Stability of domain walls coupled to Abelian gauge fields

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Abstract

Rozowsky, Volkas and Wali recently found interesting numerical solutions to the field equations for a gauged $U(1) \otimes U(1)$ scalar field model. Their solutions describe a reflection-symmetric domain wall with scalar fields and coupled gauge configurations that interpolate between constant magnetic fields on one side of the wall and exponentially decaying ones on the other side. This corresponds physically to an infinite sheet of supercurrent confined to the domain wall with a linearly rising gauge potential on one side and Meissner suppression on the other. While it was shown that these static solutions satisfied the field equations, their stability was left unresolved. In this paper, we analyse the normal modes of perturbations of the static solutions to demonstrate their perturbative stability.

1 Introduction

Topological and non-topological defects are interesting classes of solutions to study for a large range of physical systems. They are frequently manifest in cosmological models described by classical relativistic field equations and show up in quantised systems as non-perturbative effects. The more specific kind of defect known as a domain wall or kink can act as an interface, separating two regions described by the same physics but with different boundary conditions. Condensed matter physics uses domain walls to model phase transitions and large scale structures in a system. Additionally, domain wall solutions are used as a basis for brane-world models, where our universe is embedded in a higher dimensional space. This space is described throughout by the same physical model, but different asymptotic vacuum behaviour in the higher dimension induces a kink defect to which our $3 + 1$ dimensional world is confined.

In one such brane-world toy model, Rozowsky, Volkas and Wali have found interesting solutions consisting of a pair of concentric domain walls coupled to a pair of $U(1)$ gauge fields. Their solutions look physically like infinite sheets of supercurrent confined to the wall, producing a linearly increasing gauge potential on one side and Meissner suppression on the other. The purpose of this
paper is to demonstrate the perturbative stability of this configuration. Our method is based on an analysis of the normal modes of perturbations of the static solutions, and we find that for a large range of parameters these modes are oscillatory, remaining bounded in time.

In Section 2 we present the model with a slight generalisation and display the static solutions for light- and space-like gauge fields. Our stability analysis method is outlined in Section 3 followed by a full investigation demonstrating the perturbative stability of the static configurations. We give a summary of our results in Section 4.

2 The model

We present a slight generalisation of the Rozowsky et al. model which specifies two scalar fields \( \phi_i (i = 1, 2) \) each with independent local \( U(1) \) gauge symmetries with associated gauge fields \( A_i^\mu \). There is an additional discrete \( Z_2 \) symmetry interchanging \( \phi_1 \leftrightarrow \phi_2 \) and \( A_1^\mu \leftrightarrow A_2^\mu \). This model is a toy model invented to study the clash-of-symmetries mechanism in its simplest non-trivial setting; see \[8, 9, 10, 11\]. A quartic scalar potential couples the scalar fields to each other and permits domain wall solutions asymptoting to different degenerate minima. The Lagrangian density is

\[
L = (D^\mu \phi_1)^* D_\mu \phi_1 + (D^\mu \phi_2)^* D_\mu \phi_2 - \frac{1}{4} F_1^{\mu\nu} F_1_{\mu\nu} - \frac{1}{4} F_2^{\mu\nu} F_2_{\mu\nu} - V(\phi_1, \phi_2),
\]

with the scalar field potential given by

\[
V(\phi_1, \phi_2) = \lambda_1 (\phi_1^* \phi_1 + \phi_2^* \phi_2 - v^2)^2 + \lambda_2 \phi_1^* \phi_1 \phi_2^* \phi_2.
\]

We work in the \( \lambda_{1,2} > 0 \) parameter regime, where the degenerate global minima are manifestly given by

\[
|\phi_1| = v, \quad \phi_2 = 0 \quad \text{and} \quad \phi_1 = 0, \quad |\phi_2| = v.
\]

The \( U(1) \) gauge fields \( A_i^\mu \) are described in the usual way by \( F_i^{\mu\nu} \) and their appearance in the covariant derivative

\[
D^\mu = \partial^\mu - iQ_1 A_1^\mu - iQ_2 A_2^\mu,
\]

where \( Q_i \) is the charge operator associated with the \( U(1)_i \) symmetry. Keeping the discrete \( Z_2 \) symmetry, the \( U(1)_1 \otimes U(1)_2 \) charges of the scalar fields are \( \phi_1 \sim (\tilde{e}, e) \) and \( \phi_2 \sim (\check{e}, e) \), with \( e \) and \( \check{e} \) constants.\(^1\)

The Euler-Lagrange equations of motion for the scalar and gauge fields are

\[
D^\mu D_\mu \phi_i = -2\lambda_1 \phi_i (\phi_i^* \phi_i + \phi_j^* \phi_j - v^2) - \lambda_2 \phi_i^* \phi_j \phi_j, \\
\partial_\nu F_i^{\nu\mu} = 2 \text{Im} (c \phi_i^* D^\mu \phi_i + \check{c} \phi_j^* D^\mu \phi_j),
\]

where \( i = 1 \) and \( j = 2 \), or \( i = 2 \) and \( j = 1 \) (this notation is to be understood in subsequent equations). Following \[1\], we look for static solutions that depend only on \( z \) and utilise a polar decomposition for the scalar fields; \( \phi_i(z) = R_i(z) e^{i\Theta_i(z)} \). The scalar potential \( \[2\] \) allows one to construct domain

\(^1\)The Rozowsky et al. model took \( \check{e} = 0 \), so this is a slight generalisation, first introduced in \[11\].
V in an appropriate limit. Since our fields vary only with \( z \), this will be the dimension perpendicular to the wall and the limit will be as \( |z| \) tends to infinity. This gives us the boundary conditions

\[
|\phi_1| \to 0, \quad |\phi_2| \to v \quad \text{as} \quad z \to -\infty, \\
|\phi_1| \to v, \quad |\phi_2| \to 0 \quad \text{as} \quad z \to \infty, 
\]

or vice-versa.

Since we are working with gauge fields we have the freedom to choose two gauges, one for each \( A_i^{\mu} \); the Lorentz gauge \( \partial_{\mu} A_i^{\mu} = 0 \) turns out to be the most suitable choice for both. The algebra simplifies if instead of \( A_i^{\mu} \) one considers the linear combination \( A_i^{\mu} = e_i A_i^{\mu} + \bar{e}_i A_i^{\mu} \). Working with these choices, the field equations of motion (3) reduce to

\[
R_i'' = -R_i(A_i^2 - A_i^2 - A_i^{(2)^2}) + 2\lambda_1 R_i(R_i^2 + R_j^2 - v^2) + \lambda_2 R_i R_j^2, \\
(5)
\]

\[
A_i^{(t,x,y)''} = 2(e^2 + e^2)R_i A_i^{(t,x,y)} + 4\bar{e} R_j A_j^{(t,x,y)}, \\
(6)
\]

\[
A_i' = 0, \\
(7)
\]

\[
\Theta_i' = -A_i^z, \\
(8)
\]

where prime denotes differentiation with respect to \( z \). We immediately see that the \( A_i^z \) and hence the \( A_i^{\pm} \) are pure gauge and do not contribute to the physics; neither do the \( \Theta_i \).

To further simplify the problem, we note that each gauge component \( A_i^{(t,x,y)} \) exhibits the same dynamics in (6) and appears quadratically in (5). Thus, the qualitative physical behaviour depends on whether the gauge field configuration is space-like, time-like or light-like. Expressing this behaviour as the single field \( A_i \) we have

\[
R_i'' = kR_i A_i^2 + 2\lambda_1 R_i(R_i^2 + R_j^2 - v^2) + \lambda_2 R_i R_j^2, \\
(9)
\]

\[
A_i'' = 2(e^2 + e^2)R_i^2 A_i + 4\bar{e} R_j^2 A_j, \\
(10)
\]

where \( k = +1, 0, -1 \) for space, light- and time-like gauge fields respectively. One can see that equation (9) is consistent with the boundary conditions (4) so long as \( k \geq 0 \). For \( k = -1 \) the asymptotic behaviour of \( R_i \) is oscillatory and so we discard this time-like scenario.

Considering equation (10) on the side of the wall where \( R_i \to v \) and \( R_j \to 0 \), we see that \( A_i'' \to 2(e^2 + e^2)v^2 A_i \). Since physical solutions must be bounded, we conclude that \( A_i \) is exponentially suppressed. On the other side of the wall, \( R_i \to 0, R_j \to v \) and, using the previous result, \( A_j \to 0 \). Thus \( A_i'' \to 0 \) and this gauge field takes on a linear form. We note that all measurable quantities associated with \( A_i \) arise through derivatives and so this unbounded solution is still physical.

The set of equations (9) and (10) cannot, in general, be solved analytically and we use the relaxation-on-a-mesh technique to obtain numerical solutions.

In the light-like case, \( k = 0 \), and \( R_i \) can be solved for independently of \( A_i \). Typical solutions are shown in the top two plots in Figure 1. The scalar fields assume a typical domain wall configuration asymptoting to distinct minima of the potential. As the boundary conditions for the \( R_i \) are symmetric under \( z \)
Figure 1: Static solutions for the two scalar and two gauge fields in the $U(1) \otimes U(1)$ model, plotted against $z$. The top two plots are for the light-like case, the bottom two for the space-like case. All plots have $e = 1$, $\tilde{e} = \frac{1}{2}$, $\lambda_1 = 1$, $\lambda_2 = 2$ and $v = 1$. The plots on the left have symmetric boundary conditions for the gauge fields, those on the right asymmetric. In the light-like case, the scalar fields do not feel the gauge fields and thus do not depend on the choice of gauge field boundary conditions. This is unlike the space-like case where the scalar fields centre on the gauge fields to restore the reflection symmetry.

reflection, the solutions for these scalar fields are just reflections of each other. In the left plot, the boundary conditions for the two gauge fields are also reflection symmetric. In the right plot, a different boundary condition is used for $A_2$. As the scalar fields do not feel the presence of the gauge fields they have exactly the same solutions in both cases.

For the space-like case, $k = 1$, and the scalar and gauge fields are fully coupled. Solutions are shown in the bottom two plots in Figure 1 with all parameters, except $k$, mimicking the top two plots. Although they look similar, the light- and space-like plots on the left are slightly different. A more significant difference between these two scenarios is evident in the right plots where the boundary conditions for the two gauge fields are different. In the space-like case the scalar fields are influenced by the gauge fields and the favourable configuration is that with exact reflection symmetry. The boundary conditions serve to simply shift the centre of the domain wall and the right plot on the bottom is an exact translation of the left plot. Our result is contrary to the claim in [1] that asymmetric boundary conditions in the space-like case are not equivalent to spatial translations of the domain wall centre.

Disregarding the technical details, the qualitative features of this $U(1) \otimes U(1)$ model are the reflection symmetric scalar fields in a domain wall configuration and the partially suppressed gauge fields. This suppression of $A_i$ under their
respective $R_i$ is physically similar to the Meissner effect and serves to semi-localise the gauge fields. We make the physical interpretation of an infinite sheet of supercurrent confined to the wall, producing a constant magnetic field in the region opposite the suppression.

While we have shown the existence of static solutions to the model described by (1), we have not established their stability. In the next section we demonstrate that under small perturbations, the solutions to equations (9) and (10) are stable.

3 Stability

The essence of static solutions is their time independence, but a physical processes requires the underlying fields to evolve in time. We must thus ensure that the static configurations found in the previous section are not destroyed by time-dependent perturbations. In this section we add to the static solutions perturbations expressed as normal modes and arrive at a set of equations characterising these normal eigenfunctions and associated eigenvalues. We then demonstrate that the eigenvalues are all positive and hence the perturbations are oscillatory.

We begin by taking each static field, including all four gauge components $A^i_\mu$, and adding a perturbation factored as an unspecified spatial part and a time dependent complex exponential. This exponential represents an arbitrary normal mode of the perturbation, characterised by an eigenvalue which is in general complex. We express this construction in the substitutions

$$R_i \rightarrow R_i(z) + r_i(z)e^{i\omega t},$$

$$A^i_\mu \rightarrow A^i_\mu(z) + a^\mu_i(z)e^{i\omega t},$$

$$\Theta_i \rightarrow \Theta_i(z) + \theta_i(z)e^{i\omega t}. \quad (11)$$

By the choice of an explicitly complex exponential, if $\omega$ is purely real then the perturbation will be oscillatory and hence remain bounded in time. On the other hand, if $\omega$ has an imaginary component, the exponential will blow up, signifying instability of the original static solution.

We take the original field equations (3), make the substitutions given by (11) and simplify using the equations (5), (6), (7) and (8) for the static fields. We work to first order in $r_i$, $a^\mu_i$ and $\theta_i$ and consider only independent perturbations, which decouples the resulting set of equations to give

$$\left(-\partial_z^2 - (A^i_z^2 - A^x_i^2 - A^y_i^2) + 2\lambda_1(3R_i^2 + R_j^2 - v^2) + \lambda_2 R_j^2\right) r_i = \omega^2 r_i, \quad (12)$$

$$\left(-\partial_z^2 + 2(e^2 + \bar{e}^2)R_i^2\right) a^\mu_i = \omega^2 a^\mu_i, \quad (13)$$

$$\left(-\partial_z^2 - 2\frac{R_i}{R_i} R_{z^2}\right) \theta_i = \omega^2 \theta_i. \quad (14)$$

Before we continue with these equations, we must first establish a general result. Given the equation

$$f''(z) + V(z)f'(z) + W(z)f(z) = 0,$$

one can show that if $W(z) < 0$ for all $z$, then there exist no non-trivial solutions for $f(z)$ on the domain $z \in \mathbb{R}$ with $f(z) \to 0$ as $|z| \to \infty$. To see this consider
large $-z$ with $f$ taking a vanishingly small positive value. For non-trivial solutions, $f$ must increase as $z$ increases\(^2\) and so $f' > 0$. For solutions where $f$ becomes vanishingly small for large $z$, we require $f' < 0$ for some subsequent region of the $z$-axis. This change in the sign of the first derivative requires $f'' < 0$ for some region, in particular we must have $f'' < 0$ when $f' = 0$, i.e. at the turning point. But at this point we have $f'' = -W(z)f$ and since $f > 0$ and $W(z) < 0$ for all $z$ we have $f'' > 0$. Thus the function is positive with a positive gradient and can never turn back towards the $z$-axis. A similar argument holds when $f$ is below the axis; it can never turn back up. Hence there are no non-trivial bounded solutions if $W(z) < 0$ for all $z$.

We now return to the issue of stability. Consider equation (14) with $f(z) = \theta_i(z)$ and $W(z) = \omega_0^2$. If $\omega_0^2 < 0$ then one would have $W(z) < 0$ for all $z$ and by the previous result the only solution for $\theta_i$ would be the trivial one. Thus there are no negative eigenvalues for equation (14) with bounded eigenfunctions $\theta_i$. Note that the condition that eigenfunctions $\theta_i$ be bounded does not preclude the analysis of the bounded nature of the perturbations. The perturbation to the static field is given in full by $\theta_i(z)e^{i\omega t}$ where by definition of a perturbation, $\theta_i(z)$ must be small and bounded. It is the nature of the temporal part $e^{i\omega t}$, hence the eigenvalues, that determines if the fields are stable. Since we have shown that $\omega_0^2 \geq 0$ we have $\omega$ real and thus oscillatory perturbations and hence a stable static field $\Theta$.

For the gauge fields, inspection of equation (13) yields

$$W(z) = \omega_0^2 - 2(e^2 + \bar{e}^2)R_i^2.$$  

For bounded $\alpha_i^\mu(z)$ we require $W(z) \geq 0$ for some non-zero domain of $z$. This means that we need

$$\omega_0^2 \geq 2(e^2 + \bar{e}^2)R_i^2$$

for some $z$. Since $\omega_0^2$ is a constant it must be greater than or equal to the minimum of $2(e^2 + \bar{e}^2)R_i^2$, hence non-negative. Thus we have shown that the static gauge fields are stable under small time dependent perturbations.

Following a similar argument for the scalar fields, equation (12) gives us the bound on the eigenvalues as

$$\omega_i^2 \geq \min(U(z)),$$

where

$$U(z) = -(A_i^2 - A_i^2 - A_i^2) + 2\lambda_1(3R_i^2 + R_j^2 - v^2) + \lambda_2 R_i^2.$$  

(15)

It is not so clear as to the sign of this function. We analyse the light-like case first where the $A_i^\mu$ terms are absent. In this case, as can be seen from equation (9) with $k = 0$, the scalar field configuration and hence $U(z)$ depend only on the parameters $\lambda_1$, $\lambda_2$ and $v$. Since $v$ can be absorbed into a rescaling of the $R_i$, we only have two parameters to consider. A typical plot of the function $U(z)$ for the two permutations of $i$ and $j$ is shown on the left in Figure 3.

We see that $U(z) > 0$ for this choice of parameters. Figure 3 shows the minimum of $U(z)$ for a large range of values of $\lambda_1$ and $\lambda_2$. Since all minima are positive, it must be that $\omega_i^2 > 0$ and hence the static scalar fields in the light-like case are stable, at least for this range of parameters.

\(^2\)Since $f(-\infty) = 0$ there must be a region where $f$ increases if it is to attain a finite positive value.
Figure 2: The function $U(z)$ used to determine the eigenvalues of the $r_i$ perturbation in the light- (left) and space- (right) like cases, plotted as a function of $z$. The parameters and corresponding field configurations are as in the reflection symmetric cases in Figure 1. There are two plots in each graph corresponding to $U(z)$ with $i = 1$, $j = 2$ and $i = 2$, $j = 1$. 

Figure 3: The minimum of the function $U(z)$ in the light-like case plotted against $\lambda_2$. The upper curves correspond to successively larger values of $\lambda_1$, which runs from 0.2 to 2 in steps of 0.2.
Figure 4: The minimum of the function $U(z)$ in the space-like case. The plot on the left is against $\lambda_2$ with upper curves corresponding to larger values of $\lambda_1$, which runs from 0.2 to 2 in steps of 0.2. The plot on the right is against the boundary condition for the gauge field with upper curves corresponding to larger values of $v$, which runs from 0.2 to 0.8 in steps of 0.1.

In the space-like scenario, the results are similar. The plot in the right of Figure 2 shows equation (15) with $A_x^i$ and $A_y^i$ present. Figure 4 shows the minimum of $U(z)$ for various values of the parameters and boundary conditions for the gauge fields. It is clear that the minima are all positive and so the static scalar field configuration is also stable in the space-like case.

Since each field in the model permits static solutions which are independently stable, we conclude that the static configuration as a whole is a stable one. We have also verified this analysis with explicit numerical calculation of the eigenvalues.

4 Conclusion

Static solutions to a $U(1) \otimes U(1)$ gauged scalar model were recently found by Rozowsky, Volkas and Wali [1]. In this paper we have generalised the model slightly, presented the static solutions and demonstrated the stability of this field configuration. We achieved this by adding small time dependent perturbations, in the form of normal modes, to the static fields and obtaining eigenvalue equations. It was shown that these eigenvalues, corresponding to the normal modes, were positive for a large range of parameters in the model. Thus the perturbations were oscillatory and the static fields stable.

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