On the Existence of Coproducts in Categories of $q$-Matroids

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Abstract

$q$-Matroids form the $q$-analogue of classical matroids. In this paper we introduce various types of maps between $q$-matroids. These maps are not necessarily linear, but they must map subspaces to subspaces and respect the $q$-matroid structure in certain ways. The various types of maps give rise to different categories of $q$-matroids. We show that only one of these categories possesses a coproduct. This is the category where the morphisms are linear weak maps, that is, the rank of the image of any subspace is not larger than the rank of the subspace itself. The coproduct in this category is the very recently introduced direct sum of $q$-matroids.

Keywords: $q$-matroid, representability, strong maps, weak maps, coproduct.

1 Introduction

Due to their close relation to rank-metric codes, $q$-matroids have gained a lot of attention in recent years. They were first introduced in [8], and their generalization to $q$-polymatroids appeared first in [7] and [11]. While $\mathbb{F}_{q^m}$-linear rank-metric codes induce $q$-matroids, $\mathbb{F}_q$-linear rank-metric codes give rise to the more general $q$-polymatroids.

A $q$-matroid is the $q$-analogue of a classical matroid: Its ground space is a finite-dimensional vector space over some finite field $\mathbb{F}_q$, and the rank function is defined on the lattice of subspaces. Its properties are the natural generalization of the rank function for classical matroids.

While many results have already been derived about $q$-matroids, see for instance the abundance of cryptomorphisms in [2], the theory of $q$-matroids is still at an early stage. For instance, the notion of direct sum and decomposability into direct summands has not yet been fully established. Simple examples show that none of the equivalent definitions of a direct sum of classical matroids leads to a well-defined notion for $q$-matroids. Our Proposition 4.5 illustrates this fact for a very simple class of representable $q$-matroids. Just recently, a first attempt of defining a direct sum of $q$-matroids has been put forward in [3]. We will show that it plays a prominent role in our framework.

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In this paper we introduce various maps between \( q \)-matroids and study the existence of coproducts in the resulting categories. Just like for classical matroids these are maps between the ground spaces that respect the \( q \)-matroid structure in a particular way; see [12] Ch. 8 and 9 for classical matroids. We take the most general approach and do not require the maps to be linear. However, in order to be compatible with \( q \)-matroid structure, they have to map subspaces to subspaces. In other words, they induce maps between the corresponding subspace lattices. There are various ways how maps can respect \( q \)-matroid structure: A map is strong if the pre-image of a flat is a flat; it is weak (resp. rank-preserving) if the rank of a subspace is not smaller than (resp. is equal to) the rank of its image.

All of this leads to several categories with \( q \)-matroids as objects and those maps as morphisms. Motivated by the fact that for classical matroids the direct sum is a coproduct in the category with strong maps as morphisms [4], we investigate the existence of coproducts in these various categories. It turns out that the only category that admits a coproduct is the one where the morphisms are linear weak maps. In that case, the direct sum introduced in [3] is the coproduct. The non-existence of a coproduct in the category with strong maps (or linear strong maps) stands in contrast to the classical case, where the direct sum is indeed the coproduct.

**Notation:** Throughout, let \( F = \mathbb{F}_q \) and \( E \) be a finite-dimensional \( F \)-vector space. We denote by \( \mathcal{L}(E) = (\mathcal{L}(E), \leq, +, \cap) \) the subspace lattice of \( E \). A \( k \)-space is a subspace of dimension \( k \). For a matrix \( M \in \mathbb{F}^{n \times b} \) we denote by \( \text{rs}(M) \in \mathcal{L}(\mathbb{F}^b) \) and \( \text{cs}(M) \in \mathcal{L}(\mathbb{F}^a) \) the row space and column space of \( M \), respectively. For clarity we will use throughout the terminology classical matroid for the ‘non-\( q \)-version’ of matroids, that is, matroids based on subsets of a finite ground set.

### 2 Basic Notions of \( q \)-Matroids

**Definition 2.1.** A \( q \)-matroid with ground space \( E \) is a pair \((E, \rho)\), where \( \rho : \mathcal{L}(E) \rightarrow \mathbb{N}_0 \) is a map satisfying

- (R1) Dimension-Boundedness: \( 0 \leq \rho(V) \leq \dim V \) for all \( V \in \mathcal{L}(E) \);

- (R2) Monotonicity: \( V \leq W \implies \rho(V) \leq \rho(W) \) for all \( V, W \in \mathcal{L}(E) \);

- (R3) Submodularity: \( \rho(V + W) + \rho(V \cap W) \leq \rho(V) + \rho(W) \) for all \( V, W \in \mathcal{L}(E) \).

The value \( \rho(V) \) is called the rank of \( V \) and \( \rho(M) := \rho(E) \) is the rank of the \( q \)-matroid. If \( \rho \) is the zero map, we call \((E, \rho)\) the trivial \( q \)-matroid on \( E \).

A large class of \( q \)-matroids are the representable ones.

**Definition 2.2 ([8 Sec. 5]).** Consider a field extension \( \mathbb{F}_{q^m} \) of \( \mathbb{F}_q \) and let \( G \in \mathbb{F}_{q^m}^{k \times n} \) be a matrix of rank \( k \). Define the map \( \rho : \mathcal{L}(\mathbb{F}_{q^m}) \rightarrow \mathbb{N}_0 \) via

\[
\rho(V) = \text{rk}(GY^T), \quad \text{where } Y \in \mathbb{F}_q^{g \times n} \text{ such that } V = \text{rs}(Y).
\]

Then \( \rho \) satisfies (R1)–(R3) and thus defines a \( q \)-matroid \( \mathcal{M}_G = (\mathbb{F}_{q^m}, \rho) \), which has rank \( k \). We call \( \mathcal{M}_G \) the \( q \)-matroid represented by \( G \). Furthermore, a \( q \)-matroid \( \mathcal{M} = (\mathbb{F}_q^n, \rho) \) of rank \( k \) is representable over \( \mathbb{F}_{q^m} \) if \( \mathcal{M} = \mathcal{M}_G \) for some \( G \in \mathbb{F}_{q^m}^{k \times n} \) and it is representable if it is representable over \( \mathbb{F}_{q^m} \) for some \( m \).
Not every $q$-matroid is representable. The first non-representable $q$-matroid appeared in [6] Ex. 4.9. However, the smallest non-representable $q$-matroid is the following one, found just recently in [3].

**Example 2.3** ([3] Sec. 3.3). Consider $E = \mathbb{F}_2^3$ and let

$$V = \{ (1000, 0100), (0010, 0001), (1001, 0111), (1011, 0110) \}$$

(or any other partial spread of size 4). Define $\rho(V) = 1$ for $V \in V$ and $\rho(V) = \min\{2, \dim V\}$ otherwise. It follows from [6] Prop. 4.7 that this does indeed define a $q$-matroid $\mathcal{M} = (\mathbb{F}_2^3, \rho)$. In [3] Sec. 3.3 it has been shown that $\mathcal{M}$ is a non-representable $q$-matroid.

**Remark 2.4.** While in this paper we will only make use of representability in the sense of Definition 2.2 we would like to briefly discuss a broader notion of representability of $q$-matroids. Let $\mathcal{C} \leq \mathbb{F}_{n \times m}^{m \times n}$ be a linear rank-metric code (that is, a subspace of $\mathbb{F}_{n \times m}^{m \times n}$ endowed with the rank-metric). For any $V \in \mathcal{L}(\mathbb{F}^n)$ define $\mathcal{C}(V, c) = \{ M \in \mathcal{C} \mid \text{cs}(M) \leq V \}$. Then $\mathcal{C}$ gives rise to a map $\rho_{\mathcal{C}} : \mathcal{L}(\mathbb{F}^n) \rightarrow \mathbb{Q}_{\geq 0}$ via $\rho_{\mathcal{C}}(V) = (\dim \mathcal{C} - \dim \mathcal{C}(V^\perp, c))/m$, and where $V^\perp$ denotes the orthogonal of $V$ with respect to the standard dot product in $\mathbb{F}^n$. In general, this leads to a $q$-polymatroid, that is, some of the rank values are not integers (see [7] Thm. 5.3 or [6]). If the code $\mathcal{C}$ is $\mathbb{F}_{q^m}$-linear (in the sense of [6] Def. 3.6]), then $\rho_{\mathcal{C}}$ is integer-valued and thus defines a $q$-matroid $\mathcal{M}_{\mathcal{C}} = (\mathbb{F}_2^m, \rho_{\mathcal{C}})$. In fact, in this case $\rho_{\mathcal{C}}$ agrees with the rank function in Definition 2.2. But even if $\mathcal{C}$ is not $\mathbb{F}_{q^m}$-linear, then $\rho_{\mathcal{C}}$ may define a $q$-matroid $\mathcal{M}_{\mathcal{C}}$ (that is, the rank values are integers). This is for instance the case for all MRD codes if $n \leq m$; see [7] Cor. 6.6. We call a $q$-matroid $\mathcal{M}$ representable over $\mathbb{F}_{n \times m}$ if it is of the form $\mathcal{M}_{\mathcal{C}}$ for some rank-metric code $\mathcal{C} \leq \mathbb{F}_{n \times m}^{m \times n}$. In the appendix we will show that the $q$-matroid of Example 2.3 is not representable over $\mathbb{F}_{n \times m}$ for any $m$. This is in fact the first case where a $q$-matroid is proven to be non-representable in this broader sense.

We return to general $q$-matroids. In [2] (and partly in [1]) a variety of cryptomorphic definition of $q$-matroids have been derived. We will need the one based on flats. In order to formulate the corresponding results, we recall the following notions from lattice theory. Let $(\mathcal{F}, \leq)$ be any lattice. We denote by $F_1 \lor F_2$ and $F_1 \land F_2$ the join and meet of $F_1$ and $F_2$, respectively. We say that $F_1$ covers $F_2$ if $F_2 < F_1$ and there exists no $F \in \mathcal{F}$ such that $F_2 < F < F_1$. The lattice is semi-modular if it satisfies: whenever $F_1$ covers $F_1\cap F_2$, then $F_1 \lor F_2$ covers $F_2$.

**Theorem 2.5** ([1] Prop. 3.8 and Thm. 3.10]). Let $\mathcal{M} = (E, \rho)$ be a $q$-matroid. For $V \in \mathcal{L}(E)$ define the closure of $V$ as

$$\nabla = \sum_{\rho(V+X) = \rho(V)} X,$$

$V$ is called a flat if $\nabla = V$. Denote by $\mathcal{F} := \mathcal{F}_\rho$ the collection of all flats of $\mathcal{M}$. Then

(F1) $E \in \mathcal{F}$,

(F2) If $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$,

(F3) For all $F \in \mathcal{F}$ and all $v \in E \setminus F$, there exists a unique cover of $F$ in $\mathcal{F}$ containing $v$.

The converse result tells us that any collection of subspaces satisfying (F1)–(F3) gives rise to a $q$-matroid.
Theorem 2.6 (I Cor. 3.11 and Thm. 3.13]). Let \( F \subseteq \mathcal{L}(E) \) be a collection of subspaces satisfying (F1)–(F3) from Theorem 2.5. For \( V \in \mathcal{L}(E) \) set \( \overline{V} = \bigcap_{F \subseteq F} V \).

(a) \((F, \subseteq)\) is a semi-modular lattice with the meet and join of two flats given by \( F_1 \wedge F_2 = F_1 \cap F_2 \) and \( F_1 \vee F_2 = F_1 + F_2 \), respectively.

(b) By semi-modularity all maximal chains between any two fixed elements in \( F \) have the same length (Jordan-Dedekind chain condition).

(c) For any \( F \in F \) denote by \( h(F) \) the length of a maximal chain from \( 0 \) to \( F \). Define the map \( \rho_F : L(E) \rightarrow \mathbb{N}_0, V \mapsto h(\overline{V}) \). Then \((E, \rho_F)\) is a \( q \)-matroid.

The above two processes are mutually inverse in the following sense.

Theorem 2.7 (I Thm. 3.13]).

(a) Let \((E, \rho)\) be a \( q \)-matroid. Then \( \rho_{\rho_F} = \rho \).

(b) Let \( F \) be a collection of subspaces satisfying (F1)–(F3). Then \( F_{\rho_F} = F \).

This last result allows us to define \( q \)-matroids based on their flats, denoted by \((E, F)\), or based on the rank function, denoted by \((E, \rho)\).

Example 2.8. We denote by \( \mathcal{U}_k(E) \) the uniform \( q \)-matroid of rank \( k \) on \( E \), that is, its rank function is given by \( \rho(V) = \min\{k, \dim V\} \) and the flats are \( \mathcal{F} \mathcal{U}_k(E) = \{V \subseteq E | \dim V \leq k - 1\} \cup \{E\} \). Note that \( \mathcal{U}_0(E) \) is the trivial \( q \)-matroid on \( E \). It is well known (I Ex. 31 or I Cor. 6.6) that \( \mathcal{U}_k(F_q^n) \) is representable over any \( F_q^m \) with \( m \geq n \): choose any \( G \in F_q^{m \times n} \) that generates an \( F_q^m \)-linear MRD code (of rank distance \( n - k + 1 \)).

3 Maps Between \( q \)-Matroids

In this section we introduce and discuss maps between \( q \)-matroids. The candidates for such maps are (possibly nonlinear) maps between the ground spaces of the \( q \)-matroids with the property that they map subspaces to subspaces. In addition, they should respect the \( q \)-matroid structure. For the latter there are various options, which will be discussed below. We start with the first property.

Throughout this section let \( E_1, E_2 \) be finite-dimensional \( F \)-vector spaces.

Definition 3.1. Let \( \phi : E_1 \rightarrow E_2 \) be a map. We call \( \phi \) an \( \mathcal{L} \)-map if \( \phi(V) \in \mathcal{L}(E_2) \) for all \( V \in \mathcal{L}(E_1) \). The induced map from \( \mathcal{L}(E_1) \) to \( \mathcal{L}(E_2) \) is denoted by \( \phi_\mathcal{L} \). A bijective \( \mathcal{L} \)-map is called an \( \mathcal{L} \)-isomorphism. Finally, \( \mathcal{L} \)-maps \( \phi, \psi \) from \( E_1 \) to \( E_2 \) are equivalent, denoted by \( \phi \sim_\mathcal{L} \psi \), if \( \phi_\mathcal{L} = \psi_\mathcal{L} \).

Our definition of \( \mathcal{L} \)-isomorphisms is justified by the following simple fact.

Remark 3.2. Let \( \phi : E_1 \rightarrow E_2 \) be a bijective \( \mathcal{L} \)-map. Then \( \phi^{-1} \) is also an \( \mathcal{L} \)-map. To see this, note that \( \dim E_1 = \dim E_2 \) by bijectivity of \( \phi \) and thus the subspace lattices \( \mathcal{L}(E_1) \) and \( \mathcal{L}(E_2) \) are isomorphic. Hence \( \phi^{-1} \) is also an \( \mathcal{L} \)-map.
An $\mathcal{L}$-map is a possibly nonlinear map between $\mathbb{F}$-vector spaces that maps subspaces to subspaces. The induced map is clearly a lattice homomorphism (respecting join and meet and thus being order-preserving). In general, a lattice homomorphism from $\mathcal{L}(E_1)$ to $\mathcal{L}(E_2)$ is induced by a $\mathcal{L}$-map. However we have the following result, which is simply a reformulation of the Fundamental Theorem of Projective Geometry; see for instance [10, Thm. 1], where Section 2 contains a straightforward proof based solely on Linear Algebra.

**Theorem 3.3.** Let $\dim E_1 = \dim E_2 = n$, where $n \geq 3$ or $q = n = 2$. Let $\tau : \mathcal{L}(E_1) \rightarrow \mathcal{L}(E_2)$ be a lattice isomorphism. Then there exists a semi-linear map $\phi : E_1 \rightarrow E_2$ such that $\tau = \phi_\mathcal{L}$.

We return to general $\mathcal{L}$-maps. Any $\mathcal{L}$-map $\phi$ satisfies $\phi(0) = 0$ and

$$\phi((v)) = \langle \phi(v) \rangle \text{ for all } v \in E_1.$$  \hfill (3.1)

This follows from the fact that $\phi((v))$ is a subspace of dimension at most 1 containing 0 and $\phi(v)$.

Recall that $\phi$ is $\mathbb{F}$-semilinear if $\phi$ is additive and there exists $\sigma \in \text{Aut}(\mathbb{F})$ such that $\phi(cv) = \sigma(c)\phi(v)$ for all $v \in E_1$ and $c \in \mathbb{F}$. Clearly, any semi-linear map $\phi : E_1 \rightarrow E_2$ is an $\mathcal{L}$-map.

**Example 3.4.** (a) Let $\phi : \mathbb{F}_2^3 \rightarrow \mathbb{F}_2^3$ be given by $\phi(0,0) = (0,0)$ and $\phi(v) = (1,1)$ for all $v \in \mathbb{F}_2^3 \setminus (0,0)$. One easily checks that $\phi$ is a nonlinear $\mathcal{L}$-map. Even more, the pre-image of any subspace is a subspace.

(b) Let $\phi : \mathbb{F}_2^3 \rightarrow \mathbb{F}_2^3$ be given by $\phi(v_1,v_2,0) = (v_1,v_2)$ and $\phi(v_1,v_2,1) = (0,0)$ for all $v_1,v_2 \in \mathbb{F}_2$.

Again, $\phi$ is a nonlinear $\mathcal{L}$-map. In this case, the pre-image of a subspace is not necessarily a subspace, for instance $\phi^{-1}((0,0))$.

We list some basic facts about equivalent $\mathcal{L}$-maps.

**Proposition 3.5.** Let $\phi, \psi : E_1 \rightarrow E_2$ be $\mathcal{L}$-maps such that $\phi \sim_\mathcal{L} \psi$.

(a) For all $v \in E_1$ there exist $\lambda_v \in \mathbb{F}^*$ such that $\phi(v) = \lambda_v \psi(v)$.

(b) $\phi^{-1}(V) = \psi^{-1}(V)$ for all $V \in \mathcal{L}(E_2)$.

(c) If $\psi$ is an $\mathcal{L}$-isomorphism, then so is $\phi$, and $\phi^{-1} \sim_\mathcal{L} \psi^{-1}$.

**Proof.** (a) We may assume without loss of generality that $v \neq 0$. Note that $\phi(v) = 0 \iff \psi(v) = 0$, and in this case we may choose $\lambda_v = 1$. In the case $\psi(v) \neq 0 \neq \phi(v)$ the result follows from (3.1).

(b) For $v \in E_1$ let $\lambda_v \in \mathbb{F}^*$ be as in (a). For any $V \in \mathcal{L}(E_2)$ we have $\phi^{-1}(V) = \{v \in E_1 \mid \phi(v) \in V\} = \{v \in E_1 \mid \lambda_v \psi(v) \in V\} = \{v \in E_1 \mid \psi(v) \in V\} = \psi^{-1}(V)$.

(c) If $\psi$ is an $\mathcal{L}$-isomorphism, then the lattices $\mathcal{L}(E_1)$ and $\mathcal{L}(E_2)$ are isomorphic (see Remark 3.2). Next, $E_2 = \psi(E_1) = \phi(E_1)$, and thus $\phi$ is bijective as well. Furthermore, $\psi_\mathcal{L}$ and $\phi_\mathcal{L}$ are lattice isomorphisms from $\mathcal{L}(E_1)$ to $\mathcal{L}(E_2)$ satisfying $(\psi_\mathcal{L})^{-1} = (\phi_\mathcal{L})^{-1} = (\phi^{-1})_\mathcal{L}$. \hfill \Box

The following simple result allows us to construct $\mathcal{L}$-maps. It also shows that $\mathcal{L}$-maps are equivalent if they agree on the 1-dimensional spaces.

**Proposition 3.6.** Let $\psi : E_1 \rightarrow E_2$ be an $\mathcal{L}$-map and let $\phi : E_1 \rightarrow E_2$ be such that $\phi(0) = 0$ and $\phi((v)) = \psi((v))$ for all $v \in E_1$. Then $\phi$ is an $\mathcal{L}$-map and $\phi \sim_\mathcal{L} \psi$. 

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The following construction shows how we may alter an \( L \)-isomorphism without changing the induced lattice homomorphism. It will be crucial later on.

**Proposition 3.7.** Let \( \psi : E_1 \rightarrow E_2 \) be an \( L \)-isomorphism. Fix \( w \in E_1 \setminus 0 \) and \( \tau \in \mathbb{F}^* \) and set \( \hat{w} = \psi^{-1}(\tau \psi(w)) \). Then \( \langle \hat{w} \rangle = \langle w \rangle \). Define the map \( \phi : E_1 \rightarrow E_2 \) via

\[
\phi(v) = \psi(v) \quad \text{for} \quad v \in E_1 \setminus \langle w \rangle, \quad \phi(\mu w) = \psi(\mu \hat{w}) \quad \text{for} \quad \mu \in \mathbb{F}.
\]

Then \( \phi \) is an \( L \)-isomorphism and \( \phi \sim_L \psi \). If \( \tau = 1 \), we have \( \phi = \psi \).

**Proof.** Bijectivity of \( \psi \) implies \( \hat{w} \neq 0 \) and \( \psi(\hat{w}) = \tau \psi(w) \), thus \( \psi(\langle w \rangle) = \langle \psi(w) \rangle = \langle \psi(\hat{w}) \rangle = \psi(\langle \hat{w} \rangle) \). This in turn implies \( \langle w \rangle = \langle \hat{w} \rangle \). The map \( \phi \) is clearly bijective and satisfies \( \phi(0) = 0 \). We show the condition of Proposition 3.6. First let \( v \notin \langle w \rangle \). Then also \( \lambda v \notin \langle w \rangle \) for all \( \lambda \in \mathbb{F}^* \) and \( \phi(\langle v \rangle) = \{ \phi(\lambda v) \mid \lambda \in \mathbb{F} \} = \{ \psi(\lambda v) \mid \lambda \in \mathbb{F} \} = \psi(\langle v \rangle) \). Next, for \( v = \mu w \) with \( \mu \neq 0 \) we have \( \phi(\langle v \rangle) = \{ \phi(\lambda \mu v) \mid \lambda \in \mathbb{F} \} = \{ \psi(\lambda \mu \hat{w}) \mid \lambda \in \mathbb{F} \} = \psi(\langle \hat{w} \rangle) = \psi(\langle w \rangle) = \psi(\langle v \rangle) \). This verifies the conditions of Proposition 3.6 and the statement follows.

The following fact will be crucial later on.

**Theorem 3.8.** Let \( \dim E_1 = \dim E_2 = n \), where \( n \geq 3 \) or \( q = n = 2 \). Let \( \phi : E_1 \rightarrow E_2 \) be an \( L \)-map and suppose there exist vectors \( v_1, \ldots, v_n \) of \( E_1 \) such that \( \phi(v_1), \ldots, \phi(v_n) \) are linearly independent. Then \( \phi \) is bijective and there exists a semi-linear isomorphism \( \psi : E_1 \rightarrow E_2 \) such that \( \phi \sim_L \psi \). In particular, if \( q = 2 \), then \( \phi = \psi \) is a linear isomorphism.

**Proof.** Since \( \phi(E_1) \) is a subspace and contains \( \phi(v_1), \ldots, \phi(v_n) \), we conclude that \( \phi(E_1) = E_2 \). Hence \( \phi \) is a bijection. For \( n = 2 = q \) it is clear that \( \phi \) is a linear isomorphism (any nonzero vector in \( E_1 \) or \( E_2 \) is the sum of the other two nonzero vectors). For \( n \geq 3 \) we may use Theorem 3.3.

We now turn to \( L \)-maps between \( q \)-matroids. There are different options for such a map to respect the \( q \)-matroid structure.

**Definition 3.9.** Let \( \mathcal{M}_i = (E_i, \rho_i) \) be \( q \)-matroids with flats \( F_i := F(M_i) \). Let \( \phi : E_1 \rightarrow E_2 \) be an \( L \)-map. We define the following types.

(a) \( \phi \) is a strong map from \( \mathcal{M}_1 \) to \( \mathcal{M}_2 \) if \( \phi^{-1}(F) \in F(M_1) \) for all \( F \in F(M_2) \) (this implies in particular that \( \phi^{-1}(F) \) is a subspace of \( E_1 \)).

(b) \( \phi \) is a weak map from \( \mathcal{M}_1 \) to \( \mathcal{M}_2 \) if \( \rho_2(\phi(V)) \leq \rho_1(V) \) for all \( V \in L(E_1) \).

(c) \( \phi \) is rank-preserving if \( \rho_2(\phi(V)) = \rho_1(V) \) for all \( V \in L(E_1) \).

For any \( L \)-map \( \phi : E_1 \rightarrow E_2 \) we will also use the notation \( \phi : \mathcal{M}_1 \rightarrow \mathcal{M}_2 \). This allows us to discuss its type.

It should be noted that our notion of rank-preserving maps is different from rank-preserving weak maps for classical matroids: the latter are weak maps that preserve the rank of the matroid; see [12, p. 260]. However, the definition above will be convenient for us.
Clearly, the composition of maps of the same type is also a map of that type. Furthermore, equivalent \( L \)-maps are of the same type. Note, however, that if \( \phi : M_1 \rightarrow M_2 \) is a bijective strong (resp. weak) map, then \( \phi^{-1} : M_2 \rightarrow M_1 \) may not be strong (resp. weak): take for instance the identity map \( U_k(\mathbb{F}_q^2) \rightarrow U_{k-1}(\mathbb{F}_q^2) \). Being rank-preserving and being strong are not related: there exist strong maps that are not rank-preserving (e.g., the identity map from any nontrivial \( q \)-matroid \( M = (E, \rho) \) to the trivial \( q \)-matroid on \( E \)) and there exist rank-preserving maps that are not strong (e.g., \( \phi : \mathbb{F}_2^2 \rightarrow \mathbb{F}_2^2, (x, y) \mapsto (x, 0) \) and where \( M_1 \) and \( M_2 \) are the \( q \)-matroids on \( \mathbb{F}_2^2 \) of rank 1 with \( \{e_2\} \) and \( \{e_1 + e_2\} \) as the unique flat of rank 0, respectively). Unsurprisingly, weak maps are in general not strong: take for instance the identity on \( \mathbb{F}_2^2 \), which induces a weak, but not strong map from \( U_2(\mathbb{F}_2^2) \) to the \( q \)-matroid \( M \) from Example 2.3. However, it can be shown that – just like in the classical case \cite[Lemma 8.1.7]{12} – strong maps are weak. This needs some preparation and will be worked out in a different paper. We do not need this fact in this paper. We have the following positive result.

**Proposition 3.10.** Let \( M_i = (E_i, \rho_i) \) be \( q \)-matroids and \( \phi : M_1 \rightarrow M_2 \) be an \( L \)-isomorphism. Then

\[
\phi \text{ and } \phi^{-1} \text{ are weak maps } \iff \phi \text{ is rank-preserving } \iff \phi \text{ and } \phi^{-1} \text{ are strong maps.}
\]

**Proof.** Recall from Remark 3.2 that \( \phi^{-1} \) is also an \( L \)-map. The first equivalence is clear. Consider the second equivalence.

\[ \Rightarrow \] Let \( \phi \) be a rank-preserving isomorphism. Then \( \rho_2(\phi(V)) = \rho_1(V) \) and \( \dim \phi(V) = \dim V \) for all \( V \in \mathcal{L}(E_1) \). Furthermore, since \( \phi_L \) is a lattice isomorphism from \( \mathcal{L}(E_1) \) to \( (\mathcal{L}(E_2), \leq) \), we have \( \phi(V + X) = \phi(V) + \phi(X) \) for all \( V, X \in \mathcal{L}(E_1) \). Using the definition of a flat in Theorem 2.5, we conclude that \( V \) is a flat in \( M_1 \) iff \( \phi(V) \) is a flat in \( M_2 \).

\[ \Leftarrow \] Let now \( \phi \) and \( \phi^{-1} \) be strong maps. Then \( \phi(\mathcal{F}(M_1)) = \mathcal{F}(M_2) \), that is, \( \phi \) induces an isomorphism between the lattices of flats of \( M_1 \) and \( M_2 \). Using the height function on these lattices (see Theorem 2.6(c) and Theorem 2.7(a)) we conclude that \( \rho_1(F) = \rho_2(\phi(F)) \) for all \( F \in \mathcal{F}(M_1) \) and thus \( \rho_1(V) = \rho_2(\phi(V)) \) for all \( V \in \mathcal{L}(E_1) \). Hence \( \phi \) is rank-preserving. \( \square \)

We now turn to minors of \( q \)-matroids and determine the type of the corresponding maps. For restrictions and contractions of \( q \)-matroids, discussed next, see for instance \cite[Def. 5.1 and Thm. 5.2]{6}.

**Proposition 3.11.** Let \( M = (E, \rho) \) be a \( q \)-matroid and let \( X \subseteq E \).

(a) Let \( M|_X \) be the restriction of \( M \) to \( X \). Then the embedding \( \iota : X \rightarrow E, x \mapsto x \), is a linear strong and rank-preserving (hence weak) map from \( M|_X \) to \( M \).

(b) Let \( M/X \) be the contraction of \( M \) from \( X \). Then the projection \( \pi : E \rightarrow E/X, x \mapsto x+X \) is a linear strong and weak map from \( M \) to \( M/X \).

**Proof.** (a) Recall that \( M|_X = (X, \tilde{\rho}) \), where \( \tilde{\rho}(V) = \rho(V) \) for all \( V \subseteq X \). This shows that \( \iota \) is rank-preserving. Let \( F \in \mathcal{F}(M) \). Then \( \iota^{-1}(F) = F \cap X \) and \( F \cap X = \tilde{F} = F \), where \( \tilde{F} \) denotes the closure of the space \( A \) in the matroid \( M \) (for the closure see Theorem 2.5). Thus for any \( v \in X \) the identity \( \rho((F \cap X) + \langle v \rangle) = \rho(F \cap X) \) implies \( v \in F \). Hence \( F \cap X = F \cap X \) and thus \( \iota^{-1}(F) \in \mathcal{F}(M|_X) \).

(b) Recall that \( M/X = (E/X, \tilde{\rho}) \), where \( \tilde{\rho}(\pi(V)) = \rho(V + X) - \rho(X) \). Thus by submodularity
\[ \tilde{\rho}(\pi(V)) \leq \rho(V) - \rho(V \cap X), \] showing that \( \pi \) is weak. In order to show that \( \pi \) is strong, let \( F \in \mathcal{F}(\mathcal{M}/X) \). Thus \( \tilde{\rho}(F + \langle v + X \rangle) > \tilde{\rho}(F) \) for all \( v + X \in E/X \setminus F \). Since \( \pi^{-1}(F + \langle v + X \rangle) = \pi^{-1}(F) + \langle v \rangle \), this implies \( \rho(\pi^{-1}(F) + \langle v \rangle) > \rho(\pi^{-1}(F)) \) for all \( v \in E \setminus \pi^{-1}(F) \). Thus \( \pi^{-1}(F) \in \mathcal{F}(\mathcal{M}) \).

Restricting an \( \mathcal{L} \)-map to its image does not change its type.

**Proposition 3.12.** Let \( \mathcal{M}_i = (E_i, \rho_i) \) be \( q \)-matroids and \( \phi: \mathcal{M}_1 \rightarrow \mathcal{M}_2 \) be a strong (resp. weak or rank-preserving) map. Let \( X := \text{im} \phi \). Then \( X \) is a subspace of \( E_2 \) and we call the restriction \( \mathcal{M}_2|_X \) the image of \( \phi \). The map \( \hat{\phi}: E_1 \rightarrow X, v \mapsto \phi(v) \), is a strong (resp. weak or rank-preserving) map from \( \mathcal{M}_1 \) to \( \mathcal{M}_2|_X \). In other words, \( \phi \) restricts to a map \( \mathcal{M}_1 \rightarrow \mathcal{M}_2|_X \) of the same type.

**Proof.** The statement is clear for weak and rank-preserving maps. Let \( F \in \mathcal{F}(\mathcal{M}_2|_X) \). Then \( \rho_2(F + \langle v \rangle) > \rho_2(F) \) for all \( v \in X \setminus F \). Hence the closure \( \bar{F}^{\mathcal{M}_2} \) satisfies \( \bar{F}^{\mathcal{M}_2} \setminus F \subseteq E_2 \setminus X \). Using that \( \text{im} \phi = X \), we obtain \( \phi^{-1}(\bar{F}^{\mathcal{M}_2}) = \hat{\phi}^{-1}(F) \). Since the former is a flat in \( \mathcal{M}_1 \) and \( F \) was arbitrary, we conclude that \( \hat{\phi} \) is a strong map.

Unsurprisingly, representability is not preserved under strong or weak bijective maps. Take for instance the identity map on \( \mathbb{F}_2^4 \). It induces a bijective strong and weak map from the representable \( q \)-matroid \( U_4(\mathbb{F}_2^4) \) (see Example 2.8) to the non-representable \( q \)-matroid in Example 2.3.

## 4 Non-Existence of Coproducts in Categories of \( q \)-Matroids

In this section we consider categories of \( q \)-matroids with various types of morphisms. We will show that – with one exception – none of these categories has a coproduct.

**Definition 4.1.** We denote by \( q \)-Mat\(_s\), \( q \)-Mat\(_p\), \( q \)-Mat\(_w\), \( q \)-Mat\(_{l,s}\), \( q \)-Mat\(_{l,p}\), and \( q \)-Mat\(_{l,w}\) the categories with \( q \)-matroids as objects and where the morphisms are the strong, weak, and rank-preserving, linear strong, linear weak, and linear rank-preserving maps, respectively.

In this section we show that none of the first 5 categories has a coproduct, and in the next section we establish the existence of a coproduct in \( q \)-Mat\(_{l,w}\). It is in fact the direct sum as introduced recently in [3]. The non-existence of a coproduct in \( q \)-Mat\(_s\) and \( q \)-Mat\(_{l,s}\) stands in contrast to the case of classical matroids, where the direct sum (see [9 Sec. 4.2]) forms a coproduct in the category with strong maps as morphisms, see [12 Ex. 8.6, p. 244] (which goes back to [4]). We know from Proposition 3.10(b) that isomorphisms in the first three categories coincide, and so do those in the second three categories. This gives rise to the following notions of isomorphic \( q \)-matroids.

**Definition 4.2.** We call \( q \)-matroids \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) **isomorphic** if they are isomorphic in the category \( q \)-Mat\(_p\), that is, there exists a rank-preserving \( \mathcal{L} \)-isomorphism \( \phi: \mathcal{M}_1 \rightarrow \mathcal{M}_2 \) (equivalently, \( \phi \) and \( \phi^{-1} \) are strong). \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are **linearly isomorphic**, denoted by \( \mathcal{M}_1 \cong \mathcal{M}_2 \), if they are isomorphic in the category \( q \)-Mat\(_{l,p}\).

Due to Theorem 3.3 the above notion of isomorphism coincides with lattice-equivalence in [3 Def. 4].
Remark 4.3. Let $\phi : E_1 \rightarrow E_2$ be an $\mathcal{L}$-isomorphism and $\mathcal{M}_1 = (E_1, \rho_1)$ be a $q$-matroid. Define $\mathcal{M}_2 = (E_2, \rho_2)$ where $\rho_2(V) := \rho_1(\phi^{-1}(V))$ for all $V \in \mathcal{L}(E_2)$. Then $\mathcal{M}_2$ is a $q$-matroid and isomorphic to $\mathcal{M}_1$. The flats of $\mathcal{M}_2$ are given by $\mathcal{F}(\mathcal{M}_2) = \{\phi(F) \mid F \in \mathcal{F}(\mathcal{M}_1)\}$.

The following linear maps will be used throughout this paper. For $E = E_1 \oplus E_2$ let

$$\iota_i : E_i \rightarrow E, \ x \mapsto x$$

be the natural embeddings. If $E_1 = \mathbb{F}^{n_i}$ and $E = \mathbb{F}^{n_1+n_2}$ we define the maps as

$$\iota_1 : \mathbb{F}^{n_1} \rightarrow \mathbb{F}^{n_1+n_2}, \ x \mapsto (x,0), \quad \iota_2 : \mathbb{F}^{n_2} \rightarrow \mathbb{F}^{n_1+n_2}, \ y \mapsto (0,y).$$

The following construction will be useful later on. It shows that representable $q$-matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ can be embedded in a $q$-matroid $\mathcal{M}$ in such a way that the ground spaces of $\mathcal{M}_i$ form a direct sum of the ground spaces of $\mathcal{M}$. However, as we will illustrate by an example below, the resulting $q$-matroid $\mathcal{M}$ is not uniquely determined by the $q$-matroids $\mathcal{M}_1$ and $\mathcal{M}_2$, but rather depends on the representing matrices (which has also been observed in [3, Sec. 3.2]). It is exactly this non-uniqueness that allows us to prove the non-existence of coproducts.

**Proposition 4.4.** Let $G_i \in \mathbb{F}_q^{n_i \times n_i}$, $i = 1, 2$, be matrices of full row rank and set

$$G = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} \in \mathbb{F}_q^{(n_1+n_2) \times (n_1+n_2)}.$$

Denote by $\mathcal{M}_i = (\mathbb{F}^{n_i}, \rho_i)$, $i = 1, 2$, and $\mathcal{N} = (\mathbb{F}_q^{n_1+n_2}, \rho)$ the $q$-matroids represented by $G_1, G_2$, and $G$, thus $\rho_i(rs(Y)) = \text{rk}(G_iY^T)$ for $Y \in \mathbb{F}_q^{n_i \times n_i}$ and $\rho(rs(Y)) = \text{rk}(GY^T)$ for $Y \in \mathbb{F}_q^{n_1+n_2 \times n_1+n_2}$. Then $\iota_i : \mathcal{M}_i \rightarrow \mathcal{N}$, $i = 1, 2$, is a linear, rank-preserving, and strong map with image $\mathcal{N}|_{\mathbb{F}^{n_i}}$. Thus $\mathcal{N}|_{\mathbb{F}^{n_i}}$ is linearly isomorphic to $\mathcal{M}_i$ for $i = 1, 2$.

**Proof.** Let $Y \in \mathbb{F}_q^{n_i \times n_1}$ be a matrix of rank $y$. Then $\text{rs}(Y) \leq \mathbb{F}_q^{n_i}$ and $\iota_1(\text{rs}(Y)) = \text{rs}(Y \mid 0)$. Moreover,

$$\rho(\text{rs}(Y \mid 0)) = \text{rk}\begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} \begin{pmatrix} Y^T \\ 0 \end{pmatrix} = \text{rk}(G_1Y^T) = \rho(\text{rs}(Y)).$$

Thus $\iota_1$ is rank-preserving. Similarly, $\iota_2$ is rank-preserving. In order to show that $\iota_i : \mathcal{M}_i \rightarrow \mathcal{N}$ is a strong map, let us first consider the pre-images of a subspace $\text{rs}(Y)$, where $Y \in \mathbb{F}_q^{n_1+n_2 \times n_1+n_2}$. They can be computed as follows. There exist $U_i \in \text{GL}_y(\mathbb{F})$ such that

$$U_1Y = \begin{pmatrix} A_1 \\ A_3 \end{pmatrix}, \quad U_2Y = \begin{pmatrix} 0 & B_2 \\ B_3 & B_4 \end{pmatrix},$$

where the first block column consists of $n_1$ columns and the second one of $n_2$ columns, and where $A_4$ and $B_3$ have full row rank. Then $\iota_i^{-1}(\text{rs}(Y)) = \text{rs}(A_1)$ and $\iota_i^{-1}(\text{rs}(Y)) = \text{rs}(B_2)$. Suppose now that $\text{rs}(Y) \in \mathcal{F}(\mathcal{M})$. By symmetry it suffices to show that $\text{rs}(A_1) \in \mathcal{F}(\mathcal{M}_1)$. To this end let $v_1 \in \mathbb{F}_q^{n_1}$ such that $\rho_1(\text{rs}(A_1) + \langle v_1 \rangle) = \rho(\text{rs}(A_1))$. Setting $v = \iota_1(v_1) = (v_1, 0)$, we have

$$\rho_1(\text{rs}(A_1) + \langle v_1 \rangle) = \text{rk}(G_1(A_1^Tv_1^T)) \quad \text{and} \quad \rho(\text{rs}(Y) + \langle v \rangle) = \text{rk}\begin{pmatrix} G_1A_1^T & G_1v_1^T & G_1A_3^T \\ 0 & 0 & G_2A_4^T \end{pmatrix}. $$

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Let $L$ and $N$ be $m$-matroids. As a consequence, if $L$ is a $m$-matroid, then $N$ is the image of $L$ under the map $\iota_i$, and thus $\iota_i$ induces an isomorphism between $\mathcal{M}_i$ and $\mathcal{N}_i$ of the direct sum; see [9, p. 126, Ex. 7].

Proposition 4.5. Let $\mathbb{F} = \mathbb{F}_q$ and consider the field extension $\mathbb{F}_q^m$ with primitive element $\omega$. Let $i \in \{1, \ldots, q^m - 2\}$ such that $\omega^i \not\in \mathbb{F}_q$ (i.e., $i$ is not a multiple of $(q^m - 1)/(q - 1)$) and define

$$G^{(i)} = \begin{pmatrix} 1 & \omega & 0 & 0 \\ 0 & 0 & 1 & \omega^i \end{pmatrix}, \quad T_1 = \langle 1000, 0100 \rangle \quad \text{and} \quad T_2 = \langle 0010, 0001 \rangle.$$

Let $\mathcal{N}^{(i)} = (\mathbb{F}^4, \rho)$ be the $q$-matroid represented by $G^{(i)}$. Then

(a) $\rho(T_1) = \rho(T_2) = 1$ and $\rho(\mathbb{F}^4) = 2$.

(b) $\rho(V) = 1$ for all 1-spaces $V$ and $\rho(V) = 2$ for all 3-spaces $V$.

(c) Let $\mathcal{L}_2 = \{V \in \mathcal{L}(\mathbb{F}^4) \mid \dim V = 2, V \not\in \{T_1, T_2\}\}$. Then

$$\rho(V) = 2 \quad \text{for all} \quad V \in \mathcal{L}_2 \iff 1, \omega, \omega^i, \omega^{i+1} \quad \text{are linearly independent over} \quad \mathbb{F}_q.$$

As a consequence, if $m > 3$ we obtain at least two non-isomorphic $q$-matroids.

Proof. (a) Clearly $\rho(T_1) = \rho(T_2) = 1$ and $\rho(\mathbb{F}^4) = \text{rk}(G) = 2$.

(b) By assumption on $i$ the elements $1, \omega^i$ are linearly independent over $\mathbb{F}_q$. Thus, $G^{(i)}x \neq 0$ for any nonzero vector $x \in \mathbb{F}^4$ and hence $\rho(V) = 1$ for all 1-spaces $V$. Suppose there exists a 3-space $V$ such that $\rho(V) = 1$. Clearly, $V$ does not contain both $T_1$ and $T_2$. Let $T_1 \not\subseteq V$. Then $\dim(V \cap T_1) = 1$, and submodularity of $\rho$ implies $2 = \rho(\mathbb{F}^4) = \rho(V + T_1) \leq 1 + 1 - \rho(V \cap T_1) = 1$, which is a contradiction.

(c) Consider now an arbitrary 2-space $V = \langle (a_0, a_1, a_2, a_3), (b_0, b_1, b_2, b_3) \rangle$ and suppose $\rho(V) = 1$. Since $\rho(V)$ is the rank of the matrix

$$\begin{pmatrix} 1 & \omega & 0 & 0 \\ 0 & 0 & 1 & \omega^i \end{pmatrix} \begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} = \begin{pmatrix} a_0 + a_1\omega & b_0 + b_1\omega \\ a_2 + a_3\omega^i & b_2 + b_3\omega^i \end{pmatrix},$$

we conclude that its determinant is zero, thus

$$(a_0b_2 - a_2b_0) + (a_1b_2 - a_2b_1)\omega + (a_0b_3 - a_3b_0)\omega^i + (a_1b_3 - a_3b_1)\omega^{i+1} = 0. \quad (4.4)$$

“$\Leftarrow$” Suppose $1, \omega, \omega^i, \omega^{i+1}$ are linearly independent. Then all coefficients in (4.4) are zero. Now we can go through the cases. (i) If $a_0 \neq 0 = b_0$, then $b_2 = b_3 = 0$. Thus $b_1 \neq 0$ since $\dim V = 2$, and subsequently $a_2 = a_3 = 0$. But then $V = T_1$. (ii) Suppose $a_0 = b_0 = 0$. If $a_1 = 0 = b_1$, then $V = T_2$. Thus assume without loss of generality that $a_1 \neq 0$. But then $b_2 = (b_1/a_1)a_2$ and
Then \( \alpha \) is of this form. It is well-known (and straightforward to verify) that if \((M, \xi_1, \xi_2)\) is a coproduct of \(M_1\) and \(M_2\) and \(\phi : M \rightarrow M\) is an isomorphism (i.e., a bijective morphism whose inverse is also a morphism), then \((\tilde{M}, \phi \circ \xi_1, \phi \circ \xi_2)\) is also a coproduct of \(M_1\) and \(M_2\). Furthermore, every coproduct of \(M_1\) and \(M_2\) is of this form.

The rest of this section is devoted to the non-existence of coproducts in the first 5 categories of Definition 4.1. Our first result narrows down the ground space of a putative coproduct in an expected way. Furthermore, it shows that the accompanying morphisms are injective and can be chosen as linear maps. We introduce the following notation. Let \(\mathcal{T} = \{s, r, p, w, l-s, l-r, l-w\}\) be the set of types of morphisms. For any \(\Delta \in \mathcal{T}\) we denote by \(q\text{-Mat}^\Delta\) the corresponding category, and the morphisms in this category are called type-\(\Delta\) maps. Note that the maps in Proposition 4.3 and Proposition 3.11(a) are type-\(\Delta\) for each \(\Delta\).

**Theorem 4.7.** Let \(\Delta \in \mathcal{T}\). Let \(\mathcal{M}_i, i = 1, 2\), be representable \(q\)-matroids with ground spaces \(E_i\). Suppose \(\mathcal{M}_1\) and \(\mathcal{M}_2\) have a coproduct \((\mathcal{M}, \xi_1, \xi_2)\) in \(q\text{-Mat}^\Delta\). Then they have a coproduct of the form \((\mathcal{M}, \iota_1, \iota_2)\) where \(\mathcal{M}\) has ground space \(E_1 \oplus E_2\) and \(\iota_i : E_i \rightarrow E_1 \oplus E_2\) are the natural embeddings as in (1.1) (hence linear).

**Proof.** Suppose without loss of generality that \(\mathcal{M}_i\) is a \(q\)-matroid on the ground space \(\mathbb{F}^{n_i}\). Let \((\mathcal{M}, \xi_1, \xi_2)\) be a coproduct of \(\mathcal{M}_1, \mathcal{M}_2\). We may also assume that the ground space of \(\mathcal{M}\) is \(\mathbb{F}^n\) for some \(n\). Recall that \(\xi_i\) may not be semi-linear maps, but the images \(\xi_i(\mathbb{F}^{n_i})\) are subspaces of \(\mathbb{F}^n\).

We proceed in several steps.

**Claim 1:** \(\xi_i\) are injective and \(\xi_1(\mathbb{F}^{n_1}) \cap \xi_2(\mathbb{F}^{n_2}) = \{0\}\).

Since \(\mathcal{M}_i\) are representable, we may apply Proposition 4.3 and obtain the existence of a \(q\)-matroid \(\mathcal{N}\) on \(\mathbb{F}^{n_1+n_2}\) and linear maps \(\alpha_i : \mathcal{M}_i \rightarrow \mathcal{N}\) where \(\alpha_1(v_1) = (v_1, 0)\) and \(\alpha_2(v_2) = (0, v_2)\) for all \(v_i \in \mathbb{F}^{n_i}\). Thanks to Proposition 4.4 the maps \(\alpha_i\) are type-\(\Delta\). Hence the universal property of the coproduct implies the existence of a type-\(\Delta\) map \(\epsilon : \mathcal{M} \rightarrow \mathcal{N}\) such that \(\epsilon \circ \xi_i = \alpha_i, i = 1, 2\). Now injectivity of \(\xi_i\) follows from injectivity of \(\alpha_i\). Suppose \(\xi_1(v_1) = \xi_2(v_2)\) for some \(v_i \in \mathbb{F}^{n_i}\). Then \(\alpha_1(v_1) = \alpha_2(v_2)\), which means \((v_1, 0) = (0, v_2)\). Thus \(v_1 = 0\) and \(v_2 = 0\). This implies \(\xi_1(v_1) = \xi_2(v_2) = 0\), and the claim is proved.
Claim 2: $\mathbb{F}^n = \xi_1(\mathbb{F}^{n_1}) \oplus \xi_2(\mathbb{F}^{n_2})$ and thus $n = n_1 + n_2$. Set $X_i = \xi_i(\mathbb{F}^{n_i})$, which is a subspace of $\mathbb{F}^n$, and $X = X_1 \oplus X_2$. Consider the restriction $\mathcal{M}|_X$. The maps $\hat{\xi}_i : \mathcal{M} \rightarrow \mathcal{M}|_X$, $v \mapsto \xi_i(v)$, are clearly type-$\Delta$ as well; see Proposition 3.12. We want to show that $(\mathcal{M}|_X, \hat{\xi}_1, \hat{\xi}_2)$ is a coproduct as well. To do so, consider first the diagram

Since $\mathcal{M}$ is a coproduct, there is a unique type-$\Delta$ map $\hat{\epsilon}$ satisfying $\hat{\epsilon} \circ \xi_i = \hat{\xi}_i$ for $i = 1, 2$. Define $\tau : \mathcal{M}|_X \rightarrow \mathcal{M}$ via $x \mapsto x$. From Proposition 3.11(a) we know that $\tau$ is type-$\Delta$. Consider the map $\hat{\epsilon} \circ \tau : \mathcal{M}|_X \rightarrow \mathcal{M}|_X$. It satisfies $\hat{\epsilon} \circ \tau|_{X_i} = \text{id}_{X_i}$ for $i = 1, 2$. Hence Theorem 3.8 implies that $\hat{\epsilon} \circ \tau : X \rightarrow X$ is a bijective map (and equivalent to a semi-linear isomorphism).

Claim 3: $\mathcal{M}_1, \mathcal{M}_2$ have a coproduct of the form $(\tilde{\mathcal{M}}, \iota_1, \iota_2)$, where $\tilde{\mathcal{M}}$ has ground space $\mathbb{F}^{n_1+n_2}$ and $\iota_i : \mathcal{M}_i \rightarrow \tilde{\mathcal{M}}$ are as in (4.2). First of all, there exists an $\mathcal{L}$-isomorphism $\beta : X \rightarrow \mathbb{F}^{n_1+n_2}$ such that $\beta \circ \xi_i = \iota_i$, $i = 1, 2$. To see this, use again the construction in Proposition 4.3, consider any $q$-matroid $\mathcal{N}$ on $\mathbb{F}^{n_1+n_2}$ and the type-$\Delta$ maps $\iota_i : \mathcal{M}_i \rightarrow \mathcal{N}$ with $\iota_i$ as in (4.2). Since $\mathcal{M}|_X$ is a coproduct, there exists a type-$\Delta$ map $\beta : X \rightarrow \mathbb{F}^{n_1+n_2}$ such that $\beta \circ \xi_i = \iota_i$, $i = 1, 2$. Hence $e_1, \ldots, e_n$ are in the image of $\beta$ and thus Theorem 3.8 implies that $\beta$ is bijective. All of this provides us with the desired $\mathcal{L}$-isomorphism $\beta$. Now we use Remark 4.3 to define a new $q$-matroid structure on $\mathbb{F}^{n_1+n_2}$. Set
\( \hat{M} = (F^{n_1 + n_2}, \hat{\rho}) \) via \( \hat{\rho}(V) := \rho(\rho^{-1}(V)) \) with \( \rho \) being the rank function of \( \mathcal{M}|_X \). This makes sense because the inverse of an \( \mathcal{L} \)-isomorphism is again an \( \mathcal{L} \)-map. Thanks to Remark 2.3 the flats of \( \hat{M} \) are given by \( \hat{F} := \{ \beta(F) \mid F \in \mathcal{F}(\mathcal{M}|_X) \} \). Now \( \beta \) is an isomorphism in \( \mathcal{q}\text{-}{\text{Mat}}^\Delta \) from \( \mathcal{M} \) to \( \hat{M} \), and the maps \( \iota_i : \mathcal{M} \rightarrow \hat{\mathcal{M}} \) are type-\( \Delta \). Thus \( (\hat{\mathcal{M}}, \beta \circ \xi_1, \beta \circ \xi_2) = (\mathcal{M}, \iota_1, \iota_2) \) is a coproduct, as desired.

Now we are ready to show the non-existence of a coproduct in the three nonlinear categories for \( q \geq 3 \).

**Theorem 4.8.** Let \( q \geq 3 \) and \( \Delta \in \{ s, w, rp \} \). Then there exists representable \( q \)-matroids that do not have a coproduct in \( \mathcal{q}\text{-}{\text{Mat}}^\Delta \).

**Proof.** Suppose \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) have a coproduct. Let the ground space of \( \mathcal{M}_i \) be \( F^{n_i} \). By Theorem 4.7 we may assume that the coproduct is of the form \( (\mathcal{M}, \iota_1, \iota_2) \) where the ground space of \( \mathcal{M} \) is \( F^{n_1 + n_2} \) and \( \iota_i \) are as in (4.2). Thanks to Proposition 4.1 there exists a \( q \)-matroid \( \mathcal{N} \) with ground space \( F^{n_1 + n_2} \) and strong maps \( \iota_i : \mathcal{M}_i \rightarrow \mathcal{N} \). Hence there exists a type-\( \Delta \) map \( \epsilon : \mathcal{M} \rightarrow \mathcal{N} \) such that \( \epsilon \circ \iota_i = \iota_i \) for \( i = 1, 2 \). Note that \( \epsilon : F^{n_1 + n_2} \rightarrow F^{n_1 + n_2} \) is a bijective \( \mathcal{L} \)-map (see also Theorem 3.8). Choose now \( w \in F^{n_1 + n_2} \setminus \{ \iota_1(F^{n_1}) \cup \iota_2(F^{n_2}) \} \) and \( \tau \in F \setminus \{ 0, 1 \} \). Set \( \hat{w} = \epsilon^{-1}(\tau \epsilon(w)) \). By Proposition 3.7 we have \( \langle w \rangle = \langle \hat{w} \rangle \) and an \( \mathcal{L} \)-isomorphism \( \epsilon' : F^{n_1 + n_2} \rightarrow F^{n_1 + n_2} \) given \( \epsilon'(v) = \epsilon(v) \) for all \( v \in F^{n_1 + n_2} \setminus \langle w \rangle \) and \( \epsilon'(\mu v) = \epsilon(\mu \hat{w}) \). Furthermore, \( \epsilon' \sim \epsilon \). All of this implies that \( \epsilon' \) is a type-\( \Delta \) map satisfying \( \epsilon' \circ \iota_i = \iota_i \) for \( i = 1, 2 \). This contradicts the uniqueness of the map \( \epsilon \).

All other cases have to be considered individually. We start with \( q = 2 \) and (linear) strong maps.

**Theorem 4.9.** Let \( q = 2 \) and \( \Delta \in \{ s, l\text{-}s \} \). There exist representable \( q \)-matroids that do not have a coproduct in \( \mathcal{q}\text{-}{\text{Mat}}^\Delta \).

**Proof.** Let \( F = F_2 \) and \( \mathcal{M}_1 = U_1(F_2^2) = \mathcal{M}_2 \), that is \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are the uniform matroids on \( F^2 \) with rank 1, respectively. Their collections of flats are

\[
\mathcal{F}(\mathcal{M}_1) = \mathcal{F}(\mathcal{M}_2) = \{ \{0\}, F^2 \}. \tag{4.5}
\]

From Theorem 4.7 we know that if \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) have a coproduct, then it is without loss of generality of the form \( (\mathcal{M}, \iota_1, \iota_2) \), where \( \mathcal{M} \) has ground space \( F^4 \) and \( \iota_i \) are as in (4.2). We will show that such \( \mathcal{M} \) does not exist. Do so so, we construct various \( q \)-matroids \( \mathcal{N}^{(j)} \) along with linear strong maps \( \tau_i : \mathcal{M}_i \rightarrow \mathcal{N}^{(j)} \), \( i = 1, 2 \). Let \( \omega \in F_2^4 \) be a primitive element with minimal polynomial \( x^4 + x + 1 \in F_2[x] \) and let \( G^{(j)}, j \in \{1, \ldots, 14\} \), be as in Proposition 4.5. For every \( j \in \{1, \ldots, 14\} \) let \( \mathcal{N}^{(j)} = (F^4, \rho_j) \) be the associated \( q \)-matroid, thus \( \rho_j(rs(Y)) = \text{rk}(G^{(j)}Y^T) \) for \( Y \in F^{4 \times 4} \). Note that the uniform \( q \)-matroids \( \mathcal{M}_1 = \mathcal{M}_2 \) are represented by every matrix \( (1 \, \omega^j) \in F_2^{1 \times 2} \). Therefore Proposition 4.4 tells us that for every \( j \in \{1, \ldots, 14\} \) the maps \( \iota_i, i = 1, 2 \), are linear strong maps \( \iota_i : \mathcal{M}_i \rightarrow \mathcal{N}^{(j)} \). Since \( (\mathcal{M}, \iota_1, \iota_2) \) is a coproduct, there exists for every \( j \in \{1, \ldots, 14\} \) a type-\( \Delta \) map \( \epsilon_j : \mathcal{M} \rightarrow \mathcal{N}^{(j)} \) such that \( \epsilon_j \circ \iota_i = \iota_i \) for \( i = 1, 2 \). Hence \( \epsilon_j(v_1, 0) = (v_1, 0) \) and \( \epsilon_j(0, v_2) = (0, v_2) \) for all \( v_1, v_2 \in F^2 \). Now Theorem 3.8 tells us that \( \epsilon_j \) is linear (regardless of \( \Delta \)) and since it maps all standard basis vectors to itself, we conclude that \( \epsilon_j = \text{id}_{F^4} \) for all \( j \in \{1, \ldots, 14\} \).
Thus we arrive at the commutative diagrams

\[
\begin{array}{ccc}
M_1 & \xrightarrow{\iota_1} & M \\
\downarrow \iota_1 & & \downarrow \text{id} \\
N^{(j)} & \xhookrightarrow{i_1} & M_2 \\
\downarrow i_2 & & \downarrow \iota_2 \\
M_2 & & M_2
\end{array}
\]

(4.6)

Since \(\text{id} : M \to N^{(j)}\) is a strong map for all \(j \in [14]\), we conclude that the elements of

\[\mathcal{F'} := \{ F \leq \mathbb{F}_2^4 \mid \exists j \in [14] \text{ such that } F \in \mathcal{F}(N^{(j)})\}\]

have to be flats of \(M\). It turns out that \(|\mathcal{F'}| = 19\). Let \(\mathcal{F} := \mathcal{F}(M)\). Then \(\mathcal{F}\) satisfies (F1)–(F3) from Theorem 2.5. This implies in particular that (F3) has to be true for the flats in \(\mathcal{F'}\). To investigate this further we define for \(V \in \mathcal{F'}\)

\[\text{Cov}_{\mathcal{F'}}(V) = \{ F \in \mathcal{F'} \mid V < F \text{ and there is no } Z \in \mathcal{F'} \text{ such that } V < Z < F\},\]

\[\text{Cov}_{\mathcal{F}}(V) = \{ F \in \mathcal{F} \mid V < F \text{ and there is no } Z \in \mathcal{F} \text{ such that } V < Z < F\}.\]

We call the elements of \(\text{Cov}_{\mathcal{F'}}(V)\) and \(\text{Cov}_{\mathcal{F}}(V)\) covers of \(V\) in \(\mathcal{F'}\) and \(\mathcal{F}\), respectively. Of course, the covers of \(V\) in \(\mathcal{F'}\) need not be actual covers of \(V\) in the lattice \(\mathcal{F}\). However, the set

\[S := \{(V, v) \in \mathcal{F'} \times \mathbb{F}_2^4 \mid v \notin V \text{ and } V + \langle v \rangle \not\leq F \text{ for all } F \in \text{Cov}_{\mathcal{F'}}(V)\}\]

provides crucial information. Indeed, \(|S| = 90\), which means that \(\text{Cov}_{\mathcal{F'}}(V) \neq \text{Cov}_{\mathcal{F}}(V)\) for each \(V \in \mathcal{F'}\) that appears in a pair in \(S\), and in particular, \(\mathcal{F'} \neq \mathcal{F}\). Since \(\mathcal{F}\) satisfies (F3), the set \(\mathcal{F} \setminus \mathcal{F'}\) must contain at least one space from the set

\[\mathcal{Y} = \{ Y \in \mathbb{F}_2^4 \mid \exists (V, v) \in S \text{ such that } V + \langle v \rangle \leq Y \text{ and } F \not\leq Y \text{ for all } F \in \text{Cov}_{\mathcal{F'}}(V)\}\]

We obtain \(|\mathcal{Y}| = 33\). However, for each \(Y \in \mathcal{Y}\) we have \(\iota_1^{-1}(Y) \notin \mathcal{F}(M_1)\) or \(\iota_2^{-1}(Y) \notin \mathcal{F}(M_2)\); see (1.5). Hence, no space of \(\mathcal{Y}\) can be contained in \(\mathcal{F}\) and we arrived at a contradiction. All of this shows that there is no such coproduct of \(M_1\) and \(M_2\) in \(q\text{-Mat}^\Delta\). \(\Box\)

**Remark 4.10.** A similar example as in the previous proof can be constructed for \(q = 3\) and 5 (and possibly larger \(q\)). This establishes the non-existence of a coproduct in \(q\text{-Mat}^{\text{lp}}\) for (some) \(q > 2\).

The above proof can be adjusted to the case of linear rank-preserving morphisms for arbitrary \(q\). Since these are actually isomorphisms, it should be clear that they do not allow a coproduct.

**Corollary 4.11.** There exists representable \(q\)-matroids \(M_1, M_2\) that do not have a coproduct in \(q\text{-Mat}^{\text{lp}}\).
Proof. Let \( F = \mathbb{F}_q \) and \( \mathcal{M}_1 = \mathcal{M}_2 = U_1(\mathbb{F}_q^2) \), and assume that \((\mathcal{M}, \iota_1, \iota_2)\) with ground space \( \mathbb{F}_q^4 \) and \( \iota_i \) as in (4.2) is a coproduct. This is the same setting as in the proof of Theorem 4.9 but for arbitrary \( q \). Consider the first part of that proof. Let \( N^{(j)}, j = 1, \ldots, q^m - 2 \), be the \( q \)-matroids represented by \( G^{(j)} \), where \( G^{(j)} \) is as in Proposition 4.5 The maps \( \iota_i : M_i \rightarrow N^{(j)} \) are linear and rank-preserving. Thus we have linear rank-preserving maps \( \epsilon_j : M \rightarrow N^{(j)} \) satisfying \( \epsilon_j \circ \iota_i = \iota_i \) for \( i = 1, 2 \). However, linearity forces these maps to be the identity on \( \mathbb{F}_4 \). Thus we arrive at Diagram (4.6), and we conclude that \( M \) is linearly isomorphic to \( N^{(j)} \) for all \( j \), but that is a contradiction to Proposition 4.5.

It remains to discuss weak and rank-preserving morphisms for \( q = 2 \).

**Theorem 4.12.** Let \( q = 2 \) and \( \Delta \in \{ w, rp \} \). There exist representable \( q \)-matroids that do not have a coproduct in \( q \)-Mat\(^{\Delta} \).

**Proof.** Let \( F = \mathbb{F}_2 \). As in the proof of Theorem 4.9 let \( \mathcal{M}_1 = \mathcal{M}_2 = U_1(\mathbb{F}^2) \). Suppose \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) have a coproduct in \( q \)-Mat\(^{\Delta} \). Then it is of the form \( (\mathcal{M}, \iota_1, \iota_2) \) with ground space \( \mathbb{F}_4^{\Delta} \) and \( \iota_i \) as in (4.2). Denote by \( \rho \) its rank function. The first part of the proof of Theorem 4.9 works in the same way for type-\( \Delta \) maps and we arrive at (4.6) and where \( \text{id}_{\mathbb{F}_4} : M \rightarrow N^{(j)} \) is type-\( \Delta \). Now one can go through all \( N^{(j)} \) from (4.6) and obtains that

\[
\rho(V) \geq 1 \text{ for dim } V = 1 \text{ and } V \in \{ \iota_1(\mathbb{F}^2), \iota_2(\mathbb{F}^2) \}
\]

and \( \rho(V) \geq 2 \) for all other nonzero subspaces \( V \in \mathcal{L}(\mathbb{F}^4) \). Using that \( \iota_i : \mathcal{M}_i \rightarrow \mathcal{M} \) are also type-\( \Delta \), we conclude that \( \rho(\iota_1(\mathbb{F}^2)) = \rho(\iota_2(\mathbb{F}^2)) = 1 \). Thus \( \rho(V) = \text{dim } V \) for all subspaces \( V \) of dimension at most 2 except for \( V \in \{ \iota_1(\mathbb{F}^2), \iota_2(\mathbb{F}^2) \} \). Now (R2) and (R3) from Definition 2.1 imply \( 2 \leq \rho(\mathbb{F}^4) \leq 1 + 1 = 2 \), hence \( \rho(\mathbb{F}^4) = 2 \). All of this shows that \( \mathcal{M} = (\mathbb{F}^4, \rho) \), where

\[
\rho(V) = \begin{cases} 
0, & \text{if } V = 0, \\
1, & \text{if dim } V = 1 \text{ or } V \in \{ \iota_1(\mathbb{F}^2), \iota_2(\mathbb{F}^2) \}, \\
2, & \text{otherwise.}
\end{cases}
\]

Consider now the uniform \( q \)-matroid \( \mathcal{N} = U_3(\mathbb{F}^3) \) and let \( \alpha_i : \mathbb{F}^2 \rightarrow \mathbb{F}^3 \) be defined as \( \alpha_i(0) = 0 \) and

\[
\alpha_1(v) = 100 \quad \text{and} \quad \alpha_2(v) = 010 \quad \text{for all } v \in \mathbb{F}^2 \setminus 0.
\]

Then \( \alpha_1, \alpha_2 \) are nonlinear \( \mathcal{L} \)-maps and in fact rank-preserving (hence type-\( \Delta \)) from \( \mathcal{M}_i \) to \( \mathcal{N} \) for \( i = 1, 2 \). Thus, there exists a type-\( \Delta \) map \( \epsilon : \mathcal{M} \rightarrow \mathcal{N} \) such that \( \epsilon \circ \iota_i = \alpha_i \). In other words,

\[
\epsilon(x, y, 0, 0) = 100 \quad \text{and} \quad \epsilon(0, 0, x, y) = 010 \quad \text{for all } (x, y) \in \mathbb{F}^2 \setminus (0, 0).
\]

But now it is easy to verify that there is no \( \mathcal{L} \)-map \( \epsilon \) with these conditions. Indeed \( \epsilon((1000, 0010)) \) contains 000, 100, 010 and thus must contain 110. This implies \( \epsilon(1010) = 110 \). In the same way \( \epsilon(1001) = 110 \). But then \( \epsilon((1010, 1001)) = \{000, 110, 010\} \), which is not a subspace. All of this shows that \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) have no coproduct in \( q \)-Mat\(^{\Delta} \). \( \square \)
5 A Coproduct in $q$-Mat$_{lw}$

In this section we establish the existence of a coproduct in the category $q$-Mat$_{lw}$. In fact, we will show that the direct sum, introduced in [3, Sec. 7] is such a coproduct. For the sake of self-containment we will first give a concise description of the direct sum and provide the necessary proofs. The description, different from the original one in [3], will make the presentation consistent with this paper and be helpful for proving the existence of a coproduct. Alternative proofs of the statements below can be found at [3] Prop. 22, Prop. 26, Def. 40).

We need to recall the notions of independent spaces and circuits of a $q$-matroid. Let $\mathcal{M} = (E, \rho)$ be a $q$-matroid. Then $V \in \mathcal{L}(E)$ is independent if $\rho(V) = \dim V$ and dependent otherwise. A dependent space all of whose proper subspaces are independent is called a circuit. We refer to [2, Def. 7 and 14] for the properties of the collections of independent spaces and the collection of circuits of a $q$-matroid. We will only need the following two facts: (a) every subspace of an independent space is independent; (b) for every $V \in \mathcal{L}(E)$ there exists an independent space $I \leq V$ such that $\rho(I) = \rho(V)$.

We start with the following result taken from [5].

Theorem 5.1 ([5, Thm. 3.9 and its proof]). Let $\tau : \mathcal{L}(E) \to \mathbb{N}_0$ be a map satisfying (R2), (R3) from Definition 2.1. Define $r_\tau : \mathcal{L}(E) \to \mathbb{N}_0$, $V \mapsto \min\{\tau(X) + \dim V - \dim X \mid X \in \mathcal{L}(V)\}$. Then $r_\tau$ satisfies (R1)–(R3) and thus defines a $q$-matroid $\mathcal{M} = (E, r_\tau)$. Furthermore, a subspace $V$ is independent in $\mathcal{M}$ iff $\tau(W) \geq \dim W$ for all $W \in V$.

Theorem 5.2 ([3, Sec. 7]). Let $\mathcal{M}_i = (E_i, \rho_i)$, $i = 1, 2$, be $q$-matroids and set $E = E_1 \oplus E_2$. Let $\pi_i : E \to E_i$ be the corresponding projection onto $E_i$. Define $\rho'_i : \mathcal{L}(E) \to \mathbb{N}_0$, $V \mapsto \rho_i(\pi_i(V))$ for $i = 1, 2$. Then $\mathcal{M}'_i = (E, \rho'_i)$ is a $q$-matroid. Furthermore, let

$$\rho : \mathcal{L}(E) \to \mathbb{N}_0, \quad V \mapsto \min\{\rho'_1(X) + \rho'_2(X) + \dim V - \dim X \mid X \in \mathcal{L}(V)\}. \tag{5.1}$$

Then $\mathcal{M} = (E, \rho)$ is a $q$-matroid. It is called the direct sum of $\mathcal{M}_1$ and $\mathcal{M}_2$ and denoted by $\mathcal{M}_1 \oplus \mathcal{M}_2$. The collection of circuits of $\mathcal{M}_1 \oplus \mathcal{M}_2$ is given by

$$\mathcal{C}(\mathcal{M}_1 \oplus \mathcal{M}_2) = \left\{ C \in \mathcal{L}(E) \left\mid C \text{ is inclusion-minimal subject to the condition } \rho'_1(C) + \rho'_2(C) \leq \dim C - 1 \right. \right\}.$$

Proof. 1) We show that $\rho'_1$ is indeed a rank function. Using that $\rho_1$ is a rank function, we have $0 \leq \rho_1(\pi_1(V)) \leq \dim \pi_1(V) \leq \dim V$, and this shows (R1). (R2) is trivial. For (R3) let $V, W \in E$. Note first that $\pi_1(V \cap W) \subseteq \pi_1(V) \cap \pi_1(W)$. Thus

$$\rho'_1(V + W) = \rho_1(\pi_1(V + W)) = \rho_1(\pi_1(V) + \pi_1(W)) \leq \rho_1(\pi_1(V)) + \rho_1(\pi_1(W)) - \rho_1(\pi_1(V) \cap \pi_1(W)) \leq \rho_1(\pi_1(V)) + \rho_1(\pi_1(W)) - \rho_1(\pi_1(V \cap W)) = \rho'_1(V) + \rho'_1(W) - \rho'_1(V \cap W).$$

2) For the fact that $\rho$ is a rank function consider first the function $\tau(V) = \rho'_1(V) + \rho'_2(V)$. It is clearly non-negative and one straightforwardly shows that it satisfies (R2), (R3) from Definition 2.1 see also [3, Thm. 28]. Hence Theorem 5.1 implies that $\rho$ is a rank function and that the collection of circuits is as stated.
The $q$-matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ are naturally embedded in the direct sum.

**Theorem 5.3** ([3] Sec. 7). Let the data be as in Theorem 5.2 and in particular $E = E_1 \oplus E_2$. Let $\iota_i$ be as in (4.1). Let $V \in \mathcal{L}(E_i)$. Then

$$\rho'_i(\iota_i(V)) = \rho_i(V) = \rho(\iota_i(V)) \quad \text{and} \quad \rho'_j(\iota_i(V)) = 0 \text{ for } j \neq i.$$ 

As a consequence, $\iota_i$ induces linear rank-preserving isomorphisms between $\mathcal{M}_i$ and $\mathcal{M}_i|_{\iota_i(E_i)}$ and $(\mathcal{M}_i')|_{\iota_i(E_i)}$.

**Proof.** $\rho'_i(\iota_i(V)) = \rho_i(V)$ follows from $\pi_i(\iota_i(V)) = V$ and $\rho'_j(\iota_i(V)) = 0$ is clear because $\pi_j(\iota_i(V)) = 0$. We show the identity $\rho'_i(\iota_i(V)) = \rho(\iota_i(V))$ for $i = 1$. Choose $X \subseteq \iota_1(E)$ and write $\iota_1(V) = X \oplus Y$. Using that $\rho'_1$ is a rank function, we obtain $\rho'_1(\iota_1(V)) \leq \rho'_1(X) + \rho'_1(Y) \leq \rho'_1(X) + \dim Y = \rho'_1(X) + \dim X = \rho'_1(X) + \rho'_2(X) + \dim X_1 - \dim X$, where the last step follows from the fact that $X \subseteq \iota_1(E_1)$, hence $\rho'_2(X) = 0$. All of this shows that the minimum in (5.1) is attained by $\rho'_1(\iota_1(V))$. The consequence is clear. 

Thanks to the above we may and will from now on identify subspaces $V$ in $E_i$ with their image $\iota_i(V)$. Note that by the above, $E_i$ is a loop space of $\mathcal{M}_j'$ for $j \neq i$ (by definition a loop space is a space with rank 0). The process from $\mathcal{M}_1$ to $\mathcal{M}_1'$ is called adding a loop space in [3]. It is a special instance of the direct sum because $\mathcal{M}_i' \cong \mathcal{M}_1 \oplus \mathcal{U}_0(E_j)$ for $i \neq j$.

We wish to mention that the direct sum also satisfies $(\mathcal{M}_1 \oplus \mathcal{M}_2)/E_i \cong \mathcal{M}_j$ for $j \neq i$. This has been shown in [3 Thm. 47] with the aid of duality but can also be proven directly using the identity $\rho(V_1 \oplus V_2) = \rho(V_1) + \rho(V_2)$ for all subspaces $V_i \subseteq E_i$. The latter identity is not hard to derive from the definition of the rank function in (5.1).

We now turn to our main result stating that $\mathcal{M}_1 \oplus \mathcal{M}_2$ is a coproduct in $q$-Mat$^{\text{lw}}$. We need the following lemma.

**Lemma 5.4.** Let $\mathcal{M}_i = (E_i, \rho_i), i = 1, 2$, be $q$-matroids and $\phi : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ be a linear map. Suppose $\phi$ is not a weak map. Then there exists a circuit $C$ of $\mathcal{M}_1$ such that

$$\rho_2(\phi(C)) > \rho_1(C), \quad \dim C = \dim \phi(C), \quad \phi(C) \text{ is an independent space in } \mathcal{M}_2.$$ 

(5.2)

**Proof.** Since $\phi$ is not weak there exists an inclusion-minimal subspace $V \in \mathcal{L}(E_1)$ such that $\rho_2(\phi(V)) > \rho_1(V)$. Clearly $V \neq 0$. We will show that $V$ is the desired circuit and proceed in several steps.

1) We first establish the following identities

$$\rho_1(W) = \rho_1(V) = \rho_2(\phi(W)) = \rho_2(\phi(V)) - 1 \text{ for all } W \subseteq V \text{ with } \dim W = \dim V - 1.$$ 

(5.3)

Let $W \subseteq V$ with $\dim W = \dim V - 1$. Write $V = W \oplus X$, thus $X$ is a 1-space. Then $\phi(W) = \phi(W) + \phi(X)$ and therefore $\dim \phi(V) \leq \dim \phi(W) + 1$. Furthermore, by minimality of $V$ we have $\rho_2(\phi(W)) \leq \rho_1(W)$. Using the properties of rank functions, we obtain

$$\rho_2(\phi(W)) \leq \rho_1(W) \leq \rho_1(V) < \rho_2(\phi(V)),$$ 

(5.4)
and thus \( \phi(W) \leq \phi(V) \), which means \( \dim \phi(V) = \dim \phi(W) + 1 \), and in fact \( \phi(V) = \phi(W) \oplus \phi(X) \). This implies \( \rho_2(\phi(V)) \leq \rho_2(\phi(W)) + 1 \). Together with (5.4) this yields \( \rho_2(\phi(V)) = \rho_2(\phi(W)) + 1 \) as well as \( \rho_2(\phi(W)) = \rho_1(W) = \rho_1(V) \). This establishes (5.3).

2) We show that \( \phi|_V \) is injective. Assume to the contrary that there exists \( v \in V \setminus 0 \) such that \( \phi(v) = 0 \). Then \( V = W \oplus \langle v \rangle \) for some subspace \( W \) of \( V \) and thus \( \phi(V) = \phi(W) \). But this contradicts (5.3). Hence \( \phi|_V \) is injective and \( \dim V = \dim \phi(V) \).

3) We show that \( \phi(V) \) is independent in \( \mathcal{M}_2 \). Assume to the contrary that \( \rho_2(\phi(V)) < \dim \phi(V) \). Then there exists an independent subspace \( I \subseteq \phi(V) \) such that \( \rho_2(\phi(V)) = \rho_2(I) = \dim I \). Since \( \phi|_V \) is injective, there exists a subspace \( J \subseteq V \) such that \( \phi(J) = I \) and \( \dim J = \dim I \). Now we have \( \dim J = \dim I = \rho_2(I) \leq \rho_1(J) \leq \dim J \), where the first inequality follows from the minimality of \( V \) subject to \( \rho_2(\phi(V)) = \rho_1(V) \). Thus we have equality throughout, which shows that \( J \) is independent in \( \mathcal{M}_1 \). Furthermore, since \( J \leq V \) there exists a subspace \( W \subseteq V \) with \( \dim W = \dim V - 1 \) such that \( J \leq W \leq V \). Applying \( \phi \) we arrive at \( \rho_2(I) \leq \rho_2(\phi(W)) \leq \rho_2(\phi(V)) = \rho_2(I) \), and we have equality throughout. But this contradicts (5.3) and therefore \( \phi(V) \) is independent in \( \mathcal{M}_2 \).

4) It remains to show that \( V \) is a circuit. (5.3) together with 2) and 3) imply that for every hyperplane \( W \) of \( V \) we have \( \rho_1(W) = \rho_1(V) = \rho_2(\phi(V)) = \rho_2(I) \), and we have equality throughout. Hence \( \rho_2(\phi(V)) = \rho_1(V) \). Thus we have equality throughout, which shows that \( J \) is independent in \( \mathcal{M}_1 \). Since \( W \) was an arbitrary hyperplane of \( V \), we conclude that \( V \) is a circuit.

**Theorem 5.5.** Let \( \mathcal{M}_i = (E_i, \rho_i), i = 1, 2 \), be q-matroids and \( \mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 = (E, \rho) \) be the direct sum as defined in Theorem 5.2. Let \( \iota_i \) be as in (4.1). Then \((\mathcal{M}, \iota_1, \iota_2)\) is a coproduct of \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) in the category \( \text{q-Mat}^{\text{lw}} \).

**Proof.** First of all, thanks to Theorem 5.3 the maps \( \iota_i : \mathcal{M}_i \rightarrow \mathcal{M} \) are rank-preserving, thus weak. Let \( \mathcal{N} = (\mathcal{E}, \bar{\rho}) \) be a q-matroid and \( \alpha_i : \mathcal{M}_i \rightarrow \mathcal{N} \) be weak linear maps. We have to show the existence of a unique weak linear map \( \epsilon : \mathcal{M} \rightarrow \mathcal{N} \) such that \( \epsilon \circ \iota_i = \alpha_i \) for \( i = 1, 2 \). Since \( \mathcal{M} \) has ground space \( E_1 \oplus E_2 \) it is clear that the only linear map satisfying this condition is given by \( \epsilon(v_1 + v_2) = \alpha_1(v_1) + \alpha_2(v_2) \); recall that we identify \( E_i \) with \( \iota_i(E_i) \). Thus it remains to show that this map \( \epsilon \) is weak. We will use Lemma 5.4. Choose any circuit \( C \) in \( \mathcal{M} \). Theorem 5.3 tells us that \( C \) satisfies \( \rho_1'(C) + \rho_2'(C) \leq \dim C - 1 \). Denote by \( \rho'_i \) and \( \pi_i \) the maps as in Theorem 5.2. Set \( X_i = \pi_i(C) \). Then \( C \leq X_1 \oplus X_2 \) and, since \( X_i \leq E_i \), we obtain \( \epsilon(C) \leq \epsilon(X_1 \oplus X_2) = \epsilon(X_1) + \epsilon(X_2) = \alpha_1(X_1) + \alpha_2(X_2) \). Applying the rank function \( \bar{\rho} \) and using the weakness of the maps \( \alpha_i \), we compute

\[
\bar{\rho}(\epsilon(C)) \leq \bar{\rho}(\alpha_1(X_1) + \alpha_2(X_2)) \leq \bar{\rho}(\alpha_1(X_1)) + \bar{\rho}(\alpha_2(X_2)) \leq \rho_1(X_1) + \rho_2(X_2) = \rho_1'(C) + \rho_2'(C) \leq \dim C - 1.
\]

This shows that \( \epsilon(C) \) is not an independent space of \( \mathcal{N} \) with the same dimension as \( C \). Thus, no circuit in \( \mathcal{M} \) satisfies (5.2), and this shows that \( \epsilon \) is weak.

**Example 5.6.** Consider the q-matroids \( \mathcal{M}_1 = \mathcal{M}_2 = \mathcal{U}_2(\mathbb{F}_q^2) \), i.e., \( \rho_1(V) = \rho_2(V) = \min\{1, \dim V\} \) for all \( V \in \mathcal{L}(\mathbb{F}_q^2) \). The direct sum \( \mathcal{M}_1 \oplus \mathcal{M}_2 \) has been determined in [3, Ex. 48] using the definition of its rank function in (5.1). In this example we will derive the same result by making use of the fact that \( \mathcal{M}_1 \oplus \mathcal{M}_2 \) is a coproduct in \( \text{q-Mat}^{\text{lw}} \). Let \( \omega \) be a primitive element of \( \mathbb{F}_{q^4} \). Then \( G = (1, \omega^2) \in \mathbb{F}_{q^4}^{1 \times 2} \) represents \( \mathcal{M}_1 = \mathcal{M}_2 \). Consider \( G^{(2)} \) as in Proposition 4.5 and let \( \mathcal{N}^{(2)} = (\mathbb{F}_q^4, \rho^{(2)}) \) be the q-matroid generated by \( G^{(2)} \). Thanks to Proposition 4.5(c) we have

\[
\rho^{(2)}(V) = \min\{2, \dim V\} \quad \text{for} \quad V \in \mathcal{L}(\mathbb{F}_q^4) \setminus \{T_1, T_2\} \quad \text{and} \quad \rho^{(2)}(T_1) = \rho^{(2)}(T_2) = 1,
\]

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where \( T_1, T_2 \) are as in Proposition 4.5. Furthermore, by Proposition 4.4 we have rank-preserving, hence weak, maps \( \iota_i : M_i \rightarrow N^{(2)} \) for \( i = 1, 2 \). As a consequence, the map \( \epsilon : M_1 \oplus M_2 \rightarrow N^{(2)} \) from the above proof is the identity map on \( \mathbb{F}_2^4 \). It thus induces a weak map from \( M_1 \oplus M_2 \) to \( N^{(2)} \). Hence the rank function \( \rho \) of \( M_1 \oplus M_2 \) satisfies

\[
\rho(V) \geq \min\{2, \dim V\} \quad \text{for} \quad V \in \mathcal{L}(\mathbb{F}_2^4) \setminus \{T_1, T_2\}\) and \( \rho(T_1) \geq 1, \rho(T_2) \geq 1. \]

Using that \( \iota_i \) are also rank-preserving maps from \( M_i \) to \( M_1 \oplus M_2 \), see Theorem 5.3 we obtain that \( \rho(T_1) = \rho(T_1) = 1 \). Together with \( \rho(V) \leq \dim V \), this implies that \( \rho = \rho^{(2)} \) and thus \( N^{(2)} = M_1 \oplus M_2 \). Finally, we can also see that \( M_1 \oplus M_2 \) is not a coproduct in the category \( q\text{-Mat}^k \). Indeed, consider \( G^{(j)} \) as in Proposition 4.5 for other values of \( j \). It turns out that, for instance, for \( j = 3 \) the \( q\text{-matroid} \( N^{(j)} \) has 11 flats, 3 of which have dimension 2. On the other hand, \( M_1 \oplus M_2 \) has 13 flats, the only 2-dimensional ones being \( T_1 \) and \( T_2 \). Thus, the identity is not a strong map from \( M_1 \oplus M_2 \) to \( N^{(3)} \).

We close this section with the following characterization of the direct sum.

**Remark 5.7.** The fact that \( M_1 \oplus M_2 \) is a coproduct in \( q\text{-Mat}^{1\text{-w}} \) can be translated into the following characterization of the direct sum. For any finite-dimensional \( \mathbb{F} \)-vector space \( E \) define the set \( S = \{ \rho \mid \rho \text{ is a rank function on } E \} \) and the partial order \( \rho \leq \rho' \) iff \( \rho(V) \leq \rho'(V) \) for all \( V \in \mathcal{L}(E) \). Let now \( M_i = (E_i, \rho_i) \) be \( q \text{-matroids} \) and set \( E = E_1 \oplus E_2 \). Then the set \( \hat{S} = \{ \rho \in S \mid \rho|_{E_i} \leq \rho_i \} \) has a unique maximum, say \( \hat{\rho} \), and \( M_1 \oplus M_2 = (E, \hat{\rho}) \).

**Open Problems**

1. Is representability of \( q \text{-matroids} \) inherited by direct sums? More precisely (and possibly stronger), given representable \( q \text{-matroids} \) \( M_1 \) and \( M_2 \), do there exist representing matrices \( G_i \) of \( M_i \) such that \( G = \text{diag}(G_1, G_2) \) is a representing matrix of \( M_1 \oplus M_2 \)? Note that Proposition 4.5 answers this question in the affirmative for a very simple case; see also Example 5.6. In this context, it is worth mentioning that the converse is true: if \( M_1 \oplus M_2 \) is represented by the matrix \( G = (G_1 \mid G_2) \in \mathbb{F}_q^{k \times (n_1 + n_2)} \), then \( M_i \) is represented by \( G_i \). Using the non-representable \( q \text{-matroid} \) from Example 2.3 we obtain immediately infinitely many non-representable \( q \text{-matroids} \).

2. Can we characterize \( q \text{-matroids} \) that cannot be decomposed into a direct sum of smaller \( q \text{-matroids} \)? Recall classical matroids decompose into the direct sum of their connected components [9, Cor. 4.2.9]. First thoughts on connectedness of \( q \text{-matroids} \) can be found in [3, Sec. 8].

3. Of course, many other categorical questions can be posed, such as the existence of products, pullbacks and pushouts.
Appendix: A \(q\)-Matroid not representable over any \(\mathbb{F}^{n \times m}\)

Let \(\mathbb{F} = \mathbb{F}_2\). We prove that the \(q\)-matroid in Example 2.3 is not representable over \(\mathbb{F}^{4 \times m}\) for any \(m\) (see Remark 2.4 for this notion of representability). Let \(V = \{V_0, V_1, V_2, V_3\} \subseteq \mathcal{L}(\mathbb{F}^4)\), where
\[ V_0 = (1000, 0100), V_1 = (0010, 0001), V_2 = (1001, 0111), V_3 = (1011, 0110). \]

As explained in Example 2.3 we obtain a \(q\)-matroid \(\mathcal{M} = (\mathbb{F}^4, \rho)\), where
\[
\rho(V) = 1 \text{ for } V \in \mathcal{V} \quad \text{and} \quad \rho(V) = \min\{2, \dim V\} \text{ otherwise.}
\]

Assume to the contrary that there exists a rank-metric code \(C \leq \mathbb{F}^{4 \times m}\) such that
\[
\rho(V) = \frac{\dim C - \dim(C(V^\perp, c))}{m} \text{ for all } V \in \mathcal{L}(\mathbb{F}^4).
\]

Using \(V = \mathbb{F}^4\), we see that \(\dim C = 2m\). Furthermore, the above values for \(\rho(V)\) and the identity \(\mathcal{V} = \{V_0^\perp, \ldots, V_3^\perp\}\) lead to
\[
\dim C(V, c) = \begin{cases} 0, & \text{if } \dim V = 1 \text{ or } \dim V = 2 \text{ and } V \not\subseteq \mathcal{V}, \\ m, & \text{if } V \in \mathcal{V} \text{ or } \dim V = 3. \end{cases} \quad \text{(A.1)}
\]

The conditions \(\dim C(V_0, c) = \dim C(V_1, c) = m\) and \(\dim C(V, c) = 0\) if \(\dim V = 1\) imply that \(C\) has a basis of the form
\[
\begin{pmatrix} A_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} A_m \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B_1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ B_m \end{pmatrix}, \quad \text{(A.2)}
\]

where \(\mathcal{A} = \langle A_1, \ldots, A_m \rangle\) and \(\mathcal{B} = \langle B_1, \ldots, B_m \rangle\) are MRD codes in \(\mathbb{F}^{2 \times m}\). As for the spaces \(V_2\) and \(V_3\) note that
\[
V_2 = \text{rs} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} = \text{rs}(I_2 \mid S^T), \quad V_3 = \text{rs} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = \text{rs}(I_2 \mid T^T),
\]

where
\[
S = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad T = S^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Clearly, a matrix \(M \in \mathcal{C}\) is in \(\mathcal{C}(V_2, c)\) iff \(M = \begin{pmatrix} A_i \\ S_{Ai} \end{pmatrix}\) for some \(A \in \mathcal{A}\). Hence there exist linearly independent matrices \(\begin{pmatrix} \tilde{A}_i \\ S_{\tilde{A}_i} \end{pmatrix}\), \(i = 1, \ldots, m\), in \(\mathcal{C}\). But then \(\tilde{A}_1, \ldots, \tilde{A}_m \in \mathcal{A}\) must be linearly independent. Thus \(\mathcal{B} = S \mathcal{A}\). Using the space \(V_3\) we obtain similarly \(\mathcal{B} = T \mathcal{A}\). Hence \(\mathcal{A} = T^{-1} S \mathcal{A}\), and the latter is \(T \mathcal{A}\). In other words, \(\mathcal{A}\) is \(T\)-invariant. Note that \(\{0, I, T, T^2\}\) is the subfield \(\mathbb{F}_4\), and in particular, \(T^2 = I + T\). All of this shows that \(\mathcal{A}\) is an \(\mathbb{F}_4\)-vector space (thus has even dimension over \(\mathbb{F}_2\)) and has an \(\mathbb{F}_2\)-basis of the form \(A_1, \ldots, A_{\ell}, TA_1, \ldots, TA_{\ell}\), where \(\ell = m/2\). Using this basis of \(\mathcal{A}\), (A.2) reads as
\[
\begin{pmatrix} A_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} A_{\ell} \\ 0 \end{pmatrix}, \begin{pmatrix} TA_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} TA_{\ell} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ TA_1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ TA_{\ell} \end{pmatrix}, \begin{pmatrix} 0 \\ (I + T)A_1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ (I + T)A_{\ell} \end{pmatrix}.
\]

This shows that \(\mathcal{C}\) contains the matrices \(\begin{pmatrix} A_i + TA_i \\ A_i + TA_i \end{pmatrix}\) for \(i = 1, \ldots, \ell\), and therefore \(\dim \mathcal{C}(V, c) \geq \ell\) for \(V = \langle 1010, 0101 \rangle\). This contradicts (A.1) and we conclude that there is no code \(C \leq \mathbb{F}^{4 \times m}\) that represents the \(q\)-matroid \(\mathcal{M}\).
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