DERIVING LAWS FOR DEVELOPING CONCURRENT PROGRAMS IN A RELY-GUARANTEE STYLE

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ABSTRACT. This paper presents a theory for the refinement of shared-memory concurrent algorithms from specifications. We augment pre and post condition specifications with Jones’ rely and guarantee conditions, all of which are encoded as commands within a wide-spectrum language. Program components are specified using either partial or total correctness versions of postcondition specifications. Operations on shared data structures and atomic machine operations (e.g. compare-and-swap) are specified using an atomic specification command. All the above constructs are defined in terms of a simple core language, based on a small set of primitive commands and a handful of operators. A comprehensive set of laws for refining such specifications to code is derived in the theory. The approach supports fine-grained concurrency, avoiding atomicity assumptions on expression evaluation and assignment commands. The theory has been formalised in Isabelle/HOL, and the refinement laws and supporting lemmas have been proven in Isabelle/HOL.

Key words and phrases: shared-memory concurrency; rely/guarantee concurrency; concurrent refinement calculus; concurrency; formal semantics; refinement calculus; rely/guarantee program verification.

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1. Introduction

Our overall goal is to develop a theory for deriving verified shared-memory concurrent programs from abstract specifications. A set of threads running in parallel can exhibit a high degree of non-determinism due to the myriad possible interleavings of their fine-grained accesses to shared variables. The set of all threads running in parallel with a thread is referred to as its *environment* and the term *interference* refers to the changes made to the shared variables of a thread by its environment.

The rely/guarantee approach. Reasoning operationally about threads that execute under interference is fraught with the dangers of missing possible interleavings. A systematic approach to concurrency is required to manage interference. The approach taken here is based on the rely/guarantee technique of Jones [Jon81, Jon83a, Jon83b], which provides a compositional approach to handling concurrency.

To illustrate the rely/guarantee approach, we give a Jones-style specification [Jon81, Jon83a] of an operation to remove an element \(i\) from a set (1.1). The interesting aspect of the example is that in removing \(i\) from the set, interference from the environment may also remove elements from the set, possibly including \(i\). The set can be represented as a bit-map stored in an array of words. Removing an element from the set then corresponds to removing an element from one of the words. Here we focus on the interesting part from the point of view of handling interference, of removing the element \(i\) from a word \(w\), where accesses to \(w\) are atomic. Words are assumed to contain \(N\) bits and hence the maximum number of elements in a set represented by a single word is \(N\). The variable \(i\) is local and hence not subject to interference. The rely condition is an assumption that the environment may neither add elements to \(w\) nor change \(i\) (i.e. the rely condition is, \(w \supseteq w' \land i' = i\), where \(w\) refers to the initial value of \(w\) and \(w'\) to its final value and likewise for \(i\)). The remove operation guarantees that each program step never adds elements to \(w\), never removes elements other than \(i\), and does not change \(i\). That rules out an (unlikely) implementation that adds additional elements to the set and then removes them as well as \(i\). Because \(w\) only decreases and \(i\) is not modified, the precondition, \(w \subseteq \{0 \ldots N - 1\} \land i \in \{0 \ldots N - 1\}\), is an invariant. The postcondition requires that \(i\) is not in \(w\) in the final state, (i.e. \(i' \notin w'\)).

\[
\begin{align*}
\text{pre} & \quad w \subseteq \{0 \ldots N - 1\} \land i \in \{0 \ldots N - 1\} \\
\text{rely} & \quad w \supseteq w' \land i' = i \\
\text{guar} & \quad w \supseteq w' \land w - w' \subseteq \{i\} \land i' = i \\
\text{post} & \quad i' \notin w' \\
\end{align*}
\]

(1.1)

Note how the requirement to remove \(i\) and only \(i\) from \(w\) is split between the post condition and the guarantee. Compare that with the postcondition of, \(w' = w - \{i\}\), of a (sequential) operation to remove \(i\) in the context of no interference. The sequential postcondition is not appropriate in the context of concurrent interference that may remove elements from \(w\) because that interference may falsify the sequential postcondition, while the postcondition \(i' \notin w'\) is stable under the rely condition.

The semantic model represents the behaviour of a thread as a set of Aczel traces [Acz83, dR01] of the form given in Figure 1. Aczel traces distinguish atomic steps (or transitions) made by a thread itself, called *program or \(\pi\) steps* here, from atomic steps made by its environment, called *environment or \(\epsilon\) steps* here. In the rely/guarantee approach, the interference on a thread \(c\) is assumed to satisfy a rely condition \(r\), where \(r\) is a reflexive, transitive binary relation between program states that all (atomic) environment steps of
Figure 1: An execution trace of a thread consisting of a sequence of states $\sigma_1$--$\sigma_7$ with either program ($\pi$) or environment ($\epsilon$) transitions between successive states. If the execution trace is from a thread satisfying a rely/guarantee specification, then if the initial state $\sigma_0$ satisfies the precondition of the specification, $p$, and all environment transitions satisfy the rely relation $r$, then all program transitions must satisfy the guarantee relation $g$, and the postcondition relation $q$ must be satisfied between the initial ($\sigma_0$) and final ($\sigma_7$) states.

c are assumed to satisfy. Because $c$ itself is part of the environment of the other threads, $c$ is required to satisfy a guarantee $g$, which is a reflexive relation between states that all program steps of $c$ must satisfy. The guarantee condition of a thread must imply the rely conditions of all threads in its environment.

**Concurrent refinement calculus.** The sequential refinement calculus [Mor94, BvW98] makes use of a wide-spectrum language, which extends an executable imperative programming language with specification constructs that encode preconditions and postconditions as the commands, $\{p\}$ and $[q]$, respectively. A postcondition specification command, $[q]$, where $q$ is a binary relation on programs states, represents a commitment that the program will terminate and satisfy $q$ between its initial and final states overall. An assertion command, $\{p\}$, where $p$ is a set of states, represents an assumption that the initial state is in $p$; it allows any behaviour whatsoever for initial states not in $p$, and hence from initial states not satisfying $p$, there is no obligation for the program to satisfy its postcondition or terminate. If the initial state is not in $p$, we say the assertion command $\{p\}$ aborts, i.e. it behaves as Dijkstra’s abort command [Dij75, Dij76], denoted by $\frac{\top}{\bot}$ here.

We extend this approach by encoding Jones’ rely condition $r$ as the command, $\text{rely } r$, and his guarantee condition $g$ as the command, $\text{guar } g$, where $r$ and $g$ are binary relations on program states. A guarantee command, $\text{guar } g$, represents a commitment that every (atomic) program step satisfies the relation $g$ between its before and after program states. A rely command, $\text{rely } r$, represents an assumption that all (atomic) environment steps satisfy $r$. If its environment performs a step not satisfying $r$, the command, $\text{rely } r$, aborts and hence any behaviour whatsoever is allowed from that point on, in particular, there is no longer an obligation for the program to terminate or to satisfy its postcondition specification overall or satisfy its guarantee from that point on.

In order to combine these commands to form a rely/guarantee specification similar to (1.1), we make use of a weak conjunction operator ($\bowtie$) novel to our approach [Hay16, HCM+16]. A behaviour of a weak conjunction of two commands, $c \bowtie d$, must be both a behaviour of $c$ and a behaviour of $d$ up until the point that either $c$ or $d$ aborts, at which point $c \bowtie d$ aborts. If both $c$ and $d$ have no aborting behaviours, then every behaviour of $c \bowtie d$ must be a behaviour of both $c$ and $d$, that is, their strong conjunction $c \land d$. We
illustrate the difference between weak and strong conjunction with an example of combining two pre-post specifications,\(^1\) where sequential composition (\(;\)) has highest precedence.

\[
\begin{align*}
{p_1} ; \{q_1\} \cap \{p_2\} ; \{q_2\} &= \{p_1 \cap p_2\} ; \{q_1 \cap q_2\} \\
{p_1} ; \{q_1\} \land \{p_2\} ; \{q_2\} &= \{p_1 \cup p_2\} ; \{(p_1 \Rightarrow q_1) \cap (p_2 \Rightarrow q_2)\} \\
\end{align*}
\]  

(1.2) (1.3)

The weak conjunction (1.2) aborts if either component aborts, as represented by the precondition of \(p_1 \cap p_2\) on the right, and must satisfy both postconditions \(q_1\) and \(q_2\) otherwise. The strong conjunction (1.3) aborts if both can abort, as represented by the precondition of \(p_1 \cup p_2\) on the right, and from initial states that satisfy \(p_1\) it must satisfy postcondition \(q_1\) and from initial states that satisfy \(p_2\) it must satisfy postcondition \(q_2\).

**Characteristic predicates.** In Hoare logic, preconditions and postconditions are predicates that are interpreted with respect to a program state \(\sigma\) that gives the values of the program variables, e.g., \(\sigma x\) is the value of the program variable \(x\) in state \(\sigma\). The semantics of a predicate \(P\) characterising a set of states is given by \(\ll P \rr\), where we use \(\ll\) and \(\rr\) as lightweight semantic brackets and colour the predicate purple to distinguish it, for example,

\[
\ll x > 0 \rr = \{\sigma . \sigma x > 0\}.
\]

Similarly, the semantics of a predicate \(R\) characterising a binary relation between states is given by \(\ll R \rr\), where we use \(\ll\) and \(\rr\) as lightweight semantic brackets, and references to a variable \(x\) in \(R\) stand for its value in the before state, \(\sigma x\), and primed occurrences \(x'\) stand for the value of \(x\) in the after state, \(\sigma' x\), as in VDM [Jon80], Z [Hay93, WD96] and TLA\(^+\) [Lam03], for example,

\[
\ll x \geq x' \rr = \{(\sigma, \sigma') . \sigma x \geq \sigma' x\}.
\]

The theory developed in the body of this paper uses the semantic models of sets and relations directly so that preconditions use sets of states, and relies, guarantees and postconditions use binary relations on states, rather than their characteristic predicates. This approach has the advantage of making the theory independent of the particular concrete syntax used to express characteristic predicates. We hope you will excuse us not giving an explicit definition of the interpretation of the predicates used in the examples; the interpretation is straightforward and the particular notation used for predicates is not of concern for expressing the theory and laws presented in the body of the paper.

**Combining commands.** Weak conjunction (\(\cap\)) and sequential composition (\(;\)) can be used to combine commands into a specification, for example, the Jones-style specification (1.1) is represented by the following command.

\[
\begin{align*}
\text{rely} & \ll w \supseteq w' \land i' = i' \rr \\
\cap & \text{guar} \ll w \supseteq w' \land w - w' \subseteq \{i\} \land i' = i' \rr \\
\cap & \{\ll w \subseteq \{0 \ldots N - 1\} \land i \in \{0 \ldots N - 1\} \rr \} ; \ll i' \notin w' \rr
\end{align*}
\]  

(1.4)

The weak conjunction requires that both the guarantee and the postcondition are satisfied by an implementation unless either the precondition does not hold initially or the rely condition fails to hold for an environment step at some point, in which case the whole specification aborts from that point. The precondition and rely have no effect if the precondition holds initially and the rely condition holds for all environment steps and hence in this case the

---

\(^1\)Such operators has been investigated for sequential programs [War93, Gro02].

\(^2\)The syntax for set comprehension matches that of Isabelle/HOL.
behaviour must satisfy both guarantee for every program step and postcondition between the initial and final states overall.

The advantage of representing relies and guarantees as separate commands is that one can develop laws for each construct in isolation as well as in combination with other constructs. For example, Jones noted that strengthening a guarantee is a refinement. In our theory, strengthening a guarantee corresponds to the refinement of \( \text{guar} \ g_1 \) to \( \text{guar} \ g_2 \), if relation \( g_1 \) contains in \( g_2 \), i.e. \( g_1 \supseteq g_2 \). Note that this law is expressed just in terms of the guarantee command, unlike the equivalent law using four-tuples of pre/rely/guar/post conditions, which must refer to extraneous (unchanged) pre/rely/post conditions and needs to be proven in term of their semantics.\(^3\)

The core theory consists of a lattice of commands with a small set of primitive commands and operators. Other commands, including \( \{p\}, \{q\}, \text{guar} \ g, \text{rely} \ r \) and programming language constructs, are defined in terms of these primitives. The theory is built up in stages: each stage introduces a new concept or command in the wide-spectrum language along with a supporting theory of lemmas and laws. Significant contributions of this paper are the following.

- A comprehensive theory for handling postcondition specification commands in the context of interference (Sections 16 and 17). Two forms of specification command are provided: one for partial correctness, \( [q] \), and the other for total correctness, \( [q] \).
- An atomic specification command, \( \langle p, q \rangle \), suitable for defining atomic machine operations, such as compare-and-swap, and for specifying atomic operations on concurrent data structures (Sect. 21).
- A theory of expressions that makes minimal atomicity assumptions making it suitable for use in developing fine-grained concurrent algorithms (Sect. 22). A practical constraint on expressions is that only a single variable in the expression is subject to interference and that variable is referenced just once in the expression (Sect. 22.3). For example, for an integer variable \( x \), \( x + x \) may evaluate to an odd value under interference that modifies \( x \) between references, but \( 2 \times x \) has a single reference to \( x \) and hence always evaluates to an even value.
- Due to the more general treatment of expression evaluation under interference, we have been able to develop more general laws than those in the existing literature [CJ07, Col08, Din02, Jon81, Jon83a, Jon83b, Pre03, SZLY21, STE+14, Sto91, XdRH97] for introducing assignments (Sect. 23), conditionals (Sect. 24) and loops (Sect. 26).
- Recursively defined commands are supported (Sect. 25) and are used to define the while loop (Sect. 26).
- The algebraic theories have been formalised within Isabelle/HOL [NPW02], with the language primitives being defined axiomatically and other constructs defined in terms of the primitives. The laws and lemmas presented in this paper have been proven in terms of these Isabelle/HOL algebraic theories.
- The sets of traces semantic model has also been formalised in Isabelle/HOL and shown to satisfy the axioms of the algebraic theories, thus establishing the consistency of the theories. We make no claims for completeness.

\(^3\)The single law given by Jones allows preconditions and rely conditions to be weakened and guarantees and postconditions to be strengthened. We prefer to treat these as four separate laws because commonly only one of these conditions is modified.
Sect. 2 introduces our language in terms of a small set of primitive commands and a small set of operators. Following sections cover the lattice of commands (Sect. 3), sequential composition (Sect. 4), fixed points and iteration (Sect. 5), tests (Sect. 6), assertions (Sect. 7), atomic steps commands (Sect. 8), synchronisation operators, parallel and weak conjunction (Sect. 9), guarantees (Sect. 11), frames (Sect. 12), relies (Sect. 13), termination (Sect. 14), partial and total correctness (Sect. 15), specification commands (Sect. 16), stability under interference (Sect. 17), parallel (Sect. 18), optional atomic steps (Sect. 19), finite stuttering (Sect. 20), atomic specifications (Sect. 21), expressions (Sect. 22), assignments (Sect. 23), conditionals (if) (Sect. 24), recursion (Sect. 25), and while loops (Sect. 26). Sect. 27 provides an example refinement of specification (1.4) to code using the laws derived in this paper. Parts of this refinement are also used as running examples throughout the paper. Sect. 28 discusses the formalisation of the theory in Isabelle/HOL.

**Related work.** Rather than presenting the related work in one section, paragraphs labeled Related work have been included throughout the paper. This is so that the comparison of the approach used in this paper with related work can refer to the relevant details of how the individual constructs are handled in the different approaches.

There are two levels at which this paper can be read: by skipping the proofs, the reader gets an overview of the refinement calculus and its laws, while delving into the proofs gives a greater insight into how the underlying theory supports reasoning about concurrent programs and the reasons for the provisos and form of the refinement laws, some of which are quite subtle due the effects of interference.

## 2. Core language

We have previously developed a concurrency theory [Hay16, FHV16, HCM+16, HMWC19] that is used to define commands in a wide-spectrum language, develop refinement laws, and and prove them correct.

### 2.1. Semantic model

In this section we briefly describe the semantic model for our theory, which is based on that in [CHM16]. A command $c$ in our theory is modelled as a prefix-closed set of Aczel traces [Acz83, dR01]—denoted $[c]$—where each trace is of the form given in Figure 1, in which a trace is a sequence of program states (each giving values of the program variables) with transitions between states differentiated as either program steps ($\sigma \xrightarrow{\pi} \sigma'$) or environment steps ($\sigma \xrightarrow{\epsilon} \sigma'$). To allow for non-terminating computations, traces may be infinite. Three types of traces are distinguished: terminated (✓), aborting (†), and either incomplete or infinite (⊥). For $tr$ a trace, the following notation is used,

- $tr^S$ : gives the non-empty sequence of states of $tr$,
- $tr^K$ : gives the sequence of kinds of transitions ($\pi$ or $\epsilon$) of $tr$ —its length is one less than $tr^S$,
- $tr^X$ : gives the type of the trace (✓, †, or ⊥), and
- $tr^T$ : gives the sequence of transitions of $tr$, i.e. $tr^T = \{ i \mapsto (tr^S_i \xrightarrow{tr^K_i} tr^S_{i+1}) \mid i \in \text{dom } tr^K \}$. 


A trace, $tr$, is uniquely characterised by $tr^S$, $tr^K$ and $tr^X$. For the trace in Figure 1,

$$tr^S = [\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7]$$

$$tr^K = [\epsilon, \epsilon, \pi, \epsilon, \pi, \pi, \epsilon]$$

$$tr^X = \checkmark,$$

$$tr^T = [\sigma_0 \xrightarrow{\ell} \sigma_1, \sigma_1 \xrightarrow{\ell} \sigma_2, \sigma_2 \xrightarrow{\pi} \sigma_3, \sigma_3 \xrightarrow{\ell} \sigma_4, \sigma_4 \xrightarrow{\pi} \sigma_5, \sigma_5 \xrightarrow{\pi} \sigma_6, \sigma_6 \xrightarrow{\ell} \sigma_7]$$

A prefix of a trace $tr$ is a trace $tp$ such that $tp^X = \bot$, sequence $tp^S$ is a prefix of $tr^S$, and sequence $tp^K$ is a prefix of $tr^K$, where prefixes are not required to be strict. An extension of a trace $tr$ is a trace $tx$ of any type such that $tr^S$ is a prefix of $tx^S$, and $tr^K$ is a prefix of $tx^K$. The set of traces $[c]$ of a command $c$ satisfies three healthiness conditions:

**prefix closure:** if $tr$ is a trace in $[c]$, for all prefixes of $tr$ (representing its incomplete behaviours because they have type $\bot$) are also in $[c]$;

**abort closure:** if $tr$ is an aborting trace of $[c]$, i.e. $tr^X = \dag$, all possible extensions of $tr$ are also in $[c]$; and

**magic closed:** $[c]$ contains all trivial incomplete traces consisting of an initial state $\sigma_0$ and no transitions; these are the traces of the command magic introduced below.

A set of traces is **closed** if it is prefix, abort and magic closed. The semantics of commands satisfies the following properties, in which $c$ and $d$ are commands.

- The lattice partial order $c \supseteq d$ represents that $c$ is refined (or implemented) by $d$. In the semantic model refinement corresponds to superset-inclusion, that is, $[c] \supseteq [d]$.
- The command magic is the least command in the lattice (i.e. every command is refined by magic). It is infeasible in every initial state and in the semantic model it is represented by the set of all incomplete traces that consist of just an initial state and no transitions.
- The command $\sharp$ (Dijkstra’s abort) is the greatest command in the lattice (i.e. every command is a refinement of $\sharp$). It allows any behaviour whatsoever; in the semantic model it is represented by the set of all possible valid traces.
- The lattice meet $c \wedge d$, with identity $\sharp$, represents a strong conjunction of $c$ and $d$, and $\bigwedge C$ represents the strong conjunction of a set of commands $C$. In the semantic model $[c \wedge d] = [c] \cap [d]$ and $[\bigwedge C] = \bigcap_{c \in C} [c]$ and hence $\bigwedge \emptyset = \sharp$.
- The lattice join $c \vee d$, with identity magic, represents a non-deterministic choice between $c$ and $d$, and $\bigvee C$ represents a non-deterministic choice over a set of commands $C$. In the semantic model $[c \vee d] = [c] \cup [d]$ and $[\bigvee C] = \bigcup_{c \in C} [c]$, for a non-empty set of commands $C$, and $\bigvee \emptyset = \text{magic}$.
- $c ; d$ represents sequential composition of commands. In the semantic model $[c ; d]$ consists of the following traces:
  1. if there is a terminating trace $tc \in [c]$, then the trace $tc$ concatenated with any trace of $[d]$ whose initial state matches the final state of $tc$ — in the concatenation, the final state of $tc$ and the initial state of $td$ are merged into a single state;
  2. all incomplete or non-terminating (infinite) traces in $[c]$; and
  3. if there is an aborting trace $tc \in [c]$, then that aborting trace $tc$ along with all possible extensions of $tc$, so as to preserve abort closure.
- $c \otimes d$ represents the weak conjunction of commands. In the semantic model $[c \otimes d]$ consists of traces that are either:
  - traces of both $c$ and $d$ (i.e. in $[c] \cap [d]$);
which is based on that of VDM [Jon90] and Z [Hay93, WD96]. Given a set of fixed point relations \( r \) and \( s \), then \( r \) and all possible extensions of \( r \) (so as to preserve abort closure); or

- if \( td \) is an aborting trace in \([d]\) and the incomplete trace corresponding to \( td \) is also a trace in \([c]\), then \( td \) and all possible extensions of \( td \).

- \( c \| d \) represents the parallel composition of \( c \) and \( d \). In the semantic model \([c \| d]\) is defined in terms of the \textit{match} relation on traces that matches a program step of one thread with an environment step of the other to give a program step of their composition, and matches environment steps of both to give an environment step of their composition. For any traces \( tr, tc \) and \( td \) the relation \textit{match}(\( tr, tc, td \)) holds if and only if,

- \( tr, tc \), and \( td \) are all the same length (but not necessarily the same type),

- if \( tr_i^T \) is the program transition \( \sigma \xrightarrow{\pi} \sigma' \) from \( \sigma \) to \( \sigma' \) of \( tr \) then either,

  * \( tc_i^T = (\sigma \xrightarrow{\pi} \sigma') \) and \( td_i^T = (\sigma \xrightarrow{\pi} \sigma') \), or

  * \( td_i^T = (\sigma \xrightarrow{\pi} \sigma') \) and \( tc_i^T = (\sigma \xrightarrow{\pi} \sigma') \), and

- if \( tr_i^T \) is the environment transition \( (\sigma \xrightarrow{\pi} \sigma') \) then \( tr_i^T = tc_i^T = td_i^T \).

Using relation \textit{match}, we have that \([c \| d]\) consists of traces, \( tr \), such that either,

- there exist traces \( tc \in [c] \) and \( td \in [d] \), for which \textit{match}(\( tr, tc, td \)) holds and \( tr^X = tc^X = td^X \) and the traces are either terminating or incomplete, i.e. \( tr^X \in \{ \top, \bot \} \), or

- there exists an aborting trace \( tc \in [c] \) and a trace \( td \in [d] \) and a trace \( tr' \) such that \textit{match}(\( tr', tc, td \)) and \( tr \) is either \( tr' \) or some extension of \( tr' \), or

- there exists an aborting trace \( td \in [d] \) and a trace \( tc \in [c] \) and a trace \( tr' \) such that \textit{match}(\( tr', tc, td \)) and \( tr \) is either \( tr' \) or some extension of \( tr' \).

The above operators preserve closure on the sets of traces.

**Syntactic precedence of operators.** Unary operators and function application have higher precedence than binary operators. Amongst the binary operators, framing (:) has the highest precedence, followed by sequential composition (\( ; \)). Non-deterministic choice (\( \lor \)) has the lowest precedence. Otherwise no assumptions about precedence are made and parentheses are used to resolve syntactic ambiguity.

### 2.2. Function abstraction, application and fixed points

We use the usual notation for lambda abstraction and function application. Least and greatest fixed points of a function \( f \) are denoted by \( \mu f \) and \( \nu f \), respectively. Following convention, we abbreviate \( \mu (\lambda x . c) \) by \( (\mu x . c) \) and \( \nu (\lambda x . c) \) by \( (\nu x . c) \).

### 2.3. Relational notation

We briefly describe the notation used for relations in this paper which is based on that of VDM [Jon90] and Z [Hay93, WD96]. Given a set \( s \) and binary relations \( r, r_1 \) and \( r_2 \), \( \text{dom} \ r \) is the domain of the relation \( r (2.1) \), \( s \prec r \) is \( r \) restricted so that its domain is contained in the set \( s (2.2) \), \( r \circ s \) is \( r \) restricted so that its range is contained in \( s (2.3) \), \( r(\{s\}) \) is the image of \( s \) through \( r (2.4) \), \( r_1 \circ r_2 \) is the relational composition of \( r_1 \) and \( r_2 (2.5) \), and \( r^* \) is the reflexive transitive closure of \( r (2.6) \), which is defined as a least fixed point (\( \mu \)) on the lattice of relations. The universal relation over a state space \( \Sigma \) is represented by \( \text{univ} (2.7) \), and \( \tau \) is the set complement of \( r \) with respect to \( \text{univ} (2.8) \). The
identity relation is represented by \( \text{id} \) (2.9).

\[
\begin{align*}
\text{dom } r & \triangleq \{ \sigma . (\exists \sigma'. (\sigma, \sigma') \in r) \} \\
s \subset r & \triangleq \{ (\sigma, \sigma') . \sigma \in s \land (\sigma, \sigma') \in r \} \\
r \triangleright s & \triangleq \{ (\sigma, \sigma') . (\sigma, \sigma') \in r \land \sigma' \in s \} \\
r(\{s\}) & \triangleq \{ \sigma' . (\exists \sigma. (\sigma, \sigma') \in r) \} \\
\mathcal{r}_1 \# \mathcal{r}_2 & \triangleq \{ (\sigma, \sigma'') . (\exists \sigma'' . (\sigma, \sigma'') \in \mathcal{r}_1 \land (\sigma'', \sigma') \in \mathcal{r}_2) \} \\
r^* & \triangleq \mu x. \text{id} \cup r \circ x \\
\text{univ} & \triangleq \Sigma \times \Sigma \\
\tau & \triangleq \{ (\sigma, \sigma') . (\sigma, \sigma') \in \text{univ} \land (\sigma, \sigma') \notin r \} \\
\text{id} & \triangleq \{ (\sigma, \sigma') . \sigma = \sigma' \}
\end{align*}
\]

When dealing with states over a set of variables, if \( X \) is a set of variables, \( \text{id}_X \) is the identity relation on just the variables in \( X \) (2.10). For \( \text{id}_X \), \( \sigma \) and \( \sigma' \) are mappings from variables to their values, noting that mappings are a special case of binary relations, and hence one can apply the domain restriction operator.

\[
\text{id}_X \triangleq \{ (\sigma, \sigma') . \text{dom } \sigma' = \text{dom } \sigma \land X \triangleleft \sigma = X \triangleleft \sigma' \}
\]

2.4. **Primitive commands.** Let \( \Sigma \) be the (non-empty) program state space, where a state \( \sigma \in \Sigma \) gives the values of the program’s variables. Given that \( r \) is a binary relation on states (i.e. \( r \subseteq \Sigma \times \Sigma \)) and \( p \) is a set of states (i.e. \( p \subseteq \Sigma \)), the primitive commands are defined as follows.

- \( \pi r \) : represents an **atomic program step** command that can perform the transition \( \sigma \xrightarrow{r} \sigma' \) and terminate, if the two states are related by \( r \) (i.e. \( (\sigma, \sigma') \in r \)).

- \( \epsilon r \) : represents an **atomic environment step** command that can perform the transition \( \sigma \xrightarrow{r} \sigma' \) and terminate, if the two states are related by \( r \).

- \( \tau p \) : represents an **instantaneous test** command that succeeds and terminates immediately if its initial state is in \( p \), otherwise it is infeasible.

For example, \( \pi \text{id} \), where \( \text{id} \) is the identity relation on states, represents a command that performs a (stuttering) program transition \( (\sigma \xrightarrow{r} \sigma') \) that does not change the state and terminates; it differs from the command \( \tau \Sigma \), which terminates immediately without performing any program or environment transitions.

If \( \pi r \) (or \( \epsilon r \)) can make no transition for some state \( \sigma \) (i.e. \( \sigma \) is not in the domain of the relation \( r \)) it is infeasible from that state; in the semantic model the only trace of \( \pi r \) with such an initial state \( \sigma \) is the incomplete trace with no transitions. Similarly, for an initial state \( \sigma \) not in \( p \), the only trace of \( \tau p \) with initial state \( \sigma \) has no transitions, representing failure of the test. That gives the following special cases: \( \pi \emptyset = \epsilon \emptyset = \tau \emptyset = \text{magic} \), where we use \( \emptyset \) for both the empty set of states and the empty relation on states.

We define the atomic step command, \( \alpha r \), that can perform either a program step or an environment step, provided the step satisfies \( r \) (2.11). Given that \( \text{univ} = \Sigma \times \Sigma \) is the universal relation on states, the command \( \pi \) (note the bold font) can perform any program step (2.12), \( \epsilon \) can perform any environment step (2.13), \( \alpha \) can perform any program or environment step (2.14), and \( \tau \) always succeeds and terminates immediately from any state (2.15).
2.5. Axiomatisation and composite commands. Axioms of the core language are summarised in Figure 2, and explained and explored in the coming sections. Throughout the paper we introduce composite commands in terms of the primitives. For convenience these are summarised in Figure 3.

3. Lattice of commands

As a foundation of the axiomatisation in Figure 2, commands form a complete distributive lattice\(^4\) that is ordered by refinement, \(c \succeq d\), representing that \(c\) is refined (or implemented) by \(d\). Nondeterministic choice (\(\lor\)), or “choice” for short, is the lattice join (least upper bound) and strong conjunction (\(\land\)) is the lattice meet (greatest lower bound). The everywhere infeasible command \texttt{magic} is the least element of the lattice, and the immediately aborting command \texttt{halt} is the greatest element. The following lemma allows refinement of a nondeterministic choice over a set \(C\) by a choice over \(D\), provided every element of \(D\) refines some element of \(C\). Important special cases are if either \(C\) or \(D\) is a singleton set.

**Lemma 3.1** (refine-choice). [BvW98] For sets of commands \(C\) and \(D\),

\[
\begin{align*}
\lor C & \succeq \lor D & \text{if } \forall d \in D . \exists c \in C . c \succeq d \quad (3.1) \\
\lor C & \succeq d & \text{if } \exists c \in C . c \succeq d \quad (3.2) \\
c & \succeq \lor D & \text{if } \forall d \in D . c \succeq d \quad (3.3)
\end{align*}
\]

4. Sequential composition

Sequential composition is associative (2.16) and has identity the null command \(\tau\) (2.17) that terminates immediately. Sequential composition distributes over nondeterministic choice from the right (2.18) and over a non-empty nondeterministic choice from the left (2.19); \(D\) is required to be non-empty because \(\lor \emptyset = \text{magic}\) but \(\texttt{halt} : \text{magic} = \texttt{halt} \neq \text{magic}\). The binary versions (4.1) and (4.2) are derived from the fact that \(c \lor d \triangleq \lor \{c, d\}\).

\[
\begin{align*}
(c_0 \lor c_1) ; d & = c_0 ; d \lor c_1 ; d \quad (4.1) \\
c ; (d_0 \lor d_1) & = c ; d_0 \lor c ; d_1 \quad (4.2)
\end{align*}
\]

\(^4\) In the refinement calculus literature and our earlier papers [HCM+16, HMWC19] refinement is written \(c \sqsubseteq d\) but in the program algebra literature (e.g. [HMSW11]) the reverse ordering \(c \succeq d\) is used. In this paper we use the latter order (\(\succeq\)) and hence \texttt{magic} is the least element (rather than the greatest), \texttt{halt} as the greatest element (rather than the least), \(\lor\) (rather than \(\land\)) is nondeterministic choice, least (rather than greatest) fixed points give finite iteration, and greatest (rather than least) fixed points give possibly infinite iteration. While the choice of ordering is arbitrary, we feel our choice makes working with the algebra simpler because, for example, finite iteration is now treated in the same way (as a least fixed point) for both binary relations and commands, the form of the operators on commands corresponds to those for sets and relations (e.g. \(\lor\) maps to \(\lor\), and \(\land\) maps to \(\land\), rather than to the inverted forms). It also better matches the trace semantics [CHM16] as \(\succeq\) maps to \(\supseteq\), whereas in the previous work \(\sqsubseteq\) mapped to \(\supseteq\).
Atomic step commands: the functions \( \pi \), \( \tau \), Tests: the function \( c \); \( \tau \) to program, respectively, environment step commands.

Sequential composition (\( ; \))

\[
c_1 : (c_2 : c_3) = (c_1 : c_2) : c_3 \quad (2.16) \quad (\lor C) : d = \lor_{c \in C}(c : d) \quad (2.18)
\]

\[
c : \tau = c = \tau : c \quad (2.17) \quad c : (\lor D) = \lor_{d \in D}(c : d) \quad \text{if } D \neq \emptyset \quad (2.19)
\]

Tests: the function \( \tau \) forms an isomorphism from the boolean algebra of sets of states to test commands (\( T \subseteq C \)).

\[
\lor_{p \in P}(\tau p) = \tau(\bigcup P) \quad (2.20) \quad \tau \overline{p} = \tau \overline{p} \quad (2.22)
\]

\[
\land_{p \in P}(\tau p) = \tau(\bigcap P) \quad \text{if } P \neq \emptyset \quad (2.21) \quad \tau p_1 \land \tau p_2 = \tau(p_1 \cap p_2) \quad (2.23)
\]

Atomic step commands: the functions \( \pi \) and \( c \) form isomorphisms from the boolean algebra of binary relations to program, respectively, environment step commands.

\[
\lor_{r \in R}(\pi r) = \pi(\bigcup R) \quad (2.24) \quad \lor_{r \in R}(\epsilon r) = \epsilon(\bigcup R) \quad (2.29)
\]

\[
\land_{r \in R}(\pi r) = \pi(\bigcap R) \quad \text{if } R \neq \emptyset \quad (2.25) \quad \land_{r \in R}(\epsilon r) = \epsilon(\bigcap R) \quad \text{if } R \neq \emptyset \quad (2.30)
\]

\[
\pi(p \triangleright p) : \pi p = \pi(p \triangleright p) \quad (2.26) \quad \epsilon(p \triangleleft p) : \epsilon p = \epsilon(p \triangleleft p) \quad (2.31)
\]

\[
\pi r_1 \lor \epsilon r_2 = \pi r_1 \lor \epsilon r_2 \quad (2.27) \quad c \triangleright p = c \quad (2.53)
\]

Weak conjunction (\( \sqcap \))

\[
c \sqcap \text{ chaos} = c = \text{ chaos} \sqcap c \quad (2.33) \quad \pi r_1 \sqcap \pi r_2 = \pi(r_1 \cap r_2) \quad (2.36)
\]

\[
c \sqcap \emptyset = \emptyset \quad (2.34) \quad \epsilon r_1 \sqcap \epsilon r_2 = \emptyset \quad (2.37)
\]

\[
c \sqcap c = c \quad (2.35) \quad \pi r_1 \sqcap \epsilon r_2 = \text{ magic} \quad (2.38)
\]

Parallel composition (\( \parallel \))

\[
c \parallel \text{ skip} = c = \text{ skip} \parallel c \quad (2.39) \quad \pi r_1 \parallel \epsilon r_2 = \pi(r_1 \cap r_2) \quad (2.42)
\]

\[
c \parallel \emptyset = \emptyset \quad (2.40) \quad \epsilon r_1 \parallel \epsilon r_2 = \emptyset \quad (2.43)
\]

\[
(c_0 \parallel d_0) \sqcap (c_1 \parallel d_1) \supseteq (c_0 \sqcap c_1) \parallel (d_0 \parallel d_1) \quad (2.41) \quad \pi r_1 \parallel \pi r_2 = \text{ magic} \quad (2.44)
\]

Synchronisation: the following axioms hold for \( \sqcap \), \( \parallel \), or \( \land \).

\[
c_1 \sqcap (c_2 \sqcap c_3) = (c_1 \sqcap c_2) \sqcap c_3 \quad (2.45)
\]

\[
c_1 \sqcap c_2 = c_2 \sqcap c_1 \quad (2.46)
\]

\[
(\lor C) \sqcap d = \lor_{c \in C}(c \sqcap d) \quad \text{if } C \neq \emptyset \quad (2.47)
\]

\[
a_1 : c_1 \sqcap a_2 : c_2 = (a_1 \sqcap a_2) : (c_1 \sqcap c_2) \quad (2.48)
\]

\[
a_1 \sqcap a_2 \sqcap = (a_1 \sqcap a_2) \sqcap \quad (2.49)
\]

\[
a : c \sqcap \tau = \text{ magic} \quad (2.50)
\]

\[
t_1 \sqcap t_2 = t_1 \sqcap t_2 \quad (2.51)
\]

\[
t : c_1 \sqcap t : c_2 = t : (c_1 \sqcap c_2) \quad (2.52)
\]

\[
(c_0 : d_0) \sqcap (c_1 : d_1) \supset (c_0 \sqcap c_1) : (d_0 \parallel d_1) \quad (2.53)
\]

Figure 2: Axioms of the core language. The naming conventions followed in this paper are: \( c \) and \( d \) are commands; \( C \) and \( D \) are sets of commands; \( p \) is a set of states; \( P \) is a set of sets of states; \( r, g \) and \( q \) are binary relations on states; \( R \) is a set of relations; \( a \) is an atomic step command (i.e. a command of the form \( \pi r_1 \lor \epsilon r_2 \)); \( t \) is a test (i.e. a command of the form \( \tau p \)); and subscripted forms of the above names follow the same conventions.
\[ c^\star \triangleq (\mu x \cdot \tau \lor c ; x) \]  
\[ c^\omega \triangleq (\nu x \cdot \tau \lor c ; x) \]  
\[ c^\infty \triangleq (\nu x \cdot c ; x) \]  
\[ \{ p \} \triangleq \tau \lor \tau \mathcal{P} ; \mathcal{I} \]  
\[ \text{skip} \triangleq \epsilon \omega \]  
\[ \text{chaos} \triangleq \alpha \omega \]  
\[ \text{guar} g \triangleq (\pi g \lor \epsilon) \omega \]  
\[ X : c \triangleq \text{guar id} \cap X \]  
\[ \text{rely} r \triangleq (\alpha \lor \epsilon ; r) \omega \]  
\[ \text{term} \triangleq \alpha \star ; \epsilon \omega \]  
\[ q \triangleq W \sigma_0 \in \Sigma (\tau \{ \sigma_0 \} ; \text{chaos} ; \tau (q(\{\sigma_0\}))) \]  
\[ q \triangleq q \cap \text{term} \]  
\[ \text{opt} q \triangleq \pi q \lor \tau (\text{dom}(q \land \text{id})) \]  
\[ \text{idle} \triangleq \text{guar id} \cap \text{term} \]  
\[ \langle p, q \rangle \triangleq \text{idle} \cap \{ p \} \lor \text{opt} q \lor \text{idle} \]  
\[ \langle q \rangle \triangleq \langle \Sigma, q \rangle \]  
\[ \text{update} x k \triangleq \text{id} \triangleright (eq x k) \]  
\[ x \defeq e \lor \bigvee_{k \in \text{Val}} ([e]_k \lor \text{opt} (\text{update} x k) \lor \text{idle}) \]  
\[ \text{if} b \text{ then } c \text{ else } d \text{ fi} \triangleq (\mathbf{b} \text{true} ; c \lor \mathbf{b} \text{false} ; d) \lor \text{idle} \lor \bigvee_{k \in \Sigma} (\mathbf{b} \cap \mathcal{I}) \]  
\[ \text{while} b \text{ do } c \text{ od} \triangleq \nu x \cdot \text{if } b \text{ then } c \text{ else } \tau \text{ fi} \]

Figure 3: Commands defined in terms of primitives, where the command \([e]_k\) represents evaluating expression e to value k (see Sect. 22 for details).

5. Iteration

In the context of a complete lattice, we have that least (\(\mu\)) and greatest (\(\nu\)) fixed points of monotone functions are well-defined. Fixed points are used to define finite iteration zero or more times, \(c^\star \triangleq (\mu x \cdot \tau \lor c ; x)\) (2.54), possibly infinite iteration zero or more times, \(c^\omega \triangleq (\nu x \cdot \tau \lor c ; x)\) (2.55), and infinite iteration, \(c^\infty \triangleq (\nu x \cdot c ; x)\) (2.56). Iteration operators have their usual unfolding (5.1–5.3) and induction properties (5.5–5.7) [ABB+95] derived from their definitions as fixed points. Iteration satisfies the standard decomposition (5.4) and isolation (5.8) properties.

\[ c^\star = \tau \lor c ; c^\star \]  
\[ c^\star = \tau \lor c^\star ; c \]  
\[ c^\omega = \tau \lor c ; c^\omega \]  
\[ (c \lor d)^\omega = c^\omega ; (d ; c^\omega)^\omega \]  
\[ c^\omega ; d \lor c^\infty \]
Note that all the above properties of finite iteration are also properties of finite iteration of relations, \( r^* \), if \( \tau \) is replaced by the identity relation \( \text{id} \) (2.9), nondeterministic choice (\( \lor \)) is replaced by union of relations (\( \cup \)), sequential composition (\( ; \)) by relational composition (\( o \)) and \( \supseteq \) by \( \supseteq \). To avoid repeating the properties, we use the above properties of finite iteration for both commands and relations (with the above replacements made). Both form Kleene algebras [Kle56, Con71].

**Lemma 5.1 (absorb-finite-iter).** If \( c \supseteq d \), then \( c^* \cup d^* = c^* \).

**Proof.** The refinement from left to right holds because \( d^* \succ \tau \). For the refinement from right to left we have \( c^* = c^* \cup c^* \supseteq c^* \cup d^* \), using the assumption \( c \supseteq d \) in the last step. \( \square \)

### 6. Tests

We identify a subset of commands \( \mathcal{T} \) that represent instantaneous tests. \( \mathcal{T} \) forms a complete boolean algebra of commands (similar to Kozen’s Kleene algebra with tests [Koz97]). For a state space \( \Sigma \) representing the values of the program variables and \( p \) a subset of states (\( p \subseteq \Sigma \)), the isomorphism \( \tau \in \mathcal{P} \Sigma \rightarrow \mathcal{T} \) maps \( p \) to a distinct test \( \tau_p \), such that from initial state \( \sigma \), if \( \sigma \in p \), \( \tau_p \) terminates immediately (the null command) but if \( \sigma \notin p \), \( \tau_p \) is infeasible. Every test can be written in the form \( \tau_p \), for some \( p \subseteq \Sigma \).

The function \( \tau \) forms an isomorphism between \( \mathcal{P} \Sigma \) and \( \mathcal{T} \) that maps set union to nondeterministic choice (2.20); set intersection to the lattice meet (2.21); set complement to test negation (2.22); and sequential composition of tests reduces to a test on the intersection of their sets of states (2.23). From these axioms one can deduce the following properties.

\[
\begin{align*}
\tau p_1 \lor \tau p_2 &= \tau(p_1 \cup p_2) & (6.1) \\
\tau p_1 \land \tau p_2 &= \tau(p_1 \cap p_2) & (6.2) \\
\tau p_1 \succ \tau p_2 & \quad \text{if} \quad p_1 \supseteq p_2 & (6.3) \\
\tau \succ \tau p & \quad \text{if} \quad p \subseteq \Sigma & (6.4)
\end{align*}
\]

Note that by (6.3), \( \tau = \tau \Sigma \succ \tau p \succ \tau \emptyset = \text{magic} \). Because \( \tau \succ \tau p \) for any \( p \), it is a refinement to introduce a test (6.4).

**Example 6.1 (test-seq).** Using the notation from Sect. 1 to represent sets of states by characteristic predicates, \( \tau_{\land x \leq 0} \lor \tau_{\land x \geq 0} = \tau(\land x \leq 0 \lor \land x \geq 0) = \tau_{\land x = 0} \).

A choice over a set of states \( p \), of a test for a singleton set of states \( \{\sigma\} \), succeeds for any state \( \sigma \) in \( p \) and hence is equivalent to \( \tau p \).

**Lemma 6.2 (Nondet-test-set).** \( \bigvee_{\sigma \in p}(\tau\{\sigma\}) = \tau p \)

**Proof.** Using (2.20), \( \bigvee_{\sigma \in p}(\tau\{\sigma\}) = \tau(\bigcup_{\sigma \in p}\{\sigma\}) = \tau p \).

A test at the start of a non-deterministic choice restricts the range of the choice.

**Lemma 6.3 (test-restricts-Nondet).** \( \tau p \lor \bigvee_{\sigma \in \Sigma}(\tau\{\sigma\} ; c) = \bigvee_{\sigma \in p}(\tau\{\sigma\} ; c) \).
Proof.

\[
\begin{align*}
\tau p ; V_{\sigma \in \Sigma}(\tau \{\sigma\} ; c) \\
= & \quad \text{distribute test (2.19) as } \Sigma \text{ is non-empty and merge tests (2.23)} \\
= & \quad \text{split choice using Lemma 3.1 (refine-choice)} \\
= & \quad \text{as } \sigma \text{ is in } p \text{ and } \sigma_1 \text{ is not in } p \text{ and } \tau \emptyset = \text{magic} \\
= & \quad \text{as } V_{\sigma \in p}(\tau \{\sigma\} ; c) \lor V_{\sigma_1 \notin p}(\tau \{\sigma_1\} ; c) \\
= & \quad \text{as } V_{\sigma_1 \notin p}(\text{magic} ; c) \lor V_{\sigma_1 \notin p}(\text{magic} ; c) \\
\end{align*}
\]

\[\quad \text{Lemma 7.1 (assert-merge). If } \{p_1\} ; c \supseteq d \text{ and } \{p_2\} ; c \supseteq d \text{ then, } \{p_1 \cup p_2\} ; c \supseteq d. \]

Proof. Using the Galois connection between assertions and tests (7.4), the hypotheses are equivalent to \(c \supseteq \tau p_1 ; d\) and \(c \supseteq \tau p_2 ; d\) and hence by Lemma 3.1 (refine-choice) \(c \supseteq \tau p_1 ; d \lor \tau p_2 ; d = (\tau p_1 \lor \tau p_2) ; d = \tau(p_1 \lor p_2) ; d\), and hence by (7.4), \(\{p_1 \lor p_2\} ; c \supseteq d\). \n
7. Assertions

An assert command, \(\{p\} \triangleq \tau \lor \tau \overline{p} ; \xi\), aborts if \(p\) does not hold (i.e. for states \(\sigma \in \overline{p}\)) but otherwise terminates immediately (2.57). It allows any behaviour whatsoever if the state does not satisfy \(p\) \[vW04\]. It satisfies the following.

\[\{p\} = \tau p \lor \tau \overline{p} ; \xi \] (7.1)

Weakening an assertion is a refinement (7.2). Note that by (7.2), \(\xi = \emptyset \supseteq \{p\} \supseteq \Sigma = \tau\). Because \(\{p\} \supseteq \tau\) for any \(p\) by (7.3), it is a refinement to remove an assertion. Tests and assertions satisfy a Galois connection \[vW04\] (7.4). Sequential composition of assertions is intersection on their sets of states (7.5). A test dominates an assertion on the same set of states (7.6), and an assertion dominates a test on the same set of states (7.7).

\[
\begin{align*}
\{p_1\} & \supseteq \{p_2\} \quad \text{if } p_1 \subseteq p_2 \quad (7.2) \quad \{p_1\} ; \{p_2\} = \{p_1 \cap p_2\} \quad (7.5) \\
\{p\} & \supseteq \tau \quad (7.3) \quad \tau p ; \{p\} = \tau p \quad (7.6) \\
\{p\} ; c \supseteq d & \iff c \supseteq \tau p ; d \quad (7.4) \quad \{p\} ; \tau p = \{p\} \quad (7.7)
\end{align*}
\]

8. Atomic Step Commands

We identify a subset of commands, \(\mathcal{A}\), that represent atomic steps. \(^6\) \(\mathcal{A}\) forms a complete boolean algebra of commands. Both \(\pi\) and \(\epsilon\) are injective functions of type \(\mathcal{P}(\Sigma \times \Sigma) \to \mathcal{A}\), so that distinct relations map to distinct atomic step commands, and the commands generated by \(\pi\) and \(\epsilon\) are distinct except that \(\pi \emptyset = \epsilon \emptyset = \text{magic}\). Every atomic step command can

\(^5\)An alternative way to encode an assertion \(\{p\}\) in our theory is as \(\text{chaos} \lor \tau \overline{p} ; \xi\). This version is combined with other commands using weak conjunction rather than sequential composition.

\(^6\)Our atomic step commands are at a similar level of granularity to the transitions in an operational semantics, such as that given by Coleman and Jones [CJ07].
be represented in the form $\pi r_1 \lor \epsilon r_2$ for some relations $r_1$ and $r_2$. A choice over a set of program step commands is equivalent to a program step command over the union of the relations (2.24), and a strong conjunction over a set of relations corresponds to a program step command over the intersection of the relations (2.25). A test $\tau p$ preceding a program step command $\pi r$ is equivalent to a program step command with its relation restricted so its domain is included in $p$ (2.26). If a program step command with its relation restricted so that its range is in $p$ is followed by a test of $p$, that test always succeeds and hence is redundant (2.27). Note that in general $\pi r ; \tau p$ does not equal $\pi (r \triangleright p)$. Environment step commands satisfy similar axioms (2.29–2.32). Negating an atomic step command corresponds to negating the relations in its program and environment step components (2.28), for example, $\pi r = \pi r \lor \epsilon \emptyset = \pi \pi \lor \epsilon$, and $\pi = \epsilon$. If $r_1 \supseteq r_2$, both the following hold,

$$\pi r_1 \supseteq \pi r_2 \quad (8.1) \quad \epsilon r_1 \supseteq \epsilon r_2 \quad (8.2)$$

and hence by (8.1) for any relation $r$, $\pi = \pi \text{univ} \supseteq \pi r \supseteq \pi \emptyset = \text{magic}$, where $\emptyset$ is the empty relation, and similarly by (8.2), $\epsilon = \epsilon \text{univ} \supseteq \epsilon r \supseteq \epsilon \emptyset = \text{magic}$.

**Example 8.1** (test-pgm-test). By (2.26),

$$\tau \triangleleft 0 < x \uparrow; \pi ^{\tau} x \leq x' \uparrow = \pi (\tau \triangleleft 0 < x \uparrow; \pi ^{\tau} x \leq x' \uparrow) = \pi ^{\tau} 0 < x \land x \leq x'.\quad (8.3)$$

The following properties follow from (2.24) and (2.29), respectively.

$$\pi r_1 \lor \pi r_2 = \pi (r_1 \cup r_2) \quad (8.3) \quad \epsilon r_1 \lor \epsilon r_2 = \epsilon (r_1 \cup r_2) \quad (8.4)$$

Weak conjunction ($\triangleleft$) is a specification operator, such that $c \triangleleft d$ behaves as both $c$ and $d$ unless either $c$ or $d$ aborts in which case $c \triangleleft d$ aborts. The weak conjunction of two program step commands gives a program step over the intersection of their relations (2.36), and similarly for environment step commands (2.37). A weak conjunction of a program step command with an environment step command is infeasible (2.38).

Parallel composition combines a program step $\pi r_1$ of one thread with an environment step $\epsilon r_2$ of the other to form a program step of their composition that satisfies the intersection of the two relations (2.42). Parallel combines environment steps of both threads to give an environment step of the composition corresponding to the intersection of their relations (2.43). Because program steps of parallel threads are interleaved, parallel combination of two program steps is infeasible (2.44). Weak conjunction and parallel satisfy an interchange axiom (2.41).

**Example 8.2** (program-parallel-environment). By (2.42), a program step that does not increase $i$ in parallel with an environment step that does not decrease $i$ gives a program step that does not change $i$: $\pi ^{\tau} i \geq i' \triangleleft \epsilon ^{\tau} i \leq i'' = \pi ^{\tau} (i \geq i' \triangleleft i' \triangleleft i' \leq i'') = \pi ^{\tau} i' = i'.\quad (8.5)$

The atomic step command $\alpha$ is the atomic step identity of weak conjunction (8.5) and the atomic step command $\epsilon$ is the atomic step identity of parallel composition (8.6), in which $\alpha$ is any atomic step command (i.e. $\alpha = \pi \emptyset \lor \epsilon$ for some relations $\emptyset$ and $\tau$).

$$a \triangleleft \alpha = a \quad (8.5) \quad a \triangleleft \epsilon = a \quad (8.6)$$

9. **Synchronisation operators: parallel and weak/strong conjunction**

Parallel composition (\(||\)) and both weak ($\triangleleft$) and strong ($\land$) conjunction satisfy similar laws; the main differences being how they combine pairs of atomic steps, compare (2.42)–(2.44) and (2.36)–(2.38); whether or not they are abort-strict, like parallel (2.40) and weak conjunction
(2.34); and whether or not they are idempotent, like weak conjunction (2.35) and strong conjunction. To bring out the commonality between them we make use of an abstract synchronisation operator, $\otimes$, which is then instantiated to parallel ($\parallel$) and weak ($\bowtie$) and strong ($\land$) conjunction [HMWC19].

How the synchronisation operators combine atomic steps, and how they interact with aborting behaviours, influences the identity of each. Because the command $\epsilon$ (2.13) is the identity of parallel for a single atomic step (8.6) the command, $\text{skip} \equiv \epsilon^\omega$ (2.58), allows its environment to do any sequence of steps, including infinitely many, without itself introducing aborting behaviour, and hence it is defined to be the identity of parallel composition (2.39). Because the command $\text{chaos}$ is the identity of weak conjunction for a single atomic step (8.5), the command, $\text{chaos} \equiv \alpha^\omega$ (2.59), allows any number of any program or environment steps but cannot abort, and hence it is defined to be the identity of weak conjunction (2.33). For commands $c$ and $d$ that refine the identity of weak conjunction, weak conjunction simplifies to conjunction [HMWC19]:

$$\text{chaos} \equiv c \land \text{chaos} \equiv d \Rightarrow c \land d = c \bowtie d \quad \tag{9.1}$$

Strong conjunction has, from the lattice axiomatisation, identity $\dagger$.

A synchronisation operator, $\otimes$, is associative (2.45) and commutative (2.46). Non-empty non-deterministic choice distributes over a synchronisation operator (2.47). We exclude the empty choice because $\sqrt{\emptyset} = \text{magic}$ but for parallel (and weak conjunction but not strong conjunction), $(\sqrt{\emptyset}) \parallel \dagger = \dagger \neq \text{magic}$. Initial atomic steps of two commands synchronise before the remainders of the commands synchronise their behaviours (2.48). Infinite iterations of atomic steps synchronise each atomic step (2.49). A command that must perform an atomic step cannot synchronise with a command that terminates immediately (2.50). Two tests synchronise to give a test that succeeds if both tests succeed (2.51). A test distributes over synchronisation (2.52). Although synchronisation does not satisfy a distributive law with sequential, it does satisfy the weak interchange axiom (2.53). For (2.53), on the right the steps of $c_0$ synchronise with the steps of $c_1$ and they terminate together, and then the steps of $d_0$ synchronise with those of $d_1$. That behaviour is also allowed on the left but $(c_0 \parallel d_0)$ synchronising all its steps with $(c_1 \parallel d_1)$ also allows behaviours such as $c_0$ synchronising with the whole of $c_1$ and part of $d_1$, and $d_0$ synchronising with the rest of $d_1$, and vice versa (see [CHM16, Hay16] for more details). The following laws follow by the interchange axioms (2.53) and (2.41), respectively.

**Lemma 9.1** (sync-seq-distrib). [HMWC19, Law 6] If $c \geq c ; c$,

$$c \otimes (d_0 ; d_1) \geq (c \otimes d_0) ; (c \otimes d_1).$$

**Lemma 9.2** (conj-par-distrib). [Hay16, Law 12] If $c \geq c \parallel c$,

$$c \bowtie (d_0 \parallel d_1) \geq (c \bowtie d_0) \parallel (c \bowtie d_1).$$

The following laws are also derived from the axioms and properties of iterations. They hold with $\otimes$ replaced by any of $\parallel$, $\bowtie$ and $\land$. See [HMWC19] for proofs of these properties (and a range similar properties) in terms of a synchronous program algebra.

$$a^\omega ; c \otimes t = c \otimes t \quad \tag{9.2}$$

$$a_1^\omega \otimes a_2^\omega = (a_1 \otimes a_2)^\omega \quad \tag{9.3}$$

$$a_1^\omega ; c_1 \otimes a_2^\omega ; c_2 = \left( a_1 \otimes a_2 \right)^\omega ; (a_1^\omega : c_1 \otimes c_2) \lor (c_1 \otimes a_2^\omega ; c_2) \quad \tag{9.4}$$

$$a_1^\omega ; c_1 \otimes a_2^* ; c_2 = \left( a_1 \otimes a_2 \right)^* ; (a_1^\omega ; c_1 \otimes c_2) \lor (c_1 \otimes a_2^* ; c_2) \quad \tag{9.5}$$
10. Abort-strict synchronisation operators

Whether a synchronisation operator is abort strict or not influences its algebraic properties.

**Definition 10.1** (abort-strict). A binary operator $\otimes$ is *abort strict* if and only if for all commands $c, c \otimes \frac{t}{d} = \frac{t}{d}$.

Parallel composition (2.40) and weak conjunction (2.34) are abort strict but strong conjunction is not. The fact that parallel and weak conjunction are abort-strict influences how they distribute tests and assertions. For example, for strong conjunction, we trivially have that an initial test on one side of a conjunction can be treated as an initial test of the whole synchronisation, e.g. $c \land t ; d = t ; (c \land d)$. For either parallel or weak conjunction we have, taking $c$ to be $\frac{t}{d}$ and $t$ to be $\text{magic}$ as an example, that this property does not hold: $\frac{t}{d} \land (t ; \text{magic}) = \frac{t}{d} \neq t ; \frac{t}{d} = t ; (\frac{t}{d} \land \text{magic})$. For arbitrary synchronisation operators (including parallel and weak conjunction), in Lemma 10.2 (test-command-sync-command) we require that the side without the test does not abort immediately, i.e. it must either terminate immediately ($\tau$), or do a (non-aborting) step ($\alpha$) and then any behaviour is allowed, including abort. A command $c$ is not immediately aborting if $c \not\Join \text{magic} = \text{magic}$.

**Lemma 10.2** (test-command-sync-command). [HMWC19, Lemma 4] Given a test $t$, and commands $c$ and $d$, if $\neg t ; c \not\Join \text{magic} = \text{magic}$,\(^7\) then $c \otimes t ; d = t ; (c \otimes d)$.

An initial assertion on one side of an abort-strict synchronisation operator can be treated as an initial assertion of the whole synchronisation.

**Lemma 10.3** (assert-distrib). If $\otimes$ is abort strict, $c \otimes \{p\} ; d = \{p\} ; (c \otimes d)$.

*Proof.* $c \otimes \{p\} ; d$

- case analysis on test $t = \tau p$ using $c = (t \lor \top) ; c = t ; c \lor \top ; c$
- $\tau p ; (c \otimes \{p\} ; d) \lor \tau \neg p ; (c \otimes \{p\} ; d)$
- distributivity of test over synchronisation (2.52)
- $(\tau p ; c \otimes \{p\} ; d) \lor (\tau \neg p ; c \otimes \tau \neg p ; \{p\} ; d)$
- simplify $\tau p ; \{p\} = \tau p$ and $\tau \neg p ; \{p\} = \tau \neg p ; \frac{t}{d}$ by (7.1); redistribute test (2.52)
- $\tau p ; (c \otimes d) \lor \tau \neg p ; (c \otimes \frac{t}{d})$
- by Definition 10.1 (abort-strict) as $\otimes$ is abort strict
- $\tau p ; (c \otimes d) \lor \tau \neg p ; \frac{t}{d}$
- as $\frac{t}{d}$ is a left annihilator of sequential composition
- $\tau p ; (c \otimes d) \lor \tau \neg p ; \frac{t}{d} ; (c \otimes d)$
- distributivity of sequential composition (4.1) and assertion property (7.1)
- $\{p\} ; (c \otimes d)$ \qed

Supporting Lemmas 10.4 to 10.6 are used to prove Lemma 10.7 (test-suffix-interchange), which states that tests at the end of abort-strict synchronisations can be factored out.

**Lemma 10.4** (sync-test-assert). If $\otimes$ is abort strict, $\tau \otimes \{p\} = \{p\}$.

\(^7\)This condition is a slight generalisation of that used in [HMWC19, Lemma 4] but the proof there generalises straightforwardly with this more general proviso.
Proof. From (2.17), Lemma 10.3 (assert-distrib) as $\otimes$ is abort strict, and (2.51) we have that $\tau \otimes \{ p \} = \tau \otimes \{ p \} ; \tau = \{ p \} ; (\tau \otimes \tau) = \{ p \}$. □

Lemma 10.5 (test-suffix-assert). If $\otimes$ is abort strict, $c \otimes d ; \tau p = (c \otimes d ; \tau p) ; \{ p \}$.

Proof. Refinement from right to left follows as $\{ p \} \not\supseteq \tau$ by (7.3). The refinement from left to right follows because tests establish assertions (7.6), interchanging $\otimes$ with sequential composition (2.53) and Lemma 10.4 (sync-test-assert) because $\otimes$ is abort strict.

$c \otimes d ; \tau p = (c \otimes d ; \tau p) ; \{ p \} \supseteq (c \otimes d ; \tau p) ; (\tau \otimes \{ p \}) = (c \otimes d ; \tau p) ; \{ p \}$ □

Lemma 10.6 (test-suffix-test). If $\otimes$ is abort strict, $c \otimes d ; \tau p = (c \otimes d ; \tau p) ; \tau p$.

Proof. The proof uses Lemma 10.5 given that $\otimes$ is abort strict, then (7.7) and then Lemma 10.5 in the reverse direction:

$c \otimes d ; \tau p = (c \otimes d ; \tau p) ; \{ p \} = (c \otimes d ; \tau p) ; \{ p \} ; \tau p = (c \otimes d ; \tau p) ; \tau p$ □

Lemma 10.7 (test-suffix-interchange). If $\otimes$ is abort strict, $c \otimes d ; \tau p = (c \otimes d) ; \tau p$.

Proof. The refinement from left to right interchanges $\otimes$ and sequential composition (2.53) after adding a $\tau$, and uses (2.51) to show $\tau \otimes \tau p = \tau \Sigma \wedge \tau p = \tau (\Sigma \cap p) = \tau p$.

$c \otimes d ; \tau p = c ; \tau \otimes d ; \tau p \supseteq (c \otimes d) ; (\tau \otimes \tau p) = (c \otimes d) ; \tau p$

The refinement from right to left adds a test by (6.4) and then uses Lemma 10.6 (test-suffix-test), given that $\otimes$ is abort strict.

$(c \otimes d) ; \tau p \supseteq (c \otimes d ; \tau p) ; \tau p = c \otimes d ; \tau p$ □

11. Guarantees

A command $c$ satisfies a guarantee condition $g$, where $g$ is a binary relation on states, if every program step of $c$ satisfies $g$ [Jon81, Jon83a, Jon83b]. The command, $\text{guar} g \triangleq (\pi g \lor \epsilon)^\omega$, is the most general command that satisfies the guarantee relation $g$ for every program step and puts no constraints on its environment (2.60). The command, $\text{guar} g \otimes c$, behaves as both $\text{guar} g$ and as $c$, unless at some point $c$ aborts, in which case $\text{guar} g \otimes c$ aborts; note that $\text{guar} g$ cannot abort.

The term law is used for properties that are likely to be used in developing programs, while lemma is used for supporting properties used within proofs. To make it easier to locate laws/lemmas, they share a single numbering sequence. If a lemma/law has been proven elsewhere, a citation to the relevant publication follows the name of the lemma/law.

A guarantee command, $\text{guar} g_0$ ensures all program steps satisfy the relation $g_0$. For relation $g_1$ such that $g_0 \supseteq g_1$, the command $\text{guar} g_1$ ensures all program steps satisfy $g_1$ and hence every program step also satisfies $g_0$; hence $\text{guar} g_0$ is refined by $\text{guar} g_1$.

Law 11.1 (guar-strengthen). [HMWC19, Lemma 23] If $g_1 \supseteq g_2$, $\text{guar} g_1 \supseteq \text{guar} g_2$.

Weak conjoining a guarantee to a command $c$ constrains its behaviour so that all program steps satisfy the guarantee, and hence is a refinement.

Law 11.2 (guar-introduce). $c \supseteq \text{guar} g \otimes c$. 

Proof. The command chaos is the identity of weak conjunction (2.33), and chaos corresponds to a guarantee of the universal relation (univ), and hence using Law 11.1 (gur-strengthen):

\[ c = \text{chaos} \triangleleft c = \text{gur univ} \triangleleft c \geq \text{gur g} \triangleleft c. \]

Two guarantee commands weakly conjoined together ensure both relations \( g_1 \) and \( g_2 \) are satisfied by every program step, i.e. their intersection is satisfied by all program steps.

**Law 11.3** (gur-merge). [HMWC19, Lemma 24] \( \text{gur} g_1 \triangleleft \text{gur} g_2 = \text{gur} (g_1 \cap g_2) \)

Two guarantees in parallel produce program steps that satisfy either guarantee.

**Lemma 11.4** (par-guar-guar). \( \text{gur} g_1 \parallel \text{gur} g_2 = \text{gur} (g_1 \cup g_2) \)

Proof.

\[
\begin{align*}
\text{gur} g_1 \parallel \text{gur} g_2 &= \text{using the definition of a guarantee (2.60)} \\
(\pi g_1 \lor \epsilon)^\omega \parallel (\pi g_2 \lor \epsilon)^\omega &= \text{by (9.3)} \\
((\pi g_1 \lor \epsilon) \parallel (\pi g_2 \lor \epsilon))^\omega &= \text{distributing (2.47) twice and using (2.44), (2.43) and (2.42)} \\
(\pi (g_1 \cup g_2) \lor \epsilon)^\omega &= \text{by definition of a guarantee (2.60)} \\
\text{gur} (g_1 \cup g_2) & \boxed{.}
\end{align*}
\]

A guarantee command weakly conjoined with a sequential composition \((c ; d)\) ensures all program steps of both \( c \) and \( d \) satisfy the guarantee, and similarly for a guarantee weakly conjoined with a parallel composition \((c \parallel d)\).

**Law 11.5** (gur-seq-distrib). \( \text{gur} g \triangleleft (c ; d) \succ (\text{gur} g \triangleleft c) ; (\text{gur} g \triangleleft d) \)

Proof. The proof follows by Lemma 9.1 (sync-seq-distrib) for \( \otimes \) weak conjunction because from definition (2.60), a guarantee is of the form \( c^\omega \), and \( c^\omega = c^\omega \triangleleft c^\omega \) for any \( c \).

**Law 11.6** (gur-par-distrib). \( \text{gur} g \triangleleft (c \parallel d) \succ (\text{gur} g \triangleleft c) \parallel (\text{gur} g \triangleleft d) \)

Proof. The proof follows from Lemma 9.2 (conj-par-distrib) using Lemma 11.4 (par-guar-guar) to show its proviso: \( \text{gur} g = \text{gur} g \parallel \text{gur} g. \)

A guarantee combined with a test reduces to the test.

**Law 11.7** (gur-test). \( \text{gur} g \triangleleft \tau p = \tau p \)

Proof. The proof follows from the definition of a guarantee as an iteration (2.60) using (9.2) and (2.51): \( \text{gur} g \triangleleft \tau p = (\pi g \lor \epsilon)^\omega \triangleleft \tau \triangleleft \tau p = \tau \triangleleft \tau p = \tau p. \)

Guarantees combine with program steps to enforce the guarantee for the step.

**Law 11.8** (gur-pgm). \( \text{gur} g \triangleleft \pi r = \pi (g \triangleleft r) \)

Proof. The proof follows from the definition of a guarantee as an iteration (2.60), by unfolding the iteration (5.3), distributing and eliminating infeasible choices, and conjoining program steps (2.36): \( \text{gur} g \triangleleft \pi r = (\pi g \lor \epsilon) \triangleleft (\pi g \lor \epsilon)^\omega \triangleleft \pi r = \pi (g \triangleleft r). \)
An assertion \{p\} satisfies any guarantee because it makes no program steps at all unless it aborts, in which case the conjunction aborts.

**Law 11.9 (guar-assert).** \( \text{guar} \; g \; \cap \; \{p\} = \{p\} \)

*Proof.* The proof uses Lemma 10.3 (assert-distrib) and Law 11.7 (guar-test).

\[
\text{guar} \; g \; \cap \; \{p\} = \text{guar} \; g \; \cap \; \{p\} ; \tau = \{p\} ; (\text{guar} \; g \; \cap \; \tau) = \{p\}
\]

---

12. Frames

A frame \(X\) is a set of variables that a command \(c\) may modify. To restrict a command \(c\) to only modify variables in the set \(X\), the command, \(X \triangleright c \triangleq \text{guar} \; \mathbf{id}_X \cap c\), is introduced (2.61). It is defined using a guarantee that all variables other than \(X\), i.e. \(\overline{X}\), are not changed by any program steps. Recall that for a set of variables \(X\), \(\mathbf{id}_X\) is the identity relation on all variables other than \(X\); see (2.10). The binary operator \(\triangleright\) for frames has the highest precedence, in particular, it has higher precedence than sequential composition. Because frames are defined in terms of guarantees, they may be distributed over operators using (2.47) for nondeterministic choice, Law 11.5 (guar-seq-distrib), Law 11.6 (guar-par-distrib) and the fact that weak conjunction is associative, commutative and idempotent.

**Law 12.1 (distribute-frame).** \(X \triangleright (c ; d) \succ X \triangleright c ; X \triangleright d\)

*Proof.* The proof follows from the definition of a frame (2.61) and Law 11.5.

Reducing the frame of a command corresponds to strengthening its guarantee.

**Law 12.2 (frame-reduce).** For sets of identifiers \(X\) and \(Y\), \((X \cup Y) \triangleright c \succ Y \triangleright c\).

*Proof.* Expanding both sides using (2.61) the proof follows by Law 11.1 (guar-strengthen) because \(\mathbf{id}_{X \cup Y} \supseteq \mathbf{id}_Y\) as \(X \cup Y \subseteq Y\), i.e. \(X \cup Y \supseteq Y\).

---

13. Relies

In the rely/guarantee approach, the allowable interference on a thread \(c\) is represented by a rely condition, a relation \(r\) that is assumed to hold for any atomic step taken by the environment of \(c\) [Jon81, Jon83a, Jon83b]. A rely condition is an *assumption* that every step of the environment satisfies \(r\). The command, \(\text{rely} \; r \; \cap \; c\), is required to behave as \(c\), unless the environment makes a step not satisfying \(r\), in which case it allows any behaviour from that point (i.e. it aborts). The command, \(\text{rely} \; r \triangleq (\alpha \lor \epsilon \triangleright r) \omega\), allows any program or environment steps (i.e. \(\alpha\) steps) but if the environment makes a step not satisfying \(r\) (i.e. a step of \(\epsilon \triangleright r\)) it aborts (2.62). A rely command, \(\text{rely} \; r_1\), is satisfied by its environment (technically, it does not abort) if all environment steps satisfy relation \(r_1\). If all environment steps satisfy \(r_1\), then for a relation \(r_2\) that contains \(r_1\), all environment steps will also satisfy \(r_2\), and hence \(\text{rely} \; r_2\) is a refinement of \(\text{rely} \; r_1\).

**Law 13.1 (rely-weaken).** [HMWC19, Lemma 25] If \(r_1 \subseteq r_2\), then \(\text{rely} \; r_1 \succ \text{rely} \; r_2\).

The ultimate weakening is to the universal relation \(\text{univ}\), which removes the rely altogether.

**Law 13.2 (rely-remove).** \(\text{rely} \; r \; \cap \; c \succ c\).
Proof. Using Law 13.1 (rely-weaken) and noting that \textbf{chaos} is the identity of weak conjunction (2.33) and \( \epsilon \text{ univ} = \epsilon \emptyset = \text{ magic} \).

\[
\text{rely } r \cap c \gg \text{ rely univ } \cap c = (\alpha \lor \epsilon \emptyset ; \frac{\epsilon}{\emptyset})^\omega \cap c = \alpha^\omega \cap c = \text{ chaos } \cap c = c
\]

A rely of \( r_1 \) assumes all environment steps satisfy \( r_1 \), and a rely of \( r_2 \) assumes all environment steps satisfy \( r_2 \), and hence their weak conjunction corresponds to assuming all environment steps satisfy both \( r_1 \) and \( r_2 \), i.e. \( r_1 \cap r_2 \).

\textbf{Law 13.3} (rely-merge). [HMWC19, Lemma 26] \( \text{ rely } r_1 \cap \text{ rely } r_2 = \text{ rely}(r_1 \cap r_2) \)

Rely conditions may be distributed into a sequential composition.

\textbf{Law 13.4} (rely-seq-distrib). \( \text{ rely } r \cap (c ; d) \gg (\text{ rely } r \cap c) ; (\text{ rely } r \cap d) \).

\begin{proof}
\text{ The proof follows from Lemma 9.1 (sync-seq-distrib) with } \otimes \text{ weak conjunction, where the lemma's proviso that } \text{ rely } r \gg \text{ rely } r \text{ follows from the definition (2.62) and the property of iterations that } c^\omega = c^\omega ; c^\omega \text{ for any } c. \]
\end{proof}

A sequential composition within the context of a rely can be refined by refining one of its components in the context of the rely.

\textbf{Law 13.5} (rely-refine-within). \textit{If } \text{ rely } r \cap c_1 \gg \text{ rely } r \cap d, \\
\textit{ then } \text{ rely } r \cap c_0 ; c_1 ; c_2 \gg \text{ rely } r \cap c_0 ; d ; c_2.

\begin{proof}
\text{ rely } r \cap c_0 ; c_1 ; c_2 \\
\gg \text{ duplicate rely as } \otimes \text{ is idempotent and apply Law 13.4 (rely-seq-distrib) twice} \\
\text{ rely } r \cap (\text{ rely } r \cap c_0) ; (\text{ rely } r \cap c_1) ; (\text{ rely } r \cap c_2) \\
\gg \text{ from the assumption and using Law 13.2 (rely-remove) to remove three relies} \\
\text{ rely } r \cap c_0 ; d ; c_2
\end{proof}

In the parallel composition \( \text{ rely } r \parallel \text{ guar } r \), the guarantee on the right does not break the rely assumption on the left, but as the rely command allows any behaviour, including program steps that satisfy \( r \), its behaviour subsumes that which can be generated by the guarantee, and hence their parallel combination reduces to the rely.

\textbf{Law 13.6} (rely-par-guar). [HMWC19, Lemma 27] \( \text{ rely } r \parallel \text{ guar } r = \text{ rely } r \)

Relies and guarantees often appear conjoined together; Lemma 13.7 provides an expansion of their conjunction useful in a later proof.

\textbf{Lemma 13.7} (conj-rely-guar). \( \text{ rely } r \cap \text{ guar } g = (\pi g \lor \epsilon r)^\omega ; (\tau \lor \epsilon \overline{\tau} ; \frac{\epsilon}{\emptyset}) \).

\begin{proof}
\text{ rely } r \cap \text{ guar } g \\
= \text{ from definitions of rely (2.62) and guarantee (2.60)} \\
(\pi \lor \epsilon r \lor \epsilon \overline{\tau} ; \frac{\epsilon}{\emptyset})^\omega \cap (\pi g \lor \epsilon)^\omega \\
= \text{ decomposition property of iterations (5.4): } (c \lor d)^\omega = c^\omega ; (d ; c^\omega)^\omega \\
(\pi \lor \epsilon r)^\omega ; (\epsilon \overline{\tau} ; \frac{\epsilon}{\emptyset}) ; (\pi \lor \epsilon r)^\omega \cap (\pi g \lor \epsilon)^\omega \\
= \text{ as } \frac{\epsilon}{\emptyset} \text{ is a left annihilator} \\
(\pi \lor \epsilon r)^\omega ; (\epsilon \overline{\tau} ; \frac{\epsilon}{\emptyset})^\omega \cap (\pi g \lor \epsilon)^\omega
\end{proof}
= unfold iteration (5.3) and \( \notin \) is an annihilator
\[
(\pi \lor \epsilon r)^\omega ; (\tau \lor \epsilon \tau ; \notin) \sqcap (\pi g \lor \epsilon)^\omega ; \tau
\]
= by (9.4) as \( (\pi \lor \epsilon r) \sqcap (\pi g \lor \epsilon) = \pi g \lor \epsilon r \)
\[
(\pi g \lor \epsilon)^\omega ; \left( \begin{array}{c}
((\pi \lor \epsilon r)^\omega ; (\tau \lor \epsilon \tau ; \notin) \sqcap \tau) \lor \\
((\tau \lor \epsilon \tau ; \notin) \sqcap (\pi g \lor \epsilon)^\omega)
\end{array} \right)
\]
= unfolding iterations (5.3) and simplifying using (9.2) and (2.48)
\[
(\pi g \lor \epsilon)^\omega ; (\tau \lor \epsilon \tau ; \notin)
\]

A rely conjoined with a parallel composition, \( \text{rely } r \sqcap (c \parallel d) \), represents an assumption that every environment step of the whole parallel composition satisfies \( r \) but environment steps of \( c \) are either environment steps of the whole composition (assumed to satisfy \( r \)) or program steps of \( d \), which do not necessarily satisfy \( r \), but will if one imposes a guarantee on \( d \).

**Law 13.8 (rely-par-distrib).**
\[
\text{rely } r \sqcap (c \parallel d) \supseteq (\text{rely } r \sqcap \text{guar } r \sqcap c) \parallel (\text{rely } r \sqcap \text{guar } r \sqcap d)
\]

**Proof.**
\[
\text{rely } r \sqcap (c \parallel d)
= \text{duplicate the rely condition as } \sqcap \text{ is idempotent; Law 13.6 (rely-par-guar)}
\]
\[
(\text{rely } r \parallel \text{guar } r) \sqcap (\text{guar } r \parallel \text{rely } r) \sqcap (c \parallel d)
\]
\[
\supseteq \text{interchanging } \sqcap \text{ and } \parallel \text{ by (2.41)}
\[
((\text{rely } r \sqcap \text{guar } r) \parallel (\text{rely } r \sqcap \text{guar } r)) \sqcap (c \parallel d)
\]
\[
\supseteq \text{interchanging } \sqcap \text{ and } \parallel \text{ by (2.41)}
\]
\[
(\text{rely } r \sqcap \text{guar } r \sqcap c) \parallel (\text{rely } r \sqcap \text{guar } r \sqcap d)
\]
\]

14. **Termination**

A command is considered to terminate if it performs only a finite number of program steps. However, that does not preclude a terminating command being pre-empted by its environment forever. The command, \( \text{term } \triangleq \alpha^* \lor \epsilon^\omega \), can perform only a finite number of program steps but it does not constrain its environment (2.63). At first sight it may appear odd that a terminating command allows infinite behaviours but it should be emphasised that the infinite behaviour ends in an infinite sequence of environment steps, i.e. the thread is never scheduled from some point onwards. To avoid such pre-emption, fair execution can be incorporated; the reader is referred to [HM18] for a treatment of fairness in our approach. The command \( \text{term} \) satisfies the following properties.

**Law 14.1 (seq-term-term).**
\[
\text{term} ; \text{term} = \text{term}
\]
Proof. We start by expanding the definition of term (2.63) on the left.
\[
\alpha^* ; \epsilon^\omega ; \alpha^* ; \epsilon^\omega \\
= \text{expand } \epsilon^\omega ; \alpha^* ; \epsilon^\omega \text{ using } c^\omega ; d = c^* ; d \lor c^\infty (5.8) \\
\alpha^* ; (\epsilon^* ; \alpha^* ; \epsilon^\omega \lor \epsilon^\infty) \\
= \text{distributing (4.2)} \\
\alpha^* ; \epsilon^* ; \alpha^* ; \epsilon^\omega \lor \alpha^* ; \epsilon^\infty \\
= \text{as } \alpha \succ \epsilon \text{ by Lemma 5.1 (absorb-finite-iter) } \alpha^* ; \epsilon^* = \alpha^* \text{ and } c^* ; c^* = c^* \\
\alpha^* ; \epsilon^\omega \lor \alpha^* ; \epsilon^\infty \\
= \text{by (4.2) and } \epsilon^\omega \lor \epsilon^\infty = \epsilon^\omega \text{ by (5.8)} \\
\alpha^* ; \epsilon^\omega \\
= \text{by the definition of term (2.63)}
\]

Parallel composition of two terminating commands terminates.

Law 14.2 (par-term-term). [HMWC19, Lemma 20] term || term = term

15. Partial and total correctness

In Hoare logic [Hoa69] the triple, \( \{p_1\} c \{p_2\} \), represents the partial correctness assertion that if command \( c \) is started in a state satisfying predicate \( p_1 \) and \( c \) terminates, then the state on termination satisfies \( p_2 \). A total correctness interpretation of the triple requires, in addition, that \( c \) terminates from initial states satisfying \( p_1 \). Our algebraic characterisations of partial and total correctness are influenced by our ability to express and differentiate terminating, non-terminating and aborting program behaviours. We use weak correctness assertions, as introduced by von Wright [vW04], as a basis for both.

The weak correctness of the Hoare triple \( \{p_1\} c \{p_2\} \) corresponds to either of the following two equivalent algebraic conditions.

\[
\tau p_1 ; c \tau p_2 = \tau p_1 ; c \\
(15.1) \\
\tau p_1 ; c \tau p_2 \succ \tau p_1 ; c \\
(15.2)
\]

A weak correctness assertion \( \{p_1\} c \{p_2\} \) is not necessarily preserved by refinement, e.g. if \( c \succ d \), then it does not follow that \( \{p_1\} d \{p_2\} \) is also weakly correct, because the assertion permits \( c \) to abort from initial states in which \( p_1 \) holds, and \( \tau p_1 ; c \) may be refined by any possible behaviour, including behaviours that terminate in states violating \( p_2 \). Given that a program that aborts from initial state \( p_1 \) provides no guarantees about its behaviour after failure (e.g. it may terminate in a state that does not satisfy \( p_2 \)), our definition of partial correctness adds to the definition of weak correctness the requirement that \( c \) does not abort from states satisfying \( p_1 \), which can be formulated in either of the two equivalent (by (7.4)) forms,

\[
\{p_1\} ; \text{chaos} \succ c \\
(15.3) \\
\text{chaos} \succ \tau p_1 ; c \\
(15.4)
\]

Total correctness adds the even stronger requirement that \( c \) terminates from states satisfying
$p_1$, i.e. either of the following equivalent properties holds.\footnote{Our termination requirement for total correctness differs from von Wright \cite{vW04}, who defines the total correctness assertion \{p_1\} $c$ \{p_2\} to hold when $\tau_1 p_1 c \tau_2$. This is because in von Wright’s sequential theory an (everywhere) terminating command $c$ satisfies, $c$; $\text{magic} = \text{magic}$, but this does not hold in our theory, e.g. we do not have that $\text{term}; \text{magic} = \text{magic}$.}

$$\{p_1\}; \text{term} \supseteq c \quad (15.5)$$

$$\text{term} \supseteq \tau p_1 c \quad (15.6)$$

Unlike weak correctness assertions, both partial and total correctness assertions are preserved by refinement. Rather than a single-state postcondition, Coleman and Jones \cite{CJ07} make use of a relational postcondition, $q$, for their quintuple verification rules, giving rise to the following notion of a command $c$ being weakly correct with respect to a relation $q$, in which for a state $\sigma$, $q(\{\sigma\})$ is the relational image of the singleton set $\{\sigma\}$ through $q$ \eqref{eq:2.4}.

**Definition 15.1** (weakly-correct). A command $c$ is **weakly correct** with respect to a relation $q$ if and only if

$$c = \bigvee_{\sigma \in \Sigma} (\tau\{\sigma\}; c; \tau(q(\{\sigma\}))$$

**Lemma 15.2** (weakly-correct). To show that command $c$ is weakly correct with respect to relation $q$ it is enough to show that $c; \tau(q(\{\sigma\})) \supseteq \tau\{\sigma\}; c$ for all $\sigma \in \Sigma$.

**Proof.** By Lemma 6.2 (Nondet-test-set), $c = \bigvee_{\sigma \in \Sigma} \tau\{\sigma\}; c$, and hence it is sufficient to show that for all $\sigma$, $\tau\{\sigma\}; c = \tau\{\sigma\}; c; \tau(q(\{\sigma\}))$. This refinement holds from left to right because it is just adding a test \eqref{eq:6.4}, and refinement from right to left holds by the assumption that $c; \tau(q(\{\sigma\})) \supseteq \tau\{\sigma\}; c$ and \eqref{eq:2.23}. \hfill \box

We can also define partial and total correctness of a command $c$ with respect to a relation $q$.

**Definition 15.3** (partially-correct). A command $c$ is **partially correct** with respect to a relation $q$ if and only if it is weakly correct with respect to $q$ and chaos $\supseteq c$.

**Definition 15.4** (totally-correct). A command $d$ is **totally correct** with respect to a relation $q$ if and only if it is weakly correct with respect to $q$ and term $\supseteq c$.

Having a theory that supports all three notions of correctness is advantageous. First, in many cases proving either the absence of catastrophic failure or termination is straightforward and hence proofs of either partial or total correctness, respectively, can be simplified by focusing on weak correctness first. In addition, some concurrent algorithms (e.g. spin lock) do not guarantee termination, and so require a partial correctness specification, instead of a total-correctness one. In the remainder of this section we present lemmas useful for establishing that commands satisfy weak-correctness assertions (remembering that proofs for partial and total correctness can be decomposed).

A refinement of the form $c; \tau p_1 \supseteq \tau p_0 c$ asserts the weak correctness condition that when $c$ is started in a state in $p_0$, if $c$ terminates normally (i.e. not as a consequence of failure), the termination state is in $p_1$. So-called **commutativity conditions** of this form allow a post-state test of a sequential composition to be replaced by progressively earlier tests, e.g. if $c_2; \tau p_2 \supseteq \tau p_1 c_2$ and $c_1; \tau p_1 \supseteq \tau p_0 c_1$ then, $c_1; c_2; \tau p_2 \supseteq c_1; \tau p_1 c_2$ and $c_1; \tau p_1 c_2 \supseteq \tau p_0 c_1; c_2$. An important special case is when the test corresponds to an invariant, i.e. $p_0 = p_1$, because rules of this form can be applied to iterations.

A single program or environment step, $\pi r$ or $\epsilon r$, establishes postcondition $p_1$ if started in a state satisfying $p_0$, if the image \eqref{eq:2.4} of $p_0$ under $r$ is in $p_1$. \hfill \box
Lemma 15.5 (atomic-test-commute). If $r(p_0) \subseteq p_1$, then both the following hold.

\begin{align}
\pi r ; \tau p_1 & \succ \tau p_0 ; \pi r \\
\epsilon r ; \tau p_1 & \succ \tau p_0 ; \epsilon r
\end{align} (15.7)

Proof. The assumption is equivalent to $p_0 \triangleleft r \triangleright p_1 = p_0 \triangleleft r$. The proof of (15.7) uses (8.1), (2.27) and (2.26).

\begin{align}
\pi r ; \tau p_1 & \succ \pi (p_0 \triangleleft r \triangleright p_1) ; \tau p_1 = \pi (p_0 \triangleleft r) = \tau p_0 ; \pi r
\end{align}

The proof for (15.8) is similar but uses (8.2), (2.32) and (2.31).

Lemma 15.6 (nondet-test-commute). For tests $t_0$ and $t_1$, and commands $c$ and $d$, if $c ; t_1 \succ t_0 ; c$ and $d ; t_1 \succ t_0 ; d$,

\begin{align}
(c \lor d) ; t_1 & \succ t_0 ; (c \lor d).
\end{align}

Proof. From both assumptions $(c \lor d) ; t_1 = c ; t_1 \lor d ; t_1 \succ t_0 ; c \lor t_0 ; d = t_0 ; (c \lor d)$. □

Lemma 15.7 (iteration-test-commute). For any test $t$ and command $c$, if $c ; t \succ t ; c$, then both the following hold.

\begin{align}
c^\omega ; t & \succ t ; c^\omega \\
c^* ; t & \succ t ; c^*
\end{align} (15.9)

Proof. (15.9) holds by $\omega$-induction (5.7) if $t \lor c ; t ; c^\omega \succ t ; c^\omega$, which is proven using the assumption and $\omega$-folding (5.3): $t \lor c ; t ; c^\omega \succ t \lor t ; c ; c^\omega = (\tau \lor c ; c^\omega) = t ; c^\omega$. (15.10) holds by $\ast$-induction (5.6) if $c^* ; t \succ t \lor c^* ; t ; c$, which is proven using the assumption and $\ast$-folding (5.2): $t \lor c^* ; t ; c \succ t \lor c^* ; c ; t = (\tau \lor c^* ; c) ; t = c^* ; t$. □

16. Specification commands

One can define a partial (correctness) specification command,

\begin{align}
[q] \triangleq \lor_{q_0 \in \Sigma} (\tau q_0 ; \text{chaos}; \tau(q(|\{q_0\}|))).
\end{align}

It requires that if started in a state $q_0$, then if it terminates, its final state is related to $q_0$ by $q$, i.e. it is in the relational image of $\{q_0\}$ through $q$ (2.64). A total (correctness) specification command, $[q] \triangleq [q] \land \text{term}$, requires termination (2.65).\footnote{A partial specification command $[q]$ is the greatest command that is partially correct with respect to $q$ (from Lemma 16.7 (partially-correct)), and similarly a total correctness specification $[q]$ is the greatest command that is totally correct with respect to $q$ (from Lemma 16.8 (totally-correct)). Note that a weak specification command defined in this way would be uninteresting, because the greatest command that is weakly correct with respect to any $q$ is $\bot$.}
Related work. The semantics of Brookes [Bro96] makes use of a “stuttering” equivalence relation on commands that considers two commands equivalent if their sets of traces are equal when all finite sequences of stuttering steps are removed from every trace of both. Because the focus of this paper is on refining from a specification, an alternative approach is used whereby specification commands implicitly allow finite stuttering, i.e. they are closed under finite stuttering.

Brookes also makes use of “mumbling” equivalence that allows two consecutive program steps \((\pi r_1 ; \pi r_2)\) to be replaced by a single program step \(\pi(r_1 \circ r_2)\) with the same overall effect, where \(\circ\) is relational composition \((2.5)\). Again, specification commands implicitly allow all mumbling equivalent implementations. Our postcondition specification command \(c\) is semantically equivalent to \(c\) modulo finite stuttering and mumbling, then \([q]\) is also refined by \(d\). In general, it does not require that \(c\) and \(d\) are refinement equivalent.

The following lemma allows a command \(c\) synchronised using an abort strict operator \(\otimes\) to be distributed into a choice that resembles the structure used in a specification command \((2.64)\) when \(d\) is chaos.

**Lemma 16.1** (sync-distribute-relation). If \(\otimes\) is abort strict,
\[
c \otimes \bigvee_{\sigma \in \Sigma} (\tau\{\sigma\} ; d ; \tau(q\{|\sigma\}|)) = \bigvee_{\sigma \in \Sigma} (\tau\{\sigma\} ; (c \otimes d) ; \tau(q\{|\sigma\}|))
\]
Proof. The application of Lemma 10.7 in the last step requires that \(\otimes\) is abort strict.
\[
c \otimes \bigvee_{\sigma \in \Sigma} (\tau\{\sigma\} ; d ; \tau(q\{|\sigma\}|))
\]
\[
= \text{ by Lemma 6.2 (Nondet-test-set)}
\]
\[
\bigvee_{\sigma_1 \in \Sigma} (\tau\{\sigma_1\} ; (c \otimes \bigvee_{\sigma \in \Sigma} (\tau\{\sigma\} ; d ; \tau(q\{|\sigma\}|))))
\]
\[
= \text{ distribute test (2.52) and Lemma 6.3 (test-restricts-Nondet)}
\]
\[
\bigvee_{\sigma_1 \in \Sigma} (\tau\{\sigma_1\} ; c \otimes \bigvee_{\sigma \in \{\sigma_1\}} (\tau\{\sigma\} ; d ; \tau(q\{|\sigma\}|)))
\]
\[
= \text{ as } \sigma_1 \text{ is the only choice for } \sigma
\]
\[
\bigvee_{\sigma_1 \in \Sigma} (\tau\{\sigma_1\} ; c \otimes \tau\{\sigma_1\} ; d ; \tau(q\{|\sigma_1\}|))
\]
\[
= \text{ distribute test (2.52) in reverse and Lemma 10.7 (test-suffix-interchange)}
\]
\[
\bigvee_{\sigma_1 \in \Sigma} (\tau\{\sigma_1\} ; (c \otimes d) ; \tau(q\{|\sigma_1\}|)) \quad \square
\]

**Lemma 16.2** (spec-distribute-sync). If \(\otimes\) is abort strict, \(c \otimes (d \ominus \big[q\big]) = (c \otimes d) \ominus \big[q\big]\).

Proof. Note that chaos is the identity of \(\&\), and \(\&\) is abort strict as required for the first and last applications of Lemma 16.1 (sync-distribute-relation).
\[
c \otimes (d \ominus \big[q\big])
\]
\[
= \text{ by definition (2.64) and Lemma 16.1 (sync-distribute-relation) for } \&
\]
\[
c \otimes \bigvee_{\sigma \in \Sigma} (\tau\{\sigma\} ; d ; \tau(q\{|\sigma\}|))
\]
\[
= \text{ by Lemma 16.1 (sync-distribute-relation) as } \otimes \text{ is abort strict}
\]
\[
\bigvee_{\sigma \in \Sigma} (\tau\{\sigma\} ; (c \otimes d) ; \tau(q\{|\sigma\}|))
\]
\[
= \text{ by Lemma 16.1 (sync-distribute-relation) for } \& \text{ (in reverse)}
\]
\[
(c \otimes d) \& \bigvee_{\sigma \in \Sigma} (\tau\{\sigma\} ; \text{chaos} ; \tau(q\{|\sigma\}|)))
\]
by the definition of a specification (2.64)

$$(c \otimes d) \cap [q]$$

A partial specification with postcondition of the universal relation only guarantees not to abort (i.e. chaos) and a total specification only guarantees to terminate (i.e. term).

Lemma 16.3 (spec-univ). Both $[\text{univ}] = \text{chaos}$ and $[\text{univ}] = \text{term}$.

Proof. The proof expands the definition of a partial specification command (2.64), uses the fact that $\text{univ}(\{\sigma_0\}) = \Sigma$ and applies (2.18) and Lemma 6.2 (Nondet-test-set).

$$[\text{univ}] = \bigvee_{\sigma_0 \in \Sigma} (\{\sigma_0\}; \text{chaos}; \tau(\text{univ}(\{\sigma_0\}))) = (\bigvee_{\sigma_0 \in \Sigma} \tau(\{\sigma_0\}); \text{chaos}; \tau = \text{chaos}$$

For total correctness, $[\text{univ}] = [\text{univ}] \cap \text{term} = \text{chaos} \cap \text{term} = \text{term}$.

Law 16.4 (spec-strengthen). If $q_1 \supseteq q_2$, then both $[q_1] \supseteq [q_2]$ and $[q_1] \nhd [q_2]$.

Proof. The proof follows directly from the definition of either specification command because if $q_1 \supseteq q_2$, then $\tau(q_1(\{\sigma_0\})) \supseteq \tau(q_2(\{\sigma_0\}))$ by (6.3).

Lemma 16.5 (spec-introduce). Both $\text{chaos} \nhd [q]$ and $\text{term} \nhd [q]$.

Proof. Using Lemma 16.3 and Law 16.4 (spec-strengthen), $\text{chaos} = [\text{univ}] \nhd [q]$ and $\text{term} = [\text{univ}] \nhd [q]$.

The definitions of weak correctness, partial correctness and total correctness can be reformulated using the specification commands.

Lemma 16.6 (weakly-correct-spec). A command $c$ is weakly correct with respect to a relation $q$ if and only if $c \cap [q] = c$.

Proof. The proof reduces the equality $c \cap [q] = c$ to Definition 15.1 (weakly-correct).

$$c \cap [q] = c$$

$\Leftrightarrow$ by (2.64)

$$c \cap \bigvee_{\sigma \in \Sigma} (\{\sigma\}; \text{chaos}; \tau(q(\{\sigma\}))) = c$$

$\Leftrightarrow$ by Lemma 16.1 (sync-distribute-relation) and $\text{chaos}$ is the identity of $\cap$

$$\bigvee_{\sigma \in \Sigma} (\{\sigma\}; c; \tau(q(\{\sigma\}))) = c$$

Lemma 16.7 (partially-correct). A command $c$ is partially correct with respect to a relation $q$ if and only if $[q] \nhd c$.

Proof. By Definition 15.3 (partially-correct), $c$ is partially correct with respect to a relation $q$ if and only if it is weakly correct with respect to $q$ and $\text{chaos} \nhd c$, or by Lemma 16.6 (weakly-correct-spec) if $[q] \cap c = c$ and $\text{chaos} \nhd c$.

$$[q] \cap c = c$$ and $\text{chaos} \nhd c$

$\Leftrightarrow$ by Lemma 16.5 (spec-introduce) and (9.1)

$$[q] \land c = c$$ and $\text{chaos} \nhd c$

$\Leftrightarrow$ lattice property: $c_1 \land c_2 = c_2$ if and only if $c_1 \nhd c_2$ for any $c_1$ and $c_2$
Lemma 16.8 (totally-correct). A command $c$ is totally correct with respect to a relation $q$ if and only if $[q] \trianglerighteq c$.

Proof. Because $\text{chaos} \trianglerighteq \text{term}$ and $\text{term} \trianglerighteq [q]$ (from Lemma 16.5 (spec-introduce)), using (9.1) we have that

\[
([q] \trianglerighteq c) \iff ([q] \cap \text{term} \trianglerighteq c) \iff ([q] \land \text{term} \trianglerighteq c) \iff ([q] \trianglerighteq c \text{ and term} \trianglerighteq c)
\]

which is true from Lemma 16.7 (partially-correct) if and only if $c$ is weakly correct with respect to $q$, $\text{chaos} \trianglerighteq c$ and $\text{term} \trianglerighteq c$. Because $\text{chaos} \trianglerighteq \text{term}$ this is equivalent to Definition 15.4 (totally-correct).

Because specifications are defined in terms of tests, laws that combine specifications with tests are useful for manipulating specifications. These laws have corollaries that show how specifications combine with assertions. Recall that $p \prec q$ is the relation $q$ with its domain restricted to $p$ (2.2). Total specification commands ensure termination. Below we give proofs of only the partial specification command properties of the lemmas; the proofs of total specification commands just add $\text{term}$ on each side.

Lemma 16.9 (test-restricts-spec). Both $\tau p; [p \prec q] = \tau p; [q]$ and $\tau p; [p \prec q] = \tau p; [q]$.

Proof. The proof applies Lemma 16.9 using the fact that $\{p\} ; \tau p = \{p\}$ by (7.7).

Frames are included in the following law to make it more useful in practice.

Law 16.11 (spec-strengthen-under-pre). Let $X$ be a set of variables, if $p \prec q_2 \subseteq q_1$ both, $\{p\} ; X : [q_1] \trianglerighteq \{p\} ; X : [q_2]$ , and $\{p\} ; X : [q_1] \trianglerighteq \{p\} ; X : [q_2]$.
Proof.

\[
\{p\} ; X : [q_1]
\]

\[= \] by the definition of a frame (2.61) and Lemma 10.3 (assert-distrib)

\[
\{p\} ; [q_1] \cap \text{guar id}_X
\]

\[\supseteq \] by Law 16.4 (spec-strengthen) using assumption \( p \prec q_2 \subseteq q_1 \)

\[
\{p\} ; [p \prec q_2] \cap \text{guar id}_X
\]

\[= \] by Lemma 16.10 (assert-restricts-spec)

\[
\{p\} ; [q_2] \cap \text{guar id}_X
\]

\[= \] by Lemma 10.3 (assert-distrib) and the definition of a frame (2.61)

\[
\{p\} ; X : [q_2]
\]

A test can be used to restrict the final state of a specification. Recall that \( q \triangleright p \) is the relation \( q \) with its range restricted to \( p \) (2.3).

**Lemma 16.12** (spec-test-restricts). Both \([q] ; \text{null} ; [p] = [q \triangleright p]\) and \([q] ; \text{null} ; [p] = [q \triangleright p]\).

**Proof.**

\[
[q] ; \text{null} ; [p] = [q \triangleright p]
\]

\[= \] definition of a specification (2.64) and distribute test (2.18)

\[
\vee_{\sigma_0 \in \Sigma} (\tau\{\sigma_0\} ; \text{null} ; [q\{\sigma_0\}] \triangleright [p])
\]

\[= \] merging tests (2.23) and \( q\{\sigma_0\} \cap p = (q \triangleright p)\{\sigma_0\} \)

\[
\vee_{\sigma_0 \in \Sigma} (\tau\{\sigma_0\} ; \text{null} ; [q \triangleright p] \{\sigma_0\})
\]

\[= \] definition of a specification (2.64)

\[
[q \triangleright p]
\]

**Lemma 16.13** (spec-assert-restricts). \([q \triangleright p] ; \{p\} = [q \triangleright p] \) and \([q \triangleright p] ; \{p\} = [q \triangleright p] \).

**Proof.** The proof applies Lemma 16.12 using the fact that \( \tau p ; \{p\} = \tau p \) by (7.6).

\[
[q \triangleright p] ; \{p\} = [q] ; \tau p ; \{p\} = [q] ; \tau p = [q \triangleright p]
\]

A specification command \([q]\) achieves a postcondition of \( q(p) \) from any initial state in \( p \).

**Lemma 16.14** (spec-test-commute). \([q] ; \tau(q(p)) \supseteq \tau p ; [q] \) and \([q] ; \tau(q(p)) \supseteq \tau p ; [q] \).

**Proof.**

\[
[q] ; \tau(q(p))
\]

\[= \] by Lemma 16.12 (spec-test-restricts)

\[
[q \triangleright (q(p))]
\]

\[\supseteq \] by Law 16.4 (spec-strengthen) as \( p \prec q \subseteq q \triangleright (q(p)) \)

\[
[p \prec q]
\]

\[\supseteq \] introducing \( \tau p \) (6.4) and Lemma 16.9 (test-restricts-spec)

\[
\tau p ; [q]
\]
A specification with a post condition that is the composition (2.5) of two relations \( q_1 \) and \( q_2 \) may be refined by a sequential composition of one specification command satisfying \( q_1 \) and a second satisfying \( q_2 \).

**Law 16.15** (spec-to-sequential). *Both \([q_1 ; q_2] \supseteq [q_1] ; [q_2] \) and \([q_1 ; q_2] \supseteq [q_1] ; [q_2] \).*

*Proof.* From relational algebra, \((q_1 ; q_2)(p) = q_2(q_1(p)))\). This allows the proof to use two applications of Lemma 16.14 (spec-test-commute) to show that \([q_1] ; [q_2] \) establishes post-condition \((q_1 ; q_2)(\{\sigma_0\})\) from initial state \(\sigma_0\), if it terminates.

\[
[q_1 ; q_2] \\
= \text{definition of a specification command (2.64) and } (q_1 ; q_2)(\{\sigma_0\}) = q_2(q_1(\{\sigma_0\})) \\
\forall_{\sigma_0 \in \Sigma} (\tau(\{\sigma_0\}) ; \text{chaos} ; \tau(q_2(q_1(\{\sigma_0\})))) \\
\supseteq \text{as } \text{chaos} = \text{chaos} ; \text{chaos and Lemma 16.5 (spec-introduce) twice} \\
\forall_{\sigma_0 \in \Sigma} (\tau(\{\sigma_0\}) ; [q_1] ; [q_2] ; \tau(q_2(q_1(\{\sigma_0\})))) \\
\supseteq \text{by Lemma 16.14 (spec-test-commute) } \\
\forall_{\sigma_0 \in \Sigma} (\tau(\{\sigma_0\}) ; [q_1] ; \tau(q_1(\{\sigma_0\})) ; [q_2]) \\
\supseteq \text{by Lemma 16.14 (spec-test-commute) } \\
\forall_{\sigma_0 \in \Sigma} (\tau(\{\sigma_0\}) ; [q_1] ; [q_2]) \\
= \text{merging tests (2.23) and apply Lemma 6.2 (Nondet-test-set) } \\
[q_1] ; [q_2]
\]

The total-correctness version uses Law 14.1 (seq-term-term), i.e. \(\text{term} = \text{term} ; \text{term}\).  

The above lemmas can be combined to give a law for splitting a specification into a sequential composition with an intermediate assertion. To make the law more useful in practice, we include a frame specifying the variables that are allowed to be modified.

**Law 16.16** (spec-seq-introduce). *For a set of variables \(X\), sets of states \(p_1\) and \(p_2\), and relations \(q, q_1\) and \(q_2\), provided \(p_1 \triangleleft (((q_1 \triangleright p_2) \triangleright q_2) \subseteq q\), both*

\[\{p_1\} ; X ; [q] \supseteq \{p_1\} ; X ; [q_1 \triangleright p_2] ; \{p_2\} ; X ; [q_2]\]

*Proof.*

\[\{p_1\} ; X ; [q] \supseteq \{p_1\} ; X ; [(q_1 \triangleright p_2) \triangleright q_2] \supseteq \text{by Law 16.11 (spec-strengthen-under-pre) and assumption} \]

\[\{p_1\} ; X ; [(q_1 \triangleright p_2) \triangleright q_2] \supseteq \text{by Law 16.15 (spec-to-sequential)} \]

\[\{p_1\} ; X ; [(q_1 \triangleright p_2) \triangleright q_2] \supseteq \text{by Lemma 16.13 (spec-assert-restricts) and Law 12.1 (distribute-frame)} \]

\[\{p_1\} ; X ; [q_1 \triangleright p_2] ; \{p_2\} ; X ; [q_2] \]
Example 16.17 (spec-seq-introduce). The following uses two applications of Law 16.16 (spec-seq-introduce) to refine a specification to a sequence of three specifications.

\[ nw, pw, w : [\forall w \supset w' \lor i' \notin w''] \]

\[ \Rightarrow \text{ by Law 16.16 (spec-seq-introduce)} \text{ – see justification below} \]

\[ nw, pw, w : [\forall w \supset pw' \land pw' \supset w''] ; \]

\[ \{ \land pw \supset w \} ; nw, pw, w : [\forall pw \supset w' \lor i' \notin w''] \]  \hspace{2cm} (16.1)

\[ \{ \land pw \supset w \} ; nw, pw, w : [\forall pw \supset w' \lor i' \notin w''] \]  \hspace{2cm} (16.2)

The proof obligation for the application of Law 16.16 above can be shown as follows. The intermediate assertion \( \land pw \supset w \supset w' \lor i' \notin w'' \) is ensured by the postcondition of the first command.

\[ \Rightarrow \text{ by Law 16.16 (spec-seq-introduce)} \text{ – see justification below} \]

\[ \{ \land pw \supset w \supset w' \lor i' \notin w'' \} ; nw, pw, w : [\forall nw = pw - \{ i \} \land pw' = pw \land pw' \supset w' \land i' = i'] ; \]

\[ \{ \land pw \supset w \land nw = pw - \{ i \} \} ; nw, pw, w : [\forall pw \supset w' \lor i' \notin w''] \]  \hspace{2cm} (16.2)

The proof obligation for the application of Law 16.16 above can be shown as follows. The intermediate assertion \( \land pw \supset w \land nw = pw - \{ i \} \) is ensured by the postcondition of the first command.

\[ \Rightarrow \text{ by Law 16.16 (spec-seq-introduce)} \text{ – see justification below} \]

\[ \{ \land pw \supset w \land nw = pw - \{ i \} \} ; nw, pw, w : [\forall nw = pw - \{ i \} \land pw' = pw \land pw' \supset w' \land i' = i'] ; \]

\[ \{ \land pw \supset w \land nw = pw - \{ i \} \} ; nw, pw, w : [\forall pw \supset w' \lor i' \notin w''] \]  \hspace{2cm} (16.1)

The next lemma is important for introducing a parallel composition or weak conjunction of specifications to refine a single specification in Sect. 18.

Lemma 16.18 (sync-spec-spec). For \( \otimes \) either \( \parallel \) or \( \ominus \), both \([ q_0 ] \otimes [ q_1 ] = [ q_0 \cap q_1 ]\) and \([ q_0 ] \otimes [ q_1 ] = [ q_0 \cap q_1 ]\).

Proof. The application of Lemma 16.1 requires the assumption that \( \otimes \) is abort strict.

\[ [ q_0 ] \otimes [ q_1 ] \]

\[ \Rightarrow \text{ definition of } [ q_1 ] \text{ from (2.64) and Lemma 16.1 (sync-distribute-relation)} \]

\[ \bigvee_{\sigma \in \Sigma} (\tau \{ \sigma \} ; [ q_0 ] ; \tau (q_1 (\{ \{ \sigma \} \}))) \]

\[ \Rightarrow \text{ by Lemma 16.9 (test-restricts-spec) and Lemma 16.12 (spec-test-restricts)} \]

\[ \bigvee_{\sigma \in \Sigma} (\tau \{ \sigma \} ; [\{ \sigma \} < q_0 \triangleright (q_1 (\{ \{ \sigma \} \}))) \)

\[ \Rightarrow \text{ simplify relation} \]

\[ \bigvee_{\sigma \in \Sigma} (\tau \{ \sigma \} ; [\{ \sigma \} < (q_0 \cap q_1)]) \]

\[ \Rightarrow \text{ by Lemma 16.9 (test-restricts-spec)} \]

\[ (\bigvee_{\sigma \in \Sigma} \tau (\{ \sigma \})) ; [ q_0 \cap q_1 ] \]
The property for a total specification follows from that for a partial specification.

\[ [q_0 \cap q_1] \]

17. Stability under interference

Stability of a property \( p \) over the execution of a command is an important property and, in the context of concurrency, stability of a property over interference from the environment is especially important [Col08, WDP10a]. This section examines stability properties that are useful for later laws.

**Definition 17.1 (stable).** A set of states \( p \) is stable under a binary relation \( r \) if and only if \( [p] r \subseteq p \). An equivalent expression of the property is that \( p \sqsubseteq r p = p \sqsubseteq r \). If \( p \) is stable under \( r \), we also say that the test, \( \tau p \), is stable under \( r \).

**Example 17.2 (stable-pred).** The set of states \( \llcorner pw \supseteq w \llcorner \) is stable under the relation \( \llcorner w \supseteq w' \wedge pw' = pw' \).

**Lemma 17.3 (stable-transitive).** If \( p \) is stable under \( r \) then \( p \) is stable under the reflexive, transitive closure of \( r \), \( r^* \) (2.6). In fact, because \( \text{id} \subseteq r^* \) one has \( r^*([p]) = p \).

**Proof.** The second step of the proof below uses the relational equivalent of (5.6), i.e.

\[ q \uplus r^* \subseteq x \quad \text{if} \quad q \cup x \uplus r \subseteq x. \quad (17.1) \]

A general property of relational image is

\[ r([p_0]) \subseteq p_1 \iff p_0 \triangleleft r \subseteq r \triangleright p_1. \quad (17.2) \]
To show \( p \) is stable under \( r^* \), Definition 17.1 (stable) requires one to show \( r^*(|p|) \subseteq p \), or using (17.2),

\[
p < r^* \subseteq r^* \triangleright p
\]

\[\Leftrightarrow \text{property of relational algebra}\]

\[(p < \text{id}) \cup r^* \subseteq r^* \triangleright p\]

\[\Leftrightarrow \text{by least } \ast \text{-induction (17.1)}\]

\[(p < \text{id}) \cup (r^* \triangleright p) \subseteq r^* \triangleright p\]

\[\Leftrightarrow \text{properties of relational algebra}\]

\[(\text{id} \triangleright p) \cup r^* \subseteq r^* \triangleright p\]

\[\Leftrightarrow \text{as } p \text{ is stable under } r, \quad p < r \subseteq r \triangleright p\]

\[\text{id} \triangleright p \cup r^* \subseteq r^* \triangleright p\]

\[\Leftrightarrow \text{distribution of range restriction}\]

\[(\text{id} \cup r^* \triangleright r) \triangleright p \subseteq r^* \triangleright p\]

The final containment holds by unfolding as \( r^* = \text{id} \cup r^* \triangleright r \) by (5.2) for relations. \( \Box \)

**Lemma 17.4** (interference-before). If \( p < (r \triangleright q) \subseteq q \) and \( p \) is stable under \( r \), then \( p < r^* \triangleright q \subseteq q \).

**Proof.** The second step of the proof below uses the relational equivalent of (5.5), i.e.

\[(r^* \triangleright q) \subseteq x \quad \text{if } q \cup r^* \subseteq x. \quad (17.3)\]

The proof follows.

\[p < r^* \triangleright q \subseteq q\]

\[\Leftrightarrow \text{as } p \text{ is stable under } r\]

\[(p < r^* \triangleright q) \subseteq q\]

\[\Leftrightarrow \text{by } \ast \text{-induction (17.3)}\]

\[q \cup (p < r^* \triangleright q) \subseteq q\]

The latter holds from assumption \( p < (r \triangleright q) \subseteq q \). \( \Box \)

**Lemma 17.5** (interference-after). If \( p < (q \triangleright r) \subseteq q \), then \( p < q \triangleright r^* \subseteq q \).

**Proof.** The property is equivalent to \( (p < q) \triangleright r^* \subseteq p < q \), which holds by \( \ast \)-induction (17.1) if \( (p < q) \cup (p < q \triangleright r) \subseteq p < q \), which follows from the assumption \( p < (q \triangleright r) \subseteq q \). \( \Box \)

**Lemma 17.6** (guar-test-commute-under-rely). If \( p \) is stable under both \( r \) and \( g \),

\[\text{rely } r \sqcap \text{guar } g \triangleright \tau \rightarrow \text{rely } r \sqcap \tau p \triangleright \text{guar } g.\]

**Proof.** First note that because \( p \) is stable under both \( r \) and \( g \), by Definition 17.1 (stable) \( r(|p|) \subseteq p \) and \( g(|p|) \subseteq p \), and hence by Lemma 15.5 (atomic-test-commute) and Lemma 15.6 (nondet-test-commute).

\[(\pi g \cup \epsilon r) \triangleright \tau p \triangleright (\pi g \cup \epsilon r) \quad (17.4)\]
The main proof follows.

\[
\text{Lemma 10.7 (test-suffix-interchange)}
\]

\[
(\text{rely } r \cap \text{guar } g) ; \tau p
\]

\[
= (\text{by Lemma 13.7 (conj-rely-guar)}
\]

\[
(\pi g \lor \epsilon r)^\omega ; (\tau \lor \epsilon \tau ; \tau p)
\]

\[
= \text{distributing the final test (4.1)}
\]

\[
(\pi g \lor \epsilon r)^\omega ; (\tau p \lor \epsilon \tau ; \tau p)
\]

\[
\geq \text{as } \tau p = \tau p \text{ and introducing } \tau p \text{ by (6.4)}
\]

\[
(\pi g \lor \epsilon r)^\omega ; (\tau p \lor \epsilon \tau p ; \epsilon \tau ; \tau p)
\]

\[
= \text{factor out } \tau p \text{ using (4.2)}
\]

\[
(\pi g \lor \epsilon r)^\omega ; \tau p ; (\tau \lor \epsilon \tau ; \tau p)
\]

\[
\geq \text{by Lemma 15.7 (iteration-test-commute) and (17.4)}
\]

\[
\tau p ; (\pi g \lor \epsilon r)^\omega ; (\tau \lor \epsilon \tau ; \tau p)
\]

\[
= \text{by Lemma 13.7 (conj-rely-guar)}
\]

\[
\tau p ; (\text{rely } r \cap \text{guar } g)
\]

\[
= \text{by Lemma 10.2 (test-command-sync-command) for } \cap
\]

\[
\text{rely } r \cap \tau p ; \text{guar } g
\]

Coleman and Jones [CJ07] recognised that the combination of a guarantee \( g \) and a rely condition \( r \) is sufficient to deduce that the overall postcondition \((r \cup g)^*\) holds on termination because each step is either assumed to satisfy \( r \) (environment step) or guarantees to satisfy \( g \) (program step). That property is made explicit in the following lemmas.

**Lemma 17.7** (spec-trade-rely-guar). \( \text{rely } r \cap [(r \cup g)^*] \geq \text{rely } r \cap \text{guar } g \)

*Proof.* Because \( \text{rely } r \geq \text{rely } r \cap \text{guar } g \) by Law 11.2 (guar-introduce), it is enough, by Lemma 16.6 (weakly-correct-spec), to show that \( \text{rely } r \cap \text{guar } g \) is weakly correct with respect to relation \((r \cup g)^*\), which holds by Lemma 15.2 (weakly-correct) if for any state \( \sigma \),

\[
(\text{rely } r \cap \text{guar } g) ; \tau ((r \cup g)^*([\{\sigma\}]))
\]

\[
= \text{by Lemma 10.7 (test-suffix-interchange) for weak conjunction}
\]

\[
\text{rely } r \cap \text{guar } g ; \tau((r \cup g)^*([\{\sigma\}]))
\]

\[
\geq \text{by Lemma 17.6 as } (r \cup g)^*([\{\sigma\}]) \text{ is stable under both } r \text{ and } g
\]

\[
\text{rely } r \cap \tau((r \cup g)^*([\{\sigma\}])) ; \text{guar } g
\]

\[
\geq \text{as } \sigma \in (r \cup g)^*([\{\sigma\}]) \text{ follows from reflexivity of } (r \cup g)^*
\]

\[
\text{rely } r \cap \tau\{\sigma\} ; (\text{rely } r \cap \text{guar } g)
\]

\[
= \text{by Lemma 10.2 (test-command-sync-command) for } \cap
\]

\[
\tau\{\sigma\} ; (\text{rely } r \cap \text{guar } g)
\]

\[\square\]

**Law 17.8** (spec-trading). \( \text{rely } r \cap \text{guar } g \cap [(r \cup g)^* \cap q] = \text{rely } r \cap \text{guar } g \cap [q] \).
Proof. The refinement from right to left holds by Law 16.4 (spec-strengthen) and that from left to right as follows.

\[
\text{rely } r \cap \text{guar } g \cap [(r \cup g)^* \cap q] = \]

\[
\text{by the definition of a specification (2.65) and Lemma 16.18 (sync-spec-spec)}
\]

\[
\text{rely } r \cap \text{guar } g \cap [(r \cup g)^*] \cap [q] \cap \text{term}
\]

\[
\text{by Lemma 17.7 (spec-trade-rely-guar); and definition of a specification (2.65)}
\]

\[
\text{rely } r \cap \text{guar } g \cap [q]
\]

\[
\text{□}
\]

Related work. In Jones’ thesis [Jon81, Sect. 4.4.1] the parallel introduction law made use of a dynamic invariant that is a relation between the initial state of a parallel composition and all successor states (both intermediate states and the final state). A dynamic invariant, DINV, is required to be reflexive and satisfy \( \text{DINV} \supseteq r \subseteq \text{DINV} \), where \( r \) is the rely condition, and for all threads \( i \), satisfy \( \text{DINV} \supseteq g_i \subseteq \text{DINV} \), where \( g_i \) is the guarantee for thread \( i \). DINV is conjoined with the conjunction of the postconditions of all the parallel components to show the resulting postcondition holds, thus allowing a stronger overall postcondition based of the extra information in DINV. If one lets \( g \) stand for the union of all the guarantee relations of the individual threads, i.e. \( g = \bigcup_i g_i \), the conditions on DINV show that it contains \((r \cup g)^*\). Hence \((r \cup g)^*\) can be seen as the smallest relation satisfying the properties for DINV. The two-branch parallel introduction rule of Coleman and Jones [CJ07] uses \((r \cup g)^*\) in place of DINV. In both [Jon81] and [CJ07] the dynamic invariant was only used as part of the parallel introduction law, but in [HJC14] it was recognised that the dynamic invariant could be decoupled from the parallel introduction law leading to a law similar to Law 17.8 (spec-trading). Here we go one step further to factor out the more basic Lemma 17.7 (spec-trade-rely-guar) from which Law 17.8 (spec-trading) can be derived. Lemma 17.7 (spec-trade-rely-guar) is also useful in the proof of Law 20.5 (rely-idle) below.

In the context of a rely condition \( r \) and guarantee condition \( g \), the strengthening of a postcondition can also assume the transitive closure of the union of the rely and guarantee. In addition, a frame consisting of a set of variables \( X \) corresponds to an additional guarantee of \( \text{id}_X^{-} \).

Law 17.9 (spec-strengthen-with-trading). If \( p \triangleleft ((r \cup (g \cap \text{id}_X^{-}))^* \cap q_2) \subseteq q_1 \),

\[
\text{rely } r \cap \text{guar } g \cap \{p\}; X: [q_1] \Rightarrow \text{rely } r \cap \text{guar } g \cap \{p\}; X: [q_2].
\]

Proof.

\[
\text{rely } r \cap \text{guar } g \cap \{p\}; X: [q_1]
\]

\[
\Rightarrow \text{ by Law 16.11 (spec-strengthen-under-pre) using the assumption}
\]

\[
\text{rely } r \cap \text{guar } g \cap \{p\}; X: [(r \cup (g \cap \text{id}_X^{-}))^* \cap q_2]
\]

\[
\text{definition of a frame (2.61) and Law 11.3 (guar-merge)}
\]

\[
\text{rely } r \cap \text{guar}(g \cap \text{id}_X^{-}) \cap \{p\}; [(r \cup (g \cap \text{id}_X^{-}))^* \cap q_2]
\]

\[
\text{□}
\]
\[ \text{by Law 17.8 (spec-trading)} \]
\[ \text{rely } r \cap \text{guar} (g \cap \text{id}_X) \cap \{p\} ; [q_2] \]
\[ \text{by Law 11.3 (guar-merge) in reverse and definition of a frame (2.61)} \]
\[ \text{rely } r \cap \text{guar } g \cap \{p\} ; X : [q_2] \]

**Example 17.10 (loop-body).** The following application of Law 17.9 strengthens a postcondition under the assumption of both the precondition and the reflexive, transitive closure of the rely and guarantee. After the strengthening, the precondition is weakened using (7.2).

\[ \text{rely } \overset{\wedge}{\text{w}} \supseteq \overset{\wedge}{w'} \land \overset{\wedge}{i'} = \overset{\wedge}{i} \cap \text{guar } \overset{\wedge}{w} \supseteq \overset{\wedge}{w'} \land \overset{\wedge}{w} \subseteq \{i\} \cap \{w \subseteq \{0 \ldots N - 1\} \land i \in \{0 \ldots N - 1\} \land k \supseteq \overset{\wedge}{w}\} ; \]
\[ \text{w} : \overset{\wedge}{w'} \subseteq \{0 \ldots N - 1\} \land \overset{\wedge}{i'} \in \{0 \ldots N - 1\} \land (k \supseteq \overset{\wedge}{w'} \lor \overset{\wedge}{i'} \notin \overset{\wedge}{w}) \]
\[ \overset{\wedge}{g} \supseteq \text{rely } \overset{\wedge}{w} \supseteq \overset{\wedge}{w'} \land \overset{\wedge}{w} \subseteq \{i\} \cap \{w \subseteq \{0 \ldots N - 1\} \land i \in \{0 \ldots N - 1\} \land k \supseteq \overset{\wedge}{w}\} ; \]
\[ \text{w} : \overset{\wedge}{w} \supseteq \overset{\wedge}{w'} \land \overset{\wedge}{i'} \notin \overset{\wedge}{w} \]

We have \((r \cup (g \cap \text{id}_W))^* \subseteq \overset{\wedge}{w} \supseteq \overset{\wedge}{w'} \land \overset{\wedge}{i'} = \overset{\wedge}{i}, \) and so it is sufficient to show the following, which is straightforward.

\[ \overset{\wedge}{w} \subseteq \{0 \ldots N - 1\} \land \overset{\wedge}{i} \in \{0 \ldots N - 1\} \land k \supseteq \overset{\wedge}{w} \land \overset{\wedge}{w'} \subseteq \{0 \ldots N - 1\} \land \overset{\wedge}{i'} \in \{0 \ldots N - 1\} \land (k \supseteq \overset{\wedge}{w'} \lor \overset{\wedge}{i'} \notin \overset{\wedge}{w}) \]
\[ \subseteq \overset{\wedge}{w'} \subseteq \{0 \ldots N - 1\} \land \overset{\wedge}{i'} \in \{0 \ldots N - 1\} \land (k \supseteq \overset{\wedge}{w'} \lor \overset{\wedge}{i'} \notin \overset{\wedge}{w}) \]

If a rely ensures that a set of variables \(Y,\) that is not in the frame of a specification, is unchanged, that is sufficient to ensure \(Y\) is unchanged in the postcondition of the specification.

**Law 17.11 (frame-restrict).** For sets of variables \(X, Y,\) and \(Z,\) if \(Z \subseteq X\) and \(Y \subseteq Z\) and \(r \subseteq \text{id}_Y\) then, rely \(r \cap X : [\text{id}_Y \cap q] \supseteq \text{rely } r \cap Z : [q].\)

**Proof.** Because \(Y \subseteq Z,\) \(\text{id}_Z \subseteq \text{id}_Y\) and hence \((r \cup \text{id}_Z)^* \subseteq (\text{id}_Y \cup \text{id}_Y)^* = \text{id}_Y.\)

\[ \text{rely } r \cap X : [\text{id}_Y \cap q] \supseteq \text{by Law 12.2 (frame-reduce) using assumption } Z \subseteq X \]
\[ \text{rely } r \cap Z : [\text{id}_Y \cap q] \supseteq \text{by Law 17.9 (spec-strengthen-with-trading) as } (r \cup \text{id}_Z)^* \cap q \subseteq \text{id}_Y \cap q \]
\[ \text{rely } r \cap Z : [q] \]

The application of Law 17.9 (spec-strengthen-with-trading) uses the implicit guarantee of guar univ (i.e. chaos), noting that univ \(\cap \text{id}_Z = \text{id}_Z.\)

**Example 17.12 (frame-restrict).** The following example refinement applies Law 12.2 (frame-reduce) to the first and third sequentially-composed specifications and Law 17.11 (frame-restrict) to the second to restrict their frames. For the application of Law 17.11
A core law for rely/guarantee concurrency is introducing a parallel composition. The definition allows an assignment with no effect, such as either doing an assignment that assigns to its current value or doing nothing.

\[
\text{Law 18.1 (spec-introduce-par).}
\]

\[
\text{rely } r \ni [q_0 \cap q_1] \succ (\text{rely}(r \cup r_1) \cap \text{guar} r_1 \cap [q_0]) \parallel (\text{rely}(r \cup r_1) \cap \text{guar} r_0 \cap [q_1])
\]

By monotonicity, any preconditions and guarantees can be carried over from the left side of an application of Law 18.1 to the right side and then distributed into the two branches of the parallel.

19. Refining to an (optional) atomic step

The optional atomic step command, \( \text{opt } q \triangleq \pi q \lor \tau(\text{dom}(q \cap \text{id})) \), performs an atomic program step satisfying \( q \), or if \( q \) can be satisfied by not changing the state, it can also do nothing (2.66). The set \( \text{dom}(q \cap \text{id}) \) represents the set of all states from which \( q \) is satisfied by not changing the state, i.e. \( \{ \sigma : (\sigma, \sigma) \in q \} \). The optional atomic step command is used in the definition of an atomic specification command (Sect. 21) and in the definition of an assignment command (Sect. 23) to represent the step that atomically updates the variable. The definition allows an assignment with no effect, such as \( x := x \), to be implemented by either doing an assignment that assigns to \( x \) its current value or doing nothing.

\[
\text{Law 19.1 (opt-strengthen-under-pre). If } p \ll q_2 \ll q_1, \text{ then } \{p\}; \text{opt } q_1 \gg \{p\}; \text{opt } q_2.
\]
Proof.

\[
\{p\} : \text{opt } q_1 \\
= \text{ by definition of opt (2.66); distribution} \\
\{p\} : \pi \; q_1 \lor \{p\} : \tau(\text{dom}(q_1 \cap \text{id})) \\
\geq \text{ by (8.1) and (6.3) as } p \triangleleft q_2 \subseteq q_1 \text{ and } \text{dom}(p \triangleleft q_1 \cap \text{id}) = p \cap \text{dom}(q_1 \cap \text{id}) \\
\{p\} : \pi(p \triangleleft q_2) \lor \{p\} : \tau(p \cap \text{dom}(q_1 \cap \text{id})) \\
= \text{ by (2.26) and (2.23) and (7.7) and (2.66)} \\
\{p\} : \text{opt } q_2
\]

Lemma 19.2 (spec-to-pgm). \([q] \succcurlyeq \pi \; q\)

Proof. Because term \(\succcurlyeq \pi \; q\), using Lemma 16.8 (totally-correct) it is sufficient to show that \(\pi \; q \circ \tau(q(\{\sigma\}\)) \succcurlyeq \tau\{\sigma\} ; \pi \; q\) for all \(\sigma\), which follows directly using Lemma 15.5 (atomic-test-commute).

Lemma 19.3 (spec-to-test). \([q] \succcurlyeq \tau(\text{dom}(q \cap \text{id}))\)

Proof. Because term \(\succcurlyeq \tau \succcurlyeq \tau(\text{dom}(q \cap \text{id}))\), by Lemma 16.8 (totally-correct) it is sufficient to show that \(\tau(\text{dom}(q \cap \text{id}))\) is weakly correct with respect to relation \(q\). That is, for all \(\sigma_0\) it is enough to show:

\[
\tau(\text{dom}(q \cap \text{id})) ; \tau(q(\{\sigma_0\}\)) \succcurlyeq \tau\{\sigma_0\} ; \tau(\text{dom}(q \cap \text{id}))
\]

\(\Leftarrow\) merging tests (2.23) and (6.3)

\[
\text{dom}(q \cap \text{id}) \cap q(\{\sigma_0\}\) \supseteq \text{dom}(q \cap \text{id}) \cap \{\sigma_0\}
\]

\(\Leftarrow\) expanding the definitions of domain and relational image

\[
\{\sigma \cdot (\sigma, \sigma) \in q \land (\sigma_0, \sigma) \in q\} \supseteq \{\sigma \cdot (\sigma, \sigma) \in q \land \sigma_0 = \sigma\}
\]

\(\Leftarrow\) set-theoretical reasoning

\[\text{true}\]

Law 19.4 (spec-to-opt). \([q] \succcurlyeq \text{opt } q\).

Proof. The proof follows from the definition of opt (2.66) by Lemma 19.2 (spec-to-pgm) and Lemma 19.3 (spec-to-test).

A guarantee \(g\) on an optional step satisfying \(q\), strengthens the optional’s relation to satisfy \(g\).

Law 19.5 (guar-opt). If \(g\) is reflexive, \(\text{guar } g \cap \text{opt } q = \text{opt } (g \cap q)\).

Proof. Because \(g\) is reflexive, \(g \cap \text{id} = \text{id}\).

\[
\text{guar } g \cap \text{opt } q
\]

\(=\) from the definition of an optional step (2.66); distribute

\[
(\text{guar } g \cap \pi \; q) \lor (\text{guar } g \cap \tau(\text{dom}(q \cap \text{id})))
\]

\(=\) from Law 11.8 (guar-pgm) and Law 11.7 (guar-test)

\[
\text{guar}(g \cap q) \lor \tau(\text{dom}(q \cap \text{id}))
\]

\(=\) as \(q \cap \text{id} = g \cap q \cap \text{id}\) because \(g\) is reflexive; definition of opt (2.66)

\[
\text{opt}(g \cap q)
\]
Law 19.6 (spec-guar-to-opt). If \( g \) is reflexive, \( \text{guar} \ g \cap x : [q] \succ \text{opt}(\text{id}_x \cap g \cap q) \).

Proof. The proof uses the definition of a frame (2.61), Law 11.3 (guar-merge), Law 19.4 (spec-to-opt) and Law 19.5 (guar-opt) as \( g \) is reflexive.

\[
\text{guar} \ g \cap x : [q] = \text{guar}(\text{id}_x \cap g) \cap [q] \succ \text{guar}(\text{id}_x \cap g) \cap \text{opt} q = \text{opt}(\text{id}_x \cap g \cap q)
\]

\[
\square
\]

20. Handling stuttering steps

The command, \( \text{idle} \triangleq \text{guar} \text{id} \cap \text{term} \), allows only a finite number of stuttering program steps that do not change the state; \( \text{idle} \) does not constrain its environment (2.67). Two \( \text{idle} \) commands in sequence is equivalent to a single \( \text{idle} \).

Lemma 20.1 (seq-idle-idle). \( \text{idle} = \text{idle} ; \text{idle} \)

Proof. Refinement from right to left holds because \( \text{idle} \succ \tau \). For refinement from left to right, the proof makes use of Law 14.1 (seq-term-term) and Law 11.5 (guar-seq-distrib): \( \text{idle} = \text{guar} \text{id} \cap \text{term} = \text{guar} \text{id} \cap \text{term} ; \text{term} \succ (\text{guar} \text{id} \cap \text{term}) ; (\text{guar} \text{id} \cap \text{term}) = \text{idle} ; \text{idle} \).

A reflexive guarantee combined with the \( \text{idle} \) command is \( \text{idle} \).

Lemma 20.2 (guar-idle). If \( g \) is reflexive, \( \text{guar} \ g \cap \text{idle} = \text{idle} \).

Proof. Because \( g \) is reflexive \( g \cap \text{id} = \text{id} \). The proof then follows from (2.67) using Law 11.3 (guar-merge).

\[
\text{guar} g \cap \text{idle} = \text{guar}(g \cap \text{id}) \cap \text{term} = \text{guar} \text{id} \cap \text{term} = \text{idle}
\]

If \( p \) is stable under \( r \) then \( p \) is stable over the command \( \text{rely} \ r \cap \text{idle} \) because it only performs stuttering program steps that do not change the state and the environment steps are assumed to maintain \( p \).

Lemma 20.3 (rely-idle-stable). If \( p \) is stable under \( r \), \( \text{rely} \ r \cap \text{idle} ; \tau p \succ \text{rely} \ r \cap \tau p ; \text{idle} \).

Proof. Note that any property \( p \) is stable under the identity relation \( \text{id} \).

\[
\text{rely} \ r \cap \text{idle} ; \tau p
\]

\[
= \text{by definition of idle (2.67)}
\]

\[
\text{rely} \ r \cap (\text{guar} \text{id} \cap \text{term}) ; \tau p
\]

\[
= \text{by Lemma 10.7 (test-suffix-interchange)}
\]

\[
\text{rely} \ r \cap \text{guar} \text{id} ; \tau p \cap \text{term}
\]

\[
\succ \text{by Lemma 17.6 (guar-test-commute-under-rely) as } p \text{ is stable under both } r \text{ and } \text{id}
\]

\[
\text{rely} \ r \cap \tau p ; \text{guar} \text{id} \cap \text{term}
\]

\[
= \text{by Lemma 10.2 (test-command-sync-command)}
\]

\[
\text{rely} \ r \cap \tau p ; (\text{guar} \text{id} \cap \text{term})
\]

\[
= \text{by definition of idle (2.67)}
\]

\[
\text{rely} \ r \cap \tau p ; \text{idle}
\]

\[
\square
\]

Lemma 20.4 (rely-idle-stable-assert). If \( p \) is stable under \( r \) then, \( \text{rely} \ r \cap \{p\} ; \text{idle} \succ \text{rely} \ r \cap \text{idle} ; \{p\} \).
Proof. The proof introduces a test, \( \tau p \), which establishes \( p \) as an assertion (7.6), then applies Lemma 20.3 (rely-idle-stable), applies (7.7) to elide the test, and finally removes an assertion (7.3): \( \text{rely} \ r \cap \{p\} ; \text{idle} \supseteq \text{rely} \ r \cap \{p\} ; \text{idle} ; \tau \ p ; \{p\} \supseteq \text{rely} \ r \cap \{p\} ; \tau \ p ; \text{idle} ; \{p\} \supseteq \text{rely} \ r \cap \text{idle} ; \{p\} \).

The following lemma is used as part of refining a specification (of a restricted form) to an expression evaluation. The command \( \text{idle} \) refines a specification with postcondition \( r^* \) in a rely context of \( r \). In addition, if \( p \) is stable under \( r \), \( \text{idle} \) maintains \( p \). A special case of the law is if \( p \) is \( \Sigma \), i.e. \( \text{rely} \ r \cap \text{idle} \supseteq \text{rely} \ r \cap \text{idle} \).

**Law 20.5 (rely-idle).** If \( p \) is stable under \( r \), then \( \text{rely} \ r \cap \{p\} ; [r^* \triangleright p] \supseteq \text{rely} \ r \cap \text{idle} \).

*Proof.* All environment steps of the right side are assumed to satisfy \( r \) and all program steps satisfy the identity relation, and hence by Lemma 17.3 (stable-transitive) the right side maintains \( p \) and satisfies \((\text{id} \cup r)^* = r^*\).

\[
\begin{align*}
\text{rely} \ r \cap \{p\} ; [r^* \triangleright p] & \supseteq \text{rely} \ r \cap \{p\} ; [r^* \triangleright p] \supseteq \text{by Law 16.11 (spec-strengthen-under-pre); } r^*[\{p\}] \subseteq p \text{ by Lemma 17.3; (7.3)} \\
\text{rely} \ r \cap [r^*] & \supseteq \text{by the definition of a total-correctness specification command (2.65)} \\
\text{rely} \ r \cap [r^*] \cap \text{term} & \supseteq \text{by Lemma 17.7 (spec-trade-rely-guar) as } (\text{id} \cup r)^* = r^* \\
\text{rely} \ r \cap \text{guar} \ \text{id} \cap \text{term} & \supseteq \text{definition of } \text{idle} \ (2.67) \\
\text{rely} \ r \cap \text{idle} & \quad \blacksquare
\end{align*}
\]

If a specification \( \{p\} ; [q] \) is placed in a context that allows interference satisfying \( r \) before and after it, the overall behaviour may not refine the specification. If the precondition \( p \) holds initially, it must hold after any interference steps satisfying \( r \), i.e. \( p \) must be stable under \( r \). If the specification is preceded by an interference step satisfying \( r \), then a step satisfying \( r \) followed by a sequence of steps that satisfies \( q \) should also satisfy \( q \) – this leads to condition (20.1), which also assumes \( p \) holds initially. Condition (20.2) is similarly required to handle an interference step following the specification.

**Definition 20.6 (tolerates-interference).** Given a set of states \( p \) and relations \( q \) and \( r \), \( q \) tolerates \( r \) from \( p \) if, \( p \) is stable under \( r \) and

\[
\begin{align*}
p \triangleright (r \triangleright q) & \subseteq q \quad (20.1) \\
p \triangleright (q \triangleright r) & \subseteq q. \quad (20.2)
\end{align*}
\]
Example 20.7 (tolerates). The relation \( \rho_{pw \triangleright w' \lor i' \notin w'} \) tolerates the rely relation \( \rho_{w \triangleright w' \land i' = i \land nw' = nw \land pw' = pw} \) from states in \( \omega_{pw \triangleright w \land nw = pw - \{i\}^\gamma} \) because \( \rho_{pw \triangleright w \land nw = pw - \{i\}^\gamma} \) is stable under the rely and
\[
\rho_{pw \triangleright w' \lor i' \notin w'} \subseteq \rho_{pw \triangleright w \land w \triangleright w' \land i' = i \land nw' = nw \land pw' = pw} \\quad \text{because} \quad \rho_{pw \triangleright w \land nw = pw - \{i\}^\gamma} \\quad \text{is} \quad \text{stable} \quad \text{under} \quad \text{the} \quad \text{rely} \quad \text{and} \quad \text{and} \quad \text{and} \quad \text{and}
\]

and
\[
\rho_{pw \triangleright w \land nw = pw - \{i\} \land (\rho_{pw \triangleright w' \lor i' \notin w'}^\gamma)} \subseteq \rho_{w \triangleright w' \land i' = i \land nw' = nw \land pw' = pw} \quad \text{if} \quad (20.1) \quad \text{and} \quad \text{p is stable under} \quad r
\]

Related work. Definition 17.1 (stable) and (20.1) correspond respectively to conditions PR-ident and RQ-ident used by Coleman and Jones [CJ07, Sect. 3.3], in which \( r \) is assumed to be reflexive and transitive, and condition (20.2) is a slight generalisation of their condition QR-ident because (20.2) includes the restriction to the set \( p \). The conditions are also related to the concept of stability of \( p \) and \( q \) in the sense of Wickerson et al. [WDP10b, WDP10a], although in that work post conditions are treated to single-state predicates rather than relations.

Lemma 20.8 (tolerates-transitive). If \( q \) tolerates \( r \) from \( p \), then \( p \triangleleft r^* \trianglerighteq q \trianglerighteq r^* \subseteq q \).

Proof. Two auxiliary properties are derived from Definition 20.6 (tolerates-interference).
\[
p \triangleleft r^* \trianglerighteq q \subseteq q \quad \text{if} \quad (20.1) \quad \text{and} \quad \text{p is stable under} \quad r
\]
\[
p \triangleleft q \trianglerighteq r^* \subseteq q \quad \text{if} \quad (20.2)
\]
Properties (20.3) and (20.4) follow by Lemma 17.4 (interference-before) and Lemma 17.5 (interference-after), respectively. The proof of the main theorem is straightforward using (20.3) and then (20.4).
\[
p \triangleleft r^* \trianglerighteq q \trianglerighteq r^* \subseteq p \triangleleft q \trianglerighteq r^* \subseteq q
\]

Assuming the environment only performs steps satisfying \( r \), a specification that tolerates \( r \) can tolerate idle commands before and after it.

Law 20.9 (tolerate-interference). If \( q \) tolerates \( r \) from \( p \), then,
\[
r \triangleleft \text{rel} \cap \{p\}; [q] = r \triangleleft \text{rel} \cap \text{idle}; \{p\}; [q]; \text{idle}.
\]
Proof. The refinement from right to left follows as \( \text{idle} \triangleright \tau \), and the refinement from left to right holds as follows.

\[
\begin{align*}
\text{rely } r \sqcap \{p\} ; [q] & \triangleright \text{by Law 16.11 (spec-strengthen-under-pre) using Lemma 20.8 (tolerates-transitive)} \\
\text{rely } r \sqcap \{p\} ; [r^* \triangleright q \triangleright r^*] & \triangleright \text{by Law 16.15 (spec-to-sequential) twice} \\
\text{rely } r \sqcap \{p\} ; [r^*] ; [q] ; [r^*] & \triangleright \text{by Law 16.4 (spec-strengthen) and Lemma 16.13 (spec-assert-restricts)} \\
\text{rely } r \sqcap \text{idle} ; \{p\} ; [q] ; \text{idle} & \triangleright \text{by Law 13.5 (rely-refine-within); Law 20.5 (rely-idle) twice with } \Sigma \text{ for } p \text{ in second}
\end{align*}
\]

The command \text{idle} plays a significant role in the definition of expressions because every program step of an expression evaluation does not change the observable state. Lemma 20.13 (idle-test-idle) below plays a crucial role in Lemma 22.8 (eval-single-reference), which is the main lemma used for handling expressions (including boolean conditions). Lemma 20.10 (par-idle-idle) and Lemma 20.12 (test-par-idle) are used in the proof of Lemma 20.13 (idle-test-idle).

Lemma 20.10 (par-idle-idle). \( \text{idle} \parallel \text{idle} = \text{idle} \)

Proof. From \( \text{idle} \triangleright \text{skip} \) and monotonicity of parallel we have, \( \text{idle} \parallel \text{idle} \triangleright \text{idle} \parallel \text{skip} = \text{idle} \). For refinement in the other direction we show

\[
\begin{align*}
\text{idle} & = \text{definition of idle (2.67)} \\
\text{guar id} \sqcap \text{term} & = \text{by Law 14.2 (par-term-term)} \\
\text{guar id} \sqcap (\text{term} \parallel \text{term}) & \triangleright \text{by Law 11.6 (guar-par-distrib)} \\
(\text{guar id} \sqcap \text{term}) \parallel (\text{guar id} \sqcap \text{term}) & = \text{definition of idle (2.67)} \\
\text{idle} \parallel \text{idle} & \quad \square
\end{align*}
\]

Lemma 20.11 (idle-expanded). \( \text{idle} = (\pi \text{id} \lor \epsilon)^* ; \epsilon^\omega \)

Proof. From the definitions of \text{idle} (2.67), a guarantee (2.60), and \text{term} (2.63), using (9.5). \( \square \)

Finite stuttering either side of a test is equivalent to finite stuttering in parallel; the skips in the following lemma allow for environment steps corresponding to the parallel \text{idle} command.

Lemma 20.12 (test-par-idle). \( \text{idle} ; t ; \text{idle} = \text{skip} ; t ; \text{skip} \parallel \text{idle} \)
Proof.  
\[ \text{skip} ; t ; \text{skip} \parallel \text{idle} \]
\[ = \text{by the definition of skip (2.58) and Lemma 20.11 (idle-expanded)} \]
\[ e^\omega ; t ; e^\omega \parallel (\pi \text{id} \lor e)^* ; e^\omega \]
\[ = \text{by (9.5) as } e \parallel (\pi \text{id} \lor e) = (\pi \text{id} \lor e)^* \text{, and using Lemma 20.11 (idle-expanded)} \]
\[ (\pi \text{id} \lor e)^* ; ((((e^\omega ; t ; e^\omega) \parallel (t ; e^\omega \parallel \text{idle})) \]
\[ = \text{by Lemma 10.2 (test-command-sync-command)} ; e^\omega \text{ is the identity of parallel (2.39)} \]
\[ (\pi \text{id} \lor e)^* ; (e^\omega ; t ; e^\omega \lor e^* ; t ; \text{idle}) \]
\[ = \text{by Lemma 5.1, } (\pi \text{id} \lor e)^* = (\pi \text{id} \lor e)^* ; e^* \text{, distributivity (4.2), and } e^* ; e^\omega = e^\omega \]
\[ (\pi \text{id} \lor e)^* ; (e^\omega ; t ; e^\omega \lor e^* ; t ; \text{idle}) \]
\[ = \text{by } c^\omega ; d = e^* ; d \lor c^\infty \text{ by (5.8)} \]
\[ (\pi \text{id} \lor e)^* ; (e^* ; t ; e^\omega \lor c^\infty \lor e^* ; t ; \text{idle}) \]
\[ = \text{using } c^\omega ; d = e^* ; d \lor c^\infty (5.8) \]
\[ (\pi \text{id} \lor e)^* ; (e^* ; t ; e^\omega \lor e^* ; t ; \text{idle}) \]
\[ = \text{using } e^\omega \triangleright e^* \text{ and } \text{idle} \triangleright e^\omega \text{ and monotonicity to eliminate the first choice} \]
\[ (\pi \text{id} \lor e)^* ; (e^\omega ; t ; \text{idle}) \]
\[ = \text{Lemma 20.11 (idle-expanded)} \]
\[ \text{idle} ; t ; \text{idle} \]

Lemma 20.13 (idle-test-idle).  \( \text{idle} ; t ; \text{idle} \parallel \text{idle} = \text{idle} ; t ; \text{idle} \)

Proof. The proof uses Lemma 20.12 (test-par-idle), Lemma 20.10 (par-idle-idle) and Lemma 20.12 again.
\[ \text{idle} ; t ; \text{idle} \parallel \text{idle} = \text{skip} ; t ; \text{skip} \parallel \text{idle} = \text{skip} ; t ; \text{skip} \parallel \text{idle} = \text{idle} ; t ; \text{idle} \]

21. Atomic specification commands

The atomic specification command, \( \langle p, q \rangle \triangleq \text{idle} ; \{ p \} \text{; opt } q ; \text{idle} \), performs a single atomic program step or test satisfying \( q \) under the assumption that \( p \) holds in the state in which the step occurs; it allows finite stuttering before and after the step and does not constrain its environment (2.68). The default precondition is the set of all states so that \( \langle q \rangle \triangleq \langle \Sigma, q \rangle \) (2.69).

Example 21.1 (CAS). Below is an atomic specification of a compare-and-swap (CAS) machine instruction.\textsuperscript{10} The local variable \( pw \) represents the previously sampled value of \( w \) and local variable \( nw \) represents the value \( w \) is to be updated to, provided \( w \) still has the value \( pw \), otherwise \( w \) is left unchanged. Both \( pw \) and \( nw \) are intended to be local variables.

\[ \text{CAS} \triangleq w : \langle \tau (w = pw \Rightarrow w' = nw) \land (w \neq pw \Rightarrow w' = w) \rangle \quad (21.1) \]

\textsuperscript{10}CAS instructions typically have an additional local boolean variable, \textit{done}, that returns whether the update succeeded or not. That is not needed for the example used here but is trivial to add to the specification.
Related work. An atomic specification command can also be used to specify atomic operations on a data structure, as used by Dingel [Din02]. In Dingel’s work the semantics of his language considers two commands the same if they are equivalent modulo finite stuttering, whereas our definition (2.68) does not use such an equivalence but builds the stuttering into the atomic specification directly using idle commands. Note that in order for \( \langle p, q \rangle \) to be closed under finite stuttering it is defined in terms of opt \( q \) rather than \( \pi q \) because, for example, \( \pi id \) requires a single stuttering step whereas opt \( id \) allows either a single stuttering step or no steps.

The following two laws follow from the definition of an atomic specification command (2.68), (7.2) and Law 19.1 (opt-strengthen-under-pre).

**Law 21.2** (atomic-spec-weaken-pre). If \( p_0 \subseteq p_1 \) then, \( \langle p_0, q \rangle \succeq \langle p_1, q \rangle \).

**Law 21.3** (atomic-spec-strengthen-post). If \( p \prec q_2 \subseteq q_1 \) then, \( \langle p, q_1 \rangle \succeq \langle p, q_2 \rangle \).

A reflexive guarantee on an atomic specification requires the specification to satisfy the guarantee.

**Law 21.4** (atomic-guar). If \( g \) is a reflexive relation, \( \text{guar} \ g \cap \langle p, q \rangle \succeq \langle p, g \cap q \rangle \).

Proof.

\[
\text{guar} \ g \cap \langle p, q \rangle = \text{definition of atomic specification (2.68)} \\
= \text{guar} \ g \cap \text{idle} ; \{ p \} ; \text{opt} q ; \text{idle} \\
\succeq \text{Law 11.5 (guar-seq-distrib), Lemma 20.2 (guar-idle) and Law 11.9 (guar-assert)} \\
= \text{idle} ; \{ p \} ; (\text{guar} g \cap \text{opt} q) ; \text{idle} \\
= \text{by Law 19.5 (guar-opt) as } g \text{ is reflexive} \\
= \text{idle} ; \{ p \} ; \text{opt} (g \cap q) ; \text{idle} \\
= \text{definition of atomic specification (2.68)} \\
\langle p, g \cap q \rangle
\]

A specification can be refined to an atomic specification that must also satisfy any guarantee.

**Law 21.5** (atomic-spec-introduce). If \( g \) is reflexive, and \( q \) tolerates \( r \) from \( p \) then,

\[
\text{rely} \ r \cap \text{guar} \ g \cap \{ p \} ; [q] \succeq \text{rely} \ r \cap \langle p, g \cap q \rangle.
\]

Proof.

\[
\text{rely} \ r \cap \text{guar} \ g \cap \{ p \} ; [q] = \text{by Law 20.9 (tolerate-interference) as } q \text{ tolerates } r \text{ from } p \\
= \text{rely} \ r \cap \text{guar} \ g \cap \text{idle} ; \{ p \} ; [q] ; \text{idle} \\
\succeq \text{by Law 19.4 (spec-to-opt) and definition of an atomic specification (2.68)} \\
= \text{rely} \ r \cap \text{guar} \ g \cap \langle p, q \rangle \\
\succeq \text{by Law 21.4 (atomic-guar) as } g \text{ is reflexive} \\
= \text{rely} \ r \cap \langle p, g \cap q \rangle
\]
Example 21.6 (intro-CAS). Law 21.5 (atomic-spec-introduce) allows a specification to be replaced by an atomic specification, after strengthening the postcondition (with trading).

\[
\text{guar} \models w \supseteq w' \land w - w' \subseteq \{i\} \land \text{rely} \models w \supseteq w' \land i' = i \land nw' = nw \land pw' = pw^\gamma \land \\
\{\downarrow pw \supseteq w \land nw = pw - \{i\}, w : [\uparrow pw \supseteq w' \lor i' \notin w']\}
\]

\[
\Rightarrow \text{ replace } i' \notin w' \text{ by } i \notin w' \text{ using Law 17.9 (spec-strengthen-with-trading)}
\]

\[
\text{guar} \models w \supseteq w' \land w - w' \subseteq \{i\} \land \text{rely} \models w \supseteq w' \land i' = i \land nw' = nw \land pw' = pw^\gamma \land \\
\{\downarrow pw \supseteq w \land nw = pw - \{i\}, w : [\uparrow pw \supseteq w' \lor i \notin w']\}
\]

\[
\Rightarrow \text{ by Law 21.5 (atomic-spec-introduce)}
\]

\[
\text{rely} \models w \supseteq w' \land i' = i \land nw' = nw \land pw' = pw^\gamma \land \\
w : [\downarrow pw \supseteq w \land nw = pw - \{i\}, \uparrow w \supseteq w' \land w - w' \subseteq \{i\} \land (pw \supseteq w' \lor i \notin w')] \tag{21.2}
\]

The law requires that the guarantee is reflexive (which is trivial) and that \(\uparrow pw \supseteq w' \lor i' \notin w'^{\gamma}\) tolerates \(\uparrow w \supseteq w' \land i' = i \land nw' = nw \land pw' = pw^\gamma\) from \(\downarrow pw \supseteq w \land nw = pw - \{i\},\) as shown in Example 20.7 (tolerates). The atomic step may be refined using Law 21.3 (atomic-spec-strengthen-post), Law 21.2 (atomic-spec-weaken-pre) and Law 13.2 (rely-remove), to a form equivalent to the compare-and-swap (CAS) operation (21.1).

\[
\tag{21.3}
\]

The proof obligation for the application of Law 21.3 can be shown as follows; the weakenings are straightforward.

\[
\downarrow pw \supseteq w \land nw = pw - \{i\}, \uparrow (w = pw \Rightarrow w' = nw) \land (w \neq pw \Rightarrow w' = w)^\gamma
\]

\[
= \uparrow pw \supseteq w \land nw = pw - \{i\} \land (w = pw \Rightarrow w' = nw) \land (w \neq pw \Rightarrow w' = w)^\gamma
\]

\[
\subseteq \uparrow w \supseteq w' \land w - w' \subseteq \{i\} \land (pw \supseteq w' \lor i \notin w')^\gamma
\]

22. Expressions under interference

In the context of concurrency, the evaluation of an expression can be affected by interference that modifies shared variables used in the expression. For fine-grained parallelism, normally simple aspects of programs such as expression evaluation in assignments and conditionals are fraught with unexpected dangers, for example, an expression like \(x - x\) is not guaranteed to be zero if the value of \(x\) can be changed by interference between the two accesses to \(x\). In our approach, expression evaluation is not considered to be atomic and programming language expressions are not part of the core language, rather expression evaluation is defined in terms of constructs in the core language. Hence laws for reasoning about expressions (including boolean guards for conditionals) can be proven in terms of the properties of the constructs from which expressions are built.

---

11To allow for all possible implementations of expression evaluation, we allow each reference to a variable in an expression to be fetched from shared memory separately, so that different references to the same variable may have different values. If expression evaluation only accessed each variable once, no matter how many times it appears within an expression, stronger properties about expression evaluation are possible, such as \(x - x = 0\). See [HBDJ13] for a discussion of different forms of expression evaluators and their relationships.
Related work. Issues such as $x - x$ evaluating to a non-zero value can be avoided by assuming expression evaluation is atomic (as done by Xu et al. [XdRH97], Prensa Nieto [Pre03], Schellhorn et al. [STE+14], Sanán et al. [SZLY21] and Dingel [Din02]) but that leads to a theory that is less suitable for practical programming languages because their implementations do not respect such atomicity constraints.

Sect. 22.1 defines the semantics of expression evaluation under interference that may change the value of variables in the expression. Sect. 22.2 considers invariant expressions that evaluate to the same value before and after interference, and Sect. 22.3 considers the case when the evaluation of an expression is equivalent to evaluating it in one of the states during the execution of the evaluation.

22.1. Expressions. The syntax of expressions, $e$, includes constants ($\kappa$), program variables ($x$), unary operators ($\ominus$) and binary operators ($\oplus$).

$$e ::= \kappa \mid x \mid \ominus e \mid e_1 \oplus e_2$$

(22.1)

First, we give the semantics of expression evaluation in a single state; this corresponds to a side-effect-free expression’s semantics in the context of a sequential program.

**Definition 22.1** (expr-single-state). The notation $e_\sigma$ stands for the value of the expression $e$ in the state $\sigma$. Its definition is the usual inductive definition over the structure of the expression, where $\ominus$ is interpreted as the semantics of the operator $\ominus$ on values and $\oplus$ is interpreted as the semantics of $\oplus$ on values.

$$\kappa_\sigma = \kappa$$

(22.2)

$$x_\sigma = \sigma(x)$$

(22.3)

$$(\ominus e)_\sigma = \ominus e_\sigma$$

(22.4)

$$(e_1 \oplus e_2)_\sigma = e_1_\sigma \oplus e_2_\sigma$$

(22.5)

The command $[e]_k$ represents evaluating the expression $e$ to the value $k$. The evaluation of an expression $e$ to $k$ does not change any variables and may either succeed or fail. If the evaluation succeeds in evaluating $e$ to be $k$, $[e]_k$ terminates but if it fails $[e]_k$ becomes infeasible (but note that it may contribute some stuttering program steps and environment steps before becoming infeasible). Because successful expression evaluation terminates and does not change any variables, an expression evaluation $[e]_k$ refines idle, the command that does a finite number of stuttering program steps. In the definition of $[e]_k$ below these stuttering steps are represented by idle and allow for updates to variables that are not observable, such as machine registers. Expression evaluation is often used in a non-deterministic choice over all possible values for $k$, and hence just one choice of $k$ succeeds for any particular execution. Here expressions are assumed to be well defined; the semantics of Colvin et al. [CHM16] provides a more complete definition that handles undefined expressions like divide by zero. The notation $eq e_1 e_2$ stands for the set of states in which $e_1$ evaluates to the same value as $e_2$ (22.6); the set may be empty.

$$eq e_1 e_2 \triangleq \{ \sigma . e_1_\sigma = e_2_\sigma \}$$

(22.6)

**Definition 22.2** (expr-evaluation). The semantics of expression evaluation in the context of interference, $[e]_k$, is defined inductively over the structure of an expression. A constant $\kappa$ evaluates to a value $k$ if $\kappa = k$ but fails (becomes infeasible) otherwise (22.7). A program variable $x$ is similar but the value of $x$ depends on the state in which $x$ is accessed (22.8), which may not be the initial state; it is assumed that the access to $x$ is atomic. The evaluation $x_\sigma$ of a variable $x$ in state $\sigma$ is the one place in expression evaluation that is dependent on the choice of representation of the state. The unary expression $\ominus e$ evaluates to
if \( e \) evaluates to a value \( k_1 \) such that \( k = \oplus k_1 \) (22.9). The expression \( e_1 \oplus e_2 \) evaluates to \( k \) if there exist values \( k_1 \) and \( k_2 \) such that \( e_1 \) evaluates to \( k_1 \), \( e_2 \) evaluates to \( k_2 \), and \( k = k_1 \oplus k_2 \).

The evaluation of \( e_1 \) and \( e_2 \) can be arbitrarily interleaved and hence the definition represents their evaluation as a parallel composition (22.10).

\[
[k]_k \triangleq \text{idle} ; \tau(\text{eq} k \kappa) ; \text{idle} \quad (22.7)
\]

\[
[x]_k \triangleq \text{idle} ; \tau(\text{eq} k x) ; \text{idle} \quad (22.8)
\]

\[
[\oplus e]_k \triangleq \lor \{[e]_k | k_1 \cdot k = \oplus k_1 \} \quad (22.9)
\]

\[
[e_1 \oplus e_2]_k \triangleq \lor \{[e_1]_k || [e_2]_k | k_1, k_2 \cdot k = k_1 \oplus k_2 \} \quad (22.10)
\]

For a unary operator like absolute value, there may be values of \( k \) for which no value of \( k_1 \) exists, e.g. for \( k = -1 \), there is no value of \( k_1 \) such that \(-1 = \text{abs}(k_1) \) because the absolute value cannot be negative; \([\text{abs}(e)]_k \) is infeasible for such values of \( k \). If \( k \) is a positive integer, such as 5, both 5 = \text{abs}(5) and 5 = \text{abs}(-5) and hence there may be multiple values of \( k_1 \) for a single value of \( k \) in the choice within (22.9). Similarly for binary operators, there may be many pairs of values \( k_1 \) and \( k_2 \) such that \( k = k_1 \oplus k_2 \). Conditional expressions, including conditional “and” and “or”, are not treated here but can be easily defined (see [CHM16]).

**Lemma 22.3 (idle-eval).** For any expression \( e \) and value \( k \), \( \text{idle} \not\triangleright [e]_k \).

**Proof.** The proof is by induction over the structure of expressions (22.1). For the binary case it relies on Lemma 20.10 (par-idle-idle).

**Law 22.4 (guar-eval).** If \( g \) is reflexive, \( \text{guar} g \ominus [e]_k = [e]_k \).

**Proof.** By Lemma 22.3 (idle-eval), \( \text{idle} \triangleright [e]_k \) and hence \( \text{idle} \ominus [e]_k = [e]_k \), therefore using Lemma 20.2 (guar-idle) as \( g \) is reflexive,

\[
\text{guar} g \ominus [e]_k = \text{guar} g \ominus \text{idle} \ominus [e]_k = \text{idle} \ominus [e]_k = [e]_k.
\]

22.2. Expressions that are invariant under a rely. An expression \( e \) is invariant under \( r \) if the evaluation of \( e \) in each of two states related by \( r \) gives the same value.

**Definition 22.5 (invariant-under-rely).** An expression \( e \) is invariant under a relation \( r \) if and only if for all \( \sigma \) and \( \sigma' \), \((\sigma, \sigma') \in r \Rightarrow e(\sigma) = e(\sigma') \).

Obviously, if all variables used in \( e \) are unmodified by the interference \( r \), \( e \) is invariant, but there are other examples for which the expression may be invariant even though the values of its variables are modified by the interference, for example, given integer variables \( x \) and \( y \),

- the absolute value of a variable \( x \), \( \text{abs}(x) \), is invariant under interference that negates \( x \) because \( \text{abs}(-x) = \text{abs}(x) \),
- \( \text{abs}(x) + \text{abs}(y) \) is invariant under interference that may negate either \( x \) or \( y \),
- \( \text{even}(x) \) is invariant under interference that changes \( x \) by a value \( 2 \times k \) for some integer \( k \),
- \( (x \text{ mod } N) \) is invariant under interference that adds \( N \) to \( x \) because \( (x + N) \text{ mod } N = x \text{ mod } N \),
- \( x - x \) is invariant under any interference because each evaluation of \( x - x \) is performed in a single state and hence it evaluates to zero in both states,

\[\text{12}\text{Conditional “and” can be defined in terms of a conditional (Sect. 24): } [e_1 \& \& e_2]_k \triangleq \text{if } e_1 \text{ then } [e_2]_k \text{ else } [\text{false}]_k \text{.}\]
• $x \times 0$ is invariant under any interference because its value does not depend on that of $x$, and
• for an array $A$, $A_i$ is invariant under interference that modifies neither $i$ nor $A_i$, although it may modify other elements within $A$.

**Related work.** Coleman [Col08] and Wickerson et al. [WDP10b] use a stronger syntactic property that requires that no variables used within $e$ are modified; none of the examples above are handled under their definition unless all variables are assumed to be unmodified. Our approach can also handle algorithms in which threads are concurrently accessing separate elements in array, using a rely that ensures the other thread is not modifying the element being accessed but may be modifying other elements. The source of the additional generality of our definition is that it is defined in terms of the semantics of expressions rather than being based on their syntactic form. The stronger assumptions of Coleman and Wickerson et al. are important special cases of our more general properties.

**Lemma 22.6 (invariant-expr-stable).** If an expression $e$ is invariant under $r$, then for any value $k$, $(eq \ k \ e)$ is stable under $r$.

**Proof.** By Definition 22.5, $(\sigma_0, \sigma) \in r \Rightarrow e_{\sigma_0} = e_\sigma$ and using Definition 17.1 (stable).

$$
r(eq \ k \ e)
= \{ \sigma \cdot \exists \sigma_0 \cdot \sigma_0 \in eq \ k \ e \land (\sigma_0, \sigma) \in r \}
\subseteq \{ \sigma \cdot \exists \sigma_0 \cdot k = e_{\sigma_0} \land e_{\sigma_0} = e_\sigma \}
= \{ \sigma \cdot k = e_\sigma \}
= eq \ k \ e
$$

22.3. **Single-reference expressions.** Evaluating an expression in the context of interference may lead to anomalies because evaluation of an expression such as $x + x$ may retrieve different values of $x$ for each of its occurrences and hence it is possible for $x + x$ to evaluate to an odd value even though $x$ is an integer variable. However, $2 \times x$ always evaluates to an even value, even if $x$ is subject to modification. While the expression $x - x$ is invariant under any interference $r$ (because evaluating it in any single state always gives 0), its evaluation under interference that modifies $x$ may use different values of $x$ from different states and hence may give a non-zero answer. This means that normal algebraic identities like $x + x = 2 \times x$ and $x - x = 0$ are no longer valid. In fact, these equalities become refinements.\(^{13}\) $[x + x]_k \triangleright [2 \times x]_k$ and $[x - x]_k \triangleright [0]_k$. Such anomalies may be reduced if we restrict our attention to expressions that are single reference under a rely condition $r$ because the evaluation of a single-reference expression under interference $r$ is equivalent to calculating its value in one of the states during its evaluation, as is shown in Lemma 22.8 (eval-single-reference) below.

**Definition 22.7 (single-reference-under-rely).** An expression $e$ is single reference under a relation $r$ iff $e$ is
• a constant $\kappa$, or
• a program variable $x$ and access to $x$ is atomic, or
• a unary expression $\ominus e_1$ and $e_1$ is single reference under $r$, or

\(^{13}\)Hence one can define a notion of refinement between expressions $e_1$ and $e_2$ as $\forall k \cdot [e_1]_k \triangleright [e_2]_k$.  

• a binary expression $e_1 \oplus e_2$ and both $e_1$ and $e_2$ are single-reference under $r$, and at least one of $e_1$ and $e_2$ is invariant under $r$.

Under this definition, the expression $abs(x) + y$ is single-reference under interference that negates $x$ because both $abs(x)$ and $y$ are single-reference expressions and $abs(x)$ is invariant under interference that negates $x$. Note that an expression being invariant under $r$ does not imply it is single-reference under $r$, e.g. $x - x$ is invariant under any rely but it is not single-reference under a rely that allows $x$ to change. Note that by our definition, the expression $0 \ast (x + x)$ is not single-reference under a rely that allows $x$ to change (because $x + x$ is not single reference) but $0 \ast (x + x)$ can be shown to be equivalent to the expression $0$, which is single reference under any rely.\(^{14}\)

**Related work.** Coleman [Col08] and Wickerson et al. [WDP10b] use a stronger *single unstable variable* property that requires at most one variable, $x$, within $e$ is modified by the interference and $x$ is only referenced once in $e$. For example, $abs(x) + y$ does not satisfy their single unstable variable property under interference that negates $x$. Overall this gives us more general laws about single-reference expressions, which are used to handle expression evaluation within assignments (Sect. 23), conditionals (Sect. 24) and loops (Sect. 26).

If an expression is single-reference under $r$, then in a context in which all environment steps are assumed to satisfy $r$, its evaluation is equivalent to its evaluation in the single state in which the single-reference variable is accessed. Evaluating expression $e$ to the value $k$ in a single state can be represented by the test $\tau(eq\ k\ e)$, leading to the following fundamental law that is used in the proofs of laws for programming language constructs involving single-reference expressions.

**Lemma 22.8** (eval-single-reference). If $e$ is a single-reference expression under $r$, and $k$ is a value,

$$\text{rly } r \cap \text{idle} \ ; \ \tau(eq\ k\ e) \ ; \ \text{idle} \ \Rightarrow \ [e]_k.$$  \hspace{1cm} (22.11)

**Proof.** If $r$ is not reflexive, weaken $r$ to $r \cup \text{id}$ using Law 13.1 (rly-weaken). The remainder of the proof assumes $r$ is reflexive. The proof is by induction over the structure of the expression (22.1). If the expression $e$ is a constant $\kappa$ or a program variable $x$, $[e]_k = [\text{idle} \ ; \ \tau(eq\ k\ e) \ ; \ \text{idle}]$ and (22.11) holds using Law 13.2 (rly-remove). If the expression $e$ is of the form $\ominus e_1$ for some expression $e_1$, then because $e$ is single-reference under $r$, so is $e_1$, and hence the inductive hypothesis is: $\text{rly } r \cap \text{idle} \ ; \ \tau(eq\ k\ e_1) \ ; \ \text{idle} \ \Rightarrow \ [e_1]_{k_1}$, for all $k_1$. Hence

$\text{rly } r \cap \text{idle} \ ; \ \tau(eq\ k\ (\ominus e_1)) \ ; \ \text{idle} \ \Rightarrow \ [\ominus e_1]_k$

$\Leftrightarrow$ by the definition of evaluating a unary expression (22.9)

$\text{rly } r \cap \text{idle} \ ; \ \tau(eq\ k\ (\ominus e_1)) \ ; \ \text{idle} \ \Rightarrow \ \sqrt{[e_1]_{k_1} \ | \ k_1 . \ k = \ominus k_1}$

$\Leftrightarrow$ by Lemma 3.1 (refine-choice)

$\forall k_1 . \ k = \ominus k_1 \Rightarrow \text{rly } r \cap \text{idle} \ ; \ \tau(eq\ k\ (\ominus e_1)) \ ; \ \text{idle} \ \Rightarrow \ [e_1]_{k_1}$

$\Leftrightarrow$ as $k = \ominus k_1$ implies $\tau(eq\ k\ (\ominus e_1)) = \tau(eq\ (\ominus k_1) (\ominus e_1)) \Rightarrow \tau(eq\ k_1\ e_1)$

$\forall k_1 . \ \text{rly } r \cap \text{idle} \ ; \ \tau(eq\ k_1\ e_1) \ ; \ \text{idle} \ \Rightarrow \ [e_1]_{k_1}$

---

\(^{14}\)To handle this case the definition of a single reference expression could allow an alternative for binary operators of the form: $e_1$ is single reference and $\forall \sigma, v, v' . \ e_1 \circledast v = e_1 \circledast v'$. For the example $0 \ast (x + x)$, the expression $0$ is trivially single reference and $0 \ast v = 0 \ast v'$ for all values $v$ and $v'$. We do not feel such an extension is warranted because expressions such as $0 \ast (x + x)$ are not useful in practice.
which is the inductive assumption. Note that in the reasoning in the last step, multiple values of \( k_1 \) may give the same value of \( k \), so this is not in general an equality, only a refinement. For example, if \( \ominus \) is absolute value, then both the states in which \( e_1 \) evaluates to \( k_1 \) and the states in which \( e_1 \) evaluates to \( -k_1 \) satisfy \( e_1 \ominus k_1 \ominus e_1 \) but only the states in which \( e_1 \) evaluates to \( k_1 \) satisfy \( e_1 \ominus k_1 \ominus e_1 \).

If \( e \) is of the form \( e_1 \ominus e_2 \), then because \( e \) is single reference under \( r \), so are both \( e_1 \) and \( e_2 \), and hence we may assume the following two inductive hypotheses:

\[
\begin{align*}
\text{rely } r \land \text{idle} ; \tau(eq k_1 e_1) ; \text{idle} & \gg [e_1]_{k_1} \quad \text{for all } k_1 \quad (22.12) \\
\text{rely } r \land \text{idle} ; \tau(eq k_2 e_2) ; \text{idle} & \gg [e_2]_{k_2} \quad \text{for all } k_2 \quad (22.13)
\end{align*}
\]

We are required to show

\[
\begin{align*}
\text{rely } r \land \text{idle} ; \tau(eq k (e_1 \ominus e_2)) ; \text{idle} & \gg [e_1 \ominus e_2]_k \\
\iff & \text{by the definition of evaluating a binary expression (22.10)} \\
\text{rely } r \land \text{idle} ; \tau(eq k (e_1 \ominus e_2)) ; \text{idle} & \gg \{ [e_1]_{k_1} \parallel [e_2]_{k_2} \mid k_1, k_2, k = k_1 \ominus k_2 \}
\end{align*}
\]

\[
\begin{align*}
\iff & \text{by Lemma 3.1 (refine-choice)} \\
\forall k_1, k_2, k = k_1 \ominus k_2 \Rightarrow & \text{rely } r \land \text{idle} ; \tau(eq k (e_1 \ominus e_2)) ; \text{idle} \gg [e_1]_{k_1} \parallel [e_2]_{k_2} \\
\iff & \text{by Lemma 20.3 (rely-idle-stable)} \\
\forall k_1, k_2, \text{ rely } r \land \text{idle} ; \tau(eq (eq k_1 e_1) \cap (eq k_2 e_2)) ; \text{idle} \gg [e_1]_{k_1} \parallel [e_2]_{k_2} 
\end{align*}
\]

(22.14)

Let \( t_1 = \tau(eq k_1 e_1) \) and \( t_2 = \tau(eq k_2 e_2) \) and recall that \( \overline{t_1} = \tau \overline{p} \) by (22.22). As \( e \) is assumed to be single reference under \( r \), from Definition 22.7 (single-reference-under-rely) either \( e_1 \) or \( e_2 \) is invariant under \( r \). By symmetry assume \( e_1 \) is invariant under \( r \) and hence by Lemma 22.6 (invariant-expr-stable) both \( t_1 \) and \( \overline{t_1} \) are stable under \( r \). Now we show (22.14).

\[
\begin{align*}
[e_1]_{k_1} \parallel [e_2]_{k_2} \\
\ll & \text{by the inductive hypotheses (22.12) and (22.13)} \\
(\text{rely } r \land \text{idle} ; t_1 ; \text{idle}) \parallel (\text{rely } r \land \text{idle} ; t_2 ; \text{idle}) \\
= & \text{case analysis on } t_1, \text{ using } c = (t_1 \lor \overline{t_1}) ; c = t_1 ; c \lor \overline{t_1} ; c \text{ and } \parallel \text{ commutes} \\
(\text{rely } r \land \text{idle} ; t_1 ; \text{idle}) \parallel (\text{rely } r \land \text{idle} ; t_1 ; \text{idle}) \lor \\
\overline{t_1} ; (\text{rely } r \land \text{idle} ; t_1 ; \text{idle}) \parallel (\text{rely } r \land \text{idle} ; t_2 ; \text{idle}) \\
\ll & \text{by assumption } t_1 \text{ and } \overline{t_1} \text{ are stable under } r \text{ and Lemma 20.3 (rely-idle-stable)} \\
(\text{rely } r \land t_1 ; \text{idle} ; t_2 ; \text{idle}) \parallel (\text{rely } r \land \text{idle}) \lor \\
(\text{rely } r \land \overline{t_1} ; \text{idle} ; t_1 ; \text{idle}) \parallel (\text{rely } r \land \text{idle}) \\
\ll & \text{by assumption } t_1 \text{ and } \overline{t_1} \text{ are stable under } r \text{ and Lemma 20.3 (rely-idle-stable)} \\
(\text{rely } r \land \text{idle} ; t_1 ; t_2 ; \text{idle}) \parallel (\text{rely } r \land \text{idle}) \lor \\
(\text{rely } r \land \text{idle} ; \overline{t_1} ; t_1 ; \text{idle}) \parallel (\text{rely } r \land \text{idle})
\end{align*}
\]
\begin{align*}
= \text{using } t_1 = t_1 = \text{magic } & \preceq t_1; t_2 \text{ and monotonicity} \\
& (\text{rely } r \cap \text{idle }; t_1; t_2; \text{idle}) \parallel (\text{rely } r \cap \text{idle}) \\
= \text{as idle and tests guarantee id} \\
& (\text{rely } r \cap \text{guar } id \cap (\text{idle }; t_1; t_2; \text{idle})) \parallel (\text{rely } r \cap \text{guar } id \cap \text{idle}) \\
\cong \text{by Law 13.8 (rely-par-distrib) and Law 11.1 as } id \subseteq r \text{ as } r \text{ is reflexive} \\
& \text{rely } r \cap ((\text{id}; t_1; t_2; \text{idle}) \parallel \text{idle}) \\
= \text{expanding abbreviations of tests } t_1 \text{ and } t_2 \text{ and merging the tests (2.23)} \\
& \text{rely } r \cap ((\text{id}; (\tau((eq k 1 e 1) \cap (eq k 2 e 2))); \text{idle}) \parallel \text{idle}) \\
= \text{Lemma 20.13 (idle-test-idle)} \\
& \text{rely } r \cap ((\text{id}; (\tau((eq k 1 e 1) \cap (eq k 2 e 2))); \text{idle}) \parallel \text{idle}) \\
\end{align*}

The following lemma allows an expression evaluation (e.g. within an assignment or in guards of conditionals and loops) to be introduced from a specification.

**Law 22.9 (rely-eval).** For a value \( k \), expression \( e \), set of states \( p \), and relations \( r \) and \( q \), if \( e \) is single reference under \( r \), \( q \) tolerates \( r \) from \( p \), and \((p \cap eq k e) \prec id \subseteq q\),

\[ \text{rely } r \cap \{p\}; [q] \Rightarrow \{e\}_k. \]

**Proof.**

\begin{align*}
\text{rely } r \cap \{p\}; [q] & \Rightarrow \text{by Law 20.9 (tolerate-interference) as } q \text{ tolerates } r \text{ from } p \\
\text{rely } r \cap \text{idle }; \{p\}; [q]; \text{idle} & \Rightarrow \text{by Law 16.11 using assumption } (p \cap eq k e) \prec id \subseteq q; (7.3) \\
\text{rely } r \cap \text{idle}; [eq k e \prec id]; \text{idle} & \Rightarrow \text{by Lemma 19.3 (spec-to-test) as } \text{dom}((eq k e \prec id) \cap \text{id}) = eq k e \\
\text{rely } r \cap \text{idle}; \tau(eq k e); \text{idle} & \Rightarrow \text{by Lemma 22.8 (eval-single-reference) as } e \text{ is single reference under } \tau(eq k e) \text{ in } \{e\}_k. \\Box
\end{align*}

The following law is useful for handling boolean expressions used in conditionals and while loops.

**Law 22.10 (rely-eval-expr).** For a value \( k \), expression \( e \), sets of states \( p \) and \( p_0 \), and relation \( r \), if \( e \) is single reference under \( r \), \( p \) is stable under \( r \), \( p \cap eq k e \subseteq p_0 \), and \( p_0 \) is stable under \((p \prec r)'\),

\[ \text{rely } r \cap \{p\}; [r^* \triangleright (p \cap p_0)] \Rightarrow [e]_k. \]

**Proof.** Note that because \( p \) is stable under \( r \), \( p_0 \) being stable under \((p \prec r)\) is equivalent to \((p \cap p_0)\) being stable under \( r \). The proof uses Law 22.9 (rely-eval), taking \( q \) to be \( r^* \triangleright (p \cap p_0) \) because this tolerates \( r \) from \( p \), and \((p \cap eq k e) \prec id \subseteq r^* \triangleright (p \cap p_0) \) because \( id \subseteq r^* \) and \( p \cap eq k e \subseteq p \cap p_0 \). \Box
23. Assignments under interference

An assignment (non-atomically) evaluates its expression \( e \) to some value \( k \) and then atomically updates the variable \( x \) to be \( k \), as represented by the relation \( \text{update} \ x \ k \triangleq \text{id}_T \triangleright \ (eq \ x \ k) \) \text{(2.70)}. We repeat its definition \text{(2.71)}:

\[
  x := e \triangleq \bigvee_{k \in \text{Val}} ([e]_k \triangleright \text{opt(} \text{update} \ x \ k \text{)} ; \text{idle}). \tag{23.1}
\]

The non-deterministic choice allows \([e]_k\) to evaluate to any value but only one value succeeds for any particular execution. Interference from the environment may change the values of variables used within \( e \) and hence influence the value of \( k \). The command \text{opt(} \text{update} \ x \ k \text{)} atomically updates \( x \) to be \( k \) but may do nothing if \( x \) is already \( k \), so that assignments like \( x := x \) can be implemented by doing nothing at all. Interference may also change the value of \( x \) after it has been updated. The \text{idle} command at the end allows for both environment steps and any hidden (stuttering) steps in the implementation after the update has been made; hidden (stuttering) steps are also allowed by the definition of expression evaluation.

**Related work.** In terms of a trace semantics in Sect. 2.1 [CHM16], any trace that is equivalent to a trace of \( x := e \) modulo finite stuttering is also a trace of \( x := e \), i.e. definition \text{(23.1)} of \( x := e \) is closed under finite stuttering. This holds because (i) expression evaluation is closed under finite stuttering, (ii) the optional update allows a possible stuttering update step to be eliminated, and (iii) the final \text{idle} command allows stuttering steps after the update. We follow this convention for the definition of all constructs that correspond to executable code. This is in contrast to the usual approach of building finite stuttering into the underlying trace semantics [Bro96, Din02].

A number of approaches [XdRH97, Pre03, WDP10a, SZLY21, STE+14] treat a complete assignment command as a single atomic action, although they allow for interference before and after the atomic action. Such approaches do not provide a realistic model for fine-grained concurrency. Coleman and Jones [CJ07] do provide a fine-grained operational semantics that is closer to the approach used here but the laws they develop are more restrictive.

Consider refining a specification of the form \( \text{rely} \ r \cap \text{guar} \ g \cap \{p\} ; x : [q] \) to an assignment command \( x := e \), where we assume access to \( x \) is atomic, \( e \) is a single-reference expression, and \( g \) is reflexive. In dealing with an assignment to \( x \) we make use of a specification augmented with a frame of \( x \), recalling from the definition of a frame \text{(2.61)} that \( x : c = \text{guar} \text{id}_T \cap c \) and noting that guarantees distribute into other constructs. Figure 4 gives an overview of the execution of \( x := e \), and the constraints on \( q \) and \( g \) that are required to show that the assignment satisfies the specification:

- end-to-end the execution must satisfy \( q \);
- because \( e \) is single-reference under \( r \), the evaluation of \( e \) to some value \( k \) corresponds to evaluating it in one of the states \((\sigma_1)\) during its evaluation;
- the optional program step that atomically updates \( x \) between \( \sigma_2 \) and \( \sigma_3 \) must satisfy \( g \);
- the state after the update \((\sigma_3)\) satisfies \( eq \ k \ x \); and
- all the steps before \( \sigma_2 \) and after \( \sigma_3 \) are either environment steps that satisfy \( r \) or program steps that do not modify any variables and hence any subsequence of these steps satisfies \( r^* \), from which one can deduce that \( \sigma_2 \) is in \( r^*([eq \ k \ e]) \).

Because the assignment is defined in terms of an optional atomic step command, the transition from \( \sigma_2 \) to \( \sigma_3 \) may be elided, i.e. \( \sigma_3 \) is \( \sigma_2 \); in this case \( q \) must be satisfied by any
opt

σ σ σ σ
1 2 3

p eq (k, e)

r∗|eq k e|

idle idle idle

Figure 4: Execution of \( x := e \) assuming that access to \( x \) is atomic and \( e \) is a single-reference expression (noting that \( \sigma_2 \) may be \( \sigma_3 \) if the optional atomic step is instantaneous). The execution is annotated using the assumption that the initial state satisfies precondition \( p \) and that the environment steps satisfy relation \( r \), and it includes the constraints on relation \( q \) and reflexive relation \( g \) that are required to show that the assignment satisfies guarantee \( g \) and postcondition specification \( q \) under those assumptions.

Figure 5: The simplified constraints on relation \( q \) and relation \( g \) that are required to show that \( x := e \) satisfies guarantee \( g \) and postcondition specification \( q \) under precondition \( p \) and rely \( r \), assuming that access to \( x \) is atomic, \( e \) is a single-reference expression, \( g \) is reflexive, \( q \) tolerates \( r \) from \( p \), and \( p \cap eq k e \subseteq p_1 k \).

The sequence of steps satisfying \( r^* \) starting from a state satisfying \( p \), and \( g \) is satisfied because all program steps are stuttering steps and \( g \) is assumed to be reflexive.

If \( q \) is assumed to tolerate \( r \) from \( p \) (Definition 20.6) then in Figure 4 if \( q \) holds between states \( \sigma_2 \) and \( \sigma_3 \), \( q \) also holds between \( \sigma \) and \( \sigma' \). We also have that \( p \) is stable under \( r \) and hence \( p \cap eq k e \) holds in state \( \sigma_1 \). We introduce a set of states \( p_1 k \) parameterised by \( k \), such that \( p_1 k \) is stable under \( r \) and \( p \cap eq k e \subseteq p_1 k \), and hence \( p_1 k \) is established in state \( \sigma_1 \) and because it is stable under \( r \), \( p_1 k \) holds in state \( \sigma_2 \). That allows the constraints on \( q \) and \( g \) in Figure 4 to be simplified to those in Figure 5, and that leads to the following general refinement law to introduce an assignment, from which a number of special case laws are derived.

**Law 23.1 (rely-guar-assign).** Given sets of states \( p \), a set of states \( p_1 k \) parameterised by \( k \), relations \( g \), \( r \) and \( q \), a variable \( x \), and an expression \( e \), if \( g \) is reflexive, \( e \) is single reference
under \( r \), \( q \) tolerates \( r \) from \( p \), and for all \( k \), \( p_1 \) \( k \) is stable under \( r \), and
\[
\begin{align*}
p \cap \text{eq } k \text{ } e & \subseteq p_1 \text{ } k \quad (23.2) \\
p_1 \text{ } k \triangleleft \text{update } x \text{ } k & \subseteq g \cap q \quad (23.3)
\end{align*}
\]
then \( \text{rely } r \cap \text{guar } g \cap \{ p \} ; x : [ q ] \succ x := e \).

**Proof.**
\[
\begin{align*}
\text{rely } r \cap \text{guar } g \cap \{ p \} ; x : [ q ] \\
= \text{duplicate precondition; Law 20.9 (tolerate-interference) as } q \text{ tolerates } r \text{ from } p \\
\text{rely } r \cap \text{guar } g \cap \{ p \} ; \text{idle} ; \{ p \} ; x : [ q ] \text{ ; idle} \\
\succ \text{by Lemma 3.1 (refine-choice) with fresh } k, \text{ Lemma 20.1 (seq-idle-idle) and (6.4)} \\
V_{k \in \text{Val}} (\text{rely } r \cap \text{guar } g \cap \{ p \} ; \text{idle} ; \tau(\text{eq } k \text{ } e) ; \text{idle} ; \{ p \} ; x : [ q ] \text{ ; idle}) \\
\succ \text{by Lemma 20.4 (rely-idle-stable-assert) and (7.6), (7.5) and (7.3)} \\
V_{k \in \text{Val}} (\text{rely } r \cap \text{guar } g \cap \text{idle} ; \tau(\text{eq } k \text{ } e) ; \{ p \cap \text{eq } k \text{ } e \} ; \text{idle} ; x : [ q ] \text{ ; idle}) \\
\succ \text{by Lemma 20.4 (rely-idle-stable-assert) assumption (23.2); } p_1 \text{ } k \text{ stable under } r \\
V_{k \in \text{Val}} (\text{rely } r \cap \text{guar } g \cap \text{idle} ; \tau(\text{eq } k \text{ } e) ; \{ p_1 \text{ } k \} ; x : [ q ] \text{ ; idle}) \\
\succ \text{by Lemma 22.8 (eval-single-reference) as } e \text{ single reference under } r \\
V_{k \in \text{Val}} (\text{rely } r \cap \text{guar } g \cap \text{idle} ; \{ p_1 \text{ } k \} ; x : [ q ] \text{ ; idle}) \\
\succ \text{by Lemma 11.5 (guar-seq-distrib); Law 22.4 (guar-eval); Lemma 20.2 (guar-idle)} \\
V_{k \in \text{Val}} (\text{rely } r \cap \text{idle} ; \{ p_1 \text{ } k \} ; x : [ q ] \text{ ; idle}) \\
\succ \text{by Lemma 10.3 (assert-distrib) and Law 19.6 (spec-guar-to-opt)} \\
V_{k \in \text{Val}} (\text{rely } r \cap \text{idle} ; \{ p_1 \text{ } k \} ; \text{opt(id}_\tau \cap g \cap q) \text{ ; idle}) \\
\succ \text{by Law 19.1 (opt-strengthen-under-pre) and assumption (23.3)} \\
V_{k \in \text{Val}} (\text{rely } r \cap \text{idle} ; \{ p_1 \text{ } k \} ; \text{opt(update } x \text{ } k) \text{ ; idle}) \\
\succ \text{by Law 13.2 (rely-remove), (7.3) and definition of an assignment (2.71)} \\
x := e
\]

If \( e \) is both single reference and invariant under \( r \) then its evaluation is unaffected by inference satisfying \( r \).

**Law 23.2** (local-expr-assign). *Given a set of states \( p \), relations \( g \), \( r \) and \( q \), variable \( x \), and an expression \( e \) that is both single reference and invariant under \( r \), if \( g \) is reflexive, \( q \) tolerates \( r \) from \( p \), and for all \( k \), \( p \cap \text{eq } k \text{ } e \triangleleft \text{update } x \text{ } k \subseteq g \cap q \),
\[
\text{rely } r \cap \text{guar } g \cap \{ p \} ; x : [ q ] \succ x := e.
\]

**Proof.** The proof uses Law 23.1 (rely-guar-assign) taking \( p_1 \text{ } k \) to be \( p \cap \text{eq } k \text{ } e \), which is stable under \( r \) by Lemma 22.6 (invariant-expr-stable) because \( e \) is invariant under \( r \): \( r(p \cap \text{eq } k \text{ } e) \subseteq r(p) \cap r(\text{eq } k \text{ } e) \subseteq p \cap \text{eq } k \text{ } e \).
Example 23.3 (assign-nw). Law 23.2 is applied to refine a rely/guarantee specification to an assignment involving variables that are not subject to any interference.

\[ \text{guar} \uparrow w \supseteq w' \wedge w - w' \subseteq \{i\} \implies \text{rely} \uparrow w \supseteq w' \wedge i' = i \wedge \text{nw} = nw \wedge pw' = pw \supseteq \{i\} \wedge pw' \supseteq w'. \]

The provisos of the law hold as follows: the expression \( pw - \{i\} \) is single reference and invariant under the rely; the guarantee is reflexive; the postcondition \( \text{nw} = pw - \{i\} \wedge pw' \supseteq w' \) tolerates the rely from the precondition \( \sqsubseteq pw \supseteq w \); and for all \( k \),

\[ \uparrow pw \supseteq w \wedge k = pw - \{i\} \wedge w' = w \wedge pw' = pw \wedge i' = i \wedge k = \text{nw} \]

\[ \subseteq \uparrow w \supseteq w' \wedge w - w' \subseteq \{i\} \wedge \text{nw} = pw - \{i\} \wedge pw' \supseteq w'. \]

The following law allows the sampling of the value of a single-reference expression \( e \). It assumes that the interference may monotonically decrease \( e \) (or monotonically increase \( e \)) during execution and hence the sampled value (in \( x \)) must be between the initial and final values of \( e \). The notation \( ge e1 e2 \) stands for \( \{\sigma . e1 \sigma \geq e2 \sigma\} \).

Law 23.4 (rely-assign-monotonic). Given a set of states \( p \), relations \( g \) and \( r \), an expression \( e \), and a variable \( x \), if \( g \) is reflexive, \( p \) is stable under \( r \), \( x \) is invariant under \( r \), \( e \) is single-reference under \( r \), and \( \supseteq \) is a reflexive, transitive binary relation, such that for all \( k \),

\[ (p \cap ge k e) \triangleleft \text{update} x k \subseteq g \]  \hspace{1cm} (23.4)

\[ r \subseteq \{(\sigma, \sigma') . e_\sigma \geq e_{\sigma'}\} \]  \hspace{1cm} (23.5)

\[ p \triangleleft id_\sigma \subseteq \{(\sigma, \sigma') . e_\sigma \geq e_{\sigma'}\} \]  \hspace{1cm} (23.6)

then rely \( r \cap \text{guar} g \cap \{p\}; x : \{(\sigma, \sigma') . e_\sigma \geq x_{\sigma'} \geq e_{\sigma'}\} \supseteq x := e. \)

For example, the relation \( \supseteq \) may be \( \geq \) on integers with postcondition \( e_\sigma \geq x_{\sigma'} \geq e_{\sigma'} \), or \( \geq \) may be \( \leq \) on integers with postcondition \( e_\sigma \leq x_{\sigma'} \leq e_{\sigma'} \), or for a set-valued expression, \( \supseteq \) may be \( \supseteq \) with postcondition \( e_\sigma \supseteq x_{\sigma'} \supseteq e_{\sigma'} \).

Proof. In the proof, the idiom, \( \bigvee_j \tau(eq j e) ; c \), can be thought of as introducing a logical variable \( j \) to capture the initial value of \( e \), similar to the construct, let \( j = e \) in \( c \).

\[
\text{rely} \cap \text{guar} g \cap \{p\}; x : \{(\sigma, \sigma') . e_\sigma \geq x_{\sigma'} \geq e_{\sigma'}\} \\
= \text{fresh} j, \bigvee_j \tau(eq j e) = \tau(\bigcup_j eq j e) = \tau \Sigma = \tau \text{ and (7.6)} \\
\bigvee_j \tau(eq j e); (\text{rely} \cap \text{guar} g \cap \{p \cap eq j e\}; x : \{(\sigma, \sigma') . e_\sigma \geq x_{\sigma'} \geq e_{\sigma'}\}) \\
= \text{by Law 16.11 (spec-strengthen-under-pre) as } j = e_\sigma \\
\bigvee_j \tau(eq j e); (\text{rely} \cap \text{guar} g \cap \{p \cap eq j e\}; x : \{(\sigma, \sigma') . j \geq x_{\sigma'} \geq e_{\sigma'}\}) \\
= \text{weaken precondition (7.2) to } p \cap ge j e, \text{ which is stable under } r \\
\bigvee_j \tau(eq j e); (\text{rely} \cap \text{guar} g \cap \{p \cap ge j e\}; x : \{(\sigma, \sigma'). j \geq x_{\sigma'} \geq e_{\sigma'}\}) \\
= \text{by Law 23.1 (rely-guar-assign) – see below} \\
\bigvee_j \tau(eq j e); x := e \\
= \text{as } \bigvee_j \tau(eq j e) = \tau \\
x := e
\]
For the application of Law 23.1 (rely-guar-assign), \( p \) is \( p \cap ge\ j\ e \), \( p_1\ k \) is \( p \cap ge\ j\ k \cap ge\ k\ e \), and \( q \) is \( \{ (\sigma, \sigma') : j \geq x_{\sigma'} \geq e_{\sigma'} \} \). Property \( p_1\ k \) is stable under \( r \) because \( r( p \cap ge\ j\ e \cap ge\ k\ e) \subseteq r( p) \cap r( ge\ j\ e) \cap r( ge\ k\ e) \subseteq p \cap ge\ j\ e \cap ge\ k\ e \) because \( p \), \( ge\ j\ e \) and \( ge\ k\ e \) are stable under \( r \) by (23.5). Post condition \( q \) tolerates \( r \) from \( p \cap ge\ j\ e \) because \( x \) is invariant under \( r \) and (23.5). Property \( p_1\ k \) is established because \( p \cap ge\ j\ e \cap eq\ k\ e \subseteq p \cap ge\ j\ k \cap ge\ k\ e \), which follows by set theory and logic. The left side of (23.3) is contained in \( q \) by (23.4) and it is contained in \( q \) by the following reasoning, which relies upon \( \succeq \) being reflexive and transitive.

\[
(p \cap ge\ j\ k \cap ge\ k\ e) \leq \text{update } x\ k
\]

\[
\subseteq \text{rewriting as a set comprehension and (23.6)}
\]

\[
\{ (\sigma, \sigma') : j \geq k \land k \geq e_{\sigma} \land e_{\sigma} \geq e_{\sigma'} \land k = x_{\sigma'} \}
\]

\[
\subseteq \text{as } j \geq k \land k = x_{\sigma} \Rightarrow j \geq x_{\sigma'} \text{ and } x_{\sigma'} = k \land k \geq e_{\sigma} \land e_{\sigma} \geq e_{\sigma'} \Rightarrow x_{\sigma'} \geq e_{\sigma'},
\]

\[
\{ (\sigma, \sigma') : j \geq x_{\sigma'} \geq e_{\sigma'} \}
\]

\[\Box\]

**Example 23.5 (assign-pw).** Law 23.4 (rely-assign-monotonic) is applied to refine a rely/guarantee specification to an assignment under interference that may remove elements from \( w \).

\[
\text{guar } \underbrace{\gamma w \supseteq w' \land w - w' \subseteq \{ i \} \cap \text{rely } \gamma w \supseteq w' \land pw' = pw' \cap \text{id}_{pw}}_{\text{pw}} \]

\[
\gamma \supseteq pw := w
\]

The provisos of Law 23.4 hold because the guarantee \( \gamma w \supseteq w' \land w - w' \subseteq \{ i \} \cap \gamma \) is reflexive, \( \bot \) is trivially stable under any rely condition, \( pw \) is invariant under the rely \( \gamma w \supseteq w' \land pw' = pw' \cap \text{id}_{pw} \), \( w \) is single-reference under any rely condition because access to \( w \) is atomic, \( \supseteq \) is a reflexive, transitive relation, \( \text{id}_{pw} \) ensures the guarantee \( \gamma w \supseteq w' \land w - w' \subseteq \{ i \} \), and rely \( \gamma w \supseteq w' \land pw' = pw' \cap \text{id}_{pw} \) ensures \( \gamma w \supseteq w' \), as does \( \text{id}_{pw} \).

**Related work.** The assignment law of Coleman and Jones [CJ07] requires that none of the variables in \( e \) and \( x \) are modified by the interference, which is overly restrictive. Wickerson et al. [WDP10a] use an atomic assignment statement and assume the precondition and (single-state) postcondition are stable under the rely condition. Xu et al. [XDRH97], Prensa Nieto [Pre03], Schellhorn et al. [STE14], Sanán et al. [SZLY21] and Dingel [Din02]) also assume assignments are atomic.

The laws developed above make the simplifying assumption that the expression in an assignment is single reference under the rely condition \( r \). That covers the vast majority of cases one needs in practice. Although the underlying theory could be used to develop laws to handle expressions that are not single reference, the cases single reference expressions do not cover can be handled by introducing a sequence of assignments using local variables for intermediate results, such that the expression in each assignment is single reference under \( r \).

## 24. Conditionals

A conditional statement, if \( b \) then \( c \) else \( d \) fi \( \triangleq \) \( ([b]true ; c \lor [b]false ; d) ; idle \lor \bigvee_{k \in \mathbb{B}} ([b]k ; f) \), either evaluates its boolean condition \( b \) to true and executes its “then” branch \( c \), or evaluates \( b \) to false and executes its “else” branch \( d \) (2.72). Because expressions in the language are untyped the definition includes a third alternative that aborts if the guard evaluates to a
The value other than true or false, as represented here by the complement of the set of booleans, \( \mathbb{B} \). The third alternative can also be used to cope with the guard expression being undefined, e.g. it includes a division by zero. The \texttt{idle} in the definition allows steps that do not modify observable state, such as branching within an implementation.

**Law 24.1 (guar-conditional-distrib).** For any reflexive relation \( g \),

\[
\text{guar } g \cap \text{if } b \text{ then } c \text{ else } d \text{ fi } \supseteq \text{if } b \text{ then } (\text{guar } g \cap c) \text{ else } (\text{guar } g \cap d) \text{ fi }.
\]

**Proof.** The proof follows because weak conjunction distributes over non-deterministic choice (2.47) and guarantees distribute over sequential composition by Law 11.5 (guar-seq-distrib). Finally Law 22.4 (guar-eval) and Lemma 20.2 (guar-idle) are applied because expression evaluation and \texttt{idle} guarantee any reflexive guarantee. \( \square \)

To simplify the presentation in this paper, when we use a boolean expression \( b \) in a position in which a set of states is expected, it is taken to mean the corresponding set of states \( \text{eq } b \text{ true} \).

An informal motivation for the form of the law for refining to a conditional is given via the control flow graph at the right (which ignores the case when \( b \) evaluates to a non-boolean). Dashed arcs indicate that interference can occur during the transition, while un-dashed arcs indicate an instantaneous transition. At entry the precondition \( p \) is assumed to hold. The precondition is assumed to be stable under the interference \( r \) while the guard \( b \) is evaluated, and because the guard is assumed to be single reference its value is that in one of the states during its evaluation, call this its evaluation state, \( \sigma_1 \). If \( b \) evaluates to true, \( p \cap b \) holds in the evaluation state but although \( p \) is stable under \( r \), \( b \) may not be. To handle that, it is assumed there is a set of states \( b_t \) that is stable under \( r \) and such that \( p \cap b \subseteq b_t \).

If the guard evaluates to true the “then” branch of the conditional is executed and it establishes the postcondition \( q \) and terminates. The postcondition relation \( q \) is required to tolerate interference \( r \) and hence \( q \) is also established between the initial and final states of the whole conditional. The “else” branch is similar but uses a set of states \( b_f \) such that \( p \cap \overline{b} \subseteq b_f \) and \( b_f \) is stable under \( r \).

Interference from the environment may affect the evaluation of a boolean test \( b \). While each access to a variable within an expression is assumed to be atomic, the overall evaluation of an expression is not assumed to be atomic. Even if \( b \) is single reference under \( r \), it may evaluate to true in the state \( \sigma_1 \) in which the access to the single-reference variable occurs but the interference may then change the state to a new state \( \sigma_2 \) in which \( b \) no longer holds. For example, the boolean expression \( 0 \leq x \) may evaluate to true in \( \sigma_1 \) but if the interference can decrease \( x \) below zero, \( 0 \leq x \) may be false in the later state \( \sigma_2 \).

As a more complex example, consider \( y < x \land y < z \) for \( b \), where interference cannot increase \( x \) and leaves \( y \) and \( z \) unchanged.\(^{15}\) If \( y < x \land y < z \) evaluates to true in \( \sigma_1 \), \( y < z \) will still evaluate to true in state \( \sigma_2 \) after interference (because its variables are not modified) but \( y < x \) may be invalidated (because \( x \) may be decreased so that \( x \leq y \)); hence \( y < z \) can

\(^{15}\)This example boolean expression is similar to one required for Owicki’s example [Owi75] to find the least index in an array that satisfies some property (see [HJ18, p.28]).
be used for $b_t$. The negation of the above example is $y \geq x \lor y \geq z$, which can be used for $b_f$ because it is stable under interference that may only decrease $x$ and not change $y$ and $z$. Note that

$$p = (p \land b) \lor (p \land \overline{b}) \subseteq b_t \cup b_f$$

but there may be states in which both $b_t$ and $b_f$ hold. For the above example, taking $b_t$ as $y < z$ and $b_f$ as $y \geq x \lor y \geq z$, both conditions hold in states satisfying $y < z \land (y \geq x \lor y \geq z)$, i.e. states satisfying $z > y \land y \geq x$.

If the guard of the conditional evaluates to a non-boolean, the conditional aborts. To avoid this possibility the law for introducing a conditional assumes that the precondition $p$ ensures that $b$ evaluates to an element of type $\mathbb{B}$. Using the following definition, the latter is expressed as $p \subseteq \text{type-of}(b, \mathbb{B})$.

**Definition 24.2** (type-of). For expression $e$ and set of values $T$,

$$\text{type-of}(e, T) \triangleq \{ \sigma : e_\sigma \in T \}.$$ 

**Law 24.3** (rely-conditional). For a boolean expression $b$, sets of states $p$, $b_t$ and $b_f$, and relation $q$, if $b$ is single reference under $r$, $q$ tolerates $r$ from $p$, $p \land b \subseteq b_t$, $p \land \overline{b} \subseteq b_f$, $p \subseteq \text{type-of}(b, \mathbb{B})$, and both $b_t$ and $b_f$ are stable under $p \triangleleft r$,

$$\text{rely } r \cap \{ p \}; \ [q] \supseteq \text{if } b \text{ then } (\text{rely } r \cap \{ b_t \cap p \}; \ [q]) \text{ else } (\text{rely } r \cap \{ b_f \cap p \}; \ [q]) \text{ fi}.$$ 

**Proof.** For the application of Law 16.11 (spec-strengthen-under-pre) below, $p \triangleleft r^* \supseteq q \subseteq q$ by (20.3) as $q$ tolerates $r$ from $p$. The proof begins by duplicating the specification as $\lor$ is idempotent.

$$(\text{rely } r \cap \{ p \}; \ [q]) \lor (\text{rely } r \cap \{ p \}; \ [q])$$



\[ \Rightarrow \]

by Law 20.9 (tolerate-interference) as $q$ tolerates $r$ from $p$, and $\text{idle } \supseteq T$

$$(\text{rely } r \cap \{ p \}; \ [q]); \text{idle} \lor (\text{rely } r \cap \{ p \}; \ [q])$$

= by Law 13.4 (rely-seq-distrib) and Law 13.2; Law 16.11 (spec-strengthen-under-pre)

$$(\text{rely } r \cap \{ p \}; \ [r^* \supseteq q]); \text{idle} \lor (\text{rely } r \cap \{ p \}; \ [r^* \supseteq q])$$

= non-deterministic choice is idempotent

$$((\text{rely } r \cap \{ p \}; \ [r^* \supseteq q]) \lor (\text{rely } r \cap \{ p \}; \ [r^* \supseteq q])); \text{idle} \lor (\text{rely } r \cap \{ p \}; \ [r^* \supseteq q])$$

\[ \Rightarrow \]

Law 16.16 (spec-seq-introduce) three times and $\{ \emptyset \} = \emptyset$

$$((\text{rely } r \cap \{ p \}; \ [r^* \triangleright (b_t \cap p)] \cap \{ b_t \cap p \}; \ [q]) \lor$$

$$(\text{rely } r \cap \{ p \}; \ [r^* \triangleright (b_f \cap p)] \cap \{ b_f \cap p \}; \ [q])); \text{idle} \lor$$

$$(\text{rely } r \cap \{ p \}; \ [r^* \triangleright \emptyset]; \emptyset)$$

\[ \Rightarrow \]

by Law 13.5 using Law 22.10 twice and assumptions; see below for third branch

$$([], \text{true}); (\text{rely } r \cap \{ b_t \cap p \}; \ [q])); \lor$$

$$([], \text{false}); (\text{rely } r \cap \{ b_f \cap p \}; \ [q])); \lor$$

$$\lor_{k \in \pi} ([b]_k; \emptyset)$$

= definition of conditional (2.72)

if $b$ then $(\text{rely } r \cap \{ b_t \cap p \}; \ [q])$ else $(\text{rely } r \cap \{ b_f \cap p \}; \ [q])$ fi
The third branch refinement holds as follows.
\[
\text{rely } r \cap \{p\} ; [r^* \triangleright \emptyset] \trianglerighteq \bigvee_{k \in \mathbb{F}} [b]_k \\
\iff \forall k \in \mathbb{F} . \text{rely } r \cap \{p\} ; [\emptyset] \trianglerighteq [b]_k \\
\iff \forall k \in \mathbb{F} . \forall \sigma . \text{eq } k \ b = \emptyset
\]

The latter holds from the assumption \( p \subseteq \text{type of} (b, \mathbb{B}) \) and Definition 24.2 (type-of).

**Related work.** Wickerson et al. [WDP10b] develop a similar rule but instead of \( b_t \) and \( b_f \) they use \( \lceil b \rceil_r \) and \( \lceil \lnot b \rceil_r \), respectively, where they define \( \lceil b \rceil_r \) as the smallest set \( b_t \) such that \( b \subseteq b_t \) and \( b_t \) is stable under \( r \). That corresponds to requiring that \( b_t \) in Law 24.3 (rely-conditional) is the least set containing \( b \) that is stable under \( r \). Law 24.3 (rely-conditional) also takes into account that the precondition \( p \) may also be assumed to hold and hence is more flexible than the rule given by Wickerson et al. As before another difference is that Wickerson et al. use postconditions of a single state, rather than relations.

Coleman [Col08] gives a rule for a simple conditional (with no “else” part). The approach he takes is to split the guard expression \( b \) into \( b_s \wedge b_u \) in which \( b_s \) contains no variables that can be modified by the interference \( r \) and \( b_u \) has a single variable that may be modified by \( r \), and that variable only occurs once in \( b_u \). His conditions are strictly stronger than those used in Law 24.3 (rely-conditional) and hence his rule is not as generally applicable.

Xu et al. [XdRH97], Prensa Nieto [Pre03], Schellhorn et al. [STE+14], Sanán et al. [SZLY21] and Dingel [Din02]) assume guard evaluation is atomic.

### 25. Recursion

This section develops a law, Law 25.2 (well-founded-recursion), to handle recursion using well-founded induction to show termination. It uses a variant expression \( w \) and a well-founded relation \( (\_ \triangleright \_) \), and is applied in Sect. 26 to verify refinement laws for while loops, which are defined there using recursion.

The proof of supporting lemma, Lemma 25.1 below, makes use of well-founded induction, that for a property \( P(k) \) defined on values, can be stated as follows: if \( (\_ \triangleright \_) \) is well founded,
\[
(\forall k . \forall j . k \triangleright j \Rightarrow P(j) \Rightarrow P(k)) \Rightarrow (\forall k . P(k)). \tag{25.1}
\]

The notation \( ge e1 e2 \) abbreviates \( \{\sigma . e1_{\sigma} \supseteq e2_{\sigma}\} \) and \( gt e1 e2 \) abbreviates \( \{\sigma . e1_{\sigma} \supset e2_{\sigma}\} \).

**Lemma 25.1** (well-founded-variant). For a variant expression \( w \), commands \( s \) and \( c \), and a well-founded relation \( (\_ \triangleright \_) \), if for fresh \( k \)
\[
(\forall k . (\{gt k w\}; s \triangleright c) \Rightarrow (\{eq k w\}; s \triangleright c)) \Rightarrow (25.2)
\]

then \( s \triangleright c \).
Proof. The notation $\bigvee_j c_j$ stands for the nondeterministic choice over all $c_j$ such that $k \supset j$. The proof starts from the assumption (25.2).

$\forall k . \{(gt \; k \; w) ; s \succ c\} \Rightarrow (\{eq \; k \; w\} ; s \succ c)$

$\Leftrightarrow$ by Galois connection between tests and assertions (7.4) twice

$\forall k . (s \succ \tau(gt \; k \; w) ; c) \Rightarrow (s \succ \tau(eq \; k \; w) ; c)$

$\Leftrightarrow$ union of tests (2.20) as $gt \; k \; w = \bigcup_j^{k \supset j} eq \; j \; w$

$\forall k . (s \succ \bigvee_j^{k \supset j} (\tau(eq \; j \; w) ; c)) \Rightarrow (s \succ \tau(eq \; k \; w) ; c)$

$\Rightarrow$ by Lemma 3.1 (refine-choice)

$\forall k . (\forall j . k \supset j \Rightarrow (s \succ \tau(eq \; j \; w) ; c)) \Rightarrow (s \succ \tau(eq \; k \; w) ; c)$

$\Rightarrow$ by well-founded induction (25.1) as $(\_ \supset \_)$ is well founded

$\forall k . s \succ \tau(eq \; k \; w) ; c$

$\Rightarrow$ by Lemma 3.1 (refine-choice)

$s \succ \bigvee_k \tau(eq \; k \; w) ; c$

$\Leftrightarrow$ as $k$ is fresh, $\bigvee_k \tau(eq \; k \; w) = \tau(\bigcup_k eq \; k \; w) = \tau \Sigma = \tau$ by (2.20)

$s \succ c$

Law 25.2 (well-founded-recursion) applies Lemma 25.1 for $c$ in the form of the greatest fixed point, $\nu f$, of a monotone function $f$ on commands.

**Law 25.2 (well-founded-recursion).** For a set of states $p$, a variant expression $w$, a command $s$, a well-founded relation $(\_ \supset \_)$, and a monotone function on commands $f$, if

\begin{align*}
\{p\} ; s & \succ \nu f \\
\forall k . \{eq \; k \; w\} ; s & \succ f(\{gt \; k \; w \cup p\} ; s)
\end{align*}

then, $s \succ \nu f$.

The proviso (25.3) is typically used to handle the case in which $p$ holding initially ensures $\nu f$ does not utilise any recursive calls, e.g. taking $f$ to be $\lambda x . \text{if } b \text{ then } (c ; x) \text{ else } \tau \text{ fi}$, it allows one to handle the case in which the loop guard $b$ is guaranteed to evaluate to false. A special case of the law is if $p = \emptyset$, in which case proviso (25.3) holds trivially.

Proof. By Lemma 25.1 (well-founded-variant) to show $s \succ \nu f$, it suffices to show,

$\forall k . (\{gt \; k \; w\} ; s \succ \nu f) \Rightarrow (\{eq \; k \; w\} ; s \succ \nu f)$.

To show this for any $k$, assume $\{gt \; k \; w\} ; s \succ \nu f$ and show

$\{eq \; k \; w\} ; s$

$\succ$ by (25.4)

$f(\{gt \; k \; w \cup p\} ; s)$

$\succ$ by Lemma 7.1 (assert-merge) as $\{gt \; k \; w\} ; s \succ \nu f$ and (25.3); $f$ is monotone

$f(\nu f)$

$= \text{ folding fixed point}$

$\nu f$
Related work. Schellhorn et al. [STE+14] include recursion in their approach. Sanán et al. [SZLY21] allow parameterless procedures and make use of a natural number call depth bound to avoid infinite recursion. The other approaches [CJ07, Din02, Pre03, WDP10a, XdRH97] do not consider recursion, instead they define the semantics of while loops via an operational semantics, as do Sanán et al. [SZLY21].

26. While loops

The definition of a while loop, \( \text{while } b \text{ do } c \text{ od} \), is in terms of a recursion involving a conditional (2.73).\(^{16}\) As usual, a fixed point of the form \( \nu(\lambda x . \text{body}) \) is abbreviated to \( \nu x . \text{body} \). The Hoare logic rule for reasoning about a loop, \( \text{while } b \text{ do } c \text{ od} \), for sequential programs uses an invariant \( p \) that is maintained by the loop body whenever \( b \) holds initially [Hoa69]. To show termination a variant expression \( w \) is used [Gri81]. The loop body must strictly decrease \( w \) according to a well-founded relation \( \succ \) whenever \( b \) holds initially, unless the body establishes the negation of the loop guard. The relation \( \succ \) is assumed to be transitive (otherwise just take its transitive closure instead); its reflexive closure is written \( \succeq \). The notation, \( \text{dec} \succ w \), stands for the relation between states for which \( w \) decreases (i.e. \( \{ (\sigma, \sigma') . w_\sigma \succ w_{\sigma'} \} \)) and \( \text{dec} \succeq w \) stands for the relation between states for which \( w \) decreases or is unchanged (i.e. \( \{ (\sigma, \sigma') . w_\sigma \succeq w_{\sigma'} \} \)).

The law for while loops needs to rule out interference invalidating the loop invariant \( p \) or increasing the variant \( w \). The invariant \( p \) and variant \( w \) must tolerate interference satisfying the rely condition \( r \) and hence \( p \) must be stable under \( r \) while evaluation of the guard \( b \) takes place and because the guard is assumed to be single reference, its value is that in one of the states during its evaluation, call this its evaluation state, \( \sigma_1 \).

\(^{16}\)One known subtlety of fixed points is that the degenerate case of this definition while true do \( \tau \) od \( \succeq (\nu x . x) \). If either the guard of the loop or its body require at least one step the loop is no longer degenerate. This is not an issue in the context of refinement because the only specification that is refined by a degenerate loop is equivalent to \( \bot \).
If $b$ evaluates to false, $p \cap \overline{b}$ holds in the evaluation state $\sigma_1$ but although $p$ is stable under $r$, $\overline{b}$ may not be. To handle that, it is assumed there is a set of states $b_f$ that is stable under $r$ and such that $p \cap \overline{b} \subseteq b_f$. Because the loop terminates when $b$ evaluates to false, the loop establishes the postcondition $p \cap b_f$. If the guard evaluates to true, $p \cap b$ holds in $b$'s evaluation state $\sigma_1$. Again $b$ may not be stable under $r$ and so a set of states $b_t$ that is stable under $r$ is used, where $p \cap b \subseteq b_t$. That set of states is satisfied on entry to the body of the loop and the body is required to re-establish $p$, thus re-establishing the invariant for further iterations of the loop.

The loop body is also required to establish the postcondition $q^*$, which must tolerate $r$ from $p$. The reason for using $q^*$ (instead of $q$) is to allow for the case where it is the environment that reduces the variant and the loop body does nothing (the reflexive case), and the case in which the environment achieves $q$ or $q^*$ while the body is executing and the body also achieves $q$. Of course, if $q$ is reflexive and transitive, $q = q^*$.

To show termination a variant expression $w$ is used. The following version of the while loop rule allows for the body of the loop to not reduce the variant provided it stably establishes the negation of the loop guard. It makes use of an extra set of states $b_x$ that if satisfied on termination of the body of the loop ensures that the loop terminates. Because $b_x$ is stable under $r$, if it holds at the end of the body of the loop, it still holds when the loop condition is evaluated and ensures it evaluates to false. Each iteration of the loop must either establish $b_x$ or reduce $w$ according to a well-founded relation $(\supset \supset)$. The termination argument would not be valid if the environment could increase $w$, so it is assumed the environment maintains the reflexive closure of the ordering, i.e. $p \triangleleft r \subseteq \text{dec}_w$.

**Law 26.1** (rely-loop-early). Given sets of states $p$, $b_1$, $b_f$ and $b_x$, relations $q$ and $r$, reflexive relation $g$, a boolean expression $b$ that is single-reference under $r$, a variant expression $w$ and a relation $(\supset \supset)$ that is well-founded, such that $p \subseteq \text{type}_x(b, B)$, $q^* \triangleright p$ tolerates $r$ from $p$, $b_1$, $b_f$ and $b_x$ are stable under $p \triangleleft r$, and

$$p \triangleleft r \subseteq \text{dec}_w \quad p \cap b \subseteq b_1 \quad p \cap \overline{b} \subseteq b_f \quad p \cap b_x \subseteq \overline{b}$$

then if for all $k$,

$$\text{guar } g \otimes \text{rely } r \otimes \{ b_1 \cap p \cap \text{ge } k \cdot w \} ; [q^* \triangleright (p \cap (\text{gt } k \cdot w \cup b_x))] \succ c \quad (26.1)$$

then, $\text{guar } g \otimes \text{rely } r \otimes \{ p \} ; [q^* \triangleright (p \cap b_f)] \succ \text{while } b \text{ do } c \text{ od}.$

**Proof.** Because a while loop is defined in terms of recursion and a conditional command, the proof makes use of Law 25.2 (well-founded-recursion) and Law 24.3 (rely-conditional). We introduce the following two abbreviations.

$$f \triangleq \lambda x . \text{ if } b \text{ then } (c ; x) \text{ else } \tau \text{ fi}$$

$$s \triangleq \text{guar } g \otimes \text{rely } r \otimes \{ p \} ; [q^* \triangleright (p \cap b_f)]$$
The law can be restated using $s$ and proven as follows.

\[
s \succeq \text{while } b \text{ do } c \text{ od} \\
\iff \text{definitions of a while loop (2.73) and } f \\
s \succeq \nu f \\
\leq \quad \text{by Law 25.2 (well-founded-recursion)} \\
\{b_x\} ; s \succeq \nu f \land \\
\forall k : \{eq \, k \, w\} ; s \succeq \text{if } b \text{ then } (c ; \{gt \, k \, w \cup b_x\} ; s) \text{ else } \tau \text{ fi} \quad (26.2)
\]

The first condition (26.2) holds because if $b_x$ holds initially, from the assumptions $p \cap b_x \subseteq b$ and hence the conditional must take the null else branch. That allows one to choose $b_t$ to be $\emptyset$ and hence allows any command for the “then” part of the conditional so one can choose $c ; \nu f$. The detailed proof of (26.2) follows, starting with expanding the definition of $s$.

\[
\text{guar } g \cap \text{rely } r \cap \{b_x \cap p\} ; [q^* \triangleright (p \cap b_f)] \\
\gg \quad \text{by Law 24.3 with } \emptyset \text{ as its } b_t \text{ and } b_x \text{ as its } b_f; \text{ as } q^* \triangleright (p \cap b_f) \text{ tolerates } r \text{ from } b_x \cap p \\
\text{guar } g \cap \text{if } b \text{ then } (\text{rely } r \cap \{\emptyset \cap p\} ; [q^* \triangleright (p \cap b_f)])) \\
\text{else } (\text{rely } r \cap \{b_x \cap p\} ; [q^* \triangleright (p \cap b_f)]) \text{ fi} \\
\gg \quad \text{by Law 13.2 (rely-remove) twice and Law 24.1 (guar-conditional-distrib)} \\
\text{if } b \text{ then } (\text{guar } g \cap \{\emptyset\} ; [q^* \triangleright (p \cap b_f)]) \text{ else } (\text{guar } g \cap \{b_x \cap p\} ; [q^* \triangleright (p \cap b_f)]) \text{ fi} \\
\gg \quad \text{precondition } \emptyset \text{ allows any refinement; Lemma 19.3 (spec-to-test) as } p \cap b_x \subseteq p \cap b_f \\
\text{if } b \text{ then } c ; \nu f \text{ else } \tau \text{ fi} \\
\gg \quad \text{folding fixed point. i.e. } f(\nu f) = \nu f \\
\nu f
\]

To show the second condition (26.3), for any $k$ consider the following refinement with the definition of $s$ substituted in.

\[
\{eq \, k \, w\} ; (\text{guar } g \cap \text{rely } r \cap \{p\} ; [q^* \triangleright (p \cap b_f)]) \\
\gg \quad \text{by Lemma 10.3 (assert-distrib) and merging preconditions (7.5)} \\
\text{guar } g \cap \text{rely } r \cap \{p \cap eq \, k \, w\} ; [q^* \triangleright (p \cap b_f)] \\
\gg \quad \text{weaken the precondition (7.2) so that it is stable under } r \\
\text{guar } g \cap \text{rely } r \cap \{p \cap ge \, k \, w\} ; [q^* \triangleright (p \cap b_f)] \\
\gg \quad \text{by Law 24.3 (rely-conditional) as } q^* \triangleright (p \cap b_f) \text{ tolerates } r \text{ from } p, \text{ ge } \, k \, w \text{ stable} \\
\text{guar } g \cap \text{if } b \text{ then } (\text{rely } r \cap \{b_t \cap p \cap ge \, k \, w\} ; [q^* \triangleright (p \cap b_f)]) \\
\text{else } (\text{rely } r \cap \{b_f \cap p \cap ge \, k \, w\} ; [q^* \triangleright (p \cap b_f)]) \text{ fi} \\
\gg \quad \text{by Law 24.1 (guar-conditional-distrib) as } g \text{ is reflexive} \\
\text{if } b \text{ then } (\text{guar } g \cap \text{rely } r \cap \{b_t \cap p \cap ge \, k \, w\} ; [q^* \triangleright (p \cap b_f)]) \\
\text{else } (\text{guar } g \cap \text{rely } r \cap \{b_f \cap p \cap ge \, k \, w\} ; [q^* \triangleright (p \cap b_f)]) \text{ fi}
\]

To complete the refinement in (26.3) we need to show both the following.

\[
\text{guar } g \cap \text{rely } r \cap \{b_t \cap p \cap ge \, k \, w\} ; [q^* \triangleright (p \cap b_f)] \gg c ; \{gt \, k \, w \cup b_x\} ; s \quad (26.4) \\
\text{guar } g \cap \text{rely } r \cap \{b_f \cap p \cap ge \, k \, w\} ; [q^* \triangleright (p \cap b_f)] \gg \tau \quad (26.5)
\]
First, (26.4) is shown as follows.
\[
\text{guar } g \land \text{rely } r \land \{ b_i \cap p \cap ge \ k \ w \}; [q^* \triangleright (p \cap b_f)]
\]
\[
\Rightarrow \quad \text{by Law 16.16 (spec-seq-introduce)}
\]
\[
\text{guar } g \land \text{rely } r \land \{ b_i \cap p \cap ge \ k \ w \}; [q^* \triangleright (p \cap (gt \ k \ w \cup b_x))];
\]
\[
\{ p \land (gt \ k \ w \cup b_x) \}; [q^* \triangleright (p \cap b_f)]
\]
\[
\Rightarrow \quad \text{use the assumption (26.1) to refine to } c, \text{ and definition of } s
\]
\[
c \land \{ gt \ k \ w \cup b_x \}; s
\]

Second (26.5) is refined as follows.
\[
\text{guar } g \land \text{rely } r \land \{ b_f \cap p \cap ge \ k \ w \}; [q^* \triangleright (p \cap b_f)]
\]
\[
\Rightarrow \quad \text{using Law 16.11 (spec-strengthen-under-pre); remove the precondition (7.3)}
\]
\[
\text{guar } g \land \text{rely } r \land \{ \text{id} \}
\]
\[
\Rightarrow \quad \text{by Lemma 19.3 (spec-to-test) as } \text{dom(id } \cap \text{id}) = \Sigma \quad \text{and } \tau \Sigma = \tau
\]
\[
\tau
\]

Example 26.2 (while-loop). The following example uses Law 26.1 (rely-loop-early) to introduce a loop that repeatedly attempts to remove an element \(i\) from a set \(w\) under interference that may remove elements from \(w\). It is assumed the implementation uses a compare-and-swap operation that may fail due to interference. For termination, it uses the finite set \(w\) as the variant expression under the superset ordering, which is well founded on finite sets. If the loop body does not succeed in removing \(i\) due to interference, that interference must have removed some element (possibly \(i\)) from \(w\) and hence reduced the variant. The specification (1.4) from Sect. 1 is repeated here.\(^{17}\)
\[
\text{guar } \tau w \supset w' \land w - w' \subseteq \{i\} \land \text{rely } \tau w \supset w' \land i' = i \land
\]
\[
\{ \uparrow w \subseteq \{0 \ldots N - 1\} \land i \in \{0 \ldots N - 1\}; w: [\tau w' \supset w^\gamma]
\]
\[
\Rightarrow \quad \text{while } i \in w \text{ do }
\]
\[
\text{guar } \tau w \supset w' \land w - w' \subseteq \{i\} \land \text{rely } \tau w \supset w' \land i' = i \land
\]
\[
\{ \uparrow w \subseteq \{0 \ldots N - 1\} \land i \in \{0 \ldots N - 1\}; w: [\tau w' \supset w^\gamma \lor i' \notin w^\gamma]
\]
\[
\quad \text{od}
\]

The form of loop introduction rule that includes the negation of the guard as an alternative is sometimes referred to as an early-termination version. In this case the early termination version is essential. For the application of Law 26.1, the invariant \(p\) is \(\uparrow w \subseteq \{0 \ldots N - 1\} \land i \in \{0 \ldots N - 1\};\), the loop test \(b\) is \(\uparrow i \in w ;\), \(b_f\) is \(\uparrow \text{true} ;\) (as interference may remove \(i\) from \(w\)), both \(b_f\) and \(b_x\) are \(\uparrow i \notin w ;\), the postcondition \(q\) is \(\uparrow \text{true} \), and the variant expression is \(w\) under the well-founded superset ordering \(\supset \) (as \(w\) is finite). One subtlety is that it may be the interference, rather than the body of the loop, that removes \(i\) from \(w\) and hence establishes the postcondition; either way the loop terminates with the desired postcondition. Another subtlety is that interference may remove \(i\) just after \(w\) is sampled for the loop guard evaluation but before the loop body begins, and hence the loop body cannot guarantee that \(w\) is decreased either by the program or the interference. For this reason it is essential that the loop body have the (early termination)

\(^{17}\)Here we use a frame of \(w\), rather than having \(i' = i\) in the guarantee as in (1.4).
alternative \( r' \notin w'^\# \) in its postcondition. The condition \( \wedge i \in w \) is single reference under the rely \( \wedge w \supseteq w' \wedge i' = i' \wedge \) because both \( i \) and \( w \) are single reference under the rely and \( i \) is invariant under the rely. It is straightforward that the invariant implies that the type of \( \wedge i \in w \) is boolean, and that \( b_t, b_f \) and \( b_x \) are stable under the rely. Because \( q \) is \( 'true' \), the proviso that \( q^* \supset p \) tolerates \( r \) from \( p \) corresponds to \( r \) being stable under \( r \), which is straightforward. The final proof obligation (26.1) corresponds to showing the following for all \( k \).

\[
\text{guar} \! \wedge w \supseteq w' \wedge w - w' \subseteq \{ i \} \wedge \text{rely} \! \wedge w \supseteq w' \wedge i' = i' \wedge \\
\{ \wedge w \subseteq \{ 0 \ldots N - 1 \} \wedge i \in \{ 0 \ldots N - 1 \} \wedge k \supseteq w \} ; \\
w : [\wedge w \subseteq \{ 0 \ldots N - 1 \} \wedge i' \in \{ 0 \ldots N - 1 \} \wedge ( k \supseteq w' \vee i' \notin w' ) ] \supseteq \text{guar} \! \wedge w \supseteq w' \wedge w - w' \subseteq \{ i \} \wedge \text{rely} \! \wedge w \supseteq w' \wedge i' = i' \wedge \\
\{ \wedge w \subseteq \{ 0 \ldots N - 1 \} \wedge i \in \{ 0 \ldots N - 1 \} \} ; \\
w : [\wedge w \supseteq w' \vee i' \notin w' ]
\]

The refinement holds because the frame of \( w \) combined with the rely \( r' = i' \) implies \( i \) is unmodified, and Law 17.9 (spec-strengthen-with-trading) can be used to complete the proof (see Example 17.10 (loop-body)).

**Law 26.3** (rely-loop). Given set of states \( p, b_t \) and \( b_f \), relations \( q \) and \( r \), reflexive relation \( g \), a variant expression \( w \) and a transitive relation \( ( \supset ) \) that is well-founded, if \( b \) is a boolean expression that is single-reference under \( r \), \( p \subseteq \text{type_of}(b, \mathbb{B}) \), \( q^* \supset p \) tolerates \( r \) from \( p, b_t \) and \( b_f \) are stable under \( p \supset r \), and

\[
p \supset r \subseteq \text{dec}_g w \\
p \cap b \subseteq b_t \\
p \cap b \subseteq b_f
\]

then if for all \( k \),

\[
\text{guar} \! g \wedge \text{rely} \! r \wedge \{ b_t \cap p \cap g k w \} ; [q^* \supset ( p \cap g k w ) ] \supseteq c
\]

then, \( \text{guar} \! g \wedge \text{rely} \! r \wedge \{ p \} ; \left[ ( \text{dec}_g w \cap q^* ) \supset ( p \cap b_f ) \right] \supseteq \text{while } b \text{ do } c \text{ od} \).

**Proof.** The proof follows from Law 26.1 using \( \emptyset \) for \( b_x \) and \( ( \text{dec}_g w \cap q ) \) for \( q \). \( \square \)

**Related work.** The simpler Law 26.3 (rely-loop) is proved using Law 26.1 (rely-loop-early), which handles the “early termination” case when the body does not necessarily reduce the variant but instead a condition that ensures the loop guard is (stably) false is established. For sequential programs, the early termination variant is usually proved in terms of the simpler law by using a more complex variant involving a pair consisting of the loop guard and a normal variant under a lexicographical order [Gri81]; that variant decreases if the body changes the guard from true to false. Interestingly, such an approach is not possible in the case of concurrency because it may be the environment that establishes that the guard is false rather than the loop body, and in that case the body may not decrease the variant pair.

Our use of a variant expression is in line with showing termination for sequential programs in Hoare logic. It differs from the approach used by Coleman and Jones [CJ07], which uses a well-founded relation \( rw \) on the state to show termination in a manner similar to the way a well-founded relation is used to show termination of a while loop for a sequential program in VDM [Jon90]. In order to cope with interference, Coleman and Jones require that the rely condition \( r \) implies the reflexive transitive closure of \( rw \), i.e. \( r \subseteq rw^* \). However, that condition is too restrictive because, if an environment step does not satisfy \( rw^+ \), it must
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not change the state at all. That issue was addressed in [HJC14] by requiring the weaker condition $r \subseteq (rw^+ \cup id_X)$, where $X$ is the set of variables on which the well-foundedness of relation $rw$ depends (i.e. the smallest set of variables, $X$, such that $id_X \not\subseteq rw \not\subseteq id_X \subseteq rw$).

In the approach used here, the requirement on the environment is that it must not increase the variant expression. If the environment does not decrease the value of the variant expression, it must ensure that the variant is unchanged, rather than the complete state of the system is unchanged as required by Coleman and Jones [CJ07]. It is also subtly more general than the approach used in [HJC14] because here we require that the variant is not increased by the environment but allow variables referenced in the variant to change, e.g. if the variant is $(x \mod N)$, the environment may add $N$ to $x$ without changing the value of the variant. The approach using a variant expression is also easier to comprehend compared to that in [HJC14].

In the approach used by Coleman and Jones [CJ07] the postcondition for the loop body $rw$ is required to be transitive and well founded. Well foundedness is required to show termination and hence $rw$ cannot be reflexive. In the version used here, termination is handled using a variant and hence the postcondition of the loop body can be the same as the overall specification, i.e. $q^*$. Because $q \subseteq q^*$, Law 26.1 (rely-loop-early) can be weakened to a law that has a body with a postcondition of $q$.

Wickerson et al. [WDP10a, WDP10b] only consider partial correctness, as do Prensa Nieto [Pre03] and Sanán et al. [SZLY21]. Dingel [Din02] makes use of a natural number valued variant that, like here, cannot be increased by the environment. Unlike the other approaches, Law 26.1 (rely-loop-early) allows for the early termination case, which unlike in the sequential case cannot be proven from the non-early termination variant Law 26.3 (rely-loop) and hence they cannot handle the refinement in Example 26.2 (while-loop).

27. Refinement of removing element from a set

The laws developed in this paper have been used for the refinement of some standard concurrent algorithms in [HJ18] and the reader is referred there for additional examples, including a parallel version of the prime number sieve, which includes an operation to remove an element from a set, for which we present a refinement to code of the example specification (1.4), which we repeat here.

\[
\text{guar} \top w \supseteq w' \land w - w' \subseteq \{i\} \cap \text{rely} \top w \supseteq w' \land i' = i \cap \{w \subseteq \{0 \ldots N-1\} \land i \in \{0 \ldots N-1\}\} ; w : \{i' \not\in w'\} \tag{27.1}
\]

If the machine on which the operation is to be implemented provides an atomic instruction to remove an element from a set (represented as a bitmap) the implementation would be trivial. Here we assume the machine has an atomic compare-and-swap (CAS) instruction. The CAS makes use of one shared variable, $w$, the variable to be updated, and two local variables, $pw$, a sample of the previous value of $w$, and $nw$, the new value which $w$ is to be updated to. The CAS is used by sampling $w$ into $pw$, calculating the new value $nw$ in terms of $pw$, and then executing the CAS to atomically update $w$ to $nw$ if $w$ is still equal to $pw$, otherwise it fails and leaves $w$ unchanged. The CAS has the following specification, repeated from (21.1).

\[
\text{CAS} \triangleq w : \langle \top (w = pw \Rightarrow w' = nw) \land (w \neq pw \Rightarrow w' = w) \rangle \tag{27.2}
\]
Because the execution of a CAS may fail if interference modifies \( w \) between the point at which \( w \) was sampled (into \( pw \)) and the point at which the CAS reads \( w \), a loop is required to repeat the use of the CAS until \( i \not\in w \). The first refinement step is to introduce a loop that terminates when \( i \) has been removed from \( w \) using Law 26.1 (rely-loop-early) – see Example 26.2 (while-loop) for details of the application of the law.

\[
(27.1) \Rightarrow \text{while } i \in w \text{ do}
\]

\[
\begin{align*}
& (\text{guar} w \supseteq w' \land w - w' \subseteq \{ i \} \Rightarrow \text{rely} w \supseteq w' \land i' = i \Rightarrow \\
& \{ \perp w \subseteq \{0 \ldots N - 1\} \land i \in \{0 \ldots N - 1\} \}; w : [w \supseteq w' \lor i' \not\in w']
\end{align*}
\]

\[
\text{od}
\]

At this point we need additional local variables \( pw \) and \( nw \). This paper does not cover local variable introduction laws (see [MH23]). A local variable is handled here by requiring each local variable name to be fresh, adding it to the frame, and assuming it is unchanged in the rely condition.\(^{18}\) The body of the loop can be decomposed into a sequential composition of three steps: sampling \( w \) into \( pw \), calculating \( nw \) as \( pw - \{i\} \), and applying the CAS. The guarantee does not contribute to the refinement steps here, and hence only the refinement of the remaining components is shown.

\[
\text{rely} w \supseteq w' \land i' = i \land nw' = nw \land pw' = pw \Rightarrow
\]

\[
\text{nw, pw, w : } [w \supseteq w' \lor i' \not\in w']
\]

\[
\Rightarrow \text{by Law 16.16 (spec-seq-introduce) twice – see Example 16.17}
\]

\[
\begin{align*}
& \text{rely} w \supseteq w' \land i' = i \land nw' = nw \land pw' = pw \Rightarrow
\end{align*}
\]

\[
\begin{align*}
& \text{nw, pw, w : } [ w \supseteq pw' \land pw' \supseteq w'] \land \{ \perp pw \supseteq w \}; \quad (27.4) \\
& \text{nw, pw, w : } [ nw' = pw - \{i\} \land pw' = pw \land pw' \supseteq w' \land i' = i \Rightarrow \\
& \{ \perp pw \supseteq w \land nw = pw - \{i\} \}; \quad (27.5) \\
& \text{nw, pw, w : } [ pw \supseteq w' \lor i' \not\in w']
\end{align*}
\]

\[
\Rightarrow \text{by Law 17.11 (frame-restrict) thrice – see Example 17.12 (frame-restrict)}
\]

\[
\begin{align*}
& \text{rely} w \supseteq w' \land i' = i \land nw' = nw \land pw' = pw \Rightarrow
\end{align*}
\]

\[
\begin{align*}
& \text{pw : } [w \supseteq pw' \land pw' \supseteq w'] \land \{ \perp pw \supseteq w \}; \quad (27.7) \\
& \text{nw : } [nw' = pw - \{i\} \land pw' \supseteq w' \Rightarrow \\
& \{ \perp pw \supseteq w \land nw = pw - \{i\} \}; \quad (27.8) \\
& \text{w : } [pw \supseteq w' \lor i' \not\in w'] \quad (27.9)
\end{align*}
\]

The postconditions in (27.4), (27.5) and (27.6) were chosen to tolerate interference that may remove elements from \( w \), for example, in (27.4) the value of \( w \) is captured in the local variable \( pw \) but because elements may be removed from \( w \) via interference (27.4) can only ensure that \( w \supseteq pw' \land pw' \supseteq w' \).

\(^{18}\)This is equivalent to treating the local variables as global variables that are not modified by the environment of the program.
The refinement of (27.7) in the guarantee context from (27.3) uses Law 23.4 (rely-assign-monotonic). The guarantee holds trivially as \( w \) is not modified.

\[
\begin{align*}
guar \uparrow w \supseteq w' \land w - w' \subseteq \{i\} \quad &\text{by Law 23.4 (rely-assign-monotonic) – see Example 23.5 (assign-pw)} \\
pw := w
\end{align*}
\]

Specification (27.8) involves an update to a local variable (\( nw \)) to an expression involving only local variables (\( pw \) and \( i \)) and hence its refinement can use a simpler assignment law.

\[
\begin{align*}
guar \uparrow w \supseteq w' \land w - w' \subseteq \{i\} \quad &\text{by Law 23.2 (local-expr-assign) – see Example 23.3 (assign-nw)} \\
nw := pw - \{i\}
\end{align*}
\]

The refinement of (27.9) introduces an atomic specification command using Law 21.5 (atomic-spec-introduce) and then strengthens its postcondition using Law 21.3 (atomic-spec-strengthen-post) and weakens its precondition using Law 21.2 (atomic-spec-weaken-pre) and weakens its rely to convert it to a form corresponding to the definition of a CAS (27.2) — see Example 21.6 (intro-CAS) for details. Note that the atomic step of the CAS satisfies the guarantee, so the guarantee is eliminated as part of the application of Law 21.5 (atomic-spec-introduce).

\[
\begin{align*}
guar \uparrow w \supseteq w' \land w - w' \subseteq \{i\} \quad &\text{CAS} \\
while i \in w \do pw := w \land nw := pw - \{i\} ; \text{CAS od}
\end{align*}
\]

Note that if the CAS succeeds, \( i \) will no longer be in \( w \) and the loop will terminate but, if the CAS fails, \( w \neq pw \) and hence \( pw \supseteq w \) because the precondition of the CAS states that \( pw \supseteq w \) and because \( w \) can only decrease, i.e. the variant \( w \) must have decreased over the body of the loop under the superset ordering. The interference may also have removed \( i \) from \( w \) but that is picked up when the loop guard is tested.

28. ISABELLE/HOL MECHANISATION

This section discusses the formalisation of our theories in the Isabelle/HOL interactive theorem prover [NPW02]. The Isabelle theories consist of two main components: a formalisation of the trace model overviewed in Sect. 2.1 and detailed in [CHM16], and a formalisation of the concurrent refinement algebra presented in this paper, which builds upon earlier work in [HMWC19], which formalised the core rely-guarantee algebra. The current refinement algebra is built up as a hierarchy of theories based on the axiomatisation in Figure 2, where
datatype ('s, 'v) expr =
  Constant "'v" | 
  Variable "'s ⇒ 'v" | 
  UnaryOp "'v ⇒ 'v" "('s,'v) expr" | 
  BinaryOp "'v ⇒ 'v ⇒ 'v" "('s,'v) expr" "('s,'v) expr"

Figure 6: Isabelle datatype definition for expressions, where 's denotes the type of the state space and 'v denotes the type of values in the expression.

each of the axioms has been shown to hold in the semantic model. The refinement laws presented in the paper are proven on the basis of the algebraic theories.19

At a high-level, the Isabelle theories come as a set of locales (proof environments), which have been parameterised by the primitive algebraic operators (sequential composition, weak conjunction, parallel composition, and the lattice operators) and primitive commands (program-step, environment-step, test, and ⌂). The primitive algebraic operators and primitive commands are each assumed to satisfy the axioms presented in Figure 2. They are used as the basis for defining the derived commands in Figure 3, such as relies, guarantees, expressions, and programming constructs. In the theory, all properties – including those of the derived commands – are proved from the axioms of those primitives and lemmas/laws that are built up in a hierarchy of theories.

A consequence of this axiomatic, parameterised theory structure is that any semantic model providing suitable definitions of the primitives can gain access to the full library of theorems by using Isabelle’s instantiate command and dispatching the required axioms.

A new contribution in this paper is a set of laws for reasoning about expressions under interference (Sect. 22). These were also interesting from a mechanisation point of view. The syntax of expressions described in Sect. 22 is encoded as a standard (inductive) Isabelle datatype [BHL+14] with four cases: constants, variable reference, unary expression and binary expression. The unary expression and binary expression cases are recursive in that they themselves accept sub-expressions (see Figure 6). To provide as much generality to the theory as possible, the unary operator (of the Isabelle function type 'v ⇒ 'v) and the binary operator (of type 'v ⇒ 'v ⇒ 'v) within expressions are both parameters to the datatype. Moreover, the theory is polymorphic in the type of values in the expression (allowing the choice of value type to depend on the program refinement), and in the representation of the program state space (which becomes another locale parameter, in the form of a variable getter- and setter-function pair). There is a minimum requirement that the value universe contains values for true and false if using conditional or loop commands. This latter requirement is achieved by requiring the value type to implement an Isabelle type class that provides values for true and false.

The approach of encoding expressions used in this work, wherein the abstract syntax of expressions is explicit, is commonly called deep embedding [ZFF16]. It contrasts with shallow embedding, popular in some other program algebra mechanisations [FZN+19], where one omits to model the abstract syntax of expressions explicitly, and instead uses an in-built type of the theorem prover to represent an expression. For example, an expression is modelled as a function from a state to a value, where the theorem prover’s in-built

---

19 The Isabelle proofs tend to be more detailed than those in this paper; the latter are designed to be more readable.
function type is used. This shallow embedding usually makes it easier to use the expressions in a program refinement, because all the methods and theorems about functions can be used to reason about expressions. However, as appealing as this solution is, it is not satisfactory for our purposes, because the shallow embedding approach does not allow one to analyse the expression substructure. Specifically, we cannot define our expressions as an isomorphism for any function from the program state to a value, because it makes it impossible to unambiguously decompose an expression into the four expression cases: whereas the expressions \( x + x \) and \( 2 * x \) are equivalent if evaluated in a single state \( \sigma \), i.e. \( (x + x)_\sigma = (2 * x)_\sigma \), in the context of interference from concurrent threads, the expressions \( x + x \) and \( x * 2 \) are not equivalent: under concurrent evaluation, \( x + x \) may evaluate to an odd number under interference, while \( x * 2 \) is always even. In our work, expressions obtain their semantics through the evaluation command \( \llbracket e \rrbracket_k \), which turns an expression into a command. In Isabelle, this takes the form of a recursive function on the expression datatype.

29. Conclusions

Our overall goal is to provide mechanised support for deriving concurrent programs from specifications via a set of refinement laws, with the laws being proven with respect to a simple core theory. The rely/guarantee approach of Jones [Jon83b] forms the basis for our approach, but we generalise the approach as well as provide a formal foundation that allows one to prove new refinement laws. Our approach is based on a core concurrent refinement algebra [HMWC19] with a trace-based semantics [CHM16]. The core theory, semantics and refinement laws have all been developed as Isabelle/HOL theories (Sect. 28).

Precondition assertions (Sect. 2), guarantees (Sect. 11), relies (Sect. 13), and partial and total specifications (Sect. 16) are each treated as separate commands in a wide-spectrum language. The commands are defined in terms of our core language primitives allowing straightforward proofs of laws for these constructs in terms of the core theory. Our refinement calculus approach differs from that of Xu et al. [XdRH97], Prensa Nieto [Pre03] and Sanan et al. [SZLY21] whose approaches use Jones-style five-tuples similar to Hoare logic. The latter two also disallow nested parallel compositions — they allow a (multi-way) parallel at the top level only. Our approach also differs from that of Dingel [Din02] who uses a monolithic, four-component (pre, rely, guarantee, and post condition) specification command. Treating the concepts as separate commands allows simpler laws about the individual commands to be developed in isolation. The commands are combined using our base language operators, including its novel weak conjunction operator. That allows more complex laws that involve multiple constructs to be developed and proven within the theory. We support postcondition specifications (Sect. 16), which encode Jones-style postconditions within our theory and for which we have developed a comprehensive theory supporting both partial and total correctness. In addition, we have defined atomic specifications (Sect. 21), which mimics the style of specification used by Dingel [Din02] for specifying abstract operations on concurrent data structures that can be implemented as a single atomic step with stuttering allowed before and after. In practical program refinements, most steps involve refining just a pre-post specification or a combination of a rely command with a pre-post specification. Noting that guarantees distribute over programming constructs (e.g. sequential and parallel composition, conditionals, and loops), guarantees only need to be considered for refinements to assignments or atomic specifications. This makes program derivations simpler as one does not need to carry around the complete quintuple of Jones or quadruple of Dingle.
The laws in this paper are “semantic” in the sense that tests and assertions use sets of states, and the relations used in relies, guarantees and specifications are in terms of sets of pairs of states. Hence the laws can be considered generic with respect to the language (syntax) used to express sets of states (e.g., characteristic predicates on values the program variables) and relations (e.g., predicates on the before and after values of program variables). That allows the theory to be applied to standard state-based theories like B [Abr96], VDM [Jon90] and Z [Hay93, WD96]. Only a handful of the laws are sensitive to the representation of the program state Σ, and hence it is reasonably straightforward to adapt the laws to different state representations. Such a change would affect the update in an assignment, and the semantics of accessing variables within expressions in assignments and as guards in conditionals and loops.

Whereas Xu et al. [XdRH97], Prensa Nieto [Pre03], Dingel [Din02], Sanán et al. [SZLY21] and Schellhorn et al. [STE+14]) assume expression evaluation is atomic, in our approach expressions are not assumed to be atomic and are defined in terms of our core language primitives (Sect. 22). An interesting challenge is handling expression and guard evaluation in the context of interference, as is required to develop non-blocking implementations in which the values of the variables in guards or assignments may be changed by interference from other threads. Expression evaluation also leads to anomalies, such as the possibility of the expression $x = x$ evaluating to false if $x$ is modified between accesses to $x$. The approach taken here generalises that taken by others [Col08, WDP10b, CJ07, HBDJ13] who assume an expression only contains a single variable that is unmodified by the interference and that variable is only referenced once in the expression. That condition ensures that the evaluation of an expression under interference corresponds to evaluating it in one of the states during its execution. Our approach makes use of a weaker requirement that the expression is single reference under the rely condition (Definition 22.7) that also guarantees that property. Because we have defined expressions in terms of our core language primitives, we are able to prove the key lemmas about single-reference expressions and then use those lemmas to prove general laws for constructs containing expressions, including assignments (Sect. 23), conditionals (Sect. 24) and while loops (Sect. 26).

Like Schellhorn et al. [STE+14], we have included recursion in our language and use it to define while loops. Our laws for recursion and while loops are more general in that they provide for early termination of recursions and loops. The generality of the refinement laws makes them more useful in practice (see the related work sections throughout this paper).

Brookes [Bro96] and Dingel [Din02] make use of a trace semantics that treats commands as being semantically equivalent if their sets of traces are equivalent modulo finite stuttering and mumbling (see Sect. 16). The approach taken here is subtly different. Our specification command is defined so that it implicitly allows for finite stuttering and mumbling: it is closed under finite stuttering and mumbling. Further, our encoding of programming language constructs (code), such as assignments and conditionals, is defined in such a way that if $c$ and $d$ are code, and $c$ is semantically equivalent to $d$ modulo finite stuttering, then $c$ and $d$ are refinement equivalent. For example, equivalences such as if true then $c$ else $d$ fi = $c$ for $c$ and $d$ code, can be handled in the algebra. Our approach handles finite stuttering and mumbling (in a different way) while allowing refinement equivalence to be handled as equality, which allows finite stuttering and mumbling to be handled in the algebra, rather than the trace semantics.

In developing our refinement laws, we have endeavoured to make the laws as general as possible. The proof obligations for each law have been derived as part of the proof
process for the law, so that they are just what is needed to allow the proof to go through. Many of our laws are more general than laws found in the related work on rely/guarantee concurrency. Our more general laws allow one to tackle refinements that are not possible using other approaches. In practice, when applying our laws, many of our proof obligations are straightforward to prove; the difficult proof obligations tend to correspond to the “interesting” parts of the refinement, where it needs to explicitly cope with non-trivial interference. The laws in this paper are sufficient to develop practical concurrent programs but the underlying theory makes it straightforward to

- develop new laws for existing constructs or combinations of constructs,
- add new data types, such as arrays, and
- extend the language with new constructs, such as multiway parallel\(^{20}\), a switch command, a \texttt{for} command, or a simultaneous (or parallel) assignment [BBH+63], and associated laws.

The building blocks and layers of theory we have used allow for simpler proofs than those based, for example, on an operational semantics for the programming language [CJ07].

This paper is based on our concurrent refinement algebra developed in [HMWC19], which includes the proof of the parallel introduction law summarised in Sect. 18. The focus of the current paper is on laws for refining a specification in the context of interference, i.e. refining each of the threads in a parallel composition in isolation. The tricky part is handling interference on shared variables. The laws developed here have been used for the refinement of some standard concurrent algorithms in [HJ18] and the reader is referred there for additional examples.

29.1. \textbf{Future work.} We are actively pursuing adding generalised invariants [Rey81, LS85, Mor89, MV90, MV94], evolution guarantees [Jon91, CJ00], and local variable blocks [MH23] to the language, along with the necessary rely/guarantee laws to handle them. That work shares the basic theory used in this paper but extends it with additional primitive operators for handling variable localisation [MH23, CHM16, DHMS19]. Local variables also allow one to develop theories for procedure parameter passing mechanisms such as value and reference parameters, and both generalised invariants and localisation are useful tools to support data refinement.

Some concurrent algorithms (such as spin lock) do not give a guarantee of termination (under interference that is also performing locks) but do guarantee termination in the absence of interference. A partial version of an atomic specification command is more appropriate for specifying such algorithms. Future work also includes developing a while loop rule for refining from a partial specification that allows the loop to not terminate under interference but guarantees termination if the environment satisfies a temporal logic property.

Concurrent threads may need to wait for access to a resource or on a condition (e.g. a buffer is non-empty). Handling resources and termination of operations that may wait are further extensions we are actively pursuing; initial ideas for incorporating these may be found in [Hay18]. The approach in the current paper assumes a sequentially consistent memory model and hence additional “fencing” is needed for use on a multi-processor with a weak memory model. Additional work is needed to include the appropriate fencing to restore the desired behaviour. The use of a concept of a resource allows one to link control/locking variables with the data they control/lock [Hay18], thus allowing the generation of appropriate fencing.

\(^{20}\)Our Isabelle theories include this.
Conceptually, the abstract state space used within this paper could also incorporate a heap and use logics for reasoning about heaps, such as separation logic [Rey02, Bro07] and relational separation logic [Yan07], but detailed investigation of such instantiations is left for future work.

The “possible values” notation [JP11, JH16] provides a richer notation for expressing postconditions. The possible values postcondition $x' \in \overline{e}$ states that the final value of $x$ is one of the possible values of $e$ in one of the states during the execution of the command. Developing the theory to handle more expressive postconditions with possible values is left as future work because representing a postcondition as a binary relation is not expressive enough to handle possible values.

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