A POLYHEDRAL CHARACTERIZATION OF QUASI-ORDINARY SINGULARITIES

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Abstract. Given an irreducible hypersurface singularity of dimension \( d \) (defined by a polynomial \( f \in K[[x]][z] \)) and the projection to the affine space defined by \( K[[x]] \), we construct an invariant which detects whether the singularity is quasi-ordinary with respect to the projection. The construction uses a weighted version of Hironaka’s characteristic polyhedron and successive embeddings of the singularity in affine spaces of higher dimensions. When \( f \) is quasi-ordinary, our invariant determines the semigroup of the singularity and hence it encodes the embedded topology of the singularity \( \{ f = 0 \} \) in a neighbourhood of the origin when \( K = \mathbb{C} \); moreover, the construction yields the approximate roots, giving a new point of view on this subject.

Introduction

Let \( K \) be an algebraically closed field of characteristic 0 and let us denote by \( K[[x]] \) the power series ring \( K[[x_1, \ldots, x_d]] \), \( d \in \mathbb{Z}_+ \). The first objects considered in this paper are quasi-ordinary polynomials: a Weierstrass polynomial \( f = f(x, z) = z^n + f_1(x)z^{n-1} + \cdots + f_{n-1}(x)z + f_n(x) \in K[[x]][z] \) satisfying \( f(0, 0) = 0 \) is said to be quasi-ordinary if its discriminant as a polynomial in \( z \) is a monomial up to multiplication by a unit in \( K[[x]] \). A celebrated theorem by Abhyankar and Jung states that the roots of such a polynomial sit in \( K[[x_1^{1/n_1}, \ldots, x_d^{1/n_d}]] \); see \([J]\) and \([A]\) (see \([Cu2]\) for a generalization of this theorem).

From a different point of view, one can consider the formal quasi-ordinary germ \( (V = \{ f = 0 \}, 0) \); then the singularities of \( (V, 0) \) are intimately related to the roots of \( f \). When \( d = 1 \), i.e., when \( V \) is a plane curve, then the Newton algorithm for determining the roots of \( f \) gives also a resolution of singularities of \( V \), \([Cu]\). When \( d > 1 \), this assertion makes sense thanks to the notion of “characteristic exponents” introduced by Lipman \([L]\). These invariants are extracted from the roots of \( f \), knowing the fact that the latter belong to \( K[[x_1^{1/n_1}, \ldots, x_d^{1/n_d}]] \), and it was proved by Gau that they determine the topological type of the singularity \( (V, 0) \) (when \( K = \mathbb{C} \)), see \([G]\). Lipman also asked how one can construct an embedded resolution of singularities of \( (V, 0) \subset (\mathbb{A}^d, 0) \) from the characteristic exponents or equivalently from the roots of \( f \). There exist many approaches to this question, e.g. \([V]\), \([GP1]\), \([BMc]\), \([CM]\).

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This leads us to the other important objects for this paper, the invariants of resolutions of singularities. In particular, the following two approaches to prove the existence of an embedded resolution of singularities are crucial in our context: The first approach is to construct an invariant which takes values in a “well ordered set” and to prove that there exists a finite sequence of blowing ups which makes this invariant strictly decrease. Such an invariant should of course detect regularity, but should also not be too sophisticated in order to be able to follow its changes after blowing up. The work of the second author suggests that this type of invariants is very much related to polyhedral invariants, namely to Hironaka’s characteristic polyhedra [H], [S2], [CS].

The other approach is to resolve singularities by one toric morphism. This is not always possible if we do not change the ambient space. So the second approach is about finding an embedding in a higher dimensional affine space in such a way that one can resolve the singularities by one toric morphisms, [T], [M1], [M2], [LMR], [GP1].

In this paper, using a mixture of these two approaches, we build an invariant which detects whether $(V, 0)$ is a quasi-ordinary singularity. We will introduce a weighted version of Hironaka’s characteristic polyhedron and build our invariant from the weighted Hironaka polyhedra of successive embeddings of our singularity in higher dimensional affine spaces. Let us give some details about this construction.

For $c, d \in \mathbb{Z}_+$, let $W$ be a map $W : \mathbb{Z}^d_{\geq 0} \to \mathbb{Q}^c_{\geq 0}$ which is the restriction of a linear map $\mathbb{Q}^d \to \mathbb{Q}^c$. The data $W$ is equivalent to the data of a $c \times d$ matrix that we also denote by $W$. This map should be thought as a weight map on the monomials of $K[[x]]$, where we assign to $x_i$ the weight $W(e_i) \in \mathbb{Q}^c_{\geq 0}$ of the $i$–th unit vector. A special case is the identity $W_0 : \mathbb{Z}^d_{\geq 0} \to \mathbb{Q}^c_{\geq 0}$ determined by the unit matrix $Id_c$.

Let $f = \sum_{a,b} \rho_{a,b} x^a z^b \in K[[x]][z]$ be a Weierstrass polynomial of degree $n$. We associate with the projection given by the inclusion $K[[x]] \to K[[x]][z]/(f)$, and with the weight $W$, a polyhedron $\Delta^W (f; x; z)$ which is defined to be the smallest convex subset of $\mathbb{R}^c_{\geq 0}$ containing all the points of the set

\[
\left\{ \frac{W(a)}{n - b} + \mathbb{R}^c_{\geq 0} \mid \rho_{a,b} \neq 0 \land b < n \right\}.
\]

We minimize $\Delta^W (f; x; z)$ (for the inclusion) with respect to all changes of variables that respect the inclusion $K[[x]] \to K[[x]][z]/(f)$ (see section 2).

The polyhedron $\Delta^W (f; x; z)$ with respect to $W$ is closely connected to Hironaka’s characteristic polyhedron [H]. In fact, once it is minimal with respect to the choice of $z$, it is the image of Hironaka’s characteristic polyhedron under $W$ considered as a map from $\mathbb{R}^d$ to $\mathbb{R}^c$.

With these polyhedra we define the notion of $\nu$-quasi-ordinary polynomials with respect to $W$: $f$ is called a $\nu$-quasi-ordinary polynomial with respect to $W$ (and with respect to the above inclusion given by the $(x)$) if there exists some $u_0 := z + h(x)$ such that the polyhedron $\Delta^W (f; x; u_0)$ is either empty or has exactly one vertex $v$ that cannot be eliminated by a change of variable in $u_0$. This generalizes Hironaka’s notion of $\nu$-quasi-ordinary polynomials which are $\nu$-quasi-ordinary polynomials with respect to $W_0$. 

If $f$ is $\nu$-quasi-ordinary with respect to $W$ then the initial form at the unique vertex $v$, i.e., the sum of those monomials determining $v$,

$$F_{v,W} = \nu_{v,W}(f) = U_0^n - \sum_{\frac{m}{n} \nu = v} \rho_{a,b}X^n U_0^b,$$

will be of particular interest for us. If $f$ is irreducible and $W = W_0$ then it is known that $F_{v,W_0} = (U_0^n - \rho X^n)^{\epsilon}$, for certain $m, e \in \mathbb{Z}_+$, $m \cdot e = n$, $\rho \in K^\times$ and $a \in \mathbb{Z}_{\geq 0}$ with $\frac{a}{n} = v$ and gcd$(a, m) = 1$, see [GG] resp. [ACLMI], or [RS] for a recent generalization to arbitrary fields.

The construction of our invariant $\kappa(f; x; z)$ goes as follows (for a detailed version see Construction 3.5). If $f$ is not $\nu$-quasi-ordinary then set $\kappa(f; x; z) := (-1)$.

**Construction 1.** Let $f \in K[[x]][z]$ be an irreducible $\nu$-quasi-ordinary polynomial of degree $n \geq 2$. Denote by $v(1)$ the only vertex of the minimal polyhedron $\Delta^{W_0}(f; x; u_0)$, $u_0 = z + h_0(x)$, and let $F_{v(1), W_0} = (U_0^n - \rho_1 X^{a(1)})^{\epsilon_1}$, where $\rho_1 \in K^\times$, $\frac{a(1)}{n} = v(1)$, and gcd$(a(1), n_1) = 1$.

We consider the embedding associated with the map $K[[x]][u_0, z_1] \rightarrow K[[x]][u_0]$, where $z_1$ is sent to $u_0^{n_1} - \rho_1 X^{a(1)}$. We extend $W_0$ to a linear map $W_1 : \mathbb{Z}_{\geq 0}^{d+1} \rightarrow \mathbb{Q}_{\geq 0}^d$ on $K[[x]][u_0]$ by assigning

$$W_1(u_0) := v(1).$$

Let $\tilde{f}(1)$ be the unique representative of the image of $f$ in

$$R_1 := \frac{K[[x]][u_0, z_1]}{(z_1 - (u_0^{n_1} - \rho_1 X^{a(1)}))}$$

which is obtained by exchanging $u_0^{n_1}$ in $f$ by $z_1 + \rho_1 X^{a(1)}$; therefore we get that $\tilde{f}(1) \in K[[x]][u_0]_{< n_1}[z_1]$, i.e., $u_0$ appears to the power at most $n_1 - 1$. Note that $\tilde{f}(1)$ is of degree $e_1 = \frac{n}{n_1} < n$ in $z_1$. This strict inequality provides that our construction will be finite.

Let

$$u_1 := z_1 + h_1(x, u_0)$$

be the change in $z_1$ such that the polyhedron $\Delta^{W_1}(\tilde{f}(1); x, u_0; u_1)$ becomes minimal.

If $f(1)$ is not $\nu$-quasi-ordinary with respect to $W_1$ (for $(x, u_0)$) then we set

$$\kappa(f; x; z) := (v(1), -1).$$

If $\Delta^{W_1}(\tilde{f}(1); x, u_0; u_1)$ is empty then we put

$$\kappa(f; x; z) := (v(1), \infty).$$

Otherwise, we denote by $v(2)$ the unique vertex of $\Delta^{W_1}(\tilde{f}(1); x, u_0; u_1)$. The initial form $F_{v(2), W_1}$ of $f(1)$ at $v(2)$ is not necessarily of the shape

$$(U_0^{n_2} - \rho_2 X^{a(2)} U_0^{b(2)})^{\epsilon_2}, \quad \text{where } \frac{W_1(a(2), b(2))}{n_2} = v(2),$$

see Example 3.1. By using the equality $\rho_1 X^{a(1)} = u_0^{n_1} - u_1 + h_1(x, u_0)$ (which holds in $R_1$), we can determine (Proposition 3.3) a representative $f(1) \in K[[x]][u_0, u_1]$ for the class defined by $f(1)$ in $R_1$ which

(1) is still $\nu$-quasi-ordinary with respect to $W_1$. 

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(2) has \(v(2)\) as the only vertex of \(\Delta W_1( f^{(1)}; x, u_0; u_1)\), and

(3) whose initial form \(F_{v(2), W_1}^{(1)}\) is a product of powers of binomials as in (0.2).

If one of the following conditions fails to hold, then we set \(\kappa(f; x; z) := (v(1), -1)\) and stop:

\[ \begin{align*}
\text{(0) } & F_{v(2), W_1}^{(1)} \text{ is the power of precisely one binomial, } \\
\text{(1) } & v(2) \in n_1 v(1) + \mathbb{R}^d_{\geq 0}, \text{ but } v(2) \neq n_1 v(1), \text{ and } \\
\text{(2) } & v(2) \not\in \mathbb{Z}^d + \mathbb{Z} v(1).
\end{align*} \]

Therefore suppose \((\text{0) } - (\text{2})\) hold. Then we have

\[ F_{v(2), W_1}^{(1)} = (t^{n_2} - \rho_2 x^{a(2)} U_0^{b(2)})^{e_2}, \quad \text{where } \frac{W_1(a(2), b(2))}{n_2} = v(2), \]

and we can iterate the previous step.

After finitely many steps, say \(g \in \mathbb{Z}_+\), the construction ends and we define

\[ \kappa(f; x; z) := (v(1), \ldots, v(g), \xi), \]

where \(v(i)\) is the vertex of the polyhedron in the \(i\)-th step, \(\xi = 0\) if the polyhedron in the \((g + 1)\)-th step is empty, and \(\xi = -1\) else.

The main result of this article is

**Theorem 2.** Let \(f \in K[[x]]\) be an irreducible \(\nu\)-quasi-ordinary polynomial of degree \(n\). Then \(f\) is quasi-ordinary with respect to the projection given by the chosen variables \((x)\) if and only if the last entry of \(\kappa(f; x; z)\) is \(\infty\).

The key ingredient for the proof is: if the last entry is \(\infty\), then our construction provides that, after adding the variables \(u_1, \ldots, u_g\), the variety \(V = \{f = 0\}\) embedded in \(\mathbb{A}^{d+g+1}_K\) is the generic fiber of an “overweight deformation” of a toric variety. This is the monomial variety associated with the \(v(i)\) which is the closure of the orbit \((s_1^n, \ldots, s_d^n, S^{n-v(1)}, \ldots, S^{n-v(g)})\), where \(S = (s_1, \ldots, s_d)\) and \(S^{(\alpha_1, \ldots, \alpha_d)} = s_1^{\alpha_1} \cdots s_d^{\alpha_d}\). Hence we have a parametrization of the toric variety.

Let us explain the overweight deformation a little bit more in detail: From the construction of our invariant we attain a linear map \(W_*: \mathbb{Q}^{d+g}_{\geq 0} \rightarrow \mathbb{Q}^d_{\geq 0}\) which can be identified with a weight map

\[ W_* : K[[x]][[u_0, u_1, \ldots, u_{g-1}]] \rightarrow \mathbb{Q}^d. \]

Then we can identify the singularity \(\{f = 0\}\) embedded in \(\mathbb{A}^{d+g+1}_K\) with the fiber \(\mathfrak{X}_1\) above \(T = 1\) of the map \(\mathfrak{X} \rightarrow \text{Spec } K[T]\), where \(\mathfrak{X} = V(F_0, \ldots, F_{g-1})\) denotes the variety defined by \(F_t \in K[[x]][u_0, u_1, \ldots, u_{g-1}, T], t \in \{0, \ldots, g-1\}\), (using multi-index notation and the abbreviation \(u_{<t} = (u_0, u_1, \ldots, u_{t-1})\))

\[ F_t = T u_{t+1} - u_t^{n_{t+1}} + \rho_t x^a u_{<t}^b - T \cdot \left( \sum_{\alpha, \beta, \gamma} \mu_{\alpha, \beta, \gamma} x^\alpha u_{<t}^\beta u_t^\gamma \right), \]

for \(a, x, b, \beta, \gamma \in \mathbb{Z}^d_{\geq 0}, \beta, \gamma \in \mathbb{Z}^d_{\geq 0}, \) all depending on \(t\), and \(\rho_t, \mu_{\alpha, \beta, \gamma} \in K, \rho_t \neq 0\). Note: The sum \(\sum_{\alpha, \beta, \gamma} \mu_{\alpha, \beta, \gamma} x^\alpha u_{<t}^\beta u_t^\gamma\) are the terms \(h_{t+1}(x, u_{<t}, u_t)\) that we need to add in order to minimize the polyhedron in (0.1). By convention, \(u_g = 0\) and further we have that (using Notation 3 below)

\[ n_{t+1} \cdot v(t + 1) = W_*(u_t^{n_{t+1}}) = W_*(x^a u_{<t}^b) <_{\text{poly}} W_*(x^a u_{<t}^\beta u_t^\gamma), \]

\[ W_*(u_t^{n_{t+1}}) <_{\text{poly}} W_*(u_{t+1}). \]
As one can see the fiber \( X_0 \) above \( T = 0 \) yields the toric variety described before. Using the overweight deformation we can lift the parametrization of the toric variety \( X_0 \) to one of \( V = \{ f = 0 \} \) embedded in \( A^{d+g+1}_K \) after studying the equations of the space of solutions of \( V \) in \( (K[[S]])^{d+g+1} \). This yields the roots of \( f \) and hence its discriminant with respect to \( z \) which happens to be a monomial times a unit in \( K[[x]] \).

On the other hand, if \( f \) is quasi-ordinary then a direct computation of \( \kappa(f; x; z) \) using the expression of \( f \) in terms of “approximate roots”, following [GP1], gives that the last entry of \( \kappa(f; x; z) \) must be infinite, this is Proposition 3.14.

As an other application of our invariant, we prove that when \( f \) is quasi-ordinary \((v(1), v(2), \ldots, v(g)) \) determines the minimal system of generators of the semi-group of quasi-ordinary hypersurface \( \{ f = 0 \} \) and thus its topology when \( K = \mathbb{C} \).

To end this introduction, it is also important to add that the algorithm that we introduce in the construction of our invariant, gives a new way to determine the approximate roots (semi-roots or key polynomials) of a quasi ordinary singularity.

The structure of the paper is as follows: in the first section we recall some basic facts about quasi-ordinary singularities. Section 2 is devoted to weighted Hironaka’s characteristic polyhedra. In section 3, we introduce the invariant \( \kappa(f; x; z) \) and show that its last entry is \( \infty \) when \( f \) is quasi-ordinary. Section 4 is the last section and is devoted to the proof of the other direction: if the last entry of \( \kappa(f; x; z) \) is \( \infty \), \( f \) is quasi-ordinary.

We shall use the notation in bold letters for the tuples \( x = (x_1, \ldots, x_d) \), and for \( x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d} \). Throughout the whole paper the product order plays an important role. Therefore let us recall its definition.

**Notation 3.** For \( v, w \in \mathbb{Q}^d_{\geq 0} \) we have:

- \( v \geq_{\text{poly}} w \iff v \in w + \mathbb{Q}^d_{\geq 0} \).
- \( v >_{\text{poly}} w \iff v \geq_{\text{poly}} w \land v \neq w \).

Note that \( v \not\geq_{\text{poly}} w \) does not imply \( v <_{\text{poly}} w \). Further, \( v =_{\text{poly}} w \) (by which we mean \( v \geq_{\text{poly}} w \) and \( v \leq_{\text{poly}} w \)) is equivalent to \( v = w \).

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### 1. Quasi-ordinary singularities

In this section we collect some facts about quasi-ordinary hypersurface singularities. It is essential to know that the construction of our invariant is independent
of these facts and that all of these facts can be seen as corollaries of our characterization of quasi-ordinary singularities.

An equidimensional germ \((V,0)\) of dimension \(d\) is quasi-ordinary if there exists a finite projection \(\pi : (V,0) \rightarrow (\mathbb{A}^d_K,0)\) whose branch locus is a normal crossing divisor. If \((V,0)\) is a hypersurface, \((V,0) \subset (\mathbb{A}^{d+1}_K,0)\), then \(V\) is defined by a single equation \(f = 0\), where \(f \in K[[x_1, \ldots, x_d]][z]\) is a Weierstrass polynomial whose discriminant with respect to \(z\) is of the form \(\Delta_z f = x_1^{a_1} \cdots x_d^{a_d} \epsilon\), where \(\epsilon\) is a unit in \(K[[x_1, \ldots, x_d]]\) and \((\delta_1, \ldots, \delta_d) \in \mathbb{Z}_{>0}^d\). In these coordinates the projection \(\pi\) is induced by the inclusion
\[
K[[x_1, \ldots, x_d]] \rightarrow K[[x_1, \ldots, x_d]][z]/\langle f \rangle.
\]

From now on, we assume that \((V,0)\) is analytically irreducible, i.e., \(f\) is irreducible in \(K[[x_1, \ldots, x_d]][z]\). A crucial building block in the theory of quasi-ordinary singularities is

**Theorem 1.1.** (Abhyankar-Jung Theorem) Let \(f \in K[[x]][z]\) be a quasi-ordinary polynomial of degree \(n\) in \(z\). The roots of \(f\) are fractional power series. More precisely, they belong to the ring \(K[[x_1^{1/n}, \ldots, x_d^{1/n}]]\).

Here and in the whole article, we implicitly mean that \(f\) is a Weierstrass polynomial of degree \(n\) in \(z\) if we write \(f\) is of degree \(n\) in \(z\). Sometimes we even omit the reference to \(z\) if it is clear from the context.

Let \(\zeta(i), i \in \{1, \ldots, n\}\), be the roots of \(f\). The difference \(\zeta(i) - \zeta(j)\) of two different roots divides the discriminant of \(f\) in the ring \(K[[x_1^{1/n}, \ldots, x_d^{1/n}]]\). Therefore
\[
\zeta(i) - \zeta(j) = x^{\lambda_{ij}} \epsilon_{ij},
\]
where \(\epsilon_{ij}\) is a unit in \(K[[x_1^{1/n}, \ldots, x_d^{1/n}]]\). These exponents have been introduced by Lipman in [L]. It follows from Proposition 1.3 in [G] that the exponents \(\lambda_{ij}\) are well ordered with respect to the product ordering \(\leq_{poly}\) and so we name them
\[
\lambda_1 \leq_{poly} \lambda_2 \leq_{poly} \ldots \leq_{poly} \lambda_g,
\]
and we call them the **characteristic exponents**.

We can then define the lattices \(M_0 := \mathbb{Z}^d\) and \(M_i := M_{i-1} + \mathbb{Z} \lambda_i\), for \(i \in \{1, \ldots, g\}\). We have that \(M_0 \subset M_1 \subset \ldots \subset M_g\) and we set
\[
n_i := [M_i : M_{i-1}], \quad i \in \{1, \ldots, g\},
\]
where \([M_i : M_{i-1}]\) denotes the index of the subgroup \(M_{i-1}\) in \(M_i\).

The importance of these exponents comes from the fact that, when \(K = \mathbb{C}\), Gau proves that they determine the topological type of \((V,0)\).

We can also define an equivalent data to the characteristic exponents ([KM], [GP1]), as follows
\[
(1.1) \quad \gamma_1 = \lambda_1 \quad \text{and} \quad \gamma_{i+1} - n_i \gamma_i = \lambda_{i+1} - \lambda_i \quad \text{for} \ i \in \{1, \ldots, g - 1\},
\]
Again, the set \(\gamma_i, i \in \{1, \ldots, g\}\), determine the topological type of \((V,0)\).
2. Weighted characteristic polyhedra

For our characterization we introduce polyhedra with respect to a linear map $W$. These polyhedra play a crucial role in the construction of our invariant and are a generalization of Hironaka’s characteristic polyhedra. (For more details on the latter, see [CS], section 1).

Let $f \in K[[x]][z]$ a Weierstrass polynomial of degree $n \in \mathbb{Z}_+$ in $z$, i.e.,
\begin{equation}
    f(x, z) = z^n + f_1(x) z^{n-1} + \cdots + f_n(x)
\end{equation}
for some $f_i(x) \in K[[x]]$, $i \in \{1, \ldots, n\}$. Consider an expansion of $f$ of the form
\[ f = \sum_{a,b} \rho_{a,b} x^a z^b, \]
for certain $\rho_{a,b} \in K$, and $a \in \mathbb{Z}_{\geq 0}^d$, $b \in \mathbb{Z}_{\geq 0}$.

Further, let
\[ W : \mathbb{Z}_{\geq 0}^d \to \mathbb{Q}_{\geq 0}^c, \]
for some $c \in \mathbb{Z}_+$, be the map defined by
\[ W(e_i) := \alpha_1, \ W(e_2) := \alpha_2, \ldots, \ W(e_d) := \alpha_d, \]
for certain $\alpha_1, \ldots, \alpha_d \in \mathbb{Q}_{\geq 0}^c$, and, for $i \in \{1, \ldots, d\}$, $e_i$ denotes the $i$-th unit-vector of $\mathbb{Z}_{\geq 0}^d$ (i.e., $e_i = (\delta_{ij})_{j \in \{1, \ldots, d\}}$ and $\delta_{ij}$ is one if $j = i$ and zero for all $j$ with $j \neq i$).

Clearly, this is the linear map determined by the $c \times d$ matrix, also denoted by $W$, with column vectors $\alpha_1, \ldots, \alpha_d$.

\[ W = (\alpha_1 | \alpha_2 | \cdots | \alpha_d). \]

For $a = (a_1, \ldots, a_d) \in \mathbb{Z}_{\geq 0}^d$, we have
\[ W(a) = a_1 \alpha_1 + \cdots + a_d \alpha_d. \]

**Convention:** In order to avoid technical special cases that are not relevant for our considerations we make the convention that whenever we mention vectors $\alpha_i \in \mathbb{Q}_{\geq 0}^c$ as above we implicitly assume that all of them are non-zero, $\alpha_i \neq (0, \ldots, 0)$.

**Remark 2.1.** One may consider $W$ also as a map from the monomials of $K[[x]]$ to $\mathbb{Q}_{\geq 0}^c$. This means $W$ assigns to $x_i$ the “weight” $\alpha_i \in \mathbb{Q}_{\geq 0}^c$, for all $i \in \{1, \ldots, d\}$. In the following we sometimes write also $W(x^a)$ when we mean $W(a)$. We refer also to the first section in [T].

**Example 2.2.**

(1) A first example is the map given by the identity, i.e., $c = d$ and $W(a) = a$. In the following, we denote this special example by $W_0$.

(2) Let $d = 3$, and $c = 2$. Then another example for such a map is the following
\[ W : \mathbb{Z}_{\geq 0}^3 \to \mathbb{Q}_{\geq 0}^2 \]
(on $K[[x_1, x_2, x_3]]$) which is given by $W = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$, i.e.,
\begin{equation}
    W(x_1) = (1, 0), \quad W(x_2) = (0, 1), \quad W(x_3) = (2, 1).
\end{equation}

Let us now come to the definition of the polyhedron.

**Definition 2.3.** Let $f = \sum_{a,b} \rho_{a,b} x^a z^b \in K[[x]][z]$ be a polynomial of degree $n$ and let $W : \mathbb{Z}_{\geq 0}^d \to \mathbb{Q}_{\geq 0}^c$ be a linear map as before.
Remark 2.4. 

(1) Based on discussions of the authors a variant of this idea was already mentioned in [SI], Remark 5.9. Besides that there is no reference known to the authors, where such a kind of polyhedron has been considered before. A connected notion has been considered in [As].

(2) The polyhedron $\Delta^W(f; x; z)$ has finitely many vertices. (This follows with the same arguments as in [CP], Proposition 2.1).

Further, one sees easily that the associated polyhedron $\Delta^W(f; x; z)$ is the projection of the Newton polyhedron $\Delta^{N,W}(f; x, z)$ from the point $(0, 1) \in \mathbb{R}^{c+1}_+ \times \mathbb{R}^c_+$ onto $\mathbb{R}^c_+$ corresponding to the coordinates $(x)$.

(3) If we consider $W$ as a linear map from $\mathbb{R}^d$ to $\mathbb{R}^c$ then we have $W(a) = W \left( \frac{a}{n-b} \right)$. Hence

\begin{equation}
\Delta^W(f; x; z) = W(\Delta(f; x; z)) + \mathbb{R}^c_+,
\end{equation}

where $\Delta(f; x; z) = \Delta^W_a(f; x; z)$ is the polyhedron associated to $(f, x, z)$ (see also Definition 1.2 in [CS]).

Example 2.5. Let $d = 3$, $c = 2$, and $W = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ the linear map defined by (2.2). Consider

\[ f = z^2 + 2zx_1x_2^3 + x_1^2x_2^2 + x_1^3x_3 + x_2^2x_3. \]

Then we have

\[
\begin{align*}
W(x_1x_2^3) &= (1, 3), & & \Rightarrow & v_1 := (1, 3), \\
W(x_1^2x_2^2) &= (2, 6), & & \Rightarrow & v_2 := v_1 = (1, 3), \\
W(x_1^3x_3) &= (5, 1), & & \Rightarrow & v_3 := (\frac{5}{2}, \frac{1}{2}), \\
W(x_2^2x_3) &= (6, 5), & & \Rightarrow & v_4 := (3, \frac{5}{2}),
\end{align*}
\]

where $v_i$ denotes the corresponding point $\frac{W(a)}{n-b}$ in the polyhedron. Thus the vertices of $\Delta^W(f; x; z)$ are $v_1 = (1, 3)$ and $v_3 = (\frac{5}{2}, \frac{1}{2})$, whereas $v_4 = (3, \frac{5}{2})$ lies in the interior.

If we put $y := z + x_1x_2^3$, then $f = y^2 + x_1^3x_3 + x_2^2x_3$. Clearly, the point $v_1$ does not appear in $\Delta^W(f; x; y)$ which has only $v_3$ as vertex. Moreover, $v_3$ can not be eliminated by a further change of $y$. This means the polyhedron $\Delta^W(f; x; y)$ is minimal with respect to the choice of $y$, in the sense that there is no choice $\tilde{y}$ for $y$ such that $\Delta^W(f; x; \tilde{y}) \subseteq \Delta^W(f; x; y)$. 

(1) We define the \textit{Newton polyhedron of $(f, x, z)$ with respect to $W$} as the smallest convex subset of $\mathbb{R}^{c+1}_+$ containing all the points of the set

\[
\left\{ (W(a), b) + \mathbb{R}^{c+1}_+ \mid \rho_{a,b} \neq 0 \right\}
\]

and we use the notation $\Delta^{N,W}(f; x, z)$.

(2) We define the \textit{associated polyhedron for $(f, x, z)$ with respect to $W$} as the smallest convex subset of $\mathbb{R}^c_+$ containing all the points of the set

\[
\left\{ \frac{W(a)}{n-b} + \mathbb{R}^c_+ \mid \rho_{a,b} \neq 0 \land b < n \right\}
\]

and we use the notation $\Delta^W(f; x; z)$. 


Choices for $z$: As we have seen, the polyhedron $\Delta^W(f; x; z)$ depends heavily on the choice of $z$. Thus we seek for $\tilde{z}$, where $\tilde{z} = z + h(x)$, for some $h(x) \in K[[x]]$, such that $\Delta^W(f; x; \tilde{z}) \subset \mathbb{R}_{\geq 0}^c$ becomes minimal with respect to inclusion. For quasi-ordinary singularities one can achieve this by considering so-called $P$-good coordinates (see for example [ACLM2], Definition 3.1 and Lemma 4.6).

More generally, since $K$ has characteristic zero, we can choose $z$ in such a way that it has maximal contact with $f$. Thus we attain the desired change in $z$ by performing the so-called Tschirnhausen transformation: Given any $z$ such that $f$ is of degree $n$ as in (2.1), we set

$$\tilde{z} := z + \frac{1}{n} \cdot f_1(x).$$

This determines a hypersurface of maximal contact for $f$ and we can replace $z$ by $\tilde{z}$. Then we get

$$f(x, z) = \tilde{z}^n + g_2(x) \tilde{z}^{n-2} + \ldots + g_n(x)$$

for some $g_i(x) \in K[[x]]$, $i \in \{2, \ldots, n\}$, and $g_1(x) \equiv 0$. By Proposition 6.1 in [S1] the associated polyhedron $\Delta(f; x; \tilde{z})$ is minimal with respect to choices for $\tilde{z}$ and since $\Delta^W(f; x; \tilde{z}) = W(\Delta(f; x; \tilde{z})) + \mathbb{R}_{\geq 0}$, the same is true for $\Delta^W(f; x; \tilde{z})$.

**Remark 2.6.** The desired $\tilde{z}$ can also be achieved by the following step-by-step process: Consider the given coordinates $(x, z)$. We equip $\mathbb{R}_{\geq 0}^c$ with the total order determined by the lexicographical order of the entries of a vector $v \in \mathbb{R}_{\geq 0}^c$. In the following we use the words *smallest* or *smaller* always with respect to this chosen total order.

Let $v \in \mathbb{Q}_{\geq 0}^c$ be the smallest vertex of $\Delta^W(f; x; z)$. Remember that $W$ is defined by vectors $\alpha_1, \ldots, \alpha_d \in \mathbb{Q}_{\geq 0}^c$. If $v$ is contained in the semi-group generated by these vectors (i.e., if $v \in \alpha_1\mathbb{Z}_{\geq 0} + \ldots + \alpha_d\mathbb{Z}_{\geq 0} \subset \mathbb{Z}_{\geq 0}^c$) then there is the chance to eliminate $v$ by a change $z \mapsto z + h_v(x)$. Here, $h_v(x) = \sum_{\alpha} \mu_{\alpha} x^\alpha \in K[[x]]$ and those $\alpha$ with $\mu_{\alpha} \neq 0$ fulfill $W(\alpha) = \alpha_1 a_1 + \ldots + \alpha_d a_d = v$. (Note that relations as, for example, $2a_1 + 3a_2 = v = 2a_2 + a_3$ might hold, i.e., the linear equations $W(\alpha) = v \in \mathbb{Q}_{\geq 0}^c$ might have more than one solution).

Suppose it is possible to eliminate the vertex $v$. Then we perform the change $z \mapsto z' := z + h_v(x)$ as described before with appropriate constants $\mu_{\alpha}$ and we consider the polyhedron with respect to $z'$. Then we pick the smallest element in the set of vertices (which is bigger than the previous one with respect to the chosen total order) and try to eliminate this one and so on.

If it is not possible to eliminate the smallest vertex, then we go on and consider the next smallest vertex and try to eliminate that one and so on.

By doing this (possibly not finite) process we end up with a certain element $\tilde{z} = z + h(x)$, where $h(x) = \sum_{\alpha} \mu_{\alpha} x^\alpha \in K[[x]]$, such that $\Delta^W(f; x; \tilde{z})$ is minimal with respect to changes in $\tilde{z}$.

**Changes in $(x)$:** In our context a good choice for the variables $(x)$ is also important if $W = W_0$; for example, consider the polyhedron with respect to $W_0$ for $f(x, z) = z^2 + x_1^2(x_1 + x_2) + \ldots$ and compare it with the one for $f(y, z) = z^2 + y_1^2 y_2$, where $y_1 := x_1$ and $y_2 := x_1 + x_2$. Therefore one may ask if there exist choices for $(x, z)$ such that the polyhedron $\Delta_{W_0}(f; x; z)$ is minimal with respect to inclusion in $\mathbb{R}_{\geq 0}^c$. (Of course, one may ask this question in the more general case for arbitrary $W$, but since this is becoming more complicated and since it is not needed here, we do not discuss this).
We want to make only changes in \((x, z)\) which respect the projection of the singularity determined by \(f\) onto the affine space given by the variables \((x)\), i.e., on the level of rings which respect the inclusion
\[
K[[x]] \hookrightarrow K[[x]][z]/(f).
\]
Thus we seek for \((\bar{x}, \bar{z})\), where \(\bar{x}_i = x_i + h_i(x)\) and \(\bar{z} = z + h(x)\), for some \(h_i(x), h(x) \in K[[x]]\) \((i \in \{1, \ldots, d\})\), such that \(\Delta^W(f; \bar{x}; \bar{z}) \subset \mathbb{R}_{\geq 0}\) becomes minimal with respect to inclusion. (Note that we use for \((\bar{x})\) also the map \(W_0\) induced by the identity; in particular, we do not require any compatibility with the original \(W_0\) defined by \((x)\); for example, we allow a change of the form \(\bar{x}_2 := x_2 + x_1\).)

The coordinates \((\bar{x}, \bar{z})\) can be constructed in the following way: As before we equip \(\mathbb{Q}_{\geq 0}\) with the total ordering given by the lexicographical order of the entries. First, we change \(z\) to \(\bar{z}\) as above such that \(\Delta^{W_0}(f; x; \bar{z})\) becomes minimal with respect to the choice of \(\bar{z}\). If \(\Delta^{W_0}(f; x; \bar{z})\) has only one vertex or is empty, then it is minimal with respect to the choice of \((x)\) and we are done.

Suppose the latter is not the case. Consider the two smallest vertices and try to eliminate one of them by changes in \((x)\) respecting (2.4). If this is not possible, the smallest vertex is fixed and we compare the next two smallest vertices and so on.

If we can eliminate one of the vertices by changing \((x)\), then we perform this change and start over again. Note that the condition \(f_1(x) \equiv 0\) is stable under the considered changes in \((x)\).

In the case of a quasi-ordinary hypersurface singularity this process will end with a polyhedron that has exactly one vertex or is empty. Thus in our context the shape of the obtained polyhedron will be unique.

**Remark 2.7.**  
(1) Suppose \(z\) is such that \(\Delta(f; x; z) := \Delta^{W_0}(f; x; z)\) is minimal with respect to the choice for \(z\). Then the (unique!) polyhedron obtained coincides with Hironaka’s characteristic polyhedron associated to \((f, x)\). (See for example, Definition 1.7 and Theorem 1.8 in [CS]). By (2.3), this implies that for any linear map \(W\) the polyhedron \(\Delta^W(f; x; z)\) is also minimal with respect to the choice of \(z\).

(2) One could make the definition of \(\Delta^W(f; x; z)\) more general by allowing a whole system of variables \((z) = (z_1, \ldots, z_d)\) or by extending \(W\) to \(K[[x]][z]\). In fact, one might even replace \(K[[x]][z]\) by a regular local ring \(R\) with regular system of parameters \((x, z)\) and may consider instead of an element \(f\) an ideal \(J \subset R\). Similar ideas have been discussed in [S1], Remark 5.9. Since this is not important for our aim, we do not discuss these things in more detail.

For the later use (Lemma 3.7) we introduce in our special situation the notion of \(\nu\)-quasi-ordinary singularities with respect to a linear map \(W: \mathbb{Z}_{\geq 0}^d \rightarrow \mathbb{Q}_{\geq 0}^c\). This generalizes the notion of \(\nu\)-quasi-ordinary polynomial which was introduced by Hironaka; more precisely, \(f\) is \(\nu\)-quasi-ordinary if it is \(\nu\)-quasi-ordinary with respect to \(W_0\).

**Definition 2.8.** Let \(f \in K[[x]][z]\) be of degree \(n\) and let \(W: \mathbb{Z}_{\geq 0}^d \rightarrow \mathbb{Q}_{\geq 0}^c\) be a linear map. Suppose that \((x, z)\) are such that \(\Delta^W(f; x; z)\) is minimal. Then we say \(f\) is \(\nu\)-quasi-ordinary with respect to \(W\) and with respect to the projection determined by \((x)\) if \(\Delta^W(f; x; z)\) either is empty or has exactly one vertex, say \(v \in \mathbb{Q}_{\geq 0}^c\).
Let us to point out that the definition can easily be extended to arbitrary ideals \( J \) instead of a single irreducible Weierstrass polynomial \( f \). Since this general form is not needed in our context, we stay in the special case \( J = \langle f \rangle \).

Since we have assumed \( f \) to be a Weierstrass polynomial, the Newton polyhedron \( \Delta^{N,W}(f; x, z) \) must have a one-dimensional face starting from the point \((0, n) \in \mathbb{Q}_{\geq 0}^{c+1}\) and projecting down to the point \( v \). In particular, \( v \) can not be eliminated by changes in \( z \) and thus the same is true for the described face in the Newton polyhedron. Clearly, this condition on the Newton polyhedron is equivalent to the one given in the definition.

For \( W = W_0 \) we obtain the usual definition of \( \nu \)-quasi-ordinary polynomials (see for example, [ACLM2], Definition 1.11) for these special shaped \( f \). The equality (2.3) shows that a \( \nu \)-quasi-ordinary polynomial is \( \nu \)-quasi-ordinary polynomial with respect to any linear map \( W \).

On the other hand, it is not hard to give an example of a polynomial which is \( \nu \)-quasi-ordinary with respect to some \( W \neq W_0 \) but which is not \( \nu \)-quasi-ordinary in the usual sense.

**Definition 2.9.** Let \( f = \sum_{\alpha, \beta} \rho_{\alpha, \beta} x^\alpha z^\beta \in K[[x]][[z]] \) be of degree \( n \) as before. Let \( W : \mathbb{Z}^2_{\geq 0} \to \mathbb{Q}_{\geq 0}^{c} \), \( c \in \mathbb{Z}_+ \), be the linear map determined by vectors \( \alpha_1, \ldots, \alpha_d \in \mathbb{Q}_{\geq 0}^c \). Let \( v \in \Delta^W(f; x, z) \) be a vertex of the associated polyhedron.

We define the initial form (or initial part) of \( f \) at \( v \) with respect to \( W \) by

\[
F_{v,W} := \text{inv}_{v,W}(f) := Z^n + \sum_{(\alpha, \beta) : (\ast)} \rho_{\alpha, \beta} X^\alpha Z^\beta \in K[X, Z],
\]

where the sum ranges over those \((\alpha, \beta) \in \mathbb{Z}^{d+1}_{\geq 0}\) fulfilling

\[
\frac{W(\alpha)}{n - \beta} = v.
\]

Since \( F_{v,W} \) lies in the graded ring with respect to \( W \) we use capital letter. Note that \( F_{v,W} \neq Z^n \) since \( v \) is a point appearing in the polyhedron.

**Example 2.10.** Consider \( f = z^2 - x^{21} - x^{18}u^3 \in K[[x, u]][[z]] \). Let \( W : \mathbb{Z}^2_{\geq 0} \to \mathbb{Q}_{\geq 0}^{c} \) be the linear map defined by \( W(x) = 1 \) and \( W(u) = 1 \). Then \( \Delta^W(f; x, u; z) \) has only the vertex \( v = \frac{1}{2} \) and

\[
F_{v,W} = Z^2 - X^{21} - X^{18}U^3.
\]

The previous situation will appear again in Example 3.9. In particular, then it will be more clear why we choose here these names for the variables and this particular \( W \).

Of course, it was not used in the previous definition that \( f \) is irreducible. Moreover, we want to point out that all \( f \) being irreducible does not imply \( F_{v,W} = (Z^{n_1} + \rho_0 x^{a(0)})^{c_1} \), for some \( \rho_0 \in K \), \( n_1, c_1 \in \mathbb{Z}_+ \) and \( a(0) \in \mathbb{Z}^d_{\geq 0} \):

In fact, the possible shape of \( F_{v,W} \) is connected with the question how many solution the linear system \( W(\alpha) = v \in \mathbb{Q}_{\geq 0}^c \) has. If there is a unique solution, say \( a(1) \in \mathbb{Z}^d_{\geq 0} \), and if \( f \) is irreducible, then \( F_{v,W} = (Z^{n_1} + \rho_{a(1)} X^{a(1)})^{c_1} \), for some \( \rho_{a(1)} \in K \) and \( c_1, n_1 \in \mathbb{Z}_+ \) (since \( K \) is algebraically closed). For example, this is the case for the usual \( \nu \)-quasi-ordinary polynomials, i.e., which are \( \nu \)-quasi-ordinary with respect to \( W_0 \) (see for example Theorem 1.5 in [ACLM1], or Theorem 2.4 in
In the next section, we prove a similar result (Proposition 3.3) for particular linear maps \( W \) appearing in our process.

3. The invariant and the main theorem

Using the polyhedron introduced in the previous section we can now give the construction of our invariant. The main result which we also state in this section is that the invariant detects whether a given irreducible hypersurface singularity is quasi-ordinary or not. But first, we need some simple techniques.

Consider the ring
\[
R := K[[x]][u, z]/\langle z - (u^m - q(x, u)) \rangle,
\]
for some \( m \in \mathbb{Z}_+ \) and \( q(x, u) \in K[[x]][u] \) with \( \deg_u(q(x, u)) < m \). Let \( f \) be an element in \( K[[x]][u] \). There is a unique (!) representative for the class of \( f \) in \( R \) contained in
\[
K[[x]][u]_{<m}[z] := \left\{ \sum_{(a, b, c) \in \mathbb{Z}_{\geq 0}^d} \mu_{a, b, c} X^a u^b z^c \mid \mu_{a, b, c} \in K \wedge 0 \leq b < m \right\},
\]
i.e., where the only powers of \( u \), which may appear, are \( u, u^2, \ldots, u^{m-1} \); whenever \( u^m \) appears it is replaced by \( z + q(x, u) \).

Thus we can identify \( K[[x]][u]_{<m}[z] \) with \( R = K[[x]][u, z]/\langle z - (u^m - q(x, u)) \rangle \), where we choose the representative of a class uniquely as an element in (3.1).

The important case for us is when \( q \) is a monomial, say \( q = \rho_a x^a \), for some \( a \in \mathbb{Z}_{\geq 0}^d \) and \( \rho_a \in K^\times \). This case naturally arises in our characterization. Namely, there appear irreducible polynomials \( f \in K[[x]][u] \) which are \( \nu \)-quasi-ordinary with respect to some linear map \( W \) (on \( K[[x]] \)) and whose initial form at \( v \) is
\[
F_{v, W} = (U^m - \rho_a X^a)^e,
\]
for some \( \rho_a \in K^\times \) and \( e, m \in \mathbb{Z}_+ \). Here, we assume that \( \Delta^W(f; x; u) \) is minimal for the choice of \( u \) and, further, \( v \) denotes its sole vertex.

In order to detect more refined information on the variety defined by \( f \) we pass to a higher dimensional ambient space. We do this via the embedding that is given by \( K[[x]][u, z] \to K[[x]][u] \), where \( z \) is mapped to \( u^m - \rho_a X^a \). Then we consider the unique representative \( \tilde{f}^+ \in K[[x]][u]_{<m}[z] \) of the image of \( f \) in \( K[[x]][u, z]/\langle z - (u^m - \rho_a X^a) \rangle \).

We extend \( W \) to a linear map \( W_+ \) on \( K[[x]][u] \) by setting \( W_+(u) := v \). If \( f^+ \) is \( \nu \)-quasi-ordinary with respect to \( W_+ \) then we would like to repeat the previous step. The initial form of \( f^+ \) at the unique vertex is not necessarily of the shape (3.2) (as we can see in the following example), but we can choose in a canonical way a representative in \( R \) that is of the desired form.

**Example 3.1.** Let us have a look at
\[
f = (u^2 - x^3)^4 - 2x^5 u (u^2 - x^3)^2 + x^{13} \in \mathbb{C}[x, u],
\]
The polyhedron \( \Delta^W(f; x; u) \) has exactly one vertex \( v = \frac{3}{2} \) which can not be eliminated. (Recall: \( W_0 \) is given by the identity matrix). The corresponding initial form is \( F_{v, W_0} = (U^2 - X^3)^4 \) and we set \( z := u^2 - x^3 \).

Then \( f^+ = z^4 - 2x^5 uz^2 + x^{13} \) and it is \( \nu \)-quasi-ordinary with respect to \( W_+ \).
Note: \( W_+(u) = \frac{3}{2} \). Moreover, \( \nu_+ = \frac{13}{4} \) is the only vertex of the minimal polyhedron \( \Delta^W_0( x^+; u, z ; ) \), and the initial form

\[
\tilde{F}^+_{v_+, W_+} = Z^4 - 2X^5UZ^2 + X^{13}
\]

is not a binomial as desired. But, after replacing \( U \) by \( X^{\nu_+} \), in the ring whose monomials are in the positive part of the lattice \( Z + \gamma Z \), \( \gamma := v = \frac{3}{2} \), the initial form (when the polyhedron have only one vertex) can be written as a binomial to some power, namely \( (Z^2 - X^{\frac{13}{4}})^2 \). In order to obtain an honest polynomial we need to represent \( X^{\frac{13}{4}} \) by a monomial in \( K[[x]][u] <_{2} [z] \), where the weight of \( x \) is 1, the weight of \( u \) is \( 3/2 \). By construction, there is a unique such a monomial which is \( x^5u \). Then the canonical form of \( f^+ \) is given by

\[
f^+ = (z^2 - x^5u)^2 + z^4 - 2x^5uz^2 + x^{13} - (z^2 - x^5u)^2 = (z^2 - x^5u)^2 - x^{10}z,
\]

where \( (z^2 - x^5u)^2 \) is the initial part that we keep as it is and where we have put \( u^2 = z + x^3 \) in the remaining part.

Note that the monomial \( x^{10}z \) corresponds to the point \( 18 = \frac{40}{12} > \frac{39}{12} = \frac{43}{14} \) in the polyhedron and hence lies in the interior.

More generally, the arguments of the example can be applied to construct an appropriate representative for the class of \( f \). Let us explain how this works. Since we want to iterate this procedure, we introduce some notations. Set \( v(1) := v \). For \( t \geq 0 \), suppose we have given

\[
v(1), \ldots, v(t + 1) \in \mathbb{Q}_{\geq 0}^d \quad \text{and} \quad n_1, \ldots, n_{t+1} \in \mathbb{Z}_+
\]

such that, for all \( i \in \{0, \ldots, t\} \),

- \( v(i + 1) >_{poly} n_i v(i) \), if \( i > 0 \), and
- \( \nu(i + 1) \notin \mathbb{Z}^d + \mathbb{Z}v(1) + \cdots + \mathbb{Z}v(i) \).

Let us begin with a useful lemma. Let \( L \) be the lattice

\[
L := \mathbb{Z}^d + \mathbb{Z}v(1) + \cdots + \mathbb{Z}v(t + 1).
\]

Let \( h \in K[[[x^{d_{\geq 0}}]]][z] \) be a monic polynomial in \( z \) such that the monomials appearing in the coefficients of \( z^d \) have exponents in \( L_{\geq 0} \), the non-negative part of the lattice \( L \). Note that \( \mathbb{Z}^d \subset L \subset \mathbb{Q}^d \), i.e., we work with fractional exponents.

We naturally have a weight \( W_{0} : L \rightarrow \mathbb{Q}^d \) on these monomials, which is simply defined by \( W_{0}(x^n) = a \). (Since it is the extension of \( W_{0} \) on \( \mathbb{Z}^d \), we also use the name \( W_{0} \) here). We write \( h = z^n + \sum h_i z^{n_i - 1} \). As in Definition 2.3, we can associate a polyhedron that we will denote again by \( \Delta_{W_0}(h; x; z) \subset \mathbb{R}_{\geq 0}^d \).

For the case \( L = \mathbb{Z}^d \), the following lemma is proven in [GG] section 4.3, see also [RS] Corollary 2.7 iii) for a variant over arbitrary fields.

**Lemma 3.2.** With the above notations, let \( h \in K[[[x^{d_{\geq 0}}]]][z] \). If \( \Delta_{W_0}(h; x; z) \) has a unique vertex \( v(t + 2) \), then the initial part of \( h \) at \( v(t + 2) \) (with respect to \( W_{0} \)) is a product of binomials, i.e., it is of the form

\[
H_{v(t+2)} := \pi_{v(t+2), W_{0}}(h) = \prod_{i=1}^{d_{t+2}} (Z^{n_{t+2}} - \rho_{t+2,i} X^{\alpha})^{e_{t+2,i}}
\]

for some \( \alpha \in L \), and \( n_{t+2}, d_{t+2}, e_{t+2,i} \in \mathbb{Z}_+ \), and \( \rho_{t+2,i} \in K \), pairwise different.
Proof. Let \((0,\ldots,0,n)\) and \((a,i)\) be the extremities of the face (actually the segment) of the Newton polyhedron that correspond to \(v(t+2)\). Let
\[(b,c) := \frac{1}{\mu}(-a,n-i) \in L\]
where \(\mu\) is the largest positive integer such that \(\frac{1}{\mu}(-a,n-i) \in L\). Then
\[X^{-a}Z^{-b}H_{v(t+2)}(X,Z) = P(X^b Z^c),\]
for a polynomial \(P \in K[t]\). Factorizing \(P\) into a product of linear polynomials (\(K\) is algebraically closed) and multiplying the result by \(X^a Z^t\) provides the desired form. \(\square\)

Coming back to \(f \in K[[x]][z]\), we will also repeatedly pass to a higher dimensional ambient space. With the above notations, we set \(R_0 := K[[x]][u,z]\) and, for \(t \geq 0\), we define
\[R_{t+1} := R_1[(zt+1)/(zt+1 - (ut^{n+1} - qt+1(x,u_{\leq t}))]),\]
for some \(q_{t+1} \in K[[x]][u_{\leq t}]\) with \(\deg_{u_t}(q_{t+1}) < n_{t+1}\). Here and in the following, we frequently abbreviate:
\[
\begin{align*}
  u_{\leq t} &:= (u_0, u_1, \ldots, u_t) \\
  u_{\leq t-1} &:= u_{\leq t-1} \\
  x^a u^b_{\leq t} &:= x^{a(t+1)} u_0^{b(t+1)} u_1^{b(t+1)} \cdots u_{t-1}^{b(t+1)},
\end{align*}
\]
for \(a \in \mathbb{Z}_{\geq 0}^d\) and \(b \in \mathbb{Z}_{\geq 0}^d\). Note that \(u_0 = u, u_1 = z\), and \(R_1 = R\).

Similarly as we did for \(R_1\), we can identify \(R_{t+1}\) with
\[K[[x]][u_0 < n_1 < u_t ... u_{t+1} < n_{t+1}]\]
which is defined analogously to (3.1). Furthermore, we equip \(K[[x]]\) with the linear map \(W_0 : \mathbb{Z}_0^d \to \mathbb{Q}_{\geq 0}^d\) and we extend it to a linear map on \(K[[x]][u_{\leq t}]\) using \(v(1),\ldots,v(t+1)\),
\[
\begin{align*}
  W_{t+1}(x_i) &:= W_0(x_i), \quad \text{for all } 1 \leq i \leq d, \\
  W_{t+1}(u_j) &:= v(j+1) \in \mathbb{Q}_{\geq 0}, \quad \text{for all } 0 \leq j \leq t.
\end{align*}
\]
We denote by \(f(t+1)\) the unique representative of the image of \(f\) in \(R_{t+1}\) which is given by an element in \(K[[x]][u_0 < n_1 < u_2 ... u_{t+1} < n_{t+1}][zt+1]\).

Proposition 3.3. Let the notations be as before. Suppose that the associated polyhedron \(\Delta^{W_{t+1}}(f(t+1); x,u_{\leq t}; zt+1)\) is minimal with respect to the choice of \(zt+1\) and that it has a unique vertex, say
\[v(t+2) \in \Delta^{W_{t+1}}(f(t+1); x,u_{\leq t}; zt+1).\]
There exists a canonical representative \(f(t+1)\) of \(f(t+1)\) (i.e., of \(f\)) in \(R_{t+1}\) such that the initial form of \(f(t+1)\) at the vertex \(v(t+2)\) with respect to \(W_{t+1}\) is a product of powers of binomials, i.e.,
\[
F_{v(t+2),W_{t+1}}^{(t+1)} = in_{v(t+2),W_{t+1}}(f(t+1)) = \prod_{i=1}^{d_{t+2}} (Z_{t+1}^{n_{t+2,i}} - \rho_{t+2,i} X^a U^b_{\leq t+1})^{e_{t+2,i}},
\]
where \(\rho_{t+2,i}\) and \(\rho_{t+2,i} = 1\).
where \( \mathbf{a} \in \mathbb{Z}^d_{\geq 0}, \mathbf{b} \in \mathbb{Z}^{n+1}_{\geq 0} \) with \( \frac{W_{t+2}(\mathbf{a},\mathbf{b})}{n_{t+2}} = v(t+2), \) and \( \gcd(n_{t+2}, \mathbf{a}, \mathbf{b}) = 1, \) and \( n_{t+2}, d_{t+2}, c_{t+2,i} \in \mathbb{Z}_{\geq 0}, \) and \( \rho_{t+2,i} \in K, \) pairwise different.

**Proof.** Let \( h \in K[[x_1^{\geq 1}]] \) be the polynomial obtained from \( f(t+1) \) by changing the variables \( u_i \) to \( x^{v(i+1)}_i. \) Then \( h \) satisfies the hypothesis of Lemma 3.2 and we obtain a factorization of \( H_{v(t+2)} = i_{v(t+2), W_0}(h), \) i.e., using the notations of the cited lemma,

\[
H_{v(t+2)} = \prod_{i=1}^{d_{t+2}} (Z_{t+1}^{n_{t+2}} - \rho_{t+2,i} x^\alpha)^{e_{t+2,i}}.
\]

For simplicity, we set

\[
T := K[[x]][u_0 < n_1 | u_1 | < n_2 \cdots | u_t | < n_{t+1}].
\]

By the definition of \( T, \) there exists for every element \( \mathbf{a} \in L_{\geq 0} \) in the non-negative part of \( L \) a unique monomial \( M_\mathbf{a} := M_\mathbf{a}(x, u_{<t}) \) in \( T \) having the weight \( \mathbf{a}, \) \( W_{t+1}(M_\mathbf{a}) = \mathbf{a}. \) This allows us to associate with \( H_{v(t+2)} \) the following unique product of binomials in \( T, \)

\[
f^* := \prod_{i=1}^{d_{t+2}} (z_{t+1}^{n_{t+2}} - \rho_{t+2,i} M_\mathbf{a})^{e_{t+2,i}} = \prod_{i=1}^{d_{t+2}} (z_{t+1}^{n_{t+2}} - \rho_{t+2,i} x^\mathbf{a} u_{<t+1}^{\mathbf{b}})^{e_{t+2,i}} \in T,
\]

where \( \mathbf{a}, \mathbf{b}, n_{t+2}, d_{t+2}, c_{t+2,i}, \) and \( \rho_{t+2,i} \) are as in the statement of the proposition.

We define \( f(t+1) \in R_{t+1} \) as follows: first, we consider

\[
f^* + f(t+1) - f^*
\]

and then we obtain \( f(t+1) \) by replacing \( f(t+1) - f^* \) with its unique representative in \( T = K[[x]][u_0 | u_1 | < n_2 \cdots | u_t | < n_{t+1}]. \) By construction, \( f(t+1) \) is a representative of \( f \) in \( R_{t+1} \) and its initial form at the unique vertex \( v(t+2) \) with respect to \( W_{t+1} \) is as desired. \( \square \)

**Notation 3.4.** In the situation of the previous proposition, we say that **condition** \( (\ast) \) **holds** if the following are true:

\[
\begin{align*}
(\ast0) & \quad d_{t+2} = 1, \\
(\ast1) & \quad v(t+2) >_{\text{poly}} n_{t+1} v(t+1), \text{ and} \\
(\ast2) & \quad v(t+2) \not\in \mathbb{Z}^d + \mathbb{Z}v(1) + \cdots + \mathbb{Z}v(t+1).
\end{align*}
\]

Now, we can give the construction of the invariant

\[\kappa = (\kappa_1; \ldots; \kappa_g; \kappa_{g+1}), \quad g \in \mathbb{Z}_{\geq 0},\]

which we use to characterize quasi-ordinary singularities.

**Construction 3.5.** Let \( f = \sum_{\mathbf{a}, \mathbf{b}} \rho_{\mathbf{a}, \mathbf{b}} x^\mathbf{a} z^\mathbf{b} \in K[[x]][z] \) be an irreducible polynomial of degree \( n. \) First, we consider the polyhedron of \( f \) with respect to \( W = W_0 \) on \( K[[x]]. \) (Recall: \( W_0(\mathbf{a}) = \mathbf{a}. \) Set

\[
\Delta^0(f; x; z) := \Delta^{W_0}(f; x; z)
\]

As described in the previous section we minimize \( \Delta^0(f; x; z) \) with respect to the choice of \( (x, z) \) respecting the inclusion \( K[[x]] \hookrightarrow K[[x]][z]/(f). \) Hence we assume that \( (x, u_0), u_0 := z + h_0(x), \) are such that \( \Delta^0(f; x; u_0) \) is minimal.
We define the first entry of $\kappa = \kappa(f; x; z)$ by

$$
\kappa_1 := \begin{cases} 
  v(1), & \text{if } \Delta^0(f; x; u_0) \text{ has exactly one vertex } v(1), \\
  \infty, & \text{if } \Delta^0(f; x; u_0) = \emptyset \text{ is empty,} \\
  -1, & \text{else.}
\end{cases}
$$

If we are not in the first case then the construction ends and $\kappa := (\kappa_1)$.

Suppose $\Delta^0(f; x; u_0)$ has exactly one vertex $v(1)$. Then we can write the initial form of $f$ at the vertex $v(1)$ (Definition 2.9) as

$$
F_{v(1), W_0} = \left(U_0^{n_1} - \rho_1 X^{a(1)}\right)^{e_1},
$$

for certain $n_1, e_1 \in \mathbb{Z}_+$, $n = n_1 e_1$, $\rho_1 \in K^\times$ and $a(1) \in \mathbb{Z}_+^d$ the unique solution for $W_0(a(1)) = n \cdot v(1)$. Note that we have $n_1 > 1$ (and thus $e_1 < n$) because otherwise we can eliminate the vertex $v(1)$.

We consider

$$
R_1 := K[[x]][u_0, z_1]/\langle z_1 - (u_0^{n_1} - \rho_1 X^{a(1)}) \rangle
$$

In there, we have

$$
z_1 = u_0^{n_1} - \rho_1 X^{a(1)}. 
$$

Let $\tilde{f}(1) \in K[[x]][u_0] < n_1[z_1]$ be the unique representative of the class of $f$ in $R_1$, as in (3.1). Since $f$ is a polynomial of degree $n$ in $u_0$ we obtain that $\tilde{f}(1)$ is a polynomial of degree $e_1 = \frac{n}{n_1} < n$ in $z_1$. As in (3.4), we extend $W_0$ from $K[[x]]$ to a linear map $W_1: \mathbb{Z}_+^{d+1} \to \mathbb{Q}_+^d$ on $K[[x]][u_0]$ by assigning the value $v(1) \in \mathbb{Q}_+^d$ to $u_0$. This finishes the first cycle of the construction and we call $\tilde{f}(1)$ the transform of $f$ under the first cycle.

Suppose we have finished $t + 1$ cycles, $t \geq 0$, and the construction did not end so far. For notational convenience we put

$$
f^{(0)} := f \in K[[x]][u_0] =: R_0 \quad \text{and} \quad e_0 := n.
$$

Then the first $t+1$ entries of $\kappa$ are given by $(v(1); v(2); \ldots; v(t+1))$ and the given data is

$$
\tilde{f}(t+1) \in K[[x]][u_0] < n_t \subset u_1 < n_2 \subset \ldots \subset u_t < n_{t+1} \subset z_{t+1},
$$

which is of degree $e_{t+1} \in z_{t+1}$, and $e_t < e_t < \ldots < e_1 < e_0$. Note: $\tilde{f}(t+1)$ is the unique representative of $f$ in the ring

$$
R_{t+1} := R_t[z_{t+1}]/\langle z_{t+1} - (u_t^{n_t+1} - \rho_{t+1} X^{a} b_{t+1}) \rangle,
$$

similar to (3.1). (Recall the abbreviations (3.3)). Clearly, the following equality holds in $R_{t+1}$:

$$
z_{t+1} = u_t^{n_t+1} - \rho_{t+1} X^{a} b_{t+1}.
$$

By the construction, we also have:

$$
W_{t+1} = \frac{W_t(a(t+1), b(t+1))}{n_{t+1}} = v(1)
$$

Further, $W_0$ got extended to a linear map $W_{t+1}: \mathbb{Z}_+^{d+1} \to \mathbb{Q}_+^d$ on $K[[x]][u_{t+1}]$ by defining $W_{t+1}(u_j) := v(j+1)$, for $0 \leq j \leq t$, as in (3.4).

We consider the polyhedron associated to the given data and try to minimize it by changes in $z_{t+1}$,

$$
z_{t+1} \mapsto u_{t+1} := z_{t+1} + h_{t+1}(x, u_{t+1}).
$$
Suppose \( u_{t+1} \) is such that
\[
\Delta_{t+1} := \Delta_{W_{t+1}}(f^{(t+1)}; x, u_{\leq t}; u_{t+1}) \subset \mathbb{R}^d_{\geq 0}
\]
is minimal. We define
\[
\kappa_{t+2} := \begin{cases} 
  v(t + 2), & \text{if } \Delta_{t+1} \text{ has exactly one vertex } v(t + 2) \text{ and } (\ast) \text{ holds,} \\
  \infty, & \text{if } \Delta_{t+1} = \emptyset \text{ is empty,} \\
  -1, & \text{else.}
\end{cases}
\]
In the first case, we denote by \( f^{(t+1)} \) the canonical representative of the class of \( f \) in \( R_{t+1} \), as in Proposition 3.3. In particular, its initial form at \( v(t + 2) \) (with respect to \( W_{t+1} \)) is a product of powers of binomials (3.5), and (\( \ast \)) implies
\[
F^{(t+1)}\bigg|_{v(t+2),W_{t+1}} = i_{\nu(v(t+2),W_{t+1})} \left( f^{(t+1)} \right) = (U_{t+1}^{n_{t+2}} - \rho_{t+2} X^a U_{t+1}^b)^{e_{t+2}},
\]
where \( \nu_{t+2}(a,b) = v(t + 2) \), \( \gcd(n_{t+2}, a, b) = 1 \), \( d_{t+2}, e_{t+2}, i \in \mathbb{Z}_+ \).

Note that \( n_{t+2} > 1 \) and thus \( e_{t+2} < e_{t+1} \). We define the ring
\[
R_{t+2} := R_{t+1}[z_{t+2}] / (z_{t+2} - (u_{t+1}^{n_{t+2}} - \rho_{t+2} X^a u_{t+1}^b)).
\]
Finally, we denote the unique representative for the class of \( f^{(t+1)} \) in \( R_{t+2} \) (which coincides with the class of \( f \)) lying in \( K[[x]][u_0]_{<u_1} \cdots [u_{t+1}]_{<u_{t+2}}[z_{t+2}] \) by \( \overline{f}^{(t+2)} \) and we extend \( W_{t+1} \) to \( W_{t+2} \) as in (3.4) by putting \( W_{t+2}(u_{t+1}) := v(t + 2) \).

Since \( n = e_0 > e_1 > \ldots > e_t \geq 1 \), the construction ends after finitely many cycles, say after \( g \geq 1 \) cycles, and we obtain
\[
\kappa := \kappa(f; x; z) := (v(1); v(2); \ldots; v(g); \xi),
\]
with \( \xi \in \{\infty, -1\} \).

(End of Construction 3.5).

Let us point out that we perform only in the first cycle changes in \( (x) \). In particular, \( (x, u_0, \ldots, u_t) \) are fixed in the \((t + 1)\)-st cycle of the construction, \( t \geq 1 \).

Remark 3.6. (1) Roughly speaking, the characterization theorem below will state that if \( \kappa \) ends with \( \infty \), then the given \( f \) is quasi-ordinary with respect to the projection \((x, z)\).

On the other hand, if the last entry of \( \kappa \) is \(-1\) then \( f \) is not quasi-ordinary with respect to the chosen projection. But still \( f \) might be quasi-ordinary with respect to another choice of the projection, see Example 3.13.

(2) In each cycle of the construction we embed the singularity into an ambient space with one dimension more. From this we then obtain refined information on the hypersurface. This is highly motivated by Teissier’s approach to local uniformization for Abhyankar valuations, see [T]. His idea is to embed a given singularity into a suitable higher dimensional ambient space such that it becomes an overweight deformation of a toric variety. Then a local resolution of the original singularity can be obtained from that of the associated toric variety.

(3) Moreover, this construction is slightly connected to the invariant by Bierstone and Milman for constructive resolution of singularities in characteristic zero, see [BM1] or [BM2]. In [S2] the second author connected their invariant with certain polyhedra. More precisely, at the beginning
\( \delta(f) := |v(1)| = v(1)_1 + \ldots + v(1)_d \) is the resolution invariant for \( f \). Therefore, \( v(1) \in \mathbb{Q}^d_{\geq 0} \) is a refined information of \( \delta(f) \in \mathbb{Q}_{\geq 0} \) and \( v(t+1) \) can be interpreted as a generalization of \( \delta(f^{(t)}) \) to a situation with weighted coordinates.

Recall that we defined in Definition 2.8 \( f \) to be \( \nu \)-quasi-ordinary with respect to \( W \) if \( \Delta^W(( f ; x ; z )) \) has exactly one vertex. The following is an easy consequence of Construction 3.5:

**Lemma 3.7.** Let \( f \) and \( (x,z) \) be as in Construction 3.5. We have

\[
\kappa(f; x; z) = (v(1); v(2); \ldots; v(g); \infty)
\]

if and only if the successive transforms \( \tilde{f}^{(t)} \) of \( u \) (note that this is a reformulation of (3.10)) we obtain, for every \( t, \nu \) they are an extension of \( W \).

**Observation 3.8.** When the construction ends, we have

\[
\tilde{f}(g) \in K[[x]][u_0 < n_1 u_1 < n_2 \cdots [u_{t-1} < n_t z_t]
\]

and \( \Delta^W(\tilde{f}(g); x, u_0, \ldots, u_{g-1}; u_g) \) (with \( u_g \), such that the polyhedron is minimal) is either empty or has more than one vertex. If the polyhedron is empty, then \( \tilde{f}(g) = u_g \). Otherwise \( f \) would be not irreducible.

Furthermore, we constructed an extension of \( W_0 : \mathbb{Z}^d_{\geq 0} \rightarrow \mathbb{Q}^d_{\geq 0} \) (on \( K[[x]] \)) to

\[
W_* := W_0 : \mathbb{Z}^d_{\geq 0} \times \mathbb{Z}^g_{\geq 0} \rightarrow \mathbb{Q}^d_{\geq 0} \quad \text{(on \( K[[x]][u_0, u_1, \ldots, u_{g-1}] \))},
\]

\[
W_*(u_j) := W_*(u_j) = v(j+1), \quad \text{for all } 0 \leq j \leq g - 1,
\]

where \( v(j+1) \) is the unique vertex of \( \Delta^W(\tilde{f}(j); x, u_0, \ldots, u_{j-1}; u_j) \) (and \( u_j \) is assumed to be chosen such that the polyhedron is minimal). The linear map \( W_* \) is determined by the \( d \times (d + g) \) matrix with the \( d \times d \) identity matrix \( I_d \) at first and then the matrix with columns given by the \( v(j+1), 0 \leq j \leq g - 1 \), i.e.,

\[
W_* = ( I_d | v(1) | v(2) | \cdots | v(g) ) .
\]

Recall the abbreviation (3.3). Then, using (3.6) and taking into account that we do some minimizing process (3.8) for the polyhedron \( \Delta^{W_{t+1}}( f(t+1); x, u_{<t+1}; u_{t+1}) \), we obtain, for every \( t \geq 0 \),

\[
(3.10) \quad u_{t+1} = u_t^{n_{t+1}} - \rho_t x^a u_{<t}^b + \sum_{(3.12)} \mu_{\alpha, \beta, \beta_+} x^\alpha u_{<t}^\beta u_t^{\beta_+},
\]

where \( a = a(t+1) \in \mathbb{Z}^d_{\geq 0} \) and \( b = b(t+1) \in \mathbb{Z}^g_{\geq 0} \) is the unique solution for

\[
W_*(a, b) = n_{t+1} \cdot v(t+1)
\]

(note that this is a reformulation of (3.7) using \( W_*(a, b) = W_0(a, b) \) since \( W_* \) is an extension of \( W_t \) and the second sum ranges over those \( \alpha = \alpha(t + 2) \in \mathbb{Z}^d_{\geq 0} \) and \( (\beta, \beta_+) = \beta(t + 2) \in \mathbb{Z}^{d+1}_{\geq 0} \) which fulfill

\[
\frac{W_*(\alpha, \beta)}{n_{t+1} - \beta_+} \geq_{\text{poly}} v(t+1)
\]
By using $W_*(\alpha, \beta, \beta_+) = W_*(\alpha, \beta) + \beta_+ \cdot v(t + 1)$ we obtain the equivalent condition
\[(3.12) \quad W_*(\alpha, \beta, \beta_+) >_{\text{poly}} n_{t+1} \cdot v(t + 1)\]
Here we use the product order on $\mathbb{Q}_d^\ell$, see Notation 3.

To illustrate the construction of the invariant, let us do some examples before continuing with our results.

**Example 3.9.** Consider the curve determined by
\[f = [(z^7 - x^7)^2 - x^{21} - x^{18} z^3]^2 + x^{43} \in K[[x]][z]\]
The construction tells us to start with $W_0$. As one computes easily, the polyhedron $\Delta^{W_0}(f; x; z) \subset \mathbb{R}_{\geq 0}$ is minimal, hence $u_0 = z$. Moreover, it has the only vertex $v(1) = 1$ and
\[F_{v(1),W_0} = (U_0^7 - X^7)^2.\]
The attentive reader surely recognized that $\gamma_1 := v(1) \in \mathbb{Z}$, which contradicts the statement. Thus one of the assumptions must fail – namely, the given $f$ is not irreducible, see for example [RS] Theorem 2.4.

If we ignore the latter and introduce the $R_1$ as in the construction then we have $z_1 = u_0^7 - x^7$,
\[\widehat{f}(1) = [z_1^2 - x^{21} - x^{18} u_0^3]^2 + x^{43}\]
and the extension of $W_0$ to $W_1 : \mathbb{Q}_{\geq 0}^2 \to \mathbb{Q}_{\geq 0}$ is given by $W_1(x) = 1, W_1(u_0) = 1$. The polyhedron $\Delta^{W_1}(\widehat{f}(1); x, u_0; z_1) \subset \mathbb{R}_{\geq 0}$ has the single vertex $v(2) = \frac{21}{2}$ and
\[\widehat{F}(1)_{v(2),W_1} = (Z_1^2 - X^{21} - X^{18} U_0^3)^2.\]
(This looks familiar, see Example 2.10). Since we only have $x^7 = u_0^7 - z_1$ in $R_1$ we can not proceed as in Example 3.1 in order to get the desired shape of the initial form.

**Example 3.10.** Let us give an example of an irreducible hypersurface. Consider
\[f = (z^2 - x_1 x_2 x_3)^3 - x_1^6 x_2^7 x_3^3 z.\]
The polyhedron $\Delta^{W_0}(f; x; z) \subset \mathbb{R}_{\geq 0}^3$ is minimal, $u_0 = z$, and has only one vertex $v(1) = (\frac{1}{7}, \frac{1}{7}, \frac{1}{7})$. We have
\[F_{v(1),W_0} = (U_0^3 - X_1 X_2 X_3)^3.\]
Hence we have $z_1 = u_0^3 - x_1 x_2 x_3$ in $R_1$. Then we extend $W_0$ to $W_1$ by defining $W_1(u_0) := (\frac{7}{2}, \frac{7}{2}, \frac{7}{2})$. We get
\[\widehat{f}(1) = z_1^3 - x_1^6 x_2^7 x_3^3 u_0\]
and $\Delta^{W_1}(\widehat{f}(1); x, u_0; z_1) \subset \mathbb{R}_{\geq 0}^3$ has the single vertex $v(2) = (\frac{5}{7}, \frac{17}{6}, \frac{3}{7})$, $u_1 = z_1$. Furthermore, $f(1) = f(\widehat{f}(1))$ and
\[F_{v(2),W_1} = U_1^3 - X_1^6 X_2^7 X_3^3 U_0\]
Therefore we obtain $u_2 = z_2 = f(1)$ in $R_2$ and the construction ends:
\[\kappa(f; x; z) = \left( \left( \frac{1}{7}, \frac{1}{7}, \frac{1}{7} \right); \left( \frac{5}{2}, \frac{17}{6}, \frac{3}{7} \right); \infty \right).\]
Example 3.11. We encourage the studious reader to apply the construction for
\[ f = (z^3 - x_1^2 x_2)^2 + 2(z^3 - x_1^2 x_2)x_1^3 x_2 x_3 z + x_1^6 x_2^2 x_3^2 z^2 - x_1^3 x_2^3 z^3. \]

The main result of this article is the following characterization theorem

**Theorem 3.12.** Let \( f \in K[[x]][z] \) be an irreducible Weierstrass polynomial of degree \( n \). Then \( f \) is quasi-ordinary with respect to the chosen projection defined by \((x, z)\) if and only if the last entry of \( \kappa(f; x; z) \) is \( \infty \).

If the last entry of \( \kappa(f; x; z) \) is \(-1\), then the theorem implies that \( f \) is not quasi-ordinary with respect to the given projection determined by \((x, z)\). But still it is possible that there is another projection such that the given singularity is quasi-ordinary.

**Example 3.13.** For \( f = x_1^2 + z^3 + z^2 x_2 \) we have \( \kappa(f; x; z) = -1 \). On the other hand, if we pick \((y_1, y_2, w) = (z, x_2, x_1)\) then \( f = w^2 + y_1^2 + y_2^3 y_2 = w^2 + y_1^2 (y_1 + y_2) \) is quasi-ordinary, \( \kappa(f; y; w) = (1, 1, 2; \infty) \).

We begin by proving the easy direction of the theorem. The other direction is left to the next section.

**Proposition 3.14.** Let \( f \in K[[x]][z] \) be an irreducible Weierstrass polynomial of degree \( n \). If \( f \) is quasi-ordinary with respect to the chosen projection defined by \((x, z)\) then the last entry of \( \kappa(f; x; z) \) is \( \infty \).

**Proof.** The main ingredient is the description of the equation of a quasi-ordinary polynomial \( f \) in terms of its approximate roots by Gonzalez-Perez [GP1]. We take \( q_0, \ldots, q_g \) to be approximate roots of \( f \) (these are the annihilator polynomials of truncations of the roots of \( f \) [CM]). Note that \( q_g = f \) and, in our notation, \( q_0 = u_0 \).

Recall the notations \( n_j \) and \( \gamma_j \), \( j \in \{1, \ldots, g\} \), that have been defined in the first section. Indeed, it follows from [GP1] Lemma 35 that, for \( j \in \{1, \ldots, g\} \), the approximate roots \( q_j \) satisfy
\[
c_j^* q_j = q_j^{n_j} - c_j x^{a(j)} q_{<j-1} + \sum c_{\alpha, \beta} x^\alpha q_{<j}^\beta
\]
(similar to (3.3), we abbreviate \( x^{a(j)} q_{<j-1} = x_1^{a(j)_1} \cdots x_d^{a(j)_d} q_0 \cdots q_{j-2} \), etc.), where \( c_j^* \), \( c_j \in K^\times \), and \( c_{\alpha, \beta} \in K \), and \( a(j), \alpha \in \mathbb{Z}_d^0 \), and \( (b(j), 0) \), \( \beta(j) \in \mathbb{Z}_d^0 \) with \( b(j)_i, \beta(j)_i \leq n_i \) (for \( i \in \{1, \ldots, j\} \)), and, for all \( (\alpha, \beta) \) with \( c_{\alpha, \beta} \neq 0 \),
\[
n_j \gamma_j = a(j) + b(j)_1 \gamma_1 + \cdots + b(j)_{j-1} \gamma_{j-1} < \text{poly } \alpha + \beta_1 \gamma_1 + \cdots + \beta_j \gamma_j.
\]
Since \( f = q_g \), applying Construction 3.5 shows that
\[
u_i = q_i, \quad \text{for all } i \in \{0, \ldots, g\}.
\]
Hence \( \kappa(f; x; z) = (\gamma_1; \ldots; \gamma_g; \infty) \).

As a direct application of the proof of Proposition 3.14 we obtain a new algorithm to determine approximate roots (or semi-roots) of a quasi-ordinary polynomial:

**Corollary 3.15.** Let \( f \in K[[x]][z] \) be an irreducible quasi-ordinary polynomial of degree \( n \) and let
\[
\kappa(f; x; z) = (v(1); v(2); \ldots; v(g); \infty).
\]
The approximate roots of \( f \) are given by the constructed \( u_i, i \in \{0, \ldots, g\} \).
4. Computing roots by using overweight deformations

Finally, we show the remaining part for the proof of Theorem 3.12 in Proposition 4.1. Let \( f \in K[[x]][z] \) be an irreducible polynomial of degree \( n \). Suppose the last entry of \( \kappa(f; x, z) \) is \( \infty \). Then \( f \) is quasi-ordinary with respect to the chosen projection defined by \( (x, z) \).

The strategy of the proof is as follows: First, we analyse the data provided by the assumption that the last entry of our invariant is \( \infty \). More precisely, we recall briefly Observation 3.8 and explain how this is connected with an overweight deformation. From this we deduce a parametrization of the singularity which then yields a statement about the roots of \( f \) (Theorem 4.4). In the final step, we study the parametrization closely and compute the difference of two roots of \( f \) explicitly. Since the discriminant is determined by the square of these differences this implies then that \( f \) is quasi-ordinary.

As we have seen in Observation 3.8, if the last entry of \( \kappa(f; x, z) \) is \( \infty \), we constructed a linear map \( W_* \) which can be identified with a weight map

\[
W := W_* : K[[x]][u_0, u_1, \ldots, u_{g-1}] \rightarrow \mathbb{Q}^d.
\]

By (3.10), the singularity \( V(f) \) can be identified with the fiber \( x_t \) above \( T = 1 \) of the map \( x \rightarrow \text{Spec } K[T] \), for \( x = V(F_0, \ldots, F_{g-1}) \) the variety given by

\[
F_t \in K[[x]][u_0, u_1, \ldots, u_{g-1}, T], \quad t \in \{0, \ldots, g-1\},
\]

where (using the abbreviation (3.3) and Proposition 3.3)

\[
(4.1) \quad F_t = T u_{t+1} - u_{t+1}^{n_{t+1}} + \rho_t x^a u^b_{<t} - T \cdot \left( \sum_{\alpha, \beta} \mu_{\alpha, \beta} x^\alpha u^\beta_{\leq t} \right),
\]

where \( a = a(t+1) \in \mathbb{Z}_{\geq 0}^d, b = b(t+1) \in \mathbb{Z}_{\geq 0}^d, \alpha = \alpha(t+2) \in \mathbb{Z}_{> 0}^d, (\beta, \beta_+) = \beta(t+2) \in \mathbb{Z}_{> 0}^d \), and \( \rho_t, \mu_{\alpha, \beta, \beta_+} \in K \). By convention, \( u_g = 0 \) and further we have that (using Notation 3)

\[
(4.2) \quad \left\{ \begin{array}{ll}
W(u_t^{n_{t+1}}) &= W(x^a u^b_{<t}) <_{\text{poly}} W(x^\alpha u^\beta_{\leq t} u_{t+1}^\gamma), \\
W(u_t^{n_{t+1}}) &= W(u_{t+1}) <_{\text{poly}} W(u_{t+1}).
\end{array} \right.
\]

The first line follows by (3.11) and (3.12). The second can be deduced from the fact that \( v(t+1) \) is the only vertex of \( \Delta W^i(f(t); x, u_0, \ldots, u_{t-1}; u_t) \) (Construction 3.5).

For \( t \in \{0, \ldots, g-1\} \), we set

\[
\gamma_{t+1} := W(u_t) = v(t+1).
\]

By (3.9), \( W \) is determined by the matrix \( (I_{d} | \gamma_1 | \ldots | \gamma_g) \). Hence (4.2) can be rewritten as

\[
\left\{ \begin{array}{l}
\gamma_{t+1} \gamma_{t+1} = a + \sum_{i=1}^t b_i \gamma_i <_{\text{poly}} \alpha + \sum_{i=1}^t \beta_i \gamma_i + \beta_+ \gamma_{t+1}, \\
\gamma_{t+1} \gamma_{t+1} = \gamma_{t+2} <_{\text{poly}} \gamma_{t+2},
\end{array} \right.
\]

where we write \( b = (b_1, \ldots, b_t) \) and \( \beta = (\beta_1, \ldots, \beta_t) \). Note that actually \( b = b(t+1) \) and \( (\beta, \beta_+) = \beta(t+2) \) but in order to avoid too complicated expressions...
we use these references only when they are really needed. Otherwise we suppress them.

Let \( \mathcal{X}_0 \) be the special fiber which is defined by the above equations where we replace \( T \) by 0, i.e., with the notations of (4.1)
\[
(4.3) \quad u_i^{n_i+1} - \rho_t \mathbf{X}^a \mathbf{u}^b = 0 , \quad t \in \{0, \ldots, g - 1\}.
\]
Note that \( \mathcal{X}_0 \) is a toric variety. Consider the point on \( \mathcal{X}_0 \) given by
\[
(x_1, \ldots, x_d, u_0, u_1, \ldots, u_{g-1}) = (1, \ldots, 1, c_0, c_1, \ldots, c_{g-1}).
\]
By using this in (4.3), we can determine the entries \( c_0, \ldots, c_{g-1} \). Namely,
\[
(4.4) \quad \begin{cases} 
  c_0 = \eta_1 \cdot \rho_0^{\frac{1}{n_1}}, & \text{where } \eta_1^{n_1} = 1 \\
  c_t = \eta_t \cdot (\rho_t \cdot c_b^{\frac{1}{t}}), & \text{where } \eta_t^{n_t} = 1, \quad \text{for } t \geq 1,
\end{cases}
\]
and we abbreviate again \( c_b^t = c_0^{b_{(t+1)}}, \ldots, c_1^{b_{(t+1)}}, \) for \( b = b(t+1) \in \mathbb{Z}_{\geq 0}^d \).

For each root of unity \( \eta_t \), there are \( n_t \) choices and we get in total \( n_1 \cdot n_2 \cdot \ldots \cdot n_g = n \) choices for \( (c_0, \ldots, c_{g-1}) \). Later we will see that these \( n \) choices determine the \( n \) different roots of the original quasi-ordinary hypersurface \( f \).

We can assume without loss of generality that \( (1, \ldots, 1, c_0, c_1, \ldots, c_{g-1}) \) does not belong to the singular locus of \( \mathcal{X}_0 \). Let \( S = (s_1, \ldots, s_d) \) be a set of \( d \) independent variables. A parametrization of \( \mathcal{X}_0 \) is then given by
\[
(4.5) \quad \begin{cases} 
  x_i = s_i^{n_i}, & i \in \{1, \ldots, d\}, \\
  u_0 = c_0 S^{n_1}, \\
  u_t = c_t S^{n_1+1}, & t \in \{1, \ldots, g-1\},
\end{cases}
\]
Recall that we denoted for \( i \in \{1, \ldots, d\} \) by \( e_i \) be the vector in \( \mathbb{R}^d \) whose \( i \)-th coordinate is 1 and all others are 0. Then we have

**Lemma 4.2.** The germ \( \mathcal{X}_1 = V(f) \) has a parametrization of the form
\[
(4.6) \quad \begin{cases} 
  x_i = s_i^{n_i} + \sum_{M \triangleright poly \ n \cdot e_i} X_{i, M} S^M, & i \in \{1, \ldots, d\}, \\
  u_0 = c_0 S^{n_1} + \sum_{M \triangleright poly \ n \cdot \gamma_1} Z_M S^M, \\
  u_t = c_t S^{n_1+1} + \sum_{M \triangleright poly \ n \cdot \gamma_1+1} U_{t, M} S^M, & t \in \{1, \ldots, g-1\}.
\end{cases}
\]

**Proof.** We are searching for a solution of the equations (4.1) (with \( T = 1 \)) in the power series ring
\[ K[[S]] = K[[s_1, \ldots, s_d]] \]
which lifts the solution of \( \mathcal{X}_0 \) given in (4.5). Let \( f_r, r \in \{0, \ldots, g-1\} \), be the series obtained from \( F_r \) in (4.1) by replacing \( T \) by 1.

Let us denote by \( f_r((4.6)) \) the expansion which we obtain if we replace \( (x, u) \) by the right hand side of (4.6). This can be written as
\[ f_r((4.6)) = \sum_{N} f_r, N S^N, \]
where \( f_r, N \) (\( r \in \{0, \ldots, g-1\} \), \( N \in \mathbb{Z}^d_{\geq 0} \)) are polynomials in the variables \( X_{i, M} (i \in \{1, \ldots, d\}) \), \( Z_M \) and \( U_{t, M} (t \in \{0, \ldots, g-1\}) \) with \( M \in \mathbb{Z}^d_{\geq 0} \).

The weight conditions and the fact that (4.5) gives a parametrization of \( \mathcal{X}_0 \) imply that \( f_r, N \equiv 0 \) for all \( N \in \mathbb{Z}^d_{\geq 0} \) with \( N \not\equiv poly n_r+1n_{\gamma r+1} \) or \( N = n_r+1n_{\gamma r+1} \). (Recall Notation 3 in order to keep in mind which \( \not\equiv poly \) means). To conclude, we need to
Theorem 4.4. Let $f \in K[[x]][z]$ be an irreducible Weierstrass polynomial of degree $n$. Assume that the last entry of $\kappa(f; x; z)$ is $\infty$. Then the roots of $f$ as polynomial in $z$ are contained in $K[[x_1^{\frac{1}{\gamma}}, \ldots, x_d^{\frac{1}{\gamma}}]]$.

Proof. Note that we can write $x_i = s_i^n \cdot \epsilon_i$ in the parametrization found in Lemma 4.2, where $\epsilon_i$ is a unit in $K[[S]]$, for all $i \in \{1, \ldots, d\}$. Since we are in characteristic zero, we can extract an $n$-th root of $\epsilon_i$ and after a change of variables we can write $x_i = s_i^n, i \in \{1, \ldots, d\}$. Hence $s_i = x_i^{\frac{1}{\gamma}}, i \in \{1, \ldots, d\}$. After replacing all $s_i$ by $x_i^{\frac{1}{\gamma}}$ in the expansion of $z = u_0(S) + h_0(x(S))$ in Lemma 4.2, we obtain that $f$ has a root in $K[[x_1^{\frac{1}{\gamma}}, \ldots, x_d^{\frac{1}{\gamma}}]]$. But the extension of the field of fractions of $K[[x]]$ to the field of fractions of $K[[x_1^{\frac{1}{\gamma}}, \ldots, x_d^{\frac{1}{\gamma}}]]$ is Galois and hence all the other roots are also contained in $K[[x_1^{\frac{1}{\gamma}}, \ldots, x_d^{\frac{1}{\gamma}}]]$. \qed

Observation 4.5. In the previous two results we have seen that it is important to determine the coefficients $Z_M$ if we want to study the roots of $f$. Let us have a look at $f_{0,N}$ which is coming from

$$f_0 = u_1 - u_0^{n_1} + \rho_0 x^n - \left(\sum \mu_{\alpha, \beta+} x^\alpha u_0^{\beta_+}\right),$$

where the sum ranges over those $(\alpha, \beta_+) \in \mathbb{Z}^d$ such that

$$\alpha + \beta_+ + \gamma_1 > \text{poly } n \gamma_1.$$

We have seen that we can pick $x_i = s_i^n$, for all $i \in \{1, \ldots, d\}$, and by (4.6) we have to plug in

$$u_0 = c_0 S^{n \gamma_1} + \sum Z_M \text{ poly } n \gamma_1 Z_M S^M,$$

$$u_1 = c_1 S^{n \gamma_2} + \sum U_{M} \text{ poly } n \gamma_2 U_{1,M} S^M.$$
Thus
\[ f_0 = (c_1 S^{n\gamma_2} + \sum_{M > \text{poly} n^{\gamma_2}} U_{1.M} S^M) - (c_0 S^{n\gamma_1} + \sum_{M > \text{poly} n^{\gamma_1}} Z_M S^M)^{n_1} + + \rho_0 S^n - \left( \sum_{\mu_{\alpha,\beta} \neq 0} S^{n\alpha} (c_0 S^{n\gamma_1} + \sum_{M > \text{poly} n^{\gamma_1}} Z_M S^M)^{\beta} \right) \]
\[ = \sum_N f_{0,N} S^N. \]

We already mentioned that \( f_{0,N} \equiv 0 \) for \( N \not\in \text{poly} n\gamma_1 \) or \( N = n_1 n\gamma_1 \). Therefore we only need to consider \( N = n_1 n\gamma_1 + \mathbf{P} >_{\text{poly}} n_1 n\gamma_1 \). We set \( M(N) := n\gamma_1 + \mathbf{P} \) and claim
\[
(4.9) \quad f_{0,N} = \begin{cases}
-n_1 c_0^{n_1 - 1} Z_{M(N)} + h_\prec, & \text{if } N \not\in \text{poly} n\gamma_2, \\
 c_1 - n_1 c_0^{n_1 - 1} Z_{M(N)} + h_\prec, & \text{if } N = n\gamma_2, \\
 U_{1,N} - n_1 c_0^{n_1 - 1} Z_{M(N)} + h_\prec, & \text{if } N >_{\text{poly}} n\gamma_2,
\end{cases}
\]
where \( h_\prec \) is a polynomial only in variables \( Z_M \) which are already known, i.e., for which \( M <_{\text{poly}} M(N) \). The only part that one might has to think about is the property on \( h_\prec \). This can be seen with the following arguments:

- Consider the term \( (n_1 c_0^{n_1 - 1})^{1-j} S^{n\gamma_1(1-j)} \left( \sum_{M > \text{poly} n\gamma_1} Z_M S^M \right)^j \), for some \( j \in \{2, \ldots, n_1\} \). Then for each term with non-zero coefficient the exponent of \( S \) is of the form
  \[ n\gamma_1(n_1 - j) + M(1) + \ldots + M(j) = n\gamma_1 n_1 + \mathbf{P}(1) + \ldots + \mathbf{P}(j), \]
  where \( M(k) = n\gamma_1 + \mathbf{P}(k), \mathbf{P}(k) >_{\text{poly}} 0, \) for all \( k \in \{1, \ldots, j\} \). This exponent coincides with \( N = n_1 n\gamma_1 + \mathbf{P} \) if and only if
  \[ \mathbf{P} = \mathbf{P}(1) + \ldots + \mathbf{P}(j). \]
  But since all \( \mathbf{P}(k) \) are non-zero this implies \( \mathbf{P} >_{\text{poly}} \mathbf{P}(k) \), for all \( k \), which on the other hand means
  \[ M(k) = n\gamma_1 + \mathbf{P}(k) <_{\text{poly}} n\gamma_1 + \mathbf{P} = M(N), \]
  for all \( k \in \{1, \ldots, j\} \).

- By using (4.8) and the previous arguments, one can show that we also have \( \bar{M} <_{\text{poly}} M(N) \) for those \( Z_{\bar{M}} \) appearing in \( f_{0,N} \) and coming from
  \[ \sum_{\mu_{\alpha,\beta} \neq 0} S^{n\alpha} \cdot U_{1,N} \cdot U_{0,N} \]

Set \( C := n_1 c_0^{n_1 - 1} \). In fact, this is the constant \( C \) mentioned in (4.7). Since we must have \( f_{0,N} = 0 \) for all \( N \) we obtain from (4.9)
\[
Z_{M(N)} = \begin{cases}
C^{-1} \cdot h_\prec, & \text{if } N \not\in \text{poly} n\gamma_2, \\
C^{-1} \cdot (c_1 + h_\prec), & \text{if } N = n\gamma_2, \\
C^{-1} \cdot (U_{1,N} + h_\prec), & \text{if } N >_{\text{poly}} n\gamma_2,
\end{cases}
\]
Therefore we understand \( Z_{M(N)} \) quite well in the first two cases, whereas we have to determine \( U_{1,N} \) for the last one. Before we come to this, let us mention that \( N = n_1 n\gamma_2 \) is equivalent to
\[ M(N) = n\gamma_1 + (n\gamma_2 - n_1 n\gamma_1) = n(\gamma_2 - n_1 \gamma_1 + \lambda_1) = n\lambda_2, \]
where we use \( \lambda_1 := \gamma_1 \) and \( \lambda_2 := \gamma_2 - n_1 \gamma_1 + \lambda_1 \). (Compare the last definitions with (1.1)).
In general, one can show with the same arguments that, for \( r \geq 1 \),
\[
 f_{r, \mathbf{N}} = \begin{cases} 
   -n_{r+1}c_{r+1}^{-n_{r+1}-1}U_r, & \text{if } \mathbf{N} \not< \text{poly } n\gamma_{r+2}, \\
   c_{r+1} - n_{r+1}c_{r+1}^{-n_{r+1}-1}U_r, & \text{if } \mathbf{N} = n\gamma_{r+2}, \\
   U_{r+1, \mathbf{N}} - n_{r+1}c_{r+1}^{-n_{r+1}-1}U_r, & \text{if } \mathbf{N} > \text{poly } n\gamma_{r+2},
\end{cases}
\]
where \( h_{\prec} \) is a polynomial in variables that are already known. Moreover, \( \mathbf{N} = n\gamma_{r+2} \) is equivalent to
\[
 M(\mathbf{N}) = n\lambda_{r+2},
\]
where \( \lambda_{r+2} := \gamma_{r+2} - n_{r+1}\gamma_{r+1} + \lambda_{r+1} \).

As conclusion we get that
\[
 (4.10) \quad u_0 = c_0 S^{\lambda_1} + \sum (c_0) + \sum_{r=1}^{g-1} \left( d_r(c_0, \ldots, c_r) S_n^{\lambda_r+1} + \sum (c_0, \ldots, c_r) \right),
\]
where
1. \( d_r(c_0, \ldots, c_r) \) is a polynomial in \((c_0, \ldots, c_r)\) which is linear in \( c_r \) and
2. \( \sum (c_0) := \sum Z_M(c_0) S^M \) (resp. \( \sum (c_0, \ldots, c_r) = (\sum Z_M(c_0, \ldots, c_r) S^M) \)) is an abbreviation for the intermediate terms whose coefficients do only depend on \( c_0 \) (resp. \( (c_0, \ldots, c_r) \)). By this we mean those \( M \) with \( M > \text{poly } n\lambda_1 \), but \( M \not> n\lambda_2 \) (resp. with \( M > \text{poly } n\lambda_{r+1} \), but \( M \not> n\lambda_{r+2} \)).

Further, recall that \( z = u_0 + h_0(x) \).

Finally, we have determined all the coefficients \( Z_M \) of the parametrization of \( u_0 \). If we replace now each \( s_i \) by \( x_i^{\lambda_i} \), \( i \in \{1, \ldots, n\} \), then we obtain the different roots of \( f \). (Keep in mind that the different choices for \((c_0, c_1, \ldots, c_{g-1})\) determine all the roots of \( f \).

Now, we can come to the

Proof of Proposition 4.1. We want to prove that \( f \in K[[x]] [z] \) is quasi-ordinary. By the assumption that our invariant ends with \( \infty \) we obtained an overweight deformation from which we constructed the roots in the previous parts. It remains to show that the difference of the different roots are monomials times a unit.

By (4.4), a root \( \zeta \) is determined by the choice of certain roots of unity \( (\eta_1, \ldots, \eta_g) \) which determine the values \((c_0, \ldots, c_{g-1})\). Let \( \zeta_1, \zeta_2 \) be two roots of \( f \) and let \((\eta_1^{(1)}, \ldots, \eta_g^{(1)}) \) resp. \((\eta_1^{(2)}, \ldots, \eta_g^{(2)}) \) be the corresponding roots of unity. If \( \eta_1^{(1)} \neq \eta_1^{(2)} \) then it follows from (4.4) and (4.10) that
\[
 \zeta_1 - \zeta_2 = x^{\lambda_1} \cdot \epsilon_1,
\]
for a unit \( \epsilon_1 \). Note that \( \epsilon_1 \) corresponds to a unit in \( K[[S]] \). (Recall that we have to replace \( s_i \) by \( x_i^{\lambda_i} \) for all \( i \)).

Therefore suppose \( \eta_j^{(1)} = \eta_j^{(2)} \), for all \( j \in \{1, \ldots, r-1\} \), and \( \eta_r^{(1)} \neq \eta_r^{(2)} \), for some \( r \geq 1 \). Since all the terms in (4.10) before \( S^{n\lambda_r} \) do only depend on \( c_0, \ldots, c_{r-1} \) (which coincide for both roots by the assumption on the roots of unity and (4.4)) these must vanish in \( \zeta_1 - \zeta_2 \). Moreover, since \( d_r(c_0, \ldots, c_r) \) is linear in \( c_r \) we get
\[
 \zeta_1 - \zeta_2 = x^{\lambda_r} \cdot \epsilon_r,
\]
for some unit \( \epsilon_r \) as before. This implies the assertion of the proposition. \( \square \)
Further, this reveals that the invariant $\kappa(f; x; z)$ does not only tell us if a given $f$ is quasi-ordinary, but provides even more information in the quasi-ordinary case. More precisely,

**Proposition 4.6.** Let $f$ be quasi-ordinary and irreducible and

$$\kappa(f; x; z) = (v(1); v(2); \ldots; v(g); \infty).$$

The entries $(v(1), \ldots, v(g)) = (\gamma_1, \ldots, \gamma_g)$ yield the minimal system of generators of the semi-group associated to the quasi-ordinary singularity.

**Proof.** This is a direct application of the proof of Proposition 3.14. □

Besides the order of a hypersurface at a point another measure for the complexity of a singularity is the so called log canonical threshold. In [BGG], Theorem 3.1, it was proved that the log canonical threshold of a quasi-ordinary singularity can be computed from the generators of the corresponding semi-group. Therefore we obtain:

**Corollary 4.7.** If $f$ is an irreducible quasi-ordinary hypersurface singularity and $(x, z)$ such that the last entry of $\kappa(f; x; z)$ is $\infty$, then one can compute from $\kappa(f; x; z)$ the log canonical threshold of $f$.

**Remark 4.8.** Finally, let us remark that our original approach to the characterization was via [ACLM2] (see also [GV]). There, quasi-ordinary singularities are characterized using the Newton process and Newton trees. One can show that there is a one-to-one correspondence between the Newton process and our construction which may yield another proof of our characterization. Nevertheless, as it turned out, we were able to avoid this technical notation and could prove our characterization directly. And it is worth noticing that the two characterizations are different in nature. This difference may be compared to the difference between resolution of singularities of quasi-ordinary singularities by a sequence of toric maps corresponding to the Puiseux pairs, and their resolution by one toric map, see [GP1].

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