A Constructive Brownian Limit Theorem

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Abstract

In this paper, we present and prove a boundary limit theorem for Brownian motions for the Hardy space $h^p$ of harmonic functions on the unit ball in $R^m$, where $p \geq 1$ and $m \geq 2$ are arbitrary. Our proof is constructive in the sense of [Bishop and Bridges 1985, Chan 2021, Chan 2022]. Roughly speaking, a mathematical proof is constructive if it can be compiled into some computer code with the guarantee of exit in a finite number of steps on execution. A constructive proof of said boundary limit theorem is contained in [Durret 1984] for the case of $p > 1$. In this article, we give a constructive proof for $p = 1$, which then implies, via the Lyapunov’s inequality, a constructive proof for the general case $p \geq 1$. We conjecture that the result can be used to give a constructive proof of the nontangential limit theorem for Hardy spaces $h^p$ with $p \geq 1$.

We note that, ca 1970, R. Getoor gave a talk on the Brownian limit theorem at the University of Washington. We believe that the proof he presented is constructive only for the case $p > 1$ and not for the case $p = 1$. We are however unable to find a reference for his proof.

1 Preliminaries

In this section we gather together basic notions in the integration on an $(m-1)$-sphere, harmonic functions, Brownian motions, and exit times. These can serve as a basis of future constructive development in classical potential theory. We will also quote from [Chan 2021] Bishop’s maximal inequality for martingales. The Brownian limit theorem, the main theorem of this article, is then an easy application of said inequality for martingales.

1.1 Surface-area integration on an $(m-1)$-sphere

Unless otherwise defined, notations and terminologies in this article are from [Chan 2021]; they conform mostly to familiar usage in the probability literature.

If $x, y$ are mathematical objects, we write $x \equiv y$ to mean “$x$ is defined as $y$”, “$x$, which is defined as $y$”, “$x$, which has been defined earlier as $y$”, or any other grammatical variation depending on the context.

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1 Preliminaries

Following [Bishop and Bridges 1985], we define the functions \( \cos : R \to [-1, 1] \) in terms of the familiar power series, and define the constant \( \pi \) as twice the smallest positive zero of the function \( \cos \). Without further mention, we will use properties, proved in typical analysis text books, of the functions \( \cos, \sin \) and the constant \( \pi \), and of related functions.

**Definition 1. Matrix notations.** The transpose of a matrix \( \alpha \) is denoted by \( \alpha^T \). Let \( p, q \geq 1 \) be arbitrary integers. Each point \( z \equiv (z_1, \cdots, z_p) \in R^p \) will be identified with a column vector

\[
\begin{bmatrix}
z_1 \\
\vdots \\
z_p
\end{bmatrix}
\]

In other words, it is regarded as a \( p \times 1 \) matrix. Let \( \| \cdot \| \) denote the Euclidean norm on the space \( M^{p \times q} \) of \( p \times q \) matrices, defined by

\[
\| \alpha \| = \sqrt{\sum_{i=1}^{p} \sum_{j=1}^{q} a_{i,j}^2}
\]

for each \( \alpha \equiv [a_{i,j}]_{i=1, \cdots, p;j=1, \cdots, q} \in M^{p \times q} \). In particular, \( \| z \| = \sqrt{z_1^2 + \cdots + z_p^2} \) for each \( z \equiv (z_1, \cdots, z_p) \in R^p \). Thus \( M^{p \times q} \) is equipped with the Euclidean metric \( d_ecl \) defined by \( d_ecl(\alpha, \beta) \equiv \| \alpha - \beta \| \) for each \( \alpha, \beta \in M^{p \times q} \). At some small risk of confusion, we suppress the reference to \( p \) and \( q \) in the definitions of \( \| \cdot \| \) and of \( d_ecl \), and will write \( 0 \) for both the real number \( 0 \in R \) and the matrix \( 0 \in M^{p \times q} \) whose entries are all zeros. Likewise, we will write \( I \) for the identity matrix in \( M^{p \times p} \). Suppose, for each \( j = 1, \cdots, q \), \( \hat{e}_j \equiv [\hat{e}_{i,j}]_{i=1, \cdots, p} \) is a member of \( M^{p \times 1} \). Then we write

\[
[\hat{e}_1, \cdots, \hat{e}_q] \equiv [\hat{e}_{i,j}]_{i=1, \cdots, p;j=1, \cdots, q}
\]

for the matrix in \( M^{p \times q} \) whose \( j \)-th column is equal to the vector \( \hat{e}_j \).

For arbitrary \( z, x \in R^p \), the \( 1 \times 1 \) matrix \( z^T x \) is identified with its sole entry, denoted by \( z \cdot x \) and called the inner product of the two vectors \( z \) and \( x \).

\square

**Definition 2. Integration notations.** If \((S, L, \nu)\) is some complete integration space in the sense of [Bishop and Bridges 1985], and if \( g \in L \) is an arbitrary integrable function, then we write \( \nu(g) \), \( \nu(g \chi) \), \( \int g(x) \nu(dx) \), and \( \int \nu(dx) g(x) \) interchangeably. If, in addition, \( A \) is an integrable subset, then we write \( \nu(g1_A) \), \( \nu(g\chi A) \), \( \int_{x \in A} g(x) \nu(dx) \), \( \int_{x \in A} \nu(dx) g(x) \), and \( \int_{x \in A} \nu(dx) g(x) \) interchangeably, and write \( A \in L \). Moreover, if \( S = R^m \) and \( \nu \) is the Lebesgue integration, then we write \( dx \) for \( \nu(dx) \), and write

\[
\int \cdots \int_{(x(1), \cdots, x(m)) \in A} g(x_1, \cdots, x_m) dx_1 \cdots dx_m
\]

\[
= \int \cdots \int_{x \in A} g(x) dx \equiv \int_{x \in A} g(x) \nu(dx).
\]
If, in addition, \( m = 1 \) and \( A \) is a finite interval with end points \( a \leq b \), then we write \( \int_{a}^{b} g(x) \, dx \) for \( \int_{1A} g(x) \, dx \).

Suppose \( S \in L \) with \( \nu(S) = 1 \). We will then call \((S, L, \nu)\) a probability space. When the sample space \( S \) and the probability integration \( \nu \) are understood, we will also call \( L \) a probability space. We will write \( \|g\|_{L} \equiv \nu|g| \) for the \( L_{1} \)-norm of each \( g \in L \).

\[ \square \]

**Definition 3. Miscellaneous notations and conventions.** To lessen the burden on subscripts, for arbitrary expressions \( a \) and \( b \) we will write the expressions \( a \vee b \) and \( a \wedge b \) interchangeably. Moreover, if in a discussion involving a r.r.v. \( X \), the measurability of the set \((X \leq t)\) is required, it will be assumed that \( t \) has been so chosen to ensure such measurability; similar assumptions when \( \leq \) is replaced by \(<, \geq, \text{ or } >\).

For arbitrary real valued expressions \( a \) and \( b \), we will write \( a \vee b \) and \( a \wedge b \) for \( \max(a, b) \) and \( \min(a, b) \) respectively. Note that if \( a, b \) and \( c \) are real valued expressions with \( a \leq c \), then \((a \vee b) \wedge c = a \vee (b \wedge c)\). We will then omit the parentheses and write

\[ a \vee b \wedge c = (a \vee b) \wedge c = a \vee (b \wedge c). \]

Moreover, at some small risk of confusion, we sometimes write \( a = b \pm c \) for the inequality \(|a - b| \leq c\).

We will let \([\cdot]_{1}\) denote the operation which assigns to each \( a \in \mathbb{R} \) an integer \([a]_{1}\) which is in the interval \((a, a + 2)\). Note that there is no constructive proof that the operation \([\cdot]_{1}\) is a function. In other words, there is no guarantee that \([a]_{1} = [b]_{1}\) if \( a = b \).

\[ \square \]

**Definition 4. Angle.** Let \( m \geq 2 \) be arbitrary. Let \( x, y \in \mathbb{R}^{m} \) be arbitrary such that \(|x| > 0\) and \(|y| > 0\). We will let \( \angle(y, x) \) denote the unsigned angle between the vectors \( x \) and \( y \), defined as that unique real number \( \angle(y, x) \equiv \theta \in [0, \pi] \) such that

\[ \cos \theta = \frac{x \cdot y}{\|x\| \cdot \|y\|}. \]

In other words,

\[ \angle(y, x) \equiv \arccos \left( \frac{x \cdot y}{\|x\| \cdot \|y\|} \right). \]

**Lemma 5. Continuity of \( \cos \angle(y, x) \).** Let \( r > 0 \) be arbitrary. Then \( \cos \angle(y, x) \) and \( \sin \angle(y, x) \) are uniformly continuous functions on

\[ G_{r} \equiv \{(x, y) \in \mathbb{R}^{m} \times \mathbb{R}^{m} : \|x\| \geq r, \|y\| \geq r\} \]

for each \( r > 0 \).
Proof. Consider each $(x, y), (u, v) \in G_r$. Then

$$| \cos \angle(y, x) - \cos \angle(u, v) | = \left| \frac{x \cdot y}{\|x\| \cdot \|y\|} - \frac{u \cdot v}{\|u\| \cdot \|v\|} \right|$$

$$= \left| \frac{x \cdot y - x \cdot v}{\|x\| \cdot \|y\|} - \frac{x \cdot v - u \cdot v}{\|u\| \cdot \|v\|} \right| \leq \left| \frac{x \cdot y - x \cdot v}{\|x\| \cdot \|y\|} \right| + \left| \frac{x \cdot v - u \cdot v}{\|u\| \cdot \|v\|} \right|$$

$$\leq \frac{\|y - v\|}{\|y\|} + \frac{\|x - u\|}{\|u\|} \leq r^{-1} (\|y - v\| + \|x - u\|).$$

Thus the function $\cos \angle(y, x)$ is uniformly continuous on $G_r$. It follows that the function $\sin \angle(y, x) = \sqrt{1 - \cos^2 \angle(y, x)}$ is also uniformly continuous on $G_r$. \qed

Definition 6. Rotation matrix and rotation mapping. Let $m \geq 2$ be arbitrary. An $m \times m$ matrix $\alpha$ is called an orthogonal matrix if $\alpha^T \alpha = I$, where $I$ is the identity matrix. Let $O(m)$ denote the group of orthogonal matrices, with matrix multiplication as the group multiplication. $O(m)$ is called the orthogonal group. A member of $O(m)$ is called a rotation matrix if it has determinant $+1$. The subgroup $SO(m)$ of $O(m)$ consisting of rotation matrices is called the rotation group.

Let $y \in \mathbb{R}^m$ and $\alpha \in SO(m)$ be arbitrary. Define the mapping $\rho_{\alpha, y} : \mathbb{R}^m \to \mathbb{R}^m$ by

$$\rho_{\alpha, y}(z) \equiv y + \alpha \cdot (z - y)$$

for each $z \in \mathbb{R}^m$. Then $\rho_{\alpha, y}$ will be called the rotation of $\mathbb{R}^m$ about the point $y \in \mathbb{R}^m$ by the rotation matrix $\alpha \in SO(m)$. Note that $\rho_{\alpha, y}$ preserves distance, with $\rho_{\alpha, y}(y) = y$. Moreover, $\rho_{\alpha, y} \circ \rho_{\alpha^{-1}, y}$ is the identity mapping. In other words $\rho_{\alpha^{-1}, y}$ is the inverse function of $\rho_{\alpha, y}$.

\qed

Definition 7. Ball and sphere. Let $m \geq 1$, $x \in \mathbb{R}^m$, and $r > s > 0$ be arbitrary. Then the set

$$D_{x, r}^m \equiv \{ y \in \mathbb{R}^m : \|y - x\| < r \}$$

is called the open $m$-ball with center $x$ and radius $r$. The set

$$\overline{D}_{x, r}^m \equiv \{ y \in \mathbb{R}^m : \|y - x\| \leq r \}$$

is called the closed $m$-ball with center $x$ and radius $r$. The set

$$D_{x, s, r}^m \equiv \{ y \in \mathbb{R}^m : s < \|y - x\| < r \}$$
is called the open m-shell with center $x$, radius $r$, and thickness $r - s$. The set

$$D^m_{x,s,r} \equiv \{ y \in \mathbb{R}^m : s \leq \|y - x\| \leq r \}$$

is called the closed m-shell with center $x$, radius $r$, and thickness $r - s$. The set

$$\partial D^m_{x,r} \equiv \{ y \in \mathbb{R}^m : \|y - x\| = r \}$$

is called the $(m - 1)$-sphere with center $x$ and radius $r$.

Each of the above-defined sets inherits the Euclidean metric $d_{ecl}$ from $\mathbb{R}^m$.

Let $C(D^m_{x,r}), C(D^m_{x,s,r}), \text{ and } C(\partial D^m_{x,r})$ denote the spaces of uniformly continuous functions on the compact metric spaces $(D^m_{x,r}, d_{ecl})$, $(D^m_{x,s,r}, d_{ecl})$, and $(\partial D^m_{x,r}, d_{ecl})$ respectively.

When the dimension $m$ is understood, and abbreviation desired, we omit it in the subscripts and write $D_{x,r}, D_{x,s,r}, \text{ and } \partial D_{x,r}$ for $D^m_{x,r}, D^m_{x,s,r}, \text{ and } \partial D^m_{x,r}$ respectively.

□

**Definition 8. Spherical coordinates.** Let $m \geq 2$ be arbitrary. Define the open upper half space

$$R^m_+ \equiv \{ x \equiv (x_1, \cdots, x_m) \in \mathbb{R}^m : x_m > 0 \}$$

and open lower half space

$$R^m_- \equiv \{ x \equiv (x_1, \cdots, x_m) \in \mathbb{R}^m : x_m < 0 \}.$$

For each $x \equiv (x_1, \cdots, x_m) \in R^m_+ \cup R^m_- \text{ and } i = 1, \cdots, m - 1$, define

$$r(x) \equiv \|x\| > 0,$$

$$\theta_i(x) \equiv r(x)^{-1}x_i,$$

Define the functions

$$\varphi_+(x) \equiv (r(x), \theta_1(x), \cdots, \theta_{m-1}(x)) \in (0, \infty) \times D^{m-1}_{0,1}$$

for each $x \equiv (x_1, \cdots, x_m) \in R^m_+$ and

$$\varphi_-(x) \equiv (r(x), \theta_1(x), \cdots, \theta_{m-1}(x)) \in (0, \infty) \times D^{m-1}_{0,1}$$

for each $x \equiv (x_1, \cdots, x_m) \in R^m_-$. Then the function

$$\varphi_+ : R^m_+ \to (0, \infty) \times D^{m-1}_{0,1}$$

is an injection. The $m$-tuple $\varphi_+(x)$ is called the spherical coordinates of $x \in R^m_+$.

Similarly, the function

$$\varphi_- : R^m_- \to (0, \infty) \times D^{m-1}_{0,1}$$
is a injection. The $m$-tuple $\varphi_-(x)$ is called the spherical coordinates of $x \in \mathbb{R}^m$.

Let

$$\psi_+ : (0, \infty) \times D_{0,1}^m \to \mathbb{R}^m_+$$

and

$$\psi_- : (0, \infty) \times D_{0,1}^m \to \mathbb{R}^m_-$$

denote the inverses of $\varphi_+$ and $\varphi_-$ respectively.

□

Proposition 9. Surface-area integration on $(m-1)$-sphere centered at 0. Let $m \geq 2$, $y \in \mathbb{R}^m$, and $r > 0$ be arbitrary. Let $g \in C(\partial D_{0,r})$ be arbitrary. Define

$$g_+(x_1, \cdots, x_m) \equiv g(x_1, \cdots, x_m)1_{x(m) > 0}$$

and

$$g_-(x_1, \cdots, x_m) \equiv g(x_1, \cdots, x_m)1_{x(m) < 0}$$

for each $x \equiv (x_1, \cdots, x_m) \in \partial D_{0,r}$. Following page 4 of [Helms 1969], define the integral

$$\sigma_{m,0,r}(g) \equiv \int \sigma_{m,0,r}(dz)g(z)$$

$$\equiv \int \cdots \int_{(\theta_1, \cdots, \theta_{m-1}) \in D^{m-1}(0,1)} \frac{r^{m-1}}{\sqrt{1 - \theta_1^2 - \cdots - \theta_{m-1}^2}}$$

$$\times \left\{ g(r\theta_1, \cdots, r\theta_{m-1} + r, \sqrt{1 - \theta_1^2 - \cdots - \theta_{m-1}^2}) + g(r\theta_1, \cdots, r\theta_{m-1} - r, \sqrt{1 - \theta_1^2 - \cdots - \theta_{m-1}^2}) \right\} \quad (1.3)$$

Then the following conditions hold.

1. $\int \sigma_{m,0,r}(dz)g(z) = r^{m-1} \int \sigma_{m,1}(dz)g(rz)$.
2. The function $\sigma_{m,0,r} : C(\partial D_{0,r}) \to \mathbb{R}$ is an integration on $\partial D_{0,r}$. It will be called the surface-area integration on the $(m-1)$-sphere $\partial D_{0,r}$.
3. Let $s \in (0, r)$ be arbitrary. Define the constants

$$\sigma_{m,r} \equiv \sigma_{m,0,r}(1),$$

$$\nu_{m,r} \equiv \int \cdots \int_{x \in D(0,r)} 1dx$$

and

$$\nu_{m,s,r} \equiv \int_{x \in \partial D(0,s,r)} 1dx = \int_{x \in D(0,s,r)} 1dx = \nu_{m,r} - \nu_{m,s}.$$ 

Then we have

$$\sigma_{m,r}^{-1} \int_{z \in \partial D(0,r)} \sigma_{m,0,r}(dz)g(z) = \lim_{s \uparrow r} \nu_{m,s,r}^{-1} \int \cdots \int_{x \in D(0,s,r)} g\left(\frac{x}{\|x\|}\right)dx \quad (1.4)$$
for each \( g \in C(\partial D_{0,r}) \). Thus the surface distribution \( \sigma_{m,1}^{-1}, \sigma_{m,0,r} \) can be defined in terms of Lebesgue integration.

4. Let \( y \in \mathbb{R}^m \) and \( \alpha \in SO(m) \) be arbitrary. Define the surface area integration \( \sigma_{m,y,r} \) on \( \partial D_{y,r} \) by

\[
\sigma_{m,y,r}(g) \equiv \int \sigma_{m,0,r}(dz)g(z - y)
\]

for each \( g \in C(\partial D_{y,r}) \). Then a function \( g \) on \( \partial D_{y,r} \) is integrable relative to \( \sigma_{m,y,r} \) iff the function \( g \circ \rho_{\alpha,y} \) is integrable relative to \( \sigma_{m,y,r} \), in which case

\[
\sigma_{m,y,r}(g) = \sigma_{m,y,r}(g \circ \rho_{\alpha,y}).
\]

In short, the surface-area integration \( \sigma_{m,y,r} \) on the \((m-1)\)-sphere \( \partial D_{y,r} \) is invariant relative to rotations about \( y \).

**Proof.**

1. Assertion 1 can be verified directly from the defining equality 1.3.

2. From the defining equality 1.3, we see that the function \( \sigma_{m,0,r} \) is linear in \( g \in C(\partial D_{0,r}) \). Now suppose \( g \in C(\partial D_{0,r}) \) is such that \( \sigma_{m,0,r}(g) > 0 \). Then, because the Lebesgue integration

\[
\int \cdots \int_{(\theta_1, \ldots, \theta_{m-1}) \in D_{m-1}(0,1)} \cdot d\theta_1 \cdots d\theta_{m-1}
\]

is an integration, the expression in the braces on the right-hand side of equality 1.3 is positive at some \((\theta_1, \ldots, \theta_{m-1}) \in D_{m-1,0,1}\). Consequently, \( g(x) > 0 \) for some \( x \equiv (r\theta_1, \ldots, r\theta_{m-1}) \),\( + r\sqrt{1 - \theta_1^2 - \cdots - \theta_{m-1}^2} \)

or

\( x \equiv (r\theta_1, \ldots, r\theta_{m-1}) \),\( - r\sqrt{1 - \theta_1^2 - \cdots - \theta_{m-1}^2} \).

Thus we have verified the positivity condition, condition (ii), in definition 4.2.1 in [Chan 2021] for the linear function \( \sigma_{m,0,r} \) to be an integration on the compact metric space \( \partial D_{0,r} \). Assertion 2 is proved.

3. To prove Assertion 3, let \( s \in (0,r) \) be arbitrary. Define the function \( \overline{g} \in C(\overline{D}_{0,s,r}) \) by \( \overline{g}(x) \equiv g(\frac{rx}{\|x\|}) \) for each \( \overline{D}_{0,s,r} \).

4. Consider the limit

\[
\lim_{s \uparrow r} \nu_{m,s,r}^{-1} \int \cdots \int_{x \in D(0,s,r)} g(\frac{rx}{\|x\|}) dx = \lim_{s \uparrow r} \nu_{m,s,r}^{-1} \int \cdots \int_{x \in R^m \backslash D(0,s,r)} g(\frac{rx}{\|x\|}) dx + \lim_{s \uparrow r} \nu_{m,s,r}^{-1} \int \cdots \int_{x \in R^m \backslash D(0,s,r)} g(\frac{rx}{\|x\|}) dx.
\]

(1.6)
Apply a change of integration-variables to the first integral on the right-hand side, with

\[ x \equiv (x_1, \ldots, x_m) = (t\theta_1, \ldots, t\theta_{m-1}, + t\sqrt{1 - \theta_1^2 - \cdots - \theta_{m-1}^2}). \]

The Jacobian is

\[
\begin{vmatrix}
\theta_1, \theta_2, \ldots, \theta_{m-1}, \\
t, 0, \ldots, 0, \\
0, t, \ldots, 0, \\
\vdots & \vdots & \ddots & \vdots \\
0, 0, \ldots, t
\end{vmatrix} = |t^{m-1}(1 - \theta_1^2 - \cdots - \theta_{m-1}^2)^{1/2} + t^{m-1}\theta_1^2(1 - \theta_1^2 - \cdots - \theta_{m-1}^2)^{-1/2} + \cdots + t^{m-1}\theta_{m-1}^2(1 - \theta_1^2 - \cdots - \theta_{m-1}^2)^{-1/2}|.
\]

The first limit on the right-hand side of equality [1.6] then becomes

\[
\lim_{s \uparrow r} \nu_{m,s, \nu}^{-1} \int \cdots \int_{x \in R^n_D(0, s, r)} g\left(\frac{rx}{\|x\|}\right) dx
\]

\[
= \lim_{s \uparrow r} \nu_{m,s, \nu}^{-1} \int_{t=s}^{r} \cdots \int_{(\theta_1, \ldots, \theta_{m-1}) \in D^{m-1}(0,1)} dtd\theta_1 \cdots d\theta_{m-1} \frac{t^{m-1}}{\sqrt{1 - \theta_1^2 - \cdots - \theta_{m-1}^2}} g(r\theta_1, \ldots, r\theta_{m-1}, + r\sqrt{1 - \theta_1^2 - \cdots - \theta_{m-1}^2})
\]

\[
= \lim_{s \uparrow r} (r^m \nu_{m,0,1} - s^m \nu_{m,0,1})^{-1} m^{-1} (r^m - s^m)
\]

\[
\frac{1}{\sqrt{1 - \theta_1^2 - \cdots - \theta_{m-1}^2}} g(r\theta_1, \ldots, r\theta_{m-1}, + r\sqrt{1 - \theta_1^2 - \cdots - \theta_{m-1}^2})
\]

\[
= \nu_{m,0,1}^{-1} m^{-1} \int \cdots \int_{(\theta_1, \ldots, \theta_{m-1}) \in D^{m-1}(0,1)} d\theta_1 \cdots d\theta_{m-1} \frac{1}{\sqrt{1 - \theta_1^2 - \cdots - \theta_{m-1}^2}} g(r\theta_1, \ldots, r\theta_{m-1}, + r\sqrt{1 - \theta_1^2 - \cdots - \theta_{m-1}^2}).
\]
Similarly, the second limit on the right-hand side of equality 1.6 is equal to
\[ \nu_{m,0,1}^{-1} m^{-1} \int \cdots \int_{(\theta_1, \ldots, \theta_{m-1}) \in D^{m-1}(0,1)} d\theta_1 \cdots d\theta_{m-1} \frac{1}{\sqrt{1 - \theta_1^2 - \cdots - \theta_{m-1}^2}} \]
\[ g(r\theta_1, \ldots, r\theta_{m-1}, -r \sqrt{1 - \theta_1^2 - \cdots - \theta_{m-1}^2}). \]
Combining, equality 1.6 yields
\[ \lim_{s\uparrow r} \nu_{m,s,r}^{-1} m^{-1} \int \cdots \int_{x \in D(0,s,r)} g\left( \frac{r x}{\|x\|} \right) dx = \nu_{m,0,1}^{-1} m^{-1} r^{-m+1} \int \sigma_{m,0,r}(dz) g(z), \]
where the last equality is thanks to the defining equality 1.3.

For the special case where \( g \equiv 1 \), equality (1.7) reduces to
\[ 1 = \nu_{m,0,1}^{-1} m^{-1} r^{-m+1} \sigma_{m,r}. \]
Hence equality (1.7) can be rewritten as
\[ \lim_{s\uparrow r} \nu_{m,s,r}^{-1} m^{-1} \int \cdots \int_{x \in D(0,s,r)} g\left( \frac{r x}{\|x\|} \right) dx = \sigma_{m,r}^{-1} \int_{z \in \partial D(0,r)} \sigma_{m,0,r}(dz) g(z), \]
as alleged in equality 1.4 of Assertion 3.

4. Next, let \( \alpha \) be an arbitrary rotation matrix. Consider each \( g \in C(\partial D_{0,r}). \) Then equality (1.4) implies that
\[ \sigma_{m,0,r}(g \circ \alpha) = \sigma_{m,r} \lim_{s\uparrow r} \nu_{m,s,r}^{-1} m^{-1} \int \cdots \int_{x \in D(0,s,r)} g\left( \frac{r \alpha x}{\|x\|} \right) dx \]
\[ = \sigma_{m,r} \lim_{s\uparrow r} \nu_{m,s,r}^{-1} m^{-1} \int \cdots \int_{x \in D(0,s,r)} g\left( \frac{r \alpha x}{\|x\|} \right) dx \]
1 Preliminaries

\[ \sigma_{m,r} \lim_{s \uparrow r} \int \cdots \int_{x \in D(0,s,r)} g(\alpha x) \, dx \]

\[ = \sigma_{m,r} \lim_{s \uparrow r} \int \cdots \int_{z \in D(0,s,r)} g(z) \cdot |\det \alpha^{-1}| \cdot dz \cdots dz \]

\[ = \sigma_{m,r} \lim_{s \uparrow r} \int \cdots \int_{z \in D(0,s,r)} g(z) \, dz \cdots dz = \sigma_{m,0,r}(g). \]

(1.8)

5. Next let \( y \in \mathbb{R}^m \) be arbitrary. Define the integration \( \sigma_{m,y,r} \) on \( \partial D_{y,r} \) by

\[ \sigma_{m,y,r}(g) \equiv \sigma_{m,0,r}(g(y - y)) \]

(1.9)

for each \( g \in C(\partial D_{y,r}) \).

6. Now let \( y \in \mathbb{R}^m \) and \( \alpha \in SO(m) \) be arbitrary. Consider each \( g \in C(\partial D_{y,r}) \). We will prove that

\[ \sigma_{m,y,r}(g \circ \rho_{\alpha,y}) = \sigma_{m,y,r}(g). \]

(1.10)

To that end, note that the left-hand side of equality [1.10] can be written as

\[ \sigma_{m,y,r}(g \circ \rho_{\alpha,y}) \equiv \int_{z \in \partial \mathcal{D}(y,r)} \sigma_{m,y,r}(dz) g(\rho_{\alpha,y}(z)) \]

\[ = \int_{z \in \partial \mathcal{D}(y,r)} \sigma_{m,y,r}(dz) g(y + \alpha \cdot (z - y)) \]

\[ = \int_{u \in \partial \mathcal{D}(0,r)} \sigma_{m,0,r}(du) g(y + \alpha u) \]

\[ = \int_{u \in \partial \mathcal{D}(0,r)} \sigma_{m,0,r}(du) g(y + u) \]

\[ = \int_{v \in \partial \mathcal{D}(y,r)} \sigma_{m,y,r}(dv) g(v) \equiv \sigma_{m,y,r}(g), \]

where the fourth equality is by equality [1.8] and where \( g \in C(\partial D_{y,r}) \) is arbitrary. Thus the integrations \( \sigma_{m,y,r}(g \circ \rho_{\alpha,y}) \) and \( \sigma_{m,y,r} \) on \( \partial D_{y,r} \) are equal. Therefore a function \( g \) on \( \partial D_{y,r} \) is integrable relative to \( \sigma_{m,y,r} \) iff the function \( g \circ \rho_{\alpha,y} \) is integrable relative to \( \sigma_{m,y,r} \), in which case \( \sigma_{m,y,r}(g) = \sigma_{m,y,r}(g \circ \rho_{\alpha,y}) \). Assertion 4 and the proposition are proved.

Definition 10. Total surface-area of an \((m-1)\)-sphere centered at 0.

The integral \( \sigma_{m,r} \equiv \sigma_{m,0,r}(1) \) is called the total surface area of the \((m-1)\)-sphere \( \partial D_{m,0,r} \). The Lebesgue integral \( \nu_{m,r} \equiv \int_{x \in D_{m,0,r}} 1 \, dx \) is called the total volume of the unit ball \( D_{m,0,r} \).
Proposition 11. Total volume of \( m \)-ball and total surface area of \((m-1)\)-sphere. Let \( m \geq 2 \) and \( r > 0 \) be arbitrary. Then the following conditions hold.

1. \( \nu_{m,r} = r^m \nu_{m,1} \).
2. \( \sigma_{m,r} = r^{m-1} \sigma_{m,1} \).
3. \( \nu_{m,1} = \frac{1}{m} \sigma_{m,1} \).
4. \[
\sigma_{m,1} = \begin{cases} 
\frac{m^{m/2-1}}{(m/2)!} & \text{if } m \text{ is even,} \\
\frac{2^{m-1}}{(m-1)/2 \pi^{(m-1)/2} 1 \cdot 3 \cdot 5 \cdots (m-2)} & \text{if } m \text{ is odd.}
\end{cases}
\]

Proof. See pages 3, 4, and 5 of [Helms 1969]. \(/)\)

Definition 12. Uniform distribution on \((m-1)\)-sphere. The normalized surface area integration \( \sigma \equiv \sigma_{m,y,r} \equiv \sigma_{m,y,1} \) is called the uniform distribution on the \((m-1)\)-sphere \( \partial D_{y,r} \). Note that, since \( \sigma_{m,y,r} \) is invariant relative to rotations \( \rho_{\alpha,y} \) about the center \( y \), so is the uniform distribution \( \sigma_{m,y,r} \). We will write \( L(\sigma_{m,y,r}) \) for the probability space generated by the family \( C(\partial D_{y,r}) \) of continuous functions on \( \partial D_{y,r} \) relative to the uniform distribution \( \sigma_{m,y,r} \).

Theorem 13. Each distribution on the \((m-1)\)-sphere \( \partial D_{0,1} \) that is invariant relative to rotations is equal to the uniform distribution. Let \( \sigma \) be an arbitrary distribution on \( \partial D_{0,1} \) that is invariant relative to rotations. Then \( \sigma = \sigma_{m,0,1} \).

Proof. For ease of notations, we will give the proof only for the case \( m \geq 3 \). The case where \( m = 2 \) would be along similar lines and simpler. The proof will be by means of Haar measures as in chapter 8 of [Bishop and Bridges 1985].

1. Let the points \( \hat{x}_1, \ldots, \hat{x}_m \in R^m \) be the natural basis of \( R^m \). In other words, \( [\hat{x}_1, \ldots, \hat{x}_m] \) is the identity \( m \times m \) matrix \( I \).

2. First consider the case where \( y = 0 \in R^m \) and \( r = 1 \). Fix the reference point \( \hat{e} \equiv \hat{e}_1 \equiv \hat{x}_1 \equiv (1, 0, \ldots, 0) \in \partial D_{0,1} \). For each \( k \geq 1 \), define the set
\[
G_k \equiv \{ z \in \partial D_{0,1} : \| z - \hat{e} \| \geq 2^{-k} \}.
\]

Define the dense subsets
\[
G \equiv \bigcup_{k=1}^{\infty} G_k \equiv \{ z \in \partial D_{0,1} : \| z - \hat{e} \| > 0 \}
\]
and
\[
G \cup \equiv G \cup \{ \hat{e} \}.
\]
of \( \partial D_{0,1,d_{euc}} \).

3. Consider each \( z \in \partial D_{0,1} \). Let \( \theta_z \equiv \angle(\hat{e}, z) \in [0, \pi] \). Then
\[
\cos \theta_z = \frac{z \cdot \hat{e}}{\| z \| \cdot \| \hat{e} \|} = z \cdot \hat{e}.
\]
Thus $\cos \theta_z$ is a uniformly continuous function of $z \in \partial D_{0,1}$. Hence $\sin \theta_z = \sqrt{1 - \cos^2 \theta_z}$ is also a uniformly continuous function of $z \in \partial D_{0,1}$.

4. Next, let $k \geq 1$ and $z \in G_k$ be arbitrary. Write $i_1 \equiv 1$ and $i_2 \equiv 2$. Let $\hat{e}_{1,z} \equiv \hat{e}_1 \equiv \hat{e}$ and define

$$\hat{e}_{2,z} \equiv \hat{e}_{2} = \frac{z - (z \cdot \hat{e}) \hat{e}}{\|z - (z \cdot \hat{e}) \hat{e}\|} = \frac{z - (\cos \theta_z) \hat{e}}{\|z - (\cos \theta_z) \hat{e}\|}. \quad (1.11)$$

Then

$$\|z - (z \cdot \hat{e}) \hat{e}\|^2 = 1 - (z \cdot \hat{e})^2 = 2^{-1}(2 - 2(z \cdot \hat{e})^2) = 2^{-1} \|z - \hat{e}\|^2 \geq 2^{-k-1}.$$ 

Hence $\hat{e}_{2,z}$ is well defined and is a uniformly continuous function on $G_k$, where $k \geq 1$ is arbitrary. Moreover, $\hat{e}_{1} \cdot \hat{e}_{2} = 0$. Furthermore, equality (1.11) yields

$$(\cos \theta_z) \hat{e}_{1} + \|z - (\cos \theta_z) \hat{e}_1\| \hat{e}_{2} = z. \quad (1.12)$$

Since $\hat{e}_{1}, \hat{e}_{2}, z$ are unit vectors, while $\hat{e}_{1}, \hat{e}_{2}$ are mutually orthogonal, it follows that

$$(\cos \theta_z)^2 + \|z - (\cos \theta_z) \hat{e}_1\|^2 = 1,$$

whence

$$\|z - (\cos \theta_z) \hat{e}_1\| = \sqrt{1 - (\cos \theta_z)^2} = \sin \theta_z.$$ 

Therefore equality (1.12) can be restated as

$$(\cos \theta_z) \hat{e}_{1} + (\sin \theta_z) \hat{e}_{2} = z. \quad (1.13)$$

3. Let

$$V_z \equiv \{ c_1 \hat{e}_1 + c_2 \hat{e}_2 : c_1, c_2 \in \mathbb{R} \}$$

be the 2-dimensional subspace spanned by $\hat{e}_{1}, \hat{e}_{2}$. Let $V_z^\perp \equiv \{ u \in \mathbb{R}^m : u \cdot v = 0 \quad \text{for each } v \in V \}$ be the $(m - 1)$-dimensional subspace that is orthogonal to $V_z$. Let $\hat{e}_3, \ldots, \hat{e}_m$ be an arbitrary orthonormal basis of $V_z^\perp$. Then $\hat{e}_1, \ldots, \hat{e}_m$ be an arbitrary orthonormal basis of $\mathbb{R}^m$.

4. Define the rotation matrix

$$\alpha_z \equiv [\hat{e}_1, \hat{e}_2, \hat{e}_3, \cdots, \hat{e}_m]$$

$$= [\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4, \cdots, \hat{e}_m]$$

$$= \begin{bmatrix}
\cos \theta_z & -\sin \theta_z & 0 & 0 & \cdots & 0 \\
\sin \theta_z & \cos \theta_z & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\hat{e}_1^T \\
\hat{e}_2^T \\
\hat{e}_3^T \\
\hat{e}_4^T \\
\vdots \\
\hat{e}_m^T
\end{bmatrix}$$

$$= \begin{bmatrix}
\hat{e}_1^T \cos \theta_z - \hat{e}_2^T \sin \theta_z \\
\hat{e}_1^T \sin \theta_z + \hat{e}_2^T \cos \theta_z \\
\hat{e}_3^T \\
\hat{e}_4^T \\
\vdots \\
\hat{e}_m^T
\end{bmatrix}$$
\[ = \hat{e}_1(\hat{e}_1^T \cos \theta_z - \hat{e}_2^T \sin \theta_z) + \hat{e}_2(\hat{e}_1^T \sin \theta_z + \hat{e}_2^T \cos \theta_z) + \sum_{i=3}^{m} \hat{e}_i \hat{e}_i^T. \] (1.14)

Right multiplication by \( \hat{e}_1 \) yields
\[ \alpha_z \hat{e}_1 = \hat{e}_1 \cos \theta_z + \hat{e}_2 \sin \theta_z = z. \]

Right multiplication by \( \hat{e}_2 \) yields
\[ \alpha_z \hat{e}_2 = -\hat{e}_1 \sin \theta_z + \hat{e}_2 \cos \theta_z. \]

5. We will show that the matrix \( \alpha_z \) is independent of \( \hat{e}_3, \ldots, \hat{e}_m \). To that end, let \( \hat{e}_3, \ldots, \hat{e}_m \) be a second arbitrary orthonormal basis for \( V^\perp \). Then there exists an orthogonal matrix \( \beta \in O(m) \) such that \( \hat{e}_i = \tilde{e}_i \beta \) for each \( i = 1, \ldots, m \). Hence
\[ \sum_{i=3}^{m} \hat{e}_i \hat{e}_i^T = \sum_{i=3}^{m} \tilde{e}_i \beta \beta^T \tilde{e}_i^T = \sum_{i=3}^{m} \hat{e}_i \hat{e}_i^T. \]

Equality 1.14 therefore shows that the matrix \( \alpha_z \) is independent of the orthonormal basis \( \hat{e}_3, \ldots, \hat{e}_m \) of the space \( V^\perp \), and is completely determined by \( z \), where \( z \in G_k \) and \( k \geq 1 \) are arbitrary.

6. Therefore the operation \( \psi : G_v \to SO(m) \) defined on
\[ G_v \equiv \bigcup_{k=1}^{\infty} G_k \cup \{ \tilde{e} \} \equiv G \cup \{ \tilde{e} \}. \]

by
\[ \psi(z) \equiv \alpha_z \]

for each \( z \in G \) and by \( \psi(\tilde{e}) \equiv I \), is a well defined function. Moreover,
\[ \psi(z) \tilde{e} = z. \]

for each \( z \in G \).

7. We have already seen that the function \( \psi \) is uniformly continuous on each compact subset of \( G \). We will next prove that \( \psi \) is uniformly continuous on \( G_v \). To that end, it suffices to prove that \( \psi \) is continuous at \( \tilde{e} \). Note that, as \( z \to \tilde{e} \) with \( z \in G \), we have \( \cos \theta_z \to 1 \) and \( \sin \theta_z \to 0 \). Hence, equality 1.14 shows that, as \( z \to \tilde{e} \), we have
\[ \psi(z) \equiv \alpha_z \to \sum_{i=1}^{m} \tilde{e}_i \tilde{e}_i^T = I \equiv \psi(\tilde{e}). \]

Thus \( \psi \) is continuous at \( \tilde{e} \), and therefore uniformly continuous on \( G_v \).

8. Since \( G_v \) is a dense subset of \( \partial D_{0,1} \) the function \( \psi \) can be extended to a continuous function.
\[ \psi : \partial D_{0,1} \to SO(m) \]
such that $\psi(z)\hat{e} = z$ for each $z \in \partial D_{0,1}$. Let
$$SO(m) \equiv \psi(\partial D_{0,1})$$
denote the range of $\psi$. Then
$$\psi : \partial D_{0,1} \to SO(m)$$
is a continuous surjection.

9. In the other direction, define the function
$$\varphi : SO(m) \to \partial D_{0,1}$$
by $\varphi(\alpha) \equiv \alpha \hat{e} \in \partial D_{0,1}$ for each $\alpha \in SO(m)$. Then $\varphi$ is continuous and
$$\varphi \psi (z) \equiv \psi(z)\hat{e} = z$$
for each $z \in \partial D_{0,1}$. Thus $\varphi$ is the inverse of the continuous function $\psi$. Hence
$$\psi : (\partial D_{0,1}, d_{ecl}) \to (SO(m), d_{ecl})$$
is a homeomorphism. Therefore we can regard each point $z \in \partial D_{0,1}$ as a rotation matrix $\psi(z) \in SO(m)$, and write $z \cdot w$ for the matrix product $\psi(z) \cdot \psi(w)$ for each $z, w \in \partial D_{0,1}$.

10. Since $(SO(m), d_{ecl}, \cdot)$ is a compact group, the $(m-1)$-sphere $\partial D_{0,1}$ becomes a compact group, with the matrix multiplication as the group operations, and with $\hat{e} \cdot w \equiv \psi(\hat{e}) \cdot w = I \cdot w = w$ for each $w \in \partial D_{0,1}$. Thus the reference point $\hat{e}$ is the identity element in the compact group $\partial D_{0,1}$.

11. Now suppose $\sigma$ is an arbitrary distribution on $\partial D_{0,1}$ which is invariant relative to rotations. Then, for each $z \in \partial D_{0,1}$, we have
$$\int_{w \in \partial D_{0,1}} g(z \cdot w) \sigma(dw) = \int_{z \in \partial D_{0,1}} g(w) \sigma(dw).$$
Hence $\sigma$ is invariant relative to the left-multiplication by an arbitrary group element $z \in \partial D_{0,1}$.

12. In particular, since the uniform distribution $\sigma_{m,0,1}$ is invariant relative to rotations, it is invariant relative to the left-multiplication by an arbitrary group element $z \in \partial D_{0,1}$. Hence theorem (1.19) in chapter 8 of [Bishop and Bridges 1985] says that there exists a constant $c$ such that $\sigma = c \sigma_{m,0,1}$. Since $\sigma$ and $\sigma_{m,y,r}$ are both distributions, it follows that $1 = \sigma(1) = c \sigma_{m,0,1}(1) = c$. Therefore $\sigma = \sigma_{m,0,1}$, as alleged. \[Q.E.D.\]

**Corollary 14.** Each distribution on the $(m-1)$-sphere $\partial D_{y,r}$ that is invariant relative to rotations is equal to the uniform distribution. Let $y \in \mathbb{R}^m$ and $r > 0$ be arbitrary. Then the following conditions hold.

1. Let $\sigma$ be an arbitrary distribution on $\partial D_{y,r}$ that is invariant relative to rotations about the center $y$. Then $\sigma = \sigma_{m,y,r}$.
2. Suppose \( y = 0 \). Define a distribution \( \tilde{\sigma} \) on \( \partial D_{0,r} \) by

\[
\tilde{\sigma}(f) \equiv \int_{z \in \partial D} \tilde{\sigma}_{m,0,1}(dz)f(rz)
\]

for each \( f \in C(\partial D_{0,r}) \). Then \( \tilde{\sigma} = \tilde{\sigma}_{m,0,r} \).

3. A function \( f \) on \( \partial D_{0,r} \) is integrable relative to \( \tilde{\sigma}_{m,0,r} \) iff \( f(r\cdot) \) is integrable relative to \( \tilde{\sigma}_{m,0,1} \), in which case

\[
\int_{x \in \partial D(0,r)} \tilde{\sigma}_{m,0,r}(dx)f(x) = \int_{z \in \partial D} \tilde{\sigma}_{m,0,1}(dz)f(rz).
\]

Proof. 1. Let \( \sigma \) be an arbitrary distribution on \( \partial D_{y,r} \) that is invariant relative to rotations about the center \( y \). Define the distribution \( \tilde{\sigma} \) on \( \partial D_{0,1} \) by \( \tilde{\sigma}(g) \equiv \sigma(g(\frac{\cdot - y}{r})) \) for each \( g \in C(\partial D_{0,1}) \). Then \( \tilde{\sigma} \) is a distribution on \( \partial D_{0,1} \) that is invariant relative to rotations about 0. Similarly, the distribution \( \tilde{\sigma}_{m,y,r} \) is a distribution on \( \partial D_{0,1} \) that is invariant relative to rotations about 0. By Theorem 13, we have \( \tilde{\sigma} = \tilde{\sigma}_{m,y,r} \). Now let \( h \in C(\partial D_{y,r}) \) be arbitrary. Define \( g \in C(\partial D_{0,1}) \) by \( g \equiv h(y + r\cdot) \). Then

\[
\sigma(h) = \sigma(g(\frac{\cdot - y}{r})) \equiv \tilde{\sigma}(g) = \tilde{\sigma}_{m,y,r}(g) = \tilde{\sigma}_{m,y,r}(g(\frac{\cdot - y}{r})) = \tilde{\sigma}_{m,y,r}(h),
\]

where \( h \in C(\partial D_{y,r}) \) is arbitrary. Thus \( \sigma \equiv \tilde{\sigma}_{m,y,r} \) as distributions. Assertion 1 is proved.

2. Suppose \( y = 0 \). Define a distribution \( \tilde{\sigma} \) on \( \partial D_{0,r} \) by

\[
\tilde{\sigma}(f) \equiv \int_{z \in \partial D} \tilde{\sigma}_{m,0,1}(dz)f(rz)
\]

for each \( f \in C(\partial D_{0,r}) \). Then \( \tilde{\sigma} \) is a distribution on the compact space \( \partial D_{0,r} \). Let \( \alpha \) be an arbitrary \( m \times m \) rotation matrix. Let \( f \in C(\partial D_{0,r}) \) be arbitrary. Define \( g \equiv f(r\cdot) \in C(\partial D_{0,1}) \). Then

\[
\tilde{\sigma}(f \circ \alpha) = \int_{z \in \partial D} \tilde{\sigma}_{m,0,1}(dz)g \circ \alpha(z)
\]

\[
= \tilde{\sigma}_{m,0,1}(g \circ \alpha) = \tilde{\sigma}_{m,0,1}(g) = \tilde{\sigma}_{m,0,1}(f(r\cdot)) \equiv \tilde{\sigma}(f).
\]

where the third inequality is by Definition 12. Hence \( \tilde{\sigma} = \tilde{\sigma}_{m,0,r} \) by Assertion 1. This proves Assertion 2.

3. By Assertion 2

\[
\int_{x \in \partial D(0,r)} \tilde{\sigma}_{m,0,r}(dx)f(x) \equiv \int_{z \in \partial D} \tilde{\sigma}_{m,0,1}(dz)f(rz) \tag{1.15}
\]

for each \( f \in C(\partial D_{0,r}) \). By L1 continuity, equality (1.15) can be extended to each function \( f \) on \( \partial D_{0,r} \) that is integrable relative to \( \tilde{\sigma}_{m,0,r} \). Conversely, equality (1.15) can be extended to each function \( f \) on \( \partial D_{0,r} \) such that the function \( f(r\cdot) \) is integrable relative to \( \tilde{\sigma}_{m,0,1} \). Assertion 3 is proved.
1.2 Harmonic function

**Definition 15. Harmonic function.** Let $A$ be an arbitrary open subset of $\mathbb{R}^m$. A function $u \equiv u(x_1, \cdots, x_m)$ that has a second derivative which is uniformly continuous on compact subsets of $A$ is said to be *harmonic* if it satisfies the Laplace equation

$$\Delta u \equiv \sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2} = 0$$
onumber

on $A$.

□

**Definition 16. Mean value property.** Let $A$ be an arbitrary open subset of $\mathbb{R}^m$. Let $u$ be a function with domain $(u) = A$ that is uniformly continuous on compact subsets of $A$. Then the function $u$ is said to have the mean value property if one of the following two conditions hold.

(i) (Mean value property on balls). For each $y \in \mathbb{R}^m$ and $r > 0$ such that $D_{y, r} \subset A$, we have

$$u(y) = \nu_{m, r}^{-1} \int \cdots \int_{x \in D(y, r)} u(x) dx.$$  \hfill (1.16)

(ii) (Mean value property on spheres). For each $y \in \mathbb{R}^m$ and $r > 0$ such that $\overline{D}_{y, r} \subset A$, we have

$$u(y) = \sigma_{m, r}^{-1} \int_{z \in \partial D(y, r)} u(z) \sigma_{m, y, r}(dz).$$  \hfill (1.17)

The next proposition says that these two conditions are equivalent. □

**Proposition 17. Mean value property on balls is equivalent to mean value property on spheres.** Let $A$ be an arbitrary open subset of $\mathbb{R}^m$. Let $u$ be a function with domain $(u) = A$ that is uniformly continuous on compact subsets of $A$. Then Conditions (i) and (ii) in Definition 16 are equivalent.

**Proof.** 1. Suppose Condition (i) holds for the function $u$. Let $y \in \mathbb{R}^m$ and $r > s > 0$ be arbitrary such that $\overline{D}_{y, r} \subset A$. Then

$$\nu_{m, r} \cdot u(y) = \int \cdots \int_{x \in D(y, r)} u(x) dx,$$  \hfill (1.18)

with a similar equality where $r$ is replaced by $s$. Hence

$$u(y) = \lim_{s \uparrow r} \nu_{m, s, r}^{-1} (\nu_{m, r} - \nu_{m, s}) \cdot u(y)$$

$$= \lim_{s \uparrow r} \nu_{m, s, r}^{-1} \left( \int \cdots \int_{x \in D(y, r)} u(x) dx - \int \cdots \int_{x \in D(y, s)} u(x) dx \right)$$
At the same time, for each $k \nu \nu$ Then $Hence$

$1$ Preliminaries

$17$

where the second equality is due to equality $1.18$ where the fifth equality is continuous on $x \in D(0, s, r)$ and because $u$ is uniformly continuous on $x \in D(0, s, r)$, and where the sixth equality is by applying equality $1.4$ of Proposition $9$ to the function $g \equiv u(\cdot + y)$.

Thus Condition (ii) holds. We conclude that Condition (i) implies Condition (ii).

2. Conversely, suppose Condition (ii) in Definition $16$ holds. Let $y \in R^m$ and $r > s > 0$ be arbitrary such that $D_{y,r} \subset A$. Let $\varepsilon > 0$ be arbitrary. Take $K \geq 1$ so large that $rs^{-1}K^{-1}(r-s) < \delta_u(\varepsilon)$ where $\delta_u$ is a modulus of continuity of the function $u$ on $D_{y,r}$. Let $r_k \equiv s + kK^{-1}(r-s)$ for each $k = 0, \cdots, K$.

Then

$$\nu_{m,s,r}^{-1} \int \cdots \int_{x \in D(0, s, r)} u(y+x)dx = \nu_{m,s,r}^{-1} \sum_{k=1}^{K} \int \cdots \int_{x \in D(0, r(k-1), r(k))} u(y+x)dx.$$

At the same time, for each $k = 1, \cdots, K$ and $x \in D_{0,r(k-1),r(k)}$, we have

$$\left\| (y + x) - (y + r_k \frac{x}{\|x\|}) \right\| \leq \|x\| \cdot |1 - \frac{r_k}{r_{k-1}}|$$

$$\leq r \frac{r_k - r_{k-1}}{r_{k-1}} \leq r \frac{K^{-1}(r-s)}{s} < \delta_u(\varepsilon).$$

Hence

$$\nu_{m,s,r}^{-1} \int \cdots \int_{x \in D(0, s, r)} u(y+x)dx$$

$$= \nu_{m,s,r}^{-1} \sum_{k=1}^{K} \int \cdots \int_{x \in D(0, r(k-1), r(k))} u(y+x)dx$$

$$= \nu_{m,s,r}^{-1} \sum_{k=1}^{K} \int \cdots \int_{x \in D(0, r(k-1), r(k))} (u(y + r_k \frac{x}{\|x\|}) \pm \varepsilon)dx$$

$$\int \cdots \int_{x \in D(0, s, r)} u(y + r_k \frac{x}{\|x\|}) dx$$
\[ \nu_{m,s,r}^{-1} \sum_{k=1}^{K} \cdots \int_{x \in D(0,r(k-1),r(k))} u(y + r_k \frac{x}{\|x\|}) dx \pm \varepsilon. \quad (1.19) \]

By applying equality 1.4 of Proposition 9 with \( u(y + \cdot) \) in the place of the function \( g \) and with \( r_{k-1} \) and \( r_k \) in the places of \( s \) and \( r \) respectively, we obtain

\[ \int \cdots \int_{x \in D(0,r(k-1),r(k))} u(y + r_k x) dx = \nu_{m,r(k-1),r(k)}^{-1} \sigma_{m,r(k)}^{-1} \sigma_{m,0,r(k)}(u(y + \cdot)). \quad (1.20) \]

Therefore equality (1.19) reduces to

\[ \nu_{m,s,r}^{-1} \sum_{k=1}^{K} \cdots \int_{x \in D(0,s,r)} u(y + x) dx = \nu_{m,r(k-1),r(k)}^{-1} \sigma_{m,r(k)}^{-1} \sigma_{m,0,r(k)}(u(y + \cdot)) \pm \varepsilon \]

where the second to last equality is thanks to the assumed Condition (ii) in Definition 16. Since \( \varepsilon > 0 \) is arbitrarily small, we infer that

\[ \nu_{m,s,r}^{-1} \sum_{k=1}^{K} \cdots \int_{x \in D(0,s,r)} u(y + x) dx = u(y). \]

Letting \( s \downarrow 0 \), we obtain

\[ u(y) = \nu_{m,0,r}^{-1} \sum_{k=1}^{K} \cdots \int_{x \in D(0,r)} u(y + x) dx = \nu_{m,r}^{-1} \sum_{k=1}^{K} \cdots \int_{x \in D(y,r)} u(x) dx, \]

which is Condition (i) in Definition 16. Thus Condition (i) follows from Condition (ii).

**Proposition 18.** Harmonic functions have mean value property. Let \( A \) be an arbitrary open subset of \( \mathbb{R}^m \). Suppose a function \( u \equiv u(x) \equiv u(x_1, \cdots, x_m) \) is harmonic on \( A \). Suppose a ball \( D_{y,r} \subset A \) for some \( y \in \mathbb{R}^m \) and \( r > 0 \). Then

\[ u(y) = \nu_{m,r}^{-1} \sum_{k=1}^{K} \cdots \int_{x \in D(y,r)} u(x) dx. \quad (1.22) \]

Moreover,

\[ u(y) = \sigma_{m,r}^{-1} \sigma_{m,y,r} u = \int_{z \in \partial D(y,r)} u(z) \sigma_{m,y,r}(dz). \quad (1.23) \]
Proof. See theorems 1.5 and 1.6 of [Helms 1969].

Lemma 19. \( C^k \)-function smoothed by convolution with indicator of a ball is \( C^{k+1} \). Let \( u \) be an arbitrary bounded continuous function on the open set \( A \subset \mathbb{R}^m \). Let \( r > 0 \) be arbitrary. Define the function \( u_r \) on the subset

\[ A_r \equiv \{ y \in A : D_{y,r} \subset A \} \]

by

\[ u_r(y) \equiv \nu_{m,r}^{-1} \int_{D_{y,r}} u(x) \, dx \quad (1.24) \]

for each \( y \in A_r \). Then the following conditions hold.

1. \( u_r \) has continuous first derivative on \( A_r \).
2. Suppose, in addition, the function \( u \) has continuous \( k \)-th derivative on \( A \), for some \( k \geq 1 \). Then the function \( u_r \) has continuous \( (k+1) \)-st derivative on \( A_r \).
3. Suppose \( u \) has the mean value property. Then \( u \) has continuous \( (k+1) \)-st derivative on \( A_r \) for each \( k \geq 1 \).

Proof. 1. For the proof of Assertions 1 and 2, see theorem 1.14 on page 19 of [Helms 1969]  
2. To prove Assertion 3, suppose \( u \) has the mean value property. In other words, suppose \( u = u_r \) for each \( y \in A_r \). Then, Assertion 1 implies that \( u \) has continuous second derivative on \( A_r \). Assertion 2 then implies that \( u = u_r \) has continuous second derivative on \( A_r \). Recursively applying Assertion 2, we see that \( u = u_r \) has continuous \( (k+1) \)-st derivative on \( A_r \) for each \( k \geq 1 \).

Theorem 20. Divergence Theorem. Let \( A \) be an arbitrary open subset of \( \mathbb{R}^m \). Let \( F = (F_1, \ldots, F_m) : A \to \mathbb{R}^m \) be a vector field with continuous derivatives on each closed ball contained in \( A \). Then for each \( y \in \mathbb{R}^m \) and \( r > 0 \) with \( D_{y,r} \subset A \), we have defined

\[ \int_{x \in D_{y,r}} (\nabla \cdot F)(x) \, dx = \int_{x \in \partial D_{y,r}} (F \cdot n)(x) \sigma_{m,y,r}(dx), \]

where, for each \( x = (x_1, \ldots, x_m) \in \partial D_{y,r} \), we have defined (i) \( n(x) \equiv (n_1, \ldots, n_m)(x) \equiv r^{-1}(x - y) \) is the outward-pointing unit normal vector at \( x \), (ii) \( (\nabla \cdot F)(x) \equiv \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_m}{\partial x_m} \) \( (x) \), and (iii) \( (F \cdot n)(x) \equiv (F_1 n_1 + \cdots + F_m n_m)(x) \).

Proof. The reader should refer to [Rudin 1976] pp 253-275 for a treatment of differential forms and Stokes theorem and the divergence theorem in \( \mathbb{R}^n \), where \( n \geq 1 \) is arbitrary. Then the reader should refer to theorem 10.51 on page 288 of [Rudin 1976] which translates the divergence theorem in differential forms to the present divergence theorem in rectangular coordinates. The proof of theorem 10.51 in [Rudin 1976] assumes \( m = 3 \). The reader should verify that the assumption that \( m = 3 \) is however not essential to said proof.
Proposition 21. Functions with the mean value property are harmonic. Let \( A \) be an arbitrary open subset of \( \mathbb{R}^m \). Suppose a function \( u \equiv u(x) \equiv u(x_1, \cdots, x_m) \) is uniformly continuous on each closed ball contained in \( A \) and

\[
    u(y) = \int_{z \in \partial D(y,r)} u(z) \sigma_{m,y,r}(dz) \tag{1.25}
\]

for each \( y \in \mathbb{R}^m \) and \( r > 0 \) with \( \overline{D}_{y,r} \subset A \). Then \( u \) is harmonic on \( A \).

Proof. 1. By equality (1.25) in the hypothesis, the function \( u \) satisfies Condition (ii) in Definition 16. Therefore, by Proposition 17, the function \( u \) satisfies Condition (i) in Definition 16. Hence, according to Assertion 3 of Lemma 19, the function \( u \) has a second derivative which is uniformly continuous on each compact subset of \( A \). In particular, \( \Delta u \) is uniformly continuous on each compact subset of \( A \).

2. Suppose, for the sake of a contradiction, that \( \Delta u(y) < 0 \) for some \( y \equiv (y_1, \cdots, y_m) \in A \). Then there exists \( r > 0 \) with \( \overline{D}_{y,r} \subset A \) such that \( \Delta u < 0 \) on \( D_{y,r} \). Recall the function \( F : \overline{D}_{y,r} \to \mathbb{R}^m \) defined by

\[
    F(x) \equiv (F_1(x), \cdots, F_m(x)) \equiv \nabla u(x) \equiv \left( \frac{\partial u}{\partial x_1}(x), \cdots, \frac{\partial u}{\partial x_m}(x) \right)
\]

for each \( x \in \overline{D}_{y,r} \). Recall the functions \( n : \{ x \in \overline{D}_{y,r} : \| x - y \| > 0 \} \to \mathbb{R}^m \) by

\[
    n(x) \equiv \| x - y \|^{-1} (x - y)
\]

for each \( x \in \{ x \in \overline{D}_{y,r} : \| x - y \| > 0 \} \).

Consider each \( s \in (0, r] \). Then Condition (ii) in Definition 16 implies that

\[
    u(y) = \int_{x \in D(0,s)} \sigma_{m,0,s}(dx) u(y + x)
    = \int_{z \in \partial D(0,1)} \sigma_{m,0,1}(dz) u(y + sz)
\]

where the second equality is by Assertion 1 of Proposition 9. With \( y \) fixed, differentiation relative to \( s \) under the integral sign yields

\[
    0 = \sigma_{m,1}^{-1} \int_{z \in \partial D(0,1)} \sum_{i=1}^{m} \frac{\partial u}{\partial x_i}(y + sz) z_i \sigma_{m,0,1}(dz)
    = \sigma_{m,1}^{-1} \int_{z \in \partial D(0,1)} F(y + sz) \cdot n(y + sz) \sigma_{m,0,1}(dz)
    = \sigma_{m,1}s^{-m+1} \int_{z \in \partial D(0,s)} F(y + z) \cdot n(y + z) \sigma_{m,0,s}(dz)
    = \sigma_{m,1}s^{-m+1} \int_{z \in \partial D(y,s)} F(z) \cdot n(z) \sigma_{m,y,s}(dz)
\]
\[ \sigma_{m,1}^{-1} s^{-m+1} \int_{x \in D(y,s)} (\nabla \cdot \mathbf{F})(x) \, dx \]

\[ = \sigma_{m,1}^{-1} s^{-m+1} \int_{x \in D(y,s)} \triangle u(x) \, dx < 0, \quad (1.26) \]

where the third equality by applying Assertion 1 of Proposition 9 to the function 
\[ g \equiv \mathbf{F}(y+s \cdot) \cdot \mathbf{n}(y+s \cdot), \] 
where the fourth equality is by Definition 12, and where 
the fifth equality is by the Divergence Theorem, Theorem 20. Inequality 1.20
is a contradiction. We conclude that \( \triangle u(y) \geq 0 \) for each \( y \in A \). By symmetry
\( \triangle u(y) \leq 0 \) for each \( y \in A \). Thus \( \triangle u = 0 \) on \( A \). In other words, the function \( u \)
is harmonic on \( A \).

**Theorem 22.** Maximum modulus theorem for harmonic functions. Let
\( A \) be an arbitrary open subset of \( R^m \). Let \( u \) be an arbitrary harmonic function
on \( A \). Suppose \( n \geq 1, y_1, \cdots, y_n \in A \) and \( \rho_1, \cdots, \rho_n > 0 \) are such that (i)
\( K \equiv \bigcup_{i=1}^{n} \mathcal{D}_{y(i), \rho(i)} \) is a compact subset of \( A \), and (ii) \( D_{y(i), \rho(i)} \cap D_{y(i+1), \rho(i+1)} \)
is nonempty for each \( i = 1, \cdots, n-1 \). Then
\[ \sup_{x \in K} |u(x)| = \sup_{x \in \partial K} |u(x)|. \]

**Proof.** This is a special case of corollary 1.13 of [Helms 1969].

**Theorem 23.** Poisson kernel and Poisson integration on the \((m-1)\)-sphere. Let \( y \in R^m \) and \( r > 0 \) be arbitrary. Define the function \( k_{y,r} : D_{y,r} \times \partial D_{y,r} \to (0, \infty) \) by
\[ k_{y,r}(x, z) \equiv \frac{1}{r} \frac{r^2 - \|y - x\|^2}{\|z - x\|^m} \]
for each \( (x, z) \in D_{y,r} \times \partial D_{y,r} \). The function \( k_{y,r} \) is called the Poisson kernel for
the ball \( D_{y,r} \). Then the following conditions hold.
1. Let \( x \in D_{y,r} \) be arbitrary. Then \( k_{y,r}(x, \cdot) > 0 \) on \( \partial D_{y,r} \), and
\[ 1 = \int_{z \in \partial D(y,r)} k_{y,r}(x, z) \overline{\sigma}_{m,y,r}(dz). \quad (1.27) \]
2. Let \( z \in \partial D_{y,r} \) be arbitrary. Then the function \( k_{y,r}(\cdot, z) \) is harmonic on
the open ball \( D_{y,r} \).
3. Let \( x \in \overline{D}_{y,r} \) be arbitrary. Then the function \( k_{y,r}(x, \cdot) \) is integrable relative
to the uniform distribution \( \overline{\sigma}_{m,y,r} \).
4. Let \( \overline{g} \in C(\partial D_{y,r}) \) be arbitrary. Then the function
\[ g(x) \equiv \int_{z \in \partial D(y,r)} k_{y,r}(x, z) \overline{g}(z) \overline{\sigma}_{m,y,r}(dz) \quad (1.28) \]
of \( x \in D_{y,r} \) is harmonic on \( D_{y,r} \).
5. Let \( \overline{g} \in C(\partial D_{y,r}) \) be arbitrary and let \( g \in C(D_{y,r}) \) be the function defined
by equality 1.23. Then \( \lim_{x \to z} g(x) = \overline{g}(z) \) uniformly for each \( z \in \partial D_{y,r} \).
6. Let \( x \in D_{y,r} \) be arbitrary. Define the function \( \mu_x \) on \( C(\partial D_{y,r}) \) by

\[
\mu_x g \equiv \sigma_{m,y,r}(y, \cdot)g
\]

(1.29)

for each \( g \in C(\partial D_{y,r}) \). Then the triple \( (\partial D_{y,r}, C(\partial D_{y,r}), \mu_x) \) is an integration space with a complete extension \( (\partial D_{y,r}, \mathcal{L}_x, \mu_x) \), where \( \mathcal{L}_x \) is the space of integrable functions.

7. Let \( f \) be an arbitrary integrable function on \( D_{y,r} \) relative to the uniform distribution \( \sigma_{m,y,r} \). Then the function \( u \) defined on \( D_{y,r} \) by

\[
u(x) \equiv \mu_x f \equiv \sigma_{m,y,r}(y, \cdot)f
\]

for each \( x \in D_{y,r} \) is a well defined harmonic function on \( D_{y,r} \).

8. Let \( s \in (0, 1) \) be arbitrary. Let \( (x, z) \in \partial D_{0,s} \times \partial D_{0,1} \) be arbitrary. Then \( k_{0,1}(x, z) \geq 1 - s^2(1 + s)^{-m} \) (1.30)

Proof. 1 Assertion 1 follows from theorem 1.8 in [Helms 1969], where the function \( h \) is replaced by 1.

2. Assertion 2 is a special case of theorem 2.3 in [Helms 1969] where the signed measure \( \mu \) is replaced by the probability measure \( \delta_z \) which assigns probability 1 to the point \( z \).

3. Assertion 3 is a special case of corollary 2.4 in [Helms 1969] where the signed measure \( \mu \) is replaced by the surface-area integration \( \sigma_{m,y,r} \) on \( \partial D_{m,y,r} \).

4. Assertion 4 is, in essence, theorem 2.8 in [Helms 1969].

5. Assertion 5 can be verified from the proofs of the lemmas 2.5-7 in [Helms 1969].

6. To prove Assertion 6, let \( x \in D_{y,r} \) be arbitrary. Let \( g \in C(\partial D_{y,r}) \) be arbitrary. Then \( g \) is bounded and continuous. By Assertion 3, \( k_{y,r}(x, \cdot) \) is integrable relative to \( \sigma_{m,y,r} \). Hence the product \( k_{y,r}(x, \cdot)g \) is integrable relative to \( \sigma_{m,y,r} \), and \( \mu_x g \) is well defined. Moreover, since \( k_{y,r}(x, \cdot) > 0 \), it follows that \( \mu_x \) is a linear function that satisfies conditions (i) and (ii) in definition 4.2.1 of [Chan 2021] to be an integration on the compact space \( \partial D_{y,r} \). Hence, according to proposition 4.3.3 of [Chan 2021], the triple \( (\partial D_{y,r}, C(\partial D_{y,r}), \mu_x) \) is an integration space. Therefore, by proposition 4.4.2 of [Chan 2021], said triple can be extended to a complete integration space \( (\partial D_{y,r}, \mathcal{L}_x, \mu_x) \). Assertion 6 is proved.

7. It remains to prove Assertion 7. By hypothesis \( f \) is an integrable function on \( \partial D_{y,r} \) relative to the uniform distribution \( \sigma_{m,y,r} \). Therefore there exists a sequence \( \{g_k\}_{k=1}^\infty \) in \( C(\partial D_{y,r}) \) such that

\[
\sum_{k=1}^\infty |\sigma_{m,y,r} g_k| < \infty,
\]

(1.31)

\[
domain(f) = \{z \in \partial D_{y,r} : \sum_{k=1}^\infty |g_k(z)| < \infty\},
\]

(1.32)
and
\[ f(z) = \sum_{k=1}^{\infty} g_k(z) \]  
(1.33)

for each \( z \in \text{domain}(f) \). In short, the sequence \((g_k)_{k=1,2,...}\) is a representation of the integrable function \( f \) relative to \( \sigma_{m,y,r} \).

Let \( s \in (0,r) \) be arbitrary. Consider each \( x \in \overline{D}_{y,s} \subset D_{y,r} \) and \( k \geq 1 \). Then \( k_{y,r}(x,\cdot) \in C(\partial D_{y,r}) \) with \( b_s \geq k_{y,r}(x,\cdot) \geq 0 \) for some \( b_s \geq 0 \). It follows that \( k_{y,r}(x,\cdot)g_k \in C(\partial D_{y,r}) \). According to equality 1.29 for each \( x \in D_{y,r} \) the integral
\[ \mu_x g_k \equiv \sigma_{m,y,r}k_{y,r}(x,\cdot)g_k \]  
(1.34)
is well defined and, according to Assertion 4, is a harmonic function of \( x \in D_{y,r} \). Similarly,
\[ \sum_{k=1}^{\infty} \mu_x |g_k| \equiv \sum_{k=1}^{\infty} \sigma_{m,y,r}k_{y,r}(x,\cdot)|g_k| \]
\[ \leq b_s \sum_{k=1}^{\infty} \sigma_{m,y,r}|g_k| < \infty. \]  
(1.35)
where the last inequality is by inequality 1.31. Combining inequality 1.35 with equalities 1.32 and 1.33 we see that the sequence \((g_k)_{k=1,2,...}\) in \( C(\partial D_{y,r}) \) is a representation of the function \( f \) relative to the integration \( \mu_x \) on \( \partial D_{y,r} \). Hence the function \( f \) is integrable relative to the integration \( \mu_{\sigma} \). Moreover, for each \( x \in D_{y,s} \) and for each \( n \geq 1 \) we have
\[ |\mu_{\sigma}f - \sum_{k=1}^{n} \mu_{\sigma}g_k| = |\sum_{k=1}^{\infty} \mu_{\sigma}g_k - \sum_{k=1}^{n} \mu_{\sigma}g_k| \]
\[ = |\sum_{k=n+1}^{\infty} \mu_{\sigma}g_k| \leq |\sum_{k=n+1}^{\infty} \sigma_{m,y,r}k_{y,r}(\cdot,\cdot)g_k| \]
\[ \leq b_s \sum_{k=n+1}^{\infty} \sigma_{m,y,r}|g_k|. \]
Note that the last bound depends on \( s \) but is otherwise independent of \( x \in D_{y,s} \). In other words, \( \sum_{k=1}^{\infty} \mu_{\sigma}g_k \to \mu_{\sigma}f \) uniformly in \( x \in D_{y,s} \), where \( s \in (0,r) \) is arbitrary. Thus the harmonic function \( v(x) \equiv \sum_{k=1}^{\infty} \mu_x g_k \) of \( x \in D_{y,r} \) converges to the function \( \mu_{\sigma}(x) \equiv \mu_{\sigma}f \) uniformly on the compact subset \( D_{y,s} \) for each \( s \in (0,r) \), as \( n \to \infty \). At the same time, the harmonic function \( v \) has the mean value property on compact subsets of \( D_{y,r} \). Therefore, in view of the aforementioned uniform convergence, the function \( \sigma \) inherits the mean value property on compact subsets of \( D_{y,r} \). It follows from Proposition 21 that the function \( \sigma \) is a harmonic function on \( D_{y,r} \). Assertion 7 is proved.

8. Let \( s \in (0,1) \) be arbitrary. We compute
\[ \bigwedge_{(x,z) \in \partial D_{0,s} \times \partial D_{0,1}} k_{0,1}(x,z) = \bigwedge_{v \in \partial D_{0,s}} k_{0,1}(v,1) \]
Theorem 25. Strong Markov property of Brownian motion. Let $x$ be the time parameter $t$ relative to the filtration $L$. Theorem 25. Strong Markov property of Brownian motion. Let $x$ be the time relative to the filtration $L$.

Definition 24. Specification of two Brownian motions. In the remaining of this article, let $m \geq 1$ be arbitrary. Let $B, \tilde{B} : [0, \infty) \times (\Omega, L, E) \to R^m$ be two independent Brownian motions, as defined in [Chan 2021]. Thus $B_0 = \tilde{B}_0 = 0 \in R^m$. Let $\mathcal{L} \equiv \{ L^{(t)} : t \geq 0 \}$ be the right continuous extension of the natural filtration of $B$. For each $i = 1, \cdots, m$ and $t \in [0, \infty)$ let $B_{t,t}$ denote the r.r.v. that is the $i$-th component of $B_t$. Thus $B_{i,t} : [0, \infty) \times (\Omega, L, E) \to R^1$ is a Brownian motion in $R^1$ for each $i = 1, \cdots, m$. Here the dot in the subscript of $B_{i,t}$ serves as a place holder for the time parameter $t$.

Let $x \equiv (x_1, \cdots, x_m) \in R^m$ be arbitrary. Define the Brownian motion $B^x$ with initial state $x$ by $B^x \equiv x + B : [0, \infty) \times (\Omega, L, E) \to R^m$. Thus $B^x_t = x + B_t$. for each $t \geq 0$. The process $B^x$ is adapted to the filtration $\mathcal{L}$. Similarly define the Brownian motion $\tilde{B}^x \equiv x + \tilde{B}$ with initial state $x$.

Our main focus is on the process $B$; the process $\tilde{B}$ merely facilitates some presentation.

Refer to [Chan 2021] for the definition and basic properties of a stopping time relative to the filtration $\mathcal{L}$. □

Theorem 25. Strong Markov property of Brownian motion. Let $x \in R^m$ be arbitrary. Then $B^x : [0, \infty) \times (\Omega, L, E) \to R^m$ is a strong Markov process relative to the right continuous filtration $\mathcal{L}$. Specifically, let $\tau : (\Omega, L, E) \to [0, \infty)$ be an arbitrary stopping time relative to the filtration $\mathcal{L}$. Define the probability subspace

$$L^{(\tau)} \equiv \{ Y \in L : Y 1_{\tau \leq t} \in L^{(t)} \text{ for each regular point } t \text{ of } \tau \}$$

of integrable observables up to and including the stopping time $\tau$. Let $n \geq 1$, the sequence $0 \leq r_0 \leq r_1 \leq \cdots \leq r_n$, and the function $g \in C_{ub}(R^{n+1})$ be arbitrary. Then

$$E(g(B^x_{\tau+r(0)}, \cdots, B^x_{\tau+r(n)})|L^{(\tau)}) = E(g(\tilde{B}^x_{\tau}, \cdots, \tilde{B}^x_{\tau})).$$

In words, given all observables of the process $B^x$ up to and including the stopping time $\tau$, the future development is as if starting an independent Brownian motion $\tilde{B}$ anew at the current state $\tilde{B}_{\tau}$.
Proof. See [Chan 2021].

**Lemma 26. Reflection principle.** Suppose \( m = 1 \). In other words, suppose \( B \) is a real valued Brownian motion. Let \( t, \lambda > 0 \) be arbitrary. Then

\[
P( \bigvee_{s \in [0,t]} B_s \geq \lambda ) = 2(1 - \Phi_{0,1}(\frac{\lambda}{\sqrt{t}})),
\]

where \( \Phi_{0,1} \) is the standard normal CDF on \( \mathbb{R} \).

**Proof.** The specified Brownian motion \( B \) has continuous paths. Therefore theorem 2.1 in [Doob 1953], of which the present lemma is a restatement, and the proof of said theorem 2.1 in [Doob 1953] are applicable.

**Definition 27. Modulus of continuity at 0 of the standard normal CDF.** It will be convenient to fix a sequence of integers which represents the modulus of continuity at 0 of the standard normal CDF \( \Phi_{0,1} \). Specifically, let \( \tilde{r} > 0 \) be arbitrary. Then the function \( 2\Phi_{0,1}(\frac{\tilde{r}}{\sqrt{t}}) - 1 \) of \( t \in (0, \infty) \) is strictly decreasing, and converges to 0 as \( t \uparrow \infty \), i.e. as \( \frac{\tilde{r}}{\sqrt{t}} \downarrow 0 \). Hence, given an arbitrary \( \tilde{r} > 0 \), we can fix an increasing sequence \( (N_{\tilde{r},k})_{k=0,1,...} \) of positive integers such that

\[
2\Phi_{0,1}(\frac{\tilde{r}}{\sqrt{N(\tilde{r},k)}}) - 1 < 2^{-k}.
\]

for each \( k \geq 0 \).

**Lemma 28. Convention regarding stopping times.** In the remainder of this article, we make the convention that all stopping times in the discussion are r.r.v.’s with values in \([0, \infty)\). Before each stopping time is used, there will be a stated or unstated proof that it is a r.r.v. with values in \([0, \infty)\). In other words, a stopping time \( \tau \) is admissible only if \( P(\tau \leq t) \uparrow 1 \) as \( t \to \infty \). This is in contrast to some usage in the literature where the point \( \infty \) at infinity for the extended time line \([0, \infty]\) is an admissible value for a stopping time.

**Definition 29. First exit time from open ball by Brownian motion.** Let \( r > 0 \) and \( x \in D_{0,r} \) be arbitrary. Suppose \( \tau \) is a stopping time relative to \( \mathcal{L} \), with values in \((0, \infty)\). Define the function \( B^x_\tau : \Omega \to R^m \) by \( \text{domain}(B^x_\tau) \equiv \text{domain}(\tau) \) and by

\[
B^x_\tau(\omega) \equiv B^x(\tau(\omega), \omega)
\]

for each \( \omega \in \text{domain}(\tau) \). Suppose, in addition, the following three conditions hold:

(i) For each \( \omega \in \text{domain}(\tau) \), we have \( B^x_\tau(\omega) \in \partial D_{0,r} \).

(ii) The function \( B^x_\tau : (\Omega, \mathcal{L}, E) \to \partial D_{0,r} \) is a r.v.

(iii) For each \( \omega \in \text{domain}(\tau) \) and \( t \in [0, \tau(\omega)) \), we have \( B^x_t(\omega) \in D_{0,r} \).
Then we define \( \tau_{x,r} \equiv \tau \) and call \( \tau_{x,r} \) the first exit time for the process \( B^x \) to exit the open ball \( D_{0,r} \), and write \( \tau_{D_{0,r}}(B^x) \equiv \tau_{x,r} \), and call the r.v. \( B^x_{\tau_{x,r}} \) the corresponding random point of exit. If emphasis of the underlying Brownian motion \( B \) is needed, then we write \( \tau_{x,r:B} \) for \( \tau_{x,r} \). \( \square \)

**Lemma 30. Existence of certain first exit times.** Let \( \widetilde{r} > r > 0 \) and \( x \in D_{0,\widetilde{r}} \) be arbitrary. Then the following conditions hold.

1. There exists a countable subset \( H \) of \( R \) such that, for each \( a \in (\tau, \widetilde{r})H_c \), the first exit time \( \tau_{x,a} \equiv \tau_{D(0,a)}(B^x) \) exists. Here \( H_c \) denotes the metric complement of \( H \) in \( R \). Thus \( \tau_{x,a} : (\tau, \widetilde{r})H_c \times \Omega \to (0, \infty) \) is a stochastic process with nondecreasing paths.

A subsequent theorem will prove that the process \( \tau_{x,a} \) is a.u. continuous on the parameter set \( (\tau, \widetilde{r})H_c \), which is dense in the interval \( (\tau, \widetilde{r}) \). Therefore the process \( \tau_{x,a} \) is extendable to an a.u. continuous process with parameter set \( (\tau, \widetilde{r}) \).

2. The family \( \{\tau_{x,a} : x \in D_{0,\tau}; a \in (\tau, \widetilde{r})H_c\} \) is tight, with the sequence \( (N_{\tau,k})_{k=0,1,\ldots} \) of positive integers playing the role of modulus of tightness. Specifically, let \( k \geq 0 \) and \( t > N_{\tau,k} \), \( x \in D_{0,\tau} \), \( a \in (\tau, \widetilde{r})H_c \) be arbitrary. Then

\[
P(\tau_{x,a} > t) \leq 2^{-k+1}. \tag{1.38}
\]

**Proof.** 1. Let \( \widetilde{r} > r > 0 \) and \( x \equiv (x_1, \ldots, x_m) \in D_{0,\widetilde{r}} \) be arbitrary. Thus \( ||x|| \leq \widetilde{r} \). Define the function \( f : R^m \to R \) by

\[
f(z) \equiv \widetilde{r} \wedge ||z|| \tag{1.39}
\]

for each \( z \in R^m \). Then \( f \in C_{ub}(R^m) \) and \( f(x) \leq \widetilde{r} \). At the same time, a Brownian motion is a Feller process in the sense of [Chan 2021]. Hence, by theorem 11.10.7 of [Chan 2021], there exists a countable subset \( H \) of \( R \) such that, for each

\[
a \in (\tau, \widetilde{r})H_c
\]

and for each \( n \geq 1 \), the first exit time \( \tau \equiv \tau_{x,a,n} \equiv \tau_{f,a,n}(B^x) \) for the process \( B^x \) to exit the open set

\[
(f < a) \equiv \{z \in R^m : f(z) < a\}
\]

on or before time \( n \), is well defined relative to the filtration \( \mathcal{L} \), in the sense of definition 10.11.1 of [Chan 2021]. Moreover, by definition 10.11.1 of [Chan 2021], the first exit time \( \tau_{x,a,n} \) has values in \((0, n]\). Consider each \( a \in (\tau, \widetilde{r})H_c \). Then, by said definition, the function

\[
\tau \equiv \tau_{x,a,n} \equiv \tau_{f,a,n}(B^x)
\]

is a stopping time relative to \( \mathcal{L} \), with values in \((0, n]\), and the function

\[
B^x_{\tau} \equiv B^x_{\tau(x,a,n)} : \Omega \to R^m \tag{1.40}
\]

is a well-defined r.v. Furthermore, for each \( \omega \in \text{domain}(\tau) \), we have

(i) \( f(B^x(\cdot, \omega)) < a \) on the interval \([0, \tau(\omega)) \), and
(ii) $f(B_x^2(\omega)) \geq a$ if $\tau(\omega) < n$.

2. By lemma 10.11.2 of [Chan 2021], we have, for each $n' \geq n \geq 1$ and $t \in (0, n)$,

$$\tau_{x,a,n} \leq \tau_{x,a,n'},$$

(1.41)

and

$$\tau_{x,a,n} < n \subset (\tau_{x,a,n'} = \tau_{x,a,n}),$$

(1.42)

3. Note that $B_x \equiv (x_1 + B_{1,1}, \cdots, x_m + B_{m,1})$. Here

$$B_{1,1} : [0, \infty) \times \Omega \to \mathbb{R}^1$$

is the $i$-th component process of $B : [0, \infty) \times \Omega \to \mathbb{R}^m$, with the dot in the subscripts serving as a place holder for the time parameter $t$. Let $n \geq 1$ be arbitrary. Consider each $t \in (0, n)$. Then

$$P(\tau_{x,a,n} > t)$$

$$\leq P\{\omega \in \text{domain}(\tau) : f(B_x^\tau(s, \omega)) < a \text{ on } [0, t] \subset [0, \tau(\omega))\}$$

$$\leq P\{\omega : \bigvee_{s \in [0, t]} f(B_x^\tau(s, \omega)) \leq a \text{ on } [0, t]\}$$

$$\equiv P\{\omega : \bigvee_{s \in [0, t]} \bar{r} \wedge \|B_x^\tau(s, \omega)\| \leq a\}$$

$$= P\{\omega : \bigvee_{s \in [0, t]} \|B_x^\tau(s, \omega)\| \leq \|x\| + a\}$$

$$\leq P\{\omega : \bigvee_{s \in [0, t]} B_{1,1}(s, \omega) \leq \|x\| + a\}$$

$$= 2\Phi_{0,1}(\frac{\|x\| + a}{\sqrt{t}}) - 1$$

$$\leq 2\Phi_{0,1}(\frac{\bar{r} + \tilde{r}}{\sqrt{t}}) - 1,$$

(1.44)

where the first inequality is by Condition (i), and where the last equality is by the reflection principle, Lemma 26 applied to the one-dimensional Brownian motion $B_{1,1}$.

4. Next, define a function $\tau_{x,a} : \Omega \to (0, \infty)$ by

$$\text{domain}(\tau_{x,a}) \equiv \{\omega \in \bigcap_{n=1}^{\infty} \text{domain}(\tau_{x,a,n}) : \lim_{n \to \infty} \tau_{x,a,n}(\omega) \text{ exists}\}$$
and by 
\[ \tau_{x,a}(\omega) \equiv \lim_{n \to \infty} \tau_{x,a,n}(\omega) \]
for each \( \omega \in \text{domain}(\tau_{x,a}) \).

5. We will verify that \( \tau_{x,a} \) is a well defined r.r.v. and is a stopping time relative to the filtration \( L \). Recall from Definition 27 the increasing sequence \( (N_{\tilde{r},k})_{k=0,1,\ldots} \) of positive integers, with 
\[ 2\Phi_{0,1}(\frac{\bar{r}}{N(\tilde{r},k)}) - 1 < 2^{-k}. \]  
(1.45)
for each \( k \geq 0 \). Now let \( k \geq 1 \) be arbitrary. Fix a regular point \( t_k \in (N_{\tilde{r},k-1}, N_{\tilde{r},k}) \) of the stopping time \( \tau_{x,a,N(\tilde{r},k)} \). Define the measurable set 
\[ A_k \equiv (\tau_{x,a,N(\tilde{r},k)} \leq t_k) \equiv (\tau_{f,a,N(\tilde{r},k)}(B^x) \leq t_k). \]
Then, by inequality 1.44 we have 
\[ P(A_k^c) = P(\tau_{x,a,N(\tilde{r},k)} > t_k) \leq 2\Phi_{0,1}(\frac{\bar{r} + \bar{r}}{\sqrt{l(k)}}) - 1 \]
\[ \leq 2\Phi_{0,1}(\frac{\bar{r}}{N(\tilde{r},k-1)}) - 1. \]
Letting \( \bar{r} \downarrow 0 \), we obtain 
\[ P(A_k^c) \leq 2\Phi_{0,1}(\frac{\bar{r}}{N(\tilde{r},k-1)}) - 1 \leq 2^{-k+1}, \]  
(1.46)
where the last inequality is by inequality 1.37 applied to \( k-1 \). Moreover, for each \( n \geq N_{\tilde{r},k} \), we have 
\[ A_k \equiv (\tau_{x,a,N(\tilde{r},k)} \leq t_k) \]
\[ \subset (\tau_{x,a,n} = \tau_{x,a,N(\tilde{r},k)} \leq t_k) \subset (\tau_{x,a,N(\tilde{r},k)} = \tau_{x,a,n}), \]
where the first inclusion relation is due to relation 1.42. Thus 
\[ A_k \subset \bigcap_{n=N(\tilde{r},k)}^{\infty} (\tau_{x,a,N(\tilde{r},k)} = \tau_{x,a,n}), \]  
(1.47)
where \( k \geq 1 \) is arbitrary. Since \( P(A_k) \leq 2^{-k+1} \) is arbitrarily small as \( k \to \infty \), we see that \( \tau_{x,a,n} \to \tau_{x,a} \) a.u. as \( k \to \infty \), whence \( \tau_{x,a} \) is a r.r.v. Furthermore, relation 1.47 implies 
\[ A_k \subset \bigcap_{n=N(\tilde{r},k)}^{\infty} (\tau_{x,a,n} = \tau_{x,a}) \subset (\tau_{x,a,N(\tilde{r},k)} = \tau_{x,a}). \]  
(1.48)
Recall here that 
\[ a \in (\bar{\tau}, \bar{r})H_c \]
is arbitrary.

7. To prove that the r.r.v. $\tau_{x,a}$ is a stopping time, take an arbitrary regular point $t \in (0, \infty)$ of $\tau_{x,a}$. Let $(s_j)_{j=1,2,...}$ be an arbitrary decreasing sequence in $(t, \infty)$ such that $s_j \downarrow t$ and such that $s_j$ is a regular point of the r.r.v.’s $\tau_{x,a}$ and $\tau_{x,a,N(\tilde{r},k)}$ for each $k \geq 1$. Consider each $j \geq 1$. Then, in view of relation (1.48) we have

$$\left( \bigcup_{k=1}^{\infty} A_k \right) (\tau_{x,a} \leq s_j) = \bigcup_{k=1}^{\infty} A_k \left( \tau_{x,a,N(\tilde{r},k)} \leq s_j \right)$$

Note that $A \equiv \bigcup_{k=1}^{\infty} A_k$ is a full set. Hence $A \in L^{(a(j))}$. At the same time $\bigcup_{k=1}^{\infty} (\tau_{x,a,N(\tilde{r},k)} \leq s_j) \in L^{(a(j))}$ because $\tau_{x,a,N(\tilde{r},k)}$ is a stopping time relative to the filtration $\mathcal{L}$, for each $k \geq 1$. Hence equality (1.49) implies that $(\tau_{x,a} \leq s_j) \in L^{(s(j))}$, where $j \geq 1$ is arbitrary. It follows that

$$(\tau_{x,a} \leq t) = \bigcap_{j=1}^{\infty} (\tau_{x,a} \leq s_j) \in L^{(t)} = L^{(t)}$$

where the last equality is thanks to the assumed right continuity of the filtration $\mathcal{L}$. Since $t$ is an arbitrary regular point of $\tau_{x,a}$, the r.r.v. $\tau_{x,a}$ is a stopping time relative to $\mathcal{L}$, as alleged.

8. Proceed to prove that $\tau_{x,a}$ is the first exit time $\tau_{D(0,a)}(B^x)$ of the open ball $D_{0,a}$ by the Brownian motion $B^x$. To that end, recall that $\tilde{r} > r > 0$ and $a \in (r, \tilde{r})H_\nu$ are arbitrary. Write for abbreviation $\tau \equiv \tau_{x,a}$. Let $g \in C(\partial D_{0,a})$ be arbitrary and let $a' \equiv 2^{-1}a$. Define $\overline{g} \in C_{ab}(\overline{D}_{0,a'},a)$ by

$$\overline{g}(x) \equiv g(y + \frac{a(x - y)}{\|x - y\|})$$

for each $x \in \overline{D}_{0,a',a}$. Then, for each $k \geq 1$, relation (1.48) implies that

$$A_k \subset (\tau_{x,a,N(\tilde{r},k)} = \tau).$$

At the same time, $B^x_{\tau_{x,a,N(\tilde{r},k)}} : \Omega \rightarrow \mathbb{R}^m$ is a well defined r.v. according to (1.40) in Step 1. Hence $\overline{g}(B^x_{\tau_{x,a,N(\tilde{r},k)}}) \in L$. Therefore relation (1.51) implies that

$$\overline{g}(B^x_{\tau_{x,a,N(\tilde{r},k)}})1_{A_k} = \overline{g}(B^x_{\tau_{x,a,N(\tilde{r},k)}})1_{A_k} \in L.$$  

(1.52)

Since $P(A_k) \uparrow 1$ as $k \rightarrow \infty$, the Monotone Convergence Theorem implies that

$$\overline{g}(B^x_{\tau_{x,a,N(\tilde{r},k)}}) \in L.$$  

(1.53)

Next let $\omega \in \text{domain}(\tau) \cap A$ be arbitrary. Take $k \geq 1$ so large that $\tau(\omega) < N_{\tilde{r},k}$ and $\omega \in A_k$. Then, according to relation (1.51) we have $\tau_{x,a,N(\tilde{r},k)}(\omega) = \tau(\omega) < N_{\tilde{r},k}$. Hence, according to Condition (ii’) in Step 3, we have

$$B^x_{\tau_{x,a,N(\tilde{r},k)}}(\omega) = B^x_{\tau_{x,a,N(\tilde{r},k)}}(\omega) \in \partial D_{0,a}.$$
Therefore, from the defining equality 1.50 we obtain \( g(B^x) = \mathcal{T}(B^x) \) on the full set \( \text{dom}(\tau) \cap A \). Consequently, relation 1.53 implies that \( g(B^x) \in L \). Since \( g \in C(\partial D_{0,a}) \) is arbitrary, we conclude that \( B^x : (\Omega, L, E) \to \partial D_{0,a} \) is a r.v.

Summing up, Conditions (i-iii) in Definition 29 are satisfied with \( a \) in the place of \( r \). Accordingly, the stopping time \( \tau \equiv \tau_{x,a} \) is the first exit time for the process \( B^x \) to exit the open ball \( D_{0,a} \), with exit point \( B_{x,a}^x \). Assertion 1 is proved.

9. To prove the remaining Assertion 2, let \( k \geq 0 \) be arbitrary. Fix a regular point \( t_k \in (N_r,k-1, N_{r,k}) \) of the stopping time \( \tau_{x,a,N(r,k)} \). Then

\[
(\tau_{x,a} > t) \subset (\tau_{x,a} > N_{r,k}) \subset (\tau_{x,a} > N_{r,k} + A_k \cup A_k^c)
\]

\[
\subset (\tau_{x,a,N(r,k)} > N_{r,k} + A_k \cup A_k^c)
= \phi \cup A_k^c = A_k^c
\]

where the third inequality is by relation 1.38 and where the first equality is because the first exit time \( \tau_{x,a,N(r,k)} \) has values in \( (0, N_{r,k}) \). Hence

\[
P(\tau_{x,a} > t) \leq P(A_k^c) \leq 2^{-k+1}
\]

where the second inequality is by inequality 1.40. Note here that \( r > 0 \) and \( a \in (\tau, r) \) are arbitrary. Assertion 2 and the lemma are proved. \( \square \)

The next theorem strengthens the preceding Lemma 30 by removing the exceptional set \( H \) and by proving that the first hitting time \( \tau_{x,a} \) is then an a.u. continuous process with \( a \in (\tau, r) \) as parameter.

**Theorem 31. Existence and continuity of first exit time.** Let \( \tau > 0 \) and \( x \in D_{0,\tau} \) be arbitrary. Then the following conditions hold.

1. (a.u. Continuity on dense subset of the interval \( (\tau, \tau) \)) By Lemma 30, there exists a countable subset \( H \) of \( R \) such that, for each \( a \in (\tau, \tau)H_e \), the first exit time \( \tau_{x,a} \equiv \tau_{D_{0,a}}(B^x) \) exists.

Let \( k \geq 1 \) be arbitrary. Let \( a_k^e, a_k^u \in (\tau, \tau)H_e \) be such that

\[
a_k^u - a_k^e < 2^{-k-1}
\]

and such that

\[
2\Phi_{0,1}(\frac{a_k^u - a_k^e}{\sqrt{2^{-k-1}}}) - 1 < 2^{-k-1}
\]

Then there exists a measurable set \( A_k \) with \( P(A_k^c) < 2^{-k} \) such that

\[
A_k \subset (\tau_{x,a^e}(k) - \tau_{x,a^e}(k) \leq 2^{-k}).
\]

2. (Existence). Let \( r \in (\tau, \tau) \) be arbitrary. Then the first exit time \( \tau_{x,r} \equiv \tau_{D_{0,r}}(B^x) \) exists.

3. (a.u. Continuity on \( (\tau, \tau) \)). Let \( \kappa \geq 1 \) be arbitrary. Let \( r', r'' \in (\tau, \tau) \) be arbitrary with

\[
0 \leq r'' - r' < 2^{-\kappa-1}
\]
and

$$2\Phi_0,1\left(\frac{\rho''-\rho'}{\sqrt{2-\kappa-1}}\right) - 1 < 2^{-\kappa-1}. \tag{1.59}$$

Then there exists a measurable set $G_\kappa$ with $P(G_\kappa^c) < 2^{-\kappa+1}$ such that

$$G_\kappa \subset \{0 \leq \tau_{x,\rho''} - \tau_{x,\rho'} \leq 2^{-\kappa+1}\}.$$

4. (Tightness). The family $\{\tau_{x,r} : x \in D_0, r \in (\overline{r}, \bar{r})\}$ of r.v.'s is tight. Specifically, consider each $r \in (\overline{r}, \bar{r})$. Let $k \geq 1$ be arbitrary and let $N_{\overline{r},k}$ be the integer defined in (1.37) of Definition 27. Then

$$P(\tau_{x,r} > t) \leq 2^{-k+1}$$

for each $t > N_{\overline{r},k}$.

Proof. 1. To prove Assertion 1, let $k \geq 1$ and the points $a'_k, a''_k \in (\overline{r}, \bar{r})H_x$ be as given, satisfying inequalities 1.55 and 1.56. For abbreviation, write $\alpha' \equiv a'_k$, $\alpha'' \equiv a''_k$, $\alpha''' \equiv \alpha''_k$. Fix an arbitrary $\delta_k \in (2^{-k-1}, 2^{-k})$.

2. We will verify that the measurable set

$$A_k \equiv (\tau_{x,a'''}(k) \leq \tau_{x,a''(k)} + \delta_k). \tag{1.60}$$

has the desired properties in Assertion 1. Because $\delta_k < 2^{-k}$, the desired relation 1.57 is trivial. It remains to show that

$$P(A_k^c) = P(\tau_{x,a''(k)} > \tau_{x,a''(k)} + \delta_k) < 2^{-k}. \tag{1.61}$$

3. To that end, consider each $\omega \in A_k^c$ and $s \in [0, \delta_k]$. We can apply condition (iii) in Definition 29 to the time point $t \equiv \tau'(\omega) + s$ and the first exit time $\tau'' \equiv \tau_{x,a''''(k)}$, to obtain $B_{\tau''+s}(\omega) \in D_{0,a''''(k)}$. Thus

$$A_k^c \subset \bigcap_{s \in [0, \delta_k]} (B_{\tau''+s} \in D_{0,a''''(k)})$$

$$\subset (\bigcup_{s \in [0, \delta_k]} \|B_{\tau''+s}\| < a''_k). \tag{1.62}$$

4. We will prove that probability of the measurable set on the right-hand side of relation 1.62 is bounded by $2^{-k}$. For that purpose, consider the process $B_{\tau''+s} : [0, \infty) \times \Omega \rightarrow \mathbb{R}^m$

Applying condition (ii) in Definition 29 to the first exit time $\tau' \equiv \tau_{x,a''(k)}$, we see that $B_{\tau'}$ is a r.v. with values in $\partial D_{0,a''(k)}$. Hence the r.v.

$$U \equiv a''_k^{-1}B_{\tau'} \tag{1.63}$$

has values in $\partial D_{0,1}$. In term of components,

$$U \equiv (U_1, \cdots, U_m) \equiv a''_k^{-1}(B_{\tau'}^1, \cdots, B_{\tau'}^m).$$
Thus the process $\tilde{a} u. c. $ continuous, which is the characteristic function of $B$

Because the process $B_{s}$ for each set

$\sum_{j=1}^{m} U_{j}^{2} = 1$ and where $U_j \in L(\nu)$ for each $j = 1, \cdots, m.$

5. Define the process $\tilde{B} : [0, \infty) \times \Omega \to \mathbb{R}^1$ by

$$\tilde{B}_{s} \equiv U \cdot (B_{\tau}^{x} + \tilde{B}_{s}^{x} - B_{s}^{x}) = \sum_{j=1}^{m} U_{j} (B_{j, \tau}^{x} + \tilde{B}_{s}^{x} - B_{j}^{x})$$

$$= U \cdot (B_{\tau}^{x} + \tilde{B}_{s}^{x} - B_{s}^{x}) = \sum_{j=1}^{m} U_{j} (B_{j, \tau}^{x} + \tilde{B}_{s}^{x} - B_{j}^{x})$$  (1.64)

for each $s \geq 0$, where the dot between two vectors signifies inner product. Because the process $\tilde{B}$ is a.u. continuous on compact subsets of the parameter set $[0, \infty)$, so is the process $\tilde{B}$.

6. Trivially, $\tilde{B}_{0} = 0$. Let the nondecreasing sequence $(s_{1}, \cdots, s_{k})$ in $[0, \infty)$ and the sequence $(\lambda_{1}, \cdots, \lambda_{k})$ in $\mathbb{R}^m$ be arbitrary. Write $i \equiv \sqrt{-1}$ and $s_{0} \equiv 0$. Then the characteristic function

$$E(\exp \sum_{k=1}^{n} i \lambda_{k} (\tilde{B}_{s}(k) - \tilde{B}_{s}(k-1))) = E(\prod_{k=1}^{n} \exp i \lambda_{k} (\tilde{B}_{s}(k) - \tilde{B}_{s}(k-1)))$$

$$= E(E(\prod_{k=1}^{n} \exp i \lambda_{k} (\tilde{B}_{s}(k) - \tilde{B}_{s}(k-1))) | U_{1}, \cdots, U_{m}))$$

$$= E(E(\prod_{k=1}^{n} \exp i \lambda_{k} (\sum_{j=1}^{m} U_{j} (B_{j, \tau}^{x} + \tilde{B}_{s}^{x} - B_{j}^{x}) | U_{1}, \cdots, U_{m}))$$

$$= E(E(\prod_{k=1}^{n} \prod_{j=1}^{m} \exp i \lambda_{k} U_{j} (\tilde{B}_{j, s}(k) - \tilde{B}_{j, s}(k-1))))$$

$$= E(\prod_{k=1}^{n} \prod_{j=1}^{m} \exp -2^{-1} \lambda_{k}^{2} U_{j}^{2} (s_{k} - s_{k-1})$$

$$= E(\prod_{k=1}^{n} \exp -2^{-1} \lambda_{k}^{2} \sum_{j=1}^{m} U_{j}^{2} (s_{k} - s_{k-1}))$$

$$= E(\prod_{k=1}^{n} \exp -2^{-1} \lambda_{k}^{2} (s_{k} - s_{k-1}))$$

which is the characteristic function of $B_{1, s(1)} - B_{1, s(0)}, \cdots, B_{1, s(n)} - B_{1, s(n-1)}$. Thus the process $\tilde{B}$ is equivalent to the Brownian motion $B_{1}$ in $\mathbb{R}^1$. Being also a.u. continuous, $\tilde{B}$ is itself a Brownian motion in $\mathbb{R}^1$. 

1 Preliminaries

32
7. Recall from the defining equality 1.60 in Step 2 the measurable set
\[ A_k \equiv (\tau_{x,a''(k)} - \tau_{x,a'(k)} \leq \delta_k). \]
We will verified that its complement has probability bounded by \(2^{-k-1}\). Consider each \(s \in [0, \delta_k]\) and
\[ \omega \in A_k^c = (\tau_{x,a''(k)} > \tau' + \delta_k) \subset (\tau_{x,a''(k)} > \tau' + s). \]
Then \(t \equiv \tau'(\omega) + s < \tau_{x,a''(k)}(\omega)\), Therefore, \(B_t^s(\omega) \in D_{0,a''(k)}\). Consequently,
\[ \tilde{B}_s(\omega) = U(\omega) \cdot (B^s_{\tau' + s}(\omega) - B^s_{\tau'}(\omega)) \]
\[ = a^{\tau' - s} B^s_{\tau'}(\omega) \cdot B^s_{\tau'}(\omega) - U(\omega) \cdot B^s_{\tau'}(\omega) \]
\[ \leq a^{\tau' - s} \| B^s_{\tau'}(\omega) \| \cdot \| B^s_{\tau'}(\omega) \| - U(\omega) \cdot B^s_{\tau'}(\omega) \]
\[ < a^{\tau' - s} a'' - U(\omega) \cdot B^s_{\tau'}(\omega) \]
\[ \equiv a'' - U(\omega) \cdot (a''U(\omega)) = a'' - a_k' \]
where the first inequality is by the Cauchy-Schwarz inequality, and where the next to last inequality is by the defining equality 1.63. Combining, we obtain
\[ A_k^c \subset \bigcup_{s \in [0, \delta(k)]} \tilde{B}_s \leq a'' - a_k' \]
Hence
\[ P(A_k^c) \leq P \left( \bigcup_{s \in [0, \delta(k)]} \tilde{B}_s \leq a'' - a_k' \right) \]
\[ = 1 - P \left( \bigcup_{s \in [0, \delta(k)]} \tilde{B}_s > a'' - a_k' \right) \]
\[ = 2 \Phi_0,1 \left( \frac{a'' - a_k'}{\sqrt{\delta} k} \right) - 1 \]
\[ < 2 \Phi_0,1 \left( \frac{a'' - a_k'}{\sqrt{2^{-k-1}}} \right) - 1 < 2^{-k-1}. \]  
(1.65)
where the last equality by applying the reflection principle, Lemma 26 to the Brownian motion \(B\), and where the last inequality is by inequality 1.56. Inequality 1.64 and Assertion 1 are proved.

8. Next, let \(r \in (\tau, \bar{r})\) be arbitrary. Let \((a_k')_{k=1,2,\ldots}\) be an increasing sequence in \((\tau, r)H_{\epsilon}\) and let \((a_k')_{k=1,2,\ldots}\) be a decreasing sequence in \((\tau, r)H_{\epsilon}\) such that inequalities 1.55 and 1.56 hold. For each \(k \geq 1\), fix an arbitrary \(\delta_k \in (2^{-k-1}, 2^{-k})\).and define, as in equality 1.60, the measurable set
\[ A_k \equiv (\tau_{x,a''(k)} \leq \tau_{x,a'(k)} + \delta_k). \]  
(1.66)
Then \(P(A_k^c) < 2^{-k}\) and
\[ A_k \subset (\tau_{x,a''(k)} - \tau_{x,a'(k)} \leq 2^{-k}). \]  
(1.67)
9. Let $\kappa \geq 1$ be arbitrary. Define $A_{\kappa+} \equiv \bigcap_{k=\kappa}^{\infty} A_k$. From relation \ref{L66} we see that

$$A_{\kappa+} \subset \bigcap_{k=\kappa}^{\infty} \left( \tau_{x,a'}(k) \leq \tau_{x,a''}(k+1) \leq \tau_{x,a'''}(k) \leq \tau_{x,a'}(k) + \delta_k \right)$$

$$\subset \bigcap_{k=\kappa}^{\infty} \left( \tau_{x,a'}(k+1) - \tau_{x,a'}(k) \leq 2^{-k+1} \right)$$

$$\cap \bigcap_{k=\kappa}^{\infty} \left( \tau_{x,a''}(k) - \tau_{x,a''}(k+1) \leq 2^{-k+1} \right)$$

$$\cap \bigcap_{k=\kappa}^{\infty} \left( \tau_{x,a'''}(k) - \tau_{x,a'}(k) \leq 2^{-k+1} \right). \tag{1.68}$$

Since $P(A_{\kappa+}^c) \leq \sum_{k=\kappa}^{\infty} P(A_k^c) < 2^{-\kappa}$ is arbitrarily small for sufficiently large $\kappa \geq 1$, it follows that $\tau_{x,a'}(k) \uparrow \tau'$ a.u. for some r.r.v. $\tau'$, that $\tau_{x,a''}(k) \downarrow \tau''$ for some r.r.v. $\tau''$, and that $\tau' = \tau''$. Moreover, relation \ref{L68} implies that

$$A_{\kappa+} \subset (\tau' - \tau_{x,a'}(\kappa)) \leq 2^{-\kappa+2}$$

$$\cap (\tau_{x,a''}(\kappa) - \tau' \leq 2^{-\kappa+2}) \cap (\tau' = \tau'') \tag{1.69}$$

where $\kappa \geq 1$ is arbitrary.

10. Now define the r.r.v. $\tau : \Omega \to (0, \infty)$ by

$$domain(\tau) \equiv \bigcup_{\kappa=1}^{\infty} A_{\kappa+}$$

and

$$\tau(\omega) \equiv \tau'(\omega) \equiv \lim_{k \to \infty} \tau_{x,a''(k)}(\omega) \equiv \tau''(\omega) \equiv \lim_{k \to \infty} \tau_{x,a'(k)}(\omega)$$

for each $\omega \in domain(\tau)$. By Step 9, we have $\tau_{x,a'(k)} \uparrow \tau$ a.u. and $\tau_{x,a''(k)} \downarrow \tau$ a.u. Since $\tau_{x,a''(k)}$ is a stopping time relative to the right continuous filtration $\mathcal{L}$, it follows that the a.u. limit $\tau$ is a stopping time relative to $\mathcal{L}$.

11. Proceed to verify that $\tau$ is the first exit time $\tau_{D(0,r)}(B^x)$. To that end, consider each $\omega \in domain(\tau)$. Then $\tau_{x,a''(k)}(\omega) \downarrow \tau(\omega)$ as $k \to \infty$. Hence

$$B^x_{\tau}(\omega) = \lim_{k \to \infty} B^z_{\tau(x,a''(k))}(\omega) \in \bigcap_{k=1}^{\infty} D_{0,r,a''(k)} = \partial D_{0,r}.$$

Thus condition (i) of Definition 29 for the first exit time has been proved for $\tau$. Next, let $g \in C(\partial D_{0,r})$ be arbitrary. Define the function $\overline{g} \in C(\partial D_{0,r})$ by

$$\overline{g}(z) \equiv g\left(\frac{r^2}{\|z\|^2}\right) \tag{1.70}$$
for each $z \in \overline{D_0, \tau, \tilde{r}}$. Since $\tau_{x, a''(k)} < \tau$ a.u. as $k \to \infty$, we have $\mathfrak{F}(B^2_{\tau_{x, a''(k)}}) \to \mathfrak{F}(B^2_{\tau})$ a.u. Therefore $\mathfrak{F}(B^2_{\tau})$ is integrable. At the same time, by the defining equality $(1.70)$ we see that $g(B^2_{\tau}) = \mathfrak{F}(B^2_{\tau})$. Hence $g(B^2_{\tau})$ is integrable, with

$$Eg(B^2_{\tau}) = E\mathfrak{F}(B^2_{\tau})$$  \hspace{1cm} (1.71)$$

where $g \in C(\partial D_{0,r})$ is arbitrary. Thus $B^2_{\tau} : \Omega \to \partial D_{0,r}$ is a r.v. Condition (ii) of Definition $(1.69)$ is verified for the stopping time $\tau$. Finally, consider each $\omega \in \text{domain}(\tau)$ and $t \in [0, \tau(\omega))$. Then $t \in [0, \tau_{x, a''(k)}(\omega))$ for some $k \geq 1$. Hence $B^2_{\tau}(\omega) \in D_{0,a''(k)} \subset D_{0,r}$. All three conditions of Definition $(1.69)$ have been verified for the stopping time $\tau$. Accordingly, $\tau$ is the first exit time $\tau_{\tau, r} \equiv \tau_{D(0,r)}(B^2_{\tau})$. Thus Assertion 2 is proved. Moreover, relation $(1.69)$ implies that

$$A_{\kappa+} \subset (\tau_{x,r} - \tau_{x,a''(k)}) \leq 2^{-\kappa+2}$$

$$\cap (\tau_{x,a''(k)} - \tau_{x,r} \leq 2^{-\kappa+2}),$$

(1.72)

where $P(A_{\kappa+}) < 2^{-\kappa}$, and where $\kappa \geq 1$ is arbitrary.

12. We will next prove Assertion 3. To that end, let $r', r'' \in (\overline{\tau}, \overline{\tau})$ be arbitrary with $r' \leq r''$. Suppose

$$r'' - r' < 2^{-\kappa-1}$$

and

$$2\Phi_{0,1}(\frac{r'' - r'}{\sqrt{2^{-\kappa-1}}}) - 1 < 2^{-\kappa-1}$$

(1.73)

for some $\kappa \geq 1$. Then there exist $a'_k, a''_k \in (\overline{\tau}, r')H_\kappa$ such that $a''_k - a'_k < 2^{-\kappa-1}$ and such that $a'_k < r' \leq r'' < a''_k$ with $a''_k - a'_k < 2^{-\kappa-1}$ and

$$2\Phi_{0,1}(\frac{a''_k - a'_k}{\sqrt{2^{-\kappa-1}}}) - 1 < 2^{-\kappa-1}.$$  \hspace{1cm} (1.74)

As in equality $(1.69)$ of Step 2, define the measurable set

$$A_{\kappa} \equiv (\tau_{x,a''(k)} - \tau_{x,a'(k)} \leq \delta_\kappa).$$  \hspace{1cm} (1.75)

Relation $(1.72)$ applied to $r \equiv r'$ then implies that there exists a measurable set $\overline{A}_{\kappa} \subset A_{\kappa}$ with $P(A_{\kappa}) < 2^{-\kappa}$, and

$$\overline{A}_{\kappa} \subset (\tau_{x,r} - \tau_{x,a'(k)} \leq 2^{-\kappa+2})$$  \hspace{1cm} (1.76)

Relation $(1.72)$ applied to $r = r''$ similarly implies that there exists a measurable set $\overline{A}_{\kappa} \subset A_{\kappa}$ with $P(\overline{A}_{\kappa}) < 2^{-\kappa}$, and with

$$\overline{A}_{\kappa} \subset (\tau_{x,a''(k)} - \tau_{x,r''} \leq 2^{-\kappa+2}).$$  \hspace{1cm} (1.77)

Now define the measurable set

$$G_{\kappa} = \overline{A}_{\kappa} \cap \overline{A}_{\kappa}.$$
Then inequalities $1.75$, $1.76$ and $1.77$ together implies that
\[
G_n = \overline{A}_n \cap \overline{A}_k, \cap A_k \subset (\tau_{x,r''} - \tau_{x,r'}) \leq 2^{-\kappa + 2} + 2^{-\kappa + 2} + \delta_n
\]
\[
\subset (\tau_{x,r''} - \tau_{x,r'}) \leq 2^{-\kappa + 4},
\]
where
\[
P(G_n^c) = P(A_n^c \cup A_k^c) < 2^{-\kappa} + 2^{-\kappa} = 2^{-\kappa + 1}.
\]
Assertion 3 is proved.

13. It remains to prove that the family $\{\tau_{x,r} : x \in D_0, r \in (\tau, \tilde{\tau})\}$ is tight. To that end, consider each $x \in D_0$ and $r \in (\tau, \tilde{\tau})$. Let $k \geq 1$ be arbitrary. Take any $a_k', a_k'' \in (\tau, \tilde{\tau})H$ such that $a_k' < r < a_k''$ and such that inequalities $1.55$ and $1.56$ hold. Then, according to Assertion 1, there exists a measurable set $A_k$ with $P(A_k^c) < 2^{-k}$ such that
\[
A_k \subset (\tau_{x,a''}(k) - \tau_{x,a'}(k)) \leq 2^{-k}).
\]
(1.78)
Consider each $t > N_{\bar{\tau},k}$, where $N_{\bar{\tau},k}$ is the integer defined in 1.37 of Definition 27. Then, according to inequality 1.38 in Assertion 2 of Lemma 30, we have
\[
P(\tau_{x,a''}(k) > t) \leq 2^{-k+1},
\]
Since $r > a_k'$ and $\tau_{x,r} \geq \tau_{x,a''}(k)$, it follows that
\[
P(\tau_{x,r} > t) \leq 2^{-k+1}
\]
where $x \in D_0$, $\tilde{\tau} > \tau > 0$, $k \geq 1$, and $t > N_{\bar{\tau},k}$ are arbitrary. Thus the family $\{\tau_{x,r} : x \in D_0, r \in (\tau, \tilde{\tau})\}$ is tight. Assertion 4 is proved.

Lemma 32. Invariance of first exit time relative to time scaling. Let $r > 0$ be arbitrary. Then
\[
\tau_{0,\sqrt{\tau}(\bar{B})} = r\tau_{0,1}(B),
\]
(1.79)
where $\bar{B}$ is the Brownian motion defined by $\bar{B}_s \equiv \sqrt{r}B_{s/r}$ for each $s \geq 0$.

Proof. Consider each $\omega \in \Omega$. Write $t \equiv \tau_{0,\sqrt{\tau}(\bar{B})}(\omega)$. Thus $t$ is the first time when $|\bar{B}_t(\omega)| = |\sqrt{r}B_{t/r}(\omega)| = |\sqrt{r}|$. Hence $t$ is the first time when $|B_{t/r}| = 1$. Therefore $t/r = \tau_{0,1}(B)(\omega)$. In other words, $\tau_{0,1}(B)(\omega) = \tau_{0,\sqrt{\tau}(\bar{B})}(\omega)/r$, where $\omega \in \Omega$ is arbitrary. Equality $1.79$ follows.

Theorem 33. Exit distribution for the $(m-1)$-sphere. Let $x \in D_0$ and $r \in (0,1]$ be arbitrary such that $x \in D_0$. Write $\tau_{x,r} \equiv \tau_{D(0,r)}(B^x)$. Then the following conditions hold.

1. For each $f \in C(\partial D_0)$ we have
\[
Ef(B^x_{\tau_{x,r}}) = \int_{z \in \partial D(0,1)} \frac{1 - \|z\|^2}{\|x - z\|} f(z)\rho_{m,0,1}(dz).
\]
(1.80)
2. In terms of the Poisson kernel \( k_{0,1} \) introduced in Theorem 23, equality (1.80) can be restated as

\[
Ef(B^x_{\tau(x,1)}) = \int_{z \in \partial D(0,1)} k_{0,1}(x,z)f(z)\sigma_{m,0,1}(dz).
\]

Thus \( B^x_{\tau(x,1)} \) induces a distribution on \( \partial D_{0,1} \) that has the density function \( k_{0,1}(x,) \) relative to the uniform distribution \( \sigma_{m,0,1} \).

3. For each \( f \in C(\partial D_{0,r}) \) we have

\[
Ef(B_{\tau(r)}) = \int_{z \in \partial D(0,r)} f(z)\sigma_{m,0,r}(dz). \tag{1.81}
\]

In other words, the r.v. \( B_{\tau(r)} : \Omega \to \partial D_{0,r} \) induces the uniform distribution \( \sigma_{m,0,r} \) on \( \partial D_{0,r} \).

Proof. 1. For the proof of equality (1.80), see assertion (1) in section 1.10 of [Durret 1984]. Assertion 2 of the present theorem is then a trivial consequence of the definition of the Poisson kernel.

2. To prove Assertion 3, let \( \alpha \) be an arbitrary rotation matrix. Then

\[
Ef \circ \alpha(B_{\tau(r)}; B) = Ef((\alpha B)_{\tau(r)}; \alpha B) = Ef(B_{\tau(1); B}) = Ef(B_{\tau(1); B}).
\]

Thus the distribution induced on \( C(\partial D_{0,r}) \) by the random exit point \( B_{\tau(r)} = B_{\tau(1); B} \) is invariant relative to rotation about the center. By Corollary 14 we see that \( B_{\tau(r)} \) induces the uniform distribution on \( C(\partial D_{0,r}) \).

In the remainder of this article, for abbreviation we will write \( D \equiv D_{0,1} \equiv D_{m,0,1} \equiv \overline{D}_{0,1} \equiv \overline{D}_{m,0,1} \), and \( \partial D \equiv \partial D_{0,1} \equiv \partial D_{m,0,1} \). For each \( r > 0 \), we will write \( \tau_r \equiv \tau_{0,r} \) for the first exit time for the Brownian motion \( B \) to exit the open ball \( D_{0,r} \).

**Theorem 34. Dynkin’s theorem.** Let \( a > 0 \) be arbitrary. Let \( u \) be an arbitrary continuous function on \( \overline{D}_{0,a} \) which is harmonic on \( D_{0,a} \). Define the process

\[
X : [0, \infty) \times (\Omega, L, E) \to R
\]

by

\[
X_t \equiv u(B_{t\wedge \tau(a)})
\]

for each \( t \in [0, \infty) \). Then \( X \) is a martingale relative to the right continuous filtration \( L \).

Proof. The present theorem of Dynkin for the case where \( a = 1 \) is cited as assertion (5) on page 26 of [Durret 1984] along with a proof which is an application of Ito’s lemma in the theory of stochastic integration relative to the Brownian motion. The reader can convince himself or herself that both said proof and said theory of stochastic integration presented in chapter 2 of [Durret 1984] are constructive, or can easily be made so. In addition, the reader should convince himself or herself that the assumption \( a = 1 \) is not essential.
1 Preliminaries

Lemma 35. $B_{\tau(r)}$ induces an integration $\partial D_{0,r}$. Let $r > 0$ be arbitrary.

1. Let $g : \partial D \to R$ be an arbitrary function. Then $g$ is an integrable function on $\partial D$ relative to $\mathfrak{m}_{m,0,1}$ iff $g \circ (r^{-1}B_{\tau(r)})$ is integrable on $\Omega$ relative to $E$, in which case

$$\mathfrak{m}_{m,0,1}g = E(g \circ (r^{-1}B_{\tau(r)})).$$

2. Let $f$ be an arbitrary continuous function on $\partial D_{0,r}$. Define the function $g : \partial D \to R$ by

$$g \equiv f(r).$$

Then

$$\mathfrak{m}_{m,0,1}g = E(f \circ B_{\tau(r)}).$$

Proof. 1. Assertion 3 of Theorem 34 says that the random point $B_{\tau(r)} : (\Omega, L, E) \to \partial D_{0,r}$ of exit induces the uniform distribution $\mathfrak{m}_{m,0,r}$ on $\partial D_{0,r}$. Hence the r.v. $r^{-1}B_{\tau(r)} : (\Omega, L, E) \to \partial D$ induces the uniform distribution $\mathfrak{m}_{m,0,1}$ on $\partial D$. Assertion 1 of the present lemma follows.

2. Next, Let $f$ be an arbitrary continuous function on $\partial D_{0,r}$. Define the function $g : \partial D \to R$ by $g \equiv f(r)$. Then

$$\mathfrak{m}_{m,0,1}g \equiv \mathfrak{m}_{m,0,1}f(r) = \mathfrak{m}_{m,0,r}f = E(f \circ B_{\tau(r)}).$$

□

Corollary 36. Observations of a harmonic function of the Brownian motion at successive first exit times of concentric open balls constitute an a.u. continuous martingale. Let $u$ be an arbitrary harmonic function on $D \equiv D_{0,1}$. Then the following conditions hold.

1. For each $r \in [0, 1)$, we have $u(B_{\tau(s)}) \to Y_r$ a.u. as $s \downarrow r$ for some $Y_r \in L$. 

2. For each $r \in (0, 1)$, we have $Y_r = u(B_{\tau(r)})$.

3. $Y_0 = u(0)$.

4. The process $Y : [0, 1) \times (\Omega, L, E) \to R$ is an a.u. continuous martingale relative to the right continuous filtration

$$\hat{\mathcal{L}} \equiv \{L^{\tau(r)} : r \in (0, 1)\}.$$

5. The expectations $E|u(B_{\tau(r)})|$ and $\mathfrak{m}_{m,0,1}|u(r)|$ are equal and nondecreasing in $r \in (0, 1)$.

6. The expectations $E \exp(-|u(B_{\tau(r)})|)$ and $\mathfrak{m}_{m,0,1} \exp(-|u(r)|)$ are equal and nondecreasing in $r \in (0, 1)$.

Proof. 1. Let $r \in [0, 1)$ be arbitrary. Then $r \in (0, a]$ for some $a < 1$. By hypothesis, the function $u$ is continuous on $\overline{D_{0,a}}$ and is harmonic on $D_{0,a}$. Hence Theorem 34 is applicable, and implies that the process $X : [0, \infty) \times (\Omega, L, E) \to R$, defined by $X_t \equiv u(B_{t \wedge \tau(a)})$ is a martingale.

2. Because the a.u. continuous process $B_{\wedge \tau(a)}$ has values in the compact ball $D_{0,a}$ and because the function $u$ is continuous on $\overline{D_{0,a}}$, so the martingale $X \equiv u(B_{\wedge \tau(a)})$ is a.u. continuous and bounded.
3. Let $\varepsilon > 0$ be arbitrary. Then there exists $\overline{t}(\varepsilon) > 0$ so large that $P(G) < \varepsilon$ where $G \equiv (\tau_{0,a} > \overline{t}(\varepsilon))$ and

$$G^c \equiv (\tau_{0,a} \leq \overline{t}(\varepsilon)).$$

Because $X \equiv u(B_{\tau(\alpha)})$ is a.u. continuous on $[0, \overline{t}(\varepsilon)]$, there exists a measurable set $\overline{G}$ with $P(\overline{G}) < \varepsilon$ and $\delta(\varepsilon) > 0$ such that

$$\overline{G} \subset \bigcap_{t,t' \in [0,\overline{t}];|t-t'|<\overline{\delta}(\varepsilon)} (|X_t - X_{t'}| \leq \varepsilon).$$

Because $\tau_0$ is a.u. continuous on $(0,1]$, there exists a measurable set $\tilde{G}$ with $P(\tilde{G}) < \varepsilon$ and $\delta(\varepsilon) > 0$ such that

$$\tilde{G} \subset \bigcap_{s,s' \in (0,a];|s-s'|<\delta(\varepsilon)} (|\tau_{0,s} - \tau_{0,s'}| \leq \delta).$$

Consider each $\omega \in G^c \cap \overline{G}$. Let $s, s' \in (r,a]$ be arbitrary with $|s-s'| < \delta(\varepsilon)$. Then

$$\tau_{0,s}(\omega) \lor \tau_{0,s'}(\omega) \leq \tau_{0,a}(\omega) \leq \overline{t}$$

and $|\tau_{0,s}(\omega) - \tau_{0,s'}(\omega)| < \delta$, whence $|X_{\tau(s)}(\omega) - X_{\tau(s')}(\omega)| \leq \varepsilon$. Thus

$$G^c \cap \overline{G} \subset \bigcap_{s,s' \in (r,a];|s-s'|<\delta(\varepsilon)} (|X_{\tau(s)} - X_{\tau(s')}| \leq \varepsilon).$$

Since $P(G \cup \overline{G} \cup \tilde{G}) < 3\varepsilon$ is arbitrarily small, we see that $X_{\tau(s)} \to Y_r$ a.u. for some $Y_r \in \mathcal{L}$ as $s \downarrow r$. In other words, $u(B_{\tau(s)}) \to Y_r$ a.u. as $s \downarrow r$ Assertion 1 is proved.

4. Assertions 2 and 3 are then trivial.

5. Note that, for each $a > 0$ and $r \in (0,a]$, we have

$$Y_r = u(B_{\tau(r)}) = u(B_{\tau(r) \land \tau(\alpha)}) = X_{\tau(r)}.$$
7. Similarly, the expectation $E \exp(-|u(B_{\tau(r)})|) = E \exp(-|Y_r|)$ is nondecreasing because $|Y_r|$ is a wide sense submartingale and because $\exp(-y)$ is a convex function of $y \in R$. Since $B_{\tau(r)}$ induces the uniform distribution on $\partial D_{0,r}$, we have

$$E \exp(-|u(B_{\tau(r)})|) = \sigma_{m,0,r} \exp(-|u|) = \sigma_{m,0,1} \exp(-|u(r)|).$$

Hence $\sigma_{m,0,1} \exp(-|u(r)|)$ is nondecreasing in $r \in (0,1)$. Assertion 6 and the lemma are proved.

### 1.4 Maximal inequality for a martingale

**Definition 37.** The special symmetric convex function. Define the continuous function $\lambda: R \to R$ by

$$\lambda(v) \equiv e^{-|v|} - 1 + |v| = \frac{|v|^2}{2!} - \frac{|v|^3}{3!} + \cdots \quad (1.83)$$

for each $v \in R$. We will call $\lambda$ the *special symmetric convex function*. Then $\lambda$ is continuously differentiable and strictly convex on $R$, with

$$|\lambda(v)| \leq |v| \quad (1.84)$$

for each $v \in R$. □

**Theorem 38.** Maximal inequality for a martingale. Let $Z: \{0,1,\ldots,n\} \times \Omega \to R$ be an arbitrary martingale. Let $\epsilon > 0$ be arbitrary. Suppose

$$E\lambda(Z_n) - E\lambda(Z_0) < \frac{1}{6} \epsilon^3 \exp(-\frac{3}{\epsilon} (E|Z_0| \vee E|Z_n|)). \quad (1.85)$$

Then

$$P\left( \bigvee_{k=0}^n |Z_k - Z_0| > \epsilon \right) < \epsilon. \quad (1.86)$$

*Proof.* It can easily be verified that $\lambda$ satisfies the defining conditions for admissible function in the maximal inequality of Chapter 8 in [Bishop 1967], of which the present theorem is therefore a special case. □

### 2 Brownian Limit Theorem for Hardy space

**Definition 39.** Hardy space. A harmonic function $u$ on $D$ is said to be a member of the Hardy space $h^p$ if there exists $b_0 > 0$ such that

$$\int_{z \in \partial D} |u(rz)|^p \sigma_{m,0,1}(dz) < b_0 \quad (2.1)$$

for each $r \in (0,1)$. Recall here that $\sigma_{m,0,r}$ is the uniform distribution on the $(m-1)$–sphere $\partial D_{0,r}$ for each $r > 0$. Note that, by Lyapunov’s inequality, we have $h^p \subset h \equiv h^1$. Without loss of generality, we will assume that $u(0) = 0$. 

Lemma 40. Monotonicity and boundedness of certain integrals for each member of the Hardy space $h^p$. Let the dimension $m \geq 2$ and the exponent $p \geq 1$ be arbitrary, but fixed. Let $u \in h^p$ be arbitrary. Then $u \in h$ and the following conditions hold.

1. For each $r \in (0, 1)$, the function $u$ is uniformly continuous on $D_0,r$, with some modulus of continuity $\delta_{u,r}$.

2. The integral
   \[ I_{1,r} = \sigma_{m,0,1}|u(r\cdot)| = \int_{z \in \partial D} |u(rz)||\sigma_{m,0,1}(dz) \]  
   (2.2)
   is a nondecreasing function of $r \in (0, 1)$. Moreover, there exists $b_0 > 0$ such that $I_{1,r} \leq b_0$ for each $r \in (0, 1)$.

3. The integral
   \[ I_{2,r} = \sigma_{m,0,1}(e^{-|u(r\cdot)|}) = \int_{z \in \partial D} e^{-|u(rz)|}\sigma_{m,0,1}(dz) \]  
   (2.3)
   is a nondecreasing function of $r \in (0, 1)$. Moreover, $I_{2,r} \leq 1$ for each $r \in (0, 1)$.

Proof. 1. Assertions 1 and the boundedness of $I_{1,r}$ immediately follow from Definition 39 for the Hardy space.

2. According to assertion 5 of Corollary 36, the integral $I_{1,r} \equiv \sigma_{m,0,1}|u(r\cdot)|$ is nondecreasing in $r \in (0, 1)$. Since $I_{1,r}$ has just been proved to be bounded, Assertion 2 of the present lemma follows.

3. According to assertion 6 of Corollary 36, the integral $I_{2,r} \equiv \sigma_{m,0,1}e^{-|u(r\cdot)|}$ is nondecreasing. Since $e^{-|u(r\cdot)|} \leq 1$, we see that $I_{2,r} \leq 1$. Assertion 3 of the present lemma is proved.

In the following, for abbreviation, we will write $\tau_r \equiv \tau_{0,r}$ for each $r > 0$. If emphasis is needed for the underlying Brownian motion $B$, then we write $\tau_{r,B} \equiv \tau_{0,r,B}$ for each $r > 0$. Next is the main theorem of the present paper.

Theorem 41. Brownian limit theorem for the Hardy space $h$. Let the dimension $m \geq 2$ and the exponent $p \geq 1$ be arbitrary, but fixed. Let $u$ be an arbitrary member of the Hardy space $h^p$. Suppose the following conditions (i) and (ii) hold.

(i) There exists $b_1 \geq 0$ such that $I_{1,r} \equiv \sigma_{m,0,1}|u(r\cdot)| \to b_1$ as $r \to 1$ with $r \in (0, 1)$. More precisely, suppose that, for each $\varepsilon > 0$ there exists $\delta_1(\varepsilon) \in (0, 1)$ so small that
   \[ 0 \leq b_1 - I_{1,r} < \varepsilon \]  
   (2.4)
   for each $r \in (1 - \delta_1(\varepsilon), 1)$. The number $b_1$ and the operation $\delta_1$ are, respectively, the limit and the rate of convergence, of the integral $I_{1,r}$ as $r \to 1$ with $r \in (0, 1)$. By replacing $\delta_1$ if necessary, we may, without loss of generality, assume that $\delta_1(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$. 

□
(ii) There exists $b_2 \geq 0$ such that $I_{2,r} \equiv \sigma_{m,0}(e^{-|u(r)|}) \to b_2$ as $r \to 1$ with $r \in (0,1)$. More precisely, suppose that, for each $\varepsilon > 0$, there exists $\delta_2(\varepsilon) \in (0,1)$ so small that
\begin{equation}
0 \leq b_2 - I_{2,r} < \varepsilon
\end{equation}
for each $r \in (1-\delta_2(\varepsilon),1)$. The number $b_2$ and the operation $\delta_2$ are, respectively, the limit and the rate of convergence, of the integral $I_{2,r}$ as $r \to 1$ with $r \in (0,1)$. By replacing $\delta_2$ if necessary, we may, without loss of generality, assume that $\delta_2(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$.

Then there exist an increasing sequence $(r_q)_{q=1,2,\ldots}$ in $(0,1)$ with $r_q \uparrow 1$ and a r.v. $V$ such that, for each $q \geq 1$, there exists a measurable set $A_q$ with $P(A_q) < 2^{-q+4}$ and
\begin{equation}
|V - u(B_s)| \leq 2^{-q+3}
\end{equation}
provided that $1 - r < \delta_3(\varepsilon) = \delta_1(2^{-1}\varepsilon) \wedge \delta_2(2^{-1}\varepsilon)$.

In short, as $s \uparrow 1$, we have $u(B_s) \to V$ a.u.

\textbf{Proof.} 1. We will verified that the integral
\begin{equation}
I_{3,r} = \sigma_{m,0}(u(r))
\end{equation}
is a nondecreasing function of $r \in (0,1)$. To that end, we will first show that
\begin{equation}
I_{3,r} \uparrow \gamma = b_2 - 1 + b_1
\end{equation}
as $r \to 1$ with $r \in (0,1)$. More precisely, we will prove that for each $r \in (0,1)$ and $\varepsilon > 0$, we have
\begin{equation}
0 \leq \gamma - I_{3,r} < \varepsilon
\end{equation}
provided that
\begin{equation}
1 - r < \delta_3(\varepsilon) = \delta_1(2^{-1}\varepsilon) \wedge \delta_2(2^{-1}\varepsilon).
\end{equation}
Thus $\gamma$ and the function $\delta_3$ are, respectively, the limit and a rate of convergence of the integral $I_{3,r}$ as $r \to 1$. Note that both $\gamma$ and $\delta_3$ depend only on the given items $b_1, b_2, \delta_1, \delta_2$.

2. Note that
\begin{align}
I_{3,r} &= \sigma_{m,0}(u(r)) = \int_{z \in \partial D} \sigma_{m,0}(u(rz))\sigma_{m,0}(dz) \\
&= \int_{x \in \partial D(0,r)} \sigma_{m,0}(dx)\sigma_{m,0}(u(x)) \\
&= E[\lambda(u(B_{\tau(r)}))] = E[\lambda(Y_r)],
\end{align}
where the third equality is by an application of Corollary 14 where fourth equality is because the r.v. $B_{\tau(r)}$ induces the uniform distribution $\sigma_{m,0}$ on $\partial D(0,r)$ according to Assertion 3 of Theorem 33 and where the last equality is by Assertion 2 of Corollary 36 regarding the martingale $Y$. Since $\lambda$ is a convex
function, $E\overline{X}(Y_r)$ is nondecreasing in $r$. Therefore $I_{3,r}$ on the left-hand side of equality (2.10) is nondecreasing in $r$.

3. By Definition 3, the harmonic function $u$ on $D$ is uniformly continuous on compact subsets of $D$. In particular, $u$ is uniformly continuous on $D_{0,r}$ with some modulus of continuity $\delta_{u,r}$ for each $r \in (0, 1)$. Consequently, the function $u(r)$ is nondecreasing in $r$.

4. By Definition 37, we have

$$\lambda(v) \equiv (e^{-|v|} - 1 + |v|)$$

for each $v \in \mathbb{R}$. Hence

$$I_{3,r} \equiv \sigma_{m,0,1}(u(r)) = \sigma_{m,0,1}(e^{-|u(r)|} - 1 + |u(r)|) = I_{2,r} - 1 + I_{1,r} \rightarrow \gamma \equiv b_2 - 1 + b_1,$$

as $r \uparrow 1$. More precisely, for each $r \in (0, 1)$ and $\varepsilon > 0$, we have

$$0 \leq \gamma - I_{3,r} = (b_2 - I_{2,r}) + (b_1 - I_{1,r}) < 2^{-1}\varepsilon + 2^{-1}\varepsilon = \varepsilon,$$

provided that

$$1 - r < \delta_3(\varepsilon) \equiv \delta_1(2^{-1}\varepsilon) \land \delta_2(2^{-1}\varepsilon).$$

Convergence relation (2.8) and inequality (2.9) are proved.

5. Define an increasing sequence $(r_q)_{q=1,2,\ldots}$ in $(0, 1)$ by

$$r_q \equiv 1 - \delta_3\left(\frac{1}{12}2^{-q}\exp(-3 \cdot 2^q b_1)\right)$$

for each $q \geq 1$. Then $r_q \uparrow 1$ as $q \rightarrow \infty$.

6. Consider each $s, r \in (0, 1)$ with $s \leq r$. Since the pair

$$(u(B_{\tau(s)}), u(B_{\tau(r)})) \equiv (u(B_{\tau(s)}), u(B_{\tau(r)}))$$

of r.r.v.’s is a martingale according to Corollary 36, theorem 8.3.2 of [Chan 2021] (the Bishop-Jensen inequality) is applicable, to yield

$$0 \leq E\overline{X}(u(B_{\tau(r)})) - E\overline{X}(u(B_{\tau(s)})).$$

7. Let $\varepsilon > 0$ be arbitrary. Then, in view of equality (2.10) we can restate equality (2.12) as

$$0 \leq \gamma - E\overline{X}(Y_r) < \varepsilon.$$

for each $r \in (1 - \delta_3(\varepsilon), 1)$.

8. Now let $q \geq 1$ be arbitrary. Write $\varepsilon_q \equiv 2^{-q}$ for abbreviation. Then inequality (2.13) implies

$$1 - r_q < \delta_3\left(\frac{1}{6}\varepsilon_q\exp(-\frac{3}{\varepsilon_q} b_1)\right).$$
or, equivalently,

\[ 1 - r_q < \bar{\delta}_q \equiv \delta_3(\bar{\varepsilon}_q) \]

where

\[ \bar{\varepsilon}_q \equiv \frac{1}{6} \varepsilon_q \exp\left(-\frac{3}{\varepsilon_q} b_1\right). \]

9. Separately, recall from Definition [27] the integer \( N_{2,q} \), with

\[ 2\Phi_{0,1}\left(\frac{2}{\sqrt{N(2,q)}}\right) - 1 < 2^{-q}, \quad (2.17) \]

where \( \Phi_{0,1} \) is the standard normal C.D.F. Consider the first exit time \( \tau_1 \). Fix an arbitrary \( t_q > N_{2,q} \). Then, applying Assertion 4 of Theorem [31] where \( x, r, r, \bar{r}, k, t \) are replaced with \( 0, 2^{-1}, 1, 2, q, t_q \) respectively, we obtain

\[ P(\tau_1 > t_q) \leq 2^{-q+1}. \quad (2.18) \]

10. Next, note that

\[ E|Y_{r(q)}| = E|u(B_{r(r(q))})| = \overline{\sigma}_{m,0,r(q)}|u| \]

\[ = \overline{\sigma}_{m,0,1}|u(r_q)| \equiv I_{1,r(q)} \leq b_1, \quad (2.19) \]

where the inequality is by the first half of inequality [2.4]. At the same time, since

\[ 1 - r_q < \delta_3(\bar{\varepsilon}_q) \leq \delta_1(2^{-1}\bar{\varepsilon}_q) \leq \delta_1(\bar{\varepsilon}_q), \]

we have, by inequality [2.4]

\[ 0 \leq \gamma - E\overline{X}(Y_{r(q)}) < \bar{\varepsilon}_q, \]

In view of the monotonicity of the sequence \((E\overline{X}(Y_{r(q)}))_{q=1,2,...}\), this implies that

\[ 0 \leq E\overline{X}(Y_{r(q+1)}) - E\overline{X}(Y_{r(q)}) \leq 0 \]

\[ < \bar{\varepsilon}_q \leq \frac{1}{6} \varepsilon_q \exp\left(-\frac{3}{\varepsilon_q} b_1\right) \]

\[ \leq \frac{1}{6} \varepsilon_q \exp\left(-\frac{3}{\varepsilon_q} (E|Y_{r(q)}| \lor E|Y_{r(q+1)}|)\right), \quad (2.20) \]

where the last inequality is thanks to inequality [2.19].

11. Continuing, write

\[ \varepsilon_q \equiv \delta_{u,r(q+1)}(\varepsilon_q) > 0. \]

where we recall that \( \delta_{u,r(q+1)} \) is a modulus of continuity of the harmonic function \( u \) on the compact subset \( \overline{D}_{0,r(q+1)} \) of \( D_{0,1} \). Since the Brownian motion \( B : [0,t_q] \times \Omega \rightarrow \mathbb{R}^m \) is a.u. continuous on \([0,t_q]\), there exists a measurable set \( H_q \subset \Omega \) with

\[ P(H_q) < \varepsilon_q \quad (2.21) \]
and some $\delta_{B,t(q)}(\tau_q) > 0$ such that for each $\omega \in H_q^c$ and $v, s \in [0, t_q]$ with $|v - s| \leq \delta_{B,t(q)}(\tau_q)$, we have $|B_v(\omega) - B_s(\omega)| < \tau_q$.

12. Now consider an arbitrary $n \geq 1$ and arbitrary sequence $0 \equiv v_0 < v_1 < \cdots < v_n \equiv t_q$ such that

$$\bigvee_{k=1}^n |v_k - v_{k-1}| < \delta_{B,t(q)}(\tau_q).$$

(2.22)

Consider the martingale $Z_0, \cdots, Z_n, Z_{n+1}$ where

$$Z_k \equiv u(B_{\tau(r(q)) \lor v(k) \land \tau(r(q+1)))}$$

for each $k = 0, \cdots, n$, and where

$$Z_{n+1} \equiv Y_{r(q+1)} \equiv u(B_{\tau(r(q+1)))}.$$

Take an arbitrary $\varepsilon_q \in (\varepsilon_q, \varepsilon_{q-1}) \equiv (2^{-q}, 2^{-q+1})$. Then

$$0 \leq E\lambda(Z_{n+1}) - E\lambda(Z_0)$$

$$= E\lambda(Y_{r(q+1)}) - E\lambda(Y_{r(q)})$$

$$\leq \frac{1}{6} \varepsilon_q^3 \exp\left(-\frac{3}{\varepsilon_q} (E|Y_{r(q)}| \lor E|Y_{r(q+1)})\right)$$

$$\leq \frac{1}{6} \varepsilon_q^3 \exp\left(-\frac{3}{\varepsilon_q} (E|Y_{r(q)}| \lor E|Y_{r(q+1)})\right)$$

$$= \frac{1}{6} \varepsilon_q^3 \exp\left(-\frac{3}{\varepsilon_q} (E|Z_0| \lor E|Z_{n+1}())\right),$$

where the second inequality is from inequality (2.20). Thus the conditions in Theorem 38 are satisfied by the martingale $Z_0, \cdots, Z_{n+1}$ and the constant $\varepsilon_q$. Accordingly, inequality (1.86) of Theorem 38 is applicable and yields

$$P\left(\bigvee_{k=0}^{n+1} |Z_k - Z_0| > \varepsilon_q \right) < \varepsilon_q.$$

(2.23)

In other words,

$$P\left(\bigvee_{k=0}^{n+1} |u(B_{\tau(r(q)) \lor v(k) \land \tau(r(q+1)))} - u(B_{\tau(r(q)))}| > \varepsilon_q \right) < \varepsilon_q.$$

Hence

$$P(K_q) < \varepsilon_q < 2^{-q+1},$$

(2.24)

where

$$K_q \equiv \bigvee_{k=0}^n |u(B_{\tau(r(q)) \lor v(k) \land \tau(r(q+1)))} - u(B_{\tau(r(q)))}| \leq \varepsilon_q.$$

(2.25)
13. Now consider the measurable set
\[ A_q = G_q \cup H_q \cup K_q. \]
Consider each \( \omega \in A_q^c \). Then \( \omega \in G_q^c \equiv (\tau_1 \leq t_q) \subset (\tau_s(\tau_q+1) \leq t_q) \). Let
\[ s \in [\tau_s(\omega), \tau_s(\tau_q+1)(\omega)] \subset [0, t_q] \]
be arbitrary. Then there exists \( k = 0, \cdots, n \) such that
\[ |\tau_q(s) \\land v_k \land \tau_{s}(\tau_q+1)(\omega) - s| \leq |v_k - s| \leq \sum_{i=1}^{n} |v_i - v_{i-1}| < \delta_{B,t,q}(\varepsilon_q). \]

Hence, because \( \omega \in H_q^c \), we have
\[ \|B_{\tau_q} \cap v_k \land B_{s}(\omega)\| < \varepsilon_q \equiv \delta_{u,q+1}(\varepsilon_q). \]
Therefore, since \( B_{\tau_q} \cap v_k \land B_{s}(\omega) \in D_{0,t,q+1} \), we obtain
\[ |u(B_{\tau_q} \cap v_k \land B_{s}(\omega)) - u(B_{s}(\omega))| < \varepsilon_q. \]  \hfill (2.26)

Moreover, since \( \omega \in K_q^c \), it follows from equality 2.25 that
\[ |u(B_{\tau_q} \cap v_k \land B_{s}(\omega)) - u(B_{s}(\omega))| \leq \tilde{\varepsilon}_q < 2^{-q+1}. \]  \hfill (2.27)

Combining inequalities 2.26 and 2.27, we obtain
\[ |u(B_{\tau_q}) - u(B_{s})(\omega)| < \varepsilon_q + \tilde{\varepsilon}_q < 2^{-q} + 2^{-q+1} < 2^{-q+2}, \]  \hfill (2.28)
where \( s \in [\tau_q(\omega), \tau_{s}(\tau_q+1)(\omega)] \) and \( \omega \in A_q^c \) are arbitrary. Summing up,
\[ A_q^c \subset \bigcup_{s \in [\tau_q(\omega), \tau_{s}(\tau_q+1)]} |u(B_{\tau_q}(\omega)) - u(B_{s})(\omega)| \leq 2^{-q+2}. \]

and
\[ P(A_q) = P(G_q) + P(H_q) + P(K_q) < 2^{-q+1} + 2^{-q} + 2^{-q+1} < 2^{-q+3}, \]
thanks to inequalities 2.18, 2.21 and 2.24.

14. Now let \( \overline{A_q} \equiv \bigcup_{k=q}^{\infty} A_k \). Then \( P(\overline{A_q}) < 2^{-q+4} \). Moreover
\[ \overline{A_q}^c \subset \bigcap_{k=q}^{\infty} \bigcup_{s \in [\tau_q(\omega), \tau_{s}(\tau_q+1)]} |u(B_{\tau_q}(\omega)) - u(B_{s})(\omega)| \leq 2^{-k+1}. \]  \hfill (2.29)
Hence
\[ \overline{A_q}^c \subset \bigcap_{k=q}^{\infty} \bigcup_{s \in [\tau_q(\omega), \tau_{s}(\tau_q+1)]} |u(B_{\tau_q}(\omega)) - u(B_{s})(\omega)| \leq 2^{-k+1}. \]  \hfill (2.30)
Thus \( u(B_{r(k)}) \rightarrow V \) a.u. for some r.r.v. \( V \). Combining with (2.29) we obtain

\[
\mathcal{A}_q \subset \bigcap_{k=q}^{\infty} \left\{ \left\{ |u(B_{r(k)}) - V| \leq 2^{-k+2} \right\} \cap \left( \bigvee_{s \in [r(r(k)), r(r(k+1))]} |u(B_s) - u(B_{r(k)})| \leq 2^{-k+1} \right) \right\}
\]

\[
\subset \left( \bigvee_{k=q}^{\infty} \left| V - u(B_s) \right| \leq 2^{-k+2} + 2^{-k+1} \right)
\]

\[
\subset \left( \bigvee_{s \in [r(r(q)), r(1)]} |V - u(B_s)| \leq 2^{-q+3} \right)
\]

where \( P(\mathcal{A}_q) < 2^{-q+4} \), where \( q \geq 1 \) is arbitrary. The theorem is proved.

We conclude this article with a conjecture regarding the non-tangential limit of harmonic functions in the unit \( m \)-sphere.

**Definition 42. (Convex hull, Stolz domain, and Nontangential limit).**

Let \( A \) be an arbitrary subset of \( \mathbb{R}^m \). The convex hull of the set \( A \) is then defined as the set

\[
\hat{A} \equiv A^\wedge \equiv \{ \alpha_1 v_1 + \cdots + \alpha_k v_k : k \geq 1; v_1 \cdots v_k \in A; \alpha_1, \cdots, \alpha_k \geq 0; \alpha_1 + \cdots + \alpha_k = 1 \}
\]

of all convex combinations of points in \( A \). Let \( z \in \partial D_{0,1} \) and \( a \in [0, 1) \) be arbitrary. Then the subset

\[
S_{z,a} \equiv (\{ z \} \cup D_{0,a})^\wedge
\]

of \( D_{0,1} \) is called a Stolz domain.

Let \( g : D_{0,1} \rightarrow \mathbb{R} \) be an arbitrary continuous function. Then we say that the function \( g \) has a nontangential limit \( c \) at the point \( z \in \partial D_{0,1} \) if, for each \( a \in [0, 1) \), we have

\[
\lim_{x \rightarrow z; x \in S(z,a)} g(x) = c.
\]

The classical nontangential-limit theorem can be reformulated as follows.

**Theorem 43. (Classical nontangential limit theorem for Hardy spaces).**

Let \( u \) be an arbitrary member of the Hardy space \( h^p \). Suppose condition (i) in Theorem 41 holds. Then \( u \) has a nontangential limit at each point \( z \) in some measurable subset with probability 1 in \( \partial D_{0,1} \) relative to the uniform distribution \( \sigma_{m,0,1} \).

We conjecture that Theorem 43 does not have a constructive proof. We conjecture however that if condition (ii) in Theorem 41 also holds then Theorem 43 has a constructive proof.
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