GENERATING FUNCTIONS IN \( \mathbb{R}^{2n} \) AND THE HATCHER-WALDHAUSEN MAP

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Abstract. In this paper we construct a generating function quadratic at infinity for any exact Lagrangian in \( \mathbb{R}^{2n} \) equal to \( \mathbb{R}^n \) outside a compact set. This type of Lagrangian is equivalent to a Lagrangian filling in \( D^{2n} \) of the standard Legendrian unknot \( S^{n-1} \). Generating functions of the type we construct are related to the space \( \mathcal{M}_\infty \) considered by Eliashberg and Gromov. We also show that \( \mathcal{M}_\infty \) is the homotopy fiber of the so-called Hatcher-Waldhausen map. This further relates the understanding of exact Lagrangians (and Legendrians) to algebraic K-theory of spaces. As a result of this and the result by Bökstedt that the Hatcher-Waldhausen map is a rational homotopy equivalence we prove that the stable Lagrangian Gauss map (relative boundary) of the Lagrangian is homotopy trivial.

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1. Introduction

In this paper we describe how to construct a generating function (sometimes called a generating family) quadratic at infinity for any exact Lagrangian \( L_\infty \subset \mathbb{C}^n = T^* \mathbb{R}^n \) equal to the zero-section \( \mathbb{R}^n \) outside a compact set. By scaling we can assume that the nontrivial part of \( L_\infty \) is completely contained in \( T^* \mathbb{D}^n \) and consider compact \( L \subset T^* \mathbb{D}^n \) with boundary equal to \( S^{n-1} \). Alternatively, one can consider this an exact Lagrangian filling of the standard Legendrian unknot \( S^{n-1} \subset \mathbb{D}^{2n} \).

By adding a point at \( \infty \) to \( L_\infty \) we also consider the exact Lagrangian submanifold \( L^+ \subset \mathbb{R}^n \subset \mathbb{R}^{2n} \) to a Lagrangian \( K \) which contains \( L \) - then we appeal to Chaperon’s broken geodesic approach from \([6]\) to conclude that \( K \) has a generating function quadratic at infinity (extended in \([11]\) to cotangent bundles). We then use a cut and paste argument to construct one for \( L \). This last step requires some control on the primitive of the Liouville form on \( K \). Indeed, the cutting part will use fiber-wise regular values of this primitive. The construction of the isotopy is carried out in Section 2 and the cut and paste argument is carried out in Section 3.

Before formulating the precise theorem we need to define which type of generating function we are considering. Let \( N \) be a compact smooth manifold of dimension \( n \) possibly with boundary. We will say that a Lagrangian \( L \subset T^* N \) has \textit{compact support} if \( L \) agrees with the zero-section outside a compact set in \( T^* N \) which projects to the interior of \( N \). Similarly, we will say that a Hamiltonian is compactly supported if it has support contained in a compact set in \( T^* N \) which projects to the interior of \( N \).

Let \( \pi_N : N \times \mathbb{R}^{2k} \to N \) denote the projection. Let \( F : N \times \mathbb{R}^{2k} \to \mathbb{R} \) be a smooth function and let \( F_z \) for each \( x \in N \) denote \( F \) restricted to the fiber \( \pi_N^{-1}(x) \). Let \( \Sigma_F \subset N \times \mathbb{R}^{2k} \) be the zero-set of the fiber-wise differential \( d_z F = \bigcup_{z \in N} d(F_z) \) of \( F \). We have a canonical map

\[
d_h : \Sigma_F \to T^* N.
\]

Indeed, the projection to the base \( N \) is given by \( \pi_N \) above, and taking any tangent vector \( v \in T_{\pi_N(z)}N \) we can lift it to a vector \( w \in T_z(N \times \mathbb{R}^{2k}) \) and define \( d_h(v) = dF(w) \). This is independent of the lift since the fiber-wise differential \( d_z F \) is 0 at \( z \in \Sigma_F \). By a \textit{generating function} for an immersed Lagrangian \( L \to T^* N \) we will in this paper mean (unless otherwise stated) a function as above such that

G1) the set \( \Sigma_F \) is cut out transversely by the equation \( d_z F = 0 \), and

G2) the map \( d_h : \Sigma_F \to T^* N \) factors through a diffeomorphism to \( L \).

Note that the image of \( d_h \) is always an exact Lagrangian immersion since it is easy to prove that \( F \) restricted to \( \Sigma_F \) is a primitive for the canonical form on \( T^* N \) pulled back to \( \Sigma_F \). Let \( Q_k : \mathbb{R}^{2k} \to \mathbb{R} \) be the quadratic form

\[
Q_k(x, y) = -\|x\|^2 + \|y\|^2
\]

(1.1)
for \(x, y \in \mathbb{R}^k\). We will refer to this as the standard quadratic form. By abuse of notation we will also let \(Q_k\) denote the composition
\[
N \times \mathbb{R}^{2k} \rightarrow \mathbb{R}^{2k} \xrightarrow{Q_k} \mathbb{R}.
\]
In this paper we will say that a generating function is quadratic at infinity if
1. \(\text{G3) the support of } F - Q_k \text{ is compact.}\)
Thus we allow the support to be non-trivial in fibers over boundary points - even when generating the zero-section. The first result of this paper is the following.

**Theorem 1.1.** Any exact Lagrangian \(L \subset T^*D^n\) agreeing with the zero-section over a neighborhood of the boundary \(\partial D^n = S^{n-1}\) has a generating function quadratic at infinity.

This type of generating function quadratic at infinity is one of the most restrictive types of generating functions. However, in the case where \(N\) has boundary, it is not the most restrictive type. In fact, we will use the terminology that a function \(F : N \times \mathbb{R}^{2k} \rightarrow \mathbb{R}\) is quadratic at both infinities if the support of \(F - Q_k\) is compact and maps to the interior of \(N\). Given a compactly supported exact Lagrangian \(L \subset T^*N\) the difference between these two definitions leads us to consider the space \(\mathcal{M}_\infty = \lim_{k \rightarrow \infty} \mathcal{M}_{k,k}\), where \(\mathcal{M}_{k,k}\) is essentially the space of functions on \(\mathbb{R}^{2k}\) equal to \(Q\) at infinity and with only one critical point which has value 0 and is also non-degenerate (this was defined by Eliashberg and Gromov in [7]). This space essentially consists of the possible fiber functions generating the zero-section, and any generating function quadratic at infinity generating the zero-section close to \(\partial N\) defines a map
\[
\partial N \rightarrow \mathcal{M}_\infty.
\]
(1.2)
The map is given by taking the adjoint of the generating function (see Section 5).

**Remark 1.2.** We will not use the following for anything other than motivation. If the map in Equation (1.2) extends to a map \(N \rightarrow \mathcal{M}_\infty\) then one may modify the original generating function to be quadratic at both infinities by essentially fiberwise adding the inverse of this extension over \(N\). More interestingly, if the map does not extend then the Lagrangian generated cannot be compactly Hamiltonian isotopic to the zero-section. Indeed, any isotopy would by [11] lead to a generating function of the zero-section which is an extension of the map.

In general we therefore would like to understand the homotopy type of \(\mathcal{M}_\infty\). In particular, in the case \(N = D^n\), its homotopy groups are relevant. We will show that this space is closely related to the stable pseudo-isotopy group of a point \(\mathcal{P}_\infty\) (see e.g. [9]) and its delooping \(\mathcal{H}_\infty\), which is the space of stable \(h\)-cobordisms of a point. For this we will need to consider the \(J\)-homomorphism:
\[
J : O = \lim_{n \rightarrow \infty} O(n) \rightarrow F = \lim_{n \rightarrow \infty} \text{Map}_*^\wedge(S^n)
\]
where \(\text{Map}_*^\wedge\) denotes the space of based self homotopy equivalences with stabilization maps given by taking reduced suspension. The map \(J\) is defined by the based action of \(O(n)\) on the sphere \(S^n \cong \mathbb{R}^n \cup \{\infty\} = (\mathbb{R}^n)^+.\) As \(J\) is a map of monoids (in fact a map of infinite loop spaces) it has a delooping \(BJ : BO \rightarrow BF\) which sends a vector space to its associated sphere bundle. The quotient of \(J\) which is also

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1 Theorem 1.4 and [15] show that \(\mathcal{M}_\infty\) is an infinite loop space, but we will not need this.
the homotopy fiber of $BJ$ is denoted $F/O$. In the paper we essentially show that the Hatcher-Waldhausen map $\text{hw} : F/O \to \mathcal{H}_\infty$ can be defined as follows. Define

$$Q^l_k(x, y) = -\|x\|^2 + \|y\|^2$$

for $(x, y) \in \mathbb{R}^k \times \mathbb{R}^l$. Consider the set $\{Q^l_k \leq -1\}$ (red in Figure 1). For a vector space $V \subset \mathbb{R}^{k+l}$ we let $DV$ denote its disc and $SV$ its sphere. Consider the space

![Figure 1. The image of $i$ (red) and the smoothing defining hw (blue).](image)

of pairs $(V, i)$ where $V \in \text{Gr}_k(\mathbb{R}^{k+l})$ and $i : DV \to \{-1 \leq Q^l_k \leq 1\}$ is a smooth embedding such that

- $i$ is the standard inclusion of $V \subset \mathbb{R}^{k+l}$ in a neighborhood of 0.
- The interior of $DV$ is mapped to $\{-1 < Q^l_k < 1\}$.
- $i$ maps $SV$ transversely to $\{Q^l_k = -1\} \simeq S^{k-1}$ and the map is a homotopy equivalence.

The image of such an $i$ is illustrated by the line in the space between red and blue in Figure 1. We essentially prove in Section 8 that the colimit of such spaces of discs $k, l \to \infty$ is a model for $F/O$. Heuristically the idea is that increasing $l$ essentially makes the embedding and transversality conditions redundant, and the map from $SV$ to $\{Q^l_k = -1\}$ is a homotopy trivialization of the sphere bundle $SV$. We then pick a tubular neighborhoods $T$ around $\text{im} i$ (pink in the figure). We then realize $\text{hw}$ at level $(k, l)$ as a smoothing of the boundary of $\{-1 \leq Q^l_k \leq 1\} \setminus T$ (white in the figure). Indeed, this is essentially a compactly supported $h$-cobordism from $\{Q^l_k = 1\}$ to the other part of the boundary. Stably this defines and element in $\mathcal{H}_\infty$. We give details of this in Section 9.

**Remark 1.3.** As pointed out by a referee, the definition of the Hatcher-Waldhausen map is maybe a bit ambiguous in the literature. We use the definition from Corollary 3.4 in [19] as our definition of the Hatcher-Waldhausen map. In Section 9 we relate the above description to this definition. We also note that the result used in Appendix A from [16] uses this definition, and that the result in [5] does not depend on the specific definition as potential variations of the map only concerns torsion.

In sections 6, 7, 8 and 9 we prove the following theorem. In Section 6 we even prove an interesting unstable version of the theorem.

**Theorem 1.4.** There is a fibration sequence

$$\mathcal{M}_\infty \xrightarrow{N_\infty} F/O \xrightarrow{\text{hw}} \mathcal{H}_\infty.$$  (1.3)
Here the map $\tilde{N}_{\infty}$ is essentially given by taking the tangent space of the unstable manifold at the critical point (we prove that it has a canonically trivial stable sphere so that it lifts through $F/O \to BO$). The geometric idea of why the composition is homotopy trivial is essentially as follows. We will give a description of $\tilde{N}_{\infty}$ as remembering the unstable manifold disc of the unique critical point, essentially defining a point in the spaces of pairs $(V,i)$ considered above. The description above of the Hatcher-Waldhausen map now implies that we are attaching a tubular neighborhood of this unstable disc to $\{Q_k^l \leq 1\}$. However, as a function in $\mathcal{M}_{\infty}$ has only this one critical point, the produced $h$-cobordism (the white part in the figure) has a function defined on it with no critical points (and is standard at infinity). Such a “collar” essentially defines a path to the base-point of the element in $\mathcal{H}_{\infty}$.

Any Lagrangian $L \subset T^*D^n$ has a Lagrangian Gauss map $L \to U(n)/O(n)$ given at each point by the fact that the tangent space is a linear Lagrangian in $\mathbb{R}^{2n}$. The stable version of this map has target $U/O = \lim_{n \to \infty} U(n)/O(n)$. Restricting this map to $S^{n-1} = \partial L$ it is trivial and thus defines an element in $[L/S^{n-1}, U/O] \cong \pi_n(U/O)$. In section 3 we relate this to the map $\mathcal{M}_{\infty} \to F/O \to BO$ in the theorem above. In [12] Bökstedt proved that this composition is rationally trivial. We will use this to prove the following result.

**Theorem 1.5.** With $L$ as in Theorem 1.4 above the stable Lagrangian Gauss map (relative to the boundary) $L/S^{n-1} \to U/O$ is null homotopic.

It was already known from [3] that this class had to lie in the kernel of $\pi_n(U/O) \cong \pi_{n-1}(\mathbb{Z} \times BO) \cong \pi_{n-1}(\mathbb{Z} \times BJ) \cong \pi_{n-1}(\mathbb{Z} \times BF)$. However, even though the image for $n = 4k + 1$ grows large with $k$ the kernel is never 0 (except for $k = 0$). Indeed, the homotopy groups of the image are finite except when $n = 1$.

It is well-known that Gromov’s $h$-principle for Lagrangian immersions tells us that Lagrangian immersion classes of a given homotopy sphere into $T^*S^n$ is canonically identified with $\pi_n(U)$. This is a lift of the Gauss map under the canonical map $U \to U/O$, and since $\pi_{4k+1}(U) \to \pi_{4k+1}(U/O)$ is injective the above theorem implies the following corollary.

**Corollary 1.6.** For $n = 4k + 1$ the Lagrangian embedding $L^+ \subset T^*S^n$ represents the trivial immersion class $0 \in \pi_n(U)$.

Note that this is not the same as saying that $L^+$ is isotopic through immersions to the zero-section. Indeed, there is still the obstruction given by the possible exotic smooth structure on $L^+$.

It follows from Theorem 1.4 and Bökstedt’s result that $\mathcal{M}_{\infty}$ has finite homotopy groups, but more refined information about the homotopy groups of $\mathcal{M}_{\infty}$ in low degrees can be inferred from [17], [16], [4] and the recent pre-print [18]. We carry out details of this in Appendix A where we argue that the homotopy groups are

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----|---|---|---|---|---|---|---|---|---|---|
| $\pi_n(\mathcal{M}_{\infty})$ | 0 | 0 | 0 | 0 | $\mathbb{Z}/m_1$ | 0 | 0 | $\mathbb{Z}/m_2$ | $\mathbb{Z}/8$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/4$ | $\mathbb{Z}/3$ or 0 |

Table 1. Here $m_i$ are unknown odd integers.

as listed in Table 1. This table implies that the generating functions constructed in this paper can be extended over infinity generating $L^+ \subset T^*S^n$ in the cases $n \in \{1, 2, 3, 5, 6, 7\}$ (and possibly $n = 10$).
Remark 1.7. Work in progress by the present author and Eliashberg involves extending the injectivity theorem from [2] to get non-trivial results about homotopy groups of spaces of Legendrians. This is also a construction that relies heavily on algebraic K-theory of spaces, but it is not clear how this relates to the current construction (nor the newer more general construction in [2]). In fact, it seems to be of a different nature.

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2. A Hamiltonian moving part of the zero-section to L

Consider the cotangent bundle of \( \mathbb{R}^n \) restricted to the disc \( T^*D^n \to D^n \). Let \( L \subset D^{2n} \subset T^*D^n \) be an exact Lagrangian embedding equal to the zero-section near the boundary of the disc (thus having boundary \( S^{n-1} \)). In this section we construct a compactly supported Hamiltonian flow on \( \mathbb{R}^{2n} = T^*\mathbb{R}^n \) which moves the zero-section \( \mathbb{R}^n \) to a Lagrangian \( K \) which contains \( L \), and such that the other parts of the image that lands in \( T^*D^n \) has primitive values much lower than those on \( L \). Parts of this section only work when \( n \geq 5 \). In Section 4 we will describe modifications to the arguments when \( n \leq 4 \).

Let \( L_\infty = L \cup (\mathbb{R}^n - D^n) \) be the extension by the zero-section in \( \mathbb{R}^{2n} \). In this section we will denote coordinates in (this copy of) \( \mathbb{R}^{2n} \) by \( z = (x, y) \) with \( x, y \in \mathbb{R}^n \). Let \( x_0 = (-1, 0, \ldots, 0) \in \mathbb{R}^n \) and let \( f_{L_\infty} : L_\infty \to \mathbb{R} \) be the unique primitive of the standard Liouville form \( ydx \) that has compact support (equivalently \( f_{L_\infty}(x_0) = 0 \)).

Lemma 2.1. For \( n \geq 5 \) there is a diffeomorphism \( L_\infty \cong \mathbb{R}^n \) which outside of \( D_2^n \) is given by \( x \mapsto \|x\|\phi(\hat{x}) \) where \( \phi : S^{n-1} \cong S^{n-1} \) is some diffeomorphism and \( \hat{x} \) is the normalization of \( x \). Furthermore, we can assume that in a neighborhood of \( x_0 \) it is given by \( x \mapsto x - x_0 \). In particular it maps \( x_0 \) to 0.

We will call a diffeomorphism \( \mathbb{R}^n \to \mathbb{R}^n \) radial at \( x \in \mathbb{R}^n \) if it is locally of the form \( x \mapsto \|x\|\phi(\hat{x}) \) (as in the lemma).

Proof. Recall from the introduction that \( L_\infty \) is contractible and hence \( L \) is a smooth homotopy ball with standard boundary \( S^{n-1} \). Using e.g. Milnor [14] we know that there exists an orientation preserving diffeomorphism \( \varphi : L \cong D^n \) with the identification at the boundary possibly an exotic diffeomorphism \( \varphi|_{S^{n-1}} : S^{n-1} \cong S^{n-1} \). We may adjust \( \varphi \) to be radial in a neighborhood of the boundary, so that we may extend it radially to all of \( L_\infty \cong \mathbb{R}^n \). It is thus on the desired form outside of \( T^*D_2 \). We can now post compose with a ambient isotopy of \( \mathbb{R}^n \) with support in \( D_2^n \) to make it on the form \( x \mapsto x - x_0 \) in a neighborhood of \( x_0 \). \( \Box \)

We use this to induce a flat metric on \( L_\infty \) (when \( n \geq 5 \)) and give \( T^*L_\infty \) coordinates which we denote \( q, p \in \mathbb{R}^n \). By the translation property in the lemma we have that \( (x, y) = (q + x_0, p) \) in a neighborhood of the fiber \( T^*_x \mathbb{R}^n \), and the Riemannian structures agree.

Fix a \( \delta < 1 \) such that for some \( \delta' < \delta \) the Lagrangian \( L_\infty \) equals the zero section outside of \( D_{\delta'}^n \). We define \( L_{\delta} = L \cap D_{\delta}^n \), which is in the interior of \( L \). We will denote the closed unit disc and sphere bundles of a cotangent bundle \( T^*M \) by \( D^*M \)
and $S^r M$ respectively, and we will denote the discs and sphere bundles of radius $r$ by $D^r_r M$ and $S^r_r M$ respectively.

**Lemma 2.2.** For any Riemannian structure on $L_\infty$ induced by a radial diffeomorphism $\varphi : L_\infty \cong \mathbb{R}^n$ there exists an $\varepsilon > 0$ such that the embedding $i : L_\infty \subset \mathbb{R}^{2n}$ extends to a symplectic embedding

$$D^*_\varepsilon L_\infty \subset D^* \mathbb{R}^n$$

which over $L_\infty - L_\delta$ is induced by the identification $L_\infty - L_\delta = \mathbb{R}^n - D^n_\beta$ (equal as subsets inside the zero section $\mathbb{R}^n \subset \mathbb{R}^{2n}$) - hence respects fibers. Furthermore, we may assume that this is the only part of the image landing in $T^*(\mathbb{R}^n - D^n_\delta)$.

Note that even though this respects fibers outside of $D^n_\delta$ the Riemannian structures may not agree - hence the image may not be a disc bundle in the target structure. However, we made sure that in a neighborhood of $x_0$ these do in fact agree.

**Proof.** With $\delta' < \delta$ as above we first define $L_\beta = L \cap D^n_\beta$ for some $\beta \in (\delta', \delta)$ and $L_\beta^c = L_\infty - L_\beta \subset \mathbb{R}^n$. We thus have the canonical symplectic identification $T^* L_\beta^c = T^*(\mathbb{R}^n - D^n_\beta)$ which extends the embedding over $L_\beta^c$ to $\psi : D^*_\varepsilon L_\beta^c \to \mathbb{R}^{2n}$. As the diffeomorphism from $L_\infty \to \mathbb{R}^n$ inducing the Riemannian structure is radial outside of $D^n_\delta$ its derivative has a global bound and it follows that for small enough $\varepsilon$ this lands in $D^* \mathbb{R}^n$.

We may extend the domain to include a small symplectic neighborhood of $L_\beta \subset T^* L_\beta$ by using the Darboux-Weinstein symplectic neighborhood theorem on the map

$$\psi \cup i : (D^*_\varepsilon L_\beta^c) \cup L_\beta \to \mathbb{R}^{2n}.$$ 

By compactness of $L_\beta$ there is a possibly smaller $\varepsilon > 0$ such that this defines an embedding $D^*_\varepsilon L_\infty \subset D^* \mathbb{R}^n$. By the choice of $\delta > \beta$ we may shrink $\varepsilon$ a bit more to make sure that the image of $D^*_\varepsilon L_\beta$ avoids the parts over $\mathbb{R}^n - L_\delta$. \ □

Consider a non-compactly supported Hamiltonian of the type

$$H(x, y) = f(||(x, y)||)$$

defined on all of $\mathbb{R}^{2n}$.

**Lemma 2.3.** For any smooth function $\theta : [0, \infty) \to \mathbb{R}$ with $\theta(r)$ constant when $r$ is close to 0 there exists an $f : [0, \infty) \to \mathbb{R}$ such that $H(x, y) = f(||(x, y)||)$ is smooth and its Hamiltonian time 1 flow is given by $z \mapsto e^{i\theta(||z||)} z$.

We will refer to such a Hamiltonian isotopy as a *generalized rotation*.

**Proof.** It is an easy exercise to see that the Hamiltonian vector field of such an $H$ is given by:

$$(X_H)_z = \frac{f'(||z||)}{||z||} (J_0 z)$$

whose time 1 flow is given by $z \mapsto e^{i \int_0^r \theta(s) ds} z$. Given $\theta$ as in the lemma we can solve such that this factor is equal to $\theta(r)$ by putting:

$$f(r) = \int_0^r \theta(s) ds$$
which, by the assumption on $\theta$, is $f(r) = cr^2$ close to 0 so that $H$ is globally smooth.

Now fix a smooth decreasing function $\theta : [0, \infty) \to [0, \pi/2]$ so that
\begin{itemize}
  \item $\theta(r) = \pi/2$ for $0 \leq r \leq 1$,
  \item $1 < r \sin(\theta(r)) \leq 2$ for $r > 1$ and
  \item $\theta(r) = \arcsin(\frac{1}{r})$ for $r \geq 3$.
\end{itemize}

Note that the second condition is easily solved compatibly with the others by rewriting it as $\theta(r) = (\arcsin(\frac{1}{r}), \arcsin(\frac{2}{r}))$. If we apply the generalized rotation associated with this $\theta$ to the zero section $\mathbb{R}^n \subset \mathbb{C}^n$ we get a Lagrangian $\mathcal{F}'_{\infty} \subset \mathbb{R}^{2n} = \mathbb{C}^n$ (illustrated as the red line in Figure 2) which satisfies $\mathcal{F}'_{\infty} \cap D^*\mathbb{R}^n = iD^n$. Indeed,

\begin{align*}
  x &\quad y \\
  0 &\quad 2 \\
  \mathcal{F}'_{\infty} &\quad -1 \\
  \end{align*}

\textbf{Figure 2. Image of rotated zero-section for $n = 1$}

the second bullet point above implies that for $x \in \mathbb{R}^n$ with $\|x\| > 1$ the norm of the imaginary part of $e^{i\theta(\|x\|)}x$ must lie in $(1, 2)$. We also conclude that $\mathcal{F}'_{\infty} \subset D_3^2\mathbb{R}^n$.

\textbf{Lemma 2.4.} For any $R > 1$ there is a compactly supported generalized rotation of the zero-section to a Lagrangian $\mathcal{F}'_R$ such that

\[ \mathcal{F}'_R \cap T^*D_{R+1}^n = \mathcal{F}'_{\infty} \cap T^*D_{R+1}^n. \]

The Lagrangian $\mathcal{F}'_R$ is illustrated in Figure 3

\begin{align*}
  x &\quad y \\
  0 &\quad 2 \\
  \mathcal{F}'_R &\quad -1 \\
  \end{align*}

\textbf{Figure 3. Image of rotated zero-section with compact support}

\textit{Proof.} As the conditions get stronger when we make $R$ larger we may assume that $R > 2$. Define a new function $\theta_R$ by $\theta_R(r) = \theta(r)$ for $r \leq R + 2$ but bumped off for $r > R + 2$ to be equal to 0 outside a compact set. We let $\phi_R : \mathbb{C}^n \to \mathbb{C}^n$ denote the associated compactly supported generalized rotation. We define $\mathcal{F}'_R = \phi_R(\mathbb{R}^n)$ and prove in the following that this satisfies the conditions. We may of course assume that $0 \leq \theta_R \leq \theta$ which again implies that $\mathcal{F}'_R \subset D_3^2\mathbb{R}^n$.

As the norm is preserved by generalized rotations it follows that if $z$ is any point where $\mathcal{F}'_{\infty}$ and $\mathcal{F}'_R$ disagrees then $\|z\| \geq R + 2$ as the imaginary part is bounded in norm by 2 from above and $R > 2$ its real part must satisfy

\[ \|\text{Re } z\|^2 = \|z\|^2 - \|\text{Im } z\|^2 \geq ((R + 1) + 1)^2 - 2^2 > (R + 1)^2 \]
and is therefore not contained in $T^*D^n_{R+1}$.

Let $F_R$ for $R \in [2, \infty]$ be given by the shift $F'_R + (x_0, 0)$ where $F'_R$ is as in the lemma above when $R < \infty$. This is illustrated in Figure 4. The above lemma implies the following (which explains why we used $R + 1$ in the lemma above).

**Corollary 2.5.** For any $R > 1$ there is a compactly supported Hamiltonian isotopy of the zero-section to a Lagrangian $F_R$ such that

$$F_R \cap T^*D^n_R = F_\infty \cap T^*D^n_R.$$ 

In particular $F_R \cap D^*D^n_R = D^*_x \mathbb{R}^n = iD^n + (x_0, 0)$.

As $(x_0, 0)$ correspond to $(0, 0)$ in $(q, p) \in T^*L_\infty$ coordinates the plan is roughly to undo this rotation inside the neighborhood $D^*_x L_\infty \cong D^*_x \mathbb{R}^n$ (the latter using $(q, p)$ coordinates) to rotate the Lagrangian back to the zero-section $L_\infty$ - at least over the compact subset $L$. The end result of this will be a sort of “smear” of two copies of the fiber over $L$. The end result is illustrated in Figure 5. Concretely, we apply a generalized rotation in symplectic coordinates of the form $(q', p') = (cq, c^{-1}p) \in \mathbb{R}^{2n}$ depending on some large $c > 0$. We want the rotation to happen within the set where $\|p\| < \varepsilon$ which corresponds to $\|p'\| < c\varepsilon$. So we let $f_c(r)$ be a solution to the rotation from Lemma 2.3 with $\theta = \theta_c$ a smooth increasing extension of:

$$\theta_c(r) = \begin{cases} 
-\frac{\pi}{2} & r < \frac{c\varepsilon}{2} \\
0 & r \geq \frac{c\varepsilon}{2} 
\end{cases}.$$

This defines a Hamiltonian which in $(q, p)$ coordinates are given by

$$H(q, p) = f_c(\sqrt{c^{-2}\|q\|^2 + c^2\|p\|^2}) \quad (2.2)$$

In $(q, p)$ coordinates the resulting flow applied to the fiber over $q = 0$ is illustrated by the red part in Figure 5.

**Lemma 2.6.** Let $S_c$ be the result of the Hamiltonian flow of the Hamiltonian from Equation (2.2) applied to the fiber $T^*_x \mathbb{L}_\infty \subset T^*L_\infty$ and let $p_c : S_c \to \mathbb{R}$ be the unique primitive of the Liouville form $\lambda_{L_\infty}$ in $T^*L_\infty$ which has primitive equal 0 at $(q, p) = (0, 0)$. For any $C > 0$ there is $c > 0$ large enough such that (see Figure 3)

$$S_c \cap T^*L = L \sqcup K_1$$

with primitive $p_c = 0$ on $L$ and $\sup_{K_1} p_c < -C$. 

![Figure 4. Image of bumped off rotated zero-section for n = 1](image-url)
Proof. This follows by construction. In particular the statement about the primitive follows by the fact that the red part not on the zero-section stays out of the dashed ellipse in Figure 5 and thus the signed symplectic area below the curve as it swings back to $K_1$ goes to $-\infty$ as $c \to \infty$. Note in particular that the primitive is a constant depending on $c$ on the non-compact part coinciding with the fiber. □

We now have all the pieces for proving the following proposition.

**Proposition 2.7.** For $n \geq 5$ there exists a compactly supported Hamiltonian isotopy of the zero-section $\mathbb{R}^n \subset \mathbb{C}^n$ to a Lagrangian $K$ such that

$K \cap T^*D^n = L \cup K_1$ and $K \cap T^*x_0\mathbb{R}^n = \{x_0\}$

and such that for any primitive $p_K : K \to \mathbb{R}$ of the Liouville form there is a $C \in \mathbb{R}$ such that

$$\min_{z \in L} p_K(z) > C > \max_{z \in K_1} p_K(z).$$

**Proof.** All Lagrangian that we will consider in this proof are connected and will contain $x_0$ and we will thus make primitives unique by always choosing the one that is 0 at $x_0$. This only changes the wanted inequality up to a shift, which we may ignore. The Lagrangian $F_R$ from Corollary 2.5 satisfies $F_R \cap T^*D^n = F_\infty \cap T^*D^n$ and this piece of it is compact and connected. It follows that the primitive, say
$f_{F_\infty}: F_R \cap T^*D^n \to \mathbb{R}$, is independent of $R$ and that there is a $C_1 > 0$ such that $|f_{F_\infty}(z) - f_{F_\infty}(z')| < C_1$ for $z, z' \in F_\infty \cap T^*D^n$.

The difference between the two Liouville forms $\lambda_{L_\infty} - ydx$ on $D_+^*L_\infty$ is compactly supported. Indeed, in Lemma 2.2 we made sure that outside a compact set the embedding was given by the canonical identification of cotangent bundles - hence the Liouville forms agree. By exactness this difference is the differential of a function $P: D_+^*L_\infty \to \mathbb{R}$ which extends $f_{L_\infty}: L_\infty \to \mathbb{R}$. Since $f_{L_\infty}$ is compactly supported we get that $P$ is compactly supported. Hence there is a $C_2 > 0$ bounding the absolute value of $P$. Since $P(x_0) = 0$ and we are fixing primitives to be $0$ at $x_0$ we conclude that the two primitives (one for each Liouville form) on $S_c$ (from Lemma 2.6) inside $D_+^*L_\infty$ differs exactly by $P$ restricted to $S_c$. Hence they differ by at most $C_2$.

Now fix $c > 0$ in Lemma 2.6 such that, on the part of $S_c$ inside $D_+^*L$ where the primitive $p_c$ is negative it is in fact strictly less than $-2C_2 - C_1$. The Hamiltonian in that Lemma is compactly supported inside $D_+^*L_\infty$ and thus extends to a Hamiltonian on $\mathbb{R}^{2n}$ (using $(x, y)$ coordinates) with support inside $D_+^*D_R^n$ for some $R > 1$ that we now fix.

Now, we define $K'$ (almost going to be $K$) as the Lagrangian given by first using this $R$ in Lemma 2.4 to get a Lagrangian $F_R$ that coincides with the fiber at $x_0$ in $D_+^*L_\infty \cap D^*D_R^n$. Then we use Lemma 2.6 with our chosen $c > 0$ inside this to flow $F_R$ to $K'$. By construction we have $K' \cap T^*D^n = L \cup K'_1$. See figure 6 for a heuristic picture of $K'$.

By construction; the primitive (with respect to $ydx$) which equals $0$ at $x_0$ for $K' \subset \mathbb{R}^{2n}$ is given by $P = f_{L_\infty}$ on the part of $K'$ in $T^*D^n$ which coincides with $L$. This is bounded from below by $-C_2$.

We claim that the primitive on $K'_1$ is strictly bounded from above by $-C_2$. Indeed, by our choice of $c$ the primitive inside $D_+^*L_\infty$ on $K'_1$ with respect to $\lambda_{L_\infty}$ is bounded from above by $-2C_2 - C_1$. Hence by the bound on $P$ the primitive inside $D_+^*L_\infty$ with respect to $ydx$ is bounded strictly from above by $-C_2 - C_1$. This in particular implies the same bound on the boundary $S_c T_{x_0}^*L_\infty \subset F_R \cap T^*D^n$. As $F_R - D_+^*L_\infty$ is connected and the isotopy from Lemma 2.6 keeps $K'$ constantly equal to $F_R$ outside of $D_+L_\infty$ the primitive can only change by addition with a constant. Therefore, the bound $C_1$ on the variation of the primitive $f_{F_\infty}$ on $F_R \cap T^*D^n$ is still valid for the primitive on $K'$ - hence we get that the primitive on the part of $K'$ outside of $D_+^*L_\infty$ is bounded from above by $-C_2 - C_1 + C_1 = -C_2$.

The only thing in the proposition that $K'$ does not satisfy is that $K' \cap T_{x_0}^*\mathbb{R}^n$ is equal to $\{x_0\}$ union an entire annulus. However, as the green line in Figure 6 indicates one may shear close to this fiber and have the fiber completely miss this part of $K'$. To be explicit, we consider a local neighborhood around $x_0$ in $L$ where the Riemannian structures agree. In Figure 7 we see $K'$ close to the fiber $T_{x_0}^*\mathbb{R}^n$ for $n = 1$. For $n > 1$ the picture still describes all points in $K'$. Indeed, intersecting $K'$ with an affine complex line through $(x_0, 0)$ parallel to a real unit vector $v \in S^{n-1} \subset \mathbb{R}^n$ yields the same local picture independent of $v$. It follows that an arbitrarily small negative generalized rotation (around $(x_0, 0)$) would clear the fiber for all points except $x_0$. However, we do not want to change the piece that equals $L$. So, we pick a Hamiltonian $H$ which equals $H(x, y) = -|y|^2$ in a small neighborhood of the annulus, but has support in a slightly larger neighborhood (away from $L$). We define the final Lagrangian $K$ as the Hamiltonian time 1 flow.
of \( \delta H \) for small \( \delta > 0 \). For \( \delta \) small enough this does not change the strict bounds on the primitive and \( K \) does not intersect the fiber except at \( x_0 \in L \). \( \square \)

3. Cutting and pasting generating functions and proof of Theorem 1.1

In this section we perform the cut and paste argument on a generating function quadratic at infinity for \( K \) from Proposition 2.7 to get one for \( L \subset T^*D^n \) and thus proving Theorem 1.1 (in the case \( n \geq 5 \)).

Recall that \( F : \mathbb{R}^n \times \mathbb{R}^{2k} \to \mathbb{R} \) is called quadratic at both infinities if \( F = Q \) outside of a compact set. Here

\[
Q(q, x, y) = Q(x, y) = Q_k(x, y) = -\|x\|^2 + \|y\|^2
\]

where \( x, y \in \mathbb{R}^k \) is the standard quadratic form. In [6] Chaperon proved that any compactly supported Hamiltonian isotopy of \( \mathbb{R}^n \) has such a generating function. We use this and the previous constructions to prove the following lemma.

**Lemma 3.1.** There is a generating function \( F : D^n \times \mathbb{R}^{2k} \to \mathbb{R} \) quadratic at infinity and two constants \( C' < C \in \mathbb{R} \) fiber-wise regular for \( F \) such that

- \( F \) generates the zero-section in a neighborhood of \( \{x_0\} \in D^n \).
- \( F \) restricted to the set \( \{F \leq C'\} \cup \{F \geq C\} \) generates \( L \subset T^*D^n \).

Note that the set in the second bullet point has complement given by

\[
\{C' < F < C\}.
\]

In the next section we will prove the case \( n \leq 4 \).

**Proof of Lemma 3.1** for \( n \geq 5 \). Proposition 2.7 and Chaperon’s result proves that we have a generating function \( F' : \mathbb{R}^n \times \mathbb{R}^{2k} \to \mathbb{R} \) quadratic at both infinities for the Lagrangian \( K \) described in the proposition.

Now the restriction

\[
F = F'_{|D^n \times \mathbb{R}^{2k}} : D^n \times \mathbb{R}^{2k} \to \mathbb{R}
\]

generates \( K \cap T^*D^n \), which by proposition 2.7 has two pieces:

\[
K \cap T^*D^n = L \sqcup K_1
\]

and in particular has \( K \cap T^*x_0\mathbb{R}^n = \{x_0\} \).
Let $C$ be the value mentioned in the proposition which isolates the values of the primitive on these two pieces (for the specific primitive defined by restricting our generating function $F$ to $L$). This primitive describes the fiber-critical value of $F$ associated to each fiber-critical point generating a point in $K$. We may pick another constant $C'$ smaller than $C$ such that the fiber-wise critical values of $F$ are all bounded from below by $C'$. This implies that if we further restrict $F$ to

$$F| : \{F \leq C'\} \cup \{F \geq C\} \to \mathbb{R}$$

then we get a function whose fiber-wise critical points only generates $L$ and not the part $K_1$.

We now prove Theorem 1.1 given Lemma 3.1 for all $n$.

**Proof of Theorem 1.1.** The function $F$ provided by Lemma 3.1 is not defined on the full product $D^n \times \mathbb{R}^{2k}$. So, consider the closure of the missing piece

$$M = \{C' \leq F \leq C\}$$

Since $C$ and $C'$ are fiber-wise regular points this is a fiber bundle (where the fibers have boundary). Let $K \subset D^n \times \mathbb{R}^{2k}$ be compact such that $F = Q$ outside $K$. Picking a large ball $D^n_R \subset \mathbb{R}^{2k}$ containing the projection of $K$ in its interior we see that $M \cap (D^n \times (\mathbb{R}^{2k} - D^n_R))$ is a product. Since $D^n$ is contractible we can trivialize $M$ respecting the fact that it is a product outside $D^n \times D^n_R$. In fact, we will be slightly more specific about this. Pick a horizontal vector bundle (a fiber-bundle connection) for $D^n \times \mathbb{R}^{2k} \to D^n$ which satisfies:

- It is parallel to all regular level sets $\{F = a\}$ for $a$ close to $C$ or $C'$.
- Outside $D^n \times D^n_R$ it is the trivial horizontal vector bundle.

This restricts to $M$ by construction, and using parallel transport (lifting of smooth curves) we can thus trivialize the bundle.

The choice of horizontal vector field even makes it possible to fill out the missing piece without creating fiber-critical points if we can do this in a single fiber. The function $F|_{x_0 \times \mathbb{R}^{2k}}$ is already such an extension in the fiber over $x_0$. There is no problem with smoothness along the “seam” $\partial M$. Indeed, as the horizontal vector field is parallel to the values of $F$ close to this seam the pasted function fit smoothly together in a neighborhood. Furthermore, the fact that the trivialization is induced by the standard product structure outside a compact set makes the created function equal to $Q$ outside this compact set (since this was already true in the fiber over $x_0$). We therefore have our generating function proving Theorem 1.1. □

4. Extending to $n \leq 4$.

In this section we describe how to modify Section 2 and Section 3 to the case when $n \leq 4$. The difficulty is of course that the standard Morse theory arguments do not work in low dimension.

Instead of $L_\infty$ we “stabilize” and consider the Lagrangian $L'_\infty = L_\infty \times \mathbb{R}^{9-n} \subset \mathbb{C}^9$ which equals $\mathbb{R}^3$ outside of $T^*(D^n \times \mathbb{R}^{9-n})$. Similarly we let $L' = L \times \mathbb{R}^{9-n}$, and as before we let $L'_\delta = L' \cap T^*D^n_\delta$ be a slightly smaller version of $L$ and also define its stabilization $L'_\delta$. The spaces $L'$ and $L'_\delta$ are not compact but as we only care about generating the compact $L$ over $D^n \times \{0\}$ this will not present any serious problems.
The stabilization \( L'_\infty \) is of course still contractible, and the first main point is to find a flat Riemannian structure on \( L'_\infty \) and generalize Lemma 2.1 and Lemma 2.2 into the following lemma. We still denote \( x_0 = (-1, 0, \ldots, 0) \in L \times D^{9-n} \).

**Lemma 4.1.** There is a diffeomorphism \( \varphi : L'_\infty \cong \mathbb{R}^9 \) sending \( x \) to \( x - x_0 \) in a small neighborhood of \( x_0 \), and there is a symplectic embedding

\[
D^*_\varepsilon L_\infty \subset D^*_\frac{\varepsilon}{2} \mathbb{R}^n
\]
satisfying the properties listed in Lemma 2.2. Here \( L_\infty = L_\infty \times \{0\} \subset L'_\infty \) is given the restricted Riemannian structure induced by \( \varphi \). Furthermore, there is an \( \varepsilon' < \varepsilon \) small enough so that

\[
D^*_\varepsilon' L'_\infty \subset D^*_\varepsilon' L_\infty \times D^*_\frac{\varepsilon'}{2} \mathbb{R}^{9-n} \subset D^* \mathbb{R}^9.
\]

We will need the following lemma to prove this. Recall that near the boundary \( \partial L \) agrees with \( D^n \).

**Lemma 4.2.** There exists a diffeomorphism

\[
\psi : L \times D^{9-n} \cong D^n \times D^{9-n}
\]
as manifolds with corners, which is the identity near the boundary part \( \partial L \times D^{9-n} \) and on the form

\[
\psi(l, x) = (\psi_1(l, \hat{x}), ||x||\psi_2(l, \hat{x})) \tag{4.1}
\]
near the other boundary part \( L \times S^{8-n} \cong D^n \times S^{8-n} \).

**Proof.** Consider a homotopy equivalence rel boundary \( \varphi : L \to D^n \) which is the identity near \( \partial L = S^{n-1} \). We may lift this to a smooth embedding \( L \to D^n \times D^{9-n} \), which agrees with the standard embedding of \( D^n \) in a neighborhood of \( S^{n-1} \times D^{9-n} \). The normal bundle of this embedding is trivial over the boundary \( S^{n-1} \) by identifying its disc with \( D^{9-n} \) and we claim that this trivialization extends over all of \( L \). Indeed, in the cases \( n = 1 \) and \( n = 2 \) this follows as \( D^n \) and the homotopy equivalence is homotopic rel boundary to the identity, and in the case \( n = 3 \) it follows from \( \pi_3(BO(6)) = 0 \). In the case \( n = 4 \) it follows as Hirzebruch signature theorem (used after attaching a disc to get a 4-sphere) shows that the tangent bundle of \( L \) rel boundary is stably trivial.

It follows that we have an embedding \( L \times D^{9-n}_r \to D^n \times D^{9-n} \) for small \( r > 0 \) which is the standard inclusion over a neighborhood of the boundary of \( L \). The closure of the complement is (by possibly making \( r \) smaller) thus an h-cobordism rel boundary from \( L \times S^{8-n}_r \) to \( D^n \times S^{8-n} \), which by the h-cobordism theorem can be trivialized rel boundary. This means identifying it with the closure of \( L \times (D^9 - D^9_r) \). This can be pasted together with the inclusion to get the wanted diffeomorphism. The last statement follows since we may assume that the trivialization of the h-cobordism respects the radial collar close to \( D^n \times S^{8-n} \). \( \square \)

**Proof of Lemma 4.1.** Let \( \psi \) be a diffeomorphism as in the Lemma 4.2 above. As it satisfies Equation (4.1) close to \( L \) we may extend it to a diffeomorphism

\[
\psi_2 : L \times \mathbb{R}^{9-n} \cong D^n \times \mathbb{R}^{9-n}
\]
which is the identity in a neighborhood of \( S^{n-1} \times \mathbb{R}^{9-n} \). We may thus further extend by the identity to a diffeomorphism \( \Psi : L_\infty \times \mathbb{R}^{9-n} \cong \mathbb{R}^9 \). Changing this
diffeomorphism inside $D^n_2 \times D^{3-n}$ we may assume that in a neighborhood of $x_0$ it maps $x \to x - x_0$.

Finding $\varepsilon > 0$ and an embedding $D^*_n L_\infty \to T^*\mathbb{R}^n$ is proved exactly as Lemma 2.2. Except the diffeomorphism $\Psi$ is actually the identity outside of $L$, and so the induced Riemannian structure on $L_\infty$ is in this case the trivial one. As $\Psi$ is $n$-radial outside a compact set its derivatives are globally bounded. It follows that there is an $\varepsilon' > 0$ as prescribed in the lemma. \hfill $\square$

We will again employ Chaperon’s construction of a generating function over $\mathbb{R}^3$. However, as we will then restrict this to $D^n \times \{0\}$ it is important to exactly identify what such restrictions generate. This is given by a symplectic reduction. Indeed, for a generating function $F : \mathbb{R}^3 \times \mathbb{R}^{2k}$ generating a Lagrangian $K \subset T^*\mathbb{R}^3$ the restriction generates the image of

$$K \cap (T^*\mathbb{R}^3)|_{\mathbb{R}^n} \to T^*\mathbb{R}^n.$$  

Here we are restricting the entire 9 dimensional cotangent bundle over $\mathbb{R}^n$ and then orthogonally projecting to the sub-bundle. Note that, even if $K$ is embedded in $T^*\mathbb{R}^3$ the image might not be embedded in $T^*\mathbb{R}^n$. The condition that the restriction cuts out its singular set transversely is the same as the intersection above being transverse.

To finish the argument we thus need the following slight modification of Lemma 2.6. We are now using $(q, p)$ coordinates on $T^*L'_\infty$ induced by a diffeomorphism to $\mathbb{R}^3$, which later will be constructed using Lemma 4.1 above.

**Lemma 4.3.** Let $S_c$ be the result of the Hamiltonian flow of the Hamiltonian from Equation (2.2) applied to the fiber $T^*_x L'_\infty \subset T^*L'_\infty$ and let $p_c : S_c \to \mathbb{R}$ be the unique primitive of the Liouville form $\lambda'_{L'_\infty}$ in $T^*L'_\infty$ which has primitive equal 0 at $(q, p) = (0, 0)$. For any $C > 0$ there is $c > 0$ large enough such that

$$S_c \cap (T^*L')|_L = L \sqcup K_1$$

with primitive $p_c = 0$ on $L$ and $\sup_{K_1} p_c < -C$. Furthermore, the intersection is transverse along $L$.

**Proof.** The proof is exactly the same - except we apply the generalized rotation within $D^*_n L'_\infty$ from Lemma 4.1. Furthermore, even though $L'$ is non-compact we only need $L$ compact to conclude all statements in the lemma except the last. $L$ is transversely cut out since the flow of the fiber gives the zero-section in a neighborhood of $L$. \hfill $\square$

The version of Proposition 2.7 we will need is the following.

**Proposition 4.4.** There exists a compactly supported Hamiltonian isotopy of the zero-section $\mathbb{R}^9 \subset \mathbb{C}^9$ to a Lagrangian $K$ such that

$$K \cap T^*\mathbb{R}^9_{D^n \times \{0\}} = L \sqcup K_1 \quad \text{and} \quad K \cap T^*_x \mathbb{R}^9 = \{x_0\}$$

and such that for any primitive $p_K : K \to \mathbb{R}$ of the Liouville form there is a $C \in \mathbb{R}$ such that

$$\min_{z \in L} p_K(z) > C > \max_{z \in K_1} p_K(z).$$

**Proof.** The proof is the same as that of Proposition 2.7 except we use the lemmas above, and we only care about getting $K$ to look right and getting the bounds over the compact sets $L \times \{0\}$ and $D^n \times \{0\}$. \hfill $\square$
Proof of Lemma 3.1 in the case \( n \leq 4 \). The proof is essentially the same, but using the proposition above instead of Proposition 2.7 and restricting the generating function to \( D^n \times \{0\} \). Note that \( K_1 \) projected to \( T^*D^n \) need not be embedded, but the constructed generating function only generates \( L \), which is embedded. \( \square \)

5. The Space \( \mathcal{M}_\infty \) and the Lagrangian Gauss-map

In this section we define the spaces \( \mathcal{M}_k \) whose colimit will be denoted \( \mathcal{M}_\infty \). The space \( \mathcal{M}_k \) is essentially the possible fibers of generating functions quadratic at infinity that generate the zero section. This means that given such a generating function \( N \times \mathbb{R}^{2k} \rightarrow \mathbb{R} \) the adjoint defines a map

\[
N \rightarrow \mathcal{M}_k
\]

which we will make precise in this section. This is of interest since the generating function from Theorem 1.1 generates the zero-section over each point in the boundary \( S^{n-1} \) and therefore the adjoint over \( S^{n-1} \) defines a map

\[
S^{n-1} \rightarrow \mathcal{M}_k.
\]

We also consider the canonical map

\[
N_\infty : \mathcal{M}_\infty \rightarrow BO \subset \mathbb{Z} \times BO
\]

given by taking the negative eigenspace of the Hessian at the critical point. We then relate this map to the stable Lagrangian Gauss map of \( L \) by proving that the element defined in \( \pi_{n-1}(\mathbb{Z} \times BO) \cong \pi_n(U/O) \) by the composition

\[
S^{n-1} \rightarrow \mathcal{M}_k \rightarrow \mathcal{M}_\infty \rightarrow \mathbb{Z} \times BO
\]
is the stable Lagrangian Gauss map of \( L \) relative to the fact that it maps \( S^{n-1} \subset L \) to the base-point.

Let \( Q^l_k : \mathbb{R}^k \times \mathbb{R}^l \rightarrow \mathbb{R} \) by the generalization of \( Q_k = Q^k_k \) given by

\[
Q^l_k(x, y) = -\|x\|^2 + \|y\|^2
\]
for \( x \in \mathbb{R}^k \) and \( y \in \mathbb{R}^l \). To make things as explicit as possible we denote the coordinates in \( \mathbb{R}^k \times \mathbb{R}^l \) by

\[
z = (x_k, x_{k-1}, \ldots, x_1, y_1, y_2, \ldots, y_l).
\]

Let \( C^l_k \) be the set of smooth maps \( f : \mathbb{R}^{2k} \rightarrow \mathbb{R} \) such that

\[
\|f - Q^l_k\|_{C^1} = \max(\|f - Q^l_k\|_{L^\infty}, \|\nabla f - \nabla Q^l_k\|_{L^\infty}) < \infty.
\]

However, as this \( C^1 \) norm is not continuous in the \( C^\infty \) topology we topologize \( C^l_k \) by identifying it with the subspace of pairs

\[
(f, \|f - Q^l_k\|_{C^1}) \in C^\infty(\mathbb{R}^{2k}, \mathbb{R}) \times \mathbb{R}.
\]

We need this modified topology to properly control the behavior at infinity. We will also often need this bound explicitly so we define

\[
ce_f = \|f - Q^l_k\|_{C^1} + 1.
\]

To understand the homotopy types of the mapping spaces the following lemma is convenient.

**Lemma 5.1.** Given any compact set \( K \subset \mathbb{R}^{k+l} \) the subspace in \( C^l_k \) of functions \( f \) with the support of \( f - Q^l_k \) in \( K \) has the \( C^\infty \) topology.
Proof. The map \( C^\infty(\mathbb{R}^{k+l},\mathbb{R}) \to \mathbb{R} \) given by taking the \( C^1 \) norm above is continuous on the subspace of functions with support in \( K \). \( \square \)

Note, however, that the subspace of all functions with arbitrary compact support is not induced with the \( C^\infty \) topology. For that reason, we will quite often need to consider the homotopy from the identity on \( C^1_k \) that bumps the function outside a compact set to make it equal to \( Q_k^1 \) at infinity. Indeed, pick a function \( \varphi : \mathbb{R} \to [0,1] \) such that \( \varphi(t) = 0 \) when \( t \leq 0 \), \( \varphi(t) = 1 \) when \( t \geq 2 \) and \( 0 \leq \varphi'(t) \leq 1 \). We use this to define the homotopy \( B_t : C^1_k \to C^1_k \) from the identity to functions where \( f - Q_k^1 \) has compact support. We let

\[
B_t(f)(z) = (1 - t\varphi(\|z\| - c_f))f(z) + t\varphi(\|z\| - c_f)Q_k^1(z). \tag{5.3}
\]

Lemma 5.2. The homotopy \( B_t : C^1_k \to C^1_k \) is well defined and for each \( t \) the functions \( B_t(f) \) and \( f \) are equal near their critical points. Furthermore \( c_{B_t(f)} \leq c_f \) for each \( t \).

Proof. As the topology we impose on \( C^1_k \) makes taking the norm \( \|f - Q_k^1\|_{C^1} \) continuous the map defines a continuous map \( C^1_k \to C^\infty(\mathbb{R}^{2k},\mathbb{R}) \). Since the supremum norm \( \|B_t(f) - Q_k^1\|_{C^1} \) is continuous in \( t \) and \( f \) it follows that \( B_t \) is continuous.

The gradient of \( B_t(f) \) satisfies (using \( \|\varphi'\| \leq 1 \))

\[
\|\nabla B_t(f) - \nabla Q_k^1\| \leq (1 - t)\|\nabla B_t(f) - \nabla Q_k^1\| + t\|\nabla Q_k^1\| \leq \|f - Q_k^1\|_{C^1}.
\]

This bound shows that all critical points must be inside \( D^k_{c_f} \) where we are not changing the function. It also shows the last statement of the lemma. \( \square \)

In this section we will only need the following subspace.

Definition 5.3. Let \( M^k_f \subset C^1_k \) be the subspace of functions \( f \) such that

- \( f \) has only one critical point.
- This critical point is non-degenerate and has critical value 0.

We denote \( M_k = M^1_k \).

Note that the bound on the supremum \( C^1 \) norm means that the level sets at infinity are diffeomorphic to those of \( Q_k^1 \) (this is even true for \( C^1_k \)). It follows by standard Morse theory that the index of the unique critical point of \( f \in M^1_k \) must be \( k \). We will only need \( M_k \) for the rest of this section.

The adjoint of a map \( F : X \times \mathbb{R}^{2k} \to \mathbb{R} \) is the map \( F^{\text{Adj}} : X \to \text{Map}(\mathbb{R}^{2k},\mathbb{R}) \) which maps \( x \in X \) to \( F_{(x)} \). Any generating function \( F : D^n \times \mathbb{R}^{2k} \to \mathbb{R} \) from Theorem 1.1 has a single critical point over each fiber \( x \in S^{n-1} \). The critical value may be non-zero, but varying \( x \) in \( S^{n-1} \) this critical value is constant. We claim that we may assume that this constant is 0. Indeed, any \( f \in C^1_k \) has \( f + c \in C^1_k \) for any \( c \in \mathbb{R} \). So, by adding a global constant to \( F \), and bumping of fiber-wise using Lemma 5.2 to again make the function equal to \( Q_k \) at infinity shows the claim. Note in particular that according to the lemma the bumping of do not affect fiber-wise critical points - hence do not change what Lagrangian the function generates. Using compactness of \( S^{n-1} \) and the fact that the generating function equals \( Q_k \) outside a compact set we thus get a map

\[
F^{\text{Adj}} : S^{n-1} \to M_k. \tag{5.4}
\]

It is well known that having generating functions for fixed \( k \) is not a Hamiltonian invariant (can be explicitly proven by locally rotating any Lagrangian sufficiently
to get an arbitrary high range of Maslov indices when intersected with a fiber).
So, we replace these with their stable versions by defining stabilization maps \( s_k : \mathcal{M}_k \to \mathcal{M}_{k+1} \) given by adding the quadratic form (corresponding to the standard one) in the new variables. That is,

\[
s_k(f)(x_{k+1}, x, y, y_{k+1}) = -x_{k+1}^2 + f(x, y) + y_{k+1}^2.
\]

This is not exactly equal to \( Q_{k+1} \) at infinity. However, again using Lemma \[5.2\] one could bump of to get a stabilized generating function quadratic at infinity.

We thus define

\[
\mathcal{M}_\infty = \lim_{k \to \infty} \mathcal{M}_k.
\]

There are maps

\[
N_k : \mathcal{M}_k \to \text{Gr}_k(\mathbb{R}^{2k})
\]

given by taking the negative eigenspace of the Hessian at the critical points. This map commutes on the nose with stabilization maps if we use the convention that the stabilization maps \( \text{Gr}_k(\mathbb{R}^{2k}) \to \text{Gr}_{k+1}(\mathbb{R}^{2k+2}) \) takes direct sum with \( \mathbb{R} \subset \mathbb{R}^2 \) by adding coordinates \( x_{k+1} \) and \( y_{k+1} \) such that \( x_{k+1} \) is in the sub-space. So, in the limit \( k \to \infty \) we get a map

\[
N_\infty : \mathcal{M}_\infty \to BO = \{0\} \times BO \subset \mathbb{Z} \times BO.
\]

For the rest of this section we allow \( L \to T^*D^n \) to be any immersed Lagrangian disc equal to the zero section in a neighborhood of \( S^{n-1} \subset T^*D^n \). This has a Lagrangian Gauss map \( L \to U(n)/O(n) \). Indeed, the tangent space at each point is a Lagrangian in \( \mathbb{C}^n \) hence defines a point in \( U(n)/O(n) \). On the boundary \( S^{n-1} \) this is the standard \( \mathbb{R}^n \in U(n)/O(n) \), which we use as the base-point. Since \( L/S^{n-1} \simeq S^n \) we thus get an element in \( \pi_n(U(n)/O(n)) \). Stabilizing using

\[
U/O = \lim_{n \to \infty} U(n)/O(n)
\]

we get an element in \( \pi_n(U/O) \). By Bott-periodicity (see e.g. \[13\], which we partly recall in the proof below) we have that \( \Omega(U/O) \simeq \mathbb{Z} \times BO \) and thus

\[
\pi_{n-1}(\mathbb{Z} \times BO) \cong \pi_n(U/O).
\]

**Proposition 5.4.** Let \( S^{n-1} \to \mathcal{M}_k \) be the adjoint of the restriction to the boundary of a generating function \( F : D^n \times \mathbb{R}^{2k} \to \mathbb{R} \) quadratic at infinity. Assume that it generates an immersed Lagrangian disc \( L \to T^*D^n \) which agrees with the zero section in a neighborhood of \( S^{n-1} \subset T^*D^n \). The composition

\[
S^{n-1} \to \mathcal{M}_k \to \mathcal{M}_\infty \xrightarrow{N_\infty} \mathbb{Z} \times BO
\]

represents the class defined above in \( \pi_n(U/O) \) under the isomorphism induced by Bott periodicity.

**Proof.** Firstly, we note that the map \( S^{n-1} \to \mathcal{M}_\infty \) has not been constructed as a based map. This we could have done, but it makes the construction more intricate, and at this point it makes no difference since the action of \( \pi_1(BO) \) on higher homotopy groups is trivial since \( BO \) is an (infinite) loop space. The unbased nature of the map could also make it slightly ambiguous which component of \( \mathbb{Z} \times BO \) we land in. However, we defined \( N_\infty \) to land in \( \{0\} \times BO \) and the Maslov index for anything with a generating function as in the proposition is 0. This implies that the
adjoint also land in this component (only relevant when \( n = 1 \)). It is thus enough to prove that the two maps from \( S^{n-1} \) to \( BO \) are freely homotopic.

In [8] Giroux proves that if a Lagrangian has a generating function then the Gauss map is homotopy trivial, which is in no way surprising in our case since \( L \) is contractible. However, using his explicit null homotopy we will see the needed identification. Indeed consider \( L \cong D^n \) mapping into \( U/O \) and sending the boundary \( S^{n-1} \) to the base-point. Giroux’s null-homotopy is a free null-homotopy of the map from this disc. So tracing what happens at the boundary we get a map:

\[
S^{n-1} \times I \to U/O
\]

which sends \( \{0, 1\} \times S^{n-1} \) to the base-point. The adjoint of this, therefore, represents the free homotopy class of the usual adjoint of the map \( S^{n-1} \to \Omega(U/O) \) of the original stable Gauss map from the disc \( L/S^{n-1} \cong S^n \).

We, now, recall Giroux’s proof and elaborate a bit on some parts to also identify this with the negative eigen-bundle map of the generating function under Bott periodicity. His argument goes as follows: Let \( \Lambda(n, 2k) \) be the linear Lagrangians in \( \mathbb{C}^{n+2k} \) that transversely intersect \( H = \mathbb{C}^n \oplus \mathbb{R}^{2k} \). These are the linear Lagrangians that symplectically reduce to \( \mathbb{C}^n \) by transversely intersecting with \( H \) and projecting to \( \mathbb{C}^n \). The reduction map \( r: \Lambda(n, 2k) \to \Lambda(n) = U(n)/O(n) \) is a fiber-bundle with contractible fibers. We define the standard stabilization of the Gauss map as adding Lagrangian factors given by the differential of the function \( \frac{1}{2} \|x\|^2 + \frac{1}{2} \|y\|^2 \) with \((x, y) \in \mathbb{R}^{2k} \in T^*\mathbb{R}^{2k} = \mathbb{C}^{2k} \) - this adds the Lagrangian factor \((1+i)\mathbb{R}^{2k}\). When the tangent space at a point \( l \in L \) is also a graph (in \( T^*\mathbb{R}^n \)) the stabilized linear Lagrangian at that point is the graph of a block matrix:

\[
\begin{pmatrix}
0 & 0 \\
0 & ?
\end{pmatrix},
\]

where the two diagonal blocks are of size \( n \times n \) and \( 2k \times 2k \) respectively. In particular for \( l \) close to \( \partial L = S^{n-1} \) the first block “?” is 0.

Sometimes it is more natural to define these stabilizations by adding the factor \( \mathbb{R}^{2k} \) (corresponding to the second block in the matrix also being 0) or \( i\mathbb{R}^{2k} \) (not a section). However, since we want the result to both be graphical like this over \( \mathbb{R}^{2k} \) and lie in \( \Lambda(n, 2k) \) we need it to be transverse to both of these.

The fact that \( r \) has contractible fibers means that any two sections in \( \Lambda(n, 2k) \) are homotopic. In particular the stabilization is a section, but also the differential of the generating function \( F: D^n \times \mathbb{R}^{2k} \to \mathbb{R} \) defines a section and hence there is a homotopy between these two. Giroux’s final observation is now that since all the Lagrangians in \( dF \) are transverse to \( i\mathbb{R}^{n+2k} \) (\( dF \) is graphical) its Gauss map is null-homotopic.

Parsing the construction of this homotopy locally for a point in \( S^{n-1} \subset L \) where the tangent space initially agreed with the zero-section we get the following. The stabilization from the generating function is equivalent to adding the differential of another non-degenerate quadratic form. I.e. the Hessian of \( F \) which varies with the point in \( S^{n-1} \); which we, however, assume without loss of generality to have only eigenvalues \( \pm 1 \). At these points, the Hessian of \( F \) is in block form:

\[
\begin{pmatrix}
0 & 0 \\
0 & ?
\end{pmatrix},
\]
where the diagonal blocks again have size \( n \times n \) and \( 2k \times 2k \) respectively and the \( ? \) is non-degenerate symmetric with eigenvalues \( \pm 1 \). As both this "?" and "I" defining the two sections are non-degenerate we may view the lower \( 2k \times 2k \) part as defining graphs over \( i\mathbb{R}^{2k} \) (taking values in \( \mathbb{R}^{2k} \)). Hence the homotopy between the two sections in \( \Lambda(n, 2k) \) can simply be described by a convex interpolation of these graphs (keeping the 0 constant in the first \( n \) factors). This interpolation may pass through \( i\mathbb{R}^{2k} \) and hence at times it will not be graphical over the base \( \mathbb{R}^{2k} \). In fact its intersection locus with \( i\mathbb{R}^{2k} \) is precisely \( i \times \text{negative eigen bundle of Hessian} \) (since the function \( \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 \) only has positive eigenvalues). This is exactly how the space of minimal geodesics in the corresponding component of \( \Omega(U(2k)/O(2k)) \) mapping to \( \text{Gr}_k(\mathbb{R}^{2k}) \) is identified in the proof of Bott-periodicity in [13].

One might interject that the choice of initial stabilization seems important. After all we could have used the differential of another quadratic form (e.g. \( -\frac{1}{2}\|x\|^2 - \frac{1}{2}\|y\|^2 \)) and maybe gotten a different result. This will, indeed, produce other null homotopies of the map \( D^n \to U/O \). However, as \( D^n \) is contractible such only differ (up to homotopy) by the action of \( \pi_1 \), and this is precisely the ambiguity of which component of \( \mathbb{Z} \times BO \) we land in, which we addressed at the beginning of the proof.

6. The Fibration Sequence

In this section we prove the unstable version

\[
\mathcal{M}_k^l \to \mathcal{U}_k^l \to \mathcal{H}_k^l
\]

(and define these spaces) of the fibration sequence from Theorem 1.4. We also define stabilization maps between these and prove that the fibration structures are compatible with these resulting in the colimit version of the sequence:

\[
\mathcal{M}_\infty \to \mathcal{U}_\infty \to \mathcal{H}_\infty
\]

In the last sections we then identify the space \( \mathcal{U}_\infty \) as \( F/O \), the space \( \mathcal{H}_\infty \) as the stable \( h \)-cobordism space of a point, and the map \( \mathcal{U}_\infty \to \mathcal{H}_\infty \) as the Hatcher-Waldhausen map.

With \( C_k^l \) as in Section 5, let \( (C_k^l)^1 \subset C_k^l \) be the subspace of functions which has 1 as a regular value. We define

- \( \mathcal{H}_k^l \subset (C_k^l)^1 \) as the connected component containing \( Q_k^l \).
- \( \mathcal{U}_k^l \subset \mathcal{H}_k^l \) the subspace of functions where there is only one critical point with value below 1. We further require that it has critical value 0 and is non-degenerate.

Again by considering level sets the one critical point for any \( f \in \mathcal{U}_k^l \) must have Morse index \( k \).

The space \( \mathcal{H}_k^l \) is for \( k + l \geq 6 \) homotopy equivalent to the \( h \)-cobordism space denoted \( \mathcal{H}(S^{l-1} \times D^k) \) (see e.g. Waldhausen’s model in [19]). We will almost prove this in the last section. However, the heuristic argument is what was used in the description of the Hatcher-Waldhausen map in the introduction. Indeed, the level set \( \{ f = 1 \} \) is essentially the end of a compactly supported \( h \)-cobordism from \( \{ Q_k^l = C \} \cong S^{l-1} \times \mathbb{R}^k \) for some large \( C \).
We define the shrinking (expansion) \( b \circ f \) of a function \( f \in \mathcal{C}_k^l \) by a constant \( b \in (0, 1) \) \((b \geq 1)\) by the formula
\[
(b \circ f)(z) = b^2 f(b^{-1} z).
\]
(6.1)

Note that if \( f = Q_k^l \) on some set \( A \) then \( b \circ f = Q_k^l \) on \( bA \). Furthermore we have
\[
\|b \circ f - Q_k^l\|_{\mathcal{C}^1} \leq \max(b, b^2)\|f - Q_k^l\|_{\mathcal{C}^1},
\]
(6.2)

and the critical values of \( b \circ f \) are those of \( f \) scaled by \( b^2 \).

**Lemma 6.1.** The space \( \mathcal{M}_k^l \) from Definition 5.3 is a subspace in \( \mathcal{U}_k^l \).

**Proof.** The only condition we need to check is that it actually lies in the connected component of \( (\mathcal{C}_k^l)^1 \) containing \( Q_k^l \). We prove this by taking any \( f \in \mathcal{M}_k^l \) and continuously shrinking the function \( f \). Indeed, \( b \circ f \) for \( b \in [\varepsilon, 1] \) defines a path from \( \varepsilon \circ f \) to \( f \). For small \( \varepsilon \) we have \( \|\varepsilon \circ f - Q_k^l\|_{\mathcal{C}^1} \) as small as needed. We may make it small enough so that the convex interpolation \( (1 - t)(\varepsilon \circ f) + tQ_k^l \) has 1 as a regular value.

We define the subspace \( K\mathcal{C}_k^l \subset \mathcal{C}_k^l \) as those function \( f \in \mathcal{C}_k^l \) where \( f - Q_k^l \) has compact support. We define
\[
K\mathcal{M}_k^l \subset \mathcal{M}_k^l, \quad K\mathcal{U}_k^l \subset \mathcal{U}_k^l, \quad K\mathcal{H}_k^l \subset \mathcal{H}_k^l.
\]
(6.3)

using the same condition. We similarly define the subspace \( D_b\mathcal{C}_k^l \subset \mathcal{C}_k^l \) as those function \( f \in \mathcal{C}_k^l \) where the support of \( f - Q_k^l \) is contained in \( D_b^k \). We define
\[
D_b\mathcal{M}_k^l \subset \mathcal{M}_k^l, \quad D_b\mathcal{U}_k^l \subset \mathcal{U}_k^l, \quad D_b\mathcal{H}_k^l \subset \mathcal{H}_k^l.
\]
(6.4)

using the same condition. Note that \( K\mathcal{C}_k^l = \bigcup_b D_b\mathcal{C}_k^l \).

**Lemma 6.2.** The inclusions in Equation (6.3) are homotopy equivalences.

**Proof.** This follows from Lemma 5.2 and the fact that all the spaces are defined by conditions on critical points.

Consider the subspace \( (\mathcal{H}_k^l)^{\geq 1} \subset \mathcal{H}_k^l \) defined by those \( f \) where all values above 1 are also regular. We define the homotopy
\[
H_t(f) = \begin{cases} 
(1 - t + t\left(\frac{1}{c^2 + 9}\right)) \circ f & t \in [0, 1] \\
(2 - t)\left(\frac{1}{c^2 + 9}\right) \circ f + (t - 1)Q_k^l & t \in [1, 2]
\end{cases}
\]
(6.5)

where again \( c_f = \|f - Q_k^l\|_{\mathcal{C}^1} + 1 \).

**Lemma 6.3.** The homotopy \( H_t \) defines a homotopy from the identity on \( (\mathcal{H}_k^l)^{\geq 1} \) to the constant map at \( Q_k^l \). Hence \( (\mathcal{H}_k^l)^{\geq 1} \) is contractible.

**Proof.** The shrinking scales the critical values by a factor less than 1, so \( H_t(f) \) has all values in \([1, \infty)\) regular for \( t \in [0, 1] \). The bound (by Equation 6.2)
\[
\left\|\left(\frac{1}{c^2 + 9}\right) \circ f - Q_k^l\right\|_{\mathcal{C}^1} \leq \frac{c_f}{c_f^2 + 9} \leq \frac{1}{9}
\]
shows that all values in \([1, \infty)\) are regular for \( H_t(f) \) with \( t \in [1, 2] \). Indeed, any point \( z \) where \( H_t(f)(z) = 1 \) has \( Q_k^l \geq \frac{8}{9} \) and the gradient norm of \( Q_k^l \) at these points are much larger than \( \frac{1}{9} \).
The composition $\mathcal{M}_k^l \to \mathcal{U}_k^l \to \mathcal{H}_k^l$ lands in $(\mathcal{H}_k^l)^{\geq 1}$ and using the homotopy above is thus null homotopic.

**Lemma 6.4.** The sequence

$$\mathcal{M}_k^l \subset \mathcal{U}_k^l \subset \mathcal{H}_k^l$$

with the null-homotopy above is a homotopy fibration sequence.

**Proof.** Fix a $b > 1$ and let $\tilde{H}$ be the space of smooth co-dimension 1 submanifolds in $\mathbb{R}^{k+l}$ which agree with $\{Q_k^l = 1\}$ outside of $D_b^{k+l}$. We give $\tilde{H}$ the $C^\infty$ topology (using spaces of local sections in normal bundles as charts). We claim that the map $D_b\mathcal{H}_k^l \to \tilde{H}$ given by $f \mapsto \{f = 1\}$ and the composition $D_b\mathcal{U}_k^l \to \tilde{H}$ through $\mathcal{H}_k^l$ are both fiber bundles. It follows from this that the first is a homotopy equivalence since its fibers are contractible.

To prove the claim we let $M \in \tilde{H}$ be given. Note that $M = \{Q_k^l = 1\}$ outside of $D_b^{k+l}$ and is thus transverse to $S_b^{k+l-1}$. So we may pick a smooth tubular neighborhood $M \times [-2, 2] \subset \mathbb{R}^{k+l}$ which over any point $x \in M \cap S_b^{k+l-1}$ is mapped into $S_b^{k+l-1}$. Using this we can identify an open neighborhood $U$ of $M$ with smooth maps $s : M \to (-1, 1)$ with support on $M \cap D_b^{k+l}$ (sections in the normal bundle). Pick a map $\psi : [-2, 2] \to [0, 1]$ which is 1 on $[-1, 1]$ and 0 close to $\{-2, 2\}$. For each $s : M \to (-1, 1)$ let $X_s$ be the vector field on $M \times [-2, 2]$ defined by $\psi(t)s(x)\frac{\partial}{\partial t}$ at points $(x, t)$. This is constantly equal to $s(x)\frac{\partial}{\partial t}$ in the interval between $s(x)$ and 0 - hence its time 1 flow, which we denote by $\varphi^s$ on $M \times [-2, 2]$ takes the zero section $M$ to $s(M)$. As $\varphi^s$ is the identity outside of $D_b^{k+l}$ it acts continuously on $D_b\mathcal{U}_k^l$ and $D_b\mathcal{H}_k^l$ (by Lemma 5.1). We thus get trivializations of both bundles in the claim by composing functions over $s \in U$ from the right with $\varphi^s$.

The fiber over the base-point of the fiber bundle $D_b\mathcal{U}_k^l \to \tilde{H}$ consists of functions $f \in \mathcal{U}_k^l$ such that $\{f = 1\} = \{Q_k^l = 1\}$. We claim that this deformation retracts onto the subspace $W \subset D_b\mathcal{M}_k^l$ defined by those $f \in D_b\mathcal{M}_k^l$ which agrees with $Q$ on $\{Q_k^l \geq 1\}$. To prove this claim, we may first use an isotopy to make $f = Q_k^l$ at $\{Q_k^l = 1\}$ to all orders (this uses that $\{f = 1\} = \{Q_k^l = 1\}$ and that $f$ has 1 as a regular value). After this we may convexly interpolate on the set $\{Q_k^l \geq 1\}$ between $f$ and $Q_k^l$ (leaving $f$ as it is on $\{Q_k^l < 1\}$ where it had only the one critical point by the definition of $\mathcal{U}_k^l$).

We also claim that $W \subset D_b\mathcal{M}_k^l$ is a homotopy equivalence. Indeed, shrinking with factor $b^{-k}$ defines a map back $D_b\mathcal{M}_k \to W$. The compositions in both directions are homotopic to the identity using the shrinking homotopies considered above.

The final claim is that this is compatible with the null homotopy $H_t$ given in Equation 6.5 above. Indeed, we first note that the homotopy preserves $D_b(\mathcal{H}_k^l)^{\geq 1} = D_b\mathcal{H}_k^l \cap (\mathcal{H}_k^l)^{\geq 1}$. Moreover, it even preserves the property $f = Q_k^l$ on $\{Q_k^l \geq 1\}$. It follows that the composition $W \to D_b\mathcal{H}_k^l \to \tilde{H}$ constantly maps to the base-point in $\tilde{H}$ during this homotopy. This shows that the map from $W$ and hence $D_b\mathcal{M}_k^l$, to the homotopy fiber of $D_b\mathcal{U}_k^l \to \tilde{H}$ is a homotopy equivalence. Hence the induced map to the homotopy fiber of the $D_b\mathcal{M}_k^l \to D_b\mathcal{H}_k^l$ is also a homotopy equivalence. This last map to the larger homotopy fiber is compatible with unions for larger $b$ and shows that

$$K\mathcal{M}_k^l \to K\mathcal{U}_k^l \to K\mathcal{H}_k^l$$
is a homotopy fibration sequence with this particular null homotopy. As the inclusion of these spaces into the original sequence are homotopy equivalences (Lemma 6.2) and the map to the homotopy fiber is again compatible with this inclusion, the result follows.

We define stabilization maps

$$s^- : \mathcal{H}_k^l \to \mathcal{H}_{k+1}^l$$ and $$s^+ : \mathcal{H}_k^l \to \mathcal{H}_{k}^{l+1}$$

by

$$s_-(f)(x_{k+1}, x, y) = -x_{k+1}^2 + f(x, y) \quad (6.6)$$

$$s_+(f)(x, y, y_{l+1}) = f(x, y) + y_{l+1}^2. \quad (6.7)$$

Note that $$s_- \circ s_+ = s_+ \circ s_-$$.

The function $$s_-(f)$$ only has critical points of the type $$(0, x, y)$$ where $$(x, y)$$ is critical for $$f$$. Similar for $$s_+(f)$$. This shows that $$s_{\pm}$$ preserves the three types of subspaces $$\mathcal{U}_k^l, \mathcal{M}_k^l$$ and $$\mathcal{H}_k^l$$ and we get a commutative diagrams

$$\begin{array}{ccc}
\mathcal{M}_k^l & \longrightarrow & \mathcal{U}_k^l \\
\downarrow{s^\pm} & & \downarrow{s^\pm} \\
\mathcal{M}_{k'}^l & \longrightarrow & \mathcal{U}_{k'}^l \\
\end{array} \quad \begin{array}{ccc}
\mathcal{H}_k^l \quad \longrightarrow \quad \mathcal{H}_{k}^l \\
\downarrow{s^\pm} & & \downarrow{s^\pm} \\
\mathcal{H}_{k'}^l \\
\end{array}$$

We define

$$\mathcal{M}_\infty = \text{colim}_{k \to \infty, l \to \infty} \mathcal{M}_k^l, \quad \mathcal{U}_\infty = \text{colim}_{k \to \infty, l \to \infty} \mathcal{U}_k^l, \quad \mathcal{H}_\infty = \text{colim}_{k \to \infty, l \to \infty} \mathcal{H}_k^l$$

all under these stabilization maps. The stabilizations also preserve the subspaces $$(\mathcal{H}_k^l)^{\geq 1}$$, and we have a contractible subspace $$\mathcal{H}_\infty^{\geq 1} = \text{colim}_{k \to \infty, l \to \infty} (\mathcal{H}_k^l)^{\geq 1} \subset \mathcal{H}_\infty$$. The homotopy in Equation (6.5) is compatible with stabilizations and defines a null homotopy of the identity on $$\mathcal{H}_\infty^{\geq 1}$$. Defining a null homotopy of the composition

$$\mathcal{M}_\infty \to \mathcal{U}_\infty \to \mathcal{H}_\infty.$$ 

Proposition 6.5. The sequence

$$\mathcal{M}_\infty \to \mathcal{U}_\infty \to \mathcal{H}_\infty.$$ 

with the above null homotopy is a homotopy fibration sequence.

Proof. This follows as the map from each $$\mathcal{M}_k^l$$ to the homotopy fiber of $$\mathcal{U}_k^l \to \mathcal{H}_k^l$$ is a homotopy equivalence by the above lemma, and this is compatible with taking the colimit.

7. Parametrized handle attachments

In this section we give explicit constructions of functions $$f \in C_k^l$$ depending on certain maps from vector spaces (similar to $$(V, i)$$ from the introduction used to describe the Hatcher-Waldhausen map). In the case considered in the introduction these functions will have level sets $${f = 1}$$ that are smoothings of the boundary of $$\{Q_k \leq -1\} \cup T$$ where $$T$$ is a tubular neighborhood of the disc image (pink in Figure 1). The idea is to implant a standard Morse function in a certain chart of the tubular neighborhood, we refer to the resulting function as a mountain pass function.
For a vector space $V \subset \mathbb{R}^{k+l}$ we will use the coordinate notation $(v, w) \in V \times V^\perp \subset \mathbb{R}^{k+l}$. We will also consider the generalization
\[ Q_V : \mathbb{R}^{k+l} \to \mathbb{R} \]
defined by $Q_v(v, w) = -\|v\|^2 + \|w\|^2$ (which may not have signature $l-k$). We will let \( \vec{v} \) denote the (identity) vector field on $V$, that is \( \vec{v}(v) = v \in V = T_e V \). Similar we denote the identity vector field on $V^\perp$ by \( \vec{w} \). By abuse we also use this notation for the associated parallel vector fields on $\mathbb{R}^{k+l}$. We e.g. have $\nabla Q_V = -2\vec{v} + 2\vec{w}$.

**Definition 7.1.** Let $\mathcal{U}N_k^l$ be the space of triples \( \tilde{c} = (V, r, e) \) with $V \subset \mathbb{R}^{k+l}$ a vector space, $r \geq 1$ and $e = (e_x, e_y) : V \to \mathbb{R}^{k} \times D_r^l$ a smooth map such that
- $e$ is proper.
- $e$ is the linear inclusion $V \subset \mathbb{R}^{k+l}$ close to 0.
- $\vec{r}(\|e_x\|) > 0$ when $\|e_x\| \geq r$.

We topologize this using local trivializations of the canonical bundle over \( \text{Gr}_d(\mathbb{R}^{k+l}) \) and the product topology on $(r, e)$, where we use the $C^\infty$ topology on $e$.

The mountain pass functions themselves will be functions in the following space. Let $\mathcal{F}^l_k \subset (\mathcal{C}^l_k)^1$ be the subspace of those $f$ satisfying:
- There is a $V \in \text{Gr}_d(\mathbb{R}^{k+l})$ so that $f = Q_V$ close to 0.
- The only critical point with value $\leq 1$ is 0.

For later use we need to be able to adjust how wide the mountain is (using a parameter $c$) and how narrow the mountain pass is (using a parameter $\delta$). To make this precise we will need a few explicit constructions.

Pick a smooth even function $\varphi : \mathbb{R} \to \mathbb{R}$ such that
- $\varphi(t) = 0$ for $|t| \leq 1$.
- $\varphi(t) > 0$ for $t > 1$.
- $\varphi(t) = t^2$ for $t \geq 2$.

Define
\[ \hat{Q}_k^l(x, y) = -\varphi(\|x\|) + \varphi(\|y\|). \quad (7.1) \]
This has $\hat{Q}_k^l \in H^l_k$ and is 0 on $D^k \times D^l$ it has $\nabla Q_k^l$ as a weak pseudo gradient. By a weak pseudo gradient $X$ for a function $f$ we mean a vector field such that $X(f) \geq 0$ with equality only at critical points. A pseudo gradient which is 0 at critical points will be called a strict pseudo gradient.

We will need the following technical constructions to be able to implant the mountain pass.

**Lemma 7.2.** There exists $C^\infty$ tubular embeddings $\tau_e : V \times DV^\perp \to \mathbb{R}^{k+l}$ continuous in $\tilde{e} \in \mathcal{U}N_k^l$ such that
- The restriction of $\tau_e$ to $V \times \{0\}$ equals $e$.
- $\tau_e$ is the standard identification $V \oplus V^\perp = \mathbb{R}^{k+l}$ in a neighborhood of 0.
- $(c \circ \hat{Q}_k^l) \circ \tau_e$ only depends on $v \in V$ for $c \geq r$.

**Proof.** As $e$ lands in $\mathbb{R}^k \times D_r^l$ we have that
\[ (c \circ \hat{Q}_k^l) = -(c \circ \varphi)(\|x\|) \]
on $\mathbb{R}^k \times D_r^k$ for $c \geq r$. We assumed in the definition above that $e$ is strictly transverse to the foliation $S^{k-1}_a \times D_r^l$, $a \geq r$. These are level sets of $c \circ \hat{Q}_k^l$ in $\mathbb{R}^k \times D_r^l$ except that $D_r^k \times D_r^l$ is one large (degenerate) level set.
Let $\nu_c \to \text{im } e$ denote a choice of normal bundle which close to zero is $V^\perp$ and outside $D^k_e \times \mathbb{R}^l$ is contained in the foliation $S_{a}^{k-1} \times D^l_r, a \geq C(\varepsilon)$. Give $\nu$ the induced metric and define $T : D\nu \to \mathbb{R}^{k+l}$ by $T(z, w) = z + w$. Pick a smooth $\varepsilon_c : \text{im } e \to \mathbb{R}$ such that $T|_{D\varepsilon_c\nu}$ is a smooth embedding. This may be picked continuously in $e$ and such that $\varepsilon_c$ is constant close to 0. Pick a smooth family of smooth diffeomorphisms $\psi_a : [0, 1] \cong [0, a]$ such that $\psi_a(t) = t$ for $t$ close to 0, and let $\Psi_{\varepsilon_c} : D\nu \to D_{\varepsilon_c}\nu$ be the diffeomorphism that over $z \in \text{im } e$ stretches each vector to have length $\varphi_{\varepsilon_c}(\|v\|)$. Pick a family of isometric isomorphisms $\Phi_e : \nu_e \cong \text{im } e \times V^\perp$ and define

$$\tau_e : V \times D\nu^\perp \xrightarrow{c \times \text{id}} \text{im } e \times D\nu^\perp \xrightarrow{\Phi_{\varepsilon_c}^{-1}} D\nu \xrightarrow{\Psi_{\varepsilon_c}} D_{\varepsilon_c}\nu \xrightarrow{T} \mathbb{R}^{k+l}.$$  

We fix such a choice of $\tau_e$ in the following. Let $\mathcal{U}N^T_k$ be the space of tuples $(\tilde{e}, c, \delta) = (V, r, e, c, \delta)$ such that

- $\tilde{e} = (V, r, e) \in \mathcal{U}N^T_k.$
- $c \geq r.$
- $\delta \in (0, 1).$

We finally arrive at the construction of the mountain pass function.

**Lemma 7.3.** There is a map $MP : \mathcal{U}N^T_k \to \mathcal{F}_k^l$ such that with

$$W = \tau_e(V \times D\nu^\perp) \cap \{c \circ \tilde{Q}_k^l \geq -3\}$$

we have

- $MP(\tilde{e}, c, \delta)$ equals $c \circ \tilde{Q}_k^l + 2$ outside $W$.
- $MP(\tilde{e}, c, \delta)$ equals $Q_{\nu}$ in a neighborhood of 0.
- $MP(\tilde{e}, c, \delta)$ has $(d\tau_e)_* (\nabla_{Q_{\nu}})$ as a strict pseudo gradient on Int $W$.

This essentially moves up the values of $c \circ \tilde{Q}_k^l$ by 2 and implants a narrow Mountain pass along $e$ using the tubular neighborhood $\tau_e$ but depending on the parameters $c$ and $\delta$. Here $c$ controls the size of the plateau where the value is 2, and $\delta$ the width of the mountain pass.

**Proof.** Define $W' = \tau_e^{-1}(W)$. Consider the function $P : W' \to \mathbb{R}$ given by

$$P = (c \circ \tilde{Q}_k^l) \circ \tau_e|_{W'}.$$ 

The last properties of the tube from Corollary 7.2 shows that $P(v, w) = P(v, 0) = (c \circ \tilde{Q}_k^l)(v(0))$. This together with the definition of $\mathcal{U}N^T_k$ implies that $\tilde{v}(P) \leq 0$ and $\tilde{v}(P) < 0$ for $\|v\| \geq r$.

Fix a smooth family of smooth functions $\psi_\delta : [0, \infty) \to [0, 2]$ for $\delta \in (0, 1)$ such that

- $\psi_\delta(t) = -2 + t^2$ for small $t$.
- $\psi_\delta'(t) > 0$ for $t \in (0, \delta)$.
- $\psi_\delta(t) = 0$ for $t \geq \delta$.

Fix a smooth function $\overline{v} : \mathbb{R} \to [0, 1]$ so that

- $0 \leq \overline{v}'(t) < \frac{1}{2}$ for all $t$.
- $\overline{v}(t) = 0$ for $t \leq -3$.
- $\overline{v}(t) = 1$ for small $t \geq 0$.
Define the function
\[
\tilde{P}(v, w) = P(v, 0) + 2 + \tilde{\psi}(P(v, 0))\psi_\delta(\|w\|).
\]

On the boundary of \(W'\) this equals \(P + 2\) to all orders. Indeed, either \(P(v, 0) = -3\) or \(\|w\| = \delta\) on this boundary. It also equals \(\|w\|^2\) in a neighborhood of 0, and it has
\[
\tilde{v}(\tilde{P}) = \tilde{v}(P)(1 - \tilde{\psi}(P)\psi_\delta(\|w\|)) \leq 0
\]
and equality only at points where \(\tilde{v}(P) = 0\). We also have
\[
\tilde{w}(\tilde{P}(v, w)) = \tilde{\psi}(P)\psi_\delta(\|w\|)\|w\| \geq 0
\]
and equality only when \(\tilde{\psi}(P(v, 0)) = 0\) or \(\|w\| = 0\) or at the boundary of \(W'\). It follows that
\[
\nabla Q_V = -2\tilde{v} + 2\tilde{w}
\]
is a weak pseudo gradient for \(\tilde{P}\) on \(W'\). It is not a strict pseudo gradient on \(\text{Int} W\) as the function has many critical points. However, all of its critical points in \(W'\) are by the above a closed subset in \(\text{Int} W' \cap (V \times \{0\})\), and they all have critical value 0. These can be removed by adding a \(C^1\) small function \(\psi\) with support in \(\text{Int} W\) which equals \(-\|v\|^2\) in a neighborhood of 0 and has \(\tilde{v}(\psi) < 0\) on the set of critical points. Now the gradient is non-zero on the interior of \(W' \setminus \{(0, 0)\}\) and the function equals \(Q_V\) in a neighborhood of 0. Implanting the resulting function on \(W \subset \mathbb{R}^{k+t}\) using \(\tau_\varepsilon\) and extending by \(c@Q_1^k + 2\) proves the lemma. \(\square\)

We will need to adjust the parameters \(c\) and \(\delta\) depending on already given functions \(f \in F_k^4\) for which \(e\) is already an unstable disc. To make this precise we consider the space \(\mathcal{UN}F_k\) of pairs \((\hat{e}, f) = (V, r, e, f) \in \mathcal{UN}F_k^4 \times F_k\) where \((de)_*(\tilde{v})\) is a strict pseudo gradient for \(f\) restricted to the image of \(e\).

**Corollary 7.4.** There exists a continuous function \(\delta_1(\hat{e}, f) : \mathcal{UN}F_k^4 \to (0, 1]\) such that with \(\delta \leq \delta_1(\hat{e}, f)\) and \(c \geq c_f\) we have that the convex interpolation
\[
t \mapsto (1 - t)f + t \text{MP}(\hat{e}, c, \delta)
\]
defines a path in \(F_k^4\).

**Proof.** The function \(g = \tau_\varepsilon \circ f\) has by the assumptions \(\tilde{v}\) as a pseudo gradient on \(V \times \{0\}\). Close to 0 we have \(g = Q_V\). So the function \(\nabla Q_V(g)\) is positive on \(V \times \{0\}\) and equal \(4\|\tilde{v}\|^2 + 4\|\tilde{w}\|^2\) near 0. It follows that we can use bounds on the first three derivatives to find a \(\delta_1\) so that this is positive on the compact set \(W'\) defined as in the lemma above for a \(\delta \leq \delta_1\). It follows that \((d\tau_\varepsilon)_*(\nabla Q_V)\) is a strict pseudo gradient on the interior of such \(W\). It then follows that the convex interpolation cannot have any other critical points than 0 in the interior of \(W\).

As \(c \geq c_f\) we see that \(\nabla Q_1^k\) is a weak pseudo gradient for both \(f\) and \(\text{MP}(\hat{e}, c, \delta)\) on the closure of the complement of \(W\). It follows that the convex interpolation cannot have critical points which are not convex interpolations of critical points, so as both function has all critical values above 1 on this set so will the convex interpolations. \(\square\)
8. Identification of $U_\infty \simeq F/O$.

The goal of this section is to identify $U_\infty$ from the fibration sequence in Proposition 6.5 as $F/O$. We also introduce a sequence of replacement spaces $U_k^l \to U_k^l$ (compatible with stabilizations and an equivalence in the limit) which will be convenient in the next section.

We first generalize the spaces $C_k^l$ to spaces $C(X)_k^l$, which consists of pairs $(f, X)$ with $f \in U_k$ and $X$ a vector field on $\mathbb{R}^{k+l}$ such that:

- $\|X - \nabla Q^f_k\|_{L^\infty} < \infty$.
- $X(f) \geq 0$ with equality only at critical points for $f$.

Similar to before we define

$$c_{f,X} - 1 = \|(f, X) - (Q^f_k, \nabla Q^f_k)\|_{C^1} = \max(\|f - Q^f_k\|_{C^1}, \|X - \nabla Q^f_k\|_{L^\infty})$$

and topologize $C(X)_k^l$ as a subspace of the product of two $C^\infty$ spaces and $\mathbb{R}$ identifying it with triples $(f, X, c_{f,X})$ so that the $c_{f,X}$ defines a continuous function. We again define the subspace

$$D_h C(X)_k^l \subset C(X)_k^l$$

where both the support of $f - Q^f_k$ and $X - \nabla Q^f_k$ are contained in $D_h^{k+1}$, and again this inherits the $C^\infty$ topology.

Similar to before we define the homotopy $B_t = (B^f_t, B^X_t) : C(X)_k^l \to C(X)_k^l$ by

$$B^f_t(f, X)(z) = (1 - t\varphi(\|z\| - c_{f,X}))f(z) + t\varphi(\|z\| - c_{f,X})Q^f_k(z),$$

$$B^X_t(f, X) = (1 - t\varphi(\|z\| - c_{f,X}))X + t\varphi(\|z\| - c_{f,X})\nabla Q^f_k(z)$$

with $\varphi$ as used in Equation (5.5).

**Lemma 8.1.** The homotopy $B_t : C(X)_k^l \to C(X)_k^l$ is well defined and for each $t$ the functions $B^f_t(f, X)$ and $f$ are equal near their critical points.

**Proof.** The proof is almost as the proof of Lemma 5.2 (noting that $c_{f,X} \geq c_f$). The difference is that here we also need to prove that $B^X_t(f, X)$ is actually a weak pseudo gradient for $B^f_t(f, X)$.

We only need to consider points where $\|z\| > c_{f,X}$ as otherwise $B^f_t(f, X) = f$ and $B^X_t(f, X) = X$. On this set we have $\|\nabla Q^f_k\| > 2c_{f,X}$. In the old proof we saw that $\|\nabla B^f_t(f, X) - \nabla Q^f_k\| < c_f \leq c_{f,X}$. From the definition above we have $\|X - \nabla Q^f_k\| \leq c_{f,X}$. For three vectors $a_1, a_2, b$ with $\|a_i - b\| < \frac{1}{2}\|b\|$ we have that $\langle a_i, b \rangle > 0$ and $\langle a_1, a_2 \rangle > 0$. We thus see that both $\nabla Q^f_k$ and $X$ are strict pseudo gradients for $B^f_t(f, X)$ (when $\|z\| < c_{f,X}$).

We then consider the subspace $\overline{U}_k^l \subset C(X)_k^l$ defined by pairs $(f, X)$ such that

- $f$ equals a quadratic form with eigenvalues $\pm 1$ in a neighborhood of 0.
- $X = \nabla f$ in a neighborhood of 0.

The first condition implies that the unique critical point with value 0 is actually at 0, as we saw earlier that this critical point has to have Morse index $k$.

**Lemma 8.2.** The canonical forgetful map $\overline{U}_k^l \to U_k$ is a homotopy equivalence.
As the proof is standard and distracting at this point we have moved it to the end of this section. The advantages of \( U^l_k \) is that the unstable manifold emanating from 0 is flat close to 0, and the pseudo gradient adds some needed flexibility to the space.

We define the stabilization maps \( s_- : U^l_k \to U^l_{k+1} \) and \( s_+ : U^l_k \to U^{l+1}_k \) as before on the function part and on the pseudo gradient part by

\[
s_X^-(f, X) = -2 \frac{\partial}{\partial x_{k+1}} + X \quad \text{and} \quad s_X^+(f, X) = X + 2 \frac{\partial}{\partial y_{l+1}}
\]

where we by abuse of notation also denote the relevant pull back of \( X \) to \( R^{k+l+1} \) by \( X \). With this the canonical map \( U^l_k \to U^l_k \) commutes with stabilizations.

The maps \( N_k : M_k \to Gr_k(R^{2k}) \) considered in Section 5 generalizes to maps \( N^l_k : U^l_k \to Gr_k(R^{k+l}) \) which by abuse are denoted the same. These are again defined by taking the negative eigenspace of the Hessian at the unique critical point with value 0 (or equivalently the tangent space of the unstable manifold). We define \( s_- : Gr_k(R^{k+l}) \to Gr_{k+1}(R^{1+k+l+1}) \) by

\[
s_-(V) = R \oplus V \subset R \times R^{k+l} = R^{1+k+l+1}
\]

where the new coordinate is \( x_{k+1} \) and \( s_+ : Gr_k(R^{k+l}) \to Gr_k(R^{k+l+1}) \) by

\[
s_+(V) = V \subset R^{k+l} \times R = R^{k+l+1}
\]

where the new coordinate is \( y_{l+1} \). With this stabilizations commute and both version of \( N^l_k \) commute with stabilizations.

Recall that we often denote the identity vector field on a vector space by \( \vec{v} \). Using \( X \) we may construct a canonical smooth parametrization of the unstable manifold

\[
\text{un}(X) : N^l_k(f) \to \{ f \leq 1 \}
\]

defined by the conditions

- \( \text{un}(X) \) is the linear inclusion of \( N^l_k(f) \) in a neighborhood of 0.
- \( (d \text{un}(X))(\vec{v}) = -\frac{1}{2}X \).

Note that these two are not contradictory as the inclusion locally satisfies this equation. The conditions uniquely defines \( \text{un}(X) \) as each radial half line pointing away from 0 in \( R^k \) solves a specific ODE (not defined at 0), for which the local condition around zero fixes initial conditions.

Note that this defines a map \( U^l_k \to uN^l_k \) (from Definition 7.1) by

\[
(f, X) \mapsto (N^l_k(f), c_{f,X}, \text{un}(x)).
\]

Indeed, the bound \( c_{f,X} \) on \( \| \nabla Q^l_k - X \|_{L^\infty} \) proves both of the following.

- \( -X \) is strictly inwards pointing on the boundary of \( R^k \times D^l_{c_{e,f}} \), which means that \( \text{un}(X) \) lands in this set.
- \( X \) increases \( \| x \| \) to the first order when \( \| x \| \geq c_{e,f} \).

Furthermore, \( \text{un}(X) \) is proper as there are no critical points with value less than 0 that the gradient lines can flow to.

By uniqueness we get that

\[
\text{un}(s^X_-(f, X)) : R \oplus N^l_k(f) \to \{ s_-(f) \leq 1 \}
\]
Lemma 8.3. The one point compactified map \( \text{un}(X)^+ : N^l_k(f)^+ \to \{f \leq 1\}^+ \) is a based homotopy equivalence.

Proof. This follows from standard Morse theory. Indeed, the gradient flow of \(-X\) shows that \(\{f \leq -1\}^+\) is contractible, and the same flow but instantly stopped when reaching the set \(\{f \leq -1\}\) defines a deformation retraction of \(\{f \leq 1\}^+\) onto \(\{f \leq -1\}^+ \cup e\) where \(e\) is a standard handle attachment. This standard handle attachment is represented by the unstable manifold at the critical point. \(\square\)

Definition 8.4. We let \(\tilde{\mathcal{U}}^l_k \subset \mathcal{U}^l_k\) be the subspace defined by those \((f, X)\) satisfying that the weak pseudo gradient \(X\) restricted to \(\text{im}\text{ un}(X)\) points strictly towards 0 \(\in \mathbb{R}^{k+l}\) (except at 0).

Pointing strictly towards zero at some point \(z\) means that the flow strictly decreases the norm \(\|z\|\). The condition is equivalent to \(d(\text{un}(X))(\vec{v})\) pointing away from zero. We see that the stabilizations \(s_{\pm}\) preserve these subspaces and thus their restrictions define compatible stabilizations

\[
\begin{align*}
\tilde{s}_- : \tilde{\mathcal{U}}^l_k & \to \tilde{\mathcal{U}}^l_{k+1} & \text{and} & \tilde{s}_+ : \tilde{\mathcal{U}}^l_k & \to \tilde{\mathcal{U}}^l_{k+1}.
\end{align*}
\]

In the following we identify \(S^k = (\mathbb{R}^k)^+\). Let

\[
F(k) = \text{Map}_*(S^k, S^k)_{\pm 1}
\]

denote the mapping space of based maps of degree \(\pm 1\) with monoid structure given by composition. The base point is the identity. The \(J\)-homomorphism can be represented by the homomorphisms \(J_k : O(k) \to F(k)\) sending \(A \in O(k)\) to the induced map on \((\mathbb{R}^k)^+\). The delooping of this map represents

\[
B J_k : BO(k) \to BF(k)
\]

which is the map of classifying spaces which sends a metric vector bundle to its fiberwise one point compactification.

The next goal is to show that \(N^l_k\) lifts to \(F/O\). To make this lift explicit we will use the explicit model \(\text{colim}_{k+l} F(O)^k_{k+l} / F/O\) for \(F/O\) where \((F/O)^k\) consists of pairs \((V, \theta)\) such that \(V \in \text{Gr}_k(\mathbb{R}^{k+l})\) and \(\theta : V^+ \simeq S^k\) is a based homotopy equivalence. Explicitly this is the quotient of \(F(k)\) times the highly connected Stiefel manifold of \(k\) frames in \(\mathbb{R}^{k+l}\) by \(O(k)\). It thus fits in the fibration sequence

\[
F(k) \to (F/O)^k_k \to \text{Gr}_k(\mathbb{R}^{k+l}).
\]

Let

\[
\pi_k : \mathbb{R}^{k+l} \to \mathbb{R}^k
\]

be the projection onto the \(x\) coordinates.

Lemma 8.5. The map \(\pi_k \circ \text{un}(X)^+ : (N^l_k(f))^+ \to (\mathbb{R}^k)^+\) is for each \((f, X) \in \mathcal{U}^l_k\) well defined and a homotopy equivalence. This thus defines a canonical lift

\[
\begin{align*}
\begin{array}{ccc}
(F/O)^l_k & \xrightarrow{\pi_k} & (\mathbb{R}^k)^+ \\
N^l_k & \xrightarrow{\text{un}} & \text{Gr}_k(\mathbb{R}^{k+l})
\end{array}
\end{align*}
\]
given by setting $\theta = (\pi_k \circ \text{un}(X))^+$.

Proof. By the bound $\|f - Q^l_k\| \leq c_f$ we have an inclusion $\{f \leq 1\}^+ \subset \{Q^l_k \leq c_f\}^+$. We claim that this is a homotopy equivalence. Indeed, this inclusion is defined for all $f \in \mathcal{H}_k^l$ and the homotopy types are locally constant in $f$. So, the claim follows as the inclusion at $f = Q^l_k$ is a homotopy equivalence and $\mathcal{H}_k^l$ is connected.

Using Lemma 8.3 we now see that all maps in the sequence

$$N_k(f)^+ \xrightarrow{\text{un}(X)} \{f \leq 1\}^+ \subset \{Q^l_k \leq c_f\}^+ \xrightarrow{\pi_k^+} (\mathbb{R}^k)^+.$$ are homotopy equivalences. □

The stabilizations $s_- : (F/O)^l_k \rightarrow (F/O)^{l+1}_k$ and $s_+ : (F/O)^l_k \rightarrow (F/O)^{l+1}_k$ are defined by $s_-(V, \theta) = (\mathbb{R} \oplus V, \text{id}_{\mathbb{R}} \wedge \theta)$ and $s_+(V, \theta) = (V, \theta)$ (adding only an ambient $y_{l+1}$ coordinate). So, by construction the lifts $\tilde{N}_k^l$ commute with stabilizations.

**Proposition 8.6.** The colimit of the maps

$$\mathcal{U}_\infty \xrightarrow{\tilde{N}_k^l} \text{colim}_{k,l \rightarrow \infty} (F/O)^l_k \simeq F/O \quad \text{and} \quad \tilde{U}_\infty \xrightarrow{\tilde{N}_k^l} \text{colim}_{k,l \rightarrow \infty} (F/O)^l_k \simeq F/O$$

are homotopy equivalences. In particular, $\tilde{U}_\infty \rightarrow \mathcal{U}_\infty$ is a homotopy equivalence.

Before proving this we need a few constructions and lemmas. Let $\mathcal{W}_k^l \subset \mathcal{U}_k^l$ denote the subspace of those $(f, X)$ where $f = Q^l_k$ (and thus $X = \nabla Q^l_k$) in a neighborhood of 0. We have the map of fibrations

$$\begin{array}{ccc}
\mathcal{W}_k^l & \longrightarrow & \mathcal{U}_k^l \\
\downarrow i_k^l & & \downarrow \tilde{N}_k^l \\
F(k) & \longrightarrow & (F/O)^l_k \\
\end{array}$$ \hspace{1cm} (8.3)

Where $i_k^l$ is the restriction of $\tilde{N}_k^l$.

For any smooth map $\varphi : S^{k-1} \rightarrow \mathbb{R}^k - \{0\}$ we call a map

$$\Phi : \mathbb{R}^k \rightarrow \mathbb{R}^k$$

a $b$-suspension of $\varphi$ if it satisfies

$$\Phi(x) = \|x\| \varphi(\hat{x})$$ \hspace{1cm} (8.4)

when $\|x\| \varphi(\hat{x}) \geq b$. Let $D_b F(k) \subset F(k)$ be the subspace of $b$-suspensions.

**Lemma 8.7.** The map $i_k^l$ takes $D_b \mathcal{W}_k^l$ to $D_b F(k)$.

Proof. Outside of $D_b^k$ we have $d(\text{un}(X))(\bar{v}) = -\frac{1}{2} \nabla Q^l_k$, which means the projection to $\mathbb{R}^k$ satisfies

$$d(\text{un}(X))(\bar{v}) = \bar{v}$$

which implies the lemma. □

**Lemma 8.8.** The subspace inclusion $D_b F(k) \rightarrow F(k)$ is $k-2$ connected.
Proof. Given a map \( \varphi : S^{k-1} \to S^{k-1} \) we can define its suspension as the one point compactification of \( \Phi'(x) = \|x\|\varphi(\tilde{x}) \). We claim that \( D_{k, F}(k) \) deformation retracts onto the subspace of suspensions of smooth maps. Indeed, we construct such a deformation as the concatenation of two deformation retraction: the first is induced by convexly interpolating \( \Phi \) as in Equation (8.4) to the map defined globally on \( \mathbb{R}^k \) by the same formula. The second is again induced by the same formula while continuously rescaling \( \varphi \) to actually land in \( S^{k-1} \).

This means the inclusion is equivalent to the inclusion \( F^u(k-1) \to F(k) \) where \( F^u(k-1) \) denotes unbased homotopy equivalences \( S^{k-1} \to S^{k-1} \) and the map is the suspension. This map is well known to be \( k-2 \) connected. \( \square \)

Proof of Proposition 8.6. Consider the map of fibration sequences in Equation (8.3) and the version of it where \( \overline{W}_k \to \overline{U}_k \) is replaced by the subspaces \( \overline{W}'_k \to \overline{U}'_k \). They show that we only need to prove that the corresponding fiber maps \( \overline{W}_\infty \to F = \text{colim}_{k \to \infty} F(k) \) and \( W_\infty \to F \) in the colimit are homotopy equivalences. We prove both cases simultaneously.

Surjectivity on homotopy groups: Let \( \theta_w : S^n \to F \) be a based map representing a homotopy group. That is \( \theta_w \in F \) for each \( w \in S^n \) and \( \theta_{w_0} = \text{id}_{\mathbb{R}^k} \) for \( w_0 \in S^n \) the basepoint. The goal is to construct a based family \( (f_w, X_w), w \in S^n \) in \( \overline{W}'_k \) (or \( \overline{W}_k \)), which maps to this homotopy class. As \( \pi_0(F/O) = 0 \) we may assume that \( n > 0 \). We divide the argument into three steps.

Step 1: Constructing the unstable disc maps. We will denote these \( e_w : \mathbb{R}^k \to \mathbb{R}^{k+l} \) for \( w \in S^n \), and this corresponds to a suitable lift of the map to \( S^n \to \mathbb{U}N_k' \).

By picking \( k \) to be larger than \( n+3 \) we may assume by Lemma 8.8 that the homotopy group is represented by a map \( q : S^n \to F(k) \) such that \( \theta_w \) is the one point compactification of a smooth family of \( r \) suspension maps

\[ \Phi_w : \mathbb{R}^k \to \mathbb{R}^k \]

for some chosen \( r > 1 \). We may assume that \( \Phi_{w_0} \) is the identity. We may also assume that \( \Phi_w \) is the identity close to 0.

For \( l > k+2n+1 \) we may pick a lift of this family to a family of smooth embedding

\[ e_w = (\Phi_w, \Phi_{w'} : \mathbb{R}^k \to \mathbb{R}^k \times D'^l_0 \]

such that for each \( w \in S^n \) we have

\[ de_w(\tilde{v}) = -\frac{1}{2} \nabla Q'_k \]

outside \( D^k \times D'^l_0 \). We may assume that each \( e_w \) is the standard inclusion close to 0. We may also assume that \( e_{w_0} \) is the standard inclusion globally.

In the case of \( \overline{W}'_k \) we require this lift to further satisfy the condition that \( d(e_w)(\tilde{v}) \) is outwards pointing except at 0. This may be achieved by starting with a lift \( e_w \) as above, and then considering the smooth functions \( g_{\tilde{x}, w}(t) = \|e_w(t\tilde{x})\|, t \geq 0 \) for each \( \tilde{x} \in S^{k-1} \) and \( w \in S^n \). Each of these are equal to \( t \) in a neighborhood of 0 and strictly positive elsewhere. As \( e_w \) follows the flow of \( -\frac{1}{2} \nabla Q'_k \) outside the set \( D^k \times D'^l_0 \) it follows that there is a \( t_0 > 0 \) so that \( g_{\tilde{x}, w} \) is increasing on \( t \geq t_0 \) for all \( \tilde{x} \) and \( w \). Now pick another such family of smooth functions \( \tilde{g}_{\tilde{x}, w} \) which equals \( g_{\tilde{x}, w} \) close to 0 and on \( t \geq t_0 \), but which is globally strictly increasing. We may then
replace $e_w$ by a small embedded perturbation of the map

$$x \mapsto \frac{\hat{g}_{\hat{k},w}(\|x\|)}{g_{\hat{k},w}(\|x\|)}e_w(x).$$

Convex interpolation shows that this represents the same homotopy group in $F(k)$.

Consistent with the notation from the previous section we put $e_w = (\mathbb{R}, r, e_w)$. We have thus defined a lift $\hat{e}_w : S^n \to UN^d_k$. Indeed, we made sure that $e_w$ lands in $\mathbb{R}^k \times D^l$ and has $(de_w)_*(\hat{v})$ equal to $-\frac{1}{2}Q_k^l$ outside of $D^k \times D^l$ so that it points in a direction strictly increasing $\|x\|$.

**Step 2:** Constructing an unbased lift to $\mathcal{F}^l_k$. Applying Lemma 7.3 we may use $\tilde{e}_w$ to define the family of functions $f_w = MP(\tilde{e}_w, c, \delta)$ for some $c \geq r$ and $0 < \delta \leq \delta_0(\tilde{e}_w)$ for each $w \in S^n$.

We will not argue until later that this lands in $\mathcal{W}_k^l$ (or $\mathcal{W}_k^l$). We need to also define a pseudo gradient $X_w$ for $f_w$ so that $\text{un}(X_w) = e_w$. We define it to be equal to $(d_{e_w})_*(\nabla Q_k^l)$ on a neighborhood of $\text{im} e_w \cap W$ (with $W$ from Lemma 7.3). We extend (convex interpolate) this to $\mathbb{R}^{k+l}$ using the gradient of $f_w$, which by construction is equal to $\nabla Q_k^l = -2(de_w)_*(\hat{v})$ on the rest of the image of $e_w$.

**Step 3:** Basepoint. We will prove that our lift $(f_{w_1}, X_{w_1})$ over the basepoint $w_0 \in S^n$ is homotopic to $(Q_k^l, \nabla Q_k^l)$ through maps over $e_w = \text{id}_{Q_k} \in F(k)$. This implies (in both cases) that the lift can be assumed to be a based lift, which as $S^n$ is connected even implies that $(f_w, X_w)$ actually lie in $\mathcal{W}_k^l \subset \mathcal{F}^l_k$ (or $\mathcal{W}_k^l$). As this is very similar to step 3 below where we consider a relative lift we omit it here.

**Injectivity on homotopy groups.** We assume that $(g_-, Y_-) : S^{n-1} \to \mathcal{W}_k^l$ (or $\mathcal{W}_k^l$) is mapped by $i_k$ to a map $S^{n-1} \to F(k)$, which is the restriction of a map $\theta_- : D^n \to F(k)$. We again need to lift, but this time relative the lift already given at the boundary. Again we divide the argument into three steps.

**Step 1:** Extending the family of unstable disc embeddings $\text{un}(Y_w) : \mathbb{R}^k \to \mathbb{R}^{k+l}$ to $e_w$ for each $w \in D^n$. We first stabilize to get the bounds $k > n + 3$ and $l > k + 2n + 1$ used in step 1 above. Using Lemma 5.2 we may replace $(g_-, Y_-)$ by a family that lands in some $D_{b_0} \mathcal{W}_k^l$ (or $D_{b_0} \mathcal{W}_k^l$) for some $b_0 > 0$. We then pick $r$ used in step 1 above to bigger than this $b_0$. Now every part of the construction in step 1 above carries out in this relative case, where we extend $e_w = \text{un}(g_w, Y_w)$ from $S^{n-1}$ to $D^n$.

**Step 2:** Constructing an unbased lift $(f_w, X_w)$ possibly not equal to the original $(g_w, Y_w)$ over $S^{n-1}$. We may construct $(f_w, X_w)$ with $f_w \in \mathcal{F}^l_k$ for each $w \in D^n$ precisely as step 2 above. However, for the sake of step 3 we point out that since $c \geq r > b_0$ we have that $D_{b_0}^l$ contains the part where $(g_w, Y_w)$ are not equal to $(Q_k^l, \nabla Q_k^l)$.

**Step 3:** We show that the families $(g_w, Y_w)$ and $(f_w, X_w)$ for $w \in S^{n-1}$ are homotopic through such lifts. I.e. that there is a homotopy between them preserving each unstable map $\text{un}(X_w) = \text{un}(Y_w)$. This implies that we can insert this homotopy in a little collar around $S^{n-1} \subset D^n$ to get the relative lift. It also implies that each $(f_w, X_w)$ actually lie in $\mathcal{W}_k^l$ (or $\mathcal{W}_k^l$).

We note that as the unstable disc maps agree each ray must solve the same ODE and we must have $X_w = Y_w$ on the image $\text{un}(X_w) = \text{un}(Y_w) = e_w$ for $w \in S^{n-1}$. It follows by Corollary 7.4 that we could have picked $\delta$ small enough for the convex interpolation to work as a homotopy from $g_w$ to $f_w$ in either space.
Again as the choice of pseudo gradient (fixed to be $-2\varepsilon_w(\vec{v})$ on the unstable discs) is a contractible choice we may make such a choice for the convex interpolation. □

**Proof of Lemma 8.2.** Picking the pseudo gradient is a contractible choice, so it suffices to show that the inclusion of the subspace $U^q_k \subset U_k$ defined by those $f$ which is equal to a quadratic form with eigenvalues $\pm 1$ close to 0 is a homotopy equivalence.

For each $z \in \mathbb{R}^{k+l}$ we may consider the path moving the point $z$ in a straight line to 0. Using isotopy extension we may extend the family of these paths to a family of compactly supported ambient isotopies $\psi^t_z : \mathbb{R}^{k+l} \to \mathbb{R}^{k+l}$ such that $\psi^0_z = \text{id}$. Letting $z_f$ denote the unique critical point with critical value 0 for $f \in U_k$ we may consider the homotopy given by $(f, t) \mapsto f \circ \psi^t_z$. This defines a deformation retraction of $U_k$ unto the subspace where the unique critical point is 0.

Then, similarly using a local isotopy fixing 0 that stretches the eigenspaces of the Hessian $H_f$ one may adjust the function and deformation retract onto the subspace where the Hessian has eigenvalues $\pm 1$.

Now let $f$ be in this subspace. Denote the second order approximation of $f$ at 0 by $p_2 f$ (a quadratic form with eigenvalues $\pm 1$). Let $X = \nabla f$ which we know to be transverse to 0 at 0. Let $\varphi : \mathbb{R}^{2k} \to [0, 1]$ be a bump function with compact support and equal 1 close to 0. For each $a > 0$ let $\varphi_w(z) = \varphi(az)$ and consider for large $a$ the function

$$f' = \varphi_w \cdot (p_2 f) + (1 - \varphi_w) \cdot f.$$

Applying $X$ locally yields

$$X(f') = X(\varphi_w)(p_2 f - f) + \varphi_w X(p_2 f - f) + X(f) \geq \|X\|^2 - ac_1c_2\|X\|^4 - c_3\|X\|^3$$

where $c_1 \leq \|\nabla \varphi\|$ and locally we have $\|p_2 f - f\| \leq c_2\|X\|^3$ and $X(p_2 f - f) \leq c_3\|X\|^3$. The function $f'$ equals $f$ outside a neighborhood of 0 that shrinks as $a$ increases. It follows that for $a$ large enough $X$ is a strict pseudo-gradient for $f'$.

As the bounds on $a$ only depends on the first few derivatives of $f$ in a neighborhood of 0 it can be chosen continuously depending on $f$ in the $C^\infty$ topology. Interpolating from $f$ to $f'$ thus defines a homotopy from the identity to a map landing in $U^q_k$. As this homotopy preserves $U^q_k$ as a set it follows that the inclusion is a homotopy equivalence. □

9. **Identification of the Hatcher-Waldhausen map**

In this section we identify the map $U_\infty \to \mathcal{H}_\infty$ in the colimit fibration sequence in Proposition 6.5 as the Hatcher-Waldhausen map. Thereby finishing the proof of Theorem 1.4.

In [19] Waldhausen defined the so-called tube space, and all thought we will be using a variation of a slightly different model due to Rognes in [15], we will also denote this $\mathcal{T}_\infty$. There is a classifying map for the associated spherical fibration $c : \mathcal{T}_\infty \to BF$ (recalled below). In [19] Waldhausen defines the so-called rigid tube map $rt : BO \to \mathcal{T}_\infty$, which factors the $J$-homomorphism as $J : BO \xrightarrow{\Delta} \mathcal{T}_\infty \to BF$. He also identified the homotopy fiber of the map $c : \mathcal{T}_\infty \to BF$ as the stable $h$-cobordism space of a point. We will model this homotopy fiber as a space $\mathcal{H}_F_\infty$, and prove that it is also equivalent to our $\mathcal{H}_\infty$. He then considered the map, which
is now known as the Hatcher-Waldhausen map, induced on homotopy fibers

\[
\begin{array}{c}
F/O \\ \text{hw} \\
\mathcal{H}_\infty \\
\approx \\
F/O \\
\end{array}
\begin{array}{c}
BO \\ \text{rt} \\
\mathcal{T}_\infty \\
\text{c} \\
\mathcal{B} \\
\end{array}
\begin{array}{c}
BF \\
\mathcal{F}_\infty \\
\approx \\
BF \\
\end{array}
\]

We will construct a homotopy commutative diagram

\[
\begin{array}{c}
\tilde{U}_\infty \\ \approx \\
F/O \\
\end{array}
\begin{array}{c}
\mathcal{U}_\infty \\
\mathcal{H}_\infty \\
\mathcal{H}_\infty \\
\mathcal{F}_\infty \\
\end{array}
\begin{array}{c}
\rightarrow \\
\approx \\
\rightarrow \\
\rightarrow \\
\end{array}
\]

where the top map is the colimit of the previously defined (but not named until now) maps \(w_k^l : \tilde{U}_k^l \rightarrow U_k^l \subset H_k^l\) and the left vertical map is the colimit of the maps \(\tilde{N}_k^l\) from Lemma 8.5. We start by constructing this level wise and later discuss stabilizations.

9.1. Level-wise construction. Let \(T_k^l\) be a smooth compact tubular neighborhood around \(\{0\} \times S^{l-1} \subset R_k \times R_l\). We define the tube space \(T_k^l \subset \mathcal{C}_{k+l}^k\) as the component containing those \(f\) where \(\{f = 1\}\) is isotopic to \(T_k^l\). The map \(c_k^l : T_k^l \rightarrow BF(k)\) (discussed stably above) is defined to be the classifying map for the canonical based spherical fibration that has fiber \(\{f \leq 1\}^+\) over \(f\). As we did for \((F/O)_k^l\), we model the homotopy fiber \(\mathcal{H}_k^l\) of \(c_k^l : T_k^l \rightarrow BF(k)\) as the space \(\mathcal{H}_k^l\) of pairs \((f, \eta)\) where \(f \in T_k^l\) and \(\eta : \{f \leq 1\}^+ \rightarrow S^k\) is a based homotopy equivalence. We will precisely relate \(T_k^l\) (and \(\mathcal{H}_k^l\)) to Waldhausen’s construction in the next subsection.

We now set out to define the missing maps in the diagram in Equation (9.1) at level \((k, l)\). Consider any smooth even function \(\overline{q} : R \rightarrow R\) such that

- \(\overline{q}(t) = 0\) when \(0 \leq t \leq 3\).
- \(\overline{q}'(t)\) is decreasing for \(t \geq 0\).
- \(\overline{q}''(t) = -2\) for \(t \geq t_0\) for some \(t_0 > 3\).

Lemma 9.1. The function \(\overline{q}\) satisfies that for any \(t \geq 3\) where \(|\overline{q}'(t) + 2t| \leq 1\) we have \(\overline{q}(t) + t^2 \geq 9\).

Proof. The function \(g(t) = \overline{q}(t) + t^2\) has derivative \(g'(t) = \overline{q}'(t) + 2t\). This derivative is concave and it starts out as \(2t\) for \(t \in [0, 3]\). So it has maximum value \(m \geq 6\) at some \(g(t_0) = m\). Consider its tangent with slope \(-a < 0\) at the unique point

Figure 8. Red is larger than yellow and blue is smaller than green
$t > 0$ where $g'(t) = 0$. As indicated in Figure [8] the area under the curve from the maximum to this 0 is larger than \( \frac{m^2}{3} \). Similarly the area over the curve from the 0 to where $g'(t) = -1$ is less than \( \frac{1}{4} \). It follows that $g(t) \geq g(t_0) > g(3) = 9$ on the interval $t \in [t_0, t_1]$ where $g'(t_1) = -1$. In particular $g(t) > 9$ on the subinterval where $|g'(t)| \leq 1$.

We define
\[
Q^i_k(x, y) = Q^i_k(x, y) + \varphi(\|y\|)
\]
which equals $Q^i_k$ on $\mathbb{D}^{k+1}_3$ and is close to $-\|z\|^2$ at infinity in the sense that $Q^i_k \in \mathcal{C}_k$. The above lemma shows that
\[
\|Q^i_k\| \leq 1 \Rightarrow \mathcal{Q} \geq 9 \quad \text{or} \quad \|z\| \leq 1.
\]
Using again the shorthand $c_f = \|f - Q^i_k\|_{C^1} + 1 \geq 1$ we define a map
\[
h: \mathcal{C}_k \to \mathcal{C}_k
\]
by
\[
h(f) = f + (c_f \varphi)(\|y\|) = f - Q^i_k + (c_f \mathcal{Q}^i_k).
\]
This equals $f$ on $\{\|y\| \leq 3c_f\}$ where $f$ has all its critical points. Outside of this it will have other critical points. Shrinking by $c_f^{-1}$ we see that
\[
c_f^{-1}(h(f)) = c_f^{-1} + \varphi(\|y\|) = c_f^{-1}(f - Q^i_k) + \mathcal{Q}^i_k.
\]
The first term is bounded in $C^1$ norm by 1. So by the above property of $\mathcal{Q}^i_k$ any critical point outside of $\mathbb{D}^{k+1}_3$ must have
\[
(c_f^{-1}(f - Q^i_k) + \mathcal{Q}^i_k)(z) \geq -1 + 9 > 1.
\]
Expanding back, it follows that the critical values of $h(f)$ from the part where it does not equal $f$ are strictly above $c_f^2 \geq 1$. In particular if 1 is regular for $f$ then 1 is regular for $h(f)$.

We may $\mathbb{C}^2$ perturb $\varphi$ still satisfying the above such that $\varphi(t) + t^2$ has a unique maximum at some $t_0 > 3$. This means that $\mathcal{Q}^i_k = h(Q^i_k)$ has the submanifold $\{0\} \times S^{k-1} \subset \mathbb{R}^k \times \mathbb{R}$ as a Morse-Bott critical manifold of maxima, and $\{\mathcal{Q} = 1\}$ is the boundary of its tubular neighborhood $\{\mathcal{Q} \geq 1\}$ and is thus isotopic to $\partial T^i_k$. It follows that $h$ maps $\mathcal{H}^i_k$ to $T^i_k$ (even without perturbation) and we define the map
\[
h^i_k: \mathcal{H}^i_k \to T^i_k
\]
as the restriction of the above of $h$.

**Lemma 9.2.** The inclusion $\{f \leq 1\}^\uparrow \subset \{h^i_k(f) \leq 1\}^\uparrow$ is a homotopy equivalence.

**Proof.** The functions agree on the set $\{\|y\| \leq 2c_f\}$ and the common gradient $\nabla f = \nabla h(f)$ on this set points strictly out at the boundary. All the critical points with critical values below 1 are the same and contained in this set. We therefore claim that either of the one point compactified sublevel set deformations retracts onto the one point compactification of
\[
W = \{f \leq 1\} \cap \{\|y\| \leq 2c_f\} = \{h^i_k(f) \leq 1\} \cap \{\|y\| \leq 2c_f\}.
\]
In the case of $h^i_k(f)$ let $\varphi_t$ denote the negative gradient flow. Define $t: \{h^i_k \leq 1\} \to \mathbb{R}$ by $t(z) = \text{inf}\{t \geq 0 \mid \varphi_t(z) \in W\}$. This is well defined and continuous.
Indeed, the flow is defined for all times and given a sequence \( z_n \to z \) we may use convergences on compact subsets of \([0, \infty)\) for the flow lines to argue this. The map \( d_t(z) = \varphi_{\min(t, t(z))}(z)^+ \), \( t \in [0, \infty) \) thus defines such a deformation retraction.

The argument for \( f \) is similar yet easier. \( \square \)

We lift \( h^l_k \) to a map \( \hat{h}^l_k : \mathcal{H}^l_k \to \mathcal{H}F^l_k \) by putting

\[
\hat{h}^l_k(f) = (h^l_k(f), \eta_f). \tag{9.2}
\]

where \( \eta_f : \{ h^l_k(f) \leq 1 \}^+ \to S^k \) is a choice of extension of \( \pi^+_k : \{ f \leq 1 \} \to S^k \).

**Remark 9.3.** We will use several times that extending a function from a cofibrant inclusion which is a homotopy equivalence is a contractible choice.

We will also need the analogue of \( Q_V \) in \( T^l_k \). Indeed, for each \( V \in \text{Gr}_k(\mathbb{R}^{k+l}) \) define \( \overline{Q}_V : \mathbb{R}^{k+l} \to \mathbb{R} \) by

\[
\overline{Q}_V(v, w) = -\|v\|^2 + \|w\|^2 + \theta(\|w\|) \tag{9.3}
\]

where \( (v, w) \in V \oplus V^\perp = \mathbb{R}^{k+l} \). We define the rigid tube map at level \((k, l)\)

\[
rt^l_k : \text{Gr}_k(\mathbb{R}^{k+l}) \to T^l_k
\]

by \( rt^l_k(V) = \overline{Q}_V \). As the inclusion of \( V^+ \subset \{ \overline{Q}_V \leq 1 \}^+ \) is a homotopy equivalence we have an essentially canonical homotopy between the composition of \( rt^l_k \) with \( c^l_k \) and the \( J \)-homomorphism (at level \((k, l)\)). This implies that the Hatcher Waldhausen map \( hw^l_k : (F/O)^l_k \to \mathcal{H}F^l_k \) at level \((k, l)\) can be realized explicitly as

\[
hw^l_k(V, \theta) = (rt^l_k(V), \theta_V)
\]

where \( \theta_V : \{ \overline{Q}_V \leq 1 \}^+ \to S^k \) is a global choice extending \( \theta \).

We have now defined the maps in the unstable version of the diagram from Equation [9.1]:

\[
\begin{array}{ccc}
\tilde{U}^l_k & \xrightarrow{w^l_k} & \mathcal{H}^l_k \\
\downarrow \pi^l_k & & \downarrow \hat{h}^l_k \\
(F/O)^l_k & \xrightarrow{hw^l_k} & \mathcal{H}F^l_k
\end{array}
\tag{9.4}
\]

The two ways around this diagram are now somewhat different. However, as indicated in the diagram it homotopy commutes, and the rest of this subsection is devoted to constructing this homotopy \( H_t \).

Pick a smooth increasing function \( \psi : [0, 1] \to [0, 1] \) so that \( \psi(t) = 0 \) for \( t \leq \frac{1}{4} \) and \( \psi(t) = 1 \) for \( t \geq \frac{3}{4} \). We then consider the following “shrinking/expansion” homotopy

\[
\text{un}(X)_t : N^l_k(f) \to \mathbb{R}^{k+l}
\]

from the inclusion of the tangent space \( N^l_k(f) \subset \mathbb{R}^{k+l} \) to \( \text{un}(X) \) defined by

\[
\text{un}(X)_t(v) = \psi(t)^{-1} \text{un}(X)(\psi(t)v)
\]

smoothly extended to \( \psi(t) = 0 \) by the differential of \( \text{un}(X) \) at 0 which is, indeed, the standard inclusion. The image of \( \text{un}(X)_t \) is the image of \( \text{un}(X) \) scaled by \( \psi(t)^{-1} \).

As these maps are transverse outwards pointing on all spheres this defines a map

\[
\tilde{e} : \tilde{U}^l_k \times I \to \mathcal{U}N^l_{k+l} \quad \text{(from Definition 7.1)}
\]
given by $\tilde{c}(f, X, t) = (N^l_k(f), \text{un}(X), t)$. Note that this map is locally constant in the variable $t$ for $t \in [0, \frac{1}{3}] \cup \left[\frac{2}{3}, 1\right]$. Pick a function

$$\delta: \mathcal{U}_k \times \left[\frac{1}{3}, \frac{2}{3}\right] \to (0, 1)$$

such that $\delta(f, X, \frac{1}{3}) \leq \delta_1(\text{un}(X)_0, Q_{N^l_k(f)})$ and $\delta(f, X, \frac{2}{3}) \leq \delta_1(\text{un}(X)_1, h^+_k(f))$ (using $\delta_1$ from Corollary 7.4). We consider the homotopy of mountain pass functions

$$g_t = \text{MP}(\tilde{c}(f, X, t), c_{h^+_k(f)}, \delta(f, X, t)) \quad \text{for} \quad t \in \left[\frac{1}{3}, \frac{2}{3}\right].$$

defined in Lemma 7.3. We define the function part of the homotopy $H_t = (g_t, \eta_t)$ by extending this $g_t$ as follows.

- $g_t$ is the convex interpolation from $Q_{N^l_k(f)}$ to $g_\frac{1}{3}$ for $t \in [0, \frac{1}{3}]$.
- $g_t$ is the convex interpolation from $g_\frac{2}{3}$ to $h^+_k(f)$ for $t \in \left[\frac{2}{3}, 1\right]$.

By Corollary 7.4 and the lemma above this is well defined. It lands in the component of $\mathcal{F}_{k+1}$ given by $\mathcal{T}^l_k$ as e.g. $g_0$ is in this component.

Note that the inclusions

$$\text{un}(X)_t^+ : N^l_k(f)^+ \to \{g_t \leq 1\}^+$$

are all homotopy equivalences. We thus extend the one point compactification of the map

$$\text{im } \text{un}(X)_t \xrightarrow{\text{un}(X)_t^{-1}} N^l_k(f) \xrightarrow{s^+_k \text{un}(X)} S^k$$

to all of $\{g_t \leq 1\}^+$ to define $\eta_t$.

### 9.2. Stabilizations

In this subsection we define stabilizations $s_\pm : \mathcal{T}^l_k \to \mathcal{T}^l_{k'}$ and relate the limit to the tube spaces defined by Waldhausen in [19]. We lift these stabilizations to $s_\pm : \mathcal{H}\mathcal{F}^l_k \to \mathcal{H}\mathcal{F}^l_{k'}$ and prove that it is a model for the homotopy fiber of the canonical map to $BF$. However, the maps constructed in the previous subsection will not be strictly compatible with these stabilizations. We fix this problem and lift the maps in the next subsection.

For our tube spaces $\mathcal{T}^l_k$ we define stabilizations

$$s_-(f) = -x^2_{k+1} + f$$

and

$$s_+(f) = f + y^2_{l+1} + (c_f @ \eta)(y_{l+1}). \quad (9.5)$$

Note that we have $s_- \circ s_+ = s_+ \circ s_- \text{ as } c_{s_-(f)} = c_f$. This is well defined because of the following lemma and one may check that $\{s_+(Q^l_{k}) = 1\}$ is isotopic to $T^l_{k+1}$ or $T^{l+1}_k$ depending on the sign. In fact this follows from the more general convex interpolation that we will consider in Equation (9.7).

**Lemma 9.4.** The function $s_+(f)$ has $y_{l+1} = 0$ for any critical point with critical value less than or equal to 1. In particular 1 is a regular value for $f^+$. **Proof.** As $\|f - Q^l_{k+1}\| \leq c_f$ and $\|\nabla Q^l_{k+1}\| = 2\|z\|$ we get that the gradient of $f$ is strictly inwards pointing outside of $D^l_{k+1}$ it follows that there must be a point in $D^l_{k+1}$ where $f = 1$. Indeed, otherwise $\{f = 1\}$ would be transverse to an inwards pointing vector field and hence diffeomorphic to $S^{k+l-1}$ (which $T^l_k$ is not).
As $\| \nabla f \| \leq 3c_f$ on $D^{k+l}_{\partial}$ and $f = 1$ somewhere on this disc we get that
\[ f(z) > 1 - 2c_f(3c_f), \quad \text{for } \| z \| \leq c_f. \]
Any critical point $(z, y_{l+1})$ for $f^+$ must have $|y_{l+1}| = t_0c$ where $t_0 > 3$ (by definition of $\nabla f$) or $y_{l+1} = 0$, so we need to check that those with $|y_{l+1}| = t_0c_f$ are valued above 1. As $\nabla f$ is non-zero when $\| z \| > c_f$ we can assume that $\| z \| \leq c_f$, which now yields
\[ f(z) + y_l^2 + (c \otimes \bar{q})(y_{l+1}) > (1 - 6c_f^2) + t_0^2c_f^2 + 0 > 1. \]

\[ \square \]

We will need a few facts about Waldhausen’s partition spaces (see e.g. [19]). Let $M$ be any manifold possibly with boundary, and let $P(M)$ denote the space of partition in $M \times I$. I.e. sub manifolds in $M \times (0, 1)$ that agree with $M \times \{a\}$ outside a compact set in the interior of $M \times (0, 1)$ for some $a \in (0, 1)$. Waldhausen uses simplicial sets to define this space, which controls the behavior at infinity. However, we only need the fact that each partition in a compact family only differs from $M \times \{a_k\}$ inside a joint compact set (but the $a_k$ may vary). By the relative homotopy type of a partition we mean everything on one side of the partition modulo that sets end $M \times \{0\}$. In fact one should think of a partition as an embedded cobordism from $M \times \{0\}$ to the chosen submanifold. Let $P(M)_k \subset P(M)$ denote the component of the partitions containing a single standard handle attachment.

Given any codimension 0 embedding $M \subset N$ we get an inclusion $P(M) \subset P(N)$ by extending any $M \times \{a\}$ with $(N \setminus M) \times \{a\}$.

Let $R^l_k$ denote the model for $T^l_k$ given by the $C^\infty$ space of smooth compact submanifolds $M \subset \mathbb{R}^{k+l}$ isotopic to $T^l_k$. We let $M_+$ denote the bounded smooth codimension 0 manifold with boundary $M$, and we let $M_-$ denote the unbounded manifold with boundary $M$. We have the Serre fibrant homotopy equivalence
\[ T^l_k \to R^l_k \quad \text{given by} \quad f \mapsto \{ f = 1 \} \]
such that $M_- = \{ f \leq 1 \}$ and $M_+ = \{ f \geq 1 \}$. We slightly reinterpreting Waldhausen’s model in [19] for tube spaces. His model is essentially the partition space $P(\mathbb{R}^{k+l-1})_k$. So, it consists of codimension 1 smooth submanifolds in $\mathbb{R}^{k+l}$ equal to $\mathbb{R}^{k+l-1} \times \{a\}$ outside a compact set and isotopic through such to a standard handle. However, it is more convenient for us to identify the distinguished direction with the radial direction in $\mathbb{R}^{k+l}$ and define the space $W^l_k \subset R^l_k$ as codimension 1 smooth submanifolds in $\mathbb{R}^{k+l} - \{0\} \cong S^{k+l-1} \times \mathbb{R}$ agreeing with $S^{k+l-1} \times \{r\}$ in a neighborhood of $\{s_0\} \times \mathbb{R}$ where $s_0 \in S^{k+l-1}$ is a basepoint (playing the role of $\infty$ in the comparison with Waldhausen’s model). We also impose the condition that they should be isotopic to $T^l_k$ in $\mathbb{R}^{k+l}$. Note that it is not clear if this last condition is exactly the same condition as Waldhausen’s in low dimensions (i.e. that the space $W^l_k$ is connected as his is). However, this will not matter stably.

Consider also the intermediate space
\[ W^l_k \subset (W')^l_k \subset R^l_k \]
defined by those $M$ where $M_+$ contains 0.

Waldhausen also defines two types of stabilization maps $\sigma : T^l_k \to T^l_{k+1}$ and $\overline{\sigma} : T^l_k \to T^{l+1}_k$ for partition spaces. They are described in our reinterpretation of his model for tube spaces as follows. They are in fact defined on all three spaces in Equation (9.6). The stabilization $\sigma$ is defined by crossing $M_+$ with a narrow
Lemma 9.5. The inclusions in Equation [9.6] are homotopy equivalences in the limit.

Proof. The first inclusion is highly connected as this corresponds to an induced inclusion of Waldhausen’s partition spaces. Indeed, \((W')_k^l\) correspond to a partitions in \(P(S^{k+l-1})_k\). The subspace \(W'_k\) corresponds to the subspace \(P(\mathbb{R}^{k+l-1})_k \subset P(S^{k+l-1})_k\). This inclusion is highly connected for large \(l\) and \(k\) as the connectivity of the inclusion \(D^{k+l-1} \subset S^{k+l-1}\) is and the relative homotopy type is high. Indeed, given a partition in \(P(S^{k+l-1})\) it is a highly connected choice to pick a smoothly embedded path \(I \to S^{k+l-1} \times I\) from \(s_0 \times \{0\} \to s_0 \times \{1\}\) which only intersects the partition boundary once and transversely so. Picking an isotopy to the standard path and using isotopy extension proves the connectivity.

The second inclusion is highly connected as it is a highly connected choice to pick a point in \(M\) for \(M \in \mathcal{T}^l_k\). Using such a point we may simply translate to make that point 0. \(\square\)

Lemma 9.6. There is a way to make the specific choices in Waldhausen’s stabilization maps \(\sigma\) and \(\bar{\sigma}\) and picking sections \(\mathcal{R}^l_k \to \mathcal{T}^l_k\) such that the sections commute with stabilizations.

Proof. We denote the image of the sections \(g_M\) for \(M \in \mathcal{R}^l_k\) and we shorten \(c_M = c_{g_M}\). Using such sections we may define alternate stabilization maps on \(\mathcal{R}^l_k\) by

\[
\sigma_-(M) = \{g_M - x_{k+1}^l = 1\} \quad \text{and} \quad \sigma_+(M) = \{g_M + y_{l+1}^2 + (c_M \circ \bar{\sigma}) (y_{l+1}) = 1\}.
\]

We first claim that we can make the choices of all these \(g_M\) so that \(\sigma_- \circ \sigma_+ = \sigma_+ \circ \sigma_-\).

Observe that for any choice of \(g_M\) on all of \(\mathcal{T}^l_k\) the maps \(\sigma_{\pm}\) are cofibrations. Indeed, we recover \(M\) by intersecting the new manifold \(\sigma_{\pm}(M)\) with \(\mathbb{R}^{k+l}\) - so it is injective. An open neighborhood of the image is still transverse to \(\mathbb{R}^{k+l}\) and stabilizing again provides a retraction of such a neighborhood onto the image.

We first define \(g_M\) on \(\mathcal{T}^l_0\) inductively in \(l\). By making sure that \(g_{\sigma_+(M)} = g_M + y_{l+1}^2 + (c_M \circ \bar{\sigma}) (y_{l+1})\). Note that \(c_{\sigma_+(M)} > c_M\). We then inductively in \(k\) define \(g_M\) on \(\mathcal{T}^l_k\) by making sure that \(g_{\sigma_-(M)} = g_M + x_{k+1}^2\). Note that \(c_{\sigma_-(M)} = c_M\) which is the important part of realizing why these choices now makes the stabilization maps commute.

We then prove that there are choices of \(\sigma\) and \(\bar{\sigma}\) so that they are in fact equal to \(\sigma_-\) and \(\sigma_+\) respectively.

To get \(\sigma = \sigma_+\) we consider that in the definition of \(\sigma(M)\) we smoothen the boundary of

\[
W = M_+ \times [-\varepsilon, \varepsilon] \cup \{\|z\| \geq R\}
\]

for some \(\varepsilon > 0\) and some large \(R >> \varepsilon\). To smoothen the compact boundary with corners of a smooth manifold \(W \subset \mathbb{R}^{k+l}\) means to first make the contractible choice of a smooth vector field \(Y\) transverse to \(\partial W\). Then if we let \(U\) denote the open neighborhood defined by the flow for all time of \(Y\) applied to the topological manifold we get an identification of the manifold with the smooth leaves \(U/Y\).
Hence $W$ gets a smooth structure. We then make the contractible choice of a smooth section $W \to U$ and its image is then the smoothing of $W$ (see e.g. [19]).

Let $g^+ = g_M + y_{t+1}^2 + (c_M \otimes \eta)(|y_{t+1}|)$. For $R$ very large and $\varepsilon < 1$ the gradient $\nabla g^+$ is transverse to $\partial W$ and points out of $W$ (into the bounded region $W^c$ see Figure 9). Let $U$ denote the image of the flow of $\nabla g^+$ for all time on $\partial W$. We claim that for possibly larger $R$ and smaller $\varepsilon$ there is a unique section $s : \partial W \to U$ such that the image is exactly $\sigma_+(M)$, making $\sigma_+(M)$ this particular smoothing of $\partial W$.

To prove this claim we note that for large enough $R$ we have that $g^+(z) < 0$ when $\|z\| \geq R$, and it follows that at all points in $\partial W$ we have $g^+(z) < 1 + \varepsilon^2$. By the lemma above we have that all critical points of $g^+$ with value below 1 lie in the interior of $W$. We may even assume that $\varepsilon$ is so small that $[1, 1 + \varepsilon^2]$ is regular for $g_M$ and $g^+$. It follows that for each $z \in \partial W$ there is a unique time for which the gradient flow takes $z$ to a point where $g^+ = 1$. Indeed, for $g^+(z) \in [1, 1 + \varepsilon^2]$ we can flow backwards with no critical points in the way, and for $g^+(x) < 1$ the positive gradient flow takes us into $W^c$ where we also do not meet any critical points until we reach $\{g^+ = 1\}$.

Similarly, to prove that the corresponding section surjects onto $\sigma_+(M)$ we must argue that any point in $\{g^+ = 1\}$ can be taken to a point in $\partial W$ by the flow. We also divide this argument into two cases.

Case 1: $z \in W$ implies that $z \in M_- \times [-\varepsilon, \varepsilon]$. Indeed, if $\|z\| \geq R$ we would have $g^+(z) < 1$. If $y_{t+1} = 0$ it means that $g_M = g^+ = 1$ and thus $z \in \partial W$. If $y_{t+1} \neq 0$ the gradient flow will increases $y_{t+1}$ and it follows that it eventually reaches the boundary of $M_- \times [-\varepsilon, \varepsilon]$.

Case 2: $z \notin W$ means that we can use the negative gradient flow to get to $\partial W$.

To extend the above stabilizations to $\mathcal{H}^k$ we need to make a few choices. Indeed for $(f, \eta) \in \mathcal{H}^k$ we define $s_+(f, \eta) = (s'_+(f, \eta), s''_+(f, \eta))$ where

$s''_+(f, \eta) : \{s_+(f) \leq 1\}^+ \to S^k$

is a choice of extension of $\eta : \{f \leq 1\}^+ \times \{0\} \to S^k$ using that $\{f \leq 1\}^+ \subset \{s_+(f) \leq 1\}^+$ is a homotopy equivalence. Similarly for $s_-$ where

$s''_-(f, \eta) : \{s_-(f) \leq 1\}^+ \to S^k$
is a choice of extension of \( \text{id}_R \wedge \eta : \mathbb{R}^+ \wedge \{ f \leq 1 \}^+ \to S^{k+1} \). As the stabilization maps on \( 
abla_k \) are cofibrations we may pick these extensions inductively to make the stabilizations \( s_- \) and \( s_+ \) on \( HF_k \) commute.

**Corollary 9.7.** The limit \( \mathcal{T}_\infty = \text{colim}\{k \to \infty\} \mathcal{T}_k \) is equivalent to Waldhausen’s tube space, and the space \( HF_\infty \) is a model for Waldhausen’s map to \( BF \) under this equivalence.

**Proof.** The first part follows from the above discussion and lemmas. The second follows as our map to \( BF \) is on all spaces above given by \( M \mapsto M^+_l \) or \( f \mapsto \{ f \leq 1 \}^+ \), which is the same as Waldhausen’s map. The stabilizations for this map that Waldhausen (implicitly) uses are the same fact as we used above. I.e. that the inclusion into the bigger space is a canonical based homotopy equivalence hence gives coherent systems of classification maps to \( BF \).

9.3. Lifts. In this subsection we define a larger version \( HF'_k \) of \( HF_k \) and stabilizations. We then lift the maps and homotopies from the diagram in Equation (9.4) when \( k = l \) to a diagram

\[
\begin{array}{ccc}
\mathcal{U}_k & \xrightarrow{w_k} & \mathcal{H}_k \\
\downarrow S_k & & \downarrow S_k' \\
(F/O)_k & \xrightarrow{kw'_k} & HF'_k
\end{array}
\]

that strictly commutes with stabilizations. As we at each level are doing both a positive and negative stabilization we will shorten all such stabilizations \( s = s_- \circ s_+ \). So that e.g. \( s(f) \in H_{k+1} \) for \( f \in H_k \).

To make the diagram in Equation (9.4) strictly commute with stabilizations we need to consider more flexible stabilizations. For each \( (f, \eta) \in HF_k = HF'_k \) we define a *generalized stabilization* of \( (f, \eta) \) to be a path \( \sigma = (f_u, \eta_u), u \in I \) in \( HF_{k+1} \) such that

- \( f_0 = s_-(s_+(f)) = s(f) \).
- \( \eta_0|_{\mathbb{R}^+ \wedge \{ f \leq 1 \}^+ \times \{ 0 \}} = \text{id}^+_R \wedge \eta \).

We define the stabilization associated to \( \sigma \) as \( s(f, \eta) = (f_1, \eta_1) \). We then define \( HF'_k \) to consist of infinite sequences

\[
((f, \eta), \sigma_1, \sigma_2, \ldots)
\]

such that \( \sigma_i \) is a generalized stabilization of \( \sigma_{i-1}(\cdots (\sigma_1(f, \eta))) \), and we define the stabilization \( HF'_k \to HF'_{k+1} \) by sending the above sequence to

\[
(\sigma_1(f, \eta), \sigma_2, \ldots)
\]

The forget full map \( HF'_k \to HF_k \) is a Serre fibration with contractible fibers and is thus a homotopy equivalence. It follows that the maps \( HF_k \to HF'_k \) given by setting all \( \sigma_i \) equal to the constant paths at \( s_- (s_+ (f)) \) are homotopy equivalences strictly commuting with stabilizations.

**Lemma 9.8.** We have homotopies \( \alpha_u^- : s_- \circ \tilde{h}_k^l \simeq \tilde{h}_k^l \circ s_- \) such that

\[
\mathbb{R} \times \{ f \leq 1 \} \subset \{ \alpha_u (f) \leq 1 \}
\]
for each $u \in I$. We similarly have homotopies $\alpha_u^+: s_+ \circ \tilde{h}_k^l \simeq \tilde{h}_k^{l+1} \circ s_+$ such that
\[
\{f \leq 1\} \subset \{\beta_u(f) \leq 1\}
\]
for each $u \in I$.

Proof. The function part $s_- \circ h_k^l$ and $h_k^{l+1} \circ s_-$ are equal as maps into $T_{k+1}^l$. The two homotopy equivalences might not agree. However, they are both extensions of $\pi_{k+1}^+$ on the homotopy equivalent subspace
\[
(\mathbb{R} \times \{f \leq 1\})^+ \subset \{s_-(h_k^l(f)) \leq 1\}^+ = \{h_k^{l+1}(s_+(f))\}^+.
\]
As the choice of such extensions are contractible we may pick a family (in $u$) of extension to get a homotopy between them.

In the case of $s_+ \circ h_k^l \simeq h_k^{l+1} \circ s_+$ the functions are not equal. We, however, claim that the convex interpolation
\[
F_u(x, y, y_{l+1}) = f(x, y) + y_{l+1}^2 + (1-u)\left( (c_f \circ \mathfrak{q})(\|y\|) + (cc_f \circ \mathfrak{q})(y_{l+1}) \right) + u(c_f \circ \mathfrak{q})(\|y, y_{l+1}\|).
\]
is well defined in $T_{k+1}^l$. Here $c = c_{h_k^l}/c_f \geq 1$. We need to check that 1 is regular throughout. This follows as the functions
\[
g(r) = (1-u)\left( (c_f \circ \mathfrak{q})(\|ry\|) + (cc_f \circ \mathfrak{q})(ry_{l+1}) \right) + u(c_f \circ \mathfrak{q})(\|ry, ry_{l+1}\|)
\]
for each $u$ and each $\|y\|^2 + y_{l+1}^2 = c_f^2$ satisfy the conditions listed for $\mathfrak{q}$ in the beginning of this section. Hence $g(r) \geq 9$ when $r \geq 1$ and $|g'(r)| \leq 1$ by Lemma 9.1.

It follows that outside of $D_{c_f}^{k+1}$ the corresponding term in $F_u$ has value larger than $9c_f^2$ when its gradient is shorter than $c_f$, which implies that critical values from outside $D_{c_f}^{k+1}$ (where the interpolation is not constant) are all above $9c_f^2 - c_f > 1$.

We again extend the homotopy equivalence by using that the inclusion
\[
\{f \leq 1\}^+ \subset \{F_u \leq 1\}^+
\]
is in fact a homotopy equivalence for each $u \in I$.

We lift the map $\tilde{h}_k$ to a map $\tilde{h}_k' : \mathcal{H}_k \to \mathcal{H} \mathcal{F}_k'$ by defining $\sigma_1$ using the concatenated homotopy
\[
s_- \circ s_+ \circ \tilde{h}_k' \simeq s_- \tilde{h}_{k+1}^l \circ s_+ \simeq \tilde{h}_{k+1} \circ s_- \circ s_+
\]
using $\alpha_u^+$. We then define higher $\sigma_i$ inductively by the same process - so that the maps $\tilde{h}_k'$ strictly commute with stabilizations.

Lemma 9.9. We have homotopies $\beta_u^+: s_+ \circ \text{hw}_k^l \simeq \text{hw}_{k+1}^l \circ s_+$ and $\beta_u^-: s_- \circ \text{hw}_k^l \simeq \text{hw}_{k+1}^l \circ s_-$ such that for $(V, \theta) \in (F/O)_k^l$, the sphere $V^+$ (or $(\mathbb{R} \oplus V)^+$) is contained in the level set throughout and the homotopy equivalence is constantly $\theta^+$ or $(\text{id}_\mathbb{R} \wedge \theta)^+$ on this subspace.

Proof. The proof is similar to the above. Noting that for the $s_-$ case the convex interpolation function is given by
\[
F_u(v, w, y_{l+1}) = Q_V(v, w) + y_{l+1}^2 + (1-s)(\mathfrak{q}(\|w\|) + (cQ_V \circ \mathfrak{q})(y_{l+1})) + s\mathfrak{q}(\|w, y_{l+1}\|).
\]
This is well defined in $T_{k+1}^l$ for reasons similar to the above.
We lift the map $hw_k$ to a map $hw'_k : (F/O)_k \to \mathcal{H}F^l_k$ by defining $\sigma_1$ using the concatenated homotopy
\[ s_- \circ s_+ \circ hw_k \simeq s_- hw^k_{k+1} \circ s_+ \simeq hw_{k+1} \circ s_- \circ s_+ \]
from the lemma above. We then define higher $\sigma_i$ inductively by the same process so that the maps $hw'_k$ strictly commute with stabilizations.

We similarly lift the homotopy $H_i$ to the homotopy $H'_i$ by noting that $(s \circ H_i)(f, X) = (a_t, \eta_t)$ and $(H_i \circ s)(f, X) = (a'_t, \eta'_t)$ has the same unstable manifolds maps
\[ (id_\mathbb{R} \times \text{un}(X))_i^+ : (\mathbb{R} \times N_k(f))^+ \xrightarrow{\sim} \{ a_i^+ \leq 1 \}^+ \]
Even the homotopies $\alpha^+_i$ and $\beta^+_i$ constructed above constantly has $(id_\mathbb{R} \times \text{un}(X)_0)^+$ and $(id_\mathbb{R} \times \text{un}(X)_1)^+$ as such maps. It follows by Corollary 7.4 that we can find common mountain pass functions and use these to convexly interpolate between them and extend the homotopy.

9.4. Proof of Theorem 1.4. So far we have constructed the maps in the diagram in Equation (9.1) - except that $\mathcal{H}F_{\infty}$ was replaced by $\mathcal{H}F^l_{\infty}$. We saw in Proposition 6.5 that the homotopy fiber of the top horizontal map is identified with $\mathcal{M}_{\infty}$, and we saw in Proposition 8.6 that the left most map is a homotopy equivalence. In the previous subsection we proved that the diagram is homotopy commutative. The only missing piece, which we prove in this subsection, is to prove that the limit map $\tilde{h}_{\infty} : \mathcal{H}_{\infty} \to \mathcal{H}F_{\infty}$ is a homotopy equivalence.

Let $(\mathcal{H}^1_k)_k \subset \mathcal{H}^1_k$ and $(T^1_k)_k \subset T^1_k$ be the subspaces defined by those functions where \( \{ f \leq 1 \} \) contains $\mathbb{R}^k \times \{ 0 \} \subset \mathbb{R}^k \times \mathbb{R}^l$ in its interior such that the inclusion $(\mathbb{R}^k)^+ \subset \{ f \leq 1 \}^+$ is a homotopy equivalence. We may define a map $(T^1)_k \to \mathcal{H}F^l_k$ given by sending $f$ to $(f, \eta)$ where $\eta : \{ f \leq 1 \}^+ \to (\mathbb{R}^k)^+$ is any extension of the identity on $(\mathbb{R}^k)^+$, and get a homotopy commutative diagram

\[
\begin{array}{ccc}
(\mathcal{H}^1_k)_k & \xrightarrow{\subset} & \mathcal{H}^1_k \\
\downarrow \tilde{h}^1_k & & \downarrow \tilde{h}^1_k \\
(T^1)_k & \xrightarrow{\pi^1} & \mathcal{H}F^l_k
\end{array}
\]

**Lemma 9.10.** The horizontal maps in the diagram above are at least $\frac{l-k+2}{2}$ connected.

**Proof.** In the case of $\mathcal{H}^1_k$, we have the canonical homotopy equivalence $\{ f \leq 1 \}^+ \xrightarrow{\pi^+_k} (\mathbb{R}^k)^+$. On the subspace $(\mathcal{H}^1_k)_k$, the inclusion $(\mathbb{R}^k)^+ \subset \{ f \leq 1 \}^+$ is a right inverse to this. It is a contractible choice to generally pick a based right inverse $\eta : (\mathbb{R}^k)^+ \to \{ f \leq 1 \}^+$ together with a based homotopy $h_t$ of the composition $\pi^+_k \circ \eta$ to the identity. This means that we may assume we have a based homotopy equivalence

\[ \eta : (\mathbb{R}^k)^+ \to \{ f \leq 1 \}^+ \]

for all $f \in \mathcal{H}^1_k$ such that this is the standard inclusion for $f \in (\mathcal{H}^1_k)_k$. By changing $\eta$ a bit close to infinity we may assume that these are all the standard inclusion in a neighborhood of infinity.

For any compact pair map $(A, B) \to (\mathcal{H}^1_k, (\mathcal{H}^1)_k)$ where $A$ and $B$ are CW complexes of dimension less than $\frac{l-k+1}{2}$ we can perturb the associated family $\eta_a, a \in A$
and assume that each $\eta_a$ is the one point compactifications of smooth embeddings $\mathbb{R}^k \to \mathbb{R}^{k+l}$ which are standard outside a compact set. Again we may assume that $\eta_b$ is the standard inclusion for $b \in B$.

For the dimension of $A$ and $B$ less than $\frac{l-k+2}{2}$ we may pick a smooth isotopy through such embeddings $\eta_t$ such that $\eta_0 = \eta$ and $\eta_1$ is the standard inclusion. We may again assume that this is constant on $B$. These isotopies can be extended to compactly supported isotopies of all of $\mathbb{R}^{k+l}$ (isotopy extension theorem). Applying these to the family of maps $f_a$ defines a homotopy of the map from $A$ relative to $B$ to a map sending all of $A$ to $(H^1)_k$, which proves the connectivity statement.

The case of $(T^1)_k$ is similar except the homotopy equivalence is not $\pi_k$ but some more general $\eta_t$ which on the image of $(T^1)_k$ was constructed so that the inclusion is a right inverse.

By Lemma 9.8 and the definition of the lifts $\tilde{h}_k'$ and stabilizations the diagram

\[
\begin{array}{cccc}
\mathcal{H}_k & \mathcal{H}_k & \mathcal{H}_l & \\
\downarrow & \downarrow & \downarrow & \\
\mathcal{H}_k & \mathcal{H}_k & \mathcal{H}_l & \\
\tilde{h}_k & \tilde{h}_k & \tilde{h}_l & \\
\mathcal{H}_k & \mathcal{H}_k & \mathcal{H}_l & \\
\mathcal{F}_k & \mathcal{F}_k & \mathcal{F}_l & \\
\tilde{h}_k' & \tilde{h}_k' & \tilde{h}_l' & \\
\mathcal{F}_k & \mathcal{F}_k & \mathcal{F}_l & \\
\end{array}
\]

homotopy commutes. It follows by this diagram and the lemma above that we can then replace all spaces and maps in the colimit over $k = l$ with

$$(H^1)_k \to (T^1)_k$$

where we have restricted to the subspace and forgotten the maps to $S^k$. Here the prime refers to the fact that we still need a larger space for stabilizations. Indeed, we still need to convexly interpolate functions as they do not strictly commute with stabilizations. Note that it follows from Lemma 9.8 and Lemma 9.9 that the stabilizations preserve this sequence of subspaces in $\mathcal{H}_k$ and $\mathcal{F}_k'$.

Let $(T^2)_k \subset (T^1)_k$ be the subspace where also $S^{l-1} \subset \{ f \geq 1 \}$.

We finish the argument by using a similar diagram as above and proving.

**Lemma 9.11.** The horizontal maps in the diagram above are at least $\frac{k-l+2}{2}$ connected, and the left vertical map has high connectivity for high $\min(k,l)$.

**Proof.** For $f \in (T^2)_k$, we have a canonical homotopy equivalence $\{ f \geq 1 \} \to S^{l-1}$. Given by the fact that $\{ f \geq 1 \} \subset S^{k+l} - S^k \approx S^{l-1}$. As in the proof above the inclusion of $S^{l-1}$ is a left inverse to this. So we may argue the first part as in the proof above.
For the other part pick a (nice) diffeomorphism
\[ S^k \times S^{l-1} \times (0, 1) \cong T^l_k \times (0, 1) \cong S^{k+l} \setminus (S^k \cup S^{l-1}). \]

Let \( H(M) \subset P(M) \) denote Waldhausen’s partitions that are also \( h \)-cobordisms. Using the diffeomorphism we may identify the pair \((T^2)^k,(H^2)^k\) with the pair \((H(T^2)^k),H(\mathbb{R}^k \times S^{l-1})\). Note in particular that the conditions at infinity for \( f \in \mathcal{H}_k \) and the addition of \((c_f \oplus ||y||)\) to bend down the positive direction is essentially that this cobordism is trivial over \( \{s_0\} \times S^{l-1} \).

As before, the inclusion of these partition spaces are highly connected when the inclusion inducing them are highly connected and the relative homotopy type is highly connected (in this case contractible).

\[ \square \]

\section*{Appendix A. Homotopy groups of \( \mathcal{M}_\infty \)}

In \cite{Rognes} and \cite{BlumbergMandel} Rognes computed many of the homotopy groups of \( \mathcal{H}_\infty \). In \cite{BlumbergMandel} Blumberg and Mandel improved these calculation. Using their table 1 together with the recent pre-print \cite{AbouzaidKragh} showing that \( K_8(\mathbb{Z}) = 0 \) we can extract the groups listed in Table 2. Indeed, the second column in their table is the factor of the sphere spectrum in the identification \( S \vee \text{Wh}^{\text{Diff}}(\ast) \cong K(S) \), and the table consists of the remaining columns combined and shifted by 1 since \( \mathcal{H}_\infty \cong \Omega \text{Wh}^{\text{Diff}}(\ast) \).

These are very similar to the well known homotopy groups of \( F/O \) also listed in Table 2. Indeed, the only difference is that \( \pi_9(F/O) \cong (\mathbb{Z}/2)^2 \) is missing the summand of \( \mathbb{Z}/8 \). The homotopy groups of \( \mathcal{M}_\infty \) listed above and in Table 1 now follows from the long exact sequence of homotopy groups associated to the fibration in Theorem \ref{thm:main} and Theorem 7.5 in \cite{BlumbergMandel}, which states the the Hatcher-Waldhausen map is a 2 primary equivalence in degrees less than 8 and injective in degrees less than 13.

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
\( n \) & \( \leq 1 \) & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
\( \pi_n(F/O) \) & 0 & \( \mathbb{Z}/2 \) & 0 & \( \mathbb{Z}/2 \) & 0 & \( \mathbb{Z} \oplus \mathbb{Z}/2 \) & \( (\mathbb{Z}/2)^2 \) & \( \mathbb{Z}/6 \) & & \\
\hline
\( \pi_n(\mathcal{H}_\infty) \) & 0 & \( \mathbb{Z}/2 \) & 0 & \( \mathbb{Z}/2 \) & 0 & \( \mathbb{Z} \oplus \mathbb{Z}/2 \) & \( (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/8 \) & \( \mathbb{Z}/6 \) & & \\
\hline
\( \pi_{n-1}(\mathcal{M}_\infty) \) & 0 & 0 & \( \mathbb{Z}/m_1 \) & 0 & 0 & \( \mathbb{Z}/m_2 \) & \( \mathbb{Z}/8 \) or \( \mathbb{Z}/2 \oplus \mathbb{Z}/4 \) & \( \mathbb{Z}/3 \) or 0 & & \\
\hline
\end{tabular}
\end{center}

Table 2. Homotopy groups of \( \mathcal{H}_\infty, F/O \) and \( \mathcal{M}_\infty \).

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