A note on Brehm’s extension theorem

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Abstract

Brehm’s extension theorem states that a non–expansive map on a finite subset of a Euclidean space can be extended to a piecewise–linear map on the entire space. In this note, it is verified that the proof of the theorem is constructive provided that the finite subset consists of points with rational coordinates. Additionally, the initial non–expansive map needs to send points with rational coordinates to points with rational coordinates. The two–dimensional case is considered.

Keywords: constructive mathematics, extension, Euclidean space

1. Introduction

Brehm’s extension theorem is stated as follows:

Theorem 1. Let \( M \) be a finite subset of \( \mathbb{R}^n \), and \( \varphi : M \to \mathbb{R}^m, m \leq n \) a map with the property that for any \( a, b \in M \), the condition \( \| \varphi(a) - \varphi(b) \| \leq \| a - b \| \) holds. Then, there exists a piecewise–linear map \( f : \mathbb{R}^n \to \mathbb{R}^m \) such that \( \forall a \in M. f(a) = \varphi(a) \).

Here, \( \| \cdot \| \) denotes the Euclidean distance. A map with the property described is also called non–expansive. The theorem was first addressed by Kirszbraun (1934) and Valentine (1943), and then revisited by Brehm (1981). A similar proof can be found in Akopyan and Tarasov (2008) and Petrunin and Yashinski (2014, p. 21). In the present work, it is verified that the theorem admits a constructive proof in the sense of Bishop’s constructive mathematics (Bishop and Bridges, 1985) provided that \( M \) and \( \varphi(M) \) consist of points with rational coordinates. Only the planar case \( m = n = 2 \) is considered.

2. Preliminaries

In this section, selected basics of constructive mathematics are briefly discussed. For a comprehensive description, refer, for example, to Bishop and Bridges (1985), Bridges and Richman (1987), Bridges and Vita (2007), Yed (2011), Schwichtenberg (2012). Bishop’s constructive mathematics uses the notion of an operation which is an algorithm that produces a unique result in a finite number of steps for each input in its domain. For example, a real number \( x \) is a regular Cauchy sequence of rational numbers in the sense that

\[
\forall n, m \in \mathbb{N}. |x(n) - x(m)| \leq \frac{1}{n} + \frac{1}{m}
\]

where \( x(n) \) is an operation that produces the \( n \)th rational approximation to \( x \). A set is a pair of operations: \( \in \) determines that a given object is a member of the set, and \( = \) determines whenever two given set members are equal. Existence and universal quantifiers are interpreted as follows: \( \exists x \in A. \varphi[x] \) means that an operation has been derived that constructs an instance \( x \) along with a proof of \( x \in A \) and a proof of the logical formula \( \varphi[x] \) as witnesses; \( \forall x \in A. \varphi[x] \) means that an operation has been derived that proves \( \varphi[x] \) for any \( x \) provided with a witness for \( x \in A \). A set \( A \) is called inhabited if there exists an \( x \in A \). A finite set is a set that

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admits a bijection to a set \{1, 2, \ldots, n\} for some \(n \in \mathbb{N}\) which means that all its elements are enumerable. The Euclidean space \(\mathbb{R}^n\) is a normed space with the norm \(\|x\| \triangleq \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}\) where \(x_i\) is the \(i\)-th coordinate of \(x\). The metric is defined as \(\|x - y\|\) for any \(x, y \in \mathbb{R}^n\). A point \(x\) in the Euclidean space is called \emph{algebraic} if its coordinates are algebraic numbers.

A (closed) \emph{polytope} is a union of polyhedrons whereas a (closed) \emph{polyhedron} is an inhabited set of points of the Euclidean space satisfying linear inequalities \(Ax \leq b, A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^{n \times 1}\). If the entries of \(A\) and \(b\) are solely algebraic numbers, then the polyhedron is called \emph{algebraic}. If a polytope is a union of solely algebraic polyhedrons, then it is algebraic as well. A polyhedron (respectively, polytope) \(P\) is \emph{bounded} if there exists a rational number \(\bar{x}\) such that \(\|x\| \leq \bar{x}\) for any \(x\) in \(P\). An \(n\)-dimensional \emph{simplex} is a convex hull of \(n + 1\) affinely independent points. A \emph{triangulation} of a bounded algebraic \(n\)-dimensional polyhedron \(P\) is a finite set \(\{T_i\}_i\) of algebraic simplices whose intersections are at most \((n - 1)\)-dimensional and such that \(P = \bigcup_i T_i\). For example, a triangulation of a two-dimensional polyhedron is a collection of non-degenerate triangles that may have a common vertex or segment of an edge, but no two-dimensional intersection.

A \emph{motion} in the plane \(\mathbb{R}^2\) is a map \(f\) that is a composition of a translate, a rotation and a reflection. Clearly, it is distance-preserving in the sense that \(\|f(x) - f(y)\| = \|x - y\|\) for any \(x, y\). Translate, rotation and reflection can be described as linear transformations of the form \(x \mapsto Tx\) where \(T\) is the transformation matrix. A motion can be thus described by a transformation matrix as well. Notice that a motion is the corresponding transformation matrix comprises solely of algebraic entries. For example, \[ f(x) = \begin{bmatrix} \sqrt{1 - \frac{9}{25}} & \frac{3}{5} \\ -\frac{3}{5} & \sqrt{1 - \frac{9}{25}} \end{bmatrix} x \] is an algebraic motion and describes the clockwise rotation by the angle \(\arcsin \frac{3}{5}\). In contrast, \(f(x) = x + \pi\) is not an algebraic motion. Any three algebraic points forming a non-degenerate triangle can be moved by an algebraic motion to new algebraic points preserving the respective distances (Petrunin and Yashinski, 2014).

An \emph{algebraic piecewise-linear map} \(f\) on a bounded algebraic two-dimensional polyhedron \(P\) is a pair of a triangulation \(\{T_i\}_i\) of the polyhedron and a collection of algebraic motions \(\{f_i\}_i\) such that \(f|_{T_i} = f_i\). For example, folding of a piece of paper without ripping can be considered as an algebraic piecewise-linear map if foldings are performed at algebraic points. Notice that each algebraic piecewise-linear map has a triangulation and a collection of algebraic motions as witnesses. Clearly, an algebraic piecewise-linear map is non-expansive. The notion of an algebraic piecewise-linear map can be directly generalized to algebraic polytopes.

It is important to notice that, for arbitrary real numbers \(x, y\), it is not decidable whether \(x = y\) or \(x \neq y\). This limitation has a number of consequences for the theory of the Euclidean space \(\mathbb{R}^n\). In particular, no full power of set operations is available. For example, if \(A\) and \(B\) are arbitrary sets in \(\mathbb{R}^n\), it is not decidable whether \(A \cap B = \emptyset\) or \(A \cap B\) is inhabited. In this note, set operations are limited to the class of sets of the form \(\{x : \bigwedge_{i=1}^{M} \bigvee_{j=1}^{n} E_{ij}\}\) with \(E_{ij}\) being a formula of the type \(A_{ij}x \bullet b_{ij}\) or \(\|f_{ij}(x)\| \bullet \|g_{ij}(x)\|\) where “\(\bullet\)” denotes “\(<\)”、“\(\leq\)” or “\(=\)” and \(f_{ij}\) and \(g_{ij}\) are algebraic piecewise-linear maps on algebraic polytopes. Denote this class by \(\mathcal{AS}\). For example, an algebraic polytope itself belongs to \(\mathcal{AS}\). Further, the set complement of a set \(A \in \mathcal{AS}\), denoted by \(\mathbb{R}^n \setminus A\) is, again, an element of the class \(\mathcal{AS}\) (it can be done by transforming the sign “\(<\)”、“\(\leq\)” or “\(=\)” in the respective formula). Notice that if \(f_{ij}\) and \(g_{ij}\) are algebraic motions, each \(\|f_{ij}(x)\| \leq \|g_{ij}(x)\|\) is equivalent to \(\sum_{k=1}^{n}(f_{ij}(x))^2 \leq \sum_{k=1}^{n}(g_{ij}(x))^2\). The same applies if \(f_{ij}\) and \(g_{ij}\) are algebraic piecewise-linear maps by considering the inequalities on the simplices where \(f_{ij}\) and \(g_{ij}\) are both algebraic motions. If a set \(A\) has the form \(\{x : \bigwedge_{i=1}^{n} \|f_i(x)\| < \|g_i(x)\|\}\) with \(f_i\) and \(g_i\) algebraic piecewise-linear, then its boundary is defined to be the set \(\partial A \triangleq \{x : \bigwedge_{i=1}^{n} \|f_i(x)\| = \|g_i(x)\|\}\) which in turn belongs to \(\mathcal{AS}\). Lemma 4.1 from Beeson (1980, p. 8) states decidability of equality over the field of algebraic real numbers. This allows performing the ordinary set operations on the sets of the described class. In the following, the extension theorem is revisited and verified to admit a constructive proof.
3. Extension theorem

The proof of the following theorem is mostly based on [Akopyan and Tarasov, 2008; Petrunin and Yashinski, 2014].

**Theorem 2.** Let \( \{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n \) be finite subsets of points in \( \mathbb{R}^2 \) with rational coordinates such that \( \forall i, j, \|b_i - b_j\| \leq \|a_i - a_j\| \). Let \( A \) be the convex hull of \( \{a_i\}_{i=1}^n \). Then, there exists an algebraic piecewise–linear map \( f : A \rightarrow \mathbb{R}^2 \) such that \( \forall i, f(a_i) = b_i. \)

**Proof.** The theorem is proven by induction on the number of points. If \( n = 1 \), one may take \( f(x) := x + (b_1 - a_1) \) which is clearly an algebraic motion on the entire space. Suppose that an algebraic piecewise–linear map \( g : A \rightarrow \mathbb{R}^2 \), such that \( \forall i = 1, \ldots, n-1, g(a_i) = b_i \), was constructed. Define a set \( \Omega := \{ x : x \in A \land \|a_n - x\| < \|b_n - g(x)\| \} \). Since \( g \) is algebraic, it is decidable whether \( b_n = g(a_n) \) or \( b_n \neq g(a_n) \). In the former case, take \( f \) to be \( g \). In the latter, \( \Omega \) is inhabited since \( a_n \in \Omega \). Notice that if \( x \) belongs to \( \Omega \), then so does the line segment between \( a_n \) and \( x \). Take a point \( y \) in this line segment. Then,

\[
\|a_n - y\| + \|y - x\| = \|a_n - x\|.
\]

Since \( x \in \Omega \),

\[
\|a_n - x\| \leq \|b_n - g(x)\|.
\]

The map \( g \) is an algebraic piecewise–linear which implies

\[
\|g(x) - g(y)\| \leq \|x - y\|.
\]

Therefore,

\[
\|a_n - y\| = \|a_n - x\| - \|y - x\|
\leq \|b_n - g(x)\| - \|g(x) - g(y)\|
\leq \|b_n - g(y)\|
\]

where the last line follows from the triangle inequality \( \|g(x) - g(y)\| \leq \|b_n - g(y)\| + \|b_n - g(y)\| \). Since \( \|a_n - y\| < \|b_n - g(y)\| \), it follows that \( y \in \Omega \). Now, the boundary \( \partial_A \Omega := \partial \Omega \cap A \) is inspected. Let \( \{T_i\}_i \) be the triangulation of \( A \) such that \( g \) on each triangle is a motion \( g_i \). Let \( c_n := g_i^{-1}(a_n) \). Notice that \( c_n \) is an algebraic point. Since \( g_i \) is a motion and \( g|_{T_i} = g_i \), for any \( x \in T_i \), it follows that

\[
\|c_n - x\| = \|b_n - g(x)\|.
\]

Since \( \Omega \) and \( T_i \) belong to \( \mathcal{AS} \), it is decidable whether the intersection \( \Omega \cap T_i \) is inhabited. Suppose it is inhabited. Then, consider the line

\[
l_i := \{ x : \|x - a_n\| = \|x - c_n\| \}.
\]

It follows that:

\[
\partial_A \Omega \cap T_i = \{ x : \|a_n - x\| = \|b_n - g(x)\| \land x \in T_i \land x \in A \}
\]

and

\[
l_i \cap T_i \cap A = \{ x : \|x - a_n\| = \|x - c_n\| \land x \in T_i \land x \in A \}.
\]

Matching (3) with (4), one can see that \( \partial_A \Omega \cap T_i \) is a line segment. Since \( \{T_i\}_i \) is a finite set, \( \partial_A \Omega \) is a finite collection of line segments. Consider a line segment \( \omega_i \) of \( \partial_A \Omega \). Let \( \tau_i \) be the triangle formed by \( a_n \) and \( \omega_i \). Let \( f_i \) be an algebraic motion that maps \( a_n \) to \( b_n \) and the endpoints of \( \omega_i \) to their respective positions under \( g_i \). For \( x \in \omega_i \), it follows that \( g(x) = g_i(x) \) and so \( g(x) = f_i(x) \). Let \( f|_{T_i} := f_i \) and \( f|_{A \setminus \Omega} := g \). Further, since \( \partial \Omega, \partial_A \Omega \in \mathcal{AS} \), it is decidable whether \( \Delta := \partial \Omega \cap \partial_A \Omega \) is inhabited. If this is the case (for otherwise, the result is trivial), consider the algebraic polytopes \( D_k, k = 1, \ldots, m \) formed by the endpoints of \( \Delta \) that lie on \( \partial \Omega \cap A \) and the line segments from these endpoints to \( a_n \). Let \( \lambda_1 \) and \( \lambda_2 \) denote the said endpoints.
for some algebraic polytope $D_k$. Since $f$ coincides with $g$ on the line segments $[a_n, \lambda_1]$ and $[a_n, \lambda_2]$, and, moreover, it acts as algebraic motions on these line segments, and since $g$ is non–expansive, it follows that

$$
\|\lambda_1 - a_n\| = \|g(\lambda_1) - b_n\|,
\|\lambda_2 - a_n\| = \|g(\lambda_2) - b_n\|,
\|g(\lambda_1) - g(\lambda_2)\| \leq \|\lambda_1 - \lambda_2\|.
$$

The required map on $D_k$ can be constructed as follows. Translate and rotate $D_k$ so that $[a_n, \lambda_1]$ coincides with $[b_n, g(\lambda_1)]$. This can be done since the initial and the new vertices of $D_k$ are algebraic. So far, the line segment $[a_n, \lambda_2]$ “turned” around $b_n$ closer to $[a_n, \lambda_1]$. Draw a line segment from $g(\lambda_1)$ to $g(\lambda_2)$ which is the chord of the circle on which the point $\lambda_2$ slid to the new position. Take the middle point of the chord and fold $D_k$ around the ray going from $a_n$ to this middle point so that $\lambda_2$ matches with $g(\lambda_2)$. The resulting map is thus constructed by translating and rotating the whole $D_k$ and then reflecting the fragment—to–fold around the said ray which constitutes a piecewise–linear map. This map is clearly algebraic since all the points involved are algebraic.

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