A COMPARISON OF VASSILIEV AND ZIEGLER-ŽIVALJEVIĆ MODELS FOR HOMOTOPY TYPES OF SUBSPACE ARRANGEMENTS

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Abstract. In this paper we represent the Vassiliev model for the homotopy type of the one-point compactification of subspace arrangements as a homotopy colimit of an appropriate diagram over the nerve complex of the intersection semilattice of the arrangement. Furthermore, using a generalization of simplicial collapses to diagrams of topological spaces over simplicial complexes, we construct an explicit deformation retraction from the Vassiliev model to the Ziegler-Živaljević model.

1. Introduction

Goresky and MacPherson, [6, Part III], were the first to express the cohomology groups of the complement of a subspace arrangement \( \mathcal{A} \) in terms of the homology groups of the order complexes of lower intervals of the associated intersection semilattice. Following that, there was a sizable body of work studying the topological properties of the complement of subspace arrangements, or, dually, of the one-point compactification of the union of subspaces, which we denote by \( \hat{\mathcal{U}}(\mathcal{A}) \), see [1, 7, 12, 14, 15]. Especially elucidating argument can be found in [14, Chapter II.5].

In particular, two models were constructed, one by Vassiliev, [12], and one by Ziegler and Živaljević, [15], reproducing \( \hat{\mathcal{U}}(\mathcal{A}) \) up to homotopy equivalence. The Ziegler-Živaljević model is based on the notion of homotopy colimit, dating back at least to [4], but see also [13] for a fresh approach; while Vassiliev’s construction is explicitly geometrical. It was explicitly verified in [13, page 140] that the two models are homotopy equivalent.

The purpose of this paper is twofold. First we find a presentation for the Vassiliev model as a certain homotopy colimit, thus bringing the two models to a common formal framework. Second, by using a diagram-theoretic generalization of simplicial collapses, coupled with the technical machinery of Discrete Morse Theory, [5], we describe a sequence of generalized collapses leading from the Vassiliev model to the Ziegler-Živaljević model. This, in turn, connects the two models by a deformation retraction.

Acknowledgments. I am grateful to Eva-Maria Feichtner for the careful proofreading of this paper, and to Günter M. Ziegler for the useful comments on the history of the subject.

Date: March 28, 2022.
Mathematics Subject Classification (2000): Primary 52C35, Secondary 55R80.
This research was supported by the Research Grant dnr 2247/1999 of the Swedish Natural Science Research Council.
2. Background

2.1. The terminology of posets.

A poset is a set with a specified partial order. We say that a poset $P$ is a semi-lattice if for any $x, y \in P$ the sets $\{z \in P \mid x \leq z, y \leq z\}$, resp. $\{z \in P \mid x \geq z, y \geq z\}$ are either empty or have minimal, resp. maximal elements.

Let $\mathcal{P}$ denote the full subcategory of the category of all small categories consisting of posets. Here posets are viewed as categories in the standard way, i.e., with elements being the objects and order relations being the morphisms. Let furthermore $\text{Top}$ denote the category of topological spaces and continuous maps.

The definition of the nerve of a category goes back to Quillen, [10], and Segal, [11], we state it only in the special case of posets, and we also compose it at once with the functor mapping simplicial complexes to their geometric realizations.

**Definition 2.1.** The functor $\Delta : \mathcal{P} \to \text{Top}$ maps a poset $P$ to the geometric realization of the simplicial complex whose vertices are the elements of $P$ and whose simplices correspond to chains (totally ordered subsets) of $P$. $\Delta(P)$ is commonly known as the order complex of $P$.

For $x \in P$, we denote by $P_{\leq x}$ the full subposet of $P$ consisting of elements $\{y \in P \mid y \leq x\}$. Analogously, $P_{< x}$ is the full subposet of $P$ consisting of elements $\{y \in P \mid y < x\}$.

The barycentric subdivision of a poset $P$, denoted $\text{Bd}(P)$ is a poset whose elements are all non-empty chains of $P$ partially ordered by inclusion.

Given a simplicial complex $K$, we denote by $\mathcal{F}(K)$ its face poset, which is the poset consisting of all non-empty faces of $K$ partially ordered by inclusion.

For $x, y \in P$, $x \geq y$, we denote by $I(y \hookrightarrow x)$ the inclusion map of the simplicial complexes $I(y \hookrightarrow x) : \Delta(P_{\leq y}) \hookrightarrow \Delta(P_{\leq x})$.

2.2. The terminology of subspace arrangements.

A subspace arrangement is a collection $A = \{A_1, \ldots, A_k\}$ of affine linear subspaces in $\mathbb{R}^n$, such that if $A_i \subseteq A_j$, then $A_i = A_j$. To this collection we associate the following invariants:

- The intersection semilattice $\mathcal{L}(A)$ consisting of all possible non-empty intersections of $A_i$’s ordered by reverse inclusion;
- The collection $\mathcal{B}(A) = \{B(x) \mid x \in \mathcal{L}(A)\}$ of corresponding affine subspaces indexed by the elements of the intersection semilattice;
- We denote $\mathcal{U}(A) = \cup_{i=1}^k A_i$ and $\mathcal{M}(A) = \mathbb{R}^n \setminus \mathcal{U}(A)$. Let $\hat{\mathcal{U}}(A)$ denote the one-point compactification of $\mathcal{U}(A)$.

In the rest of this section, following Vassiliev and Ziegler-Živaljević, [12, 15], we define two different topological spaces both of which are homotopy equivalent to $\hat{U}(A)$ (in particular, they are of course homotopy equivalent to each other).

2.3. Homotopy colimits.

**Definition 2.2.** A diagram of topological spaces over a poset $P$, is a covariant functor from $P$ to $\text{Top}$.

If the functor is denoted by $\mathcal{D}$, and $x$ is an element of $P$, we use $\mathcal{D}(x)$ to denote the topological space associated to $x$; and if $x, y \in P$, $x \geq y$, we use $\mathcal{D}(x \rightarrow y)$ to denote
the continuous map associated to the order relation \( x \geq y \) (which is a morphism in \( P \) viewed as a category).

In this paper the topological spaces \( D(x) \) are always direct products of (geometric realizations of) simplicial complexes with linear subspaces, and the maps \( D(x \to y) \) are always inclusions.

**Definition 2.3.** The homotopy colimit of a diagram of topological spaces \( D : P \to \text{Top} \), denoted by \( \text{hocolim}(D) \), is the colimit of the functor \( \Delta(D) : \text{Bd}(P) \to \text{Top} \) defined by:

- **on the elements:** \( \Delta(D)(x_1 > \cdots > x_i) = \Delta(P_{\leq x_i}) \times D(x_1) \);
- **on the morphisms:**
  \[
  \Delta(D)((x_1 > \cdots > x_i) \to (x_{i_1} > \cdots > x_{i_p})) = I(x_t \hookrightarrow x_{i_p}) \times D(x_1 \to x_{i_1}).
  \]

One of the main sources for details on homotopy colimits is \([4]\), see also \([13]\) for many combinatorial applications of the concept.

Later on, we shall need the following explicit description of the topological space \( \text{hocolim}(D) \). Consider the disjoint union of spaces \( D(x) \), for \( x \in P \), then for any order relation \( x > y \) glue in the mapping cylinder of the map \( D(x \to y) \), taking \( D(x) \) as the source, and \( D(y) \) as the base of it; for every triple \( x > y > z \) glue in the "mapping triangle" of maps \( D(x \to y) \) and \( D(y \to z) \) and so on through the entire order complex of \( P \). Of course, while geometrically intuitive, this description follows word-by-word the definition of the colimit.

An important special example which we need in this paper is the case when \( P \) is the face poset of a simplicial complex \( K \), \( P = \mathcal{F}(K) \). In this case, we call \( D : P \to \text{Top} \), a diagram over the simplicial complex \( K \).

**Definition 2.4.** Let \( D : P \to \text{Top} \) be a diagram of topological spaces over a poset, define a diagram over the simplicial complex \( \Delta(P) \), \( \text{Bd}(D) : \text{Bd}(P) \to \text{Top} \) as follows:

- **on objects:** \( \text{Bd}(D)(x_1 > \cdots > x_k) = D(x_1) \);
- **on morphisms:** \( \text{Bd}(D)((x_1 > \cdots > x_k) \to (x_{i_1} > \cdots > x_{i_t})) = D(x_1 \to x_{i_1}) \).

As the next proposition shows (verification is left to the reader) any diagram over a poset can be replaced with a diagram over a simplicial complex.

**Proposition 2.5.** For any diagram \( D \) of topological spaces over a poset, the space \( \text{hocolim}(\text{Bd}(D)) \) is homeomorphic to \( \text{hocolim}(D) \).

3. Description of the models. Representing the Vassiliev model as a homotopy colimit

3.1. Ziegler-Živaljević model.

The following diagram was suggested for consideration in \([13]\) \([14]\).

**Definition 3.1.** Given an affine subspace arrangement \( A \) in \( \mathbb{R}^n \), the diagram \( ZZ(A) : \mathcal{L}(A) \to \text{Top} \) is defined by:

- **on objects:** \( ZZ(A)(x) = B(x) \);
- **on morphisms:** \( ZZ(A)(x \to y) \) is the corresponding inclusion map of \( B(x) \) into \( B(y) \).

It follows from Proposition \([23]\) that \( \text{hocolim}(ZZ(A)) \) is homeomorphic to the homotopy colimit of the corresponding diagram over the simplicial complex \( \Delta(\mathcal{L}(A)) \).
The following proposition is a consequence of the Projection Lemma, [4, XII.3.1(iv)], see [15, 14].

**Proposition 3.2.** For an affine subspace arrangement \( \mathcal{A} \) in \( \mathbb{R}^n \), \( \mathcal{U} \) is homotopy equivalent to \( \text{hocolim} \left( \mathbb{Z} Z(\mathcal{A}) \right) \cup \{ \infty \} \).

By using the Homotopy Lemma, [4, XII.4.2], Ziegler and Živaljević could then prove the following formula for the homotopy type of \( \mathcal{U}(\mathcal{A}) \).

**Theorem 3.3.** For an affine subspace arrangement \( \mathcal{A} \) in \( \mathbb{R}^n \)

\[
\mathcal{U}(\mathcal{A}) \simeq \bigvee_{x \in \mathcal{L}(\mathcal{A})} (\Delta(\mathcal{L}(\mathcal{A})_{<x}) * S^{\dim(B(x))}).
\]

And hence, by Alexander duality, one gets the cohomology groups of the complement, originally due to Goresky and MacPherson, [6].

**Theorem 3.4.** For an affine subspace arrangement \( \mathcal{A} \) in \( \mathbb{R}^n \)

\[
\tilde{H}^i(M(\mathcal{A}); \mathbb{Z}) \cong \bigoplus_{x \in \mathcal{L}(\mathcal{A})} \tilde{H}_{n-i-\dim(B(x))} - 2(\Delta(\mathcal{L}(\mathcal{A})_{<x}); \mathbb{Z}).
\]

### 3.2. Vassiliev model.

Vassiliev has suggested a slightly different modification of the subspace arrangement. The idea is to ”simplicially blow up” the intersections of the subspaces. Vassiliev calls it a geometric resolution.

More precisely: take \( N \) to be a sufficiently large number and embed subspaces \( A_i \) into \( \mathbb{R}^N \) in a generic position; for every \( x \in \mathcal{U}(\mathcal{A}) \), let \( V(x) \) be the convex hull of the images of \( x \) in \( \mathbb{R}^N \).

Let \( V(\mathcal{A}) = \bigcup_{x \in \mathcal{U}(\mathcal{A})} V(x) \). It is a ”resolution” of the arrangement in the following sense.

**Lemma 3.5.** [12, Lemma 1, p. 120] One can choose \( N \) sufficiently large, and the embedding sufficiently generic, so that, for every \( x \in \mathcal{U}(\mathcal{A}) \), \( V(x) \) is a simplex with vertices being the images of \( x \) in \( \mathbb{R}^N \), and, for every \( x, y \in \mathcal{U}(\mathcal{A}) \), \( x \neq y \), the simplices \( V(x) \) and \( V(y) \) do not intersect.

**Proposition 3.6.** [12, Lemma 2, p. 120] The one-point compactification of the geometric resolution \( V(\mathcal{A}) \cup \{ \infty \} = \tilde{V}(\mathcal{A}) \) is homotopy equivalent to \( \tilde{U}(\mathcal{A}) \).

Vassiliev then, by means of an explicit argument using Stratified Morse Theory of Goresky and MacPherson, [4], obtains a description for the homotopy type of \( \tilde{U}(\mathcal{A}) \) which is essentially identical to the Ziegler-Živaljević description. Amazingly both results were obtained simultaneously and independently.

An observation which both Vassiliev and Ziegler-Živaljević make is that it follows by Spanier-Whitehead duality that the stable homotopy type of \( M(\mathcal{A}) \) is defined by the combinatorial data of the arrangement (the intersection semilattice together with the dimension information), while it is well-known that the homotopy type of \( M(\mathcal{A}) \) is not a combinatorial invariant, see [12, 14], [15, Theorem 3.4].

### 3.3. Representing Vassiliev model as a homotopy colimit.

**Definition 3.7.** Given a semilattice \( P \), we define the simplicial complex \( N(P) \) as follows:

- the vertices of \( N(P) \) are the minimal elements of \( P \);
• the simplices of $\mathcal{N}(P)$ are those collections of minimal elements of $P$ which have a join in $P$.

$\mathcal{N}(P)$ is known as the nerve complex of $P$.

It was proved by Leray, [9], that the Čech homology groups of $\mathcal{N}(P)$ and of $\Delta(P)$ are equal, and by Borsuk, [3], that the two complexes are actually homotopy equivalent.

Next, we use the notion of the nerve complex of the intersection lattice to define a specific diagram of spaces associated to an affine subspace arrangement, which to our knowledge was not previously considered in the literature.

**Definition 3.8.** Let $\mathcal{A} = \{A_1, \ldots, A_k\}$ be an affine subspace arrangement, and denote the elements of $\mathcal{L}(\mathcal{A})$ corresponding to $A_1, \ldots, A_k$ by $a_1, \ldots, a_k$. We define the Vassiliev diagram $V : F(\mathcal{N}(\mathcal{L}(\mathcal{A}))) \to \text{Top}$ to be the functor specified by:

- on elements: $V(\{a_1, \ldots, a_k\}) = A_{i_1} \cap \cdots \cap A_{i_k}$;
- on morphisms: the maps are inclusions

$$V(\{a_1, \ldots, a_k\} \to \{a_{j_1}, \ldots, a_{j_q}\}) = (A_{i_1} \cap \cdots \cap A_{i_k}) \hookrightarrow (A_{j_1} \cap \cdots \cap A_{j_q}),$$

for any $\{j_1, \ldots, j_q\} \subseteq \{i_1, \ldots, i_k\}$.

**Theorem 3.9.** $\text{hocolim}(V) \cup \{\infty\}$ is homeomorphic to $\hat{\mathcal{V}}(\mathcal{A})$.

**Proof.** It is immediate from the definitions that $\text{hocolim}(V) \cup \{\infty\}$ is a "barycentric subdivision" of $\hat{\mathcal{V}}(\mathcal{A})$, that is, all the simplices which Vassiliev spans on the images of points under the generic embedding are barycentrically subdivided in $\text{hocolim}(V) \cup \{\infty\}$. Other than that, there is no difference in the construction and so we conclude that the two spaces are homeomorphic.

Again, it follows from Proposition 2.5, that $\text{hocolim}(V)$ is homeomorphic to the homotopy colimit of the corresponding diagram over the simplicial complex $\text{Bd}(\mathcal{N}(\mathcal{L}(\mathcal{A})))$.

4. **A deformation retract from the Vassiliev model to the Ziegler-Živaljević model**

4.1. **Single collapse.**

Assume that we have a diagram over a simplicial complex $K$, $\mathcal{D} : F(K) \to \text{Top}$, such that for some simplices $\sigma, \tau \in F(K)$ the following is true:

- $\sigma < \tau$, and there exists no simplex in $K$, other than $\tau$ and $\sigma$ itself, which contains $\sigma$, in particular $\tau$ is maximal; in such situation one says that removing $\sigma$ and $\tau$ from $K$ is an elementary simplicial collapse;
- $\mathcal{D}(\tau \rightarrow \sigma)$ is an identity map.

**Proposition 4.1.** In the situation above there exists a deformation retract from $\text{hocolim}\mathcal{D}$ to $\text{hocolim}\mathcal{D}'$, where $\mathcal{D}' : F(K \setminus \{\sigma, \tau\}) \to \text{Top}$ is the restriction of the functor $\mathcal{D}$.

**Proof.** The desired retract is a simple generalization of the deformation which retracts a mapping cylinder to the target space. It can be easily visualized as follows: think that we have a string connecting the unique vertex $v$ of $\tau$ which does not lie in $\sigma$ to the barycenter $w$ of $\sigma$, and that we start to shrink the string so that $w$ approaches $v$ over an interval of time $[0, 1]$ ($w$ coincides with $v$ at moment 1). We let the entire
homotopy colimit be deformed accordingly, and refer to the explicit description of homotopy colimits in Subsection 2.3 for visualizing this process.

This is clearly a retract from $\text{hocolim} \mathcal{D}$ to $\text{hocolim} \mathcal{D}'$. The continuity of this deformation at any time $0 \leq t < 1$ follows from the fact that $\mathcal{D}(\tau \to \sigma)$ is an identity map, and the continuity at $t = 1$ follows from the definition of the category $\textbf{Top}$ (the morphisms are continuous maps). □

4.2. Terminology of Discrete Morse Theory.

Although unaware of an exact reference, we are confident that it is folklore knowledge that for every finite semilattice $P$ there is a sequence of collapses leading from $\text{Bd} \left( \mathcal{N}(P) \right)$ to $\Delta(P)$. However, to use Proposition 4.1, we need to check a condition that certain maps are identities, so we will list this sequence of collapses explicitly.

It is handy to use the formal setup of Discrete Morse Theory. We provide below the necessary terminology and results for the special case that we need, see [5] for further details.

Let $K$ be a simplicial complex. A matching $W$ on $P = \mathcal{F}(K)$ (cf. [3, Definition 9.1]) is a set of disjoint pairs $(\sigma, \tau)$ such that $\tau, \sigma \in P$, $\tau \succ \sigma$, ("$\succ$" denotes the covering relation). We set

$$W^0 = \{ \sigma \in P \mid \text{there exists } \tau \text{ such that } (\sigma, \tau) \in W \},$$
$$W = \{ \tau \in P \mid \text{there exists } \sigma \text{ such that } (\sigma, \tau) \in W \}.$$

If $(\sigma, \tau) \in W$ then we set $W(\sigma) = \tau$.

**Definition 4.2.** (cf. [3, Definition 9.2]). A matching is called acyclic if it is impossible to find a sequence $\sigma_0, \ldots, \sigma_t \in \overline{W}$, such that $\sigma_0 \neq \sigma_1$, $\sigma_0 = \sigma_t$, and $W(\sigma_t) \succ \sigma_{t+1}$, for $0 \leq t \leq t - 1$.

The following proposition is the only fact that we need for our argument, see also [3, Corollary 3.5, Theorem 9.3], and [3, Theorem 3.2 (2)].

**Proposition 4.3.** Let $K$ be a simplicial complex and $P = \mathcal{F}(K)$ be its face poset. Let $W$ be an acyclic matching on $P$. If the unmatched simplices form a subcomplex $K^C$ of $K$, then there is a sequence of elementary collapses leading from $K$ to $K^C$.

4.3. An acyclic matching for our case.

Let $P$ be a semilattice. We call a set $\{a_1, \ldots, a_i\} \subseteq \text{min}(P)$ complete if

- $\sqcup_{i=1}^{t} a_i$ exists;
- if $x \leq \sqcup_{i=1}^{t} a_i$, and $x \in \text{min}(P)$, then $x \in \{a_1, \ldots, a_i\}$; in other words $\text{min}(P) \cap \{a_1, \ldots, a_i\} = \{a_1, \ldots, a_i\}$.

Otherwise a subset of $\text{min}(P)$ is called incomplete. For any subset $\{b_1, \ldots, b_q\} \subseteq \text{min}(P)$, such that $\sqcup_{j=1}^{q} b_j$ exists, we call $\text{min}(P) \cap \{P_{\leq \sqcup_{j=1}^{q} b_j}\}$ the completion of $\{b_1, \ldots, b_q\}$, and denote it by $C(\{b_1, \ldots, b_q\})$. Clearly, a set is complete iff it is equal to its own completion.

By construction, $\Delta(P)$ is the full subcomplex of $\text{Bd} \left( \mathcal{N}(P) \right)$ spanned by the vertices which are enumerated by the complete subsets of $\text{min}(P)$.

Let us now define an acyclic matching on $\text{Bd} \left( \mathcal{N}(P) \right)$. For a simplex $\Sigma = (S_1 < \cdots < S_t)$ of $\text{Bd} \left( \mathcal{N}(P) \right)$ let $\text{piv}(\Sigma)$ denote the incomplete set $S_i$ with the maximal possible index $i$, if it exists; set $\text{piv}(\Sigma) = \emptyset$ if it does not. If $\text{piv}(\Sigma) \neq \emptyset$, set $i(\Sigma)$ to be equal to the index of $\text{piv}(\Sigma)$ in $\Sigma$. Define

$$\overline{W}^0 = \{ \Sigma = (S_1 < \cdots < S_t) \mid \text{piv}(\Sigma) \neq \emptyset \text{ and } C(\text{piv}(\Sigma)) \not\in \Sigma \}.$$
Correspondingly we define
\[
\overrightarrow{W} = \{ \Sigma = (S_1 < \cdots < S_t) \mid \text{piv}(\Sigma) \neq \emptyset \text{ and } C(\text{piv}(\Sigma)) \in \Sigma \}.
\]
Finally, for \( \Sigma \in \overrightarrow{W} \) we define \( \overleftarrow{W}(\Sigma) = \Sigma \cup \{ C(\text{piv}(\Sigma)) \} \).

Proposition 4.4. \textit{The matching } \overrightarrow{W} \textit{ described above is acyclic.}

\textbf{Proof.} Assume that there exists a sequence \( \Sigma_0, \ldots, \Sigma_t \in \overrightarrow{W} \), such that \( \Sigma_0 \neq \Sigma_1, \Sigma_0 = \Sigma_t \), and \( W(\Sigma_i) > \Sigma_{i+1} \), for \( 0 \leq i \leq t-1 \). We have the following equalities and inequalities:
\[
\iota(\Sigma_0) = \iota(W(\Sigma_0)) > \iota(\Sigma_1) = \iota(W(\Sigma_1)) > \cdots > \iota(W(\Sigma_{t-1})) > \iota(\Sigma_t) = \iota(\Sigma_0),
\]
which yields a contradiction. \(\square\)

4.4. The deformation retract theorem.

Theorem 4.5. \( \text{hocolim}(\mathbb{ZZ} \cup \{ \infty \}) \) is a deformation retract of \( \text{hocolim}(V \cup \{ \infty \}) \).

\textbf{Proof.} It is enough to show that \( \text{hocolim}(\text{Bd}(\mathbb{ZZ})) \) is a deformation retract of \( \text{hocolim}(\text{Bd}(V)) \). For that we need to verify that in the matching described in Subsection 4.3 the maps within the matched pairs are always identities.

Since both diagrams are obtained by subdivisions, it follows from Definition 2.4 that the desired maps are obviously identities in all cases, except possibly when a pair \( (\Sigma, W(\Sigma)) \) is such that \( \text{piv}(\Sigma) \) is the maximal element of \( \Sigma \).

In this case, if we use the notations \( \text{piv}(\Sigma) = \{ a_1, \ldots, a_t \} \), and \( C(\text{piv}(\Sigma)) = \{ a_1, a_{t+1}, \ldots, a_{t+k} \} \), then the desired map is the inclusion \( \bigcap_{i=1}^{t+k} A_i \hookrightarrow \bigcap_{i=1}^t A_i \), which is the identity by definition of the completion (here, \( A_i \in \mathcal{A} \) denotes the subspace indexed by \( a_i \in L(A) \)).

The deformation procedure is illustrated on Figure 1 for the example of the arrangement consisting of 3 lines, all intersecting in the same point.

4.5. Final remark.

Removing the infinity throughout the paper yields the uncompactified version of the result.

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