.

Suppose $M_{12}$ is the movable part of $|12K_X'|$. Similar to (5.1), we have

$$|K_X' + \gamma 10\pi^*(K_X)| + S|_S = |K_S + \gamma 10\pi^*(K_X)| + S|$$

$$\supset |K_S + L_{10}| \supset |C + L_{10}|.$$

Using Lemma 2.7, we can easily see that $M_{12}|S \geq C + L_{10}$. Replacing $M_7$ by $M_{12}$ in (5.2), we also have

$$|K_X' + \gamma \pi^*(K_X)| + M_{12} + S|_S = |K_S + \gamma \pi^*(K_X)| + S + M_{12}|_S$$

$$\supset |K_S + \gamma \pi^*(K_X)| + S + L_{10} + C|.$$

Using the vanishing theorem once more, we have

$$|K_S + \gamma \pi^*(K_X)| + S + L_{10} + C| = |K_C + \gamma \pi^*(K_X)| + S + L_{10}|_C|.$$

Let $L'_{15}$ be the movable part of $|K_S + \gamma \pi^*(K_X)| + S + L_{10} + C|$. From Lemma 2.7, it is easy to see $M_{15}|S \geq L'_{15}$ and

$$h^0(C, L'_{15}|_C) = h^0(K_C + \gamma \pi^*(K_X)| + L_{10}|_C)$$

$$= \deg(\gamma \pi^*(K_X)| + L_{10}) + \deg(L_{10}|_C) + g(C) - 1 \geq 7.$$
where $M_{15}$ is the movable part of $|15K_{X'}|$. By R-R and the Clifford’s theorem, we see that

$$L'_{15} \cdot C \geq h^0(C, L'_{15}|C) + g(C) - 1 \geq 8.$$ 

Thus

$$15\pi^*(K_X)|S \cdot C \geq M_{15}|S \cdot C \geq L'_{15} \cdot C \geq 8.$$ 

This means that $\xi \geq \frac{8}{15} > \frac{1}{2}$, which directly induces the birationality of $\phi_9$.

We can infinitely repeat this programme, but omit the details. So we can get the following sequence

\[
\begin{align*}
n_0 &= 10, \\ n_1 &= n_0 + 5, \\ n_k &= n_{k-1} + 5, \\ \xi &= \frac{d_k}{n_k}, \text{ for all } k.
\end{align*}
\]

Thus

$$\xi \geq \lim_{k \to \infty} \frac{d_k}{n_k} = \lim_{k \to \infty} \frac{3k + 5}{5k + 10} = \frac{3}{5}.$$ 

The claim is proved.

(ii). If $p_g(X) \geq 12$, then we have $O(11) \hookrightarrow f_*\omega_{X'}$. So

$$O(1) \otimes f_*\omega_{X'/P_1}^5 = O(11) \otimes f_*\omega_{X'}^5 \hookrightarrow f_*\omega_{X'}^6.$$ 

It is easy to see that $\phi_9$ is birational for $X$ by virtue of Kollár’s technique. □

Both Proposition 5.2 and Proposition 5.3 imply

**Corollary 5.4.** Let $X$ be a smooth projective 3-fold of general type. Then $\phi_9$ is birational if $p_g(X) \geq 2$.

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