The Cremona problem in dimension 2

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Abstract. The Cremona conjecture, also called Jacobi problem, claims that a polynomial morphism \( \mathbb{C}^n \rightarrow \mathbb{C}^n \) is invertible as a polynomial morphism if its Jacobian is constant and not zero. In this paper, we show that the conjecture is true for \( n = 2 \). The starting point of our proof is an important result of Shreeram Abhyankar. Then we use a computation in rigid geometry to achieve the result.

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Introduction. A polynomial map \((f, g) : \mathbb{C}^2 \rightarrow \mathbb{C}^2\) is given by two polynomials \(f\) and \(g\) in two variables \(X\) and \(Y\) with complex scalars. We write \(f\) and \(g\) as sums of their homogenous components

\[
f = f_m + \cdots + f_{m'} \quad \text{and} \quad g = g_n + \cdots + g_{n'},
\]

where \(f_\mu\) respectively \(g_\nu\) are linear combinations of the terms of total degree \(\mu\) respectively \(\nu\). The forms \(f_m\) respectively \(g_n\) of highest degree are called the leading forms.

It was shown by Abhyankar that, for a given counterexample \((f, g)\) to the Jacobian conjecture in dimension 2, one can assume that, after a suitable transformation of variables, the leading forms of \(f\) and \(g\) have the following shape

\[
f_m = X^{m_1}Y^{m_2} \quad \text{and} \quad g_n = X^{n_1}Y^{n_2};
\]

cf. [1, Theorem 8.7] or [5, Corollary 10.2.22]. In this paper, we will show that this assumption leads to a contradiction. Thus the Jacobian conjecture is true in dimension 2.
1. Division algorithm. In this section, let $\mathbb{K}$ be an algebraically closed field of characteristic 0. The $\mathbb{K}$-algebra $L := \mathbb{K}[X,Y]_{XY}$ consists of all Laurent polynomials in two variables. It carries a canonical graduation of type $\mathbb{Z}$ given by the total degree function. Let $H_n$ be the subspace of all homogenous Laurent polynomials of degree $n$ including the zero polynomial. For $f = \sum_{\nu \in \mathbb{Z}} f_{\nu} \in L$ and $f \neq 0$, we set

$$\deg f := \sup\{\nu \in \mathbb{Z}; f_{\nu} \neq 0\}.$$ 

On $L$, we have a filtration $(L_n; n \in \mathbb{N})$ where

$$L_n := \{f \in L; \deg f \leq -n\} = \bigoplus_{\nu \leq -n} H_{\nu}.$$ 

The completion with respect to this filtration is denoted by $A := \widehat{L} = \lim \leftarrow L/L_n$; cf. [2, Chap. 3]. It consists of all series

$$\sum_{\mu = -\infty}^{m} f_{\mu} \text{ with } f_{\mu} \in H_{\mu} \text{ for some } m \in \mathbb{Z};$$

cf. [5, Prop. 10.2.8]. The degree, the multiplication, and the filtration on $A$ are declared as on $L$. The $\mathbb{K}$-algebra $A$ represents the formal functions on a neighborhood of the twice punctured projective line at infinity which behave like meromorphic functions there. The algebra $A$ has similar properties as the algebra $R^{\sim}$ defined in [5, Prop. 10.2.8]. In this section, we consider these functions without conditions of convergence; in Section 2, we will focus on that by means of rigid geometry.

Lemma 1.1. An element $g \in A$ is a unit in $A$ if and only if $g$ is of the form

$$g = c \cdot X^{n_1} Y^{n_2} \cdot (1 - v)$$

where $c \in \mathbb{K}^\times$, $n_i \in \mathbb{Z}$, $\deg(v) < 0$. Such a representation is unique. Such a unit $g$ admits a $k$-th root for $0 \neq k \in \mathbb{Z}$ if and only if $k$ divides both numbers $n_1$ and $n_2$.

Proof. The proof can be left to the reader. For example, we have for the inverse

$$g^{-1} = c^{-1} \cdot X^{-n_1} Y^{-n_2} \cdot \left(\sum_{\nu=0}^{\infty} v^{\nu}\right).$$

For $c = 1$ and $k \in \mathbb{N}$, the $k$-th root is given by

$$g^{1/k} := X^{n_1/k} Y^{n_2/k} \cdot \left(\sum_{\nu=0}^{\infty} \left(1/k\right)_{\nu} (-v)^{\nu}\right)$$

if $k$ divides $n_1$ and $n_2$. We consider this as the canonical $k$-th root of $g$. □
Corollary 1.2. Let $g = X^{n_1}Y^{n_2} \cdot (1 + v) \in A$ with $\text{deg}(v) < 0$ and $r \in \mathbb{Q}$ be such that $r \cdot n_1$ and $r \cdot n_2$ belong to $\mathbb{Z}$, then

$$g^r := X^{r-n_1}Y^{r-n_2} \cdot \left( \sum_{\nu=0}^{\infty} \binom{r}{\nu} v^\nu \right)$$

is well defined.

In the following, we denote by $\partial/\partial X$ respectively $\partial/\partial Y$ the partial derivatives of Laurent series. Obviously they give rise to $\mathbb{K}$-derivations on the $\mathbb{K}$-algebra $A$. They satisfy the usual rules for $\mathbb{K}$-derivations. Since the field $\mathbb{K}$ has characteristic 0, we have $\ker(\partial/\partial X, \partial/\partial Y) = \mathbb{K}$.

Definition 1.3. A couple $(f, g)$ of elements of $A$ is called a Jacobian couple if its Jacobian

$$\det \begin{pmatrix} \frac{\partial f}{\partial X} & \frac{\partial f}{\partial Y} \\ \frac{\partial g}{\partial X} & \frac{\partial g}{\partial Y} \end{pmatrix} = d \in \mathbb{K}^\times$$

is constant and not 0.

As for polynomials in two variables, we also have the notion of a leading form for a Laurent series in $A$.

Proposition 1.4. Consider a Jacobian couple $(f, g)$ as introduced above, where $m := \text{deg } f$ and $n := \text{deg } g$. Assume that the leading form of $g$ has the shape $g_n = X^{n_1}Y^{n_2}$.

(a) Then we always have $m + n \geq 2$.

(b) If $m + n > 2$, then $f_m^n \cdot g_m^{-m}$ is constant.

Proof. (a) The homogenous components of the Jacobian of degree $m + n > 2$ vanish and for $m + n = 2$ it is given by

$$\frac{\partial f_m}{\partial X} \cdot \frac{\partial g_n}{\partial Y} - \frac{\partial f_m}{\partial Y} \cdot \frac{\partial g_n}{\partial X}. \quad (1)$$

Since that the Jacobian is constant and the degree of a constant is 0, we see that $m + n \geq 2$.

(b) If $m + n > 2$, then the expression (1) is zero. We compute

$$\frac{\partial}{\partial X} \left( \frac{f_m^n}{g^n_m} \right) = \frac{f_m^{n-1} \cdot g_n^{m-1}}{g_n^m} \cdot \left( \frac{\partial f_m}{\partial X} \cdot n \cdot g_n - \frac{\partial g_n}{\partial X} \cdot m \cdot f_m \right). \quad (2)$$

For homogenous polynomials, we have Euler’s differential equation

$$n \cdot g_n = X \frac{\partial g_n}{\partial X} + Y \frac{\partial g_n}{\partial Y} \quad \text{and} \quad m \cdot f_m = X \frac{\partial f_m}{\partial X} + Y \frac{\partial f_m}{\partial Y}. \quad \text{(3)}$$

Then the term in parentheses of equation (2) is equal to

$$\frac{\partial f_m}{\partial X} \cdot \left( X \frac{\partial g_n}{\partial X} + Y \frac{\partial g_n}{\partial Y} \right) - \frac{\partial g_n}{\partial X} \cdot \left( X \frac{\partial f_m}{\partial X} + Y \frac{\partial f_m}{\partial Y} \right)$$

$$= Y \left( \frac{\partial f_m}{\partial X} \cdot \frac{\partial g_n}{\partial Y} - \frac{\partial f_m}{\partial Y} \cdot \frac{\partial g_n}{\partial X} \right).$$
This vanishes due to (1) since \( m + n > 2 \). Thus we see that the left hand term of equation (2) is equal to 0. Analogously, one shows

\[
\frac{\partial}{\partial Y} \left( \frac{f_m^n}{g_n^m} \right) = X \left( \frac{\partial f_m}{\partial Y} \cdot \frac{\partial g_n}{\partial X} - \frac{\partial f_m}{\partial X} \cdot \frac{\partial g_n}{\partial Y} \right) = 0. \tag{3}
\]

So the total differential of \( f_m^n \cdot g_n^{-m} \) vanishes. Thus, the function \( f_m^n \cdot g_n^{-m} \) is constant. \( \square \)

**Corollary 1.5.** Let \((f, g)\) be a Jacobian couple as in 1.4 satisfying \( g_n = X^{n_1} Y^{n_2} \) with integers \( n_1 > 0, n_2 > 0 \). If \( m + n > 2 \), then we have \( f_m = c \cdot X^{m_1} Y^{m_2} \) with a constant \( c \in \mathbb{K}^\times \). Moreover it holds

\[
\frac{m}{n} = \frac{m_1}{n_1} = \frac{m_2}{n_2}.
\]

Therefore the following expression is well defined

\[
g^{m/n} = X^{m_1} Y^{m_2} \left( 1 + \sum_{\nu=-\infty}^{n-1} g_\nu g_n^{-1} \right)^{m/n}.
\]

In particular, we have \((g^{m/n})^n = g^m\). Furthermore \( |m_1 - m_2| \neq |n_1 - n_2| \) if \( |m| \neq |n| \).

**Proof.** The first assertion follows from 1.4(b). For the second assertion, we use

\[
\frac{n_1 \cdot m}{n} = \frac{m_1}{n_1} = m_1 \quad \text{and} \quad \frac{n_2 \cdot m}{n} = \frac{m_2}{n_2} = m_2.
\]

So we see \( |m_1 - m_2| \neq |n_1 - n_2| \) if \( |m| \neq |n| \). The formula for \( g^{m/n} \) follows from 1.2. \( \square \)

Now we turn to the division algorithm.

**Proposition 1.6.** Let \((f, g)\) be as in 1.5. Then there exists a rational number \( r \in \mathbb{Q} \) such that \( r \cdot n_1 \in \mathbb{Z} \) and \( r \cdot n_2 \in \mathbb{Z} \) are integers, and a constant \( c \in \mathbb{K}^\times \) with \( \deg(f - c \cdot g^r) < m \). The couple \((c, r)\) is uniquely determined; actually we have \( r = m/n \) and \( f_m = c \cdot X^{m_1} Y^{m_2} \).

If, in addition, \( \deg(f - c \cdot g^r) = 2 - n \), then \( n_1 \neq n_2 \) and the leading form of \( h := f - c \cdot g^r \) is given by

\[
h_{2-n} = \sum_{i+j=2-n} c_{i,j} X^i Y^j,
\]

where \( c_{1-n_1,1-n_2} \neq 0 \). Furthermore there is at most one index \((i, j)\) with \((i, j) \neq (1-n_1, 1-n_2)\) with \( c_{i,j} \neq 0 \). For this index, we have

\[
\frac{i}{n_1} = \frac{j}{n_2} = \frac{2-n}{n}.
\]

**Proof.** The first assertion follows from 1.5.

For the supplement, set \( m' := \deg(f - c \cdot g^r) = 2 - n \). The Jacobian \( d \) of the couple \((h, g)\) is equal to the Jacobian of \((f, g)\). Then we have

\[
n_2 \sum_{i+j=2-n} i \cdot c_{i,j} X^{i+n_1-1} Y^{j+n_2-1} - n_1 \sum_{i+j=2-n} j \cdot c_{i,j} X^{i+n_1-1} Y^{j+n_2-1} = d.
\]
For \((i,j) = (1-n_1,1-n_2)\), it follows that
\[
n_2(1-n_1)c_{1-n_1,1-n_2} - n_1(1-n_2)c_{1-n_1,1-n_2} = d.
\]
Thus, we see \(n_1 \neq n_2\) and \(c_{1-n_1,1-n_2} \neq 0\). For all the other indices, we have
\[
n_2 \cdot i \cdot c_{i,j} - n_1 \cdot j \cdot c_{i,j} = 0.
\]
If \(c_{i,j} \neq 0\), then
\[
\frac{i}{n_1} = \frac{j}{n_2}.
\]
Moreover we know \(i + j = 2 - n\) and \(n = n_1 + n_2\). This yields
\[
\frac{i}{n_1} = \frac{j}{n_2} = \frac{2-n-i}{n_2}
\]
and \(i \cdot n_2 = (2-n-i) \cdot n_1\) and hence \(i \cdot n = i \cdot (n_2 + n_1) = (2-n) \cdot n_1\)

**Corollary 1.7.** Keep the assumptions of 1.6. Then we have \(n_1 \neq n_2\) and there exist a natural number \(s \in \mathbb{N}\), constants \(c_\sigma \in K\), and rational numbers \(r_\sigma \in \mathbb{Q}\) satisfying
\[
r_1 > r_2 > \cdots > r_s = \frac{2-n}{n}
\]
such that
\[
G = \sum_{\sigma=1}^{s} c_\sigma \cdot g^{r_\sigma}
\]
belongs to \(A\) and the leading term of \((f - G)\) fulfills
\[
(f - G)_{2-n} = c_{1-n_1,1-n_2}X^{1-n_1}Y^{1-n_2}.
\]

**Proof.** Apply 1.6 inductively. Note that \(\deg(f - G)\) is always an integer and that \((f - G, g)\) is a Jacobian couple. Therefore the procedure stops after finitely many steps until we arrive at the situation \(\deg(f - G) = 2 - n\) since there is at each step at most one term which has to be cancelled. In the case \(\deg(f - G) = 2 - n\), we apply the additional claim of 1.6. Then we obtain for the leading form
\[
(f - G)_{2-n} = c_{1-n_1,1-n_2}X^{1-n_1}Y^{1-n_2} + c_{i,j}X^{i}Y^{j},
\]
where \(i/n_1 = j/n_2 = (2-n)/n\) as follows from 1.6. Then we subtract
\[
c_{i,j} \cdot g^{(2-n)/n} := c_{i,j} \cdot \left(X^{n_1}Y^{n_2} \cdot \left(1 + \sum_{\nu=-\infty}^{n-1} g_{\nu}g_{-\nu}^{-1}\right)\right)^{(2-n)/n}
\]
which cancels the term \(c_{i,j}X^{i}Y^{j}\). Thus the assertion is proved. \(\square\)
2. Convergence of the division algorithm. In the following, we make use of some elementary results in rigid geometry; for a general reference, we cite [3] or [4]. We consider an algebraically closed field $\mathbb{K}$ which is complete with respect to a non-Archimedean valuation and which has residue characteristic 0. We assume that $\mathbb{K}$ contains the field $K$ of characteristic 0 as a subfield, where $K$ is the algebraically closed field over which the Jacobian problem is posed. Such a field can be constructed in the following way: Consider the field of fractions $K'$ of $K[[T]]$ and define $K$ as the topological algebraic closure of $K'$. The canonical valuation on $K[[T]]$ extends to a valuation of $\mathbb{K}$. Note that we write valuations in the multiplicative way. So we obtain on $\mathbb{K}$ a canonical structure of rigid space in the sense of Tate. On each subset $V \subset \mathbb{K}^2$, we have the spectral norm of functions $f$

$$|f|_V := \sup \{|f(x)|; x \in V\}.$$  

In particular, we have the notion of an affinoid domain $V \subset \mathbb{K}^2$; for example, bounded domains described by finitely many inequalities

$$V := \{x \in \mathbb{K}^2 ; 1 \leq |f_i(x)|, |g_j(x)| \leq 1 \text{ for } i = 1, \ldots, r, j = 1, \ldots, s\}$$

with polynomials $f_i, g_j \in \mathbb{K}[X,Y]$ are affinoid domains. Affinoid functions on such a domain are functions which can be uniformly approximated by rational functions without poles in $V$. Such functions are bounded and take their maximal absolute value in $V$. Thus the spectral norm $|f|_V$ is always a non-negative real number which actually lies in the value group of $\mathbb{K}$. We are mainly interested in domains of the following shape

$$W_{\varepsilon, \rho} := \{(x, y) \in \mathbb{K}^2 ; \varepsilon \leq |x| \leq \rho, \varepsilon \leq |y| \leq \rho\}$$

for values $\varepsilon \leq \rho$ belonging to the value group of $\mathbb{K}$. The affinoid functions on $W_{\varepsilon, \rho}$ are exactly the Laurent series which converge on $W_{\varepsilon, \rho}$. Of particular interest will be the following domains

$$U_{\varepsilon, \rho} := \{(x, y) \in \mathbb{K}^2 ; \varepsilon \leq |x| = |y| \leq \rho\}.$$  

These subsets are also affinoid and they are open subsets in the rigid analytic sense.

Lemma 2.1. Keep the above notations. Let $\varepsilon, \rho$ be elements of the value group $|\mathbb{K}^\times|$ with $\rho \geq \varepsilon$.

(a) If $v$ is an affinoid function on $U := U_{\varepsilon, \rho}$ with $|v|_U < 1$, then the series

$$h := \sum_{\nu=0}^{\infty} \binom{r}{\nu} v^\nu,$$

for any $r \in \mathbb{Q}$, converges uniformly on $U_{\varepsilon, \rho}$ and gives rise to an affinoid function there. In particular, $(1+v)^r$ is well-defined and affinoid on $U_{\varepsilon, \rho}$.

(b) Let $g = g_n + \cdots + g_0 \in \mathbb{K}[X,Y]$ be a polynomial with homogenous components $g_\nu$ of degree $\nu$. Assume $g_n = X^{n_1}Y^{n_2}$. Then there exists an $\varepsilon$ in $|\mathbb{K}^\times|$ such that $|g_\nu(x,y)| < |g_n(x,y)|$ for all $(x,y) \in U_{\varepsilon, \rho}$ and all $\nu = 0, \ldots, n-1$ and $\rho \geq \varepsilon$. Especially, for any $r \in \mathbb{Q}$ with $n_1 \cdot r \in \mathbb{Z}$ and
Let $n_2 \cdot r \in \mathbb{Z}$, the function $g^r$ is well-defined and affinoid on $U_{\varepsilon, \rho}$ for all $\rho \geq \varepsilon$.

**Proof.** (a) Since the residue field of $\mathbb{K}$ has characteristic 0, the absolute value $|\ell(\nu)| = 1$ is equal to 1. Therefore the series converges on $U_{\varepsilon, \rho}$ for all $\rho \geq \varepsilon$. (b) For all monomials $X^\nu Y^\mu$ of $g_\nu$ with $\nu < n$, we have $|x^\nu y^\mu| \leq |x^n y^n|$ if $(x, y) \in U_{\varepsilon, \rho}$ and $\varepsilon > 1$. If we now choose $\varepsilon \geq |c_{\nu_1, \nu_2}|$ for all the coefficients $c_{\nu_1, \nu_2}$ of $g_\nu$ for all $\nu = 0, \ldots, n - 1$, then the assertion follows by (a).

For the last assertion, note that $(X^{n_1}Y^{n_2})^r = X^{m_1}Y^{m_2}$, where $n_1 \cdot r = m_1$ and $n_2 \cdot r = m_2$ with $m_1, m_2 \in \mathbb{Z}$. Then it follows from (a). $\square$

**Proposition 2.2.** Let $(f, g)$ be a Jacobian couple of polynomials with homogenous decompositions

$$f = X^{m_1}Y^{m_2} + \sum_{\mu=0}^{m-1} f_\mu \quad \text{and} \quad g = X^{n_1}Y^{n_2} + \sum_{\nu=0}^{n-1} g_\nu$$

in $\mathbb{K}[X,Y]$ with $n_1 > 0$, $n_2 > 0$, where $m := m_1 + m_2$ and $n := n_1 + n_2$.

If we apply the division algorithm of 1.6 and 1.7 to $f$ and set $v := \sum_{\nu=0}^{n-1} g_\nu n^{-1}$, then there exists an $\varepsilon \in [\mathbb{K}^\times]$ such that the formal series $G$ defined in 1.7 converges on every affinoid domain $U_{\varepsilon, \rho}$ for all $\rho \geq \varepsilon$ and gives rise to an affinoid function there.

After a possible enlarging of $\varepsilon$, the function $(f - G)$ has the form

$$(f - G)_{U_{\varepsilon, \rho}} = eX^{1-n_1}Y^{1-n_2}(1 + u)$$

with $e \in \mathbb{K}^\times$, $\deg(u) < 0$, is affinoid on each $U_{\varepsilon, \rho}$, and satisfies $|u|_{U_{\varepsilon, \rho}} < 1$.

**Proof.** The claim follows from Lemma 2.1. $\square$

In the following, we will compute the cardinality of the fibers of $(f - G, g)$ on $U_{\varepsilon, \rho}$.

**Proposition 2.3.** Let $(f, g)$ be a Jacobian couple as in 2.2. Thus we have the map

$$(f - G, g) := (eX^{1-n_1}Y^{1-n_2}(1 + u), X^{n_1}Y^{n_2}(1 + v)) : U_{\varepsilon, \rho} \longrightarrow \mathbb{K}^2.$$

Set $k := \gcd(n_1, n_2)$. Then, for any domain $V := U_{\varepsilon', \rho'} \subset U_{\varepsilon, \rho}$, the fibers of the morphism $(f - G, g^{1/k}) |_V$ consist of exactly $|n_1 - n_2| / k$ points. The fibers of $(f, g) |_V$ consist of exactly $|n_1 - n_2|$ points.

**Proof.** We abbreviate

$$\Psi := (\psi_1, \psi_2) := (f - G, g^{1/k}) |_V.$$

Since $|u|_V < 1$ and $|v|_V < 1$, the map $\Psi$ gives rise to a map

$$|\Psi| := (|\psi_1|, |\psi_2|) : |V| := \{(|x|, |y|) \in [\mathbb{K}^\times]^2 ; (x, y) \in V \} \longrightarrow [\mathbb{K}^\times] \times [\mathbb{K}^\times],$$

$$(|x|, |y|) \mapsto \left(|e| \cdot |x|^{1-n_1}|y|^{1-n_2}, |x|^{n_1/k}|y|^{n_2/k}\right).$$

Due to the construction, all numbers $k \cdot r_\sigma$ are integers by 1.6 since $r_\sigma \cdot n_1 \in \mathbb{Z}$ and $r_\sigma \cdot n_2 \in \mathbb{Z}$. Obviously, this map is injective. So, for every $(x_0, y_0) \in V$, the map $\Psi$ induces a mapping
\[ \Psi_{(x_0,y_0)} : \{(x,y) \in V : |x| = |x_0|, |y| = |y_0|\} \]
\[ \longrightarrow \{ (x,y) \in \mathbb{K}^2 : |x| = |\psi_1(x_0)|, |y| = |\psi_2(y_0)| \} . \]

Next we will compute the cardinality of the fibers of \( \Psi_{(x_0,y_0)} \). After adjusting the radii \( |x_0| \) and \( |y_0| \) to 1 and the constant \( e \) to 1, we are concerned with a morphism of type \( \Phi : W \longrightarrow \mathcal{W} \) with

\[ W := \{ (x,y) \in \mathbb{K} \times \mathbb{K} : |x| = 1, |y| = 1 \} , \]

sending \( (x,y) \in W \) to \( (x^{1-n_1}y^{1-n_2}(1+u(x,y)), x^{m_1/k}, y^{m_2/k}, (1+v(x,y))^{1/k} \). The degree of this map can be calculated via its reduction. The algebra of affinoid functions on \( W \) which are bounded by 1 is given by \( \mathbb{K}^\circ(x,y,x^{-1},y^{-1}) \), where \( \mathbb{K}^\circ \) denotes the valuation ring of \( \mathbb{K} \). Denote by \( \hat{\mathbb{K}} \) the residue field of the valued field \( \mathbb{K} \) and by \( \hat{\mathbb{K}}^\times \) its multiplicative group. The reduction of \( W \) is given by the spectrum of the \( \hat{\mathbb{K}}^- \) algebra \( \hat{W} = \hat{\mathbb{K}}[\hat{x}, \hat{x}^{-1}, \hat{y}, \hat{y}^{-1}] \), where \( \hat{x} \) resp. \( \hat{y} \) is the reduction of \( x \) resp. \( y \). Since \( |u| < 1 \) and \( |v| < 1 \), the map of the reductions coincides with the mapping

\[ \hat{\Phi} : \hat{\mathbb{K}}^\times \times \hat{\mathbb{K}}^\times \longrightarrow \hat{\mathbb{K}}^\times \times \hat{\mathbb{K}}^\times, (\hat{x}, \hat{y}) \mapsto (\hat{x}^{1-n_1}\hat{y}^{1-n_2}, \hat{x}^{m_1/k}\hat{y}^{m_2/k}) . \]

The degree of this map is \( |n_1 - n_2|/k \) as claimed; cf. Lemma 2.4 below. This degree is the degree of \( \Phi \) since a finite generating system of the reduced module via \( \hat{\Phi} \) lifts to a generating system via \( \Phi \) due to the lemma of Nakayama [3, 1.2.4/6]. Linear independence is also preserved as one easily checks.

It remains to compute the cardinality of the fibers of \( (f,g)_V \). Recall from 1.7 that \( G(x,y) \) is a function of \( g(x,y) \). Therefore, we have that the fiber of \( (f-G,g^{1/k})|_V \) of a point \( (x,y) \in V \) with image \( (z_1,z_2) := (f-G,g^{1/k})(x,y) \) coincides with the fiber of \( (f,g^{1/k})|_V \) over the point \( (z_1+c_1,z_2) \) where \( c_1 := G(x,y) \) depends only on \( z_2 = g^{1/k}(x,y) \). Thus, we see that the cardinality of the fiber of \( \Psi \) coincides with that of \( (f,g^{1/k})|_V \). Therefore the cardinality of the fiber of \( (f,g)_V \) is equal to \( |n_1 - n_2| \) since \( \Phi \) is finite and étale.  

\[ \square \]

**Lemma 2.4.** Let \( k \) be a field and let \( m_1, m_2 \in \mathbb{Z} \) be non-zero and \( m_1 \neq m_2 \). Let \( x,y \) be variables and \( r \in \mathbb{Z} \). Then the extension of \( k \)-algebras

\[ k[x^{1-rm_1}y^{1-rm_2}, x^{rm_1-1}y^{rm_2-1}, x^{m_1}y^{m_2}, x^{-m_1}y^{-m_2}] \longrightarrow k[x,x^{-1},y,y^{-1}] \]

is finite flat of degree \( |m_1 - m_2| \).

**Proof.** Obviously we have

\[ k[x^{1-rm_1}y^{1-rm_2}, x^{rm_1-1}y^{rm_2-1}, x^{m_1}y^{m_2}, x^{-m_1}y^{-m_2}] = k[xy, x^{-1}y^{-1}, x^{m_1}y^{m_2}, x^{-m_1}y^{-m_2}] = k[xy, x^{-1}y^{-1}, y^{m_2-m_1}, y^{m_1-m_2}] . \]

Moreover, we have that the extension

\[ k[xy, x^{-1}y^{-1}, y^{m_2-m_1}, y^{m_1-m_2}] \longrightarrow k[xy, x^{-1}y^{-1}, y, y^{-1}] = k[x, x^{-1}, y, y^{-1}] \]

is finite flat of degree \( |m_1 - m_2| \).  

\[ \square \]
3. The contradiction. Now we have all the preparations to deduce the main result of our article.

**Proposition 3.1.** There does not exist a Jacobian couple $(f, g)$ of polynomials $f, g \in \mathbb{C}[X,Y]$ with homogenous decompositions

$$f = \sum_{\mu=0}^{m} f_\mu \in \mathbb{K}[X,Y] \quad \text{and} \quad g = \sum_{\nu=0}^{n} f_\nu \in \mathbb{K}[X,Y]$$

where $f_m = X^{m_1}Y^{m_2}$ and $g_n = X^{n_1}Y^{n_2}$ with $m_1m_2 \neq 0, n_1n_2 \neq 0$.

**Proof.** First of all we perform a field extension $\mathbb{C} \hookrightarrow \mathbb{K}$ as introduced in Section 2. It is clear that it suffices to show the assertion for the field $\mathbb{K}$. Assume that $(f, g)$ is such a couple in the $\mathbb{K}$-algebra $A$.

Assume first $m = n$. If $m + n = 2$, then we would have $m = m_1 = 1$ and $n = n_2 = 1$ without loss of generality. Obviously, that case cannot occur as a counterexample. If $m + n > 2$, then we have $m_1 = m_1$ and $m_2 = m_2$ due to 1.5. So we can replace $f$ by $h := f - g$. Due to 2.2, the leading form of the polynomial $h$ also has the shape $h_r = a \cdot X^{r_1}Y^{r_2}$ and $r := r_1 + r_2 < n$. Thus we can assume $m \neq n$. Moreover, we have $m_1 \neq n_1$ and $m_2 \neq n_2$ and $|m_1 - m_2| \neq |n_1 - n_2|$ due to 1.5. Thus we see that we can start just from the beginning with $m \neq n$.

Now we apply Proposition 2.3. So there exists a function in $A$

$$G := \sum_{\sigma=1}^{s} c_\sigma \cdot g^{r_\sigma}$$

as in 1.5 such that $h := (f - G)$ is of degree $(2 - n)$ and $h$ has a leading form of the shape

$$h_{2-n} := c_{1-n_1,1-n_2} X^{1-n_1}Y^{1-n_2}.$$ 

Note that all the monomials of $h$ have negative degree. Furthermore, there exists a domain $U := U_{\varepsilon, \rho}$ such that for any subdomain $V := U_{\varepsilon', \rho'} \subset U$ the restriction $(h, g)|_V$ has fibers with cardinality $n' = |n_1 - n_2|$, which coincides with the degree of the map $(f, g)|_V$; cf. 2.3.

If we interchange $f$ and $g$, then, after a possible shrinking of $U$, the degree of $(f, g)|_V$ would be $m' = |m_1 - m_2|$. This has to be equal to $n'$, but we have $m' \neq n'$ due to 1.5. Contradiction!  

Summarizing the arguments we obtain the main result. Indeed, by Abhyankar’s result, a counterexample would give rise to a Jacobian couple of the given shape, which cannot exist due to Proposition 3.1.

**Theorem 3.2.** Let $(f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial morphism. If the Jacobian of $(f, g)$ is constant and unequal zero, then $(f, g)$ is an isomorphism.

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