The stochastic nonlinear Schrödinger equations driven by pure jump noise

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Abstract
In this paper, we establish the existence and uniqueness of solutions of stochastic nonlinear Schrödinger equations with additive jump noise in $L^2(\mathbb{R}^d)$. Our results cover all either focusing or defocusing nonlinearity in the full subcritical range of exponents as in the deterministic case.

Keywords: Nonlinear Schrödinger equation; Stochastic Strichartz estimate; pure jump noise
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1. Introduction and motivation
In this paper, we consider the following stochastic nonlinear Schrödinger equations driven by pure jump noise:

$$dX(t) = i[\Delta X(t) - \lambda |X(t)|^{\alpha - 1}X(t)]dt + dL(t), \quad X(0) = x \in L^2(\mathbb{R}^d),$$

where $\lambda \in \{-1, 1\}$ and $1 < \alpha < 1 + \frac{4}{d}$ and $L = (L(t))_{t \geq 0}$ is an $L^2(\mathbb{R}^d)$-valued pure jump Lévy process defined as

$$L(t) = \int_0^t \int_B z\tilde{N}(dz, dt),$$

where $B = \{z \in L^2(\mathbb{R}^d) : 0 < \|z\|_{L^2(\mathbb{R}^d)}^2 \leq 1\}$. Here, $N$ represents a time homogeneous Poisson random measure over $(\mathfrak{B}(L^2(\mathbb{R}^d)), \mathfrak{B}(\mathbb{R}^+))$ with $\sigma$-finite intensity measure $\nu$ satisfying $\int_{L^2(\mathbb{R}^d) \setminus \{0\}} \|z\|_{L^2(\mathbb{R}^d)}^2 \wedge 1 \nu(dz) < \infty$ and $\tilde{N}(dz, dt) = N(dz, dt) - \nu(dz)dt$ denotes the corresponding compensated Poisson random measure.

The nonlinear Schrödinger equation appears in a variety of applications in nonlinear physics, including nonlinear optics, nonlinear water propagation, nonlinear acoustics, quantum condensates etc. In recent years, existence and uniqueness of solutions for the stochastic nonlinear Schrödinger equation with Gaussian noise have been investigated by many authors \cite{1, 2, 3, 4, 7, 10}. When it comes to Lévy noise, the current literature is still very limited. In \cite{7}, Brzeźniak et al. proved the existence of a martingale solution of the stochastic nonlinear Schrödinger equation with a multiplicative jump noise in the Marcus canonical form in $H^1$ on compact manifolds by using the Giaquinto-Galerkin method. De Bouard and Hausenblas investigated the existence of martingale solutions of the nonlinear Schrödinger equation with a Lévy noise with infinite activity in \cite{5} and pathwise uniqueness was studied in a separate paper \cite{6} with Ondrejat. Recently, Brzeźniak et al. established a new version of the stochastic Strichartz estimate for the stochastic convolution driven by a jump noise in \cite{8}. By applying the stochastic Strichartz estimates in a fixed point argument, they proved the existence and uniqueness of a global solution to stochastic nonlinear Schrödinger equation with a Marcus-type jump noise in $L^2(\mathbb{R}^d)$ with either focusing or defocusing nonlinearity in the full subcritical range of exponents.

The purpose of this paper is to prove the existence and uniqueness of mild solutions for the stochastic nonlinear Schrödinger equation \cite{11} with the additive jump noise. By means of the deterministic and stochastic Strichartz’s
estimates due to Brzeźniak et al. from [8], we apply the classical truncation procedure of the nonlinearities and use the well-known fixed point argument to construct a local solution. One main ingredient of establishing global solution in [8] is the mass conservation for the stochastic nonlinear Shrödinger equation. Unlike the $L^2(\mathbb{R}^d)$-norm conservation of solutions in the literature [8], we establish some uniform $L^2(\mathbb{R}^d)$-norm estimate of the local solution and select some specific stopping times that we believe are critical to the proof. Let us formulate our main result of this paper.

**Theorem 1.1.** Let $p \geq 2$, $1 < \alpha < 1 + \frac{1}{2}$, $r = \alpha + 1$ such that $(p, r)$ is an admissible pair. For any $x \in L^2(\mathbb{R}^d)$, there exists a unique global mild solution $X^x = (X^x(t), t \in [0, \infty))$ of (1) such that

$$X^x \in D([0, \infty); L^2(\mathbb{R}^d)) \cap L^p(0, T; L^r(\mathbb{R}^d)), \text{ $\mathbb{P}$-a.s. } \omega \in \Omega.$$  

Moreover, we have for all $t \in [0, T]$ and any $q \geq 2$

$$\mathbb{E} \sup_{s \in [0, t]} \|X(s)\|_{L^q(\mathbb{R}^d)}^q \leq C(q, T, \|x\|_{L^2(\mathbb{R}^d)}).$$  

(2)

2. Setting and Strichartz estimates

In this section, we introduce some notations, assumptions and solution concepts and recall deterministic and stochastic Strichartz estimates, which will be used to prove the well-posedness of the solution of the truncated equation.

For $t > 0$, let us denote $D(0, t; L^2(\mathbb{R}^d))$ the space of all right continuous functions with left-hand limits from $[0, t]$ to $L^2(\mathbb{R}^d)$ and

$$Y_t := L^2(0, t; L^2(\mathbb{R}^d)) \cap L^p(0, t; L^r(\mathbb{R}^d)).$$  

(3)

Then $Y_t$ is a Banach space with norm defined by

$$\|u\|_{Y_t} := \sup_{s \in [0, t]} \|u(s)\|_{L^2(\mathbb{R}^d)} + \left(\int_0^t \|u(s)\|_{L^r(\mathbb{R}^d)}^r \, ds\right)^{\frac{1}{r}}, \text{ for } u \in Y_t.$$  

(4)

For the details we refer to (2.10) and (2.11) in [8].

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a filtered probability space satisfying the usual conditions, where $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is the filtration.

Let $\tau > 0$ be a stopping time. We call $\tau$ an accessible stopping time if there exists an increasing sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times such that $\tau_n < \tau$ and $\tau_n \not\rightarrow \tau \text{ $\mathbb{P}$-a.s. as } n \rightarrow \infty$ and we call $(\tau_n)_{n \in \mathbb{N}}$ an approximating sequence for $\tau$.

Let $M^p_\mathbb{F}(Y_t)$ denote the space of all $L^2(\Omega; L^2(0, \tau; L^2(\mathbb{R}^d)) \cap L^p(0, \tau; L^r(\mathbb{R}^d)))$-valued $\mathbb{F}$-progressively measurable processes $u : [0, T] \times \Omega \rightarrow L^2(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)$ satisfying

$$\|u\|_{M^p_\mathbb{F}(Y_t)} := \mathbb{E}\|u\|_{Y_t}^p = \mathbb{E}\left(\sup_{s \in [0, \tau]} \|u(s)\|_{Y_t}^p + \int_0^\tau \|u(s)\|_{Y_t}^p \, ds\right) < \infty.$$

(5)

Now we introduce the definitions of local solutions and maximal local solutions, see e.g. [12] for more details.

**Definition 2.1.** A local mild solution to equation (1) is an $\mathbb{F}$-progressively measurable process $X(t), t \in [0, \tau)$, where $\tau$ is an accessible stopping time with an approximating sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times such that for every $n \in \mathbb{N}$,

(i) $(X(t))_{t \in [0, \tau_n]} \in D(0, \tau_n; L^2(\mathbb{R}^d)), \text{ $\mathbb{P}$-a.s.}$;

(ii) $(X(t))_{t \in [0, \tau_n]} \text{ belongs to } M^p_\mathbb{F}(Y_{\tau_n});$

(iii) for every $t \in [0, T]$, the following equality holds

$$X(t \wedge \tau_n) = S_{t \wedge \tau_n}x - i\lambda \int_0^{t \wedge \tau_n} S_{t \wedge \tau_n-s}(|X(s)|^{q-1}X(s)) \, ds + \int_0^{t \wedge \tau_n} \int_B 1_{[0, \tau_n]}(s) S_{t \wedge \tau_n-s}zN(dz, ds), \text{ $\mathbb{P}$-a.s.}$$
**Definition 2.2.** A local mild solution \( X = (X(t))_{t \in [0, T]} \) is called a maximal local mild solution if for any other local mild solution \((Y(t))_{t \in [0, T]}\), satisfying \( \sigma \geq \tau \) a.s. and \( Y|_{[0, \tau]} \) is equivalent to \( X \), one has \( \sigma = \tau \) a.s.

A local mild solution \((X(t))_{t \in [0, T]}\) is a global mild solution if \( \tau = T \), \( \mathbb{P}\)-a.s. and \( t \in M^2_p(Y_T) \).

Throughout the paper, the symbol \( C \) will denote a positive generic constant whose value may change from place to place. If a constant depends on some variable parameters, we will put them in subscripts.

We now state the following deterministic and stochastic Strichartz estimates, we refer the reader to [11] and [8]. Propositions 2.2 and 2.6. Let \((S_t)_{t \in \mathbb{R}}\) denote the group of isometries on \( L^2(\mathbb{R}^d) \) generated by \( i\Delta \). We say a pair \((p, r)\) is admissible if \( p, r \in [2, \infty] \) and \((p, r, d) \neq (2, 2, 2)\) satisfying \( \frac{2}{p} + \frac{d}{r} = \frac{d}{2} \) and \( 2 \leq \frac{d}{r} \leq \frac{2d}{d-2} \).

**Lemma 2.1.** Let \((p, r)\) and \((\gamma, \rho)\) be two admissible pairs and let \( \gamma', \rho' \) be conjugates of \( \gamma \) and \( \rho \). Then

1. For every \( \phi \in L^2(\mathbb{R}^d) \), the function \( t \mapsto S_t \phi \) belongs to \( L^p(\mathbb{R}; L^r(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}; L^2(\mathbb{R}^d)) \) and there exists a constant \( C \) such that
   \[
   ||S_t \phi||_{L^p(\mathbb{R}; L^r(\mathbb{R}^d))} \leq C ||\phi||_{L^2(\mathbb{R}^d)}. \tag{5}
   \]

2. Let \( I \) be an interval of \( \mathbb{R} \) and \( J = \bar{I} \) with \( 0 \in J \). Then for every \( f \in L^q(I; L^r(\mathbb{R}^d)) \), the function \( t \mapsto \Phi_t f = \int_0^t S_{t-s} f(s) ds \) belongs to \( L^p(I; L^r(\mathbb{R}^d)) \) and \( L^{\infty}(J; L^2(\mathbb{R}^d)) \) and there exists a constant \( C \) independent of \( I \) such that
   \[
   ||\Phi_t f||_{L^p(I; L^r(\mathbb{R}^d))} \leq C ||f||_{L^q(I; L^r(\mathbb{R}^d))}, \tag{6}
   \]
   \[
   ||\Phi_t f||_{L^{\infty}(J; L^r(\mathbb{R}^d))} \leq C ||f||_{L^q(I; L^r(\mathbb{R}^d))}. \tag{7}
   \]

3. For all \( q \geq 2 \) and all \( \mathbb{F} \)-predictable process \( \xi : [0, T] \times L^2(\mathbb{R}^d) \times \Omega \rightarrow L^2(\mathbb{R}^d; \mathbb{C}) \) in \( L^2(\Omega; L^q([0, T] \times L^2(\mathbb{R}^d); L^2(\mathbb{R}^d))) \), we have
   \[
   \mathbb{E} \left[ \int_0^T \int_{L^2(\mathbb{R}^d)} S_{t-s} \xi(s, z) \tilde{N}(dz, ds) \right] \leq C_q \mathbb{E} \left( \int_0^T \int_{L^2(\mathbb{R}^d)} ||\xi(s, z)||_{L^2(\mathbb{R}^d)}^2 \nu(ds) \right)^{\frac{1}{2}} + C_q \mathbb{E} \left( \int_0^T \int_{L^2(\mathbb{R}^d)} ||\xi(s, z)||_{L^2(\mathbb{R}^d)}^2 \nu(ds) \right). \tag{8}
   \]

3. **The proof of Theorem 1.1.**

In this section, we shall prove the global existence of the original equation (1). To do that, we first use the fixed point argument to show the global well-posedness of a truncated solution. Next we establish some uniform \( L^2 \)-norm estimate of the local solution which combining with some stopping time arguments would lead us to believe that solutions ought to exist globally.

**Step 1. Existence and uniqueness of a local solution:** First we define a truncation function \( \theta \). Let \( \theta : \mathbb{R} \rightarrow [0, 1] \) be a non-increasing \( C^0 \) function such that \( 1_{[0,1]} \leq \theta \leq 1_{[0,2]} \) and \( \inf_{x \in \mathbb{R}} \theta'(x) \geq -2 \). For \( R \geq 1 \), set \( \theta_R(x) = \theta(\frac{x}{R}) \).

Let us fix \( R \geq 1 \). Applying Lemma 2.1 similar to the proofs of Proposition 3.1, Proposition 3.6 and Proposition 3.7 in [8], we can prove the existence and uniqueness of the global solution \( Z^R \) to the following truncated equation

\[
Z^R(t) = S_{t-x} - i t \int_0^t S_{t-s} (\theta_R(|Z^R|_{L^2})) |Z^R|^{a-1} Z^R(s) ds + \int_0^t S_{t-s} \tilde{N}(dz, ds), \quad 0 \leq t \leq T. \tag{9}
\]

We define a stopping time \( \tau_R \) by

\[
\tau_R = \inf\{t \in [0, T] : ||Z^R||_{Y_t} > R\},
\]

with the usual convention \( \inf \emptyset = T \). Define \( X(t) = Z^R(t) \) for \( t \in [0, \tau_R] \) and \( \tau_\infty = \lim_{T \rightarrow \infty} \tau_R \). Then \( X(t)_{t \in [0, \tau_\infty]} \) is a maximal local mild solution of (1).

**Step 2. \( L^2 \)-norm estimate:** For any \( q \geq 2 \), we need to prove

\[
\mathbb{E} \sup_{n \in [0, T]} ||Z^R(t)||_{L^2(\mathbb{R}^d)}^q \leq Y. \tag{10}
\]
where $\gamma = \gamma(q, T, ||x||_{L^2(\mathbb{R}^d)})$ is independent of $R$. Thus, if $\mathbb{P}(\tau_\infty = T) = 1$, then we can obtain $[8]$ by taking $R \to \infty$.

As a weak equation in $H^{-1}(\mathbb{R}^d)$, for $0 \leq t \leq T$, we have, $\mathbb{P}$-a.s.

$$
Z^R(t) = x + i \int_0^t \Delta Z^R(s) ds - i \lambda \int_0^t \partial_R(||Z^R||_{L^2(\mathbb{R}^d)})Z^R(s)|^{q-1}Z^R(s) ds + \int_0^t \int_\mathbb{R} \zeta N(dz, ds).
$$

Applying the Itô formula, we get

$$
||Z^R(t)||^2_{L^2(\mathbb{R}^d)} = ||x||^2_{L^2(\mathbb{R}^d)} + \int_0^t \int_\mathbb{R} ||Z^R(s) + z||^2_{L^2(\mathbb{R}^d)} - ||Z^R(s-)||^2_{L^2(\mathbb{R}^d)} N(dz, ds)
$$

$$
+ \int_0^t \int_\mathbb{R} ||Z^R(s) + z||^2_{L^2(\mathbb{R}^d)} - ||Z^R(s)||^2_{L^2(\mathbb{R}^d)} - 2Re(Z^R(s), z)_{L^2(\mathbb{R}^d)} v(dz) ds,
$$

where we used the fact that $Re(Z^R(s), i\Delta Z^R(s)) = 0$, since $i\Delta$ is skew-self-adjoint in $L^2(\mathbb{R}^d)$.

Now we use the Itô formula for the real-valued process $\left(||Z^R(t)||^2_{L^2(\mathbb{R}^d)}\right)^{q/2}$ to get

$$
||Z^R(t)||^2_{L^2(\mathbb{R}^d)} = ||x||^2_{L^2(\mathbb{R}^d)} + \int_0^t \int_\mathbb{R} ||Z^R(s) + z||^2_{L^2(\mathbb{R}^d)} - ||Z^R(s-)||^2_{L^2(\mathbb{R}^d)} N(dz, ds)
$$

$$
+ \int_0^t \int_\mathbb{R} ||Z^R(s) + z||^2_{L^2(\mathbb{R}^d)} - ||Z^R(s)||^2_{L^2(\mathbb{R}^d)} - q||Z^R(s)||^2_{L^2(\mathbb{R}^d)} Re(Z^R(s), z)_{L^2(\mathbb{R}^d)} v(dz) ds
$$

$$
+ \int_0^t \int_\mathbb{R} ||Z^R(s)||^2_{L^2(\mathbb{R}^d)} ||z||^2_{L^2(\mathbb{R}^d)} v(dz) ds.
$$

By the elementary calculus, we have the following fact. For any $a, b \in L^2(\mathbb{R}^d)$,

$$
||a + b||^2_{L^2(\mathbb{R}^d)} - ||a||^2_{L^2(\mathbb{R}^d)} \leq C_d(||a||^2_{L^2(\mathbb{R}^d)} + ||b||^2_{L^2(\mathbb{R}^d)}).
$$

Hence, by the Burkholder-Davis-Gundy inequality and Gronwall’s inequality and (12), we obtain (10).

**Step 3. Global solution:** In order to obtain global solution of (1), we only need to prove $\mathbb{P}(\tau_\infty = T) = 1$. For the simplicity of presentation, we shall adopt the following notations for $t \in [0, T]$,

$$
[\Psi^R(X)](t) = -i t \int_0^t S_{t-s}(\theta_R(||Z^R||_{L^2(\mathbb{R}^d)}))Z^R(s)|^{q-1}Z^R(s) ds,
$$

$$
M(t) = \int_0^t \int_\mathbb{R} S_{t-s} \zeta N(dz, ds).
$$

Applying (3) of Lemma [2.4] for all $q \geq 2$,

$$
\mathbb{E}(||M||^q_{L^p(0,T;L^r(\mathbb{R}^d))}) \leq C_q T \left( \int_\mathbb{R} ||z||^q_{L^r(\mathbb{R}^d)} v(dz) + \left( \int_\mathbb{R} ||z||^q_{L^r(\mathbb{R}^d)} v(dz) \right)^{q/2} \right) < \infty.
$$

Let us fix $\omega \in \Omega$ and take $T_R(\omega) \in (0, T)$ whose value will be determined later on. By Lemma [2.4] and [8, Proposition 3.1], we have

$$
||Z^R(\omega)||_{L^p(0,T;L^r(\mathbb{R}^d))} \leq ||S_x||_{L^p(0,T;L^r(\mathbb{R}^d))} + ||[\Psi^R(Z^R(\omega))]|_{L^p(0,T;L^r(\mathbb{R}^d))} + ||M(\omega)||_{L^p(0,T;L^r(\mathbb{R}^d))}
$$

$$
\leq C||x||_{L^2(\mathbb{R}^d)} + CT_R^{-\frac{r+1}{r}} ||Z^R(\omega)||^r_{L^p(0,T;L^r(\mathbb{R}^d))} + ||M(\omega)||_{L^p(0,T;L^r(\mathbb{R}^d))}
$$

$$
\leq M^R(\omega) + CT_R^{-\frac{r+1}{r}} ||Z^R(\omega)||^r_{L^p(0,T;L^r(\mathbb{R}^d))},
$$

(14)
where
\[ M^T_R(\omega) := C \sup_{0 \leq s < T} \|Z^R(t, \omega)\|_{L^1(\mathbb{R}^d)} + \|M(\omega)\|_{L^p(0, T; L^1(\mathbb{R}^d))}. \]  
(15)

Also let us denote
\[ M_R(\omega) := C \sup_{0 \leq s < T} \|Z^R(t, \omega)\|_{L^1(\mathbb{R}^d)} + \|M(\omega)\|_{L^p(0, T; L^1(\mathbb{R}^d))}. \]  
(16)

The following fact will be used. Fix \( K > 0 \), consider a function \( f \) defined by
\[ f(x) = K + \frac{x^\alpha}{4(2K)^{\alpha-1}} - x, \quad x \geq 0. \]

Then there exist two points \( c_1, c_2 \) with \( 0 < c_1 < 2M < c_2 < 4\alpha - 2K < \infty \) at which \( f(c_1) = f(c_2) = 0 \), and \( f(x) \geq 0 \) if and only if \( 0 \leq x \leq c_1 \) or \( x \geq c_2 \). The proof of this fact can be found around (4.8) in [3].

Let \( T_2(\omega) = T \wedge (4C(2M_R(\omega))^{\alpha-1})^{-1} \), then
\[ CT_2(\omega)^{1 - \frac{\alpha}{1 - \alpha}} (C \sup_{0 \leq s < 2} \|Z^R(s, \omega)\|_{L^1(\mathbb{R}^d)} + \|M(\omega)\|_{L^p(0, T; L^1(\mathbb{R}^d))})^\alpha - 1 < \frac{1}{2^{\alpha+1}}. \]

Combining this with the fact above, we have
\[ \|Z^R(\omega)\|_{L^p(0, T; L^1(\mathbb{R}^d))} \leq 2M^T_R(\omega) \leq 2M_R(\omega). \]  
(17)

This inequality suggests that we can replace \( T_R \) which is not a stopping time by a stopping time \( \sigma_R \) defined by
\[ \sigma_R := \inf\{t \in [0, T] : C t^{-\frac{\alpha}{2-\alpha}} (C \sup_{0 \leq s \leq t} \|Z^R(s, \omega)\|_{L^1(\mathbb{R}^d)} + \|M(\omega)\|_{L^p(0, T; L^1(\mathbb{R}^d))})^{-1 + \alpha} > \frac{1}{2^{\alpha+1}}\}. \]  
(18)

Therefore, similar to the argument above, (17) holds with \( T_R \) replaced by \( \sigma_R \). Then we define a sequence \( \sigma_R^j \) for \( j = 1, \cdots, \) as follow. For \( j = 1 \) we put \( \sigma_R^1 = \sigma_R \).
\[ \sigma_R^{j+1} := \inf\{t \in (\sigma_R^j, T) : C t^{-\frac{\alpha}{2-\alpha}} (C \sup_{\sigma_R^{j1} \leq s \leq t} \|Z^R(s, \omega)\|_{L^1(\mathbb{R}^d)} + \|M(\omega)\|_{L^p(0, T; L^1(\mathbb{R}^d))})^{-1 + \alpha} > \frac{1}{2^{\alpha+1}}\}. \]  
(19)

By the definition of \( \sigma_R^j \), we infer \( \sigma_R^{j+1} - \sigma_R^j \geq T_R \), for each \( j \). Let \( N = \lceil \frac{T}{T_R} \rceil \). Then we can see that \( \sigma_R^{N+1} = T \). Hence, applying similar arguments as above, for \( j = 1, \cdots, N \),
\[ \|Z^R\|_{L^p(0, T; L^1(\mathbb{R}^d))} \leq 2M_R. \]

Then,
\[ \|Z^R\|_{L^p(0, T; L^1(\mathbb{R}^d))} \leq \|Z^R\|_{L^p(0, \sigma_R^N; L^1(\mathbb{R}^d))} + \sum_{j=1}^N \|Z^R\|_{L^p(\sigma_R^j; \sigma_R^{j+1}; L^1(\mathbb{R}^d))} \leq 2\left(\frac{T}{T_R} + 1\right)M_R \]
\[ = 2M_R + 2T(4C(2M_R)^{\alpha-1})^{-1 + \alpha} M_R \]
\[ \leq 2M_R + C_{T, \alpha, d}(M_R)^\theta, \]  
(20)

where \( \theta = \frac{\alpha(\alpha-1)}{4(\alpha-1)\alpha} + 1 \). By applying (10) and (13), we obtain
\[ \mathbb{E}\|Z^R\|_{L^p(0, T; L^1(\mathbb{R}^d))} \leq 2\mathbb{E}M_R + C_{T, \alpha, d}(M_R)^\theta \]
\[ \leq C_{T, \alpha, d}[\mathbb{E} \sup_{0 \leq s < T} \|Z^R(t, s, \omega)\|_{L^1(\mathbb{R}^d)} + \mathbb{E} \sup_{0 \leq s < T} \|Z^R(t, s, \omega)\|_{L^1(\mathbb{R}^d)} + 1] \]
\[ \leq C_{T, \alpha, d, \gamma}. \]  
(21)
Combining with (10) and (21), we deduce that
\[
P(\tau = T) = P\left( \sup_{0 \leq t \leq T} \|Z^R(t)\|_{L^2(\mathbb{R}^d)} + \|Z^R\|_{L^p(0,T;L^r(\mathbb{R}^d))} \leq R \right)
\geq 1 - \frac{E\sup_{0 \leq t \leq T} \|Z^R(t)\|_{L^2(\mathbb{R}^d)} + E\|Z^R\|_{L^p(0,T;L^r(\mathbb{R}^d))}}{R} \geq 1 - \frac{C_{\gamma,a,d,\theta}}{R}.
\]

Hence we have
\[
P(\tau_\infty = T) = \lim_{R \to \infty} P(\tau_\infty = T) = 1.
\]

Thus we infer that \(\tau_\infty \geq T\), \(\mathbb{P}\)-a.s.. Since \(T\) is arbitrary, this shows \(X(t), t \in [0, \infty)\) is a unique global mild solution to (1).

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