Noether’s Theorem for Fractional Optimal Control Problems∗

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Abstract

We begin by reporting on some recent results of the authors (Frederico and Torres, 2006), concerning the use of the fractional Euler-Lagrange notion to prove a Noether-like theorem for the problems of the calculus of variations with fractional derivatives. We then obtain, following the Lagrange multiplier technique used in (Agrawal, 2004), a new version of Noether’s theorem to fractional optimal control systems.

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1 Introduction

The concept of symmetry plays an important role both in Physics and Mathematics. Symmetries are described by transformations of the system, which result in the same object after the transformation is carried out. They are described mathematically by parameter groups of transformations. Their importance ranges from fundamental and theoretical aspects to concrete applications, having profound implications in the dynamical behavior of the systems, and in their basic qualitative properties.

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Another fundamental notion in Physics and Mathematics is the one of conservation law. Typical application of conservation laws in the calculus of variations and optimal control is to reduce the number of degrees of freedom, and thus reducing the problems to a lower dimension, facilitating the integration of the differential equations given by the necessary optimality conditions.

Emmy Noether was the first who proved, in 1918, that the notions of symmetry and conservation law are connected: when a system exhibits a symmetry, then a conservation law can be obtained. One of the most important and well known illustration of this deep and rich relation, is given by the conservation of energy in Mechanics: the autonomous Lagrangian \( L(q, \dot{q}) \), correspondent to a mechanical system of conservative points, is invariant under time-translations (time-homogeneity symmetry), and

\[
-L(q, \dot{q}) + \partial_2 L(q, \dot{q}) \cdot \dot{q} \equiv \text{constant (1)}
\]

follows from Noether’s theorem, i.e., the total energy of a conservative closed system always remain constant in time, “it cannot be created or destroyed, but only transferred from one form into another”. Expression (1) is valid along all the Euler-Lagrange extremals \( q(\cdot) \) of an autonomous problem of the calculus of variations. The conservation law (1) is known in the calculus of variations as the 2nd Erdmann necessary condition; in concrete applications, it gains different interpretations: conservation of energy in Mechanics; income-wealth law in Economics; first law of Thermodynamics; etc. The literature on Noether’s theorem is vast, and many extensions of the classical results of Emmy Noether are now available for the more general setting of optimal control (see (Torres, 2002; Torres, 2004) and references therein). Here we remark that in all those results conservation laws always refer to problems with integer derivatives.

Nowadays fractional differentiation plays an important role in various fields: physics (classic and quantum mechanics, thermodynamics, etc), chemistry, biology, economics, engineering, signal and image processing, and control theory (Agrawal et al., 2004; Hilfer, 2000; Klimek, 2002). Its origin goes back more than 300 years, when in 1695 L’Hospital asked Leibniz the meaning of \( \frac{d^n y}{dx^n} \) for \( n = \frac{1}{2} \). After that, many famous mathematicians, like J. Fourier, N. H. Abel, J. Liouville, B. Riemann, among others, contributed to the development of the Fractional Calculus (Hilfer, 2000; Miller and Ross, 1993; Samko et al., 1993).

The study of fractional problems of the Calculus of Variations and respective Euler-Lagrange type equations is a subject of current strong research. F. Riewe (Riewe, 1996; Riewe, 1997) obtained a version of the Euler-Lagrange equations for problems of the Calculus of Variations with fractional derivatives, that combines the conservative and non-conservative cases. In 2002 O. Agrawal proved a formulation for variational problems with right and left fractional derivatives in the Riemann-Liouville sense (Agrawal, 2002). Then these Euler-Lagrange equations were used by D. Baleanu and T. Avkar to investigate problems with Lagrangians which are linear on the velocities (Baleanu and Avkar, 2004). In

\[ \text{We use the notation } \partial_i f \text{ to denote the partial derivative of some function } f \text{ with respect to its } i\text{-th argument.} \]
(Klimek, 2001; Klimek, 2002a) fractional problems of the calculus of variations with symmetric fractional derivatives are considered and correspondent Euler-Lagrange equations obtained, using both Lagrangian and Hamiltonian formalisms. In all the above mentioned studies, Euler-Lagrange equations depend on left and right fractional derivatives, even when the problem depend only on one type of them. In (Klimek, 2005) problems depending on symmetric derivatives are considered for which Euler-Lagrange equations include only the derivatives that appear in the formulation of the problem. In (El-Nabulsi, 2005a; El-Nabulsi, 2005b) Riemann-Liouville fractional integral functionals, depending on a parameter \( \alpha \) but not on fractional-order derivatives of order \( \alpha \), are introduced and respective fractional Euler-Lagrange type equations obtained. More recently, the authors have used the results of (Agrawal, 2002) to generalize Noether’s theorem for the context of the Fractional Calculus of Variations (Frederico and Torres, 2006). Here we extend the previous optimal control Noether results in (Torres, 2002; Torres, 2004) to the wider context of fractional optimal control, making use (i) of the fractional version of Noether’s theorem obtained by the authors in (Frederico and Torres, 2006), (ii) and the Lagrange multiplier rule (Agrawal, 2004).

2 Fractional derivatives

In this section we collect the definitions of right and left Riemann-Liouville fractional derivatives and their main properties (Agrawal, 2002; Miller and Ross, 1993; Samko et al., 1993).

**Definition 1** Let \( f \) be a continuous and integrable function in the interval \([a, b] \). For all \( t \in [a, b] \), the left Riemann-Liouville fractional derivative \( aD^\alpha_t f(t) \), and the right Riemann-Liouville fractional derivative \( tD^\alpha_b f(t) \), of order \( \alpha \), are defined in the following way:

\[
aD^\alpha_t f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_a^t (t-\theta)^{n-\alpha-1} f(\theta) d\theta, \tag{2}
\]

\[
tD^\alpha_b f(t) = \frac{1}{\Gamma(n-\alpha)} \left( -\frac{d}{dt} \right)^n \int_t^b (t-\theta)^{n-\alpha-1} f(\theta) d\theta, \tag{3}
\]

where \( n \in \mathbb{N}, n - 1 \leq \alpha < n \), and \( \Gamma \) is the Euler gamma function.

**Remark 2** If \( \alpha \) is an integer, then from (2) and (3) one obtains the standard derivatives, that is,

\[
aD^\alpha_t f(t) = \left( \frac{d}{dt} \right)^\alpha f(t),
\]

\[
tD^\alpha_b f(t) = \left( -\frac{d}{dt} \right)^\alpha f(t).
\]
**Theorem 3** Let \( f \) and \( g \) be two continuous functions on \([a, b] \). Then, for all \( t \in [a, b] \), the following properties hold:

1. for \( p > 0 \),
   \[
   aD^p_t(f(t) + g(t)) = aD^p_t f(t) + aD^p_t g(t);
   \]

2. for \( p \geq q \geq 0 \),
   \[
   aD^p_t(aD^{-q}_t f(t)) = aD^{p-q}_t f(t);
   \]

3. for \( p > 0 \),
   \[
   aD^p_t(aD^{-p}_t f(t)) = f(t)
   \]

   (fundamental property of the Riemann-Liouville fractional derivatives);

4. for \( p > 0 \),
   \[
   \int_a^b (aD^p_t f(t)) g(t) \, dt = \int_a^b f(t) aD^p_t g(t) \, dt.
   \]

**Remark 4** In general, the fractional derivative of a constant is not equal to zero.

**Remark 5** The fractional derivative of order \( p > 0 \) of function \((t-a)^v, v > -1\), is given by

\[
 aD^p_t(t-a)^v = \frac{\Gamma(v+1)}{\Gamma(-p+v+1)}(t-a)^{v-p}.
\]

**Remark 6** In the literature, when one reads “Riemann-Liouville fractional derivative”, one usually means the “left Riemann-Liouville fractional derivative”. In Physics, if \( t \) denotes the time-variable, the right Riemann-Liouville fractional derivative of \( f(t) \) is interpreted as a future state of the process \( f(t) \). For this reason, the right-derivative is usually neglected in applications, when the present state of the process does not depend on the results of the future development. Following (Agrawal, 2004), in this work we only consider problems with left Riemann-Liouville fractional derivatives. Using (Frederico and Torres, 2006), the results of the paper can, however, be written for the case when both left and right fractional derivatives are present.

We refer the reader interested in additional background on fractional theory, to the comprehensive book (Samko et al., 1993).

## 3 Preliminaries

In (Agrawal, 2002) a formulation of the Euler-Lagrange equations is given for problems of the calculus of variations with fractional derivatives.

Let us consider the following fractional problem of the calculus of variations: to find function \( q(\cdot) \) that minimizes the integral functional

\[
 I[q(\cdot)] = \int_a^b L(t, q(t), aD^\alpha_t q(t)) \, dt,
\]
where the Lagrangian \( L : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) is a \( C^2 \) function with respect to all its arguments, and \( 0 < \alpha \leq 1 \).

**Remark 7** In the case \( \alpha = 1 \), problem \((4)\) is reduced to the classical problem
\[
I[q(\cdot)] = \int_a^b L(t, q(t), \dot{q}(t)) \, dt \longrightarrow \min.
\]

**Theorem 8 ((Agrawal, 2002))** If \( q \) is a minimizer of problem \((4)\), then it satisfies the fractional Euler-Lagrange equations:
\[
\partial_2 L(t, q, a_\alpha D^\alpha_t q) + \frac{t}{D^\alpha_b} \partial_3 L(t, q, a_\alpha D^\alpha_t q) = 0.
\]

The following definition is useful in order to introduce an appropriate concept of fractional conservation law.

**Definition 9 ((Frederico and Torres, 2006))** Given two functions \( f \) and \( g \) of class \( C^1 \) in the interval \([a, b]\), we introduce the following notation:
\[
\mathcal{D} \{ (fg)^\alpha_t \} = -g_\alpha D^\alpha_t f + f_\alpha D^\alpha_t g,
\]
where \( t \in [a, b] \).

**Remark 10** For \( \alpha = 1 \) operator \( \mathcal{D} \) is reduced to
\[
\mathcal{D} \{ (fg)^1_t \} = -g_1 D^1_t f + f_1 D^1_t g = \dot{f}g + f \dot{g} = \frac{d}{dt} (fg).
\]

**Remark 11** The linearity of the operators \( a_\alpha D^\alpha_t \) and \( \frac{t}{D^\alpha_b} \) imply the linearity of the operator \( \mathcal{D} \).

**Definition 12 ((Frederico and Torres, 2006))** We say that \( C_f(t, q, a_\alpha D^\alpha_t q) \), where \( C_f \) has the form of a sum of products
\[
C_f(t, q, d) = \sum_1^2 C^1_i(t, q, d) \cdot C^2_i(t, q, d)
\]
is a fractional conservation law if, and only if,
\[
\mathcal{D} \{ C_f(t, q, a_\alpha D^\alpha_t q) \} = 0 \quad (7)
\]
along all the fractional Euler-Lagrange extremals (i.e. along all the solutions of the fractional Euler-Lagrange equations \((5)\)).

**Remark 13** For \( \alpha = 1 \) \((7)\) is reduced to
\[
\frac{d}{dt} \{ C_f(t, q(t), \dot{q}(t)) \} = 0 \iff C_f(t, q(t), \dot{q}(t)) \equiv \text{constant},
\]
which is the standard meaning of conservation law: a function \( C_f(t, q, \dot{q}) \) preserved along all the Euler-Lagrange extremals \( q(t), t \in [a, b] \), of the problem. We also note that standard \((\alpha = 1)\) Noether’s conservation laws are always a sum of products, as we are assuming in \((4)\).
Definition 14 ((Frederico and Torres, 2006)) Functional (4) is said to be invariant under the one-parameter group of infinitesimal transformations

\[
\begin{align*}
\bar{t} &= t + \varepsilon \tau(t, q) + o(\varepsilon), \\
\bar{q}(\bar{t}) &= q(t) + \varepsilon \xi(t, q) + o(\varepsilon),
\end{align*}
\]

if, and only if,

\[
\int_{t_a}^{t_b} L(t, q(t), aD^\alpha_t q(t)) \, dt = \int_{\tilde{t}(t_a)}^{\tilde{t}(t_b)} L(\bar{t}, \bar{q}(\bar{t}), \bar{a}D^\alpha_{\bar{t}} \bar{q}(\bar{t})) \, d\bar{t}
\]

for any subinterval \([t_a, t_b] \subseteq [a, b]\).

Remark 15 Having in mind that condition (9) is to be satisfied for any subinterval \([t_a, t_b] \subseteq [a, b]\), we can rid off the integral signs in (9) (cf. Definition 22).

The next theorem provides the extension of Noether’s theorem for Fractional Problems of the Calculus of Variations.

Theorem 16 ((Frederico and Torres, 2006)) If functional (4) is invariant under (8), then

\[
[L(t, q, aD^\alpha_t q) - \alpha \partial_3 L(t, q, aD^\alpha_t q) \cdot aD^\alpha_t q] \tau(t, q) + \partial_3 L(t, q, aD^\alpha_t q) \cdot \xi(t, q)
\]

is a fractional conservation law (cf. Definition 12).

4 Main Result

Using Theorem 16 we obtain here a formulation of Noether’s Theorem for the fractional optimal control problems introduced in (Agrawal, 2004):

\[
I[q(\cdot), u(\cdot)] = \int_a^b L(t, q(t), u(t)) \, dt \rightarrow \min,
\]

together with some boundary conditions on \(q(\cdot)\) (which are not relevant with respect to Noether’s theorem). In problem (10), the Lagrangian \(L: [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}\) and the velocity vector \(\varphi: [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n\) are assumed to be \(C^1\) functions with respect to all the arguments. In agreement with the calculus of variations, we also assume that the admissible control functions take values on an open set of \(\mathbb{R}^m\).

Definition 17 A pair \((q(\cdot), u(\cdot))\) satisfying the fractional control system \(aD^\alpha_t q(t) = \varphi(t, q(t), u(t))\) of problem (10), \(t \in [a, b]\), is called a process.

Theorem 18 (cf. (13)-(15) of (Agrawal, 2004)) If \((q(\cdot), u(\cdot))\) is an optimal process for problem (10), then there exists a co-vector function \(p(\cdot)\) such that the following conditions hold:
• the Hamiltonian system
\[
\begin{align*}
\alpha D_\alpha q(t) &= \partial_1 \mathcal{H}(t, q(t), u(t), p(t)), \\
\beta p(t) &= \partial_2 \mathcal{H}(t, q(t), u(t), p(t)).
\end{align*}
\]

• the stationary condition
\[
\partial_3 \mathcal{H}(t, q(t), u(t), p(t)) = 0;
\]
with the Hamiltonian $\mathcal{H}$ defined by
\[
\mathcal{H}(t, q, u, p) = L(t, q, u) + p \cdot \varphi(t, q, u).
\]

Remark 19 In classical mechanics, the Lagrange multiplier $p$ is called the generalized momentum. In the language of optimal control, $p$ is known as the adjoint variable.

Definition 20 Any triplet $(q(\cdot), u(\cdot), p(\cdot))$ satisfying the conditions of Theorem \ref{thm18} will be called a fractional Pontryagin extremal.

For the fractional problem of the calculus of variations \ref{eq:10} one has $\varphi(t, q, u) = u \Rightarrow \mathcal{H} = L + p \cdot u$, and we obtain from Theorem \ref{thm18} that
\[
\begin{align*}
\alpha D_\alpha q &= u, \\
\beta p &= \partial_2 L, \\
\partial_3 \mathcal{H} &= 0 \Leftrightarrow p = -\partial_3 L \Rightarrow \beta p = -\beta D_\alpha \partial_3 L.
\end{align*}
\]
Comparing the two expressions for $\beta D_\alpha p$, one arrives to the Euler-Lagrange differential equations \ref{eq:5}:
\[
\partial_2 L = -\beta D_\alpha \partial_3 L.
\]

We define the notion of invariance for problem \ref{eq:10} in terms of the Hamiltonian, by introducing the augmented functional as in (Agrawal, 2004):
\[
J[q(\cdot), u(\cdot), p(\cdot)] = \int_a^b \left[ \mathcal{H}(t, q(t), u(t), p(t)) - p(t) \cdot \alpha D_\alpha q(t) \right] dt,
\]
where $\mathcal{H}$ is given by \ref{eq:11}.

Remark 21 Theorem \ref{thm18} is obtained applying the necessary optimality condition \ref{eq:5} to problem \ref{eq:12}.

Definition 22 A fractional optimal control problem \ref{eq:10} is said to be invariant under the $\varepsilon$-parameter local group of transformations
\[
\begin{align*}
\tilde{t} &= t + \varepsilon \tau(t, q(t), u(t), p(t)) + o(\varepsilon), \\
\tilde{q} &= q(t) + \varepsilon \xi(t, q(t), u(t), p(t)) + o(\varepsilon), \\
\tilde{u} &= u(t) + \varepsilon \sigma(t, q(t), u(t), p(t)) + o(\varepsilon), \\
\tilde{p} &= p(t) + \varepsilon \zeta(t, q(t), u(t), p(t)) + o(\varepsilon),
\end{align*}
\]
if, and only if,

\[
[H(\bar{t}, \bar{q}(\bar{t}), \bar{u}(\bar{t}), \bar{p}(\bar{t})) - \bar{p}(\bar{t}) \cdot _aD_t^\alpha \bar{q}(\bar{t})] \, dt = [H(t, q(t), u(t), p(t)) - p(t) \cdot _aD_t^\alpha q(t)] \, dt .
\]

(14)

**Theorem 23 (Fractional Noether’s theorem)** If the fractional optimal control problem (10) is invariant under (13), then

\[
[H - (1 - \alpha) p(t) \cdot _aD_t^\alpha q(t)] \tau - p(t) \cdot \xi
\]

is a fractional conservation law, that is,

\[
D \{[H - (1 - \alpha) p(t) \cdot _aD_t^\alpha q(t)] \tau - p(t) \cdot \xi\} = 0
\]

along all the fractional Pontryagin extremals.

**Remark 24** For \(\alpha = 1\) the fractional optimal control problem (10) is reduced to the classical optimal control problem

\[
I[\{q(\cdot), u(\cdot)\}] = \int_a^b L(t, q(t), u(t)) \, dt \rightarrow \min , \\
\dot{q}(t) = \varphi(t, q(t), u(t)) ,
\]

and we obtain from Theorem (16) the optimal control version of Noether’s theorem (Torres, 2002): invariance under a one-parameter group of transformations (13) imply that

\[
C(t, q, u, p) = H(t, q, u, p)\tau - p \cdot \xi
\]

is constant along any Pontryagin extremal (one obtains (16) from (15) setting \(\alpha = 1\)).

**Proof.** The fractional conservation law (15) is obtained applying Theorem (16 to the augmented functional (12).

5 An Example

Let us consider the autonomous fractional optimal control problem, i.e. the particular situation when the Lagrangian \(L\) and the fractional velocity vector \(\varphi\) do not depend explicitly on time \(t\):

\[
I[\{q(\cdot), u(\cdot)\}] = \int_a^b L(q(t), u(t)) \, dt \rightarrow \min , \\
_aD_t^\alpha q(t) = \varphi(q(t), u(t)) .
\]

(17)

For the autonomous fractional problem (17) the Hamiltonian \(H\) does not depend explicitly on time, and it is a simple exercise to check that (17) is invariant.
under time-translations: invariance condition (14) is satisfied with $\tilde{t} = t + \varepsilon$, $\tilde{q}(\tilde{t}) = q(t)$, $\tilde{u}(\tilde{t}) = u(t)$ and $\tilde{p}(\tilde{t}) = p(t)$. In fact, given that $d\tilde{t} = dt$, (14) holds trivially proving that $aD_t^\alpha \tilde{q}(\tilde{t}) = aD_t^\alpha q(t)$:

$$aD_t^\alpha \tilde{q}(\tilde{t})$$

$$= \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_{\alpha}^{\tilde{t}} (\tilde{t} - \theta)^{n-\alpha-1} \tilde{q}(\theta) d\theta$$

$$= \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_{\alpha+\varepsilon}^{\tilde{t}+\varepsilon} (t + \varepsilon - \theta)^{n-\alpha-1} \tilde{q}(\theta) d\theta$$

$$= \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{ds} \right)^n \int_{a}^{\tilde{t}} (t - s)^{n-\alpha-1} \tilde{q}(s + \varepsilon) ds$$

$$= aD_{t+\varepsilon}^\alpha \tilde{q}(t+\varepsilon) = aD_t^\alpha \tilde{q}(\tilde{t})$$

$$= aD_t^\alpha q(t) .$$

Using the notation in (13) one has $\tau = 1$ and $\xi = \sigma = \zeta = 0$. It follows from our fractional Noether’s theorem (Theorem 18) that

$$\mathcal{D} \{ \mathcal{H} - (1-\alpha) p(t) \cdot aD_t^\alpha q(t) \} = 0 .$$

In the classical framework of optimal control theory $\alpha = 1$ and our operator $\mathcal{D}$ coincides with $\frac{d}{dt}$: we then get from (18) the well known fact that the Hamiltonian is a preserved quantity along any Pontryagin extremal.

6 Conclusions

The proof of fractional Euler-Lagrange equations is a subject of strong current study (Riewe, 1996; Riewe, 1997; Klinek, 2001; Agrawal, 2002; Klimek, 2002a; Baleanu and Avkar, 2004; Klinek, 2005; El-Nabulsi, 2005a; El-Nabulsi, 2005b) because of its numerous applications. In (Frederico and Torres, 2006) a fractional Noether’s theorem is proved.

The fractional variational theory is in its childhood so that much remains to be done. This is particularly true in the area of fractional optimal control, where the results are rare. A fractional Hamiltonian formulation is obtained in (Muslih and Baleanu, 2005), but only for systems with linear velocities. The main study of fractional optimal control problems seems to be (Agrawal, 2004), where the Euler-Lagrange equations for fractional optimal control problems (Theorem 18) are obtained, using the traditional approach of the Lagrange multiplier rule. Here we use the Lagrange multiplier technique to derive, from the results in (Frederico and Torres, 2006), a Noether-type theorem for fractional optimal control systems. As an example we have considered a fractional autonomous problem, proving that the Hamiltonian defines a conservation law only in the case $\alpha = 1$. 

9
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