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Twin-width can be exponential in treewidth

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Abstract

For any small positive real $\varepsilon$ and integer $t > 1$, we build a graph with a vertex deletion set of size $t$ to a tree, and twin-width greater than $2^{\left(1-\varepsilon\right)t}$. In particular, this shows that the twin-width is sometimes exponential in the treewidth, in the so-called oriented twin-width and grid number, and that adding an apex may multiply the twin-width by at least $2 - \varepsilon$. Except for the one in oriented twin-width, these lower bounds are essentially tight.

1 Introduction

Twin-width is a graph parameter introduced by Bonnet, Kim, Thomassé, and Watrigant [12]. It is defined by means of trigraphs. A trigraph is a graph with some edges colored black, and some colored red. A (vertex) contraction consists of merging two (non-necessarily adjacent) vertices, say, $u, v$ into a vertex $w$, and keeping every edge $wz$ black if and only if $uz$ and $vz$ were previously black edges. The other edges incident to $w$ become red (if not already), and the rest of the trigraph remains the same. A contraction sequence of an $n$-vertex graph $G$ is a sequence of trigraphs $G = G_n, \ldots, G_1 = K_1$ such that $G_i$ is obtained from $G_{i+1}$ by performing one contraction. A $d$-sequence is a contraction sequence in which every vertex of every trigraph has at most $d$ red edges incident to it. The twin-width of $G$, denoted by $\text{tww}(G)$, is then the minimum integer $d$ such that $G$ admits a $d$-sequence. Figure 1 gives an example of a graph with a 2-sequence, i.e., of twin-width at most 2. Twin-width can be naturally extended to matrices (unordered [12] or ordered [9]) over a finite alphabet, and hence to any binary structures. Classes of binary structures with bounded twin-width include graphs with bounded treewidth, bounded clique-width, $K_t$-minor free graphs, posets with antichains of bounded size, strict subclasses of permutation graphs, map graphs, bounded-degree string graphs [12], segment graphs with no $K_{t,t}$ subgraph, visibility graphs of 1.5D terrains without large half-graphs, visibility graphs of simple polygons without large independent sets [6], as well as $O(\log n)$-subdivisions of $n$-vertex graphs, classes with bounded queue number or bounded stack number, and some classes of cubic expanders [7].

Despite their apparent generality, classes of bounded twin-width are small [7], $\chi$-bounded [8], even quasi-polynomially $\chi$-bounded [17], preserved (albeit with a higher upper bound) by first-order transductions [12], and by the usual graph products when one graph has bounded degree [16, 7], have VC density 1 [11, 19], admit, when $O(1)$-sequences are given, a fixed-parameter tractable first-order model checking [12], an (almost) single-exponential

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{A 2-sequence witnessing that the initial graph has twin-width at most 2.}
\end{figure}
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parameterized algorithm for various problems that are W[1]-hard in general [8], as well as a parameterized fully-polynomial linear algorithm for counting triangles [15], an (almost) linear representation [18], a stronger regularity lemma [19], etc.

In all these applications, the upper bound on twin-width, although somewhat hidden in the previous paragraph, plays a role. There is then an incentive to obtain as low as possible upper bounds on particular classes of bounded twin-width. To give one concrete algorithmic example, an independent set of size $k$ can be found in time $O(k^2d^k n)$ in an $n$-vertex graph given with a $d$-sequence [8]. This is relatively practical for moderate values of $k$, with the guarantee that $d$ is below 10, but not when $d$ is merely upperbounded by $10^{10}$. Another motivating example: triangle-free graphs of twin-width at most $d$ are $d + 2$-colorable [8], a stronger fact in the former case than in the latter.

In that line of work, Buláň and Hlinený show that posets of width $k$ (i.e., with antichains of size at most $k$) have twin-width at most $9k$ [2]. Unit interval graphs have twin-width at most 2 [8], and proper $k$-mixed-thin graphs (a recently proposed generalization of unit interval graphs) have twin-width $O(k)$ [3]. Every graph obtained by subdividing at least $2 \log n$ (throughout the paper, all logs are in base 2) times each edge of an $n$-vertex graph has twin-width at most 4 [4]. Schidler and Szeider report the (exact) twin-width of a collection of graphs [20], obtained via SAT encodings. Jacob and Pilipczuk [14] give the current best upper bound of 183 on the twin-width of planar graphs, while graphs with genus $g$ have twin-width $O(g)$ [13]. Most relevant to our paper, for every graph $G$, $\text{tww}(G) \leq 3 \cdot 2^{\text{tw}(G) - 1}$ [14], where $\text{tw}(G)$ denotes the treewidth of $G$.

Conversely, one may ask the following.

Question 1. What is the largest twin-width a graph of treewidth $k$ can have?

A lower bound of $\Omega(k)$ comes from the existence of $n$-vertex graphs with twin-width $\Omega(n)$ (since the treewidth is trivially upperbounded by $n - 1$). This is almost surely the case of graphs drawn from $G(n, 1/2)$. Alternatively, the $n$-vertex Paley graph (for a prime $n$ such that $n \equiv 1 \pmod{4}$) has precisely twin-width $(n - 1)/2$ [1]. Another example to derive the linear lower bound is the power set graph [14]. Improving on this lower bound is not obvious, and $\Theta(k)$ is indeed the answer to Question 1 within the class of planar graphs [14], or when replacing ‘treewidth’ by ‘cliquewidth’ or ‘pathwidth.’

When switching ‘twin-width’ and ‘treewidth’ in Question 1, the gap is basically as large as possible: There are $n$-vertex graphs with treewidth $\Omega(n)$ and twin-width at most 6, in the iterated 2-lifts of $K_4$ [7, 5].

An important characterization of bounded twin-width is via the absence of complex divisions of an adjacency matrix. A matrix has a $k$-mixed minor if its row (resp. column) set can be partitioned into $k$ sets of consecutive rows (resp. columns), such that each of the $k^2$ cells defined by this $k$-division contains at least two distinct rows and at least two distinct columns. The mixed number of a matrix $M$ is the largest integer $k$ such that $M$ admits a $k$-mixed minor. The mixed number of a graph $G$, denoted by $\text{mnx}(G)$, is the minimum, taken among all the adjacency matrices $M$ of $G$, of the mixed number of $M$. The following was shown.

Theorem 1 (12). For every graph $G$, $(\text{mnx}(G) - 1)/2 \leq \text{tww}(G) \leq 2^{O(\text{mnx}(G))}$. In sparse graphs (here, excluding a fixed $K_{1,4}$ as a subgraph), the previous theorem is both simpler to formulate and has a better dependency. A matrix has a $k$-grid minor if it has a $k$-division with at least one 1-entry in each of its $k^2$ cells. The grid number of a matrix and of a graph $G$, denoted by $\text{gn}(G)$, are defined analogously to the previous paragraph. We

$\text{mnx}(G) \equiv$
only state the inequality that is useful to bound the twin-width of a sparse class, but is valid in general.

▶ **Theorem 2** (follows from [12]). For every graph \( G \), \( \text{tww}(G) \leq 2^{O(\text{sn}(G))} \).

Theorems 1 and 2 allow to bound the twin-width of a class \( C \) by exhibiting, for every \( G \in C \), an adjacency matrix of \( G \) without large mixed or grid minor. Therefore one merely has to order \( V(G) \) (the vertex set of \( G \)) in an appropriate way. The double (resp. simple) exponential dependency in mixed number (resp. grid number) implies relatively weak twin-width upper bounds. For several classes whose twin-width was originally upper bounded via Theorem 1, better bounds were later given by avoiding this theorem (see [7, 2, 14, 13, 4]). Still for some geometric graph classes, bypassing Theorem 1 seems complicated (see [6]). And in general (since this theorem is at the basis of several other applications, see for instance [7, 8, 9]) it would help to have an improved upper bound of \( \text{tww}(G) \); in particular a negative answer to the following question.

▶ **Question 2.** Is twin-width sometimes exponential in mixed and grid number?

A variant of twin-width, called oriented twin-width, adds an orientation to the red edges (see [10]). The red edge (arc) is oriented away from the contracted vertex. The oriented twin-width \( d \) of a graph \( G \), denoted by \( \text{otww}(G) \), is then defined similarly as twin-width by tolerating more than \( d \) red arcs incident to a vertex, as long as at most \( d \) of them are out-going. Rather surprisingly twin-width and oriented twin-width are tied.

▶ **Theorem 3** ([10]). For every graph \( G \), \( \text{otww}(G) \leq \text{tww}(G) \leq 2^{2^{O(\text{otww}(G))}} \).

Classic results show that planar graphs have oriented twin-width at most 9 [10]. Thus it would be appreciable to lower the dependency of \( \text{tww}(G) \) in \( \text{otww}(G) \).

▶ **Question 3.** Is twin-width sometimes exponential in oriented twin-width?

An elementary argument shows that when adding an apex (i.e., an additional vertex with an arbitrary neighborhood) to a graph \( G \), the twin-width of the obtained graph is at most \( 2 \cdot \text{tww}(G) + 1 \). Again it is not clear whether this increase could be made smaller.

▶ **Question 4.** Does twin-width sometimes essentially double when an apex is added?

Note that Question 1 is asked by Jacob and Pilipczuk [14], and Question 3 is posed by Bonnet et al. [10], and is closely related to Question 2.

**Our contribution.**

With a single construction, we answer all these questions. The answer to Questions 2, 3, and 4 is affirmative, while the answer to Question 1 is \( 2^{\Theta(k)} \), which confirms the intuition of the authors of [14]. More precisely, we show the following.

▶ **Theorem 4.** For every real \( 0 < \varepsilon \leq 1/2 \) and integer \( t > 1/\varepsilon \), there is a graph \( G_{t,\varepsilon} \) with a feedback vertex set of size \( t \) and such that \( \text{tww}(G_{t,\varepsilon}) > 2^{(1-\varepsilon)t} \).

The graph \( G_{t,\varepsilon} \) has in particular treewidth at most \( t + 1 \), grid number at most \( t + 2 \), and oriented twin-width at most \( t + 1 \). Thus

\[
\begin{align*}
\text{tww}(G_{t,\varepsilon}) & > 2^{(1-\varepsilon)(\text{tw}(G_{t,\varepsilon})-1)}, \\
\text{tww}(G_{t,\varepsilon}) & > 2^{(1-\varepsilon)(\text{gn}(G_{t,\varepsilon})-2)}, \text{ and} \\
\text{tww}(G_{t,\varepsilon}) & > 2^{(1-\varepsilon)(\text{otww}(G_{t,\varepsilon})-1)}. 
\end{align*}
\]
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Hence Theorem 4 has the following consequences.

Corollary 5. For every small $\varepsilon > 0$, there is a family $\mathcal{F}$ of graphs with unbounded twin-width such that for every $G \in \mathcal{F}$: $\text{tww}(G) > 2^{(1-\varepsilon)(\text{tw}(G)-1)}$.

Up to multiplicative factors, it matches the known upper bound [14, 12], and essentially settles Question 1. The following answers Question 2.

Corollary 6. For every small $\varepsilon > 0$, there is a family $\mathcal{F}$ of graphs with unbounded twin-width such that for every $G \in \mathcal{F}$: $\text{tww}(G) > 2^{(1-\varepsilon)(\text{gm}(G)-2)}$.

The following answers Question 3.

Corollary 7. For every small $\varepsilon > 0$, there is a family $\mathcal{F}$ of graphs with unbounded twin-width such that for every $G \in \mathcal{F}$: $\text{tww}(G) > 2^{(1-\varepsilon)(\text{otww}(G)-1)}$.

The following answers Question 4.

Corollary 8. For every small $\varepsilon > 0$, there is a family $\mathcal{F}$ of graphs with unbounded twin-width such that for every $G \in \mathcal{F}$: $\text{tww}(G) > (2-\varepsilon)\text{tww}(G - \{v\})$, where $v$ is a single vertex of $G$.

We leave as an open question if the twin-width upper bound in oriented twin-width and mixed number can be made single-exponential.

2 Preliminaries

For $i$ and $j$ two integers, we denote by $[i,j]$ the set of integers that are at least $i$ and at most $j$. For every integer $i$, $[i]$ is a shorthand for $[1,i]$. We use the standard graph-theoretic notations: $V(G)$ denotes the vertex set of a graph $G$, $E(G)$ denotes its edge set, $G[S]$ denotes the subgraph of $G$ induced by $S$, etc.

We give an alternative approach to contraction sequences. The twin-width of a graph, introduced in [12], can be defined in the following way (complementary to the one given in introduction). A partition sequence of an $n$-vertex graph $G$, is a sequence $\mathcal{P}_n,...,\mathcal{P}_1$ of partitions of its vertex set $V(G)$, such that $\mathcal{P}_n$ is the set of singletons $\{\{v\} : v \in V(G)\}$, $\mathcal{P}_1$ is the singleton set $\{V(G)\}$, and for every $2 \leq i \leq n$, $\mathcal{P}_{i-1}$ is obtained from $\mathcal{P}_i$ by merging two of its parts into one. Two parts $P,P'$ of a same partition $\mathcal{P}$ of $V(G)$ are said homogeneous if either every pair of vertices $u \in P, v \in P'$ are non-adjacent, or every pair of vertices $u \in P, v \in P'$ are adjacent. Two non-homogeneous parts are also said red-adjacent. The red degree of a part $P \in \mathcal{P}$ is the number of other parts of $\mathcal{P}$ which are red-adjacent to $P$. Finally the twin-width of $G$, denoted by $\text{tww}(G)$, is the least integer $d$ such that there is a partition sequence $\mathcal{P}_n,...,\mathcal{P}_1$ of $G$ with every part of every $\mathcal{P}_i$ ($1 \leq i \leq n$) having red degree at most $d$.

The definition of the previous paragraph is equivalent to the one given in introduction, via contraction sequences. Indeed the trigraph $G_i$ is obtained from partition $\mathcal{P}_i$, by having one vertex per part of $\mathcal{P}_i$, a black edge between any fully adjacent pair of parts, and a red edge between red-adjacent parts. A partial contraction sequence is a sequence of trigraphs $G_n,...,G_i$, for some $i \in [n]$. A (full) contraction sequence is one such that $i = 1$. We naturally consider the trigraph $G_j$ to come after (resp. before) $G_j'$ if $j < j'$ (resp. $j > j'$). Thus when we write the first trigraph of the sequence $\mathcal{S}$ to satisfy $X$ (or the first time a trigraph of $\mathcal{S}$ satisfies $X$) we mean the trigraph $G_j$ with largest index $j$ among those satisfying $X$. The same goes for partition sequences.
If \( u \) is a vertex of a trigraph \( H \), then \( u(G) \) denotes the set of vertices of \( G \) eventually contracted into \( u \) in \( H \). We denote by \( P_G(H) \) (and \( P(H) \) when \( G \) is clear from the context) the partition \( \{u(G) : u \in V(H)\} \) of \( V(G) \). We may refer to a part of \( H \) as any set in \( \{u(G) : u \in V(H)\} \). We may also refer to a part of a contraction/partition sequence as any part of one its trigraphs/partitions. A contraction involves a vertex \( v \) if it produces a new part (of size at least 2) containing \( v \). In general, we use trigraphs and partitioned graphs somewhat interchangeably, when one notion appears more convenient than the other.

## 3 Proof of Theorem 4

We fix once and for all, \( 0 < \varepsilon \leq 1/2 \), a possibly arbitrarily small positive real. We build for every integer \( t > 1/\varepsilon \), a graph \( G_{t,\varepsilon} \), that we shorten to \( G_t \). We set

\[
f(t) = \left[ 2 + C_t 2^{1-\varepsilon} t (2 + C_t(2^{1-\varepsilon} t^3 + 1)) \right]
\]

where \( C_t = 2^{1-\varepsilon} t / \varepsilon \).

**Construction of \( G_t \).** Let \( T \) be the full \( 2^t \)-ary tree of depth \( f(t) \), i.e., with root-to-leaf paths on \( f(t) \) edges. Let \( X \) be a set of \( t \) vertices, that we may identify to \([t]\). The vertex set of \( G_t \), is \( X \cup V(T) \). The edges of \( G_t \) are such that \( G[X] \) is an independent set, and \( G[V(T)] = T \). The edges between \( V(T) \) are \( X \) such that

- the root of \( T \) has no neighbor in \( X \), and
- the \( 2^t \) children (in \( T \)) of every internal node of \( T \) each have a distinct neighborhood in \( X \).

Note that this defines a single graph up to isomorphism. By a slight abuse of language, we may utilize the usual vocabulary on trees directly on \( G_t \). By root, internal node, child, parent, leaf of \( G_t \), we mean the equivalent in \( T \).

We start with this straightforward observation.

**Lemma 9.** \( G_t \) has treewidth at most \( t + 1 \).

**Proof.** The set \( X \) is a feedback vertex set of \( G_t \) of size \( t \), thus \( \text{tw}(G_t) \leq \text{fvs}(G_t) + 1 \leq t + 1 \). ▶

The following is the core lemma, which occupies us for the remainder of the section.

**Lemma 10.** \( G_t \) has twin-width greater than \( 2^{(1-\varepsilon)t} \).

**Proof.** We assume, by way of contradiction, that \( G_t \) admits a \( d \)-sequence with \( d \leq 2^{(1-\varepsilon)t} \). We consider the partial \( d \)-sequence \( S \), starting at \( G_t \), and ending right before the first contraction involving a child of the root. We first show that no vertex of \( X \) can be involved in a contraction of \( S \). Note that it implies, in particular, that the root cannot be involved in a contraction of \( S \).

**Claim 11.** No part of \( S \) contains more than one vertex of \( X \).

**Proof of the Claim:** Observe that, for every \( i \neq j \in [t] \), there are \( 2^{t-1} \) sets of \( 2^t \) containing exactly one of \( i, j \): \( 2^{t-2} \) only contain \( i \), and \( 2^{t-2} \) only contain \( j \). Recall now that by assumption, in every trigraph of \( S \), every child of the root is alone in its part. Thus a part \( P \) of \( S \) such that \( |P \cap X| \geq 2 \) would have red degree at least \( 2^{t-1} > 2^{(1-\varepsilon)t} \geq d \). ◊

**Claim 12.** No part of \( S \) intersects both \( X \) and \( V(T) \).
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Proof of the Claim: For the sake of contradiction, consider the first occurrence of a part \( P \supseteq \{x, v\} \) with \( x \in X \) and \( v \in V(T) \). Vertex \( x \) is adjacent to half of the children of the root, whereas \( v \) is adjacent to at most one of them, or all of them (if \( v \) is itself the root). In both cases, this entails at least \( 2^{t-1} - 1 \) red edges for \( P \) towards children of the root. If \( v \) is not a grandchild of the root, the red degree of \( P \) is at least \( 2^{t-1} \). We thus assume that \( v \) is a grandchild of the root.

As \( t \geq 2 \), there is a \( y \in X \setminus \{x\} \). Let \( v' \) be the child of \( v \) whose neighborhood in \( X \) is exactly \( \{y\} \). This vertex exists since \( f(t) \geq 3 \). If \( P \) contains \( v' \), \( P \) is also red-adjacent to \( \{y\} \) (indeed a part, by Claim 11). If instead, \( P \) does not contain \( v' \), then \( P \) is also red-adjacent to the part containing \( v' \).

Thus, in any case, the red degree of \( P \) is at least \( 2^{t-1} > 2^{1-\epsilon}t \geq d \). ∎

From Claims 11 and 12, we immediately obtain:

▷ Claim 13. Every part of \( S \) intersecting \( X \) is a singleton.

Crucial to the proof, we introduce two properties \( \mathcal{P} \), and later \( \mathcal{Q} \), on internal nodes \( v \in V(T) \) in trigraphs \( H \in S \). Property \( \mathcal{P} \) is defined by

\[
\mathcal{P}(v, H) = \text{"At least } 2^t \text{ children of } v \text{ are in the same part of } \mathcal{P}(H)."
\]

We first remark that any internal node in a non-singleton part verifies \( \mathcal{P} \).

▷ Claim 14. Let \( H \) be any trigraph of \( S \) and \( v \) be any internal node of \( T \) whose part in \( \mathcal{P}(H) \) is not a singleton. Then \( \mathcal{P}(v, H) \) holds.

Proof of the Claim: Let \( P \) be the part of \( v \) (i.e., the one containing \( v \)) in \( \mathcal{P}(H) \), and \( u \in P \setminus \{v\} \). At least \( 2^t - 1 \) children of \( v \) are not adjacent to \( u \). Thus these \( 2^t - 1 \) vertices have to be in at most \( d + 1 \leq 2^{1-\epsilon}t + 1 \) parts. These parts are part \( P \), plus at most \( d \) parts linked to \( P \) by a red edge. Since \( (2^t - 1)(2^{1-\epsilon}t + 1) < 2^t - 1 \) (recall that \( \epsilon < 1/2 \)), one of these parts (possibly \( P \)) contains at least \( 2^t \) children of \( v \). ∎

As the merge of a singleton part \( \{v\} \) with any other part does not change the intersections of parts with the set of children of \( v \), we get a slightly stronger claim.

▷ Claim 15. Let \( v \) be an internal node of \( T \), and \( H \) be the last trigraph of \( S \) for which \( v \) is in a singleton part of \( \mathcal{P}(H) \). Then \( \mathcal{P}(v, H) \) holds.

A preleaf is an internal node of \( T \) adjacent to a leaf, i.e., the parent of some leaves. We obtain the following as a direct consequence of Claim 14.

▷ Claim 16. In any trigraph \( H \in S \), any non-preleaf internal node \( v \in V(T) \) that verifies \( \mathcal{P}(v, H) \) has at least \( 2^t \) children \( u \) verifying \( \mathcal{P}(u, H) \).

We define the property \( \mathcal{Q} \) on internal nodes \( v \) of \( T \) and trigraphs \( H \in S \) by induction:

\[
\mathcal{Q}(v, H) = \begin{cases} 
\mathcal{P}(v, H) & \text{if } v \text{ is a preleaf, and otherwise} \\
\mathcal{Q}(u_1, H) \land \mathcal{Q}(u_2, H) & \text{for some pair } u_1 \neq u_2 \text{ of children of } v.
\end{cases}
\]

That is, \( \mathcal{Q} \) is defined as \( \mathcal{P} \) for preleaves, and otherwise, \( \mathcal{Q} \) holds when it holds for at least two of its children. Observe that \( \mathcal{P} \) and \( \mathcal{Q} \) are monotone in the following sense: If \( \mathcal{P}(v, H) \) (resp. \( \mathcal{Q}(v, H) \)) holds, then \( \mathcal{P}(v, H') \) (resp. \( \mathcal{Q}(v, H') \)) holds for every subsequent trigraph \( H' \) of the partial \( d \)-sequence \( S \). We may write that \( v \) satisfies \( \mathcal{P} \) (resp. \( \mathcal{Q} \)) in \( H \) when \( \mathcal{P}(v, H) \) (resp. \( \mathcal{Q}(v, H) \)) holds, and may add for the first time if no trigraph \( H' \in S \) before \( H \) is such that \( \mathcal{P}(v, H') \) (resp. \( \mathcal{Q}(v, H') \)) holds.
Claim 17. For any trigraph $H \in \mathcal{S}$ and internal node $v$ of $T$, $\mathcal{P}(v, H)$ implies $\mathcal{Q}(v, H)$.

Proof of the Claim: This is a tautology if $v$ is a preleaf. The induction step is ensured by Claim 16, since $2^{2d} \geq 2$.

At the end of the partial $d$-sequence $\mathcal{S}$, we know, by Claim 15, that at least one child of the root satisfies $\mathcal{P}$, hence satisfies $\mathcal{Q}$, by Claim 17. Thus the first time in the partial $d$-sequence $\mathcal{S}$ that $\mathcal{Q}(v, H)$ holds, for a trigraph $H \in \mathcal{S}$ and a child $v$ of the root, is well-defined. We call $F$ this trigraph, and $v_0$ a child of the root such that $\mathcal{Q}(v_0, F)$ holds.

We now find many nodes satisfying $\mathcal{Q}$ in $F$, whose parents form a vertical path of singleton parts.

Claim 18. There is a set $Q \subset V(T)$ of at least $f(t) - 2$ internal nodes such that

- for every $v \in Q$, $\mathcal{Q}(v, F)$ holds,
- the parent of any $v \in Q$ is in a singleton part of $\mathcal{P}(F)$, and
- and no two distinct nodes of $Q$ are in an ancestor-descendant relationship.

Proof of the Claim: We construct by recurrence two sequences $(v_i)_{i \in [f(t) - 2]}$, $(q_i)_{i \in [0, f(t) - 3]}$ of internal nodes of $T$ such that for all $i \in [f(t) - 2]$, $v_i$ is a child of $v_{i-1}$, $v_{i-1}$ is in a singleton part of $\mathcal{P}(F)$, and $v_{i-1}$ has a child $q_{i-1} \neq v_i$ for which $\mathcal{Q}(q_{i-1}, F)$ holds.

Assume that the sequence is defined up to $v_i$, for some $i < f(t) - 2$. We will maintain the additional invariant that $v_i$ satisfies $\mathcal{Q}$ for the first time in $F$. This is the case for $i = 0$.

As $v_i$ is not a preleaf, it satisfies $\mathcal{Q}$ for the first time when a second child of $v_i$ satisfies $\mathcal{Q}$. Let $v_{i+1}$ be this second child, and $q_i$ be the first child to satisfy $\mathcal{Q}$ (breaking ties arbitrarily if both children satisfy $\mathcal{Q}$ for the first time in $F$). The vertex $v_{i+1}$ satisfies $\mathcal{Q}$ for the first time in $F$. Thus our invariant is preserved.

For every $i \in [f(t) - 2]$, $v_i$ is in a singleton part of $\mathcal{P}(F)$. Indeed, by Claim 15, if $v_i$ was not in a singleton part of $\mathcal{P}(F)$, $v_i$ would satisfy $\mathcal{P}$, hence $\mathcal{Q}$, in the trigraph preceding $F$; a contradiction.

The set $Q$ can thus be defined as $\{q_i : i \in [0, f(t) - 3]\}$. We already checked that the first two requirements of the lemma are fulfilled. No pair in $Q$ is in an ancestor-descendant relationship since the nodes of $Q$ are all children of a root-to-leaf path made by the $v_i$s (see Figure 2).
Let $B$ the vertices $w \in V(F)$ such that $w(G)$ contains at least $2^t$ children of the same node of $T$. Each vertex of $B$ is red-adjacent to at least $\log(2^t) = \epsilon t$ (singleton) parts of $X$. Therefore, since the red degree of (singleton) parts of $X$ is at most $2(1-\epsilon)t$:

$$|B| \leq \frac{2^{(1-\epsilon)t}}{\epsilon}.$$ 

Next we show that there is relatively large set of vertices of $F$ each corresponding to a non-singleton part that contains an internal node of $T$. 

\textbf{Claim 19.} There is a set $B' \subseteq V(F)$ of size at least

$$\frac{1}{(1-\epsilon)t} \log \left( \frac{f(t)-2}{|B|} \right) - 1$$

such that for every $b \in B'$ there is an internal node $v$ of $T$ with $v \in b(G_t)$ and $|b(G_t)| \geq 2$.

\textbf{Proof of the Claim:} Let $s := \frac{1}{(1-\epsilon)t} \log \left( \frac{f(t)-2}{|B|} \right) - 1$. Our goal is to construct a sequence $(b_i)_{i \in [0, s]}$ of distinct vertices of $F$ such that for every $i \in [s]$,

part $b_i(G_t)$ is not a singleton and contains an internal node of $T$. \hfill (1)

We first focus on finding $b_0$. Note that $b_0$ need not satisfy Invariant (1), but will be chosen to force the existence of $b_1$ itself satisfying (1) and starting the induction.

Let $Q := \{q_j : 0 \leq j \leq f(t) - 3 \} \subseteq V(T)$ be as described in Claim 18. Every $q_j \in Q$ has (at least) one descendant $q_j'$ that is a preleaf and satisfies $Q$, hence $\mathcal{P}$, in $F$. The $q_j'$s are pairwise distinct because no two nodes of $Q$ are in an ancestor-descendant relationship. We set $Q' := \{q_j' : 0 \leq j \leq f(t) - 3 \}$.

Now for every $q_j'$, at least $2^t$ of its children are in the same part of $\mathcal{P}(F)$; hence, this part corresponds to a vertex in $B$. By the pigeonhole principle, there is a $b_0 \in B$ that contains at least $2^t$ children of $q_j$ at least $(f(t) - 2)/|B|$ nodes of $Q'$.

For each $q_i$, we define $Q_i \subseteq Q$ as the set of vertices $q_j$ such that $b_i(G_t)$ contains a (not necessarily strict) descendant $z$ of $q_j$, and $b_i(G_t)$ with $i' < i$ contains a node on the path between $q_j$ and $z$ in $T$. Thus $|Q_0| \geq (f(t) - 2)/|B|$.

We now assume that $b_i \in V(F)$, for some $0 \leq i < s$, has been found with

$$|Q_i| \geq \frac{f(t) - 2}{|B| : 2^{(1-\epsilon)t}}.$$ \hfill (2)

Observe that $Q_0$ satisfies (2). We construct $b_{i+1}, Q_{i+1}$ satisfying the invariants (1) and (2).

For each $q_j \in Q_i$, consider the highest descendant $z_j$ of $q_j$ in $b_i(G_t)$, and $z_j'$ the parent of $z_j$ in $T$. By construction, the part $P_j$ of $\mathcal{P}(F)$ containing $z_j'$ is not a $b_k(G_t)$ for any $k \leq i$. Part $P_j$ is linked to $b_i(G_t)$ by a red edge. Therefore there are at most $2(1-\epsilon)t$ such parts $P_j$.

In particular, there is a $b_{i+1} \in V(F)$ such that $b_{i+1}(G_t)$ contains at least

$$\frac{|Q_i|}{d} \geq \frac{f(t) - 2}{|B| : 2^{(1-\epsilon)t}} \cdot \frac{1}{2^{(1-\epsilon)t}} = \frac{f(t) - 2}{|B| : 2^{(i+1)(1-\epsilon)t}}$$

parents $z_j'$ of highest descendants $z_j$.

Remark that $b_{i+1}(G_t)$ has size at least two while $(f(t) - 2)/(|B| : 2^{(i+1)(1-\epsilon)t}) > 1$, which holds since $i < s$. Thus $b_{i+1}(G_t)$ does not contain any parent $v_j$ of a $q_j$ (since the $v_j$s are in singleton parts). In particular, $|Q_{i+1}| \geq (f(t) - 2)/(|B| : 2^{(i+1)(1-\epsilon)t})$, and $b_{i+1}, Q_{i+1}$ satisfy (1) and (2).
Finally, the set $B' := \{b_i : 1 \leq i \leq s\}$ has the required properties. \hfill \lozenge

We can now finish the proof of the lemma.

For every $b_i \in B'$, let $u_i \in b_i(G_t)$ be an internal node of $T$. As $b_i(G_t) \geq 2$, $u_i$ satisfies $\mathcal{P}$ in $F$. This implies that $b_i$ or a red neighbor of $b_i$ is in $B$. Therefore, the total number of red edges incident to a vertex of $B$ is at least $|B'| - |B|$. Thus there is a vertex in $B$ with red degree at least $(|B'|-|B|)/|B|$. This is a contradiction since

$$\frac{|B'| - |B|}{|B|} = \frac{|B'|}{|B|} - 1 \geq \left( \frac{1}{(1-\varepsilon)t} \log \left( \frac{f(t)}{|B|} \right) - 1 \right) \cdot \frac{1}{|B|} - 1$$

$$\geq \left( \frac{1}{(1-\varepsilon)t} \log \left( 2(1-\varepsilon)t(2+C_t(2^{(1-\varepsilon)t}+1)) \right) - 1 \right) \cdot \frac{1}{|B|} - 1$$

$$= \left( (2 + C_t \cdot (2^{(1-\varepsilon)t}+1)) - 1 \right) \cdot \frac{1}{|B|} - 1 > 2^{(1-\varepsilon)t} + 1 - 1 = 2^{(1-\varepsilon)t} \geq d.$$

since, we recall, $f(t) = \left[ 2 + C_t \cdot 2^{(1-\varepsilon)t(2+C_t(2^{(1-\varepsilon)t}+1))} \right]$ and $C_t = \frac{2^{(1-\varepsilon)t}}\varepsilon \geq |B|$.

Since $X$ is a feedback vertex set of size $t$ of $G_t$, Lemma 10 implies Theorem 4, and hence Corollary 5.

As the twin-width of $T$ is 2, adding the $t$ apexes in $X$, multiplies the twin-width by at least $2^{(1-\varepsilon-\frac{1}{t})}$. Thus one apex in $X$ multiplies the twin-width by at least $2^{1-\varepsilon-\frac{1}{t}}$, which can be made arbitrarily close to 2. This establishes Corollary 8.

### 4 Oriented twin-width and grid number

In this section, we check that $G_t$ has oriented twin-width at most $t+1$, and grid number at most $t+2$.

A (partial) oriented contraction sequence is defined similarly as a (partial) contraction sequence with every red edge replaced by a red arc leaving the newly contracted vertex. Then a (partial) oriented $d$-sequence is such that all the vertices of all its ditrigraphs have at most $d$ out-going red arcs. The oriented twin-width of a graph $G$, denoted by $\text{otww}(G)$, is the minimum integer $d$ such that $G$ admits an oriented $d$-sequence.

\textbf{Lemma 20.} The oriented twin-width of $G_t$ is at most $t+1$.

\textbf{Proof.} We observe that the 2-sequence for trees [12] is an oriented 1-sequence. We contract $T$ to a single vertex (without touching $X$) in that manner. This yields a partial oriented $t+1$-sequence for $G_t$, ending on a $t+1$-vertex ditriphgraph, which can be contracted in any way. This contraction sequence witnesses that $\text{otww}(G_t) \leq t+1$. \hfill \blacktriangleleft

Thus Corollary 7 holds.

We finish by establishing Corollary 6.

\textbf{Lemma 21.} The grid number of $G_t$ is at most $t+2$.

\textbf{Proof.} Recall that $V(G_t) = X \cup V(T)$. Let $\prec$ be the total order on $V(G_t)$ that puts first all the vertices of $X$ in any order, then from left to right, all the leaves of $T$, followed by the preleaves, the nodes at depth $f(t) - 2$, the nodes at depth $f(t) - 3$, and so on, up to the root. We denote by $M$ the adjacency matrix of $G_t$ ordered by $\prec$.\hfill \blacktriangleleft
Let $M_T$ be the submatrix of $M$ obtained by deleting the $t$ rows and $t$ columns corresponding to $X$. Note that the grid number of $M$ is at most $\text{gn}(M_T) + t$. We claim that there is no 3-grid minor in $M_T$.

Indeed, in the order $\prec$, above the diagonal of $M_T$ there is no pair of 1-entries in strictly decreasing positions. Thus overall there is no triple of 1-entries in strictly decreasing positions. Thus no 3-grid minor is possible in $M_T$.

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