Asymptotic behavior of observables in the asymmetric quantum Rabi model

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Abstract

The asymmetric quantum Rabi model with broken parity invariance shows spectral degeneracies in the integer case, that is when the asymmetry parameter equals an integer multiple of half the oscillator frequency, thus hinting at a hidden symmetry and accompanying integrability of the model. We study the expectation values of spin observables for each eigenstate and observe characteristic differences between the integer and noninteger cases for the asymptotics in the deep strong coupling regime, which can be understood from a perturbative expansion in the qubit splitting. We also construct a parent Hamiltonian whose exact eigenstates possess the same symmetries as the perturbative eigenstates of the asymmetric quantum Rabi model in the integer case.

Keywords: quantum Rabi model, symmetry, degeneracy, asymptotics

(Some figures may appear in colour only in the online journal)

1. Introduction

The importance of the simplest model for light-matter interaction, introduced in its semi-classical form by Rabi eighty years ago, has been emphasized in the introductory article to this special issue [1]. Its fully quantized version, the quantum Rabi model (RM), given by the Hamiltonian

\[ H_{RM} = \omega a^\dagger a + g(a^\dagger + a)\sigma_x + \Delta \sigma_z, \]  

(1)

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was studied using the rotating-wave approximation by Jaynes and Cummings [2]. It involves a spin-\(\frac{1}{2}\) with standard Pauli matrices \(\sigma\),

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]  

(2)

representing an atomic two-level system (qubit). The qubit splitting is given by \(\Delta = \Omega/2\), where \(\Omega\) represents the qubit frequency (we set \(\hbar = 1\) throughout), or, in solid-state realizations such as circuit QED [3] as well as in the polaron picture [4], the energy gap or hybridization. The qubit couples to the radiation field (in dipole approximation), which is described by a single harmonic oscillator with number eigenstates \(|n\rangle\), where \(a^\dagger a|n\rangle = n|n\rangle\) and \([a, a^\dagger] = 1\).

Realizations of (1) with ultracold atomic gases were also proposed recently [5].

The Hamiltonian (1) has an important discrete symmetry, because it commutes with the operator \(P = \sigma_z (-1)^{a^\dagger a}\). Since \(P^2 = 1\), the symmetry group is \(\mathbb{Z}_2\), i.e. the Hilbert space \(\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2\) is the direct sum of two invariant spaces with fixed parity, \(\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-\).

\[
\mathcal{H}_p = \{|\psi_p\rangle \mid P|\psi_p\rangle = p|\psi_p\rangle\}, \quad p = \pm 1.
\]  

(3a)

This discrete symmetry renders \(H_{RM}\) integrable [6]. The rotating-wave approximation for \(H_{RM}\) keeps only the coupling of \(\sigma_\pm = \sigma_x \pm i \sigma_y\) to \(a\) and \(a^\dagger\), respectively. This extends the symmetry to a continuous \(U(1)\) symmetry, leading to the superintegrability of the Jaynes–Cummings model,

\[
H_{JCM} = \omega a^\dagger a + g(a^\dagger \sigma_- + a \sigma_+) + \Delta \sigma_z,
\]  

(4)

for which the Hilbert space separates into an infinite number of two-dimensional invariant subspaces. In the spectral graph, i.e. a plot of eigenenergies as a function of a parameter such as \(g\), the larger symmetry group of \(H_{JCM}\) creates additional level crossings, producing infinitely many two-legged ‘ladders’; for \(H_{RM}\), on the other hand, the spectral graph consists of two ladders with infinitely many legs, since there are only two invariant subspaces (each infinite dimensional) related to the two eigenvalues \(p = \pm 1\) of \(P\) [6, 7]. These two ladders intersect in the spectral graph at the so-called Juddian points [8], corresponding to quasi-exact, doubly degenerate eigenvalues of \(H_{RM}\) [9, 10].

A generalization of \(H_{RM}\) is the asymmetric quantum Rabi model (ARM) [6, 11] with Hamiltonian

\[
H_{ARM} = H_{RM} + \epsilon \sigma_z = \omega a^\dagger a + g(a^\dagger + a)\sigma_x + \epsilon \sigma_z + \Delta \sigma_z.
\]  

(5)

The term \(\epsilon \sigma_z\) corresponds to spontaneous flips of the two-level system and appears naturally in implementations with flux qubits [3, 12]. This term breaks the \(\mathbb{Z}_2\) invariance of \(H_{RM}\), so that \(H_{ARM}\) possesses no obvious symmetry and no level crossings should be expected in the spectral graph for any \(\epsilon \neq 0\). Indeed, no level crossings are observed if \(\epsilon\) is not an integer multiple of \(\omega/2\), which we call the noninteger case henceforth. Furthermore the ARM has a quasi-exact exceptional spectrum just like the RM [13], but the quasi-exact eigenstates are no longer doubly degenerate in the noninteger case.

Surprisingly, degeneracies are observed again if \(\epsilon\) is an integer multiple of \(\omega/2\), to which we refer as integer values or the integer case from now on. These degeneracies correspond to two intersecting ladders with infinitely many legs in the spectral graph, just as in the RM, thus suggesting the presence of a hidden \(\mathbb{Z}_2\) symmetry of the ARM in the integer case [6, 14]. This symmetry would make the integer case of the ARM (iARM) integrable again according to the level labeling criterion proposed in [6]. For this reason, the ARM has been under intense study [13, 15–17]. In fact, a proof for these degeneracies was developed for the case \(\epsilon = \omega/2\) [10, 18, 19] and can be generalized to higher half-integer multiples of \(\omega\).
In this paper, we investigate another aspect of the ARM, namely the effect of integer values of the asymmetry parameter on the asymptotics of spin expectation values in the energy eigenstates, which were also recently studied in [20]. We begin with numerical results for these quantities in section 2. These can be understood perturbatively for small hybridization $\Delta$ (sections 3–5). Furthermore, the lowest-order perturbative eigenstates serve as eigenstates of a parent Hamiltonian which we construct in section 6, and which might be useful in understanding the symmetries of the iARM. We close with a summary and outlook in section 7.

2. Spin expectation values

In this section, we discuss numerical results for spin expectation values in the energy eigenstates of the ARM. For a general state, 

$$|\psi\rangle = \left( |\phi^\uparrow\rangle \right) \left( |\phi^\downarrow\rangle \right), \quad (6)$$

where $|\phi_\sigma\rangle$ are states of the oscillator, expectation values of the spin operator are given by

$$\langle \sigma_x \rangle = 2 \text{Re} \langle \phi^\uparrow | \phi^\downarrow \rangle, \quad (7a)$$

$$\langle \sigma_y \rangle = 2 \text{Im} \langle \phi^\uparrow | \phi^\downarrow \rangle, \quad (7b)$$

$$\langle \sigma_z \rangle = \langle \phi^\uparrow | \phi^\uparrow \rangle - \langle \phi^\downarrow | \phi^\downarrow \rangle. \quad (7c)$$

2.1. Symmetric Rabi model

Consider first the symmetric RM, i.e. the case $\epsilon = 0$. All eigenstates $|\psi_p\rangle$ with fixed parity $p = \pm 1$ have the form

$$|\psi_+\rangle = \left( |\phi^\uparrow_{-}\rangle \right)^e, \quad |\psi_-\rangle = \left( |\phi^\downarrow_{-}\rangle \right)^o, \quad (8)$$

where

$$|\phi^e\rangle = \sum_{n=0,2,4,\ldots} a^e_n |n\rangle, \quad |\phi^o\rangle = \sum_{n=1,3,5,\ldots} a^o_n |n\rangle. \quad (9)$$

Therefore the overlap $\langle \phi^\uparrow | \phi^\downarrow \rangle$ which appears in $\langle \sigma_x \rangle$, $\langle \sigma_z \rangle$ vanishes for all eigenstates of $H_{RM}$, because Fock states with an even number of photons are orthogonal to all states with an odd number. Hence $\langle \sigma_x \rangle$ and $\langle \sigma_z \rangle$ are zero for the RM, and in particular independent of $g$ and $\Delta$.

2.2. Asymmetric Rabi model

Consider next the ARM with $\epsilon \neq 0$. In this case the parity invariance is broken and the value of $\langle \sigma_x \rangle$ and $\langle \sigma_z \rangle$ depend on the state, or, for the energy eigenstates, on $g$ and $\Delta$. ($\langle \sigma_y \rangle$ remains identically zero, because $H$ can be represented as an orthogonal matrix with real eigenvectors.) From now on we set

$$\epsilon = \frac{1}{2} M \omega, \quad (10)$$

so that the integer (noninteger) case corresponds to integer (noninteger) $M$, respectively.
Figures 1–4 show numerical exact diagonalization data for \langle \sigma_x \rangle and \langle \sigma_z \rangle as a function of \( g \) for the lowest few eigenstates. For large \( g \), we observe that for integer \( M \) the expectation value \( \langle \sigma_x \rangle \) tends to \( -1 \) only for the lowest \( M \) energy eigenstates (figures 2 and 3) and to zero otherwise, while in the noninteger case \( \langle \sigma_x \rangle \) tends to \( \pm 1 \). On the other hand, \( \langle \sigma_z \rangle \) tends to zero always, for integer and noninteger \( M \).

In the following sections, we offer two ‘physical’ explanations for this behavior. On the one hand, we will show that these asymptotics are characteristic for small hybridization in the deep strong coupling limit (see (25) below), using perturbation theory for small hybridization (sections 3–5). On the other hand we construct a a related Hamiltonian \( H' \) with similar properties (section 6).

3. Perturbation theory for weak hybridization \( \Delta \): preliminaries

In this section we deal with some preparations regarding the perturbation theory for small values of \( \Delta \). From now on the real parameters \( g, \epsilon, \Delta \) are assumed to be nonnegative without loss of generality. Moreover, we assume \( M > 0 \) (see (10)) from now on, omitting the case of the symmetric RM. The positive energy scale \( \omega \) is often set to unity, but we retain it for later convenience. Below we will also use the notation \( \tilde{g} = g/\omega \) and \( \tilde{\Delta} = \Delta/\omega \).

3.1. Rotation of spin quantization axis

For small \( \Delta \) it is preferable to work in the familiar spin-boson picture, i.e. in the eigenbasis of \( \sigma_x \). As usual we perform a rotation of the spin quantization axis by means of a unitary transformation \( \tilde{U} \).

\[
\tilde{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\]  

(11a)

\[
\tilde{\sigma} = \tilde{U}^\dagger \sigma \tilde{U}, \quad \tilde{\sigma}_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \tau_z,
\]

(11b)

\[
\tilde{\sigma}_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -\tau_y, \quad \tilde{\sigma}_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \tau_x.
\]  

(11c)

Here \( \tau \) are again the standard Pauli matrices, relabelled in order to remind us of the transformed basis. In this basis the Hamiltonian reads (removing a constant energy term \( -g^2/\omega \) and omitting the tildes on the transformed Hamiltonian \( H \) and its eigenstates, as well as on the the observables \( \tilde{\sigma} \)),

\[
H = \tilde{U}^\dagger H_{ARM} \tilde{U} + \frac{g^2}{\omega} = H_0 + V,
\]  

(12a)

\[
H_0 = \begin{pmatrix} h_+ & 0 \\ 0 & h_- \end{pmatrix}, \quad V = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix} = \Delta \tau_x,
\]  

(12b)

\[
h_\sigma = \omega a_\sigma a_\sigma + \frac{\sigma M}{2}, \quad a_\sigma = a + \sigma \tilde{g}, \quad \sigma = \pm 1,
\]  

(12c)

i.e. \( H_0 \) can be written in terms of oscillators that are shifted in opposite directions.
3.2. Unperturbed eigenstates and their overlaps

The eigenstates of $H_0$ are

$$|n\sigma(0)\rangle = \left(\frac{\delta_{\sigma+}}{\delta_{\sigma-}}\right)|n\rangle_\sigma, \quad n = 0, 1, \ldots, \sigma = \pm 1,$$  \hspace{1cm} (13)

with eigenvalues

$$E^{(0)}_{n\sigma} = \omega \left(n + \frac{\sigma M}{2}\right),$$  \hspace{1cm} (14)

which are nondegenerate for noninteger $M$, whereas for integer $M$ only the first $M$ energy levels $E^{(0)}_n (n = 0, \ldots, M-1)$ are nondegenerate, followed by the doubly degenerate energy levels $E^{(0)}_n - E^{(0)}_{n-M+} = E^{(0)}_n (n = M, M+1, \ldots)$.  

In (13) we denoted the eigenstates of the shifted number operators $\hat{a}_\sigma^+ \hat{a}_\sigma$ by $|n\rangle_\sigma$. These can be written in terms of the operators and coherent states $|z\rangle_c$ of the unshifted oscillator (denoting $\sigma = -\sigma$) as

$$|n\rangle_\sigma = \frac{(\alpha^+ + \sigma \tilde{g})^n}{\sqrt{n!}} |\sigma \tilde{g}\rangle_c,$$  \hspace{1cm} (15a)

$$|z\rangle_c = e^{\alpha^+ |0\rangle} e^{-\frac{1}{4}z^2}, \quad z \in \mathbb{C},$$  \hspace{1cm} (15b)

due to the property $a |z\rangle_c - z |z\rangle_c$ of the latter. Below we will encounter the overlap

$$-\langle n'|n\rangle_+ = F_{n'\sigma}(2\tilde{g}),$$  \hspace{1cm} (16)

which defines a function $F_{n'\sigma}(x)$ which we now calculate. We first introduce shifted coherent states, which are related to unshifted coherent states by a translation in $z$,

$$|\zeta\rangle_{c,\sigma} = e^{-|\zeta|^2/2} e^{\alpha^+ |0\rangle} |\sigma \tilde{g}\rangle_c = |z - \sigma \tilde{g}\rangle_c e^{i\sigma \tilde{g} \text{ Im} \zeta}.$$  \hspace{1cm} (17)

For shifts in the same direction $\sigma$ they have the usual nonzero overlap of coherent states,

$$c_{\sigma} (\zeta' | \zeta\rangle_{c,\sigma} = e^{-\frac{1}{2} |\zeta'|^2 - \frac{1}{2} |\zeta|^2 + \zeta' \zeta},$$  \hspace{1cm} (18)

while for opposite shifts we obtain

$$c_{-\sigma} (\zeta' | \zeta\rangle_{c,\sigma} = e^{-\frac{1}{2} |\zeta'|^2 - \frac{1}{2} |\zeta|^2 + \zeta' \zeta + 2\tilde{g}(z - \zeta') - 2\tilde{g}^2} = \sum_{n'n' = 0}^{\infty} \frac{-\langle n'|n\rangle_+ (n' | \zeta'\rangle_{c,-\sigma} c_{\sigma} | \zeta\rangle_{c,\sigma}}{\sqrt{n'n'!}} e^{-\frac{1}{2} |\zeta'|^2 - \frac{1}{2} |\zeta|^2}.$$  \hspace{1cm} (19)

It is useful to work with the following two-variable Hermite polynomials $H_{nm}(x,y)$ and their generating function [21],

$$H_{nm}(x,y) = \sum_{k \geq 0} \binom{n}{k} \binom{m}{k} k! (-1)^k x^k y^{m-k},$$  \hspace{1cm} (20)

$$e^{-uv + uv + vy} = \sum_{n,m=0}^{\infty} \frac{u^n v^m}{n! m!} H_{nm}(x,y).$$  \hspace{1cm} (21)
Taking coefficients in (19), we thus obtain
\[ F_{n'}(x) = (-1)^{n'} \sqrt{n'!} H_{n'}(x,x) e^{-x^2/2}, \tag{22a} \]
\[ \sigma \langle n' | n \rangle = F_{n'}(2\bar{g}) \sigma^{n+n'}, \quad \sigma = \pm 1. \tag{22b} \]
We note that for equal arguments the two-variable Hermite polynomials can be written as
\[ H_{nm}(x,x) = x^m \text{e}^{-x^2/2} \sum_{n'=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n \ e^{a_j} \ n!}{\prod_{j=1}^{q} (b_j)_n \ n!}, \tag{24} \]
with Pochhammer symbols expressing rising factorials, i.e. \((a)_n = \Gamma(a+n)/\Gamma(a)\) in terms of the Euler Gamma function.

3.3. A priori conditions for the validity of perturbation theory

We expect that \(V\) may be treated perturbatively if \(\Delta\) is sufficiently small, i.e. small compared to energy differences of the unperturbed energy eigenstates, which involve \(\omega\) and possibly \(\epsilon\). Another energy scale in the problem is \(g\), so that we must necessarily require \(g \gtrsim \omega\) in order to be able to expand in \(\Delta/\omega\), and not having to expand in \(g/\Delta\) first. This puts us in the deep strong coupling regime \([22]\), \(\Delta \ll \omega \lesssim g\). \(\tag{25} \)

As an opposite point of reference, we note the spectrum and expectation values when harmonic oscillator and qubit decouple for \(g = 0\),
\[ E_{n\sigma}^{(g=0)} = \omega n + \sigma \sqrt{\epsilon^2 + \Delta^2}, \quad (a)_{\sigma}^{(g=0)} = \omega n, \tag{26c} \]
which are drawn in figure 1–4 as dotted lines.

The regime (25) fits the case of integer \(M\), for which energy differences of unperturbed eigenstates are always integer multiples of \(\omega\). However, in the case of noninteger \(M\), its fractional part \(\delta M\) leads to smaller energy differences, with magnitude \(|n + \sigma M - n'| \omega\), that appear in the denominator in the corrections for \(E_{n\sigma}\). Minimizing with respect to \(n'\) we find that we must replace (25) by the stronger requirement
\[ \Delta \ll \sqrt{\frac{1 - |\xi|}{2}} \omega \lesssim g, \quad \text{where} \quad \delta M = M - |M| = \frac{1 + \xi}{2}. \tag{27} \]
For the noninteger case, our perturbation theory will thus work best if $|\xi| = 0$, i.e. if $\epsilon = \frac{1}{2}, \frac{3}{4}, \frac{5}{4} \ldots$ etc. On the other hand, for $M$ near an integer, i.e. $\xi = \pm (1 - \rho)$ with $|\rho| \ll 1$, only very small values of the hybridization, $\Delta \ll \sqrt{\rho/2} \omega$, can be expected to be accessible perturbatively.

We use standard Rayleigh–Schrödinger perturbation theory. Nondegenerate perturbation theory applies for noninteger $M$ (see section 4) and degenerate perturbation theory for integer $M$ (see section 5).

4. Perturbation theory for weak hybridization $\Delta$: the case of noninteger $M$ without degeneracies

4.1. Eigenenergy corrections

As described above, the spectrum of $H_0$ is nondegenerate for noninteger $M$. In this case we expand the energy eigenvalue $E_{n\sigma}$ and eigenstate $|n\sigma\rangle$ of $H$ as a power series in $\Delta = \Delta/\omega$,

$$E_{n\sigma} = E_{n\sigma}^{(0)} + E_{n\sigma}^{(1)} + E_{n\sigma}^{(2)} + O(\omega\Delta^3),$$

$$|n\sigma\rangle = |n\sigma^{(0)}\rangle + |n\sigma^{(1)}\rangle + O(\Delta^2).$$

From the unperturbed spectrum we obtain

$$\langle n'\sigma^{(0)}|V|n\sigma^{(0)}\rangle = \Delta_{\sigma} \langle n'|n\rangle_{\sigma} \delta_{\sigma'\sigma},$$

$$E_{n\sigma}^{(0)} - E_{n'\sigma'}^{(0)} = \omega (n - n' + \sigma M) \delta_{\sigma'\sigma}.$$ (29a)

(29b)

For the first two eigenenergy corrections we obtain

$$E_{n\sigma}^{(1)} = \langle n\sigma^{(0)}|V|n\sigma^{(0)}\rangle = 0,$$

$$E_{n\sigma}^{(2)} = \sum_{n'\sigma' \neq n\sigma} \frac{\langle n'\sigma^{(0)}|V|n\sigma^{(0)}\rangle^2}{E_{n\sigma}^{(0)} - E_{n'\sigma'}^{(0)}}$$

$$= \sum_{n' = 0}^{\infty} \omega \Delta^2 F_{n'}(2\bar{\delta})^2 = -\omega \Delta^2 F_{n}(2\bar{\delta}, -n - \sigma M),$$

where we introduced the function (for $z$ not a nonpositive integer)

$$F_n(x, z) = \sum_{m = 0}^{\infty} \frac{F_{m, x}}{m + z} = e^{-x} \sum_{m = 0}^{\infty} \frac{H_{m, x}}{m!}(m + z), \quad z \neq 0, -1, -2, \ldots,$$

(31)

for which a closed form and its asymptotics are derived below (see (41) and (42)).

4.2. Eigenstate corrections and expectation values

The first-order corrections to the eigenstates are given by

$$|n\sigma^{(1)}\rangle = \sum_{n'\sigma' \neq n\sigma} \frac{|n'\sigma'^{(0)}\rangle \langle n'\sigma'^{(0)}|V|n\sigma^{(0)}\rangle}{E_{n\sigma}^{(0)} - E_{n'\sigma'}^{(0)}}$$

$$= \sum_{n' = 0}^{\infty} \frac{\Delta F_{n'}(2\bar{\delta})}{n - n' + \sigma M} |n'\sigma^{(0)}\rangle.$$ (32)
Instead of normalizing these states it is preferable to obtain expectation values in the (normalized) perturbed eigenstates directly from (30) by taking derivatives of the (perturbed) eigenenergy, which is given by

\[ E_{n\sigma} (\omega, g, \epsilon, \Delta) = \omega n + \sigma \epsilon - \frac{\Delta^2}{\omega} F_n \left( \frac{2g}{\omega}, -n - \frac{2\sigma \epsilon}{\omega} \right) + O\left( \frac{\Delta^3}{\omega^3} \right). \]  

(33a)

We obtain, using \( X = 2g / \omega \) and \( Z = -n - \sigma M = -n - 2\sigma \epsilon / \omega \) for the repeatedly occurring arguments,

\[ \langle \sigma_x \rangle_{n\sigma} = \frac{\partial E_{n\sigma}}{\partial \epsilon} = \sigma \left( 1 + 2\Delta^2 F_n^{(0,1)}(X, Z) \right) + O(\Delta^3), \]  

(33b)

\[ \langle \sigma_z \rangle_{n\sigma} = \frac{\partial E_{n\sigma}}{\partial \Delta} = -2\Delta F_n(X, Z) + O(\Delta^2) \]  

(33c)

\[ (a^\dagger a)_{n\sigma} = \frac{\partial}{\partial \omega} \left( E_{n\sigma} - \frac{\Delta^2}{\omega} \right) = n + \Delta^2 \left[ F_n(X, Z) + 2g F_n^{(1,0)}(X, Z) - \sigma M F_n^{(0,1)}(X, Z) \right] + O(\Delta^3), \]  

(33d)

where we use \( f^{m_1, \ldots, m_N}(x_1, \ldots, x_N) \equiv \frac{\partial^{m_1}}{\partial x_{1}} \cdots \frac{\partial^{m_N}}{\partial x_{N}} f(x_1, \ldots, x_N) \) as an abbreviation for partial derivatives throughout.

### 4.3. Evaluation and asymptotics

Here we discuss the function \( F_n(x, z) \) in (31) and its large-\( x \) asymptotics, as well as the resulting asymptotics for observables depicted in figure 1. \( F_n(x, z) \), as a function of complex \( z \), has simple poles at the nonpositive integers. We first consider positive real \( z \) and obtain an explicit closed form for \( F_n(x, z) \), which can then be analytically continued for all \( z \) (except for the poles).

For real \( z > 0 \) we write

\[ F_n(x, z) = e^{x^2} \int_0^1 s^{z-1} \sum_{m=0}^{\infty} \frac{H_{m}(x, x)^2 s^{m}}{n!m!} ds \]  

(34)

and employ the generating function of the two-variable Hermite polynomials [21], specialized to equal arguments,

\[ \sum_{m=0}^{\infty} \frac{H_{n,m}(x, y)H_{m,n}(x', y')} {n!m!} = e^{sx^2} s^n L_n \left( \frac{y}{s} - x' \right) (sx - y') \]  

(35)

\[ \sum_{m=0}^{\infty} \frac{H_{n,m}(x, x)^2 s^{m}} {n!m!} = e^{sx^2} s^n L_n (-x^2 \frac{(1-s)^2}{s}) = e^{sx^2} \sum_{k=0}^{n} \binom{n}{k} \frac{x^{2k}}{k!} s^{n-k} (1-s)^{2k}, \]  

(36)

where we used the explicit form of the simple Laguerre polynomials \( L_n(x) \) in the last line. We insert this into (34) and integrate termwise, using the integral representation of the confluent hypergeometric function,
Figure 1. Eigenenergies and (left) and spin expectation values (middle and right) for the noninteger case of the ARM ($M = 0.5$, $\epsilon = 0.25$) with small hybridization parameter ($\Delta = 0.3$) in terms of $\omega = 1$ as energy unit. As discussed in the text, low-order perturbation theory in $\Delta$ (light/green) describes the exact diagonalization results (dark/blue) well provided $g$ is not too small. For large $g$, $|\langle \sigma_x \rangle|$ tends to 1 and $\langle \sigma_z \rangle$ tends to 0 for all eigenstates. The exact values for $g = 0$, given in (26), are marked by dotted lines.

Figure 2. As figure 1, but for the integer case of the ARM (iARM) with $M = 1$, $\epsilon = 0.5$, $\Delta = 0.3$, $\omega = 1$. In the iARM both $|\langle \sigma_x \rangle|$ and $|\langle \sigma_z \rangle|$ tend to 0 for large $g$, except for $M$ states (i.e. one state in the present case) for which $\langle \sigma_x \rangle$ tends to $-1$. The perturbative results for the spin expectation values are shown only for the $M+1$ lowest energy eigenstates.

Figure 3. As figure 2 for the iARM, but with $M = 2$, $\epsilon = 1$, $\Delta = 0.3$, $\omega = 1$, with two states for which $\langle \sigma_x \rangle$ tends to $-1$. 

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\[ M(a, b, z) = {_1F_1}(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!} = e^z M(b - a, b, -z) \quad (37a) \]

\[ = B(a, b - a) \int_0^1 s^{a-1} (1-s)^{b-a-1} e^z ds, \quad \text{Re} \, b > \text{Re} \, a > 0, \quad (37b) \]

where \( B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b) \) denotes the Euler Beta function. We thus obtain a finite sum of \(_1F_1\) functions,

\[ \mathcal{F}_n(x, z) = \sum_{k} \binom{n}{k} x^{2k} k! B(2k+1, z+n-k) M(2k+1, z+k+n+1, -x^2), \quad (38) \]

after using Kummer’s transformation (37a). The expressions in (38) are also obtained if one instead expands the exponential in (36) as a power series in \( s \) and integrates termwise, using the integral representation of the Euler Beta function,

\[ B(a, b) = \int_0^1 s^{a-1} (1-s)^{b-1} ds, \quad \text{Re} \, b > \text{Re} \, a > 0, \quad (39) \]

and summing the resulting series of type (37a). Next we employ the regularized confluent hypergeometric function \( M(a, b, z) \), which has the advantage that it is an entire function of the parameters \( a \) and \( b \) for fixed \( z \). Its definition and asymptotic expansion for large real argument read

\[ M(a, b, z) = \frac{1}{\Gamma(b)} M(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{\Gamma(b+n) n!} = e^z M(b - a, b, -z) \quad (40a) \]

\[ \sim e^z \sum_{k=0}^{\infty} \frac{(1-a)_k (b-a)_k}{\Gamma(a)_k k!} z^{b-a+k}, \quad z \to \infty \quad (a \neq 0, -1, -2, \ldots). \quad (40b) \]

The desired analytic continuation of (38), valid at all \( z \) except the poles at nonpositive integers, is then given by
\[ \mathcal{F}_n(x,z) = \sum_{k=0}^{n} \binom{n}{k} \frac{x^{2k}}{k!} (2k)! \Gamma(z + n - k) M(2k + 1, z + k + n + 1, -x^2), \quad (41) \]

which evaluates (31). From (40b) we find its asymptotic behavior as

\[ \mathcal{F}_n(x,z) \sim \sum_{k=0}^{\infty} \text{F}_2(-n, -k, k + 1; 1, 1 - z - n; 1) \frac{(1 - z - n)_k}{x^{2k+2}}, \quad x^2 \to \infty, \quad (42a) \]

\[ = \frac{1}{x^2} - \frac{(z + n - 1)(z - 2n)}{x^4} + O(x^{-6}). \quad (42b) \]

In particular, the leading order is asymptotic to \(1/x^2\) and independent of \(z\).

Together with (33), this yields the following scenario for the large-\(g\) asymptotics of \(\sigma_x\) and \(\sigma_z\): in this limit, due to the decay in (42) for large \(x\), we find that \(\langle \sigma_x \rangle \to \sigma = \pm 1\) and \(\langle \sigma_z \rangle \to 0\) for all energy eigenstates, as in figure 1 (where \(\epsilon = 0.25\) is far away from any half integer). As expected from the discussion in section 3, the expressions in (33) describe \(\langle \sigma_x \rangle\) and \(\langle \sigma_z \rangle\) well only if \(g\) is not too small.

### 5. Perturbation theory for weak hybridization \(\Delta\): the case of integer \(M\) with degeneracies (iARM)

#### 5.1. Perturbation-diagonal eigenbasis and energy corrections

In the case of integer \(M\) the unperturbed spectrum has nondegenerate and degenerate parts. For notational convenience we label the unperturbed energies and their eigenstates as \(E_n^{(0)}\) and \(|n; \alpha^{(0)}\rangle\), where \(\alpha\) labels a possible degeneracy.

For integer \(M \geq 1\), the lowest \(M\) unperturbed energy levels \(E_n^{(0)}\) and eigenstates are

\[ E_n^{(0)} = E_{n=0}^{(0)} = \omega \left( n - \frac{M}{2} \right), \quad 0 \leq n \leq M - 1, \quad (43) \]

\[ |n; 0^{(0)}\rangle = |n^{(0)}\rangle = \left( \begin{array}{c} 0 \\ |n\rangle_+ \end{array} \right). \quad (44) \]

whose unperturbed eigenstates (see (13)) are nondegenerate so that the degeneracy label \(\alpha\) takes only one value (zero, by convention); this part of the spectrum is present only for non-zero \(M\). The perturbation expansion for these states is denoted as in (28), and the first-order energy correction remains zero as in (30a). Next follow the doubly degenerate unperturbed energy levels

\[ E_n^{(0)} = E_{n=M+}^{(0)} = E_{n=0}^{(0)} = \omega \left( n - \frac{M}{2} \right), \quad n = M, M + 1, \ldots, \quad (45) \]

with unperturbed eigenstates \(|n - M^{(0)}\rangle\) and \(|n^{(0)}\rangle\), see (13). For the latter we introduce linear combinations that yield only diagonal matrix elements of the perturbation \(V\),

\[ |n; \alpha^{(0)}\rangle = \frac{1}{\sqrt{2}} \left( \begin{array}{c} (n - M)_+ \\ \alpha |n\rangle_- \end{array} \right), \quad \alpha = \pm, \quad n \geq M, \quad (46a) \]
\[ \langle n; \alpha^{(0)} | V | n; \alpha^{(0)} \rangle = \delta_{\alpha\alpha} E_n^{(1)}. \] (46b)

We write the perturbative expansion for small \( \Delta \) as
\[ E_{n;\alpha} = E_n^{(0)} + E_n^{(1)} + E_n^{(2)} + O(\Delta^3), \] (47a)
\[ |n; \alpha\rangle = |n; \alpha^{(0)}\rangle + |n; \alpha^{(1)}\rangle + O(\Delta^2). \] (47b)

The degeneracies are lifted completely in first order, with
\[ E_n^{(1)} = \langle n; \alpha^{(0)} | V | n; \alpha^{(0)} \rangle \]
\[ = \begin{cases} 0 & \text{if } n < M, \, \alpha = 0, \\ \alpha \omega \bar{\Delta} F_{n,n-M}(2\bar{g}) & \text{if } n \geq M, \, \alpha = \pm. \end{cases} \] (48)

The second-order correction reads
\[ E_n^{(2)} = \sum_{n',(\neq n),\alpha'} \frac{\langle n'; \alpha^{(0)} | V | n'; \alpha^{(0)} \rangle^2}{E_n^{(0)} - E_{n'}^{(0)}} \]
\[ = -\omega \bar{\Delta}^2 \sum \frac{F_n(2\bar{g}, M - n)}{2} \frac{1}{E_n^{(0)} - E_{n'}^{(0)}} \]
\[ = -\omega \bar{\Delta}^2 \left( \frac{1}{2} \sum G_n^{(n-M)}(2\bar{g}) + G_n^{(n)}(2\bar{g}) \right) \text{ if } n \geq M, \, \alpha = \pm, \] (49)

where we defined \( G_p^{(q)}(x) \) (for integer \( p, q \geq 0 \)) as
\[ G_p^{(q)}(x) = \sum_{m \geq 0} \frac{F_{pm}(x)^2}{m - q} = e^{-x^2} \sum_{m \geq 0} \frac{H_{pm}(x)^2}{p!m!(m - q)}, \] (50)
for which a closed form and its large-\( x \) asymptotics are derived below (see (58) and (59)).

### 5.2. Eigenstate corrections and expectation values

Next we determine the first-order corrections to the eigenstates. For the initially nondegenerate states (\( 0 \leq n < M, \, \alpha = 0 \)) these are
\[ |n; 0^{(1)}\rangle = \sum_{n',\alpha'} \frac{\langle n'; \alpha^{(0)} | V | n; \alpha^{(0)} \rangle}{E_n^{(0)} - E_{n'}^{(0)}} |n'; \alpha'\rangle \]
\[ = \sqrt{2} \Delta \sum_{n' \geq M} \frac{F_{n,n-M}(2\bar{g})}{n'-n} \left( \frac{|n' - M|}{0} \right), \quad 0 \leq n < M. \] (51)

For the initially degenerate states (\( n \geq M, \, \alpha = \pm \)) we have
\[ |n; \alpha^{(1)}\rangle = \sum_{n',\alpha'} \frac{\langle n'; \alpha^{(0)} | V | n; \alpha^{(0)} \rangle}{E_n^{(0)} - E_{n'}^{(0)}} |n'; \alpha'\rangle \]
\[ = \sum_{n',\alpha'} |n'; \alpha'\rangle \tilde{E}_{n',\alpha';\alpha}, \] (52)
where for different energies (\( n' \neq n, \, \alpha' = \pm \))
\[ C_{\alpha',\alpha,\sigma}^{(1)} = \frac{\langle n', \alpha'; 0 \rangle | V | n; \alpha \rangle}{E^{(0)}_n - E^{(0)}_m} \]

\[ = \frac{\tilde{\Delta}}{2(n - n')} \times \left\{ \sqrt{\frac{2}{n}} F_{n', M} - \alpha F_{n', M} + \alpha F_{n', M'} \right\} \]

if \( 0 \leq n < M, \alpha' = 0, \)

\[ \text{if } n' \geq M, n' \neq n, \alpha' = \pm. \]

(53)

while for equal energies (\( \alpha' \neq \alpha \), hence only \( \alpha' = -\alpha \equiv \bar{\alpha} \) occurs)

\[ C_{\bar{\alpha}, \alpha, \sigma}^{(1)} = \frac{1}{E^{(1)}_{n, \bar{\alpha}} - E^{(1)}_{n, \alpha'}} \sum_{m, \beta} \frac{\langle n; \alpha'; 0 \rangle | V | m; \beta \rangle \langle m; \beta \rangle | V | n; \alpha \rangle}{E^{(0)}_n - E^{(0)}_m} \]

\[ = \frac{\alpha \tilde{\Delta}}{2} C_{n, M}(2\tilde{g}) \delta_{\bar{\alpha}, \bar{\alpha}}, \]

(54a)

\[ C_{n, M}(x) = \frac{G_n^{(n-M)}(x) - G_n^{(n)}(x)}{2F_{n-M,M}(x)}. \]

(54b)

We use these eigenstate corrections to calculate the expectation value of \( \sigma_x \) and \( a^\dagger a \), whereas a derivative suffices for that of \( \sigma_z \). We obtain

\[ \langle \sigma_x \rangle_{\|} = \langle \tau_x \rangle_{\|} = \begin{cases} 1 + O(\tilde{\Delta}^2) & \text{if } n < M, \alpha = 0, \\ \alpha \tilde{\Delta} C_{n, M}(2\tilde{g}) + O(\tilde{\Delta}^2) & \text{if } n \geq M, \alpha = \pm, \end{cases} \]

(55a)

\[ \langle \sigma_z \rangle_{\|} = \langle \tau_z \rangle_{\|} = \frac{\partial E_{n, \alpha}}{\partial \Delta} = \frac{E^{(1)}_{n, \alpha}}{E^{(1)}_{n, \alpha}} + 2E^{(2)}_{n, \alpha} \frac{\omega}{\omega}, \]

\[ = \left\{ \begin{array}{ll} \alpha F_{n-M, M}(2\tilde{g}) - 2\tilde{\Delta} F_{n-M, M}(2\tilde{g}) & \text{if } n < M, \alpha = 0, \\ -\tilde{\Delta} [G_{n-M}^{(n-M)}(2\tilde{g}) + G_{n-M-M}^{(n-M-M)}(2\tilde{g})] + O(\tilde{\Delta}^2) & \text{if } n \geq M, \alpha = \pm. \end{array} \right. \]

(55b)

\[ \langle a^\dagger a \rangle_{\|} = \begin{cases} n + \tilde{g}^2 + O(\tilde{\Delta}^2) & \text{if } n < M, \alpha = 0, \\ n + \tilde{g}^2 - \frac{M}{2} (1 + \alpha \tilde{\Delta} C_{n, M}(2\tilde{g})) + O(\tilde{\Delta}) & \text{if } n \geq M, \alpha = \pm, \end{cases} \]

(55c)

where we have calculated only part of the linear order in \( \tilde{\Delta} \) in (55c), namely that which comes from \( C_{\alpha, \alpha', \bar{\alpha}}^{(1)} \).

Note that in the integer case, i.e. for a fixed integer value of \( 2\epsilon/\omega = M \), the parameter \( \omega \) now couples to the combined term \( a^\dagger a + \frac{i}{2} M \sigma_x \) in the Hamiltonian, in contrast to the noninteger case of the previous section. From the derivative with respect to \( \omega \) we thus obtain

\[ \langle a^\dagger a \rangle_{\|} + \frac{M}{2} \langle \sigma_x \rangle_{\|} = \frac{\partial}{\partial \omega} \left( E_{n, \alpha} - \frac{\tilde{g}^2}{\omega} \right) \]

\[ = \left\{ \begin{array}{ll} n + \tilde{g}^2 + O(\tilde{\Delta}^2) & \text{if } n < M, \alpha = 0, \\ n + \tilde{g}^2 - \frac{M}{2} + 2 \alpha \tilde{g} \tilde{\Delta} F_{n-M-M}(2\tilde{g}) + O(\tilde{\Delta}^2) & \text{if } n \geq M, \alpha = \pm. \end{array} \right. \]

(55d)

Note that the parts linear \( \tilde{\Delta} \) listed in (55a) and (55c) cancel in (55d), although another linear term (from \( \langle a^\dagger a \rangle \)) remains in the latter.
We can now understand the qualitative behavior of \( \langle \sigma_x \rangle \) as follows. Consider for now only the zeroth order in \( \Delta \) in (55a). In this order the lowest \( M \) eigenstates have \( \langle \sigma_x \rangle = -1 \) and \( \langle a^\dagger a \rangle = n + \frac{g^2}{2} \), but all higher eigenstates have \( \langle \sigma_x \rangle = 0 \) and \( \langle a^\dagger a \rangle = n + \frac{g^2}{2} - \frac{M}{2} \). In view of (55d) we can interpret this as due to the coupling of \( \omega \) to both these expectation values for integer \( M \), and \( \langle a^\dagger a \rangle \) (rather than \( \frac{i}{2} M \langle \sigma_x \rangle \)) absorbing the contribution \( -\frac{M}{2} \) for the states with \( n \geq M \) which are indirectly coupled by the hybridization \( \Delta \). This provides a qualitative reason why, in contrast to the noninteger case, \( \langle \sigma_x \rangle \neq 1 \) should be expected in the integer case. At the end of the next subsection we will discuss the effect of higher orders in \( \Delta \).

### 5.3. Evaluation and asymptotics

We now discuss the function \( G_{pq}^{(q)}(x) \) in (50), for nonnegative integers \( p \) and \( q \). First we express it in terms of the known function \( F_p(x,z) \) without its pole \( R_{pq}(x)/(z + q) \) at \( z = -q \),

\[
G_{pq}^{(q)}(x) = \lim_{z \to -q} \left[ F_p(x,z) - \frac{R_{pq}(x)}{z + q} \right]. \tag{56a}
\]

\[
R_{pq}(x) = \lim_{z \to -q} ((z + q) F_p(x,z)) = e^{-\frac{H_p(x,x)^2}{p q}}. \tag{56b}
\]

For \( z \) near the pole at \( -q \) we see from (41) that the Gamma function yields divergent as well as regular contributions to \( F_p(x,z) \). Specifically, for integer \( m \geq 0 \) and \( |\delta| < 1 \),

\[
\Gamma(m-q+\delta) = \begin{cases} 
\frac{1}{\delta} \left( \frac{1}{(q-m)!} \right) \left[ 1 + \psi(1+q-m) \delta + O(\delta^2) \right] & \text{if } m \leq q, \\
(m-q)! + O(\delta) & \text{if } m > q,
\end{cases} \tag{57}
\]

where \( m \) corresponds to \( n - k \) in (41). In the second case we may set \( \delta \) to zero to obtain a constant contribution (provided \( q < p \)), while in the first case we first subtract the pole \( R_{pq}(x)/\delta \) and then extract the constant term from the next order in \( \delta \) which involves \( M(a,b,x) \) and the Euler Digamma function \( \psi(z) = \Gamma'(z)/\Gamma(z) \). We thus obtain, after some rearrangement,

\[
G_{pq}^{(q)}(x) = \Theta(p > q) \sum_{k=0}^{p-q-1} \binom{p}{k} \frac{x^{2k}}{k!} B(2k+1,p-q-k) M(2k+1,p-q+1+k,-x^2)
\]

\[+ e^{-x^2} \sum_{k=\max(0,p-q)}^{p} \binom{p}{k} \frac{x^{2k}}{k!} (-1)^k \frac{1}{(k-p+q)!} \psi(k-p+q+1) \]

\[\times M(p - q - k, p - q + 1 + k, x^2) + M^{(0,0)}(p - q - k, p - q + 1 + k, x^2) + M^{(0,1,0)}(p - q - k, p - q + 1 + k, x^2) \]. \tag{58}

where the step function \( \Theta(A) \) is one or zero according to whether \( A \) is true or false, respectively; the first sum is thus absent if \( q \geq p \). In the second sum the hypergeometric functions terminate because \( k \geq p - q \). For large real \( x \) the second sum is therefore exponentially small, while the asymptotic expansion of the first sum follows again from (40). We obtain,
The following harmonic oscillator identities are straightforward to obtain:
\[
G_p^{(q)}(x) = \Theta(p > q) \sum_{k=0}^{\infty} \text{F}_2(-p, -k, k + 1; 1, q + 1 - p; 1) \frac{(q + 1 - p)k}{x^{2k+2}} + O(x^{2q}e^{-x^2})
\]

\[
= \Theta(p > q) \frac{1}{x^2} + (p + q + 1) \Theta(p > q - 1) \frac{1}{x^4} + O(x^{-6}) + O(x^{2q}e^{-x^2})
\]
for \( x^2 \to \infty \).

From (55) we thus obtain the following scenario for the spin expectation values in figures 2–4 for large \( g \). The perturbative result for \( \langle \sigma_z \rangle \) tends to zero this limit for all energy eigenstates in agreement with the numerical result, as in the noninteger case. Furthermore, numerically we observe \( \langle \sigma_z \rangle \to -1 \) for the lowest \( M \) energy eigenstates which corresponds to the (constant) perturbative result in linear order in \( \Delta \). In figures 2–4, the perturbative result (55a) captures the qualitative behavior of \( \langle \sigma_z \rangle \) for the eigenstates with \( n \geq M \) in first order in \( \Delta \) for not too small \( g \). However, it cannot be used to obtain the large-\( g \) asymptotics, because it contains \( F_{x,n-M}(x) \) in the denominator which is asymptotic to \( x^M e^{-x^2/2} \) and will thus eventually invalidate the perturbative result for large \( g \). The large-\( g \) asymptotics can thus be only partially be understood from the small-\( \Delta \) behavior in the integer case, due to the more complicated interplay of the expansion parameters \( g \) and \( \Delta \).

6. A model with number-nonconserving hybridization related to the iARM

6.1. Construction in terms of exact eigenstates

We now construct a solvable model \( H' \) which is related to the integer case of the asymmetric Rabi model \( H \) as follows. We demand that it has the states \( |m; \alpha^{(0)} \rangle \) (for which \( V \) has only diagonal expectation values) as exact eigenstates but nevertheless contains a hybridization term \( V' \), i.e.

\[
H' = H_0 + V',
\]

\[
H'|n; \alpha^{(0)}\rangle = E'_n|n; \alpha^{(0)}\rangle,
\]

where \( H_0 \) is given in (12) and \( M \) is a positive integer.

First we obtain possible forms of the operator \( V' \) that are off-diagonal like \( V \), i.e. noncommuting with \( \tau_z \). The following harmonic oscillator identities are straightforward to obtain from the Baker–Campbell–Hausdorff formulas,

\[
e^{z(a-a^\dagger)}(a^\dagger)^p = (a^\dagger + z)^p e^{z(a-a^\dagger)},
\]

\[
e^{2z(a-a^\dagger)}|z\rangle_c = |z^2\rangle_c,
\]

from which we obtain the operators \( R_{\sigma}, \sigma = \pm \), which transform between the shifted harmonic oscillators of (12),

\[
R_{\sigma}|n\rangle = |n\rangle_{\sigma}, R_{\sigma} = e^{-2\sigma \tilde{\delta}(a-a^\dagger)} = R_{\sigma}^\dagger = R_{\sigma}^{-1}.
\]

We may view \( R_{\sigma} \) as performing two successive reverse shifts \( U^{-\sigma} \) of a shifted oscillator state, one back to the original oscillator and one further shift into the other shifted oscillator, i.e.

\[
R_{\sigma} = U^{-2\sigma}, \quad U = e^{\delta(a-a^\dagger)} = (U^\dagger)^{-1}.
\]
With these operators we can transform either component of \(|n; \alpha(0)\rangle\) into the other. Namely, for \(n \geq M\),

\[
Ua^M U|n\rangle_- = a^M R_- |n\rangle = w_n |n - M\rangle_+,
\]

\[
U^\dagger (a^\dagger)^M U^\dagger |n - M\rangle_+ = R_+ (a^\dagger)^M |n - M\rangle_+ = w_n |n\rangle_-,
\]

\[
w_n = \sqrt{\frac{n!}{(n-M)!}} = \sqrt{n(n-1) \cdots (n-M+1)}.
\]

so that we have \((n \geq M, \alpha = \pm 1)\)

\[
\left( \begin{array}{c}
0 \\
U^\dagger (a^\dagger)^M U^\dagger
\end{array} \right) |n; \alpha(0)\rangle = \alpha w_n |n; \alpha(0)\rangle.
\]

Hence a rather general operator \(V'\) with the property \(V'|n; \alpha(0)\rangle = \nu'_{\alpha n} |n; \alpha(0)\rangle\) can be written as

\[
V' = \left( \begin{array}{cc}
0 & U f(a^\dagger a)^M U^\dagger \\
U^\dagger (a^\dagger)^M U^\dagger & 0
\end{array} \right),
\]

\[
\nu'_{\alpha n} = \begin{cases} 
\alpha w_n f(n-M) & n \geq M, \alpha = \pm 1 \\
0 & 0 \leq n \leq M - 1, \alpha = 0
\end{cases}
\]

\[
E'_{\alpha n} = E^{(0)}_{\alpha n} + \nu'_{\alpha n},
\]

where \(f(n)\) is an arbitrary complex function of nonnegative integer \(n\). For example, we might choose

\[
f(n) = \frac{\Delta_{n+M}}{w_{n+M}} = \frac{\Delta_{n+M}}{\sqrt{(n+1) \cdots (n+M)}},
\]

\[
\Rightarrow \nu'_{\alpha n} = \alpha \Delta_n.
\]

We note a slight resemblance of \(V'\) to the rotating-wave term in the Jaynes–Cummings Hamiltonian which also contains Hermitian conjugate oscillator operators in the upper and lower off-diagonal.

### 6.2. Making the number-conserving part number-independent

Interestingly, \(V'\) contains a number-conserving part \(\bar{V}'\), similar to \(V' - \Delta_{\tau_4}\), because \(U\) is a linear combination of arbitrary powers of \(a\) and \(a^\dagger\), some of which compensate the factor \((a^\dagger)^M\) in \(V'\). For real \(f(n)\) we obtain

\[
V' = \bar{V}' + (a^\dagger a - \text{nonconserving terms})
\]

\[
\bar{V}' = \tau_4 \sum_{n=0}^{\infty} |n\rangle \bar{f}(n) \langle n|,
\]

\[
\bar{f}(n) = \langle n| U f(a^\dagger a)^M U |n\rangle = \frac{(-1)^M \sqrt{\pi}}{n!} \sum_{m=0}^{\infty} \frac{f(m)}{m!} H_{n,m}(\tilde{g}, \tilde{g}) H_{n,m+M}(\tilde{g}, \tilde{g}).
\]
It is in fact possible to make these diagonal matrix elements of $V'$ independent of $n$, so that $f(n) = \text{const}$ for all $n$, as we now discuss. In fact, we can evaluate the series in (69c) for a function $f(n) \propto s^n$, as follows. Since a product of two Hermite polynomials appears in (69c), we first derive its generating function. Using the operator Hermite polynomial calculus of [21], we begin with

$$
\sum_{m=0}^{\infty} \frac{s^m}{m!} \overline{H_{m+n}(x, y)} : H_{m,n}(a^\dagger, a) : = \sum_{m=0}^{\infty} \frac{s^m}{m!} (a^\dagger)^m a^n \overline{H_{m+n}(x, y)}
$$

where $: \cdots :$ denotes antinormal ordering of bosonic operators and equation numbers in square brackets refer to [21]. Summing over $M$ yields

$$
\sum_{M=0}^{M} \frac{t^M}{M!} \overline{H_{m,n}(x, y - sa^\dagger)(a - sx)^n} \overset{(21)}{=} e^{i\eta}(y - t - sa^\dagger)^n(a - sx)^n
$$

$$
\overset{(22)}{=} e^{i\eta}(-s^n)H_{m,n}(i\frac{y - t}{s} - a^\dagger), (a - sx))
$$

$$
= e^{i\eta} s^n M! L_m((-\frac{y - t}{s} - a^\dagger)(a - sx)).
$$

(71)

Inside the antinormal ordering it is permissible to replace $a^\dagger$ and $a$ by scalars $x'$ and $y'$, respectively. Putting (70) and (71) together then yields

$$
\sum_{m=0}^{\infty} \frac{t^M}{M!} \overline{H_{m,n}(x, y)^{\prime} H_{m,n}(x', y') t^M s^m n! m! M!} = e^{x'x} + r_n s^n L_m((-\frac{y - t}{s} - x')(sx - y')).
$$

(72)

which is a generalization of (35) that includes the shift $M$ in one of the indices. Next we take coefficients of $t^M$ in (72), and specialize to equal arguments, $x - y = x' - y'$,

$$
\sum_{m=0}^{\infty} \frac{H_{m+n}(x, y) H_{m,n}(x', y') s^n}{m! n!} = e^{x'x} \sum_{k=0}^{n} \binom{M}{k} x^{M-k} (sx - y')^k s^{n-k} L_{n-k}^{(k)}((-\frac{y}{s} - x')(sx - y')),
$$

(73a)

$$
\sum_{m=0}^{\infty} \frac{H_{m+n}(x, x) H_{m,n}(x, x) s^n}{m! n!} = e^{x'x} x^M \sum_{k=0}^{n} \binom{M}{k} (s - 1)^k s^{n-k} L_{n-k}^{(k)}(-x^2 (1 - s)^2 s),
$$

(73b)

If we choose $f(n)$ proportional to $s^n$, with $s$ a real parameter, we thus find

$$
f(m) = \frac{s^n \Delta}{(-g)^M} \Rightarrow \tilde{f}(m) = \Delta \sum_{k=0}^{n} \binom{M}{k} (s - 1)^k s^{n-k} L_{n-k}^{(k)}(-g^2 (1 - s)^2 s).
$$

(74)

so that indeed $f(n) = \text{const}$ for $s = 1$,

$$
f(m) = \frac{\Delta}{(-g)^M} \Rightarrow \tilde{f}(m) = \Delta \Rightarrow \tilde{V}' = \Delta \delta_s = V.
$$

(75)
Remarkably, the number-conserving part $\tilde{V}'$ in $H'$ coincides with $\tilde{V}$ in $H$ for the choice (75).

We thus arrive at the following alternative picture of the iARM. Namely, its Hamiltonian $H$ (with integer $M$) may be viewed as the number-conserving part of a parent Hamiltonian $H'$ with exact eigenvalues $E'_{n\alpha}$ and exact eigenstates $|n; \alpha(0)\rangle$. The spin expectation values for $H'$ thus have the special $n$-dependent asymptotics described above, and these are unchanged in first-order perturbation theory in $(H - H')$, which yields no contribution to the eigenvalues of $H$ (and by the derivatives in (55) neither to the spin expectation values).

Compared to the results obtained from direct perturbation theory in $\Delta$, this explains the $n$-dependent asymptotics of the iARM as due to the vicinity of the parent Hamiltonian $H'$. This picture is nonperturbative in $\Delta$, in the sense that the exact eigenstates of $H'$ are known and agree with those of the iARM up to first order in $(H - H')$, and higher orders apparently do not destroy this connection.

7. Conclusion and outlook

In summary, we studied the spin expectation values in the asymmetric Rabi model as functions of the coupling $g$. We showed that the large-$g$ asymptotics can mostly be understood perturbatively for small hybridization $\Delta$. The spin expectation values tend to zero for large $g$, except in the integer case, i.e. for $\epsilon = M\omega/2$ with integer $M$, for which $\langle s_x \rangle$ tends to $-1$ for the $M$ lowest-lying eigenstates. As an alternative argument, we constructed a related Hamiltonian $H'$ with additional number-nonconserving terms, the exact eigenstates of which are those of the asymmetric Rabi model in the limit of vanishing $\Delta$.

As an outlook, we note that both methods hold some perspective for further applications. The weak-$\Delta$ perturbation theory describes the crossing of energy levels on the baselines $E_n^{(0)}$ in the integer case (left panels in Figure 2–4), as $E_n^{(1)}$ vanishes there. This regime may therefore be useful to better understand the physical origin of these degeneracies. Similarly, the parent Hamiltonian $H'$ captures some properties of the asymmetric Rabi model with integer asymmetry parameter $M$ and can serve as a starting point for further studies.

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