Optimal control on finite graphs: a reference case

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Abstract

For optimal control problems on finite graphs in continuous time, the dynamic programming principle leads to value functions characterized by systems of nonlinear ordinary differential equations. In this paper, we exhibit a family of such optimal control problems for which these nonlinear equations can be transformed into linear ones thanks to a change of variables. As a consequence, the value function associated with an optimal control problem of that family can be written in (almost-)closed form and the optimal controls characterized and computed easily. Furthermore, when the graph is (strongly) connected, we show that the asymptotic analysis of such a problem – in particular the computation of the ergodic constant – can be carried out with classical tools of matrix analysis.

Key words: Optimal control, Graphs, Asymptotic analysis, Matrix analysis.

1 Introduction

In many fields of mathematics and physics, the very existence of special cases that are easy to solve is often regarded as a blessing. These reference cases often serve indeed to better understand a more general problem or to illustrate general results. They are useful to test ideas while carrying out research. They are also extremely useful for building approximation algorithms and numerical methods: one can indeed test a general algorithm on some reference cases or “approximate” a problem by another one from the set of reference cases in order to initialize an iterative approximation algorithm.

In the vast set of problems tackled thanks to the tools of optimal control theory, linear-quadratic ones in $\mathbb{R}^d$ are known to be among the easiest to solve because they boil down to Riccati equations. In the case of the optimal control of continuous-time Markov chains on discrete state spaces, or equivalently optimal control on finite graphs in continuous time, there was no such well-known nontrivial family of cases that are easier to solve (before that of the present paper). There are several reasons for that. First, continuous-time optimal control on finite graphs can be regarded as a segment of the larger field of optimal control of point processes which has always drawn little attention from academics beyond the seminal work of Brémaud (see for instance [2, 1] in spite of numerous applications (see for instance the literature on market making [3, 6, 7]). Second, given that one typically controls the intensity of point processes with, as for any intensity, a constraint of nonnegativity, the nonlinearity in optimal control problems on finite graphs is more complicated than that of linear-quadratic problems in $\mathbb{R}^d$. Third, in spite of the discrete nature of the problem, the intricate topology and geometry of graphs (compared to that of Euclidian spaces) leaves, a priori, little hope of obtaining something comparable to linear-quadratic optimal control problems.

In this short paper, we nonetheless exhibit a large family of continuous-time optimal control problems on finite graphs for which the solution can be computed in (almost-)closed form.

For optimal control problems where reward/cost functions depend on transition intensities through some form of entropy (see Section 2 for a precise statement), the well-known duality between exponential and entropy leads indeed to a Hamilton-Jacobi equation for the value function that can be transformed into a linear system of ordinary differential equations. In particular, the value function

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The paper [8] provides general results in the case of finite graphs. Beyond optimal control, the case of mean-field games has also been addressed in [9].
and the optimal controls can be computed very easily using matrix exponentiation. Moreover, when the graph is connected, one can characterize the long-run behavior of the value function and the optimal controls in such problems using classical tools of matrix analysis (see \cite{8} for general results on the long-run behavior in optimal control problems on connected finite graphs). In particular, the ergodic constant in such optimal control problems can be characterized in a very simple manner using spectral theory.

In Section 2 we introduce the notations and the family of optimal control problems considered throughout the paper. In Section 3, we solve in (almost-)closed form the Hamilton-Jacobi equations associated with these optimal control problems and derive both the value functions and the optimal controls. In Section 4, in the particular case of a connected finite graph, we study the long-run behavior of the value function and optimal controls and derive a spectral characterization of the ergodic constant.

\section{Notations and description of the optimal control problems}

Let $T > 0$. Let \( \left( \Omega, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P} \right) \) be a filtered probability space, with \( (\mathcal{F}_t)_{t \in [0,T]} \) satisfying the usual conditions. We assume that the stochastic processes introduced in this paper are defined on \( \Omega \) and adapted to the filtration \((\mathcal{F}_t)_{t \in [0,T]}\).

We consider a finite directed graph \( G \) with no self-loop (i.e. there is no edge connecting a node to itself). The set of nodes is denoted by \( N = \{1, \ldots, N\} \) where \( N \geq 2 \) is an integer. For each node \( i \in N \), we introduce \( V(i) \subset N \setminus \{i\} \) the neighborhood of the node \( i \), i.e. the set of nodes \( j \) for which a directed edge exists from \( i \) to \( j \).

We consider an agent evolving on the graph \( G \). This agent can choose the values of the transition intensities. At any time \( t \in [0, T] \), transition intensities, i.e. instantaneous probabilities of transition, are described by a collection of feedback control functions \( (\lambda_t(i, \cdot))_{i \in N} \) where \( \lambda_t(i, \cdot) : V(i) \to \mathbb{R}_+ \).

We assume that the controls are in the admissible set \( A_T^i \) where, for \( t \in [0, T] \),

\[
A_T^i = \{(\lambda_t(i, j))_{s \in [t, T], i \in N, j \in V(i)} \text{ nonnegative, } \mathcal{F}_t\text{-measurable}\} \quad \forall i \in N, \forall j \in V(i), s \mapsto \lambda_s(i, j) \in L^1(t, T)\}

We assume that the instantaneous cost of the agent located at node \( i \) is described by a function

\[
L(i, \cdot) : (\lambda_t)_{j \in V(i)} \in \mathbb{R}^{|V(i)|} \mapsto L \left( i, (\lambda_t)_{j \in V(i)} \right) = -r(i) + \sum_{j \in V(i)} (\lambda_{ij} \log(\lambda_{ij}) + b_{ij} \lambda_{ij}),
\]

where \( |V(i)| \) is the cardinal of \( V(i) \), \( r : N \to \mathbb{R} \) a (real) reward function, and \( (b_{ij})_{i \in N, j \in V(i)} \) a family of real numbers.

At time \( T \), we consider a terminal reward for the agent. This reward depends on his position in the graph and is modelled by a function \( g : N \to \mathbb{R} \).

Let us denote by \((X^{i, t, \lambda}_s)_{s \in [t, T]}\) the continuous-time Markov chain on \( G \) starting from node \( i \) at time \( t \), with transition intensities given by \( \lambda \in A_T^i \).

Starting from a given node \( i \) at time \( t = 0 \), the optimal control problem we consider is the following:

\[
\sup_{\lambda \in A_T^i} \mathbb{E} \left[ -\int_0^T L \left( X^{0, t, \lambda}_s, \lambda_t \left( X^{0, t, \lambda}_s, j \right) \right)_{j \in V(i)} \left( X^{0, t, \lambda}_s \right) dt + g \left( X^{0, t, \lambda}_T \right) \right].
\]

For each node \( i \in N \), the value function of the agent, at state \( i \), is defined as

\[
u^T_i(t) = \sup_{\lambda \in A_T^i} \mathbb{E} \left[ -\int_t^T L \left( X^{i, t, \lambda}_s, \lambda_s \left( X^{i, t, \lambda}_s, j \right) \right)_{j \in V(i)} \left( X^{i, t, \lambda}_s \right) ds + g \left( X^{i, t, \lambda}_T \right) \right].
\]

Our goal in the next Section is to compute that value function in (almost-)closed form and to deduce the optimal controls.

\footnote{Throughout this paper, connected means strongly connected. In other words, for any two nodes there exists a path from the first to the second and a path from the second to the first.}

\footnote{Throughout this paper, the function \( x \in \mathbb{R}^+ \mapsto x \log(x) \) is prolonged by continuity to \( x = 0 \) (by the value 0).}
3 Solution of the associated Hamilton-Jacobi equation and derivation of the optimal controls

The Hamilton-Jacobi equation associated with the above optimal control problem is
\[
\forall i \in \mathcal{N}, \quad \frac{d}{dt} V_i^T(t) = - \sup_{(\lambda_{ij}) \in V(i) \in \mathbb{R}^{|V(i)|}} \left( \left( \sum_{j \in V(i)} \lambda_{ij} \left(V_j^T(t) - V_i^T(t)\right) \right) - L \left(i, (\lambda_{ij})_{j \in V(i)}\right) \right),
\]
with terminal condition
\[
\forall i \in \mathcal{N}, \quad V_i^T(T) = g(i).
\]

Let us define for all \( i \in \mathcal{N} \) the Hamiltonian function associated with the cost function \( L(i, \cdot) \):
\[
H(i, \cdot) : \mathbb{R}^{|V(i)|} \rightarrow \mathbb{R}
\]
\[
(p_{ij})_{j \in V(i)} \mapsto \sup_{(\lambda_{ij}) \in V(i) \in \mathbb{R}^{|V(i)|}} \left( \left( \sum_{j \in V(i)} \lambda_{ij} p_{ij} \right) - L \left(i, (\lambda_{ij})_{j \in V(i)}\right) \right).
\]

Given the specific entropy-like form chosen for the cost functions, it is straightforward to compute the Hamiltonian functions in closed form. This is the purpose of the following lemma.

**Lemma 1.** \( \forall i \in \mathcal{N}, \forall p = (p_{ij})_{j \in V(i)} \in \mathbb{R}^{|V(i)|}, \)
\[
H(i, p) = r(i) + \sum_{j \in V(i)} e^{-1-b_{ij}} e^{p_{ij}}.
\]

Moreover, the supremum in the definition of \( H(i, p) \) is a maximum, reached when
\[
\forall j \in V(i), \quad \lambda_{ij} = e^{-1-b_{ij}} e^{p_{ij}}.
\]

**Proof.** Let us consider \( i \in \mathcal{N} \) and \( p = (p_{ij})_{j \in V(i)} \in \mathbb{R}^{|V(i)|} \).

The function
\[
(\lambda_{ij})_{j \in V(i)} \in \mathbb{R}^{|V(i)|} \mapsto \left( \sum_{j \in V(i)} \lambda_{ij} p_{ij} \right) - L \left(i, (\lambda_{ij})_{j \in V(i)}\right)
\]
is concave. Its gradient vanishes whenever
\[
\forall j \in V(i), \quad p_{ij} - \log(\lambda_{ij}) - 1 - b_{ij} = 0.
\]
Therefore, the supremum in the definition of \( H(i, p) \) is in fact a maximum, reached when
\[
\forall j \in V(i), \quad \lambda_{ij} = e^{-1-b_{ij}} e^{p_{ij}},
\]
and
\[
H(i, p) = r(i) + \left( \sum_{j \in V(i)} p_{ij} e^{-1-b_{ij}} e^{p_{ij}} \right) - \left( \sum_{j \in V(i)} e^{-1-b_{ij}} e^{p_{ij}} (p_{ij} - 1 - b_{ij}) + b_{ij} e^{-1-b_{ij}} e^{p_{ij}} \right)
\]
\[
= r(i) + \sum_{j \in V(i)} e^{-1-b_{ij}} e^{p_{ij}}.
\]

Using the above lemma, we see that the Hamilton-Jacobi equation (2) associated with Problem (1) writes
\[
\forall i \in \mathcal{N}, \quad \frac{d}{dt} V_i^T(t) = -r(i) - \sum_{j \in V(i)} e^{-1-b_{ij}} \exp \left( V_j^T(t) - V_i^T(t)\right),
\]
with terminal condition
\[
\forall i \in \mathcal{N}, \quad V_i^T(T) = g(i).
\]
Eq. (3) with terminal condition (4) can be solved in (almost-)closed form. This is the purpose of the next theorem.
Theorem 1. Let $B = (B_{ij})_{(i,j) \in \mathbb{N}^2}$ be the matrix defined by

$$B_{ij} = \begin{cases} 
  e^{-1-b_{ij}}, & \text{if } j \in \mathcal{V}(i), \\
  r(i), & \text{if } j = i, \\
  0, & \text{otherwise}.
\end{cases}$$

Let $\mathbf{g}$ be the column vector $(e^{g(1)}, \ldots, e^{g(N)})^T$.

Then, the function $w^T : t \in [0, T] \mapsto w^T(t) = e^{B(T-t)} \mathbf{g}$ verifies

$$\forall i \in \mathcal{N}, \forall t \in [0, T], \quad w_i^T(t) > 0$$

and

$$v^T : t \in [0, T] \mapsto \left(\log(w_i^T(t))\right)_{i \in \mathcal{N}}$$

defines a solution to Eq. 47 with terminal condition 48.

Proof. Let us consider a constant $\sigma > -\min_{i \in \mathcal{N}} r(i)$. By definition, $B + \sigma I_N$ is a nonnegative matrix and so is $e^{(B+\sigma I_N)(T-t)} - I_N$ for all $t \in [0, T]$. Since $\mathbf{g}$ has positive coefficients, the vector $e^{\sigma(T-t)}w_i^T(t) = e^{(B+\sigma I_N)(T-t)} \mathbf{g} = \mathbf{g} + \left(e^{(B+\sigma I_N)(T-t)} - I_N\right) \mathbf{g}$ has positive coefficients for all $t \in [0, T]$. We deduce that

$$\forall i \in \mathcal{N}, \forall t \in [0, T], \quad w_i^T(t) > 0.$$ 

From the above positiveness result, $v^T$ is well defined and we have for all $i \in \mathcal{N}$,

$$\frac{d}{dt} w_i^T(t) = \frac{1}{w_i^T(t)} \frac{d}{dt} w_i^T(t) = \frac{1}{w_i^T(t)} \left( -r(i)w_i^T(t) - \sum_{j \in \mathcal{V}(i)} e^{-1-b_{ij}} w_j^T(t) \right) = -r(i) - \sum_{j \in \mathcal{V}(i)} e^{-1-b_{ij}} \frac{w_j^T(t)}{w_i^T(t)} = -r(i) - \sum_{j \in \mathcal{V}(i)} e^{-1-b_{ij}} \exp \left( w_j^T(t) - w_i^T(t) \right).$$

Because for all $i \in \mathcal{N}$, $w_i^T(T) = \log(e^{g(i)}) = g(i)$, $v^T$ defines a solution to Eq. 47 with terminal condition 48.

By using a standard verification argument, we obtain from Lemma 47 and Theorem 48 the solution to Problem 49.

Theorem 2. We have:

- $\forall i \in \mathcal{N}, \forall t \in [0, T], w_i^T(t) = v_i^T(t) = \log(w_i^T(t))$.

- The optimal controls for Problem 50 are given in feedback form by:

$$\forall i \in \mathcal{N}, \forall j \in \mathcal{V}(i), \forall t \in [0, T], \quad \lambda^*_T(i, j) = e^{-1-b_{ij}} \frac{w_j^T(t)}{w_i^T(t)}.$$

4 Asymptotic analysis: ergodic constant and optimal controls in the long run

The behavior of value functions when the time horizon $T$ tends to $+\infty$ is a classical topic in the optimal control literature. In $\mathbb{R}^d$, there is indeed an extensive literature on the long-run behavior of solutions of Hamilton-Jacobi equations (see for instance 11 12 13). In the case of connected graphs, the long-run behavior of value functions and the existence of asymptotic optimal controls have been studied in 14 with tools inspired from the literature on Hamilton-Jacobi equations (in particular ideas related to semi-groups). However, when $\mathcal{G}$ is connected, given the expression of $w^T$, the asymptotic analysis of Problem 50 can be carried out independently of the existing literature, with spectral tools. This is the purpose of the following theorem.

\footnote{designates the transpose operator throughout this paper.}
For optimal controls, we obtain hence the result for $\alpha$

By taking logarithms, we obtain that Therefore,

**Theorem 3.** Assume $\mathcal{G}$ is connected – i.e. $\forall i, j \in \mathcal{I}, \exists K \geq 2, \exists i_1, \ldots, i_K \in \mathcal{I}$, such that $i_1 = i$, $i_K = j$, and $\forall k \in \{1, \ldots, K - 1\}, i_{k+1} \in \mathcal{V}(i_k)$.

Let us denote by $\text{Sp}_B(B)$ the real spectrum of the matrix $B$.

$\text{Sp}_B(B)$ is a nonempty set and $\gamma = \max \text{Sp}_B(B)$ is an algebraically simple eigenvalue whose associated eigenspace is directed by a column vector with positive coefficients, hereafter denoted by $f$.

$\gamma$ is the ergodic constant associated with Problem (1) and

$$\exists \alpha \in \mathbb{R}, \forall i \in \mathcal{N}, \forall t \in \mathbb{R}, \lim_{T \to +\infty} u_i^T(t) - \gamma(T - t) = \alpha + \log(f_i).$$

Moreover, the asymptotic behavior of the optimal quotes is given by

$$\forall i \in \mathcal{N}, \forall j \in \mathcal{V}(i), \forall t \in \mathbb{R}, \lim_{T \to +\infty} \lambda_i^T(i, j) = e^{-1-b_{ij}} \frac{f_j}{f_i}$$

**Proof.** Let us consider a constant $\sigma > - \min_{i \in \mathcal{N}} r(i)$. Then, let us denote by $B(\sigma)$ the nonnegative matrix $B + \sigma I_N$.

The matrix $B(\sigma)$ defined by

$$\forall (i, j) \in \mathcal{N}^2, \quad B_{ij}(\sigma) = 1_{B_{ij}(\sigma) \neq 0}$$

is the adjacency matrix of a connected graph (the graph $\mathcal{G}$ to which self-loops have been added). Therefore, $B(\sigma)$ is an irreducible matrix, and so is $B(\sigma)$.

By Perron-Frobenius theorem, the spectral radius $\rho(\sigma)$ of $B(\sigma)$ is an algebraically simple eigenvalue of $B(\sigma)$ and the associated eigenspace is directed by a column vector $f$ with positive coefficients.

In particular $\text{Sp}_B(B)$ is a nonempty set and its maximum $\gamma$, equal to $\rho(\sigma) - \sigma$, is an algebraically simple eigenvalue of $B$ whose associated eigenspace is also directed by $f$.

Similarly, $\rho(\sigma)$ is an algebraically simple eigenvalue of $B(\sigma)'$ and the associated eigenspace is directed by a column vector $\phi$ with positive coefficients.

Using a Jordan decomposition of $B(\sigma)$, we see that $g$ can be written as $\beta f + h$ where $\beta \in \mathbb{R}$ and $h \in \text{Im}(B(\sigma) - \rho(\sigma) I_N)$.

Since $\text{Im}(B(\sigma) - \rho(\sigma) I_N) = \text{Ker}(B(\sigma)' - \rho(\sigma) I_N) = \text{span}(\phi)^\perp$, we have $g - \beta f \perp \phi$. All the coefficients of $g$, $f$, and $\phi$ being positive, we must have $\beta > 0$.

Now,

$$w_i^T(t) = e^{B(T-t)} g = e^{-\sigma(T-t)} e^{B(\sigma)(T-t)} g = \beta e^{(\rho(\sigma) - \sigma)(T-t)} f + e^{-\sigma(T-t)} e^{B(\sigma)(T-t)} h.$$ 

Therefore,

$$e^{-\gamma(T-t)} w_i^T(t) = e^{-\rho(\sigma) T} w_i^T(t) - \gamma(T - t) = \log(\beta) + \log(f_i).$$

By taking logarithms, we obtain that

$$\forall i \in \mathcal{N}, \lim_{T \to +\infty} u_i^T(t) - \gamma(T - t) = \log(\beta) + \log(f_i),$$

hence the result for $\alpha = \log(\beta)$. In particular, $\gamma$ is the ergodic constant associated with Problem (1).

For optimal controls, we obtain

$$\forall i \in \mathcal{N}, \forall j \in \mathcal{V}(i), \forall t \in [0, T], \quad \lambda_i^T(i, j) = e^{-1-b_{ij}} \frac{w_i^T(t)}{w_i^T(t)} = e^{-1-b_{ij}} \frac{e^{-\gamma(T-t)} w_i^T(t)}{e^{-\gamma(T-t)} w_i^T(t)} \to e^{-1-b_{ij}} \frac{f_j}{f_i},$$

hence the result.

This theorem states in particular that solving the ergodic problem associated with Problem (1) when $\mathcal{G}$ is connected simply boils down to finding the largest real eigenvalue – and an associate eigenvector – of a matrix that depends on the structure of the graph $\mathcal{G}$ and the parameters defining the reward/cost functions $(L(i, \cdot))_{i \in \mathcal{N}}$. 

5
References

[1] Guy Barles and Panagiotis E. Souganidis. On the large time behavior of solutions of Hamilton-Jacobi equations. *SIAM Journal on Mathematical Analysis*, 31(4):925-939, 2000.

[2] Pierre Brémaud. *Point processes and queues: martingale dynamics*, volume 50. Springer, 1981.

[3] Álvaro Cartea, Sebastian Jaimungal, and José Penalva. *Algorithmic and high-frequency trading*. Cambridge University Press, 2015.

[4] Albert Fathi. Sur la convergence du semi-groupe de Lax-Oleinik. *Comptes Rendus de l’Académie des Sciences-Series I-Mathematics*, 327(3):267-270, 1998.

[5] Olivier Guéant. Existence and uniqueness result for mean field games with congestion effect on graphs. *Applied Mathematics & Optimization*, 72(2):291–303, 2015.

[6] Olivier Guéant. *The Financial Mathematics of Market Liquidity: From optimal execution to market making*, volume 33. CRC Press, 2016.

[7] Olivier Guéant, Charles-Albert Lehalle, and Joaquin Fernandez-Tapia. Dealing with the inventory risk: a solution to the market making problem. *Mathematics and financial economics*, 7(4), 477–507, 2013.

[8] Olivier Guéant and Iuliia Manziuk. Optimal control on graphs: existence, uniqueness, and long-term behavior. *ESAIM: Control, Optimisation & Calculus of Variations*, 26:1–18, 2020.

[9] Gawtum Namah and Jean-Michel Roquejoffre. Remarks on the long time behaviour of the solutions of Hamilton-Jacobi equations. *Communications in partial differential equations*, 24(5-6):883–893, 1999.

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