THE RADIUS OF INJECTIVITY OF LOCAL RING $Q$–HOMEOMORPHISMS

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Abstract

The paper is devoted to the study of mappings with non–bounded characteristics of quasiconformality. The analog of the theorem about radius injectivity of locally quasiconformal mappings was proved for some class of mappings. There are found sharp conditions under which the so called local $Q$–homeomorphisms are injective in some neighborhood of a fixed point.

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1 Introduction

Here are some definitions. Everywhere below, $D$ is a domain in $\mathbb{R}^n$, $n \geq 2$, $m$ be a measure of Lebesgue in $\mathbb{R}^n$, and $\text{dist} (A, B)$ is the Euclidean distance between the sets $A$ and $B$ in $\mathbb{R}^n$. The notation $f : D \to \mathbb{R}^n$ assumes that $f$ is continuous on its domain. In what follows $(x, y)$ denotes the standard scalar multiplication of the vectors $x, y \in \mathbb{R}^n$, $\text{diam} A$ is Euclidean diameter of the set $A \subset \mathbb{R}^n$,

$$B(x_0, r) = \{ x \in \mathbb{R}^n : |x - x_0| < r \}, \quad \mathbb{B}^n := B(0, 1),$$

$$S(x_0, r) = \{ x \in \mathbb{R}^n : |x - x_0| = r \}, \quad \mathbb{S}^{n-1} := S(0, 1),$$

$\omega_{n-1}$ denotes the square of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, $\Omega_n$ is a volume of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$. A mapping $f : D \to \mathbb{R}^n$ is said to a local homeomorphism if for every $x_0 \in D$ there is a number $\delta > 0$ such that a mapping $f|_{B(x_0, \delta)}$ to be a homeomorphism.

Recall that a mapping $f : D \to \mathbb{R}^n$ is said to be a mapping with bounded distortion, if the following conditions hold:

1) $f \in W^1_{\text{loc}}$,
2) a Jacobian $J(x, f) := \det f'(x)$ of the mapping $f$ at the point $x \in D$ preserves the sign almost everywhere in $D$,
3) $\|f'(x)\|^n \leq K \cdot |J(x, f)|$ at a.e. $x \in D$ and some constant $K < \infty$, where

$$\|f'(x)\| := \sup_{h \in \mathbb{R}^n : |h|=1} |f'(x)h|,$$
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see., e.g., [7] § 3, Ch. I, or definition 2.1 of the section 2 Ch. I in [8]. In this case we also say that $f$ is $K$–quasiregular, where $K$ is from condition 3) meaning above.

The following result was proved in the work [5] by O. Martio, S. Rickman and J. Väisälä, see [5] Theorem 2.3 or [8] Theorem 3.4.III, see also the paper [3].

Statement 1. If $n \geq 3$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a $K$–quasiregular local homeomorphism, then $f$ is injective in a ball $B(0, \psi(n, K))$, where $\psi$ is a positive number depending only on $n$ and $K$.

A goal of the present paper is a proof of the analog of the Statement 1 for more general classes of mappings of ring $Q$–homeomorphisms. To introduce this class of the mappings, we give some definitions.

A curve $\gamma$ in $\mathbb{R}^n$ is a continuous mapping $\gamma : \Delta \to \mathbb{R}^n$ where $\Delta$ is an open, closed or half–open interval in $\mathbb{R}$. Given a family $\Gamma$ of paths $\gamma$ in $\mathbb{R}^n$, $n \geq 2$, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called admissible for $\Gamma$, abbr. $\rho \in \text{adm } \Gamma$, if

$$\int_\gamma \rho(x) |dx| \geq 1$$

for each $\gamma \in \Gamma$. The modulus of $\Gamma$ is the quantity

$$M(\Gamma) = \inf_{\rho \in \text{adm } \Gamma} \int_{\mathbb{R}^n} \rho^n(x) dm(x).$$

Given a domain $D$ and two sets $E$ and $F$ in $\mathbb{R}^n$, $n \geq 2$, $\Gamma(E, F, D)$ denotes the family of all paths $\gamma : [a, b] \to \mathbb{R}^n$ which join $E$ and $F$ in $D$, i.e., $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for $a < t < b$.

Let $D$ be a domain in $\mathbb{R}^n$, $Q : D \to [0, \infty]$ be a (Lebesgue) measurable function. Set

$$A(x_0, r_1, r_2) = \{ x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2 \}.$$ 

We say that a mapping $\mathbb{R}^n$ is a ring $Q$–mapping at a point $x_0 \in D$ if

$$M(f(\Gamma(S(x_0, r_1), S(x_0, r_2), A(x_0, r_1, r_2)))) \leq \int_{A(x_0, r_1, r_2)} Q(x) \cdot \eta^n(|x - x_0|) dm(x) \quad (1.1)$$

for every ring $A(x_0, r_1, r_2)$, $0 < r_1 < r_2 < r_0 = \text{dist}(x_0, \partial D)$, and for every Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r) dr \geq 1.$$ 

If the condition $(1.1)$ holds at every point $x_0 \in D$, then we also say that $f$ is a ring $Q$–mapping in the domain $D$, see [4] section 7.

In what follows $q_{x_0}(r)$ denotes the integral average of $Q(x)$ under the sphere $|x - x_0| = r$,

$$q_{x_0}(r) := \frac{1}{\omega_{n-1} r^{n-1}} \int_{|x - x_0| = r} Q(x) dS, \quad (1.2)$$
where $dS$ is element of the square of the surface $S$.

One of the main results of the paper is following.

**Theorem 1.1.** Let $n \geq 3$ and $f : \mathbb{B}^n \to \mathbb{R}^n$ is a local ring $Q$–homeomorphism at the point $x_0 = 0$, such that $Q \in L^1_{loc}(\mathbb{B}^n)$ and

$$
\int_0^1 \frac{dt}{tq_0^{1/(n-1)}(t)} = \infty .
$$

(1.3)

Then $f$ is injective in a ball $B(0, \delta(n, Q))$, where $\delta$ is a positive number depending only on $n$ and function $Q$. From other hand, the condition (1.3) is precise, in fact, for every $\delta > 0$ and every $Q \in L^1_{loc}(\mathbb{B}^n)$ with $Q(x) \geq 1$ a.e. and

$$
\int_0^1 \frac{dt}{tq_0^{1/(n-1)}(t)} < \infty
$$

(1.4)

there exists a mapping $f = f_Q : \mathbb{B}^n \to \mathbb{R}^n$ which is local ring $Q$–homeomorphism at the point $x_0 = 0$ and which is not injective in $B(0, \delta)$.

## 2 The main Lemma

A set $Q \subset \mathbb{R}^n$ is said to be **relatively locally connected** if every point in $\mathring{Q}$ has arbitrary small neighborhoods $U$ such that $U \cap Q$ is connected.

We also need following statements, see [8 Lemmas 3.1.III–3.3.III].

**Proposition 2.1.** Let $f : G \to \mathbb{R}^n$ be a local homeomorphism, let $Q$ be a simply connected and locally pathwise connected set in $\mathbb{R}^n$, and let $P$ be a component of $f^{-1}(Q)$ such that $\mathring{P} \subset G$. Then $f$ maps $P$ homeomorphically onto $Q$. If, in addition, $Q$ is relatively locally connected, $f$ maps $\mathring{P}$ homeomorphically onto $\mathring{Q}$.

**Proposition 2.2.** Let $f : G \to \mathbb{R}^n$ be a local homeomorphism and let $F$ be a compact set in $G$ such that $f_F$ is injective. Then $f$ is injective in a neighborhood of $F$.

**Proposition 2.3.** Let $f : G \to \mathbb{R}^n$ be a local homeomorphism, let $A, B \subset G$, and let $f$ be homeomorphic in $A$ and $B$. If $A \cap B \neq \emptyset$ and $f(A) \cap f(B)$ is connected, then $f$ is homeomorphic in $A \cup B$.

Finally, we need the following statement of P. Koskela, J. Onminen and K. Rajala, see [3 Lemma 3.1].

**Proposition 2.4.** Let $n \geq 1$ and $r > 0$. Let $a \neq b$, $a, b \in S(0, r)$. Then there exists a point $p = p(a, b) \in B(0, r)$ such that for every $t \in \left(\frac{r}{2}, \frac{\sqrt{3}r}{2}\right)$ either

$$
0, b \in B(p, t) \quad \text{and} \quad a \notin B(p, t)
$$

or

$$
a, b \in B(p, t) \quad \text{and} \quad 0 \notin B(p, t) .
$$
The following Lemma plays the main role in the following.

Lemma 2.1. Let $n \geq 3$, $Q : \mathbb{B}^n \to [0, \infty]$ and $f : \mathbb{B}^n \to \mathbb{R}^n$ is a local ring $Q$–homeomorphism at the point $x_0 = 0$. Suppose that there exist a function $\psi : (0, 1) \to [0, \infty]$ and a constant $C = C(n, Q, \psi)$ such that

$$0 < I(r_1, r_2) := \int_{r_1}^{r_2} \psi(t)dt < \infty \quad \forall r_1, r_2 \in (0, 1)$$

(2.1)

and for some $\alpha > 0$

$$\int_{r_1 < |x| < r_2} Q(x)\psi(|x|)dm(x) \leq C \cdot I^{n-\alpha}(r_1, r_2).$$

(2.2)

Let

$$I(0, 1) := \int_0^1 \psi(t)dt = \infty,$$

(2.3)

then $f$ is injective in a ball $B(0, \delta(n, Q, \psi))$, where $\delta$ is a positive number depending only on $n$, functions $Q$ and $\psi$.

Proof. The 1 step. We may assume $f(0) = 0$. Let $r_0 = \sup\{r \in \mathbb{R} : r > 0, \overline{U(0, r)} \subset \mathbb{B}^n\}$, where $U(0, r)$ is the 0–component of $f^{-1}(B(0, r))$. Clearly $r_0 > 0$. Fix $r < r_0$ and set $U = U(0, r)$,

$$l^* = l^*(0, f, r) = \inf\{|z| : z \in \partial U\},$$

$$L^* = L^*(0, f, r) = \sup\{|z| : z \in \partial U\}.$$  

By Proposition 2.1, $f$ maps $\overline{U}$ homeomorphically onto $\overline{B(0, r)}$. Thus $f$ is injective in $B(0, l^*)$ and it suffices to find a lower bound for $l^*$.

The 2 step. Note that $L^* \to 1$ as $r \to r_0$. Suppose the contrary: $L^* \not\to 1$ as $r \to r_0$.

a) Remark that $U(0, r_1) \subset U(0, r_2)$ as $0 < r_1 < r_2 < r_0$. In fact, let us assume that there exists $x \in U(0, r_1) \setminus U(0, r_2)$. Since $f(U(0, r_i)) = B(0, r_i)$, $i = 1, 2$, we have $f(x) = y \in B(0, r_1)$ and $f(z) = y \in B(0, r_1)$, $z \neq x$. However, this contradicts to the Proposition 2.3 because $f$ is homeomorphism in $U(0, r_1) \cup U(0, r_2)$ in this case.

b) It follows from a) that the function $L^*$ is increase by $r$ and, consequently, there exists the limit of $L^*$ as $r \to r_0$. Then $L^* \to \varepsilon_0$ as $r \to r_0$, where $\varepsilon_0 \in (0, 1)$. In this case, $U(0, r) \subset B(0, \varepsilon_0)$ for every $0 < r < r_0$.

c) Remark that $B(0, r_0) \subset f(B(0, \varepsilon_0))$. In fact, let $y \in B(0, r_0)$, then $y \in B(0, r_1)$ for some $r_1 \in (0, r_0)$. It follows from hence that there exists $x \in U(0, r_1) \subset f(x) = y$ and, consequently, $y \in f(B(0, \varepsilon_0))$, i.e., $B(0, r_0) \subset f(B(0, \varepsilon_0))$.

d) Remark that $\overline{B(0, r_0)} \subset f(\overline{B(0, \varepsilon_1)})$ and, consequently, by the openness of $f$, $f(B(0, \varepsilon_1))$ contains some neighborhood of $B(0, r_0)$ for every $\varepsilon_1 \in (0, \varepsilon_0)$. Thus, $U(0, r_0)$ lies inside of $B(0, \varepsilon_0)$, that contradicts to the definition of $r_0$. The contradiction obtained above implies that $L^* \to 1$ as $r \to r_0$ that is desired conclusion.
The 3 step. Pick $x$ and $y \in \partial U$ such that $|x| = L^*$ and $|y| = L^*$. Note that, by the definition of $U$, $f(x), f(y) \in S(0, r)$. By Proposition 2.4 there exists a point $p \in B(0, r)$ such that, for every $t \in \left(\frac{r}{2}, \frac{\sqrt{3}r}{2}\right)$, $f(x) \in B(p, t)$ and either $0 \in B(p, t)$ and $f(y) \notin B(p, t)$, or $0 \notin B(p, t)$ and $f(y) \in B(p, t)$. Fix such a $t$. Note that $0, f(y)$ and $f(x) \in f\left(B(0, L^*)\right)$ and, consequently, $f\left(B(0, L^*)\right) \cap B(p, t) \neq \emptyset \neq f\left(B(0, L^*)\right) \setminus B(p, t)$. Since $f\left(B(0, L^*)\right)$ is connected, this implies that there exists a point $z_t \in S(p, t) \cap f\left(B(0, L^*)\right)$, see [4] Theorem 1.I.46.5.

Let $z_t^* \in f^{-1}(z_t) \cap B(0, L^*)$. Let $C_t(\varphi) \subset S(p, t)$ be the spherical cap with center $z_t$ and opening angle $\varphi$, $C_t(\varphi) = \{y \in \mathbb{R}^n : |y - p| = t, (z_t - p, y - p) > t^2 \cos \varphi\}$.

Let $\varphi_t$ be the supremum of all $\varphi$ for which the $z_t^*$-component of $f^{-1}(C_t(\varphi))$ gets mapped homeomorphically onto $C_t(\varphi)$. Let $C_t = C_t(\varphi_t)$ and let $C_t^*$ be the $z_t^*$-component of $f^{-1}(C_t)$.

The 4 step. We claim that $C_t^*$ meets $S(0, L^*)$. Suppose this is not true.

a) Since $C_t^*$ is connected and $C_t^* \cap B(0, L^*) = \emptyset$, this implies that $C_t^* \subset B(0, L^*)$, see [4] Theorem 1.I.46.5. Remark that, in this case, $C_t^*$ is a compact subset of $U$ and by Proposition 2.1 $f$ maps $\overline{C_t^*}$ homeomorphically onto $\overline{C_t}$. (It is not true at $n = 2$ because $C_t(\pi)$ is not relatively locally connected). By Proposition 2.2 $f$ is injective in a neighborhood of $\overline{C_t}$. Thus $\varphi_t = \pi$, $\overline{C_t} = S(p, t)$ and $\overline{C_t^*}$ is a topological $(n - 1)$-sphere in $\mathbb{R}^n$. Note that bounded component $D$ of $\mathbb{R}^n \setminus \overline{C_t^*}$ contained in $B(0, L^*)$. Now $f(D)$ is a compact subset of $f(\mathbb{R}^n)$ and, since the mapping $f$ is open, $\partial f(D) \subset f(\partial D) = S(p, t)$.

b) Remark that $f(D) \subset B(p, t)$. In fact, let $f(D) \notin B(p, t)$, then there exists $y \in f(D) \setminus B(p, t)$. Now we have $\left(f(\mathbb{R}^n) \setminus B(p, t)\right) \cap f(D) \neq \emptyset$ and, since $f(D)$ is compact subdomain of $f(\mathbb{R}^n)$, $\left(f(\mathbb{R}^n) \setminus B(p, t)\right) \cap f(D) \neq \emptyset$. Since $f(\mathbb{R}^n) \setminus B(p, t)$ is connected, this implies that there exists $z \in \partial f(D) \cap f(\mathbb{R}^n) \setminus B(p, t)$, see [4] Theorem 1.I.46.5, that contradicts to the inclusion $\partial f(D) \subset S(p, t)$.

c) Now $f(D) \subset B(p, t)$. Remark that $B(p, t) \subset f(D)$. Indeed, let there exists $a \in B(p, t) \setminus f(D)$. Since $B(p, t)$ is connected and $B(p, t) \cap f(D) = \emptyset$ this implies that $\partial f(D) \cap B(p, t) \neq \emptyset$, see [4] Theorem 1.I.46.5. The last relation contradicts to the inclusion $\partial f(D) \subset S(p, t)$.

d) Thus $f(D) = B(p, t)$. By the definition, $D$ is a component of $f^{-1}(B(p, t))$. By Proposition 2.1 $f$ maps $\overline{D}$ onto $\overline{B(p, t)}$ homeomorphically.

e) Since $z_t^* \in \overline{C_t^*} \cap U$, $\overline{D} \cap \overline{U} \neq \emptyset$. Since $f$ maps $\overline{U}$ homeomorphically onto $\overline{B(0, r)}$, $f$ is injective in $\overline{U} \cup \overline{D}$ by Proposition 2.3. This is impossible, because in view of the equality $f(D) = B(p, t)$ and that $f(x) \in B(p, t)$ there exists a point $x_1 \neq x, x_1 \in D$, such that $f(x_1) = f(x)$. Thus $C_t^*$ meets $S(0, L^*)$.

The 5 step. Let $k_t^* \subset C_t^* \cap S(0, L^*)$ and $k_t = f(k_t^*)$. Let $\Gamma'_t$ be the family of all curves connecting $k_t$ and $z_t$ in $C_t$. Moreover, let $\Gamma'$ be the union of the curve families $\Gamma'_t$, $t \in \left(\frac{r}{2}, \frac{\sqrt{3}r}{2}\right)$. Denote by $f_t$ the restriction of $f$ to $C_t^*$. Then $f_t$ maps $C_t^*$ homeomorphically onto $C_t$. Furthermore, denote $\Gamma = \bigcup_{t \in \left(\frac{r}{2}, \frac{\sqrt{3}r}{2}\right)} \{f_t^{-1} \circ \gamma : \gamma \in \Gamma'_t\}$.
Since for every $t \in \left( \frac{r}{2}, \frac{\sqrt{n}}{2} \right)$, $z^*_t \in B(0, l^*)$ and $k_t \in S(0, L^*)$, be the definition of ring $Q$–mapping we have

$$M(f(\Gamma(S(0, l^*), S(0, L^*), A(0, l^*, L^*)))) \leq \int_{A(0, l^*, L^*)} Q(x) \cdot \eta^n(|x|) dm(x) \quad (2.4)$$

for every function $\eta : (l^*, L^*) \rightarrow [0, \infty]$ with

$$\int_{l^*}^{L^*} \eta(r) dr \geq 1. \quad (2.5)$$

Setting $\eta(t) = \psi(t)/I(l^*, L^*)$, where $\psi$ is the function from the condition of Lemma, we observe that $\eta$ satisfies the above condition. Now from (2.2) and (2.4) we obtain that

$$M(\Gamma') = M(f(\Gamma(S(0, l^*), S(0, L^*), A(0, l^*, L^*)))) \leq \int_{A(0, l^*, L^*)} Q(x) \cdot \eta^n(|x|) dm(x) \leq C/I^\alpha(l^*, L^*). \quad (2.6)$$

On other hand, by [12, Theorem 10.2],

$$\int_{S(p,t)} \rho^n(x) dS \geq \frac{C_n}{t} \quad (2.7)$$

for every $\rho$ for which $\int_{\gamma} \rho(x)|dx| \geq 1$ for every $\gamma \in \Gamma'_t$. The integration of (2.7) over $t$ yields

$$M(\Gamma') \geq C'_n \quad (2.8)$$

for some constant $C'_n > 0$. We obtain from (2.5) and (2.7) that

$$C_n \leq C/I^\alpha(l^*, L^*) \leq C/I^\alpha(l^*(0, f, r_0), L^*(0, f, r)) \quad (2.9)$$

because $I(\varepsilon_1, \varepsilon_2) > I(\varepsilon_3, \varepsilon_2)$ as $\varepsilon_3 > \varepsilon_1$. Letting into the limit as $r \rightarrow r_0$ in (2.8), we have

$$C_n \leq C/I^\alpha(l^*(0, f, r_0), 1). \quad (2.10)$$

First of all, from the (2.10) follows that $I(\varepsilon, 1) < \infty$ for every $\varepsilon \in (0, 1)$. Follow, let $l^*(0, f, r_0) \rightarrow 0$, then it follows from (2.3) that the right hand of (2.9) tends to zero, that contradicts to (2.9). Thus, $l^*(0, f, r_0) \geq \delta$ for all such $f$. The proof is complete. □

### 3 Proof of the main result

The following statement would be very useful, see [10, Theorem 3.15].

**Proposition 3.1.** Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, and $Q : D \rightarrow [0, \infty]$ a locally integrable measurable function. A homeomorphism $f : D \rightarrow \mathbb{R}^n$ is a ring $Q$–homeomorphism at a point $x_0$ if and only if for every $0 < r_1 < r_2 < r_0 = \text{dist}(x_0, \partial D)$,

$$M(\Gamma(f(S_1), f(S_2), f(D))) \leq \frac{\omega_{n-1}}{I^{n-1}}$$
where $\omega_{n-1}$ is the area of the unit sphere in $\mathbb{R}^n$, $q_{x_0}(r)$ is the average of $Q(x)$ over the sphere $|x - x_0| = r$, $S_j = \{x \in \mathbb{R}^n : |x - x_0| = r_j\}$, $j = 1, 2$, and

$$I = I(r_1, r_2) = \frac{\int_{r_1}^{r_2} dr}{r q_{x_0}^{-1}(r)}.$$ 

**Proof of Theorem 1.** Given $0 < r_1 < r_2 < r_0 = 1$ consider the function

$$\psi(t) = \begin{cases} \frac{1}{t q_0^{-1}(t)}, & t \in (r_1, r_2), \\ 0, & t \notin (r_1, r_2). \end{cases} \quad (3.1)$$

Note that $\psi$ satisfies all the conditions of Lemma 2.1, in particular, $\int_{r_1}^{r_2} \frac{dt}{t q_0^{-1}(t)} < \infty$ by [11], Theorem 1. and by Fubini theorem, \(\int_{r_1 < |x| < r_2} Q(x) \cdot \psi^n(|x|) \, dm(x) = \omega_{n-1} \cdot I(r_1, r_2)\). Now the first part of the Theorem follows from Lemma 2.1.

To prove second part of the Theorem, we take $\delta > 0$ and some function $Q \in L^1_{loc}(\mathbb{R}^n)$ satisfying (3.3). Set

$$f(x) = \frac{x}{|x|} \rho(|x|),$$

where

$$\rho(r) = \exp\left\{ - \int_{r}^{1} \frac{dt}{t q_0^{-1}(t)} \right\}, \quad \tilde{q}_0(r) := \frac{1}{\omega_{n-1} r^{n-1}} \int_{|x|=r} \tilde{Q}(x) \, dS,$$

$$\tilde{Q}(x) = \begin{cases} Q(x), & |x| > \delta, \\
1/K, & |x| \leq \delta, \end{cases}$$

where $K \geq 1$ would be chosen bellow. Note that a mapping $f$ is a ring $\tilde{Q}$–homeomorphism at $x_0 = 0$. In fact, we have $f(S(0, r)) = S(0, R)$, where $R := \exp\left\{ - \int_{r}^{1} \frac{dt}{t q_0^{-1}(t)} \right\}$. Now

$$f(\Gamma(S(0, r_1), S(0, r_2), A(0, r_1, r_2))) = \Gamma(S(0, R_1), S(0, R_2), A(0, R_1, R_2)),$$

where $R_i := \exp\left\{ - \int_{r_i}^{1} \frac{dt}{t q_0^{-1}(t)} \right\}$, $i = 1, 2$. Now by [12], section 7.5,

$$M(f(\Gamma(S(0, r_1), S(0, r_2), A(0, r_1, r_2)))) = \frac{\omega_{n-1}}{\left( \int_{r_1}^{r_2} \frac{dt}{t q_0^{-1}(t)} \right)^{n-1}}.$$

Now $f$ is a ring $\tilde{Q}$–homeomorphism at the point $x_0 = 0$ by Proposition 3.1 and, consequently, is a ring $Q$–mapping at 0. Note that under $\delta \to 0$ the image $f(B(0, \delta))$ includes the ball $B(0, \sigma)$, where $\sigma$ does not depend on $\delta$. Now we map the ball $B(0, \sigma)$ by some map $g$, which is $K$–quasiregular and local homeomorphism for some $K \geq 1$, but not injective in $B(0, \sigma)$; for instance, let $g$ is a winding map, whose axes of rotation does not contain a
ball $B^n = f(B(0, 1))$, see [7, section 5.1]. Remark that $K$ does not depend on $\delta$. Now we construct a local ring $K \cdot Q(x)$–homeomorphism $f_2$ at zero, $f_2 = g \circ f$, which is not injective in $B(0, \delta)$. Since $Q$ is arbitrary locally integrable function with $Q \geq 1$ satisfying (1.4), we can replace $Q$ on the $Q/K$ in the start of the second part of the proof. So, we obtain a local ring $Q(x)$–homeomorphism with the properties meaning above. The proof is complete. □

4 Corollaries

The following statement is a simple consequence from the first part of the Theorem 1.1.

**Corollary 4.1.** Let $f : B^n \to \mathbb{R}^n, n \geq 3$, be a local ring $Q$–mapping at $x_0 = 0$ such that

$$q_0(r) \leq C \cdot \log^{n-1} \frac{1}{r}$$

(4.1)

for some $C > 0$ and $r \to 0$. Then $f$ is injective in some ball $B(0, \delta(n, Q))$ where $\delta$ depends only on $n$ and $Q$.

**Proof.** The desired conclusion follows from the Theorem 1.1 in view of (4.1). In fact, it follows from the Fubini Theorem (see [9, Theorem 8.1, Ch. III]) that $Q \in L^1_{loc}(B^n)$, besides of that, if follows from (4.1) that (1.3) holds. □

Following [1], we say that a function $\varphi : D \to \mathbb{R}$ has finite mean oscillation at a point $x_0 \in D$ if

$$\limsup_{\varepsilon \to 0} \frac{1}{\Omega_{n} \cdot \varepsilon^n} \int_{B(x_0, \varepsilon)} |\varphi(x) - \tilde{\varphi}_\varepsilon| dm(x) < \infty$$

where

$$\tilde{\varphi}_\varepsilon = \frac{1}{\Omega_{n} \cdot \varepsilon^n} \int_{B(x_0, \varepsilon)} \varphi(x) dm(x)$$

is the average of the function $\varphi(x)$ over the ball $B(x_0, \varepsilon) = \{x \in \mathbb{R}^n : |x - x_0| < \varepsilon\}$.

We also say that a function $\varphi : D \to \mathbb{R}$ is of finite mean oscillation in the domain $D$, abbr. $\varphi \in FMO(D)$ or simply $\varphi \in FMO$, if $\varphi$ has finite mean oscillation at every point $x_0 \in D$. Note that $FMO$ is not $BMO_{loc}$, see examples in [6, p. 211]. It is well–known that $L^\infty(D) \subset BMO(D) \subset L^p_{loc}(D)$ for all $1 \leq p < \infty$, see e.g. [2], but $FMO(D) \nsubseteq L^p_{loc}(D)$ for any $p > 1$. The following statement can be found in [6, Lemma 6.1].

**Proposition 4.1.** Let $0 \in D \subset \mathbb{R}^n, n \geq 3, \varphi : D \to \mathbb{R}$ be a nonnegative function having a finite mean oscillation at $x_0 = 0$. Then there exists $\varepsilon_0 > 0$ with

$$\int_{B(0, \varepsilon_0)} \frac{\varphi(x) dm(x)}{|x| \log \frac{1}{|x|}} < \infty.$$

The following statement take a place.

**Theorem 4.1.** Let $g : B^n \to \mathbb{R}^n, n \geq 3$, be a local ring $Q$–mapping at $x_0 = 0$ such that $Q \in FMO(0)$. Then $g$ is injective in some ball $B(0, \delta(n, Q))$, where $\delta$ is positive number depending only on $n$ and $Q$. 
5 On normality of the families of local homeomorphisms

In what follows, we use in $\mathbb{R}^n = \mathbb{R}^n \cup \{\infty\}$ the spherical (chordal) metric $h(x, y) = |\pi(x) - \pi(y)|$ where $\pi$ is the stereographic projection of $\mathbb{R}^n$ onto the sphere $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ in $\mathbb{R}^{n+1}$, i.e.

$$h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2 \sqrt{1 + |y|^2}}}, \quad x \neq \infty \neq y,$$

$$h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}.$$

It is clear that $\mathbb{R}^n$ is homeomorphic to the unit sphere $S^n$ in $\mathbb{R}^{n+1}$.

The spherical (chordal) diameter of a set $E \subset \mathbb{R}^n$ is

$$h(E) = \sup_{x,y \in E} h(x, y).$$

Let $(X, d)$ and $(X', d')$ be metric spaces with distances $d$ and $d'$, respectively. A family $\mathfrak{f}$ of continuous mappings $f : X \rightarrow X'$ is said to be normal if every sequence of mappings $f_m \in \mathfrak{f}$ has a subsequence $f_{m_k}$ converging uniformly on each compact set $C \subset X$ to a continuous mapping. Normality is closely related to the following. A family $\mathfrak{f}$ of mappings $f : X \rightarrow X'$ is said to be equicontinuous at a point $x_0 \in X$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that $d'(f(x), f(x_0)) < \varepsilon$ for all $f \in \mathfrak{f}$ and $x \in X$ with $d(x, x_0) < \delta$. The family $\mathfrak{f}$ is equicontinuous if $\mathfrak{f}$ is equicontinuous at every point $x_0 \in X$. It is known that every normal family $\mathfrak{f}$ of mappings $f : X \rightarrow X'$ between metric spaces $(X, d)$ and $(X', d')$ is equicontinuous. The inverse conclusion is true whenever $(X, d)$ is separable and $(X', d')$ is compact metric space (see the version of the Arzela–Ascoli’s theorem mentioned above in [12, section 20.4]).

Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, $Q : D \rightarrow [0, \infty]$ be a Lebesgue measurable function. Denote by $\mathfrak{R}_{Q, \Delta}(x_0)$ the family of all ring $Q$–homeomorphisms $f : D \rightarrow \mathbb{R}^n$ at $x_0$ with $h(\mathbb{R}^n \setminus f(D)) \geq \Delta > 0$. Let $\mathfrak{R}_{Q, \Delta}(D)$ denotes a family of all homeomorphisms $f : D \rightarrow \mathbb{R}^n$ such that $f \in \mathfrak{R}_{Q, \Delta}(x_0)$ at every point $x_0 \in D$. Let us consider $\mathfrak{R}_{Q, \Delta}(x_0)$ as the family of the mapping between metric spaces $(X, d)$ and $(X', d')$, where $X = D$, $X' = \mathbb{R}^n$, $d(x, y) = |x - y|$ be Euclidean metric and $d'(x, y) = h(x, y)$ be chordal metric. The following statement take a place (see [10, Theorem 6.1, Theorem 6.5, Corollary 6.7]).

\textbf{Proposition 5.1.} The family $\mathfrak{R}_{Q, \Delta}(x_0)$ is equicontinuous at $x_0 \in D$ whenever at least one of the conditions holds: 1) The relation $\int_0^{\varepsilon(x_0)} e^{\varepsilon(x_0)\frac{dt}{Q\log(q^{(n-1)/q})}} = \infty$ take a place for some $\varepsilon(x_0) > 0$,
\[ \varepsilon(x_0) < \text{dist}(x_0, \partial D); \quad 2) \text{The relation } q_{x_0}(r) \leq C \cdot \log^{n-1} \frac{1}{r} \text{ holds as } r \to 0 \text{ and some } C > 0; \]

3) \( Q \in \text{FMO}(x_0) \). Besides of that, if at least one of the conditions 1)–3) holds for every \( x_0 \in D \), the family \( \mathfrak{R}_{Q, \Delta}(D) \) is equicontinuous (normal) in \( D \).

Denote by \( \mathfrak{F}_{Q, \Delta}(x_0) \) the family of all local ring \( Q \)–homeomorphisms \( f : D \to \mathbb{R}^n \) at \( x_0 \) with \( h(\mathbb{R}^n \setminus f(D)) \geq \Delta > 0 \). Let \( \mathfrak{F}_{Q, \Delta}(D) \) denotes a family of all local homeomorphisms \( f : D \to \mathbb{R}^n \) such that \( f \in \mathfrak{R}_{Q, \Delta}(x_0) \) at every point \( x_0 \in D \). The following statement take a place.

**Theorem 5.1.** The family \( \mathfrak{F}_{Q, \Delta}(x_0) \) is equicontinuous at \( x_0 \in D \) whenever at least one of the conditions holds: 1) The relation \( \int_0^\varepsilon(x_0) \frac{dt}{q_{x_0}(t)} = \infty \) take a place for some \( \varepsilon(x_0) > 0 \), \( \varepsilon(x_0) < \text{dist}(x_0, \partial D) \); 2) The relation \( q_{x_0}(r) \leq C \cdot \log^{n-1} \frac{1}{r} \) holds as \( r \to 0 \) and some \( C > 0 \); 3) \( Q \in \text{FMO}(x_0) \). Besides of that, if at least one of the conditions 1)–3) holds for every \( x_0 \in D \), the family \( \mathfrak{F}_{Q, \Delta}(D) \) is equicontinuous (normal) in \( D \).

**Proof.** It follows from the assumptions that every mapping \( f \in \mathfrak{F}_{Q, \Delta}(x_0) \) omits at least two values \( a_f \) and \( b_f \) in \( \mathbb{R}^n \). Let \( T_f \) be a Möbius transformation mapping the \( b_f \) to \( \infty \). Since the Möbius transformations preserve the moduli of curve’s families (see [12, Theorem 8.1]), the family of mappings \( \tilde{\mathfrak{F}}_{Q, \Delta}(x_0) = \{ \tilde{f} = T_f \circ f : f \in \mathfrak{F}_{Q, \Delta}(x_0) \} \) consists of the local ring \( Q \)–homeomorphisms \( f : D \to \mathbb{R}^n \) at \( x_0 \), which omit \( \tilde{a}_f \in \mathbb{R}^n \) and \( \infty \). Suppose that one of the cases 1), 2) or 3) take a place. Then every \( \tilde{f} \in \tilde{\mathfrak{F}}_{Q, \Delta}(x_0) \) is injective in some neighborhood of \( x_0 \) whose radius depends only on \( n \) and \( Q \) (see Theorem 4.1, Theorem 4.1 or Corollary 4.1 correspondingly). Now the family \( \tilde{\mathfrak{F}}_{Q, \Delta}(x_0) \) as well as the family \( \mathfrak{F}_{Q, \Delta}(x_0) \) is equicontinuous at \( x_0 \) by Proposition 5.1. The corresponding conclusion for the family \( \mathfrak{F}_{Q, \Delta}(D) \) follows from the proved above. \( \square \)

**Remark 5.1.** The results of the paper are not true for \( n = 2 \) that shows the example \( f_m(z) = e^{mz}, m \in \mathbb{N}, z \in \mathbb{B}^2 \).

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