The scaling limits for Wiener sausages in random environments

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Abstract

We consider the statistical mechanics of a random polymer with random walks and disorders in \( \mathbb{Z}^d \). The walk collects random disorders along the way and gets nothing if it visits the same site twice. In the continuum and weak disorder regime, the partition function as a random variable converges weakly to a Wiener Chaos expansion when the dimension is lower than the critical dimension, which is four. A finite temperature case in one dimension is also discussed. The last case suggests that the end-point behavior of the polymer is \( t^{2/3} \).

1 Introduction

1.1 The model

We consider the model that a particle walks on \( \mathbb{Z}^d \) lattice, and \( \{ \omega_x \}_{x \in \mathbb{Z}^d} \) is an i.i.d. random field with mean zero, variance one and finite exponential moment under the measure \( \mathbb{P} \). The movement of the particle is described by a simple symmetric random walk \( S_n \) with the measure \( P \), \( \mathbb{P} \) and \( P \) are independent. The particle utilizes the field when it visits a new site \( x \), then the random field no longer exists. Let \( \mathcal{R}_n \) denote the set of sites visit by the particle in the first \( n \) steps, that is,

\[
\mathcal{R}_n := \{ x \in \mathbb{Z}^d \mid S_i = x, 1 \leq i \leq n \},
\]

and \( R_n = |\mathcal{R}_n| \) is the cardinality of \( \mathcal{R}_n \).

The Hamiltonian is defined as follows

\[
H_n := \sum_{x \in \mathbb{Z}^d} (\beta \omega_x + h) \cdot 1_{x \in \mathcal{R}_n}.
\]  

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\[\beta \geq 0\] is related to the inverse temperature and \(h \in \mathbb{R}\) represents an external force. The polymer measure is

\[P_n^\omega(S) := \frac{1}{Z_n} \exp(H_n) \cdot P(S) \quad (1.2)\]

where

\[Z_n := E(\exp(H_n)) \quad (1.3)\]

is called the quenched partition function. We also define the annealed partition function

\[E \mathbb{Z}_n := E(e^{(\lambda(\beta) + h)R_n}) \quad (1.4)\]

where \(\lambda(\beta) = \log \mathbb{E} e^{\beta \omega_0}\). In the literature, the annealed model is called Wiener sausage. \([5]\) is the seminal paper when \(h < -\lambda(\beta)\), and \([3]\) considered \(h > -\lambda(\beta)\) with the random walks replaced by Brownian motions.

In this paper, we discuss the case \(h = -\lambda(\beta)\) and the intermediate regime \(\beta = \beta_n\), that is,

\[Z_n(\beta_n) := E(e^{\sum_{x \in \mathbb{Z}^d} (\beta_n \omega_x - \lambda(\beta_n)) 1_{x \in \mathcal{R}_n}}), \quad E \mathbb{Z}_n(\beta_n) = 1. \quad (1.5)\]

This intermediate regime was discussed in \([4]\). Define \(\eta_x(\beta) = \beta^{-1}(e^{\beta \omega_x - \lambda(\beta)} - 1)\) and recall that

\[1_{x \in \mathcal{R}_n} = 1_{T_x \leq n}. \quad (1.6)\]

\([4]\) consider the following expansion of \(Z_n(\beta_n)\),

\[
Z_n(\beta_n) = E \left[ \prod_{x \in \mathbb{Z}^d} \left(1 + (e^{\beta_n \omega_x - \lambda(\beta_n)} - 1) 1_{T_x \leq n}\right) \right] = E \left[ \prod_{x \in \mathbb{Z}^d} \left(1 + \beta_n \eta_x(\beta_n) 1_{T_x \leq n}\right) \right] \\
= E \left[ 1 + \beta_n \sum_{x \in \mathbb{Z}^d} \eta_x(\beta_n) 1_{T_x \leq n} + \frac{1}{2!} \beta_n^2 \sum_{x,y \in \mathbb{Z}^d; x \neq y} \eta_x(\beta_n) 1_{T_x \leq n} \eta_y(\beta_n) 1_{T_y \leq n} + \frac{1}{3!} \beta_n^3 \sum_{x,y,z \in \mathbb{Z}^d; x \neq y \neq z} \eta_x(\beta_n) 1_{T_x \leq n} \eta_y(\beta_n) 1_{T_y \leq n} \eta_z(\beta_n) 1_{T_z \leq n} + \cdots \right] \\
= 1 + \beta_n \sum_{x \in \mathbb{Z}^d} \eta_x(\beta_n) E(1_{T_x \leq n}) + \frac{1}{2!} \beta_n^2 \sum_{x,y \in \mathbb{Z}^d; x \neq y} \eta_x(\beta_n) \eta_y(\beta_n) E(1_{T_x \leq n} 1_{T_y \leq n}) + \frac{1}{3!} \beta_n^3 \sum_{x,y,z \in \mathbb{Z}^d; x \neq y \neq z} \eta_x(\beta_n) \eta_y(\beta_n) \eta_z(\beta_n) E(1_{T_x \leq n} 1_{T_y \leq n} 1_{T_z \leq n}) + \cdots \quad (\star) \]

Notice that \(\eta_x's\) are i.i.d. with \(\mathbb{E} \eta_x = 0\) and \(\sigma^2(\eta_x) \approx 1\) when \(\beta \approx 0\).

This expansion is going to converge to a non-trivial limit \(Z^W\) as \(\beta_n\) approaches 0. We need two ingredients to show the convergence. First, the scaling limits for \(k\)-point
function $E(1_{T_{\gamma_{\pi_{x_{1}}} \leq n} \cdots 1_{T_{\gamma_{\pi_{x_{k}}} \leq n}}})$. Second, $\beta_n$ is chosen such that the variance of the first-order term in $(\ast)$ is finite in the limit.

$$\text{Var}(\beta_n \sum_{x \in \mathbb{Z}^d} \eta_x E1_{x \in \mathcal{R}_n}) \sim \beta_n^2 \sum_{x \in \mathbb{Z}^d} [E1_{x \in \mathcal{R}_n}]^2 = \beta_n^2 E \times E'(S^1[1, n] \cap S^2[1, n]). \quad (1.7)$$

$E \times E'(S^1[1, n] \cap S^2[1, n])$ is called the intersection of independent ranges $[6]$. Define

$$J_n^{(p)} := \#\{S^1[1, n] \cap S^2[1, n] \cap \ldots \cap S^p[1, n]\}.$$

Let $p = 2$ and $J_n = J_n^{(2)}$,

\[
\begin{align*}
    d = 1, & \quad \frac{1}{\sqrt{n}} J_n \rightarrow_{\text{law}} \min \max_{i=1,2} B^i(s) - \max \min_{i=1,2} B^i(s), \\
    d = 2, & \quad \frac{1}{\log n} J_n \rightarrow_{\text{law}} 2\pi^2 \alpha([0, 1]^2), \\
    d = 3, & \quad \frac{1}{\sqrt{n}} J_n \rightarrow_{\text{law}} 2\gamma^2 \alpha([0, 1]^2), \\
    d = 4, & \quad \frac{1}{\log n} J_n \rightarrow_{\text{law}} 2(2\pi)^{-2} \gamma^2 N(0, 1)^2,
\end{align*}
\]

and $d \geq 5, J_n < \infty$ almost surely. $\gamma = \gamma_d$ is the escape rate of $d$-dimensional simple random walk, and $\alpha(\cdot)$ is the 2-multiple mutual-intersection local time of independent Brownian motions $B^1(t)$ and $B^2(t)$ $[6]$. Because of (1.7), $\beta_n$ is chosen as follows

\[
\begin{align*}
    d = 1, & \quad \beta_n = \frac{\hat{\beta}}{n^{1/4}}, \\
    d = 2, & \quad \beta_n = \frac{\log n}{\sqrt{n}} \hat{\beta}, \\
    d = 3, & \quad \beta_n = \frac{\beta}{n^{1/4}}. 
\end{align*}
\]

### 1.2 Main results

Denote $\bar{p}_n(x) := (\frac{d}{2\pi n})^{d/2} \exp(-\frac{d|x|^2}{2n})$. For $d = 2$, $g_t(x) := \pi \bar{p}_t(x)$, for $d = 3$, $g_t(x) := \gamma \bar{p}_t(x)$.

**Theorem 1.1.** Let $[0, t]^k := \{(t_1, t_2, \ldots, t_k); 0 \leq t_1 \leq t_2 \leq \ldots \leq t_k \leq t\}$ and $\Sigma_k$ be the permutation group over $\{1, \ldots, k\}$. Also set $t_0 = 0, x_{\sigma(0)} = 0$. The expansion $(\ast)$ of $Z_{Nt}(\beta N)$ converges weakly to a Wiener chaos expansion

$$Z_t^W := 1 + \sum_{k=1}^\infty \frac{\beta^k}{k!} \int \cdots \int_{(\mathbb{R}^2)^k} \psi_t(x_1, ..., x_k) W(dx_1) \cdots W(dx_k). \quad (1.8)$$

as $N \rightarrow \infty$. For $d=2,3$,

$$\psi_t(x_1, \ldots, x_k) := \sum_{\sigma \in \Sigma_k} \int_{[0, t]^k} \prod_{m=1}^k g_{t_{m-1}}(x_{\sigma(m)} - x_{\sigma(m-1)}) \, dt_1 \cdots dt_k. \quad (1.9)$$

For $d=1$,

$$\psi_t(x_1, \ldots, x_k) := \sum_{\sigma \in \Sigma_k} P(\min_{0 \leq s \leq t} B_s \leq x_{\sigma(1)} < \cdots < x_{\sigma(k)} \leq \max_{0 \leq s \leq t} B_s). \quad (1.10)$$
1.3 Discussions

The overlap is defined as

$$\sum_{x \in \mathbb{Z}^d} E \times E'(1_{x \in \mathcal{R}_n} 1_{x \in \mathcal{R}'_n}) = E \times E'(S^1[1,n] \cap S^2[1,n]) = J_n. \quad (1.11)$$

From the behavior of the overlap $J_n$, when $d = 1, 2, 3$, the model is called disorder relevant; $d \geq 5$ the model is disorder irrelevant.

$d = 4$ is believed to be the critical dimension. We are not going to discuss this case here. See more details about the critical dimension in [5].

There is a similar model called directed polymers in random environments (DPRE) [7]. The Hamiltonian is

$$H_n := \sum_{i=1}^n \left( \beta \omega(i,S_i) - \lambda(\beta) \right) \quad (1.12)$$

The random potential $\omega$ is defined on space and time. Unlike the model we discussed in this paper, DPRE is directed. For the one-dimensional case, the weak disorder limit is the solution of stochastic heat equation [1], however, we don’t see the analogous result here. On another hand, the critical dimension for DPRE is two, and here we have four instead.

The last comment we would like to make is that, for $d = 1$, $h = 0$ and finite $\beta$, the the scale of the end-point position of the walk seems to be $n^{2/3}$. Details are in Section 3.

2 Proof of Theorem 1.1

2.1 Preliminary results

Denote $p_n(x) := P(S_n = x)$, $G_n(x) := \sum_{m=1}^n p_m(x)$, $G_n := G_n(0)$, $\gamma_n := G_n^{-1}$.

We recall Lemma 5.1.3 in [6].

Lemma A. For any $x \in \mathbb{Z}^d$,

$$P(S_k = x) = \sum_{j=1}^k P(T_x = j) P(S_{k-j} = 0). \quad (2.1)$$

Consequently,

$$\sum_{k=1}^n P(S_k = x) = \sum_{k=1}^n P(T_x = k) G_{n-k}. \quad (2.2)$$

We first consider the one-point function. Notice that

$$1_{T_x \leq n} = \sum_{i=1}^n 1_{T_x = i}. \quad (2.3)$$
Lemma 2.1. $x \neq 0$,

\[ P(T_x \leq n) \geq G_n^{-1} \sum_{k=1}^{n} P(S_k = x) \quad (2.4) \]

and

\[ G_{\epsilon n} P(T_x \leq n) \leq \sum_{k=1}^{(1+\epsilon)n} P(S_k = x). \quad (2.5) \]

Proof. They are typical bounds. First, since $G_n$ is increasing in $n$,

\[ G_n P(T_x \leq n) \geq \sum_{k=1}^{n} P(T_x = k) G_{n-k} = \sum_{k=1}^{n} P(S_k = x) \]

and

\[ G_{\epsilon n} P(T_x \leq n) \leq \sum_{k=1}^{(1+\epsilon)n} P(T_x = k) G_{(1+\epsilon)n-k} \leq \sum_{k=1}^{(1+\epsilon)n} P(T_x = k) G_{(1+\epsilon)n-k} = \sum_{k=1}^{(1+\epsilon)n} P(S_k = x). \]

Equalities are from (2.2).

We now discuss the $k$-point function. Recall that $\Sigma_k$ is the permutation group over \{1, ..., $k$\}.

Lemma 2.2. $d = 2, 3$ and $0, x_1, ..., x_k$ are distinct points,

\[ P(T_{x_1} \leq n, T_{x_2} \leq n, ..., T_{x_k} \leq n) \leq G_n^{-k} \cdot \sum_{\sigma \in \Sigma_k} \sum_{1 \leq j_1 < \cdots < j_k \leq (1+\epsilon)n} P(S_{j_1} = x_{\sigma(1)}) P(S_{j_2-j_1} = x_{\sigma(2)} - x_{\sigma(1)}) \cdots P(S_{j_k-j_{k-1}} = x_{\sigma(k)} - x_{\sigma(k-1)}) \quad (2.6) \]

For the lower bound,

\[ G_n^{-2} \left( \sum_{1 \leq j < k \leq n} P(S_j = x_1, S_k = x_2) + \sum_{1 \leq k < j \leq n} P(S_j = x_1, S_k = x_2) \right) + \text{higher order terms}. \quad (2.7) \]

Moreover, for general $k$,

\[ G_n^{-k} \sum_{\sigma \in \Sigma_k} \sum_{1 \leq j_1 < \cdots < j_k \leq n} P(S_{j_1} = x_{\sigma(1)}) P(S_{j_2-j_1} = x_{\sigma(2)} - x_{\sigma(1)}) \cdots P(S_{j_k-j_{k-1}} = x_{\sigma(k)} - x_{\sigma(k-1)}) + \text{higher order terms}. \quad (2.8) \]

Remark. The “higher order terms” means that after we take $n \to \infty$, it at least has one more factor $\frac{1}{\sqrt{n^{d}}}$ than the first term.
Proof. First we examine the upper bound (2.6). For \( k = 2 \), exactly from (5.3.22) in [6] p. 153,

\[
P(T_{x_1} \leq n, \ T_{x_2} \leq n)
= \sum_{\sigma \in \Sigma_2} \sum_{1 \leq j_1 < j_2 \leq n} P(T_{x_{\sigma(1)}} = j_1, \ T_{x_{\sigma(2)}} = j_2)
\leq G_{en}^{-2} \cdot \sum_{\sigma \in \Sigma_2} \sum_{1 \leq j_1 < j_2 \leq (1+2\epsilon)n} P(S_{j_1} = x_{\sigma(1)}) P(S_{j_2-j_1} = x_{\sigma(2)} - x_{\sigma(1)})
\]

For general \( k \), do the induction on the \( T_{x_k} \).

For the lower bound (2.8), again the case \( k = 2 \),

\[
P(T_x \leq n, \ T_y \leq n) = \sum_{j=1}^{n} P(T_x = j, \ T_y \leq n)
= \sum_{j=1}^{n} \sum_{k=1}^{n} P(T_x = j, \ S_k = y, \ S_{k+1} \neq y, \ S_{k+2} \neq y, \ldots, S_n \neq y)
= \sum_{1 \leq j < k \leq n} P(T_x = j, \ S_k = y, \ S_{k+1} \neq y, \ S_{k+2} \neq y, \ldots, S_n \neq y)
+ \sum_{1 \leq k < j \leq n} P(T_x = j, \ S_k = y, \ S_{k+1} \neq y, \ S_{k+2} \neq y, \ldots, S_n \neq y)
= I + II.
\]

We have

\[
I = \sum_{1 \leq j < k \leq n} P(T_x = j) P(S_{k-j} = y - x) P(S_1 \neq 0, S_2 \neq 0, \ldots, S_{n-k} \neq 0) \geq \sum_{1 \leq j < k \leq n} P(T_x = j) P(S_{k-j} = y - x) \gamma_n
\geq \gamma_n G_{n}^{-1} \sum_{1 \leq j < k \leq n} P(S_j = x, \ S_k = y)
\]

and

\[
II = \sum_{1 \leq k < j \leq n} P(T_x \geq k, \ S_k = y, \ S_{k+1} \neq y, \ S_{k+2} \neq y, \ldots, S_n \neq y, \ S_{k+1} \neq x, \ldots, S_{j-1} \neq x, S_j = x)
= \sum_{1 \leq k < j \leq n} P(T_x \geq k, \ S_k = y) P(T_0 > n - k, T_{x-y} = j - k)
= \sum_{1 \leq k < j \leq n} P(T_x \geq k, \ S_k = y) P(T_0 > j - k, T_{x-y} = j - k) P(T_{y-x} > n - j)
\]
\[ \geq \sum_{1 \leq k < j \leq n} P(T_x \geq k, S_k = y)P(T_0 > j - k, T_{x-y} = j - k) \cdot P(T_{y-x} > n). \]

Apply Lemma 2.3 below to the middle term of the last line \( P(T_0 > j - k, T_{x-y} = j - k) \), then

\[ II \geq \sum_{1 \leq k < j \leq n} P(T_x \geq k, S_k = y)G_n^{-1}P(T_{x-y} = j - k) \cdot P(T_{y-x} > n). \]

Continue with steps in [6] p. 154,

\[ II \geq G_n^{-2} \sum_{1 \leq k < j \leq n} P(S_k = y, S_j = x) \cdot P(T_{y-x} > n) \]
\[ -G_n^{-1}P(T_x \leq n)P(T_{x-y} \leq n) \sum_{k=1}^{n} P(S_k = y - x) \cdot P(T_{y-x} > n). \]

Notice that \( P(T_{\sqrt{n}(y-x)} > n) = 1 - P(T_{\sqrt{n}(y-x)} \leq n) \to 1 \) when \( d = 2, 3 \) by Lemma 2.4 below. The last term has an extra \( P(\cdot) \), which gives a factor \( \frac{1}{\sqrt{n}} \) more than the previous one. This finishes the case \( k = 2 \). We use symbols \( T \) and \( S \) for \( T_x \) and \( S_j \) respectively to explain the procedure again. Let’s say the proof above for the 2-point case \( [2.4] \) is a process from \( TT \) to \( SS \). \( TT \) first splits into \( TS \) and \( ST \) right before the line \( I + II \). \( TS \) in \( I \) and \( ST \) in \( II \) give \( SS \). For the 3-point function, we fix the first hitting time \( j_1 \) of \( x_1 \) and work on the last two time spots \( j_2 \) and \( j_3 \). The 3-point case \( T[TT] \) first becomes \( T[SS] \) from the 2-point case. Then we consider the first two time spots and make \( [TS]S \) become \( SS[S \) from \( I \).

For general \( k, T \cdots TT \to T \cdots TSS, \) then continues by induction. Thus, the proof is complete. \( \square \)

**Lemma 2.3.**

\[ \sum_{k=1}^{n} P(T_x = k) \leq G_n \sum_{j=1}^{n} P(T_x = j, T_0 > j). \]

**Proof.** From [9] p. 112, \( P(S_k = x, T_0 > k) = P(T_x = k) \). So we need to prove

\[ \sum_{k=1}^{n} P(S_k = x, T_0 > k) \leq G_n \sum_{j=1}^{n} P(T_x = j, T_0 > j). \]

First,

\[ P(S_k = x, T_0 > k) = P(S_k = x, T_x \leq k, T_0 > k) = \sum_{j=1}^{k} P(S_k = x, T_x = j, T_0 > k) \]
\[ = \sum_{j=1}^{k} P(T_x = j, T_0 > j)P(S_{k-j} = 0, T_{x-j} > k-j). \]
Again,
\[
\sum_{k=1}^{n} P(S_k = x, T_0 > k) = \sum_{j=1}^{n} \sum_{k=j}^{n} P(T_x = j, T_0 > j)P(S_{k-j} = 0, T_{-x} > k-j)
\]
\[
= \sum_{j=1}^{n} P(T_x = j, T_0 > j) \sum_{i=0}^{n-j} P(S_i = 0, T_{-x} > i) \leq G_n \sum_{j=1}^{n} P(T_x = j, T_0 > j).
\]

\[\square\]

### 2.2 Capacity for \(d = 2, 3\)

Recall that \(p_n(x) := P(S_n = x)\), \(\bar{p}_n(x) := (\frac{d}{2\pi n})^{d/2} \exp(-\frac{d|x|^2}{2n})\). For \(d = 2\), \(g_t(x) := \pi \bar{p}_t(x)\), for \(d = 3\), \(g_t(x) := \gamma \bar{p}_t(x)\). Let \(k_N = \log N\), if \(d = 2\); \(k_N = \sqrt{N}\) if \(d = 3\).

**Lemma 2.4.** \(d=2,3, x_1 \neq 0\),
\[
k_N P(T_{\sqrt{N}x_1} \leq Nt_1) \to \int_0^{t_1} g_s(x_1)ds. \quad (2.9)
\]

**Proof.** It is well known that \(d = 2, G_n \sim \frac{1}{e} \log n\) and \(d = 3, G_n \to 1/\gamma\).

With Lemma 2.1, we only need to prove
\[
\sqrt{N}^{d-2} \sum_{k=1}^{Nt_1} P(S_k = \sqrt{N}x_1) \sim \int_0^{t_1} \bar{p}_s(x_1)ds.
\]

Also note that \(p_k(\sqrt{N}x_1) = \bar{p}_k(\sqrt{N}x_1) + O(\frac{1}{k^{d/2+1}})\). For \(n << N\), \(\sum_{k=1}^{n} p_k(\sqrt{N}x_1) \approx 0\), and for \(n \approx N\), \(\sum_{k=n+1}^{N} O(\frac{1}{k^{d/2+1}}) = O(1/N^{d/2})\). The rest is the Riemann sum for \(\sum \bar{p}_k(\sqrt{N}x_1)\).

\[\square\]

For the k-point function. Let \([0, t]^k = \{(t_1, t_2, ..., t_k); 0 \leq t_1 \leq t_2 \leq ... \leq t_k \leq t\}.

**Lemma 2.5.** \(d = 2,3\) and \(0, x_1, ..., x_k\) are distinct points,
\[
(k_N)^k P(T_{\sqrt{N}x_1} \leq Nt, T_{\sqrt{N}x_2} \leq Nt, ..., T_{\sqrt{N}x_k} \leq Nt)
\]
\[
\sim \sum_{\sigma \in \Sigma_k} \int_{[0,t]^k} \prod_{m=1}^{k} g_{t_m-t_{m-1}}(x_{\sigma(m)} - x_{\sigma(m-1)})dt_1 \cdots dt_k. \quad (2.10)
\]

**Proof.** Use Lemma 2.2

\[\square\]

Here is a byproduct, the case for the point-to-point function.

**Lemma 2.6.** \(d = 2,3\) and \(0, x, x_1\) are distinct points,
\[
k_N \cdot P(T_{\sqrt{N}x_1} \leq Nt \mid S_{Nt} = \sqrt{N}x) \sim \int_0^{t} g_{t_1}(x_1)g_{t-t_1}(x - x_1)dt_1/g_t(x). \quad (2.11)
\]

**Proof.**
\[
P(T_{\sqrt{N}x_1} = j_1, S_{Nt} = \sqrt{N}x)
\]
\[
= P(S_{Nt} = \sqrt{N}x \mid T_{\sqrt{N}x_1} = j_1)P(T_{\sqrt{N}x_1} = j_1) = P(S_{Nt} = \sqrt{N}x \mid S_{j_1} = \sqrt{N}x_1)P(T_{\sqrt{N}x_1} = j_1).
\]

\[\square\]
2.3 Convergence for $d = 2, 3$

Let $\beta_N = \beta a_N$, $a_N = (\sqrt{N}/\log N)^{-1}$ if $d = 2$; $a_N = N^{-1/4}$ if $d = 3$. Moreover,

\[ \psi_{N,t}(x_1, \ldots, x_k) := a_N^k P(T_{\sqrt{N}x_1} \leq Nt, \ldots, T_{\sqrt{N}x_k} \leq Nt) \]  

(2.12)

and

\[ \psi_t(x_1, \ldots, x_k) := \sum_{\sigma \in \Sigma_k} \int_{[0,t]^k} \prod_{m=1}^k g_{t_m-t_{m-1}}(x_{\sigma(m)} - x_{\sigma(m-1)}) \, dt_1 \cdots dt_k. \]  

(2.13)

The space scaling $v_N := 1/\sqrt{N}^d$, then we have $a_N/\sqrt{v_N} = k_N$. From the previous section,

\[ \lim_{N \to \infty} \frac{v_N^{-k/2}}{\psi_{N,t}(x_1, \ldots, x_k)} = \psi_t(x_1, \ldots, x_k). \]  

(2.14)

Recall that $\eta_{N,x} := \frac{1}{a_N} \left( e^{\beta N \omega_{\sqrt{N}x} - \Lambda(\beta N)} - 1 \right)$.

Let an index set $T_N = \{N^{-1/2}k : k \in \mathbb{Z}^d\} \subset \mathbb{R}^d$ and a family of polynomial chaos expansions $(\Psi_N(\eta_{N,x}))$. Let $\mu_0(x) = 0$, $\sigma_0(x) = \hat{\beta}$. We are ready to check the conditions for the case $\mu_0 = 0$ in Theorem 2.3 [4] which we state here:

(i) $\eta_{N,x}$’s are uniformly integrable, $v_N \to 0$ as $N \to \infty$ and $\lim_{N \to \infty} Var(\eta_{N,x}) = \sigma_0^2$.

(ii) There exists $\psi_t$ with $\psi_t \in L^2((\mathbb{R}^d)^k)$ for every $k \in \mathbb{N}$ such that

\[ \lim_{N \to \infty} ||v_N^{-k/2} \psi_{N,t} - \psi_t||_{L^2((\mathbb{R}^d)^k)} = 0. \]  

(2.15)

(iii) \[ \lim_{\ell \to \infty} \limsup_{N \to \infty} \sum_{I \in T_N, |I| > \ell} (\sigma_N^2)^{|I|} \psi_{N,t}(I)^2 = 0. \]  

(2.16)

Proof of Theorem 1.1

(i) $\eta_{N,x}$’s are uniform bounded.

\[ \mathbb{E}(\eta_{N,x})^4 \leq \frac{1}{a_N^4} O(\beta_N^4) = O(1) \]

And $\lim_{N \to \infty} Var(\eta_{N,x}) = \hat{\beta}^2$.

ii) Since $P(S_{Nt} = \sqrt{N}x) \leq C_N \tilde{p}_t(x)$ for all $x$ and $C_N \to 1$ as $N \to \infty$,

\[ \int |v_N^{-k/2} \psi_{N,t}(\cdot)|^2 \prod_{i=1}^k dx_i \]

\[ \leq C_N \int \prod_{i=1}^k dx_i \left[ \sum_{\sigma \in \Sigma_k} \int_{[0,t]^k} \tilde{p}_{t_j-t_{j-1}}(x_{\sigma(j)} - x_{\sigma(j-1)}) \prod_{j=1}^k dt_j \right]^2 := C_N I_{k,t}. \]
Notice that
\[
\int_{[0,t]} \bar{p}_{t_j-t_{j-1}}(x_{\sigma(j)} - x_{\sigma(j-1)}) \prod_{j=1}^{k} dt_j 
\leq \prod_{j=1}^{k} \int_{0}^{t} p_s(x_{\sigma(j)} - x_{\sigma(j-1)}) ds.
\]

So
\[
I_{k,t} \leq k! \sum_{\sigma \in \Sigma_k} \int_{0}^{t} \prod_{i=1}^{k} dx_i \left[ \prod_{j=1}^{k} \int_{0}^{t} p_{s_1}(x_{\sigma(j)} - x_{\sigma(j-1)}) ds_1 \right]^2
\]
\[
= k! \sum_{\sigma \in \Sigma_k} \int_{0}^{t} \int_{0}^{t} ds_1 ds_2 \left[ \prod_{i=1}^{k} dx_i \prod_{j=1}^{k} p_{s_1}(x_{\sigma(j)} - x_{\sigma(j-1)}) p_{s_2}(x_{\sigma(j)} - x_{\sigma(j-1)}) \right]
\]
\[
= \binom{k}{\ell} \int_{0}^{t} \int_{0}^{t} ds_1 ds_2 \left[ \int \prod_{i=1}^{k} p_{s_1}(x_{i}) p_{s_2}(x_{i}) dy_i \right]^k
\]
\[
= C(k!)^2 \int_{0}^{t} \int_{0}^{t} ds_1 ds_2 \frac{1}{(s_1 + s_2)^{d/2}}
\]
\[
\leq C(k!)^2 \int_{0}^{t} \int_{0}^{t} ds_1 ds_2 \frac{1}{s_1^{d/4} s_2^{d/4}}
\]
which is finite since \(d = 2, 3\).

iii) First, \(Var(\eta_{N,x}) = \sigma_N^2 \leq B\) for some \(B \in (0, \infty)\). And
\[
\sum_{l \in T, l > \ell} (\sigma_N^2)^l |\psi_N l(I)|^2
\]
\[
\leq \sum_{k > \ell} B^k \frac{1}{k!} \sum_{(z_1, \ldots, z_k) \in (T_N)^k} \psi_N(z)^2
\]
\[
= \sum_{k > \ell} B^k \frac{1}{k!} ||\psi_N^{-k/2} \psi_N l||^2_{L^2(\mathbb{R}^d)^k}.
\]

From [6] Theorem 2.2.3 p. 29 and p. 41 with \(p = 2\), and for each \(k\), the \(L^2\) norm has an upper bound \((k!)^{d/2} t^{1-d} \ell^{k} C\). Thus,
\[
\leq \sum_{k > \ell} B^k \frac{1}{k!} (k!)^{d/2} t^{1-d} \ell^{k} C^k.
\]
The latter term goes to 0 when \(\ell \to \infty\) for both \(d = 2, 3\). The completes the case \(d = 2, 3\) in Thoerem 1.1.

Remark. For the point to point case,
\[
(k_N)^k P(T_{\sqrt{N}x_1} \leq Nt, \ldots, T_{\sqrt{N}x_k} \leq Nt| S_{Nt} = \sqrt{N}x)
\]
\[
\sim \frac{1}{gt(x)} \sum_{\sigma \in \Sigma_k} \int_{[0,t]} \prod_{m=1}^k g_{t_m - t_{m-1}}(x_{\sigma(m)} - x_{\sigma(m-1)}) g_{t-t_k}(x - x_{\sigma(k)}) \, dt_1 \cdots dt_k.
\]

And
\[
\psi_{N,(t,x)}^c(x_1, \ldots, x_k) := a_N^k P(T_{\sqrt{N}x_1} \leq Nt, \ldots, T_{\sqrt{N}x_k} \leq Nt | S_{Nt} = \sqrt{N}x),
\]

(2.17)

\[
\psi_{t,x}^c(x_1, \ldots, x_k) := \frac{1}{gt(x)} \sum_{\sigma \in \Sigma_k} \int_{[0,t]} \prod_{m=1}^k g_{t_m - t_{m-1}}(x_{\sigma(m)} - x_{\sigma(m-1)}) g_{t-t_k}(x - x_{\sigma(k)}) \, dt_1 \cdots dt_k.
\]

(2.18)

The superscript \(c\) stands for “constrained”. We then have
\[
\lim_{N \to \infty} \psi_{N,(t,x)}^c(x_1, \ldots, x_k) = \psi_{t,x}^c(x_1, \ldots, x_k).
\]

(2.19)

In total, say \(t = 1\),
\[
Z_N(x) := E(\exp(\beta_N H_N) | S_N = \sqrt{N}x) \to \text{law} Z_{W,c}(x).
\]

One can define the point-to-point partition function,
\[
Z^W(x) := Z_{W,c}(x) \tilde{p}_1(x).
\]

2.4 \(d = 1\)

For \(d = 1\), take \(\beta_N = \hat{\beta} N^{-1/4}\). We first have
\[
P(T_{\sqrt{N}x} \leq Nt) \to P(T_x \leq t).
\]

For the k-point function
\[
P(T_{\sqrt{N}x_1} \leq Nt, T_{\sqrt{N}x_2} \leq Nt, \ldots, T_{\sqrt{N}x_k} \leq Nt) \to P(T_{x_1} \leq t, T_{x_2} \leq t, \ldots, T_{x_k} \leq t),
\]

and the last term is equal to
\[
P_0(\min_{0 \leq s \leq t} B_s \leq x_1 < x_2 < \ldots < x_k \leq \max_{0 \leq s \leq t} B_s).
\]

So it is easy to see the case \(d = 1\) in Theorem 1.1.

3 A special case for \(d = 1\) when \(\beta\) is finite

In this section, we consider the case \(h = 0\). In the mean time, we choose the underlying process as one-dimensional Brownian motion \(\{B(t)\}_{t \geq 0}\), and the random environment is modeled by a two-sided Brownian motion \(\{W(x)\}_{-\infty < x < \infty}\). \(B\) and \(W\) are independent. The Hamiltonian we have is
\[ H_t := \int_{x \in \mathbb{R}} W(dx) = \int_{m_t}^{M_t} W(dx) = W_{M_t} - W_{m_t}, \]  

(3.1)

and

\[ Z_t := E(\exp(\beta H_t)) \]  

(3.2)

where \( M_t := \max_{0 \leq s \leq t} B_s \) and \( m_t := \min_{0 \leq s \leq t} B_s \). The annealed partition function is easily computed.

\[ E Z_t := E(\exp(\frac{1}{2} \beta^2 R_t)), \]  

(3.3)

where \( R_t = M_t - m_t \) is the range of the Brownian motion \( B \) up to time \( t \). By rescaling \( W \),

\[ Z_t =^d E \exp(\beta t^{1/3} (W_{M_t/t^{2/3}} - W_{m_t/t^{2/3}})). \]

Denote \( T = t^{1/3} \),

\[ Z_t =^d E \exp(\beta T (W_{M_T/T} - W_{m_T/T})) \]

\[ = E \exp(\beta T (W_{M_T/T} - W_{m_T/T})). \]

The last equality is obtained by rescaling \( B \). [2] calculated the explicit joint density of \( (m_T, M_T) \). If the brownian motion \( B \) is ballistic, namely, of order \( T \), the price to pay is \( \exp(-cT) \), and the energy term gives \( \exp(\tilde{c}T) \) as well. So

\[ \frac{1}{T} \log E \exp(\beta T (W_{M_T/T} - W_{m_T/T})) \approx O(1). \]

Then we have

\[ t^{-1/3} \log Z_t \approx O(1) \]  

(3.4)

from above discussions. From the scaling relation \( \frac{1}{3} = 2\chi - 1 \), \( 3.4 \) suggests that the scale \( \chi \) of the end-point of the polymer in the 1D case is \( t^{2/3} \), which is the same as \((1 + 1)\)-directed polymer.

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