A correction to the uniqueness of a partial perfect locality over a Frobenius $P$-category

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Abstract: Let $p$ be a prime, $P$ a finite $p$-group and $\mathcal{F}$ a Frobenius $P$-category. In Existence, uniqueness and functoriality of the perfect locality over a Frobenius $P$-category, Algebra Colloquium, 23(2016) 541-622, we also claimed the uniqueness of the partial perfect locality $\mathcal{L}^X$ over any up-closed set $X$ of $\mathcal{F}$-selfcentralizing subgroups of $P$, but recently Bob Oliver exhibit some counter-examples, demanding some revision of our arguments. In this Note we show that, up to replacing the perfect localities by the extendable perfect localities over any up-closed set $X$ of $\mathcal{F}$-selfcentralizing subgroups of $P$, our arguments are correct, still proving the existence and the uniqueness of the perfect $\mathcal{F}^\ast$-locality, since it is extendable. We take advantage to simplify some of our arguments.

1. Introduction

1.1. Let $p$ be a prime, $P$ a finite $p$-group, $\mathcal{F}$ a Frobenius $P$-category [2] and $\mathcal{T}_P$ the category where the objects are the subgroups of $P$, the morphisms are defined by the $P$-transporters and the composition is defined by the product in $P$. Recall that, according to [3, 17.3], an $\mathcal{F}$-locality $\mathcal{L}$ is a finite category where the objects are all the subgroups of $P$, endowed with two functors

$$\tau : \mathcal{T}_P \rightarrow \mathcal{L} \quad \text{and} \quad \pi : \mathcal{L} \rightarrow \mathcal{F} \quad \text{1.1.1}$$

which are the identity on the set of objects, $\pi$ being full, and such that the composition $\pi \circ \tau$ is induced by the conjugation in $P$; we say that $\mathcal{L}$ is divisible whenever it fulfills the following condition

1.1.2. If $Q$, $R$ and $T$ are subgroups of $P$, for any $\mathcal{L}$-morphisms $x : R \rightarrow Q$ and $y : T \rightarrow Q$ such that the image of $\pi_{Q,T}(y)$ is contained in the image of $\pi_{Q,R}(x)$, there is a unique $\mathcal{L}$-morphism $z : T \rightarrow R$ such that $x \cdot z = y$.

1.2. Then, it follows from [3, Proposition 18.4 and Theorem 18.6] that a perfect $\mathcal{F}$-locality, introduced in [3, 17.13], is a divisible $\mathcal{F}$-locality $\mathcal{P}$ such that, for any subgroup $Q$ of $P$ fully normalized in $\mathcal{F}$ [3, 2.6] the finite group $\mathcal{P}(Q)$ endowed with the group homomorphims

$$\tau_Q : N_{\mathcal{P}}(Q) \rightarrow \mathcal{P}(Q) \quad \text{and} \quad \pi_Q : \mathcal{P}(Q) \rightarrow \mathcal{F}(Q) \quad \text{1.2.1}$$

is the $\mathcal{F}$-localizer of $Q$, introduced in [3, 18.5]. Actually, as we show in [3, Theorem 20.24] and, more carefully, in [5, Theorem 7.2], $\mathcal{P}$ is uniquely determined by the full subcategory $\mathcal{P}^\ast$ over the set of $\mathcal{F}$-selfcentralizing subgroups of $P$, introduced in [3, 4.8].

1.3. More generally, in order to apply inductive arguments, for any nonempty set $\mathcal{X}$ of $\mathcal{F}$-selfcentralizing subgroups of $P$ which contains any
subgroup of $P$ admitting an $F$-morphism from some subgroup in $X$, we consider the full subcategory $F_X$ of $F$ over $X$ as the set of objects and, replacing $T_P$ by its full subcategory $T_P^X$ over $X$, we may introduce the $F_X$-localities as the finite categories $L^X$ where the objects are the groups in $X$, endowed with two functors

$$\tau^X : T_P^X \to L^X$$

and

$$\pi^X : L^X \to F^X$$

which are the identity on the set of objects, $\pi^X$ being full, and such that the composition $\pi^X \circ \tau^X$ is induced by the conjugation in $P$. In particular, in [5, 2.8] we consider a perfect $F_X$-locality $P_X$ and in [5, 6.1] we claimed its existence and uniqueness.

1.4. But, recently, Bob Oliver exhibit some counter-examples to this uniqueness [1]; of course, these counter-examples demand a revision of our arguments in [5]. Our purpose in this Note is to show that, up to restricting the perfect $F_X$-localities we consider, our arguments become correct and the uniqueness of these restricted perfect $F_X$-localities, called extendable, is true; naturally, our extendable perfect $F_X$-localities include $P^{ex}$ above. Moreover, we take advantage of this revision to simplify some arguments in [5]. Notations and terminology are the same as in [5] and the main references come from [3].

2. Extendable perfect $F_X$-localities

2.1. With the notation above, let us consider a perfect $F_X$-locality $P^{ex}$; that is to say, $P^{ex}$ is a divisible $F_X$-locality such that, for any group $Q$ in $X$ fully normalized in $F$ [3, 2.6], the finite group $P^{ex}(Q)$ endowed with the group homomorphims

$$\tau^Q : N_P(Q) \to P^{ex}(Q)$$

and

$$\pi^Q : P^{ex}(Q) \to F(Q)$$

is the $F$-localizer of $Q$, introduced in [3, 18.5]; in particular, $\pi^Q$ is surjective and, since $Q$ is $F$-selfcentralizing, $\tau^Q$ is injective [3, Remark 18.7]. Moreover, note that condition 18.6.3 in [3, Theorem 18.6] implies that $Q$ fulfills equality 17.10.1 in [3, Proposition 17.10]; in particular, extending $P^{ex}$ as in [3, 17.4], it follows from [3, Proposition 17.10] that $P^{ex}$ is a coherent $F_X$-locality [3, 17.9]; that is to say, we have

$$x \cdot \tau^Q(v) = \tau^Q(\pi^Q(x)v) \cdot x$$

for any pair of subgroups $Q$ and $R$ in $X$, any $x \in P^{ex}(Q,R)$ and any $v \in R$.

2.2. Actually, considering the normalizer $N_P(Q)$ of $Q$ in $F$ [3, 2.14] which is a Frobenius $N_P(Q)$-category [3, Proposition 2.16], denoting by $X^Q$ the set of subgroups of $N_P(Q)$ belonging to $X$ and setting $P^Q = N_P(Q)$...
and $\mathcal{F}^Q = N_{\mathcal{F}}(Q)$, we can also consider the normalizer $N_{\mathcal{P}^x}(Q)$ of $Q$ in $\mathcal{P}^x$ [3, 17.4 and 17.5] and, setting $\mathcal{F}^{x,Q} = (\mathcal{F}^Q)^{x,Q}$, it is not difficult to see that $\mathcal{P}^{x,Q} = N_{\mathcal{P}^x}(Q)$ is actually a perfect $\mathcal{F}^{x,Q}$-locality.

2.3. Moreover, the $\mathcal{F}^{x,Q}$-locality $\mathcal{P}^{x,Q}$ and the group $\mathcal{P}^x(Q)$ are related throughout the transporter of the $p$-subgroups of $\mathcal{P}^x(Q)$; explicitly, let us call transporter $\mathcal{T}_{p^x(Q)}$ of $\mathcal{P}^x(Q)$ the $\mathcal{F}^Q$-locality formed by the category where the objects are all the subgroups of $\mathcal{P}^Q$, where the morphisms are defined by the equivalence $[3, 2.9]$ and, setting $\mathcal{T}_Q$ of the elements $\mathcal{F}^Q$ -locality formed by the category where the composition is defined by the product in $\mathcal{P}^Q$, endowed with the obvious functors induced by $\tau_Q^x$ and by $\pi_Q^x$. Then, denoting by $\mathcal{T}_{p^x(Q)}$ the full subcategory of $\mathcal{T}_{p^x(Q)}$ over $\mathcal{P}^x$, we claim that we have an $\mathcal{F}^Q$-locality equivalence $[3, 2.9]

\mathcal{P}^{x,Q} \cong \mathcal{T}_{p^x(Q)}  \ \ \ 2.3.1.$

Firstly, we need the following lemma which admits the same proof as in [3, Proposition 24.2].

**Lemma 2.4.** Any $\mathcal{P}^x$-morphism is a monomorphism and an epimorphism.

2.5. Now, we already know that any $\mathcal{P}^{x,Q}$-morphism $x : T \to R$ is induced by a $\mathcal{P}^x$-morphism $\hat{x} : T \cdot Q \to R \cdot Q$ which stabilizes $Q$ $[3, 2.14.1]$; then, it easily follows from the lemma above that $\hat{x}$ is uniquely determined by $x$, and the divisibility of $\mathcal{P}^x$ guarantees the existence of a unique $\hat{x}_Q \in \mathcal{P}^x(Q)$ fulfilling

$$\tau_{R \cdot Q, Q}^x (1) \cdot \hat{x}_Q = \hat{x} \cdot \tau_{T \cdot Q, Q}^x (1) \ \ \ \ 2.5.1;$$

moreover, from the coherence of $\mathcal{P}^x$ (cf. 2.1.2), for any $t \in T \subset \mathcal{P}^Q$ we get

$$\tau_{R \cdot Q, Q}^x (1) \cdot \hat{x}_Q \tau_{Q}^x (t) = \hat{x} \cdot \tau_{T \cdot Q, Q}^x (t) \cdot \tau_{T \cdot Q, Q}^x (1) \ \ \ \ 2.5.2,$$

so that from the lemma above we still get

$$\hat{x}_Q \cdot \tau_Q^x (t) \cdot (\hat{x}_Q)^{-1} = \tau_Q^x \left( \left( \tau_{R \cdot T}^x (x) (t) \right) \right) \ \ \ \ 2.5.3,$$

thus, the element $\hat{x}_Q$ belongs to the $\mathcal{P}^x(Q)$-transporter $\mathcal{T}_{p^x(Q)} (\tau_Q^x(R), \tau_Q^x(T))$.
and it is not difficult to check that the correspondence sending the $\mathcal{P}^{x,Q}$-morphism $x:T \rightarrow R$ to the $\mathcal{T}_{\mathcal{P}^{x}(Q)}$-morphism $\hat{x}_Q:T \rightarrow R$ defines a \textit{faithful $\mathcal{F}^Q$-locality functor} $\mathcal{P}^{x,Q} \rightarrow \mathcal{T}_{\mathcal{P}^{x}(Q)}$ [3, 2.9]. The “surjectivity” follows again from condition 18.6.3 in [3, Theorem 18.6].

2.6. But, for any $\mathcal{F}$-selfcentralizing subgroup $W$ of $P$ \textit{fully normalized} in $\mathcal{F}$, we still have the normalizer $\mathcal{F}^W = \mathcal{N}_\mathcal{F}(W)$; if $P^W$ belongs to $\mathcal{X}$, so that the set $\mathcal{X}^W$ of subgroups of $P^W$ belonging to $\mathcal{X}$ is not empty, then we also can consider the normalizer $\mathcal{P}^{x,w}_W = \mathcal{N}_\mathcal{P}^x(W)$, which is again a \textit{perfect $\mathcal{F}^{x,w}$-locality}, and we always have the existence of the $\mathcal{F}$-\textit{localizer} $L^W$ of $W$ [3, Theorem 18.6]; thus, we still can consider the transporter $\mathcal{T}_{\mathcal{L}^W}$ of $L^W$ as an $\mathcal{F}^W$-locality and the full subcategory $\mathcal{T}^W_{\mathcal{L}^W}$ of $\mathcal{T}^W_{\mathcal{L}^W}$ over $\mathcal{X}^W$ as the set of objects. Finally, we say that the \textit{perfect $\mathcal{F}^{x}$-locality} $\mathcal{P}^x$ is \textit{extendable} whenever for any $\mathcal{F}$-selfcentralizing subgroup $W$ of $P$ \textit{fully normalized} in $\mathcal{F}$ such that $P^W \in \mathcal{X}$ there exists an $\mathcal{F}^{x,w}$-\textit{locality isomorphism}\footnote{In [5, 6.18], arguing by induction we claim such an equivalence but, with the notation there, if the group $U$ is \textit{normal} in $\mathcal{F}$ then the induction argument cannot be applied!}

\[
\mathcal{P}^{x,w} \cong \mathcal{T}^W_{\mathcal{L}^W}
\]

Proposition 2.7. If $\mathcal{P}^x$ is an extendable perfect $\mathcal{F}^x$-locality then, for any $\mathcal{F}$-selfcentralizing subgroup $V$ of $P$ \textit{fully normalized} in $\mathcal{F}$ such that $P^V \in \mathcal{X}$, $\mathcal{P}^{x,v}$ is an extendable perfect $\mathcal{F}^{x,v}$-locality.

Proof: From our definition we have an $\mathcal{F}^v$-locality isomorphism

\[
\mathcal{P}^{x,v} \cong \mathcal{T}^v_{\mathcal{L}^v}
\]

which determines an $\mathcal{N}_{\mathcal{F}^v}(W)$-locality isomorphism

\[
\mathcal{N}_{\mathcal{P}^{x,v}}(W) \cong \mathcal{N}_{\mathcal{F}^v}(W) = \mathcal{T}^{x,v,w}_{\mathcal{N}^{x,v}}(W)
\]

where we identify $P^V$ with its image in $L^V$ and, for any $\mathcal{F}^v$-selfcentralizing subgroup $W$ of $P^V$ \textit{fully normalized} in $\mathcal{F}^v$ such that $\mathcal{N}_{\mathcal{F}^v}(W) \in \mathcal{X}^V$, we denote by $\mathcal{X}^{v,w}$ the set of subgroups of $\mathcal{N}_{\mathcal{F}^v}(W)$ belonging to $\mathcal{X}^v$.

But, it is not difficult to check that the normalizer $\mathcal{N}_{\mathcal{F}^v}(W)$, endowed with the group homomorphisms

\[
\mathcal{N}_{\mathcal{F}^v}(W) \rightarrow \mathcal{N}_{\mathcal{L}^v}(W) \quad \text{and} \quad \mathcal{N}_{\mathcal{L}^v}(W) \rightarrow \mathcal{F}^v(W)
\]
induced by the structural group homomorphisms of \( L^V_X \), is the \( J^V \)-localizer of \( W \). We are done.

3. A reduction procedure

3.1. With the notation above, recall that a basic \( P \times P \)-set \([3, 21.4]\) is a finite nonempty \( P \times P \)-set \( \Omega \) such that \( \{1\} \times P \) acts freely on \( \Omega \), that we have

\[
\Omega^0 \cong \Omega \quad \text{and} \quad |\Omega|/|P| \not\equiv 0 \mod p
\]

3.1.1

where we denote by \( \Omega^0 \) the \( P \times P \)-set obtained by exchanging both factors, and that, for any subgroup \( Q \) of \( P \) and any injective group homomorphism \( \varphi : Q \to P \) such that \( \Omega \) contains a \( P \times P \)-subset isomorphic to \( (P \times P)/\Delta_\varphi(Q) \) where we set \( \Delta_\varphi(Q) = \{(\varphi(u), u)\}_{u \in Q} \), we have a \( Q \times P \)-set isomorphism

\[
\text{Res}_{\varphi \times id_P}(\Omega) \cong \text{Res}_{\varphi \times id_P}(\Omega)
\]

3.1.2

Denoting by \( G^\alpha \) the group of automorphisms of the \( \{1\} \times P \)-set \( \text{Res}_{\{1\} \times P}(\Omega) \), it is clear that we have an injective map from \( P \times \{1\} \) in \( G^\alpha \); we identify its image with the \( p \)-group \( P \) so that, from now on, \( P \) is contained in \( G^\alpha \) and acts freely on \( \Omega \). Recall that the full subcategory of the \( G^\alpha \)-transporter over the set of subgroups of \( P \) induces a Frobenius \( P \)-category \([3, \text{Proposition 21.9}]\) and we say that \( \Omega \) is an \( F \)-basic \( P \times P \)-set if, for any pair of subgroups \( Q \) and \( R \) of \( P \), we have

\[
T_{G^\alpha}(Q, R)/C_{G^\alpha}(R) \cong F(Q, R)
\]

3.2.1

3.2. Denoting by \( G^\alpha \) the group of automorphisms of the \( \{1\} \times P \)-set \( \text{Res}_{\{1\} \times P}(\Omega) \), it is clear that we have an injective map from \( P \times \{1\} \) in \( G^\alpha \); we identify its image with the \( p \)-group \( P \) so that, from now on, \( P \) is contained in \( G^\alpha \) and acts freely on \( \Omega \). Recall that the full subcategory of the \( G^\alpha \)-transporter over the set of subgroups of \( P \) induces a Frobenius \( P \)-category \([3, \text{Proposition 21.9}]\) and we say that \( \Omega \) is an \( F \)-basic \( P \times P \)-set if, for any pair of subgroups \( Q \) and \( R \) of \( P \), we have

\[
T_{G^\alpha}(Q, R)/C_{G^\alpha}(R) \cong F(Q, R)
\]

3.2.1

3.3. Actually, it follows from \([3, \text{Proposition 21.12}]\) that an \( F \)-basic \( P \times P \)-set always exists; more precisely, we say that an \( F \)-basic \( P \times P \)-set \( \Omega \) is natural if it fulfills \([5, 3.5]\)

\[
|\Omega^\Delta(\varphi)| = |Z(\varphi)|
\]

3.3.1

for any \( F \)-selfcentralizing subgroup \( Q \) of \( P \) and any \( \varphi \in F(P, Q) \), and if it is thick \([3, 21.7]\) outside of the set of \( F \)-selfcentralizing subgroups of \( P \) — namely the multiplicity of \( (P \times P)/\Delta_\psi(R) \) is at least two if \( R \) is not \( F \)-selfcentralizing and \( \psi \) belongs to \( F(P, R) \). The existence of natural \( F \)-basic \( P \times P \)-sets follows from \([5, \text{Proposition 3.4}]\) together with \([3, \text{Proposition 21.12}]\); here, we are interested in the following form of \([5, \text{Proposition 3.7}]\)

Proposition 3.4 Let \( \Omega \) be a natural \( F \)-basic \( P \times P \)-set, \( Q \) and \( T \) a pair of \( F \)-selfcentralizing subgroups of \( P \) and \( \eta \) an element of \( F(Q, T) \). The multiplicity of \( (Q \times P)/\Delta_\eta(T) \) in \( \text{Res}_{Q \times P}(\Omega) \) is at most one, and if it is one then we have

\[
\text{Aut}_{Q \times P}((Q \times P)/\Delta_\eta(T)) \cong Z(T)
\]

3.4.1

3.5. From now on, \( \Omega \) is a natural \( F \)-basic \( P \times P \)-set. For any subgroup \( Q \) of \( P \), it is clear that \( C_{G^\alpha}(Q) \) is just the group of automorphisms of the
$Q \times P$-set $\text{Res}_{Q \times P}(\Omega)$ and it is clear that the correspondence sending $Q$ to $C_{G^0}(Q)$ induces a contravariant functor $C_{G^0}$ from $\mathcal{F}$ to the category $\mathfrak{Sr}$ of finite groups. Let us denote by $C_{\text{nsc}}^{\alpha}(Q)$ the subgroup of elements $f \in C_{G^0}(Q)$ which act trivially on all the $Q \times P$-orbits of $\Omega$ isomorphic to $(Q \times P)/\Delta (T)$ where $T$ is $\mathcal{F}$-selfcentralizing; in particular, if $Q$ is not $\mathcal{F}$-selfcentralizing then we have $C_{\text{nsc}}^{\alpha}(Q) = C_{G^0}(Q)$; in any case, $C_{\text{nsc}}^{\alpha}(Q)$ is normal in $C_{G^0}(Q)$ and, according to Proposition 3.4, the quotient $C_{G^0}(Q)/C_{\text{nsc}}^{\alpha}(Q)$ is Abelian.

3.6. More generally, for any $Q \in \mathcal{X}$ denote by $C_{G^0}^{\times}(Q)$ the subgroup of elements $f \in C_{G^0}(Q)$ which act trivially on all the $Q \times P$-orbits of $\Omega$ isomorphic to $(Q \times P)/\Delta (T)$ where $T$ belongs to $\mathcal{X}$; it is easily checked that the correspondence sending $Q \in \mathcal{X}$ to $C_{G^0}^{\times}(Q)$ defines a subfunctor $C_{G^0}^{\times} : \mathcal{F}^{\times} \to \mathfrak{Sr}$ of the restriction of $C_{G^0}$ to $\mathcal{F}^{\times}$, and we consider the quotient $\mathcal{F}^{\times}$-locality $\overline{\mathcal{T}}_{G^0} = \mathcal{T}_{G^0}/C_{G^0}^{\times} = \overline{\mathcal{T}}_{G^0}^{\times}$ — noted $\overline{\mathcal{T}}_{G^0}^{\times}$ in $[5, 5.1.2]$ — sending any pair of groups $Q$ and $R$ in $\mathcal{X}$ to

$$\overline{\mathcal{T}}_{G^0}^{\times}(Q, R) = \mathcal{T}_{G^0}(Q, R)/C_{G^0}^{\times}(R)$$

3.6.1; here we are interested in the following form of $[5$, Corollaries 5.20 and 5.21].

**Proposition 3.7.** For any perfect $\mathcal{F}^{\times}$-locality $\mathcal{P}^{\times}$ there is a unique naturally $\mathcal{F}^{\times}$-isomorphic class of faithful $\mathcal{F}^{\times}$-locality functors $\lambda^{\times} : \mathcal{P}^{\times} \to \overline{\mathcal{T}}_{G^0}^{\times}$. Moreover, if $\mathcal{P}^{\times}$ is a perfect $\mathcal{F}^{\times}$-locality which is $\mathcal{F}^{\times}$-locality isomorphic to $\mathcal{P}^{\times}$ then there is a commutative diagram of $\mathcal{F}^{\times}$-locality functors

$$\begin{array}{ccc}
\mathcal{P}^{\times} & \overset{\rho^{\times}}{\cong} & \mathcal{P}^{\times} \\
\lambda^{\times} & \nearrow & \lambda^{\times} \\
\overline{\mathcal{T}}_{G^0}^{\times} & \phantom{\cong} & \phantom{\rho^{\times}}
\end{array}$$

3.7.1.

3.8. With the notation in 2.2 above, for any $\mathcal{F}$-selfcentralizing subgroup $W$ of $P$ fully normalized in $\mathcal{F}$ such that $P^W \in \mathcal{X}$, it follows from [3, Proposition 21.11] that the subset of $\Omega$

$$\Omega_w = \bigcup_{\chi \in \mathcal{F}(W)} \Omega_{\Delta \chi(W)}$$

3.8.1 is actually an $\mathcal{F}^w$-basic $P^w \times P^w$-set; mutatis mutandis, denote by $G^{\alpha W}$ the group of $1 \times P^w$-set automorphisms of $\text{Res}_{1 \times P^w}(\Omega_w)$ and identify $P^w$ with $P^w \times \{1\}$; since the quotient $N_{G^{\alpha W}}(W)/C_{G^{\alpha W}}(W)$ is isomorphic to $\mathcal{F}(W)$ (cf. 3.2.1), it is clear that $N_{G^{\alpha W}}(W)$ stabilizes $\Omega_w$ and therefore we have a canonical group homomorphism from $N_{G^{\alpha W}}(W)$ to $G^{\alpha W}$; again, we are interested in the following form of $[5$, Proposition 6.15].
Proposition 3.9. With the notation above, for any pair of subgroups $Q$ and $R$ of $P^W$ containing $W$ and any element $\varphi$ in $F^W(Q,R)$, there exists at most one $Q \times P^W$-orbit in $\Omega_W$ isomorphic to $(Q \times P^W)/\Delta_\varphi(R)$, $\Omega_W$ is a natural $F^W$-basic $P^W \times P^W$-set and, in particular, $C_{G^\Omega W} (Q)$ is an Abelian $p$-group.

3.10. It follows from this proposition that, as in 3.6 above, if $P^W$ belongs to $X$ then we get the quotient $F^{x, W}$-locality $T^{x, W}$; actually, it follows from Propositions 3.4 and 3.9 above that, with the notation in 2.2 and 2.6 above, the canonical group homomorphism from $N_{G^\Omega}(W)$ to $G^{\Omega W}$ induces an $F^{x, W}$-locality functor

$$g^{x, W} : N_{T^{x, W}} (W) \rightarrow T^{x, W}.$$ 3.10.1;

note that, according to Proposition 3.7 above, we have faithful $F^{x, W}$-locality functors from $P^{x, W} = N_{P^x}(W)$ to both $F^{x, W}$-localities $N_{T^{x, W}} (W)$ and $T^{x, W}$ and we may assume that they agree with $g^{x, W}$.

3.11. On the other hand, let $L^W_F$ be the $F$-localizer of $W$ [3, Theorem 18.6]; that is to say, $L^W_F$ is a finite group endowed with an injective and a surjective group homomorphisms

$$\tau^W : P^W \rightarrow L^W_F \quad \text{and} \quad \pi^W : L^W_F \rightarrow F(W)$$ 3.11.1,

$\tau^W(P^W)$ is a Sylow $p$-subgroup of $L^W_F$, the composition $\pi^W \circ \tau^W$ is defined by the conjugation in $F(W)$ and we also have the exact sequence

$$1 \rightarrow Z(W) \xrightarrow{\tau^W} L^W_F \xrightarrow{\pi^W} F(W) \rightarrow 1.$$ 3.11.2.

Below, we restate [5, Proposition 6.19].

Proposition 3.12. With the notation above, there is a unique $C_{G^\Omega W} (W)$-conjugacy class of group homomorphisms

$$\lambda^W_F : L^W_F \rightarrow N_{C^\Omega W} (W)$$ 3.12.1

compatible with the structural group homomorphisms from $P^W$ and to $F(W)$.

3.13. As in 2.6 above, denote by $T^W_{L^W_F}$ the $F^W$-locality determined by $\tau^W_F$ and by the transporter of the group $L^W_F$; it is clear that any group homomorphism $\lambda^W_F : L^W_F \rightarrow N_{C^\Omega W} (W)$ in 3.12.1 above determines an $F^W$-locality functor

$$l^W_F : T^W_{L^W_F} \rightarrow T^W_{G^\Omega W}.$$ 3.13.1
and two of them are naturally $F^w$-isomorphic \[5, 2.9\]; moreover, if $P^w \in X$, it is not difficult to see that the full subcategory $T_{L^w}^{x^w}$ of $T_{P^w}$ over $X^w$ as the set of objects is a perfect $F^{x,w}$-locality, and from 3.13.1 we get an $F^{x,w}$-locality functor

$$I_{x^w} : T_{L^w}^{x^w} \longrightarrow T_{G^w}^{x^w} \quad 3.13.2.$$

4. Existence and uniqueness of an extendable perfect $F^x$-locality

4.1. With the notation in 1.3 above, our main purpose is to prove that

**Theorem.** There exists an extendable perfect $F^x$-locality $P^x$, which is unique up to $F^x$-locality isomorphisms.

The existence and the uniqueness of the $F$-localizer $L^P_F$ of $P$ \[3, Theorem 18.6\] proves the existence and the uniqueness of the extendable perfect $F^x$-locality whenever $X = \{P\}$; indeed, $L^P_F$ is actually a semidirect product $P \rtimes K$ where $K \cong F(P)/F_P(P)$ is a $p'$-group and, for any $F$-selfcentralizing normal subgroup $W$ of $P$, the $F^{x,w}$-locality equivalence 2.6.1 is obvious.

4.2. Thus, we may assume that $X \neq \{P\}$ and will argue by induction on $|X|$. Choose a minimal element $U$ in $X$ fully normalized in $F$ and set

$$\mathfrak{F} = X \setminus \{\theta(U) \mid \theta \in F(P,U)\} \quad 4.2.1;$$

then, by the induction hypothesis, we may assume that there exists an extendable perfect $F^{\mathfrak{F}}$-locality $P^{\mathfrak{F}}$, endowed with the structural functors

$$\tau^{\mathfrak{F}} : T^{\mathfrak{F}}_P \longrightarrow P^{\mathfrak{F}} \quad \text{and} \quad \pi^{\mathfrak{F}} : P^{\mathfrak{F}} \longrightarrow F^{\mathfrak{F}} \quad 4.2.2,$$

which is unique up to $F^{\mathfrak{F}}$-locality isomorphisms. At this point, according to Proposition 3.7 above, we may assume that $P^{\mathfrak{F}}$ is an $F^{\mathfrak{F}}$-sublocality of the $F^{\mathfrak{F}}$-locality $T^{x^\mathfrak{F}}_G$ introduced in 3.6 above; then, denoting by $(T^{x^\mathfrak{F}}_G)^{\mathfrak{F}}$ the full subcategory of $T^{x^\mathfrak{F}}_G$ over $\mathfrak{F}$ as the set of objects, we have an obvious functor $(T^{x^\mathfrak{F}}_G)^{\mathfrak{F}} \longrightarrow T^{\mathfrak{F}}_G$ and we look to the pull-back

$$P^{\mathfrak{F}} \subset T^{\mathfrak{F}}_G \uparrow \quad 4.2.3,$$

$$M^{\alpha,\mathfrak{F}} \subset (T^{x^\mathfrak{F}}_G)^{\mathfrak{F}} \uparrow \quad 4.2.3,$$

which defines a coherent $F^{\mathfrak{F}}$-locality $M^{\alpha,\mathfrak{F}}$ \[3, 17.9\] endowed with obvious structural functors

$$\nu^{\alpha,\mathfrak{F}} : T^{\mathfrak{F}}_P \longrightarrow M^{\alpha,\mathfrak{F}} \quad \text{and} \quad \rho^{\alpha,\mathfrak{F}} : M^{\alpha,\mathfrak{F}} \longrightarrow F^{\mathfrak{F}} \quad 4.2.4.$$
4.3. We extend $\mathcal{M}^{\Omega, x}$ to a coherent $\mathcal{F}^x$-sublocality $\mathcal{M}^{\Omega, x}$ of $\mathcal{T}^x_C$ which contains $\mathcal{M}^{\Omega, x}$ as a full subcategory over $\mathcal{Y}$ and fulfills
\[
\mathcal{M}^{\Omega, x}(Q, V) = \mathcal{T}^x_C(Q, V)
\]
for any $Q \in \mathfrak{X}$ and any $V \in \mathfrak{X} - \mathcal{Y}$, and denote by
\[
\nu^{\Omega, x} : \mathcal{T}^x_P \rightarrow \mathcal{M}^{\Omega, x} \quad \text{and} \quad \rho^{\Omega, x} : \mathcal{M}^{\Omega, x} \rightarrow \mathcal{F}^x
\]
the corresponding structural functors; finally, we consider the quotient $\mathcal{F}^x$-locality $\bar{\mathcal{M}}^{\Omega, x}$ of $\mathcal{M}^{\Omega, x}$ defined by
\[
\bar{\mathcal{M}}^{\Omega, x}(Q, R) = \mathcal{M}^{\Omega, x}(Q, R) / \tau^{\Omega, x}_R(Z(R))
\]
for any $Q, R \in \mathfrak{X}$, together with the induced natural maps — denoted by $\bar{\nu}^{\Omega, x}$ and $\bar{\rho}^{\Omega, x}$. Then, the proof of the Theorem above can be reduced to the proof of the following fact, that we prove in the next section

4.3.4. The structural functor $\bar{\rho}^{\Omega, x}$ admits an $\mathcal{F}^x$-locality functorial section.

4.4. Let us first prove this reduction. Choose an $\mathcal{F}^x$-locality functorial section $\bar{\sigma}^{\Omega, x} : \mathcal{F}^x \rightarrow \bar{\mathcal{M}}^{\Omega, x}$; for any pair of groups $Q$ and $R$ in $\mathcal{Y}$, we know that (cf. 2.1)
\[
\mathcal{F}^x(Q, R) \cong \mathcal{P}^\Omega(Q, R) / \tau^\Omega_R(Z(R))
\]
and therefore, denoting by $\bar{\mathcal{P}}^{\Omega, x}(Q, R)$ the converse image of $\bar{\sigma}^{\Omega, x}_Q(\mathcal{F}^x(Q, R))$ in $\mathcal{M}^{\Omega, x}(Q, R)$, it is clear that the canonical map $\mathcal{M}^{\Omega, x}(Q, R) \rightarrow \mathcal{P}^\Omega(Q, R)$ induces a bijection $\mathcal{P}^{\Omega, x}(Q, R) \cong \mathcal{P}^\Omega(Q, R)$; that is to say, looking to the pull-back
\[
\mathcal{F}^x \xrightarrow{\bar{\sigma}^{\Omega, x}} \bar{\mathcal{M}}^{\Omega, x} \quad \quad \quad \quad \quad \mathcal{P}^{\Omega, x} \xrightarrow{\bar{\sigma}^{\Omega, x}} \bar{\mathcal{M}}^{\Omega, x}
\]
— which defines a coherent $\mathcal{F}^x$-locality $\mathcal{P}^{\Omega, x}$ [3, 17.9] endowed with obvious structural functors
\[
\tau^{\Omega, x} : \mathcal{T}^x_P \rightarrow \mathcal{P}^{\Omega, x} \quad \text{and} \quad \pi^{\Omega, x} : \mathcal{P}^{\Omega, x} \rightarrow \mathcal{F}^x
\]
— and denoting by $(\mathcal{P}^{\Omega, x})^\mathfrak{Y}$ the full subcategory of $\mathcal{P}^{\Omega, x}$ over $\mathcal{Y}$ as the set of objects, it follows from those bijections above that we have an $\mathcal{F}^\mathcal{Y}$-locality isomorphism $(\mathcal{P}^{\Omega, x})^\mathcal{Y} \cong \mathcal{P}^\mathcal{Y}$.

4.5. That is to say, for any $Q \in \mathcal{Y}$ fully normalized in $\mathcal{F}$, we already know that $\mathcal{P}^{\Omega, x}(Q)$ is an $\mathcal{F}$-localizer of $Q$ and, for any $V \in \mathfrak{X} - \mathcal{Y}$, it follows from the pull-back 4.4.2 above that we have the exact sequence
\[
1 \rightarrow Z(V) \rightarrow \mathcal{P}^{\Omega, x}(V) \rightarrow \mathcal{F}(V) \rightarrow 1
\]
and it is easily checked that the group $\mathcal{P}^{\alpha,x}(V)$, endowed with the group homomorphisms

$$\tau^\alpha_W : N_P(V) \rightarrow \mathcal{P}^{\alpha,x}(V) \quad \text{and} \quad \pi^\alpha_W : \mathcal{P}^{\alpha,x}(V) \rightarrow \mathcal{F}(V)$$

determined by the functors $\tau^\alpha$ and $\pi^\alpha$, is actually a $\mathcal{F}$-localizer of $V$ whenever $V$ is fully normalized in $\mathcal{F}$; consequently, it follows from 2.1 above that $\mathcal{P}^{\alpha,x}$ is a perfect $\mathcal{F}$-locality.

4.6. We claim that $\mathcal{P}^{\alpha,x}$ is actually an extendable perfect $\mathcal{F}$-locality; indeed, let $W$ be an $\mathcal{F}$-selfcentralizing subgroup of $P$ fully normalized in $\mathcal{F}$ such that $P^W = N_P(W)$ belongs to $\mathcal{F}$; thus, if $P^W$ does not belong to $\mathcal{F}$ then we have $\mathcal{X}^W = \{P^W\}$ and $P^W$ is the unique object in both $\mathcal{F}^x$-localities $N_{\mathcal{P}^{\alpha,x}}(W)$ and $\mathcal{T}_{L^x}^W$: in this case, since

$$(N_{\mathcal{P}^{\alpha,x}}(W))(P^W) \cong P^W \times K \cong \mathcal{T}_{L^x}^W(P^W)$$

where $K \cong \mathcal{F}_{x}^x(P^W)/\mathcal{F}_{P^W}(P^W)$, it is clear that we get the equivalence 2.6.1. Otherwise $\mathcal{F}$ is not empty and, setting $\mathcal{P}^{\alpha,x}(W) = N_{\mathcal{P}^{\alpha,x}}(W)$ and denoting by $\mathcal{P}^{\alpha,x}(W)$ the full subcategory of $\mathcal{P}^{\alpha,x}$ over $\mathcal{F}$, from 4.4 above we get an $\mathcal{F}^{\alpha,x}$-locality isomorphism

$$\mathcal{P}^{\alpha,x} \cong N_{\mathcal{P}^{\alpha,x}}(W)$$

but, since $\mathcal{P}^{\alpha,x}$ is extendable, it follows from our definition in 2.6 above that we still get an $\mathcal{F}^{\alpha,x}$-locality isomorphism

$$N_{\mathcal{P}^{\alpha,x}}(W) \cong \mathcal{T}_{L^x}^W$$

4.7. Always assuming that $\mathcal{F}$ is not empty, note that in 3.10 above $\mathcal{T}_{L^x}^W$ sends $N_{\mathcal{P}^{\alpha,x}}(W)$ isomorphically to its image in $\mathcal{T}_{L^x}^W$ — still noted $\mathcal{P}^{\alpha,x}$; then, from this inclusion, mutatis mutandi we can define a coherent $\mathcal{F}^{\alpha,x}$-locality $\mathcal{M}^{\alpha,x}$ as in 4.2.3, and coherent $\mathcal{F}^{x}$-localities $\mathcal{M}^{\alpha,x} \subset \mathcal{T}_{L^x}^W$ and $\mathcal{M}^{\alpha,x}$ as in 4.3; moreover, it is clear that $\mathcal{P}^{\alpha,x}$ induces an $\mathcal{F}^{x}$-locality functorial section $\mathcal{P}^{\alpha,x} : \mathcal{F}^{x} \rightarrow \mathcal{M}^{\alpha,x}$ and that we can define a coherent $\mathcal{F}^{x}$-locality $\mathcal{P}^{\alpha,x}$ as in 4.4 above which still fulfills

$$(\mathcal{P}^{\alpha,x})^W \cong \mathcal{P}^{\alpha,x}$$

we denote by $\tau^{\alpha,x} : \mathcal{T}_{L^x}^W \rightarrow \mathcal{P}^{\alpha,x}$ and by $\pi^{\alpha,x} : \mathcal{P}^{\alpha,x} \rightarrow \mathcal{F}^{x}$ the structural functors.
4.8. On the other hand, since $T_{LF}^{x}$ is a perfect $F_{x}^{W}$-locality (cf. 3.13),
it follows from Proposition 3.7 (or from 3.13.2) that $T_{LF}^{x}$ is actually an
$F_{x}^{W}$-sublocality of $T_{G}^{x}$; in particular, denoting by $(T_{G}^{x})^{\mathfrak{g}}$ and by
$(T_{LF}^{x})^{\mathfrak{g}}$ the respective full subcategories of $T_{G}^{x}$ and of $T_{LF}^{x}$ over $\mathfrak{g}$,
it is easily checked that the canonical functor
\[ (T_{G}^{x})^{\mathfrak{g}} \rightarrow (T_{LF}^{x})^{\mathfrak{g}} \] sends $(T_{LF}^{x})^{\mathfrak{g}}$ isomorphically onto $T_{LF}^{x} \subset T_{G}^{x}$.

4.9. Moreover, from 4.4 we know that the canonical functor
\[ (T_{G}^{x})^{\mathfrak{g}} \rightarrow (T_{G}^{x})^{\mathfrak{g}} \] sends $(T_{G}^{x})^{\mathfrak{g}}$ isomorphically onto $T_{G}^{x}$; but, it follows from our definition
in 3.10 that, denoting by $(g_{x},\mathfrak{g})$ the restriction of $g_{x}$ to the normalizer
in $(T_{G}^{x})^{\mathfrak{g}}$ of $\mathfrak{g}$, we have a commutative diagram of functors
\[
\begin{array}{ccc}
N_{T_{G}^{x}}(W) & \xrightarrow{\mathfrak{g}_{\Omega}^{W}} & T_{G}^{x}\\
\uparrow & & \uparrow \\
N_{T_{G}^{x}}^{x}(W) & \xrightarrow{(\mathfrak{g}_{\Omega}^{x})^{\mathfrak{g}}} & (T_{G}^{x})^{\mathfrak{g}}
\end{array}
\] where the vertical arrows are defined by the functors 4.8.1 and 4.9.1; hence,
since the functor 4.9.1 sends $(P_{\mathfrak{g}_{\Omega}}^{x})^{\mathfrak{g}}$ isomorphically onto $P_{\mathfrak{g}_{\Omega}}^{x}$ (cf. 4.4),
this functor sends $N_{P_{\mathfrak{g}_{\Omega}}^{x}}(W)$ isomorphically onto $N_{P_{\mathfrak{g}_{\Omega}}^{x}}(W)$ and we already
know that $\mathfrak{g}_{\Omega}^{W}$ sends $N_{P_{\mathfrak{g}_{\Omega}}^{x}}(W)$ isomorphically onto $P_{\mathfrak{g}_{\Omega}}^{x}$ (cf. 4.7),
which is isomorphic to $T_{LF}^{x}$ (cf. 4.6.2).

4.10. At this point, it follows from Proposition 3.7 that there exist an
$F_{x}^{\mathfrak{g}}$-locality functor $\mathfrak{g}_{\mathfrak{g}}^{W} : T_{LF}^{\mathfrak{g}} \rightarrow T_{G}^{\mathfrak{g}}$ which sends $T_{LF}^{\mathfrak{g}}$ isomorphically
to $P_{\mathfrak{g}_{\Omega}}^{W}$, and that this functor is naturally $F_{x}^{\mathfrak{g}}$-isomorphic to the inclusion
$T_{LF}^{\mathfrak{g}} \subset T_{G}^{\mathfrak{g}}$ in 4.8 above; that is to say, according to our definition
in [5, 2.9] and since the kernel of the structural group homomorphism from

\[ T_n^\omega (P^\omega) \] to \[ \mathcal{F}^\omega (P^\omega) \] is the image of \( C_n^\omega (P^\omega) \subset T_n^\omega (P^\omega) \). There is \( z \in C_n^\omega (P^\omega) \) such that, denoting by \( \zeta_Q^\omega \) the image of \( z \) in \( T_n^\omega (Q) \) for any \( Q \in \mathcal{Y}^\omega \), we get
\[ P^{\omega,Y,W} (Q,R) = \zeta_Q^\omega \cdot T_n^\omega (Q,R) \cdot (\zeta_R^\omega)^{-1} \] 4.10.1
in \( T_n^\omega (Q,R) \), for any pair of groups \( Q \) and \( R \) in \( \mathcal{Y}^\omega \).

4.11. But, we also can consider the images \( \zeta_Q^\omega \) of \( z \) in \( T_n^\omega (Q) \) for any \( Q \in \mathcal{X}^\omega \). Hence, up to replacing our choice of \( T_n^\omega \) as a \( \mathcal{F}^\omega \)-sublocality of \( T_n^\omega \) by the choice of \( \zeta_Q^\omega \cdot T_n^\omega (Q,R) \cdot (\zeta_R^\omega)^{-1} \) in \( T_n^\omega (Q,R) \), for any pair of groups \( Q \) and \( R \) in \( \mathcal{X}^\omega \), in \( T_n^\omega \) we actually may assume that we get
\[ P^{\omega,X,W} = T_n^\omega \] 4.11.1.

In this situation, it follows from our definitions in 4.7 above that in \( T_n^\omega \) the coherent \( \mathcal{F}^\omega \)-sublocality \( \mathcal{M}^{\omega,X} \) contains \( T_n^\omega \).

4.12. In particular, if \( \mathcal{X}^\omega = \mathcal{Y}^\omega \) then we have
\[ P^{\omega,X,W} = P^{\omega,Y,W} = T_n^\omega = \mathcal{T}_n^\omega \] 4.12.1,
so that we are done. Assume that \( \mathcal{X}^\omega \neq \mathcal{Y}^\omega \); then, by the very definition of \( T_n^\omega \) (cf. 3.6.1 and 4.3), for any \( V \in \mathcal{X}^\omega - \mathcal{Y}^\omega \) we have
\[ \text{Ker}(\tilde{\rho}_V) = T_n^\omega (V)/C_n^\omega (V) = \prod_{\tilde{\theta} \in \mathcal{F}_V^\omega (P^\omega,V)} Z(V) \] 4.12.2
and therefore, since \( p \) does not divide \( |\mathcal{F}_V^\omega (P^\omega,V)| \) [3, Proposition 6.7], we have a surjective group homomorphism
\[ \nabla_V^{\alpha,X,W} : \text{Ker}(\tilde{\rho}_V) \to Z(V) \] 4.12.3
mapping \( z = (z_{\tilde{\theta}})_{\tilde{\theta} \in \mathcal{F}_V^\omega (P^\omega,V)} \) on
\[ \nabla_V^{\alpha,X,W} (z) = \frac{1}{|\mathcal{F}_V^\omega (P^\omega,V)|} \sum_{\tilde{\theta} \in \mathcal{F}_V^\omega (P^\omega,V)} z_{\tilde{\theta}} \] 4.12.4.
4.13. At this point, considering the contravariant functor
\[ \Omega : \mathbb{T}_{G}^{x,W} \to \text{Ab} \]
mapping any \( Q \in \mathcal{Y}^{W} \) on \{0\} and any \( V \in \mathcal{X}^{W} - \mathcal{Y}^{W} \) on \( \text{Ker}(\nabla_{V}^{x,W}) \), and the quotient \( \mathcal{F}^{x,W} \)-locality \( \mathcal{T}_{G}^{x,W} / \Omega_{G}^{x,W} \) [5, 2.10], it is easily checked that the coherent \( \mathcal{F}^{x,W} \)-localities \( \mathcal{P}^{x,W} \) (cf. 4.7) and \( \mathcal{T}_{L}^{x,W} / \Omega_{L}^{x,W} \) have the same image in this quotient; indeed, it follows from equalities 4.11.1 above that their images coincide over \( \mathcal{Y}^{W} \) and, since for any \( V \in \mathcal{X}^{W} - \mathcal{Y}^{W} \) we have
\[ (\mathcal{T}_{G}^{x,W} / \Omega_{G}^{x,W})(V) \cong L_{F}^{V} \]
\[ \mathcal{P}^{x,W}(V) \] and \( \mathcal{T}_{L}^{x,W}(V) \) map both isomorphically onto \( (\mathcal{T}_{G}^{x,W} / \Omega_{G}^{x,W})(V) \).
In particular, we get \( \mathcal{P}^{x,W} \cong \mathcal{T}_{L}^{x,W} \) since the functors from \( \mathcal{P}^{x,W} \) and \( \mathcal{T}_{L}^{x,W} \) to the quotient \( \mathcal{T}_{G}^{x,W} / \Omega_{G}^{x,W} \) are faithful. This proves our claim in 4.6.

4.14. It remains to prove the uniqueness; thus, assume that \( \mathcal{P}^{x} \) and \( \mathcal{P}'^{x} \) are two extendable perfect \( \mathcal{F}^{x} \)-localities; it follows from Proposition 3.7 that we may assume that both are \( \mathcal{F}^{x} \)-sublocalities of the \( \mathcal{F}^{x} \)-locality \( \mathcal{T}_{G}^{x} \) introduced in 3.6 above. On the other hand, since the respective full subcategories \( \mathcal{P}^{\mathbb{Y}} \) of \( \mathcal{P}^{x} \) and \( \mathcal{P}'^{\mathbb{Y}} \) of \( \mathcal{P}'^{x} \) over \( \mathbb{Y} \) as the set of objects are still two extendable perfect \( \mathcal{F}^{\mathbb{Y}} \)-localities, it follows from our induction hypothesis that they are \( \mathcal{F}^{\mathbb{Y}} \)-locality isomorphic. Consequently, considering the inclusions of \( \mathcal{P}^{\mathbb{Y}} \) and \( \mathcal{P}'^{\mathbb{Y}} \) in \( \mathcal{T}_{G}^{x} \) induced by the inclusions
\[ \mathcal{P}^{\mathbb{Y}} = (\mathcal{P}^{x})^{\mathbb{Y}} \subset (\mathcal{T}_{G}^{x})^{\mathbb{Y}} \supset (\mathcal{P}'^{x})^{\mathbb{Y}} = \mathcal{P}'^{\mathbb{Y}} \]
and by the canonical functor \( (\mathcal{T}_{G}^{x})^{\mathbb{Y}} \rightarrow \mathcal{T}_{G}^{\mathbb{Y}} \) (cf. 4.9.1), the existence of an \( \mathcal{F}^{\mathbb{Y}} \)-locality isomorphism \( \mathcal{P}^{\mathbb{Y}} \cong \mathcal{P}'^{\mathbb{Y}} \) determines two \( \mathcal{F}^{\mathbb{Y}} \)-locality functors from \( \mathcal{P}^{\mathbb{Y}} \) to \( (\mathcal{T}_{G}^{x})^{\mathbb{Y}} \); then, it follows again from Proposition 3.7 that the functors ainsi obtained are naturally \( \mathcal{F}^{\mathbb{Y}} \)-isomorphic.

4.15. That is to say, as in 4.10 above, since the kernel of the structural group homomorphism from \( \mathcal{T}_{G}^{\mathbb{Y}}(P) \) to \( \mathcal{F}^{\mathbb{Y}}(P) \) is the image of \( C_{G}^{\alpha}(P) \)
in $T_{G\alpha}^\beta(P)$, there is $z \in C_{G\alpha}(P)$ such that, denoting by $\tau^\beta_Q$ the image of $z$ in $T_{G\alpha}^\beta(Q)$ for any $Q \in \mathcal{Y}$, in $T_{G\alpha}^\beta(Q, R)$ we get
\[
P^\beta(Q, R) = \tau^\beta_Q \cdot P^\beta(Q, R) \cdot (\tau^\beta_R)^{-1}
\]
for any pair of groups $Q$ and $R$ in $\mathcal{Y}$. As above, considering the images $\tau^\times_Q$ of $z$ in $T_{G\alpha}^\times(Q)$ for any $Q \in \mathcal{X}$ and modifying our choice of $P^\times$ as a $\mathcal{F}^\times$-sublocality of $T_{G\alpha}^\times$ by the choice of $\tau^\times_Q \cdot P^\times(Q, R) \cdot (\tau^\times_R)^{-1}$ in $T_{G\alpha}^\times(Q, R)$ for any pair of groups $Q$ and $R$ in $\mathcal{X}$, we actually may assume that in $\mathcal{T}_{G\alpha}^\times$ we have $P^\times = P'^\times$.

4.16. Moreover, as in 4.12 above, by the very definition of $\mathcal{T}_{G\alpha}^\times$ (cf. 3.6.1 and 4.3), for any $V \in \mathcal{X} - \mathcal{Y}$ we have
\[
\ker(\bar{\rho}_V^\alpha) = T_{G\alpha}^\times(V)/C_{G\alpha}^\times(V) = \prod_{\tilde{\theta} \in \bar{\mathcal{F}}(P,V)} Z(V)
\]
and therefore, since $p$ does not divide $|\bar{\mathcal{F}}(P,V)|$ [3, Proposition 6.7], we have a surjective group homomorphism
\[
\nabla^\alpha_V : \ker(\bar{\rho}_V^\alpha) \rightarrow Z(V)
\]
mapping $z = (z_{\tilde{\theta}})_{\tilde{\theta} \in \bar{\mathcal{F}}(P,V)}$ on
\[
\nabla^\alpha_V(z) = \frac{1}{|\bar{\mathcal{F}}(P,V)|} \sum_{\tilde{\theta} \in \bar{\mathcal{F}}(P,V)} z_{\tilde{\theta}}
\]

4.17. At this point, considering the contravariant Dirac functor
\[
\delta^\alpha : \mathcal{T}_{G\alpha}^\times \rightarrow \text{Ab}
\]
mapping any $Q \in \mathcal{Y}$ on $\{0\}$ and any $V \in \mathcal{X} - \mathcal{Y}$ on $\ker(\nabla^\alpha_V)$, and the quotient $\mathcal{F}^\times$-locality $T_{G\alpha}^\times/\delta^\alpha$ [5, 2.10], it is easily checked that the coherent $\mathcal{F}^\times$-localities $P^\times$ and $P'^\times$ have the same image in this quotient; indeed, it follows from 4.15 above that their images coincide over $\mathcal{Y}$ and, since for any $V \in \mathcal{X} - \mathcal{Y}$ we have
\[
(T_{G\alpha}^\times/\delta^\alpha)(V) \cong L^V\mathcal{F}
\]
$P^\times(V)$ and $P'^\times(V)$ map both isomorphically onto $(T_{G\alpha}^\times/\delta^\alpha)(V)$. In particular, we get $P^\times \cong P'^\times$ since the functors from $P^\times$ and $P'^\times$ to the quotient $T_{G\alpha}^\times/\delta^\alpha$ are faithful. This proves the uniqueness.
5. Existence and uniqueness of the sections from \( F^x \) to \( M^{\alpha,x} \)

5.1. With the hypothesis and notation in 4.3 above, our purpose in this section is to prove that

**Theorem.** The structural functor \( \bar{\rho}^{\alpha,x} : M^{\alpha,x} \to F^x \) admits an \( F^x \)-locality functorial section \( \bar{\sigma}^{\alpha,x} : F^x \to M^{\alpha,x} \).

Actually, since we assume that \( U \neq P \), we also have \( U \neq N_P(U) = P^U \) and therefore \( P^U \) belongs to \( \mathcal{Y}^U \); thus, this theorem is just the existence part of [5, Theorem 6.22] but we restate the proof in our new context; indeed, here we assume that \( \mathcal{P}^\theta \) is an extendable perfect \( F^x \)-locality and therefore the \( F^x \)-locality isomorphism in [5, 6.18]

\[
\mathcal{P}^{\alpha,u} = N_P^\theta(U) \cong T_L^{\alpha,u} \quad \text{5.1.1}
\]

follows from our definition in 2.6; in particular, as in 4.11 above, in \( T_L^{\alpha,u} \) we may assume that \( \mathcal{P}^{\alpha,u} = T_L^{\alpha,u} \).

5.2. Since \( \mathcal{Y}^U \) is not empty, as in 4.7 above we can define the coherent \( F^x \)-locality \( M^{\alpha,\mathcal{Y}^U} \) via the pull-back (cf. 4.2.3)

\[
\begin{align*}
\mathcal{P}^{\alpha,u} & \subset T_L^{\alpha,u} \\
\uparrow & \\
M^{\alpha,\mathcal{Y}^U} & \subset (T_G^{\alpha,u})^{\mathcal{Y}^U}
\end{align*} \quad \text{5.2.1}
\]

and the coherent \( F^{x,u} \)-localities \( M^{\alpha,\mathcal{Y}^U} \subset T_L^{\alpha,u} \) and \( M^{\alpha,x} \) as in 4.3, with the second structural functors

\[
\rho^{\alpha,x,u} : M^{\alpha,x} \to F^{x,u} \quad \text{and} \quad \bar{\rho}^{\alpha,x,u} : \bar{M}^{\alpha,x,u} \to F^{x,u} \quad \text{5.2.2}.
\]

Now recall that, denoting by \( \bar{F}^x \) and \( \bar{F}^{x,u} \) the respective exterior quotients of \( F^x \) and \( F^{x,u} \) [3, 1.3], the coherency of \( M^{\alpha,x} \) and \( M^{\alpha,x,u} \) determines contravariant functors [5, 2.8.3]

\[
\mathcal{R}et(\rho^{\alpha,x}) : \bar{F}^x \to \text{Ab} \quad \text{and} \quad \mathcal{R}et(\bar{\rho}^{\alpha,x,u}) : \bar{F}^{x,u} \to \text{Ab} \quad \text{5.2.3};
\]

as usual, the existence of \( \bar{\sigma}^{\alpha,x} \) depends on the vanishing of the cohomology class of a suitable \( \mathcal{R}et(\bar{\rho}^{\alpha,x}) \)-valued 2-cocycle and, from the reduction procedure developed in section 3, we will move to the corresponding \( \mathcal{R}et(\bar{\rho}^{\alpha,x,u}) \)-valued 2-cocycle.
5.3. From the commutative diagram 4.2.3 we get the following commutative diagram of the normalizers of \( U \)
\[
\begin{array}{ccc}
N_{P^\#} (U) & \subset & N_{\text{gr}_U} (U) \\
\uparrow & & \uparrow \\
N_{M^\#, P^\#} & \subset & N_{\text{gr}_U} (U)
\end{array}
\]
moreover, we are setting \( N_{P^\#} (U) = P^{\#, U} \) and we have the commutative diagram 4.9.2 for \( W = U \). Consequently, the \( \mathcal{F}^{\#, U} \) and \( \mathcal{F}^{x, U} \)-locality functors (cf. 3.10.1)
\[
\begin{array}{ccc}
\mathcal{A}^{\#, U} : N_{\mathcal{F}^{\#, U}} (U) & \longrightarrow & \mathcal{T}^{\#, U} \ G \ 
\mathcal{M}^{\#, U} & \longrightarrow & \mathcal{T}^{\#, U} \\
\end{array}
\]
successively induce the new \( \mathcal{F}^{\#, U} \)-locality functor (cf. 5.2.1)
\[
\begin{array}{ccc}
l^{\#, U} : N_{M^{\#, U}} (U) & \longrightarrow & M^{\#, U} \\
\end{array}
\]
and, moreover, the \( \mathcal{F}^{x, U} \)-locality functors (cf. 4.3)
\[
\begin{array}{ccc}
l^{x, U} : M^{\#, x} & \longrightarrow & M^{\#, x} \\
\end{array}
\]
Similarly, since we are assuming that \( P^{\#, U} = T^{\#, U} \) (cf. 4.11.1), the \( \mathcal{F}^{x, U} \)-locality functor (cf. 3.13.2)
\[
\begin{array}{ccc}
l^{x, U} : M^{\#, x} & \longrightarrow & M^{\#, x} \\
\end{array}
\]
and the pull-back 5.2.1 above determine new \( \mathcal{F}^{x, U} \)-locality functors (cf. 4.3)
\[
\begin{array}{ccc}
m^{x, U} : M^{\#, x} & \longrightarrow & M^{\#, x} \\
\end{array}
\]
and therefore, for any \( n \in \mathbb{N} \), we also get a group homomorphism (cf. 5.1.2)
\[
\begin{array}{ccc}
H^n (\mathcal{F}^{x, U} , \text{Re} (\tilde{\rho}^{\#, x})) & \longrightarrow & H^n (\mathcal{F}^{x, U} , \text{Re} (\tilde{\rho}^{\#, x}) \circ \tilde{\iota}^{x, U}) \\
\end{array}
\]
moreover, \( \tilde{h}^{\#, U} \) induces a natural map [5, 2.10.1]
\[
\begin{array}{ccc}
\nu^{\#, U} : \text{Re} (\tilde{\rho}^{\#, x}) \circ \tilde{\iota}^{x, U} & \longrightarrow & \text{Re} (\tilde{\rho}^{\#, x} , U) \\
\end{array}
\]
and therefore, for any \( n \in \mathbb{N} \), we also get a group homomorphism
\[
\begin{array}{ccc}
H^n (\mathcal{F}^{x, U} , \text{Re} (\tilde{\rho}^{\#, x} , U) \circ \tilde{\iota}^{x, U}) & \longrightarrow & H^n (\mathcal{F}^{x, U} , \text{Re} (\tilde{\rho}^{\#, x} , U)) \\
\end{array}
\]
In [5, Proposition 6.9, 6.12.3 and 6.21.7] we prove that, for any \( n \in \mathbb{N} \), the composition of the homomorphisms 5.4.1 and 5.4.3 determines an isomorphism
\[
\mathbb{H}^n(F^x, \text{ker}(\rho^{\alpha,x})) \cong \mathbb{H}^n(F^x, \text{ker}(\rho^{\alpha,x,U}))
\]
5.5. Let us exploit the announced \( \text{ker}(\rho^{\alpha,x}) \)-valued 2-cocycle. For any \( F^x \)-morphism \( \varphi : R \to Q \), choose a lifting \( x_\varphi \) in \( \mathcal{M}^{\alpha,x}(Q,R) \) (cf. 4.3) and denote by \( \tilde{x}_\varphi \) the image of \( x_\varphi \) in \( \tilde{\mathcal{M}}^{\alpha,x}(Q,R) \); actually, we can do our choice in such a way that we have (cf. 4.3)
\[
\tilde{x}_{\kappa_Q^\alpha(u) \circ \varphi} = \tilde{\iota}_Q^{\alpha,x}(u) \cdot \tilde{x}_\varphi
\]
for any \( u \in Q \), where \( \kappa_Q^\alpha(u) \in F^x(Q) \) denotes the conjugation by the image of \( u \); indeed, if we have \( \kappa_Q^\alpha(u) \circ \varphi = \varphi \) then we get \( u = \varphi(z) \) for a suitable \( z \in Z(R) \); since \( \tilde{\mathcal{M}}^{\alpha,x} \) is coherent, in this case we obtain
\[
\tilde{\iota}_Q^{\alpha,x}(u) \cdot \tilde{x}_\varphi = \tilde{\iota}_Q^{\alpha,x}(\varphi(z)) \cdot \tilde{x}_\varphi = \tilde{x}_\varphi \cdot \tilde{\iota}_R^{\alpha,x}(z) = \tilde{x}_\varphi
\]
More precisely, if \( Q \) and \( R \) are contained in \( F^U \) and \( \varphi : R \to Q \) comes from an \( F^x \)-morphism, it is quite clear that we may assume that \( x_\varphi \) belongs to \( (N_{\mathcal{M}_{\alpha,x}(U)}(Q,R) \) and then that \( h_{\alpha}^{x,U}(x_\varphi) \) belongs to the image of \( T_{L^x}(Q,R) \) via \( l_{x}^{x,U} \), so that actually we have (cf. 5.3.6)
\[
\hat{h}_{\alpha}^{x,U}(\tilde{x}_\varphi) = m_{x}^{x,U}(\varphi)
\]
5.6. Then, for any triple of subgroups \( Q, R \) and \( T \) in \( \mathcal{X} \), and any pair of \( \mathcal{F} \)-morphisms \( \psi : T \to R \) and \( \varphi : R \to Q \), since \( x_\varphi \cdot x_\psi \) and \( x_{\varphi \circ \psi} \) have the same image \( \varphi \circ \psi \) in \( \mathcal{F}(Q,T) \), the divisibility of \( \mathcal{M}^{\alpha,x} \) guarantees the existence and the uniqueness of \( k_{\varphi,\psi} \in \text{ker}(\rho^{\alpha,x}_T) \) fulfilling
\[
x_{\varphi \cdot x_\psi} = x_{\varphi \circ \psi} \cdot k_{\varphi,\psi}
\]
Denote by \( \tilde{k}_{\varphi,\psi} \) the image of \( k_{\varphi,\psi} \) in \( \text{ker}(\rho^{\alpha,x}_T) \); since \( \tilde{\mathcal{M}}^{\alpha,x} \) is coherent, on the one hand for any \( u \in Q \) and any \( v \in R \) we get (cf. 5.5.1)
\[
\tilde{x}_{\kappa_Q^\alpha(u) \circ \varphi \cdot \kappa_R^\alpha(v) \circ \psi} = \left( \tilde{\iota}_Q^{\alpha,x}(u) \cdot \tilde{x}_\varphi \right) \cdot \left( \tilde{\iota}_R^{\alpha,x}(v) \cdot \tilde{x}_\psi \right)
\]
\[
= \tilde{\iota}_Q^{\alpha,x}(u \varphi(v)) \cdot \tilde{x}_\varphi \cdot \tilde{x}_\psi
\]
\[
\tilde{x}_{(\kappa_Q^\alpha(u) \circ \varphi) \circ (\kappa_R^\alpha(v) \circ \psi)} = \tilde{x}_{\kappa_Q^\alpha(u \varphi(v)) \circ \varphi \circ \psi} = \tilde{\iota}_Q^{\alpha,x}(u \varphi(v)) \cdot \tilde{x}_{\varphi \circ \psi}
\]
hence, from the divisibility of \( \tilde{\mathcal{M}}^{\alpha,x} \) we obtain
\[
\tilde{k}_{\kappa_Q^\alpha(u) \circ \varphi \cdot \kappa_R^\alpha(v) \circ \psi} = \tilde{k}_{\varphi,\psi}
\]
5.6.3.
That is to say, for any \( n \in \mathbb{N} \), setting [3, 1.5]
\[
\mathbb{C}^n \left( \tilde{F}^\Omega, \text{Ret}(\tilde{\rho}^\Omega) \right) = \prod_{\tilde{q} \in \tilde{\text{Ret}}(\Delta_2, \tilde{F}^x)} \text{Ker}(\tilde{\rho}^\Omega_{\tilde{q}(0)}) \quad 5.6.4,
\]
we have obtained an element \( \tilde{k} = \{ \tilde{k}_{\tilde{q}} \}_{\tilde{q} \in \tilde{\text{Ret}}(\Delta_2, \tilde{F}^x)} \) in \( \mathbb{C}^2 \left( \tilde{F}^\Omega, \text{Ret}(\tilde{\rho}^\Omega) \right) \) where we set \( \tilde{k}_{\tilde{q}} = \tilde{k}_{\tilde{q}(1(\bullet),\tilde{q}(\bullet \bullet))} = \tilde{k}_{\tilde{q}(1(\bullet),\tilde{q}(\bullet \bullet))} \) for some representative \( \tilde{q} : \Delta_2 \to \tilde{F}^x \) of \( \tilde{q} \).

5.7. We claim that \( \tilde{k} \) is actually a 2-cocycle; explicitly, considering the usual differential map [3, A13.11]
\[
\tilde{d}^\Omega_2 : \mathbb{C}^2 \left( \tilde{F}^x, \text{Ret}(\tilde{\rho}^\Omega) \right) \to \mathbb{C}^3 \left( \tilde{F}^x, \text{Ret}(\tilde{\rho}^\Omega) \right) \quad 5.7.1,
\]
we claim that \( \tilde{d}^\Omega_2(\tilde{k}) = 0 \); indeed, with the notation above, for a third \( \tilde{F}^x \)-morphism \( \eta : W \to T \) we get
\[
(\tilde{x}, \tilde{\eta})\cdot(\tilde{x}, \tilde{\eta}) = (\tilde{x}, \tilde{\eta}) = (\tilde{x}, \tilde{\eta}) \quad 5.7.2
\]
and the divisibility of \( \mathcal{M}^\Omega \) forces
\[
\tilde{k}_{\tilde{\varphi}, \tilde{\psi}}(\text{Ret}(\tilde{\rho}^\Omega))(\tilde{\eta}) = \tilde{k}_{\tilde{\varphi}, \tilde{\psi}} \tilde{k}_{\tilde{\varphi}, \tilde{\psi}} = 0 \quad 5.7.3
\]
since \( \text{Ker}(\tilde{\rho}^\Omega) \) is Abelian, in the additive notation we obtain
\[
0 = (\text{Ret}(\tilde{\rho}^\Omega))(\tilde{\eta}) - (\tilde{k}_{\tilde{\varphi}, \tilde{\psi}} + \tilde{k}_{\tilde{\varphi}, \tilde{\psi}} - \tilde{k}_{\tilde{\varphi}, \tilde{\psi}}) \quad 5.7.4
\]
proving our claim.

5.8. Then, in order to prove the existence of a section \( \tilde{\sigma}^\Omega \), it suffices to show that \( \tilde{k} \) is a 2-coboundary and therefore, according to isomorphism 5.4.4 above, it suffices to prove that the image via \( \nu_{\tilde{\nu}, \tilde{U}} \) (cf. 5.4.2) of the restriction of \( \tilde{k} \) to \( \tilde{F}^x \) is a 2-coboundary. But, for any pair of \( \tilde{F}^x \)-morphisms \( \varphi : R \to Q \) and \( \psi : T \to R \), we have chosen \( x_{\varphi} \) in \( (N_{\mathcal{M}^\Omega,x}(U))(Q,R) \), \( x_{\psi} \) in \( (N_{\mathcal{M}^\Omega,x}(U))(R,T) \) and \( x_{\varphi \psi \varphi} \) in \( (N_{\mathcal{M}^\Omega,x}(U))(Q,T) \), so that in equality 5.6.1 the element \( k_{\varphi, \psi} \) belongs to \( (N_{\mathcal{M}^\Omega,x}(U))(T) \) and therefore we get (cf. 5.3.4)
\[
\tilde{h}_{\tilde{\Omega}}(x_{\varphi}) \cdot \tilde{h}_{\tilde{\Omega}}(x_{\psi}) = \tilde{h}_{\tilde{\Omega}}(x_{\varphi \psi \varphi}) \tilde{h}_{\tilde{\Omega}}(k_{\varphi, \psi}) \quad 5.8.1
\]
and therefore we still get
\[
\tilde{h}_{\tilde{\Omega}}(x_{\varphi}) \cdot \tilde{h}_{\tilde{\Omega}}(x_{\psi}) = \tilde{h}_{\tilde{\Omega}}(x_{\varphi \psi \varphi}) \tilde{h}_{\tilde{\Omega}}(k_{\varphi, \psi}) \quad 5.8.2
\]
so that equalities 5.5.3 force \( h_{\alpha}^x(u, \bar{k}_\varphi, w) = 1 \); that is to say, the image via \( \nu_{\alpha}^x(u) \) of the restriction of \( \bar{k} \) to \( \mathcal{F}^x_v \) is just trivial, proving that \( \bar{k} \) is a 2-coboundary.

5.9. Thus, we have obtained a functorial section \( \bar{\sigma}^{n,x} : \mathcal{F}^x \to \mathcal{M}^{\alpha,x} \) of \( \bar{\rho}^{\alpha,x} \); actually, \( \bar{\sigma}^{n,x} \) can be modified in order to get an \( \mathcal{F}^x \)-locality functorial section [5, 2.9]. Indeed, for any \( \mathcal{F}^x_p \)-morphism \( \zeta : R \to Q \), choosing \( u_\zeta \) in \( T_P(R, Q) \) lifting \( \zeta \), both \( \bar{\mathcal{M}}^{\alpha,x} \)-morphisms \( \bar{\sigma}^{n,x}_{Q,R}(\zeta) \) and \( \bar{v}^{\alpha,x}_{Q,R}(u_\zeta) \) (cf. 4.3) lift \( \zeta \); once again, the divisibility of \( \bar{\mathcal{M}}^{\alpha,x} \) guarantees the existence and the uniqueness of \( \bar{m}_\zeta \in \text{Ker}(\bar{\rho}^{\alpha,x}_R) \) fulfilling

\[
\bar{v}^{\alpha,x}_{Q,R}(u_\zeta) = \bar{\sigma}^{n,x}_{Q,R}(\zeta) \cdot \bar{m}_\zeta
\]

5.9.1; actually, it follows easily from 5.5.1 that \( \bar{m}_\zeta \) only depends on \( \zeta \in \bar{\mathcal{F}}_P(Q, R) \) and, as above, we write \( \bar{m}_\zeta \) instead of \( m_\zeta \); moreover, for a second \( \mathcal{F}^x_p \)-morphism \( \xi : T \to R \), we get

\[
\bar{\sigma}^{n,x}_{Q,T}(\zeta \circ \xi) \cdot \bar{m}_\zeta = \bar{\sigma}^{n,x}_{Q,R}(\zeta) \cdot \bar{m}_\zeta \\
= \bar{\sigma}^{n,x}_{Q,R}(\zeta) \cdot \bar{m}_\zeta \cdot \bar{\sigma}^{n,x}_{R,T}(\xi) \cdot \bar{m}_\xi
\]

5.9.2.

5.10. Then, always the divisibility of \( \bar{\mathcal{M}}^{\alpha,x} \) forces

\[
\bar{m}_{\zeta \circ \xi} = \left( \mathcal{Ker}(\bar{\rho}^{\alpha,x}_T)(\bar{\zeta}) \right) (\bar{m}_\zeta) = \bar{m}_\zeta
\]

5.10.1; and, since \( \text{Ker}(\bar{\rho}^{\alpha,x}_T) \) is Abelian (cf. Proposition 3.4, 3.5 and 3.6), in the additive notation we obtain

\[
0 = \left( \mathcal{Ker}(\bar{\rho}^{\alpha,x}_T)(\bar{\zeta}) \right) (\bar{m}_\zeta) = \bar{m}_{\zeta \circ \xi} + \bar{m}_\xi
\]

5.10.2; that is to say, denoting by \( \hat{\mathcal{F}}^x_P \subset \mathcal{F}^x \) the obvious inclusion functor, the correspondence \( \bar{m} \) sending any \( \hat{\mathcal{F}}^x_P \)-morphism \( \hat{\zeta} : R \to Q \) to \( \bar{m}_\zeta \) defines a 1-cocycle in \( \mathcal{C}^1(\hat{\mathcal{F}}^x_P, \mathcal{Ker}(\bar{\rho}^{\alpha,x}_T) \circ \bar{\iota}_P) \); but, since the category \( \hat{\mathcal{F}}^x_P \) obviously has a final object, we actually have [3, Corollary A4.8]

\[
\mathcal{H}^1(\hat{\mathcal{F}}^x_P, \mathcal{Ker}(\bar{\rho}^{\alpha,x}_T) \circ \bar{\iota}_P) = \{0\}
\]

5.10.3; consequently, we obtain \( \bar{m} = d_0^{\alpha,x}(\bar{w}) \) for some element \( \bar{w} = (\bar{w}_Q)_{Q \in X} \) in

\[
\mathcal{C}^0(\hat{\mathcal{F}}^x_P, \mathcal{Ker}(\bar{\rho}^{\alpha,x}_T) \circ \bar{\iota}_P) = \mathcal{C}^0(\hat{\mathcal{F}}^x_P, \mathcal{Ker}(\bar{\rho}^{\alpha,x}_T))
\]

5.10.4. In conclusion, equality 5.9.1 becomes

\[
\bar{v}^{\alpha,x}_{Q,R}(u_\zeta) = \bar{\sigma}^{n,x}_{Q,R}(\zeta) \cdot \left( \mathcal{Ker}(\bar{\rho}^{\alpha,x}_T)(\bar{\zeta}) \right) (\bar{w}_Q \cdot \bar{w}_R^{-1} = \bar{w}_Q \cdot \bar{\sigma}^{n,x}_{Q,R}(\zeta) \cdot \bar{w}_R^{-1})
\]

5.10.5;
thus, the new correspondence which, for any pair of subgroups $Q$ and $R$ in $\mathfrak{X}$, sends any $\varphi \in \mathcal{F}(Q, R)$ to $\bar{w}_Q \cdot \bar{\sigma}_{Q,R}^{\mathfrak{X}}(\varphi) \cdot \bar{w}_R^{-1}$ defines an $\mathcal{F}_{\mathfrak{X}}$-locality functorial section of $\rho^x$. We are done.

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