CROSSED PRODUCT TENSOR CATEGORIES

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Abstract. A graded tensor category over a group $G$ will be called a crossed product tensor category if every homogeneous component has at least one multiplicative invertible object. Our main result is a description of the crossed product tensor categories, graded monoidal functors, monoidal natural transformations, and braidings in terms of coherent outer $G$-actions over tensor categories.

1. Introduction

A $G$-graded ring $A = \bigoplus_{\sigma \in G} A_{\sigma}$ is called a $G$-crossed product if each $A_{\sigma}$ has an invertible element. Some important classes of rings like skew group-rings and twisted group-rings are special cases of crossed product rings. One of the basic examples is the group algebra of a group $F$, it is graded by a quotient group of $F$, see [4, Subsection 11C]. In this case the theory of representation of $F$ can be analyzed using Clifford theory, see [4, Subsection 11A].

In analogy with graded rings, a $G$-graded tensor category (see Subsection 2.5) $\mathcal{C} = \bigoplus_{\sigma \in G} \mathcal{C}_{\sigma}$ will be called $G$-crossed product tensor category, if there is an invertible object in each homogeneous component of $\mathcal{C}$.

The graded tensor categories appear naturally in classification problems of fusion categories and finite tensor categories [5], [8]. One of the most interesting examples of $G$-crossed product tensor categories is the semi-direct product tensor category associated to an action of a group over a tensor category, see [14]. Semi-direct tensor product category has been used in order to solve an important open problem in semisimple Hopf algebras theory [13].

In [9] was proposed a Clifford theory categorification for crossed product tensor categories, in order to describe simple module categories in terms of subgroups and induced module categories. We stress that in [9] crossed product tensor categories were called strongly graded tensor categories. Recently, $G$-graded fusion categories (that include crossed product fusion categories) were classified by Etingof, Nikshych, and Ostrik in [7], using invertible bimodule categories over fusion categories.

The crossed product rings are commonly described using crossed systems [12]. Crossed systems can be interpreted in terms of monoidal functors in the following way: if $A$ is a ring, let we denote by $\text{Out}(A)$ the monoidal category (in fact, it is a categorical-group) of outer automorphisms, were the objects are automorphisms of $A$, and the arrows between automorphisms $\sigma$ and $\tau$ are invertible elements $a \in A$, such $a\sigma(x) = \tau(x)a$ for all $x \in A$. Given a group $G$, the data that define $G$-crossed
systems are the same as the data that define monoidal functors from $G$ to $\text{Out}(A)$, where $G$ is the discrete monoidal category associated with $G$.

We develop crossed product system theory for crossed product tensor categories using higher category theory. If $C$ is a tensor category, the bicategory $\text{Bieq}(C)$ is a monoidal bicategory (in fact, it is a weak 3-group). So, a $G$-crossed system or coherent outer $G$-action over $C$ must be a trihomomorphism from $G$ (the discrete 3-category associated to $G$, see Remark 3.5) to $\text{Bieq}(C)$.

The main goal of this paper is to describe the 2-category of $G$-crossed product tensor categories in terms of coherent outer $G$-actions over a tensor category (Theorem 4.1), and describe the braidings of $G$-graded tensor categories.

The organization of the paper is as follows: in Section 2 we recall the main definitions of bicategory theory, as well as the definitions of categorical-groups, graded tensor categories, and the monoidal structure over $\text{Bieq}(C)$. In Section 3 we define incoherent and coherent outer $G$-actions over a tensor category, and we show an explicit bijective correspondence between equivalence classes of $G$-crossed product tensor categories and coherent outer $G$-actions. In Section 4 we show a biequivalence between the 2-category of crossed product tensor categories and the 2-category of coherent outer $G$-actions. Finally, in Section 5 we give some necessary and sufficient conditions for the existence of a braiding over a crossed product tensor category.

2. PRELIMINARIES

2.1. General conventions. Throughout this article we work over a field $k$. By a tensor category $(C, \otimes, \alpha, I)$ we understand a $k$-linear abelian category $C$, endowed with a $k$-bilinear exact bifunctor $\otimes : C \times C \to C$, an object $I \in C$, and an associativity constraint $\alpha_{V,W,Z} : (V \otimes W) \otimes Z \to V \otimes (W \otimes Z)$, such that Mac Lane’s pentagon axiom holds $\alpha_{V,I,W} = \text{id}_V \otimes \text{id}_W$ for all $V, W \in C$ and $\dim_k \text{End}_C(I) = 1$.

We shall consider only monoidal categories which constraint of unit are identities. So, without loss of generality, we shall suppose that for every monoidal functor $(F, \psi) : C \to D$, we have $F(I_C) = I_D$ and $\psi_{V,I} = \psi_{I,V} = \text{id}_V$, since each monoidal functor is monoidally equivalent to one with these properties.

2.2. Bicategories. In this section we review some definitions on bicategory theory that we shall need later. We refer the reader to [4] for a detailed exposition on the subject.

Definition 2.1. A bicategory $B$ consists of the following data

- a set $\text{Obj}(B)$ (with elements $A, B, \ldots$ called 0-cells),
- for each pair $A, B \in \text{Obj}(B)$, a category $B(A, B)$ (with objects $V, W, \ldots$ called 1-cells and morphisms $f, g, \ldots$ called 2-cells),
- for each $A, B, C \in \text{Obj}(B)$, a bifunctor $\varpi^{ABC} : B(A, B) \times B(B, C) \to B(A, C)$,
- for each 0-cell $A \in \text{Obj}(B)$, a 1-cell $I_A \in B(A, A)$,
for each $A, B, C, D \in \text{Obj}(\mathcal{B})$, natural isomorphisms (constraint of associativity)

$$\alpha^{A,B,C,D} : -\otimes^ABD(-\otimes^BCD) \rightarrow (-\otimes^ABC-)^{ACD} :$$

$\mathcal{B}(A, B) \times \mathcal{B}(B, C) \times \mathcal{B}(C, D) \rightarrow \mathcal{B}(A, D)$.

Subject to the following axioms

- coherence of the associativity: if $(S, T, U, V)$ is an object in $\mathcal{B}(A, B) \times \mathcal{B}(B, C) \times \mathcal{B}(C, D) \times \mathcal{B}(D, E)$, the next diagram commutes

$$\begin{array}{ccc}
& S \otimes (T \otimes (U \otimes V)) & \\
\alpha_{S,T,U,V} & \Downarrow \text{id} \otimes \alpha_{T,U,V} & \\
(S \otimes T) \otimes (U \otimes V) & \Downarrow \alpha_{S,T,U,V} & \Downarrow \alpha_{S,T,U,V} \\
((S \otimes T) \otimes U) \otimes V & \Downarrow \alpha_{S,T,U,V} & \Downarrow \alpha_{S,T,U,V} \\
& (S \otimes (T \otimes U)) \otimes V & \\
\end{array}$$

- coherence of the unity

$$\alpha_{S,I_B,T} = \text{id}_{S \otimes T}.$$ 

If $\alpha$ is the identity, we have $(S \otimes T) \otimes U = S \otimes (T \otimes U)$ and similarly for morphisms, in this case we shall say that $\mathcal{B}$ is a 2-category.

A monoidal category $(\mathcal{C}, \otimes, I, \alpha)$ is the same as a bicategory with only one 0-cell, and in this case $\otimes = \otimes'$.

**Definition 2.2.** Let $\mathcal{B} = (\otimes, I, \alpha)$ and $\mathcal{B}' = (\otimes', I', \alpha')$ be bicategories. A pseudo-functor $\Phi = (F, \phi)$ from $\mathcal{B}$ to $\mathcal{B}'$ consists of the following data

- a function $F : \text{Obj}(\mathcal{B}) \rightarrow \text{Obj}(\mathcal{B}')$, $A \mapsto F(A)$,
- for each pair $A, B \in \text{Obj}(\mathcal{B})$, functors

$$F_{AB} : \mathcal{B}(A, B) \rightarrow \mathcal{B}'(F(A), F(B)), \quad S \mapsto F(S), \quad f \mapsto F(f),$$

- for each triple $A, B, C \in \text{Obj}(\mathcal{B})$, a natural isomorphism

$$\psi^{ABC} : F_{AC}(- \otimes^ABC-) \rightarrow F_{AB}(-)^{F(A)F(B)F(C)}F_{BC}(-),$$

Subject to the following axioms

(i) $F_{AA}(I_A) = I'_{F(A)}$. 

(ii) if \((S, T, U)\) is an object in \(\mathcal{B}(A, B) \times \mathcal{B}(B, C) \times \mathcal{B}(C, D)\), the following diagram commutes (where the indexes have been omitted)

\[
\begin{array}{ccc}
F(S\overline{STU}) & \xrightarrow{F(\alpha)} & F(S\overline{STU}) \\
\downarrow\psi & & \downarrow\psi \\
F(S\overline{F(T\overline{U})}) & \xrightarrow{\alpha'} & (F(S\overline{F(T)})\overline{F(U)})
\end{array}
\]

(iii) if \(S\) is an object in \(\mathcal{B}(A, B)\), then \(\psi_{S,I_B} = \text{id}_{F(S)}\) and \(\psi_{I_A,S} = F(S)\), for each pair of 0-cells \(A, B \in \text{Obj}(\mathcal{B})\).

**Remark 2.3.**

1. The notion of pseudofunctor can be in some manner dualized by reversing the direction of the 2-cells \(F_{A,B}\), this notion will be called op-pseudofunctor.
2. A pseudofunctor between monoidal categories is just a monoidal functor and an op-pseudofunctor is an op-monoidal functor.

**Definition 2.4.** Let \(F, G : \mathcal{B}_0 \to \mathcal{B}_1\) be pseudofunctors between bicategories \(\mathcal{B}_0\) and \(\mathcal{B}_1\). A pseudonatural transformation \(\sigma : F \to G\), consists of the following data

- for each \(A \in \text{Obj}(\mathcal{B}_0)\), 1-cells \(\sigma_A \in \mathcal{B}_1(F(A), G(A))\),
- for each pair \(A, B \in \text{Obj}(\mathcal{B}_0)\), and each 1-cell \(V \in \mathcal{B}_0(A, B)\) a natural isomorphism

\[
\sigma_V : F^{AB}(V)\overline{F(A)}F(B)G(B)\sigma_B \to \sigma_AF(A)G(B)G^{AB}(V),
\]

such that \(\sigma_I_A = \text{id}_{I_A}\) for all \(A \in \text{Obj}(\mathcal{B}_0)\) and for all \(S \in \mathcal{B}_0(A, B)\), \(T \in \mathcal{B}_0(B, C)\), the following diagram

\[
\begin{array}{ccc}
F(ST)\sigma & \xrightarrow{\sigma_{ST}} & \sigma G(ST) \\
\downarrow\psi_{F(T)} & & \downarrow\psi_{G(T)} \\
F(S)F(T)\sigma & \xrightarrow{\text{id}_{F(T)}\overline{\sigma_T}} & F(S)\sigma G(T) \xrightarrow{\sigma_S\text{id}_{G(T)}} \sigma G(S)G(T)
\end{array}
\]

commutes (where associativity constraint, indexes, and the symbols \(\overline{\cdot}\) between objects have been omitted as a space-saving measure).

**Remark 2.5.** Again, the notion of pseudonatural transformation can be dualized by reversing the order of the natural isomorphisms \(\sigma_V\), this notion will be called op-pseudonatural transformation.

Pseudonatural transformations may be composed in the obvious way. If \(\sigma : F \to G\), and \(\tau : G \to H\) are pseudonatural transformations, then we define a new pseudonatural transformation \(\sigma\overline{\tau} : F \to H\) by \((\sigma\overline{\tau})_A = \sigma_A\overline{\tau}_A\), and \((\sigma\overline{\tau})_V\) is defined by the commutativity of the diagram:
A modification between two pseudonatural transformations $\Gamma : \sigma \to \tilde{\sigma}$, consists of 2-cells $\Gamma_A : \sigma_A \to \tilde{\sigma}_A$ in $\mathcal{B}_1(F(A), G(A))$, such that for all 1-cell $V \in \mathcal{B}_0(A, B)$ the diagram

$$F^{AB}(V) \tilde{\sigma}_B \xrightarrow{\tilde{\sigma}_V} \sigma_A G^{AB}(V)$$

commutes.

2.3. The Monoidal bicategory $\text{Bieq}(\mathcal{C})$ of a tensor category. Given a pair of bicategories $\mathcal{B}$ and $\mathcal{B}'$, we can define the “functor bicategory” $[\mathcal{B}, \mathcal{B}']$, whose 0-cells are pseudofunctors $\mathcal{B} \to \mathcal{B}'$, whose 1-cells are pseudonatural equivalence, and whose 2-cells are invertible modifications.

The bicategory $[\mathcal{B}, \mathcal{B}']$ is not usually a 2-category, because composition of 1-cells in $[\mathcal{B}, \mathcal{B}']$ involves composition of 1-cells in $\mathcal{B}'$, but in the case that $\mathcal{B}'$ is a 2-category, $[\mathcal{B}, \mathcal{B}']$ is a 2-category.

When $\mathcal{B} = \mathcal{B}'$, the bicategory $[\mathcal{B}, \mathcal{B}]$ will be denoted by $\text{Bieq}(\mathcal{B})$, and it has a monoidal structure in the sense of [10]. Now, we shall describe the monoidal bicategory $\text{Bieq}(\mathcal{C})$ associated to a tensor category $(\mathcal{C}, \otimes, I)$.

The tensor product $\boxtimes$ of monoidal endofunctors is defined by the composition of monoidal functors. If $(\theta', \theta'_{(-)}) : K \to K'$, $(\theta, \theta_{(-)}) : H \to H'$ are pseudonatural transformations, the tensor product $(\theta', \theta'_{(-)}) \boxtimes (\theta, \theta_{(-)}) : K H \to K' H'$ is defined as $(\theta', \theta'_{(-)}) \boxtimes (\theta, \theta_{(-)}) := (\theta' \otimes \theta', \theta' \otimes \theta)$, where $\theta' \otimes \theta$ is given by the commutativity of the following diagram

$$K H(V) \otimes K(\theta) \otimes \theta' \xrightarrow{(\theta \boxtimes \theta')_V} K(\theta) \otimes \theta' \otimes K' H'(V)$$

The tensor product of modifications $g$ and $f$ is defined as $g \boxtimes f := K(f) \otimes g$. 

\[ F(V)[\sigma_A \tilde{\tau}_A] \xrightarrow{(\sigma \tilde{\tau})_V} (\sigma_B \tilde{\tau}_B)[\tilde{\tau}_A] \]

\[ (F(V)[\sigma_A \tilde{\tau}_A]) \xrightarrow{\sigma_B \tilde{\tau}_B} \sigma_B (\tilde{\tau}_B \tilde{\tau}_A) \]

\[ (\sigma_B \tilde{\tau}_B \tilde{\tau}_A) \xrightarrow{\alpha_{\sigma_B \tilde{\tau}_B \tilde{\tau}_A}(V)} \alpha_{\sigma_B \tilde{\tau}_B \tilde{\tau}_A}(V) \]

\[ \tilde{\tau}_A \xrightarrow{\sigma_B \tilde{\tau}_B} \sigma_B (\tilde{\tau}_B \tilde{\tau}_A) \]
If \( \chi_1 = (\theta, \theta_{(-)}) : F \to F' \) and \( \chi_2 = (\theta', \theta'_{(-)}) : H \to H' \) are pseudonatural transformations, where \( F, F', H, H' : \mathcal{C} \to \mathcal{C} \) are monoidal functors, then the comparison constraint

\[
\begin{array}{ccc}
\chi_1 \circ \text{id}_H & \xrightarrow{F' \circ H} & \text{id}_F \circ \chi_2 \\
\downarrow c_{\chi_1, \chi_2} & & \downarrow c_{\chi_1, \chi_2} \\
F \circ H & \xrightarrow{\chi_1 \circ \text{id}_H} & F' \circ H'
\end{array}
\]

is given by

\[c_{\chi_1, \chi_2} := \theta^{-1} \circ F'(\theta') \to F(\theta') \circ \theta.
\]

The constraint of associativity \( a_{f,g,h} : (f \otimes g) \otimes h \to f \otimes (g \otimes h) \) of the tensor product of pseudonatural transformations \( f : K \to K', g : H \to H', h : G \to G' \) is given by the modification

\[\psi^K_{H(h),g} \otimes \text{id}_f : KH(h) \otimes K(g) \otimes f \to KH(h) \otimes K(g) \otimes f,
\]

and it is easy to see that \( a \) satisfies the pentagonal identity.

**Remark 2.6.** The data (TD6), (TD7), and (TD8) of [10], in the monoidal bicategory \( \text{Bieq}(\mathcal{C}) \) are trivial, since we only consider monoidal functor \( (F, \psi) : \mathcal{C} \to \mathcal{C} \) such that \( F(I) = I \) and \( \psi_{V,I} = \psi_{I,V} = \text{id}_V \), for all \( V \in \mathcal{C} \).

The category \( \text{Bieq}(\mathcal{C})(\text{id}_\mathcal{C}, \text{id}_\mathcal{C}) \) is exactly the center of \( \mathcal{C} \), i.e., the braided monoidal category \( \mathcal{Z}(\mathcal{C}) \), see [11] pag. 330).

### 2.4. Categorical-groups

A categorical-group \( \mathcal{G} \) is a monoidal category where every object, and every arrow is invertible, i.e. for all \( X \in \text{Obj}(\mathcal{G}) \) there is \( X^* \in \text{Obj}(\mathcal{G}) \), such that \( X \otimes X^* \cong X^* \otimes X \cong I \). We refer the reader to [1] for a detailed exposition on the subject.

A trivial example of a categorical-group is the discrete categorical-group \( \mathcal{G} \), associated to a group \( G \). The objects of \( \mathcal{G} \) are the elements of \( G \), the arrows are only the identities, and the tensor product is the multiplication of \( G \). A more interesting examples is the following.

**Example 2.7.** Let \( G \) be a group, \( A \) a \( G \)-module, and \( \omega \in Z^3(G, A) \) a normalized 3-cocycle. We shall define the category \( \mathcal{C}(G, A, \omega) \) by:

1. \( \text{Obj}(\mathcal{C}(G, A, \omega)) = G \),
2. \( \text{Hom}_{\mathcal{C}(G, A, \omega)}(g, h) = \begin{cases} A, & \text{if } g = h \\ \emptyset, & \text{if } g \neq h, \end{cases} \)

We define a monoidal structure in \( \mathcal{C}(G, A, \omega) \) as follows:

Let \( g \in \text{End}(a) \) and \( h \in \text{End}(b) \), \( a, b \in A, g, h \in G \). Thus, \( a \otimes b = a + \phi(b) \) and \( g \otimes h = gh \). We define the associator as \( \Phi_{g,h,k} = \omega(g, h, k) \).

The 3-cocycle condition is equivalent to the pentagon axiom, and the condition of normality implies that \( e \) is the unit object for this category.

Complete invariants of a categorical-group \( \mathcal{G} \) with respect to monoidal equivalences are

\[\pi_0(\mathcal{G}), \pi_1(\mathcal{G}), \phi(\mathcal{G}) \in H^3(\pi_0(\mathcal{G}), \pi_1(\mathcal{G})),\]

where \( \pi_0(\mathcal{G}) \) is the group of isomorphism classes of objects, \( \pi_1(\mathcal{G}) \) is the abelian group of automorphisms of the unit object. The group \( \pi_1(\mathcal{G}) \) is a \( \pi_0(\mathcal{G}) \)-module in
the natural way, and \( \phi(G) \) is a third cohomology class given by the associator, see [1] Subsection 8.3 for details on how to obtain \( \phi(G) \).

If \( G \) is a categorical group by [1] Theorem 43 there is an equivalence of monoidal categories between \( G \) and \( C(\pi_0(G), \pi_1(G), \phi) \), where \( \phi \) is a 3-cocycle in the class \( \phi(G) \).

Also, it is easy to see that there is a bijective correspondence between monoidal functors

\[
F : C(G, A, \omega) \to C(G', A', \omega')
\]

and triples \( (\pi_0(F), \pi_1(F), \theta(F)) \) that consist of:

- a group morphism \( \pi_0(F) : G \to G' \),
- a \( G \)-module morphism \( \pi_1(F) : A \to A' \),
- a normalized 2-cochain \( k(F) : G^2 \to A' \), such that \( dk(F) = \pi_1(F)\omega - \omega'\pi_0(F)^3 \).

For monoidal functors \( F, F' : C(G, A, \omega) \to C(G', A', \omega') \), there is a bijective correspondence between monoidal natural isomorphisms \( \theta : F \to F' \) and normalized 1-cochains \( p(\theta) : G \to A' \), where \( dp(\theta) = k(F) - k(F') \).

The next result follows from the last discussion or from [1] Theorem 43.

**Proposition 2.8.** Let \( G \) be a categorical group and let \( f : G \to \pi_0(G) \) be a morphism of groups. Then there is a monoidal functor \( F : \underline{G} \to G \), such that \( f = \pi_0(F) \) if and only if the cohomology class of \( \phi f^3 \) is zero, where \( \phi \) is a 3-cocycle in the class \( \phi(G) \).

If \( \phi f^3 \) is zero, the classes of equivalence of monoidal functors \( F : \underline{G} \to G \) such that \( \pi_0(F) = f \) are in one to one correspondence with \( H^2(G, \pi_1((G))) \).

\( \square \)

### 2.5. Crossed product tensor categories

Let \( G \) be a group and let \( C \) be a tensor category. We shall say that \( C \) is \( G \)-graded, if there is a decomposition

\[
C = \bigoplus_{x \in G} C_x
\]

into a direct sum of full abelian subcategories, such that for all \( \sigma, x \in G \), the bifunctor \( \otimes \) maps \( C_\sigma \times C_x \) to \( C_{\sigma x} \), see [5].

**Definition 2.9.** Let \( C \) be a tensor category graded over a group \( G \). We shall say that \( C = \bigoplus_{\sigma \in G} C_\sigma \) is a crossed product tensor category over \( G \), if every \( C_\sigma \) has a multiplicatively invertible object.

Given a group \( G \), we define the 2-category of \( G \)-crossed product tensor categories. The 0-cells are crossed product tensor categories over \( G \), 1-cells are graded op-monoidal functors, i.e., op-monoidal functors \( F : C \to D \) such that \( F \) maps \( C_\sigma \) to \( D_\sigma \) for all \( \sigma \in G \), and 2-cells are monoidal natural transformations between the graded op-monoidal functors. The composition of 1-cells and 2-cells is the obvious.

**Remark 2.10.** The existence of some extra properties of a crossed product tensor category \( C \), can be deduced from the tensor subcategory \( C_e \). For example \( C \) is semisimple or rigid if and only if \( C_e \) is semisimple or rigid. However, if \( C_e \) is a braided tensor category, not necessary \( C \) is braided, see Section [5]

A crossed product tensor category \( C \) is a fusion category [5] or finite tensor category [3], if and only if \( G \) is finite, and \( C_e \) is a fusion category or a finite tensor category, respectively.
3. Outer $G$-actions over tensor categories

3.1. Incoherent outer $G$-actions. Let $\mathcal{C}$ be a tensor category. We define the categorical-group $2\text{Out}\otimes(\mathcal{C})$, where objects are monoidal autoequivalences of $\mathcal{C}$, and arrows are equivalence classes of invertible pseudonatural isomorphisms up to invertible modifications. The composition of arrows in $2\text{Out}\otimes(\mathcal{C})$ is the equivalence class of pseudonatural isomorphisms composition, and the tensor product is the composition of monoidal functors and pseudonatural transformations.

**Definition 3.1.** Let $G$ be a group and let $\mathcal{C}$ be a monoidal category. An incoherent outer $G$-action over $\mathcal{C}$, is an op-monoidal functor $*: \mathcal{G} \to 2\text{Out}\otimes(\mathcal{C})$. Two incoherent outer $G$-actions are equivalent if the associated monoidal functors are monoidally equivalent.

We shall analyze the incoherent outer $G$-action using the Subsection [24]. Complete invariants for the categorical group $2\text{Out}\otimes(\mathcal{C})$ are $\pi_0(2\text{Out}\otimes(\mathcal{C}))$ the equivalences classes of monoidal functor under invertible modification, $\pi_1(2\text{Out}\otimes(\mathcal{C})) = \text{Inv}(\mathcal{Z}(\mathcal{C}))$ the abelian group of isomorphisms classes of invertible objects of the center of $\mathcal{C}$, and a third cohomology class $\phi(2\text{Out}\otimes(\mathcal{C})) \in H^3(\pi_0(2\text{Out}\otimes(\mathcal{C})), \text{Inv}(\mathcal{Z}(\mathcal{C})))$.

Every incoherent outer $G$-action over a tensor category induces a group morphism $f : G \to \pi_0(2\text{Out}\otimes(\mathcal{C}))$. We shall say that a group morphism $f : G \to \pi_0(2\text{Out}\otimes(\mathcal{C}))$ is realizable if there is some incoherent outer $G$-action such that the induced group morphism coincides with $f$.

**Proposition 3.2.** Let $G$ be a group and let $f : G \to \pi_0(2\text{Out}\otimes(\mathcal{C}))$ be a group morphism. Then there is an incoherent outer $G$-action over $\mathcal{C}$ that realize the morphism $f$ if and only if the cohomology class of $\phi f^3$ is zero, where $\phi$ is some 3-cocycle in the class of $\phi(2\text{Out}\otimes(\mathcal{C}))$.

If $\phi f^3$ is zero, the classes of equivalence of monoidal functors $F: \mathcal{G} \to 2\text{Out}\otimes(\mathcal{C})$ such that $\pi_0(F) = f$ are in one to one correspondence with $H^2(G, \text{Inv}(\mathcal{Z}(\mathcal{C})))$.

**Proof.** See Proposition [23].

3.2. Coherent outer $G$-actions. Let $\mathcal{C}$ be a monoidal category and let $F: G \to 2\text{Out}\otimes(\mathcal{C})$ be an incoherent outer $G$-action. We define a crossed system associated to $F$ as the following data

- monoidal functors $(\sigma_*, \psi_\sigma) : \mathcal{C} \to \mathcal{C}$ for all $\sigma \in G$,
- pseudonatural isomorphisms $(U_{\sigma, \tau}, \chi_{\sigma, \tau}) : \sigma_* \circ \tau_* \to (\sigma\tau)_*$ for all $\sigma, \tau \in G$,
- invertible modifications $\omega_{\sigma, \tau, \rho} : \chi_{\sigma, \tau, \rho}^\psi(id_{\sigma \otimes \chi_{\tau, \rho}}) \to \chi_{\sigma, \tau, \rho}^\psi(id_{\sigma \otimes \chi_{\tau, \rho}})$ for all $\sigma, \tau, \rho \in G$.

that realize the incoherent outer $G$-action $F$ (recall that the symbol $\psi$ is the composition of 1-cells in the bicategory Bieq($\mathcal{C}$)).

**Remark 3.3.** (1) By abuse of notation, we write $\chi_{\sigma, \tau}$ instead of pseudonatural transformation $(U_{\sigma, \tau}, \chi_{\sigma, \tau})$, when no confusion can arise.

(2) By the definition of $2\text{Out}\otimes(\mathcal{C})$, there are several crossed systems that realize an incoherent $G$-action.

(3) For every crossed system, without loss of generality, we can and shall assume that

- $e_* = id_{\mathcal{C}}$ the monoidal identity functor,
- $\chi_{e, \sigma} = \chi_{\sigma, e} = (I, id_{\sigma})$ the identity pseudonatural isomorphism,
\[ \chi_{\sigma,\tau} = \text{id}_{\chi_{\sigma,\tau}} \] the identity modification, for all \( \sigma, \tau \in G \).

**Definition 3.4.** Let \( C \) be a tensor category and let \( F : G \to 2\text{Out}_G(C) \) be an incoherent outer \( G \)-action. A coherent outer \( G \)-action associated to \( F \), is a crossed system \( (\{ \tilde{\sigma} \}_{\sigma \in G}; \chi, \omega) \) associated to \( F \), such that for the pseudonatural isomorphisms \( \chi_{\sigma,\tau} \) and the invertible modifications \( \omega_{\sigma,\tau,\rho} \), the diagram

\[
\begin{array}{ccc}
\sigma_*(\tau_* (U_{\rho,\mu}) U_{\sigma,\tau \rho \mu}) & \xrightarrow{\chi_{\sigma,\tau}} & \sigma_*(U_{\sigma,\tau \rho \mu}) U_{\sigma,\tau \rho \mu} \\
\psi_{\tau,\rho \mu}^* & \xrightarrow{\psi_{\tau,\rho \mu}} & \psi_{\sigma,\tau,\rho \mu}^* \\
\sigma_*(\tau_* (U_{\rho,\mu}) U_{\sigma,\tau \rho \mu}) & \xrightarrow{\text{id}_{\sigma_*(U_{\sigma,\tau \rho \mu})}} & \sigma_*(U_{\sigma,\tau \rho \mu}) U_{\sigma,\tau \rho \mu} \\
\end{array}
\]

commutes for all \( \sigma, \tau, \rho, \mu \in G \) (where tensor symbols among objects and arrows have been omitted as a space-saving measure).

**Remark 3.5.** For every group \( G \), we can associate a discrete 3-category \( G \), where objects are elements of \( G \), and

\[
G(g, h) = \begin{cases} 
\{ * \}, & \text{if } g = h \\
\emptyset, & \text{if } g \neq h.
\end{cases}
\]

\[
\otimes : G \times G \to G,
\]

\[
\otimes_{g,h} : G(\sigma, \sigma) \times G(\tau, \tau) \to G(\sigma \tau, \sigma \tau).
\]

The definition of a coherence outer \( G \)-action over \( C \), is equivalent to the definition of a trihomomorphism from \( G \) to \( \text{Bieq}(C) \) (see [10] for the definition of trihomomorphism).

Given a crossed system associated to an incoherent outer \( G \)-action, we can define a monoidal bicategory. In order to describe the monoidal bicategory in a simple way we can suppose, without loss of generality that \( C \) is skeletal, so for every pair \( \sigma, \tau \in G \) there is only one pseudonatural transformation \( \chi^{-1}_{\sigma,\tau} : (\sigma \tau)_* \to \sigma_* \tau_* \), such that \( \chi^{-1}_{\sigma,\tau} \circ \chi_{\sigma,\tau} = \text{id}_{\sigma_* \tau_*} \), and \( \chi_{\sigma,\tau} \circ \chi^{-1}_{\sigma,\tau} = \text{id}_{(\sigma \tau)_*} \) for all \( \sigma, \tau \in G \). Let \( G \subseteq \text{Bieq}(C) \) be the full sub-bicategory where objects are \( \{ \sigma \}_{\sigma \in G} \). We define a homomorphism of bicategories \( \otimes_G : < G > \times < G > \to < G > \) by \( \sigma_* \otimes_G \tau_* = (\sigma \tau)_* \), and the commutativity of the diagram

\[
\begin{array}{c}
\sigma_\tau \times \sigma_\tau \\
\chi_{\sigma,\tau} \downarrow \quad \quad \quad \chi_{\sigma,\tau} \downarrow \\
(\sigma \tau)_* \quad \quad \quad (\sigma \tau)_*
\end{array}
\]

where \( f \in \text{Bieq}(\sigma_* \tau_*), g \in \text{Bieq}(C)(\tau_* \tau_*). \)
We define a pseudonatural equivalence in the bicategory \([\mathcal{C} \times \mathcal{C} \times \mathcal{C}]<G>,<G>\) by the commutativity of the diagram

\[
\begin{array}{ccc}
(\sigma \tau \rho)_s & \xrightarrow{a_{\sigma,\tau,\rho}} & (\sigma \tau \rho)_s \\
\chi_{\sigma,\tau,\rho} & & \chi_{\sigma,\tau,\rho} \\
\sigma_s \circ (\tau \rho)_s & & (\tau \rho)_s \rho_s \\
\id_s \otimes \chi_{\tau,\rho} & & \chi_{\tau,\rho} \otimes \id_s \\
\sigma_s \tau_s \rho_s & & \chi_{\sigma,\tau \rho} \\
\end{array}
\]

The diagram (3.1) defines a modification \(\pi: a_{\sigma,\tau,\rho,\mu} \circ a_{\sigma,\tau,\rho,\mu} \rightarrow a_{\sigma,\tau,\rho,\mu} \otimes \id_{\mu} \circ a_{\sigma,\tau,\rho,\mu} \circ \id_{\sigma} \otimes G a_{\tau,\rho,\mu},\)
in the bicategory \([\mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C}],<G>,<G>\).

Since \(\mathcal{C}\) is skeletal for every invertible object \(U \in \mathcal{C}\), we can identify \(\text{Aut}_\mathcal{C}(U)\) with \(\text{Aut}(I) = k^*\), so the modification \(\omega\) is identified by a map \(\omega: G \times G \rightarrow k^*\).

The modification \(\pi\) defines a map \(\pi: G \times G \times G \rightarrow k^*\), given by

\[
\pi(\sigma, \tau, \rho, \mu) = \delta(\omega)(\sigma, \tau, \rho, \mu)\psi_{\tau_s(U_{\tau,\rho})}^\sigma(\psi_{\sigma_s(U_{\sigma,\tau})}^\tau(U_{\tau,\rho}))^{-1}.
\]

It is straightforward to see that \(\pi\) is a 4-cocycle, see [7, Subsection 8.4]. It is also possible to see the 4-cocycle condition directly for the nonabelian 4-cocycle condition [10, (TA1)] in the monoidal bicategory \(<G>\). It is clear that if the chosen modification is changed, the 4-cocycle \(\pi\) only change for a 4-coboundary, so an incoherent outer \(G\)-action defines a fourth cohomology class.

**Proposition 3.6.** Let \(\mathcal{C}\) be a tensor category. An incoherent outer \(G\)-action over \(\mathcal{C}\) is coherent if and only if the associated fourth cohomology class is trivial.

**Proof.** If an outer \(G\)-action is coherent, the diagram (3.1) commutes, so the map \(\pi\) is trivial. Conversely, if there is a 3-coboundary \(\lambda: G \times G \times G \rightarrow G\), such that \(\delta(\lambda) = \pi\), then the modification defined by the map \(\lambda^{-1}\omega\) defines a coherent outer \(G\)-action. \(\square\)

**3.3. The crossed product tensor category associated to a coherent outer \(G\)-action.** If a group \(G\) acts over a monoidal category \(\mathcal{C}\), we shall define a \(G\)-crossed product tensor category associated to this action, denoted as \(\mathcal{C} \times \mathcal{C}\). We set \(\mathcal{C} \times \mathcal{C} = \bigoplus_{\sigma \in G} \mathcal{C}_\sigma\) as an abelian category, where \(\mathcal{C}_\sigma = \mathcal{C}\). We shall denote by \([V, \sigma]\) the object \(V \in \mathcal{C}_\sigma\), and a morphism from \(\bigoplus_{\sigma \in G}[V, \sigma]\) to \(\bigoplus_{\sigma \in G}[W, \sigma]\) is expressed as \(\bigoplus_{\sigma \in G}[f_\sigma, \sigma]\) with \(f_\sigma: V_\sigma \rightarrow W_\sigma\) a morphism in \(\mathcal{C}\).
The tensor product $\cdot : \mathcal{C} \times G \times \mathcal{C} \times G \to \mathcal{C} \times G$ is defined by
\[
[V, \sigma] \cdot [W, \tau] := [V \otimes \sigma_*(W) \otimes U_{\sigma, \tau}, \sigma \tau]
\]
for objects, and
\[
[f, \sigma] \cdot [g, \tau] := [f \otimes \sigma_*(g) \otimes \text{id}_{U_{\sigma, \tau}}, \sigma \tau]
\]
for morphisms.

It is easy to see that the unit object is $(I, e)$. The associativity is given by
\[
[V, \sigma] \cdot ([W, \tau] \cdot [Z, \rho]) \Rightarrow \begin{array}{c}
\alpha_{[V, \sigma], [W, \tau], [Z, \rho]} \\
\alpha_{C \times G}^{(V, \sigma, W, \tau, Z, \rho), \sigma \tau \rho}
\end{array}
\]

where $\alpha_{C \times G}^{(V, \sigma, W, \tau, Z, \rho)}$ is the composition
\[
\begin{array}{l}
V \otimes \sigma_*(W \otimes \tau_* (Z) \otimes U_{\tau, \rho}) \otimes U_{\sigma, \tau \rho} \\
\quad \text{id}_V \otimes \psi^{\tau_* (Z) \otimes U_{\tau, \rho} \otimes \text{id}_{U_{\sigma, \tau \rho}}} \\
\quad V \otimes \sigma_*(W) \otimes \sigma_* (\tau_* (Z) \otimes U_{\tau, \rho}) \otimes U_{\sigma, \tau \rho} \\
\quad \text{id}_V \otimes \psi^{\tau_* (Z) \otimes U_{\tau, \rho} \otimes \text{id}_{U_{\sigma, \tau \rho}}} \\
\quad V \otimes \sigma_*(W) \otimes \sigma_* (\tau_* (Z)) \otimes \sigma_*(U_{\tau, \rho}) \otimes U_{\sigma, \tau \rho} \\
\quad \text{id}_V \otimes \psi^{\tau_* (Z) \otimes \text{id}_{U_{\sigma, \tau \rho}}} \\
\quad V \otimes \sigma_*(W) \otimes \sigma_* (\tau_* (Z)) \otimes U_{\sigma, \tau} \otimes U_{\sigma \tau, \rho} \\
\quad \text{id}_V \otimes \psi^{\tau_* (Z) \otimes \text{id}_{U_{\sigma \tau, \rho}}} \\
\quad V \otimes \sigma_*(W) \otimes U_{\sigma, \tau} \otimes (\sigma \tau)_*(Z) \otimes U_{\sigma \tau, \rho}
\end{array}
\]

The associativity constraint has been omitted as a space-saving measure. As we shall see, the coherence condition over an outer $G$-action, is exactly the pentagonal identity for $C \times G$.

3.3.1. **Pentagonal identity for $C \times G$.** For a category $\mathcal{D}$ with a bifunctor $\otimes : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ and natural isomorphisms $\alpha_{A, B, C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$, we shall denote by $P(A, B, C, D)$ the following pentagonal diagram
\[
\begin{array}{c}
\alpha_{A, B, C \otimes D} \\
\text{id} \otimes \alpha_{B, C, D}
\end{array}
\]

\[
\begin{array}{c}
\alpha_{A \otimes B, C, D} \\
\alpha_{A, B \otimes C, D}
\end{array}
\]

\[
\begin{array}{c}
\alpha_{A, B, C \otimes \text{id}} \\
\text{id} \otimes \alpha_{B, C, D}
\end{array}
\]

\[
\begin{array}{c}
\alpha_{A \otimes B, C \otimes D} \\
\text{id} \otimes \alpha_{B, C, D}
\end{array}
\]

\[
\begin{array}{c}
\alpha_{A, B \otimes C, D}
\end{array}
\]

\[
\begin{array}{c}
\alpha_{A, B, C \otimes \text{id}}
\end{array}
\]

\[
\begin{array}{c}
\alpha_{A \otimes B, C \otimes \text{id}} \\
\alpha_{A, B \otimes C, \text{id}}
\end{array}
\]

\[
\begin{array}{c}
\alpha_{A, B \otimes C, \text{id}}
\end{array}
\]

\[
\begin{array}{c}
\alpha_{A \otimes B, C \otimes \text{id}}
\end{array}
\]

\[
\begin{array}{c}
\alpha_{A, B \otimes C, \text{id}}
\end{array}
\]
Remark 3.7. From now on, we shall denote $[V] := [V, e]$ and $[\sigma] := [I, \sigma]$, for all $V \in C$, $\sigma \in G$. Analogously, $[f] := [f, e] : [V] \to [W]$ for all arrow $f : V \to W$ in $C$. Note that $[V] \cdot [\sigma] = [V, \sigma]$ and $[\sigma] \cdot [V] = [\sigma, (V), \sigma]$ for all $V \in C$, $\sigma \in G$.

In order to prove the coherence of $C \times G$, is sufficient to see the pentagonal identity for the $[V], [\sigma], V \in C, \sigma \in G$, since every object in $C \times G$ is a direct sum of tensor products of $[V], [\sigma]$.

First, see the next equality

\[
\alpha_{[V^σ, [W, τ], [Z, ρ]]} = \text{id}_{[V]} \cdot \alpha_{[I, [σ, [W, τ]], [Z, ρ]]},
\]

so $\alpha_{[V^σ, [W, τ], [Z, ρ]]} = \text{id}$. The eight pentagonal identities of the form $P([V], \ldots)$ follows from (3.2).

The pentagon $P([σ], [V], [W], [τ])$ commutes because

\[
\alpha_{[σ, [W], [Z, ρ]]} = \alpha_{[σ, [W]], [Z]} \cdot \text{id}_{[ρ]}.\]

The pentagons $P([σ], [V], [W], [τ]), P([σ], [τ], [ρ]), P([σ], [τ], [V], [ρ])$ commute by the definition of $α$.

The following table explains the commutativity of the other pentagons

| Pentagons                  | Pentagonal identity equivalence |
|----------------------------|--------------------------------|
| $P([σ], [V], [W], [Z])$   | $(σ, ω^{σ, ω})$ is a monoidal functor |
| $P([σ], [τ], [V], [W])$   | $(χ_{σ, τ}, U_{σ, τ})$ is a pseudonatural equivalence |
| $P([σ], [τ], [ρ], [V])$   | $ω_{σ, τ, ρ}$ is a modification |
| $P([σ], [τ], [ρ], [γ])$   | commutativity of the diagram (3.1). |

3.4. The coherent outer $G$-action associated to a $G$-crossed product tensor category. Let $C$ be a $G$-crossed product tensor category. In order to show more clearly the associated coherent outer $G$-action, we shall make some reductions. Let we choose a family $\{N_σ\}_{σ \in G}$ of homogeneous invertible objects, where $N_e = I$. The family $\{N_σ\}_{σ \in G}$ defines the equivalences of categories

$N_σ \otimes (-) : C_e \to C_σ$

$V \mapsto V \otimes N_σ$

$f \mapsto f \otimes \text{id}_{N_σ}$.

Using these equivalences, we have an equivalence of categories

\[
C = \bigoplus_{σ \in G} C_σ \to \bigoplus_{σ \in G} C_σ^e,
\]

where $C_σ^e = C_e$, for all $σ \in G$.

Now, we can transport the monoidal structure of $C$ to $\bigoplus_{σ \in G} C_e$. Then, without loss of generality we can suppose that the graded tensor category $C = \bigoplus_{σ \in G} C_σ$ has the following properties:

- $C_σ = C_e$ for all $σ \in G$ (so we can and will use the same notations of the Remark 3.7).
- the objects $[σ] \in C_σ$ are invertible for all $σ \in G$,
- $[V] \cdot [W, σ] = [V \otimes W, σ]$, for all $V, W \in C_e$, $σ \in G$. 
For each pair $\sigma, \tau \in G$, we have that $[\sigma] \cdot [\tau] \in \mathcal{C}_G$, so there is a unique invertible object $U_{\sigma \tau} \in \mathcal{C}_G$, such that $[\alpha] \cdot [\tau] = ([U_{\sigma \tau}, \sigma \tau]$. Analogously, the objects $[\sigma]$ define functors $\sigma_* : \mathcal{C}_G \to \mathcal{C}_G$, $V \mapsto \sigma_*(V)$ by the rule $[\sigma] \cdot [V] = [\sigma_*(V), \sigma]$ for all $V \in \mathcal{C}_G$, and $id_{[\sigma]} \cdot [f] = [\sigma(f), \sigma]$, for all arrow $f$ in $\mathcal{C}_G$.

**Lemma 3.8.** If the category $(\mathcal{C}_G, \otimes, I)$ is skeletal, then

$$[V, \sigma] \cdot [W, \tau] = [V \otimes \sigma_*(W) \otimes U_{\sigma \tau}, \sigma \tau]$$

for all $V, W \in \mathcal{C}_G$, $\sigma, \tau \in G$.

**Proof.** Since $\mathcal{C}_G$ is skeletal, the category $\mathcal{C} = \bigoplus_{\sigma} \mathcal{C}_G$ is skeletal. Then we do not need to parenthesize tensor products for objects in $\mathcal{C}$. Also, recall that $[\sigma] \cdot [V] = [\sigma_*(V), \sigma] = [\sigma_*(V)] \cdot [\sigma]$, for all $V \in \mathcal{C}_G$, $\sigma \in G$.

Hence,

$$[V, \sigma] \cdot [W, \tau] = [V \cdot [\sigma] \cdot [W \cdot [\tau] = [V] \cdot [\sigma_*(W)] \cdot [\sigma] \cdot [\tau] = [W \otimes \sigma_*(W)] \cdot [U_{\sigma \tau}, \sigma \tau] = [W \otimes \sigma_*(W)] \cdot [U_{\sigma \tau}, \sigma \tau] = [W \otimes \sigma_*(W) \otimes U_{\sigma \tau}, \sigma \tau] = [W \otimes \sigma_*(W) \otimes U_{\sigma \tau}, \sigma \tau]$$

for all $V, W \in \mathcal{C}_G$, $\sigma \in G$.

Under this reduction and using the Lemma 3.8 we can describe the coherent outer $G$-action as the reciprocal construction of the Subsection 3.3. Suppose that $\mathcal{C}_G$ is skeletal, then the data that define the coherent outer $G$-action associated to $\mathcal{C}$ are the following:

- monoidal equivalences: $(\sigma_*, \psi^{\sigma,*}) : \mathcal{C}_G \to \mathcal{C}_G$, where $[\psi^{\sigma,*}_{W, Z}, \sigma] := [\sigma_*(W \otimes Z), \sigma] \to [\sigma_*(W) \otimes \sigma_*(Z), \sigma]$,
- pseudonatural transformations: $(U_{\sigma \tau}, \chi_{\sigma \tau}) : \sigma_* \circ \tau_* \to (\sigma \tau)_*$, where $[\chi_{\sigma \tau}(Z), \sigma \tau] := [\sigma_*(\tau_*(Z)) \otimes U_{\sigma \tau}, \sigma \tau] \to [U_{\sigma \tau} \otimes (\sigma \tau)_*(Z), \sigma \tau]$,
- modifications $\omega_{\sigma \tau, \rho} : \chi_{\sigma \tau, \rho} \circ (id_{\sigma} \odot \chi_{\tau, \rho}) \to \chi_{\sigma \tau, \rho} \odot (\chi_{\sigma \tau, \rho} \odot id_{\rho})$, where $[\omega_{\sigma \tau, \rho}, \sigma \tau \rho] := [\alpha_{\sigma, [\tau, [\rho]]} : [\sigma_*(U_{\tau \rho}) \otimes U_{\sigma \tau \rho}, \sigma \tau \rho] \to [U_{\sigma \tau} \otimes U_{\sigma \tau \rho}, \sigma \tau \rho]$.

4. Classification of crossed product tensor category

In this section, we shall see that the 2-category of crossed product tensor category over a fixed group $G$, is equivalent to the 2-category of all coherent outer $G$-actions.

The 0-cells of the 2-category of coherent outer $G$-actions are coherent outer $G$-action over a tensor category.

Let $(\{ \sigma \}_{\sigma \in G}, \chi, \omega)$ and $(\{ \rho \}_{\rho \in G}, \chi', \omega')$ be coherent outer $G$-actions over tensor categories $\mathcal{C}$ and $\mathcal{D}$, respectively. An arrow from $(\{ \sigma \}_{\sigma \in G}, \chi, \omega)$ to $(\{ \rho \}_{\rho \in G}, \chi', \omega')$, is a triple $(H, \theta, \Pi)$, where $(H, \psi^H) : \mathcal{C} \to \mathcal{D}$ is an op-monoidal functor, $(\theta^\sigma, \theta^\rho) : \sigma \circ H \to H \circ \sigma$ is a pseudonatural equivalence for each $\sigma \in G$, and $\Pi$ is a modification.
such that: \((\theta^e, \theta^e) = (I, \text{id})\), \(\Pi_{\sigma, e} = \Pi_{e, \sigma} = \text{id}_{\theta_e}\) for all \(\sigma \in G\), and the diagram

\[
\begin{array}{ccc}
\hat{\sigma}H \hat{\tau} & \xrightarrow{\text{id}_{\sigma} \otimes \theta_\tau} & \hat{\sigma} \hat{\tau}H \\
\uparrow \Pi_{\sigma, \tau} & & \uparrow H \Pi_{\hat{\sigma}, \hat{\tau}} \\
\hat{\sigma} \hat{\tau}H & \xrightarrow{\theta_{\sigma \tau}} & H \hat{\sigma} \hat{\tau}
\end{array}
\]

\[
\chi^'_{\sigma, \tau} \otimes \text{id}_H
\]

\[
\begin{array}{ccc}
\hat{\sigma}H \hat{\tau} & \xrightarrow{\text{id}_{\sigma} \otimes \theta_\tau} & \hat{\sigma} \hat{\tau}H \\
\uparrow \Pi_{\sigma, \tau} & & \uparrow H \Pi_{\hat{\sigma}, \hat{\tau}} \\
\hat{\sigma} \hat{\tau}H & \xrightarrow{\theta_{\sigma \tau}} & H \hat{\sigma} \hat{\tau}
\end{array}
\]

commutes for all \(\sigma, \tau, \rho \in G\) (where tensor symbols among objects have been omitted as a space-saving measure).

A 2-cell from \((H, \theta, \Pi)\) to \((\tilde{H}, \tilde{\theta}, \tilde{\Pi})\) consist of the data \(\{m_\sigma, m\}_{\sigma \in G}\), where \(m : H \to \tilde{H}\) is a monoidal natural transformation and \(m_\sigma : \theta_\sigma \to \tilde{\theta}_\sigma\) are morphisms in \(C\). The previous data are subject to the following axioms: \(m_e = \text{id}_t\) and the diagrams

\[
\begin{array}{ccc}
\theta_\sigma \tilde{\sigma}(\theta_\tau) U'_{\sigma, \tau} & \xrightarrow{m_\sigma \otimes \tilde{\theta}(m_\tau) \otimes \text{id}_{U'_\tau}} & \theta_\sigma \tilde{\sigma}(\tilde{\theta}(V)) U'_{\sigma, \tau} \\
\Pi_{\sigma, \tau} \uparrow & & \downarrow \Pi_{\hat{\sigma}, \hat{\tau}} \\
\theta_\sigma \tilde{\sigma}(H(U_{\sigma, \tau}, \theta_\gamma)) U'_\sigma & \xrightarrow{m_\sigma \otimes \tilde{\theta}(m_\tau) \otimes \text{id}_{U'_\tau}} & \theta_\sigma \tilde{\sigma}(H(V)) U'_\sigma
\end{array}
\]

\[
\begin{array}{ccc}
H(U_{\sigma, \tau}) \theta_{\sigma \tau} & \xrightarrow{m_{U_{\sigma, \tau}} \otimes m_{\sigma \tau}} & H(U_{\sigma, \tau}) \tilde{\theta}_{\sigma \tau} \\
\Pi_{\sigma, \tau} \uparrow & & \uparrow \Pi_{\hat{\sigma}, \hat{\tau}} \\
H(\tilde{\sigma}(V)) \theta_{\sigma} & \xrightarrow{m_{\tilde{\sigma}(V)} \otimes m_{\sigma \tau}} & H(\tilde{\sigma}(V)) \theta_{\sigma}
\end{array}
\]

commute for all \(\sigma, \tau \in G, V \in C\) (where tensor symbols among objects have been omitted).

**Theorem 4.1.** There is a biequivalence between the 2-category of coherent outer \(G\)-actions and the 2-category of \(G\)-crossed product tensor categories.
Proof. The bijective correspondence between $G$-crossed product tensor categories and coherent outer $G$-action was described in the Section 3.

If $T = (H, \theta, \Pi)$ is a 1-cell between coherent outer $G$-action $(\{T\}_{g \in \mathcal{G}}, \chi)$ and $(\{\tilde{T}\}_{g \in \mathcal{G}}, \hat{\chi}, \hat{\omega})$ over $\mathcal{C}$ and $\mathcal{D}$ respectively, then we define an op-monoidal functor $(T, \psi^T) : \mathcal{C} \times G \to \mathcal{D} \times G$ as

\[ \psi^T : T([V, \sigma]) \cdot T([W, \sigma]) \to T([V, \sigma] \cdot [W, \tau]), \]

where

\[ T([V, \sigma]) \cdot T([W, \sigma]) \xrightarrow{\psi_{[V, \sigma],[W, \tau]}^T} H(V\theta_\sigma \tilde{\sigma}(H(W)\theta_\tau)U'_{\sigma, \tau} \xrightarrow{id_{H(V)\theta_\sigma \tilde{\sigma}(H(W)\theta_\tau)U'_{\sigma, \tau}}} H(V\tilde{\sigma}(W))H(U_{\sigma, \tau})\theta_\sigma \xrightarrow{id_{H(V\tilde{\sigma}(W))H(U_{\sigma, \tau})\theta_\sigma \theta_\tau}} H(V\tilde{\sigma}(W))U_{\sigma, \tau} \xrightarrow{id_{H(V\tilde{\sigma}(W))U_{\sigma, \tau}}} H(V\tilde{\sigma}(W)U_{\sigma, \tau})\theta_\sigma \theta_\tau \]

(where tensor symbols among objects of $\mathcal{C}$ have been omitted as a space-saving measure). Conversely, given a graded op-monoidal functor $(T, \psi^T) : \mathcal{C} \times G \to \mathcal{D} \times G$, we define a 1-cell $(H, \theta, \Pi)$ as $H(V, e) = T([V, e])$, $[\theta_\sigma, \sigma] = T([I, \sigma]), [\theta_\hat{\sigma}, \sigma] = \tilde{T}_{[I, \sigma],[V, e]}, [\Pi_{\sigma, \tau}, \sigma \tau] = \tilde{T}_{[I, \sigma],[I, \tau]}$.

Given a 2-cell $\{m_\sigma, m_{\sigma}\} \in \mathcal{G}$ between 1-cells $T = (H, \theta, \Pi)$ and $T' = (H', \theta', \Pi')$, we define a monoidal natural isomorphism $m : T \to T'$ between the associated op-monoidal functors by $m_{[V, \sigma]} = [m_V \otimes m_\sigma, \sigma]$. Conversely, given a monoidal natural isomorphism $m : T \to T'$, we define a 2-cell by $[m_V, e] = m_{[V, e]}, [m_\sigma, \sigma] = m_{[I, \sigma]}$.

Finally, in order to see that the 2-categories are biequivalent, note that every crossed product tensor category is equivalent to one of the form $\mathcal{C}_c \times G$. So, every functor between $\mathcal{C} \times G$ and $\mathcal{D} \times G$ is monoidally equivalent to one induced by a 1-cell of the coherent outer $G$-action 2-category, and every monoidal natural transformation is equal to one induced by a 2-cell. \qed

5. Braided crossed product tensor categories

Recall that a braiding for a monoidal category $(\mathcal{C}, \otimes, I, \alpha)$ is a natural isomorphism $c : \otimes \to \otimes \tau$, where $\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is the flip, and the hexagons
commute for all \( U, V, W \in C \).

If a \( G \)-crossed product tensor category admits a braiding, the group \( G \) must be abelian. So, from now on we shall suppose that \( G \) is abelian.

Let \( C \) be a tensor category with a coherent outer \( G \)-action, such that the tensor category \( C \rtimes G \) admits a braiding \( c \). The braiding \( c_{[V],[\sigma]} : [V, \sigma] \to [\sigma_*(V), \sigma] \) defines natural isomorphisms \( c_{V,\sigma} : V \to \sigma_*(V) \). The commutativity of the hexagon (5.1) is equivalent to \( c_{V,\sigma} \) is a monoidal natural isomorphism from \( \text{id}_C \) to \( \sigma_* \). For that reason, if \( C \rtimes G \) has a braiding, we can suppose that \( \sigma_* = \text{id}_C \) for all \( \sigma \in G \).

**Definition 5.1.** A coherent outer \( G \)-action shall be called central if \( \sigma_* = \text{id}_C \) for all \( \sigma \in G \).

**Remark 5.2.** For a central coherent outer \( G \)-action, the pseudonatural transformations \( (\chi_{\sigma,\tau}, U_{\sigma,\tau}) \) are just elements in \( \mathcal{Z}(C) \) (the center of \( C \)), and the modifications \( \omega \) are morphisms in \( \mathcal{Z}(C) \).

**Definition 5.3.** Let \((C, c)\) be a braided tensor category, and let \( G \) be an abelian group. A braiding for a central coherent \( G \)-action over \( C \) is a triple \((\theta^e, \overline{\theta}^e, t_{\sigma,\tau})_{\sigma,\tau \in G}\), where

- \( \theta^e, \overline{\theta}^e : \text{id}_C \to \text{id}_C \) are monoidal natural isomorphisms,
- \( t_{\sigma,\tau} : \chi_{\sigma,\tau} \to \chi_{\tau,\sigma} \) are isomorphisms in \( \mathcal{Z}(C) \) for all \( \sigma, \tau \in G \),

such that \( \theta^e = \overline{\theta}^e = \text{id} \), \( \theta^e_I = \text{id}_I \), \( t_{e,\sigma} = t_{e,\tau} = \text{id}_I \), and the diagrams

\[
\begin{align*}
\text{Z} \otimes U_{\sigma,\tau} & \xrightarrow{\chi_{\sigma,\tau}(Z)} U_{\sigma,\tau} \otimes \text{Z} \\
(\overline{\theta}^e_{\text{Z}})^{-1} & \xrightarrow{(\overline{\theta}^e_{\text{Z}}) \otimes \text{id}_{U_{\sigma,\tau}}} Z \otimes U_{\sigma,\tau} \\
\end{align*}
\]
commute for all $\sigma, \tau, \rho \in G, Z \in \mathcal{C}$ (where tensor symbols among objects have been omitted as a space-saving measure).

**Theorem 5.4.** Let $(\mathcal{C}, c)$ be a braided tensor category with a coherent central outer $G$-action. Then, there is a bijective correspondence between braidings over $\mathcal{C} \times G$ and braidings over the central coherent outer $G$-action of $\mathcal{C}$.

**Proof.** Let $(\theta, \overline{\theta}, t)$ be a braiding for a central coherent outer $G$-action $(\chi, \omega)$. Then, we define a braiding over $\mathcal{C} \times G$ by

$$c_{[V, \sigma], [W, \tau]} = (c_{V, W} \circ (\theta_V \otimes \overline{\theta}_W)) \otimes t_{\sigma, \tau} = ((\overline{\theta}'_V \otimes \theta'_W) \circ c_{V, W}) \otimes t_{\sigma, \tau}.$$  

Conversely, given a braiding $\overline{c}$ over $\mathcal{C} \times G$, we define a braiding for the coherent central outer $G$-action by

$$[\theta^\sigma_V, \sigma] := c_{[V], [\sigma]}, \quad [\overline{\theta}'_V, \sigma] := c_{[\sigma], [V]}, \quad [t_{\sigma, \tau}, \sigma \tau] := c_{[\sigma], [\tau]}.$$  

Let we denote by $H(U, V, W)$ and $H'(U, V, W)$ the hexagons (5.1) and (5.2), respectively. Let $\theta^\sigma, \overline{\theta}' : \text{id}_\mathcal{C} \to \text{id}_\mathcal{C}$ be natural isomorphisms for each $\sigma \in G$, and let $t_{\sigma, \tau} : U_{\sigma, \tau} \to U_{\tau, \sigma}$ be isomorphisms in $\mathcal{C}$. If we set the following definitions of natural isomorphisms

$$c_{[V], [\sigma]} := [\theta^\sigma_V, \sigma], \quad c_{[\sigma], [V]} := [\overline{\theta}'_V, \sigma], \quad c_{[\sigma], [\tau]} := [t_{\sigma, \tau}, \sigma \tau],$$

it is easy to see that the commutativity of $H([V], [W], \sigma)$ and $H'([\sigma], [V], [W])$ is equivalent to $\theta^\sigma$ and $\overline{\theta}'$ be monoidal natural isomorphisms, respectively. The commutativity of $H'([\sigma], [\tau], [Z])$ and $H([\sigma], [V], [\tau])$ is equivalent to $t_{\sigma, \tau}$ be a morphism in $\mathcal{Z}(\mathcal{C})$. The commutativity of $H([\sigma], [\tau], [Z])$ and $H([Z], [\sigma], [\tau])$ is equivalent to the commutativity of (5.3) and (5.4), respectively. The commutativity of
$H([\sigma],[\tau],[\rho])$ and $H'([\sigma],[\tau],[\rho])$ is equivalent to the commutativity of (5.5), (5.6), respectively.

\[\square\]

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