Nonlinear QED Effects in Strong-Field Magnetohydrodynamics

Jeremy S. Heyl
Theoretical Astrophysics 130-33, California Institute of Technology, Pasadena, California 91125

Lars Hernquist
Lick Observatory, University of California, Santa Cruz, California 95064, USA

We examine wave propagation and the formation of shocks in strongly magnetized plasmas by applying a variational technique and the method of characteristics to the coupled magnetohydrodynamic (MHD) and quantum-electrodynamic (QED) equations of motion. In sufficiently strong magnetic fields such as those found near neutron stars, not only is the plasma extremely relativistic but the effects of QED must be included to understand processes in the magnetosphere. As Thompson & Blaes [1] find, the fundamental modes in the extreme relativistic limit of MHD coupled with QED are two oppositely directed Alfvén modes and the fast mode. QED introduces nonlinear couplings which affect the propagation of the fast mode such that waves traveling in the fast mode evolve as vacuum electromagnetic ones do in the presence of an external magnetic field [2]. The propagation of a single Alfvén mode is unaffected but QED does alter the coupling between the Alfvén modes.

This processes may have important consequences for the study of neutron-star magnetospheres especially if the typical magnetic field strength exceeds the QED critical value \( B_{\text{QED}} \approx 4.4 \times 10^{13} \text{ G} \) as is suspected for soft-gamma repeaters and anomalous X-ray pulsars.

I. INTRODUCTION

Ultrarelativistic plasmas play an important role in energy transmission in many astrophysical settings including neutron-star magnetospheres, black hole accretion disks and the sources of gamma-ray bursts. Additionally, in the case of neutron stars, the magnetic field may exceed the QED critical value \( \approx 4.4 \times 10^{13} \text{ G} \) and vacuum corrections may affect dynamics of the electromagnetic field.

In relativistic field theory, it is natural to study dynamics using a Lagrangian formulation. In the case of QED, the one-loop vacuum corrections may be summarized by an effective Lagrangian which includes both the classical Lagrangian and the consequences of virtual pairs [3–6]. If the separation of the charges comprising the plasma can be neglected, the dynamics of the plasma may be treated using magnetohydrodynamics [7,1].

II. THE ACTION

The Lagrangian derived by Achterberg [7] may be simplified dramatically if the inertia of the charge carriers can be neglected, i.e. in the extreme relativistic limit. Thompson & Blaes [1] present two formulations for magnetohydrodynamics (MHD) appropriate in this limit. Furthermore, they also examine the limit where \( \omega^2, k^2 \ll eB \). In this limit, the passing MHD wave cannot excite the charge carriers into the second Landau level, so they are effectively trapped along a single field line. The fermion fields are restricted to 1 + 1 dimensions, and they may be treated using the technique of bosonization. The fermion fields are replaced by a four-dimensional axion field \( \theta \) which enforces the MHD condition, i.e. \( E \cdot B = 0 \).

Thompson & Blaes [1] obtain the simple action for the electromagnetic field in the presence of the relativistic plasma:

\[
S' = \int d^4x \left[ -\frac{1}{4} I + \frac{e^2}{2} \theta J \right].
\]

We have the following additional definitions,

\[
I = F_{\mu\nu}F^{\mu\nu} \quad \text{and} \quad J = F^{\mu\nu}F_{\mu\nu}
\]

where \( F^{\mu\nu} \) is the dual to the field tensor given by
\[ F^{\mu\nu} = \frac{1}{2} \epsilon^{\rho\lambda\mu\nu} F_{\rho\lambda}. \]  
\( \epsilon^{\rho\lambda\mu\nu} \) is the completely antisymmetric Levi-Civita tensor.

Here we will examine the modified action where
\[ S'' = \int d^4x \left[ \Lambda_{\text{QED}}(I, K) + \frac{e^2}{2} \theta J \right] \]
where \( K = -J^2 \). To maintain the \( CP \) and Lorentz invariance of QED, its effective Lagrangian \( \Lambda_{\text{QED}} \) must be a function of the field scalars
\[ I = 2 \left( |B|^2 - |E|^2 \right), \]
\[ K = -4 \left( E \cdot B \right)^2, \]
rather than of the pseudoscalar \( J = -4E \cdot B \).

### III. WAVE PROPAGATION

To study the propagation of waves through the plasma we will use the formalism of Heyl & Hernquist \cite{2}. We use the results of Thompson & Blaes \cite{1} to describe the traveling modes. Specifically we designate the external magnetic field by \( B_0 \) and the electric and magnetic fields associated with the wave by \( \delta E \) and \( \delta B \) respectively. We also have the constraints \( \delta E \cdot B_0 = 0 \) and \( k \cdot \delta B = 0 \) where \( k \) is the wave vector. Fig. 1 depicts the geometry of the propagating wave and defines the three Euler angles \( \Psi, \Theta \) and \( \phi \) used to describe its configuration \cite{8}. We will take the \( x, y \) and \( z \)-axes to be aligned with \( \delta B \times k, k \) and \( \delta B \) respectively.

The definitions allow us to calculate the invariants
\[ I = 2 \left[ B_0^2 + (\delta B)^2 + 2B_0 \delta B \sin \Theta \sin \Psi - (\delta E)^2 \right], \]
\[ J = -4(\delta B)(\delta E) \cos \Theta. \]

#### A. The Lagrange Condition

To calculate the equations of motion of the wave we assume that the wave fields \( \delta B \) and \( \delta E \) and the axion field \( \theta \) are dynamic while the external magnetic field \( B_0 \) is static. Varying the action with respect to the axion field yields,
\[ \frac{\partial L}{\partial \theta} = 0 \Rightarrow J = -4E \cdot B = 0. \]

The field \( \theta \) acts as a Lagrange multiplier to enforce the MHD condition. The equations of motion for the fields \( \delta E \) and \( \delta B \) are more complicated than in the vacuum case \cite{2} because here the relationship between the wave fields and their potential is more complicated.

#### B. The four-potential

We will designate the potential of the wave by the four-vector \( \delta A_\mu = [\delta A_t, \delta A(x, y, z)] \). To within a gauge transformation the vector potential is given by \( (\delta A)_t = -\psi(y, t) \) and \( (\delta A)_y = (\delta A)_z = 0 \).

Let us now examine the electric field,
\[ \delta E = -\nabla (\delta A_t) - \frac{\partial \delta A}{\partial t}. \]
FIG. 1. The configuration of the MHD wave. N.B.: in the configuration depicted, all three Euler angles are less than zero.

A portion of the electric field may be due to the scalar potential. We define

$$\delta E_{\text{int}} = \delta E + \frac{\partial \delta A}{\partial t}. \quad (11)$$

Let us insist that the direction of $\delta E$ is constant in time and its magnitude is a function of $y$ and $t$ only. Therefore, we have $\delta E = \delta E(y, t)x$ and calculate the curl of the irrotational component to obtain that $(\delta E)_z = 0$ (i.e. $J = 0$, $\cos \Theta = 0$), unless we have $\delta E$ constant throughout space. We also find that

$$(\delta E)_x = \psi, t + f(x, z, t). \quad (12)$$

However, since $(\delta E)_x$ and $\psi$ are functions of $y$ and $t$ we find that $f(x, z, t)$ depends only on $t$. $(\delta E)_y$ is still unconstrained. However we do know that $(\delta E)_y = R(\delta E)_x$. We can immediately deduce the scalar potential obtaining

$$\delta A_t = -yRf(t) - R \int_0^y \psi, t(u, t) du. \quad (13)$$
The first group of terms yields a constant background electric field in space which can vary in time. We drop this extra field and obtain the following four-potential,

\[ \delta A_t = R \int_0^y \psi_t(u, t)du \]  
\[ \delta A_y = \delta A_z = 0 \]  
\[ \delta A_x = -\psi(y, t) \]  

where

\[ R = \frac{(\delta E)_y}{(\delta E)_x} = -\cot \phi. \]  

The four-potential derived here is more general than that calculated by Heyl & Hernquist in that it contains an electric field such that \( k \cdot \delta E \neq 0 \) which complicates the derivation of the equations of motion. In summary we have

\[ (\delta B)_x = (\delta B)_y = (\delta E)_z = 0 \]  
\[ (\delta B)_z = \psi_t, \quad (\delta E)_x = \psi_t, \quad \text{and} \quad (\delta E)_y = R\psi_t. \]  

C. Equations of Motion

The equations of motion are obtained by attempting to minimize the action with respect to the electromagnetic potential and the axion field. We found earlier that the axion field acts as a Lagrange multiplier to enforce the MHD condition. For the potential we obtain the following expression

\[ \partial_{\mu} \frac{\partial L}{\partial (\delta A_{\nu, \mu})} = 0. \]  

Substituting the MHD action yields,

\[ \partial_{\mu} \left[ \left( 2e^2 \theta - 8J \frac{\partial \Lambda_{\text{QED}}}{\partial K} \right) F^{\mu \nu} + 4 \frac{\partial \Lambda_{\text{QED}}}{\partial I} F^{\mu \nu} \right] = 0. \]  

We see that the axion field and QED alter the equations of motion for the electromagnetic field,

\[ \partial_{\mu} F^{\mu \nu} = J^\nu. \]  

where

\[ J^\nu = -\frac{1}{B} (F^{\mu \nu} \partial_{\mu} A + F^{\mu \nu} \partial_{\mu} B) \]  

with

\[ A = 2e^2 \theta - 8J \frac{\partial \Lambda_{\text{QED}}}{\partial K} \]  
\[ B = 4 \frac{\partial \Lambda_{\text{QED}}}{\partial I}. \]  

If we neglect the effects of QED on the dynamics we obtain that

\[ \rho = 2e^2 \mathbf{B} \cdot \nabla \theta \]  
\[ \mathbf{J} = 2e^2 (\mathbf{E} \times \nabla \theta - \theta \mathbf{B}). \]  

If we specialize to the geometry and potential described in the previous subsections and eliminate the axionic degrees of freedom from the equations, we find that remaining components of Eq. can be written as
\[ a_1 \psi_{,yy} + b_1 \psi_{,yt} + c_1 \psi_{,tt} = 0 \] \hspace{1cm} (28)
\[ a_3 \psi_{,yy} + b_3 \psi_{,yt} + c_3 \psi_{,tt} = 0 \] \hspace{1cm} (29)

where

\[ a_1 = - \left[ 1 + Q(\psi_{,y} + B_{0z})^2 \right] \] \hspace{1cm} (30)
\[ b_1 = -Q \psi_{,t} (\psi_{,y} + B_{0z}) \left( \frac{RB_{0x}}{B_{0y}} - 2 - R^2 \right) \] \hspace{1cm} (31)
\[ c_1 = Q \psi_{,t}^2 \left( 1 + R^2 \right) \left( \frac{RB_{0x}}{B_{0y}} - 1 \right) - \frac{RB_{0x}}{B_{0y}} + 1 \] \hspace{1cm} (32)
\[ a_3 = Q(\psi_{,y} + B_{0z}) \left( \frac{\psi_{,t}^2 R}{B_{0y}} + B_{0x} \right) \] \hspace{1cm} (33)
\[ b_3 = \frac{\psi_{,t} R}{B_{0y}} \left\{ 1 - Q \left[ \left( 1 + R^2 \right) \left( \psi_{,t}^2 + B_{0z} \frac{B_{0y}}{R} \right) + (\psi_{,y} + B_{0z})^2 \right] \right\} \] \hspace{1cm} (34)
\[ c_3 = \frac{(\psi_{,y} + B_{0z}) R}{B_{0y}} [Q \psi_{,t}^2 (1 + R^2) - 1] \] \hspace{1cm} (35)

where

\[ B_{0x} = B_0 \cos \Psi \cos \phi \] \hspace{1cm} (36)
\[ B_{0y} = B_0 \cos \Psi \sin \phi \] \hspace{1cm} (37)
\[ B_{0z} = B_0 \sin \Theta \sin \Psi \] \hspace{1cm} (38)
\[ Q = 16 \frac{\partial^2 \Lambda_{\text{QED}}}{\partial I^2} / \frac{\partial \Lambda_{\text{QED}}}{\partial I}. \] \hspace{1cm} (39)

D. The Classical Limit

If we neglect the effects of QED in the equations above, we can derive the wave equations in relativistic MHD. Specifically, we take \( Q = 0 \) (in the classical limit \( \partial \Lambda_{\text{QED}} / \partial I = -1/4 \)) to obtain

\[ v^{-2} \psi_{,tt} - \psi_{,yy} = 0 \] \hspace{1cm} (40)
\[ \frac{R}{B_{0y}} (\psi_{,t} \psi_{,yt} - \psi_{,y} \psi_{,tt} - B_{0z} \psi_{,tt}) = 0 \] \hspace{1cm} (41)

where

\[ v^{-2} = 1 - \frac{RB_{0x}}{B_{0y}}. \] \hspace{1cm} (42)

Eq. (40) is satisfied by

\[ \psi(y, t) = \psi(y \pm vt). \] \hspace{1cm} (43)

This equation also satisfies Eq. (41) if \( R = 0 \) or \( B_{0z} = 0 \); consequently, MHD supports fully nonlinear modes if \( \delta \mathbf{E} \cdot \mathbf{k} = 0 \) (fast modes) or \( \delta \mathbf{B} \cdot \mathbf{B}_0 = 0 \) (Alfvén modes).

For the fast modes, we have \( R = 0 \) and a dispersion relation of \( \omega^2 = k^2 \). For the Alfvén modes, we find that \( \mathbf{B}_0, \delta \mathbf{E} \) and \( \mathbf{k} \) all lie in the plane perpendicular to \( \delta \mathbf{B} \). Since \( \delta \mathbf{E} \cdot \mathbf{B}_0 = 0 \) we have

\[ R = \frac{\delta \mathbf{B}}{\delta \mathbf{E}} \frac{\partial \mathbf{E}}{\partial \mathbf{B}} = -\frac{B_{0x}}{B_{0y}} \Rightarrow v_A^2 = \frac{1}{1 + R^2} = \sin^2 \phi, \] \hspace{1cm} (44)

yielding a dispersion relation of \( \omega = \pm k B_0 \cdot k_{B_0} \) is the component of \( \mathbf{k} \) directed along the external magnetic field.

However, even if \( \delta \mathbf{B} \cdot \mathbf{B}_0 = 0 \) unless \( \mathbf{B}_0 \) and \( \mathbf{k} \) are parallel (i.e. \( R = 0 \)).
\[ \psi(y, t) = \psi_1(y + v_A t) + \psi_2(y - v_A t) \]  \hspace{1cm} (45)

does not satisfy Eq. 1. Two oppositely traveling Alfvén modes will interact through the cross term,

\[ \frac{Rv_A^2}{B_0 y} (\psi_{1,y} \psi_{2,y} + \psi_{2,y} \psi_{1,y}) = 0. \]  \hspace{1cm} (46)

It is also straightforward to derive equations Eq. 10 and Eq. 11 from Eq. 26 and Eq. 27 using Maxwell’s equations.

**E. The Effects of QED**

QED introduces several additional terms into the equations of motion. By restricting the geometry of the wave, we can still satisfy Eq. 29 identically.

1. Fast Modes

First, we examine the fast modes which have both \( R = 0 \) and \( B_0 x = 0 \) and therefore satisfy Eq. 29 even when QED effects are included. For these modes, we obtain

\[ b_1 = 2Q \psi_{,t} (\psi_{,y} + B_0 z) \]  \hspace{1cm} (47)

\[ c_1 = 1 - Q \psi^2_{,t} \]  \hspace{1cm} (48)

Because \( a_1 \) does not depend on \( R \) or \( B_0 x \) it is still given by Eq. 30. As in Heyl & Hernquist \[2\] we expand the coefficients to first order in the fields,

\[ a_1 = a_{1,0} + a_{1,B} \psi_{,y} + a_{1,E} \psi_{,t} + O(\delta B^2) \]  \hspace{1cm} (49)

\[ b_1 = b_{1,0} + b_{1,B} \psi_{,y} + b_{1,E} \psi_{,t} + O(\delta B^2) \]  \hspace{1cm} (50)

\[ c_1 = c_{1,0} + c_{1,B} \psi_{,y} + c_{1,E} \psi_{,t} + O(\delta B^2). \]  \hspace{1cm} (51)

Notice the sign change relative to Heyl & Hernquist \[2\]. In the previous work, we selected a gauge where \((\delta E) = -\psi_{,t}\). We obtain

\[ a_{1,0} = -4 \left[ 4B_0^2 \frac{\partial^2 \Lambda_{QED}}{\partial I^2} + \frac{\partial \Lambda_{QED}}{\partial I} \right] \]  \hspace{1cm} (52)

\[ a_{1,B} = -16 \left[ 3 \frac{\partial^2 \Lambda_{QED}}{\partial I^2} B_0 z \right] \]  \hspace{1cm} (53)

\[ b_{1,E} = 32 \frac{\partial^2 \Lambda_{QED}}{\partial I^2} B_0 z \]  \hspace{1cm} (54)

\[ c_{1,0} = 4 \frac{\partial \Lambda_{QED}}{\partial I} \]  \hspace{1cm} (55)

\[ c_{1,B} = 16 \frac{\partial^2 \Lambda_{QED}}{\partial I^2} B_0 z, \]  \hspace{1cm} (56)

and

\[ a_{1,E} = b_{1,0} = b_{1,B} = c_{1,E} = 0. \]  \hspace{1cm} (57)

If we substitute these expansions back into Eq. 28 we obtain an identical result to that of Heyl & Hernquist \[2\] in the limit of \( K = 0 \). This is not particularly surprising because the fast modes do not carry current and cannot excite the axion field. Furthermore, the results of Heyl & Hernquist \[2\] may be applied directly to understand the evolution of the fast modes including the effects of QED to one-loop order.

Heyl & Hernquist \[2\] found that an electromagnetic wave traveling through a strongly magnetized vacuum will develop discontinuities after traveling a distance proportional to its wavelength and amplitude. Furthermore, the opacity to shocking peaks near the critical field \((\approx 4.4 \times 10^{13} \text{ G})\). After the discontinuity forms, the energy of the wave is quickly dissipated most likely as electron-positron pairs. The results of this section show that the presence of the plasma does not affect the development of this nonlinearity for the fast modes; therefore, even in the plasma-filled magnetosphere surrounding a neutron star, one would expect shocks to develop as waves in the fast mode propagate.
The treatment of the current-carrying modes within QED is more subtle. We can use the results that $B_{0z} = 0$ and $R = -B_{0x}/B_{0y}$ to simplify the equations,

\begin{align}
a_1 &= -1 - Q\psi_y^2 \\
b_1 &= 2Q\psi_t\psi_y (1 + R^2) \\
c_1 &= -Q\psi_t^2 (1 + R^2)^2 + 1 + R^2 \\
a_3 &= \frac{\psi_y B_{0y}}{B_{0y}} Q (\psi_t^2 - B_{0y}^2) \\
b_3 &= \frac{\psi_t R}{B_{0y}} \left[ 1 - Q \left( (1 + R^2) (\psi_t^2 - B_{0y}^2) + \psi_y^2 \right) \right] \\
c_3 &= \frac{\psi_y R}{B_{0y}} [Q\psi_t^2 (1 + R^2) - 1].
\end{align}

To proceed we combine Eq. 28 and Eq. 29 into a single relation by eliminating $\psi_y t$,

\[-v^2 \psi_{yy} + \psi_{tt} = 0 (64)\]

where

\[v^2 = \frac{a_1 b_3 - a_3 b_1}{b_3 c_1 - b_1 c_3} = \frac{1}{1 + R^2} \frac{1 - Q\psi_y^2}{1 - Q\psi_y^2 (1 + R^2)}.\]

This elimination is impossible in pure MHD (i.e. $Q = 0$) since the term $\psi_y t$ appears only in the second of the equations of motion (Eq. 41).

Let us attempt to use a simple wave solution similar to that of MHD,

\[\psi = \psi (y \pm v_{A,QED} t).\]

If we substitute this into Eq. 64, we find

\[v_{A,QED}^2 = \frac{1}{1 + R^2} \frac{1 - Q\psi_y^2}{1 - Q\psi_y^2 (1 + R^2)}.\]

which is satisfied by

\[v_{A,QED}^2 = v_{A}^2 = \frac{1}{1 + R^2}.\]

We find that a single Alfvén mode does not suffer any nonlinearity due to QED if the QED effective Lagrangian is invariant with respect to gauge and Lorentz transformations and

\[\frac{\partial \Lambda_{QED}}{\partial t} \neq 0.\]

Eq. 44 like Eq. 41 indicates that two oppositely traveling Alfvén modes will interact. If we substitute Eq. 45 into the equations of motion for Alfvén waves including QED we obtain

\[Q\psi_{2,y}^2 \psi_{1,yy} + \left\{ 1 \leftrightarrow 2 \right\} = 0 \]

\[\frac{Rv_A^2}{B_{0y}} \psi_{2,y}^2 \psi_{1,yy} \left[ 1 + Q \left( B_0^2 + 2\psi_{2,y} (\psi_{1,y} - \psi_{2,y}) \right) \right] + \left\{ 1 \leftrightarrow 2 \right\} = 0\]

where \(\left\{ 1 \leftrightarrow 2 \right\}\) designates the same terms repeated with the functions $\psi_1$ and $\psi_2$ swapped to obtain a symmetric sum.

To lowest order in the wave fields we find that QED does not affect the coupling between the Alfvén waves. However, at higher order, vacuum processes introduce several new interaction terms.
IV. CONCLUSIONS

We find that QED affects the propagation of magnetohydrodynamic fast modes traveling through an external magnetic field. The induced nonlinearity is identical to that suffered by electromagnetic radiation traveling through an external magnetic in the absence of a plasma. For the Alfvén modes, we find that QED introduces additional couplings between oppositely directed waves, but that because of the gauge and Lorentz invariance of QED, no nonlinearities manifest themselves in the propagation of a single Alfvén mode.

These new nonlinear processes emerge in regions where the magnetic field strength is comparable to or exceeds $B_{\text{QED}} \approx 4.4 \times 10^{13}$ G. Both anomalous X-ray pulsars (AXPs) and soft-gamma repeaters (SGRs) circumstantially exhibit such strong magnetic fields [9–13]; consequently, these nonlinear processes specific to intense magnetic fields may play an important role in the magnetospheres surrounding these objects.

ACKNOWLEDGMENTS

We would like to thank Peter Goldreich and Roger Blandford for useful discussions. J.S.H. would also like to acknowledge a Lee A. DuBridge Postdoctoral Scholarship and Cal Space Grant CS-12-97.

[1] C. Thompson and O. Blaes, Phys. Rev. D 57, 3219 (1998).
[2] J. S. Heyl and L. Hernquist, Phys Rev D 58, 043005 (1998).
[3] W. Heisenberg and H. Euler, Z. Physik 98, 714 (1936).
[4] V. S. Weisskopf, Kongelige Danske Videnskabernes Selskab, Mathematiske Fysik tabelser 14, 1 (1936).
[5] J. Schwinger, Physical Review 82, 664 (1951).
[6] J. S. Heyl and L. Hernquist, Phys. Rev. D 55, 2449 (1997).
[7] A. Achterberg, Phys. Rev. A 28, 2449 (1983).
[8] H. Goldstein, Classical Mechanics, 2nd ed. (Addison-Wesley, Reading, Massachusetts, 1980).
[9] G. Vasisht and E. V. Gottelf, ApJL 486, 129 (1997).
[10] J. S. Heyl and L. Hernquist, ApJL 489, 67 (1997).
[11] J. S. Heyl and S. R. Kulkarni, ApJL 506, 61 (1998).
[12] C. Thompson and R. C. Duncan, ApJ 473, 322 (1996).
[13] C. Kouveliotou et al., Nature 393, 235 (1998).