Pseudo PT Symmetric Lattice

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Abstract. We study pseudo PT symmetry for a tight binding lattice with a general form of the modulation. Using high-frequency Floquet method, we show that the critical non-Hermitian degree for the reality of the spectrum can be manipulated by varying the parameters of the modulation. We study the effect of periodical and quasi-periodical nature of the modulation on the pseudo PT symmetry.

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1 Introduction

Bender and Boettcher showed that the usual requirement of Hermiticity for the reality of the spectrum for a given Hamiltonian could be replaced by parity-time PT symmetry [1]. Because of the antilinear character of PT operator, the eigenvectors of the Hamiltonian and PT operator are not always the same. The spectrum for a PT symmetric non-Hermitian Hamiltonian is real provided that non-Hermitian degree is below than a critical number. If it is beyond the critical number, spontaneous PT symmetry breaking occurs. It implies the eigenfunctions of the Hamiltonian are no longer simultaneous eigenfunction of PT operator and consequently the energy spectrum becomes either partially or completely complex. In the past decade, complex Hamiltonian exhibiting the PT symmetry has attracted a great deal of attention. Recently, PT symmetric optical systems with balanced gain and loss has been experimentally realized [2,3,4]. Not only the reality of spectrum but also some other physical effects are also interesting for the PT symmetric optical systems. These are, for example, unconventional beam refraction and power oscillation [5,6,7], nonreciprocal Bloch oscillations [8], unidirectional invisibility [9], an additional type of Fano resonance [10], and chaos [11]. Bendix et al. studied a disordered optical system by considering a pair of N coupled dimers with two symmetrically placed impurities [12]. They noted that their system is not PT symmetric as a whole (global symmetry), but it possesses a local PdT symmetry that admits real spectrum. Recently, the concept of pseudo PT symmetry has been introduced [13,14]. The authors considered a system modulated in such a way that the corresponding Hamiltonian is neither Hermitian nor PT symmetric. Using the high-frequency Floquet analysis, they showed that the Hamiltonian admits real energy spectrum for certain values of parameters in the Hamiltonian. In [13], the authors studied periodically modulated two channel optical coupler with balanced gain and loss. In this study, we will study pseudo PT symmetry for a tight binding lattice with a general form of the modulation. The tight binding description of PT symmetric lattice has attracted great attention in recent years [15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30] and is a good model to study pseudo PT symmetry. We consider periodical and quasi-periodical modulation and study their effects on pseudo PT symmetry.

2 Model

Consider an array of N-site modulated tight binding lattice with position dependent gain/loss and tunneling amplitude through which particles are transferred from site to site. The system is described by the following Hamiltonian in the tight binding regime

\[ H = - \sum_{n=1}^{N} T_n (|n><n+1| + |n+1><n|) \]

\[ + \sum_{n=1}^{N} (f(z)n + i \gamma_n) |n><n| \]  

where \( T_n \) is position dependent tunneling amplitude, \( f(z) \) is the real valued z-dependent potential gradient, and the complex refractive index related coefficient, \( \gamma_n \), is non-Hermitian degree describing the strength of gain/loss material that is assumed to be balanced, i.e., \( \sum_{n=1}^{N} \gamma_n = 0 \).

To study pseudo PT symmetry, consider the following
general form for the potential gradient
\[
f(z) = \omega_0 \left( l + \sum_i \kappa_i \cos(\beta_i \omega_0 z + \phi_i) \right),
\]
where \( l \neq 0 \) is an integer, \( \omega_0 \) is the modulation frequency, \( \beta_i \) are arbitrary real numbers, \( \phi_i \) are the initial phases and the constants \( \omega_0 \kappa_i \) are the amplitudes of the oscillating term (ac-like term). As can be seen, the potential gradient function is composed of dc-like plus ac-like terms. Apparently, dc-like term breaks the \( \mathcal{PT} \) symmetry of the Hamiltonian. So, we expect that the system has zero threshold where \( \phi = \omega \neq 0 \). Even in the absence of the dc-like term, the ac-like term also leads to the broken \( \mathcal{PT} \) symmetry when the relative phase between any two component, \( \phi_i - \phi_j \), is neither zero nor \( \pi \). Below we show that the pseudo \( \mathcal{PT} \) symmetry appears effectively for our non-\( \mathcal{PT} \) symmetric system.

Let us investigate the Hamiltonian \( \mathcal{H} \) in more detail. A common way in the studies of such systems is to find a \( z \)-independent effective Hamiltonian. It is well known that the tunneling parameter is replaced by an effective tunneling amplitude, \( T_n^\text{eff} \), in the high-frequency domain. With application of the high-frequency Floquet approach, the Hamiltonian \( \mathcal{H} \) can then effectively be described as
\[
H_{\text{eff}} = - \sum_{n=1}^{N} T_n^{\text{eff}} |n < n + 1| + T_n^{\text{eff},*} |n + 1 > n| + i \sum_{n=1}^{N} \gamma_n |n > n|
\]
where star denotes the complex conjugate and the effective tunneling is given by
\[
\frac{T_n^{\text{eff}}}{T_n} = \int_{0}^{z} e^{\eta} dz'.
\]
where overline denotes the average over \( z \) and \( \eta \) is given by \( \eta(z) = \int_{0}^{z} f(z')dz' \). Here, it is useful expand the oscillatory term \( e^{\eta} \) in terms of Bessel functions by using the Jacobi-Anger expansion; \( e^{i \kappa \sin(x)} = \sum_{m} J_m(\kappa) e^{imx} \), where \( J_m \) is the \( m \)-th order Bessel function of first kind.

The \( \mathcal{PT} \) symmetry breaking modulation term in the original Hamiltonian \( \mathcal{H} \) doesn’t exist in the effective Hamiltonian. The system is pseudo \( \mathcal{PT} \) symmetric if not the original Hamiltonian but the effective Hamiltonian is \( \mathcal{PT} \) symmetric. Apparently, the effective Hamiltonian is \( \mathcal{PT} \) symmetric if a precise relation between \( T_n \) and \( \gamma_n \) hold. Below, we will obtain effective tunneling amplitude for some specific cases and discuss the physical results.

### 2.1 Monochromatic Modulation

Consider first monochromatic case
\[
f(z) = \omega_0 \left( l + \kappa \cos(\omega_0 z + \phi) \right).
\]

Let us find the corresponding effective tunneling amplitude. Substitute the Jacobi-Anger expansion into the integration of the equation \( (4) \). Then we get
\[
\int_{0}^{z} e^{\eta} dz' = \sum_{m} J_m(\kappa) e^{im\phi} S_m
\]
where \( S_m = \int_{0}^{z} e^{i(m+1)\omega_0 z'} dz' \). If we take the time average of \( S_m \), i.e. \( \bar{S}_m = \delta_{-l,m} \), we get the effective tunneling expression
\[
\frac{T_n^{\text{eff}}}{T_n} = J_{-l}(\kappa) e^{-il\phi}
\]

We conclude that the presence of the monochromatic modulation corresponds to a modification of the tunneling amplitude. The Bessel function \( J_{-l}(\kappa) \) is roughly like an oscillating sine function that decay proportionally as \( \kappa \) increases. In the absence of ac-like term, \( \kappa = 0 \), the Bessel function is always zero, \( J_{-l}(0) = 0 \). Therefore, the effective tunneling is suppressed and the spectrum is complex. This result is a direct consequence of the broken \( \mathcal{PT} \) symmetry that occurs when \( l \) changes from zero to nonzero. The tunneling is partially restored and the system enters the pseudo \( \mathcal{PT} \) symmetric phase with the additional application of ac-like term. In the pseudo \( \mathcal{PT} \) symmetric region, the spectrum is real as long as \( \gamma < \gamma_{PT} \), where the critical point depends on the effective tunneling amplitude. The critical point becomes zero whenever \( \kappa \) is a root of Bessel function of order \( l \). So, we say that the pseudo \( \mathcal{PT} \) symmetry is spontaneously broken at such values of \( \kappa \).

To gain more insight, let us find the energy eigenvalues by

\[\text{Fig. 1. The parametric region for the reality of the spectrum for the parameters } \gamma^2/T^2 \text{ and } \kappa \text{ for } l = 1. \text{ The shaded (unshaded) region corresponds to parameters with complex (real) energy eigenvalues. The } \mathcal{PT} \text{ symmetry is broken when } l \text{ changes from zero to nonzero. Value of } \kappa = 0 \text{, the spectrum is complex because of the broken } \mathcal{PT} \text{ symmetry. The presence of ac-like term gives rise to the pseudo } \mathcal{PT} \text{ symmetry that guarantees the reality of the spectrum in a broad range of parameters.} \]
specifying position dependent tunneling amplitude. Analytically solvable systems are particularly interesting in the study of non-Hermitian systems. Without loss of generality, suppose that the system has two sites, \( N = 2 \) (dimer) with two impurities \( \mp i \gamma \) and \( T_n = T \). Therefore, the corresponding energy eigenvalues are given by

\[
E = \mp \sqrt{|T \mathcal{J}(\kappa)|^2 - \gamma^2}. \tag{8}
\]

Obviously, the parameters \( l \) and \( \kappa \) play vital roles on the reality of the spectrum. The energy eigenvalues are real in a broad range of the coupling strength \( \kappa \) for fixed \( l \). In the parametric region for the reality of the spectrum is shown as a function of the ratio \( \gamma^2/\kappa^2 \) and \( \kappa \) at \( l = 1 \). The dark and bright region correspond to complex and real spectrum, respectively. The boundary between the dark and bright regions determines the critical point, \( \gamma_{PT} \). Note that the Bessel function is always smaller than one, so the non-Hermitian degree must satisfy the condition \( \gamma < T \) for the reality of the spectrum. As can be seen from the figure, the critical point is zero, \( \gamma_{PT} = 0 \), at \( \kappa = 0 \) and hence the energy eigenvalues are complex. The harmonic modulation enhances the critical point, \( \gamma_{PT} \). In other words, the spectrum becomes real unless \( \gamma < \gamma_{PT} \) although the Hamiltonian is not symmetric under \( \mathcal{PT} \) transformation. This phase is called pseudo \( \mathcal{PT} \) symmetric phase \([13]\).

Consider another exactly solvable one-dimensional array with \( N = 3 \), called trimer. Suppose the non-Hermitian degree of the first and second sites are \( \gamma \) and \( s \gamma \), respectively, where \( s \) is an arbitrary real valued constant. Since the gain and loss in the system is balanced, the non-Hermitian degree of the third one is \(- (1 + s) \gamma\). Suppose \( T_1 \) and \( T_2 \) denote the tunneling amplitude between the first and second sites and the second and third sites, respectively. The corresponding energy eigenvalues satisfy the following equation

\[
E^3 - aE - ib = 0 \tag{9}
\]

where the coefficients

\[
a = (T^2_1 + T^2_2)\mathcal{J}(\kappa) - \gamma^2(1 + s + s^2) \tag{10}
\]

\[
b = \gamma(\gamma^2 s(1 + s) + (T^2_1(1 + s) - T^2_2)\mathcal{J}(\kappa)) \tag{11}
\]

Equation (9) allows us to study the reality of the energy eigenvalues. We demand that the coefficient \( b \) vanishes (a necessary condition for the reality of the spectrum). In this case, the three energy eigenvalues are given by \( E = 0 \), \( E = \mp \sqrt{a} \). Therefore the spectrum is entirely real if \( a \geq 0 \). The energy eigenvalues are degenerate at \( a = 0 \) and symmetric with respect to zero energy value when \( a > 0 \). The necessary condition \( b = 0 \) is satisfied if either one of the following relations is satisfied

\[
s = 0; T_1 = T_2 = T \tag{12}
\]

\[
\gamma = \gamma_{PT} = \mp |\mathcal{J}(\kappa)|\sqrt{\frac{T^2_2 - T^2_1(1 + s)}{s(1 + s)}} \tag{13}
\]

where we assume \( T^2_2 > T^2_1 s(1 + s) \) for the latter relation.

In the former case, \( b \) vanishes at any \( \gamma \). However increasing \( \gamma \) switches the coefficient \( a \) to a negative value at

\[
\gamma = \gamma_{PT} = \gamma \tag{14}
\]

and hence the energy spectrum becomes complex. The critical point \( \gamma_{PT} \) equals to \( \sqrt{2T\mathcal{J}(\kappa)} \) and the energy eigenvalues are given by

\[
E = 0, \quad E = \mp \sqrt{2T^2\mathcal{J}(\kappa)} - \gamma^2 \tag{15}
\]

In the latter case \([13]\), the constant \( b \) vanishes if \( \gamma = \gamma_{PT} \). It is interesting to observe that these are two distinct values that admit real spectrum. The corresponding energy eigenvalues are given by

\[
E = 0, \quad E = \mp \sqrt{T^2_1 + T^2_2 + \frac{T^2_2(1 + s) - T^2_1}{s(1 + s)}}(1 + s + s^2) \tag{16}
\]

The term in the square root must be greater or equal than zero for the reality of the spectrum. Comparing the energy expressions \([14, 15]\), we see that the parameters \( l \) and \( \kappa \) have nothing to do with the reality of the energy spectrum for the latter case. Instead, they change the impurity strength \( \gamma_{PT} \) for which the spectrum is real.

So far, we have studied the cases with \( N = 2 \) and \( N = 3 \). If the system has many sites, one can perform numerical computation to find the energy eigenvalues. We perform numerical calculation when \( N = 16 \) by supposing the impurities alternate at each site, \( \gamma_n = (-1)^n \gamma \) and \( T_n = 1 \). The Fig.(2) plots the imaginary part of energy eigenvalues at \( l = 1 \) and \( \gamma = 0.1 \). The spectrum is real for certain range of the parameter \( \kappa \) because of the pseudo \( \mathcal{PT} \) symmetric phase.

### 2.2 Bichromatic Modulation

Let us now consider a bichromatic modulation

\[
f(z) = \omega_0 (l + \kappa_1 \cos(\omega_0 z + \phi_1) + \kappa_2 \cos(\omega_0 z + \phi_2)) \tag{16}
\]
The bichromatic modulation is periodic if $\beta$ is a rational number and quasiperiodic if it is an irrational number. Note that $\beta$ can be given with a finite number of digits in a real experiment ($\beta = p/q$, where $p$, $q$ are two coprime positive integers). If $p$ and $q$ are sufficiently large, then the system becomes effectively quasi-periodic on the scale of an experiment. Let us find the corresponding effective tunneling amplitude. If we substitute the Jacobi-Anger expansion into the equation (17), we get

$$
\int_0^\infty e^{i n z'} dz' = \sum_{n,m} J_n(\kappa_1) J_m(\kappa_2/\beta) e^{i m \phi_1 + i m \phi_2} S_{n,m} \tag{17}
$$

where $S_{n,m} = \int_0^\infty e^{i (l+n+m) \omega_0 z'} dz'$. Substitution of the time average $S_{n,m} = \delta_{l-m,0} m$ into the equation (17) gives the effective tunneling

$$
T_n^{eff} = e^{-i l \phi_1} \sum_{m} e^{i m (\phi_2 - \beta \phi_1)} J_{l-m} (\kappa_1) J_m(\kappa_2/\beta) \tag{18}
$$

where the phase $-l \phi_1$ affects the effective tunneling as a whole. There are infinitely many terms in the summation. However, only a few terms are practically important for small values of $\kappa$ since the Bessel functions decrease with $m$. $J_{n-1}(\kappa_1)/J_n(\kappa_1)$.

The effective tunneling expressions (18) show significant difference between the two cases. Contrary to the monochromatic modulation, varying the initial phases $\phi_1$ and $\phi_2$ change absolute value of the effective tunneling for the bichromatic case. Therefore, it is possible to observe the spontaneous pseudo $PT$ symmetry breaking not only by changing $\kappa_1, \kappa_2$ but also by changing the initial phases at fixed $l$ and $\gamma/\Gamma$. Note that the role of the initial phases depends sensitively on the periodical nature of bichromatic modulation. If the bichromatic modulation is quasi-periodic, then only the term $J_{l-m}(\kappa_1)$ with $m = 0$ in the summation (18) survives. Hence, the absolute value of the effective tunneling for an irrational value of $\beta$ is simplified to

$$
\frac{|T_n^{eff}|}{T_n} = J_{l}(\kappa_1) J_0(\kappa_2/\beta) \tag{19}
$$

We conclude that the effect of the second ac-like term with coupling $\kappa_2$ is the reduced effective tunneling. The tunneling is lost whenever either $\kappa_1$ or $\kappa_2/\beta$ are the roots of the Bessel function of order $l$ and zero, respectively. If the bichromatic modulation is periodical, then the effective tunneling changes with the initial phases. Therefore, the reality of the spectrum can be controlled by the initial phases $\phi_1$ and $\phi_2$. The effective tunneling amplitude is not in general real. The complex effective tunneling (18) can be rewritten as $T_n^{eff} = |T_n^{eff}| e^{i \theta}$, where $\theta$ is known as the Peierls phase. Let us now study the effect of a Peierls phase on the energy spectrum. Assume that our system has periodic boundary conditions. Therefore a momentum representation is useful to study the system. Suppose that tunneling amplitude changes with the lattice number $n$ such that $T_n = T$ when $n$ is odd and $T_n = cT$ when $n$ is even. The band structure of this system with $\gamma_n = (1)^n \gamma$ over the Brillouin zone reads

$$
E = \pm \sqrt{((c-1)^2 + 4c \cos^2(\frac{q - \Theta}{2}) ) |T^{eff}|^2 - \gamma^2} \tag{20}
$$

where the effective tunneling amplitude can be found using (18). Observe that if the tunneling amplitude is constant over the whole lattice, i.e. $c = 1$, then the spectrum would be complex at any $\gamma$ since the cosine term vanishes at some particular values. In contrast, it can be seen, the effect of the Peierls phase is to shift the minimum of the band structure to a $q_{min} = \Theta$. However, the Peierls phase has nothing to do with the reality of the spectrum and the critical point, $\gamma_{PT} = (c-1) T^{eff}$. We have seen that the effect of quasi periodical modulation is to reduce the effective tunneling amplitude (19). A question arises. Does the pseudo $PT$ symmetric phase appear if the periodic part of the modulation is multicolored? To answer this question, consider $f(z) = \omega_0 (l + \sum_{n=1}^{\infty} \kappa_n \cos(n \omega_0 z + \phi_n)) + \kappa \cos(\beta \omega_0 z + \phi)$, where the first term is the Fourier series expansion of a periodic function and $\beta$ is an irrational number such that $f(z)$ is quasi-periodic. One can study a very large family of potential gradient function using the Fourier series expansion method. If we use the Jacobi-Anger expression and evaluate the integral (14), we get the absolute value of the effective tunneling amplitude

$$
|T_n^{eff}| / T_n = J_0(\kappa_1/\beta) \prod_{m=1} J_{l-m}(\kappa_m/\beta) \tag{21}
$$

where $m$ is the index of multiplication in the product symbol. Recall that the Bessel function of order $l$ is always less than one. If there are infinitely many terms in the product, the effective tunneling is zero independent of $\kappa_m$. As a result, we conclude that the pseudo $PT$ symmetric phase disappears if the quasi periodical modulation is polychromatic.

3 Conclusion

The idea of pseudo $PT$ symmetry has recently been introduced [13]. A pseudo $PT$ symmetric Hamiltonian has no $PT$ symmetry but can be transformed to a $PT$ symmetric Hamiltonian using the high-frequency Floquet approach. In this paper, we have studied pseudo $PT$ symmetry for a modulated tight binding lattice with balanced gain and loss. The Hamiltonian (1), which is neither Hermitian nor $PT$ symmetric, has been shown to admit real spectrum in a broad range of parameters. As for a $PT$ symmetric Hamiltonian, the spectrum for a pseudo $PT$ symmetric is real unless the non-Hermitian degree is below a critical number. We have shown that the critical non-Hermitian degree can be manipulated by varying the parameters of the modulation. We have analytically found
the critical number for a dimer and trimer and performed numerical calculation for a lattice with many sites. We have also investigated the effect of periodical and quasi-periodical modulation on the reality of the spectrum.

References

1. C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80, 5243 (1998).
2. C. E. Ruter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, Nat. Phys. 6, 192 (2010).
3. A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, Phys. Rev. Lett. 103, 093902 (2009).
4. A. Regensburger, C. Bersch, M. A. Miri, G. Onishchukov, D. N. Christodoulides, and U. Peschel, Nature (London) 488, 167 (2012).
5. C. E. Ruter et al., Nat. Phys. 6, 192 (2010).
6. K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z. H. Musslimani, Phys. Rev. Lett. 100, 103904 (2008).
7. J. Ctyroky, V. Kuzmiak, and S. Eyderman, Optics Express 18, 21585 (2010).
8. S. Longhi, Phys. Rev. Lett. 103, 123601 (2009).
9. Z. Lin, H. Ramezani, T. Eichelkraut, T. Kottos, H. Cao, and D. N. Christodoulides, Phys. Rev. Lett. 106, 213901 (2011).
10. A. E. Miroshnichenko, B. A. Malomed, and Yu. S. Kivshar, Phys. Rev. A 84, 012123 (2011).
11. C.T. West, T. Kottos, and T. Prosen, Phys. Rev. Lett. 104, 054102 (2010).
12. O. Bendix, R. Fleischmann, T. Kottos and B. Shapiro, J. Phys. A: Math. Theor. 43 265305 (2010).
13. X. Luo, J. Huang, H. Zhong, X. Qin, Q. Xie, Y. S. Kivshar, Chaohong Lee, Phys. Rev. Lett. 110, 243902 (2013).
14. X. Lian, H. Zhong, Q. Xie, X. Zhou, Y. Wu and W. Liao, Eur. Phys. J. D 68 189 (2014).
15. Andrey A. Sukhorukov, Sergey V. Dmitriev, Sergey V. Suchkov, and Yuri S. Kivshar, Opt. Lett. 37, 2148 (2012).
16. O. Bendix, R. Fleischmann, T. Kottos, B. Shapiro, Phys.Rev.Lett. 103, 030402, (2009).
17. Stefano Longhi, Phys. Rev. A 88 052102 (2013).
18. C. Yuce, Phys. Lett. A 378, 2024 (2014).
19. Y. N. Joglekar, D. Scott, M. Babbey, A. Saxena, Phys. Rev. A 82, 030103(R) (2010).
20. M. C. Zheng, D. N. Christodoulides, R. Fleischmann, and T. Kottos, Phys. Rev. A 82, 010103R (2010).
21. W. H. Hu, L. Jin, Y. Li, and Z. Song, Phys. Rev. A 86, 042110 (2012).
22. X. Z. Zhang, L. Jin, and Z. Song, Phys. Rev. A 85, 012106 (2012).
23. L. Jin and Z. Song, Phys. Rev. A 80, 052107 (2009).
24. S. Kalish, Z. Lin, and T. Kottos, Phys. Rev. A 85, 055802 (2012).
25. H. Vemuri, V. Vavilala, T. Bhamidipati, and Y. N. Joglekar, Phys. Rev. A 84, 043826 (2011).
26. C. T. West, T. Kottos, and T. Prosen, Phys. Rev. Lett. 104, 054102 (2010).
27. J. Wu and X. T. Xie, Phys. Rev. A 86, 032112 (2012).
28. S. Longhi, Phys. Rev. B 80, 235102 (2009).
29. R. El-Ganainy, K. G. Makris, and D. N. Christodoulides, Phys. Rev. A 86, 033813 (2012).
30. G. DellaValle and S. Longhi, Phys. Rev. A 87 022119 (2013).
31. A. Eckardt, C. Weiss, and M. Holthaus, Phys. Rev. Lett. 95, 260404 (2005).
32. A. Hemmerich, Phys. Rev. A 81, 063626 (2010).
33. C. Yuce, Europhys. Lett. 103, 30011 (2013).
34. M. C. Zheng, D. N. Christodoulides, R. Fleischmann, and T. Kottos, Phys. Rev. A 82, 010103 (2010).
35. H. Ramezani, D. N. Christodoulides, V. Kovanis, I. Vitebskiy, and T. Kottos, Phys. Rev. Lett. 109, 033902 (2012).